Abstract

Given $X \subseteq \mathbb{Z}_N$, $X$ is called a cyclic basis if $(X + X) \cup X = \mathbb{Z}_N$, symmetric if $x \in X$ implies $-x \in X$, and sum-free if $(X + X) \cap X = \emptyset$. We ask, for which $m, N \in \mathbb{Z}^+$ can the set of non-identity elements of $\mathbb{Z}_N$ be partitioned into $m$ symmetric sum-free cyclic bases? If, in addition, we require that distinct cyclic bases interact in a certain way, we get a proper relation algebra called a Ramsey algebra. Ramsey algebras (which have also been called Monk algebras) have been constructed previously for $2 \leq m \leq 7$. In this manuscript, we provide constructions of Ramsey algebras for every positive integer $m$ with $2 \leq m \leq 400$, with the exception of $m = 8$ and $m = 13$.

1 Introduction and motivation

Let $N$ be a positive integer, and let $\mathbb{Z}_N$ denote the ring of integers modulo $N$. For $X \subseteq \mathbb{Z}_N$, let

$$X + X = \{x_1 + x_2 : x_1, x_2 \in X\}.$$ 

A subset $X \subseteq \mathbb{Z}_N$ is called a cyclic basis for $\mathbb{Z}_N$ if $(X + X) \cup X = \mathbb{Z}_N$. A cyclic basis $X$ is called sum-free if $(X + X) \cap X = \emptyset$.

One interesting question about cyclic bases is how small they can be. More precisely, let $m(2, k)$ denote the largest $N$ such that there is some $A \subseteq \mathbb{Z}_N$ with $|A| = k$ and $A \cup (A + A) = \mathbb{Z}_N$ (see [4]). It is easy to see that $m(2, k) = O(k^2)$; an interesting question is how large the coefficient $\alpha$ on $k^2$ can be made so that $m(2, k) \geq \alpha k^2$. The largest $\alpha$ currently known is $1/3 - \varepsilon$ for any $\varepsilon > 0$ and for all sufficiently large $k$, due to Shen and Jia [5]. Jia’s excellent manuscript [4]
provides a great background on the topic of cyclic bases as well as a healthy list of references, and the authors wish to refer the interested reader to it.

For $m \in \mathbb{Z}^+$, a partition of $\mathbb{Z}_N \setminus \{0\}$ into sets $X_0, X_1, \ldots, X_{m-1}$ is called a \textit{sum-free cyclic multi-basis} if $X_i$ is a sum-free cyclic basis for $i = 0, 1, \ldots, m - 1$.

A subset $X \subseteq \mathbb{Z}_N$ is called \textit{symmetric} if $\forall x \in X \rightarrow -x \in X$; that is, $X$ is closed under additive inverse. We take a moment to note that if $X$ is symmetric, then $X - X = X + X$.

For ease, if a partition is a symmetric sum-free cyclic multi-basis, we shall call it an \textit{SSFCMB}.

We also desire our partitions to have one more property: $\forall i \forall j (i \neq j \rightarrow X_i + X_j = \mathbb{Z}_N \setminus \{0\})$.

If a partition has this property, we shall say that the partition satisfies the \textit{mandatory triangle condition}. (The reason for this term is due to the connection to Ramsey algebras, which will be made clear shortly.)

The smallest example of a non-trivial SSFCMB which satisfies the mandatory triangle condition has parameters $m = 2$ and $N = 5$. The partition of $\mathbb{Z}_5 \setminus \{0\}$ that we use is $X_0 = \{1, 4\}$ and $X_1 = \{2, 3\}$. We leave it to the reader to check that this partition is an SSFCMB possessing the mandatory triangle condition. The next smallest example has parameters $m = 3$ and $N = 13$. Here, the partition of $\mathbb{Z}_{13} \setminus \{0\}$ is

- $X_0 = \{1, 5, 8, 12\}$
- $X_1 = \{2, 3, 10, 11\}$, and
- $X_2 = \{4, 6, 7, 9\}$.

In [1], Comer constructs these previous two examples as well several others. The focus of this manuscript is to attack the following question: given $m \in \mathbb{Z}^+$, can we find $N \in \mathbb{Z}^+$ and a partition of $\mathbb{Z}_N \setminus \{0\}$ into $m$ parts which is an SSFCMB that satisfies the mandatory triangle condition? Our main result is summarized below as Theorem 1.

**Theorem 1.** For every positive integer $m$ with $2 \leq m \leq 400$, with the possible exception of $m = 8$ and $m = 13$, there exists a positive integer $N$ and a partition of $\mathbb{Z}_N \setminus \{0\}$ which is an SSFCMB that satisfies the mandatory triangle condition.

The proof of Theorem 1 is based on an optimized computer search, whose inception was based in the ideas of Comer and Maddux. Section 2 describes our search algorithm. Section 3 describes how we have optimized the search. Appendix A contains the results of the search, summarized as a table of values of $m$ and $N$, with enough information to recreate the partition (for the curious reader).

Before we get to the search algorithm, we wish to discuss the connection between SSFCMBs and the theory of relation algebras. A \textit{(proper) relation algebra} is an algebra $\langle A, \cup, \cdot, \circ^{-1}, Id \rangle$, where $A$ is a subset of the power set of some equivalence relation $E$ that forms a Boolean algebra $\langle A, \cup, \cdot \rangle$, the operator
is composition of relations, the operator \( -1 \) is conversion of relations, and \( Id \) is the identity subrelation of \( E \). A \textit{Ramsey algebra in \( m \) colors} is a proper relation algebra where all of the \textit{atoms} (i.e., minimal non-empty relations) \( A_0, \ldots, A_{m-1} \) distinct from \( Id \) satisfy

1. \( A^{-1}_i = A_i \);
2. \( A_i \circ A_i = A^c_i \);
3. for \( i \neq j \), \( A_i \circ A_j = Id \).

A \textit{cyclic Ramsey algebra in \( m \) colors} is a Ramsey algebra where \( E = \mathbb{Z}_N \times \mathbb{Z}_N \) for some \( N \in \mathbb{Z}^+ \), and all of the atoms \( A_i \) are defined by “difference sets” \( X_i \), so that \( A_i = \{(x, y) : x - y \in X_i \} \), where \( -X_i = X_i \). In this case, each \( X_i \subseteq \mathbb{Z}_N \) is a symmetric sum-free cyclic basis, since it must satisfy \( X_i + X_i = \mathbb{Z}_N \setminus X_i \). Furthermore, the collection \( X_0, \ldots, X_{m-1} \) is an SSFCMB for \( \mathbb{Z}_N \) that has the additional property that each sum \( X_i + X_j \) is as large as it can possibly be; that is,

\[
\forall i, X_i + X_i = \mathbb{Z}_N \setminus X_i \quad \text{and} \quad (1)
\]

\[
\forall i \neq j, X_i + X_j = \mathbb{Z}_N \setminus \{0\}. \quad (2)
\]

Thus the existence of a cyclic Ramsey algebra in \( m \) colors is equivalent to the existence of an SSFCMB in \( m \) parts satisfying (1) and (2).

Our example above with \( m = 2 \) and \( N = 5 \) is a cyclic Ramsey algebra in 2 colors. Let \( X_0 = \{1, 4\} \) and \( X_1 = \{2, 3\} \). Define two relations

\[
R = \{(x, y) \in \mathbb{Z}_5 \times \mathbb{Z}_5 : x - y \in X_0\}
\]

and

\[
B = \{(x, y) \in \mathbb{Z}_5 \times \mathbb{Z}_5 : x - y \in X_1\}.
\]

Let \( R \) and \( B \) be the two atoms besides the identity \( Id = \{(x, x) : x \in \mathbb{Z}_5\} \). They satisfy

\[
R \circ R = B \cup Id, \\
B \circ B = R \cup Id, \quad \text{and} \\
R \circ B = R \cup B.
\]

Note that \( R \circ R = B \cup Id \) follows from the fact that \( X_0 + X_0 = X_1 \cup \{0\} \), \( X_0 \) is sum-free, which means that \( R \) is “triangle-free,” as in the graph depicted in Figure 1. The graph depicts the relations \( R \) and \( B \) as sets of edges in \( K_5 \) colored red and blue, respectively.

Similarly, for \( m = 3 \) and \( N = 13 \) we can construct a cyclic Ramsey algebra in 3 colors. As above, let

\[
\begin{align*}
X_0 &= \{1, 5, 8, 12\} \\
X_1 &= \{2, 3, 10, 11\}, \quad \text{and} \\
X_2 &= \{4, 6, 7, 9\}.
\end{align*}
\]
be a partition of the non-identity elements of \( Z_{13} \). Define three relations

\[
R = \{(x, y) \in Z_{13} \times Z_{13} : x - y \in X_0\},
\]

\[
B = \{(x, y) \in Z_{13} \times Z_{13} : x - y \in X_1\}, \text{ and}
\]

\[
G = \{(x, y) \in Z_{13} \times Z_{13} : x - y \in X_2\}.
\]

See Figure 2 for a graph that depicts the relations \( R, B, \) and \( G \) as sets of edges in \( K_{13} \) colored red, blue, and green, respectively. These examples illustrate the fact that the question of the existence of Ramsey algebras can be stated in purely graph-theoretical terms.

This is where the name “Ramsey algebra” comes from — the atoms of a (cyclic) Ramsey algebra, interpreted as edge sets in a complete graph \( K_N \) instead of as symmetric binary relations, yield an edge-coloring of \( K_N \) in \( m \) colors that contains no monochromatic triangles. Note that under this interpretation, the mandatory triangle condition says that every edge participates in every possible type of triangle except for monochromatic triangles.

Let us pause briefly to discuss terminology further. The abstract-algebraic counterpart to Ramsey algebras has been used in the literature under various names for over 30 years. They were first mentioned by Maddux [9] but given no name. They have been called, variously, Monk algebras, Maddux algebras, and very recently, Ramsey algebras. In [3], Hirsch and Hodkinson use the term “Monk algebra” to refer to a more general kind of algebra in which the colors can come in different “shades”, but in their usage monochromatic triangles are still forbidden. In his 2011 talk at the AMS meeting in Iowa City [3], Maddux defined (for the first time, it would seem) a Ramsey algebra as we did above. In [6], Kowalski uses the term “Ramsey algebra” to refer to an abstract algebra, so that the Ramsey algebras of the present paper would be, in his terminology, representations of (Kowalski’s) Ramsey algebras. We choose to adopt Maddux’s
terminology, since it allows the problem of existence of Ramsey algebras to be stated in purely combinatorial terms. See Kowalski’s paper [6] for the abstract-algebraic treatment.

The question of the existence of Ramsey algebras in all numbers of colors was raised (though not in those terms) by Maddux in [9], Problem 2.7. Sometime in the mid-80s, Erdős, Szemerédi, and Trotter gave a purported proof that Ramsey algebras exists for all sufficiently large $m$. Comer told Trotter about the problem sometime in the early-to-mid 80s. Trotter sent a version of the purported proof to Comer via e-mail, and Comer sent it to Maddux [7]. Unfortunately, their “proof” was in error, as their construction did not satisfy the mandatory triangle condition. Comer produced constructions of cyclic Ramsey algebras for $m = 2, 3, 4, 5$ in 1983 [1]. In 2011, Maddux produced constructions for $m = 6, 7$ using the same method as Comer but with a 2011 computer. Maddux failed to construct a Ramsey algebra for $m = 8$. In [6], Kowalski simultaneously and independently derives results that match ours for $2 \leq m \leq 120$. In addition, he finds different constructions over finite fields of prime-power order. The present authors independently rediscovered Comer’s method of using so-called cyclotomic classes, and we show that Comer’s method does not work for $m = 8$, but does work for all $m$ between 9 and 400, except possibly for $m = 13$. Therefore, cyclic Ramsey algebras in $m$ colors exist for all $m$ between 2 and 400, except possibly 8 and 13. In addition, we found some SSFCMBs that failed to be Ramsey algebras (because they failed to satisfy the mandatory triangle condition).
2 Description of the search algorithm

To describe the algorithm we used to search for these SSFCMBs, we first bring the reader’s attention to a property of the examples mentioned for $m = 2$ and $m = 3$. Recall that if $m = 2$, we may take $N = 5$, $X_0 = \{1, 4\}$, and $X_1 = \{2, 3\}$. Notice that $2 \in \mathbb{Z}_5$ is a generator of $\mathbb{Z}^\times_5$. Modulo 5, we have $X_0 = \{2^0, 2^2\}$ and $X_1 = \{2^1, 2^3\}$.

For $m = 3$, we had $N = 13$ with
\[
X_0 = \{1, 5, 8, 12\}, \quad X_1 = \{2, 3, 10, 11\}, \quad \text{and} \quad X_2 = \{4, 6, 7, 9\}.
\]

Again, 2 is a generator of $\mathbb{Z}^\times_{13}$, and we also have
\[
X_0 = \{2^0, 2^3, 2^6, 2^9\}, \quad X_1 = \{2^1, 2^4, 2^7, 2^{10}\}, \quad \text{and} \quad X_2 = \{2^2, 2^5, 2^8, 2^{11}\}.
\]

Based on these two constructions, we tried to continue this pattern. That is, given $m$, we look at primes $N = mk + 1$ with $k$ even. We find a generator $x$ of $\mathbb{Z}_N^\times$, and construct the partition
\[
X_0 = \left\{ x^0, x^m, x^{2m}, \ldots, x^{(k-1)m} \right\}
\]
with $X_i = x \cdot X_{i-1}$, for $i = 1, 2, \ldots, m - 1$.

For ease, if a partition constructed in this fashion is an SSFCMB, we shall call it a single-generator SSFCMB. We wish to make it clear that our algorithm searches only for single-generator SSFCMBs.

2.1 The search algorithm for single-generator SSFCMBs

Below we describe the search algorithm as a series of steps.

1. Use the Sieve of Eratosthenes to generate a list $P$ of primes smaller than 2000000.

2. Fix a positive integer $m$.

3. Range over elements of $P$ until we come across a prime $N \equiv 1 \pmod{2m}$.

4. Set $k = \frac{N-1}{m}$.

5. Find the prime divisors $p_1, p_2, \ldots, p_r$ of $N - 1$.

\[\text{These two examples are well-known “folklore” among relation-algebraists, and can be found in many sources.}\]
6. Find the smallest \( x \in \mathbb{Z}_N^\times \) such that \( x^{(N-1)/p_i} \not\equiv 1 \pmod{N} \) for every \( i \in \{1, 2, \ldots, r\} \). Such \( x \) is the smallest generator of the cyclic group \( \mathbb{Z}_N^\times \).

7. Compute \( X_0 = \{ x^0, x^m, \ldots, x^{(k-1)m} \} \).

8. Check that \( X_0 \) is sum-free and that \( |X_0 + X_0| = N - k \). If it is, proceed; otherwise, discard \( N \) and keep checking the elements of \( P \).

9. For \( i = 1, 2, \ldots, m - 1 \), check that \( X_0 + X_i = \mathbb{Z}_N \setminus \{0\} \).

To see that this collection of steps is sufficient for the constructed partition to form an SSFCMB, we turn our attention to Section 3 which provides the lemmas we used to complete some of the steps.

3 Efficiency lemmas

This section consists of a collection of lemmas which are used to improve the efficiency of the search algorithm. Together, they significantly reduce the number of checks that need to be made from what would be required in a naïve approach.

Lemma 1 states that for given \( m \) and prime \( N \equiv 1 \pmod{2^m} \), it suffices to check only a single generator. Lemma 2 states that we need only to check whether the element 1 is in \( X_0 + X_0 \) to determine if \( X_0 \) is sum-free. Lemma 3 states that if \( X_0 \) is a sum-free cyclic basis, then so is \( X_i \) for \( i = 1, \ldots, m - 1 \). Lemma 4 reduces the number of calculations required to check if an SSFCMB satisfies the mandatory triangle condition from \( O(N^2) \) to \( O(N) \).

Throughout this section, we let \( m \in \mathbb{Z}^+ \) and let \( N = mk + 1 \) be a prime number. Note that a version of Lemma 1 appears in [1].

Lemma 1. If \( x \) and \( y \) are generators of \( \mathbb{Z}_N^\times \), then

\[
\left\{ x^0, x^m, x^{2m}, \ldots, x^{(k-1)m} \right\} = \left\{ y^0, y^m, y^{2m}, \ldots, y^{(k-1)m} \right\}.
\]

Proof. Suppose \( x \) and \( y \) are generators of \( \mathbb{Z}_N^\times \). We must show that every power of \( y^m \) is some power of \( x^m \).

To that end, fix a nonnegative integer \( \ell \). Since \( x \) is a generator of \( \mathbb{Z}_N^\times \), there exists an integer \( \alpha \) so that \( x^\alpha = y \). Hence,

\[
y^{\ell m} = (x^\alpha)_{\ell m} = x^{\alpha \ell m} = x^{(\alpha \ell)m},
\]

as desired. \(\square\)

Lemma 2. If \( x \) is a generator of \( \mathbb{Z}_N^\times \) and \( X_0 = \{ x^0, x^m, x^{2m}, \ldots, x^{(k-1)m} \} \), then \( X_0 \) is sum-free if and only if 1 \( \not\in (X_0 + X_0) \).
Proof. It is clear that if \( X_0 \) is sum-free, then \( \frac{1}{1} \notin (X_0 + X_0) \). For the other direction, suppose \( X_0 \) is not sum-free. This means there exist \( \alpha, \beta, \) and \( \gamma \) so that
\[
 x^{\alpha m + i} + x^{\beta m + i} = x^{\gamma m + i}.
\] (3)
If \( \min \{ \alpha, \beta, \gamma \} = \alpha \), we may factor out \( x^{\alpha m + i} \) from both sides of (3) to get
\[
 1 + x^{m(\beta - \alpha)} = x^{m(\gamma - \alpha)}, \text{ or } 1 = x^{m(\gamma - \alpha)} - x^{m(\beta - \alpha)}, \text{ so } 1 \in (X_0 - X_0) \text{. Since } X_0 \text{ is symmetric, } X_0 - X_0 = X_0 + X_0 \text{.}
\]
Similarly, if \( \min \{ \alpha, \beta, \gamma \} = \gamma \), then we may factor out \( x^{\gamma m + i} \) from both sides of (3), and get that \( 1 \in (X_0 + X_0) \).

Lemma 3. Suppose \( x \) is a generator of \( Z_N^{\times} \). For \( i \in \{0, 1, \ldots, m - 1\} \), define
\[
 X_i = \{ x^i, x^{m+i}, x^{2m+i}, \ldots, x^{(k-1)m+i} \} \text{.}
\]
If \( X_0 + X_0 = Z_N \setminus X_0 \), then
\[
 X_i + X_i = Z_N \setminus X_i
\]
for all \( i \in \{1, 2, \ldots, m - 1\} \).

Proof. First we check that each \( X_i \) is sum-free. Every element of \( X_i + X_i \) is of the form
\[
 x^{\alpha m + i} + x^{\beta m + i}
\]
for some integers \( \alpha \) and \( \beta \). If there is an integer \( q \) so that \( x^{\alpha m + i} + x^{\beta m + i} = x^{q m + i} \), then by factoring out \( x^i \), we have
\[
 x^{\alpha m} + x^{\beta m} = x^{q m},
\]
which is a contradiction, as \( X_0 + X_0 = Z_N \setminus X_0 \).

Suppose \( z \in Z_N \setminus X_i \). Recall that \( x \) is a generator of \( Z_N^{\times} \), so there exists an integer \( k \) so that \( z = x^k \). Since \( x^k \notin X_i \), we have \( x^k - i \notin X_0 \). This means there exist integers \( \alpha \) and \( \beta \) so that
\[
 x^{\alpha m} + x^{\beta m} = x^{k - i}.
\]
Multiplying both sides by \( x^i \) achieves the desired result.

Lemma 4. Suppose \( x \) is a generator of \( Z_N^{\times} \). For \( i \in \{0, 1, \ldots, m - 1\} \), define
\[
 X_i = \{ x^i, x^{m+i}, x^{2m+i}, \ldots, x^{(k-1)m+i} \} \text{.}
\]
If \( X_0 + X_i = Z_N \setminus \{0\} \) for all \( i \in \{1, 2, \ldots, m - 1\} \), then
\[
 \forall i \forall j (i \neq j \rightarrow X_i + X_j = Z_N \setminus \{0\}) \text{.}
\]
Proof. Fix \( i \) and \( j \) with \( i \neq j \). Without loss of generality, say \( j > i \). Given a nonnegative integer \( k \), we need to show that there exist integers \( \alpha \) and \( \beta \) so that

\[
x^{\alpha m + i} + x^{\beta m + j} = x^k.
\]

Since \( X_0 + X_{j-i} = \mathbb{Z}_N \setminus \{0\} \), there exist integers \( \alpha \) and \( \beta \) so that

\[
x^{\alpha m} + x^{\beta m + (j-i)} = x^{k-i}.
\]

Multiplying both sides by \( x^i \) gives the desired result. \( \square \)

4 Future directions

Although we have found constructions for many values of \( m \), we have not gained any insight into any sort of pattern, as the sequence of successive moduli is not even monotonic. If there is a pattern, it currently eludes the authors.

The recursive upper bound from [2,10] gives \( R(3, 3, 3, 3, 3, 3, 3, 3, 3) \leq 109602 \). By checking every candidate prime up through this bound, we were able to determine that there is no single-generator SSFCMB for \( m = 8 \).

**Theorem 2.** Let \( N \in \mathbb{Z}^+ \). There does not exist a partition of \( \mathbb{Z}_N \setminus \{0\} \) into 8 parts that is a single-generator SSFCMB.

For the case of \( m = 13 \), the recursive bound is too large for the computing power available to the authors to rule out existence of a single-generator SSFCMB. (The recursive upper bound is \( \approx 1.69 \cdot 10^{103} \).) However, if there is such a construction for \( m = 13 \), the modulus \( N \) must exceed 190997. Since this value is more than 100 times the size of those moduli for other similarly small values of \( m \), we conjecture that there is no such partition for \( m = 13 \).

**Conjecture 1.** Let \( N \in \mathbb{Z}^+ \). There does not exist a partition of \( \mathbb{Z}_N \setminus \{0\} \) into 13 parts that is a single-generator SSFCMB.

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A Table of \( m \) and corresponding moduli

Below we include tables containing the corresponding smallest modulus \( N \) for each value of \( m \), together with the smallest generator \( x \) of \( \mathbb{Z}_N^k \) needed to
construct a single-generator SSFCMB. Notice that $N = mk + 1$ for some positive integer $k$ in every case. To reconstruct any of the partitions, set $X_0 = \{x^0, x^m, x^{2m}, \ldots, x^{(k-1)m}\}$ and $X_i = x \cdot X_{i-1}$ for $i = 1, 2, \ldots, m - 1$. Hence, independent verification of any of the triples below is quite straightforward.

As mentioned in Section 4, the values $m = 8$ and $m = 13$ are missing from the table.

| $m$ | $N$ | $x$ | $m$ | $N$ | $x$ | $m$ | $N$ | $x$ |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 2   | 5   | 2   | 41  | 13367| 5   | 78  | 53197| 2  |
| 3   | 13  | 2   | 42  | 19993| 10  | 79  | 64781| 2  |
| 4   | 41  | 6   | 43  | 14621| 2   | 80  | 53441| 3  |
| 5   | 71  | 7   | 44  | 12497| 3   | 81  | 65287| 3  |
| 6   | 97  | 5   | 45  | 14401| 11  | 82  | 64781| 2  |
| 7   | 491 | 2   | 46  | 14537| 3   | 83  | 113213| 2 |
| 9   | 523 | 2   | 47  | 20117| 2   | 84  | 76777| 5  |
| 10  | 1181| 7   | 48  | 18913| 7   | 85  | 91121| 6  |
| 11  | 947 | 2   | 49  | 22541| 3   | 86  | 80153| 3  |
| 12  | 769 | 11  | 50  | 22901| 2   | 87  | 70123| 2  |
| 14  | 1709| 3   | 51  | 19687| 5   | 88  | 67409| 3  |
| 15  | 1291| 2   | 52  | 29537| 3   | 89  | 131543| 5 |
| 16  | 1217| 3   | 53  | 26501| 2   | 90  | 74161| 7  |
| 17  | 4013| 2   | 54  | 21493| 2   | 91  | 81173| 2  |
| 18  | 2521| 17  | 55  | 23321| 3   | 92  | 80777| 3  |
| 19  | 1901| 2   | 56  | 23297| 3   | 93  | 78307| 2  |
| 20  | 2801| 3   | 57  | 21319| 14  | 94  | 70877| 2  |
| 21  | 1933| 5   | 58  | 30509| 2   | 95  | 100511| 11|
| 22  | 3257| 3   | 59  | 28439| 11  | 96  | 136897| 5 |
| 23  | 3221| 10  | 60  | 26041| 13  | 97  | 96419| 6  |
| 24  | 4129| 13  | 61  | 45263| 5   | 98  | 105449| 6 |
| 25  | 3701| 2   | 62  | 27281| 6   | 99  | 87517| 2  |
| 26  | 4889| 3   | 63  | 30367| 5   | 100| 95801| 3  |
| 27  | 5563| 2   | 64  | 39041| 3   | 101| 154127| 5 |
| 28  | 8849| 3   | 65  | 37181| 2   | 102| 95881| 13 |
| 29  | 6323| 2   | 66  | 29569| 17  | 103| 119687| 5 |
| 30  | 5521| 11  | 67  | 38459| 2   | 104| 131249| 3 |
| 31  | 6263| 5   | 68  | 64601| 3   | 105| 89671| 6  |
| 32  | 5441| 3   | 69  | 31741| 6   | 106| 144161| 3 |
| 33  | 8779| 11  | 70  | 45641| 11  | 107| 88811| 2  |
| 34  | 7481| 6   | 71  | 36535| 3   | 108| 122041| 7 |
| 35  | 7841| 12  | 72  | 37441| 17  | 109| 128621| 2 |
| 36  | 10009| 11 | 73  | 44531| 2   | 110| 122321| 6 |
| 37  | 13469| 2 | 74  | 58313| 3   | 111| 95461| 2  |
| 38  | 12161| 3 | 75  | 48751| 3   | 112| 122753| 3 |
| 39  | 8971| 2 | 76  | 39521| 3   | 113| 120233| 3 |
| 40  | 14561| 6 | 77  | 70379| 6   | 114| 98953| 10 |
\begin{tabular}{|c|c|c|c|c|c|}
\hline
\textbf{m} & \textbf{N} & \textbf{x} & \textbf{m} & \textbf{N} & \textbf{x} \\
\hline
115 & 115001 & 3 & 152 & 213713 & 3 \\
116 & 159617 & 3 & 153 & 245719 & 11 \\
117 & 118873 & 5 & 154 & 590129 & 3 \\
118 & 159773 & 2 & 155 & 220721 & 3 \\
119 & 166601 & 6 & 156 & 254281 & 7 \\
120 & 120721 & 14 & 157 & 282287 & 5 \\
121 & 176903 & 5 & 158 & 352973 & 2 \\
122 & 160553 & 3 & 159 & 246769 & 7 \\
123 & 145879 & 13 & 160 & 281921 & 3 \\
124 & 171617 & 3 & 161 & 303647 & 7 \\
125 & 121001 & 6 & 162 & 347329 & 7 \\
126 & 165817 & 15 & 163 & 240263 & 5 \\
127 & 182627 & 2 & 164 & 278801 & 3 \\
128 & 129281 & 3 & 165 & 266641 & 19 \\
129 & 142159 & 6 & 166 & 292493 & 3 \\
130 & 225941 & 2 & 167 & 313961 & 3 \\
131 & 208553 & 3 & 168 & 294673 & 5 \\
132 & 187441 & 13 & 169 & 277499 & 2 \\
133 & 173699 & 2 & 170 & 329801 & 3 \\
134 & 243077 & 2 & 171 & 302329 & 7 \\
135 & 197101 & 2 & 172 & 320609 & 3 \\
136 & 215153 & 3 & 173 & 330431 & 23 \\
137 & 190979 & 6 & 174 & 285709 & 2 \\
138 & 156217 & 5 & 175 & 449051 & 2 \\
139 & 179033 & 3 & 176 & 375233 & 3 \\
140 & 191801 & 3 & 177 & 355063 & 7 \\
141 & 224473 & 10 & 178 & 395873 & 3 \\
142 & 218681 & 13 & 179 & 307523 & 2 \\
143 & 200201 & 3 & 180 & 361441 & 13 \\
144 & 184321 & 13 & 181 & 381911 & 17 \\
145 & 218081 & 6 & 182 & 347621 & 3 \\
146 & 257837 & 2 & 183 & 345139 & 2 \\
147 & 221677 & 2 & 184 & 315377 & 3 \\
148 & 262553 & 3 & 185 & 383321 & 3 \\
149 & 238103 & 5 & 186 & 418129 & 7 \\
150 & 199501 & 2 & 187 & 394571 & 6 \\
151 & 237977 & 3 & 188 & 429017 & 3 \\
\hline
\end{tabular}
|   m   |  N   |   x   |   m   |  N   |   x   |   m   |  N   |   x   |
|--------|------|-------|--------|------|-------|--------|------|-------|
| 226    | 539237 | 2     | 264    | 699073 | 5     | 302    | 1111361 | 3   |
| 227    | 52463 | 5     | 265    | 880331 | 7     | 303    | 948391 | 30   |
| 228    | 585049 | 7     | 266    | 1229453 | 2    | 304    | 964289 | 3    |
| 229    | 583493 | 2     | 267    | 690997 | 2     | 305    | 1087631 | 34  |
| 230    | 555221 | 10    | 268    | 941753 | 3     | 306    | 1171981 | 2   |
| 231    | 609379 | 2     | 269    | 833363 | 2     | 307    | 925913 | 3    |
| 232    | 609233 | 3     | 270    | 689581 | 10    | 308    | 1153709 | 3   |
| 233    | 642149 | 3     | 271    | 804329 | 3     | 309    | 975823 | 3    |
| 234    | 496549 | 2     | 272    | 875297 | 3     | 310    | 1009361 | 3   |
| 235    | 635441 | 12    | 273    | 716899 | 3     | 311    | 1014727 | 5   |
| 236    | 575669 | 3     | 274    | 778509 | 2     | 312    | 1129441 | 14  |
| 237    | 501493 | 2     | 275    | 929501 | 3     | 313    | 1214441 | 3    |
| 238    | 637841 | 21    | 276    | 724777 | 10    | 314    | 1366529 | 3    |
| 239    | 664421 | 2     | 277    | 916871 | 7     | 315    | 1167391 | 14  |
| 240    | 653281 | 7     | 278    | 856241 | 3     | 316    | 1216601 | 6    |
| 241    | 603947 | 2     | 279    | 921259 | 2     | 317    | 1381487 | 5    |
| 242    | 691637 | 2     | 280    | 975521 | 11    | 318    | 1176601 | 11  |
| 243    | 618679 | 3     | 281    | 911003 | 2     | 319    | 1052063 | 5    |
| 244    | 746153 | 3     | 282    | 680749 | 2     | 320    | 1210241 | 3    |
| 245    | 623771 | 2     | 283    | 946919 | 7     | 321    | 1145329 | 7    |
| 246    | 661741 | 2     | 284    | 983777 | 3     | 322    | 1409717 | 2    |
| 247    | 736061 | 2     | 285    | 949621 | 10    | 323    | 1149881 | 7    |
| 248    | 631409 | 3     | 286    | 1035893 | 2   | 324    | 1082161 | 7   |
| 249    | 761443 | 2     | 287    | 1080269 | 2   | 325    | 1066001 | 3    |
| 250    | 655501 | 2     | 288    | 816709 | 13    | 326    | 1270097 | 3    |
| 251    | 646577 | 3     | 289    | 826541 | 2     | 327    | 1043131 | 11   |
| 252    | 632521 | 11    | 290    | 1006301 | 2   | 328    | 1144721 | 3    |
| 253    | 719027 | 5     | 291    | 1230349 | 2   | 329    | 1309421 | 10   |
| 254    | 689357 | 2     | 292    | 1073393 | 3   | 330    | 1151041 | 17   |
| 255    | 632911 | 6     | 293    | 1181963 | 2   | 331    | 1397483 | 2    |
| 256    | 724481 | 3     | 294    | 981373 | 6     | 332    | 1496657 | 3    |
| 257    | 668201 | 6     | 295    | 918041 | 3     | 333    | 1235431 | 3    |
| 258    | 751297 | 5     | 296    | 877937 | 3     | 334    | 1269869 | 2    |
| 259    | 746957 | 2     | 297    | 880903 | 3     | 335    | 1345361 | 6    |
| 260    | 710321 | 3     | 298    | 1086509 | 2   | 336    | 1109473 | 5    |
| 261    | 694261 | 2     | 299    | 1288691 | 2   | 337    | 1317671 | 11   |
| 262    | 793337 | 3     | 300    | 940801 | 41    | 338    | 1643357 | 2    |
| 263    | 803729 | 3     | 301    | 1104671 | 7   | 339    | 1332949 | 6    |

12
| $m$  | $N$   | $x$ |
|------|-------|-----|
| 340  | 1247801 | 3   |
| 341  | 1434929 | 3   |
| 342  | 1240777 | 7   |
| 343  | 1423451 | 2   |
| 344  | 1922273 | 3   |
| 345  | 1267531 | 2   |
| 346  | 1325873 | 3   |
| 347  | 1345667 | 2   |
| 348  | 1251409 | 14  |
| 349  | 1741511 | 7   |
| 350  | 1378301 | 10  |
| 351  | 1308529 | 7   |
| 352  | 1490369 | 3   |
| 353  | 1650629 | 2   |
| 354  | 1215637 | 2   |
| 355  | 1392311 | 13  |
| 356  | 1536497 | 3   |
| 357  | 1391587 | 2   |
| 358  | 1644653 | 2   |
| 359  | 1482671 | 7   |
| 360  | 1204561 | 29  |
| 361  | 1608617 | 3   |
| 362  | 1755701 | 2   |
| 363  | 1577599 | 3   |
| 364  | 1486577 | 3   |
| 365  | 1658561 | 6   |
| 366  | 1630897 | 10  |
| 367  | 1551677 | 2   |
| 368  | 1389569 | 3   |
| 369  | 1461979 | 2   |

| $m$  | $N$   | $x$ |
|------|-------|-----|
| 370  | 1400081 | 3   |
| 371  | 1570073 | 3   |
| 372  | 1490233 | 7   |
| 373  | 2387201 | 3   |
| 374  | 1831853 | 2   |
| 375  | 1695751 | 3   |
| 376  | 1711553 | 3   |
| 377  | 1627133 | 2   |
| 378  | 1751653 | 2   |
| 379  | 1685511 | 11  |
| 380  | 1931921 | 3   |
| 381  | 1423417 | 11  |
| 382  | 1642601 | 3   |
| 383  | 1607069 | 2   |
| 384  | 1545217 | 15  |
| 385  | 1657811 | 2   |
| 386  | 1818833 | 3   |
| 387  | 1963639 | 3   |
| 388  | 1689353 | 3   |
| 389  | 2059367 | 5   |
| 390  | 1861861 | 2   |
| 391  | 1730567 | 5   |
| 392  | 1821233 | 3   |
| 393  | 1758283 | 3   |
| 394  | 1795853 | 2   |
| 395  | 1837541 | 3   |
| 396  | 1744777 | 7   |
| 397  | 1971503 | 5   |
| 398  | 2173877 | 2   |
| 399  | 2108317 | 2   |
| 400  | 1772801 | 3   |

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