MULTIPLE SOLUTIONS FOR A CLASS OF SEMILINEAR ELLIPTIC EQUATIONS

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Abstract. By using truncation technique, minimization method and Morse theory, we obtain three nontrivial solutions for a class of semilinear elliptic equations.

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1. Introduction

In this paper we consider semilinear elliptic boundary value problems of the form

\[
\begin{align*}
-\Delta u &= g(u), \quad \text{in } \Omega, \\
u &= 0, \quad \text{on } \partial \Omega,
\end{align*}
\]

where \( \Omega \) is a bounded smooth domain in \( \mathbb{R}^N \), \( g \) is a \( C^1 \)-function on \( \mathbb{R} \) such that

\((g)\) there are \( a^- < 0 < a^+ \) such that \( g(a^\pm) = 0 \) and \( \delta > 0 \) such that for some \( k \geq 2 \),

\[ \lambda_k \leq \frac{g(t)}{t} \leq \lambda_{k+1}, \quad \text{for } t \in (-\delta, \delta). \]

Here, \( \lambda_n \) denote the \( n \)th eigenvalue of the Laplacian operator subject to homogenious Dirichlet boundary condition. In what follows we also denote by \( \phi_n \) the eigenfunction corresponding to \( \lambda_n \) with \( |\phi_n|^2 = 1 \). Note that \((g)\) implies that \( g(0) = 0 \) and hence \( u = 0 \) is a trivial solution of \((1.1)\).

Theorem 1.1. If \( g \) satisfies \((g)\), then \((1.1)\) has at least three nontrivial solutions.

As a special case of the theorem, let \( f \in C^1(\mathbb{R}) \) be such that

\[
\lim_{|u| \to 0} \frac{f(t)}{t} = 0, \quad \lim_{|u| \to \infty} \frac{f(t)}{t} = +\infty. \tag{1.2}
\]

We have the following result.

Corollary 1.2. Suppose \( \lambda > \lambda_2, \lambda \neq \lambda_n \) for all \( n \), and \((1.2)\) holds, then the boundary value problem

\[
\begin{align*}
-\Delta u &= \lambda u - f(u), \quad \text{in } \Omega, \\
u &= 0, \quad \text{on } \partial \Omega,
\end{align*}
\]

has at least three nontrivial solutions.

2. The First Two Nontrivial Solutions

It is well known that variational methods are very useful in studying the existence and multiplicity of solutions for elliptic boundary value problems. However, since we have no restriction on the growth rate of the right hand side of the equation in \((1.1)\), there is not a functional \( \Phi : H_0^1(\Omega) \to \mathbb{R} \) whose critical points are solutions of \((1.1)\), so variational methods are not directly applicable. A well known method to overcome this is the truncation method.

Set

\[
g_+(t) = \begin{cases} 
g(t), & \text{if } 0 \leq t \leq a^+, \\
0, & \text{if } t < 0 \text{ or } t > a^+
\end{cases}
\]
and consider the truncated problem
\[
\begin{cases}
-\Delta u = g_+(u), & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega.
\end{cases}
\]
(2.1)

The following result is well known to experts, since we can not find a reference containing the proof, we would like to present the detailed proof.

**Lemma 2.1.** Suppose \( u \in C^2(\Omega) \cap C(\overline{\Omega}) \) is a classical solution of (2.1), then
\[
0 \leq u(x) \leq a^+, \quad \text{for } x \in \Omega.
\]
(2.2)

Hence \( u \) is also a classical solution of (1.1).

**Proof.** Firstly we show
\[
M = \max_{\overline{\Omega}} u \leq a^+.
\]
If this is not true, then there is \( x_0 \in \Omega \) such
\[
u(x_0) = M > a^+.
\]
Let \( U = u^{-1}(M) \), then \( U \) is a closed subset of \( \Omega \).

Now pick \( x \in U \), then
\[
u(x) = M > a^+
\]
So there is \( r > 0 \) such that
\[
u(y) > a^+, \quad \text{for all } y \in B_r(x).
\]
Hence \( g_+(u) \equiv 0 \) in \( B_r(x) \) and because \( u \) is a classical solution of (2.1), it is harmonic in \( B_r(x) \).

Therefore, since
\[
u(x) = \max_{B_r(x)} u,
\]
u must be a constant function in \( B_r(x) \). We conclude that \( B_r(x) \subset U \).

Consequently, \( U \) is not only closed but also open in \( \Omega \). Hence \( U = \Omega \). This is a contradiction to the condition that \( u = 0 \) on \( \partial \Omega \).

In a similar manner, we can show that
\[
m = \min_{\overline{\Omega}} u \geq 0,
\]
and the desired result follows. \( \square \)

Since \( g_+ \) is bounded, let
\[
G_+(t) = \int_0^t g_+(s)ds,
\]
we may define a functional \( \Phi_+: H^1_0(\Omega) \rightarrow \mathbb{R}, \)
\[
\Phi_+(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \int_\Omega G_+(u)dx.
\]
Then \( \Phi_+ \) is of class \( C^1 \) and because we have assumed \( g \in C^1(\mathbb{R}) \), the critical points of \( \Phi_+ \) are classical solutions of (2.1).

**Lemma 2.2.** \( \Phi_+ \) is coercive and attains its minimum at some \( u_+ \in H^1_0(\Omega) \), which is a nontrivial solution of (2.1).

**Proof.** Firstly, by (g) we see that, there is \( \delta > 0 \), such that if \( u \in (0, \delta) \) then
\[
G_+(u) \geq \frac{\lambda_2}{2} u^2.
\]
Now for \( s \in (0, |\phi_1|_{\infty}^{-1} \delta) \), we have
\[
0 < s\phi_1(x) < \delta, \quad \text{for } x \in \Omega.
\]
Hence
\[
\Phi_+(s\phi_1) = \frac{\lambda_1}{2} s^2 \int_\Omega |\phi_1|^2 dx - \int_\Omega G_+(s\phi_1)dx
\]
\[
< \frac{\lambda_1 s^2}{2} \int_\Omega |\phi_1|^2 \, dx - \frac{\lambda_2 s^2}{2} \int_\Omega |\phi_1|^2 \, dx \\
= \frac{\lambda_1 - \lambda_2}{2} s^2 \int_\Omega |\phi_1|^2 \, dx < 0.
\]

Consequently,
\[
\inf_{H_0^1(\Omega)} \Phi_+ < 0.
\]

Obviously, \(g_+\) is bounded, hence there exist \(C_1 > 0, C_2 > 0\) such that
\[
|G_+(u)| \leq C_1 + C_2 |u|.
\]
Hence
\[
\Phi_+(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \int_\Omega G_+(u) \, dx \\
\geq \frac{1}{2} ||u||^2 - C_1 |\Omega| - C_2 |u|_1 \\
\geq \frac{1}{2} ||u||^2 - C_2 S_2 ||u|| - C_1 |\Omega| \to +\infty,
\]
where \(S_p\) is the Sobolev constant of the embedding \(H_0^1(\Omega) \hookrightarrow L^p(\Omega)\),
\[
||u||_p \leq S_p ||u||, \quad u \in H_0^1(\Omega).
\]
We see that \(\Phi_+\) is coercive. It is then well known that there is a \(u_+ \in H_0^1(\Omega)\) such that
\[
\Phi_+(u^+) = \inf_{H_0^1(\Omega)} \Phi_+ < 0.
\]
Since \(\Phi_+(0) = 0\), it follows that \(u^+\) is a nonzero critical point of \(\Phi_+\). Thus \(u^+\) is a nontrivial solution of (2.1). \(\square\)

Actually by elliptic regularity, \(u^+\) is a classical solution of (2.1). According to Lemma 2.1, \(u^+\) satisfies (2.2) and it is a positive solution of (1.1).

In a similar manner, setting
\[
\tilde{g}_-(t) = \begin{cases} 
 g(t), & \text{if } a^- \leq t \leq 0, \\
 0, & \text{if } t > 0 \text{ or } t < a^- 
\end{cases}
\]
and consider
\[
\begin{cases} 
 -\Delta u = g_-(u), & \text{in } \Omega, \\
 u = 0, & \text{on } \partial\Omega,
\end{cases}
\]
we can obtain a negative solution \(u^-\) of the problem (1.1).

3. The third nontrivial solution

To get the third nontrivial solution, we shall apply Morse theory (see [3] or [5, Chapter 8] for a systematic exposition). The key concept in this theory is critical group.

Let \(\Phi\) be a \(C^1\)-functional in a Banach space \(X\) and \(u\) a critical point of \(\Phi\) with \(\Phi(u) = c\). We call
\[
C_q(\Phi, u) = H_q(\Phi^c, \Phi^c \setminus \{u\})
\]
the \(q\)th critical group of \(\Phi\) at \(u\), where \(q = 0, 1, \ldots, H_q\) stands for the \(q\)th singular homology group with coefficient in \(\mathbb{Z}\), and \(\Phi^c = \Phi^{-1}(\infty, c]\).

It is known that if \(u\) is a critical point produced via the mountain pass lemma of Ambrosetti and Rabinowitz [1], then
\[
C_1(\Phi, u) \neq 0.
\]

To prove our theorem, we need to truncate the problem once more. Let
\[
\tilde{g}(t) = \begin{cases} 
 g(t), & \text{if } a^- \leq t \leq a^+, \\
 0, & \text{if } t \notin [a^-, a^+] 
\end{cases}
\]
and consider
\[
\begin{cases}
-\Delta u = \tilde{g}(u), & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega.
\end{cases}
\]  
(3.1)

The solutions of (3.1) are critical points of the \( C^{2-0} \)-functional \( \Phi : H^1_0(\Omega) \to \mathbb{R} \),
\[
\Phi(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \int_\Omega G(u), \quad G(u) = \int_0^u \tilde{g}(s) \, ds.
\]

Similar to Lemma 2.1, we have the following result.

**Lemma 3.1.** Suppose \( u \in C^2(\Omega) \cap C(\overline{\Omega}) \) is a classical solution of (3.1), then
\[
a^- \leq u(x) \leq a^+, \quad \text{for } x \in \Omega.
\]  
(3.2)

Hence \( u \) is also a classical solution of (1.1).

Consider the solution \( u^+ \) of (1.1) obtained in the last section: it is a global minimizer of \( \Phi_+ \) on \( H^1_0(\Omega) \). By strong maximum principle we know that
\[
u^+ > 0 \quad \text{in } \Omega, \quad \frac{\partial u^+}{\partial \nu} > 0 \quad \text{on } \partial \Omega,
\]  
(3.3)

where \( \nu \) is the interior normal on \( \partial \Omega \). From this it follows that \( u^+ \) is an interior point of the set
\[
P = \left\{ u \in C^1(\overline{\Omega}) \big| u > 0 \text{ in } \Omega, \, u = 0 \text{ on } \partial \Omega \right\}
\]
with respect to the \( C^1 \)-topology. In fact, if this is not true, we may find \( \{v_n\} \subset C^1(\overline{\Omega}) \) and \( \{x_n\} \subset \Omega \) such that \( v_n(x_n) < 0 \) and
\[
\sup_\Omega |v_n - u^+| + \sup_\Omega |\nabla v_n - \nabla u^+| \to 0.
\]  
(3.4)

Since \( \Omega \) is bounded, we may assume \( x_n \to x^* \in \overline{\Omega} \). Because
\[
|v_n(x_n) - u^+(x^*)| \leq |v_n(x_n) - u^+(x_n)| + |u^+(x_n) - u^+(x^*)| \\
\leq \sup_\Omega |u^+ - v_n| + |u^+(x_n) - u^+(x^*)| \to 0,
\]
we see that
\[
u^+(x^*) = \lim_{n \to \infty} v_n(x_n) \leq 0.
\]

Because \( u^+ > 0 \) in \( \Omega \), we deduce \( x^* \in \partial \Omega \). Similarly, using (3.4) again we have
\[
\nabla u^+(x^*) = \lim_{n \to \infty} \nabla v_n(x_n) = 0,
\]

Thus
\[
\frac{\partial u^+}{\partial \nu} \bigg|_{x^*} = \nabla u^+(x^*) \cdot \nu = 0,
\]
a contradiction to (3.3).

Thus, there exists \( r > 0 \) such that if \( u \in C^1(\overline{\Omega}) \), \( u = 0 \) on \( \partial \Omega \) and
\[
\|u - u^+\|_{C^1} < r,
\]
then \( u > 0 \) in \( \Omega \). Consequently, \( \Phi(u) = \Phi_+(u) \). Therefore, \( u^+ \) is also a local minimizer of \( \Phi \) in \( C^1 \)-topology.

By a result of Brézis and Nirenberg [2], we conclude that \( u^+ \) is a local minimizer of \( \Phi \) in the \( H^1 \)-topology.

Similarly, the negative solution \( u^- \) of (1.1) is also a local minimizer of \( \Phi \). Because \( \Phi \) satisfies the Palais-Smale condition, from the two local minimizer \( u^+ \) and \( u^- \) we can obtain a third critical point \( u^* \) of \( \Phi : H^1_0(\Omega) \to \mathbb{R} \) via the mountain pass lemma. Since \( u^* \) is of mountain pass type, we have
\[
C_1(\Phi, u^*) \neq 0.
\]  
(3.5)

On the other hand, by our assumption (g) and [4, Proposition 1.1], we have
\[
C_q(\Phi, 0) = \begin{cases}
\mathbb{Z}, & \text{if } q = k, \\
0, & \text{if } q \neq k.
\end{cases}
\]  
(3.6)
Since \( k \geq 2 \), From (3.5) and (3.6) we see that
\[
C_1(\Phi, u^*) \neq C_1(\Phi, 0).
\]
Hence, \( u^* \neq 0 \), which is the third nontrivial solution of (1.1).

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