NONCOMMUTATIVE WEIL CONJECTURE

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Abstract. In this article, following an insight of Kontsevich, we extend the Weil conjecture, as well as the strong form of the Tate conjecture, from the realm of algebraic geometry to the broad noncommutative setting of dg categories. Moreover, we establish a functional equation for the noncommutative Hasse-Weil zeta functions, compute the \( l \)-adic and \( p \)-adic absolute values of the eigenvalues of the cyclotomic Frobenius, and provide a complete description of the category of noncommutative numerical motives in terms of Weil \( q \)-numbers. As a first application, we prove the noncommutative Weil conjecture and the noncommutative strong form of the Tate conjecture in several cases: twisted schemes, Calabi-Yau dg categories associated to hypersurfaces, noncommutative gluings of schemes, root stacks, (twisted) global orbifolds, and finite-dimensional dg algebras. As a second application, we provide an alternative noncommutative proof of the Weil conjecture and of the strong form of the Tate conjecture in the particular cases of intersections of two quadrics and linear sections of determinantal varieties.

1. Statement of results

Let \( k = \mathbb{F}_q \) be a finite field, with \( q = p^r \), \( W(k) \) the ring of \( p \)-typical Witt vectors of \( k \), and \( K := W(k)_{1/p} \) the fraction field of \( W(k) \).

Weil conjecture. Given a smooth proper \( k \)-scheme \( X \) of dimension \( d \), its zeta function is defined as the formal power series

\[
Z(X; t) := \exp(\sum_{n \geq 1} \#X(\mathbb{F}_{q^n}) t^n) \in \mathbb{Q}[t],
\]

where \( \exp(t) := \sum_{n \geq 0} \frac{t^n}{n!} \). In the same vein, given an integer \( 0 \leq w \leq 2d \), we can consider the formal power series

\[
Z_w(X; t) := \det(id - t Fr^w|H^w_{\text{crys}}(X))^{-1} \in K[[t]],
\]

where \( H^w_{\text{crys}}(X) \) stands for the crystalline cohomology \( H^w_{\text{crys}}(X/W(k)) \otimes_{W(k)} K \) of \( X \), \( Fr \) for the Frobenius endomorphism of \( X \), and \( Fr^w \) for the induced automorphism of \( H^w_{\text{crys}}(X) \). Thanks to the Lefschetz trace formula established by Grothendieck and Berthelot (see [4, Chapitre VII §3.2]), we have the following weight decomposition:

\[
Z(X; t) = \frac{Z_0(X; t)Z_2(X; t) \cdots Z_{2d}(X; t)}{Z_1(X; t)Z_3(X; t) \cdots Z_{2d-1}(X; t)}.
\]

In the late forties, Weil [56] conjectured the following\(^1\):

Conjecture \( W(X) \): The eigenvalues of the automorphism \( Fr^w \), with \( 0 \leq w \leq 2d \), are algebraic numbers and all their complex conjugates have absolute value \( q^\frac{w}{2} \).

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\(^1\)The conjecture \( W(X) \) is a modern formulation of Weil’s original conjecture; in the late forties crystalline cohomology was not yet developed.

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In the particular case of curves, this famous conjecture follows from Weil’s pioneering work [57]. Later, in the seventies, it was proved in full generality by Deligne2 [12]. In contrast with Weil’s proof, which uses solely the classical intersection theory of divisors on surfaces, Deligne’s proof makes use of several involved tools such as the theory of monodromy of Lefschetz pencils. The Weil conjecture has numerous applications. For example, when combined with (1.1), it implies that the polynomials $p_w(X; t) := \det(id - tFr^w|H^d_{\text{crys}}(X))$ have integer coefficients.

Recall that the Hasse-Weil zeta function of $X$ is defined as the (convergent) infinite product $\zeta(X; s) := \prod_{x \in X^{(d)}} (1 - (q^{\deg(x)})^{-s})^{-1}$, with $\text{Re}(s) > d$, where $X^{(d)}$ stands for the set of closed points of $X$ and $\deg(x)$ for the degree of the finite field extension $\kappa(x)/\mathbb{F}_q$. In the same vein, given an integer $0 \leq w \leq 2d$, consider the function $\zeta_w(X; s) := \det(id - q^{-s}Fr^w|H^w_{\text{crys}}(X))^{-1}$. It follows from the Weil conjecture that $\zeta(X; s) = Z(X; q^{-s})$, with $\text{Re}(s) > d$, and that $\zeta_w(X; s) = Z_w(X; q^{-s})$, with $\text{Re}(s) > \frac{w}{2}$. Thanks to (1.1), we hence obtain the weight decomposition:

$$\zeta(X; s) = \frac{\zeta_0(X; s)\zeta_1(X; s)\cdots\zeta_{2d}(X; s)}{\zeta_1(X; s)\zeta_3(X; s)\cdots\zeta_{2d-1}(X; s)}$$

$$\text{Re}(s) > d.$$  

Note that (1.2) implies automatically that the Hasse-Weil zeta function of $X$ admits a (unique) meromorphic continuation to the entire complex plane.

**Remark 1.3 (Periodicity).** The Hasse-Weil zeta function of $X$ is periodic in the sense that $\zeta(X; s) = \zeta(X; s + \frac{2\pi i}{\log q})$; similarly for $\zeta_w(X; s)$.

**Remark 1.4 (Riemann hypothesis).** The Weil conjecture $W(X)$ is also called the “Riemann hypothesis over a finite field” because it implies that if $z \in \mathbb{C}$ is a pole of $\zeta_w(X; s)$, then $\text{Re}(z) = \frac{w}{2}$. Consequently, if $z \in \mathbb{C}$ is a pole, resp. zero, of $\zeta(X; s)$, then $\text{Re}(z) \in \{0, 1, \ldots, d\}$, resp. $\text{Re}(z) \in \{\frac{1}{2}, \frac{3}{2}, \ldots, \frac{2d-1}{2}\}$.

Let $\mathcal{A}$ be a smooth proper dg category in the sense of Kontsevich; see §4.1. Examples include the (unique) dg enhancement perf$_{dg}(X)$ of the category of perfect complexes perf$(X)$ of smooth proper k-schemes $X$ (or, more generally, of smooth proper algebraic k-stacks); consult [26, 38]. As explained in §5.2 below, the topological periodic cyclic homology group $TP_0(\mathcal{A})_{1/p}$ (this is a finite-dimensional $K$-vector space), resp. the topological periodic cyclic homology group $TP_1(\mathcal{A})_{1/p}$, comes equipped with an automorphism $F_0$, resp. $F_1$, called the “cyclotomic Frobenius”. Following Kontsevich [29], we hence define the even/odd zeta function of $\mathcal{A}$ as the following formal power series:

$$Z_{\text{even}}(\mathcal{A}; t) := \det(id - tF_0|TP_0(\mathcal{A})_{1/p})^{-1} \in K[t]\]$$

$$Z_{\text{odd}}(\mathcal{A}; t) := \det(id - tF_1|TP_1(\mathcal{A})_{1/p})^{-1} \in K[t].$$

Weil’s conjecture admits the following noncommutative counterpart:

**Conjecture $W_{nc}(\mathcal{A})$: The eigenvalues of the automorphism $F_0$, resp. $F_1$, are algebraic numbers and all their complex conjugates have absolute value 1, resp. $\sqrt{q}$**.

The noncommutative Weil conjecture was originally envisioned by Kontsevich in his seminal talks [32, 33]. The next result relates this conjecture with Weil’s original conjecture:

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2Deligne worked with étale cohomology instead. However, as explained by Katz-Messing in [23], Deligne’s results hold similarly in crystalline cohomology. More recently, Kedlaya [23] gave an alternative proof of the Weil conjecture which uses solely $p$-adic techniques.
Theorem 1.5. Given a smooth proper $k$-scheme $X$, we have the equivalence of conjectures $W_{nc}(\text{perf}_{dg}(X)) \Leftrightarrow W(X)$.

Intuitively speaking, Theorem 1.5 shows that the Weil conjecture belongs not only to the realm of algebraic geometry but also to the broad noncommutative setting of dg categories.

In contrast with the commutative world, the cyclotomic Frobenius is not induced from an endomorphism\footnote{Note that in the particular case where $A$ is a $k$-algebra $A$, the Frobenius map $a \mapsto a^q$ is a $k$-algebra endomorphism if and only if $A$ is commutative.} of $A$. Consequently, in contrast with the commutative world, it is not known if the polynomials $p_0(A; t) := \det(\text{id} - tF_0|TP_0(A)_{1/p})$ and $p_1(A; t) := \det(\text{id} - tF_1|TP_1(A)_{1/p})$ have integer coefficients. Nevertheless, after choosing an embedding $\iota: K \hookrightarrow \mathbb{C}$, we can still define the even/odd Hasse-Weil zeta function of $A$ as follows:

$$
\zeta_{\text{even}}(A; s) := \det(\text{id} - q^{-s}(F_0 \otimes_{K, \iota} \mathbb{C})|TP_0(A)_{1/p} \otimes_{K, \iota} \mathbb{C})^{-1} \\
\zeta_{\text{odd}}(A; s) := \det(\text{id} - q^{-s}(F_1 \otimes_{K, \iota} \mathbb{C})|TP_1(A)_{1/p} \otimes_{K, \iota} \mathbb{C})^{-1}.
$$

Remark 1.6 (Periodicity). Similarly to Remark 1.3, note that the even/odd Hasse-Weil zeta function of $A$ is periodic of period $\frac{2\pi i}{\log(q)}$.

Remark 1.7 (Noncommutative Riemann hypothesis). Similarly to Remark 1.4, the conjecture $W_{nc}(A)$ may be called the “noncommutative Riemann hypothesis over a finite field” because it implies that if $z \in \mathbb{C}$ is a pole of $\zeta_{\text{even}}(A; s)$, resp. $\zeta_{\text{odd}}(A; s)$, then $\text{Re}(z) = 0$, resp. $\text{Re}(z) = \frac{1}{2}$ (independently of the embedding $\iota: K \hookrightarrow \mathbb{C}$).

Functional equation. Thanks to the work of Artin and Grothendieck (consult [18] and the references therein), the Hasse-Weil zeta function $\zeta(X; s)$ of a smooth proper $k$-scheme $X$ of dimension $d$ is known to satisfy the functional equation

$$
(1.8) \quad \zeta(X; s) = \pm q^{\chi(X)s} \cdot q^{-\frac{\chi_1(X)}{2}d} \cdot \zeta(X; d - s),
$$

where $\chi(X)$ stands for the Euler characteristic of $X$. Morally speaking, the equality (1.8) describes a “symmetry” of $\zeta(X; s)$ along the vertical line $\text{Re}(s) = \frac{d}{2}$. This functional equation admits the following noncommutative counterpart:

Theorem 1.9. The even/odd Hasse-Weil zeta function of a smooth proper $\text{dg category} A$ satisfies the following functional equation

$$
\begin{align*}
\zeta_{\text{even}}(A; s) &= (-1)^{\chi_0(A)} \cdot q^{\chi_0(A)s} \cdot \det(F_0 \otimes_{K, \iota} \mathbb{C}) \cdot \zeta_{\text{even}}(A; 1 - s), \\
\zeta_{\text{odd}}(A; s) &= (-1)^{\chi_1(A)} \cdot q^{-\chi_1(A)(1-s)} \cdot \det(F_1 \otimes_{K, \iota} \mathbb{C}) \cdot \zeta_{\text{odd}}(A; 1 - s),
\end{align*}
$$

where $\chi_0(A) := \dim_K TP_0(A)_{1/p}$ and $\chi_1(A) := \dim_{K} TP_1(A)_{1/p}$.

Intuitively speaking, Theorem 1.9 describes a “symmetry” of $\zeta_{\text{even}}(A; s)$, resp. $\zeta_{\text{odd}}(A; s)$, along the vertical line $\text{Re}(s) = 0$, resp. $\text{Re}(s) = \frac{1}{2}$.

Remark 1.10 (Related work). In [46] we developed a general theory of (Hasse-Weil) zeta functions for smooth proper $\text{dg categories}$ equipped with an endomorphism. Among other applications, this theory led to a far-reaching noncommutative generalization of the results of Dwork [14] and Grothendieck [18] concerning the rationality and the functional equation of the classical (Hasse-Weil) zeta function.
**$l$-adic absolute value.** Let $X$ be a smooth proper $k$-scheme of dimension $d$. Thanks to the work of Deligne [12], the eigenvalues of the automorphisms $\text{Fr}_w^*, 0 \leq w \leq 2d$, are algebraic numbers. Moreover, for every prime $l \neq p$, it is well-known that all the $l$-adic conjugates of these eigenvalues have absolute value 1.

Let $\mathcal{A}$ be a smooth proper dg category. Motivated by the aforementioned facts, Kontsevich also conjectured in [32, 33] the following:

**Conjecture $\text{W}_{nc}(\mathcal{A})$:** The eigenvalues of the automorphisms $F_0$ and $F_1$ are algebraic numbers. Moreover, for every prime $l \neq p$, all the $l$-adic conjugates of these eigenvalues have absolute value 1.

The next result (partially) solves Kontsevich’s conjecture:

**Theorem 1.11.** Assume that there exists an integer $c \in \mathbb{Z}$ (which depends on $\mathcal{A}$) such that the eigenvalues of $F_0$ and $F_1$ become algebraic integers after multiplication by $q^c$. Under this assumption, the conjecture $\text{W}_{nc}(\mathcal{A})$ holds.

As explained in Remark 6.6 below, the assumption of Theorem 1.11 holds when $\mathcal{A} = \text{perf}_{dg}(X)$ with $X$ a smooth proper $k$-scheme (with $c = d$); consult §2 below for further examples.

**$p$-adic absolute value.** Let $\mathcal{A}$ be a smooth proper dg category. The above conjecture $\text{W}_{nc}(\mathcal{A})$, resp. $\text{W}_{nc}^l(\mathcal{A})$, predicts the absolute value of the complex, resp. $l$-adic, conjugates of the eigenvalues of $F_0$ and $F_1$. The next result imposes some restrictions on the $p$-adic absolute value of the eigenvalues of $F_0$ and $F_1$:

**Theorem 1.12.** Let $\chi_0 := \dim_K \text{TP}_0(\mathcal{A})_{1/p}$ and $\chi_1 := \dim_K \text{TP}_1(\mathcal{A})_{1/p}$.

(i) The polynomial $p_0(\mathcal{A}; t)$ has $\mathbb{Q}_p$-coefficients. Moreover, there exist integers $0 < r_1 < \cdots < r_n \leq \frac{\chi_0}{2}$ and $0 > s_1 > \cdots > s_n$ such that the Newton polygon of the polynomial $p_0(\mathcal{A}; t)$ is the following:

(ii) The polynomial $p_1(\mathcal{A}; t)$ has $\mathbb{Q}_p$-coefficients. Moreover, there exist integers $0 < r_1 < \cdots < r_n - 1 < \frac{\chi_1}{2}$ and $0 > s_1 > \cdots > s_n$ such that the Newton polygon of the polynomial $p_1(\mathcal{A}; t)$ is the following:
Let us write $\nu_q(-): (\overline{\mathbb{Q}}_p)^\times \to \mathbb{Q}$ for the $q$-adic valuation on the algebraic closure of $\mathbb{Q}_p$ (with $\nu_q(p) = 1/r$). It is well-known that if $s/r$ is a slope in the Newton polygon of the polynomial $p_0(A; t)$, resp. $p_1(A; t)$, with horizontal length $m$, then there exist precisely $m$ eigenvalues $\lambda$ of $F_0$, resp. $F_1$, with $\nu_q(\lambda) = s/r$ (and vice-versa). Therefore, since the $p$-adic absolute value is defined as $|-|_p := p^{-\nu_q(-)}$, Theorem 1.12 imposes some strong “symmetry” restrictions on the possible $p$-adic absolute value of the eigenvalues of $F_0$ and $F_1$.

**Strong form of the Tate conjecture.** Given a smooth proper $k$-scheme $X$ of dimension $d$ and an integer $0 \leq i \leq d$, let us write $Z^i(X)_{\mathbb{Q}/\sim_{\text{num}}}$ for the $\mathbb{Q}$-vector space of algebraic cycles of codimension $i$ on $X$ up to numerical equivalence.

In the mid sixties, Tate [55] conjectured the following:

**Conjecture ST($X$):** The order $\text{ord}_{s=j} \zeta(X; s)$ of the Hasse-Weil zeta function $\zeta(X; s)$ at (the pole) $s = j$, with $0 \leq j \leq d$, is equal to $-\dim_{\mathbb{Q}} Z^j(X)_{\mathbb{Q}/\sim_{\text{num}}}$. This conjecture is usually called the “strong form of the Tate conjecture”. It holds for 0-dimensional schemes, for curves, for abelian varieties of dimension $\leq 3$, and also for $K3$-surfaces. Besides these cases (and some other cases scattered in the literature), it remains wide open.

Given a smooth proper dg category $A$, recall from §4.3 below the definition of the numerical Grothendieck group $K_0(A)_{\mathbb{Q}/\sim_{\text{num}}}$. The strong form of the Tate conjecture admits the following noncommutative counterpart:

**Conjecture ST$_{\text{nc}}$(A):** The order $\text{ord}_{s=0} \zeta_{\text{even}}(A; s)$ of the even Hasse-Weil zeta function $\zeta_{\text{even}}(A; s)$ at (the pole) $s = 0$ is equal to $-\dim_{\mathbb{Q}} K_0(A)_{\mathbb{Q}/\sim_{\text{num}}}$. Theorem 1.12 imposes some strong “symmetry” restrictions on the possible $p$-adic absolute value of the eigenvalues of $F_0$ and $F_1$.
equivalently, of $F_0$). Therefore, the conjecture $\text{ST}_{\text{nc}}(A)$ may be alternatively formulated as follows: \textit{the algebraic multiplicity of the eigenvalue 1 of $F_0$ agrees with $\dim Q K_0(A)/\sim_{\text{num}}$.} This shows, in particular, that the integer $\text{ord}_s = \zeta_{\text{even}}(A; s)$ is independent of the embedding $\iota: K \hookrightarrow \mathbb{C}$ used in the definition of $\zeta_{\text{even}}(A; s)$.

\textbf{Remark 1.14 (Equivalent conjectures).} As proved in Theorem 10.4 below, the noncommutative strong form of the Tate conjecture is equivalent to the noncommutative $p$-version of the Tate conjecture plus the noncommutative standard conjecture of type $D$. Moreover, when all smooth proper dg categories are considered simultaneously, the noncommutative strong form of the Tate conjecture is equivalent to the fully-faithfulness of the isocrystals realization functor; consult §10.5 below.

The next result relates the noncommutative strong form of the Tate conjecture with the strong form of the Tate conjecture:

\textbf{Theorem 1.15.} \textit{Given a smooth proper $k$-scheme $X$, we have the equivalence of conjectures $\text{ST}_{\text{nc}}(\text{perf} \text{dg}(X)) \iff \text{ST}(X)$.}

Similarly to Theorem 1.5, Theorem 1.15 shows that the strong form of the Tate conjecture belongs not only to the realm of algebraic geometry but also to the broad noncommutative setting of dg categories.

\textbf{Numerical motives.} Let $w$ be an integer. Recall first that an algebraic number $\lambda$ is called a \textit{Weil $q$-number of weight $w$} if all its complex conjugates have absolute value $q^{-w}$ and if there exists an integer $c \in \mathbb{Z}$ such that $q^c \lambda$ is an algebraic integer. Under these notations, given a smooth proper dg category $A$, the noncommutative Weil conjecture admits the following variant:

\textit{Conjecture $\mathcal{W}_{\text{nc}}(A)$: The eigenvalues of the automorphism $F_0$, resp. $F_1$, are Weil numbers of weight 0, resp. 1.}

Note that $\mathcal{W}_{\text{nc}}(A) \Rightarrow \mathcal{W}_{\text{nc}}(A)$. As explained in Remark 6.6 below, the conjecture $\mathcal{W}_{\text{nc}}(A)$ holds when $A = \text{perf} \text{dg}(X)$ with $X$ a smooth proper $k$-scheme; consult §2 below for further examples. Consider now from the category of noncommutative numerical motives $\text{NNum}(k)_{Q_p}$ (with $Q_p$-coefficients); consult §4.2. As proved in [46, Thm. 3.1], this category is abelian semi-simple. Therefore, it is completely characterized by its simple objects (up to isomorphism) and by the division $Q_p$-algebras of endomorphisms of each one of its simple objects. The next (conditional) result provides an explicit description of such data:

\textbf{Theorem 1.16.} \textit{Assume that the conjectures $\mathcal{W}_{\text{nc}}(A)$ and $\text{ST}_{\text{nc}}(A)$ hold for every smooth proper dg category $A$. Under these assumptions, the following holds:

(i) The category $\text{NNum}(k)_{Q_p}$ comes equipped with a $\otimes$-automorphism $\pi$ of the identity functor. Moreover, given an object $N \in \text{NNum}(k)_{Q_p}$, the center of the $Q_p$-algebra $\text{End}_{\text{NNum}(k)_{Q_p}}(N)$ agrees with the $Q_p$-subalgebra $Q_p[\pi_{NM}]$ generated by $\pi_{NM}$; in particular, $Q_p[\pi_{NM}]$ is a field when $N$ is simple.

(ii) Assume moreover that the conjecture $\mathcal{W}_{\text{nc}}(A)$ holds for every smooth proper dg category $A$. Under this extra assumption, we have the following bijection:

\[
\{\text{simple objs. in } \text{NNum}(k)_{Q_p}\} \overset{\text{similarity}}{\longrightarrow} \bigcup_{w=0,1} \frac{\{\text{Weil } q\text{-numbers of weight } w\}}{\text{Gal}(Q_p/Q_p)\text{-action}},
\]

Note that we are implicitly considering the set $\{\text{Weil } q\text{-numbers of weight } w\}$ as a subgroup of $(Q_p)^\times$ via any embedding $\mathbb{Q} \subset Q_p$. Note also that under such identification the subgroup $\{\text{Weil } q\text{-numbers of weight } w\}$ is stable under the action of the absolute Galois group $\text{Gal}(Q_p/Q_p)$.}
where $[\pi_{NM}]$ stands for the set of $p$-adic conjugates of $\pi_{NM}$.

(iii) Given a simple object $NM \in \text{Num}(k)_{q}$, the Hasse-invariant of the central division algebra $\text{End}_{\text{Num}(k)_{q}}(NM)$ is equal to

$$-\nu_q(\pi_{NM}) \cdot [Q_p[\pi_{NM}] : Q_p] \in \mathbb{Q}/\mathbb{Z},$$

where $\nu_q(-) : (\mathbb{Q}_p)^\times \to \mathbb{Q}$ stands for the $q$-adic valuation (with $\nu_q(p) = 1/r$).

Recall that the classical Hasse-invariant yields an isomorphism between the Brauer group $\text{Br}(Q_p[\pi_{NM}])$ of the local field $Q_p[\pi_{NM}]$ and the group $\mathbb{Q}/\mathbb{Z}$. In other words, the Hasse-invariant classifies all the central division algebras over $Q_p[\pi_{NM}]$.

2. Applications to noncommutative geometry

In this section, we prove the conjectures $C_{nc}(-)$, with $C \in \{W, W', ST, \mathbb{W}\}$, in several interesting cases.

**Twisted schemes.** Let $X$ be a smooth proper $k$-scheme and $F$ a sheaf of Azumaya algebras over $X$. Similarly to $\text{perf}_{dg}(X)$, we can also consider the smooth proper $\text{dg}$ category $\text{perf}_{dg}(X; F)$ of perfect complexes of $F$-modules.

**Theorem 2.1.** We have the following equivalences of conjectures:

$$W(X) \Leftrightarrow W_{nc}(\text{perf}_{dg}(X; F)) \quad \text{ST}(X) \Leftrightarrow \text{ST}_{nc}(\text{perf}_{dg}(X; F)).$$

Moreover, the conjectures $W_{nc}(\text{perf}_{dg}(X; F))$ and $\mathbb{W}_{nc}(\text{perf}_{dg}(X; F))$ hold.

Morally speaking, Theorem 2.1 shows that in what concerns the (noncommutative) Weil conjecture and the (noncommutative) strong form of the Tate conjecture, there is no difference between schemes and twisted schemes.

**Calabi-Yau dg categories associated to hypersurfaces.** Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree $\deg(X) \leq n + 1$. As proved by Kuznetsov in [35, Cor. 4.1], we have a semi-orthogonal decomposition:

$$(2.2) \quad \text{perf}(X) = \langle T(X), \mathcal{O}_X, \ldots, \mathcal{O}_X(n - \deg(X)) \rangle.$$

Moreover, the associated $\text{dg}$ category $T_{dg}(X)$, defined as the $\text{dg}$ enhancement of $T(X)$ induced from $\text{perf}_{dg}(X)$, is a smooth proper Calabi-Yau $\text{dg}$ category of fractional dimension $\frac{(n+1)(\deg(X)-2)}{\deg(X)}$.

**Remark 2.3** (Noncommutative K3-surfaces). In the particular case where $n = 5$ and $\deg(X) = 3$, the $\text{dg}$ categories $T_{dg}(X)$ are usually called “noncommutative K3-surfaces” because they share many of the key properties of the $\text{dg}$ categories of perfect complexes of the classical K3-surfaces. Moreover, Kuznetsov conjectured in [37] that $T(X)$ is (Fourier-Mukai) equivalent to the category of perfect complexes of a K3-surface if and only if $X$ is rational.

**Theorem 2.4.** We have the following equivalences of conjectures:

$$W(X) \Leftrightarrow W_{nc}(T_{dg}(X)) \quad \text{ST}(X) \Leftrightarrow \text{ST}_{nc}(T_{dg}(X)).$$

Moreover, the conjectures $W_{nc}(T_{dg}(X))$ and $\mathbb{W}_{nc}(T_{dg}(X))$ hold.

Similarly to Theorem 2.1, Theorem 2.4 shows that in what concerns the (noncommutative) Weil conjecture and the (noncommutative) strong form of the Tate conjecture, there is no difference between the hypersurface $X$ and the associated Calabi-Yau $\text{dg}$ category $T_{dg}(X)$.
Noncommutative gluings of schemes. Let $X$ and $Y$ be two smooth proper $k$-schemes and $B$ a perfect dg perf$_{dg}(X)$-perf$_{dg}(Y)$-bimodule. Following Orlov [45, §3.2], we can consider the gluing $X \oplus_B Y$ of perf$_{dg}(X)$ and perf$_{dg}(Y)$ via $B$ (Orlov used a different notation). This new dg category is smooth and proper.

**Theorem 2.5.** We have the following equivalences of conjectures:

\[ W(X) + W(Y) \Leftrightarrow W_{nc}(X \oplus_B Y) \quad \text{ST}(X) + \text{ST}(Y) \Leftrightarrow \text{ST}_{nc}(X \oplus_B Y). \]

Moreover, the conjectures $W_l_{nc}(X \oplus_B Y)$ and $W_{nc}(X \oplus_B Y)$ hold.

Intuitively speaking, Theorem 2.5 shows that the noncommutative Weil conjecture and the noncommutative strong form of the Tate conjecture are “additive” with respect to gluings. This implies, in particular, that the noncommutative strong form of the Tate conjecture holds for every noncommutative gluing of curves.

Root stacks. Let $X$ be a smooth proper $k$-scheme, $L$ a line bundle on $X$, $ς \in \Gamma(X, L)$ a global section, and $n \geq 1$ an integer. Following Cadman [11, Def. 2.2.1], the associated root stack is defined as the following fiber-product

\[
\begin{array}{ccc}
X & \xrightarrow{f} & \mathbb{A}^1/G_m \\
\downarrow & & \downarrow \theta_n \\
(L, ς) & \to & (\mathbb{A}^1/G_m),
\end{array}
\]

where $\theta_n$ stands for the morphism induced by the $n^{th}$ power map on $\mathbb{A}^1$ and $G_m$.

**Theorem 2.6.** Assume that the zero locus $D \hookrightarrow X$ of $ς$ is smooth. Under this assumption, we have the following equivalences of conjectures:

\[ W(X) + W(D) \Leftrightarrow W_{nc}(\text{perf}_{dg}(X)) \quad \text{ST}(X) + \text{ST}(D) \Leftrightarrow \text{ST}_{nc}(\text{perf}_{dg}(X)). \]

Moreover, the conjectures $W_l_{nc}(\text{perf}_{dg}(X))$ and $W_{nc}(\text{perf}_{dg}(X))$ hold.

Theorem 2.6 implies that the noncommutative strong form of the Tate conjecture holds, for example, for all those root stacks whose underlying scheme is a curve.

Global orbifolds. Let $G$ be a finite group of order $n$, $X$ a smooth proper $k$-scheme equipped with a $G$-action, and $X' := [X/G]$ the associated global orbifold.

**Theorem 2.7.** Assume that $p \nmid n$ ($\Leftrightarrow 1/n \in k$). Under this assumption, we have the following implications of conjectures ($\sigma$ is a cyclic subgroup of $G$):

\[
\sum_{\sigma \leq G} W(X^\sigma \times \text{Spec}(k[\sigma])) \quad \Rightarrow \quad W_{nc}(\text{perf}_{dg}(X'))
\]

\[
\sum_{\sigma \leq G} \text{ST}(X^\sigma \times \text{Spec}(k[\sigma])) \quad \Rightarrow \quad \text{ST}_{nc}(\text{perf}_{dg}(X')).
\]

Under the stronger assumption $n\mid(q - 1)$ (\(\Leftrightarrow k\) contains the $n^{th}$ roots of unity), the same implications hold with $X^\sigma \times \text{Spec}(k[\sigma])$ replaced by $X^\sigma$. Moreover, the conjectures $W_l_{nc}(\text{perf}_{dg}(X'))$ and $W_{nc}(\text{perf}_{dg}(X'))$ hold.

Since the $k$-scheme $\text{Spec}(k[\sigma])$ is 0-dimensional, Theorem 2.7 implies that the noncommutative strong form of the Tate conjecture holds, for example, for all those global orbifolds whose underlying scheme is a curve.
Remark 2.10 (McKay correspondence). A famous conjecture of Reid asserts that the category perf(\(X\)) is (Fourier-Mukai) equivalent to the category of perfect complexes of a(ny) crepant resolution \(Y\) of the (singular) geometric quotient \(X/G\). Whenever this holds, the right-hand sides of (2.8)-(2.9) may be replaced by the conjectures \(W(Y)\) and \(ST(Y)\), respectively. Reid’s conjecture has been proved in several cases; consult, for example, the work of Bezrukavnikov and Kaledin [5], Bridgeland, King and Reid [10], Kapranov and Vasserot [22], and Kawamata [24].

Twisted global orbifolds. Let \(G\) be a finite group of order \(n\), \(X\) a smooth proper \(k\)-scheme equipped with a \(G\)-action, \(X := [X/G]\) the associated global orbifold, and \(F\) a sheaf of Azumaya algebras over \(X\). Similarly to \(\text{perf} dg(X)\), we can also consider the \(dg\) category \(\text{perf} dg(X; F)\) of perfect complexes of \(F\)-modules.

**Theorem 2.11.** Assume that \(n \mid (q - 1)\) (\(\Leftrightarrow k\) contains the \(n\)th roots of unity). Under this assumption, we have the following implications of conjectures

\[
\sum_{\sigma \subseteq G} W(Y_{\sigma}) \Rightarrow W_{nc}(\text{perf}_{dg}(X; F)) \quad \sum_{\sigma \subseteq G} ST(Y_{\sigma}) \Rightarrow ST_{nc}(\text{perf}_{dg}(X; F)),
\]

where \(Y_{\sigma}\) is a certain \(\sigma\)-Galois cover of \(X_{\sigma}\) induced by the restriction of \(F\) to \(X_{\sigma}\). Moreover, the conjectures \(W_{nc}(\text{perf}_{dg}(X; F))\) and \(W_{nc}(\text{perf}_{dg}(X; F))\) hold.

Similarly to Theorem 2.7, Theorem 2.11 implies that the noncommutative strong form of the Tate conjecture holds, for example, for all those twisted global orbifolds whose underlying scheme is a curve.

**Finite-dimensional dg algebras.** Let \(A\) be a smooth finite-dimensional \(dg\) \(k\)-algebra in the sense of Orlov [45], i.e., \(\dim_k(A^i) < \infty\) for every \(i \in \mathbb{Z}\). The next result proves the noncommutative Weil conjecture(s) and the noncommutative strong form of the Tate conjecture for this (large) class of dg algebras:

**Theorem 2.12.** The conjectures \(C_{nc}(A)\), with \(C \in \{W, W^l, ST, W\}\), hold.

3. Applications to commutative geometry

Recall from §1 that both the Weil conjecture as well as the strong form of the Tate conjecture hold for curves (Weil proved his famous conjecture for curves using solely the classical intersection theory of divisors on surfaces). In this section, making use of Theorems 1.5 and 1.15, we bootstrap these results from curves to intersections of two quadrics and to linear sections of determinantal varieties. This yields an alternative noncommutative proof of the Weil conjecture and of the strong form of the Tate conjecture for all these (higher dimensional) schemes.

**Intersections of two quadrics.** Let \(X \subset \mathbb{P}^{n-1}\) be a smooth complete intersection of two quadric hypersurfaces, with \(n \geq 4\). The linear span of these two quadrics gives rise to an hypersurface \(Q \subset \mathbb{P}^1 \times \mathbb{P}^{n-1}\), and the projection onto the first factor gives rise to a flat quadric fibration \(f : Q \to \mathbb{P}^1\) of relative dimension \(n - 2\).

**Theorem 3.1.** Assume that all the fibers of \(f\) have corank \(\leq 1\). Under this assumption, the following holds:

(i) When \(n\) is even, the conjectures \(W(X)\) and \(ST(X)\) hold.

(ii) When \(n\) is odd and \(p \neq 2\), the conjectures \(W(X)\) and \(ST(X)\) hold.

---

5Equivalently, \(F\) is a \(G\)-equivariant sheaf of Azumaya algebras over \(X\).
**Linear sections of determinantal varieties.** Let $U_1$ and $U_2$ be two finite-dimensional $k$-vector spaces of dimensions $d_1$ and $d_2$, respectively, $V := U_1 \otimes U_2$, and $0 < r < \min(d_1, d_2)$ an integer. Consider the determinantal variety $Z_{d_1,d_2}^r \subset \mathbb{P}(V)$ defined as the locus of those matrices $U_2 \to U_1^*$ with rank $\leq r$.

**Example 3.2 (Segre varieties).** In the particular case where $r = 1$, the determinantal varieties reduce to the classical Segre varieties. Concretely, $Z_{d_1,d_2}^1$ is given by the image of Segre homomorphism $\mathbb{P}(U_1) \times \mathbb{P}(U_2) \to \mathbb{P}(V)$.

In contrast with the Segre varieties, the varieties $Z_{d_1,d_2}^r$, with $r \geq 2$, are not smooth. Their singular locus consists of those matrices $U_2 \to U_1^*$ with rank $< r$, i.e., it agrees with the closed subvarieties $Z_{d_1,d_2}^{r-1}$. Nevertheless, it is well-known that $Z_{d_1,d_2}^r$ admits a canonical resolution of singularities $X := X_{d_1,d_2}^r \to Z_{d_1,d_2}^r$. Dually, consider the variety $W_{d_1,d_2}^r \subset \mathbb{P}(V^*)$, defined as the locus of those matrices $U_2^* \to U_1$ with corank $\geq r$, and the associated canonical resolution of singularities $Y := Y_{d_1,d_2}^r \to W_{d_1,d_2}^r$. Finally, given a linear subspace $L \subseteq V^*$, consider the associated linear sections $X_L := X \times_{\mathbb{P}(V)} \mathbb{P}(L)$ and $Y_L := Y \times_{\mathbb{P}(V^*)} \mathbb{P}(L)$.

**Theorem 3.3.** Assume that $X_L$ and $Y_L$ are smooth, and that $\text{codim}(X_L) = \dim(L)$ and $\text{codim}(Y_L) = \dim(L^*)$. Under these assumptions (which hold for a generic choice of $L$), the following holds:

(i) When $\dim(L) = r(d_1 + d_2 - r) - 2$, the conjectures $W(Y_L)$ and $\text{ST}(Y_L)$ hold.
(ii) When $\dim(L) = 2 - r(d_1 - d_2 - r)$, the conjectures $W(X_L)$ and $\text{ST}(X_L)$ hold.

**Example 3.4 (Segre varieties).** Let $r = 1$. Thanks to Theorem 3.3(ii), when $\dim(L) = 3 - d_1 + d_2$, the conjectures $W(X_L)$ and $\text{ST}(X_L)$ hold. In all these cases, $X_L$ is a linear section of the Segre variety $Z_{d_1,d_2}^1$. Moreover, $\dim(X_L) = 2d_1 - 5$. Therefore, for example, by letting $d_1 \to \infty$ and by keeping $\dim(L)$ fixed, we obtain infinitely many examples of smooth projective $k$-schemes $X_L$, of arbitrary high dimension, satisfying the Weil conjecture and the strong form of the Tate conjecture.

**Example 3.5 (Square matrices).** Let $d_1 = d_2$. Thanks to Theorem 3.3(ii), when $\dim(L) = 2 + r^2$, the conjectures $W(X_L)$ and $\text{ST}(X_L)$ hold. In all these cases, we have $\dim(X_L) = 2r(d_1 - r) - 3$. Therefore, for example, by letting $d_1 \to \infty$ and by keeping $\dim(L)$ fixed, we obtain infinitely many examples of smooth projective $k$-schemes $X_L$, of arbitrary high dimension, satisfying the Weil conjecture and the strong form of the Tate conjecture.

### 4. Preliminaries

Throughout the article, $k = \mathbb{F}_q$ is a finite field, with $q = p^\ell$, $W(k)$ is the ring of $p$-typical Witt vectors of $k$, and $K := W(k)_{1/p}$ is the fraction field of $W(k)$.

**4.1. Dg categories.** For a survey on dg categories, we invite the reader to consult [26]. A differential graded (=dg) category $\mathcal{A}$ is a category enriched over (cochain) complexes of $k$-vector spaces. In what follows, we will write $\text{dgcat}(k)$ for the category of (essentially small) dg categories.

Let $\mathcal{A}$ be a dg category. The opposite dg category $\mathcal{A}^{\text{op}}$, resp. category $\text{H}^0(\mathcal{A})$, has the same objects as $\mathcal{A}$ and $\mathcal{A}^{\text{op}}(x,y) := \mathcal{A}(y,x)$, resp. $\text{H}^0(\mathcal{A})(x,y) := \text{H}^0(\mathcal{A}(x,y))$.

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6The linear section $X_L$ is smooth if and only if the linear section $Y_L$ is smooth.

7Consult also the pioneering work [7].
A right dg $\mathcal{A}$-module is a dg functor $M : \mathcal{A}^{\text{op}} \to \mathcal{C}_{\text{dg}}(k)$ with values in the dg category of complexes of $k$-modules. Let $\mathcal{C}(\mathcal{A})$ be the category of right dg $\mathcal{A}$-modules. Following [26, §3.2], the derived category $\mathcal{D}(\mathcal{A})$ of $\mathcal{A}$ is defined as the localization of $\mathcal{C}(\mathcal{A})$ with respect to the objectwise quasi-isomorphisms. In what follows, we will write $\mathcal{D}_{c}(\mathcal{A})$ for the subcategory of compact objects.

A dg functor $F : \mathcal{A} \to \mathcal{B}$ is called a Morita equivalence if it induces an equivalence between derived categories $\mathcal{D}(\mathcal{A}) \simeq \mathcal{D}(\mathcal{B})$; see [26, §4.6]. As explained in [50, §1.6], the category $\text{dgcat}(k)$ admits a Quillen model structure whose weak equivalences are the Morita equivalences. Let $\text{Hmo}(k)$ be the associated homotopy category.

The tensor product $\mathcal{A} \otimes_k \mathcal{B}$ of dg categories is defined as follows: the set of objects of $\mathcal{A} \otimes_k \mathcal{B}$ is the cartesian product of the sets of objects of $\mathcal{A}$ and $\mathcal{B}$ and $(\mathcal{A} \otimes_k \mathcal{B})(x, w) := \mathcal{A}(x, y) \otimes_k \mathcal{B}(w, z)$. As explained in [50, §1.1.1 and §1.6.4], this construction gives rise to a symmetric monoidal structure $- \otimes_k -$ on the category $\text{dgcat}(k)$, which descends to a symmetric monoidal category $\text{Hmo}(k)$.

A dg $\mathcal{A} \mathcal{B}$-bimodule is a dg functor $B : \mathcal{A} \otimes_k \mathcal{B}^{\text{op}} \to \mathcal{C}_{\text{dg}}(k)$. An example is the dg $\mathcal{A} \mathcal{B}$-bimodule $\mathcal{B} : (x, w) \mapsto \mathcal{B}(w, F(x))$ associated to a dg functor $F : \mathcal{A} \to \mathcal{B}$.

Finally, following [30, 31, 33, 34], a dg category $\mathcal{A}$ is called smooth if the dg $\mathcal{A} \mathcal{B}$-bimodule $\text{id}_{B}$ belongs to the subcategory $\mathcal{D}_{c}(\mathcal{A}^{\text{op}} \otimes_k \mathcal{A})$ and proper if all the complexes of $k$-vector spaces $\mathcal{A}(x, y)$ belong to the subcategory $\mathcal{D}_{c}(k)$. As explained in [50, Thm. 1.43], the smooth proper dg categories can be (conceptually) characterized as the dualizable objects of the symmetric monoidal category $\text{Hmo}(k)$. Moreover, the dual of a smooth proper dg category $\mathcal{A}$ is its opposite dg category $\mathcal{A}^{\text{op}}$.

### 4.2. Noncommutative motives

For a book, resp. survey, on noncommutative motives, we invite the reader to consult [50], resp. [48]. Recall from [50, §4.1] the construction of the category of noncommutative Chow motives $\text{NChow}(k)$. This category is $\mathbb{Q}$-linear, additive, idempotent complete, rigid symmetric monoidal\(^8\), and comes equipped with a (composed) symmetric monoidal functor:

$$U(-) : \text{dgcat}_{\text{sp}}(k) \longrightarrow \text{Hmo}_{\text{sp}}(k)_{\mathbb{Q}} \longrightarrow \text{NChow}(k)_{\mathbb{Q}}.$$  

Moreover, given smooth proper dg categories $\mathcal{A}$ and $\mathcal{B}$, we have an isomorphism:

$$\text{Hom}_{\text{NChow}(k)_{\mathbb{Q}}}(U(\mathcal{A})_{\mathbb{Q}}, U(\mathcal{B})_{\mathbb{Q}}) \simeq K_{0}(\mathcal{A}^{\text{op}} \otimes_k \mathcal{B})_{\mathbb{Q}}.$$  

Recall from [50, §4.6] the construction of the category of noncommutative numerical motives $\text{NNum}(k)$. This category is also $\mathbb{Q}$-linear, additive, idempotent complete, rigid symmetric monoidal, and comes equipped with a (quotient) $\mathbb{Q}$-linear symmetric monoidal functor $\text{NChow}(k)_{\mathbb{Q}} \to \text{NNum}(k)_{\mathbb{Q}}$.

### 4.3. Numerical Grothendieck group

Given a smooth proper dg category $\mathcal{A}$, its Grothendieck group $K_{0}(\mathcal{A}) := K_{0}(\mathcal{D}_{c}(\mathcal{A}))$ comes equipped with the Euler bilinear pairing $\chi : K_{0}(\mathcal{A}) \times K_{0}(\mathcal{A}) \to \mathbb{Z}, ([M], [N]) \mapsto \sum (-1)^{n} \dim_{k} \text{Hom}_{\mathcal{D}_{c}(\mathcal{A})}(M, N[n])$. This pairing is not symmetric neither skew-symmetric. Nevertheless, making use of the notion of Serre functor developed in [8], it can be shown that the left and right kernels of $\chi$ agree; consult [50, Prop. 4.24]. Hence, the numerical Grothendieck group $K_{0}(\mathcal{A})/_{\sim_{\text{num}}}$ is defined as the quotient of $K_{0}(\mathcal{A})$ by the kernel of $\chi$. As proved in [46, Thm. 5.1], $K_{0}(\mathcal{A})/_{\sim_{\text{num}}}$ is a finitely generated free abelian group. In

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\(^8\)Recall that a symmetric monoidal category is called rigid if all its objects are dualizable.
what follows, we will write $K_0(A)_Q/\sim_{num}$ for the finite-dimensional $Q$-vector space $K_0(A)/\sim_{num} \otimes \mathbb{Z}Q$. Finally, recall from [50, §4.6-§4.7] that we have an isomorphism

$$\text{Hom}_{\text{NNum}}(k)_q(U(A)_Q, U(B)_Q) \simeq K_0(A^{op} \otimes_k B)_Q/\sim_{num}.$$

5. **Topological periodic cyclic homology**

For recent/modern references on topological (periodic) cyclic homology, we invite the reader to consult [20, 43]. Following [1, 6], topological periodic cyclic homology gives rise to a symmetric monoidal functor $TP_*(-)_{1/p} : \text{dgcat}_{sp}(k) \to \text{mod}_Z(K[v^{\pm 1}])$ with values in the category of (degreewise finite-dimensional) $Z$-graded $K[v^{\pm 1}]$-modules, where $v$ is a variable of degree $-2$. Moreover, as explained in [46, Thm. 2.3], this functor yields a $Q$-linear symmetric monoidal functor:

$$TP_*(-)_{1/p} : \text{NChow}(k)_Q \to \text{mod}_Z(K[v^{\pm 1}]).$$ (5.1)

5.1. **Homological Grothendieck group.** Let $A$ be a smooth proper dg category. By combining the functor (5.1) with the identification (4.1) (with $A := k$ and $B := A$), we obtain an induced $Q$-linear homomorphism $\text{ch} : K_0(A)_Q \to TP_0(A)$. Under these notations, the **homological Grothendieck group** $K_0(A)_Q/\sim_{hom}$ is defined as the quotient of $K_0(A)_Q$ by the kernel of $\text{ch}$.

5.2. **Cyclotomic Frobenius.** Let $A$ be a smooth proper dg category. By construction, its topological Hochschild homology $THH(A)$ carries a canonical cyclotomic structure in the sense of [43, §2]. Using the $S^1$-action on $THH(A)$, we can consider the spectrum of homotopy orbits $THH(A)_{hS^1}$, the spectrum of homotopy fixed-points $TC_-(A) := THH(A)^{hS^1}$, and also the Tate construction $TP(A) := THH(A)^{tS^1}$ in the sense of [16]. As explained in [43, Cor. I.4.3], these spectra are related by the following cofiber sequence

$$\Sigma THH(A)_{hS^1} \xrightarrow{N} THH(A)^{hS^1} \xrightarrow{\text{can}} THH(A)^{tS^1},$$ (5.2)

where $N$ stands for the norm map. It is well-known that the abelian groups $THH_n(A)$ are $k$-linear. Hence, after inverting $p$, we have $\Sigma THH(A)_{hS^1}/[1/p] \simeq *$. Consequently, the above cofiber sequence (5.2) leads to a canonical isomorphism:

$$\text{can} : TC_-(A)_{1/p} \xrightarrow{\cong} TP_*(A)_{1/p}. $$ (5.3)

It is also well-known that the spectrum $THH(A)$ is a dualizable $THH(k)$-module spectrum. Thanks to Bökstedt’s celebrated computation $THH_n(k) \simeq k[u]$, where $u$ is a variable of degree 2, this implies that the spectrum $THH(A)$ is bounded below, i.e., there exists an integer $m \in \mathbb{Z}$ such that $THH_n(A) = 0$ for every $n < m$. Since the abelian groups $THH_n(A)$ are $k$-linear, this also implies that the spectrum $THH(A)$ is $p$-complete. Therefore, as explained in [43, Lem. II.4.2], the cyclotomic structure of $THH(A)$ yields another homomorphism:

$$\varphi_p : TC_-(A)_{1/p} \to TP_*(A)_{1/p}. $$ (5.4)

It follows from [1, Prop. 4.7] that the homomorphism (5.4) is invertible. Hence, let us write $\varphi_* := \varphi_p \circ \text{can}^{-1}$ for the induced automorphism of $TP_*(A)_{1/p}$. The automorphism $\varphi_*$ is **not** $K$-linear. Instead, it is $\sigma$-semilinear with respect to the isomorphism $\sigma : K \xrightarrow{\cong} K$ induced by the Frobenius map $\lambda \mapsto \lambda^p$ on $k$. Therefore, its $r$-fold composition $F_* := \varphi_1^r$ becomes a $K$-linear automorphism of $TP_*(A)_{1/p}$.

**Notation 5.5.** The $K$-linear automorphism $F_*$ is called the **cyclotomic Frobenius**.
Remark 5.6 (Lack of linearity). The automorphism $\varphi_*$ is not $K[v^{\pm 1}]$-linear. Instead, it is $\rho$-semilinear with respect to the isomorphism $\rho: K[v^{\pm 1}] \xrightarrow{\sim} K[v^{\pm 1}], v \mapsto pv$. In other words, we have the following commutative squares:

$$\text{(5.7)} \quad TP_n(A)_{1/p} \xrightarrow{\varphi_n \simeq} TP_{n-2}(A)_{1/p} \quad n \in \mathbb{Z}$$

Consequently, we obtain an induced $\sigma$-semilinear isomorphism:

$$\varphi_*: TP_*(A)_{1/p}^\rho := TP_*(A)_{1/p} \otimes_{K[v^{\pm 1}], \rho} K[v^{\pm 1}] \xrightarrow{\sim} TP_*(A)_{1/p}.$$ 

Since $F_* := \varphi^*_\tau$, the cyclotomic Frobenius is also not $K[v^{\pm 1}]$-linear. Instead, it is $\tau$-semilinear with respect to the isomorphism $\tau: K[v^{\pm 1}] \xrightarrow{\sim} K[v^{\pm 1}], v \mapsto qv$. In other words, we have the following commutative squares:

$$\text{(5.8)} \quad TP_n(A)_{1/p} \xrightarrow{F_n \simeq} TP_{n-2}(A)_{1/p} \quad n \in \mathbb{Z}$$

Consequently, we obtain an induced $K$-linear isomorphism:

$$F_*: TP_*(A)_{1/p}^\tau := TP_*(A)_{1/p} \otimes_{K[v^{\pm 1}], \tau} K[v^{\pm 1}] \xrightarrow{\sim} TP_*(A)_{1/p}.$$ 

Remark 5.9 (Loss of information). Similarly to §1, given an integer $n \in \mathbb{Z}$, we can consider the following Hasse-Weil zeta function:

$$\zeta_n(A; s) := \det(id - q^{-s}(F_n \otimes_{K, \tau} \mathbb{C})/TP_n(A)_{1/p} \otimes_{K, \tau} \mathbb{C})^{-1}.$$ 

Thanks to the above commutative squares (5.8), we have $\zeta_n(A; s) = \zeta_0(A; s + \frac{n}{2})$ when $n$ is even and $\zeta_n(A; s) = \zeta_1(A; s + \frac{n-1}{2})$ when $n$ is odd. Consequently, there is no loss of information in working solely with the even/odd Hasse-Weil zeta functions $\zeta_{\text{even}}(A; s) := \zeta_0(A; s)$ and $\zeta_{\text{odd}}(A; s) := \zeta_1(A; s)$ (as done in §1).

5.3. Natural transformation. Given smooth proper dg categories $A$ and $B$, we have a natural isomorphism $F_*^A \otimes_B F_*^B \cong F_*(A) \otimes_{K[v^{\pm 1}]} F_*(B)$. Therefore, by construction of the category of noncommutative Chow motives $\text{NCChow}(k)_Q$, the assignment $U(A)_{1/p} \mapsto F_*^A$ (parametrized by the smooth proper dg categories $A$) yields a $\mathbb{Q}$-linear symmetric monoidal natural transformation from the functor

$$\text{(5.10)} \quad TP_*(-)_{1/p}^\tau: \text{NCChow}(k)_Q \longrightarrow \text{mod}_2(K[v^{\pm 1}])$$

to the above $\mathbb{Q}_p$-linear symmetric monoidal functor (5.1).

Remark 5.11 (Generalization). Thanks to the aforementioned natural transformation, we can similarly formulate the conjectures $C_{nc}(NM)$, with $C \in \{W, W^l, ST, \mathbb{W}_l\}$, for every noncommutative Chow motive $NM \in \text{NCChow}(k)_Q$. In the particular case where $NM = U(A)_Q$, with $A$ a smooth proper dg category, these reduce to the conjectures $C_{nc}(A)$, with $C \in \{W, W^l, ST, \mathbb{W}_l\}$.
5.4. **Isocrystals realization functor.** The classical category of isocrystals $\text{Crys}(K)$ (consult [13, 39]) admits the following variant:

**Definition 5.12.** Let $\text{Crys}_2(K[v^{\pm 1}])$ be the category whose objects are given by the pairs $(V_*, \phi^V_*)$, where $V_*$ is a (degreewise finite-dimensional) $\mathbb{Z}$-graded $K[v^{\pm 1}]$-module and $\phi^V_* \colon V^p_* \rightarrow V_* \otimes K[v^{\pm 1}]$ is a $\sigma$-semilinear isomorphism, and the morphisms $f_* \colon (V_*, \phi^V_*) \rightarrow (W_*, \phi^W_*)$ are given by the homomorphisms of $\mathbb{Z}$-graded $K[v^{\pm 1}]$-modules $f_* \colon V_* \rightarrow W_*$ such that $f_* \circ \phi^V_* = \phi^W_* \circ f^*_*$. 

Note that the category $\text{Crys}_2(K[v^{\pm 1}])$ is $\mathbb{Q}_p$-linear (not $K$-linear) and that it comes equipped with a symmetric monoidal structure induced from the canonical symmetric monoidal structure on $\text{mod}_2(K[v^{\pm 1}])$. Given smooth proper dg categories $\mathcal{A}$ and $\mathcal{B}$, we have a natural isomorphism $\varphi^\mathcal{A} \otimes \varphi^\mathcal{B} \simeq \varphi^\mathcal{A} \otimes_{K[v^{\pm 1}]} \varphi^\mathcal{B}$. Therefore, by construction of the category of noncommutative Chow motives $\text{NChow}(k)_Q$, we obtain the following $\mathbb{Q}$-linear symmetric monoidal functor:

\[
(5.13) \quad \text{NChow}(k)_Q \longrightarrow \text{Crys}_2(K[v^{\pm 1}]) \quad U(\mathcal{A})_Q \rightarrow (T \mathcal{P}_*(\mathcal{A})_{1/p}, \varphi_*) .
\]

In what follows, we will call (5.13) the **isocrystals realization functor**.

6. **Proof of Theorem 1.5**

Following [15, Thm. 2][47, Thm. 5.2] (this is a result of Scholze), we have natural isomorphisms of finite-dimensional $K$-vector spaces:

\[
(6.1) \quad T \mathcal{P}_0(\text{perf}_{dg}(X))_{1/p} \simeq \bigoplus_{w \text{ even}} H^w_{\text{crys}}(X)
\]

\[
(6.2) \quad T \mathcal{P}_1(\text{perf}_{dg}(X))_{1/p} \simeq \bigoplus_{w \text{ odd}} H^w_{\text{crys}}(X).
\]

Moreover, following [20, §7], the cyclotomic Frobenius $F_0$ corresponds under the above isomorphism (6.1) to the following automorphism:

\[
(6.3) \quad \bigoplus_{w \text{ even}} q^{-\frac{w}{p}} F_{r^w} : \bigoplus_{w \text{ even}} H^w_{\text{crys}}(X) \xrightarrow{\sim} \bigoplus_{w \text{ even}} H^w_{\text{crys}}(X).
\]

Similarly, the cyclotomic Frobenius $F_1$ corresponds under the above isomorphism (6.2) to the following automorphism:

\[
(6.4) \quad \bigoplus_{w \text{ odd}} q^{-\frac{w-1}{p}} F_{r^w} : \bigoplus_{w \text{ odd}} H^w_{\text{crys}}(X) \xrightarrow{\sim} \bigoplus_{w \text{ odd}} H^w_{\text{crys}}(X).
\]

Given an integer $0 \leq w \leq 2d$, let us write $\{\lambda(\omega, 1), \ldots, \lambda(\omega, w)\}$ for the eigenvalues (with multiplicity) of the automorphism $F_{r^w}$, where $\beta_w := \dim K H^w_{\text{crys}}(X)$. Thanks to the identification of cyclotomic Frobenius $F_0$, resp. $F_1$, with the automorphism (6.3), resp. (6.4), the eigenvalues of $F_0$, resp. $F_1$, are given by $\bigcup_{w \text{ even}} \{q^{-\frac{w}{p}} \lambda(\omega, 1), \ldots, q^{-\frac{w}{p}} \lambda(\omega, w)\}$, resp. $\bigcup_{w \text{ odd}} \{q^{-\frac{w-1}{p}} \lambda(\omega, 1), \ldots, q^{-\frac{w-1}{p}} \lambda(\omega, w)\}$. Consequently, since $q^{-\frac{w}{p}}$, with $w$ even, and $q^{-\frac{w-1}{p}}$, with $w$ odd, are rational numbers, we conclude that the conjecture $W_{nc}(\text{perf}_{dg}(X))$ holds if and only if the conjecture $W(X)$ holds.

**Remark 6.5** (Weight normalization). Thanks to the identification of cyclotomic Frobenius $F_0$, resp. $F_1$, with the automorphism (6.3), resp. (6.4), we have the
following description of the even/odd Hasse-Weil zeta function:

\[ \zeta_{\text{even}}(\text{perf}_{\text{dg}}(X); s) = \prod_{w \text{ even}} \det(id - q^{-s + \frac{1}{2}}(F_{w, K})|H_{\text{crys}}^{w}(X) \otimes K, \mathbb{C})^{-1} \]

\[ \zeta_{\text{odd}}(\text{perf}_{\text{dg}}(X); s) = \prod_{w \text{ odd}} \det(id - q^{-s + \frac{w-1}{2}}(F_{w, K})|H_{\text{crys}}^{w}(X) \otimes K, \mathbb{C})^{-1}. \]

Using the fact that the polynomials \( p_w(X; t), 0 \leq w \leq 2d \), have integer coefficients, we hence conclude that \( \zeta_{\text{even}}(\text{perf}_{\text{dg}}(X); s) = \prod_{w \text{ even}} \zeta_w(X; s + \frac{w}{2}) \) and \( \zeta_{\text{odd}}(\text{perf}_{\text{dg}}(X); s) = \prod_{w \text{ odd}} \zeta_w(X; s + \frac{w-1}{2}) \). Roughly speaking, this shows that the even/odd Hasse-Weil zeta function of \( \text{perf}_{\text{dg}}(X) \) may be understood as the “weight normalization” of the product of the Hasse-Weil zeta functions \( \zeta_w(X; s) \).

**Remark 6.6 (Weil q-numbers).** Let \( X \) be a smooth proper \( k \)-scheme of dimension \( d \). Given an integer \( 0 \leq w \leq 2d \), let us write \( \{\lambda_{(w,1)}, \ldots, \lambda_{(w,\beta_w)}\} \) for the eigenvalues (with multiplicities) of the automorphism \( F_{w, K} \), where \( \beta_w := \dim_{K} H_{\text{crys}}^{w}(X) \). Thanks to the work of Deligne [12], the eigenvalues \( \{\lambda_{(w,1)}, \ldots, \lambda_{(w,\beta_w)}\} \) are algebraic integers and all their complex conjugates have absolute value \( q^{\frac{w}{2}} \). Consequently, they are Weil q-numbers of weight \( w \). As explained in the proof of Theorem 1.5, in the case of the dg category \( \mathcal{A} = \text{perf}_{\text{dg}}(X) \), the eigenvalues of the cyclotomic Frobenius \( F_0 \), resp. \( F_1 \), are given by \( \bigcup_{w \text{ even}} \{q^{-\frac{w}{2}} \lambda_{(w,1)}, \ldots, q^{-\frac{w}{2}} \lambda_{(w,\beta_w)}\} \), resp. \( \bigcup_{w \text{ odd}} \{q^{-\frac{w-1}{2}} \lambda_{(w,1)}, \ldots, q^{-\frac{w-1}{2}} \lambda_{(w,\beta_w)}\} \). Therefore, by taking \( c := d \), we conclude that the eigenvalues of \( F_0 \) and \( F_1 \) become algebraic integers after multiplication by \( q^c \). This shows, in particular, that both the assumption of Theorem 1.11 as well as conjecture \( W_{\text{nc}}(\mathcal{A}) \) hold for the dg category \( \mathcal{A} = \text{perf}_{\text{dg}}(X) \).

### 7. Proof of Theorem 1.9

We start with the following general result:

**Lemma 7.1.** Let \( \theta : V \otimes_{K} W \rightarrow K \) a perfect bilinear pairing of finite-dimensional \( K \)-vector spaces, \( f \) an automorphism of \( V \), \( g \) an automorphism of \( W \), and \( \lambda \in K \) a non-zero scalar, making the following diagram commute:

\[
\begin{array}{ccc}
V \otimes_{K} W & \xrightarrow{\theta} & K \\
\downarrow{f \otimes_{K} g} \simeq & \downarrow{\simeq} & \downarrow{\lambda \cdot} \\
V \otimes_{K} W & \xrightarrow{\theta} & K .
\end{array}
\]

Under these assumptions, we have the following equalities:

\[
\det(id - tg|W) = \frac{(-1)^{\dim(V)} \lambda^{\dim(V)} f^{\dim(V)}}{\det(f|V)} \cdot \det(id - \lambda^{-1} t^{-1} f|V)
\]

\[
\det(g|W) = \frac{\lambda^{\dim(V)}}{\det(f|V)}.
\]

**Proof.** A simple exercise that we leave for the reader. \( \square \)
Proposition 7.2. Given a smooth proper dg category \( A \), there exist perfect bilinear pairings \( \theta_0 \) and \( \theta_1 \) making the following diagrams commute:

\[
TP_0(A^{\text{op}})_{1/p} \otimes_K TP_0(A)_{1/p} \xrightarrow{\theta_0} K \\
TP_0(A^{\text{op}})_{1/p} \otimes_K TP_1(A)_{1/p} \xrightarrow{\theta_1} K
\]

\[
F_0 \otimes_K F_0 \simeq F_1 \otimes_K F_1 \\
\simeq q^{-}
\]

Proof. Recall from §5.3 that the assignment \( U(A)_{\mathbb{Q}} \rightarrow F^A \) (parametrized by the smooth proper dg categories \( A \)) yields a \( \mathbb{Q} \)-linear symmetric monoidal natural transformation from the functor (5.10) to the functor (5.1). Recall also from §4.1–§4.2 that \( U(A)_{\mathbb{Q}} \) is a dualizable object of the symmetric monoidal category NChow(\( k \)\( \mathbb{Q} \)) and that \( U(A^{\text{op}})_{\mathbb{Q}} \) is the dual of \( U(A)_{\mathbb{Q}} \). Hence, by applying the aforementioned natural transformation to the evaluation morphism \( U(A^{\text{op}})_{\mathbb{Q}} \otimes U(A)_{\mathbb{Q}} \rightarrow U(k)_{\mathbb{Q}} \), we obtain the following commutative diagram:

\[
(7.3) \quad TP_*(A^{\text{op}})_{1/p} \otimes_{K[v^\pm 1]} TP_*(A)_{1/p} \rightarrow TP_*(k)_{1/p} = K[v^\pm 1]^{\tau}
\]

\[
F_* \otimes_{K[v^\pm 1]} F_* \simeq F_*^+ \simeq q^{-}
\]

\[
TP_0(A^{\text{op}})_{1/p} \otimes_K TP_0(A)_{1/p} \rightarrow (TP_*(A^{\text{op}})_{1/p} \otimes_{K[v^\pm 1]} TP_*(A)_{1/p})_0 \rightarrow TP_0(k)_{1/p}
\]

where the left-hand side horizontal morphisms are induced by the monoidal structure of the category \( \text{mod}_\mathbb{Z}(K[v^\pm 1]) \) and the right-hand side horizontal morphisms are induced from (7.3). By construction, the horizontal composition(s), denoted by \( \theta_0 \), is a perfect bilinear pairing. Similarly, the right-hand side commutative diagram of Proposition 7.2 is defined as the composition

\[
TP_1(A^{\text{op}})_{1/p} \otimes_K TP_1(A)_{1/p} \rightarrow (TP_*(A^{\text{op}})_{1/p} \otimes_{K[v^\pm 1]} TP_*(A)_{1/p})_2 \rightarrow TP_2(k)_{1/p}
\]

where the left-hand side horizontal morphisms are induced by the monoidal structure of the category \( \text{mod}_\mathbb{Z}(K[v^\pm 1]) \) and the right-hand side horizontal morphisms are induced from (7.3). By construction, the horizontal composition(s), denoted by \( \theta_1 \), is a perfect bilinear pairing. \( \square \)

We now have all the ingredients necessary to conclude the proof of Theorem 1.9. By construction, we have \( TP_*(A^{\text{op}})_{1/p} = TP_*(A)_{1/p} \) (as \( \mathbb{Z} \)-graded \( K[v^\pm] \)-modules)
Now, choose an embedding $\iota: K \rightarrow \mathbb{C}$ and replace $F_0$ and $F_1$ by $F_0 \otimes_{K, \iota} \mathbb{C}$ and $F_1 \otimes_{K, \iota} \mathbb{C}$, respectively. Then, replace $t$ by $q^{-s}$ and pass to the inverse. This yields the sought functional equations of Theorem 1.9.

Remark 7.4 (Smooth proper schemes). Let $X$ be a smooth proper $k$-scheme of dimension $d$. Note that the isomorphisms (6.1)-(6.2) imply that $\chi_0(\text{perf}_{dg}(X)) = \chi_{\text{even}}(X)$ and $\chi_1(\text{perf}_{dg}(X)) = \chi_{\text{odd}}(X)$, where $\chi_{\text{even}}(X) := \sum_{w \text{ even}} \dim_K H_{\text{crys}}^w(X)$ and $\chi_{\text{odd}}(X) := \sum_{w \text{ odd}} \dim_K H_{\text{crys}}^w(X)$. Note also that the identification of the cyclotomic Frobenius $F_0$, resp. $F_1$, with the automorphism (6.3), resp. (6.4), leads to the following equalities

$$\det(F_0 \otimes_{K, \iota} \mathbb{C}) = \prod_{w \text{ even}} q^{-\frac{d}{2}\beta_w} \cdot \det(\text{Fr}^w \otimes_{K, \iota} \mathbb{C})$$

$$\det(F_1 \otimes_{K, \iota} \mathbb{C}) = \prod_{w \text{ odd}} q^{-\frac{w+1}{2}\beta_w} \cdot \det(\text{Fr}^w \otimes_{K, \iota} \mathbb{C}),$$

where $\beta_w := \dim_K H_{\text{crys}}^w(X)$. Now, recall, for example from [19, App. C Thm. 4.4], that we have moreover the following equalities

$$\det(F^{2d-w} \otimes_{K, \iota} \mathbb{C}) = \frac{q^{d\beta_w}}{\det(\text{Fr}^w \otimes_{K, \iota} \mathbb{C})} \quad 0 \leq w \leq 2d.$$ 

By combining (7.7) with the fact that $\beta_w = \beta_{2d-w}$ for every $0 \leq w \leq 2d$, we hence conclude (via a simple computation) that the square of (7.5), resp. of (7.6), is equal to 1, resp. to $q^{\chi_{\text{odd}}(X)}$. These considerations imply that the functional equations of Theorem 1.9, with $A = \text{perf}_{dg}(X)$, reduce to the functional equations:

$$\zeta_{\text{even}}(\text{perf}_{dg}(X); s) = \pm q^{\chi_{\text{even}}(X)s} \cdot \zeta_{\text{even}}(\text{perf}_{dg}(X); -s)$$

$$\zeta_{\text{odd}}(\text{perf}_{dg}(X); s) = \pm q^{\chi_{\text{odd}}(X)s} \cdot \zeta_{\text{odd}}(\text{perf}_{dg}(X); 1 - s).$$

Moreover, making use of Remark 6.5, these may be re-written as follows:

$$\prod_{w \text{ even}} \zeta_w(X; s + \frac{w}{2}) = \pm q^{\chi_{\text{even}}(X)s} \cdot \prod_{w \text{ even}} \zeta_w(X; -s + \frac{w}{2})$$

$$\prod_{w \text{ odd}} \zeta_w(X; s + \frac{w-1}{2}) = \pm q^{\chi_{\text{odd}}(X)s} \cdot \prod_{w \text{ odd}} \zeta_w(X; 1 - s + \frac{w-1}{2}).$$

8. Proof of Theorem 1.11

Let $\lambda$ be an eigenvalue of the automorphism $F_0$. Note first that if $q^c\lambda$ is an algebraic integer, then $\lambda$ is, in particular, an algebraic number. Therefore, it suffices to prove that, for every $l \neq p$, all the $l$-adic conjugates of $\lambda$ have absolute value 1.
As explained in §7, there exists a perfect bilinear pairing \( \theta_0 \) making the following diagram commute (consult Proposition 7.2 and the subsequent arguments):

\[
\begin{array}{ccc}
TP_0(A)_{1/p} \otimes_K TP_0(A)_{1/p} & \xrightarrow{\theta_0} & K \\
F_0 \otimes_K F_0 & \cong & F_0 \otimes_K F_0 \\
TP_0(A)_{1/p} \otimes_K TP_0(A)_{1/p} & \xrightarrow{\theta_0} & K \\
\end{array}
\]

Thanks to Lemma 7.1, this implies that if \( \lambda \) is an eigenvalue of \( F_0 \), then \( \frac{1}{\lambda} \) is also an eigenvalue of \( F_0 \). Let \( \mathbb{Q}(\lambda)/\mathbb{Q} \) be the (finite) field extension of \( \mathbb{Q} \) generated by \( \lambda \), \( \mathcal{O} \subset \mathbb{Q}(\lambda) \) for the associated ring of integers, and \( (q^e \lambda) = \mathfrak{p}_1 \cdots \mathfrak{p}_m \) and \( (q^\frac{1}{e} \lambda) = \mathfrak{p}'_1 \cdots \mathfrak{p}'_m \) for the (unique) prime decomposition in \( \mathcal{O} \) of the ideals generated by the algebraic integers \( q^e \lambda \) and \( q^\frac{1}{e} \lambda \), respectively. Since \( \mathcal{O} \) is a Dedekind domain, we have the (unique) prime decomposition \( (q^{2e}) = (q^e \lambda)(q^\frac{1}{e} \lambda) = \mathfrak{p}_1 \cdots \mathfrak{p}_m \mathfrak{p}'_1 \cdots \mathfrak{p}'_m \). This implies that all the prime ideals \( \mathfrak{p}_1, \ldots, \mathfrak{p}_m, \mathfrak{p}'_1, \ldots, \mathfrak{p}'_m \) lie over \( p \in \mathbb{Z} \). Consequently, since \( \lambda = (q^e)^{-1}(q^\frac{1}{e} \lambda) \), we conclude that all the \( q \)-adic conjugates of \( \lambda \) have absolute value 1. The proof is similar in the case of the automorphism \( F_1 \):

simply replace (8.1) by the following commutative diagram

\[
\begin{array}{ccc}
TP_1(A)_{1/p} \otimes_K TP_1(A)_{1/p} & \xrightarrow{\theta_1} & K \\
F_1 \otimes_K F_1 & \cong & F_1 \otimes_K F_1 \\
TP_1(A)_{1/p} \otimes_K TP_1(A)_{1/p} & \xrightarrow{\theta_1} & K \\
\end{array}
\]

and note that if \( \lambda \) is an eigenvalue of \( F_1 \), then \( \frac{\lambda}{\lambda} \) is also an eigenvalue of \( F_1 \).

9. PROOF OF THEOREM 1.12

We start by proving item (i). Recall from §5.2 that \( \sigma: K \xrightarrow{\sim} K \) stands for the isomorphism induced by the Frobenius map \( \lambda \mapsto \lambda^p \) on \( K \). Recall also from loc. cit. that the finite-dimensional \( K \)-vector space \( TP_0(A)_{1/p} \) equipped with the \( \sigma \)-semilinear automorphism \( \varphi_0: TP_0(A)_{1/p} \xrightarrow{\sim} TP_0(A)_{1/p} \) is a (classical) isocrystal. Furthermore, we have \( F_0 := \varphi_0 \). Under these notations, the polynomial \( p_0(A; t) = a_0 + a_1 t + \cdots + a_{\chi_0} t^{\chi_0} \) has \( \mathbb{Q}_p \)-coefficients if and only if \( \sigma(a_i) = a_i \) for every \( 0 \leq i \leq \chi_0 \). By definition of \( p_0(A; t) := \det(\text{id} - tF_0|TP_0(A)_{1/p}) \), we have \( a_i = (-1)^i \text{trace}(\lambda^i(F_0)) \). Therefore, using the fact that the category of (classical) isocrystals is closed under exterior products, it suffices to show that \( \sigma(\text{trace}(F_0)) = \text{trace}(F_0) \). Let us choose a basis \( B = (e_1, \ldots, e_{\chi_0}) \) of \( TP_0(A)_{1/p} \) and write \( M(\varphi_0; B, B) \), resp. \( M(F_0; B, B) \), for the matrix of \( \varphi_0 \), resp. \( F_0 \), in the basis \( B \). Note that we have the following equality

\[
M(F_0; B, B) = \sigma^{(r-1)}(M(\varphi_0; B, B)) \cdots \sigma(M(\varphi_0; B, B)) \cdot M(\varphi_0; B, B).
\]

Since \( \sigma^r = \text{id} \), by combining (9.1) with the fact that \( \text{trace}(MN) = \text{trace}(NM) \) for any two matrices \( M \) and \( N \), we hence conclude that \( \sigma(\text{trace}(F_0)) = \text{trace}(F_0) \). This proves the first claim of item (i). In what concerns the second claim, recall that the Newton polygon of the polynomial \( p_0(A; t) = a_0 + a_1 t + \cdots + a_{\chi_0} t^{\chi_0} \) is defined as the convex hull of the set of points \( \{(0, \infty), (i, \nu_q(a_i)), (\chi_0, \infty)\}_{0 \leq i \leq \chi_0} \), where \( \nu_q(-): (\mathbb{Q}_p)^\times \to \mathbb{Q} \) stands for the \( q \)-adic valuation (with \( \nu_q(p) = 1/r \)). On the one hand, we have \( a_0 = (-1)^0 \text{trace}(\lambda^0(F_0)) = 1 \). This implies that \( (0, \nu_q(a_0) =
10. Noncommutative strong form of the Tate conjecture

In this section we prove that the noncommutative strong form of the Tate conjecture is equivalent to the noncommutative p-version of the Tate conjecture plus the noncommutative standard conjecture of type D. As a byproduct of this equivalence of conjectures, we obtain a proof of Theorem 1.15 and also an alternative formulation of the noncommutative strong form of the Tate conjecture in terms of the isocrystals realization functor (consult §10.5 below).

10.1. Noncommutative standard conjecture of type D. Let \( \mathcal{A} \) be a smooth proper dg category. Recall from §5.1, resp. §4.3, the definition of the homological, resp. numerical, Grothendieck group \( K_0(\mathcal{A})_{\text{q/\hom}} \), resp. \( K_0(\mathcal{A})_{\text{q/\num}} \). The noncommutative standard conjecture of type D asserts the following:

\[\text{Conjecture } D_{\text{nc}}(\mathcal{A}) : \text{The equality } K_0(\mathcal{A})_{\text{q/\hom}} = K_0(\mathcal{A})_{\text{q/\num}} \text{ holds.}\]

Remark 10.1 (Standard conjecture of type D). Let \( X \) be a smooth proper k-scheme. As proved in [47, Thm. 1.1], we have the following equivalence of conjectures \( D_{\text{nc}}(\text{perf}_{\text{dg}}(X)) \Leftrightarrow D(X) \), where \( D(X) \) stands for the standard conjecture of type D (consult Grothendieck [17] and Kleiman [27, 28]).

10.2. Noncommutative p-version of the Tate conjecture. Let \( \mathcal{A} \) be a smooth proper dg category. As proved in [51, Lem. 3.7], the \( \mathbb{Q} \)-linear homomorphism \( \text{ch} : K_0(\mathcal{A})_{\mathbb{Q}} \to TP_0(\mathcal{A})_{1/p} \) defined in §5.1 takes values in \( TP_0(\mathcal{A})_{1/p}^{\text{q/\num}} \). Hence, we can consider the \( K \)-linear homomorphism \( \text{ch}_K : K_0(\mathcal{A})_K \to TP_0(\mathcal{A})_{1/p}^{\text{K/\num}} \) and the associated homological Grothendieck group \( K_0(\mathcal{A})_{K/\hom} \). Under these notations, the noncommutative p-version of the Tate conjecture asserts the following:

\[\text{Conjecture } T_{\text{nc}}^p(\mathcal{A}) : \text{The homomorphism } \text{ch}_K \text{ is surjective.}\]

Remark 10.2 (Alternative formulation). Thanks to Galois descent, the conjecture \( T_{\text{nc}}^p(\mathcal{A}) \) may be alternatively formulated as follows: the induced \( \mathbb{Q}_p \)-linear homomorphism \( \text{ch}_{\mathbb{Q}_p} : K_0(\mathcal{A})_{\mathbb{Q}_p} \to TP_0(\mathcal{A})_{1/p}^{\text{q/\num}} \) is surjective.

Remark 10.3 (p-version of the Tate conjecture). Let \( X \) be a smooth proper k-scheme. As proved in [51, Thm. 1.3], we have the equivalence of conjectures \( T_{\text{nc}}^p(\text{perf}_{\text{dg}}(X)) \Leftrightarrow T^p(X) \), where \( T^p(X) \) stands for the p-version of the Tate conjecture (consult Milne [41] and Tate [55]).
10.3. Equivalence of conjectures. The next result is of independent interest:

**Theorem 10.4.** Given a smooth proper dg category $\mathcal{A}$, we have the equivalence of conjectures $\text{ST}_{\text{nc}}(\mathcal{A}) \leftrightarrow T_p^{\text{nc}}(\mathcal{A}) + D_{\text{nc}}(\mathcal{A})$.

**Proof.** We start by proving the implication $\text{ST}_{\text{nc}}(\mathcal{A}) \Rightarrow T_p^{\text{nc}}(\mathcal{A}) + D_{\text{nc}}(\mathcal{A})$. Recall from Remark 1.13 that if the conjecture $\text{ST}_{\text{nc}}(\mathcal{A})$ holds, then the algebraic multiplicity of the eigenvalue $1$ of $F_0$ agrees with the dimension of the $\mathbb{Q}$-vector space $K_0(\mathcal{A})_{\mathbb{Q}/\sim\text{num}}$. Note also that the geometric multiplicity of the eigenvalue $1$ of $F_0$, which is always less (or equal) than the algebraic multiplicity, agrees with the dimension of the $\mathbb{K}$-vector space $TP_0(A)_{1/p}^{F_0}$. In order to prove the conjecture $T_p^{\text{nc}}(\mathcal{A})$, we need then to show that the dimension of the $\mathbb{K}$-vector space $TP_0(A)_{1/p}^{F_0}$ is less (or equal) than the dimension of the $\mathbb{K}$-vector space $K_0(\mathcal{A})_{\mathbb{K}/\sim\text{hom}}$. This follows from the following (in)equalities:

$$
\dim_K TP_0(A)_{1/p}^{F_0} = \text{geometric multiplicity of the eigenvalue } 1 \text{ of } F_0 \\
\leq \text{algebraic multiplicity of the eigenvalue } 1 \text{ of } F_0 \\
= \dim_{\mathbb{Q}} K_0(\mathcal{A})_{\mathbb{Q}/\sim\text{num}} \\
\leq \dim_{\mathbb{Q}} K_0(\mathcal{A})_{\mathbb{Q}/\sim\text{hom}} = \dim_K K_0(\mathcal{A})_{\mathbb{K}/\sim\text{hom}}.
$$

Similarly, since $\dim_{\mathbb{Q}} K_0(\mathcal{A})_{\mathbb{Q}/\sim\text{num}} \leq \dim_{\mathbb{Q}} K_0(\mathcal{A})_{\mathbb{Q}/\sim\text{hom}}$, in order to prove the conjecture $D_{\text{nc}}(\mathcal{A})$, we need then to show that the dimension of the $\mathbb{Q}$-vector space $K_0(\mathcal{A})_{\mathbb{Q}/\sim\text{hom}}$ is less (or equal) than the dimension of the $\mathbb{Q}$-vector space $K_0(\mathcal{A})_{\mathbb{Q}/\sim\text{num}}$. This follows from the following (in)equalities:

$$
\dim_{\mathbb{Q}} K_0(\mathcal{A})_{\mathbb{Q}/\sim\text{hom}} = \dim_K K_0(\mathcal{A})_{\mathbb{K}/\sim\text{hom}} \leq \dim_K TP_0(A)_{1/p}^{F_0} \\
= \text{geometric multiplicity of the eigenvalue } 1 \text{ of } F_0 \\
\leq \text{algebraic multiplicity of the eigenvalue } 1 \text{ of } F_0 \\
= \dim_{\mathbb{Q}} K_0(\mathcal{A})_{\mathbb{Q}/\sim\text{num}}.
$$

We now prove the implication $T_p^{\text{nc}}(\mathcal{A}) + D_{\text{nc}}(\mathcal{A}) \Rightarrow \text{ST}_{\text{nc}}(\mathcal{A})$. Note that if both conjectures $T_p^{\text{nc}}(\mathcal{A})$ and $D_{\text{nc}}(\mathcal{A})$ hold, then the geometric multiplicity of the eigenvalue $1$ of $F_0$ is equal to the dimension of the $\mathbb{Q}$-vector space $K_0(\mathcal{A})_{\mathbb{Q}/\sim\text{num}}$. Hence, in order to prove the conjecture $\text{ST}_{\text{nc}}(\mathcal{A})$, it suffices then to show that the geometric multiplicity of the eigenvalue $1$ of $F_0$ agrees with the algebraic multiplicity of the eigenvalue $1$ of $F_0$. Thanks to Lemma 10.6 below, this will follow from the injectivity of the canonical morphism $\epsilon: TP_0(A)_{1/p}^{F_0} \to (TP_0(A)_{1/p}^{F_0})_0$ induced by the identity on $TP_0(A)_{1/p}$. Recall from §4.1-§4.2 that $U(\mathcal{A})_{\mathbb{Q}}$ is a dualizable object of the symmetric monoidal category $\text{NChow}(k)_{\mathbb{Q}}$ and that $U(\mathcal{A}^{\text{op}})_{\mathbb{Q}}$ is the dual of $U(\mathcal{A})_{\mathbb{Q}}$. Consequently, by applying the functor $\text{Hom}_{\text{NChow}(k)}(U(k)_{\mathbb{Q}}, -)$ to the evaluation morphism $U(\mathcal{A}^{\text{op}})_{\mathbb{Q}} \otimes U(\mathcal{A})_{\mathbb{Q}} \to U(k)_{\mathbb{Q}}$, we obtain (from the symmetric monoidal structure of $\text{NChow}(k)_{\mathbb{Q}}$) a bilinear pairing $\psi: K_0(\mathcal{A}^{\text{op}})_{\mathbb{Q}} \otimes K_0(\mathcal{A})_{\mathbb{Q}} \to \mathbb{Q}$. Note that since the $\mathbb{Q}$-linear functor (5.1) is symmetric monoidal, we have the following commutative diagram

$$
\begin{array}{ccc}
TP_0(\mathcal{A}^{\text{op}})_{1/p}^{F_0} \otimes_K TP_0(\mathcal{A})_{1/p}^{F_0} & \xrightarrow{\delta_0} & \mathbb{K} \\
\uparrow \text{ch}_{\mathbb{K}} \otimes_K \text{ch}_{\mathbb{K}} \quad & & \quad \uparrow \psi_K \\
K_0(\mathcal{A}^{\text{op}})_{\mathbb{K}} \otimes_K K_0(\mathcal{A})_{\mathbb{K}} & \xrightarrow{\psi_K} & \mathbb{K},
\end{array}
$$
where \( \theta_0 \) stands for the perfect bilinear pairing of Proposition 7.2 and \( \psi_K \) for the \( K \)-linearization of \( \psi \). By adjunction, this yields the induced commutative diagram:

\[
\begin{array}{ccc}
TP_0(A)^{F_0}_{1/p} \ar{r}{\theta_0^0} \ar{d}{ch_K} & \text{Hom}_K(TP_0(A)^{op})^{F_0}_{1/p}, K) \ar{d}{\text{Hom}(ch_K,K)} \\
K_0(A)_{K/\sim_{\text{hom}}} \ar{r}{\psi_K} & \text{Hom}_K(K_0(A)^{op})_{K/\sim_{\text{hom}}, K}.
\end{array}
\]

Thanks to the left-hand side commutative diagram of Proposition 7.2, the morphism \( \theta_0^0 \) admits the following factorization:

\[
\theta_0^0 : TP_0(A)^{F_0}_{1/p} \rightarrow (TP_0(A)^{F_0}_{1/p})_{F_0} \rightarrow \text{Hom}_K(TP_0(A)^{op})^{F_0}_{1/p}, K).
\]

Using the fact that the left-hand side vertical morphism in (10.5) is surjective (=conjecture \( \text{TP}_{nc}(A) \)), we observe that in order to show that the canonical morphism \( \epsilon \) is injective, it suffices then to show that the morphism \( \psi_K \) is injective. As explained in [40, §6], a Grothendieck class \( \alpha \in K_0(A)_{\mathbb{Q}} \) is numerically trivial in the sense of §4.3 if and only if \( \psi(\beta, \alpha) = 0 \) for every \( \beta \in K_0(A)^{op}_{\mathbb{Q}} \). In other words, the numerical Grothendieck group \( K_0(A)_{\mathbb{Q}/\sim_{\text{num}}} \) may be identified with the quotient of \( K_0(A)_{\mathbb{Q}/\sim_{\text{hom}}} \) by the kernel of \( \theta_0^0 \). Therefore, in order to prove that \( \psi_K \) is injective, we can then consider the following commutative diagram:

\[
\begin{array}{ccc}
(K_0(A)_{\mathbb{Q}/\sim_{\text{hom}}})_K \ar{r} \ar{d} & K_0(A)_{K/\sim_{\text{hom}}} \ar{d} & \text{Hom}_K(K_0(A)^{op})_{K/\sim_{\text{hom}}, K} \\
(K_0(A)_{\mathbb{Q}/\sim_{\text{num}}})_K \ar{r} & K_0(A)_{K/\sim_{\text{num}}} & \text{Hom}_K(K_0(A)^{op})_{K/\sim_{\text{hom}}, K}.
\end{array}
\]

Since the curved morphism is injective and the vertical morphism(s) is injective (=conjecture \( \text{D}_{nc}(A) \)), we hence conclude that the morphism \( \psi_K \) is also injective. This finishes the proof of Theorem 10.4. \( \square \)

**Lemma 10.6.** Let \( f : V \rightarrow V \) be an automorphism of a finite-dimensional \( K \)-vector space \( V \). Under these assumptions, the geometric multiplicity of the eigenvalue 1 of \( f \) agrees with the algebraic multiplicity of the eigenvalue 1 of \( f \) if and only if the canonical morphism \( \epsilon : V_f \rightarrow V_f \) (induced by the identity on \( V \)) is injective.

**Proof.** Note that the geometric multiplicity of the eigenvalue 1 of \( f \) agrees with the algebraic multiplicity of the eigenvalue 1 of \( f \) if and only if \( \text{Ker}(\text{id} - f) = \text{Ker}((\text{id} - f)^2) \). Hence, the proof follows from the fact that the latter condition is equivalent to the condition \( \text{Ker}(\text{id} - f) \cap \text{Im}(\text{id} - f) = \emptyset \), i.e., to the injectivity of the canonical morphism \( V_f \rightarrow V_f \). \( \square \)

**10.4. Proof of Theorem 1.15.** As mentioned in Remarks 10.1 and 10.3, we have the following equivalences of conjectures

\[
\text{D}_{nc}(\text{perf}_{dg}(X)) \Leftrightarrow \text{D}(X) \quad \text{TP}_{nc}(\text{perf}_{dg}(X)) \Leftrightarrow \text{TP}(X).
\]

Therefore, by combining Theorem 10.4 (with \( A = \text{perf}_{dg}(X) \)) with the equivalence of conjectures \( \text{ST}(X) \Leftrightarrow \text{TP}(X) + \text{D}(X) \) established in [41, Thm. 1.11] (consult also [54, Thm. 2.9]), we obtain the sought equivalence \( \text{ST}_{nc}(\text{perf}_{dg}(X)) \Leftrightarrow \text{ST}(X) \). 


Remark 10.7 (Direct proof). A direct proof of the following implication of conjectures ST(X) ⇒ ST_{nc}(\text{perf}_{dg}(X)) can be achieved as follows: thanks to the factorization (1.2) and to Remark 6.5, we have the following equality:

$$\text{ord}_{s=0}\zeta_{\text{even}}(\text{perf}_{dg}(X); s) = \sum_{w \text{ even}} \text{ord}_{s=\frac{1}{2}}\zeta_{w}(X; s) = \sum_{0 \leq j \leq d} \text{ord}_{s=j}\zeta(X; s).$$

Therefore, since the numerical Grothendieck group $K_0(\text{perf}_{dg}(X))_{Q/\text{num}}$ is isomorphic to the direct sum $\bigoplus_{i=0}^d Z^i(X)_{Q/\text{num}}$ (consult [49, Prop. 1.7(i)]), we conclude that the conjecture ST_{nc}(\text{perf}_{dg}(X)) follows from ST(X).

10.5. Alternative formulation. The next result, which is of independent interest, provides an alternative formulation of the noncommutative strong form of the Tate conjecture (when all smooth proper dg categories are considered simultaneously):

Proposition 10.8. The conjecture ST_{nc}(\mathcal{A}) holds for every smooth proper dg category \mathcal{A} if and only if the isocrystal realization functor (5.13) yields the following $Q_p$-linear symmetric monoidal fully-faithful functor:

$$\text{NNum}(k)_{Q_p} \rightarrow \text{Crys}_{\mathbb{Q}}(K[v^{\pm 1}]) \quad U(\mathcal{A})_{Q_p} \rightarrow \langle TP_*(\mathcal{A})_{1/p}, \varphi_* \rangle.$$  

Proof. By construction, every object of $\text{NNum}(k)_{Q_p}$ is a direct summand of an object of the form $U(\mathcal{A})_{Q_p}$ for some smooth proper dg category $\mathcal{A}$; consult [50, §4.6]. This implies that the functor (5.13) descends to the category of noncommutative numerical motives $\text{NNum}(k)_{Q_p}$ if and only if the conjecture $\text{D}_{nc}(\mathcal{A})$ holds for every smooth proper dg category $\mathcal{A}$. Given smooth proper dg categories $\mathcal{B}$ and $\mathcal{C}$, note that by definition of the category $\text{Crys}_{\mathbb{Q}}(K[v^{\pm 1}])$, we have a natural isomorphism:

$$\text{Hom}_{\text{Crys}_{\mathbb{Q}}(K[v^{\pm 1}])(\langle TP_*(\mathcal{B})_{1/p}, \varphi_* \rangle, \langle TP_*(\mathcal{C})_{1/p}, \varphi_* \rangle) \simeq TP_0(\mathcal{B}^{op} \otimes_k \mathcal{C})_{1/p}.$$  

This implies that the $Q_p$-linear symmetric monoidal faithful functor (10.9) is moreover full if and only if the conjecture $\text{T}_{nc}(\mathcal{A})$ holds for every smooth proper dg category $\mathcal{A}$; consult Remark 10.2. Hence, the proof follows now from Theorem 10.4. □

11. Proof of Theorem 1.16

Remark 11.1. Similarly to Remark 5.11, thanks to the above functor (10.9), we can formulate the conjectures $C_{nc}(\mathcal{M})$, with $\mathcal{M} \in \{W, W^l, ST, \overline{W}\}$, for every noncommutative numerical motive $\mathcal{M} \in \text{NNum}(k)_{Q_p}$. In the particular case where $\mathcal{M} = U(\mathcal{A})_{Q_p}$, with $\mathcal{A}$ a smooth proper dg category, these reduce to the conjectures $C_{nc}(\mathcal{A})$, with $\mathcal{A} \in \{W, W^l, ST, \overline{W}\}$. By construction, every object of $\text{NNum}(k)_{Q_p}$ is a direct summand of an object of the form $U(\mathcal{A})_{Q_p}$ for some proper dg category $\mathcal{A}$. Therefore, if one of the conjectures $C_{nc}(-),$ with $\mathcal{A} \in \{W, W^l, ST, \overline{W}\},$ holds for every smooth proper dg category $\mathcal{A}$, then the same conjecture holds for every noncommutative numerical motive $\mathcal{M} \in \text{NNum}(k)_{Q_p}$.

Item (i). Note first that the category $\text{Crys}_{\mathbb{Q}}(K[v^{\pm 1}])$ (see Definition 5.12) comes equipped with the following $\otimes$-automorphism $\pi$ of the identity functor:

$$\langle V_*, \varphi_* \rangle \mapsto \pi_*^{(V_*, \varphi_*)} \quad \text{where} \quad \pi_n^{(V_*, \varphi_*)} := \begin{cases} \left(\frac{p^n}{2} \cdot \varphi_n^{V_*}\right)^r & n \text{ even} \\ \left(\frac{n!(p/n)^n}{2} \cdot \varphi_n^{V_*}\right)^r & n \text{ odd} \end{cases}.$$  

Making use of the $Q_p$-linear symmetric monoidal fully-faithful functor (10.9), we hence conclude that the category $\text{NNum}(k)_{Q_p}$ also comes equipped with a $\otimes$-automorphism $\pi$ of the identity functor. This proves the first claim. The second
Proof. We need to show that there are no morphisms $f$ such that $f(K)$ is the center of $K$. The proof follows now from the (classical) double centralizer theorem applied to the $K$-$A$-subalgebra $A$ with the centralizer of $K$. Given an object $p$ we have the following natural identifications:

\[ \text{End}_{\text{Crys}}(K[v^{\pm 1}])(TP_*(NM)_{1/p}, \varphi_*) \]

agrees with the $Q_p$-subalgebra $Q_p[\pi_{NM}]$ generated by $\pi_{NM}$. Proposition 11.3. Given an object $NM \in \text{NNum}(k)_{Q_p}$, the center of the $Q_p$-algebra $\text{End}_{\text{NNum}(k)_{Q_p}}(NM)$ agrees with the $Q_p$-subalgebra $Q_p[\pi_{NM}]$ generated by $\pi_{NM}$. Proof. Thanks to the $Q_p$-linear fully-faithful functor (10.9), it suffices to show that the center of the following $Q_p$-algebra

\[ \text{End}_K(TP_0(NM)_{1/p} \oplus TP_1(NM)_{1/p}) \]

and the $K$-subalgebra $A := K[v] = \left( \begin{array}{cc} F_0 & 0 \\ 0 & F_1 \end{array} \right)$ generated by the automorphism $\left( \begin{array}{cc} F_0 & 0 \\ 0 & F_1 \end{array} \right)$. Lemma 11.6. The center of $A$ in (11.5) consists of those matrices $\left( \begin{array}{cc} f_0 & 0 \\ 0 & f_1 \end{array} \right)$ such that $f_0 \circ F_0 = F_0 \circ f_0$ and $f_1 \circ F_1 = F_1 \circ f_1$. Proof. The center of $A$ in (11.5) consists of those matrices $\left( \begin{array}{cc} f_0 & f_1 \\ f_0 & f_1 \end{array} \right)$ such that $f_0 \circ F_0 = F_0 \circ f_0$, $f_1 \circ F_1 = F_1 \circ f_1$, $f_1 \circ f_0 = f_0 \circ f_1$, and $F_0 \circ f_1 = f_0 \circ F_1$. Hence, we need to show that there are no morphisms $f_{01}: TP_0(NM)_{1/p} \to TP_1(NM)_{1/p}$ and $f_{10}: TP_1(NM)_{1/p} \to TP_0(NM)_{1/p}$ such that $f_{01} \circ f_{01} = f_0 \circ F_0$ and $F_0 \circ f_{10} = f_{10} \circ F_1$. In other words, we need to show that the following automorphisms

\[ \text{Hom}_K(TP_0(NM)_{1/p}, TP_1(NM)_{1/p}) \quad f_{01} \mapsto F_1 \circ f_{01} \circ F_0^{-1} \]

\[ \text{Hom}_K(TP_1(NM)_{1/p}, TP_0(NM)_{1/p}) \quad f_{10} \mapsto F_0 \circ f_{10} \circ F_1^{-1} \]

do not have fixed points. Let $NM^\vee$ be the dual of $NM$. Since the $Q_p$-linear functor

\[ TP_*(-)_{1/p}: \text{NNum}(k)_{Q_p} \to \text{mod}_2(K[u^{\pm 1}]) \]

is symmetric monoidal, we have the following natural identifications:

\[ TP_*(NM^\vee \otimes NM)_{1/p} \simeq TP_*(NM)_{1/p} \otimes K[u^{\pm 1}] TP_*(NM)_{1/p} \]

\[ \simeq \text{Hom}(TP_*(NM)_{1/p}, TP_*(NM)_{1/p}). \]
This implies that $TP_1(\mathcal{M}^\vee \otimes \mathcal{M})_{1/p}$ is naturally isomorphic to the direct sum
\[ \operatorname{Hom}_K(\mathcal{M})_{1/p}, TP_1(\mathcal{M})_{1/p} \oplus \operatorname{Hom}_K(\mathcal{M})_{1/p}, \]
Under this latter isomorphism, the automorphism $F_1$ of $TP_1(\mathcal{M}^\vee \otimes \mathcal{M})_{1/p}$ corresponds to the following automorphisms:
\begin{align}
(11.9) & \quad \operatorname{Hom}_K(\mathcal{M})_{1/p}, TP_1(\mathcal{M})_{1/p} \quad f \mapsto F_1 \circ f \circ F_0^{-1} \\
(11.10) & \quad \operatorname{Hom}_K(\mathcal{M})_{1/p}, TP_2(\mathcal{M})_{1/p} \quad g \mapsto F_2 \circ g \circ F_1^{-1}.
\end{align}
Now, note that since the conjecture $W_{nc}(\mathcal{M}^\vee \otimes \mathcal{M})$ holds (consult Remark 11.1), we have $TP_1(\mathcal{M}^\vee \otimes \mathcal{M})_{1/p} = TP_1(\mathcal{M}^\vee \otimes \mathcal{M})_{1/p}$ and $TP_2(\mathcal{M})_{1/p} = 0$. We hence conclude from (11.9), resp. from the combination of (11.10) with the commutative square (5.8), that the above automorphism (11.7), resp. (11.8), does not have fixed points. 

**Item (ii).** Let $\mathcal{M} \in NNum(k)_{Q_p}$ be a simple object. Since the $Q_p$-linear functor (10.9) is fully-faithful, the object $(TP_0(\mathcal{M})_{1/p}, \varphi_*) \in \operatorname{Crys}_Z(K[v_{\pm 1}])$ is also simple. Therefore, we have $TP_0(\mathcal{M})_{1/p} = 0$ or $TP_1(\mathcal{M})_{1/p} = 0$. In what follows, we will assume that $TP_0(\mathcal{M})_{1/p} = 0$. Consider the finite-dimensional field extension $\mathbb{Q}_p[\pi_{\mathcal{M}}]/\mathbb{Q}_p$. The associated (irreducible) minimal polynomial agrees with the minimal polynomial of the automorphism $F_0$ of $TP_0(\mathcal{M})_{1/p}$. Since the conjecture $W_{nc}(\mathcal{M})$ holds (see Remark 11.1), this implies that the set $[\pi_{\mathcal{M}}]_p$ of $p$-adic conjugates of $\pi_{\mathcal{M}}$ consists of Weil $q$-numbers of weight 0. Moreover, given any two simple objects $\mathcal{M}, \mathcal{M}' \in NNum(k)_{Q_p}$, note that there exists a $Q_p$-algebra isomorphism $Q_p[\pi_{\mathcal{M}}] \simeq Q_p[\pi_{\mathcal{M}'}]$ sending $\pi_{\mathcal{M}}$ to $\pi_{\mathcal{M}'}$. In other words, we have $[\pi_{\mathcal{M}}] = [\pi_{\mathcal{M}'}]$. As a consequence, we obtain the following well-defined map:

\[ (11.11) \quad \begin{array}{c}
\{ \text{simple objs. in } NNum(k)_{Q_p} \} \\
\text{isomorphism}
\end{array} \quad \begin{array}{c}
\mathbb{F}_p \rightarrow [\pi_{\mathcal{M}}] \\
\text{Weil } q\text{-numbers of weight 0}
\end{array} \quad \begin{array}{c}
\text{Gal}(\mathbb{Q}_p/\mathbb{Q}_p)\text{-action}
\end{array}.
\]

We now prove that (11.11) is injective. Let $\mathcal{M}, \mathcal{M}' \in NNum(k)_{Q_p}$ be two simple objects such that $[\pi_{\mathcal{M}}] = [\pi_{\mathcal{M}'}]$, i.e., for which there exists a $Q_p$-algebra isomorphism $Q_p[\pi_{\mathcal{M}}] \simeq Q_p[\pi_{\mathcal{M}'}]$ sending $\pi_{\mathcal{M}}$ to $\pi_{\mathcal{M}'}$. We need to show that $\mathcal{M} \simeq \mathcal{M}'$. Let us assume by absurd that $\mathcal{M} \not\simeq \mathcal{M}'$. This would imply that $\operatorname{End}_{NNum(k)_{Q_p}}(\mathcal{M} \oplus \mathcal{M}') = \operatorname{End}_{NNum(k)_{Q_p}}(\mathcal{M}) \times \operatorname{End}_{NNum(k)_{Q_p}}(\mathcal{M}')$ and consequently that $Q_p[\pi_{\mathcal{M} \oplus \mathcal{M}'}] = Q_p[\pi_{\mathcal{M}'}] \times Q_p[\pi_{\mathcal{M}'}]$. However, this is a contradiction because, thanks to Proposition 11.3, the $Q_p$-algebra $Q_p[\pi_{\mathcal{M} \oplus \mathcal{M}'}]$ is contained in the graph of the isomorphism $Q_p[\pi_{\mathcal{M}}] \simeq Q_p[\pi_{\mathcal{M}'}]$ and this latter graph is strictly smaller than the product $Q_p[\pi_{\mathcal{M}}] \times Q_p[\pi_{\mathcal{M}'}]$.

We now prove that (11.11) is moreover surjective. Given a Weil $q$-number $\lambda$ of weight 0, we need to construct a simple object $\mathcal{M}' \in NNum(k)_{Q_p}$ such that $[\pi_{\mathcal{M}'}] = [\lambda]$. Consider the classical (abelian semi-simple) category of numerical motives $\operatorname{Num}(k)_{Q_p}$. Since the conjecture ST$_{nc}(A)$ holds for every smooth proper dg category $A$, it follows from Theorem 1.15 that the conjecture ST$(X)$ holds for every smooth proper $k$-scheme $X$. This has two implications. Firstly, as proved in [42, Prop. 2.4], given a simple object $M \in \operatorname{Num}(k)_{Q_p}$, the $Q_p$-subalgebra $Q_p[\operatorname{Fr}_M]$ of $\operatorname{End}_{\operatorname{Num}(k)_{Q_p}}(M)$ generated by the Frobenius automorphism $\operatorname{Fr}_M$ of $M$ is a finite-dimensional field extension of $Q_p$. Secondly, as proved in [42, Prop. 3.8], there exists a simple object $M' \in \operatorname{Num}(k)_{Q_p}$ such that $[\operatorname{Fr}_{M'}] = [\lambda]$. Now, recall from [50, §4.6] that there exists a $Q_p$-linear symmetric monoidal fully-faithful functor $\Phi$ making
the following diagram commute

\[
\begin{array}{ccc}
\text{SmProp}(k)^{\text{op}} & \xrightarrow{X \mapsto \text{perf}_{dg}(X)} & \text{dgcat}_{sp}(k) \\
\downarrow \text{Fr}_{M'} & & \downarrow \Phi \\
\text{Num}(k)_{Q_p} & \xrightarrow{\Phi} & \text{NNum}(k)_{Q_p} \\
\downarrow \text{Num}(k)_{Q_p} / - \otimes Q_p(1) & \phi & \\
\end{array}
\]

where SmProp\((k)\) stands for the category of smooth proper \(k\)-schemes, dgcat\(_{sp}(k)\) for the category of smooth proper dg categories, \(Q_p(1)\) for the Tate motive, and Num\((k)_{Q_p} / - \otimes Q_p(1)\) for the orbit category with respect to the \(\otimes\)-invertible object \(Q_p(1)\). Therefore, we can consider the noncommutative numerical motive \(\text{NM'} := \Phi(M') \in \text{NNum}(k)_{Q_p}\). We claim that \(\Phi(M')\) is a simple object and that \([\pi_{\Phi(M')}]) = [\text{Fr}_{M'}]\). In what concerns the first claim, recall that

\[\text{Hom}_{\text{Num}(k)_{Q_p} / - \otimes Q_p(1)}(M', M') := \bigoplus_{n \in \mathbb{Z}} \text{Hom}_{\text{Num}(k)_{Q_p}}(M', M' \otimes Q_p(n))^p.\]

Since the simple objects \(M'\) and \(M' \otimes Q_p(n)\), with \(n \neq 0\), are not isomorphic, we have \(\text{Hom}_{\text{Num}(k)_{Q_p}}(M', M' \otimes Q_p(n)) = 0\) for every \(n \neq 0\). Consequently, using the fact that \(\Phi\) is fully-faithful, we obtain an induced \(Q_p\)-algebra isomorphism:

\[\text{End}_{\text{Num}(k)_{Q_p}}(M') \simeq \text{End}_{\text{NNum}(k)_{Q_p}}(\Phi(M')).\]

Note that the isomorphism (11.13) implies, in particular, that the noncommutative numerical motive \(\Phi(M')\) is simple. In order to prove the second claim, note that it suffices to show that the automorphism \(\text{Fr}_{M'}\) corresponds under (11.13) to the automorphism \(\pi_{\Phi(M')}\). Since the functor (10.9) is fully-faithful and the forgetful functor \(\text{Crys}_{\mathbb{Z}}(K[v^{\pm 1}]) \to \text{mod}_{\mathbb{Z}}(K[v^{\pm 1}])\) is faithful, we have an induced injective \(Q_p\)-algebra homomorphism:

\[\text{End}_{\text{Num}(k)_{Q_p}}(\Phi(M')) \rightarrow \text{End}_{\text{mod}_{\mathbb{Z}}(K[v^{\pm 1}])(\text{TP}_0(\Phi(M'))_{1/p}).\]

Moreover, since the \(p\)-adic conjugates of \(\text{Fr}_{M'}\) are Weil \(q\)-numbers of weight 0, we have \(H^{w}_{\text{crys}}(X) = 0\) for every \(w \neq 0\). Making use of the identifications (6.1)-(6.2), we hence conclude that \(\text{TP}_0(\Phi(M'))_{1/p} \simeq H^0_{\text{crys}}(M')\) and \(\text{TP}_1(\Phi(M'))_{1/p} = 0\). This implies, in particular, that the forgetful \(Q_p\)-algebra homomorphism

\[\text{End}_{\text{mod}_{\mathbb{Z}}(K[v^{\pm 1}])(\text{TP}_0(\Phi(M'))_{1/p}) \rightarrow \text{End}_{\mathbb{K}}(\text{TP}_0(\Phi(M'))_{1/p})\]

is also injective. On the one hand, the image of \(\text{Fr}_{M}\) under the composition (11.15) \(\circ\) (11.14) \(\circ\) (11.13) corresponds to the automorphism \(\text{Fr}^0\) of \(H^0_{\text{crys}}(M')\). On the other hand, thanks to the identification (6.3), the image of \(\pi_{\Phi(M')}\) under the composition (11.15) \(\circ\) (11.14) corresponds to the automorphism \(q^{-2}\text{Fr}^0 = \text{Fr}^0\). Consequently, the proof follows now from the fact that (11.14) and (11.15) are injective.

Finally, note that the above proof of item (ii) holds similarly when \(\text{TP}_0(\text{NM})_{1/p} = 0\): simply replace the Weil \(q\)-numbers of weight 0 by the Weil \(q\)-numbers of weight 1, and the identification (6.3) by the identification (6.4).
**Item (iii).** Let $\mathcal{M} \in \text{NNum}(k)_p$ be a simple object. Since the $\mathbb{Q}_p$-linear functor (10.9) is fully-faithful, we have an induced $\mathbb{Q}_p$-algebra isomorphism:

\[(11.16) \quad \text{End}_{\text{NNum}(k)_p}(\mathcal{M}) \cong \text{End}_{\text{Crys}(K[u^{\pm 1}])}(\mathcal{M}) .\]

This implies that the object $\mathcal{M}$ is also simple for the noncommutative Chow motives. Consequently, we have $\mathcal{M}_0/(\mathcal{M})_{1/p} = 0$ or $\mathcal{M}_1/(\mathcal{M})_{1/p} = 0$. In what follows, we will assume that $\mathcal{M}_1/(\mathcal{M})_{1/p} = 0$. Consider the following forgetful functor

\[(11.17) \quad \text{Crys}(K[u^{\pm 1}]) \to \text{Crys}(K), \quad (V_*, \varphi^*_V) \mapsto (V_0, \varphi^*_0),\]

where $\text{Crys}(K)$ stands for the (classical) category of isocrystals. Note that it follows from the definition of the category $\text{Crys}(K[u^{\pm 1}])$ and from the assumption that $\mathcal{M}_1/(\mathcal{M})_{1/p} = 0$ that the functor (11.17) yields a $\mathbb{Q}_p$-algebra isomorphism:

\[(11.18) \quad \text{End}_{\text{Crys}(K[u^{\pm 1}])}(\mathcal{M}) \cong \text{End}_{\text{Crys}(K)}(\mathcal{M}).\]

As explained in the proof of Proposition 11.3, the left-hand side of (11.18) is a division $\mathbb{Q}_p$-algebra with center $\mathbb{Q}_p[\pi] \cong \mathbb{Q}_p[F_0]$. Consequently, the right-hand side is a division $\mathbb{Q}_p$-algebra with center $\mathbb{Q}_p[\varphi^*_0] = \mathbb{Q}_p[F_0]$. This implies, in particular, that $\mathcal{M}_0/(\mathcal{M})_{1/p, \varphi}$ is a simple isocrystal. Hence, the proof follows now from the combination of the isomorphisms (11.16) and (11.18) with the fact that the Hasse-invariant of the central division $\mathbb{Q}_p[F_0]$-algebra $\text{End}_{\text{Crys}(K)}(\mathcal{M})$ is known to be equal to $-\nu_q(F_0) \cdot [\mathbb{Q}_p[F_0] : \mathbb{Q}_p]$; consult [42, Prop. 2.14].

Finally, note that the above proof holds similarly when $\mathcal{M}_0/(\mathcal{M})_{1/p} = 0$: simply replace the forgetful functor (11.17) by the forgetful functor $(V_*, \varphi^*_V) \mapsto (V_1, \varphi^*_1)$.

12. PROOF OF THEOREMS 2.1, 2.4-2.7, AND 2.11-2.12

Recall from Remark 5.11 that the conjectures $C_{\text{nc}}(-)$, with $C \in \{W, W^l, ST, \overline{W}\}$, may be formulated for every $\mathcal{M} \in \text{NChow}(k)_\mathbb{Q}$. In addition, we can also formulated the conjectures $D_{\text{nc}}(-)$ and $T_{\text{nc}}^p(-)$; consult §10.1—§10.2.

**Proposition 12.1.** The conjectures $C_{\text{nc}}(-)$, with $C \in \{W, W^l, ST, \overline{W}, D, T^p\}$, are stable under direct sums and direct summands of noncommutative Chow motives.

**Proof.** The stability under direct sums is clear. The stability under direct summands is also clear for the noncommutative Weil conjecture(s), the noncommutative standard conjecture of type $D$, and the noncommutative $p$-version of the Tate conjecture. In what regards the noncommutative strong form of the Tate conjecture, it follows from the equivalence $\text{ST}_{\text{nc}}(\mathcal{M}) \Leftrightarrow T_{\text{nc}}^p(\mathcal{M}) + D_{\text{nc}}(\mathcal{M})$.

**Corollary 12.2.** Given noncommutative Chow motives $\mathcal{M}, \mathcal{M}' \in \text{NChow}(k)_\mathbb{Q}$, we have the following equivalence of conjectures:

$C_{\text{nc}}(\mathcal{M} \oplus \mathcal{M}') \leftrightarrow C_{\text{nc}}(\mathcal{M}) + C_{\text{nc}}(\mathcal{M}')$ with $C \in \{W, W^l, ST, \overline{W}\}$.

**Example 12.3 (Semi-orthogonal decompositions).** Let $\mathcal{B}, \mathcal{C} \subseteq \mathcal{A}$ be smooth proper dg categories inducing a semi-orthogonal decomposition of triangulated categories $\mathbf{H}^0(\mathcal{A}) = (\mathbf{H}^0(\mathcal{B}), \mathbf{H}^0(\mathcal{C}))$ in the sense of Bondal-Orlov [9]. As proved in [50, Prop. 2.2], the inclusions $\mathcal{B}, \mathcal{C} \subseteq \mathcal{A}$ give rise to an isomorphism of noncommutative Chow motives $U(\mathcal{A})_\mathbb{Q} \cong U(\mathcal{B})_\mathbb{Q} \oplus U(\mathcal{C})_\mathbb{Q}$. Hence, Corollary 12.2 yields the equivalences of conjectures $C_{\text{nc}}(\mathcal{A}) \leftrightarrow C_{\text{nc}}(\mathcal{B}) + C_{\text{nc}}(\mathcal{C})$, with $C \in \{W, W^l, ST, \overline{W}\}$. 

Proof of Theorem 2.1. As proved in [53, Thm. 2.1], we have an isomorphism of noncommutative Chow motives $U(\text{perf}_{dg}(X))_\mathbb{Q} \cong U(\text{perf}_{dg}(X; F))_\mathbb{Q}$. Hence, we have the equivalences of conjectures $C_{nc}(\text{perf}_{dg}(X)) \Leftrightarrow C_{nc}(\text{perf}_{dg}(X; F))$, with $C \in \{W, W^l, ST, W\}$. Therefore, the proof follows from Theorems 1.5, 1.11, and 1.15, and from Remark 6.6.

Proof of Theorem 2.4. The noncommutative Weil conjecture(s) as well as the noncommutative strong form of the Tate conjecture hold for $\text{Spec}(k)$. Consequently, an iterated application of Example 12.3 to the semi-orthogonal decomposition (2.2) yields the equivalences of conjectures $C_{nc}(\text{perf}_{dg}(X)) \Leftrightarrow C_{nc}(\text{T}_{dg}(X))$, with $C \in \{W, W^l, ST, W\}$. Therefore, the proof follows from Theorems 1.5, 1.11, and 1.15, and from Remark 6.6.

Proof of Theorem 2.5. Up to Morita equivalence, the dg category $X \otimes_B Y$ admits a semi-orthogonal decomposition whose components are (Fourier-Mukai) equivalent to $\text{perf}(X)$ and $\text{perf}(Y)$. Consequently, the proof follows from the combination of Example 12.3 with Theorems 1.5, 1.11, and 1.15, and with Remark 6.6.

Proof of Theorem 2.6. As proved by Ishii-Ueda in [21, Thm. 1.6], whenever the zero locus $D \hookrightarrow X$ of $\varsigma$ is smooth, we have a semi-orthogonal decomposition

\begin{equation}
\text{perf}(X) = \langle \text{perf}(D)_{n-1}, \ldots, \text{perf}(D)_1, f^*(\text{perf}(X)) \rangle,
\end{equation}

where all the categories $\text{perf}(D)_i$ are (Fourier-Mukai) equivalent to $\text{perf}(D)$ and $f^*(\text{perf}(X))$ is (Fourier-Mukai) equivalent to $\text{perf}(X)$. Therefore, thanks to Theorems 1.5, 1.11, and 1.15, and to Remark 6.6, the proof follows from an iterated application of Example 12.3 to the semi-orthogonal decomposition (12.4).

Proof of Theorem 2.7. Let $\varphi$ be the set of all cyclic subgroups of $G$ and $\varphi_{\sim}$ a set of representatives of conjugacy classes in $\varphi$. Since the category $\text{NChow}(k)_\mathbb{Q}$ is $\mathbb{Q}$-linear, it follows from [52, Thm. 1.1 and Rk. 1.3(iii)] that the noncommutative Chow motive $U(\text{perf}_{dg}(X'))_\mathbb{Q}$ is a direct summand of the following direct sum:

\begin{equation}
\bigoplus_{\sigma \in \varphi_{\sim}} U(\text{perf}_{dg}(X'^{\sigma} \times \text{Spec}(k[\sigma])))_\mathbb{Q}.
\end{equation}

Under the stronger assumption $n|(q-1)$, the same holds with $X'^{\sigma} \times \text{Spec}(k[\sigma])$ replaced by $X'^{\sigma}$; consult [52, Cor. 1.5(ii)]. Recall from Proposition 12.1 that the noncommutative Weil conjecture(s) and the noncommutative strong form of the Tate conjecture are stable under direct sums and direct summands. Therefore, the proof follows from the combination of (12.5) (under the stronger assumption $n|(q-1)$, replace $X'^{\sigma} \times \text{Spec}(k[\sigma])$ by $X'^{\sigma}$) with Theorems 1.5, 1.11, and 1.15, and with Remark 6.6.

Proof of Theorem 2.11. The proof of Theorem 2.11 is similar to the proof of Theorem 2.7 (in the case where $n|(q-1)$). Simply, replace $\text{perf}_{dg}(X)$ by $\text{perf}_{dg}(X'; F)$, $\text{perf}_{dg}(X'^{\sigma})$ by $\text{perf}_{dg}(Y_{\sigma})$, and [52, Cor. 1.5(ii)] by [52, Cor. 1.28(ii)].
Proof of Theorem 2.12. Following Orlov [44, §2.3], let us write \( \text{Aus}(A) \) for the smooth proper Auslander dg \( k \)-algebra associated to \( A \) (Orlov used a different notation). As proved in [44, Thms. 2.18-2.19], we have a semi-orthogonal decomposition
\[
\text{perf} (\text{Aus}(A)) = \langle \text{perf}(D_1), \ldots, \text{perf}(D_n) \rangle ,
\]
where \( D_i \) is a division \( k \)-algebra; note that since \( k \) is a finite field, the division \( k \)-algebras \( D_1, \ldots, D_n \) are just finite field extensions \( l_1, \ldots, l_n \) of \( k \). Moreover, as also proved in loc. cit., the category \( \text{perf}(A) \) can be embedded (using a Fourier-Mukai functor) into \( \text{perf}(\text{Aus}(A)) \) as an admissible triangulated subcategory. This implies that the noncommutative Chow motive \( U(A)_{\mathbb{Q}} \) is a direct summand of the direct sum \( \bigoplus_{i=1}^{n} U(l_i) \). Consequently, since the conjectures \( C_{nc}(l_i) \), with \( C \in \{ W, W^t, ST, \mathcal{W} \} \), hold, we conclude from Proposition 12.1 that the conjectures \( C_{nc}(A) \), with \( C \in \{ W, W^t, ST, \mathcal{W} \} \), also hold.

13. PROOF OF THEOREM 3.1

As proved in [36, Thm. 5.5] (see also [2, Thm. 2.3.7]), we have the following semi-orthogonal decomposition
\[
\text{perf}(X) = \langle \text{perf}(\mathbb{P}^1; Cl_0(q)), \mathcal{O}_X(1), \ldots, \mathcal{O}_X(n-4) \rangle ,
\]
where \( Cl_0(q) \) stands for the sheaf of even parts of the Clifford algebra associated to the flat quadric fibration \( f: Q \to \mathbb{P}^1 \). Consequently, since the noncommutative Weil conjecture as well as the noncommutative strong form of the Tate conjecture hold for \( \text{Spec}(k) \), an iterated application of Example 12.3 yields the following equivalences of conjectures:
\[
\begin{align*}
(13.1) \quad W_{nc}(\text{perf}_{\text{dg}}(X)) & \iff W_{nc}(\text{perf}_{\text{dg}}(\mathbb{P}^1; Cl_0(q))) \\
(13.2) \quad ST_{nc}(\text{perf}_{\text{dg}}(X)) & \iff ST_{nc}(\text{perf}_{\text{dg}}(\mathbb{P}^1; Cl_0(q))) .
\end{align*}
\]
We start by proving item (i). Following [36, §3.5] (see also [2, §1.6]), let \( Z \) be the center of \( Cl_0(q) \) and \( \text{Spec}(Z) := \mathbb{P}^1 \to \mathbb{P}^1 \) the discriminant cover of \( \mathbb{P}^1 \). As explained in loc. cit., \( \mathbb{P}^1 \to \mathbb{P}^1 \) is a 2-fold cover which is ramified over the (finite) set \( D \) of critical values of \( f \). Moreover, since \( D \) is smooth, \( \mathbb{P}^1 \) is also smooth. Let us write \( \mathcal{F} \) for the sheaf \( Cl_0(q) \) considered as a sheaf of noncommutative algebras over \( \mathbb{P}^1 \). As proved in loc. cit., since by assumption all the fibers of \( f: Q \to \mathbb{P}^1 \) have corank \( \leq 1 \), \( \mathcal{F} \) is a sheaf of Azumaya algebras over \( \mathbb{P}^1 \). Moreover, the category \( \text{perf}(\mathbb{P}^1; Cl_0(q)) \) is (Fourier-Mukai) equivalent to \( \text{perf}(\mathbb{P}^1; \mathcal{F}) \). Note that since the Brauer group of every smooth curve over a finite field \( k \) is trivial, the latter category reduces to \( \text{perf}(\mathbb{P}^1) \). Therefore, thanks to Theorem 1.5, resp. Theorem 1.15, the equivalence \( (13.1) \), resp. \( (13.2) \), reduces to \( W(X) \iff W(\mathbb{P}^1) \), resp. \( ST(X) \iff ST(\mathbb{P}^1) \). Finally, since \( \mathbb{P}^1 \) is a curve, we hence conclude that the conjectures \( W(X) \) and \( ST(X) \) hold.

We now prove item (ii). Note first that \( 1/2 \in k \). Following [36, §3.6] (see also [2, §1.7]), let \( \mathbb{P}^1 \) the discriminant stack associated to the flat quadric fibration \( f: Q \to \mathbb{P}^1 \). As explained in loc. cit., since \( 1/2 \in k \), \( \mathbb{P}^1 \) is a square root stack. Moreover, the underlying \( k \)-scheme of \( \mathbb{P}^1 \) is \( \mathbb{P}^1 \). Therefore, it follows from Theorem 2.6 that the conjectures \( W_{nc}(\text{perf}_{\text{dg}}(\mathbb{P}^1)) \) and \( ST_{nc}(\text{perf}_{\text{dg}}(\mathbb{P}^1)) \) hold. Let us write \( \mathcal{F} \) for the sheaf \( Cl_0(q) \) considered as a sheaf of noncommutative algebras over \( \mathbb{P}^1 \). As proved in loc. cit., since by assumption all the fibers of \( f: Q \to \mathbb{P}^1 \) have corank \( \leq 1 \), \( \mathcal{F} \) is a sheaf of Azumaya algebras over \( \mathbb{P}^1 \). Moreover, the category
perf(\(\mathbb{P}^1; \text{Cl}_0(q)\)) is (Fourier-Mukai) equivalent to \(\text{perf}(\hat{\mathbb{P}}^1; \mathcal{F})\). Note that since the Brauer group of every smooth curve over a finite field \(k\) is trivial, the latter category reduces to \(\text{perf}(\hat{\mathbb{P}}^1)\). Hence, thanks to Theorem 1.5, resp. Theorem 1.15, the equivalence (13.1), resp. (13.2), reduces to \(W(X) \Leftrightarrow W_{nc}(\text{perf}_{dg}(\hat{\mathbb{P}}^1))\), resp. \(ST(X) \Leftrightarrow ST_{nc}(\text{perf}_{dg}(\hat{\mathbb{P}}^1))\). Finally, since \(W_{nc}(\text{perf}_{dg}(\hat{\mathbb{P}}^1))\) and \(ST_{nc}(\text{perf}_{dg}(\hat{\mathbb{P}}^1))\) hold, we hence conclude that the conjectures \(W(X)\) and \(ST(X)\) also hold.

14. Proof of Theorem 3.3

As proved in [3, Cor. 3.7], the following holds:

(a) When \(\dim(L) < d_2 r\), the category \(\text{perf}(X_L)\) admits a semi-orthogonal decomposition with one component (Fourier-Mukai) equivalent to \(\text{perf}(Y_L)\) and with \((d_2 r - \dim(L))(d_1 r)\) exceptional objects.

(b) When \(\dim(L) = d_2 r\), the category \(\text{perf}(X_L)\) is (Fourier-Mukai) equivalent to the category \(\text{perf}(Y_L)\).

(c) When \(\dim(L) > d_2 r\), the category \(\text{perf}(Y_L)\) admits a semi-orthogonal decomposition with one component (Fourier-Mukai) equivalent to \(\text{perf}(X_L)\) and with \((\dim(L) - d_2 r)(d_1 r)\) exceptional objects.

Consequently, since the (noncommutative) Weil conjecture as well as the (noncommutative) strong form of the Tate conjecture hold for \(\text{Spec}(k)\), by combining Theorems 1.5 and 1.15 with an iterated application of Example 12.3, we hence conclude from (a)-(c) that \(W(X_L) \Leftrightarrow W(Y_L)\) and \(ST(X_L) \Leftrightarrow ST(Y_L)\). Therefore, the proof of item (i), resp. item (ii), follows from the fact that \(\dim(X_L) = r(d_1 + d_2 - r) - 1 - \dim(L)\), resp. \(\dim(Y_L) = r(d_1 - d_2 - r) - 1 + \dim(L)\).

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