Dynamos in rotating compressible convection

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Abstract. Motivated by open questions in fundamental dynamo theory, the overall aim of this paper is to investigate some of the properties of dynamo action in rotating compressible convection. We study dynamo action in a convective layer of electrically-conducting, compressible fluid, rotating about the vertical axis. In order to identify the effects of rotation, we also carry out an equivalent set of calculations of convectively-driven dynamo action in a non-rotating layer. Whether or not the layer is rotating, the convection acts as a small-scale dynamo provided that the magnetic diffusivity is small enough. Defining the magnetic Reynolds number in terms of the horizontal scales of motion, we find that rotation reduces the critical value of this parameter above which dynamo action is observed. In the nonlinear regime, a rotating dynamo calculation and a separate non-rotating simulation are found to saturate at a similar level, even though the mid-layer value of the local magnetic Reynolds number is smaller in the rotating case. We compute the Lyapunov exponents of the flow to show that the stretching properties of the convection are modified by rotation. Furthermore, rotation significantly reduces the magnetic energy dissipation in the lower part of the layer.

1. Introduction

Hydromagnetic dynamo action is a process in which kinetic energy is converted into magnetic energy by the motions of an electrically-conducting fluid. A dynamo-generated magnetic field can only be sustained if the dissipative effects of magnetic diffusion are outweighed by the inductive effects of the fluid motions. Many natural dynamos are driven by thermal convection, especially solar, stellar and planetary dynamos. Most theoretical studies of convectively-driven dynamos have focused upon local models of convection, in which a layer of fluid is heated from below and cooled from above. Models of dynamo action in Boussinesq convection have been studied numerically by Meneguzzi & Pouquet (1989) and Cattaneo (1999). Dynamos in models of this type tend to produce intermittent magnetic fields on scales smaller than (or comparable to) the characteristic scale of the velocity field. Similar behaviour has been found in more recent models of dynamo action in fully compressible convection (Vögl & Schüessler, 2007; Brummell et al., 2010; Bushby et al., 2010).

Rotation is a feature of most natural dynamos. It is well known that the Coriolis force not only inhibits convection, but also tends to reduce the preferred horizontal scale of the convective instability (Chandrasekhar, 1961). So although rotation is not a necessary ingredient for dynamo action in a convective layer, we would expect the properties of convectively-driven dynamos to be influenced by the presence of rotation. There have been several numerical and theoretical studies of dynamo action in rotating Boussinesq convection (see, for example, Childress & Soward, 1972; St. Pierre, 1993; Jones & Roberts, 2000; Cattaneo & Hughes, 2006), and this particular problem...
Table 1. The set of parameters for the two different cases.

| Run | Ra   | Ta   | θ   | κ    | Re   |
|-----|------|------|-----|------|------|
| R1  | $3 \times 10^5$ | 0    | 3   | 0.0055 | 157  |
| R2  | $4.6 \times 10^5$ | $10^5$ | 3   | 0.0044 | 153  |

is now fairly well understood. The effects of rotation have been incorporated into calculations of dynamo action in compressible convection, however existing models generally include additional physical features such as imposed shears, inclined rotation vectors or utilise more complex model atmospheres (see, for example, Brandenburg et al., 1996; Käpylä et al., 2008). In this paper, we consider the simpler problem of convection in a single polytropic layer of compressible fluid. The aim of this study is to determine the influence that rotation has upon dynamo action in models of this type.

2. Model setup, parameters and numerical methods

We consider the evolution of a layer of compressible electrically-conducting fluid. Following the approach outlined in Matthews et al. (1995), the thermal conductivity $K$, the shear viscosity $\mu$, the magnetic diffusivity $\eta$, the magnetic permeability $\mu_0$ and the specific heat capacities $c_p$ and $c_v$, are all assumed to be constant properties of the fluid. This layer is bounded above and below by two impermeable stress-free surfaces, a distance $d$ apart. These bounding surfaces are held at fixed temperatures: $T_0$ at the upper surface and $T_0 + \Delta T$ at the lower boundary. We also assume that the upper and lower boundaries are perfect electrical conductors, which implies that the horizontal components of any magnetic fields that are present must vanish at these surfaces. This choice of boundary conditions enables us to compare our results with previous Boussinesq studies (see, for example, Cattaneo & Hughes, 2006). It is worth noting here that the dynamo efficiency of Boussinesq convection appears to be only weakly dependent upon the precise choice of magnetic boundary conditions, so it is probable that these boundary conditions do not have a major influence upon the behaviour of the present model. To define the geometry of the problem, we use a Cartesian grid in which the $z$-axis points vertically downwards (parallel to the constant gravitational acceleration $g = g\hat{z}$). Hence $z = 0$ corresponds to the upper boundary whilst $z = d$ corresponds to the lower boundary. The layer is rotating about the vertical axis with a constant angular velocity $\Omega = \Omega\hat{z}$. The $x$ and $y$ axes correspond to the two horizontal directions, with the fluid occupying the region $0 < x, y < \lambda d$. All variables are assumed to be periodic in both horizontal directions. The governing equations for this model are identical to those given in Matthews et al. (1995) apart from the addition of rotation.

A number of parameters must be fixed in order to complete the specification of the model. For the polytropic index, we choose a value of $m \equiv (gd/R_\ast \Delta T) - 1 = 1$ (where $R_\ast = c_p - c_v$ represents the gas constant), whilst the ratio of specific heats is given by $\gamma \equiv c_p/c_v = 5/3$. These two parameter choices ensure that the polytropic layer is convectively unstable. We also fix the thermal stratification to be $\theta \equiv \Delta T/T_0 = 3$, which implies that the temperature varies by a factor of four across the layer. The mid-layer Taylor number is defined to be $Ta = 4(1 + \theta/2)^{2m} \rho_0^2 \Omega^2 d^4/\mu^2$, where $\rho_0$ is the density at the upper surface in the absence of convection. The non-rotating calculations clearly correspond to $Ta = 0$, whilst we choose a value of $Ta = 10^5$ for the rotating cases. For numerical convenience, we choose a Prandtl number of $\sigma \equiv \mu c_p/K = 1$. The final key parameter for hydrodynamic convection is the dimensionless thermal diffusivity, $\kappa \equiv K/d\rho_0 c_p (R_\ast T_0)^{1/2}$. However, rather than specifying
Figure 1. (a) Temperature fluctuations in compressible rotating convection (white corresponds to hot fluid whereas black-red correspond to cold fluid). $Ra = 4.6 \times 10^5$ and $Ta = 10^5$. (b) A closer view of streamlines coloured with the local value of the enstrophy (blue for small values and red for large values).

a value for $\kappa$, it is often more convenient to specify the mid-layer Rayleigh number $Ra = (m + 1 - m\gamma)(1 + \theta/2)^{2(m-1)/(m+1)}\theta^2\gamma\sigma$, which is the parameter that is usually quoted in Boussinesq studies. Given that rotation tends to inhibit convection, it does not make sense to consider the same value of the Rayleigh number in both the rotating and the non-rotating cases. Accordingly, we choose a value of $Ra = 3 \times 10^5$ in the non-rotating case and $Ra = 4.6 \times 10^5$ in the rotating calculations. A summary of our choice of parameters for each case is given in Table 1.

When comparing results from rotating and non-rotating convection, it is clearly important to ensure that the calculations are in similar parameter regimes. The Reynolds number is a natural measure of the vigour of the convective motions. Here, we define the mid-layer Reynolds number to be $Re = \rho_{mid}U_{rms}d/\mu$, where $\rho_{mid}$ is the mean density at the mid-layer of the domain and $U_{rms}$ is the rms velocity. Our choice of Rayleigh numbers ensures that $Re \approx 155$ in both the rotating and the non-rotating calculations (in the absence of a magnetic field), which suggests that these calculations are comparable. Note, however, that the effects of rotation tend to reduce the horizontal scales of motion. This implies that a (depth-dependent) definition of the Reynolds number that was based upon the horizontal integral scale, $l_0(z)$, and the horizontally-averaged rms velocity, $u_{rms}(z)$, would yield different values for the rotating and non-rotating cases (differing by almost afact of two at the mid-layer). So, even though the calculations have very similar global Reynolds numbers, the rotating calculations are (in some sense) less turbulent than the non-rotating cases. This may be a significant contributor to any differences in the dynamo properties of these flows. Similar considerations apply to the magnetic Reynolds number, which is a crucial parameter in any dynamo calculation. Although it is more natural to define this parameter in terms of the depth of the layer and the rms velocity, a local definition (based upon the horizontal integral scale) probably gives a fairer indication of the differences between the rotating and the non-rotating cases.

The governing equations are solved using a modified version of the mixed pseudospectral/finite difference code originally described by Matthews et al. (1995). Due to periodicity in the horizontal direction, horizontal derivatives are computed in Fourier space using fast Fourier transforms. In the vertical direction, a fourth-order finite difference scheme is used. The time-stepping is performed by an explicit adaptive third-order Adams-Bashforth technique. The resolution is 256 grid-points in each horizontal directions and 120 grid-points in the vertical direction.
3. Results

With the given set of parameters, hydrodynamic convection is vigorous and time-dependent. For both the rotating and the non-rotating calculations we evolve the hydrodynamic equations until the convection has reached a statistically-steady state (as illustrated in Figure 1 for case R2). At this stage, a seed magnetic field (with zero net flux) is inserted into the flow. The critical parameter in determining the evolution of this magnetic field is the magnetic Reynolds number, which is inversely proportional to the magnetic diffusivity, $\eta$. We aim to find the critical value of the magnetic Reynolds number above which dynamo action is possible. Using a global definition for the magnetic Reynolds number, we find that the critical value is rather similar in the rotating and the non-rotating cases. Differences emerge, however, when we consider the mid-layer value of the local magnetic Reynolds number, $R_M = u_{\text{rms}}(0.5)l_0(0.5)/\eta$. Without rotation, the critical value for $R_M$ is approximately 400 whereas in the rotating case, $R_{M\text{crit}} \approx 200$. So rotation actually tends to reduce the critical value of $R_M$. A more complete view of the problem can be obtained by varying $R_M$ and then computing the evolution of the magnetic energy in the kinematic regime ("turning off" the Lorentz force). The exponential growth rates of the magnetic energy (normalised by the mid-layer turnover time $l_0(0.5)/u_{\text{rms}}(0.5)$) are shown in figure 2(a). With this definition of $R_M$ we find consistently lower kinematic growth rates in the non-rotating cases, where (at high $R_M$) the growth rate appears to have a logarithmic dependence upon $R_M$. It is more difficult to fit a scaling law in the rotating cases, but a $R_M^{1/2}$ scaling may be more appropriate here.

In fully nonlinear calculations, the Lorentz force eventually halts the exponential growth. Figure 2(b) shows the time-evolution of the magnetic energy in two nonlinear calculations (one rotating, one non-rotating) in the highest $R_M$ cases. Note that if we were using a global definition for the magnetic Reynolds number (based upon the layer depth), this parameter would be approximately 480 in both simulations. In each calculation, there is a short period of kinematic growth, during which it is apparent that the growth rate of the magnetic energy is slightly larger in the rotating case. The dynamos saturate once the magnetic field becomes dynamically significant. In each case, the mean magnetic energy during the nonlinear phase is highly time-
dependent, but is never more than a few percent of the mean kinetic energy. Note that the magnetic energy in the rotating case appears to saturate at a slightly higher level, despite the fact that the mid-layer value of the local magnetic Reynolds number is considerably larger in the non-rotating calculations.

During the nonlinear saturated phase, we also release fluid particles into the flow in order to compute the maximum Lyapunov exponents. One would like to identify regions with large Lyapunov exponents since they are expected to lead to the strongest amplification of the magnetic field. Trajectories of fluid particles are computed using the following equation: \( \frac{\partial x_p}{\partial t} = u(x_p, t) \), where \( x_p \) is the position of the particle. The velocity at the particle position is interpolated from the grid values using a sixth-order Lagrangian interpolation scheme. The boundaries are treated with a decentred scheme. The short-time Lyapunov exponent \( \lambda_e \) is then calculated using the following expression: \( \lambda_e = \frac{1}{t} \log \frac{d(t)}{d_0} \), where \( d(t) \) is the distance between two neighbouring particles at time \( t \). Figure 3(a) presents the maximum Lyapunov exponent (as a function of depth) during the nonlinear phase, normalised by the mean turnover time. The first point to note is that the convection is characterised by positive exponential stretching, as needed for dynamo action, everywhere in the layer. There is clearly more stretching in the lower part of the layer, where the flow is more turbulent (the local value of the Reynolds number is increasing with depth due to the increase of density). However, \( \lambda_e \) seems to be less depth-dependent in the rotating case. Hence rotation tends to reduce stretching in the lower part of the domain.

To fully understand the differences between dynamos in rotating and non-rotating compressible convection, we must analyse all the key contributors to the evolution of the magnetic energy. The evolution equation for the total magnetic energy is:

\[
\frac{\partial M}{\partial t} = \frac{1}{2} \langle B \cdot [(B \cdot \nabla) u - (u \cdot \nabla) B - (\nabla \cdot u) B] \rangle + \frac{1}{2} \zeta_0 \kappa \langle j^2 \rangle
\]  

where \( j = \nabla \times B \) is the current density and \( M = \langle B^2 \rangle / 2 \) is the total magnetic energy. The brackets \( \langle . \rangle \) mean a statistical average over time and all spatial coordinates. The first term on the right-hand side of the equation corresponds to the production of magnetic energy due to the stretching and advection by the turbulent flow. This term is directly related to the exponential stretching quantified by the Lyapunov exponents in the previous paragraph. Note that the direct effect of the divergence of the velocity field is found to be small compared to the two other contributions, so will not be discussed further here. The final term in the magnetic energy equation corresponds to ohmic dissipation. Figure 3(b) shows the ohmic dissipation averaged over the horizontal coordinates and time during the nonlinear phase. The magnitude of the dissipation term near the top of the layer is roughly the same regardless of whether or not the layer is rotating. However, whilst the ohmic dissipation increases rapidly with depth in the non-rotating case, it increases more gradually with depth in the rotating layer. In other words rotation tends to reduce the effects of dissipation in the lower part of the domain. It is possible to observe a similar trend in the kinematic phase however the ohmic dissipation during this period is problematic to quantify, since it is difficult to define an appropriate time-average whilst the magnetic energy is growing exponentially. So it seems that any reduction in stretching in the lower part of the layer, due to the effects of rotation, is compensated for by the fact that the magnetic field organises itself in such a way that the rate of dissipation in these cases is lower than it is in the equivalent non-rotating calculations.

4. Conclusions and discussion

Small-scale dynamo action is possible in both rotating and non-rotating convection. Adopting a definition for the magnetic Reynolds number that is based upon the horizontal scales of motion,
we find that rotation reduces the critical value of this parameter for the onset of dynamo action. Furthermore, at a given value of the local mid-layer magnetic Reynolds number, higher kinematic dynamo growth rates are observed in the rotating calculations. Similar results were obtained in the nonlinear regime, where comparable levels of saturation were found despite the fact that the local mid-layer magnetic Reynolds number was much larger in the non-rotating case. These results may partly be explained by the fact that (although the global Reynolds number is roughly constant across all simulations) rotation actually reduces the local value of the mid-layer Reynolds number. Although it would, in theory, be possible to modify the parameters so as to ensure that the local mid-layer Reynolds numbers were more comparable, the global Reynolds numbers would then be different. Hence, it is not clear that the resulting set of rotating and non-rotating calculations would be any more (or, for that matter, any less) comparable than the present set of simulations. Leaving aside this particular issue, it is clear that rotation does have a profound influence upon the convection. The Lyapunov exponents of the flow suggest that the level of stretching increases much more gradually with depth in rotating convection than it does in the corresponding non-rotating case. However, from the point of view of the dynamo, any reduction in stretching in the lower part of the domain in the rotating case is compensated by a similar reduction in magnetic dissipation.

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