On the Restriction Formula

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Abstract
Let $\varphi$ be a quasi-psh function on a complex manifold $X$ and let $\mathfrak{d}$ be a linear system on $X$ with $1 \leq \dim \mathfrak{d} < \infty$ and $\text{codim}_X \Sigma \geq 2$, where $\Sigma$ is the base locus of $\mathfrak{d}$. We show that the restriction formulas for both multiplier ideal sheaves $\mathcal{I}(\varphi)|_{S \setminus \Sigma} = \mathcal{I}(\varphi|_{S \setminus \Sigma})$ and complex singularity exponents $c_x(\varphi|_{S}) = c_x(\varphi)$, $\forall x \in S \setminus \Sigma$, hold if the divisor $S \in \mathfrak{d}$ is chosen outside of a Lebesgue measure zero set. As an application, one can obtain a useful exact sequence in the inductive arguments.

Keywords  Restriction formula · Multiplier ideal sheaf · Complex singularity exponent · Jumping number · Adjunction exact sequence

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1 Introduction

Let $X$ be a complex manifold which is always assumed to be second countable throughout this paper. A quasi-plurisubharmonic (quasi-psh for short) function on $X$ is by definition a function $\varphi$ which is locally equal to the sum of a plurisubharmonic function and of a smooth function. The multiplier ideal sheaf $\mathcal{I}(\varphi)$ is the ideal subsheaf of $\mathcal{O}_X$ defined by

$$\mathcal{I}(\varphi)_x = \{ f \in \mathcal{O}_{X,x} ; |f|^2 e^{-2\varphi} \text{ is integrable in a neighborhood of } x \}.$$

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If $\varphi \equiv -\infty$ near $x$, then we put $I(\varphi)_x = 0$. It is well known that $I(\varphi)$ is a coherent analytic sheaf.

Another related invariant is complex singularity exponent. For any compact set $K \subset X$, we introduce the complex singularity exponent of $\varphi$ on $K$ to be the nonnegative number

$$c_K(\varphi) = \sup \left\{ c \geq 0; \quad e^{-2c\varphi} \text{ is } L^1 \text{ on a neighborhood of } K \right\}.$$

If $\varphi \equiv -\infty$ near some connected component of $K$, we put of course $c_K(\varphi) = 0$. Given a point $x \in X$, we write $c_x(\varphi)$ instead of $c_{\{x\}}(\varphi)$.

Multiplier ideal sheaves and complex singularity exponents associated to quasi-psh functions are basic objects in several complex variables and complex algebraic geometry. The restriction formulas for multiplier ideal sheaves and complex singularity exponents are useful in the inductive arguments. Various important and fundamental properties about them have been established by [3, 6, 7, 9–11, 15, 19, 24, 26], etc.

The following basic restriction formulas are direct consequences of the Ohsawa–Takegoshi $L^2$ extension theorem.

**Theorem 1.1** [6, 7] Let $\varphi$ be a quasi-psh function on a complex manifold $X$ and let $Y \subset X$ be a complex submanifold. Then

$$I(\varphi|_Y) \subset I(\varphi)|_Y.$$

Moreover, if $\varphi \not\equiv -\infty$ on every connected component of $Y$ and $K$ is a compact subset of $Y$, then

$$c_K(\varphi|_Y) \leq c_K(\varphi).$$

It is an interesting question whether it is possible to get equalities in the above formulas. Since it may happen that the inclusion in the above proposition is strict for some submanifolds, therefore, it is natural to consider the restriction of multiplier ideal sheaves on “general” submanifolds. In fact, there is the following problem due to Sébastien Boucksom:

**Problem 1** Let $\mathfrak{d}$ be a base point free linear system on a compact complex manifold $X$ with $\dim \mathfrak{d} \geq 1$. Let $\varphi$ be a quasi-psh function on $X$ and put

$$b = \left\{ S \in \mathfrak{d}; \quad S \text{ is smooth and } I(\varphi)|_S = I(\varphi|_S) \right\}.$$

Is $\mathfrak{d} \setminus b$ a pluripolar set?

This problem was studied by Hwang and Lazarsfeld (cf. [18]) in the algebro-geometric setting. However, in the more general analytic setting, the problem is still open and one cannot expect that $\mathfrak{d} \setminus b$ is a countable union of analytic subsets of $\mathfrak{d}$ in general. For counterexamples, one can refer to Example 3.10 in [11]. An important

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development is the result of Fujino and Matsumura: \( b \) is dense in \( \mathcal{d} \) in the classical topology (Theorem 1.10 in [11]). A relative version was given by Fujino [10]. However, their arguments give no information on the set \( \mathcal{d} \setminus b \) (cf. Remark 4.7 in [10]). Since pluripolar sets have Lebesgue measure zero, it is natural to consider the following problem:

**Problem 2** Is \( \mathcal{d} \setminus b \) of measure zero?

In the present article, we solve Problem 2 affirmatively: \( \mathcal{d} \setminus b \) has Lebesgue measure zero in \( \mathcal{d} \), i.e., the restriction formula holds generically. In fact, we will discuss the problem in a more general setting. We first give a generalized version of Bertini theorem.

**Theorem 1.2** Let \( F \) be a globally generated holomorphic vector bundle over a complex manifold \( X \), and let \( W \) be a finite dimensional subspace of \( H^0(X, F) \) such that \( W \) generates all fibers \( F_x \), \( x \in X \). Let \( P(W) \) denote the projective space of \( W \). If \( W \) has no non-vanishing sections (i.e., for any section \( s \in W \), \( s(x) = 0 \) at some point \( x \in X \)), then \( \dim X \geq \text{rank } F \) and the set

\[
Z = \left\{ [s] \in P(W); \ s^{-1}(0) \text{ is not smooth} \right\}
\]

has Lebesgue measure zero in \( P(W) \). If, moreover, \( X \) is compact, then \( Z \) is analytic in \( P(W) \).

The restriction formula for multiplier ideal sheaves is also a Bertini type result. Motivated by [3], we shall give a new approach to this problem.

**Theorem 1.3** Let \( F \) be a globally generated holomorphic vector bundle over a complex manifold \( X \), and let \( W \) be a finite dimensional subspace of \( H^0(X, F) \) such that \( W \) generates all fibers \( F_x \), \( x \in X \). Let \( \varphi \) be a quasi-psh function on \( X \). Let \( P(W) \) denote the projective space of \( W \) and let

\[
B = \left\{ [s] \in P(W); \ S = s^{-1}(0) \text{ is smooth and } \mathcal{I}(\varphi|_S) = \mathcal{I}(\varphi)|_S \right\}.
\]

If \( W \) has no non-vanishing sections, then \( P(W) \setminus B \) has measure zero in \( P(W) \).

One of the main results in this paper is the following theorem

**Theorem 1.4** Let \( \mathcal{d} \) be a linear system on a complex manifold \( X \) with \( 1 \leq \dim \mathcal{d} < +\infty \), and let \( \varphi \) be a quasi-psh function on \( X \). Suppose that the codimension of the base locus \( \Sigma \) of \( \mathcal{d} \) is at least two. If we put

\[
b = \left\{ S \in \mathcal{d}; \ S \setminus \Sigma \text{ is smooth and } \mathcal{I}(\varphi)|_{S\setminus\Sigma} = \mathcal{I}(\varphi|_{S\setminus\Sigma}) \right\},
\]

then \( \mathcal{d} \setminus b \) has measure zero in \( \mathcal{d} \).
Note that we do not need to require that $X$ is compact and $\mathfrak{d}$ is base point free. In application, Theorem 1.4 gives enough choices of hypersurfaces in the inductive arguments. Our approach is different from that of Fujino and Matsumura.

Similar arguments can be adapted to prove the following restriction formula for complex singularity exponents.

**Theorem 1.5** Let $F$ be a globally generated holomorphic vector bundle over a complex manifold $X$ and let $W$ be a finite dimensional subspace of $H^0(X, F)$ such that $W$ generates all fibers $F_x$, $x \in X$. Let $\varphi$ be a quasi-psh function on $X$. Let $P(W)$ denote the projective space of $W$ and let

$$Q = \{ [s] \in P(W); \ S = s^{-1}(0) \text{ is smooth and } c_x(\varphi|_S) = c_x(\varphi), \ \forall x \in S \}.$$

If $W$ has no non-vanishing sections, then $P(W) \setminus Q$ has measure zero in $P(W)$.

The restriction formula for multiplier ideal sheaves can be used to deduce an important exact sequence which is useful in the inductive arguments.

**Proposition 1.6** Let $L$ be a globally generated holomorphic line bundle over a complex manifold $X$ and let $W$ be a finite dimensional subspace of $H^0(X, L)$ such that $W$ generates all fibers $L_x$, $x \in X$. We denote by $P(W)$ the projective space of $W$. Let $\varphi$ be a quasi-psh function on $X$ and let $Y_i$, $i \in I$ be the analytic subsets associated to $O_X/I(\varphi)$. Suppose $L$ is not a trivial line bundle. Then

- the set

$$A = \{ [s] \in P(W); \ S = s^{-1}(0) \supset Y_i \text{ for some } i \in I \}$$

is a countable union of proper analytic subsets of $P(W)$. If, moreover, $X$ is compact, then $A$ is analytic in $P(W)$.

- the sequence

$$0 \longrightarrow I(\varphi) \otimes \mathcal{O}(-S) \longrightarrow I(\varphi) \longrightarrow I(\varphi)|_S \longrightarrow 0$$

is exact if and only if $[s] \notin A$.

**Corollary 1.7** Let $\mathfrak{d}$ be a base point free linear system on a complex manifold $X$ with $1 \leq \dim \mathfrak{d} < +\infty$, and let $\varphi$ be a quasi-psh function on $X$. Then there exists a measure zero set $n \subset \mathfrak{d}$ such that for each divisor $S \in \mathfrak{d} \setminus n$, $S$ is smooth, $I(\varphi|_S) = I(\varphi)|_S$, and the sequence

$$0 \longrightarrow I(\varphi) \otimes \mathcal{O}(-S) \longrightarrow I(\varphi) \longrightarrow I(\varphi)|_S \longrightarrow 0$$

is exact.

The above corollary can also be obtained by the approach of Cao [3]. Following the arguments of Cao, one can obtain the following result.
Theorem 1.8 Let $\varphi$ be a quasi-psh function on a complex manifold $X$ and let $\sigma_0$ be a positive continuous function on $X$ such that $I(\varphi) = I((1 + \sigma_0)\varphi)$. Suppose $L$ is a globally generated holomorphic line bundle over $X$. Let $W$ be a finite dimensional subspace of $H^0(X, L)$ such that $W$ generates all fibers $L_x$, $x \in X$. Then there exists a measure zero set $N \subset P(W)$ such that for each $[s] \in P(W) \setminus N$, the multiplier ideal sheaf $I(\varphi)$ can be written as

$$I(\varphi)_x = \left\{ f \in \mathcal{O}_{X,x}; \exists U_x \text{ such that } \int_{U_x} \frac{|f|^2}{|s|^{2(1-\varepsilon)}} e^{-2(1+\sigma)\varphi} dV < +\infty \right\}$$

for $0 < \sigma \leq \sigma_0$ and $0 < \varepsilon < \sigma$, where $dV$ is a smooth volume form on $X$.

The existence of $\sigma_0$ is guaranteed by the solution of the strong openness conjecture [13, 14].

The present note was posted on arXiv [24].

2 Bertini Type Theorem

Let us recall a well-known fact.

Theorem 2.1 (cf. [12], [28]) Let $F : X \to Y$ be a surjective holomorphic map between complex manifolds, and let $X_y = F^{-1}(y)$ be the “full” fiber over a point $y \in Y$ (i.e., $F^{-1}(y)$ is equipped with the structure sheaf coming from $\text{Im} \left( F^*(\mathbb{M}_y) \to \mathcal{O}_X \right)$). Then the set

$$Z = \{ y \in Y; X_y \text{ is not smooth} \}$$

has Lebesgue measure zero in $Y$, and the tangent map $F_* : T_X \to T_Y$ is surjective at each point of $X \setminus F^{-1}(Z)$. If, moreover, $F$ is proper, then $Z$ is analytic in $Y$.

As an application of Theorem 2.1, we can deduce the following statement.

Theorem 2.2 Let $F$ be a globally generated holomorphic vector bundle over a complex manifold $X$, and let $W$ be a finite dimensional subspace of $H^0(X, F)$ such that $W$ generates all fibers $F_x$, $x \in X$. Let $P(W)$ denote the projective space of $W$. If $W$ has no non-vanishing sections, then $\dim X \geq \text{rank} F$ and the set

$$Z = \{ [s] \in P(W); \ s^{-1}(0) \text{ is not smooth} \}$$

has Lebesgue measure zero in $P(W)$. If, moreover, $X$ is compact, then $Z$ is analytic in $P(W)$.

Proof We denote by $r$ the rank of $F$ and by $N$ the dimension of $W$. We first note that $N \geq r$ since $F$ is generated by sections in $W$. However, if $N = r$, then one can find a basis $s_1, \ldots, s_r \in W$ such that $F$ is generated by these $r$ sections. In this case, $s_1(x), \ldots, s_r(x) \in F_x$ are linearly independent at each point $x \in X$ and hence $s_1, \ldots, s_r$ are non-vanishing sections in $W$. Therefore, we may assume that $N \geq r + 1$. 

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Let \( W = X \times W \) be the trivial vector bundle of rank \( N \) over \( X \). Since \( F \) is generated by sections in \( W \), we have a surjective bundle morphism

\[
W \xrightarrow{\Phi} F \rightarrow 0.
\]

Here, the bundle morphism \( \Phi \) is given by the evaluation map

\[
(x, s) \mapsto s(x), \quad x \in X, \ s \in W.
\]

Then we have the following exact sequence of vector bundles

\[
0 \rightarrow E \rightarrow W \rightarrow F \rightarrow 0,
\]

where \( E \) is a holomorphic subbundle of \( W \) and the fibers of \( E \) are \( E_x = \{ s \in W ; \ s(x) = 0 \} \), \( x \in X \).

Let us consider the projectivized bundle

\[
P(W) = X \times P(W).
\]

The points of \( P(W) \) can be identified with the lines in the fibers of \( W \). The elements of \( P(W) \) can be written as

\[
(x, [s]) \in P(W), \quad x \in X, \ s \in W.
\]

If we define

\[
P(E) = \bigsqcup_{x \in X} P(E_x),
\]

then

\[
\pi : P(E) \rightarrow X
\]

is the projectivized bundle of \( E \). Since \( E \) is a subbundle of \( W \), we have

\[
P(E) \subset P(W)
\]

and the points of \( P(E) \) can be represented by

\[
(x, [s]) \in P(W), \quad x \in X, \ s \in W, \ s(x) = 0.
\]

The natural projection

\[
\mu : P(W) = X \times P(W) \rightarrow P(W), \quad (x, [s]) \mapsto [s]
\]
induces a holomorphic map
\[ \mu : P(E) \to P(W). \]

Thus, we obtain the following diagram
\[
\begin{array}{ccc}
P(E) & \xrightarrow{\mu} & P(W) \\
\downarrow{\pi} & & \\
X & & \\
\end{array}
\]

We claim that \( \mu \) is surjective. In fact, for any \( s \in W \), we can find a point \( x_0 \in X \) such that \( s(x_0) = 0 \) since \( W \) has no non-vanishing sections. Therefore, for any point \([s] \in P(W)\), we can find a point \((x_0, [s]) \in P(E)\) such that \( \mu(x_0, [s]) = [s] \).

The surjectivity of \( \mu \) yields \( \dim P(E) \geq \dim P(W) \). The dimensions of \( P(E) \) and \( P(W) \) are \( \dim X + N - r - 1 \) and \( N - 1 \), respectively. So we can conclude that \( \dim X \geq r \).

We next consider the fibers of \( \mu \). If \([s] \in P(W)\), then the fiber
\[ \mu^{-1}([s]) = \{(x, [s]) \in X \times P(W); \ s(x) = 0\} \]
is isomorphic to the subvariety
\[ S = \{x \in X; \ s(x) = 0\} \subset X. \]

Here, the isomorphism is given by the projection \( \pi : P(E) \to X \). In what follows we do not distinguish the fiber \( \mu^{-1}([s]) \) and the subvariety \( S = \pi(\mu^{-1}([s])) \).

If \( X \) is compact, then the map \( \mu : P(E) \to P(W) \) is proper. By Theorem 2.1, one can find a proper analytic subset \( Z \subset P(W) \) such that
\[ \mu : P(E) \setminus \mu^{-1}(Z) \to P(W) \setminus Z \]
is a submersion and the fibers \( \mu^{-1}([s]) \cong s^{-1}(0), \ [s] \in P(W) \setminus Z \) are smooth.

In the general case, the holomorphic map \( \mu : P(E) \to P(W) \) may not be proper. Let \( C \) denote the critical set of \( \mu \), which is the set of points in \( X \) at which the tangent map \( \mu_* : T_{P(E)} \to T_{P(W)} \) of \( \mu \) is not surjective. It is obvious that \( C \) is an analytic subset of \( P(E) \).

By Sard’s theorem, the set \( Z = \mu(C) \) has Lebesgue measure zero in \( P(W) \). Then \( s^{-1}(0) \subset X \) is smooth for \([s] \in P(W) \setminus Z\). Moreover, one can show that \( \mu(C) \) is a countable union of nowhere dense closed subsets of \( P(W) \). But in general, \( Z \) is not an analytic subset of \( P(W) \).

Given a globally generated holomorphic vector bundle \( F \), one can always find a finite dimensional subspace \( W \subset H^0(X, F) \) to generate all fibers of \( F \). This fact is shown by the following theorem.
Theorem 2.3 (cf. [5]) Let $F$ be a holomorphic vector bundle of rank $r$ over a complex manifold $X$. If $F$ is globally generated, then there exists a finite dimensional subspace $W \subset H^0(X, F)$, $\dim W \leq \dim X + r$, such that $W$ generates all fibers $F_x$, $x \in X$.

If $F$ has no trivial subbundle, then $W \subset H^0(X, F)$ has no non-vanishing sections. Conversely, we have the following result.

Proposition 2.4 Let $F$ be an ample vector bundle over a compact complex manifold $X$ with $\dim X \geq \text{rank } F$. Suppose $F$ can be generated by sections in $W \subset H^0(X, F)$. Then $W$ has no non-vanishing section if and only if $F$ has no trivial subbundles.

Proof If $F$ has a trivial subbundle $S$, then we have an exact sequence of vector bundles

$$0 \to S \to F \to Q \to 0.$$ 

Then $Q$ is ample and $\text{rank } Q \leq \dim X - 1$. By the vanishing theorem of Le Potier [19] and Serre’s duality,

$$H^1(X, Q^* \otimes S) \cong H^{\dim X, \dim X - 1}(X, Q \otimes S^*) = 0$$

since $S^*$ is a trivial bundle. It follows that the above exact sequence of holomorphic vector bundles splits, i.e.,

$$F \cong S \oplus Q.$$ 

Let $\Phi : F \to S$ be the natural projection.

Let $v \in S_x \subset F_x$. There exists $s \in W$ such that $s(x) = v$ since $F$ can be generated by sections in $W$. Then we can obtain a section $s_1 = \Phi \circ s \in H^0(X, S)$ such that $s_1(x) \neq 0$. Since $S$ is trivial, $s_1$ has no zeros and hence $s$ has no zeros. \qed

Remark 2.5 Let $F$ be a globally generated homomorphic vector bundle over a complex manifold $X$. If $\text{rank } F > \dim X$, then $F$ has a trivial subbundle of rank $\text{rank } F - \dim X$. This is a result due to Serre. For a proof, one may refer to [26].

3 Restriction Formula for Multiplier Ideal Sheaves

A set $P \subset X$ is pluripolar if for each $z \in P$ there is an open connected set $V \ni z$ and a plurisubharmonic function $\psi \neq -\infty$ on $V$ such that $P \cap V \subset \{ \psi = -\infty \}$.

Proposition 3.1 Let $p : X \to T$ be a proper holomorphic submersion between connected complex manifolds. Let $\varphi \neq -\infty$ be a quasi-psh function on $X$. Let $X_t = p^{-1}(t)$ and

$$N = \{ t \in T; \quad \varphi|_{X_t} \equiv -\infty \}.$$ 

Then $N \subset T$ is pluripolar. If $\{ \varphi = -\infty \}$ is closed, then $N$ is closed.
Proof Suppose \( t_0 \in N \) and \( z_0 \in X_{t_0}. \) Since \( p \) is a submersion, one can find an open neighborhood \( D \ni z_0, \) an open ball \( V \ni t_0 \) such that \( D \cong U \times V \) for some open ball \( U \) in \( \mathbb{C}^{\dim X - \dim T} \) and the map \( p|_D : D \to V \) is given by the natural projection \( U \times V \to V. \) Then

\[
(N \cap V) \subseteq \{ t \in V; \phi(x, t) = -\infty, \forall x \in U \}.
\]

Since \( \varphi \) is locally bounded from above, we can define a function

\[
\psi(t) = \sup_{x \in U} \phi(x, t), \quad t \in V.
\]

It is obvious that

\[
(N \cap V) \subseteq \{ t \in V; \psi(t) = -\infty \}.
\]

Let

\[
\psi^*(t) = \limsup_{s \to t} \psi(s)
\]

be the upper semicontinuous regularization of \( \psi(t). \) Then \( \psi^*(t) \neq -\infty \) is a quasi-psh function on \( V \) and

\[
\{ t \in V; \psi(t) = -\infty \} \subseteq \left( \{ t \in V; \psi^*(t) = -\infty \} \cup \{ t \in V; \psi(t) < \psi^*(t) \} \right).
\]

By a well-known fact of Bedford–Taylor (Theorem 7.1 in [1]), the negligible set

\[
\{ t \in V; \psi(t) < \psi^*(t) \}
\]

is pluripolar. Thus \( N \cap V \) is a pluripolar set and hence \( N \) is pluripolar.

Now suppose \( \{ \varphi = -\infty \} \) is closed. If \( t_0 \notin N, \) then \( \varphi(z) > -\infty \) for some point \( z \in X_{t_0} \) and hence \( \varphi > 0 \) on a connected open neighborhood of \( z. \) Since \( p \) is a submersion, there is an open neighborhood \( V \subset T \) of \( t_0 \) such that \( \varphi|_X \not\equiv -\infty \) for \( t \in V. \) This implies that \( N \) is closed. \( \square \)

Theorem 3.2 Let \( F \) be a globally generated holomorphic vector bundle over a complex manifold \( X, \) and let \( W \) be a finite dimensional subspace of \( H^0(X, F) \) such that \( W \) generates all fibers \( F_x, x \in X. \) Let \( \varphi \) be a quasi-psh function on \( X. \) Let \( P(W) \) denote the projective space of \( W \) and let

\[
B = \left\{ [s] \in P(W); \ S = s^{-1}(0) \text{ is smooth and } \mathcal{I}(\varphi|_S) = \mathcal{I}(\varphi)|_S \right\}.
\]

If \( W \) has no non-vanishing sections, then \( P(W) \setminus B \) has measure zero in \( P(W). \)
Proof As in the proof of Theorem 2.2, let \( E \to X \) be the holomorphic vector bundle determined by the exact sequence

\[
0 \to E \to W \to F \to 0,
\]

where \( W = X \times W \) is a trivial vector bundle over \( X \) and the surjective bundle morphism \( W \to F \) is given by the evaluation map. Let

\[
\pi : \mathbb{P}(E) \to X
\]

be the projectivized bundle of \( E \to X \). Then the natural projection induces a holomorphic map

\[
\mu : \mathbb{P}(E) \subset \mathbb{P}(W) = X \times \mathbb{P}(W) \to \mathbb{P}(W), \quad (x, [s]) \mapsto [s]
\]

The map \( \mu : \mathbb{P}(E) \to \mathbb{P}(W) \) is surjective and the fiber \( \mu^{-1}([s]) \) is isomorphic to the analytic subset \( S = s^{-1}(0) \subset X \) via \( \pi \).

Let us consider the following diagram

\[
\begin{array}{ccc}
P(E) & \longrightarrow & P(W) \\
\downarrow \mu & & \downarrow \\
X & & \\
\end{array}
\]

If \( S \) is a complex submanifold of \( X \), then the Ohsawa–Takegoshi \( L^2 \) extension theorem implies that

\[
\mathcal{I}(\varphi|_S) \subset \mathcal{I}(\varphi)|_S.
\]

For the other direction, let \( U \subset X \) be a small open subset and suppose \( f \in \mathcal{O}(U) \) with

\[
\int_U |f|^2 e^{-2\varphi} dV < +\infty.
\]

Let us consider the bundle \( \pi : \pi^{-1}(U) \to U \), where \( \pi^{-1}(U) \subset P(E) \). After shrinking \( U \), we may assume \( \pi^{-1}(U) \cong U \times \mathbb{P}^{N-r-1} \) is a trivial bundle over \( U \). Set

\[
F = f \circ \pi, \quad \tilde{\varphi} = \varphi \circ \pi.
\]

Then \( F \in \mathcal{O}(\pi^{-1}(U)) \) and \( \tilde{\varphi} \) is quasi-psh on \( \pi^{-1}(U) \). Let \( dV_{FS} \) be the volume form on \( \mathbb{P}^{N-r-1} \) associated to the Fubini–Study metric. Let \( dV_{\pi^{-1}(U)} \) be the smooth volume form on \( \pi^{-1}(U) \) induced by \( dV \) and \( dV_{FS} \). Since \( F \) and \( \tilde{\varphi} \) are constant along the fibers \( \mathbb{P}^{N-r-1} \),

\[
\int_{\pi^{-1}(U)} |F|^2 e^{-2\tilde{\varphi}} dV_{\pi^{-1}(U)} = \int_{\mathbb{P}^{N-r-1}} dV_{FS} \cdot \int_U |f|^2 e^{-2\varphi} dV < +\infty
\]
by Fubini’s theorem.

We first assume that $X$ is compact. Then the map $\mu : P(E) \to P(W)$ is proper. By Theorem 2.1, one can find a proper analytic subset $Z \subset P(W)$ such that

$$\mu : P(E) \setminus \mu^{-1}(Z) \to P(W) \setminus Z$$

is a submersion and the fibers $\mu^{-1}([s]), [s] \in P(W) \setminus Z$ are smooth.

Let $\Omega$ be a simply connected domain in $P(W) \setminus Z$. Then $\mu : \mu^{-1}(\Omega) \to \Omega$ is a submersion and hence $\mu^{-1}(\Omega) \subset P(E)$ is diffeomorphic to the product smooth manifold $\Omega \times S_0$, where $S_0$ is a complex submanifold of $X$ determined by an element in $\Omega$. Let $dV_{P(W)}$ be a smooth volume form on $P(W)$ and $dV_S$ a smooth volume form on $S_0$. These two measures induce a smooth volume form $dV_{\mu^{-1}(\Omega)}$ on $\mu^{-1}(\Omega)$.

However, by shrinking $U$ and $\Omega$ smaller, the two volume forms $dV_{\pi^{-1}(U)}$ and $dV_{\mu^{-1}(\Omega)}$ are equivalent on $\pi^{-1}(U) \cap \mu^{-1}(\Omega)$. So we can conclude

$$\int_{\pi^{-1}(U) \cap \mu^{-1}(\Omega)} |F|^2 e^{-2\varphi} dV_{\mu^{-1}(\Omega)} < +\infty.$$

By Fubini’s theorem,

$$\int_{[s] \in \Omega} \left( \int_{U \cap \{s^{-1}(0)\}} |f|^2 e^{-2\varphi} dV_S \right) dV_{P(W)}$$

$$= \int_{\pi^{-1}(U) \cap \mu^{-1}(\Omega)} |F|^2 e^{-2\varphi} dV_{\mu^{-1}(\Omega)} < +\infty$$

and hence the set

$$N(U, \Omega, f) = \left\{ [s] \in \Omega; \int_{U \cap \{s^{-1}(0)\}} |f|^2 e^{-2\varphi} dV_S = +\infty \right\}$$

has measure zero in $P(W)$. For $[s] \in \Omega \setminus N(U, \Omega, f)$, we have

$$\int_{U \cap \{s^{-1}(0)\}} |f|^2 e^{-2\varphi} dV_S < +\infty$$

and hence $f|_S \in \mathcal{I}(\varphi|_S)$.

After shrinking $U$, we may assume that $\mathcal{I}(\varphi)|_U$ is globally generated by $f_1, \ldots, f_m \in \mathcal{O}(U)$ and

$$\int_{U} |f_j|^2 e^{-2\varphi} dV < +\infty, \quad 1 \leq j \leq m.$$

The set $N(U, \Omega) = \cup_{j=1}^{m} N(U, \Omega, f_j)$ has measure zero. If $[s] \notin N(U, \Omega)$, then $f_j|_S \in \mathcal{I}(\varphi|_S)$ for all $j$ and hence $\mathcal{I}(\varphi)|_{S \cap U} = \mathcal{I}(\varphi|_{S \cap U})$. Let

$$N(U) = \{ [s] \in P(W) \setminus Z; \mathcal{I}(\varphi)|_{S \cap U} \neq \mathcal{I}(\varphi|_{S \cap U}) \}.$$
Then it is easy to see that the measure of $N(U)$ is zero and hence the set

$$N = \{[s] \in P(W) \setminus Z; \mathcal{I}(\varphi)|_S \neq \mathcal{I}(\varphi|_S)\}.$$

has measure 0. Therefore, the set

$$P(W) \setminus B = Z \cup N$$

has measure zero in $P(W)$.

In the general case, the holomorphic map $\mu : P(E) \rightarrow P(W)$ may not be proper. Let $C$ denote the critical set of $\mu$. It is an analytic subset of $P(E)$. By Sard’s theorem, the set $\mu(C)$ has measure zero in $P(W)$. Let $p \in \pi^{-1}(U) \setminus C$ be a regular point of $\mu$. Then the tangent map

$$\mu_* : T_p P(E) \rightarrow T_{\mu(p)} P(W)$$

is surjective. By the inverse function theorem, there is an open neighborhood $D_p \subset \pi^{-1}(U) \setminus C$ of $p$, an open neighborhood $\Omega_p$ of $\mu(p)$, and a domain $G_p \subset \mathbb{C}^{n-r}$ such that $D_p$ is diffeomorphic to the product space $\Omega_p \times G_p$ and the map $\mu|_{D_p}$ is given by the natural projection

$$\Omega_p \times G_p \rightarrow \Omega_p.$$

In other words, $\mu^{-1}(\{s\}) \cap D_p$ is isomorphic to $G_p$.

Suppose $dV_{P(W)}$ and $dV_{G_p}$ are smooth volume forms on $P(W)$ and $G_p$, respectively. These two measures induce a smooth volume form $dV_{D_p}$ on $D_p$. By shrinking $D_p$ smaller if necessary, the two volume forms $dV_{\pi^{-1}(U)}$ and $dV_{D_p}$ are equivalent on $D_p$. Then we can conclude

$$\int_{D_p} |F|^2 e^{-2\tilde{\varphi}} dV_{D_p} < +\infty.$$

By Fubini’s theorem,

$$\int_{\Omega_p} \left( \int_{G_p} |F|^2 e^{-2\tilde{\varphi}} dV_{G_p} \right) dV_{P(W)} = \int_{D_p} |F|^2 e^{-2\tilde{\varphi}} dV_{D_p} < +\infty.$$

Thus, the set

$$N(D_p, f) = \{[s] \in P(W) \setminus \mu(C); \int_{D_p \cap \mu^{-1}([s])} |F|^2 e^{-2\tilde{\varphi}} dV_{G_p} = +\infty\}$$

has measure zero in $P(W)$. Springer
One can define a function $\psi : \Omega_1 \rightarrow [-\infty, +\infty)$ by setting

$$\psi ([s]) = -\log \int_{D_p \cap \mu^{-1}([s])} |F|^2 e^{-2\bar{\varphi}} dV_G,$$

It is easy to see that $\psi$ is upper semicontinuous and

$$N(D_p, f) \cap \Omega_1 = \psi^{-1}(-\infty).$$

So $N(D_p, f)$ is a pluripolar set in case $\psi$ is quasi-psh (or $\psi \geq \phi$ for some quasi-psh function $\phi$). In fact, this is true if $F \equiv 1$ and the quasi-psh function $\phi$ is invariant under the actions of certain Lie groups (cf. [2, 8]).

By the second-countability, we can choose a countable collection $\{D_{pj}\}$ such that

$$\bigcup_j D_{pj} = \pi^{-1}(U) \setminus C$$

and the set

$$N(D_{pj}, f) = \left\{ [s] \in P(W) \setminus \mu(C); \int_{D_{pj} \cap \mu^{-1}([s])} |F|^2 e^{-2\bar{\varphi}} dV_G_{pj} = +\infty \right\}$$

has measure zero in $P(W)$ for all $j$. Let

$$N(f) = \bigcup N(D_{pj}, f).$$

It is obvious that $N(f)$ has measure zero.

Let $K \subseteq U$ be a compact subset. If $[s] \notin \mu(C)$, then $\mu^{-1}([s])$ is smooth and the set $\mu^{-1}([s]) \cap \pi^{-1}(K)$ is compact. Since $\mu^{-1}([s]) \cap C = \emptyset$, the compact set $\mu^{-1}([s]) \cap \pi^{-1}(K)$ can be covered by finitely many $D_{pj}$. If, moreover, $[s] \notin N(f)$, then $[s] \notin N(D_{pj}, f)$ and hence

$$\int_{D_{pj} \cap \mu^{-1}([s])} |F|^2 e^{-2\bar{\varphi}} dV_G < +\infty, \quad \forall j.$$

Let $dV_S$ be a smooth volume form on $S = s^{-1}(0) \cong \mu^{-1}([s])$. Then we have

$$\int_{S \cap K} |f|^2 e^{-2\varphi} dV_S = \int_{\mu^{-1}([s]) \cap \pi^{-1}(K)} |F|^2 e^{-2\bar{\varphi}} dV_S < +\infty.$$

In other words, $|f|^2 e^{-2\varphi}$ is locally integrable on $S \cap U$. Therefore,

$$f|_S \in \mathcal{I}(\varphi|_{S \cap U}).$$
After shrinking $U$, we may assume that $\mathcal{I}(\varphi)|_{U}$ is globally generated by $f_1, \ldots, f_m \in \mathcal{O}(U)$ and
\[
\int_U |f_k|^2 e^{-2\varphi} dV < +\infty, \quad 1 \leq k \leq m.
\]
The set $N(U) = \bigcup_{k=1}^m N(f_k)$ has measure zero. If $[s] \notin N(U)$, then $f_k|_S \in \mathcal{I}(\varphi|_{S \cap U})$ for all $k$ and hence $\mathcal{I}(\varphi)|_{S \cap U} = \mathcal{I}(\varphi|_{S \cap U})$. Finally, we may take at most countable many $U_i$ covering $X$ so that $N(U_i) \subset P(W)$ has measure zero and
\[
\mathcal{I}(\varphi)|_{s^{-1}(0) \cap U_k} = \mathcal{I}(\varphi|_{s^{-1}(0) \cap U_k})
\]
for $[s] \notin N(U_k) \cup \mu(C)$. Let
\[
N = \{ [s] \in P(W) \setminus \mu(C); \quad \mathcal{I}(\varphi)|_S \neq \mathcal{I}(\varphi|_S) \}.
\]
Then $N \subset \bigcup_k N(U_i)$ has measure 0. Therefore, the set
\[
P(W) \setminus B = \mu(C) \cup N
\]
has measure zero in $P(W)$.

**Remark 3.3** If $F$ is a line bundle, then associated to the exact sequence
\[
0 \to E \to W \to F \to 0,
\]
one can define another natural map (Kodaira map)
\[
\kappa : X \to G_1(W) \cong P(W^*), \quad x \mapsto E_x,
\]
such that the diagram
\[
\begin{array}{ccc}
F & \longrightarrow & \mathcal{O}(1) \\
\downarrow & & \downarrow \\
X & \xrightarrow{\kappa} & P(W^*)
\end{array}
\]
is commutative. Here, $G_1(W)$ is the set of all 1-codimensional vector subspaces of $W$. Then a member $S$ in the linear system corresponding to $W$ is the inverse image $\kappa^{-1}(H)$ of a hyperplane $H$ in $P(W^*)$. If $\dim P(W) = 1$, then $\kappa = \mu$. However, if $\dim P(W) \geq 2$, then $\kappa$ and $\mu$ are two different maps.

**Remark 3.4** Suppose $F_j \to X$, $1 \leq j \leq k$ are globally generated holomorphic vector bundles and $W_j \subset H^0(X, F_j)$ generate all fibers of $F_j$. Let $E_j$ be the subbundle given by
\[
0 \to E_j \to X \times W_j \to F_j \to 0,
\]
and let $\bigoplus_j P(E_j)$ be the Whitney sum (fiber product) of the projectivized bundles $P(E_j)$. Then we have a holomorphic map

$$\nu : \bigoplus_j P(E_j) \to X \times P(W_1) \times \cdots \times P(W_k) \to P(W_1) \times \cdots \times P(W_k).$$

If we assume that

$$\bigcap_j s_j^{-1}(0) \neq \emptyset \quad \forall s_j \in W_j,$$

then the map $\nu$ is surjective. Using similar arguments, one can show that $S = \bigcap_j s_j^{-1}(0)$ is smooth and $I(\varphi)|_S = I(\varphi|_S)$ if $([s_1], \ldots, [s_k]) \in P(W_1) \times \cdots \times P(W_k)$ is chosen outside a Lebesgue measure zero set.

Let $X$ be a complex manifold (not necessarily compact). A complete linear system on $X$ is defined as the set of all effective divisors linearly equivalent to some given divisor $D$. It is denoted $|D|$. Let $L$ be the line bundle associated to $D$. In the case that $X$ is compact the set $|D|$ is in natural bijection with $\left( H^0(X, L) \setminus \{0\} \right) / \mathbb{C}^*$ and is therefore a projective space.

A linear system $d$ is then a projective subspace of a complete linear system, so it corresponds to a vector subspace $W$ of $H^0(X, L)$. The dimension of the linear system $d$ is its dimension as a projective space. Hence $\dim d = \dim W - 1$.

**Corollary 3.5** Let $d$ be a base point free linear system on a complex manifold $X$ with $1 \leq \dim d < +\infty$, and let $\varphi$ be a quasi-plurisubharmonic function on $X$. If we put

$$b = \left\{ S \in d; \quad S \text{ is smooth and } I(\varphi)|_S = I(\varphi|_S) \right\},$$

then $d \setminus b$ has measure zero in $d$.

It is clear that $b$ is dense in $d$ when $d \setminus b$ has measure zero. So the above corollary implies Theorem 1.10 in [11] which says that $b$ is dense in $d$ in the classical topology in case $X$ is compact. Moreover, we have

**Theorem 3.6** Let $d$ be a linear system on a complex manifold $X$ with $1 \leq \dim d < +\infty$, and let $\varphi$ be a quasi-psh function on $X$. Suppose that the codimension of the base locus $\Sigma$ of $d$ is at least two. If we put

$$b = \left\{ S \in d; \quad S \setminus \Sigma \text{ is smooth and } I(\varphi)|_{S \setminus \Sigma} = I(\varphi|_{S \setminus \Sigma}) \right\},$$

then $d \setminus b$ has measure zero in $d$.

**Proof** Let

$$\tilde{d} = \{ S \setminus \Sigma; \quad S \in b \}. $$
Then \( \tilde{\phi} \) is a base point free linear system on \( X \setminus \Sigma \). It follows from Corollary 3.5 that there exists a measure zero set \( \tilde{a} \subset \tilde{b} \) such that for any \( \tilde{S} \in \tilde{b}\setminus\tilde{a} \), \( \tilde{S} \) is smooth and

\[
I(\phi)|_{\tilde{S}} = I(\phi|_{\tilde{S}}).
\]

Since the codimension of \( \Sigma \) is at least two, the restriction map

\[
b \to \tilde{b}; \quad S \mapsto S \setminus \Sigma
\]

is bijective. Therefore, the measure zero set \( \tilde{a} \subset \tilde{b} \) corresponds to a measure zero set \( a \subset b \) such that for any \( S \in a \setminus \Sigma \), \( S \setminus \Sigma \) is smooth and \( I(\phi)|_{S \setminus \Sigma} = I(\phi|_{S \setminus \Sigma}) \).

A quasi-psh function \( \varphi \) on \( X \) will be said to have analytic singularities if \( \varphi \) can be written locally as

\[
\varphi(z) = \frac{c}{2} \log \sum_k |f_k|^2 + O(1)
\]

with \( c > 0 \) and \( f_k \) are holomorphic functions.

**Proposition 3.7** Let \( p : X \to T \) be a proper holomorphic submersion between connected complex manifolds. Let \( \varphi \neq -\infty \) be a quasi-psh function on \( X \) with analytic singularities. Let \( X_t = p^{-1}(t) \) and

\[
N = \{ t \in T; \quad \varphi|_{X_t} \equiv -\infty \}
\]

Then \( N \) is a proper analytic subset of \( T \).

**Proof** By Proposition 3.1, \( N \) is closed. For any \( t \in T \) and for any simply connected open neighborhood \( \Omega \) of \( t \), \( p^{-1}(\Omega) \) is diffeomorphic to the product manifold

\[
\Omega \times X_t = \bigtimes_{j=1}^m \Omega \times C_j,
\]

where \( C_j, j = 1, \ldots, m, \) are connected components of \( X_t \). Therefore, we may assume that the fibers \( X_t, t \in T, \) are connected.

As in the proof of Proposition 3.1, let \( t_0 \in N \) and \( z_0 \in X_{t_0} \). Then one can find an open neighborhood \( D \ni z_0 \), an open polydisk \( V \ni t_0 \) such that \( D \cong U \times V \) for some open polydisk \( U \) and the map \( p|_D : D \to V \) is given by the natural projection \( U \times V \to V \). Then

\[
N \cap V = \{ t \in V; \quad \varphi(x, t) = -\infty, \quad \forall x \in U \}.
\]

Since \( \varphi \) has analytic singularities, the set

\[
A = \{ z \in X; \quad \varphi(z) = -\infty \}
\]
is a proper analytic subset of $X$. Then

$$N \cap V = \{ t \in V; \ U \times \{ t \} \subset A \}.$$ 

Let $\mathcal{I}_A$ be the ideal sheaf of $A$. After shrinking $D$, we may assume that $\mathcal{I}_A$ is generated by $g_1, \ldots, g_k \in \mathcal{O}(D)$. Then $U \times \{ t \} \subset A$ if and only if

$$g_j(\cdot, t) \equiv 0 \quad \text{for} \quad 1 \leq j \leq k.$$ 

If we write

$$g_j(x, t) = \sum h_{j,\alpha}(t) x^\alpha,$$

where $h_{j,\alpha}(t)$ are holomorphic function on $V$, then

$$N \cap V = \{ t \in V; \ h_{j,\alpha}(t) = 0, \ \forall \ j, \alpha \}.$$ 

Thus $N$ is analytic in $T$. □

**Remark 3.8** In the setting of Proposition 3.1, the set

$$\tilde{N} = \{ t \in T; \ \varphi \equiv -\infty \text{ on some connected component of } X_t \}$$

is also pluripolar. Moreover, if $\varphi$ has analytic singularities, then $\tilde{N}$ is an analytic subset.

The following result is essentially due to Lazarsfeld. Here we translate the theorem statement and proof into analytic language.

**Theorem 3.9** (Theorem 9.5.35 in [18]) Let $p : X \to T$ be a proper holomorphic submersion between complex manifolds. Let $\varphi$ be a quasi-psh function on $X$ with analytic singularities. Suppose that $\varphi \not\equiv -\infty$ on any fibers $X_t = p^{-1}(t)$. Then there is a non-empty Zariski open set $U \subset T$ such that $X_t$ is smooth and

$$\mathcal{I}(\varphi)|_{X_t} = \mathcal{I}(\varphi|_{X_t})$$

for every $t \in U$.

**Proof** By Hironaka [16], there is a smooth proper modification $\tau : \hat{X} \to X$ such that $D = \sum \lambda_j D_j$ is a normal crossing divisor and we can write locally

$$\tau^* \varphi = \sum \lambda_j \log |g_j| + O(1),$$

where $g_j$ are local generators of $D_j$ (cf. [4]). It follows that

$$\mathcal{I}(\tau^* \varphi) = \mathcal{O}_{\hat{X}} \left( - \sum [c\lambda_j]D_j \right).$$
Let $K_{\hat{X}/X} = K_{\hat{X}} - \tau^* K_X$, then

$$ \mathcal{I}(\varphi) = \tau_* \left( \mathcal{O}_X \left( K_{\hat{X}/X} \right) \otimes \mathcal{I}(\tau^* \varphi) \right) = \tau_* \left( \mathcal{O}_X \left( K_{\hat{X}/X} - \sum [c\lambda_j]D_j \right) \right). $$

Let $\hat{p} = \tau^* p : \hat{X} \to T$. There is an analytic subset $Z \subset T$ such that $\sum D_j$ restricts to a normal crossing divisor on $\hat{X}_t = \hat{p}^{-1}(t)$ for $t \notin Z$. Hence $\tau|_{\hat{X}_t} : \hat{X}_t \to X_t$ is a proper modification and

$$ \mathcal{I}(\varphi|_{X_t}) = \tau_* \left( \mathcal{O}_{\hat{X}_t} \left( K_{\hat{X}_t/X_t} \right) \otimes \mathcal{I}(\tau^* \varphi|_{\hat{X}_t}) \right) = \tau_* \left( \mathcal{O}_{\hat{X}_t} \left( K_{\hat{X}_t/X_t} - \sum [c\lambda_j]D_j |_{\hat{X}_t} \right) \right) $$

for $t \notin Z$. Let

$$ U = T \setminus Z. $$

We claim that $U$ is the desired Zariski open set. To verify this, let $t \in U$ and consider the following exact sequence of sheaves on $\hat{X}$

$$ 0 \to \mathcal{O}_{\hat{X}} \left( K_{\hat{X}/X} - \sum [c\lambda_j]D_j \right) \otimes \mathcal{I}_{\hat{X}_t} \to \mathcal{O}_{\hat{X}} \left( K_{\hat{X}/X} - \sum [c\lambda_j]D_j \right) \to \mathcal{O}_{\hat{X}_t} \left( K_{\hat{X}/X} - \sum [c\lambda_j]D_j \right) \to 0, $$

(3.1)

where $\mathcal{I}_{\hat{X}_t} \subset \mathcal{O}_{\hat{X}}$ is the ideal sheaf of $\hat{X}_t$. Since $p$ is a submersion, we have

$$ K_{\hat{X}/X}|_{X_t} = K_{\hat{X}_t/X_t}, $$

and hence

$$ \mathcal{I}(\varphi|_{X_t}) = \tau_* \left( \mathcal{O}_{\hat{X}_t} \left( K_{\hat{X}/X} - \sum [c\lambda_j]D_j \right) \right). $$

In a neighborhood of $t \in U$, the maximal ideal $\mathcal{M}_t \subset \mathcal{O}_T$ has a Koszul resolution constructed from a system of parameters at $t$ (cf. [29]). Since $\hat{p}$ is flat over $U$, we can obtain a resolution of $\mathcal{I}_{\hat{X}_t}$ by free sheaves on $\tau^{-1}(V) \subset \hat{X}$, where $V \subset U$ is an open neighborhood of $t$. However, for any Stein neighborhood $D$ of $p$, we have

$$ H^q \left( D, \mathcal{O}_{\hat{X}} \left( K_{\hat{X}} \right) \otimes \mathcal{I}(\tau^* \varphi) \right) = 0, \quad q \geq 1. $$

by Nadel vanishing theorem (cf. [22]). Then one can obtain the local vanishing result

$$ R^1 \tau_* \left( \mathcal{O}_{\hat{X}} \left( K_{\hat{X}/X} \right) \otimes \mathcal{I}(\tau^* \varphi) \otimes \mathcal{I}_{\hat{X}_t} \right) = 0. $$
Taking direct images of exact sequence (3.1) under $\tau$, one can obtain

$$I(\varphi)|_{X_t} = I(\varphi|_{X_t}), \quad t \in U.$$ 

\[\square\]

As an application of Theorem 3.9, one can obtain the following result. In the algebro-geometric setting, the statement is due to Hwang (cf. [18]).

**Theorem 3.10** Let $F$ be a globally generated holomorphic vector bundle over a compact complex manifold $X$, and let $W$ be a finite dimensional subspace of $H^0(X, F)$ such that $W$ generates all fibers $F_x, \; x \in X$. Let $\varphi$ be a quasi-psh function on $X$ with analytic singularities. Suppose $W$ has no non-vanishing sections. Let $P(W)$ denote the projective space of $W$ and let

$$B = \left\{ [s] \in P(W); \; S = s^{-1}(0) \text{ is smooth and } I(\varphi|_S) = I(\varphi)|_S \right\}.$$

Then one can find an analytic subset $Z_1$ in $P(W)$, an analytic subset $Z_2$ in $P(W) \setminus Z_1$, and an analytic subset $Z_3$ in $P(W) \setminus (Z_1 \cup Z_2)$ such that

$$P(W) \setminus B \subset (Z_1 \cup Z_2 \cup Z_3).$$

**Proof** As in the proof of Theorem 2.2, let $E \to X$ be the holomorphic vector bundle determined by the exact sequence

$$0 \to E \to X \times W \to F \to 0,$$

and let

$$\pi : P(E) \to X$$

be the projectivized bundle of $E \to X$. Then the natural projection induces a proper holomorphic map

$$\mu : P(E) \to P(W).$$

By Theorem 2.1, one can find a proper analytic subset $Z_1 \subset P(W)$ such that

$$\mu : P(E) \setminus \mu^{-1}(Z_1) \to P(W) \setminus Z_1$$

is a submersion. By Proposition 3.7, the set

$$Z_2 = \{ t \in P(W) \setminus Z_1; \; \varphi|_{X_t} \equiv -\infty \}$$
is analytic in $P(W) \setminus Z_1$. Let $[s] \notin (Z_1 \cup Z_2)$ and $S = s^{-1}(0)$. The Ohsawa–Takegoshi $L^2$ extension theorem implies that

$$\mathcal{I}(\varphi|_S) \subset \mathcal{I}(\varphi)|_S$$

For the other direction, let $U \subset X$ be an open subset and $f \in \mathcal{I}(\varphi)(U)$, in other words, $|f|^2 e^{-2\varphi}$ is locally integrable on $U$. Set

$$F = f \circ \pi, \quad \widetilde{\varphi} = \varphi \circ \pi.$$ 

Then $|F|^2 e^{-2\widetilde{\varphi}}$ is locally integrable on $\pi^{-1}(U)$, i.e.,

$$F \in \mathcal{I}(\widetilde{\varphi})\left(\pi^{-1}(U)\right).$$

By Theorem 3.9, there is a proper analytic subset $Z_3$ in $P(W) \setminus (Z_1 \cup Z_2)$ such that

$$\mathcal{I}(\widetilde{\varphi})|_{\mu^{-1}(S)} = \mathcal{I}(\widetilde{\varphi}|_{\mu^{-1}(S)}) \text{ for } [s] \notin Z_1 \cup Z_2.$$ 

Thus

$$F|_{\mu^{-1}(S)} \in \mathcal{I}(\widetilde{\varphi}|_{\mu^{-1}(S)})\left(\pi^{-1}(U) \cap \mu^{-1}(S)\right). \quad (3.2)$$

However, the map

$$\pi : \mu^{-1}(S) \longrightarrow S,$$

is an isomorphism, so we can conclude

$$f|_S \in \mathcal{I}(\varphi|_S)(U).$$

Therefore, $\mathcal{I}(\varphi|_S) = \mathcal{I}(\varphi)|_S$ for $[s] \notin Z_1 \cup Z_2 \cup Z_3$. \hfill \square

### 4 Restriction Formula for Complex Singularity Exponents

From now on, we discuss the generic restriction formula for complex singularity exponents.

**Theorem 4.1** Let $F$ be a globally generated holomorphic vector bundle over a complex manifold $X$, and let $W$ be a finite dimensional subspace of $H^0(X, F)$ such that $W$ generates all fibers $F_x$, $x \in X$. Let $\varphi$ be a quasi-psh function on $X$. Let $P(W)$ denote the projective space of $W$ and let

$$Q = \left\{ [s] \in P(W); \quad S = s^{-1}(0) \text{ is smooth and } c_x(\varphi|_S) = c_x(\varphi), \forall x \in S \right\}.$$ 

If $W$ has no non-vanishing sections, then $P(W) \setminus Q$ has measure zero in $P(W)$. \hfill \square
Proof Let \( \{x_j\}_{j \in J} \) be a countable dense subset of \( X \). Given a complete Riemannian metric on \( X \), the collection of open balls

\[
B_{jk} = \left\{ x \in X; \ \text{dist}(x, x_j) < \frac{1}{k} \right\}, \quad j \in J, \ k \geq 1
\]

is an open cover of \( X \) and \( B_{jk} \subset X \) is compact. The set

\[
Z = \left\{ [s] \in P(W); \ s^{-1}(0) \text{ is not smooth} \right\}
\]

has Lebesgue measure zero by Theorem 2.2. Fix \( c \in \mathbb{Q} \cap [0, c_{\bar{B}_{jk}}(\varphi)) \) and set

\[
E_{j,k,c} = \left\{ [s] \in P(W) \setminus Z; \ e^{-2c\varphi}|_S \notin L^1 \text{ on } S \cap \bar{B}_{jk} \right\},
\]

where \( S = s^{-1}(0) \subset X \).

We next show that \( E_{j,k,c} \) has measure zero. Again, let us consider the diagram

\[
\begin{array}{ccc}
P(E) & \longrightarrow & P(W) \\
\mu \downarrow & & \downarrow \\
X & & \\
\end{array}
\]

where the vector bundle \( E \to X \) is given by the exact sequence

\[
0 \to E \to X \times W \to F \to 0.
\]

Since \( c < c_{\bar{B}_{jk}}(\varphi) \), the function \( e^{-2c\varphi} \) is integrable on a neighborhood of \( \bar{B}_{jk} \). If we set \( \tilde{\varphi} = \pi^*\varphi \), then \( \tilde{\varphi} \) is quasi-psh on \( P(E) \) and \( e^{-2c\tilde{\varphi}} \) is integrable on a neighborhood of \( \pi^{-1}(\bar{B}_{jk}) \).

Let \( C \) be the critical set of \( \mu \). As in the proof of Theorem 3.2, for any point \( p \in \pi^{-1}(\bar{B}_{jk}) \setminus C \), there is an open neighborhood \( D_p \) of \( p \) such that \( D_p \) is isomorphic to a product space \( \Omega_p \times G_p \) and the map \( \mu|_{D_p} \) is given by the natural projection

\[
\Omega_p \times G_p \to \Omega_p.
\]

By shrinking \( D_p \) smaller if necessary, we may assume \( e^{-2c\tilde{\varphi}} \) is integrable on \( D_p \). Then we can apply Fubini’s theorem to conclude the set

\[
\left\{ [s] \in P(W) \setminus \mu(C); \ e^{-2c\tilde{\varphi}} \notin L^1 \left( D_p \cap \mu^{-1}([s]) \right) \right\}
\]

has measure zero. By the second-countability, \( \pi^{-1}(\bar{B}_{jk}) \setminus C \) can be covered by a countable collection \( \{D_{p\ell}\} \). So the set

\[
\left\{ [s] \in P(W) \setminus \mu(C); \ e^{-2c\tilde{\varphi}} \notin L^1 \left( \pi^{-1}(\bar{B}_{jk}) \cap \mu^{-1}([s]) \right) \right\}
\]
has measure zero too.

Note that the map
\[ \pi^{-1} \left( \overline{B}_{jk} \cap \mu^{-1}([s]) \right) \rightarrow \overline{B}_{jk} \cap s^{-1}(0) \]
is isomorphic. Thus we can conclude
\[ N_{j,k,c} = \left\{ [s] \in P(W) \setminus \mu(C); \quad e^{-2c\varphi} \notin L^1 \left( \overline{B}_{jk} \cap s^{-1}(0) \right) \right\} \]
has measure zero. If we define
\[ N = \bigcup_{j,k} \bigcup_{c} N_{j,k,c}, \quad j \in J, \ k \geq 1, \ c \in \mathbb{Q} \cap \left[ 0, c_{\overline{B}_{jk}}(\varphi) \right), \]
then \( N \cup Z \subset P(W) \) has measure zero.

For \([s] \in P(W) \setminus (Z \cup N)\), we claim that
\[ c_x(\varphi) = c_x(\varphi|_S), \quad \forall \ x \in S = s^{-1}(0). \]
To see this, let \( c \) be a rational number such that \( 0 \leq c < c_x(\varphi) \). Then there is a neighborhood \( D \) of \( x \) in \( X \) such that \( e^{-2c\varphi} \) is integrable on \( D \).

We may assume that the ball \( B \left( x, \frac{2}{k} \right) \) of center \( x \) and radius \( \frac{2}{k} \) is contained in \( D \).
Since \( \{x_j\}_{j \in J} \) is dense in \( X \), we can find an index \( j \in J \) such that \( \text{dist} \left( x_j, x \right) < \frac{1}{k} \).
Then
\[ x \in B_{jk} = B \left( x_j, \frac{1}{k} \right) \subset B \left( x, \frac{2}{k} \right) \subset D. \]
It is obvious that \( e^{-2c\varphi} \) is integrable on \( B_{jk} \). Now \( S \) is smooth since \([s] \notin Z\) and \( e^{-2c\varphi|_S} \) is integrable on \( \overline{B}_{jk} \cap S \) because \([s] \notin N_{j,k,c} \). It follows that \( e^{-2c\varphi|_S} \) is integrable in a neighborhood of \( x \in S \) and hence \( c_x(\varphi|_S) \geq c \). So we can get the desired inequality
\[ c_x(\varphi|_S) \geq c_x(\varphi). \]
Finally, we have \( c_x(\varphi|_S) = c_x(\varphi) \) because the inequality in the opposite direction is always true by Proposition 1.1. \( \square \)

We next discuss the restriction formula for jumping numbers. Let \( \varphi \) be a quasi-psh function on a complex manifold \( X \), and let \( \mathcal{I} \subset \mathcal{O}_X \) be a nonzero coherent ideal sheaf. The jumping number \( c_\mathcal{I}(\varphi) \) is defined as follows (see [17]):
\[ c_\mathcal{I}(\varphi) = \sup \left\{ c \geq 0; \quad |\mathcal{I}|^2 e^{-2c\varphi} \text{ is locally integrable at } x \right\}, \]
where \( (f_j)_{1 \leq j \leq N} \) are local generators of \( \mathcal{I} \) and \( |\mathcal{I}|^2 = \sum_{j=1}^{N} |f_j|^2 \). When \( \mathcal{I} = \mathcal{O}_X \), the jumping number reduces to the complex singularity exponent. Arguments similar to those in the proof of Theorems 3.2 and 4.1 easily yield
\[ \square \]
Theorem 4.2 Let $F$ be a globally generated holomorphic vector bundle over a complex manifold $X$, and let $W$ be a finite dimensional subspace of $H^0(X, F)$ such that $W$ generates all fibers $F_x$, $x \in X$. Let $\varphi$ be a quasi-psh function on $X$, and let $\mathcal{I} \subset \mathcal{O}_X$ be a nonzero coherent ideal sheaf. Let $P(W)$ denote the projective space of $W$. If $W$ has no non-vanishing sections, then there is a measure zero set $N \subset P(W)$ such that for every $[s] / s \in N$, the subvariety $S = s^{-1}(0)$ is smooth and

$$c^\mathcal{I}_x(\varphi) \leq c^\mathcal{I}_x|_S (\varphi|_S), \ \forall \ x \in S.$$

5 Adjunction Exact Sequence

The following theorem is a special case of Theorem 4 in [28]. We follow the exposition of [20].

Theorem 5.1 (Siu) Let $X$ be a complex space, and let $F$ be a coherent analytic sheaf on $X$. Then there is a locally finite family $\{Y_i, i \in I\}$ of irreducible analytic subsets of $X$ such that for each $x \in X$ the associated primes of $F_x$ is

$$\text{Ass}_{\mathcal{O}_{X,x}} F_x = \{p_{x,1}, \ldots, p_{x,r(x)}\},$$

where $p_{x,1}, \ldots, p_{x,r(x)}$ are the prime ideals of $\mathcal{O}_{X,x}$ associated to the irreducible components of the germs $Y_{i,x} i \in I$ with $x \in Y_i$.

Definition 5.2 (cf. [28]). The analytic subsets $Y_i$, $i \in I$ of the above theorem are called analytic subsets associated to the sheaf $F$. The index set $I$ is at most countable since the family $\{Y_i, i \in I\}$ is locally finite.

Lemma 5.3 Let $L$ be a holomorphic line bundle over a complex manifold $X$. Let $F$ be a coherent analytic sheaf on $X$, and let $Y_i$, $i \in I$ be the analytic subsets associated to $F$. Suppose $s$ is a nonzero section of $L$ and $S = s^{-1}(0)$, then the sequence

$$0 \rightarrow F \otimes \mathcal{O}(-S) \xrightarrow{\otimes s} F$$

is exact if and only if $S \not\supset Y_i$ for all $i \in I$.

Proof Locally, we may assume the section $s$ is given by a holomorphic function $f$ and the map $F \otimes \mathcal{O}(-S) \xrightarrow{\otimes s} F$ is given by $F \xrightarrow{f} F$.

Suppose $s = 0$ on $Y_i$ for some $i \in I$. Let $p_x$ be the prime ideal associated to an irreducible component of the germ $Y_{i,x}$. Since $s \not\equiv 0$, $Y_i$ is a proper analytic subset of $X$ and hence $p_x \neq 0$. We may assume $p_x = \text{Ann}(t_x)$ for some nonzero $t_x \in F_x$. Since $s$ vanishes on $Y_i$, we have $f_x \in p_x = \text{Ann}(t_x)$. Thus, $f_x \cdot t_x = 0$ and hence $\mathcal{F}_x \xrightarrow{f_x} \mathcal{F}_x$ is not injective.

For the other direction, suppose $\mathcal{F}_x \xrightarrow{f_x} \mathcal{F}_x$ is not injective at some point $x \in X$. There exists a nonzero section $\tilde{t}_x \in \mathcal{F}_x$ such that $f_x \cdot \tilde{t}_x = 0$. Then $f_x \in \text{Ann}(\tilde{t}_x)$. It
is easy to show that every maximal element of the family of ideals

\[ \{ \text{Ann}(t_x); \ 0 \neq t_x \in T_x, \ \text{Ann}(t_x) \supset \text{Ann}(r_x) \} \]

is an associated prime of \( T_x \) (cf. [21]). Suppose \( p_x \supset \text{Ann}(r_x) \) is an associated prime of \( T_x \). Then \( f_x \in p_x \). So the section \( s \) vanishes on a component of the germ \( Y_{i,x} \).

Since \( Y_i \) is an irreducible analytic subset of \( X \), we have \( s^{-1}(0) \supset Y_i \).

\[ \square \]

**Lemma 5.4** Let \( L \) be a globally generated holomorphic line bundle over a complex manifold \( X \), and let \( W \) be a finite dimensional subspace of \( H^0(X, L) \) such that \( W \) generates all fibers \( L_x, \ x \in X \). Let \( T \) be a coherent analytic sheaf on \( X \), and let \( Y_i, \ i \in I \) be the analytic subsets associated to \( T \). Let \( P(W) \) denote the projective space of \( W \). Suppose \( \dim P(W) \geq 1 \). Then the set

\[ A = \left\{ [s] \in P(W); \ s^{-1}(0) \supset Y_i \text{ for some } i \in I \right\} \]

is a countable union of proper analytic subsets of \( P(W) \). If, moreover, \( X \) is compact, then \( A \) is analytic in \( P(W) \).

**Proof** Let \( E \) be the holomorphic vector bundle given by the exact sequence

\[ 0 \to E \to X \times W \to L \to 0. \]

Let us consider the following diagram

\[ \begin{array}{ccc}
P(E) & \longrightarrow & P(W) \\
\downarrow & & \downarrow \\
X & & \end{array} \]

Then we can write

\[ A = \bigcup_{i \in I} \left( \bigcap_{y \in Y_i} \mu \left( P(E_y) \right) \right). \]

By the definition of \( \mu \), the set \( \mu \left( P(E_y) \right) = P(E_y) \subset P(W) \) is a hyperplane and hence \( \bigcap_{y \in Y_i} \mu \left( P(E_y) \right) \) is a linear subspace of \( P(W) \). Since the family \( \{Y_i, \ i \in I\} \) is locally finite, the index set \( I \) is at most countable. So \( A \) is at most a countable union of proper analytic subsets of \( P(W) \). If \( X \) is compact, then \( I \) is finite and hence \( A \) is an analytic subset of \( P(W) \).

\[ \square \]

**Theorem 5.5** Let \( L \) be a globally generated holomorphic line bundle over a complex manifold \( X \), and let \( W \) be a finite dimensional subspace of \( H^0(X, L) \) such that \( W \) generates all fibers \( L_x, \ x \in X \). We denote by \( P(W) \) the projective space of \( W \). Let \( \varphi \) be a quasi-psh function on \( X \), and let \( Y_i, \ i \in I \) be the analytic subsets associated to \( \mathcal{O}_X/I(\varphi) \). Suppose \( L \) is not a trivial line bundle. Then
• \( \text{the set} \)

\[
A = \left\{ [s] \in P(W); \quad S = s^{-1}(0) \supseteq Y_i \text{ for some } i \in I \right\}
\]

is a countable union of proper analytic subsets of \( P(W) \). If, moreover, \( X \) is compact, then \( A \) is analytic in \( P(W) \).

• \( \text{the sequence} \)

\[
0 \longrightarrow \mathcal{I}(\varphi) \otimes \mathcal{O}(-S) \longrightarrow \mathcal{I}(\varphi) \longrightarrow \mathcal{I}(\varphi)|_S \longrightarrow 0
\]

is exact if and only if \([s] \notin A\).

**Proof** By Lemma 5.4, we only need to prove the second statement. Let \( \mathcal{J} \) be the ideal sheaf of \( \mathcal{O}_X \) defined by \( s \in W \). There is a natural exact sequence

\[
0 \longrightarrow \mathcal{I}(\varphi) \otimes \mathcal{J} \longrightarrow \mathcal{I}(\varphi) \longrightarrow \mathcal{I}(\varphi) \otimes \left( \mathcal{O}_X / \mathcal{J} \right) \longrightarrow 0.
\]

By definition, the ideal sheaf \( \mathcal{I}(\varphi)|_S \) is the image of the map

\[
\rho : \mathcal{I}(\varphi) \otimes \left( \mathcal{O}_X / \mathcal{J} \right) \rightarrow \mathcal{O}_X \otimes \left( \mathcal{O}_X / \mathcal{J} \right) \xrightarrow{\cong} \mathcal{O}_X / \mathcal{J}.
\]

To obtain the desired exact sequence, we only need to consider the injectivity of the map \( \rho \). From the short exact sequence

\[
0 \longrightarrow \mathcal{I}(\varphi) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X / \mathcal{I}(\varphi) \longrightarrow 0,
\]

we can obtain a long exact sequence

\[
0 \rightarrow \text{Tor} \left( \mathcal{O}_X / \mathcal{I}(\varphi), \mathcal{O}_X / \mathcal{J} \right) \rightarrow \mathcal{I}(\varphi) \otimes \left( \mathcal{O}_X / \mathcal{J} \right) \rightarrow \mathcal{O}_X / \mathcal{J}.
\]

Thus, \( \rho \) is injective if and only if

\[
\text{Tor} \left( \mathcal{O}_X / \mathcal{I}(\varphi), \mathcal{O}_X / \mathcal{J} \right) = 0.
\]

From the short exact sequence

\[
0 \rightarrow \mathcal{J} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X / \mathcal{J} \longrightarrow 0,
\]

we have another exact sequence

\[
0 \rightarrow \text{Tor} \left( \mathcal{O}_X / \mathcal{I}(\varphi), \mathcal{O}_X / \mathcal{J} \right) \rightarrow \left( \mathcal{O}_X / \mathcal{I}(\varphi) \right) \otimes \mathcal{J} \rightarrow \mathcal{O}_X / \mathcal{I}(\varphi).
\]

Therefore, the injectivity of \( \rho \) is equivalent to the injectivity of the map

\[
\left( \mathcal{O}_X / \mathcal{I}(\varphi) \right) \otimes \mathcal{J} \rightarrow \mathcal{O}_X / \mathcal{I}(\varphi).
\]
Since the ideal sheaf $\mathcal{J} = \mathcal{O}(-S)$, the above map can be written as

$$(\mathcal{O}_X / \mathcal{I}(\varphi)) \otimes \mathcal{O}(-S) \otimes \mathcal{S} \rightarrow \mathcal{O}_X / \mathcal{I}(\varphi).$$

By Lemma 5.3, we can conclude that $\rho$ is injective if and only if $[s] \notin A$. Therefore, the sequence

$$0 \rightarrow \mathcal{I}(\varphi) \otimes \mathcal{O}(-S) \rightarrow \mathcal{I}(\varphi) \rightarrow \mathcal{I}(\varphi)|_S \rightarrow 0$$

is exact if and only if $[s] \notin A$.

\[\square\]

**Remark 5.6** Let $F$ be a holomorphic vector bundle of rank $r$ over a complex manifold $X$. Suppose $s \in H^0(X, F)$ and $S = s^{-1}(0) \subset X$ is smooth. Let $\mathcal{J}$ denote the ideal sheaf of $S$. There is a natural exact sequence

$$0 \rightarrow \mathcal{J} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X / \mathcal{J} \rightarrow 0.$$
The following result is essentially due to Cao [3] and Guan-Zhou [13].

**Theorem 5.8** Let $L$ be a globally generated holomorphic line bundle over a complex manifold $X$. Let $W$ be a finite dimensional subspace of $H^0(X, L)$ such that $W$ generates all fibers $L_x$, $x \in X$. Let $\varphi$ be a quasi-psh function on $X$. Then there exists a measure zero set $N \subset P(W)$ such that for each $[s] \in P(W) \setminus N$ the following statements hold

a) the multiplier ideal sheaf $\mathcal{I}(\varphi)$ can be written as

$$\mathcal{I}(\varphi)_x = \left\{ f \in \mathcal{O}_{X,x}; \exists U_x \text{ such that } \int_{U_x} \frac{|f|^2}{|s|^{2(1-\varepsilon)}} e^{-2(1+\sigma)\varphi} dV < +\infty \right\}$$

for $0 < \sigma \leq \sigma_0$ and $0 < \varepsilon < \sigma$, where $dV$ is a smooth volume form on $X$,

b) the divisor $S = s^{-1}(0)$ is smooth,

c) the following sequence

$$0 \longrightarrow \mathcal{I}(\varphi) \otimes \mathcal{O}(-S) \longrightarrow \mathcal{I}(\varphi) \longrightarrow \mathcal{I}(\varphi|_S) \longrightarrow 0$$

is exact.

**Proof** We can choose $s_1, s_2, \ldots, s_N$ to be a basis of $W$ so that $\sum_{j=1}^N |s_j(x)|^2 \neq 0$ for any $x \in X$. Let $(\tau_1, \tau_2, \ldots, \tau_N)$ be the coordinates of $\mathbb{C}^N$. Suppose

$$f \in \mathcal{I}(\varphi)_x = \mathcal{I}((1+\sigma)\varphi)_x, \quad 0 < \sigma < \sigma_0.$$

Then

$$\int \sum_{j=1}^N |\tau_j|^2 = 1 \frac{d\tau}{U_x} \int_{U_x} \frac{|f|^2}{\sum_{j=1}^N |\tau_j |^2 |s_j(x)|^2} e^{-2(1+\sigma)\varphi} dV$$

$$= \int_{U_x} \left[ \sum_{j=1}^N |s_j(y)|^2 \right]^{-1} e^{-2(1+\sigma)\varphi} dV \int \sum_{j=1}^N |\tau_j|^2 = 1 \frac{d\tau}{\sum_{j=1}^N |\tau_j |^2 |s_j(y)|^2} 2(1-\varepsilon)$$

$$= \int_{U_x} \left[ \sum_{j=1}^N |s_j(y)|^2 \right]^{-1} e^{-2(1+\sigma)\varphi} dV \int \sum_{j=1}^N |\tau_j|^2 = 1 \frac{d\tau}{|\tau_1 |^2} 2(1-\varepsilon) < +\infty.$$

For the last equality, one can change coordinate via unitary transformation so that

$$\int \sum_{j=1}^N |\tau_j|^2 = 1 \frac{d\tau}{\sum_{j=1}^N |\tau_j |^2 |s_j(y)|^2} 2(1-\varepsilon) = \int \sum_{j=1}^N |\tau_j|^2 = 1 \frac{d\tau}{|\tau_1 |^2} 2(1-\varepsilon) < +\infty.$$
By Fubini’s theorem, we can choose \((\tau_1, \tau_2, \cdots, \tau_N) \in \mathbb{C}^N\) outside a measure zero set such that the section \(s = \sum_{j=1}^N \tau_j s_j\) satisfy
\[
\int_{U_x} \frac{|f|^2}{|\sum_{j=1}^N \tau_j s_j|^{2(1-\varepsilon)}} e^{-2(1+\sigma)\varphi} dV < +\infty.
\]

Then we can choose \([\tau_1 : \tau_2 : \cdots : \tau_N] \in \mathbb{C}P^{N-1}\) outside a measure zero set in \(P(W)\). The first statement is proved.

By Theorems 2.2 and 3.2, we can choose \(S\) outside a measure zero set so that \(S\) is smooth and the restriction \(\mathcal{I}(\varphi) \to \mathcal{I}(\varphi|_S)\) is well defined. If \(f \in \mathcal{I}(\varphi|_S)_x\), then the Ohsawa–Takegoshi extension theorem implies that there exists a \(F \in \mathcal{I}(\varphi)_x\) such that \(F|_S = f\) near \(x\). So the the restriction \(\mathcal{I}(\varphi) \to \mathcal{I}(\varphi|_S)\) is surjective. For the exactness of the middle term, let \(f \in \mathcal{I}(\varphi)_x\) whose restriction in \(\mathcal{I}(\varphi|_S)_x\) is zero. Then \(f\) vanishes along \(S\) near \(x\), hence \(\frac{\mathcal{I}}{s} \in \mathcal{O}_x\). So we can conclude that
\[
\int_{U_x} \frac{|f|^2}{|s|^{4-\eta}} dV = \int_{U_x} \frac{|f|^2}{|s|^2} \cdot \frac{1}{|s|^{2-\eta}} dV < +\infty, \quad \eta > 0. \tag{5.1}
\]

By Hölder’s inequality, we have
\[
\int_{U_x} \frac{|f|^2}{|s|^2} e^{-2(1+\delta)\varphi} dV \leq \left( \int_{U_x} \frac{|f|^2}{|s|^{2(1-\varepsilon)}} e^{-2(1+\sigma)\varphi} dV \right)^{\frac{1+\delta}{1+\sigma}} \left( \int_{U_x} \frac{|f|^2}{|s|^\alpha} dV \right)^{\frac{\alpha-\delta}{1+\sigma}},
\]
where
\[
\alpha = \left[ 2 - 2(1-\varepsilon) \right] \frac{1+\delta}{1+\sigma} \frac{1+\sigma}{\sigma-\delta}.
\]

If we choose \(0 < \delta < \frac{\sigma-\varepsilon}{1+\varepsilon} < \sigma\), then \(\alpha < 4\) and hence
\[
\int_{U_x} \frac{|f|^2}{|s|^2} e^{-2(1+\delta)\varphi} dV < +\infty.
\]

This shows that \(\frac{\mathcal{I}}{s} \in \mathcal{I}(\varphi)_x \supseteq \mathcal{I}(\varphi|_S)_x\). \(\square\)

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References

1. Bedford, E., Taylor, B.A.: A new capacity for plurisubharmonic functions. Acta Math. 149, 1–40 (1982)
2. Berndtsson, B.: Prekopa’s theorem and Kiselman’s minimum principle for plurisubharmonic functions. Math. Ann. 312(4), 785–792 (1998)
3. Cao, J.Y.: Numerical dimension and a Kawamata–Viehweg–Nadel type vanishing theorem on compact Kähler manifolds. Compos. Math. 150, 1869–1902 (2014)
4. Demailly, J.P.: Analytic Methods in Algebraic Geometry, Surveys of Modern Mathematics, vol. 1. International Press, Somerville, Higher Education Press, Beijing (2012)
5. Demailly, J.-P.: Complex analytic and differential geometry. \( \text{https://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf} \)
6. Demailly, J.-P., Ein, L., Lazarf, R.: A subadditivity property of multiplier ideals. Mich. Math. J. 48, 137–156 (2000)
7. Demailly, J.-P., Kollár, J.: Semi-continuity of complex singularity exponents and Kähler–Einstein metrics on Fano orbifolds. Ann. Sci. Éc. Norm. Supér. (4) 34(4), 525–556 (2001)
8. Deng, F., Zhang, H., Zhou, X.: Positivity of character subbundles and minimum principle for noncompact group actions. Math. Z. 286(1–2), 431–442 (2017)
9. Esnault, H., Viehweg, E.: Lectures on Vanishing Theorems, DMV Sem., vol. 20. Birkhäuser, Basel (1992)
10. Fujino, O.: Relative Bertini type theorem for multiplier ideal sheaves. arXiv:1709.01406
11. Fujino, O., Matsumura, S.-I.: Injectivity theorem for pseudo-effective line bundles and its applications. Trans. Am. Math. Soc. Ser. B 8, 849–884 (2021)
12. Grauert, H., Peternell, T., Remmert, R. (eds.): Several complex variables, VII. Sheaf-Theoretical Methods in Complex Analysis. Encyclopaedia of Mathematical Sciences, vol. 74. Springer, Berlin (1994)
13. Guan, Q.A., Zhou, X.Y.: A proof of Demailly’s strong openness conjecture. Ann. Math. 182, 605–616 (2015)
14. Guan, Q.A., Zhou, X.Y.: Effectiveness of Demailly’s strong openness conjecture and related problems. Invent. Math. 202(2), 635–676 (2015)
15. Guan, Q.A., Zhou, X.Y.: Restriction formula and subadditivity property related to multiplier ideal sheaves. J. Reine Angew. Math. 769, 1–33 (2020)
16. Hironaka, R.: Resolution of singularities of an algebraic variety over a field of characteristic zero. Ann. Math. 79, 109–326 (1964)
17. Jonsson, M., Mustaţă, M.: An algebraic approach to the openness conjecture of Demailly and Kollár. J. Inst. Math. Jussieu 13(1), 119–144 (2014)
18. Lazarsfeld, R.: Positivity in Algebraic Geometry. II. Positivity for Vector Bundles, and Multiplier Ideals, Ergeb. Math. Grenzgeb. (3) vol. 49. Springer, Berlin (2004)
19. Le Potier, J.: Annulation de la cohomologie à valeurs dans un fibré vectoriel holomorphe positif de rang quelconque. Math. Ann. 218, 35–53 (1975)
20. Manaresi, M., Sard and Bertini type theorems for complex spaces. Ann. Mat. 131, 265–279 (1982)
21. Matsumura, H.: Commutative Algebra, Mathematics Lecture Note Series, vol. 56, 2nd edn. Benjamin/Cummings Publishing Co., Inc., Reading (1980)
22. Meng, X.K., Zhou, X.Y.: Pseudo-effective line bundles over holomorphically convex manifolds. J. Algebraic Geom. 28(1), 169–200 (2019)
23. Meng, X.K., Zhou, X.Y.: A generalization of Kawamata–Viehweg vanishing theorem. Commun. Anal. Geom
24. Meng, X.K., Zhou, X.Y.: On the restriction formula. arXiv:2101.08120
25. Nadel, A.M.: Multiplier ideal sheaves and existence of Kähler–Einstein metrics of positive scalar curvature. Ann. Math. 132, 613–625 (1990)
26. Okonek, C., Schneider, M., Spindler, H.: Vector bundles on Complex Projective Spaces (with Appendix by S. I. Gelfand). Birkhäuser/Springer (2011)
27. Remmert, R.: Holomorphe und meromorphe Abbildungen komplexer Räume. Math. Ann. 133, 328–370 (1957)
28. Siu, Y.T.: Noether–Lasker decomposition of coherent analytic subsheaves. Trans. Am. Math. Soc. 135, 375–385 (1969)
29. Weibel, C.: An Introduction to Homological Algebra, Cambridge Studies in Advanced Mathematics, vol. 38. Cambridge University Press, Cambridge (1994)

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