THE \textit{r}-MODE OSCILLATIONS IN RELATIVISTIC ROTATING STARS

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\section{INTRODUCTION}

In recent years, the \textit{r}-mode oscillations in rotating stars have been found to have significant implications. The axial oscillations are unstable by the gravitational radiation reaction (Andersson 1998; Friedman & Morsink 1998). The mechanism of the instability can be understood by the generic argument for the gravitational radiation–driven instability (the so-called CFS-instability), which is originally examined for the polar perturbations (Chandrasekhar 1970; Friedman & Schutz 1978; Friedman 1978). All rotating stars become unstable in the absence of internal fluid dissipation, irrespective of the parity modes. Viscosity, however, damps out the oscillations, and stabilizes them in general. The polar \textit{f}-mode instability is believed to act only in rapidly rotating neutron stars (Lindblom 1995). The axial instability is, however, found to set in even in much more slowly rotating cases, and to play an important role on the evolution of hot, newly born neutron stars (Lindblom, Owen, & Morsink 1998; Andersson et al. 1998). The instability carries away most of the angular momentum and rotational energy of the stars by the gravitational radiation. The gravitational wave emitted during the spin-down process is expected to be one of the promising sources for the laser interferometer gravitational wave detectors (Owen et al. 1998).

Most of the estimates for the \textit{r}-mode instability are based on the Newtonian calculations. That is, the oscillation frequencies are determined by inviscid hydrodynamics under the Newtonian gravity, and the gravitational radiation reaction is incorporated by evaluating the (current) multipole moments. Relativity has great influence on stellar structures, redshift in oscillation frequencies, frame dragging, radiation, and so on. It is important to explore the relativistic effects, which may not change the general features of the oscillations but may shift the critical angular velocity. The relativistic calculation is not an easy task, even using the linear perturbation method with regard to the oscillation amplitude. Both relativity and rotation complicate the problems considerably.

As the first step toward a clearer understanding of relativistic corrections, we have examined the \textit{r}-mode oscillations in rotating stars with uniform angular velocity. Using the slow rotation formalism and the Cowling approximation, we have derived the equations governing the \textit{r}-mode oscillations up to second order with respect to the rotation. In the lowest order, the allowed range of the frequencies is determined, but corresponding spatial function is arbitrary. The spatial function can be decomposed in the nonbarotropic region by a set of functions associated with the differential equation of the second-order corrections. The equation becomes singular, however, in the barotropic region, and a single function can be selected to describe the spatial perturbation of the lowest order. The frame-dragging effect among the relativistic effects may be significant, as it results in a rather broad spectrum of the \textit{r}-mode frequency, in contrast to the situation in the Newtonian first-order calculation.

\textit{Subject headings:} relativity — stars: oscillations — stars: rotation

\section{PULSATION EQUATIONS OF THE AXIAL MODE}

We consider a slowly rotating star with a uniform angular velocity $\Omega \sim O(\varepsilon)$. The rotational effects can be treated as the corrections to the nonrotating spherical star. We will take account of the corrections up to the third order in $\varepsilon$. The metric tensor for describing the stationary axisymmetric star is given by (Hartle 1967; Chandrasekhar & Miller 1974; Quintana 1976)
\[ ds^2 = -e^r[1 + 2(h_0 + h_2 P_2)]dt^2 + e^r \left[ 1 + \frac{2e^k}{r} (m_0 + m_2 P_2) \right]dr^2 + r^2(1 + 2k_2 P_2) \left( d\theta^2 + \sin^2 \theta \left[ d\phi - \left( \omega + W_1 - W_2 \frac{1}{\sin \theta} \frac{dP_2}{d\theta} \right) dt \right] \right)^2, \]  

where \( P_l(\cos \theta) \) \((l = 2, 3)\) denotes the Legendre polynomial of degree \(l\). The metric functions except \( g_{\phi\phi} \) should be expanded by an even power of \( \varepsilon \), and \( g_{\phi\phi} \) by an odd power of \( \varepsilon \) because of the rotational symmetry, i.e., \( \omega \sim O(\varepsilon), h_0, h_2, m_0, m_2, k_2 \sim O(\varepsilon^2), \) and \( (W_1, W_2) \sim O(\varepsilon^3) \). These quantities are functions of the radial coordinate \( r \).

The equilibrium state is assumed to be described with the perfect-fluid stress-energy tensor. The 4-velocity of the fluid element inside the star has the components, correct to \( O(\varepsilon^2) \),

\[ U^\phi = \Omega U^t, \quad U^t = (-g_{tt} - 2\Omega g_{t\phi} - \Omega^2 g_{\phi\phi})^{-1/2}. \]

The pressure and density distributions are subject to the centrifugal deformation, which is the effect of \( O(\varepsilon^2) \). These distributions are expressed as

\[ p = p_0(r) + \{p_{20}(r) + p_{22}(r)P_2(\cos \theta)\}, \]
\[ \rho = \rho_0(r) + \{\rho_{20}(r) + \rho_{22}(r)P_2(\cos \theta)\}, \]

where \( p_0 \) and \( \rho_0 \) are the values for the nonrotating star, and the quantities in the braces are the rotational corrections. The nonrotating spherical configuration and the rotational corrections in equations (1)–(4) are determined by successively solving the perturbed Einstein equations with the same power of \( \varepsilon \) (Harte 1967; Chandrasekhar & Miller 1974; Quintana 1976).

We will solve equations (11) and (13) by the expansion of \( \varepsilon \). In a spherical star, the perturbations decouple into purely polar and purely axial modes for each \( l \) and \( m \). A set of the functions \( \mathcal{P}_{lm}(r) = \delta p_{lm}, \delta \rho_{lm}, R_{lm}, V_{lm} \) describes the polar mode, while the mode can still be specified by a single \( m \) (e.g., Kojima 1992, 1997). The coupled equations are schematically expressed as

\[ 0 = [\mathcal{A}_{lm}] + \varepsilon \times [\mathcal{A}_{l \pm 1m}] + \varepsilon^2 \times [\mathcal{A}_{lm}, \mathcal{A}_{l \pm 2m}] + \cdots, \]
\[ 0 = [\mathcal{P}_{lm}] + \varepsilon \times [\mathcal{P}_{l \pm 1m}] + \varepsilon^2 \times [\mathcal{P}_{lm}, \mathcal{P}_{l \pm 2m}] + \cdots, \]

where the symbol \( \varepsilon \) means some functions of order \( \varepsilon \), and the square brackets formally represent the relation among the axial perturbation functions \( A_{lm} \), or the polar perturbation functions \( P_{lm} \) appearing therein. We also assume that the time variation
of the oscillation is slow and proportional to $\omega$, i.e., $\ddot{\epsilon} \sim \Omega \sim O(\epsilon)$. This is true in the $r$-mode oscillation, as will be confirmed soon. We look for the mode which is described by a single axial function with index $(l, m)$ in the limit of $\epsilon \to 0$. That is, the polar part should vanish, while the axial part becomes finite. Hence, the perturbation functions are expanded as

$$A_{lm} = A_{lm}(1) + \epsilon^2 A_{lm}(2) + \cdots, \quad B_{lm} = \epsilon^{(1)}[A_{lm} + \epsilon^2 A_{lm}(2) + \cdots] .$$

(16)

Substituting these functions into equations (14) and (15), and comparing the coefficients of $\epsilon^n (n = 0, 1, 2)$, we have

$$0 = [\partial_{t}^{(1)}] , \quad (17)$$

$$0 = [\epsilon \partial_{lm}^{(1)} + \epsilon \times \partial_{lm}^{(1)}] , \quad (18)$$

$$0 = [\epsilon^2 \partial_{lm}^{(2)}] + \epsilon \times [\epsilon \partial_{lm}^{(1)}] , \quad (19)$$

where we have used $\partial_{lm}^{(1)} = 0$ and relation (18) in the last part of equation (19).

Equation (17) represents the axial oscillation at the lowest order, which can be specified by $U_{lm}^{(1)}$. Equation (19) is the second-order form of it, and the term $\epsilon^2 \partial_{lm}^{(2)}$ can be regarded as the rotational corrections. We will show the explicit forms corresponding to equations (17)–(19) in subsequent sections.

3. First-Order Solution

The leading term of equation (11) is reduced to

$$U_{lm}^{(1)} - i m \chi U_{lm}^{(1)} = 0 , \quad (20)$$

where

$$\chi = \frac{2}{l(l + 1)} \sigma = \frac{2}{l(l + 1)} (\Omega - \omega) , \quad (21)$$

and a dot denotes a time derivative in the corotating frame, i.e., $\dot{U}_{lm} = (\dot{\epsilon} + i m \Omega)U_{lm}$. The evolution of the perturbation can be solved by the Laplace transformation,

$$u(s, r) = \int_{0}^{\infty} U_{lm}^{(1)}(t, r)e^{-st} dt . \quad (22)$$

The Laplace transformation of equation (20) is written as

$$[s + i m (\Omega - \chi)]u(s, r) - f_{lm}^{(1)}(r) = 0 , \quad (23)$$

where $f_{lm}^{(1)}$ describes the initial disturbance at $t = 0$. After solving for $u$ and using the inverse transformation, the solution in the $t$-domain is easily constructed as

$$U_{lm}^{(1)}(t, r) = \int f_{lm}^{(1)}(r) \frac{e^{st}}{s + i m (\Omega - \chi)} ds = f_{lm}^{(1)}(r)e^{-im(\Omega - \chi)t}H(t) , \quad (24)$$

where $H(t)$ is the Heaviside step function. We will consider the $t > 0$ region only, so that the function $H(t)$ may well be omitted from now on.

For the Newtonian star, $m(\Omega - \chi)$ becomes a constant,

$$\sigma_{N} = \left[ 1 - \frac{2}{l(l + 1)} \right] m \Omega . \quad (25)$$

This is the $r$-mode frequency measured in the nonrotating frame (Papaloizou & Pringle 1978; Provost, Berthomieu, & Rocca (1981); Saio 1982).

In the relativistic stars, $\sigma$ is a monotonically increasing function of $r$, $\sigma_{0} \leq \sigma \leq \sigma_{R}$. The possible range of the $r$-mode frequency is spread out. If one regards equation (24) as the sum of the Fourier components $e^{-i\sigma t}$, then the spectrum is continuous in the range

$$\left[ 1 - \frac{2}{l(l + 1)} \frac{\sigma_{R}}{\Omega} \right] m \Omega \leq \sigma \leq \left[ 1 - \frac{2}{l(l + 1)} \frac{\sigma_{0}}{\Omega} \right] m \Omega . \quad (26)$$

This result\(^1\) is the same even if metric perturbations are considered (Kojima 1998). The time dependence is determined, whereas the radial dependence $f_{lm}^{(1)}$ is arbitrary at this order. The function $f_{lm}^{(1)}$ in equation (24) is constrained by the equations of motion for the polar part, as will be shown in subsequent sections. In this manner, the perturbation scheme (17)–(19) is degenerate perturbation.

\(^1\) Note that this conclusion is derived within the lowest order approximation of the perturbation scheme. The consistency in the higher order will affect it for the barotropic case, as will be shown in the following sections.
4. SECOND-ORDER EQUATION

The pressure and density perturbations are induced by the rotation, while the axial mode in the nonrotating stars is never coupled to them. We may assume that the perturbation can be specified by a single function \( U_{lm}^{(1)} \) at the leading order, since the general case can be described by the linear combination. The axial part induces the polar parts with index \((l \pm 1, m)\) according to the perturbation scheme (eq. [18]). It is straightforward to solve for \( \delta p_{l \pm 1m} \) and \( \delta \rho_{l \pm 1m} \) from two components of equation (11) in terms of the first-order corrections and \( U_{lm}^{(1)} \). The explicit results are

\[
\begin{align*}
\delta p_{l \pm 1m} &= 2S_\pm \sigma U_{lm}^{(1)}, \\
\delta \rho_{l \pm 1m} &= -4S_\pm e^{- \nu/2} v' (e^{\nu/2} \sigma U_{lm}^{(1)})' + 2T_\pm e^\nu \left( r^2 \sigma e^{- \nu} \right) U_{lm}^{(1)},
\end{align*}
\]

where

\[
\begin{align*}
S_+ &= \frac{l}{l+1} Q_+, \quad S_- = \frac{l+1}{l} Q_-, \\
T_+ &= lQ_+, \quad T_- = -(l+1)Q_-, \\
Q_+ &= \left( \frac{(l+1)^2 - m^2}{(2l+1)(2l+3)} \right), \quad Q_- = \frac{l^2 - m^2}{(2l-1)(2l+1)}.
\end{align*}
\]

We here denote a derivative with respect to \( r \) by a prime. The lowest order form of equation (13) is also decoupled into the equation of each \( l, m \) component,

\[
\delta \hat{p}_{l \pm 1m} - C^2 \delta \hat{p}_{l \pm 1m} = AC^2 \left( e^{- \nu} R_{l \pm 1m} - \frac{3i \mu^2}{r^2} Q_{m} U_{lm}^{(1)} \right),
\]

with

\[
\begin{align*}
C^2 &= \frac{\Gamma p_0}{p_0 + \rho_0}, \quad A = \frac{\rho_0}{p_0 + \rho_0} - \frac{\rho_0}{\Gamma p_0}.
\end{align*}
\]

In equation (32), the displacement \( \xi_2 \) represents the quadrupole deformation of the stationary star and is related to the quantities of \( O(\varepsilon^2) \) as

\[
\xi_2 = -\frac{2}{\nu} \left( h_2 + \frac{1}{3} \sigma r^2 e^{- \nu} \right).
\]

The region for \( A > 0 \) is convectively unstable, while that for \( A < 0 \) is stably stratified. We here assume \( A \neq 0 \) in the following calculations, but we will consider separately the case \( A = 0 \) in § 5. From equation (32), the function \( R_{l \pm 1m} \) is solved by \( U_{lm}^{(1)} \) for the region \( A \neq 0 \). The function \( V_{l \pm 1m} \) describing the horizontal motion is calculated from the remaining component of equation (11). The explicit form is given by

\[
[l(l+1) V_{lm}]_{l \pm 1} = S_\pm [v_2 + l(l+1)r^2 \Omega e^{- \nu} U_{lm}^{(1)}] + T_\pm [v_1 + \frac{1}{2} r^2 \sigma e^{- \nu} U_{lm}^{(1)}] + (l+1)Q_\pm v_0,
\]

where

\[
\begin{align*}
v_2 &= 4e^{- (3\nu + \lambda)/2} \left\{ \frac{r^2 e^{\lambda/2}}{A' v'} \left[ (e^{\nu/2} \sigma U_{lm}^{(1)})' + \frac{v}{2C^2} (e^{\nu/2} \sigma U_{lm}^{(1)}) \right]' - \left( 4r^2 e^{- 3\nu/2} \right) \left( e^{\nu/2} \sigma U_{lm}^{(1)} \right)' \right\}, \\
v_1 &= 2e^{- (3\nu + \lambda)/2} \left[ \frac{d^{\lambda + \nu/2}}{A v'} \left( r^2 \sigma e^{- \nu} \right) U_{lm}^{(1)} \right] - 2e^{- \nu/2} \left( r^2 \sigma e^{- \nu} \right)' + \Omega^2 e^{- \nu} \right\} U_{lm}^{(1)}, \\
v_0 &= \frac{3}{2} \xi_2 \left( U_{lm}^{(1)} \right)' - \frac{r^2 e^{- \nu}}{2} (\sigma + 2\Omega) - \frac{3}{2} e^{\nu/2} \left( \sigma + 2\Omega \right) \left( \sigma + 2\Omega \right) \left( \xi_2 \right) U_{lm}^{(1)}.
\end{align*}
\]

We have used equation (20) to simplify equation (36). These corrections in the polar functions affect the axial parts with indices \((l \pm 2, m)\) and \((l, m)\). We are interested in the term with index \((l, m)\) as the corrections to the leading equation. In deriving the axial equation with these corrections, the terms up to \( O(\varepsilon^3) \) in the background field also affect it. We include both corrections to the axial equation and have the following form:

\[
0 = U_{lm}^{(2)} - im\lambda U_{lm}^{(2)} + \mathcal{L} [U_{lm}^{(1)}],
\]

(40)
where \( \mathcal{L} \) is the Sturm-Liouville differential operator defined by

\[
\mathcal{L}[U_{lm}^{(1)}] = 8c_3 e^{\lambda/2} (\rho_0 + p_0) \left[ \frac{r^2 e^{(\lambda-3)/2}}{4\ln^2 (\rho_0 + p_0)} (e^{\lambda/2} \sigma U_{lm}^{(1)})' \right] - (F + G)U_{lm}^{(1)},
\]

\[
F = -4c_3 m^2 e^{(\lambda-2)/3} \left( \frac{r^2 e^{(\lambda-3)/2}}{v^2} \right) - 4c_3 \sigma e^{\lambda/2} \frac{(\sigma^2 r^2 e^{\lambda/2} - 3c_1 \left[ \frac{r e^{\lambda/2} \xi_2}{2} \right] - 3 \frac{\sigma}{r} \xi_2 - k_2 + \frac{e^4}{r} m_2 + \frac{5W_5}{c} \right)}{Av}\]

\[
G = -8c_3 m^2 e^{(\lambda-2)/3} \left( \frac{r^2 e^{(\lambda-3)/2}}{v'} \right) + 4e_2 \sigma e^{\lambda/2} \frac{(\sigma^2 r^2 e^{\lambda/2} - 3c_1 \left[ \frac{r e^{\lambda/2} \xi_2}{2} \right] - 3 \frac{\sigma}{r} \xi_2 - k_2 + \frac{e^4}{r} m_2 + \frac{5W_5}{c} \right)}{Av}\]

\[
c_n = \frac{l + 1}{m} Q^2 + (\frac{1}{l + 1})^n Q^2.\]

In order to solve equation (40), we introduce a complete set of functions as for the operator \( \mathcal{L} \),

\[
\mathcal{L}[y_k] + \kappa y_k = 0,
\]

where \( \kappa \) is the eigenvalue and the eigenfunction \( y_k(r) \) is characterized by \( \kappa \), e.g., the number of the nodes. The eigenvalue is the real number of \( O(\ell^2) \), since \( \mathcal{L} \) is the Hermitean operator of \( O(\ell^2) \). The eigenvalue problem is solved with appropriate boundary conditions. For example, the function should satisfy the regularity condition at the center, and the Lagrangian pressure should vanish at the stellar surface. Any initial data \( f_{lm}^{(1)} \) in equation (24) can be decomposed by the set.

We may restrict our consideration to a single function labeled by \( \kappa \), since the general case is described by discrete summation or integration over a certain range. By putting \( y_k = im\xi f_{lm}^{(1)} \), we have

\[
\mathcal{L}[U_{lm}^{(1)}] = \mathcal{L}[im\xi f_{lm}^{(1)} e^{-im(\Omega - \chi)t}] = -im\xi f_{lm}^{(1)} e^{-im(\Omega - \chi)t},
\]

where \( e^{-im(\Omega - \chi)t} \) gives the first-order correction and is neglected here.

Using \( f_{lm}^{(1)} \), we can integrate equation (40) with respect to \( t \) and have the function \( U_{lm}^{(2)}(t, r) \) of \( O(\ell^2) \) as

\[
U_{lm}^{(2)} = (im\xi f_{lm}^{(1)} + f_{lm}^{(2)}) e^{-im(\Omega - \chi)t},
\]

where the function \( f_{lm}^{(2)} \) of \( O(\ell^2) \) is unknown at this order. The sum of the first- and second-order forms is approximated as

\[
U_{lm}^{(1)} + U_{lm}^{(2)} = [(1 + im\xi t) f_{lm}^{(1)} + f_{lm}^{(2)}] e^{-im(\Omega - \chi)t}
\]

\[
= [f_{lm}^{(1)} + f_{lm}^{(2)}] e^{-im(\Omega - (1 + \chi))t},
\]

where we have exploited the freedom of \( f_{lm}^{(2)} \) to eliminate the unphysical growing term in equation (48). The value \( \kappa \) originated from the fixing of the initial data becomes evident for large \( t \), since the accumulation of small effects from the higher order terms is no longer neglected. As a result, the frequency should be adjusted with the second-order correction to be a good approximation even for slightly large \( t \), as in equation (49). This renormalization of the frequency is closely related to treating \( t \) as a strained coordinate in the perturbation method. (See, e.g., Hinch 1991.) In this way, the specification of the initial data at the leading order has influence on the second-order correction \( \kappa \).

5. OSCILLATIONS IN BAROTROPIC STARS

The structure of the neutron stars is almost approximated to be barotropic. The pulsation equation is quite different from that of the nonbarotropic case, as in the Newtonian pulsation theory. Relation (32) for the case \( A = 0 \) is replaced by

\[
\delta p_{l1m} = C^2 \delta p_{l1m} = \frac{p_0}{p_0} \delta p_{l1m} = -\frac{\nu\rho_0}{\rho_0} \delta p_{l1m}.
\]

In the last part of equation (50), we have used the hydrostatic equation of the nonrotating star. In this case, unlike the \( A \neq 0 \) case, the function \( R_{l1m} \) is never determined through equation (32) but rather we have two restrictions to a single function \( U_{lm}^{(1)} \). These conditions are never satisfied simultaneously, unless for \( m = \pm l \), in which one condition is trivial, \( \delta p_{l1m} = \delta p_{l1m} = 0 \). The other condition for \( m = \pm l \) becomes

\[
(e^{\nu^2/2} \sigma U_{lm}^{(1)})' + \left[ \frac{\nu'}{2C^2} - \frac{l + 1}{2} \left( \frac{\sigma e^{\nu^2/2}}{\sigma e^{\nu^2/2}} \right)^2 \right] e^{\nu^2/2} \sigma U_{lm}^{(1)} = 0.
\]

Substituting equation (24) in this and neglecting the higher order term due to \( e^{-im(\Omega - \chi)t} \), we have the same differential equation for \( f_{lm}^{(1)} \). The integration with respect to \( r \) results in

\[
U_{lm}^{(1)}(t, r) = f_{lm}^{(1)} e^{-im(\Omega - \chi)t}
\]

\[
= [N_0(p_0 + p_0)]^{l+1} e^{-(\sigma e^{-\nu^2/2})} e^{-im(\Omega - \chi)t},
\]
where $N_0$ is a normalization constant. In this way, the function of the lowest order is determined.

The corresponding 3-velocity at $t \to +0$ is given by

$$\dot{\xi}_\phi = \delta v_\phi = N_0 r^{l+1} e^{-r} v^{(l-1)/2}.$$

(53)

As shown by Friedman & Morsink (1998), the canonical energy of the perturbation is negative for the Lagrange displacement $\xi_\phi$, and the solution therefore denotes unstable if the gravitational radiation reaction sets in.

We now specify $R_{l\pm 1m}$ of $O(\epsilon^2)$ to proceed to the pulsation equation with the second-order corrections. The function $X_{l\pm 1m}$ describing the radial motion is introduced as

$$R_{l\pm 1m} = X_{l\pm 1m} + \frac{3im\epsilon^2\xi_2}{r^2} Q_{l\pm 1m}^{(1)}.$$

(54)

The function $X_{l\pm 1m}$ is arbitrary at this order, but we have two special cases. One is $X_{l\pm 1m} = 0$, which is the limiting case for equation (32) in a sense. The other is chosen so as to eliminate the Lagrangian change of pressure within the entire star. The condition corresponds to $X_{l\pm 1m} = 4S\epsilon e^{-r} U^{(2)}_{lm}/v'$. For the first choice, the Lagrangian change of pressure never vanishes at the surface unless $\rho_0 = 0$ there. The second choice is the usual way treated in the Newtonian $r$-mode, as shown by Lindblom & Ipser (1998). With the second choice of $X_{l\pm 1m}$ and equation (52), the pulsation equation for $m = \pm l$ leads to

$$\dot{U}_{lm}^{(2)} - im\epsilon U_{lm}^{(2)} - G\dot{U}_{lm}^{(1)} = 0,$$

where $G$ is defined in equation (43). This equation is solved for $U_{lm}^{(2)}$ as in § 4. The solution up to $O(\epsilon^2)$ is written as

$$U_{lm}^{(1)} + U_{lm}^{(2)} = (f_{lm}^{(1)} + f_{lm}^{(2)}) e^{-im\Omega t} e^{-i\Omega (1+G)lt},$$

(56)

where the function $f_{lm}^{(2)}$ of $O(\epsilon^2)$ is unknown at this order. This expression (eq. [56]) is formally the same as in equation (49) for the nonbarotropic case with $\kappa = G$. The second-order correction in the frequency, however, depends on the position, owing to the particular choice of the function $f_{lm}^{(1)}$.

We show the second-order rotational correction $G$ for the $l = m = 2$ mode in Figure 1. We adopt the polytropic stellar model with index $n = 1$. For the Newtonian star, $G$ is a positive function, which monotonically increases from the center to the surface. The value ranges from $G = 0.55(\Omega^2 R^3/M)$ to $G = 0.75(\Omega^2 R^3/M)$, where $R$ and $M$ are the radius and the mass for the nonrotating star. These values are rather smaller than that of the incompressible case, $G = 37/27(\Omega^2 R^3/M)$. (Only for the Newtonian incompressible case is the factor $G$ a constant; see the Appendix.) The stellar deformation $\xi_2$, which is the most important contribution to $G$, diminishes in the compressible fluid. As the star becomes relativistic, other relativistic factors become significant. As a result, the factor $G$ is scaled down as a whole, and eventually becomes negative for some regions. In any case, the frequencies satisfy the criterion of the radiation reaction instability, which implies $0 < (1 + G)\Omega < 1$—that is, retrograde in the rotating frame and prograde in the inertial frame. Therefore, the second-order correction never changes the qualitative picture of the instability.

In Figure 2 we show the frequency range of the $r$-mode oscillations in the first-order rotational calculation. The upper and lower limits on the continuous spectrum in equation (26) are shown by two lines. The intermediate values between the two lines are allowed for a fixed model $M/R$.

The frequency is a single value $\sigma_\nu$, given by equation (25) in the Newtonian limit. The dragging effect relevant to the relativity broadens the allowed range as shown by equation (26). In Figure 3 the allowed range is shown including the second-order corrections for the extreme case $\Omega^2 = M/R^3$. Even in the Newtonian case, the oscillation frequency is not a single value, since the factor $G$ depends on $r$, as seen in Figure 1. The allowed range of the frequency further increases with the
Fig. 2.—Frequency range with the relativistic factor $M/R$. Two lines denote the upper and lower limits of the frequency. The intermediate values between them are allowed for a fixed model $M/R$. The frequency is normalized by the Newtonian value $\sigma_N$.

Fig. 3.—Same as Fig. 2, but the second-order correction with $\Omega^2 = M/R^3$ is included.

Fig. 4.—Fourier transform of the gravitational wave amplitude of the $l = m = 2$ mode for the stellar model $M/R = 0.2$. The frequency is normalized by the Newtonian value $\sigma_N$, and the amplitude $h(\sigma)$ is normalized by the maximum.
relativistic factor. From this result, we expect that the $r$-mode frequency ranges from $0.8\sigma_N$ to $1.2\sigma_N$ for a typical neutron star model with $M/R \sim 0.2$.

We examine the effect on the spectrum of the gravitational waves emitted by the $r$-mode oscillations. For simplicity we neglect all relativistic corrections except in the frequency, and estimate the spectrum by the Newtonian radiation theory. The dimensionless gravitational amplitude $h(t)$ at infinity is determined by evaluating the time variation of the current multipole moment, $h(t) \sim d^l S_{lm}/dt^l$ (Thorne 1980). The current multipole moment $S_{lm}$ for the $l = m = 2$ mode and the velocity in equation (53) is given by

$$S_{22} = N \int \rho_0(\sigma e^{-\gamma})^{1/2}e^{-2i\Omega t}e^{\mu \Theta} d\Omega,$$  

where the normalization $N$ also includes the constant from the integration over angular parts. The Fourier component $h(\sigma)$ can be expressed as

$$h(\sigma) = \int h(t)e^{i\sigma t} dt \propto \int \rho_0(\sigma e^{-\gamma})^{1/2}(\Omega - \chi)^2 \delta[\sigma - 2(\Omega - \chi)] e^{\mu \Theta} d\Omega.$$  

The spectrum has a finite line breadth as shown in Figure 4. The spectrum is nonzero only for $\sigma = \sigma_N$ in the Newtonian treatment, but is broad in the relativistic one. The width at half-maximum is $\Delta\sigma/\sigma_N \sim 0.1$.

6. DISCUSSION

In this paper we have calculated the $r$-mode oscillations in the relativistic rotating stars, neglecting the gravitational perturbations. The evolution can be described by oscillatory solutions that are neutral, i.e., never decay or grow in the absence of the dissipation. The oscillation is described not by a single frequency but by the frequencies of a broad range, unlike the Newtonian case. The reason is that the local rotation rate depends on the position due to the frame-dragging effect even for the Newtonian limit. The second-order correction of $h(t)$ was determined by solving the eigenvalue problem for the operator $\mathcal{L}_N = D_N - (F_N + G_N)$ in the case of $A \neq 0$,

$$\mathcal{L}_N[y] + \sigma_1 y = 0,$$  

with

$$\mathcal{L}_N[y] = 4c_3\Omega^2 \rho_0 \left( \frac{r^2}{Ag\rho_0} y' \right),$$

$$F_N = -4c_3\Omega^2 \left( \frac{r^2}{Ag\rho_0} \right)' - \frac{4c_2\Omega^2}{\rho_0} \left( \frac{\rho_0 r}{Ag} \right)' + \left( \frac{4c_1\Omega^2}{Ag} \right),$$

$$G_N = -4c_3\Omega^2 \left( \frac{r^2}{g} \right)' + \frac{8c_2\Omega^2 r}{g} + 2c_1 r x' + \frac{2m^2}{l(l + 1)} \alpha,$$

where we have used the gravitational acceleration $g = v'/2$ and ellipticity $\xi = -3\xi_2/(2r)$. The second-order correction $\sigma_1$ exactly corresponds to $\kappa$ in equation (45).

As shown previously (Provost et al. 1981; Saio 1982), the eigenvalue problem becomes singular for the barotropic case $A = 0$, since the second-order differential term vanishes. The second-order solution (eq. [56]) in the Newtonian limit reduces to

$$[e^o, e^s, (\rho_0 + p_0)/\rho_0, m(\Omega)] \rightarrow 1, \quad (m', m_2, k_2, W_1, W_3) \rightarrow 0.$$  

APPENDIX A

NEWTONIAN LIMIT

In this appendix, we will consider the $r$-mode oscillations in the Newtonian limit, in which

$$[e^o, e^s, (\rho_0 + p_0)/\rho_0, m(\Omega)] \rightarrow 1, \quad (m', m_2, k_2, W_1, W_3) \rightarrow 0.$$  

Equation (40) reduces to the equation derived by Provost et al. (1981), if the variables are matched. They solved the eigenvalue problem, assuming the form $e^{-i[m(\Omega - \kappa) + a_1]/\Omega}$, where $\sigma_0$ of $O(\kappa)$ is given by $2m\Omega/[l(l + 1)]$. The correction $\sigma_1$ of $O(\kappa^2)$ was determined by solving the eigenvalue problem for the operator $\mathcal{L}_N = D_N - (F_N + G_N)$ in the case of $A \neq 0$,

$$\mathcal{L}_N[y] + \sigma_1 y = 0,$$  

with

$$\mathcal{L}_N[y] = 4c_3\Omega^2 \rho_0 \left( \frac{r^2}{Ag\rho_0} y' \right),$$

$$F_N = -4c_3\Omega^2 \left( \frac{r^2}{Ag\rho_0} \right)' - \frac{4c_2\Omega^2}{\rho_0} \left( \frac{\rho_0 r}{Ag} \right)' + \left( \frac{4c_1\Omega^2}{Ag} \right),$$

$$G_N = -4c_3\Omega^2 \left( \frac{r^2}{g} \right)' + \frac{8c_2\Omega^2 r}{g} + 2c_1 r x' + \frac{2m^2}{l(l + 1)} \alpha,$$

where we have used the gravitational acceleration $g = v'/2$ and ellipticity $\xi = -3\xi_2/(2r)$. The second-order correction $\sigma_1$ exactly corresponds to $\kappa$ in equation (45).

As shown previously (Provost et al. 1981; Saio 1982), the eigenvalue problem becomes singular for the barotropic case $A = 0$, since the second-order differential term vanishes. The second-order solution (eq. [56]) in the Newtonian limit reduces to

$$[e^o, e^s, (\rho_0 + p_0)/\rho_0, m(\Omega)] \rightarrow 1, \quad (m', m_2, k_2, W_1, W_3) \rightarrow 0.$$  

2 Recently, Lockitch & Friedman (1998) showed the resolution of the singularity in Newtonian isentropic stars.
to

\[ U_{lm}^{(1)} + U_{lm}^{(2)} = (N_0 \rho_0 r^l + f_{lm}^{(2)}) e^{-i(l+\sigma)(1+\sigma)t}. \]  \hspace{1cm} (A6)

This equation is valid only for \( l = \pm m \), since the Newtonian counterpart of equation (50) is never satisfied otherwise. When the correction \( G_N \) is not a constant, the eigenvalue problem is ill-posed. The function \( G_N \) indeed depends on \( r \) for the compressible matter, so that Provost et al. (1981) arrived at no solution in this case. The exceptional case is the incompressible fluid, in which \( G_N \) is a constant, since \( \alpha = 5/4(\Omega^2 R^3/M) \), \( g = M/r R^3 \). The correction in the frequency for \( l = m \) is

\[ G_N = -\frac{4l}{(l+1)^3} \frac{\Omega^2 R^3}{M} + \frac{2l}{l+1} \alpha, \]  \hspace{1cm} (A7)

which should be the same value calculated by Provost et al. (1981), except for a misprint in their expression.

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