A method to detect cracks in the beams with imperfect boundary conditions

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Abstract. Non-destructive testing of structures involving vibration-based damage detection methods implies knowing the beam’s boundary conditions. For perfect boundary conditions, numerous damage detection methods are developed and ensure more or less accurate estimation of the crack type, position, and severity. On the contrary, for imperfect boundary conditions, which are the real ones, there are only a few dedicated works. This paper presents a methodology that allows the identification of the weak clamping and the crack if both exist. The weak clamping is modelled as a defect that produces an identical relative frequency shift (RFS) for all vibration modes. Therefore, to find the two defects namely the real crack and defect simulating the weakly clamped end, we apply the principle of superposition. The method is implemented as an application written in the Python programming language. Tests show that defects are successfully identified even if there are uncertainties about the fixing of the beam.

1. Introduction

When analyzing beams with fixed ends from the static or dynamic point of view, restriction of displacements and slopes are assumed, resulting in perfect boundary conditions. Since the boundary conditions influence the modal parameters of structures, vibration-based damage detection methods can fail due to the imperfection of the supports that usually are found for real systems. It is difficult to determine the boundary conditions of a real structure with precision [1]. Most researchers use the natural frequencies to characterize the imperfect boundary conditions [2-4], while others involve the mode shapes to detect the deviation from the ideal case [5], [6].

The fixed end condition restricts the displacement and the slope of the beam. Due to the particularity of the structure, aging, or damage, a deviation from the perfect clamping may occur. A popular model for the weak clamping is the linear combination of perfect clamping and perfect hinge to which a weight factor is applied [7]. To ensure this behavior, a torsional spring is added at the fixed end in the beam model to allow a small slope at the clamped end [8]. The weight factor is employed to control the deviation from the ideal condition [9]. A more complex model of the weak clamping consists of a system of linear and torsional springs [1], [10]. In this case, beside slope, also a small longitudinal or transverse displacement is made possible.

The disadvantage of having beams with weakly clamped ends when performing damage detection by involving vibration-based methods is the increased complexity, achieved because the natural frequencies depend not just on the structures’ properties, shapes, and the defects of their elements, but
also on the boundary conditions [11]. Consequently, we have to distinguish between the effect of the crack and the effect of the deviation from the perfect clamping.

In this paper we model the weak clamping as a crack and involve the principle of superposition to describe the combined effect of the real crack and that added to simulate the weak clamping. Then we highlight the particular way how damage at the fixed end contributes to the frequency decrease in the transverse vibration modes. This helps separating the effect of the real crack from the effect of the defect simulating the weak clamping. Eventually, we perform simulation for different positions and severities of the real crack, and deviations from the perfect clamping. We successfully identified the damage and estimated the boundary imperfection.

2. Theoretical background

This section aims to present the theoretical background on which the proposed damage detection method is based. First, we introduce relevant mathematical relations, some classical and other deduced in our previous research.

The frequency of the intact Euler-Bernoulli beam is:

$$f_u = \frac{\lambda_i^2}{2\pi} \sqrt{\frac{EI}{\rho AL}}$$

In (1), we use following notations: $E$ for Young’s modulus, $A$ for the cross-sectional area, $I$ for the second moment of inertia, $\rho$ for the density, $L$ for the beam length, and $\lambda_i$ for the eigenvalue for the $i$-th transverse vibration mode.

The relationship for the natural frequencies of a beam that has a crack with depth $a$ at location $x$ was deduced in [12]. It is

$$f_{id}(a,x) = f_u \left\{ 1 - \gamma(a) \left[ \phi_i^*(x) \right]^2 \right\}$$

where $f_u$ is the frequency of the undamaged beam, $f_{id}(a,x)$ is the frequency of the damaged beam, $\gamma(a)$ is the severity of the crack with depth $a$, and $\phi_i^*(x)$ is the normalized curvature. The relation was validated by two independent research groups, see papers [13], [14]. It worth mentioning that (2) is valid for any perfect boundary conditions and any transverse vibration mode if the right mathematical relation of modal curvature is used.

We can observe that the frequency decreases depending on the product between two terms, the severity and the curvature, which product we define as the pseudo-severity of the crack. It is:

$$\gamma_{id}(a,x) = \gamma(a) \left[ \phi_i^*(x) \right]^2$$

The pseudo-severity reflects the effect of the crack in any position other than that at which the greatest curvature is obtained, i.e. the clamped ends for the clamped-clamped beam.

The severity in (2) has the expression [15]:

$$\gamma(a) = \frac{\sqrt{\delta_u(a)} - \sqrt{\delta_d}}{\sqrt{\delta_u(a)}}$$

where we denoted $\delta_u$ the deflection of the undamaged beam and $\delta_d(a)$ the deflection of the damaged beam, both calculated for the dead mass. The severity is hence deduced using the energy method, and is always found for the position of the crack coinciding with the location on the beam where the biggest curvature is achieved.
The deflection of the undamaged beam is similarly calculated for any boundary conditions, using the relation:

$$\delta_U = \frac{\rho AL^3}{\kappa EI}$$  \hspace{1cm} (5)

where we denoted with $\kappa$ the constant depending on the boundary conditions, which is $\kappa = 384$ for the clamped-clamped beam. One can observe that $\kappa$ is actually simplified in (3), since it is contained in the nominator and the denominator, as well. Hence, the severity of a crack does not depend on the boundary condition [16].

The frequency shift due to a crack is deduced from (2), as:

$$\Delta f_{id}(a,x) = f_{id} - f_{id}(a,x) = \gamma(a)\left[\frac{\sigma^e_x}{\phi^e_x}(x)\right]^2 f_{id}$$  \hspace{1cm} (6)

Hence, the relative frequency shift, which is obtained by normalization (i.e. dividing the frequency drop to the frequency of the undamaged beam for the respective mode, is:

$$\text{RFS}_i(a,x) = \frac{\Delta f_{id}(a,x)}{f_{id}} = \gamma(a)\left[\frac{\sigma^e_x}{\phi^e_x}(x)\right]^2$$  \hspace{1cm} (7)

At the clamped end, a crack with depth $a_0$ produces the relative frequency shift

$$\text{RFS}_i(a_0,0) = \gamma(a_0)$$  \hspace{1cm} (8)

because $\frac{\sigma^e_x}{\phi^e_x}(0) = 1$. The similar situation is at the other clamped end, i.e. for the position $x = L$. Here,

$$\text{RFS}_i(a_L,L) = \gamma(a_L)$$  \hspace{1cm} (9)

Since the RFSs at the clamped ends are independent of the position,

$$\text{RFS}_i(a_0,0) = ... = \text{RFS}_i(a_1,0) = ... = \text{RFS}_n(a_0,0) = \gamma(a_0)$$  \hspace{1cm} (10)

and

$$\text{RFS}_i(a_L,L) = ... = \text{RFS}_i(a_1,L) = ... = \text{RFS}_n(a_L,L) = \gamma(a_L)$$  \hspace{1cm} (11)

In Figure 1 we exemplify the sequences of seven RFSs calculated for the crack located at the relative distance $x/L=0.2$, and in Figure 2 the sequence of seven RFSs derived for the crack located at one of the clamped ends. Both transverse cracks have the depth of 1 mm.

**Figure 1.** RFSs for the crack located at distance $x/L=0.2$ and having 1 mm depth

**Figure 2.** RFSs for the crack located at the clamped end and having 1 mm depth
We have proved in our research [17] that the superposition is valid if two damages affecting a beam are far enough from each other. Applying the superposition principle, which consists here in the addition of the frequency shifts due to the three cracks (one real and two simulating the boundary imperfection), the total relative frequency shifts of the cracked beam result as

$$RFS_i^c(a_0,0,a,x,a_L,L) = \gamma(a_0) + \gamma(a) \left[ \phi^*_i(x) \right]^2 + \gamma(a_L) = \gamma(\hat{a}) + \gamma(a) \left[ \phi^*_i(x) \right]^2$$

(12)

Hence, the mathematical model for the cracked beam with non-ideal clamping at one or both ends get the mathematical expression

$$f_{D_i}^m(a_0,0,a,x,a_L,L) = f_{\omega} \left\{ 1 - \gamma(\hat{a}) - \gamma(a) \left[ \phi^*_i(x) \right]^2 \right\}$$

(13)

If the natural frequencies for the cracked beam with uncertain boundary conditions $f_{D_i}^m$ are obtained from precise measurements, these have the same values as these achieved by calculus. This means, the difference between the estimated frequency drop for the cracked beam with perfect clamping and the frequency drop obtained from measurements will be

$$\Delta f_{D_i}^m - \Delta f_{D_i}^c(a,x) = \left( f_{\omega} - f_{D_i}^m \right) - \left( f_{\omega} - f_{D_i}^c(a,x) \right) \equiv -f_{D_i}^m(a_0,0,a,x,a_L,L) + f_{D_i}^c(a,x)$$

(14)

or, after substituting (2) and (13) in (14), we obtain

$$\Delta f_{D_i}^m - \Delta f_{D_i}^c(a,x) = f_{\omega} \left\{ \gamma(\hat{a}) + \gamma(a) \left[ \phi^*_i(x) \right]^2 \right\} - f_{\omega} \left\{ \gamma(a) \left[ \phi^*_i(x) \right]^2 \right\} = \gamma(\hat{a}) f_{\omega}$$

(15)

After normalization, which means dividing the difference to the frequency of the intact beam for the respective mode, we obtain

$$DIF_i(\hat{a},a,x) = RFS_i^m - RFS_i^c(a,x) = \gamma(\hat{a})$$

(16)

We observe that the difference is the same for all vibration modes, and can take advantage of this feature to find the severity and position of the crack. As (16) suggests, the crack position $x$ and depth $a$ are found if

$$DIF_i(\hat{a},a,x) = ... = DIF_i(\hat{a},a,x) = ... = DIF_i(\hat{a},a,x) \text{ for } i = 1...n$$

(17)

We can also conclude whether the clamping is perfect or not, by analyzing the difference of the relative frequency shifts: the clamping is perfect if $DIF_i = 0$, while for $DIF_i \neq 0$ we know the clamping is weak. Figure 3 shows weak clamping, reflected by the ordinate of the point in which the RFS differences intersect. However, it is not possible to distinguish which end has a weak(er) clamping, this should be found by analyzing the mode shape.

All curves in Figure 3 intersect in a unique point. Due to expected measurement errors, (17) is not always perfectly satisfied. For real measurements, the intersections will be distributed in a narrow region [18]. To get a clear estimation result, we calculate the anti-distance between all $DIF_i(\hat{a},a,x)$ for a presumed crack position and distance. To this aim, we involve the mathematical relation:

$$ADIF(a,x) = \left[ \sum_{i=0}^{n-1} \left( DIF_i(\hat{a},a,x) - DIF_{i+1}(\hat{a},a,x) \right)^2 \right]^{-0.5}$$

(18)

where $i$ is the vibration mode number. The crack depth $a$ and crack position $x$ for which the biggest $ADIF$ is found when varying $a \in [0,H]$ and $x \in [0,L]$ are the crack parameters. The damage indicator $DI$, therefore, expressed as:

$$DI = \max ADIF(a,x)$$

(19)
In Figure 4 we plot the $ADIF$ for the estimated crack depth $a$. $ADIF$ clearly indicates the crack position $x/L$ by the abscissa of the peak ($\text{max} ADIF$), even if the $DIF$ curves intersect in a narrow region and not a point. The higher the value of $\text{max} ADIF$, the closer the intersections of the $DIF$ curves are. Observe that both representations allow finding the crack position, the first one also suggesting the deviation from the perfect clamping condition by the ordinate of the intersections.

3. The proposed damage detection method

To make the damage detection process free of the influence of the operator, we developed an algorithm and implemented it in an application, namely PyLOC, written in the Python programming language. In short, the algorithm consists in selecting a very small damage depth $a_{\text{min}}$, for which we calculate $DIF$, and $ADIF$ for numerous equidistantly located positions along the beam. Afterward, we increase the damage depth, iteratively, with a step $s$, and calculate again $DIF$, and $ADIF$. The process is stopped when the imposed limit $a_{\text{max}}$ for the depth is achieved. The real crack parameters $a$ and $x$ correspond with the input data for which the biggest $ADIF$ is found. For these parameters, the $DIF$ curves should intersect in point or a narrow region. The method applies in all conditions: no weak clamping, weak clamping at one end, and weak clamping at both ends, respectively.

The code source of the application is presented in [19]. The steps performed when running the algorithm/PyLOC application are described in Table 1. One can observe that the operation has the single role to introduce the input data, and then the process continues in automatic mode. In this table we indicate, in addition to the steps performed, the mathematical relations implemented in the algorithm, and how the outputs are displayed.
Table 1. Steps performed when running the PyLOC application

| Action                                      | Responsible         | Obs. | Loop |
|---------------------------------------------|---------------------|------|------|
| Select boundary condition                   | Operator            |      |      |
| Select MS Excel file                        | Operator            |      |      |
| Select beam thickness $H$                   | Operator            |      |      |
| Select operation mode (manual/automatic)    | Operator            |      |      |
| Select $a_{\min}$ and $a_{\min}$           | Predefined setting  |      |      |
| Read from Excel file $n, f_{iU}, f_{iD}$   | Computer            |      |      |
| Calculate $RFS^m$ from measured data for $n$ modes | Computer | Eq.(7) |      |
| Calculate curvatures $\phi^*_i(x)$ for 100 locations $x$ and $n$ modes | Computer | [12] |      |
| Start loop                                  | Computer            |      |      |
| Calculate severity $\gamma(a)$ for $a = a_{\min}$ | Computer | [15] |      |
| Calculate $RFS^r(a, x)$                     | Computer            |      |      |
| Calculate differences $DIF_i(a, x)$         | Computer            | Eq.(2) |      |
| Calculate anti-distance $ADIF(a, x)$        | Computer            | Eq.(16) |      |
| Calculate $a_{\min} = a_{\min} + s$        | Computer            |      |      |
| Perform loop until $a_{\min} \leq a_{\max}$ | Computer            |      |      |
| Display DIF diagram                         | Computer            | (fig.5) |      |
| Display the $ADIF$ diagram                  | Computer            | (fig.6) |      |
| Display the report summary                  | Computer            | (fig.7) |      |

As shown, the first step implies introduction of data. The toolbar used for this aim is presented in Figure 5. The operator can select the boundary conditions that are: Clamped-Free or Clamped-Clamped. Automatically, the proper mathematical relation for the modal curvature is selected by the algorithm. In the meantime, a folder in which the natural frequencies for the intact and damaged beam are stored is opened. The operator can choose the file for the analysis.

Figure 5. The toolbar of the PyLOC application

In next step, the Thickness of the beam is introduced manually in the dedicated text field. The process can run in two ways:

- Manual, requesting to estimate and further select a Damage Depth - the loop is not activated, therefore the outcome represent a single diagram containing $DIF$ curves. The process is repeated manually until the operator guesses the real damage depth.
- Automatic, when all steps of the algorithm are followed – no supplementary setting is requested. It is possible to display the $DIF$ curves at iteration when Dynamic view is selected, or just the diagram for the best fit is displayed.
Figure 6. The section of PyLOC application presenting the estimated damage location

The results can be saved in an MSWord file if the operator clicks on the Word button. Diagrams such as these presented in Figure 6 are displayed. A pop-up window (see Figure 7) containing information about the input data and the achieved results is displayed. The information regards the boundary conditions, the natural frequencies in the undamaged and damaged state, the beam thickness, the depth and the location of the damage.

Figure 7. The pop-up window with the summary report

A Help menu is also integrated; clicking on this button explanation regarding the use of PyLOC is displayed. The toolbar contains an Exit button used to leave the application.
4. Simulation and results
In order to validate the proposed damage detection method, we have generated several scenarios reproducing the weak boundary conditions for a double clamped beam and performed modal analysis with the dedicated module from SolidWorks. The damage assessment was then made with PyLOC.

The analyzed structure is a prismatic beam made of Plain Carbon Steel with defined density \( \rho = 7800 \text{ kg/mm}^3 \), and Young modulus \( E = 2 \cdot 10^{11} \text{ N/m}^2 \), which is fixed at both ends.

The test beam has following dimensions: length \( L = 1 \text{ m} \), width \( B = 20 \text{ mm} \), and thickness \( H = 5 \text{ mm} \). Cracks for simulating the imperfect boundary conditions as well as the real defect are generated, all having the width \( w = 0.75 \) (see Figure 8). The depth of the real defect is denoted \( a \) and the depths of the cracks that replicate the weak clamping are denoted \( a_0 \) for the left end and \( a_L \) for the right end of the beam, respectively.

![Figure 8. The crack geometry](image)

The damage scenarios are presented in Table 2. We analyse all possible/typical cases, which are denoted:
- small deviation from perfect clamping – small crack – zero deviation from perfect clamping (SD-SC-ZD);
- small deviation from perfect clamping – big crack – zero deviation from perfect clamping (SD-BC-ZD);
- big deviation from perfect clamping – small crack – zero deviation from perfect clamping (BD-SC-ZD);
- big deviation from perfect clamping – big crack – zero deviation from perfect clamping (BD-BC-ZD);
- big deviation from perfect clamping – big crack – big deviation from perfect clamping (BD-BC-BD).

| Scenario no. | Damage location \( x \) | Damage depth | Scenario abbreviation |
|--------------|--------------------------|--------------|-----------------------|
| S0           | 0                        | 0 0          | Intact beam           |
| S1           | 267                      | 0.5 0.3      | SD-SC-ZD             |
| S2           | 267                      | 1.25 0.3     | SD-BC-ZD             |
| S3           | 163                      | 0.5 1.125    | BD-SC-ZD             |
| S4           | 163                      | 2.5 1.125    | BD-BC-ZD             |
| S5           | 237                      | 1.25 0.9     | BD-BC-BD             |
The natural frequencies resulted for the undamaged beam with perfect boundaries are presented in Table 3, and these for the beams with a damage and weak clamping at the ends are given in Table 4. One can observe that, for all scenarios including damage, the frequencies decrease.

**Table 3.** Natural frequencies for the undamaged beam

| Scenario no. | $f_{1U}$ [Hz] | $f_{2U}$ [Hz] | $f_{3U}$ [Hz] | $f_{4U}$ [Hz] | $f_{5U}$ [Hz] | $f_{6U}$ [Hz] | $f_{7U}$ [Hz] |
|-------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|
| S0          | 26.716        | 73.628        | 144.30        | 238.46        | 356.08        | 497.11        | 661.50        |

**Table 4.** Natural frequencies for the damaged beams with imperfect clamping

| Scenario no. | $f_{1U}$ [Hz] | $f_{2U}$ [Hz] | $f_{3U}$ [Hz] | $f_{4U}$ [Hz] | $f_{5U}$ [Hz] | $f_{6U}$ [Hz] | $f_{7U}$ [Hz] |
|-------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|
| S1          | 26.704        | 73.561        | 144.18        | 238.35        | 355.83        | 496.57        | 661.01        |
| S2          | 26.702        | 73.398        | 143.92        | 238.34        | 355.39        | 495.15        | 660.11        |
| S3          | 26.617        | 73.362        | 143.74        | 237.49        | 354.67        | 495.30        | 659.22        |
| S4          | 26.560        | 73.259        | 142.38        | 234.03        | 350.65        | 493.26        | 659.19        |
| S5          | 26.588        | 73.086        | 143.31        | 237.34        | 353.91        | 493.09        | 657.41        |

The results obtained for the five damaged scenarios by running the PyLOC application are displayed in Figures 9 to 13. Analyzing the results, we can observe that the position is accurately estimated for all scenarios, but two symmetrical damages are identified due to the structural symmetry. Taking into consideration only the real damage position, the biggest error found when estimating the crack position is 5 mm, which actually means 0.5% of the beam length. This error is achieved when a small crack is present. For all other cases, the error is 2 mm or less.

**Figure 9.** The estimated crack position for scenario S1
Figure 10. The estimated crack position for scenario S2

Figure 11. The estimated crack position for scenario S3
In Table 5 we present the estimated damage location for all scenarios compared with the position generated for the crack. The second analysis concerns the precision in determining the crack depth. The results are presented in Table 6.
Table 5. Estimated damage locations for all scenarios studied

| Scenario no. | Generated damage location x | Estimated damage location x | Error [%] | Scenario abbreviation |
|--------------|-----------------------------|-----------------------------|-----------|-----------------------|
| S1           | 267                         | 267                         | 0         | SD-SC-ZD              |
| S2           | 267                         | 268                         | 0.1       | SD-SC-ZD              |
| S3           | 163                         | 168                         | 0.5       | BD-BC-ZD              |
| S4           | 163                         | 166                         | 0.3       | BD-BC-ZD              |
| S5           | 267                         | 267                         | 0         | BD-BC-BD              |

Table 6. Estimated damage depths for all scenarios studied

| Scenario no. | a [mm] | Error [%] | Deviation from perfect boundary |
|--------------|--------|-----------|---------------------------------|
| generated    | estimated |         | Estimation | Criterion               |
| S1           | 0.5    | 0.6027    | 20.54     | Small | Severity < 0.1 |
| S2           | 1.25   | 1.39      | 11.2      | Small | Severity ≈ 0.1 |
| S3           | 0.5    | 0.6027    | 20.54     | Big   | Severity ≈ 0.7 |
| S4           | 2.5    | 2.6003    | 4.012     | Big   | Severity ≈ 0.7 |
| S5           | 1.25   | 1.39      | 11.2      | Big   | Severity ≈ 0.7 |

The error resulted when the damage depth is targeted are up to 20%, which apparently is much, but it is remarkable that a crack affecting the cross-section reduction with 10 to 50% can be localized and evaluated, even if the boundary conditions are non-ideal. The accuracy in estimating the crack depth can be increased if a more precise relation is used to make the correlation between the crack depth and the severity. Also, using a finer step s increase the estimation, but also increase the computational cost.

5. Conclusion

We propose a vibration-based damage detection method that involves the modal curvatures of the undamaged beam and the natural frequencies measured for the undamaged and damaged beam, respectively. The method allows an accurate estimation of the crack position (the biggest error found is 0.5%) and a quite good depth estimation (error up to 20%), even if the beam has not ideal boundary conditions. The method is based on the finding of the authors that the RFSs due to not perfectly clamped ends are the same for all transverse bending modes. The method consists in increasing the depth of the damage and calculating the RFSs, which are then compared with the RFSS achieved from measured data. The difference between the calculated and measured RFSs is represented graphically for several modes; the intersection of these curves in a narrow region indicating the damage depth is found. The abscissa of the point of intersection represents the crack position on the beam, while the ordinate shows if weak clamping is present. The bigger the ordinate, the weaker is the beam clamped.

Next approaches will concentrate on generalizing the method, and make it applicable for any boundary condition.

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