SMALL DOUBLINGS IN ABELIAN GROUPS
OF PRIME POWER TORSION

YIFAN JING AND SOUKTIK ROY

ABSTRACT. Let $A$ be a subset of $G$, where $G$ is a finite abelian group of torsion $r$. It was conjectured by Ruzsa that if $|A + A| \leq K|A|$, then $A$ is contained in a coset of $G$ of size at most $r^{CK}|A|$ for some constant $C$. The case $r = 2$ received considerable attention in a sequence of papers, and was resolved by Green and Tao. Recently, Even-Zohar and Lovett settled the case when $r$ is a prime. In this paper, we confirm the conjecture when $r$ is a power of prime. In particular, the bound we obtain is tight.

Keywords: sumset, abelian group, compression, doubling
MSC numbers: 11P70, 05D05

1. Introduction

The study of sums of sets inside ambient groups constitutes a fundamental aspect of additive combinatorics and number theory. Given sets $A, B$ inside an ambient abelian group $G$, the sum set of $A, B$ is defined by

$$A + B = \{a + b \mid a \in A, b \in B\}.$$ 

The doubling constant of $A$ is defined to be the quantity $|A + A|/|A|$. In a qualitative sense, a small value of this quantity points towards the set $A$ possessing some approximate algebraic structure. Depending on the ambient group one arrives at various notions of approximate algebraicity (indeed, the doubling constant makes sense even in non-abelian settings). Mathematical study along these lines can be traced back to a crucial theorem of Freiman [7] which asserts that any non-empty finite set of integers in $\mathbb{Z}$ with small sum set can be efficiently contained in a generalized arithmetic progression.

In this paper, we restrict our attention to an abelian group $G$ with finite torsion $r$, and a finite set $A$ in $G$. The affine span of $A$, denoted by $\langle A \rangle$, is defined to be the smallest subgroup or coset of a subgroup containing $A$. Since the structure of both $A + A$ and $\langle A \rangle$ remain unaffected if we translate all elements of $A$ by some fixed constant, we shall assume throughout this paper that the identity element $0$ is in $A$. Under this assumption, the affine span $\langle A \rangle$ is easily seen to be exactly the minimal subgroup of $G$ containing $A$. The spanning constant of $A$ is defined
by $|\langle A \rangle|/|A|$. The Freiman–Ruzsa Theorem \cite{Ru} explores the relation between the doubling constant and the spanning constant of $A$.

**Theorem 1.1** (Freiman–Ruzsa Theorem). Let $A$ be a finite subset of an abelian group with torsion $r$. Suppose there is a constant $K$ such that $|A + A|/|A| < K$, then

$$\frac{|\langle A \rangle|}{|A|} < K^2 r^{K^2}.$$ 

Then a natural question is to ask how tight this bound is. For applications, one may hope for polynomial dependence on $K$, for example, but Ruzsa observed that the dependence on $K$ is at least exponential. In the same paper he conjectured that this was essentially the worst case.

**Conjecture 1.2** (Ruzsa \cite{Ru}). Let $A$ be a finite subset of an abelian group with torsion $r$, and there exists a constant $K$ such that $|A + A|/|A| \leq K$. Then there exists some constant $C \geq 2$ such that

$$\frac{|\langle A \rangle|}{|A|} \leq r^{CK}.$$ 

Green and Ruzsa \cite{GR} improved the bound to $K^2 r^{2K^2 - 2}$. The special case $r = 2$ has received considerable attention, see \cite{Go, Gr, Go2, Gr3, Go4, CL, K, Go5}. In particular, Green and Tao \cite{GR} confirmed the conjecture when $r = 2$ by showing the spanning constant of $A$ is at most $2^{2K + O(\sqrt{K} \log K)}$, and the tight upper bound $\Theta(2^{2K}/K)$ was finally determined by Even-Zohar \cite{EZ}. Later, Even-Zohar and Lovett \cite{EZL} settled the conjecture when the ambient group has prime torsion. In this paper, we consider ambient groups of prime power torsion (i.e. we set $r$ to be $q = p^m$ for some prime $p$), where we exploit extremal set theoretic methods first used for Freiman type theorems in \cite{GR}. We refine the method by introducing two different total orders on $\mathbb{Z}/q\mathbb{Z}$ and considering compression operators acting on $(\mathbb{Z}/q\mathbb{Z})^n$ based on these orders, and arrive at structural results for the extremal sets of fixed size and affine span in $(\mathbb{Z}/q\mathbb{Z})^n$. Analysing these deductions about structure gives us the following main result.

**Theorem 1.3.** Let $A$ be a finite subset of an abelian group of torsion $q = p^m$, where $p > 2$ is a prime and $m$ is a positive integer. Suppose $K > K_0$ for some constant $K_0$, and $|A + A| \leq K|A|$. Then

$$\frac{|\langle A \rangle|}{|A|} \leq \frac{q^{2K - 2}}{2K - 1}.$$ 

This confirms Ruzsa’s Conjecture for prime power torsions. The constant $K_0$ in the theorem depends on the ambient group $G$, and an example in Section 5 shows that
the dependence is necessary. The following result allows us to remove the dependence on $K_0$.

**Theorem 1.4.** Let $A$ be a finite subset of an abelian group of torsion $q = p^m$, where $p > 2$ is a prime and $m$ is a positive integer. Suppose $K \geq 1$ and $|A + A| \leq K|A|$. If $p \geq 5$, we have $\frac{|\langle A \rangle|}{|A|} \leq q^{2K}/K$. If $p = 3$, we have $\frac{|\langle A \rangle|}{|A|} \leq q^{10K}/K$.

We have a similar result when $q = 2^m$.

**Theorem 1.5.** Let $A$ be a finite subset of an abelian group of torsion $q = 2^m$. Then for every $K \geq 1$, and $|A + A| \leq K|A|$, we have $\frac{|\langle A \rangle|}{|A|} \leq \Theta(q^{2K}/K)$.

The following well-known construction shows that the bound we obtain in Theorem 1.3 is tight. Let $A = \{0, e_1, \ldots, e_{2K-2}\}$, where $e_i$ is the basis of $(\mathbb{Z}/r\mathbb{Z})^{2K-2}$, where $K \in \mathbb{N}$ and $r \geq 3$. In this case, the doubling constant is $K$ and the spanning constant is $\frac{q^{2K-2}}{2K-1}$.

Note that for the group $G$ with torsion $r$, without loss of generality we may assume $G = (\mathbb{Z}/r\mathbb{Z})^n$, otherwise we can take the preimage of $A$ under the quotient map to obtain the same doubling and spanning constant.

**Notation.** In this paper, we always let $p$ be a prime, and $q = p^m$ for some integer $m$. We write $A = X \uplus Y$ if $A = X \cup Y$ and $X \cap Y = \emptyset$. Suppose $H, G$ are groups and $H \leq G$, we use $G/H$ to denote the collection of $H$-cosets.

**Overview.** The paper is organized as follows. In Section 2, we introduce two orders in $(\mathbb{Z}/q\mathbb{Z})^n$ and discuss the properties of the orders. In Section 3, we define compression operators under the orders we defined in section 2, and prove some structural results pertaining to the compressed sets. Section 4 contains the proof of our main results.

2. **Sum order and Pseudo-sum order**

We first consider the elements in $\mathbb{Z}/q\mathbb{Z}$, where $q = p^m$ is a power of prime. Since $\mathbb{Z}/q\mathbb{Z}$ contains non-trivial subgroups, the natural order used in the proofs of other cases [5, 6, 8] will not work. We define the sum order of $\mathbb{Z}/q\mathbb{Z}$ as follows. Let $x \in \mathbb{Z}/q\mathbb{Z}$, and $x = \sum_{i=1}^{m} x_ip^{m-i}$ where $x_i \in \mathbb{F}_p$. We define $x < y$ if $x_i < y_i$ for some $i$ and $x_j = y_j$ when $j > i$. Let $\ell(i)$ be the $(i+1)$-th element in $\mathbb{Z}/q\mathbb{Z}$ under this order.

**Example 2.1.** In $\mathbb{F}_8$, we have $0 < 4 < 2 < 6 < 1 < 5 < 3 < 7$, and $\ell(0) = 0$, $\ell(1) = 4$, $\ell(2) = 2$.

Let $G$ be a group such that

$$G := \bigoplus_{i=1}^{n} \mathbb{Z}/p^{m_i}\mathbb{Z},$$
where $1 \leq m_i \leq m$ are integers for every $1 \leq i \leq n$. We also define the sum order of the elements in $G$. For every $x, y \in G$, let $x = \sum_{i=1}^{n} x_i e_i$ and $y = \sum_{i=1}^{n} y_i e_i$, where $x_i, y_i \in \mathbb{Z}/p^m \mathbb{Z}$. We say $x \prec y$ if for some $i$ we have $x_i < y_i$ and $x_j = y_j$ for every $j > i$.

For every $x \in G$, we define the height function $h(x)$ the index of $x$ under the sum order. Given $A \subseteq G$, $h(A) = \sum_{x \in A} h(x)$. We define the initial segment of size $t$ of $A$, denoted by $IS(t, A)$, is the set of $t$ smallest elements in $A$. When $A = G$, we simply write $IS(t)$. The following lemma is the basic property of the sum order.

**Lemma 2.2.** Let $\mathbb{Z}/q\mathbb{Z}$ be the ambient group. Suppose $c, d$ are positive integers and $c, d < q$. Then

$$|IS(c) + IS(d)| = \min_{t \mid q} \left\{ \left( \left\lceil \frac{c}{t} \right\rceil + \left\lceil \frac{d}{t} \right\rceil - 1 \right) t \right\}.$$

**Proof.** Suppose $c \leq d$. We prove it by induction on $m$. The base case $m = 1$ follows the basic property of arithmetic progressions. Now we move to the induction step. If $c, d \leq p^{m-1}$, it is clear that the inductive hypothesis applied. We may assume $d > p^{m-1}$, and let $d = \lceil d \rceil p^{m-1} + t$, where $t < p^{m-1}$.

We consider first that $c \leq p^{m-1}$. We have

$$|IS(c) + IS(d)| = |(IS(c) + \mathbb{Z}/p^{m-1}\mathbb{Z}) \cup (IS(c) + (IS(d) \setminus \mathbb{Z}/p^{m-1}\mathbb{Z}))|$$

$$= \min_{1 \leq t \leq m-1} \left\{ \left( \left\lceil \frac{c}{p^t} \right\rceil + \left\lceil \frac{p^{m-1}}{p^t} \right\rceil - 1 \right) p^t \right\}$$

$$+ (d_1 - 1)p^{m-1} + \min\{t + c - 1, p^{m-1}\}$$

$$= \min_{1 \leq t \leq m} \left\{ \left( \left\lceil \frac{c}{p^t} \right\rceil + \left\lceil \frac{d}{p^t} \right\rceil - 1 \right) p^t \right\}.$$ 

Suppose $c > p^{m-1}$. Let $c = c_1 p^{m-1} + s$ where $s < p^{m-1}$ and $c_1 > 0$. We obtain

$$|IS(c) + IS(d)| = |(\mathbb{Z}/p^{m-1}\mathbb{Z} + \mathbb{Z}/p^{m-1}\mathbb{Z}) \cup (IS(d) + \mathbb{Z}/p^{m-1}\mathbb{Z})$$

$$\cup ((IS(c) \setminus \mathbb{Z}/p^{m-1}\mathbb{Z}) + (IS(d) \setminus \mathbb{Z}/p^{m-1}\mathbb{Z}))|$$

$$= \min\{p^{m-1} + d_1 p^{m-1} + (c_1 - 1)p^{m-1} + s + t - 1, p^m,$$

$$p^{m-1} + d_1 p^{m-1} + (c_1 - 1)p^{m-1} + p^{m-1}\}$$

$$= \min_{1 \leq t \leq m} \left\{ \left( \left\lceil \frac{c}{p^t} \right\rceil + \left\lceil \frac{d}{p^t} \right\rceil - 1 \right) p^t \right\},$$

which finishes the proof. \qed

Note that in the sum order, 0 is always the smallest element, but 1 is quite large. In fact, we have $\ell^{-1}(1) = p^{m-1}$ when $q = p^m$. Sometimes we want 1 is small as well. We define the pseudo-sum order ($\prec_p$) of $G$, for every $x = \sum_{i=1}^{n} x_i e_i$ and $y = \sum_{i=1}^{n} y_i e_i$, where $x \prec_p y$ if $x_n < y_n$, or $x_i < y_i$ for some $i \leq n - 1$ and $x_j = y_j$ for all $j > i$. Let
$H \subseteq G$, we define $IS_p(t, H)$ be the initial segment of size $t$ of $H$. It is not hard to see, Lemma 2.2 does not hold for pseudo-sum order.

3. Structure of Compressed sets

3.1. Compressions. In this section, we will use cosets to partition $G = \bigoplus_{i=1}^n \mathbb{Z}/q_i\mathbb{Z}$, where $q_i = p^{m_i}$. For every $X \subseteq G$, let $H(X)$ be the smallest subgroup (or the coset of a subgroup) containing $X$. The $X$-compression of a subset $A \subseteq G$ is

$$C_X(A) := \bigcup_{S \in G/H(X)} IS(|A \cap S|, S).$$

When $X = \{v\}$, we simply write $X$-compression of $A$ as $C_v(A)$. If $C_X(A) = A$, we say $A$ is $X$-compressed. Clearly, $C_X(A)$ is $X$-compressed, and $|C_X(A)| = |A|$.

The following theorem [13] is an analogue of Cauchy–Davenport Theorem [11, 12].

**Theorem 3.1** ([13]). Let $R, S$ be non-empty finite subsets of $\mathbb{Z}/r\mathbb{Z}$. Then

$$|R + S| \geq \min_{d|r} \left\{ \left\lceil \frac{|R|}{d} \right\rceil + \left\lceil \frac{|S|}{d} \right\rceil - 1 \right\} d.$$ 

The following lemma shows, compression operators under sum order behave well on sumsets. In our proof, it suffices to consider the case when $X$ is a single vector, and the same proof works for the general case as well.

**Lemma 3.2.** Suppose $A \subseteq G$ and $v \in G \setminus \{0\}$. Then

$$|C_v(A) + C_v(A)| \leq |A + A|.$$ 

**Proof.** Let $v = \sum_{i=1}^n v_ie_i$, where $v_i \in \mathbb{Z}/q_i\mathbb{Z}$. Note that $H(\{v\}) = \{kv \mid k \in \mathbb{Z}/q_i\mathbb{Z}\}$.

Suppose $S_x, S_y \subseteq G/H(\{v\})$, where $S_x = x + (\mathbb{Z}/q\mathbb{Z})v$ and $S_y = y + (\mathbb{Z}/q\mathbb{Z})v$. Let $t$ be the largest integer such that $v_t \neq 0$. Without loss of generality, we may assume $v_t = p^\alpha$, where $0 \leq \alpha \leq m_i - 1$. Otherwise we can apply an affine transform on $v$.

Assume $X, Y \subseteq \mathbb{Z}/q\mathbb{Z}$ and $A \cap S_x = x + Xv$, $A \cap S_y = y + Yv$. By applying Theorem 3.1 we have

$$|(A + A) \cap S_{x+y}| \geq |(A \cap S_x) + (A \cap S_y)|$$

$$\geq \min_{1 \leq j \leq m} \left\{ \left\lceil \frac{|X|}{p^j} \right\rceil + \left\lceil \frac{|Y|}{p^j} \right\rceil - 1 \right\} p^j$$

$$= IS(|X|, \mathbb{Z}/q\mathbb{Z}) + IS(|Y|, \mathbb{Z}/q\mathbb{Z})$$

$$= IS(|X|, S_x) + IS(|Y|, S_y).$$

The latter follows by Lemma 2.2 and definition of the sum order of $S_x$ at its $t$-th coordinate, and same as in $S_y$ and $S_{x+y}$. Therefore,

$$IS(|(A + A) \cap S_{x+y}|, S_{x+y}) \geq IS(|A \cap S_x|, S_x) + IS(|A \cap S_y|, S_y).$$

Lemma 3.3. Let $u \subseteq G$, and suppose for every $u \in A$, $u = \sum_{i=1}^{n} u_i e_i$, we have $u_n \in \{0, 1, \ldots, p - 1\}$. Let $v \in G \setminus \{0\}$. Then

$$|C_v(A) + C_v(A)| \leq |A + A|$$

Proof. Let $v = \sum_{i=1}^{n} v_i e_i$. Let $t$ be the largest integer such that $v_t \neq 0$, and we may assume that $v_t = p^t$, where $0 \leq \alpha \leq m_t - 1$. If $t \leq n - 1$, this case is proved in Lemma 3.2. We now consider $t = n$.

Suppose $S_x = x + (\mathbb{Z}/q\mathbb{Z})v$ and $S_y = y + (\mathbb{Z}/q\mathbb{Z})v$. Assume $X, Y \subseteq \mathbb{Z}/q\mathbb{Z}$, such that $A \cap S_x = x + Xv$ and $A \cap S_y = y + Yv$. We assume $v = 0$, otherwise $X = Y = \emptyset$. Then we have

$$|(A + A) \cap S_{x+y}| \geq |(A \cap S_x) + (A \cap S_y)|$$

$$\geq |X| + |Y| - 1$$

$$= |IS_p([X], S_x) + IS_p([Y], S_y)|.$$

By taking union of all $S_x, S_y$, we have $C_v(A) + C_v(A) \subseteq C_v(A + A)$, which implies $|C_v(A) + C_v(A)| \leq |A + A|$. \hfill \Box

Lemma 3.3 shows that when $A$ has certain structure, the compression operators under pseudo-sum order also behave well on subset of $A$.

3.2. Compressions preserve affine spanning. We first consider the compressions under sum order. Note that for every $A \subseteq G$, after we apply compression operator to $A$, Lemma 3.2 implies the doubling constant does not change. The main idea of the proof is reduction the problem to compressed sets. If we also have $|\langle C_v(A) \rangle| = |\langle A \rangle|$, we are able to apply induction on $h(A)$. However, in most of the cases, $|\langle C_v(A) \rangle| \leq |\langle A \rangle|$. In this subsection, we study the compressions which preserve the affine spanning of $A$.

Let $E = \{0, e_1, \ldots, e_n\}$ be the affine basis of $G$ and $A \supseteq E$. We say $A$ is $\langle \langle E \rangle \rangle$-compressed, if for every $v$, $E \subseteq C_v(A)$ implies $A$ is $v$-compressed. The lemmas below give us the rough structure of the $\langle \langle E \rangle \rangle$-compressed sets.

**Lemma 3.4.** Suppose $A \subseteq G = \bigoplus_{i=1}^{n} \mathbb{Z}/q_i\mathbb{Z}$ is $\langle \langle E \rangle \rangle$-compressed and $q = p^m_i$. Then for every $i \in [n]$ and $v \in \text{span}\{0, e_1, \ldots, e_{i-1}\}$, we have the following property. If $m_i \geq 2$, then $A$ is $(te_i - v)$-compressed, for every $t \in \mathbb{Z}/p^m_i\mathbb{Z} \setminus \{0\}$ which is divisible by $p$. When $m_i = 1$, $A$ is $(e_i - v)$-compressed for every $v \in A \cap \text{span}\{0, e_1, \ldots, e_{i-1}\}$. 

Proof. We first consider the case when \( m_i \geq 2 \). Let \( b_i = te_i - v \), recall that

\[
C_{b_i}(A) = \bigcup_{S \in G/H\{b_i\}} IS(|A \cap S|, S),
\]

where \( S = x + (\mathbb{Z}/q_i\mathbb{Z})b_i \) for some \( x \in G \). When \( j \neq i \), it is clear that \( e_j \) is the smallest element in the coset \( e_j + (\mathbb{Z}/q_i\mathbb{Z})b_i \) except for 0. Now we consider the coset \( e_i + (\mathbb{Z}/q_i\mathbb{Z})b_i \). By the definition of \( \ell(i) \), we can see that \( e_i \) is still the smallest element in \( e_i + (\mathbb{Z}/q_i\mathbb{Z})b_i \), since \( \ell(p^{m_i-1}) = 1 \) and \( p \mid \ell(i) \) when \( i < p^{m_i-1} \). This implies \( A \) is \( b_i \)-compressed for every \( i \).

When we have \( m_i = 1 \), then for \( j \neq i \), we still have that \( e_j \) is the smallest element in \( e_j + \mathbb{Z}/p\mathbb{Z}(e_i - v) \). For \( e_i \), note that \( e_i \) is the second smallest element in \( e_i + \mathbb{Z}/p\mathbb{Z}(e_i - v) \) while the smallest one is \( v \). Thus \( v \in A \) implies that \( A \) is \((e_i - v)\)-compressed. \( \square \)

Let \( F \leq G \) be the maximum subgroup of \( A \) such that \( F = \text{span}\{0, e_1, \ldots, e_f\} \). The following lemma gives us some information of the structure of compressed set.

**Lemma 3.5.** Suppose \( A \subseteq G \) is \( (\langle E \rangle) \)-compressed, and let \( F, f \) be defined as above. Therefore,

(i) for every \( j \geq 2 \) and \( v \in \text{span}\{0, e_1, \ldots, e_{f+j-1}\} \), we have \( v + te_{f+j} \notin A \), where \( t \in \mathbb{Z}/q_{f+j}\mathbb{Z} \) is divisible by \( p \).

(ii) for every \( j \geq 2 \) and \( v \in \text{span}\{0, e_1, \ldots, e_{f+j-1}\} \), if there is some \( i \geq tp^{m_{f+j}-1}+1 \) such that \( \ell(i)e_{f+j} + u \in A \) for some \( u \in \text{span}\{e_1, \ldots, e_{f+j-1}\} \) and \( t = 1, \ldots, p-1 \). Then for every \( tp^{m_{f+j}-1} \leq s \leq i-1 \), we have \( \ell(s)e_{f+j} + v \in A \).

**Proof.** By the way we define \( F \), it is clear that \( F + (\mathbb{Z}/q_{f+1}\mathbb{Z})e_{f+1} \not\subseteq A \). That is, there exists some \( v \in \text{span}\{0, e_1, \ldots, e_{f+1}\} \) such that \( v + e_{f+1} \notin A \). Given \( t \in (\mathbb{Z}/q_{f+1}\mathbb{Z}) \setminus \{0\} \) with \( p \mid t \) and \( j \geq 2 \), we have \( te_{f+j} + u \succ e_{f+1} + v \), for every \( u \in \text{span}\{e_1, \ldots, e_{f+j-1}\} \). Both of them lie in the coset \( e_{f+1} + v + \mathbb{Z}/q_{m+j}\mathbb{Z}(te_{f+j} - v + u - e_{f+1}) \), then Lemma 3.4 implies that \( te_{f+j} + u \notin A \).

Suppose \( j \geq 2 \), \( \ell(i)e_{f+j} + u \in A \) for some \( u \in \text{span}\{e_1, \ldots, e_{f+j-1}\} \) and \( i \geq tp^{m_{f+j}-1}+1 \). Then for every \( tp^{m_{f+j}-1} \leq s \leq i-1 \) and every \( v \in \text{span}\{e_1, \ldots, e_{f+j-1}\} \), we have \( \ell(s)e_{f+j} + v \prec \ell(i)e_{f+j} + u \). Since both of them lie on \( \ell(s)e_{f+j} + v + \mathbb{Z}/q_{f+j}\mathbb{Z}((\ell(i) - \ell(s))e_{f+j} + (u - v)) \), and \( p \mid (\ell(i) - \ell(s)) \), \( \ell(i)e_{f+j} + u \in A \) implies \( \ell(s)e_{f+j} + v \in A \). \( \square \)

In the rest of the section, we consider the pseudo-sum order of \( A \). We still assume that \( A \) contains \( E \) as a subset. The following observation provides some information the compression operators preserve affine spanning.

**Lemma 3.6.** For every \( v \in A \cap \text{span}\{0, e_1, \ldots, e_{n-1}\} \), the compression operator \( C_{e_n-v} \) preserve the affine spanning of \( A \).
Proof. Note that all the elements in the coset \( e_n + \mathbb{Z}/q\mathbb{Z}(e_n - v) \) smaller than \( e_n \) are already in \( A \). Also \( e_i \) is the smallest element in \( e_i + \mathbb{Z}/q\mathbb{Z}(e_n - v) \) when \( i \neq n \). Thus, \( C_{e_n-v} \) preserves the affine spanning of \( A \). \( \square \)

The following lemma gives us a rough structure of the compressed set under pseudo-sum order.

Lemma 3.7. Given \( A \subseteq G \) is \((e_n-v)\)-compressed for every \( v \in A \cap \text{span}\{0, e_1, \ldots, e_{n-1}\} \). Suppose \( A = \bigcup_{i=0}^{p-1} (A_i + ie_n) \), where \( A_i \subseteq \text{span}\{0, e_1, \ldots, e_{n-1}\} \) and \( v \in A_i \) when \( v + ie_n \in A \). Then we have the following properties.

(i) \( A_0 \supseteq A_1 \supseteq \cdots \supseteq A_{p-1} \).

(ii) If \( A_0 \) is not a subgroup or a coset of a subgroup, we have \(|A_1| = 1 \) and \(|A_2| = \cdots = |A_{p-1}| = 0 \).

Proof. For every \( v \neq A_i \), both of \( v + ie_n \) and \( v + (i+1)e_n \) lies on the coset \( v + ie_n + (\mathbb{Z}/q\mathbb{Z})e_n \), and \( v + ie_n \prec v + (i+1)e_n \). This means if \( v \notin A_i \), then \( v \notin A_{i+1} \). Hence we have \( A_0 \supseteq A_1 \supseteq \cdots \supseteq A_{p-1} \).

Let \( u, v \in A_0 \) and \( u + v \notin A_0 \). Consider the cosets \( u + v + \mathbb{Z}/q\mathbb{Z}(e_n - u) \) and \( u + v + \mathbb{Z}/q\mathbb{Z}(e_n - v) \), we can see that both of the \( e_n + u \) and \( e_n + v \) are not in \( A \), that is \( u, v \notin A_1 \). Then for every \( v \in A_1 \), we have \( v \in A_0 \) and \( v + A_0 = A_0 \). Since \( A_0 \) is not a subgroup or a coset of a subgroup, we only have \( 0 + A_0 = A_0 \). By the assumption, \( e_n \in A \), which implies \( 0 \notin A_1 \). Then \( |A_1| = 1 \).

Note that \( |A_0| > 1 \) since \( e_i \in A_0 \) for every \( i = 1, \ldots, n - 1 \). Then there is \( u \in A_0 \) such that \( u \notin A_1 \). Consider the coset \( e_n + u + \mathbb{Z}/q\mathbb{Z}(e_n - u) \), we have \( e_n + u \prec 2e_u \) and both of them lie in the coset. Then \( 2e_n \notin A \), which is \( 0 \notin A_2 \). Therefore, by (i) we obtain \( |A_2| = \cdots = |A_{p-1}| = 0 \). \( \square \)

4. Proof of the main results

We make use of the following results obtained by Even-Zohar [5] and by Even-Zohar and Lovett [6].

Theorem 4.1 ( [6] ). Let \( G(x) = \frac{x^2 + x + 1}{x + 1} \). For \( K \geq 1 \), denote by \( t \geq 1 \) the unique integer for which \( G(t) \leq K < G(t + 1) \). For \( A \subseteq \mathbb{F}_2^n \) such that \( |A + A|/|A| \leq K \), we have \(|\langle A \rangle|/|A| \leq F(K) \), where

\[
F(K) = \begin{cases} 
\frac{2^{t+1}}{t^{t+2}} & K \geq G(t) < \frac{t^{2}+t+1}{2t}, \\
\frac{2^{t+1}}{t^{t+1}} & \frac{t^{2}+t+1}{2t} \leq K < G(t + 1).
\end{cases}
\]

\( F(K) \) grows as \( \Theta(2^K/K) \).
Theorem 4.2 (4.2). Let $p > 2$ prime and $K > K_0$. Suppose $A$ is a subset of an abelian group of torsion $p$. If $|A + A| \leq K|A|$, then $|\langle A \rangle| \leq q^{2K-2} |A|$. Here $K_0 = 8$ is a constant.

We now have all the machinery needed to prove Theorem 1.3.

Proof of Theorem 1.3. Suppose $|\langle A \rangle| = G$ and

$$ A \subseteq G = \bigoplus_{i=1}^{n} \mathbb{Z}/q_i \mathbb{Z}, $$

where $q_i = p^{m_i}$. We may assume $1 \leq m_1 \leq m_2 \leq \cdots \leq m_n = m$. Without loss of generality, we may also assume that $A$ contains the affine basis of $G$, that is, $E = \{0, e_1, \ldots, e_n\} \subseteq A$. Suppose we have

$$ \frac{|\langle A \rangle|}{|A|} = \frac{q^{2K-2}}{2K-1} $$

for some $K \geq K_0$. We are going to show that the doubling constant of $A$ is at least $K$. The proof goes by induction on $h(A)$ under sum order.

We consider first when $A$ is not $\langle \langle E \rangle \rangle$-compressed. Then there exists $v$ such that $C_v(A) \neq A$ and $|\langle C_v(A) \rangle| = |\langle E \rangle| = |\langle A \rangle|$. Since $h(C_v(A)) < h(A)$, and by Lemma 3.2, the inductive hypothesis applied.

Now we assume that $A$ is $\langle \langle E \rangle \rangle$-compressed. We are going to prove the theorem by induction on $\sum_{i=1}^{n} m_i$. The base case is when $G$ is a finite field, and it is obtained by Theorem 4.2. It is easy to see that when $f = n$, both of the doubling constant and the spanning constant of $A$ are 1.

Suppose $f = n - 1$. In this case, we have $|A| > |\langle A \rangle|/q$, which implies

$$ \frac{q^{2K-2}}{2K-1} < q, $$

a contradiction when $K_0 > 2$.

Now we assume that $f \leq n - 2$. Note that $A$ is $\langle \langle E \rangle \rangle$-compressed, by Lemma 3.5, we have that $te_n + v \notin A$ for every $t \in \mathbb{Z}/q_0 \mathbb{Z}$ with $p \mid t$ and every $v \in \text{span}\{0, e_1, \ldots, e_{n-1}\}$. Let $ip^{m-1} \leq s_i \leq (i + 1)p^{m-1} - 1$ ($i = 1, \ldots, p - 1$) be the largest integer such that $\ell(s_i)e_n + v \in A$ for some $v \in \text{span}\{0, e_1, \ldots, e_{n-1}\}$. Let $A_{i,j}$ ($1 \leq j \leq s_i + 1 - ip^{m-1}$) be the subset of $\text{span}\{e_1, \ldots, e_{n-1}\}$ such that for every $v \in A_{i,j}$, we have $v + \ell(ip^{m-1} - 1 + j)e_n \in A$ for every $i \geq 1$, and $v \in A_0$ if $v + e_n \in A$. We write $\gamma_i := s_i + 1 - ip^{m-1}$ for the convenience.

By Lemma 3.3, we have $|A_{i,j}| = \prod_{t=1}^{n-1} q_t$ for every $1 \leq j \leq \gamma_i - 1$, and $|\langle A_0 \rangle| = \prod_{t=1}^{n-1} q_t$. We denote $A_{i,j}$ by $A'$ for $1 \leq j \leq \gamma_i - 1$ since all of them are same. We have $|\langle A \rangle| = p^m |A'|$. 
Suppose there is some \( i \in \{1, 2, \ldots, p-1\} \) such that \( s_i \geq ip^{m-1} + 1 \). We have \(|A| > |A'| = |\langle A \rangle|/p^m\). Therefore,
\[
\frac{(p^m)^{2K-2}}{2K - 1} = \frac{|\langle A \rangle|}{|A|} < p^m,
\]
which cannot happen when \( K_0 \geq 2 \).

Now we have \( s_i = ip^{m-1} \), that is, \( \ell(s_i) = i \). We write \( A_i := A_{i,1} \) for every \( i = 1, \ldots, p-1 \). That means
\[
A = \bigsqcup_{i=0}^{p-1}(A_i + ie_n).
\]

In the rest of the proof, let us consider the pseudo-sum order of \( A \). Let \( C(A) \) be the set obtained from \( A \) by applying all the possible compressions \( C_{e_n-u} \) for every \( u \in A_0 \). Note that we also have \( C(A) = \bigsqcup_{i=0}^{p-1}(C(A)_i + ie_n) \), where \( v \in C(A)_i \) if \( v + ie_n \in C(A) \). We simply write \( A_i := C(A)_i \).

Suppose \( A_0 \) is a subgroup or a coset of a subgroup of \( C(A) \). Thus
\[
\frac{q^{2K-2}}{2K - 1} = \frac{|\langle A \rangle|}{|A|} = \frac{q|\langle A_0 \rangle|}{|A|} = \frac{q|A_0|}{|A|} < q,
\]
contradicts \( K_0 \geq 2 \).

Now we apply Lemma \( 3.7 \). This gives us \( |C(A)| = |A_0| + 1 \). By induction hypothesis, there exists \( L > 0 \) such that
\[
(4.1) \quad \frac{|\langle A_0 \rangle|}{|A_0|} = \frac{q^{2L-2}}{2L - 1},
\]
and \( |A_0 + A_0| \geq L|A_0| \).

Apply Lemma \( 3.3 \) we have
\[
\frac{|A + A|}{|A|} \geq \frac{|C(A) + C(A)|}{|A|} = \frac{|A_0 + A_0| + |A_0| + 1}{|A|} \geq L \frac{|A_0|}{|A|} + 1.
\]

Note that
\[
\frac{q^{2L-2}}{2L - 1} = \frac{|\langle A_0 \rangle|}{|A_0|} = \frac{q|\langle A_0 \rangle|}{|A_0|} = \frac{|A_0|}{|A_0|} q^{2K-3} = \frac{|A_0| + 1}{|A_0|} q^{2K-3} = \frac{q^{2K-3}}{2K - 1},
\]
by the monotonicity of the function we have
\[
\frac{|A + A|}{|A|} \geq L \frac{|A_0|}{|A_0|} + 1 + 1 = \left( K - \frac{1}{2} \right) \frac{2K - 2}{2K - 1} + 1 = K.
\]
This finishes the proof. \( \square \)
In the proof of Theorem 1.3 we apply induction on \( A_0 \), so we require \( L \geq K_0 \), which implies \( K_0 \) will depend on the ambient group \( G \). This dependence is necessary, and we will discuss it in the next section. Theorem 1.4 gives us a result for all \( K \geq 1 \), which provides more information when the doubling constant is small relative to the dimension of the ambient group. Theorem 1.4 is proved identically to Theorem 1.3, the only different being our inductive step. When we apply induction, instead of using the result in Theorem 4.2 for the prime torsion case, we use the following theorem.

**Theorem 4.3.** Let \( G \) be a group of torsion \( p \) and \( A \) is a subset of \( G \) where \( p > 2 \) is a prime. Suppose there is \( K \geq 1 \) such that \( |A + A| \leq K|A| \). If \( p \geq 5 \), we have \( \frac{|A|}{|A|} \leq p^{2K}/K \). If \( p = 3 \), we have \( \frac{|A|}{|A|} \leq p^{10K}/K \).

The proof of Theorem 4.3 follows the same steps as the proof of Theorem 4.2 in [6], with a slightly different computation. We omit the further details.

Note that all the results in Section 2 and Section 3 work when \( q = 2^m \). The proof of Theorem 1.5 goes exactly the same as the proof of Theorem 1.3, except that in the induction step, the base case is by Theorem 4.1 when the ambient group is \( \mathbb{F}_2^m \) instead. By a careful computation we can also obtain a tight bound in this case since the result in Theorem 4.1 is tight. We leave the proof to the readers.

### 5. Concluding remarks

The constant \( K_0 \) in Theorem 1.3 obtained from the proof depends on the ambient group \( G \). Suppose \( G = \bigoplus_{i=1}^n \mathbb{Z}/p^{m_i}\mathbb{Z} \) and \( m_1 \leq \cdots \leq m_n \). Let \( \beta \) be the smallest integer such that \( m_\beta > 1 \), and let \( \alpha = n - \beta + 1 \). From the inductive argument in the proof above, we can see that \( K_0 \) we obtained is at least \( 8 + \frac{\alpha}{2} \), where 8 comes from Theorem 4.2. The following example shows that that dependence is needed, and \( K_0 \) we obtained is almost the best possible.

Let \( G_1 = \mathbb{Z}/3\mathbb{Z} \) and \( G_2 = (\mathbb{Z}/3^m\mathbb{Z})^\alpha \). Let \( G = G_1 \oplus G_2 \) and \( E = \{e_1, \ldots, e_{\alpha+1}\} \) is a basis of \( G \). Suppose \( A \subseteq G \) and \( A = \{0, e_1, 2e_1, e_2, e_3, \ldots, e_{\alpha+1}\} \). Then the doubling constant of \( A \) is

\[
K := \frac{|A + A|}{|A|} = \frac{3 + 3\alpha + \binom{\alpha}{2}}{3 + \alpha} = \frac{\alpha}{2} + 1.
\]

On the other hand we have

\[
\frac{|\langle A \rangle|}{|A|} = \frac{3^{m_\alpha + 1}}{3 + \alpha} > \frac{(3^m)^\alpha}{\alpha + 1} = \frac{(3^m)^{2K-2}}{2K - 1}.
\]

This fact shows that in this case \( K_0 \) should be at least \( 1 + \frac{\alpha}{2} \).
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Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL, USA

E-mail address: yifanjing17@gmail.com

Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL, USA

E-mail address: souktik2@illinois.edu