Abstract. We apply the nested algebraic Bethe ansatz to the models with $\mathfrak{gl}(2|1)$ and $\mathfrak{gl}(1|2)$ supersymmetry. We show that form factors of local operators in these models can be expressed in terms of the universal form factors. Our derivation is based on the use of the $RTT$-algebra only. It does not refer to any specific representation of this algebra. We obtain thus determinant representations for form factors of local operators in the cases where an explicit solution of the quantum inverse scattering problem is not known.

Keywords: form factors, quantum integrability (Bethe ansatz), Hubbard and related model
1. Introduction

Quantum inverse scattering method (QISM) is a powerful tool for solving quantum integrable models [1–4]. This method allows one to find spectra of quantum Hamiltonian via the algebraic Bethe ansatz. The main advantage of the algebraic Bethe ansatz is that it gives a systematic procedure to describe the spectra of the models, which might have completely different physical interpretation. This is because this method only deals with the algebra of the monodromy matrix entries, but not with its specific representation.

The QISM and the algebraic Bethe ansatz also can be used for calculation of form factors [4–7]. Similarly to the problem of the Hamiltonian spectrum, in many cases this method gives quite general results, which can be used for the study of a wide class of models. In particular, it was shown recently [8, 9] that form factors of local operators (FFLO) in the models with \( \mathfrak{gl}(3) \)-invariant \( R \)-matrix are all expressed in terms of universal form factors [10]. The latter are completely determined by the \( R \)-matrix and do not depend on the model under consideration.

Knowing FFLO one can solve the problem of correlation functions via their form factor expansion. In the models, for which an explicit solution of the quantum inverse scattering problem is known [11, 12], the FFLO are directly related to the ones of the monodromy matrix entries. Using this result, correlation functions of the XXZ spin
chain and other integrable models were studied in the series of works [6, 13–16]. It is worth mentioning, however, that the existence of an explicit solution of the quantum inverse scattering problem is based on a specific representation of the underlying $RTT$-algebra. It can be found for various spin chains, but not in general.

In the present paper we consider models described by the $\mathfrak{gl}(2|1)$ and $\mathfrak{gl}(1|2)$ superalgebras. The study of integrable models based on the high rank algebras was initiated in the works [17–19], where the method of the nested Bethe ansatz was introduced. The algebraic version of this approach within the QISM was developed in papers [20, 21]. The application of the QISM to the superalgebras was considered in [22]. Recently, integrable models with $\mathfrak{gl}(2|1)$ and $\mathfrak{gl}(1|2)$ supersymmetries were studied within the framework of the nested algebraic Bethe ansatz in a series of papers [23–25]. Determinant representations for form factors of the monodromy matrix entries in these models were found in [26]. These formulas can be directly applied to the calculation of the FFLO in the supersymmetric t-J model [27–31], because the quantum inverse scattering problem for this model was solved in [32]. Our goal is to generalize these results. Namely, we show that similarly to the $\mathfrak{gl}(N)$ case, the FFLO in the models with $\mathfrak{gl}(2|1)$ and $\mathfrak{gl}(1|2)$ supersymmetries are proportional to the universal form factors.

Our method is based on the composite model [5]. It was used in [9, 34] for the calculation of the FFLO in the $\mathfrak{gl}(3)$-based models. In the present paper we generalize this approach to the case of superalgebras. Due to an isomorphism between Yangians of $\mathfrak{gl}(2|1)$ and $\mathfrak{gl}(1|2)$ [23] we consider below the case of $Y(\mathfrak{gl}(2|1))$ only. It is worth mentioning that our method does not use any specific representation of the $RTT$-algebra. We require only the existence of the highest weight representation, which is a necessary condition for the application of the nested Bethe ansatz. Therefore, the obtained results have the same degree of generality as the Bethe equations which are used to determine the spectra of the quantum Hamiltonians.

The paper is organized as follows. In section 2 we introduce the model under consideration and describe the basic notions and the notation used in the paper. In section 3 we recall the notion of the composite model and introduce partial zero modes of the monodromy matrix. There we explain how the use of the partial zero modes allows one to relate different FFLO with each other. The main results of the paper are collected in section 4. In section 5 we prove the results of section 4. A part of the proof is placed in appendix A.

1. Basic notions and notation

In this section we briefly describe basic notions of $\mathfrak{gl}(2|1)$-based integrable models solvable by the algebraic Bethe ansatz. The reader can find a more detailed description in [22, 23, 35, 36].

2.1. Graded models and Bethe vectors

The central object of the algebraic Bethe ansatz method is a quantum monodromy matrix $T(u)$. It acts in the tensor product $V \otimes \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space of a
quantum model and $V$ is an auxiliary space. Commutation relations between the entries $T_{ij}(u)$ are gathered in an $RTT$-relation

$$R(u, v) \cdot (T(u) \otimes I) \cdot (I \otimes T(v)) = (I \otimes T(v)) \cdot (T(u) \otimes I) \cdot R(u, v).$$

Here $R(u, v)$ is an $R$-matrix acting in the tensor product $V \otimes V$ of the auxiliary vector spaces $V$. Equation (2.1) holds in the tensor product $V \otimes V \otimes \mathcal{H}$.

For $\mathfrak{g}(2|1)$-based models the auxiliary vector space $V$ is a $\mathbb{Z}_2$-graded space $\mathbb{C}^{2|1}$ with a basis $\{e_1, e_2, e_3\}$. We call the vectors $\{e_1, e_2\}$ even, while $e_3$ is odd. Respectively, we introduce a parity function on the set of indices as $[1] = [2] = 0$ and $[3] = 1$.

The $R$-matrix in (2.1) has the form

$$R(u, v) = I + g(u, v)P,$$

where $c$ is a constant, $I$ is the identity matrix in $V \otimes V$, and $P$ is the graded permutation matrix [22]. The tensor product in (2.1) is also graded leading to the set of commutation relations between the monodromy matrix entries $T_{ij}$:

$$[T_{ij}(u), T_{kl}(v)] = (-1)^{[i][k]+[j][l]} g(u, v)(T_{ij}(v)T_{kl}(u) - T_{kl}(u)T_{ij}(v)),$$

where we have introduced a graded commutator as

$$[T_{ij}(u), T_{kl}(v)] = T_{ij}(u)T_{kl}(v) - (-1)^{[i][j][k][l]} T_{kl}(v)T_{ij}(u).$$

The supertrace of the monodromy matrix

$$T(u) = \text{str}T(u) = \sum_{i=1}^{3} (-1)^{[i]} T_{ii}(u)$$

is called the transfer matrix. It is a generating function of the integrals of motion of the integrable model under consideration. The transfer matrix eigenstates are called on-shell Bethe vectors. They play an important role in the considerations below.

We assume that the space $\mathcal{H}$, in which the operators $T_{ij}$ act, contains a pseudo-vacuum vector $|0\rangle$. This vector possesses the following properties:

$$T_{ii}(u)|0\rangle = \lambda_i(u)|0\rangle,$$

$$T_{ij}(u)|0\rangle = 0, \quad i > j,$$

where $\lambda_i(u)$ are some functions of complex variable $u$. A specific choice of these functions means fixing of a specific integrable model. For us, however, they remain free functional parameters. This treatment of $\lambda_i(u)$ allows us to consider a wide class of integrable models within a common framework.

We also assume that the operators $T_{ij}$ act in the dual space $\mathcal{H}^*$ with a dual pseudo-vacuum vector $\langle 0 |$. This vector has analogous properties

$$\langle 0 | T_{ii}(u) = \lambda_i(u)\langle 0 |,$$

$$\langle 0 | T_{ij}(u) = 0, \quad i < j,$$

where $\lambda_i(u)$ are the same as in (2.6). Below we use the ratios of these functions

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Bethe vectors of \( \mathfrak{gl}(2|1) \)-invariant models are certain polynomials in operators \( T_{ij}(u) \) with \( i < j \) acting on the pseudovacuum vector. Their explicit form was found in [23] (see also [36] for the general \( \mathfrak{gl}(m|n) \) case). They depend on two sets of variables called Bethe parameters. We denote the Bethe vectors \( \mathbb{B}_{a,b}(\bar{u}; \bar{v}) \). Here the Bethe parameters are \( \bar{u} = \{ u_1, \ldots, u_a \} \) and \( \bar{v} = \{ v_1, \ldots, v_b \} \). The subscripts \( a \) and \( b \) \((a, b = 0, 1, \ldots)\) denote the cardinalities of the sets \( \bar{u} \) and \( \bar{v} \) respectively.

Similarly one can construct dual Bethe vectors in the dual space \( \mathcal{H}^* \) as polynomials in operators \( T_{ij}(u) \) with \( i > j \) acting on the dual pseudovacuum vector \( \mathbf{0} \) [23]. We denote them \( \mathbb{C}_{a,b}(\bar{u}; \bar{v}) \) with the same meaning of the arguments and the subscripts.

For generic (dual) Bethe vectors the Bethe parameters \( \bar{u} \) and \( \bar{v} \) are generic complex numbers. If these parameters satisfy a system of Bethe equations

\[
\tau(\bar{u}, \bar{v}) = \lambda_1(\bar{w}) \prod_{j=1}^{a} f(u_j, w) + \lambda_2(\bar{w}) \prod_{j=1}^{a} f(w, u_j) \prod_{k=1}^{b} f(v_k, w) - \lambda_3(\bar{w}) \prod_{k=1}^{b} f(v_k, w). \tag{2.12}
\]

Besides the monodromy matrix \( T(u) \) we also consider a twisted monodromy matrix \( T_\kappa(u) = \kappa T(u) \), where \( \kappa \) is a diagonal matrix \( \kappa = \text{diag}\{\kappa_1, \kappa_2, \kappa_3\} \), and \( \kappa_i \) are complex numbers. The supertrace \( \text{str} T_\kappa(u) \) is called the twisted transfer matrix. A generic Bethe vector becomes an eigensate of the twisted transfer matrix, if the Bethe parameters satisfy a system of twisted Bethe equations

\[
\tau(\bar{u}, \bar{v}) = \lambda_1(\bar{w}) \prod_{j=1}^{a} f(u_j, w) + \lambda_2(\bar{w}) \prod_{j=1}^{a} f(w, u_j) \prod_{k=1}^{b} f(v_k, w) - \lambda_3(\bar{w}) \prod_{k=1}^{b} f(v_k, w). \tag{2.12}
\]
The corresponding (dual) Bethe vector is then called the twisted (dual) on-shell Bethe vector. The twisted transfer matrix eigenvalue on the vector $\mathbb{B}_{a, b}(\bar{u}; \bar{v})$ is given by

$$\tau_{a}(w|\bar{u}, \bar{v}) = \kappa_{1}\lambda_{1}(w) \prod_{j=1}^{a} f(u_{j}, w) + \kappa_{2}\lambda_{2}(w) \prod_{j=1}^{a} f(w, u_{j}) + \kappa_{3}\lambda_{3}(w) \prod_{k=1}^{b} f(v_{k}, w).$$

(2.14)

The use of the twisted monodromy matrix and the twisted on-shell Bethe vectors allows us to construct a special generating functional for FFLO (see section 5).

2.2. Shorthand notation

We denote sets of variables by bar: $\bar{u}, \bar{v}$ etc. If necessary, the cardinalities of the sets are given in special comments. Individual elements of the sets are denoted by latin subscripts: $u_{j}, v_{k}$ etc. We say that $\bar{x} = \bar{x}'$, if $\#\bar{x} = \#\bar{x}'$ and $x_{i} = x'_{i}$ (up to a permutation) for $i = 1, \ldots, \#\bar{x}$. We say that $\bar{x} \neq \bar{x}'$ otherwise.

Below we consider partitions of the sets into subsets. The notation $\bar{u} \Rightarrow \{\bar{u}_{I}, \bar{u}_{II}\}$ means that the set $\bar{u}$ is divided into two disjoint subsets. As a rule, we use roman numbers for subscripts of subsets: $\bar{u}_{I}, \bar{v}_{II}$ etc. However, if we deal with a big quantity of subsets, then we use standard arabic numbers for their notation. In such cases we give special comments to avoid ambiguities.

To lighten long formulas we use a shorthand notation for products of some functions. Namely, if the functions $r_{k}$ (2.8) or the functions $g$ and $f$ depend on sets of variables, this means that one should take the product over the corresponding set. For example,

$$r_{1}(\bar{u}) = \prod_{u_{k} \in \bar{u}} r_{1}(u_{k}); \quad g(\bar{z}, \bar{w}) = \prod_{w_{j} \in \bar{w}} g(z_{j}, w_{j}); \quad f(\bar{u}, \bar{v}) = \prod_{u_{j} \in \bar{u}} \prod_{v_{k} \in \bar{v}} f(u_{j}, v_{k}).$$

(2.15)

By definition any product with respect to the empty set is equal to 1. If we have a double product, then it is also equal to 1 if at least one of the sets is empty.

In section 3 we shall introduce several new scalar functions and will extend the convention (2.15) to their products.

2.3. Universal form factors

In this paper we reduce FFLO to the universal form factors of the monodromy matrix entries. The latter are defined as follows

$$\mathcal{F}_{(i, \bar{j})}^{(a, \bar{a})}(\bar{u}^{C}, \bar{v}^{C}; \bar{u}^{B}, \bar{v}^{B}) = \frac{\mathcal{C}_{a'}',b'(\bar{u}^{C}, \bar{v}^{C}) T_{b}(z) \mathbb{B}_{a, b}(\bar{u}^{B}, \bar{v}^{B})}{\tau(z|\bar{u}^{C}, \bar{v}^{C}) - \tau(z|\bar{u}^{B}, \bar{v}^{B})}.\quad (2.16)$$

Here both $\mathcal{C}_{a', b'}(\bar{u}^{C}, \bar{v}^{C})$ and $\mathbb{B}_{a, b}(\bar{u}^{B}, \bar{v}^{B})$ are on-shell Bethe vectors, and we assume that the Bethe parameters of the two Bethe vectors are different: $\{\bar{u}^{C}, \bar{v}^{C}\} \neq \{\bar{u}^{B}, \bar{v}^{B}\}$. The parameter $z$ is an arbitrary complex number. It was proved in [26] that the ratio in the r.h.s. of (2.16) does not depend on $z$.

The form factors (2.16) are called universal, because they are completely determined by the $R$-matrix of the model. They do not depend on a specific representation of the $RTT$-algebra, in particular, they do not depend on the vacuum eigenvalues $\lambda_{i}(u)$ (2.6).
In other words, if two different integrable models are described by the $R$-matrix (2.2), then they have the same universal form factors. Explicit determinant representations for the universal form factors in the $\mathfrak{gl}(2|1)$-invariant models were obtained in [26].

3. Composite model

In order to access FFLO we introduce a composite model [5, 9, 37]. Most naturally the composite model arises in the lattice models, where the monodromy matrix $T(u)$ is equal to the product of local $L$-operators

$$T(u) = L_M(u) \cdots L_1(u).$$

(3.1)

Here $M$ is the number of the lattice sites, and every $L$-operator satisfies $RTT$-relation with $R$-matrix (2.2). Let us fix a site $m \ (1 \leq m < M)$ and define two partial monodromy matrices $T^{(1)}(u)$ and $T^{(2)}(u)$ as

$$T^{(1)}(u) = L_m(u) \cdots L_1(u), \quad T^{(2)}(u) = L_M(u) \cdots L_{m+1}(u).$$

(3.2)

Then

$$T(u) = T^{(2)}(u)T^{(1)}(u).$$

(3.3)

Every $T^{(l)}(u)$ obviously satisfies $RTT$-relation (2.1) and has its own pseudovacuum vector $|0\rangle^{(l)}$, such that $|0\rangle = |0\rangle^{(1)} \otimes |0\rangle^{(2)}$. The operators $T^{(2)}(u)$ and $T^{(1)}(v)$ supercommute with each other, as they act in different spaces.

Continuous quantum models can be obtained from the lattice ones in the limit $M \to \infty$. Obviously, the determining relation (3.3) remains unchanged. The partial monodromy matrices still satisfy the $RTT$ relation, and the entries of the different partial monodromy matrices mutually supercommute. Thus, continuous quantum models also can be considered in the framework of the composite model.

Let

$$T_{ii}^{(l)}(u)|0\rangle^{(l)} = \lambda^{(l)}(u)|0\rangle^{(l)}, \quad l = 1, 2,$$

(3.4)

where $\lambda^{(l)}(u)$ are new free functional parameters. We also introduce

$$r^{(l)}_k(u) = \frac{\lambda^{(l)}_k(u)}{\lambda^{(l)}_2(u)}, \quad l = 1, 2, \quad k = 1, 3.$$  

(3.5)

Obviously

$$\lambda(u) = \lambda^{(1)}(u)\lambda^{(2)}(u), \quad r_k(u) = r^{(1)}_k(u)r^{(2)}_k(u).$$

(3.6)

Below we express form factors in terms of $r^{(1)}_k(u)$, therefore we introduce a special notation for these functions

$$r^{(1)}_k(u) = \ell_k(u), \quad \text{and hence,} \quad r^{(2)}_k(u) = \frac{r_2(u)}{\ell_k(u)}, \quad k = 1, 3.$$  

(3.7)
We extend the convention on the shorthand notation (2.15) to the products of the functions \( \ell_k(u) \).

Any monodromy matrix in (3.3) possesses its own Bethe vectors. The total Bethe vector is a bilinear combination of the partial Bethe vectors [37]:

\[
\mathbb{B}_{\alpha, \beta}(\vec{u}; \vec{v}) = \sum r_1^{(2)}(\vec{u}_1) r_3^{(1)}(\vec{v}_1) \frac{f(\vec{u}_1, \vec{u}_1) g(\vec{v}_1, \vec{v}_1)}{f(\vec{v}_1, \vec{v}_1)} \mathbb{B}_{a_1, b_1}(\vec{u}_1; \vec{v}_1) \mathbb{B}_{a_2, b_2}(\vec{u}_1; \vec{v}_1). \tag{3.8}
\]

Here the sum is taken over all partitions \( \vec{u} \Rightarrow \{ \vec{u}_1, \vec{u}_\Pi \} \) and \( \vec{v} \Rightarrow \{ \vec{v}_1, \vec{v}_\Pi \} \). The cardinalities of the subsets satisfy \( a_1 + a_2 = a \) and \( b_1 + b_2 = b \). Recall that here we have used the convention (2.15) for the products of the functions \( r_k^{(l)}, f, \) and \( g \).

Similarly, the dual total Bethe vector is a bilinear combination of the partial dual Bethe vectors:

\[
\mathcal{C}_{\alpha, \beta}(\vec{u}; \vec{v}) = \sum r_1^{(1)}(\vec{u}_1) r_3^{(2)}(\vec{v}_1) \frac{f(\vec{u}_1, \vec{u}_1) g(\vec{v}_1, \vec{v}_1)}{f(\vec{v}_1, \vec{v}_1)} \mathcal{C}_{a_1, b_1}(\vec{u}_1; \vec{v}_1) \mathcal{C}_{a_2, b_2}(\vec{u}_1; \vec{v}_1). \tag{3.9}
\]

Here the sum is the same as in (3.8).

Observe that if the total (dual) Bethe vector is on-shell (i.e. the set \( \{ \vec{u}, \vec{v} \} \) satisfies Bethe equations), then the partial (dual) Bethe vectors generically are not on-shell, because the subsets \( \{ \vec{u}_1, \vec{v}_1 \} \) and \( \{ \vec{u}_\Pi, \vec{v}_\Pi \} \) do not satisfy Bethe equations.

Comparing these formulas with the formulas for the total (dual) Bethe vectors in \( \mathfrak{gl}(3) \)-based models [8] one can see that the difference is very small. Namely, replacing the product of functions \( g(\vec{v}_1, \vec{v}_\Pi) \) in (3.8) with the product \( f(\vec{v}_\Pi, \vec{v}_1) \) we obtain the expression for the total Bethe vector in the models with \( \mathfrak{gl}(3) \) symmetry. Similarly, for the dual Bethe vectors one should make the replacement \( g(\vec{v}_\Pi, \vec{v}_1) \rightarrow f(\vec{v}_1, \vec{v}_\Pi) \). This similarity makes it possible to calculate FFLO by the same methods as in the case of the \( \mathfrak{gl}(3) \)-based models [9].

### 3.1. Zero modes

We assume a standard representation of the local \( L \)-operators in (3.1):

\[
L_n(u) = 1 + \frac{c}{u} L_n[0] + o(u^{-1}), \quad u \rightarrow \infty. \tag{3.10}
\]

Here \( \mathbf{1} \) is the identity operator in \( \mathbb{C}^{2|1} \otimes \mathcal{H} \). The matrix elements \( (L_n[0])_{ij} \) depend on the local operators of the model. Due to (3.1) we conclude that the total monodromy matrix \( T(u) \) and both partial monodromy matrices \( T^{[l]}(u) \) have the standard expansion over \( c/u \):

\[
T(u) = 1 + \sum_{n=0}^{\infty} T[n] \left( \frac{c}{u} \right)^{n+1},
\]

\[
T^{[l]}(u) = 1 + \sum_{n=0}^{\infty} T^{[l]}[n] \left( \frac{c}{u} \right)^{n+1}, \quad l = 1, 2. \tag{3.11}
\]

The operators \( T[0] \) and \( T^{[l]}[0] \) respectively are called the total and the partial zero modes. Obviously, the partial zero mode \( T^{[1]}[0] \) is equal to
Consider a form factor of the partial zero mode $T_{ij}^{(1)}[0]$

\[
\mathcal{M}^{(i,j)} \left( m \left| \begin{array}{c} \bar{u}^C \\ \bar{v}^C \\ \bar{u}^B \\ \bar{v}^B \end{array} \right|_{\nu, b} \right) = C'_{a', b'}(\bar{u}^C; \bar{v}^C) T_{ij}^{(1)}[0] \mathbb{B}_{a, b}(\bar{u}^B; \bar{v}^B),
\]  

(3.13)

where $C'_{a', b'}(\bar{u}^C; \bar{v}^C)$ and $\mathbb{B}_{a, b}(\bar{u}^B; \bar{v}^B)$ are on-shell Bethe vectors. We have stressed that this form factor depends on the number $m$ of the bulk site in (3.2). Then due to (3.12) we obtain

\[
C'_{a', b'}(\bar{u}^C; \bar{v}^C)(L_{m[0]}(0)) \mathbb{B}_{a, b}(\bar{u}^B; \bar{v}^B) = \mathcal{M}^{(i,j)} \left( m \left| \begin{array}{c} \bar{u}^C \\ \bar{v}^C \\ \bar{u}^B \\ \bar{v}^B \end{array} \right|_{\nu, b} \right) - \mathcal{M}^{(i,j)} \left( m-1 \left| \begin{array}{c} \bar{u}^C \\ \bar{v}^C \\ \bar{u}^B \\ \bar{v}^B \end{array} \right|_{\nu, b} \right). 
\]  

(3.14)

Thus, knowing the form factors of the partial zero mode $T_{ij}^{(1)}[0]$ we can find the form factors of the local operators $(L_{m[0]}(0))$. It is clear that in the case of continuous models the finite difference in the r.h.s. of (3.14) turns into the derivative over a space variable.

The use of the zero modes also allows one to obtain simple relations between different form factors. It follows from the commutation relations (2.3) that

\[
[T_{ij}^{(0)}(0), T_{kl}(0)] = (-1)^{|i||j|+|l||l|+|j||l|}(\delta_{il} T_{kj}(0) - \delta_{kj} T_{ij}(0)),
\]

\[
[T_{ij}^{(s)}(0), T_{kl}(0)] = (-1)^{|i||j|+|l||l|+|j||l|}(\delta_{il} T_{kj}^{(s)}(0) - \delta_{kj} T_{ij}^{(s)}(0)), \quad s = 1, 2.
\]  

(3.15)

Using $T_{ij}^{(0)} = T_{ij}^{(1)}[0] + T_{ij}^{(2)}[0]$ and $[T_{ij}^{(1)}[0], T_{kl}^{(2)}[0]] = 0$ we arrive at

\[
[T_{ij}^{(1)}[0], T_{kl}(0)] = (-1)^{|i||j|+|l||l|+|j||l|}(\delta_{il} T_{kj}^{(1)}[0] - \delta_{kj} T_{ij}^{(1)}[0]).
\]  

(3.16)

Equation (3.16) yields

\[
\delta_{il} \mathcal{M}^{(k,j)} \left( m \left| \begin{array}{c} \bar{u}^C \\ \bar{v}^C \\ \bar{u}^B \\ \bar{v}^B \end{array} \right|_{\nu, b} \right) - \delta_{kl} \mathcal{M}^{(i,j)} \left( m \left| \begin{array}{c} \bar{u}^C \\ \bar{v}^C \\ \bar{u}^B \\ \bar{v}^B \end{array} \right|_{\nu, b} \right) = (-1)^{|i||j|+|l||l|+|j||l|}C'_{a', b'}(\bar{u}^C; \bar{v}^C)[T_{ij}^{(1)}[0], T_{kl}(0)] \mathbb{B}_{a, b}(\bar{u}^B; \bar{v}^B). 
\]  

(3.17)

The actions of the total zero modes $T_{kl}(0)$ onto (dual) on-shell Bethe vectors were studied in [26]. Under this action a (dual) on-shell vector either vanishes or remains on-shell. Therefore, the expectation value in the r.h.s. of (3.17) is related to the form factor of the partial zero mode $T_{ij}^{(1)}[0]$. Equation (3.17), thus, allows us to express this form factor in terms of $\mathcal{M}^{(k,j)}$ and $\mathcal{M}^{(i,j)}$. We consider specific examples of these relationships in section 5.

4. Main results

We have shown in the previous section that FFLO can be reduced to the form factors of the partial zero modes $T_{ij}^{(1)}[0]$. Studying these form factors one should distinguish between two cases. In the first case an on-shell Bethe vector $\mathbb{B}_{a, b}(\bar{u}^B; \bar{v}^B)$ and a dual on shell Bethe vector $C'_{a', b'}(\bar{u}^C; \bar{v}^C)$ correspond to different eigenvalues of the transfer

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matrix. Then we say that \( \{ \bar{u}^C, \bar{v}^C \} \approx \{ \bar{u}^B, \bar{v}^B \} \). Otherwise, if \( \{ \bar{u}^C, \bar{v}^C \} = \{ \bar{u}^B, \bar{v}^B \} \), then both vectors correspond to the same eigenvalue. The latter case occurs for form factors of the diagonal elements \( T^{ab}_{ij}[0] \) only.

**Theorem 4.1.** Let \( \mathbb{B}_{a,b}(\bar{u}; \bar{v}) \) and \( \mathbb{C}_{a',b'}(\bar{u}^C; \bar{v}^C) \) be on-shell (dual) Bethe vectors such that \( \{ \bar{u}^C, \bar{v}^C \} \approx \{ \bar{u}^B, \bar{v}^B \} \). Then

\[
\mathcal{M}^{(i,j)}(m) \begin{pmatrix} \bar{u}^C \bar{v}^B \\ \bar{v}^C \bar{u}^B \end{pmatrix}_{i,j,b} = \left( \frac{\ell_i(\bar{u}^C)\ell_b(\bar{v}^B)}{\ell_i(\bar{u}^B)\ell_b(\bar{v}^C)} - 1 \right) \mathfrak{F}^{(i,j)}( \bar{u}^C \bar{u}^B, \bar{v}^C \bar{v}^B )_{i',j'}, \quad (4.1)
\]

where \( \mathfrak{F}^{(i,j)} \) is the universal form factor of the total monodromy matrix element \( T_{ij}(z) \).

**Theorem 4.2.** Let \( \mathbb{B}_{a,b}(\bar{u}; \bar{v}) \) be an on-shell Bethe vector and \( \mathbb{C}_{a',b'}(\bar{u}^C; \bar{v}^C) \) be its dual on-shell Bethe vector. Let \( \mathbb{C}_{a,b}(\bar{u}(\bar{r}); \bar{v}(\bar{r})) \) be a twisted on-shell deformation of \( \mathbb{C}_{a,b}(\bar{u}; \bar{v}) \) such that \( \bar{u}(\bar{r}) = \bar{u}, \bar{v}(\bar{r}) = \bar{v} \) at \( \bar{r} = 1 \). Then

\[
\mathcal{M}^{(i,j)}(m) \begin{pmatrix} \bar{u} \bar{v} \\ \bar{v} \bar{u} \end{pmatrix}_{a,b,b} = \left( \lambda^{(i)}_1[0] + (-1)^{|i|} \frac{d}{d\kappa_i} \log \frac{\ell_i(\bar{u}(\bar{r}))}{\ell_i(\bar{v}(\bar{r}))} \bigg|_{\kappa_i=1} \right)^2 \mathfrak{F}^{(i,j)}(\bar{u}(\bar{r}); \bar{v}(\bar{r}))_{i',j'}, \quad (4.2)
\]

where \( \lambda^{(i)}_1[0] \) can be found from the expansion

\[
\lambda^{(i)}_k(u) = 1 + \lambda^{(i)}_k[0] \frac{c_i}{u} + \ldots, \quad u \to \infty. \quad (4.3)
\]

We prove theorems 4.1 and 4.2 in section 5. The most technical part of the proof is given in appendix A. Here we would like to mention only that the general strategy of the proof is the same as in the \( \mathfrak{g}(3) \) case [9].

Comparing these formulas with the corresponding expressions in the \( \mathfrak{g}(3) \)-based models [9] we see that the only difference is the sign factor \( (-1)^{|i|} \) in (4.2). Certainly, the determinant formulas for the universal form factors in the models with \( \mathfrak{g}(2 \, | \, 1) \) and \( \mathfrak{g}(3) \) symmetries are different, however, the relation between \( \mathcal{M}^{(i,j)} \) and \( \mathfrak{F}^{(i,j)} \) is the same (modulus the sign factor mentioned above). Most probably, the same relation takes place in the general \( \mathfrak{g}(m \, | \, n) \) case as well. One should remember, however, that in models with \( \mathfrak{g}(2 \, | \, 1) \)-invariant R-matrix there exist compact determinant representations for the universal form factors [26]. These representations can be directly used for analysis of correlation functions. At the same time, analogous determinant formulas for the universal form factors in the general \( \mathfrak{g}(m \, | \, n) \) case are unknown up to date.

Observe that the dependence on the local site \( m \) in (4.1) and (4.2) enters only the functions \( \ell_i(u) \) and \( \lambda^{(i)}_1[0] \). This dependence follows from representation (3.2)

\[
\lambda^{(i)}_k(u) = \prod_{n=1}^{m} \lambda_{k}(u|n), \quad k = 1, 2, 3, \quad (4.4)
\]

\[
\ell_i(u) = \prod_{n=1}^{m} \ell_i(u|n) = \prod_{n=1}^{m} \frac{\lambda_i(u|n)}{\lambda_i(u|n)}, \quad k = 1, 3, \quad (4.5)
\]

---

4 Here and below \( \tilde{\kappa} = \{ \kappa_1, \kappa_2, \kappa_3 \} \), and \( \tilde{\kappa} = 1 \) means \( \kappa_1 = \kappa_2 = \kappa_3 = 1 \).

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where $\lambda_k(u|n)$ are vacuum eigenvalues of the local $L$-operators entries $(L_n(u))_{kk}$. Due to (3.10) the expansion of the vacuum eigenvalues $\lambda_k(u|n)$ takes the form

$$
\lambda_k(u|n) = 1 + \lambda_k[0|n] \frac{c}{u} + \ldots, \quad u \to \infty,
$$

and hence, the coefficient $\lambda_k^{(1)}[0]$ in (4.3) is

$$
\lambda_k^{(1)}[0] = \sum_{n=1}^{m} \lambda_k[0|n].
$$

Then, due to (3.14) we find for $\{ \bar{u}^C, \bar{v}^C \} \neq \{ \bar{u}^B, \bar{v}^B \}$

$$
\mathbb{C}_{a',b'}(\bar{u}^C; \bar{v}^C)(L_m[0])_{a,b}\mathbb{B}_{a',b'}(\bar{u}^B; \bar{v}^B) = (\mathcal{L}_m - 1) \left( \prod_{n=1}^{m-1} \mathcal{L}_n \right) \mathfrak{s}^{(i,j)} \left( \bar{u}^C \bar{u}^B \right)^{e',a}_{\nu',b} ,
$$

where

$$
\mathcal{L}_n = \frac{\ell_1(\bar{u}^C|n) \ell_2(\bar{u}^B|n)}{\ell_1(\bar{u}^C|n) \ell_2(\bar{u}^C|n)}.
$$

If $\{ \bar{u}^C, \bar{v}^C \} = \{ \bar{u}^B, \bar{v}^B \} = \{ \bar{u}, \bar{v} \}$, then

$$
\mathbb{C}_{a,b}(\bar{u}; \bar{v})(L_m[0])_{a,b}\mathbb{B}_{a,b}(\bar{u}; \bar{v}) = \left[ \lambda|0|m \right] + (-1)^{i_1} \frac{d}{d\kappa_i} \log \frac{\ell_1(\bar{u}(\kappa)|m)}{\ell_2(\bar{v}(\kappa)|m)} \bigg|_{\kappa=1} \left| \mathbb{B}_{a,b}(\bar{u}; \bar{v}) \right|^2 ;
$$

where $\bar{u}(\kappa)$ and $\bar{v}(\kappa)$ are the deformations of $\bar{u}$ and $\bar{v}$ described in theorem 4.2.

5. Generating functional for form factors of partial zero modes

All the form factors of the partial zero modes $T_{ij}^{(1)|0}$ can be found from a special generating functional. Consider an operator

$$
Q_{\beta} = \sum_{i=1}^{3} (-1)^{i} |\beta_i T_{ij}^{(1)|0},
$$

where $\beta_i$ are some complex numbers.

Let $\mathbb{B}_{a,b}(\bar{u}^B; \bar{v}^B)$ be an on-shell Bethe vector. Let also $\mathbb{C}_{a,b}^{(\kappa)}(\bar{u}^C; \bar{v}^C)$ be a twisted dual on-shell Bethe vector with the twist $\kappa = \text{diag}\{\kappa_1, \kappa_2, \kappa_3\}$. We stressed this fact by adding the superscript $(\kappa)$ to the vector $\mathbb{C}_{a,b}(\bar{u}^C; \bar{v}^C)$. Suppose that $\kappa_i = e^{\beta_i}$ and consider the following expectation value

$$
\mathcal{M}^{(\kappa)} \left( m \left| \begin{array}{c} \bar{u}^C \\ \bar{v}^C \\ \bar{u}^B \\ \bar{v}^B \end{array} \right. \right) = \mathbb{C}_{a,b}^{(\kappa)}(\bar{u}^C; \bar{v}^C) \ e^{Q_{\beta}} \ \mathbb{B}_{a,b}(\bar{u}^B; \bar{v}^B).
$$

Taking the derivative of this generating functional over $\beta_i$ at $\kappa = 1$ (that is, all $\beta_j = 0$) we obtain

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\[ (-1)^{ij} \mathcal{M}^{(i,j)}\left( m \left| \begin{array}{c} \bar{u}^C \\ \bar{v}^B \\ \bar{g}^C \\ \bar{g}^B \end{array} \right. \right)_{b,b} = \frac{d}{d\beta_i} \mathcal{M}^{(k)}\left( m \left| \begin{array}{c} \bar{u}^C \\ \bar{v}^B \\ \bar{g}^C \\ \bar{g}^B \end{array} \right. \right)_{b,b} - \mathcal{C}_{a,b}(\bar{u}^C; \bar{v}^C) \mathbb{B}_{a,b}(\bar{u}^B; \bar{v}^B) \right)_{k=1}. \]

It was shown in [24] that for \( \{ \bar{u}^C, \bar{v}^C \}_{k=1} \neq \{ \bar{u}^B, \bar{v}^B \} \)
\[ \frac{d}{d\beta_i} \mathcal{C}_{a,b}(\bar{u}^C; \bar{v}^C) \mathbb{B}_{a,b}(\bar{u}^B; \bar{v}^B) \left|_{k=1} \right. = (-1)^{ij} \mathbf{F}_{i,j} \left( \bar{u}^C; \bar{v}^C \right)_{b,b}. \]

Hence, we obtain in this case
\[ \mathcal{M}^{(i,j)}\left( m \left| \begin{array}{c} \bar{u}^C \\ \bar{v}^B \\ \bar{g}^C \\ \bar{g}^B \end{array} \right. \right)_{b,b} = (-1)^{ij} \frac{d}{d\beta_i} \mathcal{M}^{(k)}\left( m \left| \begin{array}{c} \bar{u}^C \\ \bar{v}^B \\ \bar{g}^C \\ \bar{g}^B \end{array} \right. \right)_{b,b} - \mathbf{F}_{i,j} \left( \bar{u}^C; \bar{v}^C \right)_{b,b}. \]

Thus, calculating the generating functional (5.2), we can find the form factors of the diagonal partial zero modes \( T_{ij}^{(1)}[0] \) at least for \( \{ \bar{u}^C, \bar{v}^C \}_{k=1} \neq \{ \bar{u}^B, \bar{v}^B \} \). The case \( \{ \bar{u}^C, \bar{v}^C \}_{k=1} = \{ \bar{u}^B, \bar{v}^B \} \) will be considered later.

The form factors of the partial zero modes \( T_{ij}^{(1)}[0] \) with \( i \neq j \) can be obtained via relation (3.17). Let us give an example. Let \( i = j = l = 2 \) and \( k = 1 \) in (3.17). Then this formula takes the form
\[ \mathcal{M}^{(1,2)}\left( m \left| \begin{array}{c} \bar{u}^C \\ \bar{v}^B \\ \bar{g}^C \\ \bar{g}^B \end{array} \right. \right)_{b,b}^{a+1,a} = C_{a+1,0}(\bar{u}^C; \bar{v}^C)(T_{22}^{(1)}[0] T_{12}[0] - T_{12}[0] T_{22}^{(1)}[0]) \mathbb{B}_{a,0}(\bar{u}^B; \bar{v}^B). \]

Due to the results of [26] we have
\[ C_{a+1,0}(\bar{u}^C; \bar{v}^C) T_{12}[0] = 0, \quad T_{12}[0] \mathbb{B}_{a,0}(\bar{u}^B; \bar{v}^B) = \lim_{w \to \infty} \frac{w}{c} \mathbb{B}_{a+1,0}(\{ w, \bar{u}^B \}; \bar{v}^B), \]

where we used the fact that both \( C_{a+1,0}(\bar{u}^C; \bar{v}^C) \) and \( \mathbb{B}_{a,0}(\bar{u}^B; \bar{v}^B) \) are on-shell. Due to the Bethe equations (2.9) we conclude that if \( \mathbb{B}_{a,0}(\bar{u}^B; \bar{v}^B) \) is on-shell, then \( \mathbb{B}_{a+1,0}(\{ w, \bar{u}^B \}; \bar{v}^B) \) is also on-shell at \( w \to \infty \). This is because \( r_1(w) \to 1 \) at \( w \to \infty \) according to expansion (3.11). Thus, we arrive at
\[ \mathcal{M}^{(1,2)}\left( m \left| \begin{array}{c} \bar{u}^C \\ \bar{v}^B \\ \bar{g}^C \\ \bar{g}^B \end{array} \right. \right)_{b,b}^{a+1,a} = \lim_{w \to \infty} \frac{w}{c} C_{a+1,0}(\bar{u}^C; \bar{v}^C) T_{22}^{(1)}[0] \mathbb{B}_{a+1,0}(\{ w, \bar{u}^B \}; \bar{v}^B), \]

and since both vectors in the r.h.s. of (5.8) are on-shell, we obtain
\[ \mathcal{M}^{(1,2)}\left( m \left| \begin{array}{c} \bar{u}^C \\ \bar{v}^B \\ \bar{g}^C \\ \bar{g}^B \end{array} \right. \right)_{b,b}^{a+1,a} = \lim_{w \to \infty} \frac{w}{c} \mathcal{M}^{(2,2)}\left( m \left| \begin{array}{c} \bar{u}^C \\ \bar{v}^B \\ \bar{g}^C \\ \bar{g}^B \end{array} \right. \right)_{b,b}^{a+1,a+1}. \]

Thus, knowing an explicit representation for the form factor \( \mathcal{M}^{(2,2)} \) we can find the form factor \( \mathcal{M}^{(1,2)} \) sending one of the Bethe parameters to infinity. Similarly all the other form factors of the off-diagonal partial zero modes \( T_{ij}^{(1)}[0] \) can be found.

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5.1. Calculation of the generating functional

The calculation of the generating functional (5.2) is straightforward. First of all, we use explicit expressions (3.8) and (3.9) for the total (dual) Bethe vectors in terms of the partial ones. This allows us to find the action of the operator \( \exp\{Q_\beta\} \) onto the Bethe vector \( \mathbb{B}_{a,b}(\bar{\alpha}_P; \bar{\beta}_P) \). After this we obtain a sum over partitions of the Bethe parameters involving two scalar products of the partial Bethe vectors. Using an explicit representation for the scalar product of generic Bethe vectors in the \( \mathfrak{g}(2|1) \)-based models [25] we find an explicit expression for the generating functional in terms of a sum over partitions. This sum can be further simplified leading eventually to the final result.

We describe here the first steps of this derivation. The most technical part is shifted to appendix A.

We start with equation (3.8) for the total Bethe vector. If this vector is on-shell, then we can present the product of functions \( r_1^{(2)}(\bar{\alpha}_P) \) as \( r_1^{(2)}(\bar{\alpha}_P) = r_1(\bar{\alpha}_P)\ell_1^{-1}(\bar{\alpha}_P) \) (see (3.7)) and express \( r_1(\bar{\alpha}_P) \) in terms of Bethe equations. Then we obtain

\[
\mathbb{B}_{a,b}(\bar{\alpha}_P; \bar{\beta}_P) = \sum \frac{\ell_2(\bar{\alpha}_P)}{\ell_1(\bar{\alpha}_P)} f(\bar{\alpha}_P, \bar{\alpha}_P) g(\bar{\alpha}_P, \bar{\alpha}_P) f(\bar{\alpha}_P, \bar{\alpha}_P) \mathbb{B}_{a,b}^{(2)}(\bar{\alpha}_P; \bar{\alpha}_P) \mathbb{B}_{a,b}^{(1)}(\bar{\alpha}_P; \bar{\alpha}_P). \tag{5.10}
\]

Similarly, if \( \mathbb{C}_{a,b}(\bar{\alpha}_P; \bar{\beta}_P) \) is a twisted on-shell Bethe vector, then we can present the product of functions \( r_3^{(2)}(\bar{\alpha}_P) \) as \( r_3(\bar{\alpha}_P)\ell_3^{-1}(\bar{\alpha}_P) \) and express \( r_3(\bar{\alpha}_P) \) in terms of the twisted Bethe equations. Then we have

\[
\mathbb{C}_{a,b}(\bar{\alpha}_P; \bar{\beta}_P) = \sum \frac{\kappa_2}{\kappa_3} f(\bar{\alpha}_P, \bar{\alpha}_P) g(\bar{\alpha}_P, \bar{\alpha}_P) f(\bar{\alpha}_P, \bar{\alpha}_P) \mathbb{C}_{a,b}^{(1)}(\bar{\alpha}_P; \bar{\alpha}_P) \mathbb{C}_{a,b}^{(2)}(\bar{\alpha}_P; \bar{\alpha}_P). \tag{5.11}
\]

Now we should compute the action of \( \exp\{Q_\beta\} \) onto the Bethe vector \( \mathbb{B}_{a,b}(\bar{\alpha}_P; \bar{\beta}_P) \). Obviously, the partial zero modes \( T_{\alpha}^{(1)}[0] \) act only on the partial Bethe vectors \( \mathbb{B}_{a,b}^{(1)} \) and do not act on \( \mathbb{B}_{a,b}^{(2)} \). The explicit action formulas are [26]:

\[
T_{11}^{(1)}[0] \mathbb{B}_{a,b}^{(1)}(\bar{\alpha}_P; \bar{\beta}_P) = (\lambda_{1}^{(1)}[0] - a_1) \mathbb{B}_{a,b}^{(1)}(\bar{\alpha}_P; \bar{\beta}_P),
\]

\[
T_{22}^{(1)}[0] \mathbb{B}_{a,b}^{(1)}(\bar{\alpha}_P; \bar{\beta}_P) = (\lambda_{2}^{(1)}[0] + a_2 - b_2) \mathbb{B}_{a,b}^{(1)}(\bar{\alpha}_P; \bar{\beta}_P),
\]

\[
T_{33}^{(1)}[0] \mathbb{B}_{a,b}^{(1)}(\bar{\alpha}_P; \bar{\beta}_P) = (\lambda_{3}^{(1)}[0] - b_1) \mathbb{B}_{a,b}^{(1)}(\bar{\alpha}_P; \bar{\beta}_P). \tag{5.12}
\]

Here \( \mathbb{B}_{a,b}^{(1)}(\bar{\alpha}_P; \bar{\beta}_P) \) is a generic partial Bethe vector. Thus, we find

\[
e^{Q_\beta} \mathbb{B}_{a,b}(\bar{\alpha}_P; \bar{\beta}_P) = \sum e^{Q_{\beta(a_1-a_2)} + b_1(b_2-b_3)} \frac{\ell_2(\bar{\alpha}_P)}{\ell_1(\bar{\alpha}_P)} f(\bar{\alpha}_P, \bar{\alpha}_P) g(\bar{\alpha}_P, \bar{\alpha}_P) f(\bar{\alpha}_P, \bar{\alpha}_P) \mathbb{B}_{a,b}^{(2)}(\bar{\alpha}_P; \bar{\alpha}_P) \mathbb{B}_{a,b}^{(1)}(\bar{\alpha}_P; \bar{\alpha}_P), \tag{5.13}
\]

where

\[
Q_\beta = \sum_{i=1}^{3} (-1)^{i} \beta_i \lambda_i^{(1)}[0]. \tag{5.14}
\]
is the eigenvalue of the operator $Q_\beta$ on the vector $|0\rangle$. Substituting (5.11) and (5.13) into (5.2) we arrive at

$$\mathcal{M}^{(\kappa)}\left( m \left| \frac{\tilde{u}^C}{\tilde{v}^C} \frac{\tilde{u}^B}{\tilde{v}^B} \right|_b \right) = \sum e^{Q_{\delta} + a_{(\beta, -\beta)}} \left( \frac{\ell_1(\tilde{u}^C_1)\ell_3(\tilde{v}^B_1)}{\ell_1(\tilde{u}^B_1)\ell_3(\tilde{v}^C_1)} \right) f(\tilde{u}^C_1, \tilde{u}^B_1) f(\tilde{v}^C_1, \tilde{v}^B_1) g(\tilde{v}^C_1, \tilde{v}^B_1) \times f(\tilde{v}^B_1, \tilde{u}^B_1) f(\tilde{v}^C_1, \tilde{u}^B_1) \mathcal{C}^{(1)}_{a_i, b_i}(\tilde{u}^B_1; \tilde{v}^B_1) \cdot \mathcal{C}^{(2)}_{a_i, b_i}(\tilde{u}^C_1; \tilde{v}^C_1) \mathcal{B}_{a_i, b_i}(\tilde{u}^B_1; \tilde{v}^B_1). \quad (5.15)$$

Thus, the problem of calculating the generating functional is reduced to the calculation of the scalar products and further summation over partitions. Further derivation is quite technical, therefore, we give the details in appendix A. We would like to mention only that this derivation goes along the same lines as in the $\mathfrak{g}(3)$-case [9] with minor modifications. Here we formulate the final result only.

**Proposition 5.1.** Let $\mathcal{B}_{a_i, b_i}(\tilde{u}^B; \tilde{v}^B)$ be an on-shell Bethe vector and $\mathcal{C}^{(\kappa)}_{a_i, b_i}(\tilde{u}^C; \tilde{v}^C)$ be a twisted dual on-shell Bethe vector.

$$\mathcal{M}^{(\kappa)}\left( m \left| \frac{\tilde{u}^C}{\tilde{v}^C} \frac{\tilde{u}^B}{\tilde{v}^B} \right|_b \right) = e^{Q_{\delta} + a_{(\beta, -\beta)}} \left( \frac{\ell_1(\tilde{u}^C)\ell_3(\tilde{v}^B)}{\ell_1(\tilde{u}^B)\ell_3(\tilde{v}^C)} \right) \mathcal{C}^{(\kappa)}_{a_i, b_i}(\tilde{u}^C; \tilde{v}^C) \mathcal{B}_{a_i, b_i}(\tilde{u}^B; \tilde{v}^B). \quad (5.16)$$

Suppose that $\{ \tilde{u}^C, \tilde{v}^C \} = \{ \tilde{u}^B, \tilde{v}^B \}$. Then, it follows immediately from (5.16) that

$$\frac{d}{d\beta_i} \mathcal{M}^{(\kappa)}\left( m \left| \frac{\tilde{u}^C}{\tilde{v}^C} \frac{\tilde{u}^B}{\tilde{v}^B} \right|_b \right) = \left( \frac{\ell_1(\tilde{u}^C)\ell_3(\tilde{v}^B)}{\ell_1(\tilde{u}^B)\ell_3(\tilde{v}^C)} - 1 \right) \tilde{\mathcal{C}}^{(i, j)}_{a_i, b_i}(\tilde{u}^C; \tilde{v}^C) \mathcal{B}_{a_i, b_i}(\tilde{u}^B; \tilde{v}^B), \quad (5.17)$$

Indeed, for $\tilde{\kappa} = 1$ the scalar product in (5.16) turns into the scalar product of two different on-shell Bethe vectors. Hence, it vanishes. Therefore, the $\beta_i$-derivative must act on this scalar product only, otherwise we obtain zero contribution. Using then (5.4) and (5.5) we immediately obtain

$$\mathcal{M}^{(i, j)}\left( m \left| \frac{\tilde{u}^C}{\tilde{v}^C} \frac{\tilde{u}^B}{\tilde{v}^B} \right|_b \right) = \left( \frac{\ell_1(\tilde{u}^C)\ell_3(\tilde{v}^B)}{\ell_1(\tilde{u}^B)\ell_3(\tilde{v}^C)} - 1 \right) \tilde{\mathcal{C}}^{(i, j)}_{a_i, b_i}(\tilde{u}^C; \tilde{v}^C) \mathcal{B}_{a_i, b_i}(\tilde{u}^B; \tilde{v}^B), \quad (5.18)$$

for $\{ \tilde{u}^C, \tilde{v}^C \} = \{ \tilde{u}^B, \tilde{v}^B \}$. Thus, we reproduce (4.1) for $i = j$. The form factors of the off-diagonal partial zero modes then can be derived via (3.17).

Consider finally the case $\{ \tilde{u}^C, \tilde{v}^C \} = \{ \tilde{u}^B, \tilde{v}^B \} = \{ \tilde{u}, \tilde{v} \}$. Then (5.16) yields

$$\frac{d}{d\beta_i} \mathcal{M}^{(\kappa)}\left( m \left| \frac{\tilde{u}}{\tilde{v}} \frac{\tilde{u}}{\tilde{v}} \right|_b \right) = \frac{d}{d\beta_i} \mathcal{C}^{(\kappa)}(\tilde{u}; \tilde{v}) \mathcal{B}_{\tilde{u}, \tilde{v}}(\tilde{u}; \tilde{v}) \bigg|_{\kappa=1} \left( -1 \right)^{|i|} \lambda_i^{(1)}[0] + \frac{d}{d\beta_i} \log \left( \frac{\ell_1(\tilde{u})}{\ell_3(\tilde{v})} \right) \bigg|_{\kappa=1} \right) \mathcal{B}_{\tilde{u}, \tilde{v}}(\tilde{u}; \tilde{v})|^2. \quad (5.19)
Comparing this equation with (5.3) we arrive at
\[
\mathcal{M}^{(i,j)}(m\left| \delta \bar{u}, \delta \bar{v}\right. \bigg)_{\bar{b},\bar{b}}^{\bar{a},\bar{a}} = \left( \frac{d}{d\beta_i} \right) \log \left( \frac{\mathcal{E}(\delta(\bar{\kappa}))}{\mathcal{E}(\delta(\bar{\kappa}))} \right)_{\kappa=1}^{(i,j)} \right|_{\bar{b}}^{(i,j)}\right|_{\bar{b}}^{(i,j)}.
\]
Finally, using
\[
\frac{d}{d\beta_i} \bigg|_{\kappa=1} = \frac{d}{d\log \kappa_i} \bigg|_{\kappa=1} = \frac{d}{d\kappa_i} \bigg|_{\kappa=1},
\]
we reproduce (4.2).

6. Conclusion

In this paper we have calculated FFLO in the integrable models with $\mathfrak{gl}(2|1)$-invariant \( R \)-matrix. We have shown that these form factors are proportional to the universal form factors. At the same time, FFLO dependence on the local site is given by a pre-factor. This pre-factor is nothing else but the result of the translation operator action onto the local operator, therefore, this simple dependence on the lattice site is very natural. On the other hand, the appearance of universal form factors in the final results has no simple explanation to date.

Our derivation is not based on a specific representation of the $RTT$-algebra, and thus, it is valid for a wide class of integrable models solvable by the algebraic Bethe ansatz (besides the t-J model see e.g. [38–41]). Due to an isomorphism between $Y(\mathfrak{gl}(2|1))$ and $Y(\mathfrak{gl}(1|2))$ these results also can be applied to the $\mathfrak{gl}(1|2)$-invariant models.

It is worth mentioning that for calculating the FFLO it was enough to find a special generating functional (5.2). All the form factors then can be found by taking $\kappa$-derivatives of this generating functional and sending some of the Bethe parameters to infinity. These close relations between different form factors occur due to the commutation relations between zero modes $T_{ij}[0]$ in the expansion (3.11). Generically, local $L$-operators not necessarily have the form (3.10), therefore, the monodromy matrix might have an asymptotic expansion different from (3.11). This may lead to a redefinition of the zero modes and changing of the commutation relations between them, which in turn yields a modification of the relations between the FFLO (see e.g. [33, 34]). Nevertheless, the latter still are proportional to the universal form factors. Therefore, the results of this paper for the most part apply to the models in which the monodromy matrix does not have the asymptotic expansion (3.11).

Our results are in complete analogue with the ones obtained in [9] for the models with the $\mathfrak{gl}(3)$-invariant $R$-matrix. Of course, these results are also applicable to the simpler case of $\mathfrak{gl}(2)$ based models. To approach the $\mathfrak{gl}(2)$ case, it is sufficient to set $\bar{\sigma} = \emptyset$ in the formulas for form factors. FFLO in the models in a finite volume with $\mathfrak{gl}(2)$ symmetry were calculated in earlier works (see, e.g. [11, 42]). In all these studies, the overall structure of the answers was the same. Therefore, one can conjecture that the relationship between the FFLO and the universal form factors remains true in the general $\mathfrak{gl}(m|n)$ case and, possibly, its $q$-deformation. However, compact representations for the universal form factors in the $\mathfrak{gl}(m|n)$-invariant models are not known for today. Similar problem

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also arises in the study of $q$-deformation of the higher rank algebras (see e.g. [43]). At the same time, in the models with $\mathfrak{gl}(2|1)$ or $\mathfrak{gl}(1|2)$ symmetries the universal form factors were calculated in [26] in terms of determinants. These representations allow one to use our results for studying correlation functions via the form factor expansion.

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Appendix A. Summation over partitions

A.1. Scalar products

In this section we give some results of the papers [24, 25]. The scalar product of generic Bethe vectors in $\mathfrak{gl}(2|1)$-based models is given by the formula

$$
\sum_{\lambda} r_1(\tilde{u}_I^B) r_1(\bar{u}_I^B) r_3(\tilde{v}_I^C) r_3(\bar{v}_I^C) f(\tilde{u}_I^C, \bar{u}_I^C) f(\tilde{v}_I^C, \bar{v}_I^C) 
\times g(\tilde{u}_I^{B'), \bar{u}_I^{B'}) g(\tilde{v}_I^{B'), \bar{v}_I^{B'}) f(\tilde{u}_I^{B'}, \bar{u}_I^{B'}) f(\tilde{v}_I^{B'}, \bar{v}_I^{B'}) 
\times Z_{\alpha, \beta}(\tilde{u}_I^{B'}, \bar{u}_I^{B'}; \tilde{v}_I^{B'}, \bar{v}_I^{B'}) Z_{\alpha, \beta}(\tilde{u}_I^{B'}, \bar{u}_I^{B'}; \tilde{v}_I^{B'}, \bar{v}_I^{B'}).
$$

(A.1)

Here the sum is taken over the partitions

$$
\tilde{u}^C \Rightarrow \{ \tilde{u}_I^C, \bar{u}_I^C \}, \quad \tilde{v}^C \Rightarrow \{ \tilde{v}_I^C, \bar{v}_I^C \},
\tilde{u}^B \Rightarrow \{ \tilde{u}_I^B, \bar{u}_I^B \}, \quad \tilde{v}^B \Rightarrow \{ \tilde{v}_I^B, \bar{v}_I^B \}.
$$

(A.2)

The partitions are independent except that $\# \tilde{u}_I^B = \# \tilde{u}_I^C = a_I$ with $a_I = 0, \ldots, a$, and $\# \tilde{v}_I^B = \# \tilde{v}_I^C = b_I$ with $b_I = 0, \ldots, b$.

The rational functions $Z_{\alpha, \beta}(\tilde{u}_I^B, \bar{u}_I^B; \tilde{v}_I^C, \bar{v}_I^C)$ and $Z_{\alpha, \beta}(\tilde{u}_I^B, \bar{u}_I^B; \tilde{v}_I^B, \bar{v}_I^B)$ are so called highest coefficients. Explicit determinant formulas for them can be found in [25]. We do not use these explicit presentations in our calculations.

Equation (A.1) holds for arbitrary Bethe vectors. In other words, we do not assume any constraint between functional parameters $r_k$ and complex variables $\tilde{u}^{C,B}$ and $\tilde{v}^{C,B}$. However, one can consider particular cases of (A.1), where certain constraints are imposed. One of these particular cases is the scalar product of the twisted on-shell and usual on-shell Bethe vectors. Then the functional parameters $r_k$ can be expressed in terms $\tilde{u}^{C,B}$ and $\tilde{v}^{C,B}$ via (twisted) Bethe equations (2.9) and (2.13). Denoting this scalar product by $S^{(e)}_{\alpha, \beta}$ we obtain...
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\[ S_{a,b}^{(k)} = \sum_{\kappa_1} \left( \frac{\kappa_2}{\kappa_1} \right)^a \left( \frac{\kappa_2}{\kappa_3} \right)^b f(a_{\Pi}^C, \bar{a}_{\Pi}^C) f(a_{\Pi}^B, \bar{a}_{\Pi}^B) g(\bar{v}_{\Pi}^C, \bar{v}_{\Pi}^C) g(\bar{v}_{\Pi}^B, \bar{v}_{\Pi}^B) \times f(\bar{v}_{\Pi}^C, \bar{v}_{\Pi}^C) f(\bar{v}_{\Pi}^B, \bar{v}_{\Pi}^B) Z_{ab, b}(a_{\Pi}^C, \bar{a}_{\Pi}^C, \bar{v}_{\Pi}^C, \bar{v}_{\Pi}^C) Z_{ab, b}(a_{\Pi}^B, \bar{a}_{\Pi}^B, \bar{v}_{\Pi}^B, \bar{v}_{\Pi}^B). \] (A.3)

**Remark 1.** We would like to draw attention of the reader that here the parameters \( \bar{a}_{C, B} \) and \( \bar{v}_{C, B} \) still are arbitrary complex numbers, in spite of we used (twisted) Bethe equations to obtain (A.3). The matter is that the functions \( r_k \) are free functional parameters. Therefore, the (twisted) Bethe equations give the constraints for these functional parameters, but not for the Bethe parameters \( \bar{a}_{C, B} \) and \( \bar{v}_{C, B} \).

Setting \( \kappa = 1 \) we obtain the scalar product of two on-shell Bethe vectors, which vanishes for \( a + b > 0 \). Hence,

\[ \delta_{a+b,0} = \sum \frac{f(a_{\Pi}^C, \bar{a}_{\Pi}^C) f(a_{\Pi}^B, \bar{a}_{\Pi}^B) g(\bar{v}_{\Pi}^C, \bar{v}_{\Pi}^C) g(\bar{v}_{\Pi}^B, \bar{v}_{\Pi}^B) \times f(\bar{v}_{\Pi}^C, \bar{v}_{\Pi}^C) f(\bar{v}_{\Pi}^B, \bar{v}_{\Pi}^B) Z_{ab, b}(a_{\Pi}^C, \bar{a}_{\Pi}^C, \bar{v}_{\Pi}^C, \bar{v}_{\Pi}^C) Z_{ab, b}(a_{\Pi}^B, \bar{a}_{\Pi}^B, \bar{v}_{\Pi}^B, \bar{v}_{\Pi}^B).}{(A.4)} \]

**Remark 2.** We stress that (A.4) is an identity. It is clear that it holds if \( \{ \bar{a}^C, \bar{v}^C \} = \{ \bar{a}^B, \bar{v}^B \} \), because in this case the r.h.s. of (A.4) is the scalar product of two different on-shell vectors. However, even in the case \( \bar{a}^C = \bar{a}^B \) and \( \bar{v}^C = \bar{v}^B \) the equation (A.4) is still valid. This is because the equation (A.4) does not describe the norm of the on-shell vector at \( \bar{a}^C = \bar{a}^B \) and \( \bar{v}^C = \bar{v}^B \). In order to obtain the norm we first had to take the limit \( \bar{a}^C = \bar{a}^B \) and \( \bar{v}^C = \bar{v}^B \) in (A.1) and only then use Bethe equations. The way that we have used was opposite, therefore the r.h.s. of (A.4) is identically zero for \( a + b > 0 \).

The general formula (A.1) also can be applied for the scalar products of the partial Bethe vectors \( \bar{C}_{a/b}(\bar{a}^C; \bar{v}^C)\bar{B}_{a/b}(\bar{a}^B; \bar{v}^B), l = 1, 2 \). Then one should simply replace the functions \( r_k \) with the functions \( r_k^{(l)} \).

**A.2. Calculating the sum over partitions**

We begin with equation (5.15). We should substitute the formulas for the scalar products of generic Bethe vectors (A.1) into the r.h.s. of this equation. Recall that we denote \( r_k^{(1)} = \ell_k \), and thus, \( r_k^{(2)} = r_k^{(1)} \). It is clear that in the end each set of the Bethe parameters will be divided into four subsets. To avoid the cumbersome roman numbers, we use arabic subscripts to denote these subsets. Namely, we assume that

\[ \bar{a}_{C, B}^{(1)} \Rightarrow \{ \bar{a}^C, B_1, \bar{a}^C, B_3 \}, \quad \bar{a}_{C, B}^{(2)} \Rightarrow \{ \bar{a}^C, B_2, \bar{a}^C, B_4 \}, \quad \bar{v}_{C, B}^{(1)} \Rightarrow \{ \bar{v}^C, B_1, \bar{v}^C, B_3 \}, \quad \bar{v}_{C, B}^{(2)} \Rightarrow \{ \bar{v}^C, B_2, \bar{v}^C, B_4 \}. \] (A.5)

The cardinalities of the subsubsets are \( a_n = \# \bar{a}_{C, B}^{(n)}, \quad b_n = \# \bar{v}_{C, B}^{(n)}, \quad n = 1, 2, 3, 4 \). In particular, \( a_1 = a_2 + a_3 \) and \( b_1 = b_2 + b_3 \).

We have

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\[ \mathcal{M}^{(\alpha)} \left( m \left| \tilde{u}^C \quad \tilde{u}^B \right. \quad \tilde{v}^C \quad \tilde{v}^B \right)_b^a = \sum e^{Q_j + (\beta_2 - \beta_1) a} \frac{\ell_{1}(\tilde{u}_1^C) \ell_{3}(\tilde{v}_1^B)}{\ell_{1}(\tilde{u}_1^B) \ell_{3}(\tilde{v}_1^C)} \frac{f(\tilde{u}_1^B, \tilde{u}_1^C) g(\tilde{v}_1^C, \tilde{v}_1^B)}{f(\tilde{v}_1^B, \tilde{v}_1^C) g(\tilde{u}_1^C, \tilde{u}_1^B)} \times \frac{r_1(\tilde{u}_2^B) r_1(\tilde{u}_3^B) r_2(\tilde{v}_2^B) r_2(\tilde{v}_3^B)}{f(\tilde{u}_1^B, \tilde{u}_1^C) f(\tilde{v}_1^C, \tilde{v}_1^B)} \frac{f(\tilde{u}_2^B, \tilde{u}_3^B) g(\tilde{v}_2^B, \tilde{v}_3^B)}{f(\tilde{v}_2^B, \tilde{v}_3^B) g(\tilde{u}_2^B, \tilde{u}_3^B)} \frac{Z_{a_2 b_2}(\tilde{u}_2^B; \tilde{v}_2^B)}{Z_{a_3 b_3}(\tilde{u}_3^B; \tilde{v}_3^B)} Z_{a_2 b_3}(\tilde{u}_2^B; \tilde{v}_3^B) \quad (A.6) \]

Now we should express the products of the functions \( r_i \) in (A.6) via (twisted) Bethe equations. We have

\[ r_1(\tilde{u}_2^B) = \frac{f(\tilde{u}_2^B; \tilde{u}_3^B) f(\tilde{u}_2^B; \tilde{u}_3^B) f(\tilde{u}_2^B; \tilde{u}_3^B)}{f(\tilde{u}_1^B; \tilde{u}_2^B) f(\tilde{u}_3^B; \tilde{u}_2^B) f(\tilde{u}_3^B; \tilde{u}_2^B)} \frac{f(\tilde{u}_3^B; \tilde{u}_2^B)}{f(\tilde{u}_2^B; \tilde{u}_3^B)} \quad (A.7) \]

\[ r_1(\tilde{u}_4^C) = e^{a_2 - \beta_1} \frac{f(\tilde{u}_4^C; \tilde{u}_3^B) f(\tilde{u}_3^B; \tilde{u}_4^C)}{f(\tilde{u}_4^C; \tilde{u}_3^B) f(\tilde{u}_3^B; \tilde{u}_4^C)} \frac{f(\tilde{u}_3^B; \tilde{u}_4^C)}{f(\tilde{u}_4^C; \tilde{u}_3^B)} \quad (A.8) \]

\[ r_2(\tilde{v}_2^B) = f(\tilde{v}_2^B, \tilde{v}_3^B) \quad (A.9) \]

\[ r_3(\tilde{v}_3^C) = e^{b_2 - \beta_1} \frac{f(\tilde{v}_3^C, \tilde{v}_2^B)}{f(\tilde{v}_3^C, \tilde{v}_2^B)} \quad (A.10) \]

All these expressions should be substituted into (A.6). We also should write the products over subsets I and II in terms of the products over subsubsets (A.5). Then we obtain

\[ \mathcal{M}^{(\alpha)} \left( m \left| \tilde{u}^C \quad \tilde{u}^B \quad \tilde{v}^C \quad \tilde{v}^B \right. \right)_b^a = \sum e^{Q_j + (\beta_2 - \beta_1) (a - a_2) + (\beta_2 - \beta_1) b_1} \frac{\ell_{1}(\tilde{u}_1^C) \ell_{3}(\tilde{v}_1^B)}{\ell_{1}(\tilde{u}_1^B) \ell_{3}(\tilde{v}_1^C)} \times \frac{\ell_{1}(\tilde{u}_2^B) \ell_{3}(\tilde{v}_2^B)}{\ell_{1}(\tilde{u}_2^B) \ell_{3}(\tilde{v}_2^B)} \frac{Z_{a_2 b_2}(\tilde{u}_2^B; \tilde{v}_2^B)}{Z_{a_3 b_3}(\tilde{u}_3^B; \tilde{v}_3^B)} \quad (A.11) \]

Here

\[ Z = Z_{a_2 b_2}(\tilde{u}_2^B; \tilde{v}_2^B) Z_{a_3 b_3}(\tilde{u}_3^B; \tilde{v}_3^B) \quad (A.12) \]

\[ F_{uu}^C = f(\tilde{u}_1^C, \tilde{u}_1^C) f(\tilde{u}_2^C, \tilde{u}_2^C) f(\tilde{u}_1^C, \tilde{u}_2^C) f(\tilde{u}_1^C, \tilde{u}_3^C) f(\tilde{u}_1^C, \tilde{u}_3^C) \quad (A.13) \]

\[ F_{uu}^B = f(\tilde{u}_1^B, \tilde{u}_1^B) f(\tilde{u}_2^B, \tilde{u}_2^B) f(\tilde{u}_1^B, \tilde{u}_2^B) f(\tilde{u}_1^B, \tilde{u}_3^B) f(\tilde{u}_1^B, \tilde{u}_3^B) \quad (A.14) \]

\[ F_{vv}^C = g(\tilde{v}_1^C, \tilde{v}_1^C) g(\tilde{v}_2^C, \tilde{v}_2^C) g(\tilde{v}_1^C, \tilde{v}_2^C) g(\tilde{v}_1^C, \tilde{v}_3^C) g(\tilde{v}_1^C, \tilde{v}_3^C) \quad (A.15) \]

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\[ F^{B}_{vv} = g(\epsilon^B_1, \bar{\epsilon}_4^B)g(\bar{\epsilon}_1^B, \bar{\epsilon}_4^B)g(\bar{\epsilon}_2^B, \bar{\epsilon}_3^B)g(\bar{\epsilon}_3^B, \bar{\epsilon}_2^B), \]  
(A.16)

\[ F^{C}_{vv} = f(\bar{\epsilon}^C_1, \bar{\epsilon}^C_4)f(\bar{\epsilon}^C_1, \bar{\epsilon}^C_4)f(\bar{\epsilon}^C_1, \bar{\epsilon}^C_4)f(\bar{\epsilon}^C_1, \bar{\epsilon}^C_4), \]  
(A.17)

\[ F^{B}_{vu} = f(\bar{\epsilon}^B_3, \bar{\epsilon}^B_4)f(\bar{\epsilon}^B_3, \bar{\epsilon}^B_4)f(\bar{\epsilon}^B_1, \bar{\epsilon}^B_2)f(\bar{\epsilon}^B_1, \bar{\epsilon}^B_2). \]  
(A.18)

Actually it remains to combine different factors in (A.11) together. First of all we combine subsubsets into new groups:

\[
\{ \bar{a}^C_{1,B}, \bar{a}^C_{4,B} \} = \bar{a}^C_{1,B},
\{ \bar{a}^C_{2,B}, \bar{a}^C_{3,B} \} = \bar{a}^C_{3,B},
\{ \bar{v}^C_{1,B}, \bar{v}^C_{4,B} \} = \bar{v}^C_{1,B},
\{ \bar{v}^C_{2,B}, \bar{v}^C_{3,B} \} = \bar{v}^C_{3,B}.
\]  
(A.19)

Then

\[ F^{C}_{uu} = f(\bar{u}^C_1, \bar{u}^C_1)f(\bar{u}^C_3, \bar{u}^C_2)f(\bar{u}^C_1, \bar{u}^C_1), \]  
(A.20)

\[ F^{B}_{uu} = f(\bar{a}^B_1, \bar{a}^B_1)f(\bar{a}^B_3, \bar{a}^B_2)f(\bar{a}^B_1, \bar{a}^B_1), \]  
(A.21)

\[ F^{C}_{vv} = (-1)^{h_{ba}}g(\bar{v}^C_1, \bar{v}^C_1)g(\bar{v}^C_2, \bar{v}^C_2)g(\bar{v}^C_1, \bar{v}^C_1), \]  
(A.22)

\[ F^{B}_{vv} = (-1)^{h_{ba}}g(\bar{v}^B_1, \bar{v}^B_1)g(\bar{v}^B_2, \bar{v}^B_2)g(\bar{v}^B_1, \bar{v}^B_1), \]  
(A.23)

\[ F^{C}_{vu} = f(\bar{v}^C_1, \bar{v}^C_4)f(\bar{v}^C_1, \bar{v}^C_1), \]  
(A.24)

\[ F^{B}_{vu} = f(\bar{v}^B_1, \bar{v}^B_2)f(\bar{v}^B_1, \bar{v}^B_1), \]  
(A.25)

Equation (A.11) then takes the form

\[
\mathcal{M}^{(\nu)}(m) \sum_{\nu} \frac{\ell(\bar{u}^C_b)\ell(\bar{v}^B_a)}{\ell(\bar{u}^C_b)\ell(\bar{v}^B_a)} G_1(\bar{u}^C_{a,B}, \bar{v}^C_{b,B}) G_2(\bar{u}^C_{a,B}, \bar{v}^C_{b,B}) 
\times f(\bar{u}^C_1, \bar{u}^C_1)f(\bar{u}^C_3, \bar{u}^C_2)f(\bar{v}^C_1, \bar{v}^C_1)f(\bar{v}^C_1, \bar{v}^C_1)f(\bar{v}^C_1, \bar{v}^C_1)f(\bar{v}^C_1, \bar{v}^C_1), \]  
(A.26)

where the functions \( G_1 \) and \( G_2 \) in their turn are given as sums over partitions:

\[
G_1(\bar{u}^C_{a,B}, \bar{v}^C_{b,B}) = \sum_{\bar{u}^C_{a,B} \rightarrow \{ \bar{u}^C_{1,B}, \bar{u}^C_{2,B} \}} \sum_{\bar{v}^C_{b,B} \rightarrow \{ \bar{v}^C_{1,B}, \bar{v}^C_{2,B} \}} f(\bar{u}^C_1, \bar{u}^C_1)f(\bar{u}^C_3, \bar{u}^C_2)f(\bar{v}^C_1, \bar{v}^C_1)f(\bar{v}^C_1, \bar{v}^C_1)f(\bar{v}^C_1, \bar{v}^C_1)f(\bar{v}^C_1, \bar{v}^C_1), \]  
(A.27)

and

\[
G_2(\bar{u}^C_{a,B}, \bar{v}^C_{b,B}) = \sum_{\bar{u}^C_{a,B} \rightarrow \{ \bar{u}^C_{1,B}, \bar{u}^C_{2,B} \}} \sum_{\bar{v}^C_{b,B} \rightarrow \{ \bar{v}^C_{1,B}, \bar{v}^C_{2,B} \}} \times f(\bar{v}^C_1, \bar{v}^C_1)f(\bar{v}^C_1, \bar{v}^C_1)f(\bar{v}^C_1, \bar{v}^C_1)f(\bar{v}^C_1, \bar{v}^C_1)f(\bar{v}^C_1, \bar{v}^C_1)f(\bar{v}^C_1, \bar{v}^C_1), \]  
(A.28)

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Comparing the function $G_1$ (A.27) with (A.4) we see that $G_1 = 0$ unless $\vec{a}_i^{C,B} = \emptyset$ and $\vec{v}_i^{C,B} = \emptyset$. Hence, $\vec{u}_i^{C,B} = \vec{u}_i^{C,B}$, $\vec{v}_i^{C,B} = \vec{v}_i^{C,B}$, $a_1 = a_4 = 0$, $b_2 = b_3 = 0$.

Looking now at $G_2$ (A.28) and comparing it with (A.3) we see that

$$G_2 = e^{Q_i} S_{a,b}^{(\kappa)}.$$  \hspace{2cm} (A.29)

Substituting this into (A.26) and setting there $\vec{a}_i^{C,B} = \emptyset$ and $\vec{v}_i^{C,B} = \emptyset$ we immediately arrive at

$$M^{(\kappa)} \left( m_1^{\vec{u}^{C}_i, \vec{u}^{B}_i} \right) = e^{Q_i} \delta_i(\vec{v}^{C}_i) \delta_i(\vec{v}^{B}_i) S_{a,b}^{(\kappa)}.$$  \hspace{2cm} (A.30)

and thus, proposition 5.1 is proved.

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