NONEXISTENCE OF ALMOST COMPLEX STRUCTURES ON THE PRODUCT 
\( S^{2m} \times M \)

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Abstract. In this note we give a necessary condition for having an almost complex structure on the product \( S^{2m} \times M \), where \( M \) is a connected orientable closed manifold. We show that if the Euler characteristic \( \chi(M) \neq 0 \), then except for finitely many values of \( m \), we do not have almost complex structure on \( S^{2m} \times M \). In the particular case when \( M = \mathbb{C}P^n, n \neq 1 \), we show that if \( n \not\equiv 3 \pmod{4} \) then \( S^{2m} \times \mathbb{C}P^n \) has an almost complex structure if and only if \( m = 1, 3 \). As an application we obtain conditions on the nonexistence of almost complex structure on Dold manifolds.

1. Introduction

Recall that an oriented manifold \( X \) has an almost complex structure (a.c.s. for short) if there is a complex vector bundle \( \xi \) such that the underlying real bundle \( \xi_R \) of \( \xi \) is isomorphic to the tangent bundle \( \tau_X \) of \( X \) as oriented bundles.

It is well known [1] that the only even dimensional spheres admitting an a.c.s are \( S^2 \) and \( S^6 \). The product of even dimensional spheres, \( S^{2m} \times S^{2n} \) with \( m, n \neq 0 \), has an a.c.s. if and only if \((m, n) = (1, 1), (1, 3), (3, 1), (1, 2), (2, 1) \) or \((3, 3) \) (see, for example, [2] and [9]). In [11], Tang showed that \( S^{2m} \times \mathbb{C}P^2 \) has an a.c.s. if and only if \( m = 1, 3 \), and \( S^{2m} \times \mathbb{C}P^3 \) has an a.c.s. if and only if \( m = 1, 2, 3 \).

In this note we deal with the question of existence of an a.c.s. on the general product of the form \( S^{2m} \times M \), where \( M \) is a connected orientable closed manifold of even dimension and \( m \neq 0 \). We have the following main result.

Theorem 1.1. If \( S^{2m} \times M \) has an a.c.s. then \( 2^r \cdot (m-1)! \) divides the Euler characteristic \( \chi(S^{2m} \times M) \), where \( 2^r \) is the highest power of 2 dividing \( m \).

In the special case when \( M = \mathbb{C}P^n \), we have the following result.

Theorem 1.2. Let \( n > 1 \) and \( n \not\equiv 3 \pmod{4} \). Then \( S^{2m} \times \mathbb{C}P^n \) has an a.c.s. if and only if \( m = 1, 3 \).

We also observe that \( S^2 \times \mathbb{C}P^1 \) has exactly two almost complex structures whereas \( S^2 \times \mathbb{C}P^2 \) and \( S^4 \times \mathbb{C}P^3 \) have infinitely many almost complex structures. As an application we obtain a result on the nonexistence of a.c.s. on Dold manifolds which strengthens the result obtained in [12].

For any real or complex vector bundle \( \xi \) over a CW-complex \( X \), we shall denote its stable class in \( \widetilde{KO}(X) \) or \( \widetilde{K}(X) \) by \( [\xi] \). Let \( Vect_n(X) \) be the set of all isomorphism classes of \( n \)-rank complex vector bundles over \( X \). One of the stability properties of vector bundles over \( X \) is that if \( X \) is of dimension \( 2n \) then the map \( Vect_n(X) \to \widetilde{K}(X) \) which takes the complex vector bundle \( \xi \) to its stable class \( [\xi] \) is bijective (see [7] Chapter 9, Theorem 1.5). If \( \rho: \widetilde{K}(X) \to \widetilde{KO}(X) \) is the realization map then the proof of our result is based on the following theorem.

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Theorem 1.3. ([9] Theorem 1.1 and [14] Lemma 1.1) A 2n-dimensional connected oriented manifold \( X \) admits an a.c.s. if and only if there exists an element \( a \in \tilde{K}(X) \) such that \( \rho(a) = [\tau_X] \) and nth-Chern class \( c_n(a) \) equals to the Euler class \( e(X) \). \( \square \)

In this note, if \( X \) is a complex manifold then the holomorphic tangent bundle of \( X \) will be denoted by \( T_X \). All the manifolds \( M \) we consider here will be connected, closed, orientable and of dimension \( 2n \).

2. Almost Complex Structures on \( S^{2m} \times M \)

We first fix our notations. Let \( \nu \) be the canonical complex line bundle over \( S^2 \). Let \( g = (\nu - 1) \in \tilde{K}(S^2) \) be a generator. Then by the Bott-Periodicity, \( g^m \in \tilde{K}(S^{2m}) \) is a generator. We fix the generator \( y_m \in H^{2m}(S^{2m}; \mathbb{Z}) \) such that the total Chern class \( c(g^m) = 1 + (m - 1)! \cdot y_m \). The Bott Integrability Theorem [7] Chapter 20, Corollary 9.8 says that for any \( a \in \tilde{K}(S^{2m}) \), the top Chern class \( c_m(a) \) is divisible by \( (m - 1)! \). (Here divisibility is in the sense that \( c_m(a) = (m - 1)! \cdot c_i \) for some \( c_i \in H^{2m}(S^{2m}; \mathbb{Z}) \)).

In the following proposition we shall give a condition on the Chern classes of any vector bundle over the smash product \( S^{2m} \times M \).

Proposition 2.1. For an element \( a \in \tilde{K}(S^{2m} \times M) \), each Chern class \( c_i(a) \) is divisible by \( (m - 1)! \).

Proof. Let \( a \in \tilde{K}(S^{2m} \times M) \). Then by the Bott-Periodicity, \( a = g^m \otimes (\beta - n) \) for some complex vector bundle \( \beta \) of rank \( n \) over \( M \). Let \( c(\beta) = \prod_k (1 + s_k) \) be the formal factorization obtained by using the splitting principle. As in the proof of Lemma 2.1(2) [10], we can formally write the total Chern class of \( a \) as follow

\[
c(a) = c(g^m \otimes (\beta - n)) = \prod_k (1 + (m - 1)! \cdot y_m ((1 + s_k)^{-m} - 1)) \tag{1}
\]

The last equality in Equation 1 is due to the fact that \( y_m^r = 0 \). The summation \( \sum_k s_k \) can be expressed as a polynomial in Chern classes of \( \beta \) with integer coefficients (by Newton-Girard formula). From here it is clear that each Chern class \( c_i(a) \) is divisible by \( (m - 1)! \). \( \square \)

Next we shall obtain conditions on Chern classes of any element in \( \tilde{K}(S^{2m} \times M) \). Consider the following split exact sequence

\[
0 \rightarrow \tilde{K}(S^{2m} \times M) \rightarrow \tilde{K}(S^{2m} \times M) \rightarrow \tilde{K}(S^{2m}) \oplus \tilde{K}(M) \rightarrow 0.
\]

Hence any element \( a \in \tilde{K}(S^{2m} \times M) \) can be written as \( a = a_1 + a_2 + a_3 \), where \( a_1 \in \tilde{K}(S^{2m} \times M) \), \( a_2 \in \tilde{K}(S^{2m}) \) and \( a_3 \in \tilde{K}(M) \). In the following proposition we choose an orientation on \( S^{2m} \times M \).

Proposition 2.2. For any \( a \in \tilde{K}(S^{2m} \times M) \), all the Chern numbers of \( a \) are divisible by \( (m - 1)! \).

Proof. Let \( a = a_1 + a_2 + a_3 \) be as described above. Let the total Chern class \( c(a_1) = 1 + y_m x_1 + y_m x_2 + \cdots + y_m x_n \), \( c(a_2) = 1 + b y_m \) and \( c(a_3) = 1 + z_1 + z_2 + \cdots + z_n \) where \( x_i, z_i \in H^2(M; \mathbb{Z}) \) and \( b \) is an integer. Since \( a_1 \in \tilde{K}(S^{2m} \times M) \), by Proposition 2.1 it is clear that each \( y_m x_i \) is divisible by \( (m - 1)! \). Also since \( a_2 \in \tilde{K}(S^{2m}) \), by Bott Integrability Theorem, we have that \( b \) is divisible by \( (m - 1)! \). Now the proof of the proposition follows easily. \( \square \)
To prove our main result consider the following commutative diagram

\[
\begin{array}{ccc}
\tilde{K}(S^{2m} \times M) & = & \tilde{K}(S^{2m} \land M) \\
\downarrow \rho & & \downarrow \rho \\
\tilde{KO}(S^{2m} \times M) & = & \tilde{KO}(S^{2m} \land M) \\
\end{array}
\]

In view of Theorem 1.3 if \( a \in \tilde{K}(S^{2m} \times M) \) gives an a.c.s. on \( S^{2m} \times M \) then \( \rho(a) = [\tau_S \times M] \) and the top Chern class \( \tau_{m+n}(a) = e(S^{2m} \times M) \). If we write \( a = a_1 + a_2 + a_3 \) then using the fact that \( [\tau_S \times M] = [\tau_M] \), we have \( \rho(a_1) = 0, \rho(a_2) = 0, \rho(a_3) = [\tau_M] \).

**Proof of Theorem 1.1.** Suppose \( a \in \tilde{K}(S^{2m} \times M) \) gives an a.c.s. on \( S^{2m} \times M \). From Proposition 2.2 and as \( \tau_{m+n}(a) = e(S^{2m} \times M) \), it follows that \((m-1)!\) divides \( \chi(S^{2m} \times M) \). Hence in the case when \( m \) is odd, the proof of the theorem is complete.

Next assume that \( m = 2p \). As above we write \( a = a_1 + a_2 + a_3 \). First observe that the realization map \( \rho : \tilde{K}(S^{4p}) \to \tilde{KO}(S^{4p}) \) is injective, and since \( \rho(a_2) = 0 \), we have \( a_2 = 0 \). Next note that \( \rho(a_1) = 0 \) and this implies that \( a_1 + \overline{a_1} = 0 \). Hence \( 2c_2(a_1) = 0 \) for \( i > 0 \) as the non-trivial cup-products in the cohomology of the smash product \( S^{4p} \times M \) are zero. Therefore the top Chern class

\[
c_{2p+n}(a) = \sum_{2i+1+j=2p+n} c_{2i+1}(a_1)c_j(a_3).
\]

Now consider the element \( a'_1 \in \tilde{K}(S^{4p+2} \land M) \) given as \( a'_1 = g \otimes a_1 \). We can write \( a_1 = g^{2p} \otimes (\beta - n) \) for some complex vector bundle \( \beta \) of rank \( n \) over \( M \) and let \( c(a_1) = 1 + y_{2p}x_1 + y_{2p}x_2 + \cdots + y_{2p}x_n \). Then as in the proof of Proposition 2.4, the total Chern class of \( a'_1 \) will be

\[
c(a'_1) = c(g \otimes a_1) = 1 + y_{2p+1}(\sum_{i \geq 1} (2p+i)x_i).
\]

Since \( a'_1 \in \tilde{K}(S^{4p+2} \land M) \), by Proposition 2.2 we have that each \( (2p+i)y_{2p+1}x_i \) is divisible by \( (2p)! \). This immediately implies that \( (2p)! \) divides \( (2p+i)y_{2p+1}x_i \). Hence for each odd \( i \), \( (2p)! \) will divide

\[
\left( \prod_j (2p+j) \right) \cdot y_{2p}x_i,
\]

where the product varies over odd \( j \) and \( 1 \leq j \leq n \). Hence by Equation 2 we have that \( (2p)! \) divides

\[
\left( \prod_j (2p+j) \right) \cdot \chi(S^{4p} \times M),
\]

where \( j \) is odd and \( 1 \leq j \leq n \). Now the proof of the theorem follows from the fact that \( (2p-1)! \) divides \( \chi(S^{4p} \times M) \).

From Theorem 1.1 we can observe that for a given \( M \) with \( \chi(M) \neq 0 \), except for finitely many values of \( m \), we do not have a.c.s on \( S^{2m} \times M \). In particular we have the following corollary whose proof follows immediately.

**Corollary 2.3.** Let \( \chi(M) \neq 0 \) (mod 4) or \( \chi(M) \) be a power of two. Then \( S^{2m} \times M \) does not have an a.c.s. for \( m \neq 1, 2, 3 \).

The restriction on \( m \) in Corollary 2.3 is the best possible as we know that \( S^2 \times \mathbb{C}P^n \), \( S^6 \times \mathbb{C}P^n \), \( S^4 \times S^2 \) and \( S^4 \times \mathbb{C}P^3 \) (see Example 3.10 below) admit a.a.c.
3. Almost complex structure on $S^{2m} \times \mathbb{CP}^n$

In this section we shall deal with the case when $M$ is a complex projective space $\mathbb{CP}^n$. By Corollary 2.3 we know that in the case when $n \not\equiv 3 \pmod{4}$ or $n + 1 = 2k$ for some $k \geq 0$, the product $S^{2m} \times \mathbb{CP}^n$ does not admit a.c.s. for $m \not\equiv 1, 2$ and $3$. We shall prove that if $n \not\equiv 3 \pmod{4}$ and $n > 1$ then $S^4 \times \mathbb{CP}^n$ also does not admit a.c.s.

Let $H$ be the canonical line bundle over $\mathbb{CP}^n$. Let $x = c_1(H) \in H^2(\mathbb{CP}^n; \mathbb{Z})$ be a generator. Let $\eta = H - 1 \in \tilde{K}(\mathbb{CP}^n)$ and $r = \lfloor n/2 \rfloor$. For $\alpha \in \tilde{K}(S^{2m})$ and $\beta \in \tilde{K}(\mathbb{CP}^n)$ we shall write the external product $\alpha \otimes \beta \in \tilde{K}(S^{2m} \wedge \mathbb{CP}^n)$ as $\alpha \beta$. In view of Lemma 3.5 [4] we have the following lemma whose proof follows from the Bott-Periodicity.

**Lemma 3.1.** Each of the following system of elements form an integral basis of $\tilde{K}(S^{2m} \wedge \mathbb{CP}^n)$.

(i) $g^m \eta, g^m \eta(\eta + \bar{\eta}), \cdots, g^m \eta(\eta + \bar{\eta})^{r-1}, g^m(\eta + \bar{\eta}), g^m(\eta + \bar{\eta})^2, \cdots, g^m(\eta + \bar{\eta})^r$, and also in case $n$ is odd, $g^m \eta^{2r+1} = g^m \eta(\eta + \bar{\eta})^r$;

(ii) $g^m \eta, g^m \eta(\eta + \bar{\eta}), \cdots, g^m \eta(\eta + \bar{\eta})^{r-1}, g^m(\eta - \bar{\eta}), g^m(\eta - \bar{\eta})(\eta + \bar{\eta}), \cdots, g^m(\eta - \bar{\eta})(\eta + \bar{\eta})^{r-1}$, and also in case $n$ is odd, $g^m \eta^{2r+1} = g^m \eta(\eta + \bar{\eta})^r$. \hfill $\Box$

Let $w_k = g^m(H^k - 1) - g^m(H^k - 1) \in \tilde{K}(S^{2m} \wedge \mathbb{CP}^n)$. Similar to Proposition 4.3 [13], we have the following three propositions.

**Proposition 3.2.** Let $m \equiv 1, 3 \pmod{4}$. Then the kernel of the realization map $\rho : \tilde{K}(S^{2m} \wedge \mathbb{CP}^n) \to \tilde{KO}(S^{2m} \wedge \mathbb{CP}^n)$ is freely generated by $w_1, w_2, \cdots, w_r$.

**Proof.** It is clear that all $w_k$’s are in the kernel of the realization map. Further it can be proved using the fact that $\bar{g} = -g$ in $\tilde{K}(S^2)$ and using induction on $k$ that

$$g^m(\eta + \bar{\eta})^k = w_k + \text{(linear combination of } w_1, \cdots, w_{k-1}).$$

Now using the structure of $\tilde{KO}^{-2m}(\mathbb{CP}^n)$ (see [4]) and the fact that the $\mathbb{Q}$-linear map

$$\rho \otimes \text{Id} : \tilde{K}(S^{2m} \wedge \mathbb{CP}^n) \otimes \mathbb{Q} \to \tilde{KO}(S^{2m} \wedge \mathbb{CP}^n) \otimes \mathbb{Q}$$

is surjective, we can see that $w_1, \cdots, w_k$ freely generates the kernel of $\rho$. \hfill $\Box$

For next two propositions we first note that if $m$ is even then

$$g^m(\eta - \bar{\eta})(\eta + \bar{\eta})^{k-1} = w_k + \text{(linear combination of } w_1, \cdots, w_{k-1}),$$

whose proof can again be given using induction on $k$. Next consider the following commutation diagram,

$$\begin{array}{ccc} \tilde{K}(S^{2m} \wedge S^{2n}) & \longrightarrow & \tilde{K}(S^{2m} \wedge \mathbb{CP}^n) \\ \rho \downarrow & & \rho \downarrow \\ \tilde{KO}(S^{2m} \wedge S^{2n}) & \longrightarrow & \tilde{KO}(S^{2m} \wedge \mathbb{CP}^n) \\ \rho \downarrow & & \rho \downarrow \\ \tilde{KO}(S^{2m} \wedge S^{2n}) & \longrightarrow & \tilde{KO}(S^{2m} \wedge \mathbb{CP}^n) \longrightarrow \tilde{KO}(S^{2m} \wedge \mathbb{CP}^{n-1}) \end{array}$$

Now the proof of the following two propositions follow from the fact that the realization map $\rho : \tilde{K}(S^l) \to \tilde{KO}(S^l)$ is nonzero for $l \not\equiv 3 \pmod{4}$.

**Proposition 3.3.** Let $m \equiv 0 \pmod{4}$. Then the kernel of the realization map $\rho : \tilde{K}(S^{2m} \wedge \mathbb{CP}^n) \to \tilde{KO}(S^{2m} \wedge \mathbb{CP}^n)$ is freely generated by

1. $w_1, w_2, \cdots, w_r$ when $n$ is even;
2. $w_1, w_2, \cdots, w_r, g^m \eta^n$, when $n \equiv 3 \pmod{4}$;
In the above proposition, using that the fact that a binomial coefficient 
\( \binom{s}{t} \) is even if \( s \) is even and \( t \) is odd, one can see that if \( m \) is even and \( n > 1 \) then for any \( a_1 \in \tilde{K}(S^{2m} \wedge \mathbb{C}P^n) \) such that \( \rho(a_1) = 0 \) we have that \( 4 \cdot (m-1)! \) divides each Chern class \( c_i(a_1) \).

**Proposition 3.4.** Let \( m \equiv 2 \pmod{4} \). Then the kernel of the realization map \( \rho : \tilde{K}(S^{2m} \wedge \mathbb{CP}^n) \to KO(S^{2m} \wedge \mathbb{CP}^n) \) is freely generated by

1. \( w_1, w_2, \ldots, w_r \) when \( n \) is even;
2. \( w_1, w_2, \ldots, w_r, g^m \eta^n \), when \( n \equiv 1 \pmod{4} \);
3. \( w_1, w_2, \ldots, w_r, 2g^m \eta^n \), when \( n \equiv 3 \pmod{4} \).

Next we shall describe the Chern classes of elements in the Kernel of the realization map \( \rho : \tilde{K}(S^{2m} \wedge \mathbb{CP}^n) \to KO(S^{2m} \wedge \mathbb{CP}^n) \). Using Lemma 2.1(2) \([10]\) and by the fact that \( y_m^2 = 0 \), we can easily compute the total Chern class of \( w_k \). When \( m \) is even then

\[
c(w_k) = 1 - (m-1)! \sum_{i \geq 1} 2 \left( \frac{m + 2i - 2}{2i - 1} \right) k^{2i-1} y_m x^{2i-1}.
\]

When \( m \) is odd we have

\[
c(w_k) = 1 + (m-1)! \sum_{i \geq 1} 2 \left( \frac{m + 2i - 1}{2i} \right) k^{2i} y_m x^{2i}.
\]

To compute the Chern class of \( g^m \eta^n \in \tilde{K}(S^{2m} \wedge \mathbb{CP}^n) \), consider the following exact sequence

\[
0 \to \tilde{K}(S^{2m} \wedge S^{2n}) \to \tilde{K}(S^{2m} \wedge \mathbb{CP}^n) \to \tilde{K}(S^{2m} \wedge \mathbb{CP}^{n-1}).
\]

The element \( g^m \eta^n \in \tilde{K}(S^{2m} \wedge \mathbb{CP}^n) \) is a generator of the kernel of the last map in the above sequence. Hence the total Chern class of \( g^m \eta^n \) is as follows:

\[
c(g^m \eta^n) = 1 \pm (m + n - 1)! \cdot y_m x^n
\]

\[
= 1 \pm (m - 1)! \cdot n! \cdot \left( \frac{m+n-1}{n} \right) y_m x^n.
\]

We have the following proposition whose proof follows from the above discussion.

**Proposition 3.5.** The total Chern class of any element in the kernel of the realization map \( \rho : \tilde{K}(S^{2m} \wedge \mathbb{CP}^n) \to KO(S^{2m} \wedge \mathbb{CP}^n) \) is of the form

1. \( 1 + 2 \cdot (m-1)! \cdot \left[ \sum_{i \geq 1} \left( \frac{m+2i-2}{2i-1} \right) \left( \sum_{k=1}^r b_k k^{2i-1} \right) y_m x^{2i} \right] \), when \( m \) is odd.
2. \( 1 - 2 \cdot (m-1)! \cdot \left[ \sum_{i \geq 1} \left( \frac{m+2i-1}{2i} \right) \left( \sum_{k=1}^r b_k k^{2i-1} \right) y_m x^{2i-1} \right] \), when \( m, n \) are even.
3. \( 1 - (m-1)! \cdot \left[ 2 \sum_{i=1}^{r+1} \left( \frac{m+2i-2}{2i-1} \right) \left( \sum_{k=1}^r b_k k^{2i-1} \right) y_m x^{2i-1} \pm \left( \frac{m+n-1}{n} \right) \cdot n! \cdot b_{r+1} y_m x^n \right] \), when \( m \equiv 0 \pmod{4} \) and \( n \equiv 3 \pmod{4} \), or \( m \equiv 2 \pmod{4} \) and \( n \equiv 1 \pmod{4} \).
4. \( 1 - 2 \cdot (m-1)! \cdot \left[ \sum_{i=1}^{r+1} \left( \frac{m+2i-2}{2i-1} \right) \left( \sum_{k=1}^r b_k k^{2i-1} \right) y_m x^{2i-1} \pm \left( \frac{m+n-1}{n} \right) \cdot n! \cdot b_{r+1} y_m x^n \right] \), when \( m \equiv 0 \pmod{4} \) and \( n \equiv 1 \pmod{4} \), or \( m \equiv 2 \pmod{4} \) and \( n \equiv 3 \pmod{4} \),

for \( b_1, b_2, \ldots, b_r, b_{r+1} \in \mathbb{Z} \).

**Remark 3.6.** In the above proposition, using that the fact that a binomial coefficient \( \binom{s}{t} \) is even if \( s \) is even and \( t \) is odd, one can see that if \( m \) is even and \( n > 1 \) then for any \( a_1 \in \tilde{K}(S^{2m} \wedge \mathbb{CP}^n) \) such that \( \rho(a_1) = 0 \) we have that \( 4 \cdot (m-1)! \) divides each Chern class \( c_i(a_1) \).
Proposition 3.7. If $S^{4p} \times \mathbb{C}P^n$ has a.c.s. then $2 \cdot (2p-1)!$ divides $\chi(\mathbb{C}P^n) = (n+1)$.

Proof. Suppose $a \in \tilde{K}(S^{4p} \times \mathbb{C}P^n)$ gives an a.c.s. on $S^{4p} \times \mathbb{C}P^n$ with $a = a_1 + a_2 + a_3$. As we argued in the proof of Theorem 1.1, we have $a_2 = 0$. By Remark 3.6 we have that if $n > 1$ then $4 \cdot (2p-1)!$ divides each Chern class $c_i(a_1)$. Hence in the case $n > 1$, we have that $4 \cdot (2p-1)!$ divides the Euler characteristic $\chi(S^{4p} \times \mathbb{C}P^n) = 2(n+1)$. In the case when $n = 1$, we know that $S^{4p} \times \mathbb{C}P^1$ has a.c.s only when $p = 1$. This completes the proof of the proposition.

Next we recall that for any element $a_3 \in \tilde{K}(\mathbb{C}P^n)$ such that $\rho(a_3) = [\tau_{\mathbb{C}P^n}]$, the total Chern class of $a_3$ has been described on p.130 of [13] as

$$c(a_3) = (1-x)^{n+1}(1 \pm (n-1)!x^n)^{ud_{r+1}} \prod_{1 \leq k \leq r} \frac{1+kx}{1-kx} d_k,$$

where $d_i$’s are integers and

- $u = 0$ if $n$ is even,
- $u = 1$ if $n \equiv 3 \pmod{4}$,
- $u = 2$ if $n \equiv 1 \pmod{4}$.

We remark that there is a typographical error in the signs in the first two products of the right hand side of Equation 3 as expressed in [13]. For example one can easily compute $c(\eta^2) = (1 + (n-1)!x^n)$ when $n = 1, 3$ whereas $c(\eta^2) = (1 - x^2)$ when $n = 2$.

We know that $S^{2m} \times \mathbb{C}P^1$ has a.c.s. if and only if $m = 1, 2$ and 3. Next we give the proof of Theorem 1.2.

Proof of Theorem 1.2. First note that $S^2 \times \mathbb{C}P^n$, $S^6 \times \mathbb{C}P^n$ have a.c.s. By Corollary 2.3 and Proposition 3.7, only thing that remains to complete the proof of the theorem is to show that for $q > 0$ there is no a.c.s. on $S^4 \times \mathbb{C}P^{4q+1}$. Suppose $a \in \tilde{K}(S^4 \times \mathbb{C}P^{4q+1})$ gives an a.c.s. on $S^4 \times \mathbb{C}P^{4q+1}$ with $a = a_1 + a_2 + a_3$. As observed in the proof of Theorem 1.1, we have $a_2 = 0$. The total Chern class of $a_1$ is given in (3) of Proposition 3.5 as

$$c(a_1) = 1 - 2 \sum_{i=1}^{2q+1} 2i \left( \sum_{k=1}^{2q} b_k k^{2i-1} \right) y_2 x^{2i-1} \pm (4q+2)! \cdot b_{2q+1} y_2 x^{4q+1}$$

where $b_i$’s are integers. The total Chern class of $a_3$ as described in Equation 3 is as follows

$$c(a_3) = (1-x)^{4q+2} (1 \pm (4q)!x^{4q+1})^{2d_{2q+1}} \prod_{1 \leq k \leq 2q} \left( \frac{1+kx}{1-kx} \right)^{d_k},$$

where $d_i$’s are integers. After simplifying we can write

$$c(a_3) = 1 + \sum_{i=1}^{2q} \left( \frac{4q+2}{2i} \right) x^{2i} + A$$

where $A$ is a polynomial in $x$ with even coefficients. As $c(a) = c(a_1)c(a_3)$, the top Chern class

$$c_{4q+3}(a) = (-2 \sum_{i=1}^{2q+1} 2i \left( \frac{4q+2}{2i} \right) \sum_{k=1}^{2q} b_k k^{2i-1} + h) y_2 x^{4q+1}$$

where $h$ is a multiple of 8. We shall next prove that the coefficient

$$h_k := -2 \sum_{i=1}^{2q+1} 2i \left( \frac{4q+2}{2i} \right) k^{2i-1}$$
of $b_ky_2x^{4q+1}$ in Equation 4 is a multiple of 8. Clearly, when $k$ is even then $h_k$ is a multiple of 8. Next consider the case when $k$ is odd. Consider the following equality.

$$(4q + 2)(1 + k)^{4q+1} = \sum_{l=1}^{4q+2} \binom{4q+2}{l} l^{k-1}.$$ 

As $k$ is odd, the left hand summation is a multiple of 4. The right hand summation can be decomposed into three parts as follows:

$$\sum_{i=1}^{2q+1} 2i^{(4q+2)}k^{2i-1} + \sum_{j=1}^{q} \binom{4q+2}{2j-1} (2j - 1)k^{2j-2} + (4q - 2j + 3)k^{4q-2j+2} + (4q+2)k^{2q+2}.$$ 

The middle term in the above summation is a multiple of 4. To see that $(\frac{4q+2}{2q+1})$ is a multiple of 4, we write

$$\binom{4q+2}{2q+1} = \frac{(4q+2)(4q+1)}{(2q+1)^2} = \frac{4q}{2q}$$

and use the fact that $(\frac{4q}{2q})$ is even which can be seen from the following equality

$$2^{4q} = (1 + 1)^{4q} = \sum_{i=0}^{4q} \binom{4q}{i}.$$ 

This completes the proof that each $h_k$ is a multiple of 8. Since $a$ gives an a.c.s. on $S^4 \times \mathbb{CP}^{4q+1}$, the top Chern class $c_{4q+3}(a)$ is multiple of 8. This implies that 8 divides the Euler characteristic $2(4q + 2)$. But this is a contradiction. Hence there is no a.c.s. on $S^4 \times \mathbb{CP}^{4q+1}$. This completes the proof of the theorem.

The above construction helps us to give an explicit way to obtain almost complex structures on $S^{2m} \times \mathbb{CP}^n$ for few values of $m$ and $n$. In particular when $m = 2$ and $n = 3$ we show that $S^4 \times \mathbb{CP}^4$ has infinite number of almost complex structures. We fix an orientation on $S^{2m} \times \mathbb{CP}^n$ such that $e(S^{2m}) = -2y_m$ and $e(\mathbb{CP}^n) = (-1)^n(n + 1)\chi^n$. We fix an orientation on $S^{2m} \times \mathbb{CP}^n$ arising by taking the orientation on each factor.

**Example 3.8.** Let $m = 1$ and $n = 1$. From Propositions 3.2 and 3.8 we get that $a = a_1 + a_2 + a_3 \in \tilde{K}(S^2 \times \mathbb{CP}^1)$ gives an a.c.s. on $S^2 \times \mathbb{CP}^1 = S^2 \times S^2$ if and only if there are integers $d_1, d_2$ such that $a_1 = 0, a_2 = 2d_1\eta$ and $a_3 = [T_{\mathbb{CP}^1}] + 2d_2\eta$ and $c_2(a) = e(S^2 \times S^2) = (-2y_1)(-2x) = 4y_1x$. This gives the equation

$$4d_1(d_2 - 1) = 4,$$

which has the following two solutions: $d_1 = 1, d_2 = 2$ and $d_1 = -1, d_2 = 0$. Therefore we have the following two possibilities: $a = 2\eta + [T_{\mathbb{CP}^1}] + 4\eta$ and $a = -2\eta + [T_{\mathbb{CP}^1}]$. By stability property, we thus have that there are exactly two non-isomorphic a.c.s. on $S^2 \times \mathbb{CP}^1$. Hence the only two almost complex structures are $T_S^2 \oplus T_{\mathbb{CP}^1}$ and $T_{S^2} \oplus T_{\mathbb{CP}^1}$. Here $T_{S^2}$ is complex conjugate bundle of $T_{S^2}$ and $[T_{\mathbb{CP}^1}] = [T_{\mathbb{CP}^1}] + 4\eta$. We note here that if we change the orientation on $S^2 \times S^2$, a similar argument will say that again it has exactly two almost complex structures which will be given by $T_S^2 \oplus T_{\mathbb{CP}^1}$ and $T_{S^2} \oplus T_{\mathbb{CP}^1}$. The existence of exactly two almost complex structures on $S^2 \times S^2$ was also observed by Sutherland in [9].

**Example 3.9.** Let $m = 1$ and $n = 2$. As in Example 3.8 an element $a \in \tilde{K}(S^2 \times \mathbb{CP}^2)$ gives an a.c.s. on $S^2 \times \mathbb{CP}^2$ if and only if there are integers $b_1, d_1$ and $d_2$ such that $a_1 = b_1(g(H - 1) - g(H - 1))$, 

$$a_2 = 2d_1\eta + 2d_2\eta$$

and $c_2(a) = e(S^2 \times S^2) = (-2y_1)(-2x) = 4y_1x$. This
\[ a_2 = 2d_1 \eta, \ a_3 = [T_{\mathbb{C}P^2}] + d_2 (H - \bar{H}) \] and \( c_3(a) = -6y_1 x^2 \). This gives the following equations

\[ b_1 + d_1 (-4d_2 + 4\left(\frac{d_2}{2}\right) + 3) = -3 \quad \text{when} \ d_2 \geq 0 \]

\[ b_1 + d_1 (-8d_2 + 4\left(\frac{-d_2}{2}\right) + 3) = -3 \quad \text{when} \ d_2 < 0. \]

Clearly, we shall get infinite number of solutions of the above equations and by stability property, we have infinite number of almost complex structures on \( S^2 \times \mathbb{CP}^2 \). The solution \( b_1 = 0, d_1 = -1 \) and \( d_2 = 0 \) corresponds to the a.c.s. given by the holomorphic tangent bundle \( T_S^2 \oplus T_{\mathbb{C}P^2} \). Similarly, if we change the orientation on \( S^2 \times \mathbb{CP}^2 \), we shall again get infinite number of almost complex structures.

**Example 3.10.** Let \( m = 2 \) and \( n = 3 \). Going along the same line of arguments as we did in last two examples we observe that number of almost complex structures on \( S^4 \times \mathbb{CP}^3 \) is in one to one correspondence with the number of solutions of the following equation

\[ b_1 (4 - 3d_1 + 2\left(\frac{d_1}{2}\right) + 3)b_2 = -1 \quad \text{when} \ d_1 \geq 0 \]

\[ b_1 (4 - 5d_1 + 2\left(\frac{-d_1}{2}\right) + 3)b_2 = -1 \quad \text{when} \ d_1 < 0. \]

For each \( k \in \mathbb{Z} \), by taking \( b_1 = -7 + 6k, b_2 = 1 \) and \( d_1 = 1 \) we get infinite number of solutions of the above equation and thus this gives infinite number of almost complex structures on \( S^4 \times \mathbb{CP}^3 \). Again the reverse orientation on \( S^4 \times \mathbb{CP}^3 \) has infinite number of almost complex structures. We remark that the existence of an a.c.s. on \( S^4 \times \mathbb{CP}^3 \) also follows from Theorem 2 of [6] and this was observed by Tang in [11].

We end this section with a result on nonexistence of a.c.s. on the Dold manifold \( D(m, n) \). It is known (see [3]) that \( D(m, n) \) is orientable if and only if \( m + n \) is odd. So an even dimensional orientable Dold manifold is of the form \( D(2p, 2q + 1) \) for some \( p, q \geq 0 \). By considering the 2-fold covering map \( S^{2p} \times \mathbb{CP}^{2q+1} \rightarrow D(2p, 2q + 1) \), we note that the nonexistence of an a.c.s. on \( S^{2p} \times \mathbb{CP}^{2q+1} \) will imply the nonexistence of an a.c.s. on \( D(2p, 2q + 1) \). We have the following result.

**Theorem 3.11.** Let \( p > 0 \). Then \( D(2p, 2q + 1) \) has no a.c.s. if one of the following is true

1. \( p \) is odd.
2. \( p \equiv 0 \ (\text{mod} \ 4) \) and \((q + 1)\) is not a multiple of \( 2^{r-2} \cdot (p - 1)! \), where \( 2^r \) is the maximum power of \( 2 \) dividing \( p \).
3. \( p \equiv 2 \ (\text{mod} \ 4) \) and \((q + 1)\) is not a multiple of \( (p - 1)! \).
4. \( p = 2 \) and \( q \) is even.

**Proof.** Statement (1) follows from Theorem 3.2 of [12]. The proof of statements (2), (3) and (4) follow from Theorem [13] Proposition 3.7 and Theorem 1.2 respectively.

From the above theorem it is clear that \( D(2p, 2q + 1) \) does not admit an a.c.s. if \( p \) is odd or \( q \) is even.

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