Projective Hessian and Sasakian manifolds

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Abstract

The Hessian geometry is the real analogue of the Kähler one. Sasakian geometry is an odd-dimensional counterpart of the Kähler geometry. In the paper, we study the connection between projective Hessian and Sasakian manifolds analogous to the one between Hessian and Kähler manifolds. In particular, we construct a Sasakian structure on $TM \times \mathbb{R}$ from a projective Hessian structure on $M$. Especially, we are interested in the case of invariant structure on Lie groups. We define semi-Sasakian Lie groups as a generalization of Sasakian Lie groups. Then we construct a semi-Sasakian structure on a group $G \ltimes \mathbb{R}^{n+1}$ for a projective Hessian Lie group $G$. Further, we describe examples of homogeneous Hessian Lie groups and corresponding semi-Sasakian Lie groups. The big class of projective Hessian Lie groups can be constructed by homogeneous regular domains in $\mathbb{R}^{n}$. The groups SO(2) and SU(2) belong to another kind of examples. Using them, we construct semi-Sasakian structures on the group of the Euclidean motions of the real plane and the group of isometries of the complex plane.

1 Introduction

A flat affine manifold is a differentiable manifold equipped with a flat, torsion-free connection. Equivalently, it is a manifold equipped with an atlas such that all translation maps between charts are affine transformations ([FGH] or [Sh]). A Hessian manifold is an affine manifold with a Riemann metric which is locally equivalent to a Hessian of a function. Any Kähler metric can be defined as a complex Hessian of a plurisubharmonic function. Thus, the Hessian geometry is a real analogue of the Kähler one. In particular, we can construct a Kähler structure on $TM$ by a Hessian structure on $M$. For more details about Hessian geometry see [Sh].

The contact geometry is an odd-dimensional counterpart of the symplectic geometry. A manifold $M$ is contact if and only if there exists a selfsimilar symplectic form $\omega$ on the cone $M \times \mathbb{R}^{>0}$. Moreover, if there exists an $\mathbb{R}^{>0}$-invariant complex structure $I$ on $M \times \mathbb{R}^{>0}$ such that $\omega(I \cdot, \cdot)$ is positive defined, that is, there is a Kähler structure on $M \times \mathbb{R}^{>0}$ with respect to the action of $\mathbb{R}^{>0}$. Manifolds satisfied this condition are called Sasakian. The connection between Sasakian and Kähler geometries is the same as between contact and symplectic geometries.

We say that a manifold $M$ is projective Hessian if there is an $\mathbb{R}^{>0}$-invariant affine structure and selfsimilar Hessian metric on $M$. In the paper, we study the connection between projective Hessian and Sasakian manifolds analogous to the one between Hessian and Kähler manifolds.

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In particular, we construct a Sasakian structure on $TM \times \mathbb{R}$ by a projective Hessian structure on $M$ (in section 4).

Further, we work with the Lie groups and invariant structures on them. There are different descriptions of an invariant affine structure on a group $G$: a left invariant torsion-free flat connection on $G$, an étale affine representation $G \to \text{Aff}(\mathbb{R}^n)$, where $n$ is dimension of $G$, or a structure of a left symmetric algebra on the corresponding Lie algebra $\mathfrak{g}$, that is, a multiplication on $\mathfrak{g}$ satisfying

$$XY - YX = [X,Y]$$

and

$$X(YZ) - (XY)Z = Z(XY) - (ZX)Y$$

for any $X, Y, Z \in \mathfrak{g}$ (see [Bu1]).

In section 6 we describe the correspondence between invariant flat affine structures on a Lie group $G$ and invariant complex structures on the total space of the tangent bundle $TG$ satisfying a certain condition.

In section 7 we identify $TG \simeq G \ltimes_\theta \mathbb{R}^n$, where $\theta$ is an étale affine representation on $G$, and construct an invariant complex structure on the group $G \ltimes_\theta \mathbb{R}^n$ using the result of section 4. In [M], there is a construction of an invariant Kähler structure on $G \ltimes_\theta (\mathbb{R}^n)^*$ by an invariant Hessian structure on $G$. We prove the same statement for the group $G \ltimes_\theta \mathbb{R}^n$. Further, we adapt this construction to the case of contact and Hessian geometries. We say that a group $G$ is semi-Sasakian if there exist an action $\varphi: \mathbb{R}^+ \to \text{Aut}(G)$ and $G$-invariant selfsimilar Kähler metric on a semi-direct product $\mathbb{R}^+ \ltimes_\varphi G$. This definition arises by the following way: we prove that if a group $G$ is projective Hessian then the $G \ltimes_\theta \mathbb{R}^{n+1}$ is semi-Sasakian. Thus, examples of projective Hessian groups provide examples of semi-Sasakian Lie groups.

Homogeneous regular convex domains in $\mathbb{R}^n$ are the main source of examples of projective Hessian Lie groups for us. A domain $\Omega \subset \mathbb{R}^n$ is regular if $\Omega$ does not contain any straight line. Vinberg proved in [V], that for any homogeneous regular convex domain $\Omega$ there is a triangular group $T \subset \text{Aut}(\Omega)$ that acts simply transitively on $\Omega$. Moreover, he described a metric on a regular convex cone $V \subset \mathbb{R}^n$ that is invariant under $\text{SL}(\mathbb{R}^n) \cap \text{Aut}(V) = \text{Aut}_{\text{SL}}(V)$ and selfsimilar. It follows that the group $T \cap \text{Aut}_{\text{SL}}(V)$ is projective Hessian. If $T$ acts simply transitively on a convex regular domain $\Omega$ then the corresponding Lie algebra $\mathfrak{t}$ is called associated with $\Omega$. The affine structure on $\Omega$ induces an affine structure on $\mathfrak{t}$. Vinberg proved that a left symmetric algebra $\mathfrak{t}$ can be associated with a regular convex domain $\Omega$ if and only if there is a 1-form $\chi$ on $\mathfrak{t}$ such that the bilinear form $g(X,Y) = \chi(XY) = \chi(\nabla_X Y)$ is symmetric and positive defined. An algebra satisfying this condition is called clan (see [V]). Any clan is a projective Hessian Lie algebra.

The groups $\text{U}(1) = \text{SO}(2)$ and $\text{SO}(2)$ belong to another kind of projective Hessian Lie groups. Using them, we can construct a semi-Sasakian structure on the Euclidean group $\text{E}(2)$ and the group of isometries of the complex plane $\mathbb{C}^2$. Any Sasakian group is obviously semi-Sasakian. The Sasakian groups of dimension $n \leq 5$ are classified in [AFV]. Using this classification, we verify that the semi-Sasakian group $\text{E}(2)$ does not admit a Sasakian structure.

Any clan admits both structures: Hessian and projective Hessian. The group $\text{SO}(2)$ admits both structures too. However, not any projective Hessian group is Hessian. The group $\text{SU}(2)$ is not Hessian just because the sphere $S^3$ does not admit any flat affine structure. However, the group $\text{SU}(2)$ admits an invariant projective structure, since there is an invariant affine structure on $\text{SU}(2) \times \mathbb{R}^{>0}$. Thus, the natural question arises: does the existence of $G$-invariant and $\mathbb{R}^{>0}$-Hessian structure on $G \times \mathbb{R}^{>0}$ implies the existence of $G \times \mathbb{R}^{>0}$-invariant Hessian structure on $G$?
$G \times \mathbb{R}^{>0}$? The answer is positive when $G$ is a clan or $U(1)$. We show that $SU(2) \times \mathbb{R}^{>0}$ does not admit an invariant Hessian structure.

## 2 Hessian and Kähler structures

**Definition 2.1.** An affine manifold is a differentiable manifold equipped with a flat, torsion-free connection. Equivalently, it is a manifold equipped with an atlas such that all translation maps between charts are affine transformations [FGH].

**Definition 2.2.** Let $(M, \nabla)$ be a flat affine manifold. A Riemannian metric $g$ on $M$ is called to be a Hessian metric if $g$ is locally expressed by a Hessian of a function $g = \text{Hess} \varphi = \frac{\partial^2}{\partial x^i \partial x^j} dx^i dx^j$,

where $x^1, \ldots, x^n$ are flat local coordinates.

Let $U$ be an open chart on a flat affine manifold $M$, functions $x^1, \ldots, x^n$ be affine coordinates on $U$, and $x^1, \ldots, x^n, y^1, \ldots, y^n$ be corresponding coordinates on $TU$. Define the complex structure $I$ by $I(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial y^i}$. Corresponding complex coordinates are given by $z^i = x^i + \sqrt{-1} y^i$. The complex structure $I$ does not depend on a choice of flat coordinates on $U$. Thus, in this way, we get a complex structure on $TM$.

Let $\pi : TM \rightarrow M$ be a natural projection. Consider a Riemann metric $g$ given locally by $f_{i,j} dx^i dx^j$. Define the Hermitian metric $g^T$ on $TM$ by $\pi^* f_{i,j} dz^i d\bar{z}^j$.

**Proposition 2.3.** Let $M$ be flat affine, $g$ and $g^T$ be as above. Then the following conditions are equivalent:

i) $g$ is a Hessian metric.

ii) $g^T$ is a Kähler metric.

Moreover, if $g = \text{Hess} \varphi$ locally then $g^T$ is equal to a complex Hessian $g = \text{Hess}_C(4 \pi^* \varphi)$.

**Proof.** Let $g$ be a Hessian metric $g = \frac{\partial^2 \varphi}{\partial x^i \partial x^j} dx^i dx^j$.

Consider the complex Hessian

$$\text{Hess}(\pi^* \varphi) = \frac{\partial^2 (\pi^* \varphi)}{\partial z^i \partial \bar{z}^j} dz^i d\bar{z}^j = \frac{1}{4} \left( \frac{\partial}{\partial x^i} - \frac{\partial}{\partial y^i} \right) \left( \frac{\partial}{\partial x^j} + \frac{\partial}{\partial y^j} \right) \varphi dz^i d\bar{z}^j.$$

The function $\pi^* \varphi$ does not depend on $y^1, y^2, \ldots, y^n$. Thus

$$\text{Hess}(\pi^* \varphi) = \frac{1}{4} \frac{\partial^2 \varphi}{\partial x^i \partial x^j} dz^i d\bar{z}^j = \frac{1}{4} g^T.$$

Therefore, the Hermitian form $g^T$ is equal to the complex Hessian of the function $4 \pi^* \varphi$. Thus, $g^T$ is a Kähler metric. The proof in the opposite direction is analogous. \qed
Definition 2.4. The metric $g^T$ is called Kähler metric associated with $g$.

Proposition 2.5. Let $(M, g)$ be a flat Hessian manifold, $I$ a complex structure on $TM$, and $\pi : TM \to M$ a projection. Then the associated Kähler metric $g^T$ equals

$$h(X, Y) = \pi^* g(X, Y) + \pi^* g(IX, IY) + \sqrt{-1} \pi^* g(IX, Y) - \sqrt{-1} \pi^* g(X, IY). \quad (2.1)$$

Proof. If

$$g = f_{i,j} dx^i dx^j \quad (2.2)$$

then

$$g^T = \pi^* f_{i,j} dz^i d\bar{z}^j = \pi^* f_{i,j} d(x^i + \sqrt{-1} y^i) d(x^j - \sqrt{-1} y^j) =
\pi^* f_{i,j} dx^i dx^j + \pi^* f_{i,j} dy^i dy^j + \sqrt{-1} \pi^* f_{i,j} dx^i dy^j - \sqrt{-1} \pi^* f_{i,j} dx^j dy^i. \quad (2.3)$$

It is enough to check the identity

$$h(X, Y) = g^T(X, Y)$$
on pairs of basis vectors. For any $i \in \{1, \ldots, n\}$ we have

$$\pi^* g (\frac{\partial}{\partial y^i}, \ldots) = 0,$$

moreover,

$$I \frac{\partial}{\partial x^i} = \frac{\partial}{\partial y^i}.$$

Hence, by (2.1),

$$h \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) = \pi^* g \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right)$$

and, by (2.2),

$$\pi^* g \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) = f_{i,j}.$$

On the other hand, by (2.3),

$$g^T \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) = f_{i,j}.$$

Thus, we get

$$g^T \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) = h \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right).$$

Checking for the pairs $\left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j} \right)$ and $\left( \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right)$ is similar. \qed

3 Projective Hessian manifolds

Definition 3.1. Let $(M \times \mathbb{R}^{\geq 0}, g)$ be a Riemannian manifold. If $g$ satisfies $\lambda_q^* g = q^2 g$, where $\lambda_q(x \times t) = x \times qt$, then $g$ is called selfsimilar.
Lemma 3.2. Let \((M \times \mathbb{R}^>, g)\) be a Riemannian manifold and the metric \(g\) is selfsimilar. Then we have

\[ g = t^2 g_M + t \text{Sym}(dt \otimes \alpha) + f dt^2, \]

where \(t\) is a coordinate on \(\mathbb{R}^>\), \(g_M\) is a Riemannian metric on \(M\), \(\alpha\) a 1-form on \(M\), and \(f\) is a positive definite function on \(M\).

**Proof.** Let \(\pi : M \times \mathbb{R}^> \to M\) be a projection. Then we have

\[ T(M \times \mathbb{R}^>) = \pi^* M \oplus \ker \pi. \]

The Riemannian metric \(g\) lies in

\[ \text{Sym}^2 (\pi^* M \oplus \ker \pi)^* = \text{Sym}^2 (\pi^* M)^* \oplus \text{Sym}^2 (\pi^* M \otimes \ker \pi)^* \oplus \text{Sym}^2 (\ker \pi)^*. \]

Let \(\pi_{2,0}, \pi_{1,1}\), and \(\pi_{0,2}\) be the projections of \(\text{Sym}^2 (\pi^* M \oplus \ker \pi)^*\) on three summands from the decomposition above. Let \(g|_{M \times 1} = g_M\). For any \(X_1, X_2 \in \pi^* M\) we have

\[ g(X_1, X_2)(m \times t) = \lambda^*_t g(X_1, X_2)(m \times 1) = t^2 g(X_1, X_2)(m \times 1). \]

Thus

\[ \pi_{2,0} g = t^2 g_M. \]

For any \(X \in \pi^* TM\) and \(\frac{\partial}{\partial t} \in \ker \pi\), since

\[ \lambda^*_t \left( t^{-1} \frac{\partial}{\partial t} \right) = \frac{\partial}{\partial t}, \]

we have

\[ g \left( X, \frac{\partial}{\partial t} \right)(m \times t) = \lambda^*_t g \left( X, t^{-1} \frac{\partial}{\partial t} \right)(m \times 1) = t g \left( X, \frac{\partial}{\partial t} \right)(m \times 1). \quad (3.1) \]

Define

\[ \alpha = \left( \iota_{\frac{\partial}{\partial t}} g \right)|_{M \times 1}. \]

Then, by (3.1), we get

\[ g \left( X, \frac{\partial}{\partial t} \right)(m \times t) = t \alpha(X) = t \text{Sym}(dt \otimes \alpha(X)) \left( X, \frac{\partial}{\partial t} \right). \]

Hence,

\[ \pi_{1,1} g = t \text{Sym}(dt \otimes \alpha(X)), \]

Finally, we have

\[ g \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right)(m, t) = \lambda^*_t g \left( t^{-1} \frac{\partial}{\partial t}, t^{-1} \frac{\partial}{\partial t} \right)(m, 1) = g \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right)(m, 1). \]

Therefore, we get

\[ \pi_{0,2} g = f dt^2, \]

where the function

\[ f(m) = g \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right)(m, 1) \]

is a function on \(M\).
Proposition 3.3. Let \((M \times \mathbb{R}^>, g)\) be a Riemannian manifold and the metric \(g\) is selfsimilar. Then we have 
\[
g = t^2 g_M + f dt^2,  
\]
where \(t\) is a coordinate on \(\mathbb{R}^>\), \(g_M\) is a Riemannian metric on \(M\), and \(f\) is a positive definite function on \(M\).

Proof. By the previous lemma
\[
g = t^2 g_M + t \text{Sym}(dt \otimes \alpha) + f dt^2.  
\]
Suppose that \(\alpha \neq 0\). Define \(t_h = h^{-1}t\) for any positive definite function \(h\) on \(M\). Then we have
\[
g = t^2 g_M + t \text{Sym}(dt \otimes \alpha) + f dt^2 = (\varphi t_h)^2 g_M + \varphi t_h \text{Sym}(d\varphi(t_h) \otimes \alpha) + f d(\varphi t_h)^2 = 
\]
\[
= \varphi^2 t_h^2 g_M + \varphi t_h \text{Sym}(d\varphi \otimes \alpha) + \varphi^2 \text{Sym}(dt_h \otimes \alpha) + \varphi^2 f dt_h^2.  
\]
Notice that \(t_h = 1\) on \(M_h = \{(m \times h(m) \in M \times \mathbb{R}^>\}. Therefore, 
\[
g|_{M_h} = h^2 g_M + h \text{Sym}(dh \otimes \alpha)  
\]
Let \(x^1 \ldots x^n\) be local coordinates on a neighborhood of a point \(p\) in \(M_h\), and locally \(\alpha = \sum_{i=1}^n a_i dx^i\). We can assume that \(a_1 \neq 0\). Then
\[
g \left( \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^1} \right)(p) = h^2 g_M \left( \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^1} \right)(p) + h a_1 \frac{\partial h}{\partial x^1}.  
\]
There exist a positive definite function \(h\) on \(M\) such that \(h(p) = 1\) and
\[
\frac{\partial h}{\partial x^1} = -a_1^{-1} g_M \left( \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^1} \right)(p),  
\]
on a neighborhood of \(p\). For such \(\varphi\) we have
\[
g \left( \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^1} \right)(p) = 0.  
\]
But \(g\) is positive definite. Therefore \(\alpha = 0\). \(\square\)

Proposition 3.4. Let \(M\) be a manifold and \(g\) be a selfsimilar metric on \(M \times \mathbb{R}^>\), and \(t\) a coordinate on \(\mathbb{R}^>\). Then there exists a function \(h\) on \(M\) such that for \(t_0 = ht\) we have
\[
g = t_0^2 g_M + dt_0^2,  
\]
where \(g_M\) is a Riemannian metric on \(M\).

Proof. By Proposition 3.3, we have
\[
g = t^2 g'_M + f dt^2,  
\]
where \(\mathbb{R}^>, g'_M\) is a Riemannian metric on \(M\), and \(f\) is a positive definite function. Put \(h = f^{1/2}\). Then
\[
g = t^2 g'_M + f dt^2 = t_0^2 f^{-1} g'_M + dt_0^2 = t_0^2 g_M + dt_0^2,  
\]
where \(g_M = f g'_M\) be a Riemannian metric on \(M\). \(\square\)
Theorem 3.5. Let $M$ be a manifold. Consider a cone $M \times \mathbb{R}^>0$. Let $t$ be a coordinate on $\mathbb{R}^>0$. Then the following conditions are equivalent:

(i) There exists a Hessian structure $(\nabla, g)$ on $M \times \mathbb{R}^>0$ such that $\nabla$ is $\mathbb{R}^>0$-invariant and $g$ is selfsimilar.

(ii) There exists a Hessian structure $(\nabla', g')$ on $M \times \mathbb{R}^>0$ such that $\nabla'$ is $\mathbb{R}^>0$-invariant and

$$g' = t^2 g_M + dt^2,$$

where $g_M$ is a Riemannian metric on $M$.

Proof. (ii) $\Rightarrow$ (i). It is obvious that $g'$ is selfsimilar.

(i) $\Rightarrow$ (ii). By Proposition 3.4, there exists a positive definite function $h$ on $M$ such that for $t_0 = ht$ we have

$$g' = t_0^2 g_M + dt_0^2,$$

where $g_M$ is a Riemannian metric on $M$. Define the automorphism $\varphi$ of $M \times \mathbb{R}^>0$ by

$$\varphi(m \times t) = m \times h^{-1}t.$$

Let $g' = \varphi^* g$ and $\nabla' = \varphi^* \nabla$. The connection $\nabla'$ is well-defined because $\varphi$ is an automorphism. Since we have $t_0 = \varphi^* t$, we have

$$g' = \varphi^* g = \varphi^* (t_0^2 g_M + dt_0^2) = g' = t^2 g_M + dt^2.$$

Moreover, $g'$ is a Hessian metric with respect to $\nabla'$ because $g'$ is a Hessian metric with respect to $\nabla'$.

Definition 3.6. Manifolds satisfying the conditions of Theorem 3.5 are called projective Hessian.

4 Sasakian manifolds

Definition 4.1. Let $M$ be a $(2n - 1)$-dimensional manifold and $(h, I)$ be a Kähler structure $M \times \mathbb{R}^>0$. If we have $\lambda_q^* h = q^2 h$, where $\lambda_q$ is as above, then $h$ is called selfsimilar. If $h$ is selfsimilar and $I$ is $\mathbb{R}^>0$-invariant, then $M$ is called a Sasakian manifold.

There exists a decomposition

$$T(M \times \mathbb{R}^>0) = TM \times T\mathbb{R}^>0 = TM \times \mathbb{R}^>0 \times \mathbb{R}.$$

If $M \times \mathbb{R}^>0$ possess a Hessian structure then, by Proposition 2.3, $T(M \times \mathbb{R}^>0)$ admits a Kähler structure.

Proposition 4.2. Let $M$ be a manifold. Consider $N = T(M \times \mathbb{R}^>0) = TM \times \mathbb{R} \times \mathbb{R}^>0$ as a cone over $TM \times \mathbb{R}$. Suppose that there exists a selfsimilar Hessian metric $g$ on $M \times \mathbb{R}^>0$. Then the associated Kähler metric $g^T$ on $N$ is selfsimilar.
Proof. We have the commutative diagram

\[
\begin{array}{ccc}
T(M \times \mathbb{R}^0) & \xrightarrow{\mu_q} & T(M \times \mathbb{R}^0) \\
\downarrow \pi & & \downarrow \pi \\
M \times \mathbb{R}^0 & \xrightarrow{\lambda_q} & M \times \mathbb{R}^0
\end{array}
\]

where \(\mu_q\) and \(\lambda_q\) are multiplications of the coordinate on \(\mathbb{R}^0\) by \(q\). By Proposition 2.5, we have

\[
g^T(X,Y) = \pi^* g(X,Y) + \pi^* g(IX, IY) + \sqrt{-1} \pi^* g(IY, IX) - \sqrt{-1} \pi^* g(X, Y). \tag{4.1}
\]

Since the diagram is commutative, it follows that

\[
\mu_q^* \pi^* = \pi^* \lambda_q. \tag{4.2}
\]

Moreover, \(g\) is selfsimilar, that is,

\[
\lambda_q^* g = q^2 g. \tag{4.3}
\]

It follows from (4.1), (4.2), and (4.3) that

\[
\mu_q^* g^T(X,Y) = q^2 g^T(X,Y)
\]

i.e. \(g^T\) is selfsimilar. \(\square\)

Actually, we proved the following theorem.

**Theorem 4.3.** Let \((M \times \mathbb{R}^0, \nabla, g)\) be a projective Hessian manifold. Then \(TM \times \mathbb{R}\) admits a structure of a Sasakian manifold.

**Corollary 4.4.** Let \((M, \nabla, g = \nabla^2 \varphi)\) be a Hessian manifold such that the function \(\varphi\) is bounded below by a positive constant \(-c\). Then there exists a Sasakian structure on \(TM \times \mathbb{R}\).

**Proof.** The direct product of the canonical connection on \(\mathbb{R}^0\) and \(\nabla\) is a connection \(\tilde{\nabla}\) on the cone \(M \times \mathbb{R}^0\). Consider a function on \(M \times \mathbb{R}^0\)

\[
\psi = t^2 \varphi + ct^2,
\]

where \(t\) is the standard coordinate on \(\mathbb{R}^0\). For any \(V \in TM\) we have

\[
\tilde{\nabla}^2 \psi(V,V) = V(V(\psi)) = V(V(t^2 \varphi + ct^2)),
\]

Since \(V(t) = 0\), we get

\[
\tilde{\nabla}^2 \psi(V,V) = V(V(t^2 \varphi)) = t^2 V(V(\varphi)) = t^2 \nabla^2 \varphi(V,V) = t^2 g(V,V) > 0
\]

Moreover,

\[
\tilde{\nabla}^2 \psi \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = \frac{\partial^2 \psi}{\partial t^2} = \varphi + c > 0
\]

because \(\varphi\) is bounded below by \(-c\). Therefore, we have a Hessian metric \(\tilde{\nabla}^2 \psi\) on \(M \times \mathbb{R}^0\). This metric is selfsimilar so, by Theorem 4.4, we can construct a Sasakian structure on \(TM \times \mathbb{R}\). \(\square\)
5 Affine representations and flat torsion free connections on Lie groups

The group of affine transformations $\text{Aff}(\mathbb{R}^n)$ is given by the matrices of the form
\[
\begin{pmatrix}
A & a \\
0 & 1
\end{pmatrix} \in \text{GL}(\mathbb{R}^{n+1}),
\]
where $A \in \text{GL}(\mathbb{R}^n)$, and $a \in \mathbb{R}^n$ is a column vector. The corresponding Lie algebra $\text{aff}(\mathbb{R}^n)$ is given by matrices of the form
\[
\begin{pmatrix}
A & a \\
0 & 0
\end{pmatrix} \in \mathfrak{gl}(\mathbb{R}^{n+1}).
\]
The commutator of $\text{aff}(\mathbb{R}^n)$ is equal to
\[
\begin{pmatrix}
A & a \\
0 & 0
\end{pmatrix}, \begin{pmatrix}
B & b \\
0 & 0
\end{pmatrix} = \begin{pmatrix}
[A,B] & A(b) - B(a) \\
0 & 0
\end{pmatrix}.
\]

Algebra $\text{aff}(\mathbb{R}^n)$ is the semidirect product $\mathfrak{gl}(\mathbb{R}^n) \ltimes \mathbb{R}^n$, where the commutator is given by
\[
[A \ltimes a, B \ltimes b] = [A, B] \ltimes (Ab - Ba).
\]

Group $\text{Aff}(\mathbb{R}^n)$ is the semidirect product $\text{GL}(\mathbb{R}^n) \ltimes \mathbb{R}^n$, where multiplication is given by
\[
(A \ltimes a)(B \ltimes B) = AB \ltimes (a + Ab).
\]

Definition 5.1. Affine representation is called étale if there exists a point $x \in \mathbb{R}^n$ such that the orbit of $x$ is open and the stabilizer of $x$ is discrete.

Theorem 5.2. Let $G$ be a Lie group. There is a correspondence between left invariant torsion-free flat connections and étale affine representations. Moreover, if the connection is complete then the corresponding étale affine representation acts simply transitive on $\mathbb{R}^n$.

Proof. Choosing a basis, identify $\mathfrak{g}$ with $\mathbb{R}^n$. Then consider $\nabla_X$ as a linear endomorphism of $\mathbb{R}^n$ for any $X \in \mathfrak{g}$. The corresponding to $\nabla$ étale representation is given by
\[
\alpha : \mathfrak{g} \to \text{aff}(\mathbb{R}^n),
\]
\[
\alpha(X) = \begin{pmatrix}
\nabla_X & X \\
0 & 0
\end{pmatrix} \in \text{aff}(\mathbb{R}^n) \subset \mathfrak{gl}(\mathbb{R}^{n+1}) \tag{5.1}
\]
(see [Bu1]).

For any $X \in \mathfrak{g}$ we can consider $\nabla_X$ on the Lie algebras $\mathfrak{g}$ as a linear automorphism of the vector space $\mathfrak{g}$. Thus, the linear automorphism
\[
\exp \nabla_X = \text{id} + \frac{\nabla_X}{1!} + \frac{\nabla_X \nabla_X}{2!} + \frac{\nabla_X \nabla_X \nabla_X}{3!} + \ldots
\]
is well defined.
Proposition 5.3. Let $\nabla$ be a left invariant flat torsion-free connection on a simply connected Lie group $G$ and $\alpha$ be the corresponding étale affine representation. If $X \in \mathfrak{g}$ then the linear part of $\alpha(\exp X)$ is equal to $\exp \nabla X$.

Proof. By (5.1), we have

$$\alpha(\exp X) = \exp \begin{pmatrix} \nabla X & X \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \exp \nabla X & (\exp \nabla X)(X) \\ 0 & 1 \end{pmatrix}.$$ $$\exp \nabla X.$$/p>

Hence, the linear part of $\alpha(\exp X)$ is equal to $\exp \nabla X$. $\square$

6 Invariant affine structures on Lie groups

Theorem 6.1. Let $G$ be a simply connected Lie group, $\mathfrak{g}$ the corresponding Lie algebra. Then the following conditions are equivalent:

(i) The group $G$ possesses a flat torsion-free left invariant connection $\nabla$.

(ii) The algebra $\mathfrak{g}$ is a subalgebra of an algebra $\mathfrak{h}$ with an integrable complex structure $I$ such that $\mathfrak{h}$ is a direct sum of vector spaces $\mathfrak{h} = \mathfrak{g} \oplus I\mathfrak{g}$ and $I\mathfrak{g}$ is an abelian ideal of $\mathfrak{h}$.

Proof. (i) $\Rightarrow$ (ii). Define the commutator on $\mathfrak{h} = \mathfrak{g} \oplus I\mathfrak{g}$ by the rule

$$[X_1 \oplus Y_1, X_2 \oplus Y_2] = [X_1, Y_1] \oplus I(\nabla X_1 Y_2 - \nabla X_2 Y_1).$$

In other word, we have

$$[X \oplus 0, Y \oplus 0] = [X, Y] \oplus 0,$$

$$[X \oplus 0, 0 \oplus IY] = 0 \oplus I\nabla X Y,$$

$$[0 \oplus IX, 0 \oplus IY] = 0$$

for any $X, Y \in \mathfrak{g}$. Check the Jacobi identity on $\mathfrak{h}$. First, we have

$$[X \oplus 0, [Y \oplus 0, Z \oplus 0]] + [Y \oplus 0, [Z \oplus 0, X \oplus 0]] + [Z \oplus 0, [X \oplus 0, Y \oplus 0]] =$$

$$= ([X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]]) \oplus 0 = 0$$

because of the Jacobi identity on $\mathfrak{g}$. Second,

$$[X \oplus 0, [Y \oplus 0, 0 \oplus IZ]] + [Y \oplus 0, [0 \oplus IZ, X \oplus 0]] + [0 \oplus IZ, [X \oplus 0, Y \oplus 0]] =$$

$$0 \oplus I(\nabla X \nabla Y Z - \nabla Y \nabla X Z - \nabla_{[X,Y]} Z) = 0,$$

because $\nabla$ is flat. Finally,

$$[X \oplus 0, [0 \oplus Y, 0 \oplus IZ]] + [0 \oplus IY, [0 \oplus IZ, X \oplus 0]] + [0 \oplus IZ, [X \oplus 0, 0 \oplus IY]] =$$

$$[0 \oplus IY, 0 \oplus -I\nabla X Z] + [0 \oplus TZ, 0 \oplus I\nabla X Y] = 0.$$

We have proved that $\mathfrak{h}$ is actually a Lie group.

For any $X, Y \in \mathfrak{g}$, we have

$$[X \oplus IX, Y \oplus IY] = [X, Y] \oplus I(\nabla X Y - \nabla Y X).$$
Thus, the Newlander–Nirenberg condition
\[[X \oplus IX, Y \oplus IY] = [X, Y] \oplus I[X, Y]\]
is equal to the identity
\[\nabla_v u - \nabla_u v = [u, v].\]
The identity means that the torsion is equal to zero. We have deduced the second condition from the first. The proof in the opposite direction is analogous.

**Definition 6.2.** The algebra \(\mathfrak{g} \oplus I\mathfrak{g}\) from the previous theorem is called **associated with connection** \(\nabla\) and denoted by \(\mathfrak{g}_\nabla\).

For any algebra \(\mathfrak{g}\) equipped with a left invariant affine structure \(\nabla\) and corresponding étale affine representation \(\alpha\) we define a map
\[\eta : \mathfrak{g} \to \mathfrak{gl}(\mathbb{R}^n), \quad \eta(X) = \nabla_X,\]
and a homomorphism
\[\theta : G \to \text{GL}(n),\]
sending \(g\) to the linear part of \(\alpha(g)\). The connection \(\nabla\) is flat if and only if \(\eta\) is a morphism of algebras, because if one of these conditions is satisfied then
\[\eta[X, Y] = \nabla_{[X,Y]} = \nabla_X \nabla_Y - \nabla_Y \nabla_X = [\nabla_X, \nabla_Y].\]

**Proposition 6.3.** Let \(G\) be an \(n\)-dimensional Lie group equipped with a left invariant affine structure \(\nabla\) and \(\theta, \eta\) be as above. Then the associated Lie algebra \(\mathfrak{g}_\nabla = \mathfrak{g} \oplus I\mathfrak{g}\) from Theorem 6.1 is isomorphic to the semidirect product \(\mathfrak{g} \rtimes_{\eta} \mathbb{R}^n\). The corresponding Lie group \(G_\nabla\) is isomorphic to the semidirect product \(G \rtimes_{\theta} \mathbb{R}^n\).

Moreover, there is an identification \(TG = G_\nabla\) such that fields of the form \(0 \oplus I\mathfrak{g}\subset T(TG)\) are vertical, that is, they lie in \(\ker d\pi\), where \(\pi : TG \to G\) is a projection.

**Proof.** By the definition, \(\mathfrak{g}_\nabla\) is isomorphic to \(\mathfrak{g} \rtimes_{\eta} \mathbb{R}^n\). The corresponding Lie group is equal to the semidirect product \(G \rtimes \mathbb{R}^n\) with respect to the action such that the element \(\exp X\) acts on \(\mathbb{R}^n\) by \(\exp \nabla_X\). By Proposition 5.3, this action equals \(\theta\). Thus, the corresponding Lie group \(G_\nabla\) is equal to \(G \rtimes_{\theta} \mathbb{R}^n\).

Using the trivialization of the tangent bundle of \(G\) by the flat connection \(\nabla\), we identify \(TG\) with \(G \times \mathbb{R}^n\). Define the multiplication by
\[(g_1 \times X_1)(g_2 \times X_2) = (g_1g_2) \times (X_1 + \theta(g_1)(X_2)).\]
This multiplication is equal to the multiplication on \(G_\nabla = G \rtimes_{\theta} \mathbb{R}^n\). Thus, the group \(TG\) with this multiplication is isomorphic to \(G_\nabla\). Moreover, the left invariant fields corresponding to the subalgebra \(0 \oplus I\mathfrak{g}\) are actually vertical.

**Theorem 6.4.** Let \(G\) be a simply connected Lie group equipped with a left invariant affine structure, \(\mathfrak{g}\) the corresponding Lie algebra and \(\theta\) be the linear part of the corresponding affine action of \(G\). Then there exists a left invariant integrable complex structure \(I\) on the group
\[G \rtimes_{\theta} \mathbb{R}^n \simeq TG\]
defined in Theorem 6.1 such that \(I\) swaps vertical and horizontal tangent subbundles.

**Proof.** The theorem follows from Proposition 6.3 and Theorem 6.1.
7 Sasakian Lie groups

Definition 7.1. Let $G$ be a Lie group. A Hessian structure $(\nabla, g)$ is called left invariant if $\nabla$ and $g$ are invariant under the left action of $G$.

Theorem 7.2. Let $G$ be an $n$-dimensional simply connected Lie group equipped with a left invariant affine structure $\nabla$, and $\theta$ the linear part of the corresponding affine action of $G$. Then there exists a left invariant Kähler metric on $G_\nabla = G \ltimes_\theta \mathbb{R}^n = TG$.

Proof. The group $G \ltimes_\theta \mathbb{R}^n = TG$ is locally biholomorphic to $\mathfrak{g} \oplus \mathbb{R}^n$. So, we have the local coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$ on $G \ltimes_\theta \mathbb{R}^n = TG$ such that $I \frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i}$ and $x_1, \ldots, x_n$ are constant along any fiber of $\pi : TG \to G$. The Hessian metric $g$ is locally equivalent to $\text{Hess}\varphi$. Define the associated Hermitian metric $g^T$ as in Section 2. By Proposition 2.3, $g^T$ is a Hessian metric which is locally expressed by $\text{Hess}_\mathbb{C}(4\pi^*\varphi)$. The subgroup $\text{id} \ltimes \mathbb{R}^n \subset G \ltimes_\theta \mathbb{R}^n$ acts on fibers of $TG \to G$. The function $\pi^*\varphi$ is constant along the fibers hence $g^T = \text{Hess}_\mathbb{C}(4\pi^*\varphi)$ is invariant under the action of $\mathbb{R}^n \subset G \ltimes_\theta \mathbb{R}^n$. By Proposition 2.5,

$$g^T(X, Y) = \pi^*g(X, Y) + \pi^*g(IX, IY) + \sqrt{-1}\pi^*g(IX, Y) - \sqrt{-1}\pi^*g(X, IY).$$

Moreover, $g$ is invariant under the action of $G \ltimes 0 \subset G \ltimes_\theta \mathbb{R}^n$. Thus, $g^T$ is invariant under the action of $G \ltimes 0 \subset G \ltimes_\theta \mathbb{R}^n$, also. Therefore, $g^T$ is invariant under the action of the group $G \ltimes_\theta \mathbb{R}^n$. \hfill \square

Definition 7.3. A Lie algebra $\mathfrak{g}$ is called projective if there is an invariant affine structure $\nabla$ on $\mathfrak{g} \times \mathbb{R}$ such that

$$\nabla_X E = \nabla_E X = X,$$

where $X \in \mathfrak{g}$ and $E \in \mathbb{R}$. A Lie group is called projective if the corresponding Lie algebra is projective.

Notice that if $G$ is projective Lie group then there is an invariant affine structure on $G \times \mathbb{R}^2$.

Definition 7.4. Let $G$ be a Projective Lie group, $\nabla$ be an invariant torsion free flat connection on $G \times \mathbb{R}^2$. We say that $G$ is projective Hessian if there is a Hessian structure $(\nabla, g)$ on $G \times \mathbb{R}^2$ such that $g$ is $G$-invariant and selfsimilar.

Definition 7.5. We say that a Lie group $G$ is semi-Sasakian if there are an action $\theta$ of the group $\mathbb{R}^n \times G$, and a Kähler structure $(I, h)$ on $G_\theta \ltimes \mathbb{R}^n$ such that $h$ is invariant under the action of $G$ and selfsimilar, $I$ is invariant under the action of $G_\theta \ltimes \mathbb{R}^n$. If $\theta = \text{id}$ then such Kähler metric exists on $G \times \mathbb{R}^2$, and the group $G$ is called Sasakian.

Theorem 7.6. Let $G$ be an $n$-dimensional simply connected projective Hessian Lie group and $\theta$ be the linear part of the corresponding affine representation of $G \times \mathbb{R}^2$. Then there exists a structure of a semi-Sasakian Lie group on $G \ltimes_\theta \mathbb{R}^{n+1}$, corresponding to the semi-Sasakian structure on $G \times \mathbb{R}^{n+1}$. Moreover, $G \ltimes_\theta \mathbb{R}^{n+1} \simeq TG \times \mathbb{R}$.

Lemma 7.7. Let $F, G, H$ be groups, $\theta$ and $\eta$ actions of $F$ and $G$ on $H$ respectively such that for any $f \in F, g \in G$, and $h \in H$ we have

$$\theta(f)\eta(g)h = \eta(g)\theta(f)h.$$ 

Then

$$(F \times G) \ltimes_\theta \times_\eta H = F \ltimes_{\text{id} \times \theta} (G \ltimes \eta H).$$
Proof. Both semidirect products are equal to \( F \times G \times H \) as sets. Thus, it is enough to check that two multiplications coincide. Write multiplication on the group \((F \times G)_{\theta \times \eta} H\)

\[
(f_1 \times g_1) \ltimes_{\theta \times \eta} h_1)((f_2 \times g_2) \ltimes_{\theta \times \eta} h_2) = (f_1 f_2 \times g_1 g_2) \ltimes_{\theta \times \eta} h_1 \theta(f_1) \eta(g_1) h_2.
\]

Write multiplication on the group \( F \ltimes \text{id} \times \theta (G \ltimes \eta H) \)

\[
(f_1 \ltimes \text{id} \times \theta (g_1 \ltimes \eta h_1))(f_2 \ltimes \text{id} \times \theta (g_2 \ltimes \eta h_2)) = f_1 f_2 \ltimes \text{id} \times \theta (g_1 \ltimes \eta h_1)(g_2 \ltimes \eta \theta(f_1) \eta(h_1)) h_2.
\]

Two multiplications coincide.

Proof of Theorem 7.6. There exist an invariant torsion free flat connection \( \nabla \) on \( G \times R^>0 \) and a Hessian metric \( g \) invariant under the action of \( G \). Define the Kähler metric \( g^T \) on \((G \times R^>0) \ltimes_{\theta \eta} R^{n+1}\) as in the proof of Theorem 7.2. This metric is invariant under the action of \( G \ltimes_{\theta \eta} R^{n+1} \subset (G \times R^>0) \ltimes_{\theta \eta} R^{n+1} \) by the same argument as in the proof of Theorem 7.2. Moreover, by Proposition 6.3, we have a decomposition \( G \ltimes_{\theta \eta} R^{n+1} = T(G \times R^>0) = TG \times R \times R^>0 \). By Proposition 3.3, \( g^T \) is selfsimilar.

We constructed the Kähler metric \( g^T \) on \((G \times R^>0) \ltimes_{\theta \eta} R^{n+1}\) which is invariant under the action of \( G \ltimes_{\theta \eta} R^{n+1} \) and selfsimilar. Also, by Lemma 7.7, we have

\[
(G \times R^>0) \ltimes_{\theta \eta} R^{n+1} = R^>0 \ltimes (G \ltimes_{\theta \eta} R^{n+1}).
\]

Thus, \( G \ltimes_{\theta \eta} R^{n+1} \) is a semi-Sasakian Lie group.

Examples of projective Hessian Lie groups are described in the next two sections.

## 8 Convex properly projective structures and regular convex cones

**Definition 8.1.** A subset \( S \subset \mathbb{R}^n \) is called regular if \( S \) does not contain any straight line.

**Definition 8.2.** Let \( C \) be a convex cone in \( \mathbb{R}^n \) then the set

\[
C^* = \{y \in (\mathbb{R}^n)^* \mid \forall x \in C : y(x) > 0\}
\]

is called a dual cone.

**Definition 8.3.** Let \( C \subset \mathbb{R}^n \) be a regular convex cone and \( C^* \) the dual cone. Let \( dy \) be a parallel volume form on \((\mathbb{R}^n)^*\). Then the characteristic function of \( C \) is defined by

\[
\varphi(x) = \int_{C^*} e^{y(x)} dy
\]

(see [V]).
Example 8.4. Let
\[ C = \{ (x^1, \ldots, x^n) \in \mathbb{R}^n \mid \forall i \in \{1, \ldots, n\} : x^i > 0 \} . \]
Then the characteristic function is equal to
\[ \varphi(x) = \prod_{i=1}^{n} x^i \]
(see [?]).

Example 8.5. Let \( V \) be the vector space of all real symmetric matrices of rank \( n \) and \( \Omega \) be the set of all positive definite symmetric matrices in \( V \). Then \( \Omega \) is a regular convex cone and the characteristic function is equal to
\[ \varphi(x) = (\det x)^{-\frac{n+1}{2}} \varphi(e) , \]
where \( e \) is the unit matrix (see [Sh]).

Proposition 8.6. The characteristic function \( \varphi \) of a cone \( C \) is well defined on the interior of \( C \) and
\[ \lim_{x \to \partial C} \varphi(x) = +\infty . \]
Proof. \[ \Box \]

Definition 8.7. Let \( C \) be a regular convex cone. We denote the maximal subgroup of \( \text{GL}(\mathbb{R}^n) \) preserving \( C \) by \( \text{Aut}(C) \) and \( \text{Aut}(C) \cap \text{SL}(\mathbb{R}^n) \) by \( \text{Aut}_{\text{SL}}(C) \).

Definition 8.8. Let \( C \) be a regular convex cone in \( \mathbb{R}^n \) and \( \varphi \) a characteristic function of \( C \). Then the submanifold \( V_1 = \{ x \in C \mid \varphi(x) = 1 \} \) is called the characteristic hypersurface.

Proposition 8.9. Let \( dx \) be the standard volume form on \( \mathbb{R}^n \), \( C \) be a convex cone in \( \mathbb{R}^n \) and \( \varphi \) be a characteristic function of \( C \). Then \( \varphi dx \) is an \( \text{Aut}(C) \)-invariant volume form.
Proof. \[ \Box \]

Corollary 8.10. Let \( C \) be a regular convex cone in \( \mathbb{R}^n \) and \( \varphi \) a characteristic function of \( C \) then the characteristic hypersurface \( V_1 = \{ x \in C \mid \varphi(x) = 1 \} \) is invariant under \( \text{Aut}C \cap \text{SL}(n) \).

Proposition 8.11. Let \( C \) be a regular convex cone in \( \mathbb{R}^n \), \( \varphi \) be the characteristic function of \( C \), and \( D \) be the standard affine connection on \( \mathbb{R}^n \). Then Hessians \( D^2 \ln \varphi \) and \( D^2 \varphi \) are positive definite. Moreover, \( D^2 \ln \varphi \) is invariant under the action of \( \text{Aut}(C) \).

Definition 8.12. Let \( C, \varphi, D \) be as above. The metric \( g^{\text{can}} = D^2 \ln \varphi \) is called canonical metric and \( g^{\text{con}} = D^2 \varphi \) is called conical metric.

Proposition 8.13. Let \( C, \varphi, g^{\text{can}}, g^{\text{con}} \) be as above. Then the following conditions are satisfied:
\[ \text{i) The metric } g^{\text{can}} \text{ is invariant under the action of } \text{Aut}(C). \]
\[ \text{ii) The metric } g^{\text{con}} \text{ is invariant under the action of } \text{Aut}_{\text{SL}}(C). \text{ Moreover, the metric } g^{\text{con}} \text{ is selfsimilar.} \]
\[ \text{iii) Metrics } g^{\text{con}} \text{ and } g^{\text{can}} \text{ coincide on the characteristic hypersurface } V_1 . \]
**Definition 8.14.** A manifold $M$ is called properly convex $\mathbb{RP}^n$-manifold if it is given by a factor

$$M = \Omega / \Gamma,$$

where $\Omega$ is a convex bounded domain in $\mathbb{R}^n \subset \mathbb{RP}^n$ and $\Gamma$ is a discrete subgroup of $\text{PGL}(\mathbb{R}^n)$.

**Proposition 8.15.** Let $M$ be a properly convex $\mathbb{RP}^n$-manifold. Then $M$ admits a structure of a Hessian manifold.

**Lemma 8.16.** Let $\Omega$ be a convex projective domain, $C$ be the corresponding regular convex cone, and

$$\iota : \text{SL}(\mathbb{R}^n) \to \text{PGL}(\mathbb{R}^n)$$

a natural projection. Then $\iota$ induces an isomorphism of $\text{Aut}_{\text{SL}}(C)$ and $\text{Aut}_{\text{PGL}}(\Omega)$.

**Proof.** If $n$ is odd then $\iota$ is an isomorphism. In this case, a transformation $g \in \text{SL}(\mathbb{R}^n)$ preserves $C$ if and only if $\iota(g)$ preserves $\Omega$. Thus, $\text{Aut}_{\text{SL}}(C)$ is isomorphic to $\text{Aut}_{\text{PGL}}(\Omega)$ via $\iota$.

Suppose that $n$ is even. Then for any $g \in \text{Aut}_{\text{PGL}}(\Omega)$ the preimage $\iota^{-1}(g)$ consists of two elements: one preserves $C$ and the other swaps $C$ and $-C$. Thus, the group $\text{Aut}_{\text{SL}}(C)$ contains one and only one element from any preimage $\iota^{-1}(g)$, where $g \in \text{Aut}_{\text{PGL}}(\Omega)$. Moreover, $\text{Aut}_{\text{SL}}(C)$ does not contain any other elements. Therefore, $\text{Aut}_{\text{SL}}(C)$ is isomorphic to $\text{Aut}_{\text{PGL}}(\Omega)$ via $\iota$.

**Proof of statement 7.15.** Let $M = \Omega / \Gamma$, $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ be the projection, and $\pi^{-1}(\Omega) = C \cup -C$. By Lemma 3.14, we can consider $\Gamma$ as subgroup of $\text{Aut}_{\text{SL}}(C)$. Then the manifold $C/\Gamma$ is diffeomorphic to $(\Omega / \Gamma) \times \mathbb{R}^> = M \times \mathbb{R}^>$. Moreover, according to Statement 3.11, $\Gamma$ preserves the conical Hessian metric $g$ on $C$. Thus, $g$ produces a homogeneous Hessian metric on $C/\Gamma = M \times \mathbb{R}^>$. Hence $M$ is a homogeneous Hessian manifold.

**Corollary 8.17.** Let $M$ be a properly convex $\mathbb{RP}^n$-manifold. Then $TM \times \mathbb{R}$ admits a structure of a Sasakian manifold.

**Example 8.18.** Let $V$ be the vector space of all real symmetric matrices of rank $n$ and $\Omega$ be the set of all positive definite symmetric matrices in $V$. Then $\Omega$ is a regular convex cone and the group of upper triangular matrix $T(\mathbb{R}^n)$ acts simply transitively on $\Omega$ by $s(x) = sxst^T$, where $x \in \Omega$ and $s \in T(n)$. Let $\Gamma$ be a discrete subgroup of $\text{Aut}(\Omega)$. Then $\Omega / \Gamma$ be a properly convex $\mathbb{RP}^n$-manifold.

For more examples of properly convex $\mathbb{RP}^n$-manifolds see [B].

**Theorem 8.19.** A homogeneous regular convex domain $\Omega$ admit a simply transitive triangular group $T$ of automorphisms.

**Proof.**

**Definition 8.20.** The group $T$ from the previous theorem is called associated with $\Omega$. The corresponding Lie algebra $t$ is called associated with $\Omega$, too.

**Definition 8.21.** A normal left symmetric algebra $t$ is called clan if there is a 1-form on $t$ such that the bilinear form $g(X,Y) = \xi(XY) = \xi(\nabla_X Y)$ is symmetric and positive defined.
Theorem 8.22. A Lie algebra $\mathfrak{t}$ is associated with a homogeneous regular convex domain if and only if there is a structure of a clan on $\mathfrak{t}$.

Proof. \[\square\]

Proposition 8.23. Let $\mathfrak{t}$ be a clan. Then $\mathfrak{t}$ is a projective Hessian Lie algebra.

Proof. Let $\Omega$ be homogeneous regular convex domain corresponding to $\mathfrak{t}$. Let $V = \Omega \times \mathbb{R}^{>0}$ be a regular convex cone over $\Omega$. Then $T \times \mathbb{R}^{>0}$ acts on $C$ simply transitive. The conical metric $g^{con}$ is $T$-invariant and $\mathbb{R}^{>0}$ homogeneous. \[\square\]

9 Projective Hessian structure on $SO(2)$ and $SU(2)$

Example 9.1. Consider the group $\mathbb{R}$ as the universal covering of $U(1) = SO(2)$. The identification $SO(2) \times \mathbb{R}^{>0} \simeq \mathbb{R}^2 \setminus \{0\}$ sets a projective Hessian structure on $SO(2)$ and on the universal covering $\mathbb{R}$. The corresponding to $\mathbb{R}$ semi-Sasakian group $\mathbb{R} \times \mathbb{R}^2$ is the universal covering to the group of Euclidean motions $E(2) = SO(2) \ltimes \mathbb{R}^2$. Hence, the Lie algebra of Euclidean motions $\mathfrak{e}(2)$ is semi-Sasakian. Let $t, x, y, r$ be standard coordinates on the group $\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^{>0}$. The complex structure on $\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^{>0}$ is given by $I(\frac{\partial}{\partial t}) = \frac{\partial}{\partial x}$ and $I(\frac{\partial}{\partial x}) = \frac{\partial}{\partial y}$. The Kähler metric is given by $Hess_C(r^2)$. The Kähler structure on $\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^{>0}$ induces the Kähler structure on $SO(2) \ltimes \mathbb{R}^2 \times \mathbb{R}^{>0}$. Thus, the group of Euclidean motions $E(2)$ is semi-Sasakian, too.

All 3-dimensional Sasakian Lie algebras are classified.

Proposition 9.2. Any 3-dimensional Sasakian Lie algebra is isomorphic to one of the following: $\mathfrak{su}(2), \mathfrak{sl}(2, \mathbb{R}), \mathfrak{aff}(R) \times \mathbb{R}$, and the Heisenberg algebra $\mathfrak{h}_3$.

The algebras $\mathfrak{su}(2)$ and $\mathfrak{sl}(2, \mathbb{R})$ are semisimple, and the algebras $\mathfrak{aff}(R) \times \mathbb{R}$ and $\mathfrak{h}_3$ are nilpotent. The algebra $\mathfrak{e}(2)$ is solvable but not nilpotent. Therefore, the algebra $\mathfrak{e}(2)$ is semi-Sasakian but not Sasakian.

Example 9.3. There is an identifications $SU(2) \simeq S^3$ and $SU(2) \times \mathbb{R}^{>0} = \mathbb{R}^4 \setminus \{0\}$. The group structure on $S^3$ equals to the restriction on $S^3$ of the standard $SU(2)$-action on $\mathbb{C}^2 \simeq \mathbb{R}^4$. The corresponding semi-Sasakian Lie group is equal to $SU(2) \ltimes \mathbb{C}^2$.

Notice that the group $SU(2)$ is not Hessian just because the sphere $S^3$ does not admit an affine structure. However, there exists an invariant affine structures on $SU(2) \times \mathbb{R}^{>0} \simeq \mathbb{R}^4 \setminus \{0\}$. Moreover, if the group $G$ is one of the previous examples of projective Hessian Lie groups (a clan or $SU(1)$) then the group $G \ltimes \mathbb{R}^{>0}$ is Hessian. We prove that the group $SU(2) \times \mathbb{R}^{>0}$ is not Hessian.

Proposition 9.4. The manifold $S^3 \times \mathbb{R}^{>0}$ does not admit an $\mathbb{R}^{>0}$-invariant Hessian structure.

Proof. If $S^3 \times \mathbb{R}^{>0}$ admits an $\mathbb{R}^{>0}$-invariant Hessian structure then the Hopf manifold

$$ S^3 \times \mathbb{R}^{>0}/_{(x \times t) \sim (x \times 2t)} \simeq S^3 \times S^1 $$

admits a Hessian structure. In [Sh], Shima proved that if $M$ is compact Hessian manifold then the universal covering $\tilde{M}$ is convex domain in $\mathbb{R}^n$. The universal covering of the Hopf manifold is diffeomorphic to $S^3 \times \mathbb{R}^{>0}$ that cannot be diffeomorphic to any domain in $\mathbb{R}^4$. Therefore, $S^3 \times \mathbb{R}^{>0}$ does not admit an $\mathbb{R}^{>0}$-invariant Hessian structure. \[\square\]

Corollary 9.5. The group $SU(2) \times \mathbb{R}^{>0}$ is not Hessian.
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