New plots and parameter degeneracies in neutrino oscillations

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Abstract. It is shown that eightfold degeneracy in neutrino oscillations is easily seen by plotting constant probabilities in the \((\sin^2 2\theta_{13}, 1/s^2_{23})\) plane. Using this plot, we discuss how an additional long baseline measurement resolves degeneracies after the JPARC experiment measures the oscillation probabilities \(P(\nu_\mu \to \nu_e)\) and \(P(\bar{\nu}_\mu \to \bar{\nu}_e)\) at \(|\Delta m^2_{31}|L/4E = \pi/2\). By measuring \(P(\nu_\mu \to \nu_e)\) or \(P(\bar{\nu}_\mu \to \bar{\nu}_e)\), the \(\text{sgn}(\Delta m^2_{31})\) ambiguity is resolved better at longer baselines and the \(\delta \leftrightarrow \pi - \delta\) ambiguity is resolved better when \(\left| |\Delta m^2_{31}|L/4E - \pi|/2\right|\) is larger. The \(\theta_{23}\) ambiguity may be resolved as a by-product if \(\left| |\Delta m^2_{31}|L/4E - \pi|\right|\) is small and the CP phase \(\delta\) turns out to satisfy \(|\cos(\delta + |\Delta m^2_{31}|L/4E)| \sim 1\). It is pointed out that the low-energy option \((E \sim 1 \text{ GeV})\) at the off-axis NuMI experiment may be useful in resolving these ambiguities. The \(\nu_e \to \nu_\tau\) channel offers a promising possibility that it would potentially resolve all the ambiguities.
1. Introduction

From recent experiments on atmospheric [1], solar [2] and reactor [3, 4] neutrinos, we now know the approximately correct values of the mixing angles and mass squared differences for atmospheric and solar neutrino oscillations:

\[
\sin^2 2\theta_{12}, \Delta m^2_{21} \simeq (0.8, 7 \times 10^{-5} \text{ eV}^2) \quad \text{for the solar neutrino,}
\]

\[
\sin^2 2\theta_{23}, |\Delta m^2_{31}| \simeq (1.0, 2 \times 10^{-3} \text{ eV}^2) \quad \text{for the atmospheric neutrino,}
\]

where we use the standard parametrization [5] of the MNS mixing matrix

\[
U = \begin{pmatrix}
    c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\
    -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\
    s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13}
\end{pmatrix},
\]

and the case of \( \Delta m^2_{31} > 0 \) (\( \Delta m^2_{31} < 0 \)) corresponds to the normal (inverted) mass hierarchy, as shown in figure 1. In the three-flavour framework of neutrino oscillations, the oscillation parameters that are still unknown to date are the third mixing angle \( \theta_{13} \), the sign of the mass...
squared difference $\Delta m_{31}^2$ of the atmospheric neutrino oscillation and the CP phase $\delta$. It is expected that long-baseline experiments in the future will determine these three quantities.

From the work of Burguet-Castell et al [6], it is known that even if the values of the oscillation probabilities $P(\nu_\mu \rightarrow \nu_e)$ and $P(\bar{\nu}_\mu \rightarrow \bar{\nu}_e)$ are exactly given, we cannot determine uniquely the values of the oscillation parameters owing to parameter degeneracies. There are three kinds of parameter degeneracies: intrinsic ($\theta_{13}, \delta$) degeneracy [6], the degeneracy of $\Delta m_{31}^2 \leftrightarrow -\Delta m_{31}^2$ [7] and the degeneracy of $\theta_{23} \leftrightarrow \pi/2 - \theta_{23}$ [8, 9]. Intrinsic degeneracy is exact when $\Delta m_{31}^2/\Delta m_{31}^2$ is exactly zero. The $\text{sgn}(\Delta m_{31}^2)$ degeneracy is exact when $AL$ is exactly zero, where $A (\equiv \sqrt{2} G_F N_e)$ and $L$ stand for the matter effect and the baseline, respectively ($G_F$ is the Fermi constant and $N_e$ the electron density in matter). The $\theta_{23}$ degeneracy is exact when $\cos 2\theta_{23}$ is exactly zero. Each degeneracy gives a twofold solution, so that, in total, we will have an eightfold solution if all the degeneracies are exact. In this case, prediction for physics is the same for all the degenerated solutions and there is no problem. However, these degeneracies are lifted slightly in long baseline experiments, and there are in general eight different solutions [9]. When we try to determine the oscillation parameters, ambiguities arise since the values of the oscillation parameters are different for each solution. In particular, this causes a serious problem in the measurement of CP violation, which is expected to be a small effect in the long baseline experiments, and we could mistake a fake effect because of the ambiguities for nonvanishing CP violation if we do not treat the ambiguities carefully.

In previous studies [6, 7, 9], various diagrams have been given to visualize how degeneracies are lifted in the parameter space. To see how the eightfold degeneracy is lifted, it is necessary

\footnote{\text{cos} 2\theta_{23} may be exactly zero, but the present atmospheric neutrino data [10] allow the possibility of \text{cos} 2\theta_{23} \neq 0. Hence, we will assume \text{cos} 2\theta_{23} \neq 0 in general in the following discussions.}
for the plot to give eight different points for the eight different solutions. An effort was made by Minakata et al [11] to visualize the eight different points by plotting the trajectories of constant probabilities in the \((\sin^2 2\theta_{13}, s_{23}^2)\) plane. In the present paper, we propose a plot in the \((\sin^2 2\theta_{13}, 1/s_{23}^2)\) plane, which offers the simplest way to visualize how the eightfold degeneracy is lifted. As a by-product, we show how the third measurement of \(\nu_\mu \rightarrow \nu_e, \bar{\nu}_\mu \rightarrow \bar{\nu}_e\) or \(\nu_e \rightarrow \nu_\tau\) resolves the ambiguities, after the JPARC experiment [12] measures the oscillation probabilities \(P(\nu_\mu \rightarrow \nu_e)\) and \(P(\bar{\nu}_\mu \rightarrow \bar{\nu}_e)\) at the oscillation maximum, i.e. at \(|\Delta m_{31}^2| L/4E = \pi/2\). Unlike the work of Burguet-Castell et al [6], statistical and systematic errors are not taken into account in this paper, and we hope that the present formalism offers a way to understand intuitively how ambiguities appear and how they are resolved by combining other experiments.

In the following discussions, we assume that \(|\Delta m_{31}^2|, \Delta m_{21}^2\) and \(\theta_{12}\) are sufficiently precisely known. This is justified because the correlation between these parameters and the CP phase \(\delta\) is not so strong for JPARC [13], and we can safely ignore the uncertainty of these parameters to discuss the ambiguities in \(\delta\) due to parameter degeneracies.

2. Plots in the \((\sin^2 2\theta_{13}, 1/s_{23}^2)\) plane

As in [14], let us discuss the ambiguities due to degeneracies step by step in the order \((\theta_{23} - \pi/4 = 0, \Delta m_{21}^2 = 0, A = 0) \rightarrow (\theta_{23} - \pi/4 \neq 0, \Delta m_{21}^2 = 0, A = 0) \rightarrow (\theta_{23} - \pi/4 \neq 0, \Delta m_{21}^2 \neq 0, A = 0) \rightarrow (\theta_{23} - \pi/4 \neq 0, \Delta m_{21}^2 \neq 0, A \neq 0)\).

2.1. \(\cos 2\theta_{23} = 0, \Delta m_{21}^2/\Delta m_{31}^2 = 0, AL = 0\)

In this case, the oscillation probabilities \(P(\nu_\mu \rightarrow \nu_e)\) and \(P(\bar{\nu}_\mu \rightarrow \bar{\nu}_e)\) are equal and are given by

\[
P(\nu_\mu \rightarrow \nu_e) = P(\bar{\nu}_\mu \rightarrow \bar{\nu}_e) = s_{23}^2 \sin^2 2\theta_{13} \sin^2 \Delta,
\]

where we have introduced the notation

\[
\Delta \equiv \frac{|\Delta m_{31}^2| L}{4E}.
\]

To plot the line \(P(\nu_\mu \rightarrow \nu_e) = P(\bar{\nu}_\mu \rightarrow \bar{\nu}_e) = \text{const.}\) in the \((\sin^2 2\theta_{13}, 1/s_{23}^2)\) plane, let us introduce the variables

\[
X \equiv \sin^2 2\theta_{13}, \quad Y \equiv \frac{1}{s_{23}^2}.
\]

Then

\[
P = s_{23}^2 \sin^2 2\theta_{13} \sin^2 \Delta
\]

gives a straight line

\[
Y = \frac{\sin^2 \Delta}{P} X,
\]

in the \((X, Y)\) plane, where \(P\) and \(\sin^2 \Delta\) are constant. The intersection of equation (1) and \(Y = 1/s_{23}^2 = 2\) in the \((\sin^2 2\theta_{13}, 1/s_{23}^2)\) plane is a unique point that corresponds to a solution with eightfold degeneracy. The solution is depicted in figure 2(a).
Figure 2. Solutions, marked by black blobs, for the given $P(\nu_\mu \to \nu_e)$, $P(\bar{\nu}_\mu \to \bar{\nu}_e)$ and $P(\nu_\mu \to \nu_\mu)$ for $\Delta m^2_{21}/\Delta m^2_{31} = 0$ and $AL = 0$. (a) For $\cos 2\theta_{23} = 0$, the intersection of $Y \equiv 1/s^2_{23} = 2$ and the trajectory of $P(\nu_\mu \to \nu_e) = P(\bar{\nu}_\mu \to \bar{\nu}_e) = \text{const.}$ is a point with eightfold degeneracy. (b) For $\cos 2\theta_{23} \neq 0$, the intersections are two solutions with fourfold degeneracy.

2.2. $\cos 2\theta_{23} \neq 0$, $\Delta m^2_{21}/\Delta m^2_{31} = 0$, $AL = 0$

At present, the Superkamiokande atmospheric neutrino data give the allowed region $0.90 < \sin^2 2\theta_{23} \leq 1.0$ at 90% CL [10], and $\sin^2 2\theta_{23}$ can be, in general, different from 1.0. If $\sin^2 2\theta_{23}$, which is more accurately determined from the oscillation probability $P(\nu_\mu \to \nu_\mu)$ in the future long baseline experiments, deviates from 1, then we have two solutions for $Y \equiv 1/s^2_{23}$:

$$Y_+ = \frac{2}{1 - \sqrt{1 - \sin^2 2\theta_{23}}}, \quad Y_- = \frac{2}{1 + \sqrt{1 - \sin^2 2\theta_{23}}}.$$
In this case, there are two solutions, one given by equation (1) and \( Y = Y_+ \) and another given by (1) and \( Y = Y_- \). These are two solutions with fourfold degeneracy. The two solutions in the \( (\sin^2 2\theta_{13}, 1/s_{23}^2) \) plane are shown in figure 2(b). From this, we see that even if we know precisely the values of \( P(\nu_\mu \rightarrow \nu_e) \), \( P(\bar{\nu}_\mu \rightarrow \bar{\nu}_e) \) and \( P(\nu_\mu \rightarrow \nu_\mu) \), there are two sets of solutions, and this represents the ambiguity due to the \( \theta_{23} \leftrightarrow \pi - \theta_{23} \) degeneracy.

2.3. \( \cos 2\theta_{23} \neq 0, \Delta m_{21}^2/\Delta m_{31}^2 \neq 0, AL = 0 \)

If we turn on the effect of non-zero \( \Delta m_{21}^2 \) in addition to non-zero \( \cos 2\theta_{23} \), then the oscillation probabilities are as follows:

\[
\begin{align*}
\left\{ P(\nu_\mu \rightarrow \nu_e), P(\bar{\nu}_\mu \rightarrow \bar{\nu}_e) \right\} &= x^2 \sin^2 \Delta + 2xy \sin \Delta \cos(\delta \pm \Delta) + y^2 \Delta^2, \\
\end{align*}
\]

which are correct to the second-order in the small parameters \( |\Delta m_{21}^2/\Delta m_{31}^2| \) and \( \sin 2\theta_{13} \), where

\[
\begin{align*}
x &\equiv s_{23} \sin 2\theta_{13}, \\
y &\equiv \left| \frac{\Delta m_{21}^2}{\Delta m_{31}^2} \right| c_{23} \sin 2\theta_{12}.
\end{align*}
\]

In this case, the trajectory of \( P(\nu_\mu \rightarrow \nu_e) = P, P(\bar{\nu}_\mu \rightarrow \bar{\nu}_e) = \bar{P} \) (where \( P \) and \( \bar{P} \) are constant) in the \( (X \equiv \sin^2 2\theta_{13}, Y \equiv 1/s_{23}^2) \) plane is given by a quadratic curve:

\[
16C_0 X(Y - 1) \sin^2 \Delta = \frac{1}{\sin^2 \Delta} (P - \bar{P})^2 Y^2 + \frac{1}{\cos^2 \Delta} [(P + \bar{P} - 2C_0)(Y - 1) + P + \bar{P} - 2X \sin^2 \Delta]^2,
\]

where

\[
C_0 \equiv \left( \frac{\Delta m_{21}^2}{\Delta m_{31}^2} \right)^2 \Delta^2 \sin^2 2\theta_{12}.
\]

Equation (3) becomes a hyperbola for most of the region of \( \Delta \), but it becomes an ellipse for the region \( \Delta \approx \pi \).

When \( \sin^2 2\theta_{23} = 1 \), there are two solutions for the intersection of \( Y = 2 \) and equation (3). This indicates that even if we know the precise values of \( P(\nu_\mu \rightarrow \nu_e), P(\bar{\nu}_\mu \rightarrow \bar{\nu}_e) \) and \( P(\nu_\mu \rightarrow \nu_\mu) \), there are two sets of solutions for \( (\theta_{13}, \theta_{23}, \delta) \) with fourfold degeneracy when \( \sin^2 2\theta_{23} = 1 \), as shown in figure 3(a). This represents the ambiguity due to intrinsic \( (\theta_{13}, \delta) \) degeneracy. When \( \sin^2 2\theta_{23} \neq 1 \), there are four sets of solutions with twofold degeneracy, as depicted in figure 3(b).

\[2 \text{ This result is obtained by taking the limit } A \equiv \sqrt{2} G_F N_e \rightarrow 0 \text{ in equation (16) in [15].} \]
Figure 3. Solutions, marked by black blobs, for the given $P(\nu_\mu \rightarrow \nu_e)$, $P(\bar{\nu}_\mu \rightarrow \bar{\nu}_e)$ and $P(\nu_\mu \rightarrow \nu_\mu)$ for $\Delta m_{21}^2/\Delta m_{31}^2 \neq 0$ and $AL = 0$. (a) For $\cos 2\theta_{23} = 0$, the intersection of $Y \equiv 1/s_{23}^2 = 2$ and the trajectory of $P(\nu_\mu \rightarrow \nu_e)$ = const. and $P(\bar{\nu}_\mu \rightarrow \bar{\nu}_e)$ = const. are two points with fourfold degeneracy. (b) For $\cos 2\theta_{23} \neq 0$, the intersections are four solutions with twofold degeneracy.

2.4. $\cos 2\theta_{23} \neq 0$, $\Delta m_{21}^2/\Delta m_{31}^2 \neq 0$, $AL \neq 0$

Furthermore, if we turn on the matter effect $AL$, then the oscillation probabilities are given by [9, 15]

$$P(\nu_\mu \rightarrow \nu_e) = x^2 f^2 + 2xyfg \cos(\delta + \Delta) + y^2 g^2,$$

$$P(\bar{\nu}_\mu \rightarrow \bar{\nu}_e) = x^2 \bar{f}^2 + 2xy\bar{f}g \cos(\delta - \Delta) + y^2 g^2$$

(4)
for the normal hierarchy and by
\[ P(\nu_\mu \rightarrow \nu_e) = x^2 \bar{f}^2 - 2xy \bar{g} \cos(\delta - \Delta) + y^2 g^2, \]
\[ P(\bar{\nu}_\mu \rightarrow \bar{\nu}_e) = x^2 f^2 - 2xyfg \cos(\delta + \Delta) + y^2 g^2, \]
for the inverted hierarchy, where \( x \) and \( y \) are given by equation (2) and
\[
\begin{align*}
\{ f \} & \equiv \frac{\sin(\Delta + AL/2)}{(1 \mp AL/2\Delta)}, \\
g & \equiv \frac{\sin(AL/2)}{AL/2\Delta}.
\end{align*}
\]
Equations (4) and (5) are correct up to the second-order in \( |\Delta m_{31}^2/\Delta m_{21}^2| \) and \( \sin 2\theta_{13} \), and all orders in \( AL \). The trajectory of \( P(\nu_\mu \rightarrow \nu_e) = P \), \( P(\bar{\nu}_\mu \rightarrow \bar{\nu}_e) = \bar{P} \) (where \( P \) and \( \bar{P} \) are constant) in the \((X \equiv \sin^2 2\theta_{13}, Y \equiv 1/s^2_{23})\) plane is again a quadratic curve for either of the mass hierarchies:
\[
16CX(Y - 1) = \frac{1}{\cos^2 \Delta} \left[ \left( \frac{P - C}{f} + \frac{\bar{P} - C}{\bar{f}} \right) (Y - 1) - (f + \bar{f})X + \frac{P}{f} + \frac{\bar{P}}{\bar{f}} \right]^2
\]
\[
+ \frac{1}{\sin^2 \Delta} \left[ \left( \frac{P - C}{f} - \frac{\bar{P} - C}{\bar{f}} \right) (Y - 1) - (f - \bar{f})X + \frac{P}{f} - \frac{\bar{P}}{\bar{f}} \right]^2
\]
for the normal hierarchy, and
\[
16CX(Y - 1) = \frac{1}{\cos^2 \Delta} \left[ \left( \frac{P - C}{f} + \frac{\bar{P} - C}{\bar{f}} \right) (Y - 1) - (f + \bar{f})X + \frac{P}{f} + \frac{\bar{P}}{\bar{f}} \right]^2
\]
\[
+ \frac{1}{\sin^2 \Delta} \left[ \left( \frac{P - C}{f} - \frac{\bar{P} - C}{\bar{f}} \right) (Y - 1) - (f - \bar{f})X + \frac{P}{f} - \frac{\bar{P}}{\bar{f}} \right]^2
\]
for the inverted hierarchy, where
\[
C \equiv \left( \frac{\Delta m_{23}^2}{\Delta m_{31}^2} \right)^2 \frac{\sin(AL/2)}{AL/2\Delta} \sin^2 2\theta_{12}.
\]
Again, these quadratic curves become hyperbolas for most of the region of \( \Delta \), but they become ellipses for some \( \Delta \approx \pi \).
If \( \sin^2 2\theta_{23} = 1 \), there are four solutions with twofold degeneracy, as shown in figure 4(a). If we know for some reason (e.g. from reactor experiments) which solution is selected for each mass hierarchy, then there are only two solutions. This is the ambiguity due to the \( \text{sgn}(\Delta m_{31}^2) \) degeneracy. If \( \sin^2 2\theta_{23} \neq 1 \) and if we do not know which solution is favoured with respect to the intrinsic degeneracy for each hierarchy, and if we do not know \( \text{sgn}(\Delta m_{31}^2) \), then there are eight solutions without any degeneracy, as depicted in figure 4(b). The advantage of our plot is that all the eight solutions for \((\theta_{13}, \theta_{23})\) give different points, and all the lines in the \((\sin^2 2\theta_{13}, 1/s^2_{23})\) plane are described by (at most) quadratic curves so that their behaviours are easy to see.
Figure 4. Solutions, marked by black blobs, for the given $P(\nu_\mu \to \nu_e)$, $P(\bar{\nu}_\mu \to \bar{\nu}_e)$ and $P(\nu_\mu \to \nu_\mu)$ for $\Delta m^{2}_{21}/\Delta m^{2}_{31} \neq 0$ and $AL \neq 0$. (a) For $\cos 2\theta_{23} = 0$, the intersection of $Y \equiv 1/s^{2}_{23} = 2$ and the trajectory of $P(\nu_\mu \to \nu_e) = \text{const.}$ and $P(\bar{\nu}_\mu \to \bar{\nu}_e) = \text{const.}$ are four points with twofold degeneracy. (b) For $\cos 2\theta_{23} \neq 0$, the intersections are eight solutions without degeneracy.

2.5. Oscillation maximum

Finally, consider the case where experiments are done at the oscillation maximum, i.e. when the neutrino energy $E$ satisfies $\Delta \equiv |\Delta m^{2}_{31}|L/4E = \pi/2$. In this case, the probabilities become

$$P(\nu_\mu \to \nu_e) = x^2 f^2 - 2xyfg \sin \delta + y^2 g^2.$$ (11)
\[
P(\bar{\nu}_\mu \to \bar{\nu}_e) = x^2 \bar{f}^2 + 2xy \bar{f}g \sin \delta + y^2 g^2 \tag{12}
\]
for the normal hierarchy, and
\[
P(\nu_\mu \to \nu_e) = x^2 \bar{f}^2 - 2xy \bar{f}g \sin \delta + y^2 g^2, \tag{13}
\]
\[
P(\bar{\nu}_\mu \to \bar{\nu}_e) = x^2 f^2 + 2xyfg \sin \delta + y^2 g^2 \tag{14}
\]
for the inverted hierarchy, where \(x\) and \(y\) are given by equation (2), and \(f, \bar{f}, g\) in equations (6) and (7) become
\[
\left\{ \begin{array}{c}
f \\
\bar{f}
\end{array} \right\} = \pm \frac{\cos(AL/2)}{1 \mp AL/\pi}, \quad g = \frac{\sin(AL/2)}{AL/\pi}
\]
for \(\Delta = \pi/2\). The trajectory of \(P(\nu_\mu \to \nu_e) = P, P(\bar{\nu}_\mu \to \bar{\nu}_e) = \bar{P}\) in the \((X \equiv \sin^2 2\theta_{13}, Y \equiv 1/s^2_{23})\) plane becomes a straight line and is given by
\[
Y = \frac{f + \bar{f}}{P/f + \bar{P}/\bar{f} - C(1/f + 1/\bar{f})} \left( X - \frac{C}{ff} \right) \tag{15}
\]
for the normal hierarchy, and
\[
Y = \frac{f + \bar{f}}{P/f + \bar{P}/\bar{f} - C(1/f + 1/\bar{f})} \left( X - \frac{C}{ff} \right) \tag{16}
\]
for the inverted hierarchy, where \(C\) is given by equation (10). The straight lines (15) and (16) are very close to each other in relatively short long-baseline experiments such as JPARC, where the matter effect is small. As shown in appendix B, (15) and (16) have the minimum values in \(Y \equiv 1/s^2_{23}\), which is larger than the naive value 1 for either of the mass hierarchies. Since equations (15) and (16) are linear in \(X\), there is only one solution between them and \(Y = \text{const.}\). Thus the ambiguity due to the intrinsic degeneracy is solved by performing experiments at the oscillation maximum, although it is then transformed into another ambiguity due to the \(\delta \leftrightarrow \pi - \delta\) degeneracy.

If \(\sin^2 2\theta_{23} \simeq 1\), all the four solutions are basically close to each other in the \((\sin^2 2\theta_{13}, 1/s^2_{23})\) plane, and the ambiguity due to degeneracies are not serious as far as \(\theta_{13}\) and \(\theta_{23}\) are concerned (see figure 5(a)). On the other hand, if \(\sin^2 2\theta_{23}\) deviates fairly from 1, then the solutions are separated into two groups, those for \(\theta_{23} > \pi/4\) and those for \(\theta_{23} < \pi/4\) in the \((\sin^2 2\theta_{13}, 1/s^2_{23})\) plane, as shown in figure 5(b). In this case, resolution of the \(\theta_{23} \leftrightarrow \pi/2 - \theta_{23}\) ambiguity is required to determine \(\theta_{13}, \theta_{23}\) and \(\delta\).

### 2.6. Fake effects on CP violation due to degeneracies

#### 2.6.1. \(\sin^2 2\theta_{23} \simeq 1\)

If the JPARC experiment reveals from the measurement of the disappearance probability \(P(\nu_\mu \to \nu_\mu) = P\) that \(\sin^2 2\theta_{23} \simeq 1.0\) with a good approximation, then we would not have to worry very much about parameter degeneracy as far as \(\theta_{13}\) and
Figure 5. The $\theta_{23}$ ambiguity that could arise after JPARC measurements of $P(\nu_\mu \to \nu_e)$, $P(\bar{\nu}_\mu \to \bar{\nu}_e)$ and $P(\nu_\mu \to \nu_\mu)$ at the oscillation maximum. (a) If $\sin^2 2\theta_{23} \simeq 1.0$, the values of $\theta_{13}$ and $\theta_{23}$ are close to each other for all the solutions and the ambiguity is not serious as far as $\theta_{13}$ and $\theta_{23}$ are concerned. (b) If $\sin^2 2\theta_{23} < 1$, the $\theta_{23}$ ambiguity needs to be resolved to determine $\theta_{13}$ and $\theta_{23}$ to good precision.

$\theta_{23}$ are concerned, since the values of $\theta_{13}$ and $\theta_{23}$ for all the different solutions are close to each other.

On the other hand, when it comes to the value of the CP phase $\delta$, we have to be careful. From [9], the true value $\delta$ and the fake value $\delta'$ for the CP phase satisfy the following:

$$x' \sin \delta' = x \sin \delta \frac{\Delta f^2 + \bar{\Delta} f^2 - f \bar{f}}{f \bar{f}} - \frac{x^2}{\sin \Delta} \frac{f^2 + \bar{\Delta} f^2}{f \bar{f}} \frac{f - \bar{f}}{2 \bar{y} g}, \quad (17)$$
where \(x, y\) are given by equation (2), \(f, \bar{f}, g\) are given by (6) and (7) and \(x'\) is defined by

\[
x'^2 = x^2(f^2 + \bar{f}^2 - f\bar{f}) - 2yg(f - \bar{f})x \sin \delta \sin \Delta.
\]

Equation (17) indicates that, even if \(\sin \delta = 0\), we have nonvanishing fake CP violating effect

\[
\sin \delta' = -x f^2 + \bar{f}^2 \frac{f - \bar{f}}{2yg \sin \Delta} \sqrt{f^2 + \bar{f}^2 - f\bar{f}},
\]

if we fail to identify the correct sign of \(\Delta m_{31}^2\). For the JPARC experiment, equation (18) implies that

\[
\sin \delta' \simeq -2.2 \sin 2\theta_{13},
\]

which is not negligible unless \(\sin^2 2\theta_{13} \ll 10^{-2}\). Therefore we have to know the sign of \(\Delta m_{31}^2\) to determine the CP phase to good precision.

2.6.2. \(\sin^2 2\theta_{23} < 1\). As explained in section 2.5, if \(\sin^2 2\theta_{23}\) deviates fairly from 1, we need to resolve the ambiguity due to the \(\theta_{23}\) degeneracy to determine the values of \(\theta_{13}\) and \(\theta_{23}\). As for the value of the CP phase \(\delta\), we can estimate how serious the effect of the \(\theta_{23}\) ambiguity on the value of \(\delta\) could be. If the true value of \(\delta\) is zero, the CP phase \(\delta'\) for the fake solution can be estimated as [9]

\[
\sin 2\theta_{13} \sin \delta' = \frac{|\Delta m_{21}^2|}{|\Delta m_{31}^2|} \frac{g(f - \bar{f}) \sin 2\theta_{12} \cot 2\theta_{23}}{f\bar{f}} \sin \Delta,
\]

where

\[
\sin^2 2\theta_{13} = \sin^2 2\theta_{13} \tan^2 \theta_{23} + \left(\frac{\Delta m_{21}^2}{\Delta m_{31}^2}\right)^2 \frac{g^2 \sin^2 2\theta_{12}}{f\bar{f}} (1 - \tan^2 \theta_{23}),
\]

and \(f, \bar{f}, g\) are defined by (6) and (7). For JPARC, we have

\[
|\sin \delta'| \sim \frac{1}{200} \frac{1}{\cot 2\theta_{23}} \frac{1}{\sin 2\theta_{13}} \lesssim \frac{1}{500} \frac{1}{\sin^2 2\theta_{13}},
\]

where we have used the bound \(0.90 \leq \sin^2 2\theta_{23} \leq 1.0\) from the atmospheric neutrino data in the second inequality. Hence, we see that the ambiguity due to the \(\theta_{23}\) does not cause a serious problem on determination of \(\delta\) for \(\sin^2 2\theta_{13} \gtrsim 10^{-2}\). However, it should be stressed that the effect on CP violation due to the sgn\((\Delta m_{31}^2)\) ambiguity is also serious in this case.

3. Resolution of ambiguities by the third measurement after JPARC

Assuming that the JPARC experiment, which is expected to be the first superbeam experiment, measures \(P(\nu_\mu \rightarrow \nu_e)\) and \(P(\bar{\nu}_\mu \rightarrow \bar{\nu}_e)\) at the oscillation maximum \(\Delta \equiv |\Delta m_{31}^2|L/4E = \pi/2\), we
Figure 6. Four possible values for the CP phase $\delta$ at the oscillation maximum. The red (blue) line stands for the normal (inverted) hierarchy. Since we are assuming the normal hierarchy here, the red (blue) line corresponds to the correct (wrong) assumption on the mass hierarchy. The $+$ and $-$ signs stand for the choice of $s_{23}^2 = (1 \pm \sqrt{1 - \sin^2 2\theta_{23}})/2$ in the $\theta_{23}$ ambiguity and $c$ ($w$) stands for the correct (wrong) assumption on the mass hierarchy.

will discuss in this section how the third measurement after JPARC can resolve the ambiguities by using the plot in the $(\sin^2 2\theta_{13}, 1/s_{23}^2)$ plane. Resolution of the $\theta_{23}$ ambiguity has been discussed previously using the disappearance measurement of $P(\bar{\nu}_e \to \bar{\nu}_e)$ at reactors [8, 11], [16]–[18], the silver channel $\nu_e \to \nu_\tau$ at neutrino factories [19] and the $\nu_\mu \to \nu_e$ channel [21, 22]. Here, we take the following reference values for the oscillation parameters:

$$
\sin^2 2\theta_{12} = 0.8, \quad \sin^2 2\theta_{13} = 0.05, \quad \sin^2 2\theta_{23} = 0.96,
$$

$$
\Delta m_{21}^2 = 7 \times 10^{-5} \text{ eV}^2, \quad \Delta m_{31}^2 = 2.5 \times 10^{-3} \text{ eV}^2 > 0, \quad \delta = \pi/4.
$$

3.1. $\nu_\mu \to \nu_e$

Let us discuss the case in which another long-baseline experiment measures $P(\nu_\mu \to \nu_e)$. From the measurements of $P(\nu_\mu \to \nu_e)$ and $P(\bar{\nu}_\mu \to \bar{\nu}_e)$ by JPARC at the oscillation maximum, we can deduce the value of $\delta$ up to the eightfold ambiguity ($\delta \leftrightarrow \pi - \delta, \theta_{23} \leftrightarrow \pi/2 - \theta_{23}, \Delta m_{31}^2 \leftrightarrow -\Delta m_{31}^2$). As depicted in figure 6, depending on whether $s_{23}^2 - 1/2$ is positive or negative, we assign subscript $+$ or $-$, and depending on whether our ansatz for $\text{sgn}(\Delta m_{31}^2)$ is correct or wrong,

3 There have been a number of studies [20] on how to resolve parameter degeneracies using the $\nu_\mu \to \nu_e$ channel, and they have discussed mainly the intrinsic and $\text{sgn}(\Delta m_{31}^2)$ degeneracies. In [21, 22], the $\theta_{23}$ ambiguity as well as others using the $\nu_\mu \to \nu_e$ channel and its combination with $\nu_e \to \nu_\tau$ are discussed. The present scenario, in which the third experiment follows the JPARC results on $P(\nu_\mu \to \nu_e) \lor P(\bar{\nu}_\mu \to \bar{\nu}_e)$, measured at the oscillation maximum, has been discussed in [23] from a different viewpoint.

4 I thank Hiroaki Sugiyama for pointing this out to me.
we assign subscript c or w. Thus, the eight possible values of $\delta$ are given by

$$
\delta_{+w}, \delta_{c}, \delta_{-w}, \delta_{-c}, \pi - \delta_{+w}, \pi - \delta_{c}, \pi - \delta_{-w}, \pi - \delta_{-c}.
$$

(21)

Now, suppose that the third measurement gives the value $P$ for the oscillation probability $P(\nu_\mu \to \nu_e)$. Then, there are in general eight lines in the $(X \equiv \sin^2 2\theta_{13}, Y \equiv 1/s^2_{23})$ plane, given by

$$
f^2 X = [P - C + 2C \cos^2(\delta + \Delta)](Y - 1) + P
- 2 \cos(\delta + \Delta) \sqrt{C(Y - 1)} \sqrt{[P - C \sin^2(\delta + \Delta)](Y - 1) + P}
$$

(22)

for the normal hierarchy and

$$
f^2 X = [P - C + 2C \cos^2(\delta - \Delta)](Y - 1) + P
- 2 \cos(\delta - \Delta) \sqrt{C(Y - 1)} \sqrt{[P - C \sin^2(\delta - \Delta)](Y - 1) + P}
$$

(23)

for the inverted hierarchy. Here, $C$ is defined by equation (10), $\Delta (\equiv |\Delta m^2_{31}| L/4E)$ is defined for the third measurement and $\delta$ takes one of the eight values given in equation (21). The derivation of (22) and (23) is given in appendix A. It turns out that the solutions (22) and (23) are hyperbola if $\cos^2(\delta \pm \Delta) > (C - P)/P$, where $+$ and $-$ refer to the normal and inverted hierarchy respectively, and ellipse if $\cos^2(\delta \pm \Delta) < (C - P)/P$. In practice, however, the difference between hyperbola and ellipse is not so important for the present discussions, because we are only interested in the behaviours of these curves in the region $1.52 < Y \equiv 1/s^2_{23} < 2.92$, which comes from the 90% CL allowed region of the Superkamiokande atmospheric neutrino data for $\sin^2 2\theta_{23}$.

In this context, let us look at three typical cases: $L = 295, 730$ and $3000$ km, each of which corresponds to JPARC, off-axis NuMI [24] and a neutrino factory [25].

Figures 7–9 show the trajectories of $P(\nu_\mu \to \nu_e)$ obtained in the third measurement, together with the constraint of $P(\nu_\mu \to \nu_e), P(\bar{\nu}_\mu \to \bar{\nu}_e)$ and $P(\nu_\mu \to \nu_\mu)$, by JPARC for $L = 295, 730$ and $3000$ km, respectively, where $\Delta \equiv |\Delta m^2_{31}| L/4E$ takes the values $\Delta = j\pi/8$ ($j = 1, \ldots, 7, j \neq 4$). The purple (light blue) blob stands for the true (fake) solution given by the JPARC results on $P(\nu_\mu \to \nu_e), P(\bar{\nu}_\mu \to \bar{\nu}_e)$ and $P(\nu_\mu \to \nu_\mu)$. For the correct (wrong) guess on the mass hierarchy, there are in general four red (blue) curves, owing to the fact that the CP phase $\delta$, which is deduced from the JPARC results on $P(\nu_\mu \to \nu_e), P(\bar{\nu}_\mu \to \bar{\nu}_e)$ and $P(\nu_\mu \to \nu_\mu)$, is fourfold: $(\delta_{+c}, \delta_{-c}, \pi - \delta_{+c}, \pi - \delta_{-c})$ for the correct assumption on the hierarchy and $(\delta_{+w}, \delta_{-w}, \pi - \delta_{+w}, \pi - \delta_{-w})$ for the wrong assumption. In most cases, the four (red or blue) curves are separated into two pairs.

5 For $L = 3000$ km, the density of the matter may not be treated as constant, and the probability formulae (4) and (5) may no longer be valid. It turns out, however, that the approximation of the formulae becomes good if we replace $AL$ by $AL \to \int_0^L A(x) \, dx$ everywhere in the formula. In the following discussions, the replacement $AL \to \int_0^L A(x) \, dx$ is always understood for the baseline $L = 3000$ km. It should be mentioned that the neutrino energy spectrum at neutrino factories is continuous and it is assumed here that we take one particular energy bin whose energy range can be made relatively small. It should also be noted that neutrino factories actually measure the probability $P(\nu_e \to \nu_\mu)$ or $P(\bar{\nu}_e \to \bar{\nu}_\mu)$, instead of $P(\nu_\mu \to \nu_e)$ or $P(\bar{\nu}_\mu \to \bar{\nu}_e)$. Here, we discuss for simplicity the trajectory of $P(\nu_\mu \to \nu_e)$, whose feature is the same as that of $P(\nu_e \to \nu_\mu)$.
Figure 7. The trajectories of \( P(\nu_{\mu} \rightarrow \nu_e) = \text{const.} \) of the third experiment at \( L = 295 \text{ km} \) with \( \Delta \equiv |\Delta m_{31}^{2}|L/4E = (j/8)\pi \) \( (1 \leq j \leq 7, j \neq 4) \) after JPARC. True values are those in (20). The green line is the JPARC result obtained by \( P(\nu_{\mu} \rightarrow \nu_e) \) and \( P(\bar{\nu}_{\mu} \rightarrow \bar{\nu}_e) \) at the oscillation maximum. The red (blue) lines are the trajectories of \( P(\nu_{\mu} \rightarrow \nu_e) \) given by the third experiment assuming the normal (inverted) hierarchy, where \( \delta \) takes four values for each mass hierarchy.
Figure 8. The trajectories of $P(\nu_\mu \rightarrow \nu_e) = \text{const.}$ of the third experiment at $L = 730\text{ km}$ with $\Delta \equiv |\Delta m^2_{13}|L/4E = (j/8)\pi$ ($1 \leq j \leq 7$) after JPARC. True values are those in (20).
Figure 9. The trajectories of $P(\nu_\mu \rightarrow \nu_e) = \text{const.}$ of the third experiment at $L = 3000\text{ km}$ with $\Delta \equiv |\Delta m_3^2| L/4E = (j/8)\pi$ ($1 \leq j \leq 7$) after JPARC. True values are those in (20). For $\Delta \geq (3/8)\pi$, the blue curves (with the wrong assumption for the mass hierarchy) are not seen in the figure because they are far to the right.
of curves. As we will see later, the large split is due to the $\delta \leftrightarrow \pi - \delta$ ambiguity, whereas the small split is due to the $\theta_{23} \leftrightarrow \pi/2 - \theta_{23}$ ambiguity. The reason that the latter splitting is small is because the difference in values for the CP phases is small, as can be seen from (19). In some of the figures in figures 7–9, the number of red or blue curves is less than 4 since not all values of $\delta$ give consistent solutions for a set of oscillation parameters.

Let us study the ambiguities one by one.

3.1.1. $\delta \leftrightarrow \pi - \delta$ ambiguity. As mentioned above, the large splitting of four (red or blue) lines into two pair of lines is due to the $\delta \leftrightarrow \pi - \delta$ ambiguity. From (22) and (23), we see that the only difference between the solutions with $\delta$ and $\pi - \delta$ appears in $\cos(\delta \pm \Delta)$ or $\sin(\delta \pm \Delta)$. If $\Delta = \pi/2$ (i.e. the oscillation maximum), we have $\cos(\delta + \Delta) = -\sin \delta$ and $\cos(\pi - \delta + \Delta) = -\sin \delta$, so that the values of $X$ with $\delta$ and with $\pi - \delta$ are the same, i.e. at oscillation maximum there is exact degeneracy. On the other hand, if $\Delta \neq \pi/2$, we have $\cos(\delta + \Delta) \neq \cos(\pi - \delta + \Delta)$, and the values of $X$ with $\delta$ and with $\pi - \delta$ are different. Thus, to resolve the $\delta \leftrightarrow \pi - \delta$ ambiguity, it is advantageous to perform an experiment at $\Delta$ farther away from $\pi/2$. Deviation of $\Delta$ value from $\pi/2$ implies either high or low energy. In general, the number of events increases for high energy because both the cross-section and the neutrino flux increase. Therefore the high-energy option is preferred in resolving the $\delta \leftrightarrow \pi - \delta$ ambiguity.$^6$

3.1.2. $\Delta m_{31}^2 \leftrightarrow -\Delta m_{31}^2$ ambiguity. As one can easily imagine, the $\text{sgn}(\Delta m_{31}^2)$ ambiguity is resolved better with longer baselines, since the dimensionless quantity $AL = \sqrt{2} G_F N_e L \sim (L/1900 \text{ km})(\rho/2.7 \text{ g cm}^{-3})$ becomes of order one for $L \gtrsim 1000 \text{ km}$. On the other hand, from figures 8 and 9, we observe that the split of the curves with different mass hierarchies (the red versus blue curves) is larger for lower energy. Naively, this appears to be counterintuitive, since, at low energy, the matter effect is expected to be less important ($|\Delta m_{31}^2| L/4E \gg AL$). However, this is not the case since we are dealing with the value of $\sin^2 2\theta_{13}$ obtained for a given value of $P(\nu_\mu \rightarrow \nu_e)$. To see this, let us consider for simplicity the value of $X \equiv \sin^2 2\theta_{13}$ at $Y \equiv 1/s_{23}^2 = 1$, i.e. the $X$ intercept of the quadratic curves at $Y = 1$. $(\sin^2 2\theta_{13})_n((\sin^2 2\theta_{13})_i)$ at $Y = 1$ for the normal (inverted) hierarchy is given by $x^2$ by putting $y = 0$ in (4) (equation (5)):

$$
\begin{align*}
(\sin^2 2\theta_{13})_n &= \frac{P}{f^2} \\
(\sin^2 2\theta_{13})_i &= \frac{P}{f^2}
\end{align*}
$$

for $s_{23}^2 = 1$.

The ratio of these two quantities is given, for small $AL$, by

$$
\frac{(\sin^2 2\theta_{13})_n}{(\sin^2 2\theta_{13})_i} = \frac{f^2}{f^2} = \frac{\sin^2(\Delta - AL/2)}{\sin^2(\Delta + AL/2)} \left( \frac{1 + AL/2\Delta}{1 - AL/2\Delta} \right)^2 \\
\simeq 1 + 2AL \left( \frac{1}{\Delta} - \frac{1}{\tan \Delta} \right).
$$

$^6$ Resolution of $\delta \leftrightarrow \pi - \delta$ ambiguity at neutrino factories has been discussed in [13].
Hence, the larger the $\Delta$ (the smaller the neutrino energy), the larger the above ratio, as long as $\Delta$ does not exceed $\pi$. This phenomenon suggests that it is potentially possible to enhance the matter effect by performing an experiment at low energy ($\Delta > \pi/2$) even with $L = 730$ km, and it may enable us to determine the sign of $\Delta m_{31}^2$ at the off-axis NuMI experiment. Although the neutrino flux decreases for low energy at the off-axis NuMI experiment, the cross-section at $E \sim 1$ GeV is not particularly small compared with higher energy; hence, the low energy possibility at the off-axis NuMI experiment deserves serious study.

3.1.3. $\theta_{23} \leftrightarrow \pi/2 - \theta_{23}$ ambiguity. Figures 7–9, which are plotted for $\delta = \pi/4$, suggest that there is a tendency that, for high energy, the slope of the red curve, which goes through the true point (the purple blob), is almost the same as the slope of the straight green line obtained by JPARC, whereas, for low energy, the slope of the red curve is smaller than that of the JPARC green line. Here, we will discuss the $X$ intercept at $Y = 1$ instead of calculating the slope itself, since it is easier to consider the $X$ intercept and since the difference in the $X$ intercepts inevitably implies different slopes for the two lines with almost all the curves being approximately straight lines. For JPARC, the matter effect is small ($AL \simeq 0.08$), so that we can put $f \simeq \tilde{f} \simeq 1$. From equation (15), we have the $X$ intercept at $Y = 1$ as

$$X_{\text{JPARC}} = \frac{P/f + \tilde{P}/f}{f + \tilde{f}} \simeq \frac{P + \tilde{P}}{2} \simeq x^2,$$

(24)

where the term $g^2 y^2$ has been ignored for simplicity. On the other hand, for the third measurement, from (22), we have

$$X_{\text{3rd}} = \frac{P}{f^2} \simeq x^2 + 2 \frac{g}{f} xy \cos(\delta + \Delta),$$

(25)

where the term $g^2 y^2$ has been ignored again for simplicity. Equation (25) indicates that it is the second term in (25) that deviates the intercept $X_{\text{3rd}}$ of the red line from the intercept $X_{\text{JPARC}}$ of the JPARC green line. For the difference between $X_{\text{JPARC}}$ and $X_{\text{3rd}}$ to be large, $\tilde{f}$ has to be small and $|\cos(\delta + \Delta)|$ has to be large. When $AL$ is small, for $f$ to be small, $|\Delta m_{31}^2| L/4E - \pi|$ has to be small. This is one of the conditions to resolve the $\theta_{23}$ ambiguity. Here, we are using the reference value $\delta = \pi/4$; hence, the deviation becomes maximal if $|\delta + \Delta| = |\pi/4 + \Delta| \simeq \pi$. In real experiments, however, nobody knows the value of the true $\delta$ in advance and, therefore, it is difficult to design a long baseline experiment to resolve the $\theta_{23} \leftrightarrow \pi/2 - \theta_{23}$ ambiguity. If $\delta$ turns out to satisfy $|\cos(\delta + \Delta)| \sim 1$ in the results of the third experiment, then we may be able to resolve the $\theta_{23}$ ambiguity as a by-product.

3.2. $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$

It turns out that the situation does not change very much even if we use the $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ channel in the third experiment. Typical curves are given for $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ in figure 10, which are similar to those
Figure 10. The trajectories of $P(\bar{\nu}_\mu \rightarrow \bar{\nu}_e) = \text{const.}$ of the third experiment with $\Delta \equiv |\Delta m^2_{31}|L/4E = \pi/8$ after JPARC. The behaviours are almost similar to those for $P(\nu_\mu \rightarrow \nu_e) = \text{const.}$ True values are those in (20).

in figures 7–9. Thus the conclusions we drew on resolution of the ambiguities hold qualitatively for the $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ channel.

3.3. $\nu_e \rightarrow \nu_\tau$

The experiment with the channel $\nu_e \rightarrow \nu_\tau$ requires intense $\nu_e$ beams and it is expected that such measurements can be done at neutrino factories or at beta beam experiments [26]. The oscillation probability $P(\nu_e \rightarrow \nu_\tau)$ is given by

$$P(\nu_e \rightarrow \nu_\tau) = \tilde{x}^2 f^2 + 2 f g \tilde{y} \cos(\delta + \Delta) + \tilde{y}^2 g^2,$$

where

$$\tilde{x} \equiv c_{23} \sin 2\theta_{23}, \quad \tilde{y} \equiv \left| \frac{\Delta m^2_{21}}{\Delta m^2_{31}} \right| s_{23} \sin 2\theta_{12}. $$
Figure 11. The trajectories of $P(\nu_e \rightarrow \nu_\tau) = \text{const.}$ of the third experiment at $L = 2810 \text{ km}$ with $\Delta \equiv |\Delta m^2_{31}|L/4E = (j/8)\pi \ (j = 1, 2, 3)$ after JPARC. True values are those in (20). The curves intersect with the JPARC line perpendicularly; hence, this channel is advantageous to resolve the ambiguities from an experimental point of view.

and $f, g$ are defined in (6) and (7). The solution for $P(\nu_e \rightarrow \nu_\tau) = Q$, where $Q$ is constant, is given by

$$X = \frac{Q}{f^2} \left\{ \left[ 1 + \frac{2 \cos^2(\delta + \Delta)}{1 - C/Q} \right] \frac{1 - C/Q}{Y - 1} + 1 \right. \right.$$

$$- \frac{2 \cos(\delta + \Delta)}{\sqrt{1 - C/Q}} \sqrt{\left[ 1 + \frac{\cos^2(\delta + \Delta)}{1 - C/Q} \right] \frac{1 - C/Q}{Y - 1} + 1} \left. \right\}, \quad (26)$$

where $X \equiv \sin^2 2\theta_{13}, \ Y \equiv 1/s^2_{23}$ as before and $C$ is given by equation (10). Equation (26) is plotted in figure 11 for $L = 2810 \text{ km}$. From figure 11, we see that the curve $P(\nu_e \rightarrow \nu_\tau) = Q$ intersects with the JPARC green line almost perpendicularly, and it is experimentally advantageous: namely,
in real experiments, all the measured quantities have errors and the curves become thick. In this case, the allowed region is a small area around the true solution in the \((\sin^2 2\theta_{13}, 1/s^2_{23})\) plane and one expects that the fake solution with respect to the \(\theta_{23}\) ambiguity can be excluded. This contrasts with the case of the \(\nu_{\mu} \rightarrow \nu_{e}\) and \(\bar{\nu}_{\mu} \rightarrow \bar{\nu}_{e}\) channels, where the slope of the red curve is almost the same as that of the JPARC green line and the allowed region can easily contain both the true and fake solutions, so that it becomes difficult to distinguish the true point from the fake one.

Similar to the case of the \(\nu_{\mu} \rightarrow \nu_{e}\) channel, the \(\delta \leftrightarrow \pi - \delta\) ambiguity is expected to be resolved more easily for larger values of \(|\Delta - \pi/2|\) and the \(\text{sgn}(\Delta m^2_{31})\) ambiguity is resolved easily for larger baseline \(L\) (e.g. \(L \sim 3000\) km).

Thus measurement of the \(\nu_{e} \rightarrow \nu_{\tau}\) channel offers a promising possibility as a potentially powerful candidate to resolve parameter degeneracies in the future.\(^7\)

4. Discussion and conclusion

In this paper, we have shown that the eightfold parameter degeneracy in neutrino oscillations can be easily seen by plotting the trajectory of constant probabilities in the \((\sin^2 2\theta_{13}, 1/s^2_{23})\) plane. Using this plot, we have seen that the third measurement after the JPARC results on \(P(\nu_{\mu} \rightarrow \nu_{e})\) and \(P(\bar{\nu}_{\mu} \rightarrow \bar{\nu}_{e})\) may resolve the \(\text{sgn}(\Delta m^2_{31})\) ambiguity at \(L \gtrsim 1000\) km, the \(\delta \leftrightarrow \pi - \delta\) ambiguity off the oscillation maximum \((|\Delta - \pi/2| \sim \mathcal{O}(1))\) and the \(\theta_{23}\) ambiguity if \(||\Delta m^2_{31}|L/4E - \pi|\) is small and \(\delta\) turns out to satisfy \(|\cos(\delta + \Delta)| \sim 1\). In general, all these constraints on \(\Delta \equiv |\Delta m^2_{31}|L/4E\) may be satisfied by taking \(\Delta = \pi\). The condition \(\Delta = \pi\), however, actually corresponds to the oscillation minimum, and the number of events is expected to be small for a number of reasons: (i) the probability itself is small at the oscillation minimum; (ii) \(\Delta = \pi\) implies low energy and the neutrino flux decreases at low energy; and (iii) the cross-section is generally smaller at low energy than at high energy. Therefore, to gain statistics, it is presumably wise to perform an experiment at \(\pi/2 < \Delta < \pi\) after JPARC. The off-axis NuMI experiment with \(\pi/2 < \Delta < \pi\) \((E \sim 1\) GeV\) may be advantageous to resolve these ambiguities.

As seen in figures 8 and 9, experiments at the oscillation maximum do not appear to be useful after JPARC, except for the \(\text{sgn}(\Delta m^2_{31})\) ambiguity. To achieve other goals such as resolution of the \(\delta \leftrightarrow \pi - \delta\) and \(\theta_{23}\) ambiguities, it is wise to stay away from \(\Delta = \pi/2\) in experiments after JPARC.

Although only oscillation probabilities were discussed here without taking the statistical and systematic errors into account, we hope that the present paper gives some insight on how the ambiguities can be resolved in future long-baseline experiments.

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\(^7\) Recently, a high \(\gamma\) beta beam option has been proposed [27] and it is concluded that this option may be a serious competitor to neutrino factories.
Appendix A. Expression for $P(\nu_{\mu} \to \nu_e) = P$

First of all, let us derive equations (22) and (23). For the normal hierarchy, the probability $P(\nu_{\mu} \to \nu_e) = P$ is given by

$$P = P(\nu_{\mu} \to \nu_e) = x^2 f^2 + 2xyfg \cos(\delta + \Delta) + y^2 g^2 = f^2 X + 2f \sqrt{X Y} \cos(\delta + \Delta) + C \left(1 - \frac{1}{Y}\right), \quad (A.1)$$

where $X \equiv \sin^2 2\theta_{13}, Y \equiv 1/s_{23}^2$ as in the text, $f$ is defined in (6) and $C$ is given by (10). Equation (A.1) can be rewritten as

$$(P - C)(Y - 1) + P - f^2 X = 2f \sqrt{X Y} \cos(\delta + \Delta). \quad (A.2)$$

Taking the square of both left- and right-hand sides of (A.2), we obtain

$$(f^2 X)^2 - 2f^2 X [(P - C) + 2C \cos^2(\delta + \Delta)](Y - 1) + [(P - C)(Y - 1) + P]^2 = 0.$$  

Solving this quadratic equation, we obtain

$$f^2 X = [P - C + 2C \cos^2(\delta + \Delta)](Y - 1) + P \pm \sqrt{[(P - C) + 2C \cos^2(\delta + \Delta)](Y - 1) + P^2 - [(P - C)(Y - 1) + P]^2}$$

$$= [P - C + 2C \cos^2(\delta + \Delta)](Y - 1) + P \pm 2 \cos(\delta + \Delta) \sqrt{C(Y - 1)} \sqrt{[P - C + C \cos^2(\delta + \Delta)](Y - 1) + P}. \quad (A.3)$$

If $\cos(\delta + \Delta) > 0$, then from (A.2) we find that $(P + C)(Y - 1) + P - f^2 X$ has to be positive. On the other hand, (A.3) gives

$$(P + C)(Y - 1) + P - f^2 X = \mp 2 \cos(\delta + \Delta) \sqrt{C(Y - 1)} \times \sqrt{[P - C + C \cos^2(\delta + \Delta)](Y - 1) + P - \cos(\delta + \Delta) \sqrt{C(Y - 1)}}. \quad (A.4)$$

From (A.4), we conclude that we have to consider the minus sign in (A.3) for the right-hand side of (A.4) to be positive. Hence, from $P(\nu_{\mu} \to \nu_e) = P$, we obtain

$$f^2 X = [P - C + 2C \cos^2(\delta + \Delta)](Y - 1) + P - 2 \cos(\delta + \Delta) \sqrt{C(Y - 1)} \sqrt{[P - C \sin^2(\delta + \Delta)](Y - 1) + P}, \quad (A.5)$$

Here we consider, for simplicity, the case where all the arguments of the square root are positive. After we obtain the final result, we see that the final formula makes sense as long as the whole product of all the arguments is positive.
and, from \( P(\bar{\nu}_e \rightarrow \bar{\nu}_\mu) = \bar{P} \), we have

\[
f^2 X = [\bar{P} - C + 2C \cos^2(\delta - \Delta)](Y - 1) + \bar{P} \\
- 2 \cos(\delta - \Delta)\sqrt{C(Y - 1)} \sqrt{[\bar{P} - C \sin^2(\delta - \Delta)](Y - 1) + \bar{P}}. \tag{A.6}
\]

When \( P - C \sin^2(\delta + \Delta) > 0 \), equation (A.5) is a hyperbola, and the physical region for \( Y - 1 \) is \( Y - 1 \geq 0 \). On the other hand, when \( P - C \sin^2(\delta + \Delta) < 0 \), equation (A.5) becomes an ellipse and the physical region for \( Y - 1 \) is \( 0 \leq Y - 1 \leq P/[C \sin^2(\delta + \Delta) - P] \).

Similarly, we obtain for the inverted hierarchy,

\[
f^2 X = [P - C + 2C \cos^2(\delta - \Delta)](Y - 1) + P \\
+ 2 \cos(\delta - \Delta)\sqrt{C(Y - 1)} \sqrt{[P - C \sin^2(\delta - \Delta)](Y - 1) + P}, \tag{A.7}
\]

\[
f^2 X = [\bar{P} - C + 2C \cos^2(\delta + \Delta)](Y - 1) + \bar{P} \\
+ 2 \cos(\delta + \Delta)\sqrt{C(Y - 1)} \sqrt{[\bar{P} - C \sin^2(\delta + \Delta)](Y - 1) + \bar{P}}. \tag{A.8}
\]

**Appendix B. Trajectories at the oscillation maximum**

We will assume \( \Delta = \pi/2 \) throughout this appendix and \( \Delta m^2_{31} > 0 \) for most part of this appendix. From (A.5), the condition \( P(\nu_\mu \rightarrow \nu_e) = P \) for the neutrino mode alone gives

\[
f^2 X = [P - C + 2C \sin^2 \delta](Y - 1) + P \\
+ 2 \sin \delta \sqrt{C(Y - 1)} \sqrt{[P - C \cos^2 \delta](Y - 1) + P}, \tag{B.1}
\]

whereas the condition \( P(\bar{\nu}_\mu \rightarrow \bar{\nu}_e) = \bar{P} \) for the anti-neutrino mode alone gives

\[
f^2 X = [\bar{P} - C + 2C \sin^2 \delta](Y - 1) + \bar{P} \\
- 2 \sin \delta \sqrt{C(Y - 1)} \sqrt{[\bar{P} - C \cos^2 \delta](Y - 1) + \bar{P}}. \tag{B.2}
\]

When \( \delta \) ranges from \(-\pi/2\) to \(\pi/2\), equation (B.1) sweeps out the inside of a hyperbola, as depicted by the red curves in figure B.1(a), whereas (B.2) sweeps out the inside of another hyperbola for the anti-neutrino mode (cf the blue curves in figure B.1(a)). Notice that the left (right) edge of the hyperbola (B.1) for the neutrino mode corresponds to \( \delta = -\pi/2 \) (\( \delta = +\pi/2 \)), whereas the left (right) edge of the other hyperbola (B.2) for the anti-neutrino mode corresponds to \( \delta = +\pi/2 \) (\( \delta = -\pi/2 \)). Since the straight line (15) represents the intersection of the two regions (the yellow and light blue regions in figure B.1(b)), the lowest point in the straight line is obtained by putting \( \delta = +\pi/2 \) (\( \delta = -\pi/2 \)) if \( P/f^2 < \bar{P}/\bar{f}^2 \) (if \( P/f^2 > \bar{P}/\bar{f}^2 \)), respectively, depending on whether the region for the anti-neutrino mode is to the right of that for the neutrino mode. Therefore, if \( P/f^2 < \bar{P}/\bar{f}^2 \), then, putting \( \delta = +\pi/2 \) in (11) and (12) and assuming \( xf > yg \), which should

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Figure B.1. The region of constant probabilities at the oscillation maximum. (a) Each red (blue) line stands for $P(\nu_\mu \to \nu_e) = \text{const.}$ ($P(\bar{\nu}_\mu \to \bar{\nu}_e) = \text{const.}$) for a specific value of $\delta$. The red line on the right (left) edge corresponds to $\delta = +\pi/2$ ($\delta = -\pi/2$), whereas the blue line on the edge right (left) corresponds to $\delta = -\pi/2$ ($\delta = +\pi/2$). (b) When $\delta$ varies from 0 to $2\pi$, the line $P(\nu_\mu \to \nu_e) = \text{const.}$ sweeps out the yellow region, whereas the line ($\bar{\nu}_\mu \to \bar{\nu}_e) = \text{const.}$ sweeps out the light blue region. The black straight line, which is given by $P(\nu_\mu \to \nu_e) = \text{const.}$ and $P(\bar{\nu}_\mu \to \bar{\nu}_e) = \text{const.}$, lies on the overlapping green region.
hold if $\sin^2 2\theta_{13}$ is not very small, we get

$$\sqrt{P} = xf - yg = f \sqrt{\frac{X}{Y} - \sqrt{C \left(1 - \frac{1}{Y}\right)}}, \quad \sqrt{\bar{P}} = xf + yg = f \sqrt{\frac{X}{Y} + \sqrt{C \left(1 - \frac{1}{Y}\right)}}$$

which lead to the minimum value of $Y$,

$$Y^{(n)}_{\text{min}} = \left[1 - \frac{(f \sqrt{\bar{P}} - \bar{f} \sqrt{P})^2}{C(f + \bar{f})^2}\right]^{-1}$$

for the normal hierarchy. On the other hand, for the inverted hierarchy, the corresponding values of $\delta$ for the edges for the two modes are the same as those for the normal hierarchy ($\delta = \pm \pi/2$). Hence, if $P/\bar{f}^2 < \bar{P}/f^2$, then putting $\delta = +\pi/2$ in (13) and (14) and assuming $x \bar{f} > yg$, we obtain

$$\sqrt{P} = x \bar{f} - yg, \quad \sqrt{\bar{P}} = xf + yg,$$

which leads to the minimum value of $Y$,

$$Y^{(i)}_{\text{min}} = \left[1 - \frac{(\bar{f} \sqrt{\bar{P}} - f \sqrt{P})^2}{C(f + \bar{f})^2}\right]^{-1}$$

for $\Delta m_{31}^2 < 0$.

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