Crab nebula gamma-ray flares as relativistic reconnection minijets

E. Clausen-Brown* and M. Lyutikov*

Department of Physics, Purdue University, 525 Northwestern Avenue, West Lafayette, IN 47907-2036, USA

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ABSTRACT
The unusually short durations, high luminosities and high photon energies of the Crab nebula gamma-ray flares require the relativistic bulk motion of the emitting plasma. We explain the Crab flares as a result of randomly oriented relativistic ‘minijets’ originating from reconnection events in a magnetically dominated plasma. We develop a statistical model of the emission from Doppler boosted reconnection minijets and find analytical expressions for the moments of the resulting nebula light curve (e.g. time average, variance, skewness). The light curve has a flat power spectrum that transitions at short time-scales to a decreasing power law of index 2. The flux distribution from minijets follows a decreasing power law of index ∼1, implying that the average flux from flares is dominated by bright rare events. The predictions for the flares’ statistics can be tested against forthcoming observations. We find that the observed flare spectral energy distributions (SEDs) have several notable features: a hard power-law index of p ≲ 1 for accelerated particles that is expected in various reconnection models, including some evidence of a pile-up near the radiation reaction limit. Also, the photon energy at which the SED peaks is higher than that implied by the synchrotron radiation reaction limit, indicating that the flare emission regions’ Doppler factors are ≳ a few. We conclude that magnetic reconnection can be an important, if not dominant, mechanism of particle acceleration within the nebula.

Key words: magnetic reconnection – MHD – radiation mechanisms: non-thermal – pulsars: general – ISM: individual objects: Crab nebula – ISM: jets and outflows.

1 INTRODUCTION
The constancy of the high-energy Crab nebula emission has been surprisingly shown to be false by multiple day- to week-long flares, presenting a challenge to standard pulsar wind models (Kennel & Coroniti 1984). During these events, the Crab nebula gamma-ray flux above 100 MeV exceeded its average value by a factor of several or higher (Abdo et al. 2011; Buehler et al. 2012; Striani et al. 2011; Tavani et al. 2011), while in other energy bands nothing unusual was observed (e.g. Abdo et al. 2011; Tavani et al. 2011, and references therein). Additionally, subflare variability time-scales of ∼10 h have been observed (Balbo et al. 2011; Buehler et al. 2012).

There are two interesting observational facts related to the gamma-ray flares. The first one is their unusually short duration of a few days. This time-scale, on one hand, is millions of times longer than the period of the pulsar, yet on the other hand it is hundreds of times shorter than the nebula’s dynamical time of approximately a few years. We consider it unlikely that the flare is related to the changing plasma properties within the pulsar’s light cylinder, both due to the extremely large separation of temporal scales and due to the fact that no changes in the radio pulsar timing properties were seen during the flare (Espinoza et al. 2010). Thus, we associate the duration of the flare with the stochastically changing properties of plasma within the nebula. Secondly, the flaring behaviour consists of apparently isolated, intermittent events that are dominated by bright rare flares.

Two contrasting models of these flares can be envisioned. First, a flare can be due to large-scale changes in the steady nebula flow, amplified by the effects of relativistic beaming. Operating within this model, Komissarov & Lyutikov (2011) and Lyutikov, Balsara & Matthews (2011) place the flaring location in the downstream region of an oblique shock. The post-shock flow of an oblique shock can be highly relativistic, and thus Doppler boosted, with a bulk Lorentz factor of Γ ∼ φ−1, where φ is the angle between the upstream velocity and the shock plane. The short time-scale can be attributed to the shock normal changing direction (perhaps due to sausage or kink instabilities, or a corrugation perturbation), causing the post-shock velocity to sweep across the line of sight and create a short flare in Doppler boosted emission (the lighthouse effect).

Secondly, the flare can be due to a highly localized emission region, or blob, so that the flare observables determine the intrinsic properties of the emission region. The natural flaring mechanism in this category is relativistic magnetic reconnection – the

*E-mail: browner@purdue.edu (EC-B); lyutikov@purdue.edu (ML)
focus of this work – which has been invoked by Crab nebula flare models (Bednarz & Idec 2011; Uzdensky, Cerutti & Begelman 2011; Cerutti, Uzdensky & Begelman 2012) and fast flaring models in gamma-ray bursts (GRBs) and active galactic nuclei (AGNs; Lyutikov & Blackman 2001; Lyutikov & Blandford 2003; Lyutikov 2006; Giannios, Uzdensky & Begelman 2009, 2010; McKinney & Uzdensky 2012).

To see why reconnection is a natural flaring mechanism in pulsar wind nebulae (PWN), consider that the energy budget of the pulsar wind is set by the spin-down power, \( P_{\text{spin}} \), of the pulsar. In the standard model, \( P_{\text{spin}} \) is a smooth, monotonically decreasing function of time, implying that the nebula emission will track this smooth decline. In contrast, the emission from relativistic outflows in systems such as GRBs and AGNs are probably related to an irregular accretion process. Their variable accretion rates and other accretion disc instabilities may produce unsteady outflows with internal shocks, providing a viable alternative to reconnection as a flaring mechanism in the outflow (e.g. Rees & Meszaros 1994; Spada et al. 2001). In pulsar outflows, however, the internal shock scenario is more difficult to realize given the regular behaviour of \( P_{\text{spin}} \). Instead, it is more likely that day-scale flaring is due to the build-up and eventual release of free energy somewhere in the PWN.\(^1\) Magnetic reconnection is a natural candidate for this process as it is the primary mechanism by which magnetic free energy may be suddenly converted into particle kinetic energy, which occurs via the topological rearrangement of magnetic field lines. Furthermore, in the closest space laboratory available, the Solar system, this is precisely what is observed, where intermittent flaring and the production of non-thermal particles are regularly associated with magnetic reconnection (Priest & Forbes 2000).

In this work, we locate the reconnecting plasma at multiple sites in the nebula and presume that the reconnection plasma’s Alfvén velocity approaches the speed of light. The reconnection outflow speeds are then relativistic (Lyutikov & Uzdensky 2003; Lyubarsky 2005) and behave as ‘mini-jets’, a model that has been used to overcome the gamma-ray opacity problem in the context of GRBs and AGNs (Lyutikov & Blandford 2003; Lyutikov 2006; Giannios et al. 2009). In an approach similar to ours, Yuan et al. (2011) invoke a series of relativistically moving ‘knots’ (‘mini-jets’ in our terminology), with statistical properties that are compared to the Crab nebula light curve via Monte Carlo simulations (they do not identify the knots with reconnection). Our model differs from that of Yuan et al. (2011) in that it is analytical, and it presumes that relativistic beaming controls the observed statistical properties of the reconnection mini-jets.

This paper is organized as follows. In Section 2, we first focus on constraints to the emission regions’ magnetic field derived from adiabatic expansion and synchrotron cooling, and compare these to Crab nebula magnetic field estimates based on equipartition and spectral modelling. We include the possibility that the emission region has a bulk relativistic velocity with a Lorentz factor of \( \Gamma \) and the associated Doppler factor of \( \delta = (\Gamma - \sqrt{\Gamma^2 - 1} \cos \theta_{SR})^{-1} \), where \( \theta_{SR} \) is the angle between the blob’s velocity and the line of sight. Due to the high luminosity and short variability of the flare we assume \( \theta_{SR} \approx 0 \), in which case \( \delta \approx 2 \). We then speculate in Section 3 that the Crab nebula flare originates from a relativistic outflow caused by magnetic reconnection in a high-\( \sigma \) plasma (\( \sigma = \text{Poynting flux/particle flux}; \) Kennel & Coroniti 1984). Section 4 focuses on the emission region spectral energy distribution (SED) in the context of the synchrotron radiation reaction cut-off and the functional form of the observed SEDs. In Section 5, we construct an analytical statistical model which describes the statistics of both individual flares and the entire light curve’s statistical moments (e.g. average and variance) and its power spectrum. We provide a summary of our conclusions in Section 6.

2 EMISSION REGION PARAMETER ESTIMATES

In this section, the magnetic field of the emission region, or blob, is constrained under two separate assumptions: (i) adiabatic cooling or (ii) synchrotron cooling controls the flare duration. The relevant observed flare parameters are the flare duration, \( \tau = 10^{5-5} \text{yr} \), the typical photon energy, \( \epsilon_p = 100 \epsilon_{100} \text{MeV} \), the blob’s isotropic equivalent luminosity, \( L_{\text{iso}} = \frac{10^{36}}{\Omega_{\text{obs}}} \text{ergs s}^{-1} \), and the blob’s magnetic field, \( B = 0.3 \text{B}_{\text{eq}} \text{G} \), where the fiducial flare parameter values are based on the Fermi Large Area Telescope (LAT) data of the 2010 September flare (Abdo et al. 2011) and the nebula equipartition magnetic field (Trimble 1983).

Assuming that adiabatic cooling limits the flare duration, a lower limit on the magnetic field in the blob frame can be made by examining the energy content of the blob. We parametrize the total magnetic energy of the blob as being some significant fraction \( f_M \) of the blob’s total energy content, \( f_M \epsilon_b = B^2 R^3/6 \). Assuming that the blob is causally connected and is moving directly along the line of sight, the blob’s radius is \( R \lesssim c \tau \). Of the total energy content of the blob, \( E = f_M \Gamma^2 B^3 R^3/6 \), if a fraction \( f_{\text{rad}} \) is radiated away during the flare, then the total radiated power is \( L = f_{\text{rad}} E/\tau \). \( L \) is related to the isotropic equivalent luminosity, \( L_{\text{iso}} \), by assuming that all of \( L \) is beamed into a solid angle of \( \pi \theta^2 \), which leads to the relation \( L_{\text{iso}} = 4 \Gamma^2 L \). Thus, the isotropic equivalent luminosity can be expressed as

\[
L_{\text{iso}} = \frac{2}{3} f_{\text{rad}} f_M \Gamma^4 B^5 \tau^{-1} \Gamma^{-3}.
\]

In order to use equation (1) to constrain the magnetic field, \( R \) is estimated, as before, using the light crossing time \( \tau > R/\Gamma c \). This constraint may be interpreted as the blob’s slow down time (see e.g. Giannios et al. 2009) or as the adiabatic cooling time. That is, if the blob begins with an initial radius \( R' \) and expands at near light speed, \( R' = R + c \tau \), then the blob’s light crossing time, \( R'/c \), is the time the blob takes to double in radius and adiabatically cool by a significant amount. Solving equation (1) for \( R' \) and using the inequality \( R' \leq c \Gamma \) allows us to set a lower limit on \( B' \):

\[
B' > \frac{1.5(f_M f_{\text{rad}})^{1/2}}{L_{\text{iso}}^{1/2} \tau^{1/2} \Gamma^{-3}} \text{ mG}.
\]

If we assume that 10 per cent of the blob’s energy content is radiated away (\( f_{\text{rad}} = 0.1 \)) and the blob is magnetically dominated (\( f_M \sim 1 \)), then \( B' > 4.7 \text{ mG} \). Typical PWN models suggest that the pulsar wind begins magnetically dominated and somehow transitions to being cold and particle dominated near the termination shock so that the magnetization parameter is \( \sigma = B^2/(8\pi \rho c^2) \approx 3 \times 10^{-3} \), where \( \rho \) is the mass density (Kennel & Coroniti 1984). However, exactly how \( \sigma \) evolves from \( \sigma \gg 1 \) near the pulsar to \( \sigma \ll 1 \) at

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\(^1\) The wisps in the nebula that exhibit variability time-scales of approximately months are not due to an unsteady wind, but are probably related to the ion-affected structure of the wind termination shock (Gallant & Arons 1994) or a synchrotron instability (Hester et al. 2002). However, the recent discovery of the several year ~10 per cent decrease in the nebula X-ray emission (Wilson-Hodge et al. 2011) is puzzling in light of the Crab’s smooth spin-down rate, but the unknown mechanism for this emission decrease is probably unrelated to day-long gamma-ray flares, since its time-scale is several orders of magnitude longer.
the termination shock is difficult to explain, provoking competing models that suggest \( \sigma \gtrsim 1 \) at the shock (Begelman 1998; Lyutikov 2010). Thus, near the inferred location of the reverse shock, the blob could be magnetically dominated so that \( f_M \sim 1 \) [in a cold plasma, \( f_M = \sigma/(1 + \sigma) \)] and \( B' \gg \) the standard nebula value \( (0.3 \text{ mG}) \), or in the canonical model where the plasma is not magnetically dominated, \( f_M \) may be small and \( B' \) within the measured range of \(~0.1 \text{ mG}\).

The synchrotron radiative cooling time-scale sets a more robust lower limit on \( B' \) because it avoids the uncertainty in \( f_{\text{rad}} \) and \( f_M \) that appear in equation (2). The synchrotron cooling time, \( \tau_c = 5 \times 10^5 (2\gamma B')^{-1} \text{s} \), can be found given the typical synchrotron photon energy of \( \epsilon_\gamma = 21\gamma^2 heB'/(m_c) \), where \( \gamma \) is the lepton Lorentz factor, \( h \) is the Planck constant, \( e \) is the elementary charge and \( m_c \) is the lepton mass. These two expressions combinedly give a synchrotron cooling time of

\[
\tau_c = 21e^{-1/2} B_{\text{eq}}^{-3/2} \Gamma^{-1/2} \text{d}.
\] (3)

If a significant fraction of the blob’s energy is dissipated via synchrotron radiation in the flare, then \( \tau \gtrsim \tau_c \) and a lower limit can be set on the magnetic field in the blob frame:

\[
B' > 0.97\epsilon_\gamma^{-1/3} \tau_c^{-2/3} \Gamma^{-1/3} \text{mG}.
\] (4)

Note that this lower limit (which is similar to that found in Bednarek & Idec 2011) exceeds the observed values of the nebular magnetic field \((0.1–0.3 \text{ mG})\), unless relativistic motion is invoked.

The upshot of these parameter estimates is that the magnetic field of the emission region appears to be significantly higher than both nebula equipartition field estimates \((0.3 \text{ mG}; \text{Trimble 1983})\) and measurements based on modelling of the nebula SED \((0.1 \text{ mG}; \text{Abdo et al. 2010})\). This suggests that the flaring either occurred in the region of high magnetic field in the nebula or in the emission region that is moving towards Earth with \( \Gamma \gtrsim 1 \) a few, which reduces the above magnetic field estimates.

### 3 MAGNETIC RECONNECTION

Magnetic reconnection provides a natural explanation of the implied relativistic motion discussed above, the intrinsic short time-scales and the flares’ intermittency. Reconnection is a process in which the magnetic energy of a localized region, a current sheet, is suddenly converted to random particle energy and bulk relativistic motion (for studies on relativistic reconnection, see Blackman & Field 1994; Lyutikov & Uzdensky 2003; Luyubarsky 2005; Uzdensky 2011).

With regard to PWNe, reconnection has already been studied as a possible resolution of the well-known \( \sigma \) problem (Coroniti 1990; Lyubarsky & Kirk 2001). In the canonical PWN model (Rees & Gunn 1974; Kennel & Coroniti 1984), the magnetization parameter \( \sigma \) is high only for radii that are well within the wind termination shock, \( r_s \), and reduces to \( \sigma \sim 10^{-3} \)–\( 10^{-2} \) near \( r_s \). To explain how \( \sigma \) is reduced so drastically as the plasma propagates out to \( r_s \), reconnection in a striped wind has been invoked as the mechanism by which magnetic energy is transferred to particle energy, thereby reducing \( \sigma \). In a challenge to the canonical PWN model, Begelman (1998) argues that the toroidally dominated large-scale nebular magnetic field is subject to the \( \sigma = 1 \) kink mode instability near the inferred location of \( r_s \), causing the nebular field to have coherence lengths of the order of \( r_s \) instead of the size of the radio nebula as it is in canonical models. This obviates the need for such low \( \sigma \) at \( r_s \) and may cause reconnection throughout the nebula. In a similar vein, Lyutikov (2010) proposes a model in which reconnection occurs primarily along the rotation axis and equatorial region well beyond the light cylinder, thus qualitatively reproducing the jet/equatorial wisp morphology of the nebula.

In our simple model, we propose that multiple reconnection sites are located in a region of the inner nebula with a magnetization parameter \( \sigma \) of the order of approximately a few, thereby accelerating mildly relativistic outflows (Lyutikov & Uzdensky 2003; Luyubarsky 2005). We assume that the nebula magnetic field is similar to the reconnection minijet’s emission region magnetic field, \( B_{\text{neb}} \sim B' \). The observational nebular magnetic field estimates range from \( B_{\text{neb}} \sim 0.14 \) to \( 0.3 \text{ mG} \) (Trimble 1983; Abdo et al. 2010). These values of \( B_{\text{neb}} \) are clearly less than the lower limit set by the cooling equation (4) of \( B' > 1 \text{ mG} \) unless we impose relativistic motion. Solving for the Lorentz factor in equation (4), we find

\[
\Gamma > 34E_{100}^{-3/2} B_{\text{eq}}^{-3}.
\] (5)

Such a high Lorentz factor may not be required if the magnetic field is higher near the reconnection region. In the split monopole models of PWNe, the toroidally dominated magnetic field \((B_\phi \propto \sin \theta)\), where \( \theta \) is the polar angle) is highest in the equatorial regions (Michel 1973; Bogovalov 1999). Thus, if reconnection occurs in the equatorial current sheet or equatorial striped wind, then the magnetic field will indeed be higher than the nebula average of \( 0.14–0.3 \text{ mG} \). In the magnetohydrodynamic (MHD) simulations of the Crab pulsar winds, some parts of the nebula magnetic field reach \( 0.45–0.6 \text{ mG} \) (Komissarov & Luyubarsky 2004; Komissarov & Lyutikov 2011), which requires that the Lorentz factor be at least \( 4–10 \) according to equation (5). We note that it is also possible that there is no relativistic motion, and the flaring region simply has a high magnetic field of \( \gtrsim 1 \text{ mG} \), a value that is indeed found in the bright localized regions of the nebula (e.g. the inner ‘wisp’ and ‘knots’ of the nebula; Hester et al. 1995).

### 4 MINIJET SED

The SED produced in a flaring region may be significantly influenced by the proximity of the emitting particles to the synchrotron radiation reaction limit implied by MHD such that the electron distribution is either mono-energetic or a power law with an abrupt cut-off. If a mono-energetic electron distribution is not formed and magnetic reconnection is what causes the flares, then reconnection may leave a characteristic power-law index of the accelerated particles in the form of a hard electron distribution. Here we discuss these features in relation to the observed SEDs from the 2011 April flare and suggest that these observations point towards relativistic reconnection minijets.

The best Crab nebula flare SED observations to date are 11 SEDs taken during the 2011 April flare by the Fermilat/LAT team (Buehler et al. 2012). Buehler et al. (2012) fitted the SEDs with an empirical function of the form

\[
N_\nu = A \nu^{-\beta} \exp(-\nu/\nu_c),
\] (6)

where \( \nu \) represents photon energy, and different values of \( A \) and \( \nu_c \) were used for each SED fit. The parameter \( \beta \) was assumed to be constant for all of the SEDs, and its best fit was found to be \( \beta = 1.27 \pm 0.12 \). Significantly, the observed integrated flux above \( 100 \text{ MeV} \), \( F \), correlates with the flare SED peak energy, \( \nu_{\text{peak}} \), such that \( \log F / \log \nu_{\text{peak}} = 3.42 \pm 0.85 \) as expected in simple models of Doppler beaming (Lind & Blandford 1985). In the context of the above observations, we discuss the development of an effectively mono-energetic electron distribution near the radiation reaction limit from a hard electron distribution of the type expected in magnetic reconnection.
Mono-energetic distributions have already been examined in the context of the Crab flares in Uzdensky et al. (2011) and Cerutti et al. (2012), in which a specific model of reconnection causes electrostatic particle acceleration that produces an approximate mono-energetic electron distribution near the energy associated with the electrostatic potential drop. Unlike the model we discuss here, the Uzdensky et al. (2011) model avoids the MHD limit on particle energy and displays no bulk relativistic motion by invoking a specific geometry of the reconnection region. In this section, we do not make such geometrical requirements; instead we focus on a general discussion of how the radiation reaction limit can affect flare SEDs.

A critical feature in the SED of the Crab nebula flares that has already received attention is the synchrotron emission that is above the classical synchrotron limit of $\sim 10^2$ MeV (de Jager et al. 1996; Lyutikov 2010; Abdo et al. 2011). Importantly, the electron distribution function may display a power law with an excess, or pile-up, of particles near the synchrotron limit, in which case the emitting particles will display an SED that is close to the single-particle synchrotron SED. This limit comes from ideal MHD, in which the electric field, $E$, is less than the magnetic field such that $E = \eta B$, where $0 < \eta < 1$. To illustrate these points, we briefly examine the particle acceleration process. First, we assume that a particle’s energy is approximately described by

$$\frac{d\gamma}{dt} \approx \frac{eE}{m_e c} - \beta_\gamma \gamma^2$$

$$= \eta \omega - \beta_\gamma \gamma^2,$$  \hspace{1cm} (7)

where $\beta_\gamma = 2/3e^4 B_\perp^2 / (m_e c^2)$, $E = \eta B$ and $\omega = eB/m_e c$. The first term describes particle acceleration by the electric field (where $E$ is only an approximation of the true electric field since the accelerating particle velocity is not always parallel to the electric field) and the second term describes synchrotron losses. We expect the initial particle population to be accelerated to higher energies until $d\gamma/dt = 0$, where the synchrotron energy losses equal the acceleration rate such that $\eta \omega - \beta_\gamma \gamma^2 = 0$. If ideal MHD holds in the acceleration region, then $\eta < 1$, which in turn implies the existence of a maximum possible Lorentz factor allowed by the synchrotron radiation backreaction for a given magnetic field value:

$$\gamma_{\text{rad}} = \frac{3m_e^2 c^2 \eta}{2e^2 B_\perp^2} = 5 \times 10^9 \eta^{1/2} B_{-1.0}^{-1/2}.$$  \hspace{1cm} (8)

To quantify what we mean by an electron distribution function that ‘piles up’ near $\gamma_{\text{rad}}$ (e.g. Schlickeiser 1984), we examine the time evolution of the electron distribution function $N(t, \gamma)d\gamma$, which describes the number of electrons with a Lorentz factor below $\gamma$ at time $t$. The time evolution of $N(t, \gamma)$ in our toy model is described by the equation (Kirk, Rieger & Mastichiadis 1998)

$$\frac{dN}{dt} + \frac{\partial}{\partial \gamma} \left[ \frac{\gamma}{t_{\text{esc}}} - \beta_\gamma \gamma^2 \right] N + \frac{N}{t_{\text{esc}}} = Q\delta(\gamma - \gamma_0),$$  \hspace{1cm} (9)

where the acceleration time $t_{\text{acc}}$ and the escape time $t_{\text{esc}}$ are constants, $N(\gamma, 0) = 0$, $Q$ is the injection rate of particles with Lorentz factors of $\gamma_0$ and $\delta$ is the Dirac delta. Equation (9) describes the time evolution of non-thermal particles subject to mono-energetic injection, particle acceleration and escape, and synchrotron radiation. The solution to equation (9) subject to the above conditions can be found in Kirk et al. (1998). In this solution, the electron distribution, $N$, becomes mono-energetic as $t \rightarrow \infty$ [i.e. $N \sim \delta(\gamma - \gamma_{\text{rad}})$] if the particle injection process ceases and the acceleration process continues, since all the injected particles will simply be accelerated up to $\gamma_{\text{rad}}$. Even if the injection process continues during the acceleration process, the distribution can still develop a ‘pile-up’ just below $\gamma_{\text{rad}}$, effectively becoming mono-energetic as discussed below (see also Fig. 1). However, such a pile-up distribution only occurs depending on the details of the injected spectrum of particles and particle escape. In the particular case described above, a pile-up only occurs if $t_{\text{esc}} > t_{\text{acc}}$ (see also Schlickeiser 1984).

We have plotted the solution to equation (9) at two time slices, $t = 10 t_{\text{acc}}$ and $16 t_{\text{acc}}$, in Fig. 1 for the parameter choices $t_{\text{acc}} = 10 t_{\text{esc}}$ and $\gamma_{\text{rad}} = 4 \times 10^6$. As Fig. 1 illustrates, a pile-up does eventually accumulate just short of $\gamma = \gamma_{\text{rad}}$, eventually evolving to a mono-energetic distribution that produces a single-particle synchrotron SED as shown in the bottom panel. Visual inspection of Fig. 1 (bottom panel) makes it clear that the power-law distribution of $N \propto \gamma^{-1}$ compares more favourably than the mono-energetic distribution with the best fit of the 2011 April flare SEDs reported in Buehler et al. (2012).

Unlike Buehler et al. (2012), we examine only the most luminous observed SED from the 2011 April flare (figure 6, panel 7, in Buehler et al. 2012), since over the course of the nine-day flare, the functional form of the SED may have significantly changed, and this SED is least affected by the background nebula emission. As shown in Fig. 2, we fit the SED to equation (6), which yields $\gamma_f = 1.08 \pm 0.16$ with a reduced $\chi^2$ of 0.35. The shaded area in Fig. 2 is the ‘1σ error region’. It represents a subset of curves generated by

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Figure 1. Top plot: a solution for $N(\gamma, t)$ (see equation 9) is shown for $t = 10 t_{\text{acc}}$ and $16 t_{\text{acc}}$, with different normalizations to ease visual comparison. The SEDs corresponding to $N(\gamma, t)$ at $t = 10 t_{\text{acc}}$ and $16 t_{\text{acc}}$ are in the bottom plot. For $t = 10 t_{\text{acc}}$, the solution is a pure power law of $N \propto \gamma^{-1.1}$, for $t = 16 t_{\text{acc}}$, a pile-up develops short of $\gamma = \gamma_{\text{rad}}$. Bottom plot: plotted here are four different flare SEDs added on to the average nebula SED: equation (6) with the parameters reported in Buehler et al. (2012), an SED derived from a power-law electron distribution of $N = K_\gamma \gamma^{-1.1}d\gamma$ and $N = 0$ for $\gamma > \gamma_{\text{esc}}$ (i.e. the $t = 10 t_{\text{acc}}$ solution to equation 9), the SED derived from the pile-up solution to equation (9) for $t = 16 t_{\text{acc}}$, and a single-particle synchrotron SED. For visual comparison, all three SEDs are adjusted so that they have the same maximum and peak energy; otherwise the $t = 10 t_{\text{acc}}$ solution would peak at a lower energy than the solutions for $t = 16 t_{\text{acc}}$ and $t \rightarrow \infty$ (the mono-energetic solution).
The Fermi/LAT data from the most energetic part of the 2011 April flare are shown (figure 6, panel 7, of Buehler et al. 2012) with the corresponding best-fitting curve from equation (6) and SEDs from three different electron energy distributions: a $p = 1$ power law with an abrupt cut-off, the same for $p = 2$ and a mono-energetic electron distribution. All of the curves are summed with the constant nebula SED component used in Buehler et al. (2012). The shaded area represents the 1σ error region (see the text for more details).

Equation (6) is a useful SED fitting function for the relevant energy range because it has the same functional form (to within a few per cent) as SEDs derived from three different electron distributions: power-law distributions with abrupt cut-offs, power-law distributions with pile-ups (as in Fig. 1) and mono-energetic distributions. Note, however, that the functional form of SEDs derived from power-law distributions that are near the $\epsilon F_\epsilon$ peak depends sensitively on the form of the electron distribution cut-off. For exponential cut-offs to the electron distribution function, the derived SED is broader than SEDs derived from power laws with abrupt cut-offs that are approximated by equation (6). Here we presume that the radiation reaction limit is the mechanism whereby either an abrupt cut-off or a pile-up develops in the electron energy distribution. Also, near the $\epsilon F_\epsilon$ peak, the $\gamma F$ parameter cannot be interpreted in the usual way as being equal to $(p + 1)/2$ for power-law distributions or 2/3 for mono-energetic distributions. For example, SEDs corresponding to $p = 1$ and 2 are well approximated by $\gamma F \approx 1.30$ and 1.56, respectively, while a mono-energetic distribution has $\gamma F \approx 0.7$ (e.g. Melrose 1980); for the purpose of comparison, the best fits of these distributions (which only vary normalization and cut-off energy since $p$ is fixed) are plotted in Fig. 2 next to the best fit of equation (6).

The best-fitting value of $\gamma F = 1.08 \pm 0.16$ lies in between that expected from a mono-energetic distribution and the $p = 1$ distribution, while distributions for which $p \geq 2$ are unlikely to have produced the observed SED. The precise $p$ corresponding to the best fit of $\gamma F = 1.08 \pm 0.16$ is $p \approx -0.2 \pm 0.9$, a large range of values due to the calculated SED's weak dependence on $p$ for $p \lesssim 0$ for the energy range in question. The best-fitting values for $\gamma F$ may imply the development of a pile-up near $\gamma_{rad}$ from a hard power-law electron distribution of $p \sim 1$ as seen in Fig. 1 and discussed in, for example, Schlickeiser (1984). A pile-up pushes a $p = 1$ SED's $\gamma F$ value down from $\gamma F \approx 1.30$ to that of the mono-energetic distribution, $\gamma F = 0.7$. We verify this possibility through the toy model we develop from equation (9) by setting $\gamma_0 = 10^7$, $\gamma_{rad} = 4 \times 10^9$ and $t_{esc} \gg t_{acc}$, after which we find that the best-fitting $\gamma F$ can be reproduced by the resulting pile-up that develops when $t$ is between $\sim 13 t_{acc}$ and $\sim 18 t_{acc}$. Thus, a wide variety of electron distributions could have produced the observed SED shown in Fig. 2, but we suggest it was most likely a hard distribution of $p \lesssim 1$, possibly with a pile-up near the radiation reaction limit.

If the electron distribution is a hard power law of $N \propto \gamma^{-1}$, where $N = 0$ for $\gamma > \gamma_{rad}$, the peak energy of the observer frame in the $\epsilon F_\epsilon$ representation is

$$\epsilon'_{\text{peak}} \approx 0.6 \delta \omega_\gamma (\gamma_{rad}) \approx 1.35 \eta \frac{\hbar m_e c^3}{\epsilon^2} \approx 100 \text{MeV},$$

where the critical synchrotron frequency is $\omega_\gamma = 3/2 \gamma^2 e B_\perp / m_e c$ and the location of the peak energy, $0.6 \delta \omega_\gamma (\gamma_{rad})$, was determined numerically. The observed peak energy of a Doppler boosted SED is $\epsilon_{\text{peak}} = \delta \epsilon'_{\text{peak}}$, implying that any observed peak energy above 100 MeV must have a minimum Doppler factor of $\delta_{\text{min}} = \epsilon_{\text{peak}} / \epsilon'_{\text{peak}}$. Thus, for the 2011 April flare, $\delta_{\text{min}} \gtrsim 375/100 \sim 4$.

However, if the theoretical criteria discussed above are met for the electron distribution to become mono-energetic, then the SED will be effectively described by the single-particle synchrotron SED, whose form in the low-energy limit is $F_{\epsilon} \propto \epsilon^{S_{\epsilon}}$, and in the high-energy limit, $F_{\epsilon} \propto \epsilon^{\gamma_{\epsilon}} \exp(-\epsilon/\epsilon_c)$. Since the $\epsilon F_\epsilon$-mono-energetic synchrotron spectrum peaks at the angular frequency of $\omega_{\text{peak}} = 1.3 \delta \omega_\gamma$, the $\epsilon F_\epsilon$ peak from emitting particles with Lorentz factors of $\gamma_{rad}$ is located at a photon energy of

$$\epsilon'_{\text{peak}} \approx 1.3 \delta \omega_\gamma (\gamma_{rad}) \approx 2.9 \eta \frac{\hbar m_e c^3}{\epsilon^2} \approx 200 \text{MeV}.$$  

Thus, the 2011 April flare implies a minimum Doppler factor of $\delta_{\text{min}} \gtrsim 2$. Note that this is a lower estimate of $\delta_{\text{min}}$ compared to the estimate made assuming a hard power-law electron distribution.

To illustrate the importance of Doppler beaming, we have plotted in Fig. 3 the same intrinsic single-particle synchrotron SED with two different Doppler factors, $\delta = 1$ and 3. The different Doppler factors affect the intrinsic SED's photon energies, $\epsilon = \delta \epsilon'$, and normalization, $F_{\epsilon}(\epsilon) = \delta F_{\epsilon}'(\delta \epsilon')$ (Lind & Blandford 1985). We adjust the normalization of the intrinsic SED to ensure that the $\delta = 3$ flare has a maximum at $\epsilon F_\epsilon \sim 4 \times 10^{-9} \text{ergs cm}^{-2} \text{s}^{-1}$, as observed. The same intrinsic normalization is used in the $\delta = 1$ flare. The constant Crab nebula SED as described in Buehler et al. (2012) is plotted as well. Note that by using Doppler factors consistent with the SED cut-off observed during the most luminous part of the 2011 April flare ($\delta \gtrsim 2$), the unbeamed flare does not significantly increase the high-energy flux of the average nebula.

The non-detection of significant flaring by X-ray telescopes places a further constraint on the flare SEDs. If the flare SED comes from a mono-energetic electron distribution, then in the Chandra and XMM–Newton energy bands, which we define here as extending from $\epsilon_{\text{min}} = 0.1$ to $\epsilon_{\text{max}} = 10$ keV, the SED goes as $F_{\epsilon} \propto \epsilon^{4/3}$. For the single-particle synchrotron SED discussed above, where $(\epsilon F_{\epsilon})_{\text{max}} \sim 4 \times 10^{-9} \text{ergs cm}^{-2} \text{s}^{-1}$ and $\epsilon_{\text{peak}} = 375$ MeV,

$$F_{\chi} = A \int_{0.1 \text{keV}}^{10 \text{keV}} \epsilon^{4/3} d\epsilon \approx 3.24 \times 10^{-14} \text{ergs cm}^{-2} \text{s}^{-1},$$

$$N_{\chi} = B \int_{0.1 \text{keV}}^{10 \text{keV}} \epsilon^{-2/3} d\epsilon \approx 6.39 \times 10^{-6} \text{cm}^{-2} \text{s}^{-1}.$$

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where $F_X$ is the integrated X-ray flux and $N_X$ is the total photon flux. These values are much smaller than the spatially integrated Crab nebula X-ray energy flux of $\sim 10^{-7}$ ergs cm$^{-2}$ s$^{-1}$ and the photon flux of $\sim 100$ cm$^{-2}$ s$^{-1}$ (Kirsch et al. 2005). Even the photon flux of the Crab nebula in a Chandra resolution element of $\sim 1$ arcsec$^2$, which is $\sim 10^{-2}$ cm$^{-2}$ s$^{-1}$, is well above the photon flux we predict in the X-ray band. If the flare emission were from a $p = 1$ power-law electron distribution, then the estimates in equation (12) would increase by a factor of $(\gamma_{\text{rad}}/\gamma_{\text{min}})^{1/3}$, assuming that the lowest energy range of the SED goes as $F_{\gamma} \propto \epsilon^{1/3}$. Such a power law could extend down to $\gamma_{\text{min}} \sim 10^{-5} \gamma_{\text{rad}}$ before the flare was comparable to the Crab nebula flux in one Chandra resolution element. Hence, with a $\gamma_{\text{min}}$ as low as $\sim 10^2$, it is possible to explain the non-detection of the flaring events by X-ray telescopes.

Our above discussion of the flaring SEDs has two implications: (i) for hard electron distributions, the MHD radiation reaction limit can lead to the formation of a pile-up electron distribution that is effectively mono-energetic and (ii) significant emission beyond this limit implies that the emitting region is moving along the line of sight at relativistic speeds. Observations of the 2011 April flare suggest that, regarding (ii), a lower limit on the Doppler factor of $\delta \gtrsim 2$ is required. As for (i), the observed SED suggests a pile-up distribution that is not yet effectively mono-energetic. Interestingly, as Buehler et al. (2012) point out and we confirm, their data are not consistent with typical shock acceleration models, which usually produce $p \geq 2$ (e.g. Kirk et al. 2000). Instead, we suggest that their observations are consistent with harder distributions ($p \lesssim 1$) found in many magnetic reconnection models.\footnote{Note that de Gouveia dal Pino & Lazarian (2005) construct a reconnection model that produces $p = 2\pm 0.5$ power-law indices, though O’C. Drury (2012) argues against their analysis.}

5 A MINIJET STATISTICAL MODEL

To illustrate how reconnection minijets relate to the high-energy nebula flux and variability, we construct a toy model (see Fig. 4) that produces statistical predictions about the high-energy nebula light curve. We postulate that reconnection minijets are random independent events in the nebula with an associated average reconnection event rate, $n_C$, and are therefore described by Poisson statistics. A significant simplifying assumption we make is in presuming that the probability density functions (PDFs) for the intrinsic reconnection emission region time-scale, $\tau$, unbeamed intrinsic flux, $f'$, and Lorentz factor, $\Gamma$, are narrow enough to be treated as Dirac delta probability densities. Thus, because $\tau', f'$ and $\Gamma$ are constants, the statistics of the random variables $\tau$ (observed time-scale) and $f$ (observed flux) are determined by the minijet Doppler factor, $\delta$, which itself is a random variable. Another significant simplifying assumption we make is that the reconnection outflows have an isotropic angular distribution.

The following discussion is divided into two sections. Section 5.1 covers the statistics of individual minijets and Section 5.2 develops time-series statistics relevant to the nebula light curve as a whole.

5.1 Individual minijet statistics

Define a spherical coordinate system $(r, \phi, \theta)$ with the $z$-axis ($\theta = 0$) pointing along the line of sight so that the viewing angle, $\theta$, of any given jet is equal to the coordinate $\theta$ associated with its trajectory. Thus, the PDF $\rho(\theta)$ is a function of the random variable $\theta$ alone. Assuming that the minijet angular distribution is isotropic so that $\rho(\theta) \, d\theta = d(\cos \theta)$, then from the definition of the Doppler factor, \begin{equation} d(\cos \theta) = \frac{1}{\beta \Gamma \delta^2} \frac{1}{\delta} \, d\delta. \end{equation}

Therefore, the Doppler factor PDF is \begin{equation} \rho(\delta) \, d\delta = \frac{1}{\beta \Gamma \delta^2} \, d\delta. \end{equation}

Figure 3. Shown are Doppler boosted single-particle synchrotron SEDs (dotted lines) for $\delta = 1$ and 3. The normalization of the intrinsic SED and the critical frequency values are fixed so that the $\delta = 3$ SED displays the maximum reported $\epsilon F_\epsilon$ value and correct peak energy during the 2011 April flare: $(\epsilon F_\epsilon)_{\text{max}} \sim 4 \times 10^{-3}$ ergs cm$^{-2}$ s$^{-1}$ and $\epsilon_{\text{peak}} = 375$ MeV (Buehler et al. 2012). The average nebula SED (thick line) reported in Buehler et al. (2012) is summed with the flare SEDs to produce a combined SED shown by the dashed line. Note that the difference between $\delta = 1$ and 3 SEDs is enough to render the former nearly unobservable compared to the average nebula emission.

Figure 4. Cartoon schematic of reconnection sites in the nebula as viewed from above the toroidal plane (defined by the pulsar spin axis). As shown in the upper-right corner box, each reconnection site consists of plasma inflows into the reconnecting region and twin relativistic outflows ('minijets') with some Lorentz factor $\Gamma$ of the order of a few. Also shown in the schematic is a minijet directed towards the observer, which causes a flare since its emission is more highly beamed as compared to its off-axis counterparts.
where $\delta_{\text{min}} \approx (2\Gamma)^{-1}$ and $\delta_{\text{max}} \approx 2\Gamma$. We note that $\delta_{\text{min,max}}$ may take different values for cases where the flare emission could be either blue- or red-shifted out of the relevant telescopes’ bandwidth, or overwhelmed by the constant nebula emission.

Using equation (14), we now calculate observable quantities such as minijet time-scale and flux distributions. Regarding time-scales, we find $n(\tau) \, d\tau$, which is the number of minijets that activate per unit time whose observed duration is between $\tau$ and $\tau + d\tau$. This can be calculated by substituting $\delta = \tau/\tau_1$ into equation (14) and including a factor of $n_1$;

$$n(\tau) \, d\tau = \frac{n_1 \, d\tau}{\beta \Gamma \tau_1^\prime} \quad \text{for} \quad \frac{\tau'}{(1+\beta)\Gamma} \leq \tau < \frac{\tau'}{(1+\beta)\Gamma} + \tau_1^\prime. \quad (15)$$

Hence, there is equal probability that short- or long-duration minijets will be observed. To obtain the flux distribution, we assume that the intrinsic gamma-ray flux of each minijet, $f'$, is Doppler boosted such that $f = \delta^3 f'$, where $q = 3 + \alpha (F_\gamma \propto \alpha^{-\infty})$ for moving components and for bolometric flux, $q = 4$ (Lind & Blandford 1985; Jester 2008). We now substitute the Doppler boosting formula into equation (14) to find the minijet flux probability density

$$\rho(f) \, df = \frac{1}{\beta \Gamma f_1^\prime} \left( \frac{f}{f_1^\prime} \right)^{q-1} \frac{d\tau}{\tau_1^\prime} \quad (16)$$

(Note that this formula is identical to that used in the calculation of luminosity functions for Doppler beamed sources; Urry & Shafer 1984.) Therefore, $\rho(f) \propto f^{-1}$ for $q \gg 1$, implying that the minijet flux average, averaged over different flares not time, is dominated by bright rare flares.

### 5.2 Minijet time-series statistics

In this subsection, we derive quantities relevant to the entire high-energy nebula light curve. To do this, we analyse the statistics related to the random variable $F_{\text{n neb}}$, representing the total nebula high-energy flux. First, we derive the statistical behaviour of the random variable $k$ or the number of flares active at any given moment, and then apply this to the analysis of $F_{\text{n neb}}$. Finally, assuming that all nebular variability is due to minijets, we derive the nebula light curve’s power spectrum.

The statistical behaviour of the light curve depends sensitively on $\lambda$, the average number of reconnection events that are active at any given time. To derive $\lambda$, we first note that the differential flare rate, $dn/d\delta$, immediately follows from equation (14) as $dn/d\delta = n_1 \rho(\delta)$ (recall $n_1 \equiv$ average nebular reconnection rate). Thus, the differential overlap number is $d\lambda = rd\tau$ (recall that $\tau$ is the observed flare duration), since $\tau \, dn$ is the differential number of flares with Doppler factor $\delta$ that are activated within one observed flare duration $\tau$. To calculate the average overlap number over an interval of Doppler factors from $\delta_{\text{min}}$ to $\delta_{\text{max}}$, we integrate to obtain

$$\lambda = \int_{\delta_{\text{min}}}^{\delta_{\text{max}}} \frac{n_1 \tau'}{\beta \Gamma \tau_1^\prime} d\delta$$

$$\approx \frac{n_1 \tau'}{2 \beta \Gamma \delta_{\text{min}}} \quad \text{(for} \quad \delta_{\text{max}} \gg \delta_{\text{min}} \text{).} \quad (17)$$

Since flaring consists of a random process of unrelated events governed by an average overlap number $\lambda$ within the nebula, the probability that $k$ flares are active/overlapping at any given moment is governed by the Poisson distribution, $P(k) = \lambda^k e^{-\lambda}/k!$. We have verified the applicability of the Poisson distribution and our calculation of $\lambda$ (equation 17) to our model via Monte Carlo simulations with a variety of parameter values.

The duty cycle of minijets (i.e. the fraction of time when the nebula contains one or more active minijets) can now be calculated. The duty cycle of flares with $\delta_{\text{min}} < \delta < \delta_{\text{max}}$ is simply the probability that at a time $t$, one or more flares ($k \geq 1)$ are active: $f_{\text{duty}} = P(k \geq 1) = 1 - e^{-\lambda}$. In the non-overlapping limit, $\lambda \ll 1$, the duty cycle reduces to $f_{\text{duty}} \approx \lambda$.

We may now analyse the statistics of $F_{\text{n neb}}$, the random variable representing the total high-energy flux of the nebula. To this end, we introduce the random variable for the total minijet flux due to reconnection outflows, $F_\gamma$, and the constant nebula flux, $F_{\text{const}}$, possibly originating from the pulsar wind termination shock; the random variables $F_{\text{n neb}}$ and $F_\gamma$ are related by

$$F_\gamma = F_{\text{n neb}} - F_{\text{const}}. \quad (18)$$

Note that if only one minijet is active at time $t_0$, then $F(t_0) = f$. In order to derive the statistical behaviour of the directly observable $F_{\text{n neb}}$, we analyse the PDF of $F_\gamma$, $\rho(F_\gamma)$, where $\rho(F_\gamma) \, dF_\gamma$ is the probability that, at any given time, the high-energy flux due to reconnection minijets is between $F_\gamma$ and $F_\gamma + dF_\gamma$. Each minijet is presumed to have a square pulse profile. Importantly, our analysis assumes that $\rho(F_\gamma)$ and its parameters remain constant over time intervals well in excess of the light curve’s autocorrelation time, $\tau_{\text{aut}}$. If this assumption holds, then the moments of $\rho(F_\gamma)$ may be compared with the observed moments of the light curve extracted from data spanning time intervals larger than the observed $\tau_{\text{aut}}$.

The exact moment generating function (MGF) of $\rho(F_\gamma)$ can be found, which contains the same information as $\rho(F_\gamma)$ (see Appendix A for its derivation). While any moment of $\rho(F_\gamma)$ may be calculated, we provide the first three moments here:

$$\langle F_\gamma \rangle = \frac{n_1 \tau'}{q-2} \beta \Gamma \left( \int_{f_{\text{min}}}^{f_{\text{max}}^{1-2/q}} f^{q-2} \, df - f_{\text{min}}^{1-2/q} \right)$$

$$\sigma_{F_\gamma} = \frac{\beta \Gamma}{(2q-2)^{1/2}} \left( \frac{f_{\text{max}}^{2-2/q} - f_{\text{min}}^{2-2/q}}{(n_1 \tau')^{1/2}} \right)^{1/2}$$

$$\gamma_1 = \frac{(2q-2)^{1/2}}{3q-2} \beta \Gamma \left( \frac{f_{\text{max}}^{2-2/q} - f_{\text{min}}^{2-2/q}}{(n_1 \tau')^{1/2}} \right)^{1/2}. \quad (19)$$

where $\sigma_{F_\gamma} = \langle (F_\gamma^2 - \langle F_\gamma \rangle^2)^{1/2} \rangle$ is the standard deviation and $\gamma_1 = \sigma_{F_\gamma} / \langle F_\gamma \rangle$ is the skewness; the brackets $\langle \rangle$ represent a time average. As a check against our analytical work here, we have spot-checked equations (19) with Monte Carlo simulations using a variety of parameters and found them to be consistent with one another. Note that comparing the above moments to observed light curve’s moments allows one to test whether the light curve is generated by beamed minijets as per our model. To better discuss the significance of equations (19), we assume $f_{\text{max}} \gg f_{\text{min}}$, $\Gamma \gg 1$ and $f_{\text{max}} \approx (2\Gamma)^q$, so that equations (19) can be approximated as

$$\langle F_\gamma \rangle \approx \frac{2^{q/2} \Gamma^{q/2-2}}{q-2}$$

$$\sigma_{F_\gamma} \approx \frac{(2q-4)^{1/2}}{(2q-2)^{1/2}} \frac{\Gamma^{1/2}}{(n_1 \tau')^{1/2}}$$

$$\gamma_1 \approx \frac{2(2q-2)^{3/2}}{3q-2} \frac{\Gamma^{3/2}}{(n_1 \tau')^{1/2}}. \quad (20, 21, 22)$$

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The variation about the average as expressed by the relative standard deviation (\( \sigma_F / \langle F \rangle \)) can easily take small or large values depending on how \( n_f \tau' \) compares with \( \Gamma^3 \). Because \( \gamma_f \) depends on \( \Gamma, n_f \) and \( \tau' \), in the same way as \( \sigma_F / \langle F \rangle \) in this approximation, their ratio depends only on the beaming index and is expected to be of the order of unity:

\[
\frac{\sigma_F}{\langle F \rangle} \approx \frac{(2q - 4)(3q - 2)}{2(2q - 2)} \sim 1, \tag{23}
\]

where this ratio for reasonable values of the beaming factor (\( q \geq 3 \)) ranges from \( \sigma_F / \langle F \rangle \gamma_f \sim 0.44 \) to 0.75.

Another important statistical representation of the light curve is the power spectrum, \( P(\nu) \), which measures its variability on different time-scales. We compute \( P(\nu) \) for light curves composed of square pulses, exponential pulses (i.e. zero rise time and exponential decay) and Gaussian pulses, all with a time-scale of \( \tau = \tau' / \delta \). Since individual pulses are generated by a stochastic process, the phase information of the Fourier transform is unimportant, implying that the power spectra for an individual pulse, \( P(\nu) \), can be used to generate the light-curve power spectrum; thus,

\[
P(\nu) = \int_{\tau_{\text{min}}}^{\tau_{\text{max}}} n(\tau) P(\nu) \, d\tau, \tag{24}
\]

where \( n(\tau) \) is found in equation (15). For square and exponential pulses, the single-pulse power spectra are \( P^\text{sq}_1 \propto \sin(\pi \tau \nu)^2 / \nu^2 \) and \( P^\text{exp}_1 \propto (1 + 4\pi^2 \tau^2 \nu^2)^{-1} \), respectively (due to excessive length we do not include the spectrum for Gaussian pulses). For the total power spectrum, when \( \nu \ll \tau_{\text{max}} \approx (2\Gamma \tau')^{-1} \), the power spectrum is constant (i.e. white noise) since the frequency dependence drops out of the integrand. For \( \nu \gg \tau_{\text{min}} \approx 2\Gamma / \tau' \), only the power spectrum for square and exponential pulses goes as \( P(\nu) \propto \nu^{-2} \) (for the square pulses, there is oscillatory behaviour on top of the \( \nu^{-2} \) decay). This can be summarized as follows:

\[
P(\nu) = \begin{cases} 
\text{const.} & \text{for } \nu \ll (2\Gamma \tau')^{-1} \\
\text{transition to } \nu^{-2} & \text{for } (2\Gamma \tau')^{-1} < \nu < 2\Gamma / \tau' \\
\nu^{-2} & \text{for } \nu \gg 2\Gamma / \tau'.
\end{cases} \tag{25}
\]

Unlike the square and exponential pulses, Gaussian pulses do not display power law; they drop off faster (see Fig. 5). While we have found analytical expressions for \( P(\nu) \) for some beaming factors (e.g. \( q = 4 \)), we do not reproduce them here due to their excessive length. However, the solutions to equation (24) are plotted in Fig. 5. As a check against our analytical work, we include the discrete Fourier transform (DFT) of a Monte Carlo simulated light curve composed of square pulses, which clearly follows the corresponding calculated power spectrum.

5.3 Application of the statistical model to the Crab nebula

Our model may be divided into two categories: the non-overlapping regime and the overlapping regime. In the non-overlapping regime, minijets do not temporally overlap and a significant constant emission component dominates, i.e. \( \lambda \ll 1 \) and \( F_{\text{const}}(F_1) \gg 1 \). The opposite is true for the overlapping regime, where \( \lambda \gg 1 \) and \( F_{\text{const}}(F_1) \lesssim 1 \). In the non-overlapping regime, each observed flare is due to an individual minijet’s flux, \( F_1 \); hence, individual minijet statistics are directly observable. In this approximation \( F_{\text{const}}(F_1) \gg 1 \), so that \( F_{\text{const}} \) can be related to an observable quantity via \( (F_{\text{neb}}) \approx F_{\text{const}} \). Therefore, we can write \( F \approx F_{\text{flare}} - (F_{\text{neb}}) \), where \( F_{\text{flare}} \) is the nebula flux \( (F_{\text{neb}}) \) during an isolated flare.

If we insert this expression into equation (16), we obtain

\[
p(\nu_{\text{flare}}) \text{d}F_{\text{flare}} \propto (F_{\text{flare}} - (F_{\text{neb}})) \frac{\Sigma_F}{\mu} \text{d}F_{\text{flare}}, \tag{26}
\]

where again \( F_{\text{flare}} = F_{\text{neb}} \) when a single flare is active.

A separate observable in this regime is the minijet’s time-scale distribution (equation 15) which states that flares will be equally distributed across the allowed time-scales.

For the overlapping regime, the above analysis is complicated by the difficulty of associating a single flare with a single minijet. However, applicable to both regimes is the light-curve power spectrum (equation 25), which for some pulse profiles is \( \propto \nu^{-2} \) in the short variability region. However, this has been measured by Buehler et al. (2012) as being \( \propto \nu^{-1} \). Such a measurement could be explained if the transition region where \( P_\nu \) evolves from being constant to being \( \propto \nu^{-2} \) is dominating that measurement. Other observables are the light-curve moments (equations 19), such as the ratio of variability to skewness (equation 23). However, moments such as the relative standard deviation, which measures the light-curve variability, cannot be directly compared with observational data since observed light curves smooth over short time-scale variability because of time binning and the breaks in observational coverage. As an example light curve, Fig. 6 represents the non-overlapping regime of our model that qualitatively reproduces the Crab nebula light curve. The parameters used to generate Fig. 6 are \((\delta_{\text{min}}, \delta_{\text{max}}, q, n_f, \tau', \Gamma, f') = (0.56, 5.83, 4, 0.1875 \text{ d}^{-1}, 24 \text{ d}, 3, 0.15 \times 10^{-7} \text{ s}^{-1} \text{ cm}^{-2})\), which imply a flare overlap number of \( \lambda = 1.25 \).

For either regime, the energy in PWN reconnection outflows must not exceed the total energy budget set by the spin-down power of the pulsar. For example, if the non-overlapping model is true...
Figure 6. Ten-year simulated Crab nebula light curve. The ‘2011 April type flares’ represent flares with increases of \(~\sim 30\) over the nebula average as found in Buehler et al. (2012). ‘2010 September’ flares represent flux increases by \(~\sim 5\) similar to the Crab 2010 September flare (Abdo et al. 2011; Tavani et al. 2011). The time-scale assumes that the shortest flare durations are 4 d, thus \(\tau' \approx 2\Gamma \times 4\) d, so if \(\Gamma = 3\), then \(\tau' = 24\) d. The simulated light curve was binned into 4 d intervals. For an easy comparison to data, we add a steady component of \(>100\) MeV flux \(F_{\text{const}} = 6 \times 10^{-7} \text{s}^{-1} \text{cm}^{-2}\), consistent with the Crab nebula average flux as measured by Fermi/LAT (Abdo et al. 2011). The model parameters are \((\delta_{\text{min}}, \delta_{\text{max}}, q, n_{\text{c}}, \tau', \Gamma, f') = (0.56, 5.83, 4, 0.1875 \text{d}^{-1}, 24 \text{d}, 3, 0.15 \times 10^{-7} \text{s}^{-1} \text{cm}^{-2})\), with a corresponding flare overlap number of \(\lambda = 1.25\).

and approximately one flare per year is observed with a viewing angle of \(~\lesssim \Gamma^{-1}\), then \(n \sim 4\Gamma^2 \text{yr}^{-1}\). For the 2010 September flare parameters, this implies that the minijet power expended is

\[
P_t = n_t E_t = 4.0 \times 10^{35} L_{16.6}^{-t_s} \Gamma_{-1}^{-1} \text{ erg s}^{-1},
\]

which is much less than the Crab pulsar’s spin-down power, \(P_{\text{spin}} \approx 5 \times 10^{38} \text{ erg s}^{-1}\).

In applying our statistical minijet model to the Crab nebula, its simplifying assumptions that (a) the minijets’ intrinsic parameters are the same and that (b) the minijet directions are isotropically distributed are both open to criticism. Assumption (a) is challenged by the most luminous flare being longer in duration (\(~\sim 9\) d) than the 2010 September flare (\(~\sim 4\) d), since both assumption (a) and the Doppler transformations indicate that the 2011 April flare should be shorter. However, more flare observations are necessary before any firm conclusions are made regarding a correlation (or lack thereof) between observed flare luminosity and duration. Regarding assumption (b), the clear toroidal morphology of the Crab nebula (e.g. Weiskopf et al. 2000), consistent with pulsar models with a toroidal outflow containing a toroidally dominant magnetic field and a large-scale current sheet, combines to make the assumption of an isotropic angular distribution of minijets questionable. In the PWN split monopole model or the striped wind one (Coroniti 1990; Bogovalov 1999), reconnection in the implied current sheets would only produce minijets in the plane of the torus, rendering them unobservable since the pulsar spin axis is at an angle of \(~\sim 60\)° to our line of sight (Weiskopf et al. 2000). For this reason, we have assumed that the minijets are produced in a turbulent isotropic region of the nebula. Nonetheless, the large-scale anisotropic morphology and structure of the magnetic field suggest that future work on this model should take into account at least some degree of anisotropy.

6 CONCLUSIONS

We have constructed a statistical model of Crab nebula high-energy variability by assuming that magnetic reconnection sites throughout the nebula are activated randomly, and once activated, display emission characteristics controlled by the reconnection outflow Doppler factor. At each site, a magnetically dominated reconnection region launches twin relativistic outflows along a randomly aligned axis. GeV flares are observed when, by chance, a relativistic outflow is aligned with the line of sight and is thus Doppler-boosted. The observed flares’ unusually short durations and high luminosities suggest that the emitting plasma is indeed moving towards Earth at relativistic speeds.

The flares’ SEDs contain information about the particle acceleration process that suggests that non-thermal particles are generated in reconnection regions, rather than shocks, and are undergoing bulk relativistic motion along the line of sight. Models of reconnection particle acceleration tend to create a hard power-law with an index close to \(p \sim 1\) (Romani & Lovelace 1992; Zenitani & Hoshino 2001, 2007; Larrabee et al. 2003; O’C. Drury 2012).

We have shown that such distributions can easily form a pile-up near the radiation reaction limit implied by MHD, effectively becoming mono-energetic. The April 2011 flare SEDs (Buehler et al. 2012) are inconsistent with shock acceleration and are instead consistent with a hard electron distribution that may contain a not yet effectively mono-energetic pile-up. Furthermore, because the observed location of the SED peak is above that predicted by the synchrotron radiation reaction limit, the 2011 April flare’s emitting region Doppler factor is \(\gtrsim 1\).

The predictions of our statistical minijet model can be summarized as follows.

(i) When minijets do not temporally overlap one another, the PDF for the nebula’s high-energy flux during a flare, \(F_{\text{flare}}\), is

\[
\rho(F_{\text{flare}}) \sim \text{const} (F_{\text{flare}} - \langle F_{\text{neb}} \rangle)^{-1}.
\]

(ii) The first three moments of the light curve may be compared with our theoretically calculated moments in equations (19), and any higher degree theoretical moments may be easily calculated using the method described in Appendix A.

(iii) The light-curve power spectrum (equation 25) is constant (‘white noise’) for \(v \ll (\Gamma')^{-1}\) and goes as \(P(v) \propto v^{-2}\) for \(v \gg \Gamma'/r\).

Unlike the standard model for Crab nebula non-thermal emission, which involves particle acceleration at the pulsar wind termination shock (Kennel & Coroniti 1984), we have suggested here that magnetic reconnection may play an important or even dominant role. Further research will involve investigating whether reconnection can explain both the steady nebula emission and the flaring, which would preclude the need for shock emission altogether in the nebula. Our statistical model may also apply for AGNs that exhibit several minute TeV variability.

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APPENDIX A: CALCULATING THE MOMENTS OF $\rho(F_i)$

Here we exploit a special property of MGFs (e.g. Bulmer 1979) that allows us to calculate the MGF for the PDF, $\rho(F_i)$. First we state the expression for the probability that the observed minijet flux will be between $F_i$ and $F_i + dF_i$ at any given time,

$$\rho(F_i)\,dF_i = \sum_{k=0}^{\infty} P(k)\rho(F_i)\,dF_i = \delta(F_i)e^{-\lambda}dF_i + \sum_{k=1}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \rho(F_i)\,dF_i,$$

where for compactness we have replaced $F_i|k$, or ‘$F_i$ given $k’$, with $F_i$. It is important to note that $\rho(F_i)$ is the PDF for the total summed minijet flux at any given time slice, not the PDF that governs the flux of individual flares, which is described by equation (16).

The MGF for a random variable $Y$ that follows the PDF $\rho(Y)$ is defined as $M_Y = \int_{-\infty}^{\infty} \exp(Y\,x)\rho(Y)\,dY$, which can be Taylor expanded to produce

$$M_Y = 1 + \langle Y\rangle\,x + \frac{1}{2!} \langle Y^2\rangle\,x^2 + \cdots + \frac{1}{j!} \langle Y^j\rangle\,x^j,$$

where $x$ is a dummy variable such that any moment of $\rho(Y)$ may be calculated via $\langle Y^j\rangle = d^j M_Y/dx^j|_{x=0}$. Applying the definition of $M_Y$ to $\rho(F_i)$, we find

$$M_{F_i}(x) = \langle e^{F_i x} \rangle = e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \langle e^{F_i x} \rangle_k = e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} M_{F_k},$$

where $M_{F_k}$ is the MGF associated with conditional PDF, $\rho(F_k)$.

The special property of MGFs we use here is that for a random variable $F_k$ that is the sum of $k$ random variables, $F_k = \sum_{i=1}^{k} f_i$, the MGF is simply (Bulmer 1979)

$$M_{F_k} = \prod_{i=1}^{k} M_{F_i},$$

where $M_{F_i}$ is the MGF for the random variable $f_i$, the $i$th minijet flux of the $k$ minijets active at time $t$. Because the PDFs for all minijet random variables, $f_i$, are the same, so

$$M_{F_k} = M_{F_i}^k.$$

$M_i$ is easily obtained from calculating the moments of $\rho(F_{k=1})$, which is equation (16) with an extra factor of $\tau$ and a correspondingly different normalization. This extra factor of $\tau$ is necessary because longer flares are more likely to be active at a time $t$ compared with shorter

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duration flares, while equation (16) does not take a flare’s time dependence into account. We now calculate \( \langle F_1 \rangle \):

\[
\langle F_1 \rangle = \int_{f_{\text{min}}}^{f_{\text{max}}} F_{k=1} \rho(F_{k=1}) \, dF_{k=1} = A \int_{f_{\text{min}}}^{f_{\text{max}}} f^{j} \rho(f) \, df
\]

\[
= \left( \frac{2}{jq-2} \right) \left( \frac{f_{\text{max}}^{jq} - f_{\text{min}}^{jq}}{f_{\text{max}}^{2jq} - f_{\text{min}}^{2jq}} \right)
\]

where \( f \) is the single flare flux and \( A \) is a normalization constant. Now \( M_{F_1} \) can be obtained using equation (A2), such that \( M_{F_1} = \sum 1/j! \langle F_1^j \rangle \), which can be inserted into equation (A5) to produce

\[
M_{F_1} = M_{F_1}^j = \left( \sum_{j=0}^{\infty} \frac{1}{j!} \langle F_1^j \rangle \right) .
\]

Hence, the total MGF associated with \( \rho(F) \) is

\[
M_{F} = e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \left( \sum_{j=0}^{\infty} \frac{1}{j!} \langle F_1^j \rangle \right) .
\]

To find the first three moments of \( M_{F} \), we calculate the first three moments of \( M_{F_1} \) by expanding \( M_{F_1} \) to third order in the dummy variable \( x \):

\[
M_{F_1} = M_{F_1}^k \approx 1 + k \langle F_1 \rangle x + \left( \frac{k}{2} \langle F_1^2 \rangle + \frac{k(k-1)}{2} \langle F_1^3 \rangle \right) x^2 + \left( \frac{k}{6} \langle F_1^3 \rangle + \frac{k(k-1)}{2} \langle F_1 \rangle \langle F_1^2 \rangle + \frac{k(k-1)(k-2)}{6} \langle F_1 \rangle^3 \right) .
\]

Therefore, we can now read off the first three moments of the \( \rho(F_1) \) distribution,

\[
\langle F_1 \rangle = k \langle F_1 \rangle
\]

\[
\langle F_1^2 \rangle = k \langle F_1^2 \rangle + k(k-1) \langle F_1 \rangle
\]

\[
\langle F_1^3 \rangle = k \langle F_1^3 \rangle + 3k(k-1) \langle F_1 \rangle \langle F_1^2 \rangle + k(k-1)(k-2) \langle F_1 \rangle .
\]

Now, note that

\[
\langle F_1^j \rangle = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \langle F_1^j \rangle .
\]

Evaluating the infinite sum in equation (A11) for \( k = 1, 2 \) and 3 leads us to the first three moments of the \( \rho(F_1) \) distribution,

\[
\langle F_1 \rangle = \lambda \langle F_1 \rangle
\]

\[
\langle F_1^2 \rangle = \lambda \langle F_1^2 \rangle + \lambda^2 \langle F_1 \rangle^2
\]

\[
\langle F_1^3 \rangle = \lambda \langle F_1^3 \rangle + 3\lambda^2 \langle F_1 \rangle \langle F_1^2 \rangle + \lambda^3 \langle F_1 \rangle^3.
\]

Calculating higher order moments is straightforward. In general, the \( j \)th moment of \( \rho(F_1) \) can be calculated by Taylor expanding the expressions for \( M_{F_1} \) (equation A9), reading off the \( j \)th-order term in the said expansion and using this term in evaluating the infinite sum for \( \langle F_1^j \rangle \) (equation A11).

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