Modelling Complex Networks: Cameo Graphs
And Transport Processes

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Abstract

We discuss a model accounting for the creation and development of transport networks based on the Cameo principle which refers to the idea of distribution of resources, including land, water, minerals, fuel and wealth. We also give an outlook of the use of random walks as an effective tool for the investigation of network structures and its functional segmentation. In particular, we have studied the complex transport network of Venetian canals by means of random walks.

Presentation for the volume: The challenge of complex network modelling calls for the more realistic heuristic principles that could catch the main features of network creation and development. In the Cameo model which refers to the idea of distribution of resources, including land, water, minerals, fuel and wealth, the local attractiveness of a site determining the creation of new spaces of motion in that is specified by a real positive parameter \( \omega > 0 \). We have described a possible mechanism for the emergence and development of complex transport networks based on the Cameo principle. Sustained movement patterns are generated by a subset of automorphisms of the graph spanning the transport network of canals and can be naturally interpreted as random walks. Random walks assign absolute scores to all nodes of a graph and embed space syntax into Euclidean space. Namely, every route of a transport network can be represented by a vector in Euclidean space which length quantifies the segregation of the route from the rest of the graph. We have empirically observed that the distribution of lengths over the edge connectivity in
the spatial network of Venetian canals exhibits scaling invariance phenomenon. The method is applicable to any transport network.

1 Introduction

Physics (Greek: φυσις [phusis], nature) is the branch of science concerned with the characterization of universal laws of Nature portraying its logically ordered picture in agreement with experience. Theoretical physics is closely related to mathematics, which provides a language for physical theories and allows for a rationalization of thought by making it possible to formulate these laws in terms of mathematical relations. Physicists study a wide variety of phenomena creating new interdisciplinary research fields by applying theories and methods originally developed in physics in order to solve problems in economics, social science, biology, medicine, technology, etc. In their turn, these different branches of science inspire the invention of new concepts in physics. A basic tool of analysis, in such a context, is the mathematical theory of complexity concerned with the study of complex systems including human economies, climate, nervous systems, cells and living things, including human beings, as well as modern energy or communication infrastructures which are all networks of some kind.

Complex systems appear as a result of the interplay between Topology determined by a connected graph, Dynamics described by the operators invariant with respect to graph symmetry, and properties of embedding (Euclidean) space specified by a set of measures and weights assigned to elements of the graph. In the context of complex networks theory created by physicists, the non-trivial topological structure of large networks is investigated by means of various statistical distributions. The structure and the properties of complex networks essentially depend on the way how nodes get connected to each other. Random graphs with a scale free distribution for the degree seem to appear very frequently in a great variety of real life situations like the World- Wide Web, the Internet, social networks, linguistic networks, citation networks and biochemical networks.

In most of complex networks emerging in society and technology, each node has a feature which attracts the others. In a class of simple models proposed in [1], the network dynamics can be described in terms of property of the node and the affinity other nodes have towards that property (Cameo graphs). Networks built accordingly to this principle have a degree distribution with a power law tail, whose exponent is determined only by the nodes with the largest affinity values. It appears that the extremists lead the formation process of the network and manage to shape
the final topology of the system.

The exceptional events play a crucial role in the formation of network structures [2]. The dynamics of some vertices, the "hubs" which have an extremely high number of connections to other vertices, is of primary importance for complex networks. These networks are generally "scale-free"; in other words, they exhibit architectural and statistical stability as the degree distribution grows. A class of probabilistic model for a system at a threshold of instability has been studied in [3]. The distribution of residence times below the threshold characterizes the properties of such a system. Being at a threshold of instability, the system can induce various types of random graphs and the scale-free random graphs among others [4]. The priority-based scheduling rules in single-stage queuing systems (QS) also generate fat tail behavior for the task waiting time distributions induced by the waiting times of very low priority tasks that stay unserved almost forever as the task priority indices are "frozen" in time [5]. The task waiting time distributions have been studied for a population-type model with an age structure and a QS with deadlines assigned to the incoming tasks, which is operated under the "earliest-deadline-first" policy. As the aging mechanism ultimately assigns high priority to any long waiting tasks, fat tails cannot find their origin in the scheduling rule alone.

Graphs obtained by successive creation and elimination of edges into small neighborhoods of the vertices evolve towards small world graphs with logarithmic diameter, high clustering coefficients and a fat tail distribution for the degree [6]. It is important to note that it was only local edge formation processes that rise small worlds, no preferential attachment was used. Simple edge generation rules based on an inverse like mass action principle for random graphs over a structured vertex set, under very weak assumptions on the structure generating distribution, also yield a scale-free distribution for the degree [7]. A local search principle important in many social applications, "my friends are your friends" have also been introduced and studied; networks generated in accordance to such a principle have essentially high clustering coefficients.

Although investigations into the statistical properties of graphs such as a heavy-tail in the degree distribution of nodes could uncover their hierarchical structure, they are futile if the detailed information on the structure of graphs is of primary interest since many graphs characterized by similar statistics of node degrees and shortest path lengths can be of dramatically different structures. The structure and symmetry of graphs play the crucial role in behavior of dynamical systems defined on that. It was clearly demonstrated in epidemiological research describing the dynamics of sexually transmitted diseases, the Human Immune Deficiency Virus (HIV) and AIDS, in particular [2]. Mathematical modelling on the spread of sexually transmitted diseases [8]-[13] studied on various random graphs displays the importance of critical parameters such as the
transmission probability and edge creation probability for the epidemic spreading.

It has been found that the epidemic spreading in scale-free networks is very sensitive to the statistics of degree distribution, the effective spreading rate, the social strategy used by individuals to choose a partner, and the policy of administrating a cure to an infected node [14]. Depending on the interplay of these four factors, the stationary fractions of infected population as well as the epidemic threshold properties can be essentially different. For a model of scale-free graphs with biased partner choice that knowing the exponent for the degree distribution is in general not sufficient to decide epidemic threshold properties for exponents less than three [15]. Absence of epidemic threshold happens precisely when a positive fraction of the nodes form a cluster of bounded diameter. Probably, it is impossible to obtain a simple immunization program that can be simultaneously effective for all types of scale-free networks [14]. A similar approach can be applied in order to study social diseases like corruption. It has been investigated in [16] as a generalized epidemic process on the graph of social relationships. Corruption is characterized by a strong non-linear dependence of the transmission probability from the local density of corruption and the mean field influence of the overall corruption in the society. Network clustering and the degree-degree correlation play an essential role in these types of dynamics. In particular, it follows that strongly hierarchically organized societies are more vulnerable to corruption than democracies. A similar type of modelling can be applied to other social contagion spreading processes like opinion formation, doping usage, social disorders or innovation dynamics. An agent-based model of factual communication in social systems, drawing on concepts from Luhmann’s theory of social systems [17] has been studied in [18]. The agent communications are defined by the exchange of distinct messages. Message selection is based on the history of the communication and developed within the confines of the problem of double contingency. We have examined the notion of learning in the light of the message-exchange description.

Topology plays the primary role in the dynamical processes which have place on networks. The investigations in transitions to spatio-temporal intermittency in random network of coupled Chaté-Manneville maps [19] show that spatiotemporal intermittency occurs for some intervals or windows of the values of the network connectivity, coupling strength, and the local parameter of the map. Within the intermittency windows, the system exhibits periodic and other nontrivial collective behaviors. Genetic regulatory networks constitute an important example of dynamical systems defined on graphs. Local dynamics of network nodes exhibits multiple stationary states and oscillations depending crucially upon the global topology of a 'maximal' graph (comprising of all possible interactions be-
between genes in the network) [20]. The long time behavior observed in the network defined on the homogeneous 'maximal' graphs is featured by the fraction of positive interactions (activations) allowed between genes. In networks defined on the inhomogeneous directed graphs depleted in cycles, no oscillations arise in the system even if the negative interactions (inhibitions) in between genes present therein in abundance. Local dynamics observed in the inhomogeneous scalable regulatory networks is less sensitive to the choice of initial conditions.

In mathematics, the automorphism groups of a graph are studied. They characterize its symmetries, and are therefore very useful in determining certain of its properties. In particular, the Euclidean metric related to dynamics can be defined on some graphs by means of linear operators remaining invariant under the permutations of nodes and satisfying some conservation properties. These operators describe certain dynamical processes defined on graphs such as random walks and diffusions. We have studied transport through generalized trees in [21]. Trees contain the simple nodes and super-nodes, either well-structured regular subgraphs or those with many triangles. We observe super-diffusion for the highly connected nodes while it is Brownian for the rest of the nodes. Transport within a super-node is affected by the finite size effects vanishing as \( N \to \infty \). For a space of even dimensions, \( d = 2, 4, 6 \ldots \), the finite size effects break down the perturbation theory at small scales and can be regularized by using a heat-kernel expansion. Diffusion processes and Laplace operators related to them can be used in order to investigate the structure of networks in the spirit of spectral graph theory. In [22], different models of random walks on the dual graphs of compact urban structures are considered. Dual graphs have been widely used in the framework of space syntax theories [23] for the analysis of spatial configurations. The general idea is that spaces can be broken down into components, analyzed as networks of choices, and then represented as maps and graphs that describe the relative connectivity and integration of those spaces. From these components it is thought to be possible to quantify and describe how easily navigable any space is, useful for the design of museums, airports, hospitals, and other settings where way finding is a significant issue. Space syntax has also been applied to predict the correlation between spatial layouts and social effects such as crime, traffic flow, sales per unit area, etc. Analysis of access times between streets performed in [22] helps to detect the city modularity.

The aim of the present paper is twofold. First, we discuss a model which accounts for the creation and development of transport networks basing on the Cameo principle [1] which refers to the idea of distribution of resources, including land, water, minerals, fuel and wealth in general (see Sec. [2]). Second, we give an outlook of the use of random walks as an effective tool for the investigation of network structures and its functional
segmentation (Sec. 3). In the Sec. 4 we consider two examples of graphs (the modelling example of the Petersen graph and the spatial network of Venetian canals) and analyze their properties. We conclude in the last section.

2 The Cameo principle and the origin of transport networks

Among the classical models in which the degree distribution of the arising graph satisfies a power-law is the graph generating algorithms based on the preferential attachment approach firstly proposed by H. Simon [24]. Within preferential attachment algorithms, the growth of a network starts with an initial graph of $n_0 \geq 2$ nodes such that the degree of each node in the initial network is at least 1. The celebrated Barabási-Albert model [25] have been proposed in order to model the emergency and growth of scale-free complex networks. New nodes in the model [25] are added to the network one at a time. Each new node is connected to $n$ of the existing with a probability that is biased being proportional to the number of links that the existing node already has,

$$p_i = \frac{\deg(i)}{\sum_{j=1}^{N} \deg(j)}.$$  

(1)

It is clear that the nodes of high degrees tend to quickly accumulate even more links representing a strong preference choice for the emerging nodes, while nodes with only a few links are unlikely to be chosen as the destination for a new link. The preferential attachment forms a positive feedback loop in which an initial random degree variation is magnified with time, [26]. It is fascinating that the expected degree distribution in the graph generated in accordance to the algorithm proposed in [25] asymptotically approaches the cubic hyperbola,

$$\Pr [i \in G | \deg(i) = k] \simeq \frac{1}{k^3}.$$  

(2)

It is however obvious that the mechanisms governing the creation and development of transport networks certainly do not follow such a simple preferential attachment principle as that discussed in [25]. Indeed, nowadays the new transportation routes are usually created as a result of the subdivision or redevelopment of an existing transport network. Appearing due to the complicated trade-off processes between multiple objectives, they can hardly be planned in such a way as to meet the transportation routes that already have the ever maximal number of junctions with other routes in the network.
The challenge of complex network modelling calls for the more realistic heuristic principles that could catch the main features of network creation and development. It is clear that a prominent model should take into account the structure of embedding physical space: the size and shape of landscape, and the local land use patterns if a city transportation network is considered. A suitable algorithm describing the development of complex networks which takes into account the properties of the surrounding place has been recently proposed in \cite{1}. It is called the 

\textit{Cameo-principle} having in mind the attractiveness, rareness and beauty of the small medallion with a profiled head in relief called Cameo. It is exactly their rareness and beauty which gives them their high value.

In the Cameo model \cite{1}, the local attractiveness of a site determining the creation of new spaces of motion in that is specified by a real positive parameter $\omega > 0$. Indeed, it is rather difficult if ever be possible to estimate exactly the actual value $\omega(i)$ for any site $i \in \mathcal{G}$ in the urban pattern, since such an estimation can be referred to both the \textit{local believes} of city inhabitants and may be to the \textit{cultural context} of the site that may vary over the different nations, historical epochs, and even over the certain groups of population.

Therefore, in the framework of the probabilistic approach, it seems natural to consider the value $\omega$ as a real positive independent \textit{random variable} distributed over the vertex set of the graph representation of the site uniformly in accordance to a smooth monotone decreasing probability density function $f(\omega)$. Let us suggest that there is just a few distinguished sites which are much more attractive then an average one in the city, so that the density function $f$ has a right tail for large $\omega \gg \bar{\omega}$ such that $f(\omega) \ll f(\bar{\omega})$.

Each newly created space of motion $i$ (represented by a node in the dual city graph $\mathcal{G}(N)$ containing $N$ nodes) may be arranged in such a way to connect to the already existed space $j \in \mathcal{G}(N)$ depending only on its attractiveness $\omega(j)$ and is of the form

\begin{equation}
\Pr [ i \sim j \mid \omega(j) ] \simeq \frac{1}{N} \frac{1}{f^\alpha (\omega(j)) + f^\alpha (\omega(i))} \tag{3}
\end{equation}

with some $\alpha \in (0, 1)$. The assumption (3) implies that the probability to create the new space adjacent to a space $j$ scales with the rarity of sites characterized with the same attractivity $\omega$ as $j$.

The striking observation under the above assumptions is the emergence of a scale-free degree distribution independent of the choice of distribution $f(\omega)$. Furthermore, the exponent in the asymptotic degree distribution becomes independent of the distribution $f(\omega)$ provided its tail, $f(\omega) \ll f(\bar{\omega})$, decays faster then any power law.

In the model of growing networks proposed in \cite{1}, the initial graph $\mathcal{G}_0$ has $N_0$ vertices, and a new vertex of attractiveness $\omega$ taken independently
uniformly distributed in accordance to the given density $f(\omega)$ is added to the already existed network at each time click $t \in \mathbb{Z}_+$. Being associated to the graph, the vertex establishes $k_0 > 0$ connections with other vertices already present in that. All edges are formed accordingly to the Cameo principle (3).

The main result of [1] is on the probability distribution that a randomly chosen vertex $i$ which had joined the Cameo graph $G$ at time $\tau > 0$ with attractiveness $\omega(i)$ amasses precisely $k$ links from other vertices which emerge by time $t > \tau$. It is important to note that in the Cameo model the order in which the edges are created plays a role for the fine structure of the graphs. The resulting degree distribution for $t - \tau > k/k_0$ is irrelevant to the concrete form of $f(\omega)$ and reads as following

$$\Pr \left[ \sum_{j: \tau(j) > \tau} 1_{j \sim j} = k \right] \simeq \frac{k_0^{1/\alpha}}{k^{1+1/\alpha+o(1)}} \ln^{1/\alpha} \left( \frac{t}{\tau} \right). \quad (4)$$

In order to obtain the asymptotic probability degree distribution for an arbitrary node as $t \to \infty$, it is necessary to sum (4) over all $\tau < t$ that gives

$$P(k) \simeq \frac{1}{t} \sum_{0 < \tau < t} \frac{k_0^{1/\alpha}}{k^{1+1/\alpha+o(1)}} \ln^{1/\alpha} \left( \frac{t}{\tau} \right) = \frac{1}{k^{1+1/\alpha+o(1)}}. \quad (5)$$

The emergence of the power law (5) demonstrates that graphs with a scale-free degree distribution may appear naturally as the result of a simple edge formation rule based on choices where the probability to chose a vertex with affinity parameter $\omega$ is proportional to the frequency of appearance of that parameter. If the affinity parameter $\omega$ is itself power law like distributed one could also use a direct proportionality to the value $\omega$ to get still a scale free graph.

We have described a possible mechanism for the emergence and development of complex transport networks based on the Cameo principle. In the forthcoming section, we discuss the embedding of transport network into the $(N - 1)$-dimensional Euclidean space which facilitates the discovering of important nodes, their classification, and the coarse-graining.

3 Mathematics of transport networks

Any graph representation naturally arises as the outcome of a categorization, when we abstract a real world system by eliminating all but one of its features and by grouping together things (or places) sharing a common attribute. For instance, the common attribute of all open spaces in city space syntax is that we can move through them. All elements called nodes that fall into one and the same group $V$ are considered as
essentially identical; permutations of them within the group are of no consequence. The symmetric group \( \mathfrak{S}_N \) consisting of all permutations of \( N \) elements (\( N \) being the cardinality of the set \( V \)) constitute therefore the symmetry group of \( V \). If we denote by \( E \subseteq V \times V \) the set of ordered pairs of nodes called edges, then a graph is a map \( G(V,E) : E \to K \subseteq \mathbb{R}_+ \) (we suppose that the graph has no multiple edges). If two nodes are adjacent, \((i,j) \in E\) we write \( i \sim j \).

3.1 The right choice for graph representation

First, we establish a connection between transport flows on the graph \( G \) and random walks on its dual counterpart \( G^* \).

Given a connected undirected graph \( G(V,E) \), in which \( V \) is the set of nodes and \( E \) is the set of edges, we introduce the traffic function \( f : E \to (0,\infty] \) through every edge \( e \in E \). It then follows from the Perron-Frobenius theorem [27] that the linear equation

\[
f(e) = \sum_{e' \sim e} f(e') \exp(-h\ell(e')), \tag{6}
\]

where the sum is taken over all edges \( e' \in E \) which have a common node with \( e \), has a unique positive solution \( f(e) > 0 \), for every edge \( e \in E \), for a fixed positive constant \( h > 0 \) and a chosen set of positive metric length distances \( \ell(e) > 0 \). This solution is naturally identified with the traffic equilibrium state of the transport network defined on \( G \), in which the permeability of edges depends upon their lengths. The parameter \( h \) is called the volume entropy of the graph \( G \), while the volume of \( G \) is defined as the sum

\[
\text{Vol}(G) = \frac{1}{2} \sum_{e \in E} \ell(e).
\]

The volume entropy \( h \) is defined to be the exponential growth of the balls in a universal covering tree of \( G \) with the lifted metric, [28]-[31].

The degree of a node \( i \in V \) is the number of its neighbors in \( G \), \( \deg_G(i) = k_i \). It has been shown in [31] that among all undirected connected graphs of normalized volume, \( \text{Vol}(G) = 1 \), which are not cycles and for which \( k_i \neq 1 \) for all nodes (no cul-de-sacs), the minimal value of the volume entropy, \( \min(h) = \frac{1}{2} \sum_{i \in V} k_i \log(k_i - 1) \) is attained for the length distances

\[
\ell(e) = \frac{\log((k_i - 1)(k_j - 1))}{2 \min(h)}, \tag{7}
\]

where \( k_i \) and \( k_j \) are the degrees of the nodes linked by \( e \in E \). It is then obvious that substituting (7) and \( \min(h) \) into (6) the operator \( \exp(-h\ell(e')) \) is given by a symmetric Markov transition operator,

\[
f(e) = \sum_{e' \sim e} \frac{f(e')}{\sqrt{(k_i - 1)(k_j - 1)}}, \tag{8}
\]

9
where $i$ and $j$ are the nodes linked by $e' \in E$, and the sum in (8) is taken over all edges $e' \in E$ which share a node with $e$. The symmetric operator (8) rather describes time reversible random walks over edges than over nodes. In other words, we are invited to consider random walks described by the symmetric operator defined on the dual graph $G^*$. The Markov process (8) represents the conservation of the traffic volume through the transport network, while other solutions of (6) are related to the possible termination of travels along edges. If we denote the number of neighbor edges the edge $e \in E$ has in the dual graph $G^*$ as $q_e = \text{deg}_{G^*}(e)$, then the simple substitution shows that $w(e) = \sqrt{q_e}$ defines an eigenvector of the symmetric Markov transition operator defined over the edges $E$ with eigenvalue 1. This eigenvector is positive and being properly normalized determines the relative traffic volume through $e \in E$ at equilibrium.

Eq. (8) shows the essential role Markov’s chains defined on edges play in equilibrium traffic modelling and emphasizes that the degrees of nodes are a key determinant of the transport networks properties.

The notion of traffic equilibrium had been introduced by J.G. Wardrop in [32] and then generalized in [33] to a fundamental concept of network equilibrium. Wardrop’s traffic equilibrium is strongly tied to the human apprehension of space since it is required that all travellers have enough knowledge of the transport network they use. The human perception of places is not an entirely Euclidean one, but are rather related to the perceiving of the vista spaces (viewable spaces of streets and squares) as single units and to the understanding of the topological relationships between these vista spaces, [34].

The use of Eq. (8) also helps to clarify the inconsistency of the traditional axial technique widely implemented in space syntax theory. Lines of sight are oversensitive to small deformations of the grid, which leads to noticeably different axial graphs for systems that should have similar configuration properties. Long straight paths, represented by single lines, appear to be overvalued compared to curved or sinuous paths as they are broken into a number of axial lines that creates an artificial differentiation between straight and curved or sinuous paths that have the same importance in the system [35]. Eq. (8) shows that the nodes of a dual graph representing the open spaces in the spatial network of an urban environment should have an individual meaning being an entity characterized by the certain traffic volume capacity.

Decomposition of city space into a complete set of intersecting open spaces characterized by the traffic volume capacities produces a spatial network which we call the dual graph representation of a city.
3.2 The processes associated with permutational automorphisms of the graph

While analyzing a graph, whether it is primary or dual, we assign the absolute scores to all nodes based on their properties with respect to a transport process defined on that. Indeed, the nodes of \( G(V, E) \) can be weighted with respect to some measure \( m = \sum_{i \in V} m_i \delta_{ij} \), specified by a set of positive numbers \( m_i > 0 \). The space \( \ell^2(m) \) of square-summable functions with respect to the measure \( m \) is a Hilbert space \( \mathcal{H}(V) \).

Among all measures which can be defined on \( V \), the set of normalized measures (or densities),

\[
1 = \sum_{i \in V} \pi_i \delta_{ij},
\]

are of essential interest since they express the conservation of a quantity, and therefore may be relevant to a physical process.

The fundamental physical process defined on the graph is generated by the subset of its automorphisms preserving the notion of connectivity of nodes. An automorphism is a mapping of the object to itself preserving all of its structure. The set of all automorphisms of a graph forms a group, called the automorphism group. For each graph \( G(V, E) \), there exists a unique, up to permutations of rows and columns, adjacency matrix \( A \), the \( N \times N \) matrix defined by \( A_{ij} = 1 \) if \( i \sim j \), and \( A_{ij} = 0 \) otherwise. As usual \( A \) is identified with a linear endomorphism of \( C_0(G) \), the vector space of all functions from \( V \) into \( \mathbb{R} \). The degree of a node \( i \in V \) is therefore equal to

\[
k_i = \sum_{i \sim j} A_{ij}.
\]

Let us consider the set of all linear transformations defined on the adjacency matrix,

\[
Z(A)_{ij} = \sum_{s,l=1}^N F_{ijsl} A_{sl}, \quad F_{ijsl} \in \mathbb{R},
\]

generated by the subset of automorphisms of the graph \( G \).

The graph automorphisms are specified by the symmetric group \( S_N \) including all admissible permutations \( p \in S_N \) taking \( i \in V \) to \( p(i) \in V \). The representation of \( S_N \) consists of all \( N \times N \) matrices \( \Pi_p \), such that \( (\Pi_p)_{i,j} = 1 \), and \( (\Pi_p)_{i,j} = 0 \) if \( j \neq p(i) \).

The function \( Z(A)_{ij} \) should satisfy

\[
\Pi_p^\top Z(A) \Pi_p = Z(\Pi_p^\top A \Pi_p),
\]

for any \( p \in S_N \), and therefore entries of the tensor \( F \) must have the following symmetry property,

\[
F_{p(i)p(j)p(s)p(l)} = F_{ijsl},
\]
for any \( p \in S_N \). Since the action of the symmetric group \( S_N \) preserves the conjugate classes of index partition structures, any appropriate tensor \( F \) satisfying (13) can be expressed as a linear combination of the following tensors: \( \{1, \delta_{ij}, \delta_{is}, \delta_{il}, \delta_{jl}, \delta_{sl}, \delta_{ij} \delta_{is}, \delta_{ij} \delta_{il}, \delta_{ij} \delta_{jl}, \delta_{ij} \delta_{sl}, \delta_{ij} \delta_{is}, \delta_{ij} \delta_{jl}, \delta_{ij} \delta_{sl}, \delta_{ij} \delta_{il}, \delta_{ij} \delta_{is}, \delta_{ij} \delta_{jl}, \delta_{ij} \delta_{sl} \} \). Given a simple, undirected graph \( G \) such that \( A_{ii} = 0 \) for any \( i \in V \) then by substituting the above tensors into (11) and taking account on symmetries we conclude that any arbitrary linear permutation invariant function must be of the form

\[
Z(A)_{ij} = a_1 + \delta_{ij} (a_2 + a_3 k_j) + a_4 A_{ij}, \tag{14}
\]

with \( k_j = \deg_G(j) \) and \( a_{1,2,3,4} \) arbitrary constants.

If we require that the linear function \( Z \) preserves the notion of connectivity,

\[
k_i = \sum_{j \in V} Z(A)_{ij}, \tag{15}
\]

it is clear that we should take \( a_1 = a_2 = 0 \) (indeed, the contributions \( a_1N \) and \( a_2 \) are incompatible with (15)) and then obtain the relation for the remaining constants, \( 1 - a_3 = a_4 \). Introducing the new parameter \( \beta \equiv a_4 > 0 \), we write (14) as follows,

\[
Z(A)_{ij} = (1 - \beta) \delta_{ij} k_j + \beta A_{ij}. \tag{16}
\]

If we express (15) in the form of the probability conservation relation, then the function \( Z(A) \) acquires a probabilistic interpretation. Substituting (16) back into (15), we obtain

\[
1 = \sum_{j \in V} \frac{Z(A)_{ij}}{k_i} = \sum_{j \in V} \left(1 - \beta\right) \delta_{ij} + \beta \frac{A_{ij}}{k_i} \tag{17}
\]

The operator \( T^{(\beta)}_{ij} \) represents the generalized random walk transition operator if \( 0 < \beta \leq k^{-1}_{\max} \) where \( k_{\max} \) is the maximal node degree in the graph \( G \). In the random walks defined by \( T^{(\beta)}_{ij} \), a random walker stays in the initial vertex with probability \( 1 - \beta \), while it moves to another node randomly chosen among its nearest neighbors with probability \( \beta / k_i \). If we take \( \beta = 1 \), then the operator \( T^{(1)}_{ij} \) describes the usual random walks discussed extensively in the classical surveys [36]-[38].

Being defined on a connected aperiodic graph, the matrix \( T^{(\beta)}_{ij} \) is a real positive stochastic matrix, and therefore, in accordance to the Perron-Frobenius theorem [27], its maximal eigenvalue is 1, and it is simple. A left eigenvector

\[
\pi T^{(\beta)} = \pi \tag{18}
\]
associated with the eigenvalue 1 has positive entries satisfying (9). It is interpreted as a unique equilibrium state \( \pi \) (stationary distribution of the random walk). For any density \( \sigma \in \mathcal{H}(V) \),

\[
\pi = \lim_{t \to \infty} \sigma T^t. \tag{19}
\]

### 3.3 Transport network as Euclidean space

Markov’s operators on Hilbert space appear therefore as the natural language of complex network theory and space syntax theory in particular. Now we demonstrate that random walks embed connected undirected graphs into Euclidean space, in which distances and angles acquire the clear statistical interpretations.

The Markov operator \( \hat{T} \) is self-adjoint with respect to the normalized measure (9) associated to the stationary distribution of random walks \( \pi \),

\[
\hat{T} = \frac{1}{2} \left( \pi^{1/2} T \pi^{-1/2} + \pi^{-1/2} T^\top \pi^{1/2} \right), \tag{20}
\]

where \( T^\top \) is the transposed operator. In the theory of random walks defined on graphs \([36, 38]\) and in spectral graph theory \([39]\), basic properties of graphs are studied in connection with the eigenvalues and eigenvectors of self-adjoint operators defined on them. The orthonormal ordered set of real eigenvectors \( \psi_i \), \( i = 1 \ldots N \), of the symmetric operator \( \hat{T} \) defines a basis in \( \mathcal{H}(V) \).

In particular, the symmetric transition operator \( \hat{T} \) of the random walk defined on connected undirected graphs is

\[
\hat{T}_{ij} = \begin{cases} 
\frac{1}{\sqrt{k_i k_j}}, & i \sim j, \\
0, & i \sim j.
\end{cases}
\tag{21}
\]

Its first eigenvector \( \psi_1 \) belonging to the largest eigenvalue \( \mu_1 = 1 \),

\[
\psi_1 \hat{T} = \psi_1, \quad \psi_1^2 = \pi_i, \tag{22}
\]

describes the local property of nodes (connectivity), since the stationary distribution of random walks is

\[
\pi_i = \frac{k_i}{2M} \tag{23}
\]

where \( 2M = \sum_{i \in V} k_i \). The remaining eigenvectors, \( \{ \psi_s \}_{s = 2}^N \), belonging to the eigenvalues \( 1 > \mu_2 \geq \ldots \geq \mu_N \geq -1 \) describe the global connectedness of the graph. For example, the eigenvector corresponding to the second eigenvalue \( \mu_2 \) is used in spectral bisection of graphs. It is called the Fiedler vector if related to the Laplacian matrix of a graph \([39]\).
Markov’s symmetric transition operator \( \hat{T} \) defines a projection of any density \( \sigma \in \mathcal{H}(V) \) on the eigenvector \( \psi_1 \) of the stationary distribution \( \pi \),

\[
\sigma \hat{T} = \psi_1 + \sigma_\perp \hat{T}, \quad \sigma_\perp = \sigma - \psi_1,
\]

in which \( \sigma_\perp \) is the vector belonging to the orthogonal complement of \( \psi_1 \). In space syntax, we are interested in a comparison between the densities with respect to random walks defined on the graph \( G \). Since all components \( \psi_{1,i} > 0 \), it is convenient to rescale the density \( \sigma \) by dividing its components by the components of \( \psi_1 \),

\[
\tilde{\sigma}_i = \frac{\sigma_i}{\psi_{1,i}}.
\]

Thus, it is clear that any two rescaled densities \( \tilde{\sigma}, \tilde{\rho} \in \mathcal{H} \) differ with respect to random walks only by their dynamical components,

\[
(\tilde{\sigma} - \tilde{\rho}) \hat{T}^t = (\tilde{\sigma}_\perp - \tilde{\rho}_\perp) \hat{T}^t,
\]

for all \( t > 0 \). Therefore, we can define the distance \( \| \ldots \|_T \) between any two densities established by random walks by

\[
\| \sigma - \rho \|_T^2 = \sum_{t \geq 0} \left\langle \tilde{\sigma}_\perp - \tilde{\rho}_\perp \bigl| \hat{T}^t \bigl| \tilde{\sigma}_\perp - \tilde{\rho}_\perp \right\rangle.
\]

or, using the spectral representation of \( \hat{T} \),

\[
\| \sigma - \rho \|_T^2 = \sum_{t \geq 0} \sum_{s=2}^{N} \mu_s^t \frac{\langle \tilde{\sigma}_\perp - \tilde{\rho}_\perp | \psi_s \rangle \langle \psi_s | \tilde{\sigma}_\perp - \tilde{\rho}_\perp \rangle}{1 - \mu_s},
\]

where we have used Dirac’s bra-ket notations especially convenient for working with inner products and rank-one operators in Hilbert space.

If we introduce a new inner product for densities \( \sigma, \rho \in \mathcal{H}(V) \) by

\[
(\sigma, \rho)_T = \sum_{s=2}^{N} \frac{\langle \tilde{\sigma}_\perp | \psi_s \rangle \langle \psi_s | \tilde{\rho}_\perp \rangle}{1 - \mu_s},
\]

then (27) is nothing else but

\[
\| \sigma - \rho \|_T^2 = \| \sigma \|_T^2 + \| \rho \|_T^2 - 2 (\sigma, \rho)_T,
\]

where

\[
\| \sigma \|_T^2 = \sum_{s=2}^{N} \frac{\langle \tilde{\sigma}_\perp | \psi_s \rangle \langle \psi_s | \tilde{\sigma}_\perp \rangle}{1 - \mu_s}
\]

is the square of the norm of \( \sigma \in \mathcal{H}(V) \) with respect to random walks defined on the graph \( G \).
We finish the description of the \((N-1)\)-dimensional Euclidean space structure of \(G\) induced by random walks by mentioning that given two densities \(\sigma, \rho \in \mathcal{H}(V)\), the angle between them can be introduced in the standard way,

\[
\cos \angle (\rho, \sigma) = \frac{(\sigma, \rho)}{\|\sigma\|_T \|\rho\|_T}.
\]  

(31)

Random walks embed connected undirected graphs into the Euclidean space \(\mathbb{R}^{N-1}\). This embedding can be used in order to compare nodes and to construct the optimal coarse-graining representations.

Namely, in accordance to (30), the density \(\delta_i\), which equals 1 at \(i \in V\) and zero otherwise, acquires the norm \(\|\delta_i\|_T\) associated to random walks defined on \(G\). In the theory of random walks [36], its square,

\[
\|\delta_i\|^2_T = \frac{1}{\pi_i} \sum_{s=2}^{N} \frac{\psi^2_{s,i}}{1 - \mu_s},
\]  

(32)

gets a clear probabilistic interpretation expressing the access time to a target node quantifying the expected number of steps required for a random walker to reach the node \(i \in V\) starting from an arbitrary node chosen randomly among all other nodes with respect to the stationary distribution \(\pi\).

The Euclidean distance between any two nodes of the graph \(G\) calculated as the distance (27) between the densities \(\delta_i\) and \(\delta_j\) induced by random walks,

\[
K_{i,j} = \|\delta_i - \delta_j\|^2_T = \sum_{s=2}^{N} \frac{1}{1 - \mu_s} \left(\frac{\psi_{s,i}}{\sqrt{\pi_i}} - \frac{\psi_{s,j}}{\sqrt{\pi_j}}\right)^2,
\]  

(33)

quantifies the commute time in theory of random walks being equal to the expected number of steps required for a random walker starting at \(i \in V\) to visit \(j \in V\) and then to return back to \(i\), [36].

It is important to mention that the cosine of an angle calculated in accordance to (31) has the structure of Pearson’s coefficient of linear correlations that reveals its natural statistical interpretation. Correlation properties of flows of random walkers passing by different paths have been remained beyond the scope of previous studies devoted to complex networks and random walks on graphs. The notion of angle between any two nodes of the graph arises naturally as soon as we become interested in the strength and direction of a linear relationship between two random variables, the flows of random walks moving through them. If the cosine of an angle (31) is 1 (zero angles), there is an increasing linear relationship between the flows of random walks through both nodes. Otherwise, if it is close to -1 (\(\pi\) angle), there is a decreasing linear relationship. The correlation is 0 (\(\pi/2\) angle) if the variables are linearly independent. It
is important to mention that as usual the correlation between nodes does not necessarily imply a direct causal relationship (an immediate connection) between them.

4 Examples: Petersen graph and the network of Venetian Canals

In the present section, we construct the Euclidean embedding of two graphs. One graph we study is the Petersen graph of 10 nodes (see Fig. 1). Another example is the spatial network of 96 Venetian canals which serve the function of roads in the ancient city that stretches across 122 small islands (see Fig. 2). While identifying a canal over the plurality of water routes on the city map of Venice, the canal-named approach has been used, in which two different arcs of the city canal network were assigned to the same identification number provided they have the same name. The Petersen graph is a regular graph, $k_i = 3, i = 1, \ldots, 10$, so that

\[
\sum k_i = 30, \quad \text{and the stationary distribution of random walks is uniform,} \quad \pi_i^{(\text{Pet})} = 0.1.
\]

The spectrum of the random walk transition operator (21) contains the Perron eigenvalue 1 which is simple, then the eigenvalue $1/3$ of multiplicity 5, and $-2/3$ of multiplicity 4. Therefore, there are just 3 linearly independent eigenvectors, and two eigensubspaces for which the orthonormal basis vectors can be calculated, so that the resulting matrix of eigenvectors and basis vectors which we use in (32-33) always has full column dimension. Random walks embed the Petersen graph into a 9-dimensional Euclidean space, in which all nodes have equal norm $\|i\|_T = 3.14642654$ that means the expected number of steps a random

\[
\begin{align*}
3 & \quad 10 \quad 2 \\
4 & \quad 7 & \quad 1 \\
5 & \quad 9 & \quad 6
\end{align*}
\]

Figure 1: The Petersen graph.
walker starting from a node chosen randomly with probability $p = 0.1$ reaches any node in the Petersen graph equals 9.9. Indeed, the structure of 9-dimensional vector space induced by random walks defined on the Petersen graph cannot be represented visually, however if we choose a node as a point of reference, we can draw its 2-dimensional projection by arranging other nodes at the distances calculated accordingly to (33) and under the angles found from (31) they are with respect to the chosen reference node (see Fig. 3).

It is expected that a random walker starting at #1 visits any periphery node (#2, 3, 4, 5) and then returns back in 18 random steps, while it is required 24 random steps in order to visit any internal node in the deep of the graph (#6, 7, 8, 9, 10). It is also obvious that while the linear relationship between the random walks flows through #1 and those through the periphery nodes is positive, it is negative with respect to the flows passing through the internal nodes. Due to the symmetry of the Petersen graph, the figure displayed on Fig. 3 is essentially the same if we draw it with respect to any other periphery node (#2, 3, 4, 5). It is also important to note that it turns to be mirror-reflected if we draw it with respect to any internal node (#6, 7, 8, 9, 10). Therefore, we can conclude that the Petersen graph contains two groups of nodes, at the periphery and in deep,
which appears to be "a quarter" higher segregated from each other (18 random steps vs. 24 random steps). It is clear that the 9-dimensional embedding of the Petersen graph into Euclidean space is characterized by the highest degree of symmetry.

The graph representation of the spatial network of Venetian canals (Fig 2) is much more complicated than the Petersen graph. The graph is far from being regular, so that the stationary distribution of random walks defined on it is not uniform. In [22], we have discussed that it is not evident that the degree distributions in compact urban patterns and in Venice, in particular, follow a power law. The spectrum of the Markov transition operator (21) defined on that is presented in Fig. 4. The matrix (21) for the canal network in Venice is strongly defective. In particular, it contains the eigenvalue $\mu = 0$ of multiplicity 22. This degeneracy indicates the presence of the complete bipartite subgraph in the spatial network of Venice shown in Fig. 2. The norms of canals with respect to random walks are different and scales with their connectivity (see Fig. 5). The notion of spatial segregation acquires a statistical interpretation with respect to random walks by means of (32). In urban spatial networks encoded by their dual graphs, the access times strongly vary from one open space to another and could be very large for statistically segregated spaces. Three data points characterized by the shortest access times shown in Fig. 5 are due to the Lagoon of Venice, the Giudecca canal, and the Grand canal - the most central water routes in the city canal network. Four data points characterized by the worst accessibility represent the canal subnetwork of Venetian Ghetto. The slope of the regression line equals 2.07.

The 2-dimensional projection of the Euclidean space of 96 Venetian canals set up by random walks drawn for the the Grand Canal of Venice (the point (0, 0)) is shown in Fig. 6. Nodes of the dual graph representation of the canal network in Venice are shown by disks with radiiuses taken
equal to the degrees of the nodes. All distances between the chosen origin and other nodes of the graph (Fig. 2) have been calculated in accordance to (33) and (31) has been used in order to compute angles between nodes. Canals negatively correlated with the Grand Canal of Venice are set under negative angles (below the horizontal), and under positive angles (above the horizontal) if otherwise.

It is evident from Fig. 6 that disks of smaller radiuses demonstrate a clear tendency to be located far away from the origin being characterized by the excessively long commute times with the reference point (the Grand canal of Venice), while the large disks which stand in Fig. 6 for the main water routes are settled in the closest proximity to the origin that intends an immediate access to them.

5 Discussion and Conclusion

In the present paper, we have developed a self-consistent approach to modelling of complex networks. We have discussed the possible creation algorithm (see Sec. 2) which refers to the idea of distribution of resources, including land, water, minerals, fuel and wealth in general rather than to the popularity driven preferential attachment approach proposed in [25, 26]. We have also demonstrated that random walks are the effective tool for investigation of the graph structure since they embed a connected graph into Euclidean space, in which the distances and angles acquire the
Figure 5: The scatter plot of the connectivity vs. the norm a node in the dual graph representation of 96 Venetian canals acquires with respect to random walks. Three data points characterized by the shortest access times represent the main water routes of Venice: the Lagoon of Venice, the Giudecca canal, and the Grand canal. Four data points of the worst accessibility are for the canal subnetwork of Venetian Ghetto. The slope of the regression line equals 2.07.

Probably, the most important conclusion of space syntax theory is that the adequate level of the positive relationship between the connectivity of city spaces and their integration property (vs. segregation) called intelligibility encourages peoples way-finding abilities [23]. Intelligibility of Venetian canal network reveals itself quantitatively in the scaling of the norms of nodes with connectivity shown in Fig. 5 and qualitatively in the tendency of smaller disks to be located on the outskirts of the Venetian space syntax displayed in Fig 6.

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Figure 6: The 2-dimensional projection of the 95-dimensional Euclidean spaces associated to random walks defined on the city canal network built from the perspective of the Grand canal of Venice chosen as the origin. The labels of the horizontal axes display the expected number of random walk steps. The labels of the vertical axes show the degree of nodes (radiuses of the disks).

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