Sampling theorems for the Heisenberg group

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Abstract

In the first part of the paper a general notion of sampling expansions for locally compact groups is introduced, and its close relationship to the discretisation problem for generalised wavelet transforms is established. In the second part, attention is focussed on the simply connected nilpotent Heisenberg group $H$. We derive criteria for the existence of discretisations and sampling expansions associated to lattices in $H$. Analogies and differences to the sampling theorem over the reals are discussed, in particular a notion of bandwidth on $H$ will figure prominently. The main tools for the characterisation are the Plancherel formula of $H$ and the theory of Weyl-Heisenberg frames. In the last section we compute an explicit example.

1 Introduction

Let us first introduce some notation and definitions. $G$ denotes a locally compact topological group, with left Haar measure $\mu_G$, and associated $L^2$-space $L^2(G)$. The left regular representation $\lambda_G$ acts on $L^2(G)$ by left translation. For a function $g$ on $G$, $g^*$ is defined as $g^*(x) = g(x^{-1})$. Representations of $G$ are understood to be strongly continuous and unitary.

In this paper we wish to discuss the existence of sampling theorems analogous to the famous Whittaker-Shannon-Kotelnikov Theorem. The setting we study is group-theoretic: Given a discrete subset $\Gamma \subset G$, we wish to reconstruct certain functions $g \in L^2(G)$ from their values sampled at $\Gamma$. We formalise this in the following notion.

Definition 1.1 Let $G$ be a locally compact group, $\Gamma \subset G$. Let $\mathcal{H} \subset L^2(G)$ be a leftinvariant closed subspace of $L^2(G)$ consisting of continuous functions. We call $\mathcal{H}$ a sampling space (with respect to $\Gamma$) if it has the following two properties:

(i) There exists a constant $c_{\mathcal{H}} > 0$, such that for all $f \in \mathcal{H},$

$$\sum_{\gamma \in \Gamma} |f(\gamma)|^2 = c_{\mathcal{H}} \|f\|_2^2.$$ 

In other words, the restriction mapping $R_{\Gamma} : \mathcal{H} \ni f \mapsto (f|_\Gamma) \in \ell^2(\Gamma)$ is a scalar multiple of an isometry.
(ii) There exists $S \in H$ such that every $f \in H$ has the expansion

$$f(x) = \sum_{\gamma \in \Gamma} f(\gamma) S(\gamma^{-1} x) ,$$

(1)

with convergence both in $L^2$ and uniformly.

The function $S$ from condition (ii) is called **sinc-type function**. \(\square\)

The definition is modelled after the following, prominent example:

**Example 1.2** [Whittaker, Shannon, Kotel’nikov] Let $G = \mathbb{R}$, $\Gamma = \mathbb{Z}$ and

$$H = \{ f \in L^2(\mathbb{R}) : \text{supp}(\hat{f}) \subset [-0.5, 0.5] \} .$$

Then $H$ is a sampling subspace, with

$$S(x) = \text{sinc}(x) = \frac{\sin(\pi x)}{\pi x} .$$

\(\square\)

One of the aspects of the Whittaker-Shannon-Kotel’nikov sampling theorem has not been covered by Definition 1.1: The sampling expansion (1) can be read as interpolating between values of $g$ at $\Gamma$. The original sampling theorem allows to interpolate arbitrary $\ell^2$-sequences by elements of $H$, i.e., the restriction map is **onto**. We do not require this for sampling subspaces. It will become apparent in the second part of the paper that the Heisenberg group allows a variety of sampling spaces, but none with arbitrary interpolation.

Sampling spaces will be discussed in connection with notions coming from wavelet theory, such as admissible vectors and tight frames. A system $(\eta_i)_{i \in I}$ of vectors in a Hilbert space $H$ is a **tight frame** if the **coefficient operator**

$$T : H \ni \varphi \mapsto (\langle \varphi, \eta_i \rangle)_{i \in I} \in \ell^2(I)$$

(2)

is isometric up to a constant, or equivalently, if

$$\varphi = \frac{1}{c} \sum_{i \in I} \langle \varphi, \eta_i \rangle \eta_i ,$$

holds for every $\varphi \in H$. The sum converges (unconditionally) in the norm. A tight frame is called **normalised** if the constant $c$ equals 1. It is clear that a tight frame is total in $H$. The following proposition collects some basic used facts about tight frames. The facts and their proofs are widely known and just repeated for the sake of completeness.

**Proposition 1.3** Let $(\eta_i)_{i \in I} \subset H$ be a tight frame with frame constant $c$.

(a) If $H' \subset H$ is a closed subspace and $P : H \rightarrow H'$ is the projection onto $H'$, then $(P \eta_i)_{i \in I}$ is a tight frame of $H'$ with frame constant $c$.

(b) Suppose that $c = 1$. Then $(\eta_i)_{i \in I}$ is an orthonormal basis iff $\| \eta_i \| = 1$, for all $i \in I$.

(c) If $\| \eta_i \| = \| \eta_j \|$, for all $i, j \in I$, then $\| \eta_i \|^2 \leq c$. 

2
(d) For an arbitrary system \((\phi_i)_{i \in I} \subset \mathcal{H}\) define the coefficient operator \(T\) analogously to (3), with maximal possible domain, i.e. \(\text{dom}(T) = \{ z \in \mathcal{H} : (z,\phi_i)_{i \in I} \in \ell^2(I) \}\) (which may be trivial). Then \(T\) is a closed operator.

(e) \((\eta_i)_{i \in I}\) is an orthonormal basis iff \(c = 1\) and the coefficient operator is onto.

**Proof.** Part (a) follows from the fact that on \(\mathcal{H}'\) the coefficient map associated to \((P\eta_i)_{i \in I}\) coincides with the coefficient map associated to \((\eta_i)_{i \in I}\). The “only if”-part of (b) is clear. The “if”-part follows from

\[
1 = \|\eta_i\|^2 = \sum_{i \in I} |\langle \eta_i, \eta_j \rangle|^2 = 1 + \sum_{i \neq j} |\langle \eta_i, \eta_j \rangle|^2,
\]

whence \(\langle \eta_i, \eta_j \rangle\) vanishes for \(i \neq j\). Part (c) follows from a similar argument. The proof of part (d) is a straightforward application of the Cauchy-Schwarz inequality. The “only if” part of (e) is obvious. For the converse let \(\delta_i \in \ell^2(I)\) be the Kronecker delta at \(i\). Then \(\langle T^*\delta_i, \varphi \rangle = \langle \delta_i, T\varphi \rangle = \langle \eta_i, \varphi \rangle\) for all \(\varphi \in \mathcal{H}\) implies \(T^*\delta_i = \eta_i\), or \(T\eta_i = \delta_i\) (\(T\) is by assumption unitary), which is the desired orthonormality relation. □

In this paper, all frames of interest will be of the form

\[\pi(\Gamma)\eta = (\pi(\gamma)\eta)_{\gamma \in \Gamma},\]

where \(\pi\) is a representation of \(G\) (usually the restriction of \(\lambda_G\) to some left-invariant closed subspace \(\mathcal{H} \subset L^2(G)\)), and \(\Gamma\) a suitable discrete subset. The associated coefficient map is usually called “discrete wavelet transform”.

Let us give a short outline of the paper: We first establish a close connection between the discretisation of (generalised) wavelet transforms and sampling spaces, for general locally compact groups. The discretisation problem for wavelet transforms and the construction of sampling spaces are essentially equivalent. We then address these problems for a concrete example, the three-dimensional Heisenberg group \(\mathbb{H}\). Section 3 contains the basic results on the Heisenberg group which are used in the following. In Section 4 we formulate and discuss our main results. Our discussion aims at exposing analogies as well as differences to the sampling theorem over the reals. In particular, a notion of bandwidth will turn out to play a prominent role. Sections 5 through 7 contain the proofs of the main results. The main technical devices used for the proofs are the Plancherel formula for the Heisenberg group and the theory of Weyl-Heisenberg frames. In Section 8 we illustrate our techniques by explicitly computing a sinc-type function on \(\mathbb{H}\). An appendix contains notations and results used in connection with the Plancherel formula, concerning Hilbert-Schmidt operators, direct integrals and the like.

For locally compact abelian groups, the paper by Klusváné [14] is considered to contain the definitive form of the sampling theorem. In chapter 10 of [12], which covers these results, extensions to nonabelian groups are mentioned as a natural but difficult question. To our knowledge, the paper by Dooley [6] is the only source dealing with sampling expansions for nonabelian groups. As was observed in [14, chapter 10], a straightforward generalisation of Klusváné’s approach is not possible, since the dual lattice, which is the central device in the abelian case, is not available in the context of nonabelian groups (the dual is not a group). However Dooley’s results indicated that the Plancherel formula of a nonabelian group could be used to study sampling expansions, and our paper provides further evidence.
The initial purpose of this paper was to investigate whether the Plancherel formula could be used for the discretisation of continuous wavelet transforms. It was shown recently in [1, 9, 10] that Plancherel theory provides a natural framework for a unified treatment of continuous wavelet transforms and Wigner functions, and extending this framework to discretisation problems seems to be an attractive project. Moreover, the similarity between the discretisation problem and the Shannon sampling theorem has been observed, more or less explicitly, by various authors, most notably by Feichtinger and Gröchenig in their series of papers (see the bibliography of [11]). The approach via the Plancherel formula allows to address this analogy explicitly. We chose the Heisenberg group mainly as a test case which should provide some orientation for more general settings.

2 Discretised wavelet transforms and sampling expansions

In this section we establish a close connection between sampling expansions and the problem of discretising generalised wavelet transforms.

**Definition 2.1** Let \((\pi, \mathcal{H}_\pi)\) be a representation of \(G\). \(\eta \in \mathcal{H}_\pi\) is called **admissible** if the coefficient mapping

\[
V_\eta : \varphi \mapsto V_\eta(\varphi) ,
\]

where

\[
V_\eta(\varphi)(x) = \langle \varphi, \pi(x)\eta \rangle ,
\]

is an isometry \(\mathcal{H}_\pi \to L^2(G)\). The operator \(V_\eta\) is then called **(generalized) continuous wavelet transform**, it intertwines \(\lambda_G\) with \(\pi\). If an admissible vector exists, \(\pi\) is called **square-integrable**. The admissibility condition implies the following reconstruction formula (to be read in the weak sense)

\[
\phi = \int_G V_\eta(\varphi) \pi(x)\eta \, d\mu_G(x) .
\]

Note that there exist various definitions of square-integrable representations in the literature. Perhaps the most common one is that \(\pi\) is in the **discrete series**, i.e., \(\pi\) is irreducible and has admissible vectors. Square-integrable representations, as defined here, are (equivalent to) subrepresentations of the regular representation; the converse does not hold in general, see [10]. If \(\pi\) is the restriction of \(\lambda_G\) to a leftinvariant subspace, then \(V_\eta\phi = \phi \ast \eta^*\). For groups with a type I regular representation there exist precise criteria for square-integrability which employ the Plancherel measure of such groups, see [10]. For the Heisenberg group they are formulated in Theorem 3.4.

The connection between generalised wavelet transforms and sampling spaces is realised via the space \(V_\eta(\mathcal{H}_\pi)\). The space is characterised by the **reproducing kernel relation**

\[
f = f \ast V_\eta\eta ,
\]

which follows by elementary computation from the isometry property of \(V_\eta\). The convolution kernel \(S = V_\eta\eta\) is a **(right) selfadjoint convolution idempotent** in \(L^2(G)\), i.e. it fulfills \(S = S \ast S = S^*\). Considering subspaces of the form \(\mathcal{H} = V_\eta(\mathcal{H}_\pi)\), for admissible \(\eta\), is equivalent to studying selfadjoint convolution idempotents in \(L^2(G)\). Indeed, given such an
idempotent $S$, it is straightforward to check that $f \mapsto f * S$ is the orthogonal projection onto a closed, leftinvariant subspace $\mathcal{H} = L^2(G) * S$. With suitable identifications this space is easily seen to be the image of a generalised wavelet transform: Simply pick $S = S * S \in \mathcal{H}$ as admissible vector for the restriction of $\lambda_G$ to $\mathcal{H}$, then the generalised wavelet transform $V_S$ is the inclusion map.

The discretisation problem for a generalised wavelet transform can be phrased as follows: Find a suitable (admissible) $\eta$ and a sampling subset $\Gamma \subset G$, such that $\pi(\Gamma)\eta$ is a tight frame. Now the definition of a sampling space implies that it is enough to ensure that $V_\eta(\mathcal{H}_\pi)$ is a sampling subspace. Indeed, the coefficient operator associated to $\pi(\Gamma)\eta$ factors into the continuous wavelet transform associated to $\eta$, followed by the restriction map $R_\Gamma$. Conversely, discretisations give rise to sampling subspaces, as the following proposition shows.

**Proposition 2.2** Let $(\pi, \mathcal{H}_\pi)$ be a square-integrable representation of $G$, and let $\eta$ be an admissible vector. Assume in addition that $\pi(\Gamma)\eta$ is a tight frame of $\mathcal{H}_\pi$ with frame constant $c_\eta$. Then $\mathcal{H} = V_\eta(\mathcal{H}_\pi)$ is a sampling space, and $S = \frac{1}{c_\eta}V_\eta\eta$ is the associated $\Gamma$-sinc-type function for $\mathcal{H}$.

**Proof.** Clearly $V_\eta(\mathcal{H}_\pi)$ consists of continuous functions. Using the isometry property of $V_\eta$ together with the tight frame property of $\pi(\Gamma)\eta$, we obtain for all $f = V_\eta\phi \in \mathcal{H}$

$$
V_\eta\phi = V_\eta \left( \frac{1}{c_\eta} \sum_{\gamma \in \Gamma} \langle \phi, \pi(\gamma)\eta \rangle \pi(\gamma)\eta \right)
= \sum_{\gamma \in \Gamma} \frac{1}{c_\eta} V_\eta \phi(\gamma) V_\eta(\pi(\gamma)\eta) = \sum_{\gamma \in \Gamma} f(\gamma) S(\gamma^{-1} \cdot),
$$

with convergence in $\| \cdot \|_2$. In order to obtain uniform convergence, we find that for $g = V_\eta\psi \in \mathcal{H}$,

$$
|f(x) - g(x)| = |\langle \phi - \psi, \pi(x)\eta \rangle| \leq \|\phi - \psi\| \|\eta\| = \frac{1}{c_\eta} \|f - g\|_2 \|\eta\|,
$$

i.e., $L^2$-convergence entails uniform convergence. □

**Remarks 2.3** 1. The original sampling theorem over the reals can be seen to fit into this setting. If we pick $\eta$ to be the sinc-function, we find that $V_\eta : \mathcal{H} \to L^2(\mathbb{R})$ is just the inclusion map, hence $\eta$ is admissible. Moreover, it is immediately checked on the Fourier transform side, that $(\lambda_R(n)\eta)_{n \in \mathbb{Z}}$ is an ONB of $\mathcal{H}$.

2. The proposition shows that various results on the relation between discrete wavelet or Weyl-Heisenberg systems and continuous ones give rise to sampling theorems: For the wavelet case, the underlying group is the $ax + b$-group. A result by Daubechies [4] ensures that every wavelet giving rise to a frame is in fact an admissible vector, hence we are precisely in the setting of the proposition. Similarly for discrete Weyl-Heisenberg system, where the underlying group is the Weyl-Heisenberg group (a quotient of the Heisenberg group by a discrete central subgroup). Here admissibility of the window function is trivial, as is always the case for irreducible square-integrable representations of unimodular groups. Again the expansion coefficients are sampled values of the windowed Fourier transform, which is the underlying (generalised) continuous wavelet transform. □
Our next aim is to show that, at least for unimodular groups, the discretisation problem and construction of sampling spaces are equivalent problems. We first need an auxiliary result ensuring the existence of sufficiently many convolution idempotents. It immediately follows from [16, Theorem 2.3].

**Proposition 2.4** Let $G$ be unimodular, let $\{0\} \neq \mathcal{H} \subset L^2(G)$ be closed and leftinvariant. Then $\mathcal{H}$ contains a nontrivial selfadjoint convolution idempotent.

The following theorem serves various purposes: First of all it shows that, at least for a large class of settings, the definition of a sampling space is redundant: Property $(ii)$ follows from property $(i)$. Secondly it shows that every sampling space can be obtained from the construction in Proposition 2.2, hence the construction of sampling subspaces and the discretisation problem are (in a somewhat abstract sense) equivalent.

**Theorem 2.5** Assume that $G$ is unimodular. Let $\mathcal{H} \subset L^2(G)$ be a leftinvariant closed subspace consisting of continuous functions, and assume that it has property $(i)$ of a sampling space. Then $\mathcal{H}$ is a sampling subspace. More precisely, there exists a selfadjoint convolution idempotent $S$, such that $\frac{1}{c_H} S$ is the associated sinc-type function, and in addition $\mathcal{H} = L^2(G) * S$. In particular,

$$\forall f \in \mathcal{H}, \forall \gamma \in \Gamma : f(\gamma) = (f, \lambda_G(\gamma) S) ,$$

and thus $\lambda_G(\Gamma) S$ is a tight frame for $\mathcal{H}$. The restriction map $R_\Gamma$ is onto iff $\lambda_G(\Gamma) S$ is an orthonormal basis of $\mathcal{H}$.

**Proof.** Let $R_\Gamma : \mathcal{H} \to \ell^2(\Gamma)$ denote the restriction map, and define

$$S_\gamma = R^*_\Gamma(\delta_\gamma) ,$$

where $\delta_\gamma \in \ell^2(\Gamma)$ is the Kronecker delta at $\gamma$. Then $\frac{1}{c_H} R^*_\Gamma R_\Gamma = \text{Id}_\mathcal{H}$ shows that

$$f = \sum_{\gamma \in \Gamma} f(\gamma) \frac{1}{c_H} S_\gamma , \quad (3)$$

with convergence in the norm. The orthogonal projection $P : \ell^2(\Gamma) \to R_\Gamma(\mathcal{H})$ is given by $P = \frac{1}{c_H} R_\Gamma R^*_\Gamma$. Hence, using polarisation we compute

$$f(\gamma) = (R_\Gamma f, \delta_\gamma) = \frac{1}{c_H} (R_\Gamma f, R_\Gamma R^*_\Gamma \delta_\gamma) = (f, R^*_\Gamma \delta_\gamma) = (f, c_H S_\gamma) . \quad (4)$$

Next pick a maximal family $(\mathcal{H}_i)_{i \in I}$ of nontrivial pairwise orthogonal closed subspaces of the form $\mathcal{H}_i = L^2(G) * S_i$, where the $S_i$ are selfadjoint convolution idempotents in $L^2(G)$. Then Proposition 2.4 implies that

$$\mathcal{H} = \bigoplus_{i \in I} \mathcal{H}_i .$$

Since right convolution with $S_i$ is the orthogonal projection onto $\mathcal{H}_i$, equation [3] implies, for all $f \in \mathcal{H}_i$,

$$\langle f, S_\gamma * S_i \rangle = \langle f * S_i, S_\gamma \rangle = \langle f, S_\gamma \rangle = f(\gamma) = \langle f, \lambda_G(\gamma) S_i \rangle .$$
As a consequence, \( S_\gamma \ast S_i = \lambda_G(\gamma)S_i \). Letting
\[
S = \sum_{i \in I} S_i
\]
we find that
\[
S_\gamma = \sum_{i \in I} S_\gamma \ast S_i = \sum_{i \in I} \lambda_G(\gamma)S_i = \lambda(\gamma)S.
\]
(Note that equation (5) implies that the sum defining \( S \) converges in the norm.) The self-adjointness of the \( S_i \) implies the same property for \( S \). Moreover, for all \( f \in H \),
\[
(f \ast S^*)(x) = \langle f, \lambda_G(x)S \rangle = \langle f, \sum_{i \in I} \lambda_G(x)S_i \rangle = \left( \sum_{i \in I} f \ast S_i \right)(x) = f(x).
\]
Hence \( H = L^2(G) \ast S \). Now (3) and (5) shows that for \( \frac{1}{c_H} \) to be the associated sinc-type function, only the uniform convergence of the sampling expansion remains to be shown. For the latter we note once again that
\[
|f(x) - g(x)| = |\langle f - g, \lambda_G(x)S \rangle| \leq \|S\|_2 \|f - g\|_2,
\]
i.e., norm-convergence entails uniform convergence. The remaining statements of the theorem are obvious.

Proposition 2.6 Let \( G \) be a unimodular group, \( \Gamma < G \) a discrete subgroup and \( H \subset L^2(G) \) a sampling subspace for \( \Gamma \). Then \( \Gamma \) is a lattice, with \( \text{covol}(\Gamma) = \frac{1}{c_H} \).

Proof. A lattice is by definition a discrete subgroup with finite covolume, the latter being defined as \( \text{covol}(\Gamma) = \mu_G(A) \), where \( A \) is a measurable set of representatives of the right coset space \( G/\Gamma \). It is straightforward to check that the definition does not depend on the choice of \( A \). If \( f \in H \) is any nonzero vector, we can therefore compute
\[
\|f\|^2 = \int_A \sum_{\gamma \in \Gamma} |f(\gamma x)|^2 d\mu_G(x) = \int_A c_H \|\lambda_G(x^{-1})f\|^2 d\mu_G(x) = \|f\|^2 c_H \text{covol}(\Gamma).
\]

3 The Heisenberg group

In this section we collect the relevant results concerning the Heisenberg group. The Heisenberg Lie algebra \( \mathfrak{h} \) is a three-dimensional Lie algebra, with basis \( P, Q, Z \) and defining relations
\[
[P, Q] = Z,
\]
all other commutators vanish. It is a one-step nilpotent Lie algebra, and we equip it with the Campbell-Baker-Hausdorff product to make it a Lie group \( \mathbb{H} \). It is a unimodular group, with
the usual Lebesgue measure on $\mathbb{H} \cong \mathbb{R}^3$ as Haar measure. In the coordinates with respect to the above basis, the product reads

$$(p, q, t) \ast (p', q', t') = (p + p', q + q', t + t' + (pq' - qp')/2).$$

The center of $\mathbb{H}$ is given by $Z(\mathbb{H}) = \{(0, 0, t) : t \in \mathbb{R}\}$. We denote the group of topological automorphisms of $\mathbb{H}$ by $\text{Aut}(\mathbb{H})$. There exists a family of irreducible, pairwise inequivalent representations $\rho_h$ ($h \neq 0$) on $L^2(\mathbb{R})$, namely the Schrödinger representations acting via

$$[\rho_h(p, q, t)f](x) = e^{2\pi i h t} e^{2\pi i qx} e^{\pi i hp} f(x + hp).$$

The Schrödinger representations do not exhaust the dual of $\mathbb{H}$, which in addition contains the characters of the abelian factor group $\mathbb{H}/Z(\mathbb{H})$. However, for the decomposition of the regular representation of $\mathbb{H}$, we may concentrate on the Schrödinger representations. This is a consequence of the Plancherel Theorem, which we state next. The proof may be found in [7]. Some notation and definitions in connection with direct integrals and Hilbert-Schmidt operators, which we will use without further comment throughout the paper, can be found in the appendix.

**Theorem 3.1** Define, for $f \in L^2(\mathbb{H}) \cap L^1(\mathbb{H})$ the Fourier transform as the operator field

$$\mathcal{P}(f) := (\rho_h(f))_{h \in \mathbb{R}'} := \left( \int_{\mathbb{H}} f(x) \pi(x) d\mu_{\mathbb{H}}(x) \right)_{h \in \mathbb{R}'}. $$

Then $\mathcal{P}(f)(h) \in B_2(L^2(\mathbb{R}))$, the space of Hilbert-Schmidt operators on $L^2(\mathbb{R})$, and we have the Parseval formula

$$\|f\|_2^2 = \int_{\mathbb{R}'} \|\rho_h(f)\|_{B_2}^2 |h| dh.$$

By continuity this mapping extends to a unitary equivalence

$$\mathcal{P} : L^2(\mathbb{H}) \rightarrow \int_{\mathbb{R}'} B_2(L^2(\mathbb{R})) |h| dh,$$

which is the Plancherel transform on $\mathbb{H}$. It intertwines the two-sided regular representation with the direct integral representation $\int_{\mathbb{R}'} \rho_h \otimes \overline{\rho_h} |h| dh$.

In the following we shall use $\hat{f}$ to denote the Plancherel transform of an $L^2$-function $f$. Besides the decomposition into irreducibles, the Plancherel decomposition provides a simple characterization of leftinvariant operators on $L^2(\mathbb{H})$. We remark that the following proposition can be formulated for general unimodular groups with type I regular representation [5].

**Proposition 3.2** Let $T : L^2(\mathbb{H}) \rightarrow L^2(\mathbb{H})$ be a bounded leftinvariant operator. Then there exists a measurable field $\hat{T}_h$ of uniformly bounded operators on $L^2(\mathbb{R})$ such that for all $f \in L^2(\mathbb{H})$ and almost all $h \in \mathbb{R}'$, $\hat{T}f(h) = \hat{f}(h) \circ \hat{T}_h$.

In particular, given any closed leftinvariant subspace $\mathcal{H} \subset L^2(\mathbb{H})$, the orthogonal projection $P$ onto $\mathcal{H}$ decomposes on the Plancherel transform side into a measurable field $(\hat{P}_h)_{h \in \mathbb{R}'}$ of orthogonal projections on $L^2(\mathbb{R})$.

The proposition motivates the following definitions:
Definition 3.3 For a leftinvariant subspace $\mathcal{H}$ let $(\widehat{P}_h)_{h \in \mathbb{R}'}$ denote the associated field of projection operators, and define

$$\Sigma(\mathcal{H}) = \{ h \in \mathbb{R}' : \widehat{P}_h \neq 0 \}$$

and similarly, $\Sigma(g) = \{ h \in \mathbb{R}' : \widehat{g}(h) \neq 0 \}$, for $g \in L^2(\mathbb{H})$. Both are defined only up to a set of measure zero. We call $\mathcal{H}$ (resp. $g$) bandlimited if $\Sigma(\mathcal{H})$ (resp. $\Sigma(g)$) is a bounded set in $\mathbb{R}$. In addition, define

$$m(h) := \text{rank}(\widehat{P}_h),$$

the multiplicity function of $\mathcal{H}$. $\mathcal{H}$ is called multiplicity-free if $m(h) \in \{0, 1\}$, for almost every $h$. □

In the abelian case, say over the reals, $\Sigma(\mathcal{H})$ has a natural counterpart, and it plays a central role for the characterisation of sampling spaces. The multiplicity function provides more detailed information, which in the abelian case is superfluous, since $\lambda_\mathbb{R}$ is multiplicity-free.

The following theorem characterises admissible vectors for leftinvariant subspaces via their Plancherel transform. See [10] for the proof of the general version for unimodular groups with type I regular representation. The (rather obvious) part (iii) is not proved there; it provides a description of selfadjoint convolution idempotents on the Plancherel transform side.

Theorem 3.4 Let $\mathcal{H} \subset L^2(\mathbb{H})$ be leftinvariant, and let $(\widehat{P}_h)_{h \in \mathbb{R}'}$ be the associated field of projection operators.

(i) $f \in \mathcal{H}$ is admissible for $\mathcal{H}$ iff $\widehat{f}(h)^*$, restricted to $\widehat{P}_h(L^2(\mathbb{R}))$, is an isometry.

(ii) There exist admissible vectors for $\mathcal{H}$ iff $\int_{\mathbb{R}'} m(h)|h|dh < \infty$.

(iii) The necessary and sufficient condition of (ii) is equivalent to the property that the operator field $(\widehat{P}_h)_{h \in \mathbb{R}'}$ is the Plancherel transform of a selfadjoint convolution idempotent $S \in L^2(\mathbb{H})$. Accordingly, $\mathcal{H} = L^2(\mathbb{G}) \ast S$.

To close our survey of the Heisenberg group, we cite a result classifying the lattices of $\mathbb{H}$. We associate to such a lattice $\Gamma$ two numbers $d(\Gamma) \in \mathbb{N}'$, $r(\Gamma) \in \mathbb{R}^+$ which contain sufficient information for our purposes. Both parameters can be interpreted as a measure of the density of $\Gamma$ in $\mathbb{H}$. We first single out a particular family of lattices, which turns out to be exhaustive (up to automorphisms of $\mathbb{H}$).

Definition 3.5 For any positive integer $d$ let $\Gamma_d$ be the subgroup generated by $P, dQ, Z$. $\Gamma_d$ is a lattice, with

$$\Gamma_d = (Z \cdot Z) \ast (Z \cdot P) \ast (dZ \cdot Q) = \left\{ (m, dk, \ell + \frac{1}{2}dmk) : m, k, \ell \in \mathbb{Z} \right\}.$$ 

It is convenient to introduce the reduced lattice $\Gamma_d^r$ which is the subset

$$\Gamma_d^r = (Z \cdot P) \ast (dZ \cdot Q) = \left\{ (m, dk, dmk/2) : m, k \in \mathbb{Z} \right\}.$$ 

Note that $\Gamma_d^r$ is not a lattice, not even a subgroup. □
Let us next give a classification of lattices. It has been attributed (in more generality) to Maltsev. Since we were not able to locate a source, we sketch a short proof for the sake of completeness.

**Theorem 3.6** Let $\Gamma$ be a lattice of $\mathbb{H}$. Then there exists a strictly positive integer $d$ and $\alpha \in \text{Aut}(\mathbb{H})$ with $\alpha(\Gamma_d) = \Gamma$. The integer $d$ is uniquely determined by these properties.

**Proof.** By [3, Theorem 5.1.6], there exist a basis $\bar{P}, \bar{Q}, \bar{Z}$ of $\mathfrak{h}$ with $\bar{Z} \in Z(\mathbb{H})$, and $\Gamma = \mathbb{Z}\bar{Z} \ast \mathbb{Z}\bar{P} \ast \mathbb{Z}\bar{Q}$. Now $[\bar{P}, \bar{Q}] = \bar{P} \bar{Q}^{-1} \bar{Q}^{-1} \in \Gamma \cap Z(\mathbb{H}) = \mathbb{Z}\bar{Z}$ implies $[\bar{P}, \bar{Q}] = d\bar{Z}$ for some $d \in \mathbb{Z}$, w.l.o.g. $d \geq 0$ (otherwise exchange $\bar{Q}, \bar{P}$). In fact, $d > 0$ since $\mathfrak{h}$ is not abelian. It is immediately checked that the linear isomorphism defined by $P \mapsto \bar{P}$, $dQ \mapsto \bar{Q}$, $Z \mapsto \bar{Z}$ is in Aut($\mathbb{H}$). That $d$ is unique follows from the fact that each automorphism $\alpha$ mapping $\Gamma_d$ to $\Gamma_d'$ maps $Z$ onto $\pm Z$. From Proposition 7.2 (a) below follows that $\alpha$ leaves the Haar measure of $\mathbb{H}$ invariant, and this implies that $\text{covol}(\Gamma_d) = \text{covol}(\Gamma_d')$. On the other hand, $\text{covol}(\Gamma_d) = d$, hence $d = d'$.

We denote by $d(\Gamma)$ the unique integer $d$ from the theorem. For the definition of $r(\Gamma)$ we take $r \in \mathbb{R}^+$ with $\Gamma \cap Z(\mathbb{H}) = r(\Gamma)\mathbb{Z}\mathbb{Z}$.

**4 Main results**

Now we can state the main results of this paper. In this section, $\mathcal{H}$ always denotes a closed, leftinvariant subspace of $L^2(\mathbb{H})$, and $\Gamma < \mathbb{H}$ a lattice. Recall from Theorem 2.5 that we may assume $\mathcal{H} = L^2(G) \ast S$, where $S$ is a selfadjoint convolution idempotent, and that $\mathcal{H}$ is a sampling space if and only if $\lambda_{\mathbb{H}}(\Gamma)S$ is a tight frame of $\mathcal{H}$. The main theorem characterises the subspaces admitting tight frames.

**Theorem 4.1** (i) There exists a tight frame of the form $\lambda_{\mathbb{H}}(\Gamma)\Phi$ with suitable $\Phi \in \mathcal{H}$ iff the multiplicity function $m$ associated to $\mathcal{H}$ satisfies

$$m(h) \cdot |h| + m \left( h - \frac{1}{r(\Gamma)} \right) \cdot \left| h - \frac{1}{r(\Gamma)} \right| \leq \frac{1}{d(\Gamma)r(\Gamma)} \quad \text{(almost everywhere).} \quad (6)$$

In particular, $\Sigma(\mathcal{H}) \subset \left[ -\frac{1}{d(\Gamma)r(\Gamma)} \cdot \frac{1}{d(\Gamma)r(\Gamma)} \right]$ (up to a set of measure zero).

(ii) If $\lambda_{\mathbb{H}}(\Gamma)\Phi$ is a tight frame of $\mathcal{H}$, then $\sqrt{\frac{1}{dr(\Gamma)}}\Phi$ is admissible for $\mathcal{H}$.

(iii) There does not exist an orthonormal basis of the form $\lambda_{\mathbb{H}}(\Gamma)\Phi$ for $\mathcal{H}$.

**Remark 4.2** Note that for $d(\Gamma) > 1$, inequality (5) simplifies to

$$m(h) \cdot |h| \leq \frac{1}{d(\Gamma)r(\Gamma)} \quad \text{(almost everywhere).} \quad (7)$$
Corollary 4.3 Assume that the multiplicity function $m$ associated to $H$ is bounded. There exists a lattice $\Gamma$ and a $\Phi \in H$ such that $\lambda_H(\Gamma)\Phi$ is a tight frame of $H$ iff $H$ is bandlimited. 

The following is a rephrasing for square-integrable representations:

Corollary 4.4 Let $(\pi, H_\pi)$ be a square-integrable representation of $\mathbb{H}$ with bounded multiplicity. Associate the set $\Sigma(\pi) \subset \mathbb{R}'$ by picking an admissible vector $\eta \in H_\pi$ and letting $\Sigma(\pi) := \Sigma(V_\eta \eta)$; up to a null set, this is independent of the choice of $\eta$. There exists a lattice $\Gamma$ and a vector $\eta$ such that $\pi(\Gamma)\eta$ is a tight frame iff $\Sigma(\pi)$ is bounded. Any such $\eta$ is admissible.

That bounded multiplicity cannot be dispensed with in the last corollary is shown by the next result:

Corollary 4.5 There exists a bandlimited leftinvariant subspace $H = L^2(G) * S$, with a selfadjoint convolution idempotent $S \in L^2(H)$, admitting no tight frame of the form $\lambda_H(\Gamma)\Phi$.

With regard to the existence of sampling subspaces, we have:

Corollary 4.6 Not every space admitting a tight frame of the form $\lambda_H(\Gamma)S$ is a sampling subspace for $\Gamma$. However, for such a space $H$ there exists $\Phi \in H$ such that $f \mapsto f * \Phi^*$ is an isometry on $H$, mapping $H$ onto a sampling space. There does not exist a sampling space $H$ for which the restriction map $R_\Gamma$ is onto.

The proofs for these results will be given in Section 7 below. The following remarks discuss similarities and differences to the case of the reals:

Remarks 4.7

1. The main similarity lies in the notion of bandwidth, and the fact that it can be interpreted as inversely proportional to the density of the lattice. Note that over $\mathbb{H}$ the bandwidth restriction is much more rigid: The set $\Sigma(H)$ is contained in a fixed interval, whereas the analog of that set in the real case can be shifted arbitrarily and still give a sampling subspace.

2. Corollaries 4.3 and 4.6 mark important differences between the sampling theories of $\mathbb{H}$ and $\mathbb{R}$. None of the counterexamples given in the corollaries has an analog in the real setting. In particular, the question whether a given space is a sampling space is much more subtle than deciding whether it has a frame. For the first problem, a close inspection of the projection operator field $(\hat{P}_h)_{h \in \mathbb{R}'}$ is necessary, for the second, only the ranks of these operators are needed. By contrast, for the reals it can be shown that every subspace of $L^2(\mathbb{R})$ admitting a frame obtained from the action of a lattice is already a sampling subspace.

3. While Theorem 4.1 shows that the Plancherel transform can be used to characterise sampling spaces and frames, it is not clear how it can be generalised to a larger class of locally compact groups. Indeed, as far as we are aware, among the entities entering the central relation (6), only the multiplicity function $m$ has an abstract interpretation. A possible starting point for more general considerations could be to study multiplicity-free subspaces and try to come up with criteria involving the natural topology on the dual.
4. We have used lattices as sampling sets simply because they are easily accessible. In particular, we have not at all exploited the representation theory of the lattices. An alternative approach along these lines could provide valuable additional information.

\[\blacksquare\]

5 Reduction to Weyl-Heisenberg systems

In this section we start the discussion of normalised tight frames for left-invariant subspaces. On the Plancherel transform side, the space \(\mathcal{H}\) under consideration decomposes into a direct integral. In this section, we reduce the complexity of the problem in two ways: We get rid of the direct integral on the one hand, and the central variable of the lattice on the other, and are faced with the problem of constructing certain normalised tight frames in the fibres, arising from the action of the reduced lattice. The latter problem is equivalent to the construction of Weyl-Heisenberg (super-)frames, which allows to finish our proof.

**Proposition 5.1** Let \(\Phi \in \mathcal{H}\) be such that \(\lambda(\Gamma)\Phi\) is a normalised tight frame of \(\mathcal{H}\). Then, for almost every \(h \in \Sigma(\mathcal{H})\), the reduced lattice satisfies the following condition:

\[
\left(\|h\|^{1/2}\rho_h(\gamma)\hat{\Phi}(h)\right)_{\gamma \in \Gamma^r} \text{ is a normalised tight frame of } \mathcal{B}_2(L^2(\mathbb{R})) \circ \hat{P}_h.
\] (8)

Conversely, if both (8) (for almost every \(h\)) and the support condition

\[
\forall m \in \mathbb{Z} \setminus \{0\} : \Sigma(\mathcal{H}) \cap m + \Sigma(\mathcal{H}) \text{ has measure zero}
\] (9)

hold, then \(\lambda_G(\Gamma)\Phi\) is a normalised tight frame of \(\mathcal{H}\).

**Proof.** We calculate

\[
\sum_{\gamma \in \Gamma} |\langle f, \lambda_H(\gamma)\Phi \rangle|^2 = \sum_{\gamma \in \Gamma} \left| \int_{\Sigma(f)} \langle \hat{f}(h), \rho_h(\gamma)\hat{\Phi}(h) \rangle \|h\|dh \right|^2
\]

\[
= \sum_{\gamma \in \Gamma^r} \sum_{\ell \in \mathbb{Z}} \left| \int_{\Sigma(f)} e^{-2\pi i \ell h} \langle \hat{f}(h), \rho_h(\gamma)\hat{\Phi}(h) \rangle \|h\|dh \right|^2
\]

\[
= \sum_{\gamma \in \Gamma^r} \int_{\Sigma(f)} \left| \langle \hat{f}(h), \rho_h(\gamma)\hat{\Phi}(h) \rangle \|h\|^2dh \right|
\]

\[
= \int_{\Sigma(f)} \sum_{\gamma \in \Gamma^r} \left| \langle \hat{f}(h), \rho_h(\gamma)|h|^{1/2}\hat{\Phi}(h) \rangle \right|^2 |h|dh.
\]

Here we used the assumption on \(\Sigma(f)\) to apply the Plancherel Theorem on \(\Sigma(f)\) and thereby discard the summation over \(\ell\). On the other hand, the tight frame condition together with the Plancherel formula for \(\mathbb{H}\) implies that

\[
\sum_{\gamma \in \Gamma} |\langle f, \lambda(\gamma)s \rangle|^2 = \int_{\Sigma(f)} \|\hat{f}(h)\|^2 |h|dh
\]

and thus

\[
\int_{\Sigma(f)} \sum_{\gamma \in \Gamma^r} \left| \langle \hat{f}(h), \rho_h(\gamma)|h|^{1/2}\hat{\Phi}(h) \rangle \right|^2 |h|dh = \int_{\Sigma(f)} \|\hat{f}(h)\|^2 |h|dh.
\] (10)
Replacing \( f \) by \( g \) with \( \tilde{g}(h) = \chi_B(h)\tilde{f}(h) \), we see that we may replace \( \Sigma(f) \) in (10) by any Borel subset \( B \). Hence the integrands must be equal almost everywhere:

\[
\sum_{\gamma \in \Gamma^r} \left| \langle \tilde{f}(h), \rho_h(\gamma)|h|^{1/2} \hat{\Phi}(h) \rangle \right|^2 = \|\tilde{f}(h)\|^2. \tag{11}
\]

Writing an arbitrary \( f \) as orthogonal sum of functions \( g \) fulfilling the initial support condition we see that (11) holds for every \( f \) and almost every \( h \in \mathbb{R}' \). However, it remains to show that the relation holds for all \( h \) in a common conull subset, independent of \( f \). For this purpose we pick a countable dense \( \mathbb{Q} \)-subspace \( A \subset L^2(\mathbb{H}) \). Then there exists a conull subset \( C \subset \mathbb{R}' \) such that, for all \( h \in C \), \( \{\tilde{f}(h) : f \in A \} \) is dense in \( \mathcal{B}_2(L^2(\mathbb{R})) \circ \hat{P}_h \), and in addition (11) holds for all \( f \in A \). Now, for every \( h \in C \), the coefficient map

\[
\tilde{f}(h) \mapsto \left( \langle \tilde{f}(h), \rho_h(\gamma)|h|^{1/2} \hat{\Phi}(h) \rangle \right)_{\gamma \in \Gamma^r}
\]

is a closed linear operator, by Proposition 1.3 (d), coinciding with an isometry on a dense subset, hence it is an isometry.

Finally, we note that the argument can be reversed to prove the sufficiency of condition (11) under the additional assumption (6). \( \square \)

6 Weyl-Heisenberg frames

By a Weyl-Heisenberg system \( \mathcal{G}(\alpha, \beta, g) \) of \( L^2(\mathbb{R}) \) we mean a family \( (g_{k,m})(k,m) \in \mathbb{Z}^2 \)

\[
g_{k,m}(x) = e^{2\pi i \beta k x} g(x + \alpha m)
\]

resulting from a single function \( g \in L^2(G) \). A (normalised, tight) Weyl-Heisenberg frame is a Weyl-Heisenberg system which is a (normalised, tight) frame of \( L^2(\mathbb{R}) \). For any \( g \in L^2(\mathbb{R}) \), the operation of the reduced lattice \( \Gamma^r \) on \( g \) via \( \rho_h \) gives the system

\[
(\rho_h(m, dk, dmk/2)g)(x) = e^{\pi i hmdk} e^{2\pi i km} g(x + hm),
\]

hence \( \rho_h(\Gamma^r)g \) and the Weyl-Heisenberg system \( \mathcal{G}(h, d, g) \) only differ up to the phase factor \( e^{\pi i hmk} \). Clearly the phase factor does not influence any normalised tight frame or ONB properties of the system, hence we may and will switch freely between the Weyl-Heisenberg system and the orbit of the reduced lattice.

The central results concerning Weyl-Heisenberg frames are contained in the following.

Theorem 6.1 There exists a normalised tight Weyl-Heisenberg frame \( \mathcal{G}(h, d, g) \) of \( L^2(\mathbb{R}) \) iff \( |h|d \leq 1 \). For any such frame we have \( \|g\|^2 = |h|d \).

Proof. The “only-if”-part is [1], Corollary 7.5.1. The “if”-part follows from [1], Theorem 6.4.1, applied to a suitably chosen characteristic function. The norm equality is due to [1], Corollary 7.3.2. \( \square \)

In dealing with subspaces of Hilbert-Schmidt spaces, we have to consider a more general setting: We are interested in normalised tight frames of \( (L^2(\mathbb{R}))^r \) consisting of vectors of the type

\[
g_{k,m} = (e^{2\pi i dkx} g_j^i(x + h_j m))_{j = 1, \ldots, r} = (g_{k,m}^j)_{j = 1, \ldots, r} \tag{12}
\]
where $g = (g^i)_{j=1,...,r} \in (L^2(\mathbb{R}))^r$ is suitably chosen, and $h = (h_j)_{j} \in \mathbb{R}^r$ is a vector of nonzero real numbers. This problem has already been considered by other authors, see [2] and the references therein. Following [2], we call a system of the type (12) a Weyl-Heisenberg superframe. The following two lemmata extend the results on $L^2(\mathbb{R})$ to the more general situation. The first one is quite obvious and does not reflect the special structure of Weyl-Heisenberg frames. An alternate version (for arbitrary frames) is given in [2].

**Lemma 6.2** Let $h = (h_1, \ldots, h_r)$ and $g = (g^i)_{j=1,...,r} \in (L^2(\mathbb{R}))^r$. Then $(g_k,m)_{k,m\in\mathbb{Z}}$, defined as in equation (13), is a normalised tight frame of $(L^2(\mathbb{R}))^r$ iff

(i) for $j = 1, \ldots, r$, $G(h_j,d,g^i)$ is a normalised tight frame of $L^2(\mathbb{R})$; and

(ii) for $i \neq j$, and for all $f_1, f_2 \in L^2(\mathbb{R})$,

\[
\left\langle \left( (f_1, g^i_m)_m, (f_2, g^i_m)_m \right) \right\rangle = 0 .
\]

i.e., the coefficient operators belonging to $G(h_j,d,g^i)$ and $G(h_i,d,g^i)$ have orthogonal ranges in $\ell^2(\mathbb{Z} \times \mathbb{Z})$.

**Proof.** Consider the subspace $\mathcal{H}_j \subset (L^2(\mathbb{R}))^r$ whose elements are nonzero at most on the $j$th component. The necessity of property (i) follows immediately from Proposition 1.3 (a), applied to the $\mathcal{H}_j$. Property (ii) is necessary because the (pairwise orthogonal) $\mathcal{H}_j$ need to have orthogonal images in $\ell^2(\mathbb{Z} \times \mathbb{Z})$. The converse is clear.

Necessary and sufficient conditions for the existence of such frames are given in the next proposition.

**Proposition 6.3** Let $(h_j)_{j=1,...,r}$, $d \in \mathbb{N}'$ be given.

(a) There exists a normalised tight frame of $(L^2(\mathbb{R}))^r$ of the form (12) iff $d \sum_{j=1}^r |h_j| \leq 1$.

(b) Assume that $h_j = h$, for all $j = 1, \ldots, r$, and $g = (g^i)_{j=1,...,r}$ is such that (12) is a normalised tight frame. Then $g^i \perp g^j$, for $i \neq j$.

**Proof.** For the necessity in part (a), observe that Lemma 6.2 together with Theorem 6.1 yields that $\|g^i\|^2 = |h_j|d$, and thus $\|g\|^2 = d \sum_{j=1}^r |h_j|$. Now Proposition 1.3 (c) entails the desired inequality.

The proof for sufficiency is a slight modification of a construction given by Balan [2, Example 13]. Define $c_i = \sum_{j=1}^r |h_j|$, and let $g^i = \sqrt{d} \chi_{[c_{i-1},c_i]}$. Given $f = (f^i) \in (L^2(\mathbb{R}))^r$, we compute

\[
\langle f, g^i_k, m \rangle = \sum_{i=1}^r (f^i, g^i_k, m) = \sum_{i=1}^r \sqrt{d} \int_{c_{i-1}}^{c_i} e^{-2\pi imdx} f^i(x + \beta_i k) dx = \sqrt{d} \int_0^{1/d} e^{-2\pi imdx} H_k(x) dx ,
\]

where

\[
H_k(x) = \begin{cases} f^i(x - h_i k) & x \in [c_{i-1},c_i] \\ 0 & \text{elsewhere} \end{cases} .
\]
Fixing $k$, we compute
\[
\sum_{m \in \mathbb{Z}} |\langle f, g_{k,m} \rangle|^2 = \sum_{m \in \mathbb{Z}} d \int_0^{1/d} e^{-2\pi i m dx} H_k(x) dx \int_0^{1/d} |H_k(x)|^2 dx = \sum_{i=1,\ldots,r} \int_{c_{i-1}}^{c_i} |f^i(x - h_i k)|^2 dx .
\]

Since the $h_i \mathbb{Z}$-translates of $[c_{i-1}, c_i]$ tile $\mathbb{R}$, summing over $k$ yields the desired norm equality. This closes the proof of (a).

For the proof of (b), pick $f_1, f_2 \in L^\infty(\mathbb{R})$ with supports in $[0, |h|]$. Then we calculate
\[
\sum_{m,k \in \mathbb{Z}} \langle f_1, g_{m,k}^j \rangle \langle f_2, g_{m,k}^j \rangle = \sum_{m \in \mathbb{Z}} \int_{\mathbb{R}} \left( \sum_{k \in \mathbb{Z}} \langle f_1 g_{m,0}^j, e^{2\pi i dk} e^{2\pi i dx} \rangle \right) f_2(x) g^j(x + hm) dx
= \sum_{m \in \mathbb{Z}} d^{-1} \int_0^{|h|} f_1(x) \overline{f_2(x)} g^j(x + hm) g^j(x + hm) dx
= d^{-1} \int_0^{|h|} \left( \sum_{m \in \mathbb{Z}} g^j(x + hm) g^j(x + hm) \right) f_1(x) \overline{f_2(x)} dx
\]
Here the Fourier series
\[
\sum_{k \in \mathbb{Z}} \langle f_1 g_{m,0}^j, e^{2\pi i dk} e^{2\pi i dx} \rangle = d^{-1} f_1(x) g^j(x + hm)
\]
is valid on $[0, |h|]$, at least in the $L^2$-sense, because of $|h| \leq d^{-1}$, the latter being a consequence of Theorem 6.1. Now, for arbitrary $f_1, f_2$, the scalar product we started with has to be zero, whence we obtain for almost every $x \in [0, |h|]$,
\[
\sum_{m \in \mathbb{Z}} g^j(x + hm) g^j(x + hm) = 0 .
\]
Integrating over $[0, |h|]$ and applying Fubini’s theorem yields $\langle g^j, g^j \rangle = 0$. \hfill \Box

Remark 6.4 Note that the vectors $(g^j_i)_{i=1,\ldots,r}$ constructed in the proof of part (a) depend measurably on $h$, i.e., if we let $(g_h^j)$ be the vector of functions constructed from $h$, then $(x, h) \mapsto (g_h^j(x))_{i=1,\ldots,r}$ is a measurable mapping. \hfill \Box

7 Proofs of the main results

The general proof strategy consists in explicit calculation for the $\Gamma_d$ and then transferring the results to arbitrary lattices by the action of Aut($\mathbb{H}$). For this purpose we need a more detailed description of Aut($\mathbb{H}$) and its action on the Plancherel transform side. Most of the results are standard, and we only sketch the proofs.
Proposition 7.1  (a) For \( r > 0 \) let \( \alpha_r(p,q,t) := (\sqrt{r}p, \sqrt{r}q, rt) \). Then \( \alpha_r \in \text{Aut}(\mathbb{H}) \). In addition, \( \alpha_{\text{inv}} : (p,q,t) \mapsto (q,p,-t) \) defines an involutory automorphism of \( \mathbb{H} \).

(b) Each \( \alpha \in \text{Aut}(\mathbb{H}) \) can be written uniquely as \( \alpha = \alpha_r\alpha_{\text{inv}}\alpha' \), where \( r \in \mathbb{R}' \), \( i \in \{0,1\} \) and \( \alpha' \) leaves the center of \( \mathbb{H} \) pointwise fixed.

(c) Suppose that \( \alpha(\Gamma_d) = \Gamma \) for some \( d, \alpha \), and let \( \alpha = \alpha_\gamma\alpha_{\text{inv}}\alpha' \) be the decomposition from part (b). Then \( r(\Gamma) = s \).

Proof. For parts (a), (b) see \cite{[3]} Theorem 1.22. Part (c) follows directly from the definition of \( r(\Gamma) \) and the fact that \( \alpha' \) and \( \alpha_{\text{inv}} \) map every discrete subgroup of \( Z(\mathbb{H}) \) onto itself. \( \square \)

Next let us consider the action on the Fourier transform side.

Proposition 7.2  (a) Define \( \Delta : \text{Aut}(\mathbb{H}) \to \mathbb{R}^+ \) by

\[
\Delta(\alpha) = \frac{\mu_{\mathbb{H}}(\alpha(B))}{\mu_{\mathbb{H}}(B)},
\]

where \( B \) is a measurable set of positive Haar measure. \( \Delta \) does not depend on the choice of \( B \), and it is a continuous group homomorphism. For \( \alpha = \alpha_r\alpha_{\text{inv}}\alpha' \) as in \cite{[3]} (b), \( \Delta(\alpha) = r^2 \).

(b) For \( \alpha \in \text{Aut}(\mathbb{H}) \), let \( \mathcal{D}_\alpha : L^2(\mathbb{H}) \to L^2(\mathbb{H}) \) be defined as \( (\mathcal{D}_\alpha f)(x) := \Delta(\alpha)^{1/2} f(\alpha(x)) \). This defines a unitary operator.

(c) Let \( \mathcal{H} \subset L^2(G) \) be a closed, leftinvariant subspace with multiplicity function \( m \). Then \( \mathcal{H} = \mathcal{D}_\alpha(\mathcal{H}) \) is closed and leftinvariant as well. Let \( \tilde{m} \) denote the multiplicity function related to \( \mathcal{H} \). If \( \alpha = \alpha_r\alpha_{\text{inv}}\alpha' \) then \( \tilde{m} \) satisfies

\[
\tilde{m}(h) = m((-1)^i r^{-1} h) \text{ (almost everywhere)}.
\]  (14)

(d) Let \( \Gamma \) be a lattice, \( \alpha \in \text{Aut}(\mathbb{H}) \) such that \( \alpha(\Gamma_d) = \Gamma \). Let \( \mathcal{H} \subset L^2(\mathbb{H}) \) be a closed, leftinvariant subspace. Then \( \lambda_\mathbb{H}(\Gamma_d) \Phi \) is a normalised tight frame (an ONB) for \( \mathcal{H} \) iff \( \lambda_\mathbb{H}(\Gamma_d)(\mathcal{D}_\alpha \Phi) \) is a normalised tight frame (an ONB) for \( \mathcal{D}_\alpha(\mathcal{H}) \).

Proof. Parts (a) and (b) are standard results concerning the action of automorphisms on locally compact groups, see \cite{[3]}. The explicit formula for \( \Delta(\alpha) \) follows from the fact that every automorphism leaving the center invariant factors into an inner and a symplectic automorphism \cite{[3]} Theorem 1.22; both do not affect the Haar measure.

For part (c), we first note that by the Stone-von Neumann theorem \cite{[3]} Theorem 1.50, any automorphism \( \alpha' \) keeping the center pointwise fixed acts trivially on the dual of \( \mathbb{H} \). Hence,

\[
(D_{\alpha'f})(h) = U_{\alpha',h} \circ \widehat{f}(h) \circ U_{\alpha',h}^*,
\]

where \( U_{\alpha',h} \) is a unitary operator on \( L^2(\mathbb{R}) \). Hence the action of \( \alpha' \) does not affect the multiplicity function, and from now on, we only consider \( \alpha = \alpha_r\alpha_{\text{inv}}^i \). In this case, letting

\[
(D_r f)(x) = r^{1/2} f(rx),
\]

we obtain by straightforward computation that

\[
(D_\alpha f)^\wedge(h) = r^{-1} \cdot D_r \circ \widehat{f}((-1)^i r^{-1} h) \circ D_r^*.
\]  (15)
This immediately implies (14).

To prove (d), observe that the unitarity of \( D_\alpha \) implies that \( D_\alpha (\lambda_{\Xi}(\Gamma)) \) is a normalised tight frame of \( D_\alpha (H) \), and check the equality

\[
D_\alpha (\lambda_{\Xi}(x)S) = \lambda_{\Xi}(\alpha^{-1}(x))(D_\alpha S) .
\]

\[\square\]

**Proof of Theorem 4.1.** We first prove the theorem for the case \( \Gamma = \Gamma_d \). Writing

\[
\widehat{\Phi}(h) = \sum_{i \in I_h} \varphi_i^h \otimes \eta_i^h ,
\]

we find by Proposition 5.1 that for almost every \( h \), \( (\rho_h(\gamma) \circ |h|^{1/2} \widehat{\Phi}(h))_{\gamma \in \Gamma} \) has to be a normalised tight frame of \( L^2(\mathbb{R}) \circ \widehat{P}_h \), or equivalently, that the vector \( (\varphi_i^h)_{i=1, \ldots, m(h)} \) generates a Weyl-Heisenberg superframe of \( (L^2(\mathbb{R}))^{m(h)} \), for \( h = (h, \ldots, h) \). (For the equivalence, confer \([22]\) in the appendix.) Then 6.2 (a) implies that \( G(h, d, |h|^{1/2} \varphi_i^h) \) is a normalised tight frame of \( L^2(\mathbb{R}) \). In particular, Theorem 6.3 entails

\[
\|\varphi_i^h\|^2 = d ,
\]

as well as \( \Sigma(H) \subseteq \left[ -\frac{1}{d}, \frac{1}{d} \right] \). Moreover, Proposition 5.3 (b) entails that the \( \varphi_i^h \) are mutually orthogonal (for \( h \) fixed). This shows that

\[
\widehat{\Phi}(h)^* = \sum_{i \in I_h} \eta_i^h \otimes \varphi_i^h
\]

is \( \sqrt{d} \) times an isometry on \( \widehat{P}_h(L^2(\mathbb{R})) \), and thus \( \frac{1}{\sqrt{d}} \Phi \) is admissible for \( H \), by Theorem 4.4 (ii). This proves part (ii) of the theorem. If \( d > 1 \), the support condition (3) in Proposition 5.1 is fulfilled. Hence Proposition 5.3 (a), applied to \( h = (h, \ldots, h) \), shows that (7) is necessary and sufficient for the existence of a normalised tight frame for \( H \). (Note that by Remark 5.4, 6.3 (a) provides a *measurable* field of operators.)

The case \( d = 1 \) requires a somewhat more involved argument. Assume that \( \lambda_{\Xi}(\Gamma) \Phi \) is a normalised tight frame, and let \( f \in H \). Condition (8) from Proposition 5.1 yields

\[
\|f\|^2 = \int_0^1 \left( \sum_{\gamma \in \Gamma^r} \left| \langle \hat{f}(h), \rho_h(\gamma) \widehat{\Phi}(h) \rangle \right|^2 |h|^2 + \left| \langle \hat{f}(h-1), \rho_{h-1}(\gamma) \widehat{\Phi}(h-1) \rangle \right|^2 |h-1|^2 \right) dh .
\]

On the other hand,

\[
\|f\|^2 = \sum_{\gamma \in \Gamma^r} \sum_{i \in I} \left( \int_0^1 e^{-2\pi ih\ell} \left( \langle \hat{f}(h), \rho_h(\gamma) \widehat{\Phi}(h) \rangle |h| + \langle \hat{f}(h-1), \rho_{h-1}(\gamma) \widehat{\Phi}(h-1) \rangle |h-1| \right) dh \right)^2
\]

\[
= \int_0^1 \left( \sum_{\gamma \in \Gamma^r} \left| \langle \hat{f}(h), \rho_h(\gamma) \widehat{\Phi}(h) \rangle |h| + \langle \hat{f}(h-1), \rho_{h-1}(\gamma) \widehat{\Phi}(h-1) \rangle |h-1| \right|^2 \right) dh .
\]

As in the proof of Proposition 5.1, the fact that the two equations hold for all \( f \in H \) allows to equate the integrands of (17) and (18). But this implies the orthogonality of the coefficient families:

\[
\left\langle \left( \langle \hat{f}(h), \rho_h(\gamma) \widehat{\Phi}(h) \rangle \right)_{\gamma \in \Gamma^r} , \left( \langle \hat{f}(h-1), \rho_{h-1}(\gamma) \widehat{\Phi}(h-1) \rangle \right)_{\gamma \in \Gamma^r} \right\rangle_{\ell^2(\Gamma^r)} = 0 .
\]
Plugging this fact, together with condition (8) from Proposition 5.1, into Proposition 6.2, we finally realise that the system

$$\left((\rho_{h-1}(\gamma)|h-1|^{1/2}\varphi_{1}^{h-1},\ldots,\rho_{h-1}(\gamma)|h-1|^{1/2}\varphi_{1}^{h-1});\rho_{h}(\gamma)|h|^{1/2}\varphi_{1}^{h};\ldots;\rho_{h}(\gamma)|h|^{1/2}\varphi_{m(h)}^{h}\right)_{\gamma\in\Gamma^{r}}$$

has to be a normalised tight frame of \((L^{2}(\mathbb{R}))^{m(h)+m(h-1)}\). An application of Proposition 6.3 (a) with \(h = (h-1,\ldots,h-1,h,\ldots,h)\) yields that such a frame exists if \(m(h)|h| + m(h-1)|h-1| \leq 1\). This shows the necessity of (8). The sufficiency is obtained by running the proof backward; the measurability of the constructed operator field is again ensured by Remark 6.4.

For the proof of \((iii)\) we need to show, by 1.3(c), that \(\|\Phi\| < 1\), for every \(\Phi\) for which \(\lambda_{\mathcal{H}}(\Gamma)\Phi\) is a normalised tight frame. Recalling that

$$|h|\|\tilde{\Phi}(h)\|_{B_{2}}^{2} = |h|m(h)d,$$

and using the fact that the inequality \(m(h)|h|d + m(h-1)|h-1|d \leq 1\) is strict almost everywhere (say, for \(h\) irrational) we can estimate

$$\|\Phi\|^{2} = \int_{-1}^{1}\|\tilde{\Phi}(h)\|_{B_{2}}^{2} |h|dh$$

$$= \int_{0}^{1} m(h)|h|d + m(h-1)|h-1|d dh$$

$$< 1.$$  

This closes the proof for \(\Gamma = \Gamma_{d}\). For \(\Gamma = \alpha(\Gamma_{d})\), write \(\alpha = \alpha_{r(\Gamma)}\alpha_{inv}\alpha_{1}\) as in Proposition 7.3 (c). By 7.2 (d), we may consider \(\tilde{\mathcal{H}} = \mathcal{D}_{\alpha}(\mathcal{H})\) instead of \(\mathcal{H}\) and \(\mathcal{H}\). Parts \((iii)\) immediately follows from this observation. For part \((i)\), we find by Proposition 7.3(c) that the associated multiplicity function \(\hat{m}\) fulfills \(\hat{m}(h) = m((-1)^{r(\Gamma)}h)\). Hence, (9) for \(\Gamma_{d}\), \(\mathcal{H}\) becomes

$$m((-1)^{r(\Gamma)}h)|h| + m((-1)^{r(\Gamma)}(h-1)|h-1| \leq \frac{1}{d} \quad \text{(almost everywhere)}$$

which after dividing both sides by \(r(\Gamma)\) and passing to the variable \(\tilde{h} = (-1)^{r(\Gamma)}h\) is the desired inequality (9).

For part \((ii)\) it remains to show that \(\frac{1}{\sqrt{d(\Gamma)^{r(\Gamma)}}}\tilde{\Phi}(h)\) is an isometry on \(\tilde{P}_{h}(L^{2}(\mathbb{R}))\), by Theorem 8.4(b). Clearly the only problem is showing that the normalisation is correct. Part \((ii)\) for \(\Gamma_{d(\Gamma)}\) implies that \(\frac{1}{C(\Gamma_{d})}(\mathcal{D}_{\alpha}\Phi)^{\wedge}(h)\) is a partial isometry, for almost every \(h\). Now relation (13) implies that \(\frac{1}{\sqrt{d(\Gamma)^{r(\Gamma)}}}\tilde{\Phi}(h)\) is a partial isometry as well. □

**Proof of Corollary 4.3.** The assumptions imply that \(m(h)|h| \leq c\), for all \(h \in \mathbb{R}\), and \(c\) a constant. Hence picking \(s \geq \frac{2c}{\pi}\) and defining \(\Gamma = \alpha_{s}(\Gamma_{d})\) ensures that (9) is fulfilled. □

**Proof of Corollary 4.4.** For two admissible vectors \(\eta,\eta'\), the spaces \(V_{\eta}(\mathcal{H}_{\eta})\) and \(V_{\eta'}(\mathcal{H}_{\eta})\) carry equivalent subrepresentations of \(\lambda_{\mathcal{H}}\). The intertwining operator \(T\) between these subspaces decomposes on the Plancherel transform side, by Proposition 3.2, and maps \(V_{\eta}\eta\) to \(V_{\eta'}\eta'\), and thus \(\Sigma(V_{\eta}\eta) = \Sigma(V_{\eta'}\eta')\). This shows the well-definedness of \(\Sigma(\pi)\). The remaining statements are then obtained by transferring the corresponding results from Theorem 4.1 and Corollary 4.3 back to \(\mathcal{H}_{\eta}\) via \(V_{\eta}^{-1}\). □
Proof of Corollary 4.5. Pick any measurable function \( m : [-1, 1] \to \mathbb{N}' \) such that \( h \mapsto m(h)|h| \) is integrable but unbounded. There exists a closed, leftinvariant space \( \mathcal{H} \) with multiplicity function \( m \), by Lemma A.1 below. \( \mathcal{H} \) is of the desired form, but violates (R), for all lattices \( \Gamma \).

Proof of Corollary 4.6. To give an example proving the first statement, let \( \Gamma = \Gamma_d \); using the appropriate \( \alpha \in \text{Aut}(\mathbb{H}) \) the argument can be adapted to suit any other lattice. For \( h \in \left[0, \frac{1}{2}\right] \), define

\[
\eta^h = \frac{1}{\sqrt{|h/2|}} \chi_{[0,h/2]}.
\]

and \( S \in \mathcal{L}_2(\mathbb{H}) \) with \( \widehat{S}(h) = \eta_h \otimes \eta_h \). Then \( S \) is a selfadjoint convolution idempotent, and \( \mathcal{H} = \mathcal{L}_2(\mathbb{H}) \ast S \) has a tight frame of the form \( \lambda_\mathbb{H}(\Gamma)\Phi \). However, for \( \mathcal{H} \) to be a sampling space, \( \lambda_\mathbb{H}(\Gamma)S \) must be a tight frame, and condition (R) implies that \( \mathcal{G}(h,d,\eta_h) \) is a tight frame of \( \mathcal{L}_2(\mathbb{R}) \), for almost every \( h \). But \( \chi_{[h/2,h]} \) has disjoint support with all elements of that system, hence \( \mathcal{G}(h,d,\eta_h) \) is not even total.

The second statement is obvious from Theorem 4.1 (ii) and Proposition 2.2. The last statement follows from Theorem 4.1 (iii). \( \square \)

8 A concrete example

In this section we explicitly compute a sinc-type function for \( \Gamma = \Gamma_1 \). The construction proceeds backwards, starting on the Plancherel transform side by giving a field of rank-one projection operators fulfilling the additional requirements for the sampling space property. Fourier inversion yields the sinc-type function \( S \). As a consequence, the sampling space is given as \( \mathcal{L}_2(\mathbb{H}) \ast S \). In order to minimise tedium, we have drastically shortened some of the more straightforward calculations. The three steps carry out the abstract program developed above.

1. Construction on the Plancherel transform side: For \( h \in [-0.5, 0.5] \) let \( \eta_h = |h|^{-1/2}\chi_{[-|h|/2,|h|/2]} \), and

\[
\widehat{S}(h) = \eta_h \otimes \eta_h,
\]

and let \( \widehat{S} \) be zero outside of \( [-0.5, 0.5] \). \( \widehat{S} \) is a measurable field of rank-one projection operators, with integrable trace, hence has an inverse image \( S \in \mathcal{L}_2(\mathbb{H}) \) which is a selfadjoint convolution idempotent. Moreover, it is straightforward to check that \( \rho_h(\Gamma_1^e)|h|^{1/2}\eta_h = \rho_h(\Gamma_1^e)\chi_{[-0.5,0.5]} \) is a normalised tight frame of \( \mathcal{L}_2(\mathbb{R}) \), (compare the proof of Proposition 6.3 (a)). Hence, by Proposition 5.1, \( \lambda_\mathbb{H}(\Gamma)S \) is a normalised tight frame of \( \mathcal{H} = \mathcal{L}_2(\mathbb{H}) \ast S \), and \( \mathcal{H} \) is a sampling space.

2. Plancherel inversion: We use the inversion formula

\[
f(x) = \int_G \text{trace}(\hat{f}(\sigma)\sigma(x)^*)(d\nu_G(\sigma),
\]

for all \( f \in \mathcal{L}_2(G) \), for which the formula makes sense; i.e., for all \( f \) such that \( \hat{f}(\sigma) \) is trace-class (\( \nu_G \)-almost everywhere) and in addition \( \int_G \text{trace}|\hat{f}(\sigma)||d\nu_G(\sigma) < \infty \). The formula was proved by Lipsman \([13]\). In our concrete situation, it is immediately checked that the integrability condition is fulfilled, and we obtain

\[
S(p,q,t) = \int_{-0.5}^{0.5} \langle \eta_h, \rho_h(p,q,t)\eta_h \rangle |h|dh.
\]
Then \( S \) is a sampling function for \( \Gamma \), with \( c_H = 1 \), and \( S \) is the associated sinc-type function. \( \lambda_H(\Gamma_1)S \) is a normalised tight frame, but not an orthonormal basis of \( H \), because of \( \|S\|_2 = \frac{1}{2} \).

### A Hilbert-Schmidt operators and direct integrals

In this section we collect a few technical details concerning Hilbert-Schmidt operators and direct integral Hilbert spaces. If \( K \) is a Hilbert space, then \( B_2(K) \) denotes the space of bounded operators \( T \) for which \( T^*T \) is trace-class. \( B_2(K) \) is a Hilbert space, with scalar
product $\langle S, T \rangle = \text{tr}(T^* S)$- The finite-rank operators are dense in $B_2(\mathcal{K})$. We use the notation $\varphi \otimes \eta$ to denote the rank-one operator $z \mapsto \langle z, \eta \rangle \varphi$. For the purposes of computation with Hilbert-Schmidt operators, the formulae

$$(\varphi \otimes \eta)^* = \eta \otimes \varphi$$

and

$$S(\varphi \otimes \eta)T = (S \varphi) \otimes (T^* \eta)$$

are very convenient. As a matter of fact, all calculations in the algebra $B_2(\mathcal{K})$ can be carried out using these relations, since the rank-one operators span a dense subspace. Given any projection $P$ and any orthonormal basis $(\eta_i)_{i \in I}$ of $P(\mathcal{K})$, the operators $T \in B_2(\mathcal{K}) \circ P$ can be shown to have the form

$$T = \sum_{i \in I} \varphi_i \otimes \eta_i \ ,$$

with $\|T\|_2^2 = \sum_{i \in I} \|\varphi_i\|^2$. Moreover, for $S = \sum_{i \in I} \psi_i \otimes \eta_i$, we compute

$$\langle S, T \rangle = \sum_{i \in I} \langle \psi_i, \varphi_i \rangle \ .$$

The Hilbert space $B_2(\mathcal{K}) \circ P$ can be thought of as a tensor product space or a direct sum of copies of $\mathcal{K}$:

$$B_2(\mathcal{K}) \circ P \simeq \mathcal{K} \otimes l^2(I) \simeq (\mathcal{K})^{[I]} \ .$$

This is particularly useful when representations are considered: If $\pi$ is any representation of the group $G$ on $\mathcal{K}$, the representation $\pi \otimes \overline{\pi}$ of $G \times G$ operates on $B_2(\mathcal{K})$ by

$$(\pi \otimes \overline{\pi})(x, y)T = \pi(x) T \pi(y)^* \ .$$

Denote by $\pi \otimes 1$ the restriction of this representation to $G \times \{1\} \simeq G$. The space $B_2(\mathcal{K}) \circ P$ is invariant under $\pi \otimes 1$, and in the decomposition (21) the action is given as

$$(\pi \otimes 1)(x)T = \sum_{i \in I} (\pi(x) \varphi_i) \otimes \eta_i \ .$$

This shows that $\pi \otimes 1$, restricted to $B_2(\mathcal{K}) \circ P$, is equivalent to the direct sum of rank($P$) copies of $\pi$. Hence rank($P$) is the multiplicity of $\pi$ in the restriction of $\pi \otimes 1$ to $B_2(\mathcal{K}) \circ P$. In particular, given a set $\Gamma \subset G$, (22) entails the equivalence

$$(\pi \otimes 1)(\Gamma)T \subset B_2(\mathcal{K}) \circ P \text{ is a tight frame} \iff ((\pi(\gamma) \varphi_i)_{i \in I})_{\gamma \in \Gamma} \subset \mathcal{K}^{[I]} \text{ is a tight frame} \ .$$

with the same frame constant. Let us next turn to direct integrals. It is most convenient to define measurable vector fields first and then measurable operator fields in terms of the former. Since all Schrödinger representations live on the same space, the direct integral Hilbert space of measurable vector fields is easily identified with an ordinary $L^2$-space:

$$\int_{\mathbb{R}'} L^2(\mathbb{R}|h)dh \simeq L^2(\mathbb{R}, dx) \otimes L^2(\mathbb{R}', |h|dh) \simeq L^2(\mathbb{R} \times \mathbb{R}', dx|h|dh) \ .$$

Hence measurable vector fields $(\eta^h)_{h \in \mathbb{R}'}$ may be identified with families of functions such that $(x, h) \mapsto \eta^h(x)$ is measurable and square-integrable with respect to $dx|h|dh$. Measurable
operator fields are operators on $L^2(\mathbb{R}, dx) \otimes L^2(\mathbb{R}', |h|dh)$ of the form $(\eta^h) \rightarrow (T^h \eta^h)_h$. Now let $\mathcal{H} \subset L^2(\mathbb{H})$ be leftinvariant, with associated field of projection operators $(\mathcal{P}_h)_{h \in \mathbb{R}}$ and multiplicity function $m(h) = \text{rank}(\mathcal{P}_h)$. Define $I_h = \{1, 2, \ldots, m(h)\}$; by convention, $I_h = \mathbb{N}$ for $m(h) = \infty$. It is possible to pick a family of vector fields $(\eta^h)_{h \in \mathbb{R}'}$, with the property that for almost all $h$, $\{\eta^h_1, \ldots, \eta^h_{m(h)}\}$ is an orthonormal basis of $L^2(\mathbb{R})$, and $\eta^h_n = 0$ for $n > m(h)$ [8, Proposition 7.27]. Then the elements $f \in \mathcal{H}$ are characterised on the Plancherel transform side by

$$\hat{f}(h) = \sum_{i \in I_h} \psi^h_i \otimes \eta^h_i,$$

with measurable vector fields $(\psi^h_i)_{h \in \mathbb{R}'}$. Hence, in constructing such elements on the Plancherel transform side, the measurability of the vector fields $(\psi^h_i)_{h \in \mathbb{R}'}$ (with $i \in \mathbb{N}$) ensures the measurability of the resulting operator field. As a particular application we note the following lemma:

**Lemma A.1** If $m : \mathbb{R}' \rightarrow \mathbb{N}$ is measurable, there exists a closed leftinvariant subspace $\mathcal{H} \subset L^2(\mathbb{H})$ with $m$ as multiplicity function.

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