The maximum mutual information between the output of a binary symmetric channel and a Boolean function of its input

Septimia Sarbu, Student Member, IEEE

Abstract

We prove the Courtade-Kumar conjecture, which states that the mutual information between any Boolean function of an \( n \)-dimensional vector of independent and identically distributed inputs to a memoryless binary symmetric channel and the corresponding vector of outputs is upper-bounded by \( 1 - H(p) \), where \( H(p) \) represents the binary entropy function. That is, let \( X = [X_1 \ldots X_n] \) be a vector of independent and identically distributed Bernoulli random variables, which are the input to a memoryless binary symmetric channel, with the error probability equal to \( 0 \leq p \leq \frac{1}{2} \), and \( Y = [Y_1 \ldots Y_n] \) the corresponding output. Let \( f : \{0,1\}^n \rightarrow \{0,1\} \) be an \( n \)-dimensional Boolean function. Then, \( \text{MI}(f(X), Y) \leq 1 - H(p) \). We provide the proof for the most general case of the conjecture, that is for any \( n \)-dimensional Boolean function \( f \) and for any value of the error probability of the binary symmetric channel, \( 0 \leq p \leq \frac{1}{2} \). Our proof employs only basic concepts from information theory, probability theory and transformations of random variables and vectors.

Index Terms

mutual information, Boolean functions, binary entropy function, memoryless binary symmetric channel

I. INTRODUCTION

BOOLEAN functions represent a fundamental mathematical formalism used to analyse and provide solutions to a wide range of problems in digital circuit design, theoretical computer science, logic, combinatorics, game theory, reliability theory, artificial intelligence, cryptography, coding theory [1]. More recently, Boolean networks have been successfully employed in the modelling and the analysis of complex biological systems, such as gene regulatory networks [2], [3]. In the effort to understand the organizational principles of such complex systems, several information-theoretic studies of Boolean networks have been carried out [4], [5], [6]. In information theory, a recent conjecture, termed the Courtade-Kumar conjecture, was stated in [7], involving the mutual information between any Boolean function of \( n \) independent and identically distributed inputs to a memoryless binary symmetric channel and the \( n \) outputs of the channel. Several proofs have appeared in the literature, for particular cases of this conjecture, but the most general case has remained unsolved. We prove the Courtade-Kumar conjecture [7], in the most general context, that is, for all Boolean functions and for all error probabilities of the memoryless binary symmetric channel, \( 0 \leq p \leq \frac{1}{2} \). As we have proven the conjecture, we state it as Theorem [1].

Our paper is structured as follows: we start the introductory section with our contributions, followed by the prior results that have been obtained so far in the literature, in the effort to solve the Courtade-Kumar conjecture. We also mention several generalizations of this conjecture. In the beginning of Section [II] we introduce the mathematical notation we used throughout this article. We continue this section with the description of the fundamental mathematical concepts from the hypothesis of this conjecture and the ones we used for its proof: the binary symmetric channel, the entropy of the Bernoulli random variable, the mutual information, concepts from probability theory and transformations of random variables. The essence of this paper, the proof of the Courtade-Kumar conjecture, is given in Section [III]. We present the conclusions of this study in Section [IV].

A. Our contributions

**Theorem 1:** Let \( X_i \) be a Bernoulli random variable, with the probability of success \( q_X = \frac{1}{2} \) and the input to a discrete memoryless binary symmetric channel, without feedback and with error probability \( 0 \leq p \leq \frac{1}{2} \). Let \( Y_i \) be the output of such a channel, when \( X_i \) is given as its input. Let \( X = [X_1 \ X_2 \ldots X_n] \) be an \( n \)-dimensional random vector of such \( X_i \) i.i.d. Bernoulli random variables and \( Y = [Y_1 \ Y_2 \ldots Y_n] \) the result of sending \( X \) through the binary symmetric channel. Let \( f : \{0,1\}^n \rightarrow \{0,1\} \) be an \( n \)-dimensional Boolean function. Let \( \text{MI}(f(X), Y) \) denote the mutual information between the result of the Boolean function of the input vector to the binary symmetric channel, \( X \), and its output, \( Y \). Let \( H(p) \) denote the binary entropy function. Then, the following inequality holds, for any \( n \)-dimensional Boolean function, \( f \), and any value of the error probability, \( 0 \leq p \leq \frac{1}{2} \):

\[
\text{MI}(f(X), Y) \leq 1 - H(p).
\]  

(1)

S. Sarbu is with the Department of Signal Processing, Tampere University of Technology, Tampere, Finland, sarbus.papers@gmail.com.
We give the proof our the Courtade-Kumar conjecture in section III. The novelty of our work is represented by the proof of
the Courtade-Kumar conjecture for the most general case, which has not been achieved so far. In addition, the proof employs
fundamental, straightforward mathematical techniques, such as information-theoretic equations, conditional probabilities and
transformations of random variables and vectors, whereas, the intermediate results in proving the Courtade-Kumar conjecture
and other related work, as in [8], [9], [6], [7], [10], [11], [12] and [13] make use of more complicated mathematical concepts, such
as the lemma by Wyner and Ziv [14], Fourier analysis of Boolean functions, the theory of lex functions and hypercontractivity.

B. Prior work related to the Courtade-Kumar conjecture

In the article [7], the authors introduce the Courtade-Kumar conjecture that gives the upper bound on the mutual information
between a Boolean function of a random vector of inputs to a memoryless binary symmetric channel and the vector of the
outputs. The mutual information is computed between a Boolean function of \(n\) independent and identically distributed (i.i.d.)
Bernoulli random variables, with success probability, \(q = \frac{1}{2}\), and the output of a memoryless binary symmetric channel, with
error probability, \(0 \leq p \leq \frac{1}{2}\), when this vector of Bernoulli random variables is passed as its input. The conjecture states that
this upper bound is equal to \(1 - H(p)\), where \(H(p)\) denotes the binary entropy function.

**Definition 1 (Courtade-Kumar conjecture [7]):** Let \(X = [X_1 \ X_2 \ \ldots \ X_n]\) be a vector of \(n\) i.i.d. Bernoulli random variables,
with success probability \(q = \frac{1}{2}\). Let \(Y = [Y_1 \ Y_2 \ \ldots \ Y_n]\) be the vector of outputs, when \(X\) is given as an input random vector
to a memoryless binary symmetric channel, with error probability \(0 \leq p \leq \frac{1}{2}\). Let \(f\) be an \(n\)-dimensional Boolean function,
\(f : \{0, 1\}^n \rightarrow \{0, 1\}\). Then, for any Boolean function \(f\) and any \(0 \leq p \leq \frac{1}{2}\), the following bound holds
\[
\text{MI}(f(X), Y) \leq 1 - H(p). \tag{2}
\]

As preliminary steps in proving the Courtade-Kumar conjecture, the authors state other conjectures and prove several weaker
theorems. In Theorem 1 of [7], they prove that, if \(f(X)\) is an equiprobable Boolean function, then the following inequality holds
\[
\sum_{i=1}^{n} \text{MI}(f(X), Y_i) \leq 1 - H(p). \tag{3}
\]

A Boolean function \(f\) is termed equiprobable, if the probability of the function being equal to 1 is equal to the probability of
the function being equal to 0, for any combination of the input values. Both of these probabilities are equal to \(\frac{1}{2}\), as the
function can have only two values, 0 and 1. Balanced functions, equiprobable functions and functions with the expectation
equal to \(\frac{1}{2}\) are equivalent descriptions for such Boolean functions.

The authors give two more conjectures, formulated as Conjecture 2 and 3, which, if proven, would facilitate the proof of
the Courtade-Kumar conjecture. In Definition 1 and 2, the authors introduce the concept of lex functions, which are a subset
of Boolean functions.

**Definition 2 (Conjecture 2 of [7]):** Let \(p_b\) be the bias of the Boolean function \(f\), that is, \(p_b = P(f(X) = 0)\). Given \(n\) and
\(p_b\), such that, \(H(p_b) \geq 1 - H(p)\), then, the functions, \(f\), which are lex, minimize the conditional entropy, \(H(f(X)|Y)\).

A subset of Boolean functions, \(S_n\), is defined in Theorem 3 and Corollary 1 of [7]. Here, the mutual information corresponding
to this subset is proven to be greater than the mutual information corresponding to any other Boolean function, with a given
number of inputs \(n\). For the functions in the subset \(S_n\), the authors have numerically verified that the Courtade-Kumar conjecture
and Conjecture 2 hold, for \(n \leq 7\).

**Definition 3 (Conjecture 3 of [7]):** \(H(p)\) denotes the binary entropy function and \(H(f(X))\) denotes the entropy of the random
variable \(f(X)\). The following inequality holds, for the subset of Boolean functions, termed lex functions,
\[
H(f(X)|Y) \geq H(f(X)) \cdot H(p). \tag{4}
\]
The authors numerically verify that Conjecture 3 holds, for all values of \(n\) and for all \(p\) in the interval \([0 \ \frac{1}{2}]\), using increments
of 0.001.

In [8], the author relates problems in financial investments to the rate-distortion theory and derives upper bounds on functions
describing such investments, which involve the maximization of the mutual information between various random variables
describing such processes. We would like to point out that this reference is incorrect in the articles [7], [11], where it is cited
as the best known bound on the mutual information under study in the Courtade-Kumar conjecture
\[
\text{MI}(f(X), Y) \leq (1 - 2 \cdot p^2). \tag{5}
\]
From a mathematical point of view, the problem studied in [Ch 3, Th. 3, Th. 4, Th. 5, [8]] is different from the one in the
Courtade-Kumar conjecture [7]. In [8], the mathematical model is a cascade of two binary symmetric channels that form a
Markov chain, whereas, in [11], it is a binary symmetric channel and a transformation of its input random vector by a Boolean
function. The author of [8] proves that the derivative of the maximum mutual information between the input to the first binary symmetric channel and the output of the last binary symmetric channel, subject to several constraints, as can be found in [Ch 3, Corollary 1, 39] is upper bounded by \((1 - 2 \cdot p)^2\), where \(1 - p\) is the error probability of the last channel. Unless a proof is presented that relates the mutual information from the Courtade-Kumar conjecture, to the one studied in [Ch 3, Th. 3, Th. 4, Th. 5, 38], we cannot draw the conclusion of 5. The results shown in the PhD dissertation [8] have been published in [15].

Using Fourier analysis for Boolean functions, the authors of [6] investigate the mutual information between a Boolean function \(f\) of \(n\) i.i.d. inputs, defined as \(X = [X_1 X_2 \ldots X_n]\) and one of the inputs, \(X_i\), that is \(\text{MI}(f(X), X_i)\). They show that this mutual information between a function \(f\) that produces an output with fixed mean, \(\mu = \mathbb{E}[f(X)]\), and one input variable, \(X_i\), is maximized, if the function \(f\) is canalizing in the variable \(X_i\). A canalizing \(n\)-dimensional function represents a Boolean function, for which, whenever one of the \(n\) input variables has a particular value, the output of the function will have a certain value, corresponding to this input, regardless of the combination of the values of the other \(n - 1\) input variables [6]. The authors of [6] prove this theorem in the case when the input binary vector \(X\) is uniformly distributed and in the case when it is product distributed, with some constraints on the canalizing input and the canalizing value of the function. If the mean \(\mu\) of the output produced by the function \(f\) is not fixed, then the dictatorship function is the maximizing function of this mutual information, in the case when the input binary vector \(X\) is uniformly distributed and in the case when it is product distributed. The dictatorship function is an \(n\)-dimensional Boolean function, such that \(f(X) = f(X_1, \ldots, X_n) = X_i\) or \(f(X) = f(X_1, \ldots, X_n) = X_i\). The authors also investigated the mutual information, \(\text{MI}(f(X), X_T)\), in the case of several inputs, defined as \(X_T = \{X_i : i \in T\}\), with \(|T| \leq n\), where the symbol \(|\cdot|\) denotes the cardinality of a set. They found that, when the input binary vector \(X\) is product distributed and the output of the function has a fixed expectation, \(\mu = \mathbb{E}[f(X)]\), the mutual information \(\text{MI}(f(X), X_T)\) is maximized when the function \(f\) is jointly canalizing in the set \(T\).

More recent results include several preprints. The authors of [11] employ Fourier analysis to prove the bound stated in Theorem 1, in the case of balanced or equiprobable Boolean functions and \(p\) in the range \(\frac{1}{2} \leq p \leq \frac{3}{4}\)

\[
\text{MI}(f(X), Y) \leq \frac{\log(e)}{2} \cdot (1 - 2 \cdot p)^2 + 9 \left(1 - \frac{\log(e)}{2}\right) \cdot (1 - 2 \cdot p)^4. \tag{6}
\]

In Corollary 1, the Courtade-Kumar conjecture is shown to hold for the dictatorship function, as a special case of equiprobable Boolean functions, when \(p \to \frac{1}{2}\). This region is termed the noise interval \(p \in [\frac{1}{2} - \frac{\sqrt{2}}{2} \frac{1}{2}]\), where \(\frac{1}{p}\) is defined as \(\frac{1}{p} = \frac{1}{2^n}\). In Theorem 1.14, the author of [12] extends this result of [11], to prove that the Courtade-Kumar conjecture holds for any balanced Boolean function, for any value of the probability \(p\) in the interval \(p \in [\frac{1}{2} - \frac{\sqrt{2}}{2} \frac{1}{2}]\), with the constant \(\delta > 0\). The author refers to \(p\) as the noise element and terms this interval as the region of high noise: \(0 \leq p \leq \frac{1}{2}\), such that \((1 - 2 \cdot p)^2 \leq \delta\), with the constant \(\delta > 0\). In addition, the author provides an improvement of Theorem 1 derived by Wyner and Ziv in [14], known as Mrs. Gerber’s Lemma. This original result of Wyner and Ziv has been employed in [8], for the proof of Theorem 4, which we mentioned in the above paragraphs as previous work on this topic. In the preprint [13], the result of [12] is extended, from balanced Boolean functions to all Boolean functions, under the same requirements of high noise. In [13], in Theorem 1.2, the Courtade-Kumar conjecture is proven to hold for any Boolean function, for any value of the probability \(p\) in the interval \(p \in [\frac{1}{2} - \frac{\sqrt{2}}{2} \frac{1}{2}]\), with the constant \(\delta > 0\). Similarly to [12], the author refers to \(p\) as the noise element and terms this interval as the region of high noise: \(0 \leq p \leq \frac{1}{2}\), such that \((1 - 2 \cdot p)^2 \leq \delta\), with the constant \(\delta > 0\).

We mention here several studies of generalizations of the Courtade-Kumar conjecture. An extension of the Courtade-Kumar conjecture to two \(n\)-dimensional Boolean functions, is hypothesized to hold in [9]. This conjecture states that, for any Boolean functions \(f, g : \{0, 1\}^n \to \{0, 1\}\), the mutual information \(\text{MI}(f(X), g(Y)) \leq 1 - H(p)\). For several specific cases of the joint probability mass function of the binary random variables \(f(X)\) and \(g(Y)\), the authors analytically prove another conjecture, termed Conjecture 4, which implies this generalization of the Courtade-Kumar conjecture. A similar form of Conjecture 4 of [9] is analytically proved in [10], in a more general context than that of the results of [9]. In section V of [10], the authors prove that the mutual information \(\text{MI}(B, \hat{B}) \leq 1 - H(p)\), for Boolean functions, \(B = f(X)\) and \(\hat{B} = g(Y)\), an estimator of \(Y\), with fixed mean \(\mathbb{E}(B) = \mathbb{E}(\hat{B}) = a\) and \(P(B = \hat{B} = 0) \geq a^2\). The Courtade-Kumar conjecture is generalized to continuous random variables in the preprint [15]. Here, the aim is to maximize \(\text{MI}(f(X), Y)\), where the function \(f\) takes as input \(n\)-dimensional real vectors and produces as output values from the set \(\{0, 1\}\). The authors investigate two cases: when \(X\) and \(Y\) are \(n\)-dimensional correlated Gaussian random vectors and when \(X\) and \(Y\) are correlated random vectors from the unit sphere.

II. MATHEMATICAL BACKGROUND

A. Mathematical notations and symbols

Throughout this article, we use the following mathematical notations and symbols:

- \(X_i\) denotes a discrete random variable, with ensemble \(\mathcal{E}_{X_i}\),
- \(X\) denotes a discrete \(n\)-dimensional random vector, \(X = [X_1 X_2 \ldots X_n]\), with ensemble \(\mathcal{E}_X\),
- \(p\) is the error probability of the binary symmetric channel,
The probability of error is denoted as $P_e$ without feedback: when the binary symmetric channel is used with consecutive inputs, $\mathbb{P}(X_i = 0)$ is the probability that the discrete random variable $X_i$ is equal to 0, $\mathbb{P}(Y_i = 0|X_i = 0)$ is the conditional probability that the discrete random variable $Y_i$ is equal to 0, given that the discrete random variable $X_i$ is equal to 0, $p_{X_i}(x_i)$ is the probability at the value $X_i = x_i$. We may omit the index $X_i$, $p(x_i)$ to refer to the same quantity. To avoid confusion, we use the index whenever probability mass functions for different random variables or vectors appear in the same derivations.

$p_{Y_i|X_i}(y_i|x_i)$ is the conditional probability at the value $Y_i = y_i$, given that $X_i = x_i$. Similarly, we use the index whenever conditional probability mass functions for different random variables or vectors appear in the same derivations.

$H(p)$ represents the binary entropy function.

The remainder of this section pertains to transformations of random variables and vectors. We present the definition of how to obtain the probability mass function of a random variable $Z$, which has been transformed by a unidimensional function, $f$.

**B. The binary symmetric channel**

*Definition 4 (Binary symmetric channel):* The binary symmetric channel is defined as having the input and output modelled as Bernoulli random variables with success probabilities, $q_X$ and $q_Y$: $X \in \mathcal{X} = \{0, 1\}, X \sim \text{Bernoulli}(q_X)$ and $Y \in \mathcal{Y} = \{0, 1\}, Y \sim \text{Bernoulli}(q_Y)$ [Ch 7 of [17]]. In our problem, $q_X = \frac{1}{2}$. The conditional probabilities describing the relationship between the input and output random variables are as follows:

\[
\begin{align*}
p_{Y|X}(0|0) &= \mathbb{P}(Y = 0|X = 0) = 1 - p \\
p_{Y|X}(0|1) &= \mathbb{P}(Y = 0|X = 1) = p \\
p_{Y|X}(1|0) &= \mathbb{P}(Y = 1|X = 0) = p \\
p_{Y|X}(1|1) &= \mathbb{P}(Y = 1|X = 1) = 1 - p.
\end{align*}
\]

The probability of error is denoted as $p$ and is in the range $0 \leq p \leq \frac{1}{2}$. This channel is characterized as memoryless and without feedback: when the binary symmetric channel is used with consecutive inputs, $\forall i = 1 : n$, it has no memory, that is $p(y_i|x_1, \ldots, x_i, y_1, \ldots, y_{i-1}) = p(y_i|x_i)$ and no feedback, that is $p(x_i|x_1, \ldots, x_{i-1}, y_1, \ldots, y_{i-1}) = p(x_i|x_1, \ldots, x_{i-1})$ [Ch 7 of [17]]. For completeness, in Appendix A using these two properties, we prove by induction the known result [Ch 7 of [17]] that

\[
\begin{align*}
p(y_{k+1}, y_k, \ldots, y_1|x_{k+1}, x_k, \ldots, x_1) &= \prod_{i=1}^{k+1} p(y_i|x_i), \forall k = 1, n - 1, \\
p(x_{k+1}, x_k, \ldots, x_1, y_{k+1}, y_k, \ldots, y_1) &= \prod_{i=1}^{k+1} p(x_i, y_i), \forall k = 1, n - 1.
\end{align*}
\]

**C. Probability theory and transformations of random variables**

Given two events, $A$ and $B$, the following fundamental results are known from probability theory [Ch 1 section 3 of [18]]:

\[
\begin{align*}
\mathbb{P}(A|B) &= \frac{\mathbb{P}(A, B)}{\mathbb{P}(B)} \\
\mathbb{P}(B|A) &= \frac{\mathbb{P}(A, B)}{\mathbb{P}(A)}.
\end{align*}
\]

Using these equations, we obtain the joint probability mass functions of the input and the output of the binary symmetric channel as:

\[
\begin{align*}
p_{XY}(0, 0) &= \mathbb{P}(X = 0, Y = 0) = \mathbb{P}(Y = 0|X = 0) \cdot \mathbb{P}(X = 0) = \frac{1}{2}(1 - p) \\
p_{XY}(0, 1) &= \mathbb{P}(X = 0, Y = 1) = \mathbb{P}(Y = 1|X = 0) \cdot \mathbb{P}(X = 0) = \frac{1}{2}p \\
p_{XY}(1, 0) &= \mathbb{P}(X = 1, Y = 0) = \mathbb{P}(Y = 0|X = 1) \cdot \mathbb{P}(X = 1) = \frac{1}{2}p \\
p_{XY}(1, 1) &= \mathbb{P}(X = 1, Y = 1) = \mathbb{P}(Y = 1|X = 1) \cdot \mathbb{P}(X = 1) = \frac{1}{2}(1 - p).
\end{align*}
\]
from another random variable $X$, that is $Z = f(X)$ [Ch 5, section 5 of \[13\]]. The probability mass function $p_X(x)$ is known. We also present the definition of how to obtain the joint probability mass function of a random variable $Z$ and the random vector $Y$, when $Z$ is any function of a random vector $X$, that is, $Z = f(X)$ [Ch 5, section 6 of \[13\]]. The joint probability mass function $p_{XZ}(x, z)$ is known.

Definition 5 (Probability mass function of transformations of random variables): Let $X, Z$ be two discrete random variables and $f$ be an $n$-dimensional function, such that $Z = f(X)$. Then, the probability mass function of $Z$, $p_Z(z)$, is obtained from the probability mass function of $X$, $p_X(x)$, as

$$p_Z(z) = \sum_{x \in \mathcal{X}, f(x)=z} p_X(x). \quad (11)$$

Definition 6 (Probability mass function of transformations of random vectors): Let $X$ be an $n$-dimensional discrete random vector, $X = [X_1 \ X_2 \ldots \ X_n]$, $Z$ a discrete random variable and $f$ be an $n$-dimensional function, such that $Z = f(X)$. Then, the probability mass function of $Z$, $p_Z(z)$, is obtained from the probability mass function of $X$, $p_X(x)$, as

$$p_Z(z) = \sum_{x \in \mathcal{X}, f(x)=z} p_X(x). \quad (12)$$

Let $X, Y$ be two $n$-dimensional discrete random vectors, $Z$ a discrete random variable and an $n$-dimensional function $f$, such that $Z = f(X)$. Let $T, U$ be two random vectors and $g$ be a multidimensional function, such that

$$g(X, Y) = \begin{bmatrix} g_1(X, Y) \\ g_2(X, Y) \\ g_3(X, Y) \end{bmatrix}, \quad \begin{bmatrix} T \\ U \\ Z \end{bmatrix} = \begin{bmatrix} g_1(X, Y) = Y \\ g_2(X, Y) = X \\ g_3(X, Y) = f(X) \end{bmatrix} \quad (13)$$

Then, the random vector $\begin{bmatrix} T \\ U \\ Z \end{bmatrix}$ is the transformed random vector $\begin{bmatrix} X \\ Y \end{bmatrix}$, by the function $g$. Its joint probability mass function is equal to

$$p_{TUZ}(t, u, z) = \sum_{x \in \mathcal{X}, y \in \mathcal{Y}, g_1(x, y) = t, g_2(x, y) = u, g_3(x, y) = z} p_{XY}(x, y) = p_Y(z).$$

Therefore, we have the following properties:

If $0 \leq x \leq \frac{1}{2}$ and $x \geq p \Rightarrow H(x) \geq H(p)$,\quad (16)

because the entropy function is an increasing function of its argument, on this interval.

If $\frac{1}{2} \leq x \leq 1$ and $x \leq 1 - p \Rightarrow H(x) \geq H(1 - p) = H(p)$,\quad (17)

because the entropy function is a decreasing function of its argument, on this interval.

E. The mutual information

Definition 7 (Mutual information): Let $X$ and $Y$ be two discrete random vectors, with the joint probability mass function denoted by $p_{XY}(x, y)$ and their marginal probability mass functions denoted by $p_X(x)$ and $p_Y(y)$. Then, the mutual information between $X$ and $Y$ is defined as \[17], \[19], \[20]\n
$$\text{MI}(X, Y) = \sum_x \sum_y p_{XY}(x, y) \cdot \log \frac{p_{XY}(x, y)}{p_X(x) \cdot p_Y(y)}. \quad (18)$$
III. PROOF OF THE COURTADE-KUMAR CONJECTURE [7]

We give the proof of the Courtade-Kumar conjecture [7], for the most general case. We stated our result as Theorem [1] in the section I-A.

Proof:

A. Part 1 of the proof:

We prove that $\text{MI}(Y, Z) = 1 - H(p)$, for two special cases: $Z = f(X_1, X_2, \ldots, X_n) = X_i$ and $Z = f(X_1, X_2, \ldots, X_n) = \overline{X_i}$, that is, for the dictatorship function.

1) Case 1: $Z = f(X_1, X_2, \ldots, X_n) = X_i$: The mutual information can be written as [Ch 2 [17]]

$$\text{MI}(Y, Z) = \text{MI}(Y_1, Y_2, \ldots, Y_n, Z) = H(Y_i) - H(Y_i|Y_1, \ldots, Y_{i-1}, Y_{i+1}, \ldots, Y_n, X_i).$$

We will now prove that $H(Y_i|Y_1, \ldots, Y_{i-1}, Y_{i+1}, \ldots, Y_n, X_i) = H(Y_i|X_i)$, which implies that

$$\text{MI}(Y, Z) = H(Y_i) - H(Y_i|X_i) = 1 - H(Y_i|X_i).$$

Then, we will prove that $H(Y_i|X_i) = H(p)$, yielding that

$$\text{MI}(Y, Z) = 1 - H(p).$$

For the simplicity of notation, let the random vector $W$ be equal to $W = [Y_1, Y_2, \ldots, Y_{i-1}, Y_{i+1}, \ldots, Y_n, X_i]$ and one of its values be $w = [y_1, y_2, y_{i+1}, \ldots, y_n, x_i]$. Then, the conditional entropy becomes $H(Y_i|Y_1, \ldots, Y_{i-1}, Y_{i+1}, \ldots, Y_n, X_i) = H(Y_i|W)$. The conditional entropy is equal to [Ch 2 [17]]

$$\begin{align*}
H(Y_i|X_i) &= - \sum_{x_i} \left[ p_{X_i}(x_i) \cdot \sum_{y_i} p_{Y_i|X_i}(y_i|x_i) \cdot \log p_{Y_i|X_i}(y_i|x_i) \right], \\
H(Y_i|W) &= - \sum_{w} \left[ p_{W}(w) \cdot \sum_{y_i} p_{Y_i|W}(y_i|w) \cdot \log p_{Y_i|W}(y_i|w) \right].
\end{align*}$$

$$p(y_i|w) = p(y_i|y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n, x_i) = \frac{p(x_i, y_1, \ldots, y_n)}{p(x_i, y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n)} = \frac{\sum_{x_1} \ldots \sum_{x_{i-1}} \sum_{x_{i+1}} \ldots \sum_{x_n} p(x, y)}{\sum_{x_1} \ldots \sum_{x_{i-1}} \sum_{x_{i+1}} \ldots \sum_{x_n} p(x, y)} = \frac{\sum_{x_1} \ldots \sum_{x_{i-1}} \sum_{x_{i+1}} \ldots \sum_{x_n} \prod_{k=1}^{n} p(x_k, y_k)}{\sum_{y_1} \ldots \sum_{y_{i-1}} \sum_{y_{i+1}} \ldots \sum_{y_n} \prod_{k=1}^{n} p(x_k, y_k)} = \frac{\prod_{k=1, k\neq i}^{n} p(y_k) \cdot p(x_i, y_i)}{\prod_{k=1, k\neq i}^{n} p(y_k) \cdot p(x_i)} = p(y_i|x_i), \quad (23)
$$

$$p_{w}(w) = \sum_{x_1} \ldots \sum_{x_{i-1}} \sum_{x_{i+1}} \ldots \sum_{x_n} p(x, y) = \sum_{y_1} \sum_{x_1} \ldots \sum_{x_{i-1}} \sum_{x_{i+1}} \ldots \sum_{x_n} \prod_{k=1}^{n} p(x_k, y_k) = \prod_{k=1, k\neq i}^{n} p(y_k) \cdot p(x_i). \quad (24)$$
Then, we will prove that
\[
H(Y_i | W) = - \sum_{w} \left[ p_W(w) \cdot \sum_{y_i} p_{Y_i|W}(y_i|w) \cdot \log p_{Y_i|W}(y_i|w) \right]
\]
\[
= - \sum_{x_i} \sum_{y_i} \sum_{y_{i-1} y_{i+1}} \ldots \sum_{y_n} \left[ \prod_{k=1, k \neq i}^{n} p(y_k) \cdot p(x_i) \cdot \sum_{y_i} p_{Y_i|X_i}(y_i|x_i) \cdot \log p_{Y_i|X_i}(y_i|x_i) \right]
\]
\[
= \left[ \sum_{y_i} \sum_{y_{i-1} y_{i+1}} \ldots \sum_{y_n} \prod_{k=1, k \neq i}^{n} p(y_k) \right] \cdot \left[ - \sum_{x_i} p(x_i) \cdot \sum_{y_i} p_{Y_i|X_i}(y_i|x_i) \cdot \log p_{Y_i|X_i}(y_i|x_i) \right]
\]
\[
= - \sum_{x_i} p(x_i) \cdot \sum_{y_i} p_{Y_i|X_i}(y_i|x_i) \cdot \log p_{Y_i|X_i}(y_i|x_i) = H(Y_i | X_i).
\] (25)

\[
H(Y_i | X_i) = - \sum_{x_i} \left[ p_X(x_i) \cdot \sum_{y_i} p_{Y_i|X_i}(y_i|x_i) \cdot \log p_{Y_i|X_i}(y_i|x_i) \right]
\]
\[
= - \sum_{x_i} \left\{ p_X(x_i) \cdot \left[ p_{Y_i|X_i}(0|x_i) \cdot \log p_{Y_i|X_i}(0|x_i) + p_{Y_i|X_i}(1|x_i) \cdot \log p_{Y_i|X_i}(1|x_i) \right] \right\}
\]
\[
= - p_X(0) \cdot \left[ (1 - p) \cdot \log (1 - p) + p \cdot \log p \right] - p_X(1) \cdot \left[ p \cdot \log p + (1 - p) \cdot \log (1 - p) \right]
\]
\[
= \frac{1}{2} \cdot 2 \cdot \left[ p \cdot \log p + (1 - p) \cdot \log (1 - p) \right]
\]
\[
= H(p)
\]
\[
\Rightarrow \text{MI}(Y, Z) = 1 - H(p).
\] (26)

2) Case 2: \(Z = f(X_1, X_2, \ldots, X_n) = \overline{X_i} \).

\[
\text{MI}(Y, Z) = \text{MI}(Y_1, Y_2, \ldots, Y_n, Z) = H(Y_i) - H(Y_i | Y_1, \ldots, Y_{i-1}, Y_{i+1}, \ldots, Y_n, \overline{X_i}).
\] (27)

We will now prove that \(H(Y_i | Y_1, \ldots, Y_{i-1}, Y_{i+1}, \ldots, Y_n, \overline{X_i}) = H(Y_i | \overline{X_i})\), which implies that
\[
\text{MI}(Y, Z) = H(Y_i) - H(Y_i | \overline{X_i}) = 1 - H(Y_i | \overline{X_i}).
\] (28)

Then, we will prove that \(H(Y_i | \overline{X_i}) = H(p)\), yielding that
\[
\text{MI}(Y, Z) = 1 - H(p).
\] (29)

For the simplicity of notation, let the random vector \(W\) be equal to \(W = [Y_1, \ldots, Y_{i-1}, Y_{i+1}, \ldots, Y_n, \overline{X_i}]\) and one of its values be \(w = [y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n, \overline{x_i}]\). Then, the conditional probability becomes \(p(y_i | y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n, \overline{x_i}) = p(y_i | w)\) and the conditional entropy becomes \(H(Y_i | Y_1, \ldots, Y_{i-1}, Y_{i+1}, \ldots, Y_n, \overline{X_i}) = H(Y_i | W)\). The conditional entropy is equal to [Ch 2(17)]

\[
H(Y_i | \overline{X_i}) = - \sum_{x_i} \left[ p_X(x_i) \cdot \sum_{y_i} p_{Y_i|\overline{X_i}}(y_i|x_i) \cdot \log p_{Y_i|\overline{X_i}}(y_i|x_i) \right]
\]
\[
H(Y_i | W) = - \sum_{w} \left[ p_W(w) \cdot \sum_{y_i} p_{Y_i|W}(y_i|w) \cdot \log p_{Y_i|W}(y_i|w) \right].
\] (30)

We need to show that \(p(y_i | w) = p(y_i | \overline{x_i})\). We will prove it, using the theory of transformations of random variables. We introduced this notation earlier: \(X = [X_1 \ldots X_i \ldots X_n]\) and \(Y = [Y_1 \ldots Y_i \ldots Y_n]\). Let \(x^* = [x_1, \ldots, x_{i-1}, \overline{x_i}, x_{i+1}, \ldots, x_n]\). Let \(Z\) be a random variable equal to \(Z = \overline{X_i}\), \(U\) be a random vector equal to \(U = [X_1 \ldots X_{i-1} X_{i+1} \ldots X_n Y]\) and \(g\) a multidimensional function, such that

\[
g(X, Y) = \begin{bmatrix} g_1(X, Y) \\ g_2(X, Y) \end{bmatrix}
\]
\[
[ Z ] = \begin{bmatrix} g_1(X, Y) = \overline{X_i} \\ g_2(X, Y) = [X_1 \ldots X_{i-1} X_{i+1} \ldots X_n Y] \end{bmatrix}
\] (31)
The random vector \( \begin{bmatrix} X \\ Y \end{bmatrix} \) is transformed, by the multidimensional function \( g \), into the random vector \( \begin{bmatrix} Z \\ U \end{bmatrix} \). Then, the joint probability mass function, \( p_{ZU}(z, u) \) is obtained as

\[
p_{ZU}(z, u) = \sum_{x \in E} \sum_{y \in \mathcal{Y}} p_{XY}(x, y) \quad \text{such that} \quad g_1(x, y) = z, g_2(x, y) = u.
\]

\[
p_{ZU}(0, u) = \sum_{x \in E} \sum_{y \in \mathcal{Y}} p_{XY}(x, y) = \sum_{x \in E} \sum_{y \in \mathcal{Y}} p_{XY}(x, y) \quad \text{such that} \quad g_1(x, y) = 0, g_2(x, y) = u.
\]

\[
p_{ZU}(1, u) = \sum_{x \in E} \sum_{y \in \mathcal{Y}} p_{XY}(x, y) = \sum_{x \in E} \sum_{y \in \mathcal{Y}} p_{XY}(x, y) \quad \text{such that} \quad g_1(x, y) = 1, g_2(x, y) = u.
\]

\[
p_{ZU}(z, u) = p_{X,Y}(1, y_i) \cdot \prod_{k=1, k \neq i} p_{X,Y_k}(x_k, y_k)
\]

\[
\Rightarrow p_{ZU}(z, u) = p_{X,Y}(1, y_i) \cdot \prod_{k=1, k \neq i} p_{X,Y_k}(x_k, y_k)
\]

\[
\Rightarrow p(x^*, y) = p_{X,Y}(1, y_i) \cdot \prod_{k=1, k \neq i} p_{X,Y_k}(x_k, y_k).
\]

\[
\Rightarrow p(y_i | w) = \frac{p(y_i | y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n, x_i)}{p(x^*, y)} = \frac{p(x^*, y)}{\sum_{x_1} \sum_{x_{i-1}} \sum_{x_{i+1}} \sum_{x_n} p(x^*, y)}
\]

\[
= \frac{\sum_{x_1} \sum_{x_{i-1}} \sum_{x_{i+1}} \sum_{x_n} p_{X,Y}(x_i, y_i) \cdot \prod_{k=1, k \neq i} p(x_k, y_k)}{\sum_{x_1} \sum_{x_{i-1}} \sum_{x_{i+1}} \sum_{x_n} p_{X,Y}(x_i, y_i) \cdot \prod_{k=1, k \neq i} p(x_k, y_k)}
\]

\[
= \frac{\prod_{k=1, k \neq i} p(y_k) \cdot p(x_i, y_i)}{\prod_{k=1, k \neq i} p(y_k) \cdot p(x_i)} = p(y_i | x_i).
\]

\[
p(w) = \sum_{x_1} \sum_{x_{i-1}} \sum_{x_{i+1}} \sum_{x_n} p(x^*, y) = \sum_{x_1} \sum_{x_{i-1}} \sum_{x_{i+1}} \sum_{x_n} p_{X,Y}(x_i, y_i) \cdot \prod_{k=1, k \neq i} p(x_k, y_k)
\]

\[
= \prod_{k=1, k \neq i} p(y_k) \cdot p(x_i).
\]

\[
\Rightarrow H(Y_i | W) = - \sum_{w} \left[ p(w) \cdot \sum_{y_i} p_{Y_i}(y_i | w) \cdot \log p_{Y_i}(y_i | w) \right]
\]

\[
= - \sum_{x_1} \sum_{x_{i-1}} \sum_{x_{i+1}} \sum_{x_n} \left[ \prod_{k=1, k \neq i} p(y_k) \cdot p(x_i) \cdot \sum_{y_i} p_{X,Y}(y_i | x_i) \cdot \log p_{Y_i}(y_i | x_i) \right]
\]

\[
= \left[ \sum_{y_1} \sum_{y_{i-1}} \sum_{y_{i+1}} \sum_{y_n} \prod_{k=1, k \neq i} p(y_k) \right] \cdot \left[ - \sum_{x_i} p(x_i) \cdot \sum_{y_i} p_{X,Y}(y_i | x_i) \cdot \log p_{Y_i}(y_i | x_i) \right]
\]
\[ \log p_{X_i|ZU}(x_i|z,u) = \sum_{y_i, \bar{y}_i} p_{Y_i|X_i}(y_i|x_i) \cdot \log \frac{p_{Y_i|X_i}(y_i|x_i)}{p_{Y_i|\bar{X}_i}(y_i|\bar{x}_i)} \]  \hspace{1cm} (35)

We need to find the expression of the conditional probability density, \( p_{Y_i|X_i}(y_i|x_i) \). To this end, we use the definitions from the transformation of random variables and vectors, the definitions 5, 6. Let \( X_i, Y_i, Z \) and \( U \) be random variables and \( g \) a multidimensional function, such that

\[ g(X_i, Y_i) = \begin{bmatrix} g_1(X_i, Y_i) \\ g_2(X_i, Y_i) \end{bmatrix} \]
\[ \begin{bmatrix} Z \\ U \end{bmatrix} = \begin{bmatrix} g_1(X_i, Y_i) = X_i \\ g_2(X_i, Y_i) = Y_i \end{bmatrix} \]  \hspace{1cm} (36)

Then, the joint probability mass function of the transformed random vector \( \begin{bmatrix} X_i \\ Y_i \end{bmatrix} \), by the function \( g \), is the joint probability mass function of the random vector \( \begin{bmatrix} Z \\ U \end{bmatrix} \):

\[
p_{ZU}(z, u) = \sum_{x_i, y_i \in X_i, Y_i} \sum_{x, y} p_{X_i, Y_i}(x_i, y_i)
\Rightarrow p_{ZU}(0, 0) = \sum_{x_i, y_i \in X_i, Y_i, g_1(x_i, y_i) = z, g_2(x_i, y_i) = u} p_{X_i, Y_i}(x_i, y_i) = \frac{1}{2} \cdot p
\]
\[
p_{ZU}(0, 1) = \sum_{x_i, y_i \in X_i, Y_i, g_1(x_i, y_i) = z, g_2(x_i, y_i) = u} p_{X_i, Y_i}(x_i, y_i) = \frac{1}{2} \cdot (1 - p)
\]
\[
p_{ZU}(1, 0) = \sum_{x_i, y_i \in X_i, Y_i, g_1(x_i, y_i) = z, g_2(x_i, y_i) = u} p_{X_i, Y_i}(x_i, y_i) = \frac{1}{2} \cdot (1 - p)
\]
\[
p_{ZU}(1, 1) = \sum_{x_i, y_i \in X_i, Y_i, g_1(x_i, y_i) = z, g_2(x_i, y_i) = u} p_{X_i, Y_i}(x_i, y_i) = \frac{1}{2} \cdot p \]  \hspace{1cm} (37)

\[
Z = g_1(X_i, Y_i) = X_i; p_{Z}(z) = \sum_{u \in U} p_{ZU}(z, u)
\Rightarrow p_Z(0) = \sum_{u \in U} p_{ZU}(0, u) = \frac{1}{2} \cdot p + \frac{1}{2} \cdot (1 - p) = \frac{1}{2}
\]
\[
p_Z(1) = \sum_{u \in U} p_{ZU}(1, u) = \frac{1}{2} \cdot (1 - p) + \frac{1}{2} \cdot p = \frac{1}{2}
\]

\[
U = g_2(X_i, Y_i) = Y_i; p_{U}(u) = \sum_{z \in Z} p_{ZU}(z, u)
\Rightarrow p_U(0) = \sum_{z \in Z} p_{ZU}(z, 0) = \frac{1}{2} \cdot p + \frac{1}{2} \cdot (1 - p) = \frac{1}{2}
\]
\[
p_U(1) = \sum_{z \in Z} p_{ZU}(z, 1) = \frac{1}{2} \cdot (1 - p) + \frac{1}{2} \cdot p = \frac{1}{2} \]  \hspace{1cm} (38)

\[
p_{U|Z}(u|z) = \frac{p_{ZU}(z, u)}{p_Z(z)}
\Rightarrow p_{U|Z}(0|0) = \frac{p_{ZU}(0, 0)}{p_Z(0)} = p
\]
\[
p_{U|Z}(0|1) = \frac{p_{ZU}(1, 0)}{p_Z(1)} = 1 - p
\]
\[
p_{U|Z}(1|0) = \frac{p_{ZU}(0, 1)}{p_Z(0)} = 1 - p
\]
\[
p_{U|Z}(1|1) = \frac{p_{ZU}(1, 1)}{p_Z(1)} = p \]  \hspace{1cm} (39)
The mutual information is equal to

\[ H(Y_i|Z) = \sum_z \left[ p_z(z) \cdot \sum_u p_{U|Z}(u|z) \cdot \log p_{U|Z}(u|z) \right] \]

\[ = - \sum_z \left\{ p_z(z) \cdot \left[ p_{U|Z}(0|z) \cdot \log p_{U|Z}(0|z) + p_{U|Z}(1|z) \cdot \log p_{U|Z}(1|z) \right] \right\} \]

\[ = - p_z(0) \cdot \left[ p \cdot \log p + (1-p) \cdot \log (1-p) \right] \]

\[ = - \frac{1}{2} \cdot 2 \cdot \left[ p \cdot \log_2 p + (1-p) \cdot \log_2 (1-p) \right] \]

\[ = H(p) \]

\[ \Rightarrow \text{MI}(Y, Z) = 1 - H(p). \] (40)

**B. Part 2 of the proof:**

We prove that \( \text{MI}(Y, Z) \leq 1 - H(p) \), for all \( n \)-dimensional Boolean functions and all values of the error probability \( 0 \leq p \leq \frac{1}{2} \). This case includes the dictatorship function, shown in part 1 of this proof. We presented the proof for the dictatorship function, separately, to show how the equality is achieved.

From the theory of multidimensional transformations of random variables, presented in the definition 6, we know that

\[ p_{Z|YX}(0, y, x) = \mathbb{P}(Z = 0, Y = y, X = x) = \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}, g_1(x, y) = z = f(x) = 0, g_2(x, y) = y, g_3(x, y) = x} p_{XY}(x, y) \]

\[ = p_{XY}(x, y) \cdot (1 - f(x)) \] (41)

\[ p_{Z|YX}(1, y, x) = \mathbb{P}(Z = 1, Y = y, X = x) = \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}, g_1(x, y) = z = f(x) = 1, g_2(x, y) = y, g_3(x, y) = x} p_{XY}(x, y) \]

\[ = p_{XY}(x, y) \cdot f(x) \] (42)

\[ p_{YZ}(y, z) = \sum_{x \in \mathcal{X}} p_{Z|YX}(z, y, x) \] (43)

\[ \Rightarrow p_{YZ}(y, 0) = \mathbb{P}(Y = y, Z = 0) = \sum_{x \in \mathcal{X}} p_{Z|YX}(0, y, x) \]

\[ = \sum_{x \in \mathcal{X}} p_{XY}(x, y) \cdot (1 - f(x)) \]

\[ \Rightarrow p_{YZ}(y, 1) = \mathbb{P}(Y = y, Z = 1) = \sum_{x \in \mathcal{X}} p_{Z|YX}(1, y, x) \]

\[ = \sum_{x \in \mathcal{X}} p_{XY}(x, y) \cdot f(x) \] (44)

For the simplicity of notation, let the random vector \( U \) be equal to \( U = [Y_1, \ldots, Y_{i-1}, Y_{i+1}, \ldots, Y_n, Z] \) and one of its values be \( u = [y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n, z] \). Let \( u_0 = [y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n, 0] \) and \( u_1 = [y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n, 1] \).

The mutual information is equal to \( \text{Ch} 2 \) [

\[ \text{MI}(Y, Z) = H(Y_i) - H(Y_i|Y_1, \ldots, Y_{i-1}, Y_{i+1}, \ldots, Y_n, Z) = 1 - H(Y_i|U) \]

\[ H(Y_i|U) = - \sum_u \left[ p_U(u) \cdot \sum_{y_i} p_{Y_i|U}(y_i|u) \cdot \log p_{Y_i|U}(y_i|u) \right] \]

\[ = - \sum_{y_1} \ldots \sum_{y_{i-1}} \sum_{y_{i+1}} \ldots \sum_{y_n} \left[ p_U(u_0) \cdot \sum_{y_i} p_{Y_i|U}(y_i|u_0) \cdot \log p_{Y_i|U}(y_i|u_0) \right] + \]

\[ + p_U(u_1) \cdot \sum_{y_i} p_{Y_i|U}(y_i|u_1) \cdot \log p_{Y_i|U}(y_i|u_1) \]

\[ = \sum_{y_1} \ldots \sum_{y_{i-1}} \sum_{y_{i+1}} \ldots \sum_{y_n} \left[ p_U(u_0) \cdot H(p_{Y_i|U}(0|u_0)) + p_U(u_1) \cdot H(p_{Y_i|U}(0|u_1)) \right]. \] (45)
Using the properties of the binary entropy function, from the section II-D, we will show that $H(Y_i|U) \geq H(p)$. From the properties of the memoryless binary symmetric channel, we know that

$$p_X(x, y) = \prod_{k=1}^{n} p_{X_k,Y_k}(x_k, y_k). \tag{46}$$

$$p_U(u) = \sum_{y_i} p_{U,Y}(y, z)$$

$$p_U(u_0) = \sum_{y_i} p_{U,Y}(y, 0) = \sum_{y_i} \sum_{x} p_{X,Y}(x, y) \cdot (1 - f(x)) = \sum_{x} \prod_{k=1}^{n} p_{X_k,Y_k}(x_k, y_k) \cdot (1 - f(x))$$

$$= \sum_{x} p_{X_1}(x_i) \cdot \prod_{k=1, k \neq i}^{n} p_{X_k,Y_k}(x_k, y_k) \cdot (1 - f(x)) = \frac{1}{2} \sum_{x=[x_1, \ldots, x_n]} \prod_{k=1, k \neq i}^{n} p_{X_k,Y_k}(x_k, y_k) \cdot (1 - f(x))$$

$$p_U(u_1) = \sum_{y_i} p_{U,Y}(y, 1) = \sum_{y_i} \sum_{x} p_{X,Y}(x, y) \cdot f(x) = \sum_{x} \prod_{k=1}^{n} p_{X_k,Y_k}(x_k, y_k) \cdot f(x)$$

$$= \sum_{x} p_{X_1}(x_i) \cdot \prod_{k=1, k \neq i}^{n} p_{X_k,Y_k}(x_k, y_k) \cdot f(x) = \frac{1}{2} \sum_{x=[x_1, \ldots, x_n]} \prod_{k=1, k \neq i}^{n} p_{X_k,Y_k}(x_k, y_k) \cdot f(x). \tag{47}$$

We will separately derive the values of the following conditional probabilities: $p_{Y|U}(0|u_0)$, $p_{Y|U}(0|u_1)$, $p_{Y|U}(1|u_0)$ and $p_{Y|U}(1|u_1)$.

Let $N_i$ be the number of values of $x$, out of the $2^n$ possible values, for which $f(x) = 1 - f(x)$ is equal to 1 and, equivalently, $f(x)$ is equal to 0. Let $x^{(r)}$ denote the $r^{th}$ value of the vector $x$, out of the $2^n$ possible values. Thus, we have $x^{(r)} = [x^{(r)}_1 \ldots x^{(r)}_i \ldots x^{(r)}_n]$, $\forall r = 1 \ldots 2^n$.

$$p_{Y|U}(y|u) = \frac{p_{Y,U}(y,u)}{p_U(u)} = \frac{p_{Y,Z}(y,z)}{\sum_{y_i} p_{Y,Z}(y,z)}$$

$$p_{Y|U}(0|u_0) = \frac{p_{Y,Z}(0,u_0)}{p_U(u_0)}$$

$$p_{Y|U}(0|u_0) = \sum_{x=[x_1, \ldots, x_n]} p_{X,Y}(x,0) = \prod_{k=1, k \neq i}^{n} p_{X_k,Y_k}(x_k, y_k) \cdot (1 - f(x))$$

$$= p(x^{(1)}_i,0) \cdot \prod_{k=1, k \neq i}^{n} p(x^{(1)}_k, y_k) \cdot (1 - f(x^{(1)}_i)) + \ldots +$$

$$+ p(x^{(r)}_i,0) \cdot \prod_{k=1, k \neq i}^{n} p(x^{(r)}_k, y_k) \cdot (1 - f(x^{(r)}_i)) + \ldots + p(x^{(2^n)}_i,0) \cdot \prod_{k=1, k \neq i}^{n} p(x^{(2^n)}_k, y_k) \cdot (1 - f(x^{(2^n)}_i))$$

$$= p(x^{(l_1)}_i,0) \cdot \prod_{k=1, k \neq i}^{n} p(x^{(l_1)}_k, y_k) + \ldots + p(x^{(l_{N_i})}_i,0) \cdot \prod_{k=1, k \neq i}^{n} p(x^{(l_{N_i})}_k, y_k) \tag{48}$$

Out of the $N_i$ values of $x_i^{(r)} = [x^{(r)}_1 \ldots x^{(r)}_i \ldots x^{(r)}_n]$, $\forall r = 1 \ldots 2^n$, for which the function $f(x^{(r)})$ is equal to 0, let $N_i^{(1)}$ be the number of the values that have the element $x_i^{(r)} = 0$. Let $N_i^{(2)}$ be the number of the values that have the element $x_i^{(r)} = 1$. We have that $N_i^{(1)} + N_i^{(2)} = N_i$. We use the indices $[m_1 \ldots m_{N_i^{(1)}}]$ and $[l_1 \ldots l_{N_i^{(2)}}]$, to denote the subset of joint probabilities, for which $x_i^{(r)} = 0$ and $x_i^{(r)} = 1$, respectively. The union of these two subsets is the original set indexed by $[l_1 \ldots l_{N_i}]$.

Using the properties of the binary symmetric channel (10), we obtain

$$p_{Y|U}(0|u_0) = \frac{1}{2} \cdot (1 - p) \cdot \left[ \prod_{k=1, k \neq i}^{n} p(x^{(m_1)}_k, y_k) + \ldots + \prod_{k=1, k \neq i}^{n} p(x^{(m_{N_i^{(1)})}}_k, y_k) \right] +$$

$$+ \frac{1}{2} \cdot p \cdot \left[ \prod_{k=1, k \neq i}^{n} p(x^{(l_1)}_k, y_k) + \ldots + \prod_{k=1, k \neq i}^{n} p(x^{(l_{N_i^{(2)})}}_k, y_k) \right]$$
Let \( A_1 = \prod_{k=1, k \neq i}^n p(x_k^{(m)}, y_k) + \ldots + \prod_{k=1, k \neq i}^n p(x_k^{(n)}, y_k) \)

\[ B_1 = \prod_{k=1, k \neq i}^n p(x_k^{(1)}, y_k) + \ldots + \prod_{k=1, k \neq i}^n p(x_k^{(2)}, y_k) \]

\[ \Rightarrow A_1 + B_1 = \prod_{k=1, k \neq i}^n p(x_k^{(1)}, y_k) + \ldots + \prod_{k=1, k \neq i}^n p(x_k^{(n)}, y_k) \]

\[ p_{U}(u_0) = \frac{1}{2} \cdot \sum_{x=[x_1 \ldots x_n]}^n \prod_{k=1, k \neq i}^n p_{X_k Y_k}(x_k, y_k) \cdot (1 - f(x)) \]

\[ = \frac{1}{2} \cdot \left[ \prod_{k=1, k \neq i}^n p(x_k^{(1)}, y_k) \cdot (1 - f(x^{(1)})) + \ldots + \prod_{k=1, k \neq i}^n p(x_k^{(r)}, y_k) \cdot (1 - f(x^{(r)})) \right] + \ldots + \prod_{k=1, k \neq i}^n p(x_k^{(2^n)}, y_k) \cdot (1 - f(x^{(2^n)})) \]

\[ = \frac{1}{2} \cdot \left[ \prod_{k=1, k \neq i}^n p(x_k^{(1)}, y_k) + \ldots + \prod_{k=1, k \neq i}^n p(x_k^{(n)}, y_k) \right] = \frac{1}{2} \cdot (A_1 + B_1) \]

\[ \Rightarrow p_{Y, |U}(0|u_0) = \frac{(1 - p) \cdot A_1 + p \cdot B_1}{A_1 + B_1} = \frac{A_1 + p \cdot (B_1 - A_1)}{A_1 + B_1} \quad \text{(49)} \]

1) If \( B_1 - A_1 \leq 0 \), then we have that

\begin{align*}
0 \leq p \leq \frac{1}{2} & \Rightarrow A_1 + \frac{1}{2} \cdot (B_1 - A_1) \leq A_1 + p \cdot (B_1 - A_1) \Rightarrow \frac{1}{2} \geq \frac{A_1 + p \cdot (B_1 - A_1)}{A_1 + B_1} \Rightarrow p_{Y, |U}(0|u_0) \geq \frac{1}{2} \\
0 \leq p \leq 1 - p \leq \frac{1}{2} & \Rightarrow p_{Y, |U}(0|u_0) \leq \frac{(1 - p) \cdot A_1 + (1 - p) \cdot B_1}{A_1 + B_1} = 1 - p.
\end{align*}

Using the properties of the binary entropy function [17], this equation yields \( H(p_{Y, |U}(0|u_0)) \geq H(p) \).

2) If \( B_1 - A_1 \geq 0 \), then we have that

\begin{align*}
0 \leq p \leq \frac{1}{2} & \Rightarrow A_1 + \frac{1}{2} \cdot (B_1 - A_1) \geq A_1 + p \cdot (B_1 - A_1) \Rightarrow \frac{1}{2} \geq \frac{A_1 + p \cdot (B_1 - A_1)}{A_1 + B_1} \Rightarrow p_{Y, |U}(0|u_0) \geq \frac{1}{2} \\
0 \leq p \leq 1 - p \leq \frac{1}{2} & \Rightarrow p_{Y, |U}(0|u_0) \geq \frac{p \cdot A_1 + p \cdot B_1}{A_1 + B_1} = p.
\end{align*}

Using the properties of the binary entropy function [16], this equation yields \( H(p_{Y, |U}(0|u_0)) \geq H(p) \).

\[ p_{Y, |U}(1|u_0) = \frac{p_{YZ}(1, u_0)}{p_{U}(u_0)} \]

\[ p_{YZ}(1, u_0) = \sum_{x=[x_1 \ldots x_n]}^n p_{X, Y}(x_1, 1) \cdot \prod_{k=1, k \neq i}^n p_{X_k Y_k}(x_k, y_k) \cdot (1 - f(x)) \]

\[ = p(x_1^{(1)}, 1) \cdot \prod_{k=1, k \neq i}^n p(x_k^{(1)}, y_k) \cdot (1 - f(x^{(1)})) + \ldots + p(x_1^{(r)}, 1) \cdot \prod_{k=1, k \neq i}^n p(x_k^{(r)}, y_k) \cdot (1 - f(x^{(r)})) + \ldots + p(x_1^{(2^n)}, 1) \cdot \prod_{k=1, k \neq i}^n p(x_k^{(2^n)}, y_k) \cdot (1 - f(x^{(2^n)})) \]

\[ = p(x_1^{(1)}, 1) \cdot \prod_{k=1, k \neq i}^n p(x_k^{(1)}, y_k) + \ldots + p(x_1^{(n)}, 0) \cdot \prod_{k=1, k \neq i}^n p(x_k^{(n)}, y_k) \quad \text{(52)} \]

\[ \Rightarrow p_{Y, |U}(1|u_0) = \frac{1}{2} \cdot p \cdot \left[ \prod_{k=1, k \neq i}^n p(x_k^{(m)}, y_k) + \ldots + \prod_{k=1, k \neq i}^n p(x_k^{(n)}, y_k) \right] + \]

\[ + \frac{1}{2} \cdot (1 - p) \cdot \left[ \prod_{k=1, k \neq i}^n p(x_k^{(1)}, y_k) + \ldots + \prod_{k=1, k \neq i}^n p(x_k^{(2)}, y_k) \right] \]
We have that
\[
\Pr(Y_i \mid U) = \frac{p \cdot A_1 + (1 - p) \cdot B_1}{A_1 + B_1} = \frac{A_1 + p \cdot (B_1 - A_1)}{A_1 + B_1}.
\] (53)

We obtain that \(\Pr(Y_i \mid U) + \Pr(Y_i \mid U) = 1\), which is true, given these are conditional probabilities, conditioned on the same event, \(U = u_0\).

Let \(N_j\) be the number of values of \(x\), out of the \(2^n\) possible values, for which \(f(x)\) is equal to 1. Let \(x^{(r)}\) denote the \(r\)th value of the vector \(x\), out of the \(2^n\) possible values. Thus, we have \(x^{(r)} = [x_1^{(r)} \ldots x_{i-1}^{(r)} x_i^{(r)} x_{i+1}^{(r)} \ldots x_n^{(r)}], \forall r = 1 : 2^n\).

\[
\Pr(Y_i \mid U) = \frac{\Pr(Y_i, U)}{\Pr(U)} = \frac{\sum_{x=x_1 \ldots x_n} \Pr(Y_i, x_i = 0) \cdot \prod_{k=1,k \neq i}^n \Pr(Y_k, x_k, y_k) \cdot f(x)}{\sum_{x=x_1 \ldots x_n} \prod_{k=1,k \neq i}^n \Pr(Y_k, x_k, y_k) \cdot f(x)}
\]

\[
= \Pr(x_i^{(1)}, 0) \cdot \prod_{k=1,k \neq i}^n \Pr(x_k^{(1)}, y_k) \cdot f(x^{(1)}) + \ldots + \Pr(x_i^{(r)}, 0) \cdot \prod_{k=1,k \neq i}^n \Pr(x_k^{(r)}, y_k) \cdot f(x^{(r)}) + \ldots + \Pr(x_i^{(2^n)}, 0) \cdot \prod_{k=1,k \neq i}^n \Pr(x_k^{(2^n)}, y_k) \cdot f(x^{(2^n)})
\]

\[= \Pr(x_i^{(0)}, 0) \cdot \prod_{k=1,k \neq i}^n \Pr(x_k^{(0)}, y_k) + \ldots + \Pr(x_i^{(I_j)}, 0) \cdot \prod_{k=1,k \neq i}^n \Pr(x_k^{(I_j)}, y_k)\] (54)

Out of the \(N_j\) values of \(x^{(r)} = [x_1^{(r)} \ldots x_{i-1}^{(r)} x_i^{(r)} x_{i+1}^{(r)} \ldots x_n^{(r)}]\), for which the function \(f(x^{(r)})\) is equal to 1, let \(N_j^{(1)}\) be the number of the values that have the element \(x_i^{(r)} = 0\). Let \(N_j^{(2)}\) be the number of the values that have the element \(x_i^{(r)} = 1\). We have that \(N_j^{(1)} + N_j^{(2)} = N_j\). We use the indexes \([m_1 \ldots m_{N_j^{(1)}}]\) and \([t_1 \ldots t_{N_j^{(2)}}]\), to denote the subset of joint probabilities, for which \(x_i^{(r)} = 0\) and \(x_i^{(r)} = 1\), respectively. The union of these two subsets is the original set indexed by \([l_1 \ldots l_{N_j}]\). Using the properties of the binary symmetric channel (10), we obtain

\[
\Rightarrow \Pr(Y_i \mid U) = \frac{1}{2} \cdot (1 - p) \cdot \left[ \prod_{k=1,k \neq i}^n \Pr(x_k^{(m_{N_j^{(1)}})} y_k) + \prod_{k=1,k \neq i}^n \Pr(x_k^{(m_{N_j^{(2)}})} y_k) \right] + \frac{1}{2} \cdot p \cdot \left[ \prod_{k=1,k \neq i}^n \Pr(x_k^{(t_{N_j^{(1)}})} y_k) + \prod_{k=1,k \neq i}^n \Pr(x_k^{(t_{N_j^{(2)}})} y_k) \right]
\]

Let \(A_2 = \prod_{k=1,k \neq i}^n \Pr(x_k^{(m_j)} y_k) + \prod_{k=1,k \neq i}^n \Pr(x_k^{(m_{N_j^{(1)}})} y_k)\)

\(B_2 = \prod_{k=1,k \neq i}^n \Pr(x_k^{(t_1)} y_k) + \prod_{k=1,k \neq i}^n \Pr(x_k^{(t_{N_j^{(1)}})} y_k)\)

\[
\Rightarrow A_2 + B_2 = \prod_{k=1,k \neq i}^n \Pr(x_k^{(l_1)} y_k) + \prod_{k=1,k \neq i}^n \Pr(x_k^{(l_{N_j^{(1)}})} y_k)
\]

\[
\Pr(U) = \frac{1}{2} \cdot \sum_{x=x_1 \ldots x_n} \prod_{k=1,k \neq i}^n \Pr(Y_k, x_k, y_k) \cdot f(x)
\]

\[
= \frac{1}{2} \cdot \left[ \prod_{k=1,k \neq i}^n \Pr(x_k^{(l_1)} y_k) \cdot f(x^{(1)}) + \ldots + \prod_{k=1,k \neq i}^n \Pr(x_k^{(l_{N_j^{(1)}})} y_k) \cdot f(x^{(2^n)}) \right]
\]

\[
= \frac{1}{2} \cdot \left[ \prod_{k=1,k \neq i}^n \Pr(x_k^{(l_1)} y_k) + \ldots + \prod_{k=1,k \neq i}^n \Pr(x_k^{(l_{N_j^{(1)}})} y_k) \right] = \frac{1}{2} \cdot (A_2 + B_2)
\]

\[
\Rightarrow \Pr(Y_i \mid U) = \frac{(1 - p) \cdot A_2 + p \cdot B_2}{A_2 + B_2} = \frac{A_2 + p \cdot (B_2 - A_2)}{A_2 + B_2}.
\] (55)
1) If \( B_2 - A_2 \leq 0 \), then we have that
\[
0 \leq p \leq \frac{1}{2} \Rightarrow A_2 + \frac{1}{2} \cdot (B_2 - A_2) \leq A_2 + p \cdot (B_2 - A_2) \Rightarrow \frac{1}{2} \leq \frac{A_2 + p \cdot (B_2 - A_2)}{A_2 + B_2} \Rightarrow p_{Y_i|U}(0|u_1) \geq \frac{1}{2}.
\]

Using the properties of the binary entropy function [17], this equation yields \( H(p_{Y_i|U}(0|u_1)) \geq H(p) \).

2) If \( B_2 - A_2 \geq 0 \), then we have that
\[
0 \leq p \leq \frac{1}{2} \Rightarrow A_2 + \frac{1}{2} \cdot (B_2 - A_2) \geq A_2 + p \cdot (B_2 - A_2) \Rightarrow \frac{1}{2} \geq \frac{A_2 + p \cdot (B_2 - A_2)}{A_2 + B_2} \Rightarrow p_{Y_i|U}(0|u_1) \leq \frac{1}{2}.
\]

Using the properties of the binary entropy function [16], this equation yields \( H(p_{Y_i|U}(0|u_1)) \geq H(p) \).

\[
p_{Y_i|U}(1|u_1) = \frac{p_{YZ}(1, u_1)}{p_{U}(u_1)}
\]

\[
p_{YZ}(1, u_1) = \sum_{x = [x_1, \ldots, x_n]} p_{X,Y_i}(x_i, 1) \cdot \prod_{k=1, k \neq i}^n p_{X_k,Y_k}(x_k, y_k) \cdot f(x)
\]

\[
= p(x_i^{(1)}, 1) \cdot \prod_{k=1, k \neq i}^n p(x_k^{(1)}, y_k) \cdot f(x^{(1)}) + \ldots + \]

\[
+ p(x_i^{(r)}, 1) \cdot \prod_{k=1, k \neq i}^n p(x_k^{(r)}, y_k) \cdot f(x^{(r)}) + \ldots + p(x_i^{(2^n)}, 1) \cdot \prod_{k=1, k \neq i}^n p(x_k^{(2^n)}, y_k) \cdot f(x^{(2^n)})
\]

\[
= p(x_i^{(1)}, 1) \cdot \prod_{k=1, k \neq i}^n p(x_k^{(1)}, y_k) + \ldots + p(x_i^{(t_{N_i})}, 0) \cdot \prod_{k=1, k \neq i}^n p(x_k^{(t_{N_i})}, y_k)
\]

Using the properties of the binary symmetric channel [10], we obtain
\[
\Rightarrow p_{Y_i|U}(1|u_1) = \frac{1}{2} \cdot p \cdot \left[ \prod_{k=1, k \neq i}^n p(x_k^{(m_1)}, y_k) + \ldots + \prod_{k=1, k \neq i}^n p(x_k^{(m_{N_i})}, y_k) \right] + \]

\[
+ \frac{1}{2} \cdot (1 - p) \cdot \left[ \prod_{k=1, k \neq i}^n p(x_k^{(t_1)}, y_k) + \ldots + \prod_{k=1, k \neq i}^n p(x_k^{(t_{N_i})}, y_k) \right]
\]

\[
\Rightarrow p_{Y_i|U}(1|u_1) = \frac{p \cdot A_2 + (1 - p) \cdot B_2}{A_2 + B_2} = \frac{A_2 + p \cdot (B_2 - A_2)}{A_2 + B_2}.
\]

We obtain that \( p_{Y_i|U}(0|u_0) + p_{Y_i|U}(1|u_1) = 1 \), which is true, given these are conditional probabilities, conditioned on the same event, \( U = u_1 \).

We have proven that \( H(p_{Y_i|U}(0|u_0)) \geq H(p) \) and \( H(p_{Y_i|U}(0|u_1)) \geq H(p) \).

\[
\Rightarrow H(Y_i|U) = \sum_{y_1} \ldots \sum_{y_{i-1}} \sum_{y_{i+1}} \ldots \sum_{y_n} \left[ p_{U}(u_0) \cdot H(p_{Y_i|U}(0|u_0)) + p_{U}(u_1) \cdot H(p_{Y_i|U}(0|u_1)) \right]
\]

\[
\geq \sum_{y_1} \ldots \sum_{y_{i-1}} \sum_{y_{i+1}} \ldots \sum_{y_n} \left[ p_{U}(u_0) \cdot H(p) + p_{U}(u_1) \cdot H(p) \right]
\]

\[
= H(p) \cdot \left[ \sum_{y_1} \ldots \sum_{y_{i-1}} \sum_{y_{i+1}} \ldots \sum_{y_n} p_{Y_i \ldots Y_{i-1}Y_{i+1} \ldots Y_n}(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n) \right]
\]

\[
= H(p)
\]

\[
\Rightarrow H(Y_i|U) \geq H(p)
\]

\[
\Rightarrow MI(Y, Z) \leq 1 - H(p).
\]
IV. Conclusions

We have proven the Courtade-Kumar conjecture [7], for all $n$-dimensional Boolean functions and for all values of the error probability of the binary symmetric channel, $0 \leq p \leq \frac{1}{2}$. We divided the proof into two parts: in the first part, we demonstrated that equality is achieved in the Courtade-Kumar conjecture, in the case of the dictatorship function. This equality result has been known in the literature. However, we provided a comprehensive proof of its validity and showed that our proposed methodology facilitates the proof of this equality. In the second part of the proof, we showed that the conjecture holds for all Boolean functions and for all values of $0 \leq p \leq \frac{1}{2}$. The proof from the second part includes the dictatorship function. Our proofs relied on information theory, conditional probability theory and transformations of random variables and vectors. We found the expressions of several conditional probability mass functions, related them to their corresponding entropies and, then, showed that these entropies are greater than or equal to the binary entropy function, $H(p)$. By using fundamental mathematical concepts, we have derived a clear and straightforward proof of the Courtade-Kumar conjecture, in the most general context.

APPENDIX A

Properties of the binary symmetric channel

Using the properties of no memory and no feedback, we prove, by induction, the following equations describing the conditional and joint mass functions, for several inputs to the binary symmetric channel,

$$p(y_{k+1}, y_{k}, \ldots, y_{1}|x_{k+1}, x_{k}, \ldots, x_{1}) = \prod_{i=1}^{k+1} p(y_{i}|x_{i}), \forall k = 1, n - 1,$$

$$p(x_{k+1}, x_{k}, \ldots, x_{1}, y_{k+1}, y_{k}, \ldots, y_{1}) = \prod_{i=1}^{k+1} p(x_{i}, y_{i}), \forall k = 1, n - 1. \quad (61)$$

**Proof:**

Step 1: Verify that the identity holds for $k = 1$ and $k = 2$.

$k = 1$

$$p(y_{1}|x_{1}) = p(y_{1}|x_{1}). \quad (62)$$

This statement is true.

$k = 2$

$$p(y_{2}, y_{1}|x_{2}, x_{1}) = \frac{p(x_{1}, x_{2}, y_{1}, y_{2})}{p(x_{1}, x_{2})} = \frac{p(y_{2}|x_{2}, x_{1}, y_{1}) \cdot p(x_{2}, x_{1}, y_{1})}{p(x_{1}, x_{2})}. \quad (63)$$

We use the property that the channel has no memory, that is $p(y_{2}|x_{2}, x_{1}, y_{1}) = p(y_{2}|x_{2})$, and the property that it has no feedback, that is $p(x_{2}|x_{1}, y_{1}) = p(x_{2}|x_{1})$.

$$\Rightarrow p(y_{2}, y_{1}|x_{2}, x_{1}) = \frac{p(y_{2}|x_{2}) \cdot p(x_{2}|x_{1}, y_{1}) \cdot p(x_{1}, y_{1})}{p(x_{1}) \cdot p(x_{2})} \cdot \frac{p(x_{2})}{p(x_{1})} \cdot \frac{p(y_{1}|x_{1})}{p(x_{1})} \cdot p(x_{1}) \cdot p(y_{1}|x_{1}) \cdot p(x_{2}). \quad (64)$$

From the fact that $X_{1}, X_{2}$ are i.i.d

$$p(x_{2}|x_{1}) = \frac{p(x_{1}, x_{2})}{p(x_{1})} = \frac{p(x_{1}) \cdot p(x_{2})}{p(x_{1})} = p(x_{2}). \quad (65)$$

$$\Rightarrow p(y_{2}, y_{1}|x_{2}, x_{1}) = p(y_{2}|x_{2}) \cdot p(y_{1}|x_{1}). \quad (66)$$

Step 2: $\forall 1 \leq k \leq n - 1$, assume that the equation

$$p(y_{k}, y_{k-1}, \ldots, y_{1}|x_{k}, x_{k-1}, \ldots, x_{1}) = \prod_{i=1}^{k} p(y_{i}|x_{i}) \quad (67)$$

holds and prove that this implies that the equation

$$p(y_{k+1}, y_{k}, \ldots, y_{1}|x_{k+1}, x_{k}, \ldots, x_{1}) = \prod_{i=1}^{k+1} p(y_{i}|x_{i}) \quad (68)$$
holds.
\[
p(y_{k+1}, y_k, \ldots, y_1| x_{k+1}, x_k, \ldots, x_1) = \frac{p(y_{k+1}, y_k, \ldots, y_1| x_{k+1}, x_k, \ldots, x_1)}{p(x_{k+1}, x_k, \ldots, x_1)} = \frac{p(y_{k+1}| x_{k+1}, x_k, \ldots, x_1, y_k, \ldots, y_1) \cdot p(x_{k+1}, x_k, \ldots, x_1, y_k, \ldots, y_1)}{p(x_{k+1}, x_k, \ldots, x_1)}.
\]

(69)

We use the property that the channel has no memory, that is
\[
p(y_{k+1}| x_{k+1}, x_k, \ldots, x_1, y_k, \ldots, y_1) = p(y_{k+1}| x_{k+1})
\]

(70)

and the property that it has no feedback, that is
\[
p(x_{k+1}| x_k, \ldots, x_1, y_k, \ldots, y_1) = p(x_{k+1}| x_k, \ldots, x_1).
\]

(71)

\[
\Rightarrow p(y_{k+1}, y_k, \ldots, y_1| x_{k+1}, x_k, \ldots, x_1) = \frac{p(y_{k+1}| x_{k+1}) \cdot p(x_{k+1}, x_k, \ldots, x_1, y_k, \ldots, y_1)}{p(x_{k+1}) \cdot p(x_k, \ldots, x_1)}
\]

\[
= \frac{p(y_{k+1}| x_{k+1}) \cdot p(x_{k+1}| x_k, \ldots, x_1, y_k, \ldots, y_1) \cdot p(x_k, \ldots, x_1, y_k, \ldots, y_1)}{p(x_{k+1}) \cdot p(x_k, \ldots, x_1)}
\]

\[
= \frac{p(y_{k+1}| x_{k+1}) \cdot p(x_{k+1}| x_k, \ldots, x_1, y_k, \ldots, y_1) \cdot p(y_k, \ldots, y_1| x_k, \ldots, x_1)}{p(x_{k+1})}.
\]

(72)

From the fact that \(X_1, X_2, \ldots, X_{k+1}\) are i.i.d
\[
p(x_{k+1}| x_k, \ldots, x_1) = \frac{p(x_{k+1}, x_k, \ldots, x_1)}{p(x_k, \ldots, x_1)} = \frac{p(x_{k+1}) \cdot \prod_{i=1}^{k} p(x_i)}{\prod_{i=1}^{k} p(x_i)} = p(x_{k+1}).
\]

(73)

\[
\Rightarrow p(y_{k+1}, y_k, \ldots, y_1| x_{k+1}, x_k, \ldots, x_1) = p(y_{k+1}| x_{k+1}) \cdot p(y_k, \ldots, y_1| x_k, \ldots, x_1).
\]

(74)

Then, from our assumption that
\[
p(y_k, \ldots, y_1| x_k, \ldots, x_1) = \prod_{i=1}^{k} p(y_i| x_i),
\]

\[
\Rightarrow p(y_{k+1}, y_k, \ldots, y_1| x_{k+1}, x_k, \ldots, x_1) = \prod_{i=1}^{k+1} p(y_i| x_i).
\]

(75)

\[
\Rightarrow p(x_{k+1}, x_k, \ldots, x_1, y_{k+1}, y_k, \ldots, y_1) = \prod_{i=1}^{k+1} p(y_i| x_i) \prod_{j=1}^{k+1} p(x_j)
\]

\[
= \prod_{i=1}^{k+1} p(y_i| x_i) \cdot p(x_{k+1}, x_k, \ldots, x_1)
\]

\[
= \prod_{i=1}^{k+1} p(y_i| x_i) \cdot p(x_i)
\]

\[
= \prod_{i=1}^{k+1} p(x_i, y_i).
\]

(76)

\[\square\]

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