IRREDUCIBLE SUBQUOTIENTS OF GENERIC GELFAND-TSETLIN MODULES OVER $U_q(\mathfrak{gl}_n)$

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Abstract. We provide a classification and explicit bases of tableaux of all irreducible subquotients of generic Gelfand-Tsetlin modules over $U_q(\mathfrak{gl}_n)$ where $q \neq \pm 1$.

1. Introduction

Recently there has been a breakthrough in the theory of Gelfand-Tsetlin modules in the papers [7], [9], [10], [8]. In these papers new classes of simple $\mathfrak{gl}_n$-modules were constructed generalising a classical Gelfand-Tsetlin basis [14], [22] for finite-dimensional representations. These new representations also have a basis consisting of Gelfand-Tsetlin tableaux but such tableaux are not necessarily eigenvectors of the Gelfand-Tsetlin subalgebra [5]. This fact requires a modified action of the generators of the Lie algebra on this basis. Gelfand-Tsetlin representations are related to the theory of integrable systems [19], [20], [1], [2], [3], [4], general hypergeometric functions on the complex Lie group $GL(n)$, [15], [16]; solutions of the Euler equation, [6], [25] among the others.

The purpose of current paper is to study the Gelfand-Tsetlin basis for quantum $\mathfrak{gl}_n$ aiming to generalize the constructions above in the quantum case. Previously, partial results were obtained for example in [21], [23], [24], [12]. A general theory of Gelfand-Tsetlin modules for quantum $\mathfrak{gl}_n$ was developed in [11]. Even though quantization of the Gelfand-Tsetlin basis for generic module in the non-root of unity case may seem straightforward it does require a very careful treatment which was done in this paper. We also include a root of unity case.

Our main result is Theorem 6.2 which provides explicit construction of all irreducible generic Gelfand-Tsetlin modules with tableaux realization. In Section 7 we consider $q$ a root of unity and apply our construction in this case. It yields new explicit constructions of some finite dimensional irreducible modules.

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2. Notation and conventions

Throughout the paper we fix an integer $n \geq 2$. The ground field will be $\mathbb{C}$. For $a \in \mathbb{Z}$, we write $\mathbb{Z}_{\geq a}$ for the set of all integers $m$ such that $m \geq a$. Similarly, we define $\mathbb{Z}_{< a}$, etc. By $U_q$ we denote the quantum enveloping algebra of $\mathfrak{gl}(n)$. We
fix the standard Cartan subalgebra \( \mathfrak{h} \), the standard triangular decomposition and
the corresponding basis of simple roots of \( U_q \). The weights of \( U_q \) will be written as
\( n \)-tuples \( (\lambda_1, \ldots, \lambda_n) \). For a commutative ring \( R \), by \( \text{Specm} R \) we denote the set of
maximal ideals of \( R \). We will write the vectors in \( \mathbb{C}^{n(n+1)/2} \) in the following form:
\[
L = (l_{ij}) = (l_{n1}, \ldots, l_{nn} \mid l_{n-1,1}, \ldots, l_{n-1,n-1} \mid \cdots | l_{21}, l_{22}, l_{11}).
\]
For \( 1 \leq j \leq i \leq n \), \( \delta_{ij} \in \mathbb{Z}^{n(n+1)/2} \) is defined by \( (\delta_{ij})_{ij} = 1 \) and all other \( (\delta_{ij})_{k\ell} \) are zero. For \( i > 0 \) by \( S_i \) we denote the \( i \)th symmetric group. Let \( 1(q) \) be the set of all
complex \( x \) such that \( q^x = 1 \). Finally, for any complex number \( x \), we set
\[
(x)_q = \frac{q^x - 1}{q - 1}, \quad [x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}.
\]

3. Gelfand-Tsetlin modules

Let \( P \) be the free \( \mathbb{Z} \)-lattice of rank \( n \) with the canonical basis \( \{\epsilon_1, \ldots, \epsilon_n\} \), i.e.
\( P = \bigoplus_{i=1}^n \mathbb{Z}\epsilon_i \), endowed with symmetric bilinear form \( \langle \epsilon_i, \epsilon_j \rangle = \delta_{ij} \). Let \( \Pi = \{\alpha_j = \epsilon_j - \epsilon_{j+1} \mid j = 1, 2, \ldots\} \) and \( \Phi = \{\epsilon_i - \epsilon_j \mid 1 \leq i \neq j \leq n - 1\} \). Then \( \Phi \) realizes the
root system of type \( A_{n-1} \) with \( \Phi \) a basis of simple roots.

We define \( U_q \) as a unital associative algebra generated by \( e_i, f_i (1 \leq i \leq n) \) and \( q^h (h \in P) \) with the following relations:
\[
\begin{align*}
(1) & \quad q^0 = 1, \quad q^h q^{h'} = q^{h+h'} (h, h' \in P), \\
(2) & \quad q^h e_i q^{-h} = q^{\langle h, \alpha_i \rangle} e_i, \\
(3) & \quad q^h f_i q^{-h} = q^{-\langle h, \alpha_i \rangle} f_i, \\
(4) & \quad e_i f_j - f_j e_i = \delta_{ij} \frac{q^\alpha_i - q^{-\alpha_i}}{q - q^{-1}}, \\
(5) & \quad e_i^2 - (q + q^{-1}) e_i + q\epsilon_i = 0 \quad (|i-j| = 1), \\
(6) & \quad f_i^2 - (q - q^{-1}) f_i + q\epsilon_i = 0 \quad (|i-j| = 1), \\
(7) & \quad e_i^2 = (q^2 - 1) e_i f_i - f_i e_i. \quad (|i-j| > 1).
\end{align*}
\]
The quantum special linear algebra \( U_q(sl_n) \) is the subalgebra of \( U_q \) generated by \( e_i, f_i, q^{\pm \epsilon_i} (i = 1, 2, \ldots, n - 1) \).

**Remark 3.1** ([3], Theorem 12). \( U_q \) has an alternative representation. It is iso-

\[
\begin{align*}
(8) & \quad RL_i^+ L_j^+ = L_j^+ L_i^+ R, \\
(9) & \quad RL_i^+ L_j^- = L_j^- L_i^+ R
\end{align*}
\]
where \( R = q \sum_{i \neq j} e_{ii} \otimes e_{jj} + \sum_{i \neq j} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{i < j} e_{ij} \otimes e_{ji} \). The isomorphism
between this two representations is given by
\[
\begin{align*}
l_{ii}^- = q^{\epsilon_i}, \\
l_{i,i+1}^+ = (q - q^{-1}) q^{\epsilon_i}, \\
l_{i+1,i}^- = (q - q^{-1}) f_i q^{\epsilon_i}.
\end{align*}
\]

Let \( m \leq n \), \( \mathfrak{g}l_m \) be the Lie subalgebra of \( \mathfrak{g}l(n) \) spanned by \( \{E_{ij} \mid i, j = 1, \ldots, m\} \). We have the following chain
\[
\mathfrak{g}l_1 \subset \mathfrak{g}l_2 \subset \ldots \subset \mathfrak{g}l_n.
\]
It induces the chain $U_1 \subset U_2 \subset \ldots \subset U_n$ for the universal enveloping algebras $U_m = U(\mathfrak{gl}_m)$, $1 \leq m \leq n$. If we denote by $(U_m)_q$ the quantum universal enveloping algebra of $\mathfrak{gl}_m$. We have the following chain $(U_1)_q \subset (U_2)_q \subset \ldots \subset (U_n)_q$. Let $Z_m$ denote the center of $(U_m)_q$. The subalgebra of $(U_1)_q$ generated by $\{Z_m \mid m = 1, \ldots, n\}$ will be called the Gelfand-Tsetlin subalgebra of $(U_1)_q$ and will be denoted by $\Gamma_q$.

**Theorem 3.2** ([13], Theorem 14). The center of $U_q(\mathfrak{gl}_m)$ is generated by the following $m + 1$ elements

$$c_{mk} = \sum_{\sigma, \sigma' \in S_m} (-q)^{l(\sigma)+l(\sigma')} l^+_{\sigma(1), \sigma'(1)} \cdots l^+_{\sigma(k), \sigma'(k)} l^-_{\sigma(k+1), \sigma'(k+1)} \cdots l^-_{\sigma(m), \sigma'(m)},$$

where $0 \leq k \leq m$.

**Definition 3.3.** A finitely generated $U$-module $M$ is called a Gelfand-Tsetlin module (with respect to $\Gamma_q$) if

$$M = \bigoplus_{m \in \text{Spec} \Gamma_q} M(m),$$

where $M(m) = \{v \in M \mid m^kv = 0 \text{ for some } k \geq 0\}$. Equivalently,

$$M = \bigoplus_{\chi \in \Gamma_q^*} M(\chi)$$

where $M(\chi) = \{v \in M : \forall g \in \Gamma_q, \exists k \in \mathbb{Z}_{>0} \text{ such that } (g - \chi(g))^kv = 0\}$.

The Gelfand-Tsetlin support of $M$ is the set $\text{Supp}_{GT}(M) := \{\chi \in \Gamma_q^* : M(\chi) \neq 0\}$.

**Lemma 3.4.** Any submodule of a Gelfand-Tsetlin module over $U_q$ is a Gelfand-Tsetlin module.

**Proof.** Analogous to [9] Lemma 3.2. □

4. **Finite dimensional modules of $U_q$**

In this section we recall the quantum version of a classical result of Gelfand and Tsetlin which provides an explicit basis for every irreducible finite dimensional $U_q$-module.

**Definition 4.1.** For a vector $L = (l_{ij})$ in $\mathbb{C}^{n(n+1)/2}$, by $T(L)$ we will denote the following array with entries $\{l_{ij} : 1 \leq j \leq i \leq n\}$

$$
\begin{array}{cccccccc}
 l_{n1} & l_{n2} & \cdots & l_{nn} \\
 l_{n-1,1} & \cdots & l_{n-1,n-1} \\
 \cdots & \cdots & \cdots \\
 l_{11} & l_{21} & l_{22} \\
\end{array}
$$

such an array will be called a Gelfand-Tsetlin tableau of height $n$. A Gelfand-Tsetlin tableau of height $n$ is called standard if $l_{ki} - l_{k-1,i} \in \mathbb{Z}_{\geq 0}$ and $l_{k-1,i} - l_{k,i+1} \in \mathbb{Z}_{>0}$ for all $1 \leq i \leq k \leq n - 1$. 
The following theorem describes the Gelfand-Tsetlin approach for simple finite dimensional $U_q$ modules with a given highest weight.

**Theorem 4.2** ([24], Theorem 2.11). Let $L(\lambda)$ be the finite dimensional irreducible module over $U_q$ of highest weight $\lambda = (\lambda_1, \ldots, \lambda_n)$. Then there exist a basis of $L(\lambda)$ consisting of all standard tableaux $T(L)$ with fixed top row $l_{n,j} = \lambda_j - j$. Moreover, the action of the generators of $\Gamma_q$ on $L(\lambda)$ is given by the Gelfand-Tsetlin formulae:

$$
q^{k}(T(L)) = q^{a_k}T(L), \quad a_k = \sum_{i=1}^{k} l_{k,i} - \sum_{i=1}^{k-1} l_{k-1,i} + k, \quad k = 1, \ldots, n,
$$

(12)

$$
e_k(T(L)) = -\sum_{j=1}^{k} \frac{\prod_{i\neq j}^{|l_{k+1,i} - l_{k,j}|} q}{\prod_{i\neq j}^{|l_{k,i} - l_{k,j}|} q} T(L + \delta^{kj}),
$$

(13)

$$
f_k(T(L)) = \sum_{j=1}^{k} \frac{\prod_{i\neq j}^{|l_{k-1,i} - l_{k,j}|} q}{\prod_{i\neq j}^{|l_{k,i} - l_{k,j}|} q} T(L - \delta^{kj}).
$$

The next proposition gives the explicit action of the generators of $\Gamma_q$.

**Proposition 4.3.** The generator $c_{nk}$ of $\Gamma_q$ acts on $T(L)$ as a scalar multiplication by

$$
\gamma_{nk}(L) = (k)_{q-2}!(n-k)_{q-2}! q^{k(k+1)+\frac{n(n-3)}{2}} \sum_{\tau} q^{\sum_{i=1}^{n} l_{n,\tau(i)} - \sum_{i=k+1}^{n} l_{n,\tau(i)}}
$$

where $\tau \in S_n$ is such that $\tau(1) < \cdots < \tau(k), \tau(k+1) < \cdots < \tau(n)$.

**Proof.** Analogous to [?] Theorem 5.1. Choose a lowest weight vector $v = T(L)$ in $L(\lambda)$, the entries of $T(L)$ should satisfy $l_{i,j} = l_{i+1,j+1} + 1$ for any $i, j$. Note that the generators $l_{ij}^+, l_{ij}^-$ belong to the upper and lower Borel subalgebra generated by $e_i, f_i$ respectively. The element $l_{\sigma(k+1), \sigma'(k+1)} \cdots l_{\sigma(n), \sigma'(n)}$ kills $v$ unless $\sigma_{k+1} = \sigma'_{k+1}, \ldots, \sigma_n = \sigma'_n$. But $\sigma_1 \leq \sigma'_1, \ldots, \sigma_k \leq \sigma'_k$, so one must have $\sigma_i = \sigma'_i$ for all $1 \leq i \leq n$ in the action of $c_{nk}$ on $v$. We thus have

$$
c_{nk}v = \sum_{\sigma \in S_n} q^{2l(\sigma)} q^{a_{\sigma(1)} + \cdots + a_{\sigma(k)} - a_{\sigma(k+1)} - \cdots - a_{\sigma(n)}} v.
$$

where

$$
a_{\sigma(i)} = \sum_{j=1}^{\sigma(i)} l_{\sigma(i),j} - \sum_{j=1}^{\sigma(i)-1} l_{\sigma(i)-1,j} + \sigma(i)
$$

$$
= l_{\sigma(i),1} + 1
$$

$$
= \lambda_{n+1-\sigma(i)}.
$$

Then

$$
\gamma_{nk}(\lambda) = \sum_{\sigma \in S_n} q^{2l(\sigma)} q^{a_{\sigma(1)} + \cdots + a_{\sigma(k)} - a_{\sigma(k+1)} - \cdots - a_{\sigma(n)}}
$$

$$
= \sum_{\sigma \in S_n} q^{l(\sigma)(n-1)-2l(\sigma)} q^{\lambda_{n+1-\sigma(1)} + \cdots + \lambda_{n+1-\sigma(k)} - \lambda_{n+1-\sigma(k+1)} - \cdots - \lambda_{n+1-\sigma(n)}}
$$

$$
= \sum_{\sigma \in S_n} q^{n(n-1)-2l(\sigma)} q^{\lambda_{\sigma(1)} + \cdots + \lambda_{\sigma(k)} - \lambda_{\sigma(k+1)} - \cdots - \lambda_{\sigma(n)}}.
$$
Let \( \tau \) be a permutation in \( S_n \) such that \( \tau(1) < \cdots < \tau(k), \ \tau(k+1) < \cdots < \tau(n) \). One has that
\[
\gamma_{nk}(\lambda) = (k)_{q-1} (n-k)_{q-1} q^{n(n-1)/2} \sum_{\tau} q^{-2(\tau(1)+\cdots+\tau(k)+\cdots+\tau(n))} q^{\lambda_{\tau(1)}+\cdots+\lambda_{\tau(n)}}
\]
\[
= (k)_{q-1} (n-k)_{q-1} q^{n(n-1)/2} \sum_{\tau} q^{-2 \sum_{i=1}^{k} (\tau(i)-1)+\sum_{i=1}^{k} (l_{n\tau(i)}+\tau(i)) - \sum_{i=k+1}^{n} (l_{n\tau(i)}+\tau(i))}
\]
\[
= (k)_{q-1} (n-k)_{q-1} q^{k(k+1)+\sum_{i=1}^{n-k} l_{n\tau(i)} - \sum_{i=k+1}^{n} l_{n\tau(i)}}
\]
\[

\square
\]

Corollary 4.4. The generator \( c_{mk} \) of \( \Gamma_q \) acts on \( T(L) \) as multiplication by
\[
(14) \quad \gamma_{mk}(\lambda) = (k)_{q-1} (m-k)_{q-1} q^{k(k+1)+\sum_{i=1}^{m-k} l_{m\tau(i)} - \sum_{i=m+k}^{m} l_{m\tau(i)}}
\]
where \( \tau \in S_m \) is such that \( \tau(1) < \cdots < \tau(k), \ \tau(k+1) < \cdots < \tau(m) \).

Proof. Follows directly from Theorem 4.3 and the fact that eigenvalues of \( \gamma_{nk} \) depend only on the \( n \)-th row of the tableau. \( \square \)

5. Generic Gelfand-Tsetlin modules of \( U_q \)

Recall that \( 1(q) \) stands for the set of all complex \( x \) such that \( q^n = 1 \).

Definition 5.1. A Gelfand-Tsetlin tableau \( T(L) \) is called \( q \)-generic if it satisfies the following defining conditions:
\[
l_{ij} - l_{ik} \neq \frac{1(q)}{2} + \mathbb{Z} \text{ for all } 1 \leq i \leq n \text{ and } k \neq j.
\]
By \( B(T(L)) \) we will denote the set of all Gelfand-Tsetlin tableaux \( T(R) \) of height \( n \) satisfying \( r_{nj} = l_{nj} \) and \( r_{ij} - l_{ij} \in \mathbb{Z} \) for \( 1 \leq j \leq i \leq n-1 \).

Theorem 5.2 (21) Theorem 2. Let \( T(L) \) be a generic tableau, the vector space \( V(T(L)) = \text{span} B(T(L)) \) has a structure of a \( U_q \)-module of finite length with action of the generators of \( U_q \) given by the Gelfand-Tsetlin formulae (12).

Proposition 5.3. The Gelfand-Tsetlin subalgebra \( \Gamma_q \) separate the tableaux in \( V(T(L)) \). That is, for any two different tableaux in \( V(T(L)) \), there exists an element in \( \Gamma_q \) with different eigenvalues corresponding to the tableaux.

Proof. Let \( T(R) \) and \( T(S) \) be two tableaux with different \( m \)-th row. Assume \( T(R) \) and \( T(S) \) have the same eigenvalue for any element in \( \Gamma_q \). It is easy to see from (4.3) that \( (q^{2r_{m1}}, \ldots, q^{2r_{mm}}) \) is a permutation of \( (q^{2s_{m1}}, \ldots, q^{2s_{mm}}) \). Therefore, for any \( r_{mi} \), there exist \( j \) such that \( q^{2r_{mi}} = q^{2s_{mj}} \), which implies that \( r_{mi} - s_{mj} \in \frac{1(q)}{2} \). This lead to \( i = j \) and \( r_{mi} = s_{mj} \) which is a contradiction. \( \square \)

5.1. Classification of irreducible generic Gelfand-Tsetlin \( U_q \)-modules. We recall the following result of Mazorchuk and Turowska.

Theorem 5.4 (21) Proposition 2. If \( n \in \text{Spec} \Gamma \) is generic, then there exists a unique irreducible Gelfand-Tsetlin module \( N \) such that \( N(n) \neq 0 \).
Definition 5.5. If $T(R)$ is a $q$-generic tableau and $r \in \text{Spec} \Gamma_q$ corresponds to $R$ then, the unique module $N$ such that $N(r) \neq 0$ is called the irreducible Gelfand-Tsetlin module containing $T(R)$, or simply, the irreducible module containing $T(R)$.

This section is devoted to an explicit construction of the irreducible Gelfand-Tsetlin module containing $T(R)$ for every $q$-generic tableau $T(R)$.

For convenience we introduce and recall some notation.

Notation 5.6. Let $T(L)$ be a fixed tableau of height $n$.

(i) $\mathcal{B}(T(L)) := \{T(L + z) : z \in \mathbb{Z}^{\pi(n-1)}\}$.

(ii) $V(T(L)) = \text{span} \mathcal{B}(T(L))$.

(iii) For any $T(R) \in \mathcal{B}(T(L))$ and for any $1 \leq p \leq n$, $1 \leq s \leq p$ and $1 \leq u \leq p - 1$ we define:

(a) $\omega_{p,s,u}(T(R)) := r_{p,s} - r_{p-1,u}$.

(b) $\Omega(T(R)) := \{(p, s, u) : \omega_{p,s,u}(T(R)) \in \frac{1}{q} + \mathbb{Z}\}$

(c) $\Omega^+(T(R)) := \{(p, s, u) : \omega_{p,s,u}(T(R)) \in \frac{1}{q} + \mathbb{Z}_{\geq 0}\}$

(d) $\mathcal{N}(T(R)) := \{T(S) \in \mathcal{B}(T(L)) : \Omega^+(T(R)) \subseteq \Omega^+(T(S))\}$

(e) $W(T(R)) := \text{span} \mathcal{N}(T(R))$

(f) $U_q \cdot T(R)$: the $U_q$-submodule of $V(T(L))$ generated by $T(R)$

5.2. Submodule generated by a single tableau. In order to find an explicit basis of every irreducible generic module, we first find a basis of $U_q \cdot T(R)$ for any tableau $T(R)$ in $\mathcal{B}(T(L))$.

Definition 5.7. Given $T(Q)$ and $T(R)$ in $\mathcal{B}(T(L))$, we write $T(R) \preceq_{(1)} T(Q)$ if there exist $g \in \mathfrak{gl}(n)$ such that $T(Q)$ appears with nonzero coefficient in the decomposition of $g \cdot T(R)$ into a linear combination of tableaux. For any $p \geq 1$ we write $T(R) \preceq_{(p)} T(Q)$ if there exist tableaux $T(L^{(1)}), \ldots, T(L^{(p)})$, such that

$T(R) = T(L^{(0)}) \preceq_{(1)} T(L^{(1)}) \preceq_{(1)} \ldots \preceq_{(1)} T(L^{(p)}) = T(Q)$.

As an immediate consequence of the definition of $\preceq_{(p)}$ we have the following.

Lemma 5.8. If $T(Q), T(Q^{(0)}), T(Q^{(1)})$, and $T(Q^{(2)})$ are tableaux in $\mathcal{B}(T(L))$ then:

(i) $T(Q^{(0)}) \preceq_{(p)} T(Q^{(1)})$ and $T(Q^{(1)}) \preceq_{(q)} T(Q^{(2)})$ imply $T(Q^{(0)}) \preceq_{(p+q)} T(Q^{(2)})$.

(ii) $T(Q) \preceq_{(1)} T(Q)$.

The next theorem describes the submodule of $V(T(L))$ generated by a fixed tableau $T(R)$.

Theorem 5.9. Let $T(L)$ be $q$-generic tableau, $T(R)$ and $T(S)$ be in $\mathcal{B}(T(L))$.

(i) The Gelfand-Tsetlin formulas endow $W(T(R))$ with a $U_q$-module structure.

(ii) $U_q \cdot T(R) = W(T(R))$. In particular, $\mathcal{N}(T(R))$ forms a basis of $U_q \cdot T(R)$, and the action of $U_q$ on $U_q \cdot T(R)$ is given by the Gelfand-Tsetlin formulas.

(iii) $U_q \cdot T(R) = U_q \cdot T(S)$ if and only if $\Omega^+(T(S)) = \Omega^+(T(R))$.

(iv) $U_q \cdot T(R) = V(T(L))$ whenever $\Omega^+(T(R)) = \emptyset$.

(v) Every submodule of $V(T(L))$ is finitely generated.

Proof. (i) In order to prove that $W(T(R))$ is a submodule, it is enough to prove $U \cdot T(S) \subseteq W(T(R))$ for any $T(S) \in \mathcal{N}(T(R))$. We will show $g \cdot T(S)$ is in $W(T(R))$ for every (standard) generator of $U_q$. 


Suppose \( g = e_k \) for some \( 1 \leq k \leq n - 1 \). By the Gelfand-Tsetlin formulas, we have

\[
e_k(T(S)) = -\sum_{j=1}^{k} \frac{\prod_{i \neq j} [s_{ki} - s_{kj}]_q}{\prod_{i \neq j} [s_{i,i} - s_{k,j}]_q} T(S + \delta^k)
\]

If \( e_k(T(S)) \notin W(T(R)) \), then there exist \( k \) and \( j \) such that \( T(S) \in \mathcal{N}(T(R)) \) but \( T(S + \delta^k) \notin \mathcal{N}(T(R)) \). That implies

\[
\Omega^+(T(R)) \subseteq \Omega^+(T(S)), \quad \text{and} \quad \Omega^+(T(R)) \notin \Omega^+(T(S + \delta^k)).
\]

Hence, there exists \((p, s, u) \in \Omega^+(T(R))\) such that \( \omega_{p,s,u}(T(S)) \in \frac{1}{2} + \mathbb{Z}_{\geq 0}\) and \( \omega_{p,s,u}(T(S + \delta^k)) \notin \frac{1}{2} + \mathbb{Z}_{\geq 0}\). The latter holds only in two cases:

\[
(p, s, u) \in \{(k, j, u), (k + 1, s, j) : 1 \leq u \leq k - 1; 1 \leq s \leq k + 1\}.
\]

Note that if neither of these two cases hold, we have \( \omega_{p,s,u}(T(R + \delta^k)) = \omega_{p,s,u}(T(S)) \).

We consider now each of the two cases separately.

(a) Suppose \((p, s, u) = (k, j, u)\). Then \( \omega_{k,j,u}(T(S)) = s_{kj} - s_{k-1,u} \in \frac{1}{2} + \mathbb{Z}_{\geq 0}\) \nand \( \omega_{k,j,u}(T(S + \delta^k)) = (s_{kj} + 1) - s_{k-1,u} \notin \frac{1}{2} + \mathbb{Z}_{\geq 0} \), which is impossible.

(b) Suppose \((p, s, u) = (k+1, s, j)\). Then \( \omega_{k+1,s,j}(T(S)) = s_{k+1,s} - s_{k,j} \in \frac{1}{2} + \mathbb{Z}_{\geq 0}\) \nand \( \omega_{k+1,s,j}(T(S + \delta^k)) = s_{k+1,s} - (s_{k+1} + 1) \notin \frac{1}{2} + \mathbb{Z}_{\geq 0} \). Hence \( s_{k+1,s} - s_{k,j} = 0 \) and then the coefficient of \( T(S + \delta^k) \) in the decomposition of \( e_k(T(S)) \) is

\[
-\frac{\prod_{i \neq j} [s_{ki} - s_{kj}]_q}{\prod_{i \neq j} [s_{i,i} - s_{k,j}]_q} = 0.
\]

Therefore, the tableaux that appear with nonzero coefficients in the decomposition of \( e_k(T(S)) \) are elements of \( \mathcal{N}(T(R)) \). Hence, \( e_k(T(S)) \in W(T(R)) \).

The proof that \( f_k(T(S)) \in W(T(R)) \) is analogous to the one of \( e_k(T(S)) \in W(T(R)) \). The case \( g^k \) is trivial because \( g^k \) acts as a multiplication by a scalar on \( T(S) \) and \( T(S) \in \mathcal{N}(T(R)) \subseteq W(T(R)) \).

(ii) The Gelfand-Tsetlin subalgebra separate tableaux in \( B(T(L)) \), it is sufficient to prove that for any \( T(S) \in W(T(R)) \), \( T(S) \preceq_{(p)} T(R) \) for some \( p \in \mathbb{Z}_{>0} \). Let \( T(S) = T(R + z) \), we prove the statement by induction on \( \sum_{1 \leq j \leq n} |z_{ij}| \) when \( \sum_{1 \leq j \leq n} |z_{ij}| = 1 \), there exist \( z_{ij} = \pm 1 \), all other entries are zero. We consider each case separately.

(a) Suppose \( z_{ij} = 1 \). Then the coefficient of \( T(S) \) in \( e_i T(R) \) is

\[
\frac{\prod_{j \neq i} [r_{i+1,k} - r_{i,j}]_q}{\prod_{j \neq i} [r_{j,k} - r_{i,j}]_q}
\]

If there exist \( [r_{i+1,k} - r_{i,j}]_q = 0 \), one has \( s_{i+1,k} - s_{i,j} = \frac{1}{2} - 1 \), then \( T(S) \notin W(T(R)) \). Thus \( r_{i+1,k} - r_{i,j} \neq 0 \) for any \( k \) which implies \( T(S) \preceq_{(1)} T(R) \).

(b) Suppose \( z_{ij} = -1 \). Similarly the coefficient of \( T(S) \) in \( f_j T(R) \) is not zero.

When \( \sum_{1 \leq j \leq n} |z_{ij}| > 1 \), it is sufficient to prove the following statement.

Let \( z_{i_0,j_0}, z_{i_0+1,j_0}, \ldots, z_{i_t,j_t} \) be the nonzero elements such that \( r_{i_t,j_t} - r_{i_t+1,j_t} \in \frac{1}{2} + \mathbb{Z} \) for any \( 1 \leq t, t' \leq k \), then there exist \( T(S') = T(R + z') \) such that \( \Omega^+(T(S) \subseteq \Omega^+(T(S') \subseteq \Omega^+(T(R)) \) and \( |z'_{ij}| \leq |z_{ij}| \). Let \( t \) be the maximal number such that the all the numbers \( z_{i_0,j_0}, \ldots, z_{i_t,j_t} \) have the same sign, then \( \Omega^+(T(S) \subseteq \Omega^+(T(S - \sum_{s=0}^{t} \delta^{s}x_{j_t}))) \subseteq \Omega^+(T(S)) \) if the sign is positive. \( \Omega^+(T(R)) \subseteq \Omega^+(T(S+ \ldots \Omega^+(T(S))) \subseteq \Omega^+(T(S)) \).
Proof. (i) For each tableau $U$, for any $T$ where $\sum_{VYACHESLAV FUTORNY, LUIS ENRIQUE RAMIREZ, AND JIAN ZHANG}$

Also, for any $T \in B(T(L))$, $1 < p \leq n$ and $1 \leq u \leq p-1$, define $d_{pu}(T(R))$ to be the number of distinct elements in $\{v_{p,s,u}(T(R)) \mid (p,s,u) \in \Omega(T(R))\}$, where $\omega_{p,s,u}(T(R)) = u_{p,s,u}(T(R)) + v_{p,s,u}(T(R))$, with $u_{p,s,u}(T(R)) \in \frac{1}{pq}$ and $v_{p,s,u}(T(R)) \in \mathbb{Z}$.

Now we are ready to give the main theorem in the paper.

Theorem 6.2. Let $T(L)$ be $q$-generic tableau, $T(R) \in B(T(L))$.

(i) The irreducible module containing $T(R)$ has a basis of tableaux

$$\mathcal{I}(T(R)) = \{T(S) \in B(T(R)) : \Omega^+(T(S)) = \Omega^+(T(R))\}.$$  

The action of $U_q$ on this irreducible module is given by the Gelfand-Tsetlin formulas (17).

(ii) The number of irreducible modules in the block associated with $T(L)$ is:

$$\prod_{1 \leq u \leq p-1 < n} (d_{pu}(T(L)) + 1).$$

In particular, $V(T(L))$ is irreducible if and only if $d_{pu}(T(L)) = 0$ for any $p$ and $u$, or equivalently, if and only if $\Omega(T(L)) = 0$.

Proof. (i) For each tableau $T(R)$, we have an explicit construction of the module containing $T(R)$ (recall Definition 6.1):

$$M(T(R)) := U \cdot T(R) / \left( \sum U \cdot T(S) \right)$$

where the sum is taken over tableaux $T(Q)$ such that $\Omega^+(T(R)) \subseteq \Omega^+(T(S))$ and $U \cdot T(S)$ is a proper submodule of $U \cdot T(R)$. The module $M(T(R))$ is simple. Indeed, this follows from the fact that for any nonzero tableau $T(S)$ in $M(T(R))$ we have $U \cdot T(S) = U \cdot T(R)$ and, hence, $T(S)$ generates $M(T(R))$. By Theorem 5.9, $\mathcal{I}(T(R))$ is a basis for $M(T(R))$.

(ii) The irreducible modules are in one-to-one correspondence with the subsets of $\Omega(T(L))$ of the form $\Omega^+(T(L + z))$. For any $T(R) \in B(T(L))$, we can decompose $\Omega(T(R))$ into a disjoint union $\Omega(T(R)) = \bigsqcup_{p,u} \Omega_{p,u}(T(R))$, where

$$\Omega_{p,u}(T(R)) = \{(p,1,u), (p,2,u), \ldots, (p,p,u)\} \cap \Omega(T(R)).$$

Now, if $\Omega^+_{p,u}(T(R)) := \Omega_{p,u}(T(R)) \cap \Omega^+(T(R))$, one can write $\Omega^+(T(R)) = \bigsqcup_{p,u} \Omega^+_{p,u}(T(R))$. For $p, u$ fixed, let us denote by $s_{p,u}$ the number of different subsets of the form $\Omega^+_{p,u}(T(R))$. So, the number of different subsets of the form $\Omega^+(T(R))$ is $\prod_{p,u} s_{p,u}$. It is easy to see that $s_{p,u} = d_{pu}(T(L)) + 1$. □
7. Root of unity case

This section is devoted to describing the irreducible module of the quantum enveloping algebra $U_q$ when the complex parameter $q$ is a root of unity. In this case denote by $d$ its order. Since $q \neq \pm 1$, we must have $d > 2$.

**Theorem 7.1.** \cite{18} When $q$ is a root of unity, any irreducible module of $U_q$ is finite dimensional.

Denote

$$e = \begin{cases} 
    d & \text{if } d \text{ is odd} \\
    d/2 & \text{if } d \text{ is even}
\end{cases}$$

It is easy to verify that

$$[x]_q = 0 \iff x = 0 \mod e.$$

**Remark 7.2.** In the Gelfand-Tsetlin formulae \cite{12}, none of the $[l_{ki} - l_{kj}]_q$ is zero if $l_{n1} - l_{nn} \leq e$. So when $q$ is a root of unity, Theorem 4.2 holds if $\lambda_1 - \lambda_n \leq e + 1 - n$. For a generic tableau $T(L)$ all $[l_{ki} - l_{kj}]_q$ are not zero. Hence, Theorem 5.2 holds when $q$ is a root of unity.

Quantum Gelfand-Tsetlin subalgebra $\Gamma_q$ separates the tableaux in the following sense.

**Theorem 7.3.** Let $q$ be a root of unity, $T(L)$ a generic tableau. If $T(R), T(S) \in V(T(L))$ and $r_{ij} - s_{ij} \neq 0 \mod e, 1 \leq j \leq i < n$, then $\Gamma_q$ separates $T(R)$ and $T(S)$.

**Proof.** Let $T(R)$ and $T(S)$ be two tableaux with two different $m$-th row. Assume $T(R)$ and $T(S)$ have the same eigenvalue for any element in $\Gamma_q$. It is easy to see from \cite{13} that $(q^{2^m_1}, \ldots, q^{2^m_m})$ is a permutation of $(q^{2r_m_1}, \ldots, q^{2r_m_m})$. For any $r_{mi}$ there exist $j$ such that $q^{2r_m_j} = q^{2s_{mj}}$. We have that $r_{mi} - s_{mj} \in \frac{1(q)}{2}$. $T(L)$ is $q$-generic, one has that $i = j$. Since $r_{ij} - s_{ij} \neq 0 \mod e$, then $r_{mi} = s_{mj}$ which is a contradiction. \quad \Box

**Proposition 7.4.** Let $T(R)$ be a tableau in $V(T(L))$ and $N$ the submodule of $V(T(L))$ generated by $T(R)$. If $g \cdot T(R) = \sum c_i T(R_i)$ for some distinct tableaux $T(R_i)$ in $B(T(L))$ and nonzero $c_i \in \mathbb{C}$, we have $T(R_i) \in N$ for all $i$.

**Proof.** Suppose $g = e_k$ for some $1 \leq k \leq n - 1$. By the Gelfand-Tsetlin formulas, we have

$$e_k(T(R)) = -\sum_{j=1}^{k} \prod_{i \neq j} \left[ \frac{r_{k+1,i} - r_{k,j}}{r_{k,i} - r_{k,j}} \right]_q T(R + \delta^{kj})$$

Let $T(R_1)$ and $T(R_2)$ be any two tableaux in the summation with nonzero coefficients, then $(r_1)_{ij} - (r_2)_{ij} = 0$ or $\pm 1$ for any $1 \leq j \leq i < n$. It follows from Theorem 7.3 that $\Gamma_q$ separate these two tableaux. Thus $T(R_i) \in N$ for all $i$.

The proof that $g_k(T(R)) \in W(T(R))$ is analogous to the one of $e_k(T(R))$. The case $q^k$ is trivial because $q^k$ acts as a multiplication by a scalar on $T(R)$. \quad \Box

7.1. Submodule generated by a single tableau.

**Notation 7.5.** Let $T(R)$ be a fixed tableau of height $n$. We set
Theorem 7.7. Let \(T(L)\) be a \(q\)-generic Gelfand-Tsetlin tableau, \(T(R)\) and \(T(S)\) be in \(B(T(L))\).

\(\begin{aligned}
(i) & \quad \Omega(T(R)) := \{(p, s, u) : \omega_{p, s, u}(T(R)) \in \frac{1}{2} + \mathbb{Z}\} \\
(ii) & \quad U_q \cdot T(R) = W(T(R)). \text{ In particular, } N(T(R)) \text{ forms a basis of } U_q \cdot T(R), \text{ and the action of } U_q(\mathfrak{gl}(n)) \text{ on } U_q \cdot T(R) \text{ is given by the Gelfand-Tsetlin formulas [12].} \\
(iii) & \quad U_q \cdot T(R) = U_q \cdot T(S) \text{ if and only if } u_{p, s, u}(T(S)) = u_{p, s, u}(T(R)) \text{ for all } (p, s, u) \in \Omega(T(R)).
\end{aligned}\)

Proof. (i) In order to prove that \(W(T(R))\) is a submodule, it is enough to show \(g \cdot T(S)\) is in \(W(T(R))\) for every generator of \(U_q\). The proof is similar to theorem 7.6 (i).

(ii) Similar to theorem 5.9 (ii), it is sufficient to prove that for any \(T(S) \in W(T(R))\), \(T(S) \preceq (p) T(R)\) for some \(p \in \mathbb{Z}_{\geq 0}\).

(iii) It follows from (i) and (ii).

\[\square\]

7.2. New constructions of irreducible Gelfand-Tsetlin modules. In this section we use Gelfand-Tsetlin basis to give a new realisation of some irreducible Gelfand-Tsetlin modules in root of unity case. We assume \(d\) to be odd.

Let \(p = (p_{ij}), 1 \leq j \leq i \leq n\) with nonzero entries in \(C\), \(W_{ij}(R)\) be the submodule generated by \(T(R+\delta^0) - p_{ij}T(R)\). By Theorem 7.6 a basis for \(W_{ij}(R)\) is the set \(\{T(S+d\delta^0) - p_{ij}T(S) \mid T(S) \in W(T(R))\}\).

Let \(N = \sum_{T(R) \in B(T(L))} W_{ij}(R), \text{ and } M = \text{span}_R N\).

**Theorem 7.7.** \(M\) is an irreducible module with dimension \(d^{\frac{n(n-1)}{2}}\). Moreover, \(M\) has a basis of tableaux \(T(L + m_{ij}\delta^0), 0 \leq m_{ij} < d, 1 \leq j \leq i < n\).

Proof. The submodule \(N\) has a basis \(\{T(R+\delta^0) - p_{ij}T(R) : R \in B(T(L)), 1 \leq j \leq i < n\}\). So the subquotient \(M\) has basis \(T(L + m_{ij}\delta^0), 0 \leq m_{ij} < d, 1 \leq j \leq i < n\). We denote the basis of \(M\) by \(I\). Suppose \(M_1\) is a nonzero submodule of \(M\), by Proposition 7.3 the basis of \(M_1\) is a subset of \(I\). From theorem 7.6 and the relations in quotient module \(M\), one has that \(I \subseteq U_qT(R)\) for any tableau \(T(R)\) in \(I\), . Thus \(M_1 = M\) and \(M\) is irreducible.

\[\square\]

**Remark 7.8.** This module is similar to the module constructed in \S 7.5.5 of [18].

From now on we will denote by \(\Lambda\) the following set

\[\{(i, j) \mid (i + 1, s, j) \in \Omega(T(R)) \text{ for some } 1 \leq s \leq i + 1\}.\]
**Definition 7.9.** For any \( T(R) \in \mathcal{B}(T(L)) \), \( 1 < p \leq n \) and \( 1 \leq u \leq p - 1 \), for \((i, j) \in \Lambda\) define \( a_{ij}(T(R)) \) and \( b_{ij}(T(R)) \) as follows

\[
\begin{align*}
    a_{ij}(T(R)) &= \min\{v_{i+1,s} \mid (i + 1, s, j) \in \Omega(T(R))\}, \\
    b_{ij}(T(R)) &= \min\{d - v_{i+1,s} \mid (i + 1, s, j) \in \Omega(T(R))\}.
\end{align*}
\]

Define

\[
t_{ij}(T(R)) = \begin{cases} 
    a_{ij}(T(R)) + b_{ij}(T(R)) & \text{for } (i, j) \in \Lambda \\
    d & \text{for } (i, j) \notin \Lambda.
\end{cases}
\]

**Definition 7.10.** Let \( \Lambda_1 \) be a subset of \( \Lambda \), \( \Lambda_2 = \Lambda \setminus \Lambda_1 \), \( \tilde{M}(T(R)) \) de quotient of \( U_q \cdot T(R) \) by

\[
\left( \sum_{(i,j) \notin \Lambda} W_{ij}(R) + \sum_{T(S_1)} U_q(T(S_1)) + \sum_{T(S_2)} U_q(T(S_2') - p_{ij}T(S_2)) \right),
\]

where \( T(S_1), t = 1, 2 \) run through over the set of tableaux in \( N(T(R)) \) such that \((i, j) \in \Lambda_1, \omega_{1,s,j}T(S_2')-\omega_{1,s,j}(T(S_2')) = d \) for some \((i - 1, s, j) \in \Omega(T(R))\), \( \omega_{p,s,u}(T(S_2') - \omega_{p,s,u}(T(S_2')) = 0 \) for any \((p, s, u) \neq (i - 1, s, j)\).

**Theorem 7.11.** \( \tilde{M}(T(R)) \) is an irreducible module of dimension \( \prod_{1 \leq j \leq n} t_{ij}(T(R)) \).

**Proof.** The subquotient \( U_q \cdot T(R)/\sum_{T(S)} U_q(T(S)) \) has basis

\[
I = \{T(S) \mid u_{p,s,u}(T(S)) = u_{p,s,u}(T(R)) \text{ for all } (p, s, u) \in \Omega(T(L))\}.
\]

The module \( \tilde{M}(T(R)) \) can be regarded as the subquotient of \( U_q \cdot T(R)/\sum_{T(S)} U_q(T(S)) \). Then it has basis: \( \{T(S) \in I \mid s_{ij} = r_{ij} + m_{ij}, 0 \leq m_{ij} < d, (i, j) \notin \Lambda\} \). Similar to theorem 7.7, \( \tilde{M}(T(R)) \) is irreducible. For any \((i, j) \in \Lambda\), if we fix the \( i + 1 \)-th row of the tableau, the number of distinct \( s_{ij} \) in \( I \) is \( t_{ij}(T(R)) \). For \((i, j) \notin \Lambda\), there are \( d \) different \( s_{ij} \). Thus the dimension of \( \tilde{M}(T(R)) \) is \( \prod_{1 \leq j \leq n} t_{ij}(T(R)) \).

**7.3. Example.** Recall two families \( d \)-dimensional modules of \( U_q(sl_2) \). The first depends on three complex numbers \( \lambda, a \) and \( b \). We assume \( \lambda \neq 0 \). Consider the \( d \)-dimensional vector space with a basis \( \{v_0, v_1, \ldots, v_{d-1}\} \). for \( 0 \leq p \leq d - 1 \), set

\[
K v_p = \lambda q^{-2p} v_p,
\]

\[
E v_{p+1} = (\frac{q^{-p} - q^{-1}}{q - q^{-1}})[p + 1]_q + ab) v_p,
\]

\[
F v_p = v_{p+1},
\]

and \( E v_0 = av_{d-1}, F v_{d-1} = bv_0 \), and \( K v_{-1} = \lambda q^{-2(d-1)}v_p \). These formulas endow the vector space with a \( U_q \)-module structure, denoted by \( V(\lambda, a, b) \).

The second family depends on two scalars \( \mu \neq 0 \) and \( c \). Let \( E, F, K \) act on the vector space with basis \( \{v_0, v_1, \ldots, v_{d-1}\} \) by

\[
K v_p = \mu q^{2p} v_p,
\]

\[
F v_{p+1} = \frac{q^{-p} - q^{-1}}{q - q^{-1}}[p + 1]_q v_p,
\]

\[
E v_p = v_{p+1},
\]
and $Fv_0 = 0, E v_{d-1} = cv_0,$ and $K v_{d-1} = \mu q^{-2} v_{d-1}$. These formulas endow the vector space with a $U_q$-module structure, denoted by $\hat{V}(\mu, c)$.

Theorem 7.12. \cite{7} Any irreducible $U_q$ module of dimension $d$ is isomorphic to one of the following list:

(i) $V(\lambda, a, b)$ with $b \neq 0$,

(ii) $V(\lambda, a, 0)$ where $\lambda$ is not of the form $\pm q^{j-1}$ for any $1 \leq j \leq d - 1$,

(iii) $\hat{V}(\pm q^{j-1}, c)$ with $c \neq 0$ and $1 \leq j \leq d - 1$.

In the following we will compare above modules with modules in \cite{7} and \cite{77}.

Let $x, y, z$ be three complex number, $v_p = (x, y|z - p), 0 \leq p \leq d - 1$. Consider the vector space with basis of tableaux $\{T(v_p) : 0 \leq p \leq d - 1\}$. Theorem \cite{7} endows the vector space with a $U_q$-module structure. The actions of $E, F, K$ are given by

\begin{align*}
(21) & \quad K T(v_p) = q^{2z-(x+y+1)} q^{-2p} T(v_p), \\
(22) & \quad E T(v_{p+1}) = -[x + p + 1 - z]_q [y + p + 1 - z]_q T(v_p), \\
(23) & \quad F T(v_p) = T(v_{p+1}),
\end{align*}

and $E T(v_0) = -s(x-z)_q [y-z]_q T(v_{d-1}), F T(v_{d-1}) = \frac{1}{q} T(v_0)$. Let $\lambda = q^{2z-(x+y+1)}, b = \frac{1}{q}, a = -s(x-z)_q [y-z]_q v_{d-1},$ this module is isomorphic to $V(\lambda, a, b)$ with $b \neq 0$.

Let $x, y, z$ be three complex number with $x - z$ or $y - z \in \mathbb{N}$, Consider be the vector space with basis of tableaux $\{T(v_p) : 0 \leq p \leq d - 1\}$, where $v_p = (x, y|z - p), 0 \leq p \leq d - 1$. Theorem \cite{7} endows the vector space with a $U_q$-module structure. The actions of $E, F, K$ are given by

\begin{align*}
(24) & \quad K T(v_p) = q^{2z-(x+y+1)} q^{-2p} T(v_p), \\
(25) & \quad E T(v_{p+1}) = -[x + p + 1 - z]_q [y + p + 1 - z]_q T(v_p), \\
(26) & \quad F T(v_p) = T(v_{p+1}),
\end{align*}

and $Ev_0 = 0, F v_{d-1} = s v_0$. This module is isomorphic to $V(\lambda, 0, s), \lambda = q^{2z-(x+y+1)}$.

There exist an algebra endomorphism of $U_q(sl_2)$ such that $E \mapsto F, F \mapsto E, K \mapsto K^{-1}$. $V(\lambda, a, 0)$ and $\hat{V}(\mu, c)$ can be obtained from $V(\lambda, 0, 0)$ by the algebra endomorphism.

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