Potts model on infinite graphs and the limit of chromatic polynomials

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Abstract
Given an infinite graph $\mathcal{G}$ quasi-transitive and amenable with maximum degree $\Delta$, we show that reduced ground state degeneracy per site $W_r(\mathcal{G}, q)$ of the $q$-state antiferromagnetic Potts model at zero temperature on $\mathcal{G}$ is analytic in the variable $1/q$, whenever $|2\Delta e^3/q| < 1$. This result proves, in an even stronger formulation, a conjecture originally sketched in [10] and explicitly formulated in [14], based on which a sufficient condition for $W_r(\mathcal{G}, q)$ to be analytic at $1/q = 0$ is that $\mathcal{G}$ is a regular lattice.

Keywords: Potts model, chromatic polynomials, cluster expansion

§1 Introduction

The Potts model with $q$ states (or $q$ “colors”) is a system of random variables (spins) $\sigma_x$ sitting in the vertices $x \in \mathcal{V}$ of a locally finite graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$, and taking values in the set of integers $\{1, 2, \ldots, q\}$. Usually the graph $\mathcal{G}$ is a regular lattice, such as $\mathbb{Z}^d$ with the set of edges $\mathcal{E}$ being the set on nearest neighbor pairs, but of course more general situations can be considered. A configuration $\sigma_\mathcal{V}$ of the system is a function $\sigma_\mathcal{V} : \mathcal{V} \rightarrow \{1, 2, \ldots, q\}$ with $\sigma_x$ representing the value of the spin at the site $x$. We denote by $\Gamma_\mathcal{V}$ the set of all spin configurations in $\mathcal{V}$. If $\mathcal{V} \subseteq \mathcal{V}$ we denote $\sigma_\mathcal{V}$ the restriction of $\sigma_\mathcal{V}$ to $\mathcal{V}$ and by $\Gamma_\mathcal{V}$ the set of all spin configurations in $\mathcal{V}$.

Let $\mathcal{V} \subseteq \mathcal{V}$ and let $\mathcal{G}|_\mathcal{V} = (\mathcal{V}, \mathcal{E}|_\mathcal{V})$, where $\mathcal{E}|_\mathcal{V} = \{\{x, y\} \in \mathcal{E} : x \in \mathcal{V}, y \in \mathcal{V}\}$. Then for $\mathcal{V} \subseteq \mathcal{V}$ finite, the energy of the spin configuration $\sigma_\mathcal{V}$ in $\mathcal{G}|_\mathcal{V}$ is defined as

$$H_{\mathcal{G}|_\mathcal{V}}(\sigma_\mathcal{V}) = -J \sum_{\{x,y\} \subseteq \mathcal{E}|_\mathcal{V}} \delta_{\sigma_x \sigma_y}$$

(1.1)

where $\delta_{\sigma_x \sigma_y}$ is the Kronecker symbol which is equal to one when $\sigma_x = \sigma_y$ and zero otherwise. The coupling $J$ is called ferromagnetic if $J > 0$ and anti-ferromagnetic if $J < 0$.

The statistical mechanics of the system can be done by introducing the Boltzmann weight of a configuration $\sigma_\mathcal{V}$, defined as $\exp\{-\beta H_{\mathcal{G}|_\mathcal{V}}(\sigma_\mathcal{V})\}$ where $\beta \geq 0$ is the inverse temperature. Then the probability to find the system in the configuration $\sigma_\mathcal{V}$ is given by

$$\text{Prob}(\sigma_\mathcal{V}) = \frac{e^{-\beta H_{\mathcal{G}|_\mathcal{V}}(\sigma_\mathcal{V})}}{Z_{\mathcal{G}|_\mathcal{V}}(q)}$$

(1.2)
The normalization constant in the denominator is called the *partition function* and is given by

\[ Z_{G|V}(q, \beta) = \sum_{\sigma_V \in \Gamma_V} e^{-\beta H_{G|V}(\sigma_V)} \] (1.3)

The case \( \beta J = -\infty \) is the *anti-ferromagnetic* and *zero temperature* Potts model with \( q \)-states. In this case configurations with non zero probability are only those in which adjacent spins have different values (or colors) and \( Z_{G|V}(q) \) becomes simply the number of all allowed configurations.

The *thermodynamics* of the system at inverse temperature \( \beta \) and “volume” \( V \) is recovered through the *free energy per unit volume* given by

\[ f_{G|V}(q, \beta) = \frac{1}{|V|} \ln Z_{G|V}(q) \] (1.4)

where \( |V| \) denotes the cardinality of \( V \). All thermodynamic functions of the system can be obtained via derivative of the free energy. In the zero temperature anti-ferromagnetic case the function \( f_{G|V}(q, \beta) \) is usually called the *ground state entropy* of the system.

The Potts model, despite its simple formulation, is a intensely investigated subject. Besides its own interest as a statistical mechanics model, it has deep connections with several areas in theoretical physics, probability and combinatorics.

In particular, Potts models on general graphs are strictly related to a typical combinatorial problem. As a matter of fact, the partition function of the Potts model with \( q \) state on a finite graph \( G \), is equal, in the zero temperature antiferromagnetic case, to the number of proper coloring with \( q \) colors of the graph \( G \), where proper coloring means that adjacent vertices of the graph must have different colors. This number viewed as a function of the number of colors \( q \) is actually a polynomial function in the variable \( q \) which is known as *chromatic polynomial*. On the other hand, the same partition function in the general case can be related to more general chromatic type polynomials, known as *Tutte polynomials* [19]. This beautiful connection between statistical mechanics and graph coloring problems, first discussed by Fortuin and Kasteleyn [7], has been extensively studied and continues to attract many researchers till nowadays (see e.g. [1], [6], [13], [16], [17], [18] and reference therein).

One of the main interests in statistical physics is to establish whether or not a given system exhibits *phase transitions*. This means to search for points in the interval \( \beta \in [0, \infty] \) where some thermodynamic function (like e.g. the free energy defined above) is non analytic. Now, functions as (1.3) and (1.4) are manifestly analytic as long as \( V \) is a finite set. Hence phase transition (i.e. non-analyticity) can arise only in the so called *infinite volume limit* or *thermodynamic limit*. That is, the graph \( G \) is some countably infinite graph, usually a regular lattice, and the infinite volume limit

\[ f_G(q, \beta) = \lim_{N \to \infty} \frac{1}{|V_N|} \ln Z_{G|V_N}(q, \beta) \] (1.5)

is taken along a sequence \( V_N \) of finite subsets of \( \mathbb{V} \) such that, roughly speaking, \( G|V_N \) increase in size equally in all directions. Typically, when \( \mathbb{V} = \mathbb{Z}^d \), \( V_N \) are usually cubes of increasing size \( L_N \). There is a considerable amount of rigorous results about thermodynamic limit and phase transitions for the Potts model on \( \mathbb{Z}^d \) and other regular lattices, see e.g. the reviews [21] and, more recently, [18].
On the other hand, the study of thermodynamic limits of spin systems on infinite graphs which are not usual lattices has recently drove the attention of many researcher (e.g. [4], [8], [9], [11] and references therein).

Concerning specifically the antiferromagnetic Potts model and/or chromatic polynomial on infinite graphs, the problem of the thermodynamic limit was first considered by Biggs [3] with further discussions in [4] and in [10]. Very recently Sokal [17] has shown that for any finite graphs, the problem of the thermodynamic limit was first considered by Biggs [3] with further references therein. An important extension of this result would be to prove the existence and analyticity of the limiting free energy per unit volume (1.5) for a suitable class, as wide as possible, of infinite graphs. Such a generalization would be relevant from the statistical mechanics point of view, since it would imply that anti-ferromagnetic Potts model on such class of infinite graphs, if $q$ is sufficiently large, does not present a phase transition at zero temperature (and hence at any temperature).

To this respect, Shrock and Tsai have explicitly formulated a conjecture [14] (see also [10]), based on which the ground state entropy per unit volume of the antiferromagnetic Potts model at zero temperature on an infinite graph $\mathbb{G}$ should be analytic in the neighbor of $1/q = 0$ whenever $\mathbb{G}$ is a regular lattice.

In this paper we actually prove that this conjecture is true not only for regular lattices, but even for a wide calls of graphs. In particular we prove that the ground state zero entropy is analytic near $1/q = 0$ for all infinite graphs which are quasi transitive and amenable, and the limit may be evaluated along any Følner sequence in $\mathbb{V}$. We stress that this result proves the Schrock conjecture in a considerably stronger formulation, since all regular lattices, either with the elementary cell made by one single vertex or by more than one vertex, are indeed quasi-transitive amenable graphs but actually the class of quasi-transitive amenable graphs is much wider than that of regular lattices.

The paper is organized as follows. In section 2 we introduce the notations used along the paper, and we enunciate the main result (theorem 2). In section 3 we rephrase the problem in term of polymer expansion and prove a main technical result (lemma 4). In section 4 we prove a graph theory property (lemma 6) concerning quasi-transitive amenable graphs. Finally in section 5 we give the proof of the main result of the paper, i.e. theorem 2.

§2. Some further notations and statement of the main result

In general, if $V$ is any finite set, we denote by $|V|$ the number of elements of $V$. The set $\{1, 2, \ldots, n\}$ will be denoted shortly $I_n$. We denote by $P_2(V)$ the set of all subsets $U \subset V$ such that $|U| = 2$ and by $P_{\geq 2}(V)$ the set of all finite subsets $U \subset V$ such that $|U| \geq 2$.

Given a countable set $V$, and given $E \subset P_2(V)$, the pair $G = (V, E)$ is called a graph in $V$. The elements of $V$ are called vertices of $G$ and the elements of $E$ are called edges of $G$. Given two graphs $G = (V, E)$ and $G' = (V', E')$ in $V$, we say that $G' \subset G$ if $E' \subset E$ and $V' \subset V$.

Given a graph $G = (V, E)$, two vertices $x$ and $y$ in $V$ are said to be adjacent if $\{x, y\} \in E$. The degree $d_x$ of a vertex $x \in V$ in $G$ is the number of vertices $y$ adjacent to $x$. A graph $G = (V, E)$ is said locally finite if $d_x < +\infty$ for all $x \in V$, and it is said bounded degree if $\max_{x \in V} \{d_x\} \leq \Delta < \infty$. A graph $G = (V, E)$ is said to be connected if for any pair $B, C$ of subsets of $V$ such that $B \cup C = V$ and $B \cap C = \emptyset$, there is an edge $e \in E$ such that $e \cap B \neq \emptyset$ and $e \cap C \neq \emptyset$.

We denote by $\mathcal{G}_V$ the set of all connected graphs with vertex set $V$. If $V = I_n$ we use the notation $\mathcal{G}_n$ in place of $\mathcal{G}_I_n$. A tree graph $\tau$ on $V$ is a connected graph $\tau \in \mathcal{G}_V$ such that $|\tau| = |V| - 1$. We denote by $\mathcal{T}_V$ the set of all tree graphs of $V$ and shortly $\mathcal{T}_n$ in place of $\mathcal{T}_I_n$. 

Let $R_n = (R_1, \ldots, R_n)$ an ordered n-ple of non empty sets, then we denote by $E(R_n)$ the set $\subset P_2(I_n)$ defined as $E(R_n) = \{\{i, j\} \in P_2(I_n) : R_i \cap R_j \neq \emptyset\}$. We denote $G(R_n)$ the graph $(I_n, E(R_n))$.

Given two distinct vertices $x$ and $y$ of $G = (V, E)$, a path $\tau(x, y)$ joining $x$ to $y$ is a tree sub-graph of $G$ with $d_x = d_y = 1$ and $d_z = 2$ for any vertex $z$ in $\tau(x, y)$ distinct from $x$ and $y$. We define the distance between $x$ and $y$ as $|x - y| = \min\{|\tau(x, y)| : \tau(x, y)$ path in $G\}$. Remark that $|x - y| = 1 \iff \{x, y\} \in E$.

Given $G = (V, E)$ connected and $R \subset V$, let $E|_R = \{\{x, y\} \in E : x \in R, y \in R\}$ and define the graph $G|_R = (R, E|_R)$. Note that $G|_R$ is a sub-graph of $G$. We call $G|_R$ the restriction of $G$ to $R$. We say that $R \subset V$ is connected if $G|_R$ is connected. For any non void $R \subset V$, we further denote by $\partial R$ the external boundary of $R$ which is the subset of $V \setminus R$ given by

$$\partial R = \{y \in V \setminus R : \exists x \in R : |x - y| = 1\} \quad (2.1)$$

An automorphism of a graph $G = (V, E)$ is a bijective map $\gamma : V \to V$ such that $\{x, y\} \in E \Rightarrow \{\gamma x, \gamma y\} \in E$.

A graph $G = (V, E)$ is called transitive if, for any $x, y$ in $V$, it exists an automorphism $\gamma$ on $G$ which maps $x$ to $y$. The graph $G$ is called quasi-transitive if $V$ can be partitioned in finitely many sets $O_1, \ldots, O_s$ (vertex orbits) such that for $\{x, y\} \in O_i$ it exists an automorphism $\gamma$ on $G$ which maps $x$ to $y$ and this holds for all $i = 1, \ldots, s$. If $x \in O_i$ and $y \in O_i$ we say that $x$ and $y$ are equivalent. Remark that a locally finite quasi-transitive graph is necessarily bounded degree.

Roughly speaking in a transitive graph any vertex of the graph is equivalent; in other words $G$ “looks the same” by observers sitting in different vertices. In a quasi-transitive graph there is a finite number of different type of vertices and $G$ “looks the same” by observers sitting in vertices of the same type.

As a immediate example all periodic lattices with the elementary cell made by one site (e.g. square lattice, triangular lattice, hexagonal lattice, etc.) are transitive infinite graphs, while periodic lattices with the elementary cell made by more than one site are quasi-transitive infinite graphs.

Let $G = (V, E)$ be a connected infinite graph. $G$ is said to be amenable if

$$\inf \left\{ \frac{|\partial W|}{|W|} : W \subset V, \ 0 < |W| < +\infty \right\} = 0$$

A sequence $\{V_N\}_{N \in \mathbb{N}}$ of finite sub-sets of $V$ is called a Følner sequence if

$$\lim_{N \to \infty} \frac{|\partial V_N|}{|V_N|} = 0 \quad (2.2)$$

From now on $G = (V, E)$ will denote a connected locally finite infinite graph and $V_N \subset V$ a finite subset.

The partition function of the antiferromagnetic Potts model with $q$ colours on $G|_{V_N}$ at zero temperature can be rewritten (in a slightly different notation respect (1.1)) as

$$Z_{G|_{V_N}}(q) = \sum_{\sigma \in \sigma_N} \exp \left\{ - \sum_{\{x, y\} \in P_2(V_N)} J_{xy} \delta_{\sigma_x, \sigma_y} \right\} \quad (2.3)$$
where

\[ J_{xy} = \begin{cases} +\infty & \text{if } |x - y| = 1 \\ 0 & \text{otherwise} \end{cases} \quad (2.4) \]

We stress again that, due to assumption (2.4) (i.e. antiferromagnetic interaction + zero temperature), the function \( Z_{G|V_N}(q) \) represents the number of ways that the vertices \( x \in V_N \) of \( G|V_N \) can be assigned "colors" from the set \( \{1, 2, \ldots, q\} \) in such way that adjacent vertices always receive different colors. We also recall that the function \( Z_{G|V_N}(q) \) is called, in the graph theory language, the chromatic polynomial of \( G|V_N \).

**Definition 1.** Let \( G = (V, E) \) connected and locally finite infinite graph and let \( \{V_N\}_{N \in \mathbb{N}} \) be a Følner sequence of subsets of \( V \). Then we define, if it exists, the ground state specific entropy of the antiferromagnetic Potts model at zero temperature on \( G \) as

\[ S_G(q) = \lim_{N \to \infty} \frac{1}{|V_N|} \ln Z_{G|V_N}(q) \quad (2.5) \]

We also define the reduced ground state degeneracy per site as

\[ W_r(G, q) = \frac{1}{q} \lim_{N \to \infty} \left[ Z_{G|V_N}(q) \right]^{\frac{1}{|V_N|}} \quad (2.6) \]

The ground state specific entropy \( S_G(q) \) and the reduced ground state degeneracy \( W_r(G, q) \) are directly related by the identity

\[ S_G(q) = \ln W_r(G, q) + \ln q \quad (2.7) \]

We can now state our main result.

**Theorem 2.** Let \( G = (V, E) \) a locally finite connected quasi-transitive amenable infinite graph with maximum degree \( \Delta \), and let \( \{V_N\}_{N \in \mathbb{N}} \) a Følner sequence in \( G \). Then, \( W_r(G, q) \) exists, is finite, is independent on the choice of the sequence \( \{V_N\}_{N \in \mathbb{N}} \), and is analytic in the variable \( 1/q \) whenever \( |1/q| < 1/2e^3\Delta \) (\( e \) being the basis of natural logarithm).

Again we stress that this result proves the Schrock conjecture in a considerably stronger formulation, since any regular lattice is a quasi-transitive amenable graph but the class of quasi-transitive amenable graphs is actually much wider than that of regular lattices.

We remark also that the proof of analyticity of \( W_r(G, q) \) requires to prove the analyticity and boundness of the function \( |V_N|^{-1} \ln Z_{G|V_N}(q) \) for any finite graph \( G|V_N \) in a disk \( |1/q| < 1/C\Delta \) uniformly in the volume \( V_N \), which is a stronger statement than theorem 5.1 in [17], claiming that the zeros of the function \( Z_{G|V_N}(q) \) lie in the disk \( |q| < C\Delta \) for any \( G|V_N \) finite with maximum degree \( \Delta \).

§3. **Polymer expansion and analyticity**

We first rewrite the partition function of the Potts model on a generic finite graph \( G = (V, E) \) as a hard core Polymer gas grand canonical partition function. Without loss in generality, we will assume in this section that \( G \) is a sub-graph of a bounded degree infinite graphs \( G = (V, E) \) with maximum degree \( \Delta \). Denote by \( \pi(V) \) the set of all unordered partitions of \( V \), i.e. an element of \( \pi(V) \) is an unordered n-ples \( \{R_1, R_2, \ldots, R_n\} \), with \( 1 \leq n \leq |V| \), such that, for \( i, j \in I_n, R_i \subset V \),
$R_i \neq \emptyset$, $R_i \cap R_j = \emptyset$, and $\cup_{i=1}^{n_i} R_i = V$. Then, by writing the factor $\exp\{-\sum_{\{x,y\}\subset V} \delta_{x,y} J_{xy}\}$ in (2.3) as $\prod_{\{x,y\}\subset V} \{\exp\{-\delta_{x,y} J_{xy}\} - 1\} + 1$ and developing the product (a standard Mayer expansion procedure, see e.g. [5]) we can rewrite the partition function on $G$ (2.3) as

$$Z_G(q) = q^{\lvert V \rvert} \Xi_G(q)$$

(3.1)

where

$$\Xi_G(q) = \sum_{n \geq 1} \sum_{\{R_1,\ldots,R_n\} \in \pi(V)} \rho(R_1) \cdots \rho(R_n)$$

(3.2)

with

$$\rho(R) = \begin{cases} 1 & \text{if } |R| = 1 \\ q^{-|R|} \sum_{\sigma_R \in \Gamma_R} \sum_{E' \subset \nu_2(R)} \prod_{\{x,y\} \subset E'} [e^{-\delta_{x,y} J_{xy}} - 1] & \text{if } |R| \geq 2 \text{ and } G|_R \in \mathcal{G}_R \\ 0 & \text{if } |R| \geq 2 \text{ and } G|_R \notin \mathcal{G}_R \end{cases}$$

(3.3)

Observe that the sum in l.h.s. of (3.3) runs over all possible connected graphs with vertex set $R$. The r.h.s. of (3.2) can be written in a more compact way, by using the short notations

$$R_n \equiv (R_1,\ldots,R_n) \quad ; \quad \rho(R_n) \equiv \rho(R_1) \cdots \rho(R_n)$$

as

$$\Xi_G(q) = 1 + \sum_{n \geq 1} \frac{1}{n!} \sum_{R_n \in [P_{\geq 2}(V)]^n} \rho(R_n)$$

(3.4)

where $[P_{\geq 2}(V)]^n$ denote the $n$-times Cartesian product of $P_{\geq 2}(V)$ (which, we recall, denotes the set of all finite subsets of $V$ with cardinality greater than 2).

It is also convenient to simplify the expression for the activity (3.3) by performing the sum over $\sigma_R$. As a matter of fact

$$q^{-|R|} \sum_{\sigma_R \in \Gamma_R} \sum_{E' \subset \nu_2(R)} \prod_{\{x,y\} \subset E'} [e^{-\delta_{x,y} J_{xy}} - 1] = q^{-|R|} \sum_{\sigma_R \in \Gamma_R} \sum_{E' \subset \nu_2(R)} \prod_{\{x,y\} \subset E'} \delta_{x,y} [e^{-J_{xy}} - 1] =$$

$$= q^{-|R|} \sum_{E' \subset \nu_2(R)} \left[ \sum_{\sigma_R \in \Gamma_R} \prod_{\{x,y\} \subset E'} \delta_{x,y} \right] \prod_{\{x,y\} \subset E'} [e^{-J_{xy}} - 1]$$

But now, for any connected graph $(R, E') \in \mathcal{G}_R$

$$\sum_{\sigma_R \in \Gamma_R} \prod_{\{x,y\} \subset E'} \delta_{x,y} = q$$

Hence we get, for $|R| > 1$

$$\rho(R) = \begin{cases} q^{-|R|-1} \sum_{E' \subset \nu_2(R)} \prod_{\{x,y\} \subset E'} [e^{-J_{xy}} - 1] & \text{if } G|_R \in \mathcal{G}_R \\ 0 & \text{otherwise} \end{cases}$$

(3.5)
By definitions (3.5) or (3.3), the polymer activity $\rho(R)$ can be viewed as a real valued function defined on any finite subset $R$ of $V$. Of course this function depends on the “topological structure” of $G$. We remark that if $\gamma$ is an automorphism of $G$, then (3.5) clearly implies that $\rho(\gamma R) = \rho(R)$. In other words the activity $\rho(R)$ is invariant under automorphism of $G$.

The function $\Xi_G(q)$ is the standard grand canonical partition function of an hard core polymer gas in which the polymers are finite subsets $R \in V$ with cardinality greater than 2, with activity $\rho(R)$, and submitted to an hard core condition ($R_i \cap R_j = \emptyset$ for any pair $\{i, j\} \in I_n$).

Note that by (3.1) and definitions (2.6)-(2.7) we have

$$W_\gamma(G, q) = \exp \left\{ \lim_{N \to \infty} \frac{1}{|V_N|} \ln \Xi_{G|V_N}(q) \right\} \quad (3.6)$$

It is a well known fact in statistical mechanics that the natural logarithm of $\Xi$ can be rewritten as formal series, called the Mayer series (see e.g. [5]) as

$$\ln \Xi_G(q) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{R_n \in |P_{\geq 2}(V)|^n} \phi^T(R_n) \rho(R_n) \quad (3.7)$$

where

$$\phi^T(R_n) = \left\{ \begin{array}{ll} \sum_{E' \subset E(R_n)} \prod_{\{i, j\} \in E'} (-1)^{|E'|} & \text{if } G(R_n) \in \mathcal{G}_n \\ 0 & \text{otherwise} \end{array} \right. \quad (3.8)$$

and $G(R_n) \equiv G(R_1, \ldots, R_n)$ defined at the beginning of section 2. The reader should note that the summation in the l.h.s. of (3.4) is actually a finite sum. On the contrary, the summation in the l.h.s. of (3.7) is an infinite series.

We conclude this section showing two important technical lemmas concerning precisely the convergence of the series (3.8). In the proof of both lemma we will use a well known combinatorial inequality due to Rota [12], which states that if $G = (V, E)$ is a connected graph, i.e. $G \in \mathcal{G}_V$, then

$$\left| \sum_{E' \subset E : \langle V, E' \rangle \in \mathcal{G}_V} (-1)^{|E'|} \right| \leq \sum_{E' \subset E : \langle V, E' \rangle \in \mathcal{T}_V} 1 = N_{\mathcal{T}_V}[G] \quad (3.9)$$

where $N_{\mathcal{T}_V}[G]$ is the number of tree graphs with vertex set $V$ which are sub-graphs of $G$.

**Lemma 3.** Let $G = (V, E)$ a bounded degree infinite graph with maximum degree $\Delta$, and let, for any $R \in V$ such that $|R| \geq 2$, the activity $\rho(R)$ be given as in (3.5). Then, for any $n \geq 2$

$$\sup_{x \in V} \sum_{R \in \mathcal{P}_{|x|} : |R| = n} |\rho(R)| \leq \left[ \frac{e\Delta}{|q|} \right]^{n-1} \quad (3.10)$$

**Proof.** By definition

$$\sup_{x \in V} \sum_{R \in \mathcal{P}_{|x|} \text{ s.t. } |R| = n} |\rho(R)| = |q|^{-(n-1)} \sup_{x \in V} \sum_{R \in \mathcal{P}_{|x|} \text{ s.t. } |R| = n} \sum_{E' \subset E_{x\gamma}} \prod_{\{x, y\} \in E'} [e^{-J_{xy}} - 1] \quad (3.11)$$
Using thus the Rota inequality (3.9), recalling that $\mathbb{E}|_R = \{(x, y) \in \mathbb{E} : x \in R, y \in R\}$, and observing that $e^{-J_{xy}} - 1 = -1$ if $|x - y| = 1$ and $e^{-J_{xy}} - 1 = 0$ otherwise, we get

$$\left| \sum_{E' \subseteq P_2(I_n)} \prod_{\{x,y\} \in E'} [e^{-J_{xy}} - 1] \right| = \sum_{E' \subseteq \mathbb{E}|_R} (-1)^{|E'|} \Delta_{|E'|} \leq \sum_{E' \subseteq \mathbb{E}|_R} 1 = \sum_{E' \subseteq P_2(I_n)} \prod_{\{x,y\} \in E'} \delta_{|x-y|1}$$

where $\delta_{|x-y|1} = 1$ if $|x - y| = 1$ and $\delta_{|x-y|1} = 0$ otherwise. Hence

$$\sup_{x \in \mathbb{V}} \sum_{R \subseteq \mathbb{V}, \{x \in R| |R||n = n, R \notin \emptyset|} |\rho(R)| \leq |q|^{-(n-1)} \sup_{x \in \mathbb{V}} \sum_{R \subseteq \mathbb{V}, \{x \in R| |R||n = n, R \notin \emptyset|} \prod_{\{x,y\} \in E'} \delta_{|x-y|1} \leq \frac{|q|^{-(n-1)}}{(n-1)!} \sum_{E' \subseteq \mathbb{P}_2(I_n)} \left( \sup_{x=x', (x_2, \ldots, x_n) \in \mathbb{N}^{n-1}} \prod_{i,j \in I_n} \delta_{|x_i-x_j|1} \right)$$

It is now easy to check that, for any $E' \subseteq \mathbb{P}_2(I_n)$ such that $(I_n, E')$ is a tree, it holds

$$\sup_{x \in \mathbb{V}} \sum_{E' \subseteq \mathbb{P}_2(I_n)} \prod_{\{x,y\} \in E'} \delta_{|x-y|1} \leq \frac{\Delta^{n-1}}{(n-1)!}$$

and since, by Cayley formula, $\sum_{E' \subseteq \mathbb{P}_2(I_n)} 1 = n^{n-2}$, we get

$$\sup_{x \in \mathbb{V}} \sum_{R \subseteq \mathbb{V}, |R||n = n} |\rho(R)| \leq \left( \frac{\Delta}{|q|} \right)^{n-1} \frac{n^{n-2}}{(n-1)!} \leq \left[ \frac{e^{\Delta}}{|q|} \right]^{n-1}$$

□

To enunciate the second lemma we need to introduce a formal series more general that l.h.s. of (3.7). Let thus $U \subseteq \mathbb{V}$ finite and let $m$ a positive integer. We define

$$S_U^m(G, q) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{R_n \in [P_{\geq 2}(U)]^n} \phi^T(R_n) \rho(R_n)$$ (3.12)

where $|R_n| = \sum_{i=1}^{n} |R_i|$ and recall that $P_{\geq 2}(\mathbb{V})$ denotes the set of all finite subsets of $\mathbb{V}$ with cardinality greater or equal than 2 and $[P_{\geq 2}(U)]^n$ denote the $n$-times Cartesian product. We will now prove the following:

**Lemma 4.** Let $G = (\mathbb{V}, E)$ a locally finite infinite graph with maximum degree $\Delta$. Let $U \subseteq \mathbb{V}$ finite and let $m$ a positive integer. Then $S_U^m(G, q)$ defined in (3.12) exists and is analytic as a function of $1/q$ in the disk $|2\Delta e^3/q| < 1$. Moreover it satisfies the following bound

$$|S_U^m(G)| \leq |U| \frac{1}{1 - \sqrt{2e^3|\Delta/q|}} \left| \frac{2e^3\Delta}{q} \right|^{m/2}$$
Proof.
We will prove the theorem by showing directly that the r.h.s. of (3.12) converge absolutely when $|1/q|$ is sufficiently small. Let us define

$$|S|^m_U(\mathcal{G}) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathcal{R}_n \in [P_{\geq 2}(V)]^n} |\phi^T(\mathcal{R}_n)| \rho(\mathcal{R}_n)$$

(3.13)

then $|S|^m_U(\mathcal{G}) \leq |S|^m_U(\mathcal{G})$. We now bound $|S|^m_U(\mathcal{G})$. We have:

$$|S|^m_U(\mathcal{G}) \leq \sum_{s=m}^{\infty} \sum_{n=1}^{\lfloor s/2 \rfloor} \frac{1}{n!} \sum_{\mathcal{R}_n \in [P_{\geq 2}(V)]^n} |\phi^T(\mathcal{R}_n)| \rho(\mathcal{R}_n) = \sum_{s=m}^{\infty} \sum_{n=1}^{\lfloor s/2 \rfloor} \frac{1}{n!} \sum_{k_n \in \mathbb{N}^n, k_i \geq 2} B_n(k_n)$$

where $k_n = (k_1, \ldots, k_n)$, $\mathbb{N}^n$ denotes the $n$-times Cartesian product of $\mathbb{N}$, $\lfloor s/2 \rfloor = \max\{\ell \in \mathbb{N} : \ell \leq s/2\}$, and

$$B_n(k_n) = \sum_{\mathcal{R}_n \in [P_{\geq 2}(V)]^n, \mathcal{R}_1 \cap U \neq \emptyset, |\mathcal{R}_n| = s} |\phi^T(\mathcal{R}_n)| \rho(\mathcal{R}_n)$$

recalling now (3.8) and using again the Rota bound (3.9) we get

$$|\phi^T(\mathcal{R}_n)| \begin{cases} \leq N_{\mathcal{T}_n}[G(\mathcal{R}_n)] & \text{if } G(\mathcal{R}_n) \in \mathcal{G}_n \\ = 0 & \text{otherwise} \end{cases}$$

Hence

$$B_n(k_n) \leq \sum_{G \in \mathcal{G}_n} N_{\mathcal{T}_n}[G] \sum_{\mathcal{R}_n \in [P_{\geq 2}(V)]^n, \mathcal{R}_1 \cap U \neq \emptyset, G(\mathcal{R}_n) = G} |\rho(\mathcal{R}_n)|$$

(3.14)

Observing now that

$$\sum_{G \in \mathcal{G}_n} N_{\mathcal{T}_n}[G] (\cdots) = \sum_{\tau \in \mathcal{T}_n} \sum_{G \in \mathcal{G}_n : G \supseteq \tau} (\cdots)$$

We can rewrite

$$B_n(k_n) \leq \sum_{\tau \in \mathcal{T}_n} B_n(\tau, k_n)$$

(3.15)

where

$$B_n(\tau, k_n) = \sum_{\mathcal{R}_n \in [P_{\geq 2}(V)]^n, \mathcal{R}_1 \cap U \neq \emptyset, G(\mathcal{R}_n) \supseteq \tau} |\rho(\mathcal{R}_n)|$$

Note now that for any non negative function $F(\mathcal{R})$ it holds

$$\sum_{\mathcal{R} \in \mathcal{V}, \mathcal{R} \cap \mathcal{R}' \neq \emptyset, |\mathcal{R}| = k} F(\mathcal{R}) \leq |\mathcal{R}'| \sup_{x \in \mathcal{V}} \sum_{\mathcal{R} \in \mathcal{V}, \mathcal{R} \cap x = \mathcal{R}} F(\mathcal{R})$$

(3.16)
Hence we can now estimates $B_n(\tau, k_n)$ for any fixed $\tau$ by explicitly perform the sum over polymers $R_n$ submitted to the constraint that $g(R_n) \supset \tau$, summing first over the “outermost polymers”, i.e. those polymers $R_i$ such that $i$ is a vertex of degree 1 in $\tau$, and using repetively the bounds (3.16). Then one can easily check that

$$B_n(\tau, k_n) \leq |U| \sup_{x \in V} \sum_{R_1 \in V} \cdots \sup_{x \in V} \sum_{R_k \in V} |\rho(R_1)||R_1|^{d_1} \prod_{i=2}^k |R_i|^{d_i} - |\rho(R_i)|$$

(3.17)

where $d_i$ is the degree of the vertex $i$ of $\tau$. Recall that, for any tree $\tau \in T_n$, it holds $1 \leq d_i \leq n - 1$ and $d_1 + \ldots + d_n = 2n - 2$. Now, by lemma 3, (3.10), we can bound

$$B_n(\tau, k_n) \leq |U| e^{k_1-1} k_1^{d_1} \prod_{i=2}^k \left( k_i^{d_i-1} e^{k_n-1} \right)$$

(3.18)

where we have put for simplicity $\varepsilon = e^{\Delta} / |q|$. Noting that estimates in l.h.s. of (3.17) depends only on the degrees $d_1, \ldots, d_n$ of the vertices in $\tau$, we can now easily sum over all connected tree graphs in $T_n$ and obtain

$$B_n(k_n) \leq \sum_{\tau \in T_n} B_n(\tau, k_n) = \sum_{\tau \in T_n} \sum_{r_1, \ldots, r_n=2n-2} \sum_{d_1, \ldots, d_n=r_n} B_n(\tau, k_n) \leq$$

$$\leq |U| \sum_{d_1, \ldots, d_n \text{ fixed}} (n-2)! k_1^{n} \prod_{i=1}^k \left( k_i^{r_i-1} \right)$$

where in the second line we used the bound (3.17) and Cayley formula

$$\sum_{\tau \in T_n} 1 = \frac{(n-2)!}{\prod_{i=1}^k (d_i - 1)!}$$

(3.19)

Now, recalling that $k_1 + \ldots + k_n = s$ and using the Newton multinomial formula, we get

$$B_n(k_n) \leq |U| k_1 s^{n-2} \varepsilon s^{-n} \leq |V| s^n \varepsilon s^{-n}$$

thus, since $\sum_{k_1, \ldots, k_n: k_i \geq 2} 1 \leq 2^{s-n}$, we obtain

$$|S_{U}^n(G)| \leq |U| \sum_{s=m}^{\infty} \sum_{n=1}^{[s/2]} \frac{s^n}{n!} \varepsilon s^{-n} \leq 1 \leq |U| \sum_{s=m}^{\infty} \sum_{n=1}^{[s/2]} \frac{s^n}{n!} [2\varepsilon]^{s-n}$$

The series above converge if $\varepsilon < \frac{1}{2e}$ and we get the bound

$$|S_{U}^n(G)| \leq |U| \sum_{s=m}^{\infty} \sum_{n=1}^{[s/2]} \frac{s^n}{n!} [2\varepsilon]^{s-n} \leq \sum_{s=m}^{\infty} [2\varepsilon]^{s-[s/2]} \sum_{n=1}^{s} \frac{s^n}{n!} \leq \sum_{s=m}^{\infty} [2e^2\varepsilon]^{s/2} \leq$$
\[ \left| \frac{1}{|V|} \log \Xi_G(q) \right| \leq \frac{1}{1 - \sqrt{2e^3 \Delta/q}} \left| \frac{2e^3 \Delta}{q} \right| \]  \quad (3.20)

**Proof.** For any \( G = (V, E) \subset \mathbb{G} = (\mathbb{V}, \mathbb{E}) \) with \( V \) finite, by definition (3.7) and (3.12), it holds that \( |\ln \Xi_G(q)| \leq |\mathcal{S}|_r^2(\mathbb{G}, q) \) and one can thus apply lemma 4. □

\section*{§4. A graph theory lemma}

**Lemma 6.** Let \( \mathbb{G} = (\mathbb{V}, \mathbb{E}) \) be a locally finite quasi-transitive infinite graph and let \( \{V_N\}_{N \in \mathbb{N}} \) be a Følner sequence of finite subsets of \( \mathbb{V} \). Then, for every vertex orbit \( O \subset \mathbb{V} \) of \( \text{Aut}(\mathbb{G}) \), there exists a non-zero finite limit

\[ \lim_{N \to \infty} \frac{|O \cap V_N|}{|V_N|} \]  \quad (4.1)

and it is independent on the choice of the sequence \( \{V_N\}_{N \in \mathbb{N}} \).

**Proof.** For a natural \( r \) and a finite set \( F \subset \mathbb{V} \) denote by \( B_r(F) \) the set

\[ B_r(F) = \{ x \in \mathbb{V} : \exists y \in F \ | x - y| \leq r \} \quad (4.2) \]

Thus, for a single-point set \( \{y\} \), \( B_r(\{y\}) \) is the ball of radius \( r \) centered at \( y \). Moreover we have the bound

\[ |B_r(F)| \leq |F|(1 + \Delta + \ldots + \Delta^r) \leq \Delta^{r+1}|F| \quad (4.3) \]

Let \( O_1, \ldots, O_s \) be the complete list of vertex orbits of \( \text{Aut}(\mathbb{G}) \) in the set \( \mathbb{V} \) and let \( A_0 \subset \mathbb{V} \) be a set with exactly one element in common with every orbit. Denote by \( d \) the diameter of \( A_0 \). Consider the orbit \( \mathcal{A} = \{ gA_0 : g \in \text{Aut}(\mathbb{G}) \} \) of \( A_0 \). A set \( A \subset \mathbb{V} \) is therefore an element of \( \mathcal{A} \) if it exists a \( g \in \text{Aut}(\mathbb{G}) \) such that \( A = gA_0 \). For any set \( U \subset \mathbb{V} \) we denote \( \mathcal{A}_U = \{ A \in \mathcal{A} : A \subset U \} \). Note that for any set \( A \in \mathcal{A} \) and any vertex orbit \( O \), we have that \( |A \cap O| = 1 \), hence for a fixed vertex orbit \( O_i \) we can define the function \( \varphi_i \) as follows.

\[ \varphi_i : \mathcal{A} \rightarrow O_i : A \mapsto A \cap O_i \]

The function \( \varphi_i \) is a surjection and for \( x \in O_i \) the number \( k_i = |\varphi_i^{-1}(x)| \) is finite and does not depend on the choice of \( x \in O_i \). For the sets

\[ V_N^- = V_N \setminus B_d(\partial V_N), \quad V_N^+ = V_N \cup B_d(\partial V_N) \quad (4.4) \]
we have $V_N^{-} \subset V_N \subset V_N^{+}$ and

\[
\varphi_i^{-1}(V_N^{-} \cap O_i) \subset \mathcal{A}_N \subset \varphi_i^{-1}(V_N \cap O_i) \subset \mathcal{A}_N^{+}
\]  

(4.5)

Indeed, suppose that $A \in \varphi_i^{-1}(V_N^{-} \cap O_i)$ and $A \not\subset V_N$. Let $a_1 = \varphi_i(A) \in V_N^{-} \subset V_N$ and let $a_2 \in A \setminus V_N$. There exists a path $\tau(a_1, a_2)$ in $G$ of length $\leq \Delta d$. This chain must have at least one point in $\partial V_N$. This implies $a_1 \in B_d(\partial V_N)$ contradicting the assumption $a_1 \in V_N^{-}$. The second inclusion of (4.5) is obvious and the third one is true by the same reason as the first one. (4.5) implies

\[
k_i |V_N^{-} \cap O_i| \leq |\mathcal{A}_N| \leq k_i |V_N \cap O_i| \leq |\mathcal{A}_N^{+}|
\]  

(4.6)

and

\[
|V_N^{-} \cap O_i| \leq \frac{1}{k_i} |\mathcal{A}_N| \leq |V_N \cap O_i| \leq \frac{1}{k_i} |\mathcal{A}_N^{+}|
\]  

(4.7)

By taking sum over $i$ we get

\[
|V_N^{-}| \leq \alpha |\mathcal{A}_N| \leq |V_N| \leq \alpha |\mathcal{A}_N^{+}|
\]  

(4.8)

where $\alpha = \sum_{i=1}^s \frac{1}{k_i}$. On the other hand $\mathcal{A}_N^{+} \setminus \mathcal{A}_N \subset \mathcal{A}_{B_d(\partial V_N)}$ and, by (4.3),

\[
|\mathcal{A}_N^{+}| \leq |\mathcal{A}_N| + |\mathcal{A}_{B_d(\partial V_N)}| \leq |\mathcal{A}_N| + k \Delta^{d+1} |\partial V_N|
\]  

(4.9)

where $k = \max\{k_i : i \in \{1, \ldots, s\}\}$. From the first inequality of (4.8) we have

\[
|\mathcal{A}_N| \geq \frac{1}{\alpha} |V_N^{-}| \geq \frac{1}{\alpha} (|V_N| - |B_d(\partial V_N)|) \geq \frac{1}{\alpha} (|V_N| - \Delta^{d+1} |\partial V_N|)
\]  

(4.10)

If $\frac{|\partial V_N|}{|V_N|} \leq \varepsilon$ then, by (4.9) and (4.10),

\[
1 \leq \frac{|\mathcal{A}_N^{+}|}{|\mathcal{A}_N|} \leq 1 + \frac{\Delta^{d+1} |\partial V_N|}{\frac{1}{\alpha} (|V_N| - \Delta^{d+1} |\partial V_N|)} \leq 1 + \frac{\Delta^{d+1} \varepsilon}{\frac{1}{\alpha} (1 - \Delta^{d+1} \varepsilon)}
\]

This proves that

\[
\lim_{N \to \infty} \frac{|\mathcal{A}_N^{+}|}{|\mathcal{A}_N|} = 1
\]  

(4.11)

By (4.8) and (4.10) we also have

\[
\lim_{N \to \infty} \frac{|\mathcal{A}_N|}{|V_N|} = \frac{1}{\alpha}
\]  

(4.12)

Dividing (4.7) by $|V_N|$ and using (4.12) we obtain

\[
\lim_{N \to \infty} \frac{|O_i \cap V_N|}{|V_N|} = \frac{1}{k_i \alpha}
\]

and the lemma is proved. □
§5. Potts model on infinite graphs: proof of theorem 2.

Let $G = (V,E)$ infinite bounded degree and let $x \in V$. Then we define

$$f_G(x) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{R_n \in [P_{\geq 2}(V)]^n} \phi^T(R_n) \frac{\rho(R_n)}{|R_1|}$$

(5.1)

We stress that, by construction, $f_G(x)$ is invariant under automorphism. I.e. if $x \in V$ and $y \in V$ are equivalent (i.e. it exists $\gamma$ automorphism of $G$ such that $y = \gamma x$) then $f_G(x) = f_G(y)$.

Given now a finite set $V_N \subset V$, we define

$$F(V_N) = \frac{1}{|V_N|} \sum_{x \in V_N} f_G(x)$$

(5.2)

The numbers $F(V_N)$ are actually functions of $q$. As a trivial corollary of lemma 4 we can state the following

**Lemma 7.** Let $G = (V,E)$ infinite bounded degree. Then for any $V_N \subset V$ finite, the functions $f_G(x)$ and $F(V_N)$ defined in (5.1) and (5.2) are analytic in the variable $1/q$ for $|1/q| < 1/2e^3\Delta$ and bounded by $|2e^3\Delta/q|/(1 - \sqrt{2e^3}\Delta/q)$ uniformly in $N$.

**Proof.** Comparing l.h.s. of (3.12) with l.h.s. of (5.1) we have that $|f_G(x)| \leq |S|^2_{(x)}(G)$, hence one can again use lemma 4 and get immediately the proof. $\Box$.

From lemma 6 and lemma 7 it follows:

**Proposition 8.** Let $G = (V,E)$ be a locally finite quasi-transitive infinite graph and let $\{V_N\}_{N \in \mathbb{N}}$ be a sequence of finite subsets of $V$ such that $|\partial V_N|/|V_N| \to 0$ as $N \to \infty$. Let $\Delta$ be the maximum degree of $G$, then the limits

$$\lim_{N \to \infty} F(V_N) = F_G(q)$$

(5.3)

exists, is finite, is independent on the sequence $\{V_N\}_{N \in \mathbb{N}}$, and is analytic as a function of $1/q$ for $|1/q| < 1/2e^3\Delta$.

**Proof.** If the limit (5.3) exists, then by lemma 7 it is clearly bounded by $|2e^3\Delta/q|/(1 - \sqrt{2e^3}\Delta/q)$ and it analytic in $1/q$ for $|1/q| < 1/2e^3\Delta$. To prove the existence of the limit (5.3) we proceed as follows.

Since $G$ is quasi-transitive then $V$ can be partitioned into orbits $O_1, \ldots, O_s$ of $\text{Aut}(G)$ such that for two any vertices $x,y$ in the same orbit $O_i$ there is an automorphism of $G$ which maps $x$ to $y$. Hence for such a pair we have $f_G(x) = f_G(y)$ and we can conclude that $f_G(x)$ has value in a finite set $\{f_1, \ldots, f_s\}$ with $f_i = f_G(x)$ where $x$ is any vertex $x \in O_i$.

Thus for any finite connected $V_N$ and any $j \in \{1,2,\ldots,s\}$ we have

$$\frac{1}{|V_N|} \sum_{x \in V_N} f_G(x) = \frac{|V_N \cap O_1|}{|V_N|} f_1 + \ldots + \frac{|V_N \cap O_s|}{|V_N|} f_s$$

hence

$$\lim_{N \to \infty} F(V_N) = f_1 \lim_{N \to \infty} \frac{|V_N \cap O_1|}{|V_N|} + \ldots + f_s \lim_{N \to \infty} \frac{|V_N \cap O_s|}{|V_N|}$$

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and by lemma 6 the limit above exists. □

We are at last in the position to prove the main results of the paper, namely the theorem 2 enunciated at the end of section 2.

**Proof of theorem 2.** We will prove that \( \lim_{N \to \infty} |V_N|^{-1} \log \mathbb{E}_{|V_N|}(q) = F_G(q) \) where \( F_G(q) \) is the function defined in (5.3) and then use definition (3.6).

\[
\log \mathbb{E}_{|V_N|} - \sum_{x \in V_N} f_G(x) = \sum_{n=1}^{\infty} \frac{1}{n!} \left[ \sum_{R_n \in [P_{\geq 2}(V_N)]^n} \phi^T(R_n) \rho(R_n) - \sum_{x \in V_N} \sum_{R_n \in [P_{\geq 2}(V)]^n} \phi^T(R_n) \frac{\rho(R_n)}{|R_1|} \right]
\]

Now note that
\[
\sum_{R_n \in [P_{\geq 2}(V)]^n} (\cdots) = \sum_{R_n \in [P_{\geq 2}(V_N)]^n} (\cdots) + \sum_{R_n \in [P_{\geq 2}(V)]^n} (\cdots)
\]

moreover
\[
\sum_{x \in V_N} \sum_{R_1 \in V_N} (\cdots) = \sum_{R_1 \in V_N} |R_1|(\cdots), \quad \sum_{x \in V_N} \sum_{R_1 \in V} (\cdots) = \sum_{R_1 \in V} |R_1 \cap V_N|(\cdots)
\]

hence, using also that \( |R_1 \cap V_N|/|R_1| \leq 1 \) we get
\[
\left| \log \mathbb{E}_{|V_N|} - \sum_{x \in V_N} f_G(x) \right| \leq \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{R_n \in [P_{\geq 2}(V)]^n} |\phi^T(R_n) \rho(R_n)|
\]

Let now choose \( p > \ln \Delta \) and define
\[
m_N^p = \frac{1}{p} \log \left[ \frac{|V_N|}{|\partial V_N|} \right]
\]

remark that, since by the hypothesis the sequence \( V_N \) is Følner and hence (2.2) holds, then \( \lim_{N \to \infty} m_N^p = \infty \), for any integer \( p \). We now can rewrite
\[
\sum_{R_1 \cap V_N \neq \emptyset, \exists R_1, R_1 \notin V_N} (\cdots) = \sum_{R_1 \cap V_N \neq \emptyset, |R_n| \geq m_N^p} (\cdots) + \sum_{R_1 \cap V_N \neq \emptyset, |R_n| < m_N^p} (\cdots)
\]

Hence
\[
\left| \log \mathbb{E}_{|V_N|} - \sum_{x \in V_N} f_G(x) \right| \leq \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{R_n \in [P_{\geq 2}(V)]^n} |\phi^T(R_n) \rho(R_n)| +
\]
\[ + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{R_n \in [P^2(V)]^n \\cap V_N \neq \emptyset, |R_n| < m^p_N \\exists R_i, R_i \not\subset V_N.}} \left| \phi^T(R_n) \rho(R_n) \right| \] (5.5)

but, concerning the first sum, recalling definition (3.13), we have

\[
\sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{R_n \in [P^2(V)]^n \\cap V_N \neq \emptyset, |R_n| < m^p_N \\exists R_i, R_i \not\subset V_N.}} \left| \phi^T(R_n) \rho(R_n) \right| \leq \left| S \right|_{V_N}^m (G, q) \leq \text{Const.} |V_N| \varepsilon^{m^p_N/2}
\]

where \( \varepsilon = q/2e^3\Delta < 1 \) by hypothesis, which, divided by \( |V_N| \), converge to zero as \( N \to \infty \) because by hypothesis \( m^p_N \to \infty \) as \( N \to \infty \).

On the other hand, recalling that due to the factor \( \phi^T(R_n) \) the sets \( R_i \) must be pair-wise connected, we have that \( | \cup_i R_i | < \sum_i |R_i| \). So, since \( | \cup_i R_i | < m^p_N \) and at least one among \( R_i \) intersects \( \partial V_N \), this means that all polymers \( R_i \) must lie in the set

\[ B_{m^p_N}(\partial V_N) = \{ x \in V : \exists v \in \partial V_N : |x - v| \leq m^p_N \} \]

Recalling (4.2) we have

\[ |B_{m^p_N}(\partial V_N)| \leq |\partial V_N| \Delta m^p_N + 1 \]

Hence we have, again recalling (3.13), that second sum in l.h.s. of (5.5) is bounded by

\[
\sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{R_n \in [P^2(V)]^n \\cap V_N \neq \emptyset, |R_n| < m^p_N \\exists R_i, R_i \not\subset V_N.}} \left| \phi^T(R_n) \rho(R_n) \right| \leq |S|^2_{B_{m^p_N}(\partial V_N)} (G, q) \leq \\
\leq \text{Cost.} |B_{m^p_N}(\partial V_N)| \varepsilon \leq \text{Const.} |\partial V_N| \Delta m^p_N \varepsilon
\]

Thus recalling definition (5.4), we have

\[
\left\lfloor \frac{1}{|V_N|} \log \Xi_{G|V_N} - \frac{1}{|V_N|} \sum_{x \in V_N} f_G(x) \right\rfloor = \left\lfloor \frac{1}{|V_N|} \log \Xi_{G|V_N} - F_G(q) \right\rfloor \leq \\
\leq \text{Const.} \left[ \frac{|\partial V_N|}{|V_N|} \right]^{\frac{1 - \ln \Delta}{\rho}} + \text{Const.} \left[ \frac{|\partial V_N|}{|V_N|} \right]^{1 - \frac{\ln \Delta}{\rho}}
\]

Since by hypothesis \( |\partial V_N|/|V_N| \to 0 \) as \( N \to \infty \), we conclude that the quantity above is as small as we please for \( N \) large enough. This ends the proof of the theorem. \( \square \)

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References

[1] Baxter, R. J.: *Dichromatic Polynomials and Potts Models Summed Over Rooted Maps*. Annals of Combinatorics, 5 (2001) 17–36.

[2] Benjamini, I. and Schramm, O.: *Percolation beyond \( \mathbb{Z}^d \), many questions and a few answers*. Electr. Comm. Probab., 1 (1996), 71–82.

[3] Biggs, N. L.: *Chromatic and thermodynamic limits*. J. Phys. A, 8 (1975), no. 10, L110–L112.

[4] Biggs, N. L.; Meredith, G. H. J.: *Approximations for chromatic polynomials*. J. Combinatorial Theory Ser. B, 20 (1976), no. 1, 5–19.

[5] Cammarota, C.: *Decay of Correlations for Infinite range Interactions in unbounded Spin Systems*. Comm. Math Phys., 85 (1982), 517–528

[6] Chang, Shu-Chiuan; Shrock, Robert: *Structural properties of Potts model partition functions and chromatic polynomials for lattice strips*. Phys. A, 296 (2001), no. 1-2, 131–182.

[7] Fortuin, C. M.; Kasteleyn, P. W.: *On the random-cluster model. I. Introduction and relation to other models*. Physica 57 (1972), 536–564.

[8] Håggström, O.; Schonmann, R. H.; Steif, J. E.: *The Ising model on diluted graphs and strong amenability*. Ann. Probab. 28 (2000), no. 3, 1111–1137.

[9] Jonasson, J.: *The random cluster model on a general graph and a phase transition characterization of nonamenability*. Stochastic Process Appl., 79 (1999), 335–534.

[10] Kim, D.; Enting, I. G.: *The limit of chromatic polynomials*. J. Combin. Theory Ser. B, 26 (1979), no. 3, 327–336.

[11] Lyons, R.: *Phase transitions on nonamenable graphs*. Probabilistic techniques in equilibrium and nonequilibrium statistical physics. J. Math. Phys., 41 (2000), no. 3, 1099–1126.

[12] Rota, G.: *On the foundations of combinatorial theory. I. Theory of Möbius functions*. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, 2 (1964), 340–368.

[13] Salas, J.; Sokal, A. D.: *Transfer matrices and partition-function zeros for antiferromagnetic Potts models. I. General theory and square-lattice chromatic polynomial*. J. Statist. Phys., 104 (2001), no. 3-4, 609–699.

[14] Shrock R., Tsai S.H.: *Families of graphs with \( W_r(G,q) \) functions that are nonanalytic at \( 1/q=0 \)*. Phys. Rev. E, 56 (1997) n. 4, 3935–3943.

[15] Shrock R., Tsai S.H.: *Ground-state degeneracy of Potts antiferromagnets: homeomorphic classes with noncompact \( W \) boundaries*. Physica A, 265 (1999), no. 1-2, 186-223

[16] Shrock, R.: *Chromatic polynomials and their zeros and asymptotic limits for families of graphs*. 17th British Combinatorial Conference (Canterbury, 1999). Discrete Math., 231 (2001), no. 1-3, 421–446.
[17] Sokal, A. D.: Bounds on the complex zeros of (di)chromatic polynomials and Potts-model partition functions. Combin. Probab. Comput., 10 (2001), no. 1, 41–77.

[18] Sokal, A. D.: A personal list of unsolved problems concerning lattice gases and antiferromagnetic Potts models. Inhomogeneous random systems (Cergy-Pontoise, 2000). Markov Process. Related Fields, 7 (2001), no. 1, 21–38.

[19] Tutte, W. T.: A contribution to the theory of chromatic polynomials. Canadian J. Math., 6 (1954), 80–91.

[20] Welsh D. J. A., Merino C.: The Potts model and the Tutte polynomial. J. Math. Phys. 41 (2000 ), no. 3 1127–1152.

[21] Wu, F. Y.: The Potts model. Rev. Modern Phys., 54 (1982), no. 1, 235–268