Thermally fluctuating superconductors in two dimensions

Subir Sachdev and Oleg A. Starykh
Department of Physics, Yale University
P.O. Box 208120, New Haven, CT 06520-8120, USA
(January 1, 2000)

We describe the different regimes of finite temperature dynamics in the vicinity of a zero temperature superconductor to insulator quantum phase transition in two dimensions. New results are obtained for a low temperature phase-only hydrodynamics, and for the intermediate temperature quantum-critical region. In the latter case, we obtain a universal relationship between the frequency-dependence of the conductivity and the value of the d.c. resistance.

In many interesting two-dimensional superconducting systems, such as Josephson junction arrays, granular superconducting films, and the high temperature superconductors, it appears that the electrons bind into Cooper pairs below a pairing temperature \(T_p\) that is well above the Kosterlitz-Thouless transition, at \(T_{KT}\), to long-range superconducting order. It is natural to search for a direction in parameter space where \(T_{KT}\) vanishes at a \(T = 0\) superconductor-insulator quantum phase transition, and to then expand in the deviation from the quantum critical point—it is not necessary for this point to be experimentally accessible for such an approach to be valuable, as it offers a controlled description of an intermediate coupling regime. We describe crossovers in the dynamics near the critical point: new results are obtained for a low temperature phase-only regime, and for the intermediate temperature ‘quantum-critical’ region. In the latter regime we describe the frequency \(\omega\) dependent conductivity \(\sigma(\omega)\), in terms of a single dimensionless parameter, \(\gamma(T)\), which determines the d.c. conductivity; \(\gamma(T)\) can also be determined by separate static measurements.

For clarity, we will present our results in the context of a familiar microscopic model for the superconductor-insulator transition; however, our results generalize to a much wider class of systems, and this will be discussed towards the end of the paper. We consider a array of superconducting quantum dots at the sites, \(i\), of a regular two-dimensional lattice. The operator \(\hat{n}_i\) measures the number of electron pairs on dot \(i\), and \(\hat{\varphi}_i\) is its canonical conjugate phase \(\langle \hat{n}_i, \hat{\varphi}_j \rangle = \delta_{ij}\). We consider the Hamiltonian

\[
H = \left( \frac{E_C}{2} \right) \sum_i (\hat{n}_i - N_0)^2 - E_J \sum_{<ij>} \cos(\hat{\varphi}_i - \hat{\varphi}_j), \tag{1}
\]

where \(N_0\) is the mean integer number of Cooper pairs on each dot, \(<ij>\) represents nearest neighbor pairs, \(E_C\) is the charging energy of a dot, and \(E_J\) is the Josephson tunnel coupling between dots. This model exhibits a superconductor to insulator transition as the dimensionless parameter \(g = E_C/E_J\) is increased through a critical value \(g_c\), the phase diagram in the \(T, g\) plane is summarized in Fig 1. We will be mainly concerned with the region \(g \leq g_c\). For \(T < T_{KT}\), in the superconducting phase, the conductivity has the delta-function contribution \(\text{Re}[\sigma] = (\pi^2/\nu k_B^2 \rho_s(T) / \hbar^2) \delta(\omega)\), \((\nu = 2e)\), which defines the superfluid stiffness \(\rho_s(T)\) (in units of energy). The hyperscaling property of the quantum critical point implies that all dynamic properties in its vicinity are entirely characterized by a single energy, \(\rho_s(0)\), which vanishes at \(g = g_c\). In particular, the conductivity obeys

\[
\sigma(\omega, T) = \frac{\rho_s(0)}{k_B T} \left( \frac{\hbar \omega}{k_B T} \right) \tag{2}
\]

where \(\Sigma\) is a completely universal scaling function. Such a two argument scaling form is not overly constraining, and permits a rich variety of behavior which we shall describe here as a function of the second argument \(\lambda \equiv \rho_s(0)/k_B T\). Earlier studies of superfluid dynamics (made without reference to quantum phase transitions) emerge in limiting regimes, with \(\lambda > 0\) only placing restrictions on certain parameter values.

Demanding consistency of \(\Sigma\) with the definition of \(\rho_s(T)\) above immediately leads to interesting conclusions: (i) \(\rho_s(T)/\rho_s(0)\) is a universal function of \(\lambda\); (ii) the Kosterlitz Thouless transition occurs at
the universal value $\lambda = \lambda_c$, where $\Sigma$ is singular; (iii) $T_{KT} = \rho_s(0)/\lambda_c$. Aspects of the data on the high temperature superconductors are consistent with such trends suggesting the proximity of an insulator of Cooper pairs (with stripe order).

We now describe the evolution of dynamic properties with decreasing $\lambda$.

Phase hydrodynamics: At very large $\lambda$ ($k_B T \ll \rho_s(0)$) vortex excitations are exponentially suppressed, and we can derive an effective quantum action, $S_\varphi$ for the continuum phase variable $\varphi(x, \tau)$ (where $x$ is a two-dimensional spatial co-ordinate and $\tau$ is imaginary time) in a gradient expansion; the resulting action is valid only at the longest scales, and is in the spirit of the ‘chiral lagrangians’ of particle physics:

$$S_\varphi = \int d^2x d\tau \left[ \frac{\rho_s(0)}{2} (\partial_\mu \varphi)^2 - \frac{A_1(h v)^2}{2 \rho_s(0)} (\partial_\mu \varphi)^2 (\partial_\nu \varphi)^2 \right].$$

Here $\mu, \nu$ are spacetime indices with $\partial_\mu = (\nabla_x, \partial_\tau/v)$, and $v$ is the velocity of the ‘spin-wave’ excitations produced by the harmonic terms in $S_\varphi$ ($v$ remains non-singular through the quantum-critical point). The non-linear term, arises from integrating out amplitude fluctuations, and leads to spin-wave scattering: consistency demands (universal) $A_1$ for the dimensionless number $A_1$ is universal. In general spatial dimension, $d$, $A_1$ multiplies $(h v)^2/(d-1)\rho_s(0)^2(d-3)/(d-1)$, from which it is evident that the universality of $A_1$ holds only for $1 < d < 3$; we computed $A_1$ for $H$ in an expansion in $\epsilon = 3 - d$ and obtained

$$A_1 = \left[ \frac{10(4\pi)^{(d+1)/2}}{\Gamma((d+1)/2)} \right]^{2/(d-1)} \left( 1 - \frac{11\epsilon}{30} + O(\epsilon^2) \right).$$

Determination of the $T > 0$ transport properties of $S_\varphi$ requires solution of the appropriate quantum kinetic equations—we will not do this here. The procedure is closely analogous to early work on phonon transport in Galilean-invariant superfluids; an important difference is that these systems had a cubic non-linearity which is forbidden in our case by particle-hole symmetry.

Vortex hydrodynamics: In the shaded region of Fig. 1, in the vicinity of $T_{KT}$, the vortices proliferate, and the dynamics can be described by a well-developed classical theory for the vortices alone. This theory is contained within $\Sigma$ for $\lambda \approx \lambda_c$, and compatibility constrains various prefactors. So e.g. for $T > T_{KT}$ the response of the free vortices is controlled by their diffusivity, $A_2 h v^2/\rho_s(0)$, and their screening length, $A_3 (h v/\rho_s(0)) e^{\lambda \lambda_c^{1/2}}$; in general the $A_{2-4}$ are arbitrary dimensionless scale factors—however, near $g = g_c$ they become universal numbers.

Quantum critical: Discussion of this small $\lambda$ regime will occupy the remainder of the paper. Previous work relied upon expansions in either small $3 - d$, or small $d - 1$, or large $N$ (the number of real order parameter components). Here we shall present a theory directly for the physical case $N = 2, d = 2$.

Unlike the two previous regimes discussed above, it is no longer possible to decouple the spin-wave and vortex degrees of freedom. We will therefore use the complex superconducting order parameter $\psi(x, t)$ ($t$ is real time) which is the continuum limit of the lattice operator $e^{i\varphi}$, and allow for both amplitude and phase fluctuations in $\psi$. Our theory follows from two hypotheses: (i) The equal-time correlations of $\psi$ are controlled by a Gaussian effective action. Evidence for this hypothesis emerged in detailed studies of order parameter correlations in the quantum-critical region—the non-Gaussian components of the $\psi$ correlations are weak because the order parameter anomalous dimension at the quantum-critical point, $\eta \approx 0.03$, is so small. (This small $\eta$ also shows why $\psi$ is preferred over the ‘dual order parameter’ measuring vorticity, the latter has an appreciable anomalous dimension.) (ii) The time evolution of $\psi(x, t)$ is described by classical equations of motion. The characteristic relaxation (phase coherence) time in the quantum-critical regime is of order $\hbar/k_B T$; however the dominant spectral weight is at frequencies $\omega < k_B T$, and there is good evidence that the errors made by focusing on this low-frequency classical regime are quite small. We note that a different classical dynamic model was considered in Ref. 2, but it does not apply to transport properties.

We now define our model for quantum-critical transport, and then present its exact (numerical) solution. In addition to the field $\psi(x, t)$, we will need the canonically conjugate variable measuring density fluctuations, $\delta n(x, t)$, which is the continuum limit of $\delta n_i - N_0$. The equal-time correlations of $\psi$ and $\delta n$ are described by the partition function

$$Z = \int D\psi(x) D\delta n(x) \exp(- (\mathcal{H}_1 + \mathcal{H}_2)/k_B T),$$

$$\mathcal{H}_1 = \int d^2 x \left[ |\nabla \psi|^2 + \xi^{-2}(T)|\psi|^2 \right],$$

$$\mathcal{H}_2 = (1/[2\chi_u(T)]) \int d^2 x (\delta n(x))^2.$$
\[ \{\delta n(x), \psi(x')\}_{\text{P.B.}} = (i/\hbar)\psi(x)\delta^2(x - x'), \] (6)

which is the continuum, classical limit of the commutator between \( \dot{\varphi} \) and \( \dot{\phi}_i \). Notice that \( \hbar \) appears on the r.h.s. even though we are considering classical equations. The equations of motion implied by (3)-(6) are simple and familiar. They are the continuity equation

\[ \partial \delta n(x,t) / \partial t + \nabla \cdot J(x,t) = 0, \] (7)

where \( J = -(i/\hbar)(\psi^* \nabla \psi - \psi \nabla \psi^*) \), and

\[ \partial \psi(x,t) / \partial t = -(i/\hbar)\Phi(x,t)\psi(x,t), \] (8)

which is the Josephson equation, with the electrochemical potential \( \Phi(x,t) = \delta n(x,t)/\chi_0(T) \). We restate our dynamical model for quantum critical transport: choose a set of equal-time initial conditions for \( \psi \) and \( \delta n \) from the thermal ensemble defined by \( Z \), and evolve them deterministically by (5) and (6). Correlation functions are determined by the average over initial conditions.

We will shortly present convincing numerical evidence that (3)-(6) define a sensible continuum theory free of short distance or short time (‘ultraviolet’) divergences; this is supported by perturbative and renormalization group arguments, which we do not describe here. So unlike \( S_{\varphi} \), the present theory describes the couplings of vortex and spin-wave fluctuations at different length scales. It is helpful to visualize the continuum theory by coarse-graining to a lattice spacing, \( a \): the shorter distance degrees of freedom lead to fluctuations in the phase and amplitude of \( \psi \), but the long distance transport properties are insensitive to the value of \( a \). The absence of factors of \( a \) in the final results allows us to deduce their functional dependence on \( \xi(T) \), \( \chi_0(T) \) by simple engineering dimensional analysis. In this manner we conclude that

\[ \sigma(\omega, T) = \frac{e^2}{\hbar} \gamma(T)\Sigma_c \left( \gamma(T) \frac{\hbar \omega}{k_B T} \right), \] (9)

where \( \Sigma_c \) is a universal function (‘exact’ numerical results are below), and the dimensionless \( \gamma(T) \) is defined by

\[ \gamma(T) \equiv [k_B T \chi_0(T) \xi^2(T)]^{1/2}. \] (10)

Consistency of (3) with (2) only requires that \( \gamma(T) \) is a universal function of \( \lambda \) (also recall that the present quantum-critical theory is valid only for small \( \lambda \)). The result (3) has a clear experimental signature: it implies a correlation between the value of the d.c. conductivity and the inverse frequency-width of \( \sigma(\omega) \), as they are both determined by \( \gamma(T) \). More stringent comparisons can be made by using the measured d.c. conductivity to determine the unknown \( \gamma(T) \), and then using our numerical results below for \( \Sigma_c \) to determine the \( \omega \) dependence of \( \sigma \). It should be noted that our present computations for \( \Sigma_c \) will not apply for very large \( \hbar \omega/k_B T \), for then a full quantum theory is necessary, and we crossovers to the phase-coherent regime discussed earlier.

Exactly at \( g = g_c \), hyperscaling arguments imply that \( \xi(T) \sim T^{-1/2} \) and \( \chi_0(T) \sim T^{-(2- \gamma)/2} \); so \( \gamma(T) \) is \( T \)-independent for any \( z \), and is expected to be a universal number. For the model \( H \) in (3), this universal number can be computed in a 1/N expansion:

\[ \gamma(T)|_{g=g_c} = \left[ \frac{\sqrt{5}}{4\pi \ln((\sqrt{5} + 1)/2)} \right]^{1/2} \left( 1 - \frac{0.5468}{N} \right). \] (11)

We turn to our numerical results for \( \Sigma_c \). The simulations were carried out on an \( N \times N \) square lattice of spacing \( a \) with periodic boundary conditions, and \( a \ll \xi(T) \ll Na \). For each \( a \), successively larger values of \( N \) were used, until the results became \( N \)-independent, and the continuum limit was then approached as \( a \to 0 \). The lattice form of \( H_1 + H_2 \) was obtained by mapping the momentum \( (k_x, k_y) \) dependence of the couplings of the continuum Hamiltonian under \( k^2_x \to (4/a^2)^2[k_1 \sin^2(k_x a/2) + k_2^2 \sin^2(k_x a) + k_3^2 \sin^2(k_y a/2)] \), and similarly for \( k_y \) (for \( |\nabla \psi|^2 \), this amounts to including first, second, and third neighbor couplings between the \( \psi \)’s). We chose two different sets of values for \( K_{1,2,3} \) to test the independence of the continuum theory on the lattice realization—\( A \) the familiar \( K_1 = 1, K_2 = K_3 = 0 \), and \( B \) \( K_1 = 3/2, K_2 = -3/20, K_3 = 1/90 \) for which \( k_x^2 \to k_x^2 (1 + O(k_x a)^6) \). The initial conditions specified by \( Z \) form a Gaussian ensemble, and are easily generated in a single sweep. The time evolution was carried out by a fourth order predictor-corrector algorithm with a time step determined to conserve total energy to a relative accuracy better than \( 10^{-5} \). We measured the autocorrelation function of the total current, \( C_J \), by averaging over 3000 initial conditions; \( C_J \) is normalized such that

\[ \Sigma_c(\omega) = \int_0^\infty d\tau C_J(T)e^{i\omega \tau}, \] (12)

where \( \omega = \gamma(T)\hbar \omega/(k_B T) \) and \( \tau = k_B T/(\hbar \gamma(T)) \).

Our results for \( C_J \) and \( \Sigma_c \) are in Fig. 3. Notice the \( a \) dependence at \( \tau = 0 \)—this is expected as an analytic calculation of equal-time correlations shows that \( C_J(0) = (1/\pi) \ln(1/a) + \ldots \). However, the \( a \) dependence quickly disappears at small non-zero \( \tau \) and then a universal continuum value obtains; this is strong evidence for the existence of the continuum theory (3). We obtained \( \Sigma_c(0) \approx 1.85 \); for the model \( H \) in (3), this combines with (3) to yield \( \sigma(0,T) = 0.82\hbar^2e^2/k_B T \) at \( g = g_c \).

The experiments of Rimberg et al. appear to be a convenient testing ground for our theory. The metallic gate screens the long-range Coulomb interactions and so justifies the short-range coupling in \( H_2 \). However, it also introduces dissipation, which will induce additional terms in the equations of motion e.g. (3) is modified to

\[ \partial \psi / \partial t = -\Gamma \delta H_1 / \delta \psi^* - (i/\hbar)\Phi \psi + \zeta, \] (13)

where \( \Gamma \) is a damping co-efficient and \( \zeta \) is the associated random noise. Fortunately, it can be shown, as in
of long-range Coulomb interactions. This will modify
ments apply to the following paragraph.)
unchanged, but with a new \( \Sigma \); the structure of the phase
unchanged, but with \( \Phi(\cdots) \) for \( a/\xi(T) = 1/16 \); for large
we have \( \text{Re} \Sigma_c \sim 1/|\omega| \) and \( \text{Im} \Sigma_c \sim \text{ln}(|\omega|)/\omega \).

Refs 23, that for large \( \xi(T) \), such modifications leave the
dynamic function \( \Sigma_c \) unchanged. The damping can be a relevant perturbation on the \( T = 0 \) quantum critical point\( \text{ii} \) but this manifested only via changes in \( \xi(T) \), \( \chi_c(T) \), and \( \gamma(T) \). (Outside the quantum critical region, the change in quantum universality leaves the form \( \text{iii} \) unchanged, but with a new \( \Sigma \); the structure of the phase
hydrotamodynamics, \( S_\sigma \) can also be modified. Similar com-
ments apply to the following paragraph.)

As a second modification, we consider the inclusion of long-range Coulomb interactions. This will modify \( \mathcal{H}_2 \) to \( e^2 \int d^2x d^2x' \delta n(x) \delta n(x')/2|x - x'| \), and leave \( \text{iv} \) unchanged, but with \( \Phi(x, t) = e^2 \int d^2x' \delta n(x', t)/|x - x'| \). Assuming the existence of the continuum limit of the classical
dynamical theory, we again obtain \( \text{v} \), but with \( \gamma(T) = (k_B T \xi(T)/e^2)^{1/2} \); the functional form of \( \Sigma_c \) will of
course be changed and requires a separate numerical simulation. Note that at \( g = g_c \), \( \gamma(T) \) is a \( T \)-independent
universal number only if \( z = 1 \), the value expected for long-range interactions\( \text{vi} \).

The experiments of Corson et al.\( \text{vii} \) on an underdoped
high temperature superconductor observe scaling closely
related to \( \text{vii} \), with the frequency scale proportional to the
resistivity scale, and a scaling function with large \( \omega \) behavior similar to that in Fig 2. However their fits
use a prefactor \( T^0_c \) whose weak \( T \)-dependence disagrees with our continuum two-dimensional theory. We spec-
tulate that this can be explained by corrections to scaling
which are appreciable because both modifications dis-
cussed above are present here – damping from fermionic
excitations and long-range Coulomb interactions. Corson
et al.\( \text{viii} \) motivated the scaling using the vortex theory\( \text{vii} \), but did not consider bound vortex pairs, whose
contributions do not obey their scaling assumptions.

In summary, this paper has delineated the distinct
dynamic regimes of thermally fluctuating superconduc-
tors: the low \( T \) phase-only hydrodynamics, the classi-
ical vortex hydrodynamics in the vicinity of \( T_KT \), and
the quantum-critical region where phase and vortex fluct-
uations strongly coupled—here we proposed a dynamical
model in which non-linear ‘mode-coupling’ terms dem-
anded by the Poisson bracket\( \text{vi} \) dominate the univer-
sal, low-frequency, dissipative dynamics. Our approach
could be extended to other two-dimensional systems: (i)
the magnetic field-tuned superconductor-insulator transi-
tion, where a coupling to the external field would be
required in \( \text{vii} \), and (ii) quantum Hall transitions.

We thank J. Orenstein, D. Grempel, and A. Sudbø for
useful discussions. This research was supported by NSF
Grant No DMR 96–23181.

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