Information Design in Large Anonymous Games

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Abstract

We consider anonymous Bayesian cost games with a large number of players, i.e., games where each player aims at minimizing a cost function that depends on the action chosen by the player, the distribution of the other players’ actions and an unknown parameter. We study the nonatomic limit versions of these games. In particular, we introduce the concepts of correlated and Bayes correlated Wardrop equilibria, which extend the concepts of correlated and Bayes correlated equilibria to nonatomic games. We prove that (Bayes) correlated Wardrop equilibria are indeed limits of action flow distributions induced by (Bayes) correlated equilibria of the game with a large finite set of small players. For nonatomic games with complete information admitting a convex potential, we show that the set of correlated Wardrop equilibria is the set of probability distributions over Wardrop equilibria. Then, we study how to implement optimal Bayes correlated Wardrop equilibria and show that in games with a convex potential, every Bayes correlated Wardrop equilibrium can be fully implemented.

Keywords: Bayes correlated Wardrop equilibrium, Bayesian Wardrop equilibrium, Bayesian persuasion, congestion games, information design, nonatomic games, potential games, selfish routing.

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1 Introduction

1.1 The problem

In many economic environments with a large number of participants, single agents are impacted only by their own choice, the distribution of actions in the population, and the state of the world. For instance, the time a morning commuter spends on the road to go from home to office depends on the chosen route, the number of commuters on the different roads, and the presence of accidents; the benefit of adopting a new technological standard or subscribing to a social network depends on the quality of the standard or network, and on the proportions of adopters or subscribers; the impact of deciding to self isolate or to get vaccinated depends on the number of people who respect lockdown measures or are vaccinated, on the efficiency of vaccines, and on the dangerousness of the virus. Such situations can be represented by anonymous games where the utility that a player enjoys, or the cost that a player incurs, depends on this player’s action and on the distribution of other players’ actions. A widely studied class of games of this type is given by congestion games, where players use resources whose cost increases with the number of users. Prominent applications are routing games, where players travel through a network of roads with the objective of reaching destination as fast as possible. When the number of players is large and each individual has a small impact on the overall distribution, these games can be approximated by nonatomic games, which are often more tractable (see, e.g., Roughgarden, 2007).

When deciding which road to take, commuters rely on public and private information about traffic conditions provided by radio channels or apps such as Google Maps or Waze. Providing information helps the individual driver to avoid traffic jams and may reduce the global congestion. Yet, some paradoxical phenomena of negative information value can occur. For example, in some road networks, if all players discover the existence of a fast shortcut, too many of them may take it, thereby increasing the total congestion as in the Braess paradox. It is thus important to carefully study the design of information structures and the efficiency of the induced equilibria.

In this paper, we consider a model of anonymous nonatomic Bayesian games. These games are to be interpreted as a suitable limit of anonymous games with a large number of players who choose actions from a finite set. Each agent incurs a cost depending on the individual action, on the action distribution in the population, and on an a priori unknown state parameter. A prominent instance are Bayesian routing games and, in line with the literature on this topic, we take the convention that players minimize cost functions, which simply are negatives of utility functions. Yet, our class of games is much larger than routing games or congestion games and encompasses population games, as described in Sandholm (2010). These include some oligopoly games and random matching games. We extend this class of games to the Bayesian setting.

Our main focus is a designer who can choose how to correlate players’ information to the
state of the world, with the goal of minimizing a cost function, such as the total cost across players. We aim at studying how this designer’s objective can be achieved through information design. For instance, is it optimal to provide full information, public information, conditionally independent signals, or signals which are partially correlated across players and with the state? To answer such questions, we take the point of view of Bergemann and Morris (2016) who consider all possible equilibrium outcomes for all possible information structures, in the context of Bayesian games with finitely many players. This approach immediately faces the daunting task of describing all information structures for a continuum of players. To circumvent this problem, we adopt the point of view of action flows and Wardrop equilibria (called Nash equilibria of population games in Sandholm, 2010) where strategy profiles mapping players to actions are not explicitly defined. Instead, we consider the distributions of action flows in the population that result from equilibrium behavior.

1.2 Our contributions

We study two solution concepts, which we call correlated Wardrop equilibrium (CWE) and Bayes correlated Wardrop equilibrium (BCWE). They are the analogs of correlated equilibrium (Aumann, 1974) and Bayes correlated equilibrium (Bergemann and Morris, 2016) and are expressed in terms of probability distributions over action flows (conditional on the state), instead of probability distributions over action profiles. We also introduce the concept of Bayes deterministic Wardrop equilibrium, which corresponds to a Bayes correlated Wardrop equilibrium with deterministic flows: it assigns a flow over actions to each state of the world. Under complete information, a Bayes deterministic Wardrop equilibrium (BDWE) is a Wardrop equilibrium (WE). Our first results motivate the model of nonatomic games as a limit of games with large but finite sets of players with small weights. We show that every converging sequence of Bayes correlated equilibrium outcomes of finite $n$-player games, converges to a BCWE when $n$ tends to infinity (Proposition 1). Conversely, we show that for every BCWE, there exists a sequence of (approximate) Bayes correlated equilibrium outcomes that converges to that BCWE (Proposition 2). Furthermore, we prove that Bayes deterministic Wardrop equilibria correspond to limits of Bayes correlated equilibrium outcomes with conditionally independent signals (Proposition 3).

Next, we consider games with complete information that admit a potential: the cost functions are the partial derivatives of the potential function. Our main result in this part is that when the potential is convex, every CWE is a mixture of WE, and all WE have the same cost profiles (Proposition 4). As a consequence, all CWE have the same cost profiles as well. This result applies in particular to congestion games with increasing resource costs, which admit a convex

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1We are not the first ones studying nonatomic versions of correlated and Bayes correlated equilibria. Closely related notions appear under different names in similar settings (see, e.g., Díaz, Mitsche, Rustagi, and Saia, 2009; Tavafoghi and Tenekeetzis, 2018; Zhu and Savla, 2020, see also Section 1.3).
potential. The implication is that there is no use for information design in games with complete information with a convex potential. Combined with our convergence result, this result implies that all flow distributions of correlated equilibria in games with finitely many players, converge to Wardrop equilibria (Corollary 1). These results do not hold without a convex potential (see Example 1).

Then, we study how to implement a designer-optimal BCWE as a Bayesian Wardrop equilibrium with an appropriate information structure (as defined in Wu, Amin, and Ozdaglar, 2021). We prove that an optimal BCWE with finite support exists and we bound its cardinality (Proposition 5). Except under specific conditions, the optimal BCWE is not always a convex combination of BDWE, even if the game has a convex potential in each state. We also prove a version of the revelation principle for nonatomic Bayesian games and show that if a game admits a (strictly) convex potential in each state, then, for every information structure, Bayesian Wardrop equilibrium flows (costs) are unique (Proposition 10). Therefore, an optimal BCWE can be fully implemented (Corollary 2).

We illustrate our concepts and compute the optimal solution for the designer in several examples and in a class of binary-action and binary-state congestion games. In this class of games, we emphasize the differences between optimal public information structures, optimal BDWE and optimal BCWE, depending on the designer’s objective and players’ cost functions.

1.3 Related literature

First, our paper is related to nonatomic games, among them notably congestion and routing games. Congestion games with a finite number of players are introduced by Rosenthal (1973), their relations with potential games are studied by Monderer and Shapley (1996). Wardrop (1952) studies a strategic model of traffic where each agent has a negligible weight and introduces the principle now known as Wardrop equilibrium. Beckmann, McGuire, and Winsten (1956) characterize Wardrop equilibria as the solution of a convex optimization program. The idea that equilibria of routing games are inefficient goes back to Pigou (1920). Measuring this inefficiency through the price of anarchy (Koutsoupias and Papadimitriou, 1999) is the object of a huge literature in algorithmic game theory (see, e.g., Roughgarden, 2007). The works of Sandholm (2001, 2010) give a general model of games with nonatomic players, beyond congestion games.

Related to our convergence results, the relationships between Nash equilibria and Wardrop equilibria in congestion games are studied by Haurie and Marcotte (1985) in the atomic splittable case, where the finitely many players can split their weights among several routes, and by Cominetti, Scarsini, Schröder, and Stier-Moses (2022) in the atomic nonsplittable case. Sandholm (2001) studies the convergence of the potential of finite potential games to the potential of nonatomic potential games.
Next, our paper is related to information design. The introduction of correlated equilibria by Aumann (1974, 1987) can be seen as a first seminal step in this field. Properties of the set of correlated equilibria in both finite and infinite games are studied by Hart and Schmeidler (1989). Various extensions of correlated equilibria to incomplete information are analyzed by Myerson (1982), Forges (1993, 2006) and Bergemann and Morris (2016). Information design for a single player is also known as Bayesian persuasion (Kamenica and Gentzkow, 2011). Several papers consider extensions of Bayesian persuasion to multiple decision makers, among them Arieli and Babichenko (2019), and Mathevet, Perego, and Taneva (2020). Full implementation via information design has been recently analyzed by, e.g., Mathevet et al. (2020) and Morris, Oyama, and Takahashi (2020). This literature is surveyed in Bergemann and Morris (2019), Kamenica (2019) and Forges (2020). Dughmi (2017) surveys the algorithmic literature on information design.

The following papers are related to the difference between correlated equilibria and Nash equilibrium outcomes. Neyman (1997) proves that in games with a finite number of players, convex strategy sets, and a convex potential, every correlated equilibrium is a mixture of Nash equilibria. Ashlagi, Monderer, and Tennenholtz (2008) introduce the mediation value—i.e., the ratio between the maximal welfare in a correlated equilibrium to the maximal welfare in a mixed-strategy equilibrium—and show that the mediation value can be arbitrarily high in congestion games with finitely many players. Relatedly, Roughgarden (2015) studies conditions under which bounds for the price of anarchy extend from pure to mixed and correlated equilibria. A result closely related to our Proposition 4 is in Díaz et al. (2009) who show that correlation does not decrease total cost compared to Nash equilibria in congestion games with parallel edges and nonatomic players.³ This result does not extend beyond congestion games as shown by the El Farol example given in Díaz et al. (2009) and Mitsche, Saad, and Saia (2013), which we recall in Example 1.

There is a small literature on information design for congestion games with an unknown state. Arnott, de Palma, and Lindsey (1991) study the relevance of full or partial information in a simple routing model with a finite number of agents who choose one of two parallel routes and their departing time. Vasserman, Feldman, and Hassidim (2015) consider a nonatomic Bayesian routing game with \( m \) parallel edges and incomplete information on the cost functions. The mediation ratio is computed for various classes of cost functions. Bhaskar, Cheng, Ko, and Swamy (2016) consider hardness results for Bayesian routing games where a principal uses partial information revelation to minimize the average latency of the equilibrium flow. Castiglioni, Celli, Marchesi, and Gatti (2021) study efficient computation of optimal signaling schemes for Bayesian

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²Neyman (1997) considers players who maximize utilities in a game with a concave potential. For the equivalent game where players minimize costs, the potential is convex.

³Díaz et al. (2009) use a definition of correlated equilibrium which is equivalent to a correlated Wardrop equilibrium with finite support.
atomic network congestion games. Das, Kamenica, and Mirka (2017) study optimal information design in some examples of nonatomic routing games in which i.i.d. signals are optimal. Zhu and Savla (2020) study the computational issues of finding the optimal information structure and prove that i.i.d. signals are optimal for routing games with two routes and affine costs. As Proposition 3 shows, this amounts to focusing on Bayes deterministic Wardrop equilibria. This may entail loss of optimality, as we show in Example 4 and Example 6.

For our implementation results, we use the notion of Bayesian Wardrop equilibrium introduced by Wu et al. (2021) for Bayesian routing games. This concept is based on information structures where players are partitioned in a finite number of populations, all players in the same population observe the same signals, and signals are correlated across populations. Wu et al. (2021) determine the effect of population sizes on the relative value of information. Relatedly Liu, Amin, and Schwartz (2016) study Bayesian Wardrop equilibrium for two populations with heterogenous information, and Wu and Amin (2019) characterize the optimal information structure for binary states, routes and affine cost functions.

In this paper, we have chosen to abstract away from the description of the set of players as a measurable space. Instead, we have focused on action flows, as is common in the literature on congestion games (Roughgarden, 2007), and as is done in the systematic study of population games (Sandholm, 2010). There exists a huge literature on games with measurable set of players, and Hart and Schmeidler (1989) have extended the definition of correlated equilibrium to measurable sets of players, using finitely additive measures. The approach of describing general information structures for arbitrary measurable sets of players inevitably faces the technical hurdle of uncountable families of independent random variables. We have purposefully chosen to avoid those difficulties. For a recent contribution on related topics, see Hellwig (2022) and references therein.

1.4 Organization of the paper

Section 2 presents the model of Bayesian nonatomic games and defines the concept of Bayes correlated Wardrop equilibria. Section 3 shows in which sense this equilibrium concept is a reasonable approximation of Bayes correlated equilibria in games with large finite sets of players. Section 4 gives some results specific to games with complete information. Section 5 studies the design of optimal information structures for Bayesian nonatomic games, Section 6 deals with their implementation. Section 7 provides some concluding remarks.

To simplify the exposition, the model and results are presented in the main text for symmetric players. Yet, the analysis and most of the results extend to a multi-population model, as described in Appendix A, where the action sets and cost functions are population specific. The
proofs are in Appendix B. Appendix C analyzes a class of binary congestion games in details. Appendix D provides additional results and examples.

1.5 Notation

For any compact set \( \mathcal{X} \), we let \( \Delta(\mathcal{X}) \) be the set of Borel probability distributions over \( \mathcal{X} \). The symbol \( \delta_x \) denotes the Dirac mass on \( x \). Given probability distributions \( \{\mu^n\}_{n \in \mathbb{N}}, \mu \in \Delta(\mathcal{X}) \), the notation \( \mu^n \rightharpoonup^* \mu \) denotes weak* convergence of \( \mu^n \) to \( \mu \) i.e., \( \int f \, d\mu^n \to \int f \, d\mu \) for every continuous function \( f : \mathcal{X} \to \mathbb{R} \). When the set \( \mathcal{X} \) is finite, \( |\mathcal{X}| \) denotes its cardinality and \( \Delta(\mathcal{X}) \) is identified with the simplex:

\[
\Delta(\mathcal{X}) = \left\{ y \in \mathbb{R}^{|\mathcal{X}|} : \forall k \in \mathcal{X}, y_k \geq 0, \sum_{k \in \mathcal{X}} y_k = 1 \right\} .
\] (1.1)

2 Anonymous nonatomic games

We study nonatomic games with a finite number of actions, where the total mass of players is normalized to 1. The analog of a strategy profile in this context is a distribution of actions that we call a flow. Throughout we consider cost minimization games whereby players choose actions in order to minimize cost functions. Also, we focus on anonymous games where costs depend only on other players choices via the flow of actions and some state parameter in the case of incomplete information. This section formally presents the model and our solution concepts.

2.1 Complete information

A nonatomic game \( \Gamma = (\mathcal{A}, c) \) consists of a finite set of actions \( \mathcal{A} \) and a profile of continuous cost functions \( c = (c_a)_{a \in \mathcal{A}} \), where \( c_a : \mathcal{Y} \to \mathbb{R} \) for each action \( a \), and \( \mathcal{Y} := \Delta(\mathcal{A}) \) denotes the set of flows over actions. In words, the cost \( c_a \) of taking action \( a \) is a function of the flow \( y \) over the action set \( \mathcal{A} \).

Definition 1. A Wardrop equilibrium (WE) of the nonatomic game \( \Gamma \) is a flow \( y \in \mathcal{Y} \) such that, for all \( a, b \in \mathcal{A} \), we have

\[
y_a c_a(y) \leq y_a c_b(y).
\] (2.1)

A correlated Wardrop equilibrium (CWE) of the nonatomic game \( \Gamma \) is a distribution \( \mu \in \Delta(\mathcal{Y}) \) over flows such that, for all \( a, b \in \mathcal{A} \), we have

\[
\int y_a c_a(y) \, d\mu(y) \leq \int y_a c_b(y) \, d\mu(y).
\] (2.2)
The symbols $\text{WE}(\Gamma)$ and $\text{CWE}(\Gamma)$ denote the sets of Wardrop equilibria and correlated Wardrop equilibria of $\Gamma$, respectively.

The definition of WE is the usual one: only actions with the smallest cost receive positive flow. A CWE can be interpreted as follows. A mediator draws a flow vector $\mathbf{y}$ according to $\mu$. For each action $a$, a fraction $y_a$ of the population is drawn at random to play action $a$. If we divide Eq. (2.2) by the total probability $\int y_a \, d\mu(\mathbf{y})$ of action $a$, then the left-hand-side is the expected cost of playing $a$, conditionally of being recommended $a$, the right-hand-side is the expected cost of playing $b$, conditionally of being recommended $a$. In a CWE, no player has a profitable deviation from any recommended action.

### 2.2 Incomplete information

A Bayesian nonatomic game $\Gamma = (A, \Theta, p, c)$ is given by a finite set of actions $A$, a finite set of states $\Theta$, a full-support probability distribution over states $p \in \Delta(\Theta)$ and, for each $a \in A$, a continuous cost function $c_a : \mathcal{Y} \times \Theta \to \mathbb{R}$. A transition probability $\mu : \Theta \to \Delta(\mathcal{Y})$ which associates a distribution over flows to each state, is called an outcome of the game.

**Definition 2.** A Bayes deterministic Wardrop equilibrium (BDWE) of a Bayesian nonatomic game $\Gamma$ is a mapping $\mathbf{y}(\cdot) : \Theta \to \mathcal{Y}$ such that, for all $a, b \in A$,

$$
\sum_{\theta \in \Theta} p(\theta) y_a(\theta) c_a(\mathbf{y}(\theta), \theta) \leq \sum_{\theta \in \Theta} p(\theta) y_a(\theta) c_b(\mathbf{y}(\theta), \theta). \quad (2.3)
$$

A Bayes correlated Wardrop equilibrium (BCWE) of a Bayesian nonatomic game $\Gamma$ is an outcome $\mu : \Theta \to \Delta(\mathcal{Y})$ such that, for all $a, b \in A$,

$$
\sum_{\theta \in \Theta} p(\theta) \int y_a c_a(\mathbf{y}, \theta) \, d\mu(\mathbf{y} \mid \theta) \leq \sum_{\theta \in \Theta} p(\theta) \int y_a c_b(\mathbf{y}, \theta) \, d\mu(\mathbf{y} \mid \theta). \quad (2.4)
$$

For example, if $\mathbf{y}(\theta)$ is a WE of the nonatomic game $(A, (c_a(\cdot, \theta))_{a \in A})$ for every $\theta$, then $\mathbf{y}(\cdot)$ is a Bayes deterministic Wardrop equilibrium and $\mu(\theta) = \delta_{\mathbf{y}(\theta)}$ is a “fully revealing” Bayes correlated Wardrop equilibrium. Alternatively, if $\bar{\mathbf{y}}$ is a WE of the average game with complete information $(A, (\bar{c}_a)_{a \in A})$, with $\bar{c}_a(\cdot) = \sum_{\theta} p(\theta) c_a(\cdot, \theta)$, then $\mathbf{y}(\cdot) = \bar{\mathbf{y}}$ is a BDWE and $\mu(\theta) = \delta_{\bar{\mathbf{y}}}$ is a “non-revealing” BCWE. In Section 6.1, we formally describe a general family of information structures and show how equilibrium outcomes of the game extended with any information structure induce BCWE (Proposition 7).

The interpretation of Bayes correlated Wardrop equilibrium is similar to the interpretation of correlated Wardrop equilibrium: for each state $\theta \in \Theta$, a mediator who knows the state draws a flow vector $\mathbf{y}$ at random according to the distribution $\mu(\cdot \mid \theta)$ and for each $a \in A$, recommends
a random fraction $y_a$ of players to play $a$. The outcome is a BCWE if there is no incentive to deviate from the mediator’s recommendation. A BDWE is a BCWE where in each state, the distribution $\mu(\cdot \mid \theta)$ is a Dirac distribution $\delta_{y(\theta)}$ on some flow $y(\theta)$. The BDWE inequalities can be interpreted as incentive constraints when players get what Das et al. (2017) call “i.i.d. signals,” whereby in each state $\theta$, the proportion of players receiving $a$ is $y_a(\theta)$ and the ex-ante probability of receiving $a$ is $\sum_{\theta} p(\theta) y_a(\theta)$. In Proposition 3, we provide a formal link between BDWE and conditionally independent signals for large finite sets of players. Note that, under complete information, the definition of BDWE reduces to WE and the definition of BCWE reduces to CWE.

We call information design the optimization problem of minimizing an information designer’s expected cost over equilibrium outcomes. Throughout the paper, the symbol $\text{DC}: \mathcal{Y} \times \Theta \to \mathbb{R}$ denotes a designer cost function, which measures the impact of flows in each state on the cost of the designer. A commonly used functional form when the designer cares about welfare is the total cost

$$\text{DC}(y, \theta) = \text{TC}(y, \theta) := \sum_{a \in \mathcal{A}} y_a c_a(y, \theta),$$

(2.5)

i.e., the total sum of individual costs weighted by the flow over actions.

3 Convergence of atomic to nonatomic games

In this section we explore the relationships between Bayesian anonymous games with finitely many players and Bayesian nonatomic games. The aim is to show that our solution concepts correspond to limits of equilibrium outcomes of Bayesian games with $n$ players as $n$ goes to infinity.

Given a Bayesian nonatomic game $\Gamma = (\mathcal{A}, \Theta, p, c)$, we define the associated $n$-player Bayesian game

$$\Gamma^n = ([n], \mathcal{A}, \Theta, p, (C_i)_{i \in [n]}),$$

(3.1)

where $[n] = \{1, \ldots, n\}$. Each player has the same action set $\mathcal{A}$. Every action profile $a \in \mathcal{A}^n$ induces a flow vector $y(a) = (y_a)_{a \in \mathcal{A}}$ defined by

$$y_a = \frac{1}{n} \sum_{i \in [n]} 1 \{a_i = a\}.$$  

(3.2)

The cost function $C_i: \mathcal{A}^n \times \Theta \to \mathbb{R}$ of player $i$ is given by

$$C_i(a, \theta) = c_{a_i}(y(a), \theta).$$

(3.3)
The game $\Gamma^n$ is anonymous: Player $i$’s cost is a function of player $i$’s action, the distribution of players’ actions, and the state. We now recall the definition of Bayes correlated equilibrium for finite games (Bergemann and Morris, 2016) within the present context.

**Definition 3.** A Bayes correlated equilibrium (BCE) of the $n$-player game $\Gamma^n$ is a mapping $\beta^n : \Theta \to \Delta(A^n)$ such that for all $i \in [n]$ and for all $a, b \in A$,

$$\sum_{\theta \in \Theta} \sum_{a_{-i} \in A_i} p(\theta) \beta^n(a, a_{-i} | \theta)c_a(y(a, a_{-i}), \theta) \leq \sum_{\theta \in \Theta} \sum_{a_{-i} \in A_i} p(\theta) \beta^n(a, a_i | \theta)c_b(y(b, a_{-i}), \theta). \quad (3.4)$$

When this inequality is satisfied up to $\varepsilon > 0$, for all $i, a, b$,

$$\sum_{\theta \in \Theta} \sum_{a_{-i} \in A_i} p(\theta) \beta^n(a, a_{-i} | \theta)c_a(y(a, a_{-i}), \theta) \leq \sum_{\theta \in \Theta} \sum_{a_{-i} \in A_i} p(\theta) \beta^n(a, a_{-i} | \theta)c_b(y(b, a_{-i}), \theta) + \varepsilon, \quad (3.5)$$

the mapping $\beta^n$ is called an $\varepsilon$-Bayes correlated equilibrium.

Given a mapping $\beta^n : \Theta \to \Delta(A^n)$, the induced outcome is $\mu^n : \Theta \to \Delta(Y)$ defined as:

$$\mu^n(y | \theta) = \sum_{a : y(a) = y} \beta^n(a | \theta). \quad (3.6)$$

In words, there is mediator who knows the state, draws an action profile $a = (a_i, a_{-i})$, and privately recommends action $a_i$ to player $i$. Eq. (3.4) is the BCE constraint stating that player $i$ who is recommended $a_i = a$, prefers to play $a$ over any other action $b$. Similarly to Eq. (2.2) and Eq. (2.4), dividing both sides by the probability of recommending $a$, Eq. (3.4) compares the expected costs of playing $a$ or $b$, conditionally on being recommended to play $a$. Bergemann and Morris (2016) proved that the set of BCE outcomes is the set of all Bayesian Nash equilibrium outcomes, for all possible information structures over the state set $\Theta$.

The next result shows that the BCE outcomes of $\Gamma^n$ converge to the BCWE of the nonatomic game $\Gamma$ as the number of players tends to infinity.

**Proposition 1.** Let $\mu^n$ be a BCE outcome of the game $\Gamma^n$ for each integer $n$. Then, any weak accumulation point $\mu$ of the sequence $\mu^n$ is a BCWE of the Bayesian nonatomic game $\Gamma$.

Proposition 1 is proved in Appendix B for a more general class of multi-population games with heterogenous players who have possibly different weights (as defined in Appendix A), when all weights tend to zero as $n$ tends to infinity. The argument is that the incentive conditions of the BCE of the $n$-player game depend on the distribution of flows and carry over to the limit as the number of players grows. This proposition extends the convergence result of Cominetti et al. (2022), which proves the convergence of Nash equilibria of congestion games with finitely many players to Wardrop equilibria of the corresponding nonatomic congestion games.
The next result complements Proposition 1 by showing that any BCWE is a limit of approximate BCE of the games with \( n \) players.

**Proposition 2.** Let \( \mu \) be a BCWE of the Bayesian nonatomic game \( \Gamma \). There exists a sequence \( \varepsilon^n \downarrow 0 \) and a sequence of \( \varepsilon^n\)-BCE outcomes \( \mu^n \) of \( \Gamma^n \) such that \( \mu^n \) weak* converges to \( \mu \) as \( n \) tends to \( \infty \).

The structure of the proof is as follows. First, we approximate a BCWE \( \mu \) by a transition probability with finite support whose probabilities are rational numbers with common denominator \( n \). Then, we consider a mediator who draws a flow from \( \mu \) and assigns actions to subsets of players drawn according to the above rational probabilities. If the approximation is fine enough, this yields an approximate BCE.

Next, we show how a BDWE of a nonatomic game is related to a BCE with conditionally independent signals in \( n \)-player games. A BCE \( \beta^n \) of the \( n \)-player game has the conditional independence property if for all \( a \) and \( \theta \), \( \beta^n(a \mid \theta) = \prod_i \beta_i^n(a_i \mid \theta) \) for some \( \beta_i^n : \Theta \rightarrow \Delta(A_i) \). That is, recommended actions are independent, conditional on the state. A BCE with the conditional independence property is a Bayesian Nash equilibrium induced by conditionally independent signals.

**Proposition 3.** (a) Let \( \mu^n \) be a BCE outcome of the game \( \Gamma^n \) with the conditional independence property for each integer \( n \). Then, any weak* accumulation point \( \mu \) of the sequence \( \mu^n \) is a BDWE of the Bayesian nonatomic game \( \Gamma \).

(b) Let \( \mu \) be a BDWE of the Bayesian nonatomic game \( \Gamma \). There exists a sequence \( \varepsilon^n \downarrow 0 \) and a sequence of \( \varepsilon^n\)-BCE outcomes \( \mu^n \) of \( \Gamma^n \) with the conditional independence property such that \( \mu^n \) weak* converges to \( \mu \) as \( n \) tends to \( \infty \).

Compared with the proof of Proposition 1, the additional ingredient to get the first point is that, from the weak law of large numbers, conditional independence implies that distributions of flows converge to their expectation. Thus, the limits of expected flows define a BDWE.

The second part also rests on the law of large numbers: the mediator sends i.i.d. recommendations to players. In the limit, the proportion of players recommended a given action coincides with the flow on this action given by the BDWE.

### 4 Complete information

This section presents results that are specific to games with complete information. Remark that existence of WE is obtained by a standard Kakutani fixed point argument (see Sandholm, 2010, theorem 2.1.1, page 24) and it is immediate to see that

\[
\Delta(\text{WE}(\Gamma)) \subseteq \text{CWE}(\Gamma).
\]
As shown by the next example, taken from Mitsche et al. (2013), the inclusion can be strict.

**Example 1** (El Farol tapas bar). In this nonatomic game, the action set $\mathcal{A}$ is $\{a, b\}$ which represent staying home and going to the bar, respectively. The cost functions are:

$$c_a(y) = 1, \quad c_b(y) = \max\{2 - 4y, 4y - 2\}. \quad (4.2)$$

The idea is that it is nice to go to the bar when there is some crowd, but not when the crowd is either too large or too small. The game $\Gamma$ admits three WE: $(1, 0)$, $(3/4, 1/4)$, and $(1/4, 3/4)$, and the total cost is equal to 1 for all of them.

Take now $y^1 = (1, 0)$ and $y^2 = (1/2, 1/2)$. The distribution $\mu$ such that

$$\mu(y^1) = \frac{1}{3}, \quad \mu(y^2) = \frac{2}{3}, \quad (4.3)$$

is a CWE, since it satisfies the equilibrium constraints

$$\frac{2}{3} = \frac{1}{3}y^1_ac_a(y^1) + \frac{2}{3}y^2_bc_b(y^2) \leq \frac{1}{3}y^1_ac_b(y^1) + \frac{2}{3}y^2_bc_a(y^2) = \frac{2}{3},$$

$$0 = \frac{1}{3}y^1_bc_b(y^1) + \frac{2}{3}y^2_ac_b(y^2) \leq \frac{1}{3}y^1_bc_a(y^1) + \frac{2}{3}y^2_ac_b(y^2) = \frac{1}{3}.$$

This CWE is not a mixture of WE. Therefore $\Delta(\text{WE}(\Gamma)) \nsubseteq \text{CWE}(\Gamma)$. Moreover, the total cost of this CWE is 2/3, which is lower than the total cost of each WE.

### 4.1 Potential games

Now we show that equality holds in Eq. (4.1) for an important class of nonatomic games.

**Definition 4.** A nonatomic game $\Gamma = (\mathcal{A}, c)$ is a **potential game** if there exists $\Phi: \mathcal{Y} \to \mathbb{R}$ such that, for every $a \in \mathcal{A}$ and $y \in \mathcal{Y}$, we have

$$\frac{\partial \Phi(y)}{\partial y_a} = c_a(y). \quad (4.4)$$

The function $\Phi$ is called the potential of $\Gamma$.

**Example 2.** Congestion games are an extensively studied class of nonatomic potential games,\(^5\) (see e.g., Roughgarden, 2007). One of the main applications of congestion games is traffic routing, as is reflected in the terminology, yet the model is more general. There is a finite set of resources $\mathcal{E}$, a set of actions $\mathcal{A} \subseteq 2^\mathcal{E}$ and, for each resource $e$, a continuous nondecreasing cost function $c_e: \mathbb{R}_+ \to \mathbb{R}_+$. For each resource $e$, the load on resource $e$ is $x_e = \sum_{a \in e} y_a$. The symbols

\(^5\)Other examples of nonatomic potential games include Cournot competition, random matching games and all games with only two actions (see Sandholm, 2001).
\( y = (y_a)_{a \in A} \) and \( x = (x_e)_{e \in E} \) denote the flow and load profiles, respectively. The cost of using resource \( e \) is \( c_e(x_e) \) and the cost of choosing action \( a \) is obtained additively: \( c_a(y) = \sum_{e \in a} c_e(x_e) \), with a common abuse of notation. In routing models there is an underlying oriented network with two nodes that represent origin and destination, the resources are the edges of the network and the actions are paths from the origin to the destination.\(^6\)

Congestion games are potential games and the potential is given by

\[
\Phi(y) = \sum_{e \in E} \int_0^{\sum_{a \ni e} y_a} c_e(u) \, du. \tag{4.5}
\]

Since the cost functions of resources are nondecreasing, this function is convex. It is well known that all minimizers of this function have the same cost profiles \((c_e(x_e))_{e \in E}\). In other words, Wardrop equilibria of a congestion game are not necessarily unique, but all have the same costs.

**Proposition 4.** If a nonatomic game \( \Gamma \) has a convex potential, then

\[
\text{CWE}(\Gamma) = \Delta(\text{WE}(\Gamma)). \tag{4.6}
\]

In addition, all WE (and therefore all CWE) have the same costs profiles: all actions with positive flow have the same cost in all equilibria.

Proposition 4 shows that, for nonatomic games with a convex potential, distributions over WE exhaust the set of CWE.\(^7\) Further, since equilibrium costs are unique, it is enough to focus on WE to minimize any expected designer’s cost function over the set of CWE: mediation does not help as a tool for inducing good equilibria. Díaz et al. (2009, theorem 1) showed that in a particular class of routing games (essentially assuming parallel routes), the smallest total cost achieved by a CWE cannot be smaller than the smallest total cost achieved by a WE. Proposition 4 significantly generalizes the result of Díaz et al. (2009) in two directions. First, Proposition 4 applies to all games with a convex potential; hence, to all routing games. Second, it applies to any designer’s cost function, not only for the total cost, as defined in Eq. (2.5).

Note that the El Farol game (Example 1) is a potential game, but the potential is

\[
\Phi(1 - y_b, y_b) = \begin{cases} 
1 + y_b - 2y_b^2 & \text{if } y_b \leq \frac{1}{2}, \\
2 - 3y_b + 2y_b^2 & \text{if } y_b \geq \frac{1}{2}, 
\end{cases} \tag{4.7}
\]

\(^6\)In this description, we assume that all players have the same origin and destination. However, the model and our results extend to multiple populations with different origins and destinations, see Appendix A.

\(^7\)The proof of Proposition 4 also applies to a weaker version of CWE that requires each player to be weakly better off by complying to the recommendation system than by choosing an action independently, without observing any recommendation. With a finite set of players, this weaker version of CWE corresponds to the coarse correlated equilibrium defined by Moulin and Vial (1978).
which is not convex. We observe that $\Delta(\text{WE}(\Gamma)) \subsetneq \text{CWE}(\Gamma)$ and there is a CWE that Pareto dominates all WE.

The proof of Proposition 4 relies on convex analysis methods. Showing that any CWE is a distribution of WE follows similar steps as the proof of Theorem 1 in Neyman (1997). Neyman’s result states that correlated equilibria are distributions of Nash equilibria in potential games with finitely many players, convex sets of actions and convex potential. There are several differences between Proposition 4 and Neyman (1997): finite vs. infinite set of players, convex vs. finite set of actions. Also, the games in Neyman (1997) are potential games in the sense of Monderer and Shapley (1996): differences of cost functions along unilateral deviations are equal to differences of the potential. To the best of our knowledge, we provide the first non-trivial class of games with finite action sets where correlated equilibria coincide with convex combinations of Nash equilibrium outcomes. This property does not hold even for two-player zero-sum games (see Forges, 1990).

Combining Proposition 1 (applied to games with complete information) and Proposition 4 we get the following corollary:

**Corollary 1.** Let the nonatomic game $\Gamma$ have a convex potential and for each integer $n$, let $\mu^n$ be a correlated equilibrium outcome of $\Gamma^n$. Then any weak* accumulation point $\mu$ of the sequence $\mu^n$ belongs to $\Delta(\text{WE}(\Gamma))$.

This corollary shows that the value of mediation, that is, the additional welfare due to correlation, tends to zero when the weight of each player tends to zero, i.e., when the impact of each player’s action on other players’ costs becomes negligible. Ashlagi et al. (2008) showed that correlated equilibria may significantly decrease the total cost over Nash equilibria (the value of mediation is positive) in $n$-player congestion games, even for a large number of players. However, in their examples, the weights of some players do not tend to zero.

### 5 Optimal information design

We now consider Bayesian nonatomic games and study cost minimization over the set of BCWE. Let $DC : \mathcal{Y} \times \Theta \to \mathbb{R}$ be a continuous cost function whose expectation is the objective the designer aims at minimizing over the set of BCWE.
5.1 Problem formulation

The designer’s optimization program is:

\[
\min_{\mu \in \Delta(Y)} \sum_{\theta \in \Theta} p(\theta) \int DC(y, \theta) \, d\mu(y \mid \theta) \\
\text{s.t.} \sum_{\theta \in \Theta} p(\theta) \int y_a c_a(y, \theta) \, d\mu(y \mid \theta) \leq \sum_{\theta \in \Theta} p(\theta) \int y_a c_b(y, \theta) \, d\mu(y \mid \theta), \quad \forall a, b \in A.
\]

\[(P)\]

The next proposition proves the existence of an optimal BCWE with finite support, whose cardinality is upper bounded.

**Proposition 5.** Program \((P)\) admits a solution \(\mu\) with finite support, whose cardinality is at most \((|A|^2 + 1)|\Theta|\).

The consequence is that program \((P)\) reduces to an optimization problem for which we can bound the number of variables and constraints. The problem can then be decomposed in two parts, identifying the support and solving a standard linear programming problem. Such problems belong to the class of Generalized Problem of Moments, (see, e.g., Lasserre, 2008; Zhu and Savla, 2020, for a formulation in terms of semi-definite programming and computational results).

The proof follows from a use of Caratheodory’s theorem which is common in the theory of moment problems (see, e.g., Lasserre, 2008). The argument is based on the observation that the objective function and the constraints depend only on the expected values of the following functions of flows:

\[\sum_{\theta \in \Theta} p(\theta) DC(y, \theta) \text{ and } \sum_{\theta \in \Theta} p(\theta) y_a c_a(y, \theta), \text{ with } a, b \in A.\]

Thus, program \((P)\) can be expressed via the convex hull of the joint range of those functions.

The dimension of this set gives the upper bound on the cardinality of the support. The use of Caratheodory’s theorem for bounding the number of signals is commonly used in information design (see, e.g., Kamenica and Gentzkow, 2011).

There are two natural subsets of BCWE over which the designer could minimize its expected cost. The optimal BDWE is the solution of the program:

\[
\min_{y(\cdot) \in \mathcal{Y}} \sum_{\theta \in \Theta} p(\theta) DC(y(\theta), \theta) \\
\text{s.t.} \sum_{\theta \in \Theta} p(\theta) y_a(\theta) c_a(y(\theta), \theta) \leq \sum_{\theta \in \Theta} p(\theta) y_a(\theta) c_b(y(\theta), \theta), \quad \forall a, b \in A.
\]

\[(D)\]

We call *public* any BCWE where the designer sends a public message to all players. Consider a finite set of messages \(\mathcal{M}\) and a transition probability \(\zeta: \Theta \to \Delta(\mathcal{M})\). This induces a public
information structure that announces message $m$ with probability $\zeta(m \mid \theta)$ in state $\theta$. This in turn induces posterior distributions $p_m \in \Delta(\Theta)$, where

$$p_m(\theta) = \frac{p(\theta) \zeta(m \mid \theta)}{\lambda_m} \quad \text{and} \quad \lambda_m = \sum_\theta p(\theta) \zeta(m \mid \theta),$$

(5.1)

hence $p = \sum_m \lambda_m p_m$. Given message $m$ and public belief $p_m$, players choose a flow of actions which is a Wardrop equilibrium of the complete information game with cost functions $\sum_\theta p_m(\theta) c_a(\cdot, \theta)$. This gives rise to the following minimization problem:

$$\min \sum_m \lambda_m \sum_\theta p_m(\theta) DC(y_m, \theta)$$

(Pub)

s.t. $\lambda_m \geq 0, \forall m \in \mathcal{M}$, $\sum_m \lambda_m = 1$, $\sum_m \lambda_m p_m = p$, $y_m \in WE(p_m),$

where $WE(q)$ denotes the set of Wardrop equilibria of the game with complete information and cost functions $\sum_\theta q(\theta) c_a(\cdot, \theta)$ for $q \in \Delta(\Theta)$. If we let

$$C(q) := \min_{y \in WE(q)} \sum_\theta q(\theta) DC(y, \theta),$$

(5.2)

this problem can be reformulated as

$$\text{vex } C(p) := \min \left\{ \sum_m \lambda_m C(p_m) \text{ s.t. } \lambda_m \geq 0, \forall m \in \mathcal{M}, \sum_m \lambda_m = 1, \sum_m \lambda_m p_m = p \right\},$$

(5.3)

whose value $\text{vex } C$ is the convexification (or convex envelope) of the function $C$, i.e., the largest convex function below $C$.

In the next section, we show by examples that BDWE and public BCWE are not comparable: one can find cases where either or none of these classes is optimal.

### 5.2 Bayesian potential games

We define Bayesian potential games as follows.

**Definition 5.** A Bayesian nonatomic game $\Gamma = (\mathcal{A}, \Theta, p, c)$ is a potential game if for every $\theta \in \Theta$, there exists a function $\Phi_\theta : \mathcal{Y} \to \mathbb{R}$ such that, for every $a \in \mathcal{A}$ and $y \in \mathcal{Y}$, we have

$$\frac{\partial \Phi_\theta(y)}{\partial y_a} = c_a(y, \theta).$$

(5.4)

For instance, congestion games with state-dependent cost functions are Bayesian potential
games, and so is any game with two actions. Since the concept of Bayes deterministic Wardrop equilibrium is an extension of Wardrop equilibrium to Bayesian games, we ask whether Proposition 4 can be extended to this more general framework. That is, we ask whether it is enough to focus on BDWE to minimize a designer cost function. To this end, we consider the following parametrized game, studied in details in Appendix C.

**Leading example.** A unitary mass of agents can choose one of two actions $a, b$, which represent choices of locations, road itineraries or online servers. The flow on action $b$ is denoted by $y_b = 1 - y_a$. There are two possible states $Θ = \{θ, \bar{θ}\}$, with prior distribution $(p(θ), p(\bar{θ}))$. The cost of action $a$ only depends only on the state, whereas the cost of action $b$ does not depend on the state and is (weakly) increasing in $y_b$:

$$c_a(y, θ) = \mathbb{1}\{θ = \bar{θ}\}, \quad c_b(y, θ) = αy_b + β, \quad \text{with}\; α, β ≥ 0. \quad (5.5)$$

**Example 3.** For instance, suppose that $p(θ) = 1/2$, $c_b(y, θ) = 2y_b + 1/3$, and the designer aims at minimizing the total cost $TC(y, θ) = y_a c_a(y, θ) + y_b c_b(y, θ)$. This specification is the example found in Das et al. (2017). Then the global minimum of $TC(y, θ)$ over all flows is achieved by $(y_a(θ), y_b(θ)) = (1, 0)$ in state $θ$ and $(y_a(θ), y_b(θ)) = (5/6, 1/6)$ in state $\bar{θ}$. This first best outcome is a BDWE, since it satisfies the incentive constraints:

$$p(θ)y_a(θ)0 + p(\bar{θ})y_a(\bar{θ})1 = \frac{5}{12} < p(θ)y_a(θ)\left(\frac{1}{3} + 2y_b(θ)\right) + p(\bar{θ})y_a(\bar{θ})\left(\frac{1}{3} + 2y_b(\bar{θ})\right) = \frac{16}{36},$$

$$p(θ)y_b(θ)\left(\frac{1}{3} + 2y_b(θ)\right) + p(\bar{θ})y_b(θ)\left(\frac{1}{3} + 2y_b(\bar{θ})\right) = \frac{1}{18} < p(θ)y_b(θ)0 + p(\bar{θ})y_b(\bar{θ})1 = \frac{1}{12}. \quad (5.6)$$

However, the optimal BCWE is not necessarily a BDWE, as the next example shows.

**Example 4.** Consider the case where $p(θ) = 1/2$, $c_b(y, θ) = 1/3$, and $DC(y, θ) = g(y_b)$ with $g$ increasing, strictly concave, and such that $g(0) = 0, g(1) = 1$. In Appendix C we calculate the best BDWE and show that it is given by

$$(y_a(θ), y_b(θ)) = (1, 0), \quad (y_a(\bar{θ}), y_b(θ)) = \left(\frac{1}{2}, \frac{1}{2}\right). \quad (5.7)$$

Thus, the designer’s expected cost is

$$\frac{1}{2}g\left(\frac{1}{2}\right) + \frac{1}{2}g(0) = \frac{1}{2}g\left(\frac{1}{2}\right). \quad (5.8)$$

Consider now public information. It is easy to see that for each $q$, the Wardrop equilibrium is $y_b(q) = 0$ if $q(\bar{θ}) < 1/3$ and $y_b(q) = 1$ if $q(\bar{θ}) > 1/3$; if $q(\bar{θ}) = 1/3$, any flow is an equilibrium.
Thus,
\[
C(q) = \begin{cases} 
  g(0) = 0 & \text{for } q(\bar{\theta}) \leq 1/3, \\
  g(1) = 1 & \text{for } q(\bar{\theta}) > 1/3,
\end{cases}
\] (5.9)

and
\[
vex C(p) = \begin{cases} 
  0 & \text{for } q(\bar{\theta}) \leq 1/3, \\
  \frac{q(\bar{\theta}) - \frac{1}{3}}{1 - \frac{1}{3}} & \text{for } q(\bar{\theta}) > 1/3.
\end{cases}
\] (5.10)

Since \(g\) is strictly concave, we have \(g(1/2) > 1/2\) and
\[
\frac{1}{2} g \left( \frac{1}{2} \right) > \frac{1}{4} = vex C \left( \frac{1}{2} \right).
\] (5.11)

Therefore, public information dominates BDWE in this case. To keep the example simple, we have considered a limit case where costs do not depend on flows (\(\alpha = 0\)). However, this is also the case for \(\alpha > 0\) and small enough, as shown in Appendix C. There, we also show that there exist parameters \(\alpha, \beta\) and designer cost functions for which the optimal BCWE is neither deterministic nor public.

In Appendix D, we provide the more intricate Example 6, a Bayesian routing game that has a BCWE for which the expected total cost is strictly smaller than in every BDWE. This example shows that, even when minimizing the total cost in Bayesian routing games with incomplete information, restricting to BDWE (i.e., to independent signals) may induce loss of optimality.

Our next result gives conditions on games with binary actions under which the minimum of the designer cost over the set of BCWE is achieved with a BDWE.

**Proposition 6.** Consider a Bayesian nonatomic game \(\Gamma\) such that \(A\) contains two actions and, for every \(a \in A\) and \(\theta \in \Theta\), the function \(y \mapsto c_a(y, \theta)\) is concave and the function \(y \mapsto y_a c_a(y, \theta)\) is convex. Suppose that the designer’s cost function \(DC(y, \theta)\) is convex for every \(\theta \in \Theta\). Then, for every BCWE of \(\Gamma\), there exists a BDWE whose expected designer cost is weakly smaller in each state \(\theta\).

Notice that under the above conditions on the cost functions, the total cost \(\sum_a y_a c_a(y, \theta)\) is convex and the result applies to its minimization. The main argument is that the convexity and concavity conditions in Proposition 6 are such that the expected flows of a BCWE is necessarily a BDWE by Jensen’s inequality. A special case of Proposition 6 with affine cost functions appears in Zhu and Savla (2020). Our leading example with convex cost functions also falls into this category. Remark that for games with complete information, Proposition 6 is not a corollary of Proposition 4 because the assumptions of Proposition 6 do not necessarily imply convexity of the potential function.
6 Implementation via finite information structures

6.1 Implementation and a Revelation Principle

Next, we consider finite information structures and Bayesian Wardrop equilibria as defined in Wu et al. (2021). We prove that any Bayesian Wardrop equilibrium outcome is a BCWE and that any BCWE with finite support and rational flows can be obtained as a Bayesian Wardrop equilibrium outcome for some finite information structure. The information structure we construct is a direct system of action recommendations and equilibrium strategies are obedient, as in the revelation principle for Bayesian games with finitely many players (Myerson, 1982). While Wu et al. (2021) introduced the notion of Bayesian Wardrop equilibrium in the context of Bayesian routing games, their definition is easily adapted beyond routing games to our more general model. The following definitions provide a nonatomic analog of the classical Bayesian Nash equilibrium in games with finitely many players.

**Definition 6** (Wu et al. (2021)). An information structure \((\gamma, \mathcal{T}, \pi)\) for the Bayesian nonatomic game \(\Gamma = (\mathcal{A}, \Theta, p, c)\) is given by: a finite set \(K\) of populations, where the size of population \(k\) is \(\gamma_k > 0\), and \(\sum_{k \in K} \gamma_k = 1\); for each population \(k\), a finite set of types \(\mathcal{T}^k\); and a mapping \(\pi : \Theta \rightarrow \Delta(\mathcal{T})\), where \(\mathcal{T} = \times_{k \in K} \mathcal{T}^k\).

For each state \(\theta \in \Theta\), the type profile \(\tau \in \mathcal{T}\) is drawn with probability \(\pi(\tau | \theta)\) and the entire population \(k\) observes the type \(\tau_k \in \mathcal{T}^k\). For \(k \in K\), consider the map \(y_k(\cdot) : \mathcal{T}^k \rightarrow \Delta(\mathcal{A}) := \left\{ y : \forall a \in \mathcal{A}, y_a \geq 0, \sum_{a \in \mathcal{A}} y_a = \gamma_k \right\}\), and let \(y^k(\tau^k)\) denote the flow of players who observe type \(\tau^k\) and chose action \(a\). Define

\[
y_a(\tau) = \sum_{k \in K} y^k_{a}(\tau^k) \quad \text{and} \quad y(\tau) = (y_a(\tau))_{a \in \mathcal{A}}. \tag{6.1}
\]

**Definition 7** (Wu et al. (2021)). A Bayesian Wardrop equilibrium is a profile \((y^k(\cdot))_{k \in K}\) such that, for all \(k \in K\), for all \(\tau^k \in \mathcal{T}^k\), and for all \(a, b \in \mathcal{A}\), if \(y^k_a(\tau^k) > 0\), then

\[
\sum_{\theta \in \Theta} \sum_{\tau^{-k} \in \mathcal{T}^{-k}} p(\theta) \pi(\tau^k, \tau^{-k} | \theta) c_a(y(\tau), \theta) \leq \sum_{\theta \in \Theta} \sum_{\tau^{-k} \in \mathcal{T}^{-k}} p(\theta) \pi(\tau^k, \tau^{-k} | \theta) c_b(y(\tau), \theta). \tag{6.2}
\]

That is, for each population \(k\) and type \(\tau^k\), a positive mass chooses action \(a\) only if it has the least expected cost, conditional on \(\tau^k\).

A mapping \(\mu : \Theta \rightarrow \Delta(\mathcal{Y})\) is a Bayesian Wardrop equilibrium outcome of \(\Gamma = (\mathcal{A}, \Theta, p, c)\) with information structure \((\gamma, \mathcal{T}, \pi)\) if there exists a Bayesian Wardrop equilibrium \((y^k(\cdot))_{k \in K}\)
such that for every $\theta \in \Theta$ and for every $z \in \mathcal{Y}$,

$$
\mu(z \mid \theta) = \sum_{\tau : y(\tau) = z} \pi(\tau \mid \theta).
$$

(6.3)

**Proposition 7.** Given a Bayesian nonatomic game $\Gamma$, for every information structure, a Bayesian Wardrop equilibrium outcome is a BCWE of $\Gamma$.

This proposition makes explicit how to construct some BCWEs: choose a finite information structure, deliver information through it to players and let them play a Bayesian Wardrop equilibrium of the Bayesian game extended with that information structure. The intuition of the proof of Proposition 7 is derived from the revelation principle: if a player ends up choosing action $a$ in equilibrium, then this player should be willing to choose $a$ when it is recommended by a mediator who uses the equilibrium to generate the distribution of flows.

Then, we study how a given BCWE, in particular an optimal one, can be implemented as a Bayesian Wardrop equilibrium of the original Bayesian game with some appropriate information structure. First, we consider the case where every flow in the support of $\mu$ is a vector with rational components.

**Proposition 8.** If $\mu$ is a BCWE with finite support and rational flows, then there exists an information structure $(\gamma, T, \pi)$ and a Bayesian Wardrop equilibrium with outcome $\mu$.

In addition, the information structure is direct: for every $k \in K$, $T^k = A$; and the Bayesian Wardrop equilibrium is obedient: the whole population $k$ plays $a$ upon observing the type $\tau^k = a$.

The combination of Propositions 7 and 8 can be seen as an extension of the revelation principle to anonymous nonatomic Bayesian games: whatever the information structure, a Bayesian Wardrop equilibrium induces a BCWE, and for every BCWE there exists a direct recommendation system that implements this BCWE with obedient equilibrium strategies. This revelation principle also applies to games with complete information: if a game with complete information is extended by some information structure, then a Bayesian Wardrop equilibrium of the extended game induces a CWE. Conversely, for every CWE of a game with complete information, there exists a direct recommendation system that implements this CWE with obedient equilibrium strategies. For instance, in Example 1, the cost minimizing CWE $\mu(1, 0) = \frac{1}{3}$ and $\mu\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{2}{3}$ is implemented as a Bayesian Wardrop equilibrium with the following information structure: there are two populations of size $\gamma^1 = \gamma^2 = \frac{1}{2}$, and $\pi(a, a) = \pi(a, b) = \pi(b, a) = \frac{1}{3}$, and the incentive compatibility conditions of the CWE are equivalent to the obedience constraints.

In the Bayesian game of Example 3, the BDWE $(y_a(0), y_b(0)) = (1, 0)$ and $(y_a(1), y_b(1)) = \left(\frac{5}{6}, \frac{1}{6}\right)$ is implemented as a Bayesian Wardrop equilibrium with the following information structure: there are six populations of size $\gamma^k = 1/6$, for $k = 1, \ldots, 6$, $\pi(a, a, a, a, a, a \mid \theta = 0) = 1$.
and
\[
\begin{align*}
\pi(b, a, a, a, a, a | \theta = 1) &= \pi(a, b, a, a, a, a | \theta = 1) = \pi(a, a, b, a, a, a | \theta = 1) = \\
\pi(a, a, b, a, a, a | \theta = 1) &= \pi(a, a, a, b, a, a | \theta = 1) = \pi(a, a, a, a, b, a | \theta = 1) = \frac{1}{6}.
\end{align*}
\]

As before, the incentive compatibility conditions of the Bayesian Wardrop equilibrium are equivalent to the obedience constraints.

The next proposition provides an approximate implementation result for all BCWE.

**Proposition 9.** Let \( \mu \) be a BCWE of the Bayesian nonatomic game \( \Gamma \). Then there exist a sequence \( \varepsilon^k \searrow 0 \), a sequence of information structures with \( k \) populations, and corresponding Bayesian Wardrop \( \varepsilon^k \)-equilibrium outcomes \( \mu^k \) such that \( \mu^k \xrightarrow{\text{w}} \mu \).

### 6.2 Full implementation

In this section, we study full implementation of BCWE. That is, as in Morris et al. (2020), implementation of a BCWE \( \mu \) in such a way that every Bayesian Wardrop equilibrium of a given information structure induces \( \mu \). When a BCWE is fully implementable, this allows to solve the information design problem in a way that is robust to adversarial equilibrium selection. Our full implementation results concern games with (strictly) convex potential in each state. This encompasses for instance routing games with parallel networks and (strictly) increasing costs.

**Proposition 10.** Consider a potential Bayesian nonatomic game \( \Gamma \). If the potential is convex in each state, then for every information structure, there exists a unique Bayesian Wardrop equilibrium cost profile. If the potential is strictly convex in each state, then for every information structure there exists a unique Bayesian Wardrop equilibrium.

To prove Proposition 10, we first show that a Bayesian Wardrop equilibrium of a Bayesian game is equivalent to a WE of a game with complete information and multiple populations. If the original game has a (strictly) convex potential in each state, then the associated game with complete information also has a (strictly) convex potential. Then, uniqueness of WE costs follows from Proposition 11 in the appendix, which is a generalization of Proposition 4 to multiple populations. As a consequence, we get uniqueness of Bayesian Wardrop equilibrium cost profiles in the original game. Uniqueness of the Bayesian Wardrop equilibrium when the potential is strictly convex follows from the fact that the WE of the associated game with complete information is the (unique) minimizer of the potential (see Lemma 3 in Appendix B).

From Propositions 8 and 10 we obtain the following full implementation result for games with a (strictly) convex potential.
Corollary 2. Consider a Bayesian nonatomic game $\Gamma$ with convex potential in each state and let $\mu$ be a BCWE with finite support and rational flows. There exists an information structure $(\gamma, T, \pi)$ such that the ex-ante total cost of every Bayesian Wardrop equilibrium of the extended game is the same as the ex-ante total cost of $\mu$. If the potential is strictly convex, then the Bayesian Wardrop equilibrium is unique and implements the BCWE $\mu$. The implementation can be achieved with $T^k = A$ and obedient equilibrium strategies.

Corollary 2 is a powerful result that applies to all congestion games. For instance, in Example 3, it implies that there exists an information structure (the one described on page 20) that implements the optimal BCWE in every Bayesian Wardrop equilibrium of the game extended with this information structure.

7 Concluding remarks

The contributions of this paper demonstrate that information design can be tractably analyzed in large anonymous games. We have shown that framing the problem in terms of nonatomic games is appropriate and provides important new results. The set of equilibrium outcomes that an information designer is able to implement can be characterized through distributions of action flows with the notion of Bayes correlated Wardrop equilibrium; furthermore, a revelation principle applies. Regardless of the objective, the designer’s problem can be written as an optimization program for which the number of variables and constraints is bounded, given the number of actions and states. In the class of games with a convex potential, such as congestion games, we have shown that information design is useless when the state is commonly known.

We have provided a class Bayesian games with binary actions, where the designer who aims at minimizing the total cost, can issue deterministic recommendations conditional on the state, without loss of optimality. However, this result does not extend to more general classes of games, or to more general objective functions.

We have also shown that information design is extremely powerful in Bayesian games with convex potential. In such games, the optimal solution of the designer can be fully implemented: there exists a direct recommendation system such that the solution of the designer is implemented regardless of the equilibrium play. To the best of our knowledge, these results have no analog in games with a finite number of players.

References

ARIELI, I. AND Y. BABICHENKO (2019): “Private Bayesian persuasion,” *J. Econom. Theory*, 182, 185–217.
ARNOTT, R., A. DE PALMA, AND R. LINDSEY (1991): “Does providing information to drivers reduce traffic congestion?” Transportation Res. Part A, 25, 309–318.

ASHLAGI, I., D. MONDERER, AND M. TENNEHOLTZ (2008): “On the value of correlation,” J. Artificial Intelligence Res., 33, 575–613.

AUMANN, R. J. (1974): “Subjectivity and correlation in randomized strategies,” J. Math. Econom., 1, 67–96.

——— (1987): “Correlated equilibrium as an expression of Bayesian rationality,” Econometrica, 55, 1–18.

AUMANN, R. J. AND M. B. MASCHLER (1995): Repeated Games with Incomplete Information, Cambridge, MA: MIT Press.

BECKMANN, M. J., C. McGUIRE, AND C. B. WINSTEN (1956): Studies in the Economics of Transportation, New Haven, CT: Yale University Press.

BERGEMANN, D. AND S. MORRIS (2016): “Bayes correlated equilibrium and the comparison of information structures in games,” Theor. Econ., 11, 487–522.

——— (2019): “Information design: A unified perspective,” J. Econ. Lit., 57, 44–95.

BHASKAR, U., Y. CHENG, Y. K. KO, AND C. SWAMY (2016): “Hardness results for signaling in Bayesian zero-sum and network routing games,” in Proceedings of the 2016 ACM Conference on Economics and Computation, New York, NY, USA: Association for Computing Machinery, EC ’16, 479–496.

BLUM, A., E. EVEN-DAR, AND K. LIGETT (2010): “Routing without regret: on convergence to Nash equilibria of regret-minimizing algorithms in routing games,” Theory Comput., 6, 179–199.

CASTIGLIONI, M., A. CELLI, A. MARCHESI, AND N. GATTI (2021): “Signaling in Bayesian network congestion games: the subtle power of symmetry,” Proceedings of the AAAI Conference on Artificial Intelligence, 35, 5252–5259.

COMINETTI, R., M. SCARSONI, M. SCHRÖDER, AND N. STIER-MOSES (2022): “Approximation and convergence of large atomic congestion games,” Math. Oper. Res., forthcoming.

DAS, S., E. KAMENICA, AND R. MIRKA (2017): “Reducing congestion through information design,” in 2017 55th Annual Allerton Conference on Communication, Control, and Computing (Allerton), 1279–1284.
Díaz, J., D. Mitsche, N. Rustagi, and J. Saia (2009): “On the power of mediators,” in Internet and Network Economics, ed. by S. Leonardi, Springer Berlin Heidelberg, 455–462.

Dughmi, S. (2017): “Algorithmic information structure design: a survey,” ACM SIGecom Exchanges, 15, 2–24.

Fenchel, W. (1929): “Über Krümmung und Windung geschlossener Raumkurven,” Math. Ann., 101, 238–252.

Forges, F. (1990): “Correlated equilibrium in two-person zero-sum games.” Econometrica, 58, 515.

——— (1993): “Five legitimate definitions of correlated equilibrium in games with incomplete information,” Theory and Decision, 35, 277–310.

——— (2006): “Correlated equilibrium in games with incomplete information revisited,” Theory and Decision, 61, 329–344.

——— (2020): “Games with incomplete information: From repetition to cheap talk and persuasion,” Ann. Econ. Stat., 137, 3–30.

Hart, S. and D. Schmeidler (1989): “Existence of correlated equilibria,” Math. Oper. Res., 14, 18–25.

Haurie, A. and P. Marcotte (1985): “On the relationship between Nash-Cournot and Wardrop equilibria,” Networks, 15, 295–308.

Hellwig, M. F. (2022): “Incomplete-information games in large populations with anonymity,” Theor. Econ., 17, 461–506.

Kamenica, E. (2019): “Bayesian persuasion and information design,” Ann. Rev. Econ., 11, 249–272.

Kamenica, E. and M. Gentzkow (2011): “Bayesian persuasion,” Amer. Econ. Rev., 101, 2590–2615.

Koutsoupias, E. and C. H. Papadimitriou (1999): “Worst-case equilibria,” in STACS ’99: Proceedings of the 16th Annual Symposium on Theoretical Aspects of Computer Science.

Lasserre, J. B. (2008): “A semidefinite programming approach to the generalized problem of moments,” Math. Program., 112, 65–92.

Liu, J., S. Amin, and G. Schwartz (2016): “Effects of information heterogeneity in Bayesian routing games,” Tech. rep., arXiv:1603.08853.
Mathevet, L., J. Perego, and I. Taneva (2020): “On information design in games,” *J. Pol. Econ.*, 128, 1370–1404.

Mertens, J.-F., S. Sorin, and S. Zamir (2015): *Repeated Games*, New York: Cambridge University Press.

Mitsche, D., G. Saad, and J. Saia (2013): “The power of mediation in an extended El Farol game,” in *Algorithmic Game Theory*, Springer, Heidelberg, vol. 8146 of *Lecture Notes in Comput. Sci.*, 50–61.

Monderer, D. and L. S. Shapley (1996): “Potential games,” *Games Econom. Behav.*, 14, 124–143.

Morris, S., D. Oyama, and S. Takahashi (2020): “Implementation via Information Design in Binary-Action Supermodular Games,” *Available at SSRN*.

Moulin, H. and J.-P. Vial (1978): “Strategically zero-sum games: the class of games whose completely mixed equilibria cannot be improved upon,” *Internat. J. Game Theory*, 7, 201–221.

Myerson, R. B. (1982): “Optimal coordination mechanisms in generalized principal-agent problems,” *J. Math. Econom.*, 10, 67–81.

Neyman, A. (1997): “Correlated equilibrium and potential games,” *Internat. J. Game Theory*, 26, 223–227.

Pigou, A. C. (1920): *The Economics of Welfare*, London: Macmillan and Co., 1st ed.

Rockafellar, R. T. (1970): *Convex Analysis*, Princeton Mathematical Series, No. 28, Princeton University Press, Princeton, N.J.

Rosenthal, R. W. (1973): “A class of games possessing pure-strategy Nash equilibria,” *Internat. J. Game Theory*, 2, 65–67.

Roughgarden, T. (2007): “Routing games,” in *Algorithmic Game Theory*, Cambridge: Cambridge Univ. Press, 461–486.

——— (2015): “The price of anarchy in games of incomplete information,” *ACM Trans. Econ. Comput.*, 3, Art. 6, 20.

Sandholm, W. H. (2001): “Potential games with continuous player sets,” *J. Econom. Theory*, 97, 81–108.

——— (2010): *Population Games and Evolutionary Dynamics*, Economic Learning and Social Evolution, MIT Press, Cambridge, MA.
TAVAFOGHI, H. AND D. TENEKETZIS (2018): “Strategic information provision in routing games,” Mimeo.

VASSERMAN, S. S., M. FELDMAN, AND A. HASSIDIM (2015): “Implementing the wisdom of Waze,” in International Joint Conference on Artificial Intelligence (IJCAI), Buenos Aires, Argentina, vol. 24.

WARDROP, J. G. (1952): “Some theoretical aspects of road traffic research,” in Proceedings of the Institute of Civil Engineers, Part II, vol. 1, 325–378.

WU, M. AND S. AMIN (2019): “Information design for regulating traffic flows under uncertain network state,” in 2019 57th Annual Allerton Conference on Communication, Control, and Computing (Allerton), IEEE, 671–678.

WU, M., S. AMIN, AND A. E. OZDAGLAR (2021): “Value of information in Bayesian routing games,” Oper. Res., 69, 148–163.

ZHU, Y. AND K. SAVLA (2020): “A semidefinite approach to information design in non-atomic routing games,” Tech. rep., arXiv:2005.03000.
A Multiple Populations

There is a finite set of populations $\mathcal{K}$, each population $k \in \mathcal{K}$ has size $\gamma^k > 0$. For each population $k \in \mathcal{K}$, there is a finite set of actions $\mathcal{A}^k$. Let $\mathcal{Y}^k = \Delta(\mathcal{A}^k) := \{y^k: \forall a \in \mathcal{A}^k, y^k_a \geq 0, \sum_{a \in \mathcal{A}^k} y^k_a = \gamma^k\}$ be the set of flows over $\mathcal{A}^k$ and $\mathcal{Y} = \times_k \mathcal{Y}^k$. A flow profile is denoted by $y = (y^k)_{k \in \mathcal{K}}$. For each $(k, a)$ with $a \in \mathcal{A}^k$, there is a continuous cost function $c^k_a: \mathcal{Y} \times \Theta \to \mathbb{R}$. The cost for an individual in population $k$ who chooses action $a$ is $c^k_a(y, \theta)$. For each state $\theta$, a WE of the game with complete information at $\theta$ is a flow profile $y = (y^k)_{k \in \mathcal{K}}$ such that for all $k$, and all $a, b \in \mathcal{A}^k$

$$y^k_a c^k_a(y, \theta) \leq y^k_b c^k_b(y, \theta).$$

(A.1)

In other words, $\forall k, a, y^k_a > 0 \implies c^k_a(y, \theta) = \min_b c^k_b(y, \theta)$ (this equilibrium is called Nash equilibrium of population game in Sandholm (2010) where existence is proved in theorem 2.1.1, page 24).

The definition of Bayes correlated Wardrop equilibrium extends as follows:

**Definition 8.** A Bayes correlated Wardrop equilibrium of a multi-population game is a mapping $\mu: \Theta \to \Delta(\mathcal{Y})$ such that for every $k \in \mathcal{K}$ and $a, b \in \mathcal{A}^k$:

$$\sum_{\theta} p(\theta) \int y^k_a c^k_a(y, \theta) \, d\mu(y | \theta) \leq \sum_{\theta} p(\theta) \int y^k_b c^k_b(y, \theta) \, d\mu(y | \theta).$$

(A.2)

The extended definition of correlated Wardrop equilibrium is obtained by considering a single state $|\Theta| = 1$, the extension of Bayes deterministic Wardrop equilibrium is obtained by considering a deterministic mapping $y(\cdot): \Theta \to \mathcal{Y}$ instead of $\mu: \Theta \to \Delta(\mathcal{Y})$.

The multi-population game is a potential game if for every $\theta$ there exists a function $\Phi_\theta: \mathcal{Y} \to \mathbb{R}$ such that $\forall k, a, \frac{\partial \Phi_\theta}{\partial y^k_a} = c^k_a(y, \theta)$. In the case of complete information, as in games with a single population, if $\Phi$ is convex, then the set of WE is the set of minimizers of $\Phi$, there is a unique WE cost for each population, and we have $\text{CWE}(\Gamma) = \Delta(\text{WE}(\Gamma))$ (see Proposition 11 in Appendix B which extends Proposition 4 to multi-population games). The convergence results of Section 3 also extend to multi-population games (see the proofs of Propositions 1–3 in Appendix B). The implementation results of BCWE as Bayesian Wardrop equilibria (BWE) could potentially be extended to multi-population games at the price of extending the Bayesian Wardrop equilibrium of Wu et al. (2021) to heterogeneous action spaces and cost functions. We have not included this extension to avoid simple but cumbersome calculations. However, the consideration of multi-population is useful for our implementation results, since we interpret a Bayesian Wardrop equilibrium as a WE of a multi-population game (see the proof of Proposition 10 in Appendix B).

Multi-population routing games where populations only differ with respect to their origins and destinations have a convex potential (see, e.g., Roughgarden, 2007, proposition 18.11).
Hence, Proposition 4 applies to such games, and therefore mediation has no value. However, multi-population routing games in which the cost functions differ across populations do not necessarily have a convex potential. In the multi-population routing game of Example 5 in Appendix D, with two populations with different cost functions, we show that there exists a CWE that Pareto dominates the unique WE.

B Proofs

We start this section by proving Proposition 5, since it is independent from the rest and it implies Lemma 1, which is used in other proofs. The argument is borrowed from the theory of Generalized Problem of Moments (see, e.g., Lasserre, 2008).

**Proof of Proposition 5.** We consider the general case of several populations \( k \in K \). First, we remark that a BCWE \( \mu \in \Delta(Y^\Theta) \) can equivalently represented by some \( \tilde{\mu} \in \Delta(Y^\Theta) \). That is, the mediator draws a vector of flows \( \tilde{y} = (y(\theta))_{\theta \in \Theta} \) according to the distribution \( \tilde{\mu} \) over \( Y^\Theta \) defined as the product of the distributions \( \mu(\cdot | \theta), \theta \in \Theta \).

\[
\text{Program (P)} \text{ can be written as}
\]

\[
\min \int \sum_{\theta} p(\theta) \text{DC}(y(\theta), \theta) \, d\tilde{\mu}(\tilde{y}),
\]

\[
\text{s.t. } \forall k \in K, \forall a, b \in A^k \int \sum_{\theta} p(\theta)y^k_a(\theta)c^k_b(y(\theta), \theta) \, d\tilde{\mu}(\tilde{y}) \leq \int \sum_{\theta} p(\theta)y^k_a(\theta)c^k_b(y(\theta), \theta) \, d\tilde{\mu}(\tilde{y}).
\]

Define for \( k \in K \) and \( a, b \in A^k \),

\[
\tilde{z}^k_{a,b}(\tilde{y}) = \sum_{\theta} p(\theta)y^k_a(\theta)c^k_b(y(\theta), \theta),
\]

\[
\tilde{z}(\tilde{y}) = (\tilde{z}^k_{a,b}(\tilde{y}))_{k \in K, (a,b) \in A^k \times A^k} \in \mathbb{R}^{1_k(A^k \times A^k)},
\]

\[
\text{DC}(\tilde{y}) = \sum_{\theta} p(\theta) \text{DC}(y(\theta), \theta),
\]

\[
Z = \left\{ (z, c) \in \mathbb{R}^{1_k(A^k \times A^k)} \times \mathbb{R} : \exists y \in Y^\Theta \text{ s.t. } (z, c) = (\tilde{z}(\tilde{y}), \text{DC}(\tilde{y})) \right\}.
\]

The objective function and the constraints are linear with respect to \( (\tilde{z}(\tilde{y}), \text{DC}(\tilde{y})) \) so Program (P) is equivalent to

\[
\min \left\{ c : (z, c) \in \text{co}(Z) \text{ and } \forall k \in K, \forall a, b \in A^k, z^k_{a,a} \leq z^k_{a,b} \right\}.
\]

\(8\)This is an application of Kuhn’s Theorem of equivalence between mixed and behavior strategies, to the one player game tree where \( \theta \) is observed before choosing \( y \). Since the set \( Y \) is infinite, a reference beyond finite trees is in Mertens, Sorin, and Zamir (2015, Theorem II.1.6 page 63).
Consider a game \( \Gamma_n \) which contradicts Proposition 5. Suppose that there exists a BCWE \( \mu \) Proof. We prove the result for multi-population games and general weights. Proof of Proposition 1. Lemma 1. Any point in \( \text{co}(\mathcal{Z}) \) can be obtained as a convex combination of \( |\cup_k(\mathcal{A}^k \times \mathcal{A}^k)| + 1 \) points, using Caratheodory's theorem and the fact that \( \mathcal{Z} \) is connected (Fenchel, 1929). Since any \( \hat{y} \) induces at most \( |\Theta| \) different flows, the designer cost of any BCWE can be obtained with a BCWE that randomizes over a set of flows with cardinality at most \( |\Theta|(|\cup_k(\mathcal{A}^k \times \mathcal{A}^k)| + 1) \)

Proposition 5 has the following implication.

**Lemma 1.** The set of BCWE with finite support is dense in the set of BCWE.

**Proof.** Suppose that there exists a BCWE \( \mu \) which does not belong to the closure \( \mathcal{C} \) of the set of BCWE with finite support. This set \( \mathcal{C} \subseteq (\Delta(\mathcal{Y}))^\Theta \) is convex and closed. From the separation theorem, for each \( \theta \), there exists a continuous function \( f(\cdot, \theta) : \mathcal{Y} \rightarrow \mathbb{R} \) such that

\[
\sum_{\theta} p(\theta) \int f(y, \theta) \, d\mu(y | \theta) < \inf_{\nu \in \mathcal{C}} \left\{ \sum_{\theta} p(\theta) \int f(y, \theta) \, d\nu(y | \theta) \right\},
\]

which contradicts Proposition 5.

**Proof of Proposition 1.** We prove the result for multi-population games and general weights. Consider a game \( \Gamma_n \) with \( |\mathcal{K}| \) populations, where there are \( n^k \) players in population \( k \) and \( n = \sum_k n^k \). Each player \( i \) in population \( k \) has a weight \( w_i^k \) with \( \sum_{i=1}^{n^k} w_i^k = \gamma^k \) for each \( k \). We show that for every sequence of weights \( \{w(n)\} \) such that \( \max_{k,i} w_i^k(n) \rightarrow_n 0 \) and for every sequence \( \{\mu^n\} \) of BCE outcomes of the \( n \)-player games, any weak* accumulation point \( \mu \) of \( \{\mu^n\} \) is a BCWE. The result of the proposition corresponds to the particular case of a single population with weight \( w_i^k = 1/n \) for every player.

Given an action profile \( a = (a^k)_k \), the induced flow \( y(a) = (y^k(a^k))_k \) is such that for each \( k \) and \( a \in \mathcal{A}^k \), we have \( y_i^k(a^k) = \sum_{i=1}^{n^k} w_i^k(n) \mathbb{1}\{a_i^k = a\} \). A BCE of the \( n \)-player game is \( \beta^n : \Theta \rightarrow \Delta(\times_k(\mathcal{A}^k)^{n^k}) \) such that \( \forall k, \forall i = 1, \ldots, n^k, \forall a^k, b^k \in \mathcal{A}^k \), we have

\[
\sum_{\theta, a_{-i}} p(\theta) \beta^n(a^k, a_{-i}^k, a^{-k} | \theta) c_a(y^k(a^k, a_{-i}^k), y^{-k}(a^{-k}), \theta)
\leq \sum_{\theta, a_{-i}} p(\theta) \beta^n(a^k, a_{-i}^k, a^{-k} | \theta) c_a(y^k(b^k, a_{-i}^k), y^{-k}(a^{-k}), \theta).
\]

Eq. (B.6) can be rewritten as follows:

\[
E[\mathbb{1}\{a_i^k = a\} c_a(y(a), \theta)] \leq E[\mathbb{1}\{a_i^k = a\} c_a(y^k(a^k) + w_i^k(n)(\delta_a^k - \delta_a^k), y^{-k}(a^{-k}), \theta)],
\]

where \( \delta_a^k \) is the flow on \( \mathcal{A}^k \) such that \( y_a^k = 1 \). Multiplying by \( w_i^k(n) \) and summing over \( i = 29 \)
Consider a sequence \( \{ \mu^k \} \) where the left-hand side of Eq. (B.8) is
\[
\mathbb{E} \left[ \sum_i w_i^k(n) \mathbb{I} \{ a_i^k = a \} c_a^k(y(a), \theta) \right] 
\leq \mathbb{E} \left[ \sum_i w_i^k(n) \mathbb{I} \{ a_i^k = a \} c_b^k(y^k(a^k) + w_i^k(n)(\delta_b^k - \delta_a^k), y^{-k}(a^{-k}), \theta) \right].
\] (B.8)

The left-hand side of Eq. (B.8) is
\[
\mathbb{E} \left[ \sum_i w_i^k(n) \mathbb{I} \{ a_i^k = a \} c_a^k(y(a), \theta) \right] 
= \sum_{\theta, y} P(\theta, y(a) = y) \mathbb{E} \left[ \sum_i w_i^k(n) \mathbb{I} \{ a_i^k = a \} c_a^k(y(a), \theta) \mid y(a) = y, \theta \right] 
= \sum_{\theta, y} P(\theta, y(a) = y) \mathbb{E} \left[ \sum_i w_i^k(n) \mathbb{I} \{ a_i^k = a \} \mid y(a) = y \right] c_b^k(y, \theta) 
= \sum_{\theta, y} P(\theta, y(a) = y) y_a^k c_a^k(y, \theta) 
= \sum_{\theta} P(\theta) \mu^n(y \mid \theta) y_a^k c_a^k(y, \theta) 
= \sum_{\theta} P(\theta) \int y_a^k c_a^k(y, \theta) \, d\mu^n(y \mid \theta),
\] (B.9)

where \( \mu^n(\cdot \mid \theta) \in \Delta(\mathcal{Y}) \) is the (marginal) distribution of the flow \( y(a) \) induced by \( \beta^n(\cdot \mid \theta) \).

Consider a sequence \( \{ \beta^n \} \) of BCE of the \( n \)-player game such that for each \( \theta \), \( \mu^n(\cdot \mid \theta) \) weak* converges to some \( \mu(\cdot \mid \theta) \) in \( \mathcal{Y} \) (take a subsequence if needed) and suppose that \( \max_{k,i} w_i^k(n) \to n \) 0. Then, the expression in (B.9) tends to \( \sum_{\theta} P(\theta) \int y_a^k c_a^k(y, \theta) \, d\mu(y \mid \theta) \).

The right-hand side of Eq. (B.8) is
\[
\mathbb{E} \left[ \sum_i w_i^k(n) \mathbb{I} \{ a_i^k = a \} c_b^k(y(a), \theta) \right] 
= \sum_{\theta} P(\theta) \int y_a^k c_b^k(y, \theta) \, d\mu^n(y \mid \theta) + \text{error term}.
\] (B.10)

The first term in Eq. (B.10) tends to \( \sum_{\theta} P(\theta) \int y_a^k c_b^k(y, \theta) \, d\mu(y \mid \theta) \). For the second term, note that the finite family of functions \( \{ y \mapsto c_b^k(y, \theta) \}_{k, \theta, b} \) is uniformly equicontinuous on the compact \( \mathcal{Y} \); that is, \( \forall \varepsilon > 0, \exists \bar{w}, \text{s.t.} \forall w \leq \bar{w}, \forall k, \theta, a, b, \forall y \in \mathcal{Y}, \text{we have} |c_b^k(y^k + w(\delta_b^k - \delta_a^k), y^{-k}, \theta) - \)
\[ c_k^*(\mathbf{y}, \theta) \leq \varepsilon. \] Thus for all \( \varepsilon > 0 \), there exists \( \tilde{n} \) such that \( \forall n \geq \tilde{n}, \forall k, i \), we have \( w_i^k(n) \leq \tilde{w}; \) therefore, \( |\text{error term}| \leq \varepsilon \) for all \( n \geq \tilde{n} \). Therefore, the r.h.s. of Eq. (B.8) tends to \( \sum_{\theta} p(\theta) \int y_{\theta}^k c_k^*(\mathbf{y}, \theta) \, d\mu(\mathbf{y} | \theta) \), which proves that \( \mu \) is a BCWE. \hfill \Box

**Proof of Proposition 2.** We prove the result for games with a set \( \mathcal{K} \) of different populations with sizes \( \gamma^k > 0 \), \( \sum_k \gamma^k = \gamma \). We consider a game with \( |\mathcal{K}|n \) players with \( n \) players per population and where each player in population \( k \) has weight \( \gamma^k/n \). From Lemma 1 it is enough to prove the result for a BCWE \( \mu \) with finite support. Recall the BCWE inequality:

\[
\forall k \in \mathcal{K}, \forall a, b \in \mathcal{A}^k, \sum_{\theta, \mathbf{y}} p(\theta) \mu(\mathbf{y} | \theta) y_{\alpha}^k c_k^*(\mathbf{y}, \theta) = \sum_{\theta, \mathbf{y}} p(\theta) \mu(\mathbf{y} | \theta) y_{\alpha}^k c_k^*(\mathbf{y}, \theta). \tag{B.11}
\]

Equivalently,

\[
\forall k \in \mathcal{K}, \forall a, b \in \mathcal{A}^k, \sum_{\theta, \mathbf{y}} p(\theta) \mu(\mathbf{y} | \theta) \frac{y_{\alpha}^k}{\gamma^k} c_k^*(\mathbf{y}, \theta) = \sum_{\theta, \mathbf{y}} p(\theta) \mu(\mathbf{y} | \theta) \frac{y_{\alpha}^k}{\gamma^k} c_k^*(\mathbf{y}, \theta). \tag{B.12}
\]

Let \( \mathcal{Y}^* := \bigcup_{\alpha} \text{supp} \mu(\cdot | \theta) \). We approximate the numbers \( y_{\alpha}^k/\gamma^k \) with sequences of rationals for \( k \in \mathcal{K}, a \in \mathcal{A}^k \) and \( \mathbf{y} \in \mathcal{Y}^* \). For every integer \( n \), there exist integers \( (N_{k,a}^n(\mathbf{y}))_{k \in \mathcal{K}, a \in \mathcal{A}^k, \mathbf{y} \in \mathcal{Y}^*} \) such that for all \( \mathbf{y} \in \mathcal{Y}^*, k \in \mathcal{K}, a \in \mathcal{A}^k, \)

\[
\sum_{a \in \mathcal{A}^k} N_{k,a}^n(\mathbf{y}) = n \quad \text{and} \quad \left| \frac{N_{k,a}^n(\mathbf{y})}{n} - \frac{y_{\alpha}^k}{\gamma^k} \right| \leq \eta_n, \tag{B.13}
\]

with \( \lim_{n \to \infty} \eta_n = 0 \). The flow profile

\[
\left( \gamma^k \frac{N_{k,a}^n(\mathbf{y})}{n} \right)_{k \in \mathcal{K}, a \in \mathcal{A}^k} \tag{B.14}
\]

is denoted by \( \mathbf{y}_n \).

We construct \( \beta^n : \Theta \to \Delta(\times_k (\mathcal{A}^k)^n_k) \) as follows: Conditionally on state \( \theta \), the mediator draws \( \mathbf{y} \in \mathcal{Y}^* \) with probability \( \mu(\mathbf{y} | \theta) \), then, for each population \( k \), recommends action \( a \) to a subset of players of cardinality \( N_{k,a}^n(\mathbf{y}) \), chosen uniformly from population \( k \). Conditionally on \( (\theta, \mathbf{y}) \), the probability that player \( i \) in population \( k \) is recommended \( a \) is \( N_{k,a}^n(\mathbf{y})/n \). The total probability that player \( i \) in population \( k \) is recommended \( a \) is

\[
P^n_k(a) = \sum_{\theta, \mathbf{y}} p(\theta) \mu(\mathbf{y} | \theta) N_{k,a}^n(\mathbf{y}). \tag{B.15}
\]

Notice that \( P^n_k(a) \to_n \sum_{\theta, \mathbf{y}} p(\theta) \mu(\mathbf{y} | \theta) y_{\alpha}^k/\gamma^k =: P^k(a) \). If \( P^k(a) = 0 \), then \( P^n_k(a) \) is arbitrarily small for large \( n \), thus deviating after being recommended \( a \) cannot bring a profit larger than \( \varepsilon \)
for such \( n \). If \( P^k(a) > 0 \), then \( P^k_n(a) > 0 \) for all \( n \) large enough. Consider such an action \( a \) from now on. Conditionally on being recommended \( a \), the expected cost of player \( i \) in population \( k \) who plays \( b \) is

\[
\frac{1}{P^k_n(a)} \sum_{\theta, y} p(\theta) \mu(y \mid \theta) \frac{N^k_{a,\theta}(y)}{n} c_{\theta}^k \left( y_n^{-k}, y_n^k + \frac{\gamma}{n} (\delta_n^k - \delta_n^a), \theta \right).
\]

(B.16)

We have

\[
\left| \frac{N^k_{a,\theta}(y)}{n} c_{\theta}^k \left( y_n^{-k}, y_n^k + \frac{\gamma}{n} (\delta_n^k - \delta_n^a), \theta \right) - \frac{y_n^k}{\gamma} c_{\theta}^k(y, \theta) \right| \leq \left| \frac{N^k_{a,\theta}(y)}{n} c_{\theta}^k \left( y_n^{-k}, y_n^k + \frac{\gamma}{n} (\delta_n^k - \delta_n^a), \theta \right) - c_{\theta}^k(y_n, \theta) \right| + \left| \frac{N^k_{a,\theta}(y)}{n} c_{\theta}^k(y_n, \theta) - \frac{y_n^k}{\gamma} c_{\theta}^k(y, \theta) \right|
\]

(B.17)

Thus,

\[
\frac{N^k_{a,\theta}(y)}{n} c_{\theta}^k \left( y_n^{-k}, y_n^k + \frac{\gamma}{n} (\delta_n^k - \delta_n^a), \theta \right) - \frac{y_n^k}{\gamma} c_{\theta}^k(y, \theta) \leq \omega \left( \frac{\gamma}{n} \right) + \omega(\eta) =: \epsilon_n,
\]

(B.18)

where \( \omega(\cdot) \) is a modulus of continuity \( \lim_{\eta \downarrow 0} \omega(\eta) = 0 \) common to all mappings \( y \mapsto c_{\theta}^k(y, \theta), y \mapsto y_n^k c_{\theta}^k(y, \theta) \), for \( k \in K \) and \( (a, b) \in A^k \times A^k \). This exists since all these mappings are uniformly continuous on the compact \( \mathcal{Y} \). It follows from Eq. (B.12) that any unilateral deviation cannot lead to a profit larger that \( \epsilon_n \).

Proof of Proposition 3. We prove this result for games with multi-populations. For the first part, consider a general sequence of weights \( \{w(n)\} \) such that \( \max_{k,i} w_i^k(n) \to_n 0 \). The convergence result follows from Proposition 1 and the following lemma.

**Lemma 2.** Any weak* accumulation point \( \mu \) of \( \mu^n(\cdot \mid \theta) \) is a mapping \( y : \Theta \to \mathcal{Y} \). In other words, the distribution of flows \( \mu(\cdot \mid \theta) \) in state \( \theta \) is a Dirac mass on some flow \( y(\theta) \).

Proof. Consider a sequence of BCE \( \beta^n \) with outcome \( \mu^n \) which has the conditional independence property for each \( n \) and, up to extracting a subsequence, assume that \( \mu^n(\cdot \mid \theta) \) converges to \( \mu(\cdot \mid \theta) \) for each \( \theta \in \Theta \). For each \( k \) and action \( a \in A^k \), let \( Y^k_a(n) := \sum_{i=1}^n w_i^k(n) \mathbb{1} \{a_i^k = a\} \) denote the random flow of population \( k \) on action \( a \). We have \( \mathbb{E}[Y^k_a(n) \mid \theta] = \int y_a^k \, d\mu^{
u}(y \mid \theta) \to \int y_a^k \, d\mu(y \mid \theta) =: y_a^k(\theta) \). To prove the result, it is enough to show that conditionally on \( \theta \), the variance of \( Y^k_a(n) \) tends to 0.

\[
\mathbb{Var}_\theta(Y^k_a(n)) \leq \frac{\sum_i (w_i^k(n))^2}{4} \leq \frac{(\max_i w_i^k(n)) \sum_i w_i^k(n)}{4} = \frac{\max_i w_i^k(n) \gamma_k}{4} \to 0.
\]

(B.19)
Thus, conditionally on \( \theta \), the random variable \( Y^k_a(n) \) converges in distribution to \( y^k_a(\theta) \), which means that \( \mu^n(\cdot \mid \theta) \) weak* converges to \( \delta_{y(\theta)} \).

For the second part, we consider multi-populations \( k \in K \) with sizes \( \gamma^k/n \) in each population. Take a BDWE \( y(\cdot) \), fix a number of players \( n \) for each population \( k \) and construct \( \beta^n \) as follows. For each state \( \theta \) and each population \( k \), each player is recommended action \( a \) with probability \( y^k_a(\theta)/\gamma^k \); recommendations are i.i.d. across players in each population and independent across populations. Let \( Z^k_a(n) = (\gamma^k/n) \sum_{i=1}^n 1\{a_i^k = a\} \) denote the weighted average number of players in population \( k \) who are recommended action \( a \) and let \( Z^k_k(n) := (Z^k_a(n))_{a \in A^k} \). Under \( \beta^n \), the expected cost \( C^k_{a,b}(\beta^n) \) of a player \( i \) in population \( k \) who plays \( b \), conditionally on being recommended \( a \), is:

\[
C^k_{a,b}(\beta^n) = \frac{1}{P^k(a)} \sum_\theta p(\theta) E[1\{a_i^k = a\}c^k_b(Z^{-k}(n), Z^k(n) + (\gamma^k/n)(\delta^k_b - \delta^k_a), \theta)],
\]

which is

\[
P^k(a) := \sum_\theta p(\theta) \frac{y^k_a(\theta)}{\gamma^k}.
\]

The product \( P^k(a)C^k_{a,b}(\beta^n) \) is the expectation \( E[1\{a_i^k = a\}(\text{cost of playing } b)] \). By the law of large numbers, this converges to

\[
\sum_\theta p(\theta) \frac{y^k_a(\theta)}{\gamma^k} c^k_b(y^{-k}(\theta), y^k(\theta), \theta).
\]

Since the set of triples \((k, a, b)\) is finite, for any \( \varepsilon > 0 \), there exists \( \bar{n} \) such that for all \( n \geq \bar{n} \), for all \( k \in K \) and all \( a, b \in A^k \), we have

\[
\left| P^k(a)C^k_{a,b}(\beta^n) - \sum_\theta p(\theta) \frac{y^k_a(\theta)}{\gamma^k} c^k_b(y^{-k}(\theta), y^k(\theta), \theta) \right| \leq \frac{\varepsilon}{2}.
\]

Since \( y(\cdot) \) is a BDWE, from Eq. (B.12), we get that \( \beta^n \) is an \( \varepsilon \)-BCE.

The next result is a generalization of Proposition 4 to general multi-population games.

**Proposition 11.** If a multi-population game has a convex potential, then

\[
\text{CWE}(\Gamma) = \Delta(\text{WE}(\Gamma)).
\]

In addition, all WE (and therefore all CWE) have the same costs profiles: all actions with positive flow have the same cost in all equilibria.
Proof. We first recall the following lemma.

Lemma 3. If a multi-population game has a convex potential, then the set of WE is the set of minimizers of $\Phi$.

Proof. This result follows from Sandholm (2001, proposition 3.1), who proves that the set of WE is the set of Karush–Kuhn–Tucker (KKT) points of the minimization problem $\min \{ \Phi(y) : y \in Y \}$. Since $\Phi$ is convex, the KKT points are the minimizers of $\Phi$.

From Lemma 3 we have $\Delta(\arg \min \Phi) = \arg \min_{\mu \in \Delta(Y)} E_\mu \Phi \subseteq \text{CWE}(\Gamma)$. Conversely, take a CWE $\mu$ and suppose that there exists $z \in \Delta(Y)$ such that $\Phi(z) < \int \Phi(y) \, d\mu(y)$. Using the fact that $\Phi(z) - \Phi(y) \geq \nabla \Phi(y)(z - y)$ and integrating, we obtain

$$\int \nabla \Phi(y)(z - y) \, d\mu(y) \leq \Phi(z) - \int \Phi(y) \, d\mu(y) < 0.$$ 

We have

$$\nabla \Phi(y)(z - y) = \sum_{k \in K} \sum_{a \in A^k} (z^k_a - y^k_a)c^k_a(y),$$

so

$$\sum_{k \in K} \sum_{a \in A^k} \int y^k_ac^k_a(y) \, d\mu(y) > \sum_{k \in K} \sum_{a \in A^k} z^k_a \int c^k_a(y) \, d\mu(y) \geq \sum_{k \in K} \min_{a \in A^k} \int c^k_a(y) \, d\mu(y).$$

Thus, there exist $k \in K$ such that

$$\sum_{a \in A^k} \int y^k_ac^k_a(y) \, d\mu(y) > \sum_{a \in A^k} z^k_a \int c^k_a(y) \, d\mu(y) \geq \min_{a \in A^k} \int c^k_a(y) \, d\mu(y),$$

and $b \in A^k$ such that

$$\sum_{a \in A^k} \int y^k_bc^k_a(y) \, d\mu(y) > \sum_{a \in A^k} z^k_a \int c^k_a(y) \, d\mu(y) \geq \int c^k_b(y) \, d\mu(y).$$

Since

$$\int c^k_b(y) \, d\mu(y) = \sum_{a \in A^k} \int y^k_ac^k_b(y) \, d\mu(y),$$

we get a contradiction with Eq. (A.2) and $\mu$ cannot be a CWE.

Next, we show that all WE have the same costs. We know from Lemma 3 that the WE minimize the potential, i.e., they are the solutions of the following convex optimization problem:

$$\min \left\{ \Phi(y) : \forall k \in K, \sum_a y^k_a = 1, \forall a \in A, \forall k \in K, y^k_a \geq 0 \right\}.$$
The Lagrangian of this problem is

\[ L(y, \lambda) = \Phi(y) - \sum_{k \in K} \lambda^k \left( \sum_{a \in A} y^k_a \right) - \sum_{a \in A} \sum_{k \in K} \lambda_{a,k} y_a, \]

with \( \lambda = \left( (\lambda^k)_{k \in K}, (\lambda_{a,k})_{a \in A, k \in K} \right) \). From the KKT theorem, \( \hat{y} \) is a solution if and only if there exists \( \hat{\lambda} \) such that \( (\hat{y}, \hat{\lambda}) \) satisfies the KKT conditions: For all \( a \in A, k \in K \),

\[ \frac{\partial \Phi}{\partial y^k_a}(\hat{y}) = \hat{\lambda}^k + \hat{\lambda}_a^k; \quad \hat{\lambda}_a^k \cdot \hat{y}_a^k = 0; \quad \hat{\lambda}_a^k \geq 0; \quad \sum_{a} \hat{y}_a^k = 1. \]

This condition is satisfied if and only if for all \( (y, \lambda) \),

\[ L(y, \lambda) \leq L(\hat{y}, \lambda) \leq L(\hat{y}, \hat{\lambda}), \]

(see Rockafellar, 1970, theorem 28.3, page 281). From the exchange property, if \( (\hat{y}, \hat{\lambda}) \) and \( (\bar{y}, \bar{\lambda}) \) are such saddle points, then \( (\hat{y}, \bar{\lambda}) \) and \( (\bar{y}, \hat{\lambda}) \) are also saddle points.

Consider the relative interior of the set \( \text{WE}(\Gamma) \). For each \( k \in K \) there exists a subset of actions \( B^k \subseteq A^k \) such that \( \times_{k \in K} B^k \) is the support of all flows in this relative interior. To see this, notice that for each pair of WE \( y, z \) such that \( y \) is in the relative interior and for each \( t \in (0, 1] \), we have that \( ty + (1 - t)z \) is also in the relative interior. Therefore, for each \( a, k \), whenever \( y_a^k > 0 \) for some WE, this must be true for all points in the relative interior of WE.

Consider now two points \( \hat{y}, \bar{y} \) in the relative interior of \( \text{WE}(\Gamma) \). For every \( a \in B^k \), we have \( \bar{y}_a^k > 0, \bar{y}_a^k > 0 \) and

\[ \frac{\partial \Phi}{\partial y^k_a}(\hat{y}) = \hat{\lambda}_a^k = \bar{\lambda}_a^k = \frac{\partial \Phi}{\partial y^k_a}(\bar{y}). \]

Thus, for every \( k \in K \) and \( a \in B^k \), \( c^k_a(\hat{y}) = c^k_a(\bar{y}) \) and all points in the relative interior of \( \text{WE}(\Gamma) \) have the same costs. By continuity, all points in \( \text{WE}(\Gamma) \) have the same costs (notice that the support can only shrink when approaching the boundary).

**Proof of Proposition 4.** Directly from Proposition 11.

**Proof of Proposition 6.** Consider the problem of minimizing the expected designer cost,

\[ \min_{\mu} \sum_{\theta} p(\theta) \int \text{DC}(y, \theta) \; d\mu(y \mid \theta), \]

\[ \text{s.t. } \forall a, b \sum_{\theta} p(\theta) \int y_a c_a(y, \theta) \; d\mu(y \mid \theta) \leq \sum_{\theta} p(\theta) \int y_a c_b(y, \theta) \; d\mu(y \mid \theta). \]

Since there are two actions, \( y_b = 1 - y_a \), the incentive constraint where \( a \) is the recommended
action and \( b \) is the deviation is
\[
\sum_{\theta} p(\theta) \int y_a c_a(y, \theta) \, d\mu(y \mid \theta) \leq \sum_{\theta} p(\theta) \int (1 - y_b) c_b(y, \theta) \, d\mu(y \mid \theta),
\]
which can be written as
\[
\sum_{\theta} p(\theta) \int y_a c_a(y, \theta) \, d\mu(y \mid \theta) + \sum_{\theta} p(\theta) \int y_b c_b(y, \theta) \, d\mu(y \mid \theta) \leq \sum_{\theta} p(\theta) \int c_b(y, \theta) \, d\mu(y \mid \theta).
\]
(B.26)

For each action \( a \) and state \( \theta \), define \( \bar{y}_a(\theta) = \int y_a \, d\mu(y \mid \theta) \) as the expected flow. Since the mappings \( y \mapsto y_a c_a(y, \theta) \) are convex and \( y \mapsto c_a(y, \theta) \) are concave, we have
\[
\sum_{\theta} p(\theta) \int y_a c_a(y, \theta) \, d\mu(y \mid \theta) \geq \sum_{\theta} p(\theta) \bar{y}_a(\theta) c_a(\bar{y}(\theta), \theta),
\]
and
\[
\sum_{\theta} p(\theta) \int c_b(y, \theta) \, d\mu(y \mid \theta) \leq \sum_{\theta} p(\theta) c_b(\bar{y}(\theta), \theta).
\]
Therefore, Eq. (B.26) implies that \( \theta \mapsto \bar{y}(\theta) \) is a BDWE. Further, it yields a weakly smaller designer cost since \( \text{DC}(\cdot, \theta) \) is convex and from Jensen’s inequality:
\[
\sum_{\theta} p(\theta) \int \text{DC}(y, \theta) \, d\mu(y \mid \theta) \geq \sum_{\theta} p(\theta) \text{DC}(\bar{y}(\theta), \theta).
\]

Proof of Proposition 7. Let \( y(\cdot) \) be a BWE. Multiplying Eq. (6.2) by \( y^k_a(\tau^k) \) gives \( \forall k, \forall \tau^k, \forall a, b, \)
\[
\sum_{\theta \in \Theta} p(\theta) \sum_{\tau^k \in T^{-k}} y^k_a(\tau^k) \pi(\tau^k, \tau^{-k} \mid \theta) c_a(y(\tau), \theta) \leq \sum_{\theta \in \Theta} p(\theta) \sum_{\tau^k \in T^{-k}} y^k_a(\tau^k) \pi(\tau^k, \tau^{-k} \mid \theta) c_b(y(\tau), \theta).
\]
Summing over \( k \) and then over \( \tau^k \) we get
\[
\forall a, b, \sum_{\theta} p(\theta) \sum_{\tau} \pi(\tau \mid \theta) y_a(\tau) c_a(y(\tau), \theta) \leq \sum_{\theta} p(\theta) \sum_{\tau} \pi(\tau \mid \theta) y_a(\tau) c_b(y(\tau), \theta).
\]
Letting \( \mu(y \mid \theta) = \sum_{\tau : y(\tau) = y} \pi(\tau \mid \theta) \), we conclude that \( \mu \) is a BCWE.

Proof of Proposition 8. Proposition 8 directly follows from the following lemma.

Lemma 4. Let \( \mu \) be a BCWE with finite support. Then there exist a sequence \( \varepsilon^k \downarrow 0 \), a sequence of information structures, and associated Bayesian Wardrop \( \varepsilon^k \)-equilibrium flows \( \mu^k \), such that
\( \mu^k \xrightarrow{w^*} \mu \). Furthermore, if all flows in the support of \( \mu \) are rational, then we can choose \( \varepsilon^k = 0 \) and \( \mu^k = \mu \).

**Proof.** Let \( \mu \) be a BCWE with finite support \( \mathcal{Y}^* \). Then, for all \( a, b \in \mathcal{A} \),

\[
\sum_{\theta \in \Theta} \sum_{y \in \mathcal{Y}^*} p(\theta) \mu(y \mid \theta) y_a c_a(y, \theta) \leq \sum_{\theta \in \Theta} \sum_{y \in \mathcal{Y}^*} p(\theta) \mu(y \mid \theta) y_a c_b(y, \theta). \tag{B.27}
\]

We use vectors with rational components to approximate flows in \( \mathcal{Y}^* \). For every \( y \in \mathcal{Y}^* \), \( a \in \mathcal{A} \), and \( k \in \mathbb{N} \), there exist \( N^k(y_a) \in \mathbb{N} \) and \( \eta^k > 0 \) such that

\[
\sum_{a \in \mathcal{A}} N^k(y_a) = k \quad \text{and} \quad \left| \frac{N^k(y_a)}{k} - y_a \right| \leq \eta^k, \tag{B.28}
\]

and \( \eta^k \downarrow 0 \). Let \( y^k \) denote the flow \( (y^k_{a})_{a \in \mathcal{A}} \) with \( y^k_a = N^k(y_a)/k \).

We construct an information structure as follows. There are \( k \) population of players, each population has the same size \( \gamma^k = 1/k \), the set of types is \( \mathcal{T}^k = \mathcal{A} \) for each population \( k \). Conditionally on state \( \theta \), the mediator draws \( y \in \mathcal{Y}^* \) with probability \( \mu(y \mid \theta) \), then recommends action \( a \) to a subset of populations of size \( N^k(y_a) \) chosen uniformly, so that, conditionally on \( y \) and \( \theta \), the probability that population \( k \) is recommended \( a \) is \( y^k_a \). We let \( \mu^k \) be the induced outcome which puts probability \( \mu(y \mid \theta) \) on \( y^k \) in state \( \theta \).

The expected cost of playing \( b \) conditional of being recommended \( a \) (multiplied by the total probability of \( a \)) is

\[
\sum_{\theta \in \Theta} \sum_{y \in \mathcal{Y}^*} p(\theta) \mu(y \mid \theta) y_a c_b(y^k, \theta). \tag{B.29}
\]

The mapping \( y \mapsto y_a c_b(y, \theta) \) is uniformly continuous on \( \mathcal{Y} \) for each pair of actions \( (a, b) \), thus, there exists a common modulus of continuity \( \omega(\cdot) \) with \( \lim_{\eta \downarrow 0} \omega(\eta) = 0 \) such that, \( \forall \theta \in \Theta, \forall (a, b) \in \mathcal{A} \times \mathcal{A}, \forall (y, z) \in \mathcal{Y} \times \mathcal{Y} \),

\[
|y_a c_b(y, \theta) - z_a c_b(z, \theta)| \leq \omega(\max_b |y_b - z_b|). \tag{B.30}
\]

It follows from Eqs. (B.27), (B.28) and (B.30) that

\[
\sum_{\theta \in \Theta} \sum_{y \in \mathcal{Y}^*} p(\theta) \mu(y \mid \theta) y_a c_a(y^k, \theta) \leq \sum_{\theta \in \Theta} \sum_{y \in \mathcal{Y}^*} p(\theta) \mu(y \mid \theta) y_a c_b(y^k, \theta) + 2\omega(\eta^k).
\]

This way, we have constructed a Bayesian Wardrop \( \varepsilon^k \)-equilibrium with \( \varepsilon^k = 2\omega(\eta^k) \). The induced outcome weak* converges to \( \mu \) because, for every continuous \( f : \mathcal{Y} \times \Theta \to \mathbb{R} \), we have

\[
\sum_{\theta \in \Theta} \sum_{y \in \mathcal{Y}^*} \mu(y \mid \theta) f(y^k, \theta) \xrightarrow{k \to \infty} \sum_{\theta \in \Theta} \sum_{y \in \mathcal{Y}^*} \mu(y \mid \theta) f(y, \theta). \tag{B.31}
\]
If flows in $Y^∗$ have rational coefficients with common denominator $k$, then we can choose $y^k = y$, $\eta^k = 0$ and we have a Bayesian Wardrop 0-equilibrium.

**Proof of Proposition 9.** Proposition 9 follows directly from Lemmas 1 and 4.

**Proof of Proposition 10.** Consider a Bayesian game and an information structure $(\gamma^k)_{k \in K}$, $T = \times_{k \in K} T^k$ and $\pi : \Theta \to \Delta(T)$. Consider the multi-population auxiliary “agent-normal-form” game with complete information where populations are indexed by $(k, \tau^k)$, for $k \in K, \tau^k \in T^k$ with flows $y_a^{k, \tau^k}$, $y = (y_a^{k, \tau^k})_{k, \tau^k, a}$ and costs

$$c^{k, \tau^k}_a(y) = \sum_{\theta, \tau^{-k}} p(\theta)\pi(\tau^k, \tau^{-k} | \theta) c_a\left(\left(\sum_{l} y_{b}^{l, \tau^l}\right)_{b \in A}, \theta\right). \quad (B.32)$$

We can see that a WE of this auxiliary game is a Bayesian Wardrop equilibrium for this information structure, with $y_a^{k} (\tau^k) = y_a^{k, \tau^k}$ (as given by Eq. (6.1)), for every $a \in A, \tau^k \in T^k, k \in K$.

Suppose that there is a potential $\Phi_\theta$ in each state $\theta$, then

$$\Phi_\pi(y) := \sum_{\theta, \tau} p(\theta)\pi(\tau | \theta)\Phi_\theta\left(\left(\sum_{k} y_a^{k, \tau^k}\right)_{a \in A}\right) \quad (B.33)$$

is a potential for the auxiliary game: $\frac{\partial \Phi_\pi(y)}{\partial y_a^{k, \tau^k}} = c^{k, \tau^k}_a(y)$. If $\Phi_\theta$ is convex for each $\theta$, then so is $\Phi_\pi$ and by Proposition 4 the equilibrium cost profile is unique for any information structure. If $\Phi_\theta$ is strictly convex for each $\theta$, then so is $\Phi_\pi$ and, by Lemma 3, equilibrium flows are unique for any information structure.

38
C Binary congestion games

In this section we illustrate our concepts in a class of binary-action and binary-state congestion games. We also characterize and compute the optimal solution for the designer in some enlightening special cases, to illustrate the differences between optimal public information structures, optimal BDWE and optimal BCWE, depending on the designer’s objective and players’ cost parameters.

We assume $\mathcal{A} = \{a, b\}$ and $y_b = 1 - y_a$, where $y_a$ is the flow on action $a$. The state space is $\Theta = \{\theta, \bar{\theta}\}$, with $p(\theta) = p$ and $p(\bar{\theta}) = \bar{p}$. The cost of action $a$ only depends on the state. The cost of action $b$ does not depend on the state and is increasing in the flow on action $b$:

$$
c_a(y, \theta) = \mathbb{1}\{\theta = \bar{\theta}\}, \quad c_b(y, \theta) = \alpha y_b + \beta, \quad \text{with } \alpha, \beta \geq 0. \quad (C.1)
$$

The designer’s cost function in state $\theta$ is denoted by $DC(y, \theta)$. In what follows we will consider as special cases the two following cost functions for the designer. The first is simply the total cost:

$$
DC(y, \theta) = TC(y, \theta) := (1 - y_b)c_a(y, \theta) + y_bc_b(y, \theta) = (1 - y_b)\theta + y_b(\alpha y_b + \beta). \quad (C.2)
$$

The second is a state-independent, increasing and concave function of the flow on action $b$:

$$
DC(y, \theta) = g(y_b), \quad (C.3)
$$

where $g$ is concave and strictly increasing, $g(0) = 0$ and $g(1) = 1$. In Section C.3 we will consider a generalization of the previous cost function that allows the ideal flow of the designer to be state-dependent. For all these special cases, the ideal flow of the designer is $0$ in state $\theta = \theta$, so there is no conflict of interest between players and the designer in that state.

C.1 Public Information

Consider first the case in which the designer can only disclose public information, so that a belief-based approach can be used, as in repeated games with incomplete information (Aumann and Maschler, 1995) and in sender-receiver Bayesian persuasion problems (Kamenica and Gentzkow, 2011). If players get public information, then for every signal they have a common posterior $q$. Given this common posterior $q$, they play the (symmetric information) average game with cost $q(\bar{\theta}) = \bar{q}$ for action $a$ and cost $\alpha y_b + \beta$ for action $b$. It is immediate that the Wardrop equilibrium
of this average game is characterized by the following flow on action $b$:

$$y_b(q) = \begin{cases} 
0 & \text{if } \bar{q} \leq \beta, \\
\frac{\bar{q} - \beta}{\alpha} & \text{if } \beta < \bar{q} < \alpha + \beta, \\
1 & \text{if } \alpha + \beta \leq \bar{q},
\end{cases} \quad (C.4)$$

when either $\bar{q} \neq \beta$ or $\alpha \neq 0$. If $\bar{q} = \beta$ and $\alpha = 0$, then every flow constitutes a Wardrop equilibrium, and in this case the designer-preferred Wardrop equilibrium is selected. The symbol $C(q)$ denotes the indirect cost for the designer at $q$, that is, the expected cost of the designer when the common posterior is $q$ and players play the (designer-preferred) Wardrop equilibrium at $q$:

$$C(q) := \bar{q} \text{DC}(y(\bar{q}), \theta) + (1 - \bar{q}) \text{DC}(y(q), \theta). \quad (C.5)$$

The optimal public information structure for the designer is characterized by a splitting $\{q_k, \lambda_k\}_k$ of $p$, i.e., a probability distribution over posterior beliefs where $\lambda_k$ is the probability of posterior $q_k$ and $\sum_k \lambda_k q_k = p$, that minimizes the ex-ante expected indirect cost for the designer:

$$\sum_k \lambda_k C(q_k).$$

The minimal ex-ante expected cost of the designer under public information is denoted by $C^{\text{Pub}} = \text{vex} C(p)$, where $\text{vex} C(\cdot)$ is defined as in Eq. (5.3).

**Minimizing total cost.** Consider the case in which the designer’s cost function is given by Eq. (C.2), so that the designer aims at minimizing the total cost. The indirect cost function is then given by

$$C(q) = \begin{cases} 
\bar{q} & \text{if } \bar{q} \leq \alpha + \beta, \\
\alpha + \beta & \text{if } \bar{q} > \alpha + \beta.
\end{cases} \quad (C.6)$$

Then, $\text{vex} C(p) = (\alpha + \beta)p$ if $\alpha + \beta < 1$, in which case the unique optimal public information structure is full information, and $\text{vex} C(p) = p$ if $\alpha + \beta \geq 1$, in which case every information structure is optimal.

**Minimizing the flow on action $b$.** Consider the case in which the designer’s cost function is given by Eq. (C.3), so that the designer would like to minimize the flow on action $b$, whatever the state. The indirect cost function is then given by $C(q) = g(y_b(q))$, which is equal to 0 for
\( \bar{q} \leq \beta \), is concave and strictly increasing for \( \beta \leq \bar{q} \leq \alpha + \beta \), and is equal to 1 for \( \bar{q} \geq \alpha + \beta \):

\[
C(q) = \begin{cases} 
  g(0) = 0 & \text{if } \bar{q} \leq \beta, \\
  g(\frac{\bar{q} - \beta}{\alpha}) & \text{if } \beta < \bar{q} < \alpha + \beta, \\
  g(1) = 1 & \text{if } \alpha + \beta \leq \bar{q}.
\end{cases}
\] (C.7)

Hence, the optimal expected cost of the designer with public information is

\[
\text{vex } C(p) = \begin{cases} 
  0 & \text{if } \bar{p} \leq \beta, \\
  \frac{\bar{p} - \beta}{1 - \beta} & \text{if } \bar{p} \geq \beta \text{ and } \alpha + \beta \leq 1, \\
  g\left(\frac{1 - \beta}{\alpha}\right) \frac{\bar{p} - \beta}{1 - \beta} & \text{if } \bar{p} \geq \beta \text{ and } \alpha + \beta > 1.
\end{cases}
\] (C.8)

The optimal information structure is no information disclosure if \( \bar{p} \leq \beta \). If \( \bar{p} > \beta \), then the designer splits the prior \( \bar{p} \) to the posterior \( q = 1 \) with probability \( \frac{\bar{p} - \beta}{1 - \beta} \) and to the posterior \( q = \beta \) with the complementary probability. Notice that when \( \alpha = 0 \), i.e., when there is no cost externalities, this example is analogous to the judge-prosecutor example in Kamenica and Gentzkow (2011).

\section{C.2 Private Information with Deterministic Flows: BDWE}

Consider now the case in which the designer can use private information structures, but is restricted to deterministic flows. That is, the minimization of the designer’s expected cost is achieved with the choice of a BDWE (see Eq. (2.3)). A BDWE is given by a flow \( \bar{y}_b \) on action \( b \) in state \( \bar{\theta} \) and a flow \( y_b \) on action \( b \) in state \( \theta \) satisfying the following incentive-compatibility constraints, obtained from Eq. (2.3) in the definition of BDWE:

\[
\begin{align*}
\alpha(\bar{p}y_b^2 + py_b^2) + \beta(\bar{p}y_b + py_b) - \bar{p}y_b &\leq \beta - \bar{p} + \alpha(\bar{p}y_b + py_b), \\
\alpha(\bar{p}y_b^2 + py_b^2) + \beta(\bar{p}y_b + py_b) - \bar{p}y_b &\leq 0.
\end{align*}
\] (ICa) (ICb)

Therefore, the problem is to choose the values \( \bar{y}_b \) and \( y_b \) that minimize the designer’s ex-ante expected cost

\[
C_{\text{BDWE}}(\bar{y}_b, y_b) := \bar{p} DC(\bar{y}_b, \bar{\theta}) + p DC(y_b, \theta),
\] (C.9)

under the constraints (ICa) and (ICb).

**Minimizing total cost.** Consider the case in which the designer wants to minimize the total cost (see Eq. (C.2)). The designer’s expected cost given \( \bar{y}_b \) and \( y_b \) becomes

\[
C_{\text{BDWE}}(\bar{y}_b, y_b) = \alpha(\bar{p}y_b^2 + py_b^2) + \beta(\bar{p}y_b + py_b) - \bar{p}y_b + \bar{p}.
\] (C.10)
Observe that by minimizing $C_{\text{BDWE}}$ the designer is also minimizing the LHS of (ICb) so the constraint (ICb) can be ignored. Hence, the designer’s problem is to minimize $C_{\text{BDWE}}(\bar{y}_b, y_b)$ under the constraint (ICa). Clearly, the designer’s first-best is given by $y_b = 0$, which produces

$$C_{\text{BDWE}}(\bar{y}_b, y_b) = \alpha \bar{p} \bar{y}_b^2 + \beta \bar{p} \bar{y}_b - \bar{p} \bar{y}_b + \bar{p}. \quad (C.11)$$

The value of $\bar{y}_b$ that minimizes (C.11) is

$$\bar{y}_b^{\text{FB}} := \begin{cases} 
0 & \text{if } \beta \geq 1 \\
\frac{1 - \beta}{2\alpha} & \text{if } 1 - 2\alpha \leq \beta \leq 1 \\
1 & \text{if } \beta \leq 1 - 2\alpha.
\end{cases} \quad (C.12)$$

It can be easily verified that this first-best outcome is incentive compatible when it is not interior, i.e., when $\beta \geq 1$ or $\beta \leq 1 - 2\alpha$. Assume now that it is interior, i.e., $1 - 2\alpha < \beta < 1$. At this first-best, the incentive-compatibility condition becomes

$$\alpha \bar{p} \bar{y}_b^2 + \beta \bar{p} \bar{y}_b - \bar{p} \bar{y}_b \leq \beta - \bar{p} + \alpha \bar{p} \bar{y}_b. \quad (C.13)$$

Observe that this condition is always satisfied when $\bar{p}$ is sufficiently small. Replacing $\bar{y}_b$ with $(1 - \beta)/2\alpha$ and simplifying, the incentive-compatibility condition can be rewritten as

$$\bar{p}(4\alpha + (1 - \beta)(\beta - 2\alpha - 1)) \leq 4\alpha \beta. \quad (C.14)$$

If this condition is satisfied and $1 - 2\alpha < \beta < 1$, then it is possible to achieve the designer’s first-best, which is strictly smaller than what can be obtained with the best public information structure. This is the case, for instance, in one of the example of Das et al. (2017) (see Example 3 in the core of the text). In that example, $\alpha = 2$, $\beta = 1/3$ and $\bar{p} = 1/2$, so $\bar{y}_b^{\text{FB}} = 1/6$ and the previous inequality is equivalent to $\bar{p} \leq 6/11$. The total cost is

$$C_{\text{BDWE}}(\frac{1}{6}, 0) = \bar{p} - \frac{1}{18} \bar{p} < \text{vex } C(p) = \bar{p}. \quad (C.15)$$

**Minimizing the flow on action $b$.** Assume that the cost of the designer is given by Eq. (C.3). The designer’s expected cost given $\bar{y}_b$ and $y_b$ is then

$$C_{\text{BDWE}}(\bar{y}_b, y_b) = \bar{p}g(\bar{y}_b) + pg(y_b). \quad (C.16)$$

If $\beta \geq \bar{p}$, then designer’s first best is achieved with $\bar{y}_b = y_b = 0$, which is the same as under public information because it corresponds to no information disclosure. Assume now that $\beta < \bar{p}$.
Constraint (ICA) is now the binding constraint at the optimum, so the program of the designer is to minimize \( \bar{p}g(\bar{y}_b) + pg(y_b) \) under the constraint

\[
\alpha(\bar{p}\bar{y}_b^2 + p y_b^2) + \beta(\bar{p}\bar{y}_b + p y_b) - \bar{p}y_b = \beta - \bar{p} + \alpha(\bar{p}y_b + p y_b).
\]  
(C.17)

As an illustration, let \( \alpha = 0 \). Then the binding constraint becomes

\[
\beta(\bar{p}\bar{y}_b + p y_b) - \bar{p}y_b = \beta - \bar{p}.
\]  
(C.18)

Then, at the optimum we have

\[
y_b = 0 \quad \text{and} \quad \bar{y}_b = \frac{\bar{p} - \beta}{\bar{p}(1 - \beta)},
\]  
(C.19)

so

\[
C_{BDWE}(\bar{y}_b, y_b) = \bar{p}g\left(\frac{\bar{p} - \beta}{\bar{p}(1 - \beta)}\right).
\]  
(C.20)

Observe that when \( g \) is strictly concave we have

\[
C_{BDWE}(\bar{y}_b, y_b) = \bar{p}g\left(\frac{\bar{p} - \beta}{\bar{p}(1 - \beta)}\right) > \frac{\bar{p} - \beta}{1 - \beta} = C_{Pub}.
\]  
(C.21)

Hence, the designer is strictly better off with a public information structure than with a private information structure and deterministic flows, and the BCWE induced by the optimal public information structure is not a convex combination of BDWE (see Fig. 1). We get Example 4 in the core of the text when \( \bar{p} = 1/2, \alpha = 0 \) and \( \beta = 1/3 \), in which case \( \bar{y}_b = 1/2 \).

![Figure 1: Graph of \( C_{BDWE} \) (red) and \( C_{Pub} \) (blue) as a function of \( \beta \), when \( \bar{p} = 1/2 \) and \( g(\cdot) = \sqrt{\cdot} \).](image-url)
C.3 General information structures

In the class of problems studied in this section we know from Proposition 6 that if the designer aims at minimizing the total cost, then deterministic flows suffice. However, we have seen above that if the designer has a different objective, then the best BDWE could be dominated by a BCWE. In particular, we have seen that if the designer wants to minimize the flow on action $b$ and has a strictly concave cost function (see Eq. (C.3)), then the best BDWE is dominated by a public information structure. We provide below a class of examples in which the designer could improve upon all BDWE and all public information structures. That is, there exists a BCWE which is strictly better for the designer than every equilibrium induced by public information or deterministic flows.

Consider a generalization of the designer’s cost function of Eq. (C.3), where the designer has a state-dependent ideal flow:

$$DC(y_b, \bar{\theta}) = g\left(|y_b - \bar{y}^D_b|\right), \quad DC(y_b, \theta) = g(y_b),$$

(C.22)

where $g$ is strictly concave and strictly increasing, with $g(0) = 0$ and $g(1) = 1$. In state $\theta$ the designer’s ideal flow on action $b$ is still 0, but in state $\bar{\theta}$ the designer’s ideal flow is

$$\bar{y}^D_b \in \left[0, \frac{\bar{p} - \beta}{\bar{p}(1 - \beta)}\right),$$

(C.23)

where we assume that $\beta < 1/2$, $\bar{y}^D_b < 1/2$ and $0 < \beta < \bar{p}$. When $\bar{y}^D_b = \bar{y}^{FR}_b$, the designer’s ideal flow coincides with the flow that minimizes the total cost, and if $\bar{y}^D_b = 0$, then the designer wants to minimize the flow on action $b$, whatever the state. To simplify the presentation we further assume that $\alpha = 0$, i.e., the cost of action $b$ is constant.

We first characterize the optimal designer cost with public information. The indirect cost function is

$$C(q) = qg(y_b(q) - \bar{y}^D_b) + (1 - q)g(y_b(\bar{q})), \quad \text{(C.24)}$$

where the designer’s preferred equilibrium flow with common posterior $q$ is

$$y_b(q) = \begin{cases} 
0 & \text{if } \bar{q} \leq \beta \\
1 & \text{if } \bar{q} > \beta.
\end{cases} \quad \text{(C.25)}$$

Hence, the indirect cost function is given by

$$C(q) = \begin{cases} 
\bar{q}g(\bar{y}^D_b) & \text{if } \bar{q} \leq \beta \\
1 - \bar{q}(1 - g(1 - \bar{y}^D_b)) & \text{if } \bar{q} > \beta.
\end{cases} \quad \text{(C.26)}$$
Using the assumption \( y^D_b < 1/2 \) and \( \beta < 1 \) we have
\[
g(1 - y^D_b) > \beta g(y^D_b) \quad \text{and} \quad \frac{g(1 - y^D_b)}{1 - \beta} > g(y^D_b).
\]
(C.27)

The convexification of \( C \) yields:
\[
vex C(q) = \begin{cases} 
\bar{q}g(y^D_b) & \text{if } \bar{q} \leq \beta \\
\frac{g(1 - y^D_b) - \beta g(y^D_b)}{1 - \beta} \bar{q} - \frac{\beta}{1 - \beta} (g(1 - y^D_b) - g(y^D_b)) & \text{if } \bar{q} > \beta.
\end{cases}
\]
(C.28)

For \( \beta < \bar{p} < 1 \), the optimal public information structure for the designer is therefore the splitting of the prior \( \bar{p} \) to the posterior \( q = \beta \) with probability \( \lambda = (1 - \bar{p})(1 - \beta) \) and to the posterior \( q = 1 \) with probability \( 1 - \lambda \). The optimal expected cost of the designer under public information is then
\[
C_{Pub} = vex C(p) = \lambda \beta g(y^D_b) + (1 - \lambda) g(1 - y^D_b).
\]
(C.29)

Next, consider the case in which the designer chooses the best BDWE. The designer chooses the flow \( \bar{y}_b \) in state \( \bar{\theta} \) and \( y_b \) in state \( \theta \) to minimize
\[
C_{BDWE}(y_b, \bar{y}_b) = \bar{p}g(\bar{y}_b - y^D_b) + pg(y_b), \quad \text{u.t.c. (ICA) and (ICb).}
\]
(C.30)

Observe that (ICA) implies (ICb) because \( \beta < \bar{p} \). Moreover, \( y_b = 0 \) so the designer minimizes
\[
\bar{p}g(\bar{y}_b - y^D_b), \quad \text{u.t.c. } \beta \bar{p} \bar{y}_b - \bar{p} \bar{y}_b \leq \beta - \bar{p}.
\]
(C.31)

The constraint can be rewritten as \( \bar{y}_b \geq (\bar{p} - \beta)/(\bar{p}(1 - \beta)) \). Since \( y^D_b < (\bar{p} - \beta)/(\bar{p}(1 - \beta)) \) by assumption, the constraint is binding at the optimum, so the best BDWE for the designer is
\[
\bar{y}_b = \frac{\bar{p} - \beta}{\bar{p}(1 - \beta)}, \quad y_b = 0,
\]
(C.32)
as in the previous section, when \( y^D_b = 0 \). The induced cost is
\[
C_{BDWE} = \bar{p}g(\bar{y}_b - y^D_b), \quad \text{where } \bar{y}_b = \frac{\bar{p} - \beta}{\bar{p}(1 - \beta)}.
\]
(C.33)

As already observed earlier, if \( y^D_b = 0 \), then
\[
C_{BDWE} = \bar{p}g\left(\frac{\bar{p} - \beta}{\bar{p}(1 - \beta)}\right) > C_{Pub} = \frac{\bar{p} - \beta}{1 - \beta}.
\]
(C.34)

On the other hand, if \( y^D_b = \bar{y}_b = (\bar{p} - \beta)/(\bar{p}(1 - \beta)) \), then \( C_{BDWE} = 0 < C_{Pub} \). More generally,
$C_{BDWE}$ is decreasing in $\bar{y}_b^D$ while $C_{Pub}$ is increasing in $\bar{y}_b^D$, so if $\bar{y}_b^D$ is low then public information is better than the BDWE, and if $\bar{y}_b^D$ is high then public information is worse than the BDWE.

Finally, consider the following BCWE, which can neither be implemented with public signals nor as a BDWE outcome: in state $\bar{\theta}$ the flow is $\bar{y}_b^D$ with probability $\gamma$ and 1 with probability $1 - \gamma$, and in state $\tilde{\theta}$ the flow is 0, where

$$\gamma = \frac{1 - \bar{y}_b}{1 - \bar{y}_b^D} \quad \text{and} \quad \bar{y}_b = \frac{\bar{p} - \beta}{\bar{p}(1 - \beta)}.$$  \hspace{1cm} (C.35)

Observe that the expected flow in each state is the same as under the above-characterized best BDWE: $\gamma \bar{y}_b^D + (1 - \gamma)1 = \bar{y}_b$. Hence, players’ expected costs in each state are the same as in the optimal BDWE characterized before, and therefore the outcome is a BCWE. The designer’s expected cost under this outcome is however different from the expected cost at the optimal BDWE and is given by

$$C_{BCWE} = \bar{p}(1 - \gamma)g(1 - \bar{y}_b^D), \quad \text{where} \quad \gamma = \frac{1 - \bar{y}_b}{1 - \bar{y}_b^D}. \hspace{1cm} (C.36)$$

Notice that $C_{BCWE} = C_{Pub}$ if $\bar{y}_b^D = 0$, and $C_{BCWE} = C_{BDWE}$ if $\bar{y}_b^D = \bar{y}_b$.

Under the above assumptions (i.e., $\alpha = 0$, $0 < \beta < \bar{p}$, $\beta < 1/2$, $0 < \bar{y}_b^D < \bar{y}_b$, $\bar{y}_b^D < 1/2$), we conclude that

$$C_{BCWE} < \min\{C_{Pub}, C_{BDWE}\}. \hspace{1cm} (C.37)$$

Indeed, we have:

$$C_{BCWE} < C_{BDWE} \iff \bar{p} \frac{\bar{y}_b - \bar{y}_b^D}{1 - \bar{y}_b} g(1 - \bar{y}_b^D) < \bar{p} g(\bar{y}_b - \bar{y}_b^D) \iff \frac{g(1 - \bar{y}_b^D)}{1 - \bar{y}_b^D} < \frac{g(\bar{y}_b - \bar{y}_b^D)}{\bar{y}_b - \bar{y}_b^D}, \hspace{1cm} (C.38)$$

which is satisfied because $g$ is strictly concave. Moreover,

$$C_{BCWE} < C_{Pub} \iff \bar{p} \frac{\bar{y}_b - \bar{y}_b^D}{1 - \bar{y}_b^D} g(1 - \bar{y}_b^D) < \lambda \beta g(\bar{y}_b^D) + (1 - \lambda) g(1 - \bar{y}_b^D). \hspace{1cm} (C.39)$$

This inequality holds if

$$\bar{p} \frac{\bar{y}_b - \bar{y}_b^D}{1 - \bar{y}_b^D} \leq 1 - \lambda = \frac{\bar{p} - \beta}{1 - \beta}. \hspace{1cm} (C.40)$$

Because $(\bar{y}_b - \bar{y}_b^D)/(1 - \bar{y}_b^D)$ is decreasing in $\bar{y}_b^D$, a sufficient condition for this inequality to be satisfied is

$$\bar{p} \bar{y}_b \leq \frac{\bar{p} - \beta}{1 - \beta}, \quad \text{i.e.,} \quad \bar{p} \frac{\bar{p} - \beta}{\bar{p}(1 - \beta)} \leq \frac{\bar{p} - \beta}{1 - \beta}, \hspace{1cm} (C.41)$$

which holds as an equality. Fig. 2 shows the graph of $C_{Pub}$, $C_{BDWE}$, and $C_{BCWE}$ as functions of $\bar{y}_b^D$.  

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Figure 2: Graph of $C_{\text{Pub}}$ (red), $C_{BDWE}$ (blue) and $C_{BCWE}$ (green) as a function of $y_D$, when $\beta = 1/4$, $\lambda = 1/2$, $\bar{p} = 1/2$, and $g(\cdot) = \sqrt{\cdot}$.

D Additional results and examples

D.1 Coarse correlated equilibria

Our work is also related to the literature on no-regret procedures. Blum, Even-Dar, and Ligett (2010) consider nonatomic congestion games and show that if a sequence of flows has no external regret, then its average cost converges to the Wardrop equilibrium cost. Our approach allows us to generalize this result. For the sake of simplicity, we consider a single population.

Definition 9. A coarse correlated Wardrop equilibrium of the nonatomic game $\Gamma$ is a distribution $\mu \in \Delta(Y)$ over flows such that, for all $b \in A$, we have

$$\int \sum_a y_a c_a(y) \, d\mu(y) \leq \int c_b(y) \, d\mu(y).$$

This definition is the adaptation of the concept of coarse correlated equilibrium to nonatomic games: the mediator recommends actions in such a way that obedience yields an ex-ante expected cost which is less or equal to the cost of playing $b$ unconditionally on the recommendation.

The proof of Proposition 4 actually shows the following:

Proposition 12. If a nonatomic game $\Gamma$ has a convex potential, then

$$CCWE(\Gamma) = \Delta(WE(\Gamma)),$$

where $CCWE(\Gamma)$ denotes the set of coarse correlated Wardrop equilibria of $\Gamma$.

Now, define a no-regret sequence of flows as follows.
**Definition 10.** A sequence of flows \( y^t, \ t \in \mathbb{N}, \) has no regret if

\[
\frac{1}{T} \sum_{t=1}^{T} \sum_a y^t_ac_a(y^t) \leq \min_b \frac{1}{T} \sum_{t=1}^{T} c_b(y^t) + \varepsilon_T, \tag{D.3}
\]

for every \( T \in \mathbb{N}, \) with \( \lim_{T \to +\infty} \varepsilon_T = 0. \)

The link with coarse correlated Wardrop equilibrium is immediate. Passing to the limit in Eq. (D.3) gives the following:

**Lemma 5.** Any weak* accumulation point \( \mu \) of \( \frac{1}{T} \sum_{t=1}^{T} \delta y^t \) is a coarse correlated Wardrop equilibrium.

We thus get the following generalization of Theorem 4.1 in Blum et al. (2010):

**Corollary 3.** Consider a game \( \Gamma \) with convex potential. If a sequence of flows \( y^t \) has no regret, then its total cost converges to the Wardrop equilibrium cost.

### D.2 Additional examples

**Example 5.** We give an example of a nonatomic routing game with complete information and two populations in which there exists a CWE that Pareto dominates the unique WE. As a consequence, this multi-population routing game does not admit a convex potential. The set of actions is \( \mathcal{A} = \{a, b\} \) for both populations. We assume that it is a strictly dominant action for population 2 to choose action \( b, \) so the flow of population 2 on action \( b \) is 1 in every CWE. With a slight abuse of notation, we can express costs as functions of the flow \( y \in [0, 1] \) of population 1 on action \( b. \) The costs functions in population 1 are \( c^1_a(y) = 1 \) and \( c^1_b(y) = 2y. \) The costs functions in population 2 are

\[
c^2_a(y) = \begin{cases} 
2y & \text{if } y \leq 1/2 \\
1 & \text{if } y > 1/2.
\end{cases}
\tag{D.4}
\]

The unique WE is \( y = 1/2. \) Consider now the CWE in which the flow of population 1 on action \( b \) is equal to 1 with probability 1/2 and is equal to 0 with probability 1/2. The total cost of population 1 is unchanged, but the expected total cost of population 2 is strictly lower than in the WE:

\[
\frac{1}{2}c^2_b(1) + \frac{1}{2}c^2_b(0) = 1/2 < c^2_b(1/2) = 1.
\]

**Example 6.** We examine a Bayesian routing game in which the expected total cost in a BCWE is strictly smaller than the expected total cost in every BDWE. The set of actions is \( \mathcal{A} = \{a, b, c', c''\}. \) The state space is \( \Theta = \Theta_1 \times \Theta_2 = \{0, 3\} \times \{\theta'_2, \theta''_2\} \) and the prior probability distribution is uniform. The network of the routing game is shown in Fig. 3, where the different
actions correspond to the combinations of the following edges: $a = \{e_a\}, \ b = \{e_{bc}, e_b\}, \ c = \{e_{bc}, e_{c'}\}, \ c'' = \{e_{bc}, e_{c''}\}$.

**Figure 3:** Routing game of Example 6.

Take a parameter $\nu > 4$ and let $\varepsilon = \frac{1}{2(\nu+1)}$. Let

$$g(x) = \begin{cases} 
\frac{1}{2} & \text{if } x \leq \frac{\nu}{\nu+1}, \\
\frac{1}{2} + (\nu + 1)x - \nu & \text{if } \frac{\nu}{\nu+1} \leq x \leq \frac{\nu}{\nu+1} + \varepsilon, \\
1 & \text{if } x \geq \frac{\nu}{\nu+1} + \varepsilon,
\end{cases}$$  \hspace{1cm} (D.5)

and let

$$f(x) = \begin{cases} 
-10 & \text{if } x \leq \frac{\nu}{\nu+1} \\
\frac{20}{\varepsilon}x - \frac{20\nu}{(\nu+1)^2} - 10 & \text{if } \frac{\nu}{\nu+1} \leq x \leq \frac{\nu}{\nu+1} + \varepsilon \\
10 & \text{if } x \geq \frac{\nu}{\nu+1} + \varepsilon.
\end{cases}$$  \hspace{1cm} (D.6)

The costs of each edge as a function of the state $\theta = (\theta_1, \theta_2)$ and the load $x$ on that edge are
given as follows:\footnote{These cost functions admit negative values. If suffices to add +10 to all cost functions to get non-negative values.}

\[
\begin{align*}
    c_{e_a}(x, \theta) &= \theta_1, \\
    c_{e_{bc}}(x, \theta) &= g(x), \\
    c_{e_b}(x, \theta) &= 0, \\
    c_{e_{c'}}(x, \theta) &= \begin{cases} 
        f(x) & \text{if } \theta_2 = \theta'_2, \\
        30 & \text{if } \theta_2 = \theta''_2,
    \end{cases} \\
    c_{e_{c''}}(x, \theta) &= \begin{cases} 
        30 & \text{if } \theta_2 = \theta'_2, \\
        f(x) & \text{if } \theta_2 = \theta''_2.
    \end{cases}
\end{align*}
\]

Hence, the costs of each action as a function of the state \(\theta = (\theta_1, \theta_2)\) and the flow profile \(y\) are

\[
\begin{align*}
    c_a(y, \theta) &= \theta_1, \\
    c_b(y, \theta) &= g(y_b + y_{c'} + y_{c''}), \\
    c_{c'}(y, (\theta_1, \theta'_2)) &= c_{c'}(y, (\theta_1, \theta'_2)) = g(y_b + y_{c'} + y_{c''}) + 30, \\
    c_{c''}(y, (\theta_1, \theta''_2)) &= \begin{cases} 
        g(y_b + y_{c'} + y_{c''}) - 10 & \text{if } y_{c'} \leq \frac{\nu}{\nu + 1}, \\
        g(y_b + y_{c'} + y_{c''}) + 20 \frac{\nu}{\nu + 1} y_{c'} - \frac{20 \nu}{(\nu + 1)\varepsilon} - 10 & \text{if } \frac{\nu}{\nu + 1} \leq y_{c'} \leq \frac{\nu}{\nu + 1} + \varepsilon, \\
        g(y_b + y_{c'} + y_{c''}) + 10 & \text{if } y_{c'} \geq \frac{\nu}{\nu + 1} + \varepsilon,
    \end{cases}
\end{align*}
\]

With these costs, it is optimal to always send a fraction \(\frac{\nu}{\nu + 1}\) of players on action \(c'\) or \(c''\) in order for them to get the lowest cost \(-10\). The remaining players should be sent on action \(a\), even when the cost of action \(a\) is equal to \(\theta_1 = 3\) because by choosing another action (and hence using \(e_{bc}\)) they impose a strong negative externality on the remaining \(\frac{\nu}{\nu + 1}\) players who use edge \(e_{bc}\).

Claim 1. For every \(\theta\), the flow profile that minimizes the total cost is given by:

\[
y_a(\theta_1, \theta_2) = \frac{1}{\nu + 1}, \quad y_b(\theta_1, \theta_2) = 0, \quad y_{c'}(\theta_1, \theta'_2) = y_{c''}(\theta_1, \theta''_2) = \frac{\nu}{\nu + 1}.
\]

Proof. It is immediate to see that the total cost is minimized for \(y_{c'}(\theta_1, \theta'_2) = y_{c''}(\theta_1, \theta''_2) = \frac{\nu}{\nu + 1}\) for every \(\theta_1\). It is also immediate that \(y_a(0, \theta_2) = \frac{1}{\nu + 1}\). Hence, \(y_a(3, \theta_2) = \frac{1}{\nu + 1} - y_b(3, \theta_2)\) and it
remains to characterize the flow \( y_b(3, \theta_2) \in [0, \frac{1}{\nu + 1}] \) that minimizes the total cost when \( \theta_1 = 3 \). Let \( y_b = y_b(3, \theta_2) \). The total cost when \( \theta_1 = 3 \) is equal to

\[
3 \left( \frac{1}{\nu + 1} - y_b \right) + g \left( y_b + \frac{\nu}{\nu + 1} \right) y_b + \left( g \left( y_b + \frac{\nu}{\nu + 1} \right) - 10 \right) \frac{\nu}{\nu + 1},
\]

which is minimized at \( y_b = 0 \) when \( \nu > 4 \).

This first-best outcome is not a BDWE (or a BCWE) because it is not incentive compatible: when action \( a \) is recommended, the expected cost of action \( a \) is \( \frac{3}{2} \), whereas the cost of action \( b \) is \( \frac{1}{2} < \frac{3}{2} \). Therefore, to get a BDWE we have to decrease \( y_a(\theta_1, \theta_2) \) and increase \( y_b(\theta_1, \theta_2) \) when \( \theta_1 = 3 \).

**Claim 2.** The optimal BDWE is as follows:

\[
y_a(0, \theta_2) = \frac{1}{\nu + 1}; \quad y_b(3, \theta_2) = \frac{1}{\nu + 1}; \quad y_{c'}(\theta_1, \theta_2') = \frac{\nu}{\nu + 1}; \quad y_{c''}(\theta_1, \theta_2'') = \frac{\nu}{\nu + 1}.
\]

**Proof.** In order to minimize the total cost, the BDWE must be such that \( y_{c'}(\theta_1, \theta_2') = y_{c''}(\theta_1, \theta_2'') = \frac{\nu}{\nu + 1} \) for every \( \theta_1 \), \( y_a(0, \theta_2) = \frac{1}{\nu + 1} \) and \( y_b(3, \theta_2) = \frac{1}{\nu + 1} - y_b(3, \theta_2) \).

To see this, first notice that it is optimal that as many players as possible enjoy the least cost \( \frac{1}{2} - 10 \). Thus, the optimal BDWE should recommend \( c' \) to a mass of \( \frac{\nu}{\nu + 1} \) players in state \( \theta_2' \) and \( c'' \) to a mass of \( \frac{\nu}{\nu + 1} \) players in state \( \theta_2'' \). Further, it is optimal to send no more than a mass of \( \frac{\nu}{\nu + 1} \) on \( c' \) or \( c'' \). If we send an additional small mass \( m \), this increases the total cost for the first \( \frac{\nu}{\nu + 1} \) players by \( \frac{20}{\varepsilon} m \), thus this increases the total cost by \( \frac{20}{\varepsilon} m \). The decrease in the total cost for the \( m \) players is at most \( (9.5 - 3) m \). Since \( \varepsilon = \frac{1}{2(\nu + 1)} \), \( \frac{\nu}{\nu + 1} \frac{20}{\varepsilon} m > (9.5 - 3) m \), so we have increased the total cost. Since \( f \) is piecewise linear, we cannot improve for any \( m \leq \varepsilon \). There is also no point in increasing \( m \) further, additional players get a cost of 10 on \( c' \) or \( c'' \) whereas they could get 0 on \( b \).

Such recommendations can be made incentive compatible. A player recommended \( c' \) or \( c'' \) is sure to get the least possible cost. Also, recommendations of playing \( a, b \) can be chosen to depend only on \( \theta_1 \), thus to be independent from \( \theta_2 \). Then, a player would never want to play \( c' \) or \( c'' \) unless being recommended to: given that \( a \) or \( b \) is recommended, \( c' \) and \( c'' \) are equally likely, thus there is probability \( \frac{1}{2} \) to get a cost of 30 by deviating to \( c' \) or \( c'' \).

Then, necessarily \( y_a(0, \theta_2) = \frac{1}{\nu + 1} \) in the optimal BDWE. It is socially optimal to send all the remaining \( \frac{1}{\nu + 1} \) player on \( a \) in state \( \theta_1 = 0 \) and it gives the maximal incentive to play \( a \) when it is recommended.

Let then \( y_b := y_b(3, \theta_2) \in [0, \frac{1}{\nu + 1}] \) the only remaining free variable and consider minimizing
the total cost in state $\theta_1 = 3$
\[
3 \left( \frac{1}{\nu + 1} - y_b \right) + g \left( y_b + \frac{\nu}{\nu + 1} \right) y_b + \left( g \left( y_b + \frac{\nu}{\nu + 1} \right) - 10 \right) \frac{\nu}{\nu + 1},
\]
under the incentive compatibility constraints. The only constraint to check is the condition that a player who is recommended to play a should not have an incentive to deviate to b. We must have:
\[
\sum_{\theta \in \Theta} p(\theta)g_a(\theta, y(\theta), \theta) \leq \sum_{\theta \in \Theta} p(\theta)g_a(\theta, y(\theta), \theta),
\]
\[
\iff 3 \left( \frac{1}{\nu + 1} - y_b \right) \leq \frac{1}{2} \frac{1}{\nu + 1} + g \left( \frac{\nu}{\nu + 1} + y_b \right) \left( \frac{1}{\nu + 1} - y_b \right),
\]
\[
\iff \frac{5 - 6(\nu + 1)y_b}{1 - (\nu + 1)y_b} \leq 2g \left( \frac{\nu}{\nu + 1} + y_b \right),
\]
which implies $y_b > \frac{1}{2(\nu + 1)}$. Under this constraint, the total cost in state $\theta_1 = 3$ given by Eq. (D.8) is minimized at $y_b = \frac{1}{\nu + 1}$.

Now, take a parameter $\alpha \in [0, 1]$ and define a BCWE $\mu$ as follows:

- It coincides with the previous BDWE in state $\theta_1 = 0$;
- In state $\theta_1 = 3$, it induces the flow $(y_a = \frac{1}{\nu + 1}, y_b = 0)$ with probability $\alpha$ and $(y_a = 0, y_b = \frac{1}{\nu + 1})$ with probability $1 - \alpha$;
- The flows on $c'$ and $c''$ are as in the optimal BDWE.

Claim 3. This is a BCWE for $\alpha \leq \frac{1}{5}$.

Proof. The only condition to check is the incentive to play a. When recommended to play a, a player’s posterior belief about state $\theta_1 = 3$ is $\frac{\alpha}{1 + \alpha}$. Hence, playing a is optimal if and only if,
\[
3 \frac{\alpha}{1 + \alpha} \leq g \left( \frac{\nu}{\nu + 1} \right) = \frac{1}{2},
\]
that is, $\alpha \leq \frac{1}{3}$.

To show that this BCWE improves the total cost upon the optimal BDWE, we only need to compare total costs in state $\theta_1 = 3$. The total cost of the optimal BDWE in state $\theta_1 = 3$ is
\[
\frac{1}{\nu + 1}(1 + 0) + \frac{\nu}{\nu + 1}(1 - 10).
\]
Under $\mu$, the total cost in state $\theta_1 = 3$ is a convex combination of the first-best total cost and of the total cost of the optimal BDWE:

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\[ \alpha \left[ \frac{1}{\nu + 1} \nu \left( \frac{1}{2} - 10 \right) \right] + (1 - \alpha) \left[ \frac{1}{\nu + 1} (1 + 0) + \frac{\nu}{\nu + 1} (1 - 10) \right]. \]

This improves the total cost provided that

\[ \frac{1}{\nu + 1} \nu \left( \frac{1}{2} - 10 \right) < \frac{1}{\nu + 1} (1 + 0) + \frac{\nu}{\nu + 1} (1 - 10), \]

which holds when \( \nu > 4 \). We conclude that for \( \nu > 4 \), the BCWE that minimizes total cost strictly improves upon all BDWE.