Asymptotic Analysis of a Viscous Fluid in a Curved Pipe with Elastic Walls

G. Castiñeira\textsuperscript{a}, J. M. Rodríguez\textsuperscript{b}

\textsuperscript{a}Departamento de Matemática Aplicada, Univ. de Santiago de Compostela, Spain
\textsuperscript{b}Departamento de Métodos Matemáticos e Representación, Univ. da Coruña, Spain

Abstract

This communication is devoted to the presentation of our recent results regarding the asymptotic analysis of a viscous flow in a tube with elastic walls. This study can be applied, for example, to the blood flow in an artery. With this aim, we consider the dynamic problem of the incompressible flow of a viscous fluid through a curved pipe with a smooth central curve. Our analysis leads to obtain an one dimensional model via singular perturbation of the Navier-Stokes system as $\varepsilon$, a non dimensional parameter related to the radius of cross-section of the tube, tends to zero. We allow the radius depend on tangential direction and time, so a coupling with an elastic or viscoelastic law on the wall of the pipe is possible.

To perform the asymptotic analysis, we do a change of variable to a reference domain where we assume the existence of asymptotic expansions on $\varepsilon$ for both velocity and pressure which, upon substitution on Navier-Stokes equations, leads to the characterization of various terms of the expansion. This allows us to obtain an approximation of the solution of the Navier-Stokes equations.

Keywords: Asymptotic Analysis, Blood flow, Navier-Stokes equations.

1. Introduction

Last decades, applied mathematics have been involved in some new fields where they had not been applied before. One of these fields is biomedicine, from which new methods to improve the diagnosis and treatment of different diseases are demanded. In particular, in the case of cardiovascular problems, modeling the blood flow in veins and arteries is a difficult problem.

A large number of articles have studied the flow of a viscous fluid through a pipe. For example, in \cite{2, 5, 11} the flow behavior inside the pipe is related with the curvature and torsion of its middle line. In \cite{2} the main term of the asymptotic expansion of the solution is compared with a Poiseuille flow inside a pipe with rigid walls. In \cite{8}, the same problem but with visco-elastic walls is considered, leading to a fluid-structure problem. In \cite{3} the
2. Setting the problem in a reference domain

Let us suppose that central curve of the pipe is parametrized by \( c(s) \), where \( s \in [0, L] \) is the arc-length parameter, and the interior points of the pipe are given by

\[
(x, y, z) = c(s) + \varepsilon R(t, s) \left[ (\cos \theta)N(s) + (\sin \theta)B(s) \right],
\]

where \( r \in [0, 1], \theta \in [0, 2\pi] \), \( \{T = c', N, B\} \) is the Frenet-Serret frame of \( c \), and \( \varepsilon R(t, s) \) is the radius of the cross-section of the pipe at point \( c(s) \) and time \( t \). The non dimensional parameter \( \varepsilon \) represents the different scale of magnitude between the pipe diameter and its length, so we shall assume that \( \varepsilon << 1 \).

Let us introduce the following notation, \( s_1 := s, s_2 := \theta, s_3 := r \) for the variables, and \( \{v_1 := T, v_2 := N, v_3 := B\} \), for the Frenet-Serret frame of \( c \). This new notation will allow us to use Einstein summation convention in what follows.

Let be the subsets of \( \mathbb{R}^3 \) defined by \( \Omega^\varepsilon = [0, L] \times [0, 2\pi] \times [0, \varepsilon] \) and \( \Omega = [0, L] \times [0, 2\pi] \times [0, 1] \). We define the maps \( \phi_1^\varepsilon: \Omega \to \Omega^\varepsilon, \phi_2^\varepsilon: \Omega^\varepsilon \to \hat{\Omega}_i^\varepsilon \), where \( \phi_1^\varepsilon \) and \( \phi_2^\varepsilon \) are given by the expressions,

\[
\phi_1^\varepsilon(s_1, s_2, s_3) = (s_1, s_2, \varepsilon s_3) =: (s_1^\varepsilon, s_2^\varepsilon, s_3^\varepsilon),
\]

\[
\phi_2^\varepsilon(s_1^\varepsilon, s_2^\varepsilon, s_3^\varepsilon) = c(s_1^\varepsilon) + s_3^\varepsilon R(t, s_1^\varepsilon) \left[ (\cos s_2^\varepsilon) v_2(s_1^\varepsilon) + (\sin s_2^\varepsilon) v_3(s_1^\varepsilon) \right],
\]  

(2.1)

and \( \hat{\Omega}_i^\varepsilon = \phi_2^\varepsilon(\phi_1^\varepsilon(\Omega)) \) represents the interior points of the pipe.

We can then introduce the change of variable from the reference domain \( \Omega \),

\[
\phi^\varepsilon = (\phi_2^\varepsilon \circ \phi_1^\varepsilon): \Omega \to \hat{\Omega}_i^\varepsilon,
\]

\[
\phi^\varepsilon(s_1, s_2, s_3) = c(s_1) + \varepsilon s_3 R(t, s_1) \left[ (\cos s_2) v_2(s_1) + (\sin s_2) v_3(s_1) \right] =: (x_1^\varepsilon, x_2^\varepsilon, x_3^\varepsilon).
\]  

(2.2)
Let us consider the incompressible Navier-Stokes equations in the domain $\hat{\Omega}_t^\varepsilon$ given by,

$$\frac{\partial \mathbf{u}^\varepsilon}{\partial t} + (\nabla \mathbf{u}^\varepsilon)\mathbf{u}^\varepsilon = \frac{1}{\rho_0} \text{div} \mathbf{T}^\varepsilon + \mathbf{b}_0^\varepsilon,$$  \hspace{1cm} (2.3)

$$\text{div} \mathbf{u}^\varepsilon = 0,$$  \hspace{1cm} (2.4)

where $\mathbf{u}^\varepsilon$ stands for the velocity field, $\mathbf{b}_0^\varepsilon$ is the density of body forces and $\mathbf{T}^\varepsilon$ is the stress tensor given by

$$\mathbf{T}^\varepsilon = -p^\varepsilon \mathbf{I} + 2\mu \Sigma^\varepsilon,$$

where $p^\varepsilon$ is the pressure field, $\mu$ the dynamic viscosity and $\Sigma^\varepsilon = \frac{1}{2} (\nabla \mathbf{u}^\varepsilon + (\nabla \mathbf{u}^\varepsilon)^T)$.

Let $\nu = \mu/\rho_0$ be the kinematic viscosity, so we can write these equations,

$$\frac{\partial \mathbf{u}^\varepsilon}{\partial t} + (\nabla \mathbf{u}^\varepsilon)\mathbf{u}^\varepsilon + \frac{1}{\rho_0} \nabla p^\varepsilon - \nu \Delta \mathbf{u}^\varepsilon = \mathbf{b}_0^\varepsilon,$$  \hspace{1cm} (2.5)

$$\text{div} \mathbf{u}^\varepsilon = 0.$$  \hspace{1cm} (2.6)

We shall consider continuity between the fluid and the wall of the pipe displacements. Let us suppose that only radial displacements of the wall are allowed. Then the boundary condition at the interface of the fluid and the wall of the pipe can be expressed as

$$\mathbf{u}^\varepsilon = \left( \varepsilon \frac{\partial R}{\partial t} \right) \mathbf{n}^\varepsilon \text{ at } s_3^\varepsilon = \varepsilon,$$  \hspace{1cm} (2.7)

where $\mathbf{n}^\varepsilon$ is the outward unitary normal at $s_3^\varepsilon = \varepsilon$.

The next step is to write the equations of the problem in the reference domain $\Omega$. Taking into account the change of variable (2.2), we can associate to each vector field $\mathbf{w}^\varepsilon$ in $\hat{\Omega}_t^\varepsilon$, a new vector field $\mathbf{w}(\varepsilon)$ defined in $\Omega$, as follows

$$w_i^\varepsilon = \mathbf{w}^\varepsilon \cdot e_i = (w_k^\varepsilon e_k) \cdot e_i = (w_k^\varepsilon) e_k \cdot e_i =: w_k^\varepsilon v_{ki},$$  \hspace{1cm} (2.8)

where $\{e_1, e_2, e_3\}$ is an orthonormal basis, we are using the Einstein summation convention (where latin indices indicate sum from 1 to 3), and we denote $v_{ki} := v_k \cdot e_i$.

2.1. Computing the Jacobian of the inverse mapping of the change of variable

As first step, we shall need to study the inverse mapping of the change of variable (2.2), in particular, of its Jacobian, which terms will be needed to write Navier-Stokes equations in the reference domain. Let us consider the mapping:

$$\tilde{\phi}^\varepsilon : [0,T] \times \Omega \longrightarrow [0,T] \times \hat{\Omega}_t^\varepsilon$$

$$\tilde{\phi}^\varepsilon(t, s_1, s_2, s_3) := (t^\varepsilon, x^\varepsilon) = (t^\varepsilon, \phi^\varepsilon(s_1, s_2, s_3)), $$  \hspace{1cm} (2.9)

$$\frac{\partial \mathbf{u}^\varepsilon}{\partial t} + (\nabla \mathbf{u}^\varepsilon)\mathbf{u}^\varepsilon = \frac{1}{\rho_0} \text{div} \mathbf{T}^\varepsilon + \mathbf{b}_0^\varepsilon,$$  \hspace{1cm} (2.3)

$$\text{div} \mathbf{u}^\varepsilon = 0,$$  \hspace{1cm} (2.4)

where $\mathbf{u}^\varepsilon$ stands for the velocity field, $\mathbf{b}_0^\varepsilon$ is the density of body forces and $\mathbf{T}^\varepsilon$ is the stress tensor given by

$$\mathbf{T}^\varepsilon = -p^\varepsilon \mathbf{I} + 2\mu \Sigma^\varepsilon,$$

where $p^\varepsilon$ is the pressure field, $\mu$ the dynamic viscosity and $\Sigma^\varepsilon = \frac{1}{2} (\nabla \mathbf{u}^\varepsilon + (\nabla \mathbf{u}^\varepsilon)^T)$.

Let $\nu = \mu/\rho_0$ be the kinematic viscosity, so we can write these equations,

$$\frac{\partial \mathbf{u}^\varepsilon}{\partial t} + (\nabla \mathbf{u}^\varepsilon)\mathbf{u}^\varepsilon + \frac{1}{\rho_0} \nabla p^\varepsilon - \nu \Delta \mathbf{u}^\varepsilon = \mathbf{b}_0^\varepsilon,$$  \hspace{1cm} (2.5)

$$\text{div} \mathbf{u}^\varepsilon = 0.$$  \hspace{1cm} (2.6)

We shall consider continuity between the fluid and the wall of the pipe displacements. Let us suppose that only radial displacements of the wall are allowed. Then the boundary condition at the interface of the fluid and the wall of the pipe can be expressed as

$$\mathbf{u}^\varepsilon = \left( \varepsilon \frac{\partial R}{\partial t} \right) \mathbf{n}^\varepsilon \text{ at } s_3^\varepsilon = \varepsilon,$$  \hspace{1cm} (2.7)

where $\mathbf{n}^\varepsilon$ is the outward unitary normal at $s_3^\varepsilon = \varepsilon$.

The next step is to write the equations of the problem in the reference domain $\Omega$. Taking into account the change of variable (2.2), we can associate to each vector field $\mathbf{w}^\varepsilon$ in $\hat{\Omega}_t^\varepsilon$, a new vector field $\mathbf{w}(\varepsilon)$ defined in $\Omega$, as follows

$$w_i^\varepsilon = \mathbf{w}^\varepsilon \cdot e_i = (w_k^\varepsilon e_k) \cdot e_i = (w_k^\varepsilon) e_k \cdot e_i =: w_k^\varepsilon v_{ki},$$  \hspace{1cm} (2.8)

where $\{e_1, e_2, e_3\}$ is an orthonormal basis, we are using the Einstein summation convention (where latin indices indicate sum from 1 to 3), and we denote $v_{ki} := v_k \cdot e_i$.

2.1. Computing the Jacobian of the inverse mapping of the change of variable

As first step, we shall need to study the inverse mapping of the change of variable (2.2), in particular, of its Jacobian, which terms will be needed to write Navier-Stokes equations in the reference domain. Let us consider the mapping:

$$\tilde{\phi}^\varepsilon : [0,T] \times \Omega \longrightarrow [0,T] \times \hat{\Omega}_t^\varepsilon$$

$$\tilde{\phi}^\varepsilon(t, s_1, s_2, s_3) := (t^\varepsilon, x^\varepsilon) = (t^\varepsilon, \phi^\varepsilon(s_1, s_2, s_3)), $$  \hspace{1cm} (2.9)
hence, the associated Jacobian denoted by $J_\phi$ is

$$
J_\phi = 
\begin{pmatrix}
\frac{\partial t^\varepsilon}{\partial t} & \frac{\partial s_1}{\partial t} & \frac{\partial s_2}{\partial t} & \frac{\partial s_3}{\partial t} \\
\frac{\partial x_1^\varepsilon}{\partial t} & \frac{\partial x_1^\varepsilon}{\partial s_1} & \frac{\partial x_2^\varepsilon}{\partial s_2} & \frac{\partial x_3^\varepsilon}{\partial s_3} \\
\frac{\partial x_2^\varepsilon}{\partial t} & \frac{\partial x_2^\varepsilon}{\partial s_1} & \frac{\partial x_2^\varepsilon}{\partial s_2} & \frac{\partial x_3^\varepsilon}{\partial s_3} \\
\frac{\partial x_3^\varepsilon}{\partial t} & \frac{\partial x_3^\varepsilon}{\partial s_1} & \frac{\partial x_3^\varepsilon}{\partial s_2} & \frac{\partial x_3^\varepsilon}{\partial s_3}
\end{pmatrix}
= 
\begin{pmatrix}
\frac{\partial t^\varepsilon}{\partial t} & \frac{\partial s_1}{\partial t} & \frac{\partial s_2}{\partial t} & \frac{\partial s_3}{\partial t} \\
\frac{\partial x_1^\varepsilon}{\partial t} & \frac{\partial x_1^\varepsilon}{\partial s_1} & \frac{\partial x_2^\varepsilon}{\partial s_2} & \frac{\partial x_3^\varepsilon}{\partial s_3} \\
\frac{\partial x_2^\varepsilon}{\partial t} & \frac{\partial x_2^\varepsilon}{\partial s_1} & \frac{\partial x_2^\varepsilon}{\partial s_2} & \frac{\partial x_3^\varepsilon}{\partial s_3} \\
\frac{\partial x_3^\varepsilon}{\partial t} & \frac{\partial x_3^\varepsilon}{\partial s_1} & \frac{\partial x_3^\varepsilon}{\partial s_2} & \frac{\partial x_3^\varepsilon}{\partial s_3}
\end{pmatrix}
= 
\begin{pmatrix}
\frac{\partial t^\varepsilon}{\partial t} & \frac{\partial s_1}{\partial t} & \frac{\partial s_2}{\partial t} & \frac{\partial s_3}{\partial t} \\
\frac{\partial x_1^\varepsilon}{\partial t} & \frac{\partial x_1^\varepsilon}{\partial s_1} & \frac{\partial x_2^\varepsilon}{\partial s_2} & \frac{\partial x_3^\varepsilon}{\partial s_3} \\
\frac{\partial x_2^\varepsilon}{\partial t} & \frac{\partial x_2^\varepsilon}{\partial s_1} & \frac{\partial x_2^\varepsilon}{\partial s_2} & \frac{\partial x_3^\varepsilon}{\partial s_3} \\
\frac{\partial x_3^\varepsilon}{\partial t} & \frac{\partial x_3^\varepsilon}{\partial s_1} & \frac{\partial x_3^\varepsilon}{\partial s_2} & \frac{\partial x_3^\varepsilon}{\partial s_3}
\end{pmatrix}
\begin{pmatrix}
\nabla_x x^\varepsilon
\end{pmatrix},
$$

where $s = (s_1, s_2, s_3)$. Since

$$
t^\varepsilon = t, \\
x^\varepsilon = c(s_1) + \varepsilon s_3 R(t, s_1)((\cos s_2)v_2(s_1) + (\sin s_2)v_3(s_1)),
$$

then,

$$
\frac{\partial t^\varepsilon}{\partial t} = 1, \\
\frac{\partial t^\varepsilon}{\partial s_i} = 0, \\
\frac{\partial x^\varepsilon}{\partial t} = \varepsilon s_3 \frac{\partial R}{\partial t}((\cos s_2)v_2(s_1) + (\sin s_2)v_3(s_1)), \\
\frac{\partial x^\varepsilon}{\partial s_1} = c'(s_1) + \varepsilon s_3 \frac{\partial R}{\partial s_1}((\cos s_2)v_2(s_1) + (\sin s_2)v_3(s_1)) + \varepsilon s_3 R((\cos s_2)v'_2(s_1) + (\sin s_2)v'_3(s_1)), \\
\frac{\partial x^\varepsilon}{\partial s_2} = \varepsilon s_3 R(-(\sin s_2)v_2(s_1) + (\cos s_2)v_3(s_1)), \\
\frac{\partial x^\varepsilon}{\partial s_3} = \varepsilon R((\cos s_2)v_2(s_1) + (\sin s_2)v_3(s_1)).
$$

The inverse mapping $(\tilde{\phi})^{-1} : [0, T] \times \tilde{\Omega} \rightarrow [0, T] \times \Omega$ is such that its Jacobian, denoted by $J_{\tilde{\phi}}^{-1}$, is

$$
J_{\tilde{\phi}}^{-1} = 
\begin{pmatrix}
\frac{\partial t}{\partial t^\varepsilon} & \frac{\partial t}{\partial x_1^\varepsilon} & \frac{\partial t}{\partial x_2^\varepsilon} & \frac{\partial t}{\partial x_3^\varepsilon} \\
\frac{\partial s_1}{\partial t^\varepsilon} & \frac{\partial s_1}{\partial x_1^\varepsilon} & \frac{\partial s_1}{\partial x_2^\varepsilon} & \frac{\partial s_1}{\partial x_3^\varepsilon} \\
\frac{\partial s_2}{\partial t^\varepsilon} & \frac{\partial s_2}{\partial x_1^\varepsilon} & \frac{\partial s_2}{\partial x_2^\varepsilon} & \frac{\partial s_2}{\partial x_3^\varepsilon} \\
\frac{\partial s_3}{\partial t^\varepsilon} & \frac{\partial s_3}{\partial x_1^\varepsilon} & \frac{\partial s_3}{\partial x_2^\varepsilon} & \frac{\partial s_3}{\partial x_3^\varepsilon}
\end{pmatrix}
= 
\begin{pmatrix}
1 & 0 & 0 & 0 \\
\frac{\partial s}{\partial t^\varepsilon} & \nabla_x x^\varepsilon & s
\end{pmatrix}.
$$

4
Therefore, since \( J_\phi J_\phi^{-1} = I \), we find the following relations,

\[
\nabla x s = (\nabla x s^e)^{-1},
\]

\[
\frac{\partial s}{\partial t^e} = - (\nabla x s^e) \frac{\partial x^e}{\partial t^e}.
\]

(2.11)

(2.12)

In order to compute \( \frac{\partial s_i}{\partial x^e} \), let us write \( \frac{\partial s_i}{\partial x^e} = \alpha_i v_1 + \beta_i v_2 + \gamma_i v_3 \).

We know, by Frenet-Serret formulas, that the following equalities hold

\[
\begin{align*}
\mathbf{v}'_1(s_1) &= \kappa(s_1) \mathbf{v}_2(s_1), \\
\mathbf{v}'_2(s_1) &= -\kappa(s_1) \mathbf{v}_1(s_1) + \tau(s_1) \mathbf{v}_3(s_1), \\
\mathbf{v}'_3(s_1) &= -\tau(s_1) \mathbf{v}_2(s_1),
\end{align*}
\]

(2.13)

where the functions \( \kappa \) and \( \tau \) denote the curvature and torsion of the middle line of the curved pipe. Now, by (2.11), we have that

\[
\frac{\partial s_i}{\partial x^e} \cdot \frac{\partial x^e}{\partial s_j} = \delta_{ij},
\]

(2.14)

where \( \delta_{ij} \) is the Kronecker’s delta. For \( i = 1 \) we find that

\[
1 = (\alpha_1 v_1 + \beta_1 v_2 + \gamma_1 v_3) \cdot \left( v_1 + \varepsilon s_3 \frac{\partial R}{\partial s_1} (\cos s_2 v_2 + \sin s_2 v_3) \right.
\]

\[
+ \varepsilon s_3 R (\cos s_2 v'_2 + \sin s_2 v'_3) \right)
\]

\[
= \alpha_1 (1 + \varepsilon s_3 R (\cos s_2 (v'_2 \cdot v_1) + \sin s_2 (v'_3 \cdot v_1)))
\]

\[
+ \beta_1 \varepsilon s_3 \left( \frac{\partial R}{\partial s_1} \cos s_2 + R \sin s_2 (v'_3 \cdot v_2) \right)
\]

\[
+ \gamma_1 \varepsilon s_3 \left( \frac{\partial R}{\partial s_1} \sin s_2 + R \cos s_2 (v'_2 \cdot v_3) \right),
\]

(2.15)

since \( v_1 = c' \), and

\[
0 = (\alpha_1 v_1 + \beta_1 v_2 + \gamma_1 v_3) \cdot (\varepsilon s_3 R (-\sin s_2 v_2 + \cos s_2 v_3))
\]

\[
= -\beta_1 \varepsilon s_3 R \sin s_2 + \gamma_1 \varepsilon s_3 R \cos s_2,
\]

(2.16)

\[
0 = (\alpha_1 v_1 + \beta_1 v_2 + \gamma_1 v_3) \cdot (\varepsilon R (\cos s_2 v_2 + \sin s_2 v_3))
\]

\[
= \beta_1 \varepsilon R \cos s_2 + \gamma_1 \varepsilon R \sin s_2.
\]

(2.17)

From (2.16)–(2.17) is easy to check that \( \beta_1 = \gamma_1 = 0 \). Hence, from (2.15) and (2.13), we obtain that

\[
\alpha_1 = \frac{1}{1 - \varepsilon \kappa(s_1) s_3 R(t, s_1) \cos s_2}
\]

(2.18)
Now, for $i = 2$ in (2.14) we find, on one hand, that

$$
0 = \alpha_2 \left(1 + \varepsilon s_3 R \left(\cos s_2 (v'_2 \cdot v_1) + \sin s_2 (v'_3 \cdot v_1)\right)\right)
+ \beta_2 \varepsilon s_3 \left(\frac{\partial R}{\partial s_1} \cos s_2 + R \sin s_2 (v'_3 \cdot v_2)\right)
+ \gamma_2 \varepsilon s_3 \left(\frac{\partial R}{\partial s_1} \sin s_2 + R \cos s_2 (v'_2 \cdot v_3)\right),
$$

(2.19)

and, on the other hand, that

$$
1 = -\beta_2 \varepsilon s_3 R \sin s_2 + \gamma_2 \varepsilon s_3 R \cos s_2
0 = \beta_2 \varepsilon R \cos s_2 + \gamma_2 \varepsilon R \sin s_2.
$$

where we deduce that,

$$
\beta_2 = -\frac{\sin s_2}{\varepsilon s_3 R(t, s_1)}, \quad \gamma_2 = \frac{\cos s_2}{\varepsilon s_3 R(t, s_1)}.
$$

(2.20)

Therefore, from (2.13) and (2.19), we obtain that

$$
\alpha_2 = -\frac{\tau(s_1)}{1 - \varepsilon \kappa(s_1) s_3 R(t, s_1) \cos s_2}.
$$

(2.21)

Finally, for $i = 3$ in (2.14) we find, on one hand, that

$$
0 = \alpha_3 \left(1 + \varepsilon s_3 R \left(\cos s_2 (v'_2 \cdot v_1) + \sin s_2 (v'_3 \cdot v_1)\right)\right)
+ \beta_3 \varepsilon s_3 \left(\frac{\partial R}{\partial s_1} \cos s_2 + R \sin s_2 (v'_3 \cdot v_2)\right)
+ \gamma_3 \varepsilon s_3 \left(\frac{\partial R}{\partial s_1} \sin s_2 + R \cos s_2 (v'_2 \cdot v_3)\right),
$$

(2.22)

and, on the other hand, that

$$
0 = -\beta_3 \varepsilon s_3 R \sin s_2 + \gamma_3 \varepsilon s_3 R \cos s_2
1 = \beta_3 \varepsilon R \cos s_2 + \gamma_3 \varepsilon R \sin s_2.
$$

where we deduce that,

$$
\beta_3 = \frac{\cos s_2}{\varepsilon R(t, s_1)}, \quad \gamma_3 = \frac{\sin s_2}{\varepsilon R(t, s_1)}.
$$

(2.23)

Therefore, from (2.13) and (2.22), we obtain that

$$
\alpha_3 = -\frac{s_3}{R(t, s_1) (1 - \varepsilon \kappa(s_1) s_3 R(t, s_1) \cos s_2)} \left(\frac{\partial R}{\partial s_1}\right)(t, s_1).
$$

(2.24)
To sum up, we have obtained that
\[
\frac{\partial s_1}{\partial x^\varepsilon} = \frac{1}{1 - \varepsilon \kappa(s_1)s_3 R(t, s_1) \cos s_2} v_1(s_1),
\]
\[
\frac{\partial s_2}{\partial x^\varepsilon} = -\frac{1 - \varepsilon \kappa(s_1)s_3 R(t, s_1) \cos s_2}{\varepsilon s_3 R(t, s_1)} v_1(s_1) - \frac{\sin s_2}{\varepsilon s_3 R(t, s_1)} v_2(s_1)
\]
\[
+ \frac{\cos s_2}{\varepsilon s_3 R(t, s_1)} v_3(s_1),
\]
\[
\frac{\partial s_3}{\partial x^\varepsilon} = -\frac{s_3}{R(t, s_1)} (1 - \varepsilon \kappa(s_1)s_3 R(t, s_1) \cos s_2) \frac{\partial R}{\partial s_1}(t, s_1)v_1(s_1)
\]
\[
+ \frac{\cos s_2}{\varepsilon R(t, s_1)} v_2(s_1) + \frac{\sin s_2}{\varepsilon R(t, s_1)} v_3(s_1),
\]
and from the relation found in (2.12), we deduce that
\[
\frac{\partial s}{\partial t} = -\frac{s_3}{R(t, s_1)} \frac{\partial R}{\partial t}(t, s_1)v_3(s_1).
\]

2.2. Writing Navier-Stokes equations into the reference domain

We are now in conditions to find the expressions of the fields in (2.5)-(2.6) in the reference domain. Firstly, from (2.8), the chain rule and (2.28), we find that
\[
\frac{\partial u^i}{\partial t^e} = \frac{\partial u^e}{\partial t^e} \cdot e_i = \frac{\partial (u^e_i e_k)}{\partial t^e} \cdot e_i = \left( \frac{\partial (u^e_i v_k)}{\partial t} + \frac{\partial (u^e_k v_i)}{\partial s_3} \frac{\partial \varepsilon}{\partial t} \right) \cdot e_i
\]
\[
= \left( \frac{\partial u^e_k}{\partial t} - \frac{s_3}{R(t, s_1)} \frac{\partial R}{\partial t} \frac{\partial u^e_k}{\partial \varepsilon} \right) (v_k \cdot e_i) = (D_t u^e_k)v_{ki},
\]
where $D_t$ is the operator defined by
\[
D_t := \left( \frac{\partial}{\partial t} - \frac{s_3}{R(t, s_1)} \frac{\partial R}{\partial t} \frac{\partial}{\partial s_3} \right).
\]

The components of the non-linear term in (2.5), from (2.8) and the chain rule, can be written as follows,
\[
\frac{\partial u^e_i}{\partial x^j} u^e_j = \frac{\partial (u^e_i e_k)}{\partial x^j} (u^e_k e_i) = \frac{\partial ((u^e_k e_k) \cdot e_i)}{\partial x^j} ((u^e_m e_m) \cdot e_j)
\]
\[
= \frac{\partial ((u^e_k (\varepsilon) v_k) \cdot e_i)}{\partial x^j} ((u^e_m (\varepsilon) v_m) \cdot e_j) = \left( \frac{\partial (u^e_k (\varepsilon) v_k)}{\partial s_3} \frac{\partial s_3}{\partial x^j} \right) (u^e_m (\varepsilon) v_{mj}).
\]

The Laplacian term in (2.5), from (2.8) and the chain rule, leads to
\[
\Delta u^e_i = \Delta (u^e_i \cdot e_i) = \Delta (u^e_k e_k) \cdot e_i = \Delta (u^e_k (\varepsilon) v_k) \cdot e_i = \frac{\partial^2}{\partial (x^j)^2} (u^e_k (\varepsilon) v_{ki})
\]
\[
= \frac{\partial}{\partial x^j} \left( \frac{\partial (u^e_k (\varepsilon) v_k)}{\partial s_3} \frac{\partial s_3}{\partial x^j} \right) = \frac{\partial}{\partial s_m} \left( \frac{\partial (u^e_k (\varepsilon) v_k)}{\partial s_3} \frac{\partial s_3}{\partial x^j} \right) \frac{\partial s_m}{\partial x^j}.
\]
In the same way we obtain the components of the pressure gradient and of the volume forces as follows,
\[
\frac{\partial p}{\partial x_i^\varepsilon} = \frac{\partial p}{\partial s_q} \frac{\partial s_q}{\partial x_i^\varepsilon},
\]
\[
(b_0^\varepsilon)_i = b_0^\varepsilon \cdot e_i = ((b_0^\varepsilon)_k e_k) \cdot e_i = ((b_0(\varepsilon))_k v_k) \cdot e_i = b_{0k}(\varepsilon) v_{ki},
\]
where \(b_{0k}(\varepsilon) := (b_0(\varepsilon))_k\).

Finally, the incompressibility equation (2.6) in the reference domain, using (2.8) and the chain rule, has the following expression,
\[
\text{div } u^\varepsilon = \frac{\partial (u^\varepsilon \cdot e_j)}{\partial x_j^\varepsilon} = \frac{\partial (u_k^\varepsilon (\varepsilon) v_k^\varepsilon) \cdot e_j}{\partial x_j^\varepsilon},
\]
\[
\text{div } u^\varepsilon = \frac{\partial (u_{k}(\varepsilon) v_{kj})}{\partial s_q} \frac{\partial s_q}{\partial x_j^\varepsilon}.
\]

With these considerations, the incompressible Navier-Stokes equations in the reference domain can be written as
\[
D_t(u_k(\varepsilon) v_{ki}) + \left( \frac{\partial (u_k(\varepsilon) v_{ki})}{\partial s_q} \frac{\partial s_q}{\partial x_j^\varepsilon} \right) (u_m(\varepsilon) v_{mj}) - \nu \frac{\partial \left( \frac{\partial (u_k(\varepsilon) v_{ki})}{\partial s_q} \frac{\partial s_m}{\partial x_j^\varepsilon} \right)}{\partial x_j^\varepsilon} = - \frac{1}{\rho_0} \frac{\partial p(\varepsilon)}{\partial s_q} \frac{\partial s_q}{\partial x_i^\varepsilon} + b_{0k}(\varepsilon) v_{ki}, \tag{3.22}
\]
\[
\frac{\partial}{\partial s_q} (u_k(\varepsilon) v_{kj}) \frac{\partial s_q}{\partial x_j^\varepsilon} = 0. \tag{3.23}
\]

Let \( n^\varepsilon = (\cos s_2) v_2(s_1) + (\sin s_2) v_3(s_1) \) the outward unit normal vector at \( s_3 = 1 \). Then, from the boundary condition (2.7) at \( s_3 = \varepsilon \), we have that
\[
\text{div } u^\varepsilon = u_1^\varepsilon \cdot e_1 = u_1(\varepsilon) v_1 = \varepsilon \frac{\partial R}{\partial t} ((\cos s_2) v_2(s_1) + (\sin s_2) v_3(s_1)), \tag{3.34}
\]
hence, we obtain the following boundary conditions for the scaled components of velocity:
\[
\begin{cases}
  u_1(\varepsilon) = 0 & \text{at } s_3 = 1, \\
  u_2(\varepsilon) = \varepsilon \frac{\partial R}{\partial t} \cos s_2 & \text{at } s_3 = 1, \\
  u_3(\varepsilon) = \varepsilon \frac{\partial R}{\partial t} \sin s_2 & \text{at } s_3 = 1.
\end{cases} \tag{3.35}
\]

3. Asymptotic expansion of the solution

3.1. Expansion of the solution on powers of \( \varepsilon \)

Following [4], we assume that the solution of (2.32)-(2.33) admits a formal expansion on powers of \( \varepsilon \), so the components of velocity and pressure fields can be written,
\[
u_k(\varepsilon) = u_k^0 + \varepsilon u_k^1 + \varepsilon^2 u_k^2 + \ldots \tag{3.1}
\]
\[
p(\varepsilon) = \frac{1}{\varepsilon^2} p^0 + \frac{1}{\varepsilon} p^1 + p^2 + \ldots \tag{3.2}
\]
We must remark that this assumption implies, as we shall see later, that the pressure gradient determines the velocity field. Other assumptions can be considered by choosing different order of \( \varepsilon \) in the pressure and velocity fields in (3.1)-(3.2), leading to different conclusions, but we consider that this is the most interesting case.

Substituting (3.1)-(3.2) in the boundary conditions in the reference domain (see (2.35)), we obtain the following boundary conditions for the terms of the asymptotic expansion,

\[
\begin{aligned}
&u_1^k = 0, \quad k \geq 0, \quad \text{at } s_3 = 1, \\
&u_2^0 = u_3^0 = 0 \quad \text{at } s_3 = 1, \\
&u_2^1 = \frac{\partial R}{\partial t} \cos s_2, \quad u_3^1 = \frac{\partial R}{\partial t} \sin s_2 \quad \text{at } s_3 = 1, \\
&u_\alpha^k = 0, \quad k \geq 2, \quad \alpha = 2, 3 \quad \text{at } s_3 = 1.
\end{aligned}
\]

(3.3)

We need to write (2.25)-(2.27) as expansions of \( \varepsilon \). If we remark that

\[
\frac{1}{a + \varepsilon b} = c_0 + c_1 \varepsilon + c_2 \varepsilon^2 + ...
\]

where \( a, b \in \mathbb{R} \), such that \( a \neq 0 \), then is easy to check that

\[
c_k = (-1)^k \frac{b^k}{a^{k+1}}, \quad k \geq 0.
\]

Therefore, with \( a = 1 \) and \( b = -\kappa s_3 R \cos s_2 \) in (2.25)-(2.27), we find that

\[
\frac{\partial s_q}{\partial x^\varepsilon} \cdot e_j = \frac{\partial s_q}{\partial x^\varepsilon} = \frac{1}{\varepsilon} d_{-1j}^q + \varepsilon d_{1j}^q + \varepsilon^2 d_{2j}^q + ...
\]

(3.4)

with \( q = 1, 2, 3 \) and where,

\[
d_{-1j}^q = 0, \\
\begin{aligned}
d_{-1j}^1 &= \frac{\sin s_2}{R s_3} v_{2j} + \frac{\cos s_2}{R s_3} v_{3j}, \\
d_{-1j}^2 &= \frac{\cos s_2}{R} v_{2j} + \frac{\sin s_2}{R} v_{3j}, \\
d_{-1j}^3 &= \frac{\cos s_2}{R} v_{2j} + \frac{\sin s_2}{R} v_{3j},
\end{aligned}
\]

(3.5)

\[
\begin{aligned}
d_{kj}^1 &= (-1)^k \frac{b^k}{a^{k+1}} v_{1j}, \\
d_{kj}^2 &= (-1)^k \frac{(-\tau) b^k}{a^{k+1}} v_{1j}, \\
d_{kj}^3 &= (-1)^{k+1} \frac{s_3}{R} \frac{\partial R}{\partial s_1} \frac{b^k}{a^{k+1}} v_{1j},
\end{aligned}
\]

(3.6)

with \( k \geq 0 \).

Also, we assume that the applied forces admit an asymptotic expansion of the form

\[
b_{0k}(\varepsilon) = b_{0k}^0 + \varepsilon b_{0k}^1 + \varepsilon^2 b_{0k}^2 + ...
\]
We substitute (3.1)-(3.2) into the equations (2.32)-(2.33) and use (3.4). Hence, we obtain the incompressible Navier-Stokes equations in the reference domain in powers of $\varepsilon$:

$$
D_t((u^0_k + \varepsilon u^1_k + \varepsilon^2 u^2_k + ...)v_{ki}) + \left( \frac{\partial((u^0_k + \varepsilon u^1_k + \varepsilon^2 u^2_k + ...)v_{ki})}{\partial s_q} \right) \left( \frac{1}{\varepsilon} d^q_{1i} + d^q_{0j} + \varepsilon d^q_{1j} + \varepsilon^2 d^q_{2j} + ... \right)
+ \nu \frac{\partial}{\partial s_m} \left( \frac{\partial((u^0_k + \varepsilon u^1_k + \varepsilon^2 u^2_k + ...)v_{ki})}{\partial s_q} \right) \left( \frac{1}{\varepsilon} d^q_{1i} + d^q_{0j} + \varepsilon d^q_{1j} + \varepsilon^2 d^q_{2j} + ... \right)
+ \varepsilon^2 d^q_{2j} + ... \right)
+ (b^0_{0k} + \varepsilon b^1_{0k} + \varepsilon^2 b^2_{0k} + ...) \ n_{ki},
$$

$$
\frac{\partial}{\partial s_q} \left( ((u^0_k + \varepsilon u^1_k + \varepsilon^2 u^2_k + ...)v_{kj}) \right) \left( \frac{1}{\varepsilon} d^q_{1j} + d^q_{0j} + \varepsilon d^q_{1j} + \varepsilon^2 d^q_{2j} + ... \right) = 0.
$$

(3.8)

(3.9)

3.2. Asymptotic expansion of the flow

In the next proposition we shall show some conditions satisfied by the asymptotic expansion of the flow, that will be used later to characterize the terms of the asymptotic expansion of the solution. Firstly, let $Q^\varepsilon(t, s)$ denotes the flow at position $s_1 = s$ and at time $t$, defined in the original domain by

$$
Q^\varepsilon(t, s) := \int_{s_1=s} u^\varepsilon \cdot n_1 dA.
$$

(3.10)

where $\int_{s_1=s} \phi dA$ represents the surface integral of $\phi$ on the transversal section of $\Omega^\varepsilon_t$ at $s_1 = s$.

Using the mapping (2.2), we define the flow in the reference domain by $Q(\varepsilon)$, that is

$$
Q^\varepsilon(t, s_1) = \varepsilon^2 Q(\varepsilon)(t, s_1),
$$

(3.11)

where

$$
Q(\varepsilon) = R^2 \int_0^{2\pi} \int_0^1 s_3 u_1(\varepsilon) ds_3 ds_2.
$$

(3.12)

We also define the cross-sectional area in the original domain by

$$
A^\varepsilon := \varepsilon^2 A_0 = \varepsilon^2 \pi R^2,
$$

(3.13)

where $A_0$ denotes the cross-sectional area in the reference domain, $A_0 = \pi R^2$. 

10
Proposition 3.1. Let us consider a fluid inside the curved pipe $\hat{\Omega}_t$, which movement is described by the incompressible Navier-Stokes equations (2.5)–(2.6) with the boundary condition (2.7). Let us assume that there exists an asymptotic expansion of the form (3.1)–(3.2) of the problem in the reference domain. Then there exists an asymptotic expansion for the scaled flow in the reference domain of the form

$$Q(\varepsilon) = Q^0 + \varepsilon Q^1 + \varepsilon^2 Q^2 + ...$$

where the term $Q^k$ is defined by

$$Q^k = R^2 \int_0^{2\pi} \int_0^1 s_3 u^k_1 ds_3 ds_2.$$ (3.14)

Moreover, the following relations hold:

$$\frac{\partial Q^0}{\partial s_1} + \frac{\partial A^0}{\partial t} = 0, \quad \frac{\partial Q^k}{\partial s_1} = 0 \quad (k \geq 1).$$ (3.15)

Proof. Let $\tilde{\Omega}_t$ be a portion of the original domain $\hat{\Omega}_t$ between $s_1 = a$ and $s_1 = b$ ($a < b$). From (2.6) and the Gauss Theorem, we deduce that

$$0 = \int_{\tilde{\Omega}_t} \text{div} u^\varepsilon dV = \int_{\partial \tilde{\Omega}_t} u^\varepsilon \cdot n^\varepsilon dA$$

$$= \int_{s_1 = a} u^\varepsilon \cdot n^\varepsilon dA + \int_{s_1 = b} u^\varepsilon \cdot n^\varepsilon dA + \int_{s_3^\varepsilon = \varepsilon R(t, s_1)} u^\varepsilon \cdot n^\varepsilon dA.$$ (3.16)

At the beginning and end of $\tilde{\Omega}_t$ we have that

$$u^\varepsilon \cdot n^\varepsilon = (u_k(\varepsilon)v_k) \cdot (-v_1) = -u_1(\varepsilon) \quad \text{at } s_1 = a,$$

$$u^\varepsilon \cdot n^\varepsilon = (u_k(\varepsilon)v_k) \cdot (v_1) = u_1(\varepsilon) \quad \text{at } s_1 = b.$$ Therefore, from (3.10) and (3.16) we obtain

$$0 = -Q^\varepsilon(t, a) + Q^\varepsilon(t, b) + \int_{s_3^\varepsilon = \varepsilon R(t, s_1)} u^\varepsilon \cdot n^\varepsilon dA.$$ (3.17)

At $s_3^\varepsilon = \varepsilon R$, we must consider continuity between the fluid and the wall of the pipe displacements (see (2.7)), hence

$$u^\varepsilon \cdot n^\varepsilon = \frac{\partial}{\partial t} [c(s_1) + \varepsilon R(t, s_1)(\cos s_2^\varepsilon v_2(s_1) + \sin s_2^\varepsilon v_3(s_1))] \cdot n^\varepsilon = \varepsilon \frac{\partial R}{\partial t},$$

and then,

$$\int_{s_3^\varepsilon = \varepsilon R(t, s_1)} u^\varepsilon \cdot n^\varepsilon dA = \int_{s_1 = a}^{s_1 = b} \int_{s_2 = 0}^{s_2 = 2\pi} \varepsilon^2 R \frac{\partial R}{\partial t} ds_2 ds_1 = \int_{a}^{b} 2\pi \varepsilon^2 R \frac{\partial R}{\partial t} ds_1.$$
Substituting this in (3.17) and dividing the expression by \( b - a \), we obtain that
\[
0 = \frac{Q^\varepsilon(t, b) - Q^\varepsilon(t, a)}{b - a} + \frac{1}{b - a} \int_a^b 2\pi \varepsilon^2 R \frac{\partial R}{\partial t} \, ds.
\]

Taking the limit when \( b \) tends to \( a \) and using (3.13), we obtain the following relation,
\[
0 = \frac{\partial Q^\varepsilon}{\partial s_1} + 2\pi \varepsilon^2 R \frac{\partial R}{\partial t} = \frac{\partial Q^\varepsilon}{\partial s_1} + \frac{\partial A^\varepsilon}{\partial t}.
\]

Now, since \( A^0 = A^\varepsilon/\varepsilon^2 = \pi R^2 \) and \( Q(\varepsilon) = Q^\varepsilon/\varepsilon^2 \), we deduce that
\[
\frac{\partial Q(\varepsilon)}{\partial s_1} + \frac{\partial A^0}{\partial t} = 0. \tag{3.18}
\]

On the other hand, taking into account the expansion for \( u_1(\varepsilon) \) in (3.1) and (3.12), we can deduce that there exists an asymptotic expansion of the form
\[
Q(\varepsilon) = Q^0 + \varepsilon Q^1 + \varepsilon^2 Q^2 + \ldots \tag{3.19}
\]
where
\[
Q^k = R^2 \int_0^{2\pi} \int_0^1 s_3 u^k_1 \, ds_3 \, ds.
\]

Finally, upon substitution of (3.19) in (3.18), we conclude that
\[
\frac{\partial Q^0}{\partial s_1} + \frac{\partial A^0}{\partial t} = 0, \quad \frac{\partial Q^k}{\partial s_1} = 0 \quad (k \geq 1). \tag{3.18}
\]

**Remark 3.2.** The previous result is a direct consequence of the law of conservation of mass. Similar results can be found in previous works, for instance, see [1].

### 3.3. Identifying the terms of the asymptotic expansion of the solution

In order to identify some of the terms of the asymptotic expansion proposed in (3.1)-(3.2), we shall group the terms multiplied by the same power of \( \varepsilon \) in (3.8)-(3.9), obtaining in this way new equations, easier than the original one, that can be solved to identify the mentioned terms of the asymptotic expansion. With this aim, we shall recall a result from [12] (Theorem 2.4), that will be used in the following.

**Theorem 3.3.** Let \( \Omega \) be an open bounded set of class \( C^2 \) in \( \mathbb{R}^n \) and \( \Gamma = \partial \Omega \). Let there be given \( f \in H^{-1}(\Omega), g \in L^2(\Omega), \varphi \in H^{1/2}(\Gamma) \), such that
\[
\int_{\Omega} g \, dx = \int_{\Gamma} \varphi \cdot n \, d\Gamma,
\]

12
Then there exists \( u \in H^1(\Omega), p \in L^2(\Omega) \), which are solutions of the Stokes problem

\[
\begin{align*}
-\nu \Delta u + \nabla p &= f \text{ in } \Omega, \\
\text{div } u &= g \text{ in } \Omega, \\
u \quad \quad u &= \varphi \quad \text{ on } \Gamma.
\end{align*}
\]

\( u \) is unique and \( p \) is unique up to the addition of a constant.

Let us introduce the local cartesian coordinates of the cross section of the pipe at \( s_1 \), as the points \( z \in \mathbb{R}^2 \) defined by

\[
z = (z_2, z_3) = (s_3 \cos s_2, s_3 \sin s_2), \tag{3.20}
\]

and let \( \omega = \{(z_2, z_3) \in \mathbb{R}^2/ z_2^2 + z_3^2 < 1\} \). We can prove now the following theorem, where the first terms of the asymptotic expansion are identified.

**Theorem 3.4.** Let us assume that there exists an asymptotic expansion of the form (3.1)-(3.2). Then:

(i) The term of order zero of velocity, \( u^0 \), verifies

\[
\begin{align*}
u_1^0 &= \frac{R^2}{4 \rho_0 \nu} \frac{\partial p^0}{\partial s_1} (s_3^2 - 1), \\
u_2^0 &= u_3^0 = 0,
\end{align*}
\]

while zeroth order term of pressure, \( p^0 \), is the solution of the problem,

\[
\frac{\partial}{\partial s_1} \left( R^4 \frac{\partial p^0}{\partial s_1} \right) = 16 \nu \rho_0 R \frac{\partial R}{\partial t}, \tag{3.23}
\]

with suitable boundary conditions.

(ii) The components of the first order term of velocity, \( u^1 \), are

\[
\begin{align*}
u_1^1 &= \left[ \frac{3R^3 s_3 \cos s_2}{16 \nu \rho_0} \frac{\partial p^0}{\partial s_1} + \frac{R^2}{4 \nu \rho_0} \frac{\partial p^1}{\partial s_1} \right] (s_3^2 - 1), \\
u_2^1 &= \frac{s_3 R}{16 \rho_0 \nu} \left[ 2 \frac{\partial}{\partial s_1} \left( R^2 \frac{\partial p^0}{\partial s_1} \right) - R^2 s_3^2 \frac{\partial^2 p^0}{\partial s_1^2} \right] \cos s_2, \\
u_3^1 &= \frac{s_3 R}{16 \rho_0 \nu} \left[ 2 \frac{\partial}{\partial s_1} \left( R^2 \frac{\partial p^0}{\partial s_1} \right) - R^2 s_3^2 \frac{\partial^2 p^0}{\partial s_1^2} \right] \sin s_2.
\end{align*}
\]

The first order term of pressure, \( p^1 \), is the solution of the problem,

\[
\frac{\partial}{\partial s_1} \left( R^4 \frac{\partial p^1}{\partial s_1} \right) = 0, \tag{3.27}
\]

with the appropriate boundary conditions.
(iii) The first component of the second order term of velocity, \( u^2 \), is given by

\[
u_1 = \frac{R^2}{16} \left[ \frac{R^2}{4 \rho_0 \nu^2} \frac{\partial^2 p^0}{\partial t \partial s_1} - \frac{R^4 \rho_0 \nu^3 \partial^2 p^0}{16 \rho_0 \nu \partial s_1 \partial s_1^2} - \frac{R^2 \partial^3 p^0}{2 \rho_0 \nu \partial s_1^3} + \frac{11 \kappa^2 R^2 \partial^2 p^0}{8 \rho_0 \nu \partial s_1^3} \right] (s_3^4 - 1)
+ \frac{R^2}{4} \left[ -\frac{1}{4 \rho_0 \nu^2} \frac{\partial}{\partial t} \left( \frac{R^2 \partial p^0}{\partial s_1} \right) + \frac{R^2}{16 \rho_0 \nu^3} \frac{\partial}{\partial s_1} \left( \frac{R^2 \partial p^0}{\partial s_1} \right) \right] \
+ \frac{1}{4 \rho_0 \nu \partial s_1^2} \left( R^2 \partial p^0 \right) - \frac{7 \kappa^2 R^2 \partial p^0}{16 \rho_0 \nu \partial s_1} + \frac{1}{\rho_0 \nu} \frac{\partial \rho}{\partial s_1} - \frac{b_{01}}{\nu} \right] (s_3^2 - 1)
+ \frac{R^6}{1152 \rho_0 \nu^3 \partial s_1 \partial s_1^3} (s_3^3 - 1) + \frac{3 \kappa R^3 \partial p^1}{16 \rho_0 \nu s_1} (s_3^3 - s_3) \cos s_2 \\
+ \frac{5 \kappa^2 R^4 \partial p^0}{64 \rho_0 \nu \partial s_1} (s_3^3 - s_3^2) \cos(2s_2),
\]

and the second order term of pressure, \( p^2 \), is

\[
p^2 = -\frac{R^2 \partial^2 p^0}{4 \partial s_1^2} s_3^2 + p_0^2(t, s_1),
\]

where \( p_0^2(t, s_1) \) is the solution, with the adequate boundary conditions, of the problem

\[
\frac{\partial}{\partial s_1} \left( R^4 \frac{\partial p^0}{\partial s_1} \right) = \frac{\partial}{\partial s_1} \left[ -\frac{3 R^8}{64 \rho_0 \nu^2 \partial s_1 \partial s_1^2} - \frac{R^8 \partial^3 p^0}{12 \partial s_1^4} - \frac{\kappa^2 R^6 \partial p^0}{48 \partial s_1} + \frac{R^5 \partial R \partial p^0}{2 \nu \partial t \partial s_1} \right] \\
- \frac{R^7}{8 \rho_0 \nu^2 \partial s_1} \left( \frac{\partial p^0}{\partial s_1} \right)^2 - \frac{R^4}{2} \left( \frac{\partial R}{\partial s_1} \right)^2 \frac{\partial p^0}{\partial s_1} - \frac{R^5 \partial^2 R \partial p^0}{2 \partial s_1^2} - \frac{R^6 \partial^2 p^0}{6 \nu \partial t \partial s_1} + R^4 \rho_0 b_{01}. \]

Let us consider the local cartesian coordinates at cross section of the pipe at \( s_1 \), defined by \( z = (z_2, z_3) = (s_3 \cos s_2, s_3 \sin s_2) \) (and then, \( s_3, s_2 \) are the local polar coordinates at the same cross section). Let be \( U^2 = (u^2_2, u^2_3) \). Then \( (U^2, p^3) \) is the solution of the following problem

\[
\begin{cases}
\Delta_z U^2 = \frac{R}{\rho_0 \nu} \nabla_z p^3 + F & \text{in } \omega, \\
\text{div}_z U^2 = g & \text{in } \omega, \\
U^2 = 0 & \text{on } \partial \omega,
\end{cases}
\]

where \( \omega = \{(z_2, z_3)/z_2^2 + z_3^2 < 1\} \) and the fields \( g \) and \( F \) are defined, respectively, by

\[
g = \left( -\frac{\kappa R^4 \partial p^0}{2 \rho_0 \nu \partial s_1^2} - \frac{3 \kappa^2 R^6 \partial p^0}{16 \rho_0 \nu \partial s_1} \right) z_2 (z_2^2 + z_3^2) - \frac{3 \kappa R^4 \partial p^0}{16 \rho_0 \nu \partial s_1} z_3 (z_2^2 + z_3^2) \\
+ \left( \frac{9 \kappa R^4 \partial R \partial p^0}{8 \rho_0 \nu \partial s_1 \partial s_1} + \frac{9 \kappa^2 R^6 \partial p^0}{16 \rho_0 \nu \partial s_1} + \frac{3 \kappa^2 R^4 \partial p^0}{16 \rho_0 \nu \partial s_1} \right) z_2 \\
+ \frac{3 \kappa R^4 \partial p^0}{16 \rho_0 \nu \partial s_1} z_3 - \frac{R^3}{4 \rho_0 \nu \partial s_1^2} (z_2^2 + z_3^2) + \frac{R}{4 \rho_0 \nu \partial s_1} \left( R^2 \partial p^1 \partial s_1 \right),
\]

(3.32)
\[
\mathbf{F} = \left( \frac{\kappa R^6}{16 \rho_0^2 \nu^3} \left( \frac{\partial p^0}{\partial s_1} \right)^2 \left( (z_2^2 + z_3^2)^2 + 1 \right) + \frac{\kappa' R^4 \partial p^0}{4 \rho_0 \nu} \frac{\partial}{\partial s_1} + \frac{5 R^2 \kappa}{8 \rho_0 \nu} \frac{\partial}{\partial s_1} \left( R^2 \frac{\partial p^0}{\partial s_1} \right) \right)
\]
\[
+ \left( -\frac{\kappa R^6}{8 \rho_0^2 \nu^3} \left( \frac{\partial p^0}{\partial s_1} \right)^2 - \frac{9 R^4 \kappa \partial^2 p^0}{16 \rho_0 \nu} \frac{\partial}{\partial s_1} - \frac{\kappa' R^4 \partial p^0}{4 \rho_0 \nu} \frac{\partial}{\partial s_1} \right) (z_2^2 + z_3^2) - \frac{\kappa R^4 \partial^2 p^0}{8 \rho_0 \nu} \frac{\partial}{\partial s_1} z_3^2
\]
\[
- \frac{R^2}{\nu} b_{02}, - \frac{\kappa \tau R^4 \partial p^0}{4 \rho_0 \nu} \frac{\partial}{\partial s_1} (z_2^2 + z_3^2 - 1) - \frac{2 R^4 \kappa \partial^2 p^0}{16 \rho_0 \nu} \frac{\partial}{\partial s_1} z_2 z_3 - \frac{R^2}{\nu} b_{03} \right). \quad (3.33)
\]

Problem (3.31) has a unique solution \((U^2, p^3)\), with \(U^2\) unique and \(p^3\) unique up to a function depending on \(t\) and \(s_1\), that can be computed explicitly (see appendix B).

Proof. After substitution of (3.1)–(3.2) in (2.32)–(2.33), we have obtained equations (3.8)–(3.9), and we proceed now to identify the terms of the asymptotic expansion.

(i) Grouping the terms multiplied by \(\varepsilon^{-3}\) in the equation (3.8), we find these two equations related with the zeroth-order term of pressure,
\[
- \frac{\sin s_2}{s_3} \frac{\partial p^0}{\partial s_2} + (\cos s_2) \frac{\partial p^0}{\partial s_3} = 0, \quad \frac{\cos s_2}{s_3} \frac{\partial p^0}{\partial s_2} + (\sin s_2) \frac{\partial p^0}{\partial s_3} = 0.
\]
Therefore, it is clear that \(\frac{\partial p^0}{\partial s_2} = \frac{\partial p^0}{\partial s_3} = 0\), so
\[
p^0 = p^0(t, s_1). \quad (3.34)
\]

This is, the zeroth-order term of pressure does not depend on the cross-sectional variables and only depends on time and on the point \(s_1\) of the middle line of the curved pipe.

If we group now the terms multiplied by \(\varepsilon^{-2}\) in the equation (3.8), we obtain the equations
\[
\frac{1}{(R s_3)^2} \frac{\partial^2 u_1^0}{\partial s_2^2} + \frac{1}{R^2 s_3} \frac{\partial u_1^0}{\partial s_3} + \frac{1}{R^2} \frac{\partial^2 u_1^0}{\partial s_3^2} = \frac{1}{\nu \rho_0} \frac{\partial p^0}{\partial s_1}, \quad (3.35)
\]
\[
\frac{1}{(R s_3)^2} \frac{\partial^2 u_2^0}{\partial s_2^2} + \frac{1}{R^2 s_3} \frac{\partial u_2^0}{\partial s_3} + \frac{1}{R^2} \frac{\partial^2 u_2^0}{\partial s_3^2} = \frac{1}{\nu \rho_0} \left( -\frac{\sin s_2}{R s_3} \frac{\partial p^1}{\partial s_2} + \frac{\cos s_2}{R} \frac{\partial p^1}{\partial s_3} \right), \quad (3.36)
\]
\[
\frac{1}{(R s_3)^2} \frac{\partial^2 u_3^0}{\partial s_2^2} + \frac{1}{R^2 s_3} \frac{\partial u_3^0}{\partial s_3} + \frac{1}{R^2} \frac{\partial^2 u_3^0}{\partial s_3^2} = \frac{1}{\nu \rho_0} \left( \frac{\cos s_2}{R s_3} \frac{\partial p^1}{\partial s_2} + \frac{\sin s_2}{R} \frac{\partial p^1}{\partial s_3} \right). \quad (3.37)
\]

Using the change of variable (3.20) in (3.35), and taking into account the boundary condition in (3.3), we obtain the following problem for the axial component of the zeroth-order term of velocity:
\[
\begin{cases}
\Delta u_1^0 = \frac{R^2}{\nu \rho_0} \frac{\partial p^0}{\partial s_1} \quad \text{in } \omega, \\
u_1^0 = 0 \quad \text{on } \partial \omega.
\end{cases} \quad (3.38)
\]

15
The problem \((3.38)\) has a unique solution, which expression is
\[
u^0_1 = \frac{R^2}{4\rho_0 \nu} \frac{\partial p^0}{\partial s_1} (s_3^2 - 1).
\]

Now, from the first relation in \((3.15)\), we have that
\[
\frac{\partial}{\partial s_1} \left( R^2 \int_0^{2\pi} \int_0^1 s_3 u^0_1 ds_3 ds_2 \right) = -2\pi R \frac{\partial R}{\partial t}.
\]

Hence, using the expression for \(u^0_1\) that we have obtained, we deduce that the left-hand side of last equality verifies
\[
\frac{\partial}{\partial s_1} \left( R^2 \int_0^{2\pi} \int_0^1 R^{2} \frac{\partial p^0}{\partial s_1} s_3(s_3^2 - 1) ds_3 ds_2 \right) = \frac{\partial}{\partial s_1} \left( -\frac{2\pi R^4}{16\rho_0 \nu} \frac{\partial p^0}{\partial s_1} \right)
\]

Since \(p^0\) does not depend on the cross-sectional variables (see \((3.34)\)), we obtain that \(p^0\) satisfies the following equation,
\[
\frac{\partial}{\partial s_1} \left( R^2 \frac{\partial p^0}{\partial s_1} \right) = 16\nu \rho_0 R \frac{\partial R}{\partial t},
\]
that has a unique solution with the appropriate initial and boundary conditions.

If we now group the terms multiplied by \(\varepsilon^{-1}\) in the equation \((3.9)\), we find that
\[
-\frac{\sin s_2}{s_3} \frac{\partial u^0_2}{\partial s_2} + (\cos s_2) \frac{\partial u^0_2}{\partial s_3} + \frac{\cos s_2}{s_3} \frac{\partial u^0_3}{\partial s_2} + (\sin s_2) \frac{\partial u^0_3}{\partial s_3} = 0.
\]

Using the change of variable \((3.20)\) in this last equation and in \((3.36) - (3.37)\), and considering the boundary conditions in \((3.3)\), the cross-sectional components of the zeroth-order term of velocity, denoted by \(U^0 = (u^0_2, u^0_3)\), and the first order term of pressure, \(p^1\), are solution of the problem,
\[
\begin{align*}
\Delta_x U^0 &= \frac{R}{\nu \rho_0} \nabla_x p^1 \quad &\text{in } \omega, \\
\text{div}_x U^0 &= 0 \quad &\text{in } \omega, \\
U^0 &= 0 \quad &\text{on } \partial \omega,
\end{align*}
\]

Applying Theorem \(3.3\) this problem has a unique solution (up to an arbitrary function depending only on \(t\) and \(s_1\), for the pressure term), and this solution is
\[
u^0_2 = \nu^0_3 = 0, \quad p^1 = p^1(t, s_1).
\]
(ii) Grouping the terms multiplied by $\varepsilon^{-1}$ in the equation \[ 3.8 \], we obtain

\[
\frac{1}{(R_s)^2} \frac{\partial^2 u_1^1}{\partial s_2^2} + \frac{1}{R^2} \frac{\partial^2 u_1^1}{\partial s_3^2} + \frac{1}{R^2} \frac{\partial^2 u_1^1}{\partial s_1^2} = \frac{1}{\nu \rho_0} \left( \frac{\partial p^1}{\partial s_1} + R \kappa s_3 \cos s_2 \frac{\partial p^0}{\partial s_1} \right) + \frac{\kappa \cos s_2}{R} \frac{\partial u_1^0}{\partial s_3},
\tag{3.39}
\]

\[
\frac{1}{(R_s)^2} \frac{\partial^2 u_2^1}{\partial s_2^2} + \frac{1}{R^2} \frac{\partial^2 u_2^1}{\partial s_3^2} + \frac{1}{R^2} \frac{\partial^2 u_2^1}{\partial s_1^2} = \frac{1}{\nu \rho_0} \left( \sin s_2 \frac{\partial p^2}{\partial s_1} + \cos s_2 \frac{\partial p^2}{\partial s_1} \right),
\tag{3.40}
\]

\[
\frac{1}{(R_s)^2} \frac{\partial^2 u_3^1}{\partial s_2^2} + \frac{1}{R^2} \frac{\partial^2 u_3^1}{\partial s_3^2} + \frac{1}{R^2} \frac{\partial^2 u_3^1}{\partial s_1^2} = \frac{1}{\nu \rho_0} \left( \cos s_2 \frac{\partial p^2}{\partial s_1} + \sin s_2 \frac{\partial p^2}{\partial s_1} \right),
\tag{3.41}
\]

Using (3.21) in (3.39), and then the change of variable (3.20) and the boundary condition (3.3), we obtain that $u_1^1$ is the unique solution of the problem

\[
\begin{aligned}
\Delta_x u_1^1 &= \frac{R^2}{\nu \rho_0} \left( \frac{\partial p^1}{\partial s_1} + \frac{3R \kappa}{2} \frac{\partial p^0}{\partial s_1} \right) \quad \text{in } \omega, \\
\left. u_1^1 \right|_{\partial \omega} &= 0.
\end{aligned}
\tag{3.42}
\]

Now, it is easy to check (for example, by substitution in (3.39)) that the unique solution is

\[
u_1^1 = \left[ \frac{3R^3 \kappa s_3 \cos s_2}{16 \nu \rho_0} \frac{\partial p^0}{\partial s_1} + \frac{R^2}{4 \nu \rho_0} \frac{\partial p^1}{\partial s_1} \right] (s_3^2 - 1).
\]

From the second relation in (3.15), for $k = 1$, we have that

\[
\frac{\partial}{\partial s_1} \left( R^2 \int_0^{2\pi} \int_0^1 s_3 u_1^1 ds_3 ds_2 \right) = 0,
\]

and, substituting the expression of $u_1^1$, we obtain that

\[
0 = \frac{\partial}{\partial s_1} \left( R^2 \int_0^{2\pi} \int_0^1 \left[ \frac{3R^3 \kappa s_3 \cos s_2}{16 \nu \rho_0} \frac{\partial p^0}{\partial s_1} + \frac{R^2}{4 \nu \rho_0} \frac{\partial p^1}{\partial s_1} \right] s_3 (s_3^2 - 1) ds_3 ds_2 \right)
\]

\[
= \frac{\partial}{\partial s_1} \left( R^2 \int_0^{2\pi} \int_0^1 \frac{R^2}{4 \rho_0 \nu} \frac{\partial p^1}{\partial s_1} s_3 (s_3^2 - 1) ds_3 ds_2 \right)
\]

\[
= \frac{\partial}{\partial s_1} \left( \frac{2\pi R^4}{16 \rho_0 \nu} \frac{\partial p^1}{\partial s_1} \right),
\]

so we conclude that $p^1$ (remember that it is only function of $t$ and $s_1$) is solution of

\[
\frac{\partial}{\partial s_1} \left( R^4 \frac{\partial p^1}{\partial s_1} \right) = 0,
\]

with suitable initial and boundary conditions.
If we now group the terms multiplied by $\varepsilon^0$ in (3.9), we obtain

$$
\begin{align*}
- \frac{s_2}{s_3} \frac{\partial u_2^1}{\partial s_2} + (\cos s_2) \frac{\partial u_2^1}{\partial s_3} + \frac{\cos s_2}{s_3} \frac{\partial u_3^1}{\partial s_2} + (\sin s_2) \frac{\partial u_3^1}{\partial s_3} & \\
+ R \left( \frac{\partial u_1^0}{\partial s_1} - \tau \frac{\partial u_1^0}{\partial s_2} - \frac{s_3}{R} \frac{\partial R}{\partial s_1} \frac{\partial u_1^0}{\partial s_3} \right) & = 0.
\end{align*}
$$

Let be $U_1 = (u_1^1, u_1^2)$. Applying the change of variable (3.20) to the previous equation, to (3.40)–(3.41) and taking into account the boundary conditions (3.3), we find that $(U_1, p^2)$ is solution of the problem

$$
\begin{align*}
\Delta z U_1^1 &= R \nu \rho_0 \nabla z p^2 \quad \text{in } \omega, \\
\text{div}_z U_1^1 &= \frac{R}{4\rho_0 \nu} \left( \frac{\partial}{\partial s_1} \left( R^2 \frac{\partial p^0}{\partial s_1} \right) - (z_2^2 + z_3^2) R^2 \frac{\partial^2 p^0}{\partial s_1^2} \right) =: g^1 \quad \text{in } \omega, \\
U_1^1 &= \frac{\partial R}{\partial t} \left( \cos s_2, \sin s_2 \right) =: \varphi^1 \quad \text{on } \partial \omega.
\end{align*}
$$

Theorem 3.3 ensures the existence and uniqueness of solution of this problem (up to an arbitrary function depending only on $t$ and $s_1$, for the pressure term) if the compatibility condition given by

$$
\int_\omega g^1 = \int_{\partial \omega} \varphi^1 \cdot n,
$$

where $n = (\cos s_2, \sin s_2)$ is the unit outward normal vector on $\partial \omega$, is fulfilled. On one hand, we have

$$
\begin{align*}
\int_\omega g^1 &= \int \frac{R}{4\rho_0 \nu} \left( \frac{\partial}{\partial s_1} \left( R^2 \frac{\partial p^0}{\partial s_1} \right) - (z_2^2 + z_3^2) R^2 \frac{\partial^2 p^0}{\partial s_1^2} \right) dz_2 dz_3 \\
&= \int_0^1 \int_0^{2\pi} s_3 R \left( \frac{\partial}{\partial s_1} \left( R^2 \frac{\partial p^0}{\partial s_1} \right) - s_2^2 R^2 \frac{\partial^2 p^0}{\partial s_1^2} \right) ds_2 ds_3 \\
&= \frac{2\pi R}{4\rho_0 \nu} \int_0^1 \left( s_3 \frac{\partial}{\partial s_1} \left( R^2 \frac{\partial p^0}{\partial s_1} \right) - s_2^2 R^2 \frac{\partial^2 p^0}{\partial s_1^2} \right) ds_3 \\
&= \frac{2\pi R}{16\rho_0 \nu} \left( 2 \frac{\partial}{\partial s_1} \left( R^2 \frac{\partial p^0}{\partial s_1} \right) - R^2 \frac{\partial^2 p^0}{\partial s_1^2} \right).
\end{align*}
$$

On the other hand,

$$
\int_{\partial \omega} \varphi^1 \cdot n = \int_0^{2\pi} \frac{\partial R}{\partial t} ds_2 = 2\pi \frac{\partial R}{\partial t}.
$$

Therefore, the compatibility condition (3.44) is equivalent to

$$
\frac{R}{16\rho_0 \nu} \left( 2 \frac{\partial}{\partial s_1} \left( R^2 \frac{\partial p^0}{\partial s_1} \right) - R^2 \frac{\partial^2 p^0}{\partial s_1^2} \right) = \frac{\partial R}{\partial t}.
$$
and it is easy to deduce from (3.23) that (3.45) is verified. Then, we can conclude that there exists a solution \((\mathbf{U}^1, p^2)\) of (3.43) such that \(\mathbf{U}^1\) is unique and \(p^2\) is unique up to a function depending on \(t\) and \(s_3\). This solution can be computed explicitly (see appendix A for the details), and it is

\[
\mathbf{U}^1 = \frac{R}{16 \rho_0 \nu} \left( 2 \frac{\partial}{\partial s_1} \left( R^2 \frac{\partial p_0}{\partial s_1} \right) - (z_2^2 + z_3^2) \frac{R^2 \partial^2 p_0}{\partial s_1^2} \right) (z_2, z_3),
\]

\[
p^2 = -\frac{R^2 \partial^2 p_0}{4 \partial s_3^2} (z_2^2 + z_3^2) + p_0^2 (t, s_3),
\]

where the unknown function \(p_0^2\) will be determined later (see (3.51)).

(iii) Grouping the terms multiplied by \(\varepsilon^0 = 1\) in (3.8), we find

\[
\frac{\partial (u_k^0 v_{ki})}{\partial t} - s_3 \frac{\partial R}{\partial t} \frac{\partial (u_k^0 v_{ki})}{\partial s_3} + \left( - \frac{\sin s_2}{R} \frac{u_2^1}{u_3} + \frac{\cos s_2}{R} \frac{u_3^1}{u_3} \right) \frac{\partial (u_k^0 v_{ki})}{\partial s_3} + \frac{\partial (u_k^0 v_{ki})}{\partial s_3} - \tau \frac{\partial (u_k^0 v_{ki})}{\partial s_3} \frac{u_0^1}{s_3} - \tau \frac{\partial (u_k^0 v_{ki})}{\partial s_3} \frac{u_0^1}{s_3}
\]

\[
- \nu \left( \kappa^2 \cos s_2 \sin s_2 \frac{\partial (u_k^0 v_{ki})}{\partial s_3} - \kappa^2 s_3 \cos^2 s_2 \frac{\partial (u_k^0 v_{ki})}{\partial s_3} + \frac{\partial^2 (u_k^0 v_{ki})}{\partial s_3^2} \right)
\]

\[
- \tau' \frac{\partial (u_k^0 v_{ki})}{\partial s_2} - s_3 \frac{\partial R}{\partial s_1} \left( \frac{\partial^2 R}{\partial s_1^2} - \frac{2 \left( \frac{\partial R}{\partial s_1} \right)^2}{R} \right) \frac{\partial (u_k^0 v_{ki})}{\partial s_3} - 2 \tau' \frac{\partial (u_k^0 v_{ki})}{\partial s_1} \frac{u_0^1}{s_3}
\]

\[
- 2 s_3 \frac{\partial R}{\partial s_1} \frac{\partial^2 (u_k^0 v_{ki})}{\partial s_1 \partial s_3} + 2 \tau s_3 \frac{\partial R}{\partial s_1} \frac{\partial^2 (u_k^0 v_{ki})}{\partial s_3^2} + \tau' \frac{\partial^2 (u_k^0 v_{ki})}{\partial s_2^2}
\]

\[
+ \frac{s_3^2}{R^2} \left( \frac{\partial R}{\partial s_1} \right)^2 \frac{\partial^2 (u_k^0 v_{ki})}{\partial s_2^2} + \kappa \sin s_2 \frac{\partial (u_k^0 v_{ki})}{\partial s_2} - \kappa \cos s_2 \frac{\partial (u_k^0 v_{ki})}{\partial s_3}
\]

\[
+ \frac{1}{(R s_3)^2} \frac{\partial^2 (u_k^0 v_{ki})}{\partial s_2^2} + \frac{1}{R^2 s_3} \frac{\partial (u_k^0 v_{ki})}{\partial s_3} + \frac{1}{R^2} \frac{\partial^2 (u_k^0 v_{ki})}{\partial s_3^2} \right) = DP_k v_{ki} + b_0 v_{ki}, \quad (3.46)
\]
where,
\[
DP_1^0 = -\frac{1}{\rho_0} \left( \kappa s_3^2 R^2 \cos^2 \frac{s_2}{\rho_0} \frac{\partial p_0}{\partial s_1} + \kappa s_3 R \cos s_2 \frac{\partial p_1}{\partial s_1} - \tau \kappa s_3 R \cos s_2 \frac{\partial p_1}{\partial s_2} \right. \\
\left. - \kappa s_3^2 \cos s_2 \frac{\partial R \partial p_1}{\partial s_3} + \frac{\partial p_2}{\partial s_1} - \tau \frac{\partial p_2}{\partial s_2} - \frac{s_3}{R} \frac{\partial R}{\partial s_1} \frac{\partial p_2}{\partial s_3} \right),
\]
\[
DP_2^0 = -\frac{1}{\rho_0} \left( -\sin s_2 \frac{\partial p_3}{\partial s_2} + \cos s_2 \frac{\partial p_3}{\partial s_2} \right),
\]
\[
DP_3^0 = -\frac{1}{\rho_0} \left( \frac{\cos s_2 \partial p_3}{\partial s_2} + \sin s_2 \frac{\partial p_3}{\partial s_2} \right).
\]

Now, we use the expressions obtained in steps (i) and (ii) (see (3.21)–(3.27) and (3.29)), and we replace them into equation (3.46). Since \{v_1, v_2, v_3\} is an orthonormal basis, we obtain three equations by grouping the terms multiplied by each vector in (3.46). Therefore, from the terms multiplied by \(v_1\) in (3.46), we obtain
\[
\frac{1}{(Rs_3)^2} \frac{\partial^2 u_1^2}{\partial s_2^2} + \frac{1}{Rs_3} \frac{\partial u_1^2}{\partial s_3} + \frac{1}{R^2} \frac{\partial^2 u_1^2}{\partial s_3^2} = \left( \frac{R^2}{4\rho_0 \nu^2} \frac{\partial^2 p_0}{\partial t \partial s_1} - \frac{R^4}{16 \rho_0 \nu^3} \frac{\partial^4 p_0}{\partial s_1^4} - \frac{R^2}{2 \rho_0 \nu} \frac{\partial^2 p_0}{\partial s_1^2} + \frac{7 \kappa^2 R^2}{16 \rho_0 \nu} \frac{\partial p_0}{\partial s_1} \right) s_3^2 \\
+ \frac{R^4}{32 \rho_0^2 \nu^3} \frac{\partial^4 p_0}{\partial s_1^4} s_3^4 - \frac{1}{4 \rho_0 \nu} \frac{\partial}{\partial t} \left( R^2 \frac{\partial p_0}{\partial s_1} \right) + \frac{R^2}{16 \rho_0 \nu} \frac{\partial^2 p_0}{\partial s_1^2} \frac{R^2}{\partial s_1^2} \frac{\partial^2 p_0}{\partial s_1^2} \\
+ \frac{1}{4 \rho_0 \nu} \frac{\partial^2}{\partial s_1^2} \left( R^2 \frac{\partial p_0}{\partial s_1} \right) - \frac{7 \kappa^2 R^2}{16 \rho_0 \nu} \frac{\partial p_0}{\partial s_1} + \frac{1}{\rho_0} \frac{\partial p_0}{\partial s_1} \\
+ \frac{3 \kappa R^2}{2 \rho_0 \nu} \frac{\partial^2 p_1}{\partial s_1^2} s_3 \cos s_2 + \frac{15 \kappa^2 R^2}{8 \rho_0 \nu} \frac{\partial^2 p_0}{\partial s_1^2} s_3^2 \cos^2 s_2 - \frac{b_01}{\nu}.
\]

This equation, together with the boundary condition at \(s_3 = 1\) (see (3.3)), and using the change of variable (3.20), shows that \(u_1^2\) is the unique solution of the problem
\[
\Delta u_1^2 = \left( \frac{R^4}{4 \rho_0 \nu^2} \frac{\partial^2 p_0}{\partial t \partial s_1} - \frac{R^6}{16 \rho_0^2 \nu^3} \frac{\partial^4 p_0}{\partial s_1^4} - \frac{R^4}{2 \rho_0 \nu} \frac{\partial^2 p_0}{\partial s_1^2} + \frac{7 \kappa^2 R^4}{16 \rho_0 \nu} \frac{\partial p_0}{\partial s_1} \right) (z_2^2 + z_3^2) \\
+ \frac{R^6}{32 \rho_0^2 \nu^3} \frac{\partial^4 p_0}{\partial s_1^4} (z_2^2 + z_3^2)^2 - \frac{R^2}{4 \rho_0 \nu^2} \frac{\partial}{\partial t} \left( R^2 \frac{\partial p_0}{\partial s_1} \right) + \frac{R^4}{16 \rho_0^2 \nu^3} \frac{\partial^4 p_0}{\partial s_1^4} \frac{R^2}{\partial s_1^2} \frac{\partial^2 p_0}{\partial s_1^2} \\
+ \frac{7 \kappa^2 R^4}{16 \rho_0 \nu} \frac{\partial p_0}{\partial s_1} \frac{3 \kappa R^3}{2 \rho_0 \nu} \frac{\partial^4 p_1}{\partial s_1^4} z_2 + \frac{15 \kappa^2 R^4}{8 \rho_0 \nu} \frac{\partial^2 p_0}{\partial s_1^2} z_2^2 - \frac{R^2 b_01}{\nu} \right) \text{ in } \omega,
\]
\[
u u_1^2 = 0 \text{ on } \partial \omega.
\]
The solution of (3.47) must be polynomial on \(z_2\) and \(z_3\) and, by inspection in this kind of functions, we can find that \(u_1^2\) is

\[
\begin{align*}
\frac{u_1^2}{16} &= \frac{R^2}{4\rho_0\nu^2} \left( \frac{R^2}{4\rho_0\nu^2} \frac{\partial^2 p_0}{\partial \theta \partial s_1} - \frac{R^4}{16\rho_0^3 \nu^3 s_1 \partial^2 s_1^2} - \frac{R^2}{2\rho_0 \nu} \frac{\partial^2 p_0}{\partial s_1^2} + \frac{11\kappa^2 R^2 \partial p_0}{8\rho_0 \nu} \right) ((z_2^2 + z_3^2)^2 - 1) \\
&\quad + \frac{R^2}{4} \left( -\frac{1}{4\rho_0 \nu^2} \frac{\partial}{\partial t} \left( R^2 \frac{\partial p_0}{\partial s_1} \right) + \frac{R^2}{16\rho_0^2 \nu^3 s_1 \partial s_1^2} \frac{\partial^2 p_0}{\partial s_1^2} + \frac{1}{4\rho_0 \nu} \frac{\partial^2 p_0}{\partial s_1^2} \right) \\
&\quad - \frac{7\kappa^2 R^2 \partial p_0}{16\rho_0 \nu} \frac{\partial}{\partial s_1} \left( \frac{1}{\rho_0 \nu} \frac{\partial \rho_0}{\partial s_1} - \frac{b_{01}}{\nu} \right) (z_2^2 + z_3^2 - 1) + \frac{R^6}{1152\rho_0^2 \nu^4} \frac{\partial p_0 \partial^2 p_0}{\partial s_1^2} ((z_2^2 + z_3^2)^3 - 1) \\
&\quad + \frac{3\kappa R^3 \partial^3 p_1}{16\rho_0 \nu} (z_2^2 + z_3^2 - 1) z_2 + \frac{5\kappa^2 R^4 \partial p_0}{64\rho_0 \nu} (z_2^2 + z_3^2 - 1)(z_2^2 - z_3^2), \\
&\quad (3.48)
\end{align*}
\]

that, using the change of variable (3.20), can be written

\[
\begin{align*}
\frac{u_1^2}{16} &= \frac{R^2}{4\rho_0\nu^2} \left( \frac{R^2}{4\rho_0\nu^2} \frac{\partial^2 p_0}{\partial \theta \partial s_1} - \frac{R^4}{16\rho_0^3 \nu^3 s_1 \partial^2 s_1^2} - \frac{R^2}{2\rho_0 \nu} \frac{\partial^2 p_0}{\partial s_1^2} + \frac{11\kappa^2 R^2 \partial p_0}{8\rho_0 \nu} \right) (s_4^2 - 1) \\
&\quad + \frac{R^2}{4} \left( -\frac{1}{4\rho_0 \nu^2} \frac{\partial}{\partial t} \left( R^2 \frac{\partial p_0}{\partial s_1} \right) + \frac{R^2}{16\rho_0^2 \nu^3 s_1 \partial s_1^2} \frac{\partial^2 p_0}{\partial s_1^2} + \frac{1}{4\rho_0 \nu} \frac{\partial^2 p_0}{\partial s_1^2} \right) \\
&\quad - \frac{7\kappa^2 R^2 \partial p_0}{16\rho_0 \nu} \frac{\partial}{\partial s_1} \left( \frac{1}{\rho_0 \nu} \frac{\partial \rho_0}{\partial s_1} - \frac{b_{01}}{\nu} \right) (s_2^2 - 1) + \frac{R^6}{1152\rho_0^2 \nu^4} \frac{\partial p_0 \partial^2 p_0}{\partial s_1^2} (s_2^2 - 1) \\
&\quad + \frac{3\kappa R^3 \partial^3 p_1}{16\rho_0 \nu} (s_2^2 - s_3) \cos s_2 + \frac{5\kappa^2 R^4 \partial p_0}{64\rho_0 \nu} (s_2^2 - s_3^2) \cos(2s_2), \\
&\quad (3.49)
\end{align*}
\]

Using now (3.15) for \(k = 2\), we have that

\[
\frac{\partial}{\partial s_1} \left( R^2 \int_0^{2\pi} \int_0^1 s_3 u_1^2 ds_3 ds_2 \right) = 0, \\
(3.50)
\]

and, if we substitute the expression of \(u_1^2\) given by (3.49) in (3.50), we obtain that \(p_0^2\) is the solution of the problem

\[
\begin{align*}
\frac{\partial}{\partial s_1} \left( R^4 \frac{\partial p_0^2}{\partial s_1} \right) &= \frac{\partial}{\partial s_1} \left( \frac{3R^8}{64\rho_0^2 \nu^2 s_1 \partial^2 s_1^2} - \frac{R^6 \partial^2 p_0^2}{12 \partial s_1^2} - \frac{\kappa^2 R^6 \partial p_0^2}{48 \partial s_1^2} + \frac{R^5 \partial R \partial p_0^2}{8 \rho_0 \nu} \right) \\
&\quad - \frac{R^7}{8\rho_0^2 \nu^2} \frac{\partial R}{\partial s_1} \left( \frac{\partial p_0^2}{\partial s_1} \right)^2 - \frac{R^4}{2} \left( \frac{\partial R}{\partial s_1} \right)^2 \frac{\partial p_0^2}{\partial s_1} - \frac{R^5 \partial^2 R \partial p_0^2}{2 \partial s_1^2} \\
&\quad - \frac{R^6 \partial R \partial^2 p_0^2}{\partial s_1^2 \partial s_1^2} + \frac{R^6}{6\nu} \frac{\partial R}{\partial s_1} + \frac{R^4 \rho_0 b_{01}}{2}, \\
&\quad (3.51)
\end{align*}
\]

with the adequate initial and boundary conditions. With this equation, we have completed the description of \(p^2\) (see (3.29)–(3.30)).
Identifying now the terms multiplied by \( v_2 \) in \((3.46)\), we obtain

\[
\frac{1}{(R_3 s)^2} \frac{\partial^2 u_2^3}{\partial s_2^2} + \frac{1}{R^2 s^3} \frac{\partial u_2^3}{\partial s_3} + \frac{1}{R^2 s^3} \frac{\partial^2 u_2^3}{\partial s_3^2} = \kappa R^4 \frac{(\partial p^0)}{(\partial s_1)}^2 (s_3^4 + 1)
+ \frac{\kappa R^2}{4 \rho_0 \nu} \frac{\partial p^0}{\partial s_1} + \frac{5 \kappa}{8 \rho_0 \nu} \frac{\partial}{\partial s_1} \left( \frac{R^2 \partial p^0}{\partial s_1} \right)
+ \left( - \frac{\kappa R^4}{8 \rho_0^3 \nu^3} \frac{(\partial p^0)}{(\partial s_1)}^2 - \frac{\kappa}{2 \rho_0 \nu} \frac{\partial}{\partial s_1} \left( \frac{R^2 \partial p^0}{\partial s_1} \right) - \frac{\kappa R^2}{4 \rho_0 \nu} \frac{\partial p^0}{\partial s_1} - \frac{\kappa R^2}{16 \rho_0 \nu} \frac{\partial^2 p^0}{\partial s_1^2} \right) s_3^2
- \frac{\kappa R^2}{8 \rho_0 \nu} \frac{\partial^2 p^0}{\partial s_1^2} s_3^2 \cos^2 s_2 + \frac{1}{\rho_0 \nu} \left( - \frac{\sin s_2 \partial p^3}{R s_3} \frac{\partial p^3}{\partial s_2} + \frac{\cos s_2 \partial p^3}{R} \frac{\partial p^3}{\partial s_3} \right) - b_{02} \frac{\nu}{\nu},
\]

and if we identify the terms of \((3.46)\) multiplied by \( v_3 \),

\[
\frac{1}{(R_3 s)^2} \frac{\partial^2 u_3^3}{\partial s_2^2} + \frac{1}{R^2 s^3} \frac{\partial u_3^3}{\partial s_3} + \frac{1}{R^2 s^3} \frac{\partial^2 u_3^3}{\partial s_3^2} = -\kappa \sin s_2 \cos s_2 \frac{s_3}{16 s_3 \rho_0 \nu} \left( 2 \frac{\partial}{\partial s_1} \left( \frac{R^2 \partial p^0}{\partial s_1} \right) s_3 - \frac{R^2 \partial^2 p^0}{\partial s_1^2} s_3^3 \right)
+ \frac{\kappa \sin s_2 \cos s_2}{16 \rho_0 \nu} \left( 2 \frac{\partial}{\partial s_1} \left( \frac{R^2 \partial p^0}{\partial s_1} \right) - 3 R^2 \frac{\partial^2 p^0}{\partial s_1^2} s_3^2 \right) - \frac{\kappa \tau R^2}{4 \rho_0 \nu} \frac{\partial p^0}{\partial s_1} (s_3^4 - 1)
+ \frac{1}{\rho_0 \nu} \left( \frac{\cos s_2 \partial p^3}{R s_3} \frac{\partial p^3}{\partial s_2} + \frac{\sin s_2 \partial p^3}{R} \frac{\partial p^3}{\partial s_3} \right) - b_{03} \frac{\nu}{\nu}.
\]

Using the change of variable \((3.20)\) and doing some simplifications, we deduce that \( u_2^3 \) and \( u_3^3 \) verify

\[
\Delta u_2^3 = \frac{\kappa R^6}{16 \rho_0^3 \nu^3} \left( \frac{\partial p^0}{\partial s_1} \right)^2 (z_2^2 + z_3^2 + 1) + \frac{\kappa R^4}{8 \rho_0 \nu} \frac{\partial}{\partial s_1} \left( \frac{R^2 \partial p^0}{\partial s_1} \right)
+ \left( - \frac{\kappa R^6}{8 \rho_0^3 \nu^3} \frac{(\partial p^0)}{(\partial s_1)}^2 - \frac{9 \kappa R^4}{16 \rho_0 \nu} \frac{\partial^2 p^0}{\partial s_1^2} - \frac{\kappa R^4}{4 \rho_0 \nu} \frac{\partial p^0}{\partial s_1} \right) (z_2^2 + z_3^2)
- \frac{\kappa R^4}{8 \rho_0 \nu} \frac{\partial^2 p^0}{\partial s_1^2} z_2 + \frac{R}{\rho_0 \nu} \frac{\partial p^3}{\partial z_2} - \frac{R^2}{\nu} b_{02},
\]

\[
\Delta u_3^3 = -\frac{\kappa R^4}{4 \rho_0 \nu} \frac{\partial p^0}{\partial s_1} (z_2^2 + z_3^2 - 1) - \frac{2 \kappa R^4}{16 \rho_0 \nu} \frac{\partial^2 p^0}{\partial s_1^2} z_2 z_3 + \frac{R}{\rho_0 \nu} \frac{\partial p^3}{\partial z_3} - \frac{R^2}{\nu} b_{03}.
\]

Now, if we group the terms multiplied by \( \varepsilon \) in \((3.9)\), we obtain

\[
- \frac{\sin s_2 \partial u_2^3}{R s_3} \frac{\partial u_2^3}{\partial s_2} + \frac{\cos s_2 \partial u_2^3}{R s_3} \frac{\partial u_2^3}{\partial s_3} + \frac{\cos s_2 \partial u_2^3}{R} \frac{\partial u_2^3}{\partial s_3} + \frac{\sin s_2 \partial u_2^3}{R} \frac{\partial u_2^3}{\partial s_3} = -\kappa s_3 R \cos s_2 \left( \frac{\partial u_1^0}{\partial s_1} - \frac{s_3}{R} \frac{\partial R \partial u_1^0}{\partial s_1 \partial s_3} \right)
- \frac{\partial u_1^1}{\partial s_1} + \kappa u_2^1 + \frac{\tau}{\partial s_2} + \frac{s_3}{R} \frac{\partial R \partial u_1^1}{\partial s_1 \partial s_3}.
\]
and using in this equation the expressions obtained in steps (i) and (ii) (see (3.21)–(3.27)),

\[- \sin s_2 \frac{\partial u_2^2}{\partial s_2} + \frac{\cos s_2 \partial u_2^2}{\partial s_2} + (\cos s_2) \frac{\partial u_2^2}{\partial s_2} + (\sin s_2) \frac{\partial u_2^2}{\partial s_2} = -\kappa R^2 s_3 \cos s_2 \left( \frac{1}{4 \rho_0 \nu \frac{\partial}{\partial s_1}} \left( \frac{\partial^2 p_0}{\partial s_1^2} \right) \left( s_3^2 - 1 \right) - \frac{R}{2 \rho_0 \nu \frac{\partial}{\partial s_1}} \left( \frac{\partial R \frac{\partial p_0}{\partial s_1^2}}{\partial s_1} \right) \right) \]

\[- 3R \frac{\partial}{\partial s_1} \left( \kappa R^3 \frac{\partial p_0}{\partial s_1} \right) s_3 (s_3^2 - 1) \cos s_2 - \frac{R}{4 \rho_0 \nu \frac{\partial}{\partial s_1}} \left( \frac{\partial^2 p_1}{\partial s_1^2} \right) \left( s_3^2 - 1 \right) \]

\[+ \frac{\kappa R^2}{16 \rho_0 \nu} \left( 2 \frac{\partial}{\partial s_1} \left( \frac{\partial^2 p_0}{\partial s_1^2} \right) - \frac{\partial^2 p_0}{\partial s_1^2} s_3 \cos s_2 - \frac{3 \kappa R^4 \frac{\partial p_0}{\partial s_1}}{16 \rho_0 \nu \frac{\partial}{\partial s_1}} s_3 (s_3^2 - 1) \sin s_2 \right) \]

\[+ \frac{3 \kappa R^3}{16 \rho_0 \nu} \frac{\partial R p_0}{\partial s_1} (s_3^2 - 1) s_3 \cos s_2 + \frac{R^2}{2 \rho_0 \nu \frac{\partial}{\partial s_1}} \frac{\partial R \frac{\partial p_1}{\partial s_1}}{\partial s_1} s_3^2 =: g. \quad (3.54) \]

Let be \( U^2 = (u_2^2, u_3^2) \). Using the change of variable (3.20), we obtain from the equations (3.52), (3.53), (3.54) and the boundary conditions (3.3), that \((U^2, p^3)\) solves the following problem,

\[
\begin{align*}
\Delta_{z} U^2 &= \frac{R}{\rho_0 \nu} \nabla_{z} p^3 + F \quad \text{in} \quad \omega, \\
\text{div} U^2 &= g \quad \text{in} \quad \omega, \\
U^2 &= 0 \quad \text{on} \quad \partial \omega,
\end{align*}
\]

where,

\[F := \left( \frac{\kappa R^6}{16 \rho_0 \nu^3} \left( \frac{\partial p_0}{\partial s_1} \right)^2 \left( z_2^2 + z_3^2 \right)^2 + 1 \right) + \frac{\kappa R^4}{8 \rho_0 \nu \frac{\partial}{\partial s_1}} \left( \frac{\partial R \frac{\partial p_0}{\partial s_1}}{\partial s_1} \right) \left( z_2^2 + z_3^2 \right) \]

\[+ \left( - \frac{\kappa R^6}{8 \rho_0 \nu^3} \left( \frac{\partial p_0}{\partial s_1} \right)^2 - \frac{9 R^4 \kappa}{16 \rho_0 \nu \frac{\partial}{\partial s_1}} \frac{\partial^2 p_0}{\partial s_1^2} - \frac{\kappa R^4 \frac{\partial p_0}{\partial s_1}}{4 \rho_0 \nu \frac{\partial}{\partial s_1}} \right) \left( z_2^2 + z_3^2 \right) \]

\[- \frac{\kappa R^4}{8 \rho_0 \nu \frac{\partial}{\partial s_1}} \left( z_2^2 - \frac{R^2}{\nu} b_0 \right) - \kappa R^4 \frac{\partial p_0}{\partial s_1} (z_2^2 + z_3^2 - 1) - \frac{2 \kappa R^4}{16 \rho_0 \nu \frac{\partial}{\partial s_1}} z_2 z_3 - \frac{R^2}{\nu} b_0, \quad (3.55) \]

and \( g \) is given by (3.32) (from the definition of \( g \) in (3.54), and using (3.20) after some simplification (see (3.3)), we can obtain that \( g \) is given by (3.32)).

Applying Theorem 3.3, this problem has a unique solution if the compatibility condition

\[
\int_{\omega} g \, dz_2 dz_3 = 0, \quad (3.55)
\]

is fulfilled, or equivalently, using the change of variable (3.20), if

\[
\int_{0}^{2\pi} \int_{0}^{1} s_3 g \, ds_3 ds_2 = 0.
\]
Taking into account the definition of $g$ (see the right-hand side of (3.54)), we obtain that
\[
\int_0^{2\pi} \int_0^1 s_3 g ds_3 ds_2 = \int_0^{2\pi} \int_0^1 s_3 g ds_3 ds_2
= \pi R \frac{\partial}{\partial s_1} \left( R^2 \frac{\partial p_1}{\partial s_1} \right) + \frac{\pi R^2 \frac{\partial R}{\partial s_1} \frac{\partial p_1}{\partial s_1}}{4 \rho_0 \nu} + \frac{\pi R^3 \frac{\partial^2 p_1}{\partial s_1^2}}{8 \rho_0 \nu \frac{\partial s_1^2}{s_1^2}}.
\]

From (3.27) we deduce that
\[
4R^3 \frac{\partial R}{\partial s_1} \frac{\partial p_1}{\partial s_1} + R^4 \frac{\partial^2 p_1}{\partial s_1^2} = 0,
\]
so the compatibility condition (3.55) is verified and the problem (3.31) has uniqueness of solution. Furthermore, since $g$ and $F$ are polynomial on $s_3$ (see (3.32)–(3.33)), the solution of (3.31) must be also polynomial on $s_3$ and we can compute it explicitly (see appendix B for the details).

\[\square\]

4. Behavior of the wall of the pipe

We need to close the equations of the model presented here (see theorem 3.4) with a law describing the behavior of the wall of the pipe, that is, with an equation that allows us to determine $R$. There are different possibilities: a rigid wall, elastic or viscoelastic laws, etc.

The simplest case is when we consider that the wall of the pipe is rigid. We have assumed that $R(t, s_1)$ is given in (2.1)–(2.2), so it is enough to suppose that
\[
\frac{\partial R}{\partial t} = 0
\]
to obtain a rigid wall. If we consider the steady case and a rigid wall of the pipe, our model reduces to the model obtained in [4].

Other simple case is when we consider an algebraic elastic law (see [1]):
\[
p^0 - p_e = \frac{E h_0}{R_0^2} (R - R_0)
\]
where $E$ is the Young modulus of the wall, $h_0$ its thickness, $R_0$ the radius of the cross-section at rest, and $p_e$ is the external pressure.

More complex (elastic or viscoelastic) laws can also be considered (see [1] again).

5. Some numerical examples

In this section we shall present some numerical examples in some representative cases in order to illustrate the behavior of the approximated solution obtained in the previous sections.
We start plotting the main tangential velocity $u_0^1$ and its corrections $u_1^1$ and $u_2^1$. We observe in Figure 1 that $u_0^1$ is a Poiseuille flow (other works as [2, 8] have also shown this behavior).

![Figure 1: Plot of $u_0^1$ field.](image1)

In Figure 2 we can see that $u_1^1$ is a correction of $u_0^1$ that takes into account the curvature of the middle line (the fluid is faster in the side of the cross section of the pipe pointing to N).

![Figure 2: Plot of $u_1^1$ field.](image2)

The correction of order two $u_2^1$, has a complex dependence on various terms (see (3.28)), but it is also similar to a Poiseuille flow (see Figure 3).
We have seen at (3.22) that, at order zero, the transversal velocity is zero, so the tangential velocity is dominant. The first order correction, $U^1 = (u_1^1, u_3^1)$, is related with the expansion and contraction of the pipe wall in radial direction. We can see in Figure 4 different cases depending on the value of $\frac{\partial p^0}{\partial s_1}$ (dp1), $\frac{\partial^2 p^0}{\partial s_1^2}$ (dp2) and $\frac{\partial r}{\partial s_1}$ (dr).

The second order correction of transversal velocity, $U^2 = (u_2^2, u_3^2)$, is related with the recirculation of the fluid in the cross section of the pipe, as we can see in Figure 5 where we show different cases depending on the curvature (k), its derivative (dk) and the torsion (tau) of the middle line of the pipe.

6. Conclusions

A transient model for a newtonian fluid through a curved pipe with moving walls has been obtained. The asymptotic expansions have allowed us to find out the main components of velocity and their corrections. Furthermore, our model reduces to the obtained in [4], when steady case and rigid walls are considered. Plots presented here (see figures 1-5) compare very well with real patterns of fluid flow through a curved pipe and agree with the data available in the literature. A simple algebraic elastic law for the pipe wall has been considered in [4.1], but other more general laws can be used.

7. Acknowledgements

This research was partially supported by Ministerio de Economía y Competitividad under grant MTM2012-36452-C02-01 with the participation of FEDER.
Figure 4: Plot of $(u_2, u_3)$ field.
Figure 5: Plot of \((u_2^2, u_3^2)\) field.
A. Computing $U^1$ and $p^2$

Let us consider $(U^1, p^2)$, the solution of problem (3.43),

\[
\begin{cases}
\Delta z U^1 = \frac{R}{\nu \rho_0} \nabla z p^2 \quad \text{in} \ \omega, \\
\text{div}_z U^1 = \frac{R}{4 \rho_0 \nu} \left( \frac{\partial}{\partial s_1} \left( R^2 \frac{\partial p^0}{\partial s_1} \right) - (z_2^2 + z_3^2) R^2 \frac{\partial^2 p^0}{\partial s_1^2} \right) =: g^1 \quad \text{in} \ \omega, \\
U^1 = \frac{\partial R}{\partial t} (\cos s_2, \sin s_2) =: \varphi^1 \quad \text{on} \ \partial \omega,
\end{cases}
\]

that we have seen in the proof of theorem 3.4 that has a unique solution (in the case of $p^2$, up to an arbitrary function of $t$ and $s_1$). In order to compute $U^1$ and $p^2$, we are going to consider some easier auxiliary problems. First, let us consider the problem

\[
\begin{cases}
\Delta z \varphi = g^1 \quad \text{in} \ \omega, \\
\frac{\partial \varphi}{\partial n} = \varphi^1 \cdot n = \frac{\partial R}{\partial t} \quad \text{on} \ \partial \omega,
\end{cases}
\]

(A.1)

which has a unique solution (up to an arbitrary function of $t$ and $s_1$), since the compatibility condition (3.44) is verified. This problem can be written, using change of variable (3.20), as follows

\[
\begin{cases}
\frac{1}{s_2^3} \frac{\partial^2 \varphi}{\partial s_2^2} + \frac{1}{s_3^3} \frac{\partial \varphi}{\partial s_3} + \frac{\partial^2 \varphi}{\partial s_3^2} = \frac{R}{4 \rho_0 \nu} \left( \frac{\partial}{\partial s_1} \left( R^2 \frac{\partial p^0}{\partial s_1} \right) - s_3^3 R^2 \frac{\partial^2 p^0}{\partial s_1^2} \right) \quad \text{in} \ \omega, \\
\frac{\partial \varphi}{\partial s_3} = \frac{\partial R}{\partial t} \quad \text{on} \ \partial \omega.
\end{cases}
\]

(A.2)

If we look for a solution of the form

\[ \varphi = a(t, s_1) s_2^4 + b(t, s_1) s_3^4 + c(t, s_1), \]

we can identify $a$ and $b$ substituting in (A.2). Taking into account (3.23) to verify that the boundary condition is fulfilled, we find that

\[ \varphi(t, s_1, s_2, s_3) = \frac{s_2^4 R}{16 \rho_0 \nu} \left( \frac{\partial}{\partial s_1} \left( R^2 \frac{\partial p^0}{\partial s_1} \right) - s_3^4 R^2 \frac{\partial^2 p^0}{\partial s_1^2} \right) + c(t, s_1) \]

is the solution of (A.2). Using again (3.20), we obtain that

\[ \varphi(t, s_1, z_2, z_3) = \frac{z_2^4 + z_3^4}{16 \rho_0 \nu} \left( \frac{\partial}{\partial s_1} \left( R^2 \frac{\partial p^0}{\partial s_1} \right) - \frac{s_3^4 R^2}{4} \frac{\partial^2 p^0}{\partial s_1^2} \right) + c(t, s_1). \]

(A.3)

Let be $V = U^1 - \nabla z \varphi$, and $\chi = (-\sin s_2, \cos s_2)$ the unit tangent vector on $\partial \omega$. Then, from (3.43) and (A.1), we obtain that $(V, p^2)$ satisfies the problem

\[
\begin{cases}
\Delta z V = \frac{R}{\nu \rho_0} \nabla z p^2 - \Delta z (\nabla z \varphi) \quad \text{in} \ \omega, \\
\text{div}_z V = 0 \quad \text{in} \ \omega, \\
V \cdot n = U^1 \cdot n - \nabla z \varphi \cdot n = 0 \quad \text{on} \ \partial \omega, \\
V \cdot \chi = U^1 \cdot \chi - \nabla z \varphi \cdot \chi = -\frac{\partial \varphi}{\partial \chi} \quad \text{on} \ \partial \omega.
\end{cases}
\]

(A.5)
Since on $\partial \omega$ we have that $U^1 \cdot \chi = 0$, $\nabla z \varphi \cdot \chi = \frac{\partial \varphi}{\partial \chi} = \frac{\partial \varphi}{\partial s_2} = 0$, and $\varphi$ verifies in $\omega$ that

$$
\nabla z \varphi = \frac{2R}{16 \rho_0 \nu} \left( \frac{\partial}{\partial s_1} \left( R^2 \frac{\partial p^0}{\partial s_1} \right) - \frac{R^2 \partial^2 p^0}{2 \partial s_1^2} \right) (z_2, z_3),
$$

$$
\Delta z (\nabla z \varphi) = -\frac{R^3}{2 \rho_0 \nu} \frac{\partial^2 p^0}{\partial s_1^2} (z_2, z_3),
$$

we obtain that $(V, p^2)$ satisfies the problem

\[
\begin{cases}
\Delta z V = \frac{R}{\nu \rho_0} \nabla z p^2 + \frac{R^3}{2 \rho_0 \nu} \frac{\partial^2 p^0}{\partial s_1^2} (z_2, z_3) & \text{in } \omega, \\
\text{div} z V = 0 & \text{in } \omega, \\
V = 0 & \text{on } \partial \omega,
\end{cases}
\]

(A.6)

Then, by Theorem 3.3, problem (A.6) has a unique solution (up to an arbitrary function of $t$ and $s_1$, in the case of $p^2$), which expression is

$$
V = 0,
$$

$$
p^2 = -\frac{R^2 \partial^2 p^0}{4 \partial s_1^2} (z_2^2 + z_3^2) + p_0^2(t, s_1),
$$

(A.7)

where $p_0^2(t, s_1) = c(t, s_1)$ is a smooth function, which is determined in (3.51). Finally, since $U^1 = V + \nabla z \varphi$, we have that

$$
U^1 = \frac{R}{16 \rho_0 \nu} \left( 2 \frac{\partial}{\partial s_1} \left( R^2 \frac{\partial p^0}{\partial s_1} \right) - \frac{R^2 \partial^2 p^0}{2 \partial s_1^2} \right) (z_2, z_3).
$$

B. Computing $U^2$ and $p^3$

Let us consider $(U^2, p^3)$, solution of the problem

\[
\begin{cases}
\Delta z U^2 = \frac{R}{\rho_0 \nu} \nabla z p^3 + \mathbf{F} & \text{in } \omega, \\
\text{div} U^2 = g & \text{in } \omega, \\
U^2 = 0 & \text{on } \partial \omega,
\end{cases}
\]

(B.1)

where $\mathbf{F}$ and $g$ are given, respectively, by (3.33) and (3.54).

In order to compute $(U^2, p^3)$, we shall consider a decomposition of this problem in some easier ones (as done to compute $(U^1, p^2)$ in appendix A).

First, let us consider the problem,

\[
\begin{cases}
\Delta z \varphi = g & \text{in } \omega, \\
\frac{\partial \varphi}{\partial n} = 0 & \text{on } \partial \omega,
\end{cases}
\]

(B.2)
which has unique solution, since the compatibility condition (3.55) is verified.

Simplifying from (3.54), we have

\[ g = \left( -\frac{\kappa R^4}{2\rho_0 \nu} \frac{\partial^2 p^0}{\partial s_1^2} - \frac{3\kappa' R^4}{16\rho_0 \nu} \frac{\partial p^0}{\partial s_1} \right) s_3^3 \cos s_2 - \frac{3\kappa R^4}{16\rho_0 \nu} \frac{\partial p^0}{\partial s_1} s_3^3 \sin s_2 \]

\[ + \left( \frac{3\kappa R^4}{8\rho_0 \nu} \frac{\partial R}{\partial s_1} \frac{\partial p^0}{\partial s_1} + \frac{9\kappa R^4}{16\rho_0 \nu} \frac{\partial^2 p^0}{\partial s_1^2} + \frac{3\kappa' R^4}{16\rho_0 \nu} \frac{\partial p^0}{\partial s_1} \right) s_3 \cos s_2 \]

\[ + \frac{3\kappa R^4}{16\rho_0 \nu} \frac{\partial p^0}{\partial s_1} s_3 \sin s_2 - \frac{R^3}{4\rho_0 \nu} \frac{\partial^2 p^1}{\partial s_1^2} s_3^3 + \frac{R}{4\rho_0 \nu} \frac{\partial}{\partial s_1} \left( R^2 \frac{\partial p^1}{\partial s_1} \right), \]

that, using (3.20), can be written as in (3.32). From (B.2), and using again (3.20), we have that \( \varphi \) is solution of

\[ \begin{cases} 
1 \frac{\partial^2 \varphi}{s_3^2 \partial s_2^2} + \frac{1}{s_3} \frac{\partial \varphi}{\partial s_3} + \frac{\partial^2 \varphi}{\partial s_3^2} = g \quad \text{in } \omega, \\
\frac{\partial \varphi}{\partial s_3} = 0 \quad \text{on } \partial \omega. 
\end{cases} \]

(B.4)

If we look for a solution of the form

\[ \varphi = a(t, s_1) s_3^5 \cos s_2 + b(t, s_1) s_3^3 \sin s_2 + c(t, s_1) s_3^2 \cos s_2 + d(t, s_1) s_3^2 \sin s_2 
\]

\[ + e(t, s_1) \cos s_2 + f(t, s_1) s_3 \sin s_2 + j(t, s_1) s_3^2 \sin s_2 \]

\[ + h(t, s_1) s_3^2 \cos s_2 + i(t, s_1), \]

(B.5)

where \( a, b, c, d, e, f, g, h, i \) are smooth unknown functions, and we substitute in (B.4), we find that

\[ \varphi(t, s_1, s_2, s_3) = \left( -\frac{R^4}{384 \rho_0 \nu} \left( 8\kappa \frac{\partial^2 p^0}{\partial s_1^2} + 3\kappa' \frac{\partial p^0}{\partial s_1} \right) \cos s_2 - \frac{\kappa R^4}{128 \rho_0 \nu} \frac{\partial p^0}{\partial s_1} \sin s_2 \right) s_3^5 
\]

\[ - \frac{R^3}{64 \rho_0 \nu} \frac{\partial^2 p^1}{\partial s_1^2} s_3^3 + \left( \frac{3R^3}{128 \rho_0 \nu} \left( 6\kappa \frac{\partial R}{\partial s_1} \frac{\partial p^0}{\partial s_1} + 3\kappa R \frac{\partial^2 p^0}{\partial s_1^2} + \kappa' R \frac{\partial p^0}{\partial s_1} \right) \cos s_2 \right) s_3^3 
\]

\[ + \frac{3\kappa R^4}{128 \rho_0 \nu} \frac{\partial p^0}{\partial s_1} \sin s_2 \right) s_3^3 + \frac{R}{16 \rho_0 \nu} \frac{\partial}{\partial s_1} \left( R^2 \frac{\partial p^1}{\partial s_1} \right) s_3^2 
\]

\[ + \left( \left( \frac{5R^4}{384 \rho_0 \nu} \left( 8\kappa \frac{\partial^2 p^0}{\partial s_1^2} + 3\kappa' \frac{\partial p^0}{\partial s_1} \right) - \frac{9R^3}{128 \rho_0 \nu} \left( 6\kappa \frac{\partial R}{\partial s_1} \frac{\partial p^0}{\partial s_1} + \kappa' R \frac{\partial p^0}{\partial s_1} \right) + \kappa R \frac{\partial p^0}{\partial s_1} \right) \cos s_2 \right) s_3 + i(t, s_1), \]

(B.6)

where equation (3.27) has been used to guarantee that the boundary condition in (B.4) is
verified. Coming back to the local cartesian coordinates, we can write (B.6)

\[ \partial \omega (\psi) = \left( -\frac{R^4}{384 \rho_0 \nu} \left( \frac{8 \kappa \partial^2 p^0}{\partial s_1^2} + 3 \kappa \frac{\partial p^0}{\partial s_1} \right) z_2 - \frac{\kappa R^4}{128 \rho_0 \nu} \frac{\partial p^0}{\partial s_1} z_3 \right) \]

\[ - \frac{R^3}{64 \rho_0 \nu} \frac{\partial^2 p^0}{\partial s_1^2} \left( z_2^2 + z_3^2 \right) + \left( \frac{3 R^3}{128 \rho_0 \nu} \left( \frac{6 \kappa \partial R}{\partial s_1} \frac{\partial^2 p^0}{\partial s_1} + 3 \kappa R \frac{\partial^2 p^0}{\partial s_1^2} + \kappa R^2 \frac{\partial^2 p^0}{\partial s_1^3} \right) \right) z_2 \]

\[ + 3 \kappa R^4 \frac{\partial p^0}{\partial s_1} z_3 + \frac{R}{16 \rho_0 \nu} \left( \frac{R^2 \partial p^0}{\partial s_1} \right) \left( z_2^2 + z_3^2 \right) \]

\[ + \left( \frac{5 R^4}{384 \rho_0 \nu} \left( 8 \kappa \frac{\partial^2 p^0}{\partial s_1^2} + 3 \kappa \frac{\partial p^0}{\partial s_1} \right) - \frac{9 R^3}{64 \rho_0 \nu} \left( 6 \kappa \frac{\partial R}{\partial s_1} \frac{\partial^2 p^0}{\partial s_1} + \kappa R \frac{\partial^2 p^0}{\partial s_1} + 3 \kappa R^2 \frac{\partial^2 p^0}{\partial s_1^3} \right) \right) z_2 \]

\[ - \frac{4 \kappa R^4 \partial p^0}{128 \rho_0 \nu} z_3 + i(t, s_1). \] (B.7)

Now, let us consider \( \mathbf{V} = \mathbf{U}^2 - \nabla x \varphi \), and \( \mathbf{X} = (-\sin s_2, \cos s_2) \) the unit tangent vector on \( \partial \omega \). Then, from (B.1) and (B.2), we obtain that \( (\mathbf{V}, p^3) \) satisfies the problem

\[
\begin{align*}
\Delta_x \mathbf{V} &= \frac{R}{\rho_0 \nu} \nabla_x p^3 + \mathbf{F} - \Delta_x (\nabla_x \varphi) \quad \text{in } \omega, \\
\text{div}_x \mathbf{V} &= 0 \quad \text{in } \omega, \\
\mathbf{V} \cdot \mathbf{n} &= \mathbf{U}^2 \cdot \mathbf{n} - \nabla_x \varphi \cdot \mathbf{n} = 0 \quad \text{on } \partial \omega, \\
\mathbf{V} \cdot \mathbf{X} &= \mathbf{U}^2 \cdot \mathbf{X} - \nabla_x \varphi \cdot \mathbf{X} = -\frac{\partial \varphi}{\partial \mathbf{X}} \quad \text{on } \partial \omega.
\end{align*}
\] (B.8)

Let us consider an arbitrary smooth function \( \psi \) and let be \( \mathbf{W} = \mathbf{V} - \left( \frac{\partial \psi}{\partial z_3}, -\frac{\partial \psi}{\partial z_2} \right) \). Then \( \mathbf{W} \) is solution of the problem

\[
\begin{align*}
\Delta_x \mathbf{W} &= \frac{R}{\rho_0 \nu} \nabla_x p^3 + \mathbf{F} - \Delta_x (\nabla_x \varphi) - \Delta_x \left( \frac{\partial \psi}{\partial z_3}, -\frac{\partial \psi}{\partial z_2} \right) \quad \text{in } \omega, \\
\text{div}_x \mathbf{W} &= 0 \quad \text{in } \omega, \\
\mathbf{W} \cdot \mathbf{n} &= \left( \frac{\partial \psi}{\partial z_3}, -\frac{\partial \psi}{\partial z_2} \right) \quad \text{n} = -\nabla_x \psi \cdot \mathbf{X} = -\frac{\partial \psi}{\partial \mathbf{X}} \quad \text{on } \partial \omega, \\
\mathbf{W} \cdot \mathbf{X} &= \left( \frac{\partial \varphi}{\partial \mathbf{X}} + \left( \frac{\partial \psi}{\partial z_3}, -\frac{\partial \psi}{\partial z_2} \right) \right) \cdot \mathbf{X} = -\frac{\partial \varphi}{\partial \mathbf{X}} + \frac{\partial \psi}{\partial \mathbf{n}} \quad \text{on } \partial \omega.
\end{align*}
\] (B.9)

If we are able to find a function \( \psi \) such that

\[
\frac{\partial \psi}{\partial \mathbf{X}} = 0 \quad \text{on } \partial \omega, \quad \frac{\partial \psi}{\partial \mathbf{n}} = \frac{\partial \varphi}{\partial \mathbf{X}} \quad \text{on } \partial \omega,
\] (B.10)

then problem (B.9) will be equivalent to

\[
\begin{align*}
\Delta_x \mathbf{W} &= \frac{R}{\rho_0 \nu} \nabla_x p^3 + \mathbf{F} - \Delta_x (\nabla_x \varphi) - \Delta_x \left( \frac{\partial \psi}{\partial z_3}, -\frac{\partial \psi}{\partial z_2} \right) \quad \text{in } \omega, \\
\text{div}_x \mathbf{W} &= 0 \quad \text{in } \omega, \\
\mathbf{W} &= 0 \quad \text{on } \partial \omega,
\end{align*}
\] (B.11)
and we shall have, by Theorem 3.3, uniqueness of solution \((W, p^3)\) \((W \text{ is unique and } p^3 \text{ is unique up to a function depending on } t \text{ and } s_1)\).

The next step is, following (B.10), finding a function \(\psi\) such that
\[
\frac{\partial \psi}{\partial \chi} = \frac{\partial \psi}{\partial s_2} = 0 \text{ on } \partial \omega, \tag{B.12}
\]
\[
\frac{\partial \psi}{\partial n} = \frac{\partial \psi}{\partial s_3} = \frac{\partial \varphi}{\partial \chi} = \frac{\partial \varphi}{\partial s_2} \text{ on } \partial \omega. \tag{B.13}
\]

From (B.6) and (3.27), we deduce that
\[
\frac{\partial \psi}{\partial s_3} = \left( -\frac{4R^4}{384\rho_0\nu} \left( \frac{8\kappa}{\partial s_1^2} + \frac{3\kappa}{\partial s_1} \right) + \frac{6R^3}{128\rho_0\nu} \left( \frac{6\kappa}{\partial s_1} \frac{\partial p^0}{\partial s_1} + \frac{6\kappa}{\partial s_1} \frac{\partial p^0}{\partial s_1} + 3\kappa R \frac{\partial^2 p^0}{\partial s_1^2} \right) \right) \sin s_2 - \frac{2\kappa R^4}{128\rho_0\nu} \frac{\partial p^0}{\partial s_1} \cos s_2 \text{ on } \partial \omega,
\]
so, one possible election for \(\psi\) is
\[
\psi(t, s_1, s_2, s_3) = \frac{R^3}{384\rho_0\nu} \left( \left( 22\kappa R \frac{\partial^2 p^0}{\partial s_1^2} + 6\kappa R \frac{\partial p^0}{\partial s_1} + 108\kappa \frac{\partial R}{\partial s_1} \frac{\partial p^0}{\partial s_1} \right) \sin s_2 - 6\kappa R \frac{\partial p^0}{\partial s_1} \cos s_2 \right) s_3(s_3 - 1)\frac{2}{2},
\]
that, applying (3.20), can be written
\[
\psi(t, s_1, z_2, z_3) = (\psi(t, s_1)z_2 + \psi_3(t, s_1)z_3) \frac{z_2^2 + z_3^2 - 1}{2} \tag{B.14}
\]
where,
\[
\psi_3(t, s_1) = \frac{R^3}{384\rho_0\nu} \left( 22\kappa R \frac{\partial^2 p^0}{\partial s_1^2} + 6\kappa R \frac{\partial p^0}{\partial s_1} + 108\kappa \frac{\partial R}{\partial s_1} \frac{\partial p^0}{\partial s_1} \right), \tag{B.15}
\]
\[
\psi_2(t, s_1) = -\frac{\kappa R^4}{64\rho_0\nu} \frac{\partial p^0}{\partial s_1}. \tag{B.16}
\]

Then, we have
\[
\left( \frac{\partial \psi}{\partial z_3}, -\frac{\partial \psi}{\partial z_2} \right) = \left( \frac{\psi_3}{2} \left( z_2^2 + 3z_3^2 - 1 \right) + \psi_2 z_2 z_3, -\frac{\psi_2}{2} \left( 3z_2^2 + z_3^2 - 1 \right) - \psi_3 z_2 z_3 \right), \tag{B.17}
\]
and
\[
-\Delta_z \left( \frac{\partial \psi}{\partial z_3}, -\frac{\partial \psi}{\partial z_2} \right) = (-4\psi_3, 4\psi_2). \tag{B.18}
\]
Hence, from (B.11), $W$ is solution of the problem

$$\begin{align*}
\Delta_\omega W &= \nabla_\omega q^2 + F \quad \text{in } \omega, \\
\text{div}_\omega W &= 0 \quad \text{in } \omega, \\
W &= 0 \quad \text{on } \partial\omega,
\end{align*}$$

where

$$q^2 = \frac{R}{\rho_0' \nu} p^3 - g - 4\psi_3z_2 + 4\psi_2z_3 + q_0^2,$$

with $q_0^2 = q_0^2(t, s_1)$ an arbitrary smooth function of $t$ and $s_1$. From Theorem 3.3, we have the existence and uniqueness (up to an arbitrary function of $t$ and $s_1$, in the case of $q^2$) of $(W, q^2)$.

Once we have computed $W$ and $q^2$ (see below), we obtain $U^2$ and $p^3$ in the following way,

$$U^2 = W + \nabla_\omega \varphi + \left( \frac{\partial\psi}{\partial z_3}, -\frac{\partial\psi}{\partial z_2} \right),$$

$$p^3 = \frac{\rho_0' \nu}{R} \left( q^2 + g + 4\psi_3z_2 - 4\psi_2z_3 - q_0^2 \right),$$

where $\varphi$, $\psi$, $\psi_3$ and $\psi_2$ are given, respectively, by (B.7), (B.14), (3.32), (B.15) and (B.16).

Let us now compute $(W, q^2)$. In order to make such computation, let us remark that $F = (F_2, F_3)$ is polynomial in $z_2$ and $z_3$. Developing, from (3.33), $F_2$ and $F_3$ in powers of $z_2$ and $z_3$, we obtain that

$$F_2 = f_2^{00} + f_2^{20}z_2^2 + f_2^{02}z_2^2 + f_2^{22}z_2^2 + f_2^{10}z_3^4 + f_2^{04}z_3^4,$$

$$F_3 = f_3^{00} + f_3^{11}z_2z_3 + f_3^{20}z_2^2 + f_3^{02}z_2^2,$$

where $f_2^{mn}$ denotes the coefficient that multiplies $z_2^m z_3^n$ in $F_2$. Since $F$ is polynomial in $z_2$ and $z_3$, $W = (W_2, W_3)$ and $q^2$ must be also polynomial in $z_2$ and $z_3$, so let us suppose that

$$W_2 = \left( w_2^{00} + w_2^{10}z_2 + w_2^{01}z_2 + w_2^{20}z_2^2 + w_2^{11}z_2z_3 + w_2^{02}z_2^2 + w_2^{21}z_2^2z_3 + w_2^{30}z_3 + w_2^{12}z_2z_3 + w_2^{03}z_3 ight. + w_2^{22}z_2^2z_3 + w_2^{31}z_2z_3 + w_2^{32}z_2z_3 + w_2^{13}z_2z_3 + w_2^{04}z_3^4 \bigg) \left( z_2^2 + z_3^2 - 1 \right),$$

$$W_3 = \left( w_3^{00} + w_3^{10}z_2 + w_3^{01}z_2 + w_3^{20}z_2^2 + w_3^{11}z_2z_3 + w_3^{02}z_2^2 + w_3^{21}z_2^2z_3 + w_3^{30}z_2 + w_3^{12}z_2z_3 + w_3^{03}z_2 ight. + w_3^{22}z_2^2z_3 + w_3^{31}z_2z_3 + w_3^{32}z_2z_3 + w_3^{13}z_2z_3 + w_3^{04}z_3^4 \bigg) \left( z_2^2 + z_3^2 - 1 \right),$$

$$q^2 = q_0^{00} + q_0^{10}z_2 + q_0^{01}z_3 + q_0^{20}z_2^2 + q_0^{11}z_2z_3 + q_0^{02}z_2^2 + q_0^{21}z_2^2z_3 + q_0^{12}z_2z_3 + q_0^{03}z_3^3 + q_0^{30}z_2 + q_0^{22}z_2^2 + q_0^{13}z_2z_3 + q_0^{04}z_2 + q_0^{40}z_3 + q_0^{14}z_2z_3 + q_0^{05}z_3^3 + q_0^{23}z_2^2z_3 + q_0^{15}z_2z_3 + q_0^{06}z_3^5.$$  

By substitution in (B.19), we obtain a linear system with a unique solution, so $(W, q^2)$ was correct.
In this way we find that,

\[
\begin{align*}
  w_2^{00} &= \frac{1}{192} (f_3^{11} - f_2^{04}) - \frac{1}{1152} f_2^{22} - \frac{1}{96} f_2^{02}, \\
  w_2^{02} &= \frac{5}{96} f_2^{02} + \frac{13}{96} f_2^{04} + \frac{31}{576} f_2^{02} - \frac{5}{192} f_3^{11}, \\
  w_2^{20} &= \frac{1}{96} f_2^{02} + \frac{7}{96} f_2^{04} - \frac{11}{576} f_2^{22} - \frac{1}{192} f_3^{11}, \\
  w_2^{40} &= \frac{1}{360} f_2^{22} - \frac{1}{480} f_2^{04}, \\
  w_3^{11} &= \frac{1}{480} f_2^{22} - \frac{1}{40} f_2^{04} - \frac{1}{24} f_2^{02} + \frac{1}{48} f_3^{11}, \\
  w_3^{13} &= \frac{1}{80} f_2^{04} - \frac{1}{240} f_2^{22}, \\
  w_3^{31} &= \frac{1}{80} f_2^{04} - \frac{1}{60} f_2^{22},
\end{align*}
\]

(B.26)

while

\[
\begin{align*}
  w_2^{01} &= w_2^{03} = w_2^{10} = w_2^{12} = w_2^{21} = w_2^{30} = w_2^{31} = w_3^{20} = w_2^{31} = w_3^{40} = 0, \\
  w_3^{13} &= w_3^{31} = w_3^{10} = w_3^{13} = w_3^{20} = w_3^{31} = w_3^{3} = w_3^{31} = 0
\end{align*}
\]

(B.27)

and

\[
\begin{align*}
  q^{10} &= \frac{1}{12} f_3^{11} - \frac{1}{6} f_2^{02} - \frac{1}{16} f_2^{04} - \frac{1}{96} f_2^{22} - f_2^{00}, \\
  q^{12} &= \frac{3}{40} f_2^{22} - \frac{3}{20} f_2^{04} - \frac{1}{4} f_2^{02} - \frac{3}{8} f_3^{11}, \\
  q^{14} &= -\frac{1}{16} f_2^{04} - \frac{5}{96} f_2^{22}, \\
  q^{30} &= \frac{1}{12} f_2^{22} + \frac{1}{20} f_2^{04} - \frac{1}{3} f_2^{02} - \frac{1}{40} f_2^{22} - \frac{1}{24} f_3^{11}, \\
  q^{32} &= \frac{1}{8} f_2^{04} - \frac{11}{48} f_2^{22}, \\
  q^{50} &= \frac{11}{480} f_2^{22} - \frac{1}{80} f_2^{04} - \frac{1}{5} f_2^{22},
\end{align*}
\]

(B.28)

while

\[
q^{02} = q^{04} = q^{05} = q^{11} = q^{13} = q^{20} = q^{22} = q^{23} = q^{31} = q^{30} = q^{41} = 0,
\]

(B.29)

and $q^{00}$ is an arbitrary smooth function depending only on $t$ and on $s_1$.

References

[1] L. Formaggia, D. Lamponi, and A. Quarteroni. One-dimensional models for blood flow in arteries. Journal of Engineering Mathematics, 47:251–276, 2003.

[2] D. Gammack and P. E. Hydon. Flow in pipes with non-uniform curvature and torsion. J. Fluid Mech., 433:357–382, 2001.
[3] W. H. Lyne. Unsteady viscous flow in a curved pipe. *J. Fluid. Mech.*, 45:13–31, 1970.
[4] E. Marušić-Paloka. The effects of flexion and torsion on a fluid flow through a curved pipe. *Appl. Math. Optim.*, 44:245–272, 2001.
[5] E. M. Marušić-Paloka and I. Pažanin. Fluid flow through a helical pipe. *Z. angew. Math. Phys.*, 58:81–89, 2007.
[6] G. Panasenko and K. Pileckas. Asymptotic analysis of the non-steady Navier-Stokes equations in a tube structure. I. The case without boundary-layer-in-time. *Nonlinear Analysis*, 122:125–168, 2015.
[7] G. Panasenko and K. Pileckas. Asymptotic analysis of the non-steady Navier-Stokes equations in a tube structure. II. General case. *Nonlinear Analysis*, 125:582–607, 2015.
[8] G. P. Panasenko and R. Stavre. Asymptotic analysis of a periodic flow in a thin channel with visco-elastic wall. *Journal de Mathématiques Pures et Appliquées*, 85:558–579, 2006.
[9] T. J. Pedley. Mathematical modelling of arterial fluid dynamics. *Journal of Engineering Mathematics*, 47:419–444, 2003.
[10] N. Riley. Unsteady fully-developed flow in a curved pipe. *Journal of Engineering Mathematics*, 34:131–141, 1998.
[11] F. T. Smith. Fluid flow into a curved pipe. *Proc. R. Soc. Lond. A.*, 351:71–87, 1976.
[12] R. Temam. *Navier-Stokes Equations: Theory and Numerical Analysis*. AMS Chelsea Publishing, 2000.