A New Variant of the Casimir Effect and Its Exact Evaluation.

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Abstract

A new version of the Casimir effect where the two plates conduct in specific, different, directions is considered. By direct functional integration the evaluation of the Casimir energy as a function of the angle between the conduction directions is reduced to quadratures. Other applications of the method and a magnetic Casimir variant are mentioned.
The Casimir force per unit area [1]

\[ \frac{F_{\text{cas}}(a)}{A} = -\frac{\pi^2}{240} \frac{\hbar c}{a^4} \]  

(1)

attracts two parallel conducting plates at a distance \( a \) apart in vacuum. Its independence of atomic and QED parameters reflects the perfect conductor idealization where all details are subsumed into boundary conditions \( E_x = 0, \quad E_y = 0 \) at \( z = 0 \) or \( z = a \). These, in turn, quantize the \( z \) component of the wave number vector for modes in the region between the plates, \( k_z = \frac{n\pi}{a} \).

The problem then reduces to evaluating the change in vacuum energy of all the transverse modes inside this region:

\[ \frac{E_{\text{cas}}}{A} = \int d\varepsilon_x d\varepsilon_y \left\{ \sum_n \sqrt{k_x^2 + k_y^2 + \left( \frac{n\pi}{a} \right)^2} - \frac{a}{\pi} \int d\varepsilon_z \sqrt{k_x^2 + k_y^2 + k_z^2} \right\} \]  

(2)

A careful regularization of this formally divergent expression yields [2] \( E_{\text{cas}}(a)/A = -\pi^2 \hbar c/(720a^3) \) and \( F_{\text{cas}} = -\frac{d}{da} E_{\text{cas}}(a) \).

The tiny Casimir forces are elusive. Past efforts [3] verified Eq. (1) rather roughly and only recently a 5% precision experiment was done [4].

In this paper we focus mainly on a new variant of the Casimir effect: each of the two plates conduct in a specific direction: \( \hat{e}_1 \) for plate one and \( \hat{e}_2 \) on plate two so that only the components \( \vec{E} \cdot \hat{e}_1 \) and \( \vec{E} \cdot \hat{e}_2 \) need to vanish on plates I and II respectively, and \( \hat{e}_1 \cdot \hat{e}_2 \equiv \cos \beta \) is arbitrary.

The two polarizations contribute equally to \( F_{\text{cas}} \). By the above “twist” these polarizations can generate a controlled \( W_{\text{cas}}^{(\beta)}, F_{\text{cas}}^{(\beta)} \) and a Casimir torque, \( \tau_{\text{cas}}^{(\beta)} \).

The Casimir force can be derived also [5] by evaluating the pressure imbalance due to reflection of “vacuum modes” off the outside surface of the plate and of the (fewer) internal modes off the inside surfaces.

If the two plates are replaced by arrays of only vertical (or horizontal) conducting wires (mimicking the anisotropic conductivities) then only the \( \hat{e}_y(\hat{e}_x) \) polarized modes will be reflected suggesting that we have half the Casimir force. We also expect further reduction [6] of the Casimir force as \( \beta \), the angle between the directions of the two sets of wires (or the directions of conductivity in the two plates) increases from \( \beta = 0 \) to \( \beta = 90 \).

We will next present an exact evaluation of \( W_{\text{cas}}^{(\beta,a)} \) for general \( \beta \)'s using an altogether different method [7]. We add to the free electromagnetic
action \( S_{em} = -\frac{1}{4} \int F^2 d^4 x \) a Lagrange multiplier term \( \int J \cdot A d^3 x \) (where the last integral is only over the area-time of the plates). The functional integration over \( DJ \) ensures the vanishing of the transverse electric field \( E_T \) over the plates. Since we want the \( A_\mu \) field to vanish only up to gauge transformations we integrate only over conserved currents \( J \). Also since only transverse components of \( E \) have to vanish we allow \( J \) to have components only in the three-dimensional area-time of the plates i.e. \( J = (J_x, J_y, J_t) \). Thus we will write for the partition function

\[
Z = \int DADJ \exp \left( -i \int d^4 x \frac{1}{4} F^2 + i \int d^3 x A \cdot J \right)
\]

(3)

The only difference in the case of interest, where each plate conducts in a specific direction, is that only one component of \( E_T \) has to vanish on each plate. This can be achieved by further restricting the allowed currents \( J \). Hence Eq. (6) and its consequences remain correct as long as we remember to interpret \( \int DJ \) differently. Changing the order of integration and doing first the standard \( DA \) integration yields:

\[
Z = \int DJ \exp \left( -\frac{i}{2} \int d^3 \vec{x} \int d^3 \vec{y} \left[ \frac{\vec{J}_1(x) \cdot \vec{J}_1(y) + \vec{J}_2(x) \cdot \vec{J}_2(y)}{(x-y)^2 - i\epsilon} + 2\frac{\vec{J}_1(x) \cdot \vec{J}_2(y)}{(x-y)^2 - a^2 - i\epsilon} \right] \right)
\]

(4)

where \( \Delta_F(x-y) = \frac{1}{4\pi^2 (x-y)^2 - i\epsilon} \) is the massless Feynman propagator [8] (constant’s coefficients such as the \( 1/4\pi^2 \) above contribute only an overall multiplicative factor or an additive term to the energy and will be discarded henceforth). Denoting the currents on the first and second plates by \( J_1 \) and \( J_2 \), we have the more explicit expression for \( Z \):

\[
\int DJ \cdot \exp \left( -i \int d^3 \vec{x} d^3 \vec{y} \left[ \frac{\vec{J}_1(x) \cdot \vec{J}_1(y) + \vec{J}_2(x) \cdot \vec{J}_2(y)}{(x-y)^2 - i\epsilon} + 2\frac{\vec{J}_1(x) \cdot \vec{J}_2(y)}{(x-y)^2 - a^2 - i\epsilon} \right] \right)
\]

(5)

or after a Wick rotation

\[
\int DJ \cdot \exp \left[ -i \int d^3 \vec{x} d^3 \vec{y} \left( \frac{\vec{J}_1(x) \cdot \vec{J}_1(y) + \vec{J}_2(x) \cdot \vec{J}_2(y)}{(x-y)^2} + 2\frac{\vec{J}_1(x) \cdot \vec{J}_2(y)}{(x-y)^2 + a^2} \right) \right]
\]

(6)

where we think of \( \vec{J}_{1,2} \) as of ordinary 3-vectors in ordinary 3-dimensional Euclidean space (although it is actually spanned by \( x, y, t \)). Fourier trans-
forming in $\vec{x} = (x, y, t)$ this becomes:

$$\int DJ(\vec{k}) \exp -\int d^3k \left( \frac{J_1(\vec{k}) \cdot J_1(-\vec{k}) + J_2(\vec{k}) \cdot J_2(-\vec{k})}{k} + 2 \frac{J_1(\vec{k}) \cdot J_2(-\vec{k}) e^{-ka}}{k} \right)$$

(7)

where $\vec{k} = (k_x, k_y, k_t)$, $k = |\vec{k}|$ and we used translation invariance. In the usual case of two conducting plates both $J_1(k)$ and $J_2(k)$ have two transverse degrees of freedom fixed by the current conservation condition: $\vec{k} \cdot \vec{J} = 0$. In the case of specific conduction directions, $J_1(k)$, and likewise $J_2(k)$, have only one allowed non-zero component determined by current conservation and by the demand that its spatial part $(J_x, J_y)$ is along the direction of conduction. Let us denote the cosine of the angle between the directions of $J_1(\vec{k})$ and $J_2(\vec{k})$ by $\alpha(\vec{k})$ (with $\vec{J}_1, \vec{J}_2$ an ordinary Euclidean vector). Then we can write for $Z$:

$$\int DJ(\vec{k}) \exp -\int d^3k \left( \frac{J_1(\vec{k}) J_1(-\vec{k}) + J_2(\vec{k}) J_2(-\vec{k})}{k} + 2 \frac{J_1(\vec{k}) J_2(-\vec{k}) \alpha(k)}{k} e^{-ka} \right)$$

(8)

where the $J_i(\vec{k})\hat{J}_i$ are scalars and the reality of $J(x)$ implies:

$$[J_i(\vec{k})]^* = J_i(-\vec{k})$$

(9)

Since the action is quadratic, $Z$ is given by the corresponding determinant which is just the product of the two-dimensional determinants corresponding to the various value of $\vec{k}$. Hence

$$Z = \prod_{\vec{k}} \det \left( \frac{1}{\alpha(\vec{k})} \frac{\alpha(k)}{k} e^{-ka} \frac{1}{k} \right)^{-1/2}$$

(10)

$$\ln Z = -\frac{1}{2} \sum_{\vec{k}} \ln \det(\ldots)$$

$$= -\frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \ln \left| \frac{1}{\alpha(\vec{k})} \frac{\alpha(k)}{k} e^{-ka} \frac{1}{k} \right|$$

$$= -\frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \ln \left( \frac{1 - \alpha^2 e^{-2ka}}{k^2} \right)$$

$$= -\frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \ln(1 - \alpha(k)^2 e^{-2ka}) + \text{const.}$$

(11)
where the area-time $AT$ came from density of states factor [9]. It corresponds to having

$$\sum_{k} \rightarrow V \int \frac{d^3k}{(2\pi)^3}$$

(12)

the usual quantization of continuous modes in a box of volume $V$. Note that the last integral in Eq. (11) is well defined and convergent. To obtain it, we discarded the infinite

$$AT \int \frac{d^3k}{(2\pi)^3} \ln k^2$$

(13)

term which does not depend on $a$ or the angle $\beta$ between the directions of conductivity in the two plates and hence does not contribute to Casimir forces/torque [9]. Identifying $\ln Z = -ET$ we get finally:

$$E = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \ln(1 - \alpha(\hat{k})^2 e^{-2ka})$$

(14)

or using integration by parts (with respect to $k$ after separating $\int d^3k = \int d\Omega \int k^2dk$):

$$E = -\frac{ka}{3} \int \frac{d^3k}{(2\pi)^3} \frac{\alpha(\hat{k})^2 e^{-2ka}}{1 - \alpha(\hat{k})^2 e^{2ka}}$$

$$= -\frac{1}{48a^2} \int \frac{d^3k}{(2\pi)^3} \frac{k\alpha(\hat{k})^2}{e^k - \alpha(\hat{k})^2}$$

(15)

where in the last step we multiplied numerator and denominator by $e^{2ka}$ and changed variables $2ka \rightarrow k$.

We next find an explicit expression for $\alpha(\hat{k})$. To this end, let us denote by $\hat{n}_1, \hat{n}_2$ the two unit vectors in the planes of plates and perpendicular to the direction of conduction in the first and second plate respectively. The direction $\hat{J}_i$ is then determined by the conditions $\hat{k} \cdot \hat{J}_i = \hat{n}_1 \cdot \hat{J}_i = 0$. Hence

$$\alpha = \cos(\hat{J}_1, \hat{J}_2) = \cos(\hat{k} \times \hat{n}_1, \hat{k} \times \hat{n}_2) = \frac{(\hat{k} \times \hat{n}_1) \cdot (\hat{k} \times \hat{n}_2)}{|\hat{k} \times \hat{n}_1| |\hat{k} \times \hat{n}_2|}$$

(16)

Choosing $\hat{n}_1 = (1, 0, 0), \hat{n}_2 = (\cos \beta, \sin \beta, 0)$ and using polar coordinates decomposition of $k$: $\hat{k} = (k \sin \theta \cos \varphi, k \sin \theta \sin \varphi, k \cos \theta)$ we find

$$\alpha^2 = \frac{[\cos \beta - \sin^2 \theta \cos \varphi \cos(\varphi - \beta)]^2}{(1 - \sin^2 \theta \cos^2 \varphi)(1 - \sin^2 \theta \cos^2(\varphi - \beta))}$$

(17)
Figure 1: The Casimir energy versus the angle between the directions of conduction.

For $\beta = 0$ the integral can be calculated analytically

$$\frac{E_0}{A} = -\frac{\pi^2}{1440a^3}$$

Equation (18)

In this case $\alpha \equiv 1$ and Eq. (14) yields

$$\frac{E}{A} = -\frac{1}{48a^3} \int \frac{4\pi}{(2\pi)^3} \frac{k^3 dk}{e^k - 1} = -\frac{1}{96a^3\pi^2} \sum_{n} \int_{0}^{\infty} k^3 e^{-nk} dk$$

$$= -\frac{6 \sum_{n} \frac{1}{n^3}}{96a^3\pi^2} = -\frac{6\pi^4}{90} = -\frac{\pi^2}{1440a^3}$$

This is exactly half the usual Casimir energy, as expected since only one polarization contributes providing a nice check of the calculation.

For general $\beta$ numerical integration of Eq. (15) yields $E(a, \beta)/E(a, 0)$ as plotted in Fig. 1.
In addition to the $\beta$ dependent $F_{\text{cas}} = -\frac{\partial}{\partial a} E(a, \beta)$,

$$- \frac{\partial}{\partial \beta} E_{\text{cas}}(a, \beta) = \tau_{\text{cas}}(a, \beta)$$

is a Casimir torque tending to align the plates so that $\beta = 0$, i.e. parallel conductivity directions and minimal energy are achieved. Since torques are easier to measure, $\tau_{\text{cas}}(a, \beta)$ could perhaps be tested to a better accuracy than $F_{\text{cas}}(a)$ or $F_{\text{cas}}(a, \beta)$—though extreme, global, flatness will be required to avoid friction due to microscopic roughness, as the circular plates rotate relative to each other.

We note that for non-circular plates, or when we have broad conducting stripes, another aligning torque results from the preference to have maximal adjacent plate area. Clearly, this trivial geometrical effect is not the subject of interest here. The following comments are in order:

(i) The mode sum/integral in Eq. (2) involved in the usual derivation is very different from that in the present derivation. In particular, beyond our discarding of the $a, \beta$ independent infinite part in the last step of Eq. (11), no regularization is required here. Also extending the standard method to derive our result for $\beta \neq 0$, seems rather difficult.

(ii) The only relevant distance in this problem is the space-like plate separation involved here. This problem as well as any other static Casimir calculation can therefore be done directly by the Euclidean path integral with $Z = e^{-ET}$ and no Wick rotation is needed.

(iii) The present method was applied also to other geometries (spherical, cylindrical, etc.) The known Casimir energies were retrieved though not with extra ease or rigor.

(iv) For the case of two magnetic/electric polarizable objects much smaller than their separation the present formalism readily yields the usual Casimir–Polder result for the potential between polarizable atoms [10].

(v) Finite temperature can be easily incorporated into the present approach, by replacing the $K_t$ integration in Eq. (14) by a discrete sum. We can also generalize to the case of $n \neq 3$ dimensions, by replacing $d^3k$ there by $d^n k$. 

6
The present approach can be readily generalized to time dependent boundary condition. As noted also by Golestanian and Kardar (see Ref. 7 above), this can result in “Casimir radiation”. In particular we find that the lowest order (two-photon amplitude is simply given by

\[ \langle 0|k_1\epsilon_1,k_2\epsilon_2\rangle = \alpha w_1 w_2 (\epsilon_1 \cdot \epsilon_2) \int d\tau \exp[i(k^{\mu}_1 + k^{\mu}_2) \cdot x_\mu(\tau)] \] (20)

with \( \alpha \) the polarizability of a neutral and small (relative to \( \lambda \)) object which moves along a trajectory \( x_\mu(\tau) \) with \( \tau \) the proper time.

All of the points (iii)-(iv) will be elaborated in a longer paper by O. Kenneth.

Returning to the main theme of this Letter, we note that the naive argument of Ref. (6) that \( W_{cas}(\beta = 90^\circ) \) should vanish [due to the fact that for two arrays of orthogonal wires, the \( \hat{x} \) polarized modes are free to escape from the left, say, and the \( \hat{y} \) polarized modes from the right] fails. As indicated in Fig. 1, this is definitely not the case. Indeed one simple approach views the ordinary attractive Casimir effect as the attraction between patches of charges formed on one plate by charge separation due to quantum fluctuations and patches of opposite sign, “Image”, charges induced on the other plate. Clearly this mechanism can operate, albeit with reduced strength, even if the two plates are made of conducting stripes pointing in orthogonal directions.

This brings us to the final subject that we would like to mention, namely, an analog interpretation of magnetic Casimir attraction between two conducting rings.

Consider then two parallel conducting rings of size \( a \) and at a distance \( a \) apart. The magnetic vacuum fluctuations include closed B field lines which link both rings. These will induce, by Faraday’s law, parallel currents in the two rings. Thus, regardless of the sign of the B fluctuation and of the ensuing circulating current, the resulting current current forces will be attractive [12].

The fluctuations of interest are of scale \( \lambda \approx a \) when the above current - current forces on the various segments of the rings add coherently corresponding to the net current flow in the rings \( R_1, R_2 \). If there is no net global flux change in the rings due to the vacuum fluctuation there will be - in this approximation - no net current and no net force.

How will this force be modified if the rings become superconducting?
Ideally, the superconducting rings impose a new integral constraint, namely that the total fluxes threading the various superconducting rings must be integer multiples of the flux quantum: \( \Phi = n\Phi_0 = \frac{n\pi h}{e} \).

This implies however a strong exponential suppression. Thus if we have a fluctuation with roughly constant \( B \) on scale \( a \):

\[ \pi Ba^2 \approx n\Phi_0 \approx \frac{n\hbar}{e} \]

(21)

The action of such a configuration will therefore be:

\[ A = \int (cB)^2 d^4x dt \approx \pi^2 c^2 B^2 a^3 \frac{\alpha}{c} = \pi^2 c (Ba)^2 = \frac{cn^2h^2}{e^2} \]

(22)

The exponential suppression \( \exp \left[ -\frac{A}{\hbar} \right] \approx \exp \left[ -\frac{n^2}{\alpha_{em}} \right] \) renders such fluctuation and the attendant magnetic Casimir forces completely negligible.

The above considerations suggest that if the Casimir force between conducting rings is constantly monitored as the temperature of the system is lowered below the superconducting critical temperature, then the quenching of part of the magnetic Casimir force reduces the observed effect. Hopefully, this amusing effect can eventually be observed, but we will not elaborate here on the conditions necessary for this.

**Acknowledgment**

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**References and Remarks**

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[2] See, e.g., C. Itzykson and J. B. Zuber, *Quantum Field Theory*, McGraw Hill (1985), p. 137.

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[6] S. Nussinov, “Polarized/Magnetic Versions of the Casimir Forces”, hep-ph/9509228 (September 1995).

[7] This method, utilizing direct path integration, has been independently discovered by one of us (O. Keneth (1996), unpublished), while attempting an exact evaluation of the Casimir effect for different directions of conductivity suggested in Ref. (6) above. We have found that M. Kardar and collaborators (H. Li, R. Golestanian, M. Goulian) suggested this method sometime ago and applied it to a broad range of dynamical and other Casimir effects—though not to the specific electromagnetic effects of interest here; see, e.g. H. Li and M. Kardar, Phys. Rev. Lett. \textbf{67}, 3275 (1991), Phys. Rev. A\textbf{46}, 6490 (1992), R. Golestanian and M. Kardar, quant-ph/9802017 (February 1998).

[8] Since the current $J_\mu$ is conserved, we can adopt this particular choice.

[9] Notice that the original one-half factor from Eq. (10) survives; we have to integrate over both the Real and Imaginary parts of $J_i(\vec{k})$ but thanks to Eq. (9), only over half the $\vec{k}$ values say those with $k_t$ larger than 0.

[10] It should be emphasized though, that this finite result is not obtained by discarding what formally looks like “self-interactions” pertaining to each plate separately, namely, the $J_1 \cdot J_1$ and $J_2 \cdot J_2$ terms. Indeed, these diagonal terms are crucial for making the Gaussian integral convergent and meaningful.

[11] H. G. B. Casimir and D. Polder, Phys. Rev. \textbf{73}, 369 (1948). For a more recent, detailed, derivation, see G. Feinberg and J. Sucher, Phys. Rev. A\textbf{2}, 2395 (1970).

[12] There are also figure eight-type configurations in which the B field lines thread the two conducting rings in opposite directions and these would induce a repulsive interaction. However, the B line cannot self-intersect. The need for spatial avoidance causes the flux lines to be longer and the action suppression to be stronger for such configurations.
Figure Captions

Fig. [1] The Casimir energy versus the angle between the conducting directions.