WEIGHTED MORREY SPACES RELATED TO CERTAIN NONNEGATIVE POTENTIALS AND RIESZ TRANSFORMS

HUA WANG

ABSTRACT. Let \( L = -\Delta + V \) be a Schrödinger operator, where \( \Delta \) is the Laplacian on \( \mathbb{R}^d \) and the nonnegative potential \( V \) belongs to the reverse Hölder class \( RH_q \) for \( q \geq d \). The Riesz transform associated with the operator \( L = -\Delta + V \) is denoted by \( R = \nabla(-\Delta + V)^{-1/2} \) and the dual Riesz transform is denoted by \( R^* = (-\Delta + V)^{-1/2}\nabla \). In this paper, we first introduce some kinds of weighted Morrey spaces related to certain nonnegative potentials belonging to the reverse Hölder class \( RH_q \) for \( q \geq d \). Then we will establish the boundedness properties of the operators \( R \) and its adjoint \( R^* \) on these new spaces. Furthermore, weighted strong-type estimate and weighted endpoint estimate for the corresponding commutators \([b, R]\) and \([b, R^*]\) are also obtained. The classes of weights, the classes of symbol functions as well as weighted Morrey spaces discussed in this paper are larger than \( A_p \), \( BMO(\mathbb{R}^d) \) and \( L^{p,\kappa}(w) \) corresponding to the classical Riesz transforms \((V \equiv 0)\).

1. Introduction

1.1. The critical radius function \( \rho(x) \). Let \( d \geq 3 \) be a positive integer and \( \mathbb{R}^d \) be the \( d \)-dimensional Euclidean space. A nonnegative locally integrable function \( V(x) \) on \( \mathbb{R}^d \) is said to belong to the reverse Hölder class \( RH_q \) for some exponent \( 1 < q < \infty \), if there exists a positive constant \( C > 0 \) such that the following reverse Hölder inequality

\[
\left( \frac{1}{|B|} \int_B V(y)^q \, dy \right)^{1/q} \leq C \left( \frac{1}{|B|} \int_B V(y) \, dy \right)
\]

holds for every ball \( B \) in \( \mathbb{R}^d \). For given \( V \in RH_q \) with \( q \geq d \), we introduce the critical radius function \( \rho(x) = \rho(x, V) \) which is given by

\[
\rho(x) := \sup_{r > 0} \left\{ r : \frac{1}{r^{d-2}} \int_{B(x,r)} V(y) \, dy \leq 1 \right\}, \quad x \in \mathbb{R}^d,
\]

where \( B(x, r) \) denotes the open ball centered at \( x \) and with radius \( r \). It is well known that \( 0 < \rho(x) < \infty \) for any \( x \in \mathbb{R}^d \) under our assumption (see [9]). We need the following known result concerning the critical radius function.

2010 Mathematics Subject Classification. Primary 42B20; 35J10; Secondary 46E30; 47B47.

Key words and phrases. Schrödinger operators; Riesz transforms; commutators; weighted Morrey spaces; \( A_p^{\infty} \) weights; \( BMO_p(\mathbb{R}^d) \).
Lemma 1.1 ([9]). If $V \in RH_q$ with $q \geq d$, then there exist two constants $C > 0$ and $N_0 \geq 1$ such that
\begin{equation}
1 + \frac{|x - y|}{\rho(x)}^{\frac{-N_0}{q - d + 1}} \leq \frac{\rho(y)}{\rho(x)} \leq C \left(1 + \frac{|x - y|}{\rho(x)}\right)^{\frac{N_0}{q - d + 1}},
\end{equation}
for all $x, y \in \mathbb{R}^d$. As a straightforward consequence of (1.2), we have that for all $k = 1, 2, 3, \ldots$, the following estimate
\begin{equation}
1 + 2^{k} r \frac{\rho(y)}{\rho(x)} \geq C \left(1 + \frac{r}{\rho(x)}\right)^{\frac{-N_0}{q - d + 1}} \left(1 + 2^{k} \frac{r}{\rho(x)}\right),
\end{equation}
is valid for any $y \in B(x, r)$ with $x \in \mathbb{R}^d$ and $r > 0$.

1.2. Schrödinger operators. On $\mathbb{R}^d$, $d \geq 3$, we consider the Schrödinger operator
\[ L := -\Delta + V, \]
where $V \in RH_q$ for $q \geq d$. The Riesz transform associated with the Schrödinger operator $L$ is defined by
\begin{equation}
\mathcal{R} := \nabla L^{-1/2},
\end{equation}
and the associated dual Riesz transform is defined by
\begin{equation}
\mathcal{R}^* := L^{-1/2} \nabla.
\end{equation}
Boundedness properties of $\mathcal{R}$ and its adjoint $\mathcal{R}^*$ have been obtained by Shen in [9], where he showed that they are all bounded on $L^p(\mathbb{R}^d)$ for any $1 < p < \infty$ when $V \in RH_q$ with $q \geq d$. Actually, $\mathcal{R}$ and its adjoint $\mathcal{R}^*$ are standard Calderón-Zygmund operators in such a situation. The operators $\mathcal{R}$ and $\mathcal{R}^*$ have singular kernels that will be denoted by $K(x, y)$ and $K^*(x, y)$, respectively. For such kernels, we have the following key estimates, which can be found in [9] and [2, 3].

Lemma 1.2. Let $V \in RH_q$ with $q \geq d$. For any positive integer $N$, there exists a positive constant $C_N > 0$ such that
\begin{align*}
K(x, y) &\leq C_N \left(1 + \frac{|x - y|}{\rho(x)}\right)^{-N} \frac{1}{|x - y|^d}; \\
K^*(x, y) &\leq C_N \left(1 + \frac{|x - y|}{\rho(x)}\right)^{-N} \frac{1}{|x - y|^d}.
\end{align*}

1.3. $A^p_{\infty}$ weights. A weight will always mean a nonnegative function which is locally integrable on $\mathbb{R}^d$. Given a Lebesgue measurable set $E$ and a weight $w$, $|E|$ will denote the Lebesgue measure of $E$ and
\[ w(E) = \int_E w(x) \, dx. \]
Given $B = B(x_0, r)$ and $t > 0$, we will write $tB$ for the $t$-dilate ball, which is the ball with the same center $x_0$ and with radius $tr$. In [1] (see also [2, 3]), Bongioanni, Harboure and Salinas introduced the following classes of weights that are given in terms of the critical radius function (1.1). Following the terminology of [1], for given $1 < p < \infty$, we define

$$A^{p, \infty}_p := \bigcup_{\theta > 0} A^{p, \theta}_p,$$

where $A^{p, \theta}_p$ is the set of all weights $w$ such that

$$\left( \frac{1}{|B|} \int_B w(x) \, dx \right)^{1/p} \left( \frac{1}{|B|} \int_B w(x)^{-p'/p} \, dx \right)^{1/p'} \leq C \cdot \left( 1 + \frac{r}{\rho(x_0)} \right)^{\theta}$$

holds for every ball $B = B(x_0, r) \subset \mathbb{R}^d$ with $x_0 \in \mathbb{R}^d$ and $r > 0$, where $p'$ is the dual exponent of $p$ such that $1/p + 1/p' = 1$. For $p = 1$ we define

$$A^{1, \infty}_1 := \bigcup_{\theta > 0} A^{1, \theta}_1,$$

where $A^{1, \theta}_1$ is the set of all weights $w$ such that

$$\frac{1}{|B|} \int_B w(x) \, dx \leq C \cdot \left( 1 + \frac{r}{\rho(x_0)} \right)^{\theta} \text{ess inf}_{x \in B} w(x)$$

holds for every ball $B = B(x_0, r)$ in $\mathbb{R}^d$. For $\theta > 0$, let us introduce the maximal operator that is given in terms of the critical radius function (1.1).

$$M_{\rho, \theta} f(x) := \sup_{r > 0} \left( 1 + \frac{r}{\rho(x)} \right)^{-\theta} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y)| \, dy, \quad x \in \mathbb{R}^d.$$

Observe that a weight $w$ belongs to the class $A^{1, \infty}_1$ if and only if there exists a positive number $\theta > 0$ such that $M_{\rho, \theta} w \leq Cw$, where the constant $C > 0$ is independent of $w$. Since

$$1 \leq \left( 1 + \frac{r}{\rho(x_0)} \right)^{\theta_1} \leq \left( 1 + \frac{r}{\rho(x_0)} \right)^{\theta_2}$$

for $0 < \theta_1 < \theta_2 < \infty$, then for given $p$ with $1 \leq p < \infty$, one has

$$A_p \subset A^{p, \theta_1}_p \subset A^{p, \theta_2}_p,$$

where $A_p$ denotes the classical Muckenhoupt’s class (see [1 Chapter 7]), and hence $A_p \subset A^{p, \infty}_p$. In addition, for some fixed $\theta > 0$,

$$A^{p, \theta}_p \subset A^{p, \theta}_{p_1} \subset A^{p, \theta}_{p_2}$$

whenever $1 \leq p_1 < p_2 < \infty$. Now, we present an important property of the classes of weights in $A^{p, \theta}_p$ with $1 \leq p < \infty$, which was given by Bongioanni et al. in [1, Lemma 5].
Lemma 1.3. If \( w \in A_p^\rho \) with \( 0 < \theta < \infty \) and \( 1 \leq p < \infty \), then there exist positive constants \( \epsilon, \eta > 0 \) and \( C > 0 \) such that
\[
(1.6) \quad \left( \frac{1}{|B|} \int_B w(x)^{1+\epsilon} \, dx \right)^{\frac{1}{1+\epsilon}} \leq C \left( \frac{1}{|B|} \int_B w(x) \, dx \right)^{\frac{1}{1+\epsilon}} \left( 1 + \frac{r}{\rho(x_0)} \right)^{\eta}
\]
for every ball \( B = B(x_0, r) \) in \( \mathbb{R}^d \).

As a direct consequence of Lemma 1.3, we have the following result.

Lemma 1.4. If \( w \in A_p^\rho \) with \( 0 < \theta < \infty \) and \( 1 \leq p < \infty \), then there exist two positive numbers \( \delta > 0 \) and \( \eta > 0 \) such that
\[
(1.7) \quad w(E) \leq C \left( \frac{|E|}{|B|} \right)^{\delta} \left( 1 + \frac{r}{\rho(x_0)} \right)^{\eta}
\]
for any measurable subset \( E \) of a ball \( B = B(x_0, r) \), where \( C > 0 \) is a constant which does not depend on \( E \) and \( B \).

For any given ball \( B = B(x_0, r) \) with \( x_0 \in \mathbb{R}^d \) and \( r > 0 \), suppose that \( E \subset B \), then by Hölder’s inequality with exponent \( 1 + \epsilon \) and (1.6), we can deduce that
\[
w(E) = \int_B \chi_{E}(x) \cdot w(x) \, dx
\]
\[
\leq \left( \int_B w(x)^{1+\epsilon} \, dx \right)^{\frac{1}{1+\epsilon}} \left( \int_B \chi_{E}(x)^{\frac{1+\epsilon}{\epsilon}} \, dx \right)^{\frac{\epsilon}{1+\epsilon}}
\]
\[
\leq C |B|^{\frac{1}{1+\epsilon}} \left( \frac{1}{|B|} \int_B w(x) \, dx \right)^{\frac{\epsilon}{1+\epsilon}} \left( 1 + \frac{r}{\rho(x_0)} \right)^{\eta} |E|^{\frac{\epsilon}{1+\epsilon}}
\]
\[
= C \left( \frac{|E|}{|B|} \right)^{\frac{\epsilon}{1+\epsilon}} \left( 1 + \frac{r}{\rho(x_0)} \right)^{\eta}.
\]
This gives (1.7) with \( \delta = \epsilon/(1 + \epsilon) \).

Given a weight \( w \) on \( \mathbb{R}^d \), as usual, the weighted Lebesgue space \( L^p(w) \) for \( 1 \leq p < \infty \) is defined to be the set of all functions \( f \) such that
\[
\| f \|_{L^p(w)} := \left( \int_{\mathbb{R}^d} |f(x)|^p w(x) \, dx \right)^{1/p} < \infty.
\]

We also denote by \( WL^1(w) \) the weighted weak Lebesgue space consisting of all measurable functions \( f \) for which
\[
\| f \|_{WL^1(w)} := \sup_{\lambda > 0} \lambda \cdot w \left( \left\{ x \in \mathbb{R}^d : |f(x)| > \lambda \right\} \right) < \infty.
\]

Recently, Bongioanni et al. \cite{1} obtained weighted strong-type and weak-type estimates for the operators \( R \) and \( R^* \) defined in (1.4) and (1.5). Their results can be summarized as follows:
Theorem 1.5 ([1]). Let $1 < p < \infty$ and $w \in A^p_{\infty}$. If $V \in RH_q$ with $q \geq d$, then the operators $R$ and $R^*$ are all bounded on $L^p(w)$.

Theorem 1.6 ([1]). Let $p = 1$ and $w \in A^p_{1,\infty}$. If $V \in RH_q$ with $q \geq d$, then the operators $R$ and $R^*$ are all bounded from $L^1(w)$ into $WL^1(w)$.

1.4. The space $\text{BMO}_{\rho,\infty}(\mathbb{R}^d)$. We denote by $\mathcal{T}$ either $R$ or $R^*$. For a locally integrable function $b$ on $\mathbb{R}^d$ (usually called the symbol), we will also consider the commutator operator

$$[b, \mathcal{T}]f(x) := b(x) \cdot \mathcal{T}f(x) - \mathcal{T}(bf)(x), \quad x \in \mathbb{R}^d.$$  

Recently, Bongioanni et al. [3] introduced a new space $\text{BMO}_{\rho,\infty}(\mathbb{R}^d)$ defined by

$$\text{BMO}_{\rho,\infty}(\mathbb{R}^d) := \bigcup_{\theta > 0} \text{BMO}_{\rho,\theta}(\mathbb{R}^d),$$

where for $0 < \theta < \infty$ the space $\text{BMO}_{\rho,\theta}(\mathbb{R}^d)$ is defined to be the set of all locally integrable functions $b$ satisfying

$$\frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} |b(x) - b_{B(x_0, r)}| \, dx \leq C \left(1 + \frac{r}{\rho(x_0)}\right)^\theta,$$

for all $x_0 \in \mathbb{R}^d$ and $r > 0$, $b_{B(x_0, r)}$ denotes the mean value of $b$ on $B(x_0, r)$, that is,

$$b_{B(x_0, r)} := \frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} b(y) \, dy.$$

A norm for $b \in \text{BMO}_{\rho,\theta}(\mathbb{R}^d)$, denoted by $\|b\|_{\text{BMO}_{\rho,\theta}}$, is given by the infimum of the constants satisfying (1.9), or equivalently,

$$\|b\|_{\text{BMO}_{\rho,\theta}} := \sup_{B(x_0, r)} \left(1 + \frac{r}{\rho(x_0)}\right)^{-\theta} \left(\frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} |b(x) - b_{B(x_0, r)}| \, dx\right),$$

where the supremum is taken over all balls $B(x_0, r)$ with $x_0 \in \mathbb{R}^d$ and $r > 0$. With the above definition in mind, one has

$$\text{BMO}(\mathbb{R}^d) \subset \text{BMO}_{\rho,\theta_1}(\mathbb{R}^d) \subset \text{BMO}_{\rho,\theta_2}(\mathbb{R}^d)$$

for $0 < \theta_1 < \theta_2 < \infty$, and hence $\text{BMO}(\mathbb{R}^d) \subset \text{BMO}_{\rho,\infty}(\mathbb{R}^d)$. Moreover, the classical BMO space [5] is properly contained in $\text{BMO}_{\rho,\infty}(\mathbb{R}^d)$ (see [2, 3] for some examples). We need the following key result for $\text{BMO}_{\rho,\theta}(\mathbb{R}^d)$, which was proved by Tang in [10].

Proposition 1.7 ([10]). Let $b \in \text{BMO}_{\rho,\theta}(\mathbb{R}^d)$ with $0 < \theta < \infty$. Then there exist two positive constants $C_1$ and $C_2$ such that for any given ball $B(x_0, r)$
in $\mathbb{R}^d$ and for any $\lambda > 0$, we have
\[
\left| \left\{ x \in B(x_0, r) : |b(x) - b_{B(x_0, r)}| > \lambda \right\} \right| 
\leq C_1 |B(x_0, r)| \exp \left[ - \left( 1 + \frac{r}{\rho(x_0)} \right)^{-\theta^*} \frac{C_2 \lambda}{\|b\|_{\text{BMO}}^{\rho, \theta}} \right],
\]}

(1.10)

where $\theta^* = (N_0 + 1)\theta$ and $N_0$ is the constant appearing in Lemma 1.1.

As a consequence of Proposition 1.7 and Lemma 1.4, we have the following result:

**Proposition 1.8.** Let $b \in \text{BMO}_{\rho, \theta}(\mathbb{R}^d)$ with $0 < \theta < \infty$ and $w \in A_{p, \infty}$ with $1 \leq p < \infty$. Then there exist positive constants $C_1, C_2$ and $\eta > 0$ such that for any given ball $B(x_0, r)$ in $\mathbb{R}^d$ and for any $\lambda > 0$, we have
\[
\left| \left\{ x \in B(x_0, r) : |b(x) - b_{B(x_0, r)}| > \lambda \right\} \right| 
\leq C_1 w(B(x_0, r)) \exp \left[ - \left( 1 + \frac{r}{\rho(x_0)} \right)^{-\theta^*} \frac{C_2 \lambda}{\|b\|_{\text{BMO}}^{\rho, \theta}} \right] \left( 1 + \frac{r}{\rho(x_0)} \right)^\eta,
\]

(1.11)

where $\theta^* = (N_0 + 1)\theta$ and $N_0$ is the constant appearing in Lemma 1.1.

**1.5. Orlicz spaces.** In this subsection, let us give the definition and some basic facts about Orlicz spaces. For more information on this subject, the reader may consult the book [8]. Recall that a function $A : [0, \infty) \to [0, \infty)$ is called a Young function if it is continuous, convex and strictly increasing with $A(0) = 0$ and $\lim_{t \to \infty} A(t) = \infty$.

An important example of Young function is $A(t) = t \cdot (1 + \log^+ t)^m$ with some $1 \leq m < \infty$. Given a Young function $A$ and a function $f$ defined on a ball $B$, we consider the $A$-average of a function $f$ given by the following Luxemburg norm:
\[
\|f\|_{A, B} := \inf \left\{ \lambda > 0 : \frac{1}{|B|} \int_B A \left( \frac{|f(x)|}{\lambda} \right) dx \leq 1 \right\}.
\]

Associated to each Young function $A$, one can define its complementary function $\bar{A}$ as follows:
\[
\bar{A}(s) := \sup_{t > 0} \left\{ st - A(t) \right\}.
\]

Such a function $\bar{A}$ is also a Young function. It is well known that the following generalized Hölder inequality in Orlicz spaces holds for any given ball $B \subset \mathbb{R}^d$:
\[
\frac{1}{|B|} \int_B |f(x) \cdot g(x)| \, dx \leq 2 \|f\|_{A, B} \|g\|_{\bar{A}, B}.
\]
In particular, for the Young function $\mathcal{A}(t) = t \cdot (1 + \log^+ t)$, the Luxemburg norm will be denoted by $\| \cdot \|_{L, \log L, B} = \| \cdot \|_{\mathcal{A}, B}$. A simple computation shows that the complementary Young function of $\mathcal{A}(t) = t \cdot (1 + \log^+ t)$ is $\overline{\mathcal{A}}(t) = \exp(t) - 1$. The corresponding Luxemburg norm will be denoted by $\| \cdot \|_{\exp, L, B} = \| \cdot \|_{\mathcal{A}, B}$. In this situation, we have

$$
\tag{1.12}
\frac{1}{|B|} \int_B |f(x) \cdot g(x)| \, dx \leq 2 \|f\|_{L, \log L, B} \|g\|_{\exp, L, B}.
$$

We next define the weighted $\mathcal{A}$-average of a function $f$ over a ball $B$. Given a Young function $\mathcal{A}$ and a weight function $w$, let (see \cite{8} for instance)

$$
\|f\|_{\mathcal{A}(w), B} := \inf \left\{ \lambda > 0 : \frac{1}{w(B)} \int_B \mathcal{A} \left( \frac{|f(x)|}{\lambda} \right) \cdot w(x) \, dx \leq 1 \right\}.
$$

When $\mathcal{A}(t) = t$, we denote $\| \cdot \|_{L(w), B} = \| \cdot \|_{\mathcal{A}(w), B}$, and when $\Phi(t) = t \cdot (1 + \log^+ t)$, we denote $\| \cdot \|_{L, \log L(w), B} = \| \cdot \|_{\Phi(w), B}$. Also, the complementary Young function of $\Phi$ is given by $\overline{\Phi}(t) = e^t - 1$ with corresponding Luxemburg norm denoted by $\| \cdot \|_{\exp, L(w), B}$. Given a weight $w$ on $\mathbb{R}^d$, we can also show the weighted version of \textbf{(1.12)}. That is, the following generalized Hölder inequality in the weighted setting

$$
\tag{1.13}
\frac{1}{w(B)} \int_B |f(x) \cdot g(x)| w(x) \, dx \leq C \|f\|_{L, \log L(w), B} \|g\|_{\exp, L(w), B}
$$

holds for every ball $B$ in $\mathbb{R}^d$. It is a simple but important observation that for any ball $B$ in $\mathbb{R}^d$,

$$
\|f\|_{L(w), B} \leq \|f\|_{L, \log L(w), B}.
$$

This is because $t \leq t \cdot (1 + \log^+ t)$ for all $t > 0$. So we have

$$
\tag{1.14}
\|f\|_{L(w), B} = \frac{1}{w(B)} \int_B |f(x)| \cdot w(x) \, dx \leq \|f\|_{L, \log L(w), B}.
$$

In [2], Bongioanni et al. obtained weighted strong $(p, p)$, $1 < p < \infty$, and weak $L \log L$ estimates for the commutators of the Riesz transform and its adjoint associated with the Schrödinger operator $\mathcal{L} = -\Delta + V$, where $V$ satisfies some reverse Hölder inequality. Their results can be summarized as follows:

**Theorem 1.9** ([2]). Let $1 < p < \infty$ and $w \in A_p^{p, \infty}$. If $V \in RH_q$ with $q \geq d$, then the commutator operators $[b, \mathcal{R}]$ and $[b, \mathcal{R}^*]$ are all bounded on $L^p(w)$, whenever $b$ belongs to $\text{BMO}_{p, \infty}(\mathbb{R}^d)$.

**Theorem 1.10** ([2]). Let $p = 1$ and $w \in A_1^{p, \infty}$. If $V \in RH_q$ with $q \geq d$ and $b \in \text{BMO}_{p, \infty}(\mathbb{R}^d)$, then for any given $\lambda > 0$, there exists a positive constant...
$C > 0$ such that for those functions $f$ such that $\Phi(|f|) \in L^1(w)$,
\[
  w\left(\{x \in \mathbb{R}^n : |[b, \mathcal{R}]f(x)| > \lambda\}\right) \leq C \int_{\mathbb{R}^d} \Phi\left(\frac{|f(x)|}{\lambda}\right) \cdot w(x) \, dx,
\]
and
\[
  w\left(\{x \in \mathbb{R}^n : |[b, \mathcal{R}^*]f(x)| > \lambda\}\right) \leq C \int_{\mathbb{R}^d} \Phi\left(\frac{|f(x)|}{\lambda}\right) \cdot w(x) \, dx,
\]
where $\Phi(t) = t \cdot (1 + \log^+ t)$ and $\log^+ t := \max\{\log t, 0\}$; that is,
\[
  \log^+ t = \begin{cases} 
    \log t, & \text{as } t > 1; \\
    0, & \text{otherwise}.
  \end{cases}
\]

In this paper, firstly, we will define some kinds of weighted Morrey spaces related to certain nonnegative potentials. Secondly, we prove that the Riesz transform $\mathcal{R}$ and its adjoint $\mathcal{R}^*$ are both bounded operators on these new spaces. Finally, we also obtain the weighted boundedness for the commutators $[b, \mathcal{R}]$ and $[b, \mathcal{R}^*]$ defined in (1.8).

Throughout this paper $C$ denotes a positive constant not necessarily the same at each occurrence, and a subscript is added when we wish to make clear its dependence on the parameter in the subscript. We also use $a \approx b$ to denote the equivalence of $a$ and $b$; that is, there exist two positive constants $C_1, C_2$ independent of $a, b$ such that $C_1 a \leq b \leq C_2 a$.

### 2. Our Main Results

In this section, we introduce some types of weighted Morrey spaces related to the potential $V$ and then give our main results.

**Definition 2.1.** Let $1 \leq p < \infty$, $0 \leq \kappa < 1$ and $w$ be a weight. For given $0 < \theta < \infty$, the weighted Morrey space $L_{p, \kappa}^{\rho, \theta}(w)$ is defined to be the set of all $L^p$ locally integrable functions $f$ on $\mathbb{R}^d$ for which
\[
  \left( \frac{1}{w(B)^\kappa} \int_B |f(x)|^p w(x) \, dx \right)^{1/p} \leq C \cdot \left( 1 + \frac{r}{\rho(x_0)} \right)^{\theta}
\]
for every ball $B = B(x_0, r)$ in $\mathbb{R}^d$. A norm for $f \in L_{p, \kappa}^{\rho, \theta}(w)$, denoted by $\|f\|_{L_{p, \kappa}^{\rho, \theta}(w)}$, is given by the infimum of the constants in (2.1), or equivalently,
\[
  \|f\|_{L_{p, \kappa}^{\rho, \theta}(w)} := \sup_B \left( 1 + \frac{r}{\rho(x_0)} \right)^{-\theta} \left( \frac{1}{w(B)^\kappa} \int_B |f(x)|^p w(x) \, dx \right)^{1/p} < \infty,
\]
where the supremum is taken over all balls $B$ in $\mathbb{R}^d$, $x_0$ and $r$ denote the center and radius of $B$ respectively. Define
\[
  L_{p, \kappa}^{\rho, \infty}(w) := \bigcup_{\theta > 0} L_{p, \kappa}^{\rho, \theta}(w).
\]
Note that this definition does not coincide with the one given in [7] (see also [11] for the unweighted case), but in view of the space $\text{BMO}_{\rho,\infty}(\mathbb{R}^d)$ defined above it is more natural in our setting. Obviously, if we take $\theta = 0$ or $V \equiv 0$, then this new space is just the weighted Morrey space $L^{p,\kappa}(w)$, which was first defined by Komori and Shirai in [6] (see also [12]).

**Definition 2.2.** Let $p = 1$, $0 \leq \kappa < 1$ and $w$ be a weight. For given $0 < \theta < \infty$, the weighted weak Morrey space $\text{WL}^{1,\kappa}_{\rho,\theta}(w)$ is defined to be the set of all measurable functions $f$ on $\mathbb{R}^d$ for which
\[
\frac{1}{w(B)} \sup_{\lambda > 0} \lambda \cdot w \left( \{ x \in B : |f(x)| > \lambda \} \right) \leq C \cdot \left( 1 + \frac{r}{\rho(x_0)} \right)^{\theta}
\]
for every ball $B = B(x_0, r)$ in $\mathbb{R}^d$, or equivalently,
\[
\|f\|_{\text{WL}^{1,\kappa}_{\rho,\theta}(w)} := \sup_B \left( 1 + \frac{r}{\rho(x_0)} \right)^{-\theta} \frac{1}{w(B)} \sup_{\lambda > 0} \lambda \cdot w \left( \{ x \in B : |f(x)| > \lambda \} \right) < \infty.
\]
Correspondingly, we define
\[
\text{WL}^{1,\kappa}_{\rho,\infty}(w) := \bigcup_{\theta > 0} \text{WL}^{1,\kappa}_{\rho,\theta}(w).
\]
Clearly, if we take $\theta = 0$ or $V \equiv 0$, then this space is just the weighted weak Morrey space $\text{WL}^{1,\kappa}(w)$ (see [13]). According to the above definitions, one has
\[
\begin{cases}
L^{p,\kappa}(w) \subset L^{p,\kappa}_{\rho,\theta_1}(w) \subset L^{p,\kappa}_{\rho,\theta_2}(w); \\
\text{WL}^{1,\kappa}(w) \subset \text{WL}^{1,\kappa}_{\rho,\theta_1}(w) \subset \text{WL}^{1,\kappa}_{\rho,\theta_2}(w).
\end{cases}
\]
for $0 < \theta_1 < \theta_2 < \infty$. Hence $L^{p,\kappa}(w) \subset L^{p,\kappa}_{\rho,\infty}(w)$ for $(p, \kappa) \in [1, \infty) \times [0, 1)$ and $\text{WL}^{1,\kappa}(w) \subset \text{WL}^{1,\kappa}_{\rho,\infty}(w)$ for $0 \leq \kappa < 1$.

The space $L^{p,\kappa}_{\rho,\theta}(w)$ (or $\text{WL}^{1,\kappa}_{\rho,\theta}(w)$) could be viewed as an extension of weighted (or weak) Lebesgue space (when $\kappa = \theta = 0$). Naturally, one may ask the question whether the above conclusions (i.e., Theorems 1.5 and 1.6 as well as Theorems 1.9 and 1.10) still hold if replacing the weighted Lebesgue spaces by the weighted Morrey spaces. In this work, we give a positive answer to this question. Our main results in this work are presented as follows:

**Theorem 2.3.** Let $1 < p < \infty$, $0 < \kappa < 1$ and $w \in A^{p,\infty}_q$. If $V \in RH_q$ with $q \geq d$, then the operators $\mathcal{R}$ and $\mathcal{R}^*$ are both bounded on $L^{p,\kappa}_{\rho,\infty}(w)$.

**Theorem 2.4.** Let $p = 1$, $0 < \kappa < 1$ and $w \in A^{p,\infty}_1$. If $V \in RH_q$ with $q \geq d$, then the operators $\mathcal{R}$ and $\mathcal{R}^*$ are both bounded from $L^{1,\kappa}_{\rho,\infty}(w)$ into $\text{WL}^{1,\kappa}_{\rho,\infty}(w)$. 
Let \( 1 < p < \infty, 0 < \kappa < 1 \) and \( w \in A_p^{\rho, \infty} \). If \( V \in RH_q \) with \( q \geq d \), then the commutator operators \([b, \mathcal{R}]\) and \([b, \mathcal{R}^*]\) are both bounded on \( L^{p, \infty}_\rho(w) \), whenever \( b \in \text{BMO}_{p, \infty}(\mathbb{R}^d) \).

To deal with the commutators in the endpoint case, we need to consider a new kind of weighted Morrey spaces of \( L \log L \) type.

**Definition 2.6.** Let \( p = 1, 0 \leq \kappa < 1 \) and \( w \) be a weight. For given \( 0 < \theta < \infty \), the weighted Morrey space \((L \log L)^{1, \kappa}_{\rho, \theta}(w)\) is defined to be the set of all locally integrable functions \( f \) on \( \mathbb{R}^d \) for which

\[
\|f\|_{(L \log L)^{1, \kappa}_{\rho, \theta}(w)} := \sup_B \left( 1 + \frac{r}{\rho(x_0)} \right)^{-\theta} w(B)^{-1-\kappa} \|f\|_{L^{1, \kappa}_{\rho, \theta}(w)},
\]

for every ball \( B = B(x_0, r) \) in \( \mathbb{R}^d \), or

\[
\|f\|_{(L \log L)^{1, \kappa}_{\rho, \theta}(w)} := \sup_B \left( 1 + \frac{r}{\rho(x_0)} \right)^{-\theta} w(B)^{-1-\kappa} \|f\|_{L^{1, \kappa}_{\rho, \theta}(w)},
\]

Concerning the continuity properties of \([b, \mathcal{R}]\) and \([b, \mathcal{R}^*]\) in the weighted Morrey spaces of \( L \log L \) type, we have

**Theorem 2.7.** Let \( p = 1, 0 < \kappa < 1 \) and \( w \in A_1^{\rho, \infty} \). If \( V \in RH_q \) with \( q \geq d \) and \( b \in \text{BMO}_{p, \infty}(\mathbb{R}^d) \), then for any given \( \lambda > 0 \) and any given ball \( B = B(x_0, r) \) of \( \mathbb{R}^d \), there exist some constants \( C > 0 \) and \( \theta > 0 \) such that the following inequalities

\[
\frac{1}{w(B)^\kappa} \cdot w\left( \{ x \in B : |[b, \mathcal{R}]f(x)| > \lambda \} \right) \leq C \left( 1 + \frac{r}{\rho(x_0)} \right)^\theta \left\| \Phi\left( \frac{|f|}{\lambda} \right) \right\|_{(L \log L)^{1, \kappa}_{\rho, \theta}(w)},
\]

\[
\frac{1}{w(B)^\kappa} \cdot w\left( \{ x \in B : |[b, \mathcal{R}^*]f(x)| > \lambda \} \right) \leq C \left( 1 + \frac{r}{\rho(x_0)} \right)^\theta \left\| \Phi\left( \frac{|f|}{\lambda} \right) \right\|_{(L \log L)^{1, \kappa}_{\rho, \theta}(w)},
\]

hold for those functions \( f \) such that \( \Phi(|f|) \in (L \log L)^{1, \kappa}_{\rho, \theta}(w) \) with some fixed \( \theta > 0 \), where \( \Phi(t) = t \cdot (1 + \log^+ t) \).

If we denote

\[
(L \log L)^{1, \kappa}_{\rho, \infty}(w) := \bigcup_{\theta > 0} (L \log L)^{1, \kappa}_{\rho, \theta}(w),
\]

then Theorem 2.7 now tells us that the commutators \([b, \mathcal{R}]\) and \([b, \mathcal{R}^*]\) are both bounded from \((L \log L)^{1, \kappa}_{\rho, \infty}(w)\) into \(WL^{1, \kappa}_{\rho, \infty}(w)\), when \( b \) is in \( \text{BMO}_{p, \infty}(\mathbb{R}^d) \).

### 3. Proofs of Theorems 2.3 and 2.4

In this section, we will prove the conclusions of Theorems 2.3 and 2.4.
Proof of Theorem 2.3. We denote by $\mathcal{T}$ either $\mathcal{R}$ or $\mathcal{R}^*$.

By definition, we only have to show that for any given ball $B = B(x_0, r)$ of $\mathbb{R}^d$, there is some $\vartheta > 0$ such that

$$
(3.1) \quad \left( \frac{1}{w(B)^\kappa} \int_B |\mathcal{T} f(x)|^p w(x) \, dx \right)^{1/p} \leq C \cdot \left( 1 + \frac{r}{\rho(x_0)} \right)^\vartheta
$$

holds for any $f \in L_{\rho,\infty}^{p,\kappa}(w)$ with $1 < p < \infty$ and $0 < \kappa < 1$. Suppose that $f \in L_{\rho,\theta}^{p,\kappa}(w)$ for some $\theta > 0$ and $w \in A_{p,\theta'}^{p,\kappa}$ for some $\theta' > 0$. We decompose $f$ as

$$
f = f_1 + f_2 \in L_{\rho,\theta}^{p,\kappa}(w);
$$

$$
f_1 = f \cdot \chi_{2B};
$$

$$
f_2 = f \cdot \chi_{(2B)^c},
$$

where $2B$ is the ball centered at $x_0$ and radius $2r > 0$, and $\chi_{2B}$ is the characteristic function of $2B$. Then by the linearity of $\mathcal{T}$, we write

$$
\frac{1}{w(B)^\kappa} \int_B |\mathcal{T} f(x)|^p w(x) \, dx \leq \left( \frac{1}{w(B)^\kappa} \int_B |\mathcal{T} f_1(x)|^p w(x) \, dx \right)^{1/p} + \left( \frac{1}{w(B)^\kappa} \int_B |\mathcal{T} f_2(x)|^p w(x) \, dx \right)^{1/p} := I_1 + I_2.
$$

We now analyze each term separately. By Theorem 1.5, we get

$$
I_1 = \left( \frac{1}{w(B)^\kappa} \int_B |\mathcal{T} f_1(x)|^p w(x) \, dx \right)^{1/p} \leq C \cdot \frac{1}{w(B)^{\kappa/p}} \left( \int_{\mathbb{R}^d} |f_1(x)|^p w(x) \, dx \right)^{1/p} = C \cdot \frac{1}{w(B)^{\kappa/p}} \left( \int_{2B} |f(x)|^p w(x) \, dx \right)^{1/p} \leq C \left\| f \right\|_{L_{\rho,\theta}^{p,\kappa}(w)} \cdot \frac{w(2B)^{\kappa/p}}{w(B)^{\kappa/p}} \cdot \left( 1 + \frac{2r}{\rho(x_0)} \right)^\theta.
$$

Since $w \in A_{p,\theta'}^{p,\kappa}$ with $1 < p < \infty$ and $0 < \theta' < \infty$, then we know that the following inequality

$$
(3.2) \quad w(2B(x_0, r)) \leq C \cdot \left( 1 + \frac{2r}{\rho(x_0)} \right)^{p\theta'} w(B(x_0, r))
$$
is valid. In fact, for $1 < p < \infty$, by Hölder’s inequality and the definition of $A_p^{\rho,\theta}$, we have

$$
\frac{1}{|2B|} \int_{2B} |h(x)| \, dx = \frac{1}{|2B|} \int_{2B} |h(x)| w(x)^{1/p} w(x)^{-1/p} \, dx
$$

$$
\leq \frac{1}{|2B|} \left( \int_{2B} |h(x)|^p w(x) \, dx \right)^{1/p} \left( \int_{2B} w(x)^{-q/p} \, dx \right)^{1/q'}
$$

$$
\leq \frac{C_w}{w(2B)^{1/p}} \left( \int_{2B} |h(x)|^p w(x) \, dx \right)^{1/p} \left( 1 + \frac{2r}{\rho(x)} \right)^{q'}.
$$

If we take $h(x) = \chi_B(x)$, then the above expression becomes

$$(3.3) \quad \frac{|B|}{2B} \leq C_w \cdot \frac{w(B)^{1/p}}{w(2B)^{1/p}} \left( 1 + \frac{2r}{\rho(x)} \right)^{q'},$$

which in turn implies (3.2). Therefore,

$$I_1 \leq C \|f\|_{L_{p,\rho,\theta}^\infty(w)} \cdot \left( 1 + \frac{2r}{\rho(x)} \right)^{(p\theta')-(\kappa/p)} \cdot \left( 1 + \frac{2r}{\rho(x)} \right) \theta
$$

$$= C \|f\|_{L_{p,\rho,\theta}^\infty(w)} \cdot \left( 1 + \frac{2r}{\rho(x)} \right)^{q'} \leq C \cdot \left( 1 + \frac{r}{\rho(x)} \right)^{q'},$$

where $q' := \kappa \cdot \theta' + \theta$. For the other term $I_2$, notice that for any $x \in B$ and $y \in (2B)^c$, one has $|x - y| \approx |x_0 - y|$. It then follows from Lemma 1.2 that for any $x \in B(x_0, r)$ and any positive integer $N$,

$$(3.4) \quad |T f_2(x)| \leq \int_{(2B)^c} |K(x, y)| \cdot |f(y)| \, dy \quad (or \ K^*(x, y))
$$

$$\leq C_N \int_{(2B)^c} \left( 1 + \frac{|x - y|}{\rho(x)} \right)^{-N} \frac{1}{|x - y|^d} \cdot |f(y)| \, dy
$$

$$\leq C_{N,d} \int_{(2B)^c} \left( 1 + \frac{|x_0 - y|}{\rho(x)} \right)^{-N} \frac{1}{|x_0 - y|^d} \cdot |f(y)| \, dy
$$

$$= C_{N,d} \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} \left( 1 + \frac{|x_0 - y|}{\rho(x)} \right)^{-N} \frac{1}{|x_0 - y|^d} \cdot |f(y)| \, dy
$$

$$\leq C \sum_{k=1}^{\infty} \frac{1}{|2^{k+1}B|} \int_{2^{k+1}B \setminus 2^k B} \left( 1 + \frac{2^k r}{\rho(x)} \right)^{-N} |f(y)| \, dy.$$
In view of (1.3) in Lemma 1.1 we further obtain
\[
|Tf_2(x)| \leq C \sum_{k=1}^{\infty} \frac{1}{|2^{k+1}B|} \int_{2^{k+1}B} \left( 1 + \frac{r}{\rho(x_0)} \right)^{\frac{N-N_0}{N_0+1}} \left( 1 + \frac{2^kr}{\rho(x_0)} \right)^{-N} |f(y)| \, dy
\]
(3.5)
\[
\leq C \sum_{k=1}^{\infty} \frac{1}{|2^{k+1}B|} \int_{2^{k+1}B} \left( 1 + \frac{r}{\rho(x_0)} \right)^{\frac{N-N_0}{N_0+1}} \left( 1 + \frac{2^{k+1}r}{\rho(x_0)} \right)^{-N} |f(y)| \, dy.
\]
Moreover, by using H"older’s inequality and $A_p^{\theta'}$ condition on $w$, we get
\[
\frac{1}{|2^{k+1}B|} \int_{2^{k+1}B} |f(y)| \, dy
\leq \frac{1}{|2^{k+1}B|} \left( \int_{2^{k+1}B} |f(y)|^pw(y) \, dy \right)^{1/p} \left( \int_{2^{k+1}B} w(y)^{-p'/p} \, dy \right)^{1/p'}
\leq C\|f\|_{L_{p',\theta}(w)} \cdot \frac{w(2^{k+1}B)^{\kappa/p}}{w(2^{k+1}B)^{1/p}} \left( 1 + \frac{2^{k+1}r}{\rho(x_0)} \right)^{\theta} \left( 1 + \frac{2^{k+1}r}{\rho(x_0)} \right)^{\theta'}.
\]
Hence,
\[
I_2 \leq C\|f\|_{L_{p',\theta}(w)} \cdot \frac{w(B)^{1/p}}{w(B)^{1/p}} \sum_{k=1}^{\infty} \frac{w(2^{k+1}B)^{\kappa/p}}{w(2^{k+1}B)^{1/p}} \left( 1 + \frac{r}{\rho(x_0)} \right)^{\frac{N-N_0}{N_0+1}} \left( 1 + \frac{2^{k+1}r}{\rho(x_0)} \right)^{-N+\theta+\theta'}
= C\|f\|_{L_{p',\theta}(w)} \left( 1 + \frac{r}{\rho(x_0)} \right)^{\frac{N-N_0}{N_0+1}} \sum_{k=1}^{\infty} \frac{w(B)^{(1-\kappa)/p}}{w(2^{k+1}B)^{(1-\kappa)/p}} \left( 1 + \frac{2^{k+1}r}{\rho(x_0)} \right)^{-N+\theta+\theta'}.
\]
Recall that $w \in A_p^{\theta'}$ with $0 < \theta' < \infty$ and $1 < p < \infty$, then there exist two positive numbers $\delta, \eta > 0$ such that (1.7) holds. This allows us to obtain
\[
I_2 \leq C\|f\|_{L_{p',\theta}(w)} \left( 1 + \frac{r}{\rho(x_0)} \right)^{\frac{N-N_0}{N_0+1}} \sum_{k=1}^{\infty} \left( \frac{|B|}{|2^{k+1}B|} \right)^{\delta(1-\kappa)/p} \frac{2^{k+1}r}{\rho(x_0)} \left( \frac{2^{k+1}r}{\rho(x_0)} \right)^{-N+\theta+\theta'}
= C\|f\|_{L_{p',\theta}(w)} \left( 1 + \frac{r}{\rho(x_0)} \right)^{\frac{N-N_0}{N_0+1}} \sum_{k=1}^{\infty} \left( \frac{|B|}{|2^{k+1}B|} \right)^{\delta(1-\kappa)/p} \left( 1 + \frac{2^{k+1}r}{\rho(x_0)} \right)^{-N+\theta+\theta'+\eta(1-\kappa)/p}.
\]
Thus, by choosing $N$ large enough so that $N > \theta + \theta' + \eta(1-\kappa)/p$, we then have
\[
I_2 \leq C\|f\|_{L_{p',\theta}(w)} \left( 1 + \frac{r}{\rho(x_0)} \right)^{\frac{N-N_0}{N_0+1}} \sum_{k=1}^{\infty} \left( \frac{|B|}{|2^{k+1}B|} \right)^{\delta(1-\kappa)/p}
\leq C \left( 1 + \frac{r}{\rho(x_0)} \right)^{\frac{N-N_0}{N_0+1}}.
\]
Summing up the above estimates for $I_1$ and $I_2$ and letting $\vartheta = \max \{ \vartheta', N \cdot \frac{N}{N_0+1} \}$, we obtain our desired inequality (3.1). This completes the proof of Theorem 2.3.

Proof of Theorem 2.4. We denote by $\mathcal{T}$ either $\mathcal{R}$ or $\mathcal{R}^\ast$. To prove Theorem 2.4, by definition, it suffices to prove that for any given ball $B = B(x_0, r)$ of $\mathbb{R}^d$, there is some $\vartheta > 0$ such that

$$
(3.6) \quad \frac{1}{w(B)^\kappa} \sup_{\lambda > 0} \lambda \cdot w(\{ x \in B : |\mathcal{T} f(x)| > \lambda \}) \leq C \cdot \left( 1 + \frac{r}{\rho(x_0)} \right)^{\vartheta}
$$

holds for any $f \in L^1_{\rho, \infty}(w)$ with $0 < \kappa < 1$. Now suppose that $f \in L^1_{\rho, \kappa}(w)$ for some $\vartheta > 0$ and $w \in A^\vartheta_{\kappa}$ for some $\vartheta' > 0$. We decompose $f$ as

$$
\begin{align*}
\begin{cases}
  f = f_1 + f_2 \in L^1_{\rho, \kappa}(w); \\
  f_1 = f \cdot \mathcal{A}_2 B; \\
  f_2 = f \cdot \mathcal{A}_2 B^{\kappa}.
\end{cases}
\end{align*}
$$

Then for any given $\lambda > 0$, by the linearity of $\mathcal{T}$, we can write

$$
\frac{1}{w(B)^\kappa} \lambda \cdot w(\{ x \in B : |\mathcal{T} f(x)| > \lambda \}) \leq \frac{1}{w(B)^\kappa} \lambda \cdot w(\{ x \in B : |\mathcal{T} f_1(x)| > \lambda/2 \})
$$

$$
+ \frac{1}{w(B)^\kappa} \lambda \cdot w(\{ x \in B : |\mathcal{T} f_2(x)| > \lambda/2 \})
$$

$$
= I'_1 + I'_2.
$$

We first give the estimate for the term $I'_1$. By Theorem 1.6, we get

$$
I'_1 = \frac{1}{w(B)^\kappa} \lambda \cdot w(\{ x \in B : |\mathcal{T} f_1(x)| > \lambda/2 \})
$$

$$
\leq C \cdot \frac{1}{w(B)^\kappa} \left( \int_{\mathbb{R}^d} |f_1(x)| w(x) \, dx \right)
$$

$$
= C \cdot \frac{1}{w(B)^\kappa} \left( \int_{2B} |f_1(x)| w(x) \, dx \right)
$$

$$
\leq C \| f \|_{L^1_{\rho, \kappa}(w)} \cdot \frac{w(2B)^\kappa}{w(B)^\kappa} \cdot \left( 1 + \frac{2r}{\rho(x_0)} \right)^{\vartheta}.
$$

Since $w \in A^\vartheta_{\kappa}$ with $0 < \vartheta' < \infty$, similar to the proof of (3.2), we can also show the following estimate as well.

$$
(3.7) \quad w(2B(x_0, r)) \leq C \cdot \left( 1 + \frac{2r}{\rho(x_0)} \right)^{\vartheta'} w(B(x_0, r)).
$$

In fact, by the definition of $A^\vartheta_{\kappa}$, we can deduce that

$$
\frac{1}{|2B|} \int_{2B} |h(x)| \, dx \leq \frac{C_w}{w(2B)} \cdot \text{ess inf}_{x \in 2B} w(x) \left( \int_{2B} |h(x)| \, dx \right) \left( 1 + \frac{2r}{\rho(x_0)} \right)^{\vartheta'}
$$

$$
\leq \frac{C_w}{w(2B)} \left( \int_{2B} |h(x)| w(x) \, dx \right) \left( 1 + \frac{2r}{\rho(x_0)} \right)^{\vartheta'}.
$$
If we choose $h(x) = \chi_B(x)$, then the above expression becomes

$$\left| \frac{|B|}{2|B|} \right| \leq C_w \cdot \frac{w(B)}{w(2B)} \left( 1 + \frac{2r}{\rho(x_0)} \right)^{\vartheta'},$$

which in turn implies (3.7). Therefore,

$$I_1' \leq C \cdot \left( 1 + \frac{2r}{\rho(x_0)} \right)^{\vartheta'} \cdot \left( 1 + \frac{2r}{\rho(x_0)} \right)^{\vartheta} \leq C \cdot \left( 1 + \frac{r}{\rho(x_0)} \right)^{\vartheta'},$$

where $\vartheta' := \vartheta' \cdot \kappa + \vartheta$. As for the other term $I_2'$, by using the pointwise inequality (3.5) and Chebyshev’s inequality, we deduce that

$$I_2' = \frac{1}{w(B)\kappa} \cdot w \left( \left\{ x \in B : |Tf_2(x)| > \lambda/2 \right\} \right)$$

$$\leq \frac{2}{w(B)\kappa} \left( \int_B |Tf_2(x)| w(x) \, dx \right)$$

$$\leq C \cdot \frac{w(B)}{w(2^k B)\kappa} \sum_{k=1}^\infty \frac{1}{2^k B} \int_{2^k B} \left( 1 + \frac{r}{\rho(x_0)} \right)^{\vartheta'} \left( 1 + \frac{2^{k+1}r}{\rho(x_0)} \right)^{-N} |f(y)| \, dy.$$

Moreover, by the $A_1^{\vartheta', \vartheta'}$ condition on $w$, we compute

$$\frac{1}{|2^{k+1} B|} \int_{2^{k+1} B} |f(y)| \, dy$$

$$\leq \frac{C_w}{w(2^{k+1} B)\kappa} \cdot \underset{y \in 2^{k+1} B}{\text{ess inf}} w(y) \left( \int_{2^{k+1} B} |f(y)| \, dy \right) \left( 1 + \frac{2^{k+1}r}{\rho(x_0)} \right)^{\vartheta'}$$

$$\leq \frac{C_w}{w(2^{k+1} B)} \left( \int_{2^{k+1} B} |f(y)| w(y) \, dy \right) \left( 1 + \frac{2^{k+1}r}{\rho(x_0)} \right)^{\vartheta'}$$

$$\leq C \| f \|_{L_p^{1, \vartheta'}(w)} \cdot \frac{w(2^{k+1} B)\kappa}{w(2^{k+1} B)\kappa} \left( 1 + \frac{2^{k+1}r}{\rho(x_0)} \right)^{\vartheta'} \left( 1 + \frac{2^{k+1}r}{\rho(x_0)} \right)^{-N + \vartheta' + \vartheta}.$$

Consequently,

$$I_2' \leq C \| f \|_{L_p^{1, \vartheta'}(w)} \cdot \frac{w(B)}{w(B)\kappa} \sum_{k=1}^\infty \frac{w(2^{k+1} B)\kappa}{w(2^{k+1} B)\kappa} \left( 1 + \frac{r}{\rho(x_0)} \right)^{N \cdot \frac{N}{N_0 + 1}} \left( 1 + \frac{2^{k+1}r}{\rho(x_0)} \right)^{-N + \vartheta' + \vartheta'}$$

$$= C \| f \|_{L_p^{1, \vartheta'}(w)} \left( 1 + \frac{r}{\rho(x_0)} \right)^{N \cdot \frac{N}{N_0 + 1}} \sum_{k=1}^\infty \frac{w(B)\kappa}{w(2^{k+1} B)\kappa} \left( 1 + \frac{2^{k+1}r}{\rho(x_0)} \right)^{-N + \vartheta' + \vartheta'}.$$
Recall that $w \in A_{1,\theta'}^p$ with $0 < \theta' < \infty$, then there exist two positive numbers $\delta', \eta' > 0$ such that (1.7) holds. Therefore,

$$I_2' \leq C\|f\|_{L^1_{\rho,\theta}(w)} \left(1 + \frac{r}{\rho(x_0)}\right)^{N\frac{N_0}{N_0+1}} \sum_{k=1}^{\infty} \left(\frac{|B|}{|2^{k+1}B|}\right)^{\delta'(1-\kappa)} \times \left(1 + \frac{2^{k+1}r}{\rho(x_0)}\right)^{\eta'(1-\kappa)}$$

$$= C\|f\|_{L^1_{\rho,\theta}(w)} \left(1 + \frac{r}{\rho(x_0)}\right)^{N\frac{N_0}{N_0+1}} \times \sum_{k=1}^{\infty} \left(\frac{|B|}{|2^{k+1}B|}\right)^{\delta'(1-\kappa)} \left(1 + \frac{2^{k+1}r}{\rho(x_0)}\right)^{-N+\theta+\theta'+\eta'(1-\kappa)}.$$

By selecting $N$ large enough so that $N > \theta + \theta' + \eta'(1-\kappa)$, we thus have

$$I_2' \leq C \left(1 + \frac{r}{\rho(x_0)}\right)^{N\frac{N_0}{N_0+1}} \sum_{k=1}^{\infty} \left(\frac{|B|}{|2^{k+1}B|}\right)^{\delta'(1-\kappa)} \left(1 + \frac{2^{k+1}r}{\rho(x_0)}\right)^{\eta'(1-\kappa)}.$$

Let $\vartheta = \max\{\vartheta', N \cdot \frac{N_0}{N_0+1}\}$. Here $N$ is an appropriate constant. Summing up the above estimates for $I_1'$ and $I_2'$, and then taking the supremum over all $\lambda > 0$, we obtain our desired inequality (3.6). This finishes the proof of Theorem 2.4.

4. Proofs of Theorems 2.5 and 2.7

For the results involving commutators, we need the following properties of $\text{BMO}_{\rho,\infty}(\mathbb{R}^d)$ functions, which are extensions of well-known properties of $\text{BMO}(\mathbb{R}^d)$ functions.

**Lemma 4.1.** If $b \in \text{BMO}_{\rho,\infty}(\mathbb{R}^d)$ and $w \in A_{p,\infty}^p$ with $1 \leq p < \infty$, then there exist positive constants $C > 0$ and $\mu > 0$ such that for every ball $B = B(x_0, r)$ in $\mathbb{R}^d$, we have

$$(4.1) \quad \left(\int_B |b(x) - b_B|^p w(x) \, dx\right)^{1/p} \leq C \cdot w(B)^{1/p} \left(1 + \frac{r}{\rho(x_0)}\right)^\mu,$$

where $b_B = \frac{1}{|B|} \int_B b(y) \, dy$. 

Proof. We may assume that \( b \in \text{BMO}_{\rho,\theta}(\mathbb{R}^d) \) with \( 0 < \theta < \infty \). According to Proposition 1.8 we can deduce that

\[
\left( \int_B |b(x) - b_B|^p w(x) \, dx \right)^{1/p} \\
= \left( \int_0^\infty p\lambda^{p-1} \left\{ x \in B : |b(x) - b_B| > \lambda \right\} \, d\lambda \right)^{1/p} \\
\leq C_1^{1/p} \cdot w(B)^{1/p} \left\{ \int_0^\infty p\lambda^{p-1} \exp \left[ - \left( 1 + \frac{r}{\rho(x_0)} \right)^{-\theta^*} \frac{C_2\lambda}{\|b\|_{\text{BMO}_{\rho,\theta}}} \right] \left( 1 + \frac{r}{\rho(x_0)} \right)^{\eta/p} \, d\lambda \right\}^{1/p} \\
= C_1^{1/p} \cdot w(B)^{1/p} \left\{ \int_0^\infty p\lambda^{p-1} \exp \left[ - \left( 1 + \frac{r}{\rho(x_0)} \right)^{-\theta^*} \frac{C_2\lambda}{\|b\|_{\text{BMO}_{\rho,\theta}}} \right] \, d\lambda \right\}^{1/p} \left( 1 + \frac{r}{\rho(x_0)} \right)^{\eta/p}.
\]

Making change of variables, then we get

\[
\left( \int_B |b(x) - b_B|^p w(x) \, dx \right)^{1/p} \\
\leq C_1^{1/p} \cdot w(B)^{1/p} \left( \int_0^\infty p\lambda^{p-1} e^{-s} \, ds \right)^{1/p} \left( \frac{\|b\|_{\text{BMO}_{\rho,\theta}}}{C_2} \right) \left( 1 + \frac{r}{\rho(x_0)} \right)^{\theta^*+\eta/p} \\
= [C_1p\Gamma(p)]^{1/p} \left( \frac{\|b\|_{\text{BMO}_{\rho,\theta}}}{C_2} \right) \cdot w(B)^{1/p} \left( 1 + \frac{r}{\rho(x_0)} \right)^{\theta^*+\eta/p},
\]

which yields the desired inequality if we choose \( C = [C_1p\Gamma(p)]^{1/p} \left( \frac{\|b\|_{\text{BMO}_{\rho,\theta}}}{C_2} \right) \) and \( \mu = \theta^* + \eta/p \). \( \square \)

**Lemma 4.2.** If \( b \in \text{BMO}_{\rho,\theta}(\mathbb{R}^d) \) with \( 0 < \theta < \infty \) and \( w \in A_1^{p,\infty} \), then there exist positive constants \( C, \gamma > 0 \) and \( \eta > 0 \) such that for every ball \( B = B(x_0, r) \) in \( \mathbb{R}^d \), we have

\[
(4.2) \quad \left( \int_B \left\{ \exp \left[ \left( 1 + \frac{r}{\rho(x_0)} \right)^{-\theta^*} \frac{\gamma}{\|b\|_{\text{BMO}_{\rho,\theta}}} |b(x) - b_B| \right] - 1 \right\} w(x) \, dx \right) \\
\leq C \cdot w(B) \left( 1 + \frac{r}{\rho(x_0)} \right)^{\eta},
\]

where \( b_B = \frac{1}{|B|} \int_B b(y) \, dy \) and \( \theta^* = (N_0 + 1)\theta \) and \( N_0 \) is the constant appearing in Lemma 1.1.

**Proof.** Recall the following identity (see Proposition 1.1.4 in [4])

\[
\left( \int_B \left\{ \exp \left[ f(x) \right] - 1 \right\} w(x) \, dx \right) = \int_0^\infty e^{\lambda} w\left( \{ x \in B : |f(x)| > \lambda \} \right) \, d\lambda.
\]
Using this identity and Proposition 1.8, we obtain
\[
\left( \int_B \left\{ \exp \left[ \left(1 + \frac{r}{\rho(x_0)} \right)^{-\theta^*} \gamma \frac{\|b\|_{BMO,\rho,\theta}}{b(x) - b_B} \right] - 1 \right\} w(x) \, dx \right) = \int_0^\infty e^{\lambda} w \left( \{ x \in B : |b(x) - b_B| > \lambda^* \} \right) d\lambda
\]
\[
\leq C_1 \cdot w(B) \int_0^\infty e^{\lambda} \exp \left[ - \left(1 + \frac{r}{\rho(x_0)} \right)^{-\theta^*} C_2 \lambda^* \right] d\lambda \left(1 + \frac{r}{\rho(x_0)} \right)^{\eta}
\]
\[
= C_1 \cdot w(B) \int_0^\infty e^{\lambda} \cdot e^{- C_2 \lambda \gamma} d\lambda \left(1 + \frac{r}{\rho(x_0)} \right)^{\eta},
\]
where \( \lambda^* \) is given by
\[
\lambda^* = \frac{\lambda \|b\|_{BMO,\rho,\theta}}{\gamma} \left(1 + \frac{r}{\rho(x_0)} \right)^{\theta^*}.
\]
If we take \( \gamma \) small enough so that \( 0 < \gamma < C_2 \), then the conclusion follows immediately.

**Lemma 4.3.** If \( b \in BMO_{\rho,\theta}(\mathbb{R}^d) \) with \( 0 < \theta < \infty \), then for any positive integer \( k \), there exists a positive constant \( C > 0 \) such that for every ball \( B = B(x_0, r) \) in \( \mathbb{R}^d \), we have
\[
|b_{2^{k+1}B} - b_B| \leq C \cdot (k + 1) \left(1 + \frac{2^{k+1}r}{\rho(x_0)} \right)^{\theta}.
\]

**Proof.** For any positive integer \( k \), we have
\[
|b_{2^{k+1}B} - b_B| \leq \sum_{j=1}^{k+1} |b_{2^jB} - b_{2^{j-1}B}|
\]
\[
= \sum_{j=1}^{k+1} \left| \frac{1}{|2^jB|} \int_{2^{j-1}B} [b_{2^jB} - b(y)] \, dy \right|
\]
\[
\leq \sum_{j=1}^{k+1} \frac{2^d}{|2^jB|} \int_{2^jB} |b(y) - b_{2^jB}| \, dy
\]
\[
\leq C_{b,d} \|b\|_{BMO,\rho,\theta} \sum_{j=1}^{k+1} \left(1 + \frac{2^j r}{\rho(x_0)} \right)^{\theta}.
\]
Since for any \( 1 \leq j \leq k + 1 \), the following estimate
\[
\left(1 + \frac{2^j r}{\rho(x_0)} \right)^{\theta} \leq \left(1 + \frac{2^{k+1} r}{\rho(x_0)} \right)^{\theta},
\]
holds trivially, and hence
\[
|b_{2^{k+1}B} - b_B| \leq C \sum_{j=1}^{k+1} \left(1 + \frac{2^{k+1} r}{\rho(x_0)} \right)^{\theta} = C \cdot (k + 1) \left(1 + \frac{2^{k+1} r}{\rho(x_0)} \right)^{\theta}.
\]
We obtain the desired result. This completes the proof.

Now, we are in a position to prove our main results in this section.

Proof of Theorem 2.5. We denote by \([b, T]\) either \([b, R]\) or \([b, R^*]\). By definition, we only need to show that for any given ball \(B = B(x_0, r)\) of \(\mathbb{R}^d\), there is some \(\vartheta > 0\) such that

\[
\left( \frac{1}{w(B)^\kappa} \int_B [b, T] f(x)^p w(x) \, dx \right)^{1/p} \leq C \cdot \left( 1 + \frac{r}{\rho(x_0)} \right)^\vartheta
\]

holds for any \(f \in L^{p,\kappa}_{\rho,\kappa}(w)\) with \(1 < p < \infty\) and \(0 < \kappa < 1\), whenever \(b\) belongs to \(\text{BMO}_{\rho,\infty}(\mathbb{R}^d)\). Suppose that \(f \in L^{p,\kappa}_{\rho,\theta}(w)\) for some \(\theta > 0\), \(w \in A^\rho_{\theta'}\) for some \(\theta' > 0\) as well as \(b \in \text{BMO}_{\rho,\theta''}(\mathbb{R}^d)\) for some \(\theta'' > 0\). We decompose \(f\) as

\[
\begin{cases}
  f = f_1 + f_2 \\
  f_1 = f \cdot \chi_{2B} \\
  f_2 = f \cdot \chi_{(2B)^c}.
\end{cases}
\]

Then by the linearity of \([b, T]\), we write

\[
\left( \frac{1}{w(B)^\kappa} \int_B [b, T] f(x)^p w(x) \, dx \right)^{1/p} \leq \left( \frac{1}{w(B)^\kappa} \int_B [b, T] f_1(x)^p w(x) \, dx \right)^{1/p} + \left( \frac{1}{w(B)^\kappa} \int_B [b, T] f_2(x)^p w(x) \, dx \right)^{1/p} := J_1 + J_2.
\]

Now we give the estimates for \(J_1\), \(J_2\), respectively. According to Theorem 1.9 we have

\[
J_1 \leq C \cdot \frac{1}{w(B)^{\kappa/p}} \left( \int_{\mathbb{R}^d} |f_1(x)|^p w(x) \, dx \right)^{1/p} = C \cdot \frac{1}{w(B)^{\kappa/p}} \left( \int_{2B} |f(x)|^p w(x) \, dx \right)^{1/p} \leq C \|f\|_{L^{p,\kappa}_{\rho,\theta}(w)} \cdot \frac{w(2B)^{\kappa/p}}{w(B)^{\kappa/p}} \cdot \left( 1 + \frac{2r}{\rho(x_0)} \right)^\vartheta.
\]

Moreover, in view of the inequality (3.2), we get

\[
J_1 \leq C \|f\|_{L^{p,\kappa}_{\rho,\theta}(w)} \cdot \left( 1 + \frac{2r}{\rho(x_0)} \right)^{(\rho \theta')(\kappa/p)} \cdot \left( 1 + \frac{2r}{\rho(x_0)} \right)^\vartheta \leq C \cdot \left( 1 + \frac{r}{\rho(x_0)} \right)^\vartheta',
\]
where \( \vartheta' := \vartheta' \cdot \kappa + \vartheta \). On the other hand, by the definition (1.8), we can see that for any \( x \in B(x_0, r) \),

\[
(4.4)
\]

\[
\| [b, \mathcal{T}] f_2 (x) \| \leq \int_{\mathbb{R}^d} |b(x) - b(y)| |\mathcal{K}(x, y) f_2 (y)| \, dy \quad \text{(or } \mathcal{K}^*(x, y) \text{)}
\]

\[
\leq |b(x) - b_B| \int_{\mathbb{R}^d} |\mathcal{K}(x, y) f_2 (y)| \, dy + \int_{\mathbb{R}^d} |b(y) - b_B| |\mathcal{K}(x, y) f_2 (y)| \, dy
\]

\[
:= \xi(x) + \zeta(x).
\]

So we can divide \( J_2 \) into two parts:

\[
J_2 \leq \left( \frac{1}{w(B)^\kappa} \int_B |\xi(x)|^p w(x) \, dx \right)^{1/p} + \left( \frac{1}{w(B)^\kappa} \int_B |\zeta(x)|^p w(x) \, dx \right)^{1/p}
\]

\[
:= J_3 + J_4.
\]

From the pointwise estimate (3.5) and (4.1) in Lemma 4.1, it then follows that

\[
J_3 \leq C \cdot \frac{1}{w(B)^{\kappa/p}} \left( \int_B |b(x) - b_B|^p w(x) \, dx \right)^{1/p}
\]

\[
\times \sum_{k=1}^{\infty} \frac{1}{|2^{k+1} B|} \int_{2^{k+1} B} \left( 1 + \frac{r}{\rho(x_0)} \right)^{N - \frac{N_0}{N_0 + 1}} \left( 1 + \frac{2^{k+1} r}{\rho(x_0)} \right)^{-N} |f(y)| \, dy
\]

\[
\leq C_b \cdot \frac{w(B)^{1/p}}{w(B)^{\kappa/p}} \left( 1 + \frac{r}{\rho(x_0)} \right)^\mu
\]

\[
\times \sum_{k=1}^{\infty} \frac{1}{|2^{k+1} B|} \int_{2^{k+1} B} \left( 1 + \frac{r}{\rho(x_0)} \right)^{N - \frac{N_0}{N_0 + 1}} \left( 1 + \frac{2^{k+1} r}{\rho(x_0)} \right)^{-N} |f(y)| \, dy.
\]

Following along the same lines as that of Theorem 2.3 we are able to show that

\[
J_3 \leq C \| f \|_{L^{p, \kappa}_\rho(w)} \left( 1 + \frac{r}{\rho(x_0)} \right)^\mu \left( 1 + \frac{r}{\rho(x_0)} \right)^{N - \frac{N_0}{N_0 + 1}}
\]

\[
\times \sum_{k=1}^{\infty} \frac{w(B)^{(1-\kappa)/p}}{w(2^{k+1} B)^{(1-\kappa)/p}} \left( 1 + \frac{2^{k+1} r}{\rho(x_0)} \right)^{-N+\vartheta+\vartheta'}
\]

\[
\leq C \| f \|_{L^{p, \kappa}_\rho(w)} \left( 1 + \frac{r}{\rho(x_0)} \right)^{\mu+N - \frac{N_0}{N_0 + 1}}
\]

\[
\times \sum_{k=1}^{\infty} \frac{|B|^{\delta(1-\kappa)/p}}{|2^{k+1} B|^{\delta(1-\kappa)/p}} \left( 1 + \frac{2^{k+1} r}{\rho(x_0)} \right)^{-N+\vartheta+\vartheta'+\eta(1-\kappa)/p}.
\]

The last inequality is obtained by using (1.7). For any \( x \in B(x_0, r) \) and any positive integer \( N \), similar to the proof of (3.4) and (3.5), we can also
deduce that

\[ (4.5) \]

\[ \zeta(x) = \int_{(2B)^c} |b(y) - b_B||K(x, y) f(y)| \, dy \]
\[ \leq C_N \int_{(2B)^c} |b(y) - b_B| \left( 1 + \frac{|x - y|}{\rho(x)} \right)^{-N} \frac{1}{|x - y|^d} \cdot |f(y)| \, dy \]
\[ \leq C_{N,d} \sum_{k=1}^{\infty} \int_{2B \setminus 2B} |b(y) - b_B| \left( 1 + \frac{|x_0 - y|}{\rho(x_0)} \right)^{-N} \frac{1}{|x_0 - y|^d} \cdot |f(y)| \, dy \]
\[ \leq C_{N,d} \sum_{k=1}^{\infty} \frac{1}{2k+1} \int_{2B \setminus 2B} |b(y) - b_B| \left( 1 + \frac{2k^r}{\rho(x_0)} \right)^{-N} \frac{1}{\rho(x_0)} f(y) \, dy \]
\[ \leq C \sum_{k=1}^{\infty} \frac{1}{2k+1} \int_{2B \setminus 2B} |b(y) - b_B| \left( 1 + \frac{2k+1}{\rho(x_0)} \right)^{-N} |f(y)| \, dy, \]

where in the last inequality we have used (1.3) in Lemma 1.1. Hence, by the above pointwise estimate for \( \zeta(x) \),

\[ \int_{2B} |b(y) - b_B||f(y)| \, dy \]
\[ \leq C \cdot w(B)^{(1-\kappa)/p} \sum_{k=1}^{\infty} \left( 1 + \frac{r}{\rho(x_0)} \right)^{N_{0+1}} \left( 1 + \frac{2k+1}{\rho(x_0)} \right)^{-N} \]
\[ \times \frac{1}{2k+1} \int_{2B \setminus 2B} |b(y) - b_B||f(y)| \, dy. \]

Moreover, for each integer \( k \geq 1 \),

\[ (4.6) \]
\[ \int_{2B \setminus 2B} |b(y) - b_B||f(y)| \, dy \]
\[ \leq \frac{1}{2k+1} \int_{2B \setminus 2B} |b(y) - b_{2k+1B}||f(y)| \, dy \]
\[ \leq 1 \int_{2k+1B} |b(y) - b_{2k+1B}||f(y)| \, dy \]
\[ \leq 1 \int_{2k+1B} |b_{2k+1B} - b_B||f(y)| \, dy. \]

By using Hölder’s inequality, the first term of the expression (4.6) is bounded by

\[ \frac{1}{2k+1} \left( \int_{2k+1B} |f(y)|^p w(y) \, dy \right)^{1/p} \left( \int_{2k+1B} |b(y) - b_{2k+1B}|^{p'} w(y)^{-p'/p} \, dy \right)^{1/p'} \]
\[ \leq C \|f\|_{L_{p', \theta}^{k/p} \cdot w(2k+1B)^{k'/p}} \cdot \frac{w(2k+1B)^{-k/p}}{[2k+1B]} \left( 1 + \frac{2k+1}{\rho(x_0)} \right)^{\theta} \left( \int_{2k+1B} |b(y) - b_{2k+1B}|^{p'} w(y)^{-p'/p} \, dy \right)^{1/p'}. \]

Since \( w \in A_{p}^{p', \theta} \) with \( 0 < \theta' < \infty \) and \( 1 < p < \infty \), then by the definition of \( A_{p}^{p', \theta} \), it can be easily shown that \( w \in A_{p}^{p', \theta} \) if and only if \( w^{-p'/p} \in A_{p}^{p', \theta} \),
where \( 1/p + 1/p' = 1 \) (see [10]). If we denote \( v = w^{-p'/p} \), then \( v \in A_{p'}^{p', \theta} \).
This fact together with Lemma 4.1 implies
\[
\left( \int_{2^{k+1}B} |b(y) - b_{2^{k+1}B}|^{p'} v(y) \, dy \right)^{1/p'} \\
\leq C_b \cdot v(2^{k+1}B)^{1/p'} \left( 1 + \frac{2^{k+1}r}{\rho(x_0)} \right)^\mu \\
= C_b \cdot \left( \int_{2^{k+1}B} w(y)^{-p'/p} \, dy \right)^{1/p'} \left( 1 + \frac{2^{k+1}r}{\rho(x_0)} \right)^\mu \\
\leq C_{b,w} \cdot \frac{2^{k+1}|B|}{w(2^{k+1}B)^{1/p}} \left( 1 + \frac{2^{k+1}r}{\rho(x_0)} \right)^{\theta' + \theta'' + \mu}.
\]

Therefore, the first term of the expression (4.6) can be bounded by a constant times
\[
\frac{w(2^{k+1}B)^{\kappa/p}}{w(2^{k+1}B)^{1/p}} \left( 1 + \frac{2^{k+1}r}{\rho(x_0)} \right)^{\theta' + \theta'' + \mu}.
\]

Since \( b \in \text{BMO}_{\rho,\theta''}(\mathbb{R}^d) \) with \( 0 < \theta'' < \infty \), then by Lemma 4.3, Hölder’s inequality and the \( A_{p,\theta'}^\rho \) condition on \( w \), the latter term of the expression (4.6) can be estimated by
\[
C_b(k+1) \left( 1 + \frac{2^{k+1}r}{\rho(x_0)} \right)^{\theta''} \frac{1}{|2^{k+1}B|} \int_{2^{k+1}B} |f(y)| \, dy \\
\leq C_b(k+1) \left( 1 + \frac{2^{k+1}r}{\rho(x_0)} \right)^{\theta''} \frac{1}{|2^{k+1}B|} \left( \int_{2^{k+1}B} |f(y)|^p w(y) \, dy \right)^{1/p} \left( \int_{2^{k+1}B} w(y)^{-p'/p} \, dy \right)^{1/p'} \\
\leq C \| f \|_{L_{\rho,\theta''}^p(w)} \cdot (k+1) \frac{w(2^{k+1}B)^{\kappa/p}}{w(2^{k+1}B)^{1/p}} \left( 1 + \frac{2^{k+1}r}{\rho(x_0)} \right)^{\theta' + \theta'' + \mu}.
\]

Consequently,
\[
\frac{1}{|2^{k+1}B|} \int_{2^{k+1}B} |b(y) - b_B||f(y)| \, dy \\
\leq C \| f \|_{L_{\rho,\theta''}^p(w)} \cdot (k+1) \frac{w(2^{k+1}B)^{\kappa/p}}{w(2^{k+1}B)^{1/p}} \left( 1 + \frac{2^{k+1}r}{\rho(x_0)} \right)^{\theta' + \theta'' + \mu}.
\]

Thus, in view of (4.7),
\[
J_4 \leq C \| f \|_{L_{\rho,\theta''}^p(w)} \cdot w(B)^{(1-\kappa)/p} \sum_{k=1}^{\infty} (k+1) \left( 1 + \frac{r}{\rho(x_0)} \right)^{N \cdot \frac{N_0}{N_0+1}} \left( 1 + \frac{2^{k+1}r}{\rho(x_0)} \right)^{-N} \\
\times \frac{1}{w(2^{k+1}B)^{(1-\kappa)/p}} \left( 1 + \frac{2^{k+1}r}{\rho(x_0)} \right)^{\theta' + \theta'' + \mu} \\
= C \| f \|_{L_{\rho,\theta''}^p(w)} \left( 1 + \frac{r}{\rho(x_0)} \right)^{N \cdot \frac{N_0}{N_0+1}} \sum_{k=1}^{\infty} (k+1) \frac{w(B)^{(1-\kappa)/p}}{w(2^{k+1}B)^{(1-\kappa)/p}} \left( 1 + \frac{2^{k+1}r}{\rho(x_0)} \right)^{-N + \theta' + \theta'' + \mu}.
\]
Notice that $w \in A_{p,\theta'}^\rho$ with $0 < \theta' < \infty$. A further application of (1.7) yields

$$J_4 \leq C \|f\|_{L_{p,\rho}^{\theta}(w)} \left( 1 + \frac{r}{\rho(x_0)} \right)^{N - \frac{N_0}{N_0 + 1}} \sum_{k=1}^{\infty} (k + 1) \left( \frac{|B|}{2^{k+1}B} \right)^{(1 - \kappa)/p} \times \left( 1 + \frac{2^{k+1}r}{\rho(x_0)} \right)^{- \eta(1 - \kappa)/p} \left( 1 + \frac{2^{k+1}r}{\rho(x_0)} \right)^{- N + \theta + \theta' + \theta'' + \mu} \times \left( 1 + \frac{2^{k+1}r}{\rho(x_0)} \right)^{- N + \theta + \theta' + \theta'' + \mu + \eta(1 - \kappa)/p}.$$ 

Combining the above estimates for $J_3$ and $J_4$, we get

$$J_2 \leq J_3 + J_4 \leq C \|f\|_{L_{p,\rho}^{\theta}(w)} \left( 1 + \frac{r}{\rho(x_0)} \right)^{\mu + N - \frac{N_0}{N_0 + 1}} \times \sum_{k=1}^{\infty} (k + 1) \left( \frac{|B|}{2^{k+1}B} \right)^{(1 - \kappa)/p} \left( 1 + \frac{2^{k+1}r}{\rho(x_0)} \right)^{- N + \theta + \theta' + \theta'' + \mu + \eta(1 - \kappa)/p}.$$ 

By choosing $N$ large enough so that $N > \theta + \theta' + \theta'' + \mu + \eta(1 - \kappa)/p$, we thus have

$$J_2 \leq C \left( 1 + \frac{r}{\rho(x_0)} \right)^{\mu + N - \frac{N_0}{N_0 + 1}} \sum_{k=1}^{\infty} (k + 1) \left( \frac{|B|}{2^{k+1}B} \right)^{(1 - \kappa)/p} \leq C \left( 1 + \frac{r}{\rho(x_0)} \right)^{\mu + N - \frac{N_0}{N_0 + 1}}.$$ 

Finally, collecting the above estimates for $J_1$, $J_2$, and letting $\vartheta = \max \{ \vartheta', \mu + N - \frac{N_0}{N_0 + 1} \}$, we obtain the desired result (4.3). The proof of Theorem 2.5 is finished.

**Proof of Theorem 2.7.** We denote by $[b, \mathcal{T}]$ either $[b, \mathcal{R}]$ or $[b, \mathcal{R}^+]$. We are going to prove that for any given $\lambda > 0$ and any given ball $B = B(x_0, r)$ of $\mathbb{R}^d$, there is some $\vartheta > 0$ such that the following inequality

$$\frac{1}{w(B)^{\kappa}} w(\{x \in B : \|[b, \mathcal{T}]f(x)\| > \lambda\}) \leq C \left( 1 + \frac{r}{\rho(x_0)} \right)^{\vartheta} \left\| \Phi\left( \frac{|f|}{\lambda} \right) \right\|_{(L \log L)_{p,\theta}^{1,\kappa}(w)}$$

holds for those functions $f$ such that $\Phi(|f|) \in (L \log L)_{p,\theta}^{1,\kappa}(w)$ with some fixed $\theta > 0$. Now assume that $w \in A_{1,\theta'}^{\rho}$ for some $\theta' > 0$ and $b \in \text{BMO}_{p,\theta''}(\mathbb{R}^d)$ for some $\theta'' > 0$. As before, we decompose $f$ as

$$f = f_1 + f_2; \quad f_1 = f \cdot \chi_{2B}; \quad f_2 = f \cdot \chi_{(2B)^c}.$$
A further application of (1.14) yields

\[ \vartheta \]

where

\[ \text{Then for any given } \lambda > 0, \text{ by the linearity of } [b, \mathcal{T}], \text{ we can write} \]

\[ \frac{1}{w(B)} \cdot w\left( \left\{ x \in B : |[b, \mathcal{T}]f(x)| > \lambda \right\} \right) \]

\[ \leq \frac{1}{w(B)} \cdot w\left( \left\{ x \in B : |[b, \mathcal{T}](f_1)(x)| > \lambda/2 \right\} \right) \]

\[ + \frac{1}{w(B)} \cdot w\left( \left\{ x \in B : |[b, \mathcal{T}](f_2)(x)| > \lambda/2 \right\} \right) \]

\[ := J'_1 + J'_2. \]

Let us first estimate the term \( J'_1 \). By using Theorem 1.10 we get

\[ J'_1 = \frac{1}{w(B)} \cdot w\left( \left\{ x \in B : |[b, \mathcal{T}](f_1)(x)| > \lambda/2 \right\} \right) \]

\[ \leq C \cdot \frac{1}{w(B)} \left[ \int_{\mathbb{R}^d} \Phi \left( \frac{|f_1(x)|}{\lambda} \right) \cdot w(x) \, dx \right] \]

\[ = C \cdot \frac{1}{w(B)} \left[ \int_{2B} \Phi \left( \frac{|f(x)|}{\lambda} \right) \cdot w(x) \, dx \right]. \]

A further application of (1.14) yields

\[ J'_1 \leq C \cdot \frac{w(2B)}{w(B)} \left\| \Phi \left( \frac{|f|}{\lambda} \right) \right\|_{L \log L(w), 2B} \]

\[ \leq C \cdot \frac{w(2B)}{w(B)} \left( 1 + \frac{2r}{\rho(x_0)} \right)^{\vartheta'} \left\| \Phi \left( \frac{|f|}{\lambda} \right) \right\|_{(L \log L)^{1,\kappa}(w)} \]

\[ \leq C \cdot \left( 1 + \frac{2r}{\rho(x_0)} \right)^{\vartheta'} \cdot \left( 1 + \frac{2r}{\rho(x_0)} \right)^{\vartheta} \left\| \Phi \left( \frac{|f|}{\lambda} \right) \right\|_{(L \log L)^{1,\kappa}(w)}, \]

where the last inequality is due to (3.7). If we denote \( \vartheta' = \kappa \cdot \vartheta' + \theta \), then

\[ J'_1 \leq C \cdot \left( 1 + \frac{2r}{\rho(x_0)} \right)^{\vartheta'} \left\| \Phi \left( \frac{|f|}{\lambda} \right) \right\|_{(L \log L)^{1,\kappa}(w)} \leq C \cdot \left( 1 + \frac{r}{\rho(x_0)} \right)^{\vartheta'} \Phi \left( \frac{|f|}{\lambda} \right) \left\|_{(L \log L)^{1,\kappa}(w)} \right. \]

as desired. Next let us deal with the term \( J'_2 \). Taking into account of (4.4), we can divide it into two parts, namely,

\[ J'_2 = \frac{1}{w(B)} \cdot w\left( \left\{ x \in B : |[b, \mathcal{T}](f_2)(x)| > \lambda/2 \right\} \right) \]

\[ \leq \frac{1}{w(B)} \cdot w\left( \left\{ x \in B : |\xi(x)| > \lambda/4 \right\} \right) + \frac{1}{w(B)} \cdot w\left( \left\{ x \in B : |\zeta(x)| > \lambda/4 \right\} \right) \]

\[ := J'_3 + J'_4, \]

where

\[ \xi(x) = |b(x) - b_B| \int_{\mathbb{R}^d} |\mathcal{K}(x, y) f_2(y)| \, dy, \]

\[ \& \zeta(x) = \int_{\mathbb{R}^d} |b(y) - b_B| |\mathcal{K}(x, y) f_2(y)| \, dy. \]
Since \( b \in \text{BMO}_{\rho,\theta}(\mathbb{R}^d) \) for some \( \theta'' > 0 \), from the pointwise inequality (3.5) and Chebyshev’s inequality, we then have
\[
J'_3 \leq \frac{1}{w(B)^{\kappa}} \cdot \frac{4}{\lambda} \left( \int_B |\xi(x)| w(x) \, dx \right)
\leq C \cdot \frac{1}{w(B)^{\kappa}} \left( \int_B |b(x) - b_B| w(x) \, dx \right)
\times \sum_{k=1}^{\infty} \frac{1}{|2^{k+1}B|} \int_{2^{k+1}B} \left( 1 + \frac{r}{\rho(x_0)} \right)^{N \cdot \frac{\theta''}{\theta_0+1}} \left( 1 + \frac{2^{k+1}r}{\rho(x_0)} \right)^{-N \cdot \frac{\theta''}{\theta_0+1}} \frac{|f(y)|}{\lambda} \, dy
\leq C_b \cdot \frac{w(B)}{w(B)^{\kappa}} \left( 1 + \frac{r}{\rho(x_0)} \right)^\mu
\times \sum_{k=1}^{\infty} \frac{1}{|2^{k+1}B|} \int_{2^{k+1}B} \left( 1 + \frac{r}{\rho(x_0)} \right)^{N \cdot \frac{\theta''}{\theta_0+1}} \left( 1 + \frac{2^{k+1}r}{\rho(x_0)} \right)^{-N \cdot \frac{\theta''}{\theta_0+1}} \frac{|f(y)|}{\lambda} \, dy,
\]
where in the last inequality we have used (4.1) in Lemma 4.1. Moreover, it follows directly from the condition \( A_\mu^{1,\theta''} \) that for each integer \( k \geq 1 \),
\[
\frac{1}{|2^{k+1}B|} \int_{2^{k+1}B} \frac{|f(y)|}{\lambda} \, dy
\leq \frac{C_w}{w(2^{k+1}B)} \cdot \text{ess inf}_{y \in 2^{k+1}B} w(y) \left( \int_{2^{k+1}B} \frac{|f(y)|}{\lambda} \, dy \right) \left( 1 + \frac{2^{k+1}r}{\rho(x_0)} \right)^{\theta''}
\leq \frac{C_w}{w(2^{k+1}B)} \left( \int_{2^{k+1}B} \frac{|f(y)|}{\lambda} \, w(y) \, dy \right) \left( 1 + \frac{2^{k+1}r}{\rho(x_0)} \right)^{\theta''}.
\]
Notice also that trivially
\[
(4.9) \quad t \leq t \cdot (1 + \log^+ t) = \Phi(t), \quad \text{for any } t > 0.
\]
This fact along with (1.14) implies that for each integer \( k \geq 1 \),
\[
\frac{1}{|2^{k+1}B|} \int_{2^{k+1}B} \frac{|f(y)|}{\lambda} \, dy
\leq \frac{C_w}{w(2^{k+1}B)} \left( \int_{2^{k+1}B} \Phi \left( \frac{|f(y)|}{\lambda} \right) \cdot w(y) \, dy \right) \left( 1 + \frac{2^{k+1}r}{\rho(x_0)} \right)^{\theta''}
\leq C \cdot \left( 1 + \frac{2^{k+1}r}{\rho(x_0)} \right)^{\theta''} \left\| \Phi \left( \frac{|f|}{\lambda} \right) \right\|_{L(\log L(w)),2^{k+1}B}
\leq C \cdot \frac{1}{w(2^{k+1}B)^{1-\kappa}} \left( 1 + \frac{2^{k+1}r}{\rho(x_0)} \right)^{\theta} \left( 1 + \frac{2^{k+1}r}{\rho(x_0)} \right)^{\theta'} \left\| \Phi \left( \frac{|f|}{\lambda} \right) \right\|_{L(\log L)^{1,\theta''}_{\rho,\theta}(w)}.
\]
Consequently,

\[ J'_3 \leq C \left(1 + \frac{r}{\rho(x_0)}\right)^\mu \left(1 + \frac{r}{\rho(x_0)}\right)^{N\frac{N_0}{N_0+1}} \]
\[ \times \sum_{k=1}^\infty \frac{w(B)^{1-\kappa}}{w(2^{k+1}B)^{1-\kappa}} \left(1 + \frac{2^{k+1}r}{\rho(x_0)}\right)^{-N+\theta+\theta'} \left\| \Phi \left(\frac{|f|}{\lambda}\right) \right\|_{(L \log L)^{1,\kappa}(w)} \]

Since \( w \in A^{1,\theta'}_1 \) with \( 0 < \theta' < \infty \), then there exist two positive numbers \( \delta', \eta' > 0 \) such that (1.7) holds. Therefore,

\[ J'_3 \leq C \left(1 + \frac{r}{\rho(x_0)}\right)^\mu N \frac{N_0}{N_0+1} \]
\[ \times \sum_{k=1}^\infty \left(\frac{|B|}{|2^{k+1}B|}\right)^{\delta'(1-\kappa)} \left(1 + \frac{2^{k+1}r}{\rho(x_0)}\right)^{\eta'(1-\kappa)} \left(1 + \frac{2^{k+1}r}{\rho(x_0)}\right)^{-N+\theta+\theta'} \left\| \Phi \left(\frac{|f|}{\lambda}\right) \right\|_{(L \log L)^{1,\kappa}(w)} \]

On the other hand, it follows from the pointwise inequality (4.5) and Chebyshev’s inequality that

\[ J'_4 \leq \frac{4}{w(B)^\kappa} \left(\int_B |\zeta(x)| w(x) \, dx\right) \]
\[ \leq C \cdot \frac{w(B)}{w(B)^\kappa} \sum_{k=1}^\infty \left(1 + \frac{r}{\rho(x_0)}\right)^{N\frac{N_0}{N_0+1}} \left(1 + \frac{2^{k+1}r}{\rho(x_0)}\right)^{-N} \]
\[ \times \frac{1}{|2^{k+1}B|} \int_{2^{k+1}B} |b(y) - b_B| \frac{|f(y)|}{\lambda} \, dy \]
\[ \leq C \cdot \frac{w(B)}{w(B)^\kappa} \sum_{k=1}^\infty \left(1 + \frac{r}{\rho(x_0)}\right)^{N\frac{N_0}{N_0+1}} \left(1 + \frac{2^{k+1}r}{\rho(x_0)}\right)^{-N} \]
\[ \times \frac{1}{|2^{k+1}B|} \int_{2^{k+1}B} |b(y) - b_B| \Phi \left(\frac{|f(y)|}{\lambda}\right) \, dy, \]
where the last inequality follows from (4.9). Furthermore, by the definition of $A_1^{p',q}$, we compute

\begin{equation}
\frac{1}{|2^{k+1}B|} \int_{2^{k+1}B} |b(y) - b_B| \Phi \left( \frac{|f(y)|}{\lambda} \right) \, dy
\end{equation}

\begin{align*}
&\leq \frac{C_w}{w(2^{k+1}B)} \cdot \text{ess inf}_{y \in 2^{k+1}B} w(y) \int_{2^{k+1}B} |b(y) - b_B| \Phi \left( \frac{|f(y)|}{\lambda} \right) \, dy \left( 1 + \frac{2^{k+1}r}{\rho(x_0)} \right)^{q'} \\
&\leq \frac{C_w}{w(2^{k+1}B)} \int_{2^{k+1}B} |b(y) - b_{2^{k+1}B}| \Phi \left( \frac{|f(y)|}{\lambda} \right) w(y) \, dy \left( 1 + \frac{2^{k+1}r}{\rho(x_0)} \right)^{q'} \\
&\quad + \frac{C_w}{w(2^{k+1}B)} \int_{2^{k+1}B} |b_{2^{k+1}B} - b_B| \Phi \left( \frac{|f(y)|}{\lambda} \right) w(y) \, dy \left( 1 + \frac{2^{k+1}r}{\rho(x_0)} \right)^{q'} .
\end{align*}

By using generalized Hölder inequality (1.13), the first term of the expression (4.10) is bounded by

\begin{equation*}
C \left\| b - b_{2^{k+1}B} \right\|_{\exp L(w), 2^{k+1}B} \left\| \Phi \left( \frac{|f|}{\lambda} \right) \right\|_{L \log L(w), 2^{k+1}B} \left( 1 + \frac{2^{k+1}r}{\rho(x_0)} \right)^{q'}
\end{equation*}

where in the last inequality we have used the fact that

\begin{equation*}
\left\| b - b_B \right\|_{\exp L(w), B} \leq C \left( 1 + \frac{r}{\rho(x_0)} \right)^{q'}, \quad \text{for any ball } B = B(x_0, r) \subset \mathbb{R}^n,
\end{equation*}

which is equivalent to the inequality (1.2) in Lemma 4.2. By Lemma 4.3 and (1.14), the latter term of the expression (4.10) can be estimated by

\begin{equation*}
C_b \frac{(k+1)}{w(2^{k+1}B)} \int_{2^{k+1}B} \Phi \left( \frac{|f(y)|}{\lambda} \right) w(y) \, dy \left( 1 + \frac{2^{k+1}r}{\rho(x_0)} \right)^{q''} \left( 1 + \frac{2^{k+1}r}{\rho(x_0)} \right)^{q'}
\end{equation*}

Consequently,

\begin{equation*}
J'_4 \leq C \cdot \frac{w(B)}{w(B)^{\kappa}} \sum_{k=1}^{\infty} \left( 1 + \frac{r}{\rho(x_0)} \right)^{N \cdot \frac{N_0}{N_{\kappa+1}}} \left( 1 + \frac{2^{k+1}r}{\rho(x_0)} \right)^{-N} \times (k+1) \left\| \Phi \left( \frac{|f|}{\lambda} \right) \right\|_{L \log L(w), 2^{k+1}B} \left( 1 + \frac{2^{k+1}r}{\rho(x_0)} \right)^{q' + q'' + q'''}
\end{equation*}

\begin{equation*}
\leq C \left( 1 + \frac{r}{\rho(x_0)} \right)^{N \cdot \frac{N_0}{N_{\kappa+1}}} \sum_{k=1}^{\infty} (k+1) \frac{w(B)^{(1-\kappa)}}{w(2^{k+1}B)^{(1-\kappa)}} \left\| \Phi \left( \frac{|f|}{\lambda} \right) \right\|_{(L \log L)^{1,\kappa}(w)}
\end{equation*}
\[ \times \left(1 + \frac{2^{k+1}r}{\rho(x_0)}\right)^{-N + \eta' + \theta + \theta' + \theta''} \leq C \left(1 + \frac{r}{\rho(x_0)}\right)^{N - N_0 + 1} \sum_{k=1}^{\infty} (k + 1) \left(\frac{|B|}{2^{k+1}B}\right)^{\delta'(1-\kappa)} \times \left(1 + \frac{2^{k+1}r}{\rho(x_0)}\right)^{-N + \eta' + \theta + \theta' + \theta'' + \eta'(1-\kappa)} \| \Phi \left(\frac{|f|}{\lambda}\right) \|_{(L \log L)^{1, \tilde{b}, \theta}(w)}. \]

Hence, combining the above estimates for \( J'_3 \) and \( J'_4 \), we have
\[
J'_2 \leq J'_3 + J'_4 \leq C \left(1 + \frac{r}{\rho(x_0)}\right)^{\mu + N - N_0 + 1} \sum_{k=1}^{\infty} (k + 1) \left(\frac{|B|}{2^{k+1}B}\right)^{\delta'(1-\kappa)} \times \left(1 + \frac{2^{k+1}r}{\rho(x_0)}\right)^{-N + \eta' + \theta + \theta' + \theta'' + \eta'(1-\kappa)} \| \Phi \left(\frac{|f|}{\lambda}\right) \|_{(L \log L)^{1, \tilde{b}, \theta}(w)}. \]

Now \( N \) can be chosen sufficiently large such that \( N > \eta' + \theta + \theta' + \theta'' + \eta'(1-\kappa) \), and hence the above series is convergent. Finally,
\[
J'_2 \leq C \left(1 + \frac{r}{\rho(x_0)}\right)^{\mu + N - N_0 + 1} \sum_{k=1}^{\infty} (k + 1) \left(\frac{|B|}{2^{k+1}B}\right)^{\delta'(1-\kappa)} \| \Phi \left(\frac{|f|}{\lambda}\right) \|_{(L \log L)^{1, \tilde{b}, \theta}(w)} \leq C \left(1 + \frac{r}{\rho(x_0)}\right)^{\mu + N - N_0 + 1} \| \Phi \left(\frac{|f|}{\lambda}\right) \|_{(L \log L)^{1, \tilde{b}, \theta}(w)}. \]

Fix this \( N \) and set \( \vartheta = \max \{ \vartheta', \mu + N - \frac{N_0}{N_0 + 1} \} \). Thus, combining the above estimates for \( J'_1 \) and \( J'_2 \), the inequality (4.1) is proved and then the proof of Theorem 2.7 is finished. \( \square \)

The higher order commutators formed by a BMO\(_{\rho, \infty}(\mathbb{R}^d)\) function \( b \) and the operators \( \mathcal{R} \) and its adjoint \( \mathcal{R}^* \) are usually defined by
\[
(4.11) \quad \begin{cases} 
[ b, \mathcal{R} ]_m f(x) := \int_{\mathbb{R}^n} \left[ b(x) - b(y) \right]^m \mathcal{K}(x, y) f(y) \, dy, & x \in \mathbb{R}^d; \\
[ b, \mathcal{R}^* ]_m f(x) := \int_{\mathbb{R}^n} \left[ b(x) - b(y) \right]^m \mathcal{K}^*(x, y) f(y) \, dy, & x \in \mathbb{R}^d; \\
m = 1, 2, 3, \ldots. 
\end{cases}
\]

Let \( \mathcal{T} \) denote \( \mathcal{R} \) or \( \mathcal{R}^* \). Obviously, \( [ b, \mathcal{T} ]_1 = [ b, \mathcal{T} ] \) which is just the linear commutator (1.8), and
\[
[ b, \mathcal{T} ]_m = [ b, [ b, \mathcal{T} ]_{m-1} ], \quad m = 2, 3, \ldots.
\]

By induction on \( m \), we are able to show that the conclusions of Theorems 2.5 and 2.7 also hold for the higher order commutators \( [ b, \mathcal{T} ]_m \) with \( m \geq 2 \). The details are omitted here.
Theorem 4.4. Let $1 < p < \infty$, $0 < \kappa < 1$ and $w \in A^{p,\infty}_\rho$. If $V \in RH_q$ with $q \geq d$, then for any positive integer $m \geq 2$, the higher order commutators $[b, R]_m$ and $[b, R^*_m]$ are all bounded on $L^{p,\kappa}_\rho(w)$, whenever $b \in \text{BMO}_{\rho,\infty}(\mathbb{R}^d)$.

Theorem 4.5. Let $p = 1$, $0 < \kappa < 1$ and $w \in A^{1,\infty}_\rho$. If $V \in RH_q$ with $q \geq d$ and $b \in \text{BMO}_{\rho,\infty}(\mathbb{R}^d)$, then for any given $\lambda > 0$ and any given ball $B = B(x_0, r)$ of $\mathbb{R}^d$, there exist some constants $C > 0$ and $\vartheta > 0$ such that the following inequalities
\[
\frac{1}{w(B)} \cdot w \left( \left\{ x \in B : |[b, R]_m f(x)| > \lambda \right\} \right) \leq C \left( 1 + \frac{r}{\rho(x_0)} \right)^\vartheta \left\| \Phi_m \left( \frac{|f|}{\lambda} \right) \right\|_{(L \log L)^{1,\kappa}_\rho(w)},
\]
\[
\frac{1}{w(B)} \cdot w \left( \left\{ x \in B : |[b, R^*_m f(x)] > \lambda \right\} \right) \leq C \left( 1 + \frac{r}{\rho(x_0)} \right)^\vartheta \left\| \Phi_m \left( \frac{|f|}{\lambda} \right) \right\|_{(L \log L)^{1,\kappa}_\rho(w)}
\]
hold for those functions $f$ such that $\Phi_m(|f|) \in (L \log L)^{1,\kappa}_\rho(w)$ with some fixed $\theta > 0$, where $\Phi_m(t) = t \cdot (1 + \log^+ t)^m$, $m = 2, 3, \ldots$.

Acknowledgment

The author would like to thank Professor L. Tang for providing the paper [10].

References

[1] B. Bongioanni, E. Harboure, O. Salinas, Classes of weights related to Schrödinger operators, J. Math. Anal. Appl., 373 (2011), 563–579.
[2] B. Bongioanni, E. Harboure, O. Salinas, Weighted inequalities for commutators of Schrödinger-Riesz transforms, J. Math. Anal. Appl., 392 (2012), 6–22.
[3] B. Bongioanni, E. Harboure, O. Salinas, Commutators of Riesz transforms related to Schrödinger operators, J. Fourier Anal. Appl., 17 (2011), 115–134.
[4] L. Grafakos, Classical Fourier Analysis, Third Edition, Springer-Verlag, 2014.
[5] F. John and L. Nirenberg, On functions of bounded mean oscillation, Comm. Pure Appl. Math, 14 (1961), 415–426.
[6] Y. Komori and S. Shirai, Weighted Morrey spaces and a singular integral operator, Math. Nachr, 282 (2009), 219–231.
[7] G. X. Pan and L. Tang, Boundedness for some Schrödinger type operators on weighted Morrey spaces
[8] M. M. Rao and Z. D. Ren, Theory of Orlicz Spaces, Marcel Dekker, New York, 1991.
[9] Z. W. Shen, $L^p$ estimates for Schrödinger operators with certain potentials, Ann. Inst. Fourier (Grenoble), 45 (1995), 513–546.

[10] L. Tang, Weighted norm inequalities for Schrödinger type operators, Forum Math., 27 (2015), 2491–2532.

[11] L. Tang and J. F. Dong, Boundedness for some Schrödinger type operators on Morrey spaces related to certain nonnegative potentials, J. Math. Anal. Appl., 355 (2009), 101–109.

[12] H. Wang, Intrinsic square functions on the weighted Morrey spaces, J. Math. Anal. Appl., 396 (2012), 302–314.

[13] H. Wang, Weak type estimates for intrinsic square functions on weighted Morrey spaces, Anal. Theory Appl., 29 (2013), 104–119.

College of Mathematics and Econometrics, Hunan University, Changsha, 410082, P. R. China, & Department of Mathematics and Statistics, Memorial University, St. John’s, NL A1C 5S7, Canada

E-mail address: wanghua@pku.edu.cn