A Universal Magnification Theorem II. Generic Caustics up to Codimension Five

A. B. Aazami
Department of Mathematics, Duke University, Science Drive, Durham, NC 27708

A. O. Petters
Departments of Mathematics and Physics, Duke University, Science Drive, Durham, NC 27708

We prove a theorem about magnification relations for all generic general caustic singularities up to codimension five: folds, cusps, swallowtail, elliptic umbilic, hyperbolic umbilic, butterfly, parabolic umbilic, wigwam, symbolic umbilic, 2nd elliptic umbilic, and 2nd hyperbolic umbilic. Specifically, we prove that for a generic family of general mappings between planes exhibiting any of these singularities, and for a point in the target lying anywhere in the region giving rise to the maximum number of real pre-images (lensed images), the total signed magnification of the pre-images will always sum to zero. The proof is algebraic in nature and makes repeated use of the Euler trace formula. We also prove a general algebraic result about polynomials, which we show yields an interesting corollary about Newton sums that in turn readily implies the Euler trace formula. The wide field imaging surveys slated to be conducted by the Large Synoptic Survey Telescope are expected to find observational evidence for many of these higher-order caustic singularities. Finally, since the results of the paper are for generic general mappings, not just generic lensing maps, the findings are expected to be applicable not only to gravitational lensing, but to any system in which these singularities appear.

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I. INTRODUCTION

One of the key signatures of gravitational lensing is the occurrence of multiple images of lensed sources. The magnifications of the images in turn are also known to obey certain relations. These relations fall into two types: “global” and “local.” “Global” magnification relations involve all the images of a given source, but they are not universal because the relations depend on the specific class of lens models used. Examples of such relations can be found in Petters et al. 2001 [21, p. 191], Witt & Mao 1995 [28], Rhie 1997 [23], Dalal 1998 [8], Witt & Mao 2000 [29], Dalal & Rabin 2001 [9], and Hunter & Evans 2001 [12]. As shown in Werner 2007 [26], such relations are in fact topological invariants.

By contrast, “local” magnification relations are universal, but they apply only to a subset of the total number of images produced. Two well-known examples of local magnification relations are the fold and cusp relations. For a source near a fold or cusp caustic, the resulting images close to the critical curve are close doublets and triplets whose signed magnifications always sum to zero (e.g., Blandford & Narayan 1986 [6], Schneider & Weiss 1992 [25], Zakharov 1999 [31], [21, Chap. 9]). The universality of these relations means that they hold independently of the choice of lens model. In addition, the fold and cusp relations have been shown to provide powerful diagnostic tools for detecting dark substructure on galactic scales using quadruple lensed images of quasars (e.g., Mao & Schneider 1998 [16], Keeton, Gaudi & Petters 2003 and 2005 [13, 14]).

Recently, Aazami & Petters 2009 [1], which we consider Paper I, established a universal magnification theorem for some of the higher-order caustics beyond folds and cusps, namely, the swallowtail, elliptic umbilic, and hyperbolic umbilic singularities. These are generic caustic surfaces or big caustics occurring in a three-parameter space. Slices of the big caustics give rise to generic caustic metamorphoses (e.g., [21, Chapters 7 and 9), all of which occur in gravitational lensing (e.g., Blandford 1990 [5], Petters 1993 [20], Schneider, Ehlers, & Falco 1992 [24], and [21]). It was shown in [1] that for lensing maps close to elliptic umbilic and hyperbolic umbilic caustics, and for general mappings exhibiting swallowtail, elliptic umbilic, and hyperbolic umbilic caustics, the total signed magnification for a source lying anywhere in the region giving rise to the maximum number of lensed images, is identically zero. As an application, they used the hyperbolic umbilic to show how such magnification relations may be used for substructure...
studies of four-image lens galaxies.

The proof of these relations in [1] was elementary, but long, and thus was not amenable to higher-order caustics beyond the three mentioned above. An elegant geometric technique, based on Lefschetz fixed point theory, has since been employed on these three singularities by Werner 2009 [22]. The aim of our current paper is to extend these results to all the remaining higher-order generic general singularities up to codimension 5. In other words, our findings are expected to be applicable not only to gravitational lensing, but to any system where these singularities appear.

We prove that to each generic general caustic singularity of codimension up to 5—not just the fold, cusp, swallowtail, elliptic umbilic, and hyperbolic umbilic, but also the butterfly, parabolic umbilic, wigwam, symbolic umbilic, 2nd elliptic umbilic, and 2nd hyperbolic umbilic—is associated a magnification sum relation of the form

\[ \sum_i m_i = 0. \]

In other words, for generic families of general mappings between planes exhibiting these singularities, and for a point anywhere in the region of the target space giving rise to the maximum number of lensed images, the total signed magnification is identically zero. Furthermore, as emphasized in [1], magnification sum relations are in fact geometric invariants, because they are the reciprocals of Gaussian curvatures at critical points.

Shin & Evans 2007 [17] constructed a realistic lens model for the Milky Way Galaxy and showed that it exhibited butterfly caustics (see also Evans & Witt [18] for another class of lens models that exhibit butterfly caustics). More recently, Orban de Xivry & Marshall 2009 [19] created an atlas of predicted gravitational lensing due to galaxies and clusters of galaxies that can exhibit several of these higher-order caustic singularities, and estimated the probabilities for their occurrence. They showed how a galaxy lens with a misaligned disk and bulge can generate swallowtails and butterfly, two merging galaxies or galactic binaries can produce elliptic umbilics, and galactic clusters can create hyperbolic umbilics. These lensing effects are expected to be seen by the Large Synoptic Survey Telescope.

Concerning the tools of the paper, we mention that the Euler trace formula was employed in [9] to determine “global” magnification relations for special classes of lens models. They used an analytical approach whereby they derived the Euler trace formula using residue calculus. We show that the Euler trace formula also lends itself quite naturally to “local” magnification relations for generic general caustics, not just those occurring in gravitational lensing. In fact, along with our main theorem, we also prove a general algebraic theorem about polynomials (Proposition 2) and derive a result about Newton sums as a corollary, which we show implies the Euler trace formula. Our main theorem is not a direct consequence of the Euler-Jacobi formula, of multi-dimensional residue integral methods, or of Lefschetz fixed point theory, because some of the singularities have fixed points at infinity.

The outline of the paper is as follows. Section II reviews the necessary singular-theoretic terminologies and results. Section III states our main theorem, which is for general mappings. As preparation for the proof of our main theorem, in Section IV we establish a recurrence relation for the coefficients of the unique polynomials in cosets of certain quotient rings and show that this relation yields a fact about Newton sums which can be employed to readily obtain the Euler trace formula. We then use the results of Section IV to prove the main theorem in Section V.

II. HIGHER-ORDER CAUSTIC SINGULARITIES

In what follows, the term “universal” or “generic” is used to denote a property that holds for an open, dense subset of mappings in the given space of mappings. With that said, consider a smooth, n-parameter family \( F_{c,s}(x) \) of functions on an open subset of \( \mathbb{R}^d \) that induces a smooth \((n-2)\)-parameter family of mappings \( f_c(x) \) between planes \((n \geq 2)\). Given \( f_c(x) = s \), call \( x \) the pre-image of the target point \( s \). Critical points of \( f_c \) are those \( x \) for which \( \det(\text{Jac} f_c)(x) = 0 \). Generically, the locus of critical points forms curves called critical curves. The value \( f_c(x) \) of a critical point \( x \) under \( f_c \) is called a caustic point. These typically form curves, but could be isolated points. Varying \( c \) causes the caustic curves to evolve with \( c \). This traces out a caustic surface, called a big caustic, in the \( n\)-dimensional space \( \{c,s\} = \mathbb{R}^{n-2} \times \mathbb{R}^2 \). Beyond the familiar folds and cusps, these surfaces form higher-order caustics that are classified into universal or generic types for locally stable families \( f_c \). Generic \( c \)-slices of these big caustics are commonly called caustic metamorphoses.

The universal form of the \((n-2)\)-parameter family \( f_c \) is obtained by using \( F_{c,s} \) to construct catastrophe manifolds that are projected into the space \( \{c,s\} = \mathbb{R}^{n-2} \times \mathbb{R}^2 \) to obtain local coordinates for \( f_c \) (e.g., Majhay 1985 [15], Castrigiano & Hayes 1993 [3], Golubitsky & Guillemin 1973 [11]). These projections of the catastrophe manifolds are called catastrophe maps or Lagrangian maps, and they are differentiably equivalent to \( f_c \) (see [21], pp. 273-275). The locally stable families \( F_{c,s} \) and their induced maps \( f_c \) are generic for \( n \leq 5 \), and have caustic singularities that are classified according to the parameter \( n \). For \( n = 3 \), the singularities generically divide into three types: swallowtails, elliptic umbilics, and hyperbolic umbilics. When \( n = 4 \), the singularities generically divide into two types: butterflies and parabolic umbilics. For \( n = 5 \), they divide into four types: wigwams, symbolic umbilics, 2nd elliptic umbilics, and...
III. STATEMENT OF MAIN THEOREM

For the generic families of functions $F_{c,s}$ in Table I, we define the magnification $\mathcal{M}(x_i; s)$ at a critical point $x_i$ of $F_{c,s}$ relative to $x = (x, y)$ by the reciprocal of the Gaussian curvature at the point $(x_i, F_{c,s}(x_i))$ in the graph of $F_{c,s}$:

$$\mathcal{M}(x_i; s) = \frac{1}{\text{Gauss}(x_i, F_{c,s}(x_i))}.$$ 

This makes it clear that the magnification invariants established in our theorem are geometric invariants.

**Theorem 1.** For any of the universal, smooth $n$-parameter family of general functions $F_{c,s}$ (or general mappings $f_c$) in Table I, and for any non-caustic point $s$ (light source position) in the indicated region, the following results hold for $\mathcal{M}_i = \mathcal{M}(x_i; s)$:

1. $A_2$ (Fold) Magnification relations in two-image region:
   $$\mathcal{M}_1 + \mathcal{M}_2 = 0.$$

2. $A_3$ (Cusp) Magnification relations in three-image region:
   $$\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 = 0.$$

3. $A_4$ (Swallowtail) Magnification relation in four-image region:
   $$\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 + \mathcal{M}_4 = 0.$$

4. $D_4^-$ (Elliptic Umbilic) Magnification relations in four-image region:
   $$\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 + \mathcal{M}_4 = 0.$$

5. $D_4^+$ (Hyperbolic Umbilic) Magnification relations in four-image region:
   $$\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 + \mathcal{M}_4 = 0.$$

6. $A_5$ (Butterfly) Magnification relation in five-image region:
   $$\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 + \mathcal{M}_4 + \mathcal{M}_5 = 0.$$

7. $D_5$ (Parabolic Umbilic) Magnification relations in five-image region:
   $$\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 + \mathcal{M}_4 + \mathcal{M}_5 = 0.$$

8. $A_6$ (Wigwam) Magnification relations in six-image region:
   $$\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 + \mathcal{M}_4 + \mathcal{M}_5 + \mathcal{M}_6 = 0.$$

9. $E_6$ (Symbolic Umbilic) Magnification relation in six-image region:
   $$\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 + \mathcal{M}_4 + \mathcal{M}_5 + \mathcal{M}_6 = 0.$$

10. $D_6^-$ (2$^{nd}$ Elliptic Umbilic) Magnification relations in six-image region:
    $$\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 + \mathcal{M}_4 + \mathcal{M}_5 + \mathcal{M}_6 = 0.$$

2$^{nd}$ hyperbolic umbilics; see Table I. A detailed treatment of these issues can be found in Poston and Stewart 1978 [22], Gilmore 1981 [10], [15], Arnold 1986 [3], and [21, Chap. 7].
| Singularity | Universal Local Forms |
|-------------|----------------------|
| Fold (2D)   | \( F_a(x, y) = -s_1 x + s_2 y + \frac{1}{2} x^2 - \frac{1}{2} y^3 \)  
\( f(x, y) = (x, y^2) \) |
| Cusp (2D)   | \( F_a(x, y) = -s_1 x + s_2 y + \frac{1}{2} x^2 - \frac{1}{2} s_1 y^2 - \frac{1}{2} y^4 \)  
\( f(x, y) = (x, xy + y^3) \) |
| Elliptic Umbilic (3D) | \( F_{c,a}(x, y) = s_1 x + s_2 y + c(x^2 + y^2) + x^3 - 3xy^2 \)  
\( f_c(x, y) = (3y^2 - 3x^2 - 2cx, 6xy - 2cy) \) |
| Hyperbolic Umbilic (3D) | \( F_{c,a}(x, y) = s_1 x + s_2 y + cyx + x^3 + y^3 \)  
\( f_c(x, y) = (-3x^2 - cy, -3y^2 - cx) \) |
| Swallowtail (3D) | \( F_{c,a}(x, y) = s_1 x - s_2 y - \frac{1}{3} s_2 x^2 + \frac{1}{3} y^2 - \frac{1}{3} x^3 - \frac{1}{3} y^2 \)  
\( f_c(x, y) = (xy + cx^2 + x^4, y) \) |
| Butterfly (4D) | \( F_{c,a}(x, y) = x^6 + c_1 x^4 + c_2 x^3 + s_2 x^2 + s_1 x + \frac{1}{2} y^2 - s_2 y \)  
\( f_c(x, y) = (-2xy - 3c_2 x^2 - 4c_1 x^3 - 6x^5, y) \) |
| Parabolic umbilic (4D) | \( F_{c,a}(x, y) = x^2 y + y^4 + c_1 x^2 + c_2 y^2 - s_1 x - s_2 y \)  
\( f_c(x, y) = (2c_1 x + 2xy, 2c_2 y + x^2 + 4y^3) \) |
| Wigwam (5D) | \( F_{c,a}(x, y) = x^7 + c_1 x^5 + c_2 x^4 + c_3 x^3 + s_2 x^2 + s_1 x + \frac{1}{2} y^2 - s_2 y \)  
\( f_c(x, y) = (-2xy - 3c_3 x^2 - 4c_2 x^3 - 5c_1 x^4 - 7x^6, y) \) |
| Symbolic umbilic (5D) | \( F_{c,a}(x, y) = x^3 + y^4 + c_1 xy^2 + c_2 xy + c_3 y^2 + s_2 y + s_1 x \)  
\( f_c(x, y) = (-3x^2 - c_1 y^2 - c_2 y, -4y^3 - 2c_1 xy - c_2 x - 2c_3 y) \) |
| 2\textsuperscript{nd} Elliptic umbilic (5D) | \( F_{c,a}(x, y) = x^2 y - y^5 + c_1 y^4 + c_2 y^3 + c_3 y^2 + s_2 y + s_1 x \)  
\( f_c(x, y) = (-2xy, -x^2 + 5y^4 - 4c_1 y^3 - 3c_2 y^2 - 2c_3 y) \) |
| 2\textsuperscript{nd} Hyperbolic umbilic (5D) | \( F_{c,a}(x, y) = x^2 y + y^5 + c_1 y^4 + c_2 y^3 + c_3 y^2 + s_2 y + s_1 x \)  
\( f_c(x, y) = (-2xy, -x^2 - 5y^4 - 4c_1 y^3 - 3c_2 y^2 - 2c_3 y) \) |

**TABLE I:** For each type of caustic singularity listed, the second column shows the universal local forms of the smooth \( n \)-parameter family of general functions \( F_{c,a} \), along with their \((n-2)\)-parameter family of induced general maps \( f_c \) between planes. The numbers 2D, 3D, etc., denote the codimension of the given singularity.
11. \( D_6^+ \) (2nd Hyperbolic Umbilic) \textit{Magnification relations in six-image region:}

\[
\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 + \mathcal{M}_4 + \mathcal{M}_5 + \mathcal{M}_6 = 0.
\]

We use the A, D, E classification notation of Arnold 1973 \cite{2} in the theorem. This notation highlights a deep link between the above singularities and Coxeter-Dynkin diagrams appearing in the theory of simple Lie algebras. As mentioned in the introduction, the fold and cusp magnification relations are known \cite{6, 21, 22, 34}. The magnification relations for the swallowtail, elliptic umbilic, and hyperbolic umbilic were discovered recently in \cite{1}.  

Remark. The results of Theorem \cite{1} actually apply even when the non-caustic point \( s \) is not in the maximum number of pre-images region. However, complex pre-images will appear, which are unphysical in gravitational lensing.

IV. A RECURSIVE RELATION FOR COEFFICIENTS OF COSET POLYNOMIALS

In this section, we present some notation and a proposition about polynomials that will yield the Euler trace formula as a corollary. The notation and the latter are used in the proof of Theorem \cite{1}. The proposition itself is proved in the Appendix.

We begin with some notation. Let \( \mathbb{C}[x] \) be the ring of polynomials over \( \mathbb{C} \) and consider a polynomial

\[
\varphi(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{C}[x].
\]

Suppose that the \( n \) zeros \( x_1, \ldots, x_n \) \( \text{pf} \varphi(x) \) are distinct (generically, the roots of a polynomial are distinct) and let \( \varphi'(x) \) be the derivative of \( \varphi(x) \). Also, let \( R \subset \mathbb{C}(x) \) denote the subring of rational functions that are defined at the roots \( x_i \) of \( \varphi(x) \):

\[
R = \left\{ \frac{p(x)}{q(x)} : p(x), q(x) \in \mathbb{C}[x] \text{ and } q(x_i) \neq 0 \text{ for all roots } x_i \right\}.
\]

Let \((\varphi(x))\) be the ideal in \( R \) generated by \( \varphi(x) \) and denote the cosets of the quotient ring \( R/(\varphi(x)) \) using an overbar. Below are two basic results that we prove in the Appendix (see Claim 2) for the convenience of the reader:

- Members of the same coset in \( R/(\varphi(x)) \) agree on the roots \( x_i \) of \( \varphi(x) \), that is, if \( h_1(x) \) and \( h_2(x) \) belong to the same coset, then \( h_1(x_i) = h_2(x_i) \).

- Every rational function \( h(x) \in R \) has in its coset \( \overline{h(x)} \in R/(\varphi(x)) \) a unique polynomial representative \( h_\ast(x) \) of degree less than \( n \).

\textbf{Proposition 2.} Consider any polynomial \( \varphi(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{C}[x] \) with distinct roots and any rational function \( h(x) \in R \). Let

\[
h_\ast(x) = c_{n-1} x^{n-1} + \cdots + c_1 x + c_0
\]

be the unique polynomial representative of the coset \( \overline{h(x)} \in R/(\varphi(x)) \) and let

\[
r(x) = b_{n-1} x^{n-1} + \cdots + b_1 x + b_0
\]

be the unique polynomial representative of the coset \( \overline{\varphi'(x) h(x)} \in R/(\varphi(x)) \). Then the coefficients of \( r(x) \) are given in terms of the coefficients of \( h_\ast(x) \) and \( \varphi(x) \) through the following recursive relation:

\[
b_{n-i} = c_{n-1} b_{n-i,n-1} + \cdots + c_1 b_{n-i,1} + c_0 b_{n-i,0} \quad i = 1, \ldots, n,
\]

(1)

with

\[
\left\{
\begin{array}{l}
b_{n-i,0} = (n - (i - 1)) a_n a_{n-(i-1)} \cdot \quad i = 1, \ldots, n, \\
b_{n-i,k} = -\frac{a_{n-i}}{a_n} b_{n-1,k-1} + b_{n-(i+1),k-1} \cdot \quad i = 1, \ldots, n, \quad k = 1, \ldots, n-1,
\end{array}
\right.
\]

(2)

where \( b_{-1,k-1} \equiv 0 \).
By Proposition\textsuperscript{2} if \( r_k(x) \) is the unique polynomial representative of the coset \( \varphi'(x)x^k \in R/\langle \varphi(x) \rangle \), then
\[
    r_k(x) = b_{n-1,k}x^{n-1} + \cdots + b_{1,k}x + b_{0,k},
\]
where its coefficients are given in terms of the coefficients of \( \varphi(x) \) through \textsuperscript{2}.

**Corollary 3.** Assume the hypotheses and notation of Proposition\textsuperscript{2}. Given the distinct roots \( x_1, \ldots, x_n \) of \( \varphi(x) \), the Newton sums \( N_k \equiv \sum_{i=1}^{n} (x_i)^k \) satisfy:
\[
    N_k = \frac{b_{n-1,k}}{a_n}, \quad k = 0, 1, \ldots, n - 1.
\]
In other words, the quantity \( a_n N_k \) equals the \((n - 1)\text{st} \) coefficient of the unique polynomial representative \textsuperscript{3} of the coset \( \varphi'(x)x^k \) in \( R/\langle \varphi(x) \rangle \).

**Proof.** Note that for \( k = 0 \), eqn. \textsuperscript{2} in Proposition\textsuperscript{2} yields
\[
    b_{n-1,0} = n a_n = N_0 a_n.
\]
For \( 1 \leq k \leq n - 1 \), there is a known recursive relation for \( N_k \), in terms of \( N_1, N_2, \ldots, N_{k-1} \); see, e.g., Barbeau 1989 [4] p. 203. It is given by
\[
    k a_{n-k} + a_{n-k+1} N_1 + a_{n-k+2} N_2 + \cdots + a_{n-1} N_{k-1} + a_n N_k = 0.
\]
We proceed by induction on \( k \) for \( 1 \leq k \leq n - 1 \). For \( k = 1 \), eqn. \textsuperscript{3} implies \( N_1 = -\frac{a_{n-1}}{a_n} \), while eqn. \textsuperscript{2} gives \( b_{n-1,1} = -a_{n-1} = a_n N_1 \), which agrees with eqn. \textsuperscript{4}. Now assume that \( b_{n-1,j} = a_n N_j \) for \( j = 1, \ldots, k - 1 \). To establish the result for \( j = k \), we shall repeatedly apply Proposition\textsuperscript{2}
\[
    b_{n-1,k} = \frac{a_{n-1}}{a_n} b_{n-1,k-1} + b_{n-2,k-1} = -\frac{a_{n-1}}{a_n} b_{n-1,k-1} + \left[ -\frac{a_{n-2}}{a_n} b_{n-1,k-2} + b_{n-3,k-2} \right] = \frac{a_{n-1}}{a_n} b_{n-1,k-1} - \frac{a_{n-2}}{a_n} b_{n-1,k-2} + \left[ -\frac{a_{n-3}}{a_n} b_{n-1,k-3} + b_{n-4,k-3} \right],
\]
\[
    \vdots
\]
\[
    = -\frac{a_{n-1}}{a_n} b_{n-1,k-1} - \frac{a_{n-2}}{a_n} b_{n-1,k-2} - \frac{a_{n-3}}{a_n} b_{n-1,k-3} - \cdots - \frac{a_{n-(k-1)}}{a_n} b_{n-1,1} - \frac{a_{n-k}}{a_n} b_{n-1,0} + b_{n-(k+1),0}.
\]
\[
    = - (a_{n-1} N_{k+1} + a_{n-2} N_{k+2} + a_{n-3} N_{k+3} + \cdots + a_{n-(k-1)} N_1 + k a_{n-k}) = a_n N_k,
\]
where \( b_{n-1,0} = n a_n \) and \( b_{n-(k+1),0} = (n - k) a_{n-k} \) follow from eqn. \textsuperscript{2} in Proposition\textsuperscript{2} and the last equality is due to \textsuperscript{3}. \( \square \)

**Corollary 4 (Euler Trace Formula).** Assume the hypotheses and notation of Proposition\textsuperscript{2}. For any rational function \( h(x) \in R \), the following holds:
\[
    \sum_{i=1}^{n} h(x_i) = \frac{b_{n-1}}{a_n},
\]
where \( b_{n-1} \) is the \((n - 1)\text{st} \) coefficient of the unique polynomial representative \( r(x) \) of the coset \( \varphi'(x)h(x) \in R/\langle \varphi(x) \rangle \) and \( a_n \) the \( n \)th coefficient of \( \varphi(x) \).

**Proof.** Let \( h_\ast(x) \) be the unique polynomial representative of the coset \( h(x) \in R/\langle \varphi(x) \rangle \). First note that, since \( h(x) \) and \( h_\ast(x) \) belong to the same coset, we have \( h(x_i) = h_\ast(x_i) \). The Euler trace formula now proceeds from a simple
application of Proposition 2 and Corollary 3:
\[
\sum_{i=1}^{n} h(x_i) = \sum_{i=1}^{n} h_s(x_i) = \sum_{i=1}^{n} \sum_{j=0}^{n-1} c_j \cdot (x_i)^j = \sum_{i=1}^{n} c_j \sum_{j=0}^{n-1} (x_i)^j = \sum_{j=0}^{n-1} c_j N_j
\]
\[
= c_{n-1} N_{n-1} + \cdots + c_1 N_1 + c_0 N_0
\]
\[
= c_{n-1} \left( \frac{b_{n-1,n-1}}{a_n} \right) + \cdots + c_1 \left( \frac{b_{n-1,1}}{a_n} \right) + c_0 \left( \frac{b_{n-1,0}}{a_n} \right) \quad \text{(by Corollary 3)}
\]
\[
= \frac{b_{n-1}}{a_n} \quad . \quad \text{(by Proposition 2)} \quad \square
\]

Remark. Dalal & Rabin 2001 gave a different proof of the Euler trace formula, one employing residues.

V. PROOF OF THE MAIN THEOREM

We begin by establishing some preliminaries before starting the computational part of the proof. Given a family of functions \( F_{c,s} \), a parameter vector \((c_0, s_0)\) is called a caustic point of the family if there is at least one critical point \(x_0\) of \( F_{c_0,s_0} \) (i.e., \( x_0 \) satisfies \( \text{grad} F_{c_0,s_0}(x_0) = 0 \)) such that the Gaussian curvature at \((x_0, F_{c,s}(x_0))\) in the graph of \( F_{c,s} \) vanishes. Furthermore, for the list of singularities in Table I the mappings \( f_c \) are induced by the families \( F_{c,s} \). In fact, a direct computation shows that for all the singularities, we can obtain \( f_c \) through the gradient of \( F_{c,s} \) as follows:

\[
\text{grad} F_{c,s}(x) = 0 \iff f_c(x) = s .
\]

We can also express the magnification in terms of the general mappings \( f_c \) induced from the \( n \)-parameter family of functions \( F_{c,s} \). To do so, recall that the Gaussian curvature at a point \((x, F_{c,s}(x))\) in the graph of \( F_{c,s} \) is given by

\[
\text{Gauss}(x, F_{c,s}(x)) = \frac{\det(\text{Hess} F_{c,s}(x))}{1 + |\text{grad} F_{c,s}(x)|^2} .
\]

At a critical point \( x_0 \), the magnification of \( x_0 \) is then given by

\[
\mathcal{M}(x_0; s) = \frac{1}{\text{Gauss}(x_0, F_{c,s}(x_0))} = \frac{1}{\det(\text{Hess} F_{c,s}(x_0))} .
\]

Note that caustics are characterized by the family \( F_{c,s} \) having at least one infinitely magnified critical point. A computation also shows that for all the singularities in Table I the following holds:

\[
\det(\text{Jac} f_c) = \det(\text{Hess} F_{c,s}) .
\]

Consequently, we can also express the magnification at a pre-image \( x_i \) of \( s \) under \( f_c \) as

\[
\mathcal{M}_i = \mathcal{M}(x_i; s) = \frac{1}{\det(\text{Jac} f_c)(x_i)} , \quad f_c(x_i) = s .
\]

Observe that caustics in the target plane of \( f_c \) are given equivalently as points \( s \) where the Jacobian determinant of \( f_c \) vanishes. Now, given an induced mapping \( f_c \) and a target point \( s = (s_1, s_2) \), we can use the pair of equations

\[
(s_1, s_2) = f_c(x, y) = (f_{1,c}(x, y), f_{2,c}(x, y))
\]

to solve for \((x, y)\) in terms of \((s_1, s_2)\), which will give the pre-images \( x_i = (x, y_i) \) of \( s \) under \( f_c \).

For the singularities in Table I we shall see that the pre-images can be determined from solutions of a polynomial in one variable, which is obtained by eliminating one of the pre-image coordinates, say \( y \). In doing so we obtain a polynomial \( \varphi(x) \in \mathbb{C}[x] \) whose roots will be the \( x \)-coordinates \( x_i \) of the different pre-images under \( f_c \):

\[
\varphi(x) = a_n x^n + \cdots + a_1 x + a_0 .
\]
Generically, we can assume that the roots of $\varphi(x)$ are distinct, an assumption made throughout the paper.

We would then be able to express the magnification $\mathcal{M}(x, y; s)$ at a general pre-image point $(x, y)$ as a function of one variable, in this case $x$, so that

$$
\mathcal{M}(x, y(x); s) = \frac{1}{J(x, y(x))} \equiv \frac{1}{J(x)} \equiv \mathcal{M}(x),
$$

where $J \equiv \det(\text{Jac } f_s)$ and the explicit notational dependence on $s$ is dropped for simplicity. Since we shall consider only non-caustic target points $s$ giving rise to pre-images $(x_1, y(x_1))$, we know that $J(x_1) \neq 0$. Furthermore, we shall only consider non-caustic points that yield the maximum number of pre-images. In addition, for the singularities in Table I, the rational function $\mathcal{M}(x)$ is defined at the roots of $\varphi(x)$, i.e., $\mathcal{M}(x) \in R$. Now, denote by $m(x)$ the unique polynomial representative in the coset $\overline{\varphi(x)\mathcal{M}(x)} \in R / (\varphi(x))$, and let $b_{n-1}$ be its $(n-1)$st coefficient. In the notation of Proposition 2, we have $h(x) \equiv \mathcal{M}(x)$ and $r(x) \equiv m(x)$. Euler’s trace formula (Corollary 3) then tells us immediately that the total signed magnification satisfies

$$
\sum_i \mathcal{M}_i = \frac{b_{n-1}}{a_n} .
$$

It therefore remains to determine the coefficient $b_{n-1}$ for each caustic singularity in Table I. Next to each singularity below we indicate the value of $n - 1$, which is the codimension of the singularity.

Finally, we mention that the full theorem is not a direct consequence of the Euler-Jacobi formula, of multidimensional residue integral methods, or of Lefschetz fixed point theory, because some of the singularities have fixed points at infinity.

1. **Fold (1):** Its corresponding induced map $f$ is

$$
f(x, y) = (x, y^2) .
$$

Let $s = (s_1, s_2)$ be a non-caustic point and let us determine the maximum number of images of $s$. Setting

$$
f(x, y) = (s_1, s_2) ,
$$

we obtain $x = s_1$ and find that the $y$-coordinates of the pre-images are the two real zeros of the polynomial

$$
\varphi(y) \equiv y^2 - s_2 .
$$

Consequently, there is a maximum of two pre-images. The magnification, expressed in the one variable $y$, is given by $\mathcal{M}(y) = 1 / J(y) = 1 / 2y$. Since $\varphi'(y) = 2y = J(y)$, we have

$$
\varphi'(y)\mathcal{M}(y) = J(y)\mathcal{M}(y) = 1 .
$$

But this implies that the unique polynomial representative in the coset $\overline{\varphi'(y)\mathcal{M}(y)}$ is the polynomial $m(y) \equiv 1$. Since the $(n-1)$st coefficient is $b_1 = 0$, we conclude via (7) that the total signed magnification in the two-image region is zero:

$$
\mathcal{M}_1 + \mathcal{M}_2 = \sum_{i=1}^{2} \mathcal{M}(y_i) = \frac{b_1}{a_2} = 0 .
$$

2. **Cusp (2):** Its corresponding induced map $f$ is

$$
f(x, y) = (x, xy + y^3) .
$$

As with the fold, let $s = (s_1, s_2)$ be a non-caustic point and let us determine the maximum number of images of $s$. Setting

$$
f(x, y) = (s_1, s_2) ,
$$

we obtain $x = s_1$ and find that the $y$-coordinates of the pre-images are the three real zeros of the polynomial

$$
\varphi(y) \equiv y^3 + s_1 y - s_2 .
$$
So, there is a maximum of three pre-images. The magnification, expressed in the one variable $y$, is given by
\[
\mathcal{M}(y) = 1/J(y) = 1/(3y^2 + s_1) .
\]
Once again we have $\varphi'(y) = J(y)$, so that
\[
\varphi'(y)\mathcal{M}(y) = J(y)\mathcal{M}(y) = 1 .
\]
As with the fold, it follows that $m(y) \equiv 1$ and $b_{n-1} = b_2 = 0$, so that the total signed magnification in the three-image region is zero:
\[
\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 = \sum_{i=1}^{3} \mathcal{M}(y_i) = \frac{b_2}{a_3} = 0 .
\]
3. **Swallowtail** (3): Its corresponding 1-parameter induced map $f_c$ is
\[
f_c(x, y) = (xy + cx^2 + x^4, y) .
\]
Let $s = (s_1, s_2)$ be a non-caustic point and let us determine the maximum number of images of $s$. Setting
\[
f(x, y) = (s_1, s_2) ,
\]
we obtain $y = s_2$ and find that the $x$-coordinates of the pre-images are the four real zeros of the polynomial
\[
\varphi(x) \equiv x^4 + cx^2 + s_2x - s_1 ,
\]
which gives a maximum of four pre-images. The magnification is $\mathcal{M}(x) = 1/J(x) = 1/(4x^3 + 2cx + s_2)$ and $\varphi'(x) = J(x)$, so that
\[
\varphi'(x)\mathcal{M}(x) = J(x)\mathcal{M}(x) = 1 .
\]
We thus have $m(x) \equiv 1$, $b_{n-1} = b_3 = 0$, and the total signed magnification in the four-image region is zero:
\[
\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 + \mathcal{M}_4 = \frac{b_3}{a_4} = 0 .
\]
4. **Elliptic Umbilic** (3): Its corresponding 1-parameter induced map $f_c$ is
\[
f_c(x, y) = (3y^2 - 3x^2 - 2cx, 6xy - 2cy) .
\]
Setting $f(x, y) = (s_1, s_2)$ for a non-caustic point $(s_1, s_2)$, we get $y = s_2/(6x - 2c)$, which we use to get the following degree 4 polynomial for the $x$-coordinates of the pre-images:
\[
\varphi(x) \equiv 4c^2 s_1 - 3s_2^2 + (8c^3 - 24cs_1)x - (36c^2 - 36s_1) x^2 + 108x^4 .
\]
Hence there is a maximum of four pre-images. The magnification is $\mathcal{M}(x, y) = 1/(4c^2 - 36x^2 - 36y^2)$, which becomes a function of $x$:
\[
\mathcal{M}(x) = \frac{1}{J(x)} = \frac{1}{4c^2 - 12s_1 - 24cx - 72x^2} .
\]
This time $\varphi'(x) \neq J(x)$, but the situation is remedied if we multiply through by $2c - 6x$ to get
\[
\varphi'(x)\mathcal{M}(x) = [J(x)\mathcal{M}(x)](2c - 6x) = 2c - 6x .
\]
Thus the unique polynomial representative in the coset $\varphi'(x)\mathcal{M}(x)$ is the polynomial $m(x) \equiv -6x + 2c$. Since $b_{n-1} = b_3 = 0$, we have
\[
\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 + \mathcal{M}_4 = 0 .
\]
5. **Hyperbolic Umbilic** (3): Its corresponding 1-parameter induced map $f_c$, for a given target point $(s_1, s_2)$ lying in the four-image region, is
\[
f_c(x, y) = (-3x^2 - cy , -3y^2 - cx) = (s_1, s_2) .
\]
We eliminate \( y \) to obtain a polynomial in the variable \( x \), given by
\[
\varphi(x) = -3s_1^2 - c^2 s_2 - c^3 x - 18s_1 x^2 - 27x^4.
\]
The magnification is \( \mathcal{M}(x, y) = 1/(c^2 + 36xy) \). Substituting for \( y \) via \( f_c(x, y) = (s_1, s_2) \), we obtain
\[
\mathcal{M}(x) = \frac{1}{J(x)} = \frac{c}{-c^3 - 36s_1 x - 108x^3}.
\]
It follows that \( cJ(x) = \varphi'(x) \), so that
\[
\varphi'(x)\mathcal{M}(x) = [J(x)\mathcal{M}(x)] c = c.
\]
Thus the unique polynomial representative in the coset \( \varphi'(x)\mathcal{M}(x) \) is the polynomial \( m(x) \equiv c \). Since \( b_{n-1} = b_4 = 0 \), we have
\[
\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 + \mathcal{M}_4 + \mathcal{M}_5 = 0.
\]

6. **Butterfly (4):** Its corresponding 2-parameter induced map \( f_c \), for a given target point \((s_1, s_2)\) lying in the five-image region, is
\[
f_c(x, y) = (-2xy - 3c_2x^2 - 4c_1x^3 - 6x^5, y) = (s_1, s_2).
\]
We eliminate \( y \) and are left with a polynomial in the variable \( x \),
\[
\varphi(x) \equiv -s_1 - 2s_2 x - 3c_2 x^2 - 4c_1 x^3 - 6x^5,
\]
whose roots are the \( x \)-coordinates of the five pre-images. In this case \( J(x) = \varphi'(x) \), so that
\[
\varphi'(x)\mathcal{M}(x) = 1.
\]
Thus \( m(x) \equiv 1 \) is the unique polynomial representative in the coset \( \varphi'(x)\mathcal{M}(x) \). Since \( b_{n-1} = b_4 = 0 \), it follows that
\[
\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 + \mathcal{M}_4 + \mathcal{M}_5 = 0.
\]

7. **Parabolic Umbilic (4):** Its corresponding 2-parameter induced map \( f_c \), for a given target point \((s_1, s_2)\) lying in the five-image region, is
\[
f_c(x, y) = (2c_1 x + 2xy, 2c_2 y + x^2 + 4y^3) = (s_1, s_2).
\]
We eliminate \( x \) to obtain a polynomial in the variable \( y \),
\[
\varphi(y) \equiv -s_1^2 + 4c_2^2 s_2 - (8c_1^2 c_2 - 8c_1 s_2)y - (16c_1 c_2 - 4s_2)y^2 - (16c_1^2 + 8c_2)y^3 - 32c_1 y^4 - 16y^5.
\]
The magnification is \( \mathcal{M}(x, y) = 1/(4c_1 c_2 - 4x^2 + 4c_2 y + 24c_1 y^2 + 24y^3) \). Substituting for \( x^2 \) via the equations \( f_c(x, y) = (s_1, s_2) \), we obtain
\[
\mathcal{M}(y) = \frac{1}{J(y)} = \frac{1}{4c_1 c_2 - 4s_2 + 12c_2 y + 24c_1 y^2 + 40y^3}.
\]
Although \( \varphi'(y) \neq J(y) \), the situation is remedied if we multiply through by \(-2c_1 - 2y\):
\[
\varphi'(y)\mathcal{M}(y) = [J(y)\mathcal{M}(y)] (-2c_1 - 2y) = -2c_1 - 2y,
\]
from which we immediately conclude that \( m(y) = -2c_1 - 2y \) is the unique polynomial representative in the coset \( \varphi'(y)\mathcal{M}(y) \). Since \( b_{n-1} = b_4 = 0 \), we have
\[
\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 + \mathcal{M}_4 + \mathcal{M}_5 = 0.
\]
8. **Wigwan** (5): Its corresponding 3-parameter induced map $f_c$, for a given target point $(s_1, s_2)$ lying in the six-image region, is

\[
f_c(x, y) = (-2xy - 3c_1x^2 - 4c_2x^3 - 5c_1x^4 - 7x^6, y) = (s_1, s_2).
\]

We eliminate $y$ and are left with a polynomial in the variable $x$,

\[
\varphi(x) = -2sx - 3c_3x^2 - 4c_2x^3 - 5c_1x^4 - 7x^6 - s_1,
\]

whose roots are the $x$-coordinates of the six pre-images. In this case $J(x) = \varphi'(x)$, so that

\[
\varphi'(x)M(x) = 1.
\]

Therefore, as with the fold, cusp, swallowtail, and butterfly, we conclude that $m(x) \equiv 1$ is the unique polynomial representative in the coset $\varphi'(x)M(x)$. Since $b_{n-1} = b_5 = 0$, it follows that

\[
M_1 + M_2 + M_3 + M_4 + M_5 + M_6 = 0.
\]

9. **Symbolic Umbilic** (5): Its corresponding 3-parameter induced map $f_c$, for a given target point $(s_1, s_2)$ lying in the six-image region, is

\[
f_c(x, y) = (-3x^2 - c_1y^2 - c_2y, -4y^3 - 2c_1xy - c_2x - 2c_3y) = (s_1, s_2).
\]

We eliminate $x$ to obtain a polynomial in $y$,

\[
\varphi(y) = -c_2^2s_1 - 3c_3^2 - 3c_2y - 4c_1c_2s_1y - 12c_3s_2y - 5c_1c_2^2y^2 - 12c_2^2y^2 - 4c_1^3y^2
\]

\[
- 8c_1^2c_2y^3 - 24c_2y^3 - 4c_1^3y^4 - 48c_3y^4 - 48y^6.
\]

The magnification is $M(x, y) = 1/(c_2^2 + 12c_3x + 12c_1x^2 - 4c_1c_2y - 4c_1^2y^2 + 72xy^2)$. Substituting for $x$ via the equations $f_c(x, y) = (s_1, s_2)$ gives

\[
M(y) = \frac{1}{J(y)} = \frac{c_2 + 2c_1y}{-c_2^3 - 4c_1c_2s_2 - 12c_3s_2 - (10c_1c_2^2 + 24c_3^2 + 8c_1s_2y) - (24c_1c_2 + 72s_2)y^2 - (16c_1^2 + 192c_3)y^3 - 288y^6}.
\]

It is not difficult to check that $\varphi'(y) = J(y)(2c_1y + c_2)$, so that

\[
\varphi'(y)M(y) = [J(y)M(y)](2c_1y + c_2) = 2c_1y + c_2.
\]

Then $m(y) = 2c_1y + c_2$ and $b_{n-1} = b_5 = 0$, so that

\[
M_1 + M_2 + M_3 + M_4 + M_5 + M_6 = 0.
\]

10. **2nd Elliptic Umbilic** (5): Its corresponding 3-parameter induced map $f_c$, for a given target point $(s_1, s_2)$ lying in the six-image region, is

\[
f_c(x, y) = (-2xy, -x^2 + 5y^4 - 4c_1y^3 - 3c_2y^2 - 2c_3y) = (s_1, s_2).
\]

Eliminating $x$, we obtain the polynomial

\[
\varphi(y) = -s_1^2 - 4s_2y^2 - 8c_3y^3 - 12c_2y^4 - 16c_1y^5 + 20y^6.
\]

The magnification is $M(x, y) = 1/(-4x^2 + 4c_3y + 12c_2y^2 + 24c_1y^3 - 40y^4)$. Substituting for $x^2$ via the equations $f_c(x, y) = (s_1, s_2)$ gives

\[
M(y) = \frac{1}{J(y)} = \frac{1}{4s_2 + 12c_3y + 24c_2y^2 + 40c_1y^3 - 60y^4}.
\]

One can check directly that $\varphi'(y) = J(y)(-2y)$, so that

\[
\varphi'(y)M(y) = [J(y)M(y)](-2y) = -2y.
\]

Then $m(y) = -2y$ and $b_{n-1} = b_5 = 0$, so that

\[
M_1 + M_2 + M_3 + M_4 + M_5 + M_6 = 0.
\]
11. 2\textsuperscript{nd} Hyperbolic Umbilic (5): Its corresponding 3-parameter induced map $f_c$, for a given target point $(s_1, s_2)$ lying in the six-image region, is

$$f_c(x, y) = (-2xy, -x^2 - 5y^4 - 4c_1y^3 - 3c_2y^2 - 2c_3y) = (s_1, s_2).$$

Eliminating $x$, we obtain the polynomial

$$\varphi(y) \equiv -s_1^2 - 4s_2y^2 - 8c_3y^3 - 12c_2y^4 - 16c_1y^5 - 20y^6.$$  

The magnification is $M(x, y) = 1/(-4x^2 + 4c_3y + 12c_2y^2 + 24c_1y^3 + 40y^4)$. Substituting for $x^2$ via the equations $f_c(x, y) = (s_1, s_2)$ gives

$$M(y) = \frac{1}{J(y)} = \frac{1}{4s_2 + 12c_3y + 24c_2y^2 + 40c_1y^3 + 60y^4}.$$  

Once again, it is easy to check that $\varphi'(y) = J(y)(-2y)$, so that

$$\varphi'(y)M(y) = [J(y)M(y)](-2y) = -2y.$$  

Then $m(y) \equiv -2y$ and $b_{n-1} = b_5 = 0$, so that

$$M_1 + M_2 + M_3 + M_4 + M_5 + M_6 = 0.$$  

This completes the proof. \hfill \square

VI. CONCLUSION

The paper presented a theorem about the magnification pre-images for caustic singularities up to codimension five. We proved that for generic families of general mappings between planes locally exhibiting such singularities, and for any point in the target lying in the region giving rise to the maximum number of real pre-images, the total signed magnification of the pre-images sums to zero. The signed magnifications are invariants as they are Gaussian curvatures at critical points. Our result extends earlier work that considered the case of singularities through to codimension three. The proof of the theorem is algebraic in nature and utilizes the Euler trace formula. In fact, we established a proposition that relates the coefficients of the unique polynomial in the coset of certain rational functions to Newton sums. It was then shown that the Euler trace formula follows readily as a corollary of our proposition. The findings of the paper are expected to be relevant to the study of dark matter substructures on galactic scales using gravitational lensing. In addition, since the results hold for generic general mappings, they are applicable to any system in which stable caustic singularities appear.

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APPENDIX A: PROOF OF PROPOSITION [2]

For convenience, we restate the result:

**Proposition.** Consider any polynomial $\varphi(x) = a_nx^n + \cdots + a_1x + a_0 \in \mathbb{C}[x]$ with distinct roots and any rational function $h(x) \in R$. Let

$$r(x) = b_{n-1}x^{n-1} + \cdots + b_1x + b_0$$

be the respective unique polynomial representative of the coset $\varphi'(x)h(x)$ in $R/(\varphi(x))$. Then the coefficients of $r(x)$ are given in terms of the coefficients of $h_*(x)$ and $\varphi(x)$ through the following recursive relation:

$$b_{n-i} = c_{n-1}b_{n-i,n-1} + \cdots + c_1b_{n-i,1} + c_0b_{n-i,0}, \quad i = 1, \ldots, n, \quad (A1)$$
with
\[
\begin{align*}
    b_{n-i,0} &= (n - (i - 1)) a_{n-(i-1)} , & i &= 1, \ldots, n , \\
    b_{n-i,k} &= -\frac{a_{n-i}}{a_n} b_{n-1,k-1} + b_{n-(i+1),k-1} , & i &= 1, \ldots, n , & k &= 1, \ldots, n-1 ,
\end{align*}
\] (A2)
where \( b_{-1,k-1} \equiv 0 \).

Proof of Proposition.
We begin with some preliminaries about quotient rings to make the proof more self-contained. Let \( \mathbb{C}[x] \) be the ring of polynomials over \( \mathbb{C} \) and let \( \mathbb{C}(x) \) be the field of rational functions formed from quotients of polynomials in \( \mathbb{C}[x] \).

The \( n \) zeros \( x_1, \ldots, x_n \) of \( \varphi(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{C}[x] \) are assumed to be distinct (generically, the roots of a polynomial are distinct). Let \( \langle \varphi(x) \rangle \) denote the ideal in \( \mathbb{C}[x] \) generated by \( \varphi(x) \), and consider the quotient ring \( \mathbb{C}[x]/\langle \varphi(x) \rangle \), whose cosets we denote by \( g(x) \). This quotient ring has two important properties:

- **Property 1:** If \( \overline{g_1(x)} = \overline{g_2(x)} \), then by definition \( g_1(x) - g_2(x) = h(x) \varphi(x) \) for some \( h(x) \in \mathbb{C}[x] \), from which it follows that \( g_1(x_i) = g_2(x_i) \) for all \( n \) roots \( x_i \) of \( \varphi(x) \). Thus members of the same coset must agree on the roots of \( \varphi(x) \), so that, in particular, \( \sum_{i=1}^{n} g_1(x_i) = \sum_{i=1}^{n} g_2(x_i) \).

- **Property 2:** Each coset \( \overline{g(x)} \) has a unique representative of degree at most \( n - 1 \), as follows: by the division algorithm in \( \mathbb{C}[x] \), there exist polynomials \( q(x) \) and \( r(x) \) such that
\[
    g(x) = q(x) \varphi(x) + r(x) ,
\]
where \( \deg r < \deg \varphi = n \). Passing to the quotient ring \( \mathbb{C}[x]/\langle \varphi(x) \rangle \), we see that \( \overline{g(x)} = \overline{r(x)} \). Suppose now that there exists another polynomial \( p(x) \) of degree less than \( n \) with \( g(x) = p(x) \). Then \( \overline{p(x)} = \overline{r(x)} \), so that
\[
    p(x) - r(x) = h(x) \varphi(x)
\]
for some \( h(x) \in \mathbb{C}[x] \). If \( h(x) \neq 0 \), then \( \deg h \varphi = n \), while the degree of the left-hand side is less than \( n \). We must therefore have \( h(x) \equiv 0 \) and \( p(x) = r(x) \). We may thus represent every coset by its unique polynomial representative of degree less than \( n \), which in turn implies that \( \mathbb{C}[x]/\langle \varphi(x) \rangle \) is a vector space of dimension \( n \), with basis \( \{ \mathbf{1}, \pi, \pi^2, \ldots, \pi^{n-1} \} \).

The next result will be used to show that Properties 1 and 2 also hold for a certain subset of rational functions in \( \mathbb{C}(x) \) (see Claim 2 below).

**Claim 1.** Let \( x_1, \ldots, x_n \in \mathbb{C} \) be distinct. Let \( c_1, \ldots, c_n \in \mathbb{C} \), not necessarily distinct. Then there exists a unique polynomial \( H(x) \in \mathbb{C}[x] \) with \( \deg h < n \) such that \( H(x_i) = c_i \).

**Proof (Claim 1).** Induction on \( n \). For \( n = 1 \), define \( H(x) \equiv c_1 \). Now assume that the result is true for \( n - 1 \), and consider a set of \( n \) distinct complex numbers \( x_1, \ldots, x_n \). By the induction hypothesis, there exists a polynomial \( h(x) \in \mathbb{C}[x] \) with \( \deg h < n - 1 \) such that \( h(x_i) = c_i \) for \( i = 1, \ldots, n - 1 \). Now define
\[
    H(x) = h(x) + \frac{(x-x_1)(x-x_2)\cdots(x-x_{n-1})}{(x_n-x_1)(x_n-x_2)\cdots(x_n-x_{n-1})} (c_n - h(x_n)) .
\]
It follows that \( H(x) \in \mathbb{C}[x] \) has degree less than \( n \), and \( H(x_i) = c_i \) for all \( i = 1, \ldots, n \). (As a simple example to show that \( H(x) \) need not be unique if the \( x_1, \ldots, x_n \) are not distinct, consider the numbers 2, 2, 3, 3 all being mapped to 0.) Then the polynomials \( H_1(x) = (x-2)^2(x-3) \), \( H_2(x) = (x-2)(x-3)^2 \), and \( H_3(x) = (x-2)(x-3) \) all satisfy the assumptions of the lemma.) Suppose that there exist two polynomials \( H_1(x) \) and \( H_2(x) \) with \( H_1(x_i) = c_i = H_2(x_i) \). By the division algorithm in \( \mathbb{C}[x] \), there are unique polynomials \( q(x) \) and \( r(x) \) such that
\[
    H_1(x) - H_2(x) = q(x) [(x-x_1)(x-x_2)\cdots(x-x_n)] + r(x) ,
\]
where \( \deg r < n \). If \( q(x) \neq 0 \), then the degree of the polynomial on the right-hand side is at least \( n \), whereas \( H_1(x) - H_2(x) \) has degree less than \( n \). We must therefore have \( q(x) \equiv 0 \). Moreover, if \( r(x) \neq 0 \), then \( H_1(x_i) = H_2(x_i) \) gives that \( r(x_i) = 0 \) for all \( x_1, \ldots, x_n \). This implies, however, that \( r(x) \) has \( n \) distinct zeros and so must have degree \( n \), a contradiction. Thus \( H_1(x) = H_2(x) \). \( \square \) (Claim 1)
Let $R \subset \mathbb{C}(x)$ denote the subring of rational functions that are defined at the roots $x_i$ of $\varphi(x)$,

$$R = \left\{ \frac{p(x)}{q(x)} : p(x), q(x) \in \mathbb{C}[x] \text{ and } q(x_i) \neq 0 \text{ for all roots } x_i \right\},$$

and consider the quotient ring $R/(\varphi(x))$. The next claim states that the ring $R/(\varphi(x))$ satisfies Properties 1 and 2.

**Claim 2.** Members of the same coset in $R/(\varphi(x))$ agree on the roots $x_i$ of $\varphi(x)$, that is, if $g_1(x)$ and $g_2(x)$ belong to the same coset, then $g_1(x_i) = g_2(x_i)$, and so $\sum_{i=1}^{n} g_1(x_i) = \sum_{i=1}^{n} g_2(x_i)$. In addition, any rational function $h(x) \in R$ will have in its coset $\bar{h}(x) \in R/(\varphi(x))$ a unique polynomial representative $r(x)$ of degree less than $n$.

**Proof (Claim 2).** Notice that, if $\bar{h}_1(x) = \bar{h}_2(x) \in R/(\varphi(x))$, then by definition there exists a rational function $h(x) \in R$ such that

$$h_1(x) - h_2(x) = h(x)\varphi(x),$$

so that $h_1(x_i) = h_2(x_i)$ for all the zeros $x_i$ of $\varphi(x)$. In other words, $R/(\varphi(x))$ also satisfies Property 1. It turns out that when the zeros $x_1, \ldots, x_n$ of $\varphi(x)$ are distinct, as we are assuming they are, then $R/(\varphi(x))$ also satisfies Property 2 (in fact $R/(\varphi(x))$ and $\mathbb{C}[x]/(\varphi(x))$ will be isomorphic as rings). For given a coset $\bar{h}(x) \in R/(\varphi(x))$, Claim 1 shows that there is a unique polynomial $g(x) \in \mathbb{C}[x]$ of degree less than $n$ whose values at the $n$ roots $x_i$ are $h(x_i)$. Then the rational function $g(x) - h(x) \in R$ vanishes at every $x_i$, and a simple application of the division algorithm applied to the numerator of $g(x) - h(x)$ shows that $g(x) = h(x) \in R/(\varphi(x))$. Thus any rational function $h(x) \in R$ will have in its coset $\bar{h}(x) \in R/(\varphi(x))$ a unique polynomial representative $r(x)$ of degree less than $n$. □ (Claim 2)

We now begin the proof of the Proposition by establishing the following Lemma:

**Lemma.** Let $\varphi(x) = a_n x^n + \cdots + a_1 x + a_0$ and consider the quotient ring $R/(\varphi(x))$. For any $1 \leq k \leq n - 1$, let

$$r_k(x) = b_{n-1,k} x^{n-1} + \cdots + b_{1,k} x + b_{0,k},$$

be the unique polynomial representative in the coset $\overline{\varphi'(x)x^k}$. Then the following recursive relation holds:

$$
\begin{align*}
b_{n-i,0} &= (n - (i - 1)) a_{n-(i-1)}, & i &= 1, \ldots, n, \\
b_{n-i,k} &= \frac{a_{n-i}}{a_n} b_{n-1,k-1} + b_{n-(i+1),k-1}, & i &= 1, \ldots, n, & k &= 1, \ldots, n-1,
\end{align*}
$$

(A3)

where $b_{-1,k-1} \equiv 0$.

**Proof of Lemma.** The existence and uniqueness of the polynomial

$$r_k(x) = b_{n-1,k} x^{n-1} + \cdots + b_{n-i,k} x^{n-i} + \cdots + b_{1,k} x + b_{0,k},$$

where

$$\overline{\varphi'(x)x^k} = \overline{r_k(x)} = b_{n-1,k} x^{n-1} + \cdots + b_{n-i,k} x^{n-i} + \cdots + b_{1,k} \overline{x} + b_{0,k} \overline{1},$$

(A4)

were established in Claim 2. Also, note that since $\overline{\varphi(x)} = \overline{0} \in R/(\varphi(x))$, we have

$$\overline{x^i} = -\frac{a_{n-i}}{a_n} \overline{x^{n-i}} - \cdots - \frac{a_1}{a_n} \overline{x} - \frac{a_0}{a_n} \overline{1}.$$

(A5)

Case $k = 0$: By (A4), we get

$$\overline{\varphi'(x)x^0} = \overline{r_0(x)} = b_{n-1,0} x^{n-1} + \cdots + b_{n-i,0} x^{n-i} + \cdots + b_{1,0} \overline{x} + b_{0,0} \overline{1}.$$

However,

$$\overline{\varphi'(x)x^0} = \overline{\varphi'(x)} = n a_n x^{n-1} + \cdots + (n - (i - 1)) a_{n-(i-1)} x^{n-i} + \cdots + 2 a_2 \overline{x} + a_1 \overline{1}.$$
Consequently,

\[ b_{n-i,0} = (n - (i - 1)) a_{n-(i-1)} , \quad i = 1, \ldots, n. \]  \hspace{1cm} (A6)

Case \( k = 1, \ldots, n - 1 \): Equations (A4) and (A5) yield

\[
\varphi'(x)^{x^k} = b_{n-1,k} x^{n-1} + \cdots + b_{n-i,k} x^{n-i} + \cdots + b_{1,k} \varphi + b_{0,k} \Gamma \\
= x \varphi'(x)^{x^{k-1}} \\
= \varphi \left[ b_{n-1,k-1} x^{n-1} + b_{n-2,k-1} x^{n-2} + \cdots + b_{1,k-1} \varphi + b_{0,k-1} \Gamma \right] \\
= b_{n-1,k-1} x^n + b_{n-2,k-1} x^{n-1} + \cdots + b_{1,k-1} x^2 + b_{0,k-1} \varphi \\
= b_{n-1,k-1} \left[ \frac{a_n-1}{a_n} b_{n-1,k-1} + b_{n-(i+1),k-1} \right] x^{n-i}.
\]

The coefficients of (A4) are then related to the coefficients of \( a_i \) of \( \varphi(x) \) as follows:

\[ b_{n-i,k} = \frac{a_n-1}{a_n} b_{n-1,k-1} + b_{n-(i+1),k-1} , \quad i = 1, \ldots, n , \quad k = 1, \ldots, n - 1 , \]

where the coefficients \( b_{n-i,0} \) are given by (A6). Note that \( b_{n,k} = 0 \) since the unique polynomial goes up to degree \( n - 1 \). \hspace{1cm} \Box (Lemma)

We now complete the proof of the Proposition. If \( h_1(x) \) and \( h_2(x) \) are the unique polynomial representatives of the cosets \( h_1(x) \) and \( h_2(x) \), respectively, then by uniqueness, the sum \( h_1(x) + h_2(x) \) is the unique polynomial representative of the coset \( h_1(x) + h_2(x) \). With that said, we note that, since \( h(x) = h_*(x) \), it follows that \( r(x) = \varphi'(x) h(x) = \varphi'(x) h_*(x) \). We thus have

\[
\frac{r(x)}{x} = \varphi'(x) h_*(x) \\
= c_{n-1} \varphi'(x) x^{n-1} + \cdots + c_1 \varphi'(x) x + c_0 \varphi(x) \\
= c_{n-1} r_{n-1}(x) + \cdots + c_1 r_1(x) + c_0 r_0(x) \\
= c_{n-1} \sum_{i=1}^n b_{n-i,n-1} x^{n-i} + \cdots + c_1 \sum_{i=1}^n b_{n-i,1} x^{n-i} + c_0 \sum_{i=1}^n b_{n-i,0} x^{n-i} \\
= \sum_{i=1}^n \left( c_{n-1} b_{n-i,n-1} + \cdots + c_1 b_{n-i,1} + c_0 b_{n-i,0} \right) x^{n-i} \\
= \sum_{i=1}^n b_{n-i} x^{n-i} . \hspace{1cm} \Box (Proposition)
\]

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