GROUPOIDS AND THE INTEGRATION OF LIE ALGEBROIDS

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Abstract. We show that a Lie algebroid on a stratified manifold is integrable if, and only if, its restriction to each strata is integrable. These results allow us to construct a large class of algebras of pseudodifferential operators. They are also relevant for the definition of the graph of certain singular foliations of manifolds with corners and the construction of natural algebras of pseudodifferential operators on a given complex algebraic variety.

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Introduction

Differentiable groupoids appear in geometry in various instances, for example in the theory of connections and parallel transport on fiber bundles, or in the theory of pseudogroups of transformations. The concept of groupoid is a generalization of the concept of group, the main difference being that not any two elements of a groupoid are composable. Intuitively, it is convenient to think of a groupoid as a set of arrows between various points, called units, two arrows being composable if, and only if, their ends match. (See Section 1 for precise definitions.)

One of the main features of differential groupoids is that they are geometric objects that interpolate between differentiable manifolds and Lie groups: differentiable manifolds have many units and few arrows; whereas Lie groups have many arrows and few units—actually only one. This “interpolation” property is valid also at the level of algebras: to a compact smooth manifold $M$ one associates the commutative algebra $C^\infty(M)$ of its differentiable functions; whereas to a Lie group $G$ one associates the convolution algebra $C^\infty_c(G)$ of compactly supported smooth functions on the group, which is usually highly non-commutative. In this way, differentiable groupoids provide a link between geometry and harmonic analysis.

The algebras $C^\infty(M)$ and $C^\infty_c(G)$ are particular cases of the convolution algebra of a differential groupoid, and this feature makes groupoids a favorite toy model in

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non-commutative geometry. From this point of view, the results of this paper are a first step towards a generalization of the results of [2], from étale groupoids to general differential groupoids.

Recall that a Lie algebroid is a vector bundle $A \to M$ on a differentiable manifold $M$ together with a Lie algebra structure on the space $\Gamma(A)$ of its smooth sections and a Lie algebra morphism $\Gamma(A) \to \Gamma(TM)$, defined by a structural vector bundle morphism $q : A \to TM$, called “the anchor map.” The Lie algebroid $A$ is called regular [resp. transitive] if, and only if, the anchor map $q : A \to TM$ has locally constant rank [resp. it is onto]. From a classical, differential geometric point of view, to a differential groupoid there is associated a “Lie algebroid,” which is some sort of “infinitesimal form” of the groupoid, generalizing both the Lie algebra of a group and the tangent space to a differentiable manifold.

In [5] it was proved that any Lie algebroid such that $q = 0$ is the Lie algebroid of a differentiable groupoid, in other words, it is integrable. This result can be regarded as a generalization of “Lie’s third theorem,” which states that every finite dimensional Lie algebra is the Lie algebra of a Lie group. This is relevant because, unlike (finite dimensional) Lie algebras, Lie algebroids do not always correspond to differential groupoids, that is, they are not always integrable. Actually, for a transitive groupoid, one can define an obstruction to integrability, see [8]. Nevertheless, it is still interesting to construct differentiable groupoids that integrate specific Lie algebroids. Some examples can be found in [14]. In that paper, to a differentiable groupoid $G$ there was associated an algebra of pseudodifferential operators $\Psi^*(G)$, such that, for $G^{(0)}$ compact, all first order differential operators in $\Psi^*(G)$ are linear combinations of sections of $A(G)$ and operators of multiplication by functions. The integration of Lie algebroids is thus a first step toward constructing algebras of pseudodifferential operators, and this explains why we are interested in the problem of integrating Lie algebroids. See [3, 4, 11, 13, 14, 21] for more on the question of constructing pseudodifferential operators.

In this paper, we approach the problem of integrating Lie algebroids from an abstract point of view, looking for some general methods to integrate Lie algebroids. Since arbitrary Lie algebroids $\pi : A \to M$ behave rather wildly, we make two assumptions. First, we assume that the manifold $M$ has a stratification $M = \cup S$ into disjoint strata, each of which is invariant with respect to the diffeomorphisms generated by the sections of $A$ and, second, we assume that the restrictions $A_S = A|_S$, which are Lie algebroids precisely because the strata are invariant, are regular. (If $M$ satisfies the first assumption, we say that it has an “$A$-invariant regular stratification.”)

For the class of groupoids with $A$-invariant regular stratifications, one can approach the “integration problem” in two steps. First, a necessary condition for the integrability of $A$ is the integrability of each of the restrictions $A_S$, and hence the first step will be to integrate each of these restrictions (see also below). Assume then that we can find a differentiable groupoid $G_S$ that integrates $A_S$, for each $S$. The second step is to “glue” the resulting groupoids $G_S$. Surprisingly enough, this
naïve approach actually works in most cases. It works for example if we choose the
integrating groupoids $G_S$ to be maximal in a suitable sense ($d$-simply connected),
and this is our main general result on the integration of differential groupoids. As in
the paper of Douady and Lazard [5], we obtain in general non-Hausdorff groupoids.

Since transitive Lie algebroids are a particular case of regular Lie algebroids, the
first part of the problem—that is, integrating regular Lie algebroids—is similar to
the problem of integrating transitive Lie algebroids, and probably can be handled
similarly. In particular, it is clear that not all regular Lie algebroids are integrable.
It is not our purpose in this paper to study the integration of general regular
algebroids, but we do show how to integrate particular classes of regular algebroids.
For example, we show that a regular algebroid $A$ is integrable if $\ker(q)$, the kernel
of $q$, consists of semisimple Lie algebras, or if $A$ is a semi-direct product.

As for the second part of the problem, it turns out that there exists at most one
way to glue the groupoids $G_S$ that integrate $A_S = A|_S$, assuming that they exist.
The problem is that the resulting glued space (a groupoid) is not always a smooth
manifold, so this procedure does not lead directly to a differentiable groupoid.
However, we show that this procedure does lead to a differentiable groupoid that
integrates $A$, provided that all groupoid $G_S$ are $d$-simply connected. We do not
assume that the strata are regular here. We thus obtain the following result.

**Theorem 1.** A Lie algebroid $\pi : A \to M$ on a manifold $M$ with an $A$-invariant
stratification is integrable if, and only if, it is integrable along each stratum.

These results provide us with an explicit way of integrating many Lie algebroids.
As an application of the theorem, we prove the integrability of certain Lie algebroids
on foliated manifolds with corners. A foliated manifold with corners is a manifold
with corners, each of whose open faces is a foliated manifold, the foliations be-
ing required to satisfy certain compatibility relations. This result generalizes a
construction due to Winkelnkemper [22]. Previously, Melrose [10] and Mazzeo–
Melrose [9] have shown how to integrate certain particular algebroids. However,
their framework was different from ours.

The results of this paper, together with the results of [14], can be used to con-
struct a natural algebra of pseudodifferential operators on a complex algebraic
variety endowed with a $\mathbb{C}^\infty$-resolution of singularities,” thus making a substantial
step towards a solution of the problem stated in [11]. Then the methods of [12]
can presumably be applied to study the resulting algebras of operators. Algebras
of pseudodifferential operators on groupoids are also a natural framework to study
adiabatic limits [14, 23]. The problem of associating an algebra of pseudodifferen-
tial operators to a groupoid was first formally stated in a paper by Weinstein, [21].
However, before that, in [3], Alain Connes has constructed algebras of pseudodif-
ferential operators on foliations, which in our setting corresponds to the case of a
regular groupoid with discrete holonomy (see also [4]). His methods have played a
role in inspiring the constructions of [14].

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1. **Basic concepts**

We begin this section by fixing notation and recalling some of the basic concepts
used in this paper.
In the following, we shall use the framework of \cite{14}. In particular, a manifold is a smooth manifold, possibly with corners. By definition, every point \( m \) in a manifold with corners \( M \) has a coordinate neighborhood diffeomorphic to \([0,1)^k \times \mathbb{R}^{n-k}\), such that the transition functions are smooth (including on the boundary). Let \( k(m) \) be the least \( k \) such that \( m \) is in a set diffeomorphic to \([0,1)^k \times \mathbb{R}^{n-k}\), and let \( \partial_k(M) \) be the set of points \( m \) for which \( k(m) = k \). A component of \( \partial_k(M) \) is called an open face of codimension \( k \). A face of \( M \) is the closure of an open face (of \( M \)). A hyperface of \( M \) is a face \( H \) of \( M \) of codimension one.

It is customary to assume that any hyperface \( H \) of a manifold with corners \( M \) is of the form \( H = \{ x_H = 0 \} \), where \( x_H \) is a smooth positive function on \( M \) such that \( dx_H \neq 0 \) on \( H \). If this is the case, \( x_H \) is called a defining function of \( H \). We shall also assume in this paper that each hyperface \( H \) has a defining function, although, most of our results are true even without this assumption. An interior point of \( M \) is a point of \( M \) that belongs to no hyperface.

A submersion \( f : M \to N \) between two manifolds with corners is a differentiable map with surjective differential at each point, such that a non-zero tangent vector to \( M \) points inward if, and only if, its image in \( TN \) is non-zero and points inward. By definition, if \( f : M \to N \) is a submersion, then the fibers \( f^{-1}(y) \) are smooth manifolds without corners. A submanifold with corners \( N \subset M \) is a closed submanifold of \( M \) such that each face of \( N \) is locally the transverse intersection of \( N \) with a face of \( M \).

We now define groupoids. Recall that a small category is a category whose class of morphisms is a set. By definition, a groupoid is a small category \( G \) in which every morphism is invertible. General results on groupoids can be found in \cite{17}.

We now fix some notation and make the definition of a groupoid more explicit. The set of objects (or units) of \( G \) is denoted by \( G^{(0)} \). The set of morphisms (or arrows) of \( G \) is denoted by \( G^{(1)} = \text{Mor}(G) \). We shall sometimes write \( G \) instead of \( G^{(1)} \), by abuse of notation. For example, when we consider a space of functions on \( G \), we actually mean a space of functions on \( G^{(1)} \). We will denote by \( d(g) \) [respectively \( r(g) \)] the domain [respectively, the range] of the morphism \( g : d(g) \to r(g) \). We thus obtain functions

\[
d, r : G^{(1)} \longrightarrow G^{(0)}
\]

that will play an important role. The multiplication \( \mu : (g, h) \to \mu(g, h) = gh \) is defined on the set \( G^{(2)} \) of composable pairs of arrows:

\[
\mu : G^{(2)} = G^{(1)} \times_M G^{(1)} := \{(g, h) : d(g) = r(h)\} \longrightarrow G^{(1)}.
\]

The inversion operation is a bijection \( \iota(g) = g^{-1} \) of \( G^{(1)} \). Denoting by \( u(x) \) the identity morphism of the object \( x \in G^{(0)} \), we obtain an inclusion \( u : G^{(0)} \to G^{(1)} \). We see that a groupoid \( G \) is completely determined by the spaces \( G^{(0)} \) and \( G^{(1)} \) and by the structural morphisms \( d, r, \mu, u, \iota \). We sometimes write

\[
G = (G^{(0)}, G^{(1)}, d, r, \mu, u, \iota).
\]

The structural maps satisfy the following properties:

(i) \( r(gh) = r(g), d(gh) = d(h) \) for any pair \((g, h) \in G^{(2)}\), and the partially defined multiplication \( \mu \) is associative.

(ii) \( d(u(x)) = r(u(x)) = x, \forall x \in G^{(0)}, u(r(g))g = g, \) and \( gu(d(g)) = g, \forall g \in G^{(1)}, \)
and $u : \mathcal{G}^{(0)} \to \mathcal{G}^{(1)}$ is injective.

(iii) $r(g^{-1}) = d(g)$, $d(g^{-1}) = r(g)$, $gg^{-1} = u(r(g))$, and $g^{-1}g = u(d(g))$.

By definition, a \textit{differentiable groupoid} is a groupoid such that $\mathcal{G}^{(0)}$ and $\mathcal{G}^{(1)}$ are smooth manifolds with corners, $\mathcal{G}^{(0)}$ is smooth, all structural morphisms are differentiable, and $d$, the domain map, is a submersion.

We observe that $\iota$ is a diffeomorphism and hence $d$ is a submersion if, and only if, $r = d \circ \iota$ is a submersion. Also, it follows from the definition of submersions of manifolds with corners that each fiber $\mathcal{G}_x = d^{-1}(x) \subset \mathcal{G}^{(1)}$ is a smooth manifold without corners whose dimension $n$ is constant on each connected component of $\mathcal{G}^{(0)}$. The étale groupoids considered in \cite{bl} are extreme examples of differentiable groupoids (corresponding to $\dim \mathcal{G}_x = 0$). Note that we allow $\mathcal{G}^{(1)}$ to be non-Hausdorff. If we want to make more precise the space of units $\mathcal{G}^{(0)}$ of $\mathcal{G}$, we say that “$\mathcal{G}$ is a differentiable groupoid on $\mathcal{G}^{(0)}$.”

We now recall the definition of a Lie algebroid \cite{bl}. See also \cite{ho}.  

\textbf{Definition 1.} A \textit{Lie algebroid} over a manifold $M$ is a vector bundle $A$ over $M$ together with a Lie algebra structure on the space $\Gamma(A)$ of smooth sections of $A$, and a bundle map $q : A \to TM$, extended to a map between sections of these bundles, such that:

(i) $q([X,Y]) = [q(X),q(Y)]$, and
(ii) $[X,qY] = q([X,Y]) + (q(X))fY$,

for all smooth sections $X$ and $Y$ of $A$ and all smooth function $f$ on $M$.

The morphism $q$ is called the \textit{anchor map}. Note that we allow the base $M$ in the definition above to be a manifold with corners. This is necessary in the most interesting examples related to singular spaces. We did not include in the definition the condition that $q(\Gamma(A))$ consist of vector fields tangent to each face of $M$, but it will be satisfied in all the cases we consider.

To a differential groupoid $\mathcal{G}$ there is naturally associated a Lie algebroid $A(\mathcal{G})$ as follows \cite{bl,bl}. Consider first the vertical tangent bundle of $\mathcal{G}$ along the fibers of the domain map $d$:

$$T_d\mathcal{G} = \ker(d_*) = \bigcup_{x \in \mathcal{G}^{(0)}} T\mathcal{G}_x \subset T\mathcal{G}^{(1)}.$$  

By definition, $A(\mathcal{G})$ is the restriction $T_d\mathcal{G}|_{\mathcal{G}^{(0)}}$ of $T_d\mathcal{G}$ to the set of units of $\mathcal{G}$. The space of $d$-vertical vector fields invariant with respect to right translations is closed with respect to the Lie bracket and identifies canonically with $\Gamma(A)$. Thus we obtain a Lie algebra structure on $\Gamma(A)$. The action on functions on the base is obtained by lifting a function on $\mathcal{G}^{(0)}$ to a function on $\mathcal{G}^{(1)}$ via $r$. Consequently, the anchor map $q : A(\mathcal{G}) \to T\mathcal{G}^{(0)}$ is obtained by restricting the differential of $r$ to $A(\mathcal{G})$.

If $A$ is a Lie algebroid and $\mathcal{G}$ is a smooth groupoid such that $A(\mathcal{G}) \cong A$, then we say that $\mathcal{G}$ \textit{integrates} $A$. Not every Lie algebroid is integrable (see \cite{bl} for an example). Nevertheless, it is important to provide examples of general methods to integrate Lie algebroids.

A differentiable groupoid is called \textit{$d$-connected} [respectively, \textit{$d$-simply connected}] if, and only if, each set $\mathcal{G}_x$ is path connected [respectively, path connected and simply connected]. As for Lie groups, all $d$-simply connected differential groupoids with isomorphic Lie algebroids are isomorphic.

If $\mathcal{G}$ is a differentiable groupoid, then there exists a $d$-simply connected groupoid $\mathcal{P}\mathcal{G}$, uniquely determined up to isomorphism, with the same Lie algebroid as $\mathcal{G}$; it
is called the \textit{path groupoid} associated to \mathcal{G}. As a set, \( PG \) consists of fixed end-point homotopy classes of \( \gamma : [0, 1] \to \mathcal{G} \) such that \( d(\gamma(t)) \) is constant and \( \gamma(0) \) is a unit. This result is due to Moerdijk in general (see also \cite{Moerdijk} for a particular case).

2. A Glueying Theorem

Let \( M \) be a manifold with corners. In this paper, by \textit{stratified manifold} we mean a smooth manifold \( M \), possibly with corners, together with a disjoint union decomposition \( M = \cup S \) of \( M \) by a locally finite family of submanifolds \( S \) without corners, called \textit{open strata}, such that the closure (in \( M \)) of each stratum \( S \) is a submanifold with corners and each \( S \) is contained in a unique open face of \( M \). (This notion is slightly stronger than that of a stratified space which is also a manifold with corners. Most of our results however hold in this greater generality.)

Consider a differentiable Lie algebroid \( A \) with anchor map \( q : A \to TM \) on a manifold \( M \). A stratification \( M = \cup S \) of \( M \) is called \( A \)--\textit{invariant} if, and only if, for each point \( x \in S \), the range of \( A_x \to T_xM \) (from the fiber of \( A \) at the point \( x \) to the tangent space to \( M \) at \( x \)) is contained in \( T_xS \) (i.e., \( q(A_x) \subset T_xS \)). The condition that

\[
q(A_x) \subset T_xS,
\]

for all \( x \in S \), is equivalent to the condition that each local diffeomorphism of the form \( \exp(q(X)) \), for some smooth section \( X \) of \( A \), preserve the strata of \( M \), or, to the condition that the restriction \( A_S \) of \( A \) to \( S \) be a Lie algebroid on \( S \), for each \( S \).

Define, for any subset \( S \subset M \) of the set of units of \( \mathcal{G} \), the groupoid

\[
\mathcal{G}_S := d^{-1}(S) \cap r^{-1}(S),
\]

called \textit{the reduction of \( \mathcal{G} \) to \( S \)}. If \( M = \cup S \) is an \( A \)--invariant stratification of \( M \) and, moreover, \( \mathcal{G} \) is a differentiable groupoid on \( M \) that integrates \( A \), then \( d^{-1}(S) = r^{-1}(S) \) and \( \mathcal{G}_S = d^{-1}(S) \) satisfies \( A_S \simeq A(\mathcal{G}_S) \). This shows that, in order to integrate \( A \), we need to integrate each restriction \( A_S \). The main point of this section is that this is also enough (Theorem \ref{thm:main}).

Recall from \cite{Moerdijk} that an \textit{admissible section} of a differential groupoid \( \mathcal{G} \) on \( M \) is a differentiable map

\[
\sigma : M \to \mathcal{G}
\]

such that \( d(\sigma(x)) = x \) and the map \( M \ni x \to r(\sigma(x)) \in M \) is a diffeomorphism. Then \( \sigma \) also defines a diffeomorphism

\[
\mathcal{G} \ni g \to \sigma g := \sigma(r(g))g \in \mathcal{G}.
\]

The main example of an admissible section is

\[
\sigma(x) = \exp(X_m) \ldots \exp(X_1)x,
\]

for suitable, smooth sections \( X_1, \ldots, X_m \) of \( A \). We will discuss this type of admissible sections in more detail below.

A \textit{differentiable family} of admissible sections is a family \( \sigma_s : M \to \mathcal{G} \), \( s \in [0, 1] \), of maps such that each \( \sigma_s \), \( s \in [0, 1] \), is an admissible section and the induced map \([0, 1] \times M \ni (s, x) \to \sigma_s(x) \in \mathcal{G} \) is differentiable.

\textbf{Lemma 1.} Let \( \mathcal{G} \) be a differentiable groupoid on \( M \) and let \( \sigma_s : M \to \mathcal{G} \) be a differentiable family of local admissible sections. Then there exists a section \( X \) of...
A such that
\begin{equation}
\partial_s f(\sigma_s g)|_{s=0} = Xf(\sigma_0 g), \quad g \in \mathcal{G}.
\end{equation}

Proof. Since the map \([0, 1] \ni s \mapsto \sigma_s g := \sigma_s(r(g))g\) is differentiable for all \(g\), there exists a vector field \(X\) on \(\mathcal{G}\) satisfying (5). We need to check that \(X\) is \(d\)-vertical and right invariant. We have that
\[d(\sigma_s g) = d(g),\]
which proves that \(X\) is \(d\)-vertical, by definition. Also,
\[\sigma_s(gh) = \sigma_s(r(gh))gh = \sigma_s(r(g))gh = (\sigma_s g)h,\]
which proves that \(X\) is also right invariant.

In the following, the section \(X \in \Gamma(A(\mathcal{G})), \) defined in the above lemma, will be denoted \(\partial_s \sigma_s|_{s=0}.\) We define in the same way \(\partial_s \sigma_s\) for all values of \(s.\)

We shall repeatedly use the exponential map, and hence we shall to consider diffeomorphisms obtained by integrating vector fields. Recall that a smooth vector field \(X\) on a manifold \(M\) is called complete if, and only if, there exists a differentiable map \(\phi : \mathbb{R} \times M \to M\) such that
\[X(\phi(t, m)) = \partial_t(\phi(t, m)),\]
for all \((t, m) \in \mathbb{R} \times M.\) We then define
\[\exp(X)m := \phi(1, m).\]
Note that a complete vector field on a manifold with corners \(M\) is necessarily tangent to each face of \(M.\)

As for manifolds without corners, it follows from the definition and basic results on ordinary differential equations that, if \(X\) is complete, then \(tX\) is also complete, for all \(t \in \mathbb{R},\) and
\[\exp((t + s)X) = \exp(tX) \exp(sX),\]
for all \(s\) and \(t.\) Consequently, \(\exp(X)\) is a diffeomorphism.

Lemma 2. Let \(\mathcal{G}\) be a differentiable groupoid on \(M\) with Lie algebroid \(A = A(\mathcal{G}).\) If \(X \in \Gamma(A)\) is a section such that \(q(X)\) is complete, then \(X,\) regarded as a \(d\)-vertical vector field on \(\mathcal{G},\) is also complete.

Proof. For manifolds without corners, this is a result from Kumpera and Spencer \([7, Appendix].\) For manifolds with corners the proof is the same.

Proposition 1. Let \(\pi : A \to M\) be a Lie algebroid with anchor map \(q : A \to TM,\) and let \(X \in \Gamma(A)\) be a section such that \(q(X)\) is complete. Then there exists a uniquely determined isomorphism \(E_X : A \to A\) of Lie algebroids satisfying (i) and (ii):
\[(i) \quad \pi \circ E_X = \exp(q(X)) \circ \pi; \]
\[(ii) \quad \partial_t E_X(Y)|_{t=0} = [X, Y].\]
(iii) If, moreover, \(\mathcal{G}\) is a differentiable groupoid integrating \(A\) and \(X, Y \in \Gamma(A)\) are both complete, then \(E_X\) also satisfies
\[\exp(X) \exp(Y) = \exp(E_X(Y)) \exp(X),\]
as admissible sections.
4.12 in [8]. For local groupoids the proof is the same.

A family \( Y_1, \ldots, Y_n \) of smooth sections of \( A \) is a local basis of \( A \) at \( y \in M \) if \( Y_1(y), \ldots, Y_n(y) \) is a basis of \( A_y \). If \( t = (t_1, \ldots, t_n) \in \mathbb{R}^n \) and \( Y_1, \ldots, Y_n \) are sections of \( A \), then we denote

\[
\text{Exp}(t, Y) := \exp(t_1 Y_1 \exp(t_2 Y_2) \cdots \exp(t_n Y_n)).
\]

Also, let

\[
B_\epsilon = \{ t = (t_1, \ldots, t_n) \in \mathbb{R}^n, \sum_{i=1}^n |t_i| < \epsilon \}.
\]

Recall that a differentiable local groupoid \( \mathcal{U} \) on \( M \) satisfies all the axioms of a differential groupoid, except that the multiplication \( \gamma \gamma' \) is not defined for all pairs \( (g, g') \) such that \( d(g) = r(g') \), but only in a neighborhood \( \mathcal{U}_2 \) of the diagonal in the set \( \{ (g, g'), d(g) = r(g') \} \). See [13, 14] for details.

**Lemma 3.** Let \( \mathcal{G} \) be a local differentiable groupoid on \( M \), and let \( \sigma : M \rightarrow \mathcal{G} \) be an admissible section with \( r(\sigma(x_0)) = y_0 \). Also, let \( Y_1, \ldots, Y_n \) be a local base of \( A \) at \( y_0 \). Then, for some small \( \epsilon > 0 \) and a relatively compact open neighborhood \( U \) of \( x_0 \) in \( M \), the map

\[
\psi^\sigma_\gamma : \mathbb{R}^n \times M \ni (t, x) \rightarrow \text{Exp}(t, Y) \sigma(x) \in \mathcal{G}
\]

is a diffeomorphism from \( B_\epsilon \times U \) to a neighborhood of \( \sigma(x_0) \) in \( \mathcal{G} \).

If \( \sigma = \exp(X_m) \cdots \exp(X_1) \), for some integrable \( X_1, \ldots, X_m \in \Gamma(A) \), then we denote \( \psi^\sigma_\gamma = \psi^X_\gamma \).

**Proof.** If \( \mathcal{G} \) is a differentiable groupoid (not just a local one), this is Proposition 4.12 in [8]. For local groupoids the proof is the same. \( \square \)

Let \( \mathcal{G} \) be a differentiable groupoid, and let \( X_1, \ldots, X_m \) be smooth complete sections of \( A(\mathcal{G}) \). Also, let \( Y_i \) be smooth, complete sections of \( A(\mathcal{G}) \) that form a local basis at some point \( x_0 \in M \), as above. For simplicity, we shall sometimes assume that the \( Y_i \) are compactly supported, which is a stronger assumption than completeness. Since the admissible section

\[
\sigma(x) = \exp(X_m) \cdots \exp(X_1)x
\]

defines a diffeomorphism \( \sigma : \mathcal{G} \rightarrow \mathcal{G} \), equation (6), we obtain that the map

\[
\phi^\gamma_X : \mathbb{R}^n \times M \ni (t, x) \rightarrow \text{Exp}(t, Y) \exp(X_m) \cdots \exp(X_1) \text{Exp}(t, Y)x \in \mathcal{G}
\]

is also a diffeomorphism from a set of the form \( B_\epsilon \times U \) to its image, for some small \( \epsilon > 0 \) and some small open subset \( U \subset M \). The maps \( \phi^\gamma_\gamma \) are slightly more convenient to work with, in what follows, than the maps \( \psi_\gamma \) of the previous lemma.

Throughout the rest of this section, \( M \) will be a smooth manifold and \( A \rightarrow M \) will be a Lie algebroid. Moreover, \( M = \bigcup S \) is an \( A \)-invariant stratification of \( M \), \( \mathcal{G}_S \) is a differentiable groupoid integrating \( A_S \), and \( \mathcal{G} = \bigcup \mathcal{G}_S \) (disjoint union).

On \( \mathcal{G} = \bigcup \mathcal{G}_S \) we can then define uniquely a natural groupoid structure on \( \mathcal{G} \) such that the structural morphisms of each \( \mathcal{G}_S \) are obtained from those of \( \mathcal{G} \) by restriction. In particular, if two arrows \( g \in \mathcal{G}_S \) and \( g' \in \mathcal{G}_{S'} \) are composable, then \( S = S' \), also, each \( \mathcal{G}_S \) is a subgropoid of \( \mathcal{G} \).
A crucial observation is that we need not use the full differentiable structure on \( \mathcal{G} \) to define the maps \( \psi^X_Y \) of Lemma 3, or the maps \( \phi^X_Y \) of equation (7), for that matter. To define \( \phi^X_Y \), it is enough to use only the smooth structure on each \( \mathcal{G}_S \). This will allow to extend the definition of \( \phi^X_Y \) to our case, when the groupoid \( \mathcal{G} \) is obtained by glueing differential groupoids. This can be done as follows.

Let \( M = \cup S \) be an \( A \)-invariant stratification of \( M \). Also, let for each \( S \mathcal{G}_S \) be a differentiable groupoid integrating \( A_S = A|_S \) and \( \mathcal{G} = \cup \mathcal{G}_S \), as above. If \( X_i \) and \( Y_j \) are sections of \( A \) on \( M = \mathcal{G}^{(0)} \) such that the vector fields \( q(X_i) \) and \( q(Y_j) \) are integrable, then the restriction of these vector fields to each strata is again integrable, and hence the restriction of \( X_i \) and \( Y_j \) to each strata are integrable as vertical vector fields on \( \mathcal{G}_S \). Then the map \( \phi^X_Y \) is defined on each \( \mathbb{R}^n \times S \) (with values in \( \mathcal{G}_S \)), by glueing these maps, we obtain the desired definition of \( \phi^X_Y : \mathbb{R}^n \times M \to \mathcal{G} \), for \( \mathcal{G} = \cup \mathcal{G}_S \).

Since the maps \( \phi^X_Y \) and \( \psi^X_Y \) play an important role in what follows, we now spell out their properties in more detail.

**Lemma 4.** (i) The maps \( \phi^X_Y \) and \( \psi^X_Y \) are related by
\[
\phi^X_Y = \psi^X_Y,
\]
where \( Y'_j = E_{X_m} \circ E_{X_{m-1}} \circ \ldots \circ E_{X_1}(Y_j) \).

(ii) Let \( Y''_j = -Y_{n+1-j} \) and \( X'_i = -X_{m+1-i} \). Then
\[
\phi^X_Y(x)^{-1} = \psi^{X''}_Y(\alpha(x)),
\]
where \( \alpha \) is the diffeomorphism \( \exp(q(X_m)) \ldots \exp(q(X_1)) \).

**Proof.** Because each \( \mathcal{G}_S \) is a differential groupoid, (i) follows from Proposition 4 on each \( \mathcal{G}_S \). From this we obtain the desired relation everywhere on \( \mathcal{G} \).

(ii) follows from the relation
\[
(\exp(X)x)^{-1} = \exp(-X)(\exp(q(X))x),
\]
valid on each \( \mathcal{G}_S \), and hence everywhere on \( \mathcal{G} \).

We let as above \( X_1, \ldots, X_m, Y_1, \ldots, Y_n \) be integrable sections of \( A \). We shall also use the following lemma.

**Lemma 5.** (i) Fix \( t \in \mathbb{R}^n \) and \( x_0 \in M \). Then we can find \( \epsilon > 0 \), a neighborhood \( U \) of \( x \) in \( M \), and a differentiable map
\[
\tau : B_\epsilon \times U \to \mathbb{R}^n \times M, \ \tau(0, x_0) = (0, x_0),
\]
which is a diffeomorphism onto its image, such that
\[
\phi^X_Y(t+s, x) = \phi^X_Y(\tau(s, x)),
\]
on \( B_\epsilon \times U \), where \( X'_j = X_j \), for \( j = 1, \ldots, m \), and \( X'_{j+m} = t_j Y_j \), for \( j = 1, \ldots, n \).

(ii) For all \( Y_1, \ldots, Y_n, Y'_1, \ldots, Y'_n \) and \( x_0 \in M \), there exist \( \delta > \epsilon > 0 \), an open neighborhood \( U \) of \( x_0 \), and a differentiable map
\[
m_1 : B_\epsilon \times B_\delta \times U \to B_\delta \times M, \ m_1(0, 0, x) = (0, x),
\]
such that
\[
\Exp(t, Y)x(\Exp(t', Y')x)^{-1} = \Exp(m(t, t', x)).
\]
Proof. We begin by writing \( \exp((t_i + s_i)X_i) = \exp(t_iX_i)\exp(s_iX_i) \). Then, using Proposition \( \ref{prop:section} \) we can find sections \( Y^*_i \) of \( A \), \( s \in \mathbb{R} \), \( Y^0_i = Y_i \), depending smoothly on \( s \), such that
\[
\phi_Y^X(t + s, x) = \phi_Y^{Y^*}(\exp(s, Y^*)x).
\]
To complete (i), we now use a local groupoid \( U \) integrating \( A \). The existence of such a \( U \) is ensured by \( \ref{lem:local_groupoid} \). By replacing \( U \) with an open neighborhood of \( M \), if necessary, we may identify \( U \) with a subset of \( G \), using the exponential map. Then, for small \( \epsilon \) and a relatively compact neighborhood \( U \) of \( x_0 \), the maps
\[
B_\epsilon \times U \ni (x, s) \to f_1(x, s) := \exp(s, Y^*),
\]
and
\[
B_\epsilon \times U \ni (x, s) \to f_2(x, s) := \exp(s, Y)x
\]
are diffeomorphisms onto neighborhoods of \( x_0 \) in \( U \), by Lemma \( \ref{lem:exp} \). The desired map \( \tau \) is obtained from \( f_2^{-1} \circ f_1 \).

The proof of (ii) is similar, using the differentiability of the multiplication in a local groupoid.

The above result suggest to introduce the following family of maps.

The family \( \Phi \): Let \( f : V_0 \to V \) be a diffeomorphism (i.e., coordinate chart) from an open subset of \( \mathbb{R}^l \) to an open, relatively compact subset of \( G^{(0)} = M \). Because the partition \( M = \cup S \) is locally finite, we can find, using Lemma \( \ref{lem:partition} \) an \( \epsilon > 0 \) and sections \( X_i, Y_j \in \Gamma(A) \), for which \( \phi_Y^X \) defines a diffeomorphism from \( B_\epsilon \times (V \cap S) \to G_S \), for all \( S \) such that the intersection \( S \cap V \) is not empty. The family \( \Phi \) then consists of all maps of the form
\[
\varphi(t, y) = \phi_Y^X(t, f(y)) : B_\epsilon \times V_0 \to G.
\]
Recall that a differentiable atlas on a set \( M_0 \) is a family of injective maps
\[
\varphi : V_\varphi \to M_0,
\]
defined on an open subset of \( \mathbb{R}^l \), for some fixed \( l \), such that \( \varphi(V_\varphi) \) is a covering of \( M_0 \); \( \varphi^{-1}(\varphi_1(V_{\varphi_1})) \) is an open subset of \( V_\varphi \), and the map \( \varphi_1^{-1}\varphi \) is differentiable on \( V \).

We are ready now to prove the following theorem.

Theorem 2. Let \( M = \cup S \) be an \( A \)-invariant stratification of \( M \) and, for each \( S \), let \( G_S \) be a \( d \)-connected differentiable groupoid on \( M \) integrating \( A_S = A|_S \). Then \( G \) has a differentiable structure making it a differentiable groupoid with \( A(G) \simeq A \) if, and only if, the family \( \Phi \), consisting of the maps in equation \( \ref{eq:phi} \), is a differentiable atlas.

Proof. Suppose first that the family \( \Phi \) is a differentiable atlas. We begin by showing that the groupoid structure on \( G \), induced from the groupoids \( G_S \), is compatible with the differentiable structure defined by \( \Phi \). That is, we need to check that the structural morphisms are differentiable. We now check this.

First, the domain map is differentiable and submersive because \( d(\phi_Y^X(t, x)) = x \), for all \( X, Y \), and \( x \in M \). Next, it is enough to check that the map \( (g', g) \to g'g^{-1} \), defined on
\[
\{(g', g), d(g') = d(g) \},
\]
is differentiable. Let \( (g', g) \) be such that \( d(g) = d(g') = x_0 \), and let \( \phi_Y^X : \mathbb{R}^n \times M \to G \) and \( \phi_Y^X' : \mathbb{R}^n \times M \to G \) be two maps such that \( \phi_Y^X(t, x_0) = g \) and \( \phi_Y^X(t', x_0) = g' \),
and such that their restrictions to some small set of the form \( B_x \times U, t, t' \in B_x \), is in the family \( \Phi \). We need to show that the induced map

\[
\mu_1 : B_x \times B_x \times U \ni (t, t', x) \mapsto \phi^X(t', x)(\phi^Y(t, x))^{-1} \in \mathcal{G},
\]

is differentiable. It is enough to prove this in a small neighborhood of \( x_0 \). Because \( \Phi \) forms an atlas, using Lemma \( \text{(ii)} \), we see that we may assume \( t = t' = 0 \), eventually by changing \( X, X', Y, \) or \( Y' \). The differentiability of \( \mu_1 \) then follows by combining Lemma \( \text{(ii)} \), and Lemma \( \text{(i)} \). This is enough to conclude that \( \mathcal{G} \) is a differentiable groupoid whenever \( \Phi \) is an atlas.

Let \( A(\mathcal{G}) \) be the Lie algebroid of \( \mathcal{G} \) corresponding to the differentiable structure defined by \( \Phi \). We need to check that \( A(\mathcal{G}) \simeq A \). We have that \( A(\mathcal{G})|_S \simeq A_S \), by construction, and hence we can identify as a set \( A(\mathcal{G}) \) with the disjoint union of the restrictions \( A_S \). The two differentiable structures on \( \bigcup A_S \) (the first induced from \( A \) and the second induced from \( A(\mathcal{G}) \)) are the same because the differential of the map \( \phi^g_\sigma \) of Lemma \( \text{(ii)} \), \( \sigma = \text{id} \), canonically identifies \( A|_U \) and \( A(\mathcal{G})|_U \), if \( U \) is as in that lemma. The Lie algebra structures on \( \Gamma(A) \) and \( \Gamma(A(\mathcal{G})) \) also coincide because the two possible brackets of two vector fields coincide on each strata \( S \), and hence they coincide everywhere. We have thus proved that, if \( \Phi \) forms an atlas, then \( \mathcal{G} \) is a differentiable groupoid with \( A(\mathcal{G}) \simeq A \).

Conversely, suppose now that \( \mathcal{G} \) is endowed with a differentiable structure, and let \( g \in \mathcal{G}_S \). Since \( \mathcal{G}_S \) is \( d \)-connected, we can choose vector fields \( X_1, \ldots, X_m \) such that

\[
g = \exp(X_m) \ldots \exp(X_1)x_0,
\]

for some \( x_0 \in S \). If \( Y_1, \ldots, Y_n \) are chosen to form a basis of \( A_{x_0} \), as in Lemma \( \text{(ii)} \), then the map \( \phi^X_\sigma \) must be a diffeomorphism of a set of the form \( B_x \times U \) onto an open neighborhood of \( g \), for some open subset \( U \subset M \). We obtain that if \( \mathcal{G} = \bigcup \mathcal{G}_S \) is a differential groupoid such that \( A(\mathcal{G}) \simeq A \), then the family \( \Phi \) is an atlas.

In particular, we immediately obtain from the above theorem and its proof that the differentiable structure on \( M \) and the groupoid structure on \( \mathcal{G} \) uniquely determine the differentiable structure on \( \mathcal{G} \) satisfying \( A(\mathcal{G}) \simeq A \). Indeed, the differentiable structure on \( \mathcal{G} \) is determined by the family \( \Phi \).

We now turn to the main theorem of this paper, Theorem \( \text{(i)} \), which shows that the assumptions of Theorem \( \text{(ii)} \) are satisfied provided that the groupoids \( \mathcal{G}_S \) are \( d \)-simply connected. We shall need an extension of the concept of differentiable family of sections to our case, \( \mathcal{G} = \bigcup \mathcal{G}_S \), when \( \mathcal{G}_S \) are smooth groupoids but \( \mathcal{G} \) is not endowed with any differentiable structure.

A differentiable family of admissible sections is a family \( \sigma_s : M \to \mathcal{G}, s \in [0,1], \) of maps such that each \( \sigma_s, s \in [0,1], \) is an admissible section, the induced map \( [0,1] \times S \ni (s, x) \mapsto \sigma_s(x) \in \mathcal{G}_S \) is differentiable for each \( S \), and the sections \( \partial_s(\sigma_s|_S) \) of \( A_S \) can be glued to a smooth section of \( A \) on \( [0,1] \times M \). This definition of a differentiable family of smooth section is thus very similar to the corresponding definition in the case when \( \mathcal{G} \) has a smooth structure, except that we replace the condition on the smoothness of the map \( [0,1] \times M \to \mathcal{G} \) by the existence and smoothness of the derivatives \( \partial_s \sigma_s \) on \( [0,1] \times M \).

The following lemma is an important technical part of the proof of Theorem \( \text{(i)} \). It achieves a continuous and smooth deformation of a local admissible section of \( \mathcal{G} \) to a local identity section. We denote by \( \Gamma_c(E) \) the space of compactly supported, smooth sections of a vector bundle \( E \).
Lemma 6. Let \( \mathcal{G} = \cup \mathcal{G}_S \) be a union of \( d \)-simply connected differentiable groupoids with \( \text{Ad}(\mathcal{G}_S) = A_S \), as in the statement of Theorem 3, and let \( x_0 \in M \). Suppose that \( X_1, \ldots, X_m \in \Gamma_c(A) \) satisfy
\[
\exp(X_m) \ldots \exp(X_1)x_0 = x_0.
\]
Then we can find a differentiable family of admissible sections \( \sigma_s : M \to \mathcal{G} \) such that:
(i) \( \sigma_1(x) = \exp(X_m) \ldots \exp(X_1)x \), on \( M \);
(ii) \( \sigma_0(x) = x \), for all \( x \in M \);
and, most importantly,
(iii) \( \sigma_s(x_0) = x_0 \), for all \( s \in [0, 1] \).

This lemma remains true if we replace the condition that \( X_i \) be compactly supported by the condition that they be integrable.

Proof. Consider the curve
\[
\phi(t) = \exp(t'X_k)\exp(X_{k-1}) \ldots \exp(X_1)x_0,
\]
where \( t' = (mt - k + 1) \) and \( k \) is chosen such that \( 0 \leq t' \leq 1 \). By assumption, \( \phi \) is a closed curve on \( \mathcal{G}_{x_0} \), and hence, by the assumption that \( \mathcal{G} \) is \( d \)-simply connected, we can continuously deform this curve to the constant curve \( x_0 \), within \( \mathcal{G}_{x_0} \), through closed curves based at \( x_0 \). More precisely, we can find
\[
\eta : [0, 1] \times [0, 1] \to \mathcal{G}_{x_0}
\]
such that \( \eta(t, 1) = \phi(t) \) and \( \eta(t, 0) = \eta(0, s) = \eta(1, s) = x_0 \), for all \( t, s \in [0, 1] \).

By an approximation argument, we can assume that \( \eta(\frac{k}{m}, s) \) depends smoothly on \( s \), for each integer \( k \). Moreover, after replacing \( m \) by a large multiple \( lm \) and each \( X_k \), \( k = 1, \ldots, m \), by \( l^{-1}X_k \) repeated \( l \) times, we can assume that there exist compactly supported sections \( X^j_k \in \Gamma(A) \), depending smoothly on \( s \), such that \( X^1_k = X_k \), \( X^0_k = 0 \), and
\[
\eta(\frac{k}{m}, s) = \exp(X^j_k)\eta(\frac{k - 1}{m}, s).
\]
The desired deformation is obtained by letting
\[
\sigma_s(x) = \exp(X^j_m) \ldots \exp(X^j_1)x.
\]
To see that \( \sigma_s \) is a smooth family of admissible sections, we use Proposition 3 for each \( \mathcal{G}_S \) and the fact that each \( \exp(X^j_k) \) is a smooth admissible section. \( \Box \)

We continue to assume that \( \mathcal{G} = \cup \mathcal{G}_S \) is as in the statement of Theorem 3.

Proposition 2. Let \( \sigma_s \), \( s \in [0, 1] \), be a differentiable family of local admissible sections. Assume that \( \sigma_s(x_0) = x_0 \), for all \( s \in [0, 1] \), and that \( \sigma_0(x) = x \), for all \( x \in M \). Let \( Y_1, \ldots, Y_n \) be a local basis of \( A \) at \( x_0 \). Then there exist \( \epsilon > \delta > 0 \), a neighborhood \( U \) of \( x_0 \) in \( M \), and a differentiable map
\[
\tau : [0, 1] \times B_\delta \times U \to B_\epsilon
\]
such that:
(i) \( \tau(s, 0, x_0) = 0 \),
(ii) for each fixed \( s \in [0, 1] \) and \( x \in U \), the map \( B_\delta \ni t \to \tau(s, t, x) \in B_\epsilon \) is a diffeomorphism onto its image, and
(iii) \( \sigma_s \exp(t, Y)x = \exp(\tau(s, t, x), Y)x \),
for all \( s, t \in [0, 1] \), and \( x \in U \).
Proof. Let $U$ be a local groupoid integrating $A$. Choose $\epsilon > 0$ and a neighborhood $U_1$ of $x_0$ small enough such that

$$
\eta: B_{2\epsilon} \times U_1 \to \mathcal{U}, \quad \eta(t,x) = \text{Exp}(t,Y)x,
$$

is a diffeomorphism onto an open neighborhood $V$ of $x_0$ in $\mathcal{U}$. Let $U_0 \subset U_1$ be a compact neighborhood of $x_0$. Because $K := \exp(B_{\epsilon} \times U_0)$ is a compact subset of the open set $V$ and $\sigma$ is continuous, $\sigma_s(x_0) = x_0$, we can find a neighborhood $U$ of $x_0$ such that $\sigma_s(k)$ is defined in $U$ whenever $k \in K$, and $d(k) \in U$, and such that

$$
\sigma_s(U_1)K \subset V.
$$

Then we can define

$$
\tau(s,t,x) = \eta^{-1}(\sigma_s \text{Exp}(t,Y)x).
$$

Because $\sigma_s \text{Exp}(t,Y)x \in V \subset \mathcal{U}$ and $\eta$ is a diffeomorphism, we obtain that $\tau$ is smooth also.

**Theorem 3.** Let $A$ be a Lie algebroid on a manifold with corners $M$. Suppose that $M$ has an $A$-invariant stratification $M = \bigcup S$ such that, for each stratum $S$, the restriction $A_S$ is integrable, then $A$ is integrable.

More precisely, let $\mathcal{G}_S$ be $d$-simply connected differential groupoids such that $A(\mathcal{G}_S) \simeq A_S$. Then the disjoint union $\mathcal{G} = \bigcup \mathcal{G}_S$ is naturally a differentiable groupoid such that $A(\mathcal{G}) \simeq A$.

**Proof.** By Theorem 2, we see that we it is enough to show that the family $\Phi$, defined using the maps $\phi_Y$ of equation (7), is an atlas. Let $\phi_Y$ and $\phi_Y'$ be two such maps, defined on $B_{\epsilon} \times U$ and, respectively, on $B_{\epsilon'} \times U'$, such that their images intersect. Let $g \in \mathcal{G}$ be an element of this intersection. As in the proof of Theorem 2 we can arrange that $\phi_Y(0,x_0) = \phi_Y'(0,x_0) = g$. Then

$$
\phi_Y(0,x_0) = \exp(X_0)\ldots\exp(X_1)x_0 = \exp(X'_0)\ldots\exp(X'_1)x_0 = \phi_Y'(0,x_0),
$$

and hence

$$
x_0 = \exp(-X'_1)\ldots\exp(-X'_m)\exp(X_0)\ldots\exp(X_1)x_0,
$$

so we may also assume that $X'_0 = 0$ (and hence $g = x_0$). By replacing $V$ and $V'$ with some smaller, relatively compact neighborhoods, if necessary, we may further assume that the sections $X_i$ are compactly supported. Let

$$
\sigma(x) = \exp(X_0)\ldots\exp(X_1)x.
$$

Since each $\mathcal{G}_S$ is $d$-simply connected, by Lemma 3 we can find a smooth family $\sigma_s : M \to \mathcal{G}$, $s \in [0,1]$, such that $\sigma_0$ is the identity, $\sigma_1 = \sigma$, the restriction to each groupoid $G_S$ is differentiable. The smoothness of the family $\sigma_s$ in this case is reduces to showing that $\partial_s \sigma_s$ is a smooth section of $A$ over $M$. Let $\tau$ be the map defined in Proposition 2 using the admissible sections $\sigma_s$, and let $\tau_1(t,x) = \tau(1,t,x)$. Then

$$
\phi_Y^X(t,x) = \phi_Y^{X'}(\tau_1(t,x),x),
$$

in a small neighborhood of $x_0$. Since $\tau_1$ is a local diffeomorphism for each fixed $x$, this proves the theorem.

\[ \square \]
3. Applications to foliations

Recall that a Lie algebroid $\pi : A \to M$, with anchor map $q : A \to TM$, is called regular if, and only if, the range of $q$ has locally constant rank. Then the sections of $q(A)$ are the sections of the tangent bundle to a foliation $\mathcal{F}_A$ whose tangent bundle, a sub-bundle of $TM$, is denoted $T\mathcal{F}_A$. If, moreover, there exists a morphism $\rho : \Gamma(T\mathcal{F}_A) \to \Gamma(A)$ such that $q \circ \rho = id$, then we may assume that $A$ is a semi-direct product.

Also, if $A$ is regular, the kernel of $q$ is a bundle of Lie algebras. See also [6, 20]. Recall that a result of Douady and Lazard from [5] states that every bundle of Lie algebras is integrable. In particular the kernel $\ker(q)$ is integrable. (I am grateful to Alan Weinstein for pointing out this reference to me.)

We say that $A$ is (isomorphic to) the semi-direct product $\ker(q) \rtimes q(A)$ if there exists a morphism of Lie algebras $\rho: \Gamma(q(A)) \to \Gamma(A)$ that is a right inverse to $q$.

**Proposition 3.** Let $A$ be a regular algebroid with anchor map $q : A \to TM$. Assume that $A$ is the semi-direct product $\ker(q) \rtimes q(A)$. Then $A$ is integrable.

Before proceeding to the proof, we first introduce some terminology and make some comments. If $E \to M$ is a vector bundle on a foliated manifold $M$, then a leafwise connection on $E$ is a linear map $\nabla : \Gamma(E) \to \Gamma(E \otimes T^*\mathcal{F})$, satisfying the Leibnitz identity. Here $\mathcal{F}$ is the foliation of $M$, $T\mathcal{F}$ is the tangent bundle to this foliation, and $T^*\mathcal{F}$ is the dual of this bundle. The equivalent definitions of a connection in terms of parallel transport or equivariant splittings of tangent spaces of principal bundles extend to the “leafwise” setting also.

The structure of regular, semi-direct product Lie algebroids is as follows. First, there exists a leafwise flat connection $\nabla$, $\nabla_Z(X) = [\rho(Z), X]$, on $\ker(q)$, which preserves its Lie bundle structure, that is

$$\nabla_Z \nabla_{Z'} - \nabla_{Z'} \nabla_Z = \nabla_{[Z,Z']},$$

and

$$\nabla_Z([X,Y]) = [\nabla_Z(X), Y] + [X, \nabla_Z(Y)],$$

for all $Z, Z' \in \Gamma(T\mathcal{F}_A)$ and $X, Y \in \Gamma(\ker(q))$. Then

$$A \simeq T\mathcal{F}_A \oplus \ker(q)$$

as vector bundles, with anchor map given by the projection onto the first component, and with Lie bracket on $\Gamma(A)$ defined by

$$[(Z, X), (Z', X')] = ([Z, Z'], \nabla_Z(X') - \nabla_{Z'}(X) + [X, X']),$$

for all $Z, Z' \in \Gamma(T\mathcal{F}_A)$ and $X, X' \in \Gamma(\ker(q))$.

With these comments, we are now ready to prove Proposition 3.

**Proof.** Recall that $\mathcal{P}\mathcal{F}$, the path groupoid of the foliation $\mathcal{F}$, consists of fixed end point homotopy classes of paths $\gamma$ that are fully contained in a single leaf, with respect to homotopies within that leaf. As explained above, the morphism $\rho$ defines a leafwise flat connection on $\ker(q)$ that preserves the Lie bracket. Let $\mathcal{K}$ be a $d$-simply connected groupoid that integrates $\ker(q)$.

We define then a groupoid $\mathcal{G}$ that integrates $A$ as follows. As a smooth manifold,

$$\mathcal{G} = \{(g, \gamma) \in \mathcal{K} \times \mathcal{P}\mathcal{F}, d(g) = \gamma(1)\}.$$
To define the multiplication, observe first that the leafwise flat connection on \( \ker(q) \) defines a parallel transport map
\[
\rho(\gamma) : \ker(q)_{\gamma(0)} \rightarrow \ker(q)_{\gamma(1)},
\]
which is a Lie algebra isomorphism for any path \( \gamma \) fully contained in a leaf. Since \( \mathcal{K}_x \) is simply connected for each \( x \), we obtain by exponentiation a group morphism
\[
\rho(\gamma) : \mathcal{K}_{\gamma(0)} \rightarrow \mathcal{K}_{\gamma(1)}.
\]
That is, the leafwise flat connection on \( \ker(q) \) lifts to a leafwise flat connection on \( \mathcal{K} \) that preserves the Lie group structure on the fibers.

We are ready now to define the groupoid structure on \( G \). Note first that \( d(g) = r(g) \) for \( g \in K \). Then \( d(g, \gamma) = \gamma(0), \quad r(g, \gamma) = r(g) \), and the product on \( G \) is given by the formula
\[
(g, \gamma)(g', \gamma') = (g \rho(\gamma')(g'), \gamma \gamma'),
\]
where the composition of paths is given by concatenation. The flatness of \( \nabla \) gives that \( \rho(\gamma \gamma') = \rho(\gamma) \rho(\gamma') \), which guarantees the associativity of the product.

In certain cases we get integrability without assuming that \( A \) is a semi-direct product.

**Proposition 4.** Let \( A \) be a regular Lie algebroid with anchor map \( q : A \rightarrow TM \) such that \( \ker(q) \) is a bundle of semisimple Lie algebras. Then \( A \) is integrable.

**Proof.** We may assume that \( M \) is connected. Since semisimple Lie algebras are rigid, all fibers of \( \ker(q) \) will be isomorphic Lie algebras. Fix one of these algebras and denote it by \( g \). Also, let \( G_0 \) be the group of automorphisms of \( g \). Then, if we define
\[
P = \cup_x \text{Iso}(g, \ker(q)_x),
\]
(the fibers are the sets of Lie algebra isomorphisms \( g \rightarrow \ker(q)_x \)), we obtain a \( G_0 \)-principal bundle on \( M \), which acquires by pull-back a foliation \( \mathcal{F} \) of the same codimension as \( \mathcal{F}_A \). The path groupoid \( \mathcal{P}\mathcal{F} \) of the foliation \( \mathcal{F} \) has an induced free action of \( G_0 \), and we define the groupoid \( \mathcal{G} \) by \( \mathcal{G} = \mathcal{P}\mathcal{F}/G_0 \). The composition of two paths in \( \mathcal{G} \) is obtained by choosing composable liftings in \( \mathcal{P}\mathcal{F} \). Because the Lie algebra of \( G_0 \) is \( g \), we obtain that \( \mathcal{G} \) integrates \( A \).

With the above results in mind, we now define \( A \)-invariant regular stratifications.

**Definition 2.** Let \( \pi : A \rightarrow M \) be a Lie algebroid with anchor map \( q : A \rightarrow TM \) on the manifold with corners \( M \). An invariant stratification \( M = \cup S \) is called regular if, and only if, the restriction \( A_S := A|_S \) is regular for each \( S \).

Although almost all interesting Lie algebroids have invariant regular stratifications, this is not true in general. Consider, for example, a closed subset \( B \subset \mathbb{R}^n \) with empty interior, which is not a manifold. Let \( \phi \geq 0 \) be a smooth function that vanishes exactly on \( B \), and let \( \mathcal{V}(\mathbb{R}^n) \) be the Lie algebra of vector fields on \( \mathbb{R}^n \). Since \( \phi \mathcal{V}(\mathbb{R}^n) \) is a free \( C^\infty(\mathbb{R}^n) \) module, using the Serre-Swan theorem, we can define a vector bundle \( A \) such that \( \Gamma(A) = \phi \mathcal{V}(\mathbb{R}^n) \). Moreover,
\[
[\phi X, \phi Y] = \phi ([X(\phi)Y - Y(\phi)X + \phi[X,Y]]),
\]
for all vector fields \( X \) and \( Y \) on \( \mathbb{R}^n \), which shows that \( A \) is a Lie algebroid. It is not difficult to see that \( A \) has no invariant regular stratification.
Although not all Lie algebroids have regular stratifications, this notion is useful because it is easier to integrate regular algebroids than general Lie algebroids, and we know that in order to integrate a Lie algebroid, it is enough to integrate it over each strata (Theorem 3).

**Theorem 4.** Let $A$ be a Lie algebroid with anchor map $q : A \to TM$ on a manifold $M$ with a regular $A$-invariant stratification $M = \cup S$. Assume, for each $S$, that either $A_S$ is the semi-direct product $\ker(q)_S \rtimes q(A_S)$, or that $\ker(q)|_S$ is a bundle of semisimple Lie algebras. Then $A$ is integrable.

**Proof.** Using Proposition 3 or Proposition 4, we see that each of the algebroids $A_S$, obtained by restricting $A$ to the stratum $S$, is integrable. By Theorem 3, the Lie algebroid $A$ is then integrable.

Let us consider now in greater detail a class of examples that is useful for the construction of pseudodifferential operators on complex algebraic varieties. It is possible to associate several “natural” algebroids to a complex algebraic variety, so the following constructions lead to a family of algebras of pseudodifferential operators associated to a complex algebraic variety. The details of this construction will be presented in a future paper. We do not know at this point which of the many algebras that one obtains is the “right” algebra to associate to a complex algebraic manifold, but we hope to address this question in a future paper.

Let $M$ be a manifold with corners, such that each hyperface (i.e., face of maximal dimension) $H \subset M$ is given by $H = \{x_H = 0\}$ for some function $x_H \geq 0$ with $dx_H \neq 0$ on $H$, (i.e., $x_H$ is a defining function of $H$.) If $F \subset M$ is an arbitrary face of $M$ of codimension $k$, then $F$ is an open component of the intersection of the hyperfaces containing it. The set $x_1, \ldots, x_k$ of defining functions of these hypersurfaces are called the defining functions of $F$; thus $F$ is a connected component of $\{x_1 = x_2 = \ldots = x_k = 0\}$.

We first introduce the class of Lie algebroids we are interested in on a manifold with corners $M$. We call these algebroids “quasi-homogeneous,” and we now proceed to construct them. First we need some related definitions that will make it easier to describe our settings.

**Definition 3.** A **Lie flag** on a manifold $M$ is an increasing finite sequence of sub-bundles
\[ E_0 \subset E_1 \subset \ldots \subset E_l \subset E_\infty := TM \]
such that $[\Gamma(E_i), \Gamma(E_j)] \subset \Gamma(E_{i+j})$.

It follows from the definition that, if $E_0 \subset \ldots \subset E_l \subset TM$ is a Lie flag on $M$, then the bundles $E_0$ and $E_l$ are integrable. We do not assume the above inclusions to be strict.

We now describe the type of behavior we want for the vector fields that are sections of a quasi-homogeneous Lie algebroid. Let $p_i$ denote the projection onto the $i$th component of a product.

**Definition 4.** Let $H = \{x_H = 0\} \subset M$ be a hyperface, and let
\[ \phi_H = (\pi_H, x_H) : V_H \to H \times [0, \epsilon) \]
be a diffeomorphism defined in a neighborhood $V_H$ of $H$ in $M$. Also, let
\[ E_0^H \subset \ldots \subset E_l^H \subset TH \]
be a Lie flag on $H$ and $d_H \in \mathbb{N} \cup \{\infty\}$. Then a vector field $X$ on $V_H$ is called $(E_i, \phi_H, d_H)$-adapted if, and only if,

$$X \in \sum_{j=0}^{d_H} \pi^*_H(E_j|V_H) + C^\infty(V_H) x_H^{d_H+1} \partial x_H,$$

(we agree that $x_H^\infty = 0$).

Finally, the sections of our quasi-homogeneous Lie algebroids will consist of vector fields that are “adapted” to each hyperface, but the data at each hyperface must be compatible.

**Definition 5.** A boundary Lie datum $\mathcal{D} = (E_i^H, \phi_H, d_H)$ on a manifold with corners $M$, where $H$ ranges through the set of hyperfaces of $M$, consists of:

(i) Lie flags $E_0^H \subset \ldots \subset E_{d_H}^H \subset \mathcal{F}$ such that all intersections $E_{i_1}^H \cap \ldots \cap E_{i_t}^H$, $i_j \in \{0, 1, \ldots, l_H, \infty\}$, as well as all finite sums of such intersections, have constant rank on the set where they are defined (and hence they are vector bundles), and form a distributive lattice.

(ii) Diffeomorphisms

$$\phi_H = (\pi_H, x_H) : V_H \to H \times [0, \epsilon)$$

such that $\pi_H \pi_H = \pi_H \pi_H$ and $x_H \pi_H = x_H$ on $V_H \cap V_{H^\prime}$, for all hyperfaces $H$ and $H^\prime$.

(iii) Degrees $d_H \in \mathbb{N} \cup \{\infty\}$.

If $\mathcal{D} = (E_i^H, \phi_H, d_H)$ is a boundary Lie datum on $M$, then the diffeomorphism $\phi_H$ defines a complement $N_H$ to $\mathcal{F}$ in $TM|_H$.

**Proposition 5.** Let $\mathcal{D} = (E_i^H, \phi_H, d_H)$ be a boundary Lie datum on the manifold with corners $M$. Suppose that $\mathcal{F}$ is a foliation of $M$ such that

$$T\mathcal{F}|_H = \begin{cases} E_{i_H}^H + N_H, & \text{if } d_H < \infty, \\ E_{i_H}^H, & \text{if } d_H = \infty. \end{cases}$$

If we define

$$\mathcal{A}_D := \{X \in \Gamma(T\mathcal{F}), X \text{ is } (E_i^H, \phi_H, d_H)\text{-adapted, for each hyperface } H\}$$

and $[x_j \partial x_j, \mathcal{A}_D] \subset \mathcal{A}_D$, then $\mathcal{A}_D$ is a Lie algebra and a projective $C^\infty(M)$-module.

**Proof.** The set of points satisfying (10) at a face $H$ is closed under the Lie bracket, by definition of a Lie flag. Since $\Gamma(T\mathcal{F})$ is also closed under the Lie bracket, it follows that $\mathcal{A}_D$ is a Lie algebra.

Fix a point $x_0$ in the interior of a face $F$ of codimension $k$ with defining functions $x_1, \ldots, x_k$. Let $H_1, \ldots, H_k$ be the corresponding hyperfaces, $H_j = \{x_j = 0\}$, and let $d_i = d_{H_i}$. Also let

$$\pi = \pi_{H_1} \ldots \pi_{H_k} : V := \cap V_{H_j} \to F,$$

where the order of composition is not important because $\pi_H \pi_H = \pi_H \pi_H$, by the definition of boundary Lie datum.

Let $\nu = (\nu_1, \ldots, \nu_k)$, $0 \leq \nu_i \leq l_{H_i}$, be a multi-index and

$$E_\nu = \pi^*(E_{\nu_1}^{H_1} \cap \ldots \cap E_{\nu_k}^{H_k}),$$

which, we recall, is a vector bundle on $H_1 \cap \ldots \cap H_k$, again by the definition of a boundary Lie datum. Also, let

$$\nu^{(i)} = (\nu_1, \ldots, \nu_{i-1}, \nu_i - 1, \nu_{i+1}, \ldots, \nu_k)$$

and choose, arbitrarily, a complement $Y_\nu$ to $\sum E_{\nu^{(i)}}$ in $E_{\nu}$.

Let $Z$ be the set of vector fields $\{x^1, \ldots, x_k\}$ defined using the diffeomorphism $(\pi, x_1, \ldots, x_k) : V = \cap V_{H_i} \to F \times [0, \epsilon)^k$, for some small $\epsilon$. Then the restriction of $A_D$ to $V$ is

$$C^\infty(V)A_D|_V = \left( \sum_{\nu=(\nu_1, \ldots, \nu_k)} x_1^{\nu_1} \ldots x_k^{\nu_k} \Gamma(\pi^*(Y_\nu)) \right) \bigoplus \left( \bigoplus_{x_i \in Z} C^\infty(V)Z_i \right)$$

$$\simeq \Gamma(\oplus \nu \nu Y_\nu \oplus \mathbb{R}^\nu).$$

Since this is a $C^\infty(V)$-projective module—the module of sections of a bundle isomorphic to the direct sum of $\oplus \nu \nu Y_\nu$ and the trivial bundle generated by the set $x^1 \ldots x_k < \infty$, we obtain that $A_D$ is a projective $C^\infty(M)$–module, as desired.

Let us examine now a particular case of this construction. Assume that in the definition above $F = M$, that is, that there exists a single leaf, and that in the boundary Lie data $d_H = 0$ and $E^H = TH$. The only choice then is that of the integrable bundles $E_0$, because the choice of the diffeomorphisms $\phi_H$ is not important. The conditions that these bundles have to satisfy are that, for each face $F$ (contained in $k$ distinct hyperplanes $H$, where $k$ is the codimension of $F$), there exist $2k$ commuting surjective projections

$$p_{FH} : TM|_F \to E^H_0|_F, \quad \text{and} \quad q_{FH} : TM|_F \to TH|_F.$$ 

We denote by $A_D$ the vector bundle on $M$ with sections $A_D$, defined by the above proposition. Because $A_D$ is a Lie algebra, $A_D$ is a Lie algebroid. A Lie algebroid $A$ is called quasi-homogeneous if it is isomorphic to one of the form $A_D$ obtained as above.

**Proposition 6.** Let $A \subset TM$ be a quasi-homogeneous algebroid. Then the set of open faces of $M$ defines an $A$-invariant regular stratification of $M$. Moreover, if $S = \text{Int}(F)$ is an open face of $M$, then the Lie algebroid $q_S : A_S := A|_S \to TS$ is integrable.

**Proof.** Let $F$ and $(E^H_i, \phi_H, d_H)$ be the foliation and, respectively, the boundary Lie data defining $A$. Observe that the sections of $A$ are vector fields that are tangent to all faces of $M$, and hence each open face of $M$ is $A$-invariant. This means that the stratification of $M$ by open faces is an $A$-invariant stratification of $M$.

Fix a face $F$ of codimension $k$ with defining functions $x_1, \ldots, x_k$, such that $H_j = \{x_j = 0\}$. Let $S := \text{Int}(F)$ be the interior of $F$, and denote $A_S := A|_S$.

For $\nu_0 = (0, \ldots, 0)$, the intersection

$$E_{\nu_0} := E^H_0 \cap \ldots \cap E^H_k \subset TS$$

is an integrable sub-bundle of $TS$ such that $q_S : A_S \to TS$ maps $\Gamma(A_S)$ surjectively onto $\Gamma(E_{\nu_0})$. Moreover, the map $\pi = \pi_{H_1} \circ \ldots \circ \pi_{H_k}$ defines a splitting of $A_S$, that is, a Lie algebra morphism $\Gamma(E_{\nu_0}) \to \Gamma(A_S)$. (This splitting is implicit in the description of $A_D$ given in (11).) Consequently, $A_S$ is integrable, by Proposition 3.

\[\square\]
Theorem 5. Every quasi-homogeneous algebroid is integrable.

Proof. This follows from Proposition 3 and Theorem 4. □

Because the construction of the differentiable groupoid integrating a regular, semi-direct product Lie algebroid depends on the construction of a differentiable groupoid $K$ integrating $\ker(q_S)$, it is useful to have an explicit construction of $K$ in the above proposition. To this end, we use the notation in the proof of Proposition 6. Let $Z_0 \subset Z := \{x^i \partial_i\}$ be the set of normal vectors (with respect to the decomposition induced by $(x, x_1, \ldots, x_k)$) such that $d_i = d_{H_i} = 0$. Also, let $A_0$ be the sub-bundle of $A|_S$ generated by $Z_0$ and $A_1$ be the vector bundle generated by the complement of $Z_0$ in $Z$ and by $E_\nu$, for $\nu \neq (0, \ldots, 0)$. We then obtain the split exact sequence

$$0 \to A_1 \to \ker(q_S) \to A_0 \to 0.$$ 

Now, each of the bundles $A_0$ and $A_1$ is a Lie algebroid with vanishing anchor map. In fact, each of $A_0$ and $A_1$ are bundles of commutative Lie algebras, and hence are integrable: to integrate them, we just consider each $A_0$ and $A_1$ as a bundle of commutative Lie groups. Moreover, $A_1$ is an ideal (in Lie algebra sense) of $\ker(q_S)$, and the sub-bundle $A_1$ acts by derivations on $A_0$ (the weights are exactly given by the exponents $\nu = (\nu_i)$). This action by derivations of each of the fibers $(A_1)_x$ on $(A_0)_x$ exponentiates to an action of the Lie group with Lie algebra $(A_1)_x$ on the Lie group with Lie algebra $(A_0)_x$. Denote the resulting semidirect product by $K_x$. In this way we obtain on $K = K_x$ a Lie algebra bundle structure, which is isomorphic to $\ker(q_S)$, as a fiber bundle, via the exponential map.

If $A$ is an integrable Lie algebroid, then we denote by $\mathcal{G}_A$ a $d$-simply-connected differential groupoid that integrates $A$, which is unique up to isomorphism. If $A$ is a quasi-homogeneous algebroid and $\mathcal{G}$ is any differential groupoid on $M$ with Lie algebroid $A$, then we call $\mathcal{G}$ a quasi-homogeneous groupoid.

We now make some elementary remarks on quasi-homogeneous algebroids and groupoids. If $A \subset M$ is a quasi-homogeneous algebroid and $F \subset M$ is a face of $M$, then $A_F$ is not a quasi-homogeneous algebroid unless $F = M$. Let $N_F \subset TM$ be the normal bundle to $F$, with the trivialization given by the boundary Lie data used in the definition of $A$. Then $A_F$ is the semidirect product of $A_F \cap TF$ and $N$. The same will be true of the integrating groupoids. Thus, assume that $F$ has codimension $k$. Then there exists an action of $\mathbb{R}^k$ on $\mathcal{G}_{A|_F \cap TF}$, which fixes $F$, the set of units of $\mathcal{G}_{A|_F \cap TF}$, such that

$$\mathcal{G}_{A_F} \simeq \mathcal{G}_{A|_F \cap TF} \times \mathbb{R}^k. \tag{12}$$

This relates the differential groupoids $\mathcal{G}_{A_F}$, associated to the faces of $M$, to the quasi-homogeneous differential groupoids $\mathcal{G}_{A|_F \cap TF}$.

Let us now take a closer look at the simplest example of a quasi-homogeneous, non-regular algebroid, the algebroid $\mathcal{V}_{b}(M)$ of all vector fields tangent to the boundary $\partial M \neq \emptyset$ of a manifold with boundary $M$. The two strata of $M$ are $\partial M$ and $\text{Int}(M) = M \setminus \partial M$. If $X$ is a topological space, we denote by $\mathcal{P}_X$ its path groupoid. Then the $d$-simply-connected differential groupoids $\mathcal{G}_1$ and $\mathcal{G}_2$ that integrate $\mathcal{V}_{b}(M)|_{\partial M} = T(\partial M) \oplus N_{\partial M}$ and $\mathcal{V}_{b}(M)|_{\text{Int}(M)}$ are, up to isomorphism,

$$\mathcal{G}_1 \simeq \mathcal{P}_{\partial M} \times \mathbb{R}, \quad \text{and} \quad \mathcal{G}_2 \simeq \mathcal{P}_{\text{Int}(M)}.$$
By Theorem 3, or directly, the two groupoids $\mathcal{G}_1$ and $\mathcal{G}_2$ above can be smoothly glued to form a differentiable groupoid

$$\mathcal{G}_{\mathcal{V}_b(M)} = \mathcal{G}_1 \cup \mathcal{G}_2$$

that integrates $\mathcal{V}_b(M)$. This groupoid will be Hausdorff if, and only if, the morphism $\pi_1(\partial M) \to \pi_1(M)$ is injective. The domain map

$$d : \mathcal{G}_{\mathcal{V}_b(M)} \to M$$

is a fibration (that is, it has the homotopy lifting property) if, and only if, $\pi_1(\partial M) \to \pi_1(M)$ is surjective.

In general, the groupoid $\mathcal{G}_{\mathcal{V}_b(M)}$ is much larger than the “stretched product” $M_b^2$ considered by Melrose [10], which also gives a groupoid that integrates $\mathcal{V}_b(M)$ after we remove its off-diagonal faces. We get the same groupoid only if both $\pi_1(\partial M)$ and $\pi_1(M)$ are trivial. Unlike our $d$-simply connected groupoid $\mathcal{G}_{\mathcal{V}_b(M)}$, the groupoid obtained from $M_b^2$ is always Hausdorff and the domain map $d$ is a fibration.

Let $r \geq 2$ be an integer. Then the same discussion applies to $\mathcal{V}_{b,k}(M)$, the Lie algebra of vector fields that at the boundary are of the form $x^i \partial_x + \sum_1^n \partial_{y_j}$, for a suitable coordinate systems $(x, y_1, \ldots, y_{n-1})$ in a neighborhood of a point of the boundary $\partial M = \{ x = 0 \}$ and a suitable choice of a complement to $T(M)|_{\partial M}$.

As a final remark, we now use the algebroid $\mathcal{V}_b(M)$ to show that some conditions on the groupoids $\mathcal{G}_S$ are necessary in Theorem 3. Otherwise the glued groupoid might not be a smooth manifold. Indeed, consider the same groupoid $\mathcal{G}_2 \simeq P_{\text{Int}(M)}$ for the big-open strata $\text{Int}(M)$, but a smaller one for the boundary:

$$\mathcal{G}_1 \simeq (\partial M \times \partial M) \times \mathbb{R}.$$ 

Then $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2$ is not a smooth manifold if $\pi_1(M)$ is non-trivial.

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