Particle Size Analysis by Transmission Fluctuation Spectrometry in the Two-Phase Flow Irradiated by a Rectangular Narrow Beam

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Abstract. The theory of transmission fluctuation spectrometry (TFS) has been developed as a new method of particle analysis in the two-phase flow. In our earlier publications, a circular beam is used whose intensity is uniform or of a Gaussian profile. In this work, the TFS theory is studied for the case of a rectangular narrow beam. The signal process of the transmission fluctuations is performed in the time and frequency domains and the corresponding analytical expression expressed in terms of the expectancy of the transmission square (ETS) is obtained. In addition, the correlation of the fluctuating transmission signals is studied, expressed in terms of the expectancy of the transmission product (ETP). Numerical calculation shows that the transition function of the transmission fluctuation spectrum is sensitive to both the ratio of beam size to particle size and the shape of the beam cross section.

1. INTRODUCTION

The transmission signal measured in a flowing particle suspension with high spatial and temporal resolution shows significant fluctuations, which include complete information on the particle size distribution and particle concentration. In the last few years, the theory of transmission fluctuation spectrometry (TFS) has been developed as a new method of particle analysis (Breitenstein et al., 1999, 2001; Shen et al., 2003a, b, c, 2005a). After the transmission signal of a narrow beam has been subjected to a procedure of variable spatial and/or temporal averaging, the transmission fluctuations are expressed in terms of the expectancy of the transmission square (ETS). By changing the averaging parameter, the ETS is obtained as a spectrum, which is related to the physical properties of the suspension and the process of signal averaging (i.e. spatial averaging, temporal averaging and their combination).

The theory is based on the assumptions of geometric ray propagation and completely absorbent particles. This implies that the particles should be much larger than the wavelength and the transmission signal should be detected within the near field range so that effects from diffraction could be disaccounted for. In this sense, the TFS can also be realized with a wide parallel beam and a photo detector with a very small receiving area, which is equivalent to the optical arrangement of a narrow beam. The receiving area of the detector could be either circular or rectangular. In our earlier publications, the theory of TFS was developed for the circular beam whose intensity is uniform or of
Gaussian profile and measurements were realized by using a convergent Gaussian beam with its beam waist being located in the middle of the measuring region (Shen et al., 2003b, 2005b; Yu et al. 2008). This work will discuss the case of the rectangular beam.

2. FORMULATION OF THE ETS

The transmission fluctuations through a 3-dimensional particle system can be derived from the transmission fluctuations through a thin suspension layer. As already applied successfully in the previous theories on the transmission of radiation through concentrated disperse systems (Riebel et al., 1994), the present model is based on a suspension monolayer, where the layer thickness corresponds to the particle diameter. Assuming geometric ray propagation of the radiation and perfectly absorbent (black) particles, the transmission behaviour of a monolayer can be described by the transparency function (see Fig.1):

\[
T_{\text{ML}}(r) = 1 - \sum_{k=1}^{N} \text{Heav} \left( \frac{d}{2} - |r - r_k| \right)
\]

(1)

Fig. 1 Schematic of the monolayer

Here, \( T_{\text{ML}}(r) \) is the local value of the transmission, assuming a value of 1 or 0, respectively. \( d \) is the particle diameter and \( r_k \) is the centre coordinate of the particle. \( N \) is the number of particles in the monolayer. \( \text{Heav} \) is the Heaviside function describing the absorption of the radiation by the particle, which is defined as:

\[
\text{Heav}(u) = \begin{cases} 
1 & u \geq 0 \\
0 & u < 0
\end{cases}
\]

(2)

According to the concept of the monolayer, a coincidence or superposition of particles is not possible within the layer.

In a first step, we shall consider the process of spatial averaging produced by the cross section of the beam. For a beam crossing the monolayer at \( r_0 \) (coordinate of the beam centre), the transmission is equal to the average of the transparency function over the beam cross section:
Here, $B(\mathbf{r}_0, \mathbf{r})$ is the normalized beam function describing the shape of the beam and its intensity profile. For a circular Gaussian beam, the beam function is:

$$B_{CG}(\mathbf{r}_0, \mathbf{r}) = \frac{8}{\pi D^2} \exp \left[ -\frac{8(r-r_0)^2}{D^2} \right]$$  \hspace{1cm} (4)

It can be expressed as:

$$B_{CU}(\mathbf{r}_0, \mathbf{r}) = \text{rect}_{D/2}(r-r_0)$$  \hspace{1cm} (5)

for a circular uniform beam and

$$B_{RL}(\mathbf{r}_0, \mathbf{r}) = \frac{1}{H \cdot W} \text{rect}_{H/2}(r-x_0) \cdot \text{rect}_{W/2}(y-y_0)$$  \hspace{1cm} (6)

for a rectangular uniform beam. Here, $\text{rect}_{L/2}(u)$ is the rect function with the width of $L$:

$$\text{rect}_{L/2}(u) = \begin{cases} 
1 & \text{if } |u| \leq \frac{L}{2} \\
0 & \text{else}
\end{cases}$$  \hspace{1cm} (7)

The variables in the beam functions are shown in Fig. 2. The flow velocity of the particle suspension is $\mathbf{v}$ in y-direction. $D$ is the diameter of the circular beam. $H$ and $W$ are the edge lengths of the rectangular beam perpendicular and parallel to the flow direction respectively.

So, Eq.(3) can be expressed as the convolution of the transparency function $T_{ML}(\mathbf{r})$ and the beam function $B(\mathbf{r}_0, \mathbf{r})$:
Provided that the fluctuating transmission signal is averaged in the time domain and the time constant of temporal averaging is $\tau$, the transmission averaged across the trajectory $L = vt$ of the beam is described as:

$$T_{ML,B,r}(r_0) = \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} T_{ML,B}(r_0 - e_y \cdot vt) dt$$

or equivalently

$$T_{ML,B,r}(r_0) = \int \int T_{ML,B}(r_0 - r) \cdot Z_L(r) dr$$

where $Z_L(r)$ is defined as

$$Z_L(r) = \frac{1}{L} \delta(x) \cdot \text{rect}_{L/2}(y)$$

Combining with Eq.(8), Eq.(10) can be written as the convolution of the functions $T_{ML}(r)$, $B(r_0, r)$ and $Z_L(r)$:

$$T_{ML,B,r}(r_0) = (T_{ML} \otimes B \otimes Z_L)_{r_0}$$

The expectancy of the transmission square (ETS) is defined as

$$e\{T_{ML,B,r}^2\} = \lim_{A \rightarrow \infty} \frac{1}{A} \iint_{r_0 \in A} \left| T_{ML,B,r}(r_0) \right|^2 dr_0$$

Here, $A$ is the area of the monolayer. Using Parseval’s theorem of the Fourier transform, the ETS is obtained as:

$$e\{T_{ML,B,r}^2\} = \lim_{A \rightarrow \infty} \frac{1}{4\pi^2 A} \iiint |\mathcal{F}\{T_{ML,B,r}(r_0)\}|^2 d\omega$$

The Fourier transform of the fluctuating transmission $T_{ML,B,r}(r_0)$ is:

$$\mathcal{F}\{T_{ML,B,r}(r_0)\} = \mathcal{F}\{T_{ML}\} \cdot \mathcal{F}\{B\} \cdot \mathcal{F}\{Z_L\}$$

where $\mathcal{F}\{T_{ML}\}$, $\mathcal{F}\{B\}$ and $\mathcal{F}\{Z_L\}$ are the Fourier transforms of the functions $T_{ML}(r)$, $B(r_0, r)$ and $Z_L(r)$ respectively.
\[ \mathcal{F} \{ T_{ML} \} = 4\pi^2 \delta(\omega) - \frac{\pi d}{|\omega|} J_1 \left( \frac{d}{2} |\omega| \right) \sum_{k=1}^{N} \exp(-\omega \cdot r_k) \]
\[ \mathcal{F} \{ Z_L \} = \text{sinc} \left( \omega_L / 2 \right) \quad (16) \]

\( J_1 \) is the 1st order Bessel function. The unnormalized sinc function is defined as
\[ \text{sinc}(u) = \frac{\sin(u)}{u} \quad (17) \]

The Fourier transform of the beam function depends on the beam function. The beam considered here is rectangular and is uniform in intensity. Thus the Fourier transform is:
\[ \mathcal{F} \{ B_R \} = \text{sinc} \left( \frac{\omega_H H}{2} \right) \text{sinc} \left( \frac{\omega_W W}{2} \right) \quad (18) \]

The further solution of the integral given in Eq.(14) involves a large number of rather complicated mathematical operations (Shen, 2003d). The final result, which is obtained without any further simplifications or assumptions, is written as
\[ e^{\{ T_{ML,B,T}^2 \}} = 1 - \beta \cdot (2 - \chi) + \beta^2 \cdot (1 + \varepsilon) \quad (19) \]

in which the analytical expressions of the transition functions \( \chi \) and \( \varepsilon \) are
\[ \chi = \int_0^{\infty} F \left( \Lambda_H, \Lambda_W, \Theta, u \right) \cdot \frac{2J_1^2(u)}{u} \cdot du \]
\[ \varepsilon = \int_0^{\infty} F \left( \Lambda_H, \Lambda_W, \Theta, u \right) \cdot \frac{2J_1^2(u)}{u} \cdot G_{ML} \left( u, \beta \right) \cdot du \quad (20) \]

\[ F \left( \Lambda_H, \Lambda_W, \Theta, u \right) = \frac{2}{\pi} \int_0^{\pi/2} \begin{bmatrix} \text{sinc} \left( u \Lambda_H \cos \varphi \right) \lambda^2 \\ -\text{sinc} \left( u \Lambda_W \sin \varphi \right) \\ \text{sinc} \left( u \Theta \sin \varphi \right) \end{bmatrix} \cdot d\varphi \quad (21) \]

\( G_{ML} \) is a function describing the monolayer structure (Shen, 2003d). \( \beta \) is the monolayer density, which is defined as the ratio of the projected area covered by particles in the monolayer.
\[ \beta = \frac{\pi d^2}{4} \cdot \frac{N}{A} \quad (22) \]

\( \Lambda_H \), \( \Lambda_W \) and \( \Theta \) are the dimensionless parameters, defined as:
The dimensionless sizes of the rectangular beam $\Lambda_H$ and $\Lambda_W$ describe the effect of spatial averaging of the transmission signal within the beam cross section and the dimensionless time interval $\Theta$ describes the effect of the temporal averaging.

The 3-dimensional particle suspension is modelled as a pile of independent monolayers of spherical particles. The transmission of the whole particle suspension is found to be the product of the transmissions through the individual monolayers and so is the expectancy of the transmission square (ETS) (Breitenstein et al., 1999):

$$e^{\{\mathbb{T}_{b,r}^2\}} = e^{\{\mathbb{T}_{ML,r}^2\}}^{N_{ML}}$$

For a dilute particle suspension, the monolayer density is so small that the logarithm of the ETS of the particle suspension can be approximated to:

$$\ln e^{\{\mathbb{T}_{b,r}^2\}} \approx -N_{ML} \beta \cdot (2 - \chi) = -1.5\Delta Z \cdot C_v \cdot (2 - \chi)$$

Here, $N_{ML}$ is the number of monolayers of the particle suspension, $\Delta Z$ is the pathlength of the suspension along the beam propagation and $C_v$ is the particle volume concentration (i.e. the ratio of particles' volume in the suspension).

### 3. NUMERICAL RESULTS

When the particle concentration is low, the transition function $\varepsilon$ is not important so that only the function $\chi$ is considered in this work. As far as we know, the transition function $\chi$ can be calculated numerically only. Fig.3 shows some examples of the numerical results. It can be found that the transition function $\chi$ decreases gradually along with the increase of the parameter $\Theta$, which means an increasing effect of the temporal averaging. The value of the transition function is between 0 and 1. The turning point is in the vicinity of $\Theta = v\tau/d = 1$, which gives the information of particle size as long as the flow velocity is known.

The limiting value of the transition function $\chi$ at large averaging temporal parameter $\Theta$ equals 0, i.e. $\lim_{\Theta \to \infty} \chi = 0$. This corresponds to the traditional extinction measurement, in which the transmission signal is averaged in the low spatial and/or temporal resolutions. However, when $\Theta$ tends to 0 (corresponding to the highest temporal resolution), the limiting value depends on the spatial averaging parameters:

$$X(\Lambda_H, \Lambda_W) = \lim_{\Theta\to\infty} \chi = \int_0^\infty F(\Lambda_H, \Lambda_W, \Theta, u) \frac{2J^2(u)}{u} \cdot du$$

(26)
\[
\lim_{\Theta \to \infty} F\left(\Lambda_H, \Lambda_W, \Theta, u\right) = \frac{2}{\pi} \int_0^{\pi/2} \left(\frac{\text{sinc}(u \Lambda_H \cos \varphi)}{-\text{sinc}(u \Lambda_W \sin \varphi)}\right)^2 \, d\varphi
\]  
\tag{27}

Here, \(X(\Lambda_H, \Lambda_W)\) describes the stepheight of the transition function, i.e. the difference between the limiting values of \(\chi\), as is denoted in Fig.3. The numerical results are plotted in Fig.4. Similar to the temporal averaging, the increase of the spatial averaging parameters diminishes the fluctuation of the transmission signal.

Fig. 3 The transition function \(\chi\) at variant beam parameters \(\Lambda_H\) & \(\Lambda_W\)

Fig. 4 The stepheight of the transition function \(X(\Lambda_H, \Lambda_W)\)

To study the effect of the beam shape on the transmission fluctuation spectrum, the stepheight of the transition function \(X(\Lambda_H, \Lambda_W)\) is calculated via the dimensionless beam cross section, which is defined as the ratio of the beam cross section to the projected area of the spherical particle (i.e.
The beam shape is expressed with the ratio of the edge lengths $\eta = W/H = \Lambda_H/\Lambda_W$ . Numerical results are shown in Fig.5. As a comparison, the stepheight of the transition function of a circular uniform beam is plotted too (Shen et al., 2003c). The dimensionless beam cross section is defined as the ratio of the beam cross section to the projected area of the spherical particle (i.e. $\Sigma = \Lambda_0^2$) for the circular uniform beam.

It can be found that the stepheight of the transition function is sensitive to the beam shape. For a fixed value of the dimensionless beam cross section, the stepheight is different depending on the ratio of the edge lengths $\eta$. The stepheight of the transition function reaches the maximum when $H/W = \Lambda_H/\Lambda_W$ (i.e. the case of a uniform beam in foursquare), which is close to that of the circular uniform beam. Therefore, in order to obtain the higher resolution of the spatial resolution in the transmission fluctuation spectrometry, it is better to use a narrow foursquare beam.

![Graph](image)

Fig. 5 The stepheight of the transition function $X(\Lambda_H, \Lambda_W)$ via the beam cross section

4. FURTHER DISCUSSIONS

The TFS theory presented in section 2 is developed in the time domain, i.e. the fluctuating transmission signal is averaged in a variable time interval. Similar to the work given by Shen et al. 2003a, the TFS theory can also be extended to the frequency domain, replacing the temporal averaging with the signal filtering. For example, as the 1st order low-pass filter is used, the function given in Eq.(21) is replaced with:

$$F(\Lambda_H, \Lambda_W, \Psi, u) = \frac{2}{\pi} \int_0^{\pi/2} \frac{\text{sinc}(u\Lambda_H \cos \phi) \cdot \text{sinc}(u\Lambda_W \sin \phi)}{1 + \left(\frac{u \cdot \sin \phi}{\Psi}\right)^2} \, d\phi$$

Here, the dimensionless temporal averaging parameter $\Theta$ is replaced with the dimensionless cut-off frequency, which is defined as $\Psi = \pi f_0/\nu$ ($f_0$ is the cut-off frequency of the 1st order low-pass filter). The deduction of the theoretical expression is very similar to that presented above and is hence not given here.
The theory may also be extended into the TFS with signal correlation. The temporal averaging expressed in Eq.(9) is replaced with the signal correlation and correspondingly the ETS defined in Eq.(14) is replaced with the ETP:

$$ e \{ T_{ML,B} T_{ML,B,r} \} = \lim_{A \to \infty} \frac{1}{A} \int_{r_0 \in A} T_{ML,B,r} \left( r_0 + v \tau \right) dr_0 $$

The flow velocity $v$ is in $y$-direction (see Fig.2). After a large number of complicated mathematical operations, the function given in Eq.(21) is replaced with

$$ F \left( \Lambda_H, \Lambda_W, \Psi, u \right) = \frac{2}{\pi} \int_0^{\pi/2} \left\{ \frac{\cos(u \Lambda_H \cos \varphi)}{\sin(u \Lambda_W \sin \varphi)} \right\}^2 \cos(2u \Gamma \sin \varphi) d \varphi $$

Here, $\tau$ is the correlation time and the corresponding dimensionless correlation parameter is defined as $\Gamma = v \tau / d$.

5. CONCLUSIONS

In this work, we present the fundamentals of the transmission fluctuation spectrometry by using a rectangular uniform beam, which can be realized with a wide beam combining a tiny photo-detector.

An analytical expression of the expectancy of the transmission square (ETS) through a monolayer of spherical particles is obtained in the time domain and for a rectangular uniform beam. When the particle concentration is low, the transmission fluctuation spectrum can be approximately described by the monolayer density $\beta$ and the transition function $\chi$. The transition function $\chi$ depends on the spatial averaging parameters $\Lambda_H$ and $\Lambda_W$ and the temporal averaging parameter $\Theta$. The maximum of the transition function $\chi = 1$ is found at the highest spatial and temporal resolutions (i.e. the limiting case of $\Lambda_H \to 0$, $\Lambda_W \to 0$ and $\Theta \to 0$). The minimum of the transition function is $\chi = 0$, corresponding to the case of the very low spatial or temporal resolution or both. Furthermore, the transition function $\chi$ is found to be related closely to the beam shape. For example, with a fixed value of the beam cross section, the stepheight of the transition function varies according to the ratio of the edge lengths of the rectangular beam and it reaches its maximum when the beam is foursquare. In addition, the TFS theory is also extended to the transmission fluctuation spectrometry in the frequency domain and that with signal correlation.

Based on the layer model, the theory presented here can be further extended to a 3-dimensional particle system and thus can be applied for real application of two-phase flow measurements.

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## Nomenclature

| Symbol | Definition                                                                 | Unit |
|--------|---------------------------------------------------------------------------|------|
| $A$    | area of the monolayer                                                    | $[m^2]$ |
| $B$    | beam function                                                             |      |
| $C_v$  | particle volume concentration                                            |      |
| $D$    | diameter of the circular beam                                            | $[m]$ |
| $d$    | particle diameter                                                         | $[m]$ |
| $e^\{\cdots\}$ | expectancy of                                                             |      |
| $f_c$  | cutoff frequency of the filter                                            | $[1/s]$ |
| $\mathcal{F}\{\cdots\}$ | Fourier transform of                                                     |      |
| $G_{ML}$ | function describing the monolayer structure                               |      |
| $H$    | edge length of the rectangular beam perpendicular to the flow direction  | $[m]$ |
| $\text{Heav}$ | Heaviside function                                                      |      |
| $J_1$  | the 1st order Bessel function                                             |      |
| $N$    | number of particles in the monolayer                                      |      |
| $N_{ML}$ | number of monolayers in the particle suspension                          |      |
| $r$    | coordinates in the monolayer                                              | $[m]$ |
| $r_c$  | coordinates of the particle center                                       | $[m]$ |
| $r_0$  | coordinates of the beam center                                           | $[m]$ |
| $\text{rect}$ | rect function                                                              |      |
| $\text{sinc}$ | sinc function                                                             |      |
| $T$    | transmission                                                              |      |
| $v$    | flow velocity                                                             | $[m/s]$ |
| $W$    | edge length of the rectangular beam parallel to the flow direction       | $[m]$ |
| $Z_L$  | temporal averaging function                                               |      |

## Greek Letters

| Symbol | Definition                                                                 |
|--------|---------------------------------------------------------------------------|
| $\beta$ | monolayer density                                                        |
| $\Gamma$ | dimensionless correlation time                                            |
| $\delta$ | Dirac delta function                                                      |
| $\Delta Z$ | pathlength of the particle suspension                                    |
| $\eta$  | ratio of the edge lengths of the rectangular beam                        |
| $\Theta$ | dimensionless temporal averaging parameter                               |
| $\Lambda_H$, $\Lambda_W$ | dimensionless spatial averaging parameters                              |
| $\Sigma$ | dimensionless beam cross section                                         |
| $\tau$  | averaging time interval or correlation time                               | $[s]$ |
| $\varphi$ | integrand variable                                                       |
| $\chi$, $\epsilon$ | transition functions                                                     |
| $\Psi$  | dimensionless cutoff frequency of the filter                             |
| $\omega$ | variable in the Fourier space                                            | $[1/m]$ |

## Subscripts

| Subscript | Definition                                                                 |
|-----------|---------------------------------------------------------------------------|
| $CG$      | circular Gaussian beam                                                    |
| $CU$      | circular uniform beam                                                     |
| $k$       | $k$-th particle in the monolayer                                           |
| $ML$      | monolayer                                                                 |
| $R$       | rectangular beam                                                          |

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