FINITE QUOTIENTS OF GALOIS PRO-\(p\) GROUPS
AND RIGID FIELDS

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Abstract. For a prime number \(p\), we show that if two certain canonical
finite quotients of a finitely generated Bloch-Kato pro-\(p\) group \(G\) coin-
cide, then \(G\) has a very simple structure, i.e., \(G\) is a \(p\)-adic analytic pro-\(p\)
group (see Theorem A). This result has a remarkable Galois-theoretic
consequence: if the two corresponding canonical finite extensions \(F^{(3)}/F\)
and \(F^{(3)}/F\) of a field \(F\) – with \(F\) containing a primitive \(p\)-th root of
unity – coincide, then \(F\) is \(p\)-rigid (see Corollary B). The proof relies
only on group-theoretic tools, and on certain properties of Bloch-Kato
pro-\(p\) groups. This paper will appear on the Annales mathématiques du
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1. Introduction

Let \(p\) be a prime number, and let \(G\) be a pro-\(p\) group. The Frattini sub-
group \(\Phi(G)\) of \(G\) is the closed subgroup of \(G\) generated by the \(p\)-powers and
the commutators of the elements of \(G\). In particular, the quotient \(G/\Phi(G)\)
is an elementary abelian \(p\)-group. Let \(\Phi_2(G)\) be the Frattini subgroup of
the Frattini subgroup of \(G\), i.e., \(\Phi_2(G) = \Phi(\Phi(G))\).

Also, let \(P_n(G)\), \(n \geq 1\), denote the \(p\)-descending central series of \(G\). In
particular, one has \(P_2(G) = \Phi(G)\) and \(P_3(G) = \Phi(G)^p[G, \Phi(G)] \supseteq \Phi_2(G)\).
For the class of finitely generated Bloch-Kato pro-\(p\) groups, we prove the
following result.

Theorem A. One has the equality \(\Phi_2(G) = P_3(G)\) if, and only if, \(G\) is
\(p\)-adic analytic.

In this case the group \(G\) has a very simple structure, as it is meta-abelian
and it is possible to provide an explicit presentation for \(G\) (cf. [13 The-
orem 4.6]).

One has also the following Galois-theoretic consequence. Let \(F\) be a field
containing a primitive \(p\)-th root of unity. By \(F^\times\) we denote the (multiplica-
tive) group of non-zero elements of \(F\). We consider the Galois extension
\(F^{(3)}\) of \(F\) obtained by first taking \(F^{(2)}\) to be the compositum over \(F\) of all
extensions of \(F\) of degree \(p\), and then taking \(F^{(3)}\) to be the compositum over

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of all the extensions of $F^{(2)}$ of degree $p$ that are Galois over $F$. We also denote by $F^{(3)}$ the compositum over $F^{(2)}$ of all extensions of $F^{(2)}$ of degree $p$ (cf. [3 § 2.3]). Thus

$$F^{(3)} = (F^{(2)})^{(2)}.$$ 

Then one may characterize those fields $F$ with the property that $F^{(3)} = F^{(3)}$. In fact, from Theorem A we shall obtain the following result.

**Corollary B.** Let $F$ be a field containing a primitive $p$-th root of unity, and assume that the quotient $F^\times/(F^\times)^p$ is finite. (Assume further that $\sqrt{-1} \in F$ if $p = 2$). Then $F^{(3)} = F^{(3)}$ if, and only if, $F$ is $p$-rigid; (For the definition of $p$-rigid field, see Section 4.)

Bloch-Kato pro-$p$ groups were introduced in [2] and studied first in [13]. A Bloch-Kato pro-$p$ group is a pro-$p$ group which satisfies the conclusion of the Rost-Voevodsky theorem (formerly known as the Bloch-Kato conjecture), i.e., such that the cohomology ring of every closed subgroup of $G$ with coefficients in the finite field $\mathbb{F}_p$ is a quadratic algebra over $\mathbb{F}_p$. For example, absolute Galois groups of fields which are pro-$p$ and Galois groups of the maximal $p$-extension of certain fields are Bloch-Kato pro-$p$ groups. Thus, a Bloch-Kato pro-$p$ group is a very natural “candidate” for being realized as absolute Galois group, and this shows the relevance of Bloch-Kato pro-$p$ groups for Galois theory.

The problem to characterize a field $F$ yielding the equality

$$F^{(3)} = F^{(3)}$$

arises rather naturally, and the case when equality (1.1) holds is considered very significant in field theory. Indeed, such problem has been widely studied in the past: in the case $p = 2$ Corollary B was proved in [1 Theorem 3.1], with arguments which make use of Galois cohomology, and later in [8, Theorem A], with arguments relying on the theory of quadratic forms. For $p$ odd, Corollary B was proved in [3 Theorem A], and the proof relies on certain properties of Bloch-Kato pro-$p$ groups, together with an essential arithmetic argument (cf. [3 Theorem 4.3]).

The above results provide a motivation for the paper, as Theorem A is the “group-theoretic translation”, and it is in fact more general, as it holds for Bloch-Kato pro-$p$ groups, and not only for Galois groups of maximal $p$-extensions. Moreover, part of the interest of this result lies in the fact that the proof is purely group-theoretical, and it does not rely on results from field theory. Further, the proof makes use of the Zassenhaus filtration of pro-$p$ groups, which is gaining increasing importance as tool for the study of Galois groups (see, e.g., [5 and [11]).

The paper is organized as follows. In the second section, we state a number of properties on pro-$p$ groups and on their descending series. In section 3 we prove Theorem A, and in section 4 we provide the “arithmetic translation” of Theorem A, and we prove Corollary B.

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2. Preliminaries on pro-$p$ groups

Throughout this paper, subgroups of pro-$p$ groups are assumed to be closed (in the pro-$p$ topology), and every generator is to be intended as topological generator. In particular, given two (closed) subgroups $H_1$ and $H_2$ of a pro-$p$ group $G$, the subgroup $[H_1, H_2]$ is the (closed) subgroup of $G$ generated by the commutators $[g_1, g_2]$, with $g_i \in H_i$ for $i = 1, 2$. Also, for a positive integer $n$, $G^n$ denotes the (closed) subgroup of $G$ generated by the $n$-powers of the elements of $G$.

For a finitely generated pro-$p$ group $G$, let $d(G)$ denote the minimal number of generators of $G$. In particular, $d(G)$ is the dimension of the quotient $G/\Phi(G)$ as vector space over the finite field $\mathbb{F}_p$ (cf. [4, Prop. 1.14]). Then, one defines the rank of a pro-$p$ group $G$ to be the number

$$\text{rk}(G) = \sup\{d(H) \mid H \leq G \text{ closed} \} \in \mathbb{N} \cup \{\infty\}$$

(cf. [4, Definition 3.12]).

For a pro-$p$ group $G$, the lower $p$-central series of $G$ is the series $P_n = P_n(G) = [G, P_{n-1}]$, $n \geq 1$, of characteristic subgroups defined by $P_1 = G$ and $P_{n+1} = P_n^p[P_n, G]$. In particular, one has that $P_2(G)$ is the Frattini subgroup $\Phi(G)$, and $[P_i, P_j] \leq P_{i+j}$ for every $i, j \geq 1$. Moreover, if $G$ is finitely generated, then the lower $p$-central series is a base of neighbourhoods of 1 in $G$ (cf. [4, Prop. 1.16]).

**Definition.** A pro-$p$ group $G$ is said to be powerful if $G/G^p$ is abelian, if $p$ is odd, or if $G/G^4$ is abelian, if $p = 2$.

In particular, one has the following (cf. [4, Theorems 3.6, 3.8]).

**Proposition 2.1.** Let $G$ be a powerful pro-$p$ group.

(1) $P_n(G) = G^{p^n-1}$ for every $n \geq 1$.

(2) If $G$ is finitely generated, then $\text{rk}(G) = d(G)$.

Another important descending series of pro-$p$ groups is the Zassenhaus filtration. For an arbitrary group $G$, the Zassenhaus filtration of $G$ is the series $D_n = D_n(G)$, $n \geq 1$, of characteristic subgroups defined by $D_1 = G$ and

$$D_n = D_{\lceil n/p \rceil} \prod_{i+j=n} [D_i, D_j],$$

where $\lceil n/p \rceil$ is the least integer $m$ such that $mp \geq n$. In particular, the Zassenhaus filtration is the fastest descending series starting at $G$ such that $[D_i, D_j] \leq D_{i+j}$ and $D_i^p \leq D_i$ for every $i, j \geq 1$. For computational purposes, one has the formula

$$D_n = \prod_{ip^h \geq n} \gamma_i(G)^{p^h},$$

established by M. Lazard (cf. [4, Theorem 11.2]), where the $\gamma_i(G)$’s are the elements of the descending central series of $G$ (i.e., $\gamma_1(G) = G$ and
\[ \gamma_{i+1}(G) = [G, \gamma_i(G)] \text{ for every } i \geq 1. \] Thus, if \( G \) is a (pro-)p group, then \( D_2(G) \) is the Frattini subgroup \( \Phi(G) \).

For the Zassenhaus filtration of a pro-p group, one has the following remarkable result (cf. [4, Theorem 11.4]).

**Theorem 2.2.** Let \( G \) be a finitely generated pro-p group. Then \( G \) has finite rank if, and only if, \( D_n(G) = D_{n+1}(G) \) for some \( n \geq 1 \).

**Definition.** A topological group \( G \) is a \( p \)-adic analytic group if \( G \) has the structure of analytic manifold over the field of \( p \)-adic numbers \( \mathbb{Q}_p \) with the properties

1. the multiplication function \( G \times G \to G \) given by \( (x,y) \mapsto xy \) is analytic;
2. the inversion function \( G \to G \) defined by \( x \mapsto x^{-1} \) is analytic.

Powerful pro-p groups and \( p \)-adic analytic groups are tightly related. Indeed, a topological group \( G \) has the structure of a \( p \)-adic analytic group if, and only if, \( G \) has an open subgroup which is a powerful finitely generated pro-p group (cf. [4, Theorem 8.1]). In the case of Bloch-Kato pro-p groups, \( p \)-adic analytic groups have a rather simple structure, as stated by the following (cf. [13, Theorem 4.8]).

**Theorem 2.3.** Let \( G \) be a finitely generated Bloch-Kato pro-p group, and assume further that \( G \) is torsion-free, if \( p = 2 \). The following are equivalent.

1. \( G \) has finite rank.
2. \( G \) is \( p \)-adic analytic.
3. \( G \) is powerful.
4. \( G \) has a presentation

\[
G = \left\langle \sigma, \tau_1, \ldots, \tau_d \left| \sigma \tau_i \sigma^{-1} = \tau_i^{1+p^k}, \tau_i \tau_j = \tau_j \tau_i \forall i, j \right. \right\rangle,
\]

with \( d = d(G) - 1 \), for some \( k \geq 1 \) (\( k \geq 2 \), if \( p = 2 \)).

3. **Proof of Theorem A**

**Lemma 3.1.** If \( G \) is a powerful Bloch-Kato group, then \( \Phi_2(G) = P_3(G) \).

**Proof.** Recall first that if \( G \) is a Bloch-Kato pro-p group, then every closed subgroup of \( G \) is again a Bloch-Kato pro-p group. By Proposition 2.1, one has \( \Phi(G) = G^p \) and \( P_3(G) = G^{p^2} \). Since \( \text{rk}(G) \) is finite, also \( \text{rk}(\Phi(G)) \) is finite, thus \( \Phi(G) \) is powerful by Theorem 2.3. Therefore,

\[
\Phi_2(G) = \Phi(\Phi(G)) = \Phi(G)^p = G^{p^2},
\]

and this yields the claim. \( \square \)

**Proof of Theorem A.** Assume that \( G \) is a finitely generated \( p \)-adic analytic Bloch-Kato group. Then, the claim holds by Theorem 2.3 and Lemma 3.1.
Conversely, assume that $\Phi_2(G) = P_3(G)$. Since $[D_2, D_2] \leq D_4$ and $D_2^p \leq D_{2p}$, one has $\Phi_2(G) = D_2^p [D_2, D_2] \leq D_4$, as $\Phi(G) = D_2$. Moreover, one has the inclusion $\gamma_3(G) \leq P_3(G)$. Therefore, one has the chain of inclusions
\[ \gamma_3(G) \leq P_3(G) = \Phi_2(G) \leq D_4. \]

We shall split the proof of this implication in three cases.

(1) Assume $p > 3$. By (2.2), one has
\[ D_3 = \prod_{i^h \geq 3} \gamma_i(G)^{p^h} = \gamma_3(G) \cdot G^p \]
and
\[ D_4 = \prod_{i^h \geq 4} \gamma_i(G)^{p^h} = \gamma_4(G) \cdot G^p. \]

Therefore, (3.1) implies
\[ D_3(G) = \gamma_3(G) \cdot G^p \leq P_3(G) = \Phi_2(G) \cdot G^p \leq D_4, \]
as $G^p \leq D_4$. Thus, one has the equality $D_3 = D_4$. Hence, Theorem 2.2 implies that $\text{rk}(G)$ is finite, and thus by Theorem 2.3 $G$ is a $p$-adic analytic Bloch-Kato pro-$p$ group.

(2) Assume $p = 2$. From (2.2) one obtains
\[ D_3 = \prod_{i^{2^h} \geq 3} \gamma_i(G)^{2^h} = \gamma_3(G) \cdot \gamma_2(G)^2 \cdot G^4 \]
and
\[ D_4 = \prod_{i^{2^h} \geq 4} \gamma_i(G)^{2^h} = \gamma_4(G) \cdot \gamma_2(G)^2 \cdot G^4. \]

Therefore, (3.1) implies
\[ D_3 = \gamma_3(G) \cdot \gamma_2(G)^2 \cdot G^4 \leq \Phi_2(G) \cdot \gamma_2(G)^2 \cdot G^4 \leq D_4, \]
as $\gamma_2(G)^2 G^4 \leq D_4$. Thus, one has the equality $D_3 = D_4$. Hence, Theorem 2.2 implies that $\text{rk}(G)$ is finite, and thus by Theorem 2.3 $G$ is a $p$-adic analytic Bloch-Kato pro-$p$ group.

(3) Assume $p = 3$. By (2.2), one has
\[ D_4 = \prod_{i^{3^h} \geq 4} \gamma_i(G)^{3^h} = \gamma_4(G) \cdot \gamma_2(G)^3 \cdot G^9 \]
and
\[ D_5 = \prod_{i^{3^h} \geq 5} \gamma_i(G)^{3^h} = \gamma_5(G) \cdot \gamma_2(G)^3 \cdot G^9. \]

Therefore, from (3.1) one obtains the chain of inclusions
\[ \gamma_4(G) = [G, \gamma_3(G)] \leq [G, D_4] = [D_1, D_4] \leq D_5, \]
which implies
\[ D_4 = \gamma_4(G) \cdot \gamma_2(G)^3 \cdot G^9 \leq D_5, \]
as $G^9, \gamma_2(G)^3 \leq D_5$. Thus, one has the equality $D_4 = D_5$. Hence, Theorem 2.2 implies that $\text{rk}(G)$ is finite, and thus by Theorem 2.3 $G$ is a $p$-adic analytic Bloch-Kato pro-$p$ group.
This establishes the theorem. □

Note that if $G$ is a finitely generated pro-$p$ group, then $\Phi_2(G)$ is an open subgroup of $G$. Thus, the quotient $G/\Phi_2(G)$ is finite, and one may reduce the equality $\Phi_2(G) = P_3(G)$ to a condition on finite $p$-groups, as done in [3 Corollary 4.15].

**Corollary 3.2.** A finitely generated Bloch-Kato pro-$p$ group $G$ is $p$-adic analytic if, and only if, $\Phi(G)/\Phi_2(G)$ is contained in the centre of $G/\Phi_2(G)$.

**Proof.** Assume that $G$ is $p$-adic analytic. Then Theorem A yields the equality $\Phi_2(G) = P_3(G)$. Since $[G, P_2] = [P_1, P_2] \leq P_3$, one has $[G, \Phi(G)] \leq \Phi_2(G)$, and $\Phi_2(G)/\Phi_2(G)$ is central in $G/\Phi_2(G)$.

Conversely, assume that $\Phi(G)/\Phi_2(G)$ is central in $G/\Phi_2(G)$. Hence the commutator subgroup $[G, \Phi(G)]$ is contained in $\Phi_2(G)$. Since $\Phi(G) p \leq \Phi_2(G)$ and $P_3 = \Phi(G)[G, \Phi(G)]$, it follows that $\Phi_2(G)$ contains $P_3(G)$, and thus the two subgroups are equal. Therefore $G$ is $p$-adic analytic by Theorem A. □

4. Proof of Corollary B

Throughout this section, a field $F$ is always assumed to contain a primitive $p$-th root of unity (and also $\sqrt{-1}$, if $p = 2$). Also, $F^\times$ denotes the multiplicative group of non-zero elements of $F$, and $(F^\times)^p$ is the subgroup of $p$-powers of $F^\times$.

**Definition.** Let $N$ denote the norm map $N: F(\sqrt[p]{a}) \to F$ of the $p$-cyclic extension $F(\sqrt[p]{a})/F$. An $p$-power-free unit $a \in F^\times$ is said to be $p$-rigid if

$$b \in N\left(F(\sqrt[p]{a})\right) \text{ if, and only if, } b \in \bigcup_{k=0}^{p-1} a^k (F^\times)^p$$

for every $b \in F^\times \setminus (F^\times)^p$. The field $F$ is called $p$-rigid if every element of $F^\times \setminus (F^\times)^p$ is $p$-rigid.

Recall from the Introduction that $F^{(2)} = F(\sqrt[p]{F})$ is the compositum over $F$ of all extensions $F(\sqrt[p]{a})$ with $a \in F^\times$. Also,

- $F^{(3)} = F^{(2)}(\sqrt[p]{F^{(2)}})$ is the compositum over $F^{(2)}$ of all the extensions $F^{(2)}(\sqrt[p]{a})$ with $a \in (F^{(2)})^\times$,
- $F^{(3)}$ is the compositum over $F^{(2)}$ of all the extensions $F^{(2)}(\sqrt[p]{a})$ such that $F^{(2)}(\sqrt[p]{a})/F$ is Galois.

Therefore, both $F^{(3)}/F$ and $F^{(3)}/F$ are Galois extensions, and $F^{(3)} \subseteq F^{(3)}$ (cf. [3 § 2.3]).

Let $G$ be the maximal pro-$p$ Galois group of $F$, i.e.,

$$G = G_F(p) = \text{Gal}(F(p)/F),$$
where $F(p)$ is the maximal $p$-extension of $F$. Recall that the maximal pro-$p$ Galois group of a field containing a primitive $p$-th root of unity is a Bloch-Kato pro-$p$ group (cf. \[13\] § 2).

By Kummer theory, one has that the Galois group of $F^{(2)}/F$ is the quotient $G/\Phi(G)$. Note that $G$ is finitely generated if, and only if, the quotient $F^\times/(F^\times)^p$ is finite (and in this case $d(G) = \dim(F^\times/(F^\times)^p)$), as $G/\Phi(G)$ and $F^\times/(F^\times)^p$ are isomorphic as discrete groups of exponent $p$. Moreover,

\begin{equation}
\text{Gal}(F^{(3)}/F) = G/P_3(G) \quad \text{and} \quad \text{Gal}(P_3^{(3)}/F) = G/\Phi_2(G)
\end{equation}

(cf. \[3\] § 4.1, see also \[1\] § 2).

**Remark 4.1.** In the case $p = 2$, the Galois groups $\text{Gal}(F^{(3)}/F)$ and $\text{Gal}(P_3^{(3)}/F)$ are called $W$-group, resp. $V$-group, of the field $F$, for the relations with the Witt ring of $F$ (cf. \[10\] and \[1\]).

**Proof of Corollary B.** Let $G$ be the maximal pro-$p$ Galois group $G_F(p)$. By hypothesis, $G$ is finitely generated. Moreover, $G$ is torsion free, since we are assuming that $\sqrt{-1} \in F$ for $p = 2$.

Assume first that the equality $F^{(3)} = F^{(3)}$ holds. Then, by (4.1) one has also the equality $\Phi_2(G) = P_3(G)$, and thus Theorem A implies that $G$ is a $p$-adic analytic Bloch-Kato pro-$p$ group, and Theorem \[2.3\] implies that $G$ is powerful. Therefore, by \[3\] Proposition 3.8 the field $F$ is $p$-rigid.

Conversely, assume that $F$ is $p$-rigid. Then, again by \[3\] Proposition 3.8 the Galois group $G$ is powerful, and thus $p$-adic analytic by Theorem \[2.3\]. Therefore, Theorem A implies the equality $\Phi_2(G) = P_3(G)$, and the equality $F^{(3)} = F^{(3)}$ follows by (4.1).

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