All Subgraphs of a Wheel are 5-Coupled-Choosable

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Abstract

A wheel graph consists of a cycle along with a center vertex connected to every vertex in the cycle. In this paper we show that every subgraph of a wheel graph has list coupled chromatic number at most 5. We further show that this bound is tight for every wheel graph with at least 5 vertices.

1 Introduction

In this paper we study the problem of coupled choosability, the problem of finding a valid coloring given list assignments to every vertex and face of a planar graph. The problem is of great relevance to list coloring 1-planar graphs, as list coupled coloring a planar graph corresponds to list coloring an optimal 1-planar graph. (Detailed definitions will be given in Section 2.) Wang and Lih [6] show that every planar graph is 7-coupled-choosable, and hence every optimal 1-planar graph is 7-choosable. It is an open problem whether this bound holds for all 1-planar graphs. They further show that maximal planar graphs are 6-coupled-choosable, planar graphs of maximum degree 3 are 6-coupled-choosable, and both series-parallel graphs and outerplanar graphs are 5-coupled-choosable.

It is not hard to characterize \( k \)-coupled-choosability for small values of \( k \). Trivially, only the empty graph is 1-coupled-choosable, and (among the connected graphs) only the single-vertex graph is 2-coupled-choosable. A connected graph is 3-coupled-choosable if and only if it is a tree, for at any edge \( e \) incident to a cycle we would require 4 colors for the endpoints of \( e \) and the two (distinct) incident faces of \( e \).

The above result by Wang and Lih settles the coupled choosability for planar partial 2-trees (which are the same as the series-parallel graphs). Initially wishing to investigate the coupled choosability of planar partial 3-trees, in this paper we investigate the coupled choosability of wheel graphs and subgraphs of wheel graphs. In Theorem 4.3 we show that any subgraph of a wheel is 5-coupled-choosable. Furthermore, in Theorem 4.4 we characterize the coupled choosability of wheel graphs by showing that 5 is tight for wheel graphs with at least 5 vertices. In the last section of the paper, we touch upon how these results could be relevant in finding the coupled choosability of planar partial 3-trees.

As for related results, the (non-coupled) choosability of wheel graphs was characterized in a different paper by Wang and Lih [7]: wheels of even order have list chromatic number 4, while wheels of odd order have list chromatic number 3. This stands in contrast to our result, as the parity of the number of vertices in the graph does not affect the coupled choosability of wheel graphs. They also show that Halin graphs that are not wheels have list chromatic number 3, while in Theorem 5.3 we prove the existence of a Halin graph that is not 5-coupled-choosable (in fact, it is not 5-coupled-colorable).

Our paper is structured as follows: In Section 2 we will go over the necessary definitions and terminology for graphs and graph coloring. In Section 3 we investigate the coupled choosability of wheel graphs. In Section 4 we extend this analysis to subgraphs of wheels, along with lower-bounding the coupled choosability of wheel graphs. In Section 5 we go over several possible extensions to our results and some related open problems.

2 Definitions

We assume basic familiarity with graph theory (see [2]). In this paper all graphs are finite and connected.

We recall that a graph \( G \) is called planar if it can be drawn in the plane without edges crossing, and plane if a specific planar drawing \( \Gamma \) is given. The maximal regions of \( \mathbb{R} \setminus \Gamma \) are called faces; the unbounded
region is known as the outer face and all other faces are inner faces. A bigon is a face that is bounded by two duplicate edges between a pair of vertices. For a plane graph \( G \), we use \( V(G) \), \( E(G) \), and \( F(G) \) to denote the set of vertices, the set of edges, and the set of faces of \( G \), respectively. The dual graph \( G^* \) of a plane graph is obtained by exchanging the roles of vertices and faces, i.e., \( G^* \) has a vertex for every face of \( G \), and an edge \((f_1, f_2)\) for every common edge of the two corresponding faces \( f_1, f_2 \) in \( G \).

A list assignment is a map \( L \) that assigns a set of colors for each vertex or face in \( V(G) \cup F(G) \). A coupled coloring with respect to \( L \) is a map \( c \) such that \( c(x) \in L(x) \) for every \( x \in V(G) \cup F(G) \), and \( L(x) \neq L(y) \) for incident or adjacent elements \( x, y \in V(G) \cup F(G) \). If such a map \( c \) exists, then we say that \( G \) is \( L \)-coupled-choosable. If \( G \) is \( L \)-coupled-choosable for every \( L \) such that \(|L(x)| = k\) for every \( x \in V(G) \cup F(G) \), then we say that \( G \) is \( k \)-coupled-choosable. The smallest integer \( k \) such that \( G \) is \( k \)-coupled-choosable is called the list coupled chromatic number of \( G \) and denoted \( \chi_{c}^L(G) \). Observe that a list coupled coloring of a graph \( G \) implies a list coupled coloring of the dual graph \( G^* \), since the roles of the vertices and the faces is exchanged but incidences/adjacencies stay the same. Hence, we have \( \chi_{c}^L(G) = \chi_{c}^L(G^*) \).

A natural way to express the list coupled chromatic number is to define a new graph \( X(G) \) with vertices for all vertices and faces of \( G \) and edges whenever the vertices and faces are adjacent/incident. This graph \( X(G) \) is 1-planar, i.e., can be drawn in the plane with at most one crossing per edge. In fact, if \( G \) is 3-connected then \( X(G) \) is an optimal 1-planar graph, i.e., it is simple and has the maximum-possible \( 4n - 8 \) edges, and all optimal 1-planar graphs can be obtained in this fashion. A coupled coloring of \( G \) corresponds to a vertex coloring of \( X \), i.e., a coloring of the vertices such that adjacent vertices have different colors. When restricting a vertex coloring to given lists \( L \), then the respective terms are \( L \)-choosable, \( k \)-choosable, and the list chromatic number \( \chi^L(X) \).

The wheel graph \( W_n \) is formed by starting with a cycle \( C_{n-1} \) on \( n - 1 \) vertices (the outer cycle), adding a center vertex inside the cycle and adding an edge from the center vertex to every vertex on the cycle. We will label the center vertex and the outer face of the wheel graph as \( x_0 \) and \( f_0 \), respectively. We further label the vertices in the outer cycle as \( x_1, \ldots, x_{n-1} \), and label the inner faces as \( f_1, \ldots, f_{n-1} \) such that \( x_i \) is incident to \( f_i \) and \( f_{i+1} \) for \( 1 \leq i < n - 1 \), and \( x_{n-1} \) is adjacent to \( f_{n-1} \) and \( f_1 \) (see Figure 1).

An outerplanar graph is a graph that can be drawn in the plane such that every vertex is on the outer face. A subdivision of a graph \( G \) is formed by repeatedly taking some edge \( uv \in E(G) \), removing \( e \) from \( G \), adding a new vertex \( x \), and adding edges \( ux \) and \( xv \).

### 3 Coupled Choosability of Wheel Graphs

In order to prove the desired result for all subgraphs of the wheel graph, we first determine the coupled choosability of the wheel graph itself. It will be helpful to recall the following result relating the choosability of a graph to the maximum degree: it is an analogue to Brook’s theorem and similarly upper-bounds the chromatic number of a graph by its maximum degree.

**Lemma 3.1.** (Erdős, Rubin, and Taylor [2]) Let \( G \) be a connected graph that is neither an odd cycle nor a complete graph. Then \( G \) is \( \Delta(G) \)-choosable.

Our main result in this section is:

**Lemma 3.2.** Every wheel graph \( W_n \), \( n \geq 4 \), is 5-coupled-choosable.

**Proof.** For \( n = 4 \), \( W_4 \) is the complete graph \( K_4 \). Wang and Lih [6] proved that \( \chi^L_{c}(K_4) = 4 \), so we assume \( n \geq 5 \). Let \( L \) be a color assignment for \( W_n \) such that \(|L(y)| = 5\) for every \( y \in V(W_n) \cup F(W_n) \). Our goal is to find a coupled coloring with respect to \( L \). Since \( x_0 \) and \( f_0 \) are both adjacent to all remaining vertices, we will color them first and then color the rest. This rest defines the following graph \( X_n \):

- The vertices of \( X_n \) correspond to vertices \( x_1, \ldots, x_{n-1} \) and faces \( f_1, \ldots, f_{n-1} \) of \( W_n \).
- There is an edge \( x_ix_j \) if the vertices \( x_i \) and \( x_j \) are adjacent in \( W_n \).
- There is an edge \( f_if_j \) if the faces \( f_i \) and \( f_j \) are adjacent in \( W_n \).
- There is an edge \( x_if_j \) if the vertex \( x_i \) is incident to the face \( f_j \) in \( W_n \).
Observe that $|V(X_n)| = 2n - 2$ and that $X_n$ is 4-regular (see Figure 1). Furthermore, it suffices to find a vertex-colouring of $X_n$ with respect to $L$, plus two suitable colors in $L(x_0)$ and $L(f_0)$ for $x_0$ and $f_0$. We have two cases:

Figure 1: The graph $W_9$ (left) and $X_9$ (right). Circled numbers indicate a lower bound on the list-length in $L'$.

**Case 1:** $L(x_0) \cap L(f_0) \neq \emptyset$. Let $a \in L(x_0) \cap L(f_0)$, and assign $a$ to $x_0$ and $f_0$. Observe that $|L(y) \setminus \{a\}| \geq 4$ for every $y \in V(X_n)$ and $X_n$ has maximum degree 4. Moreover $|X_n| = 2n - 2$ is even, so $X_n$ is not an odd cycle. Also $x_1$ and $x_3$ are not adjacent by $n \geq 5$, so $X_n$ is not a complete graph. Therefore, by Lemma 3.1 we have a list coloring of the vertices of $X_n$ that only uses colors in $L \setminus \{a\}$, which in turn implies an $L$-list-coloring of the vertices and faces of $W_n$.

**Case 2:** $L(x_0) \cap L(f_0) = \emptyset$. We find suitable colors for $x_0$ and $f_0$ by imitating the method used for $K_4$ in [4] (but adapted here to 5 colors). Define color-pairs $S := \{\{a,b\} : a \in L(x_0), b \in L(f_0)\}$. By case-assumption $|S| = 25$.

We claim that $|\{s \in S : s \subseteq L(y)\}| \leq 6$ for any $y \in V(X_n)$. To see this, let $y \in V(X_n)$, and consider the disjoint sub-lists $L_1 := L(y) \cap L(x_0)$ and $L_2 := L(y) \cap L(f_0)$. Since $|L_1| + |L_2| \leq |L(y)| = 5$, and $|L_1|$ and $|L_2|$ are integers, we have

$$|\{s \in S : s \subseteq L(y)\}| = |L_1 \times L_2| = |L_1| \cdot |L_2| \leq 6.$$

Therefore, color-pairs of $S$ appear as subsets of lists in $X_n$ at most

$$\sum_{y \in X_n} |\{s \in S : s \subseteq L(y)\}| \leq (2n - 2) \cdot 6 = 12n - 12$$

times. By $|S| = 25$, some element $\{a',b'\}$ of $S$ appears at most

$$\frac{12n - 12}{25} < \frac{n - 1}{2}$$
times as a subset of a list in $X_n$. Color $x_0$ with $a'$ and $f_0$ with $b'$. For $y \in V(X_n)$, define $L'(y) := L(y) \setminus \{a',b'\}$. For any $y \in V(X_n)$, we have $3 \leq |L'(y)| \leq 5$. We call $y$ a 3-vertex if $|L'(y)| = 3$ (this implies $\{a',b'\} \subset L(y)$), and a 4-vertex otherwise. From our choice of colors $a'$ and $b'$, we have

$$\frac{|\{y \in V(X_n) : y \text{ is a 3-vertex}\}|}{|V(X_n)|} < \frac{(n - 1)/2}{2n - 2} = \frac{1}{4}$$

Therefore, more than three quarters of the vertices of $X_n$ are 4-vertices. Consider the cyclic enumeration

$$\sigma := \langle f_1, x_1, f_2, x_2, \ldots, f_{n-1}, x_{n-1} \rangle$$
of the vertices of $X_n$. Since strictly more than $\frac{3}{4}|V(X_n)|$ of the vertices are 4-vertices, we have four consecutive 4-vertices in $\sigma$. Up to exchange of $f_i$ and $x_i$ and renumbering, we may assume that $f_1, x_1, f_2,$ and $x_2$ are 4-vertices. Figure 1 (right) illustrates the lower bounds on the size of $L'$. 

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We next color \( f_{n-1}, x_{n-1} \) and \( x_1 \) and have two sub-cases. If \( L'(f_{n-1}) \cap L'(x_1) \neq \emptyset \), then color \( f_{n-1} \) and \( x_1 \) with the same color. Otherwise, since \( |L'(f_{n-1}) \cup L'(x_1)| \geq 7 > |L(f_1)| \), there are colors \( p \) and \( q \) for \( f_{n-1} \) and \( x_1 \) respectively such that at least one of them is not in \( L(f_1) \), i.e., \( |L(f_1) \cap \{p,q\}| \leq 1 \). Pick these colors for \( f_{n-1} \) and \( x_1 \). In either case, two vertices adjacent to \( x_{n-1} \) have been colored, and \( |L'(x_{n-1})| \geq 3 \), so \( x_{n-1} \) will have at least one valid color left, and we pick this color for \( x_{n-1} \).

We now have colors \( p, q, \) and \( r \) for \( f_{n-1}, x_1, \) and \( x_{n-1} \) (respectively) such that \( |L'(f_1) \cap \{p,q,r\}| \leq 2 \). Removing these colors from the lists of their neighbors produces new lists \( L'' \) such that

\[
\begin{align*}
|L''(f_1)| &= |L'(f_1) \setminus \{p,q,r\}| \geq 4 - 2 = 2 \\
|L''(f_2)| &= |L'(f_2) \setminus \{q\}| \geq 4 - 1 = 3 \\
|L''(x_2)| &= |L'(x_2) \setminus \{q\}| \geq 4 - 1 = 3 \\
|L''(x_{n-2})| &= |L'(x_{n-2}) \setminus \{p,r\}| \geq 3 - 2 = 1 \\
|L''(f_{n-2})| &= |L'(f_{n-2}) \setminus \{p\}| \geq 3 - 1 = 2 \\
|L''(x_i)| &\geq 3 \quad \text{(for all } 3 \leq i \leq n - 3) \\
|L''(f_i)| &\geq 3 \quad \text{(for all } 3 \leq i \leq n - 3)
\end{align*}
\]

(see also Figure 2).

![Figure 2: The graph \( X'_n \) (solid). Dotted edges show the rest of the graph \( X_n \), along with a dashed edge to the vertex \( f_1 \) which remains to be colored. Numbers indicate lower bounds on the list length in \( L'' \).](image)

Let \( X'_n := X_n \setminus \{f_{n-1}, x_{n-1}, f_1, x_1\} \) and color it with respect to list assignment \( L'' \). This is feasible since \( X'_n \) is outerplanar (see Figure 2), and outerplanar graphs are 3-choosable even if the colors of two consecutive vertices on the outer face are fixed [4] (here we fix the colors for \( x_{n-2} \) and \( f_{n-2} \)). This colors all vertices except for \( f_1 \), but \( |L''(f_1)| \geq 2 \) and \( f_1 \) has only one neighbor in \( X'_n \), so we can give it a color not used by \( f_2 \). Therefore, we have a list vertex-coloring of \( X_n \) that is compatible with the colors for \( x_0, f_0 \) chosen earlier and so implies a list coupled coloring of \( W_n \).

\[ \Box \]

### 4 Subgraphs of Wheel Graphs

Now we turn to graphs that are subgraphs of wheels. In contrast to list coloring the vertices of the graph, there is no clear relationship between the list coupled chromatic number of a graph and the list coupled chromatic number of its subgraphs. Indeed, it is possible for the subgraph to have a larger list coupled chromatic number.

**Observation 1.** There exists a plane graph \( G \) with subgraph \( H \subseteq G \) such that \( \chi^L_{vf}(H) > \chi^L_{vf}(G) \)

**Proof.** Consider the graph \( K_4 \), and the subgraph \( H \) obtained by deleting one edge; see Figure 3. From Theorem 10 of [6], we know that the graph \( K_4 \) is 4-coupled-choosable, i.e \( \chi^L_{vf}(K_4) = 4 \). But in graph \( H \), observe that the incidences and adjacencies between \( x_0, x_1, x_2, f_2 \), and \( f' \) form a \( K_5 \), and therefore

\[ \chi^L_{vf}(H) \geq 5 > 4 = \chi^L_{vf}(K_4) \]
Let \( H \) be a list assignment for \( H \) such that \( |L(x)| = \max\{5, \chi^L_{ij}(G)\} \) for every vertex and face of \( H \). We prove the statement by induction on the number of subdivisions performed on \( G \) to obtain \( H \). If \( H \) is the result of subdividing the edges of \( G \) zero times, then \( H = G \) and so trivially any \( L \)-coupled-coloring of \( G \) is an \( L \)-coupled-coloring of \( H \).

Otherwise, \( H \) was the result of performing \( k+1 \) subdivisions on \( G \) for some \( k \geq 0 \). In particular, \( H \) is the result of subdividing a single edge of some graph \( H' \), where \( H' \) was the result of performing \( k \) subdivisions on \( G \). Let \( uw \in E(H') \) be the edge of \( H' \) that was subdivided, and let \( x \) be the vertex which was added. By the inductive hypothesis, \( H' \) is \( \max\{5, \chi^L_{ij}(G)\} \)-coupled-choosable. Color the faces of \( H \) and the vertices \( V(H) \setminus \{x\} \) according to how they would be colored in \( H' \). Then we only need to color the remaining vertex \( x \). Note that \( x \) has degree two with neighbors \( u \) and \( v \). Let \( f_1 \) and \( f_2 \) be the two faces adjacent to the edge \( uv \) in \( H' \). Then \( u, v, f_1, \) and \( f_2 \) are the only vertices and faces that are adjacent (respectively incident) to \( x \). Hence, after coloring the vertices and faces from \( H' \), \( x \) still has at least

\[ |L(x)| - 4 \geq 5 - 4 = 1 \]

color left. Therefore \( x \) can always be colored.

This implies another results that we will not need but find interesting. For a planar graph \( G \), subdividing an edge corresponds in the dual graph \( G^\star \) to duplicating edges to form bigons. Since \( \chi^L_{ij}(G) = \chi^L_{ij}(G^\star) \) we therefore have:

**Corollary 4.2.** Let \( G \) be a plane graph, and \( H \) the result of duplicating some edges of \( G \) to form bigons. If \( \chi^L_{ij}(G) \geq 5 \), then \( H \) is \( \chi^L_{ij}(G) \)-coupled-choosable. If \( \chi^L_{ij}(G) \leq 5 \), then \( H \) is 5-coupled-choosable.

Now we can show our main result.

**Theorem 4.3.** Let \( G \) be a subgraph of a wheel graph \( W_n \), \( n \geq 4 \). Then \( G \) is 5-coupled-choosable.

**Proof.** We examine several possibilities of the structure of \( G \).

**Case 1:** \( G = W_n \). Then by Lemma 3.2 \( G \) is 5-coupled-choosable.

**Case 2:** \( G \) is the result of deleting at least one edge or vertex of \( W_n \) that is on the outer face. Then \( G \) is outerplanar, and so by Theorem 14 in [6] \( G \) is 5-coupled-choosable.

**Case 3:** \( G \) is the result of removing the center vertex of \( W_n \). Then \( G = C_{n-1} \) has only two faces. Color them arbitrarily, which leaves at least three colors for every vertex of \( C_{n-1} \). But \( C_{n-1} \) is outer-planar and hence 3-choosable.

**Case 4:** None of the above. Then all vertices of \( W_n \) belong to \( G \), but we deleted some edges which were not on the outer face. So \( G \) is the result of deleting some of edges incident to the center vertex (spoke edges). If at most one spoke remains, then \( G \) is outer-planar (after inverting the drawing so that the face incident
to the center vertex becomes the outer-face), and therefore $G$ is 5-coupled choosable. If exactly two spokes remain, then $G$ is a subdivision of a triangle edge. A triple edge has two vertices and three faces and therefore is 5-coupled choosable; by Lemma 4.1, so is $G$. Finally if at least three spokes remain, then $G$ is a subdivision of some $W_k$ for $k \geq 4$, and by Lemmas 3.2 and 4.1 $G$ is 5-coupled-choosable.

Following the steps of our proof, one can easily verify that we can find the $L$-coupled-coloring in linear time. Finding the coloring from Lemma 3.1 is known to be linear time [5].

Having established an upper bound on the list coupled chromatic number of wheel graphs in Lemma 3.2, one might wonder whether this bound is tight or not. In [6], it is shown that the graph $K_4 = W_4$ is 4-coupled-choosable. In fact, this is the only wheel graph which is 4-coupled-choosable. For all other wheel graphs, the bound of 5-coupled-choosability is tight.

**Theorem 4.4.** $\chi_{L \ell f}^c(W_n) = 5$, for $n \geq 5$.

**Proof.** From Lemma 3.2 we know that wheel graphs are always 5-coupled-choosable. It remains to show that they are not 4-coupled-choosable for $n \geq 5$.

For $n = 5, 6$, we consider the list assignment $L$ such that $L(y) = \{1, 2, 3, 4\}$ for every $y \in V(W_n) \cup F(W_n)$. (So these graphs are not even 4-coupled-colorable.) Assume for contradiction that we have an $L$-coupled-coloring $c$ of $W_n$. If $c(x_0) \neq c(f_0)$, then this leaves two colors for coloring the triangle $x_1, f_1, f_2$ in $X_n$, impossible. Hence $c(x_0) = c(f_0)$, say they are both colored 4. Then we have an $L'$-coloring of $X_n$ with lists $L'(y) := L(y) \setminus \{4\} = \{1, 2, 3\}$.

Observe that for $X_5$ and $X_6$, any putative $L'$-coloring would be unique up to renaming the colors, since once we have colored one triangle, every other vertex can be reached via a sequence of triangles. One verifies that for these graphs (and indeed every $X_k$ where $k - 1$ is not divisible by 3), attempting such a 3-coloring leads to a contradiction (see Figure 4). This proves Theorem 4.4 for $n = 5, 6$.

For $n \geq 7$, we construct a list assignment $L$ such that $W_n$ is not $L$-coupled-choosable. Set $L(x_0) = \{1, 2, 3, 4\}$ and $L(f_0) = \{5, 6, 7, 8\}$. We further define:

- $L(f_1) = L(x_1) = L(f_2) = \{1, 2, 5, 6\}$
- $L(x_2) = L(f_3) = L(x_3) = \{1, 2, 7, 8\}$
- $L(f_4) = L(x_4) = L(f_5) = \{3, 4, 5, 6\}$
- $L(x_5) = L(f_6) = L(x_6) = \{3, 4, 7, 8\}$

Observe that each of these triples forms a triangle in $X_n$, and for any $a \in \{1, 2, 3, 4\}$ and $b \in \{5, 6, 7, 8\}$, one of these triangles has colors $\{a, b, x, y\}$ for some colors $x, y$. Assume for contradiction that we have an $L$-coupled-coloring $c$ of $W_n$. Up to symmetry, assume $c(x_0) = 1$ and $c(f_0) = 5$. But then $f_1, x_1, f_2$ have two colors left, and therefore cannot be colored, a contradiction.

![Figure 4: The graphs $X_5$ (left) and $X_6$ (right).](image)

With this, we have a characterization of the coupled choosability of wheel graphs.

**Corollary 4.5.** For a wheel graph $W_n$, we have

$$
\chi_{L \ell f}^c(W_n) = \begin{cases} 
4 & n = 4 \\
5 & n \geq 5 
\end{cases}
$$
5 Future Work

Our investigation of wheel graphs was motivated by wanting to determine the coupled choosability number of planar partial 3-trees. To define these, we first define *Apollonian networks* recursively as follows. A triangle is an Apollonian network. If $G$ is an Apollonian network, and $f$ is a face of $G$ (necessarily a triangle) that is not the outer-face, then the graph obtained by stellating face $f$ is also an Apollonian network. Here *stellating* means the operation of inserting a new vertex $v$ inside face $f$ and making it adjacent to all vertices of $f$. A *planar partial 3-tree* is a graph that is a subgraph of an Apollonian network (see Figure 5). (This definition is different, but equivalent, to the “standard” definition of partial 3-trees via treewidth or via chordal supergraphs with clique-size 4 [1].) We offer the following conjecture:

**Conjecture 5.1.** Every planar partial 3-tree is 6-coupled-choosable.

![Figure 5: A planar partial 3-tree. Dotted edges show the Apollonian network.](image)

Note that the conjecture holds for Apollonian networks, since these are maximal planar graphs and these are known to be 6-coupled-choosable (Wang and Lih [6]). But this does not imply 6-coupled-choosability of subgraphs, and so the conjecture remains open.

Towards the conjecture, we studied several graph classes that are planar partial 3-trees (and generalize wheels). One such class of graphs are the *Halin graphs*, which are defined by starting with a tree $T$ and adding a cycle between the leaves of $T$. See also the solid edges in Figure 6. Wheel graphs are the special case of Halin graphs where $T$ is a star graph. A second class of planar partial 3-trees are the *stellated outer-planar graphs*, obtained by starting with some outerplanar graph $G$, and stellating the outer-face. See also the dashed edges in Figure 6. Wheel graphs are the special case of stellated outerplanar graphs where the outerplanar graph is a cycle. These two classes are closely related.

**Lemma 5.2.** For $n \geq 2$, Halin Graphs are exactly the duals of stellated outerplanar graphs.

We suspect that this result was known before, but have not been able to find a reference and therefore provide a proof here.

**Proof.** Let $G$ be a Halin graph. Every face of $G$ that is not the outer face is adjacent to the outer face. Therefore, the *weak dual* (i.e., the dual graph with the vertex representing the outer face removed) of $G$ is an outerplanar graph. Then adding the outer face and its adjacencies creates a stellated outerplanar graph.

Let $G$ be a stellated outerplanar graph. The faces of the outerplanar graph form a tree $T$ in the dual. The faces incident to the vertex that stellated the outer face form a cycle $L$, and every such face of $L$ shares an edge with some face in $T$. Hence we can view $L$ as a set of leaves attached to $T$ and then further connected with a cycle, so this is a Halin-graph.

Therefore, any list coupled coloring of a stellated outerplanar graph corresponds to a list coupled coloring of a Halin graph. Unfortunately, our upper bound for the coupled choosability of wheel graphs does not in general extend to Halin graphs.

**Theorem 5.3.** There exists a stellated outerplanar graph (equivalently a Halin graph) that is not 5-coupled-choosable.
Proof. The Halin-graph \( G \) is the triangular prism, see Figure 6 where we also show the dual graph \( G^* \) and the 1-planar graph \( X(G) \). We claim that \( X(G) \) is not 5-colorable; therefore \( G \) is not 5-coupled-colorable and in particular not 5-coupled-choosable.

Assume for contradiction that \( X(G) \) had a 5-coloring; up to symmetry we may assume that the triangle formed by the three degree-4-faces of \( G \) is colored 1, 2, 3. Let \((t, t')\) be the edge that crosses the edge colored with 2 and 3. Vertices \( t, t' \) are colored with 1, 4 or 5; up to renaming of colors 4 and 5 hence one of them is colored 4.

Starting with this coloring, propagate restrictions on the possible colors to other vertices of \( X \) along the numerous copies of \( K_4 \) (note that all vertices other than \( t, t' \) are adjacent to the one colored 1). This leads to a triangle that has only two possible colors left, a contradiction.

Figure 6: A Halin-graph \( G \) (black solid; the tree is bold), and the dual graph \( G^* \) (blue dashed) which is a stellated outerplanar graph (the outerplanar graph is bold). Taking both, and adding the face-vertex incidences (red dotted) gives graph \( X(G) \). We also show the only possible 5-coloring (up to symmetry) of \( X(G) \), which leads to a contradiction since a triangle must be colored with 2 colors.

In particular, this shows that we cannot replace ‘6’ by ‘5’ in Conjecture 5.1.

So wheels are strictly better (as far as coupled choosability is concerned) than Halin-graphs. Now we study a second graph class that lies between the wheels and the planar partial 3-trees. These are the IO-graphs, which are the planar graphs that can be obtained by adding an independent set to the interior faces of an outerplanar graph (see Figure 7). Certainly any subgraph of a wheel is an IO graph.

**Conjecture 5.4.** Every IO-graph is 5-coupled choosable.

We studied subgraphs of wheel graphs because they may be an important stepping stone towards Conjecture 5.4. In particular, consider some IO-graph \( G \) obtained from an outerplanar graph \( O \) and independent set \( I \). Let \( G^+ \) be a maximal IO-graph containing \( G \), i.e., add edges to \( G \) for as long as the result is simple and an IO-graph. Then \( G^+ \) is a tree of wheels, where each wheel consists of a vertex \( x \in I \) with its neighbours, and the wheels have been glued together at edges. Correspondingly \( G \) is a tree of subgraphs of wheels. It may be possible to use Theorem 4.3 (enhanced with further restrictions on the coloring of some parts) to prove Conjecture 5.4 by building a coloring of \( G \) incrementally in this tree, but this remains future work.

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Figure 7: An IO graph $G$ consists of an outerplanar graph (circles) and an independent set (squares). Dotted edges are added to obtain $G^+$, and some of the wheels used to build $G^+$ are shaded.

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