GLOBAL WELL-POSEDNESS FOR THE CUBIC FRACTIONAL NLS ON THE UNIT DISK

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Abstract. In this paper, we show that the cubic nonlinear Schrödinger equation with the fractional Laplacian on the unit disk is globally well-posed for certain radial initial data below the energy space and establish a polynomial bound of the global solution. The result is proved by extending the I-method in the fractional nonlinear Schrödinger equation setting.

Keywords: I-method, global well-posedness, fractional NLS, compact manifold

Mathematics Subject Classification (2020): 35Q55, 35R01, 37K06, 37L50

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1. Introduction

We consider the two dimensional defocusing, cubic fractional nonlinear Schrödinger equations (FNLS)

\begin{equation}
\begin{aligned}
\frac{i}{\partial t} u - (-\Delta)^\alpha u &= |u|^2 u, \quad \alpha \in (0, 1], \\
u(0, x) &= \phi(x),
\end{aligned}
\end{equation}

posed on the unit disk \( \Theta = \{ x \in \mathbb{R}^2 \mid |x| < 1 \} \), where \( u = u(t, x) \) is a complex-valued function in spacetime \( \mathbb{R} \times \Theta \). We assume the radial symmetry on the initial datum \( u_0 \) and the Dirichlet boundary condition:

\[ u \big|_{\partial \Theta} = 0. \]

Note that when \( \alpha = 1 \), this is the classical nonlinear Schrödinger equation (NLS)

\begin{equation}
\frac{i}{\partial t} u + \Delta u = |u|^2 u,
\end{equation}

and for \( \alpha \in (0, 1) \), this is where our main interest located – fractional nonlinear Schrödinger equations (1.1).

Similar as in the NLS setting, the FNLS model conserves its energy and mass in the following forms

\begin{equation}
M(u) = \frac{1}{2} \int_{\Theta} |u|^2 \, dx,
\end{equation}
\begin{equation}
E(u) = \int_{\mathbb{R}^2} \frac{1}{2} |\nabla|^\alpha u|^2 + \frac{1}{4} |u|^4 \, dx,
\end{equation}

where $|\nabla| = \sqrt{-\Delta}$. Conservation laws above give the control of the $L^2$ and $H^\alpha$ norms of the solutions, respectively. Moreover, the scaling of (1.1) is given by

$$s_c = 1 - \alpha.$$ 

1.1. Motivation. In recent decades, there has been of great interest in using fractional Laplacians to model physical phenomena. The fractional quantum mechanics was introduced by Laskin [36] as a generalization of the standard quantum mechanics. This generalization operates on the Feynman path integral formulation by replacing the Brownian motion with a general Levy flight. As a consequence, one obtains fractional versions of the fundamental Schrödinger equation. That means the Laplace operator $(-\Delta)$ arising from the Gaussian kernel used in the standard theory is replaced by its fractional powers $(-\Delta)^\alpha$, where $0 < \alpha < 1$, as such operators naturally generate Levy flights. The general physical motivation of introducing fractional models is to deal with the so-called anomalous diffusion which arises in many complex systems - in physics, chemistry, biology, social sciences. While the Brownian diffusion, as mathematically studied by Smoluchowski [41] and Einstein [24], displays a Gaussian statistics, in an anomalous context the statistics usually follows a power law. This results in heavier distribution queues matching better the long range interaction phenomena. It turns out that the equation (1.1) and its discrete versions are relevant in molecular biology as they have been proposed to describe the charge transport between base pairs in the DNA molecule where typical long range interactions occur (in particular due to intersegmental jumps) [38]. The continuum limit for discrete FNLS was studied rigorously first in [35] and see also [28, 29, 32] for the recent works on the continuum limits.

In this paper, we study the global well-posedness theory\footnote{With local/global well-posedness, we mean local/global in time existence, uniqueness and continuous dependence of the datum to solution map.} of FNLS model on the unit disk. The reason why we consider this model is that (1) first, due to the lack of strong dispersion in FNLS (compared to NLS), the global well-posedness theory of FNLS is under development: (2) second, the compact manifold setting is interesting since it allows weaker dispersion than Euclidean spaces (hence less favorable). Apart from the challenges behind weak dispersion mentioned above, we should be aware of another difficulty on the unit disk – the lack of good Fourier convolution theorem. This theorem is fundamental in analyzing nonlinearities of the equation, the absence of which is due to the different Fourier transform on the unit disk (compared to those in Euclidean spaces) and will cause great difficulties in understanding the nonlinear term in the equation.

Our goal in this paper is to prove the global well-posedness of (1.1) with the regularity of the initial datum below the energy space $H^\alpha$. Before we present our result, let us first view related works in both NLS and FNLS settings.

1.2. History and related works. Let us start from the related works in NLS (that is, $\alpha = 1$ in (1.1)). Recall that in Euclidean spaces $\mathbb{R}^d$, the scaling of (1.2) is given by

$$s_c = \frac{d}{2} - 1.$$ 

The problem (1.2) is called subcritical when the regularity of the initial datum is smoother than the scaling $s_c$ of (1.2). We will adopt the language in the scaling context in other general manifolds.

In the subcritical regime ($s > s_c$), it is well-known that the initial value problem (1.2) is locally well-posed [8]. Thanks to the conservation laws of energy and mass defined in (1.3) and (1.4) (with $\alpha = 1$), the $H^1$-subcritical initial value problem and the $L^2$-subcritical initial value problem are globally well-posed in the energy space $H^1$ and mass space $L^2$, respectively. In fact, these two initial value problems are also shown to scatter\footnote{In general terms, with scattering we intend that the nonlinear solution as time goes to infinity approaches a linear one.}. However, one does not expect the scattering phenomenon in compact manifolds.

In the Euclidean space, the very first global well-posedness result in the subcritical case between the two (mass and energy) conservation laws was given by Bourgain in [5], where he developed the high-low method...
to prove global well-posedness for the cubic NLS in two dimensions for initial data in $H^s$, $s > \frac{3}{5}$. Bourgain’s method is known as high-low method since this method consists of estimating separately the evolution of the low frequencies and of the high frequencies of the initial datum. In fact, informally ‘dispersion’ will refer to the fact that the components of the wave packets tend to move at different speeds proportional to the size of the frequencies they are localized on. Hence in the study of dispersive equations (of course FNLS is a member in this family), understanding the behavior of high and low frequencies are very much needed.

To start the high-low method, the initial datum is decomposed into a (smoother) low frequency part and a (rougher) high frequency part. The reason below this cutoff here is that due to the datum belonging to the space below, its energy is infinity and it is impossible to employ the energy conservation law when iterating the local solutions up to arbitrarily large time intervals. The choice of the threshold between high and low modes will depend on the regularity of the datum and the arbitrarily large time interval that the solution lives in. After the cutoff, the low frequency part has finite energy, hence its evolution is globally existed. Separately, the nonlinear evolution (which is the Duhamel term in the integral equation) to the difference equation for the rougher part has small energy in an interval of time that is inverse proportional to the size of the low frequency part of the initial datum. Such smallness of the Duhamel term in the smoother energy space allows one to continue with an iteration by merging this smoother part with the evolution of the low frequency part of the datum. Let us remark that when estimating of the evolution of the rougher part of the datum, Bourgain used a Fourier transform based space $X^{s,b}$ that captures particularly well the behavior of solutions with low regularity initial datum. At last, let us also mention that the regularity restriction $s > \frac{3}{5}$ is derived by keeping the accumulation of energy controlled. As a result, in [5] the author obtained a polynomial bound of the sub-energy Sobolev norm of the global solution.

In [9], Colliander-Keel-Staffilani-Takaoaka-Tao improved the global well-posedness index of the initial datum to $H^s$, $s \geq \frac{7}{10}$ by introducing a different method, now known as I-method. Let us recall the I-method mechanism in [9] in the next paragraph since it is the method that we will use in this paper. This is also based on an iterative argument. As we mentioned in the high-low method, in the case of infinite energy solutions, it is hopeless to make good use of the conserved energy. However, one wishes to make some suitable modification of the energy, so that one could still have a good control on a substantial portion of the energy (which is known as the almost conservation laws). The modification in the method is to pick up all the low frequencies and certain amount of high frequencies in the datum. Similarly, the choice of the threshold between high and low frequencies depend on the regularity of the datum and the arbitrarily large time interval that one iterates the local solutions up to.

A standard I-method argument consists of three main steps. (1) One first defines a suitable Fourier multiplier (which is also called ‘I-operator’) that smooths out the initial datum from a rough Sobolev space into the energy space. The significance of this operator is that it allows one to grab and make use of the energy of the equation in some modified form. (2) Under this modification, the energy of the smoothed solution is not conserved any more (recall that the energy of the original solution is conserved, but infinity, hence impossible to be employed). Such modification is a trade-off between the conservation of infinity energy and its potential to play a role in the argument. With this being said, one can actually prove that the energy of the I-operator modified solution is almost conserved, that is, at each iteration the growth of such modified energy is uniformly small. (3) Last, iteration of the local well-posedness argument will give a global solution, and the regularity range $s > \frac{5}{6}$ is derived by keeping the accumulation of energy controlled. In fact, such almost conservation law gives alternative way to measure the growth of the Sobolev norm that the solution lives. As a byproduct, in [9] the authors obtained a polynomial bound of the sub-energy Sobolev norm of the global solution. The cubic NLS in $\mathbb{R}^3$ was also considered in [9] and the global well-posedness range is given by $s > \frac{4}{5}$. Later, in [12] by combining the Morawetz estimate with the I-method and a bootstrapping argument, the same authors were able to lower the global well-posedness range to $s > \frac{4}{5}$ and proved, for the first time\textsuperscript{3}, that the global solution also scatters for data in $H^s$, $s > \frac{4}{5}$.

The high-low method and I-method have been widely adapted into other dispersive settings and more general manifolds. For instance, [34] showed the global well-posedness for nonlinear wave equations using\textsuperscript{3}

\textsuperscript{3}Actually in [6] Bourgain proved the global well-posedness for general data in $H^s$, $s > \frac{11}{15}$ and scattering for radial data in $H^s$, $s > \frac{4}{5}$.
the high-low frequency decomposition of Bourgain and [40] applied I-method to nonlinear wave equations. [31, 51] studied the global well-posedness of the cubic NLS on closed manifolds without boundary using the I-method. As for the FNLS setting, using the high-low method, [19] was able to show the global well-posedness for FNLS on the one-dimensional torus. However, the higher dimensional analogue is still open and challenging. We will, in fact, investigate the the global behavior of FNLS in this paper in the higher dimensions. More results on the high-low method and the I-method for NLS can be found in [13, 15, 17, 18, 20, 21, 22, 23, 25, 30, 45, 48] in Euclidean spaces and [16, 37, 44] in non-Euclidean settings, and for other dispersive models [10, 11, 26, 39, 50]. The high-low method and the I-method are frequently employed in weak turbulence theory [14, 42, 43], with which the authors proved the growth of higher Sobolev norms. The tools in this paper also can be potentially applied to the weak turbulence theory.

Now let us present the main result in this paper.

1.3. Main result.

**Theorem 1.1.** The initial value problem (1.1) with \( \alpha \in \left( \frac{2}{3}, 1 \right] \) is globally well-posed for radial data \( u_0 \in H^s_{\text{rad}}(\Theta) \), where

\[
s > s^\ast(\alpha) = \max \left\{ \frac{1}{4} \left( \frac{4\alpha^2 - \alpha - 1}{2\alpha - 1} + \sqrt{\frac{5\alpha^2 - 4\alpha + 1}{(2\alpha - 1)^2}} \right), \frac{1}{4} \left( \frac{\alpha^2 + \alpha - 1}{2\alpha - 1} + \sqrt{\frac{\alpha^4 + 10\alpha^3 - 5\alpha^2 - 2\alpha + 1}{(2\alpha - 1)^2}} \right) \right\}.
\]

Moreover, we establish the polynomial bound of the solution

\[
\|u(T)\|_{H^s_{\text{rad}}(\Theta)} \lesssim T^{\frac{1}{4}(\alpha-s)p},
\]

where the power \( p \) above is given by

\[
N^p := \min \left\{ N(\alpha-s)(\frac{\alpha}{2} - 4 - \frac{2\alpha^2}{2\alpha - 1}) + \frac{1}{2}, N(\alpha-s)(\frac{\alpha}{2} - 4) + 3\alpha - 2 + \right\}.
\]

**Remark 1.2.** We note that \( s^\ast(\alpha) < \alpha \). Actually \( s^\ast(\alpha) \) looks very complicated and it might be hard for readers to see the behavior from its expression. Here is a quick plot of \( s^\ast(\alpha) \) and \( \alpha \).

\[0.7 0.75 0.8 0.85 0.9 0.95 1] \]

\[0.7 0.8 0.9 1] \]

\[s^\ast(\alpha)
\]

\[\alpha
\]

**Remark 1.3.** Using the bilinear estimates in Section 3 and similar strategy in Section 4, it is easy to obtain a local well-posedness argument for FNLS with \( \alpha \in \left( \frac{1}{2}, 1 \right) \). Then thanks to the conservation of energy, it is standard to show the global well-posedness for \( \alpha \in \left( \frac{1}{2}, 1 \right) \) with data in energy space by iterating the local theory. This is the reason why in the following plot, we can extend our global index in \( \alpha \in \left( \frac{1}{2}, \frac{2}{3} \right] \) trivially and the global well-posedness curve for \( \alpha \in \left( \frac{1}{2}, \frac{2}{3} \right] \) agrees with the energy line.
1.3.1. Discussion on the setting and the difficulties.

- **Compact manifold.** Compact domains usually allow weaker dispersion than Euclidean spaces do. Mathematically we can observe this phenomenon (‘loss of regularity’) in the Strichartz estimates on the bounded manifolds. For example in [3], the loss of \( \frac{1}{p} \) derivatives was established for the classical NLS posed on the compact Riemannian manifold \( \Omega \) with boundary

\[
\|e^{it\Delta}f\|_{L^p([0,T];L^q(\Omega))} \leq C \|f\|_{H^s(\Omega)}
\]

for fixed finite \( T, p > 2, q < \infty \) and \( \frac{2}{p} + \frac{d}{q} = \frac{d}{2} \). We expect that a similar loss of regularity phenomenon happens in the FNLS setting. To beat the weaker dispersion caused by the compact domain, we assume the radial symmetry on the initial datum. Under this assumption, we can benefit a lot from the decay of the radial Laplace operator. More precisely, the radial eigenfunctions of the Laplace operator \(-\Delta\) with Dirichlet boundary conditions behave like

\[
e_n(r) \sim \cos((n - \frac{1}{4})\pi r - \frac{\pi}{4}) \sqrt{r}.
\]

(1.5)

(where \( r = |x| \)) and their associated eigenvalues are \( z_n^2 \sim n^2 \) (see Subsection 2.3 for more detailed discussion on \( e_n \)'s and \( z_n \)'s). Relying on the decay of \( e_n \)'s, we are able to derive a bilinear Strichartz estimate for a product of two functions that are localized in high and low frequencies respectively. The benefit of the bilinear Strichartz estimate is that the ‘loss of regularity’ falls on the term with low frequency instead of on both terms (if naively splitting two functions in the bilinear form into two separate estimates then applying the Strichartz estimates), which is crucial to make up for the lack of dispersion.

- **Absence of Fourier convolution theorem.** Note that the Fourier convolution theorem plays an essential rule in I-method, since the convolution theorem translates the Fourier transform of a product into the convolution of the convolution transforms. Combining this fundamental fact with Littlewood-Paley decomposition, we can interpret the nonlinear term \( |u|^2 u \) as the sum of the interaction between functions \( u_1, u_2, u_3 \) with frequencies localized at \( \xi_i \) (\( i = 1, 2, 3 \)) on the Fourier side. For example, let the output frequency of in the nonlinearity \( |u|^2 u \) to be \( \xi \) and each function \( u \) is frequency localized at \( \xi_1, \xi_2, \xi_3 \). This convolution theorem implies that \( \xi_1 - \xi_2 + \xi_3 = \xi \), which means that this connection in \( \xi_1, \xi_2, \xi_3 \) does not allow the existence of any extremely huge frequency (compared to the output frequency \( \xi \)). However, on the unit disk, we lose such control in the highest frequency due to the absence of Fourier convolution theorem. This causes great difficulty in summing over the frequencies produced from Littlewood-Paley decomposition back to the original nonlinearity.

Let us mention that in a recent work [44], where the authors extended the high-low method of Bourgain in the hyperbolic setting. They had similar issue with the convolution theorem, and they managed to recover...
the smoothing estimate on the Duhamel term via the local smoothing estimate combining the radial Sobolev embedding. However, one does not expect to hold such local smoothing estimates on the compact domains.

Back to our unit disk setting, in order to make up for this absence, let us first take a closer look at the eigenfunctions. In the approximate expression of $\epsilon_n$ (1.5), we see nothing but trigonometric functions. This suggests in some sense the existence of certain type of weak interaction between functions whose frequencies are far from each other. Another hope for us to expect such ‘convolution’ type control is behind the following result. It is shown in [7] that in the compact domain without boundary (for example $S^2$), the weak interaction functions with separated frequencies. More precisely, for any $j = 1, 2, 3$, $z_n \ll z_n$ (recall that $z_n$’s are eigenvalues corresponding to eigenfunctions $\epsilon_n$’s), then for every $p > 0$ there exists $C_p > 0$ such that for every $w_j \in L^2(S^2)$, $j = 0, 1, 2, 3$,

\begin{equation}
(1.6) \quad \left| \int_{S^2} P_{n_0} w_0 P_{n_1} w_1 P_{n_2} w_2 P_{n_3} w_3 \, dx \right| \leq C_p z_n^{-p} \prod_{j=0}^3 \| w_j \|_{L^2}.
\end{equation}

Note that the factor $z_n^{-p}$ above can be understood as the weak interaction in their setting. Hence to obtain a similar weak interaction in our domain with boundary, we develop Proposition 5.2, which essentially captures the features in (1.6). That is,

\begin{equation}
(1.7) \quad \left| \int_{R \times \Theta} P_{n_0} w_0 P_{n_1} w_1 P_{n_2} w_2 P_{n_3} w_3 \, dx dt \right| \leq \frac{z_n^{2z_n} z_n^{3z_n}}{z_n^{2z_n} z_n^{3z_n}} \frac{1}{n_0^{z_n}} \prod_{j=0}^3 \| P_{n_j} w_j \|_{X^{0,b}}
\end{equation}

for $z_n \geq 2z_n, z_n \geq z_n$ (see Subsection 2.5 for the definition of $X^{0,b}$ norms). Here the factor $\frac{z_n^{2z_n} z_n^{3z_n}}{z_n^{2z_n} z_n^{3z_n}} \frac{1}{n_0^{z_n}}$ serves as a similar role of $z_n^{-p}$ in (1.6). It also should be pointed out that this is the key that allows us to sum up decomposed functions with frequencies greatly separated.

Now let us give the main ideas of the proofs.

1.3.2. Outline of the proofs. In this subsection we summarize the main three parts in the proof of the main Theorem 1.1.

In the first part of the proof we present the local theory of the I-operator modified FNLS. In this local theory, as one did in the NLS case, we need a Strichartz-type estimate to run the contraction mapping argument. To this end, we adapt the proof of bilinear estimates for NLS on the unit ball in [1] in Section 3 (see also [47] for the multilinear estimates for NLS on the unit ball). However, it is worth pointing out that due to the fractionality of the dispersion operator, it is impossible for us to periodize the time in the bilinear estimates and count the integer points on its Fourier characteristic surface. Instead, we have to count the integer points near the characteristic surface, which results in the local well-posedness index not as good as one obtained in the NLS setting. As for the proof of the local theory, with the help of the bilinear estimates in Section 3, we are able to obtain an estimate on the nonlinear term, hence obtain the local well-posedness via a standard contraction argument. Let us also mention that since this counting argument does not see the difference in the fractional power $\alpha$ of Laplacian, the local well-posedness index is in fact uniform for all power $\alpha \in (\frac{1}{4}, 1)$.

Following the I-method mechanism in [9], the second part of the proof deals with the analysis of the energy increment of the modified equation. A typical strategy to follow is that one dyadically decomposes all the functions in the change of energy, then proceeds the analysis in different localized frequency scenarios, and in the end sums all the decomposed frequencies back to the original form. In order to sum up all the decomposed functions in frequencies, we require a good control on the highest frequency, whose range is usually governed by the Fourier convolution theorem. However, such nice control in the highest frequency does not hold on the disk due to different format of eigenfunctions of the radial Dirichlet Laplacian. Hence a different analysis

\footnote{Roughly speaking, this $X^{s,b}$ norm is defined based on a spacetime Fourier transformation, and is very adapted to the dispersive context as it is constructed upon the underlying dispersive operator. In a perturbative regime (subcritical nonlinearities), the Fourier transform of the solutions is supported around the characteristic surface given by the linear operator, hence the $X^{s,b}$ spaces capture efficiency this clustering.}
is needed. Instead of the dyadic decomposition, we make a finer and delicate decomposition on frequencies, which allows us to observe a very weak interaction between functions localized in incomparable frequencies. Fortunately, this treatment fulfills the role of convolution theorem and allows to sum the frequency localized functions in a proper way, which is presented in (1.7).

At last, we iterate the local theory obtained in the first part, hence obtain the global solution. It should be noted that in this argument, to make the iteration work, we need to guarantee that the accumulated energy increment does not surpass the size of the initial energy of the modified initial datum, which ensures that the initial setup remains the same in the next iteration. As a byproduct of the method, one obtains that the global solutions satisfy polynomial-in-time bounds.

1.4. Organization of the paper. In Section 2, we introduce the notations, eigenfunctions and eigenvalues of the radial Dirichlet Laplacian and the functional spaces with their properties that we will use in this paper. In Section 3, we prove bilinear Strichartz estimates, which is an important tool in the proof of the energy increment in Section 6. In Section 4, we first define the I-operator in our setting and present a local theory based on the I-operator modified equation. In Section 5, we discuss the weak interaction between functions whose frequencies are localized far away. Then in Section 6, we compute the energy increment of the modified energy on small time intervals. Finally, in Section 7, we show the global well-posedness and establish the polynomial bound for the global solutions in Theorem 1.1.

Acknowledgement. X.Y. is funded in part by the Jarve Seed Fund and an AMS-Simons travel grant. Both authors would like to thank Gigliola Staffilani for very insightful comments on a preliminary draft of this paper. The authors are very grateful to the anonymous referees for valuable comments and suggestions.

2. Preliminaries

In this section, we first discuss notations used in the rest of the paper, provide the properties of Bessel functions that will be used in later sections, and recall the behaviors of eigenfunctions and eigenvalues of the radial Dirichlet Laplacian. Then we introduce the function spaces ($H^s$ and $X^{s,b}$ spaces) that we will be working on and list some useful inequalities from harmonic analysis.

2.1. Notations. We define

$$
\|f\|_{L^q_t L^r_x(I \times \Theta)} := \left[ \int_I \left( \int_{\Theta} |f(t,x)|^r \, dx \right)^{\frac{q}{r}} \, dt \right]^{\frac{1}{q}},
$$

where $I$ is a time interval.

For $x \in \mathbb{R}$, we set $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$. We adopt the usual notation that $A \lesssim B$ or $B \gtrsim A$ to denote an estimate of the form $A \leq CB$, for some constant $0 < C < \infty$ depending only on the a priori fixed constants of the problem. We write $A \sim B$ when both $A \lesssim B$ and $B \lesssim A$.

2.2. Bessel functions and their properties. The Bessel function of order $n$, $J_n(x)$, is defined by

$$
J_n(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j! \Gamma(j + n + 1)} \left( \frac{x}{2} \right)^{2j+n}.
$$

In fact, we will only need Bessel functions of order zero and order one, that is, $J_0(x)$ and $J_1(x)$.

$$
J_0(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(j!)^2} \left( \frac{x}{2} \right)^{2j},
$$

$$
J_1(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(j + 1)!} \left( \frac{x}{2} \right)^{2j+1}.
$$
Moreover, the derivatives of \( J_0(x) \) and \( J_1(x) \) satisfy
\[
\frac{d}{dx} J_0(x) = -J_1(x), \tag{2.1}
\]
\[
\frac{d}{dx} J_1(x) = x J_0(x). \tag{2.2}
\]
We also have the following approximation formulas for \( n = 0, 1 \)
\[
J_n(x) = \begin{cases} \frac{1}{n! \pi} x^n + O(x^{n+2}), & \text{when } |x| < 1, \\ \sqrt{\frac{2}{\pi x}} \cos \left( \frac{n \pi}{2} - \frac{x}{2} \right) + O(x^{-2}), & \text{when } |x| \geq 1. \end{cases} \tag{2.3, 2.4}
\]

2.3. Eigenfunctions and eigenvalues of the radial Dirichlet Laplacian. We denote \( e_n(r) \) (where \( r = |x| \)) to be the eigenfunctions of the radial Laplace operator \(-\Delta\) with Dirichlet boundary condition \( \Theta \), and the eigenvalues associated to \( e_n \) are \( z_n^2 \). Both \( e_n \)'s and \( z_n \)'s are defined via Bessel functions.

Recall that \( J_0 \) is the Bessel function of order zero
\[
J_0(x) = \sqrt{\frac{2}{\pi x}} \cos \left( \frac{x}{2} \right) + O(x^{-2}). \tag{2.5}
\]
Let \( z_n \)'s be the (simple) zeros of \( J_0(x) \) such that \( 0 < z_1 < z_2 < \cdots < z_n < \cdots \). It is known that \( z_n \) satisfies
\[
z_n = \pi \left( n - \frac{1}{4} \right) + O \left( \frac{1}{n} \right). \tag{2.6}
\]
Also \( J_0(z_n r) \) are eigenfunctions of the Dirichlet self adjoint realization of \(-\Delta\), corresponding to eigenvalues \( z_n^2 \). Moreover any \( L^2(\Theta) \) radial function can be expanded with respect to \( J_0(z_n r) \). Let us set
\[
e_n := e_n(r) = \| J_0(z_n \cdot) \|_{L^2(\Theta)}^{-1} J_0(z_n r). \tag{2.7}
\]
A direct computation gives
\[
\| J_0(z_n \cdot) \|_{L^2(\Theta)}^{-1} \sim z_n^{\frac{1}{2}} \sim n^{-\frac{1}{2}},
\]
then combining with (2.5), (2.6) and (2.7) we have
\[
e_n(r) \sim \frac{\cos((n - \frac{1}{4})\pi r - \frac{\pi}{4})}{\sqrt{r}}.
\]
In Lemma 2.5 in [2], one also has
\[
\| e_n \|_{L^p(\Theta)} \lesssim \begin{cases} 1, & \text{if } 2 \leq p < 4, \\ \ln(1 + n)^{\frac{1}{2}}, & \text{if } p = 4, \\ n^{\frac{1}{2} - \frac{1}{p}}, & \text{if } p > 4. \end{cases} \tag{2.9}
\]

2.4. \( H^s_{\text{rad}} \) spaces. Recall that \( (e_n)^{\infty}_{n=1} \) form an orthonormal bases of the Hilbert space of \( L^2 \) radial functions on \( \Theta \). That is,
\[
\int e_n^2 dL = 1,
\]
where \( dL = \frac{1}{\pi} r d\theta dr \) is the normalized Lebesgue measure on \( \Theta \). Therefore, we have the expansion formula for a function \( u \in L^2(\Theta) \),
\[
u = \sum_{n=1}^{\infty} \langle u, e_n \rangle e_n.
\]
For \( s \in \mathbb{R} \), we define the Sobolev space \( H^s(\Theta) \) on the closed unit ball \( \Theta \) as
\[
H^s_{\text{rad}}(\Theta) := \left\{ u = \sum_{n=1}^{\infty} c_n e_n, c_n \in \mathbb{C} : \| u \|_{H^s(\Theta)}^2 = \sum_{n=1}^{\infty} z_n^{2s} |c_n|^2 < \infty \right\}.
\]
We can equip $H_{rad}^{s}(\Theta)$ with the natural complex Hilbert space structure. In particular, if $s = 0$, we denote $H_{rad}^{0}(\Theta)$ by $L^{2}_{rad}(\Theta)$. For $\gamma \in \mathbb{R}$, we define the map $\sqrt{-\Delta}$ acting as isometry from $H_{rad}^{s}(\Theta)$ and $H_{rad}^{-\gamma}(\Theta)$ by

$$\sqrt{-\Delta} \left( \sum_{n=1}^{\infty} c_{n} e_{n} \right) = \sum_{n=1}^{\infty} z_{n}^{\gamma} c_{n} e_{n}.$$ 

We denote

$$S_{\alpha}(t) = e^{-it(-\Delta)^{\alpha}}$$

the flow of the linear Schrödinger equation with Dirichlet boundary conditions on the unit ball $\Theta$, and it can be written into

$$S_{\alpha}(t) \left( \sum_{n=1}^{\infty} c_{n} e_{n} \right) = \sum_{n=1}^{\infty} e^{-iz_{n}^{2} t} c_{n} e_{n}.$$ 

2.5. $X_{rad}^{s,b}$ spaces. Using again the $L^{2}$ orthonormal basis of eigenfunctions $\{e_{n}\}_{n=1}^{\infty}$ with their eigenvalues $z_{n}^{2}$ on $\Theta$, we define the $X_{rad}^{s,b}$ spaces of functions on $\mathbb{R} \times \Theta$ which are radial with respect to the second argument.

**Definition 2.1** ($X_{rad}^{s,b}$ spaces). For $s \geq 0$ and $b \in \mathbb{R}$,

$$X_{rad}^{s,b}(\mathbb{R} \times \Theta) = \{ u \in S'(\mathbb{R}, L^{2}(\Theta)) : \| u \|_{X_{rad}^{s,b}(\mathbb{R} \times \Theta)} < \infty \},$$

where

$$\| u \|_{X_{rad}^{s,b}(\mathbb{R} \times \Theta)}^{2} = \sum_{n=1}^{\infty} \left\| \left\langle x + z_{n}^{2} t \right\rangle^{b} \left\langle z_{n}^{s} \right\rangle c_{n}(\tau) \right\|_{L^{2}(\mathbb{R})}^{2},$$

and

$$u(t) = \sum_{n=1}^{\infty} c_{n}(t)e_{n}.$$ 

Moreover, for $u \in X_{rad}^{0,\infty}(\Theta) = \bigcap_{b \in \mathbb{R}} X_{rad}^{0,b}(\Theta)$ we define, for $s \leq 0$ and $b \in \mathbb{R}$, the norm $\| u \|_{X_{rad}^{s,b}(\mathbb{R} \times \Theta)}$ by (2.10). Equivalently, we can write the norm (2.10) in the definition above into

$$\| u \|_{X_{rad}^{s,b}(\mathbb{R} \times \Theta)} = \| S_{\alpha}(-t)u \|_{H_{rad}^{s,b}(\mathbb{R} \times \Theta)}.$$ 

For $T > 0$, we define the restriction spaces $X_{T}^{s,b}(\Theta)$ equipped with the natural norm

$$\| u \|_{X_{T}^{s,b}(\Theta)} = \inf \{ \| \tilde{u} \|_{X_{rad}^{s,b}(\mathbb{R} \times \Theta)} : \tilde{u}\big|_{(-T,T) \times \Theta} = u \}.$$ 

**Lemma 2.2** (Basic properties of $X_{rad}^{s,b}$ spaces). (1) We have the trivial nesting

$$X_{rad}^{s,b} \subset X_{rad}^{s',b'}$$

whenever $s' \leq s$ and $b' \leq b$, and

$$X_{T}^{s,b} \subset X_{T'}^{s,b}$$

whenever $T' \leq T$.

(2) The $X_{rad}^{s,b}$ spaces interpolate nicely in the $s,b$ indices.

(3) For $b > \frac{1}{2}$, we have the following embedding

$$\| u \|_{L^{\infty}_{x} H^{b}_{y}(\mathbb{R} \times \Theta)} \leq C \| u \|_{X_{rad}^{s,b}(\mathbb{R} \times \Theta)}.$$

(4) An embedding that will be used frequently in this paper

$$X_{rad}^{0,b} \hookrightarrow L^{4}_{x} L^{2}_{z}.$$
Note that
\[ \| f \|_{L_t^1L_x^2} = \| S_\alpha(t)f \|_{L_t^1L_x^2} \leq \| S_\alpha(t)f \|_{H_x^{s,b}L_t^2} = \| f \|_{X^{s,b} \Theta}. \]

**Lemma 2.3.** Let \( b, s > 0 \) and \( u_0 \in H_{rad}^s(\Theta) \). Then there exists \( c > 0 \) such that for \( 0 < T \leq 1 \),
\[ \| S_\alpha(t)u_0 \|_{X^{s,b}_r([-T,T] \times \Theta)} \leq c \| u_0 \|_{H_{rad}^s(\Theta)}. \]

The proofs of Lemma 2.2 and Lemma 2.3 can be found in [1].

We also recall the following lemma in [4, 27]

**Lemma 2.4.** Let \( 0 < b' < \frac{1}{2} \) and \( 0 < b < 1 - b' \). Then for all \( f \in X_{\theta}^{s-b'}(\Theta) \), we have the Duhamel term
\( w(t) = \int_0^t S_\alpha(t-s)f(\tau)ds \in X_{\theta,b}(\Theta) \) and moreover
\[ \| w \|_{X_{\theta,b}(\Theta)} \leq CT^{1-b-b'} \| f \|_{X_{\theta}^{s-b}(\Theta)}. \]

### 2.6. Useful inequalities.

#### Lemma 2.5 (Gagliardo-Nirenberg interpolation inequality). Let \( 1 < p < q \leq \infty \) and \( s > 0 \) be such that
\( \frac{1}{q} = \frac{1}{p} - \frac{a}{d} \) for some \( 0 < \theta = \theta(d, p, q, s) < 1 \). Then for any \( u \in \dot{W}^{s,p}(\mathbb{R}^d) \), we have
\[ \| u \|_{L^q(\mathbb{R}^d)} \leq L_{p,q,s} \| u \|_{L^p(\mathbb{R}^d)}^{\frac{\theta}{\theta}} \| u \|_{\dot{W}^{s,p}(\mathbb{R}^d)} \| u \|_{\dot{W}^{s,p}(\mathbb{R}^d)}. \]

#### Lemma 2.6 (Sobolev embedding). For any \( u \in C_0^\infty(\mathbb{R}^d) \), \( \frac{1}{p} - \frac{1}{q} = \frac{s}{d} \) and \( s > 0 \), we have
\( \dot{W}^{s,p}(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d). \)

From now on, for simplicity of notation, we write \( H^s \) and \( X^{s,b} \) for the spaces \( H_{rad}^s \) and \( X_{rad}^{s,b} \) defined in Section 2.

### 3. Bilinear Strichartz estimates

In this section, we prove the bilinear estimates that will be heavily used in the rest of this paper. The proof is adapted from [1] with two dimensional modification and a different counting lemma.

#### 3.1. Bilinear Strichartz estimates for FNLS.

**Lemma 3.1 (Bilinear estimates).** Consider \( \alpha \in [\frac{1}{2}, 1) \). For \( j = 1, 2, N_j > 0 \) and \( u_j \in L^2(\Theta) \) satisfying
\[ 1 - \Delta x \in [N_j, 2N_j]u_j = u_j, \]
we have the following bilinear estimates.

1. **The bilinear estimate without derivatives.** Without loss of generality, we assume \( N_1 \geq N_2 \), then for any \( \varepsilon > 0 \)
\[ \| S_\alpha(t)u_1 S_\alpha(t)u_2 \|_{L_{t,x}^{2,\varepsilon}(0,1) \times \Theta} \lesssim N_2^{\frac{1}{2} + \varepsilon} \| u_1 \|_{L^2(\Theta)} \| u_2 \|_{L^2(\Theta)}. \]

2. **The bilinear estimate with derivatives.** Moreover, if \( u_j \in H_0^0(\Theta) \) and assume \( N_1 \geq N_2 \), then for any \( \varepsilon > 0 \)
\[ \| \nabla S_\alpha(t)u_1 S_\alpha(t)u_2 \|_{L_{t,x}^{2,\varepsilon}(0,1) \times \Theta} \lesssim N_1 N_2^{\frac{1}{2} + \varepsilon} \| u_1 \|_{L^2(\Theta)} \| u_2 \|_{L^2(\Theta)}. \]

**Remark 3.2.** Lemma 3.3 also holds for \( S_\alpha(t)u_0 \overline{S_\alpha(t)v_0} \). In fact,
\[ \| S_\alpha(t)u_0 S_\alpha(t)v_0 \|_{L_{t,x}^2}^2 = \| S_\alpha(t)u_0 S_\alpha(t)v_0 \|_{L_{t,x}^2} \| S_\alpha(t)u_0 S_\alpha(t)v_0 \|_{L_{t,x}^2} = \| S_\alpha(t)u_0 S_\alpha(t)v_0 \|_{L_{t,x}^2}^2. \]
Proposition 3.3 (Lemma 2.3 in [7]: Transfer principle). For any $b > \frac{1}{2}$ and for $j = 1, 2$, $N_j > 0$ and $f_j \in X^{0,b}(\mathbb{R} \times \Theta)$ satisfying
\[
1_{\sqrt{-\Delta} \in [N_j, 2N_j]} f_j = f_j,
\]
on one has the following bilinear estimates.

1. The bilinear estimate without derivatives.
   Without loss of generality, we assume $N_1 \geq N_2$, then for any $\varepsilon > 0$
   \[
   \|f_1 f_2\|_{L_x^2((0,1) \times \Theta)} \lesssim N_2^{\frac{1}{2} + \varepsilon} \|f_1\|_{X^{0,b}((0,1) \times \Theta)} \|f_2\|_{X^{0,b}((0,1) \times \Theta)}.
   \]

2. The bilinear estimate with derivatives.
   Moreover, if $f_j \in H^1_0(\Theta)$ and assume $N_1 \geq N_2$, then for any $\varepsilon > 0$
   \[
   \|\nabla f_1 f_2\|_{L_x^2((0,1) \times \Theta)} \lesssim N_1 N_2^{\frac{1}{2} + \varepsilon} \|f_1\|_{X^{0,b}((0,1) \times \Theta)} \|f_2\|_{X^{0,b}((0,1) \times \Theta)}.
   \]

Remark 3.4 (Interpolation of bilinear estimates). In fact, using Hölder inequality, Bernstein inequality and Lemma 2.2, we write
\[
\|f_1 f_2\|_{L_x^2((0,1) \times \Theta)} \lesssim \|f_1\|_{L_t^1 L_x^{4/3}((0,1) \times \Theta)} \|f_2\|_{L_t^1 L_x^{4/3}((0,1) \times \Theta)} \lesssim \|f_1\|_{X^{0,0}((0,1) \times \Theta)} N_2 \|f_2\|_{L_x^2((0,1) \times \Theta)} \lesssim N_2 \|f_1\|_{X^{0,0}((0,1) \times \Theta)} \|f_2\|_{X^{0,0}((0,1) \times \Theta)}.
\]
Then interpolating it with (3.3), for $b = \frac{1}{2} + (1 - \beta)\frac{1}{2}$, we obtain
\[
\|f_1 f_2\|_{L_x^2((0,1) \times \Theta)} \lesssim N_2^b \|f_1\|_{X^{0,0}((0,1) \times \Theta)} \|f_2\|_{X^{0,0}((0,1) \times \Theta)},
\]
where $b(\beta) = \frac{1}{2} + (1 - \beta)\frac{1}{2}$, $\beta \in (\frac{1}{2}, 1]$. Moreover, if restricting in the time interval $[0, \delta]$, we have by Hölder inequality
\[
\|f_1 f_2\|_{L_x^2((0,\delta) \times \Theta)} \lesssim N_2^b \|f_1\|_{X^{0,0}(0,\delta)} \|f_2\|_{X^{0,0}(0,\delta)}.
\]

Proof of Lemma 3.1. First we write
\[
\begin{align*}
  u_1 &= \sum_{n_1 \sim N_1} c_{n_1} e_{n_1}(r), & u_2 &= \sum_{n_2 \sim N_2} d_{n_2} e_{n_2}(r),
\end{align*}
\]
where $e_{n_1} = (u_1, e_{n_1})_{L^2}$ and $d_{n_2} = (u_2, e_{n_2})_{L^2}$. Then
\[
S_\alpha(t) u_1 = \sum_{n_1 \sim N_1} e^{-i\tau_{n_1}^2} c_{n_1} e_{n_1}(r), & S_\alpha(t) u_2 = \sum_{n_2 \sim N_2} e^{-i\tau_{n_2}^2} d_{n_2} e_{n_2}(r).
\]
Therefore, the bilinear objects that one needs to estimate are the $L_x^2$ norms of
\[
E_0(N_1, N_2) = \sum_{n_1 \sim N_1} \sum_{n_2 \sim N_2} e^{-i\tau_{n_1}^2 + \tau_{n_2}^2} (c_{n_1} d_{n_2}) (c_{n_1} e_{n_2}),
\]
\[
E_1(N_1, N_2) = \sum_{n_1 \sim N_1} \sum_{n_2 \sim N_2} e^{-i\tau_{n_1}^2 + \tau_{n_2}^2} (c_{n_1} d_{n_2}) (\nabla c_{n_1} e_{n_2}).
\]
Let us focus on (3.1) first.
\[
(LHS \text{ of } (3.1))^2 = \|E_0(N_1, N_2)\|^2_{L^2((0,1) \times \Theta)}
\]
\[
= \int_{\mathbb{R} \times \Theta} \left| \sum_{n_1 \sim N_1} \sum_{n_2 \sim N_2} e^{-i\tau_{n_1}^2 + \tau_{n_2}^2} (c_{n_1} d_{n_2}) (c_{n_1} e_{n_2}) \right|^2 dx dt
\]
Here we employ a similar argument used in the proof of Lemma 2.6 in [46]. We fix $\eta \in C_0^\infty((0, 1))$, such that $\eta|_I \equiv 1$ where $I$ is a slight enlargement of $(0, 1)$. Thus we continue from (3.5)

$$
(3.5) \leq \int_{\mathbb{R} \times \Theta} \eta(t) \left| \sum_{n_1 \sim N_1, n_2 \sim N_2} e^{-i(tz_{n_1}^\alpha + z_{n_2}^\alpha)}(c_{n_1} d_{n_2})(e_{n_1} e_{n_2}) \right|^2 \ dx \ dt
$$

By expanding the square above and using Plancherel in time, we have

$$
(3.6) = \int_{\mathbb{R} \times \Theta} \eta(t) \left| \sum_{\tau, n_1 \sim N_1, n_2 \sim N_2} e^{-i(tz_{n_1}^\alpha + z_{n_2}^\alpha)}(c_{n_1} d_{n_2})(e_{n_1} e_{n_2}) \right|^2 \ dx \ dt,
$$

where

$$
\#A_{N_1, N_2, \tau} = \# \{(n_1, n_2) \in \mathbb{N}^2 : n_1 \sim N_1, n_2 \sim N_2, \left| z_{n_1}^\alpha + z_{n_2}^\alpha - \tau \right| \leq \frac{1}{2} \}.
$$

By expanding the square above and using Plancherel in time, we have

$$
(3.6) = \sum_{\tau, \tau' \sim N_1, N_2, \tau'} \sum_{(n_1, n_2) \in A_{N_1, N_2, \tau}} \sum_{(n_1', n_2') \in A_{N_1, N_2, \tau'}} \hat{\eta}(z_{n_1}^\alpha + z_{n_2}^\alpha) - \hat{\eta}(z_{n_1}^\alpha) \cdot \hat{\eta}(z_{n_2}^\alpha) \int_{\Theta} \left| e_{n_1} e_{n_2} \right| \left| e_{n_1'} e_{n_2'} \right| \ dx \ dt
$$

$$
(3.7) \lesssim \sum_{\tau, \tau' \sim N_1, N_2, \tau'} \sum_{(n_1, n_2) \in A_{N_1, N_2, \tau}} \sum_{(n_1', n_2') \in A_{N_1, N_2, \tau'}} \left| c_{n_1} d_{n_2} \right| \left| e_{n_1} e_{n_2} \right| L^2(\Theta) \left| e_{n_1'} e_{n_2'} \right| L^2(\Theta).
$$

Then by Schur's test, we arrive at

$$
(3.7) \lesssim \sum_{\tau \sim N_1, N_2, \tau} \left( \sum_{(n_1, n_2) \in A_{N_1, N_2, \tau}} \left| c_{n_1} d_{n_2} \right| \left| e_{n_1} e_{n_2} \right| L^2(\Theta) \right)^2
$$

$$
(3.8) \lesssim \sum_{\tau \sim N_1, N_2, \tau} \left| c_{n_1} d_{n_2} \right|^2 \left| e_{n_1} e_{n_2} \right|^2 L^2(\Theta).
$$

We claim that

**Claim 3.5.**

1. $\#A_{N_1, N_2, \tau} = O(N_2)$;
2. $\|e_{n_1} e_{n_2}\|^2 L^2(\Theta) \lesssim N_2^2$.

Assuming Claim 3.5, we see that

$$
(3.8) \lesssim \sum_{\tau \sim N_1, N_2, \tau} \left| c_{n_1} d_{n_2} \right|^2 \leq N_2^{1+\varepsilon} \left\| u_1 \right\|^2 L^2(\Theta) \left\| u_2 \right\|^2 L^2(\Theta).
$$

Therefore, (3.1) follows.

Now we are left to prove Claim 3.5.

**Proof of Claim 3.5.** In fact, (2) is due to Hölder inequality and the logarithmic bound on the $L^p$ norm of $e_n$ in (2.9).

For (1), we have that for fixed $\tau \in \mathbb{N}$ and fixed $n_2 \sim N_2$

$$
\left| z_{n_1}^\alpha + z_{n_2}^\alpha - \tau \right| \leq \frac{1}{2} \implies z_{n_1} \in \left[ (\tau - \frac{1}{2} - z_{n_2}^\alpha)^{\frac{1}{\alpha}}, (\tau + \frac{1}{2} - z_{n_2}^\alpha)^{\frac{1}{\alpha}} \right].
$$

There are at most 1 integer $z_{n_1}$ in this interval by concavity

$$
(\tau + \frac{1}{2} - z_{n_2}^\alpha)^{\frac{1}{\alpha}} - (\tau - \frac{1}{2} - z_{n_2}^\alpha)^{\frac{1}{\alpha}} \leq 1^{\frac{1}{\alpha}} = 1.
$$
Let us remark that the restriction $\alpha \geq \frac{1}{2}$ on the fractional Laplacian in this section comes from the concavity that we used here.

Then

$$\# \Lambda_{N_1, N_2, \tau} = \# \{(n_1, n_2) \in \mathbb{N}^2 : n_1 \sim N_1, n_2 \sim N_2, |z_{n_1}^{2\alpha} + z_{n_2}^{2\alpha} - \tau| \leq \frac{1}{2} \} \sim O(N_2).$$

We finish the proof of Claim 3.5. \qed

The estimation of (3.2) is similar, hence omitted.

The proof of Lemma 3.1 is complete now. \qed

**Remark 3.6.** One may guess that the bilinear Strichartz could be done via the radial Sobolev embedding,

$$\left| |x| \frac{1}{2} f \right| \lesssim \| f \|_{\dot{H}^{\frac{1}{2}}}^2,$$

however it is not clear how to deal with the weight on the left hand side. If there were no such weight, it should be sufficient to prove the bilinear estimate using the radial Sobolev embedding.

### 3.2. Bilinear Strichartz estimates for NLS

The computation above also holds for the classical NLS ($\alpha = 1$). However, we have a slightly better local well-posedness index because of the following better bilinear estimate.

**Lemma 3.7** (Bilinear estimates for classical NLS). Under the same setup as in Lemma 3.1, the bilinear estimate for classical NLS is given by

$$\| S_1(t)u_1 S_1(t)u_2 \|_{L^2_x((0,1) \times \Theta)} \lesssim N_1^2 \| u_1 \|_{L^2_x(\Theta)} \| u_2 \|_{L^2_x(\Theta)},$$

$$\| \nabla S_1(t)u_1 S_1(t)u_2 \|_{L^2_x((0,1) \times \Theta)} \lesssim N_1 N_2 \| u_1 \|_{L^2_x(\Theta)} \| u_2 \|_{L^2_x(\Theta)}.$$

The proof of Lemma 3.7 can be found in [49]. See also [1] for its extension in the unit ball setting.

It is worth pointing out that in the proof of Lemma 3.7, for example in a similar step like (3.5), one can periodize the time and use the following stronger counting lemma to estimate the number of integer points on the characteristic surface instead of counting the integer points in a thin neighborhood of the characteristic surface

**Lemma 3.8** (Lemma 3.2 in [7]). Let $M, N \in \mathbb{N}$, then for any $\varepsilon > 0$, there exists $C > 0$ such that

$$\# \{(k_1, k_2) \in \mathbb{N}^2 : N \leq k_1 \leq 2N, k_1^2 + k_2^2 = M \} \leq CN^\varepsilon.$$

However, we will not distinguish $\alpha = 1$ case from other fractional ones in the following sections. This is because the dominated term in the energy increment (the term that will give the largest energy increment and then determine the global well-posedness index in Section 7) will be the almost the same even if we take this better bilinear estimate into consideration.

### 4. I-operator and a modified local theory

In this section, we first define the I-operator in our setting and then present a local well-posedness argument for the I-operator modified equation.

#### 4.1. Definition of I-operator

**Definition 4.1** (I-operator). For $N \gg 1$, and a function $u = \sum_{n=1}^{\infty} c_n e_n$, define a smooth operator $I_N$, such that

$$I_N u = \sum_{n=1}^{\infty} m_N(z_n) c_n e_n.$$
where $m_N$ is a smooth function satisfying

$$m_N(\xi) = \begin{cases} 1, & |\xi| \leq N \\ \left(\frac{\eta}{N}\right)^{s-\alpha}, & |\xi| \geq 2N. \end{cases}$$

**Remark 4.2.** The I-operator defined above is the analogue of the one in [9] in the physical space. In the rest of this section, we will adopt the name ‘multiplier’ of $m$ from the context in [9]. It is easy to check that for $N \gg 1$

$$\|u\|_{H^s} \lesssim \|I_N u\|_{H^s} \lesssim N^{\alpha-s} \|u\|_{H^s},$$
$$\|u\|_{X^s,1+b} \lesssim \|I_N u\|_{X^{\alpha,1+b}} \lesssim N^{\alpha-s} \|u\|_{X^{\alpha,1+b}}.$$

A standard I-method argument usually comes in three major parts.

**Part 1** (Subsection 4.2) a well-adapted local theory for the I-operator modified fractional NLS.
**Part 2** (Section 6) the almost conservation law argument addressing the energy increment on each iteration,
**Part 3** (Section 7) an iterative globalization argument giving the global index and a polynomial bound of the $H^s$ norm of the solution.

### 4.2. A local theory based on $I_N$-operator

Consider the following $I_N$-operator modified FNLS equation with initial data also being smoothed into the energy space.

\[ (i\partial_t - (-\Delta)^\alpha)I_N u = I_N(|u|^2 u), \]
\[ I_N u(0) = I_N u_0. \]

Note that $I_N u \in H^\alpha$ and $\|I_N u_0\|_{H^\alpha} \lesssim N^{\alpha-s}$.

To keep our notation compact, we will write $I$ and $m$ instead of $I_N$ and $m_N$ as in Definition 4.1.

The main result in this subsection is the following local well-posedness theory.

**Proposition 4.3** (Local well-posedness). For $s \in (\frac{1}{2}, \alpha]$ and $I_N u_0 \in H^\alpha$, \(4.1\) is locally well posed. That is, there exists $\delta \sim \|I_N u_0\|_{H^\alpha}^{\frac{2(\alpha-s)}{2+2(\alpha-s)}} \gtrsim N^{-\frac{2(\alpha-s)}{2+2(\alpha-s)}} \), where $b(s) = \frac{1}{4} + \frac{3}{4}(1-s) +$, such that $Iu \in C([0,\delta], H^\alpha(\Theta))$ solves (4.1) on $[0,\delta]$ and satisfies

$$\|Iu\|_{X^{\alpha,1+b}_s} \lesssim \|Iu_0\|_{H^\alpha} \lesssim N^{\alpha-s}.$$

We will prove Proposition 4.3 by a standard contraction mapping argument. Note that the key step to close such argument is the following nonlinear estimate lemma.

**Lemma 4.4** (Nonlinear estimates). For $s > \frac{1}{2}$, there exist $b,b' \in \mathbb{R}$ satisfying

$$0 < b' < \frac{1}{2} < b, \quad b + b' < 1,$$

such that for every triple $(u_1, u_2, u_3)$ in $X^{\alpha,b}(\mathbb{R} \times \Theta)$,

$$\|I(|u|^2 u)\|_{X^{s,-b'}(\mathbb{R} \times \Theta)} \lesssim \prod_{j=1}^3 \|Iu\|_{X^{\alpha,b}(\mathbb{R} \times \Theta)}^3.$$

Assuming Lemma 4.4, we can easily finish the proof of Proposition 4.3.

**Proof of Proposition 4.3.** Using Lemma 2.4 and Proposition 4.3, we have the following standard contraction mapping calculation

$$\|Iu\|_{X^{\alpha,b}_s} \lesssim \|Iu_0\|_{H^\alpha} + \delta^{1-b-b(s)} \|I(|u|^2 u)\|_{X^{s,-b(s)}_s} \lesssim \|Iu_0\|_{H^\alpha} + \delta^{1-b-b(s)} \|Iu\|_{X^{\alpha,b(s)}_s}^3.$$
where \( b = \frac{1}{3} + \). By choosing \( \delta^{3+2b-4b(s)} \sim \| Iu_0 \|^{-2}_{H^{\alpha}} \gtrsim N^{-2(\alpha-s)} \) as what one did in a standard contraction mapping proof and a continuity argument we have that

\[
\| Iu \|_{X_{\delta}^{\alpha,b}} \lesssim \| Iu_0 \|_{H^{\alpha}} \lesssim N^{\alpha-s} \]

and

\[
\delta \gtrsim N^{-\frac{2(\alpha-s)}{4\alpha-4b(s)}}. \]

Now we are left to prove the key Lemma 4.4 in this section.

**Proof of Lemma 4.4.** It is sufficient to show the following nonlinear estimate: for \( \frac{1}{3} < s < 1 \), \( b(s) = \frac{1}{3} + \frac{1}{2}(1-s) + \) and \( u \in X_{\delta}^{s,b(s)} \)

\[
(4.2) \quad \| I(|u|^2 u) \|_{X_{\delta}^{\alpha-b(s)}} \lesssim \| Iu \|_{X_{\delta}^{\alpha,b(s)}}^3.
\]

By duality argument, it is sufficient to prove for \( v \in X^{-\alpha,b(s)} \)

\[
\left| \int_{\mathbb{R} \times \Theta} \overline{\pi I(|u|^2 u)} \, dx \, dt \right| \lesssim \| v \|_{X^{-\alpha,b(s)}} \| Iu \|_{X_{\delta}^{\alpha,b(s)}}^3.
\]

We will frequently make use of a dyadic decomposition in frequency using the orthonomal basis \( e_n \)'s of the radial Dirichlet Laplacian \(-\Delta\), writing

\[
v_0 = \sum_{N_0 \leq \langle z_0 \rangle < 2N_0} P_n v,
\]

and

\[
u_i = \sum_{N_i \leq \langle z_0 \rangle < 2N_i} P_n u, \quad \text{for } i = 1, 2, 3.
\]

After the dyadic frequency decomposition, we take a typical term and compute for the quadruple \( N = (N_0, N_1, N_2, N_3) \)

\[
L(N) := \left| \int_{\mathbb{R} \times \Theta} \overline{\pi v_i} I(u_1 \overline{u_2} u_3) \, dx \, dt \right|.
\]

In order to distribute the I-operator inside the nonlinear term, we first move the I-operator on \( v_0 \), then introduce \( m(N_i) \) (instead of I-operator) into each \( u_i \).

\[
(4.4) \quad L(N) = \left| \int_{\mathbb{R} \times \Theta} \overline{\pi v_0} (u_1 \overline{u_2} u_3) \, dx \, dt \right| = \frac{1}{m(N_1)m(N_2)m(N_3)} \left| \int_{\mathbb{R} \times \Theta} \overline{\pi v_0} \cdot m(N_1)u_1 \cdot m(N_2)u_2 \cdot m(N_3)u_3 \, dx \, dt \right|.
\]

We will explain the reason why we brought in \( m \) instead of I-operator in this calculation in Remark 4.5.

By symmetry argument and because the presence of complex conjugates will play no role here, we can assume \( N_1 \geq N_2 \geq N_3 \). Then we can reduce the sum into the following two cases:

1. \( N_0 \lesssim N_1 \)
2. \( N_0 \gtrsim N_1 \).

**Case 1:** \( N_0 \lesssim N_1 \)

Recall Remark 3.4 where by taking \( \beta = s \), we have

\[
(4.5) \quad \| f_i f_j \|_{L^2_{t,x}((0,\delta) \times \Theta)} \lesssim \min\{N_i, N_j\}^s \| f_i \|_{X_{\delta}^{0,b(s)}} \| f_j \|_{X_{\delta}^{0,b(s)}}.
\]
where
\[ b(s) = \frac{1}{4} + \frac{1}{2}(1 - s^+), \quad s \in \left(\frac{1}{2}, 1\right]. \]

Using (4.5) and Definition 2.1, we write (4.4) as
\[
L(N) \lesssim \frac{m(N_0)}{m(N_1)m(N_2)m(N_3)} \int_{\mathbb{R} \times \Theta} |m(N_1)u_1 \cdot m(N_2)u_2 \cdot m(N_3)u_3| \, dx dt
\]
\[
\lesssim \frac{m(N_0)}{m(N_1)m(N_2)m(N_3)} \|v_0 m(N_2)u_2\|_{L^2_{t,x}} \|m(N_1)u_1 \cdot m(N_3)u_3\|_{L^2_{t,x}}
\]
\[
\lesssim \frac{m(N_0)}{m(N_1)m(N_2)m(N_3)} (N_2 N_3)^\alpha \|v_0\|_{X^{\alpha, b(x)}} \prod_{i=1}^{3} \|Iu_i\|_{X^{\alpha, b(x)}}
\]
(4.6)
\[
\lesssim \frac{m(N_0)}{m(N_1)m(N_2)m(N_3)} (N_0 N_1^{-1})^\alpha (N_2 N_3)^{\alpha - \alpha} \|v_0\|_{X^{\alpha, b(x)}} \prod_{i=1}^{3} \|Iu_i\|_{X^{\alpha, b(x)}}.
\]

Note that here we used \( \|Iu_i\|_{X^{\alpha, b}} \sim \|m(N_i)u_i\|_{X^{\alpha, b}} \).

To continue the computation, we then consider the following two scenarios for \( N_2 \) and \( N_3 \)
\[
N_2^{s - \alpha} = \begin{cases} N_1^{s - \alpha} & \text{if } N_2 \leq N \\ (N_1^{-1} N_1)^{\alpha - s} N_1^{s - \alpha} = N^{s - \alpha} & \text{if } N_2 > 2N.
\end{cases}
\]

This observation implies that \( L(N) \) is summable in \( N_2 \) and \( N_3 \). That is, by taking out the terms in \( L(N) \)
in (4.6) that only depend on frequencies \( N_2 \) and \( N_3 \), we see
\[
\sum_{N_3 \leq N_2} \frac{(N_2 N_3)^{s - \alpha}}{m(N_2)m(N_3)} \|Iu_2\|_{X^{\alpha, b(x)}} \|Iu_3\|_{X^{\alpha, b(x)}} \lesssim \|Iu\|^2_{X^{\alpha, b(x)}}.
\]

Now we focus on the sum over \( N_0 \) and \( N_1 \) in (4.6).

First write
\[
\frac{m(N_0)}{m(N_1)} N_0^\alpha \leq \begin{cases} \left(\frac{N_0}{N_1}\right)^\alpha & \text{if } N_0 \lesssim N_1 \leq N \\ \left(\frac{N_0}{N_1}\right)^{\alpha - s} \left(\frac{N_0}{N_1}\right)^\alpha & \text{if } N_0 \leq N \leq N_1 \\ \left(\frac{N_0}{N_1}\right)^{\alpha - s} & \text{if } N \leq N_0 \leq N_1 \\ \left(\frac{N_0}{N_1}\right)^{\alpha - s} & \text{if } N \leq N_0 \leq N_1
\end{cases}
\]
then the sum over \( N_0 \) and \( N_1 \) in (4.6) becomes
\[
\sum_{N_0 \lesssim N_1} \frac{m(N_0)}{m(N_1)} \left(\frac{N_0}{N_1}\right)^\alpha \|v_0\|_{X^{-\alpha, b(x)}} \|Iu_1\|_{X^{\alpha, b(x)}} \lesssim \sum_{N_0 \lesssim N_1} \left(\frac{N_0}{N_1}\right)^{\alpha - s} \|v_0\|_{X^{-\alpha, b(x)}} \|Iu_1\|_{X^{\alpha, b(x)}}
\]
\[
\lesssim \|v\|_{X^{-\alpha, b(x)}} \|Iu\|_{X^{\alpha, b(x)}}.
\]

Here is a quick remark on the calculation in (4.4) and what follows. We originally planned to introduce
the I-operator into (4.4) instead of \( m(N_i) \). But this needs to bring the absolute value sign inside of the
integral in (4.4).

**Case 2:** \( N_0 \gtrsim N_1 \).

First recall Green's theorem,
\[
\int_\Omega \Delta f g - f \Delta g \, dx = \int_\partial \Omega \frac{\partial f}{\partial n} g - f \frac{\partial g}{\partial n} \, d\sigma.
\]
Note that
\[ -\Delta e_k = z_k^2 e_k, \]
where \( z_k^2 \)'s are the eigenvalues defined in (2.6). Then we write
\[
Iv_0 = -\frac{\Delta}{N_0^2} \sum_{z_n \sim N_0} c_{n_0} \left( \frac{N_0}{z_n} \right)^2 e_{n_0},
\]
where \( c_{n_0} = m_N(z_n) c_n' \) and \( c_n' = \langle v_0, e_n \rangle_{L^2} \). Here \( c_{n_0} \) is the coefficient in front of eigenfunction \( e_n \) for \( Iv_0 \) while \( c_n' \) is the coefficient in front of eigenfunction \( e_n \) for \( v_0 \).

Define
\[
T(Iv_0) = \sum_{z_n \sim N_0} c_{n_0} \left( \frac{N_0}{z_n} \right)^2 e_{n_0}, \quad V(Iv_0) = \sum_{z_n \sim N_0} c_{n_0} \left( \frac{z_n}{N_0} \right)^2 e_{n_0}.
\]
It is easy to see that for all \( s \)
\[
TV(Iv_0) = VT(Iv_0) = Iv_0,
\]
\[
\|T(Iv_0)\|_{H^s_z} \sim \|Iv_0\|_{H^s_z} \sim \|V(Iv_0)\|_{H^s_z}.
\]
Using this notation, we write
\[
Iv_0 = -\frac{\Delta}{N_0^2} T(Iv_0)
\]
and by Green's theorem
\[
L(a) \lesssim \frac{1}{m(N_1)m(N_2)m(N_3)} \frac{1}{N_0^2} \int_{\mathbb{R} \times \Theta} T(Iv_0) \Delta (\prod_{j=1}^{3} m(N_j)u_j).
\]

By the product rule and the assumption that \( N_1 \geq N_2 \geq N_3 \), we only need to consider the two largest cases of \( \Delta(u_1u_2u_3) \). They are

(1) \( (\Delta u_1)u_2u_3 \),
(2) \( (\nabla u_1) \cdot (\nabla u_2)u_3 \).

We denote
\[
J_{11}(a) = \int_{\mathbb{R} \times \Theta} T(Iv_0)(\Delta m(N_1)u_1)(m(N_2)u_2)(m(N_3)u_3),
\]
\[
J_{12}(a) = \int_{\mathbb{R} \times \Theta} T(Iv_0)(\nabla m(N_1)u_1) \cdot (\nabla m(N_2)u_2)(m(N_3)u_3).
\]
Using \( \Delta u_i = -N_i^2 V u_i \) and (4.5) under the similar calculation as in (4.6), we obtain
\[
\frac{1}{N_0^2} |J_{11}(a)| \lesssim m(N_0) \left( \frac{N_1}{N_0} \right)^2 (N_0 N_1^{-1})^\alpha (N_2 N_3)^{s-\alpha} \|v_0\|_{X^{-\alpha, \Theta}(a)} \prod_{i=1}^{3} \|u_i\|_{X^{-\alpha, \Theta}(a)}. \]

Now for \( |J_{12}(a)| \), we estimate it in a similar fashion and obtain that
\[
\frac{1}{N_0} |J_{12}(a)| \lesssim m(N_0) \frac{N_1 N_2}{N_0} (N_0 N_1^{-1})^\alpha (N_2 N_3)^{s-\alpha} \|v_0\|_{X^{-\alpha, \Theta}(a)} \prod_{i=1}^{3} \|u_i\|_{X^{-\alpha, \Theta}(a)}.
\]

Therefore, we have
\[
L(a) \lesssim \frac{m(N_0)}{m(N_1)m(N_2)m(N_3)} \left( \frac{N_1}{N_0} \right)^2 (N_0 N_1^{-1})^\alpha (N_2 N_3)^{s-\alpha} \|v_0\|_{X^{-\alpha, \Theta}(a)} \prod_{i=1}^{3} \|u_i\|_{X^{-\alpha, \Theta}(a)}. \]
To sum $N_2$ and $N_3$, we can do exactly the same thing as in Case 1. Then the sum over $N_0$ and $N_1$ becomes

$$
\sum_{N_0 \geq N_1} \frac{m(N_0)}{m(N_1)} \left( \frac{N_0}{N_1} \right) ^ \alpha \left( \frac{N_1}{N_0} \right) ^ 2 \| v_0 \| _{X^{\alpha,b(x)}} ^ {\alpha} \| Iu_1 \| _{X^{\alpha,b(x)}} ^ {\alpha} \lesssim \sum_{N_0 \geq N_1} \left( \frac{N_1}{N_0} \right) ^ {2-\alpha} \| v_0 \| _{X^{\alpha,b(x)}} \| Iu_1 \| _{X^{\alpha,b(x)}} .
$$

Therefore

$$
\left| \int _ {\mathbb{R} \times \Theta} \mathfrak{I}(\ |u|^2 u) \, dxdt \right| \lesssim \sum _ {\mathcal{N}} L(\mathcal{N}) \lesssim \| v \| _{X^{\alpha,b(x)}} \| Iu \| _{X^{\alpha,b(x)}} ^ 3 .
$$

which implies (4.2).

This finishes the proof of Lemma 4.4. \[ \square \]

**Remark 4.5.** Here is a quick remark on the calculation in (4.4) and what follows. We originally planned to introduce the $I$-operator into (4.4) instead of $m(N_i)$. But this needs to bring the absolute value sign inside of the integral in (4.4), which may ruin the Green’s theorem. So with the calculation in this proof, we justify the ‘legality’ of bringing $I$-operators inside. In the rest of this paper, we will use a slight abuse of calculation – moving the $I$-operator without justification. For example, in (4.4)

$$
\left| \int _ {\mathbb{R} \times \Theta} \mathfrak{I}(\ |u|^2 u) \, dxdt \right| \lesssim \frac{m(N_0)}{m(N_1)m(N_2)m(N_3)} \left| \int _ {\mathbb{R} \times \Theta} \mathfrak{I} \cdot Iu_1 \cdot Iu_2 \cdot Iu_3 \, dxdt \right| .
$$

5. **Weak interaction between functions localized in uncomparable frequencies**

Before we start the $I$-method argument, let us first understand the interaction between the functions at separate frequencies, which will be heavily used our proof and simplifies a lot of the case classification in Section 6.

Recall that we have no similar convolution properties as what we have on $\mathbb{R}^d$, which results in losing control of the frequency connection. In fact, after taking Fourier transformation on the nonlinear term, the convolution property implies that the frequencies in each function are linearly connected $\xi = \xi_1 - \xi_2 + \xi_3$. However this is no longer true under our setting, which means that for instance the maximum frequency could be extremely large instead of being controlled by a linear form of other lower frequencies. To deal with this bad scenario, let us take a closer look at the interaction among the nonlinearity.

This weak interaction is inspired by Lemma 2.6 in [7], that is

**Lemma 5.1.** There exists $C > 0$ such that, if for any $j = 1, 2, 3$, $C_z z_{n_j} \leq z_{n_0}$, then for every $p > 0$ there exists $C_p > 0$ such that for every $w_j \in L^2(M)$, $j = 0, 1, 2, 3$,

$$
\left| \int _ M P_{n_0} w_0 P_{n_1} w_1 P_{n_2} w_2 P_{n_3} w_3 \, dx \right| \leq C_p z_{n_0} ^ {-p} \prod _ {j=0} ^ 3 \| w_j \| _ {L^2} .
$$

Notice that on the right-hand side of (5.1), the factor $z_{n_0} ^ {-p}$ gives a huge decay, which means that the interaction in fact is weak. Now let us present our version of such weak interaction.

**Proposition 5.2 (Weak interaction).** For the frequency quadruple $\underline{n} = (n_0, n_1, n_2, n_3) \in \mathbb{N}^4$ and $n_0 \gg n_1 \geq n_2 \geq n_3$ ($n_0 \gg n_1'$ means $n_0 \geq 2n_1$), we consider the functions $w, f, g, h \in X^{0,b}$, $b = \frac{1}{4}$, with their frequencies localized at $n_0, n_1, n_2, n_3$. More precisely,

$$
w(t,x) = w_{n_0}(t)e_{n_0}(x), \quad f(t,x) = f_{n_1}(t)e_{n_1}(x), \quad g(t,x) = g_{n_2}(t)e_{n_2}(x), \quad h(t,x) = h_{n_3}(t)e_{n_3}(x).
$$

Define the interaction between these functions as follows

$$
J(\underline{n}) := \left| \int _ {\mathbb{R} \times \Theta} \mathfrak{I} \cdot f \cdot g \cdot h \, dxdt \right| .
$$
Then this interaction $J$ satisfies

\begin{equation}
J(\mathfrak{w}) \lesssim \frac{n_2^5 \sqrt{n_3}}{n_0^3} \|w\|_{X^{0,b}} \|f\|_{X^{0,b}} \|g\|_{X^{0,b}} \|h\|_{X^{0,b}} \langle n_0^{2\alpha} - n_1^{2\alpha} + n_2^{2\alpha} - n_3^{2\alpha} \rangle^b.
\end{equation}

Proof of Proposition 5.2. Notice that (5.2) can be written as

\[ J(\mathfrak{w}) = \int_{\mathbb{R} \times \Theta} w_n(t) e_{n_0}(x) \cdot f_n(t) e_{n_1}(x) \cdot g_n(t) e_{n_2}(x) \cdot h_n(t) e_{n_3}(x) dx dt \]

\[ = \left( \int_{\mathbb{R}} w_n(t) \cdot f_n(t) \cdot g_n(t) \cdot h_n(t) dt \right) \left( \int_{\Theta} e_{n_0}(x) e_{n_1}(x) e_{n_2}(x) e_{n_3}(x) dx \right) =: A \times B. \]

We will estimate the contributions of the two terms in (5.2) in Lemma 5.3 and Lemma 5.4 separately.

Lemma 5.3 (Weak interaction among separated eigenfunctions). If $n_0 \gg n_1 \geq n_2 \geq n_3$, then

\[ B := \int_{\Theta} e_{n_0} e_{n_1} e_{n_2} e_{n_3} dx < \mathcal{O} \left( \frac{n_2^5 \sqrt{n_3}}{n_0^3} \right). \]

Lemma 5.4 (Interaction between frequency localized functions). Under the same assumption as in Proposition 5.2, the interaction between their coefficients at frequencies $n_0, n_1, n_2, n_3$ satisfies

\[ A := \int_{\mathbb{R}} w_n(t) f_n(t) g_n(t) h_n(t) dt \lesssim \frac{\|w\|_{X^{0,b}} \|f\|_{X^{0,b}} \|g\|_{X^{0,b}} \|h\|_{X^{0,b}}}{\langle n_0^{2\alpha} - n_1^{2\alpha} + n_2^{2\alpha} - n_3^{2\alpha} \rangle^b}. \]

Assuming the two lemmas above, it is easy to see the weak interaction as in (5.3).

Now we are left to present the proofs of Lemma 5.3 and Lemma 5.4.

Proof of Lemma 5.3. Before proving it, let us first make an observation. In fact, a naive estimate from (2.9) gives that

\[ \left| \int_{\Theta} e_{n_0} e_{n_1} e_{n_2} e_{n_3} dx \right| \lesssim \prod_{i=0}^3 \|e_{n_i}\|_{L^4} \lesssim n_0^+ n_1^+ n_2^+ n_3^+. \]

However, we should expect a much weaker interaction when the frequencies are separate ($n_0 \gg n_1 \geq n_2 \geq n_3$).

To prove this lemma, we first write

\[ \phi(r) = e_{n_2}(r) e_{n_3}(r), \]

\[ F(r) = \int_0^r \gamma e_{n_0}(\gamma) e_{n_1}(\gamma) d\gamma, \]

then performing an integration by parts, we see that

\[ \int_{\Theta} e_{n_0} e_{n_1} e_{n_2} e_{n_3} dx \sim \int_0^1 (r e_{n_0}(r) e_{n_1}(r)) \phi(r) dr = F(r) \phi(r) \bigg|_0^1 - \int_0^1 F(r) \phi'(r) dr. \]

To see the weak interaction, we claim that

Claim 5.5.

(1) The boundary terms are zeros

\[ F(1) \phi(1) = F(0) \phi(0) = 0; \]

(2) Approximate formula for $F$

\[ F(r) \sim \frac{\sqrt{z_{n_0} z_{n_2}}}{z_{n_0} - z_{n_1}^2} \left( z_{n_0} J_1(z_{n_0} r) J_0(z_{n_1} r) - z_{n_1} J_0(z_{n_0} r) J_1(z_{n_1} r) \right); \]
(3) Approximate formula for $\phi'$

$$\phi'(r) \sim \sqrt{z_n^2n_3} (z_n^2J_1(z_n^2r)J_0(z_n^3r) + z_n^3J_0(z_n^2r)J_1(z_n^3r));$$

(4) With the assumption $n_0 \gg n_1 \geq n_2 \geq n_3$ (recall that $n_0 \gg n_1$ means $n_0 \geq 2n_1$), we obtain the weak interaction using the approximate formula for $F$ and $\phi'$, that is,

$$\left| \int_0^1 e_n e_n e_n e_n dx \right| = \left| \int_0^1 F(r)\phi'(r) dr \right| < O\left( \frac{r^2}{z_n^2n_3} \right) \sim O\left( \frac{n_3}{n_0} \right).$$

Proof of Claim 5.5. For (1), the zero boundary conditions can be justified since $F(r)$ is a zero at $r = 0$ and $\phi(r) = 0$ at $r = 1$ (recall $e_n(1) = 0$).

For (2), recall $F(r) = \int_0^r e_n e_n e_n e_n d\gamma$, and $e_n(r) = \|J_0(z_nr)\|^{-1}_{L^2(\Theta)}J_0(z_nr)$, then we write

$$F(r) = \int_0^r e_n e_n e_n e_n d\gamma = \|J_0(z_nr)\|^{-1}_{L^2(\Theta)}\|J_0(z_nr)\|^{-1}_{L^2(\Theta)} \int_0^r e_n e_n e_n e_n d\gamma.$$

Combining the following anti-derivative (for detailed calculation, see Lemma A.1 in Appendix A)

$$\int_0^r e_n e_n e_n e_n d\gamma = \frac{1}{z_n^2n_3 - z_n^2n_1} r[z_n^2J_1(z_n^2r)J_0(z_n^3r) - z_n^3J_0(z_n^2r)J_1(z_n^3r)];$$

we have

$$F(r) = \frac{\|J_0(z_nr)\|^2_{L^2(\Theta)}\|J_0(z_nr)\|^2_{L^2(\Theta)}}{z_n^2n_3 - z_n^2n_1} r[z_n^2J_1(z_n^2r)J_0(z_n^3r) - z_n^3J_0(z_n^2r)J_1(z_n^3r)];$$

where in the last step, we used $\|J_0(z_nr)\|^2_{L^2(\Theta)} \sim \frac{1}{z_n^2n_3}$ in (2.8).

For (3), by (2.7) and (2.1), we have

$$\phi'(r) = (e_n e_n e_n e_n)' = \|J_0(z_n^2r)\|^2_{L^2(\Theta)}\|J_0(z_n^3r)\|^2_{L^2(\Theta)} (J_0(z_n^2r)J_0(z_n^3r))'$$

$$\geq \|J_0(z_n^2r)\|^2_{L^2(\Theta)}\|J_0(z_n^3r)\|^2_{L^2(\Theta)} (z_n^2J_1(z_n^2r)J_0(z_n^3r) + z_n^3J_0(z_n^2r)J_1(z_n^3r))$$

$$\geq \sqrt{z_n^2z_n^3} (z_n^2J_1(z_n^2r)J_0(z_n^3r) + z_n^3J_0(z_n^2r)J_1(z_n^3r))$$

where in the approximation above, we used (2.8) again. Since $J_0(x)$, $J_1(x)$ are bounded, we see that

$$|\phi'(r)| \lesssim \sqrt{z_n^2z_n^3}(z_n^2 + z_n^3).$$

Now let us move on to (4).

Intuitively, we can think of $\gamma e_0(\gamma) e_1(\gamma)$ as a product of two trigonometric functions, which is essentially $\cos((z_n^2 \pm z_n^3)r)$ using the product to sum identities. The integral of $F$ is basically the twice integration of $\cos((z_n^2 \pm z_n^3)r)$, which will bring out a factor of $\frac{1}{z_n^2z_n^3}$ by changing of variables. Putting together the bound of $\phi'$ that we observed in (2), which is $\frac{1}{z_n^2z_n^3}$, we should be able to see the estimate in (4). However, we need to justify this bound in the rest of this proof.

Recall (2.3) and (2.4). We will use the first formula for $|x| < 1$ and the second one for $|x| \geq 1$, where $n = 0, 1$.

$$J_n(x) = \frac{1}{n!2^{n+1}}x^n + O(x^{n+2}),$$

$$J_n(x) = \sqrt{\frac{2\cos(x - \frac{n\pi}{2} - \frac{\pi}{4})}{\pi}} + O(x^{-\frac{3}{2}})$$
Combining (2) and (3), we write
\[
\int_0^1 F(r)\phi'(r)\,dr \sim \int_0^1 \frac{\sqrt{z_{n_0} - z_{n_1}}}{z_{n_0}^2 - z_{n_1}^2} r\left[z_{n_0}J_1(z_{n_0}r)J_0(z_{n_1}r) - z_{n_1}J_0(z_{n_0}r)J_1(z_{n_1}r)\right]
\times \frac{\sqrt{z_{n_2} - z_{n_3}}}{z_{n_2}^2 - z_{n_3}^2} \left[z_{n_2}J_1(z_{n_2}r)J_0(z_{n_3}r) + z_{n_3}J_0(z_{n_2}r)J_1(z_{n_3}r)\right] \, dr
\]

To estimate \(\int_0^1 F(r)\phi'(r)\,dr\), let us consider the following term as an example:
\[
\int_0^1 \frac{\sqrt{z_{n_0} - z_{n_1}}}{z_{n_0}^2 - z_{n_1}^2} r z_{n_0}J_1(z_{n_0}r)J_0(z_{n_1}r)\sqrt{z_{n_2} - z_{n_3}} z_{n_2}J_1(z_{n_2}r)J_0(z_{n_3}r) \, dr
\]
\[= \frac{\sqrt{z_{n_0} - z_{n_1}}}{z_{n_0}^2 - z_{n_1}^2} \int_0^1 r J_1(z_{n_0}r)J_0(z_{n_1}r)J_1(z_{n_2}r)J_0(z_{n_3}r) \, dr.\]
\[(5.4)\]

In fact, since \(n_0 \geq n_1\) and \(n_2 \geq n_3\), this term will contribute the most in the estimate of \(\int_0^1 F(r)\phi'(r)\,dr\).

We will use the different approximations of \(J_0(x)\) and \(J_1(x)\) for \(|x| < 1\) and \(|x| \geq 1\), hence we consider the the following five cases.

- **Case I**: \(0 \leq r < \frac{1}{z_{n_0}}\);
- **Case II**: \(\frac{1}{z_{n_0}} \leq r < \frac{1}{z_{n_1}}\);
- **Case III**: \(\frac{1}{z_{n_1}} \leq r < \frac{1}{z_{n_2}}\);
- **Case IV**: \(\frac{1}{z_{n_2}} \leq r < \frac{1}{z_{n_3}}\);
- **Case V**: \(\frac{1}{z_{n_3}} \leq r \leq 1\).

We will focus on only the main terms in the approximation formulas, and the control of error terms can be found in Subsection A.2.

**Case I**: \(0 \leq r < \frac{1}{z_{n_0}}\).

In this case, all the arguments in \(J_0\) and \(J_1\) in (5.4) will be smaller than 1, hence we will need four approximations around the origin, that is,
\[
\sqrt{z_{n_0} - z_{n_1}} z_{n_2} z_{n_3} z_{n_0} z_{n_2} \int_0^{\frac{1}{z_{n_0}}} \frac{r J_1(z_{n_0}r)J_0(z_{n_1}r)J_1(z_{n_2}r)J_0(z_{n_3}r)}{z_{n_0}^2 - z_{n_1}^2} \, dr
\]
\[
\sim \sqrt{z_{n_0} - z_{n_1}} \frac{z_{n_2} z_{n_3} z_{n_0}}{z_{n_0}^2 - z_{n_1}^2} \int_0^{\frac{1}{z_{n_0}}} r J_0(z_{n_0}r)J_0(z_{n_2}r) \, dr
\]
\[
\sim \sqrt{z_{n_0} - z_{n_1}} \frac{z_{n_2} z_{n_3} z_{n_0}}{z_{n_0}^2 - z_{n_1}^2} \int_0^{\frac{1}{z_{n_0}}} r J_0(z_{n_0}r)J_0(z_{n_2}r) \, dr
\]
\[
= \sqrt{z_{n_0} - z_{n_1}} \frac{z_{n_2} z_{n_3} z_{n_0}}{z_{n_0}^2 - z_{n_1}^2} \int_0^{\frac{1}{z_{n_0}}} r \left(\frac{z_{n_0} - \frac{r}{z_{n_0}}}{z_{n_0}^2 - \frac{r^2}{z_{n_0}^2}}\right) \, dr
\]
\[
= \sqrt{z_{n_0} - z_{n_1}} \frac{z_{n_2} z_{n_3} z_{n_0} z_{n_2}}{z_{n_0}^2 - z_{n_1}^2} \int_0^{\frac{1}{z_{n_0}}} \frac{r^4}{z_{n_0}^2 - \frac{r^2}{z_{n_0}^2}} \, dr
\]
where in the last step, we used (2.6).

**Case II**: \(\frac{1}{z_{n_0}} \leq r < \frac{1}{z_{n_1}}\).

In this case, the argument \(z_{n_0}r\) turns to be larger than 1 in (5.4), hence we need its asymptotic approximation, while the other three are treated the same as in the previous case.
\[
\sqrt{z_{n_0} - z_{n_1}} z_{n_2} z_{n_3} z_{n_0} z_{n_2} \int_0^{\frac{1}{z_{n_0}}} \frac{r J_1(z_{n_0}r)J_0(z_{n_1}r)J_1(z_{n_2}r)J_0(z_{n_3}r)}{z_{n_0}^2 - z_{n_1}^2} \, dr
\]
\[
\sim \sqrt{z_{n_0} - z_{n_1}} \frac{z_{n_2} z_{n_3} z_{n_0}}{z_{n_0}^2 - z_{n_1}^2} \int_0^{\frac{1}{z_{n_0}}} \sin\left(z_{n_0}r - \frac{z_{n_0}^2}{z_{n_0}^2 - \frac{r^2}{z_{n_0}^2}}\right) \, dr
\]
\[
\sim \sqrt{z_{n0}z_{n1}z_{n2}z_{n3}z_{n0}z_{n2}} \frac{z_{n2}}{z_{n0}^2 - z_{n1}^2} \int_{z_{n0}}^{z_{n1}} r^{3/2} \sin(z_{n0}r - \pi/4) \, dr
\]
\[
\leq \sqrt{n_2 n_3 n_0 z_{n0} z_{n2}} \frac{z_{n2}}{z_{n0}^2 - z_{n1}^2} \left( \frac{1}{z_{n0}^2} \right) \frac{1}{z_{n2}} \sim \frac{n_2 n_3 n_0 z_{n0}^2}{n_1^2 (n_0^2 - n_1^2)}.
\]

Note that the last inequality above, we used (see Lemma A.2 in Appendix A for detailed computation)
\[
\left| \int_a^b x^p \sin x \, dx \right| \lesssim a^p + b^p.
\]

**Case III:** \( \frac{1}{z_{n1}} \leq r < \frac{1}{z_{n2}} \).

Now we have two asymptotic approximations in (5.4)
\[
\sqrt{z_{n0}z_{n1}z_{n2}z_{n3}z_{n0}z_{n2}} \int_{z_{n0}}^{z_{n1}} r J_1(z_{n0}r) J_0(z_{n1}r) J_1(z_{n2}r) J_0(z_{n3}r) \, dr
\]
\[
\sim \sqrt{z_{n0}z_{n1}z_{n2}z_{n3}z_{n0}z_{n2}} \int_{z_{n0}}^{z_{n1}} r \sin(z_{n0}r - \pi/4) \cos(z_{n1}r - \pi/4) \cos(z_{n2}r) \, dr
\]
\[
\sim \sqrt{z_{n0}z_{n1}z_{n2}z_{n3}z_{n0}z_{n2}} \frac{1}{z_{n2}} \int_{z_{n0}}^{z_{n1}} r \sin(z_{n0}r - \pi/4) \cos(z_{n1}r - \pi/4) \, dr \sim \frac{1}{z_{n2} z_{n0}}.
\]

where in the last inequality, we used Lemma A.3 to obtain
\[
\int_{z_{n0}}^{z_{n1}} r \sin(z_{n0}r - \pi/4) \cos(z_{n1}r - \pi/4) \, dr \lesssim \frac{1}{z_{n2} z_{n0}}.
\]

**Case IV:** \( \frac{1}{z_{n2}} \leq r < \frac{1}{z_{n3}} \).

Now we have two asymptotic approximations in \( F(r) \), and one more from \( \phi'(r) \). To reduce the number of trig functions in (5.4), we use the trigonometric identity \( \sin \alpha \cos \beta = \frac{1}{2}(\sin(\alpha + \beta) + \sin(\alpha - \beta)) \) to re-write
\[
F(r) \sim \sqrt{z_{n0}z_{n1}} \frac{\sin(z_{n0}r - \pi/4) \cos(z_{n1}r - \pi/4)}{\sqrt{z_{n0}r} \sqrt{z_{n1}r}} - \sqrt{z_{n0}z_{n1}z_{n2}} \frac{\cos(z_{n0}r - \pi/4)}{\sqrt{z_{n0}r} \sqrt{z_{n1}r}} \frac{\sin(z_{n1}r - \pi/4)}{\sqrt{z_{n1}r}}
\]
\[
= \frac{z_{n0}}{z_{n0} - z_{n1}} \sin(z_{n0}r - \pi/4) \cos(z_{n1}r - \pi/4) - \frac{z_{n1}}{z_{n0} - z_{n1}} \cos(z_{n0}r - \pi/4) \sin(z_{n1}r - \pi/4)
\]
\[
\sim - \frac{1}{z_{n0} - z_{n1}} \cos((z_{n0} + z_{n1})r) + \frac{1}{z_{n0} + z_{n1}} \sin((z_{n0} - z_{n1})r).
\]

Recall
\[
\phi'(r) \sim \sqrt{z_{n2}z_{n3}}(z_{n2}J_1(z_{n2}r) J_0(z_{n3}r) + z_{n3} J_0(z_{n3}r) J_1(z_{n2}r))
\]
then we write
\[
(5.5) \int_{z_{n2}}^{z_{n3}} F(r) \phi'(r) \, dr \sim \int_{z_{n2}}^{z_{n3}} \left( - \frac{1}{z_{n0} - z_{n1}} \cos((z_{n0} + z_{n1})r) + \frac{1}{z_{n0} + z_{n1}} \sin((z_{n0} - z_{n1})r) \right) \phi'(r) \, dr.
\]

Take the following term as an example (since it will contribute the most in (5.5), hence (5.4))
\[
\int_{z_{n2}}^{z_{n3}} \frac{1}{z_{n0} - z_{n1}} \cos((z_{n0} + z_{n1})r) \sqrt{z_{n2}z_{n3}z_{n2}} (z_{n2}J_1(z_{n2}r) J_0(z_{n3}r)) \, dr
\]
\[ \sim \frac{\sqrt{z_{n_3}z_{n_1}z_{n_2}}}{z_{n_0} - z_{n_1}} \int_{1/n_3}^{1} \cos((z_{n_0} + z_{n_1})r) \frac{\sin(z_{n_2}r - \frac{\pi}{4})}{\sqrt{z_{n_2}r}} \, dr \]
\[ \lesssim \frac{\sqrt{z_{n_2}z_{n_3}z_{n_2}}}{z_{n_0} - z_{n_1}} \int_{1/n_3}^{1} r^{-\frac{1}{2}} \cos((z_{n_0} + z_{n_1})r) \sin(z_{n_2}r - \frac{\pi}{4}) \, dr \]
\[ \lesssim \frac{\sqrt{z_{n_2}z_{n_3}z_{n_2}}}{z_{n_0} - z_{n_1}} \frac{1}{z_{n_0} + z_{n_1}} \sim \frac{n_2^\frac{1}{2} n_3^\frac{1}{2}}{n_0^\frac{1}{2} - n_1^\frac{1}{2}}, \]
where in the last inequality, we used Lemma A.3 to obtain
\[ \int_{1/n_3}^{1} r^{-\frac{1}{2}} \cos((z_{n_0} + z_{n_1})r) \sin(z_{n_2}r - \frac{\pi}{4}) \, dr \sim \frac{\sqrt{z_{n_3}}}{z_{n_0} + z_{n_1}}. \]

Case V: \( \frac{1}{z_{n_3}} \leq r \leq 1. \)

In the last case, we need asymptotic approximations for all four terms, and write
\[ \int_{1/n_3}^{1} \frac{1}{z_{n_0} - z_{n_1}} \cos((z_{n_0} + z_{n_1})r) \sqrt{z_{n_2}z_{n_3}z_{n_2}} J_1(z_{n_2}r) J_0(z_{n_3}r) \, dr \]
\[ \sim \frac{\sqrt{z_{n_2}z_{n_3}z_{n_2}}}{z_{n_0} - z_{n_1}} \int_{1/n_3}^{1} \cos((z_{n_0} + z_{n_1})r) \frac{\sin(z_{n_2}r - \frac{\pi}{4}) \cos(z_{n_2}r - \frac{\pi}{4})}{\sqrt{z_{n_2}r}} \, dr \]
\[ \lesssim \frac{z_{n_2}}{z_{n_0} - z_{n_1}} \int_{1/n_3}^{1} r^{-1} \cos((z_{n_0} + z_{n_1})r) \cos((z_{n_2} + z_{n_3})r) \, dr \]
\[ \lesssim \frac{z_{n_2}}{z_{n_0} - z_{n_1} z_{n_0} + z_{n_1}} \sim \frac{n_2 n_3}{n_0^\frac{1}{2} - n_1^\frac{1}{2}}, \]
where in the last inequality, we used again Lemma A.3 to obtain
\[ \int_{1/n_3}^{1} r^{-1} \cos((z_{n_0} + z_{n_1})r) \cos((z_{n_2} + z_{n_3})r) \, dr \lesssim \frac{z_{n_3}}{z_{n_0} + z_{n_1}}. \]

Gathering the bounds in all these five cases, we see that (4) follows, hence the proof of Claim 5.5 is complete. \( \square \)

Hence we finish the proof of Lemma 5.3. \( \square \)

Let us move on to the proof of Lemma 5.4.

Proof of Lemma 5.4. Using Plancherel and convolution theorem, we write
\[ A := \int_{\mathbb{R}} w_n(t) f_n(t) g_n(t) h_n(t) \, dt \]
\[ = \int_{\mathbb{R}} \hat{w}_n(\tau) (f_n \hat{g}_n \hat{h}_n)^\wedge(\tau) \, d\tau \]
\[ = \int_{\mathbb{R}} \hat{w}_n(\tau) \int_{\tau - t_2 + t_3 = \tau} \hat{f}_n(t_1) \hat{g}_n(t_2) \hat{h}_n(t_3) \, dt_1 dt_2 dt_3 d\tau. \]

To make up \( X^{0,b} \) norms out of \( A \), we introduce the following new notations
\[ \tilde{w}_n(\tau, n_0) = \hat{w}_n(\tau) \langle \tau - n_0^2 \rangle^b \]
\[ \tilde{f}_n(t_1, n_1) = \hat{f}_n(t_1) \langle t_1 - n_1^2 \rangle^b \]
\[ \tilde{g}_n(t_2, n_2) = \hat{g}_n(t_2) \langle t_2 - n_2^2 \rangle^b \]
\[ \tilde{h}_n(t_3, n_3) = \hat{h}_n(t_3) \langle t_3 - n_3^2 \rangle^b. \]

Note that the \( L^2_{t_1} L^\infty_\tau \) norms of the functions above are in fact the \( X^{0,b} \) norms of \( w, f, g, h. \)
Then using the new notations and Hölder inequality, we obtain

\[
A = \int_{\mathbb{R}} \int_{\tau_1 - \tau_2 + \tau_3 = \tau} \left( \frac{\tilde{w}_{n_0}(\tau, n_0)}{\tilde{f}_{n_1}(\tau_1, n_1)} \tilde{g}_{n_2}(\tau_2, n_2) \tilde{h}_{n_3}(\tau_3, n_3) \right)^2 \, d\tau_1 \, d\tau_2 \, d\tau_3 \, d\tau \n
\leq \left\| \tilde{w}_{n_0} \right\|_{L^2} \left\| \left( \frac{\tilde{f}_{n_1} * \tilde{g}_{n_2} * \tilde{h}_{n_3}}{2} \right)^2 \right\|_{L^2} \n
= \left\| \tilde{w}_{n_0} \right\|_{L^2} \left\| \tilde{f}_{n_1} \right\|_{L^2} \left\| \tilde{g}_{n_2} \right\|_{L^2} \left\| \tilde{h}_{n_3} \right\|_{L^2} \n
= \|w\|_{X^{0,b}} \|f\|_{X^{0,b}} \|g\|_{X^{0,b}} \|h\|_{X^{0,b}}.
\]

For the \( A_2 \) term, a quick observation gives the integrability of the integrand

\[ A_2 \lesssim 1, \]

since \( b = \frac{1}{2} + \).

To be more precise, we estimate of the term \( A_2 \) using the following lemma from [19]. See also the related treatment for similar integrals in [33].

**Lemma 5.6** (Lemma 1 in [19]). If \( \gamma \geq 1 \), then

\[ \int_{\mathbb{R}} \frac{1}{(\tau - k_1)^\gamma (\tau - k_2)^\gamma} \, d\tau \lesssim (k_1 - k_2)^{-\gamma}. \]

Continuing the computation using the summing lemma above, we arrive at

\[
A_2 = \sup_{\tau} \left( \int_{\tau_1 - \tau_2 + \tau_3 = \tau} \frac{1}{(\tau - n_0^2)^{2b} (\tau_1 - n_1^2)^{2b} (\tau_2 - n_2^2)^{2b} (\tau_3 - n_3^2)^{2b}} \, d\tau_1 \, d\tau_2 \, d\tau_3 \right)^{\frac{1}{2}} \n
\leq \left( \int_{\tau_2 - \tau_3 + n_0^2 - n_1^2 = \tau} \frac{1}{(\tau_2 - \tau_3 + n_0^2 - n_1^2)^{2b} (\tau_2 - n_2^2)^{2b} (\tau_3 - n_3^2)^{2b}} \, d\tau_2 \, d\tau_3 \right)^{\frac{1}{2}} \n
\leq \left( \int_{\tau_3 - n_0^2 + n_1^2 - n_2^2 = \tau} \frac{1}{(\tau_3 - n_0^2 + n_1^2 - n_2^2)^{2b} (\tau_3 - n_3^2)^{2b}} \, d\tau_3 \right)^{\frac{1}{2}} \n
\leq \frac{1}{(n_0^2 - n_1^2 + n_2^2 - n_3^2)^{\gamma}}. \]

Now putting the calculation on \( A_1 \) and \( A_2 \) together, we finish the proof of Lemma 5.4. \( \square \)

**Remark 5.7** (Variations on assumptions in Proposition 5.2).  
(1) In fact, if assuming \( n_0 \geq n_1 \geq n_2 \geq n_3 \) and \( n_0 \geq 2n_3 \) in Proposition 5.2, we should expect a very similar bound with slight modification.
in the proof of Claim 5.5
\[
J(\hat{u}) = \int_{\mathbb{R} \times \Theta} \varpi \cdot f \cdot \overset{\cdot}{g} \cdot h \, dx \, dt \lesssim \frac{n_3^2}{n_0^2} \frac{n_3^2}{n_0^2} \|w\|_{X^{0,b}_\delta} \|f\|_{X^{0,b}_\delta} \|g\|_{X^{0,b}_\delta} \|h\|_{X^{0,b}_\delta}.
\]

(2) Instead of \(X^{0,b}\), we can take \(w \in L^2_{t,x}\), in which case, the only difference will be in the change of variables in (5.6). In fact, by not touching on \(\tilde{w}_{n_0}(\tau, n_0) = \tilde{w}_{n_0}(\tau)\), the rest of the argument follows perfectly, but resulting the appearance of \(L^2_{t,x}\) norm of \(w\) in the bound
\[
J(\hat{u}) = \int_{\mathbb{R} \times \Theta} \varpi \cdot f \cdot \overset{\cdot}{g} \cdot h \, dx \, dt \lesssim \frac{n_3^2}{n_0^2} \|w\|_{L^2_{t,x}} \|f\|_{X^{0,b}_\delta} \|g\|_{X^{0,b}_\delta} \|h\|_{X^{0,b}_\delta}.
\]

6. Energy increment

In this section, we compute the energy increment of the I-operator modified equation on a short time interval. This will be the key ingredient in this iterative argument in Section 7.

**Proposition 6.1** (Energy increment). Consider \(u\) as in (4.1) with \(\alpha \in (\frac{1}{2}, 1]\) defined on \([0, \delta] \times \Theta\), then for \(s > \frac{1}{2}\) and sufficiently large \(N\), the solution \(u\) satisfies the following energy increment
\[
E(Iu(\delta)) - E(Iu(0)) \lesssim N^{\frac{1}{2} - \alpha} + N^{\frac{1}{2} - 3\alpha} \|Iu_0\|^2_{H^\alpha} + N^{2 - 3\alpha} \|Iu_0\|^4_{H^\alpha} + \|Iu_0\|^6_{H^\alpha} + N^{2 - 4\alpha} \|Iu_0\|^6_{H^\alpha}.
\]

**Proof of Proposition 6.1.** We start first with writing the energy conservation
\[
\frac{d}{dt} E(u(t)) = \Re \int_{\Theta} \varpi(|u|^2 u + (-\Delta)^{\alpha} u) \, dx = \Re \int_{\Theta} \varpi(|u|^2 u + (-\Delta)^{\alpha} u - iu_4) \, dx = 0.
\]

Similarly we can compute the rate of change in the energy of the modified equation (4.1).
\[
\frac{d}{dt} E(Iu(t)) = \Re \int_{\Theta} \varpi(|Iu|^2 Iu + (-\Delta)^{\alpha} Iu - iIu_4) \, dx = \Re \int_{\Theta} \varpi(|Iu|^2 Iu - I(|u|^2 u)) \, dx = \Im \int_{\Theta} (-\Delta)^{\alpha} Iu(|Iu|^2 Iu - I(|u|^2 u)) \, dx + \Im \int_{\Theta} I(|u|^2 u)(|Iu|^2 Iu - I(|u|^2 u)) \, dx.
\]

Then by the fundamental theorem of calculus, we obtain
\[
E(Iu(\delta)) - E(Iu(0)) = \Im \int_{0}^{\delta} \int_{\Theta} (-\Delta)^{\alpha} Iu(|Iu|^2 Iu - I(|u|^2 u)) \, dx \, dt + \Im \int_{0}^{\delta} \int_{\Theta} I(|u|^2 u)(|Iu|^2 Iu - I(|u|^2 u)) \, dx \, dt
\]
(6.1) =: Term I + Term II.

To conclude the energy increment in this proposition, we just need to estimate the two terms in (6.1).

6.1. **Estimate on Term I.** First we decompose each \(u\) in Term I as we did in (4.3) in Lemma 4.4, then write for the quadruple \(N = (N_0, N_1, N_2, N_3)\),
\[
\text{Term I } \sim \sum_{N} \int_{0}^{\delta} \int_{\Theta} (-\Delta)^{\alpha} Iu_0(Iu_1 Tu_2 Tu_3 - I(u_1 \varpi_2 u_3)) \, dx \, dt,
\]
where \(u_i = \sum_{N_i \leq (z_i) \approx 2N_i} P_{n_i} u, \ i = 0, 1, 2, 3\).

Let
\[
\text{Term I}(N) := \int_{0}^{\delta} \int_{\Theta} (-\Delta)^{\alpha} Iu_0(Iu_1 Tu_2 Tu_3 - I(u_1 \varpi_2 u_3)) \, dx \, dt.
\]

Without loss of generality, we assume \(N_1 \geq N_2 \geq N_3\), and analyze the following different scenarios. Let us outline the cases that we will be considering.
- **Case I-1**: trivial cases;
- **Case I-2**: the maximum frequency is much larger than the second highest frequency;
- The largest two frequencies are comparable;
  * **Case I-3**: the largest two frequencies are $N_1 = N_0$,
  * **Case I-4**: the largest two frequencies are $N_1 = N_2$.

**Case I-1**: The trivial cases. After the decomposition in frequencies, we have the following two trivial cases:

- All frequencies are not comparable to $N$, that is $N_0, N_1, N_2, N_3 \ll N$,
- $N_0 = N_1 \geq N \geq N_2 \geq N_3$.

In both cases, we have

$$\text{Term I}(N) = 0,$$

hence

$$\sum_{N \in \text{Case I-1}} \text{Term I}(N) = 0.$$

Now we focus on the regime where at least the maximum frequency is larger than $N$, that is $\max\{N_0, N_1, N_2, N_3\} \gtrsim N$.

Using the abuse the notation in Remark 4.5, we write

$$\text{Term I}(N) \leq \left| \int_\Theta^\delta (\Delta)\alpha T_{u_0}(Iu_1Tu_2Tu_3) \, dxdt \right| + \left| \int_\Theta^\delta (\Delta)\alpha T_{u_0}I(u_1Tu_2u_3) \, dxdt \right|$$

$$\leq \left| \int_\Theta^\delta (\Delta)\alpha T_{u_0}(Iu_1Tu_2Tu_3) \, dxdt \right| + \frac{m(N_0)}{m(N_1)m(N_2)m(N_3)} \left| \int_\Theta^\delta (\Delta)\alpha T_{u_0}I(u_1Tu_2u_3) \, dxdt \right|$$

\[= (1 + \frac{m(N_0)}{m(N_1)m(N_2)m(N_3)}) \left| \int_\Theta^\delta (\Delta)\alpha T_{u_0}(Iu_1Tu_2Tu_3) \, dxdt \right| \]

\[
(6.2)
\]

=: $M(N) \times \text{Term I}'(N)$.

**Case I-2**: The maximum frequency is much larger than the second highest frequency.

**Case I-2a**: $\max\{n_0, n_1, n_2, n_3\} = n_0 \gtrsim N$ and $n_0 \gg n_1$.

Take **Term I’(N)** in (6.2) first. Applying the weak interaction between frequency localized functions in Proposition 5.2 (where we take $w = P_{n_0}u$, $f = P_{n_1}u$, $g = P_{n_2}u$ and $h = P_{n_3}u$) and taking out the derivative on $P_{n_0}u$, we are able to write

$$\text{Term I}'(n) \lesssim n_0^{n_1 n_2 n_3} \| P_{n_0}u \|_{X^0, b} \| P_{n_1}u \|_{X^0, b} \| P_{n_2}u \|_{X^0, b} \| P_{n_3}u \|_{X^0, b},$$

where $b = \frac{1}{2} +$. Note that in the rest of the proof we will constantly use the notation $b = \frac{1}{2} +$ for simplicity.

Then we estimate $M(n)$ by

$$M(n) \lesssim \begin{cases} 
(n_{n_0})^{\alpha-s} & \text{if } n_3 \leq n_2 \leq n_1 \leq N \lesssim n_0 \\
(n_{n_0})^{\alpha-s} & \text{if } n_3 \leq n_2 \leq N \leq n_1 \lesssim n_0 \\
(n_{n_0}^{n_2 n_3})^{\alpha-s} & \text{if } n_3 \leq N \leq n_2 \leq n_1 \lesssim n_0 \\
(n_{n_0}^{n_2 n_3})^{\alpha-s} & \text{if } N \leq n_3 \leq n_2 \leq n_1 \lesssim n_0.
\end{cases}$$

(6.3)
Now summing over all $n_i$ using Cauchy-Schwarz inequality, Bernstein inequality and Definition 2.1, we have
\[
\sum_{\mathcal{U}} \text{Term I}(\mathcal{U}) \lesssim \sum_{\mathcal{U}} M(\mathcal{U}) \cdot n_0^{\frac{3}{2}} \left(\frac{n^3}{n_0} \right)^{\alpha-s} \left(\frac{n_0^2}{n_0^3} \right)^{\alpha-s} \sum_{n_1=1}^{n_0^2-2s} \sum_{n_2}^{n_0^2-2s} \sum_{n_3}^{n_0^2-2s} \left(\frac{n_1}{n_0}\right)^{\alpha-s} \left(\frac{n_2}{n_0}\right)^{\alpha-s} \left(\frac{n_3}{n_0}\right)^{\alpha-s}
\]
\[
\lesssim N^{2\alpha-2s} \sum_{n_0}^{n_0^2-2s} \left(\frac{n_0^3}{n_0^4} \right)^{\alpha-s} \left(\frac{n_0^3}{n_0^4} \right)^{\alpha-s} \left(\frac{n_0^3}{n_0^4} \right)^{\alpha-s} \lesssim N^{4\alpha-6s}.
\]

The same bound holds for other cases of $M(\mathcal{U})$ in (6.3).

Therefore,
\[
\sum_{\mathcal{U} \in \text{Case I-2a}} \text{Term I}(\mathcal{U}) \lesssim N^{2\alpha-3s} \|Iu\|^4_{X^{\alpha,b}_s}.
\]

**Case I-2b:** $\max\{n_0, n_1, n_2, n_3\} = n_1 \geq N$ and $n_1 \gg \max\{n_0, n_2\}$.

This case is in fact similar to the previous **Case I-2a**. Using
\[
M(\mathcal{U}) \lesssim \begin{cases} 
\left(\frac{n_0}{n_0}\right)^{\alpha-s} & \text{if } n_3, n_2, n_0 \leq N \lesssim n_1 \\
\left(\frac{n_0}{n_0}\right)^{\alpha-s} & \text{if } n_3 \leq n_2 \leq N \lesssim n_0 \leq n_1 \\
\left(\frac{n_0}{n_0}\right)^{\alpha-s} & \text{if } n_3, n_0 \leq N \lesssim n_2 \leq n_1 \\
\left(\frac{n_0}{n_0}\right)^{\alpha-s} & \text{if } n_3 \leq N \lesssim n_2 \leq n_1, N \leq n_0 \\
\left(\frac{n_0}{n_0}\right)^{\alpha-s} & \text{if } n_0 \leq N \lesssim n_3 \leq n_2 \leq n_1 \\
\left(\frac{n_0}{n_0}\right)^{\alpha-s} & \text{if } N \lesssim n_3 \leq n_2 \leq n_1, N \leq n_0.
\end{cases}
\]

and similar calculation in **Case I-2a**, we see that
\[
\sum_{\mathcal{U} \in \text{Case I-2b}} \text{Term I}(\mathcal{U}) \lesssim N^{2\alpha-3s} \|Iu\|^4_{X^{\alpha,b}_s}.
\]

Therefore, by Cauchy-Schwarz inequality and Definition 2.1 we have in **Case I-2**
\[
\sum_{\mathcal{N} \in \text{Case I-2}} \text{Term I}(\mathcal{N}) \lesssim N^{2\alpha-3s} \|Iu\|^4_{X^{\alpha,b}_s}.
\]

Now we focus on the case when the largest two frequencies are comparable. Under our assumption on $N_0, N_1, N_2, N_3$, there are only following two possibilities and we will discuss them separately.

- **Case I-3:** $N_1 = N_0 \geq N$ and $N_1 \geq N_2 \geq N_3$;
- **Case I-4:** $N_1 = N_2 \geq N$ and $N_1 \geq N_0$.

**Case I-3:** The largest two frequencies are comparable. $N_1 = N_0 \geq N$. 

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In this case, we write

\[ M(N) \lesssim \frac{m(N_0)}{m(N_1)m(N_2)m(N_3)} \approx \frac{1}{m(N_2)m(N_3)}. \]

Recall the bilinear estimate that we obtained in (3.4),

\[ \|f_1 f_2\|_{L^2_x \times y ((0, \delta) \times \Theta)} \lesssim N^{\beta_2 \delta^2 (b - L(\beta))} \| f_1 \|_{X^{\alpha,b}_\delta(\Theta)} \| f_2 \|_{X^{\alpha,b}_\delta(\Theta)}. \]

for \( b(\beta) = \frac{1}{4} + (1 - \beta) \frac{1}{2} +, \beta \in (\frac{1}{4}, 1) \). Taking \( \beta = \alpha \) and combining with \( \delta \sim N^{-\frac{2(\alpha - s)}{1 + 2 \alpha - 4(\alpha - 1)}} \) in Proposition 4.3, we have

\[ \|f_1 f_2\|_{L^2_x \times y ((0, \delta) \times \Theta)} \lesssim N^{2\alpha} \frac{N^{-4(\alpha - s) (1 - \beta)/1 + 2 \alpha - 4(\alpha - 1)}}{N_2^\alpha N_3^{\alpha - 1} N_2^\alpha N_3^{\alpha}} \prod_{i = 0}^{3} \| Iu_i \|_{X^{\alpha,b}_\delta}. \]

Case I-3a: \( N_1 = N_0 \gtrsim N \geq N_2 \geq N_3 \). This is trivial as showed in Case I-1.

Case I-3b: \( N_1 = N_0 \geq N \geq N_2 \geq N_3 \).

Using

\[ M(N) \lesssim (N^{-1} N_2)^{\alpha - s}. \]

with Hölder inequality, Proposition 3.3, (6.5) and Bernstein inequality, we have

\[
\begin{align*}
\text{Term I}(N) &\lesssim N^{2\alpha} (N^{-1} N_2)^{\alpha - s} \| Iu_0 Iu_2 \|_{L^2_x \times y ((0, \delta) \times \Theta)} \| Iu_1 Iu_3 \|_{L^2_x \times y ((0, \delta) \times \Theta)} \\
&\lesssim N^{2\alpha} (N^{-1} N_2)^{\alpha - s} N_2^{\alpha} N_3^{\alpha} N^{-\frac{4(\alpha - s) (1 - \beta)/1 + 2 \alpha - 4(\alpha - 1)}} \prod_{i = 0}^{3} \| Iu_i \|_{X^{\alpha,b}_\delta} \\
&\lesssim \frac{1}{N^{\alpha - s}} \frac{N^{2\alpha} N_2^{\alpha} N_3^{\alpha} N^{-\frac{4(\alpha - s) (1 - \beta)/1 + 2 \alpha - 4(\alpha - 1)}} \prod_{i = 0}^{3} \| Iu_i \|_{X^{\alpha,b}_\delta}}{N_0^{\alpha} N_1 N_2^{\alpha} N_3^{\alpha}} \\
&\lesssim N^{\frac{1}{2} - 2 \alpha + s} N^{-\frac{4(\alpha - s) (1 - \beta)/1 + 2 \alpha - 4(\alpha - 1)}} N_2^0 N_3^0 \left( \frac{N_0}{N_1} \right)^{\alpha} \prod_{i = 0}^{3} \| Iu_i \|_{X^{\alpha,b}_\delta}. 
\end{align*}
\]

Therefore, by Cauchy-Schwarz inequality and Definition 2.1 we have

\[ \sum_{N \in \text{Case I-3b}} \text{Term I}(N) \lesssim N^{\frac{1}{2} - 2 \alpha + s} N^{-\frac{4(\alpha - s) (1 - \beta)/1 + 2 \alpha - 4(\alpha - 1)}} \| Iu \|_{X^{\alpha,b}_\delta}^3. \]

Case I-3c: \( N_1 = N_0 \geq N \geq N_2 \geq N_3 \).

Using

\[ M(N) \lesssim (N^{-2} N_2 N_3)^{\alpha - s}. \]

Hölder inequality, Proposition 3.3 and Bernstein inequality, we have

\[
\begin{align*}
\text{Term I}(N) &\lesssim N^{2\alpha} (N^{-2} N_2 N_3)^{\alpha - s} \| Iu_0 Iu_2 \|_{L^2_x \times y ((0, \delta) \times \Theta)} \| Iu_1 Iu_3 \|_{L^2_x \times y ((0, \delta) \times \Theta)} \\
&\lesssim N^{2\alpha} (N^{-2} N_2 N_3)^{\alpha - s} (N_2 N_3)^{\alpha} \prod_{i = 0}^{3} \| Iu_i \|_{X^{\alpha,b}_\delta} \\
&\lesssim N^{-2(\alpha - s)} N_0^{2\alpha} N_1 N_2^{\alpha} N_3^{\alpha} (N_2 N_3)^{\alpha - s} \prod_{i = 0}^{3} \| Iu_i \|_{X^{\alpha,b}_\delta} \\
&\lesssim N^{1 - 2 \alpha +} N_2^0 N_3^0 \left( \frac{N_0}{N_1} \right)^{\alpha} \prod_{i = 0}^{3} \| Iu_i \|_{X^{\alpha,b}_\delta}. 
\end{align*}
\]

Therefore, by Cauchy-Schwarz inequality and Definition 2.1 we have

\[ \sum_{N \in \text{Case I-3c}} \text{Term I}(N) \lesssim N^{1 - 2 \alpha +} \| Iu \|_{X^{\alpha,b}_\delta}^3. \]
To sum up, the bound in **Case I-3** is given by

$$\sum_{N \in \text{Case I-3}} \text{Term I}(N) \lesssim N^{\frac{1}{2} - 2\alpha + s} \cdot N^{-\frac{4(\alpha + 1) \cdot (b + k(\alpha - 1))}{1 + 2b + 4(\alpha - 1)}} \| Iu \|_{X^{\alpha,b}_a}^4 + N^{1 - 2\alpha} \| Iu \|_{X^{\alpha,b}_a}^4.$$ 

**Case I-4:** The largest two frequencies are comparable. $N_1 = N_2 \gtrsim N$ and $N_1 \geq N_0$.

In this case, we write

$$M(N) \lesssim \frac{m(N_0)}{m(N_1)m(N_2)m(N_3)} \lesssim \frac{m(N_0)}{m(N_3)}(N^{-2}N_1N_2)^{\alpha - s},$$

where

$$\left(\frac{m(N_0)}{m(N_3)}\right)^{\alpha - s} \lesssim \begin{cases} 1 & \text{if } N_0, N_3 \lesssim N \\ \left(\frac{N_0}{N_3}\right)^{\alpha - s} & \text{if } N_3 \lesssim N \lesssim N_0 \\ \left(\frac{N_3}{N_0}\right)^{\alpha - s} & \text{if } N_0 \lesssim N \lesssim N_3 \\ \left(\frac{N_0}{N_3}\right)^{\alpha - s} & \text{if } N \lesssim N_0, N_3. \end{cases}$$

(6.6)

Take the first case in (6.6), then by Hölder inequality, Proposition 3.3, (6.5) and Bernstein inequality, we write

$$\text{Term I}(N) \lesssim N_0^{2\alpha}(N^{-2}N_1N_2)^{\alpha - s} \| Iu_0Iu_2 \|_{L^2_{t,x}([0, \delta] \times \Theta)} \| Iu_1Iu_3 \|_{L^2_{t,x}([0, \delta] \times \Theta)}$$

$$\lesssim N_0^{2\alpha}(N^{-2}N_1N_2)^{\alpha - s} N_0^{\frac{1}{2}} N_3^{\alpha} N_3^{\frac{1}{2}} N_2^{-\alpha} N_2^{-\frac{1}{2}} N_3^{-\alpha} N_3^{-\frac{1}{2}} N_2^{\alpha} \prod_{i=0}^{3} \| Iu_i \|_{X^{\alpha,b}_a}$$

$$\lesssim N^{\frac{1}{2} - \alpha} N^{-\frac{4(\alpha + 1) \cdot (b + k(\alpha - 1))}{1 + 2b + 4(\alpha - 1)}} N_0^0 N_1^0 N_2^0 N_3^0 \prod_{i=0}^{3} \| Iu_i \|_{X^{\alpha,b}_a}.$$ 

The same bound hold for the second case in (6.6).

For the third case in (6.6), using Hölder inequality, Proposition 3.3 and Bernstein inequality, we have

$$\text{Term I}(N) \lesssim N_0^{2\alpha}(N^{-2}N_1N_2)^{\alpha - s} \left(\frac{N_2}{N} \right)^{\alpha - s} \| Iu_0Iu_2 \|_{L^2_{t,x}([0, \delta] \times \Theta)} \| Iu_1Iu_3 \|_{L^2_{t,x}([0, \delta] \times \Theta)}$$

$$\lesssim N_0^{2\alpha}(N^{-2}N_1N_2)^{\alpha - s} \left(\frac{N_2}{N} \right)^{\alpha - s} N_0^{\frac{1}{2}} N_3^{\frac{1}{2}} N_2^{\alpha} \prod_{i=0}^{3} \| Iu_i \|_{X^{\alpha,b}_a}$$

$$\lesssim N^{-2(\alpha - s)} N_0^{2\alpha}(N_1N_2N_3)^{\alpha - s} N_0^{\frac{1}{2}} N_3^{\frac{1}{2}} N_2^{\alpha} \prod_{i=0}^{3} \| Iu_i \|_{X^{\alpha,b}_a}$$

$$\lesssim N^{1 - 2\alpha} N_0^0 N_1^0 N_2^0 \prod_{i=0}^{3} \| Iu_i \|_{X^{\alpha,b}_a}.$$ 

The same bound hold for the forth case in (6.6).

Therefore, by Cauchy-Schwarz inequality and Definition 2.1 we have that in **Case I-4**.

$$\sum_{N \in \text{Case I-4}} \text{Term I}(N) \lesssim N^{\frac{1}{2} - \alpha} N^{-\frac{4(\alpha + 1) \cdot (b + k(\alpha - 1))}{1 + 2b + 4(\alpha - 1)}} \| Iu \|_{X^{\alpha,b}_a}^4 + N^{1 - 2\alpha} \| Iu \|_{X^{\alpha,b}_a}^4.$$ 

Now we summarize the estimation on **Term I**.

$$\text{Term I} \lesssim \left( \sum_{N \in \text{Case I-2}} + \sum_{N \in \text{Case I-3}} + \sum_{N \in \text{Case I-4}} \right) \text{Term I}(N).$$
In both cases, we have

Case II-2a: The maximum frequency is much larger than the minimum frequency.

Case II-2: The trivial cases. We have two trivial cases as in Case II-2.

Case II-3: All frequencies are comparable.

6.2. Estimate on Term II. Now let us focus on Term II:

$$\text{Im} \int_0^\delta \int_{\Theta} I(|u|^2 u)(|u|^2 Iu - I(|u|^2 u)) \, dx.$$  

Similarly, we decompose each \( u \) in Term II using the orthonomal basis \( e_n \)'s of the radial Dirichlet Laplacian \(-\Delta\). That is, for the quadruple \( \mathcal{N} = (N_0, N_1, N_2, N_3) \)

$$\text{Term II} \sim \sum_{\mathcal{N}} \int_0^\delta \int_{\Theta} IP_{N_0}(|u|^2 u)(Iu_1\overline{u_2}Iu_3 - I(u_1\overline{u_2}u_3)) \, dx \, dt,$$

where \( u_i = \sum_{N_i \leq \varepsilon N_i < 2N_i} P_n u_i, \ i = 1, 2, 3. \)

Let

$$\text{Term II}(\mathcal{N}) := \int_0^\delta \int_{\Theta} IP_{N_0}(|u|^2 u)(Iu_1\overline{u_2}Iu_3 - I(u_1\overline{u_2}u_3)) \, dx \, dt.$$  

Without loss of generality, we assume \( N_1 \geq N_2 \geq N_3 \). Let us outline the cases that we will be considering.

- **Case II-1**: trivial cases;
- **Case II-2**: the maximum frequency is much larger than the minimum frequency;
- **Case II-3**: all frequencies are comparable.

Again, we start with the trivial cases.

**Case II-1**: The trivial cases. We have two trivial cases as in **Term I**

- All frequencies are not comparable to \( N \), that is \( N_0, N_1, N_2, N_3 \ll N \).
- \( N_0 = N_1 \geq N \geq N_2 \geq N_3 \)

In both cases, we have

$$\sum_{\mathcal{N} \in \text{Case II-1}} \text{Term II}(\mathcal{N}) = 0.$$  

This implies that at least the maximum frequency is larger than the cutoff frequency \( N \), that is \( \max\{N_0, N_1, N_2, N_3\} \geq N \).

Similar as in (6.2), we write with abuse notation in Remark 4.5.

$$\text{Term II}(\mathcal{N}) \lesssim (1 + \frac{m(N_0)}{m(N_1)m(N_2)m(N_3)}) \left| \int_0^\delta \int_{\Theta} IP_{N_0}(|u|^2 u)(Iu_1\overline{u_2}Iu_3) \, dx \, dt \right|$$

$$=: M(\mathcal{N}) \times \text{Term II'}(\mathcal{N}).$$

**Case II-2**: The maximum frequency is much larger than minimum frequency.

**Case II-2a**: \( \max\{n_0, n_1, n_2, n_3\} = n_0 \geq N \) and \( n_0 \gg n_3 \).

In this case, using the same estimate of \( M(\mathcal{N}) \) as in (6.3) and Remark 5.7, we write

$$\text{Term II}(\mathcal{N}) \lesssim M(\mathcal{N}) \left\| IP_{n_0}(|u|^2 u) \right\|_{L^2_t} \prod_{i=1}^3 \left\| P_{n_i} u \right\|_{X^{a,b}}.$$  

In the following claim, we compute the term \( \left\| IP_{n_0}(|u|^2 u) \right\|_{L^2_t} \) separately.
Claim 6.2.

\[ B := \left\| IP_{n_0}(|u|^2 u) \right\|_{L^6_{t,x}} \lesssim m(n_0)B(N) \left\| Iu \right\|_{X^{\alpha,b}}^3, \]

where

\[ B(N) := N^{-\frac{1}{1+2\alpha+\alpha_s}} + N^{-\frac{3}{2} - 3\alpha} + N^{-\alpha}. \]

Assuming Claim 6.2, we can write

\[ \text{Term II}(u) \lesssim M(u) \frac{\alpha^4 \alpha^4}{n_0} \frac{m(n_0)}{n_1 n_2 n_3} \prod_{i=1}^3 \| P_n Iu \|_{X^{\alpha,b}} \left( B(N) \left\| Iu \right\|_{X^{\alpha,b}}^3 \right). \]

Using Cauchy-Schwarz inequality, (6.3) and Definition 2.1, we obtain that

\[ \sum_{n \in \text{Case II-2a}} \text{Term II}(u) \lesssim \left( \sum_n \left( \frac{\alpha^4 \alpha^4 M(u) \cdot m(n_0)}{n_0^2 n_1^2 n_2^2 n_3^2} \right)^2 \right)^{\frac{1}{2}} B(N) \left\| Iu \right\|_{X^{\alpha,b}}^6 \lesssim N^{-\frac{3}{2} - 2\alpha} B(N) \left\| Iu \right\|_{X^{\alpha,b}}^6. \]

Now we present the calculation for Claim 6.2.

Proof of Claim 6.2. Decompose B into dyadic frequencies, then we have

\[ \left\| IP_{n_0}(u_4 \overline{u_5} u_6) \right\|_{L^6_{t,x}} \sim \frac{m(n_0)}{m(N_4)m(N_5)m(N_6)} \| Iu_4 Iu_5 Iu_6 \|_{L^6_{t,x}} =: m(n_0) \times B(N_{456}). \]

Let us focus on B(N_{456}). We first rewrite it using Hölder inequality and Sobolev embedding

\[ B(N_{456}) \lesssim \frac{1}{m(N_4)m(N_5)m(N_6)} \| Iu_4 Iu_5 \|_{L^6_{t,x}} \| Iu_6 \|_{L^6_{t,x}} \lesssim \frac{1}{m(N_4)m(N_5)m(N_6)} N_6^{-\alpha} \| Iu_4 Iu_5 \|_{L^6_{t,x}} \| Iu_6 \|_{X^{\alpha,b}}. \]

Without loss of generality, we assume \( N_4 \geq N_5 \geq N_6. \)

Similar as the computation that we did earlier, we will consider the following two cases

- **Case B-1**: all frequencies are smaller than \( N; \)
- **Case B-2**: at least on frequency is larger than \( N. \)

**Case B-1**: \( N_6 \leq N_5 \leq N_4 \ll N. \)

Using

\[ \frac{1}{m(N_4)m(N_5)m(N_6)} = 1, \]

with (6.5) and Bernstein inequality, we obtain

\[ B(N_{456}) \lesssim N_5 N^{-\frac{\alpha}{1+2\alpha+\alpha_s}} N_6^{-1-\alpha} \| Iu_4 \|_{X^{\alpha,b}} \| Iu_5 \|_{X^{\alpha,b}} \| Iu_6 \|_{X^{\alpha,b}} \]

\[ \lesssim N^{-\frac{\alpha}{1+2\alpha+\alpha_s}} N_5 N_6^{1-\alpha} \prod_{i=4}^6 \| Iu_i \|_{X^{\alpha,b}}. \]

Then by Cauchy-Schwarz inequality with Definition 2.1,

\[ \sum_{N_4, N_5, N_6 \in \text{Case B-1}} B(N_{456}) \lesssim N N^{-\frac{\alpha}{1+2\alpha+\alpha_s}} \left\| Iu \right\|_{X^{\alpha,b}}^3. \]

**Case B-2**: \( N \lesssim N_4. \)
We compute

\[
\frac{1}{m(N_4) m(N_5) m(N_6)} \lesssim \begin{cases} 
(N^{-1} N_4)^{\alpha - s} & \text{if } N_6 \lesssim N_5 \ll N \ll N_4 \\
(N^{-2} N_4 N_5)^{\alpha - s} & \text{if } N_6 \ll N \ll N_5 \ll N_4 \\
(N^{-3} N_4 N_5 N_6)^{\alpha - s} & \text{if } N \ll N_5 \ll N_6 \ll N_4.
\end{cases}
\]

Now take the first case in (6.7), with the bilinear estimate in Proposition 3.3, we then obtain

\[
B(N_{456}) \lesssim (N^{-1} N_4)^{\alpha - s} \left( N^\frac{4}{6} N_5^{\frac{4}{6}} - N_6^{1 - \alpha} \right) \| | u_4 \|_{X^0_{a,b}} \| | u_5 \|_{X^0_{a,b}} \| | u_6 \|_{X^0_{a,b}} \\
\lesssim (N^{-1} N_4)^{\alpha - s} \frac{N^\frac{4}{6} N_5^{\frac{4}{6}} - N_6^{1 - \alpha}}{N_4^2 N_5^2 N_6^2} \prod_{i=1}^{6} \| | u_i \|_{X^0_{a,b}} \\
\lesssim \begin{cases} 
N^\frac{4}{6} - 3\alpha + N_4^0 - N_5^0 - N_6^0 \prod_{i=1}^{6} \| | u_i \|_{X^0_{a,b}} & \text{if } \alpha < \frac{3}{4} \\
N^{-\alpha} + N_4^0 - N_5^0 - N_6^0 \prod_{i=1}^{6} \| | u_i \|_{X^0_{a,b}} & \text{if } \alpha \geq \frac{3}{4}
\end{cases}
\]

\[= \max\{N^\frac{4}{6} - 3\alpha +, N^{-\alpha} \} \prod_{i=1}^{6} \| | u_i \|_{X^0_{a,b}}.\]

Similarly, the second and third cases are bounded respectively by

\[
\max\{N^\frac{4}{6} - 3\alpha +, N^{-2\alpha + s} \} \prod_{i=1}^{6} \| | u_i \|_{X^0_{a,b}}
\]

and

\[
\max\{N^\frac{4}{6} - 3\alpha +, N^{-3\alpha + 2s} \} \prod_{i=1}^{6} \| | u_i \|_{X^0_{a,b}}.
\]

Then by Cauchy-Schwarz inequality with Definition 2.1,

\[
\sum_{N_4, N_5, N_6 \in \text{Case B-2}} B(N_{456}) \lesssim \max\{N^\frac{4}{6} - 3\alpha +, N^{-\alpha} \} \| | u \|_{X^3_{a,b}}^3.
\]

Therefore, by putting the two cases together, we finish the proof of Claim 6.2.

\[
\sum_{N_4, N_5, N_6} B(N_{456}) \lesssim \left( \sum_{N_4, N_5, N_6 \in \text{Case B-1}} + \sum_{N_4, N_5, N_6 \in \text{Case B-2}} \right) B(N_{456}) \\
\lesssim N^{-\frac{2\alpha}{1+2\alpha-4\alpha s}} \| | u \|_{X^3_{a,b}}^3 + N^{\frac{4}{6} - 3\alpha +} \| | u \|_{X^3_{a,b}}^3 + N^{-\alpha} \| | u \|_{X^3_{a,b}}^3.
\]

**Case II-2b:** $\max\{n_0, n_1, n_2, n_3\} = n_1 \gtrsim N$ and $n_1 \gg \min\{n_0, n_3\}$

Writing

\[
M(n) \lesssim \frac{m(n_0)}{m(n_1)m(n_2)m(n_3)}
\]

and using Remark 5.7 and Claim 6.2, we have

\[
\text{Term II}(N) \lesssim M(n) \frac{n_2^2 n_3^2}{n_1^2} \frac{1}{n_1^2 n_2^2 n_3^2} \| | IP_{n_0}(|u|^2 u) \|_{L^2_{t, x}} \prod_{i=1}^{3} \| | u_i \|_{X^0_{a,b}} \\
\lesssim M(n) \frac{n_2^2 n_3^2}{n_1^2} \frac{m(n_0)}{n_1^2 n_2^2 n_3^2} \prod_{i=1}^{3} \| | u_i \|_{X^0_{a,b}} \left( B(N) \| | u \|_{X^3_{a,b}}^3 \right).
\]

Applying the analysis as in (6.4) and same calculation as in **Case II-2a**, we obtain the bound

\[
\sum_{n \in \text{Case II-2b}} \text{Term II}(n) \lesssim N^{2-3\alpha} B(N) \| | u \|_{X^3_{a,b}}^6.
\]
Now we have the only case left.

**Case II-3:** All frequencies are comparable. \( N_0 = N_1 = N_2 = N_3 \gtrsim N \).

In this last case, we have
\[
M(\mathcal{N}) \lesssim \frac{m(N_0)}{m(N_1)m(N_2)m(N_3)} \sim \frac{1}{m(N_2)m(N_3)} \lesssim (N^{-2}N_2N_3)^{\alpha-s}.
\]

Then using the \( M(\mathcal{N}) \) bound above, Proposition 3.3 and Claim 6.2, we write
\[
\text{Term II}(\mathcal{N}) \lesssim (N^{-2}N_2N_3)^{\alpha-s} \left| \int_0^\delta \int_\Theta IP_N(|u|^2u)(Iu_1Iu_2Iu_3) \, dxdt \right|
\]
\[
\lesssim (N^{-2}N_2N_3)^{\alpha-s} ||Iu_1Iu_2Iu_3||_{L^2_t} \left\| \left[ \sum_{\mathcal{N} \in \text{Case II-3}} IP_N(|u|^2u) \right] \right\|_{L^2_t,x}
\]
\[
\lesssim \left( \frac{N_2N_3}{N^2} \right)^{\alpha-s} \left( \frac{N_2^2+1}{N_1^\alpha N_2^\alpha} \prod_{i=1}^3 ||Iu_i||_{X^{\alpha,b}_t} \right) \left( m(N_0)B(N) \right) \left( ||Iu||^3_{X^{3,b}_t} \right).
\]

Then by Cauchy-Schwarz inequality with Definition 2.1,
\[
\sum_{\mathcal{N} \in \text{Case II-3}} \text{Term II}(\mathcal{N}) \lesssim N^{\frac{3}{2}-3\alpha}B(N) \left\| lu \right\|_{X^{3,b}_t}^6.
\]

Therefore, we summarize the estimation on all the cases in **Term II**, \( \sum_{\mathcal{N} \in \text{Case II-3}} \text{Term II}(\mathcal{N}) \)
\[
\lesssim N^{2-3\alpha}B(N) \left\| lu \right\|_{X^{\alpha,b}_t}^6 \lesssim N^{2-3\alpha}B(N) \left\| lu_0 \right\|_{H^\alpha}^6.
\]

Let us finish the calculation on the energy increment by combining the computation for both **Term I** and **Term II**
\[
E(Iu(\delta)) - E(Iu(0)) \\
\lesssim N^{\frac{1}{2}-\alpha} + \frac{4(\alpha-\alpha)}{1+2\alpha-4\alpha} \left\| lu_0 \right\|_{H^\alpha}^4 + N^{2-3\alpha} \left\| lu_0 \right\|_{H^\alpha}^4
\]
\[
+ N^{2-3\alpha} \left\| lu_0 \right\|_{H^\alpha}^4 + N^{2-3\alpha} \left\| lu_0 \right\|_{H^\alpha}^4
\]
\[
\lesssim N^{\frac{1}{2}-\alpha} + \frac{4(\alpha-\alpha)}{1+2\alpha-4\alpha} \left\| lu_0 \right\|_{H^\alpha}^4 + N^{2-3\alpha} \left\| lu_0 \right\|_{H^\alpha}^4
\]
\[
+ N^{2-3\alpha} \left\| lu_0 \right\|_{H^\alpha}^4 + N^{\frac{1}{2}-3\alpha} \left\| lu_0 \right\|_{H^\alpha}^4.
\]

The last inequality follows from Remark 4.2. Then we finish the proof of Proposition 6.1.

7. **Global well-posedness**

In this section, we finally show the global well-posedness result stated in Theorem 1.1 by iteration. We also will see the choices of the parameters in previous sections and the constraint on the regularity. As a consequence of these choices, we obtain a polynomial bound of the solution as presented in Theorem 1.1.

**Proof of Theorem 1.1.** By the definition of energy and the Gagliardo–Nirenberg interpolation inequality, we have
\[
E(Iu_0) = \frac{1}{2} \left\| lu_0 \right\|_{H^\alpha}^2 + \frac{1}{4} \left\| lu_0 \right\|_{L_t^4}^4 \lesssim \left\| lu_0 \right\|_{L_t^2}^2 \left\| lu_0 \right\|_{H^\alpha}^2 \lesssim N^{(\alpha-s)^{\frac{1}{2}}}.
\]
Then the energy increment obtained in Proposition 6.1 becomes

\[
E(Iu(\delta)) - E(Iu(0)) \lesssim N^{\frac{1}{2} - \alpha} N^{-\frac{4(a-3)(b-\alpha)}{12(\alpha-1)}} \|Iu_0\|_{H^\alpha}^4 + N^{2-3\alpha} - \|Iu_0\|_{H^\alpha}^4 + N^{2-3\alpha} N^{-\frac{4(\alpha-3)(b-\alpha)}{12(\alpha-1)}} N^4(\alpha-s) + N^{2-3\alpha} N^4(\alpha-s)
\]

\[
+ N^{2-3\alpha} N^{-\frac{4(\alpha-3)(b-\alpha)}{12(\alpha-1)}} N^4(\alpha-s) + N^{2-3\alpha} N^{6(\alpha-s)} + N^{\frac{7}{2} - 6\alpha + N^{6(\alpha-s)} + N^{2-4\alpha} N^6(\alpha-s)}.
\]

(7.1)

In order to reach a fixed time \(T \gg 1\), the number of steps in the iteration is at most,

(7.2)

\[
\frac{T}{\delta} \sim TN^{\frac{2(a-s)}{12(\alpha-1)}} \sim TN^\frac{2(a-s)}{12(\alpha-1)}.
\]

Combining (7.1) and (7.2), we write the modified energy at time \(T\) as

\[
E(Iu(T)) \lesssim E(Iu(0)) + \frac{T}{\delta} \left( N^{\frac{1}{2} - \alpha} N^{-\frac{4(a-3)(b-\alpha)}{12(\alpha-1)}} N^4(\alpha-s) + N^{2-3\alpha} N^4(\alpha-s) \right) + \frac{T}{\delta} \left( N^{2-3\alpha} N^{-\frac{4(a-3)(b-\alpha)}{12(\alpha-1)}} N^4(\alpha-s) + N^{2-3\alpha} N^4(\alpha-s) \right)
\]

\[
\lesssim N^{(\alpha-s)\frac{2}{3}} + TN^{\frac{2(a-s)}{12(\alpha-1)}} \left( N^{\frac{1}{2} - \alpha} N^{-\frac{4(a-3)(b-\alpha)}{12(\alpha-1)}} N^4(\alpha-s) + N^{2-3\alpha} N^4(\alpha-s) \right) + T N^{\frac{2(a-s)}{12(\alpha-1)}} \left( N^{2-3\alpha} N^{-\frac{4(a-3)(b-\alpha)}{12(\alpha-1)}} N^4(\alpha-s) + N^{2-3\alpha} N^4(\alpha-s) \right)
\]

\[
+ T N^{\frac{2(a-s)}{12(\alpha-1)}} \left( N^2 - 3\alpha N^{-\frac{4(a-3)(b-\alpha)}{12(\alpha-1)}} N^6(\alpha-s) + N^{\frac{7}{2} - 6\alpha + N^6(\alpha-s)} + N^{2-4\alpha} N^6(\alpha-s) \right).
\]

In order to keep this iteration valid at each step, we have to ensure that the total energy increment is always being controlled by the initial energy \(E(Iu(0))\), that is

\[
TN^{\frac{2(a-s)}{12(\alpha-1)}} \left( N^{\frac{1}{2} - \alpha} N^{-\frac{4(a-3)(b-\alpha)}{12(\alpha-1)}} N^4(\alpha-s) + N^{2-3\alpha} N^4(\alpha-s) \right) + T N^{\frac{2(a-s)}{12(\alpha-1)}} \left( N^{2-3\alpha} N^{-\frac{4(a-3)(b-\alpha)}{12(\alpha-1)}} N^4(\alpha-s) + N^{2-3\alpha} N^4(\alpha-s) \right)
\]

\[
\lesssim E(Iu(0)) \lesssim N^{(\alpha-s)\frac{2}{3}}.
\]

The requirement above implies the following five inequalities

\[
\begin{align*}
& TN^{\frac{2(a-s)}{12(\alpha-1)}} N^{\frac{1}{2} - \alpha} N^{-\frac{4(a-3)(b-\alpha)}{12(\alpha-1)}} N^4(\alpha-s) \lesssim N^{(\alpha-s)\frac{2}{3}} \\
& TN^{\frac{2(a-s)}{12(\alpha-1)}} N^{2-3\alpha} N^4(\alpha-s) \lesssim N^{(\alpha-s)\frac{2}{3}} \\
& TN^{\frac{2(a-s)}{12(\alpha-1)}} N^{2-3\alpha} N^{-\frac{4(a-3)(b-\alpha)}{12(\alpha-1)}} N^6(\alpha-s) \lesssim N^{(\alpha-s)\frac{2}{3}} \\
& TN^{\frac{2(a-s)}{12(\alpha-1)}} N^{\frac{7}{2} - 6\alpha + N^6(\alpha-s)} \lesssim N^{(\alpha-s)\frac{2}{3}} \\
& TN^{\frac{2(a-s)}{12(\alpha-1)}} N^{2-4\alpha} N^6(\alpha-s) \lesssim N^{(\alpha-s)\frac{2}{3}}.
\end{align*}
\]

whose solutions are given by

\[
\begin{align*}
s > s_1(\alpha) & \colon \frac{1}{8} \left( \frac{4a^2 - 8 \alpha - 1}{2a-1} + \frac{16a^3 - 4a + 1}{(2a-1)^2} \right), \quad \alpha > \frac{2}{3} \\
s > s_2(\alpha) & \colon \frac{1}{8} \left( \frac{\alpha^2 + 2 \alpha - 1}{2a-1} + \sqrt{\frac{\alpha^2 + 10 \alpha - 3 + 2a - 1}{(2a-1)^2}} \right), \quad \alpha > \frac{2}{3} \\
s > s_3(\alpha) & \colon \frac{1}{8} \left( \frac{6a^2 + 5a - 2}{3a-1} + \sqrt{\frac{9a^2 - 5a^2 + 4a - 1}{(3a-1)^2}} \right), \quad \alpha > \frac{2}{3} \\
s > s_4(\alpha) & \colon \frac{1}{8} \left( \frac{7a^2 - 2}{3a-1} + \sqrt{\frac{9a^2 - 5a^2 + 4a - 1}{(3a-1)^2}} \right), \quad \alpha > \frac{2}{3} \\
s > s_5(\alpha) & \colon \frac{2a^2 + 2a - 1}{6a^2 - 1},
\end{align*}
\]

Let us remark here that the restriction \(\alpha \in (\frac{2}{3}, 1]\) on the fractional Laplacian in this paper comes from the solutions above. If we track further back to the place where we obtained the second and the third conditions, we see that in Case I-2, the bounds of both Case I-2a and Case I-2b are \(N^{2-3\alpha} \|Iu\|_{X^\alpha,s}^4\). In the almost conservation law of energy, we anticipate a small factor \(N^{2-3\alpha}\), hence \(2 - 3\alpha < 0\) and \(\alpha > \frac{2}{3}\).

We can see the global well-posedness index easily from the picture below.
Therefore, the global well-posedness index that we obtain in this work is

\[ s > s_*(\alpha) = \max\{s_1(\alpha), s_2(\alpha)\}. \]

Moreover, with the choice of \( N \) solved from \( s_1(\alpha), s_2(\alpha) \), we have

\[ T \sim \min\{N^{(\alpha-s)(\alpha-4-\frac{2\alpha+1}{3})+\alpha-\frac{4}{3}}, N^{(\alpha-s)(\alpha-4)+3\alpha-2}\} =: N^p. \]

Then as a consequence of the I-method, we establish the following polynomial bound of the global solution

\[ \|u(T)\|_{H^s(\Theta)} \lesssim \|Iu(T)\|_{H^s(\Theta)} \lesssim E(Iu(T))^{\frac{1}{\alpha}} \sim N^{(\alpha-s)^{\frac{1}{\alpha}}} = T^{(\alpha-s)\frac{1}{\alpha}}. \]

Now we finish the proof of Theorem 1.1. \( \square \)

**Appendix A. Some calculations needed in the proof of Claim 5.5**

A.1. **Anti-derivative of** \( F(r) \). Recall that in Claim 5.5, we wrote \( F \) function in the following form

\[ F(r) = \int_0^r \gamma \ e_{n_0}(\gamma) e_{n_1}(\gamma) \ d\gamma = \|J_0(z_{n_0})\|_{L^2(\Theta)}^{-1} \|J_0(z_{n_1})\|_{L^2(\Theta)}^{-1} \int_0^r \gamma J_0(z_{n_0}) J_0(z_{n_1}) \ d\gamma. \]

Now we compute the anti-derivative of \( F(r) \).

**Lemma A.1.** Let \( a \neq b \), then

\[ \int_0^r \gamma J_0(a\gamma) J_0(b\gamma) \ d\gamma = \frac{1}{a^2 - b^2} (arJ_1(ar) J_0(br) - brJ_1(br) J_0(ar)). \]

**Proof of Lemma A.1.** Using (2.1) and (2.2) with integration by parts, we write

\[ \int_0^r a^2 \gamma J_0(a\gamma) J_0(b\gamma) \ d\gamma = a \int_0^r J_0(b\gamma) d(\gamma J_1(a\gamma)) = arJ_1(ar) J_0(br) + ab \int_0^r J_1(a\gamma) J_1(b\gamma) \ d\gamma \]

\[ \int_0^r b^2 \gamma J_0(a\gamma) J_0(b\gamma) \ d\gamma = b \int_0^r J_0(a\gamma) d(\gamma J_1(b\gamma)) = brJ_1(br) J_0(ar) + ab \int_0^r J_1(a\gamma) J_1(b\gamma) \ d\gamma. \]

Then taking the difference of the equations above yields the following anti-derivative

\[ \int_0^r \gamma J_0(a\gamma) J_0(b\gamma) \ d\gamma = \frac{1}{a^2 - b^2} (arJ_1(ar) J_0(br) - brJ_1(br) J_0(ar)). \]

\( \square \)
A.2. Control of error terms. Recall that we used the following formulas in (2.3) and (2.4)

\[ J_n(x) = \begin{cases} \frac{1}{n!}2^n x^n + O(x^{n+2}), & |x| < 1, \\ \sqrt{\frac{2}{\pi}} \cos(\pi - \frac{\pi}{n}) + O(x^{-\frac{1}{2}}), & |x| \geq 1, \end{cases} \]

to approximate \( J_0 \) and \( J_1 \) in the proof of Claim 5.5. We dealt with the contribution from the main terms above, and in this subsection we will verify that the error terms (namely, \( O(x^{n+2}) \) when \( |x| < 1 \) and \( O(x^{-\frac{1}{2}}) \) when \( |x| \geq 1 \)) share the same bounds.

Case I: \( 0 \leq r < \frac{1}{z_{n_0}} \).

The error term in the case will be dominated by the error in estimating \( J_1(z_{n_0} r) \), which is \( O((z_{n_0} r)^3) \), hence it is bounded by

\[
\frac{1}{z_{n_0}} \int_0^{\frac{1}{z_{n_0}}} r(z_{n_0} r)^3 J_0(z_{n_0} r) J_1(z_{n_2} r) J_0(z_{n_3} r) dr 
\approx \frac{1}{z_{n_0}^3} \int_0^{\frac{1}{z_{n_0}}} r(z_{n_0} r)^3 J_0(z_{n_1} r) J_1(z_{n_2} r) J_0(z_{n_2} r) dr
\approx \frac{1}{z_{n_0}^3} \int_0^{\frac{1}{z_{n_0}}} r(z_{n_0} r)^3 J_0(z_{n_1} r) J_1(z_{n_2} r) J_0(z_{n_2} r) dr
\approx \frac{1}{z_{n_0}^3} \int_0^{\frac{1}{z_{n_0}}} \frac{n_1^2 n_2^2 n_3^2}{n_0 (n_0^2 - n_1^2)}.
\]

Case II: \( \frac{1}{z_{n_0}} \leq r < \frac{1}{z_{n_1}} \).

The error term from estimating \( J_1(z_{n_0} r) \) is \( O((z_{n_0} r)^{-\frac{3}{2}}) \), and its contribution is

\[
\frac{1}{z_{n_0}} \int_0^{\frac{1}{z_{n_0}}} r(z_{n_0} r)^{-\frac{3}{2}} J_0(z_{n_1} r) J_1(z_{n_2} r) J_0(z_{n_2} r) dr
\approx \frac{1}{z_{n_0}^2} \int_0^{\frac{1}{z_{n_0}}} r(z_{n_0} r)^{-\frac{3}{2}} J_0(z_{n_1} r) J_1(z_{n_2} r) J_0(z_{n_2} r) dr
\approx \frac{1}{z_{n_0}^2} \int_0^{\frac{1}{z_{n_0}}} \frac{n_1^2 n_2^2 n_3^2}{n_0^3 (n_0^2 - n_1^2)}.
\]

The error term from estimating \( J_0(z_{n_1} r) \) is \( O((z_{n_1} r)^2) \), and its contribution is

\[
\frac{1}{z_{n_0}} \int_0^{\frac{1}{z_{n_0}}} r J_1(z_{n_0} r) (z_{n_1} r)^2 J_0(z_{n_2} r) J_0(z_{n_2} r) dr
\approx \frac{1}{z_{n_0}^2} \int_0^{\frac{1}{z_{n_0}}} r J_1(z_{n_0} r) (z_{n_1} r)^2 J_0(z_{n_2} r) J_0(z_{n_2} r) dr
\approx \frac{1}{z_{n_0}^2} \int_0^{\frac{1}{z_{n_0}}} \frac{2 n_1^2 n_2^2 n_3^2}{n_0^3 (n_0^2 - n_1^2)}.
\]

Case III: \( \frac{1}{z_{n_1}} \leq r < \frac{1}{z_{n_2}} \).
The error term from estimating $J_0(z_{n_1} r)$ is $O((z_{n_1} r)^{-\frac{3}{2}})$, and its contribution is

$$\frac{\sqrt{\pi n_0} z_{n_1}^{-1} z_{n_2}^{-1} z_{n_3}^{-1} z_{n_0}^{-1} z_{n_2}}{z_{n_0}^{-2} - z_{n_1}^{-2}} \int_{\frac{1}{z_{n_1}}}^{1/z_{n_2}} r J_1(z_{n_0} r)(z_{n_1} r)^{-\frac{3}{2}} J_1(z_{n_2} r) J_0(z_{n_3} r) \, dr$$

$$\sim \frac{\sqrt{\pi n_0} z_{n_1}^{-1} z_{n_2}^{-1} z_{n_3}^{-1} z_{n_0}^{-1} z_{n_2}}{z_{n_0}^{-2} - z_{n_1}^{-2}} \int_{\frac{1}{z_{n_1}}}^{1/z_{n_2}} r \sin(z_{n_0} r - \frac{\pi}{4})(z_{n_1} r)^{-\frac{3}{2}} (z_{n_2} r) \, dr$$

$$\sim \frac{\sqrt{\pi n_0} z_{n_1}^{-1} z_{n_2}^{-1} z_{n_3}^{-1} z_{n_0}^{-1} z_{n_2}}{z_{n_0}^{-2} - z_{n_1}^{-2}} \frac{1}{\sqrt{z_{n_0} r}} \sin(z_{n_0} r - \frac{\pi r}{4}) \, dr$$

$$\sim \frac{\sqrt{\pi n_0} z_{n_1}^{-1} z_{n_2}^{-1} z_{n_3}^{-1} z_{n_0}^{-1} z_{n_2}}{z_{n_0}^{-2} - z_{n_1}^{-2}} \frac{1}{\sqrt{z_{n_0} r}} \sin(z_{n_0} r - \frac{\pi r}{4}) \sim \frac{n_3^{\frac{3}{4} n_1^{\frac{1}{4}}}}{n_1 (n_0^2 - n_1^2)}.$$

The error term from estimating $J_1(z_{n_2} r)$ is $O((z_{n_2} r)^3)$, and its contribution is

$$\frac{\sqrt{\pi n_0} z_{n_1}^{-1} z_{n_2}^{-1} z_{n_3}^{-1} z_{n_0}^{-1} z_{n_2}}{z_{n_0}^{-2} - z_{n_1}^{-2}} \int_{\frac{1}{z_{n_1}}}^{1/z_{n_2}} r \frac{J_1(z_{n_0} r) J_0(z_{n_1} r) (z_{n_2} r)^3 J_0(z_{n_3} r) \, dr}{\sqrt{z_{n_0} r}}$$

$$\sim \frac{\sqrt{\pi n_0} z_{n_1}^{-1} z_{n_2}^{-1} z_{n_3}^{-1} z_{n_0}^{-1} z_{n_2}}{z_{n_0}^{-2} - z_{n_1}^{-2}} \int_{\frac{1}{z_{n_1}}}^{1/z_{n_2}} r \frac{\sin(z_{n_0} r - \frac{\pi}{4}) \cos(z_{n_1} r - \frac{\pi}{4})(z_{n_2} r)^3 \, dr}{\sqrt{z_{n_0} r}}$$

$$\sim \frac{\sqrt{\pi n_0} z_{n_1}^{-1} z_{n_2}^{-1} z_{n_3}^{-1} z_{n_0}^{-1} z_{n_2}}{z_{n_0}^{-2} - z_{n_1}^{-2}} \frac{1}{\sqrt{z_{n_0} r}} \sin(z_{n_0} r - \frac{\pi}{4}) \cos(z_{n_1} r - \frac{\pi}{4}) \, dr$$

$$\sim \frac{\sqrt{\pi n_0} z_{n_1}^{-1} z_{n_2}^{-1} z_{n_3}^{-1} z_{n_0}^{-1} z_{n_2}}{z_{n_0}^{-2} - z_{n_1}^{-2}} \frac{1}{\sqrt{z_{n_0} r}} \sin(z_{n_0} r - \frac{\pi}{4}) \cos(z_{n_1} r - \frac{\pi}{4}) \sim \frac{n_3^{\frac{3}{4} n_1^{\frac{1}{4}}}}{n_1 (n_0^2 - n_1^2)}.$$

where in the last inequality, we used Lemma A.3 to obtain

$$\int_{\frac{1}{z_{n_1}}}^{1/z_{n_2}} r^3 \sin(z_{n_0} r - \frac{\pi}{4}) \cos(z_{n_1} r - \frac{\pi}{4}) \, dr \lesssim \frac{1}{z_{n_0} n_2^{\frac{3}{2}}}.$$

Case IV: $\frac{1}{z_{n_2}} \leq r < \frac{1}{z_{n_3}}$

Recall

$$\int_{\frac{1}{z_{n_2}}}^{1/z_{n_3}} F(r) \phi'(r) \, dr \sim \int_{\frac{1}{z_{n_2}}}^{1/z_{n_3}} [-\frac{1}{z_{n_0} - z_{n_1}} \cos((z_{n_0} + z_{n_1}) r) + \frac{1}{z_{n_0} + z_{n_1}} \sin((z_{n_0} - z_{n_1}) r)] \phi' \, dr.$$

The error term from estimating $J_1(z_{n_2} r)$ is $O((z_{n_2} r)^{-\frac{3}{2}})$, and its contribution is bounded by

$$\int_{\frac{1}{z_{n_2}}}^{1/z_{n_3}} \frac{1}{z_{n_0} - z_{n_1}} \cos((z_{n_0} + z_{n_1}) r) \sqrt{z_{n_2} z_{n_3} z_{n_2}} (z_{n_2} r)^{-\frac{3}{2}} J_0(z_{n_3} r) \, dr$$

$$\sim \frac{\sqrt{z_{n_2} z_{n_3} z_{n_2}}}{z_{n_0} - z_{n_1}} \int_{\frac{1}{z_{n_2}}}^{1/z_{n_3}} \cos((z_{n_0} + z_{n_1}) r) (z_{n_2} r)^{-\frac{3}{2}} \, dr$$

$$\lesssim \frac{\sqrt{z_{n_3}}}{z_{n_0} - z_{n_1}} \int_{\frac{1}{z_{n_2}}}^{1/z_{n_3}} \cos((z_{n_0} + z_{n_1}) r) r^{\frac{1}{2}} \, dr \lesssim \frac{z_{n_3}}{z_{n_0} - z_{n_1}} (z_{n_0} + z_{n_1})^{\frac{1}{2}} \sqrt{z_{n_0}} \sim \frac{n_3^{\frac{3}{4} n_1^{\frac{1}{4}}}}{(n_0 - n_1)^{\frac{3}{2}}}.$$

where in the last inequality, we used Lemma A.2 and a change of variables.

The error term from estimating $J_0(z_{n_3} r)$ is $O((z_{n_3} r)^2)$, and its contribution is

$$\int_{\frac{1}{z_{n_2}}}^{1/z_{n_3}} \frac{1}{z_{n_0} - z_{n_1}} \cos((z_{n_0} + z_{n_1}) r) \sqrt{z_{n_2} z_{n_3} z_{n_2}} J_1(z_{n_2} r) (z_{n_3} r)^2 \, dr.$$
The cosine integral follows similarly.

Case V: \( \frac{1}{z_{n3}} \leq r \leq 1 \).

The error term in the case will be dominated by the error in estimating \( J_0(z_{n3}r) \), which is \( O(z_{n3}r)^{-\frac{3}{2}} \), hence it is bounded by

\[
\int_{\frac{1}{z_{n3}}}^{1} \frac{1}{z_{n0} - z_{n1}} \cos((z_{n0} + z_{n1})r) \sqrt{z_{n2}z_{n3}z_{n2}} J_1(z_{n2}r)(z_{n3}r)^{-\frac{3}{2}} dr
\]

\[
\sim \sqrt{z_{n2}z_{n3}z_{n2}} \left( \frac{z_{n0} - z_{n1}}{z_{n0} - z_{n1}} \right) \int_{\frac{1}{z_{n3}}}^{1} \frac{1}{r^2} \cos((z_{n0} + z_{n1})r) \sin(z_{n2}r - \frac{\pi}{4}) dr
\]

\[
\leq \frac{z_{n2}}{(z_{n0} - z_{n1})z_{n3}} \left( \frac{z_{n0} - z_{n1}}{z_{n0}} \right) \frac{n_2 n_3}{n_0},
\]

where in the last inequality, we used Lemma A.3 to obtain

\[
\int_{\frac{1}{z_{n3}}}^{1} \frac{1}{r^2} \cos((z_{n0} + z_{n1})r) \sin(z_{n2}r - \frac{\pi}{4}) dr \leq \frac{z_{n3}^{\frac{3}{2}}}{z_{n0}}.
\]

Now we conclude that all the error terms in estimating (5.4) are controlled by the same bound in (4) in Claim 5.5, which is \( O\left( \frac{n_2^{\frac{1}{2}} n_3^{\frac{1}{2}}}{n_0} \right) \).

### A.3. Estimates on some trigonometric integrals.

**Lemma A.2.** For \( a, b > 0 \) and \( p \neq 0 \), we have

\[
\left| \int_a^b x^p \sin x \, dx \right| + \left| \int_a^b x^p \cos x \, dx \right| \lesssim a^p + b^p.
\]

**Proof of Lemma A.2.** We integrate by parts and write

\[
\int_a^b x^p \sin x \, dx = -x^p \cos x \bigg|_a^b + p \int_a^b x^{p-1} \cos x \, dx,
\]

\[
\left| \int_a^b x^p \sin x \, dx \right| \lesssim |b^p \cos b - a^p \cos a| + \left| \int_a^b x^{p-1} \, dx \right| \leq a^p + b^p.
\]

The cosine integral follows similarly. \( \square \)
Lemma A.3. For $a, b > 0$, $m \leq n$ and $p \neq 0$, we have

$$\left| \int_a^b x^p \sin(mx) \cos(nx) \, dx \right| + \left| \int_a^b x^p \cos(mx) \sin(nx) \, dx \right| + \left| \int_a^b x^p \sin(mx) \sin(nx) \, dx \right| + \left| \int_a^b x^p \cos(mx) \cos(nx) \, dx \right| \lesssim \frac{n}{n^2 - m^2} (a^p + b^p)$$

Proof of Lemma A.3. We only present the calculation for the bound of the first term in this lemma. Similar calculation works for other terms.

Performing integration by parts, we write

$$\int_a^b x^p \sin(mx) \cos(nx) \, dx = \frac{1}{n} \int_a^b x^p \sin(mx) \, d\sin(nx)$$

$$= \frac{1}{n} \left( x^p \sin(mx) \sin(nx) \bigg|_a^b - p \int_a^b x^{p-1} \sin(mx) \sin(nx) \, dx \right) - \frac{m}{n} \int_a^b x^p \cos(mx) \sin(nx) \, dx.$$ 

Take the second term above and integrate by parts again, then we have

$$\int_a^b x^p \cos(mx) \sin(nx) \, dx = -\frac{1}{n} \int_a^b x^p \cos(cx) \, d\cos(nx)$$

$$= -\frac{1}{n} \left( x^p \cos(mx) \cos(nx) \bigg|_a^b - p \int_a^b x^{p-1} \cos(mx) \cos(nx) \, dx \right) - \frac{m}{n} \int_a^b x^p \sin(mx) \cos(nx) \, dx.$$ 

Now combining the copulation above, we arrive at

$$A := \int_a^b x^p \sin(mx) \cos(nx) \, dx$$

$$= \frac{1}{n} \left( x^p \sin(mx) \sin(nx) \bigg|_a^b - p \int_a^b x^{p-1} \sin(mx) \sin(nx) \, dx \right) - \frac{m}{n} \int_a^b x^p \cos(mx) \sin(nx) \, dx$$

$$= \frac{1}{n} \left( x^p \sin(mx) \sin(nx) \bigg|_a^b - p \int_a^b x^{p-1} \sin(mx) \sin(nx) \, dx \right)$$

$$+ \frac{m}{n^2} \left( x^p \cos(mx) \cos(nx) \bigg|_a^b - p \int_a^b x^{p-1} \cos(mx) \cos(nx) \, dx \right) + \frac{m^2}{n^2} A.$$ 

Therefore, solving for $A$ gives us

$$(1 - \frac{m^2}{n^2}) A = \frac{1}{n} \left( x^p \sin(mx) \sin(nx) \bigg|_a^b - p \int_a^b x^{p-1} \sin(mx) \sin(nx) \, dx \right)$$

$$+ \frac{m}{n^2} \left( x^p \cos(mx) \cos(nx) \bigg|_a^b - p \int_a^b x^{p-1} \cos(mx) \cos(nx) \, dx \right),$$

then the triangle inequality and boundedness of trig functions yield the desired bound

$$|A| \lesssim \frac{n^2}{n^2 - m^2} \frac{1}{n} \left| x^p \sin(mx) \sin(nx) \bigg|_a^b - p \int_a^b x^{p-1} \sin(mx) \sin(nx) \, dx \right|$$

$$|A| \lesssim \frac{n}{n^2 - m^2} (a^p + b^p).$$
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