A new ordering principle for the classical statistical analysis of Poisson processes with background

C. Giunti

INFN, Sezione di Torino, and Dipartimento di Fisica Teorica, Università di Torino,
Via P. Giuria 1, I–10125 Torino, Italy

Abstract

Inspired by the recent proposal by Feldman and Cousins of a “unified approach to the classical statistical analysis of small signals” based on a choice of ordering in Neyman’s construction of classical confidence intervals, I propose a new ordering principle for the classical statistical analysis of Poisson processes with background which minimizes the effect on the resulting confidence intervals of the observation of less background events than expected. The new ordering principle is applied to the calculation of the confidence region implied by the recent null result of the KARMEN neutrino oscillation experiment.

PACS numbers: 06.20.Dk, 14.60.Pq

Published in Phys. Rev. D 59, 053001 (1999).
I. INTRODUCTION

In a very interesting paper [1] Feldman and Cousins have recently proposed a “unified approach to the classical statistical analysis of small signals” that allows to construct classical confidence belts which “unify the treatment of upper confidence limits for null results and two-sided confidence intervals for non-null results”. This unified approach is very attractive because the transition from two-sided intervals to upper limits is automatic and the undercoverage introduced by a choice based on the data (“flip-flopping”) is avoided by construction. Indeed, the unified approach has promptly been adopted by the Particle Data Group as the new standard method for statistical inference [3].

The probability to observe a number \( n \) of events in a Poisson process consisting in signal events with mean \( \mu \) and background events with known calculated mean \( b \) is

\[
P(n|\mu, b) = \frac{1}{n!} (\mu + b)^n e^{-(\mu+b)}.
\]

(1.1)

The classical method for obtaining the confidence interval for the unknown parameter \( \mu \) is based on Neyman’s method to construct a confidence belt. This confidence belt is the region in the \( \mu \)-\( n \) plane lying between the two curves \( n_1(\mu, \alpha) \) and \( n_2(\mu, \alpha) \) such that for each value of \( \mu \)

\[
P(n \in [n_1(\mu, \alpha), n_2(\mu, \alpha)]|\mu, b) = \alpha ,
\]

(1.2)

where

\[
P(n \in [n_1, n_2]|\mu, b) = \sum_{n=n_1}^{n_2} P(n|\mu, b)
\]

(1.3)

and \( \alpha \) is the desired confidence level. The two curves \( n_1(\mu, \alpha) \) and \( n_2(\mu, \alpha) \) are required to be monotonic functions of \( \mu \) and can be inverted to yield the corresponding curves \( \mu_1(n, \alpha) \) and \( \mu_2(n, \alpha) \). Then, if a number \( n_{\text{obs}} \) of events is measured, the confidence interval for \( \mu \) is \([\mu_2(n_{\text{obs}}, \alpha), \mu_1(n_{\text{obs}}, \alpha)]\). This method guarantees by construction the correct coverage, i.e. the fact that the resulting confidence interval \([\mu_2(n_{\text{obs}}, \alpha), \mu_1(n_{\text{obs}}, \alpha)]\) is a member of a set of confidence intervals obtained with an ensemble of identical experiments that contain the true value of \( \mu \) with a probability \( \alpha \). Actually, in the case of a Poisson process, since \( n \) is an integer, the relation (1.2) can only be approximately satisfied and in practice the chosen acceptance intervals \([n_1(\mu, \alpha), n_2(\mu, \alpha)]\) are the smallest intervals such that

\[
P(n \in [n_1(\mu, \alpha), n_2(\mu, \alpha)]|\mu, b) \geq \alpha .
\]

(1.4)

This choice introduces an overcoverage for some values of \( \mu \) and the resulting confidence intervals are conservative. As emphasized in Ref. [1], conservativeness is an undesirable but unavoidable property of the confidence intervals in the case of a Poisson process (it is undesirable because it implies a loss of power in restricting the allowed range for the parameter \( \mu \)).

The construction of Neyman’s confidence belt is not unique, because in general there are many different couples of curves \( n_1(\mu, \alpha) \) and \( n_2(\mu, \alpha) \) that satisfy the relation (1.2).
Hence, an additional criterion is needed in order to define uniquely the acceptance intervals \([n_1(\mu, \alpha), n_2(\mu, \alpha)]\). The two common choices are

\[
P(n < n_1(\mu, \alpha)|\mu, b) = P(n > n_2(\mu, \alpha)|\mu, b) = \frac{1 - \alpha}{2},
\]

which leads to central confidence intervals and

\[
P(n < n_1(\mu, \alpha)|\mu, b) = 1 - \alpha,
\]

which leads to upper confidence limits.

Feldman and Cousins [1] proposed an ordering principle based on likelihood ratios that produces an automatic transition from a central confidence interval to an upper limit when the number of observed events in a Poisson process with background is of the same order or less than the expected background. The acceptance interval for each value of \(\mu\) is calculated assigning at each value of \(n\) a rank obtained from the relative size of the ratio

\[
R(n) = \frac{P(n|\mu, b)}{P(n|\mu_{\text{best}}, b)},
\]

where \(\mu_{\text{best}} = \mu_{\text{best}}(n)\) (for a fixed \(b\)) is the non-negative value of \(\mu\) that maximizes the probability \(P(n|\mu, b)\):

\[
\mu_{\text{best}}(n) = \max[0, n - b].
\]

As emphasized in Ref. [1], “\(R\) is a ratio of two likelihoods: the likelihood of obtaining \(n\) given the actual mean \(\mu\), and the likelihood of obtaining \(n\) given the best-fit physically allowed mean”. The rank of each value of \(n\) is assigned in order of decreasing value of the ratio \(R(n)\): the value of \(n\) which has bigger \(R(n)\) has rank one, the value of \(n\) among the remaining ones which has bigger \(R(n)\) has rank two and so on. The acceptance interval for each value of \(\mu\) is calculated by adding the values of \(n\) in increasing order of rank until the condition (1.4) is satisfied.

It is clear that the Feldman and Cousins ordering principle guarantees an automatic transition from two-sided confidence intervals to upper confidence limits for \(n \lesssim b\). Indeed, since \(\mu_{\text{best}}(n \leq b) = 0\), the rank of \(n \leq b\) for \(\mu = 0\) is one, implying that the interval \(0 \leq n \leq b\) for \(\mu = 0\) is guaranteed to lie in the confidence belt.

Although the unified approach solves brilliantly the problem of obtaining a transition with correct coverage from two-sided confidence intervals to upper confidence limits for \(n \lesssim b\), it has the undesirable feature that even when \(n \lesssim b\) the upper bound \(\mu_1(n, \alpha)\) decreases when \(b\) increases. From a physical point of view this is rather disturbing, because a stringent upper bound for \(\mu\) obtained by an experiment which has observed a number of events significantly smaller than the expected background is not due to the fact that the experiment is very sensitive to small values of \(\mu\), but to the fact that less background events than expected have been observed. Hence, two different experiments observing the same Poisson process with mean \(\mu\) and measuring the same number of events \(n_{\text{obs}}\) but expecting a different value for the background \(b\) lead to two different upper limits for \(\mu\) and the upper limit established by the experiment that expect a larger background can be significantly smaller than the other one. This is illustrated in Figs. 1 and 2.
In Fig. 1 I have plotted the 90% CL confidence belts obtained with the unified approach for \( b = 2, 3, 4, 5 \). One can see that four experiments expecting, respectively, 2, 3, 4 and 5 background events and observing, for example, \( n_{\text{obs}} = 1 \) lead to the 90% CL upper confidence limit \( \mu_1 = 2.53, 1.88, 1.34, 1.20 \), respectively.

In Fig. 2 I have plotted the upper end \( \mu_1 \) of the 90% CL confidence intervals obtained with the unified approach for \( n = 0, 1, 2, 3, 4, 5 \) as a function of the mean expected background \( b \). One can see that for each fixed value of \( n \) the upper end of the confidence intervals tends to decrease rapidly as the mean expected background \( b \) increases for small values of \( b \) and stabilizes around a value close to 0.8 for large values of \( b \). The decrease of \( \mu_1 \) is not monotonic, because of the discreteness of the observable \( n \). Figure 2 differs from Fig.8 of Ref. [1] for the fact that I have not forced \( \mu_1 \) to be a non-increasing function of \( b \), because, as discussed above, the decrease of the upper confidence limit when \( b \) increases is an undesirable feature from the physical point of view. Therefore, there is no reason to force \( \mu_1 \) to be a non-increasing function of \( b \). The only effect of this imposition is to introduce an overcoverage and then to increase the degree of conservativeness of the upper confidence limits, which is an undesirable effect.

The decrease of the upper confidence limit for a given \( n \) as the mean expected background \( b \) increases is an unavoidable feature of the unified approach. However, I will show in the next Section that this decrease can be weakened with the choice of \textit{a new ordering principle} for the construction of the confidence intervals.

\section*{II. A NEW ORDERING PRINCIPLE}

The ordering principle introduced by Feldman and Cousins [1] is useful in order to guarantee an automatic transition from two-sided confidence intervals to upper confidence limits. However, \textit{it is not unique}. Therefore, one can investigate the possibility to find a new ordering principle which minimizes the decrease of the upper confidence limit for a given \( n \) as the mean expected background \( b \) increases. With this aim one can notice that the value (1.8) of \( \mu_{\text{best}}(n) \) decreases proportionally to \( n \) until it vanishes and then it is forced to be non-negative “by hand”. It is clear that it would be preferable to have a reference value for \( \mu \) which is automatically always positive. Such a positive reference value would imply that for small but not vanishing values of \( \mu \) the values of \( n \) smaller than \( b \) have a high rank. As a consequence, the decrease of the upper confidence limit for a given \( n \) as the mean expected background \( b \) increases is weakened with respect to the one obtained with the Feldman and Cousins ordering principle.

The new ordering principle that I propose here is based on the choice for the reference value for \( \mu \) of the bayesian expectation value:

\[
\mu_{\text{ref}}(n) = \int_0^\infty \mu P(\mu|n, b) \, d\mu , \tag{2.1}
\]

where \( P(\mu|n, b) \) is the bayesian probability distribution for \( \mu \) calculated assuming\footnote{This assumption is arbitrary, but it seems to be the most reasonable one if there is no prior} a constant
prior for $\mu \geq 0$ (see, for example, [3]):

$$P(\mu | n, b) = (b + \mu)^n e^{-\mu} \left( \frac{n!}{\sum_{k=0}^{n} \frac{b^k}{k!}} \right)^{-1}. \quad (2.2)$$

The result of the integral (2.1) calculated by parts is

$$\mu_{\text{ref}}(n) = n + 1 - \left( \sum_{k=0}^{n} \frac{k b^k}{k!} \right) \left( \sum_{k=0}^{n} \frac{b^k}{k!} \right)^{-1}. \quad (2.3)$$

The obvious inequality $\sum_{k=0}^{n} k b^k/k! \leq n \sum_{k=0}^{n} b^k/k!$ implies that $\mu_{\text{ref}}(n) \geq 1$. Therefore, $\mu_{\text{ref}}(n)$ represents a reference value for $\mu$ that not only is non-negative, as desired in order to have an automatic transition from two-sided intervals to upper limits, but is even bigger or equal than one. This is a desirable characteristics in order to obtain a weak decrease of the upper confidence limit for a given $n$ when the expected background $b$ increases.

The implementation of the new ordering principle for the calculation of the confidence belt is identical to the implementation of the Feldman and Cousins ordering principle, except for the fact that in the calculation of the likelihood ratio $R(n)$, which determines the rank of each value of $n$ for a fixed value of $\mu$, $\mu_{\text{best}}(n)$ is replaced by $\mu_{\text{ref}}(n)$:

$$R(n) = \frac{P(n | \mu, b)}{P(n | \mu_{\text{ref}}, b)}. \quad (2.4)$$

The resulting 90% CL confidence belts for $b = 2, 3, 4, 5$ are shown in Fig. 3.

Confronting Fig. 3 with Fig. 1 one can see that the confidence belts are similar for $n \gg b$, but the new upper confidence limits $\mu_1$ for $n \lesssim b$ are larger than the corresponding ones obtained with the Feldman and Cousins ordering principle. For example, four experiments expecting, respectively, 2, 3, 4 and 5 background events and observing $n_{\text{obs}} = 1$ lead to the 90% CL upper confidence limits $\mu_1 = 2.99, 2.43, 2.28, 2.16$, respectively, which can be confronted with the respective values $\mu_1 = 2.53, 1.88, 1.34, 1.20$ obtained with the Feldman and Cousins ordering principle. It is clear that the decrease of $\mu_1$ as $b$ is increased is milder with the new ordering principle. This is fully illustrated in Fig. 4 where I have plotted the upper end $\mu_1$ of the 90% CL confidence intervals obtained with the new ordering principle for $n = 0, 1, 2, 3, 4, 5$ as a function of the mean expected background $b$. One can see that for small values of $b$ the decrease of the upper end $\mu_1$ of the confidence intervals as a function of $b$ is weaker than in Fig. 2 and for large values of $b$ the value of $\mu_1$ stabilizes around a value close to 1.7, which is higher than in Fig. 2.

Since the new ordering principle gives upper bounds that are higher than those obtained with the Feldman and Cousins ordering principle, it must give also higher lower bounds. This is confirmed by a comparison of Figs. 1 and 3 and is illustrated in detail in Fig. 5, where the values of the lower confidence limit $\mu_2$ obtained with the two methods for $n = 3, \ldots, 9$ are

---

information on the value of $\mu$, which is the parameter probed directly in the experiment.
plotted as functions of the mean expected background \( b \). One can see that for each value of \( n \) the maximum value of \( b \) for which the lower confidence limit is different from zero is the same in the two methods. This is a consequence of the fact (clearly seen with a comparison of Figs. [1] and [3]) that the minimum value of \( n \) for which there is a lower bound different from zero is the same in the two methods (as must be, since in both approaches the lower limit of the acceptance interval for \( \mu = 0 \) is at \( n = 0 \)). Hence, the experimental possibility to claim a positive effect is the same in the two methods. Furthermore, Fig. [5] shows that the difference between the lower limits obtained with the two methods is moderate and tends to vanish for \( n \gg b \) (indeed, for \( n \gg b \) we have \( \mu_{\text{ref}} \simeq n = \mu_{\text{best}} \) and the two methods are practically equivalent).

### III. APPLICATION TO THE RESULT OF THE KARMEN EXPERIMENT

In this Section I discuss the application of the new ordering principle proposed in Section [1] to the analysis of the recent null result of the KARMEN neutrino oscillation experiment [2].

The KARMEN experiment is searching for neutrino oscillations (see, for example, [3]) in the \( \bar{\nu}_\mu \rightarrow \bar{\nu}_e \) channel with a sensitivity in the region of the neutrino mixing parameters \( \sin^2 2\theta \) and \( \Delta m^2 \) which is allowed by the positive results of the LSND experiment [6]. So far the KARMEN experiment measured no events, with an expected background of 2.88 \( \pm 0.13 \) events [2]. In this case the unified approach leads to an upper confidence limit of 1.1 events for the mean \( \mu \) of neutrino oscillation events. On the other hand, the sensitivity of the experiment, defined as “the average upper limit one could get from an ensemble of experiments with the expected background and no true signal”, is of 4.4 events, four times bigger than the upper confidence limit (notice that the average upper limit is calculated using the unified approach). This is clearly a pathological case in which the observation of less background events than expected leads to a stringent upper confidence limit \( \mu_1 \), even if the experiment is not sensitive to the corresponding small values of \( \mu \).

The exclusion curves (the confidence region lies on the left of the exclusion curves) in the \( \sin^2 2\theta - \Delta m^2 \) plane corresponding to the upper confidence limit of 1.1 events obtained with the unified approach and to the sensitivity of the experiment (4.4 events) are reproduced in Fig. [6] (the solid curves passing through the filled and empty circles, respectively). The solid curve passing through the filled squares corresponds to the upper limit of 2.3 signal events obtained with the bayesian approach. The shadowed area in Fig. [6] is the region allowed at 90\% CL by the results of the LSND experiment [6] and the dashed, dash-dotted and dash-dot-dotted curves are the 90\% CL exclusion curves of the Bugey [4], BNL E776 [8] and CCFR [9] experiments, respectively.

The stringent limit for the neutrino mixing parameters represented by the unified approach exclusion curve strongly depends on the fact that none of the expected background events has been observed. Indeed, the unified approach exclusion curve is based on the 90\% CL exclusion of a mean signal of 1.1 events, but the signal that would have been excluded if KARMEN had observed, for example, two (background) events would have been 3.1 events and the corresponding exclusion curve in the \( \sin^2 2\theta - \Delta m^2 \) plane would lie on the right of the bayesian exclusion curve. In other words, the fact that the unified approach exclusion curve is very stringent is not due to the fact that no neutrino oscillations were observed, but
to the fact that no background event has been observed. This is clearly undesirable from the physical point of view.

The new ordering principle proposed in the previous Section allows to obtain an exclusion curve with the correct coverage that minimizes the effect of the fact that no background event has been observed in the KARMEN experiment. The new ordering principle gives a 90% CL upper confidence limit of 1.9 events for the mean $\mu$ of neutrino oscillation events. This limit is close to the upper limit of 2.3 events obtained in the bayesian approach. As shown in Fig. 6, the exclusion curve corresponding to the upper confidence limit obtained with the new ordering principle (the solid curve passing through the filled triangle) lies close to the bayesian exclusion curve.

The sensitivity of the KARMEN experiment calculated with the new ordering principle is of 4.7 events, rather close to the sensitivity of 4.4 events calculated with the unified approach. The corresponding curve is shown in Fig. 6 (the solid curve passing through the empty triangles).

Looking at Fig. 6 one can see that the new ordering principle allows to obtain from the results of the KARMEN experiment an exclusion curve that is more reliable than the one obtained with the unified approach, because the discrepancy between the exclusion curve and the corresponding sensitivity curve is smaller. Furthermore, since the new ordering principle implies an exclusion curve close to that obtained with the bayesian approach, we have a nice and desirable agreement of the results obtained with different statistical methods, one of which (the new ordering principle) guarantees a correct coverage.

In conclusion of this Section, I would like to emphasize that, in spite of the improvement for the physical interpretation of the result of the KARMEN experiment obtained with the introduction of the new ordering principle, the fact that the KARMEN exclusion curves obtained with different statistical methods are significantly different shows that no firm physical conclusion can be inferred from the null result of the KARMEN experiment. Since the KARMEN experiment is continuing to take data, some background events consistent with the expected rate should be observed in the near future and the different curves obtained with different statistical methods should converge. Only in that case it will be possible to confirm or exclude the LSND indication in favor of neutrino oscillations.

**IV. CONCLUSIONS**

The new ordering principle for the classical statistical analysis of Poisson processes with background proposed here allows to improve the unified approach recently proposed by Feldman and Cousins [1].

The confidence intervals obtained with the new ordering principle take advantage of the desirable features of the unified approach and of the bayesian approach. The choice of the bayesian expectation value $\mu_{\text{ref}}(n)$ as the reference value $\mu_{\text{ref}}(n)$ for the new ordering guarantees a reference value bigger or equal than one. The implementation of the new ordering principle in the same way as the Feldman and Cousins ordering principle is implemented in the unified approach guarantees that the resulting confidence intervals have the correct coverage and that there is an automatic transition from two-sided intervals to upper limits for $n \lesssim b$. The fact that the reference value $\mu_{\text{ref}}(n)$ is $\geq 1$ implies that the decrease of the upper confidence
limit $\mu_1$ for the mean signal $\mu$ as the mean expected background $b$ increases is weaker with the new ordering principle than with the Feldman and Cousins ordering principle. This is a desirable feature from a physical point of view, because it allows to minimize the influence that the observation of less background events than expected has on the upper confidence limit for the mean true signal events.

If one does not like the fact that in the new ordering principle the reference value $\mu_{\text{ref}}(n)$ has been obtained using the bayesian method, one can consider Eq. (2.3) as the formula that allows to define $\mu_{\text{ref}}(n)$ in order to obtain the desired result (as the conditions (1.5) and (1.6) allow to obtain central confidence intervals and upper confidence limits, respectively). After all, the construction of the confidence belt, including the ordering principle, is not uniquely defined from first principles and a scientist can choose the method which maximizes the scientific meaning of the resulting confidence intervals, as long as the correct coverage is guaranteed.

In Section III, I have applied the new ordering principle to the analysis of the null result of the KARMEN neutrino oscillation experiment [2]. The new ordering principle gives an exclusion curve in the plane of the neutrino mixing parameters $\sin^2 2\theta$ and $\Delta m^2$ which lies close to that obtained with the bayesian approach and is significantly less stringent than the one obtained with the unified approach. Since the stringency of the unified approach exclusion curve is due to the fact that no background events have been observed (with 2.88 mean expected background events) and not to the sensitivity of the experiment to a small true neutrino oscillation signal, I think that the exclusion curve obtained with the new ordering represents more appropriately the physical implications of the null result of the KARMEN experiment.

Finally, I would like to notice that the new ordering principle has been formulated here for the case of a Poisson process with background, but can be generalized to other controversial cases, as that of gaussian errors with a bounded physical region [10].

ACKNOWLEDGMENTS

I would like to thank K. Eitel for sending me useful information on the statistical analysis performed by the KARMEN collaboration and C.W. Kim for useful discussions. I would like to express my gratitude to the Korea Institute for Advanced Study (KIAS) for the kind hospitality during the initial stages of this work.
REFERENCES

[1] G.J. Feldman and R.D. Cousins, Phys. Rev. D 57, 3873 (1998).
[2] B. Zeitnitz, Talk presented at Neutrino ’98 [11]; KARMEN WWW page: http://www-ik1.fzk.de/www/karmen/karmen_e.html.
[3] C. Caso et al, Eur. Phys. J. C 3, 1 (1998).
[4] G. D’Agostini, preprint DESY 95-242 [hep-ph/9512293].
[5] S.M. Bilenky and B. Pontecorvo, Phys. Rep. 41, 225 (1978); S.M. Bilenky and S.T. Petcov, Rev. Mod. Phys. 59, 671 (1977); R.N. Mohapatra and P.B. Pal, Massive Neutrinos in Physics and Astrophysics, World Scientific Lecture Notes in Physics, Vol.41 (World Scientific, Singapore, 1991); C.W. Kim and A. Pevsner, Neutrinos in Physics and Astrophysics, Contemporary Concepts in Physics, Vol.8 (Harwood Academic Press, Chur, Switzerland, 1993).
[6] C. Athanassopoulos et al., Phys. Rev. Lett. 77, 3082 (1996); D.H. White, Talk presented at Neutrino ’98 [11].
[7] B. Achkar et al., Nucl. Phys. B 434, 503 (1995).
[8] L. Borodovsky et al., Phys. Rev. Lett. 68, 274 (1992).
[9] A. Romosan et al., Phys. Rev. Lett. 78, 2912 (1997).
[10] C. Giunti, in preparation.
[11] Neutrino ’98 WWW page: http://www-sk.icrr.u-tokyo.ac.jp/nu98.
FIGURES

Fig. 1. Confidence belts for 90% CL obtained with the unified approach for a Poisson process with background \( b = 2 \) (region between the two solid lines), \( b = 3 \) (region between the two dashed lines), \( b = 4 \) (region between the two dash-dotted lines), \( b = 5 \) (region between the two dash-dot-dotted lines).

Fig. 2. Upper end of the 90% CL confidence intervals obtained with the unified approach for \( n = 0, 1, 2, 3, 4, 5 \) as a function of the mean expected background \( b \).

Fig. 3. Confidence belts for 90% CL obtained with the new ordering principle for a Poisson process with background \( b = 2 \) (region between the two solid lines), \( b = 3 \) (region between the two dashed lines), \( b = 4 \) (region between the two dash-dotted lines), \( b = 5 \) (region between the two dash-dot-dotted lines).

Fig. 4. Upper end of the 90% CL confidence intervals obtained with the new ordering principle for \( n = 0, 1, 2, 3, 4, 5 \) as a function of the mean expected background \( b \).

Fig. 5. Lower end of the 90% CL confidence intervals obtained with the unified approach and with the new ordering principle for \( n = 3, \ldots, 9 \) (the solid line, \ldots, the dash-dot-dotted line) as a function of the mean expected background \( b \).

Fig. 6. 90% CL exclusion curves in the plane of the neutrino oscillation parameters \( \sin^2 2\theta - \Delta m^2 \) corresponding to the null result of the KARMEN experiment [2]. The solid curves passing through the filled squares, circles and triangles are obtained with the bayesian approach, the unified approach and the new ordering principle, respectively. The solid curves passing through the empty circles and triangles are the sensitivity curves obtained with the unified approach and the new ordering principle, respectively. The shadowed area is the region allowed at 90% CL by the results of the LSND experiment [8] and the dashed, dash-dotted and dash-dot-dotted curves are the 90% CL exclusion curves of the Bugey [7], BNL E776 [8] and CCFR [9] experiments, respectively.
Figure 1

Unified Approach

- $b = 2$
- $b = 3$
- $b = 4$
- $b = 5$

$n$

Figure 1
Mean expected background $b$

Figure 2
New Ordering

- $b = 2$
- $b = 3$
- $b = 4$
- $b = 5$

Figure 3
Mean expected background $b$

Figure 4
Figure 5
Figure 6