Giant fluctuations in the flow of fluidised soft glassy materials: an elasto-plastic modelling approach

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Abstract
In this work we study the effect of an additional mechanical noise on the rheological features of yield stress materials that exhibit non-homogeneous steady flows. Using a mesoscale elasto-plastic model accounting for a viscosity bifurcation in the flow response to an external shear stress, we find that additional sources of noise can lead to a fluidisation effect. As we increase the noise intensity we evidence a transition from a non-monotonic to a monotonic rheology, associated with giant fluctuations of the macroscopic shear rate and long-time correlated dynamics. Although distinct noise models can lead to different rheological behaviours in the low stress regime, we show that the noise-induced transition from shear-localised to homogeneous flow at higher stresses appears very generic. The observed dynamical features can be interpreted as a result of an out-of-equilibrium phase transition, for which we estimate the critical exponents that appear to be independent of the specific choice of the noise implementation for the microscopic dynamics.

1. Introduction

Dense disordered materials such as emulsions, foams, colloidal suspensions, or granular materials exhibit rich rheological behaviours. These materials have in common that once deformed beyond their solid elastic regime they yield plastically towards a complex flow regime with a shear rate dependent viscosity [1, 2]. The steady state flow behaviour of these yield stress fluids (YSFs) can be described at the continuum level using empirical laws like for example the Herschel–Bulkley relationship [3], or continuum descriptions, such as visco–elasto-plastic [4, 5] and fluidity models [6, 7]. While these types of descriptions account well for the average flow behaviour at a coarse grained scale, it has appeared that some flow features of YSFs are dominated by giant fluctuations of the macroscopic stress or shear rate [8–13]. This can for example lead to non-local, strongly system–size dependent, transport coefficients for the material dynamics [14–16]. Accordingly, understanding the role of mechanical noise and its spatio-temporal features has not only attracted a strong fundamental interest [2] but is also of direct importance in rheological applications [1].

Part of the mechanical fluctuations in driven disordered materials are usually generated by the flow itself, for example resulting from the elastic response of the material to localised plastic events [17, 18]. They are therefore very different in nature from thermally generated fluctuations [19] and must be incorporated into modelling approaches in a self-consistent manner [20].

Interestingly, flow-induced fluctuations can be associated in some cases with a self-fluidisation of the material, i.e. a decrease in shear stress with increasing shear rate. This leads to non-monotonic rheological constitutive curves [7, 21, 22], that can be associated with flow instabilities, potentially leading to shear localisation, metastability and hysteresis [23]. In the case of granular materials, this self-fluidisation process finds its origin in sliding frictional contacts [23–25]. In non frictional YSFs, non monotonic flow curves can be explained by mechanisms such as inertia [26] or local softening following structural rearrangements [27, 28].
Besides this self-generated mechanical noise there can be additional external sources of noise, which can be regarded in a first order approximation as independent of the shear-induced one. This is for example the case of irreversible deformations induced by thermal activation [29, 30] or mechanical vibrations [31–36]. Other mostly rate-independent fluctuations can also result from local processes such as coarsening in foams [37], or internal activity [38–40]. One important aspect of such external noise sources is that they can induce a fluidisation of the system at small imposed external stresses.

An even more interesting case is the scenario where fluctuations result from the interplay of a self-fluidising and an external source of noise. Upon an increase of the external noise magnitude, it will dominate at some critical value over the self-fluidising noise. This scenario can induce a change from a non-monotonic constitutive behaviour to a simple monotonic one, due to the fluidisation effect of the external noise. The associated transition has been investigated in the context of non-equilibrium phase transitions, both experimentally in the case of frictional granular materials [24] and theoretically in a generic elasto-plastic model [41].

In this work, we consider two different models for an external fluidising noise, and show that although they lead to distinct rheological behaviours at low stress, both induce a transition from a non-monotonic to a monotonic flow curve, associated with the transition from shear banded to homogeneous flow. We evidence that the competition between the endogenous noise and an external fluidising noise leads to giant fluctuations in the flow of soft glassy materials, that become relevant on the rheological scale. When interpreting the transition between the self-fluidised and the externally fluidised regimes upon increasing noise magnitude as a critical phenomenon [24, 41], we find that critical exponents do not depend on the model of external noise. This suggests that this type of transition might be very generic, independent of the microscopic details in the underlying dynamics.

2. Elasto-plastic models

2.1. Principle
For this study we chose to use coarse-grained elasto-plastic models (EPMs), which provide a generic framework to describe the flow of soft glassy materials (for a review see [2] and references therein). In EPMs, the material first responds elastically to a global uniform driving (either by controlling the strain or the stress). The deformation or stress can then locally induce a plastic event, associated with a rearrangement of particles on the microscopic level [17] if a local threshold \( \sigma_p \) is overcome. Such a plastic event causes a local relaxation of the stress and an elastic response of the surrounding material, and can trigger other local plastic events. The resulting macroscopic plastic flow is thus the consequence of these sequences of local rearrangements. The particle rearrangements can be seen as plastic inclusions embedded in an elastic matrix, and are described as force quadrupoles (Eshelby problem for over-damped dynamics). The elastic propagation kernel is thus described using the Eshelby propagator [42], with an asymptotic power-law decay \( \sim 1/r^d \) (\( d \) being the spatial dimension) and a quadrupolar symmetry.

2.2. Numerical model
The spatially-resolved models considered in this work are extended from previous versions used to describe both steady-state flows of YSFs using a shear-imposed protocol model [2] and transient (creep) flow using a stress-imposed protocol model [43–46].

We model an amorphous medium as a collection of mesoscopic blocks, each block being represented as a node \((i,j)\) of a square lattice of size \(L \times L\) (the lattice indices \(i,j\) represent the discretised coordinates along \(x\) and \(y\) directions respectively). The mesh size corresponds to the typical size of a cluster of particles undergoing a plastic rearrangement. These local plastic transformations are assumed to have the same geometry as the globally applied simple shear, i.e. we consider a scalar model. To describe the deformation of the amorphous system, we decompose the total deformation of each node \((i,j)\) into a local plastic strain \(\gamma_{ij}^{pl}\), which is, in general, heterogeneous, and an elastic strain \(\gamma_{ij}^{el}\).

2.2.1. Stress-imposed model
When controlling the global stress in the system as described in [45, 46], we also decompose the local stress into two parts, \(\sigma_{ij} = \sigma_{ij}^{ext} + \sigma_{ij}^{int}\), where \(\sigma_{ij}^{ext}\) is the externally applied uniform stress, and \(\sigma_{ij}^{int}\) describes the stress fluctuations resulting from the elastic interactions between regions undergoing plastic deformation (i.e. particle rearrangements). The stress contribution \(\sigma_{ij}^{int}\) is computed as the convolution product of the plastic deformation field \(\gamma_{ij}^{pl}\) with an interaction kernel \(G\) that accounts for long range elastic interactions in the system:
We study two different models for the activation of plastic events by an external noise:

\[ \sigma_{ij}^{\text{int}} = \mu \sum_{i'j'} G_{ij,i'j'}^* \gamma_{ij}^{\text{pl}}, \]

(1)

where \( \mu \) is the elastic modulus. The interaction kernel, \( G^* \), is of Eshelby’s type [42], i.e. it has a quadrupolar symmetry and a \( 1/r^d \) power law decay. It is expressed, in Fourier space, as: \( \hat{G}^*(\mathbf{q}) = -4i\frac{\mathbf{q}^2}{q^6} \) for \( q \neq 0 \) and \( \hat{G}^*(\mathbf{0}) = 0 \) so that \( \sigma_{ij}^{\text{int}} \) describes the local stress fluctuations in a macroscopically stress-free state (or equivalently, the space-averaged response to a plastic deformation is equal to zero). Applying a macroscopic driving stress \( \sigma_{ij}^{\text{ext}} \) induces a uniform shift of the local stresses without altering internal fluctuations described by \( \sigma_{ij}^{\text{int}} \).

When deforming plastically, a mesoscopic block is modelled as a viscoelastic Maxwell element with an elastic modulus \( \mu \) and a mechanical relaxation time \( \tau \). The strain rate \( \frac{d}{dt} \gamma_{ij}^{\text{pl}} \) produced by a plastic rearrangement occurring at a site \((i,j)\) thus reads:

\[ \frac{d}{dt} \gamma_{ij}^{\text{pl}} = n_{ij} \sigma_{ij}^{\text{ext}} \mu \tau = n_{ij} \sigma_{ij}^{\text{ext}} + \sigma_{ij}^{\text{int}}. \]

(2)

The state variable \( n_{ij} \) indicates whether the site deforms plasticly \((n_{ij} = 1)\) or elastically \((n_{ij} = 0)\), and has its own stochastic dynamics that will be described below.

2.2.2. Strain-imposed model

Another widely used protocol (both experimentally and numerically) consists in imposing the global strain rate \( \langle \dot{\gamma}(t) \rangle \) to the system [1, 2]. In order to impose a global deformation rate \( \langle \dot{\gamma}(t) \rangle \) to the system, one needs to adjust the value of the imposed stress depending upon the total deformation rate [46]:

\[ \dot{\sigma}^{\text{ext}}(t) = \mu \langle \dot{\gamma}(t) \rangle - \langle \dot{\gamma}^{\text{pl}}(t) \rangle. \]

(3)

The dynamics for the local stress thus reads:

\[ \frac{\partial \sigma(x, t)}{\partial t} = \sigma^{\text{ext}}(t) + \sigma^{\text{int}}(x, t) = \mu \left( \langle \dot{\gamma}(t) \rangle - \langle \dot{\gamma}^{\text{pl}}(t) \rangle \right) + \frac{\partial}{\partial t} \int d^d x' G^*(x - x') \gamma^{\text{pl}}(x', t). \]

(4)

By expliciting \( \langle \dot{\gamma}^{\text{pl}}(t) \rangle \), we get:

\[ \frac{\partial \sigma(x, t)}{\partial t} = \mu \langle \dot{\gamma}(t) \rangle + \mu \int d^d x' G^*(x - x') - \frac{1}{L^d} \frac{\partial}{\partial t} \int d^d x' \gamma^{\text{pl}}(x', t). \]

(5)

We rewrite the above expression with discretized spatial coordinates \((i,j)\) using \( G = G^* - 1/L^d \) and replacing \( \langle \dot{\gamma}(t) \rangle \) by \( \dot{\gamma} \) to lighten notations:

\[ \frac{d}{dt} \sigma_{ij} = \mu \dot{\gamma} + \mu \sum_{i'j'} G_{ij,i'j'} \frac{d}{dt} \gamma_{ij}^{\text{pl}} \]

(6)

with \( d\gamma_{ij}^{\text{pl}}/dt = n_{ij} \sigma_{ij}/\mu \tau \). The interaction kernel \( G \) is also of Eshelby’s type [42], but unlike in the stress-controlled case, \( \hat{G}(0) = 0 \) is determined by the integral over the whole system of the elastic response to a localised plastic strain.

2.2.3. Stochastic dynamics for the plastic activity

Besides the dynamics described in either equation (2) or equation (6) depending upon the driving protocol, each node alternates between a local plastic state \((n_{ij} = 1)\) and a local elastic state \((n_{ij} = 0)\). Since the detailed local yielding rules will depend sensitively on the microscopic model and since its detailed characterisation is still missing, various different phenomenological transition rules between elastic and plastic states have been proposed in the literature [19, 47–49]. In this study, we use the rules introduced by Picard in [47]. Upon loading, once the local stress has overcome a threshold, \( |\sigma_{ij}| > \sigma_{y} \), it yields plastically \((n_{ij} = 0 \rightarrow 1)\) with a rate \( 1/\tau_{pl} \). Once a site has become plastic, it relaxes stress with a relaxation time \( \tau \) and becomes elastic again \((n_{ij} = 1 \rightarrow 0)\) after a typical time \( \tau_{el} \). We consider in this work that a fluidising noise induces additional plastic events \((n_{ij} = 0 \rightarrow 1)\) with a ‘vibration rate’ \( k_{vib} \). The three different types of transitions between elastic and plastic states are summarised below:

\[
\begin{align*}
n_{ij}(t) & : 0 \xrightarrow{1/\tau_{pl}} 1 \quad \text{if} \quad |\sigma_{ij}| > \sigma_{y} \\
n_{ij}(t) & : 0 \xrightarrow{k_{vib}} 1 \quad \forall \sigma_{ij} \\
n_{ij}(t) & : 1 \xrightarrow{1/\tau_{el}} 0
\end{align*}
\]

(7)

We study two different models for the activation of plastic events by an external noise:
(i) Model 1: constant activation rate: \( k_{vib} = 1/\tau_{vib} \) for any value of the local stress \( \sigma_i \)

(ii) Model 2: Arrhenius-like activation: \( k_{vib} = k_0 e^{\lambda_{vib}(\sigma_i - \sigma)} \) with \( k_0 \) a pre-factor kept constant (\( k_0 = 1 \)) in our study and \( \lambda_{vib} \) controlling the magnitude of the noise.

These activated events have the same properties as the ones induced by shear, i.e. they lead to a redistribution of stress in the system through the Eshelby propagator.

In the following, the values of stress, strain rate and time are respectively given in units of \( \sigma_f, \sigma_f/\mu \tau \) and \( \tau \). We set \( \tau_{el} = 1 \) and the restructuring time \( \tau_{el} = 10 \) is chosen large compared to the other timescales in the system in order to induce local softening. Long restructuring times lead to non-monotonic flow curves [27] and are associated with permanent shear bands when imposing the shear rate in the system [28].

We study the influence of an external noise by varying the value of the vibration rate \( k_{vib} \), either varying \( \tau_{vib} = k_{vib}^{-1} \) for Model 1 (random activation), or \( \lambda_{vib} \) for Model 2 (Arrhenius-type activation) using both shear rate and stress controlled driving protocols, as they give access to different flow features in the case of non-monotonic flow curves. As we are interested in bulk quantities, we simulate the above elasto-plastic model using periodic boundary conditions in all directions. We perform large scale simulations of the elasto-plastic model, using a GPU-based parallel implementation to integrate the dynamics [50], with an integration time step \( dt = 10^{-2} \), at least two orders of magnitude smaller than all other relevant timescales in the dynamics.

### 3. Average flow features

#### 3.1. Rheology: flow curves

We compute the average steady state shear stress as a function of the imposed shear rate (flow curve). In the absence of a fluidising noise (\( k_{vib} = 0 \)) the elasto-plastic model exhibits a dynamical yield-stress [47], as shown by the finite stress plateau at low shear rate in the upper dark blue curve in figure 1(a). With the value of restructuring time \( \tau_{el} = 10 \) used in this work, the rheology in the absence of fluidising noise is non-monotonic, i.e. the underlying constitutive flow curve has a minimum which is set by the choice of \( \tau_{el} \) [28]. In the mean-field approximation of the model the minimum of the constitutive flow curve for \( \tau_{el} = \tau = 1 \) and \( \tau_{el} = 10 \) can be estimated to a shear rate of about \( \dot{\gamma} \approx 0.06 \) [28]. For finite dimensions and sufficiently large systems, this (unstable) non-monotonic flow curve cannot be observed in simulations, due to the existence of shear bands (leading to a stress plateau, as shown for intermediate shear rates in the top curve of figure 1(a)).

Flow curves obtained for the two models of noise are shown in figure 1(a), with magnitudes \( k_{vib} = 3.3 \times 10^{-4} \) for Model 1 (random activation) and \( \lambda_{vib} = 3.3 \times 10^{-2} \) for Model 2 (Arrhenius) respectively. Let us first describe the generic features of the flow curves for dynamics with an additional fluidising noise (\( k_{vib} = 0 \)). First, the system exhibits a fluid-like behaviour at low stress (and \( \dot{\gamma} < 10^{-3} \)), as shown by the absence of stress plateau at low shear rate. This regime is followed by a stress-plateau (\( 10^{-3} < \dot{\gamma} < 10^{-1} \)), associated with a shear banding instability (see figure 3). This shear banding instability emerges due to an underlying non-monotonic constitutive flow curve, analogous to the case when the fluidising noise is absent. For finite dimensional systems the negative-slope in the flow curve is mechanically unstable [51], leading to a shear-band instability, if the system size is large enough to accommodate the wavelength associated with the
Figure 2. Shear stress \( \Sigma \) as a function of the imposed shear rate \( \dot{\gamma} \) for various noise magnitude for (a) the random activation model (Model 1, \( k_{vib} \) ranging from \( 10^{-3} \) to \( 3.3 \times 10^{-3} \)) and (b) the Arrhenius activation model (Model 2, \( \lambda_{vib}^{-1} \) ranging from \( 2 \times 10^{-3} \) to \( 10^{-4} \)), for a system size \( L = 256 \). The upper curve in (a) and (b) is obtained in absence of noise.

instability [52]. Note that negative slopes can still be observed in the flow curves resulting from our simulations (around \( \dot{\gamma} = 10^{-3} \)) due to finite-size effects. Sheared banded profiles will be further discussed in section 3.2. The last regime (\( \dot{\gamma} > 10^{-1} \)) corresponds to a stable homogeneous flow.

While the global shape of the flow curve is the same, the rheological behaviour in the low stress regime differs between the two models of activation. The random activation rule (Model 1) leads to a Newtonian behaviour at low shear rates, with a constant viscosity \( \eta = \Sigma/\dot{\gamma} \) as shown in figure 1 (b), whereas the Arrhenius-like rule (Model 2) leads to a logarithmic-like flow behaviour, reminiscent of experiments on vibrated granular media (see figure 1 in the work by Wortel et al. [24]). This can be understood from the activation rule, using a simplified argument in the low shear-rate regime, as detailed in appendix D. This calculation is based on two simplifying assumptions: (i) that the internal mechanical noise is negligible, in this regime, with respect to the external noise, and (ii) that the plastic relaxation fully relaxes the local stress and that its duration is negligible with respect to the duration of the elastic phase. Within this simplified dynamics, we obtain for Model 1 the linear behaviour

\[
\Sigma = \frac{\sigma_c}{2} + \frac{1}{2\lambda} \ln \left( \frac{\lambda \mu}{k_0 \dot{\gamma}} \right).
\]

(8)

Flow curves for various noise magnitudes are depicted in figure 2 for the two models of activation (by varying either \( k_{vib} = 1/\lambda_{vib} \) in Model 1 or \( \lambda_{vib}^{-1} \) in Model 2). For the two models, the effect of noise is (i) a fluidisation (vanishing yield stress) at any value of the noise magnitude and (ii) a transition from a non-monotonic to a monotonic flow curve at a noise magnitude, \( k_{vib}^{-1} = (1.3 \pm 0.2) \times 10^{-3} \) or \( \lambda_{vib}^{-1} = 20 \pm 2 \) (the thick black lines in figure 2 correspond to \( k_{vib}^{-1} = 1.25 \times 10^{-3} \) and \( \lambda_{vib}^{-1} = 20 \)).

3.2. Shear rate profiles

Figure 3 depicts profiles of shear rate in steady state averaged over a strain window of 1000, for the two models of noise. In the low noise regime (non-monotonic flow curves) the system separates into two flowing regions (blue curves in figure 3), where the minimum and the maximum of the shear rate profile are determined by the boundaries of the stress plateau in figure 2. This is similar to the shear bands that occur in the mesoscale elastoplastic model in the absence of fluidising noise, leading to the coexistence of a flowing and an arrested region instead [28]. In our model, the difference in shear rate between the two bands decreases as the magnitude of the noise is increased, until reaching a stable homogeneous flow regime as shown by the flat profiles (light purple curves in figure 3) corresponding to a monotonic flow curve.

The transition from a phase separated flow to a homogeneous flow can thus be characterised using the difference in shear rate between the two flowing bands. We define the order parameter of this transition as the logarithm of the ratio of shear rates in the two flowing bands:

\[
\text{Order parameter} = \log \left( \frac{\dot{\gamma}_{\text{fast}}}{\dot{\gamma}_{\text{slow}}} \right) = S_{\text{fast}} - S_{\text{slow}}
\]

(9)

with \( S = \log(\dot{\gamma}) \).
4. Giant shear-rate fluctuations

We now investigate in more details the high noise regime corresponding to $k_vib > k^{\text{r}}_vib$ (Model 1) or $\lambda_vib^{-1} > \lambda_vib^{-1}$ (Model 2), where the flow curve is monotonously increasing, so that the homogeneous flow is stable. In this regime, using shear- or stress-controlled protocols leads to the same average rheological behaviour. Interesting properties are rather to be found in shear-rate fluctuations (using a stress-controlled protocol), that we first describe qualitatively.

We measure the macroscopic shear rate in the system as a function of time in steady state, for two different values of $k_vib$ (for Model 1)(figure 4), choosing stress values corresponding to the inflexion point of the flow curve. For large values of the noise magnitude ($k_vib = 5 \times 10^{-3}$, lower red curve in figure 4), the fluctuations of shear rate $\dot{\gamma}$ are relatively small (variations of about 10% of the mean value for a system size $N = 512^2$) and not correlated in time. When decreasing the noise towards its value at the transition between monotonic and non-monotonic flow curves ($k_vib = 1.32 \times 10^{-3}$, see figure 2), the fluctuations of $\dot{\gamma}$ increase (50%) and become correlated in time, as it can be seen on the upper (blue) curve of figure 4.

4.1. Rescaled shear-rate distributions

To perform a quantitative analysis, we compute the distributions of macroscopic shear rate in steady state for various system sizes $N$ and noise magnitudes $k_vib$, again for a stress value corresponding to the inflexion point of
the flow curves. From the central limit theorem, one expects relative fluctuations of the shear rate to scale as $1/\sqrt{N}$ for large system sizes. Figure 5 depicts the centred distributions of $\dot{\gamma}$ computed for Model 1 (random activation) at the inflexion point of the flow curve for various system sizes (from $L = 32$ to $L = 512$), with the x-axis rescaled by multiplying $\dot{\gamma}$ by the linear system size $L = \sqrt{N}$ and the y-axis by dividing $P(\dot{\gamma})$ by $\sqrt{N}$, in the stable flow phase (monotonic flow curve), for decreasing values of the noise magnitude: (a) $k_{\text{vib}} = 5 \times 10^{-3}$, (b) $k_{\text{vib}} = 2 \times 10^{-3}$, (c) $k_{\text{vib}} = 1.35 \times 10^{-3}$ and (d) $k_{\text{vib}} = 1.32 \times 10^{-3}$. Inset of (a): Lin-log plot of the distribution for $L = 256$ with Gaussian fit (red). Inset of (d): Finite size data collapse of the shear rate distributions using an exponent $x = 0.275$.

Figure 5. Centred distributions of macroscopic shear rate $\dot{\gamma}$ computed for Model 1 (random activation) at the inflexion point of the flow curve for various system sizes (from $L = 32$ to $L = 512$), with the x-axis rescaled by multiplying $\dot{\gamma}$ by the linear system size $L = \sqrt{N}$ and the y-axis by dividing $P(\dot{\gamma})$ by $\sqrt{N}$, in the stable flow phase (monotonic flow curve), for decreasing values of the noise magnitude: (a) $k_{\text{vib}} = 5 \times 10^{-3}$, (b) $k_{\text{vib}} = 2 \times 10^{-3}$, (c) $k_{\text{vib}} = 1.35 \times 10^{-3}$ and (d) $k_{\text{vib}} = 1.32 \times 10^{-3}$. Inset of (a): Lin-log plot of the distribution for $L = 256$ with Gaussian fit (red). Inset of (d): Finite size data collapse of the shear rate distributions using an exponent $x = 0.275$. 

The flow curves. From the central limit theorem, one expects relative fluctuations of the shear rate to scale as $1/\sqrt{N}$ for large system sizes. Figure 5 depicts the centred distributions of $\dot{\gamma}$ rescaled by $\sqrt{N}$ for different system sizes $N$. In this representation, curves collapse if relative fluctuations scale like $1/\sqrt{N}$. For large noise magnitudes (figure 5(a), $k_{\text{vib}} = 5 \times 10^{-3}$), the data for all system sizes collapse onto the same curve, indicating that the shear rate fluctuations obey the central limit theorem. Unsurprisingly, the shear rate fluctuations in this regime follow a Gaussian distribution as shown by the fit in the inset of figure 5(a). As the noise magnitude is decreased (figure 5(b)–(d)), the rescaled distributions widen and a systematic dependence with the system size appears. This indicates a deviation from the central limit theorem at the approach of the transition, associated with growing spatial correlations of the macroscopic shear rate. Moreover, the maximum system sizes for which finite size effects are observed in figure 5 give an estimate of the correlation length $\xi$ in the system as the noise is varied (in figure 5(b) $32 < \xi < 64$ for $k_{\text{vib}} = 2 \times 10^{-3}$, (c) $64 < \xi < 128$ for $k_{\text{vib}} = 1.35 \times 10^{-3}$ and (d) $\xi > 512$ for $k_{\text{vib}} = 1.32 \times 10^{-3}$). Note that for a linear system size $L (= \sqrt{N})$ much larger than the correlation length of the system, the standard $1/\sqrt{N}$ scaling is recovered, together with a Gaussian shape of the distribution. This crossover is visible in figures 5(b) and (c). The increase of the correlation length when decreasing the noise indicates a possibly diverging length scale in the system at the transition, which is consistent with the existence of a critical point [24, 41]. We show in the inset of figure 5(d) that the distributions for $k_{\text{vib}} = 1.32 \times 10^{-3}$ can be approximately collapsed by rescaling the shear rate by $N^x$ with $x \approx 0.275$ (instead of 1/2 far from the transition). This shows that relative fluctuations of the shear rate decay approximately as $1/N^{0.275}$ with system size, that is,
much more slowly than the standard $1/\sqrt{N}$ scaling corresponding to the central limit theorem. This slower decay of relative fluctuations with system size can be described as the presence of giant shear-rate fluctuations. These giant shear-rate fluctuations are directly visible in an experimental context [24], and their characterisation is thus of interest.

Instead of looking at the dependence of fluctuations on system size, one may also look at their dependence on stress. In figure 6, we depict the variance of $S = \log(\Sigma)$, $\Delta S^2$, as a function of the distance (in stress) to the inflexion point of the flow curve, $\Sigma - \Sigma_i$, for various noise magnitudes and for a system size $L = 128$. (a) Model 1, for values of $k_{\text{rel}}$, ranging from $1.29 \times 10^{-3}$ (upper purple curve) to $1.75 \times 10^{-3}$ (lower red curve). (b) Model 2, for values of $\lambda_{\text{vib}}^{-1}$ ranging from $5.12 \times 10^{-2}$ to $6.6 \times 10^{-1}$.

Figure 6. Variance of $S = \log(\Sigma)$, $\Delta S^2$, as a function of the distance (in stress) to the inflexion point of the flow curve, $\Sigma - \Sigma_i$, for various noise magnitudes and for a system size $L = 128$. (a) Model 1, for values of $k_{\text{rel}}$, ranging from $1.29 \times 10^{-3}$ (upper purple curve) to $1.75 \times 10^{-3}$ (lower red curve). (b) Model 2, for values of $\lambda_{\text{vib}}^{-1}$ ranging from $5.12 \times 10^{-2}$ to $6.6 \times 10^{-1}$.

4.2. Origin of non-standard fluctuations
The reason for the emergence of fluctuations with non-trivial statistics is actually the presence of a critical point, around which the system becomes correlated over a large length scale. Quite interestingly, non-trivial scalings are found close to the critical point not only for fluctuations, but also for average quantities, which may make some of the critical properties easier to measure. Both average quantities and fluctuations exhibit power-law scalings at the transition, and the critical point is described by a set of critical exponents, which are supposed not to depend on details of the system, but rather only on generic properties shared by a broad class of systems. For instance, one may expect that both Model 1 and Model 2 share the same critical properties. This statement, though, does not rest on any firm theoretical consideration, and should be tested numerically. This is one of the goals of the next section. In a previous work [41], we characterised in detail this critical point for the random activation model (Model 1). In the next section, we perform a similar analysis for the Arrhenius activation model (Model 2), and show that the scaling of both average quantities and fluctuations does not depend on the model of noise in the critical regime.

5. Generic critical point at finite shear and vibration rates
The following analysis of the critical-like behaviour at the transition point is organised into two parts. In a first part, we study the evolution of average steady state quantities (flow curve and order parameter) as the noise magnitude is varied across the transition from a phase separated to a homogeneous flow. The second part is dedicated to the scaling of shear rate fluctuations. The aim of these measurements is to understand how generic the emerging critical features are, given that they can be interpreted as resulting from an out-of-equilibrium phase transition.
In the following, the noise magnitude is designated by the relative distance to the critical point, which reads, for the two models of noise:

\[
\varepsilon = \frac{k_{\text{vib}} - k_{\text{vib}}^c}{k_{\text{vib}}^c} \quad \text{(Model 1)},
\]

\[
\varepsilon = \frac{\lambda_{\text{vib}}^{-1} - \lambda_{\text{vib}}^{-1}^c}{\lambda_{\text{vib}}^{-1}^c} \quad \text{(Model 2)}.
\]

The stable flow (noise dominated) regime thus corresponds to \( \varepsilon > 0 \) and the phase separated regime to \( \varepsilon < 0 \). We summarise below the list of scalings expected at the transition.

In the regime \( \varepsilon < 0 \), the order parameter \( S_{\text{fast}} - S_{\text{slow}} \) vanishes as the noise magnitude is increased towards its critical value as a power law with an exponent \( \beta \):

\[
S_{\text{fast}} - S_{\text{slow}} \sim |\varepsilon|^\beta \quad \text{for} \quad \varepsilon < 0.
\]

The critical point corresponds to the transition from a monotonic to a non-monotonic flow curve, and hence the slope of the flow curve at the inflexion point vanishes at the critical point. It can be interpreted as an inverse susceptibility \( \chi \), expected to scale as a power law of the distance to the critical point with an exponent \( \gamma \):

\[
\chi \sim \varepsilon^{-\gamma} \quad \text{for} \quad \varepsilon > 0.
\]

At the critical point (\( \varepsilon = 0 \)), the flow curve exhibits a zero slope and \( S \) is expected to vary as a power law of the imposed shear stress (after centring the flow curve using the coordinates of the critical point \( (S_c, \Sigma_c) \)) with an exponent \( 1/\delta \):

\[
|S - S_c| \sim |\Sigma - \Sigma_c|^{1/\delta} \quad \text{for} \quad \varepsilon = 0.
\]

The variance of \( S \) diverges at the critical point when varying the imposed stress:

\[
\Delta S^2 \sim |\Sigma - \Sigma_c|^{-\delta} \quad \text{for} \quad \varepsilon = 0.
\]

The variance and the correlation time of \( S \) diverge as the critical point is approached from the high noise regime:

\[
\Delta S^2 \sim \varepsilon^{-\gamma\delta} \quad \text{for} \quad \varepsilon > 0.
\]

\[
\tau_{\text{corr}} \sim \varepsilon^{-\mu} \quad \text{for} \quad \varepsilon > 0.
\]

5.1. Scaling of average quantities

5.1.1. Scaling of the flow curves and susceptibility

Using a stress-controlled protocol, we compute the steady state flow curve in the stable flow regime (\( \varepsilon > 0 \)), for the two models of fluidising noise and investigate the scaling of the stress \( \Sigma \) as a function of \( S = \log(\gamma) \). The data for the two models can be well fitted to a Landau type expansion in the critical regime:

\[
\Sigma = \Sigma_i + a(S - S_i)^{\delta} + b(S - S_i),
\]

where \( a, b, S_i \) and \( \Sigma_i \) are fitting parameters shown in appendix C (figure C1) for various values of \( \varepsilon \) and system sizes \( L \).

Using these fitting parameters, the data from the two models can be collapsed onto the same master curve, as shown in figure 7. \( a(\varepsilon) \) is roughly constant (figure C1(a)). The coordinates of the inflexion point \( (S(\varepsilon),\Sigma(\varepsilon)) \) evolve monotonously as the noise is varied, and describe the analogous of the super-critical liquid-gas boundary in equilibrium phase transitions (figures C1(c)–(d)). The pre-factor \( b \) of the linear term in equation (18), (figure C1(b)), decreases linearly with \( \varepsilon \) and vanishes at the critical point. As it describes the slope of the flow curve at the inflexion point, it is interpreted as the inverse susceptibility \( b = 1/\chi \), which diverges at the critical point with an exponent \( \gamma \approx 1 \) (equation (13)) (see figure 10). At the critical point, \( b = 0 \) and hence the flow curve (‘critical isotherm’) is well described by equation (14), with \( \delta = 3 \).

5.1.2. Critical point location

To locate the critical value of noise, we fit the data of figure C1(b) to extract the value of \( k_{\text{vib}} \) and \( \chi_{\text{vib}} \) for which \( b = 0 \) (diverging susceptibility). While significant finite size effects are observed for Model 1 (random) (and discussed in detail in [41]), the finite size effects on the value of \( b \) remain within the error bars for Model 2. We find \( k_{\text{vib}} = (1.35 \pm 0.01) \times 10^{-3} \) for Model 1 [41] and \( \chi_{\text{vib}}^{-1} = (5.1 \pm 0.2) \times 10^{-2} \) for Model 2.

5.1.3. Order parameter

In figure 8, we investigate the scaling of the order parameter \( S_{\text{fast}} - S_{\text{slow}} \), extracted from the shear rate profiles (figure 3) at the approach of the transition, in the regime \( \varepsilon < 0 \). The order parameter decreases as \( \varepsilon \) is decreased, and scales as a power law of the distance to the critical point (fit of the form \( S_{\text{fast}} - S_{\text{slow}} = A|\varepsilon|^\delta \)). In the fit of
We get $\beta_1 = 0.58 \pm 0.07$ (Model 1) and $\beta_2 = 0.49 \pm 0.04$ (Model 2), with $k_{\text{vib}} = (1.21 \pm 0.05) \times 10^{-3}$ and $(\lambda_{\text{vib}})^{-1} = 4.95 \pm 0.05) \times 10^{-2}$. For the two models, the critical exponent $\beta$ is close to 0.5, which is consistent with the Landau-type scaling in the regime $\varepsilon > 0$.

The critical noise magnitudes $(k_{\text{vib}}, (\lambda_{\text{vib}})^{-1})$ obtained from the fit for a system size $L = 512$, although they are not far from the previous estimate, are slightly underestimated. In fact, extracting the order parameter from the flow profiles is a difficult task near the critical point, as the coarsening time of the shear bands increases (and diverges for $\varepsilon = 0$). As a consequence, we cannot access steady state profiles in the critical regime. The configurations associated with the closest data points to the critical point in figure 8 are not coarsened yet, thus
Formally, we performed direct measurements of the critical noise magnitudes. Using a stress-controlled protocol in our previous work leading to large error bars and possibly explaining the slight underestimate of \( \Delta S \) as a function of \( |\Sigma - \Sigma_c| \) for values of the noise magnitude close to the critical value \( (\varepsilon \approx 0) \). Data for Model 1 (black empty data points) and Model 2 (red solid points) and various system sizes (circles: \( L = 128 \), squares: \( L = 256 \), triangles: \( L = 512 \)). Dashed lines: power-law guides to the eye with exponents 0.6 and 0.9.

5.1.4. Finite size effects
Let us now recall some of the main results of a former finite size data analysis that we performed for the random activation model (Model 1) [41], and that we will use to perform the finite size data collapse in this section. At the critical point, the correlation length of the system diverges as:

\[
\zeta \sim \varepsilon^{-\nu} \text{ for } \varepsilon > 0.
\]  

(19)

Formally, we performed direct measurements of \( \zeta \) for Model 1 [41] and found \( \nu = 0.95 \pm 0.05 \). This was consistent with the finite size shift of the transition, which yielded a value of \( \nu \approx 1 \).

Using \( \nu = 1 \), we show in figure 10(a) the data for the susceptibility \( \chi = 1/b \) rescaled by a factor \( L^{-\gamma/\nu} \) as a function of the scaled distance to the critical point \( \varepsilon L^{1/\nu} \). \( \varepsilon \) is expressed using the above value \( \kappa_{rb} = (1.35 \pm 0.01) \times 10^{-3} \) and refining the value of \( \lambda_{rb}^{-1} \) to get the best data collapse \( (\lambda_{rb}^{-1} = 5.15 \times 10^{-2}) \). The ability to collapse the data for all system sizes for the two models of noise suggests that they share similar critical exponents for the susceptibility and the correlation length \( (\gamma \approx 1 \text{ and } \nu \approx 1) \).

5.2. Scaling of fluctuations
In equilibrium systems, fluctuations are related to average quantities like the susceptibility, and at a critical point, the divergences of both quantities are related. In this part we investigate the scaling of the fluctuations of \( S \) when varying the imposed stress and the noise magnitude in the critical regime.

5.2.1. Varying the imposed stress in the critical regime:
As shown in figure 6, the variance of the order parameter increases as the stress approaches its value at the inflexion point of the flow curve, where it is maximal. In figure 9, we report, in a log–log plot, for the value of the noise closest to the critical point \( (\varepsilon \approx 0) \), the value of the variance of \( S \) as a function of the distance to the inflexion point, for various system sizes. Let us point out that the scattering of data near the maximum is due to finite size effects, where the smallest values correspond to the smallest systems. We find that the variance of \( S \) varies as a power law of the distance to the critical stress, with exponents \( \kappa_1 = 0.82 \pm 0.12 \) (Model 1) and \( \kappa_2 = 0.73 \pm 0.15 \) (Model 2), that do not depend significantly on the noise model. From the Landau fit of equation (18), the susceptibility \( (\chi = \partial S / \partial \Sigma \text{ for } \varepsilon = 0) \) varies as \( (\Sigma - \Sigma_c)^{1/2} \), which is within the error bars of our estimate of \( \kappa \). Note that a similar scaling is found for the correlation time of the fluctuations (see appendix B, figure B1(b)).
In this work, we studied two models of a fluidising noise leading to a transition between a self-fluidised regime and an externally fluidised regime in the flow of soft glassy materials. Upon increase of the external noise amplitude we evidence the vanishing of shear bands, which is an indication of a transition from an unstable shear-localised flow to a stable homogeneous flow. We find that a decrease of the fluidising noise amplitude in the homogeneous flow regime yields increasingly large and long lived fluctuations of the macroscopically measurable shear rate. In this scenario a single trajectory of the system can strongly differ from its average behaviour as given by the constitutive flow curve, thus leading to several difficulties for experimental characterisations of the flow behaviour.

5.2.2. Varying the noise magnitude $\varepsilon$

We now investigate the scaling of fluctuations at the inflexion point of the flow curve ($\Sigma = \Sigma_i$) when varying the noise magnitude $\varepsilon$. We compute the variance of the fluctuations and extract their correlation time from an exponential fit of the auto-correlation of $S$. A finite size data collapse is performed using $\nu = 1$. We find, for the two models of noise, a power law increase of both the variance (Figure 10(b)) and the correlation time (figure 5(c)) of the fluctuations when approaching the critical point, with the exponents $\gamma_S \approx 0.9$ and $\mu \approx 1$ respectively.

Let us now discuss the scaling of shear rate distributions depicted in figure 5(d) in the light of the critical exponents estimated in this section. In figure 5(d), the shear rate distributions at the transition could be approximately collapsed by rescaling the shear rate with a factor $N^x (N = L^2)$ and the best collapse was found with an exponent $x = 0.275$. In other words, the width of the distribution of shear rate scales as:

$$\Delta \gamma \sim L^{-2x} \sim L^{-0.55}.$$  

For sufficiently small fluctuations of the shear rate, the fluctuations of $S = \log(\gamma)$, $\Delta S$, correspond approximately to relative fluctuations of $\log(\gamma)$, $\Delta \log(\gamma)$, with $\gamma$ the critical shear rate (approximately independent of system size, see appendix C). Hence similar scalings for $\Delta S$ and $\Delta \log(\gamma)$ with the system size would be expected. From the data collapse of figure 5(d), the scaling for the shear rate variance is $L^2 \Delta \gamma^2 \sim L^{0.9}$. This is consistent with the scaling of figure 10(b), where data collapse for the variance of $S$ suggests a scaling form:

$$L^2 \Delta S^2 \sim L^{x/\nu},$$  

with $\gamma_S / \nu = 0.9$.

Note that, due to finite time limitations of our simulations in the critical region, the data for the correlation time of the fluctuations (figure 10(c)) is restricted to intermediate system sizes ($L \leq 512$) and exhibit strong scattering (see appendix B, figure B1(a)). An approximate collapse can still be performed with an exponent $\mu = 1.0 \pm 0.2$ and seems to be independent of the model of noise. As noted in [41], this value would correspond to a dynamic scaling exponent $z = \mu / \nu \approx 1$, far from the equilibrium mean-field value $z = 2$ obtained for non-conserved scalar order parameters [53].

6. Conclusion

While the extent of the power law spans two decades in the ‘shear dominated’ regime (for $\Sigma > \Sigma_i$, lower data points in figure 9), the range is reduced in the ‘noise dominated’ regime (for $\Sigma < \Sigma_i$, upper data points in figure 9). This asymmetry is likely to be due to the different origins of mechanical noise in these two regimes, where it arises mainly due to activated events in the ‘noise dominated’ regime.

Figure 10. (a) Susceptibility $\chi = 1/b$ rescaled by $L^{-\gamma_S / \nu}$ as a function of the scaled distance to the critical point $x = L^{z / \nu}$ for Model 1 (unfilled black symbols) and Model 2 (solid red symbols). (b) Rescaled variance $\Delta S^2$ versus $x = L^{z / \nu}$. (c) Rescaled correlation time $\tau_{\text{corr}}(S) = S^{-\mu / \nu}$ versus $x = L^{z / \nu}$.
Since the correlation time of the fluctuations becomes increasingly large at the transition point, averaging values in the steady state becomes tedious and care has to be taken in the data interpretation. This is a situation where conventional continuum descriptions of the flow tend to break down and call for more sophisticated modelling approaches that allow for incorporating the spatio-temporal features of the fluctuations in the rheologically relevant quantities.

Another important message regarding experimental and numerical studies is that, when using shear-imposed protocols, the flow can be heterogeneous due to the non-monotonicity of the constitutive flow curve. In the vicinity of the transition, the time for the shear bands to coarsen becomes increasingly large, and the steady state of the system may be out of reach on any experimentally relevant timescale.

All of the above features can be accounted for in the framework of non-equilibrium phase transitions, as resulting from an out-of-equilibrium critical transition point. The properties of this critical point, studied in detail for the random activation model in a previous work [41] appear to remain unchanged for the Arrhenius activation model. While the rheology at low stress differs between the two models, we find a generic scaling of the flow curves in the critical regime that is well fitted by a Landau-type expansion, as in [24]. The scaling of the order parameter (power law scaling with an exponent \( \beta \approx 0.5 \)), as determined from the flow profiles in the phase coexistence regime, is consistent with the scaling of the flow curves in the stable flow regime. We also find power law scalings of the susceptibility and the shear rate fluctuations, with exponents being independent of the model of noise.

In conclusion, the finite shear-rate critical point studied here in a minimal elasto-plastic model suggests that a generic critical behaviour arises in systems combining a non-monotonic flow curve with a fluidisation process, irrespective of the detailed physical mechanisms at play. This interpretation is reinforced when comparing the results of our model to an experimental work on granular media [24]. Here, the mechanism at the origin of self-fluidisation is inter-particle friction, and the external fluidisation is due to mechanical vibrations. Although our elasto-plastic model does not take into account any frictional dynamics and the fluidisation mechanism is simplified by introducing ad hoc rules for local yielding instead of details of an additional mechanical noise, we find similar critical exponents at the transition. Indeed our numerical values coincide within the error bars with the experimental results on granular material, suggesting a possible generic scenario, despite all these differences in the microscopic dynamics.

We discussed the consequences of competing self-fluidisation and external fluidisation mechanisms on the rheology of soft glassy materials and in the following we would like to suggest situations where this phenomenon might be of importance. As shown experimentally, this transition can easily arise in frictional granular materials [24]. But one could also speculate about other systems where such mechanisms could be at play. The minimal ingredients for the emergence of this critical point in systems sheared at a finite strain rate are: (i) a microscopic mechanism at the origin of self-fluidisation (such as an intrinsic timescale for restructuration in the material [27]), and (ii) a source of mechanical noise independent of the flow.

The order of magnitude of the critical shear rate \( \dot{\gamma}^c \) is mainly set by the shear rate corresponding to the minimum of the non-monotonic constitutive flow curve in the absence of fluidising noise \( \dot{\gamma}_{\text{min}} \) (in our case, \( \dot{\gamma}_{\text{min}} \approx 6 \times 10^{-2} \) [28] and \( \dot{\gamma} \approx 2 \times 10^{-2} \)). To observe a competition between self-fluidisation and external fluidisation (leading to critical dynamics), the rates of plastic activation due to shear (in the self-fluidisation regime) and due to the external activation must be of the same order of magnitude. For a real system with a given self-fluidisation mechanism, the shear rate corresponding to the minimum of the constitutive flow curve sets the order of magnitude of the external activation rate needed to observe critical dynamics. However, it is usually difficult in real systems to estimate how a noise magnitude translates into a plastic activation rate.

An example for a class of materials, that provides naturally various sources of additional mechanical noise with a fluidisation effect, are dense active systems, such as biological tissues [38–40]. Depending on the details of microscopic interactions, self-fluidisation could arise from a competition between intrinsic timescales in the system and shear; hence such systems could naturally display the emergence of critical dynamics accompanied by giant fluctuations as described in this work.

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Appendix A. Data analysis

We perform simulations of the 2d elasto-plastic model using shear- and stress-controlled protocols. Using a shear-imposed protocol, we measure the average steady-state stress in the system to compute the flow curve, by averaging over a strain window $\gamma = 50$. To compute the shear rate profiles in the shear banding regime, we average the profiles along the direction in which the flow is homogeneous and over a strain window $\gamma = 1000$. Using a stress-controlled protocol, we analyse time-series of the average shear rate in the system (of average duration $T = 2 \times 10^6$ for $L = 128$ and $L = 256$, $T = 6 \times 10^5$ for $L = 512$ and $T = 10^5$ for $L = 1024$, corresponding to strains ranging from $\gamma = 2000$ to $4 \times 10^4$). We compute $\langle S \rangle = \log(\dot{\gamma})$, as well as its variance $\Delta S^2$.

Appendix B. Temporal autocorrelation

To extract the correlation time $\tau_{\text{corr}}$, we fit the auto-correlation function $C(\tau) = \langle \Delta S(t + \tau) \Delta S(t) \rangle$ to an exponential and extract the characteristic time $\tau_{\text{corr}}$ from the fit. We report in figure B1(a) the correlation time as a function of the distance (in stress) to the inflexion point of the flow curve for various values of $\lambda_{\text{vib}}$ (Model 2), which exhibits a sharp peak at the critical point. We report in a log–log plot the correlation time as a function of the absolute distance to the inflexion point $|\Sigma - \Sigma_i|$, for a noise magnitude close to the critical point ($\varepsilon \approx 0$), for the two models of noise. Although the data is more noisy than for the variance of the fluctuations (see main text, figure 6), we can see there is a power-law scaling of the correlation time as well, with an exponent close to that of the variance.

![Figure B1](image_url)

**Figure B1.** (a) Auto-correlation timescale $\tau_{\text{corr}}$ as a function of $\Sigma - \Sigma_i$ for various values of noise magnitude $\lambda_{\text{vib}}$ (Model 2). (b) Log–log plot of the correlation time $\tau_{\text{corr}}$ as a function of $|\Sigma - \Sigma_i|$ for $\varepsilon \approx 0$ (‘critical isotherm’) for the two models of noise.
Appendix C. Landau expansion fit

Figure C1 depicts the fitting parameters from the Landau-like expansion fit (equation (18)): $a(\varepsilon)$, $b(\varepsilon)$, $S_i(\varepsilon)$ and $\Sigma_i(\varepsilon)$ for the two models of noise.

![Figure C1](image)

**Figure C1.** Parameters from the Landau-like expansion fit for various values of the noise $\varepsilon$ and various system sizes $L$, for Model 1 (black empty squares, dashed line) and Model 2 (red dot, solid line). (a) Pre-factor $a$; (b) inverse susceptibility $b$; (c) and (d) coordinates of the inflexion point shifted by the critical point location, respectively $S_i - S_c(c)$ and $\Sigma_i - \Sigma_c(d)$. 
Appendix D. Estimation of the flow curve at low shear rate

We provide here a simplified calculation of the flow curve \( \Sigma(\dot{\gamma}) \) in the low shear-rate regime, under several simplifying assumptions. First, we assume that in this regime, the internal mechanical noise can be neglected with respect to the external noise. Second, we assume that the local stress fully relaxes during a plastic event, and that the duration of plastic events can be neglected as compared to the duration of elastic phases.

Under these strong assumptions, the local elastic stress is expressed as \( \sigma = \mu \dot{\gamma} \Delta t \), where \( \Delta t \) is the time elapsed since the last plastic event, which relaxed the stress to \( \sigma = 0 \). Introducing the probability density function \( p(\Delta t) \) of the time interval \( \Delta t \), we get

\[
\Sigma = \mu \dot{\gamma} \int_0^{\Delta t_{\text{max}}} \Delta t \, p(\Delta t) \, d\Delta t. \tag{D.1}
\]

The upper bound \( \Delta t_{\text{max}} \) may be estimated by the condition \( \mu \dot{\gamma} \Delta t_{\text{max}} = \sigma \), from which we deduce \( \Delta t_{\text{max}} = \sigma / (\mu \dot{\gamma}) \). It follows that \( \Delta t_{\text{max}} \to \infty \) when \( \dot{\gamma} \to 0 \), and we can thus take the upper bound of the integral of equation (D.1) as infinite in the low shear-rate regime.

We now need to evaluate \( p(\Delta t) \) for both models. On general grounds, for an activation rate \( k_{\text{vb}}(\sigma) \), one can write using \( \sigma = \mu \dot{\gamma} \Delta t \),

\[
\frac{dp}{d\Delta t} = -k_{\text{vb}}(\mu \dot{\gamma} \Delta t) \, p(\Delta t), \tag{D.2}
\]

from which we get

\[
p(\Delta t) = p_0 \exp \left[ -\int_0^{\Delta t} k_{\text{vb}}(\mu \dot{\gamma} t') \, dt' \right] \tag{D.3}
\]

where \( p_0 \) is a normalization constant.

For model 1, where \( k_{\text{vb}} \) is a constant, we get

\[
\Sigma = \frac{\mu}{k_{\text{vb}}} \tag{D.4}
\]

For model 2, we have

\[
k_{\text{vb}}(\sigma) = k_0 e^{\lambda(\sigma-\sigma_1)} \tag{D.5}
\]

and thus

\[
p(\Delta t) = p_0 \exp \left[ -a(e^{\lambda \mu \dot{\gamma} \Delta t} - 1) \right], \tag{D.6}
\]

where we have defined

\[
a = \frac{k_0 \, e^{-\lambda \sigma}}{\lambda \mu \dot{\gamma}}. \tag{D.7}
\]

After some straightforward algebra, we find that the normalization constant is given by \( p_0 = \lambda \mu \dot{\gamma} \, e^{-a}/I(a) \), and the average stress by \( \Sigma = J(a)/[\lambda I(a)] \), where we have defined the auxiliary integrals

\[
I(a) = \int_1^\infty \frac{du}{u} e^{-au}, \quad J(a) = \int_1^\infty \frac{du}{u} \ln u \, e^{-au} \tag{D.8}
\]

(the integration variable \( u \) is obtained from the change of variable \( u = e^{\lambda \mu \dot{\gamma} \Delta t} \)). To get a more explicit expression in terms of \( \dot{\gamma} \), we try to perform an asymptotic expansion of the integrals \( I(a) \) and \( J(a) \). At first sight, one would guess that \( a \) is large because \( a \propto 1/\dot{\gamma} \) and \( \dot{\gamma} \) is small. However, looking explicitly at the typical values of the parameter \( a \) in the shear rate regimes explored numerically, we find that \( a \ll 1 \) as long as

\[
\dot{\gamma} \gg \frac{k_0 \, e^{-\lambda \sigma}}{\lambda \mu \dot{\gamma}} \sim 10^{-14} \tag{D.9}
\]

with the parameter values used in the simulation (the extremely small value \( 10^{-14} \), comes from the factor \( e^{-\lambda \sigma} \sim e^{-30} \)). In the regime \( a \ll 1 \), the exponential factor \( e^{-au} \) acts as a cut-off, thus providing an effective upper bound \( \sim 1/a \) in the integrals that would otherwise diverge logarithmically without the exponential factor.

As a simple approximation, we may thus write

\[
I(a) \approx \int_1^{1/a} \frac{du}{u} = \ln \frac{1}{a}, \quad J(a) \approx \int_1^{1/a} \frac{du}{u} \ln u = \frac{1}{2} \left( \ln \frac{1}{a} \right)^2. \tag{D.10}
\]

We thus eventually find

\[
\Sigma = \frac{J(a)}{\lambda I(a)} \approx \frac{1}{2 \lambda} \ln \frac{1}{a} = \frac{\sigma_1}{2} + \frac{1}{2 \lambda} \ln \left( \frac{\lambda \mu \dot{\gamma}}{k_0} \right). \tag{D.11}
\]
This logarithmic regime is valid in a finite shear-rate window: \( \gamma \) has to be larger than the (very small) lower bound given in equation (D.9), but it has to be small enough for the duration of plastic events and for the internal mechanical noise to be neglected.

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**References**

[1] Bonn D, Denn M M, Berthier L, Divoux T and Manneville S 2017 *Rev. Mod. Phys.* **89** 035005  
[2] Nicolas A, Ferrero E E, Martens K and Barrat JL 2018 *Rev. Mod. Phys.* **90** 045006  
[3] Herschel W H and Bulkley R 1926 *Colloid Polym. Sci.* **39** 291–300  
[4] Marmottant P and Graner F 2007 *Eur. Phys. J.* E **23** 337–47  
[5] Saramito P 2007 *J. Non-Newton. Fluid Mech.* **145** 1–14  
[6] Bocquet L, Colin A and Ajdari A 2009 *Phys. Rev. Lett.* **103** 036001  
[7] Fielding SM 2014 *Rep. Prog. Phys.* **77** 102601  
[8] Coussot P, Nguyen Q D, Huynh H and Bonn D 2002 *Phys. Rev. Lett.* **88** 175501  
[9] Lootens D, Van Damme H and Hebraud P 2003 *Phys. Rev. Lett.* **90** 178301  
[10] Cantat I and Pitois O 2006 *Phys. Fluids* **18** 083302  
[11] Barès J, Wang D, Wang D, Bertrand T, O’Hern C S and Behringer R P 2017 *Phys. Rev. E* **96** 052902  
[12] Pastore R, Ciamarra M P and Coniglio A 2011 *Phil. Mag.* **91** 2006–13  
[13] Srivastava I, Silbert L E, Grest G S and Lechman J B 2019 *Phys. Rev. Lett.* **122** 048003  
[14] Lemaitre A and Caroli C 2009 *Phys. Rev. Lett.* **103** 065501  
[15] Martens K, Bocquet L and Barrat J L 2011 *Phys. Rev. Lett.* **106** 155601  
[16] Tsukida B, Vandembroucq D and Maloney CE 2018 *Phys. Rev. Lett.* **121** 145501  
[17] Aragon A and Kuo H 1979 *Mater. Sci. Eng.* **39** 101–9  
[18] Schall P, Weitz D A and Spaepen F 2007 *Science* **318** 1895–9  
[19] Nicolas A, Martens K and Barrat J L 2014 *Europhys. Lett.* **107** 44003  
[20] Hebraud P and Lequeux F 1998 *Phys. Rev. Lett.* **81** 2934  
[21] Schall P and van Hecke M 2010 *Annu. Rev. Fluid Mech.* **42** 67–88  
[22] Mansard V, Colin A, Chauduri P and Bocquet L 2011 *Soft Matter* **7** 5524–7  
[23] Wortel G H, Dijkstra J A and van Hecke M 2014 *Phys. Rev. E* **89** 012202  
[24] Wortel G, Dauchot O and van Hecke M 2016 *Phys. Rev. Lett.* **117** 198002  
[25] DeGiuli E and Wyatt M 2017 *Proc. Natl Acad. Sci.* **114** 9284–9  
[26] Nicolas A, Barrat J L and Rottler J 2016 *Phys. Rev. Lett.* **116** 058303  
[27] Coussot P and Ovarez G 2010 *Eur. Phys. J.* E **33** 183–8  
[28] Martens K, Bocquet L and Barrat J L 2012 *Soft Matter* **8** 4197–205  
[29] Chattoraj J, Caroli C and Lemaitre A 2010 *Phys. Rev. Lett.* **105** 266001  
[30] Ikeda A, Berthier L and Sollich P 2012 *Phys. Rev. Lett.* **109** 018301  
[31] D’anna G, Mayor P, Barrat A, Loreto V and Nori F 2003 *Nature* **424** 909  
[32] Caballero-Robledo G A and Clément E 2009 *Eur. Phys. J.* E **30** 395  
[33] Jia X, Brunet T and Laurent J 2011 *Phys. Rev. E* **84** 020301  
[34] Hantot C, De Richter S K, Marchal P, Michot L J and Baravin C 2012 *Phys. Rev. Lett.* **108** 198301  
[35] Gnoli A, Lasanta A, Saracino A and Puglisi A 2016 *Sci. Rep.* **6** 38604  
[36] Gibaud T et al 2020 Rheoaoustic gels: tuning mechanical and flow properties of colloidal gels with ultrasonic vibrations *Physical Review X* **10** 011028  
[37] Cohen-Addad S, Höhler R and Khidas Y 2004 *Phys. Rev. Lett.* **93** 028302  
[38] Mandal R, Bhuyan P J, Rao M and Dasgupta C 2016 *Soft Matter* **12** 6268–76  
[39] Thijung E and Berthier L 2017 *Phys. Rev. E* **96** 050601  
[40] Matoz-Fernandez D, Agoritsas E, Barrat J L, Bertin E and Martens K 2017 *Phys. Rev. Lett.* **118** 158105  
[41] Le Goff M, Bertin E and Martens K 2019 *Phys. Rev. Lett.* **123** 108003  
[42] Elshelby J D 1957 *Proc. R. Soc.* **241** 376–96  
[43] Bouttes D and Vandembroucq D 2013 *AIP Conf. Proc.* **1518** 481–6  
[44] Merabia S and Detcheverry F 2016 *Europhys. Lett.* **116** 46003  
[45] Liu C, Ferrero E E, Martens K and Barrat J L 2018 *Soft Matter* **14** 8306–16  
[46] Liu C 2016 Critical dynamics at the yielding transition and creep behavior of amorphous systems: mesoscopic modeling PhD Thesis University Grenoble-Alpes  
[47] Picard G, Ajdari A, Lequeux F and Bocquet L 2005 *Phys. Rev. E* **71** 010501  
[48] Lin J, Saade A, Lerner E, Rosso A and Wyatt M 2014 *Europhys. Lett.* **105** 26003  
[49] Ferrero E and Juga E 2019 *Soft Matter* **15** 9041–55  
[50] Liu C, Ferrero E E, Puosi F, Barrat J L and Martens K 2016 *Phys. Rev. Lett.* **116** 065501  
[51] Yerushalmi J, Katz S and Shimmar R 1970 *Chem. Eng. Sci.* **25** 1891–902  
[52] Vassil V V, Le Goﬀ M, Martens K and Barrat J L 2018 arXiv:1812.03948  
[53] Hohenberg P C and Halperin B I 1977 *Rev. Mod. Phys.* **49** 435