Quantum Isometry group of dual of finitely generated discrete groups and quantum groups
Debashish Goswami and Arnab Mandal
Indian Statistical Institute
203, B. T. Road, Kolkata 700108
Email: goswamid@isical.ac.in

Abstract
We study quantum isometry groups, denoted by $Q(\Gamma, S)$, of spectral triples on $C^*_r(\Gamma)$ for a finitely generated discrete group coming from the word-length metric with respect to a symmetric generating set $S$. We first prove a few general results about $Q(\Gamma, S)$ including the polynomial growth property of the dual of $Q(\Gamma, S)$ for a group $\Gamma$ with this property under the assumption of full spectrum of the action of $Q(\Gamma, S)$ on $C^*_r(\Gamma)$, and the isomorphism $Q(\Gamma, S) \cong QISO(\hat{\Gamma}, d)$ for abelian $\Gamma$ for a suitable metric $d$ on the dual compact abelian group. We then carry out explicit computations of $Q(\Gamma, S)$ for several classes of examples including free and direct product of cyclic groups, Baumslag-Solitar group, Coxeter groups etc. In particular, we have computed quantum isometry groups of all finitely generated abelian groups which do not have factors of the form $\mathbb{Z}_2^k$ or $\mathbb{Z}_4^l$ for some $k, l$ in the direct product decomposition into cyclic subgroups. We also extend the formulation of $Q(\Gamma, S)$ to discrete quantum groups which are duals of compact matrix quantum groups, with respect to a given finite dimensional fundamental representation, and in this context, study the effect of cocycle-deformation or more general twists of quantum groups. As a concrete case-study, the quantum isometry group of the dual of $SU_q(2)$ with respect to the usual 2-dimensional fundamental representation has been identified with a $q$-deformation of $SO(4, \mathbb{R})$.

1 Introduction
It is a very interesting problem, both from the physical and mathematical viewpoint, to understand and classify quantum symmetries of possibly noncommutative $C^*$ algebras (usually with further structures), i.e. possible actions of quantum groups on them. In [26], this problem was considered in an algebraic and categorical setting, leading to the realization of some of the well known (algebraic) quantum groups such as $SL(2, q)$ as universal objects in some category of quantum groups acting on matrix algebras. S.Wang [31] took up a similar problem in the analytical framework of compact quantum groups acting on $C^*$ algebras. Later on, a number of mathematicians including Wang, Banica, Bichon and others ([31], [1], [15]) developed a beautiful theory of quantum automorphism groups of finite dimensional $C^*$ algebras as well as quantum isometry groups of finite metric spaces and finite graphs. In [23] the first named author of the present article extended such constructions to the set up of possibly infinite dimensional $C^*$ algebras, and more interestingly, that of spectral

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triples a la Connes [18], by defining and studying quantum isometry groups of spectral triples. This led to the study of such quantum isometry groups by many authors including Goswami, Bhowmick, Skalski, Banica, Bichon, Soltan, Das, Joardar and others. In the present article, our focus is on a rather special yet interesting and important class of spectral triples, namely those coming from the word-length metric of finitely generated discrete groups with respect to some given symmetric generating set. There have been several articles already on computations and study of the quantum isometry groups of such spectral triples, e.g. [14], [30], [24], [7], [5] and references therein. However, we plan to initiate a systematic and unified study of quantum isometry groups of such spectral triples.

We begin by proving some general facts about the quantum isometry group $Q(\Gamma, S)$ of a discrete group $\Gamma$ with a finite symmetric generating set $S$. We prove, among other things, the following two interesting results:

(i) If $\Gamma$ has polynomial growth and the action of $Q(\Gamma, S)$ on $C^*_r(\Gamma)$ has full spectrum, then the dual (discrete quantum group) of $Q(\Gamma)$ has polynomial growth.

(ii) In case $\Gamma$ is abelian, there is a metric on the dual compact abelian group ˆ$\Gamma$ such that the corresponding quantum isometry group in the metric space sense (as in [24]) exists and coincides with $Q(\Gamma, S)$.

Next we carry out several explicit computations. We have given special emphasis on groups of the form $\Gamma_1 \ast \Gamma_2 \ast \cdots \ast \Gamma_k$ or $\Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_k$, where $\Gamma_i = \mathbb{Z}_{n_i}$ for some $n_i$. We have proved that in few cases the quantum isometry group of these groups turn out to be the free or tensor product of the quantum isometry groups of the factors $\Gamma_i$’s. Here is a brief list of groups at which we have computed the quantum isometry groups in this paper:

1. All finitely generated abelian groups which do not have factors of the form $\mathbb{Z}_2$ or $\mathbb{Z}_4$ in the direct product decomposition into cyclic subgroups.
2. Free product of all cyclic groups which do not have factors of the form $\mathbb{Z}_2$ or $\mathbb{Z}_4$.
3. Some special cases of direct or free products of cyclic groups having $\mathbb{Z}_2$ or $\mathbb{Z}_4$ on factors.
4. Coxeter group.
5. Baumslag-Solitar group.
6. Dihedral, Tetrahedral, Icosahedral and Octahedral groups.

It is worth mentioning that in cases (5) and (6) the quantum isometry group turns out to be the doubling of the reduced group C$^*$ algebra. In a forthcoming paper by the second author some more groups including Braid groups, $\mathbb{Z}_4 \ast \mathbb{Z}_4 \ast \cdots \ast \mathbb{Z}_4$ etc. will be discussed.

Moreover, in the last section we have made a brief excursion to the more general situation of defining and studying quantum isometry groups of spectral triples defined by a quantum analogue of the length function on compact quantum groups with a finite dimensional fundamental representation as in [10]. As a special case, we show that the quantum isometry group of a compact connected Lie group $G$ w.r.t. a given finite dimensional fundamental representation turns
out to be a group, namely $C(K)$ where $K$ is the group of isometries of $(G, d)$ for some suitable metric $d$ on $G$ coming from the fundamental representation. We also study the effect of cocycle-twists and prove that the quantum isometry group of cocycle-twist $Q_\sigma$ of a quantum group $Q$ by a dual unitary 2 cocycle $\sigma$ on it is isomorphic with a cocycle-twist of the quantum isometry group of $Q$. We postpone the discussion on similar examples coming from more general Drinfeld twists for a separate article, but describe in details an interesting concrete example (without proof) by identifying the quantum isometry group for $SU_q(2)$ with the usual 2-dimensional fundamental representation with $SO_q(4, \mathbb{R})$.

2 Quantum isometry group of $C_{\text{r}}^*(\Gamma)$: existence and some generalities

We begin with a few basic definitions and facts about quantum isometry group of spectral triples defined by Bhowmick and Goswami in [12]. We denote the algebraic tensor product, spatial (minimal) $C^*$ tensor product and maximal $C^*$ tensor product by $\otimes$, $\hat{\otimes}$ and $\otimes^{\text{max}}$ respectively.

2.1 Basic definitions

Definition 2.1 A Compact quantum group (CQG for short) is a pair $(Q, \Delta)$ where $Q$ is a unital $C^*$ algebra where $\Delta : Q \to \hat{Q} \hat{\otimes} Q$ is a unital $C^*$ homomorphism satisfying two conditions. 1. $(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta$ (co-associativity). 2. Each of the linear spans of $\Delta(Q)(1 \otimes Q)$ and that of $\Delta(Q)(Q \otimes 1)$ is norm dense in $\hat{Q} \hat{\otimes} Q$.

Definition 2.2 We say that CQG $(Q, \Delta)$ acts on a unital $C^*$ algebra $B$ if there is a unital $C^*$ homomorphism (called action) $\alpha : B \to \hat{B} \otimes Q$ satisfying the following. 1. $(\alpha \otimes \text{id})\alpha = (\text{id} \otimes \Delta)\alpha$. 2. Linear span of $\alpha(B)(1 \otimes Q)$ is norm dense in $\hat{B} \hat{\otimes} Q$.

Definition 2.3 (Unitary representation): Let $(Q, \Delta)$ be a CQG. A unitary representation of $Q$ on a Hilbert space $H$ is a $C^*$ linear map from $H$ to the Hilbert module $\mathcal{H} \hat{\otimes} Q$ such that 1. $\langle \langle U(\xi), U(\eta) \rangle \rangle = \langle \xi, \eta \rangle_1 Q$ where $\xi, \eta \in \mathcal{H}$. 2. $(U \otimes \text{id})U = (\text{id} \otimes \Delta)U$.

Here $\langle \langle \cdot, \cdot \rangle \rangle$ is the $C^*$ valued inner product and $\mathcal{H} \hat{\otimes} Q$ denotes the completion of $\mathcal{H} \otimes Q$ with respect to the natural $Q$ valued inner product. Given such a unitary representation we have a unitary element $U$ belonging to $\mathcal{M}(K(\mathcal{H}) \hat{\otimes} Q)$ given by $U(\xi \otimes b) = U(\xi)b, (\xi \in \mathcal{H}, b \in Q)$ satisfying $(\text{id} \otimes \Delta)(U) = U^{12}U^{13}$. Here we used the leg-numbering notation, consider the multiplier algebra $\mathcal{M}(K(\mathcal{H}) \hat{\otimes} Q)$, it has two natural embeddings into $\mathcal{M}(K(\mathcal{H}) \hat{\otimes} Q \hat{\otimes} Q)$. The first one is obtained by extending the map $x \mapsto x \otimes 1$. The second one is obtained by composing this map with the flip on the last two factors. We will write $\omega^{12}$ and $\omega^{13}$ for the images of an element $\omega \in \mathcal{M}(K(\mathcal{H}) \hat{\otimes} Q)$ by these two maps respectively.
Given two CQG’s $Q_1, Q_2$ the free product $Q_1 \ast Q_2$ as well as the maximal tensor product $Q_1 \otimes^\text{max} Q_2$ admit the natural CQG structures, as given in \cite{32, 34}. Moreover, they have the following universal properties (see \cite{32, 34}).

**Proposition 2.4** (i) The canonical injections, say $i_1, i_2 \ (j_1, j_2 \text{ respectively})$ from $Q_1$ and $Q_2$ to $Q_1 \ast Q_2$ and $Q_1 \otimes^\text{max} Q_2$ respectively are CQG morphisms.

(ii) Given any CQG $C$ and morphisms $\pi_1 : Q_1 \to C$ and $\pi_2 : Q_2 \to C$ there always exists a unique morphism denoted by $\pi := \pi_1 \ast \pi_2$ from $Q_1 \ast Q_2$ to $C$ satisfying $\pi \circ i_k = \pi_k$ for $k = 1, 2$.

(iii) Furthermore, if the ranges of $\pi_1$ and $\pi_2$ commute, i.e. $\pi_1(\{a\})\pi_2(\{b\}) = \pi_2(\{b\})\pi_1(\{a\}) \forall a \in Q_1, b \in Q_2$, we have a unique morphism from $\pi' := Q_1 \otimes^\text{max} Q_2$ to $C$ satisfying $\pi' \circ j_k = \pi_k$ for $k = 1, 2$.

(iv) The above conclusions hold for free or maximal tensor product of any finite number of CQG’s as well.

**Definition 2.5** Let $(A^\infty, \mathcal{H}, D)$ be a spectral triple of compact type (a la Connes). Consider the category $Q(D) = Q(A^\infty, \mathcal{H}, D)$ whose objects are triples $(Q, \Delta, U)$ where $(Q, \Delta)$ is a CQG having a unitary representation $U$ on the Hilbert space $\mathcal{H}$ and $U$ commutes with $(D \otimes 1_Q)$ . Morphism between two such objects $(Q, \Delta, U)$ and $(Q', \Delta', U')$ is a CQG morphism $\psi : Q \to Q'$ such that $U' = (id \otimes \psi)U$. If a universal object exists in $Q(D)$ then we denote it by $QISO^+(A^\infty, \mathcal{H}, D)$ and the corresponding largest Woronowicz subalgebra for which $ad_Q$ is faithful (where $U$ is the unitary representation of $QISO^+(A^\infty, \mathcal{H}, D)$ ) is called the quantum group of orientation preserving isometries and denoted by $QISO^+(A^\infty, \mathcal{H}, D)$.

Here we state the Theorem 2.23 of \cite{12} for existing $QISO^+(A^\infty, \mathcal{H}, D)$.

**Theorem 2.6** Let $(A^\infty, \mathcal{H}, D)$ be a spectral triple of compact type. Assume that $D$ has one dimensional kernel spanned by a vector $\xi \in \mathcal{H}$ which is cyclic and separating for $A^\infty$ and each eigenvector of $D$ belongs to $A^\infty \xi$. Then $QISO^+(A^\infty, \mathcal{H}, D)$ exists.

Now we discuss a special case of our interest. Let $\Gamma$ be a finitely generated discrete group with generating set $S = \{a_1, a_1^{-1}, a_2, a_2^{-1}, \ldots a_k, a_k^{-1}\}$. We make the convention of choosing the generating set to be symmetric, i.e. $a_i \in S$ implies $a_i^{-1} \in S \forall i$. In case some $a_i$ has order 2, we include only $a_i$, i.e. not count it twice. Define length function on the group as $l(g) = \min \{r \in \mathbb{N}, g = h_1 h_2 \cdots h_r\}$ where $h_i \in S$ i.e. for each $h_i = a_j$ or $a_j^{-1}$ for some $j$. Notice that $S = \{g \in \Gamma, l(g) = 1\}$, using this length function we can define a metric on $\Gamma$ by $d(a,b) = l(a^{-1}b) \forall a,b \in \Gamma$. This is called the word metric. Now consider the algebra $C^*_r(\Gamma)$, which is the $C^*$ completion of the group ring $\mathbb{C} \Gamma$ viewed as a subalgebra of $B(l^2(\Gamma))$ in the natural way via left regular representation. We define the Dirac operator $D_\Gamma(\delta_g) = l(g)\delta_g$. In general, $D_\Gamma$ is an unbounded operator.

$$\text{Dom}(D_\Gamma) = \{\xi \in l^2(\Gamma) : \sum_{g \in \Gamma} l(g)^2|\xi(g)|^2 < \infty\}.$$
Here, $\delta_g$ is the vector in $L^2(\Gamma)$ which takes value 1 at the point $g$ and 0 at all other points. Natural generators of the algebra $\mathbb{C}\Gamma$ (images in the left regular representation) will be denoted by $\lambda_g$, i.e. $\lambda_g(\delta_h) = \delta_{gh}$. It is easy to check that $(\mathbb{C}\Gamma, L^2(\Gamma), D)$ is a spectral triple. Now take $A = C^*_\text{r}(\Gamma), \mathcal{A} = \mathbb{C}\Gamma, \mathcal{H} = L^2(\Gamma)$ and $\mathcal{D} = D_\Gamma$ as before, $\delta_\Gamma$ is cyclic separating vector for $\mathbb{C}\Gamma$ then $\mathcal{Q} \mathcal{ISO}(\mathbb{C}\Gamma, L^2(\Gamma), D_\Gamma)$ exists by Theorem 2.6. As the object depends on the generating set of $\Gamma$ it is denoted by $\mathcal{Q}(\Gamma,S)$). Most of the time we denote it by $\mathcal{Q}(\Gamma)$ if $S$ is understood from the context. Now as in [14] Its action $\alpha$ (say) on $C^*_\text{r}(\Gamma)$ is determined by

$$\alpha(\lambda_\gamma) = \sum_{\gamma' \in S} \lambda_{\gamma'} \otimes q_{\gamma',\gamma},$$

where the matrix $[q_{\gamma',\gamma}]_{\gamma',\gamma \in S}$ is called the fundamental unitary in $M_{\text{card}(S)}(\mathbb{Q}(\Gamma,S))$. Moreover if $\Gamma$ is commutative, the maximal commutative quantum subgroup of $\mathcal{Q}(\Gamma)$ is denoted by $\mathcal{C}(\mathcal{ISO}(\Gamma))$. In section 3 we will see many examples for which $\mathcal{Q}(\Gamma) \cong \mathcal{C}(\mathcal{ISO}(\Gamma))$. Now we fix some notational conventions which will be useful in later sections. Note that the action $\alpha$ is of the form

$$\alpha(\lambda_{a_1}) = \lambda_{a_1} \otimes A_{11} + \lambda_{a_1^{-1}} \otimes A_{12} + \lambda_{a_2} \otimes A_{13} + \lambda_{a_2^{-1}} \otimes A_{14} + \cdots + \lambda_{a_k} \otimes A_{1(2k-1)} + \lambda_{a_k^{-1}} \otimes A_{1(2k)},$$

$$\alpha(\lambda_{a_1^{-1}}) = \lambda_{a_1} \otimes A_{12}^* + \lambda_{a_1^{-1}} \otimes A_{11}^* + \lambda_{a_2} \otimes A_{14}^* + \lambda_{a_2^{-1}} \otimes A_{13}^* + \cdots + \lambda_{a_k} \otimes A_{1(2k)}^* + \lambda_{a_k^{-1}} \otimes A_{1(2k-1)}^*,$$

$$\alpha(\lambda_{a_2}) = \lambda_{a_2} \otimes A_{21} + \lambda_{a_2^{-1}} \otimes A_{22} + \lambda_{a_3} \otimes A_{23} + \lambda_{a_3^{-1}} \otimes A_{24} + \cdots + \lambda_{a_k} \otimes A_{2(2k-1)} + \lambda_{a_k^{-1}} \otimes A_{2(2k)},$$

$$\alpha(\lambda_{a_2^{-1}}) = \lambda_{a_2} \otimes A_{22}^* + \lambda_{a_2^{-1}} \otimes A_{21}^* + \lambda_{a_3} \otimes A_{24}^* + \lambda_{a_3^{-1}} \otimes A_{23}^* + \cdots + \lambda_{a_k} \otimes A_{2(2k)}^* + \lambda_{a_k^{-1}} \otimes A_{2(2k-1)}^*,$$

$$\vdots$$

$$\alpha(\lambda_{a_k}) = \lambda_{a_k} \otimes A_{k1} + \lambda_{a_k^{-1}} \otimes A_{k2} + \lambda_{a_2} \otimes A_{k3} + \lambda_{a_3} \otimes A_{k4} + \cdots + \lambda_{a_k} \otimes A_{k(2k-1)} + \lambda_{a_k^{-1}} \otimes A_{k(2k)},$$

$$\alpha(\lambda_{a_k^{-1}}) = \lambda_{a_k} \otimes A_{k2}^* + \lambda_{a_k^{-1}} \otimes A_{k1}^* + \lambda_{a_2} \otimes A_{k4}^* + \lambda_{a_3} \otimes A_{k3}^* + \cdots + \lambda_{a_k} \otimes A_{k(2k)}^* + \lambda_{a_k^{-1}} \otimes A_{k(2k-1)}^*.$$
From this we get the unitary corepresentation

\[
U = \begin{pmatrix}
A_{11} & A_{12} & A_{13} & A_{14} & \cdots & A_{1(2k-1)} & A_{1(2k)} \\
A_{21}^* & A_{22}^* & A_{23}^* & A_{24}^* & \cdots & A_{2(2k-1)}^* & A_{2(2k)}^* \\
A_{k1} & A_{k2} & A_{k3} & A_{k4} & \cdots & A_{k(2k-1)} & A_{k(2k)} \\
A_{k2}^* & A_{k1}^* & A_{k4}^* & A_{k3}^* & \cdots & A_{k(2k-1)}^* & A_{k(2k)}^*
\end{pmatrix}.
\]

The coefficients \(A_{ij}\) and \(A_{ij}^*\)'s generate a norm dense subalgebra of \(Q(\Gamma, S)\). In fact it is easy to see that \(Q(\Gamma, S)\) is the CQG generated by \(A_{ij}\) as above subject to the relation \(U\) is a unitary and \(\alpha\) given above \(*\) homomorphism on \(C^*_r(\Gamma)\).

**Proposition 2.7** For any group \(\Gamma\), we have \(C^*_r(\Gamma)\) as a quantum subgroup of its QISO. i.e. \(C^*_r(\Gamma)\) always acts on itself isometrically and faithfully.

**Proof:** The usual coproduct of \(C^*_r(\Gamma)\) gives us action on itself given by \(\Delta(\lambda_g) = \lambda_g \otimes \lambda_g, \forall g \in \Gamma\). It is clear that action is isometric and faithful. So by the definition of QISO, there is surjective morphism from \(Q(\Gamma)\) to \(C^*_r(\Gamma)\) sending the entries \(A_{i(2i-1)}\) of the fundamental unitary mentioned before to \(\lambda_{a_i}\), others are going to zero. \(\square\)

The above proposition tells us for a nonabelian group \(\Gamma\), the quantum isometry group \(Q(\Gamma)\) is always genuine CQG, i.e. non-commutative as \(C^*\) algebra.

### 2.2 Recollection of some known facts

To the best of our knowledge, first computations of \(Q(\Gamma)\) were done in [14]. Thereafter, several articles by different authors were devoted to computations of \(Q(\Gamma)\) for concrete groups. In [14] quantum isometry groups of cyclic groups (except \(\mathbb{Z}_4\)) were shown to be commutative. In case of \(\mathbb{Z}_4\) it turns out to be noncommutative and infinite dimensional. It is in fact isomorphic with \(C^*(D_\infty \times \mathbb{Z}_2)\) as a \(C^*\) algebra (see [7]). Later in [5] it was identified with \(\mathbb{Z}_2 \ast \mathbb{Z}_2\) as a quantum group.

The authors of [20] introduced the doubling procedure, and moreover they showed that for the symmetric group \(S_n\) with standard generating sets taken as \((n-1)\) transpositions, the quantum isometry group coincides with the doubling of the group algebra. The same result holds for \(D_{2(2n+1)}\) as well (see [30]). We will also briefly discuss the doubling procedure in Subsection 2.4.

In [6], Banica and Skalski introduced two parameter families \(H^*_n(p, q)\), of quantum symmetry groups and they studied quantum isometry groups of duals of free product of cyclic groups for several cases in [7]. They showed that

\[H^*_n = Q(\mathbb{Z}_2 \ast \mathbb{Z}_2 \cdots \mathbb{Z}_2),\]

\(n\) copies
2.3 The case when $\Gamma$ is a free or direct product

**Theorem 2.8** Let $\Gamma_1, \Gamma_2, \cdots, \Gamma_k$ be finitely generated discrete groups with the symmetric generating sets $S_1, S_2, \cdots, S_k$ respectively. Consider $\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_k$ with the generating set $S = S_1 \cup S_2 \cup \cdots \cup S_k$. Then $Q(\Gamma, S)$ has $Q(\Gamma_1, S_1) \otimes_{\text{max}} Q(\Gamma_2, S_2) \otimes_{\text{max}} \cdots \otimes_{\text{max}} Q(\Gamma_k, S_k)$ as quantum subgroup.

**Proof:**
Let $S_i = \{a_{ij}, j = 1, 2, \cdots, k_i\}$ where $i = 1, 2, \cdots, k$ and $((u_{jk}^{(i)}))$ be the fundamental unitary representation for the action of $Q_i \equiv Q(\Gamma_i, S_i)$ on $C^*_r(\Gamma_i)$. Consider the action $\alpha : C^*_r(\Gamma) \rightarrow C^*_r(\Gamma) \otimes Q$ where $Q = Q_1 \otimes_{\text{max}} Q_2 \otimes_{\text{max}} \cdots \otimes_{\text{max}} Q_k$ given by $\alpha(\lambda_{a_{ij}}) = \sum_j \lambda_{a_{ij}} \otimes 1(1) \otimes 1(2) \otimes \cdots \otimes 1(i-1) \otimes u_{jk}^{(i)} \otimes 1(i+1) \otimes \cdots \otimes 1(k)$, where $1(i)$ denotes the identity element of the $C^*$ algebra $Q_i$. It is easy to verify that this gives an isometric action of $Q$ on $C^*_r(\Gamma)$, hence by the universality of $Q(\Gamma, S)$ we get a surjective morphism from $Q(\Gamma, S)$ to $Q$. $\square$

**Remark 2.9** In the set up of the previous theorem, replace the direct product by free product, i.e. take $\Gamma = \Gamma_1 \ast \Gamma_2 \ast \cdots \ast \Gamma_k$. Then $Q(\Gamma_1) \ast Q(\Gamma_2) \ast \cdots \ast Q(\Gamma_k)$ is quantum subgroup of $Q(\Gamma)$. The proof is very similar and hence omitted.

**Remark 2.10** We give an example to show that $Q(H)$ may not be a quantum subgroup of $Q(\Gamma)$ in general for a subgroup $H$ of $\Gamma$, when $\Gamma$ is neither $H \times K$ nor $H \ast K$ for some $K$. Take $S_4$ with generating sets $(12), (23), (34)$ and $H$ be the subgroup of it defined by $< (12), (34) >$. $H$ is clearly isomorphic with $\mathbb{Z}_2 \times \mathbb{Z}_2$. We know that $Q(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is infinite dimensional (see [4]) but $Q(S_4)$ is $C^*(S_4) \oplus C^*(S_4)$ from [20].

**Remark 2.11** We’ll be usually interested to see whether $Q(\Gamma_1) \ast Q(\Gamma_2) \ast \cdots \ast Q(\Gamma_k)$ or $Q(\Gamma_1) \otimes_{\text{max}} Q(\Gamma_2) \otimes_{\text{max}} \cdots \otimes_{\text{max}} Q(\Gamma_k)$ coincides with $Q(\Gamma)$ (in section 2 and 3) whenever $\Gamma$ is either $\Gamma_1 \ast \Gamma_2 \ast \cdots \ast \Gamma_k$ or $\Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_k$ respectively. In this context the following observation will be useful.

**Lemma 2.12** Let $((u_{ij}))$ be the fundamental representation of $Q(\Gamma)$ on $C^*_r(\Gamma)$. Also assume $\Gamma = \Gamma_1 \ast \Gamma_2 \ast \cdots \ast \Gamma_k$. Then $Q(\Gamma) \cong Q(\Gamma_1) \ast Q(\Gamma_2) \ast \cdots \ast Q(\Gamma_k) \cong Q(\Gamma_1) \otimes_{\text{max}} Q(\Gamma_2) \otimes_{\text{max}} \cdots \otimes_{\text{max}} Q(\Gamma_k)$.

$$H^+(n, 0) \cong Q(\mathbb{Z} \oplus \mathbb{Z} \cdots \oplus \mathbb{Z})_{\text{n copies}},$$
$$H^+_s(n, 0) \cong Q(\mathbb{Z}_s \oplus \mathbb{Z}_s \cdots \oplus \mathbb{Z}_s)_{\text{n copies}},$$
where $s \neq 2, 4$. It was observed in [5]
if \((u_{ij})\) is of the block diagonal form \(U = \begin{pmatrix} C_1 & 0 & \cdots & 0 \\ 0 & C_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_k \end{pmatrix}\) with respect to

the decomposition of the generating set \(S\) of \(\Gamma\) into \(S_1 \cup S_2 \cup \cdots \cup S_k\).

**Proof:**

Let us write \(Q\) gives a morphism from \(\Gamma\). The above lemma is true if we replace the free product by direct product, then we get the desired result. Below we give a sufficient condition for the quantum isometry group to be unitary being \(C\).

**Remark 2.13** The above lemma is true if we replace the free product by direct product, then \(Q(\Gamma) \cong Q(\Gamma_1) \otimes_{\text{max}} Q(\Gamma_2) \otimes_{\text{max}} \cdots \otimes_{\text{max}} Q(\Gamma_k)\) if and only if \((u_{ij})\) is of the block diagonal form, and moreover entries of one such block commute with any entries of any other block. The proof of that is very similar to Lemma 2.12 hence omitted.

### 2.4 \(Q(\Gamma)\) as the doubling of \(C^*_\Gamma(\Gamma)\)

We briefly recall the doubling procedure of the group algebra from [20, 29]. Let \((Q, \Delta)\) be a CQG with a CQG-automorphism \(\theta\) such that \(\theta^2 = \text{id}\). The doubling of this CQG, say \((\tilde{Q}, \tilde{\Delta})\) given by \(\tilde{Q} := Q \oplus Q\) (direct sum as a \(C^*\) algebra), and the coproduct is defined by the following, where we have denoted the injections of \(Q\) onto the first and second coordinate in \(\tilde{Q}\) by \(\xi\) and \(\eta\) respectively, i.e. \(\xi(a) = (a, 0), \eta(a) = (0, a), \) (\(a \in Q\)).

\[
\tilde{\Delta} \circ \xi = (\xi \otimes \xi \otimes [\eta \circ \theta]) \circ \Delta,
\]

\[
\tilde{\Delta} \circ \eta = (\xi \otimes \eta \otimes [\xi \circ \theta]) \circ \Delta.
\]

Below we give a sufficient condition for the quantum isometry group to be the doubling of the group algebra. For this, it is convenient to use a slightly different notational convention: let \(U_{2i-1,j} = A_{ij}\) for \(i = 1, \ldots, k, j = 1, \ldots, 2k\) and \(U_{2i,2l} = A_{i(2l-1)}\) for \(i = 1, \ldots, k, l = 1, \ldots, k\).

**Lemma 2.14** Let \(\Gamma\) be a group with \(k\) generators \(\{a_1, a_2, \ldots, a_k\}\) (say) and \(\theta\) be an automorphism of order 2 of the group algebra which gives a permutation \(\sigma\) on the set \(\{1, 2, \ldots, 2k - 1, 2k\}\). Now assume the following:

1. \(B_i := U_{i, \sigma(i)} \neq 0 \forall i, \) and \(U_{i,j} = 0 \forall j \in \{\sigma(i), i\}\),
2. \(A_i B_j = B_j A_i = 0, \forall j\) such that \(\sigma(j) \neq j\), where \(A_i = U_{i,i}\),
3. All \(U_{i,j} U_{i,j}^*\) are central projections,
4. There are well defined C* isomorphisms $\pi_1, \pi_2$ from $C^*_r(\Gamma)$ to $C^*_r\{A_i, i = 1, 2, \ldots, 2k\}$ and $C^*_r\{B_i, i = 1, 2, \ldots, 2k\}$ respectively such that

$$\pi_1(\lambda_{a_i}) = A_i, \pi_2(\lambda_{a_i}) = B_i \forall i.$$  

Then $Q(\Gamma)$ is doubling corresponding to the given automorphism $\theta$. Moreover the fundamental unitary takes the following form

$$
\begin{pmatrix}
A_1 & 0 & 0 & 0 & \cdots & 0 & B_1 \\
0 & A_2 & 0 & 0 & \cdots & B_2 & 0 \\
0 & 0 & A_3 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & A_4 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & B_{2k-1} & 0 & 0 & \cdots & A_{2k-1} & 0 \\
B_{2k} & 0 & 0 & 0 & \cdots & 0 & A_{2k}
\end{pmatrix}.
$$

Proof:
For simplicity, we assume that none of the generators are of order 2. The case when some of them are of order 2 can be dealt with in a very similar way, with some more care of notations. We introduce $\gamma_i, i = 1, \ldots, 2k$ given by $\gamma_{2l-1} := a_l$, $\gamma_{2l} := a_l^{-1} \forall l = 1, 2, \cdots, k$. Now the C* algebra $C^*_r(\Gamma) \oplus C^*_r(\Gamma)$ is generated by $(\lambda_{a_i} \oplus 0), (0 \oplus \lambda_{a_i}), i = 1, \ldots, k$. We define a -homomorphism $\pi$ from $C^*_r(\Gamma) \oplus C^*_r(\Gamma)$ to $Q(\Gamma)$ given by

$$
\pi(\lambda_{a_i} \oplus 0) = A_i, \quad \pi(0 \oplus \lambda_{\gamma_l}) = B_i, \quad \text{if } \sigma(i) \neq i,
$$

$$
\pi(\lambda_{\gamma_l} \oplus \lambda_{\gamma_l}) = A_i = B_i, \quad \text{otherwise}.
$$

It is easy to verify that this is a CQG isomorhism. □

2.5 $Q(\Gamma, S)$ as a quantum isometry group of metric space

In this subsection, our aim is to identify $Q(\Gamma, S)$ with the quantum isometry group of some metric space in the sense of \cite{24, 13}, in case $\Gamma$ is abelian. We first recall the definition of quantum isometry group in the purely metric space setting. Given a compact metric space $(X, d)$, we say an action $\alpha$ of a CQG on it is isometric if the action $\alpha_r = (id \otimes \pi_r) \circ \alpha$ of the reduced CQG $Q_r$ (where $\pi_r : Q \to Q_r$ is the canonical map from $Q$ to reduced CQG $Q_r$) satisfies

$$
\alpha_r(d_x(y)) = \kappa(\alpha_r(d_y)(x)),
$$

$\forall x, y \in X$, where $d_r(.) \equiv d(x, .)$ and $\kappa$ denotes the (norm bounded) antipode of $Q_r$. It is shown in \cite{24} that in case $X \subseteq \mathbb{R}^n$ isometrically embedded, the above condition is equivalent to the following :

$$
\sum_i(F_i(x) - F_i(y))^2 = \sum_i(x_i - y_i)^2.
$$
where \( F_i(x) = \alpha(X_i)(x) \) and where \( X_i \) denotes the \( i \)th coordinate function of \( \mathbb{R}^n \), restricted to \( X \). It can be easily seen, by almost verbatim adaptation of the arguments in [24] that a similar result would hold if we replaced \( \mathbb{R}^n \) by \( \mathbb{C}^n \). That is, for \( X \subseteq \mathbb{C}^n \) isometrically, with the metric \( d(z, w)^2 = \Sigma_i |z_i - w_i|^2 \), a CQG action \( \alpha : \mathcal{C}(X) \to \mathcal{C}(X) \hat{\otimes} \mathcal{Q} \) is isometric in the metric space sense if and only if
\[
\sum_i (F_i(z) - F_i(w))^* (F_i(z) - F_i(w)) = d^2(z, w)1
\]
\( \forall z, w \in X \). Moreover, it is clear that a sufficient condition for the above is that
\[
\sum_i F_i(z)^* F_i(w) = \langle z, w \rangle + 1 \equiv \sum_i \epsilon_i w_i 1.
\]

We can also prove, as in [24], that for metric spaces \((X, d)\) isometrically embedded in \( \mathbb{C}^n \), there exists a universal CQG acting isometrically on it, to be denoted by \( \text{QISO}(X, d) \), and its action is affine in the sense that \( \alpha(z_i) = \Sigma_j z_j \otimes q_{ji} + 1 \otimes R_i \) for some \( q_{ji} \) and \( R_i \).

Let us now consider a finitely generated abelian group \( \Gamma \) with a symmetric generating set \( S = \{\gamma_1, \gamma_2 \ldots, \gamma_n\} \) and let \((z_{ji})\) be the fundamental unitary for \( \text{QISO}(\Gamma, S) \). Now consider the dual group of \( \Gamma \) say \( G = \hat{\Gamma} \), which is a compact topological group. Moreover, \( \hat{\Gamma} \) can be identified with a compact subset \( X \) of \( \mathbb{C}^n \) via the map \( \chi \mapsto (\chi(\gamma_1), \chi(\gamma_2), \ldots, \chi(\gamma_n)) \). There is a natural Euclidean metric \( d_S \) on \( \hat{\Gamma} \) given by \( d_S(\chi, \chi')^2 = \Sigma_i |\chi(\gamma_i) - \chi'(\gamma_i)|^2 \).

**Theorem 2.15** With the above set-up, we have
\[
\text{QISO}(\Gamma, S) \cong \text{QISO}(\hat{\Gamma}, d_S).
\]

**Proof:**

It is well-known that \( C^*(\Gamma) \) is isomorphic with \( C(\hat{\Gamma}) \) via the Fourier transform \( F \), extended as a unitary from \( l^2(\Gamma) \) to \( L^2(\widehat{\Gamma}) \). i.e. \( C(\Gamma) = FC^*(\Gamma)F^{-1} \subseteq B(L^2(\widehat{\Gamma})) \). Let \( U \) denote the unitary representation of \( \text{QISO}(\Gamma, S) \) on \( C^*(\Gamma) \), and let \( U' \) and \( \alpha' \) be the corresponding representation and action of \( \text{QISO}(\Gamma, S) \) on \( C(\Gamma) \) respectively, i.e. \( U' = FU^{-1} \) and \( \alpha'(.)(F^{-1} \otimes id) \). As \( \lambda_{\gamma_1}, \lambda_{\gamma_2}, \ldots, \lambda_{\gamma_n} \) generate \( C^*(\Gamma) \) as a \( C^* \) algebra, under the Fourier transform \( C(\Gamma) \) is generated by \( \bar{\lambda}_{\gamma_1}, \bar{\lambda}_{\gamma_2}, \ldots, \bar{\lambda}_{\gamma_n} \). We have \( z_i = \bar{\lambda}_{\gamma_i} \) as the coordinate functions restricted to \( \hat{\Gamma} \subseteq \mathbb{C}^n \). Clearly, \( C(\hat{\Gamma}) \cong C^*(z_1, z_2, \ldots, z_n) \subseteq C_0(\mathbb{C}^n) \). Let \( F(z) = (F_1(z), F_2(z), \ldots, F_n(z)) \) for \( z \in \hat{\Gamma} \), where \( F_i(z) := \alpha'(\gamma_i)(z) \). Now it is clear that \( F_i \equiv \alpha'(z_i) = \Sigma_j z_j \otimes q_{ji} \), and as \((q_{ji})\) is unitary, it is easy to verify \( \Sigma_i F_i(z)^* F_i(w) = \Sigma_i \epsilon_i w_i \otimes 1_Q \). Thus \( \alpha' \) is an isometric action of \( \text{QISO}(\Gamma, S) \) on \( C(\hat{\Gamma}) \). i.e. \( \text{QISO}(\Gamma, S) \) is a quantum subgroup of \( \text{QISO}(\hat{\Gamma}, d_S) \).

Conversely, assume that the isometric action of \( \text{QISO}(\hat{\Gamma}, d_S) \) is given by \( \beta(z_i) = \Sigma_j z_j \otimes b_{ji} + 1 \otimes R_i \) for some \( b_{ji} \) and \( R_i \). Consider any Borel probability measure on \( \hat{\Gamma}, d_S \) and convolving with the Haar state of \( \text{QISO}(\hat{\Gamma}, d_S) \), we get a \( \text{QISO}(\hat{\Gamma}, d_S) \) invariant probability measure say \( \mu \) on \( \hat{\Gamma} \). As \( \hat{\Gamma} \) itself acts isometrically

\[
10
\]
on $(\hat{\Gamma}, d_S)$, $C^*_r(\hat{\Gamma}) \cong C(\hat{\Gamma})$ is a quantum subgroup of $Q'$, hence $\mu$ is $\hat{\Gamma}$ invariant probability measure on $\hat{\Gamma}$. Therefore $\mu$ must be the (unique) Haar measure of $\hat{\Gamma}$. As the state corresponding to the Haar measure say $\phi$, maps each $z_i$ to zero, we conclude using the $\beta$ invariance of $\phi$ i.e. $(\phi \otimes \text{id})\beta(z_i) = \phi(z_i)1_{Q'}$, that $R_i = 0$ for each $i$. Thus $\beta$ is linear, i.e. $\beta(z_i) = \sum j z_j \otimes b_{ji}$. The $*$-homomorphism $(F^{-1} \otimes \text{id})\beta(F \otimes \text{id}) : C^*_r(\hat{\Gamma}) \to C^*_r(\hat{\Gamma}) \hat{\otimes} Q'$ maps $\lambda_{a_i}$ to $\sum j \lambda_{a_j} \otimes b_{ji}$ and is clearly an action. Finally, as $\beta$ preserves the Haar state $\mu$, it extends to a unitary representation on the $L^2$-space, hence in particular, $((b_{ji}))$ is the matrix coefficients of a unitary representation. In other words, $((b_{ji}))$ is a unitary element of $M_n(Q')$. This shows by the definition of $Q(\Gamma, S)$ that $Q'$ is a quantum subgroup of $Q(\Gamma, S)$, with the surjective morphism $q_{ji}$ to $b_{ji}$. 

2.6 Polynomial growth of $Q(\Gamma)$

We briefly discuss some sufficient conditions for the quantum group $Q(\Gamma)$ to have polynomial growth property in the sense of [10], when $\Gamma$ has polynomial growth.

For this, we need to make a definition.

**Definition 2.16** We say that an action of a CQG on a $C^*$ algebra has full spectrum if the spectral subspace corresponding to each irreducible representation of the CQG is nonzero.

We now state and prove the main result of this subsection.

**Theorem 2.17** Let $\Gamma$ be a finitely generated discrete group with polynomial growth and assume that the action of $Q(\Gamma)$ on $C^*_r(\Gamma)$ has full spectrum in the sense discussed above. Then the dual discrete quantum group of $Q(\Gamma)$ also has polynomial growth property.

**Proof:**

Let $S$ be the finite generating set for $\Gamma$. As the group $\Gamma$ has polynomial growth, there is some polynomial $p$ of one variable such that the cardinality of the set \{ $g_1g_2 \cdots g_m : g_i \in S \text{ } \forall \text{ } i, m \leq n$ \} is bounded by $p(n)$ for each $n$. That is, the dimension of the vector space say $V_n$ spanned by elements of the form $\lambda_{a_1} \cdots \lambda_{a_m}$ where $a_i \in S$ and $m \leq n$, has dimension less than or equal to $p(n)$. But this space is clearly left invariant by the action of $Q(\Gamma)$. Let us denote the restriction of this action to $V_n$, which is a finite dimensional unitary representation, by $\pi_n$. Moreover, by the assumption of minimality, every irreducible representation of $Q(\Gamma)$ must be a sub-representation of some $\pi_n$ for sufficiently large $n$. This allows us to define a central length function $l$ on the set of irreducible representation of $Q(\Gamma)$ (in the sense of [10]) by setting $l(\pi)$ to be equal to the smallest value of $n$ for which $\pi$ is a sub-representation of $\pi_n$. Clearly,

$$\sum_{\pi: l(\pi) \leq n} d_\pi^2 \leq p(n),$$

where $d_\pi$ denotes the dimension of the irreducible $\pi$. From this, it is easily seen that this length function satisfies the criteria of Definition 4.1 of [10], hence the
dual of $Q$ has polynomial growth. \(\square\)

2.7 The structure of the maximal commutative subgroup of $Q(\Gamma)$ for $\Gamma = Z_n^k$

Proposition 2.18 Let $\Gamma = (Z_n \times Z_n \times \cdots \times Z_n)$ where $n \neq 2, 4$ then $C(\text{ISO}(\Gamma))$

will be $C((Z_n \times Z_n \times \cdots \times Z_n) \times (Z_2^k \times S_k))$. Whenever $n = 2$ then we have $C(\text{ISO}(\Gamma)) \cong C((Z_2 \times Z_2 \times \cdots Z_2) \times S_k)$, where $S_k$ is the group of permutation of $k$ elements.

Proof:
Let $a_1, a_2, \ldots, a_k$ be the usual generating sets for $\Gamma$. The fundamental unitary

$$U = \begin{pmatrix}
A_{11} & A_{12} & A_{13} & A_{14} & \cdots & A_{1(2k-1)} & A_{1(2k)} \\
A_{12} & A_{11} & A_{14} & A_{13} & \cdots & A_{1(2k-1)} & A_{1(2k-1)} \\
A_{21} & A_{22} & A_{23} & A_{24} & \cdots & A_{2(2k-1)} & A_{2(2k)} \\
A_{22} & A_{21} & A_{24} & A_{23} & \cdots & A_{2(2k-2)} & A_{2(2k-1)} \\
& & & & & & \\
A_{k1} & A_{k2} & A_{k3} & A_{k4} & \cdots & A_{k(2k-1)} & A_{k(2k)} \\
A_{k2} & A_{k1} & A_{k4} & A_{k3} & \cdots & A_{k(2k-2)} & A_{k(2k-1)}
\end{pmatrix}
$$

is of the form $U = (u_{ij})$ (say). However we have the additional condition that $u_{ij}$’s commute among themselves. Our first claim is $A_{ij}A_{ik} = 0 = A_{ji}A_{jk}, j \neq k \forall i, j, k$.

We break it into two cases.

Case 1: $n = 2$ Consider the term $\alpha(\lambda_{a_i}) = \lambda_e \otimes 1_Q \forall i$. Now comparing the coefficients of $\lambda_{a_1a_m}$ \forall $l \neq m$ on both sides of the equation we obtain $A_{il}A_{im} = 0 \forall l \neq m$. Applying the antipode we find $A_{il}A_{ml} = 0 \forall l \neq m$.

Case 2: $n \neq 2$ Using the relation $\alpha(\lambda_{a_j})\alpha(\lambda_{a_{j+1}}) = \alpha(\lambda_{a_{j+1}})\alpha(\lambda_{a_j}) = \alpha(\lambda_e) = \lambda_e \otimes 1_Q$ comparing the coefficients of $\lambda_{a_i}^2$ and $\lambda_{a_{i-2}}$ on both sides we must have $A_{i(2j-1)}A_{i(2j)}^* = A_{i(2j)}^*A_{i(2j-1)} = 0$. Applying the antipode we get that $A_{i(2j-1)}^*A_{i(2j)} = A_{i(2j)}^*A_{i(2j-1)} = 0$, this shows $A_{i(2j)}A_{i(2j-1)} = A_{i(2j-1)}A_{i(2j)} = 0$, for each $i,j$ this is true. Now we show $A_{i(2j)}A_{im} = A_{i(2j-1)}A_{im} = 0$ where $m \neq 2j, (2j - 1)$. Using the condition $\alpha(\lambda_{a_i})\alpha(\lambda_{a_{i+1}}) = \alpha(\lambda_{a_{i+1}})\alpha(\lambda_{a_i}) = \alpha(\lambda_e) = \lambda_e \otimes 1_Q$ comparing the coefficients of $\lambda_{a_1a_i}, \lambda_{a_{i-1}}a_i$ where $i \neq l$ one can get $A_{i(2j-1)}A_{i(2j)}^* + A_{i(2j-1)}A_{i(2j)}^* = A_{i(2j)}A_{i(2j-1)}^* + A_{i(2j)}A_{i(2j-1)}^* = 0$, thus $A_{i(2j-1)}A_{i(2j)}^*A_{i(2j-1)} = A_{i(2j)}A_{i(2j-1)}A_{i(2j)} = 0$, hence $A_{i(2j-1)}A_{i(2j)} = A_{i(2j)}A_{i(2j-1)} = 0$. Similarly we have $A_{i(2j)}A_{i(2j-1)} = A_{i(2j-1)}A_{i(2j)} = 0$. This proves the claim by using antipode. Moreover, unitary of $U$ together with this
condition gives for finite n we have $A_{ij}^* = A_{ij}^{n-1}$.

Now we produce the explicit isomorphism. Identify $C((\mathbb{Z}_n \times \mathbb{Z}_n \times \cdots \times \mathbb{Z}_n) \times (\mathbb{Z}_2^k \times S_k))$ as a $C^*$ algebra in a natural way with $C(\mathbb{Z}_n \times \mathbb{Z}_n \times \cdots \times \mathbb{Z}_n) \otimes C(\mathbb{Z}_2^k) \otimes C(S_k)$.

Let $\chi_{ij} \in C(S_k)$ be the characteristic function of the set of these permutations which map $i$ to $j$ and also $\eta_{0,i}, \eta_{1,i} \in C(\mathbb{Z}_2^k)$ be characteristic functions of sets which have respectively 0 or 1 in the $i$th coordinate. One can easily check as in Theorem 6.1 of [6] the map

$$A_{i(2j-1)} \mapsto z_i \otimes \chi_{ij} \otimes \eta_{0,i}$$
$$A_{i(2j)} \mapsto z_i \otimes \chi_{ij} \otimes \eta_{1,i}$$

gives the isomorphism between $C(\text{ISO}(\Gamma))$ and $C((\mathbb{Z}_n \times \mathbb{Z}_n \times \cdots \times \mathbb{Z}_n) \times (\mathbb{Z}_2^k \times S_k))$, and that it preserves the respective coproducts. So it becomes a CQG isomorphism. □

3 Computations for free product of cyclic groups

Theorem 3.1 Let $\Gamma = \Gamma_1 * \Gamma_2 * \cdots * \Gamma_l$ where $\Gamma_i = (\mathbb{Z}_{n_1} * \mathbb{Z}_{n_2} * \cdots * \mathbb{Z}_{n_k})$. Also assume $n_1 \neq n_2 \neq \cdots \neq n_l$ and $n_i \neq 2, 4 \forall i$, then $Q(\Gamma)$ will be $H_{n_1}^+ (k_1, 0) * H_{n_2}^+ (k_2, 0) * \cdots * H_{n_l}^+ (k_l, 0)$. i.e. $Q(\Gamma) \cong Q(\Gamma_1) * Q(\Gamma_2) * \cdots * Q(\Gamma_l)$.

Proof:

For simplicity of notation we present the case when all $k_i = 1$. The general case will follow by exactly same argument. So now $\Gamma = \mathbb{Z}_{n_1} * \mathbb{Z}_{n_2} * \cdots * \mathbb{Z}_{n_l}$, with $\{ a_1, a_2, \cdots, a_l \}$ being the standard generating sets for this group. Without loss of generality we assume $n_1 < n_2 < \cdots < n_l$ and $o(a_i) = n_i \forall i$. Now the fundamental unitary of $Q(\Gamma)$ is of the form

$$U = \begin{pmatrix}
A_{11} & A_{12} & A_{13} & A_{14} & \cdots & A_{1(2l-1)} & A_{1(2l)} \\
A_{12}^* & A_{11}^* & A_{14}^* & A_{13}^* & \cdots & A_{1(2l-1)}^* & A_{1(2l)}^* \\
A_{21} & A_{22} & A_{23} & A_{24} & \cdots & A_{2(2l-1)} & A_{2(2l)} \\
A_{22}^* & A_{21}^* & A_{24}^* & A_{23}^* & \cdots & A_{2(2l-1)}^* & A_{2(2l)}^* \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
A_{l1} & A_{l2} & A_{l3} & A_{l4} & \cdots & A_{l(2l-1)} & A_{l(2l)} \\
A_{l2}^* & A_{l1}^* & A_{l4}^* & A_{l3}^* & \cdots & A_{l(2l-1)}^* & A_{l(2l)}^*
\end{pmatrix}$$
Our aim is to show that the fundamental co-representation actually is of the form

\[
\begin{pmatrix}
A_{11} & A_{12} & 0 & 0 & \cdots & 0 & 0 \\
A_{12}^* & A_{11}^* & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & A_{23} & A_{24} & \cdots & 0 & 0 \\
0 & 0 & A_{24}^* & A_{23}^* & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & A_{l(2l-1)} & A_{l(2l)} \\
0 & 0 & 0 & 0 & \cdots & A_{l(2l)^*} & A_{l(2l-1)^*}
\end{pmatrix}
\]

i.e. only the diagonal \((2 \times 2)\) block will survive and others become zero. By Lemma (2.12) this will complete the proof.

We break the proof into a number of lemmas.

**Lemma 3.2** \(A_{i(2j)}A_{i(2j-1)} = A_{i(2j-1)}A_{i(2j)} = 0 \ \forall \ i,j.\)

**Proof:**
Using the relation \(\alpha(\lambda_{ij})\alpha(\lambda_{ij}^{-1}) = \alpha(\lambda_{ij}^{-1})\alpha(\lambda_{ij}) = \alpha(\lambda_i) = \lambda_i \otimes 1_Q\) and comparing the coefficients of \(\lambda_{ij}\) and \(\lambda_{ij}^{-1}\) on both sides we have \(A_{i(2j-1)}A_{i(2j)}^* = A_{i(2j)}^*A_{i(2j-1)} = 0.\) Applying the antipode one can get that \(A_{i(2j-1)}A_{i(2j)}^* = A_{i(2j)}^*A_{i(2j-1)} = 0\) thus \(A_{i(2j)}A_{i(2j-1)} = A_{i(2j-1)}A_{i(2j)} = 0.\)

**Lemma 3.3** \(A_{1j} = 0 \ \forall \ j > 2.\)

**Proof:**
we know that \(\alpha(\lambda_{n1}^{-1}) = \alpha(\lambda_{a_i^{-1}})\). In L.H.S length of the elements increase gradually up to \((n_1 - 1)/2\) or \(n_1/2\) step according as \(n_1\) is odd or even and then decrease to 1 as length of the element of \(a_i^{-1}\) is 1. In the expression of \(\alpha(\lambda_{a_i^{-1}})\) all the elements are of \((n_1 - 1)\) length string. Now except \(a_1\) if we get other \(a_i\) from the \((n_1 - 1)\) length string the terms \((a_ja_j^{-1})\) or \((a_j^{-1}a_j)\) should come as \(n_1 < n_2 \cdots < n_l.\) \(a_1, a_1^{-1}\) can be written as \((a_1^{-1})^{n_1-1}, (a_1)^{n_1-1}\) respectively, but for other \(a_i\) this would not be possible as \(n_1 < n_2 \cdots < n_l.\) Further using the Lemma 3.2 and the equation \(\alpha(\lambda_{a_i^{-1}}) = \alpha(\lambda_{a_i^{-1}})\) (by comparing the coefficients of \(\lambda_{a_i}\) and \(\lambda_{a_i^{-1}}\) on both sides) we deduce that \(A_{1j} = 0 \ \forall \ j > 2.\)

Now applying the antipode we get that \(A_{i1} = A_{i2} = 0 \ \forall \ i > 1.\)

The structure of the unitary matrix reduces to the form
As in the proof of Theorem 3.1 we assume

Proof: Theorem 3.4

Theorem 3.1 is valid also for the case when \( n \in \mathbb{N} \). We’ll now show how to extend Theorem 3.1 in some cases when 2 or 4 can occur in \( n_1, n_2, \ldots, n_l \).

**Theorem 3.4** Theorem 3.1 is valid also for the case when \( n_1 = 2 \) and \( 2 \neq n_2 \neq \cdots \neq n_l \) where \( n_i \neq 4, \infty \forall i \).

**Proof:**

As in the proof of Theorem 3.1 we assume \( k_i = 1 \forall i \) for simplicity of exposition. From the relation \( \alpha(\lambda_i^2) = \lambda_i \otimes 1_Q \) we find that \( A_{22} = A_{13} = \cdots = A_{1(2l-2)} = A_{1(2l-1)} = 0 \) (comparing the coefficients of \( \lambda_i^2, \lambda_i \lambda_i^2, \lambda_i^3 \lambda_i^2, \cdots \lambda_i^{n_i-2} \lambda_i \lambda_i^{n_i-2} \) on both sides), which implies \( A_{21} = (A_{22})^2 = \cdots = A_{2(l-1)}^2 = (A_{2l})^2 = 0 \), by applying the antipode. Now consider \( \alpha(\lambda_i^{n_i-1}) = \alpha(\lambda_i) \). In the L.H.S if \( a_1, a_3, a_1^{-1}, \cdots, a_l, a_l^{-1} \) come from the \((n_2-1)\) length string then the string must contain at most one of the three types of words \( a_1, a_1^{-1} \) and \( a_i, a_i^{-1} \) \( \forall i > 1 \). Now using Lemma 3.2 we have \( A_{22} A_{2j} = A_{2j} A_{2j+1} = 0 \forall j \) and \( A_{21} = 0 \forall i > 1 \) from the above discussion. So comparing the coefficients of \( \lambda_i, \lambda_i^{-1}, \lambda_i^{n_i-1} \) on both sides of the relation \( \alpha(\lambda_i^{n_i-1}) = \alpha(\lambda_i) \) one can deduce that \( A_{21} = A_{2k} = 0 \forall k > 3 \). Applying the antipode we have, \( A_{12} = A_{13} = 0 \) and \( A_{12} = A_{3k} = 0 \forall k > 2 \).

So the fundamental Unitary is reduced to the form

\[
\begin{pmatrix}
A_{11} & 0 & 0 & \cdots & A_{1(2l-2)} & A_{1(2l-1)} \\
0 & A_{22} & A_{23} & \cdots & 0 & 0 \\
0 & 0 & A_{33} & A_{32} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
A_{l1} & 0 & 0 & \cdots & A_{l(2l-2)} & A_{l(2l-1)} \\
A_{l1} & 0 & 0 & \cdots & A_{l(2l-1)} & 0 
\end{pmatrix}
\]

Repeating similar arguments with \( a_3, a_4, \cdots, a_l \) we will get the desired block di-
Theorem 3.7. The results of Theorem 3.1 remain true for $n_1 = 4, 4 \neq n_2 \neq \cdots \neq n_l$ and $n_i \neq 2 \forall i$.

Proof: We follow the same strategy of Theorem 3.1 to complete the proof. \(\square\)

Remark 3.5. From the last line of the proof we see that finiteness of order of $a_i$ is needed. But in Theorem 2.1 this condition is not used. In fact it won't be needed in any of the results later on. So this is the only case where the finiteness condition is necessary.

Theorem 3.6. The results of Theorem 3.1 remain true for $n_1 = 4, 4 \neq n_2 \neq \cdots \neq n_l$ and $n_i \neq 2 \forall i$.

Proof: We continue using the notation and convention of Theorem 3.1 and without loss of generality assume $k_i = 1 \forall i$ and $o(a_1) = 4$. First consider the term $\alpha(\lambda_{a_1})$ and note that the coefficient of $\lambda_i$ in the expression of $\alpha(\lambda_{a_1})$ must be zero.

Thus $A_{12}A_{11} + A_{11}A_{12} + A_{13}A_{14} + A_{14}A_{13} + \cdots + A_{1(2l-1)}A_{1(2l)} + A_{1(2l)}A_{1(2l-1)} = 0$.

Hence $A_{12}A_{11} + A_{11}A_{12} = 0$, as $A_{1(2i-1)}A_{1(2i)} = A_{1(2i)}A_{1(2i-1)} = 0 \forall i > 1$

using the idea of the proof of Lemma 3.2.

Next we want to show that $A_{1i} = 0 \forall i > 2$. Now from the equation $\alpha(\lambda_{a_1}) = \alpha(\lambda_{a_1-1})$ comparing the coefficients of $\lambda_2, \lambda_{a_2-1}, \lambda_3, \lambda_{a_3-1}, \cdots \lambda_l, \lambda_{a_l-1}$, on both sides we deduce,

$$A_{1i(2i)} = A_{1(2i-1)}(A_{12}A_{11} + A_{11}A_{12} + A_{13}A_{14} + A_{14}A_{13} + \cdots + A_{1(2i-1)}A_{1(2i)} + \cdots + A_{1(2i-1)}A_{1(2l)} + A_{1(2l)}A_{1(2i-1)}),$$

$$A_{1i(2i-1)} = A_{1(i+2)}(A_{12}A_{11} + A_{11}A_{12} + A_{13}A_{14} + A_{14}A_{13} + \cdots + A_{1(2i-1)}A_{1(2i)} + \cdots + A_{1(2i-1)}A_{1(2l)} + A_{1(2l)}A_{1(2i-1)}) \forall i > 1.$$

So $A_{1i} = 0 \forall i > 2$. The fundamental unitary matrix is of the form

$$\begin{pmatrix}
A_{11} & A_{12} & 0 & 0 & \cdots & 0 & 0 \\
A_{12} & A_{11} & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & A_{23} & A_{24} & \cdots & A_{2(2i-1)} & A_{2(2i)} \\
0 & 0 & A_{24} & A_{23} & \cdots & A_{2(2i)} & A_{2(2i-1)} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & A_{l3} & A_{l4} & \cdots & A_{l(2i-1)} & A_{l(2i)} \\
0 & 0 & A_{l4} & A_{l3} & \cdots & A_{l(2i)} & A_{l(2i-1)}
\end{pmatrix}.$$ 

We follow the same strategy of Theorem 3.1 to complete the proof. \(\square\)

Theorem 3.7. The results of Theorem 3.1 hold also for $n_1 = 2, n_2 = 4$ where $2 \neq 4 \neq n_3 \neq n_4 \neq \cdots \neq n_l$ and $n_i \neq \infty \forall i$.

Proof: With the notation and convention as before, we get from the condition $\alpha(\lambda_{a_2}) = \alpha(\lambda_{a_2-1})$ comparing the coefficient of $\lambda_{a_1}$ on both sides,
\[ A_{21}^* = A_{21} (A_{22} A_{23} + A_{23} A_{22}) \] (As \( A_{2(2i+1)} A_{2(2i)} = A_{2(2i)} A_{2(2i+1)} = 0 \) \( \forall \ i > 1 \)).

Again \( (A_{21})^2 + A_{22} A_{23} + A_{23} A_{22} = 0 \) as it is the coefficient of \( \lambda_e \) in the expression of \( \alpha(\lambda_{n2}^2) \) and also using the fact \( A_{2(2i)} A_{2(2i+1)} = A_{2(2i+1)} A_{2(2i)} = 0 \) \( \forall \ i > 1 \).

Thus we obtain that \( A_{21}^* = (-A_{21})^3 \) using the above equations. On the other hand, using the condition \( \alpha(\lambda_{n2}) \alpha(\lambda_{n2}^{-1}) = \lambda_e \otimes 1_Q \), we deduce that \( A_{23} A_{21}^* = A_{22} A_{21}^* = 0 \), hence \( A_{21}^* A_{21} = A_{21} (A_{22} A_{23} + A_{23} A_{22}) A_{21}^* = 0 \) using the fact \( A_{23} A_{21}^* = A_{22} A_{21}^* = 0 \). Now as \( A_{21}^* = (-A_{21})^3 \), this implies \( A_{21}^* = A_{21} = 0 \) and also we have \( A_{12} = A_{13} = 0 \) applying \( \kappa \). This reduces the fundamental unitary to

\[
\begin{pmatrix}
A_{11} & 0 & 0 & \cdots & A_{1(2l-2)} & A_{1(2l-1)} \\
0 & A_{22} & A_{23} & \cdots & A_{2(2l-2)} & A_{2(2l-1)} \\
0 & A_{23} & A_{22} & \cdots & A_{2(2l-2)} & A_{2(2l-1)}^* \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
A_{l1} & A_{l2} & A_{l3} & \cdots & A_{l(2l-2)} & A_{l(2l-1)} \\
A_{l1}^* & A_{l2}^* & A_{l3}^* & \cdots & A_{l(2l-2)}^* & A_{l(2l-1)}^*
\end{pmatrix}
\]

The rest of the proof will be similar to Theorems 3.6 and 3.4, hence omitted. \( \square \)

**Remark 3.8** The above results allow us to compute the cases \( \Gamma = (Z_{n_1} \ast Z_{n_2} \cdots \ast Z_{n_l}) \)

\[
\ast (Z_{n_2} \ast Z_{n_2} \cdots \ast Z_{n_2}) \ast \cdots \ast (Z_{n_l} \ast Z_{n_l} \cdots \ast Z_{n_l}) \text{ where } n_1 \neq n_2 \neq \cdots \neq n_l
\]

except the case \( n_1 = 2, n_l = \infty \). From (7) we also came to know about \( Q \) \( (Z_n \ast Z_n \ast \cdots \ast Z_n) \) except the case \( n = 4 \). We will discuss the \( n = 4 \) case in a forthcoming article.

### 4 Computations for direct product of cyclic groups

In this section we will give a necessary and sufficient condition on groups such that \( Q(\Gamma) \) will be a commutative \( C^* \)-algebra.

**Theorem 4.1** If \( \Gamma = (Z_n \times Z_n \times \cdots \times Z_n) \) where \( n \neq 2, 4 \) then \( Q(\Gamma) \) will be \( C(ISO(\Gamma)) \).

**Proof:**

For simplicity we present the case \( k = 2 \). The general case will follow by using the similar arguments. Write the fundamental unitary as
$$U = \left( \begin{array}{cccc}
A & B & C & D \\
B^* & A^* & D^* & C^* \\
E & F & G & H \\
F^* & E^* & H^* & G^* 
\end{array} \right) = ((u_{ij}))$$

We will prove the result by the help of few lemmas.

**Lemma 4.2** $AB = BA = CD = DC = EF = FE = GH = HG = 0$.

**Proof:**
Using the relation $\alpha(\lambda_a)\alpha(\lambda_{a-1}) = \alpha(\lambda_{a-1})\alpha(\lambda_a) = \lambda_c \otimes 1_Q$, comparing the coefficients of $\lambda_{a^2}, \lambda_{a^2-1}, \lambda_{a^2}, \lambda_{a^2}$ we have, $AB^* = BA^* = A^*B = CD^* = DC^* = C^*D = D^*C = 0$.

Applying the antipode on these previous relations one can deduce, $AB = BA = EF = FE = 0$.

Now following the same argument using the condition $\alpha(\lambda_b)\alpha(\lambda_{b-1}) = \alpha(\lambda_{b-1})\alpha(\lambda_b) = \lambda_c \otimes 1_Q$ we get, $CD = DC = GH = HG = 0$.\[]

**Lemma 4.3** Product of any two different elements of each row and column of the unitary $U$ is zero. i.e. $u_{ij}u_{ik} = 0 \forall j \neq k$ and $u_{ij}u_{ki} = 0 \forall j \neq k$.

**Proof:**
Using the fact $ab = ba$ we have $\alpha(\lambda_{ab}) = \alpha(\lambda_{ba})$ and from this we get, $AE = EA$ comparing the coefficient of $\lambda_{a^2}$ on both sides. Hence $C^*A^* = A^*C^*$, this shows that $AC = CA$ (applying $\kappa$ and then taking adjoint). Further using $\alpha(\lambda_a)\alpha(\lambda_{a-1}) = \lambda_c \otimes 1_Q$ one can get, $AC^* + DB^* = 0$ comparing the coefficient of $\lambda_{ab-1}$ on both sides of the above equation. So $AC^*A^* = 0$ as $B^*A^* = 0$. Now $(AC)(AC)^* = (AC)(C^*A^*) = (CA)(C^*A^*) = C(AC^*A^*) = 0$ as $AC = CA$ and $AC^*A^* = 0$. Thus, we have proved that $AC = CA = 0$ and $AE = EA = 0$ (taking $\kappa$ and $\ast$). Again using the fact $\alpha(\lambda_{ab-1}) = \alpha(\lambda_{ab-1}) (ab = ba$ this shows that $ab^{-1} = b^{-1}a$) and comparing the coefficient of $\lambda_{a^2}$ we get, $AF^* = F^*A$ this gives $AD = DA$ taking $\kappa$ and $\ast$. We next obtain, $AD^* + CB^* = 0$ comparing the coefficient of $\lambda_{ab}$ on both sides of the equation $\alpha(\lambda_{ab})\alpha(\lambda_{ab-1}) = \lambda_c \otimes 1_Q$, which implies $AD^*A^* = 0$ (as $B^*A^* = 0$).

Now $(AD)(AD)^* = (AD)(D^*A^*) = (DA)(D^*A^*) = D(AD^*A^*) = 0$ using $AD = DA$ and $AD^*A^* = 0$, So we have proved that $AD = DA = 0$ and $AF^* = F^*A = 0$ (taking $\kappa$ and $\ast$).

Next we show that $BD = DB = BC = CB = 0$. From $\alpha(\lambda_{ab}) = \alpha(\lambda_{ba})$ and comparing the coefficient of $\lambda_{a-2}$ we deduce, $BF = FB$ this gives $DB = BD$ (taking $\kappa$), Again $BD^* + CA^* = 0$ hence $BD^*B^* = 0$. Then $(BD)(BD)^* = D(BD^*B^*) = 0$ as $BD = DB$ and $BD^*B^* = 0$. So we get that $BD = DB = 0$. Similarly we have $BC = CB = 0$.

We also obtain similar relations replacing $A, B, C, D$ by $E, F, G, H$ respectively.\[]
Lemma 4.4 All the entries of $U$ are normal and partial isometries. i.e. $u_{ij} u_{ij}^* = u_{ij}^* u_{ij} = u_{ij}^* u_{ij} = u_{ij} \forall i, j$.

Proof: The unitary of $U$ gives us $AA^* + BB^* + CC^* + DD^* = 1$ and $A^*A + B^*B + C^*C + D^*D = 1$. Using lemma 4.3 we have $A^*A^2 = A$ and $A(A^*)^2 = A^*$. Now $AA^* = A^*A^2A^* = A^*A$, i.e. $A$ is a normal element. We can prove normality of the other elements by exactly same argument. Now we are going to show that all of them are partial isometries. We already know that $AB^* = 0$. We will show now $AC^* = AD^* = 0$, then with the help of the equation $A^*A + B^*B + C^*C + D^*D = 1$, multiplying A on the left side it’ll follow that $A$ is a partial isometry. Note that $(AC^*)(AC^*)^* = AC^*CA^* = ACC^*A^* = 0$, hence $AC^* = 0$. By the same argument $AD^* = 0$.

So A is a normal partial isometry. Similar arguments will work for the others. \Box

Lemma 4.5 \{A, B\} commute with \{G, H, G^*, H^*\}. \{E, F\} commute with \{C, D, C^*, D^*\}.

Proof: From the relation $\alpha(\lambda_{ab}) = \alpha(\lambda_{ba})$ we get that $AG + CE = GA + EC$ comparing the coefficient of $\lambda_{ab}$ on both sides. This gives $A^2G + ACE = AGA + AEC$ multiplying A on the right side, hence $A^2G = AGA$ as $AC = AE = 0$. Now multiply A on the right side of the same equation to get $AGA = GA^2$. So we get $AGA = GA^2 = A^2G$ i.e. G commutes with $A^2$. Now $A^*A^2G = A^*(GA^2) = A^*AGA$ by multiplying A on the left of the above equation. Thus, $AG = (A^*)^2(A^2)GA$ as $A^*A^2 = A$ and using the fact that A is a normal partial isometry. Now $AG = G(A^*)^2(A^2)A = GA^*A^2 = GA$ as G commutes with $A^2$ and $(A^*)^2$. So we get that $AG = GA$. We obtain the remaining relations in a similar way. \Box

By the above lemmas together with Proposition 2.18 the result is completed.

Remark 4.6 From the proof it is easily seen that we can take also $n = \infty$. For finite $n$ we get one extra relation $u_{ij}^* = u_{ij}^{n-1}$ where $((u_{ij}))$ denotes the fundamental unitary. A close look at the proof will reveal that the fact $a_1^2, a_1^{-2}, a_2^2, a_2^{-2}, \ldots, a_k^2, a_k^{-2}$ are different elements plays a crucial role here. For this reason it does not work for the cases $n = 2, 4$.

Theorem 4.7 Let $\Gamma = \Gamma_1 \times \Gamma_2 \cdots \Gamma_l$ where $\Gamma_i = (\underbrace{\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_1} \cdots \times \mathbb{Z}_{n_1}}_{k_1 \text{ copies}}) \otimes^{\text{max}} C(\underbrace{\text{ISO} (\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_1} \cdots \times \mathbb{Z}_{n_1})}_{k_1 \text{ copies}})$ \otimes^{\text{max}} \cdots \otimes^{\text{max}} C(\underbrace{\text{ISO} (\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_1} \cdots \times \mathbb{Z}_{n_1})}_{k_1 \text{ copies}})$ \otimes^{\text{max}} \cdots \otimes^{\text{max}} C(\underbrace{\text{ISO} (\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_1} \cdots \times \mathbb{Z}_{n_1})}_{k_1 \text{ copies}})$, i.e. $Q(\Gamma) \cong Q(\Gamma_1) \otimes^{\text{max}} Q(\Gamma_2) \otimes^{\text{max}} \cdots \otimes^{\text{max}} Q(\Gamma_l)$. As all the pieces are commutative then $Q(\Gamma)$ is commutative in this case.
Proof:
As before we give the proof for the case where all \( k_i = 1 \) to avoid notational problem. Now \( \Gamma = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_l} \) where \( a_1, a_2, \cdots a_l \) are the standard generating sets for direct product of cyclic groups, \( o(a_i) = n_i \).

Fundamental co-representation is of the form

\[
U = \begin{pmatrix}
A_{11} & A_{12} & A_{13} & A_{14} & \cdots & A_{1(2l-1)} & A_{1(2l)} \\
A^*_{12} & A^*_{11} & A_{14} & A_{13} & \cdots & A^*_{1(2l-1)} & A^*_{1(2l-1)} \\
A_{21} & A_{22} & A_{23} & A_{24} & \cdots & A_{2(2l-1)} & A_{2(2l)} \\
A^*_{22} & A^*_{21} & A_{24} & A_{23} & \cdots & A^*_{2(2l-1)} & A^*_{2(2l-1)} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
A_{l1} & A_{l2} & A_{l3} & A_{l4} & \cdots & A_{l(2l-1)} & A_{l(2l)} \\
A^*_{l2} & A^*_{l1} & A_{l4} & A_{l3} & \cdots & A^*_{l(2l-1)} & A^*_{l(2l-1)} \\
\end{pmatrix}.
\]

First we want to reduce it to a block diagonal form. By remark \[2.13\] this will complete the proof.

Here we use the Lemma \[4.3\] of Theorem \[4.1\]. Thus we can get,

\[
\alpha(\lambda_{a_1^{-1}}) = \lambda_{a_1^{-1}} \otimes A_{11}^{\alpha^{-1}} + \lambda_{a_2^{-1}} \otimes A_{12}^{\alpha^{-1}} + \lambda_{a_3^{-1}} \otimes A_{13}^{\alpha^{-1}} + \lambda_{a_4^{-1}} \otimes A_{14}^{\alpha^{-1}} + \lambda_{a_5^{-1}} \otimes A_{15}^{\alpha^{-1}} + \cdots + \lambda_{a_l^{-1}} \otimes A_{1l}^{\alpha^{-1}}
\]

(As \( A_{1i} = A_{1j} = 0 \forall i \neq j \)). Now using the relation \( \alpha(\lambda_{a_i^{-1}}) = \alpha(\lambda_{a_i^{-1}}) \) we get, \( A^*_{12} = A_{12}^{\alpha^{-1}}, A^*_{11} = A_{11}^{\alpha^{-1}}, A^*_{1i} = 0 \forall i > 2 \) by comparing the coefficients of \( \lambda_{a_1}, \lambda_{a_2}, \lambda_{a_3}, \lambda_{a_4}, \cdots \lambda_{a_l} \) on both sides of \( \alpha(\lambda_{a_i^{-1}}) = \alpha(\lambda_{a_i^{-1}}) \). Applying the antipode we have \( A_{11} = A_{22} = 0 \forall i > 1 \). Now the fundamental unitary is of the form

\[
\begin{pmatrix}
A_{11} & A_{12} & 0 & 0 & \cdots & 0 & 0 \\
A^*_{12} & A^*_{11} & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & A_{23} & A_{24} & \cdots & A_{2(2l-1)} & A_{2(2l)} \\
0 & 0 & A^*_{24} & A^*_{23} & \cdots & A^*_{2(2l-1)} & A^*_{2(2l-1)} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & A_{l3} & A_{l4} & \cdots & A_{l(2l-1)} & A_{l(2l)} \\
0 & 0 & A^*_{l4} & A^*_{l3} & \cdots & A^*_{l(2l-1)} & A^*_{l(2l-1)} \\
\end{pmatrix}
\]

Proceeding in a similar way we get the desired block diagonal form. Remember the last relation \( \alpha(\lambda_{a_i^{-1}}) = \alpha(\lambda_{a_i^{-1}}) \) will not be needed. For this reason we can include the case \( n_l = \infty \). \( \square \)

**Theorem 4.8** The conclusion of Theorem \[4.7\] remain valid if \( n_1 = 2 \) where \( 2 < n_2 < n_3 \cdots < n_l \leq \infty \) and \( n_i \neq 4 \forall i \).

Proof:
For simplicity consider \( k_i = 1 \forall i \) so that \( \Gamma = \mathbb{Z}_2 \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_l} \) and
We only show how to reduce the fundamental unitary to the form given below, rest of the arguments are exactly same as of Theorem 4.7.

Note that product of any two different elements of each row and column is zero (except 1st row and 1st column) following the lines of arguments of Lemma 4.3. First we show that $A_{kl} = 0 \ \forall \ k > 1$. Using the condition $\alpha(\lambda_k \alpha(\lambda_k^{-1})) = \lambda_k \otimes 1_A \ \forall \ k > 1$, one can deduce $A_{k1} + A_{k2} + \cdots + A_{k(2l-1)} = 1$, also we know that $A_{k1} = 0 \ \forall \ k > 1$. Again we have $A_{kl} = 0 \ \forall \ k > 1$ considering the co-efficient of $\lambda_k$ in the expression of $\alpha(\lambda_k)$. Thus, $A_{k1} + A_{k2} + \cdots + A_{k(2l-1)} = 1$, $A_{k1} = 0 \ \forall \ k > 1$, hence $A_{12} = A_{13} = A_{14} = A_{15} = \cdots A_{l(2l-2)} = A_{l(2l-1)} = 0$ by applying the antipode. This gives one-step reduction of the fundamental unitary to the following form

$$U = \begin{pmatrix}
A_{11} & 0 & 0 & \cdots & 0 & 0 \\
0 & A_{22} & A_{23} & \cdots & A_{2(l-2)} & A_{2(l-1)} \\
0 & A_{23}^* & A_{22}^* & \cdots & A_{2(l-1)}^* & A_{2(l-2)}^* \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & A_{12} & A_{13} & \cdots & A_{l(2l-2)} & A_{l(2l-1)} \\
0 & A_{13}^* & A_{12}^* & \cdots & A_{l(2l-1)}^* & A_{l(2l-2)}^*
\end{pmatrix}$$

Then we proceed similarly to achieve the reduction. □

**Remark 4.9** Notice that unlike the free case, the above proof even works for the case when $o(a_1) = \infty$.

**Theorem 4.10** The conclusion of Theorem 4.7 is valid for $n_1 = 4$ where $4 \neq n_2 \neq \cdots \neq n_l$ and $n_i \neq 2 \ \forall \ i$. However, in this case $Q(\Gamma)$ is non-commutative.

**Proof:**
As before assume without loss of generality all $k_i = 1$ and $o(a_1) = 4$. Here the product of any two different elements of each row and column is zero (except the 1st and 2nd rows and columns) by using the arguments of Lemma 4.3. So all entries except the 1st and 2nd rows and columns are normal. Now our claim
is that $A_{k1} = A_{k2} = 0 \forall k > 1$. If we consider the term $\alpha(\lambda_{a1}^k) \forall k > 1$ then we have, $(A_{k1}^2 + A_{k2}^2) A_{k1} = (A_{k1}^2 + A_{k2}^2) A_{k2} = 0$, comparing the coefficients of $\lambda_{a1}, \lambda_{a1}^{-1}$ in the expression of $\alpha(\lambda_{a1}^k)$. This gives $A_{k1}^2 = A_{k2}^2 = 0$, hence $A_{k1} = A_{k2} = 0 \forall k > 1$, as they are normal. Applying antipode we deduce $A_{1k} = 0 \forall k > 2$.

Rest of the proof is very similar to the Theorem 4.7 hence omitted. \qed

Remark 4.11 Combining the above theorems and propositions 4.7 and 4.8 we get the following necessary and sufficient condition for $Q(\Gamma)$ to be commutative $C^*$ algebra, If $Q(\Gamma)$ is commutative then $\Gamma$ must be of the form $(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_1} \times \ldots \times \mathbb{Z}_{n_1})$ $k_1$ copies $\times (\mathbb{Z}_{n_2} \times \mathbb{Z}_{n_2} \times \ldots \times \mathbb{Z}_{n_2})$ $k_2$ copies $\times \ldots \times (\mathbb{Z}_{n_l} \times \mathbb{Z}_{n_l} \times \ldots \times \mathbb{Z}_{n_l})$ $k_l$ copies where $n_i \neq 4 \forall i$ and if $n_j = 2$ for some $j$ then $k_j$ must be 1.

5 Examples of groups where QISO is doubling of the group algebra

We already observe in (29) if there exists a non trivial automorphism of order 2 which preserves the generating set, then doubling of that group algebra (29, 20) will be always a quantum subgroup of $Q(\Gamma)$. In (14, 29, 30) the authors could show that $Q(\Gamma)$ coincides with doubled group algebra for some examples. In this section we produce more examples of groups where this occurs.

5.1 Dihedral groups with two different generating sets

Dihedral group has two presentations

$D_{2n} = < a, b | a^2 = b^n = e, ab = b^{-1}a > \cdot \cdot \cdot (1)$

$D_{2n} = < s, t | s^2 = t^2 = (st)^n = e > \cdot \cdot \cdot (2)$

where $e$ denotes the identity element of the group. In (30) the authors calculated for $D_{2(2n+1)}$ taken as the presentation (1). Let us give a proof for $D_{2n}$ with presentation (2).

Theorem 5.1 Let $D_{2n} = < s, t | s^2 = t^2 = (st)^n = e >$, then its QISO will be doubling of the group algebra.

Proof:

The action is defined by,

$$\alpha(\lambda_s) = \lambda_s \otimes A + \lambda_t \otimes B,$$
\[ \alpha(\lambda_t) = \lambda_s \otimes C + \lambda_t \otimes D. \]

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\]

be the corresponding unitary corepresentation.

Here we present the case for odd \( n = 2k + 1 \), the proof is almost same for even \( n \).

We note the simple observation that given \( s^2 = t^2 = e \), the condition \((st)^n = e\) is equivalent to \((st)^k s = (ts)^k t\). From the relation \( \alpha(\lambda_s) = \alpha(\lambda_{s^{-1}}) \) we have \( A = A^*, B = B^* \).

Again applying \( \alpha(\lambda_{s^2}) = \lambda_e \otimes 1 \) we get \( A^2 + B^2 = 1, AB = BA = 0 \).

Similarly \( C = C^*, D = D^* \) and \( C^2 + D^2 = 1, CD = DC = 0 \).

Applying the antipode of the above equations we find that \( A^2 + C^2 = 1, AC = C A = 0, B^2 + D^2 = 1, BD = DB = 0 \).

So we obtain \( A^2 = D^2, B^2 = C^2 \) and clearly \( A^2, B^2, C^2, D^2 \) are central projections, i.e. they commute with all elements of \( C^* \{A, B, C, D\} \).

Now we are going to use relation \((st)^n = (st)^{2k+1} = e\), from this we deduce, \((st)^k s = (ts)^k t \ldots (1)\)

We want to show that \((A, D)\) and \((B, C)\) satisfy the same relation like \( s, t \).

Using relation (1) we have \((AD)^k A + (BC)^k B = (DA)^k D + (CB)^k C\).

This gives \((AD)^k A + (BC)^k C = (DA)^k D + (BC)^k B\) applying \( \kappa \) on both sides. From both the equations we have \((AD)^k A = (DA)^k D, (CB)^k C = (BC)^k B\).

Now using the Lemma 2.14 we see that \( \mathbb{Q}(D_{2n}) \) coincides with doubling of the group algebra corresponding to the automorphism given by \( \theta(s) = t, \theta(t) = s \).

\[ \square \]

Remark 5.2 We get the same result for \( D_{2n} \) with presentation 1 except \( n = 4 \) case. The proof is almost similar to \( \mathbb{Z}_2 \otimes \mathbb{Z}_{2n} \), hence omitted. This extends the result of [30].

5.2 Baumslag-Solitar group

The group has the presentation \( \Gamma = \langle a, b | b^{-1}ab = a^2 \rangle \).

First we deduce some relations among the generators.

\begin{align*}
    b^{-1}ab & = a^2, \quad b^{-1}a^{-1}b = a^{-2}, \\
    ab & = ba^2, \quad b^{-1}a = a^2b^{-1}, \quad a^{-1}b^{-1}a = ab^{-1}, \\
    b^{-1}a^{-1} & = a^{-2}b^{-1}, \quad a^{-1}b = ba^{-2}, \quad ab = ba^2, \quad aba^{-1} = ba.
\end{align*}

Write the fundamental unitary as

\[
\begin{pmatrix}
A & B & C & D \\
B^* & A^* & D^* & C^* \\
E & F & G & H \\
F^* & E^* & H^* & G^*
\end{pmatrix}
\]

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Our aim is to show $C = D = E = F = 0$.

Using relation $\alpha(\lambda_{ab}) = \alpha(\lambda_{ba})$, in L.H.S there are terms $\lambda_{b2} \otimes (CG)$, $\lambda_{b-2} \otimes (DH)$ but there are no terms $\lambda_{b2} \otimes (.)$ and $\lambda_{b-2} \otimes (.)$ in R.H.S.

So we must have $CG = DH = 0$, applying antipode $EG = HF = 0$.

Further using the condition $\alpha(\lambda_{a-1b-1}) = \alpha(\lambda_{ab})$ comparing both sides, $CH^* = DG^* = 0$; thus $EH = GF = 0$.

So $E$ is reduced to the form

\[
\begin{pmatrix}
A & B & 0 & 0 \\
B^* & A^* & 0 & 0 \\
0 & 0 & G & H \\
0 & 0 & H^* & G^*
\end{pmatrix}
\]

Moreover, we claim that $H = 0$. Again using $\alpha(\lambda_{ab}) = \alpha(\lambda_{ba})$ we have $AH = BH = 0$, equating the coefficients of $\lambda_{ab}, \lambda_{a-1b-1}$ on both sides, $(A^*A + B^*B)H = H$, hence $H = 0$.

Final form of the fundamental unitary is

\[
\begin{pmatrix}
A & B & 0 & 0 \\
B^* & A^* & 0 & 0 \\
0 & 0 & G & 0 \\
0 & 0 & 0 & G^*
\end{pmatrix}
\]

Using the relations between the generators we deduce that $G^*AG = A^2$, $G^*BG = B^2$, $AG = GA^2$, $BG = GB^2$.

Also we find that $AA^*, BB^*$ are central projections of the algebra.

Thus, applying same argument like the previous one it is again doubling of the group algebra with respect to the automorphism given by $a \mapsto a^{-1}, b \mapsto b$. $\blacksquare$

### 5.3 Some groups of the form $< a, b | o(a) = 2, o(b) = 3 >$

First we conclude a lemma which will be useful for the proof of the result.

**Lemma 5.3** If $\Gamma = < a, b | o(a) = n, o(b) = 3 >$ where $n \neq 3$, generating set is minimal for $\Gamma$ then its unitary co-representation must be of the form
\[
\begin{pmatrix}
A & 0 & 0 \\
0 & E & F \\
0 & F^* & E^*
\end{pmatrix}
\]
when \(n = 2\).

If \(n > 3\) then its form will be
\[
\begin{pmatrix}
A & B & 0 & 0 \\
B^* & A^* & 0 & 0 \\
0 & 0 & G & H \\
0 & 0 & H^* & G^*
\end{pmatrix}
\]

**Proof:**

Using the relation \(\alpha(\lambda b^2) = \alpha(\lambda b^{-1})\), and comparing the coefficients of \(\lambda a\) and \(\lambda b^{-1}\) from both sides we will get the reduced block diagonal form. \(\square\)

Now we use the above lemma to compute quantum isometry groups of some concrete examples.

**Theorem 5.4** The quantum isometry group of \(\Gamma\) is doubling of the group algebra for the following three examples of groups generated by \(a, b\) satisfying the conditions of 5.3:

1. \((ab)^3 = 1\) (Tetrahedral)
2. \((ab)^4 = 1\) (Octahedral)
3. \((ab)^5 = 1\) (Icosahedral)

**Proof:**

In each of these cases, we can apply Lemma 5.3 to get \(A, E, F\) such that the action \(\alpha\) is given by,

\[
\alpha(\lambda a) = \lambda a \otimes A,
\alpha(\lambda b) = \lambda b \otimes E + \lambda^{-1} \otimes F,
\alpha(\lambda b^{-1}) = \lambda b \otimes F^* + \lambda^{-1} \otimes E^*.
\]

Also \(A^2 = 1, A = A^*\), applying \(\alpha(\lambda a) = \alpha(\lambda a^{-1})\) and \(\alpha(\lambda b^2) = \lambda e \otimes 1_Q\).

Similarly \(E^2 = E^*, F^2 = F^*\) using the condition \(\alpha(\lambda b^2) = \alpha(\lambda b^{-1})\). We also get \(EF = FE = 0\) by arguments used in the proof of earlier computations.

Now consider the relation \((ab)^n = 1\) \((n = 3, 4, 5\) respectively), which gives \(ab = (b^{-1}a)^{n-1}\), and \((ba) = (ab^{-1})^{n-1}\). Using the above relations we deduce \(AE = (E^* A)(E^* A) \cdots (E^* A)\), \(EA = (AE^*)(AE^*) \cdots (AE^*)\). Now \(A(EE^*) = \)

(n-1) times

(n-1) times
\[(E^* A)(E^* A) \cdots (E^* A) = E^* (A E^*) (A E^*) \cdots (A E^*) = E^* (E A) = (E^* E) A = (EE^*) A.\]

So \(EE^*\) is a central projection. Applying the same trick it can be proved that \(FF^*\) is also central projection. The rest of the proof is very similar to the proofs for the dihedral or Baumslag-solitar group, hence omitted. □

6 Coxeter groups as examples of \(\Gamma\) such that \(Q(\Gamma) \cong C_r^*(\Gamma)\)

In this section we will compute QISO for certain classes of Coxeter groups. The Coxeter group with parameters \((l, m, n)\) has presentation \(\Gamma = < a, b, c | o(a) = o(b) = o(c) = 2, (ac)^l = (ab)^m = (bc)^n = e >\)

Here we take one special class, namely \(l = 2, m = 3, n \neq 2, 3, 6.\)

**Theorem 6.1** Let \(\Gamma\) be the coxeter group with the 3 generators as above. Then its QISO is isomorphic with \(C_r^*(\Gamma)\) itself.

**Proof:**
The fundamental unitary is of the form

\[
\begin{pmatrix}
  A & B & C \\
  D & E & F \\
  G & H & K
\end{pmatrix}
\]

We divide the proof into a number of lemmas.

**Lemma 6.2** \(B = D = H = F = 0.\)

**Proof:**
We know that \(\alpha(\lambda_{ac}) = \alpha(\lambda_{ca})\) as \((ac)^2 = e.\)

\[
\begin{align*}
\alpha(\lambda_{ac}) &= \lambda_c \otimes (AG + BH + CK) + \lambda_{ac} \otimes (AK + CG) + \lambda_{ab} \otimes AH + \lambda_{bc} \otimes BK + \lambda_{ba} \otimes BG + \lambda_{cb} \otimes CH, \\
\alpha(\lambda_{ca}) &= \lambda_c \otimes (GA + HB + KC) + \lambda_{ac} \otimes (KA + GC) + \lambda_{ab} \otimes GB + \lambda_{bc} \otimes HC + \lambda_{ba} \otimes HA + \lambda_{cb} \otimes KB.
\end{align*}
\]

From the above equations we have, \(AH = GB, BK = HC, HA = BG, CH = KB,\)
and as the action is length preserving, we also get \((AG + BH + CK) = (GA + HB + KC) = 0.\)

Using the condition \(\alpha(\lambda_{a^2}) = \alpha(\lambda_c)\) we obtain, \(DF + FD = 0\) comparing the coefficient of \(\lambda_{ac}\) on both sides, and hence, \(HB + BH = 0\) (applying \(\kappa\)).
This gives \(HBH + BH^2 = 0, HBH + H^2B = 0.\)
Again using $\alpha(\lambda_{a^2}) = \alpha(\lambda_a)$ we have, $G^2 + H^2 + K^2 = 1$, hence $BG^2 + BH^2 + BK^2 = B = G^2B + H^2B + K^2B$.

Now $(AG + BH + CK) = 0$ implies $HAG + HBH + HCK = 0$.

Thus $(BG)G + HBH + (BK)K = 0$ so $BG^2 - BH^2 + BK^2 = 0$.

Now comparing it with $BG^2 + BH^2 + BK^2 = B$ we deduce $B = 2BH^2$.

We will show now $BH^2 = 0$. From the equation $(AG + BH + CK) = 0$ we have, $AGH + BH^2 + CKH = 0$, where $GH = KH = 0$ using $\alpha(\lambda_{a^2}) = \alpha(\lambda_a)$ and comparing the coefficients of $\lambda_{ab}, \lambda_{bc}$ respectively.

So $BH^2 = 0$, hence $B = 0$, and $D = 0$ applying antipode.

From earlier relations $HA = HC = 0$. We also have $H = H(A^2 + C^2) = 0$ as $(A^2 + C^2) = 1$, similarly one can get $F = 0$. □

**Lemma 6.3** $G = C = 0$.

**Proof:**

First we need to derive some more relations among the generators from the defining ones.

We have $(ab)^3 = e$ this shows that $aba = bab$. So using $\alpha(\lambda_{bab}) = \alpha(\lambda_{aba})$ we obtain,

$\alpha(\lambda_{bab}) = \lambda_{bab} \otimes EAE + \lambda_{bcb} \otimes ECE$,

$\alpha(\lambda_{aba}) = \lambda_{aba} \otimes AEA + \lambda_{abc} \otimes AEC + \lambda_{cba} \otimes CEA + \lambda_{cbc} \otimes CEC$.

Now $bcb \neq aba$ as $aba = bab$, $abc \neq bcb$ as $(bc)^3 \neq e$, if $bcb = abc$ then $(abc)^2 = e$, so $abc = cba$. Hence we get $bcb = abc = cba$. This implies $(bc)^2 = ab$, thus $(bc)^6 = e$. Now we get the contradiction from the assumption on $n$, hence the term $bcb$ is not equal to any of the terms $aba, abc, cba, cbc$. Comparing the both sides of $\alpha(\lambda_{bab}) = \alpha(\lambda_{aba})$ we find, $ECE = 0$, thus $C = E^2CE^2 = E(ECE)E = 0$ as $E^2 = 1, ECE = 0$. Hence we get $G = 0$ applying $\kappa$. □

**Proof of Theorem 6.1**

By means of the above lemmas, we have reduced the fundamental unitary to the following form:

$$
\begin{pmatrix}
A & 0 & 0 \\
0 & E & 0 \\
0 & 0 & K
\end{pmatrix}
$$

So there is a morphism from $C_\tau^r(\Gamma)$ to $\mathbb{Q}(\Gamma)$ sending $\lambda_a, \lambda_b, \lambda_c$ to $A, E, K$ respectively. In fact it is easy to check that this gives the isomorphism.

**Remark 6.4** This is the first non trivial example (other one is $\mathbb{Z}_2$) where we see that $\mathbb{Q}(\Gamma)$ coincides with the group algebra.
7 An excursion to QISO of matrix quantum groups

7.1 The formulation

In this last section, we want to extend the formulation of quantum isometry group to the realm of quantum groups. Let us consider a compact matrix quantum group \((Q, \Delta)\) which has a finite (say \(n\)) dimensional unitary fundamental representation \(\pi\), say, with \(\{(\pi_{ij})\}\) be the corresponding unitary in \(M_n(Q)\). Indeed, by definition, every irreducible representation of \(Q\) is a sub-representation of tensor copies of \(\pi\) and \(\bar{\pi}\), so as in Subsection 2.6, we may consider a central length function \(l\) which takes an irreducible say \(\alpha\) to the smallest nonnegative integer \(k\) such that \(\alpha \subset \alpha_1 \otimes \cdots \otimes \alpha_k\) where each \(\alpha_i\) is either \(\pi\) or \(\bar{\pi}\). As shown in [10], this gives rise to a spectral triple, generalising the construction of \(D_{\Gamma}\) for a finitely generated discrete group in Subsection 2.1. Moreover, this spectral triple satisfies the condition of Theorem 2.6, hence the quantum isometry group exists. Let us denote the quantum isometry group by \(QISO(\hat{Q}, \{\pi_{ij}\})\), indeed, as in the group case, the action of \(QISO(\hat{Q})\), say \(\beta\), is determined by \(\beta_{ij}\) such that

\[
\beta(u_{ij}) = \sum_{kl} u_{kl} \otimes q_{kl}^i\]

In other words, the quantum isometry group is generated by \(q_{kl}^i\) subject to the relations that make it a unitary and also make the above \(\beta\) a *-homomorphism from \(Q\) to \(Q\otimes Q(\hat{Q})\).

As a first illustration, consider a compact (but possibly noncommutative) Lie group \(G\), viewed as a matrix group with a finite dimensional fundamental unitary representation \(\{(u_{ij})\}\). Then it is possible to consider the corresponding \(Q(\hat{C}(G))\). Let us denote it by \(Q(\hat{G})\) for notational brevity. Using this fundamental unitary we get an embedding of \(G\) into \(U(n)\) (the group \(n \times n\) unitary matrices) and consider \(U(n) \subset C^{n^2}\) with the usual metric discussed in Subsection 2.5. In fact, this is given by \(d^2(u, v) := \text{Tr}((u-v)^*(u-v)) = \sum_{ij} |u_{ij} - v_{ij}|^2\), for \(u, v \in U(n)\). It is easy to see that the left action of \(U(n)\) is isometric, hence so is the \(G\)-action, viewing \(G\) as a subgroup of \(U(n)\). Now the action of \(Q(\hat{G})\) is clearly linear in the ‘coordinate functions’ \(g_{ij} := u_{ij}(g), \ g \in G\), and the arguments of the main theorem of Subsection 2.5 apply verbatim to give us the following:

**Theorem 7.1** The quantum group \(Q(\hat{G})\) is isomorphic with the quantum isometry group \(QISO(G, d)\) of the metric space \((G, d)\) discussed above, using the embedding \(G \subset U(n)\). Moreover, if \(G\) is connected, the above quantum isometry group is actually a group, i.e. \(C(ISO(G, d))\).

**Proof:**

We need to prove only the second statement. For this, note that the action of \(Q(\hat{G})\), being linear in the coordinates, is clearly a smooth action on \(C(G)\) in the
sense of [19]. But there is no genuine CQG acting faithfully and smoothly on a compact connected manifold by Corollary 11.10 of that paper, which completes the proof. □

It is interesting to note that the formulation for quantum isometry group for a matrix quantum group allows us to consider even the group algebras with a set of generators which are not necessarily of the form \( \chi_g \) for elements \( g \) of the group, i.e. not necessarily group-like elements of the group \( C^* \) algebra. This flexibility of choice can have quite interesting implications in the resulting quantum isometry groups, as illustrated by the example below. We consider the group \( \Gamma = \mathbb{Z} \times \mathbb{Z}_2 \). It has a natural set of generators consisting of group-like elements as in Theorem 4.8, where the resulting QISO turned out to be \( Q(\mathbb{Z}) \otimes Q(\mathbb{Z}_2) \). It can be identified as also the doubling of the group algebra. However, we can also view it as a matrix quantum group with a fundamental unitary whose entries are not group-like elements. More precisely, note that \( C^*_r(\mathbb{Z} \times \mathbb{Z}_2) \) is isomorphic with \( C(T) \oplus C(T) \) as a \( C^* \) algebra, and it can be described as the \( C^* \) algebra \( C^* \{\gamma, \gamma'\} \) where \( \gamma, \gamma' \) denotes the canonical generators of the two copies of \( C(T) \). They satisfy the following relations (and in fact the \( C^* \) algebra is the universal \( C^* \) algebra with two generators satisfying these relations):

\[
\gamma \cdot \gamma^* = \gamma^* \cdot \gamma, \quad \gamma' \gamma'^* = \gamma'^* \gamma',
\gamma \cdot \gamma' = \gamma' \gamma = 0,
\gamma \gamma^* + \gamma' \gamma'^* = 1.
\]

From the group structure of \( \Gamma \), it is easy to see that \( \mathcal{F} := \{\gamma, \gamma^*, \gamma', (\gamma')^*\} \) gives the matrix coefficient of a 2-dimensional fundamental unitary representation, not consisting of group-like elements. We have the following description of the quantum isometry group for the generating set \( \mathcal{F} \), which is again a doubling, but not of the group algebra itself.

**Theorem 7.2** \( Q(C^*_r(\mathbb{Z} \times \mathbb{Z}_2), \mathcal{F}) \) is isomorphic with the doubling of the quantum group \( Q(\mathbb{Z}) \ast Q(\mathbb{Z}) \).

**Proof:**

Suppose that corresponding to the action (say) \( \alpha \) on \( Q(\mathbb{Z}) \) the fundamental unitary is,

\[
\begin{pmatrix}
(a_{11}^{11})^* & a_{11}^{11} & a_{12}^{11} & a_{22}^{12}
(a_{21}^{11})^* & (a_{11}^{11})^* & (a_{22}^{12})^* & (a_{11}^{22})^*
(a_{21}^{21})^* & a_{21}^{21} & a_{22}^{22} & a_{12}^{22}
(a_{21}^{21})^* & (a_{11}^{21})^* & (a_{22}^{22})^* & (a_{12}^{22})^*
\end{pmatrix}.
\]

We break the proof into a number of steps.
Step 1: Using the condition $\alpha(\gamma\gamma^\ast) = \alpha(\gamma^\ast\gamma)$, comparing the coefficients of $\gamma^2, (\gamma^\ast)^2$ on both sides we find that $a_{11}^{11}(a_{11}^{11})^* + a_{21}^{11}(a_{21}^{11})^* = (a_{11}^{11})^*a_{11}^{11}$, $a_{21}^{11}(a_{11}^{11})^* = (a_{11}^{11})^*a_{21}^{11}$. Now we have $a_{11}^{11}(a_{11}^{11})^* + a_{21}^{11}(a_{21}^{11})^* = (a_{21}^{11})^*a_{21}^{11} + (a_{11}^{11})^*a_{11}^{11}$ by comparing the coefficient of $\gamma\gamma^*$ on both sides of the same relation, applying the antipode we obtain $a_{11}^{11}(a_{11}^{11})^* + (a_{21}^{11})^*a_{21}^{11} = a_{11}^{11}(a_{21}^{11})^* + (a_{11}^{11})^*a_{11}^{11}$. Thus using the above equations we can conclude that both the elements $a_{11}^{11}, a_{21}^{11}$ are normal. Hence the $C^*$ algebra $C^*\{a_{11}^{11}, a_{21}^{11}\}$ is commutative. Applying the same argument replacing $\gamma$ by $\gamma'$ one can deduce that $C^*\{a_{21}^{21}, a_{21}^{21}\}$ is commutative. Again from the fact $\alpha(\gamma\gamma^*) = \alpha(\gamma^*\gamma)$, comparing the coefficients of $(\gamma)^2$ and $(\gamma^*)^2$ on both sides we have $a_{11}^{12}(a_{12}^{12})^* = (a_{12}^{12})^*a_{11}^{12}, a_{21}^{12}(a_{12}^{12})^* = (a_{12}^{12})^*a_{21}^{12}$. Now comparing the coefficient of $\gamma\gamma^*$ of the same condition we get that $a_{12}^{12}(a_{12}^{12})^* + a_{22}^{12}(a_{22}^{12})^* = (a_{22}^{12})^*a_{12}^{12} + (a_{12}^{12})^*a_{22}^{12}$, applying $\kappa$ we have $a_{21}^{22}(a_{21}^{21})^* + (a_{21}^{21})^*a_{22}^{21} = (a_{21}^{21})^*a_{22}^{21} + (a_{21}^{21})^*a_{22}^{21}$ by comparing the coefficient of $\gamma\gamma^*$ on both sides from $\alpha(\gamma')\alpha(\gamma^*) = \alpha(\gamma^*)\alpha(\gamma')$. Hence, by the previous equation one can conclude that both the elements $a_{21}^{21}, a_{22}^{21}$ are normal. Using the antipode $a_{12}^{12}, a_{22}^{12}$ are normal. Thus, using the condition $a_{22}^{12}(a_{22}^{12})^* = (a_{22}^{12})^*a_{22}^{12}$ it is easily seen that the $C^*$ algebra generated by $\{a_{12}^{12}, a_{22}^{12}\}$ is commutative. So $C^*\{a_{21}^{21}, a_{22}^{21}\}$ is commutative by the help of antipode.

Step 2: From the given fact $\alpha(\gamma_\gamma') = 0$, comparing the coefficients of $\gamma^2$ and $(\gamma^*)^2$ we obtain $a_{11}^{11}a_{12}^{21} = a_{11}^{11}a_{22}^{21} = 0$. Now using the antipode and normality of the elements we get that $a_{11}^{11}a_{12}^{21} = a_{21}^{11}a_{22}^{21} = 0$. Applying the similar arguments from the relation $\alpha(\gamma\gamma^*) = 0$ one can easily check that $a_{12}^{12}a_{22}^{21} = a_{12}^{12}a_{21}^{21} = 0$. It can also be shown that $a_{22}^{12}a_{22}^{21} = a_{22}^{12}a_{22}^{21} = a_{22}^{22}a_{22}^{21} = a_{22}^{22}a_{22}^{21} = 0$ by following the same kind of arguments.

Step 3: Using the relation $\alpha(\gamma\gamma^* + \gamma'\gamma'^*) = 1 \otimes 1$ comparing the coefficient of $\gamma^2$ we have $a_{11}^{11}(a_{11}^{11})^* + a_{21}^{11}(a_{21}^{11})^* = 0$. Then we get that $a_{11}^{11}a_{11}^{11} = 0$ by multiplying $a_{11}^{11}$ on the right side (as $(a_{21}^{11})^*a_{11}^{11} = 0$). Hence $a_{11}^{11}(a_{11}^{11})^* = 0$, this shows that $a_{11}^{11}a_{11}^{11} = 0$, as they are normal elements and also $a_{21}^{11}a_{21}^{11} = 0$. Applying the same argument by comparing $(\gamma')^2$ from the fact $\alpha(\gamma\gamma^* + \gamma'\gamma'^*) = 1 \otimes 1$ one can show that $a_{11}^{11}a_{11}^{11} = a_{22}^{12}a_{22}^{21} = 0$.

Now using the above steps combining with unitary condition of the fundamental unitary the proof is completed. □

Remark 7.3 As $\mathbb{Q}(\mathbb{Z}) \ast \mathbb{Q}(\mathbb{Z})$ is noncommutative, it is clear that the quantum isometry group of $C^*_7(\mathbb{Z} \times \mathbb{Z}_2)$ with the new generating set $F$ differs from the previous one., calculated in Theorem 4.9 with group like elements. Moreover, $\mathbb{Q}(C^*_7(\mathbb{Z} \times \mathbb{Z}_2), F)$ can also be identified with $K_7^+$ (for more details see Section 5 of [2]). Moreover, we will describe another description of $K_7^+$ in the forthcoming article.
7.2 Cocycle twist

Given a dual unitary 2-cocycle on a CQG in the sense of [25], [28] and references therein, one can deform or twist the original CQG to get a new one, which has the same coalgebra structure but a new (twisted) algebra structure. A dual unitary 2-cocycle $\sigma$ on some quantum subgroup $S$ of a CQG $U$ induces a cocycle on $U$ (see [25] for more details) and we follow the convention of [25] to denote this induced 2-cocycle again by $\sigma$, with a slight abuse of notation. Moreover, given a spectral triple $(A, H, D)$ of compact type such that $\tilde{QISO}^+(A, H, D)$ exists, and a dual unitary 2 cocycle $\sigma$ on $QISO^+(A, H, D)$, a ‘deformed’ spectral triple $(A_\sigma, H, D)$ of compact type has been constructed in [25]. Moreover, the following is proved.

**Proposition 7.4** If $\tilde{QISO}^+(A_\sigma, H, D)$ also exists, we have,

$$QISO^+(A_\sigma, H, D) \cong (QISO^+(A, H, D))_\sigma.$$  

In the case of our interest, note that $Q$ is a quantum subgroup of $\hat{Q}$ as in the group case, hence any dual unitary 2-cocycle on $Q$ induces a cocycle on $Q(\hat{Q})$, and we can apply the deformation of spectral triple discussed above. With this, we conclude the following.

**Theorem 7.5** Given a 2-cocycle $\sigma$ on $Q$, we have

$$Q(\hat{Q}_\sigma) \cong (Q(\hat{Q}))_\sigma.$$  

**Proof:**

Indeed, to apply Proposition 7.4, we need to verify only that the deformed spectral triple admits existence of $\tilde{QISO}^+$. For this, it is enough to check the conditions of Theorem 7.4. But note that in this case, $H$ is the GNS space of $Q$ w.r.t. the Haar state, and 1 of $Q$ is the choice of the cyclic separating vector required in Theorem 7.4. But it is also the identity element of $Q_\sigma$ (see [25] and references therein), hence cyclic and separating for $Q_\sigma$ too, and it is easy to see the conditions of Theorem 2.6 do hold. $\Box$

Applying this to the special case of Rieffel-Wang deformation, and also combining with Theorem 7.4, we get the following corollary, the proof of which is quite straightforward and hence omitted.

**Corollary 7.6** Let $G$ be a compact connected Lie group with some even-dimensional toral subgroup and let $G_\theta$ be the Rieffel-Wang deformation in the sense of [33] for a suitable skew-symmetric matrix $\theta$. Then $Q(G_\theta) \cong K_\theta$, where $K := ISO(G, d)$ as in Theorem 7.1.

7.3 More general deformation

We now discuss an example of more general deformation by Drinfeld-Jimbo (DJ) type q-deformation which do not come from cocycles. We’ll take up the general theory for computing the quantum isometry groups of such DJ twisted group algebras elsewhere, and confine ourselves to the most well known yet very interesting concrete case of $SU_\mu(2)$, where $\mu = e^{-h}$ for a real parameter $h$.  

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We’ll use the notation and formalism of the book by Chari and Pressley (see, e.g. page 220-238 and page 435-436 of [14]). It is the universal unital algebra generated by $a, b, c, d$ subject to the relations:

\[ ab = \mu ba, bd = \mu db, ac = \mu ca, cd = \mu dc, bc = cb, ad - da + (\mu - \mu^{-1})bc = 0, ad - \mu bc = 1, \]

along with the involution (\*) given by \( a* = d, b* = -\mu c \).

Moreover, from the formulae for the Haar state given in page 456 of [17], we see that \( \sqrt{1 + \mu^2} \{ \mu a, b, \mu c, d \} \) is an orthonormal set.

Before we enter into the quantum case, let us discuss the classical case, i.e. \( \mu = 1 \), in some details. There is a natural action of \( SU(2) \times SU(2) \) on the manifold \( SU(2) \) (by diffeomorphism only, not group automorphisms) given by \( T_{U,V}(A) := UAV \), where \( U, V, A \in SU(2) \). The map \( (U, V) \mapsto T_{U,V} \in \text{Diff}(SU(2)) \) has the kernel consisting of \( \{ \pm (I, I) \} \cong \mathbb{Z}_2 \), and quotienting by this kernel we get a group \( G_1 = (SU(2) \times SU(2))/\mathbb{Z}_2 \). There is also a canonical map \( \tau : T \mapsto T' \) (\( t \) denotes the transpose) on \( SU(2) \) which corresponds to a \( \mathbb{Z}_2 \)-action, and let the group generated by \( G_1 \) and \( \tau \) in \( \text{Diff}(SU(2)) \) be \( G \). This is a semidirect product of \( G_1 \) by \( \mathbb{Z}_2 \) in the obvious way. Clearly, this is a subgroup of \( \hat{Q}(SU(2)) \), as the actions \( T_{U,V} \) and \( \tau \) map the span of the canonical matrix coordinates \( \{ a, b, c, d \} \) into itself. Moreover, using the natural identification of \( SU(2) \) with \( S^3 \subset \mathbb{R}^4 \), we can realize \( G \) as a subgroup of \( O(4, \mathbb{R}) \) and it can be proved by a dimension comparison that it actually coincides with \( O(4, \mathbb{R}) \), and in this identification \( G_1 \) corresponds to \( SO(4, \mathbb{R}) \). In other words, for \( \mu = 1 \), \( \hat{Q}(SU(2)) \) is isomorphic with \( C(O(4, \mathbb{R})) \).

In the quantum case, there is a natural analogue of the \( SU(2) \times SU(2) \) action, namely a natural action by \( SU_\mu(2) \otimes SU_\mu(2)^{op} \) on the \( C^* \) algebra of \( SU_\mu(2) \) which can also be viewed as a \( \mu \)-twisted quaternionian algebra as in [22], where \( SU_\mu(2)^{op} \) is same as \( SU_\mu(2) \) as a \( C^* \) algebra and coproduct is given by \( \Delta^{op} = \sigma \circ \Delta, \) \( \sigma \) is the flip map. In fact, this gives a quantum subgroup of \( \hat{Q}(SU_\mu(2)) \) isomorphic to \( SO_\mu(4, \mathbb{R}) \cong (SU_\mu(2) \times SU_\mu(2)^{op})/\mathbb{Z}_2 \) as in [22]. However, there is a problem to extend the analogue of \( \tau \) to the quantum case. Indeed, there is an order 2 automorphism \( \tau_\mu \) of the algebra of \( SU_\mu(2) \) without considering the \( * \)-structure, which keeps \( a, d \) fixed and interchanges \( b \) and \( c \), as the defining relations among the generators are satisfied by their images under \( \tau_\mu \). But \( \tau_\mu \) is not a \(*\)-homomorphism. Thus there is a ‘dimension-drop’ in some sense and we get

**Theorem 7.7** There is a sufficiently small \( r > 0 \) such that for all \( \mu \in (1 - r, 1 + r) \) we have

\[
\hat{Q}(SU_\mu(2)) \cong SO_\mu(4, \mathbb{R}) \cong (SU_\mu(2) \otimes SU_\mu(2)^{op})/\mathbb{Z}_2.
\]

We’ll publish elsewhere the details of the proof as a part of the general principle based on an adaptation of techniques and results of [21], allowing inhomogeneous or filtered quadratic algebras.

**Remark 7.8** The above result tells us that the Theorem 4.16 of [22] is not true for general deformation by twists not coming from cocycles.
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