New Physics Effects and Hadronic Form Factor Uncertainties in $B \to K^\ast \ell^+\ell^-$

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It is well known that New Physics can contribute to weak decays of heavy mesons via virtual processes during its decays. The discovery of New Physics, using such decays is made difficult due to intractable strong interaction effects needed to describe it. Modes such as $B \to K^\ast \ell^+\ell^-$ offer an advantage as they provide a multitude of observables via angular analysis. We show how the multitude of “related observables” obtained from $B \to K^\ast \ell^+\ell^-$, can provide many new “clean tests” of the Standard Model. The hallmark of these tests is that several of them are independent of the unknown universal form factors that describe the decay in heavy quark effective theory. We derive a relation between observables that is free of form factors and Wilson coefficients, the violation of which will be an unambiguous signal of New Physics. We also derive relations between observables and form factors that are independent of Wilson coefficients and enable verification of hadronic estimates. We show how form factor ratios can be measured directly from helicity fraction with out any assumptions what so ever. We find that the allowed parameter space for observables is very tightly constrained in Standard Model, thereby providing clean signals of New Physics. We examine in detail both the large-recoil and low-recoil regions of the $K^\ast$ meson and point out special features and derive relations between observables valid in the two limits. In the large-recoil regions several of the relations are unaffected by corrections to all orders in $\alpha_s$. We present yet another new relation involving only observables that would verify the validity of the relations between form-factors assumed in the low-recoil region. The several relations and constraints derived will provide unambiguous signals of New Physics if it contributes to these decays.

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I. INTRODUCTION

It is well known that physics beyond the Standard Model referred to as New Physics can either be discovered by direct production of new particles at high energies or by indirect searches at high luminosity facilities where new physics can contribute virtually to loop processes. The most well known example of the latter kind is the muon magnetic moment. Unfortunately, even though muon is a lepton, hadronic contributions have to be estimated and turn out to be the limiting factor in the search for New Physics. Indirect searches for New Physics often involve precision measurement of a single quantity as in the case of muon magnetic moment. The single measurement is compared to a theoretical estimate that needs to be accurately calculated. There are however, certain decays which involve measurement of several related observables. Well known examples of such decays are $B \to V_1 V_2$ where $B$ decays to two vector mesons $V_1$ and $V_2$ and the semi-leptonic penguin decay $B \to K^\ast \ell^+\ell^-$. The heavy meson decays to such modes occur in multiple partial waves and allow a measurement of a multitude of related observables. In this paper we will show how the observables obtained from an angular analysis of $B \to K^\ast \ell^+\ell^-$ allow for a cleaner signals of New Physics if it exists.

It is hoped that flavor changing neutral current transitions in $b \to s$ and $b \to d$ will be altered by physics beyond the Standard Model (SM) and their study would reveal possible signal of new physics (NP) if it exists. However, understanding the hadronic flavor changing neutral current decays requires estimating hadronic effects which cannot be completely accurately done. Experimental data collected by the Belle and Babar collaborations at the B-factories, CLEO, Tevatron and now LHCb seems to indicate that new physics does not show up as a large and unambiguous effects in flavor physics. This has bought into focus the need for theoretically cleaner observables, i.e. observables that are relatively free from hadronic uncertainties. In the search for new physics it is therefore crucial to effectively separate the effect of new physics from hadronic uncertainties that can contribute to the decay.

One of the modes that is regarded as significant in this attempt is $B \to K^\ast \ell^+\ell^-$ angular analysis of which is known to result in a multitude of observables $[1-3]$, each of which are function of invariant dilepton mass $q^2$. Throughout, our discussions we will neglect the lepton and s-quark masses, ignore the very small $CP$ violation arising within the Standard Model $[1, 2]$ and exclude studying the resonant region in $q^2$. In this limit, any observable that is chosen may eventually be expressed in terms of six real transversity amplitudes that correspond to the three states of polarizations of $K^\ast$ and the left or right chirality of the lepton $\ell^-$. We can hence have at best six independent observables. Several different experiments BaBar, Belle, CDF and LHCb have studied the mode $B \to K^\ast \ell^+\ell^-$ $[4-10]$, providing valuable data as a function of the dilepton invariant mass $q^2$ by studying uni-angular distributions. The partial branching fraction is measured in chosen $q^2$ bins by performing a complete angular integration. A study of the angular distribution of the direction of the lepton in an appropriately chosen frame (see Section III) has also already been done by all
the four experiments to measure the forward-backward asymmetry $A_{FB}$ and the longitudinal polarization fraction $F_1$, in terms of integrated dilepton invariant mass regions of $q^2$. CDF and LHCb have in addition performed an angular study of the azimuthal angle defined as the angle between the planes formed by the leptons and the decay products of $K^*$ i.e. $K, \pi$. We will show that the $F_1$ helicity fraction can be obtained from a uni-angular distribution of azimuthal angle. Future experimental studies at LHC-B, Belle II and Super-B will enable the study this mode with significantly larger statistics and make possible the analysis with multi-angular distributions and the measurement of all the observables.

In a recent paper [11] it was shown that the multitude of related observables obtained via an angular analysis in $B \rightarrow K^* \ell^+ \ell^-$, can provide many “clean tests” of the Standard Model. The hallmark of these tests is that several of them are independent of the universal form factors $\xi_\parallel$ and $\xi_\perp$ required to describe the decay using heavy quark effective theory. Indeed, in the large recoil region considered in Ref. [11], these relations are even more interesting as they are unaffected by corrections to all orders in $\alpha_s$. We will refer to such relations that are independent of universal form factors and are unaffected by corrections to all orders in $\alpha_s$ as “clean relations.” A variety of relations were derived which included relations between observables and form factors that are independent of Wilson coefficients. Such relations are inherently clean and important as they enable verification of hadronic estimates. We show how form factor ratios can be measured directly from ratios of helicity amplitudes measured at the zero crossings of asymmetries with out any assumptions what so ever. Another achievement is the derivation of a relation between observables alone, based entirely on the assumption that the amplitudes have form given by the Standard Model, but which is never-the-less independent of form factors and Wilson coefficients. This relation would provide an unambiguous test of the standard model relying purely on observables. We also presented a clean expression for the “effective photon vertex” involving the same operator that also contributes to the process $B \rightarrow K^\gamma$. We emphasize that the amplitude for $B \rightarrow K^\gamma$ involves the universal form factor $\xi_\parallel$ and is inherently not clean. It is hence some what surprising that the same vertex can be expressed independent of the universal form-factors in heavy quark effective theory in a way that is valid at order $1/m_b$ to all orders in $\alpha_s$. While $C_9$ and $C_{10}$ individually depend on form factors, we find that the expression for the ratio $C_9/C_{10}$ is clean. Based purely on the signs of the form factors and the fact that zero crossing of the forward backward asymmetry has been observed, we convincingly concluded that the signs of the Wilson coefficients are in agreement with Standard Model. We found that there exist three sets of equivalent solutions to each of the three Wilson coefficients involving different observables. However, only two of the sets are independent. It was shown that the allowed parameter space for observables is very tightly constrained in Standard Model, thereby providing clean signals of New Physics.

In this paper we not only derive all the expressions presented in Ref. [11] in detail but also derive several new expression and constraints. We extend the analysis to examine in detail both the large-recoil and low-recoil regions of the $K^*$ meson and probe special features and relations valid in the two limits. We present yet another new relation involving only observables that would verify the validity of the relations between form-factors assumed in the low-recoil region. Under this approximation mentioned earlier we have the six real presumably non-zero amplitudes that are described in terms of eight combinations of form factors and Wilson coefficients. We elaborate in this paper how the six observables can be used to verify Standard Model and to distinguish possible new physics contributions from hadronic effects which in the usual approach hinder the discovery of new physics. This is made possible by fortunate advances in our understanding of these form-factors that permit us to make two reliable inputs in terms of ratios of form-factors which are well predicted at order $1/m_b$ to all orders in $\alpha_s$ and are free from universal form factors $\xi_\parallel$ and $\xi_\perp$ in heavy quark effective theory.

In this paper we briefly review the theoretical framework of $B \rightarrow K^* \ell^+ \ell^-$ in the Standard Model in Section II. In Section III we express the differential decay distribution in terms of angular variables and helicity amplitudes. We also define observables that are directly measurable by angular analysis. The helicity amplitudes are expressed in terms of form factors IV, where we also setup essential notations used throughout the paper. While most of our analysis is independent of the values of form factors we do use the various symmetries possible in the heavy quark limit to emphasize the variety of interesting results possible with $B \rightarrow K^* \ell^+ \ell^-$. In Section IV A we discuss the symmetry relations among form factors arising in the large recoil limit of the $K^*$ meson. A similar discussion for the low recoil limit is presented in Section IV B. Our model independent analysis is described in details in Section V, where we also derive the bulk of new relations. The results presented in this section are in general valid for all $q^2$. The large recoil limit is obtained simply by assigning the form factors the expressions or values valid in this limit. The low recoil limit requires special attention and is discussed in Section VI, where we consider special features and derive more new interesting new results. In Section VII we summarize the main results of the paper. The derivation involved in the solutions of Wilson coefficients are given in Appendix A and the numerical values of form factors and inputs used by us are presented in Appendix B.

II. THEORETICAL FRAMEWORK

The decay $B(p) \rightarrow K^*(k)\ell^-(q_1)\ell^-(q_2)$ is described within the standard model by an effective Hamiltonian
\( \mathcal{H}_{\text{eff}} \) that involves separation of long-distance QCD effects from the short distance QCD and weak interaction effects. The effective short distance Hamiltonian for \( b \rightarrow s \ell^+ \ell^- \) transition is well understood and given by: [2, 3, 12]

\[
\mathcal{H}_{\text{eff}} = \frac{G_F \alpha}{\sqrt{2\pi}} V_{tb} V_{ts}^{*} \left[ C_9 (\bar{s} \gamma_\mu P_L b) \bar{\ell} \gamma^\mu \ell + C_{10} (\bar{s} \gamma_\mu P_L b) \bar{\ell} \gamma^\mu \gamma_5 \ell \right] - \frac{2 C_7^{\text{eff}}}{q^2} ( \bar{s} \sigma_{\mu \nu} q' m_b P_R b ) \bar{\ell} \gamma^\mu \ell \]

(1)

where \( q_\nu = q_1 + q_2 = p_\nu - k_\nu \) and we have defined

\[
P_{L,R} = \frac{1 \mp \gamma_5}{2}
\]

and \( q^2 \) is the dilepton invariant mass squared. In the above we have ignored the \( s \) quark mass and throughout this paper we will ignore the lepton mass. The Wilson coefficients \( C_7^{\text{eff}}, C_{9,10} \) are evaluated at the scale \( \mu = m_b = 4.8 \text{GeV} \) at NNLL accuracy [3]:

\[
C_7^{\text{eff}} = -0.304, \quad C_9^{\text{eff}} = 4.211 + Y(q^2), \quad C_{10} = -4.103
\]

where,

\[
C_7^{\text{eff}} = C_7 - \frac{1}{3} C_3 - \frac{4}{9} C_4 - \frac{20}{3} C_5 - \frac{80}{9} C_6
\]

and the function \( Y(q^2) \) is given by [3, 13–15]

\[
Y(q^2) = h(q^2, m_c) \left( \frac{4}{3} C_1 + C_2 + 6 C_3 + 60 C_5 \right)
\]

The complete angular distribution requires the polarization of \( K^* \) or a study of the angular distribution of the \( K^* \) decay into \( K \pi \). This is readily done in the narrow width approximation for the \( K^* \) since the decay of \( K^* \rightarrow K \pi \) is itself well understood in terms of an effective Lagrangian. The resulting matrix elements is described in a model independent approach in terms of three reliably calculable effective Wilson coefficients that represent short distance contributions and six (in the limit of vanishing lepton mass) \( B \rightarrow K^* \) form factors. The \( B \rightarrow K \pi \ell^+ \ell^- \) matrix element can hence, be written as

\[
\langle K^*(k) | \bar{s} \gamma_\mu (1 - \gamma_5) b | B(p) \rangle = -i \epsilon_\mu (m_B + m_{K^*}) A_1(q^2) + p_\mu (\epsilon^* \cdot q) \frac{2 A_2(q^2)}{m_B + m_{K^*}}
\]

\[
+ i \epsilon_{\mu \nu \rho \sigma} \epsilon^{\nu \rho} q^\sigma \frac{2 V(q^2)}{m_B + m_{K^*}}
\]

\[
\langle K^*(k) | \bar{s} \sigma_{\mu \nu} q' (1 + \gamma_5) b | B(p) \rangle = i \epsilon_{\mu \nu \rho \sigma} \epsilon^{\nu \rho} q^\sigma T_1(q^2) + T_2(q^2) \epsilon_\mu^\nu (m_B^2 - m_{K^*}^2) - 2 (\epsilon^* \cdot q) p_\mu
\]

where we have defined \( y = 4m_q^2/q^2 \), and we have neglected the small weak phase.

The function \( h(q^2, m_q) \) reads as:

\[
h(q^2, m_q) = -4 \left[ \ln \frac{m_q^2}{\mu^2} - \frac{2}{3} - y \right] - \frac{4}{9} (2 + y) \sqrt{|y - 1|}
\]

\[
\times \left[ \Theta(1 - y) \left( \ln \frac{1 + \sqrt{1 - y}}{\sqrt{y}} - i \pi \right) + \Theta(y - 1) \arctan \left( \frac{1}{\sqrt{y - 1}} \right) \right]
\]

where \( q_\nu = p_\nu - k_\nu \). We have dropped terms proportional to \( q_\mu \) since the terms \( q_\mu \bar{\ell} \gamma^\mu \gamma_5 \ell \) and \( q_\mu \bar{\ell} \gamma^\mu \ell \) do not contribute in the limit of vanishing lepton mass. The form factors have been studied using QCD sum rules on the light cone. QCD factorization in the heavy quark limit, soft-collinear theory and using operator product expansion that is valid for large dilepton mass \( \sqrt{q^2} \). The decay \( B \rightarrow K^* \ell^+ \ell^- \) has the advantage that it can be studied as a function of the dilepton mass or \( q^2 \). If one excludes the resonant region and the very small CP violation arising within SM, all the Wilson coefficients and form factors contributing to the decay are real. In this paper as mentioned above we will make both these assumptions.

The complete angular distribution requires the polarization of \( K^* \) or a study of the angular distribution of the \( K^* \) decay into \( K \pi \). This is readily done in the narrow width approximation for the \( K^* \) since the decay of \( K^* \rightarrow K \pi \) is itself well understood in terms of an effective Lagrangian. The resulting matrix elements is described in a model independent approach in terms of three reliably calculable effective Wilson coefficients that represent short distance contributions and six (in the limit of vanishing lepton mass) \( B \rightarrow K^* \) form factors. The \( B \rightarrow K \pi \ell^+ \ell^- \) matrix element can hence, be written as
\[
M = \frac{G_F}{\sqrt{2}\pi} V_{tb} V_{ts} \left\{ C_9 \langle \bar{K}\pi|\bar{s}\gamma^\mu P_L b|\bar{B}\rangle \bar{\gamma}_\mu l + C_{10} \langle \bar{K}\pi|\bar{s}\gamma^\mu P_L b|\bar{B}\rangle \bar{\gamma}_\mu \gamma_5 l - \frac{2C_7 m_b}{q^2} \langle \bar{K}\pi|\bar{s}\sigma_{\mu\nu} q^\nu P_R b|\bar{B}\rangle \bar{\gamma}_\mu l \right\}. 
\]

(4)

### III. Angular Distribution and Observables

The decay \( \bar{B}(p) \to K^*(k)\ell^+\ell^-(q_1) \) with \( K^*(k) \to K(k_1)\pi(k_2) \) on the mass shell, is completely described by four independent kinematic variables. These are the lepton-pair invariant mass squared \( q^2 = (q_1 + q_2)^2 \), the angle \( \phi \) between the decay planes formed by \( \ell^+\ell^- \) and \( K\pi \) respectively and the angles \( \theta_\ell \) and \( \theta_K \) defined as follows: assuming that the \( K^* \) has a momentum along the positive \( z \) direction in \( B \) rest frame, \( \theta_K \) is the angle between the \( K \) and the \( +z \) axis and \( \theta_\ell \) is the angle of the \( \ell^- \) with the \( +z \) axis. The differential decay distribution of \( B \to K^*\ell^+\ell^- \) can be written as

\[
\frac{d^4\Gamma(B \to K^*\ell^+\ell^-)}{dq^2\,d\cos\theta_\ell\,d\cos\theta_K\,d\phi} = \frac{9}{32\pi} I_1^1 \sin^2\theta_K + I_1^2 \cos^2\theta_K + (I_2^1 \sin^2\theta_K + I_2^2 \cos^2\theta_K) \cos 2\theta_\ell \\
+ I_3^1 \sin^2\theta_K \sin^2\theta_\ell \cos 2\phi + I_4^1 \sin 2\theta_K \sin 2\theta_\ell \cos 2\phi + I_5^1 \sin^2\theta_K \sin 2\theta_\ell \sin 2\phi + I_6^1 \sin^2\theta_K \cos 2\phi + I_7^1 \sin 2\theta_K \cos 2\theta_\ell \cos 2\phi + I_8^1 \sin^2\theta_\ell \sin 2\phi.
\]

(5)

We note that \( I_1^1 \) are \( q^2 \) dependent but we have chosen to suppress the explicit \( q^2 \) dependence for simplicity. Throughout the paper we will not explicitly state the \( q^2 \) dependence of observables and variables, however, the dependence is implicit. A study of the angular distribution of the decay will allow us to measure all the \( I_1^1 \)s. Since the \( K^* \) in \( B \to K^*\ell^+\ell^- \) decay, is created on shell it has three polarization states. Hence, we can express \( I_1^1 \)s explicitly in terms of the six transversity amplitudes \( A_{L,R}^{1,\perp,0} \), where \( \perp, \| \) and 0 represent the polarizations and \( L, R \) denote the chirality of the lepton \( \ell^- \). We can write the nine observables explicitly in terms of the six transversity amplitudes \( A_{L,R}^{1,\perp,0} \) as:

\[
I_1^1 = \frac{3}{4} \left[ |A_1^1|^2 + |A_0^0|^2 + (L \to R) \right],
\]

\[
I_1^2 = \frac{1}{4} \left[ |A_1^1|^2 + (L \to R) \right],
\]

\[
I_2^1 = - \frac{1}{4} \left[ |A_0^0|^2 + |A_0^1|^2 + (L \to R) \right],
\]

\[
I_2^2 = - \frac{1}{4} \left[ |A_1^0|^2 + (L \to R) \right],
\]

\[
I_3^1 = \frac{1}{2} \left[ |A_1^1|^2 - |A_0^1|^2 + (L \to R) \right],
\]

\[
I_4^1 = \frac{1}{\sqrt{2}} \left[ \text{Re}(A_0^0 A_1^1) + (L \to R) \right],
\]

\[
I_5^1 = \sqrt{2} \left[ \text{Im}(A_0^0 A_1^1) - (L \to R) \right],
\]

\[
I_6^1 = \frac{2}{\sqrt{2}} \left[ \text{Im}(A_0^0 A_1^1) + (L \to R) \right],
\]

\[
I_7^1 = \frac{1}{\sqrt{2}} \left[ \text{Im}(A_0^0 A_1^1) + (L \to R) \right],
\]

\[
I_8^1 = \left[ |A_1^1|^2 + (L \to R) \right],
\]

\[
I_9^1 = \left[ \text{Re}(A_0^0 A_1^1) + (L \to R) \right].
\]

In the above we have assumed the lepton mass. As mentioned above we will assume that the resonant region is excluded in the analysis and that \( CP \) violation arising within the standard model is negligible. In the absence of \( CP \) violation the conjugate mode \( B \to K^*\ell^+\ell^- \) has an identical decay distribution except that \( I_{5,6,8,9} \) switch signs to become \( -I_{5,6,8,9} \) in the differential decay distribution [1, 2]. The helicity amplitudes \( A_{L,R}^{1,\perp,0} \) are then all real and only six of the \( I_1^1 \)s can be non-zero and independent. In fact, it is easy to see that \( I_7, I_8 \) and \( I_9 \) must vanish in the limit of vanishing \( CP \) violation.

The explicit form of these transversity amplitudes are
\[ A_{0}^{L,R} = -\frac{N}{2m_{K}} \sqrt{q^2} \left( [C_{\text{eff}}^{L_0} + C_{\text{eff}}^{R_0}] \times [(m_{B}^2 - m_{K}^2 - q^2)(m_{B} + m_{K})A_{1}(q^2) - \lambda(m_{B}^2, m_{K}^2, q^2) \frac{A_{2}(q^2)}{m_{B} + m_{K}}] + 2m_{0}C_{0}^{R} \left[m_{B}^2 + 3m_{K}^2 - q^2 \right] T_{2}(q^2) - \frac{\lambda(m_{B}^2, m_{K}^2, q^2)}{m_{B} - m_{K}} \right) \tag{6c} \]

where,
\[ N = V_{ub}V_{us}^{*} \left[ \frac{G_{F}^{2} \alpha^{2}}{3 \cdot 2^{10} \pi^{5} m_{B}^{2}} \right] q^2 \sqrt{\lambda(m_{B}^2, m_{K}^2, q^2)} \right]^{1/2}, \tag{7} \]

with \( \lambda(m_{B}^2, m_{K}^2, q^2) = m_{B}^4 + m_{K}^4 + q^4 - 2(m_{B}^2 m_{K}^2 + m_{B}^2 q^2 + m_{K}^2 q^2) \). We note that the helicity amplitudes \( A_{0}^{L,R} \) are functions of \( q^2 \), for simplicity we have suppressed the explicit dependence on \( q^2 \).

\[ I(q^2, \theta_{\ell}, \theta_{K}, \phi) = \frac{9}{16\pi} \left[ \left( |A_{L1}^0|^2 + |A_{L1}^R|^2 + |A_{L2}^0|^2 + |A_{L2}^R|^2 \right) \frac{\sin^2 \theta_{K}(1 + \cos^2 \theta_{\ell}) + (|A_{L2}^0|^2 + |A_{L2}^R|^2) \cos^2 \theta_{K} \sin^2 \theta_{\ell}}{4} \right. \]
\[ + \left. \left( |A_{L1}^0|^2 + |A_{L1}^R|^2 - |A_{L2}^0|^2 - |A_{L2}^R|^2 \right) \frac{\cos \phi \sin \theta_{K} \cos \theta_{\ell} + \Re(A_{L1}^0 A_{L2}^R - A_{L1}^R A_{L2}^0) \cos \theta_{K} \sin \theta_{\ell}}{4} \right) \cos \phi \sin \theta_{\ell} \sin(2\theta_{K}) + \frac{\Re(A_{L1}^0 A_{L2}^R - A_{L1}^R A_{L2}^0)}{2\sqrt{2}} \cos \phi \sin(2\theta_{K}) \sin(2\theta_{K}) \right]. \tag{8} \]

It is easy to see that integration over \( \cos \theta_{K}, \cos \theta_{\ell} \) and \( \phi \) results in the differential decay rate with respect to the invariant lepton mass, which is given by the sum of the modulus squared of all the transversity amplitudes at the same invariant lepton mass:

\[ \frac{d\Gamma}{dq^2} = \sum_{\lambda=0,\|,\perp} (|A_{\lambda}^0|^2 + |A_{\lambda}^R|^2) \tag{9} \]

It is obvious from Eq. (8) that a complete study of the angular distribution will allow us to measure six observables. We define the relevant observables to be the three helicity fractions defined as follows:

\[ F_{L} = \frac{|A_{0}^L|^2 + |A_{0}^R|^2}{\Gamma_{f}}, \tag{10a} \]
\[ F_{\|} = \frac{|A_{L2}^0|^2 + |A_{L2}^R|^2}{\Gamma_{f}}, \tag{10b} \]

where \( \Gamma_{f} = \sum_{\lambda}(|A_{\lambda}^0|^2 + |A_{\lambda}^R|^2) \) and, \( F_{L} + F_{\|} + F_{\perp} = 1 \).

The well known forward–backward asymmetry \( A_{FB} \),
\[ A_{FB} = \frac{1}{\Gamma_{f}} \int_{0}^{1} \int_{-1}^{0} d\cos \theta_{\ell} \frac{d^2(T + \bar{T})}{dq^2 d\cos \theta_{\ell}} \int_{-1}^{1} d\cos \theta_{K} \frac{d^2(T + \bar{T})}{dq^2 d\cos \theta_{K} d\phi} \tag{11} \]

and two new angular asymmetries,
\[ A_{4} = \frac{\int_{\pi/2}^{\pi/2} d\phi - \int_{-\pi/2}^{\pi/2} d\phi}{\int_{0}^{1} d\cos \theta_{K} - \int_{-1}^{0} d\cos \theta_{K}} \left[ \int_{0}^{1} d\cos \theta_{\ell} - \int_{-1}^{0} d\cos \theta_{\ell} \right] \frac{d^4(T - \bar{T})}{dq^2 d\cos \theta_{\ell} d\cos \theta_{K} d\phi}, \tag{12} \]
\[ A_{5} = \frac{\int_{1}^{1} d\cos \theta_{\ell} \int_{0}^{\pi/2} d\phi \int_{-1}^{1} d\cos \theta_{K} \int_{0}^{1} d\cos \theta_{\ell} \frac{d^4(T + \bar{T})}{dq^2 d\cos \theta_{\ell} d\cos \theta_{K} d\phi}}{\int_{-1}^{1} d\cos \theta_{\ell} \int_{0}^{\pi/2} d\phi \int_{-1}^{1} d\cos \theta_{K} \frac{d^4(T + \bar{T})}{dq^2 d\cos \theta_{\ell} d\cos \theta_{K} d\phi}}, \tag{13} \]

\[ A_{FB}, A_{4} \text{ and } A_{5} \text{ can be written directly in terms of the transversity amplitudes as follows:} \]
\[ A_{4} = \frac{\sqrt{2} \Re(A_{0}^L A_{L2}^R) + \Re(A_{0}^R A_{L2}^R)}{\Gamma_{f}}, \tag{14} \]
\[ A_{5} = \frac{3}{2\sqrt{2}} \Re(A_{0}^L A_{L2}^R - A_{0}^R A_{L2}^R) \frac{\Gamma_{f}}{\Gamma_{f}}. \tag{15} \]
$$A_{FB} = \frac{3 \text{Re}(A_L^L A_L^R - A_R^L A_R^R)}{2 \Gamma_f}. \quad (16)$$

A complete angular analysis requires much larger data set than are currently analyzed, hence angular distributions in terms of only one angular variable have been studied. The angular distribution as a function of $q^2$ and $\cos \theta_t$ with $\phi$ and $\cos \theta_K$ integrated out is given by:

$$\frac{d^2 \Gamma}{dq^2 d \cos \theta_t} = \Gamma \left[ A_{FB} \cos \theta_t + \frac{3}{8} (1 - F_L) (1 + \cos^2 \theta_t) + \frac{3}{4} F_L (1 - \cos^2 \theta_t) \right]. \quad (17)$$

Angular analysis in terms of $\cos \theta_t$ enables the measurement of both $F_L$ the longitudinal helicity fraction and the forward–backward asymmetry $A_{FB}$. The other helicity fractions $F_\perp$ or $F_{||}$ can be measured from the angular distributions as well but it has been believed that one need to perform a full angular analysis. It is, however, easy to see that a combination of $F_\perp$ and $F_{||}$ can be measured if the angular distribution in terms of $\phi$ is studied. The angular distribution in $\phi$ is given by:

$$\frac{d^2 \Gamma}{dq^2 d \phi} = \frac{\Gamma}{2\pi} \left[ 1 - \frac{1 - F_L - 2F_\perp}{2} \cos 2\phi + i\theta I_9 \sin 2\phi \right]. \quad (18)$$

The distribution in $\phi$ allows us to measure $1 - F_L - 2F_\perp$. If $F_L$ is measured independently one can obtain $F_\perp$. The distribution also allows us to measure $I_9$, which is immeasurably small in SM [1], and assumed to be zero in our study. Recently the angular analysis in $\phi$ has been studied [9, 16] by CDF and LHCb collaborations. In the next section we will show that $1 - F_L - 2F_\perp$ is also small in the SM as a consequence of heavy quark effective theory. We will conclude in Sec. II that the angular distribution will be almost constant for $q^2 \approx 0$, with small variation in $\cos \phi$ at large $q^2$.

There is yet another technique to measure $F_\perp$ which involves studying angular distributions in terms of only one angular variable. However, this approach requires independent analysis in the transversity frame defined with $J/\psi$ at rest. In this frame the lepton makes an angle $\theta_{tr}$ with the z-axis. The expression for the differential decay rate as a function of $\cos \theta_{tr}$ is given by:

$$\frac{d\Gamma}{dq^2 d \cos \theta_{tr}} = \Gamma \left[ \frac{3}{8} (1 - F_\perp) (1 + \cos^2 \theta_{tr}) + \frac{3}{4} F_\perp (1 - \cos^2 \theta_{tr}) \right]. \quad (19)$$

IV. NOTATION: OBSERVABLES IN TERMS OF FORM-FACTORS

The six transversity amplitudes in Eq. (6a) to (6c) are written in terms of the Wilson coefficients and the form factors in most general form as,

$$A_L^{L,R} = C_{L,R} F_L - \widetilde{G}_L \quad (20a)$$

$$A_{||}^{L,R} = C_{L,R} F_{||} - \widetilde{G}_{||} \quad (20b)$$

$$A_0^{L,R} = C_{L,R} F_0 - \widetilde{G}_0 \quad (20c)$$

where to leading order, $C_{L,R} = C_{7}^{\text{eff}} \mp C_{10}$ and $\widetilde{G}_L = C_{7}^{\text{eff}} G_{\lambda}$. $C_{7}^{\text{eff}}$, $C_{9}^{\text{eff}}$ and $C_{10}$ are the Wilson coefficients that represent short distance corrections. $F_L$ and $\widetilde{G}_L$ are defined below in terms of $q^2$-dependent QCD form factors that parameterize the $B \to K^*$ matrix elements [3] and are suitably defined to include both factorizable and non-factorizable contributions at any given order. The treatment of the form factors depends largely on the recoil energy of the $K^*$ or equivalently $q^2$ and will have to be treated differently in the limit of heavy quark effective theory. In the large recoil limit (see Section IV A) the next to leading order effects including factorizable and non-factorizable corrections can be parametrically included by replacements $C_{9}^{\text{eff}} \to C_9$ and defining $\widetilde{G}_L = C_{7}^{\text{eff}} G_{\lambda} + \cdots$, with the dots representing the next to leading and higher order terms. Hence the Wilson coefficient and form factor can be lumped together into a single factor $\widetilde{G}_L$. We note that even at leading order it is impossible to determine $C_{7}^{\text{eff}}$ with the value of $G_{\lambda}$ being determined. The treatment of form factors in the low recoil limit (see Section IV B for details) differs significantly from the large recoil. In the low recoil limit the leading corrections are the non-perturbative effects up to and including terms suppressed by $\Lambda_{QCD}/Q$ (where $Q = \{m_b, \sqrt{q^2}\}$) and include the next-to-leading order corrections from the charm quark mass $m_c$ and the strong coupling at $O(m_c^2/Q^2, \alpha_s)$.

The form factors $F_L$ and $\widetilde{G}_L$ can be related to the form factors $V, A_{1,2}$ and $T_{1,2,3}$ introduced in Eqs. (6a) – (6c) [3] by comparing these expressions for $A_{L,R}^{L,R}$ with those in Eqs. (20a) – (20c). Including higher order QCD correction and non-factorizable corrections $\Gamma_L$ and $\widetilde{G}_L$ can be written as:

$$\widetilde{G}_L = -N \sqrt{2} \lambda(m_B^2, m_{K^*}^2, q^2) \frac{2m_B}{q^2} C_{7}^{\text{eff}} T_1(q^2) + \cdots \quad (21a)$$

$$\widetilde{G}_{||} = N \sqrt{2} (m_B^2 - m_{K^*}^2) \frac{2m_B}{q^2} C_{7}^{\text{eff}} T_2(q^2) + \cdots \quad (21b)$$

$$F_L = N \sqrt{2} \lambda(m_B^2, m_{K^*}^2, q^2) \frac{V(q^2)}{m_B + m_K^*} \quad (21c)$$

$$F_{||} = -N \sqrt{2} (m_B + m_{K^*}) A_1(q^2) \quad (21d)$$

$$F_0 = \frac{-N}{2m_{K^*} \sqrt{q^2}} \left[ (m_B^2 - m_{K^*}^2 - q^2)(m_B + m_{K^*}) A_1(q^2) \right]$$
\[- \lambda(m_B^2, m_{K^*}^2, q^2) \frac{A_2(q^2)}{m_B + m_{K^*}} \]

\[
\tilde{g}_0 = \frac{N}{2m_{K^*}\sqrt{q^2}} - \frac{2m_B}{4m_B + 2m_{K^*}} \left[ (m_B^2 + 3m_{K^*}^2 - q^2)C_7T_2(q^2) \right]
\]

\[- \lambda(m_B^2, m_{K^*}^2, q^2) \frac{C_7T_3(q^2)}{m_B^2 - m_{K^*}^2} + \cdots \]  

(21f)

With the help of Eq. (20a) to (20c) the observables \( F_L, F_{\perp}, A_{FB}, A_4 \) and \( A_5 \) can be written in terms of the Wilson coefficients and from factors as:

\[
F_L \Gamma_f = 2(C_3^f + C_9^f)F_2^\perp + 2\tilde{G}_0^\perp - 4C_9^f F_0 \tilde{G}_0^0 \]  

(22a)

\[
F_{\perp} \Gamma_f = 2(C_3^f + C_9^f)F_{\perp}^\parallel + 2\tilde{G}_0^\perp - 4C_9^f F_{\perp}^0 \tilde{G}_0^0 \]  

(22b)

\[
F_{\perp} \Gamma_f = 2(C_3^f + C_9^f)F_{\perp}^\parallel + 2\tilde{G}_0^\perp - 4C_9^f F_{\perp}^0 \tilde{G}_0^0 \]  

(22c)

\[
\frac{\pi A_4 \Gamma_f}{2\sqrt{2}} = \tilde{g}_{\parallel}^0 + (C_2^f + C_{10}^f)F_0 \tilde{F}_{\parallel}\]  

(22d)

\[
\frac{\sqrt{2}A_3 \Gamma_f}{3} = C_{10}(F_{\perp}^\parallel \tilde{G}_0^0 + \tilde{G}_\perp^\perp \tilde{F}_0^0) - 2C_0 C_{10} F_0 \tilde{G}_0^0 \]  

(22e)

\[
\frac{A_{FB} \Gamma_f}{3} = C_{10}(F_{\perp}^\parallel \tilde{G}_\perp^\perp + F_{\parallel} F_{\perp}^0) - 2C_0 C_{10} F_{\parallel} F_{\perp}^0 \]  

(22f)

We use Eq. (22a)–(22f) to solve the Wilson coefficients in terms of the observables and the form factors. This solutions are achieved by defining new variables

\[
r_{\parallel} = \frac{\tilde{g}_{\parallel}^0}{F_{\parallel}^0} - C_9, \]  

(23a)

\[
r_{\perp} = \frac{\tilde{g}_{\perp}^0}{F_{\perp}^0} - C_9, \]  

(23b)

\[
r_0 = \frac{\tilde{G}_0^0}{F_0^0} - C_9, \]  

(23c)

\[
r_{\perp} = \frac{\tilde{G}_\perp f}{F_0} = \frac{\tilde{G}_\perp f}{F_0} - C_9. \]  

(23d)

In terms of these new variables \( r_{\parallel}, r_{\perp}, r_0 \) and \( r_{\perp} \) the observables in Eq. (22a)–(22f) can be written conveniently as:

\[
F_{\parallel} \Gamma_f = 2F_{\parallel}^0 (r_{\parallel}^2 + C_{10}^f) \]  

(24a)

\[
F_{\perp} \Gamma_f = 2F_{\perp}^0 (r_{\perp}^2 + C_{10}^f) \]  

(24b)

\[
F_L \Gamma_f = 2F_L^0 (r_{\perp}^2 + C_{10}^f) \]  

(24c)

\[
(F_L + F_{\parallel}) + \sqrt{2}\pi A_4 \Gamma_f = 2(F_0 + F_{\parallel})^2 (r_{\perp}^2 + C_{10}^f) \]  

(24d)

\[
\sqrt{2}A_5 \Gamma_f = 3F_{\perp} F_0 C_{10} (r_0 + r_{\perp}) \]  

(24e)

\[
A_{FB} \Gamma_f = 3F_{\parallel} F_0 C_{10} (r_{\parallel} + r_{\perp}) \]  

(24f)

\[
(A_{FB} + \sqrt{2}A_5) \Gamma_f = 3F_{\perp} (F_0 + F_{\parallel}) C_{10} (r_{\perp} + r_{\perp}) \]  

(24g)

It is easy to see that only six of the seven equations above are independent; the last Eq. (24g) is easily obtained from Eqs. (24e) and (24f). Considerable notational simplification is achieved by defining the following six ratios of form factors:

\[
P_1 = \frac{F_{\parallel}}{F_{\parallel}}, \quad P_2 = \frac{F_{\perp}}{F_{\parallel}}, \quad P_3 = \frac{F_{\perp}}{F_0 + F_{\perp}}, \quad P_4 = \frac{P_1 P_2}{P_1 + P_2} \]  

(25)

\[
P_1' = \frac{\tilde{G}_{\perp} f}{\tilde{G}_{\parallel} f}, \quad P_2' = \frac{\tilde{G}_{\perp} f}{\tilde{G}_{\parallel} f}, \quad P_3' = \frac{\tilde{G}_{\perp} f + \tilde{G}_{\parallel} f}{P_1' P_2'} \]  

(26)

Clearly, \( r_{\perp} \) introduced in Eq. (23) is not independent and is easily obtained from a combination of \( r_{\parallel} \) and \( r_0 \). The expression for \( r_{\perp} \) in terms of \( r_{\parallel} \) and \( r_0 \) and form factors ratios \( P_1 \) and \( P_2 \) is easily derived to be,

\[
r_{\perp} = \frac{r_0 P_2 + r_0 P_1}{P_2 + P_1} \]  

(27)

Naively we have nine theoretical parameters, the three Wilson coefficients \( C_7, C_9 \) and \( C_{10} \) and the six form factors \( F_0, F_{\parallel}, F_{\perp}, G_{\parallel}, G_{\perp} \) and \( G_0 \), describing the six observables \( \Gamma_f, F_L, F_{\perp}, A_{FB}, A_4 \) and \( A_5 \). As mentioned earlier \( C_7 f \) and \( C_8 \) cannot be distinguished and they are lumped together beyond leading order, so that we have only eight independent theoretical parameters, the two Wilson coefficients \( C_9 \) and \( C_{10} \) and six form factors \( F_0, F_{\parallel}, F_{\perp}, G_{\parallel}, G_{\perp} \) and \( G_0 \). It is obvious that with two theoretical inputs in addition to the observables we should in principle be able to solve for the remaining six theoretical parameters purely in terms of these two reliable inputs and observables. Fortunately, advances in our understanding of these form-factors permit us a judicious choice of the two reliable inputs which depends on the energy of recoiling \( K^* \) (or equivalent \( q^2 \)). At large recoil the two inputs are the ratios of form-factors \( P_1 \) and \( P_1' \) which are well predicted at next to leading order in QCD corrections and free from form factors \( \xi_\parallel \) and \( \xi_\perp \) in heavy quark effective theory. While the choice of \( P_1 \) and \( P_1' \) works well at low \( q^2 \), at low recoil another condition equating the three ratios \( \tilde{G}_\lambda/F_\lambda \) for \( \lambda = \{0, ||, \perp\} \) is needed.

The decay mode \( B \to K^\ast \ell^+ \ell^- \) have been studied with form factors calculated in different models. For example, in Refs. [17] the mode have been studied using light-cone hadron distribution amplitudes [18] combined with QCD sum rules on the light cone [19]. In Refs. [20] the model was studied using naive factorization and QCD sum rules on the light cone. In in Refs. [15, 21, 22] it has been studied in the heavy quark limit using QCD factorization [23, 24]. Soft-collinear effective theory [25–29] that is valid for small \( q^2 \) (large recoil of \( K^* \)) has been used to study the decay in Ref. [30], while operator product expansion that is valid for large \( q^2 \) (low recoil) has been studied in [31].

In the next two subsections that follow, we will digress to consider the \( B \to K^\ast \ell^+ \ell^- \) form factors and their relations in the two limits of the \( K^* \) meson recoil energy. We will present our model independent analysis in the next section (Sec. V). We will assume \( P_1 \) \( P_1' \) as inputs for most of the paper as the results are valid throughout the
q^2 domain, except when P_1 = P'_1. We will show that the validity of the large recoil limit approximation can be verified by a direct measurement of P_1 in terms of helicity fractions, at the zero-crossing point of A_{FB}, i.e., at A_{FB} = 0. The low recoil limit is considered at the end in Sec. VI, where we will also examine the special case P_1 = P'_1. The validity of the low recoil limit can also be tested through a relation derived purely between observables which is valid only in the low recoil limit. In both the recoil regions we derive several important relations between observables, Wilson coefficients and form factors. We find that the six observables are not independent as there exists one constraint relation that involves observables alone and hence free from the details of recoil energy approximation as well. As a consequence we find that F|| cannot be solved for and must be taken as an additional input as well.

A. Form factor in the large recoil limit

In B → K* transition at low q^2, the light meson K* carries a large energy E_{K*}. Since the initial B meson contains the heavy b quark, in this limit the form factors can be expanded in small ratios of \Lambda_{QCD}/m_b and \Lambda_{QCD}/E_{K*} [32]. This reduces the independent B → K* form factors from seven to two universal form factors \xi_\perp and \xi_\parallel. In terms of these two form-factors, the seven form factors can be written up to 1/m_b and \alpha_s corrections as [32, 33]:

\begin{align}
A_1(q^2) &= \frac{2E_{K^*}}{m_B + m_{K^*}} \xi_\perp(E_{K^*}) \quad (28a) \\
A_2(q^2) &= \frac{m_B}{m_B - m_{K^*}} [\xi_\perp(E_{K^*}) - \xi_\parallel(E_{K^*})] \quad (28b) \\
A_0(q^2) &= \frac{E_{K^*}}{m_B} \xi_\parallel(E_{K^*}) \quad (28c) \\
V(q^2) &= \frac{m_B + m_{K^*}}{m_B} \xi_\perp(E_{K^*}) \quad (28d) \\
T_1(q^2) &= \xi_\perp(E_{K^*}) \quad (28e) \\
T_2(q^2) &= \frac{2E_{K^*}}{m_B} \xi_\parallel(E_{K^*}) \quad (28f) \\
T_3(q^2) &= \xi_\perp(E_{K^*}) - \xi_\parallel(E_{K^*}), \quad (28g)
\end{align}

where, E_{K^*} is the energy of the K* meson,

\[ E_{K^*} = \frac{m_{K^*}^2 + m_{K^*}^2 - q^2}{2m_B}. \]

We note that the form factor A_0 does not appear in our expressions in the massless lepton limit. In the large recoil limit T_2/T_1 and V/A_1 are well predicted and reduce to the simple form

\begin{align}
\frac{T_2}{T_1} &= \frac{2E_{K^*}}{m_B} \quad (29) \\
\frac{V}{A_1} &= \frac{(m_B + m_{K^*})^2}{2E_{K^*}m_B}. \quad (30)
\end{align}

Note that these ratios are independent of the universal form factors \xi_\parallel and \xi_\perp and are valid to all orders in the strong coupling constant [15].

In addition to the order \alpha_s corrections to the hadronic form factors, there also exist “non-factorizable” corrections, which can be significant in the heavy quark and large recoil limit. Following Ref. [15], these non-factorizable corrections can be incorporated in next to leading order in QCD by the following transformations

\begin{align}
C_{T_1}^{\text{eff}} T_1 &\rightarrow T_1 \quad (31a) \\
C_9^{\text{eff}} C_9 &\rightarrow C_9 \quad (31b)
\end{align}

where the Wilson Coefficients are taken at the next-to-next-to leading order, and the \mathcal{T}_i are defined as

\[ \mathcal{T}_i = \mathcal{T}_i \quad (32) \]

The complete expressions of \mathcal{T}_i are given in Ref. [15].

The form factor ratios P_3/2 and P'_3/2 can be written with the help of the Eqs. (28a)-(28g). The expressions for the ratio’s P_1 and P'_1 are of particular interest, since these form factor ratio do not receive any QCD correction in the heavy quark effective theory and are independent of the both form factor \xi_\parallel and \xi_\perp to all orders in \alpha_s and to leading order in the 1/m_b expansion. We will take expressions for P_1 and P'_1 as input and find that they are given by the simple form,

\begin{align}
P_1 &= \frac{F_\parallel}{F_||} = \sqrt{\frac{\lambda(m_{B}, m_{K^*}, q^2)}{(m_B + m_{K^*})^2}} \frac{\mathcal{V}(q^2)}{A_1(q^2)} \\
&= \left[ \frac{-\sqrt{\lambda(m_{B}, m_{K^*}, q^2)}}{2E_{K^*}m_B} \right], \quad (33a) \\
P'_1 &= \frac{\bar{F}_\parallel}{\bar{F}_||} = \frac{-\lambda(m_{B}, m_{K^*}, q^2)}{m_B - m_{K^*}} \frac{T_1}{T_2} \\
&= \left[ \frac{-\sqrt{\lambda(m_{B}, m_{K^*}, q^2)}}{2E_{K^*}(m_B^2 - m_{K^*}^2)} \right] \quad (33b)
\end{align}

It may be noted that the form factor ratios P_1 and P'_1 do not depend on the universal form factors \xi_\parallel and \xi_\perp and are unaltered by the inclusion of non-factorizable corrections and higher order corrections in QCD. P_1 and P'_1 are hence used by us as reliable theoretical inputs. On the other hand it is easy to see that P_{2,3} and P'_{2,3} depend on universal form factors and hence receive corrections from higher order and non-factorizable contributions that results in a more complicated expression. In our approach P_{2,3} and P'_{2,3} will be obtained in terms of observables and P_1 and P'_1 in Eqs. (87), (88), (90) and (91).

The expressions (33a) and (33b) are valid for large recoil region where q^2 is small and are usually considered extremely accurate for q^2 between 1 GeV and 6 GeV [33]. The region q^2 < 1 GeV is ignored to eliminate resonance.
contributions which might not only introduce uncertainties but also introduce complex contributions which we have assumed are absent. Unless otherwise stated, large recoil region would mean $0.10 \text{GeV}^2 \leq q^2 \leq 12.86 \text{GeV}^2$.

We stress that once the non-factorizable corrections are taken into account, the Wilson coefficient $C_7$ can no longer be separated from the hadronic form factor. The $C_7$ and the hadronic form factors lump together into effective photon vertex $\tilde{G}_\lambda$, which as we will show, can be expressed in terms of observables and the form factors $P_1$ and $P'_1$.

B. Form factor in the low recoil limit

A model independent description for the case of low recoil energy of the $K^*$ in $B \to K^{*} \ell^{+} \ell^{-}$ decay was put forward by Grinstein and Pirjol [31] in the modified Heavy Quark Effective Theory framework. In this approach [31], “near the zero point $q^2 \approx (m_B - m_{K^*})^2$, the long distance contributions to $B \to K^{*} \ell^{+} \ell^{-}$ can be computed as short distance effect using simultaneously heavy quark and operator product expansion in $1/Q$ with $Q = \{m_b, \sqrt{q^2}\}.” In view of this the subleading $m_{K^*}/m_B$ terms are neglected and non-factorizable corrections are ignored. An elaborate study of the predictions for $B \to K^{*} \ell^{+} \ell^{-}$ was undertaken in Ref. [35] where the next-to-leading order corrections from the charm quark mass $m_c$ and strong coupling at $O(m_c/Q^2, \alpha_s)$ were included. The result is a relations between the $B \to K^{*} \ell^{+} \ell^{-}$ form factors that reduces the number of independent hadronic form factors reduces to only three, i.e., $V, A_1$ and $A_2$ can be expressed in terms of the form factors $T_1, T_2, T_3$ as:

$$T_1(q^2) = \kappa V(q^2)$$

$$T_2(q^2) = \kappa A_1(q^2)$$

$$T_3(q^2) = \kappa A_2(q^2) \frac{m_B^2}{q^2}$$

where, the expression of $\kappa$ is given in [35].

In the low recoil limit the non-factorizable corrections and higher order corrections in $\alpha_s$ are ignorable hence we have $\tilde{G}_\lambda = C^\text{eff}_\lambda G_\lambda$ for all $\lambda = \{\parallel, \perp\}$. The condition in Eq. (34) together with Eq. (21) on ignoring $m_{K^*}/m_B$ terms can be recast as

$$\frac{G_\parallel}{F_\parallel} = \frac{G_\perp}{F_\perp} \equiv \hat{\kappa} = -\frac{2m_Bm_b}{q^2}. \quad (35)$$

This can easily be seen to imply that,

$$P_1 = P'_1, \quad P_2 = P'_2, \quad P_3 = P'_3 \quad (36)$$

and hence

$$r_\parallel = r_\perp = r_0 = r_\lambda \equiv r. \quad (37)$$

In the low recoil limit the form factor ratios $P_1$ and $P'_1$ are easily derived to be

$$P_1 = P'_1 = -\frac{\sqrt{\lambda(m_B^2, m_{K^*}^2, q^2)}}{(m_B + m_{K^*})^2} V(q^2) A_1(q^2). \quad (38)$$

Note that in this limit as well the two ratios $P_1$ and $P'_1$ are independent of the universal form factors $\xi_\parallel$ and $\xi_\perp$.

The low-recoil approximation is expected to work well in region $14.18 \text{GeV}^2 \leq q^2 \leq 19 \text{GeV}^2$. Conventionally the low-recoil region is meant to imply this range of $q^2$. In Sec. VI we will reconsider the low-recoil region to study the special feature that emerge in the low-recoil region. In the low recoil limit we need to take special care of the fact that $P_1 = P'_1$.

V. MODEL INDEPENDENT ANALYSIS

In this section we present a new model independent approach that offers a possibility of isolating hadronic effects from genuine new physics contributions. We begin by deriving the solutions for the Wilson coefficients $C_9, C_{10}$ and the effective photon vertex $\tilde{G}_\lambda$, in terms of observables and the minimum number of required form factor ratios, some of which are more or less independent of hadronic uncertainties.

The first set of solutions are obtained using three independent Eqs. (24a), (24b) and (24f), and one easily solves (see Appendix (A)) for $r_\parallel + r_\perp$ to be

$$r_\parallel + r_\perp = \pm \frac{\sqrt{\lambda}}{\sqrt{2F_\perp}} \left( P_\parallel^2 F_\parallel + F_\perp \pm P_1 \sqrt{4F_\parallel F_\perp - \frac{16}{9} A_3^2 F_\parallel^2} \right)^{1/2} \quad (39)$$

However, $r_\parallel + r_\perp = 0$ when $A_{FB} = 0$ from Eq. (24f). The term inside the round bracket of the above equation becomes a whole square if $A_{FB} = 0$, hence,

$$r_\parallel + r_\perp \bigg|_{A_{FB} = 0} = \pm \frac{\sqrt{\lambda}}{\sqrt{2F_\perp}} (\sqrt{F_\perp} \pm P_1 \sqrt{F_\parallel}) = 0 \quad (39)$$

Since, the expression for $r_\parallel + r_\perp$ should be valid for all values of the observables, the right hand side could go to zero only if positive sign ambiguity is chosen, taking into account that $P_1$ is negative. This fixes the sign ambiguity inside the round bracket. The condition $r_\parallel + r_\perp = 0$ gives us the familiar relation for the zero crossing of $A_{FB}$. The definitions of $r_\parallel$ and $r_\perp$ straight forwardly imply that $A_{FB} = 0$ at:

$$2C_9 = C_7 \left( \frac{\tilde{G}_\parallel}{F_\parallel} + \frac{G_\parallel}{F_\parallel} \right),$$

$$= -\frac{4m_B C_7 T_1(q^2)}{q^2} \left( m_B + m_{K^*} \right) \left( 1 - \frac{m_{K^*}^2}{2m_B^2} \right),$$

$$= -\frac{4m_B m_{K^*} C_7}{q^2} \left( 1 - \frac{m_{K^*}^2}{2m_B^2} \right) + O(\alpha_s) \quad (40)$$

where we have used Eqns. (29) and (30). The $O(\alpha_s)$ dependence arises from the ratio $T_1(q^2)/V(q^2)$, which also depends on $\xi_\perp(q^2)$ [33].
Notice that, Eq. (39) implies that when $A_{FB} = 0$ we must have an exact equality

$$P_1 = -\frac{\sqrt{F_{\parallel}}}{\sqrt{F_{\perp}}} \bigg|_{A_{FB}=0}$$

(41)

enabling a measurement of $P_1$ in terms of the ratio of helicity fractions. If zero crossing were to occur it would provide an interesting test of our understanding of form factors. Very recently LHCb has confirmed [16] zero crossing of $A_{FB}$ for the first time. The zero crossing is observed at $q^2 = 4.9^{+1.3}_{-1.0}$ GeV$^2$, which is consistent with the predictions of the Standard Model and lies in the large recoil region. Eq. (41) can hence be used to measure $P_1$ at the zero crossing of $A_{FB}$. A confirmation of the estimate of $P_1$ with direct helicity measurements would leave no doubt on the reliable predictability HQET in the large recoil region.

The solution of $C_{10}$ in terms of the observables and hadronic form factors now reads as:

$$C_{10} = \frac{\sqrt{15}}{\sqrt{2} F_{\parallel}} \frac{2}{3} \left[ \pm \frac{A_{FB}}{P_1 F_{\perp} + F_{\perp} + P_1 Z_1} \right].$$

(42)

where $Z_1$ is defined as,

$$Z_1 = \sqrt{4 F_{\parallel} F_{\perp} - \frac{16}{9} A_{FB}^2}$$

(43)

This solution allows us to measure $C_{10}$ directly in terms of observables, “clean” form factors $P_1$, $P_1'$ and on $F_{\parallel}$. In Tables I and II we have present the predicted values of $F_{\perp}$ and $C_{10}$ using $F_L$ and $A_{FB}$ values from [10] and [16] respectively. In Table II we also estimate $F_{\perp}$ which is computed directly from data using Eq. (18) and the value of $S_3$ quoted in Ref. [16].

A rather unexpected observation is that as long as $4 F_{\parallel}^2 F_{\perp} \geq \frac{16}{9} A_{FB}^2$, the term $P_1 F_{\perp} + F_{\perp} + P_1 Z_1$ is always positive. This is easily seen by an (infinite) series expansion in $A_{FB}$:

$$P_1 F_{\perp} + F_{\perp} + P_1 Z_1 = (P_1 \sqrt{F_{\parallel}} + \sqrt{F_{\perp}})^2 - \frac{4 A_{FB}^2 P_1}{9 \sqrt{F_{\parallel} F_{\perp}}} \left[ \frac{4 A_{FB}^4 P_1^2}{81 (F_{\parallel} F_{\perp})^{1/2}} + O(A_{FB}^6) \right] \geq 0,$$

(44)

where every term is positive since $P_1$ is negative. Since, the Wilson coefficient $C_{10}$ is real in the Standard model, $Z_1$ must be real restricting the observables $F_{\parallel}$, $F_{\perp}$ and $A_{FB}$ such that:

$$4 F_{\parallel} F_{\perp} \geq \frac{16}{9} A_{FB}^2.$$

(45)

The violation of this condition will be a clear signal of new physics. On the other hand, if the experiments find a real value that does not agree with the estimates of Standard model value, it could either be a signal of new physics or of the uncertainties in form factor estimations. The Wilson coefficient $C_9$, can also be solved (see Appendix A) in terms of observables and form factor ratios,

$$C_9 = \frac{\sqrt{15}}{\sqrt{2} F_{\parallel}} \frac{(P_1 F_{\parallel} - F_{\perp})}{\left[ \pm (P_1 - P_1') \sqrt{P_1^2 F_{\perp} + F_{\perp} + P_1 Z_1} \right]}.$$

(46)

All the discussions following Eq.(42) equally is applicable to this solutions. The way the matrix element decomposition is defined in the heavy quark and large energy limit, at next-to-leading logarithmic order [15], does not allow us to factor out the Wilson coefficient $C_9^{eff}$ from the hadronic form factors $T_i$. Hence, the solution of $C_7$ is not possible. However we can solve for the effective photon vertex $\tilde{G}_\parallel$, which we can express in terms of the observables and $P_1$, $P_1'$. The solution of $\tilde{G}_\parallel$ is:

$$\tilde{G}_\parallel = \frac{\sqrt{15}}{\sqrt{2}} \frac{(P_1^2 F_{\parallel} - F_{\perp})}{\left[ \pm (P_1 - P_1') \sqrt{P_1^2 F_{\perp} + F_{\perp} + P_1 Z_1} \right]},$$

(47)

To obtain the three expressions, Eqs. (42), (46) and (47) we removed the sign ambiguities in the solution by looking at the behavior of the solutions at the $A_{FB}$ zero crossing points. All our solutions for the Wilson coefficients depend explicitly on the assumption that $A_{FB} \neq 0$, hence, the Wilson coefficients can be determined at any $q^2$ except at the zero crossing of $A_{FB}$. The denominator of $\tilde{G}_\parallel$ and $C_9$ depend on $P_1 - P_1'$, so the behavior of the Wilson coefficients at the point $P_1 \rightarrow P_1'$ needs careful examination. Unlike the zeros of $A_{FB}$, which can be experimentally determined and hence avoided, the crossing point for $P_1$ and $P_1'$, a priori, can only be determined based on calculations and hence may be uncertain. We note that in this limit we have $r_1 - r_\perp = 0$, where as in the limit $A_{FB} = 0$ we had $r_1 + r_\perp = 0$. Naively, $C_9$ and $\tilde{G}_\parallel$ appear to be divergent in the limit $P_1 \rightarrow P_1'$, as can be seen from Eqs. (46) and (47) and indeed Eq. (A8) cannot be used to determine the Wilson coefficients $C_7$ and $C_9$. However, it is easily seen that the Wilson coefficients are finite when $P_1 \rightarrow P_1'$. Consider the combination $\tilde{G}_\parallel - F_{\parallel} C_9$, which is seen from Eqs. (47) and (46) to be manifestly finite in the limit $P_1' \rightarrow P_1$:

$$\tilde{G}_\parallel - F_{\parallel} C_9 = \sqrt{\frac{15}{2}} \frac{F_{\parallel} P_1 + \frac{1}{2} Z_1}{\sqrt{P_1^2 F_{\parallel} + F_{\perp} + P_1 Z_1}}.$$

(48)

We will show that the combination $\tilde{G}_\parallel - F_{\parallel} C_9$ can be determined and indeed if $F_{\parallel}$ is assumed $\tilde{G}_\parallel$ and $C_9$ can be individually determined and are finite.

In the limit $P_1' = P_1$, Eq. (23) implies

$$r^2_{\parallel} + C^2_{10} = r^2_{\perp} + C^2_{10} = \frac{F_{\parallel} \Gamma_{\parallel}}{2 F_{\parallel}^2} = \frac{F_{\parallel} \Gamma_{\perp}}{2 F_{\perp}}.$$

(49)
\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
q^2 (\text{GeV}^2) & 0.10-2.00 & 2.00-4.30 & 4.30-8.68 & 10.09-12.86 & 14.18-16.00 & 16.09-19.00 & 1-6 \\
\hline
F_\perp (T) & 0.44\pm0.01 & 0.14\pm0.06 & 0.19\pm0.03 & 0.25\pm0.04 & - & 0.14\pm0.016 & 0.21\pm0.05 \\
C_{10} (T) & 14.36\pm1.68 & 2.81\pm0.78 & 3.00\pm0.384 & 2.34\pm0.372 & - & 3.11\pm0.39 & 3.81\pm0.58 \\
\hline
\end{array}
\]

TABLE I. The predictions for \(F_\perp\) (Eq. (52)) and \(C_{10}\) (Eq. (42)) using 0.37 fb\(^{-1}\) LHCb [10] data for \(F_L\), \(A_{FB}\) and \(d\tau/dq^2\). “(T)” in the first column indicates the values quoted are theoretical estimates. The form factor \(F_\perp\) and the ratios \(P_1\) and \(P_2\) are averaged over each \(q^2\) bin using heavy-to-light form factor at large recoil (for 0.10 GeV\(^2\) \(\leq q^2 \leq 12.86\) GeV\(^2\)) and heavy-to-light form factor at low recoil (for 16 GeV\(^2\) \(\leq q^2 \leq 19\) GeV\(^2\)). The region 14.18 GeV\(^2\) \(\leq q^2 \leq 16\) GeV\(^2\) is neglected as the form factors can not be calculated reliably in this region. The unusual large value of \(C_{10}\) in the 0.10 GeV\(^2\) \(\leq q^2 \leq 2\) GeV\(^2\) region could be due to failure in estimating \(F_\perp\) or perhaps be a signal new physics. It is unlikely [36, 37] that such a large effect can be due to the contributions from low lying resonances in the experimental data. It may be noted that estimate of \(F_\perp\) does not depend on universal form factors and is clean in the low recoil limit.

\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
q^2 (\text{GeV}^2) & 0.10-2.00 & 2.00-4.30 & 4.30-8.68 & 10.09-12.86 & 14.18-16.00 & 16.09-19.00 & 1-6 \\
\hline
F_\perp (E) & 0.36^{+0.14}_{-0.11} & 0.11^{+0.09}_{-0.15} & 0.31 \pm 0.09 & 0.145^{+0.12}_{-0.13} & 0.35 \pm 0.13 & 0.08^{+0.13}_{-0.14} & 0.22^{+0.10}_{-0.11} \\
F_\perp (T) & 0.31\pm0.03 & 0.15\pm0.04 & 0.20\pm0.03 & 0.22\pm0.03 & - & 0.12\pm0.01 & 0.17\pm0.03 \\
C_{10} (T) & 12.91\pm1.07 & 2.60\pm0.779 & 2.88\pm0.32 & 2.0\pm0.25 & - & 2.55\pm0.29 & 3.26\pm0.45 \\
\hline
\end{array}
\]

TABLE II. The same as Table 1 but with 1.0 fb\(^{-1}\) LHCb data [16]. “(E)” in the first column indicates the values quoted are experimental estimates. \(F_\perp\) (E) is computed directly from data using Eq. (18) and the value of \(S_L\) quoted in Ref. [16]. The values of \(C_{10}\) seem to decrease with the larger data set used and are marginally lower than theoretical estimates. Unfortunately, the cause of discrepancy in \(C_{10}\) can not be fixed, it could either be due to failure in estimating \(F_\perp\) or perhaps be a signal new physics. Note that in the 0.10 GeV\(^2\) \(\leq q^2 \leq 2\) GeV\(^2\) region \(C_{10}\) is still large even with improved statistics. We emphasize that the two values of \(F_\perp\) are in good agreement almost throughout the \(q^2\) region.

We thus have

\[
P_1^2 = P_1'^2 = \frac{F_\perp}{F_\|} = \frac{F_\perp^2}{F_\|},
\]

which enables a measurement of \(P_1\). Indeed if the hadronic estimate \(P_1^2 = F_\perp^2/F_\|\) is verified by measurement even when \(A_{FB} \neq 0\), we can conclude with certainty that \(P_1 = P_1'\). Hadronic estimates can thus be verified experimentally. Note that a similar condition at \(A_{FB} = 0\) also provided a measurement of \(P_1\) in Eq. (41).

Many more important results can be derived from the expressions derived so far. We can use Eqs. (42) and (46) to obtain the ratio \(C_9/C_{10}\):

\[
R \equiv \frac{C_9}{C_{10}} = \frac{2(F_\|P_1P_1' - F_\perp)}{\frac{4}{3}A_{FB}(P_1 - P_1')} . \quad (51)
\]

\[
F_\perp = -\frac{4RA_{FB}(P_1 - P_1')(1 + P_1P_1') + 3(1 - F_L)(P_1 + P_1')^2 - (P_1 - P_1')\sqrt{T_\perp}}{3(1 + P_1^2)(1 + P_1'^2)} . \quad (52)
\]

The sign of the term containing \(\sqrt{T_\perp}\) could either be positive or negative. Of the two possible solutions for \(F_\perp\), in Eq. (52) we have chosen the solution which gives the correct value of \(R\) obtained from Eq. (51). This solution corresponds to the one with the negative ambiguity as shown in Eq. (52). As can be seen from the Eq. (52), the transversity amplitude \(F_\perp\) is expressed in terms of two observables \(F_L\) and \(A_{FB}\) which has already been measured. Using the measured values of \(F_L\) and \(A_{FB}\) from Ref. [10] and [16] we have tabulated the predicted values of \(F_\perp\) in Tables I and II respectively.
FIG. 1. The allowed $F_L - A_{FB}$ parameter space depicted by the solid (blue) triangle for $R = -1$, is obtained by demanding that $F_\perp$ (see Eq. 52) is real valued. In the figure to the left $P_1$ and $P'_1$ values are averaged over 1 GeV$^2$ to 6 GeV$^2$ using heavy-to-light form factor at large recoil (see Sec. (IV A)), and in the figure to the right we’ve used $P_1$ and $P'_1$ values averaged over 16 GeV$^2$ to 19 GeV$^2$ using heavy-to-light form factor at low recoil (see Sec. (IV B)). Inside the triangles, the solid (black) lines correspond to the $F_\perp$ values the dashed (blue) lines correspond to the $C_{10}$ values as function of $F_L$ and $A_{FB}$.

FIG. 2. The same as Fig. 1, except that the variation of the parameter space is studied as a function of $R$. The large-dashed (red) triangle and the identical lines correspond to $R = 10$. The $R = -10$ case is depicted by small dashed (blue) line. $R = -1$ case is shown for reference with solid (black) lines. In the figure to the left $P_1$ and $P'_1$ values are averaged over 1 GeV$^2$ to 6 GeV$^2$ and in the figure to the right we’ve used $P_1$ and $P'_1$ values averaged over 16 GeV$^2$ to 19 GeV$^2$.

In order that $F_\perp$ take real values, $T_\perp$ must be positive. The positivity of $T_\perp$ imposes constraints on the possible values for $F_L$ and $A_{FB}$, which cannot therefore be arbitrarily chosen. The requirement for real solution for $F_\perp$ hence
implies a constraint on $A_{FB}$ in terms of $P_1, P'_1, R$ and observable $F_L$

\[
\frac{-3(1 - F_L)}{4} T_- \leq A_{FB} \leq \frac{3(1 - F_L)}{4} T_+ 
\]

\[
T_\pm = \frac{(P_1 + P'_1)^2}{\sqrt{(1 + P_1^2)(1 + P'_1^2)}} \sqrt{(P_1 + P'_1)^2 + R^2(P_1 - P'_1)^2 \mp (1 - P_1 P'_1)(P_1 - P'_1) R}
\]

(53)

It is easy to see that $T_\pm \approx 1$ when $P_1 \approx P'_1 \approx -1$. Given the values of $P_1$ and $P'_1$ from Table III, we expect $T_\pm \approx 1$. The allowed domain for $A_{FB}$ is hence almost free from $R$ as long as $P_1 \approx P'_1 \approx -1$. In Fig. 1, we have depicted the permitted $F_L - A_{FB}$ parameter space by the solid (blue) triangle and the identical lines correspond to the $C_{10}$ values as function of $F_L$ and $A_{FB}$. In Fig. 2 the variation of the parameter space is studied as a function of $R$. The large-dashed (red) triangle and the identical lines correspond to $R = 10$. The $R = -10$ case is depicted by small dashed (blue) line. $R = -1$ case is shown for reference with solid (black) lines. In the figure to the left $P_1$ and $P'_1$ values are averaged over $1 \text{ GeV}^2$ to $6 \text{ GeV}^2$ using heavy-to-light form factor at low recoil (see Sec. (IV A)). Inside the triangles, the solid (black) lines correspond to the $F_\perp$ values the dashed (blue) lines correspond to the $C_{10}$ values as function of $F_L$ and $A_{FB}$.

Interestingly, Eq. (51) can also be inverted to express $A_{FB}$ in terms of $P_1, P'_1$ and $R$:

\[
A_{FB} = \frac{3 \left( RX - \sqrt{Y(P_1 - P'_1)^2(1 + R^2) - X^2} \right)}{4(P_1 - P'_1)(1 + R^2)}
\]

(54)

where, $X = 2(F_1 P_1 P'_1 - F_\perp)$ and $Y = 4F_1 F_\perp$. Note that the Eq. (51) is quadratic in $A_{FB}$, and should have resulted in a two-fold ambiguity in the solution. One easily confirms that only the solution with positive sign in front of the square root is valid by substituting the observables in terms form-factors and the Wilson coefficients. The usefulness of the result in Eq. (54) is that it constrains the $F_L - F_\perp$ parameter space. This is easily derived by requiring that $A_{FB}$ in Eq. (54) is real, implying the positivity of the argument of the radical in the expression for $A_{FB}$:

\[
\frac{P_1^2 + P'_1^2 + R^2(P_1 - P'_1)^2 - (P_1 - P'_1) \sqrt{R^2 + 1} \sqrt{R^2(P_1 - P'_1)^2 + (P'_1 + P_1)^2}}{2P_1^2 P'_1^2} 
\]

\[
\leq 1 - \frac{F_\perp}{F_L} \leq 1 + \frac{P_1^2 + P'_1^2 + R^2(P_1 - P'_1)^2 + (P_1 - P'_1) \sqrt{R^2 + 1} \sqrt{R^2(P_1 - P'_1)^2 + (P'_1 + P_1)^2}}{2P_1^2 P'_1^2}
\]

(55)

The constraint implied by this bound is depicted in Figs. 3 and 4 where we have considered two different values corresponding to different bins of averaged $q^2$ values. $P_1$ and $P'_1$ are averaged over the $q^2$ region as described in the figure caption. The reader will note the rigorous constraint within Standard model i.e. $R = -1$, depicted in the figures by the diagonal thick solid (blue) line that predicts $F_\perp$ to lie in a very narrow region, well approximated by a line that is a function of $F_L$ and with the slope depending on the domain of $q^2$. It is obvious from Eq. (55) that as $R^2$ increases from unity a wider region around this solid line is allowed. The allowed $F_L - F_\perp$ parameter space for $|R| = 10$ are also depicted as a wedge of dashed (blue) lines. In Fig. 4 on the right we have shown an enlarged region where for $|R| = 10$ we have plotted the values of $A_{FB}$ evaluated using Eq. (54). As the figures shows $F_L - F_\perp$ correlation is not particularly sensitive to $R$. Also plotted in these figures are the constraints on the $F_L - F_\perp$ parameter space arising from $Z^*_1 > 0$ for different values of $A_{FB}$. The plots also include other details which will be discussed later.

It is interesting to note that irrespective of the value of $R$, in the limit $P'_1 \to P_1$ one obtains $(1 - F_L)/F_\perp = 1 + 1/P_1^2$. In the limit $m_B \to \infty$ and the energy of the $K^+, E_K \to \infty$, it is easy to see that $P_1 = P'_1 \to -1$, and we find that $F_L = F_\perp$. In this limit Eq. (18) will result in a constant distribution in $\phi$. Since $P_1$ and $P'_1$ values differ slightly we expect only a very small coefficient of $\cos \phi$.

The measurements of $F_L$ and $F_\perp$ must be consistent
FIG. 3. The constraints on $F_L - F_\perp$ parameter space arising from Eq. (55) with the value of $P_1$ and $P_1'$ averaged over $1 \text{ GeV}^2 \leq q^2 \leq 6 \text{ GeV}^2$. The allowed region for $R = -1$ is depicted by the diagonal thick solid (blue) line that predicts $F_\perp$ to lie in a very narrow region, well approximated by a line. The allowed $F_L-F_\perp$ parameter space for $|R| = 10$ are also depicted as a wedge of dashed (blue) lines. The shaded region in the left figure is forbidden by $F_L + F_\perp + F_\parallel = 1$. In the figure on the left the thick dashed (red) line correspond to the solution of $A_{FB}$ relative to $C_9/C_{10}$ and $C_7/C_{10}$ as depicted in the figure. The additional curves in the right figure correspond to the constraint on $F_L - F_\perp$ arising from $P_2 > 0$ for different values of $A_{FB}$: $0.05, 0.25, 0.5, 0.7$, where all the regions to the left of these curves are allowed.

FIG. 4. The same as in Fig. 3 except that $P_1$ and $P_1'$ averaged over $16 \text{ GeV}^2 \leq q^2 \leq 19 \text{ GeV}^2$. The figure to the right is the inset of the figure to the left. In this figure the solid and the dashed diagonal (blue) lines are the same as in the figure to the right. The dot-dash (red) lines labeled by “a,b,c,d” correspond to $A_{FB} = 0.5, 0.3, 0, -0.3$ respectively for $R = -10$. The line ‘c” (for $A_{FB} = 0$) divides the domain and corresponds to the thick dashed (red) line in Fig. 3. The $A_{FB}$, $F_L$, and $F_\perp$ must be consistent as shown by the dot-dash lines. For $R = -1$ similar lines exist for different value of $A_{FB}$ but overlap with the solid blue line. Hence they are not depicted in the figure.

with value of $A_{FB}$ and there exists a domain of $R$, $P_1$ and $P_1'$ for which the consistency may hold. These values must be verified to be consistent with the values of observables. Bounds on $P_2^1$ can be obtained from the equations derived so far, in terms of observables alone. Extrmizing $P_2^1$ in terms of all the non observables in
Eq. (42), we get following bounds on $P_1^2$

$$P_1^2 \leq \frac{4F_\parallel F_\perp - 16A_{FB}^2}{F_\parallel^2} \quad \forall \frac{F_\parallel F_\perp}{F_\parallel} \leq \frac{2}{7} \left(\frac{4A_{FB}}{3}\right)^2 \quad (56)$$

For $A_{FB} = 0$, we have already noted the exact equality $P_1^2 = F_\perp/F_\parallel$. Analytical bound on $P_1^2$ are also possible, but are harder to obtain.

We now derive some useful relations that involve $C_7$ and are hence valid only at the leading order. Eqs. (42) and (47) can be re-expressed in this limit as:

$$\frac{C_7}{C_{10}} = -\frac{3}{2} \frac{F_\parallel}{G_\parallel} \frac{P_1^2 F_\parallel - F_\perp}{A_{FB} (P_1 - P_1^*)} \quad (57)$$

where we have used the fact that $\frac{G_\parallel}{C_7} = C_7 G_\parallel$ at leading order. We emphasize that $C_7/C_{10}$ is not as clean as $C_9/C_{10}$, which is expressed in Eq. (51) in terms of observables and ratio's of two form factors which are predicted exactly in heavy quark effective theory. $C_7/C_{10}$ on the other-hand depends on $F_\parallel/G_\parallel$ which in turn depends on the heavy quark effective theory form factor $\xi$. It may nevertheless be noted that the sign of $F_\parallel/G_\parallel$ is quite accurately predicted to be negative, since $A_1(q^2)$ and $T_2(q^2)$ are always positive. Eq. (57) directly implies a constraint on the sign of $C_7/C_{10}$. It is easy to conclude that $(C_7/C_{10})A_{FB} \gtrless 0$ only if $P_1^2 \lessgtr F_\perp/F_\parallel$ when $P_1 - P_1^* > 0$. Eq. (56) together with Eq. (57) can be used to obtain more useful bounds that are purely in terms of observables alone, albeit they are not completely exhaustive. Eq. (56) implies:

$$P_1^2 F_\parallel - F_\perp \lessgtr \frac{Z_1 - F_\perp F_\parallel}{F_\parallel} \quad \forall \frac{F_\parallel F_\perp}{F_\parallel} \leq \frac{2}{7} \left(\frac{4A_{FB}}{3}\right)^2 \quad (58)$$

which in turn implies for $(P_1^2 F_\parallel - F_\perp) < 0$ that,

$$\frac{C_7}{C_{10}} A_{FB} > 0 \quad \forall \frac{F_\parallel F_\perp}{F_\parallel} < \frac{32}{63} A_{FB}^2 \quad (59)$$

If, however, $(P_1^2 F_\parallel - F_\perp) > 0$ we obtain an analogous condition

$$\frac{C_7}{C_{10}} A_{FB} < 0 \quad \forall \frac{F_\parallel F_\perp}{F_\parallel} > \frac{16}{27} A_{FB}^2 \quad (60)$$

The above bounds have nothing to say on the sign of $C_7/C_{10}$ in the region,

$$\frac{32}{63} A_{FB}^2 \leq F_\parallel F_\perp \leq \frac{16}{27} A_{FB}^2 \quad (61)$$

and may not be particularly useful in general. One can nevertheless draw conclusions on the signs of the Wilson coefficients by combining Eq. (51) together with Eq. (57) to write:

$$\left(\frac{2}{3} \frac{C_9}{C_{10}} - \frac{4}{3} \frac{C_7}{C_{10}} P_1^*\right) A_{FB} = \left(P_1^2 F_\parallel + F_\perp + P_1 Z_1\right) > 0, \quad (62)$$

where, $P_1^* = (G_\parallel/F_\parallel)$ $(P_1 + P_1^*) > 0$ since each of $(G_\parallel/F_\parallel)$, $P_1$ and $P_1^*$ are always negative. Defining,

$$E_1 = \frac{C_9}{C_{10}} A_{FB}, \quad E_2 = \frac{C_7}{C_{10}} A_{FB}, \quad (63)$$

for convenience, Eq. (62) reads

$$\frac{2}{3} P_1^* E_1 - \frac{4}{3} P_1 E_2 > 0 \quad (64)$$

In SM, $C_7/C_{10} > 0$ and $C_9/C_{10} < 0$, hence the sign of $E_2$ ($E_1$) will be same (opposite) to that observed for $A_{FB}$. If for any $q^2$ we find $A_{FB} > 0$, Eq. (64) cannot be satisfied unless the contribution from the $E_2$ term exceeds the $E_1$ term, or the sign of the $E_2$ term is wrong in SM. In the SM the $E_2$ term dominates at large recoil i.e. small $q^2$, hence, $A_{FB}$ must be positive at small $q^2$ to be consistent with SM. If $A_{FB} < 0$ is observed for all $q^2$ i.e. no zero crossing of $A_{FB}$ is seen, one can convincingly conclude that $C_7/C_{10} < 0$ in contradiction with SM. However, if zero crossing of $A_{FB}$ is confirmed with $A_{FB} > 0$ at small $q^2$ it is possible to conclude that the signs $C_7/C_{10} > 0$ and $C_9/C_{10} < 0$ are in conformity with SM, as long as other constraints like $Z_1^2 > 0$ hold. In Ref. [16] the zero crossing is indeed seen. However, in the $2\text{GeV}^2 \leq q^2 \leq 4.3\text{GeV}^2$ bin, $Z_1^2 > 0$ is only marginally satisfied. We emphasize that these conclusions drawn from Eq. (62) are exact and not altered by any hadronic uncertainties.

As mentioned in the text earlier, there are three set of solutions of Wilson coefficients $C_9$ and $C_{10}$ and the effective photon vertices $G_\parallel$ and $G_\parallel + G_\parallel$. We next discuss the second and the third set of solutions. The method of solutions is identical to first set of solutions (see Eqs. (42), (46) and (47)) and has been discussed in Appendix A. Using Eqs. (24b), (24c) and (24e) we can easily solve for $r_0 + r_\perp$ as

$$r_0 + r_\perp = \pm \sqrt{-\frac{1}{2} \frac{\Gamma}{F_\parallel}} \left(P_2^2 F_L + F_\perp + P_2 Z_2\right)^{1/2} \quad (65)$$

where we have defined

$$Z_2 = \sqrt{4F_\parallel F_\perp - \frac{32}{9} A_{FB}^2} \quad (66)$$

and the form factor ratios $P_2$ has been previously defined in Eq. (25). It is easy to derive that

$$r_0 + r_\perp \Big|_{A_5=0} = \pm \sqrt{-\frac{1}{2} \frac{\Gamma}{F_\parallel}} \left(\sqrt{F_\perp} \pm P_2 \sqrt{F_L}\right) = 0 \quad (67)$$

since, Eq. (24e) implies that we have $r_0 + r_\perp = 0$ at $A_5 = 0$. Once again, repeating the arguments made when $A_{FB} = 0$, the expression of $r_0 + r_\perp = 0$ is valid for all values of the observables. The right hand side of Eq. (67) can be zero only when positive sign ambiguity is chosen, since $P_2$ is negative. At the zero crossing points of $A_5$ we also have the following exact equality,

$$P_2 = -\sqrt{\frac{F_\perp}{F_L}} \Big|_{A_5=0} \quad (68)$$
enabling measurements of form factor ratio $P_2$ in terms of observables, as long as the zero crossing of $A_5$ occurs in the large recoil region (we have verified at leading order that this is indeed true). We now write the second set of solutions of Wilson coefficients $C_9$ and $C_{10}$ and the effective photon vertex $\tilde{g}_0$:

$$C_{10} = \frac{\sqrt{7} \sqrt{2}}{2F_0 3} \left[ \frac{\sqrt{2}}{2} A_5 \left( \pm \frac{P^2 F_L + F_\perp + P_2 Z_2}{2} \right) \right]$$  \hspace{1cm} (69)

$$C_9 = \frac{\sqrt{7} \sqrt{2}}{2F_0} \left[ \frac{1}{2} (P_2 - P_2') \sqrt{P^2 F_L + F_\perp + P_2 Z_2} \right]$$  \hspace{1cm} (70)

$$\tilde{g}_0 = \frac{\sqrt{7} \sqrt{2}}{2} \left[ \frac{1}{2} (P^2 F_L - F_\perp) \right]$$  \hspace{1cm} (71)

It is easy to derive these relations which are identical to the ones derived in Eqs. (42), (46) and (47) except for the replacements: $F_L \rightarrow F_L$, $A_{FB} \rightarrow \sqrt{2} A_5$, $F_\perp \rightarrow F_0$, $g_0 \rightarrow g_0$, which also imply that $r_\parallel \rightarrow r_\parallel$, $P_1 \rightarrow P_2$ and $P_1' \rightarrow P_2'$. Straightforward extremization with respect to all the non observables in Eq. (69) gives the following bounds on the form factor ratios $P_2$

$$P_2 \leq \frac{4F_L F_\perp - \frac{32}{3} A_5^2}{F_L^2} \hspace{1cm} \forall \hspace{0.1cm} F_L F_\perp \leq \frac{2}{7} \left( \frac{4\sqrt{2} A_5}{3} \right)^2$$

Eq. (69)–(71) give:

$$C_9 = \frac{\sqrt{7} \sqrt{2}}{4F_L P_2' - (P_2 - P_2') \left( 2 \sqrt{2} A_5 (P_2 - P_2') \right)}$$  \hspace{1cm} (72)

$$\frac{\tilde{g}_0}{C_{10}} = \frac{3}{2F_0} \left( P^2 F_L - F_\perp \right)$$  \hspace{1cm} (73)

Eq. (72) can be inverted to obtain expressions for $A_5$ akin to the expression for $A_{FB}$ obtained in Eq. (54). One easily derives:

$$\sqrt{2} A_5 = \frac{3(2F_2 - \sqrt{2} Y (P_2 - P_2'))^2 (1 + R^2) - X^2_2}{4(P_2 - P_2')(1 + R^2)}$$  \hspace{1cm} (74)

where $X_2 = 2(F_L P_2 P_2' - F_\perp)$ and $Y_2 = 4F_L F_\perp$. Eqs. (72) and (73) can be combined to obtain

$$\left( \frac{2 C_7}{3 C_{10}} P_2' - \frac{4 C_9}{3 C_{10}} P_2 \right) A_5 = \frac{P^2 F_L + F_\perp + P_2 Z_2}{\sqrt{2}}$$  \hspace{1cm} (75)

where $P_2' = (g_0/F_0) (P_2 + P_2') > 0$, since $g_0/F_0$, $P_2$ and $P_2'$ are all negative. While this is not easily seen as in the case of $P_1^*$, we have numerically verified at leading order that this is true for the entire $q^2$ domain. We have shown earlier by doing a power expansion in $A_{FB}$, that $(P^2 F_L + F_\perp + P_2 Z_2)$ is always positive. It is easy to see that similar arguments can be made for the positivity of $(P^2 F_L + F_\perp + P_2 Z_2)$ by considering expansions in $A_5$. Hence if the term in the bracket must be positive $A_5$ must be positive. At large recoil the term in the bracket is expected to be positive.

The arguments made above for $r_\parallel + r_\perp$ can be repeated for $r_\perp + r_\perp$. One easily solves using Eqs. (24b), (24d) and (24g):

$$r_\perp + r_\perp = \pm \frac{\sqrt{7} \sqrt{2}}{2F_\perp} \left( P^2 F_L + F_\perp + \sqrt{2} A_5 \right)^{1/2}$$  \hspace{1cm} (76)

where, $P_3$ has been defined in Eq. (25) and we have defined

$$Z_3 = \sqrt{4(F_L + F_\perp + \sqrt{2} \pi A_4)} F_\perp - \frac{16}{9} (A_{FB} + \sqrt{2} A_5)^2.$$  \hspace{1cm} (77)

Eq. (24g) implies that $r_\perp + r_\perp = 0$ when $A_{FB} + \sqrt{2} A_5 = 0$, hence,

$$r_\perp + r_\perp \bigg|_{A_{FB} + \sqrt{2} A_5 = 0} = \pm \frac{\sqrt{7} \sqrt{2}}{2F_\perp} \left( \sqrt{F_\perp \pm \sqrt{2} \pi A_4 \pm P_3} \right) = 0$$  \hspace{1cm} (78)

Once again we choose the positive sign to fix the sign ambiguity since $P_3$ is negative. At the zero crossing points of $A_{FB} + \sqrt{2} A_5$ we hence have the equality,

$$P_3 = -\frac{\sqrt{F_\perp}}{\sqrt{F_L + F_\perp + \sqrt{2} \pi A_4}} \bigg|_{A_{FB} + \sqrt{2} A_5 = 0}$$  \hspace{1cm} (79)

Hence, the zero crossing of $A_{FB} + \sqrt{2} A_5$ enables the measurement of form factor ratio $P_3$ as well, in terms of observables. Note however, that $P_3$ is not independent and related to $P_1$ and $P_2$ (see Eq. (25). The consequences of this relation will be discussed later. The new solutions to $C_{10}$, $C_9$ and $\tilde{g}_0$|| have $g_0$:
FIG. 5. The requirement that $Z_1, Z_2, Z_3$ must be real, for any consistent set of independent observables $A_{FB}, F_L, F_\perp$ and $A_5$ constrains the allowed $F_L-F_\perp$ parameter space to lie only within the solid black lines. $A_4$ is given by Eq. (97). Even with in the allowed $F_L-F_\perp$ domain only the region on the right is allowed depending on the values of $A_{FB}$ and $A_5$. In the four figures we have sampled values of $A_{FB}$ and $A_5$ are as depicted. There is no hadronic assumption made in obtaining the constraints depicted in these plots.

\begin{align}
C_{10} &= \frac{\sqrt{17}}{\sqrt{2}(F_0 + F_\parallel)} \frac{2}{3} \left( A_{FB} + \sqrt{2}A_5 \right) \left[ \pm \sqrt{P_3^2(F_L + F_\parallel + \sqrt{2}\pi A_4) + F_\perp + P_3 Z_3} \right], \\
C_9 &= \frac{\sqrt{17}}{\sqrt{2}(F_0 + F_\parallel)} \left( (F_L + F_\parallel + \sqrt{2}\pi A_4)P_3P'_3 - F_\perp \right) - \frac{1}{2}(P_3 - P'_3)Z_3, \\
\bar{G}_\parallel + \bar{G}_0 &= \frac{\sqrt{17}}{\sqrt{2}} \left[ \pm (P_3 - P'_3) \sqrt{P_3^2(F_L + F_\parallel + \sqrt{2}\pi A_4) + F_\perp + P_3 Z_3} \right].
\end{align}

While these solutions may look more complicated they can also be obtained from Eqs. (42), (46) and (47) by the
replacements $F_\parallel \to F_L + F_\parallel + \sqrt{2}\pi A_4$, $A_{FB} \to A_{FB} + \sqrt{2}A_5$, $F_\parallel \to F_\parallel + F_0$, $\tilde{G}_\parallel \to \tilde{G}_\parallel + \tilde{G}_0$, which also imply \( r_\parallel \to r_\perp \), \( P_1 \to P_3 \) and \( P'_1 \to P'_3 \).

Once again straightforward extremization with respect to all the non observables in Eq. (80) results in the following bounds on the form factor ratio $P_3$,

\[
P_3 \leq 4(F_L + F_\parallel + \sqrt{2}\pi A_4)F_\perp - \frac{16}{3}(A_{FB} + \sqrt{2}A_5)^2 \quad \land \quad (F_L + F_\parallel + \sqrt{2}\pi A_4)F_\perp \leq \frac{2}{3}(4(A_{FB} + \sqrt{2}A_5))^2
\]

These bounds are a very good test of our understanding of the form factors. Similar relations that can be derived from Eqs. (80)–(82):

\[
\begin{align*}
\frac{C_9}{C_{10}} &= \frac{2((F_L + F_\parallel + \sqrt{2}\pi A_4)P_3P'_3 - F_\perp - (P_3 - P'_3)Z_3)}{4(A_{FB} + \sqrt{2}A_5)(P_3 - P'_3)}, \\
\frac{\tilde{G}_\parallel + \tilde{G}_0}{C_{10}} &= \frac{3}{2}(F_\parallel + F_0)(P_3(F_L + F_\parallel + \sqrt{2}\pi A_4) - F_\perp)(A_{FB} + \sqrt{2}A_5)(P_3 - P'_3),
\end{align*}
\]

Eqs. (83) can be inverted to obtain expressions for $A_{FB} + \sqrt{2}A_5$ akin to the expression for $A_{FB}$ obtained in Eq. (54). One easily derives:

\[
A_{FB} + \sqrt{2}A_5 = \frac{3(RX_3 - \sqrt{Y_3(P_3 - P'_3)^2}(1 + R^2) - X_3^2)}{4(P_3 - P'_3)(1 + R^2)}
\]

where \( X_2 = 2(F_LP_3P'_3 - F_\perp) \), \( Y_2 = 4F_LF_\perp \), \( X_3 = 2((F_L + F_\parallel + \sqrt{2}\pi A_4)P_3P'_3 - F_\perp) \) and \( Y_3 = 4(F_L + F_\parallel + \sqrt{2}\pi A_4)F_\perp \).

From Eqs. (83), (84) we can obtain yet another important relation, which is of the same kind as we obtained earlier in Eqs. (62) and (75)

\[
\left( \frac{2}{3} \frac{C_9}{C_{10}} P_3 - \frac{4}{3} \frac{C_9}{C_{10}} P_3 \right)(A_{FB} + \sqrt{2}A_5) = \left[ (P_3(F_L + F_\parallel + \sqrt{2}\pi A_4) + F_\perp + P_3Z_3 \right] > 0
\]

where $P_3'' = (\tilde{G}_0 + \tilde{G}_\parallel)/(F_\parallel + F_\perp)(P_3 + P'_3) > 0$. This is easily verified to be true at leading order for the entire $q^2$ domain. We have shown earlier by doing a power expansion in $A_{FB}$ and $A_5$, that respectively $(P_1^2F_\parallel + F_\perp + P_1Z_1)$ and $(P_3^2F_L + F_\parallel + P_3Z_2)$ are always positive. It is easy to see that similar arguments can be made for the positivity of $(P_3(F_L + F_\parallel + \sqrt{2}\pi A_4) + F_\perp + P_3Z_3)$ by considering expansions in $A_{FB} + \sqrt{2}A_5$. These equations are equally useful to determine the sign of $C_7$ as discussed earlier, however, the form factors involved are not completely free from HQET form factor.

Eq. (69)–(71) and Eq. (80)–(82) have been expressed in terms of from factor ratios $P_2$, $P_2'$, $P_3$, $P_3'$, which are not completely free from the hadronic form factors, both at large and at low recoil. The form factor ratios $P_1$ and $P_1'$ on the other hand is completely free from the Isgur-Wise form factors $\xi_\parallel$ and $\xi_\perp$, in the limit of heavy quark and large recoil of the vector meson. We can express the form factor ratios $P_2$, $P_2'$, $P_3$, $P_3'$ in terms of $P_1$ and $P_1'$.

Equating the relations obtained for $C_9/C_{10}$ and $C_7/C_{10}$ in Eqs. (51), (57) with those in Eqs. (72), (73) we obtain relations only between form $P_1$, $P_1'$, $P_2$, $P_2'$ and observables. The two equations so obtained can be used to solve for $P_2$ and $P_2'$ in terms of $P_1$ and $P_1'$.

\[
P_2 = \frac{2P_1A_{FB}F_\perp}{\sqrt{2}A_5(2F_\perp + Z_1P_1) - Z_2P_1A_{FB}}
\]

\[
P_2' = \frac{\sqrt{2}A_5(F_\perp - F_\parallel P_1)P_2P_1'}{A_{FB}T_2(P_1 - P_1') + \sqrt{2}A_5(F_\perp - F_\parallel P_1')P_2P_1'}
\]

where

\[
T_2 = P_1(F_\perp - F_\parallel P_2')
\]

We emphasize that while $P_2$ and $P_2'$ on the left hand side depend on the Isgur-Wise wave functions $\xi_\parallel$ and $\xi_\perp$, $P_1$ and $P_1'$ are independent of them. These two equations can be used to obtain information about the wave functions. Eq. (87) is also very important in the sense that the domain of observables is itself constrained by the terms under the radical signs must be positive to obtain real $P_2$. Similar relations for $P_3$ and $P_3'$ in terms of $P_1$ and $P_1'$ can be obtained by using Eq. (51), (57) and Eq. (83), (84) to get:

\[
P_3 = \frac{2P_1A_{FB}F_\perp}{(A_{FB} + \sqrt{2}A_5)(2F_\perp + Z_1P_1) - Z_3P_1A_{FB}},
\]

\[
P_3' = \frac{(A_{FB} + \sqrt{2}A_5)(F_\perp - F_\parallel P_1')P_3P_1'}{A_{FB}T_3(P_1 - P_1') + \sqrt{2}A_5(F_\perp - F_\parallel P_1')P_3P_1'}
\]
where
\[ T_3 = P_1 \left[ F_\perp (1 + P_3^2) - P_2^2 (1 + \sqrt{2} \pi A_4) \right] \]  
(92)

As emphasized earlier the Wilson coefficients are real constants except for the non resonant regions. This implies that just like \( Z_1 \), both \( Z_2 \) and \( Z_3 \) are always real if resonant regions and CP violation are excluded:
\[ 4 F_L F_\perp \geq \frac{16}{9} (\sqrt{2} A_5)^2 \]  
(93)
\[ 4 (F_L + F_\parallel + \sqrt{2} \pi A_4) F_\perp \geq \frac{16}{9} (A_{FB} + \sqrt{2} A_5)^2 \]  
(94)

The combination of bounds in Eqs. (45) and (93) results in yet another interesting bound among observables alone but involving only \( A_{FB}^2, A_5^2 \) and \( F_\perp \):
\[ 4 (1 - F_\perp) F_\perp \geq \frac{16}{9} (A_{FB}^2 + 2 A_5^2). \]  
(95)

In Fig. 6 we depict the constraint on \( A_{FB}, A_5 \) and \( F_\perp \) arrived at by Eq. (95). We emphasize that like the bounds derived in Eqs. (45), (93) and (94) this bound is also completely free from any hadronic uncertainty.

In Eq. (25) we showed that \( P_3 \) is not independent but related to \( P_1 \) and \( P_2 \). \( P_3 \) and \( P_2 \) are themselves expressed in terms of observables and \( P_3 \) in Eqs. (90) and (87) respectively. This constraint results in an interesting relation that depends on observables alone:
\[ Z_3 = Z_1 + Z_2. \]  
(96)

We use this relation to solve for \( A_4 \) leading to
\[ A_4 = \frac{8 A_5 A_{FB}}{9 \pi F_\perp} + \sqrt{2} \frac{\sqrt{F_L F_\perp - \frac{8}{9} A_5^2 \sqrt{F_\parallel F_\perp - \frac{4}{9} A_{FB}^2}}}{\pi F_\perp} \]  
(97)

Since \( F_\perp \) is already predicted in Eq. (52) in terms of the already measured observables \( F_L \) and \( A_{FB} \) and \( P_1 \) and \( P_1' \) and \( R \), we can estimate \( A_4 \) in terms of \( A_5 \). The correlations predicted by Eq. (97) would have to hold unless NP contributes. In Fig. 7 we present the correlation between the observables. It may be noted that Eq. (97) is a relation involving only observables without any assumptions of hadronic form factors, hence its violation must be an unambiguous signal of NP.

Let us summarize the approach that has led to these solutions. We have six observables, the decay width of \( B \to K^{*} \ell^{+} \ell^{-}, \Gamma_f \), the helicity fractions \( F_L \) and \( F_\perp \) and the angular asymmetries \( A_{FB}, A_4 \) and \( A_5 \). These six observables are expressed in terms of eight theoretical parameters in the most general approach. The parameters being the six effective form factors \( \mathcal{F}_0, \mathcal{F}_\parallel, \mathcal{F}_\perp, \tilde{G}_0, \tilde{G}_\parallel \) and \( \tilde{G}_\perp \) and the two Wilson coefficients \( C_9 \) and \( C_{10} \). Three of the observables \( \Gamma_f, F_L \) and \( A_{FB} \) have already been measured by several experiments. We assume three further inputs– the ratio \( R = C_9/C_{10} \) as it is theoretically reliably estimated in SM and the ratios \( P_1 \) and \( P_1' \) of form factors as defined in section. \( P_1 \) and \( P_1' \) are accurately predicted theoretically in the heavy quark limit to be free from higher order corrections and the known universal form factors \( \xi_\parallel \) and \( \xi_\perp \). These inputs allow us to estimate \( F_\perp \). We find that making assumption of one further observable \( A_3 \) we are able to predict the only remaining observable \( A_4 \), completely free from hadronic parameters or estimate of \( R \). Clearly only five of the observables are independent in SM and \( F_\parallel \) remains unsolved given all the observables possible. It has also been realized earlier [38] following a different approach that there exist symmetries in the angular distribution which reduce the number of independent observable. We emphasize that in our approach, \( C_9/C_{10} \) and all the expressions independent of Wilson coefficients are “clean” in the large recoil limit.

VI. THE LOW RECOIL LIMIT

In Sec IVB we found that (see Eqs. (35) and (37)) in the low-recoil limit the form-factors satisfied the condi-
FIG. 7. The allowed region in the $F_L$-$F_L$ parameter space, shaded as gray, for $R = -1$ and different values of $A_5$. The values of $P_1$ and $P'_1$ are averaged over $1 \text{GeV}^2 \leq q^2 \leq 6 \text{GeV}^2$. The blue lines correspond to the value of $A_4$ that is estimated using Eq. (97).

In this section we derive a new relations among observables that will test the validity of the assumption on the form factors in the low recoil limit. We will also elaborate on various other constraints in this limit.

We begin by considering Eq. (A1), (A2) and (A3) in the low recoil limit. Clearly since $r^2 + C_{10}^2$ is independent of helicity, Eqs. (A1) and (A2) reduce to the same equation, hence, we have

$$r^2 + C_{10}^2 = \frac{\Gamma_f}{2F_{\|}^2} = \frac{\Gamma_f}{2F_{\perp}^2} = \frac{\hat{F}I_f}{2}$$

(98a)

$$4rC_{10} = \frac{2A_{FB}I_f}{3F_{\|}F_{\perp}} = \frac{4A_{FB}I_f}{3\sqrt{F_{\|}F_{\perp}}} = \frac{\hat{F}I_f}{2}$$

(98b)

where

$$\hat{F} = \frac{F_{\|}}{F_{\|}^2} = \frac{F_{\perp}}{F_{\perp}^2}.$$  

(99)
In order to satisfy the limit derived by appropriate combination of Eqs. (47) and (46), the value $C_{10}^2 F_\|^2$ can be expressed in terms of observables alone. In the large recoil case $C_{10}^2 F_\|^2$ depended on $P_1$ and $P_2$. The form factor $P_1 = P_1'$ can be measured, enabling a possibility of verifying the estimates of presented in Table IV. To derive a relation between observables that is valid at low recoil and tests the validity of the approximation we note Eq. (98) leads to the generalized relation

$$\frac{r^2 + C_{10}^2}{2rC_{10}} = \frac{2}{3} \frac{A_{FB}}{F_{\|}} = \frac{2}{3} \sqrt{2} A_5 \frac{F_{\perp}}{\sqrt{F_{\|}}}$$

(105)

The equalities on the left side of the above equation yields two interesting relations

$$\sqrt{2} A_5 = A_{FB} \frac{\sqrt{F_{\perp}}}{\sqrt{F_{\|}}}$$

(106)

$$A_4 = \frac{\sqrt{2}}{\pi} \sqrt{F_{\perp} F_{\|}}$$

(107)

It is easily seen by direct substitution of Eq. (106) in Eq. (97) that it reduces to Eq. (107), hence it is not independent. It is emphasized that a reasonable validity of the low recoil approximation requires large $q^2$ and not the exact equality of form factors as derived Eq. (104). Even though the values of the form factors depicted in Table IV are not exactly equal, the low recoil approximation works well as can be seen from Fig. 9 where we plot the left hand and right hand of Eqs. (106) and (107). These figures demonstrate the domain of validity of the low recoil approximation and the region where new physics can be tested. The values of observables are estimated using the form factors given in Table IV.

We emphasize that the relation derived in Eqs. (106) and (107) are extremely important both in testing the validity of the low recoil approximation and the presence of New Physics. The value of $A_5$ predicted by these relations tests the validity of the low recoil approximation, whereas the value of $A_4$ verifies the validity of SM. If both the relations are found to be valid it would prove both the validity of the low recoil limit and the absence of New Physics. On the other hand if both the relations fail we must conclude that low recoil limit is not valid. The presence of New Physics could still be tested by the validity Eq. (97) even in this large $q^2$ domain. The remaining meaningful possibility is that Eq. (106) holds and (107) is violated. This would imply validity of low recoil limit but signal the presence of New Physics. It is interesting to note that one should expect from Eqs. (106) and (107) a very tiny product of asymmetries $A_4 A_5$,

$$A_4 A_5 = \frac{A_{FB} F_{L}}{\pi}$$

(108)
Since the right hand side $A_{FB}$ and $F_L$ have already been measured. We emphasize that even in the low recoil limit, $C_9/C_{10}$ and all the expressions independent of Wilson coefficients are independent of the universal form factors $\xi_\parallel$ and $\xi_\perp$.

**VII. CONCLUSIONS**

In this paper we have derived several important new results. After a brief introduction, we discuss the differential decay distribution of $B \to K^* \ell^+ \ell^-$ and introduced the observables $\Gamma_f$, $F_L$, $F_\perp$, $A_{FB}$, $A_4$ and $A_5$. While the partial decay rate $\Gamma_f$ can be measured by angular integration, the other observables require a study of angular distributions. We showed how uniaxial distributions in the azimuthal angle $\phi$ can be used to measure the helicity fraction $F_\perp$. $F_L$ and $A_{FB}$ have already been measured by studying the uniaxial distribution in $\theta_\ell$. $A_4$ and $A_5$ can only be measured by a complete angular analysis involving $\theta_\ell$ and $\phi$ requiring higher statistics. After setting up our notation and defining the observables in terms of form factors, we expressed the amplitude in the most general form within the Standard Model as $A_{L,R}^\lambda = C_{L,R} F_\lambda - \tilde{G}_\lambda$, where $\lambda = \{0, \perp, \parallel\}$ is the helicity of the $K^*$, $C_{L,R} = C^{\text{def}}_{9,10}$ and $L, R$ defines the chirality of the $\ell^-$. The form factors $F_\lambda$ and $\tilde{G}_\lambda$ are expressed in terms of conventional $B \to K^*$ form factors $V, A_{1,2}$ and $T_{1,2,3}$. To be exact $\tilde{G}_\lambda \equiv C_7 G_\lambda + \cdots$ with the dots representing the higher order and non-factorizable contributions and only at leading order $G_\lambda$’s are related to $T_{1,2,3}$. It may be noted that even at leading order $C_7$ and $G_\lambda$ cannot be separated and $C_7$ can only be defined at leading order on assuming $G_\lambda$. The six observables are thus defined in terms eight parameters, the six form factors $F_\lambda, \tilde{G}_\lambda$ and two Wilson Coefficients $C_{9,10}$. Hence only six theoretical parameters can be eliminated in terms of observables and a minimum of two reliable theoretical inputs are needed, to resolve between new physics and hadronic contributions. This is made possible by the significant advances in our understanding of form-factors that permit us to make truly these reliable inputs. One of our achievements are derivations of “clean relations” that permit the verifications of these hadronic inputs.

The $B \to K^*$ form factors are estimated using heavy quark effective theory and the treatment varies on the recoil energy of the $K^*$. At large recoil the ratio of the form factors $P_1 = F_\perp/F_\parallel$ and $P'_1 = \tilde{G}_\perp/\tilde{G}_\parallel$ are reliably evaluated at $\mathcal{O}(1/m_b)$ to be free from universal wave functions and are unaltered by non-factorizable contributions and higher order corrections in $\alpha_s$. In the large recoil limit we therefore choose $P_1$ and $P'_1$ as the two inputs in addition to observables. In the low recoil limit the relations $P_1 = P'_1$ between the form factors serves as an additional input.

We summarize briefly a few significant new results. The simple analytic derivation and solutions to the Wilson coefficients in terms of the observables and “clean” form factors was achieved by defining new variables $r_\lambda = \tilde{G}_\lambda/F_\lambda - C_9$. These enable solutions to $C_9$ and $C_{10}$ in terms of observables, $P_1$, $P'_1$ and the form factor $F_\parallel$ to
be

\[ C_9 = \frac{\sqrt{T_f}}{\sqrt{2}F_\parallel} \frac{(F_1 |P_1P_1^\prime - F_\perp) - \frac{1}{2}(P_1 - P_1^\prime)Z_1}{(P_1 - P_1^\prime)\sqrt{\left| P_1^2 F_\parallel + F_\perp + P_1 Z_1 \right|}}, \]

\[ C_{10} = \frac{\sqrt{T_f}}{\sqrt{2}F_\parallel} \frac{2}{3} \frac{A_{FB}}{\sqrt{P_1^2 F_\parallel + F_\perp + P_1 Z_1}} \]

where \( Z_1 \) is expressed in terms of observables in Eq. (43). Two additional solutions for \( C_9 \) and \( C_{10} \) can be obtained in terms of different observables. These are obtained by the replacements

- \( F_\parallel \rightarrow F_L, A_{FB} \rightarrow \sqrt{2}A_5, F_\parallel \rightarrow F_0, G_\parallel \rightarrow G_0, \) which also imply that \( r_\parallel \rightarrow r_0, P_1 \rightarrow P_2 \) and \( P_1^\prime \rightarrow P_2^\prime. \)

- \( F_\parallel \rightarrow F_L + F_\parallel + \sqrt{2}\pi A_4, A_{FB} \rightarrow A_{FB} + \sqrt{2}A_5, F_\parallel \rightarrow F_\parallel + F_0, G_\parallel \rightarrow G_\parallel + G_0, \) which also imply \( r_\parallel \rightarrow r_0, P_1 \rightarrow P_3 \) and \( P_1^\prime \rightarrow P_3^\prime. \)

We found that the form factor ratios \( P_1, P_2 \) and \( P_3 \) can be directly measured in terms of the ratio of helicity fractions at \( q^2 \) corresponding to the cross sections of asymmetries \( A_{FB}, A_5 \) and \( A_{FB} + \sqrt{2}A_5 \) respectively by the relations:

\[ P_1 = -\frac{\sqrt{F_\parallel}}{\sqrt{F_L}} \bigg|_{A_{FB}=0}, \quad P_2 = -\frac{\sqrt{F_\parallel}}{\sqrt{F_L}} \bigg|_{A_5=0}, \quad P_3 = -\frac{\sqrt{F_\parallel}}{\sqrt{F_L + F_\perp + \sqrt{2}\pi A_4}} \bigg|_{A_{FB} + \sqrt{2}A_5=0}. \]

Since we have neglected the tiny \( CP \) violation in the standard model, we find that the observables must satisfy the following inequalities which are completely free from any hadronic uncertainties and hence clean. These relations are,

\[ 4F_\parallel F_\perp \geq \frac{16}{9} A_{FB}^2, \]

\[ 4F_L F_\perp \geq \frac{16}{9} (\sqrt{2}A_5)^2. \]

\[ 4(1 - F_\perp) F_\perp \geq \frac{16}{9} (A_{FB}^2 + 2A_5^2), \]

\[ 4(F_L + F_\perp + \sqrt{2}\pi A_4) F_\perp \geq \frac{16}{9} (A_{FB}^2 + \sqrt{2}A_5)^2. \]

In Fig. 5 we have plotted the constraints on \( F_L - F_\perp \) that depends only on observables. The condition \( 4F_\parallel F_\perp \geq 16/9A_{FB}^2 \) implies that if \( |A_{FB}| \) is large \( F_\parallel \) must be small so that \( 4F_\parallel F_\perp \) can be sufficiently large. Our approach is sensitive enough to already show tension in the data [16].

Clearly, expressions for \( C_9 \) and \( C_{10} \) are not “clean.” However, the ratio \( C_9/C_{10} \) is obtained as a “clean expression.” Assuming the theoretical estimate of \( C_9/C_{10} \) which is reliably evaluated at NNLL in Standard Model we “cleanly” predicted \( F_\perp \) in Eq. (52). The correlation between \( A_{FB}, F_L \) and \( F_\perp \) have been plotted in Figs. 1, 2, 3 and 4. We showed that the valid domain of \( A_{FB} \) is constrained in terms of \( F_L \) as follows:

\[ -3(1 - F_L) T_\pm \leq A_{FB} \leq \frac{3(1 - F_L)}{4} T_+, \]

where \( T_\pm \) is given in terms of \( P_1, P_1^\prime \) and \( R \) in Eq. (53). It is interesting to note that \( F_L \) and \( F_\perp \) are constrained with Standard Model to lie in a very narrow region, well approximated by a line as shown in Fig. 3 and 4. The effective photon vertex \( \bar{G}_\parallel \) and \( \bar{G}_0 \) can also be expressed as a “clean expression”.

The \( C_9/C_{10} \) and \( C_7/C_{10} \) ratios in Eqs. (51) ratio in (57) were combined to obtain

\[ \frac{2}{3} C_9 - \frac{4}{3} C_7 C_{10} P_1 A_{FB} = (P_1^2 F_\parallel + F_\perp + P_1 Z_1) > 0. \]

If the \( A_{FB} \) zero crossing is confirmed [16] with \( A_{FB} > 0 \) at small \( q^2 \), then based on the signs of the form factors it is unambiguously concluded that the signs of \( C_7/C_{10} \) and \( C_9/C_{10} \) are in agreement with the Standard Model, i.e. \( C_7/C_{10} > 0 \) and \( C_9/C_{10} > 0 \) as long as other constraints like \( Z_1^2 > 0 \) hold. In Ref. [16] the zero crossing is indeed seen. However, in the \( 2 \text{GeV}^2 < q^2 < 4.3 \text{GeV}^2 \) bin \( Z_1^2 > 0 \) is only marginally satisfied. These conclusions are exact and not altered by any hadronic uncertainties.

We have obtained three sets of \( C_9/C_{10} \) and \( C_7/C_{10} \) solutions involving difference observables and form factor ratios. Since, the form factor ratios \( P_1 \) and \( P_1^\prime \) are the ones that are most reliably estimated in both large recoil and low recoil limits, we obtain relations for \( P_2, P_2^\prime \) and \( P_3, P_3^\prime \) in terms of \( P_1, P_1^\prime \) and observables. Equating the relations obtained for \( C_9/C_{10} \) and \( C_7/C_{10} \) in Eqs. (51), (57) with those in Eqs. (72), (73) and Eqs. (83), (84) we get:

\[ P_2 = \frac{2P_1 A_{FB} F_L}{\sqrt{2}A_5(2F_\perp + Z_1 P_1) - Z_3 P_1 A_{FB}}, \]

\[ P_2^\prime = \frac{A_{FB} T_2(P_1 - P_1^\prime + \sqrt{2}A_5 F_\perp - F_\parallel P_1^2) P_2^2 P_1^2}{A_{FB} T_2 P_1^2 P_2}, \]

\[ P_3 = \frac{(A_{FB} + \sqrt{2}A_5)(2F_\perp + Z_1 P_1) - Z_3 P_1 A_{FB}}{A_{FB} T_3 P_1^2 P_2^2 P_3^2 P_1^2}, \]

where \( T_2 = P_1(F_\parallel - F_\perp P_2^2) \) and \( T_3 = P_1(F_\parallel(1 + P_2^2) - P_2^2(1 + \sqrt{2}A_4)). \) Even though \( P_2, P_2^\prime \) and \( P_3, P_3^\prime \) inherently depend on \( \xi_2 \) and \( \xi_3 \) we have expressed them in terms of “clean relations” above. Hence, in our approach, all the expressions for observables are “clean,” with only the Wilson coefficients \( C_7, C_9 \) and \( C_{10} \) being expressed in terms of only one form factor \( G_\parallel \) or \( F_\parallel \).

We have derived significant constraints between observables that can be used to test for New Physics. The constraint purely in terms of observables arises since \( P_2 \) and \( P_3 \) are expressed in terms of observables and \( P_1 \) while
P_3 \text{ itself is related in Eq. (25) to } P_1 \text{ and } P_2. \text{ We obtain the interesting constraint (97) among observables: }

\[ A_4 = \frac{8A_5A_{FB}}{9\pi F_{\perp}} + \sqrt{2} \frac{\sqrt{F_L F_{\perp} - \frac{8}{9} A_3^2 \sqrt{F_{\parallel} F_{\perp} - \frac{4}{9} A_{FB}^2}}}{\pi F_{\perp}}. \]

The observables \( A_4 \) and \( A_5 \) also impose constraints on the parameter space. In Fig. 5 we plot constraints on the parameter space of \( F_L - F_{\perp} \) that depend purely on observables \( A_{FB} \) and \( A_4 \) with \( A_4 \) being calculated in terms of the above relation between observables. As can be seen the parameter space is highly constrained in the Standard Model.

We introduced six observables of which three \( \Gamma_7, F_L \) and \( A_{FB} \) have already been measured. We showed that \( F_{\perp} \) can expressed in terms of \( P_1, P'_1 \) and the ratio \( C_9/C_{10} \). If we further choose a value for \( A_4, A_5 \) \( A_4 \) can be obtained. In Fig. 7 we depict the constraints in the \( A_{FB} - F_L \) parameter space. These constraints and the constraints obtained in Fig. 1 fix completely the parameter space and predict the values of yet unmeasured observables.

We pay special attention to the low recoil limit and derive two new relations

\[ \sqrt{2} A_5 = A_{FB} \frac{\sqrt{F_L}}{\sqrt{F_{\parallel}}} \]

\[ A_4 = \sqrt{2} \frac{\sqrt{F_L F_{\parallel}}}{\pi} \]

in terms of observables alone. These two relations allow us to test not only the validity of the low recoil approximation but also the presence of New Physics. The value of \( A_5 \) predicted by these relations tests the validity of the low recoil approximation, whereas the value of \( A_4 \) verifies the validity of SM. If both relations hold we verify that the low recoil approximation is correct and that no new physics can exist. If both relation fail we can conclude that the low recoil approximation fails but \textit{one can never-the-less still test for new physics} by Eq. (97), which is valid in general. If \( A_5 \) is accurately predicted but \( A_4 \) does not have the value given by these two relations one can conclude that there is new physics and that the low recoil limit is accurate.

In this paper we re-examined the new physics discovery potential of the mode \( B \rightarrow K^* \ell^+ \ell^- \). This modes has an advantage as a multitude of observables can be measured via angular analysis. We showed how the multitude of related observables obtained from \( B \rightarrow K^* \ell^+ \ell^- \) can provide many new clean tests of the Standard Model and discriminate new physics contributions from hadronic effects. The hallmark of these tests is that most of them are independent of the unknown form factors \( \xi_{\parallel} \) and \( \xi_{\perp} \) in heavy quark effective theory. In the large recoil limit (at \( O(1/m_d) \)) these relations are valid to all orders in \( \alpha_s \). We derive a relation between observables that is free of form factors and Wilson coefficients, the violation of which will be an unambiguous signal of New Physics. We also obtained for the first time relations between observables and form factors that are independent of Wilson coefficients and enable verification of hadronic estimates. We show how form factor ratios can be measured directly from helicity fraction with out any assumptions what so ever. We find that the allowed parameter space for observables is very tightly constrained in Standard Model, thereby providing clean signals of New Physics. We examine in detail both the large-recoil and low-recoil regions of the \( K^* \) meson and probe special features valid in the two limits. Another new relation involving only observables that would verify the validity of the relations between form-factors assumed in the low-recoil region was also derived. The several relations and constraints derived will provide unambiguous signals of New Physics if it contributes to these decays. We emphasize that in our approach, \( C_9/C_{10} \) and all the expressions independent of Wilson coefficients are “clean” in the large recoil limit and in the low recoil limit they are reliably calculated as they do not depend on the universal form factors \( \xi_{\parallel} \) and \( \xi_{\perp} \).

Appendix A: Derivation of Wilson Coefficients

Below we present the solution of \( r_{\parallel} + r_{\perp} \). The solutions of \( r_0 + r_{\perp} \) and \( r_{\parallel} + r_{\perp} \) are identical.

We start with the expression involving \( r_{\parallel} \) and \( r_{\perp} \) in terms of observables as expressed in Eqs. (24a), (24c) and (24f):

\[ r_{\parallel}^2 + C_{10}^2 = \frac{F_L \Gamma_7}{2 F_{\parallel}} \]

\[ r_{\perp}^2 + C_{10}^2 = \frac{F_{\perp} \Gamma_7}{2 F_{\perp}} \]

\[ 2C_{10}(r_{\parallel} + r_{\perp}) = \frac{2 A_{FB} \Gamma_7}{3 F_{\perp}} \]

We can write

\[ \frac{F_{\parallel} F_{\perp} \Gamma_7^2}{4 F_{\parallel}^2 F_{\perp}^2} = (r_{\parallel} r_{\perp} - C_{10})^2 + C_{10}^2 \]

\[ = (r_{\parallel} r_{\perp} - C_{10})^2 + \frac{A_{FB}^2 \Gamma_7^2}{9 F_{\perp}^2} \]

hence

\[ r_{\parallel} r_{\perp} - C_{10}^2 = \pm \frac{\Gamma_7}{2 F_{\parallel} F_{\perp}} \sqrt{F_{\parallel} F_{\perp} - \frac{4 A_{FB}^2}{9}}. \]

Now we can express \( C_{10}^2 \) in terms of \( r_0^2 \) using Eq. (A1) or in terms of \( r_{\perp}^2 \) using Eq. (A2), to re-express \( w - C_{10}^2 \):

\[ 2r_{\parallel} r_{\perp} - 2C_{10}^2 = 2r_{\parallel} r_{\perp} - \left( \frac{F_{\parallel} \Gamma_7}{2 F_{\parallel}} - r_{\perp}^2 \right) - \left( \frac{F_{\perp} \Gamma_7}{2 F_{\perp}} - r_{\parallel}^2 \right) \]

\[ = \left( r_{\parallel} + r_{\perp} \right)^2 - \frac{F_{\parallel} \Gamma_7}{2 F_{\parallel}^2} - \frac{F_{\perp} \Gamma_7}{2 F_{\perp}^2} \]

(A5)
Equating Eqs. (A4) and (A5) we get
\[
 r_\parallel + r_\perp = \pm \left[ \frac{F_\parallel \Gamma_f}{2 F_\perp^2} + \frac{F_\perp \Gamma_f}{2 F_\parallel^2} \pm \frac{\Gamma_f}{2 F_\parallel F_\perp} Z_1 \right]^{1/2}
\]
\[
 = \pm \sqrt{\frac{\Gamma_f}{2 F_\perp}} \left[ p_\perp^2 F_\parallel + F_\perp \mp p_\parallel Z_1 \right]^{1/2}
\]
(A6)

where \( Z_1 = \sqrt{4 F_\parallel F_\perp - \frac{16}{9} A_{PFB}} \). Now, Eqs. (A1) and (A2) imply:
\[
 r_\parallel^2 - r_\perp^2 = \frac{F_\parallel \Gamma_f}{2 F_\parallel^2} - \frac{F_\perp \Gamma_f}{2 F_\perp^2},
\]
(A7)

which gives \( r_\parallel - r_\perp \) to be,
\[
 r_\parallel - r_\perp = \pm \sqrt{\frac{\Gamma_f}{2 F_\perp}} \left[ p_\perp^2 F_\parallel - F_\perp \pm p_\parallel Z_1 \right]^{1/2}
\]
(A8)

\( C_1 \) is readily solved using Eq. (A3) and the expression for \( r_\parallel + r_\perp \) obtained above. \( C_2 \) and \( C_3 \) are also easily solved using Eq. (23) and the expressions for \( r_\parallel - r_\perp \). The solutions for \( C_7 \), \( C_9 \) and \( C_{10} \) are presented in Eqs. (47), (46) and (42) respectively.

**Appendix B: Form Factor Calculations**

In this appendix we discuss the calculations of form factors and the form factor ratios. In our numerical analysis we have calculated the average value of the form factor \( F_\parallel \) and the two form factor ratios \( P_1 \) and \( P'_1 \) in different \( q^2 \) regions.

As has already been discussed in Sec.(IV A), at large recoil region the heavy quark symmetry applies and the seven form factors \( V, A_{1,2,3}, T_{1,2,3} \) are functions of Isgur-Wise form factors \( \xi_\parallel(q^2) \) and \( \xi_\perp(q^2) \) [39]. These two form factors are parameterized as [15]

\[
 \xi_\perp(q^2) = \xi_\perp(0) \left( 1 - \frac{q^2}{m_B^2} \right)^2
\]

\[
 \xi_\parallel(q^2) = \xi_\parallel(0) \left( 1 - \frac{q^2}{m_B^2} \right)^3
\]

where \( \xi_\perp(0) = 0.266 \pm 0.032 \) and \( \xi_\parallel(0) = 0.118 \pm 0.008 \) [3]. The two ratios \( P_1, P'_1 \) (see Eqs 33a and 33b) are independent of Isgur-Wise form factors, and only \( F_\parallel \) (see Eq. 21) is dependent on \( \xi_\perp \). In Table. (III) we have calculated the the values of \( P_1, P'_1 \) and \( F_\parallel \) averaged over each \( q^2 \) bin used by the recent experiments [10].

At low recoil the seven form factors \( V, A_{1,2,3}, T_{1,2,3} \) are parameterized [40] as:

\[
 V(q^2) = \frac{V_1}{1 - q^2/m_{V}^2} + \frac{V_2}{1 - q^2/m_{V}^2}^2
\]

\[
 A_2(q^2) = \frac{A_2}{1 - q^2/m_{A_2}^2} + \frac{A_3}{1 - q^2/m_{A_3}^2}^2
\]

\[
 T_1(q^2) = \frac{T_1}{1 - q^2/m_{T_1}^2} + \frac{T_2}{1 - q^2/m_{T_2}^2}^2
\]

\[
 T_3(q^2) = \frac{T_3}{1 - q^2/m_{T_3}^2}^2
\]

where \( T_3 \) has same parameterization as \( A_1 \). The parameters \( r_1, r_2, m_{V}, m_{A_2}, m_{A_3}, m_{T_1}, m_{T_2} \) for each of the above form factors has been taken from [40]. Following the above parameterization, the ratios \( P_1, P'_1 \) and \( F_\parallel \) has been calculated in the low recoil region, averaged over each \( q^2 \) bins and has been shown in Table. (IV).

| GeV^2   | 14.18-16 | 16-19  |
|---------|----------|--------|
| P_1     | -0.6836  | -0.4719 |
| P'_1    | -0.7093  | -0.4952 |
| F_\parallel(10^{-12}) | -27.8735 | -25.0050 |

**TABLE IV.** The form factor ratios \( P_1, P'_1 \) and \( F_\parallel \) averaged over different \( q^2 \) bins at low recoil.
