Quantum Bäcklund transformation
for the integrable DST model

V B Kuznetsov,† M Salerno‡ and E K Sklyanin§

†Department of Applied Mathematics, University of Leeds, Leeds LS2 9JT, UK
‡Department of Theoretical Physics, University of Salerno 84081, Salerno, Italy
§ Laboratoire de Physique, Groupe de Physique Théorique, ENS Lyon, 46 allée
d’Italie, 69364 Lyon 07, France

Abstract

For the integrable case of the discrete self-trapping (DST) model we construct a Bäcklund transformation. The dual Lax matrix and the corresponding dual Bäcklund transformation are also found and studied. The quantum analog of the Bäcklund transformation ($Q$-operator) is constructed as the trace of a monodromy matrix with an infinite-dimensional auxiliary space. We present the $Q$-operator as an explicit integral operator as well as describe its action on the monomial basis. As a result we obtain a family of integral equations for multivariable polynomial eigenfunctions of the quantum integrable DST model. These eigenfunctions are special functions of the Heun class which is beyond the hypergeometric class. The found integral equations are new and they shall provide a basis for efficient analytical and numerical studies of such complicated functions.

31 July 1999

Submitted to J. Phys. A: Math. Gen.

1E-mail: vadim@amsta.leeds.ac.uk
2On leave from: Steklov Mathematical Institute at St. Petersburg, Fontanka 27, St. Petersburg 191011, Russia. E-mail: sklyanin@pdmi.ras.ru
3UMR 5672 du CNRS et de l’ENS Lyon
1. Introduction

The discrete self–trapping (DST) equation was introduced by Eilbeck, Scott and Lomdhal [1] to model the nonlinear dynamics of small molecules, such as ammonia, acetylene, benzene, as well as large molecules, such as acetanilide. In simple terms, it consists of a set of \( n \) nondissipative anharmonic oscillators coupled through dispersive interactions. Due to the nonlinearity this system can have complicated dynamical behaviour going from quasiperiodic motion to chaos [2, 3]. The DST equation is also found in connection with physical problems in different areas such as stabilization of high frequency vibrations in biomolecular dynamics [4], arrays of coupled nonlinear waveguides in nonlinear optics [5], quasiparticle motion on a dimer [6]. In the case of two degrees of freedom \( n = 2 \) (DST dimer) the system is integrable having, besides the Hamiltonian (energy), another conserved quantity, the norm (number of particles in the quantum case). The integrability properties of the classical and quantum DST dimer were studied in detail by several methods such as the number state method [7], the algebraic Bethe ansatz [8], and the method of separation of variables [9]. For more than two degrees of freedoms an integrable case of the DST system was found and studied in [10]. This integrable case is close to the Toda lattice and coincides for \( n = 2 \) with the usual DST dimer.

The quantum Hamiltonian \( H \) of the integrable DST model contains \((n+1)\) parameters \( c_1, \ldots, c_n, b \) and is defined as a second-order differential operator (here \( \partial_i \equiv \partial/\partial x_i \))

\[
H = \sum_{i=1}^{n} \left( \frac{1}{2} x_i^2 \partial_i^2 \right) + \left( c_i + \frac{1}{2} \right) x_i \partial_i + b x_{i+1} \partial_i
\]  

(1.1)

acting in the space \( \mathbb{C}[\vec{x}] \) of polynomials of \( n \) variables \( \{x_1, \ldots x_n\} \equiv \vec{x} \). In (1.1) and other similar formulas we always assume the periodic boundary conditions \( x_{n+1} \equiv x_1 \).

The Hamiltonian \( H \) obviously commutes with the number-of-particles operator \( N \)

\[
N = \sum_{i=1}^{n} x_i \partial_i.
\]

As shown in section , \( H \) and \( N \) can be included in a commutative ring of differential operators generated by a basis of \( n \) operators, the fact allowing to claim the quantum integrability of the system.

The multiplication operators \( x_i \) and the respective differentiations \( \partial_i \) can be considered as generators of a Heisenberg algebra (creation/annihilation operators). There exists a well-known scalar product on \( \mathbb{C}[\vec{x}] \) (holomorphic representation) such that \( x_i \) and \( \partial_i \) become mutually adjoint \( \partial_i^\dagger = x_i \). The corresponding Hamiltonian \( H \) is self-adjoint, however, only in the dimer case \( n = 2 \). In the general \( n > 1 \) case, no involution rendering \( H \) self-adjoint is known. The Hilbert space structure is, however, quite irrelevant for the kind of problems we are interested in and will be completely ignored throughout the paper.

The DST-chain can be considered as a degenerate case of the Heisenberg magnetic chain, though not as degenerate as Toda lattice. It makes the DST-chain a good tool.
for studying various techniques applicable to integrable models since it requires more effort than Toda lattice but still is simpler than the generic magnetic chain.

The main purpose of the present paper is to construct an analog of Baxter’s $Q$-operator \cite{11} for the integrable DST model. The $Q$-operator $Q_\lambda$ by definition shares the set of eigenvectors with the Hamiltonians $H_i$, and its eigenvalues are polynomials in $\lambda$ satisfying a finite-difference equation known as Baxter or separation equation. As was shown in \cite{12} on the example of the periodic Toda lattice, the similarity transformation $O \mapsto Q_\lambda O Q_\lambda^{-1}$ turns in the classical limit into a classical Bäcklund transformation that is a one-parametric family of canonical transformations preserving the commuting Hamiltonians. Later, in \cite{13} for the classical Bäcklund transformations the property of spectrum was described which is the classical counterpart of the separation equation for the eigenvalues of $Q_\lambda$. In the present paper we follow the approach of \cite{13} studying first the classical case and paying the special attention to the spectrum property of the corresponding Bäcklund transformation.

Our main result (see sections 4–7) is the following integral equation

$$
\int_\gamma d\xi_1 \ldots \int_\gamma d\xi_n \left[ \prod_{j=1}^n \frac{i}{2\pi} \Gamma(\lambda + 1 - c_j) e^{-\xi_j (-\xi_j)^{c_j} - \lambda^{-1}} \right] \psi(\ldots, y_k \xi_k + b y_{k+1}, \ldots) \quad (1.3)
$$

\begin{equation}
q(\lambda) \psi(x_1, \ldots, x_n), \quad q(\lambda) \in \mathbb{C}[\lambda], \quad (1.4)
\end{equation}

for the polynomial eigenfunctions $\psi \in \mathbb{C}[\vec{x}]$ of the Hamiltonian (1.1)

$$
H \psi(x_1, \ldots, x_n) = h \psi(x_1, \ldots, x_n). \quad (1.5)
$$

The structure of the paper is the following. In section 2 we consider the classical version of the integrable DST-chain and describe its relation to the Toda lattice and the isotropic Heisenberg magnetic chain. Our construction of Bäcklund transformation generalizes well known results for the Toda lattice. Following \cite{13}, we also study the dual Lax matrix and the corresponding dual Bäcklund transformation in section 3.

In section 4 we discuss the quantization of the integrable DST model and present a list of properties of Baxter’s $Q$-operator. In section 5, following the approach of \cite{14}, we construct a $Q$-operator $Q_\lambda$ for the quantum DST chain as the trace of a monodromy matrix with an infinite-dimensional auxiliary space. In the spirit of \cite{12}, we consider $Q_\lambda$ as an integral operator in $\mathbb{C}[\vec{x}]$ and find in section 6 its kernel and contour of integration. In the same section we study analyticity properties of $Q_\lambda$ in the parameter $\lambda$, prove that its matrix elements in the monomial basis are polynomials in $\lambda$ and give explicit formulas for its action on polynomials. We consider in details the simplest $n = 1$ case where the $Q$-operator provides an integral representation for classical orthogonal polynomials (Charlier polynomials). In section 7 we prove that $Q_\lambda$ satisfies a finite-difference equation in the parameter $\lambda$. In the last, 8th section we discuss possible generalizations and applications of our results.
2. Classical case

In this section we consider the classical integrable DST-chain \([10]\). The model is described in terms of \(n\) pairs of canonical variables \((X_i, x_i), i = 1, \ldots, n\)

\[
\{X_i, X_j\} = \{x_i, x_j\} = 0, \quad \{X_i, x_j\} = \delta_{ij}
\]

(2.1)

(the periodicity convention \(x_{i+n} \equiv x_i, X_{i+n} \equiv X_i\) is always assumed for the indices of \(x_i\) and \(X_i\)).

The canonical momenta \(X_i\) replace in the classical case the differential operators \(\partial_i\). As mentioned before, in the quantum case we do not make any assumptions about self-adjointness of the observables. Respectively, we allow the classical variables \((X_i, x_i)\) to be complex.

To construct \(n\) commuting Hamiltonians we introduce the Lax matrix \(L(u)\) (monodromy matrix) as a product of \(n\) local Lax matrices \(\ell_i(u)\)

\[
L(u) = \ell_n(u) \ldots \ell_2(u) \ell_1(u)
\]

(2.2)

\[
\ell_i(u; x_i, X_i) = \begin{pmatrix}
u - c_i - x_i X_i & b x_i \\
-X_i & b
\end{pmatrix}
\]

(2.3)

where \(b, c_i \in \mathbb{C}\) are parameters of the model, and \(u\) is the so-called spectral parameter of the Lax matrix.

Denoting by \(\text{id}_2\) the unit \(2 \times 2\) matrix and introducing notations for the tensor products \(1 \equiv \ell \otimes \text{id}_2, 2 \equiv \text{id}_2 \otimes \ell\) one establishes the \(r\)-matrix identity \([15]\)

\[
\{\ell_i(u_1), \ell_j(u_2)\} = [r_{12}(u_1 - u_2), \ell_i(u_1) \ell_j(u_2)] \delta_{ij}, \quad r_{12}(u) = -\frac{1}{u} P_{12}
\]

(2.4)

where \(P_{12}\) is the permutation operator in \(\mathbb{C}^2 \otimes \mathbb{C}^2\). From (2.4) the corresponding identity for the monodromy matrix

\[
\{L(u_1), L(u_2)\} = [r_{12}(u_1 - u_2), L(u_1) L(u_2)]
\]

(2.5)

is derived in the standard way \([15]\) which, in turn, ensures the commutativity of the spectral invariants \(t(u)\) and \(d(u)\) of the matrix \(L(u)\) defined as coefficients of its characteristic polynomial

\[
det(v - L(u)) = v^2 - t(u)v + d(u).
\]

(2.6)

Since \(\det \ell_i(u) = b(u - c_i)\) the determinant \(d(u) \equiv \det L(u) = \prod_{i=1}^n b(u - c_i)\) is scalar, and the only nontrivial spectral invariant is the trace \(t(u)\)

\[
t(u) \equiv \text{tr} L(u) = L_{11}(u) + L_{22}(u)
\]

(2.7)

which serves as a generating function of commuting independent Hamiltonians \(H_i\)

\[
t(u) = u^n + \sum_{i=1}^n (-1)^i H_i u^{n-i}.
\]

(2.8)
As a corollary of (2.3) we have the commutativity of \( t(u) \)

\[
\{t(u_1), t(u_2)\} = 0
\]

(2.9)

and, consequently, the commutativity \( \{H_i, H_j\} = 0 \) of the Hamiltonians \( H_i \).

A direct calculation shows that

\[
H_1 = N + \sum_{i=1}^{n} c_i, \quad H_2 = \frac{1}{2}H_1^2 - H - \frac{1}{2} \sum_{i=1}^{n} c_i^2,
\]

(2.10)

where

\[
N = \sum_{i=1}^{n} x_i X_i, \quad H = \sum_{i=1}^{n} \left( \frac{1}{2}x_i^2 X_i^2 + c_i x_i X_i + bx_{i+1}X_i \right)
\]

(2.11)

ensuring that the polynomial ring of commuting Hamiltonians contains the number of particles \( N \) and the Hamiltonian \( H \).

Note that the \( r \)-matrix \( r_{12}(u) \) in (2.4) is the same as for the isotropic Heisenberg magnetic chain and Toda lattice [15] which puts these integrable models into the same class. Indeed, the Toda lattice is a degenerate case of the DST chain. To demonstrate it, it is sufficient to make a constant shift \( u \mapsto u + b^{-1} \) of the spectral parameter in \( \ell_i(u) \) given by (2.3) and take the limit

\[
b \to 0, \quad x_j = e^{q_j}(b^{-1} + p_j) + O(b), \quad X_j = e^{-q_j},
\]

(2.12)

contracting the ‘oscillator’ algebra \((x_i, X_i, x_iX_i)\) into the Euclidian Lie algebra \((e^{\pm q_j}, p_i)\). In the limit \( \ell_i(u) \) turns into the elementary \( \ell \)-matrix for the Toda lattice:

\[
\ell_i(u) \to \begin{pmatrix} u - c_i - p_i & e^{q_i} \\ -e^{-q_i} & 0 \end{pmatrix}
\]

(2.13)

(the shifts \( c_i \) become irrelevant since they can be absorbed into a simple canonical transformation \( p_i \mapsto p_i - c_i \)). On the other hand, the DST model, in turn, is a degenerate case of the Heisenberg XXX magnet corresponding to the contraction of the \( su(2) \) Lie algebra into the oscillator algebra. The DST model holds an intermediate place between Heisenberg and Toda models.

In the present paper we take the Hamiltonian point of view on Bäcklund transformation according to which the Bäcklund transformation \( B_\lambda \) is a one-parameter family of simplectic maps from the canonical variables \((\vec{X}, \vec{x})\) to the canonical variables \((\vec{Y}, \vec{y})\) possessing certain characteristic properties (see [13] for a detailed discussion). For Hamiltonian integrable systems allowing a description in terms of the \( r \)-matrix algebra (2.3) there has recently been found an algorithmic method for constructing a Bäcklund transformation [16, 17]. The method having been described in detail in the cited papers, we present here only the results.

As in the case of the periodic Toda lattice [12, 13], it is convenient to describe the canonical transformation \( B_\lambda \) in terms of the generating function

\[
F_\lambda(\vec{y} | \vec{x}) = n\lambda + \sum_{i=1}^{n} \left( \frac{x_i - by_{i+1}}{y_i} + (\lambda - c_i) \ln \frac{by_{i+1} - x_i}{(\lambda - c_i)by_i} \right).
\]

(2.14)
\[ X_i = \frac{\partial F_\lambda}{\partial x_i} = \frac{1}{y_i} + \frac{\lambda - c_i}{x_i - by_{i+1}} \]  \hspace{1cm} (2.15a) \\
\[ Y_i = -\frac{\partial F_\lambda}{\partial y_i} = bX_{i-1} + \frac{x_i - by_{i+1}}{y_i} X_i. \]  \hspace{1cm} (2.15b)

To prove that \( B_\lambda \) preserves the Hamiltonians \( H_i \)

\[ H_i(\vec{X}, \vec{x}) = H_i(\vec{Y}, \vec{y}) \]  \hspace{1cm} (2.16)

we proceed in the same manner as in \([12, 13]\) for the periodic Toda lattice. Introducing the matrices

\[ M_i(u) = \begin{pmatrix} 1 & -by_{i+1} \\ X_i & u - \lambda - by_{i+1}X_i \end{pmatrix}, \]  \hspace{1cm} (2.17)

one then directly verifies the equality

\[ M_i(u)\ell_i(u; X_i, x_i) = \ell_i(u; Y_i, y_i)M_{i-1}(u) \]  \hspace{1cm} (2.18)

from which it follows that \( B_\lambda \) preserves the spectrum of the Lax matrix \( L(u) \)

\[ M_n(u, \lambda)L(u; \vec{X}, \vec{x}) = L(u; \vec{Y}, \vec{y})M_n(u, \lambda) \]

which, in turn, ensures the invariance of \( t(u) \) and, therefore, of \( H_i \) (2.16).

To formulate the spectrality property \([13]\) of the Bäcklund transformation we introduce the quantity \( \mu \) canonically conjugated, in a sense, to \( \lambda \)

\[ \ln \mu = -\frac{\partial F_\lambda}{\partial \lambda} = \sum_{i=1}^{n} \ln \frac{\lambda - c_i}{by_{i+1} - x_i}, \quad \mu = \prod_{i=1}^{n} \frac{\lambda - c_i}{by_{i+1} - x_i}. \]  \hspace{1cm} (2.19)

The spectrality of the Bäcklund transformation means that the pair \((\lambda, \mu)\) lies on the spectral curve of the Lax matrix

\[ \det(\mu - L(\lambda)) = 0. \]  \hspace{1cm} (2.20)

To prove it, we again follow \([13]\). We observe that for \( u = \lambda \) the matrix \( M_i(u) \) degenerates

\[ M_i(\lambda) = \begin{pmatrix} 1 & -by_{i+1} \\ X_i & -by_{i+1}X_i \end{pmatrix} = \begin{pmatrix} 1 & -by_{i+1} \\ X_i & -by_{i+1}X_i \end{pmatrix}, \]  \hspace{1cm} (2.21)

and its null-vector \( \omega_i \) can be found explicitly:

\[ M_i(\lambda)\omega_i = 0, \quad \omega_i = \begin{pmatrix} by_{i+1} \\ 1 \end{pmatrix}. \]  \hspace{1cm} (2.22)

Noting then the identity

\[ \ell_i(\lambda)\omega_{i-1} = \frac{(\lambda - c_i)by_i}{by_{i+1} - x_i}\omega_i, \]  \hspace{1cm} (2.23)
we conclude that
\[ L(\lambda)\omega_n = \mu\omega_n, \quad (2.24) \]
whence (2.20) follows immediately.

The commutativity \( B_{\lambda_1} \circ B_{\lambda_2} = B_{\lambda_2} \circ B_{\lambda_1} \) is an immediate consequence of the invariance of Hamiltonians and their completeness, see [13].

Note that \( M_{i-1}^{-1}(u) \) and \( \ell_i(u) \) have, as functions of \( u \), essentially the same structure, up to a shift of \( u \) and a scalar factor. The fact is by no means a coincidence, see [17] for a detailed explanation.

3. Dual Lax matrix

We conclude the study of the classical case with presenting the dual Lax matrix and the dual Bäcklund transformation for the DST model. In [10] two different Lax matrices were found for the integrable DST system, the \( 2 \times 2 \) Lax matrix \( L(u) \) and also the \( n \times n \) Lax matrix. This bigger Lax matrix did not contain a spectral parameter. Here we present an \( n \times n \) Lax matrix \( \mathcal{L}(v) \) containing a spectral parameter \( v \) which is dual to \( L(u) \) in the sense that the corresponding spectral curves are equivalent up to interchanging the spectral parameters \( u \) and \( v \)

\[ (b^n - v) \det (u - \mathcal{L}(v)) = \det (v - L(u)). \quad (3.1) \]

To produce the dual Lax matrix \( \mathcal{L}(v) \) we take an eigenvector \( \theta_1(u) \) of \( L(u) \) corresponding to the eigenvalue \( v \) (for brevity, we will not mark the dependence on \( u \) in \( \theta \))

\[ L(u)\theta_1 = v\theta_1 \quad (3.2) \]

and define by induction \( \theta_i \) as

\[ \theta_{i+1} = \ell_i(u)\theta_i, \quad i = 1, \ldots, n. \quad (3.3) \]

From (3.2) it follows that \( \theta_{n+1} = v\theta_1 \). The function \( \theta_i(u) \), when properly normalized, is called Baker-Akhiezer function. Denoting the components of the vector \( \theta_i \) as \( \varphi_i \) and \( \psi_i \) we write down (3.3) explicitly as

\[ \begin{pmatrix} \varphi_{i+1} \\ \psi_{i+1} \end{pmatrix} = \begin{pmatrix} u - c_i - x_i X_i & bx_i \\ -X_i & b \end{pmatrix} \begin{pmatrix} \varphi_i \\ \psi_i \end{pmatrix}. \quad (3.4) \]

Then, splitting the components and taking into account the quasiperiodicity condition \( \theta_{n+1} = v\theta_1 \) we arrive to the following linear equations for \( \varphi_i \) and \( \psi_i \):

\[ u\varphi_i = \varphi_{i+1} + (c_i + x_i X_i)\varphi_i - bx_i\psi_i, \quad i = 1, \ldots, n - 1 \quad (3.5a) \]

\[ u\varphi_n = v\varphi_1 + (c_n + x_n X_n)\varphi_n - bx_n\psi_n, \quad (3.5b) \]
\[
\psi_{i+1} = -X_i \varphi_i + b \psi_i, \quad i = 1, \ldots, n - 1 \tag{3.6a}
\]
\[
v \psi_1 = -X_n \varphi_n + b \psi_n. \tag{3.6b}
\]

Eliminating \( \psi_i \) we can write down the linear problem for the vector \( \Phi \) with the components \( \varphi_i \) in the matrix form:

\[
L(v) \Phi = u \Phi, \quad \Phi = \begin{pmatrix}
\varphi_1 \\
\vdots \\
\varphi_n
\end{pmatrix} \tag{3.7}
\]

where the matrix \( L(v) \) defined as

\[
L(v) = (v - b^n)^{-1} \sum_{j,k=1}^{n} b^{n+j-k} x_j X_k E_{jk} + v E_{n1} + \sum_{j \geq k} b^{j-k} x_j X_k E_{jk} + \sum_{j=1}^{n} c_j E_{jj} + \sum_{j=1}^{n-1} E_{j,j+1} \tag{3.8}
\]

is the dual Lax matrix we were looking for. Here \( E_{jk} \) is the \( n \times n \) matrix with the only non-zero entry \( (E_{jk})_{jk} = 1 \). The proof of the identity (3.1) is an exercise which we leave to the reader. For the case \( b = 1 \) and \( v = -1 \) the dual Lax matrix for the DST model was first found in [10]. For examples of Lax matrices duality in other integrable models see [18].

The Bäcklund transformation \( B_\mu \) corresponding to the dual Lax operator \( L(v) \) is given by the same equations (2.15) and (2.19). The important difference, however, is that now \( \mu \) is a free numerical parameter of Bäcklund transformation whereas \( \lambda \) becomes a dynamical variable determined from the equation (2.19). The equality (2.19) is now reinterpreted as the equation defining the variable \( \lambda \). The generating function of \( B_\mu \) is the Legendre transform of \( F_\lambda(\vec{y} | \vec{x}) \) with respect to \( \lambda \).

The properties of the dual Bäcklund transformation \( B_\mu \) are proved in the same manner as those of \( B_\lambda \) (see also [13] for the Toda lattice case). For the proof we need a matrix \( M(v) \) playing for \( L(v) \) the same role that \( M_n(u) \) played for \( L(u) \).

Let \( \tilde{\theta}_i \) be defined as \( \tilde{\theta}_i = M_{i-1} \theta_i \). From (2.18) it follows that \( \tilde{\theta}_i \) is a Baker-Akhiezer function for \( \ell_i(u; Y_i, y_i) \). The first component of the equality \( \tilde{\theta}_i = M_{i-1} \theta_i \) reads \( \tilde{\varphi}_i = \varphi_i - b y_i \psi_i \). Substituting \( \psi_i \) from the solution to the system (3.6) we obtain the correspondence \( \tilde{\Phi} = M(v) \Phi \) with the matrix \( M(v) \) defined as

\[
M(v) = (v - b^n)^{-1} \sum_{j,k=1}^{n} b^{n+j-k} x_j X_k E_{jk} + \sum_{j > k} b^{j-k} y_j X_k E_{jk} + \sum_{j=1}^{n} E_{jj}. \tag{3.9}
\]

The invariance of the spectrum of \( L(v) \) follows from the identity

\[
M(v) L(v; \vec{X}, \vec{x}) = L(v; \vec{Y}, \vec{y}) M(v). \tag{3.10}
\]

The spectrality is expressed as the identity

\[
\det(\lambda - L(\mu)) = 0. \tag{3.11}
\]
To prove (3.11) it is sufficient to notice that the matrix $\mathcal{M}(v)$ degenerates as $v = \mu$
\[
\det \mathcal{M}(\mu) = 0,
\]
and the corresponding null-vector $\Omega$ defined by the recurrence relation
\[
\frac{\Omega_{i+1}}{\Omega_i} = \frac{b(c_i - \lambda)y_{i+1}}{x_i - by_{i+1}}, \quad i = 1, \ldots, n - 1
\]
is, by virtue of (3.10), also an eigenvector of $\mathcal{L}(\mu)$ corresponding to the eigenvalue $\lambda$
\[
\mathcal{L}(\mu)\Omega = \lambda\Omega.
\]

Since the Toda lattice is a degenerate case of the DST model, the $n \times n$ Lax matrix
for the Toda lattice can be obtained, as one could expect, from our $\mathcal{L}(v)$ matrix in
the limit $b \to 0$, as in (2.12). The result is a variant of the standard $n \times n$ Lax matrix
for the periodic Toda lattice [19]:
\[
\mathcal{L}(v) = b^{-1} + \mathcal{L}^{TL}(v) + O(b),
\]
\[
\mathcal{L}^{TL}(v) = v^{-1}e^{q_n-q_1}E_{1n} + vE_{n1} + \sum_{j=1}^{n} (p_j + c_j)E_{jj} + \sum_{j=1}^{n-1} e^{q_j-q_{j+1}}E_{j+1,j} + \sum_{j=1}^{n-1} E_{j,j+1}.
\]

Similarly, from $\mathcal{M}(v)$ one obtains the corresponding matrix for the Toda lattice,
see [13].

The Poisson brackets for both dual Lax matrices $\mathcal{L}(v)$ can be expressed in the
generalized $r$-matrix form [20]
\[
\{\hat{\mathcal{L}}(v_1), \hat{\mathcal{L}}(v_2)\} = [r_{12}(v_1, v_2), \hat{\mathcal{L}}(v_1)] - [r_{21}(v_1, v_2), \hat{\mathcal{L}}(v_2)],
\]
the ‘non-unitary’ $r$-matrix having the form:
\[
r_{12}(v_1, v_2) = \frac{1}{v_1 - v_2} \left( v_2 \sum_{k \geq j} + v_1 \sum_{k < j} \right) E_{jk} \otimes E_{kj}
\]
and $r_{21}(v_1, v_2) = \mathcal{P}r(v_2, v_1)\mathcal{P}$, where $\mathcal{P} = \sum_{j,k=1}^n E_{jk} \otimes E_{kj}$ is the permutation matrix
in $\mathbb{C}^n \otimes \mathbb{C}^n$.

The non-unitary $r$-matrix (3.18) in case of Toda’s Lax matrix can be unitarized
by a gauge transformation:
\[
\mathfrak{L}(v) = V\mathcal{L}^{TL}(v)V^{-1}, \quad V = \sum_{j=1}^{n} e^{q_j/2}E_{jj}
\]
getting for the new Lax matrix $\mathfrak{L}(v)$ the standard unitary $A_{n-1}$-type $r$-matrix
\[
\mathbf{r}_{12}(v_1, v_2) = \frac{v_1 + v_2}{v_1 - v_2} \sum_{j=1}^{n} E_{jj} \otimes E_{jj} + \frac{1}{v_1 - v_2} \left( v_2 \sum_{k \geq j} + v_1 \sum_{k < j} \right) E_{jk} \otimes E_{kj},
\]
\[
\mathbf{r}_{12}(v_1, v_2) = -\mathbf{r}_{21}(v_1, v_2)
\]
\[
\{\mathfrak{L}(v_1), \mathfrak{L}(v_2)\} = [\mathbf{r}_{12}(v_1, v_2), \mathfrak{L}(v_1)] + \mathfrak{L}(v_2)],
\]
see, for instance, [19].
4. Quantization

In the quantum case the canonical momenta \( X_i \) are replaced with the differentiations \( \partial_i \equiv \partial/\partial x_i \) (having no intent to discuss the conjugation properties of the observables, we discard the factor \( i\hbar \) to simplify the notation). To preserve the commutativity of the Hamiltonians \( H_i \) upon quantization one needs to choose the operator ordering in a special way.

The necessary algebraic framework is given by the Quantum Inverse Scattering or the \( R \)-matrix \([11, 21]\) method. Defining the local quantum Lax matrix as

\[
\ell_i(u) = \begin{pmatrix}
    u - c_i - x_i\partial_i & bx_i \\
    -\partial_i & b
\end{pmatrix}
\]  

(4.1)

one verifies the commutation relation

\[
R_{12}(u_1 - u_2)\ell(u_1)\ell(u_2) = \ell(u_2)\ell(u_1)R_{12}(u_1 - u_2)
\]

(4.2)

where

\[
R_{12}(u) = u + \mathcal{P}_{12}
\]

(4.3)

is the standard \( SL(2) \)-invariant solution to the quantum Yang-Baxter equation. The quantum Lax operator, or monodromy matrix, \( L(u) \) and its trace \( t(u) \) are defined then by the same formulas \((2.2)\) and \((2.7)\) as in the classical case. From \((4.2)\) one then derives in a standard way the similar commutation relation

\[
R_{12}(u_1 - u_2)\ell(u_1)L(u_2) = L(u_2)\ell(u_1)R_{12}(u_1 - u_2)
\]

(4.4)

for \( L(u) \) from which the commutativity of \( t(u) \)

\[
[t(u_1), t(u_2)] = 0
\]

(4.5)

follows immediately. The commutative quantum Hamiltonians \( H_i \) are defined then, like in the classical case \((2.8)\), as coefficients of the polynomial \( t(u) \). It is easy to see that \( H_i \) is a differential operator of order \( i \) leaving invariant the space \( \mathbb{C}[\vec{x}] \) of polynomials of \( x_1, \ldots, x_n \). In particular, \( H_1 \) and \( H_2 \) are given by the formulas \((2.10)\) with \( N \) and \( H \) given by \((1.2)\) and \((1.1)\), respectively.

The main problem in the quantum case is the spectral problem for commuting differential operators, quantum Hamiltonians \( \{H_i\}_{i=1}^n \):

\[
H_i\psi(x_1, \ldots, x_n) = h_i\psi(x_1, \ldots, x_n), \quad \psi(x_1, \ldots, x_n) \in \mathbb{C}[\vec{x}].
\]

(4.6)

One can describe the spectrum and eigenvectors of \( H_i \), or, equivalently, \( t(u) \) using the well-developed machinery of algebraic Bethe Ansatz \([21]\). Defining the vacuum state \( |0\rangle \) as the unit function \( |0\rangle \ (x) \equiv 1 \) in \( \mathbb{C}[\vec{x}] \) we note that

\[
L_{21} |0\rangle = 0, \quad L_{11}(u) |0\rangle = \alpha_{11}(u) |0\rangle, \quad L_{22}(u) |0\rangle = \alpha_{22}(u) |0\rangle,
\]

(4.7)
where
\[ \alpha_{11}(u) = \prod_{i=1}^{n} (u - c_i), \quad \alpha_{22}(u) = b^n. \] (4.8)

Defining the Bethe vector \( \psi_\vec{v}(x_1, \ldots, x_n) \in \mathbb{C}[\vec{x}] \) parametrized by \( m \) complex numbers \( v_j \) as
\[ \psi_\vec{v}(x_1, \ldots, x_n) \equiv |v_1, \ldots, v_m\rangle = L_{12}(v_1) \ldots L_{12}(v_m)|0\rangle \] (4.9)
one can prove [21], using the commutation relations (4.4), that \(|v_1, \ldots, v_m\rangle\) is an eigenvector of \( t(u) \), for any \( u \in \mathbb{C} \), if and only if the parameters \( v_j \) satisfy the system of algebraic Bethe equations
\[ \prod_{j=1}^{m} v_k - v_j + 1 \frac{v_k - v_j - 1}{v_k - v_j} = -\frac{\alpha_{11}(v_k)}{\alpha_{22}(v_k)}, \quad k = 1, \ldots, m \] (4.10)
and the corresponding eigenvalue \( \tau(u) \) of \( t(u) \)
\[ t(u) |v_1, \ldots, v_m\rangle = \tau(u) |v_1, \ldots, v_m\rangle \] (4.11)
is given by the formula
\[ \tau(u) = \alpha_{11}(u) \prod_{j=1}^{m} \frac{u - v_j - 1}{u - v_j} + \alpha_{22}(u) \prod_{j=1}^{m} \frac{u - v_j + 1}{u - v_j}. \] (4.12)

It is usually assumed that Bethe eigenvectors are complete, at least for generic values of parameters. The proof of the conjecture is, however, a difficult task, and is available only for a few models, see [21] for a discussion.

In his seminal study [11] of the integrable XYZ and XXZ spin chains R J Baxter has pointed out that the equations similar to ours (4.10) and (4.12) can be reformulated equivalently as a finite-difference equation in a certain class of holomorphic functions. Adapting his reasoning to our case we introduce the polynomial \( \phi(\lambda; \vec{v}) \) in \( \lambda \) whose zeros are the Bethe parameters \( v_j \):
\[ \phi(\lambda; \vec{v}) = \prod_{j=1}^{m} (\lambda - v_j), \quad \lambda \in \mathbb{C}. \] (4.13)

It is easy to see then that the following finite-difference equation of second order for \( \phi(\lambda; \vec{v}) \)
\[ \phi(\lambda; \vec{v}) \tau(\lambda) = \alpha_{11}(\lambda) \phi(\lambda - 1; \vec{v}) + \alpha_{22}(\lambda) \phi(\lambda + 1; \vec{v}) \] (4.14)
is equivalent to the system of equations (4.10) for \( \{v_j\}_{j=1}^{m} \) and to the equation (4.12) for \( \tau(\lambda) \). To show this, it is sufficient to divide both sides of (4.14) by \( \phi(\lambda) \) and take residues at \( \lambda = v_j \). The equation (4.14) is called Baxter’s or separation equation. The reason for the latter name is that an identical equation arises when solving the model via the separation of variables method (see [13] for more on relation between \( Q \)-operator and quantum separation of variables).

Now we are able to describe the problem we are going to study in the remaining sections of this paper. We are looking for a one-parameter family of operators \( Q_\lambda \)
acting in \( C[\vec{x}] \) such that \( Q_\lambda \) shares with \( t(u) \) the same set of Bethe eigenvectors, and the eigenvalues \( q(\lambda) \) of \( Q_\lambda \)

\[
Q_\lambda |v_1, \ldots, v_m\rangle = q(\lambda) |v_1, \ldots, v_m\rangle \tag{4.15}
\]

are polynomials in \( \lambda \) satisfying Baxter’s equation (4.14). Up to a normalization coefficient \( \kappa_{\vec{v}} \), depending on the eigenvector, the polynomials \( q(\lambda) \) are proportional to the polynomials \( \phi(\lambda; \vec{v}) \) defined by (4.13):

\[
q(\lambda) = \kappa_{\vec{v}} \phi(\lambda; v_1, \ldots, v_m) = \kappa_{\vec{v}} \lambda^m + O(\lambda^{m-1}), \quad \lambda \to \infty. \tag{4.16}
\]

Instead of dealing with eigenvectors and eigenvalues it is more convenient to characterize \( Q_\lambda \) by the following operator identities which are equivalent to the above characterization, assuming the completeness of Bethe eigenvectors. We demand that \( Q_\lambda \) commute with \( t(u) \)

\[
[t(u), Q_\lambda] = 0 \tag{4.17a}
\]

and self-commute

\[
[Q_{\lambda_1}, Q_{\lambda_2}] = 0, \tag{4.17b}
\]

as well as satisfy the finite-difference equation

\[
Q_\lambda t(\lambda) = Q_{\lambda-1} \prod_{i=1}^n (\lambda - c_i) + b^n Q_{\lambda+1}. \tag{4.17c}
\]

In addition, the eigenvalues of \( Q_\lambda \) should be polynomial in \( \lambda \)

\[
q(\lambda) \in C[\lambda]. \tag{4.17d}
\]

The above conditions by no means define \( Q_\lambda \) uniquely. Apparently, one can construct infinitely many \( Q \)-operators just by fixing arbitrary normalization coefficients \( \kappa_{\vec{v}} \) for each eigenvector \( |\vec{v}\rangle \) in (4.16). The difficult problem is to find an explicit expression for a \( Q \)-operator. Baxter succeeded in solving the problem in case of XYZ and XXZ spin chains, having given an expression for \( Q_\lambda \) as a trace of a monodromy matrix \([11]\). However, his formulas do not survive passing to the limiting case of the XXX spin chain, governed by the \( SL(2) \) invariant \( R \) matrix (4.3).

In the case of quantum periodic Toda lattice, which is another model governed by the \( R \) matrix (4.3), a solution was found by Pasquier and Gaudin \([12]\). Instead of trying to construct \( Q_\lambda \) as trace of a monodromy matrix, they considered \( Q_\lambda \) as an integral operator

\[
Q_\lambda : \psi(\vec{x}) \mapsto \int dx_1 \ldots \int dx_n \, Q_\lambda(\vec{y} | \vec{x}) \, \psi(\vec{x}) \tag{4.18}
\]

having given an explicit expression for its kernel \( Q_\lambda(\vec{y} | \vec{x}) \). They also discovered an important relation between the kernel \( Q_\lambda(\vec{y} | \vec{x}) \) and the generating function \( F_\lambda(\vec{y} | \vec{x}) \) of the classical Bäcklund transformation expressed by the semiclassical formula

\[
Q_\lambda(\vec{y} | \vec{x}) \sim \exp \left( -\frac{i}{\hbar} F_\lambda(\vec{y} | \vec{x}) \right), \quad \hbar \to 0. \tag{4.19}
\]
The classical Bäcklund transformation $B_\lambda$ is thus the classical limit of the similarity transformation $O \mapsto Q_\lambda Q_\lambda^{-1}$.

Recently, it was found [14] how the original Baxter’s construction [11] can be generalized to produce $Q$-operators for the models governed by the $A_1$-type $R$ matrices, such as the XXZ spin chain and sine-Gordon model. According to [14], $Q_\lambda$ is constructed as the trace of a monodromy matrix built of the local Lax operators corresponding, in the auxiliary space, to the special infinite-dimensional representations of the quantum group $U_q[\hat{\mathfrak{sl}}_2]$ ($q$-oscillator representations).

In the subsequent sections we construct a $Q$-operator for the quantum DST model and prove its characteristic properties. Our approach combines those of [12] and of [14]. Similarly to [14], we construct our $Q_\lambda$ as an integral operator acting in $\mathbb{C}[\vec{x}]$ and present several equivalent expressions for it.

The $Q$-operator being found as an integral operator will give integral equations for the eigenfunctions $\psi_{\vec{v}}$. The advantage of this transformation of the differential spectral problem into integral spectral problem is that it gives an alternative to Bethe representation of multivariable special functions. The general approach of constructing a $Q$-operator for a given integrable system will be of even greater importance in situations when Bethe ansatz does not work.

### 5. Construction of the $Q$-operator

The structure of $Q_\lambda$ is similar to that of $t(u)$ given by (2.2) and (2.7). We construct $Q_\lambda$ as the trace of a monodromy matrix built of the elementary blocks $R^{(i)}_{\lambda - c_i}$. Suppose that $R_\lambda$ is a linear operator from $\mathbb{C}[s, x]$ into $\mathbb{C}[t, y]$. The spaces $\mathbb{C}[x]$ and $\mathbb{C}[y]$ are referred to as quantum spaces and $\mathbb{C}[s]$ and $\mathbb{C}[t]$, respectively, as auxiliary ones (see [11]). To construct $Q_\lambda$ we introduce $n$ copies $R^{(i)}_{\lambda - c_i}$ of $R_\lambda$ assuming that $R^{(i)}_{\lambda - c_i} : \mathbb{C}[s_i, x_i] \mapsto \mathbb{C}[s_{i+1}, y_i]$ (remember the periodicity convention $n + 1 \equiv 1$) and extending $R^{(i)}_{\lambda - c_i}$ on $\mathbb{C}[x_j]$ $(j \neq i)$ as the unit operator. The monodromy matrix $R^{(n)}_{\lambda - c_n} \ldots R^{(1)}_{\lambda - c_1}$ acts then from $\mathbb{C}[s_1, \vec{x}]$ into $\mathbb{C}[s_1, \vec{y}]$, and $Q_\lambda$ is obtained by taking trace in the auxiliary space $\mathbb{C}[s_1]$: \[ Q_\lambda = \text{tr}_{s_1} R^{(n)}_{\lambda - c_n} \ldots R^{(1)}_{\lambda - c_1}. \quad (5.1) \]

Supposing $R_\lambda$ to be an integral operator \[ R_\lambda : \psi(s, x) \mapsto \int dx \int ds R_\lambda(t, y \mid s, x) \psi(s, x) \quad (5.2) \]
we have for the kernel $Q_\lambda(\vec{y} \mid \vec{x})$ of $Q_\lambda$ \[ Q_\lambda(\vec{y} \mid \vec{x}) = \int ds_n \ldots \int ds_1 \prod_{i=1}^{n} R_{\lambda - c_i}(s_{i+1}, y_i \mid s_i, x_i). \quad (5.3) \]
To ensure the commutativity \([t(u), Q_\lambda] = 0\) it is sufficient to demand that \(\mathbb{R}_\lambda\) intertwines

\[ M(u - \lambda; t, \partial_t)\ell(u; y, \partial_y)\mathbb{R}_\lambda = \mathbb{R}_\lambda\ell(u; x, \partial_x)M(u - \lambda; s, \partial_s) \quad (5.4) \]

the local Lax operator \(\ell(u)\) and some other representation \(M(u - \lambda)\) of the same algebra (1.2)

\[ R(u_1 - u_2)M(u_1)\mathcal{M}(u_2) = \mathcal{M}(u_2)M(u_1)R(u_1 - u_2) \quad (5.5) \]

with the same \(R\) matrix (4.3). The proof of (4.17a) follows then by a standard argument [13, 21].

Similarly, to prove \([Q_{\lambda_1}, Q_{\lambda_2}] = 0\) (4.17b) it is sufficient to establish the Yang-Baxter identity

\[
\int dt_1 \int dt_2 \int dy \tilde{R}_{\lambda_1 - \lambda_2}(w_1, w_2 \mid t_1, t_2)R_{\lambda_1}(t_1, z \mid s_1, y)R_{\lambda_2}(t_2, y \mid s_2, x) = \int dt_1 \int dt_2 \int dy R_{\lambda_2}(w_2, z \mid t_2, y)R_{\lambda_1}(w_1, y \mid t_1, x)R_{\lambda_1 - \lambda_2}(t_1, t_2 \mid s_1, s_2) \quad (5.6)
\]

with some kernel \(\tilde{R}_{\lambda}\).

The representation \(M(u - \lambda)\) of the algebra (5.4) should be chosen in such a way that the resulting \(Q_\lambda\), as a function of \(\lambda\), satisfy Baxter’s finite-difference equation (4.17c) and have polynomial eigenvalues (4.17d). As we shall show, for this purpose one can take

\[
M(u; s, \partial_s) = \begin{pmatrix} u - s\partial_s & s \\ -\partial_s & 1 \end{pmatrix} \quad (5.7)
\]

coinciding essentially with \(\ell(u)\) with \(b = 1\) and \(c_i = 0\). The representation (5.7) plays for the Yangian \(\mathcal{Y}[sl_2]\) the same role as the \(q\)-oscillator representation plays for the quantum group \(\mathcal{U}_q[sl_2]\) in [14]. Having fixed \(M(u)\) by (5.7) we get from (5.4) a set of differential equations for the kernel \(R_\lambda(t, y \mid s, x)\) of \(\mathbb{R}_\lambda\)

\[
\begin{pmatrix} u - \lambda - t\partial_t & t \\ -\partial_t & 1 \end{pmatrix} \begin{pmatrix} u - y\partial_y & by \\ -\partial_y & b \end{pmatrix} R_\lambda(t, y \mid s, x) = \begin{pmatrix} u + 1 + x\partial_x & bx \\ \partial_x & b \end{pmatrix} \begin{pmatrix} u - \lambda + 1 + s\partial_s & s \\ \partial_s & 1 \end{pmatrix} R_\lambda(t, y \mid s, x) \quad (5.8)
\]

(in the right-hand-side we have used integration by parts and the identities \(\partial_x^* = -\partial_x\), \((x\partial_x)^* = -\partial_x x = -1 - x\partial_x\)). The equations (5.8) determine \(R_\lambda\) up to a scalar factor \(\rho_\lambda:\)

\[
R_\lambda(t, y \mid s, x) = \rho_\lambda \delta(s - by) y^{-1} \exp \left(\frac{t - x}{y}\right) \left(\frac{t - x}{y}\right)^{-\lambda - 1}. \quad (5.9)
\]

It remains to choose the factor \(\rho_\lambda\) in (5.3) and the integration contour in (5.2) in such a way that \(\mathbb{R}_\lambda : \mathbb{C}[s, x] \mapsto \mathbb{C}[t, y, \lambda]\).

We describe first the final formula for \(\mathbb{R}_\lambda\) and equivalent expressions and then prove the polynomiality property. As the basic definition of \(\mathbb{R}_\lambda\) we choose the following formula:

\[
\mathbb{R}_\lambda : \psi(s, x) \mapsto \frac{i}{2\pi} \Gamma(\lambda + 1) \int_\gamma d\xi e^{-\xi} (-\xi)^{-\lambda - 1} \psi(by, y\xi + t). \quad (5.10)
\]
The infinite integration contour \( \gamma \) is shown on Figure 1. The branch of the many-valued function \((-\xi)^{-\lambda-1}\) in (5.10) is fixed by making a cut along \((0, \infty)\) and assuming that \(-\pi \leq \arg(-\xi) \leq \pi\).

\[
\int_\gamma d\xi e^{-\xi}(-\xi)^{\nu-1} = -\frac{2\pi i}{\Gamma(1-\nu)}.
\] (5.12)
Using the Pochhammer symbol \((c)_m \equiv \Gamma(c+m)/\Gamma(c) = c(c+1)\ldots(c+m-1)\)
one can write down the result as
\[
\mathbb{R}_\lambda : s^k x^j \mapsto \sum_{m=0}^j \binom{j}{m} (-\lambda)_m t^{j-m} y^{m+k} b^k = t^k b^k C_j(\lambda; t/y),
\]
where \(C_m(\lambda; b)\) are the so-called Charlier polynomials \(\text{[22, 23]}\).

The formula (5.13) proves the polynomiality \(\mathbb{R}_\lambda : \mathbb{C}[s, x] \mapsto \mathbb{C}[t, y, \lambda]\). Note that the normalization of \(\mathbb{R}_\lambda\) is chosen in such a way that \(\mathbb{R}_\lambda : 1 \mapsto 1\).

The action of \(\mathbb{R}_\lambda\) on polynomials can be described in an even more compact way. Substituting \(\psi(s, x) = s^k(x-t)^j\) into (5.10) and using again Hankel’s formula (5.12) one obtains the most economic description of \(\mathbb{R}_\lambda\)
\[
\mathbb{R}_\lambda : s^k(x-t)^j \mapsto y^{j+k} (-\lambda)_j b^k.
\]

In the end of the next section we will discuss the formula (5.14) and similar ones in more detail.

### 6. Analytical properties of the \(Q\)-operator

To produce a description of \(Q_\lambda\) as an integral operator (4.18) we substitute the expression (5.9) for the kernel \(\mathcal{R}_\lambda\) found in the previous section into the formula (5.3). The integration in \(s_i\) is easily performed due to the delta-function factors in \(\mathcal{R}_\lambda\) and, corresponding to the two choices of the factor \(\rho_\lambda\) in (5.9) and integration contour in (5.2), we obtain two equivalent descriptions of \(Q_\lambda\).

The first formula for \(Q_\lambda\) is given by (4.18) with the kernel \(\bar{Q}_\lambda(\vec{y} \mid \vec{x})\)
\[
Q_\lambda(\vec{y} \mid \vec{x}) = \prod_{i=1}^n w_i(\lambda; y_{i+1}, y_i, x_i)
\]
where
\[
w_i(\lambda; y_{i+1}, y_i, x_i) = \frac{1}{2\pi i} \Gamma(\lambda + 1 - c_i) y_{i-1} \left( \frac{by_{i+1} - x_i}{y_i} \right)^{c_i-\lambda-1} \exp \left( \frac{by_{i+1} - x_i}{y_i} \right)
\]
and integration in \(x_i\) is taken over the contour \(\gamma_i = y_i \gamma + by_{i+1}\) whereas the contour \(\gamma\) is defined in the previous section.

The alternative formula is given again by (4.18) with the kernel \(\bar{Q}_\lambda(\vec{y} \mid \vec{x})\)
\[
\bar{Q}_\lambda(\vec{y} \mid \vec{x}) = \prod_{i=1}^n \bar{w}_i(\lambda; y_{i+1}, y_i, x_i)
\]
where
\[
\tilde{w}_i(\lambda; y_{i+1}, y_i, x_i) = \frac{y_i^{-1}}{\Gamma(c_i - \lambda)} \left( \frac{x_i - by_{i+1}}{y_i} \right)^{c_i - \lambda - 1} \exp \left( \frac{by_{i+1} - x_i}{y_i} \right)
\]  
(6.4)

and integration in \( x_i \) is taken over the straight ray starting from \( x_i = by_{i+1} \) and extending to infinity in the direction \( y_i/|y_i| \).

Note that the kernels \( Q_\lambda(\vec{y} | \vec{x}) \) and \( \tilde{Q}_\lambda(\vec{y} | \vec{x}) \) satisfy the semiclassical condition (4.19) which, taking into account our quantization convention \(-i\hbar = 1\), takes the following form (up to insignificant \( \lambda \)-depending factors):
\[
Q_\lambda(\vec{y} | \vec{x}) \simeq \exp(-F_\lambda(\vec{y} | \vec{x}))
\]  
(6.5)

valid for any complex \( \lambda \), except the poles \( \lambda = c_i - k \) (\( i = 1, \ldots, n; k = 1, 2, \ldots \)) of \( \Gamma(\lambda + 1 - c_i) \). The branch of each of many-valued functions \((-\xi_i)^{c_i - \lambda - 1}\) in (6.6) is fixed by making a cut along \((0, \infty)\) and assuming that \(-\pi \leq \arg(-\xi_i) \leq \pi\).

The analog of (5.11) is the formula
\[
Q_\lambda : \psi(\vec{x}) \mapsto \int_0^\infty d\xi_1 \ldots \int_0^\infty d\xi_n \tilde{W}_\lambda(\vec{\xi}) \psi(\ldots, y_i\xi_i + by_{i+1}, \ldots),
\]  
(6.8)

valid for \( \text{Re} \lambda < \min \text{Re} c_i \). Together, the formulas (6.6) and (6.8) define \( Q_\lambda \) as a holomorphic function of \( \lambda \in \mathbb{C} \).

In the rest of this section we will show, that \( Q_\lambda \) maps polynomials in \( x \) into polynomials in \( y \) and \( \lambda \) and derive explicit formulas for its action on the monomial basis in \( \mathbb{C}[\vec{x}] \).
Before considering the general case we will give a brief account of the simplest $n = 1$ case. In this case we have only one variable $x \equiv x_1$, the Lax matrix simplifies to $L(u) = \ell(u)$, so, without loss of generality, one can put $c_1 = 0$. Trace of $L(u)$ gives rise to only one integral of motion (number of particles $N$)

$$t(u) \equiv \text{tr} L(u) = u - N + b, \quad N = x\partial. \quad (6.10)$$

We assume that $N$ acts in the space $\mathbb{C}[x]$ of polynomials of $x$ spanned by the eigenbasis $\{x^m\}_{m=0}^{\infty}$ of $N$

$$N : x^m \mapsto mx^m, \quad m = 0, 1, 2, \ldots \quad (6.11)$$

For $n = 1$ and $c = 0$ the formula (6.6) defining the $Q$ operator turns into

$$Q_\lambda : \psi(x) \mapsto \frac{i}{2\pi} \Gamma(\lambda + 1) \int d\xi e^{-\xi} (-\xi)^{-\lambda-1} \psi(y(\xi + b)), \quad \lambda \neq -1, -2, \ldots \quad (6.12)$$

and (6.8), respectively, into

$$Q_\lambda : \psi(x) \mapsto \frac{1}{\Gamma(-\lambda)} \int_0^\infty d\xi e^{-\xi} \xi^{-\lambda-1} \psi(y(\xi + b)), \quad \text{Re} \lambda < 0. \quad (6.13)$$

Similarly, from (6.1) and (6.3) one gets, respectively,

$$Q_\lambda : \psi(x) \mapsto \frac{ie^b}{2\pi} \Gamma(\lambda + 1) \int_{\gamma'} dx y^{-1} \left( b - \frac{x}{y} \right)^{-\lambda-1} e^{-x/y} \psi(x), \quad \gamma' = y(\gamma + b) \quad (6.14)$$

and

$$Q_\lambda : \psi(x) \mapsto \frac{e^b}{\Gamma(-\lambda)} \int_{by}^\infty dx y^{-1} \left( \frac{x}{y} - b \right)^{-\lambda-1} e^{-x/y} \psi(x), \quad y > 0. \quad (6.15)$$

To calculate explicitly the action of $Q_\lambda$ on the basis $x^m$ one puts $\psi(x) = x^m$ in (6.12), then expands the binomial $(\xi + b)^m$ and applies, termwise, Hankel’s integral formula (5.12). This calculation is very similar to the calculation of $R_\lambda s^k x^j$ given by the formula (5.13). The result is that the monomials $\{x^m\}_{m=0}^{\infty}$ are the eigenvectors of $Q_\lambda$:

$$Q_\lambda : x^m \mapsto q_m(\lambda)y^m, \quad (6.16)$$

the corresponding eigenvalues $q_m(\lambda)$ being polynomials in $\lambda$ of degree $m$ expressed in terms of the Charlier polynomials $C_m(\lambda; b)$ as

$$q_m(\lambda) = \sum_{j=0}^{m} \binom{m}{j} (-\lambda)^j b^{m-j} = b^m C_m(\lambda; b) \quad (6.17)$$

(compare to (5.13)).

As an immediate consequence, we have the commutativity $[Q_\lambda, N] = 0$, as well as (1.17a) and (4.17b). Another corollary is that $Q_\lambda$ maps $\mathbb{C}[x]$ into $\mathbb{C}[y, \lambda]$. Note that formula (1.17) implies the normalization $Q_\lambda : 1 \mapsto 1$. 

18
One can use the integral operator $Q_\lambda$ to derive few well known formulas for the orthogonal Charlier polynomials. For instance, putting $\psi(x) = x^m$ and $y = 1$ in (6.12) or in (6.13) one obtains integral representations for Charlier polynomials:

$$C_m(\lambda; b) = \frac{i}{2\pi} \Gamma(\lambda + 1) \int_\gamma d\xi \, e^{i\xi} \xi^{\lambda-1} \left(1 + \frac{\xi}{b}\right)^m,$$

and, respectively,

$$C_m(\lambda; b) = \frac{1}{\Gamma(-\lambda)} \int_0^\infty d\xi \, e^{-\xi} \xi^{\lambda-1} \left(1 + \frac{\xi}{b}\right)^m$$

(see [22]).

Equating $\psi(x)$ in (6.12) or (6.13) with the generating function $e^{tx} = \sum_{m=0}^\infty x^m t^m/m!$ of the monomials $x^m$ and taking the integral one gets the generating function of Charlier polynomials

$$e^t \left(1 - \frac{t}{b}\right)^\lambda = \sum_{m=0}^\infty \frac{t^m}{m!} C_m(\lambda; b).$$

(6.20)

The recurrence relation for the Charlier polynomials [23, 22] is equivalent to the finite-difference equation for the polynomials $q_i(\lambda)$

$$(\lambda - i + b) q_i(\lambda) = bq_i(\lambda + 1) + \lambda q_i(\lambda - 1),$$

which coincides with Baxter’s equation (4.14) for $n = 1$ and proves, for $n = 1$, the operator relation (4.17c).

From the explicit expression (6.17) for the polynomials $q_m(\lambda)$ we conclude that they are normalized by the condition $q_m(0) = b^m$, or, alternatively, $q_m(\lambda) = (-\lambda)^m + O(\lambda^{m-1})$, as $\lambda \to \infty$. In terms of the operator $Q_\lambda$ it is equivalent to

$$Q_0 = b^N$$

(see (6.10) for the definition of $N$) and, respectively, to

$$Q_\lambda = (-\lambda)^N + O(\lambda^{N-1}).$$

(6.23)

The generalization of the above results to the multivariable case is quite straightforward. To calculate explicitly the action of $Q_\lambda$ on the monomial basis $x_1^{m_1} \dots x_n^{m_n}$ in $\mathbb{C}[\vec{x}]$ one substitutes $\psi(\vec{x}) = x_1^{m_1} \dots x_n^{m_n}$ into (6.9), then expands the binomials $(y_i \xi_i + by_{i+1})^{m_i}$ and uses termwise Hankel’s integral formula (5.12). Recalling the definition (6.17) of Charlier polynomials, one obtains the following expression:

$$Q_\lambda : x_1^{m_1} \dots x_n^{m_n} \mapsto \prod_{i=1}^n b^{m_i} y_{i+1}^{m_i} C_{m_i}(\lambda - c_i; by_{i+1}/y_i),$$

(6.24)

from which it follows immediately that the normalization condition $Q_\lambda : 1 \mapsto 1$ holds and that $Q_\lambda$ maps $\mathbb{C}[\vec{x}]$ into $\mathbb{C}[\vec{y}, \lambda]$. The polynomiality of matrix elements of $Q_\lambda$ combined with the commutativity $[Q_{\lambda_1}, Q_{\lambda_2}]$ (4.17b) proves the polynomiality (4.17d) of the eigenvalues of $Q_\lambda$. 

19
The formula (6.24) also allows to determine the normalization (4.16) of the eigenvalues of $Q_{\lambda}$. Taking the limit $\lambda \to \infty$ in (6.24) and using the asymptotics $C_n(\lambda; b) = (-\lambda/b)^n + O(\lambda^{m-1})$ we conclude that, as in the $n = 1$ case, $Q_\lambda$ has the asymptotics (6.23) with the operator $N$ given by (1.2). In contrast, the equality (6.22), generally speaking, cannot be generalized to $n > 1$, with the exception of the homogeneous chain case $c_i \equiv 0$, $i = 1, \ldots, n$, when it is replaced by

$$Q_0 = b^N U$$

(6.25)

where $U$ is the translation operator $U : x_i \to y_{i+1}$.

As a final remark of this section, we point out yet another way of writing down the action of $Q_{\lambda}$. Substituting $\psi(\vec{x})$ in (6.6) with the polynomials $\omega_{\vec{m}}^{\vec{y}} \in \mathbb{C}[\vec{x}]$

$$\omega_{\vec{y}}^{\vec{m}}(\vec{x}) = \prod_{i=1}^{n}(x_i - by_{i+1})^{m_i}$$

parametrized by the multi-index $\vec{m} = (m_1, \ldots, m_n)$ and a vector $\vec{y} = (y_1, \ldots, y_n)$ we obtain, after performing the integrations, an elegant formula for the action of $Q_{\lambda}$ on $\omega_{\vec{y}}^{\vec{m}}$:

$$Q_{\lambda} : \prod_{i=1}^{n}(x_i - by_{i+1})^{m_i} \mapsto \prod_{i=1}^{n}(c_i - \lambda)^{m_i}$$

(6.26)

The formula (6.26) seems to provide the most compact way to encode the action of $Q_{\lambda}$ on polynomials (compare to the formula (5.14) for the action of $R_{\lambda}$). Some caution is necessary, however, when using it, since the parameters $\vec{y}$ in $\omega_{\vec{y}}^{\vec{m}}$ coincide with the variables in the target space $\mathbb{C}[\vec{y}]$ of $Q_{\lambda}$. One way of interpreting (6.26) is to consider its left-hand-side as a shorthand notation for $[Q_{\lambda}\omega_{\vec{y}}^{\vec{m}}]_{\vec{y}=\vec{y}}$. Another possibility is to extend the operator $Q_{\lambda}$ onto the polynomial ring $\mathbb{C}[\vec{x}, \vec{y}]$ assuming that it acts on $\vec{y}$ trivially: $Q_{\lambda}(\psi(\vec{x})\varphi(\vec{y})) = \varphi(\vec{y})Q_{\lambda}(\psi(\vec{x}))$. Formulas similar to (6.26) also arise in the separation of variables for Macdonald polynomials [24].

It is a challenging problem to take the formulas (5.14) and (6.26) as definitions of $R_{\lambda}$ and $Q_{\lambda}$, respectively, and to build the theory of $Q_{\lambda}$ in an entirely algebraic way.

7. Baxter’s equation

In the previous sections we have proved all the properties of $Q_{\lambda}$ from the list given in section except Baxter’s difference equation (1.17c). In this section we give a proof of the identity (1.17c) based on the ideas of [12].

The best suited for our purposes realization of $Q_{\lambda}$ is that given by the formulas (6.18) and (6.1). Recalling that $t(u) = \text{tr} L(u)$ and that $L(u)$ is a $2 \times 2$ matrix whose entries are differential operators in $x_i$, we can transform the left-hand-side of (4.17c) as follows

$$[Q_{\lambda}t(\lambda)\psi](\vec{y}) = \text{tr}[Q_{\lambda}L(\lambda)\psi](\vec{y}) = \text{tr} \int \text{d}x^n Q_{\lambda}(\vec{y} \mid \vec{x})L(\lambda)\psi(\vec{x}).$$

20
Performing integration by parts we obtain

$$[Q_{\lambda}t(\lambda)\psi](\bar{y}) = \text{tr} \int dx^n [L^*(\lambda)Q_{\lambda}(\bar{y} \mid \bar{x})]\psi(\bar{x})$$  \hspace{1cm} (7.1)$$

where $L^*(\lambda)$ is the matrix composed of adjoint differential operators $(L_{jk})^* = L_{jk}^*$. For example, $\partial^* = -\partial, (x\partial)^* = -\partial x = -x\partial - 1$, and so on.

Using the factorization (2.2) of $L(\lambda)$ into the product of elementary Lax matrices $\ell_i(\lambda)$ and the factorization (6.1) of the kernel $Q_{\lambda}(\bar{y} \mid \bar{x})$ into the factors $w_i$ (5.2) we can represent the kernel of the integral operator $Q_{\lambda}t(\lambda)$ as

$$[Q_{\lambda}t(\lambda)(\bar{y} \mid \bar{x}) = \text{tr} \ell_n^*(\lambda) \ldots \ell_1^*(\lambda) \prod_{i=1}^{n} w_i = \text{tr}\left(\ell_n^*(\lambda)w_n\right) \ldots \left(\ell_1^*(\lambda)w_1\right),$$  \hspace{1cm} (7.2)

where

$$\ell_i^*(\lambda) = \left(\begin{array}{c}
\lambda - c_i + 1 + x_i\partial x_i & bx_i \\
\partial x_i & b
\end{array}\right).$$  \hspace{1cm} (7.3)$$

The possibility of the factorization (7.2) of $[Q_{\lambda}L(\lambda)](\bar{y} \mid \bar{x})$ depends crucially on the fact that the factors $w_i$ (5.2) depend each only on one variable $x_i$. That is why we take the left-hand-side of (1.17) to be $Q_{\lambda}t(\lambda)$ rather than $t(\lambda)Q_{\lambda}$.

The rest of the proof parallels that of the spectrality property for the classical case given in section. Introducing matrices $\tilde{\ell}_i(\lambda)$ by the equality $\tilde{\ell}_i^*(\lambda)w_i = w_i\ell_i(\lambda)$ and noting that

$$\partial x_i \ln w_i(y_{i+1}, y_i, x_i) = \frac{c_i - \lambda - 1}{x_i - by_{i+1}} - \frac{1}{y_i},$$  \hspace{1cm} (7.4)$$

we obtain

$$\tilde{\ell}_i(\lambda) = \left(\begin{array}{c}
\lambda - c_i + 1 + x_i\partial x_i & bx_i \\
\partial x_i \ln w_i & b
\end{array}\right) = \left(\begin{array}{c}
\frac{b(c_i - \lambda - 1)y_{i+1}}{x_i - by_{i+1}} - \frac{x_i}{y_i} & bx_i \\
\frac{c_i - \lambda - 1}{x_i - by_{i+1}} - \frac{1}{y_i} & b
\end{array}\right).$$  \hspace{1cm} (7.5)$$

and

$$[Q_{\lambda}t(\lambda)](\bar{y} \mid \bar{x}) = Q_{\lambda}(\bar{y} \mid \bar{x}) \text{tr} \tilde{\ell}_n(\lambda) \ldots \tilde{\ell}_1(\lambda) \equiv Q_{\lambda}(\bar{y} \mid \bar{x}) \text{tr} \tilde{L}(\lambda).$$  \hspace{1cm} (7.6)$$

We note then that the gauge transformation $\tilde{\ell}_i(\lambda) \mapsto N^{-1}_{i+1}\tilde{\ell}_i(\lambda) N_i$ with the gauge matrix

$$N_i = \left(\begin{array}{cc}
1 & by_i \\
0 & 1
\end{array}\right)$$  \hspace{1cm} (7.7)$$

leaves $\text{tr} \tilde{L}(\lambda)$ invariant while making $\tilde{\ell}_i(\lambda)$ and, consequently, $\tilde{L}(\lambda)$ triangular:

$$N^{-1}_{i+1}\tilde{\ell}_i(\lambda) N_i = \left(\begin{array}{cc}
\frac{-x_i - by_{i+1}}{y_i} & 0 \\
\frac{c_i - \lambda - 1}{x_i - by_{i+1}} - \frac{1}{y_i} & \frac{b(c_i - \lambda - 1)y_{i+1}}{x_i - by_{i+1}}
\end{array}\right) = \left(\begin{array}{cc}
\frac{(\lambda - c_i)w_i(\lambda - 1)}{w_i(\lambda)} & 0 \\
\frac{c_i - \lambda - 1}{x_i - by_{i+1}} - \frac{1}{y_i} & \frac{bw_i(\lambda + 1)}{w_i(\lambda)}
\end{array}\right).$$  \hspace{1cm} (7.8)$$
where we used the identities

\[
\frac{w_i(\lambda + 1)}{w_i(\lambda)} = \frac{(c_i - \lambda - 1)y_i}{x_i - by_{i+1}}, \quad \frac{w_i(\lambda - 1)}{w_i(\lambda)} = \frac{x_i - by_{i+1}}{(c_i - \lambda)y_i}.
\]  
(7.9)

As a result, we get the equality

\[
\text{tr} \tilde{L}(\lambda) = b^n n \prod_{i=1}^{n} \frac{w_i(\lambda + 1)}{w_i(\lambda)} + \prod_{i=1}^{n} \frac{(\lambda - c_i)}{w_i(\lambda)} \frac{w_i(\lambda - 1)}{w_i(\lambda)}.
\]  
(7.10)

which, obviously, proves (4.17c).

**8. Discussion**

On the example of the quantum integrable DST model we have shown that the construction of $Q$-operator as an integral operator, in the style of the paper [12], and as the trace of a monodromy matrix with a special representation of the quantum group corresponding to the auxiliary space, in the style of papers [14], can be combined naturally within an unified approach. The same approach can be applied to other integrable models which are more general than DST model, such as the generic XXX magnetic chain. This work is in progress and the results will be reported in a separate paper. For a particular case of the homogeneous XXX chain a $Q$-operator was recently constructed in [25].

Another interesting problem is to build the theory of $Q$-operator in a purely algebraic manner starting from the formulas (5.14) and (6.26).

In [14] it is argued that for the models governed by the $A_1$ type $R$ matrices there exist two different $Q$-operators corresponding to two different $q$-oscilator representations of $U_q[\hat{sl}_2]$. Their eigenvalues correspond, respectively, to two linearly independent solutions of Baxter’s difference equations analogous to (4.14). In the case of DST model the second $Q$-operator can be obtained if we choose in the formula (5.4) another representation $\hat{M}(u-\lambda)$ of the algebra (5.3), namely $\hat{M}(u-\lambda) \sim -\hat{M}^{-1}(\lambda - u)$

\[
\hat{M}(u; s, \partial_s) = \begin{pmatrix}
-1 & s \\
-\partial_s & u + s\partial_s
\end{pmatrix}.
\]  
(8.1)

The corresponding $Q$-operator has, however, more complex nature than the one studied in the present paper. Its eigenvalues, for example, are not polynomial in $\lambda$. The problem is currently under study.

We can point out the following application of our results to the theory of special functions of many variables. Notice that the eigenfunctions of the quantum DST Hamiltonians are multivariable polynomials. The obtained family of integral equations for those polynomials provided by the $Q_\lambda$-operator supplements their representation as Bethe vectors and can be used in efficient numerical calculations of these special functions, for instance, solving integral equations by iterations. Simple considerations of the $n = 2$ case show that we deal with the discrete multivariable analogues of
the Heun polynomials. For special functions of such complexity the found integral equations might be the only explicit representations to exist because there is no hope to get, for instance, an integral representation. So, the found integral equations for the special functions which were initially defined as eigenfunctions of the commuting differential operators can be used first, as already remarked, for generating advanced numerical methods of their calculation, and, secondly, for finding various asymptotics. These applications are being worked on.

Acknowledgements

The authors wish to acknowledge the support of EPSRC and INTAS.

References

[1] Eilbeck J C, Lomdahl P S, and A C Scott A C 1985 The discrete self-trapping equation Physica D 16 318–338

[2] De Filippo S, Fusco Girard M and Salerno M 1989 Nonlinearity 2 477

[3] Cruzeiro-Hansson L, Feddersen H, Flesch R, Christiansen P L, Salerno M and Scott A C 1990 Classical and quantum analysis of chaos in the discrete self-trapping equation Phys. Rev. B 42 522–526

[4] Davydov A S and Kislukha N I 1973 Phys. Status Solidi B 59 465

[5] Finlayson N and Stegeman G I 1990 Spatial switching, instabilities, and chaos in a 3-wave-guide nonlinear directional coupler Appl. Phys. Lett. 56 2276–2278

[6] Kenkre V M and Campbell D K 1986 Self-trapping on a dimer — time-dependent solutions of a discrete nonlinear Schrödinger equation Phys. Rev. B 34 4595–4961

[7] Scott A C and Eilbeck J C 1986 The quantized discrete self-trapping equation Phys. Lett. A, 119 60–64

[8] Enolskii V Z, Salerno M, Kostov N A and Scott A C 1991 Alternative quantizations of the discrete self-trapping dimer Physica Scripta 43 229–235

[9] Enol’skii V Z, Kuznetsov V B and Salerno M 1993 On the quantum inverse scattering method for the DST dimer Physica D 68 138–152

[10] Christiansen P L, Jørgensen M F and Kuznetsov V B 1993 On integrable systems close to the Toda lattice Lett. Math. Phys. 29 165–173

[11] Baxter R I 1982 Exactly Solved Models in Statistical Mechanics (London: Academic) ch 9–10
[12] Pasquier V and Gaudin M 1992 The periodic Toda chain and a matrix generalization of the Bessel function recursion relations *J. Phys. A: Math. Gen.* **25** 5243–5252

[13] Kuznetsov V B and Sklyanin E K 1998 On Bäcklund transformations for many-body systems *J. Phys. A: Math. Gen.* **31** 2241–2251

[14] Bazhanov V, Lukyanov S and Zamolodchikov A 1997 Integrable Structure of Conformal Field Theory II. Q-operator and DDV equation *Commun. Math. Phys.* **190** 247–278

Antonov A and Feigin B 1997 Quantum Group Representations and Baxter Equation *Phys. Lett.* **B392** 115-122

[15] Faddeev L D and Takhtajan L A 1987 *Hamiltonian Methods in the Theory of solitons* (Berlin: Springer-Verlag)

[16] Sklyanin E K 1999a Canonicity of Bäcklund transformation: $r$-matrix approach. I, preprint LPENSL-Th 05/99; [solv-int/9903016](https://arxiv.org/abs/solv-int/9903016).

[17] Sklyanin E K 1999b Canonicity of Bäcklund transformation: $r$-matrix approach. II, preprint LPENSL-Th 06/99; [solv-int/9903017](https://arxiv.org/abs/solv-int/9903017); to be published in: Trudy MIAN, v. 226 (1999), Moscow, Nauka.

[18] Adams M R, Harnad J and Hurtubise J 1990 Dual moment maps to loop algebras *Lett. Math. Phys.* **20** 294–308

[19] Jimbo M 1985 Quantum $R$ matrix for the generalized Toda system *Commun. Math. Phys.* **102** 527–547

[20] Semenov-Tian-Shansky M A 1983 What is classical $r$-matrix? *Funct. Anal. Appl.* **17** 259–272

[21] Korepin V A, Bogoliubov N M and Izergin A G 1993 *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge: Cambridge University Press)

[22] Erdelyi A et al. 1953 *Higher transcendental functions* (NY: McGrow Hill)

[23] Koekoek R and Swarttouw R F 1994 *The Askey-scheme of hypergeometric orthogonal polynomials and its $q$-analogue* Report 94-05, Delft University of Technology

[24] Kuznetsov V B and Sklyanin E K 1996 Separation of variables for $A_2$ Ruijsenaars model and new integral representation for $A_2$ Macdonald polynomials *J. Phys. A: Math. Gen.* **29** 2779–2804

[25] Derkachov S E 1999 Baxter’s Q-operator for the homogeneous XXX spin chain [solv-int/9902013](https://arxiv.org/abs/solv-int/9902013), submitted to *JPA*