Anyons in 1 + 1 Dimensions

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Abstract

The possibility of excitations with fractional spin and statistics in 1 + 1 dimensions is explored. The configuration space of a two-particle system is the half-line. This makes the Hamiltonian self-adjoint for a family of boundary conditions parametrized by one real number $\gamma$. The limit $\gamma \to 0, (\infty)$ reproduces the propagator of non-relativistic particles whose wavefunctions are even (odd) under particle exchange. A relativistic ansatz is also proposed which reproduces the correct Polyakov spin factor for the spinning particle in 1 + 1 dimensions. These checks support validity of the interpretation of $\gamma$ as a parameter related to the “spin” that interpolates continuously between bosons ($\gamma = 0$) and fermions ($\gamma = \infty$). Our approach can thus be useful for obtaining the propagator for one-dimensional anyons.

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1. Introduction

Physics in $2+1$ dimensions has been extensively studied in the past few years in connection with topological field theory \[1\] and with the hope of finding interesting applications in condensed matter systems \[2\]. In this last context, the interest arises from experimental observations such as the quantum Hall effect or high-$T_c$ superconductivity in systems that are intrinsically two-dimensional and whose quantum excitations are expected to be anyonic.

A system of two particles in a plane is the manifold $\mathbb{R}^2 - \{0\}$ and for this reason the configuration space of a system of many particles in $2+1$ dimensions is multiply connected. In $2+1$ dimensions a particle interacts with another as with a composite object made out of a particle and a magnetic flux. The dynamics of a system of particles in the plane can thus be understood via an Aharonov-Bohm mechanism and the fractional statistics can be seen to emerge as a consequence of this fact. Furthermore, in $2+1$ dimensions, the rotation group is abelian and therefore the angular momentum need not be quantized \[3\].

In $1+1$ dimensions the situation is topologically quite different and the existence of unusual statistics is hardly mentioned. A common procedure aimed at extracting information about the dynamics of (1+1)-dimensional systems is bosonization \[4\], i.e., a transformation that expresses the fermions in terms of bosonic fields and vice-versa. It is tacitly assumed, however, that the quantum excitations can be either bosonic or fermionic, and nothing else. Nevertheless, a minor modification of the usual Bose-Fermi transformations can lead to fields with intermediate statistics.

Furthermore, in one spatial dimension the rotation group is discrete and abelian so that, strictly speaking, spin does not exist. \[5\] In complete analogy with the $(2+1)$-dimensional case, one could also expect a continuous interpolation of spin and statistics between bosons and fermions.

\[1\] The rotation group in 1 spatial dimension reduces to the discrete group of reflections on a plane, $\mathbb{Z}_2$, so that the notion of angular momentum as related to the dimension of its irreducible representations becomes meaningless.
and fermions because the configuration space for a system of particles is also multiply connected and the group of reflections is abelian as well.

The possibility of having unusual spin and statistics in 1+1 dimensions is theoretically very attractive also because there exist one-dimensional systems (quantum wires) where properties like fractional spin could perhaps be experimentally observed [4].

Although the continuous interpolation of spin and statistics in 1 + 1 dimensions was suggested by Leinaas and Myrheim [3] (see also [5] and [6]), the discussion found in the literature still remains rather formal. (Recently, the possibility of fractional statistics in 1+1 dimensions has been considered following the definition of statistics proposed by Haldane [8], [9]. In this article we shall not adopt that approach.)

In (2+1) dimensions the effective theory obtained after integrating the Chern-Simons field yields an explicit relation between the spin and the Chern-Simons coefficients. There is no analogous construction in (1+1) dimensions because there is no Chern-Simons action in this case.

The purpose of this note is to present a 1+1 dimensional model with the properties mentioned above and to show how the notion of spin can be related to the physical parameters of the interaction.

2. Non-relativistic Model

Let us consider two non-relativistic identical particles moving on a line. For this system the configuration space is multiply connected (the real line minus the origin) because the point where the particles collide is singular. Classically the particles cannot go through each other, bouncing off elastically every time they meet. Thus, the action of this system is defined on the half-line [3,5], i.e.

$$S = \int_{t_1}^{t_2} dt \frac{1}{2} \dot{x}^2, \quad (0 < X < \infty),$$

where $x$ is the relative position of the two particles. As it is well known, the Hamiltonian associated to (1) is not self-adjoint on the naive Hilbert space because there is no conservation of probability at $x = 0$. The Hamiltonian for (1), however, can be made self-adjoint by
adopting a class of boundary conditions for all the states in the Hilbert space of the form

\[ \psi'(0) = \gamma \psi(0), \] 

where \( \gamma \) is an arbitrary real parameter.

The calculation of the propagator to go from an initial relative position \( x_1 \) to a final one \( x_2 \) for the above problem gives

\[ G_\gamma[x(t_2), x(t_1)] = G_0(x_2 - x_1) + G_0(x_2 + x_1) - 2\gamma \int_0^\infty d\lambda e^{-\gamma\lambda} G_0(x_2 + x_1 + \lambda), \] 

where \( G_0 \) is the Green function for a free non-relativistic particle, \( i.e \)

\[ G_0(x - y) = \frac{1}{\sqrt{2\pi it}} e^{i(x-y)^2/2t}. \] 

Although in one spatial dimension it is not possible to rotate particles, they can be exchanged and their “spin” and statistics can be determined by the (anti-) symmetry of the wave function. This (anti-) symmetry in turn depends on the values of the parameter \( \gamma \).

This last fact can be seen by taking the limits \( \gamma = 0 \) and \( \gamma = \infty \) of (3)

\[ G_{\gamma=0,\infty} = G_0(x_2 - x_1) \pm G_0(x_2 + x_1), \] 

Under exchange of the positions of two particles in initial or final states, \( G_{\gamma=0} \) is even and \( G_{\gamma=\infty} \) is odd. Thus, for \( \gamma = 0 \) (\( \gamma = \infty \)) and the particles behave as bosons (fermions). The cases \( 0 < \gamma < \infty \) give particles with fractional spin and statistics.

The propagator (3) can also be obtained in the path integral representation, summing over all paths \( -\infty < x(t) < \infty \), but in the presence of a repulsive potential \( \gamma \delta(x) \). This problem was considered in \( [12] - [13] \) and the result is

\[ G_\gamma[x(t_2), x(t_1)] = \int Dx(t) e^{iS}, \] 

with

\[ S = \int_{t_1}^{t_2} dt \left( \frac{1}{2} \dot{x}^2 + \gamma \delta(x(t)) \right), \]
Here $Dx(t)$ is the usual functional measure. The potential term $\gamma \delta(x(t))$ can be interpreted as a semi-transparent barrier at $x = 0$ that allows the possibility of tunneling to the other side of the barrier. This is just another way of expressing the possibility of interchanging the (identical) particles.

It is also interesting to note here that although in $(1+1)$ dimensions the rotation group is discrete and the definition of the spin is a matter of convention, one may nevertheless view the one-dimensional motion on the half-line as a radial motion with orbital angular momentum $l = 0$ \[ 14 \] in a central potential. This gives rise to another possible definition of spin by taking the following representation for the $\delta$-function

\[
\delta(x) = \lim_{\epsilon \to 0} \frac{\sqrt{\epsilon}}{x^2 + \epsilon}.
\] (8)

Making a series expansion around $\epsilon = 0$, the leading term $\gamma \sqrt{\epsilon}/x^2$ is analogous to the centrifugal potential for the radial equation in a spherically symmetric system, with $\sqrt{\epsilon} \gamma$ playing the role of an intrinsic angular momentum squared. Thus the spin of the system ($s$) can be defined by

\[
s^2 = \sqrt{\epsilon} \gamma.
\] (9)

For real $s$ (9) only makes sense when $\gamma > 0$. This definition is consistent with the bosonic limit $\gamma = 0$. For the fermionic case, the limit $\gamma = \infty$ mentioned above is to be interpreted as simultaneous with the limit $\epsilon \to 0$ so that $\sqrt{\epsilon} \gamma = 1/4$. It is in this sense that the non-relativistic quantum mechanics on the half-line describes one-dimensional anyons. However, the normalization $s = 1/2$ for fermions is conventional.

3. Relativistic Model

In this section we generalize the previous results to the relativistic case. In order to do this one may either canonically quantize two free relativistic particles moving on the line, or use path integrals methods.

Let us start discussing the first approach. In analogy with the non-relativistic case, one may write the relativistic generalization of (1) as
\[ S = \int_{\tau_a}^{\tau_b} d\tau \left[ \frac{1}{2N} \dot{X}^2 - \frac{m^2}{2} N \right], \quad (10) \]

where \( N \) is the einbein along the worldline, \( X^\mu = X_2^\mu - X_1^\mu \) is a spacelike four vector whose spatial component is restricted to be positive. In (10) the “conformal gauge” \( \tau_1 = \tau_2 = \tau \) has been chosen.

The next step is to construct a covariant generalization of (2), namely

\[ \partial_\mu \psi = \gamma_\mu \psi, \quad (11) \]

where \( \gamma_\mu = (\gamma_0, \gamma_1) \) is a two-vector whose components gives the two parameters that make the operator \( \partial_0^2 - \partial_1^2 + m^2 \) self-adjoint, and \( \psi \) is a solution of the Klein-Gordon equation

\[ (\partial_0^2 - \partial_1^2 + m^2)\psi = 0. \quad (12) \]

Conditions (11) are consistent only for spacelike \( \gamma_\mu \). This is because for timelike \( \gamma \) there exists a reference frame in which the particle would “bounce” at \( X^0 = 0 \) reversing its time direction and violating energy conservation. Thus, one concludes that in an appropriate (“rest”) frame, \( \gamma_\mu = (0, \gamma) \), or equivalently,

\[ \frac{\partial \psi}{\partial t} \big|_{t=0} = 0, \]

\[ \frac{\partial \psi}{\partial x} \big|_{x=0} = \gamma \psi(x = 0), \quad (13) \]

are consistent boundary conditions.

The calculation of the relativistic propagator is straightforward. In the proper-time gauge \( \dot{N} = 0 \), it gives

\[ G[X(\tau_b), X(\tau_a)] = \int_0^\infty dT \int_{-\infty}^\infty dp_0 dp_1 e^{-i\frac{T}{2}(p_0^2 - p_1^2 + m^2)} \psi^*(X(\tau_b)) \psi(X(\tau_a)). \quad (14) \]

Here \( \psi \) are solutions of (12) with the boundary condition (11), namely

\[ \psi = e^{ip_0X^0} \left[ e^{ip_1X^1} + Re^{-ip_1X^1} \right], \quad (15) \]

where \( R \) is the reflection coefficient defined as
\[ R = \frac{ip_1 - \gamma}{ip_1 + \gamma} \]  

(16)

Inserting (15) in (14) one finds the relativistic generalization of (3):

\[
G_\gamma[X(\tau_b), X(\tau_a)] = N \int_0^\infty dT T^{-1} \exp \left[ i \frac{(\Delta X^0)^2}{2T} - im^2 \frac{T}{2} \right] \left( \exp \left[ -i \frac{(X^1_b - X^1_a)^2}{2T} \right] \right. 
\exp \left[ -i \frac{(X^1_b + X^1_a)^2}{2T} \right] \left. \right) - 2\gamma F_\gamma, \tag{17}
\]

where \( N \) is a normalization constant, \( \Delta X = X^0_b - X^0_a \)

\[
F_\gamma = \int_0^\infty d\lambda \int_0^\infty dT T^{-1} e^{i(\Delta X^0)^2/2T - im^2 T/2 - \gamma \lambda + i(X^1_a + X^1_b + \lambda)^2/2T} \]

\[
= \int_0^\infty d\lambda e^{-\gamma \lambda} G_0[X(\tau_b) + X(\tau_a) + \lambda] \tag{18}
\]

and \( G_0[X(\tau_b) - X(\tau_a)] \) is the free relativistic propagator,

\[
G_0[X(\tau_b) - X(\tau_a)] = \int_0^\infty dT T^{-1} e^{i(\Delta X)^2/2T - im^2 T/2}. \tag{19}
\]

Now let us consider the path integral formulation. The generalization of (7) is the action

\[
S = \frac{1}{2} \int_{\tau_a}^{\tau_b} d\tau \left[ \frac{1}{N} X^2 + m^2 N + NV \right], \tag{20}
\]

where \( V \) is a contact potential chosen so that in an instantaneous rest frame the singularity occurs along the spatial direction,

\[
V|_{\text{rest}} = \gamma \delta(X^1(\tau)). \tag{21}
\]

In a generic frame the contact term would have a different expression but its physical meaning should remain unchanged. The presence of a \( \delta \) function in the action allows the inclusion in the sum over histories of trajectories that extend to \( X^1 < 0 \). These paths can be taken into account if for each one that bounces off at \( X = 0 \) one also includes the path produced by the reflection of the bounce to \( X^1 < 0 \). This means changing the final state \( X_b \) according to \( X^1_b \rightarrow -X^1_b \), which is just an exchange of the particles in the final state.

\[2\]For simplicity we work here in Euclidean space
Taking into account the contact terms, the propagator of the relativistic particle on the
half-line in the proper time gauge ($\dot{N} = 0$), is obtained:

$$G_{\gamma}[X(\tau_b), X(\tau_a)] = \int_0^\infty dN(0) \exp \left[ -\frac{m^2N(0) \Delta t}{2} \right] \int D X^0 \exp \left[ - \int_{\tau_a}^{\tau_b} d\tau \frac{1}{2N(0)} (\dot{X}^0)^2 \right]$$

$$\times \int D X^1 \exp \left[ - \int_{\tau_a}^{\tau_b} d\tau \left( \frac{1}{2N(0)} (\dot{X}^1)^2 + i\gamma\delta(X^1(\tau)) \right) \right],$$

where $N(0)$ is the zero mode of $N(\tau)$.

The path integral over $X^0$ can be computed in analogy with a non-relativistic particle
with ‘mass’ $N^{-1}(0)$ in the interval $-\infty < X^0 < +\infty$. The result is

$$\int D X^0 \exp \left[ - \int_{\tau_a}^{\tau_b} d\tau \frac{(\dot{X}^0)^2}{2N(0)} \right] = \frac{1}{\sqrt{2\pi iT}} \exp \left[ -\frac{(\Delta X^0)^2}{2T} \right]$$

with $T = N(0) \Delta \tau$.

The integral in $X^1$ can now be computed as the path integral for a non-relativistic particle
with ‘mass’ $N^{-1}(0)$ moving on the interval $-\infty < X^1 < \infty$. The relativistic propagator is then

$$G_{\gamma}[X(\tau_b), X(\tau_a)] = G_0[X(\tau_b) - X(\tau_a)] + G_0[X(\tau_b) + X(\tau_a)]$$

$$- 2\gamma \int_0^\infty d\lambda e^{-\gamma\lambda} G_0[X(\tau_b) + X(\tau_a) + \lambda].$$

One may now check the consistency if (24) by taking the non-relativistic limit of (17).

The integration in $T$ is straightforward and the result is

$$G_{\gamma}[X(t_b), X(t_a)] = N \left[ K_0 \left( mc\Delta t \sqrt{1 - \frac{(X^1_b - X^1_a)^2}{c^2\Delta t^2}} \right) + K_0 \left( mc\Delta t \sqrt{1 - \frac{(X^1_b + X^1_a)^2}{c^2\Delta t^2}} \right) \right]$$

$$- 2\gamma \int_0^\infty d\lambda e^{-\gamma\lambda} K_0 \left( mc\Delta t \sqrt{1 + \frac{(X^1_b + X^1_a + \lambda)^2}{c^2\Delta t^2}} \right),$$

where we have inserted explicitly the speed of light ($c$) and $N$ is a normalization constant.

In the limit $c \to \infty$ the Bessel functions $K_\nu$ become

$$K_\nu(z) \to \sqrt{\frac{\pi}{z}} e^{-z}, \quad z >> 1,$$
for any order $\nu$. Using (26) in Minkowski space, (26) becomes\footnote{The exponential $e^{-mc^2t}$ (or $e^{imc^2t}$, when rotated back to Minkowski space) can be absorbed in a renormalization of the energy by $H \to H - mc^2$ or by a constant shift in the Lagrangian $L \to L + mc^2$.}

\[ G_\gamma[X(t_b), X(t_a)] = G_0(X_b - X_a) + G_0(X_b + X_a) \]

\[ -2\gamma \int_0^\infty d\lambda e^{-\gamma\lambda + imc^2\Delta t}G_0(X_b + X_a + \lambda), \]

which is the correct integral representation for the non-relativistic propagator on the half-line given in eq. (3).

Finally, one can observe that, after eliminating the Lagrange multiplier $N(\tau)$ from the relativistic action –by solving its own equation of motion–, the effective action for the system of two particles in a line can be written as

\[ S[X(\tau_2), X(\tau_1)] = m \int_{\tau_1}^{\tau_2} d\tau \sqrt{(\dot{X})^2 + \gamma \times n}, \]

where $n$ is the number of times $X^1(\tau)$ vanishes in the interval $\tau_1 < \tau < \tau_2$. In other words, the contact term (21) merely counts the number of crossings at $X^1 = 0$ of each trajectory.

4. Conclusions

a. We have constructed a model for anyons in 1 + 1 dimensions that is the analog of the (2 + 1)-dimensional case, both in the non-relativistic limit, as well as in its relativistic extension. The model is based on the observation that classically a system of two particles on a line is equivalent to that of one particle on a half-line. The statistics of the system can be read off from the propagator (3) in the non-relativistic case, and from (17) (or (26)) in the relativistic case, which is also the exact formula for a free scalar field theory on the half-line. The model can be considered also as the relativistic generalization of the problem discussed by Clark, Menikoff and Sharp \cite{12} and Farhi and Gutmann \cite{13}.

b. The parameter $\gamma$, which is related to the properties of the contact interaction between identical particles, is the analogue of the Chern-Simons coefficient $\sigma$. The (2+1)-dimensional
expression for the spin, \( s = \sigma/4\pi \), is replaced in \( 1 + 1 \) dimensions by \( s = \sqrt{\epsilon \gamma} \). In the non-relativistic case, our model provides a new interpretation of the model discussed in [12,13]. The interpretation of \( \gamma > 0 \) as the spin of the particle remains valid in the relativistic case because (9) is the right expression for the coefficient of the centrifugal potential for a relativistic particle in a spherically symmetric field.

**c.** The non-relativistic quantum system on the half-line is a particular case of the Yang-Yang model for two particles [16], and the relativistic result presented here can be seen as the corresponding generalization of that model. This may also be seen as the reason *in profundis* for the appearance of anyons in \( 1 + 1 \) dimensions.

**d.** The relativistic propagator in \( 1+1 \) dimensions can also be obtained for the special case of spin \( 1/2 \) by computing the Polyakov spin factor of the spinning particle. This calculation is straightforward [17] and gives the effective action \( m\sqrt{X^2} + \nu \), where \( \nu \) is the number of self-intersections of the wordline [18].

Our results seem to agree in spirit with the spin factor calculation of [17], but with \( \nu = \gamma n \) [c.f. eq. (28)], where \( n \) is the number of collisions between the particles [e.g., collisions with the barrier at \( X^1 = 0 \), in the formulation of the problem with the restriction \( X^1 > 0 \), or the number of crossings through the barrier at \( X^1 = 0 \) in the formulation without that restriction (using the contact potential)]. The connection with the spin factor calculation can be made more manifest by the following argument: Take all the nonintersecting paths that join a given pair of points in \( X^1 \)-space, and for each of them, take also its reflection about \( X^1 = 0 \). Then the number of self intersections of the loop formed by the two paths (joining their ends at \( t = -\infty \) and at \( t = +\infty \)), is just \( n \). So, modulo a normalization, the effective action obtained by Polyakov’s approach is the same as the one given here. There remains a puzzling point, however, because the two actions would exactly match for \( \gamma = 1 \), which is neither the bosonic \( (\gamma = 0) \) nor the fermionic case \( (\gamma = \infty) \).
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