GEOMETRIC LAGRANGIANS FOR MASSIVE HIGHER-SPIN FIELDS

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ABSTRACT

Lagrangians for massive, unconstrained, higher-spin bosons and fermions are proposed. The idea is to modify the geometric, gauge invariant Lagrangians describing the corresponding massless theories by the addition of suitable quadratic polynomials. These polynomials provide generalisations of the Fierz-Pauli mass term containing all possible traces of the basic field. No auxiliary fields are needed.
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1 Introduction

The central object in the theories of massless spin-1 and spin-2 fields is the curvature. It contains the informations needed for the classical description of the dynamics, and reflects its geometrical meaning. For higher-spin gauge fields\footnote{For reviews see [1]. An overview on the subject can also be found in the Proceedings of the First Solvay Workshop on Higher-Spin Gauge Theories [2], available on the website http://www.ulb.ac.be/sciences/ptm/pmif/Solvay1proc.pdf.} such a description of the dynamics based on curvatures is missing\footnote{We are referring here to a “metric-like” formulation, generalisation of the corresponding formulation for Gravity where the basic role is played by the metric tensor $g_{\mu\nu}$. Vasiliev’s construction of non-linear equations of motion for higher-spin gauge fields [3] represents a generalisation of the “frame-like” formulation of Einstein’s theory.} at the full interacting level, but it can exhibited at least for the linear theory, where it already displays a sensible amount of non-trivial features [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15].

In this work we would like to show that the geometric Lagrangians proposed in [4, 6, 13] admit relatively simple quadratic deformations, so as to provide a consistent description of the corresponding massive theory.

In the case of symmetric, rank-$s$ Lorentz tensors $\varphi_{\mu_1...\mu_s}$ and spinor-tensors $\psi^a_{\mu_1...\mu_s}$, to which we shall restrict our attention in this work, the conditions to be met by these fields in order to describe the free propagation of massive, irreducible representations of the Poincaré group are contained in the Fierz systems [16]:

\[
(\Box - m^2) \varphi_{\mu_1...\mu_s} = 0, \quad (i \gamma^\alpha \partial_\alpha - m) \psi^a_{\mu_1...\mu_s} = 0, \\
\partial^\alpha \varphi_{\alpha \mu_2...\mu_s} = 0, \quad \partial^\alpha \psi^a_{\alpha \mu_2...\mu_s} = 0, \\
\varphi^a_{\alpha \mu_3...\mu_s} = 0, \quad \gamma^a \psi^a_{\alpha \mu_3...\mu_s} = 0. \tag{1.1}
\]

The quest for a Lagrangian description of these systems has been a basic field-theoretical issue since Fierz and Pauli proposed it in [17], and several approaches and solutions are known up to now, both for flat and, more generally, for maximally symmetric spaces, [18, 19, 21, 20, 22, 23, 24, 25, 26, 27] (for other results on massive higher-spins see [28]). Typically in these solutions auxiliary fields are present for spin $s \geq \frac{5}{2}$ and, because of the interplay among the various fields, the proposed Lagrangians in general do not look like simple quadratic deformations of the corresponding massless ones.

This is to be contrasted with what happens for spin $s \leq 2$. Indeed, the Maxwell Lagrangian supplemented with a mass term

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 A_\mu A^\mu, \tag{1.2}
\]

gives equations of motion that are easily shown to imply $\partial_\mu A^\mu = 0$, the only condition needed to recover in this case the Fierz system. A bit less direct, and more instructive for us, is the corresponding result for the massive graviton [29]. Consider the linearised Einstein-Hilbert Lagrangian, deformed by the introduction of a so-called Fierz-Pauli mass term [17]:

\[
\mathcal{L} = \frac{1}{2} h^{\mu\nu} \{ \mathcal{R}_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \mathcal{R} - m^2 (h_{\mu\nu} - \eta_{\mu\nu} h^a_a) \}, \tag{1.3}
\]

where $\mathcal{R}_{\mu\nu}$ and $\mathcal{R}$ indicate the linearised Ricci tensor and Ricci scalar respectively. The key point is that the divergence of the corresponding equation of motion, because of the Bianchi
identity \( \partial^\alpha \{ R_{\alpha \mu} - \frac{1}{2} \eta_{\alpha \mu} R \} \equiv 0 \), implies an on-shell constraint on the mass term of the form
\[
\partial^\alpha h_{\alpha \mu} - \partial_\mu h^\alpha = 0, \tag{1.4}
\]
which in the following we shall refer to as the Fierz-Pauli constraint. It is then simple to see that the double divergence of the mass term is proportional to the Ricci scalar, so that the divergence of (1.4) implies
\[
\mathcal{R} = 0. \tag{1.5}
\]
The trace of the resulting equation
\[
\mathcal{R}_{\mu \nu} - m^2 ( h_{\mu \nu} - \eta_{\mu \nu} h^\alpha_{\alpha} ) = 0, \tag{1.6}
\]
implies in turn \( h^\alpha_{\alpha} = 0 \), and then the full Fierz system (1.1) can be recovered noticing that, under the conditions of vanishing trace and vanishing divergence of \( h_{\mu \nu} \), the Ricci tensor reduces to \( \Box h_{\mu \nu} \). In \cite{29, 30} it was shown that the Fierz-Pauli mass term in (1.3) defines the unique quadratic deformation of the linearised Einstein-Hilbert action free of tachyons or ghosts.

To summarise, both for spin 1 and spin 2 the key idea is to exploit the Bianchi identities of the “massless sector” of the equations of motion in order to derive the on-shell conditions (1.4) from the divergence of a properly chosen mass term. These conditions reveal necessary and sufficient to recover the Fierz system (1.1), and in this sense it is clear that such an approach cannot be tried, without modifications, in the absence of a divergenceless Einstein tensor.

This is the reason why this idea cannot work in the constrained description of massless higher-spin bosons given by Fronsdal in \cite{31}, which is known, on the other hand, to correctly describe the free propagation of gauge fields of integer spin. In that framework indeed the analogue of the Ricci tensor, defined as
\[
\mathcal{F}_{\mu_1 \ldots \mu_s} \equiv \Box \varphi_{\mu_1 \ldots \mu_s} - \partial_{\mu_1} \partial^\alpha \varphi_{\alpha \mu_2 \ldots \mu_s} + \ldots + \partial_{\mu_1} \partial_{\mu_2} \varphi^\alpha_{\alpha \mu_3 \ldots \mu_s} + \ldots, \tag{1.7}
\]
where the dots indicate symmetrization over the set of \( \mu \)-indices, is used to build an Einstein tensor of the form
\[
\mathcal{E}_{\mu_1 \ldots \mu_s} = \mathcal{F}_{\mu_1 \ldots \mu_s} - \frac{1}{2} ( \eta_{\mu_1 \mu_2} \mathcal{F}^\alpha_{\alpha \mu_3 \ldots \mu_s} + \ldots ), \tag{1.8}
\]
whose divergence gives identically
\[
\partial^\alpha \mathcal{E}_{\alpha \mu_2 \ldots \mu_s} \equiv - \frac{3}{2} ( \partial_{\mu_2} \partial_{\mu_3} \partial_{\mu_4} \varphi^\alpha_{\alpha \beta \mu_5 \ldots \mu_s} + \ldots ) - \frac{1}{2} ( \eta_{\mu_2 \mu_3} \partial^\alpha \mathcal{F}^\beta_{\alpha \beta \mu_4 \ldots \mu_s} + \ldots ). \tag{1.9}
\]
It is then clear that, even under the condition of vanishing double trace assumed in that context,
\[
\varphi^\alpha_{\alpha \beta \mu_5 \ldots \mu_s} \equiv 0, \tag{1.10}
\]
the divergence of \( \mathcal{E}_{\mu_1 \ldots \mu_s} \) is not zero, but still retains a trace part.

In strict analogy, in the Fang-Fronsdal theory of massless fermionic fields \cite{32} a constraint is assumed on the triple gamma-trace of the basic field\(^3\)
\[
\gamma^\alpha \gamma^\beta \gamma^\gamma \psi_{\alpha \beta \gamma \mu_4 \ldots \mu_s} \equiv 0, \tag{1.11}
\]
and the dynamics is described in terms of a generalisation of the Dirac-Rarita-Schwinger tensor, having the form
\[
\mathcal{S}_{\mu_1 \ldots \mu_s} \equiv i \{ \gamma^\alpha \partial_\alpha \psi_{\mu_1 \ldots \mu_s} - ( \partial_{\mu_1} \gamma^\alpha \psi_{\alpha \mu_2 \ldots \mu_s} + \ldots ) \}. \tag{1.12}
\]
\(^3\)From now on we drop the spinor index \( a \) in \( \psi^a_{\mu_1 \ldots \mu_s} \).
Similarly to (1.9), the divergence of the corresponding Einstein tensor
\[ G_{\mu_1 \ldots \mu_s} \equiv S_{\mu_1 \ldots \mu_s} - \frac{1}{2} (\gamma_{\mu_1} S_{\alpha_2 \ldots \mu_s} + \ldots) - \frac{1}{2} (\eta_{\mu_1 \mu_2} S^\alpha_{\alpha \mu_3 \ldots \mu_s} + \ldots), \] 
(1.13)
does not vanish even if (1.11) is imposed, but still retains $\gamma$-trace parts, according to the general identity
\[ \partial^\alpha G_{\alpha \mu_2 \ldots \mu_s} = -\frac{1}{2} (\eta_{\mu_2 \mu_3} \partial^\beta S^\alpha_{\alpha \beta \mu_4 \ldots \mu_s} + \ldots) - \frac{1}{2} (\gamma_{\mu_2} \gamma^\alpha S^\beta_{\alpha \beta \mu_3 \ldots \mu_s} + \ldots) + i (\partial_{\mu_2} \partial^\alpha \gamma^\beta \gamma^\gamma \psi_{\alpha \beta \gamma \mu_4 \ldots \mu_s} + \ldots). \] 
(1.14)

On the other hand, the Lagrangians proposed in [4, 6, 13] are based on identically divergenceless Einstein tensors, for which it is possible in principle to try the extension of the spin-1 and spin-2 results. This construction is the object of this work.

The bosonic and the fermionic cases are presented separately, in Section 2 and Section 3 respectively. In particular, Section 2.1 contains a review of the construction of geometric Lagrangians for bosons proposed in [4, 6, 13], with focus on the main conceptual ideas behind it. Accordingly, in Section 2.1.1 the definition of higher-spin curvatures following de Wit and Freedman [33] is recalled, and in Section 2.1.2 it is discussed how to derive from these curvatures candidate “Ricci tensors” $A_{\mu_1 \ldots \mu_s}$, supposed to define basic equations of motion in the form $A_{\mu_1 \ldots \mu_s} = 0$. In Section 2.1.3 the problem of constructing identically divergenceless Einstein tensors, and corresponding gauge-invariant Lagrangians, is reviewed.

The main issue to be stressed, following [13], is that this geometric program has infinitely many solutions. Nonetheless, it was shown in [13] that problems arise when a coupling with an external current is turned on, and a closer analysis proves that only one geometric theory passes the test of defining consistent current exchanges. This theory also results to be the only one for which a clear local counterpart exists, described by the local Lagrangians proposed in [10, 12].

For both bosons and fermions the geometric Lagrangians can not be standard second-order or first-order ones since, for $s \geq \frac{5}{2}$, the curvatures are higher-derivative tensors, and the corresponding equations unavoidably contain non-localities, if one wishes to preserve at least formally the dimensions of canonical, relativistic wave operators. Because of this, one of the key issues is to prove their compatibility with the (Fang-)Fronsdal theory, synthetically defined by (1.7) (1.8) (1.9) (1.10) for bosons, and (1.11) (1.12) (1.13) (1.14) for fermions, together with the conditions that the abelian gauge transformations of the fields
\[ \delta \varphi_{\mu_1 \ldots \mu_s} = \partial_{\mu_1} \Lambda_{\mu_2 \ldots \mu_s} + \ldots, \quad \delta \psi_{\mu_1 \ldots \mu_s} = \partial_{\mu_1} \epsilon_{\mu_2 \ldots \mu_s} + \ldots, \] 
(1.15)
be understood in terms of $(\gamma\gamma)$ traceless gauge parameters
\[ \Lambda^\alpha_{\alpha \mu_3 \ldots \mu_{s-1}} \equiv 0, \quad \gamma^\alpha \epsilon_{\alpha \mu_2 \ldots \mu_{s-1}} \equiv 0, \] 
(1.16)
as needed for the gauge invariance of the equations
\[ F_{\mu_1 \ldots \mu_s} = 0, \quad S_{\mu_1 \ldots \mu_s} = 0, \] 
(1.17)
respectively. This is achieved by showing that the partial gauge-fixing needed to remove all the singularities from the geometric equations of motion simultaneously reduces those equations to the local, constrained form (1.17).
The counterpart of the singularities of the geometric theory on the side of the local, unconstrained Lagrangians proposed in \([10, 12]\), and briefly recalled in Section 2.1.3, can be recognised in the presence of higher-derivative terms in the kinetic operator of a “compensator” field, \(\alpha_{\mu_1 \ldots \mu_{s-3}}\), introduced in that context in order to eliminate the non-localities of the irreducible formulation. Since this field represents a pure-gauge contribution to the equations of motion, needed to guarantee invariance under a wider symmetry than the constrained one of (1.16), the corresponding higher-derivative terms, although they could in general be the source of problems, both at the classical and at the quantum level, should not interfere in this case with the physical content of the theory itself.

Nonetheless, on the side of the interesting quest for an ordinary-derivative formulation of the same dynamics, an important result was recently achieved in \([34]\), where a two-derivative formulation for unconstrained systems of arbitrary-spin fields was proposed, in a rather economical description involving for any spin only a limited (and fixed) number of fields.\(^4\) Another relevant feature of this work is the stress put on the close relationship with the triplet systems of Open String Field Theory \([36]\) previously investigated from this point of view in \([6, 37]\)\(^5\).

Here we propose a similar result, for the bosonic case, showing in Section 2.1.4 that the same dynamical content of the local Lagrangians of \([10, 12]\) can also be expressed in terms of an ordinary-derivative Lagrangian, at the price of enlarging the field content of the theory from the minimal set of three fields of \([10, 12]\), to a slightly bigger one involving five fields altogether, for any spin. This result, with respect to the one presented in \([34]\), looks more closely related to the underlying geometric description of \([4, 6, 13]\), which, in a sense, should provide its ultimate meaning.

In the discussion of fermions, in Section 3, more space is devoted to the analysis of the fermionic geometry, which is less straightforward than the corresponding bosonic one, and for which less details were given in \([4, 6]\), where the construction of gauge-invariant, non-local, kinetic tensors for fermions was substantially deduced from the knowledge of the corresponding bosonic ones. The analysis of the fermionic geometry performed in Sections 3.1.1 and 3.1.2 provides first of all the explicit expression of the kinetic tensors of \([4, 6]\) in terms of curvatures, and shows the existence of a wider range of possibilities for the construction of basic candidate “Dirac” tensors, whose meaning is clarified in Section 3.1.3, under the requirement that a Lagrangian derivation of the postulated equations of motion be possible.

The structure of the mass deformation and the proof that the Fierz systems (1.1) are actually recovered on-shell are given in Section 2.2 for bosons and in Section 3.2 for fermions. The main result will be that the Fierz-Pauli mass term (1.3) is actually the beginning of a sequence involving all possible \((\gamma-)\)-traces of the field, according to eqs. (2.99) and (3.66), whereas the guiding principle in the construction will be to recognise the need for the implementation of the Fierz-Pauli constraint (1.4) (and of its fermionic counterpart) for any spin.

To give an idea of the outcome, the generalised Fierz-Pauli mass term for spin 4 has the

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\(^4\)The first description of unconstrained, ordinary-derivative, higher-spin dynamics was given in \([35]\). The BRST construction proposed in those works involves an unconstrained spin-\(s\) symmetric tensor together with additional fields, whose number grows proportionally to \(s\).

\(^5\)See also \([38]\) for related contributions, and \([39, 40]\) for recent developments.
As a matter of principle, all of the infinitely many geometric Lagrangians describing the same free dynamics, being built from divergenceless Einstein tensors, are equivalently amenable to the quadratic deformation by means of the generalised Fierz-Pauli mass terms given in (2.99) and (3.66). In this sense, an issue of uniqueness is present at the massive level as well, as discussed in Section 4, and the analysis of the current exchange in this case is not sufficient to provide a selection principle, as showed in the same Section by the example of spin 4.

Nonetheless, we believe that a direct relation exist between the massive theories proposed in this work and the geometric theory described in [13], uniquely selected by the requirement of consistency with the coupling to an external source. To support this hypothesis, in Section 4 we make an analysis of the form of the geometric solution introduced in [13], and use the corresponding result to perform explicit computations of the quantities of interest. Still, we were not able to give a complete proof of this conjecture, which so far is only supported by checks up to spin 11.

In this work we shall mainly resort to a compact notation introduced in [4, 6], in which all symmetrized indices are left implicit. Traces will be indicated by “primes” (\( h^\alpha_\alpha \rightarrow h' \)) and divergences by the symbol “\( \partial \)” (\( \partial^\alpha h_{\alpha \mu} \rightarrow \partial \cdot h \)), so that, for example, the Fierz-Pauli constraint (1.4) could be written in the index-free form

\[
\partial \cdot h - \partial h' = 0.
\]  

The precise definitions and the corresponding computational rules needed in order to take the combinatorics into account are given in Appendix A.

An exception is made when there will be the need to describe more than one group of symmetrized indices. In particular, since the de Wit-Freedman connections, to be introduced in Section 2.1.1, are rank-(\( m + s \)) tensors where two group of \( m \) and \( s \) indices are separately symmetrized, we found it more appropriate to use a kind of “mixed-symmetric” notation. The presence of different groups of symmetrized indices will then be explicitly displayed by choosing the same letter for each index within a symmetric group. So for example the rank-(2 + 2) tensor indicated with \( \partial^\nu \partial_\mu \varphi_{\mu \nu} \) is to be understood as follows:

\[
\partial^\nu \partial_\mu \varphi_{\mu \nu} \rightarrow \partial^\nu_1 \partial_\mu_1 \varphi_{\mu_2 \nu_2} + \partial^\nu_2 \partial_\mu_1 \varphi_{\mu_2 \nu_1} + \partial^\nu_1 \partial_\mu_2 \varphi_{\mu_1 \nu_2} + \partial^\nu_2 \partial_\mu_2 \varphi_{\mu_1 \nu_1},
\]  

whereas in the general case of two groups of symmetrized indices, we shall use the shortcut notation

\[
\varphi_{\mu_1 \ldots \mu_m, \rho_1 \ldots \rho_s} \rightarrow \varphi_{\mu m, \rho s}.
\]  

The rules of symmetric calculus listed in Appendix A apply in this notation separately for each set of symmetric indices.

---

6With \( M_\varphi \) we indicate here the linear combination of traces of \( \varphi \) entering the Lagrangian in the schematic form \( L = \frac{1}{2} \varphi \{ \mathcal{E}_\varphi - m^2 M_\varphi \} \), where \( \mathcal{E}_\varphi \) is the Einstein tensor of the corresponding massless theory. Consequently, the equations of motion will appear in the form \( \mathcal{E}_\varphi - m^2 M_\varphi = 0 \). Similarly for the fermionic mass-term, \( M_\psi \).
2 Bosons

2.1 Geometry for higher-spin bosons

2.1.1 Bosonic curvatures

The starting point of the full construction is the definition of higher-spin curvatures given by de Wit and Freedman in [33] (see also [41]) and reviewed in [4, 6, 11]. These curvatures are the top elements of a hierarchy of generalised “Christoffel connections” $\Gamma^{(m)}_{\mu_1...\mu_m,\nu_1...\nu_s}$ built from derivatives of the gauge field. Roughly speaking, the rationale behind the construction is to choose a linear combination of multiple gradients of $\varphi$ ($m$ gradients, for the $m$-th connection $\Gamma^{(m)}_{\mu_1...\mu_m,\nu_1...\nu_s}$) such that, under the transformation $\delta \varphi = \partial \Lambda$, the gauge variation of the connections themselves become simpler and simpler, with increasing $m$. Specifically, in the mixed-symmetric notation, the $m$-th connection is

$$\Gamma^{(m)}_{\mu_m,\nu_s} = \sum_{k=0}^{m} \frac{(-1)^k}{\binom{m}{k}} \partial^{m-k}_\mu \partial^k_\nu \varphi_{\mu_k \nu_{s-k}} ,$$

(2.1)

whose gauge transformation is

$$\delta \Gamma^{(m)}_{\mu_m,\nu_s} = (-1)^m (m + 1) \partial^{m+1}_\nu \Lambda_{\mu_m,\nu_{s-m-1}} .$$

(2.2)

After $s$ steps, the resulting rank-$(s + s)$ tensor is identically gauge invariant, and it is called a (linearised) curvature. For spin 1 the outcome is simply the Maxwell field strength

$$R_{\mu\nu} = \partial_\mu \varphi_\nu - \partial_\nu \varphi_\mu ,$$

(2.3)

whereas for spin 2 a linear combination of the usual linearised Riemann tensors is obtained, that in terms of the rank-2 field can be simply expressed as

$$R_{\mu\nu,\nu\nu} = \partial^2_\mu \varphi_{\nu\nu} - \frac{1}{2} \partial_\nu \partial_\mu \varphi_{\mu\nu} + \partial^2_\nu \varphi_{\mu\mu} ,$$

(2.4)

where, as indicated in the Introduction, symmetrization is here assumed separately within the $\mu$-group and the $\nu$-group of indices. In addition $R_{\mu\nu,\nu\nu}$ is symmetric under the exchange of the two pairs, and cyclic. The same kind of symmetries are displayed in the generalisation to higher-spins of this pattern, so that for example for spin 3 the resulting curvature

$$R_{\mu\mu\mu,\nu\nu\nu} = \partial^3_\mu \varphi_{\nu\nu\nu} - \frac{1}{3} \partial^2_\mu \partial_\nu \varphi_{\nu\nu\nu} + \frac{1}{3} \partial_\mu \partial^2_\nu \varphi_{\nu\mu\nu} - \partial^3_\nu \varphi_{\mu\mu\mu} ,$$

(2.5)

is such that $R_{\mu\mu\mu,\nu\nu\nu} = -R_{\nu\nu\nu,\mu\mu\mu}$, while in general for spin $s$ it can be checked that the tensors

$$R_{\mu_s,\nu_s} = \sum_{k=0}^{s} \frac{(-1)^k}{\binom{s}{k}} \partial^{s-k}_\mu \partial^k_\nu \varphi_{\mu_k \nu_{s-k}} ,$$

(2.6)

defining proper generalisations of the linearised Riemann curvature, satisfy the constraints

$$R_{\mu_s,\nu_s} = (-)^s R_{\nu_s,\mu_s} , \quad R_{\mu_{s-1},\nu_s} = 0 ,$$

(2.7)
the latter being the expression of the generalised cyclic identity\(^7\). Finally, generalised uncontracted Bianchi identities hold for the full set of connections:

\[
\begin{align*}
\partial_\lambda \{ \Gamma^{(m)}_{\mu m-1, \nu s-1} - \Gamma^{(m)}_{\nu m-1, \lambda s-1} \} + \\
\partial_\nu \{ \Gamma^{(m)}_{\lambda m-1, \nu s-1} - \Gamma^{(m)}_{\mu m-1, \lambda s-1} \} + \\
\partial_{\hat{\mu}} \{ \Gamma^{(m)}_{\nu m-1, \lambda s-1} - \Gamma^{(m)}_{\nu m-1, \lambda s-1} \} & \equiv 0. \\
\end{align*}
\]

\[
(2.8)
\]

2.1.2 Generalised Ricci tensors

For spin 1 and spin 2 the basic kinetic tensors (or “Ricci” tensors, with some abuse of terminology), needed to define proper equations of motion, are obtained taking one divergence and one trace of the corresponding curvatures, respectively. For integer spin \(s\), starting from (2.6), the simplest possibility in order to define a tensor with the same symmetries of the gauge potential \(\varphi\) is to saturate all indices belonging to the same group taking traces, while also taking one divergence for the leftover index for odd spins. In all cases, however, in order to restore the dimension of a relativistic wave operator, it is necessary to act with inverse powers of the D’Alembertian operator, thus introducing possible non-localities in the theory. For instance, the simplest choices for candidate Ricci tensors for spin 3 and 4 are\(^8\)

\[
\mathcal{F}_2 \equiv \begin{cases} \\
\frac{1}{4} \partial \cdot \mathcal{R}' & s = 3, \\
\frac{1}{4} \mathcal{R}'' & s = 4. \\
\end{cases}
\]

(2.9)

On the other hand, once it is recognised that from the chosen viewpoint non-localities are unavoidable, there is no reason in principle to discard the possibility that other tensorial structures could contribute. These must be selected taking into account possible identities between apparently independent tensors. For the case of \(s = 3\), for example, out of the five possibilities

\[
\begin{align*}
\frac{\partial^2}{\Box} \mathcal{F}'_2, & \quad \frac{\partial}{\Box} \partial \cdot \mathcal{F}_2, & \quad \frac{\partial^3}{\Box^2} \partial \cdot \mathcal{F}'_2, & \quad \frac{\partial^2}{\Box^2} \partial \cdot \partial \cdot \mathcal{F}_2, & \quad \frac{\partial^3}{\Box^3} \partial \cdot \partial \cdot \partial \cdot \mathcal{F}_2, \\
\end{align*}
\]

(2.10)

it turns out that only the first really defines a new structure, because of the Bianchi identity

\[
\partial \cdot \mathcal{F}_2 = \frac{1}{4} \partial \mathcal{F}'_2
\]

(2.11)

verified by \(\mathcal{F}_2\). Thus, we can define for spin 3 a one-parameter class of candidate Ricci tensors\(^9\)

\[
\mathcal{A}_\varphi(a_1) = \mathcal{F}_2 + a_1 \frac{\partial^2}{\Box} \mathcal{F}'_2,
\]

(2.12)

\(^7\)Other definitions of “curvature” tensors, displaying different symmetry properties, are possible. A curl on each index of \(\varphi_{\mu_1...\mu_s}\) defines a gauge-invariant, rank-(\(s + s\)) tensor \(\mathcal{R}_{[\mu_1\nu_1]...[\mu_s\nu_s]}\) antisymmetric in each pair \((\mu_i, \nu_i)\), more closely resembling the symmetry of the Riemann tensor of gravity. This is the preferred choice in the works [5, 7]. Similarly, identical gauge invariance can also be reached taking a curl on one index of the \(\Gamma^{[s-1]}\) connection. All these possibilities can be shown to be related by linear combinations, and in this sense none can be considered \textit{a priori} preferable [41].

\(^8\)The notation \(\mathcal{F}_2\) refers to the fact that for spin 1 and spin 2 the corresponding tensor is simply the standard Fronsdal tensor (1.7), indicated with \(\mathcal{F} \equiv F_1\).

\(^9\)The subscript “\(\varphi\)” in \(\mathcal{A}_\varphi(a_1)\) in the notation of [13]) is used to distinguish the non-local Ricci tensors from their local analogue to be introduced in Section 2.1.3. Those will be indicated with the symbol \(\mathcal{A}\), without subscripts, and will depend on the field \(\varphi\) and an auxiliary field \(\alpha\).
and the very first issue to be clarified is whether the corresponding postulated equations of motion
\[ A_\varphi(a_1) = 0, \tag{2.13} \]
can be shown to be consistent, at least for some choices of the parameter \( a_1 \), with the Fronsdal equation \( \mathcal{F} = 0 \), with \( \mathcal{F} \) defined in (1.7). The same issue presents itself in generalised form in the case of spin \( s \). The idea is to consider the “order zero” Ricci tensors introduced in [4, 6]
\[ F_{n+1} = \begin{cases} \frac{1}{n} R^{[n+1]} & s = 2(n + 1), \\ \frac{1}{n^2} \partial \cdot R^{[n]} & s = 2n + 1, \end{cases} \tag{2.14} \]
that might be also defined recursively, according to the relation
\[ F_{n+1} = F_n + \frac{1}{(n+1)(2n+1)} \frac{\partial^2}{\Box} F_n' - \frac{\partial}{\Box} \partial \cdot F_n, \tag{2.15} \]
and notice that, if the rank of \( \varphi \) is either \( s = 2(n + 1) \) or \( s = 2n + 1 \), they satisfy the series of identities
\[ \partial \cdot F_{n+1}^{[k]} = \frac{1}{2(n - k + 1)} \partial F_{n+1}^{[k+1]}. \tag{2.16} \]
In particular, in the odd case, \( \partial \cdot F_{n+1}^{[n]} \equiv 0 \). These identities imply that divergences of the tensors \( F_{n+1} \) can always be expressed in terms of traces, so that the only independent structures that we can consider are
\[ F_{n+1}, \; F_{n+1}', \; F_{n+1}', \; \ldots, \; F_{n+1}^{[k]}, \; \ldots, \; F_{n+1}^{[q]}, \tag{2.17} \]
where \( q = n + 1 \) or \( q = n \) depending on whether the rank is \( s = 2(n + 1) \) or \( s = 2n + 1 \). The most general candidate for a possible Ricci tensor for spin \( s \) is then given by a linear combination of all structures available, with coefficients which are arbitrary, up to an overall normalisation
\[ A_\varphi (\{a_k\}) = F_{n+1} + \ldots + a_k \frac{\partial^{2k}}{\Box} F_{n+1}^{[k]} + \ldots + \begin{cases} a_{n+1} \frac{\partial^{2(n+1)}}{\Box} F_{n+1}^{[n+1]} & s = 2(n + 1), \\ a_n \frac{\partial^{2n}}{\Box} F_{n+1}^{[n]} & s = 2n + 1. \end{cases} \tag{2.18} \]

The crucial point to be stressed at this level is that, for infinitely many choices of the coefficients \( a_1 \ldots a_n \), the postulated equation \( A_\varphi (\{a_k\}) = 0 \) can be shown to imply an equation of the form
\[ \mathcal{F} - 3 \partial^3 \alpha_\varphi (\{a_k\}) = 0, \tag{2.19} \]
where \( \alpha_\varphi (\{a_k\}) \) is a non-local tensor whose gauge transformation is a shift in the trace of the gauge parameter,
\[ \delta \alpha_\varphi (\{a_k\}) = \Lambda', \tag{2.20} \]
in such a way that, after a suitable gauge-fixing, infinitely many distinct non-local equations can be reduced to the Fronsdal form.\(^\text{11}\)
\(^\text{10}\)Here the initial condition is, as already specified, \( F_1 = \mathcal{F} \). It is possible to notice anyway that the sequence could also formally start with \( F_0 \equiv \Box \varphi \), in which case it would produce the Fronsdal tensor \( \mathcal{F} \) as a result of the first iteration.
\(^\text{11}\)Moreover, as shown in [13], choosing the first coefficient in (2.18) to be \( a_1 = \frac{n}{n+1} \), the resulting set of tensors \( A_\varphi (a_2 \ldots a_n) \) can be shown to be already in the compensator form (2.19) which, in this sense, it is shown to be highly not unique.
In particular, the reduction to the compensator form (2.19) of the class of equations \( F_n = 0 \), produces a specific \( \alpha \), that we shall denote \( H\varphi [4, 6] \):

\[
F_n = 0 \Rightarrow F - 3 \partial^3 H\varphi = 0.
\]  
(2.21)

An idea of the mechanism by which (2.21) is realised can be easily obtained for \( s = 3 \), making the form of \( F_2 \) explicit,

\[
F_2 = F + \frac{1}{6} \frac{\partial^2}{\Box} F' - \frac{1}{2} \frac{\partial \cdot F}{\Box} = 0,
\]  
(2.22)

and observing that, by virtue of the identity (1.9), the same equation can be written

\[
F_2 = F - \frac{1}{3} \frac{\partial^2}{\Box} F'.
\]  
(2.23)

The trace of (2.22) implies \( F' = \frac{\partial}{\Box} \partial \cdot F' \) which, upon substitution in (2.23) imply (2.21) with

\[
H\varphi = \frac{1}{3} \frac{\partial^2}{\Box} \partial \cdot F'.
\]  
(2.24)

The general mechanism for (2.21) is discussed for fermions in Section 3.1.3, and can be adapted to the bosonic case with minor adjustments.\(^{12}\) The main conclusion of this Section is that, starting from the curvatures (2.6), it is possible to define infinitely many Ricci-like tensors, according to (2.18), all of them unavoidably non-local. Nonetheless, it is worth stressing that, under the further requirement that the kinetic tensors have the lowest possible degree of singularity, the corresponding “order-zero” definition (2.14) turns out to be unique. This will be no more true in general for fermions, as we shall see in Section 3.1.2. Here we ask ourselves how to obtain a Lagrangian description for the geometric bosonic theory.

### 2.1.3 Geometric Lagrangians

It is a general result that the equation \( A_\varphi (\{a_k\}) = 0 \) is not a Lagrangian equation, but it can be derived from a Lagrangian via a multiple-step procedure, once an identically divergenceless Einstein tensor is constructed from \( A_\varphi (\{a_k\}) \) and its traces. For the simplest choice (2.14) this was shown to be possible in [4, 6] where these “order zero” Einstein tensors were explicitly constructed and look

\[
E_\varphi = \sum_{p \leq n} \frac{(-1)^p}{2^p p!} \eta^p F_n^{[p]}.
\]  
(2.25)

Thus, starting with the corresponding Lagrangian

\[
\mathcal{L} = \frac{1}{2} \varphi E_\varphi,
\]  
(2.26)

it is possible to show that subsequent traces of the Lagrangian equation \( E_\varphi = 0 \) imply the condition \( F_n = 0 \) and finally, as recalled in the previous Section, after some manipulation involving the identities (2.16) and the gauge-fixing of all non-localities to zero, the Fronsdal equation \( F = 0 \).

\(^{12}\) Analogous results, together with a discussion of the mixed-symmetric case, were found in [5, 7, 8, 9, 11]. In particular in [7, 9] Bekaert and Boulanger, inspired by previous works [41, 42], showed that an equation of the same form of (2.21) could be deduced from the vanishing of the trace of the curvature, as a consequence of the generalised Poincaré lemma.
It could be possible to proceed similarly for the generalised Ricci tensors defined in (2.18), and in this way infinitely many non-local, geometric Lagrangians would be defined, all describing the same free dynamics. In order to better understand their meaning, and to look for a selection principle (if any) inside this class of theories, let us make some further observations.

As already stressed, the curvatures (2.6) define identically gauge-invariant tensors, without the need for algebraic trace conditions on fields or on gauge parameters. This means that the geometric description of higher-spin dynamics is related to the removal of the constraints (1.10) and (1.16) assumed in the Fronsdal theory, the main drawback being the very introduction of non-localities.

One different possibility to remove constraints without introducing non-localities is to replace them, in some sense, with auxiliary fields. After the first results in this direction [35], already recalled in the Introduction, more recently, a “minimal” local formulation of the same dynamics was proposed in [10, 12], whose building block is in the definition of fully gauge invariant, local, kinetic tensors introduced\(^{13}\) in [4, 6, 37]. In this setting, the geometrical meaning of the unconstrained theory, obscured by the presence of the auxiliary fields, could still be recovered if a clear map between local theory and non-local ones could be established. This was done in [13], and it is briefly reviewed in the following, with focus on the bosonic case.

For a rank-\(s\) fully symmetric tensor one can begin by considering the Fronsdal tensor \(\mathcal{F}\) introduced in (1.7) and compensating its gauge transformation

\[
\delta \mathcal{F} = 3 \partial^3 \Lambda',
\]

(2.27)

with the introduction of a spin-\((s - 3)\) compensator \(\alpha\) transforming as

\[
\delta \alpha = \Lambda',
\]

(2.28)

so that the local kinetic tensor\(^{14}\)

\[
\mathcal{A} = \mathcal{F} - 3 \partial^3 \alpha,
\]

(2.29)

be identically gauge-invariant. The Bianchi identity

\[
\partial \cdot \mathcal{A} - \frac{1}{2} \partial \mathcal{A}' = - \frac{3}{2} \partial^3 \left( \varphi'' - 4 \partial \cdot \alpha - \partial \alpha' \right),
\]

(2.30)

is the other main ingredient needed to show that a gauge-invariant local Lagrangian can be written in the compact form

\[
\mathcal{L} = \frac{1}{2} \varphi \left( \mathcal{A} - \frac{1}{2} \eta \mathcal{A}' \right) - \frac{3}{4} \left( \frac{s}{3} \right) \alpha \partial \cdot \mathcal{A}' + 3 \left( \frac{s}{4} \right) \beta \left[ \varphi'' - 4 \partial \cdot \alpha - \partial \alpha' \right],
\]

(2.31)

where the Lagrange multiplier \(\beta\) transforms as\(^{15}\)

\[
\delta \beta = \partial \cdot \partial \cdot \partial \cdot \Lambda,
\]

(2.32)

\(^{13}\)For the spin-3 case an analogous result had already been found by Schwinger [43].

\(^{14}\)As previously advertised, we distinguish the local tensor \(\mathcal{A}\), function of \(\varphi\) and \(\alpha\), without subscripts, from the non-local tensors \(\mathcal{A}_\varphi(\{a_k\})\) defined in (2.18), function of \(\varphi\) only. Similarly, \(\alpha\) without subscript is an independent field, whereas \(\alpha_\varphi\) indicates in general a non-local tensor, function of \(\varphi\), with the property (2.20).

\(^{15}\)It is possible to show that \(\beta\) is necessary in order for the double trace not to propagate [13], so that the field content represented by the triple \(\varphi, \alpha, \beta\) is the minimal one needed to remove the constraints from the Fronsdal Lagrangian [10].
and the tensor $C \equiv \varphi'' - 4 \partial \cdot \alpha - \partial \alpha'$ is identically gauge-invariant.

In order to review in which sense a link between the local theory described by (2.31) and the geometric Lagrangians can be established, it will be sufficient to analyse the case of spin 4. A more complete discussion can be found in [13].

In the spin-4 case, the Lagrangian equations\(^{16}\) coming from (2.31) can be simplified to

$$
\begin{align*}
A - \frac{1}{2} \eta A' &= 0, \\
\partial \cdot A' &= 0, \\
\varphi'' &= 4 \partial \cdot \alpha.
\end{align*}
$$

(2.33)

We can invert the second equation to find $\tilde{\alpha}_\varphi$ as the non-local solution for the compensator in terms of the basic field; the result is

$$
\tilde{\alpha}_\varphi = \frac{1}{3} \Box^2 \partial \cdot F' - \frac{1}{4} \Box^3 \partial \cdot \partial \cdot F',
$$

(2.34)

which, upon substitution in (2.33) implies the non-local system

$$
\begin{align*}
\tilde{A}_\varphi - \frac{1}{2} \eta \tilde{A}' &= 0, \\
\varphi'' &= 4 \partial \cdot \tilde{\alpha}_\varphi,
\end{align*}
$$

(2.35)

where $\tilde{A}_\varphi = F - 3 \partial^3 \tilde{\alpha}_\varphi$. From the discussion of the previous Section we also know that, starting with the geometric theory defined in terms of the simplest Ricci tensor (2.9), the result is an equation of motion of the form $F_2 = 0$, that can be shown to imply the compensator-like equation

$$
F - 3 \partial^3 \mathcal{H}_\varphi = 0,
$$

(2.36)

with $\mathcal{H}_\varphi$ different from $\tilde{\alpha}_\varphi$ in (2.34), and given by

$$
\mathcal{H}_\varphi = \frac{1}{3} \Box \partial \cdot F' - \frac{1}{4} \Box^2 \varphi''.
$$

(2.37)

The very fact that $\tilde{\alpha}_\varphi$ and $\mathcal{H}_\varphi$ do not coincide, their difference being a gauge invariant tensor, is another way to see the infinite degeneracy of the free geometric theory. Actually, both the first equation in (2.35) and (2.36) can be made Lagrangian by the construction of suitable Einstein tensors, say $\mathcal{E}_\tilde{a}$ and $\mathcal{E}_\mathcal{H}$, but then (almost) any linear combination of these two Einstein tensors with coefficients $a$ and $b$ such that $a + b = 1$, is again an allowed tensor for a non-local Lagrangian whose dynamics will be equivalent to the Fronsdal one.

From the viewpoint of the connection we are after, between local and non-local theories, it is important to stress that a proper counterpart of the local theory should involve a Ricci tensor $\tilde{A}_\varphi$ having the properties encoded in the system (2.35), and that neither $\tilde{A}_\varphi$ alone nor $F - 3 \partial^3 \mathcal{H}_\varphi$ satisfy this requirement. For instance, whereas $\tilde{A}_\varphi''$ and $(F - 3 \partial^3 \mathcal{H}_\varphi)''$ do not vanish, it is possible to show that the second equation in (2.35) effectively implies that the tensor $\tilde{A}_\varphi$ is identically doubly traceless regardless the fact that the first equation be satisfied or not, i.e. even if $\varphi$ is off-shell.

\(^{16}\)The “compensator” tensor $A = F - 3 \partial^3 \alpha$ also plays a role in the linearised Vasiliev’s equations, if a suitable, unusual, projection is performed [44]. More recently, in a novel approach to the quest for a higher-spin action principle, proposed in terms of a Chern-Simons theory, it has been shown that the equation $A = 0$ is the natural outcome, at least for the spin-3 case, of the linearisation procedure in that dynamical framework [45, 15].
One possibility to get closer to (2.35) is then to select the particular combination of $E_{\tilde{\alpha}}$ and $E_H$ such that in the resulting Einstein tensor the non-local “compensator block”

$$A_\varphi \equiv \mathcal{F} - 3 \partial^3 \gamma_\varphi$$

(2.38)

be identically doubly-traceless. The unique solution gives in this particular case an Einstein tensor of the form

$$E_\varphi = \frac{4}{3} \tilde{E}_{\tilde{\alpha}} - \frac{1}{3} E_H = A_\varphi - \frac{1}{2} \eta A'_\varphi + \eta^2 B_\varphi,$$

(2.39)

with $B_\varphi$ such that

$$\partial B_\varphi = \frac{1}{2} \partial \cdot A'_\varphi,$$

(2.40)

and $\gamma_\varphi$ given by

$$\gamma_\varphi = \frac{1}{3} \Box^2 \partial \cdot \mathcal{F}' - \frac{1}{3} \frac{\partial}{\Box^3} \partial \cdot \partial \cdot \mathcal{F}' + \frac{1}{12} \frac{\partial}{\Box^2} \mathcal{F}''.$$

(2.41)

This analysis of the spin-4 case suggests that, among the infinitely many geometric theories dynamically equivalent to the Fronsdal constrained system, only one should be identified as the proper non-local counterpart of the local theory defined by the Lagrangian (2.31).

A crucial observation in order to corroborate this hypothesis, and to generalise the result to all spins, is related to the analysis of the current exchange in the presence of weak external sources performed in [13]. There it was shown that the correct structure of the propagator is guaranteed if and only if the Einstein tensor has the form, for any spin $s$,

$$E_\varphi = A_\varphi - \frac{1}{2} \eta A'_\varphi + \eta^2 B_\varphi,$$

(2.42)

with $A_\varphi$ given by (2.38), and satisfying the two identities

$$\partial \cdot A_\varphi - \frac{1}{2} \partial A'_\varphi \equiv 0,$$

$$A''_\varphi \equiv 0,$$

(2.43)

whereas the general requirement that the Einstein tensor be divergenceless fixes the tensor $B_\varphi$ in terms of $A_\varphi$. The explicit dependence of $A_\varphi$ on the curvatures has been given in [13] and looks

$$A_\varphi = \mathcal{F} - 3 \partial^3 \gamma_\varphi = \sum_{k=0}^{n+1} (-1)^{k+1} (2k - 1) \left\{ \frac{n + 2}{n - 1} \prod_{j=-1}^{k-1} \frac{n + j}{n - j + 1} \right\} \partial^{2k} \mathcal{F}_{n+1},$$

(2.44)

with $\mathcal{F}_{n+1}$ defined in (2.14). Details about the dependence of $A_\varphi$ and $B_\varphi$ on the Fronsdal tensor $\mathcal{F}$ are given in Section 4 and in Appendix B, when discussing the relation between this geometric solution and the generalised Fierz-Pauli mass terms introduced in Section 2.2.3.

Here we further observe that this result is particularly meaningful, in that not only it implies the existence of a clear, one-to-one map between minimal local theory and one geometric formulation, but it also gives a physical meaning to this map, in terms of consistency of the coupling with external sources.

\footnote{We correct here a misprint in the corresponding equation (4.67) in [13].}
Nonetheless, the local counterpart of the geometric theory, defined by (2.31), contains higher derivatives, in the kinetic operator of the compensator field $\alpha$, which could be seen as another facet of the difficulties met in a geometric-inspired description of the dynamics.

On the other hand, the very fact that the field $\alpha$ can be removed from the equations of motion by going to the “Fronsdal” gauge, where the parameter $\Lambda$ is traceless, indicates that the physical content of the theory should be safe from difficulties related to the higher-derivative terms.

Thus, to give further support to this viewpoint, before discussing the issue of constructing suitable mass deformations for the geometric theory, we shall show how the local counterpart of the geometric description can be put in more conventional form, constructing an equivalent, but ordinary-derivative, Lagrangian.

### 2.1.4 Ordinary-derivative Lagrangians for unconstrained bosons

We would like to investigate the possibility of eliminating the higher-derivative terms in the minimal Lagrangians, while still retaining their dynamical content.

The basic idea is to look for “compensators” transforming as gradients of the trace of the gauge parameter$^{18}$. This choice does not lead to a straightforward solution, to begin with just because the new compensators are no more pure-gauge fields, and it is not obvious how to avoid their propagation.

The solution is simply to include a constraint in the Lagrangian so as to “remember” that the lower-derivative compensator actually “is” a gradient of $\alpha$, whereas some more attention has to be paid to the role of the Lagrange multipliers of the theory, in order to make sure that they do not propagate extra degrees of freedom.

The starting point is the gauge transformation of the Fronsdal tensor:

\[ \delta \mathcal{F} = 3 \partial^3 \Lambda'. \]  

(2.45)

In order to define an unconstrained, local kinetic tensor, instead of the field $\alpha$, let us consider the alternative possibility

\[ \theta : \delta \theta = \partial \Lambda', \]  

(2.46)

and define the corresponding gauge-invariant tensor according to$^{19}$

\[ A_{\varphi,\theta} = \mathcal{F} - \partial^2 \theta. \]  

(2.47)

Clearly, this choice runs into the trouble of introducing in the theory a field which is not a pure shift. This problem can be solved by simply including in the Lagrangian a suitable constraint relating the field $\theta$ and the compensator $\alpha$. We can start with a trial Lagrangian of the form,

\[ \mathcal{L}_0 = \frac{1}{2} \varphi \{ A_{\varphi,\theta} - \frac{1}{2} \eta A'_{\varphi,\theta} \} + \left( \frac{s}{2} \right) \gamma (\theta - \partial \alpha), \]  

(2.48)

with $\gamma$ a gauge-invariant Lagrange multiplier, whose normalisation has been chosen for future purposes.

---

18 I am grateful to J. Mourad for discussions about this point.

19 Here we use a double subscript, to avoid possible confusion with the local tensor $A = \mathcal{F} - 3 \partial^3 \alpha$, and with the non-local tensor, function of $\varphi$ alone, $A_{\varphi} = \mathcal{F} - 3 \partial^3 \gamma_{\varphi}$ given by (2.44).
We can already notice the (obvious) point which will play a crucial role in what follows: since \( \alpha \) is anyway present in \( \mathcal{L} \), we are free to use it in other combinations, if needed.

The second structure we need is the Bianchi identity for \( A_{\phi,\theta} \)

\[
\partial \cdot A_{\phi,\theta} - \frac{1}{2} \partial A'_{\phi,\theta} = -\frac{1}{2} \partial \{ \partial^2 \phi'' + \Box \theta - \partial \partial \cdot \theta - \partial^2 \theta' \} \equiv -\frac{1}{2} \partial \hat{C},
\]

from which it is already possible to observe the second (and more delicate) difficulty of this approach: the structure of the gauge-invariant combination of fields to be compensated in the variation of \( \mathcal{L}_0 \) involves a \( \Box \) of the field \( \theta \). This implies that the corresponding Lagrange multiplier that one would introduce by analogy with (2.31) would appear as a propagating field in the equations of motion for the \( \theta \) itself.

More explicitly, let us compute the variation of the trial Lagrangian (2.48),

\[
\delta \mathcal{L}_0 = -\frac{1}{4} \left( \frac{s}{2} \right) \partial \Lambda' A'_{\phi,\theta} - \frac{1}{2} \left( \frac{s}{2} \right) \partial \cdot \hat{C},
\]

and to begin with, in order to make the problem related with this choice explicit, let us try to compensate the \( \hat{C} \)-term introducing a multiplier \( \hat{\beta} \):

\[
\delta \hat{\beta} = \partial \cdot \Lambda,
\]

allowing to complete the construction of a gauge invariant Lagrangian according to

\[
\mathcal{L} = \frac{1}{2} \phi \{ A_{\phi,\theta} - \frac{1}{2} \eta A'_{\phi,\theta} \} + \frac{1}{4} \left( \frac{s}{2} \right) \theta A'_{\phi,\theta} + \frac{1}{2} \left( \frac{s}{2} \right) \hat{\beta} \hat{C} + \left( \frac{s}{2} \right) \gamma (\theta - \partial \alpha).
\]

The corresponding equations of motion are

\[
\begin{align*}
E_{\phi} : & \quad A_{\phi,\theta} - \frac{1}{2} \eta (A'_{\phi,\theta} - \frac{1}{2} \hat{C}) + \eta^2 \hat{B} = 0, \\
E_{\theta} : & \quad 2A_{\phi,\theta} - \hat{C} - \Box \hat{D} + \partial \partial \cdot \hat{D} - 2\eta \hat{B} + 4\gamma = 0, \\
E_{\alpha} : & \quad \partial \cdot \gamma = 0, \\
E_{\beta_1} : & \quad \hat{C} = 0, \\
E_{\gamma} : & \quad \theta - \partial \alpha = 0,
\end{align*}
\]

where the various tensors are defined by

\[
\begin{align*}
A_{\phi,\theta} &= \mathcal{F} - \partial^2 \theta, \\
\hat{B} &= \partial \cdot \partial \cdot \hat{\beta} - \frac{1}{2} (\partial \cdot \partial \cdot \phi' - \partial \cdot \partial \cdot \theta), \\
\hat{C} &= \partial^2 \phi'' + \Box \theta - \partial \partial \cdot \theta - \partial^2 \theta', \\
\hat{D} &= \phi' - \theta - 2 \hat{\beta},
\end{align*}
\]

and the gauge transformations of the fields are

\[
\begin{align*}
\delta \phi &= \partial \Lambda, \\
\delta \theta &= \partial \Lambda', \\
\delta \hat{\beta} &= \partial \cdot \Lambda, \\
\delta \alpha &= \Lambda', \\
\delta \gamma &= 0.
\end{align*}
\]
Now, given that in the gauge $\alpha = 0$ the equation for $\varphi$ is manifestly consistent\(^{20}\) on the other hand it is difficult to avoid the conclusion that $\hat{\beta}$ is a propagating field as well, given that it is not possible to express it completely in terms of the other fields, and because of the presence of $\Box \hat{D}$ in $E_\theta$\(^{21}\).

Nonetheless, we can think of a possible way out, substituting $\hat{\beta}$ with the following combination\(^{22}\)

\[
\hat{\beta} \rightarrow \frac{1}{2} (\varphi' - \theta).
\]

(2.56)

together with the addition of the further coupling of the form $\sim \beta \{\varphi'' - 4 \partial \cdot \alpha - \partial \alpha'\}$, meant to provide the correct meaning of the double-trace of $\varphi$, that would be no more under control in the absence of a true constraint equation.

The complete Lagrangian is

\[
\mathcal{L} = \frac{1}{2} \varphi \{A_{\varphi, \theta} - \frac{1}{2} \eta A'_{\varphi, \theta}\} + \frac{1}{4} \left(\frac{s}{2}\right) \theta A'_{\varphi, \theta} + \frac{1}{4} \left(\frac{s}{2}\right) (\varphi' - \theta) \hat{C} + \left(\frac{s}{2}\right) \gamma (\theta - \partial \alpha) + 3 \left(\frac{s}{4}\right) \beta \{\varphi'' - 4 \partial \cdot \alpha - \partial \alpha'\},
\]

(2.57)

with $\beta$ a gauge-invariant multiplier. The corresponding equations are

\[
\begin{align*}
E_{\varphi} & : \quad A_{\varphi, \theta} - \frac{1}{2} \eta (A'_{\varphi, \theta} - \hat{C}) + \eta^2 \beta = 0, \\
E_\theta & : \quad A'_{\varphi, \theta} - \hat{C} + 2 \gamma = 0, \\
E_\alpha & : \quad \partial \cdot \gamma + \partial \beta + \frac{1}{2} \eta \partial \cdot \beta = 0, \\
E_\beta & : \quad \varphi'' - 4 \partial \cdot \alpha - \partial \alpha' = 0, \\
E_\gamma & : \quad \theta - \partial \alpha = 0,
\end{align*}
\]

(2.58)

and the set of gauge transformations is given by

\[
\begin{align*}
\delta \varphi &= \partial \Lambda, \\
\delta \theta &= \partial \Lambda', \\
\delta \alpha &= \Lambda', \\
\delta \beta &= 0, \\
\delta \gamma &= 0.
\end{align*}
\]

(2.59)

Let us make a few comments:

→ the equation for $\gamma$ transforms $A_{\varphi, \theta}$ and $\hat{C}$ in the corresponding quantities of (2.31):

\[
E_\gamma \Rightarrow \begin{cases} 
A_{\varphi, \theta} \rightarrow F - 3 \partial^3 \alpha, \\
\hat{C} \rightarrow \varphi'' - 4 \partial \cdot \alpha - \partial \alpha',
\end{cases}
\]

(2.60)

\(^{20}\)In this gauge $A''_{\varphi, \theta} = 0$, and the double trace of $E_\varphi$, implying $\hat{B} = 0$, simply fixes the double divergence of $\hat{\beta}$ in terms of $\varphi$.

\(^{21}\)From this point of view, the cancellation of the terms in $\Box \alpha$ in the Bianchi identity of the minimal theory (2.31), that would have led to the same problem, without possible solutions, looks somewhat magical. Of course, the “magic” is in the quasi-conservation of the Einstein tensor in the constrained setting, implying that only gradients of $\varphi''$ can appear, and then the structure in $\alpha$ follows from the gauge transformation of $\varphi''$.

\(^{22}\)And not $\hat{\beta} \rightarrow \frac{1}{2} (\varphi' - \partial \alpha)$, that would give a higher-derivative term in $\mathcal{L}$. 

17
this in its turn implies $E_\varphi \to A - \frac{1}{2} \eta A' + \eta^2 \beta = 0$, which is in fact the Lagrangian equation of (2.31), with some specification to be given about the multiplier $\beta$.

$\rightarrow \beta$ is a gauge-invariant tensor$^{23}$. Since $C = 0 \rightarrow A'' = 0$, it is simple to realise that multiple traces of $E_\varphi$ imply that all traces of $\beta$, and finally $\beta$ itself, vanish in the free case. In the presence of a current $J$, $\beta$ would be fixed in terms of $J''$.

$\rightarrow \gamma$ is determined in terms of the other fields, and in particular in the gauge $\alpha = 0$ it is proportional to $F'$.

Consistency with gauge-invariance can be expressed by the identity

$$\partial \cdot E_\varphi = \eta \left\{ \frac{1}{3} \left( \begin{array}{c} \varphi \\ \frac{\varphi'}{3} \end{array} \right) E_\alpha - \frac{1}{\left( \begin{array}{c} \varphi \\ \frac{\varphi'}{2} \end{array} \right)} \partial \cdot E_\theta \right\},$$

where prefactors coming from the variation of (2.57), neglected in (2.58), have also been taken into account.

In this sense, at the price of enlarging the field content of the minimal theory to include the new fields $\theta$ and $\gamma$ we can characterise the same dynamics as (2.31) by means of the ordinary-derivative Lagrangian (2.57). As already recalled in the Introduction, the difference with respect to the recent result found in [34], similar in spirit and in the total number of fields involved, is that the Lagrangian (2.57) somehow represents the ordinary-derivative version of the geometric theory synthetically described by (2.42), (2.43) and (2.44)$^{24}$.

In the next Section we turn our attention again to the geometry, and start to investigate the possibility of using the geometric Lagrangians for the study of the massive representations.

2.2 Mass deformation

We look for a massive Lagrangian for higher-spin bosons of the form

$$L = \frac{1}{2} \varphi \left\{ E_\varphi - m^2 M_\varphi \right\},$$

with $E_\varphi$ a generic member in the class of divergence-free Einstein tensors discussed in Section 2.1.3, and $M_\varphi$ a linear function of $\varphi$ to be determined.

The main idea is that $M_\varphi$ should be a linear combination of all the traces of $\varphi$, starting with the Fierz-Pauli mass term (1.3), that we rewrite here for convenience

$$M_{FP} = \varphi - \eta \varphi'.$$  

(2.63)

Qualitatively speaking this is plausible, in the sense that there is no reason in principle to assume that only order-zero and order-one traces should contribute in the unconstrained case. In general,

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$^{23}$Playing somehow the role of the tensor $B$ of the minimal theory of [13].

$^{24}$It is conceivable, and it represents an interesting issue to be clarified, that the Lagrangians (2.31) and (2.57), together with the corresponding one introduced in [34], all encoding the same irreducible dynamics, might be related by some kind of field redefinition. On the other hand, the very fact that the field content is similar, but not identical (the Lagrangian of [34] involving a total of six fields), makes it not directly obvious which could be the possible redefinition allowing to switch among these possibilities, off-shell. I would like to thank the Referee for stimulating this comment.
however, the coefficients of the various terms in the sequence could be spin-dependent, so that, for instance in the spin-3 case, the mass term could take the form

\[ M_{s=3} = \varphi - k \eta \varphi', \tag{2.64} \]

with a given constant \( k \). For the \textit{constrained} case it was shown in [21] that \( k = 1 \) is the only acceptable value for all spins. This is particularly clear if one considers the dimensional reduction of the constrained massless theory from \( D + 1 \) to \( D \) dimensions. In that framework indeed the very form of the Fronsdal Lagrangian

\[ \mathcal{L} = \frac{1}{2} \varphi \{ \mathcal{F} - \frac{1}{2} \eta \mathcal{F}' \} \sim \frac{1}{2} \varphi \{ \Box (\varphi - \eta \varphi') + \ldots \} \tag{2.65} \]

implies that, under the formal substitution

\[ \Box \rightarrow \Box - m^2, \tag{2.66} \]

the mass term will appear exactly in the Fierz-Pauli form, for all spins. The same result can be found in the Kaluza-Klein reduction of the unconstrained, local theory of [10, 12, 13], with a richer structure of Stueckelberg fields, as expected in order to account for the wider gauge symmetry allowed in that context.

The passage from the local description to the non-local one can be roughly described by the requirements that the compensator \( \alpha \) be replaced by a suitable non-local tensor with the same gauge transformation, and that higher-traces of the field enter the Lagrangian, to replace the equation of motion for the multiplier \( \beta \), ensuring in the local setting that \( \varphi'' \) be pure gauge. No modifications are expected for the local, lower-trace parts of the theory, and in this sense we do not expect the Fierz-Pauli term (2.63) to be modified, if not for the contribution of further traces of the field.

More quantitatively, we shall see that, starting from the equation

\[ \mathcal{E}_\varphi - m^2 M_\varphi = 0, \tag{2.67} \]

a necessary condition in order to recover the Fierz system (1.15) will be that \( A'_\varphi (\{ a_k \}) \), as can be computed from (2.18), vanish on-shell, and to this end we shall need \textit{exactly} the Fierz-Pauli constraint

\[ \partial \cdot \varphi - \partial \varphi' = 0, \tag{2.68} \]

whereas any other deformation of that condition, of the type

\[ \partial \cdot \varphi - k \partial \varphi' = 0, \tag{2.69} \]

that would come from a different form of the mass term, with \( k \neq 1 \), would not work.

Once it is recognised that the crucial condition to reach is (2.68), the whole remainder of the sequence in \( M_\varphi \) has to be fixed in such a way that the equation

\[ \partial \cdot M_\varphi = 0 \tag{2.70} \]

yield (2.68), \textit{together with all the consistency conditions} coming from the traces of (2.68) itself.

We begin by displaying the strategy in the simpler cases of spin 3 and spin 4, to move then in Section 2.2.3 to the general case.
2.2.1 Spin 3

Since for spin 3 there are no further traces after the first, we assume the Fierz-Pauli mass term, and consider the massive Lagrangian

\[ \mathcal{L} = \frac{1}{2} \phi \{ \mathcal{E}_\phi - m^2 (\phi - \eta \phi') \}. \]  

(2.71)

Taking a divergence, and thereafter a trace of the corresponding equation of motion

\[ \mathcal{E}_\phi - m^2 (\phi - \eta \phi') = 0, \]  

(2.72)

we obtain first

\[ \partial \cdot \phi' = 0, \]  

(2.73)

and then, as desired, (2.68). These two conditions imply, in this case, that the trace of the Fronsdal tensor \( \mathcal{F} \)

\[ \mathcal{F}' = 2 \Box \phi' - 2 \partial \cdot \partial \phi + \partial \partial \cdot \phi', \]  

(2.74)

vanishes on-shell, together with the trace of the “elementary” Ricci tensor (2.9), \( \mathcal{F}_2' \), that always contains at least one trace of \( \mathcal{F} \), as it is obvious from (2.23).

Given the form (2.12) of the general candidate Ricci tensor for spin 3, that we report here for simplicity

\[ A_{\phi}(a_1) = \mathcal{F}_2 + a_1 \frac{\partial^2}{\Box} \mathcal{F}_2', \]  

(2.75)

the conclusion is that \textit{whatever geometric Einstein tensor we choose in} (2.71), after the implementation of the Fierz-Pauli constraint the resulting equation will anyway be

\[ \mathcal{F}_2 - m^2 (\phi - \eta \phi') = 0, \]  

(2.76)

whose trace implies \( \phi' = 0 \), then \( \partial \cdot \phi = 0 \), and finally the Klein-Gordon equation, given that under these conditions the full Ricci tensor \( \mathcal{F}_2 \) reduces to \( \Box \phi \). It is already possible to appreciate the special role played by (2.68): in order to make \( \mathcal{F}' \) vanish, we \textit{need} the condition

\[ \Box \phi' - \partial \cdot \partial \cdot \phi = 0, \]  

(2.77)

which can only be a consequence of (2.68), and cannot be derived from any other different relation of proportionality between \( \partial \cdot \phi \) and \( \partial \phi' \).

2.2.2 Spin 4

For spin \( s \geq 4 \), as previously discussed, we consider reasonable to include in the mass term further traces of the field. We assume then, for \( s = 4 \), the Lagrangian

\[ \mathcal{L} = \frac{1}{2} \phi \{ \mathcal{E}_\phi - m^2 M_\phi \}, \]  

(2.78)

where, in general,

\[ M_\phi = \phi + a \eta \phi' + b \eta^2 \phi''. \]  

(2.79)

Again, we would like to fix the coefficients in the mass term so that, on-shell,

\[ A_\phi'(a_1, a_2) = (1 + a_1) \mathcal{F}_2' + (3 a_1 + a_2) \frac{\partial^2}{\Box} \mathcal{F}_2'' = 0, \]  

(2.80)
at least for some choices of $a_1$ and $a_2$. On the other hand, from the explicit form of $\mathcal{F}_2$

$$\mathcal{F}_2 = \Phi - \frac{1}{3} \nabla^2 \Phi' + \frac{1}{12} \nabla^4 \Phi'',$$

(2.81)

it is possible to see that $\mathcal{A}_x'$ starts with $\mathcal{F}'$ together with terms containing at least one divergence of $\mathcal{F}'$. As a consequence of this fact, in $\mathcal{F}'$

$$\mathcal{F}' = 2 \nabla \Phi' - 2 \partial \partial \cdot \Phi' + \partial \partial \cdot \Phi' + \partial^2 \Phi'',$$

(2.82)

the first two terms cannot be compensated by anything in the remainder of $\mathcal{A}_x'$. This means that, in order for the program to be realised, the combination

$$\nabla \Phi' - \partial \partial \cdot \Phi'$$

has to be expressible in terms of higher traces and divergences of $\Phi'$, as a consequence of the equations of motion. This kind of condition, in turn, is implemented by the Fierz-Pauli constraint, and would not hold if the constraint had the more general form (2.69) with $k \neq 1$.

If we then assume to have fixed the coefficients $a$ and $b$ so that $\partial \partial \cdot M\Phi = 0$ implies (2.68), the following consequences can be shown to hold:

$$\mathcal{F} = \Phi - \Phi^2 \Phi', \quad \mathcal{F}' = 3 \Phi^2 \Phi'', \quad \mathcal{F}_2 = \Phi - 3 \Phi^4 \Phi'', \quad \mathcal{F}_2' = 5 \Phi^4 \Phi'^{[3]},$$

(2.84)

where in particular the last one guarantees that, for spin 4, $\mathcal{F}_2' = 0$. This has the consequence that $\mathcal{A}_x'(a_1, a_2) = 0, \forall a_1, a_2$, and consequently any Lagrangian equation will be reduced on-shell to the form

$$\mathcal{F}_2 - m^2 M\Phi = 0,$$

(2.85)

It is not difficult to find that the right choice of $a$ and $b$ to guarantee that $\partial \partial \cdot M\Phi = 0$ imply (2.68), together with its consistency condition $\partial \partial \cdot \Phi' = - \partial \Phi''$ is

$$M\Phi = \Phi - \eta \Phi' - \eta^2 \Phi''.$$

(2.86)

Taking first a double trace and then a single trace of (2.85) we find in this way $\Phi'' = 0$ and then $\Phi' = 0$, which once again ensure that the Fierz system is recovered.

### 2.2.3 Spin $s$

In the general case, we look for a quadratic deformation of the geometric Lagrangians giving rise to equations of motion of the schematic form

$$\mathcal{E}_\Phi - m^2 M\Phi = 0,$$

(2.87)

where $\mathcal{E}_\Phi$ is a generic member in the class of divergence-free Einstein tensors recalled in Section 2.1.3. Again, all traces of $\Phi$ are expected to contribute to $M\Phi$, so that, for $s = 2n$ or $s = 2n + 1$, it can generally be written as

$$M\Phi = \Phi + b_1 \eta \Phi' + b_2 \eta^2 \Phi'' + \ldots + b_k \eta^k \Phi^{[k]} + \ldots + b_n \eta^n \Phi^{[n]}.$$

(2.88)

The same argument seen for spin 4 applies also in this case: we look for coefficients $b_1 \ldots b_n$ such that $\mathcal{A}_x'$ vanishes on-shell, as a consequence of $\partial \partial \cdot M\Phi = 0$. Given that no choice of the
coefficients in $M_\varphi$ exists such that $\partial \cdot M_\varphi = 0$ implies $\mathcal{F}' = 0$ altogether, for the reasons discussed in the previous Section we are led to recover the Fierz-Pauli constraint (2.68), as a necessary condition to relate the first two terms of $\mathcal{F}'$ with the remainder of $A'_\varphi$.

To this end we look for coefficients $b_1, \ldots, b_n$ such that the divergence of (2.87) imply (2.68) together with its consistency conditions

\[ \partial \cdot \varphi[k] = -\frac{1}{2k-1} \varphi^{[k+1]}, \quad k = 1 \ldots n. \quad (2.89) \]

By this we mean that, if we write the divergence of $M_\varphi$ in the form

\[ \partial \cdot M_\varphi = \partial \cdot \varphi + b_1 \partial \varphi' + \ldots + \eta^k (b_k \partial \cdot \varphi[k] + b_{k+1} \partial \varphi^{[k+1]}) + \ldots, \quad (2.90) \]

and we define

\[ \mu_\varphi \equiv \partial \cdot \varphi - \partial \varphi', \quad (2.91) \]

then we would like to rearrange (2.90) as

\[ \partial \cdot M_\varphi = \mu_\varphi + \lambda_1 \eta \mu_\varphi' + \ldots + \lambda_k \eta \mu_\varphi^{[k]} + \ldots. \quad (2.92) \]

In this fashion, subsequent traces of (2.92) would imply $\mu_\varphi^{[k]} = 0$, for $k = n, n-1 \ldots$ and then finally $\mu_\varphi = 0$, as desired\textsuperscript{25}. The form of $\mu_\varphi$ immediately fixes the first coefficient to be $b_1 = -1$, whereas consistency with (2.92) requires

\[ \lambda_k = -\frac{b_k}{2k-1}, \quad (2.93) \]

\[ b_{k+1} = \frac{b_k}{2k-1}, \quad (2.94) \]

whose unique solution is

\[ b_{k+1} = -\frac{1}{(2k-1)!!}. \]

The following relations are then fulfilled, on-shell:

\[ \mathcal{F} = \Box \varphi - \partial^2 \varphi', \quad \mathcal{F}' = 3 \partial^2 \varphi'', \]
\[ \mathcal{F}_2 = \mathcal{F} - 3 \partial^4 \varphi'', \quad \mathcal{F}_2' = 5 \partial^4 \varphi^{[3]}, \]
\[ \mathcal{F}_3 = \mathcal{F}_2 - 5 \partial^6 \varphi^{[3]}, \quad \mathcal{F}_3' = 7 \partial^6 \varphi^{[4]}, \quad (2.95) \]
\[ \ldots \]
\[ \mathcal{F}_n = \mathcal{F}_{n-1} - (2n - 1) \partial^{2n} \varphi^{[n]}, \quad \mathcal{F}_n' = (2n + 1) \partial^{2n} \varphi^{[n+1]}. \]

The relevant point in this series of equations is that the Fierz-Pauli constraint implies that $\mathcal{F}_n' = 0$ for $s = 2n$, $s = 2n + 1$, and consequently $A'_\varphi = 0$ for any Ricci tensor defined in (2.18). This means that any Lagrangian equation reduces on-shell to the form

\[ \mathcal{F}_n - m^2 (\varphi - \eta \varphi' - \ldots - \frac{1}{(2s-3)!!} \eta^k \varphi^{[k]} \ldots) = 0. \quad (2.96) \]

\textsuperscript{25}This of course given that the coefficients $\lambda_k$ do not imply any identical cancellations among the traces of $\partial \cdot M_\varphi$.\textsuperscript{22}
As usual, subsequent traces of (2.96) imply that \( \varphi^{[k]} = 0 \) \( \forall k = n, n-1, \ldots 1 \). Finally the Fierz system is recovered by observing that, as a consequence of the recursion relation

\[
\mathcal{F}_n = \Box \varphi - \sum_{k=1}^{n+1} (2k - 1) \frac{\partial^{2k}}{\Box^k} \varphi^{[k]},
\]

(2.97)

the tensors \( \mathcal{F}_n \) reduce to \( \Box \varphi \), once all traces of \( \varphi \) are set to zero.

The final result is that consistent massive Lagrangians describing a spin-\( s \) boson are given by

\[
\mathcal{L} = \frac{1}{2} \varphi \{ \mathcal{E}_{\varphi} - m^2 M_{\varphi} \},
\]

(2.98)

with \( \mathcal{E}_{\varphi} \) any of the Einstein tensors constructed from the Ricci tensors (2.18), and where

\[
M_{\varphi} = \varphi - \eta \varphi' - \eta^2 \varphi'' - \frac{1}{3} \eta^3 \varphi''' - \cdots - \frac{1}{(2k - 3)!!} \eta^k \varphi^{[k]} - \cdots,
\]

(2.99)

is the generalised Fierz-Pauli mass term, for arbitrary integer spin.

3 **Fermions**

3.1 **Geometry for higher-spin fermions**

3.1.1 **Fermionic curvatures**

To describe fermions in a geometrical fashion, it is possible to reproduce the construction of the hierarchy of connections sketched in Section 2.1.1, with the only modification that the fundamental field

\[
\psi_{\mu s} \equiv \psi_{\mu_1 \ldots \mu_s},
\]

(3.1)

be understood as carrying a spinor index as well. We consider this field subject to the second of the transformations laws (1.15), that in symmetric notation reads

\[
\delta \psi = \partial \epsilon,
\]

(3.2)

but with an *unconstrained* parameter \( \epsilon \).

The whole construction then amounts to a rephrasing of the bosonic case [33], the main result being that one can define for a rank-\( s \) spinor-tensor the generalised connections

\[
\Gamma_{\mu_1 \ldots \mu_m}^{(m)} \psi_{\nu s} = \sum_{k=0}^{m} \frac{(-1)^k}{m!} \partial_{\nu}^{m-k} \partial_{\mu}^{k} \psi_{\mu_1 \ldots \mu_{s-k}} \psi_{\nu s},
\]

(3.3)

whose gauge transformations are

\[
\delta \Gamma_{\mu_1 \ldots \mu_m}^{(m)} \psi_{\nu s} = (-1)^m (m + 1) \partial_{\nu}^{m+1} \epsilon_{\mu_1 \ldots \mu_{s-k}} \psi_{\nu s}.
\]

(3.4)

A fully gauge invariant tensor is first reached at the \( s \)-th step, and it is called a “curvature” for fermionic gauge fields:

\[
\mathcal{R}_{\mu s, \nu s} = \sum_{k=0}^{s} \frac{(-1)^k}{s!} \partial_{\nu}^{s-k} \partial_{\mu}^{k} \psi_{\mu k \nu s},
\]

(3.5)

\[26\] In the following, whereas this would not be source of confusion, and in order to simplify the language, we shall refer to the spinor-tensors \( \psi \) loosely as “tensors”.
3.1.2 Generalised Dirac tensors

In analogy with the bosonic case, we would like to make use of the curvatures (3.5) to construct generalised “Dirac-Rarita-Schwinger” tensors sharing the symmetries of the field \( \psi \). If we insist that these tensors have the dimensions of a first-order relativistic wave operator it is unavoidable to introduce non-localities, in the same fashion already reviewed for bosons in Section 2.1.2, where in particular “order-zero” candidate Ricci tensors were uniquely defined by eq. (2.14) requiring their degree of singularity to be the lowest possible. In the fermionic case, however, more possibilities are allowed, and a more refined analysis is needed to give an exhaustive description of the linear theory, and in particular to uncover the geometric meaning of the unconstrained equations proposed in [4, 6].

From the technical viewpoint the basic novelty is that, while for bosons the only ways to saturate indices are provided by traces and divergences, in the fermionic case we can also take \( \gamma \)-traces. In order to keep the degree of singularity as low as possible, a first definition of generalised Dirac tensors can then be obtained starting with the Ricci tensors (2.14), interpreted as functions of \( \psi \), and replacing divergences with \( \gamma \)-traces, while also formally acting with the operator \( \partial / \Box \) in the case of even rank, as summarised in the following table:

| Curvature | spin | “Ricci” | spin | “Dirac” |
|-----------|------|---------|------|---------|
| \( R_0 \sim \psi \) | 0 | \( \Box R_0 \) | \( 1/2 \) | \( \emptyset R_0 \equiv D_0 \) |
| \( R_1 \sim \partial \psi \) | 1 | \( \partial \cdot R_1 \) | \( 3/2 \) | \( R_1 \equiv D_1 \) |
| \( R_2 \sim \partial^2 \psi \) | 2 | \( R_2' \) | \( 5/2 \) | \( \emptyset \Box R_2' \equiv D_2 \) (3.6) |
| \( R_3 \sim \partial^3 \psi \) | 3 | \( \frac{1}{\Box} \partial \cdot R_3' \) | \( 7/2 \) | \( \frac{1}{\Box} R_3' \equiv D_3 \) |
| \( R_4 \sim \partial^4 \psi \) | 4 | \( \frac{1}{\Box^2} R_4'' \) | \( 9/2 \) | \( \frac{1}{\Box^2} \partial \cdot R_4'' \equiv D_4 \) |

The possibility of interchanging divergences and \( \gamma \)-traces can also be used to replace one Lorentz-trace with a \( \gamma \)-trace together with a divergence, according to the formal substitution

\[
\eta_{\mu\nu} = \frac{1}{2} \{ \gamma_\mu, \gamma_\nu \} \rightarrow \partial_\mu \gamma_\nu \,,
\]

while still ensuring that the total number of derivatives at the numerator be odd. Whereas for the case of odd rank the corresponding tensors would be more singular than the ones defined in (3.6), and for this reason we neglect them as a first choice, in the case of even rank only the minimum number of inverse powers of \( \Box \) is needed to restore dimensions, and in this sense the tensors defined in this manner are \emph{a priori} equivalent candidates for the description of the dynamics:

| \( R_2 \sim \partial^2 \psi \) | 2 | \( R_2' \) | \( 5/2 \) | \( \frac{1}{\Box} \partial \cdot R_2 \equiv D_2 \) |
| \( R_4 \sim \partial^4 \psi \) | 4 | \( \frac{1}{\Box^2} R_4'' \) | \( 9/2 \) | \( \frac{1}{\Box^2} \partial \cdot R_4'' \equiv D_4 \) (3.8) |
It is then clear that, in the fermionic case, keeping the singularity of the candidate tensors as low as possible it is not a sharp enough criterion to allow the identification of a unique geometric theory. Rather, when the rank of $\psi$ is even, say $2n$, the most general candidate has the form

$$D_{2n}(a_{2n}) = a_{2n} i D_{2n} + (1 - a_{2n}) i \bar{D}_{2n}. \tag{3.9}$$

We would like to clarify the meaning of this lack of uniqueness in the definition of the basic tensors, and in particular to explain the role played in this context by the gauge-invariant, unconstrained, non local tensors proposed in [4, 6], defined for spin $s = 2n + \frac{1}{2}$ and $s = 2n + \frac{3}{2}$ by the following recursion relations:

$$S_{n+1} = \frac{1}{n(2n+1)} \partial^2 \Box S_n' - \frac{2}{2n+1} \partial \cdot S_n, \tag{3.10}$$

where

$$S_1 \equiv S \tag{3.11}$$

is the Fang-Fronsdal tensor (1.12).

Moreover, for the generalised tensors (3.9) one should also discuss the basic consistency issue of compatibility with the Fang-Fronsdal theory, namely that the postulated equation of motion

$$D_{2n}(a_{2n}) = 0, \tag{3.12}$$

imply a compensator-like equation of the form

$$S = 2 i \partial^2 K_{\psi}(a_{2n}), \tag{3.13}$$

where $K_{\psi}(a_{2n})$ should be a non-local tensor shifting as $\epsilon$ under the transformation $\delta \psi = \partial \epsilon$, in order to compensate the unconstrained gauge variation of the Fang-Fronsdal tensor (1.12)

$$\delta S = -2 i \partial^2 \epsilon'. \tag{3.14}$$

In the remainder of this Section we shall give an answer to the first question, making the geometrical meaning of (3.10) explicit. The main tool we shall resort to will be the comparison between the gauge transformations (3.4) of the connections defined in Section 3.1.1 and those of the kinetic tensors $S_{n+1}$ defined in (3.10) (which are not gauge-invariant, if $s > 2n + \frac{3}{2}$).

In the next Section we shall discuss the role played by these tensors, under the criterion that (3.12) be deducible from a Lagrangian. For the subclass of tensors meeting this requirement it will be easy to show consistency with the Fang-Fronsdal theory.

To begin with, let us observe that no ambiguity manifests itself in the odd-rank case, where it is possible to show that (3.6) and (3.10) actually coincide:

$$S_n = i D_{2n-1}. \tag{3.15}$$

In order to clarify this identity it is useful to compare the gauge transformations (3.4) of the de Wit-Freedman connections with those of the tensors $S_n$,

$$\delta S_n = -2 i n \partial^{2n} n^{-1} \epsilon^{[n-1]}, \tag{3.16}$$

Barring a possible overall normalisation.
and to observe that, if the same gauge transformation is implemented by one of the tensors \( S_n \) and one connection \( \Gamma \), suitably modified in order for indices and dimensions to match, we can infer that these two quantities actually define the same tensor. Indeed, they could only differ by gauge invariant quantities, but by construction the kinetic tensors, as well as the connections, do not contain gauge-invariant “sub-tensors”.

This justifies the following identification:

\[
\delta S_n = \frac{i}{\Box n-1} F^{(2n-1)[n-1]} \quad \rightarrow \quad S_n = \frac{i}{\Box n-1} F^{(2n-1)[n-1]}. \tag{3.17}
\]

This last equality makes it clear that it is only in the odd-rank case that the tensors (3.10) can be given a straightforward geometric interpretation. Indeed, considering the first value of \( n \) such that the two tensors are gauge-invariant, (3.17) automatically reduces to (3.15).

On the other hand, for the even-rank case, the kinetic tensors (3.10) can be expressed as linear combinations of the geometric ones defined in (3.6) and (3.8). For instance, in the case of spin \( s = \frac{5}{2} \), writing all three tensors involved in terms of the Fang-Fronsdal tensor \( S \),

\[
\begin{align*}
iD_2 &= S + \frac{\partial^2}{\Box} S' - \frac{\partial}{\Box} \cdot S, \\
i\hat{D}_2 &= S - \frac{1}{2} \frac{\partial}{\Box} \cdot S, \\
S_2 &= S + \frac{1}{3} \frac{\partial^2}{\Box} S' - \frac{2}{3} \frac{\partial}{\Box} \cdot S,
\end{align*}
\tag{3.18}
\]

it is easy to realize that \( S_2 \) is just a member of the class defined in (3.9) corresponding to the case \( a_2 = 1/3 \).

To make the general validity of this observation explicit, let us recall the recursive definition of the connections [33]:

\[
\Gamma^{(m)}_{\sigma \rho m-1, \mu s} = \partial_\sigma \Gamma^{(m-1)}_{\rho m-1, \mu s} - \frac{1}{m} \partial_\mu \Gamma^{(m-1)}_{\rho m-1, \sigma \mu s-1}. \tag{3.19}
\]

If \( m = 2n \), this last equality relates \( \Gamma^{(2n)} \) to \( \Gamma^{(2n-1)} \), which, in its turn, can be related to the tensors \( S_n \), according to (3.17). As a consequence, the following equalities hold:

\[
\begin{align*}
i \frac{\Box}{\Box n} \partial \cdot F^{(2n)[n-1]} &= S_n - \frac{1}{2n} \frac{\partial}{\Box} \cdot S_n, \\
i \frac{\Box}{\Box n} \Gamma^{(2n)}[n] &= S_n - \frac{1}{2n} \frac{\partial}{\Box} \cdot S_n. \tag{3.21}
\end{align*}
\]

and using the generalised Bianchi identities verified by \( S_n \) [4, 6],

\[
\partial \cdot S_n - \frac{1}{2n} \partial S_n' - \frac{1}{2n} \partial \cdot S_n = i \frac{\partial^{2n}}{\Box^{n-1}} \psi[n], \tag{3.22}
\]

we can rewrite (3.21) as follows:

\[
i \frac{\Box}{\Box n} \Gamma^{(2n)[n]} = S_n - \frac{\partial}{\Box} \cdot S_n + \frac{1}{n} \frac{\partial^2}{\Box} S_n' + i(2n+1) \frac{\partial^{2n+1}}{\Box^n} \psi[n]. \tag{3.23}
\]
Next, looking for a combination of (3.20) and (3.23) such as to reproduce $S_{n+1}$, modulo the term in $\psi^{[n]}$, we find

$$
\frac{i}{2n + 1} \left[ \frac{\square}{\Box^n} \Gamma^{(2n)} [n] \right] + \frac{2n}{2n + 1} \left[ \frac{i}{\Box^n} \partial \cdot \Gamma^{(2n)} [n-1] \right] = S_{n+1} + i \frac{\partial^{2n+1}}{\Box^n} \psi^{[n]}.
$$

(3.24)

If $s = 2n + \frac{1}{2}$ then $\psi^{[n]}$ is not present, and the tensors on the l.h.s. just reproduce $D_{2n}$ and $\hat{D}_{2n}$ respectively; this proves that $S_{n+1}$ is a linear combination of the form (3.9) according to

$$
S_{n+1} = \frac{i}{2n + 1} D_{2n} + i \frac{2n}{2n + 1} \hat{D}_{2n}.
$$

(3.25)

To summarise, the definition of non-local, least singular, generalised Dirac tensors is unique only for odd-rank spinor-tensors, and is given by (3.15). For even-rank spinor-tensors we have in principle a one-parameter family of candidates given by (3.9), and the kinetic tensors (3.10), first introduced in [4, 6], are just one specific member in this family, as indicated by (3.25).

A higher degree of non-uniqueness could also be considered if, in analogy with the discussion of the bosonic case of Section 2.1.2, we take into account the possibility of introducing more singular contributions, involving further $\gamma-$ traces of the tensors given by (3.9) and (3.15). By including in the definition of the $D_n (a_n)$ the odd-rank case, with the corresponding coefficient always to be chosen as $a_{2n+1} = 1$, we can define in general the fermionic analogue of (2.18) by

$$
W_\psi (a_n ; \{c_k, d_k\}) = D_n (a_n) + \frac{1}{\Box} (c_1 \partial \\square \rho_n (a_n) + d_1 \partial^2 D_n^\rho (a_n)) + \ldots
$$

$$
+ \frac{1}{\Box^k} (c_k \partial^{2k-1} \\square \rho_n^{[k-1]} (a_n) + d_k \partial^{2k} D_n^{[k]} (a_n)) + \ldots.
$$

(3.26)

Finally, of course, the question remains whether one or more representatives among the whole family of generalised Dirac tensors could play any special role. As we shall see in the next Section, the requirement that the equation

$$
D_n (a_n) = 0,
$$

(3.27)

could be derived from a Lagrangian introduces a great simplification in the full description, but still does not imply the selection of a unique representative among the $D_n (a_n)$.

### 3.1.3 Geometric Lagrangians

We now look for a Lagrangian derivation of the equation

$$
D_n (a_n) = 0,
$$

(3.28)

with $D_n (a_n)$ defined in (3.9) for even $n$, and in (3.15) for odd $n$, where $n$ is the rank of $\psi$.

In the odd-rank case we already know the solution, since the Einstein tensors for (3.15) were constructed in [4, 6], and will be recalled later. In the even-rank case, the one in which a true ambiguity exists, the final outcome will be that only two tensors, among the infinitely many defined in (3.9), can be used to write a gauge-invariant Lagrangian.
For instance, in the first non-trivial case of spin $s = 5/2$, if we try to construct a divergenceless Einstein tensor from (3.9) in the form
\[ G_{a_2} (k, \lambda) = D_2 (a_2) + k \gamma \mathcal{D}_2 (a_2) + \lambda \eta D'_2 (a_2), \] (3.29)

it is possible to verify that the condition
\[ \partial \cdot G_{a_2} (k, \lambda) \equiv 0, \] (3.30)

admits, together with the known solution given by [4, 6]
\[ G_{1/3} (-1/4, -1/4) = S_2 - \frac{1}{4} \gamma S - \frac{1}{4} \eta S', \] (3.31)

only a second solution, namely
\[ G_1 (0, -1/2) = D_2 - \frac{1}{2} \eta D'_2. \] (3.32)

With hindsight, the existence of this second possibility is not surprising, since the algebraic properties of $D_2 = \frac{\partial}{\Box} R'_2$ are the same as for the corresponding bosonic tensor, and then, in this case, we could have expected to find the fermionic counterpart of the linearised Einstein tensor of Gravity.

Maybe it might be less clear what is the obstruction for the other equations in (3.28) to be derived from a Lagrangian, but indeed there is a simple algebraic reason, that can be explained looking at the full general case.

In order to find solutions to the equation\(^{28}\)
\[ \partial \cdot G_{a_{2n}} ([k_i, \lambda_i]) \equiv 0, \] (3.33)

where
\[ G_{a_{2n}} ([k_i, \lambda_i]) = D_{2n} (a_{2n}) + \sum_i [k_i \gamma \eta^{i-1} \mathcal{D}^{[i-1]}_{2n} (a_{2n}) + \eta^i \lambda_i D^{[i]}_{2n} (a_{2n})], \] (3.34)

we must look for coefficients $a_{2n}, k_1, \ldots, k_{n-1}, \lambda_1, \ldots, \lambda_n$ such to imply the needed chain of cancellations among equivalent tensorial structures. On the other hand, in (3.33) an isolated contribution will appear of the form
\[ \partial \cdot G_{a_{2n}} ([k_i, \lambda_i]) \sim \eta^{n-1} \gamma \partial \cdot \mathcal{D}^{[n-1]}_{2n} (a_{2n}), \] (3.35)

that consequently must vanish identically, thus defining a linear equation in the coefficient $a_{2n}$ alone, that can admit at most one solution. This solution can only correspond to the Einstein tensors generated from (3.10) (since we know that such a solution exists), that we report here in the general case of any rank [4, 6]
\[ G_n = S_n + \sum_{0 < p \leq n} \frac{(-1)^p}{2^p p!} \eta^{p-1} \left[ \eta S^{[p]}_n + \gamma S^{[p-1]}_n \right]. \] (3.36)

The only possible exception to this argument could be that the term
\[ \eta^{n-1} \gamma \mathcal{D}^{[n-1]}_{2n} (a_{2n}), \] (3.37)

\(^{28}\)We reintroduce the label $2n$ to stress that the following considerations refer to the even-rank case.
is not present at all in \( \mathcal{G}_{2n} (\{ k_i, \lambda_i \}) \), as it is the case if the coefficient \( k_{n-1} \) is chosen to be zero. This is possible, but then the argument can be iterated backwards, and leads to the conclusion that all \( k_i \) coefficients must vanish, that is to say no bare \( \gamma \)'s should appear in \( \mathcal{G}_{2n} (\{ k_i, \lambda_i \}) \). At that point (3.33) would be of the form

\[
\partial \cdot \mathcal{G}_{2n} (\{ k_i \equiv 0, \lambda_i \}) = \partial \cdot D_{2n} + \lambda_1 \partial D'_{2n} + \eta (\lambda_1 \partial \cdot D'_{2n} + \lambda_2 \partial \cdot D''_{2n}) + \ldots ,
\]

(3.38)

where it is important to notice that the two contributions contained in \( D_{2n} \) must undergo separate cancellations. Indeed, from (3.9)

\[
D_{2n} (a_{2n}) = a_{2n} D_{2n} + (1 - a_{2n}) \hat{D}_{2n} ,
\]

(3.39)

it is possible to appreciate that the bosonic-like contribution given by \( D_{2n} \) cannot be used to compensate terms in \( \hat{D}_{2n} \sim \partial \cdot \mathcal{R}^{[n-1]}_2 \), because of the terms in another only present in this second tensor. This means that, for instance, the following cancellations should occur simultaneously, if the Einstein tensor has to be divergenceless:

\[
a_{2n} \{ \partial \cdot D_{2n} + \lambda_1 \partial D'_{2n} \} = 0 ,
\]

(3.40)

while more generally (3.38) splits into two series of independent conditions, one for \( D_{2n} \) and its traces, and another one for \( \hat{D}_{2n} \) and its traces.

Now, whereas the first equation admits a known solution, given by \( \lambda_1 = -\frac{1}{2n} \) [4, 6], (as can be deduced from the bosonic identities (2.16)), it is possible to check that the second equation in (3.40) actually admits no solutions at all, as for instance can be verified explicitly if \( s = \frac{5}{7} \), from the expression of \( \hat{D}_2 \) given in (3.18).

The conclusion of this analysis is that the ambiguity in the definition of Dirac tensors in the even-rank case contained in (3.9) actually persists at the level of the construction of Lagrangians, but simplifies to only two options. One option is given by the tensors (3.10), that also provide the unique solution in the odd-rank case, and whose geometrical meaning is encoded in (3.15) and (3.25). The other possibility, which is competitive with the first one only in the even-rank case, consists in simply reinterpretting the bosonic field in (2.15) as carrying a spinor index as well, thus getting (3.8).

We expect these ambiguities, together with the full degeneracy appearing when the more singular possibilities (3.26) are considered, to disappear when the coupling with an external current is turned on, in the same fashion discussed for bosons in [13]. Nonetheless, this analysis has not yet been performed, and is left for future work. As we shall see, this will not prevent us from analysing the mass deformation of the fermionic Lagrangian in its full generality.

Before turning to the analysis of the massive case we shall give an argument to show that the Lagrangian equations of this Section imply the compensator equations (3.13). We do this for the equations given by \( \mathcal{G}_{n+1} = 0 \), with \( \mathcal{G}_n \) defined in (3.36). The same argument, with minor modifications, also applies to the other option described in this Section (and to the bosonic tensors (2.15) as well).

The Lagrangian equations defined by (3.36) can be easily shown to imply \( \mathcal{S}_{n+1} = 0 \). This equation in turn, using (3.10) and the Bianchi identity (3.22) can be cast it in the form

\[
\mathcal{S}_n + a_n \frac{\partial}{\partial \hat{S}_n} \hat{S}_n + b_i \frac{\partial^2}{\partial S'_{[n]} + c_n i \frac{\partial^{2n+1}}{\partial \psi^{[n]}} \psi^{[n]} = 0 .
\]

(3.41)
Computing the $\gamma$-trace of (3.41) in order to express $S_n$ as the gradient of a tensor, it is possible to rewrite it in the form

$$S_n = 2i \partial^2 K_n.$$ \hspace{1cm} (3.42)

This procedure can be iterated by making repeated use of (3.10) and (3.22). For instance at the second step the result looks

$$S_{n-1} = 2i \partial^2 \{K_{n-1} + K_n\},$$ \hspace{1cm} (3.43)

while after $n - 1$ iterations one would find the desired expression

$$S = 2i \partial^2 K_\psi,$$ \hspace{1cm} (3.44)

in which all non-localities are in the pure-gauge term $K_\psi$, and can then be eliminated using the trace of the gauge parameter, reducing in this way the non-local dynamics to the local Fang-Fronsdal form.

Given the formal consistency of these theories at the free massless level, in the next Section we shall investigate the possibility of finding a proper quadratic deformation that would extend their meaning to the massive case.

### 3.2 Mass deformation

In this Section we wish to reproduce and adapt to the fermionic case the results discussed in Section 2.2 concerning the massive phase of the bosonic theory. Since the basic ideas and the methodology strictly resemble what we already discussed in that context, here the presentation will be more concise.

The Fierz-Pauli constraint (2.68) was the building-block of the full construction in the bosonic case. Hence, the first piece of relevant information we have to obtain is about the analogous condition for fermions.

As was the case for the Proca theory, briefly recalled in the Introduction, the analysis of its direct counterpart, the massive Rarita-Schwinger theory, only furnishes an incomplete information. For spin $\frac{3}{2}$ indeed, the massive deformation of the geometric Lagrangian is

$$L = \frac{1}{2} \bar{\psi} \{S - \frac{1}{2} \gamma \ S + m (\psi - \gamma \ \psi') \} + h.c.,$$ \hspace{1cm} (3.45)

whose equation of motion can be easily reduced to the Fierz system (1.1), via the basic on-shell condition that it implies

$$\partial \cdot \psi - \partial^\prime \psi = 0.$$ \hspace{1cm} (3.46)

Nonetheless, this constraint would not be the correct one in the general case, and to uncover the full Fierz-Pauli constraint for fermions it is necessary to analyse the example of spin $\frac{5}{2}$.

#### 3.2.1 Spin 5/2

Let us consider the two different geometric formulation defined by (3.31) and (3.32). Starting from (3.31), we write a tentative massive equation in the generic form

$$S_2 - \frac{1}{4} \gamma S_2 - \frac{1}{4} \eta S_2' - m (\psi + a \gamma \ \psi' + b \eta \psi') = 0,$$ \hspace{1cm} (3.47)
where we have included all possible terms in the definition of $M_\psi$. We look for coefficients $a$ and $b$ such that
\[ \partial \cdot M_\psi = 0 \rightarrow S_2 = 0, \tag{3.48} \]
which is a necessary conditions in order to recover the Fierz system (1.1). From the explicit form of $S_2$
\[ S_2 = \frac{4}{3} (\partial \cdot \psi - \psi \partial \cdot \psi - \psi \partial \partial \cdot \psi), \tag{3.49} \]
we can see that actually if we could get (3.46) then $S_2$ would vanish. On the other hand, from
the divergence of the mass term
\[ \partial \cdot M_\psi (a, b) = \partial \cdot \psi + a \psi + a \gamma \partial \partial \cdot \psi + b \partial \psi', \tag{3.50} \]
we see that the only way to get rid of $\gamma \partial \cdot \psi$ in this expression is to show that it vanishes. The
computation of the $\gamma$-trace of $\partial \cdot M_\psi (a, b)$
\[ \gamma \cdot (\partial \cdot M_\psi (a, b)) = [1 + a (D + 2)] \partial \cdot \psi + (b - a) \psi' = 0, \tag{3.51} \]
makes it manifest that the desired condition is only achieved if $a = b$, which already tells us that
the mass term cannot have the same form as in the spin-$\frac{3}{2}$ case. Some further manipulations
allow to conclude that to get rid of $S_2$ the following condition is needed:
\[ \partial \cdot \psi - \psi \partial \partial \cdot \psi - \psi' = 0, \tag{3.52} \]
which actually will represent the fermionic Fierz-Pauli constraint for any spin. The same condition allows to write a consistent massive theory starting from the alternative geometric option
found in Section 3.1.2 and given for this case by the tensor $D_2$ in (3.6). Actually, it is possible
to show that (3.52) imply the following consequences
\[ i D_2 = i \left( \partial \psi - \partial \cdot \psi - \partial \partial \cdot \psi' \right), \tag{3.53} \]
\[ i D_2 = i \left( \partial \partial \cdot \psi' + 2 \partial \partial \cdot \psi'' \right). \]
It is simple to conclude that the consistency condition (3.52) can be obtained in this case by
choosing the mass term in the form
\[ M_\psi = \psi - \gamma \partial \partial \cdot \psi - \eta \psi'. \tag{3.54} \]

3.2.2 Spin 7/2

The basic observation, to be stressed once again, is that whereas generality suggests that all
possible $\gamma$-traces of the field enter the mass term $M_\psi$, this criterion should not be applied directly
to the Fierz-Pauli constraint (3.52), that instead we want to be reproduced without modification.
The main reason which can be given at this level is that otherwise it would not be possible to
obtain the condition that $S_2 = 0$ on-shell. In order to find the proper generalisation of $M_\psi$
consistent with (3.52) we then to look for a combination
\[ M_\psi (a, b, c) = \psi + a \gamma \psi + b \eta \psi' + c \eta \gamma \psi', \tag{3.55} \]
such that its divergence could be cast in the form

\[
\partial \cdot M_\psi (a, b, c) = \partial \cdot \psi - \hat{\partial} \psi - \partial \psi'
+ \lambda_1 \gamma \cdot (\partial \cdot \psi - \hat{\partial} \psi - \partial \psi')
+ \lambda_2 \eta (\partial \cdot \psi - \hat{\partial} \psi - \partial \psi')',
\]

(3.56)

This will guarantee that the divergence of the Lagrangian equation will produce (3.52) among its consequences. Since in the following Section we shall give a complete treatment of the general case, here we just report the result

\[
M_\psi = \psi - \gamma \psi' - \eta \gamma \psi'.
\]

(3.57)

### 3.2.3 Spin \( s + 1/2 \)

In the general case, in order to derive the Fierz system (1.1) from a quadratic deformation of the geometric Lagrangians, we look for a linear combination \( M_\psi \) of \( \gamma \)-traces of \( \psi \) such that, starting from

\[
\mathcal{L} = \bar{\psi} \{ \mathcal{E}_\psi - m M_\psi \} + h.c.,
\]

(3.58)

the divergence of the corresponding equations of motion will imply the Fierz-Pauli constraint for fermions

\[
\mu_\psi \equiv \partial \cdot \psi - \hat{\partial} \psi - \partial \psi' = 0,
\]

(3.59)

together with its consistency conditions

\[
\mu_\psi^{[n]} = - [(2n + 1) \partial \cdot \psi^{[n]} + \partial \psi^{[n+1]}] = 0,
\]

(3.60)

\[
\mu_\psi^{[n]} = - [(2n - 1) \partial \cdot \psi^{[n]} + \hat{\partial} \psi^{[n]} + \partial \psi^{[n+1]}] = 0.
\]

(3.61)

Let us consider then the general linear combination of \( \gamma \)-traces of \( \psi \) written in the form

\[
M_\psi = \psi - \sum_{j=0}^{\lfloor \frac{s}{2} \rfloor} a_{2j+1} \gamma \eta^j \psi^{[j]} - \sum_{i=1}^{\lfloor \frac{s}{2} \rfloor} a_{2i} \eta^i \psi^{[i]},
\]

(3.62)

and let us compute its divergence

\[
\partial \cdot M_\psi = \partial \cdot \psi - \sum_{j=0}^{\lfloor \frac{s}{2} \rfloor} a_{2j+1} \{ \hat{\partial} \eta^j \psi^{[j]} + \gamma \eta^j \partial \psi^{[j]} + \gamma \eta^j \partial \psi^{[j]} \}
\]

(3.63)

\[
- \sum_{i=1}^{\lfloor \frac{s}{2} \rfloor} a_{2i} \{ \eta^{i-1} \partial \psi^{[i]} + \eta^i \partial \psi^{[i]} \}.
\]

In order for it to be rearranged as

\[
\partial \cdot M_\psi = \mu_\psi + \lambda_1 \gamma \hat{\mu}_\psi + \lambda_2 \eta \mu_\psi' + \ldots + \lambda_{2k} \eta^k \mu_\psi^{[k]} + \lambda_{2k+1} \gamma \eta^k \hat{\mu}_\psi^{[k]} \ldots,
\]

(3.64)

the solution for the coefficient \( a_k \) is unique and has the form

\[
a_{2k+2} = \frac{1}{(2k-1)!!},
\]

\[
a_{2k+3} = \frac{1}{(2k+1)!!},
\]

(3.65)
so that the generalised Fierz-Pauli mass-term for fermions is\(^{30}\)

\[
M_\psi = \psi - \sum_{j=0}^{[s-1]} \frac{1}{(2j-1)!!} \gamma \eta^j \psi^{[j]} - \sum_{i=1}^{[\frac{3}{2}]} \frac{1}{(2i-3)!!} \eta^i \psi^{[i]}.
\]  

(3.66)

Again, the equation \(\partial \cdot M_\psi = 0\) implies that all \(\gamma\)-traces of \(\mu_\psi\), and then \(\mu_\psi\) itself, vanish on-shell. This in turn implies the following consequences

\[
\begin{align*}
S_1 &= i (\bar{\psi} \psi - \bar{\psi} \psi) , \\
S_2 &= S_1 - i (\frac{\partial^2}{\Box} \bar{\psi} \psi' + 3 \frac{\partial^3}{\Box} \psi'''), \\
&\quad \ldots \\
S_{n+1} &= S_n - i ((2n-1) \frac{\partial^{2n}}{\Box^n} \bar{\psi} \psi^{[n]} + (2n+1) \frac{\partial^{2n+1}}{\Box^{n+1}} \psi^{[n+1]}), \\
S_{n+1} &= i (2n+1) \frac{\partial^{2n+1}}{\Box^{n+1}} \psi^{[n+1]}. 
\end{align*}
\]  

(3.67)

In particular from the equation for \(S_{n+1}\) we deduce that any Ricci tensor of the form

\[
S_{n+1} + b_1 \gamma S_{n+1} + \ldots ,
\]  

(3.68)

reduces to \(S_{n+1}\), for \(s = 2n + \frac{1}{2}\) and \(s = 2n + \frac{3}{2}\), and then the Lagrangian equations of motion

\[
\mathcal{E}_\psi - m M_\psi = 0 ,
\]  

(3.69)

reduce to

\[
S_{n+1} - m M_\psi = 0 .
\]  

(3.70)

As usual, successive traces of this equation allow to conclude that all \(\gamma\)-traces of \(\psi\) vanish on-shell, which in turn implies, because of the recursive relations (3.67), that \(S_{n+1} = \bar{\psi} \psi\), and the Fierz system (1.1) is finally recovered.

As already stressed for the example of spin \(s = 5/2\), no conceptual differences are present in the construction of the massive theory for the tensors \(D_{2n}\) defined in (3.6), which were shown in Section 3.1.3 to constitute an alternative possibility for the formulation of the dynamics in the even-rank case. So for instance, in the spin \(s = 9/2\) case the Fierz-Pauli constraint implies the following analogues of the relations given in table (3.67):

\[
D_4 = D_2 + i (-3 \frac{\partial^3}{\Box} \psi' + 10 \frac{\partial^5}{\Box^2} \psi'' + 3 \frac{\partial^4}{\Box^2} \bar{\psi} \psi'') ,
\]  

(3.71)

showing that, under the constraint (3.59), the trace of \(D_4\) vanishes, and the usual argument leading to (1.1) can be applied.

If, on the one hand, this observation might be taken as an argument in favour of the correctness of this kind of description of the massive theory, it must be also admitted that it raises for fermions an issue of uniqueness, of the same kind as the one already observed for bosons.

We shall propose an interpretation of this open point in the following Section, focusing on the bosonic case.

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\(^{30}\)Just for the sake of keeping the formula compact, we are defining here \((-1)!! = 1\).
4 THE ISSUE OF UNIQUENESS

4.1 Setting of the problem

The construction of geometric theories for higher-spin fields proposed in [4, 6, 13], and reviewed in Sections 2.1.2 and 2.1.3 does not produce a unique answer, in the sense that infinitely many gauge-invariant Lagrangians are actually available, whose corresponding free equations can all be shown to imply the (Fang-)Fronsdal ones, after a suitable, partial gauge-fixing is performed.

Nonetheless, in [13] it was shown that the analysis of the on-shell behavior of the geometric theories was biased by the absence of couplings. Indeed, turning on even a non-dynamical source, and thus performing a deeper check of the consistency of those Lagrangians against the structure of the corresponding propagators, allows to restore uniqueness, as only one theory was proved to survive this most stringent test.

For the massive Lagrangians proposed in this work a similar issue is to be discussed. Indeed, even if the generalised mass terms found by imposing consistency with the Fierz-Pauli constraint are unique, still they can be used to describe the mass deformations of any of the geometric theories available at the free level, providing massive Lagrangians all implying the Fierz systems (1.1), on-shell.

We thus wonder whether other criteria might suggest a selection principle among those theories, and in particular what is the memory, if any, that the massive deformation keeps of the unique Einstein tensor selected at the massless level.

The most obvious thing to try would be to see whether, even in this case, turning on couplings with external sources could give any indications about the existence of some “preferred” choice. To show that this is not the case, it is sufficient to analyse the example of spin 4. Let us consider the massive, geometric theory for this case, in the presence of an external, conserved current $J$:

$$L = \frac{1}{2} \varphi \{ E_\varphi - m^2 (\varphi - \eta \varphi' - \eta^2 \varphi'') \} - \varphi \cdot J, \quad (4.1)$$

where $E_\varphi$ is a generic member in the class of geometric Einstein tensors recalled in Section 2.1.3. The key point is that conservation of currents guarantees that the equations of motion still imply the Fierz-Pauli constraint (2.68), and then, as discussed in Section 2.2.2 for spin 4, any Lagrangian equation will reduce on-shell to the form

$$\mathcal{F}_2 - m^2 (\varphi - \eta \varphi' - \eta^2 \varphi'') = J, \quad (4.2)$$

with $\mathcal{F}_2$ defined in (2.9). Moreover, again because of (2.68), the structure of $\mathcal{F}_2$ considerably simplifies to (2.97)

$$\mathcal{F}_2 = \Box \varphi - \partial^2 \varphi' - 3 \partial^4 \varphi'', \quad (4.3)$$

where it is to be noted that, in the computation of the current exchange, both the contributions in $\partial^2 \varphi'$ and $\partial^4 \varphi''$ vanish when contracted with a conserved $J$.

This observation, which is easily generalised to all spins by means of (2.95) (and (3.67), if we wish to apply the same argument to fermions), implies that, from the viewpoint of the coupling with conserved sources, the full structure of the geometric part of the equations of motion kind of disappears behind the mass term $M_\varphi$, the only remaining contribution of this sector of the Lagrangian being the term in $\Box \varphi$.
On the other hand this means that, in the computation of the current exchange, the full responsibility of giving the correct propagator is now in the structure of the mass term $M_{\phi}$, thus providing a non-trivial consistency check of its validity, coefficient by coefficient, so to speak. This test gives a positive answer, since for example for the case of spin 4 the result is\(^{31}\)

$$\mathcal{J} \cdot \varphi = \frac{1}{p^2 - m^2} \left\{ \mathcal{J} \cdot \mathcal{J} - \frac{6}{D + 3} \mathcal{J}' \cdot \mathcal{J}' + \frac{3}{(D + 1)(D + 3)} \mathcal{J}'' \cdot \mathcal{J}'' \right\}, \quad (4.5)$$

whose correctness can be checked by comparison with the corresponding computation performed in the local setting [13]. However, for the same reasons, the coupling with external sources in the massive case does not yield any indications at all on the existence of a possible preferred theory.

A different criterion might be suggested by the analogy with the example of spin 2.

We have already observed, at the beginning of Section 2.2, that the origin of the Fierz-Pauli mass term can be traced back to the Kaluza-Klein reduction of the massless theory from $D + 1$ to $D$ dimensions, the very form of the mass term itself being simply encoded in the coefficient of the D'Alembertian operator in the Einstein tensor,

$$\mathcal{R} - \frac{1}{2} \eta \mathcal{R} \sim \Box (h - \eta h') + \ldots. \quad (4.6)$$

On the other hand, it is worth stressing again that, once the mass term is fixed to have the Fierz-Pauli form, the structure of the “ancestor” geometric theory stays “hidden” behind it, in the sense that neither the free equations of motion, nor the computation of the current exchange actually allow to keep memory of it. Thus, for instance, we could consider the following non-local equation for the description of the massive graviton

$$\mathcal{R} - \frac{1}{2} \eta \mathcal{R}' + a (\eta - \partial^2 \Box) \mathcal{R}' - m^2 (h - \eta h') = 0, \quad (4.7)$$

whose geometric part is still in terms of a divergenceless (and gauge-invariant, at the massless level) tensor, which would clearly describe an inconsistent massless theory, to begin with since the current exchange would give in this case a wrong result, for generic real $a$.

Notwithstanding this deficiency, whenever this “massive graviton” is coupled to a conserved current, the implementation of the Fierz-Pauli constraint would still imply that the Ricci scalar vanishes on-shell, and the computation of the massive propagator would then furnish the correct result.

To summarise, for spin 2 (but not only for spin 2) in the massive case the basic information is encoded in the mass term, whose role mainly is to guarantee that the Fierz-Pauli constraint (2.68) be enforced on-shell. This constraint, on the other hand, is strong enough to obscure the detailed structure of the massless sector of the Lagrangian, that appears from this viewpoint of relative importance. Nonetheless, a clear link between the massive theory and the correct massless one can be traced back to the structure of the coefficient of the D'Alembertian operator in the Einstein tensor of the proper geometric theory, which is ultimately responsible for the form of the mass term, upon Kaluza-Klein reduction from $D + 1$ to $D$ dimensions.

\(^{31}\)The corresponding computation for the massless case, using the correct theory, gives

$$\mathcal{J} \cdot \varphi = \frac{1}{p^2} \left\{ \mathcal{J} \cdot \mathcal{J} - \frac{6}{D + 2} \mathcal{J}' \cdot \mathcal{J}' + \frac{3}{D(D + 2)} \mathcal{J}'' \mathcal{J}'' \right\}, \quad (4.4)$$

thus showing the generalisation of the Van Dam-Veltman-Zakharov discontinuity [46], already noticed in [13].
Even if it is not straightforward to establish such a direct link in the non-local setting, because of the inverse powers of \( \frac{1}{m} \to \frac{1}{\Box_{-m}} \) that would appear in the reduction of the non-local theory, but that are not present in our construction, we are anyhow led to conjecture that the generalised Fierz-Pauli mass term (2.99) bears a direct relationship with the correct geometric theory synthetically described by eqs. (2.42) and (2.43), of which it should simply represent the coefficient of the D’Alembertian operator.

We would like to stress that, since the mass terms (2.99) and (3.66) proposed in this work have been found following a path completely independent of any detailed knowledge of the underlying geometric theory, we think it is fair to say that to verify this conjecture would represent a robust check of the internal consistency of the whole construction.

In the following Section we shall build the setup to quantitatively discuss this conjecture. This will allow us to take a closer look at the structure of the explicit solution to the identities (2.43), and to check our hypothesis for the first few cases. Anyway, even if we regard the support provided from these explicit computations as a strong indication of its validity, still we have not yet a proof of the conjecture in its full generality.

4.2 Testing the uniqueness conjecture

We would like to show that the coefficient of the naked D’Alembertian in the Einstein tensor (2.42) has the same form of the generalised mass term (2.99). Namely

\[
\mathcal{A}_\varphi - \frac{1}{2} \eta \mathcal{A}_\varphi' + \eta^2 \mathcal{B}_\varphi \sim \Box M_\varphi + \ldots ,
\]

where

\[
M_\varphi = \varphi - \eta \varphi' - \eta^2 \varphi'' - \ldots - \frac{1}{(2k-3)!!} \eta^k \varphi^{[k]} - \ldots .
\]

We recall that \( \mathcal{A}_\varphi \) has the compensator structure (2.38),

\[
\mathcal{A}_\varphi = \mathcal{F} - 3 \partial^3 \gamma_\varphi ,
\]

and satisfies the two identities (2.43), that we report here for simplicity,

\[
\partial \cdot \mathcal{A}_\varphi - \frac{1}{2} \partial \mathcal{A}_\varphi' \equiv 0 ,
\]

\[
\mathcal{A}_\varphi'' \equiv 0 .
\]

More details on the explicit solution for \( \mathcal{A}_\varphi \) are given in Appendix B.

The tensor \( \mathcal{B}_\varphi \), defined as the solution to the equation

\[
\partial \cdot \{ \mathcal{A}_\varphi - \frac{1}{2} \eta \mathcal{A}_\varphi' + \eta^2 \mathcal{B}_\varphi \} \equiv 0 ,
\]

can be decomposed in terms of a sequence of the form [13]

\[
\mathcal{B} = \mathcal{B}_0 + \eta \mathcal{B}_1 + \ldots + \eta^k \mathcal{B}_k + \ldots \eta^p \mathcal{B}_p ,
\]

with

\[
p = \text{integer part of } \left\{ \frac{s - 4}{2} \right\} \equiv \left\lfloor \frac{p - 4}{2} \right\rfloor .
\]
In this way the condition implied by (4.12) can thus be turned into the system

\[
\begin{align*}
\partial B_0 &= \frac{1}{2} \partial \cdot A'_{\varphi}, \\
\cdots, \\
\partial B_k &= -\frac{k}{k+2} \partial \cdot B_{k-1}, \\
\cdots
\end{align*}
\] (4.14)

The double tracelessness of \( A_{\varphi} \) allows in this way to deduce from the first of (4.14) the set of identities

\[
\partial \cdot B_0^{[k]} = -\frac{1}{2(k+1)} \partial B_0^{[k+1]},
\] (4.15)

which, in turn, give the following relations among the tensors \( B_k \) and the traces of \( B_0 \):

\[
B_k = \frac{1}{2k-1(k+2)!} B_0^{[k]}.
\] (4.16)

These relations, together with the explicit solution\(^{32}\) for \( B_0 \),

\[
B_0 = \frac{1}{2} \sum_{k=0}^{n-1} \frac{1}{2k+1} \{ a_k \frac{2k+1}{2(n-k)} \frac{n+1+k}{n+1-k} + a_{k+1} \frac{n+4k+5}{2(n-k)} + a_{k+2} \} \partial^{2k} \mathcal{F}_{n+1}^{[k+2]},
\] (4.17)

allow to complete the construction of the Einstein tensor in the non-local, geometric case \([13]\).

Let us finally notice that, using (4.16), it is possible to write the conjectured equality (4.8) in the more explicit form

\[
A_{\varphi} - \frac{1}{2} \eta A'_{\varphi} + \sum_{k=0}^{[\frac{s-4}{2}]} \frac{1}{2k} \frac{1}{k!} \eta^{k+2} B_0^{[k]} \sim \Box \{ \varphi - \eta \varphi' - \sum_{k=0}^{[\frac{s-4}{2}]} \frac{1}{(2k+1)!} \eta^{k+2} \varphi^{[k+2]} \},
\] (4.18)

where \( \frac{s-4}{2} = n - 1 \) or \( n - 2 \) depending on whether the spin is \( s = 2n + 2 \) or \( s = 2n + 1 \) respectively.

It is then clear from the form of \( A_{\varphi} \) that the first two terms of \( M_{\varphi} \) are correctly reproduced by \( A_{\varphi} - \frac{1}{2} \eta A'_{\varphi} \), and the non-trivial part of the calculation is to check whether the traces of \( B_0 \) satisfy

\[
B_0^{[k]} = -\frac{2^k k!}{(2k+1)!!} \Box \varphi^{[k+2]} + \ldots.
\] (4.19)

Since we have no explicit formula for the tensors \( \mathcal{F}_{n+1} \) in terms of \( \mathcal{F} \) or \( \varphi \), in order to compute the contribution of \( \Box \varphi^{[k+2]} \) in \( B_0^{[k]} \) we shall not make use of (4.17). Rather, we shall exploit the results collected in Appendix B, about the structure of the tensor \( \gamma_{\varphi} \) in \( A_{\varphi} \) which are relevant to the present calculation.

We would like to stress that we shall not attempt here to uncover the full structure of \( B_0 \). Rather, keeping in mind our present goal, we shall systematically discard contributions involving divergences of the field \( \varphi \) and of its traces, that in our formulas will be collectively gathered under the label “\( \text{irr} \)” to indicate that they are irrelevant for the present purpose.

\(^{32}\)We are correcting here a misprint in the corresponding formula, (4.69), in [13].
4.2.1 Evaluation of $B_0^{[k]}$

The defining equation for $B_0$ is the first of (4.14), that can be written more explicitly as

$$2 \partial B_0 = \partial \cdot F' - 3(\gamma_\varphi + 3 \partial \partial \cdot \gamma_\varphi + 2 \partial^2 \partial \cdot \gamma_\varphi + \partial^2 \gamma'_\varphi + \partial^3 \partial \cdot \gamma'_\varphi),$$

(4.20)

where the general form of $\gamma_\varphi$, given in eq. (B.3) in Appendix B, together with the explicit knowledge of its first coefficient, as can be read from (B.1), allow to express the first two terms in (4.20) as the gradient of a tensor

$$\partial \cdot F' - 3 \gamma_\varphi = \partial \Delta,$$

(4.21)

where

$$\Delta = -3 \sum_{q,l,m} a_{qlm} \partial^{l-1} \partial^m \mathcal{F}^{[q]},$$

(4.22)

with range of variation of the indices in the sum given in (B.4). A further simplification comes from the observation that the basic identity (B.15) satisfied by $\gamma_\varphi$ can be also read as

$$\partial \cdot \gamma_\varphi = \frac{1}{4} (\varphi'' - \partial \gamma'),$$

(4.23)

which, in turn, implies the set of relations

$$\partial \cdot \gamma_\varphi^{[k]} = \frac{1}{2 (k + 1)} (\varphi^{[k+2]} - \partial \gamma^{[k+1]}).$$

(4.24)

As a consequence of (4.22) and (4.24) the equations for $B_0$ can be written in the following, relatively simple, form

$$2 B_0 = \Delta - \frac{9}{4} \varphi'' + \frac{1}{12} \partial^2 \varphi^{[3]} + \frac{3}{2} \partial \gamma'_\varphi - \frac{1}{4} \partial^3 \gamma''_\varphi + \text{irr},$$

(4.25)

whereas for the $k$-th trace of $B_0$ one finds

$$2 B_0^{[k]} = \Delta^{[k]} + a_k \varphi^{[k+2]} + b_k \partial^2 \varphi^{[k+3]} + c_k \partial \gamma^{[k+1]} + d_k \partial^3 \gamma_\varphi^{[k+2]} + \text{irr},$$

(4.26)

with coefficients $a_k - d_k$ recursively defined by the following system

$$a_{k+1} = a_k + b_k + c_k \frac{1}{k+3},$$

$$b_{k+1} = b_k + \frac{1}{k+4} d_k,$$

$$c_{k+1} = c_k \frac{k+2}{k+3} + d_k,$$

$$d_{k+1} = d_k \frac{k+1}{k+4}.$$
whose solution reads

\[ a_k = - \left( 1 + \frac{3k + 5}{2(k + 1)(k + 2)} \right), \]
\[ b_k = \frac{1}{2(k + 1)(k + 2)(k + 3)}, \]
\[ c_k = \frac{3}{2(k + 1)}, \]
\[ d_k = - \frac{3}{2(k + 1)(k + 2)(k + 3)}. \]  

(4.28)

The evaluation of the traces of \( \Delta \) is discussed in Appendix C. Here we only report the result:

\[ \Delta^{[k]} = -3 \sum_{q,l,m,t} \frac{a_{qlm}}{l} \{ \alpha_{k,t} \partial^{l-1-2k+2t} \partial^m \mathcal{F}^{[q+t]} + \beta_{k,t} \partial^{l-2k+2t} \partial^{m+1} \mathcal{F}^{[q+t]} \\
+ \gamma_{k,t} \partial^{l+1-2k+2t} \partial^{m+2} \mathcal{F}^{[q+t]} + i \text{rr} \}, \]

(4.29)

where the coefficients \( \alpha_{k,t}, \beta_{k,t} \) and \( \gamma_{k,t} \) are given by the relations

\[ \alpha_{k,t} = \binom{k}{t}, \]
\[ \beta_{k,t} = 2(t + 1) \binom{k}{t + 1}, \]
\[ \gamma_{k,t} = 2(t + 1)(t + 2) \binom{k}{t + 2}. \]  

(4.30)

Eqs. (4.26), (4.28) and (4.30), together with the coefficients (B.11), (B.12) and (B.13), and also together with the relation

\[ \partial^m \mathcal{F}^{[q]} = \frac{m(m - 1)}{2} \Box^2 \partial^{m-2} \varphi^{[q+1]} + [m(2q - 1) + (q + 1)] \Box \partial^m \varphi^{[q]} + i \text{rr}, \]  

(4.31)

giving the contribution to \( \Box \varphi^{[q]} \) and \( \Box \varphi^{[q+1]} \) from the m-th divergence and the q-th trace of the Fronsdal tensor, collectively represent the solution to our problem, and allow the explicit computation of the contribution in \( \Box \varphi^{[k+2]} \) contained in \( \mathcal{B}^{[k]}_0 \).

By means of those formulas it is possible to check the conjecture up to any desired order (we did it up to spin 11). Nonetheless, it is still to be proved that the sum of all the coefficients gives (4.19) for an arbitrary value of \( k \).

5 Conclusions

In this work we have proposed a Lagrangian description of massive higher-spin fields, based on the massless, unconstrained Lagrangians introduced in [4, 6, 13]. The main results of the present analysis are eqs. (2.99) and (3.66), providing the generalised Fierz-Pauli mass terms for bosons.
and fermions respectively. To the best of our knowledge, this is the first description of massive higher-spin theories which does not involve any auxiliary fields.

The algebraic meaning of the generalised mass terms can be traced to the necessity to recover the Fierz-Pauli constraints (2.68) and (3.52), in a context where neither auxiliary fields are introduced nor algebraic constraints are assumed, that might otherwise help in removing the lower-spin parts from the tensors used in the description.

On the background of this work there is the general motivation to try to investigate the possibility that quantities amenable to a geometric interpretation could play some meaningful role in the theory of higher-spin fields. In order to keep as symmetric as possible the description of integer and half-integer spins, here we also proposed an account of the fermionic theory in which, as for the bosonic case, all quantities of dynamical interest can be defined in terms of curvatures.

The issue of uniqueness, analysed in [13] for the massless theory, is met again in the present treatment, since the generalised Fierz-Pauli mass terms are such that any geometric Lagrangian can be promoted to a consistent massive theory by means of the same quadratic deformation. On the other hand, this appears to be more an algebraic consequence of the strength of the constraints (2.68) and (3.52) rather than a deep issue. In our opinion indeed, as accounted in Section 4, there are indications that the mass term (2.99) (and (3.66), in a possible extension of these arguments to fermions) actually bears a direct relationship with the “preferred” geometric theory, selected in [13] on the basis of the requirement that the coupling with an external source be consistent.

The main drawback of the use of curvatures for the description of higher-spin dynamics is the presence of singularities in the Lagrangian, in the form of inverse powers of the D’Alembertian operator. These possible non-localities might be either the signal that there could be some intrinsic obstacle to such a geometric description, or, more optimistically, the degenerate effect at the linear level of some unusual feature of the full theory.

About this point we can observe that, in the massive equations described in this work, the presence of the singularities plays a milder role. Actually, as can be better appreciated by working in Fourier transform, in the massive equations there are at least no poles on the mass-shell and, in this sense, the generalised Fierz-Pauli mass terms (2.99) and (3.66) provide a kind of “regularisation mechanism” for the non-local, massless theory. As the massless limit is taken, the poles move towards the physical region, but simultaneously the theory develops a gauge symmetry, and once the limit is fully performed, all of them appear to be secluded in a pure-gauge sector of the equations of motion.

Whether this should be taken as an indication that the non-localities are actually harmless is far from obvious. Nonetheless, a stronger argument is given by the possibility of rephrasing all the properties of the geometric, non-local theory in the local setting introduced in [10, 12], (and extended to (A)dS spaces in [13]), which results to be the minimal local setting where the description of higher-spin fields in an unconstrained fashion is at all possible.

In these local Lagrangians the role of the singular terms of the geometric description is replaced by an auxiliary field, that acts as a “compensator”, to balance the gauge variation of the terms in the equations of motion which are usually discarded in the constrained, non-geometric approaches. The “memory” of the singular nature of this field is in the presence of higher derivatives in its kinetic term which again, notwithstanding the possibility of fixing a gauge where this field vanishes, might be interpreted as the signal of something odd in the
general approach.

As an answer to this objection, we have proposed in this work a simple generalisation of the local Lagrangians of [10, 12] in which, at the price of introducing two more fields, for a total of five fields for all spins, any higher-derivative term disappears and the same dynamics of [10, 12], and then, from our viewpoint, the full geometrical picture underlying the linear theory, appears to be described in a completely conventional fashion.

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A Notation and conventions

The space-time metric is the flat, mostly-positive one in dimension $D$. If not otherwise specified, symmetrized indices are always left implicit. In addition, traces are denoted by “primes” or by a number in square brackets: $\varphi'$ is thus the trace of $\varphi$, $\varphi''$ is its double trace and $\varphi^{[n]}$ is the $n$-th trace.

This notation results in an effective calculational procedure, whose basic rules are summarised in a number of identities, which reflect some simple combinatorics. These rest on our convention of working with symmetrized objects not of unit strength, which is convenient in this context but is not commonly used. For instance, given the pair of vectors $A_\mu$ and $B_\nu$, $AB$ here stands for $A_\mu B_\nu + A_\nu B_\mu$, without additional factors of two. The key identities are then:

$$
\begin{align*}
(\partial^p \varphi)' &= \Box \partial^{p-2} \varphi + 2 \partial^{p-1} \partial \cdot \varphi + \partial^p \varphi', \\
\partial^p \partial^q &= \left( \frac{p+q}{p} \right) \partial^{p+q}, \\
\partial \cdot (\partial^p \varphi) &= \Box \partial^{p-1} \varphi + \partial^p \partial \cdot \varphi, \\
\partial \cdot \eta^k &= \partial \eta^{k-1}, \\
(\eta^k \varphi)' &= [D + 2(s+k-1)] \eta^{k-1} \varphi + \eta^k \varphi', \\
(\varphi \psi)' &= \varphi' \psi + \varphi \psi' + 2 \varphi \cdot \psi, \\
\eta \eta^{n-1} &= n \eta^n, \\
\gamma \cdot (\gamma \psi) &= (D + 2s) \psi - \gamma \psi', 
\end{align*}
$$

(A.1)

where in particular in the last equality $\psi$ is a rank-$s$ spinor-tensor. As anticipated, the basic ingredient in these expressions is the combinatorics, which is simply determined by the number of relevant types of terms on the two sides. Thus, for a pair of flat derivatives, $\partial \partial = 2 \partial^2$ reflects the fact that, as a result of their commuting nature, the usual symmetrization is redundant precisely by the overall factor of two that would follow from the second relation. In a similar fashion, for instance, the identity $\eta \eta^{n-1} = n \eta^n$ reflects the different numbers of terms generated by the naive total symmetrization of the two sides: $\left( \frac{n}{2} \right) \times (2n-1)!!$ for the expression on the l.h.s, and $(2n+1)!!$ for the expression on the r.h.s.

B On the explicit form of $A_\varphi$

The form of $\gamma_\varphi$ for the first cases of spin $s = 3, 4, 5, 6$

$$
\begin{align*}
\gamma_3 &= \frac{1}{3} \Box \partial \cdot \mathcal{F}', \\
\gamma_4 &= \gamma_3 - \frac{1}{3} \frac{\partial}{\Box^3} \partial \cdot \mathcal{F}' + \frac{1}{12} \frac{\partial}{\Box^2} \mathcal{F}'' - \frac{1}{12} \frac{\partial^2}{\Box^3} \partial \cdot \mathcal{F}'', \\
\gamma_5 &= \gamma_4 + \frac{2}{5} \frac{\partial^2}{\Box^4} \partial \cdot \partial \cdot \mathcal{F}' - \frac{1}{5} \frac{\partial^2}{\Box^3} \partial \cdot \mathcal{F}'', \\
\gamma_6 &= \gamma_5 - \frac{8}{15} \frac{\partial^3}{\Box^5} \partial \cdot \mathcal{F}' + \frac{2}{5} \frac{\partial^3}{\Box^4} \partial \cdot \mathcal{F}'' - \frac{1}{30} \frac{\partial^3}{\Box^3} \mathcal{F}'''.
\end{align*}
$$

(B.1)
suggests the recursive relation (whose general validity was proven in [13])

$$\gamma_s = \gamma_{s-1} + \partial^{s-3} \left\{ a_1 \frac{1}{s-1} \partial s^{s-2} \mathcal{F}' + \ldots + a_k \frac{1}{s-k} \partial s^{s-2k} \mathcal{F}[k] + \ldots + a_n \left\{ \frac{1}{s+n+r} \partial \cdot \mathcal{F}[n] \right\} \right\} \tag{B.2}$$

where the last two options refers to the cases of spin $s = 2n$ and $s = 2n + 1$, respectively. The crucial point is that one coefficient, regardless of the spin, is uniquely associated to each structure. It is also important that in every term in $\gamma_s - \gamma_{s-1}$ the Fronsdal tensor $\mathcal{F}$ is fully saturated.

The general form of $\gamma_s$, with a more appropriate definition of the coefficients, can be written

$$\gamma_s = \sum_{k,l,m} a_{klm} \frac{\partial^l}{m+k} \partial^m \mathcal{F}[k], \tag{B.3}$$

where

$$k = 1, \ldots, \left\lfloor \frac{s}{2} \right\rfloor, \quad l = 0, \ldots, s - 3, \quad m = l + 3 - 2k, \quad m \geq 0. \tag{B.4}$$

We would like to find $a_{klm}$ for generic values of $k,l,m$.

The idea is to exploit the identities (2.43) satisfied by $\mathcal{A}_\varphi$. As we shall see, they play quite different roles, the main reason being that, in order to deduce from those conditions equations for the coefficients $a_{klm}$, one has to make sure that there are no cancellations between terms in the field $\varphi$ contained in different structures. This is true for the double-tracelessness condition, but it is not true for the Bianchi identity, but for one single exception, that will be anyway important.

Let us also observe that the general strategy will be to consider $\gamma_{s-1}$ as known, and to try and determine all coefficients for the structures in $\gamma_s - \gamma_{s-1}$. This implies for example that our unknowns will always be of the form $a_{k,s-3,s-2k}$.

**Double-tracelessness**

The condition

$$\mathcal{A}_\varphi'' \equiv 0, \tag{B.5}$$

is equivalent to

$$\left( \partial^3 \gamma_s \right)'' \equiv \frac{1}{3} \mathcal{F}'', \tag{B.6}$$

where

$$\left( \partial^3 \gamma_s \right)'' = 4 \Box \partial \cdot \gamma_s + 4 \partial \partial \cdot \partial \cdot \gamma_s + 4 \partial^2 \partial \cdot \gamma_s' + 2 \Box \partial \gamma_s' + \partial^3 \gamma_s'' . \tag{B.7}$$

---

33 Since in this Section we want to keep track of the spin $s$, we switch to a more explicit notation, renaming $\gamma_\varphi \rightarrow \gamma_s$. Remember that $\gamma_s$, referring to the case of spin $s$, is a rank-$(s-3)$ tensor.

34 Here, as in the rest of this Appendix, for the sake of brevity we shall simply call “structure” a generic term in $\gamma_s$ of the form $\partial^l \partial^m \mathcal{F}[k]$. 

43
Computing each term in (B.7) by means of (B.3) and imposing (B.6) leads to the system\(^{35}\)

\[
\begin{align*}
\left(\begin{array}{c}
s \\
3
\end{array}\right) a_{k,s-3,s-2k} + 4 \left(\begin{array}{c}
s -1 \\
3
\end{array}\right) a_{k,s-4,s-2k-1} + 4 \left(\begin{array}{c}
s -2 \\
3
\end{array}\right) a_{k,s-5,s-2k-2} + \\
2 \left(\begin{array}{c}
s -2 \\
3
\end{array}\right) a_{k-1,s-5,s-2k} + 4 \left(\begin{array}{c}
s -3 \\
3
\end{array}\right) a_{k-1,s-6,s-2k-1} + 4 \left(\begin{array}{c}
s -4 \\
3
\end{array}\right) a_{k-2,s-7,s-2k} &= 0.
\end{align*}
\]  
(B.8)

Let us notice that in each term we have one independent variable \((k)\) and one parameter \((s)\); keeping this in mind, we can simplify the notation defining

\[
\left(\begin{array}{c}
s \\
3
\end{array}\right) a_{t,s-3,m} \equiv b_{s,m},
\]  
(B.9)

which is free of ambiguities, since for a given \(s\) there is a one-to-one relation between \(t\) and \(m\). In this notation the system can be written in the more compact form

\[
b_{s,m} + 2 b_{s-2,m} + b_{s-4,m} = -4 \{b_{s-1,m-1} + b_{s-2,m-2} + b_{s-3,m-1}\}.
\]  
(B.10)

In particular the system written as in (B.10) is ready to be solved ‘by traces’, keeping the number of divergences fixed. Of course, it would also be possible to keep the number of traces fixed and to solve the system ‘by divergences’.

There are two kinds of difficulties:

→ the need for an infinite list of initial conditions,
→ the iteration of the r.h.s. of (B.10).

To begin with, we can solve a few cases by making use of the list (B.1) to get the initial conditions. We find in this way the coefficients referring to structures involving zero, one and two divergences\(^{36}\):

\[
a_{s,2,s-3,0} = \frac{(-1)^{\frac{s}{2}}}{s(s-1)},
\]  
(B.11)

\[
a_{s-1,2,s-3,1} = \frac{(-1)^{\frac{s+1}{2}}}{s},
\]  
(B.12)

\[
a_{s-2,2,s-3,2} = (-1)^{\frac{s}{2}+1} \frac{s-2}{2(s-1)}.
\]  
(B.13)

These coefficients will be the ones actually needed to check the conjecture presented in Section 4. As a matter of principle, if we want the coefficient for a structure with a certain fixed (as a number, not as a parameter) number of divergences, we could compute explicitly, by hand, the \(\gamma_s\) where this structure first appears, and use this initial condition to solve the system in general, finding the corresponding solution for any (allowed) \(s\). In this sense, at the price of performing

\(^{35}\)Since the coefficients of each structure in \(\gamma_s\) are spin-independent, and (B.6) must hold for any value of \(s\), the only possibility is that in this relation the coefficients of the various structures all vanish.

\(^{36}\)Here the label "\(s\)" refers to the value of the spin in correspondence of which the structure multiplied by \(a_{klm}\) first appears. We would like to stress once again that every structure appears for the first time for a value of the spin such that the Fronsdal tensor is completely saturated. This implies for example that for \(m = 0\) we must saturate all indices by traces, and then the spin must be even, when the corresponding structure first appears. Similar reasonings explain why for \(m = 1\) and \(m = 2\) there is no need to take the integer parts of \(\frac{s-1}{2}\) and \(\frac{s-2}{2}\) respectively.
one explicit computation up to a certain value of the spin, we are then in the position to find a class of coefficients, as for the cases displayed above.

On the other hand, at least to allow in principle the possibility of finding the general solution in closed form, we have to ask ourselves whether we can find an infinite list of initial data. This is the point where the Bianchi identity helps.

Bianchi identity

From the first of (2.43)

$$\partial \cdot A_\varphi - \frac{1}{2} \partial A'_\varphi \equiv 0,$$

we find the following condition on $\gamma_s$

$$\varphi'' = 4 \partial \cdot \gamma_s + \partial \gamma'_s,$$

which can be regarded as the solution to the problem of inverting $F' (\varphi, \varphi', \varphi'')$ w.r.t. $\varphi''$. In order to better explain its meaning, we would like to stress two points:

→ the identity (B.15), that could naively look in some sense expected (because of the gauge transformation of $\varphi''$), actually represents a non trivial relation, given that, as stressed in the footnote at page 10, there are infinitely many other $\gamma$’s such that $\delta \gamma = \Lambda'$, none of which would satisfy it.

→ Moreover, (B.15) makes it manifest that in the Bianchi identity cancellations must occur among different structures; indeed, the very presence of a term in the naked double trace of $\varphi$ will necessarily imply a chain of compensations among the various structures.

This last observation can be easily made more concrete via a specific example. Consider the case of spin 6, and use the form of $\gamma_6$ given in (B.1) to compute the two contributions to (B.15):

\[
\begin{align*}
4 \partial \cdot \gamma_6 &= \frac{4}{15} \partial \partial \cdot 3 \mathcal{F}' - \frac{8}{15} \partial^2 \partial \cdot 4 \mathcal{F}' + \frac{1}{3} \mathcal{F}'' \\
&\quad - \frac{7}{15} \partial \partial \cdot \mathcal{F}'' + \frac{4}{5} \partial^2 \partial \cdot 2 \mathcal{F}'' - \frac{2}{15} \partial^2 \mathcal{F}''', \\
\partial \gamma'_6 &= -\frac{4}{15} \partial \partial \cdot 3 \mathcal{F}' + \frac{8}{15} \partial^2 \partial \cdot 4 \mathcal{F}' + \frac{3}{10} \partial \partial \cdot \mathcal{F}'' \\
&\quad - \frac{2}{3} \partial^2 \partial \cdot 2 \mathcal{F}'' + \frac{1}{10} \partial^2 \mathcal{F}'''.
\end{align*}
\]

We can see that in general the single structures in (B.15) will not have vanishing coefficients. For example the term $\frac{4}{3} \mathcal{F}''$ in $4 \partial \cdot \gamma_6$ contains the only term in $\varphi''$ “naked” of the whole r.h.s. of (B.15), to be matched with the l.h.s. of the same expression. On the other hand it also contains contributions in $\partial \cdot \varphi''$ and $\varphi'''$ that will cancel because of analogous terms contained in other structures. For this reason, (B.14) could not have been used to derive a system like (B.10).

It is then remarkable that structures involving one trace of $\mathcal{F}$ exactly compensate each other. This is a general result, and it is due to the fact that a structure of the form $\partial \cdot m \mathcal{F}'$ contains a contribution in $\partial \cdot m+2 \varphi$ that cannot be compensated by anything else in (B.15), given that $\mathcal{F}^{[k]}$ contains at least $\varphi^{[k-1]}$, $\forall k \geq 2$. This means that it is possible to use the Bianchi identity to find an equation for structures involving one trace of $\mathcal{F}$, together with an arbitrary number
of divergences. The solutions to this equation would provide exactly the list of initial conditions we need for the system (B.10). We determine in this way a new class of coefficients:

\[ a_{1,s-3,s-2} = (-1)^{s+1} \frac{2^s}{4s(s-1)}, \]

allowing in principle to find the general solution to (B.10), for an arbitrary number of traces and divergences of the corresponding structure. Even if such a general solution so far is not known, the coefficients explicitly determined are the only ones needed for the discussion of the relation between the geometric Einstein tensor (2.42) and the generalised Fierz-Pauli mass term (2.99).

## C Evaluation of \( \Delta^{[k]} \)

Starting from the general form of \( \Delta \), given in (4.22)

\[ \Delta = -3 \sum_{q,l,m} \frac{a_{qlm}}{l} \partial^{l-1} \partial^m \mathcal{F}[q], \]  
(C.1)

the computations of the \( k \)-th trace will generate contributions containing, for a given coefficient \( a_{qlm} \), all powers of gradients from \( l-1 \) to \( l-1-2k \). We have to take into account the general result (4.31) for an arbitrary trace and an arbitrary divergence taken on the Fronsdal tensor, that we report here for simplicity

\[ \partial^m \mathcal{F}[q] = \frac{m(m-1)}{2} \Box^2 \partial^{m-2} \varphi^{[k+1]} + [m(2q-1) + (q+1)] \Box \partial^m \varphi[q] + \text{irr}, \]  
(C.2)

showing that, for our present purposes, all multiple divergences of \( \mathcal{F} \) with \( m \geq 3 \) can be classified as irrelevant, since they will contain at least one divergence of \( \varphi \) and thus cannot contribute to \( \Box \varphi^{[k+2]} \) in (4.19). Under this condition, it is possible to observe that the various contribution to \( \Delta^{[k]} \) for a given \( a_{qlm} \) display a “reflection symmetry”, as can be appreciated for instance in the explicit example of \( \Delta^{[4]} \)

\[ \Delta^{[4]} = -3 \sum_{q,l,m} \frac{a_{qlm}}{l} \left\{ \partial^{l-9} \partial^m \mathcal{F}[q] + 8 \partial^{l-8} \partial^{m+1} \mathcal{F}[q] + 4 \partial^{l-7} \partial^m \mathcal{F}[q+1] + 
\right. 
\left. 24 \partial^{l-7} \partial^{m+2} \mathcal{F}[q] + 24 \partial^{l-6} \partial^{m+1} \mathcal{F}[q+1] + 6 \partial^{l-5} \partial^m \mathcal{F}[q+2] + 
\right. 
\left. 48 \partial^{l-5} \partial^{m+2} \mathcal{F}[q+1] + 24 \partial^{l-4} \partial^{m+1} \mathcal{F}[q+2] + 4 \partial^{l-3} \partial^m \mathcal{F}[q+3] + 
\right. 
\left. 24 \partial^{l-3} \partial^{m+2} \mathcal{F}[q+2] + 8 \partial^{l-2} \partial^{m+1} \mathcal{F}[q+3] + \partial^{l-1} \partial^m \mathcal{F}[q+4] \right\} + \text{irr} \]  
(C.3)

in the sense that, for a fixed number of divergences, there is coincidence between the coefficient of the term in \( \mathcal{F}[q+t] \) and the coefficient of \( \mathcal{F}[q+4-t] \). Because of this symmetry we can limit ourselves to the computation of the coefficients for \( t \leq \left\lfloor \frac{k+1}{2} \right\rfloor \). To clarify the notation, we introduce three kinds of coefficients, according to the following table:

\[ \alpha_{k,t} \rightarrow \text{terms with no extra divergences, and } t \text{ extra traces} \rightarrow \partial^m \mathcal{F}[q+t], \]  
(C.4)

\[ \beta_{k,t} \rightarrow \text{terms with one extra divergence, and } t \text{ extra traces} \rightarrow \partial^{m+1} \mathcal{F}[q+t], \]  
(C.4)

\[ \gamma_{k,t} \rightarrow \text{terms with two extra divergences, and } t \text{ extra traces} \rightarrow \partial^{m+2} \mathcal{F}[q+t], \]  
(C.4)
and write $\Delta^{[k]}$ according to the formula

$$
\Delta^{[k]} = -3 \sum_{q, l, m, t} \frac{a_{q, l, m}}{l} \left\{ \alpha_{k, t} \partial^{l-1-2k+2t} \partial^m \mathcal{F}^{[q+t]} + \beta_{k, t} \partial^{l-2k+2t} \partial^{m+1} \mathcal{F}^{[q+t]} + \gamma_{k, t} \partial^{l+1-2k+2t} \partial^{m+2} \mathcal{F}^{[q+t]} + \text{irr} \right\},
$$

where

$$
q = 1, \ldots, \left[ \frac{s}{2} \right],
$$

$$
l = 1, \ldots, s - 3,
$$

$$
m = l + 3 - 2q,
$$

$$
t = 0 \ldots k.
$$

It is then possible to show that the coefficients satisfy the recursive system

$$
\begin{align*}
\alpha_{k, t} &= \alpha_{k-1, t} + \alpha_{k-1, t+1}, \\
\beta_{k, t} &= \beta_{k-1, t} + \beta_{k-1, t-1} + 2\alpha_{k-1, t}, \\
\gamma_{k, t} &= \gamma_{k-1, t} + \gamma_{k-1, t-1} + 2\beta_{k-1, t},
\end{align*}
$$

whose solution is

$$
\begin{align*}
\alpha_{k, t} &= \frac{1}{t!} k \left( k - 1 \right) \ldots (k - t + 1) = \binom{k}{t}, \\
\beta_{k, t} &= \frac{2}{t!} k \left( k - 1 \right) \ldots (k - t + 1) (k - t) = 2(t + 1) \binom{k}{t+1}, \\
\gamma_{k, t} &= \frac{2}{t!} k \left( k - 1 \right) \ldots (k - t) (k - t - 1) = 2(t + 1)(t + 2) \binom{k}{t+2}.
\end{align*}
$$
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