TWO-SIDED ESTIMATES FOR ORDER STATISTICS OF LOG-CONCAVE RANDOM VECTORS

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Abstract. We establish two-sided bounds for expectations of order statistics (k-th maxima) of moduli of coordinates of centered log-concave random vectors with uncorrelated coordinates. Our bounds are exact up to multiplicative universal constants in the unconditional case for all \( k \) and in the isotropic case for \( k \leq n - cn^{5/6} \). We also derive two-sided estimates for expectations of sums of \( k \) largest moduli of coordinates for some classes of random vectors.

1. Introduction and main results

For a vector \( x \in \mathbb{R}^n \) let \( k \)-\( \max x_i \) (or \( k \)-\( \min x_i \)) denote its \( k \)-th maximum (respectively its \( k \)-th minimum), i.e. its \( k \)-th maximal (respectively \( k \)-th minimal) coordinate. For a random vector \( X = (X_1, \ldots, X_n) \), \( k \)-\( \min X_i \) is also called the \( k \)-th order statistic of \( X \).

Let \( X = (X_1, \ldots, X_n) \) be a random vector with finite first moment. In this note we try to estimate \( \mathbb{E}k\max_i |X_i| \) and

\[
\mathbb{E} \max_{|I|=k} \sum_{i \in I} |X_i| = \mathbb{E} \sum_{l=1}^{k} l\max_i |X_i|.
\]

Order statistics play an important role in various statistical applications and there is an extensive literature on this subject (cf. [2, 5] and references therein).

We put special emphasis on the case of log-concave vectors, i.e. random vectors \( X \) satisfying the property \( \mathbb{P}(X \in \lambda K + (1 - \lambda)L) \geq \mathbb{P}(X \in K) \lambda \mathbb{P}(X \in L)^{1-\lambda} \) for any \( \lambda \in [0, 1] \) and any nonempty compact sets \( K \) and \( L \). By the result of Borell [3] a vector \( X \) with full dimensional support is log-concave if and only if it has a log-concave density, i.e. the density of a form \( e^{-h(x)} \) where \( h \) is convex with values in \((-\infty, \infty]\). A typical example of a log-concave vector is a vector uniformly distributed over a convex body. In recent years the study of log-concave vectors attracted attention of many researchers, cf. monographs [1, 4].

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To bound the sum of $k$ largest coordinates of $X$ we define

$$(1) \quad t(k, X) := \inf \left\{ t > 0 : \frac{1}{k} \sum_{i=1}^{n} \mathbb{E}|X_i|1_{\{|X_i| \geq t\}} \leq k \right\}.$$ 

and start with an easy upper bound.

**Proposition 1.** For any random vector $X$ with finite first moment we have

$$(2) \quad \mathbb{E} \max_{|I|=k} \sum_{i \in I} |X_i| \leq 2kt(k, X).$$

**Proof.** For any $t > 0$ we have

$$\max_{|I|=k} \sum_{i \in I} |X_i| \leq tk + \sum_{i=1}^{n} |X_i|1_{\{|X_i| \geq t\}}. \quad \square$$

It turns out that this bound may be reversed for vectors with independent coordinates or, more generally, vectors satisfying the following condition

$$(3) \quad \mathbb{P}(|X_i| \geq s, |X_j| \geq t) \leq \alpha \mathbb{P}(|X_i| \geq s)\mathbb{P}(|X_j| \geq t) \quad \text{for all } i \neq j \text{ and all } s, t > 0.$$

If $\alpha = 1$ this means that moduli of coordinates of $X$ are negatively correlated.

**Theorem 2.** Suppose that a random vector $X$ satisfies condition (3) with some $\alpha \geq 1$. Then there exists a constant $c(\alpha) > 0$ which depends only on $\alpha$ such that for any $1 \leq k \leq n$,

$$c(\alpha)kt(k, X) \leq \mathbb{E} \max_{|I|=k} \sum_{i \in I} |X_i| \leq 2kt(k, X).$$

We may take $c(\alpha) = (288(5 + 4\alpha)(1 + 2\alpha))^{-1}$.

In the case of i.i.d. coordinates two-sided bounds for $\mathbb{E} \max_{|I|=k} \sum_{i \in I} |a_iX_i|$ in terms of an Orlicz norm (related to the distribution of $X_i$) of a vector $(a_i)_{i \leq n}$ where known before, see [7].

Log-concave vectors with diagonal covariance matrices behave in many aspects like vectors with independent coordinates. This is true also in our case.

**Theorem 3.** Let $X$ be a log-concave random vector with uncorrelated coordinates (i.e. $\text{Cov}(X_i, X_j) = 0$ for $i \neq j$). Then for any $1 \leq k \leq n$,

$$ckt(k, X) \leq \mathbb{E} \max_{|I|=k} \sum_{i \in I} |X_i| \leq 2kt(k, X).$$

In the above statement and in the sequel $c$ and $C$ denote positive universal constants. The next two examples show that the lower bound cannot hold if $n \gg k$ and only marginal distributions of $X_i$ are log-concave or the coordinates of $X$ are highly correlated.

**Example 1.** Let $X = (\varepsilon_1 g, \varepsilon_2 g, \ldots, \varepsilon_n g)$, where $\varepsilon_1, \ldots, \varepsilon_n, g$ are independent, $\mathbb{P}(\varepsilon_i = \pm 1) = 1/2$ and $g$ has the normal $\mathcal{N}(0, 1)$ distribution. Then $\text{Cov}X = \text{Id}$ and it is not hard to check that $\mathbb{E} \max_{|I|=k} \sum_{i \in I} |X_i| = k\sqrt{2/\pi}$ and $t(k, X) \sim \ln^{1/2}(n/k)$ if $k \leq n/2$. 

Example 2. Let $X = (g, \ldots, g)$, where $g \sim \mathcal{N}(0, 1)$. Then, as in the previous example, $E_{\max_{|I| = k}} k \sum_{i \in I} |X_i| = k \sqrt{2/\pi}$ and $t(k, X) \sim \ln^{1/2}(n/k)$.

**Question 1.** Let $X' = (X'_1, X'_2, \ldots, X'_n)$ be a decoupled version of $X$, i.e. $X'_i$ are independent and $X'_i$ has the same distribution as $X_i$. Due to Theorem 2 (applied to $X'$), the assertion of Theorem 3 may be stated equivalently as

$E_{\max_{|I| = k}} k \sum_{i \in I} |X'_i| \sim E_{\max_{|I| = k}} k \sum_{i \in I} |X_i|.$

Is the more general fact true that for any symmetric norm and any log-concave vector $X$ with uncorrelated coordinates $E\|X\| \sim E\|X'\|$?

Maybe such an estimate holds at least in the case of unconditional log-concave vectors?

We turn our attention to bounding $k$-maxima of $|X_i|$. This was investigated in [8] (under some strong assumptions on the function $t \mapsto P(|X_i| \geq t)$) and in the weighted i.i.d. setting in [7, 9, 15]. We will give different bounds valid for log-concave vectors, in which we do not have to assume independence, nor any special conditions on the growth of the distribution function of the coordinates of $X$. To this end we need to define another quantity:

$t^* (p, X) := \inf \left\{ t > 0 : \sum_{i=1}^{n} P(|X_i| \geq t) \leq p \right\}$ for $0 < p < n$.

**Theorem 4.** Let $X$ be a mean zero log-concave $n$-dimensional random vector with uncorrelated coordinates and $1 \leq k \leq n$. Then

$E_{\max_{i \leq n}} |X_i| \geq \frac{1}{2} \text{Med} \left( k-\max_{i \leq n} |X_i| \right) \geq ct^* \left( k - \frac{1}{2}, X \right).$

Moreover, if $X$ is additionally unconditional then

$E_{\max_{i \leq n}} |X_i| \leq C t^* \left( k - \frac{1}{2}, X \right).$

The next theorem provides an upper bound in the general log-concave case.

**Theorem 5.** Let $X$ be a mean zero log-concave $n$-dimensional random vector with uncorrelated coordinates and $1 \leq k \leq n$. Then

(4) \[ P \left( k-\max_{i \leq n} |X_i| \geq C t^* \left( k - \frac{1}{2}, X \right) \right) \leq 1 - c \]

and

(5) \[ E_{\max_{i \leq n}} |X_i| \leq C t^* \left( k - \frac{1}{2} k^{5/6}, X \right). \]

In the isotropic case (i.e. $E X_i = 0, \text{Cov} X = \text{Id}$) one may show that $t^* (k/2, X) \sim t^* (k, X) \sim t(k, X)$ for $k \leq n/2$ and $t^* (p, X) \sim \frac{n-p}{n}$ for $p \geq n/4$ (see Lemma 24 below). In particular $t^* (n - k + 1 - (n - k + 1)^{5/6}/2, X) \sim k/n + n^{-1/6}$ for $k \leq n/2$. This together with the two previous theorems implies the following corollary.
Corollary 6. Let $X$ be an isotropic log-concave $n$-dimensional random vector and $1 \leq k \leq n/2$. Then
\[ \mathbb{E}k\cdot \max_{i \leq n} |X_i| \sim t^*(k, X) \sim t(k, X) \]
and
\[ \frac{c}{n} k \leq \mathbb{E}k\cdot \min_{i \leq n} |X_i| = \mathbb{E}(n - k + 1)\cdot \max_{i \leq n} |X_i| \leq C \left( \frac{k}{n} + n^{-1/6} \right). \]
If $X$ is additionally unconditional then
\[ \mathbb{E}k\cdot \min_{i \leq n} |X_i| = \mathbb{E}(n - k + 1)\cdot \max_{i \leq n} |X_i| \sim \frac{k}{n}. \]

Question 2. Does the second part of Theorem 4 hold without the unconditionality assumptions? In particular, is it true that in the isotropic log-concave case $\mathbb{E}k\cdot \min_{i \leq n} |X_i| \sim k/n$ for $1 \leq k \leq n/2$?

Notation. Throughout this paper by letters $C, c$ we denote universal positive constants and by $C(\alpha), c(\alpha)$ constants depending only on the parameter $\alpha$. The values of constants $C, c, C(\alpha), c(\alpha)$ may differ at each occurrence. If we need to fix a value of constant, we use letters $C_0, C_1, \ldots$ or $c_0, c_1, \ldots$. We write $f \sim g$ if $c f \leq g \leq C g$. For a random variable $Z$ we denote $\|Z\|_p = (\mathbb{E}|Z|^p)^{1/p}$. Recall that a random vector $X$ is called isotropic, if $\mathbb{E}X = 0$ and $\text{Cov}X = \text{Id}$.

This note is organised as follows. In Section 2 we provide a lower bound for the sum of $k$ largest coordinates, which involves the Poincaré constant of a vector. In Section 3 we use this result to obtain Theorem 3. In Section 4 we prove Theorem 2 and provide its application to comparison of weak and strong moments. In Section 5 we prove the first part of Theorem 4 and in Section 6 we prove the second part of Theorem 4, Theorem 5, and Lemma 24.

2. Exponential concentration

A probability measure $\mu$ on $\mathbb{R}^n$ satisfies exponential concentration with constant $\alpha > 0$ if for any Borel set $A$ with $\mu(A) \geq 1/2$,
\[ 1 - \mu(A + uB^a_n) \leq e^{-u/\alpha} \quad \text{for all } u > 0. \]
We say that a random $n$-dimensional vector satisfies exponential concentration if its distribution has such a property.

It is well known that exponential concentration is implied by the Poincaré inequality
\[ \text{Var}_\mu f \leq \beta \int |\nabla f|^2 d\mu \quad \text{for all bounded smooth functions } f : \mathbb{R}^n \mapsto \mathbb{R} \]
and $\alpha \leq 3\sqrt{\beta}$ (cf. [12, Corollary 3.2]).

Obviously, the constant in the exponential concentration is not linearly invariant. Typically one assumes that the vector is isotropic. For our purposes a more natural normalization will be that all coordinates have $L_1$-norm equal to 1.

The next proposition states that bound (2) may be reversed under the assumption that $X$ satisfies the exponential concentration.
Proposition 7. Assume that $Y = (Y_1, \ldots, Y_n)$ satisfies the exponential concentration with constant $\alpha > 0$ and $\mathbb{E}|Y_i| \geq 1$ for all $i$. Then for any sequence $a = (a_i)_{i=1}^n$ of real numbers and $X_i := a_i Y_i$ we have

$$
\mathbb{E} \max_{|I|=k} \sum_{i \in I} |X_i| \geq \left(8 + 64 \frac{\alpha}{\sqrt{k}}\right)^{-1} kt(k, X),
$$

where $t(k, X)$ is given by (1).

We begin the proof with a few simple observations.

Lemma 8. For any real numbers $z_1, \ldots, z_n$ and $1 \leq k \leq n$ we have

$$\max_{|I|=k} \sum_{i \in I} |z_i| = \int_0^\infty \min\left\{k, \sum_{i=1}^n 1_{\{|z_i| \geq s\}}\right\} ds.$$

Proof. Without loss of generality we may assume that $z_1 \geq z_2 \geq \ldots \geq z_n \geq 0$. Then

$$\int_0^\infty \min\left\{k, \sum_{i=1}^n 1_{\{|z_i| \geq s\}}\right\} ds = \sum_{l=1}^{k-1} \int_{z_{l+1}}^{z_k} l ds + \sum_{l=1}^{k-1} \int_0^{z_k} k ds = \sum_{l=1}^{k-1} \left(l(z_l - z_{l+1}) + kz_{k+1}\right) = z_1 + \ldots + z_k = \max_{|I|=k} \sum_{i \in I} |z_i|. \quad \Box$$

Fix a sequence $(X_i)_{i \leq n}$ and define for $s \geq 0$,

$$N(s) := \sum_{i=1}^n 1_{\{|X_i| \geq s\}}.$$ 

(6)

Corollary 9. For any $k = 1, \ldots, n$,

$$\mathbb{E} \max_{|I|=k} \sum_{i \in I} |X_i| = \int_0^\infty \sum_{l=1}^k \mathbb{P}(N(s) \geq l) ds,$$

and for any $t > 0$,

$$\mathbb{E} \sum_{i=1}^n |X_i| 1_{\{|X_i| \geq t\}} = t\mathbb{E} N(t) + \int_t^\infty \sum_{l=1}^\infty \mathbb{P}(N(s) \geq l) ds.$$

In particular

$$\mathbb{E} \sum_{i=1}^n |X_i| 1_{\{|X_i| \geq t\}} \leq \mathbb{E} \max_{|I|=k} \sum_{i \in I} |X_i| + \sum_{l=k+1}^\infty \left(t\mathbb{P}(N(t) \geq l) + \int_t^\infty \mathbb{P}(N(s) \geq l) ds\right).$$

Proof. We have

$$\int_0^\infty \sum_{l=1}^k \mathbb{P}(N(s) \geq l) ds = \int_0^\infty \mathbb{E} \min\{k, N(s)\} ds = \mathbb{E} \int_0^\infty \min\{k, N(s)\} ds$$

$$= \mathbb{E} \max_{|I|=k} \sum_{i \in I} |X_i|,$$
where the last equality follows by Lemma 8.

Moreover,

\[ t\mathbb{E}(N(t)) + \int_t^\infty \sum_{l=1}^\infty \mathbb{P}(N(s) \geq l) ds = t\mathbb{E}(N(t)) + \int_t^\infty \mathbb{E}(N(s)) ds \]

\[ = \mathbb{E} \sum_{i=1}^n \left( t \mathbbm{1}_{\{|X_i| \geq t\}} + \int_t^\infty \mathbbm{1}_{\{|X_i| \geq s\}} ds \right) \]

\[ = \mathbb{E} \sum_{i=1}^n |X_i| \mathbbm{1}_{\{|X_i| \geq t\}}. \]

The last part of the assertion easily follows, since

\[ t\mathbb{E}(N(t)) = t \sum_{l=1}^n \mathbb{P}(N(t) \geq l) \leq \int_0^t \sum_{l=1}^k \mathbb{P}(N(s) \geq l) ds + \sum_{l=k+1}^\infty t\mathbb{P}(N(t) \geq l). \]

Proof of Proposition 7. To shorten the notation put \( t_k := t(k, X). \) Without loss of generality we may assume that \( a_1 \geq a_2 \geq \ldots \geq a_n \geq 0 \) and \( a_{\lfloor k/4 \rfloor} = 1. \) Observe first that

\[ \mathbb{E} \max_{|I|=k} \sum_{i \in I} |X_i| \geq k/4 \sum_{i=1}^n a_i \mathbb{E} |Y_i| \geq k/4, \]

so we may assume that \( t_k \geq 16\alpha/\sqrt{k}. \)

Let \( \mu \) be the law of \( Y \) and

\[ A := \left\{ y \in \mathbb{R}^n : \sum_{i=1}^n \mathbbm{1}_{\{|a_i y_i| \geq \frac{1}{2} t_k\}} < \frac{k}{2} \right\}. \]

We have

\[ \mathbb{E} \max_{|I|=k} \sum_{i \in I} |X_i| \geq \frac{k}{4} t_k \mathbb{P} \left( \sum_{i=1}^k \mathbbm{1}_{\{|a_i Y_i| \geq \frac{1}{2} t_k\}} \geq \frac{k}{2} \right) = \frac{k}{4} t_k (1 - \mu(A)), \]

so we may assume that \( \mu(A) \geq 1/2. \)

Observe that if \( y \in A \) and \( \sum_{i=1}^n \mathbbm{1}_{\{|a_i z_i| \geq s\}} \geq k \) for some \( s \geq t_k \) then

\[ \sum_{i=1}^n (z_i - y_i)^2 \geq \sum_{i=\lfloor k/4 \rfloor} (a_i z_i - a_i y_i)^2 \geq (l - 3k/4)(s - t_k/2)^2 > l s^2/16. \]

Thus we have

\[ \mathbb{P}(N(s) \geq l) \leq 1 - \mu \left( A + \frac{s \sqrt{l}}{4} B_2^n \right) \leq e^{-\frac{s \sqrt{l}}{4\alpha}} \text{ for } l > k, \]

so

\[ \int_{t_k}^\infty \mathbb{P}(N(s) \geq l) ds \leq \int_{t_k}^\infty e^{-\frac{s \sqrt{l}}{4\alpha}} ds = \frac{4\alpha}{\sqrt{l}} e^{-\frac{t_k \sqrt{l}}{4\alpha}} \text{ for } l > k. \]
and
\[
\sum_{l=k+1}^{\infty} \left( t_k \mathbb{P}(N(t_k) \geq l) + \int_{t_k}^{\infty} \mathbb{P}(N(s) \geq l) ds \right) \leq \sum_{l=k+1}^{\infty} \left( t_k + \frac{4\alpha}{\sqrt{l}} \right) e^{-\frac{t_k \sqrt{\pi}}{4\alpha}} \]
\[
\leq \left( t_k + \frac{4\alpha}{\sqrt{k+1}} \right) \int_{t_k}^{\infty} e^{-\frac{t_k \sqrt{\pi}}{4\alpha}} du \leq \left( t_k + \frac{4\alpha}{\sqrt{k+1}} \right) e^{-\frac{t_k \sqrt{\pi}}{4\alpha}} \int_{t_k}^{\infty} e^{-\frac{t_k \sqrt{\pi} \alpha}{4\alpha}} du
\]
\[
= \left( t_k + \frac{4\alpha}{\sqrt{k+1}} \right) \frac{64\alpha^2}{t_k} e^{-\frac{t_k \sqrt{\pi} \alpha}{4\alpha}} \leq \left( t_k + \frac{1}{4} t_k \right) \frac{k}{4} \leq \frac{1}{2} kt_k,
\]
where to get the next-to-last inequality we used the fact that \( t_k \geq 16\alpha/\sqrt{k} \).

Hence Corollary 9 and the definition of \( t_k \) yields
\[
kt_k \leq \mathbb{E} \sum_{i=1}^{n} |X_i| \mathbf{1}_{\{|X_i| \geq t_k\}}
\]
\[
\leq \mathbb{E} \max_{|I|=k} \sum_{i \in I} |X_i| + \sum_{l=k+1}^{\infty} \left( t_k \mathbb{P}(N(t_k) \geq l) + \int_{t_k}^{\infty} \mathbb{P}(N(s) \geq l) ds \right)
\]
\[
\leq \mathbb{E} \max_{|I|=k} \sum_{i \in I} |X_i| + \frac{1}{2} kt_k,
\]
so \( \mathbb{E} \max_{|I|=k} \sum_{i \in I} |X_i| \geq \frac{1}{2} kt_k \). \( \Box \)

We finish this section with a simple fact that will be used in the sequel.

**Lemma 10.** Suppose that a measure \( \mu \) satisfies exponential concentration with constant \( \alpha \). Then for any \( c \in (0,1) \) and any Borel set \( A \) with \( \mu(A) > c \) we have
\[
1 - \mu(A + uB_2^n) \leq \exp\left(-\left(\frac{u}{\alpha} + \ln c\right)\right) \text{ for } u \geq 0.
\]

**Proof.** Let \( D := \mathbb{R}^n \setminus (A + rB_2^n) \). Observe that \( D + rB_2^n \) has an empty intersection with \( A \) so if \( \mu(D) \geq 1/2 \) then
\[
c < \mu(A) \leq 1 - \mu(D + rB_2^n) \leq e^{-r/\alpha},
\]
and \( r < \alpha \ln(1/c) \). Hence \( \mu(A + \alpha \ln(1/c)B_2^n) \geq 1/2 \), therefore for \( s \geq 0 \),
\[
1 - \mu(A + (s + \alpha \ln(1/c))B_2^n) = 1 - \mu((A + \alpha \ln(1/c)B_2^n) + sB_2^n) \leq e^{-s/\alpha},
\]
and the assertion easily follows. \( \Box \)

3. Sums of largest coordinates of log-concave vectors

We will use the regular growth of moments of norms of log-concave vectors multiple times. By [4, Theorem 2.4.6], if \( f : \mathbb{R}^n \to \mathbb{R} \) is a seminorm, and \( X \) is log-concave, then
\[
(\mathbb{E} f(X)^p)^{1/p} \leq C_1 \frac{p}{q} (\mathbb{E} f(X)^q)^{1/q} \quad \text{for } p \geq q \geq 1,
\]
where \( C_1 \) is a universal constant.
We will also apply a few times the functional version of the Grünbaum inequality (see [14, Lemma 5.4]) which states that

\[ P(Z \geq 0) \geq \frac{1}{e} \]

for any mean-zero log-concave random variable \( Z \).

Let us start with a few technical lemmas. The first one will be used to reduce proofs of Theorem 3 and lower bound in Theorem 4 to the symmetric case.

**Lemma 11.** Let \( X \) be a log-concave \( n \)-dimensional vector and \( X' \) be an independent copy of \( X \). Then for any \( 1 \leq k \leq n \),

\[ \mathbb{E} \max_{|I|=k} \sum_{i \in I} |X_i - X'_i| \leq 2 \mathbb{E} \max_{|I|=k} \sum_{i \in I} |X_i|, \]

(9)

\[ t(k, X) \leq et(k, X - X') + 2 \max_{|I|=k} \sum_{i \in I} \mathbb{E}|X_i|, \]

and

(10)

\[ t^*(2k, X - X') \leq 2t^*(k, X). \]

**Proof.** The first estimate follows by the easy bound

\[ \mathbb{E} \max_{|I|=k} \sum_{i \in I} |X_i - X'_i| \leq \mathbb{E} \max_{|I|=k} \sum_{i \in I} |X_i| + \mathbb{E} \max_{|I|=k} \sum_{i \in I} |X'_i| = 2 \mathbb{E} \max_{|I|=k} \sum_{i \in I} |X_i|. \]

To get the second bound we may and will assume that \( \mathbb{E}|X_1| \geq \mathbb{E}|X_2| \geq \ldots \geq \mathbb{E}|X_n| \).

Let us define \( Y := X - \mathbb{E}X \), \( Y' := X' - \mathbb{E}X \) and \( M := \frac{1}{k} \sum_{i=1}^{k} \mathbb{E}|X_i| \geq \max_{i \geq k} \mathbb{E}|X_i| \). Obviously

(11)

\[ \sum_{i=1}^{k} \mathbb{E}|X_i|1_{(|X_i| \geq t)} \leq kM \quad \text{for } t \geq 0. \]

We have \( \mathbb{E}Y_i = 0 \), thus \( P(Y_i \leq 0) \geq 1/e \) by (8). Hence

\[ \mathbb{E}Y_i1_{\{Y_i > t\}} \leq e\mathbb{E}Y_i1_{\{Y_i \leq 0\}} \leq e\mathbb{E}|Y_i - Y'_i|1_{\{Y_i - Y'_i > t\}} = e\mathbb{E}|X_i - X'_i|1_{\{|X_i - X'_i| > t\}} \]

for \( t \geq 0 \). In the same way we show that

\[ \mathbb{E}|Y_i|1_{\{|Y_i| > t\}} \leq e\mathbb{E}|Y_i|1_{\{|Y_i| \leq 0\}} \leq e\mathbb{E}|X_i - X'_i|1_{\{|X_i - X'_i| > t\}} \]

Therefore

\[ \mathbb{E}|Y_i|1_{\{|Y_i| > t\}} \leq e\mathbb{E}|X_i - X'_i|1_{\{|X_i - X'_i| > t\}}. \]
Lemma 12. Suppose that $\mathbb{E}|X_i|1_{\{|X_i| > et(k, X - X') + M\}} \leq \sum_{i=k+1}^{n} \mathbb{E}|X_i|1_{\{|Y_i| > et(k, X - X')\}}$

$$\leq \sum_{i=k+1}^{n} \mathbb{E}|Y_i|1_{\{|Y_i| > t(k, X - X')\}} + \sum_{i=k+1}^{n} |\mathbb{E}X_i|\mathbb{P}(|Y_i| > et(k, X - X'))$$

$$\leq e\sum_{i=1}^{n} \mathbb{E}|X_i - X'_i|1_{\{|X_i - X'_i| > t(k, X - X')\}} + M\sum_{i=1}^{n} \mathbb{P}(|Y_i| > et(k, X - X'))$$

$$\leq et(k, X - X') + M\sum_{i=1}^{n} (et(k, X - X'))^{-1}\mathbb{E}|Y_i|1_{\{|Y_i| > et(k, X - X')\}}$$

$$\leq et(k, X - X') + Mt(k, X - X')^{-1}\sum_{i=1}^{n} \mathbb{E}|X_i - X'_i|1_{\{|X_i - X'_i| > t(k, X - X')\}}$$

$$\leq et(k, X - X') + kM.$$ 

Together with (11) we get

$$\sum_{i=1}^{n} \mathbb{E}|X_i|1_{\{|X_i| > et(k, X - X') + M\}} \leq k(et(k, X - X') + 2M)$$

and (9) easily follows.

In order to prove (10), note that for $u > 0$,

$$\mathbb{P}(|X_i - X'_i| \geq 2u) \leq \mathbb{P}(\max\{|X_i|, |X'_i|\} \geq u) \leq 2\mathbb{P}(|X_i| \geq u),$$

thus the last part of the assertion follows by the definition of parameters $t^*$.

\[ \square \]

**Lemma 12.** Suppose that $V$ is a real symmetric log-concave random variable. Then for any $t > 0$ and $\lambda \in (0, 1]$,

$$\mathbb{E}|V|1_{\{|V| \geq t\}} \leq \frac{4}{\lambda} \mathbb{P}(|V| \geq t)^{1-\lambda} \mathbb{E}|V|1_{\{|V| \geq \lambda t\}}.$$ 

Moreover, if $\mathbb{P}(|V| \geq t) \leq 1/4$, then $\mathbb{E}|V|1_{\{|V| \geq t\}} \leq 4t\mathbb{P}(|V| \geq t)$.

**Proof.** Without loss of generality we may assume that $\mathbb{P}(|V| \geq t) \leq 1/4$ (otherwise the first estimate is trivial).

Observe that $\mathbb{P}(|V| \geq s) = \exp(-N(s))$ where $N\colon [0, \infty) \rightarrow [0, \infty]$ is convex and $N(0) = 0$. In particular

$$\mathbb{P}(|V| \geq \gamma t) \leq \mathbb{P}(|V| \geq t)^{\gamma} \quad \text{for } \gamma > 1$$

and

$$\mathbb{P}(|V| \geq \gamma t) \geq \mathbb{P}(|V| \geq t)^{\gamma} \quad \text{for } \gamma \in [0, 1].$$
We have

\[ E|V| \mathbf{1}_{\{|V| \geq t\}} \leq \sum_{k=0}^{\infty} 2^{k+1} t \mathbb{P}(\{|V| \geq 2^k t\}) \leq 2t \sum_{k=0}^{\infty} 2^k \mathbb{P}(\{|V| \geq t\})^{2^k} \]

\[ \leq 2t \mathbb{P}(\{|V| \geq t\}) \sum_{k=0}^{\infty} 2^k 4^{1-2^k} \leq 4t \mathbb{P}(\{|V| \geq t\}). \]

This implies the second part of the lemma.

To conclude the proof of the first bound it is enough to observe that

\[ E|V| \mathbf{1}_{\{|V| \geq \lambda t\}} \geq \lambda t \mathbb{P}(\{|V| \geq \lambda t\}) \geq \lambda t \mathbb{P}(\{|V| \geq t\})^\lambda. \]

\[ \square \]

**Proof of Theorem 3.** By Proposition 1 it is enough to show the lower bound. By Lemma 11 we may assume that \( X \) is symmetric. We may also obviously assume that \( \|X_i\|_2 = \mathbb{E}X_i^2 > 0 \) for all \( i \).

Let \( Z = (Z_1, \ldots, Z_n) \), where \( Z_i = X_i/\|X_i\|_2 \). Then \( Z \) is log-concave, isotropic and, by (7), \( \mathbb{E}|Z_i| \geq 1/(2C_1) \) for all \( i \). Set \( Y := 2C_1Z \). Then \( X_i = a_i Y_i \) and \( \mathbb{E}|Y_i| \geq 1 \). Moreover, since any \( m \)-dimensional projection of \( Z \) is a log-concave, isotropic \( m \)-dimensional vector, we know by the result of Lee and Vempala [13], that it satisfies the exponential concentration with a constants \( C_2^{m/4} \). (In fact an easy modification of the proof below shows that for our purposes it would be enough to have exponential concentration with a constant \( C_2^{m/4} \gamma \) for some \( \gamma < 1/2 \), so one may also use Eldan’s result [6] which gives such estimates for any \( \gamma > 1/3 \).) So any \( m \)-dimensional projection of \( Y \) satisfies exponential concentration with constant \( C_2^{m/4} \).

Let us fix \( k \) and set \( t := t(k, X) \), then (since \( X_i \) has no atoms)

\[ \sum_{i=1}^{n} \mathbb{E}|X_i| \mathbf{1}_{\{|X_i| \geq t\}} = kt. \]  

For \( l = 1, 2, \ldots \) define

\[ I_l := \{i \in [n]: \beta^{l-1} \geq \mathbb{P}(\{|X_i| \geq t\}) \geq \beta^l\}, \]

where \( \beta = 2^{-8} \). By (12) there exists \( l \) such that

\[ \sum_{i \in I_l} \mathbb{E}|X_i| \mathbf{1}_{\{|X_i| \geq t\}} \geq kt2^{-l}. \]

Let us consider three cases.

(i) \( l = 1 \) and \( |I_1| \leq k \). Then

\[ \mathbb{E} \max_{|I|=k} \sum_{i \in I} |X_i| \geq \sum_{i \in I_1} \mathbb{E}|X_i| \mathbf{1}_{\{|X_i| \geq t\}} \geq \frac{1}{2} kt. \]

(ii) \( l = 1 \) and \( |I_1| > k \). Choose \( J \subset I_1 \) of cardinality \( k \). Then

\[ \mathbb{E} \max_{|I|=k} \sum_{i \in I} |X_i| \geq \sum_{i \in J} \mathbb{E}|X_i| \geq \sum_{i \in J} t \mathbb{P}(\{|X_i| \geq t\}) \geq \beta kt. \]
(iii) $l > 1$. By Lemma 12 (applied with $\lambda = 1/8$) we have

$$\sum_{i \in I_t} \mathbb{E}[X_i | 1_{\{|X_i| \geq t/8\}}] \geq \frac{1}{32} \beta^{-7(l-1)/8} \sum_{i \in I_t} \mathbb{E}[X_i | 1_{\{|X_i| \geq t\}}] \geq \frac{1}{32} \beta^{-7(l-1)/8} 2^{-l} kt.$$  

Moreover for $i \in I_t$, $\mathbb{P}(|X_i| \geq t) \leq \beta^{l-1} \leq 1/4$, so the second part of Lemma 12 yields

$$4t|I_t| \beta^{l-1} \geq \sum_{i \in I_t} \mathbb{E}[X_i | 1_{\{|X_i| \geq t\}}] \geq kt 2^{-l}$$

and $|I_t| \geq \beta^{l-2} 2^{-l/2} k = 2^{7l-10} k \geq k$.

Set $k' := \beta^{-7l/8} 2^{-l/2} k = 2^{6l} k$. If $k' \geq |I_t|$ then, using (13), we estimate

$$\mathbb{E} \max_{|I| = k} \sum_{i \in I} |X_i| \geq \frac{k}{|I|} \sum_{i \in I} \mathbb{E}[X_i] \geq \beta^{7l/8} 2^l \sum_{i \in I} \mathbb{E}[X_i | 1_{\{|X_i| \geq t/8\}}] \geq \frac{1}{32} \beta^{7l/8} 2^l t = 2^{-12} kt.$$

Otherwise set $X' = (X_i)_{i \in I_t}$ and $Y' = (Y_i)_{i \in I_t}$. By (12) we have

$$kt \geq \sum_{i \in I_t} \mathbb{E}[X_i | 1_{\{|X_i| \geq t\}}] \geq |I_t| |t| \beta^t,$$

so $|I_t| \leq k \beta^{-l}$ and $Y'$ satisfies exponential concentration with constant $\alpha' = C_2 k^{1/4} \beta^{-l/4}$.

Estimate (13) yields

$$\sum_{i \in I_t} \mathbb{E}[X_i | 1_{\{|X_i| \geq 2^{-12} t\}}] \geq \sum_{i \in I_t} \mathbb{E}[X_i | 1_{\{|X_i| \geq t/8\}}] \geq 2^{-12} k' t,$$

so $t(k', X') \geq 2^{-12} t$. Moreover, by Proposition 7 we have (since $k' \leq |I_t|$)

$$\mathbb{E} \max_{I \subset I_t, |I| = k'} \sum_{i \in I} |X_i| \geq \frac{1}{8 + 64 \alpha' / \sqrt{k'}} k' t(k', X').$$

To conclude observe that

$$\frac{\alpha'}{\sqrt{k'}} = C_2 2^{-l} k^{-1/4} \leq \frac{C_2}{4}$$

and since $k' \geq k$,

$$\mathbb{E} \max_{|I| = k} \sum_{i \in I} |X_i| \geq \frac{k}{k'} \mathbb{E} \max_{I \subset I_t, |I| = k'} \sum_{i \in I} |X_i| \geq \frac{1}{8 + 16 C_2} 2^{-12} tk.$$  

\[ \Box \]

4. Vectors satisfying condition (3)

Proof of Theorem 2. By Proposition 1 we need to show only the lower bound. Assume first that variables $X_i$ have no atoms and $k \geq 4(1 + \alpha)$.

Let $t_k = t(k, X)$. Then $\mathbb{E} \sum_{i=1}^n |X_i| 1_{\{|X_i| \geq t_k\}} = k t_k$. Note, that (3) implies that for all $i \neq j$ we have

$$\mathbb{E}[X_i X_j | 1_{\{|X_i| \geq t_k, |X_j| \geq t_k\}}] \leq \alpha \mathbb{E}[X_i | 1_{\{|X_i| \geq t_k\}}] \mathbb{E}[X_j | 1_{\{|X_j| \geq t_k\}}].$$

We may assume that $\mathbb{E} \max_{|I| = k} \sum_{i \in I} |X_i| \leq \frac{1}{6} k t_k$, because otherwise the lower bound holds trivially.
Let us define
\[ Y := \sum_{i=1}^{n} |X_i|1_{\{|X_i| \geq t_k\}} \] and \( A := (\mathbb{E}Y^2)^{1/2} \).

Since
\[ \mathbb{E}\max_{|I|=k} \sum_{i \in I} |X_i| \geq \mathbb{E}\left[ \frac{1}{2} kt_k 1_{\{Y \geq kt_k/2\}} \right] = \frac{1}{2} kt_k \mathbb{P}\left( Y \geq \frac{kt_k}{2} \right), \]
it suffices to bound below the probability that \( Y \geq kt_k/2 \) by a constant depending only on \( \alpha \).

We have
\[
A^2 = \mathbb{E}Y^2 \leq \sum_{i=1}^{n} \mathbb{E}X_i^2 1_{\{|X_i| \geq t_k\}} + \sum_{i \neq j} \mathbb{E}|X_i| \mathbb{E}|X_j| 1_{\{|X_i| \geq t_k, |X_j| \geq t_k\}}
\]
\[
\leq kt_k \mathbb{E}Y + \alpha \sum_{i \neq j} \mathbb{E}|X_i| \mathbb{E}|X_j| 1_{\{|X_i| \geq t_k\}} \mathbb{E}|X_j| 1_{\{|X_j| \geq t_k\}}
\]
\[
\leq kt_k A + \alpha \left( \sum_{i} \mathbb{E}|X_i| 1_{\{|X_i| \geq t_k\}} \right)^2 \leq \frac{1}{2} (k^2 t_k^2 + A^2) + \alpha k^2 t_k^2.
\]

Therefore \( A^2 \leq (1 + 2\alpha)k^2 t_k^2 \) and for any \( l \geq k/2 \) we have
\[
\mathbb{E}Y 1_{\{Y \geq kt_k/2\}} \leq ut_k \mathbb{P}(Y \geq kt_k/2) + \frac{1}{l t_k} \mathbb{E}Y^2
\]
\[
\leq ut_k \mathbb{P}(Y \geq kt_k/2) + (1 + 2\alpha)k^2 l^{-1} t_k.
\]

By Corollary 9 we have (recall definition (6))
\[
\sum_{i=1}^{n} \mathbb{E}|X_i| 1_{\{|X_i| \geq t_k\}} \leq \mathbb{E}\max_{|I|=k} \sum_{i \in I} |X_i| + \sum_{l=k+1}^{\infty} \left( k t_k \mathbb{P}(N(kt_k) \geq l) + \int_{kt_k}^{\infty} \mathbb{P}(N(s) \geq l) ds \right)
\]
\[
\leq \frac{1}{6} k t_k + \sum_{l=k+1}^{\infty} \left( k t_k \mathbb{E}N(kt_k)^2 l^{-2} + \int_{kt_k}^{\infty} \mathbb{E}N(s)^2 l^{-2} ds \right)
\]
\[
\leq \frac{1}{6} k t_k + \frac{1}{k} \left( k t_k \mathbb{E}N(kt_k)^2 + \int_{kt_k}^{\infty} \mathbb{E}N(s)^2 ds \right).
\]

Assumption (3) implies that
\[
\mathbb{E}N(s)^2 = \sum_{i=1}^{n} \mathbb{P}(|X_i| \geq s) + \sum_{i \neq j} \mathbb{P}(|X_i| \geq s, |X_j| \geq s)
\]
\[
\leq \sum_{i=1}^{n} \mathbb{P}(|X_i| \geq s) + \alpha \left( \sum_{i=1}^{n} \mathbb{P}(|X_i| \geq s) \right)^2.
\]
Moreover for \( s \geq k t_k \) we have
\[
\sum_{i=1}^{n} \mathbb{P}(|X_i| \geq s) \leq \frac{1}{s} \sum_{i=1}^{n} \mathbb{E}|X_i|1_{\{|X_i| \geq s\}} \leq \frac{kt_k}{s} \leq 1,
\]
so
\[
\mathbb{E}N(s)^2 \leq (1 + \alpha) \sum_{i=1}^{n} \mathbb{P}(|X_i| \geq s) \quad \text{for } s \geq k t_k.
\]

Thus
\[
k t_k \mathbb{E}N(k t_k)^2 \leq k t_k (1 + \alpha) \sum_{i=1}^{n} \mathbb{P}(|X_i| \geq k t_k) \leq (1 + \alpha) \sum_{i=1}^{n} \mathbb{E}|X_i|1_{\{|X_i| \geq k t_k\}},
\]
and
\[
\int_{k t_k}^{\infty} \mathbb{E}N(s)^2 ds \leq (1 + \alpha) \sum_{i=1}^{n} \int_{k t_k}^{\infty} \mathbb{P}(|X_i| \geq s) ds \leq (1 + \alpha) \sum_{i=1}^{n} \mathbb{E}|X_i|1_{\{|X_i| \geq k t_k\}}.
\]
This together with (16) and the assumption that \( k \geq 4(1 + \alpha) \) implies
\[
\sum_{i=1}^{n} \mathbb{E}|X_i|1_{\{|X_i| \geq k t_k\}} \leq \frac{1}{3} k t_k
\]
and
\[
\mathbb{E}Y = \sum_{i=1}^{n} \mathbb{E}|X_i|1_{\{|X_i| \geq k t_k\}} - \sum_{i=1}^{n} \mathbb{E}|X_i|1_{\{|X_i| \geq k t_k\}} \geq \frac{2}{3} k t_k.
\]
Therefore
\[
\mathbb{E}Y 1_{\{Y \geq k t_k/2\}} \geq \mathbb{E}Y - \frac{1}{2} k t_k \geq \frac{1}{6} k t_k.
\]
This applied to (15) with \( l = (12 + 24 \alpha) k \) gives us \( \mathbb{P}(Y \geq k t_k/2) \geq (144 + 288 \alpha)^{-1} \) and in consequence
\[
\mathbb{E}\max_{|I|=k} \sum_{i \in I} |X_i| \geq \frac{1}{288(1 + 2 \alpha)} k t(k, X).
\]
Since \( k \mapsto k t(k, X) \) is non-decreasing, in the case \( k \leq \lceil 4(1 + \alpha) \rceil =: k_0 \) we have
\[
\mathbb{E}\max_{|I|=k} |X_i| \geq \frac{k}{k_0} \mathbb{E}\max_{|I|=k_0} |X_i| \geq \frac{k}{5 + 4 \alpha} \cdot \frac{1}{288(1 + 2 \alpha)} k_0 t(k_0, X)
\geq \frac{1}{288(5 + 4 \alpha)(1 + 2 \alpha)} k t(k, X).
\]
The last step is to loose the assumption that \( X_i \) has no atoms. Note that both assumption (3) and the lower bound depend only on \((|X_i|)_{i=1}^{n}\), so we may assume that \( X_i \) are nonnegative almost surely. Consider \( X^e := (X_i + e Y_i)_{i=1}^{n} \), where \( Y_1, \ldots, Y_n \) are i.i.d.
nonnegative r.v’s with \( \mathbb{E}X_i < \infty \) and a density \( g \), independent of \( X \). Then for every \( s, t > 0 \) we have (observe that (3) holds also for \( s < 0 \) or \( t < 0 \)).

\[
\mathbb{P}(X_i^\varepsilon \geq s, X_j^\varepsilon \geq t) = \int_0^\infty \int_0^\infty \mathbb{P}(X_i + \varepsilon y_i \geq s, X_j + \varepsilon y_j \geq t) g(y_i)g(y_j) dy_i dy_j
\]

\[\leq \alpha \int_0^\infty \int_0^\infty \mathbb{P}(X_i \geq s - \varepsilon y_i)\mathbb{P}(X_j \geq t - \varepsilon y_j) g(y_i)g(y_j) dy_i dy_j \]

\[= \alpha \mathbb{P}(X_i^\varepsilon \geq s)\mathbb{P}(X_j^\varepsilon \geq t).
\]

Thus \( X^\varepsilon \) satisfies assumption (3) and has the density function for every \( \varepsilon > 0 \). Therefore for all natural \( k \) we have

\[
\mathbb{E}\max_{|I|=k} \sum^n_{i=1} X_i^\varepsilon \geq c(\alpha)kt(k, X^\varepsilon) \geq c(\alpha)kt(k, X).
\]

Clearly, \( \mathbb{E}\max_{|I|=k} \sum^n_{i=1} X_i^\varepsilon \) \( \rightarrow \) \( \mathbb{E}\max_{|I|=k} \sum^n_{i=1} X_i \) as \( \varepsilon \rightarrow 0 \), so the lower bound holds in the case of arbitrary \( X \) satisfying (3).

We may use Theorem 2 to obtain a comparison of weak and strong moments for the supremum norm:

**Corollary 13.** Let \( X \) be an \( n \)-dimensional centered random vector satisfying condition (3). Assume that

\[
\|X_i\|_p \leq \beta \|X_i\|_p \quad \text{for every } p \geq 2 \text{ and } i = 1, \ldots, n.
\]

Then the following comparison of weak and strong moments for the supremum norm holds: for all \( a \in \mathbb{R}^n \) and all \( p \geq 1 \),

\[
\left( \mathbb{E}\max_{i \leq n} |a_iX_i|^p \right)^{1/p} \leq C(\alpha, \beta) \left[ \mathbb{E}\max_{i \leq n} |a_iX_i| + \max_{i \leq n} \left( \mathbb{E}|a_iX_i|^p \right)^{1/p} \right],
\]

where \( C(\alpha, \beta) \) is a constant depending only on \( \alpha \) and \( \beta \).

**Proof.** Let \( X' = (X'_i)_{i \leq n} \) be a decoupled version of \( X \). For any \( p > 0 \) a random vector \((|a_iX_i|^p)_{i \leq n}\) satisfies condition (3), so by Theorem 2

\[
\left( \mathbb{E}\max_{i \leq n} |a_iX_i|^p \right)^{1/p} \sim \left( \mathbb{E}\max_{i \leq n} |a_iX'_i|^p \right)^{1/p}
\]

for all \( p > 0 \), up to a constant depending only on \( \alpha \). The coordinates of \( X' \) are independent and satisfy condition (17), so due to [11, Theorem 1.1] the comparison of weak and strong moments of \( X' \) holds, i.e. for \( p \geq 1 \),

\[
\left( \mathbb{E}\max_{i \leq n} |a_iX'_i|^p \right)^{1/p} \leq C(\beta) \left[ \mathbb{E}\max_{i \leq n} |a_iX'_i| + \max_{i \leq n} \left( \mathbb{E}|a_iX'_i|^p \right)^{1/p} \right],
\]

where \( C(\beta) \) depends only on \( \beta \). These two observations yield the assertion. \( \square \)
ORDER STATISTICS OF LOG-CONCAVE VECTORS 15

5. LOWER ESTIMATES FOR ORDER STATISTICS

The next lemma shows the relation between \( t(k, X) \) and \( t^*(k, X) \) for log-concave vectors \( X \).

**Lemma 14.** Let \( X \) be a symmetric log-concave random vector in \( \mathbb{R}^n \). For any \( 1 \leq k \leq n \) we have

\[
\frac{1}{3} \left( t^*(k, X) + \frac{1}{k} \max_{|I|=k} \sum_{i \in I} \mathbb{E}|X_i| \right) \leq t(k, X) \leq 4 \left( t^*(k, X) + \frac{1}{k} \max_{|I|=k} \sum_{i \in I} \mathbb{E}|X_i| \right).
\]

**Proof.** Let \( t_k := t(k, X) \) and \( t_k^* := t^*(k, X) \). We may assume that any \( X_i \) is not identically equal to 0. Then \( \sum_{i=1}^n \mathbb{P}(|X_i| \geq t_k^*) = k \) and \( \sum_{i=1}^n \mathbb{E}|X_i| 1_{|X_i| \geq t_k} = kt_k \).

Obviously \( t_k^* \leq t_k \). Also for any \( |I| = k \) we have

\[
\sum_{i \in I} \mathbb{E}|X_i| \leq \sum_{i \in I} (t_k + \mathbb{E}|X_i| 1_{|X_i| \geq t_k}) \leq |I| t_k + kt_k = 2kt_k.
\]

To prove the upper bound set

\[
I_1 := \{ i \in [n] : \mathbb{P}(|X_i| \geq t_k^*) \geq 1/4 \}.
\]

We have

\[
k \geq \sum_{i \in |I_1|} \mathbb{P}(|X_i| \geq t_k^*) \geq \frac{1}{4} |I_1|,
\]

so \( |I_1| \leq 4k \). Hence

\[
\sum_{i \in I_1} \mathbb{E}|X_i| 1_{|X_i| \geq t_k^*} \leq \sum_{i \in I_1} \mathbb{E}|X_i| \leq 4 \max_{|I|=k} \sum_{i \in I} \mathbb{E}|X_i|.
\]

Moreover by the second part of Lemma 12 we get

\[
\mathbb{E}|X_i| 1_{|X_i| \geq t_k^*} \leq 4t_k^* \mathbb{P}(|X_i| \geq t_k^*) \quad \text{for } i \notin I_1,
\]

so

\[
\sum_{i \notin I_1} \mathbb{E}|X_i| 1_{|X_i| \geq t_k^*} \leq 4t_k^* \sum_{i=1}^n \mathbb{P}(|X_i| \geq t_k^*) \leq 4kt_k^*.
\]

Hence if \( s = 4t_k^* + \frac{1}{k} \max_{|I|=k} \sum_{i \in I} \mathbb{E}|X_i| \) then

\[
\sum_{i=1}^n \mathbb{E}|X_i| 1_{|X_i| \geq s} \leq \sum_{i=1}^n \mathbb{E}|X_i| 1_{|X_i| \geq t_k^*} \leq 4 \max_{|I|=k} \sum_{i \in I} \mathbb{E}|X_i| + 4kt_k^* = ks,
\]

that is \( t_k \leq s \). \( \square \)

To derive bounds for order statistics we will also need a few facts about log-concave vectors.

**Lemma 15.** Assume that \( Z \) is an isotropic one- or two-dimensional log-concave random vector with a density \( g \). Then \( g(t) \leq C \) for all \( t \). If \( Z \) is one-dimensional, then also \( g(t) \geq c \) for all \( |t| \leq t_0 \), where \( t_0 > 0 \) is an absolute constant.
Lemma 16. Let $Y$ be a mean zero log-concave random variable and let $\mathbb{P}(|Y| \geq t) \leq p$ for some $p > 0$. Then

$$\mathbb{P}\left(|Y| \geq \frac{t}{2}\right) \geq \frac{1}{\sqrt{ep}} \mathbb{P}(|Y| \geq t).$$

Proof. By the Grünbaum inequality (8) we have $\mathbb{P}(Y \geq 0) \geq 1/e$, hence

$$\mathbb{P}\left(Y \geq \frac{t}{2}\right) \geq \mathbb{P}(Y \geq t) \mathbb{P}(Y \geq 0) \geq \frac{1}{\sqrt{e}} \mathbb{P}(Y \geq t) \geq \frac{1}{\sqrt{ep}} \mathbb{P}(Y \geq t).$$

Since $-Y$ satisfies the same assumptions as $Y$ we also have

$$\mathbb{P}\left(-Y \geq \frac{t}{2}\right) \geq \frac{1}{\sqrt{ep}} \mathbb{P}(-Y \geq t). \quad \square$$

Lemma 17. Let $Y$ be a mean zero log-concave random variable and let $\mathbb{P}(|Y| \geq t) \geq p$ for some $p > 0$. Then there exists a universal constant $C$ such that

$$\mathbb{P}(|Y| \leq \lambda t) \leq \frac{C\lambda}{\sqrt{p}} \mathbb{P}(|Y| \leq t) \quad \text{for} \quad \lambda \in [0, 1].$$

Proof. Without loss of generality we may assume that $EY^2 = 1$. Then by Chebyshev’s inequality $t \leq p^{-1/2}$. Let $g$ be the density of $Y$. By Lemma 15 we know that $\|g\|_{\infty} \leq C$ and $g(t) \geq c$ on $[-t_0, t_0]$, where $c, C$ and $t_0 \in (0, 1)$ are universal constants. Thus

$$\mathbb{P}(|Y| \leq t) \geq \mathbb{P}(|Y| \leq t_0 \sqrt{pt}) \geq 2ct_0 \sqrt{pt},$$

and

$$\mathbb{P}(|Y| \leq \lambda t) \leq 2\|g\|_{\infty} \lambda t \leq 2C\lambda t \leq \frac{C\lambda}{ct_0 \sqrt{p}} \mathbb{P}(|Y| \leq t). \quad \square$$
Now we are ready to give a proof of the lower bound in Theorem 4. The next proposition is a key part of it.

**Proposition 18.** Let $X$ be a mean zero log-concave $n$-dimensional random vector with uncorrelated coordinates and let $\alpha > 1/4$. Suppose that

$$P(|X_i| \geq t^*(\alpha, X)) \leq \frac{1}{C_3} \text{ for all } i.$$ 

Then

$$P\left(|4\alpha| \cdot \max_i |X_i| \geq \frac{1}{C_4} t^*(\alpha, X)\right) \geq \frac{3}{4}.$$

**Proof.** Let $t^* = t^*(\alpha, X)$, $k := |4\alpha|$ and $L = \lceil \sqrt{C_3}/4 \rceil$. We will choose $C_3$ in such a way that $L$ is large, in particular we may assume that $L \geq 2$. Observe also that $\alpha = \sum_{i=1}^n P(|X_i| \geq t^*(\alpha, X)) \leq nC_3^{-1}$, thus $Lk \leq \frac{C_3^{1/2} e^{-1/2}}{\alpha} \leq e^{-1/2}C_3^{-1/2} n \leq n$ if $C_3 \geq 1 > \frac{1}{e}$. Hence

\begin{equation}
(18) \quad k \cdot \max_i |X_i| \geq \frac{1}{k(L - 1)} \sum_{l=k+1}^{Lk} l \cdot \max_i |X_i| = \frac{1}{k(L - 1)} \left( \sum_{|I|=Lk} \max_{i \in I} |X_i| - \max_{|I|=k} \sum_{i \in I} |X_i| \right).
\end{equation}

Lemma 16 and the definition of $t^*(\alpha, X)$ yield

$$\sum_{i=1}^n P\left(|X_i| \geq \frac{1}{2} t^*\right) \geq \sqrt{\frac{C_3}{e}} \alpha \geq Lk.$$

This yields $t(Lk, X) \geq t^*(Lk, X) \geq \frac{t^*}{2}$ and by Theorem 3 we have

$$E \max_{|I|=Lk} \sum_{i \in I} |X_i| \geq c_1 Lk \frac{t^*}{2}.$$

Since for any norm $P(\|X\| \leq tE\|X\|) \leq Ct$ for $t > 0$ (see [10, Corollary 1]) we have

\begin{equation}
(19) \quad P\left(\max_{|I|=Lk} \sum_{i \in I} |X_i| \geq c_2 Lkt^*\right) \geq \frac{7}{8}.
\end{equation}

Let $X'$ be an independent copy of $X$. By the Paley-Zygmund inequality and (7), $P(|X_i| \geq \frac{1}{2} E|X_i|) \geq \frac{\left(\frac{1}{2} E|X_i|\right)^2}{\frac{1}{4} E|X_i|^2} > \frac{1}{C_3}$ if $C_3 > 16C_1^2$, so $\frac{1}{2} E|X_i| \leq t^*$. Moreover it is easy to verify that $k = |4\alpha| > \alpha$ for $\alpha > 1/4$, thus $t^*(k, X) \leq t^*(\alpha, X) = t^*$. Hence Proposition 1, Lemma 14, and inequality (10) yield

$$E \max_{|I|=k} \sum_{i \in I} |X_i| = E \max_{|I|=k} \sum_{i \in I} |X_i - E X_i'| \leq E \max_{|I|=k} \sum_{i \in I} |X_i - X_i'| \leq E \max_{|I|=2k} \sum_{i \in I} |X_i - X_i'| \leq 4kt(2k, X - X') \leq 16k(t^*(2k, X - X') + \max_i E|X_i - X_i'|) \leq 16k(2t^*(k, X) + 2 \max_i E|X_i|) \leq 96kt^*.$$
Therefore
\begin{equation}
\mathbb{P}
\left(\max_{|i|=k} \sum_{i \in I} |X_i| \geq 800kt^*\right) \leq \frac{1}{8}.
\end{equation}

Estimates (18)-(20) yield
\[ \mathbb{P}
\left(k \max_i |X_i| \geq \frac{1}{L-1}(c_2L - 800)t^*\right) \geq \frac{3}{4}, \]
so it is enough to choose $C_3$ in such a way that $L \geq 1600/c_2$.

Proof of the first part of Theorem 4. Let $t^* = t^*(k - 1/2, X)$ and $C_3$ be as in Proposition 18. It is enough to consider the case when $t^* > 0$, then $\mathbb{P}(|X_i| = t^*) = 0$ for all $i$ and $\sum_{i=1}^n \mathbb{P}(|X_i| \geq t^*) = k - 1/2$. Define
\begin{align*}
I_1 := \left\{ i \leq n: \mathbb{P}(|X_i| \geq t^*) \leq \frac{1}{C_3} \right\}, & \quad \alpha := \sum_{i \in I_1} \mathbb{P}(|X_i| \geq t^*), \\
I_2 := \left\{ i \leq n: \mathbb{P}(|X_i| \geq t^*) > \frac{1}{C_3} \right\}, & \quad \beta := \sum_{i \in I_2} \mathbb{P}(|X_i| \geq t^*).
\end{align*}

If $\beta = 0$ then $\alpha = k - 1/2$, $|I_1| = \{1, \ldots, n\}$, and the assertion immediately follows by Proposition 18 since $4\alpha \geq k$.

Otherwise define
\[ \tilde{N}(t) := \sum_{i \in I_2} 1_{\{|X_i| \leq t\}}. \]

We have by Lemma 17 applied with $p = 1/C_3$
\[ \mathbb{E}\tilde{N}(\lambda t^*) = \sum_{i \in I_2} \mathbb{P}(|X_i| \leq \lambda t^*) \leq C_5\lambda \sum_{i \in I_2} \mathbb{P}(|X_i| \leq t^*) = C_5\lambda(|I_2| - \beta). \]

Thus
\[ \mathbb{P}\left(\lfloor \beta \rfloor \max_{i \in I_2} |X_i| \leq \lambda t^*\right) = \mathbb{P}(\tilde{N}(\lambda t^*) \geq |I_2| + 1 - \lfloor \beta \rfloor) \leq \frac{1}{|I_2| + 1 - \lfloor \beta \rfloor} \mathbb{E}\tilde{N}(\lambda t^*) \leq C_5\lambda. \]

Therefore
\[ \mathbb{P}\left(\lfloor \beta \rfloor \max_{i \in I_2} |X_i| \geq \frac{1}{4C_5}t^*\right) \geq \frac{3}{4}. \]

If $\alpha < 1/2$ then $\lfloor \beta \rfloor = k$ and the assertion easily follows. Otherwise Proposition 18 yields
\[ \mathbb{P}\left(\lfloor 4\alpha \rfloor \max_{i \in I_1} |X_i| \geq \frac{1}{C_4}t^*\right) \geq \frac{3}{4}. \]

Observe that for $\alpha \geq 1/2$ we have $\lfloor 4\alpha \rfloor + \lfloor \beta \rfloor \geq 4\alpha - 1 + \beta \geq \alpha + 1/2 + \beta = k$, so
\[ \mathbb{P}\left( k-\max_i |X_i| \geq \min\left\{ \frac{t^*}{C_4}, \frac{t^*}{4C_5} \right\} \right) \geq \mathbb{P}\left( |4\alpha| \cdot \max_i |X_i| \geq \frac{1}{C_4} t^*, \left[ \beta \right] \cdot \max_i |X_i| \geq \frac{1}{4C_5} t^* \right) \geq \frac{1}{2}. \]

**Remark 19.** A modification of the proof above shows that under the assumptions of Theorem 4 for any \( p < 1 \) there exists \( c(p) > 0 \) such that

\[ \mathbb{P}\left( k-\max_{i \leq u} |X_i| \geq c(p)t^*(k-1/2, X) \right) \geq p. \]

### 6. Upper estimates for order statistics

We will need a few more facts concerning log-concave vectors.

**Lemma 20.** Suppose that \( X \) is a mean zero log-concave random vector with uncorrelated coordinates. Then for any \( i \neq j \) and \( s > 0 \),

\[ \mathbb{P}(|X_i| \leq s, |X_j| \leq s) \leq C_0 \mathbb{P}(|X_i| \leq s) \mathbb{P}(|X_j| \leq s). \]

**Proof.** Let \( C_7, C_3 \) and \( t_0 \) be the constants from Lemma 15. If \( s > t_0 \|X_i\|_2 \) then, by Lemma 15, \( \mathbb{P}(|X_i| \leq s) \geq 2c_3 t_0 \) and the assertion is obvious (with any \( C_0 \geq (2c_3 t_0)^{-1} \)). Thus we will assume that \( s \leq t_0 \min\{\|X_i\|_2, \|X_j\|_2\} \).

Let \( \tilde{X}_i = X_i/\|X_i\|_2 \) and let \( g_{ij} \) be the density of \((\tilde{X}_i, \tilde{X}_j)\). By Lemma 15 we know that \( \|g_{i,j}\|_\infty \leq C_7 \), so

\[ \mathbb{P}(|X_i| \leq s, |X_j| \leq s) = \mathbb{P}(|\tilde{X}_i| \leq s/\|X_i\|_2, |\tilde{X}_j| \leq s/\|X_j\|_2) \leq C_7 \frac{s^2}{\|X_i\|_2 \|X_j\|_2}. \]

On the other hand the second part of Lemma 15 yields

\[ \mathbb{P}(|X_i| \leq s) \mathbb{P}(|X_j| \leq s) \geq \frac{4c_3^2 s^2}{\|X_i\|_2 \|X_j\|_2}. \] \( \square \)

**Lemma 21.** Let \( Y \) be a log-concave random variable. Then

\[ \mathbb{P}(|Y| \geq ut) \leq \mathbb{P}(|Y| \geq t)^{(u-1)/2} \text{ for } u \geq 1, t \geq 0. \]

**Proof.** We may assume that \( Y \) is non-degenerate (otherwise the statement is obvious), in particular \( Y \) has no atoms. Log-concavity of \( Y \) yields

\[ \mathbb{P}(Y \geq t) \geq \mathbb{P}(Y \geq -t)^{\frac{u+1}{u+1}} \mathbb{P}(Y \geq ut)^{\frac{2}{u+1}}. \]

Hence

\[\begin{align*}
\mathbb{P}(Y \geq ut) & \leq \left( \frac{\mathbb{P}(Y \geq t)}{\mathbb{P}(Y \geq -t)} \right)^\frac{u+1}{2} \mathbb{P}(Y \geq -t) = \left( 1 - \frac{\mathbb{P}(|Y| \leq t)}{\mathbb{P}(Y \geq -t)} \right)^\frac{u+1}{2} \mathbb{P}(Y \geq -t) \\
& \leq (1 - \mathbb{P}(|Y| \leq t))^{\frac{u+1}{4}} \mathbb{P}(Y \geq -t) = \mathbb{P}(|Y| \geq t)^{\frac{u+1}{4}} \mathbb{P}(Y \geq -t).
\end{align*}\]
Since $-Y$ satisfies the same assumptions as $Y$, we also have
\[
\mathbb{P}(Y \leq -ut) \leq \mathbb{P}(|Y| \geq t)^{u+1} \mathbb{P}(Y \leq t).
\]
Adding both estimates we get
\[
\mathbb{P}(|Y| \geq ut) \leq \mathbb{P}(|Y| \geq t)^{u+1} (1 + \mathbb{P}(|Y| \leq t)) = \mathbb{P}(|Y| \geq t)^{u+1} (1 - \mathbb{P}(|Y| \leq t)^2).
\]

**Lemma 22.** Suppose that $Y$ is a log-concave random variable and $\mathbb{P}(|Y| \leq t) \leq \frac{1}{10}$. Then $\mathbb{P}(|Y| \leq 21t) \geq 5\mathbb{P}(|Y| \leq t)$.

**Proof.** Let $\mathbb{P}(|Y| \leq t) = p$ then by Lemma 21
\[
\mathbb{P}(|Y| \leq 21t) = 1 - \mathbb{P}(|Y| > 21t) \geq 1 - \mathbb{P}(|Y| > t)^{10} = 1 - (1 - p)^{10} \geq 10p - 45p^2 \geq 5p.
\]

Let us now prove (4) and see how it implies the second part of Theorem 4. Then we give a proof of (5).

**Proof of (4).** Fix $k$ and set $t^* := t(k - 1/2, X)$. Then $\sum_{i=1}^n \mathbb{P}(|X_i| \geq t^*) = k - 1/2$. Define
\[
I_1 := \left\{ i \leq n : \mathbb{P}(|X_i| \geq t^*) \leq \frac{9}{10} \right\}, \quad \alpha := \sum_{i \in I_1} \mathbb{P}(|X_i| \geq t^*),
\]
\[
I_2 := \left\{ i \leq n : \mathbb{P}(|X_i| \geq t^*) > \frac{9}{10} \right\}, \quad \beta := \sum_{i \in I_2} \mathbb{P}(|X_i| \geq t^*).
\]

Observe that for $u > 3$ and $1 \leq l \leq |I_1|$ we have by Lemma 21
\[
\mathbb{P}(l \max_{i \in I_1} |X_i| \geq ut^*) \leq \mathbb{E}\left[ \frac{1}{l} \sum_{i \in I_1} 1_{|X_i| \geq ut^*} \right] = \frac{1}{l} \sum_{i \in I_1} \mathbb{P}(|X_i| \geq ut^*) \leq \frac{1}{l} \sum_{i \in I_1} \mathbb{P}(|X_i| \geq t^*)^{(u-1)/2} \leq \frac{\alpha}{l} \left( \frac{9}{10} \right)^{(u-3)/2}.
\]

Consider two cases.

**Case 1.** $\beta > |I_2| - 1/2$. Then $|I_2| < \beta + 1/2 \leq k$, so $k - |I_2| \geq 1$ and
\[
\alpha = k - \frac{1}{2} - \beta \leq k - |I_2|.
\]

Therefore by (23)
\[
\mathbb{P}(k \max_{i \in I_1} |X_i| \geq 5t^*) \leq \mathbb{P}\left( (k - |I_2|) \max_{i \in I_1} |X_i| \geq 5t^* \right) \leq \frac{9}{10}.
\]

**Case 2.** $\beta \leq |I_2| - 1/2$. Observe that for any disjoint sets $J_1$, $J_2$ and integers $l, m$ such that $l \leq |J_1|$, $m \leq |J_2|$ we have
\[
(l + m - 1) \max_{i \in J_1 \cup J_2} |x_i| \leq \max \left\{ l \max_{i \in J_1} |x_i|, m \max_{i \in J_2} |x_i| \right\} \leq l \max_{i \in J_1} |x_i| + m \max_{i \in J_2} |x_i|.
\]

Since
\[
[\alpha] + [\beta] \leq \alpha + \beta + 2 < k + 2
\]
we have $\lceil \alpha \rceil + \lceil \beta \rceil \leq k + 1$ and, by (24),
\[ k - \max_{i} |X_i| \leq \lceil \alpha \rceil - \max_{i \in I_1} |X_i| + \lceil \beta \rceil - \max_{i \in I_2} |X_i|. \]

Estimate (23) yields
\[
P\left( \lceil \alpha \rceil - \max_{i \in I_1} |X_i| \geq ut^* \right) \leq \left( \frac{9}{10} \right)^{(u-3)/2} \text{ for } u \geq 3.
\]

To estimate $\lceil \beta \rceil - \max_{i \in I_2} |X_i| = (|I_2| + 1 - \lceil \beta \rceil) - \min_{i \in I_2} |X_i|$ observe that by Lemma 22, the definition of $I_2$ and assumptions on $\beta$,
\[
\sum_{i \in I_2} \mathbb{P}(|X_i| \leq 21t^*) \geq 5 \sum_{i \in I_2} \mathbb{P}(|X_i| \leq t^*) = 5(|I_2| - \beta) \geq 2(|I_2| + 1 - \lceil \beta \rceil).
\]

Set $l := (|I_2| + 1 - \lceil \beta \rceil)$ and
\[
\tilde{N}(t) := \sum_{i \in I_2} 1_{\{|X_i| \leq t\}}.
\]

Note that we know already that $\mathbb{E} \tilde{N}(21t^*) \geq 2l$. Thus the Paley-Zygmund inequality implies
\[
P\left( \lceil \beta \rceil - \max_{i \in I_2} |X_i| \leq 21t^* \right) \geq \mathbb{P}(\tilde{N}(21t^*) \geq l) \geq \mathbb{P}\left( \tilde{N}(21t^*) \geq \frac{1}{2} \mathbb{E} \tilde{N}(21t^*) \right) \geq \frac{1}{4} \frac{(\mathbb{E} \tilde{N}(21t^*))^2}{\mathbb{E} \tilde{N}(21t^*)^2}.
\]

However Lemma 20 yields
\[
\mathbb{E} \tilde{N}(21t^*)^2 \leq \mathbb{E} \tilde{N}(21t^*) + C_6(\mathbb{E} \tilde{N}(21t^*))^2 \leq (C_6 + 1)(\mathbb{E} \tilde{N}(21t^*))^2.
\]

Therefore
\[
P\left( k - \max_{i} |X_i| > (21 + u)t^* \right) \leq \mathbb{P}\left( \lceil \alpha \rceil - \max_{i \in I_1} |X_i| \geq ut^* \right) + \mathbb{P}\left( \lceil \beta \rceil - \max_{i \in I_2} |X_i| > 21t^* \right) \leq \left( \frac{9}{10} \right)^{(u-3)/2} + 1 - \frac{1}{4(C_6 + 1)} \leq 1 - \frac{1}{5(C_6 + 1)}
\]

for sufficiently large $u$. \hfill \Box

The unconditionality assumption plays a crucial role in the proof of the next lemma, which allows to derive the second part of Theorem 4 from estimate (4).

**Lemma 23.** Let $X$ be an unconditional log-concave $n$-dimensional random vector. Then for any $1 \leq k \leq n$,
\[
P\left( k - \max_{i \leq n} |X_i| \geq ut \right) \leq \mathbb{P}\left( k - \max_{i \leq n} |X_i| \geq t \right)^u \text{ for } u > 1, t > 0.
\]
Proof. Let $\nu$ be the law of $(|X_1|, \ldots, |X_n|)$. Then $\nu$ is log-concave on $\mathbb{R}^+_n$. Define for $t > 0$,

$$A_t := \left\{ x \in \mathbb{R}^+_n : k\max_{i \leq n} |x_i| \geq t \right\}.$$  

It is easy to check that $\frac{1}{u}A_{ut} + (1 - \frac{1}{u})\mathbb{R}^+_n \subset A_t$, hence

$$\mathbb{P}\left(k\max_{i \leq n} |X_i| \geq t\right) = \nu(A_t) \geq \nu(A_{ut})^{1/u} \nu(\mathbb{R}^+_n)^{1-1/u} = \mathbb{P}\left(k\max_{i \leq n} |X_i| \geq ut\right)^{1/u}. \quad \square$$

Proof of the second part of Theorem 4. Estimate (4) together with Lemma 23 yields

$$\mathbb{P}\left(k\max_{i \leq n} |X_i| \geq \text{Cut}^*(k - 1/2, X)\right) \leq (1 - c)^u \quad \text{for } u \geq 1,$$

and the assertion follows by integration by parts. \quad \square

Proof of (5). Define $I_1$, $I_2$, $\alpha$ and $\beta$ by (21) and (22), where this time $t^* = t^*(k - k^{5/6}/2, X)$. Estimate (23) is still valid so integration by parts yields

$$\mathbb{E}k\max_{i \in I_1} |X_i| \leq \mathbb{E}(\beta + \frac{1}{2}k^{5/6}) t^*.$$  

Set

$$k_\beta := \left\lceil \beta + \frac{1}{2}k^{5/6} \right\rceil.$$

Observe that

$$[\alpha] + k_\beta \leq \alpha + \beta + \frac{1}{2}k^{5/6} + 2 = k + 2.$$  

Hence $[\alpha] + k_\beta \leq k + 1$.

If $k_\beta > |I_2|$, then $k - |I_2| \geq [\alpha] + k_\beta - 1 - |I_2| \geq [\alpha]$, so

$$\mathbb{E}k\max_i |X_i| \leq \mathbb{E}(k - |I_2|)\max_{i \in I_1} |X_i| \leq \mathbb{E}[\alpha] \max_{i \in I_1} |X_i| \leq 23t^*.$$  

Therefore it suffices to consider case $k_\beta \leq |I_2|$ only.

Since $[\alpha] + k_\beta \leq |I_2|$, we have by (24),

$$\mathbb{E}k\max_i |X_i| \leq \mathbb{E}[\alpha] \max_{i \in I_1} |X_i| + \mathbb{E}k_\beta \max_{i \in I_2} |X_i| \leq 23t^* + \mathbb{E}k_\beta \max_{i \in I_2} |X_i|.$$  

Since $\beta \leq k - \frac{1}{2}k^{5/6}$ and $x \to x - \frac{1}{2}x^{5/6}$ is increasing for $x \geq 1/2$ we have

$$\beta \leq \beta + \frac{1}{2}k^{5/6} - \frac{1}{2}\left(\beta + \frac{1}{2}k^{5/6}\right)^{5/6} \leq k_\beta - \frac{1}{2}k_\beta.$$  

Therefore, considering $(X_i)_{i \in I_2}$ instead of $X$ and $k_\beta$ instead of $k$ it is enough to show the following claim:

Let $s > 0$, $n \geq k$ and let $X$ be an $n$-dimensional log-concave vector with uncorrelated coordinates. Suppose that

$$\sum_{i \leq n} \mathbb{P}(|X_i| \geq s) \leq k - \frac{1}{2}k^{5/6} \quad \text{and} \quad \min_{i \leq n} \mathbb{P}(|X_i| \geq s) \geq 9/10.$$
where to get the last inequality we used that $x^{5/6}$ is concave on $\mathbb{R}_+$, so $(1-t)^{5/6} \leq 1 - \frac{5}{6}t$ for $t = 1/k$. Therefore by the induction assumption applied to $(X_i)_{i \neq i_0}$,

$$\mathbb{E}k \cdot \max_{i \leq n} |X_i| \leq \mathbb{E}(k-1) \cdot \max_{i \neq i_0} |X_i| \leq C_8 s.$$  

**Case 2.** $\mathbb{P}(|X_i| \leq s) \geq \frac{5}{12} k^{-1/6}$ for all $i$. Applying Lemma 15 we get

$$\frac{5}{12} k^{-1/6} \leq \mathbb{P}

\begin{align*}
\frac{|X_i|}{\|X_i\|_2} \leq \frac{s}{\|X_i\|_2} \leq C \frac{s}{\|X_i\|_2},
\end{align*}

so $\max_i \|X_i\|_2 \leq C k^{1/6} s$. Moreover $n \leq \frac{16}{9} k$. Therefore by the result of Lee and Vempala [13] $X$ satisfies the exponential concentration with $\alpha \leq C_9 k^{5/12} s$.

Let $l = \lfloor k - \frac{1}{2}(k^{5/6} - 1) \rfloor$ then $s \geq t \cdot (l - 1/2, X)$ and $k - l + 1 \geq \frac{1}{2}(k^{5/6} - 1) \geq 1/2 k^{5/6}$. Let

$$A := \left\{ x \in \mathbb{R}^n : l \cdot \max_i |x_i| \leq C_{10} s \right\}.$$  

By (4) (applied with $l$ instead of $k$) we have $\mathbb{P}(X \in A) \geq c_4$. Observe that

$$k \cdot \max_i |x_i| \geq C_{10} s + u \Rightarrow \text{dist}(x, A) \geq \sqrt{k - l + 1} u \geq \frac{1}{3} k^{5/12} u.$$  

Therefore by Lemma 10 we get

$$\mathbb{P}

\begin{align*}
\frac{k \cdot \max_i |X_i| \geq C_{10} s + 3C_9 u s}{\leq \exp \left(-u + \ln c_4 \right)}.
\end{align*}

Integration by parts yields

$$\mathbb{E}k \cdot \max_i |X_i| \leq (C_{10} + 3C_9 (1 - \ln c_4)) s$$

and the induction step is shown in this case provided that $C_8 \geq C_{10} + 3C_9 (1 - \ln c_4)$. □

To obtain Corollary 6 we used the following lemma.

**Lemma 24.** Assume that $X$ is a symmetric isotropic log-concave vector in $\mathbb{R}^n$. Then

$$t^*(p, X) \sim \frac{n - p}{n} \text{ for } n > p \geq n/4,$$

and

$$t^*(k/2, X) \sim t^*(k, X) \sim t(k, X) \text{ for } k \leq n/2.$$
Proof. Observe that
\[ \sum_{i=1}^{n} \mathbb{P}(|X_i| \leq t^*(p, X)) = n - p. \]

Thus Lemma 15 implies that for \( p \geq c_5 n \) (with \( c_5 \in (\frac{1}{2}, 1) \)) we have \( t^*(p, X) \sim \frac{n-p}{n} \). Moreover, by the Markov inequality
\[ \sum_{i=1}^{n} \mathbb{P}(|X_i| \geq 4) \leq \frac{n}{16}, \]
so \( t^*(n/4, X) \leq 4 \). Since \( p \mapsto t^*(p, X) \) is non-increasing, we know that \( t^*(p, X) \sim 1 \) for \( n/4 \leq p \leq c_5 n \).

Now we will prove (26). We have
\[ t^*(k, X) \leq t^*(k/2, X) \leq t(k/2, X) \leq 2t(k, X), \]
so it suffices to show that \( t^*(k, X) \geq ct(k, X) \). To this end we fix \( k \leq n/2 \). By (25) we know that \( t := C_{11} t^*(k, X) \geq C_{11} t^*(n/2, X) \geq \epsilon \), so the isotropicity of \( X \) and Markov’s inequality yield \( \mathbb{P}(|X_i| \geq t) \leq e^{-2} \) for all \( i \). We may also assume that \( t \geq t^*(k, X) \).

Integration by parts and Lemma 21 yield
\[ \mathbb{E}|X_i| \mathbf{1}_{(|X_i| \geq t)} \leq 3t \mathbb{P}(|X_i| \geq t) + t \int_0^\infty \mathbb{P}(|X_i| \geq (s+3)t) ds \]
\[ \leq 3t \mathbb{P}(|X_i| \geq t) + t \int_0^\infty \mathbb{P}(|X_i| \geq t) e^{-s} ds \leq 4t \mathbb{P}(|X_i| \geq t). \]

Therefore
\[ \sum_{i=1}^{n} \mathbb{E}|X_i| \mathbf{1}_{(|X_i| \geq t)} \leq 4t \sum_{i=1}^{n} \mathbb{P}(|X_i| \geq t) \leq 4t \sum_{i=1}^{n} \mathbb{P}(|X_i| \geq t^*(k, X)) \leq 4kt, \]
so \( t(k, X) \leq 4C_{11} t^*(k, X) \). \( \square \)

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