Optimization in designing heat flux concentrators

Gennady Alekseev¹,² and Dmitry Tereshko¹
¹ Institute of Applied Mathematics FEB RAS, 7, Radio St., Vladivostok, Russia
² School of Natural Sciences, Far Eastern Federal University, Vladivostok, Russia
E-mail: alekseev@iam.dvo.ru

Abstract. Inverse problems for a 2D heat conduction model connected with developing technologies of designing thermal concentrators are considered. The shells are assumed consisting of a finite number of sectors every of which is filled with homogeneous isotropic medium. By optimization method these problems are reduced to finite-dimensional extremum problems for the solution of which the particle swarm optimization is used. Rigorous optimization analysis shows that concentrators designed using optimization method have the highest efficiency in the class of devices under consideration and are easy for implementation.

1. Introduction
In recent years, much attention has been devoted to the development of methods for solving the design problems of cloaking shells, concentrators, rotators and other special functional devices used to control thermal fields. In the first works in this field [1–3], the authors used the transformation optics (TO) method developed in pioneering papers [4, 5] to solve the problems of electromagnetic cloaking. The solutions obtained using this method play an important role in theoretical analysis, but they also have a number of drawbacks. The difficulty of technical implementation is one of the main disadvantages.

That is why other methods for solving these design problems began to develop in addition to the TO method. Among them, a special role is played by the optimization method, which, starting with the fundamental works of A.N. Tikhonov [6], is widely used in solving applied inverse problems. A number of papers have been devoted to the application of the optimization method to solve the design problems of thermal functional devices (see, for example, [7–18]). Methods of topological optimization or particle swarm optimization were used in the cited papers for the numerical solution of the corresponding extremum problems. An alternative approach is used in [19,20] when solving problems of acoustic cloaking.

In this paper, the optimization method is used to solve two inverse problems for the 2D heat conduction model associated with the concentration or concentration and cloaking of heat fluxes. Based on the analysis of computational experiments performed using the particle swarm optimization according to the scheme proposed in [15], we show that the developed method allows us to design concentrators that have the highest efficiency in the considered class of devices and the simplicity of technical implementation.

2. Statement of direct heat conduction problem
Let us start with the formulation of the direct heat conduction problem, considered in a rectangle \( D = \{ x \equiv (x, y) : |x| < x_0, |y| < y_0 \} \). We will assume that the background field \( T^b \) is created
by two vertical plates \( x = \mp x_0 \) heated to different temperatures \( T_1 \) and \( T_2 \), while the lower and upper boundaries are thermally insulated. It is also assumed that inside \( D \) there is a shell \((\Omega, k)\) where \( \Omega \) is a circular layer \( \{a < |x| < b\} \) with boundaries \( \Gamma_i, \Gamma_e \) and \( k \) is the thermal conductivity of the inhomogeneous medium filling \( \Omega \). We suppose that the interior \( \Omega_i: |x| < a \) and the exterior \( \Omega_e: |x| > b \) of the domain \( \Omega \) are filled with the same homogeneous medium having the constant thermal conductivity \( k_b \) (see Figure 1).

In this case, the direct problem of heat conduction consists of determination of three functions, namely \( T_i \) in \( \Omega_i \), \( T \) in \( \Omega \), and \( T_e \) in \( \Omega_e \), that satisfy the equations

\[
k_b \Delta T_i = 0 \quad \text{in} \quad \Omega_i, \quad \text{div} (k \, \text{grad} \, T) = 0 \quad \text{in} \quad \Omega, \quad k_b \Delta T_e = 0 \quad \text{in} \quad \Omega_e,
\]

with boundary conditions

\[
T_e|_{x=-x_0} = T_1, \quad T_e|_{x=x_0} = T_2, \quad \frac{\partial T_e}{\partial y}|_{y=\pm y_0} = 0,
\]

and the matching conditions on the boundaries \( \Gamma_i \) and \( \Gamma_e \) of the shell \( \Omega \)

\[
T_i = T, \quad k_b \frac{\partial T_i}{\partial n} = k \frac{\partial T}{\partial n} \quad \text{on} \quad \Gamma_i, \quad T_e = T, \quad k_b \frac{\partial T_e}{\partial n} = k \frac{\partial T}{\partial n} \quad \text{on} \quad \Gamma_e.
\]

3. Statements of control problems

We assume that the shell \( \Omega \) is symmetric with respect to the coordinate axes and consists of \( 4M \) sectors \( \Omega_j \) filled with homogeneous isotropic materials, where \( M \) is the number of sectors in the first quarter \( \Omega^* \). Then, thermal conductivity \( k(x) \) can be represented as

\[
k(x) = \sum_{j=1}^{M} k_j \chi_j(x), \quad x \in \Omega^*.
\]
Here, $\chi_j(x)$ is a characteristic function of the sector $\Omega_j \in \Omega^*$, and $k_j$ is the thermal conductivity of the medium filling $\Omega_j$, $j = 1, 2, \ldots, M$.

The problem of pure concentration consists in finding the parameters of the medium filling $\Omega$ from the condition of the maximum concentration of the heat flux in $\Omega_i$. The second scenario also requires that the concentrator distort the field $T^b$ in $\Omega_c$ as little as possible (see [9]). These problems reduce to finding the numbers $k_j$ by solving extremum problems. To formulate them, we introduce an $M$-dimensional vector $k = (k_1, \ldots, k_M)$ and define a set $K$ by the formula

$$K = \{k = (k_1, \ldots, k_M) : k_{\min} \leq k_j \leq k_{\max}, j = 1, 2, \ldots, M\}.$$  

(5)

Here $k_{\min}$ and $k_{\max}$ are given positive constants.

Let $T[k]$ be a solution of problem (1)-(3) for function $k(x)$ in (4). We introduce the functionals

$$J_i(k) = \frac{a}{b} \|\nabla T[k]\|_{L^2(\Omega_i)}, \quad J_e(k) = \frac{\|T[k] - T^b\|_{L^2(\Omega_c)}}{\|T^b\|_{L^2(\Omega_c)}}.$$  

(6)

The maximum of the functional $J_i(k)$ on the set $K$ does not exceed 1. Therefore, $J_i(k)$ has the meaning of a normalized measure of the concentration efficiency of the shell $(\Omega, k)$; the closer $J_i(k)$ to unity, the higher the efficiency of the shell $(\Omega, k)$. Similarly, $J_e$ is a measure of the perturbation of the background field $T^b$ in $\Omega_c$. The smaller $J_e(k)$, the higher the external cloaking effect created by the shell $(\Omega, k)$ (see [9]).

The problem of concentrating the heat flux in $\Omega_i$ reduces to the problem of maximizing the functional $J_i(k)$, which is equivalent to the minimization problem

$$-J_i(k) \rightarrow \min, \quad k \in K.$$  

(7)

Problem (7) is aimed at designing a device with the highest concentration efficiency, but its solution may have no cloaking effect. Therefore, we will also consider a more general problem:

$$J_\alpha(k) = \alpha J_e(k) - J_i(k) \rightarrow \min, \quad k = (k_1, \ldots, k_M) \in K, \quad \alpha = \text{const} \geq 0.$$  

(8)

To solve problems (7) and (8), we use particle swarm optimization (PSO) [21].

4. Simulation results

As in [13], we choose the values $x_0 = 4.5$ cm, $y_0 = 9$ cm, $a = 1$ cm, $b = 3.5$ cm, $T_1 = 321.25$ K, $T_2 = 283.15$. Our first choice of materials is described by following parameters:

$$k_b = 16 k_0, \quad k_{\min} = 0.03 k_0, \quad k_{\max} = 116 k_0, \quad k_{\max}/k_{\min} = 3867,$$  

(9)

where $k_0 = 1$ W/(m K). These parameters correspond to stainless steel, polystyrene and zinc, respectively. In (9), along with $k_b$, $k_{\min}$, and $k_{\max}$, we also give the contrast $k_{\max}/k_{\min}$. The number of controls $M$ was even and ranged from 4 to 20.

The use of PSO for solving problem (7) showed that, for all $M$, the controls $k^{opt}_j$ take one of the values $k_{\min}$ or $k_{\max}$ and

$$k^{opt}_1 = k^{opt}_2 = \ldots = k^{opt}_p = k_{\max}, \quad k^{opt}_{p+1} = \ldots = k^{opt}_M = k_{\min}.$$  

(10)

Here, $p \in [1, M]$ is a certain number depending on $M$. It follows from (10) that the designed shell $(\Omega, k^{opt})$ consists of four global sectors shown for $M = 10$ and $p = 6$ in Figure 2. The calculated values of $p$, $J_i(k^{opt})$ and $J_e(k^{opt})$ are given in Table 1.
Figure 2. Structure of the optimal concentrator for $\alpha = 0$.

Figure 3. Temperature field for optimal concentrator ($\alpha = 0$, $M = 16$).

Table 1. Numerical results for $\alpha = 0$, $k_b = 16 k_0$, $k_{min} = 0.03 k_0$, $k_{max} = 116 k_0$.

| $M$ | $p$ | $J_i(k^{opt})$ | $J_e(k^{opt})$ | CE |
|-----|-----|----------------|----------------|----|
| 4   | 3   | 0.863          | $1.95 \times 10^{-3}$ | 0.815 |
| 8   | 5   | 0.868          | $1.75 \times 10^{-3}$ | 0.826 |
| 12  | 8   | 0.869          | $1.79 \times 10^{-3}$ | 0.826 |
| 16  | 11  | 0.869          | $1.83 \times 10^{-3}$ | 0.827 |
| 20  | 13  | 0.869          | $1.77 \times 10^{-3}$ | 0.828 |

In addition, Table 1 gives values of the quantity

$$CE = \frac{|T^{opt}(a, 0) - T^{opt}(-a, 0)|}{|T^{opt}(b, 0) - T^{opt}(-b, 0)|},$$

used in [11, 22, 23] as a measure of the concentration efficiency of the designed concentrators. Note that, in our work, we use as a measure of the efficiency of the shell $(\Omega, k)$ the quantity $J_i(k)$ defined in (6). In the general case, the values of CE and $J_i(k)$ do not coincide.

In the second test we use the parameters

$$k_b = 16 k_0, \quad k_{min} = 0.03 k_0, \quad k_{max} = 427 k_0, \quad k_{max}/k_{min} = 14233,$$

that correspond to stainless steel, polystyrene and silver, respectively. The values of $p$, $J_i(k^{opt})$, $J_e(k^{opt})$ and CE are given in Table 2. Temperature field for $M = 16$ is shown in Figure 3.

Comparison of Tables 1 and 2 shows that an increase in the contrast $k_{max}/k_{min}$ (from 3867 to 14233) significantly increases the efficiency of the designed concentrators. However, their cloaking efficiency described by the value $J_e(k^{opt}) \approx 10^{-3}$ is not high.

Now we turn to problem (8) and discuss the results obtained for $\alpha = 10^3$. In this case, in contrast to (10), the solutions satisfy the relations

$$k_1^{opt} = k_3^{opt} = \ldots = k_{M-1}^{opt} = k_{max}, \quad k_2^{opt} = k_4^{opt} = \ldots = k_M^{opt} = k_{min},$$

(13)
Table 2. Numerical results for $\alpha = 0$, $k_b = 16 k_0$, $k_{\text{min}} = 0.03 k_0$, $k_{\text{max}} = 427 k_0$.

| $M$ | $p$ | $J_i(k^{\text{opt}})$ | $J_e(k^{\text{opt}})$ | CE  |
|-----|-----|----------------------|----------------------|-----|
| 4   | 3   | 0.953                | $2.22 \times 10^{-3}$ | 0.939 |
| 8   | 5   | 0.956                | $1.81 \times 10^{-3}$ | 0.946 |
| 12  | 8   | 0.957                | $1.92 \times 10^{-3}$ | 0.944 |
| 16  | 11  | 0.957                | $1.93 \times 10^{-3}$ | 0.943 |
| 20  | 13  | 0.957                | $1.87 \times 10^{-3}$ | 0.945 |

It can be seen from Table 3 that $J_i(k^{\text{opt}})$ increases and $J_e(k^{\text{opt}})$ decreases with increasing $M$. 

Table 3. Numerical results for $\alpha = 10^3$, $k_b = 16 k_0$, $k_{\text{min}} = 0.03 k_0$, $k_{\text{max}} = 427 k_0$.

| $M$ | $J_i(k^{\text{opt}})$ | $J_e(k^{\text{opt}})$ | CE  |
|-----|----------------------|----------------------|-----|
| 4   | 0.901                | $7.87 \times 10^{-4}$ | 0.942 |
| 8   | 0.913                | $3.67 \times 10^{-4}$ | 0.937 |
| 12  | 0.921                | $2.63 \times 10^{-4}$ | 0.936 |
| 16  | 0.924                | $2.45 \times 10^{-4}$ | 0.935 |
| 20  | 0.928                | $2.19 \times 10^{-4}$ | 0.934 |

Figure 4. Structure of the optimal concentrator for $\alpha = 10^3$.

Figure 5. Temperature field for optimal concentrator ($\alpha = 10^3$, $M = 16$).
Thus, an increase in $M$ leads to an increase in both the concentration and cloaking efficiency of the shells. The corresponding temperature field for $M = 16$ is shown in Figure 5.

5. Conclusion
In this paper we studied control problems for 2D model of heat conduction (1)–(3) associated with designing cylindrical thermal concentrators. Using particle swarm optimization, we showed that for the construction of highly efficient concentrators it is sufficient to use only two materials with high contrast conductivities $k_{\text{min}}$ or $k_{\text{max}}$. These results can be widely used in creating easily implemented heat fluxes concentrators of a new type, which store thermal energy with high efficiency.

References
[1] Guenneau S, Amra C and Veynante D 2012 Opt. Express 20 8207
[2] Narayana S and Sato V 2012 Phys. Rev. Lett. 108 214303
[3] Han T, Yuan T, Li B and Qiu C W 2013 Sci. Repuk. 3 1593
[4] Pendry J B, Schurig D and Smith D R 2006 Science 312 1780–2
[5] Leonhardt U 2006 Science 312 1777–80
[6] Tikhonov A N and Arsenin V Y 1977 Solutions of Ill-Posed Problems (New York: Winston)
[7] Dede E M, Nomura T and Lee J 2014 Struct. Multidiscip. 49 59–68
[8] Alekseev G V and Levin V A 2016 Dokl. Phys. 61 546–50
[9] Alekseev G V, Levin V A and Tereshko D A 2017 Dokl. Phys. 62 71–5
[10] Alekseev G V, Levin V A and Tereshko D A 2017 Dokl. Phys. 62 465–9
[11] Peralta I, Fachinotti V D and Ciarbonetti A A 2017 Sci. Repuk. 7 40591
[12] Fujii G, Akimoto Y and Takahashi M 2018 Appl. Phys. Lett. 112 061108
[13] Fachinotti V D, Ciarbonetti A A, Peralta I and Rintoul I 2018 Int. J. Therm. Sci. 128 38–48
[14] Alekseev G V 2018 Comput. Math. Math. Phys. 58 478–92
[15] Alekseev G V and Tereshko D A 2019 Int. J. Heat Mass Tran. 135 1269–77
[16] Alekseev G V, Levin V A and Tereshko D A 2019 J. Appl. Mech. Tech. Phys. 60 323–31
[17] Alekseev G and Tereshko D 2019 J. Phys.: Conf. Ser. 1268 012004
[18] Alekseev G V, Levin V A and Tereshko D A 2020 Doklady Phys. 65 115–8
[19] Romanov V G 2010 Doklady Math. 81 238–40
[20] Romanov V G and Chirkunov Yu A 2013 Doklady Math. 87 73–5
[21] Poli R, Kennedy J and Blackwel T 2007 Swarm Intel. 1 33–57
[22] Chen F and Lei D Y 2015 Sci. Rep. 5 11552
[23] Xu G, Zhou X and Zhang J 2019 Int. J. Heat Mass Transf. 142 118434