Vortex lattice solutions of the ZHK Chern-Simons equations

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March 11, 2019

Supervised by Professor I. M. Sigal. Thanks to Li Chen, Dmitri Chouchkov and Afroditi Talidou for useful discussions and support. Thanks also to Professor Bruchard for help improving this presentation.
Chern-Simons action

The (abelian) Chern-Simons (CS) action on $\mathbb{R}^3$ is [CS71]

$$S_{CS}(a) = -\frac{1}{2} \int_{\mathbb{R}^3} a \wedge da$$

where $a$ is a 1-form.

It is one of the two gauge theories occurring in odd dimensional space-times, the other being the Maxwell action

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The CS action is gauge invariant on $\mathbb{R}^3$, and in general whenever boundary terms could be neglected.
Motivation - Condensed Matter Physics

The CS action is known to be related to topological invariants of three manifolds, however its Euler-Lagrange equations are trivial. We will couple it with matter fields to get non-trivial Euler-Lagrange equations.
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Abrikosov lattice solutions

Stability of Abrikosov lattice solutions

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- The CS term occurs specifically in planar physics, and there are general arguments showing that it can be used to attach non-trivial (fractional) quantum statistics to particles [Wil90].
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- We study a theory involving the Chern-Simons term, a constant external magnetic field and a double well potential, common in physics. This theory was first written down by Zhang, Hanson and Kivelson and is called the ZHK model [ZHK89].
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► We study a theory involving the Chern-Simons term, a constant external magnetic field and a double well potential, common in physics. This theory was first written down by Zhang, Hanson and Kivelson and is called the ZHK model [ZHK89].

► The ZHK model appears in the study of the fractional quantum hall effect in condensed matter physics [ZHK89].
The ZHK Chern-Simons action

The matter action we study, in the variables $(x_0, x_1, x_2) = (t, x_1, x_2)$, is

$$S_{\text{mat}}(\psi, a, A^b) = \int_{\mathbb{R}^3} \left( i \bar{\psi} D_0 \psi - \frac{1}{2} |\nabla_{a + A^b} \psi|^2 - \frac{g}{2} (|\psi|^2 - 1)^2 \right) dt dx$$

where $D_0 \psi = \partial_0 \psi + i(a_0 + A^b_0)\psi$, $\nabla_{a + A^b} \psi = \nabla \psi + i(a + A^b)\psi$ is the covariant derivative, $A^b = \frac{b}{2} (-x_2, x_1)$ satisfies $\text{curl} A^b = b > 0$ and $g > 0$.

We study the Euler-Lagrange equations of the ZHK action, which is

$$S_{\text{matter}}(\psi, a, A^b) + S_{\text{CS}}(a)$$
The Zhang-Hanson-Kivelson equations

We define $A = A^b + a$, then the Euler-Lagrange equations of the above action in terms of $A = (A_0, A) = (A_0, A_1, A_2)$ are [ZHK89]

$$i\partial_t \psi = -\frac{1}{2}\Delta_A \psi + A_0 \psi + g(|\psi|^2 - 1)\psi$$

$$0 = \text{curl} A + |\psi|^2 - b$$

$$\star \partial_t A = -\text{curl}^* A_0 + \text{Im}(\bar{\psi} \nabla A \psi)$$

where $-\Delta_A = \nabla_A^* \nabla_A$, $\text{curl} A = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}$, $\text{curl}^* A_0 = \left(\frac{\partial A_0}{\partial x_2}, -\frac{\partial A_0}{\partial x_1}\right)$ is the adjoint of curl, and $\star$ denotes the Hodge star.
The Ginzburg-Landau equations

$$\partial_t \psi = \Delta_A \psi - A_0 \psi + \kappa^2 (1 - |\psi|^2) \psi$$

$$\partial_t A = - \text{curl}^* \text{curl} A - \nabla A_0 + \text{Im}(\bar{\psi} \nabla A \psi) \quad \text{(GL)}$$
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\[ \partial_t \Psi = \Delta_A \Psi - A_0 \Psi + \kappa^2 (1 - |\Psi|^2) \Psi \]
\[ \partial_t A = - \text{curl}^* \text{curl} A - \nabla A_0 + \text{Im}(\bar{\Psi} \nabla A \Psi) \quad \text{(GL)} \]

These equations describe superconductors near phase transitions.
Gauge equivariance

For any function $\eta : C^\infty(\mathbb{R}^2) \rightarrow \mathbb{R}$, and any solution $(\Psi(x), A(x))$ of the ZHK (GL) equations, the state $T^\text{gauge}_\eta(\Psi(x), A(x))$ defined by

$$T^\text{gauge}_\eta(\Psi(x), A(x)) = (e^{i\eta(x)}\Psi(x), A(x) + \nabla\eta(x))$$

is also a solution of the ZHK (GL) equations.
Energy and ground state

\[ E(\Psi, A) = \int_{\mathbb{R}^2} \frac{1}{2} (|\nabla A\Psi|^2 + g(|\Psi|^4 - 2|\Psi|^2)) \, dx \]

The gauge invariant ground state \((\Psi, A) = (0, (0, A^b))\) where \(\text{curl } A^b = b\), is called the \textit{normal state} \(u_0\).
Brief History

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1989 The Zhang-Hanson-Kivelson equations were first written down to describe the fractional quantum hall effect.

2011 Tzaneteas and Sigal rigorously proved the existence of Abrikosov lattice solutions of the GL equations.
Abrikosov lattice states in Superconductivity

Image of vortex lattice of a superconductor NbSe$_2$
Sketch of a lattice \( \mathcal{L} \).

Lattices and fundamental cells
Abrikosov lattice state

We want time-independent solutions \((\psi, (A_0, A))\) of the ZHK equations such that the quantities

\[
\rho = |\psi|^2 \quad J = \text{Im}(\bar{\psi} \nabla_A \psi) \quad B = \text{curl } A \quad A_0
\]

are periodic with respect to a lattice \(\mathcal{L}\). Such states \(u = (\psi, (A_0, A))\) are called **Abrikosov lattice states**.
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are periodic with respect to a lattice \(\mathcal{L}\). Such states \(u = (\psi, (A_0, A))\) are called Abrikosov lattice states.

These states are more general than requiring \(\psi\) and \(A\) to be \(\mathcal{L}\)-periodic.
Abrikosov lattice solutions

At $b = b_0 := 2g$ an Abrikosov lattice state bifurcates from the normal state, as the following theorem states.

**Theorem (Existence of a Bifurcation [RS18])**

For any $g > 0$ and some $b$ satisfying $0 < |2g - b| \ll 1$

1. There exists an Abrikosov lattice state $u_b$, in a neighbourhood of the normal branch $u_0$, which solves the ZHK equations.
2. If $g < \frac{1}{2}$, the hexagonal lattice minimizes the average energy per lattice cell.
Abrikosov lattice states in Superconductivity

Image of vortex lattice of a superconductor NbSe$^2$
Abrikosov lattice solutions on Riemann surfaces

It turns out that $(\Psi, (A_0, A))$ is an Abrikosov lattice state iff $\Psi$ lives on a line bundle over the $\mathbb{T}^2 = \mathbb{R}^2 / \mathcal{L}$, and $A$ is a connection on it. This latter viewpoint generalizes to arbitrary Riemann surfaces, and so does the bifurcation result.

**Theorem (Existence of a Bifurcation [RS18])**

Let $g > 0$ and suppose $b$ satisfies $0 < |2g - b| \ll 1$. Then on a Riemann surface of genus $h$, as long as the first Chern number $n$ of the line bundle satisfies $1 \leq n \leq h$, there exists an Abrikosov lattice state $u_b$, in a neighbourhood of the normal branch $u_0$. 
Orbital stability of Abrikosov lattice solutions

Theorem ([Raj18])

For any solution $\Psi(t) \in C^1(\mathbb{R}^+, H^1(\mathbb{T}^2))$ of the ZHK equations, we can show that

$$\|\Psi_b - \Psi(0)\| < \delta \Rightarrow \|e^{-i\gamma(t)}\Psi_b - \Psi(t)\| < \epsilon \text{ for all } t$$

and some function $\gamma(t)$. 
Key ideas behind Bifurcation theorem

- Write the time-independent ZHK equations as

\[ F(b, u) = 0 \]  \hspace{1cm} (3)
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- Write the time-independent ZHK equations as

\[ F(b, u) = 0 \]  \hspace{1cm} (3)

- A bifurcation point \( b_0 \) occurs when \( \frac{dF_u(b_0, u)}{db} \) is not invertible.

- The map \( \frac{dF_u(b_0, u)}{db} \) is always non-invertible. However, this is fixed by working in the Coloumb gauge.
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- A bifurcation point \( b_0 \) occurs when \( dF_u(b_0, u_0) \) is not invertible.

- The map \( dF_u(b_0, u_0) \) is always non-invertible. However, this is fixed by working in the Coloumb gauge.

- The change in invertibility of \( d_u F(b, u_0) \) is controlled by the following operator

\[-\frac{1}{2} \Delta A^b - g\]

Its spectrum shall be studied using a Weitzenböck-type identity.
Weitzenböck-type identity

First we define

\[
\begin{align*}
\partial_A^b &= \partial - iA_c^b \\
A_c^b &= \frac{1}{2}(A_1^b - iA_2^b)
\end{align*}
\]

\[
\begin{align*}
\partial_A^* &= \bar{\partial} - i\bar{A}_c^b \\
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\end{align*}
\]
Weitzenböck-type identity

First we define

\[
\partial_{A^b} = \partial - iA^b_c \\
A^b_c = \frac{1}{2}(A^b_1 - iA^b_2)
\]

Then the Weitzenböck identity states that

\[
-\frac{1}{2}\Delta_{A^b} = 2\partial_{A^b} \partial^*_{A^b} + \frac{\text{curl } A^b}{2}
\]
Weitzenböck-type identity

First we define

\[ \partial_{A_b} = \partial - iA^b_c \]
\[ A^b_c = \frac{1}{2}(A^b_1 - iA^b_2) \]

Then the Weitzenböck identity states that

\[ -\frac{1}{2}\Delta_{A^b} = 2\partial_{A^b}\partial^*_{A^b} + \frac{\text{curl } A^b}{2} \]

Since \( \partial_{A^b}\partial^*_{A^b} \geq 0 \) and \( \text{curl } A^b = \frac{\partial A^b_1}{\partial x^2} - \frac{\partial A^b_2}{\partial x^1} = b \), we have

\[ -\frac{1}{2}\Delta_{A^b} - g \geq b - \frac{b}{2} - g \]
Hamiltonian form of ZHK Equations

The ZHK equations can be written in Hamiltonian form using the energy functional

\[ E(\Psi, A) = \int_{\mathbb{T}^2} \frac{1}{2}(|\nabla_A \Psi|^2 + g(|\Psi|^2 - 1)^2) + A_0(-\text{curl } A + |\Psi|^2 + b) \, dx \]

Then letting \( J(\Psi, A) = (i\Psi, \star A) \), the ZHK equations become

\[ J \partial_t \begin{pmatrix} \Psi \\ A \end{pmatrix} = \nabla_{\Psi, A} E(\Psi, A_0, A) \]

\[ 0 = \nabla_{A_0} E(\Psi, A_0, A) \]
Stability analysis

If we let $H = \text{Hess} \ E(u_b)$, then to leading order in $b$, we have

$$H = \begin{pmatrix} -\Delta_{A^b_0} - b_0 & 0 & 0 \\ 0 & 0 & -\text{curl} \\ 0 & -\text{curl}^* & 0 \end{pmatrix}$$

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$$H = \begin{pmatrix} -\Delta A^b - b_0 & 0 & 0 \\ 0 & 0 & -\text{curl} \\ 0 & -\text{curl}^* & 0 \end{pmatrix}$$  \hspace{1cm} (4)$$

The operator

$$M = \begin{pmatrix} 0 & -\text{curl} \\ -\text{curl}^* & 0 \end{pmatrix}$$  \hspace{1cm} (5)$$

has an infinite number of negative eigenvalues, which makes stability impossible.
Modified equations

However we can use the second and third ZHK equations to solve for $A$ and $A_0$ as a function of $\Psi$. Substituting these into the energy functional, we obtain

$$E(\Psi) = \int_{\mathbb{T}^2} \frac{1}{2} (|\nabla_A(\Psi)|^2 + g(|\Psi|^2 - 1)^2) dx$$

whose Euler-Lagrange equation is

$$i \partial_t \Psi = -\frac{1}{2} \Delta_{A^b + A(|\Psi|)} \Psi + A_0(|\Psi|) \Psi + g(|\Psi|^2 - 1) \Psi$$

hereafter called the non-local ZHK equations.
The Hessian of the non-local energy to leading order in the bifurcation parameter is

\[-\Delta_{A_{b_0}} - b_0\]

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The Hessian of the non-local energy to leading order in the bifurcation parameter is

$$-\Delta_{A^{b_0}} - b_0$$

which is non-negative.

For the new equation, we can prove orbital stability.

**Theorem ([Raj18])**

*For any solution $\Psi(t) \in C^1(\mathbb{R}^+, H^1(\mathbb{T}^2))$ of the non-local ZHK equations, we can show that*

$$\|\Psi_{b} - \Psi(0)\| < \delta \Rightarrow \|e^{-i\gamma(t)}\Psi_{b} - \Psi(t)\| < \epsilon \text{ for all } t$$

*and some function $\gamma(t)$.*
Thank you for listening!
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