Pion-nucleon sigma term revisited in covariant baryon chiral perturbation theory

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We study the latest \( N_f = 2 + 1 + 1 \) and \( N_f = 2 \) ETMC lattice QCD simulations of the nucleon masses and extract the pion-nucleon sigma term utilizing the Feynman-Hellmann theorem in SU(2) baryon chiral perturbation theory with the extended-on-mass-shell scheme. We find that the lattice QCD data can be described quite well already at the next-to-next-to-leading order. The overall picture remains essentially the same at the next-to-next-to-next-to-leading order. Our final result is \( \sigma_{\pi N} = 50.2(1.2)(2.0) \text{ MeV} \), or equivalently, \( f_{\pi N}^u/d = 0.0535(13)(21) \), where the first uncertainty is statistical and second is theoretical originated from chiral truncations, which is in agreement with that determined previously from the \( N_f = 2 + 1 + 1 \) twisted mass fermions. Since the dynamical strange and charm quarks have minor impact on the ETMC nucleon masses, in a recent work, Alexandrou et al. (ETMC) performed a combined fit to the 17 sets of the \( N_f = 2 + 1 + 1 \) nucleon masses and one \( N_f = 2 \) physical ensemble using SU(2) BChPT 1, and predicted a pion-nucleon sigma term \( 64.9(1.5)(13.2) \text{ MeV} \) [27]. This value is much larger than that obtained from the direct method with the ensemble at the physical point by the same collaboration, \( \sigma_{\pi N} = 37.2(2.6)(4.7) \) [12]. However, ones should note that the large \( \sigma_{\pi N} \) of Ref. [27] was obtained in the spectrum method using the heavy baryon (HB) chiral perturbation theory, which is known to perform sometimes badly in terms of convergence (see, e.g., Ref. [30, 31]). Particularly, it was shown in Ref. [27] that at next-to-next-to-leading order (NNLO) the best fit yields a \( \chi^2/d.o.f. \approx 1.6 \) while only at “next-to-next-to-next-to leading order (N^3LO)” 2, a \( \chi^2/d.o.f. \approx 1.1 \) can be achieved.

Since the determination of the pion-nucleon sigma term via the Feynman-Hellmann theorem is sensitive to the extracted pion-mass dependence of the nucleon mass from the lattice QCD data, a better description of the ETMC data is needed. Therefore, it is timely and worthy to reanalyze the same lat-

1 In principle, the twisted-mass ChPT [28, 29] is more suitable for the analysis of the ETMC data.
2 One should note that this is not a complete N^3LO study in HB ChPT, since the contributions from the \( \mathcal{O}(p^4) \) tadpole and mass-insertion loop diagrams were not included.

I. INTRODUCTION

In recent years, the pion-nucleon sigma term has attracted much attention, partly because of its role in predicting the cross section of certain candidate dark matter particles interacting with the nucleons [1]. Historically, a “canonical value” of the pion-nucleon sigma term \( \sigma_{\pi N} = m_N \langle N |\bar{u}u + \bar{d}d |N \rangle \sim 45 \text{ MeV} \) was derived in Ref. [2] from the pion-nucleon scattering data. Later, an updated analysis of \( \pi N \) scattering yielded a larger value \( \sigma_{\pi N} = 64(8) \text{ MeV} \) [3]. In the past few years, several phenomenological studies of pion-nucleon scattering using chiral perturbation theory (ChPT) and/or Roy-Steiner equations, e.g. Refs. [4–8], have derived a \( \sigma_{\pi N} \) around 60 MeV. In the meantime, the pion-nucleon sigma term has also been extensively studied in lattice quantum chromodynamics (lattice QCD) by either computing three-point (the direct method) [9–13] or two-point correlation functions (the so-called spectrum method) [14–25]. Due to the many systematic and statistical uncertainties inherent in these studies, no consensus has been reached on the precise value of the pion-nucleon sigma term, although several recent studies seem to prefer a small value \( 40 \text{ MeV} \) [11–13, 25]. Apparently, there exists a tension between the pion-nucleon sigma term determined from the phenomenological studies and that from the lattice QCD simulations.

As stressed in Ref. [23], two key factors are important in a reliable and accurate determination of the pion-nucleon sigma term using the lattice nucleon mass data with the spectrum method, i.e., lattice QCD simulations with various setups and configurations and a proper formulation to parameterize the pion-mass dependence of the nucleon mass. For the later, baryon chiral perturbation theory (BChPT), an effective field theory of low-energy QCD, provides a model-independent framework to study the pion-mass dependence of the nucleon mass. In the last few years, the European Twisted Mass Collaboration (ETMC) has performed several lattice QCD studies to extract the nucleon mass with the \( N_f = 2 \) [26, 27] and \( N_f = 2 + 1 + 1 \) twisted mass fermions. Since the dynamical strange and charm quarks have minor impact on the ETMC nucleon masses, in a recent work, Alexandrou et al. (ETMC) performed a combined fit to the 17 sets of the \( N_f = 2 + 1 + 1 \) nucleon masses and one \( N_f = 2 \) physical ensemble using SU(2) BChPT 1, and predicted a pion-nucleon sigma term 64.9(1.5)(13.2) MeV [27]. This value is much larger than that obtained from the direct method with the ensemble at the physical point by the same collaboration, \( \sigma_{\pi N} = 37.2(2.6)(4.7) \) [12]. However, ones should note that the large \( \sigma_{\pi N} \) of Ref. [27] was obtained in the spectrum method using the heavy baryon (HB) chiral perturbation theory, which is known to perform sometimes badly in terms of convergence (see, e.g., Ref. [30, 31]). Particularly, it was shown in Ref. [27] that at next-to-next-to-leading order (NNLO) the best fit yields a \( \chi^2/d.o.f. \approx 1.6 \) while only at “next-to-next-to-next-to leading order (N^3LO)” 2, a \( \chi^2/d.o.f. \approx 1.1 \) can be achieved.

Since the determination of the pion-nucleon sigma term via the Feynman-Hellmann theorem is sensitive to the extracted pion-mass dependence of the nucleon mass from the lattice QCD data, a better description of the ETMC data is needed. Therefore, it is timely and worthy to reanalyze the same lat-

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tice QCD data as Ref. [27] using covariant baryon chiral perturbation theory with the extended-on-mass-shell (EOMS) scheme [32], which has shown a number of both formal and practical advantages and has solved a number of long-existing puzzles in the one-baryon sector [33]. Furthermore, the applications of the EOMS BChPT in the studies of the lattice QCD octet baryon masses turn out to be very successful as well [20, 21, 34, 35]. Therefore, in this work, we employ the two-flavor covariant BChPT to calculate the nucleon mass up to \( \mathcal{O}(3) \). It is shown that we can achieve a better description of the 18 sets of ETMC data, i.e. \( \chi^2/\text{d.o.f.} \lesssim 1.0 \), in comparison with the study in the HB scheme [27]. With the obtained LECs, we predict a pion-nucleon sigma term, \( \sigma_{\pi N} = 50.2(1.2)(2.2) \) MeV, using the Feynman-Hellmann theorem.

This paper is organized as follows. In Section II, we briefly summarize the theoretical ingredients needed to analyze the ETMC lattice QCD data. In Section III, we perform fits to them following the strategy of Ref. [27] and predict the pion-nucleon sigma term using the Feynman-Hellmann theorem. The so-obtained low-energy constants (LECs) are then used to calculate the scattering length as well as the pion-nucleon sigma term with the Cheng-Dashen theorem. In Section IV, a short summary is given.

\section{II. THEORETICAL FRAMEWORK}

The nucleon mass has been calculated up to \( \mathcal{O}(p^4) \) both in the two-flavor sector [21] and in the three-flavor sector [20] in covariant BChPT with the EOMS scheme. To make the present work self-consistent, we spell out the nucleon mass up to \( \mathcal{O}(p^4) \), which in the isospin symmetric limit reads

\[
m_N = m_0 - 4c_1m_\pi^2 + \alpha m_\pi^4 + 3c_2m_\pi^4 \frac{2}{128\pi^2 f_\pi^2} \left[ \right.
\]

\[
- \frac{3}{64\pi^2 f_\pi^2} (8c_1 - c_2 - 4c_3)m_\pi^4 \left( 1 + \log \frac{\mu^2}{m_\pi^2} \right) + \frac{3g_\pi^2}{4(4\pi f_\pi)^2} \left[ H_N^{(3)}(m_0, m_\pi, \mu) + H_N^{(4)}(m_0, (-4c_1m_\pi^2), m_\pi, \mu) \right],
\]

where \( f_\pi \) is the pion decay constant in the chiral limit, and \( g_A \) is the axial coupling. There are four LECs, \( c_1, c_2, c_3, \) and \( \alpha \). The two loop functions, \( H_N^{(3)} \) and \( H_N^{(4)} \), are the contributions of the \( \mathcal{O}(p^3) \) and \( \mathcal{O}(p^4) \) one-loop diagrams with the power-counting breaking terms subtracted [20, 21]

\[
H_N^{(3)} = -\frac{2m_\pi^2}{m_0} \left[ \frac{m_\pi}{2} \log \frac{m_\pi^2}{m_0^2} + \sqrt{4m_0^2 - m_\pi^2} \times \right.
\]

\[
\left( \arctan \frac{m_\pi}{\sqrt{4m_0^2 - m_\pi^2}} - \arctan \frac{m_\pi^2 - 2m_\pi^2}{m_\pi \sqrt{4m_0^2 - m_\pi^2}} \right),
\]

\[
H_N^{(4)} = \frac{2m_\pi^2}{m_0^2} \left( 4c_1m_\pi^4 \right) \arccos \frac{m_\pi^2}{2m_0} - m_\pi^2 \left[ \frac{4c_1m_\pi^2}{m_0^2} \log \frac{m_\pi^2}{m_0^2} - 8c_1m_\pi^2 \log \frac{m_\pi^2}{m_0^2} \right],
\]

which are calculated in the dimensional regularization scheme with the renormalization scale \( \mu \). Following Ref. [40], we take \( f_\pi = 0.0871 \) GeV, \( g_A = 1.267 \), and \( \mu = 1.0 \) GeV in our numerical study, unless otherwise specified.

In principle, the four LECs \( (c_i, \alpha) \) can be calculated directly from QCD. However, because of the nonperturbative nature of QCD at low energies, one usually determines their value by performing a least-square fit to the lattice QCD nucleon masses and/or experimental data. It was shown in Refs. [20, 22] that finite volume corrections need to be taken into account, particularly for the \( N_f \) ensemble, in order to describe the lattice QCD data with a \( \chi^2 \approx 1.0 \). In the present case, since some of the ETMC results are obtained with \( m_\pi L < 4 \), we take the finite volume corrections into account

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
Set No. & Volume & Statistics & \( a_{\mu\mu} \) & \( a_{\mu N} \) \\
\hline
1 & \( 32^3 \times 64 \) & 2960 & 0.0030 & 0.1240 & 0.5239(87) \\
2 & \( 32^3 \times 64 \) & 6224 & 0.0040 & 0.1414 & 0.5192(112) \\
3 & \( 32^3 \times 64 \) & 1548 & 0.0050 & 0.1580 & 0.5422(62) \\
4 & \( 48^3 \times 48 \) & 7664 & 0.0060 & 0.1728 & 0.5722(48) \\
5 & \( 48^3 \times 48 \) & 7184 & 0.0080 & 0.1988 & 0.5898(50) \\
6 & \( 48^3 \times 48 \) & 8016 & 0.0100 & 0.2299 & 0.6206(43) \\
7 & \( 48^3 \times 48 \) & 2468 & 0.0040 & 0.1493 & 0.5499(195) \\
\hline
\end{tabular}
\caption{Eighteen sets of the \( N_f = 2 + 1 + 1 \) and one \( N_f = 2 \) ETMC data of Ref. [27].}
\end{table}

\footnote{\textsuperscript{3} It has been extended to heavy flavor sectors in recent years, see, e.g., Refs. [36–39].}
TABLE II. Fitted LECs of the $O(p^3)$ and $O(p^4)$ EOMS BChPT, as well as the predicted $\sigma_{\pi N}$. The numbers in the parentheses are the statistical uncertainties at the 68.3% confidence level.

|            | $\chi^2$/d.o.f. | $m_0$ (GeV) | $c_1$ (GeV$^{-1}$) | $\alpha$ (GeV$^{-2}$) | $c_2$ (GeV$^{-2}$) | $c_3$ (GeV$^{-3}$) | $\sigma_{\pi N}$ (MeV) |
|------------|-----------------|-------------|-------------------|----------------------|-------------------|-------------------|---------------------|
| $O(p^3)$   | 0.87            | 0.882 ± 0.002 | -0.95 ± 0.02      | -                    | -                 | -                 | 50.2 ± 1.2          |
| $O(p^4)$   | 0.75            | 0.879 ± 0.010 | -1.03 ± 0.20      | 7.31 ± 9.43          | -2.34 ± 4.14      | -2.67 ± 1.60      | 52.2 ± 6.6          |

account up to $O(p^4)$, which read

$$\delta m_N = \frac{3\sigma_0}{4f} \left( \delta H_N^{(3)} + \delta H_N^{(4)} \right) + \frac{3}{2f^2} \left[ 2c_1m_N^2\delta_{1/2}(m_N^2) - c_2\delta_{-1/2}(m_N^2) - c_3m_N^2\delta_{1/2}(m_N^2) \right] \quad (4)$$

with

$$\delta_r(M^2) = \frac{2^{-1/2-r}(\sqrt{M^2})^{3-2r}}{\pi^{3/2}\Gamma(r)} \times \sum_{\tilde{n}\neq 0} (M^2 |\tilde{n}>)^{-3/2+r}K_{\delta/2-r}(M^2 |\tilde{n}>) \quad (5)$$

where $K_n(z)$ is the modified Bessel function of the second kind, and

$$\sum_{\tilde{n} \neq 0} = \sum_{n_x=-\infty}^{\infty} \sum_{n_y=-\infty}^{\infty} \sum_{n_z=-\infty}^{\infty} (1 - \delta(|\tilde{n}|, 0)), \quad (6)$$

with $\tilde{n} = (n_x, n_y, n_z)$. The finite volume correction of the one-loop diagrams, $\delta H_N^{(3)}$ and $\delta H_N^{(4)}$, are calculated in Refs. [20, 42] and read

$$\delta H_N^{(3)} = -\int_0^1 dx \left\{ \begin{array}{l} 1 \frac{1}{2} m_0(2x + 1)\delta_{1/2}(\mathcal{M}_N^2) \\ -\frac{1}{4} m_0 \left( m_0^2 x^3 + \mathcal{M}_N^2 (x + 2) \right) \delta_{3/2}(\mathcal{M}_N^2) \end{array} \right\}, \quad (7)$$

and

$$\delta H_N^{(4)} = -\int_0^1 dx \left\{ \begin{array}{l} -\frac{1}{2} \delta_{1/2}(\mathcal{M}_N^2)(2x + 1)m_N^2 \\ -\frac{1}{4} \delta_{3/2}(\mathcal{M}_N^2) \left[ m_N^2 (x - 1) (x + 2) m_N^2 \right] \\ -2x m_0^2 (5x^2 + 4x) m_N^2 \right\}, \quad (8)$$

with the leading order correction to the nucleon self-energy $m_N^{(2)} = -4c_1m_N^2$ and $\mathcal{M}_N^2 = x^2m_0^2 + (1 - x)m_N^2 - i\epsilon$.

Once we obtain the nucleon mass, the pion-nucleon sigma term can be predicted utilizing the Feynman-Hellmann theorem [41], which dictates that in the isospin symmetric limit the $\sigma_{\pi N}$ can be calculated from the light quark mass or equivalently the pion mass dependence of the nucleon mass, $m_N$, in the following way

$$\sigma_{\pi N} = m_\pi \langle N | \bar{u}u + \bar{d}d | N \rangle = m_\pi \frac{\partial m_N}{\partial m_{\pi}}, \quad (9)$$

where leading order ChPT has been used to relate the light quark mass with the pion mass. $^4$

III. RESULTS AND DISCUSSIONS

A. Pion-nucleon sigma term from Feynman-Hellmann theorem

In this subsection, we perform a least-square fit to the 18 sets of lattice QCD data (in the lattice unit) of Ref. [27], which are summarized in Table I, together with the nucleon mass $m_N = 0.938$ GeV at the physical pion mass $m_\pi = 0.135$ GeV.

At NNLO, one only has two LECs, namely $m_0$ and $c_1$, as shown in Eq. (1). On the other hand, the four lattice spacings $a$ should also be determined self-consistently, according to Ref. [27]. As a result, in total we have 6 free parameters to describe the ETMC nucleon masses. The best fit yields a $\chi^2$/d.o.f. = 0.87, which is already smaller than 1.0, contrary to the HB ChPT case [27]. The values of $m_0$ and $c_1$ are tabulated in Table II, and the four lattice spacings are

$$a_{N_f=2+1+1, \beta=1.90} = 0.0964(12) \text{ fm},$$
$$a_{N_f=2+1+1, \beta=1.95} = 0.0855(9) \text{ fm},$$
$$a_{N_f=2+1+1, \beta=2.10} = 0.0661(7) \text{ fm},$$
$$a_{N_f=2+1+1, \beta=2.10} = 0.0933(3) \text{ fm}. \quad (10)$$

We note that the so-determined lattice spacings are in good agreement with those determined in the HB ChPT fit with the small scale expansion scheme up to “$N^3LO$” [27]. In our studies up to $N^3LO$, there are three more LECs, namely $c_2$, $c_3$, and $\alpha$, resulting in a total of 9 parameters. We note, however, that the ETMC data cannot unambiguously fix the 5 LECs and the lattice spacings simultaneously. As a result, we chose to fix the lattice spacings at the values determined at the NNLO. The resulting fit is shown in Table II, and the description of the ETMC data is slightly improved in comparison with that of NNLO. One can see that the values of $m_0$ and $c_1$ are consistent with the ones from the $O(p^4)$ fit with slightly larger uncertainties. We note that our $c_1$ is almost the same as that given in the studies of pion-nucleon scattering [4, 6, 44, 45]. However,$^4$ We have checked that using the next-to-leading order ChPT instead of the leading order ChPT does not yield quantitatively different results [23, 40].
at N^4LO, the three LECs c_2, c_3, and a can not be determined precisely. Compared with those of Refs. [4, 6, 44, 45], the values of c_2 and c_3 are different, particularly, c_2 is negative. In addition, if the LECs c_2 and c_3 were fixed at those of Ref. [44], the fit-\chi^2/\text{d.o.f.} would increase to more than one, but the corresponding pion-nucleon sigma term would not change much.

In Fig. 1, we show the pion mass dependence of the nucleon mass as predicted by the O(p^3) and O(p^4) BChPT. Clearly, the agreement with data in both cases are of the same quality. Volume corrections have been subtracted, which can reach as large as a few tens of MeV for lattice QCD simulations with large \( m_\pi \) and small \( \pi \sigma \). In this case, we prefer to take the NNLO prediction because of the agreement with data in both cases are of the same quality.

Since the ETMC data can be well described with \( \chi^2/\text{d.o.f.} < 1.0 \) up to NNLO and N^4LO in covariant BChPT, we take the result of O(p^3) as the central value, \( \sigma_{\pi N} = 50.2(1.2)(2.0) \text{ MeV} \), where the first uncertainty is statistical and second is theoretical originated from chiral truncations. We could as well choose the N^3LO prediction as our central value and obtain \( \sigma_{\pi N} = 52.2(6.6)(2.0) \text{ MeV} \). In the present case, we prefer to take the NNLO prediction because the ETMC data do not constrain very well the LECs at N^3LO.

In Refs. [6, 21, 34, 45], the virtual \( \Delta(1232) \) was found to be able to improve the convergence of BChPT in certain cases. Thus, following Ref. [21], we take the contribution of the virtual \( \Delta(1232) \) to the nucleon mass into account up to N^4LO and study its effect on the description of the ETMC data and on the prediction of the pion-nucleon sigma term. The pertinent LECs are fixed in the following way: \( \hbar_A = 2.85 \) [21] and the mass splitting \( \delta = m_{\Delta 0} - m_0 = 0.292 \text{ GeV} \). At N^3LO, the value of \( c_{\Delta 1} \) is fixed by fitting the NLO delta-isobar mass \( m_\Delta = m_{\Delta 0} - 4c_{\Delta 1}m_0^2 \) to its physical value, yielding \( c_{\Delta 1} = (m_0 - 0.942)/(4m_0^2) \). We take the lattice spacings as given in Eq. (10) and present the fitting results in Table III. At NNLO, including the \( \Delta(1232) \) contribution increases the fit-\chi^2/\text{d.o.f.} to about 2.8, similar to what happened in Refs. [21, 34]. While, at O(p^4), the description of the lattice data is almost the same as that without the \( \Delta(1232) \) contribution, and the obtained pion-nucleon sigma term is \( \sigma_{\pi N} = 53.0(6.8) \text{ MeV} \), which agrees with the one obtained without the \( \Delta(1232) \) contribution within uncertainties. One may conclude that the contribution of the \( \Delta(1232) \) can be absorbed by the LECs of the nucleon only case up to N^4LO, consistent with the finding in the SU(3) study [34].

One must note that we did not provide a comprehensive assessment of theoretical uncertainties and they can be much larger. On one hand, they could come from the use of the SU(2) BChPT to study the \( N_f = 2 + 1 + 1 \) lattice QCD data and from our neglect of the lattice spacing artifacts. On the other hand, they can also come from our choice for the decay constant \( f_0 \) and the renormalization scale \( \mu \). For instance, we noticed that instead of \( f_0 = 0.0871 \text{ GeV} \), the choice of \( f_0 = 0.0922 \text{ GeV} \) [43] decreases the central value of \( \sigma_{\pi N} \) by 2 \~ 3 MeV at both NNLO and N^4LO. Nonetheless, such a choice still yields a \( \chi^2/\text{d.o.f.} < 1 \) and therefore cannot be distinguished from our original choice.

A recent study of the pion-nucleon scattering with the Roy-Steiner equations [7] found that the effects of isospin breaking on the pion-nucleon sigma term is around 3 MeV, which is comparable to the uncertainty from chiral truncations. Therefore, the isospin breaking effects on the \( \sigma_{\pi N} \) should be carefully investigated. However, the ETMC data is obtained in the isospin limit and therefore cannot determine the four LECs \( c_0, f_1, f_2, f_3 \), needed to parametrize the leading order isospin breaking between the \( u \) and \( d \) quarks [8].

It is interesting to note that the central value of our predicted \( \sigma_{\pi N} \) is smaller than the sigma term, 64.9 MeV, obtained in Ref. [27] by fitting to the same lattice QCD data. This difference can be traced back to the fact that HB ChPT can only describe the ETMC data with a \( \chi^2/\text{d.o.f.} \approx 1.6 \) at NNLO. Taking into account the uncertainty of chiral truncations, our result is consistent with the pion-nucleon sigma

\[ \sigma_{\pi N} = 53.0(6.8) \text{ MeV} \]
TABLE III. Fitted LECs of the $O(p^3)$ and $O(p^4)$ EOMS BCHPT with the delta-isobar contribution, as well as the predicted $\sigma_{\pi N}$. The numbers in the parentheses are the statistical uncertainties at the 68.3% confidence level.

| $O(p^3)$ | $O(p^4)$ |
|----------|----------|
| $\chi^2$/d.o.f. | $m_\pi$ (GeV) | $c_1$ (GeV$^{-1}$) | $c_2$ (GeV$^{-3}$) | $c_3$ (GeV$^{-3}$) | $\sigma_{\pi N}$ (MeV) |
| 2.77 | 0.868 $\pm 0.002$ | $-1.17 \pm 0.05$ | $- \sim$ | $- \sim$ | $- \sim$ | $59.7 \pm 0.4$ |
| 0.78 | 0.877 $\pm 0.010$ | $-1.10 \pm 0.22$ | $20.40 \pm 12.07$ | $-8.79 \pm 5.27$ | $-2.30 \pm 1.78$ | $53.0 \pm 6.8$ |

FIG. 2. Pion-nucleon sigma term, $\sigma_{\pi N}$, obtained in the phenomenological approaches, the lattice direct and spectrum methods, respectively. The light-blue band represents the result obtained in the present work. The rest data are taken from [2] (Gasser et al., 91), [3] (Pavan et al., 02), [4] (Alarcon et al., 12), [5] (Chen et al., 12), [7] (Hoferichter et al., 15) in the phenomenological studies, [9] (QCDSF 12), [11] ($\chi$QCD 16), [12] (ETMC 16), [13] (RQCD 16) with the lattice direct method, [14] (JLQCD 08), [15] (Young et al. 09), [17] (QCDSF 12), [16] (BMWC 12), [19] (Shanahan et al. 12), [21] (Alvarez et al. 13), [22] (Lutz et al. 14), [23] (Ren et al. 14), [24] (ETMC 14), [25] (BMWC 15), and [27] (Alexandrou et al. 17) with the spectrum method.

term, 64.9(1.5)(13.2) MeV, of Ref. [27]. Since at “$N^3LO$”, a $\chi^2$/d.o.f. $\approx 1.1$ can be achieved [27], it is more reasonable to take the “$N^3LO$” prediction as the central value. In this case, one would obtain $\sigma_{\pi N} = 51.7(4.3)(13.2)$ MeV, whose central value is in better agreement with our result.

It should be noted that our predicted pion-nucleon sigma term is in between that from the latest phenomenological studies, $\sigma_{\pi N} \sim 60$ MeV, and that from the recent LQCD calculations, $\sigma_{\pi N} \sim 40$ MeV, as shown in Fig. 2. In addition, the $\sigma_{\pi N}$ is consistent with the value determined from our three-flavor study [23]. This is not surprising since as shown in Ref. [40] the SU(3) and SU(2) BCHPT are consistent with each other within uncertainties, particularly for $m_\pi < 300$ MeV. Second, one should note that our results are tied to the quality of the lattice QCD data that we fitted. Nevertheless, our present study provides a further consistency check on the covariant BCHPT we employed, which in many cases is essential to the determination of the $\sigma_{\pi N}$ via the spectrum method.

B. Connection to pion-nucleon scattering

The LECs constants, $c_1$, $c_2$, $c_3$, determined in the present study can be used as inputs to perform a partial pion-nucleon scattering analysis and calculate the pion-sigma term with the Cheng-Dashen theorem and the scattering lengths. Such studies could provide a useful crosscheck on the reliability of the determination of the pion-nucleon sigma term using the Feynman-Hellmann theorem.

According to the Cheng-Dashen theorem [48], the pion-nucleon sigma term reads

$$\sigma_{\pi N} = \Sigma_d + \Delta_D - \Delta_{\sigma} - \Delta_R,$$

where $\Delta_D - \Delta_{\sigma} = (-1.8 \pm 0.2)$ MeV [49], $|\Delta_R| < 2$ MeV [50], $\Sigma_d = f_\pi^2 (d_{00}^d + 2 m_\pi^2 d_{01}^d)$ with $d_{00}^d$ and $d_{01}^d$ the sub-threshold parameters of pion-nucleon scattering. Up to $O(p^3)$ [51] $d_{00}^d$ and $d_{01}^d$ are solely determined by $c_1$ and $c_3$.

6 Since there this no counter terms at $O(p^3)$, the results of $d_{00}^d$ and $d_{01}^d$ in infrared ChPT are the same as the ones from the EOMS scheme.
as
\[
\begin{align*}
  a_{00}^+ &= -\frac{2m_c^2}{f_\pi^2} (2c_1 - c_3) + \frac{g_\alpha^2 (3 + 8g_\alpha^2)m_c^3}{64\pi f_\pi^4}, \\
  a_{01}^+ &= -\frac{c_3}{f_\pi^2} - \frac{g_\alpha^2 (77 + 48g_\alpha^2)m_c}{768\pi f_\pi^4}.
\end{align*}
\]

With \(c_1 \) and \(c_3 \) in Table II, we obtain the sigma term as 45.6(2.2) MeV and 51.8(2.2) MeV at NNLO and \(N^3LO, \) respectively. It is clear that these values are consistent with the pion-nucleon sigma terms determined by fitting to the ETMC data within uncertainties.

Recently, Hoferichter et al. [7] proposed a relationship between the pion-nucleon sigma term and the \(S\)-wave scattering lengths, \(a^{1/2}\) and \(a^{3/2},\)

\[
\sigma_{\pi N} = (59.1 \pm 3.1) \text{ MeV} + \sum_{I_s} c_I (a_I^R - \bar{a}_I^R),
\]

based on Roy-Steiner equations. In Refs. [46, 47], they showed that a small \(\sigma_{\pi N} \) is related to a even smaller value of the \(\pi N\) isoscalar scattering length, \(a^+.\) With our \(c_1, c_2, c_3 \) tabulated in Table II, we obtain \(a^+ = -130.5 \pm 195.1 \) \(10^{-3}m_c^{-1}\), using the chiral expansions of Ref. [6]. The central value is much smaller than the one obtained from the pion-nucleon scattering analysis, \(a^+ = -14.8 \) \(10^{-3}m_c^{-1}\) [45], but consistent within uncertainties. Such a difference is partly due to the negative \(c_2 \) obtained in our study in comparison with the positive one from \(\pi N\) scattering and partly due to the fact that at \(O(p^4)\), we could not constrain well \(c_2 \) and \(c_3 \) simply by fitting to the ETMC nucleon masses, consistent with the finding of Ref. [21].

IV. SUMMARY

We have reanalyzed the latest ETMC simulations of the nucleon mass and extracted the eagerly wanted pion-nucleon sigma term. We showed that because of the use of the covariant baryon chiral perturbation theory, we were able to minimize theoretical uncertainties and obtain a pion-nucleon sigma term, \(\sigma_{\pi N} = 50.2(1.2)(2.0) \text{ MeV}, \) consistent with those determined from the \(N_f = 2 + 1 \) and \(N_f = 2 \) analyses, although more lattice QCD data on the nucleon mass, and even on some complementary observables, are still needed to further reduce theoretical uncertainties.

With the LECs \(c_1, c_2, \) and \(c_3 \) determined by fitting to the ETMC nucleon masses, we also predicted the pion-nucleon sigma term using the Cheng-Dashen theorem and the scattering length \(a^+ \) of pion-nucleon scattering. The pion-nucleon sigma term is consistent with that determined from the Feynman-Hellmann theorem, but the scattering length only marginally agrees with the one from the phenomenological studies.

In order to better understand the current tension between the pion-nucleon sigma terms from the lattice QCD calculations and those from the pion-nucleon scattering analyses and to better constrain the values of \(c_2 \) and \(c_3 \), a combined study of the lattice QCD nucleon masses and the pion-nucleon scattering data in the same framework, such as the present one, is in urgent need.

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