NUCLEATION RATES IN FLAT AND CURVED SPACE

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Abstract

Nucleation rates for tunneling processes in Minkowski and de Sitter space are investigated, taking into account one loop prefactors. In particular, we consider the creation of membranes by an antisymmetric tensor field, analogous to Schwinger pair production. This can be viewed as a model for the decay of false (or true) vacuum at zero temperature, in the thin wall limit. Also considered is the spontaneous nucleation of strings, domain walls and monopoles during inflation. The instantons for these processes are spherical world-sheets or world-lines embedded in flat or de Sitter backgrounds. We find the contribution of such instantons to the semiclassical partition function, including the one loop corrections due to small fluctuations around the spherical worldsheet. We suggest a prescription for obtaining, from the partition function, the distribution of objects nucleated during inflation. This can be seen as an extension of the usual formula, valid in flat space, according to which the nucleation rate is twice the imaginary part of the free energy. In addition, we use the method of Bogolubov transformations to compute the rate of pair production by an electric field in 1+1 dimensional de Sitter space, and compare the results to those obtained using the instanton method. Both results agree where they are expected to, not only in the exponential dependence but also in the prefactor, confirming the validity of instanton techniques in de Sitter space. Throughout the paper, both the gravitational field and the antisymmetric tensor field are assumed external.
1 Introduction

A wide class of non-perturbative phenomena in field theory can be understood in terms of quantum tunneling. A well known example is the decay of false vacuum: the materialization of bubbles of true vacuum in first order phase transitions [1, 4]. Lower dimensional versions of this process have been used to model the decay of metastable topological defects, such as domain walls and strings [3].

A closely related phenomenon is the neutralization of the cosmological constant through membrane creation. In a spacetime of dimension $d = N + 1$, an antisymmetric tensor field $A$ of rank $N$ induces a cosmological constant. This is because the corresponding field strength $F = dA$ has only one independent component, which has to be constant in the absence of sources. Just as an electric field decays through Schwinger pair creation, this cosmological constant decays through membrane creation if $A$ is coupled to a membrane, a process first described by Brown and Teitelboim [4].

Such tunneling processes can happen in flat as well as in curved spacetime. In addition, in curved spacetime, new effects can arise. It has been shown that topological defects such as circular loops of string, spherical domain walls and monopole-antimonopole pairs can spontaneously nucleate during inflation in the early universe [5]. These nucleations are somewhat analogous to particle production by an external gravitational field. Another consequence of spacetime curvature is the possibility of true vacuum decay [6], through nucleation of false vacuum bubbles.

Nucleation processes can be described using the instanton methods [2]. The instantons are classical solutions of the Euclidean equations of motion, with appropriate boundary conditions. They are saddle points of the Euclidean path integral, and as such they provide the basis for a semiclassical evaluation of the partition function. The contribution of one instanton to the path integral has the form

$$A e^{-S_E},$$

(1)

where $S_E$ is the Euclidean action of the instanton, and the prefactor $A$ arises from Gaussian integration over small fluctuations around the instanton. The main part of this paper will be devoted to the calculation of the prefactors $A$ for the class of processes mentioned above.
The instanton methods can be applied in flat and in curved backgrounds. One limitation of the formalism is, however, that the spacetime under consideration has to have real Euclidean sections. Here, we shall concentrate on de Sitter and Minkowski space. One reason for studying de Sitter space is that it describes the geometry of spacetime during inflation. A de Sitter space of dimension \( d \) can be defined as a hyperboloid embedded in a Minkowski space of dimension \( d + 1 \),

\[
\eta_{AB} X^A X^B = H^{-2}.
\]  

(2)

Here \( X^A \) are the coordinates in the embedding Minkowski space \((A = 0, \ldots, d)\), \( H \) is the expansion rate during inflation and \( \eta_{AB} \) is the Minkowski metric. We use the metric convention \((-,-,+\ldots,+)\). The Euclidean section of de Sitter space is obtained by analytically continuing the temporal coordinate \( X^0 \) to imaginary values

\[
X^0 = -i X^0_E
\]

(with \( X^0_E \) a real number). With this rotation the hyperboloid (2) becomes a \( d \)-sphere of radius \( H^{-1} \) embedded in flat Euclidean space (see Fig. 1).

In flat space, at zero or at finite temperature, the nucleation rates can be related to the imaginary part of the free energy \([7, 2]\), and they are essentially given by an expression of the form (1). In curved space, it is believed that the nucleation rates also have the exponential dependence (1), although the theory has not been developed to the same level of rigor than in flat space. When the size of the instantons is very small compared with \( H^{-1} \), one expects that the usual flat space formulas should apply. However, for the spontaneous nucleation of topological defects and for the nucleation of false vacuum bubbles, the size of the instanton is comparable to the horizon scale \( H^{-1} \). In such cases it is not clear how one should compute the nucleation rate, and one may even question whether such tunnelings can occur.

In this paper we take the heuristic point of view that these nucleations can indeed occur. We suggest a prescription for computing, from the semiclassical partition function, the equilibrium distribution of created membranes, bubbles and topological defects during inflation. This can be seen as a generalization of the formulas that one uses in flat space.

In 1+1 dimensions the process of membrane (or bubble) creation reduces to that of particle creation in an external field. In that case, the predictions of the instanton method can be compared with the results that one obtains by using the better understood method of Bogolubov transformations. As
we shall see, the results of both methods agree (in the limit where they are valid), even in the case when the size of the instanton is comparable to the horizon scale. Then, at least in 1+1 dimensions, the instanton prescription seems to be valid, and there is no reason to believe that it will not be valid in higher dimensions.

Finally, we should mention that throughout the paper, both the gravitational and the antisymmetric tensor fields are assumed to be external. As we shall see, the backreaction of the membrane on the antisymmetric field can be neglected when the charge $e$ of the membrane is very small compared to the field strength. In the context of false vacuum decay, this means that the difference in energy density between the vacua in the two sides of the membrane will be much smaller than the overall cosmological constant. Also, in order that gravitational field of the membrane be negligible, the mass scale of the membrane should be sufficiently small (see e.g. [3] for a comparison of instantons with and without self gravity). Gravitational backreaction effects are interesting in their own right, leading to qualitatively different behaviour at large mass scales, but introducing new complications and problems [8, 9]. These will be left as subject for future research.

The plan of the paper is the following. In section 2 we study the instantons for membrane production. These are the same as the ones studied in Ref. [4], in the limit of negligible self gravity and small charge. These instantons are essentially spherical Euclidean worldsheets of dimension $N$ and radius $R_0$ (representing the membrane) embedded in Euclidean de Sitter space, which is itself a sphere of radius $H^{-1}$ and dimensionality $d = N + 1$. The radius $R_0$ is determined by the strength of the antisymmetric field, the charge of the membrane, its surface tension and the expansion rate $H$. Special attention is paid to the analytic continuation of the instantons to Lorentzian signature, and the effect of de Sitter transformations on the resulting solutions. These Lorentzian solutions describe the motion of the membranes after nucleation.

As mentioned above, to calculate the prefactor $A$ we need to integrate over small fluctuations around the instanton. In Section 3 the theory of such fluctuations is reviewed, with an emphasis in the so-called zero modes. The zero modes are perturbations which do not change the shape of the instanton, but correspond to infinitessimal translations of the solution as a whole. We make use of the covariant formalism developed in Refs. [10, 11, 12], according to which the worldsheet fluctuations are represented by a scalar field $\phi$ ‘living’ in the unperturbed worldsheet. This scalar field has the meaning of a normal
displacement of the worldsheet.

In Section 4, we briefly review the instantons for the spontaneous nucleation of topological defects during inflation. These can be seen as a limiting case of the ones for membrane creation, when the external antisymmetric field is switched off. However, in this case the co-dimension of the defect’s worldsheet can be larger than one and additional perturbations have to be considered. For later use in the paper, we also discuss the instantons for a massive particle at finite temperature and for pair production in 3+1 dimensions. We study fluctuations around these instantons and give the normalization of the zero modes.

Section 5 is devoted to the semiclassical evaluation of the partition function. It is shown that the evaluation of the prefactor is formally equivalent to the evaluation of the effective action for a free scalar field in curved space-time (this curved space-time is the worldsheet of the instanton; a sphere, in our case). To illustrate the method, the instanton formalism is used to compute the partition function for a gas of massive particles at finite temperature, in the semiclassical limit $\mathcal{M} \gg T$, where $\mathcal{M}$ is the particle’s mass and $T$ is the temperature.

In section 6 we compute nucleation rates in flat space, recovering known results for pair creation in 1+1 and 3+1 dimensions, and for bubble formation in 2+1 and 3+1 dimensions. The question of renormalization is briefly discussed in analogy with the renormalization of the effective action for a scalar field in curved space.

In section 7 we discuss the nucleation rates in curved spacetime. We find the size distribution of created membranes, bubbles and defects during inflation. We also give the momentum distribution for the case of pair creation.

In section 8 we study the quantization of a charged scalar field interacting with an external electric field in 1+1 dimensional de Sitter space. The spectrum of particles created by the electric and gravitational field is computed using the method of Bogolubov transformations.

Some conclusions are summarized in Section 9. The computation of functional determinants on the N-sphere, necessary for the evaluation of the instanton prefactors, is done in the Appendix.
2 Production of membranes by an antisymmetric tensor field.

In this section we describe the instantons for the creation of membranes by an antisymmetric tensor field and their analytic continuation to Lorentzian signature. The instantons discussed in subsection 2.3 also represent the formation of true (or false) vacuum bubbles in the thin wall limit.

2.1 Particle coupled to an electric field in 1+1 dimensions

For the main part of the paper, the antisymmetric tensor field will be assumed external. However, it is instructive to start our discussion with the 1+1 dimensional case and treating the electric field as dynamical. The action for a spinless particle of mass $M$ and charge $e$ interacting with a Maxwell field $A_\mu$ in 1+1 dimensions is given by

$$S = -M \int_\Sigma ds + e \int_\Sigma A_\mu dx^\mu - \frac{1}{4} \int d^2 x \sqrt{-g} F_{\mu\nu} F^{\mu\nu} + \int d^2 x \partial_\mu (\sqrt{-g} F^{\mu\nu} A_\nu).$$

Here $x^\mu(s)$ is the particle’s trajectory and $ds$ is the proper time interval. The particle’s world-line is indicated by $\Sigma$ and $g$ denotes the determinant of the spacetime metric $g_{\mu\nu}$. The last term is a boundary term, included to yield a well defined equation for $A_\mu$. (This is necessary since we shall consider situations in which the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ does not vanish at infinity.)

Using the notation of forms, $A = A_\mu dx^\mu$, the field strength is $F = dA$. In 1+1 dimensions $F$ has only one independent component, the electric field $E$,

$$F = -E \tilde{\epsilon}$$

where

$$\tilde{\epsilon} = \sqrt{|g|} \epsilon_{\mu\nu} dx^\mu \wedge dx^\nu.$$

Eq.(4) is just saying that in 1+1 dimensions any 2-form is proportional to the “volume” form $\tilde{\epsilon}$. (Here $\epsilon_{\mu\nu}$ is the antisymmetric symbol, with $\epsilon_{01} = 1$.)
Variation of (3) with respect to $A_\mu$ yields the Maxwell equation

$$\partial_\mu(\sqrt{-g}F^{\mu\nu}) = -e \int ds \delta^2(x - x(s)) \frac{dx^\nu(s)}{ds}.$$  

(6)

This equation implies that in the absence of sources the electric field is constant

$$E = \text{const.}$$

whereas the effect of a charged particle is to produce a discontinuity

$$\Delta E = e$$

in the electric field as we cross from one side of the worldline to the other. Therefore, the only equation for $A_\mu$ is Gauss’ law, which is an equation of constraint [13]. In this sense $A_\mu$ does not actually have any field degrees of freedom in 1+1 dimensions.

As is well known, a constant electric field will decay by producing pairs of charged particles, which will then accelerate away from each other, screening the electric field away. The rate of pair production can be computed using the instanton methods. For this, it is necessary to solve the Euclidean equations of motion. The Euclidean action $S_E$ can be found by complexifying the temporal coordinate $x^0 \rightarrow -ix^0_E$, $ds \rightarrow -ids_E$ and leaving the field strength $F^{\mu\nu} \rightarrow F_E^{\mu\nu}$ unchanged [4]. From $F^{01} = \partial^0 A^1 - \partial^1 A^0$ one has to complexify the vector potential $A^0 \rightarrow A^0_E$, $A^1 \rightarrow iA^1_E$. With such rotations the Euclidean action is found to be $(iS \rightarrow -S_E)$,

$$S_E = M \int \Sigma ds + e \int \Sigma A_\mu dx^\mu - \frac{1}{4} \int d^2 \sqrt{g} F_{\mu\nu} F^{\mu\nu} + \int dx^2 \partial_\mu(\sqrt{-g}F^{\mu\nu} A_\nu).$$  

(7)

Here we have dropped the Euclidean subscripts. Expanding the derivative in the last term and using (3) one has

$$S_E = M \int \Sigma ds + \frac{1}{4} \int d^2 x \sqrt{g} F_{\mu\nu} F^{\mu\nu},$$

which is positive definite.

Unlike the Lorentzian case, in Euclidean space we can consider closed worldlines, and actually these will be the ones relevant to pair production. A closed world-line divides spacetime into two regions. Following [4] we denote
them as the “inside” and the “outside” of the worldline (see Fig. 1). Note that in flat space there is a natural way to assign these labels, but in de Sitter space (that is on the 2-sphere) these denominations are just conventional. We denote by “outside” the region in which, upon analytic continuation, the electric field does not change when the pair is created, maintaining its initial value \( E_0 \). By Gauss’ law, the field in the inside region is given by \( E_i = E_0 \pm e \). The double sign reflects the fact that the worldline has to be assigned a direction in which the charge flows, and in principle both directions are possible. Without loss of generality, we take the minus sign, \( E_i = E_0 - e \), since a change in the direction of the worldline amounts to a change in the sign of \( e \).

Using \( F_{\mu \nu} F^{\mu \nu} = 2E^2 \), we have

\[
S_E = \mathcal{M} \int_\Sigma ds - (eE_0 - \frac{e^2}{2}) \int_\mathcal{V} \dot{\epsilon}.
\]

Here \( \mathcal{V} \) is the volume of the “inside” region. In (9) we have dropped the constant term

\[
\frac{1}{4} \int d^2x \sqrt{g} 2E_0^2.
\]

This term is just the Euclidean action for the background configuration, without the instanton. Of course this constant is physically meaningless because a constant can always be added to the action. It is customary to choose the constant so that the action for the background configuration vanishes [2]. To summarize, the action (9) is proportional to the length of the worldline minus a term proportional to the area enclosed by the worldline.

### 2.2 Antisymmetric tensor field coupled to a membrane

Let us now consider the natural generalization of (3), which describes an antisymmetric tensor field of rank \( N \)

\[
A = A_{[\mu...\rho]} dx^\mu \wedge ... \wedge dx^\rho
\]

interacting with the \( N \) dimensional worldsheet of a charged membrane in a spacetime of dimension \( d = N + 1 \) [4]. To keep the discussion simple, we shall
consider the situation in which the field $A$ is external, so that backreaction of the membrane on the field is ignored. This will facilitate also the comparison with the method of Bogolubov coefficients in 1+1 space-time dimensions, since with that method the electric field has to be treated as external anyway.

The action is now given by

$$S = -\mathcal{M} \int_\Sigma \sqrt{-\gamma} d^N \xi + e \int_\Sigma A.$$  \hspace{1cm} (10)$$

The first term is the Nambu action, proportional to the area of the worldsheets $\Sigma$, where $\gamma$ is the determinant of the worldsheet metric $\gamma_{ab}$, and $\xi^a$ ($a = 0, ..., N - 1$) is a set of coordinates on $\Sigma$. In (10) $\mathcal{M}$ is just a constant. For $N=1$ this constant is the particle mass, for $N=2$ it is the tension of a string, and for $N = 3$ it is the surface tension of a membrane. The second term in (10) is the generalization of the electromagnetic coupling $e \int A_\mu d\xi^\mu$ that we used in (3). Eq. (10) can be Euclideanized using prescriptions analogous to the ones that led from (3) to (7), yielding

$$S_E = \mathcal{M} \int_\Sigma \sqrt{-\gamma} d^N \xi + e \int_\Sigma A.$$  \hspace{1cm} (11)$$

As with the 1+1 dimensional case, the field strength associated with $A$, $F = dA$, has only one independent component, and can be written in the form (4), where now the antisymmetric symbol $\epsilon$ has $N + 1$ indices.

For closed worldsheets we can use Stokes’ theorem and (4) in (11) to find

$$S_E = \mathcal{M} \int_\Sigma \sqrt{-\gamma} d^N \xi - e E_0 \int_V \tilde{\epsilon},$$  \hspace{1cm} (12)$$

where we have used a constant external electric field $E_0$. Without loss of generality we take $E_0 > 0$.

The integral in the last term is just the volume of the space-time region “inside” the closed worldsheet. Eq. (12) has the same form as (9) if the term proportional to $e^2$ is neglected. Therefore one can neglect backreaction when $|e| << E_0$.

Notice that (12) is proportional to the area of the worldsheets minus a term proportional to the volume enclosed by the worldsheets. This has exactly the same form as the Euclidean action for the process of false vacuum decay.
through bubble nucleation \[2\], in the limit in which the thickness of the wall separating the true from the false vacua is much smaller than the radius of the bubble. Both processes are similar in many respects \[4\], the main difference being that membrane production by an antisymmetric tensor field can occur repeatedly at any given point in space, whereas vacuum decay occurs only once.

### 2.3 The instantons

The equation of motion following from (12) has been given for instance in Refs \[10, 12\]

\[
K_a^a = \gamma^{ab} K_{ab} = -\frac{eE_0}{\mathcal{M}}. \tag{13}
\]

where \(K_{ab}\) is the extrinsic curvature of the Euclidean worldsheet

\[
K_{ab} \equiv -e_{b\mu} \nabla_a n^\mu. \tag{14}
\]

Here \(n^\mu\) is the normal to the worldsheet, and \(e^a_b = \partial_b x^a(\xi)\) are the tangent vectors (our sign convention is that \(n^\mu\) points towards the outside region.)

In 1+1 dimensional flat spacetime, the only solution of (13) is a circular worldline of radius \(R_0 = \mathcal{M}/eE_0\),

\[
(x_E^0)^2 + (x^1)^2 = R_0^2. \tag{15}
\]

Since we have chosen \(E_0 > 0\), in order for \(R_0\) to be positive we need

\(e > 0\).

From (8), this means that the electric field inside the circle will be smaller than outside.

The evolution of the pair after nucleation is given by the analytic continuation of (13) to Minkowski space

\[
-(x^0)^2 + (x^1)^2 = R_0^2. \tag{16}
\]

This hyperbola has two branches. The branch on the right represents a particle of charge \(e\) moving forward in time. The one on the left represents a particle of charge \(-e\) moving backward in time. This is interpreted in the usual way as an antiparticle of charge \(-e\) moving forward in time. The particle and
antiparticle pair nucleate at time $x^0 = 0$, separated by a distance $2R_0$ and with zero velocity. After that, due to the constant force exerted by the field, they start moving away from each other with constant proper acceleration $R_0^{-1}$.

In higher dimensions there are also $N$–spherical worldsheets which are extrema of the action. These represent the nucleation of spherical membranes \[4\]. Let us consider directly the instantons in de Sitter space of radius $H^{-1}$. The flat space instantons can be obtained as the limiting case $H \to 0$.

With a spherical ansatz for the worldsheet the action \[12\] takes the form

$$S_E = \mathcal{M}S_N(R_0) - eE_0\mathcal{V}_N(\theta_0).$$  \hspace{1cm} (17)

Here

$$S_N(R_0) = \frac{2\pi^{N+1}}{\Gamma \left( \frac{N+1}{2} \right)} R_0^N$$  \hspace{1cm} (18)

is the surface of a worldsheet of radius $R_0$, $\theta_0$ is the polar angle on the $d$-sphere of radius $H^{-1}$ (see Fig. 1) such that

$$R_0 = H^{-1} \sin \theta_0,$$

and $\mathcal{V}_N(\theta_0)$ is the volume of the $d$-sphere that is enclosed by the worldsheet of radius $R_0$:

$$\mathcal{V}_N(\theta_0) = \frac{2\pi^{N+1}}{\Gamma \left( \frac{N+1}{2} \right)} H^{-(N+1)} \int_0^{\theta_0} \sin^N \theta d\theta.$$  \hspace{1cm} (19)

Extremizing (17) with respect to $\theta_0$

$$\frac{dS_E}{d\theta_0} = 0$$

we find

$$\tan \theta_0 = NH \frac{\mathcal{M}}{eE_0},$$  \hspace{1cm} (20)

which means that the radius of the Euclidean worldsheet is

$$R_0 = \frac{\mathcal{N} \mathcal{M}}{(N^2H^2\mathcal{M}^2 + e^2E_0^2)^{1/2}}.$$  \hspace{1cm} (21)
Substituting (21) back into (17) we find the Euclidean action for the instantons

\[ S_E = 2\pi H^{-2} \left[ (\mathcal{M}^2 H^2 + e^2 E_0^2)^{1/2} - eE_0 \right] \quad (N = 1) \]  

\[ S_E = 4\pi H^{-3} \left[ \mathcal{M}H - \frac{eE_0}{2} \arctan \left( \frac{2HM}{eE_0} \right) \right] \quad (N = 2) \]  

\[ S_E = \frac{2}{3} \pi^2 H^{-4} \left[ \frac{9H^2 \mathcal{M}^2 + 2e^2 E_0^2}{(9H^2 \mathcal{M}^2 + e^2 E_0^2)^{1/2}} - 2eE_0 \right] \quad (N = 3) \]  

An interesting feature of Equations (22-24) is that one finds instantons of finite action both for \( e > 0 \) and \( e < 0 \).

For \( e > 0 \) the electric field in the inside region decreases with respect to the initial value [see(8)]. From (20) this corresponds to \( \theta_0 < \frac{\pi}{2} \). For \( e < 0 \) the electric field in the inside region actually increases with respect to the initial value. This corresponds to \( \theta_0 > \frac{\pi}{2} \). In this case the “inside” region is actually larger than the “outside” one (see Fig. 1). Strictly speaking, if the electric field is treated as external, we should not say that the field increases or decreases in the inside region. However, both cases should still be distinguished. For instance, in 1+1 dimensions, the instanton with \( e > 0 \) corresponds to the creation of a pair with the “screening” orientation. That is, after nucleation, the + charge is to the right of the inside region and the − charge is to the left (recall our convention \( E_0 > 0 \)). On the other hand, for \( e < 0 \) the pair has the “anti-screening” orientation, with the + charge to the left and the − charge to the right. Similarly membranes can nucleate with two different orientations depending on the sign we take for \( e \).

It might appear that the particles in pairs with the anti-screening orientation would move towards each other after nucleation, and eventually annihilate each other. However, as we shall see, the distance between both particles actually grows with time due to the inflationary expansion. We should emphasize that the anti-screening instantons are just as physical as the screening ones. In order to find agreement with the results obtained using Bogolubov transformations, both instantons will have to be included. In the context of vacuum decay, the case \( e > 0 \) corresponds to the ordinary transition from false to true vacuum, whereas the case \( e < 0 \) corresponds to the decay of the true vacuum through nucleation of false vacuum bubbles [6].

In the limit when the electric field is switched off, \( E_0 \to 0 \), the instantons become spheres of maximal radius \( R_0 \to H^{-1} \), and the action (22-24) reduces
to

\[ S_E = M S_N(H^{-1}). \]  

(25)

These are the instantons for the spontaneous nucleation of defects during inflation \( [5] \), which we shall consider in more detail later on.

In general, for finite \( E_0 \), the action for \( eE_0 > 0 \) is always smaller than that for \( eE_0 < 0 \). This means that it is more probable to nucleate a screening membrane than an anti-screening one, in agreement with naive expectations.

In the flat space limit \( H \to 0 \), the anti-screening process has infinite action, and therefore it is not possible. Only the action for the screening instanton \( e > 0 \) remains finite. For \( e > 0 \) and \( H \to 0 \) we have

\[ R_0 = \frac{N M}{e E_0} \]  

(26)

and

\[ S_E = \frac{M}{N + 1} S_N(R_0). \]  

(27)

This expression reproduces the thin wall instanton action for vacuum decay in flat space \( [2] \), where \( M \) is the tension of the wall and \( e E_0 \) is the difference in energy density between the false and true vacua.

2.4 Analytic continuation

The evolution of the membranes after nucleation is given by the analytic continuation of the instantons back to Lorentzian signature. We have seen that, for the \( d = 1 + 1 \) case in flat space the instanton is a circle, and the analytic continuation is a hyperbola representing the world-line of the particle and antiparticle accelerating away from each other. Note that the hyperbola \( (16) \) is centered at the origin \( x^1 = 0 \), but of course pairs can nucleate at other locations too. If we act on the instanton \( (15) \) with a space-time translation, the resulting trajectory

\[ (x^0_E - a_E)^2 + (x^1 - b)^2 = R_0^2 \]

is also an instanton. Thus \( (15) \) is just one solution out of a two parameter family. By analytically continuing \( x^0_E \to i x^0 \) and \( a_E \to i a \), we obtain a two parameter family of Lorentzian solutions

\[ -(x^0 - a)^2 + (x^1 - b)^2 = R_0^2, \]  

(28)

13
which represent pairs nucleating at any space-time point \( x^\mu = (a, b) \).

Similar steps have to be taken to analytically continue the instantons describing the creation of membranes in de Sitter. As mentioned before these are \( N \)-spherical worldsheets of radius \( R_0 \) embedded in \( d \)-sphere of radius \( H^{-1} \). The instantons can be represented as the intersection of the \( d \)-sphere with a hyperplane at a distance

\[
\omega_0 \equiv H^{-1} \cos \theta_0
\]

from the origin, where \( \theta_0 \) is given by (20), see Fig. 1. In the representation (2), the instanton is given by

\[
(X_E^0)^2 + \sum_{J=1}^{d} (X^J)^2 = H^{-2} \]

\[
X^d = \omega_0, \tag{29}
\]

where \( X^A \) are the coordinates in a ‘fictitious’ embedding Euclidean space.

Following [3], to analytically continue this solution we choose the flat Friedman-Robertson-Walker (FRW) coordinates in de Sitter space, \((t, \bar{x})\). In these, the metric takes the form

\[
ds^2 = -dt^2 + e^{2Ht} d\bar{x}^2. \tag{30}\]

These coordinates are related to \( X^A \) through the relations

\[
X^0 = H^{-1} \sinh Ht + \frac{1}{2} H \bar{x}^2 e^{Ht},
\]

\[
X^d = H^{-1} \cosh Ht - \frac{1}{2} H \bar{x}^2 e^{Ht}, \tag{31}
\]

\[
\vec{X} = \bar{x} e^{Ht},
\]

where the vector \( \vec{X} \) has components \( X^J \), \((J = 1, ..., d - 1)\). The coordinates \((t, \bar{x})\) cover only half of the hyperboloid (2).

Taking \( X^d = \omega_0 \) in (31), the worldsheet of the membrane after nucleation is given by

\[
\bar{x}^2 = H^{-2}(1 + e^{-2Ht}) - 2H^{-1}\omega_0 e^{-Ht}. \tag{32}\]
This solution represents a spherical membrane which is expanding in time, with physical radius given by
\[ R^2 = H^{-2}(e^{2Ht} + 1) - 2H^{-1}\omega_0 e^{Ht}. \] (33)

Note that \( \text{sign}(\omega_0) = \text{sign}(e) \), but \( R \) never vanishes for either sign of \( e \). In both cases, the radius grows like the scale factor at late times. For \( N = 1 \) the spherical "membrane" reduces to a pair of points, whose worldline is given by \( x = \pm(\vec{x}^2)^{1/2} \), with \( \vec{x}^2 \) given by (32). In this case \( R \) is one half of the physical distance between the particle and antiparticle in the pair.

As with the flat space case discussed above, the solution (32) belongs to a family of solutions which can be obtained from a \( d \)-parameter family of instantons. This family is obtained by applying \( O(d + 1) \) rotations to the instanton (29). The group \( O(d + 1) \) has \( d(d + 1)/2 \) generators. Of these, \( d(d - 1)/2 \) leave the instanton invariant. They correspond to rotations in the space \( (X^0_E, \vec{X}) \). The remaining \( d \) generators correspond to rotations in the \( (X^d, X^0_E) \) plane or in any of the \( (X^d, X^{i'}) \) planes (\( i = 1, \ldots, d - 1 \)). These generators which do not leave the instanton invariant are the so called zero modes. Their effect is to rotate the hyperplane \( X^d = \omega_0 \) in (29), effectively translating the center of the worldsheet to a new location on the \( d \)-sphere.

Upon analytic continuation the parameters corresponding to rotations in the \( (X^0_E, X^J) \) plane (\( J \neq 0 \)) have to be complexified along with \( X^0_E \) in order for the resulting solutions to be real. Recall that even in flat space, the parameter \( a_E \) had to be complexified to obtain (28). In the present case, rotations turn into boosts in the \( (X^0, X^J) \) plane when the angle \( \alpha_E \) of rotation is complexified,
\[ \alpha_E = i\alpha. \] (34)

In this way, the group of rotations \( O(d + 1) \) becomes the group of de Sitter transformations \( O(d, 1) \), which can also be thought of as the group of Lorentz transformations in the Minkowski space in which the hyperboloid (3) is embedded.

Let us consider the case \( d = 1 + 1 \) in some detail. The general worldline after nucleation is obtained by taking \( X^0_E \rightarrow iX^0 \) in (29) and then applying a boost in the \( (X^0, X^2) \) plane followed by a rotation in the \( (X^1, X^2) \) plane
\[ X'^0 = X^0 \cosh \alpha + X^2 \sinh \alpha \] \[ X'^1 = -(X^0 \sinh \alpha + X^2 \cosh \alpha) \sin \beta + X^1 \cos \beta \] (35)
\[ X'^2 = (X^0 \sinh \alpha + X^2 \cosh \alpha) \cos \beta + X^1 \sin \beta. \]

Here \( \alpha \) and \( \beta \) are arbitrary parameters.

Eliminating \( X^1 \) and \( X^0 \) from the previous equations one has

\[ X'^2 \cosh \alpha \cos \beta - X'^1 \cosh \alpha \sin \beta - X^0 \sinh \alpha = X^2. \]

Taking \( X^2 = \omega_0 \), dropping primes and using the transformations (31) we find, after some algebra, that the general world-line after nucleation is given by

\[ R^2 \equiv e^{2Ht}(x - x_0)^2 = H^{-2}(e^{2H(t-t_0)} + 1) - 2H^{-1}\omega_0 e^{H(t-t_0)}, \quad (36) \]

where

\[ x_0 = H^{-1}\frac{\cosh \alpha \sin \beta}{\cosh \alpha \cos \beta + \sinh \alpha}, \quad (37) \]

\[ t_0 = H^{-1}\ln(\cosh \alpha \cos \beta + \sinh \alpha). \quad (38) \]

Eq. (36) represents a pair centered at the point \( x_0 \). The parameter \( t_0 \) shall be referred to as the time of nucleation. For \( R_0 \sim H^{-1} \) this is somewhat conventional, since there is no precise instant of time at which the pair nucleates [5]. The problem is that the concept of simultaneity becomes blurry at distances comparable to the horizon. Strictly speaking, all we can say is that solutions with different values of \( t_0 \) are time translations of one another. From a geometric point of view, however, it is clear that \((x_0, t_0)\) represents the center of symmetry of the worldsheet, and in this sense it is natural to think of it as the nucleation event.

Note also that there is no absolute value sign in the logarithm in (38). For \( \omega_0 \neq 0 \), the solutions with \( \cosh \alpha \cos \beta + \sin \alpha > 0 \) are qualitatively different from the ones with \( \cosh \alpha \cos \beta + \sinh \alpha < 0 \). Actually, we shall see that the latter ones are unphysical. They correspond to pairs whose “inside” region is centered at spatial infinity, so upon nucleation the electric field would change over an infinite (and disconnected) region of space. To find agreement with the Bogolubov method these solutions have to be discarded.

The same arguments can be repeated for \( N > 1 \). The general Lorentzian solution is a spherical membrane of physical radius (33) centered at any spacetime point [5].
3 Perturbations and zero modes

To compute the contribution of the instantons to the partition function we need to study small fluctuations of the instanton worldsheet. For this it is very useful to adopt the covariant formalism developed in Refs. [10, 11, 12, 14], according to which the worldsheet perturbations are represented by a scalar field “living” on \( \Sigma \), which has the meaning of normal displacement.

Denoting by \( x^\mu \) the coordinates in de Sitter space, the instanton configuration will be denoted by \( x^\mu(\xi^a) \). Consider now a slight deformation of the worldsheet

\[
\tilde{x}^\mu(\xi^a) = x^\mu(\xi^a) + \delta x^\mu(\xi^a). \tag{39}
\]

Since only deformations orthogonal to the worldsheet are physically meaningful, we can set

\[
\delta x^\mu(\xi^a) = M^{-1/2} \phi(\xi^a)n^\mu. \tag{40}
\]

Here \( n^\mu \) is the normal to the worldsheet and \( \phi(\xi^a) \) has the meaning of a normal displacement. The factor \( M^{-1/2} \) is inserted so that \( \phi \) has the correct dimensions for a scalar field in \( N \) dimensions, \([\phi] = (\text{mass})^{(N/2)-1}\).

Actually, the equation of motion for \( \phi \) can be derived from kinematical considerations. Since \( \phi \) is a scalar on the worldsheet, it has to satisfy a covariant equation. The only tensors available in \( \Sigma \) are the metric \( \gamma_{ab} \) and the extrinsic curvature \( K_{ab} \), but because of the symmetries of our problem they are proportional to each other \( K_{ab} \propto \gamma_{ab} \) (see e.g. [5]). The only covariant second order differential equation that we can write down with such ingredients is

\[
- \Delta \phi + M^2 \phi = 0, \tag{41}
\]

where \( \Delta \) is the Laplacian on the spherical worldsheet. By symmetry, \( M \) has to be constant.

To determine the value of \( M \) we can use “known” solutions of (41): the zero modes. These are field modes for \( \phi \) which do not correspond to true perturbations of \( \Sigma \), but to the infinitessimal version of the rotations considered in the previous section. These solutions can be found from geometric considerations. Let the instanton be given by [see (29)]

\[
(X^0_E)^2 + \sum_{J=1}^{d} (X^J)^2 = H^{-2} \tag{42}
\]
\[ X^d = \omega_0 = (H^{-2} - R_0^2)^{1/2}, \]

where, as usual, \( X^A \) are cartesian coordinates in the flat space in which the Euclidean de Sitter space is embedded.

The vector \( n^\mu \), orthogonal to \( \Sigma \), can be thought of as a vector in this embedding flat space, with components \( n^A \). As such, it is tangent to the \( d-1 \)-sphere of radius \( H^{-1} \) centered at the origin, and orthogonal to the \( N-1 \)-sphere of radius \( R_0 \) centered at \( X^d = \omega_0, X^I = 0 \) \( (I \neq d) \). It is easy to see that the normal vector is given by

\[ n^A = HR_0^{-1}(\vec{X}_\omega_0, -R_0^2). \]

Here \( \vec{X} \) has components \( X^I \), \( (I \neq d) \). A small rotation of angle \( \alpha \) in the \( (X^J, X^d) \) plane induces the change

\[ \delta X^J = \alpha X^d, \]
\[ \delta X^d = -\alpha X^J. \]

This transforms \( \Sigma \) into a new worldsheet which is also a solution of the equations of motion. Therefore, taking \( X^d = \omega_0 \) in the equations above, the field

\[ \mathcal{M}^{-1/2}\phi(\xi^a) \equiv n^A\delta X_A(\xi^a) = \alpha(HR_0)^{-1}X^J(\xi^a) \]

has to be a solution of (41), for any \( J = 0, \ldots, d-1 \). As is well known, the cartesian components \( X^J(\xi^a) \) of the points on the \( N-1 \)-sphere are linear combinations of the spherical harmonics with \( L = 1 \). The spherical harmonics are eigenfunctions of the Laplacian with eigenvalue

\[ \lambda_L = -L(L + N - 1)R_0^{-2}, \quad (L = 0, \ldots, \infty) \]

so, taking \( L = 1 \), we have \( \Delta \phi = -NR_0^{-2}\phi \). Comparing with (41) we have the mass that we were looking for

\[ M^2 = -NR_0^{-2}. \]

For \( E_0 \to 0 \) this reduces to \( M^2 = -NH^2 \), a result which was found already in [5].

Of course, equations (41) and (45) can also be derived from a perturbative expansion of the action. Introducing (39) in the action and expanding to
second order in $\phi$, the action for the perturbed worldsheet $\tilde{\Sigma}$ can be written as \[10, 11, 12\],

$$S_E[\tilde{\Sigma}] = \bar{S}_E[\Sigma] + S_E^{(2)}[\phi],$$

where $\bar{S}_E[\Sigma]$ is the action for the unperturbed instanton \[22-24\], and

$$S_E^{(2)}[\phi] = \frac{1}{2} \int d^N \xi \sqrt{\gamma}(-\phi \Delta \phi + M^2 \phi^2), \quad (46)$$

whith $\Delta$ the Laplacian on $\Sigma$ and

$$M^2 = \mathcal{R}^{(N)} - R^{(d)}_{\mu \nu} (g^{\mu \nu} - n^\mu n^\nu) - \left(\frac{eE_0}{M}\right)^2. \quad (47)$$

Here $\mathcal{R}^{(N)}$ is the Ricci scalar on $\Sigma$, and $R^{(d)}_{\mu \nu}$ and $g_{\mu \nu}$ are the Ricci tensor and the metric in the embedding de Sitter space [actually Eqs. (46) and (47) are valid for perturbations to any worldsheet solution, embedded in an arbitrary curved spacetime of dimension $d = N + 1$]. In de Sitter space (i.e. on the $d$-sphere) $R^{(d)}_{\mu \nu} = H^2 (d - 1) g_{\mu \nu}$, whereas on the $N$-spherical worldsheet of radius $R_0$, $\mathcal{R}^{(N)} = \gamma^{ab} R^{(N)}_{ab} = N(N - 1)R_0^{-2}$. Substituting in (47) and using (21) for $R_0$, the effective mass $M^2$ simplifies to

$$M^2 = -NR_0^{-2},$$

in agreement with (45).

For later convenience we expand an arbitrary perturbation $\phi$ in terms of the (real) spherical harmonics on the $N$-sphere, $\phi_{LJ}$

$$\phi(\xi^a) = \sum_{LJ} C_{LJ} \phi_{LJ}(\xi^a). \quad (48)$$

These satisfy $\Delta \phi_{LJ} = \lambda_L \phi_{LJ}$, with $\lambda_L$ given by (44) and

$$\int_{\Sigma} \phi_{LJ}^2 \sqrt{\gamma} d^N \xi = 1. \quad (49)$$

The index $J$ labels the degeneracy for given $L$

$$J = 0, ..., g_L - 1,$$

where

$$g_L = \frac{(2L + N - 1)(N + L - 2)!}{L!(N - 1)!}. \quad (50)$$
For $L = 1$ we have, with the normalization (49),

$$\phi_{1J} = \left( \frac{N + 1}{R_0^2 S_N(R_0)} \right)^{1/2} X^J(\xi),$$  \hspace{1cm} (51)

where $S_N$ is given by (18).

Comparing (43) with (48) and using (51) we find that for an infinitesimal rotation of angle $d\alpha_J$ in the $(X^j, X^n)$ plane

$$dC_{1J} = H^{-1} d\alpha_J \left( \frac{MS_N(R_0)}{N + 1} \right)^{1/2}.$$ \hspace{1cm} (52)

This equation is often referred to as the normalization of the zero modes, and it will be important in order to interpret certain divergences in the semiclassical evaluation of the partition function. In the flat space limit, the square root in the right hand side of (52) reduces to the familiar expression $S_E^{1/2}$.

4 Related instantons

The formalism used in the previous sections can be easily extended to study a few more instantons which are closely related to the ones we have seen so far, and which will be relevant for future discussion.

4.1 Nucleation of topological defects during inflation

We saw in Subsection 2.3 that for $H \neq 0$ the action for the instantons remains finite when the external field is switched off. This corresponds to the spontaneous nucleation of membranes (or topological defects), due to the gravitational field alone.

When the external field $E_0$ is zero, there is no need to restrict ourselves to co-dimension 1. Thus monopole pairs, strings and domain walls can spontaneously nucleate in $d$-dimensional de Sitter space, with $d > N$. The corresponding instantons are found by intersecting the $d$-sphere with the necessary number of hyperplanes through the origin:

$$\sum_{A=0}^{d} X^A X^A = H^{-2}$$ \hspace{1cm} (53)
\[ X^i = 0, \quad (i = N + 1, \ldots, n), \]
which gives N-spheres of radius \( H^{-1} \). (here we use the lower case latin index \( i \) for later notational convenience).

When studying small perturbations around the instanton for co-dimension larger than 1, we will have more scalar fields ‘living’ on the worldsheet, one for each normal direction. The generalization of (41) is

\[
\delta x^\mu (\xi^a) = M^{-1/2} \sum_{i=N+1}^{d} \phi^{(i)} (\xi^a) n^{(i)\mu}. \tag{54}
\]

Taking the normal vectors to be perpendicular to the hyperplanes \( X^i = 0 \), the effective action for \( \phi^{(i)} \) is [5, 10, 14]

\[
S^{(2)}[\phi] = \frac{1}{2} \sum_{i=N+1}^{d} \int dN \xi \sqrt{\gamma} \left[ \phi^{(i)}(-\Delta + M^2)\phi^{(i)} \right], \tag{55}
\]

where \( M^2 \) is still given by (45) with \( R_0 = H^{-1} \), i.e.

\[ M^2 = -NH^2. \]

The fields can be expanded in terms of spherical harmonics on the \( N \)-sphere, \( \phi^{(i)} = \sum C_{LJ}^{(i)} \phi_{LJ} \), and the normalization of the zero modes can be worked out in exactly the same way as before. For an infinitesimal rotation in the \((X^J, X^i)\) plane (where \( J = 0, \ldots, N, \quad i = N + 1, \ldots, d \)) of angle \( d\alpha_i^{(i)} \) we have

\[
dC_{1J}^{(i)} = H^{-1} d\alpha_i^{(i)} \left( \frac{MS_N(H^{-1})}{N+1} \right)^{1/2}. \tag{56}
\]

Rotations of the \( X^J \) among themselves or of the \( X^i \) among themselves leave the instanton invariant, so these will not correspond to zero modes. Then, the total number of zero modes is \((N + 1)(d - N)\).

In general, of the \((N + 1)(d - N)\) zero modes, \( d \) will correspond to space-time translations and the remaining \( N(d - N - 1) \) correspond to rotations in the spatial orientation of the defect [5]. For instance, monopole pairs in 3+1 dimensional de Sitter space can nucleate with all possible orientations of the relative position, and loops of string can nucleate with all possible orientations of the plane of the loop.

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4.2 Particle at finite temperature in flat space

For a massive particle at finite temperature the Euclidean action is

\[ S_E = M \int_0^\beta (1 + \dot{x}_E(t_E))^1/2 dt_E, \]  

(57)

where \( t_E \) is the Euclidean time and \( x(t_E) \) is the Euclidean trajectory, which has to be periodic \( x(t_E) = x(t_E + \beta) \). Here \( \beta \) is the inverse of the temperature. The instantons for this system are straight lines wrapping around the compact temporal dimension

\[ x(t_E) = x_0 = \text{const}. \]  

(58)

The second variation of the action is found by substituting \( x(t_E) = x_0 + \mathcal{M}^{-1/2} \tilde{\phi}(t_E) \) in (57),

\[ S_E^{(2)} = \frac{1}{2} \int_0^\beta (\dot{\tilde{\phi}})^2 dt_E. \]  

(59)

The fluctuations \( \tilde{\phi} \) can be thought of as a set of massless scalar fields in 0+1 dimensions. From (59), \( \tilde{\phi} \) satisfies \( \ddot{\tilde{\phi}} = 0 \). The zero mode solution \( \tilde{\phi} = \text{const.} \) amounts to a spatial translation of the original solution \( x = x_0 \).

Expanding \( \tilde{\phi} \) in “spherical harmonics” on the 1-sphere

\[ \tilde{\phi} = \beta^{-1/2} \tilde{C}_0 + \sqrt{\frac{2}{\beta}} \sum_{L=1}^\infty \left\{ \tilde{C}_L \cos \left[ \frac{L 2\pi t_E}{\beta} \right] + \tilde{D}_L \sin \left[ \frac{L 2\pi t_E}{\beta} \right] \right\}, \]  

(60)

we see that the zero mode is in the \( L = 0 \) sector. An infinitesimal translation \( d\tilde{x}_0 \) corresponds to

\[ d\tilde{C}_0 = d\tilde{x}_0 \sqrt{\mathcal{M} \beta}. \]  

(61)

The instanton (58) and the previous equation will be used to derive the partition function for a gas of massive particles at finite temperature.

4.3 Pair production in more dimensions

In curved space, \( E = \text{const} \) is a solution of the homogeneous Maxwell’s equations only in 1+1 dimensions. However, in flat space one can take \( E = \text{const} \) in arbitrary number of dimensions. For completeness, we shall briefly
consider the well known 3+1 dimensional case in flat space. The action for a particle in an external field is still given by

\[ S_E = \mathcal{M} \oint ds + e \oint A_\mu dx^\mu, \]  

(62)

where we are integrating over a closed worldline. If the electric field \( E \) is in the \( x \) direction \( \vec{E} = (E, 0, 0) \), we can take \( A_y = A_z = 0 \). It is clear from the previous sections that there will be an instanton solution which is a circle of radius \( R_0 = \mathcal{M}(eE)^{-1} \) in the \((t_E, x)\) plane, where \( t_E \) is Euclidean time. To find the second variation of the action, we take a perturbation to the instanton of the form \((\mathfrak{F})\). One of the normal vectors \( n^{(r)} \) is chosen in the outward radial direction from the origin of the \((t_E, x)\) plane. The corresponding perturbation is denoted by \( \phi_r(s) \), where \( s \equiv R_0 \arctan(t_E/x) \) parametrizes the worldline. The second variation of \( S_E \) with respect to \( \phi_r \) is independent of the existence of the two orthogonal directions \( y \) and \( z \), so from subsection a, the quadratic action for \( \phi_r \) is that of a scalar field of mass \( M^2 = -R_0^{-2} \) in 0+1 dimensions.

The other two normals are chosen in the \( y \) and \( z \) directions. The associated perturbations \( \phi_y(s) \) and \( \phi_z(s) \) do not change the second term in \((\mathfrak{F})\), since \( A_y = A_z = 0 \). Therefore, from the previous subsection these perturbations will be massless fields. Putting it all together

\[ S_E^{(2)} = \frac{1}{2} \int_0^{2\pi R_0} ds \left[ (\dot{\phi}_r^2 - R_0^{-2} \phi_r^2) + \dot{\phi}_y^2 + \dot{\phi}_z^2 \right]. \]  

(63)

The zero modes for the massless fields are in the \( L = 0 \) sector. They correspond to translations in the \( y \) and \( z \) directions and can be normalized from \((\mathfrak{F})\), with \( \beta = 2\pi R_0 \),

\[ \frac{dC_0^{(y)}}{dy} = \frac{dC_0^{(z)}}{dz} = \sqrt{2\pi R_0 \mathcal{M}}, \]  

(64)

The zero modes for \( \phi_r \) are in the \( L = 1 \) sector and they can be normalized from \((\mathfrak{F})\)

\[ \frac{dC_1^{(r)}}{dt_E} = \frac{dC_1^{(r)}}{dx} = \sqrt{\pi R_0 \mathcal{M}}, \]  

(65)

where we have replaced \( H^{-1}d\alpha_J \) by the corresponding flat space limit \( dt_E \) or \( dx \).
5 Semiclassical partition function

For systems at finite temperature, the lifetime of a metastable state is related to the imaginary part of the free energy [7, 2] [see Eq. (86) below]. The free energy is defined as

\[ F \equiv -\beta^{-1} \ln Z, \]  

(66)

where \( \beta^{-1} \) is the temperature and \( Z \) is the partition function

\[ Z \equiv \text{tr}[e^{-\beta \hat{H}}], \]  

(67)

with \( \hat{H} \) the Hamiltonian of the system. The key to the semiclassical evaluation of \( Z \) is to first express it as a path integral. For instance, for the case of a single non-relativistic particle in flat space moving in a potential \( V(x) \), one has (see e.g. ref. [15] for a nice discussion)

\[ Z = \int_{x(0)=x(\beta)} Dx(t) e^{-S_E}, \]  

(68)

where the integral is over all paths which are periodic in Euclidean time with periodicity \( \beta \), and

\[ S_E = \int_0^\beta dt \left[ \frac{1}{2} \dot{x}^2(t) + V(x(t)) \right], \]  

(69)

is the Euclidean action.

As is well known [10], de Sitter space behaves in some respects like a system at finite temperature \( \beta^{-1} = H/2\pi \). One difference with flat space is, however, that on the sphere all directions are compact. Therefore, once in Euclidean there is no actual distinction between temporal and spatial directions. In our case, the Euclidean action is given by (12) and the natural generalization of (68) is

\[ Z = \int D\Sigma(\xi^a) e^{-S_E[\Sigma]}, \]  

(70)

where now the integral is taken over all closed worldsheets (or worldlines for \( N = 1 \)).

In the semiclassical limit, \( Z \) will be dominated by the stationary points of \( S_E \), and so it will be a sum of contributions from multi-instanton configurations \( Z = Z_0 + Z_1 + \ldots \). Here \( Z_k \) is the contribution of a configuration with \( k \)
widely separated instantons. This configuration has action \(k\bar{S}_E\), where \(\bar{S}_E\) is the action for one instanton. Hence \(Z_k\) will have the exponential dependence

\[ Z_k \propto e^{-k\bar{S}_E}. \]  

(71)

To find the preexponential factor, one has to integrate over small fluctuations around the stationary points. Eq. (70) is rather formal, because we have not specified the measure of integration (this can actually be quite complicated, due to reparametrization invariance.) However, if all we are interested in are small fluctuations around the instanton solutions, the integration over neighboring worldsheets amounts to an ordinary path integral over the perturbation fields \(\phi(\xi^a)\) that we introduced in section 3. Therefore, we can write

\[ Z = \sum_{k=0}^{\infty} \frac{e^{-k\bar{S}_E}}{k!} \left( \int \mathcal{D}\phi e^{-S_E^{(2)}(\phi)} \right)^k, \]  

(72)

where \(S_E^{(2)}\) is the second variation of the action, given in (46). The sum is over multi-instanton configurations. The path integrals over \(\phi\) give the contribution of fluctuations around each one of the spherical worldsheets. The \(k!\) in the denominator can be understood as follows [2]. When integrating over \(\phi\) we are integrating also over the zero modes. That means that we are integrating over all possible locations of the instantons in Euclidean space. Since the instantons are identical, we must divide by \(k!\) to avoid overcounting.

Eq. (72) can be rewritten as

\[ Z = e^{Z_1}, \]  

(73)

where

\[ Z_1 \equiv e^{-\bar{S}_E} \int \mathcal{D}\phi \exp \left( \int \phi(\Delta + M^2)\phi\sqrt{\gamma}d^N\xi \right). \]  

(74)

This is just the path integral for a free scalar field on a curved background (the sphere), and can be calculated using well known manipulations.

Following [17], the field is expanded in spherical harmonics \(\phi_{LJ}\) [see (48)]. The integral over \(\phi\) can then be expressed in terms of the coefficients \(C_{LJ}\)

\[ \mathcal{D}\phi = \prod_{LJ} \mu \frac{dC_{LJ}}{(2\pi)^{1/2}}. \]  

(75)
Note that $Z_1$ is dimensionless, whereas $C_{LJ}$ has dimensions of $(mass)^{-1}$. To render $D\phi$ dimensionless one has to introduce the parameter $\mu$ with dimensions of mass. Using (48) and (49) one has

$$Z_1 = e^{-S_E} \prod_{LJ} \mu \left( \frac{dC_{LJ}}{(2\pi)^{1/2}} \right) \exp \left( -\frac{1}{2} \sum_{LJ} (M^2 - \lambda_L) C_{LJ}^2 \right),$$

where $\lambda_L$ is given by (44). After Gaussian integration one obtains

$$Z_1 = e^{-S_E} \prod_L \left[ (\mu R_0) \Lambda_L^{-1/2} \right]^{\mu L} \equiv e^{-S_E} (\det[(\mu R_0)^{-2} \hat{O}])^{-1/2},$$

where $g_L$ is given by (50). Here we have introduced the dimensionless operator $\hat{O} \equiv R_0^2(-\Delta + M^2)$, with eigenvalues $\Lambda_L = R_0^2(M^2 - \lambda_L)$, [see (44)].

Since the eigenvalues are known, the determinant can be calculated with the $\zeta$-function regularization method. In terms of the generalized $\zeta$-function,

$$\zeta(z) \equiv \sum_L g_L \Lambda_L^{-z},$$

the determinant can be expressed as

$$\det[(\mu R_0)^{-2} \hat{O}] = (\mu R_0)^{-2\zeta(0)} e^{-\zeta'(0)}.$$

The values of $\zeta(0)$ and $\zeta'(0)$ for the operator $\hat{O}$ on the $N$-sphere are calculated in the Appendix. For $N = 1$ we obtain $\zeta(0) = 0$ and $\zeta'(0) = -2 \ln[2 \sinh(\pi R_0 M)]$, so that

$$Z_1 = \frac{e^{-S_E}}{2 \sinh(\pi R_0 M)}. \quad (N = 1)$$

Note that, since $\zeta(0) = 0$, the dependence on the arbitrary renormalization scale $\mu$ disappears.

For worldsheets of dimension $N > 1$ it turns out that $\zeta(0) = 0$ only for odd $N$ [18]. In general, for even $N$ the determinant will depend on the renormalization scale $\mu$. As mentioned before, the calculation of $Z_1$ is the same as the calculation of the effective action for a free scalar field in a curved spacetime of dimension $N$. The appearance of a renormalization scale in that context is well known [19, 17] (in particular, this scale is responsible
for the so-called trace anomaly, the quantum mechanical breakdown of conformal invariance). The difference between even and odd dimensions can be understood from the fact that in dimensional regularization the infinities come from the poles of the gamma function \( \Gamma[j - (N/2)] \), where \( j \) is an integer. As a result, for odd \( N \) the effective action is finite after dimensional regularization, and the usual \( \log \mu \) terms do not appear. In general, for even \( N \) we have to live with the arbitrary scale \( \mu \), unless \( \zeta(0) \) happens to vanish accidentally for our particular worldsheet geometry and mass of the scalar field \( \phi \). It is well known that \( \zeta(0) \) can be computed in terms of geometrical invariants. For \( N = 2 \) we have

\[
\zeta(0) = \frac{1}{4\pi} \int d^2 \xi \sqrt{\gamma} \left[ -m^2 + \left( \frac{1}{6} - \xi \right) R^{(2)} \right], \tag{81}
\]

where \( m^2 + \xi R^{(2)} \) is the quantity that we have denoted as \( M^2 \) [see Eq. (17)]. Notice that both in flat and de Sitter backgrounds, the term involving the external Ricci tensor in (17) is constant, and can be included in \( m^2 \). We shall come back to the discussion of \( \mu \) and its role in the context of renormalization in the next Section.

Our derivation of the semiclassical partition function has been rather formal. As a check, let us apply these ideas to a simple example where the result is known: the case of free particles at finite temperature in flat space. This will also illustrate the general procedure for dealing with the zero modes.

For particles at finite temperature, the action for the instantons (discussed in subsection 4.2) is equal to the mass \( M \) of the particle times the length of a worldline that wraps around the compact temporal dimension \( S_E = M \beta \). The perturbation field \( \phi \) is massless, \( M = 0 \), and so the expression (80) diverges because of the vanishing denominator. This is to be expected because the operator \( \hat{O} \) has a zero eigenvalue corresponding to the translational zero mode discussed in section 3 (for simplicity we start by considering only one spatial dimension transverse to the worldline, hence only one field \( \phi \) and one zero mode).

Noting that \( (2\pi)^{-1/2} \int dC_0 \exp(-M^2 C_0^2/2) = M^{-1} \), the divergence at \( M \to 0 \) can be avoided by leaving the integral over \( dC_0 \) undone and by excluding the factor \( M^{-1} \) from the determinantal factor. Then we can write

\[
dZ_1 = (\text{det}'[(\mu R_0)^{-2}\hat{O}])^{-1/2} e^{-M\beta} \frac{dC_0}{(2\pi)^{1/2}}, \tag{82}
\]
where
\[
det^\prime[(\mu R_0)^{-2} \mathcal{O}] \equiv \lim_{M \to 0} \frac{\det[(\mu R_0)^{-2} \mathcal{O}]}{M^2} = (2\pi R_0)^2
\] (83)
is the determinant without the zero eigenvalue.

Using (61) and \(2\pi R_0 = \beta\) we have
\[
dZ_1 = dx_0 \left( \frac{M}{2\pi \beta} \right)^{1/2} e^{-M\beta}.
\]
This can be cast in a more familiar form by setting \(T \equiv \beta^{-1}\) and increasing the number of transverse dimensions to three. Each transverse dimension brings an additional power to the preexponential factor. Interpreting \(d^3x_0\) as the volume element we have
\[
Z_1 = V \left( \frac{MT}{2\pi} \right)^{3/2} e^{-M/T}.
\] (84)
This is the correct expression for the “one particle” partition function of an ideal gas with the Maxwell-Boltzmann distribution [note that the instanton method is only valid when the exponent in (84) is large, in which case there is no difference between bosonic and fermionic distributions]. The grand canonical partition function is obtained, according to (73), by exponentiating this expression.

Actually, Eq. (73) gives the partition function for the case of vanishing chemical potential, something that we have tacitly assumed in our derivation. The effect of a chemical potential \(\tilde{\mu}\) is to replace \(Z_1\) in (73) by \(e^{\tilde{\mu}\beta} Z_1\). The number of particles \(N\) in the volume \(V\) is then given by
\[
N = \beta^{-1} \left( \frac{\partial \ln Z}{\partial \tilde{\mu}} \right)_{\beta,V} = e^{\tilde{\mu}\beta} Z_1.
\]
For vanishing \(\tilde{\mu}\), we have
\[
N = Z_1.
\] (85)
Therefore \(Z_1\) is also the equilibrium number of particles.
6 Nucleation rates in flat space

In flat space and at sufficiently low temperatures, the decay rate of a metastable state is given by [7]

\[ \Gamma = 2 |\text{Im } F| = 2\beta^{-1} |\text{Im } Z_1|, \]  

(86)

(For temperatures \( \beta^{-1} > R_0^{-1} \), this formula has to be modified [7]. We defer the consideration of the high temperature regime to a later paper.) As shown in the previous section, the calculation of \( Z_1 \) reduces to the calculation of a functional determinant. The fact that \( F \) has an imaginary part is due to the fact that the action \( S^{(2)} \) in (10) has a negative mode, corresponding to \( L = 0 \), so the determinant will be negative. Upon taking the square root a factor of \( i \) will emerge.

The membrane creation rate can be expressed as

\[ d\Gamma = \frac{1}{2} \times 2\beta^{-1} |\text{det}'[(\mu R_0)^{-2}\hat{O}]|^{-1/2} e^{-S_E} |J| dV dt_E, \]  

(87)

where the Jacobian is given by

\[ |J| = \frac{\prod_{l=1}^{d-1} (2\pi)^{-1/2} dC_{1J}}{dV dt_E}. \]  

(88)

This equation follows from (86) and (77). As before [see (83)] the prime in the determinant means that the \( (N+1) \) zero modes (which are now in the \( L = 1 \) sector) are omitted, because the integration over \( dC_{1J} \) is left undone

\[ \text{det}'[(\mu R_0)^{-2}\hat{O}] \equiv \lim_{M^2 \to -NR_0^2} \frac{(\mu R_0)^{-2\zeta(0)} e^{-\zeta'(0)}}{(M^2 + NR_0^2)^{N+1}}. \]  

(89)

The Jacobian, which is needed to change variables from \( dC_{1J} \) to the Euclidean space-time volume element \( dV dt_E \), can be read off from (52) [for obvious reasons, in the limit \( H \to 0 \) we replace \( H^{-1}d\alpha_J \) by \( dt_E \) (for \( J = 0 \)) or by \( d\vec{x} \) (for \( J = 1, \ldots, d-1 \))],

\[ |J| = \left( \frac{M S_N(R_0)}{2\pi(N+1)} \right)^{N+1}. \]

The overall factor of 1/2 in the r.h.s. of (87) is explained in Refs. [20, 4]. It arises because the free energy of an unstable state can only be defined by
analytic continuation from a Hamiltonian in which the same state is stable. As a result, the contour of integration over the negative mode $dC_0$ has to be deformed into the complex plane in such a way that only half of the saddle point contributes to $\text{Im } F$. We should note, however, that in the derivation of (86) given in [7], these considerations are not really relevant; and $2|\text{Im } F|$ is essentially a convention to denote the r.h.s. of (87).

Integration over Euclidean time $t_E$ cancels the factor of $\beta^{-1}$, and we are left with the rate per unit volume

$$\frac{dN}{dtdV} = \frac{\Gamma}{V} = \left(\frac{\mathcal{M}S_N(R_0)}{2\pi(N+1)}\right)^{\frac{N+1}{2}} |\text{det}'[(\mu R_0)^{-2}\hat{O}]|^{-1/2}e^{-S_E},$$

where $N$ is the number of membranes created. Let us now evaluate this rate for space-times of different dimensionalities.

### 6.1 Pair creation in 1+1 dimensions

From (89) and (A7) the primed determinant is

$$\lim_{M^2 \to -R_0^2} \frac{4\sinh^2(\pi R_0 M)}{(M^2 + R_0^{-2})^2} = -\pi^2 R_0^4,$$

hence, from (90) the creation rate per unit length is

$$\frac{\Gamma}{L} = \frac{\mathcal{M}}{2\pi R_0} e^{-S_E}.$$  

Taking $R_0 = \mathcal{M}/eE_0$, we have

$$\frac{\Gamma}{L} = \frac{eE_0}{2\pi} \exp \left(-\frac{\pi\mathcal{M}^2}{eE_0}\right).$$

This can be compared with the results of Stone [21], who computed the vacuum decay rate in the sine-Gordon theory with non-degenerate vacua. His one loop result was

$$\frac{\Gamma}{L} = \frac{eE_0}{2\pi} \left| \ln(1 - e^{-\frac{\pi\mathcal{M}^2}{eE_0}}) \right|.$$
where $\mathcal{M}$ is the mass of the kink and $eE_0$ is the vacuum energy density gap between neighboring vacua. As expected, the instanton calculation gives a good approximation when the Euclidean action is large $\bar{S}_E >> 1$.

Eq. (91) also gives the rate at which a metastable cosmic string will break up by nucleating pairs of monopoles \[3\]. In that case $eE_0$ should be replaced by the string tension and $\mathcal{M}$ by the mass of the monopoles.

### 6.2 Pair creation in 3+1 dimensions

As explained in subsection \[4.3\] in 3+1 dimensions in addition to the radial perturbations of mass $M^2 = -R_0^2$, we have the perturbations $\phi_y$ and $\phi_z$ which are transverse to the plane of the instanton. These behave like massless fields $M^2 = 0$. Each field contributes its own determinantal factor, which for $\phi_y$ and $\phi_z$ is given by (83). Also, from (64), each contributes a factor $(R_0 \mathcal{M})^{1/2}$ to the Jacobian. With this, Eq. (91) is modified into

$$\Gamma V = \left(\frac{eE_0}{8\pi^3} e^{-\frac{\bar{S}_E}{eE_0}}\right).$$

(92)

This coincides with the rate of production of charged bosons in scalar electrodynamics

$$\Gamma V = \left(\frac{eE_0}{8\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} e^{-\frac{n\bar{S}_E}{eE_0}} [1 + O(e^2)]\right),$$

(93)

in the limit $S_E >> 1$. (For a more thorough account of monopole production by a magnetic field, and pair production in the strong coupling regimes see Refs. [22, 23]).

### 6.3 String creation in 2+1 dimensions

From (41) and (A10),

$$\Gamma V = \left(\frac{\mathcal{M}_S(R_0)}{6\pi}\right)^{3/2} (\mu R_0)^{7/3} R_0^{-3} e^{-S_E}.$$

(94)

Note that, since $\zeta(0) \neq 0$, the determinant depends explicitly on the renormalization scale $\mu$ [the exponent 7/3 can also be derived from (81)]. Because
of the arbitrariness in $\mu$ we cannot give an absolute estimate of the nucleation rate the way we do for $N=1$ or $N=3$. However, since $\mu$ is a constant, we can still predict how the rate changes when we change the external field $E_0$ (that is, when we change $R_0$.) It is seen that the dependence of the prefactor on $R_0$ is more complicated than what one would have guessed form dimensional analysis.

So far, by using the $\zeta$-function method, we have avoided the question of ultraviolet divergences, since they are automatically removed by analytic continuation [19] (see also Ref. [24] for a recent discussion, and references therein). However, we should recall that the functional determinants contain such divergences, and that these can be eliminated by suitable counterterms in the action. For $N = 2$, all divergences can be removed by counterterms of the form [19]

$$c \int d^2 \xi \sqrt{\gamma} + d \int d^2 \xi \sqrt{\gamma} R.$$

The first term is a contribution to the membrane tension. The second is a topological invariant which does not contribute to the equations of motion. Note, form (79) and (81), that a rescaling of the arbitrary parameter $\mu$ can be reabsorbed in a redefinition of $c$ and $d$. Then, we can eliminate the renormalized $c$ by rescaling $\mu$, and the renormalized $d$ by shifting the string tension $\mathcal{M}$.

In Ref. [25] a different approach was followed in which the product over all eigenvalues was cut off at some physical scale $\mu$. This method also produced the factor $(\mu R_0)^{7/3}$, which was referred to as the “universal term” [25, 26]. The introduction of a physical cut-off is not unreasonable when dealing with an effective theory. In deriving the action for the perturbations we have only expanded to second order in $\phi$. This is justified at low momenta $L$, but in the ultraviolet limit, higher order terms become important. To realize that this is so, let us take the case of a flat membrane and $E_0 = 0$ (this is actually quite representative of the general case). The exact Nambu action is then

$$S_E = \mathcal{M} \int d^2 \xi \sqrt{1 + \mathcal{M}^{-1} (\partial_a \phi \partial^a \phi)}.$$

When $\partial_a \phi$ is of order $\mathcal{M}^{1/2}$ higher order terms become important and the quadratic approximation fails. This suggests taking $\mu \sim \mathcal{M}^{1/2}$ as a cut-off, which is essentially what was done in [25].

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6.4 Membrane creation in 3+1 dimensions

Using (90) and (A14) we have

\[
\Gamma = \left( \frac{\mathcal{M}S_3(R_0)}{8\pi} \right)^2 \frac{4R_0^{-4}}{\pi^2} e^{\xi_0(-2)} e^{-S_\xi}.
\]

(95)

This can be compared with the results of Affleck [26]. He studied the decay rate of false vacuum in the theory of a scalar field with a symmetry breaking potential and nondegenerate vacua, in the limit in which the thickness of the wall separating the true from the false vacuum is much smaller than the radius of the bubble at nucleation.

The first factor in the r.h.s. of (95) is the Jacobian \(|J|\) that comes from the normalization of the zero modes. The rest of the pre-exponential factor coincides with what Affleck calls the “universal terms”. These are due to fluctuations of the worldsheet and therefore are independent of the details of the field theoretic model. That is, they are independent of the internal structure of the “kink” or domain wall. The evaluation of quantum corrections to the effective action due to finite thickness of the wall is in itself an interesting subject, and the corrections can be important in realistic theories [26, 22, 23, 27]. However, since these corrections are not essential to the physics of nucleation here we shall concentrate on infinitely thin membranes, without internal structure. We will only mention that the fluctuations around thick walls can be treated very naturally within the formalism described in this paper. In addition to the tachyonic field of perturbations that we have considered, there will be an infinite tower of massive fields \(\phi_q\) ‘living’ in the N-sphere [28, 11], each one contributing its own determinantal prefactor. We plan to return to this question in a later paper where the thermal properties of the fields \(\phi_q\) (with temperature \(2\pi R_0\)) are emphasized [29].

7 Nucleation rates in curved space

Although the calculation of determinants for the instantons in de Sitter space offers no special problems (we only need to substitute the appropriate values of \(R_0\) in the expressions found in the Appendix); the definition of a nucleation rate in de Sitter is a more subtle issue which has often been eluded in the literature. One possibility would be to simply use Eq. (90), interpreting \(dV\)
as the physical volume element at the time of nucleation \(dV_0\)

\[
d\Gamma = |\lambda|dV_0,
\]

where \(|\lambda|\) is defined as the r.h.s. of (94).

Although we believe that this expression is correct (when properly interpreted) it clearly needs further justification. First of all, the physical volume element, \(dV_0\), is proportional to a power of the scale factor \(e^{Ht_0}\) at the time of nucleation. If Eq. (90) was derived from a purely Euclidean calculation, how does the exponential of a Lorentzian time find its way into the r.h.s. of (96)? Also, as pointed out in [5], the time of nucleation in de Sitter space is a somewhat ambiguous concept when the size of the instantons is comparable to the horizon, and in principle it is not clear what time one should use in \(dV_0\).

These difficulties prompt us to search for an alternative way of calculating nucleation rates in de Sitter space, which does not take (86) as the starting point. Recalling that de Sitter space behaves in some respects like a thermodynamical system, one can try to estimate directly the equilibrium distribution of membranes. For this one can use Eq. (85), with \(N\) the number of membranes and \(Z_1\) given by (74). It is clear that

\[
dN = (det'(\mu R_0)^{-2\hat{\mathcal{O}}})^{-1/2}e^{-\tilde{S}_E} \prod_{J=0}^{d-1} (2\pi)^{-1/2}dC_{1J}.
\]

Using (52), we have

\[
dN = \lambda H^{-d} \prod_{J=0}^{d-1} d\alpha_J
\]

where

\[
\lambda = \left( \frac{\mathcal{M}S_N(R_0)}{2\pi(N+1)} \right)^{\frac{N+1}{2}} (det'(\mu R_0)^{-2\hat{\mathcal{O}}})^{-1/2}e^{-\tilde{S}_E}. \quad (97)
\]

Eq. (52) is valid only for infinitesimal rotations, and in that case \(\prod d\alpha_J\) can be identified with the differential solid angle on the d-sphere, \(d\Omega\), within which the center of the instanton worldsheet is to be found,

\[
dN = \lambda H^{-d}d\Omega. \quad (98)
\]

One can interpret this equation as the “equilibrium distribution of instantons” in Euclidean space.
Of course, an equilibrium distribution of instantons is not a measurable object. However, it can be given meaning by conjecturing that the equilibrium distribution of membranes in the Lorentzian section is given by the analytic continuation of the previous object to real time. This prescription is just heuristic, and we do not know how to justify it further, except by saying that it reduces to (90) in flat space and that it is quite natural from the mathematical point of view.

To see how the analytic continuation is done, let us consider, for simplicity, the 1+1 dimensional case (higher dimensional cases are completely analogous). Then (98) reads
\[ dN = \lambda H^{-2} \cos \alpha_E d\alpha_E d\beta, \]
where \( \alpha_E \) and \( \beta \) are polar and azimuthal angles on the 2-sphere. Upon analytic continuation \( \alpha_E = i\alpha \) [see (34)],
\[ dN = |\lambda| H^{-2} \cosh \alpha e^{-Ht_0} d\alpha d\beta. \]
Note that the factor of \( i \) from \( d\alpha_E \) cancels the imaginary factor from the square root of the determinant in \( \lambda \), so that \( dN \) is actually real. Using Eqs. (37) and (38), one can change variables from \((\alpha, \beta)\) to \((x_0, t_0)\). The Jacobian is
\[ \left| \frac{\partial(t_0, x_0)}{\partial(\alpha, \beta)} \right| = H^{-2} \cosh \alpha e^{-Ht_0}, \]
and therefore
\[ dN = |\lambda| e^{Ht_0} dx_0 dt_0. \]  \hfill (99)

Generalizing to spacetimes of arbitrary dimension we have
\[ dN = |\lambda| e^{NHt_0} d\vec{x}_0 dt_0, \]  \hfill (100)
which is very similar to (96), but now all ambiguities in \( dV_0 \) have been resolved.

An equation of the same form as the previous one was given in [5], based on kinematical considerations. However, the parameter \( \lambda \) was left unspecified. Like in [5], Eq. (100) is a distribution in the space of parameters \( x_0 \) and \( t_0 \), and it is therefore independent of the time of observation.

Following [5], we can use (36) to express \( t_0 \) in terms of the physical radius \( R \), and thus find the size distribution of membranes (or bubbles),
\[ \frac{dN}{dV_{phys}} = |\lambda| \frac{R(R^2 - R_0^2)^{-1/2}}{H^d \left[ \omega_0 + (R^2 - R_0^2)^{1/2} \right]^d} dR, \]
where $dV_{\text{phys}} = \exp(NHt)d^N\vec{x}_0$. Notice that the distribution diverges at the lower end $R \to H^{-1}$ when $\omega_0 \leq 0$. This divergence was interpreted in \[6\]. For large radii, one finds the scale invariant distribution

$$\frac{dN}{dV_{\text{phys}}} \approx \frac{|\lambda| dR}{H^d R^d},$$

(101)

which depends on the external field $E_0$ and membrane tension only through the coefficient $\lambda$. The fact that the size distributions are time independent is easy to understand. As the bubbles are created, they are stretched and diluted by the inflationary expansion, giving rise to a stationary distribution of sizes \[5\].

The “nucleation rates” $|\lambda|$ for $d = 2, 3$ and $4$ can be read off from the r.h.s. of (91), (94) and (95) respectively, where $R_0$ is given by (21) and $\bar{S}_E$ by (22-24). This covers the case of co-dimension one, $d = N + 1$. For completeness, we shall also consider the case of strings and monopole pairs spontaneously nucleating in $3+1$ dimensions.

For the case of strings, the co-dimension is 2, and so there will be two independent perturbation fields $\phi^{(i)}$ and two determinantal prefactors. From (74) it is clear that $\lambda$ will contain one factor of $\mathcal{M}^3$ and a factor of $\mu^{14/3}$, where $\mu$ is the renormalization scale. Taking $R_0 = H^{-1}$, the rest will be a numerical factor (which can be absorbed in $\mu$), times the appropriate power of $H$ necessary to give $\lambda$ the dimensions of $(mass)^4$:

$$|\lambda| = \mu^{14/3} \mathcal{M}^3 H^{-11/3} e^{-\bar{S}_E}.$$  

(102)

Because of the arbitrary renormalization scale, we cannot obtain an absolute estimate for $|\lambda|$, but only its dependence on the expansion rate $H$. Like in subsection (6.3), to obtain a crude absolute estimate one can set $\mu \sim \mathcal{M}^{1/2}$, which gives

$$|\lambda| \sim (\frac{\mathcal{M}}{H})^{23/3} H^4 \exp(-4\pi\mathcal{M}H^{-2}).$$

(102)

For $\mathcal{M} >> H$ this can be considerably larger than the naive dimensional estimate $|\lambda| \sim H^4 \exp(-\bar{S}_E)$ mentioned in \[5\], or even the more sophisticated $|\lambda| \sim \mathcal{M}^3 H \exp(-\bar{S}_E)$. Unfortunately, the existence of an arbitrary renormalization scale leaves us quite uncertain as to the overall normalization of (102).
Let us now consider the case of pairs spontaneously nucleating in $3 + 1$ dimensions. The co-dimension is 3 and so there will be 3 perturbation fields $\phi^{(i)}$, each one of them with mass $M^2 = -H^{-2}$, contributing to the determinantal prefactor. Each field has 2 zero modes [in the $L = 1$ sector, see (53)], which makes a total of 6. Four of them correspond to space-time translations and two of them to changes in the orientation of the monopole pair in three dimensional space. Thus we have

$$|\lambda| = \left(\frac{MH}{2\pi}\right)^3 \left(\frac{A\pi}{H^2}\right) e^{-\bar{S}_{E}}.$$  

The factor of $(MH/2\pi)^3$ comes from taking the third power of the prefactor in (91), with $R_0 = H^{-1}$. The factor $4\pi H^{-2}$ is a correction due to the fact that two of the zero modes represent changes in the angular orientation of the pair, rather than translations. For each angular variable, the Jacobian (88) has an extra power of $H$ in the denominator [see (53)]. Integration over all possible orientations gives the factor of $4\pi$. To summarize,

$$|\lambda| = \frac{H M^3}{2\pi^2} \exp \left(-\frac{2\pi M}{H}\right), \quad (d = 3 + 1)$$  

(103)

where $M$ is the mass of the monopole.

For the case of pair creation a 'size' distribution such as (101) is not very useful. Instead, it is more convenient to find the momentum distribution of particles. Let us find the conserved momentum as a function of the time of nucleation $t_0$. In 1+1 de Sitter space, the vector potential

$$A_\mu = -H^{-1} E_0 e^{H t} \delta_{\mu x}$$  

(104)

represents a constant electric field (note that $F_{\mu\nu} F^{\mu\nu} = 2 E_0^2$). With this, the action for the point particle coupled to $A_\mu$ reads

$$S = -\mathcal{M} \int \left(\dot{x}^2 e^{2H t}\right)^{1/2} d\tau - \frac{e E_0}{H} \int e^{H t} \dot{x} d\tau,$$  

(105)

where $\tau$ is the proper time and a dot denotes derivative with respect to $\tau$. Since the Lagrangian does not depend on $x$, the momentum

$$k \equiv \frac{\partial L}{\partial \dot{x}} = \mathcal{M} \dot{x} e^{2H t} - e E_0 e^{H t},$$

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is conserved. Setting \( x_0 = 0 \) in (36) we have

\[
\frac{dx}{dt} = \frac{H \omega_0 e^{-H(t_0 + t)} - e^{-2Ht}}{Hx}
\]

and from \( \dot{t}^2 - x^2 e^{2Ht} = 1 \), it follows that \( \dot{t} = e^{Ht} H |x|(1 - H^2 \omega_0^2)^{-1/2} \). Therefore,

\[
k = -\frac{M}{HR_0} e^{Ht_0} \text{sign}(x),
\]

where we have taken into account that the particle to the left of the inside region has charge of opposite sign.

Using this equation in (99), with \( |\lambda| \) given by the right hand side of (91), we have

\[
\frac{dN}{dx_0} = e^{-S_E} \frac{dk}{2\pi}. \quad (d = 1 + 1)
\]

In 3+1 dimensions (without electric field), using (100), (103) and (106), we have

\[
\frac{dN}{d^3x_0} = \exp\left(-\frac{2\pi M}{H}\right) \frac{k^2 dk}{2\pi^2}. \quad (d = 3 + 1)
\]

Note that the momentum distributions are flat, as expected on the grounds of scale invariance.

At any given time \( t \) we only need to integrate up to a cut-off momentum

\[
|k| \sim \frac{M}{HR_0} e^{Ht},
\]

since particles with higher value of the coordinate momentum have not been created yet [see (106)]. Then

\[
n \sim \frac{1}{2\pi HR_0} \frac{M}{e^{-S_E}}, \quad (d = 1 + 1)
\]

and

\[
n \sim \frac{M^3}{6\pi^2} e^{-\frac{3\pi M}{2}}. \quad (d = 3 + 1),
\]

where \( n \) is the number density of particles per unit physical volume. As noted in [4], for the case of vanishing electric field the distribution of particles contains a Boltzmann factor \( \exp(-M/T) \) where \( T = H/2\pi \) is the Gibbons-Hawking temperature [10].
8 Charged scalar field in 1+1 dimensional de Sitter space

The Klein-Gordon equation for a charged scalar field $\varphi$ coupled to an external electromagnetic field $A_\mu$ is

$$- g^{\mu\nu} (\nabla_\mu - ieA_\mu)(\nabla_\nu - ieA_\nu)\varphi + \mathcal{M}^2 \varphi = 0. \tag{109}$$

As we shall see below, for a constant electric field in 1+1 de Sitter this equation can be solved in terms of special functions. Then the problem of particle creation is amenable for calculation using Bogolubov transformations (see e.g. [19]).

To apply this method it is necessary to specify an ‘in’ state and an ‘out’ vacuum. The ‘in’ state, $|in\rangle$, is the physical quantum state of our system, fixed by initial conditions. The ‘out’ vacuum, $|0\rangle_{out}$, is a different quantum state in the Hilbert space, whose choice amounts to a definition of particles at late times. Whether or not one can unambiguously specify $|0\rangle_{out}$ depends on whether or not it is physically reasonable to define particles at late times. One way to guarantee a reasonable definition is to switch off the gravitational and the electric fields at late times (although this may not be necessary).

To illustrate the procedure, consider the case of vanishing electric field first [30]. In terms of the conformal time $\eta \equiv -H^{-1} e^{-Ht}$, and with $\varphi = \varphi_k(\eta)e^{ikx}$, Eq. (109) reads

$$\varphi''_k + \left( k^2 + \frac{\mathcal{M}^2}{H^2 \eta^2} \right) \varphi_k = 0, \tag{110}$$

where a prime denotes derivative with respect to $\eta$. This equation is symmetric in $k$, and following the usual convention, we take $k > 0$ (For $E_0 = 0$, the results for $k < 0$ are the same). Eq. (110) has the general solution

$$\varphi_k(\eta) = \left( \frac{\eta}{8} \right)^{1/2} \left[ A_k H^{(2)}_\nu(k\eta) + B_k H^{(1)}_\nu(k\eta) \right], \tag{111}$$

where

$$\nu = \left( \frac{1}{4} - \frac{\mathcal{M}^2}{H^2} \right)^{1/2}.$$
and $H^{(i)}$ are the Hankel functions. For the ‘in’ state, we shall take the Bunch-Davies vacuum \[30\]. This is characterized by positive frequency modes of the form (111) with $A_k = 1$, $B_k = 0$

$$\varphi_{in,k}(\eta) = \left(\eta \over 8\right)^{1/2} H^{(2)}_\nu(k\eta). \tag{112}$$

The choice of this vacuum as the physical ‘in’ state can be motivated from many different points of view, and it is a clear favourite in studies of inflation. In the open coordinate system that we are using, this is the only truly de Sitter invariant vacuum \[31\]. The two-point function in this state coincides with the Euclidean two-point function \[32\], which has the important property of having the Hadamard form (roughly speaking, this means that it has similar ultraviolet behaviour as the two-point function in flat space). Also, it is believed that if the Universe nucleated from ‘nothing’ into a de Sitter phase, then this is the quantum state that the fields would be in after nucleation \[33, 34\].

To define the ‘out’ vacuum it is convenient to write down the equation for the scalar field in terms of the cosmological time $t$,

$$\ddot{\varphi}_k + H\dot{\varphi}_k + \left(\mathcal{M}^2 + \frac{k^2}{a^2(t)}\right) \varphi_k = 0, \tag{113}$$

where a dot denotes $d/dt$ and $a(t) = \exp(Ht)$. Introducing $\varphi_k = a^{-1/2}\psi_k$ we have

$$\ddot{\psi}_k + \left[\mathcal{M}^2 - \frac{H^2}{4} + \frac{k^2}{a^2}\right] \psi_k = 0. \tag{114}$$

Let us take $\mathcal{M} >> H$ and define $t_k$ as the time when the physical wavelength $a(t_k)k^{-1}$ is equal to the particle’s Compton wavelength

$$k e^{-Ht_k} = \mathcal{M}. \tag{115}$$

For $t >> t_k$ the $k^2a^{-2}$ term is negligible compared to $\mathcal{M}^2$. Then we will have approximate solutions of the form

$$\varphi_k \approx a^{-1/2} \left[C_k e^{-iwt} + D_k e^{iwt}\right]$$

where

$$w = \left(\mathcal{M}^2 - \frac{1}{4}H^2\right)^{1/2}. \tag{116}$$
Since $M >> H$, the exponentials oscillate very fast compared to the rate at which $a^{1/2}$ changes, and so we will have an approximate definition of positive and negative frequency ‘out’ modes. For $t >> t_k$, 

$$\varphi_{out,k}^{(\pm)} \propto a^{-1/2} e^{\mp iwt}. \quad (117)$$

This definition is not very natural for $M^2 << H^2$, so we shall not be considering this limit.

A purist would object that we cannot define particles unless the expansion of the Universe is switched off. Consider then a FRW model in which the expansion rate $H \equiv \dot{a}/a$ is time dependent. Then $\psi_k$ satisfies the equation

$$\ddot{\psi}_k + \left[ M^2 - \frac{\mathcal{H}^2}{4} + \frac{k^2}{a^2} - \frac{1}{2} \mathcal{H} \right] \psi_k = 0. \quad (118)$$

After a sufficiently long period of inflation, $a \propto \exp(\mathcal{H}t)$, the expansion is adiabatically switched off starting at time $t_s$, in such a way that $\mathcal{H} << M^2$, until $a(t)$ reaches a constant value. The space-time is Minkowski in the asymptotic future. It is clear from (118) that if $t_s > t_k$, the mixing between positive and negative frequency modes will be negligible during the period in which the expansion is being switched off. This means that for modes such that $t_s > t_k$ the number of particles that is calculated using the definition (117) for positive and negative frequency modes is the same that the number of particles that would be found in the ‘out’ Minkowski region.

The ‘in’ positive frequency mode can be expressed as a linear combination of the ‘out’ positive and negative frequency modes

$$\varphi_{in,k}^{(+)} = \alpha_k \varphi_{out,k}^{(+)} + \beta_k \varphi_{out,k}^{(-)}, \quad (119)$$

The coefficients $\alpha_k$ and $\beta_k$ are the so-called Bogoliubov coefficients. Using the asymptotic expression for the Hankel functions at late cosmological times $t \to \infty$ (i.e. $\eta \to 0$)

$$\varphi_{in,k}^{(+)}(k\eta) = -\left(\frac{\eta}{8}\right)^{1/2} \frac{i}{\nu \pi} \left[ \left( \frac{|k\eta|}{2} \right)^{\nu} \frac{\Gamma(1 + \nu)e^{-i\pi \nu/2} - \left( \frac{|k\eta|}{2} \right)^{-\nu} \Gamma(1 + \nu)e^{i\pi \nu/2}}{1 + O(k\eta)} \right] \times$$

$$[1 + O(k\eta)],$$

and the relation

$$|\alpha_k|^2 - |\beta_k|^2 = 1, \quad (120)$$
one readily finds
\[ |\beta_k|^2 = |\exp(2\pi w H^{-1}) - 1|^{-1}, \] (121)
where \( w \) is given by (116). This equation was found e.g. in Refs. 35, 36 (see also references therein), using a different coordinate system.

The number density of particles per unit coordinate volume is then
\[ \frac{dN}{dx} = \frac{|\beta_k|^2}{2\pi} \approx e^{-2\pi M_H^{-1} \frac{dk}{2\pi}}, \quad (d = 1 + 1), \] (122)
where we have used \( M \gg H \). The same calculation can be done in \( d \) spacetime dimensions. The only difference in the final result is that
\[ w = \left( M^2 - \frac{(d-1)^2}{4} h^2 \right)^{1/2} \]
in Eq. (121). Using \( M^2 \gg H^2 \) we have
\[ \frac{dN}{d^3x} = |\beta_k|^2 \frac{d^3k}{(2\pi)^3} \approx \exp \left( -\frac{2\pi M}{H} \right) \frac{k^2 dk}{2\pi^2}. \quad (d = 3 + 1) \] (123)
Comparing (122) and (123) with (107) and (108) we find complete agreement, not only in the exponential behaviour, but also in the the preexponential factor.

Let us now consider the case of non-vanishing electric field in 1+1 dimensions. In conformal time, the Klein Gordon equation with \( A_\mu \) given by (104) reads (note that \( \nabla_\mu A^\mu = 0 \))
\[ H^2 \eta^2 \varphi''_k + \left( k^2 H^2 \eta^2 - 2eE_0 k\eta + M^2 + \frac{e^2 E_0^2}{H^2} \right) \varphi_k = 0. \] (124)
This is the Whittaker equation, having the general solution
\[ \varphi_k = A_k W_{\lambda,\sigma}(2ik\eta) + B_k W_{-\lambda,\sigma}(-2ik\eta), \] (125)
where \( W_{\lambda,\sigma} \) are the Whittaker functions, with \( \lambda = +ieE_0 H^{-2} \) and
\[ \sigma = \left( \frac{1}{4} - \frac{M^2}{H^2} - \frac{e^2 E_0^2}{H^4} \right)^{1/2}. \] (126)
At early times \((\eta \to -\infty)\), Eq. (124) is very similar to (110), and both reduce to
\[
\varphi''_k + k^2 \varphi_k = 0.
\]
Using the asymptotic expression for \(H_{\nu}^{(2)}(k\eta)\) at large \(\eta\), the ‘in’ positive frequency mode in the Bunch-Davies vacuum has the form \(\varphi_{m,k}^{(+)} \sim (4\pi k)^{-1/2} \exp(-i k \eta)\), and so it is positive frequency with respect to the conformal time. With the electric field switched on, we would like to choose a positive frequency mode which has similar behaviour at \(\eta \to -\infty\). Noting that
\[
W_{\lambda,\sigma}(2ik\eta) \sim e^{-ik\eta} (ik\eta)^{\lambda},
\]
and since \(W^{*}_{\lambda,\sigma}(z) = W_{-\lambda,\sigma}(-z)\), it is clear that we have to take \(B_k = 0\) in (125),
\[
\varphi_{m,k}^{(+)} = (4\pi k)^{-1/2} W_{\lambda,\sigma}(2i k \eta),
\]
where the normalization is due to the usual Wronskian condition.

It is quite straightforward to show that the state defined by the modes (127) is a Hadamard vacuum. For this, one simply writes the two point function as a sum over the modes (127). Similarly, one can write the Bunch-Davies two-point function as a sum over the modes (112). Because of the similarity in the ultraviolet behavior of both sets of modes, it is easy to show that the difference between both two-point functions is finite in the limit of coincident points, which simply means that both two-point functions have the same singularity structure.

To define particles at late times \((\eta \to 0)\) we observe that, in this limit, Eqs. (110) and (124) are again very similar, so the procedure that worked there will work here too. In terms of cosmological time \(t\) and the variable \(\psi_k = a^{1/2} \varphi_k\) we have
\[
\ddot{\psi}_k + \left( \mathcal{M}^2 + \frac{e^2 E_0^2}{H^2} - \frac{H^2}{4} + \frac{k^2}{a^2} - \frac{2eE_0 k}{Ha} \right) \psi_k = 0.
\]
Let us denote by \(t_k\) the time at which the physical wavelength of the mode is equal to the effective Compton wavelength
\[
k e^{-H t_k} = \left( \mathcal{M}^2 + \frac{e^2 E_0^2}{H^2} \right)^{1/2}.
\]
Provided that
\[ w \equiv \left( M^2 + \frac{e^2 E_0^2}{H^2} - \frac{H^2}{4} \right)^{1/2} >> H, \]  

(130)
it is clear that for \( t >> t_k \) we can define particles in the mode \( k \), using as positive and negative frequency modes the solutions
\[ \varphi_{\text{out},k}^{(\pm)} \propto a^{-1/2} e^{\mp iwt}. \]  

(131)
As before, these definitions are not meaningful for \( w << H \), and we shall not consider this limit.

Let us introduce the new Whittaker function \( M_{\lambda,\sigma} \),
\[ M_{\lambda,\sigma}(z) \equiv \Gamma(2\sigma + 1)e^{i\pi\lambda} \left[ e^{-i\pi(\sigma + \frac{1}{2})} \frac{W_{\lambda,\sigma}(z)}{\Gamma(\sigma + \lambda + \frac{1}{2})} + \frac{W_{-\lambda,\sigma}(-z)}{\Gamma(\sigma - \lambda + \frac{1}{2})} \right]. \]  

(132)
For small \( z = 2ik\eta \) we have [37]
\[ M_{\lambda,\sigma} = z^{\frac{1}{2} + \sigma} [1 + O(z)], \]
and so this behaves like \( \varphi_{\text{out},k}^{(+) \pm} \) in (131) provided that we take \( \sigma = +i|\sigma| \) (recall that \( \sigma \) is pure imaginary). Then, using (127), the Bogolubov coefficients can be read off directly from (132). Using (120), \( W_{\lambda,\sigma}^*(z) = W_{-\lambda,\sigma}(-z) \), and the relation
\[ |\Gamma(\frac{1}{2} + iy)|^2 = \pi (\cosh\pi y)^{-1} \]
we have
\[ |\beta_k|^2 = \frac{\cosh\pi|\sigma - \lambda|}{e^{2\pi|\sigma|}\cosh\pi|\sigma + \lambda| - \cosh\pi|\sigma - \lambda|}. \]  

(133)
For \( |\sigma \pm \lambda| >> 1 \),
\[ |\beta_k|^2 \approx e^{-2\pi|\sigma + \lambda|}, \]  

(134)
where
\[ |\sigma \pm \lambda| = H^{-2} \left[ (M^2H^2 + e^2E_0^2)^{1/2} \pm eE_0 \right] + O(H^2w^{-2}). \]

As mentioned above, in keeping with standard notation we have taken \( k > 0 \). The result for \( k < 0 \) is obtained by changing the sign of \( e \) in the final...
expression, since the differential equation only depends on the relative sign of $k$ and $e$. Therefore

$$|\beta_k|^2 \approx \exp \left(-\frac{2\pi}{H^2} \left[ (\mathcal{M}^2 H^2 + e^2 E_0^2)^{1/2} + eE_0 \right] \right), \quad (k > 0)$$

$$|\beta_k|^2 \approx \exp \left(-\frac{2\pi}{H^2} \left[ (\mathcal{M}^2 H^2 + e^2 E_0^2)^{1/2} - eE_0 \right] \right), \quad (k < 0) \quad (135)$$

In the instanton calculation we took the particle with charge $e$ to be the one to the right of the inside region, and the particle with charge $-e$ to be the one to the left. From (106), the particle to the right has $k < 0$, so we have to use the second equation in order to compare with the instanton results. From (22) and (135), we have

$$|\beta_k|^2 \approx e^{-S_E}.$$

Therefore

$$\frac{dN}{dx} = |\beta_k|^2 \frac{dk}{2\pi} \approx e^{-S_E} \frac{dk}{2\pi} \quad (136)$$

in agreement with (107).

Apart from the agreement between these momentum distributions and between (108) and (123), the time $t_k$ at which the definition of particle starts being meaningful [see (129)], is the same as the time of nucleation $t_0$ in the instanton formalism [see (106)], a suggestive coincidence.

9 Summary and conclusions

We have computed the nucleation rates for the process of membrane creation by an antisymmetric tensor field in a spacetime of dimension $d = N + 1$, for $N = 1, 2, 3$.

To this end, we have evaluated the contribution of the relevant instantons to the semiclassical partition function. These instantons are $N$-spherical worldsheets of radius $R_0$ [given by (21)] embedded in a Euclidean de Sitter background, which is itself a $d$-sphere of radius $H^{-1}$. The flat space instantons are obtained by taking $H \to 0$. We have discussed the analytic continuation of the instantons, describing the motion of the membranes after nucleation. The Lorentzian solutions are spherical membranes which at late times expand like the scale factor in the flat inflationary FRW model.
The effect of de Sitter transformations on a given solution corresponds to space-time translations of the solution.

To evaluate the preexponential factor of the instanton contribution, it is necessary to study small fluctuations of the instanton worldsheet. We have reviewed the covariant theory of such perturbations, according to which the normal displacement of the worldsheet is viewed as a scalar field $\phi$ living on the unperturbed worldsheet. We have given a kinematical derivation of the equations of motion for $\phi$, based on the fact that the zero modes, which correspond to infinitessimal translations of the instanton, have to be solutions.

The evaluation of the prefactor is seen to be equivalent to the calculation of the effective action for a free scalar field (the field $\phi$ mentioned above) living in a curved background (the $N$-sphere). The functional determinants that arise form Gaussian integration can be explicitly calculated using $\zeta$-function regularization. This method automatically removes all ultraviolet divergences. In the case $N = 2$, an arbitrary renormalization scale $\mu$ appears in the final result. Rescalings of $\mu$ are seen to be equivalent to a finite renormalization of the membrane tension and of a new term which has to be added to the action. This new term is just the Einstein-Hilbert action on the world-sheet, which for $N = 2$ is a topological invariant and hence does not contribute to the equations of motion.

In flat space, the nucleation rates are obtained using the standard formula which relates them to the imaginary part of the free energy [see 86]). We have recovered known results for the production of kinks in 1 + 1 dimensions, charged pairs in 3 + 1 dimensions and ‘bubble’ formation in 2+1 and 3+1 dimensions. Our results apply only to the case when the membranes are infinitely thin, having no internal structure. The effect of finite thickness of the membrane can be important in realistic field theories [26, 22, 23, 27]. We plan to return to this question in a forthcoming paper, where the thermal properties of the nucleated objects are studied [29].

In de Sitter space, there is no standard procedure for the calculation of nucleation rates. We have introduced a heuristic prescription to obtain the distribution of nucleated objects during inflation, following ideas related to the treatment of this problem in Ref. [3]. This distribution is obtained from the one instanton contribution to the partition function. The parameters corresponding to the zero mode rotations, which form a subgroup of $O(d + 1)$, have to be analytically continued along with the instanton so that
they correspond to a subgroup of the de Sitter group $O(d,1)$. With these manipulations, the one instanton contribution to the partition function lends itself to interpretation as a distribution of membranes in a space of parameters. The parameters can be chosen to be the place and time of nucleation. In flat space this prescription reduces to the standard formula (90) for the calculation of nucleation rates. From the parameter distribution one can easily find the size distribution of membranes in the inflationary universe, which turns out to be stationary and nearly scale invariant [see (101)].

For the case of pair creation, we have given the distribution in terms of the conserved coordinate momentum $k$. This distribution turns out to be independent of $k$ (as expected on the grounds of scale invariance) with an upper cut-off $k^2_{\text{phys}} \approx M^2 + \left(eE_0^2/H^2\right)$, where $e$ is the charge of the particle, $E_0$ is the electric field, and $k_{\text{phys}} = e^{2Ht}k$ is the physical momentum. This cut-off corresponds to the momentum of the particle at the time of nucleation $t_0$ (the physical momentum is subsequently redshifted).

For the case of pairs, the calculations can be repeated using a completely different approach. In 1+1 dimensional de Sitter space, the Klein-Gordon equation for a charged scalar field coupled to a constant electric field can be solved in terms of Whittaker functions. We find the Bogoliubov transformations between an ‘in’ state which is analogous to the Bunch-Davies vacuum and an ‘out’ state which corresponds to the definition of particles at late times. This definition of particles at late times is natural and unambiguous when the effective mass $M^2 + (e^2E^2/H^2)$ is much larger than the expansion rate (this is essentially why in everyday experiments one does not care whether we live in flat space or in a de Sitter space of very small $H$). In this limit, the results can be compared with those obtained using the instanton formalism. Both methods agree, not only in the exponential dependence but also in the prefactors. This is true even when the size of the instantons is comparable to the horizon size.

To summarize, our results seem to indicate that the spontaneous nucleation of defects during inflation and the decay of false or true vacuum through nucleation of true or false vacuum bubbles is well described by the instanton formalism, at least in 1+1 dimensions. Also, that our prescription for finding the equilibrium distribution of nucleated objects from the semiclassical partition function is correct. Although we have not attempted to give a rigorous justification to this prescription, we hope that with the examination of further examples a clearer picture will emerge. After this paper was completed,
we became aware of Refs. [38], dealing with the semiclassical approximation to the path integral. The methods developed in these references may be useful to provide a rigorous foundation to our prescription. Work along these lines is currently under way.

Finally, some comments on negative modes. Coleman has shown [39] that in flat space and at zero temperature an instanton describing the decay of a metastable state has one and only one negative mode. As noted in [1], the instantons describing the spontaneous nucleation of defects during inflation can have more than one negative mode. Indeed, we have seen that the scalar fields $\phi^{(i)}$ describing the transverse displacements of the worldsheet are tachyonic, and that as a result each field has a negative mode with $L = 0$. Therefore, the number of negative modes for such instantons is equal to the co-dimension of the worldsheet. It was shown in [3] that this is not in direct contradiction with Coleman’s theorem, which does not apply in de Sitter space (or in flat space at finite temperature). Still, the question remains of whether the wrong number of negative modes renders the instanton ‘unphysical’. From the analysis of the previous Section, we think that this is not the case. Using the method of Bogoliubov transformations, we find that particles are produced in de Sitter space for arbitrary co-dimensionality of the world-line, and that the results are always in agreement with the instanton results. Also, a nice feature of the prescription given in Section 7 is that the distribution $dN$ is always real regardless of the co-dimension, because the extra factors of $i$ coming from additional negative modes are compensated by imaginary factors coming from the complexification of additional boost zero modes.

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Appendix

In this Appendix we evaluate the functional determinants for a free scalar field $\phi$ on the $N$-sphere, for $N = 1, 2$ and 3. As explained in section 3, this scalar field represents small fluctuations of the worldsheet of the instanton, and the determinant gives the pre-exponential factor in the semiclassical partition function. The basic idea is to define a generalized zeta function (78) and then express the determinant in terms of $\zeta(0)$ and $\zeta'(0) \equiv d\zeta/dz|_{z = 0}$ [see Eq. (79)]. For the evaluation of $\zeta(z)$ we follow the method of Ref.[40], where the case $N = 4$ was studied (for fields of arbitrary spin).

A1 Determinant on the circle

This is the case $N = 1$. From (44) and (77) the eigenvalues are

$$\Lambda_L = L^2 + M^2 R_0^2 \equiv L^2 + x^2, \quad (L = 0, \ldots, \infty).$$

For $L \neq 0$ the degeneracy is 2 and for $L = 0$ it is 1, so

$$\zeta(z) = x^{-2z} + \sum_{L=1}^{\infty} 2L^{-2z} \left(1 + \frac{x^2}{L^2}\right)^{-z}.$$  \hfill (A1)

Expanding the binomial term in powers of $x/L$

$$\left(1 + \frac{x^2}{L^2}\right)^{-z} = \sum_{k=0}^{\infty} c_k \left(\frac{-x^2}{L^2}\right)^k,$$  \hfill (A1)

we have

$$\zeta(z) = x^{-2z} + \sum_{k=0}^{\infty} 2c_k x^{2k} (-1)^k \zeta_R(2z + 2k),$$  \hfill (A2)

were $\zeta_R(z) \equiv \sum_{L=1}^{\infty} L^{-z}$ is the usual Riemann’s zeta function. Of the coefficients $c_k$ all we need to know is that

$$c_0 = 1,$$  \hfill (A3)

$$c_k = \frac{z}{k} + O(z^2). \quad (k \geq 1)$$

Then it is clear that

$$\zeta(0) = 1 + 2\zeta_R(0) = 0,$$
since $\zeta_R(0) = -1/2$.

To evaluate (79) we also need $\zeta'(0)$. Expanding (A2) in powers of $z$ we have

$$\zeta(z) = -2z\ln(2\pi x) + \sum_{k=1}^{\infty} \frac{z^k}{k} x^{2k} \zeta_R(2k)(-1)^k + O(z^2),$$

where we have used $\zeta_R(0) = -1/2$, $\zeta_R'(0) = -(1/2)\ln(2\pi)$. As a result,

$$\zeta'(0) = -\ln(2\pi x)^2 + \sum_{k=1}^{\infty} (-1)^k \frac{2}{k} x^{2k} \zeta_R(2k). \quad (A4)$$

Now we need a technique to sum the series.

For convenience one introduces the notation

$$\zeta_R(z, \alpha) = \sum_{k=\alpha}^{\infty} L^{-z},$$

so that $\zeta_R(z, 1) = \zeta_R(z)$. Using [10]

$$\frac{d^n \Psi(\alpha)}{d\alpha^n} = (-1)^{n+1} n! \zeta_R(n+1, \alpha),$$

one easily arrives at the expressions

$$\sum_{n=0}^{\infty} \zeta_R(2n+1, \alpha) z^{2n} = -\frac{1}{2} [\Psi(\alpha + z) + \Psi(\alpha - z)], \quad (A5)$$

and

$$\sum_{n=1}^{\infty} \zeta_R(2n, \alpha) z^{2n} = \frac{z}{2} [\Psi(\alpha + z) - \Psi(\alpha - z)], \quad (A6)$$

where $\Psi(z) = d\ln \Gamma(z)/dz$ is the digamma function.

Now for the evaluation of (A4). Differentiating with respect to $x$ and using (A6) we have

$$\frac{d}{dx} \zeta'(0) = -\frac{2}{x} + 2i[\Psi(1+ix) - \Psi(1-ix)].$$

Upon integration we obtain the result

$$\zeta'(0) = -2\ln(2\sinh \pi x). \quad (A7)$$

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In the last step, the constant of integration is chosen so that (A7) agrees with (A4) when $x \to 0$. Restoring $x = MR_0$ we obtain the desired result (80).

Note that, since $\zeta(0) = 0$ the determinant does not depend on the arbitrary parameter $\mu$. This was to be expected, since after all for $N = 1$ the path integral is just equivalent to the path integral for a non-relativistic quantum mechanical oscillator at finite temperature. That is, (76) is equivalent to (68) with $V(x) = M^2 x^2/2$ and $\beta = 2\pi R_0$. In (68) the right hand side is shorthand for

$$\lim_{n \to \infty} \int \prod_{i=1}^{n} \frac{dx_i}{(2\pi \epsilon)^{1/2}} \exp \left(-\epsilon \sum_i \left[ \frac{1}{2} \left( \frac{x_{i+1} - x_i}{\epsilon} \right)^2 + V(x_i) \right] \right),$$

where $\epsilon = \beta/n$ and $x_{n+1} = x_1$. Here, there is no arbitrariness in the definition of the measure of integration, and so we do not expect any arbitrary scales to show up in the final result. The integral (A8) can be done using the methods of Ref. [41] applied to the case of periodic boundary conditions, and one recovers the result (80).

### A2 Determinant on the 2-sphere

In this case the eigenvalues are given by

$$\Lambda_L = L(L + 1) + M^2 R_0^2 \equiv (L + \frac{1}{2} + u)(L + \frac{1}{2} - u),$$

where $u^2 \equiv (1/4) - M^2 R_0^2$. The degeneracies are given by $g_L = 2L + 1$. It follows that

$$\zeta(z) = \sum_{n=1/2}^{\infty} 2n^{1-2z} \left[ 1 - \frac{u^2}{n^2} \right]^{-z},$$

where $n$ runs over the positive half integers. Expanding the binomial term in powers of $(u/n)$ one has

$$\zeta(z) = \sum_{k=0}^{\infty} 2c_k u^{2k} \zeta_R(2z + 2k - 1, \frac{1}{2}).$$

Expanding in the neighborhood of $z = 0$ we obtain

$$\zeta(z) = \frac{1}{12} + u^2 + z[4\zeta_R(-1, \frac{1}{2}) + Q] + O(z^2),$$

(A9)
where
\[ Q \equiv \sum_{k=2}^{\infty} \frac{2u^{2k}}{k} \zeta_R(2k - 1, \frac{1}{2}) - 2u^2 \Psi \left( \frac{1}{2} \right). \]

Here we have used
\[ \zeta_R(2z + 1, \alpha) = \frac{1}{2z} - \Psi(\alpha) + O(z), \]
and the relation
\[ \zeta_R(-1, \alpha) = -\frac{1}{2} \alpha^2 + \frac{1}{2} \alpha - \frac{1}{12}. \]

It is clear that \( \zeta(0) = (1/12) + u^2. \)

To evaluate \( \zeta'(0) \) we need to find \( Q \). Using (A5) one easily arrives at
\[ \frac{dQ}{du} = -2u[\Psi(\frac{1}{2} + u) + \Psi(\frac{1}{2} - u)]. \]

After a bit of algebra
\[ Q = -i\pi - 3 \ln \left( \frac{3}{2} - u \right) + C + O(2u - 3), \]
where \( C \) is a numerical constant of order unity. The last term, indicated as \( O(2u - 3) \) vanishes when \( u \to 3/2 \), i.e., when \( M^2 \to -2R_0^{-2} \), the case we are interested in.

To summarize, we have
\[ \zeta(0) = \frac{1}{12} + u^2 \]
\[ \zeta'(0) = -i\pi - 3 \ln \left( \frac{3}{2} - u \right) + \zeta' + O(2u - 3), \]
where we have absorbed the term \( 4\zeta'_R(-1, 1/2) \) in a new constant \( \zeta' \). Using (89) one finds, in the limit \( u \to 3/2 \),
\[ (det'[\mu R_0^{-1}]^{-1/2} = (\mu R_0)^{7/3} R_0^{-3}. \] (A10)

Some numerical constants have been absorbed in a redefinition of the renormalization scale \( \mu \).
A3 Determinant on the 3-sphere

In this case, from (44) and (77), with
\[ y^2 \equiv 1 - R_0^2 M^2, \]
\[ \Lambda_L = (L + 1)^2 - y^2. \]

The degeneracy is given by \( g_L = (L + 1)^2 \), so
\[ \zeta(z) = \sum_{n=1}^{\infty} \frac{n^2 - 2z}{n^2} = \sum_{k=0}^{\infty} c_k y^{2k} \zeta_R(-2 + 2z + 2k), \]
where \( c_k \) are given by (A3). Expanding around \( z = 0 \) we have
\[ \zeta(z) = \zeta_R(-2) + z[2\zeta_R(-2) + Q] + O(z^2), \]
where
\[ Q \equiv \sum_{k=1}^{\infty} \frac{y^{2k}}{k} \zeta_R(2k - 2). \]

It is clear that \( \zeta(0) = \zeta_R(-2) = 0 \).

Also, \( \zeta'(0) = 2\zeta_R'(-2) + Q \). To evaluate \( Q \) we first differentiate (A12) and then use (A6) to find, after some algebra,
\[ \frac{dQ}{dy} = -y^2 \frac{d}{dy} \left[ \ln(\sin \pi y) \right]. \]

Integrating,
\[ Q = -y^2 \ln(\sin \pi y) + \frac{2}{\pi^2} \int_0^{\pi y} x \ln(\sin x) dx, \]
where the constant of integration is chosen so that \( Q(y = 0) = 0 \), in order to agree with (A12).

The integral in (A13) cannot be done analytically for arbitrary \( \pi y \). However, this is not a problem, since we want to calculate the determinant only for \( M^2 \to -3R_0^{-2} \), which implies \( y = 2 \). Then the second term in (A13) is a definite integral which can be found in the tables [37], and we have
\[ Q(y) = -y^2 \ln(\sin \pi y) - 4 \ln 2 + 3i\pi + O(y - 2). \]

Putting it all together we have
\[ \zeta(0) = 0, \]
\[ \zeta'(0) = 2\zeta_R'(-2) - 4 \ln 2 + 3i\pi - y^2 \ln(\sin \pi y) + O(y - 2), \]
where \( y \equiv (1 - R_0^{-2} M^2)^{1/2} \).
Figure caption

- **Fig.1** Euclidean de Sitter space is a $d$-sphere of radius $H^{-1}$. The instantons for membrane creation and spontaneous nucleation of defects can be seen as spherical worldsheets of dimension $N$ and radius $R_0$ embedded in the $d$-sphere. For the case of spontaneous nucleation, the instantons have maximal radius $R_0 = H^{-1}$. For the case of membrane creation, the co-dimension of the worldsheet is one, $d = N + 1$, and the instanton can be obtained by intersecting the $d$-sphere with a hyperplane at a distance $\omega_0$ from the origin. The worldsheet of the membrane divides the $d$-sphere into two regions, which we conventionally denote as the inside and the outside of the membrane. The value of the electric field in the outside region is taken to be equal to the background electric field before nucleation $E_0$. By Gauss’ law, the electric field in the inside region is $E_0 - e$.

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