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Intensity–intensity correlations determined by dimension of quantum state in phase space: \( P \)-distribution

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Abstract

We use the \( P \)-distribution to show that the familiar values 1, 2 and 3 of the normalized second order correlation function at equal times \( g_{ij}(0) \) corresponding to a coherent state, a thermal state and a highly squeezed vacuum are a consequence of the number of dimensions these states take up in quantum phase space. Whereas the thermal state exhibits rotational symmetry and thus extends over two dimensions, the squeezed vacuum factorizes into two independent one-dimensional phase space variables, and in the limit of large squeezing is therefore a one-dimensional object. The coherent state is a point in the phase space of the \( P \)-distribution and thus has zero dimensions. The fact that for photon number states the \( P \)-distribution is even narrower than that of the zero-dimensional coherent state suggests the notion of ‘negative’ dimensions.

Keywords: correlation function, quantum phase space, \( P \)-distribution, dimension of quantum state

1. Introduction

A virtual photon borrowed from the vacuum cannot create a photoelectron. For this reason a photodetector measures \(^1\)normally ordered products of annihilation and creation operators of the electromagnetic field \(^2\). In this choice of operator ordering annihilation operators always stand to the right of creation operators. Since coherent states are eigenstates of the annihilation operator it is convenient to evaluate expectation values of such products with the help of a diagonal representation \(^3, 4\) of the density operator in terms of coherent states. The \( P \)-distribution is such a representation.

Unfortunately, this simplification in the evaluation comes at the price of a more sophisticated distribution function. Indeed, even for an elementary state, such as a squeezed state the corresponding \( P \)-distribution involves \(^5\) an infinite number of derivatives of delta functions. Despite this complication a careful analysis of the emerging integrals...
associated with the expectation values of normally ordered operators yields [5] meaningful results.

An important example for such an expectation value is the normalized correlation function $g^{(2)}(0)$ of second order at equal times. This quantity has become an important tool to characterize single photon light sources [6].

In the present article we show using the familiar expressions [5] for $g^{(2)}(0)$ corresponding to a thermal state, a coherent state, a squeezed vacuum and photon number states that the $P$-distribution reflects the number of dimensions the underlying quantum state takes up in phase space. Although the calculations are well-known [5] we have not been able to find this interpretation of $g^{(2)}(0)$ in the literature.

In a recent article [7] we have derived a rather complicated expression for $g^{(2)}(0)$ in terms of the Wigner function [2] and have argued that $g^{(2)}(0)$ is a measure of the dimensionality of the quantum state. We now show that this interpretation also holds true for the $P$-distribution.

Our article is organized as follows: in section 2 we first represent $g^{(2)}(0)$ as the ratio of two phase space integrals and then evaluate them for several examples of states. Whereas for a thermal state and a coherent state the corresponding $P$-distributions are well-behaved and given by a Gaussian and a Dirac delta function, respectively, the $P$-distributions of the squeezed vacuum and a photon number state involve [2] derivatives of delta functions. Nevertheless, the corresponding integrals yield finite results when evaluated with care.

The purpose of our exercise in rederiving these well-known expressions [5] is not only to emphasize the usefulness and the power of the $P$-distribution but to bring out most clearly the role of the dimensionality of the quantum state in determining the value of $g^{(2)}(0)$. In this way we demonstrate that the result $g^{(2)}(0) = 1$ originates from the fact that the $P$-distribution of a coherent state is a zero-dimensional object. Likewise, the expression $g^{(2)}(0) = 2$ for a thermal state reflects the two-dimensional nature of the corresponding $P$-distribution. A highly squeezed vacuum is one-dimensional and thus leads to $g^{(2)}(0) = 3$. For the corresponding analysis of the general case of arbitrary squeezing we refer to the appendix.

The most intriguing case with respect to $g^{(2)}(0)$ as a measure of dimensionality is the class of photon number states. They have the same rotational symmetry as a thermal state but involve a finite number of derivatives of delta functions. In this sense the corresponding $P$-distributions are narrower than coherent states. This feature suggests even ‘negative dimensions’. We emphasize that these results are in complete accordance with our analysis [7] using the Wigner function.

Section 3 represents a brief summary of our results and provides an outlook.

2. Phase space of $P$-distribution

In this section we first recall [5] the expression for $g^{(2)}(0)$ in terms of the $P$-distribution. We then evaluate the corresponding phase space integrals for a selection of quantum states that highlights in a striking way the crucial influence of the dimension of the $P$-distribution on $g^{(2)}(0)$. In order to focus on the key ideas we restrict ourselves to elementary examples but conclude by briefly discussing number states suggesting the notion of ‘negative’ dimensions.

2.1. General expression for $g^{(2)}(0)$

It is convenient to evaluate the normalized correlation function $g^{(2)}(0)$ of second order at equal times defined [1] by the ratio

$$g^{(2)}(0) \equiv \frac{\langle \hat{a}^{\dagger} \hat{a} \rangle^2}{\langle \hat{a}^{\dagger} \hat{a} \rangle}$$

of expectation values of powers of normally ordered creation and annihilation operators $\hat{a}^{\dagger}$ and $\hat{a}$ of a single mode of the radiation field with the help of the $P$-distribution [3, 4], which allows for a diagonal representation of the density operator

$$\hat{\rho} \equiv \int d^2 \alpha \; P(\alpha) |\alpha\rangle \langle \alpha|$$

in terms of coherent states $|\alpha\rangle \equiv |\alpha \; e^{i\phi}\rangle$. Here $P(\alpha)$ depends on the two real phase space variables $\alpha_\theta \equiv \text{Re}(\alpha)$ and $\alpha_\phi \equiv \text{Im}(\alpha)$ with the two-dimensional area element $d^2 \alpha \equiv d\alpha_\theta d\alpha_\phi \equiv d|\alpha| d\varphi$ and $|\alpha|^2 \equiv \alpha_\theta^2 + \alpha_\phi^2$.

Indeed, with equation (2) we can immediately express the expectation value

$$\langle (\hat{a}^{\dagger})^j (\hat{a})^k \rangle = \int d^2 \alpha (\alpha^\dagger)^j \alpha^k P(\alpha)$$

as a two-dimensional integral of $(\alpha^\dagger)^j \alpha^k$ weighted by the $P$-distribution of the quantum state and equation (1) takes the form

$$g^{(2)}(0) = \frac{\int d^2 \alpha |\alpha|^4 P(\alpha)}{\int d^2 \alpha |\alpha|^2 P(\alpha)} = \frac{\langle |\alpha|^4 \rangle}{\langle |\alpha|^2 \rangle^2} = \frac{1}{\langle \hat{n} \rangle} \langle |\alpha|^4 \rangle.$$ (4)

In the last step we have introduced the average number $\langle \hat{n} \rangle = \langle \hat{a}^{\dagger} \hat{a} \rangle$ of photons.

This formulation brings out most clearly that $g^{(2)}(0)$ probes the rotational symmetry of the quantum state represented by the $P$-distribution. Indeed, the quantities to be averaged in equation (4), that is the moments $\langle |\alpha|^2 \rangle$ and $\langle |\alpha|^4 \rangle$ display rotational symmetry. Hence, the value of $g^{(2)}(0)$ depends crucially on the symmetry of the $P$-distribution.

However, the symmetry of the state is only one factor governing $g^{(2)}(0)$, the functional form of $P$ is another. This fact stands out most clearly in the comparison of a thermal state and a photon number state whose $P$-distributions $P_\text{th}$ and $P_\text{th}$ both display rotational symmetry. Nevertheless, the values of $g^{(2)}(0)$ are utterly different due to the radial functional dependence of the corresponding $P$-distributions. Whereas $P_\text{th}$ is a Gaussian, $P_\text{th}$ involves [5] a finite number of derivatives of a delta function giving rise to $g^{(2)}(0) = 2$ and $0 \leq g^{(2)}(0) < 1$, respectively.
2.2. Examples

We now illustrate this interplay between symmetry and form of $P$ determining the value of $g^{(2)}(0)$ using various states. In particular, we address the individual examples in the order of increasing complexity in their respective $P$-distribution. We start with the Gaussian of a thermal state and then turn to the Dirac delta function of a coherent state. This class [1] marks the border between the classical and the quantum domain. Indeed, the $P$-distributions of the next two cases, that is a strongly squeezed vacuum and the photon number states contain derivatives of delta functions. This feature signifies the non-classicality [8] of these states somewhat analogous to the negativity of the Wigner function [2].

Although the number of derivatives in a photon number state is finite, in contrast to the squeezed vacuum where it is infinite, we analyze this case last. The reason for this violation of the order of complexity is the fact that this example points to the intriguing concept of ‘negative’ dimensions. For the general case of a squeezed vacuum with arbitrary squeezing we refer to the appendix.

2.2.1. Thermal state. Since for a thermal state with an average number of photons $\langle \hat{n} \rangle$ the $P$-distribution is a rotationally symmetric two-dimensional Gaussian [2]

$$P_{th}(\alpha) \equiv \frac{1}{\pi \langle \hat{n} \rangle} e^{-|\alpha|^2 / \langle \hat{n} \rangle}$$  \hspace{1cm} (5)

equation (4) reduces to

$$g^{(2)}_{th}(0) = \int_0^\infty d\xi \xi^2 e^{-\xi}.$$  \hspace{1cm} (6)

Here we have introduced the new integration variable $\xi \equiv |\alpha|^2 / \langle \hat{n} \rangle$ which transforms $P_{th}$ into a one-dimensional exponential. Moreover, due to the scaling of $\xi$ with $\langle \hat{n} \rangle$ and the prefactor $1/\langle \hat{n} \rangle^2$ of $\langle |\alpha|^4 \rangle$ in equation (4) the resulting expression for $g^{(2)}_{th}(0)$ is independent of $\langle \hat{n} \rangle$.

When we evaluate the remaining integral we arrive at the familiar expression

$$g^{(2)}_{th}(0) = 2.$$ \hspace{1cm} (7)

This calculation shows that equation (7) is the consequence of three facts: (i) the quantity $(\alpha^2)^2 = |\alpha|^4$ to be averaged is rotationally symmetric, (ii) the $P$-distribution of the thermal state given by equation (5) is rotationally symmetric as well, and (iii) the two-dimensional phase space integration in equation (4) reduces to a one-dimensional only.

2.2.2. Coherent state. For a coherent state $|\alpha_0\rangle$ the $P$-distribution is a two-dimensional Dirac delta function located at the point $\alpha_0$ of phase space, that is

$$P_{coh}(\alpha) = \delta^{(2)}(\alpha - \alpha_0)$$  \hspace{1cm} (8)

and equation (4) reduces to

$$g^{(2)}_{coh}(0) = 1.$$ \hspace{1cm} (9)

Hence, the origin of this value is the delta function, that is the fact that in the quantum phase space corresponding to the $P$-distribution a coherent state is a zero-dimensional object.

2.2.3. Strongly squeezed vacuum. Next we turn to the case of a squeezed vacuum where the $P$-distribution is a highly singular function given by infinitely many derivatives acting on a delta function located at the origin of phase space.

Indeed, we recall [2] the expression

$$P_{sq}(\alpha) \equiv \left[ e^{A_r} \delta(\alpha_r) \right] \left[ e^{A_i} \delta(\alpha_i) \right]$$  \hspace{1cm} (10)

where the operators

$$A_r \equiv \frac{1 - s}{8s} \frac{\partial^2}{\partial \alpha_r^2}$$  \hspace{1cm} (11)

and

$$A_i \equiv \frac{1 - s}{8s} \frac{\partial^2}{\partial \alpha_i^2}$$  \hspace{1cm} (12)

contain second derivatives with respect to the two cartesian phase space coordinates $\alpha_r$ and $\alpha_i$ and $0 < s$ denotes the squeezing parameter.

Hence, a quasi-Laplacian consisting of second derivatives with respect to $\alpha_r$ and $\alpha_i$ given by equations (11) and (12) in the argument of an exponential function acting on a delta function provides us according to equation (10) with $P_{sq}$.

In contrast, the corresponding Wigner function [2]

$$W_{sq}(\alpha_r, \alpha_i) \equiv \frac{2}{\pi} e^{-2|\alpha|^2} e^{-2\alpha_i^2/s}$$  \hspace{1cm} (13)

is just the product of two Gaussians of different widths.

However, common to both distributions is the property that they separate into products of two functions each with only one coordinate. This feature is crucial for our discussion of the connection between $g^{(2)}(0)$ and the dimensionality of the quantum state which stands out most clearly when we now consider the case of a strongly squeezed vacuum which is clearly a one-dimensional object. For the evaluation of $g^{(2)}_{sq}(0)$ in the presence of arbitrary squeezing we refer to the appendix.

The limit of strong squeezing corresponds to $s \to \infty$ or $s \to 0$, where either the fluctuations in the phase space coordinates $\alpha_r$ or $\alpha_i$ are suppressed. From the form of the operators $A_r$ and $A_i$ together with the definition, equation (10), of $P_{sq}$ we note that indeed the two processes only interchange the roles of $\alpha_r$ and $\alpha_i$.

For the sake of simplicity we now confine ourselves to $s \to \infty$, that is squeezing in $\alpha_r$. In this case the main contribution to the variables

$$|\alpha|^2 = \alpha_r^2 + \alpha_i^2$$  \hspace{1cm} (14)

and

$$|\alpha|^4 = \alpha_r^4 + 2\alpha_r^2 \alpha_i^2 + \alpha_i^4$$  \hspace{1cm} (15)

originating from the integration over phase space with the distribution $P_{sq}$ given by equation (10) arises from $\alpha_r^2$ and $\alpha_i^2$.
which yields the moments
\[
\langle |\alpha|^2 \rangle \approx \int_{-\infty}^{\infty} d\alpha_i \, e^{i\alpha} \delta(\alpha_i) \int_{-\infty}^{\infty} d\alpha_i \, \alpha_i^2 e^{-i\alpha} \delta(\alpha_i),
\]
(16)

or
\[
\langle |\alpha|^2 \rangle \approx \int_{-\infty}^{\infty} d\alpha_i \, \delta(\alpha_i) \exp\left(\frac{s}{8} \frac{\partial^2}{\partial \alpha_i^2}\right) \alpha_i^2.
\]
(17)

In the last step we have used partial integration to transfer the action of the exponential operators from the delta functions to the variables \(\alpha_i^2\). This procedure has also allowed us to evaluate the integral over \(\alpha_i\) which due to the normalization of \(P_{sq}\) is unity.

At this point we realize that it is the partial integration which transforms the highly singular \(P\)-distribution into an ordinary Dirac delta function. We emphasize that this reduction of complexity is only possible since we deal with expectation values, that is with integrals. Hence, the \(P\)-distribution, and for that matter any quantum phase space function, is only meaningful in connection with a phase space integration. In this sense we have only applied the familiar concept of a distribution, that is a generalized function, which is defined by an integral.

We now continue the evaluation of the expectation value \(\langle |\alpha|^2 \rangle\) given by equation (17). For this purpose we expand the remaining exponential operator and arrive at the expression
\[
\langle |\alpha|^2 \rangle \approx \int_{-\infty}^{\infty} d\alpha_i \, \delta(\alpha_i) \left[1 + \frac{s}{8} \frac{\partial^2}{\partial \alpha_i^2} + \frac{s^2}{128} \frac{\partial^4}{\partial \alpha_i^4} + \ldots \right] \alpha_i^2.
\]
(18)

which leads us to
\[
\langle \hat{\theta} \rangle \equiv \langle |\alpha|^2 \rangle \approx \frac{s}{4}
\]
(19)

and
\[
\langle |\alpha|^4 \rangle \approx \frac{3}{16} s^2.
\]
(20)

With the help of the representation equation (4) of \(g^{(2)}(0)\) we find the value
\[
g^{(2)}_{sq}(0) \approx 3,
\]
(21)

which is in complete accordance with the exact result derived in the appendix.

This calculation is different from the corresponding one of a thermal state in three respects: (i) the \(P\)-distribution given by equation (10) does not display rotational symmetry but factorizes into a product of two distributions along the two orthogonal axis of quantum phase space, (ii) the resulting moments \(\langle |\alpha|^2 \rangle\) and \(\langle |\alpha|^4 \rangle\) determine \(g^{(2)}_{sq}(0)\) result from one-dimensional integrations as shown by equation (16), and (iii) the value 3 of \(g^{(2)}_{sq}(0)\), equation (21), can be traced back to the moment \(\langle |\alpha|^4 \rangle\), equation (20), and is therefore a consequence of the effective one-dimensional nature of the strongly squeezed vacuum.

2.2.4. Photon number states. We conclude by addressing the case of a number state with \(1 \leq n\) photons where the \(P\)-distribution
\[
P_n(\alpha) \equiv L_n\left[-\frac{1}{4} \left(\frac{\partial^2}{\partial \alpha_i^2} + \frac{\partial^2}{\partial \alpha_j^2}\right)\right] \delta(\alpha)
\]
(22)

involves all powers of the two-dimensional Laplacian up to the \(n\)th order as expressed by \(n\)th Laguerre polynomial [9]
\[
L_n(\alpha) \equiv \sum_{m=0}^{n} (-1)^m \binom{n}{m} \frac{x^m}{m!}.
\]
(23)

When we substitute this expression together with the formula equation (22) for \(P_n\) into the representation equation (4) of \(g^{(2)}(0)\), make the identification \(\langle \hat{\theta} \rangle = \langle |\alpha|^2 \rangle = n\), and integrate by parts we arrive at
\[
g^{(2)}_{n}(0) = \frac{1}{n^2} \sum_{m=0}^{n} \frac{n!}{(n-m)!m!} \left(\frac{1}{4}\right)^m \int_{-\infty}^{\infty} d\alpha_i \int_{-\infty}^{\infty} d\alpha_j
\times \left(\frac{\partial^2}{\partial \alpha_i^2} + \frac{\partial^2}{\partial \alpha_j^2}\right) \left(\alpha_i^2 + \alpha_j^2\right) \delta(\alpha_i) \delta(\alpha_j).
\]
(24)

Here we have also recalled the definition
\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}
\]
(25)
of the binomial coefficient.

Due to the differentiation and the presence of the two delta functions only the term \(m = 2\) creates a non-vanishing contribution. As a result, we obtain immediately the familiar expression
\[
g^{(2)}_{1}(0) = 0
\]
(26)
corresponding to the phenomenon of anti-bunching [10–13] which is associated with a single photon.

For \(1 < n\) the integral in equation (24) reduces to
\[
g^{(2)}_{n}(0) = \frac{1}{n^2} \frac{n!}{(n-2)!} \left(\frac{1}{2}\right)^2 \cdot 4 \cdot 4 \cdot \left(4 + 2^4 + 4!\right),
\]
(27)

and takes with the help of the identity \(n! = (n-2)! (n-1) \cdot n\) the form
\[
g^{(2)}_{n}(0) = 1 - \frac{1}{n} < 1 = g^{(2)}_{coh}(0).
\]
(28)

We note that this result also contains the case \(n = 1\) given by equation (26).

The phase space representation, equation (24), of \(g^{(2)}_{n}(0)\) in terms of the \(P\)-distribution brings out three characteristic features: (i) for \(n = 1\) the integral vanishes because the number of derivatives is smaller than the powers of the phase space variables, (ii) only for \(2 \leq n\) do we find a non-zero result which according to equation (28) approaches the value of a coherent state given by unity from below, and (iii) the power \(n\) at which the action of the Laplacian produces a non-
trivial expression is dictated by the power of the quantity to be averaged, that is, by $|\alpha|^2$.

2.3. Negative dimension?

The observations of the preceding section suggest that the $P$-distribution $P_n$ of a photon number state is always narrower than that of a coherent state. Moreover, $P_1$ is narrower than any other $P_n$ and the width increases monotonously with $n$. We emphasize that these properties depend crucially on our specific choice of the ‘width’ of the distribution which in the present discussion is given by the moment $\langle |\alpha|^n \rangle$.

Next we recall that a coherent state is a zero-dimensional object in the phase space defined by the $P$-distribution. Combined with the fact that $P_n$ is always narrower than $P_{coh}$ this feature suggests that $P_n$ might have a dimension smaller than zero, that is a ‘negative’ dimension.

It is at this point that we again [7] recognize the need for a rigorous definition of the dimension of a phase space distribution. Although, the moments $\langle |\alpha|^2 \rangle$ and $\langle |\alpha|^4 \rangle$ probe aspects of dimensionality by their rotational symmetry they are no substitute for a proper definition. Unfortunately, this question goes beyond the scope of the present article and will be addressed in a future publication.

However, it is worth mentioning that we have already encountered the concept of negative dimensions in the context of photon number states. Indeed, based on the fact that the corresponding Wigner functions assume negative values in annular domains of phase space combined with an expression for $g^{(2)}(0)$ dictated by symmetric ordering we have argued [7] that we can associate a negative dimension with these states. Hence, in this approach negative dimensions arise from ‘negative’ quasi probabilities.

In contrast, the normally ordered expression, equation (1), for $g^{(2)}(0)$ central to the present article, enforces the action of a finite number of differentiations on a delta function to achieve values of $g^{(2)}(0)$ smaller than unity. Thus, in the quantum phase space of the $P$-distribution the ‘sharpening’ of a distribution, and the resulting negative dimensions are a consequence of the differentiations rather than negative values of the distribution.

3. Conclusions and outlook

In conclusion, the normalized second-order correlation function $g^{(2)}(0)$ at equal times is a measure of the dimensionality of the quantum state in phase space. Indeed, the $P$-distributions of a coherent state, a highly squeezed vacuum and a thermal state are zero-, one- and two-dimensional objects, respectively, and the corresponding values of $g^{(2)}(0)$ are an immediate consequence of this dimensionality.

The situation is more complicated for photon number states where the corresponding $P$-distribution is narrower than that of a coherent state suggesting a ‘negative’ dimension of $P_n$. However, we suspect that this counter-intuitive notion depends crucially on our choice of measure of dimensionality. Indeed, for the case of $g^{(2)}(0)$ it is based on the moment $\langle |\alpha|^4 \rangle$ which is sensitive to the dimension $D$ of the underlying quantum state represented by its $P$-distribution. Nevertheless, it is not a direct measure of $D$. Only when we have found an appropriate definition for $D$ will we be able to establish a formula expressing $g^{(2)}(0)$ in terms of $D$. The examples presented in this article will serve as guide on our path towards this goal.

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Appendix. Squeezed vacuum

In this appendix we evaluate for the sake of completeness the integrals determining $g^{(2)}_{sq}(0)$ for an arbitrary value of the squeezing parameter. This approach allows us to make contact with the approximate calculation in the main body of this article and to trace again the value of $g^{(2)}(0)$ back to the dimension of the underlying quantum state expressed by the $P$-distribution.

Due to the separation of the phase space distribution $P_{sq}$ given by equation (10) into the phase space variables $\alpha_i$ and $\alpha_t$ the moment $\langle |\alpha|^s \rangle$ following from equation (15) factors into the sum

$$\langle |\alpha|^s \rangle = \langle \alpha_i^s \rangle \langle \alpha_t^0 \rangle + 2 \langle \alpha_i^2 \rangle \langle \alpha_t^0 \rangle + \langle \alpha_i^0 \rangle \langle \alpha_t^2 \rangle \tag{A1}$$

of products of averages

$$\langle \alpha_i^s \rangle = \int_{-\infty}^{\infty} \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \\
\therefore \text{with respect to one-dimensional distributions. In the last step we have used integration by parts to let the operators act on the powers } \alpha_i^2 \text{ and } \alpha_t^{2k} \text{ instead of the delta functions.}

When we expand the exponential operators, perform the differentiations and recall the definitions, equations (11) and (12) of $A_s$ and $A_t$, we find

$$\langle \alpha_i^{2j} \rangle = \left( \frac{1 - s}{8} \right)^j \frac{(2j)!}{j!} \tag{A4}$$

and

$$\langle \alpha_t^{2k} \rangle = \left( \frac{1 - s}{8} \right)^k \frac{(2k)!}{k!} \tag{A5}$$
leading us to

\[
\langle |\alpha|^4 \rangle = 3 \left[ \frac{1}{4} \left( \frac{1}{s} + s \right) - \frac{1}{2} \right]^2 + \frac{1}{4} \left( \frac{1}{s} + s \right) - \frac{1}{2} .
\]  

(A6)

Similarly, we derive from equation (3) together with equation (14) for the average number of photons in the squeezed vacuum the formula

\[
\langle |\alpha|^2 \rangle = \frac{1}{4} \left( s + \frac{1}{s} \right) - \frac{1}{2} = \langle \hat{n} \rangle ,
\]  

(A7)

which allows us to express the moment \( \langle |\alpha|^4 \rangle \) given by equation (A6) in terms of \( \langle \hat{n} \rangle \), that is

\[
\langle |\alpha|^4 \rangle = 3 \langle \hat{n} \rangle^2 + \langle \hat{n} \rangle .
\]  

(A8)

Together with the representation, equation (4) of \( g^{(2)}(0) \), we arrive at the well-known result

\[
g_{sq}^{(2)}(0) = 3 + \frac{1}{\langle \hat{n} \rangle} ,
\]  

(A9)

which in the limit of \( 1 \ll \langle \hat{n} \rangle \) reduces to equation (21). 

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