ON MULTIPLE AND POLYNOMIAL RECURRENT EXTENSIONS
OF INFINITE MEASURE PRESERVING TRANSFORMATIONS

TOM MEYEROVITCH

Abstract. We prove that multiple-recurrence and polynomial-recurrence of
invertible infinite measure preserving transformations are both properties which
pass to extensions.

1. Introduction and statement of result

A non-singular transformation $T : X \to X$ of a measure space $(X, \mathcal{B}, \mu)$ is called
$d$-recurrent $(d \in \mathbb{N})$ if for any $A \in \mathcal{B}$ of positive measure, there exists an integer
$k \geq 1$ such that

\begin{equation}
\mu(\bigcap_{i=0}^{d} T^{-ik} A) > 0
\end{equation}

$T$ is multiply recurrent if it is $d$-recurrent for all $d \geq 1$.

$T$ is called polynomially-recurrent if for any $d \geq 1$ and any polynomials $p_1, \ldots, p_d \in \mathbb{Z}[x]$ such that $p_i(0) = 0$ for all $1 \leq i \leq d$, and any $A \in \mathcal{B}$ of positive measure, there
exists an integer $k \neq 0$ such that

\begin{equation}
\mu(\bigcap_{i=0}^{d} T^{-p_i(k)} A) > 0
\end{equation}

We say that $T : X \to X$ is an extension of a measure preserving transformation
$S : Y \to Y$ (of the measure space $(Y, \mathcal{C}, \nu)$) if there is a measurable $\pi : X \to Y$ such
that $\pi \circ T = S \circ \pi$ and $\mu \circ \pi^{-1} = \nu$.

Furstenburg [5] gave an ergodic-theoretical proof that any finite-measure pre-
serving system is multiply recurrent, giving an alternative proof of Szemerédi’s
theorem [8]. Bergelson and Leibman proved that any finite-measure preserving sys-
tem is multiply recurrent [2], and deduced a theorem about existence of polynomial
configurations in subsets of positive density.

Combinatorial results about arithmetical progressions and polynomial configu-
rations in zero density sets (for example, the primes) are generally more difficult to
obtain using methods of classical ergodic theory. There are old speculations about
the relevance of infinite-measure ergodic theory for such problems. In this direction,
Aaronson and Nakada [1] formulated a conjecture on infinite-measure preserving
transformations which holds assuming a positive solutions to a long standing con-
juncture of Erdős.

Several authors have studied multiple and polynomial properties of infinite-
measure preserving transformations: Eigen-Hajian-Halverson [4] constructed for
each $d > 0$ an ergodic infinite measure-preserving transformation is that is $d$-
recurrent but not $(d + 1)$-recurrent. Aaronson and Nakada [1] give necessary and
sufficient conditions for \(d\)-recurrence of Markov-Shifts. Danilenko and Silva \cite{3} constructed measure preserving group actions with various multiple and polynomial recurrence properties. In particular, there exist infinite-measure preserving transformations which are polynomially recurrent, and also measure preserving transformations which are multiply-recurrent and not polynomially recurrent.

We prove the following results:

**Theorem 1.1.** If an invertible measure preserving transformation \(S\) is multiply recurrent, so is any measure preserving extension \(T\) of \(S\).

**Theorem 1.2.** If an invertible measure preserving transformation \(S\) is polynomially recurrent, so is any measure preserving extension \(T\) of \(S\).

Theorem 1.1 answers a question raised by Aaronson and Nakada \cite{1}. A partial result was previously obtained by Inoue \cite{7} for isometric extensions. It is worth noting that Inoue’s result on multiple-recurrence for isometric extensions does not require that transformations involved be invertible.

It unknown if for some \(k \geq 2\) there exist \(n_k\) such that any extension of any \(n_k\)-recurrent transformation is \(k\)-recurrent.

We remark that these results generalize essentially without modification to measure preserving \(\mathbb{Z}^d\) actions, using the multidimensional Szemerédi theorem and the Bergelson-Leibman multidimensional polynomial-recurrence theorem.

Using an argument similar to the proof of our theorem \cite{1} Furstenberg and Glasner obtained a Szemerédi-type theorem for ‘\(SL(2, \mathbb{R})\) ‘\(m\)-stationary systems’. This result is described in \cite{6}, and is based on a certain structure theory of ‘\(m\)-systems’, plus a specific “multiple-recurrence property” related to \(SL(2, \mathbb{R})\). The results of this chapter were obtained independently of \cite{6}.

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2. **Proof of theorem 1.1**

Recall the following formulation of Szemerédi’s theorem:

**Theorem. (Szemerédi’s theorem - finitary version)** Let \(l \in \mathbb{N}\) and \(\delta > 0\). For any sufficiently large \(L \in \mathbb{N}\), any \(E \subset \{1, \ldots, L\}\) with \(|E| > \delta L\) contains a non-trivial \(l\)-term arithmetic progression.

Suppose \(S : Y \to Y\) is an invertible multiply recurrent measure preserving transformation, and \(T\) is an extension. We need to show that any set \(A \in \mathcal{B}\) with \(0 < \mu(A) < \infty\) is multiply recurrent.

Denote by \(\mu_y\) the conditional measure of \(\mu\) given \(y \in Y\). This is \(\nu\)-almost everywhere defined by requiring that \(y \to \mu_y\) be \(\mathcal{C}\)-measurable and

\[
\int_B \mu_y(A) d\nu(y) = \mu(A \cap \pi^{-1} B) \quad \forall B \in \mathcal{C}, A \in \mathcal{B}
\]

Since \(\mu(T^{-1} A \cap T^{-1} B) = \mu(A \cap B)\) and \(T^{-1} \mathcal{C} = \mathcal{C}\), it follows that \(\mu_{S_y}(A) = \mu_y(T^{-1} A)\) for almost any \(y \in Y\).

Let \(A \in \mathcal{B}\) with \(0 < \mu(A)\), and let

\[
B = B_\epsilon = \{y \in Y : \mu_y(A) > \epsilon\}
\]

We set some \(\epsilon > 0\), so that \(\mu(B_\epsilon) > 0\). Note that \(B \in \mathcal{C}\), since \(y \to \mu(A)\) is \(\mathcal{C}\)-measurable.

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By multiple recurrence of $S$, for any $M \in \mathbb{N}$ there exist $n \in \mathbb{N}$ such that
$$\nu(\bigcap_{j=0}^{M} S^{-jn} B) > 0.$$  
For $y \in \bigcap_{j=0}^{M} S^{-jn} B$ and $0 \leq j \leq M$, we have
$$\mu_y(T^{-j} A) = \mu_{S^{jn} y}(A) > \epsilon.$$  
Thus, for $y \in \bigcap_{j=0}^{M} S^{-jn} B$,
$$\int X \sum_{j=0}^{M} 1_{T^{-jn} A} d\mu_y(x) > \epsilon M,$$
and so
$$\int \bigcap_{j=0}^{M} S^{-jn} B \sum_{j=0}^{M} 1_{T^{-jn} A}(x) d\nu(y) > \epsilon M \nu(\bigcap_{j=0}^{M} S^{-jn} B)$$
It follows that there is a set $E \subset \{1, \ldots, M\}$ with $|E| \geq \epsilon M$ with
$$\mu(\bigcap_{j \in E} T^{-jn} A) > 0.$$  
Choose $M$ above large enough so that by Szemerédi’s theorem $E$ contains an arithmetic progression $(a + R, a + 2R, \ldots, a + lR)$ of length $l$. It follows that
$$\mu(\bigcap_{j=0}^{l-1} T^{-jR} A) > 0,$$
and so $T$ is multiply recurrent.

3. Proof of theorem 1.2

Our proof is based on the following theorem of V. Bergelson and A. Leibman [2], of which we state a finitary version:

**Theorem. (Bergelson-Leibman theorem, finitary version)** Let $\{P_{i,j}(x)\}_{1 \leq i \leq k, 1 \leq j \leq d}$ be any polynomials with rational coefficients taking on integer values in the integers and satisfying $P_{i,j}(0) = 0$, and $\epsilon > 0$. For any sufficiently large $N$, and any set $E \subset \{1, \ldots, N\}^d$ with $|E| \geq \epsilon N^d$, there exists an integer $n$ and a vector $\vec{a} \in \mathbb{Z}^d$ such that $\vec{a} + \sum_{j=1}^{k} P_{i,j}(x)e_d \in S$ for all $1 \leq i \leq k$, where $\{e_1, \ldots, e_d\}$ are the standard basis of $\mathbb{Z}^d$.

Suppose $S$ is an invertible polynomially-recurrent measure preserving transformation, and $(X, B, \mu, T)$ is an extension. We need to prove that for any $A \in B$ with $\mu(A) > 0$ and any polynomials $p_1, \ldots, p_d \in \mathbb{Z}[x]$ with $p_i(0) = 0$, there exists $k \in \mathbb{Z} \setminus \{0\}$ such that equation (2) holds.

Write $p_i(x) = \sum_{j=1}^{l} a_{i,j} x^j$. For $\vec{k} = (k_1, \ldots, k_l)$, let $q_{\vec{k}}(x) = \sum_{j=1}^{l} k_j x^j$.

Find $B \in C$ with $0 < \nu(B) < +\infty$ and $\mu_y(A) > \epsilon$ for all $y \in B$.

By polynomial-recurrence of $S$, there exists $n \in \mathbb{Z}$ such that
$$\nu(\bigcap_{k \in [N]^l} S^{\vec{k}n} B) > 0,$$
with $N$ large enough so that the conclusion of the Bergelson-Leibman theorem applies. Repeating the argument of the previous proof, there exists a set $E \subset [N]^l$ of density at least $\epsilon$ and a set $A' \subset A$ of positive measure such that:

$$\mu\left(\bigcap_{\tau \in E} T^{\pi(n)} A'\right) > 0$$

By the Bergelson-Leibman theorem, if $N$ is sufficiently large, there exist $\pi \in [N]^l$ and $r \in \mathbb{Z}$ such that $\pi + \sum_{j=1}^l a_{i,j} r^j e_j \in E$ for every $1 \leq i \leq d$. It follows that

$$\mu\left(\bigcap_{i=1}^d T^{-p_i(rn)+U(n)} A\right) > 0,$$

where $U(x) = \sum_{i=1}^d u_i x^i$, and so:

$$\mu\left(\bigcap_{i=1}^d T^{-p_i(rn)} A\right) > 0$$

**References**

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