The Partition Function of a Wilson Loop in a Strongly Coupled
$\mathcal{N} = 4$ Supersymmetric Yang-Mills Plasma with Fluctuations

De-fu Hou, James T. Liu, and Hai-cang Ren

\textit{1Institute of Particle Physics, Huazhong Normal University, Wuhan, 430079, China, Key Laboratory of Quark and Lepton Physics (Huazhong Normal University), Ministry of Education, China}

\textit{2Michigan Center for Theoretical Physics, Randall Laboratory of Physics, The University of Michigan, Ann Arbor, MI 48109–1040, USA}

\textit{3Physics Department, The Rockefeller University, 1230 York Avenue, New York, NY 10021–6399}

\textit{4Institute of Particle Physics, Huazhong Normal University, Wuhan, 430079, China}

Abstract

We examine the one-loop partition function describing the fluctuations of the superstring in a Schwarzschild-AdS$_5 \times S^5$ background. On the bosonic side, we demonstrate the one-loop equivalence of the Nambu-Goto action and the Polyakov action for a general worldsheet, while on the fermionic side, we consider the reduction of the ten-dimensional Green-Schwarz fermion action to a two-dimensional worldsheet action. We derive the partition functions of the worldsheets corresponding to both straight and parallel Wilson lines. We discuss the cancellation of the UV divergences of the functional determinants in the thermal AdS background.

PACS numbers: 11.25.-w, 11.15.-q
I. INTRODUCTION

AdS/CFT duality \[1, 2, 3\] opens a new avenue towards a qualitative or even semi-quantitative approach to exploring the strong interaction regime of some quantum field theories. Based on the isomorphism between the conformal group in four dimensions and the isometry group of AdS$_5$, the most familiar example of AdS/CFT is the duality between string theory formulated on AdS$_5 \times S^5$ and $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM) on the boundary. Working in a Poincaré patch of AdS$_5$ of radius $L$, we may take the metric

\[
ds^2 = \frac{L^2}{z^2}(-dt^2 + dx^2 + dz^2) + L^2 d\Omega_5^2, \tag{1}\]

with boundary at $z = 0$. The $\mathcal{N} = 4$ supersymmetry is required to ensure conformal invariance at the quantum level, and the isometry group of $S^5$ corresponds to the global $SU(4)$ $R$-symmetry of the SYM theory. In particular, a weakly coupled string theory at large AdS radius is dual to $\mathcal{N} = 4$ SYM at large number of colors, $N_c$, and large ’t Hooft coupling

\[
\lambda \equiv g_{\text{YM}}^2 N_c = \frac{L^4}{\alpha'}. \tag{2}\]

In spite of the differences between $\mathcal{N} = 4$ SYM and QCD, AdS/CFT has been successfully applied to the understanding of a wide spectrum of phenomenology of the quark gluon plasma (QGP) created by heavy ion collisions. A nonzero temperature $T$ of the quantum field theory on the boundary may be implemented by introducing a Schwarzschild black hole in the AdS bulk. This renders the metric to be

\[
ds^2 = \frac{L^2}{z^2} \left[-f dt^2 + dx^2 + dz^2 + \frac{d^2}{f} \right] + L^2 d\Omega_5^2, \tag{3}\]

where $f = 1 - z^4/z_h^4$ with $z_h = 1/(\pi T)$. Among the salient results in this regard are the equation of state \[4\], the shear viscosity \[5\] and the energy loss of quarks \[6, 7\] to the leading order in large $\lambda$, extracted from stress tensor correlators or from Wilson loops. The best agreement between the predictions of AdS/CFT and the observations are achieved in the range

\[
5.5 < \lambda < 6\pi, \tag{4}\]

of the ’t Hooft coupling, providing evidence for a strongly coupled QGP.

While the above results were originally derived at infinite ’t Hooft coupling, it is important to examine the finite coupling corrections in order to assess the robustness of the leading
order results. One source of such corrections comes from the $\alpha'$ expansion of the low energy effective string theory. In the present maximally supersymmetric case, the first correction terms enter at the $\alpha'^3$ level, corresponding to $\mathcal{O}(\lambda^{-3/2})$ relative to the leading order in the $\mathcal{N} = 4$ SYM theory. This correction has been calculated for the equation of state [8, 9], the shear viscosity [10, 11], the jet quenching parameter [12] and for the drag force [13]. In case of the Wilson loops, however, there is another source of corrections, namely the quantum fluctuations of the string world sheet which contribute a term of $\mathcal{O}(\lambda^{-1/2})$. In the case of jet quenching and the drag force, this formally dominates over the $\mathcal{O}(\lambda^{-3/2})$ corrections of [12, 13] computed from the $\alpha'^3$ modified supergravity background.

The expectation value of a Wilson loop operator in QCD is an important quantity that probes the confinement mechanism in the hadronic phase and meson melting and energy loss in the plasma phase. In the case of $\mathcal{N} = 4$ SYM, the theory is non-confining, and we are thus only concerned with the plasma phase. Following the AdS/CFT dictionary, the Wilson loop expectation value is dual to the path integral of a first-quantized superstring worldsheet in the AdS$_5 \times$ S$^5$ bulk spanned by the loop on the boundary [14, 15]. In particular

$$W[C] \equiv \langle P e^{i\oint_{C} A_{\mu} dx^\mu} \rangle = \text{const.} \int [dX][d\theta] e^{\frac{i}{\pi\alpha'} S[X, \theta]},$$

where $A_{\mu}$ is the gauge potential, $S[X, \theta]$ is the superstring action with bosonic (fermionic) coordinates $X$’s ($\theta$’s) and the symbol $P$ indicates path ordering. The strong coupling expansion of the SYM Wilson loop corresponds to the semi-classical expansion of the string theory action, where the inverse string tension $\alpha'$ is related to the ’t Hooft coupling $\lambda$ through (2).

We have, apart from an additive constant,

$$\ln W[C] = i\sqrt{\lambda} \left( S[\bar{X}, 0] + \frac{S_1[C]}{\sqrt{\lambda}} + \cdots \right),$$

where $\bar{X}$ denotes the solution of the classical equation of motion, $S_1[C]$ is the one-loop correction of the worldsheet fluctuations and $\cdots$ represents the multi-loop contributions.

The classical solutions for simple geometries of the Wilson loop $C$ (a straight line, a circle, a pair of parallel lines, etc.) have been obtained explicitly for both vacuum AdS [11] and Schwarzschild-AdS [8] metrics. The one-loop fluctuations of the superstring in the vacuum AdS$_5 \times$ S$^5$ background have been discussed extensively in [16] and [17], and fluctuations in a Schwarzschild-AdS background were considered in [18] and [19]. In this work, we complete the analysis of the one-loop string partition function in the Schwarzschild-AdS background.
using a systematic treatment of both bosonic and fermionic worldsheet fluctuations along the lines of [16, 17]. On the gauge theory side, these results allow for a systematic treatment of the Wilson loop at a nonzero temperature.

In order to properly treat the string worldsheet in a Ramond-Ramond background, we use the Green-Schwarz action for $S[X, \theta]$ of (5). The Green-Schwarz action was originally formulated in a ten-dimensional Minkowski background in [20, 21] and subsequently generalized to the full IIB supergravity background in [22] using superspace techniques. While formally elegant, this IIB superspace formulated action is not so practical to work with. Instead, a complementary approach to constructing the full Green-Schwarz action in an AdS$_5 \times S^5$ background was taken in [23] using supercoset methods based on the symmetries of the background. Even in this background, the action is rather complicated, although simplifications may be obtained after gauge fixing the $\kappa$-symmetry [24, 25, 26].

Since we are interested in a Schwarzschild-AdS background, the construction of [23] is not directly applicable. However, all that is needed for obtaining the one-loop partition function is the form of the Green-Schwarz action up to quadratic order in the fermions. This was obtained in component form in [27] by dimensionally reducing and T-dualizing the eleven-dimensional supermembrane action. At quadratic order, the action is a straightforward generalization of [20, 21], using the pullback of the IIB supercovariant derivative in place of the ordinary covariant derivative. Furthermore, it is valid in any background satisfying the IIB equations of motion. Although the black hole background will break the supersymmetry, the $\kappa$ symmetry still holds trivially to quadratic order (and is expected to hold to all orders) in the fermions.

For the simple Wilson loops considered in this paper, a straight line and a pair of parallel lines, the gauge fixed ten-dimensional Green-Schwarz fermion action reduces to a sum of worldsheet fermionic actions of an equal mass that depends only on the worldsheet curvature. The masses of the bosonic fluctuations, on the other hand, receive nontrivial contributions from the Weyl tensor of the non-extremal metric [9]. However they do not add any new UV divergences. Upon computing the first quantum correction to the path integral, the partition function consists of a set of $1 \times 1$ functional determinants which may be evaluated by the recently suggested method of [28].

This paper is organized as follows. In the next section, we examine the relation between the fluctuations of the Nambu-Goto action and the Polyakov action for a general worldsheet,
and show that they are equivalent at the one-loop level. We furthermore reduce the ten-dimensional Green-Schwarz fermion action to a worldsheet fermion action. Following this general analysis, we turn to the computation of the Wilson loops. The partition function corresponding to a straight line Wilson loop is derived in section III and that corresponding to the parallel lines Wilson loop is derived in the section IV. In the final section, we examine the cancellation of the UV divergences of the functional determinants in the non-extremal background as well as the computation algorithm of these determinants and some open issues. Additional technical details on longitudinal bosonic fluctuations are given in Appendix A, and we give our fermion conventions in Appendix B.

Our notation largely follows that of [17]. Without a special declaration, Latin letters with hats, \( \hat{a}, \hat{b}, \hat{c}, \ldots \), labels the 10 Lorentz components of the target space metric, while Latin letters without hats, \( a, b, c, \ldots \), pertain to the AdS\(_5\) sector only. The Greek letters, \( \mu, \nu, \rho, \lambda, \ldots \), label the 10 dimensional curved-space coordinates. On the string worldsheet, the Latin letters \( i, j, k, l, \ldots \), label the curved-space coordinates, while the Greek letters, \( \alpha, \beta, \ldots \), stand for Lorentz indexes.

II. GENERAL FORMULATION

A careful study of the one-loop partition function of the Green-Schwarz superstring in \( \text{AdS}_5 \times S^5 \) was done by Drukker, Gross and Tseytlin in [17]. The basic procedure is to start with a classical string configuration, and then to develop the quadratic fluctuations about this classical worldsheet configuration. At this order, the bosonic and fermionic fluctuations decouple, and hence can be considered separately.

A. Bosonic contribution

The bosonic part of the string action describes the embedding \( X^\mu(\sigma^i) \) of a two-dimensional worldsheet (parameterized by \( \sigma^i \) with \( i = 0, 1 \)) into \( D \)-dimensional curved spacetime endowed with a metric

\[
ds^2 = G_{\mu\nu}dX^\mu dX^\nu.
\]
This embedding may be accomplished by either the Nambu-Goto or the Polyakov action. The Nambu-Goto action is given by

$$S_{NG}[X^\mu] = \frac{1}{2\pi \alpha'} \int d^2 \sigma \sqrt{|g|}$$  \hspace{1cm} (8)

where $g$ is the determinant of the induced metric

$$g_{ij} = G_{\mu\nu} \frac{\partial X^\mu}{\partial \sigma^i} \frac{\partial X^\nu}{\partial \sigma^j}. \hspace{1cm} (9)$$

In this case, the action principle directly corresponds to the extremization of the worldsheet volume, and is a straightforward generalization of the relativistic particle action.

The Nambu-Goto action is nonlinear and often difficult to work with because of the presence of the square root. For this reason, the approach of the Polyakov action is often preferred. This action is given by

$$S_P[g_{ij}, X^\mu] = \frac{1}{4\pi \alpha'} \int d^2 \sigma \sqrt{|g|} g^{ij} G_{\mu\nu} \frac{\partial X^\mu}{\partial \sigma^i} \frac{\partial X^\nu}{\partial \sigma^j}, \hspace{1cm} (10)$$

where this time the worldsheet metric $g_{ij}(\sigma^k)$ is taken to be independent of $X^\mu$. It is well known that the Nambu-Goto and Polyakov actions are equivalent at the tree-level. To see this, one notes that the classical equation of motion for $g_{ij}$ obtained from $S_P$ gives a result which is conformally equivalent to the induced metric (9). Substituting this back in then reproduces the Nambu-Goto action, (8).

It is generally expected that the classical equivalence of (8) and (10) will be maintained at the quantum level, although actually proving this equivalence does not appear to be straightforward. One-loop equivalence of the actions for a Minkowski target space was demonstrated [29]. Furthermore, both forms of the action were considered in [17] when computing the bosonic fluctuations of the superstring in a curved target space. For non-trivial string configurations, the results (when written in terms of bosonic determinants) do not appear manifestly identical, and hence equivalence of the actions implies the existence of hidden relations among the determinants.

Here we revisit the quadratic bosonic fluctuations considered in [17] and in addition demonstrate the equivalence of the Nambu-Goto and Polyakov actions at the one-loop order. We begin with the Nambu-Goto action (8) and first consider the background worldsheet-configuration. This is given by $\bar{X}^\mu(\sigma)$, which satisfies the classical equation of motion

$$g^{ij} \nabla_i n^\mu_j = 0,$$  \hspace{1cm} (11)
where $\eta^\mu_i \equiv \partial X^\mu / \partial \sigma^i$ is both a target space vector and a worldsheet vector. Its covariant derivative takes the form
\begin{equation}
\nabla_i \eta^\mu_j = \frac{\partial \eta^\mu_i}{\partial \sigma^i} + \Gamma^\mu_{\rho \lambda} \eta^\rho_i \eta^\lambda_j - \gamma^k_{ij} \eta^\mu_k
\end{equation}
with $\Gamma^\mu_{\rho \lambda}$ the Christoffel connection of the target space and $\gamma^k_{ij}$ that of the worldsheet. The background worldsheet metric reads
\begin{equation}
\bar{g}_{ij} = G_{\mu \nu} \eta^\mu_i \eta^\nu_j
\end{equation}
and the classical value of the action is simply the area of the minimal surface described by $\bar{X}^\mu(\sigma)$
\begin{equation}
S_0 = \frac{1}{2\pi \alpha'} \int d^2 \sigma \sqrt{\bar{g}}.
\end{equation}

We now consider the fluctuations $X^\mu = \bar{X}^\mu + \delta X^\mu$ about the classical background. In order to do so, we expand the Nambu-Goto action, \[ S_0 \] , to the quadratic order in $\delta X^\mu$.
\begin{equation}
\delta g_{ij} \equiv g_{ij} - \bar{g}_{ij} = \delta_1 g_{ij} + \delta_2 g_{ij}
\end{equation}
where $\delta_1 g_{ij}$ and $\delta_2 g_{ij}$ denote terms linear and quadratic in $\delta X^\mu$, respectively
\begin{equation}
\delta_1 g_{ij} = 2G_{\mu \nu} \eta^\mu_i \nabla_j \delta X^\nu,
\end{equation}
\begin{equation}
\delta_2 g_{ij} = G_{\mu \nu} \nabla_i \delta X^\mu \nabla_j \delta X^\nu - R_{\rho \mu \lambda \nu} \eta^\rho_i \eta^\lambda_j \delta X^\mu \delta X^\nu.
\end{equation}

Introducing the target space D-beins $G_{\mu \nu} = E_a^\mu E_b^\nu \eta_{ab}$ allows us to define target space tangent coordinates $\zeta^a = E_a^\mu \delta X^\mu$. In this case, the above fluctuations may be re-expressed as
\begin{equation}
\delta_1 g_{ij} = 2E^a_{\mu} \eta^\mu_i \nabla_j \zeta^a,
\end{equation}
\begin{equation}
\delta_2 g_{ij} = \nabla_i \zeta^a \nabla_j \zeta^a - R_{\rho \mu \lambda \nu} \eta^\rho_i \eta^\lambda_j E_a^\mu E_b^\nu \zeta^a \zeta^b.
\end{equation}

It follows that, to quadratic order in $\delta X^\mu$, the Nambu-Goto action becomes
\begin{equation}
S_{\text{NG}} = S_0 + S_{\text{NG}}^{(2)} = S_0 + \frac{1}{2\pi \alpha'} (I - J),
\end{equation}
where
\begin{align*}
I &= \frac{1}{2} \int d^2 \sigma \sqrt{\bar{g}} \bar{g}^{ij} \delta_2 g_{ij},
J &= \frac{1}{8} \int d^2 \sigma \sqrt{\bar{g}} (\bar{g}^{ik} \bar{g}^{jl} + \bar{g}^{il} \bar{g}^{jk} - \bar{g}^{ij} \bar{g}^{kl}) \delta_1 g_{ij} \delta_1 g_{kl}.
\end{align*}
Note that, at this quadratic order, we may symbolically write

\[ I = \frac{1}{2} \tilde{\zeta} A \zeta, \quad J = \frac{1}{2} \tilde{\zeta} A' \zeta, \]

(20)

where \( A \) and \( A' \) are \( D \times D \) matrices of functional operators.

We now decompose the fluctuation \( \delta X^\mu \) into longitudinal and transverse components, \( \delta_\text{l} X^\mu \) and \( \delta_\text{tr} X^\mu \), respectively, according to

\[ \delta_\text{l} X^\mu = \eta^\mu_k \epsilon^k, \quad G_{\mu \nu} \delta_\text{tr} X^\mu \delta_\text{l} X^\nu = 0. \]

(21)

Here \( \epsilon^k \) is a worldsheet vector specifying the two independent longitudinal fluctuations. It follows from (16) that a longitudinal fluctuation generates a world sheet diffeomorphism

\[ \delta_1 g_{ij} = \nabla_i \epsilon_j + \nabla_j \epsilon_i, \]

(22)

and hence is a zero mode of the fluctuation action (18) and (19). (See Appendix A for a detailed proof.) In this case, we may perform a local \( SO(1, D - 1) \) rotation \( \mathcal{R} \) to highlight the transverse fluctuations

\[ \begin{pmatrix} \zeta_\text{tr} \\ \zeta_\text{l} \end{pmatrix} = \mathcal{R} \zeta, \]

(23)

so that

\[ I - J = \frac{1}{2} \tilde{\zeta} (A - A') \zeta, \]

(24)

where

\[ A - A' = \mathcal{R}^{-1} \begin{pmatrix} A_\text{tr} & 0 \\ 0 & 0 \end{pmatrix} \mathcal{R}. \]

(25)

with \( A_\text{tr} \) a \( (D - 2) \times (D - 2) \) matrix of functional operators. This highlights the well known feature that only the \( D - 2 \) transverse degrees of freedom are physical in the fundamental string action.

Having examined the Nambu-Goto form of the string action, we now turn to the Polyakov action, (10). In this case, the solution to the equations of motion gives rise to the same worldsheet embedding profile \( \bar{X}^\mu \) and, up to a conformal factor, the same induced metric \( \bar{g}_{ij} \). In conformal gauge, where the fluctuation of \( g_{ij} \) is restricted to \( \delta g_{ij} \propto \bar{g}_{ij} \), the expansion of the Polyakov action to quadratic order in \( \delta X^\mu \) yields

\[ S_P = S_0 + S_P^{(2)} = S_0 + \frac{1}{2 \pi \alpha'} I, \]

(26)
where $I$ was given previously in (19). At this point, we note that the quadratic fluctuations of the Nambu-Goto and Polyakov actions differ in that $S_{\text{NG}}$ contains the additional $J$ given in (19). On the other hand, gauge fixing the Polyakov action gives rise to a Fadeev-Popov ghost determinant, which is not present in the above treatment of the Nambu-Goto action. In order to demonstrate the equivalence of these two formulations, we must show that evaluation of this ghost determinant gives the same result as the addition of the $J$ term in $S_{\text{NG}}$.

Accounting for the ghosts, the partition function for bosonic fluctuations in the Polyakov action is given by the path integral

$$Z_{\text{bs}} = (\det A_{\text{gh}})^{1/2} \int [d\zeta] e^{iS_{\text{P}}^{(2)}} = (\text{const.}) \frac{(\det A_{\text{gh}})^{1/2}}{(\det A)^{1/2}},$$

where the $2 \times 2$ ghost operator is

$$A_{\text{gh}} = -\nabla^j \nabla_j - \frac{1}{2} R^{(2)}.$$  

Here $\nabla_j$ is the covariant derivative acting on a worldsheet Lorentz vector and $R^{(2)}$ is the worldsheet scalar curvature. Note that we have dropped the path integral over the conformal factor, which only decouples in the critical dimension.

For simple embeddings, such as the minimal surface corresponding to a straight Wilson line or that of a circular Wilson loop on the AdS$_5$ boundary, explicit computations indicate that the bosonic determinant factorizes according to

$$\det \mathcal{A} = \det A_{\text{tr}} \det A_{\text{gh}}.$$  

This, however, is not obvious with more complicated embeddings, such as the worldsheet generated by parallel lines on the boundary. In the rest of this subsection, we give an explicit demonstration that this is the case in general. Moreover, we note that this factorization does not require a simple relationship among the eigenvalues of $\mathcal{A}$, $A_{\text{tr}}$, and $A_{\text{gh}}$.

To begin with, we note that $J$, given in (19), depends only on the traceless part of $\delta_1 g_{ij}$. In other words,

$$J = \frac{1}{4} \int d^2 \sigma \sqrt{|\bar{g}|} \bar{g}^{ij} \bar{g}^{kl} h_{ij} h_{kl},$$

where $h_{ij}$ is the traceless part of $\delta_1 g_{ij}$, given by

$$h_{ij} \equiv \delta_1 g_{ij} - \frac{1}{2} \bar{g}_{ij} g^{kl} \delta_1 g_{kl}.$$
Decomposing $\delta_1 g_{ij}$ into longitudinal and transverse components

$$
\delta_1 g_{ij} = \nabla_i \epsilon_j + \nabla_j \epsilon_i + \delta g_{ij}^{\text{tr}},
$$

allows us to write

$$
h_{ij} = \nabla_i \epsilon_j + \nabla_j \epsilon_i - \bar{g}_{ij} \bar{g}^{kl} \nabla_k \epsilon_l + h_{ij}^{\text{tr}},
$$

where $h_{ij}^{\text{tr}}$ are both transverse and traceless. While there are $D-2$ transverse components $\zeta_{\text{tr}}^a$, we note that there are only two independent components of $h_{ij}^{\text{tr}}$. In this case, we may find a worldsheet vector $\chi^k$ that satisfies

$$
\nabla_i \chi_j + \nabla_j \chi_i - \bar{g}_{ij} \bar{g}^{kl} \nabla_k \chi_l = h_{ij}^{\text{tr}}.
$$

The formal solution may be expressed as

$$
\chi^i = B^i_a \zeta_{\text{tr}}^a,
$$

where $B$ is a nonlocal linear operator representing the inversion of (34). We thus end up with

$$
J = \frac{1}{2} \int d^2 \sigma \sqrt{|g|} \left( \nabla_i u^i \nabla^i u_j - \frac{1}{2} R^{(2)} u^i u_i \right) = \frac{1}{2} \int d^2 \sigma u^i A_{\text{gh}} (u_i),
$$

where $u^i = \epsilon^i + \chi^i$.

Following the definition of (19), the above expression for $J$ implies that the matrix operator $A'$ has the form

$$
A' = \mathcal{R}^{-1} \begin{pmatrix} 1 & \hat{B} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ A_{\text{gh}} & B \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathcal{R}.
$$

By adding (25) and (37), we may obtain an expression for $A$

$$
A = \mathcal{R}^{-1} \begin{pmatrix} 1 & \hat{B} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_{\text{tr}} & 0 \\ 0 & A_{\text{gh}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathcal{R}.
$$

As a result, the factorization (29) follows, and we are left with the bosonic partition function

$$
Z_{\text{bs}} = (\text{const.}) (\det A_{\text{tr}})^{-1/2},
$$

This result is independent of whether the Nambu-Goto or the Polyakov action was taken as the starting point of the computation. Finally, we note that the nonlocal operator $B$ vanishes for the long string and circular loop configurations. This is the reason why the factorization (29) was already manifest for such simple embeddings. However the operator $B$ may be highly nontrivial for generic string configurations. In such cases, we often find it more convenient to work with the Nambu-Goto action, where we do not have to concern ourselves with the added complication of gauge fixing and ghosts.
B. Fermionic contribution

For the bosonic fluctuations, we have shown above that the equivalence of working with the Nambu-Goto and Polyakov actions holds for arbitrary target spacetimes. We note, however, that because of the UV divergence, the path integral over the conformal factor only decouples in the critical dimension. This is not a concern for us since our primary interest is to consider the superstring in the critical dimension, and in particular in a Schwarzschild AdS$_5 \times S^5$ background.

To calculate the fermionic contribution to the one-loop partition function, we work with the Green-Schwarz action up to quadratic order in the fermions. Turning on only a background metric and 5-form field strength, the quadratic action takes the form

$$S_F = \frac{i}{2\alpha'} \int d^2\sigma \sqrt{|g|} \left[ \bar{\theta}^I P^{ij}_- \eta^I_\mu (D_j \theta)^I_\mu + \bar{\theta}^I P^{ij}_+ \eta^I_\mu (D_j \theta)^I_\mu \right],$$

(40)

where the fermionic coordinates $\theta^I (I = 1, 2)$ are 16 component positive chirality Majorana-Weyl spinors. We have also introduced the worldsheet projection tensors

$$P^{ij}_\pm = \bar{g}^{ij} \pm \frac{\epsilon^{ij}}{\sqrt{|g|}},$$

(41)

and the pullback of the IIB supercovariant derivative

$$D_i^{IJ} = \delta^{IJ} D_i + \frac{1}{16 \cdot 5!} \epsilon^{IJ}_{\nu_1 \cdots \nu_5} \Gamma_\nu_1 \cdots \nu_5 \eta^I_\mu \eta^J_\mu,$$

(42)

where

$$D_i = \frac{\partial}{\partial \sigma^i} + \frac{1}{4} \eta^{I}_\mu \Omega^a_\mu \Gamma_a \hat{a}.$$

(43)

Our notation follows that of [17, 23], and is detailed in Appendix B.

The $S^5$ reduction of the IIB theory is given by

$$d_{10}^2 = d_5^2 + L^2 d\Omega_5^2, \quad F_5 = \frac{4}{L} (1 + \ast) d\Omega_5,$$

(44)

and leads to a natural $5 + 5$ split of spacetime. This motivates a split of the ten-dimensional Dirac matrices into two sets of commuting $16 \times 16$ Dirac matrices, $\gamma^a$ and $\gamma^a'$, satisfy the relations

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}, \quad \{\gamma^a', \gamma^b'\} = 2\delta^{a'b'},$$

(45)
where \( a, b \) are local Lorentz indexes on \( \text{AdS}_5 \), and \( a', b' \) that on \( S^5 \). We now introduce worldsheet pullbacks of the Dirac matrices

\[
\begin{align*}
\rho_i &= \eta^\mu_i (E_a^\mu \gamma_a + i E_a'^\mu \gamma_a'), \\
\bar{\rho}_i &= \eta^\mu_i (E_a^\mu \gamma_a - i E_a'^\mu \gamma_a').
\end{align*}
\]

It follows from (45) that

\[
\bar{\rho}_i \rho_j + \bar{\rho}_j \rho_i = 2 \bar{g}_{ij},
\]

where, as we recall, \( \bar{g}_{ij} \) is the background worldsheet metric.

Making this 5 + 5 split explicit, we find that the action (40) takes the form

\[
S_F = \frac{i}{2\pi \alpha'} \int d^2 \sigma \sqrt{|\bar{g}|} [\bar{\theta}^I P_{ij} \rho_i (D_j \theta)^{1} + \bar{\theta}^2 P_{ij} \rho_i (D_j \theta)^2]
\]

where the Majorana condition is

\[
\bar{\theta}^I = \tilde{\theta}^I C \times C.
\]

Here \( C \) is the \( 4 \times 4 \) charge conjugation matrix (see Appendix B) and is a worldsheet scalar. The covariant derivative is given by

\[
D^{IJ}_j = \delta^{IJ} D_j - \frac{i}{2L} \epsilon^{IJ} \rho_j,
\]

where

\[
D_j = \frac{\partial}{\partial \sigma^j} + \frac{1}{4} \epsilon^\mu_j (\Omega_{\mu}^{ab} \gamma_{ab} + \Omega_{\mu}^{a'b'} \gamma_{a'b'}),
\]

with \( \Omega_{\mu}^{ab} \) and \( \Omega_{\mu}^{a'b'} \) the spin connections of the target spacetime. Note that, to quadratic order in the fermions, this action has the same functional form as that obtained in [23] in the vacuum \( \text{AdS}_5 \times S^5 \) background. Of course, the full Green-Schwarz action in a non-trivial background is rather complicated, and it is not at all clear whether the higher order in fermion terms would continue to match up. One way to address this question may be to return to the IIB superspace formulation of the Green-Schwarz action in [22]. However, for the present work it is sufficient to note that the quadratic action (40) with the supercovariant derivative given by (42) is valid in an arbitrary 5-form background.

To proceed, we note that the fermionic action (48) is invariant under the \( \kappa \)-transformation

\[
\delta \theta^I = 2i \bar{\rho}_j \kappa^I_j,
\]
where $\kappa^{ij}$ is a local spacetime spinor and a worldsheet vector satisfying the duality relations

$$P_\pm^{ij}\kappa_j^1 = \kappa_i^1, \quad P_\pm^{ij}\kappa_j^2 = \kappa_i^2.$$  

(53)

This can be verified with the aid of (47) and (49) and the properties of the projection tensors

$$P_{ij} = P_{ji}, \quad P_{ik} P_{jl} = P_{il} P_{jk}.$$  

(54)

We now fix the $\kappa$-symmetry by choosing the gauge $\theta^1 = \theta^2 \equiv \theta$. The action (48) then reduces to

$$S_F = \frac{i}{2\pi \alpha'} \int d^2\sigma (2\sqrt{|\bar{g}|} \bar{g}^{ij}\bar{\theta}_i \rho_j \bar{D}_j \theta - \frac{i}{L} \epsilon^{ij}\bar{\theta}_i \rho_j \theta).$$  

(55)

We furthermore consider only string worldsheets that are embedded in Schwarzschild AdS$_5$ (and which are localized on a point in $S^5$). In this case, the worldsheet Dirac matrices reduce to

$$\bar{\rho}_i = \rho_i = \eta_i^\mu E_{a}^{\gamma\alpha}.$$  

(56)

As is natural in the Green-Schwarz formulation, the action (55) is for spacetime fermions $\theta$. To reduce this to an action for worldsheet fermions, we employ a local rotation to line up the first two fünfbeins along the worldsheet directions

$$\eta_i^\mu E_{a}^{\gamma\alpha} = e_i^\alpha, \quad \eta_i^\mu E_k^p = 0,$$  

(57)

with $\alpha = 0, 1$ and $p = 2, 3, 4$. In this case

$$\bar{g}^{ij}\eta_i^\mu \eta_j^\nu = G^{\mu\nu} - E_p^{\mu} F_p^{\mu\nu}.$$  

(58)

In what follows, we shall use Greek letters $\alpha, \beta$ for the first two Lorentz indexes, and Latin letters $p, q, r, s$ for the remaining three. Introducing the worldsheet projection of the target space spin connection, $\Omega_{ij}^{ab} \equiv \eta_j^\mu \Omega_{ij}^{a\mu}$, it is straightforward to verify that

$$\Omega_j^{\alpha\beta} = \omega_j^{\alpha\beta},$$  

(59)

where $\omega_j^{\alpha\beta}$ is the ordinary worldsheet spin connection. Furthermore, we have $\rho_j = e_j^\alpha \gamma_\alpha$, and the Dirac operator may be rewritten as

$$\bar{g}^{ij}\rho_i \bar{D}_j = e^{\alpha j} \gamma_\alpha \bar{D}_j' + D_1 + D_2.$$  

(60)

Here

$$\bar{D}_j' = \frac{\partial}{\partial \xi^j} + \frac{1}{4} \omega_j^{\alpha\beta} \gamma_\alpha \gamma_\beta,$$  

(61)
and the additional terms $D_1$ and $D_2$ of (60) are given by

$$D_1 = 2e^{\alpha j} \Omega^{\beta p} \gamma_\alpha \gamma_\beta \gamma_p, \quad D_2 = e^{\alpha j} \Omega^{\beta p} \gamma_\alpha \gamma_p. \quad (62)$$

In general, the $D_1$ and $D_2$ terms could be nontrivial. However, for the class of minimal surfaces that we are interested in, the spin connections satisfy

$$\Omega^{\mu \nu} = 0, \quad (63)$$

as well as the symmetry property

$$e^{\alpha j} \Omega^{\beta p} = e^{\beta j} \Omega^{\alpha p}. \quad (64)$$

It immediately follows that $D_2 = 0$ and that $D_1 = 2e^{\alpha j} \Omega^{\beta p} \gamma_p$. Working out the covariant derivative of $\eta^\mu E^a_\mu$, and using the static gauge property (57) as well as the background equation of motion (11), we may then show that $D_1 = 0$.

Under the simplification $D_1 = D_2 = 0$, the fermionic action reduces to

$$S_F = \frac{i}{2\pi \alpha'} \int d^2 \sigma \sqrt{|g|} (2e^{\alpha j} \bar{\psi} \gamma_\alpha D_j \theta - i L \bar{\psi} \sigma_3 \psi), \quad (65)$$

We now take the representation where $\gamma_0 = -i \sigma_1 \times I_8$ and $\gamma_1 = \sigma_2 \times I_8$ (with $\sigma_1$ and $\sigma_2$ being the ordinary Pauli matrices). Decomposing the $16 \times 1$ Majorana spinor $\theta$ into eight $2 \times 1$ real spinors $\psi_n$, we finally obtain the worldsheet fermion action

$$S_F = \frac{i}{\pi \alpha'} \sum_{n=1}^8 \int d^2 \sigma \sqrt{|g|} (e^{\alpha j} \bar{\psi} \tau_0 \hat{\nabla}_j \psi_n - i L \bar{\psi} \sigma_3 \psi_n), \quad (66)$$

where $\tau_0 = i \sigma_1$, $\tau_1 = \sigma_2$ and the worldsheet covariant derivative is

$$\hat{\nabla}_j = \frac{\partial}{\partial \sigma_j} + \frac{1}{4} \omega^{\alpha \beta}_j \tau_{\alpha \beta}. \quad (67)$$

The fermionic path integral then gives

$$Z_{fm} = \int [d\theta] e^{iS_F} = \text{(const.)}(\det A_F)^4 = \text{(const.)}(\det A_F^2)^2, \quad (68)$$

where $A_F$ is the $2 \times 2$ operator

$$A_F = e^{\alpha j} \tau_0 \hat{\nabla}_j - \frac{i}{L} \sigma_3, \quad (69)$$

with

$$A_F^2 = -\hat{\nabla}_j \hat{\nabla}_j + \frac{1}{4} R^{(2)} + \frac{1}{L^2}. \quad (70)$$
Combining the bosonic and fermionic contributions (39) and (68), we now obtain the complete one-loop partition function

$$Z = Z_{bs} Z_{fm} = \left(\det A_2^2\right)^2 \left(\det A_{tr}\right)^{1/2}.$$  (71)

In the next two sections, we will examine this partition function for the cases of the long string and parallel lines Wilson loops. Before proceeding, however, we note that for a general embedding, the conditions (63) and (64) may not be satisfied. An example where $\Omega_{pq} \neq 0$ is given by the light-like parallel lines configuration underlying the AdS/CFT dual calculation of the jet-quenching parameter measured at RHIC. In such cases, we would end up with a fermionic action of the form

$$S_F = \frac{i}{\pi \alpha'} \sum_{n,n'} \int d^2 \sigma \bar{\psi}_n \left[ (\bar{e}^{\alpha j} \tau_\alpha \nabla_j - \frac{i}{L} \sigma_3) \delta_{nn'} + M_{nn'} \right] \psi_{n'}.$$  (72)

### III. A LONG STRING

In this section, we evaluate the first quantum correction to the long string solution in a Schwarzschild-AdS$_5$ background. To be concrete, we take the spacetime metric to be given by

$$ds^2 = \frac{1}{z^2} \left[ -f \, dt^2 + d\vec{X}^2 + \frac{dz^2}{f} \right] + d\Omega_5^2,$$  (73)

where

$$f = 1 - \frac{z^4}{z_h^4}.$$  (74)

(Note that we have set the AdS$_5$ radius to unity, i.e. $L = 1$.) This metric corresponds to a planar horizon black hole in Poincaré coordinates. This is of course the appropriate gravity dual of $\mathcal{N} = 4$ super-Yang-Mills gauge theory on $\mathbb{R}^1 \times \mathbb{R}^3$ or $S^1 \times \mathbb{R}^3$, where $S^1$ of the latter corresponds to the Euclidean time of period $1/(\text{temperature}) = \pi z_h$. The $z$ coordinate is such that $z = 0$ corresponds to the boundary of AdS$_5$ and $z = z_h$ the horizon.

The Riemann tensor in the AdS$_5$ sector may be written as

$$R_{\mu\nu\rho\lambda} = -(G_{\mu\rho} G_{\nu\lambda} - G_{\mu\lambda} G_{\nu\rho}) + C_{\mu\nu\rho\lambda},$$  (75)

where the nonzero components of the Weyl tensor are (up to symmetries)

$$C_{00ij} = \frac{f}{z_h^4} \delta_{ij}, \quad C_{44ij} = -\frac{1}{z_h^4 f} \delta_{ij},$$

$$C_{0404} = -\frac{3}{z_h^4}, \quad C_{ijkl} = \frac{1}{z_h^4} (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}).$$  (76)
Here $i, j, k, l$ labels the coordinates $X^1, X^2$ and $X^3$, while $X^0 = t$ and $X^4 = z$.

When computing both bosonic and fermionic fluctuations, we introduce the natural fünfbeins of the spacetime metric (73)

\[
E^0 = \frac{1}{z} \sqrt{f} dt, \quad E^1 = \frac{1}{z} dX^1, \quad E^2 = \frac{1}{z} dX^2, \quad E^3 = \frac{1}{z} dX^3, \quad E^4 = \frac{1}{z} \sqrt{f} dz. \tag{77}
\]

Since the worldsheet configurations that we are interested in reside at a single point in $S^5$, we shall describe the fluctuations in $S^5$ in terms of the cartesian coordinates of the $\mathbb{R}^5$ tangent space to this point. These coordinates will be labeled by the letter $s$ with $s = 5, 6, 7, 8, 9$, and the corresponding fünfbeins $E^s_\mu$ correspond to those of a unit radius $S^5$ (corresponding to our setting the AdS$_5$ radius to unity). Corresponding to this choice of fünfbeins, the nontrivial components of the spacetime spin connection in the AdS$_5$ sector are

\[
\Omega^0_{\tau} = -\frac{1}{z} \left( 1 + \frac{z^4}{z_h^4} \right), \quad \Omega^0_{\sigma} = -\frac{\sqrt{f}}{z} \delta^0_s. \tag{78}
\]

with $f$ is given in (74). The spin connection in the $S^5$ sector vanishes.

**A. The classical solution**

The long straight string corresponds to a worldsheet oriented along the time and radial ($z$) directions. It is a trivial solution of the classical equation of motion (11). In static gauge, we take

\[
\bar{X}^\mu(\tau, \sigma) = (\tau, 0, 0, 0, \sigma; 0, 0, 0, 0, 0). \tag{79}
\]

The induced worldsheet metric is given simply by the $t$ and $z$ components of (73)

\[
ds^2 = \frac{1}{\sigma^2} \left( -f d\tau^2 + \frac{1}{f} d\sigma^2 \right), \tag{80}
\]

where $f = 1 - \sigma^4/z_h^4$, and the corresponding worldsheet curvature scalar is

\[
R^{(2)} = -2 \left( 1 - \frac{3\sigma^4}{z_h^4} \right). \tag{81}
\]

**B. Bosonic fluctuations**

To compute the bosonic fluctuations about the classical embedding (79), we introduce the fluctuation coordinates $\xi^\mu(\tau, \sigma) \equiv \delta X^\mu(\tau, \sigma)$ along with their tangent space counterparts
\( \zeta^a = E^a_{\mu} \xi^\mu \). Here, \( \zeta^0 \) and \( \zeta^4 \) are longitudinal and may be set to zero in static gauge. Adding the transverse fluctuations to the background configuration (79), we have

\[
X^\mu(\tau, \sigma) = (\tau, \xi^1(\tau, \sigma), \xi^2(\tau, \sigma), \xi^3(\tau, \sigma), \sigma; \xi^4(\tau, \sigma)) = (\tau, \sigma \xi^1(\tau, \sigma), \sigma \xi^2(\tau, \sigma), \sigma \xi^3(\tau, \sigma), \sigma; \xi^4(\tau, \sigma)).
\] (82)

This gives

\[
dX^\mu = (d\tau, \dot{\xi}^1 d\tau + \xi^1' d\sigma, \dot{\xi}^2 d\tau + \xi^2' d\sigma, \dot{\xi}^3 d\tau + \xi^3' d\sigma, \dot{\xi}^4 d\tau + \xi^4' d\sigma),
\] (83)

where \( \dot{\cdot} = \partial / \partial \tau \) and \( \cdot' = \partial / \partial \sigma \). Substituting this into the Nambu-Goto action and expanding to the second order, we obtain

\[
S_{NG}^{(2)} = \frac{1}{4\pi \alpha'} \int d^2 \sigma \sqrt{|\bar{g}|} \left[ \sum_{a=1}^3 \left( \bar{g}^{\tau \tau} (\dot{\xi}^a)^2 + \bar{g}^{\sigma \sigma} (\xi^a')^2 + M^2 (\xi^a)^2 \right) + \sum_{s=5}^9 \left( \bar{g}^{\tau \tau} (\xi^s)^2 + \bar{g}^{\sigma \sigma} (\xi^s')^2 \right) \right],
\] (84)

where we have assumed Dirichlet boundary conditions and have dropped the surface terms associated with integration by parts. The metric coefficients \( \bar{g}^{\tau \tau} \) and \( \bar{g}^{\sigma \sigma} \) refer to the line element (80). The (position dependent) ‘mass’ of the three fluctuation coordinates in the AdS\(_5\) sector reads

\[
M^2 = 2 \left( 1 + \frac{\sigma^4}{z_h^4} \right) = \frac{8}{3} + \frac{R^{(2)}}{3},
\] (85)

while the fluctuations on \( S^5 \) have vanishing worldsheet masses. It is now clear that the operator \( A_{tr} \) of (25), which governs the bosonic fluctuations, takes the form

\[
A_{tr} = \begin{pmatrix}
\left( -\nabla^2 + \frac{8}{3} + \frac{1}{3} R^{(2)} \right) I_3 & 0 \\
0 & -\nabla^2 I_5
\end{pmatrix}.
\] (86)

We may obtain the vacuum AdS solution by taking the limit \( z_h \to \infty \). In this case, \( R^{(2)} \to -2 \), and this reproduces the result of [17] for the BPS configuration of the long straight string in AdS.

### C. Fermionic fluctuations

Turning to the fermionic sector, since the worldsheet is oriented along the 0 and 4 directions, the pullback of the fünfbeins (77) to the worldsheet according to (57) gives the zweibeins

\[
e^0 = \frac{\sqrt{f}}{\sigma} d\tau, \quad e^1 = \frac{1}{\sigma \sqrt{f}} d\sigma.
\] (87)
In addition, (59) yields the worldsheet spin connection

$$\omega^0_1 = \Omega^0_4, \quad \omega^0_\sigma = 0,$$

(88)

where $$\Omega^0_4$$ is given by (78) with $$z$$ replaced by $$\sigma$$. It follows from (51) that

$$D_\tau = \frac{\partial}{\partial \tau} - \frac{1}{2\sigma} \left( 1 + \frac{\sigma^4}{z_h^4} \right) \gamma^0\gamma^4 = \hat{\nabla}_\tau \times I_8,$$

$$D_\sigma = \frac{\partial}{\partial \sigma} = \hat{\nabla}_\sigma \times I_8,$$

(89)

where $$\hat{\nabla}_i$$ is the worldsheet covariant derivative acting on spinors. In other words, the additional $$D_1$$ and $$D_2$$ terms of (60) are absent, and the fermionic operator is given simply by $$A_F$$ of (69).

Finally, substituting the bosonic (86) and fermionic (69) operators into (71), we obtain the one-loop partition function for the long straight string in a Schwarzschild-AdS$_5$ background

$$Z = \frac{\det^2(-\nabla^2 + 1 + \frac{1}{4}R^{(2)})}{\det^2(-\nabla^2 + \frac{8}{3} + \frac{1}{3}R^{(2)}) \det^2(-\nabla^2)} = \frac{\det^2(-\nabla^2_+ + 1 + \frac{1}{4}R^{(2)})\det^2(-\nabla^2_- + 1 + \frac{1}{4}R^{(2)})}{\det^2(-\nabla^2 + \frac{8}{3} + \frac{1}{3}R^{(2)}) \det^2(-\nabla^2)},$$

(90)

where we have factorized the fermionic determinant into its two diagonal (chiral) components, i.e. $$\hat{\nabla}^2 = \text{diag}(\nabla^2_+, \nabla^2_-)$$. We have

$$\nabla^2_\pm = \tilde{g}^{\tau\tau} \left( \frac{\partial}{\partial \tau} \pm \frac{1}{2} \omega^0_\tau \right)^2 + \frac{1}{\sqrt{|\tilde{g}|}} \frac{\partial}{\partial \sigma} \left( \sqrt{|\tilde{g}|} g^{\sigma\sigma} \frac{\partial}{\partial \sigma} \right),$$

(91)

with $$\tilde{g}^{ij}$$ given by (80) and $$\omega^a_\beta$$ by (88).

IV. THE PARALLEL LINES WILSON LOOP

The long straight string considered in the previous section corresponds to the minimal surface extending from a single static quark into the bulk of AdS$_5$. The standard AdS/CFT computation of the quark-quark potential involves a ‘parallel lines’ Wilson loop where a string worldsheet is stretched between two quarks on the AdS boundary. In this section, we begin with an overview of the classical solution, and then turn to the first quantum corrections.
A. The classical solution

For the parallel lines Wilson loop, we choose the quarks to be separated in the $X^1$ direction in the spacetime metric of (73). The string minimal surface then stretches in the time direction and forms a curve in the $X^1$-$z$ plane. We choose the gauge

$$\bar{X}^\mu(\tau, \sigma) = (\tau, \sigma, 0, 0, z(\sigma); 0, 0, 0, 0),$$

(92)

where $z(\sigma)$, which specifies the string profile, may be determined by the classical equation of motion (11). To calculate the induced metric, we note that

$$d\bar{X}^\mu = (d\tau, d\sigma, 0, 0, z'd\sigma; 0, 0, 0, 0).$$

(93)

Using (73) for the spacetime metric, we find

$$ds^2 = \frac{1}{z^2} \left[ -f\, d\tau^2 + \left( 1 + \frac{z'^2}{f} \right) d\sigma^2 \right].$$

(94)

The Nambu-Goto action then takes the form

$$S_{NG} = \frac{T}{2\pi\alpha'} \int d\sigma \sqrt{f/z^4 + z'^2/z^4},$$

(95)

where $T$ is the time period.

The equation of motion for $z(\sigma)$ following from the above action is

$$z'' = \frac{f_z}{2} - \frac{2f}{z} - \frac{2}{z^2} z'^2 + \frac{f_z}{f} z'^2,$$

(96)

where

$$f_z = \frac{\partial f}{\partial z} = -4 \frac{z^3}{z^4_h}.$$  

(97)

Although this equation is rather non-trivial, it admits a first integral

$$z'^2 + f = \frac{z_0^4 f^2}{f_0 z^4},$$

(98)

Here $z_0$ is the maximum value of $z$ reached by the minimal surface. This corresponds to the value of $z$ where the string makes its closest approach to the horizon. We have also defined $f_0 = f(z_0) = 1 - z_0^4/z^4_h$. This allows us to write the on-shell induced metric as

$$ds^2 = \frac{1}{z^2} \left[ -f\, d\tau^2 + \frac{z_0^4 f^2}{f_0 z^4} d\sigma^2 \right].$$

(99)
where we ought to keep in mind that \( z(\sigma) \) is a solution to the first order equation (98) with the Dirichlet conditions \( z(\tau, -r^2) = z(\tau, r^2) = 0 \). The curvature scalar for this induced metric reads

\[
R^{(2)} = \frac{4(f - 2) + 2z^2(2 - 3f)}{(f + z^2)} = 2 \left[ \frac{4(z_h^4 - z_0^4)}{z_0^4 f^2} - \frac{4(z_h^4 - z_0^4)}{z_0^4 f} + \frac{z_0^4 - 3z_h^4}{z_h^4} + \frac{z_h^4}{z_0^4} \right].
\]

(100)

B. Bosonic fluctuations

To compute the bosonic fluctuations about the classical solution, we introduce the fluctuation coordinates

\[
\xi^\mu(\tau, \sigma) = \delta X^\mu(\tau, \sigma)
\]

and their tangent space projections \( \zeta^\alpha = \mathcal{E}_\mu^\alpha \xi^\mu \). While the string is oriented along the 0, 1 and 4 directions in the tangent space defined by the fünfbeins \( (\mathcal{E}_\mu^a) \), a simple rotation in the 1-4 plane \( [16] \)

\[
\begin{pmatrix}
\mathcal{E}_\mu^1 \\
\mathcal{E}_\mu^4
\end{pmatrix}
= \begin{pmatrix}
\cos \phi & \sin \phi \\
-\sin \phi & \cos \phi
\end{pmatrix}
\begin{pmatrix}
E_\mu^1 \\
E_\mu^4
\end{pmatrix},
\]

\[
\mathcal{E}_\mu^\alpha = E_\mu^\alpha \quad \text{for} \quad \hat{a} \neq 1, 4,
\]

(102)

with \( \tan \phi = z'/\sqrt{f} \), allows us to align the longitudinal worldsheet directions with the 0 and 1 directions in the modified fünfbeins basis defined by \( \mathcal{E}_\mu^\alpha \). In particular, it follows that

\[
\eta_\mu^a \mathcal{E}_\mu^\alpha = 0, \quad \hat{a} = 2, 3, \ldots, 9,
\]

(103)

where

\[
\eta_0^\mu = (1, 0, 0, 0, 0, 0, 0, 0, 0),
\eta_1^\mu = (0, 1, 0, 0, z', 0, 0, 0, 0),
\]

(104)

are the tangent vectors on the worldsheet. Therefore a longitudinal fluctuation takes the form

\[
\xi_i^\mu = (z\sqrt{f}\zeta^0, z\zeta^1 \cos \phi, 0, 0, z\sqrt{f}\zeta^1, 0, 0, 0, 0, 0)
\]

(105)

and is a zero mode of the quadratic action of the fluctuations as is shown in the appendix A. A transverse fluctuation reads

\[
\xi_{tr} = (0, -z\zeta^4 \sin \phi, z\zeta^2, z\zeta^3, z\sqrt{f}\zeta^1 \cos \phi, \xi^*)
\]

(106)
and represent a physical degree of freedom. Without losing generality, we choose the static gauge by taking the superposition of $\xi^\mu = \xi_{tr}^\mu + \xi_t^\mu$, such that $\xi^0 = \xi^1 = 0$. This amounts to setting $\zeta^0 = 0$ and $\zeta^1 = \zeta^4 \tan \phi$ and the amount of computation is reduced.

Including the fluctuations parameterized this way, the classical embedding profile (92) is modified to

$$X^\mu(\tau, \sigma) = (\tau, \sigma, \xi^2(\tau, \sigma), \xi^3(\tau, \sigma), z(\sigma) + \xi^4(\tau, \sigma), \xi^s(\tau, \sigma))$$

with

$$\xi^2 = z \zeta^2, \quad \xi^3 = z \zeta^3, \quad \xi^4 = z \sqrt{f + z'^2 \zeta^4} \quad \text{and} \quad \xi^s = \zeta^s$$

so that

$$dX^\mu = (d\tau, d\sigma, \dot{\xi^2} d\tau + \xi^2' d\sigma, \dot{\xi^3} d\tau + \xi^3' d\sigma, \dot{\xi^4} d\tau + (z' + \xi^4')d\sigma; \dot{\xi^s} d\tau + \xi^s' d\sigma).$$

In order to evaluate the Nambu-Goto action to second order in the fluctuations, we must keep in mind that the $X^4$ coordinate is now $X^4 = z + \xi^4$. This means that the metric function $f(X^4)$ ought to be expanded as

$$f(X^4) = f(z + \xi^4) = f + \xi^4 f_z + \frac{1}{2} (\xi^4)^2 f_{zz} + \cdots,$$

where for simplicity $f$ always denotes the function $f(z)$ given by (14) unless $f(X^4)$ is indicated explicitly. Expanding $\sqrt{|g|}$ to quadratic order in the small fluctuations is tedious but straightforward. The result is

$$\sqrt{-g} = \frac{1}{z^2} \sqrt{f + z'^2} \left[ 1 + \frac{1}{2} \xi^4 \left( \frac{f_z}{f + z'^2} - \frac{4}{z} \right) + \frac{f}{2(f + z'^2)^2} (\xi^4)^2 \right]$$

$$+ \frac{1}{2f(f + z'^2)} ((\xi'^a)^2 + z^2 (\xi'^s)^2) + \frac{1}{2f(f + z'^2)^2} (\xi^4)^2$$

$$- \frac{1}{2f} ((\xi'^a)^2 + z^2 (\xi'^s)^2) - \frac{1}{2f(f + z'^2)} (\xi^4)^2$$

$$+ (\xi^4)^2 \left( \frac{3}{z^2} + \frac{f_{zz}/4 - f_z/z}{f + z'^2} - \frac{f_z^2}{8(f + z'^2)^2} \right)$$

$$+ \xi^4 \xi^4' \left( - \frac{2z'}{z(f + z'^2)} - \frac{f_{zz}'/2}{2(f + z'^2)^2} \right).$$

The $a$ index corresponds to the 2 and 3 directions, while the $s$ index denotes tangent-space indices on $S^5$.

After an integration by parts, the linear term in $\xi^4$ gives the equation of motion (96). The quadratic terms are the ones that we are interested in. Upon substitution of (108) and
integration by parts for the terms containing $\xi^a \xi^a$ or $\xi^4 \xi^4$, we end up with

$$S_{NG}^{(2)} = \frac{1}{4\pi \alpha'} \int d^2 \sigma \sqrt{|g|} \sum_{a=2}^{3} \left[ \tilde{g}^{\tau \tau} (\dot{\zeta}^a)^2 + g^{\sigma \sigma} (\zeta^a)^2 - \frac{z}{2} \left( f_z - \frac{4f}{z} + \frac{f_z z'^2}{f + z'^2} \right) (\zeta^a)^2 \right]$$

$$+ \frac{z}{2 f + z'^2} \left( f f_z - f_z z + \frac{4f z'^2}{z} - 2f z'^2 + f f_z z \right) (\zeta^4)^2$$

$$+ \sum_{s=5}^{9} \left[ \tilde{g}^{s s} (\dot{\zeta}^s)^2 + g^{s s} (\zeta^s)^2 \right],$$

where we have freely integrated by parts because of the Dirichlet boundary conditions. The metric coefficients $\tilde{g}^{\tau \tau}$ and $g^{\sigma \sigma}$ here correspond to the worldsheet line element (99). Just as in the long string case, the fluctuating modes on $S^5$ have vanishing worldsheet masses associated with them. The remaining three transverse fluctuations in AdS$_5$ have (position dependent) ‘masses’

$$M_2^2 = M_3^2 = - \frac{z}{2} \left( f_z - \frac{4f}{z} + \frac{f_z z'^2}{f + z'^2} \right),$$

$$M_4^2 = \frac{z}{2 f + z'^2} \left( f f_z - f_z z + \frac{4f z'^2}{z} - 2f z'^2 + f f_z z \right).$$

Substituting in the explicit form of $f$ from (74) then gives

$$M_2^2 = M_3^2 = \frac{2f - 2z'^2(f - 2)}{f + z'^2} = 2 - \frac{2z'^2(f - 1)}{f + z'^2},$$

$$M_4^2 = \frac{8(f - 1) - 2z'^2(f - 2)}{f + z'^2} = 4 + R^{(2)} + \frac{4z'^2(f - 1)}{f + z'^2}.$$ (113)

In the above, we have been able to simplify the expression for $M_4^2$ by substituting in the worldsheet curvature scalar (100). Even with this substitution, however, these individual masses have an extra term

$$\delta \equiv - \frac{2z'^2(f - 1)}{f + z'^2},$$

which may be related to the components of the Weyl tensor (76) according to

$$\delta = \eta^{\mu \nu} \eta^{\nu \rho} E^{2 \rho} E^{2 \lambda} C_{\rho \mu \lambda \nu} = \eta^{\mu \nu} \eta^{\nu \rho} E^{3 \rho} E^{3 \lambda} C_{\rho \mu \lambda \nu}.$$ (116)

This indicates that $\delta$ only vanishes in the vacuum AdS$_5$ geometry when $f = 1$ and the Weyl tensor vanishes identically. Nevertheless, we note that the masses obey the sum rule

$$\sum_i M_i^2 = 8 + R^{(2)}.$$ (117)
which also holds for the long straight string configuration of the previous section. Finally, this mass spectrum allows us to write the bosonic fluctuation operator $A_{tr}$ as
\[
A_{tr} = \begin{pmatrix}
-\nabla^2 + 4 + R^{(2)} - 2\delta & 0 & 0 \\
0 & (-\nabla^2 + 2 + \delta)I_2 & 0 \\
0 & 0 & -\nabla^2I_5
\end{pmatrix}.
\] (118)
This reduces to the result found in [17] in the vacuum $\text{AdS}_5$ (i.e. $\delta \to 0$) limit.

C. Fermionic fluctuations

In order to consider the fermionic fluctuations, we start with the worldsheet metric (94) and choose the natural zweibeins
\[
e^0_\tau = \sqrt{f} \frac{dz}{z}, \quad e^1_\sigma = \frac{1}{z} \sqrt{1 + \frac{z'^2}{f}} d\sigma.
\] (119)
The corresponding worldsheet spin connection reads
\[
\omega^{01}_\tau = -\frac{2\sqrt{f_0} z}{z'^2 f} \left(1 + \frac{z^4}{z'^4 h}\right) z', \quad \omega^{01}_\sigma = 0.
\] (120)
Just as we have oriented the bosonic worldsheet using the tangent space rotation (102), we do the same for the Green-Schwarz fermions. In this case, the spinor representation of the above rotation is implemented by the $16 \times 16$ matrix $S = e^{-\frac{i}{2} \phi \gamma^1 \gamma^4}$ such that
\[
S\gamma^0 S^{-1} = \gamma^0,
S\gamma^1 S^{-1} = \gamma^1 \cos \phi + \gamma^4 \sin \phi.
\] (121)
The projection of the spacetime spin connection post rotation can be read off from the transformation of the covariant derivative (51). Starting with
\[
D_\tau = \frac{\partial}{\partial \tau} - \frac{1}{2z} \left(1 + \frac{z'^2}{f}\right) \gamma^0 \gamma^4,
D_\sigma = \frac{\partial}{\partial \sigma} - \frac{1}{2z} \sqrt{f} \gamma^1 \gamma^4,
\] (122)
we find
\[
S^{-1} D_\tau S = \frac{\partial}{\partial \tau} - \frac{1}{2z} \left(1 + \frac{z'^2}{z'^4 h}\right) \left(\gamma^0 \gamma^1 \sin \phi + \gamma^0 \gamma^4 \cos \phi\right),
S^{-1} D_\sigma S = \frac{\partial}{\partial \sigma} - \frac{1}{2} \left(\phi' + \frac{\sqrt{f}}{z}\right) \gamma^0 \gamma^4.
\] (123)
Therefore the nontrivial components of the worldsheet projection of the spacetime spin connection with respect to the rotated fünfbeins are

\[
\begin{align*}
\Omega_{01}^\tau &= -\frac{1}{z} \left(1 + \frac{z^4}{z_4^4}\right) \sin \phi = \omega_{01}^\tau, \\
\Omega_{04}^\tau &= -\frac{1}{z} \left(1 + \frac{z^4}{z_4^4}\right) \cos \phi, \\
\Omega_{04}^\sigma &= \phi' + \frac{1}{z} \sqrt{f}. \tag{124}
\end{align*}
\]

Comparing this with the general expansion (60), we find that \(D_2 = 0\) and that the quantity \(e^{\alpha j} \Omega_j^{\beta 4}\) is diagonal with respect to the indexes \(\alpha\) and \(\beta\). In this case, the \(D_1\) term vanishes following the general arguments given previously. This can also be verified explicitly with the aid of the equation of motion (96) and the expressions for \(\sin \phi\) and \(\cos \phi\). The Green-Schwarz fermionic action is then reduced to a worldsheet fermionic action for eight two-component Majorana spinors, and hence the general expression (69) for the fermionic operator \(A_F\) remains valid in this parallel lines Wilson loop case.

To summarize, by combining the bosonic (118) and fermionic (69) expressions, we have obtained the one-loop partition function for the parallel lines configuration in the Schwarzschild-AdS\(_5\) background

\[
Z = \frac{\det^2(-\nabla_+^2 + 1 + \frac{1}{4} R^{(2)}) \det^2(-\nabla_-^2 + 1 + \frac{1}{4} R^{(2)})}{\det^2(-\nabla_+^2 + 4 + R^{(2)} - 2\delta) \det(-\nabla_-^2 + 2 + \delta) \det^2(-\nabla_-^2)}, \tag{125}
\]

where \(\nabla_\pm\) is given by the same expression as (91) but with \(\bar{g}^{ij}\) referring to the metric (99) and \(\omega_j^{\alpha \beta}\) to the spin connection (120).

V. DISCUSSION

In this paper, we have explored some general properties of string worldsheet fluctuations in a Schwarzschild-AdS\(_5\) \(\times S^5\) background. For the physically interesting cases of the worldsheets of a long string on the boundary and of two parallel lines on the boundary, we have expressed the fluctuation partition functions in terms of a set of determinants of a \(1 \times 1\) Laplacian with a mass term. The same method applies to a circular Wilson loop which we did not address in this paper. Several issues remain to be addressed, however, before these determinants may be evaluated.
There are two types of divergences associated to the Wilson loop analysis. One is the UV divergence of the string $\sigma$-model underlying the determinants in the partition function and the other is caused by placing the physical 3-brane at the AdS boundary. (The latter is already present at the classical level.) To address the first one, we regularize the determinants with a heat kernel expansion, while the second one is regularized by pulling the 3-brane slightly off the AdS boundary. Using the heat kernel expansion, we write

$$\ln \det O = -\text{Tr} \int_{\epsilon}^{\infty} \frac{dt}{t} e^{-t O},$$

(126)

where $O$ is the operator $A_{tr}$ of (39) or the square of the operator $A_F$ of (48) and $\epsilon$ is a cutoff parameter with dimensions of length squared. We have [17, 30]

$$\ln \det O = \frac{a}{\epsilon} \int_W d^2\sigma \sqrt{g} + \ln \epsilon \left( b \chi_E + \int_W d^2\sigma \sqrt{g} M^2 \right) + \text{finite terms},$$

(127)

for Dirichlet or Neumann boundary conditions, where $W$ denotes the string worldsheet, $M$ corresponds to the mass term of $O$ and $a$, $b$ and $c$ are numerical constants. The Euler character of the worldsheet is given by

$$\chi_E = \frac{1}{4\pi} \int_W d^2\sigma \sqrt{g} R^{(2)} + \frac{1}{2\pi} \int_{\partial W} ds K,$$

(128)

where $\partial W$ is the boundary of the string worldsheet, $s$ is the proper length along $\partial W$ and $K$ is the geodesic curvature of $\partial W$.

The cancellation of the UV divergence of (127) for vacuum AdS$_5 \times S^5$ with a general embedding was discussed in [17], and the results can be readily carried over to the Schwarzschild-AdS$_5 \times S^5$ case. The $1/\epsilon$ term cancels between the bosonic and fermionic determinants in the partition function. The logarithmically divergent term proportional to the Euler character cancels in the critical dimension $D = 10$ independent of the details of the target space. Finally, the cancellation of the logarithmic divergence associated to the mass term requires the target space to satisfy the IIB Einstein equation of motion,

$$R_{\mu\nu} = \frac{1}{2} \frac{1}{2} \frac{1}{4!} F_{\mu \rho_1 \rho_2 \rho_3 \rho_4} F^{\rho_1 \rho_2 \rho_3 \rho_4}_{\nu},$$

(129)

which is the case for Schwarzschild-AdS$_5 \times S^5$. The only temperature effect, the nonzero Weyl tensor, does not contribute the Ricci tensor. Take the parallel lines Wilson loop as an example: the contribution of the $\delta$ terms from each determinant in the denominator of (125) to the $\ln \epsilon$ term cancels in the product following the sum rule (117).
Turning to the reduced partition functions (90) and (125), one has to keep in mind a subtlety pointed out in [17]. The logarithmic divergence of the fermionic determinant proportional to the Euler character is one quarter of that extracted from the fermion action (55) before transforming the ten-dimensional Majorana-Weyl spinor $\theta$ into worldsheet Majorana spinors because of the singularity of the transformation. However, this will not have an impact on the ratio of two determinants with identical world sheet topology. For example, the finite temperature correction to the heavy quark potential, $\Delta V$ is given by the ratio of the fluctuation determinant with a black hole to that without a black hole, and can be evaluated unambiguously within the framework of (125). In particular, we have

$$V = -T \ln Z = -T(\ln Z_{bs} + \ln Z_{fm}) = T \sum_n \left[ \ln Z_{bs}(\omega_n) + \ln Z_{fm}(\omega_{n+\frac{1}{2}}) \right],$$

(130)

where we have factorized the partition function into components of different Matsubara frequencies through the substitutions of $\frac{\partial}{\partial \tau} \rightarrow -\omega_n \equiv -2\ln \pi T$ for bosons and $\frac{\partial}{\partial \tau} \rightarrow -\omega_{n+\frac{1}{2}} \equiv -2i(n + \frac{1}{2})\pi T$ for fermions. Using the Poisson formula

$$\sum_n f(n) = \int_{-\infty}^{\infty} dx f(x) + 2\Re \sum_{m=0}^{\infty} \int_{-\infty}^{\infty} dx e^{-2im\pi x} f(x),$$

(131)

we find that

$$V = -\int_{-i\infty}^{i\infty} \frac{d\omega}{2i\pi} \left[ \ln Z_{bs}(\omega) + \ln Z_{fm}(\omega) \right]$$

$$+ 2\Re \sum_{m=1}^{\infty} \left[ \int_{-i\infty}^{i\infty} \frac{d\omega}{2i\pi} e^{-m\omega} \left[ \ln Z_{bs}(\omega) + (-1)^m \ln Z_{fm}(\omega) \right] \right]$$

$$= -\int_{-i\infty}^{i\infty} \frac{d\omega}{2i\pi} \left[ \ln Z_{bs}(\omega) + \ln Z_{fm}(\omega) \right] + \frac{2}{\pi} \int_0^{\infty} d\omega \left[ \frac{\ln Z_{bs}(\omega)}{e^{\frac{\omega}{T}} - 1} - \frac{\ln Z_{fm}(\omega)}{e^{\frac{\omega}{T}} + 1} \right],$$

(132)

where in the last step, we have deformed the integration contour of the second term on the RHS to wrap around the positive real axis along which the singularities of $\ln Z_{bs}(\omega)$ and $\ln Z_{fm}(\omega)$ reside. Only the first integral carries the logarithmic divergence, which is cancelled by subtracting the heavy quark potential at zero temperature, $T = 0$,

$$V_0 = -\int_{-i\infty}^{i\infty} \frac{d\omega}{2i\pi} \left[ \ln Z_{bs}^{(0)}(\omega) + \ln Z_{fm}^{(0)}(\omega) \right].$$

(133)

The difference $V - V_0$ may suffer from the second type of divergence mentioned at the beginning of this section, i.e. the divergence when the 3-brane is pushed back to the AdS boundary. This divergence should be cancelled by subtracting the contributions from two
straight strings, as in the classical limit. The details of the cancellation remains to be examined carefully before the correction to the screening length can be extracted.

Despite the simpler partition function of a straight string, the evaluation of the partition function is actually more challenging in this case. Consider the ratio of the partition function, \( Z/Z_0 \) with \( Z \) given by (90) and \( Z_0 \) its counterpart in the absence of the black hole. Although both world sheets of \( Z \) and \( Z_0 \) share the same boundary on the AdS boundary, their extensions to the AdS bulk are quite different; one terminates at the Schwarzschild horizon and the other at the AdS horizon. The boundary condition at the Schwarzschild horizon has to be examined carefully to guarantee the divergence free condition of the ratio. Once the ratio is made cutoff independent, dimensional arguments suggest that \( Z/Z_0 = 1 \). Since \( Z_0 = 1 \) [17], this would imply that \( Z = 1 \) as well. This is, however, not at all obvious, since the spectral equations of the operators \(-\nabla^2 + \frac{8}{3} + \frac{1}{3} R^{(2)}\) and \(-\hat{\nabla}^2 + 1 + \frac{1}{4} R^{(2)}\) appear highly nontrivial in the presence of a black hole. We hope to be able to report our progress in this direction in the near future.

In the case of more complicated embeddings, such as the worldsheet of boosted parallel Wilson lines, the Schwarzschild-AdS\(_5\) × \( S^5 \) metric in the rest frame of the corresponding heavy quarks acquires cross terms and the operators underlying the functional determinant may no longer be reduced to \( 1 \times 1 \) structures. This also covers the case of light-like parallel Wilson lines that gives rise to the jet quenching parameter from AdS/CFT duality [7]. Also, in the latter case, the string worldsheet becomes space-like with respect to real time, making it difficult to define the path integral because of the non-positivity of the fluctuation action. Thus we expect the evaluation of the \( \mathcal{O}(\lambda^{-1/2}) \) corrections to the jet quenching parameter and the drag force arising from worldsheet fluctuations to be somewhat of a challenge. However, even without a complete computation, it would be of interest to obtain at least a qualitative estimate of the size of such corrections and whether they result in an enhancement or suppression of the drag force.

**Acknowledgments**

We thank H. Liu, L. Pando Zayas and A. Tseytlin for valuable discussions. DFH acknowledges the hospitality of the Institute of Nuclear Physics at the University of Washington where part of this work was completed. The work of DFH and HCR is supported in part by
NSFC under grant Nos. 10575043 and 10735040. The work of DFH is also supported in part by Educational Committee of China under grant NCET-05-0675 and project No. IRT0624. JTL acknowledges the hospitality of the Institute of Particle Physics at Huazhong Normal University where part of this work was completed. This work was supported in part by the US Department of Energy under grant DE-FG02-95ER40899.

APPENDIX A

In this appendix, we present the details of the proof that a longitudinal fluctuation is a zero mode of the Nambu-Goto action. The mode equation with an eigenvalue $E$ reads

$$K_1\mu + K_2\mu = E\delta X\mu, \quad (A1)$$

where $K_1$ and $K_2$ arise from the variation of $I$ and $J$ of (19), respectively. Explicitly,

$$K_1\mu = -g^{ij}\nabla_i\nabla_j\delta X\mu - g^{ij}R_{\rho\mu\lambda\nu}\eta^{\rho}_i\eta^{\lambda}_j\delta X^\nu,$$

$$K_2\mu = \frac{1}{2}(g^{ik}g^{jl} + g^{il}g^{jk} - g^{ij}g^{kl})\nabla_i(\eta_{\mu j}\delta_1 g_{kl}), \quad (A2)$$

where we have suppressed the overlines on the induced metric. For a longitudinal fluctuation

$$\delta X^\mu = \eta^\mu_k\epsilon_k \quad (A3)$$

which generates a worldsheet diffeomorphism

$$\delta_1 g_{ij} = \nabla_j\epsilon_i + \nabla_i\epsilon_j, \quad (A4)$$

we find

$$K_1\mu = -g^{ij}G_{\mu\nu}(\eta^k_i\nabla_i\nabla_j\epsilon_k + 2\nabla_i\eta^\nu_k\nabla_j\epsilon^k + \epsilon^k\nabla_i\nabla_j\eta^\nu_k) + R_{\rho\mu\lambda\nu}\eta^{\rho}_i\eta^{\lambda}_j\epsilon^\nu,$$

$$= -g^{ij}G_{\mu\nu}(\eta^k_i\nabla_i\nabla_j\epsilon_k + 2\nabla_i\eta^\nu_k\nabla_j\epsilon^k + \epsilon^k[\nabla_i, \nabla_j]\eta^\nu_k) + R_{\rho\mu\lambda\nu}\eta^{\rho}_i\eta^{\lambda}_j\epsilon^\nu. \quad (A5)$$

Here we have used the symmetry

$$\nabla_i\eta^\mu_j = \nabla_j\eta^\mu_i \quad (A6)$$

and the equation of motion (11). It follows from the commutation relation

$$[\nabla_i, \nabla_j]\eta^\mu_k = R^{\mu}_{\lambda\nu\rho}\eta^\nu_i\eta^\rho_j\eta^\lambda_k - R^{(2)}_{\lambda\nu\rho}\eta^\lambda_k, \quad (A7)$$

28
that

\[ K_{1\mu} = -g^{ij}G_{\mu\nu}(\eta^i_k \nabla_i \nabla_j \epsilon^k + 2\nabla_i \eta^i_k \nabla_j \epsilon^k + \frac{1}{2}R^{(2)}\eta^i_k \epsilon^j). \]  

(A8)

The term \( K_{2\mu} \) is simplified similarly. We have

\[
K_{2\mu} = (\nabla^i \eta^i_\mu)(\nabla_j \epsilon_i + \nabla_i \epsilon_j) + \eta^i_\mu \nabla^i (\nabla_j \epsilon_i + \nabla_i \epsilon_j) - \eta^i_\mu \nabla_i \nabla_j \epsilon^j \\
= 2\nabla_i \eta^i_\mu \nabla_j \epsilon^i + \eta^i_\mu \nabla^i \epsilon_j + \eta^i_\mu [\nabla_j, \nabla_i] \epsilon^j \\
= 2\nabla_i \eta^i_\mu \nabla_j \epsilon^i + \eta^i_\mu \nabla^i \epsilon_j - R^{(2)ij}_{\ kij}\eta^i_\mu \epsilon^j \\
= 2\nabla_i \eta^i_\mu \nabla_j \epsilon^i + \eta^i_\mu \nabla_i \nabla_j \epsilon_j + \frac{1}{2}R^{(2)}\eta^i_\mu \epsilon_i. 
\]  

(A9)

Finally, by adding (A8) and (A9) together, we end up with

\[ K_{1\mu} + K_{2\mu} = 0, \]  

(A10)

and hence the longitudinal fluctuation is indeed a zero mode.

**APPENDIX B**

Here we review our spinor conventions. Following [17, 23], we use the letters \( a, b, c, \ldots \) to label the AdS\(_5\) dimensions and \( a', b', c', \ldots \) to label the S\(^5\) dimensions. The 32 \( \times \) 32 Dirac matrices in \( D = 10 \) may be written as

\[
\Gamma^a = \gamma^a \times I_4 \times \sigma_1, \\
\Gamma^{a'} = I_4 \times \gamma^{a'} \times \sigma_2,
\]  

(B1)

where the 4 \( \times \) 4 gamma matrices \( \gamma^a \) and \( \gamma^{a'} \) satisfy the anticommutation relations

\[
\{ \gamma^a, \gamma^b \} = 2\eta^{ab} = (-, +, +, +, +), \\
\{ \gamma^{a'}, \gamma^{b'} \} = 2\delta^{a'b'} = (+, +, +, +, +).
\]  

(B2)

Taken together, this gives

\[
\{ \Gamma^{\hat{a}}, \Gamma^{\hat{b}} \} = 2\eta^{\hat{a}\hat{b}} = (-, +, \cdots, +), 
\]  

(B3)

where the superscripts \( \hat{a} \) and \( \hat{b} \) label all ten dimensions. The charge conjugation matrix reads

\[ C = C \times C \times i\sigma_2, \]  

(B4)
where $C$, the four-dimensional charge conjugation matrix, satisfies the conditions $\tilde{C} = -C$, $C\tilde{\gamma}^a C^{-1} = \gamma^a$ and $C\tilde{\gamma}^{a'} C^{-1} = \gamma^{a'}$.

The ten-dimensional analog of $\gamma_5$ is

$$
\Gamma^{11} = \Gamma^0 \Gamma^1 \cdots \Gamma^9 = I_4 \times I_4 \times \sigma_3.
$$

(B5)

A Weyl spinor is an eigenstate of $\Gamma^{11}$:

$$
\Gamma^{11} \Psi_{\pm} = \pm \Psi_{\pm},
$$

(B6)

and the Majorana condition reads

$$
\bar{\Psi} = \Psi C,
$$

(B7)

with $\bar{\Psi} = \Psi^\dagger \Gamma^0$. A positive chirality IIB Weyl spinor takes the form

$$
\Psi = \psi \times \begin{pmatrix} 1 \\ 0 \end{pmatrix},
$$

(B8)

where $\psi$ is a 16 component spinor. For two such spinors, $\Psi$ and $\Psi'$, we have

$$
\bar{\Psi}' \Gamma^a \Psi = \bar{\psi}' \gamma^a \psi, \quad \bar{\Psi}' \Gamma^{a'} \Psi = i \bar{\psi}' \gamma^{a'} \psi,
$$

(B9)

where $\gamma^a$ and $\gamma^{a'}$ designate the $16 \times 16$ matrices $\gamma^a \times I_4$ and $I_4 \times \gamma^{a'}$. This 16-component formulation is adopted in this paper. In terms of a 16 component spinor $\psi$, the Majorana condition (B7) reads

$$
\bar{\psi} = \bar{\psi} C \times C,
$$

(B10)

where $\bar{\psi} = \psi^\dagger \gamma^0$.

[1] J. M. Maldacena, *The large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. 2, 231 (1998) [Int. J. Theor. Phys. 38, 1113 (1999)] [hep-th/9711200].

[2] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, *Gauge theory correlators from non-critical string theory*, Phys. Lett. B 428, 105 (1998) [hep-th/9802109].

[3] E. Witten, *Anti-de Sitter space and holography*, Adv. Theor. Math. Phys. 2, 253 (1998) [hep-th/9802150].

[4] E. Witten, *Anti-de Sitter space, thermal phase transition, and confinement in gauge theories*, Adv. Theor. Math. Phys. 2, 505 (1998) [hep-th/9803131].
[5] G. Policastro, D. T. Son and A. O. Starinets, *The shear viscosity of strongly coupled $N = 4$ supersymmetric Yang-Mills plasma*, Phys. Rev. Lett. **87**, 081601 (2001) [hep-th/0104066].

[6] C. P. Herzog, A. Karch, P. Kovtun, C. Kozcaz and L. G. Yaffe, *Energy loss of a heavy quark moving through $N = 4$ supersymmetric Yang-Mills plasma*, JHEP **0607**, 013 (2006) [hep-th/0605158].

[7] H. Liu, K. Rajagopal and U. A. Wiedemann, *Calculating the jet quenching parameter from AdS/CFT*, Phys. Rev. Lett. **97**, 182301 (2006) [hep-ph/0605178].

[8] S. S. Gubser, I. R. Klebanov and A. A. Tseytlin, *Coupling constant dependence in the thermodynamics of $N = 4$ supersymmetric Yang-Mills theory*, Nucl. Phys. B **534**, 202 (1998) [hep-th/9805156].

[9] J. Pawelczyk and S. Theisen, *AdS$_5 \times S^5$ black hole metric at $O(\alpha'^3)$*, JHEP **9809**, 010 (1998) [hep-th/9808126].

[10] A. Buchel, J. T. Liu and A. O. Starinets, *Coupling constant dependence of the shear viscosity in $N = 4$ supersymmetric Yang-Mills theory*, Nucl. Phys. B **707**, 56 (2005) [hep-th/0406264].

[11] A. Buchel, *Resolving disagreement for $\eta/s$ in a CFT plasma at finite coupling*, [arXiv:0805.2683 [hep-th]].

[12] N. Armesto, J. D. Edelstein and J. Mas, *Jet quenching at finite 't Hooft coupling and chemical potential from AdS/CFT*, JHEP **0609**, 039 (2006) [hep-ph/0606245].

[13] J. F. Vazquez-Poritz, *Drag force at finite 't Hooft coupling from AdS/CFT*, [arXiv:0803.2890 [hep-th]].

[14] J. M. Maldacena, *Wilson loops in large $N$ field theories*, Phys. Rev. Lett. **80**, 4859 (1998) [hep-th/9803002].

[15] S. J. Rey, S. Theisen and J. T. Yee, *Wilson-Polyakov loop at finite temperature in large $N$ gauge theory and anti-de Sitter supergravity*, Nucl. Phys. B **527**, 171 (1998) [hep-th/9803135].

[16] S. Forste, D. Ghoshal and S. Theisen, *Stringy corrections to the Wilson loop in $N = 4$ super Yang-Mills theory*, JHEP **9908**, 013 (1999) [hep-th/9903042].

[17] N. Drukker, D. J. Gross and A. A. Tseytlin, *Green-Schwarz string in AdS$_5 \times S^5$: Semiclassical partition function*, JHEP **0004**, 021 (2000) [hep-th/0001204].

[18] J. Greensite and P. Olesen, *Worldsheet fluctuations and the heavy quark potential in the AdS/CFT approach*, JHEP **9904**, 001 (1999) [arXiv:hep-th/9901057].

[19] S. Naik, *Improved heavy quark potential at finite temperature from anti-de Sitter supergravity,*
[20] M. B. Green and J. H. Schwarz, *Covariant Description Of Superstrings*, Phys. Lett. B 136, 367 (1984).

[21] M. B. Green and J. H. Schwarz, *Properties Of The Covariant Formulation Of Superstring Theories*, Nucl. Phys. B 243, 285 (1984).

[22] M. T. Grisaru, P. S. Howe, L. Mezincescu, B. Nilsson and P. K. Townsend, *N = 2 Superstrings In A Supergravity Background*, Phys. Lett. B 162, 116 (1985).

[23] R. R. Metsaev and A. A. Tseytlin, *Type IIB superstring action in AdS_5 x S^5 background*, Nucl. Phys. B 533, 109 (1998) [hep-th/9805028].

[24] I. Pesando, *A kappa gauge fixed type IIB superstring action on AdS_5 x S^5*, JHEP 9811, 002 (1998) [arXiv:hep-th/9808020].

[25] R. Kallosh and J. Rahmfeld, *The GS string action on AdS_5 x S^5*, Phys. Lett. B 443, 143 (1998) [arXiv:hep-th/9808038].

[26] R. Kallosh and A. A. Tseytlin, *Simplifying superstring action on AdS_5 x S^5*, JHEP 9810, 016 (1998) [arXiv:hep-th/9808088].

[27] M. Cvetic, H. Lu, C. N. Pope and K. S. Stelle, *T-Duality in the Green-Schwarz Formalism, and the Massless/Massive IIA Duality Map*, Nucl. Phys. B 573, 149 (2000) [arXiv:hep-th/9907202].

[28] M. Kruczenski and A. Tirziu, *Matching the circular Wilson loop with dual open string solution at 1-loop in strong coupling*, JHEP 0805, 064 (2008) [0803.0315 [hep-th]].

[29] E. S. Fradkin and A. A. Tseytlin, *On Quantized String Models*, Annals Phys. 143, 413 (1982).

[30] O. Alvarez, *Theory Of Strings With Boundaries: Fluctuations, Topology, And Quantum Geometry*, Nucl. Phys. B 216, 125 (1983).