ON MAXIMAL TORI IN THE CONTACTOMORPHISM GROUPS OF REGULAR CONTACT MANIFOLDS

EUGENE LERMAN

Abstract. By a theorem of Banyaga the group of diffeomorphisms of a manifold $P$ preserving a regular contact form $\alpha$ is a central $S^1$ extension of the commutator of the group of symplectomorphisms of the base $B = P/S^1$. We show that if $T$ is a Hamiltonian maximal torus in the group of symplectomorphism of $B$, then its preimage under the extension map is a maximal torus not only in the group $Diff(P, \alpha)$ of diffeomorphisms of $P$ preserving $\alpha$ but also in the much bigger group of contactomorphisms $Diff(P, \xi)$, the group of diffeomorphism of $P$ preserving the contact distribution $\xi = \ker \alpha$. We use this (and the work of Hausmann, and Tolman on polygon spaces) to give examples of contact manifolds $(P, \xi = \ker \alpha)$ with maximal tori of different dimensions in their group of contactomorphisms.

1. INTRODUCTION AND THE MAIN RESULT

Let $(B, \omega)$ be an integral symplectic manifold. Assume also that $B$ is compact and simply connected. Since the class $[\omega] \in H^2(B, \mathbb{R})$ is integral, there exists a principal circle bundle $S^1 \to P \xrightarrow{\pi} B$ with Euler class $[\omega]$. Moreover there exists a connection 1-form $\alpha$ on $P$ with $d\alpha = \pi^* \omega \ [BW]$. Consequently $\alpha$ is a contact form and $\xi := \ker \alpha$ is a contact distribution. In this note we exploit a relationship between the group $Diff(P, \xi)$ of diffeomorphisms of $P$ preserving the contact distribution and the group $Diff(B, \omega)$ in order to translate statements about maximal tori in $Diff(B, \omega)$ into statements about maximal tori in $(Diff(P, \xi)$. By a theorem of Banyaga the group $Diff(P, \alpha)$ of strict contactomorphism is a central extension (possibly nontrivial) of $Diff(B, \omega)$ by $S^1$. However the group $Diff(P, \xi)$ is much bigger than $Diff(P, \alpha)$.

Remark 1. When one talks about “maximal tori” in $Diff(P, \xi)$ or in $Diff(B, \omega)$, one can mean two different things:

1. A torus $T$ in a group $G$ is maximal if for any torus $T' \subset G$ with $T \subset T'$ we have $T = T'$.
2. A torus $T \subset Diff(B, \omega)$ satisfies $\dim T \leq \frac{1}{2} \dim B$ so a torus $T \subset Diff(B, \omega)$ of dimension $\frac{1}{2} \dim B$ is maximal.
3. Similarly, a torus $T \subset Diff(P, \xi)$ satisfies $\dim T \leq \frac{1}{2}(\dim P + 1)$ so a torus $T \subset Diff(P, \xi)$ of dimension $\frac{1}{2}(\dim P + 1)$ is maximal.

Since there exist compact simply connected symplectic four-manifolds $(B, \omega)$ that admit no symplectic circle actions $[HK]$ (1) and (2a) are quite different. We will see that (1) and (2b) are different as well, so we will use (1) as our definition of a maximal torus.

The main result of the note is the following observation:

Theorem 1. Let $(B, \omega)$ be a compact simply connected integral symplectic manifold, $S^1 \to P \xrightarrow{\pi} B$ the principal circle bundle with Euler class $[\omega]$, $\alpha \in \Omega^1(P)$ a connection 1-form with $d\alpha = \pi^* \omega$ and $\xi = \ker \alpha$ the corresponding contact structure on $P$. If $T \subset Diff(B, \omega)$ is a maximal torus then its preimage $T'$ under the surjection $\tau: Diff(P, \alpha) \to Diff(B, \omega)$ is a maximal torus in $Diff(P, \xi)$.

Remark 2. The fact that the map $\tau: Diff(P, \alpha) \to Diff(B, \omega)$ in Theorem 1 above exists and is a surjective homomorphism is a theorem of Banyaga $[B$, Theorem 1$]$. Our proof of Theorem 1 is a combination of two Lemmas below. Note that there is a distinct $S^1$ in $Diff(P, \alpha) \subset Diff(P, \xi) \subset Diff(P)$. It is the circle action that makes $P$ a principal $S^1$ bundle over $B$. We will refer to this subgroup as the $S^1$ in $Diff(P)$.

Lemma 2. The preimage of a torus $T \subset Diff(B, \omega)$ in $Diff(P, \alpha)$ under $\tau$ is indeed a torus. In fact the action of $T$ on $B$ lifts to an action of $T$ on $P$ preserving $\alpha$ and commuting with the action of the $S^1$. 

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Lemma 3. If \( H \subset \text{Diff}(P, \xi) \) is a torus containing the \( S^1 \) then \( H \subset \text{Diff}(P, \alpha) \).

Lemma 3 is almost certainly not new and must exist somewhere in the literature on pre-quantization of group actions. Unfortunately I have been unable to find a good reference for the result. It is also possible that one can deduce it from [Banyaga, Theorem 1], but I am not whether the group \( \tau^{-1}(T) \) inherits from \( \text{Diff}(P, \alpha) \) the structure of a Lie group making \( \tau : \tau^{-1}(T) \to T \) a surjective Lie group homomorphism (Banyaga is not very explicit about the topologies and smooth structures of the groups involved in [Banyaga, Theorem 1]). It is easy to prove that if a Lie group \( G \) is a central extension of a torus \( T \) by \( S^1 \) as a Lie group and not just as an abstract group, then \( G \) is \( S^1 \times T \). On the other hand, Lemma 3 has a very easy proof.

Proof of Lemma 3. Let \( R \) denote the vector field generating the \( S^1 \) action on \( P \). Then \( \alpha(R) = 1 \) and \( L_{BR} = 0 \) since \( \alpha \) is a connection. Since \( H \) contains the \( S^1 \), for any vector \( X \) in the Lie algebra \( \mathfrak{h} \) of \( H \) the vector field \( X_P \) induced by \( X \) on \( P \) commutes with \( R \). Also, since \( H \subset \text{Diff}(P, \xi = \ker \alpha) \), \( L_{X_P} \alpha = f\alpha \) for some function \( f \in C^\infty(P) \) that may depend on \( X \). We want to show that \( f \equiv 0 \). Now

\[
0 = L_{X_P} 1 = L_{X_P} (\iota(R)\alpha) = \iota(L_{X_P} R)\alpha + \iota(R) (L_{X_P} \alpha) = 0 + \iota(R) f\alpha = f.
\]

\[\square\]

Lemma 3 is a consequence of the following slightly more general proposition. We drop the assumption that \( B \) is simply connected and replace it with the assumption that the action of \( T \) is Hamiltonian. One may also drop the assumption that \( B \) is compact, but we won’t do it. I am grateful to Anton Alekseev for telling me how to prove the proposition.

Proposition 4. Let \( (B, \omega) \) be a compact integral symplectic manifold. Let \( S^1 \to P \xrightarrow{\pi} B \) be the principal \( S^1 \) bundle with first Chern (Euler) class \( c_1(P) = [\omega] \). Let \( \alpha \) be the connection 1-form on \( P \) with \( d\alpha = \pi^* \omega \). Suppose there is a Hamiltonian action of a torus \( T \) on \( B \) with an associated moment map \( \Phi : B \to \mathfrak{t}^* \).

Then there is an action of \( T \) on \( P \) preserving \( \alpha \) and making \( \pi \) equivariant.

Proof. It is well known how to lift the action of the Lie algebra \( \mathfrak{t} \) of \( T \) on \( B \) to an action on \( P \) preserving \( \alpha \) (see, for example [S]): Given \( X \in \mathfrak{t} \) the induced vector field \( X_P \) on \( P \) is defined by

\[
(1.1) \quad X_P := X_B^h = (\pi^* \Phi X) R,
\]

where \( X_B \) is the vector field induced by \( X \) on \( B \), \( X_B^h \) denotes its horizontal lift to \( P \), \( \Phi^X = \langle \Phi, X \rangle \) is the \( X \)-component of the moment map \( \Phi \), and \( R \) is the vector field generating the action of \( S^1 = \mathbb{R}/\mathbb{Z} \). Thus we have an action of the universal cover \( \tilde{T} \) of \( T \) on \( P \). The point of the proposition is that for a suitable normalization of the moment map \( \Phi \), the action of \( \tilde{T} \) descends to an action of \( T \). Note that the proposition is false if the group in question is not a torus: there is no way to lift the standard action of \( SO(3) \) on \( S^2 \) to an action on \( S^3 \).

Our normalization is as follows. Since \( B \) is compact and the action of \( T \) is Hamiltonian, the set of \( T \)-fixed points \( B^T \) is non-empty. Pick a point \( b_0 \in B^T \) and normalize \( \Phi \) by requiring that \( \Phi(b_0) = 0 \). We claim for any vector \( X \) in the integral lattice \( \mathbb{Z}_T := \ker \{ \exp : \mathfrak{t} \to T \} \) the flow \( \{ \exp tX_P \} \) defined by (1.1) is periodic of period 1. Indeed, since the vector fields \( X_B^h \) and \( (\pi^* \Phi^X) R \) commute,

\[
\exp tX_P = (\exp tX_B^h) \circ \exp (-t(\pi^* \Phi^X) R).
\]

For a point \( p \in P \),

\[
\exp(-t(\pi^* \Phi^X) R)(p) = e^{-2\pi i \Phi^X(b)},
\]

where \( b = \pi(p) \) and where we identified \( \mathbb{R}/\mathbb{Z} \) with \( U(1) \) by \( \theta \mod \mathbb{Z} \mapsto e^{2\pi i \theta} \).

The curve \( \gamma(t) = (\exp tX_B)(b) \) is a loop in \( B \) since \( X \in \mathbb{Z}_T \). Hence \( \exp tX_B^h(p) = H(\gamma) \cdot p \) where \( H(\gamma) \) denotes the holonomy of \( \gamma \). On the other hand,

\[
H(\gamma) = e^{2\pi i \int_D \omega}
\]

for any disk \( D \subset B \) with boundary \( \gamma \). (Note that if \( D' \) is another disk with boundary \( \gamma \), then \( \int_{D'} \omega = \int_D \omega \in \mathbb{Z} \), since \( [\omega] \) is integral, and consequently \( e^{2\pi i \int_D \omega} \) is well-defined.)

The curve \( \gamma \) always bounds a disk: since \( B \) is connected, there is a path \( \tau : [0, 1] \to B \) with \( \tau(0) = b_0 \) and \( \tau(1) = b \). The disk

\[
D_\tau = \{ (\exp tX_B) \cdot \tau(s) \mid 0 \leq t \leq 1, \, 0 \leq s \leq 1 \}
\]
is a desired disk. Moreover,
\[
\int_{D_r} \omega = -\int_{\tau([0,1])} \langle \iota(X_B)\omega \rangle = \int_0^1 d\Phi^X(\tau(s))
\]
\[= \Phi^X(\tau(1)) - \Phi^X(\tau(0)) = \Phi^X(b).
\]
Thus \((\exp X_B^h) \cdot p = H(\gamma) \cdot p = e^{2\pi i \Phi^X(b)} \cdot p\), and therefore
\[
(\exp X_P)(p) = \exp X_B^h \cdot e^{-2\pi i \Phi^X(b)} \cdot p = e^{2\pi i \Phi^X(b)} e^{-2\pi i \Phi^X(b)} \cdot p = p.
\]

\section{Examples}

\textbf{Example 1.} Hausmann and Knutson [HK] constructed a symplectic form \(\omega\) on \(B = \mathbb{C}P\#3\mathbb{C}P^2\) which admits no Hamiltonian (hence symplectic) circle actions. The form \(\omega\) may be taken to be integral. The manifold \((B, \omega)\) is a pentagon space. Consider the corresponding contact manifold \((P, \xi = \ker \alpha)\), where as above \(\pi: P \to B\) is the principal \(S^1\) bundle with \(c_1(P) = [\omega]\) and \(\alpha\) is a connection 1-form with \(d\alpha = \pi^*\omega\). By Theorem 1 the \(S^1\) in the contactomorphism group \(\text{Diff}(P, \xi)\) is a maximal torus. Note that \(\dim P = 5\), so the maximal possible dimension of a maximal torus in \(\text{Diff}(P, \xi)\) is 3.

\textbf{Example 2.} Hausmann and Tolman [HT] constructed a number of polygon spaces \((B, \omega)\) with the property that the symplectomorphism group \(\text{Diff}(B, \omega)\) contains maximal tori of different dimensions. For example the group of symplectomorphisms of the heptagon space \(\text{Pol}(1,1,2,2,3,3,3)\) (we use the notation of [HT]) contains maximal tori of dimensions 2, 3 and 4. Hence the contactomorphism group \(\text{Diff}(P, \xi)\) of the corresponding principal circle bundle \(P \to \text{Pol}(1,1,2,2,3,3,3)\) contains maximal tori of dimension \(2 + 1, 3 + 1\) and \(4 + 1\).

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Department of Mathematics, University of Illinois, Urbana, IL 61801

E-mail address: lerman@math.uiuc.edu