Fano’s inequality is a mistake

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Introduction

Let us consider the simplest Fano threefold, that is the three-dimensional projective space \( \mathbb{P}^3 \). For this threefold, Fano’s inequality looks as follows.

For any Cremona transformation \( f : \mathbb{P}^3 \rightarrow \mathbb{P}^3 \) defined by four homogeneous polynomials of the same degree \( d \) and without a common non-constant factor,

\[
x'_i = f_i(x_0, x_1, x_2, x_3), \quad i = 0, 1, 2, 3,
\]

either there exists a point \( P \in \mathbb{P}^3 \) such that

\[
\text{mult}_P(f_i) > d/2
\]

for every \( i = 0, 1, 2, 3 \), or there exists an irreducible curve \( C \subset \mathbb{P}^3 \) such that

\[
\text{mult}_C(f_i) > d/4
\]

for every \( i \).

One can remark that for the first time, these inequalities were indicated by Margherita Piazzola-Beloch in [1]. She was a pupil of G. Castelnuovo, her paper presents the text of her thesis, G. Castelnuovo was the adviser of the thesis. Thus all (including G. Fano) the subsequent authors of the variants or generalizations of the Fano inequality are out of the historical responsibility for the mistake explained below.
The goal of my article is to show that these inequalities do not take place for a Cremona transformation of degree 13, that is I write down the formulas for a Cremona transformation of degree 13 such that for the forms $f_0, \ldots, f_4$ defining the transformation, for any point $P \in \mathbb{P}^3$ and for any curve $C \subset \mathbb{P}^3$ one can see that

$$\min_i(\operatorname{mult}_P(f_i)) \leq 6, \quad \min_i(\operatorname{mult}_C(f_i)) \leq 3.$$ 

The construction of the example

Let us consider the homogeneous coordinates $x_0, x_1, x_2, x_3$ for $\mathbb{P}^3$ as the normalized coefficients of a binary cubic form $F(T_0, T_1)$,

$$F(T_0, T_1) = x_0 T_0^3 + 3 x_1 T_0^2 T_1 + 3 x_2 T_0 T_1^2 + x_3 T_1^3.$$ 

Let $D = D(x_0, x_1, x_2, x_3)$ be the discriminant of the binary cubic,

$$D = x_0^2 x_3^2 - 3 x_1^2 x_2^2 - 6 x_0 x_1 x_2 x_3 + 4 x_0 x_2^3 + 4 x_3 x_1^3.$$ 

Let us fix a parameter $t$ and consider four following forms of degree 13.

$$(f_t)_0 = x_0 D^3,$$

$$(f_t)_1 = x_1 D^3 + tx_0^7 D^2,$$

$$(f_t)_2 = x_2 D^3 + 2tx_1 x_0^4 D^2 + t^2 x_0^9 D,$$

$$(f_t)_3 = x_3 D^3 + 3tx_2 x_0^4 D^2 + 3t^2 x_1 x_0^8 D + t^3 x_0^{13}.$$ 

These four forms define a one-parameter family of rational maps

$$g_t : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3.$$ 

If $t = 0$, then all the four forms have a common factor, the factor is $D^3$, after the cancellation we see that $g_0$ is the identity transformation. For our example we need non-zero values of $t$. If $t$ is not zero, then it is clear that the four forms are without a common non-constant factor. Further,

$$D((f_t)_0, (f_t)_1, (f_t)_2, (f_t)_3) = D(x_0, x_1, x_2, x_3)^{13},$$
this identity actually expresses the invariant property (with respect to the triangular transformation of variables $T_0, T_1$) of the discriminant. Using the latter identity, it is not hard to see that

$$(f_{-t})_i((f_t)_0, (f_t)_1, (f_t)_2, (f_t)_3) = x_i D^{42},$$

that is

$$g_{(-t)} \circ g_t = \text{the identity transformation}.$$ 

Thus $g_t$ is rationally invertible and is a Cremona transformation. More generally,

$$g_{s} \circ g_t = g_{s+t},$$

and we get a one-parameter group of Cremona transformations. These transformations induce biregular automorphisms of an affine open subset of the projective space, the subset is the complement to the discriminant quartic surface $D = 0$. Indeed, the above formula of the discriminant transformation proves it (moreover, one can see below the exact calculation of the fundamental points of such a transformation).

The formulas for $g_t$ (or more general formulas for an infinite-dimensional family of automorphisms of the complement to the discriminant surface) were written down on page 8 of the Max-Planck-Institute preprint [2].

Let us fix a nonzero value of the parameter $t$, for example put $t = 1$, and consider the corresponding Cremona transformation

$$x_0' = x_0 D^3,$$

$$x_1' = x_1 D^3 + x_0^5 D^2,$$

$$x_2' = x_2 D^3 + 2 x_1 x_0^4 D^2 + x_0^9 D,$$

$$x_3' = x_3 D^3 + 3 x_2 x_0^4 D^2 + 3 x_1 x_0^8 D + x_0^{13}.$$ 

First of all, we will find the points $P$ where the multiplicities of every of the right hand sides are positive (that is the set of all common zeros of these right hand sides, or, in other words, the fundamental points of the transformation).

The first right hand side vanishes if either $x_0 = 0$, or $D = 0$, or simultaneously $x_0 = 0, D = 0$.

If $x_0 = 0$, but $D \neq 0$, then using other three formulas, one can see that for other three coordinates of a fundamental point, the equalities $x_1 = x_2 = x_3 = 0$ take place, that is in this case we are out of the projective space.
The case $D = 0$, but $x_0 \neq 0$ is also impossible for a fundamental point. 

Thus, the set of fundamental points consists of the solutions of the following system of equations

$$x_0 = 0, \quad D = 0,$$

or equivalently,

$$x_0 = 0, \quad x_1^2(4x_3x_1 - 3x_2^2) = 0.$$

We see that the set of fundamental points is the union of two curves, the first curve is line $L$,

$$L : \quad x_0 = 0, \quad x_1 = 0,$$

the second curve is conic $C$,

$$C : \quad x_0 = 0, \quad 4x_3x_1 - 3x_2^2 = 0.$$

The discriminant surface $D = 0$ has double points disposed on the twisted cubic $T$ having the following homogeneous parameterization,

$$x_0 = t_0^3, \quad x_1 = t_0^2t_1, \quad x_2 = t_0t_1^2, \quad x_3 = t_1^3.$$

More precisely, the singular locus of the discriminant surface is $T$, and $\text{mult}_P(D) = 2$ for every $P \in T$. Therefore general points of the line $L$ and of the conic $C$ are the points of multiplicity one on the discriminant,

$$\text{mult}_L(x_1D^3 + x_0^5D^2) = 3,$$

$$\text{mult}_C(x_1D^3 + x_0^5D^2) = 3.$$

More generally, if a point $P$ of the intersection of discriminant surface with the plane $x_0 = 0$ is located out of the twisted cubic $T$, then at least one of the multiplicities $\text{mult}_P(x_1D^3)$ is equal to 3. The last hope to get a point of higher multiplicity is to consider the point of intersection of the twisted cubic $T$ with the union of curves $L$ and $C$. It is obvious that

$$T \cap (L \cup C) = T \cap L \cap C = \{Q\}, \quad Q = (0 : 0 : 0 : 1),$$

but it clear that

$$\text{mult}_Q(x_3D^3 + 3x_2x_0^4D^2 + 3x_1x_0^5D + x_0^{13})$$

$$= \text{mult}_Q(D^3) = 6.$$
References

[1] M. Piazzola-Beloch (= M. Beloch ), *Title??* Annali di Matematica, vol. 5, 1910(?) or 1905 . Or Selected Papers of M. Piazzola-Beloch, vol.I, the opening article.

(Sorry, unfortunately other details of this remarkable paper are out of my memory.)

[2] M.Gizatullin. *Examples of m-algebras*, Max-Planck-Institut für Mathematik, Preprint Series, 2000 (50).