The Symplectic Geometry of a Parametrized Scalar Field on a Curved Background

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(October, 1995)

Abstract

We study the real, massive Klein-Gordon field on a $C^\infty$ globally-hyperbolic background space-time with compact Cauchy hypersurfaces. In particular, the parametrization of this system as initiated by Dirac and Kuchař is put on a rigorous basis. The discussion is focussed on the structure of the set of spacelike embeddings of the Cauchy manifold into the space-time, and on the associated e-tensor density bundles and their tangent and cotangent bundles. The dynamics of the field is expressed as a set of automorphisms of

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the space of initial data in which each pair of embeddings defines one such automorphism. Using these results, the extended phase space of the system is shown to be a weak-symplectic manifold, and the Kuchař constraint is shown to define a smooth constraint submanifold which is foliated smoothly by the constraint orbits. The pull-back of the symplectic form to the constraint surface is a presymplectic form which is singular on the tangent spaces to the constraint orbits. Thus, the geometric structure of this infinite-dimensional system is analogous to that of a finite-dimensional, first-class parametrized system, and hence many of the results for the latter can be transferred to the infinite-dimensional case without difficulty.
I. INTRODUCTION

The long history of studies of quantum field theory in a background space-time peaked sharply in the seventies (for reviews from that era, see \([1,2]\)) following Hawking’s discovery of the quantum radiation produced by a black hole \([3]\). In those days, the main aim was to find a direct quantization of the true physical degrees of freedom of the system. However, much earlier, Dirac had reformulated this system in a parametrized form so that it could be used as a model for general relativity proper \([4]\). The idea is to treat embeddings of Cauchy hypersurfaces in the space-time as additional degrees of freedom of the system. Together with their conjugate momenta and the Cauchy data of the scalar field on the embeddings, they define an extended phase space. To retrieve the original dynamics one has to impose constraints. This procedure was studied in some detail in the references \([5,6]\) and \([7]\); a shorter exposition can be found in \([8]\). The resulting system can be classified as a ‘first-class parametrized system’.

A method of quantizing first-class parametrized systems was initiated by Dirac; in particular, he studied a system of massive particles in Minkowski space-time \([9]\). A generalization of this method to any finite-dimensional first-class parametrized system was given by Hájíček \([10]\) using a combination of Dirac’s ideas with the group quantization method of Isham \([11]\) and the algebraic quantization method of Ashtekar \([12]\). In what follows, this generalization will be referred to as the ‘perennial formalism’.

If one is interested in extending the perennial formalism to infinite-dimensional systems, the scalar field on a fixed space-time offers itself naturally as a well-understood and much-discussed model. However, the perennial formalism is based on a geometrical form of Hamiltonian dynamics, whereas the existing formulations of the parametrized scalar field are non-geometrical. The main purpose of the present paper is to recast the classical theory of a parametrized scalar field in a fixed background into the geometrical, Hamiltonian form for infinite-dimensional systems developed by Marsden and collaborators (see \([13]\)). The ensuing results are of interest in themselves and are also used in the accompanying paper.
dealing with the application of the perennial formalism to the quantum theory of a field propagating on a curved background.

Marsden’s techniques have been applied to a variety of systems, including general relativity itself [13]. In all these examples, the extended phase space is an (open) subset of a linear space, and hence the tangent space at any point in the phase space can be naturally identified with this linear space. The resulting significant simplifications have been thoroughly exploited in the literature (for example, see [13]). However, in our case, the space of embeddings is a genuine manifold, not just a subspace of a linear space. Of course, the rough idea of how the theory is to be applied in general to such cases is well-known [13], but—as we shall see—the specific system of interest to us possesses some crucial additional structure that is very helpful in the detailed analysis.

The plan of the paper is as follows. In section II, the theory of the Cauchy problem of a hyperbolic, partial differential equation (as reviewed, for example, in [13]) is applied to the dynamics of a massive scalar field in a globally hyperbolic space-time. The Cauchy hypersurfaces are assumed to be compact, but it seems likely that—if desired—the proofs (which are relegated to the Appendix) could be adapted to deal with asymptotically flat Cauchy hypersurfaces. The set of all Cauchy data is given the structure of a topological vector space $\Gamma_\phi$, and an automorphism of this space is associated with each oriented pair of embeddings of the Cauchy manifold in the space-time, thereby producing a rather generalised concept of ‘time evolution’.

In section III, we extend the phase space from $\Gamma_\phi$ to $\Gamma_\phi \times T^*E$, where $E$ is the space of embeddings introduced by Kuchař in [3]. This way of parametrizing the system was discussed in detail by Isham and Kuchař [7]. The structure of the $c$-tensor density bundles over $E$ is described, and their tangent and cotangent bundles are studied; this is where we lay the foundation for the subsequent mathematical developments. Our approach to the resulting infinite-dimensional Hamiltonian system is based on the geometrical ideas of Chernov, Fischer, and Marsden [13,15]. The constructions of the phase space manifold, the symplectic structure, and the Poisson brackets, are all described explicitly.
In section [IV], we show that (i) the constraint set is a submanifold of the phase space; (ii) the constraint orbits are submanifolds of the constraint surface; and (iii) the vector space defined by the right hand side of the evolution equation coincides with the tangent space of a constraint orbit. For the model we study, the proofs (which are adapted from [15]) are straightforward. The resulting structure is analogous to a high degree to that of a finite first-class parametrized system, and hence the application of the perennial formalism is relatively unproblematic.

II. FIELD DYNAMICS

We are interested in the theory of a relativistic field propagating on a fixed background space-time $\mathcal{M}$. The associated dynamical equation defines an evolution map between Cauchy data along two arbitrary Cauchy hypersurfaces. We shall need various properties of these maps that can be derived from results concerning the Cauchy problem for linear hyperbolic systems as described—for example—in [15,16].

The following properties of the space-time $(\mathcal{M}, g)$ will be assumed:

1. The space-time $\mathcal{M}$ is equipped with a $C^\infty$ differential structure. In particular, this means that diffeomorphisms do not mix the different Sobolev structures that will be placed on various function spaces associated with $\Sigma$ and $\mathcal{M}$.

2. The Lorentzian metric $g$ is such that the pair $(\mathcal{M}, g)$ is globally hyperbolic. Thus the four-manifold $\mathcal{M}$ is necessarily diffeomorphic to $\Sigma \times \mathbb{R}$ where the three-manifold $\Sigma$ is a model for any Cauchy hypersurface in $\mathcal{M}$.

3. The three-manifold $\Sigma$ is compact. This assumption is made for the sake of simplicity, but we expect that similar results can be obtained for asymptotically flat space-times using analogous—but more laborious—methods; see [13]).

The relativistic wave equation of interest is the Klein-Gordon equation for a real scalar field:

\[ \Box \phi = 0. \]
\[ |\det g|^{-1/2} \partial_\mu (|\det g|^{1/2} g^{\mu \nu} \partial_\nu \phi) + m^2 \phi = 0 \]  

where the real, non-negative constant \( m \) is the mass parameter.

A central role in the theory is played by \( C^{r+1} \) embeddings \( X : \Sigma \to M \) that are spacelike with respect to \( g \). We shall refer to any such \( X \) simply as an ‘embedding’, and denote by \( \text{Emb}_g(\Sigma, M) \) the space of all such (see [7] and [5]). Each embedding \( X \) determines a positive-definite \( C^r \)-metric \( \gamma \) on \( \Sigma \) as the pull-back \( X^* g \) of \( g \) by \( X \); i.e., in local coordinates on both \( \Sigma \) and \( M \),

\[
\gamma_{ab}(x) := g_{\mu \nu}(X(x)) X^\mu_a(x) X^\nu_b(x). 
\]

The map \( X \) also determines a future-oriented, unit normal vector \( n(X(x)) \) at each point of the hypersurface \( X(\Sigma) \) in \( M \).

Any embedding \( X \in \text{Emb}_g(\Sigma, M) \) defines a ‘Cauchy datum’ for \( \phi \) along the hypersurface \( X(\Sigma) \). This is a pair \((\varphi, \pi)\) of fields on \( \Sigma \), where the scalar \( \varphi \) and the density (of weight \( w = 1 \)) \( \pi \) are defined by

\[
\varphi(x) := \phi(X(x)), \quad (3)
\]

\[
\pi(x) := (\det \gamma)^{1/2}(x) n^\mu(X(x)) \partial_\mu \phi(X(x)), \quad (4)
\]

for all \( x \in \Sigma \), and where \( \gamma \) is defined in Eq. (2).

We shall need certain Sobolev spaces of tensor-density fields on \( \Sigma \) (for more details see [16]). The first step is to introduce a fixed, auxiliary \( C^r \) Riemannian metric \( f_{kl} \) on \( \Sigma \). Let \( I, J, K, \ldots \) denote multiple indices, and let \(|I|\) be the number of simple indices within \( I \). Let \( f_{IJ} \) be the abbreviation for the tensor product of \(|I| = |J|\) copies of the covariant tensors \( f_{ij} \), and—similarly—\( f^{IJ} \) denotes the appropriate tensor product of contravariant fields \( f^{ij} \).

If \( T^J_I \) is a \( C^r \) tensor-density field on \( \Sigma \) of type \((|I|, |J|, w)\) (\( w \) is the weight), \( T^J_{I|K} \) will denote the tensor-density field obtained from \( T^J_I \) by a \(|K|\)-fold covariant derivative (covariant with respect to the auxiliary metric \( f \)). Then the Sobolev space scalar product \( (T, S)_s^I \) between two tensor-density fields \( T^I_J \) and \( S^I_J \) is defined by

\[
(T, S)_s^I := \int_\Sigma \frac{1}{\sqrt{\det f}} \sqrt{\det g} |T^I_J|^{s_1} |S^I_J|^{s_2} \sqrt{\gamma} \sqrt{\gamma}^T dx. 
\]
\[ (T, S)_f^s := \sum_{|M| = |N| = 0}^s \int_{\Sigma} d^3 x (\det f)^{1/2 - \nu} f^{IK} f^{MN} f_{JL} T^I_{LM} S^K_{LN} \] (5)

where \( s \leq r \). We denote the corresponding Sobolev space by \( H_{|I|,|J|w}(\Sigma) \), or simply \( H_w^s(\Sigma) \) if no confusion can result. Note that the topology of these spaces is independent of the auxiliary metric \( f \) provided that \( \Sigma \) is compact (which we are assuming).

Using this notation, we can introduce the space \( \Gamma^s_\phi \) of Cauchy data:

\[ \Gamma^s_\phi := H^s_0(\Sigma) \times H^{s-1}_1(\Sigma) \] (6)

with \( \varphi \in H^s_0(\Sigma) \) and \( \pi \in H^{s-1}_1(\Sigma) \). The dynamics of the field \( \phi \) as determined by Eq. (1) defines maps between the spaces of Cauchy data corresponding to different embeddings. The relevant facts about these maps (which are more or less well-known; see [15]) are listed in Appendix 1, and culminate in the following theorem:

**Theorem 1** Let \( X_1 \) and \( X_2 \) be two arbitrary \( C^{r+1} \) embeddings with \( r \geq 4 \). Then each pair \((\varphi_1, \pi_1) \in \Gamma^s_\phi \) (with \( 2 \leq s \leq r - 2 \)) defines a unique solution \( \phi \) of the field equation (1) such that

1. the Cauchy datum associated with the embedding \( X_1 \) is the given pair \((\varphi_1, \pi_1)\);
2. the embedding \( X_2 \) gives a well-defined Cauchy datum \((\varphi_2, \pi_2)\) that belongs to \( \Gamma^s_\phi \);
3. the map \( \rho_{X_1, X_2} : \Gamma^s_\phi \to \Gamma^s_\phi \) thus defined is an automorphism of the Sobolev space \( \Gamma^s_\phi \).

Note that we obtain a maximal classical solution in the sense that the embedding \( X_2 \) can be chosen to map \( \Sigma \) so that it passes through any given point of \( \mathcal{M} \).

The theorem above implies that a differentiable solution is obtained if \( s \) is sufficiently large. Indeed, the famous Sobolev lemma asserts that \( H^s_w(\Sigma) \subset C^{s'}_w(\Sigma) \) if \( s > s' + \frac{1}{2}\dim \Sigma \). For example, \( \phi \) will be \( C^2 \) if \( s = 4 \). The index \( s \) can be taken as large as one wishes if \( r \) is sufficiently large. In particular, for \( r = \infty \), one can take the intersection \( \Gamma^\infty_\phi \) of the Hilbert spaces \( \Gamma^s_\phi \), \( s = 4, 5, \ldots \), to give the space of all pairs of \( C^\infty \)-functions and densities that is equipped with the structure of a countably Hilbert nuclear space (see [17]).
III. THE EXTENDED PHASE SPACE

The theory of a scalar field on a curved background was rewritten in the form of a parametrized system in [7,15]. In this section, we shall reformulate a part of this work so that it becomes compatible with the mathematical formalism of Fischer and Marsden [15]. First however, the studies in [5,7] of the differential geometry of the space of embeddings \( \mathcal{E} \) must be extended to include certain bundles over \( \mathcal{E} \).

The construction of a smooth differential structure on a space of continuous maps between two finite-dimensional manifolds was described as early as 1958 by Eells [19], but we shall use the method developed more recently in [20]. Recall that, in a pair of local charts 

\[(U,h)\) of \( \Sigma \) and \((\bar{V},\bar{h})\) of \( \mathcal{M} \) \hfill (7)

(where \( X(U) \cap \bar{V} \neq \emptyset \)), a given embedding \( X : \Sigma \to \mathcal{M} \) can be represented by the function \( \bar{h} \circ X \circ h^{-1} : h(U) \to \mathbb{R}^4 \). We say that \( X \) belongs to the space \( H^s(\Sigma, \mathcal{M}) \) if these local representatives are in the Sobolev space \( H^s(h(U), \mathbb{R}^4) \) for all such pairs of local charts. This notion can be shown to be atlas-independent for \( s > 3/2 \) (which means that \( X \) is continuous since, according to the Sobolev lemma, \( X \in C^r(\mathcal{N}, \mathcal{M}) \) if \( s > r + \text{dim}(\mathcal{N})/2 \).

In what follows, a major role is played by the tangent and cotangent bundles to the infinite-dimensional manifold of embeddings. In the differential geometry of a finite-dimensional manifold \( \mathcal{N} \), a tangent vector \( \tau \) at a point \( p \in \mathcal{N} \) can be defined in several different ways. One algebraic approach is to view \( \tau \) as a derivation at \( p \) of the ring \( C^\infty(\mathcal{N}) \) of smooth functions on \( \mathcal{N} \): \( i.e., \tau : C^\infty(\mathcal{N}) \to \mathbb{R} \) is a linear map with the property that if \( f, g \in C^\infty(\mathcal{N}) \) then \( \tau(fg) = f(p)\tau(g) + g(p)\tau(f) \). A more geometrical approach is to define \( \tau \) as an equivalence class of local curves \( \sigma : (-\epsilon, \epsilon) \to \mathcal{N} \) where (i) \( \epsilon > 0 \) (and can be \( \sigma \)-dependent); (ii) \( \sigma(0) = p \); and (iii) two local curves \( \sigma_1 \) and \( \sigma_2 \) are regarded as being equivalent if their tangent vectors at \( p \in \mathcal{N} \) are equal as computed in a local coordinate system around \( p \) (the equivalence classes are independent of choice of coordinate system).

In the finite-dimensional case, these two definitions can be shown to be equivalent. However, the situation in infinite dimensions is quite different since complicated functional-
analytical problems need to be resolved before the algebraic definition can even be posed. Fortunately, the geometrical definition of a tangent vector as an equivalence class of curves still works well and, when applied to the case of interest, leads naturally to the definition of the tangent space $T_X H^s(\Sigma, \mathcal{M})$ to $H^s(\Sigma, \mathcal{M})$ at the embedding $X$ as the set of maps $V : \Sigma \rightarrow T\mathcal{M}$ with the property that the image $V(x)$ of a point $x \in \Sigma$ is a tangent vector on $\mathcal{M}$ at the point $X(x)$; i.e., $V(x) \in T_{X(x)}\mathcal{M}$; more formally, $V$ satisfies the relation $X = \chi \circ V$, where $\chi : T\mathcal{M} \rightarrow \mathcal{M}$ is the bundle projection of the tangent bundle $T\mathcal{M}$ of $\mathcal{M}$.

This defining property of $V \in T_X H^s(\Sigma, \mathcal{M})$ can be expressed in another way that will be useful later. Namely, we recall that if $\rho : E \rightarrow \mathcal{M}$ is the projection map of any fibre bundle $E$ over a manifold $\mathcal{M}$, and if $f : N \rightarrow \mathcal{M}$ is a map from another manifold $N$ into $\mathcal{M}$, then the pull-back bundle $f^* E$ over $N$ is defined as

$$f^* E := \{(x, e) \in N \times E \mid f(x) = \rho(e)\},$$

and a cross-section of this bundle is given by any map $\psi : N \rightarrow E$ such that $\rho(\psi(x)) = f(x)$. It follows therefore that a vector $V \in T_X H^s(\Sigma, \mathcal{M})$ can be regarded as a cross-section of the bundle $X^*(T\mathcal{M})$.

Note that the vector space $T_X H^s(\Sigma, \mathcal{M})$ can be given an $H^s$-structure so that the function space $H^s(\Sigma, \mathcal{M})$ becomes a Banach manifold modelled on the Banach space $T_X H^s(\Sigma, \mathcal{M})$ (for example, via an exponential map in $\mathcal{M}$). We shall assume from now on that this has been done, and we shall consider only embeddings that lie in $H^s(\Sigma, \mathcal{M})$, and (with the value $s$ of the Sobolev class understood) denote the set of all such by $\mathcal{E}$; i.e., $\mathcal{E} := \text{Emb}_g(\Sigma, \mathcal{M}) \cap H^s(\Sigma, \mathcal{M})$. It can be shown that $\mathcal{E}$ is an open subset of $H^s(\Sigma, \mathcal{M})$, and hence $\mathcal{E}$ is a Banach manifold with the same tangent spaces and analogous manifold structure as that of $H^s(\Sigma, \mathcal{M})$ itself.

The above definition of a tangent vector leads immediately to the definition of the tangent bundle $T\mathcal{E}$ of $\mathcal{E}$. More generally, if $\mathcal{E}$ is the tensor bundle of type $(R, S)$ over $\mathcal{M}$ ($R$ times contravariant and $S$ times covariant), we obtain an ‘$e$-tensor bundle’ $\mathcal{E}$ over $\mathcal{E}$ by defining an $e$-tensor at the point $X \in \mathcal{E}$ to be a map $\psi : \Sigma \rightarrow T\mathcal{E}$ such that $\psi(x) \in T_X \mathcal{M}$;
i.e., $X = \chi \circ \psi$, where $\chi$ now denotes the bundle projection of $\mathcal{R}_s^s T \mathcal{M}$. Equivalently, $\psi$ can be regarded as a cross-section of the bundle $X^*(\mathcal{R}_s^s T \mathcal{M})$ over $\Sigma$.

We shall need an even more general $e$-tensor of the type defined in [5] which transforms as a tensor density of type $(r, s, w)$ with respect to a coordinate change in $\Sigma$ around $x$, and as a tensor of type $(R, S)$ with respect to a coordinate change in $\mathcal{M}$ around the point $X(x) \in \mathcal{M}$. The precise definition of such an $e$-tensor is that it is a cross-section $\psi$ of the bundle $r, w \mathcal{T}_x \Sigma \otimes X^* (\mathcal{R}_s^s T \mathcal{M})$ over $\Sigma$, where $r, w \mathcal{T}_x \Sigma$ is the bundle of tensors over $\Sigma$ that are $r$ times contravariant, $s$ times covariant, and of tensor-density weight $w$. It follows that, for all $x \in \Sigma$, we have $\psi(x) \in r, w \mathcal{T}_x \Sigma \otimes \mathcal{R}_s^s T \mathcal{X}(x) \mathcal{M}$, and hence $\psi(x)$ can be represented by its components $\psi_{k_1 \ldots k_r \mu_1 \ldots \mu_R}^{l_1 \ldots l_s \nu_1 \ldots \nu_S} (x)$ on $\Sigma$ defined with respect to an appropriate pair of coordinate charts on $\Sigma$ and $\mathcal{M}$.

There is another way of representing $e$-tensors which is particularly useful for discussing tangent vectors to the collection of all $e$-tensors of a certain type. Namely, recall that if $V$ and $W$ are any pair of finite-dimensional vector spaces then there is a canonical isomorphism of $V^* \otimes W$ with the space $L(V, W)$ of linear maps from $V$ to $W$ in which the linear map $L_{\ell \otimes w}$ associated with $\ell \otimes w \in V^* \otimes W$ is defined by $L_{\ell \otimes w}(v) := \langle \ell, v \rangle w$ for all $v \in V$. In particular, we note that $r, w \mathcal{T}_x \Sigma$ is the algebraic dual of $s, -w \mathcal{T}_x \Sigma$, and hence it follows that if $\psi$ is an $e$-tensor with $\psi(x) \in r, w \mathcal{T}_x \Sigma \otimes \mathcal{R}_s^s T \mathcal{X}(x) \mathcal{M}$, then we can identify $\psi(x)$ as a linear map $\psi(x) : s, -w \mathcal{T}_x \Sigma \to \mathcal{R}_s^s T \mathcal{X}(x) \mathcal{M}$. This can be summarised rather neatly by defining an $e$-tensor to be a vector bundle map $\psi$ from $s, -w \mathcal{T} \Sigma$ to $\mathcal{R}_s^s T \mathcal{M}$, i.e., the following diagram is commutative

\[
\begin{array}{ccc}
\mathcal{R}_s^s T \Sigma & \xrightarrow{\psi} & \mathcal{R}_s^s T \mathcal{M} \\
\downarrow \rho & & \downarrow \chi \\
\Sigma & \xrightarrow{X} & \mathcal{M}
\end{array}
\]

where $\rho$ and $\chi$ are the projection maps in the indicated tensor bundles over $\Sigma$ and $\mathcal{M}$ respectively. Thus if $\tau \in s, -w \mathcal{T}_x \Sigma$, we have $\psi(\tau) \in \mathcal{R}_s^s T \mathcal{X}(x) \mathcal{M}$.

We shall denote the bundle of such $e$-tensors by $\mathcal{R}_s^s T \mathcal{E}$. The linear space $\mathcal{R}_s^s T \mathcal{X} \mathcal{E}$ can be given a Sobolev structure with an arbitrary Sobolev class (not necessarily the same as
that of $\mathcal{E}$). An easy, ‘covariant’ method for doing so can be obtained by using an auxiliary Riemannian metric on $\mathcal{M}$ in addition to that on $\Sigma$. This enables covariant derivatives of $e$-tensors to be defined (see [5]). However, a ‘coordinate dependent’ method is even easier to define, and is more general in the sense that it can also be used for objects of higher rank, like the elements of $T_{(X,P)}(T^*\mathcal{E})$ (see below). This is based on pairs of charts Eq. (7), which will associate a map of $h(X^{-1}(X(U) \cap \bar{V})) \subset \mathbb{R}^3$ into $\mathbb{R}^m$ with any $e$-tensor. One can then define a Sobolev scalar product by patching together the integrands within each set $h(X^{-1}(X(U) \cap \bar{V}))$ with the aid of a partition of unity corresponding to a covering of $\Sigma$ by these sets. This scalar product depends on the system of charts chosen, but the topology does not and is equivalent to that obtained using the covariant method.

It is clear that the usual tensor operations like linear combination, tensor product and contraction at a point $x \in \Sigma$ or $X(x) \in \mathcal{M}$, define corresponding operations on the $e$-tensors (for details see [4]). The result is an $e$-tensor whose Sobolev class coincides with the lowest class involved in the operation.

There is one more operation of importance that we shall call ‘pairing’. Let $\xi \in \mathcal{S}_{s,s}^{R,r,w}T_X\mathcal{E}$ and $\eta \in \mathcal{S}_{s}^{R,r,1-w}T_X\mathcal{E}$. The pairing $\langle \xi, \eta \rangle$ is defined by

$$
\langle \xi, \eta \rangle := \int_{\Sigma} d^3 x \xi(x) \cdot \eta(x)
$$

where $\xi(x) \cdot \eta(x)$ denotes the contractions at $x$ and $X(x)$ such that all indices of $\xi$ are contracted with those of $\eta$ in the order in which they appear. Then $\xi(x) \cdot \eta(x)$ is a scalar density on $\Sigma$, and hence $\langle \xi, \eta \rangle$ is a coordinate independent real number.

A particularly important example of a bundle of $e$-tensors is the tangent bundle $T\mathcal{E} := \mathcal{T}^0_0\mathcal{E}$ whose points $\xi$ are pairs $(X,V)$ where $X \in \mathcal{E}$, and $V$ is a $T\mathcal{M}$-valued function on $\Sigma$ with $V(x) \in T_{X(x)}\mathcal{M}$; equivalently, $V$ is a cross-section of $X^*(T\mathcal{M})$. Even more important for our purposes is the cotangent bundle $T^*\mathcal{E}$. To define cotangent vectors we have to identify $T^*_X\mathcal{E}$ with a particular topological vector space of real-valued, linear functions on $T_X\mathcal{E}$. The appropriate choice for our purposes is the space $\mathcal{S}^{0,1}_{1,0}T_X\mathcal{E}$ of Sobolev class $s'$, the linear operation being the pairing (the class parameter $s'$ must satisfy the condition $s' \leq s$, where
s is the class parameter of $E$, but can otherwise be arbitrary). Thus, $P \in T^*_X E := \mathbb{T}^*_0 T^*_X E$ is a cross-section of the bundle $D^1 \Sigma \otimes X^*(T^*M)$ where, in general, $D^w \Sigma$ is shorthand for the real-line bundle $^0_0 T^w \Sigma$ of scalar densities on $\Sigma$ of weight $w$. Thus $P(x) \in D^1_x \otimes T^*_X(x) M$; equivalently, $P$ is a bundle map from $D^1 \Sigma$ to $T^*_M$ that ‘covers’ $X$ in the sense that $P(\tau) \in T^*_X M$ for all $\tau \in D^1_x \Sigma$.

With respect to a local coordinate chart $y^\mu$, $\mu = 0, 1, 2, 3$, on $M$, the pair $(X, P) \in T^*_E$ is represented by local functions $(X^\mu, P_\nu)$ on $\Sigma$, where $X^\mu$ is defined by

$$X^\mu(x) := y^\mu(X(x)), \quad (11)$$

and where the four scalar-density functions $P_\nu, \nu = 0, 1, 2, 3$, on $\Sigma$ are defined by the expansion

$$P(x) = P_\nu(x) dy^\nu|_{X(x)} \quad (12)$$

using the differentials $dy^\nu$ associated with the local coordinate system $y^\mu$ on $M$ (as usual, summation over repeated indices is understood) and using a local coordinate system on $\Sigma$ to locally-trivialise the line bundle $D^{-1} \Sigma$. In terms of these local functions, the pairing operation is

$$\langle (X, P), (X, V) \rangle = \int_\Sigma d^3 x P_\mu(x) V^\mu(x) \quad (13)$$

where the functions $V^\mu$ on $\Sigma$ are defined by the equation

$$V(x) = V^\mu(x) \left( \frac{\partial}{\partial y^\mu} \right)_{X(x)}. \quad (14)$$

The $\mathbb{R}$-tensor bundles can be given a manifold structure based on that of $E$. A point in such a bundle $^R_{\mathbb{R}, \mathbb{R}} T^*_S E$ is represented by a pair $(X, \psi)$ where $X \in E$, and $\psi \in ^R_{\mathbb{R}, \mathbb{R}} T^*_X E$ is a cross-section of the bundle $^R_{\mathbb{R}, \mathbb{R}} T^\Sigma \otimes X^*(^R_{\mathbb{R}} T^*_M)$ or—better for our purposes—the pair $(X, \psi)$ fits into the commutative diagram Eq. (9). A tangent vector is defined as an equivalence classes of curves $t \mapsto (X_t, \psi_t)$, and it is clear from this geometrically that a tangent vector at $(X, \psi)$ consists of a pair of objects $(V, W)$ where (i) $V \in T^*_X E$ (i.e., $V(x) \in T^*_X M$);
(ii) \( W \) is a bundle map from \( \xi^{-w}T\Sigma \) to \( T(\xi S T\mathcal{M}) \) such that, for all \( \tau \in \xi^{-w}T_x\Sigma \), we have \( W(\tau) \in T_{\psi(\tau)}(\xi S T\mathcal{M}) \); and (iii) \( \chi_s(W(\tau)) = V(x) \) where \( \chi_s : T(\xi S T\mathcal{M}) \rightarrow T\mathcal{M} \) is induced from the bundle projector \( \chi : \xi S T\mathcal{M} \rightarrow \mathcal{M} \) in Eq. (9). Note that \( V \) and \( W \) both take their values in vector spaces, and hence the collection of all such pairs \((V,W)\) can be given an appropriate Sobolev structure to become Banach spaces. By this means, the bundle of \( e \)-tensors over \( \mathcal{E} \) becomes a Banach manifold.

Now, in general, if \( E \) is any vector bundle over \( \mathcal{M} \), the tangent space \( T_p E \) splits into a direct sum \( E_{\pi(p)} \oplus T_{\pi(p)} \mathcal{M} \) where \( E_{\pi(p)} \) denotes the fiber of \( E \) over the point \( \pi(p) \in \mathcal{M} \). However, in the absence of any connection on \( E \) there is no natural way of performing such a split. Note that in our case, where \( E = \xi S T\mathcal{M} \), this (non-canonical) split means that the vector \( W(\tau) \in T_{\psi(\tau)}(\xi S T\mathcal{M}) \) (with \( \tau \in \xi^{-w}T_x\Sigma \)) can be written as a sum of an element of \( \xi S T_{X(x)} \mathcal{M} \) (the analogue of \( E_{\pi(p)} \)) and an element (in fact, \( V(x) \)) of \( T_{X(x)} \mathcal{M} \). This means that, using a local coordinate system on \( \mathcal{M} \), the vector \( W(\tau) \) can be written as a collection of numbers associated with the point \( X(x) \in \mathcal{M} \), namely the components \( V^\mu(x) \) of the vector \( V(x) \), and the components of the space-time object in \( \xi S T_{X(x)} \mathcal{M} \), which we shall write as \( W_{\mu_1...\mu_R}(\tau) \).

A particularly simple example is the tangent bundle \( T(T^*\mathcal{E}) \). The bundle \( T^*\mathcal{E} \) consists of pairs \((X,P)\) where \( X \in \mathcal{E} \), and where \( P \) is a bundle map \( P : D^{-1}\Sigma \rightarrow T^*\mathcal{M} \) that covers \( X : \Sigma \rightarrow \mathcal{M} \), i.e., \( P(\tau) \in T_{X(\tau)}^*\mathcal{M} \) for all \( \tau \in D_{x}^{-1}\Sigma \). Then, according to the discussion above, a tangent vector to \( T^*\mathcal{E} \) at the point \((X,P)\) consists of a pair \((V,W)\) where \( V \in T_X \mathcal{E} \) and where \( W : D^{-1}\Sigma \rightarrow T(T^*\mathcal{M}) \) satisfies \( W(\tau) \in T_{P(\tau)}(T^*\mathcal{M}) \) with \( \pi_s(W(\tau)) = V(x) \) for all \( \tau \in D_{x}^{-1}\Sigma \), where \( \pi : T^*\mathcal{M} \rightarrow \mathcal{M} \) is the bundle projection.

The problem occasioned by the non-canonical split can be seen by looking at the situation using a local coordinate system. The element \((X,P) \in T^*\mathcal{E} \) is represented by the local functions \((X^\mu,P_\nu)\) on \( \Sigma \), and a transformation of coordinates in \( \mathcal{M} \) from \( y^\mu \) to \( y'^\alpha \) leads to new functions \((X'^\alpha,P'^\mu)\) satisfying

\[
X'^\alpha(x) = y'^\alpha(X(x)) \tag{15}
\]
\[ P^\mu_\beta(x) = J^{\nu}_{\beta}(X(x)) P_\nu(x), \tag{16} \]

where \( J^{\nu}_{\beta} \) denotes the matrix \( \partial y^{\mu}/\partial y^{\nu} \). As emphasised earlier, a tangent vector in infinite-dimensional differential geometry is defined as an equivalence class of curves and, with respect to the pair of charts Eq. (7), a curve in \( T^*E \) is represented by functions \((t, x) \mapsto (X^\mu(x,t), P_\nu(x,t))\). Hence the tangent vector to this curve is represented by the functions \( \dot{X}^\mu \) and \( \dot{P}_\nu \) on \( \Sigma \) where

\[
\dot{X}^\mu(x) := \frac{\partial X^\mu(x,t)}{\partial t} \bigg|_{t=0}, \quad \dot{P}_\nu(x) := \frac{\partial P_\nu(x,t)}{\partial t} \bigg|_{t=0}. \tag{17}
\]

Note that \( P_\mu(x,t) \) are components with respect to coordinates at the point \( X(x,t) \) of \( M \) while \( P_\mu(x,t+dt) \) are those at another point \( X(x,t+dt) \) of \( M \).

Differentiating formula Eq. (15) along the curve with respect to \( t \), we obtain

\[
\dot{X}^{\nu}(x) := \frac{\partial X^{\nu}(x,t)}{\partial t} \bigg|_{t=0} = \frac{\partial y^{\nu}(X(x),t)}{\partial t} \bigg|_{t=0} \frac{\partial y^{\nu}(X(x))}{\partial y^{\mu}(X(x),t)} \bigg|_{t=0} = J^{\nu}_{\mu}(X(x)) \dot{X}^\mu(x), \tag{18}
\]

which shows that, as expected, the derivatives \( \dot{X}^\mu(x) \) transform on \( M \) in a tensorial way.

On the other hand, the analogous calculations for Eq. (14) yield

\[
\dot{P}^{\nu}_\beta(x) = J^{\nu}_{\beta\nu}(X(x)) \dot{X}^\mu(x) P_\mu(x) + J^{\nu}_{\gamma}(X(x)) P_\gamma(x) \dot{P}_\nu(x) \tag{19}
\]

where \( J^{\nu}_{\beta\nu} \) denotes the derivative of the matrix \( J^{\nu}_{\beta} \) with respect to the original coordinates \( y^{\mu} \). This shows that the four quantities \( \dot{P}^{\nu}_\beta(x) \) cannot be regarded as the components of any tensorial object on \( M \). In general, a tangent vector to \((X, P) \in T^*E \) can be represented by the functions \((V^{\mu}, W_\nu)\) on \( \Sigma \) which transform under a change of coordinates on \( M \) as

\[
\begin{pmatrix}
V^{\nu}(x) \\
W^{\nu}_\beta(x)
\end{pmatrix} = \begin{pmatrix}
J^{\nu}_{\mu}(X(x)) & 0 \\
J^{\nu}_{\beta\mu}(X(x)) P_\mu(x) & J^{\nu}_{\gamma}(X(x))
\end{pmatrix} \begin{pmatrix}
V^{\mu}(x) \\
W_\nu(x)
\end{pmatrix}. \tag{20}
\]

We shall often need to work with the cotangent vectors from \( T^*(T^*E) \). Let us describe their most important properties. As discussed above, a vector in \( T_{(X,P)}(T^*E) \) is a pair \((V,W)\) where \( V \in T_XE \), and \( W : D^{-1}\Sigma \rightarrow T(T^*M) \) is such that \( \pi_* (W(\tau)) = V(\tau) \in T_{X(\tau)}M \) for
all \( \tau \in \mathcal{D}_x^{-1}\Sigma \), where \( \pi : T^*\mathcal{M} \to \mathcal{M} \) is the bundle projection. By definition, the kernel of the map \( \pi_* : T(T^*\mathcal{M}) \to T\mathcal{M} \) is the set of vertical vectors in \( T(T^*\mathcal{M}) \), and at each \( k \in T^*\mathcal{M} \) there is a natural isomorphism \( \iota \) of \( V_k(T^*\mathcal{M}) \) (the vertical tangent vectors at \( k \)) with \( T^*_{\pi(k)}\mathcal{M} \). In fact, at each \( k \in T^*\mathcal{M} \), the map \( \pi_* \) fits into the short exact sequence

\[
0 \longrightarrow T^*_{\pi(k)}\mathcal{M} \overset{\iota}{\longrightarrow} T_k(T^*\mathcal{M}) \overset{\pi_*}{\longrightarrow} T_{\pi(k)}\mathcal{M} \longrightarrow 0. \tag{21}
\]

The dual of this sequence is the short exact sequence

\[
0 \longrightarrow T^*_{\pi(k)}\mathcal{M} \overset{\pi^*_k}{\longrightarrow} T^*_k(T^*\mathcal{M}) \overset{\iota^*}{\longrightarrow} T^*_{\pi(k)}\mathcal{M} \longrightarrow 0 \tag{22}
\]

where a ‘\(^*\)’ superscript denotes the adjoint of the linear map to which it is attached. Note that, since \( \mathcal{M} \) is finite-dimensional, the third term in Eq. \((22)\), \( i.e. \), \( T^*_{\pi(k)}\mathcal{M} \) is (non-canonically) isomorphic to \( T_{\pi(k)}\mathcal{M} \).

Using the ideas above, it is natural to define an element of \( T^*_{(X,P)}(T^*\mathcal{E}) \) as a pair \((A, B)\) where \( B : \Sigma \to T^{**}\mathcal{M} \simeq T\mathcal{M} \), and where the bundle map \( A : \mathcal{D}^{-1}\Sigma \to T^*(T^*\mathcal{M}) \) satisfies \( A(\tau) \in T^*_{\pi(\tau)}(T^*\mathcal{M}) \) with \( \iota^*(A(\tau)) = B(x) \) for all \( \tau \in \mathcal{D}_x^{-1}\Sigma \). In local coordinates on \( \mathcal{M} \), an element \((A, B) \in T^*_{(X,P)}(T^*\mathcal{E})\) can be represented by a set of local functions \( (A_\mu, B^\nu) \) on \( \Sigma \) (each \( A_\mu \) is actually a scalar density on \( \Sigma \); \( B^\nu \) are genuine scalar functions). The pairing of such an object with a vector \((V, W)\) in \( T_{(X,P)}(T^*\mathcal{E}) \) is defined by

\[
\langle (A, B), (V, W) \rangle := \int_\Sigma d^3x \left( A_\mu(x)V^\mu(x) + B^\nu(x)W_\nu(x) \right). \tag{23}
\]

The condition that the result be independent of coordinates on \( \mathcal{M} \) determines the transformation of \((A_\mu(x), B^\nu(x))\) to be

\[
\begin{pmatrix}
A'_\alpha(x) \\
B'^\beta(x)
\end{pmatrix} = \begin{pmatrix}
J^\mu_{\alpha'} - J^\mu_{\alpha \nu}(X(x)) P_\nu(x) \\
0 & J^\beta_{\nu}(X(x))
\end{pmatrix} \begin{pmatrix}
A_\mu(x) \\
B^\nu(x)
\end{pmatrix}. \tag{24}
\]

Thus \( V^\mu(x) \) and \( B^\mu(x) \) represent tensorial objects on \( \mathcal{M} \), but \( W_\mu(x) \) and \( A_\mu(x) \) do not.

A key observation is the following. If the transformation law Eq. \((20)\) is compared with Eq. \((24)\), it is clear that the quantity \((G^\mu, -F_\mu)\) formed from the ‘components’ of a covector
on $T^*\mathcal{E}$ transforms as a tangent vector on $T^*\mathcal{E}$. Thus we have a map $J_\mathcal{E}: T^*(T^*\mathcal{E}) \rightarrow T(T^*\mathcal{E})$ defined in local coordinates on the component functions by

$$J_\mathcal{E}(A_\alpha, B^\nu) = (B^\mu, -A_\alpha). \quad (25)$$

We can choose the Sobolev classes so that the classes of the functions $V^\alpha$, $X^\alpha$ and $B^\alpha$ coincide, and so do those of $P_\beta$, $W_\beta$ and $A_\beta$. Then $J_\mathcal{E}$ is a Sobolev space isomorphism.

More abstractly, we recall that if $Q$ is any finite-dimensional manifold there is a canonical isomorphism $j: T^*_p(T^*Q) \rightarrow T_p(T^*Q)$ defined on any cotangent vector $\ell$ at the point $p$ in $T^*Q$ by

$$\omega(j(\ell), v) = \langle \ell, v \rangle_p, \text{ for all } v \in T_p(T^*Q) \quad (26)$$

where $\omega$ is the canonical two-form on the cotangent bundle $T^*Q$. In the context of the embedding space, if $(A, B) \in T^*_\mathcal{E}(T^*\mathcal{E})$ then $A(\tau) \in T^*_\mathcal{M}(T^*\mathcal{M})$, and hence we can use the isomorphism $j: T^*_\mathcal{M}(T^*\mathcal{M}) \rightarrow T^*_\mathcal{M}(T^*\mathcal{M})$ to define a map $J: T^*_\mathcal{E}(T^*\mathcal{E}) \rightarrow T^*_\mathcal{M}(T^*\mathcal{E})$ by requiring that $J(A, B)(\tau) := j(A(\tau))$. This is the coordinate-free definition of the object given in Eq. (25).

The phase space $\Gamma_\phi$ of the scalar field $\phi$ was introduced in section II. We note now that $\Gamma_\phi$ can be considered as the cotangent bundle $T^*Q$, where $Q$ is the space $H^\infty_0$ of all $C^\infty$-scalar fields on $\Sigma$. As $Q$ is a linear space, there is a natural identification between the spaces $T^*Q$ and $Q \times T^*_\phi Q$ for any $\varphi \in Q$. Also, $T^*_\phi Q = H^\infty_1$ is itself a linear space, and so there is an identification between the spaces $T_{(\varphi, \pi)}(T^*Q)$ and $T^*Q \simeq H^\infty_0 \times H^\infty_1$. Similarly, $T_{(\varphi, \pi)}(T^*Q) \simeq H^\infty_1 \times H^\infty_0$. More precisely, if $(\xi, \eta) \in T_{(\varphi, \pi)}(T^*Q)$ and $(f, h) \in H^\infty_1 \times H^\infty_0$, then the pairing $(f, h): T_{(\varphi, \pi)}(T^*Q) \rightarrow \mathbb{R}$ is defined by

$$\langle (f, h), (\xi, \eta) \rangle := \int_\Sigma d^3x (f \xi + h \eta). \quad (27)$$

The extended phase space $\Gamma$ of the parametrized scalar field of $[\mathbb{I}]$ is defined as $\Gamma := \Gamma_\phi \times T^*\mathcal{E}$. Hence $\Gamma \simeq T^*(Q \times \mathcal{E})$, so that the extended phase space is again a cotangent bundle. In the context of the full space $\Gamma$, a tangent vector from $T_{(\varphi, \pi, X, P)}\Gamma$ can be specified
by its ‘components’ \((\Phi, \Pi, V, W)\), where \((V, W) \in T_{(X,P)}(T^*E)\) and \((\Phi, \Pi) \in T_{(\varphi, \pi)}(T^*Q)\). Similarly, a cotangent vector from \(T^*_{(\varphi, \pi, X, P)}\Gamma\) can be specified by \((A_\varphi, A_\pi, A, B)\), where \((A, B) \in T^*_{(X,P)}(T^*E)\) and \((A_\varphi, A_\pi) \in T^*_{(T^*Q)}\). The natural pairing is

\[
\langle (A_\varphi, A_\pi, A, B), (\Phi, \Pi, V, W) \rangle := \int_\Sigma d^3x (A_\varphi \Phi + A_\pi \Pi + A_\mu V^\mu + B^\mu W_\mu).
\] (28)

There is an isomorphism \(J : T^*_{(\varphi, \pi, X, P)}\Gamma \to T_{(\varphi, \pi, X, P)}\Gamma\) (if the Sobolev classes are chosen to match each other) given by

\[
J(A_\varphi, A_\pi, A_X, A_P) := (A_\pi, -A_\varphi, A_P, -A_X).
\] (29)

Using this isomorphism, a symplectic structure \(\Omega\) on \(\Gamma\) can be defined as follows. Let \(v_1\) and \(v_2\) be two vectors in \(T^*_{(\varphi, \pi, X, P)}\Gamma\). Then

\[
\Omega(v_1, v_2) := -\langle J^{-1}v_1, v_2 \rangle
\] (30)

or, in ‘component’ form,

\[
\Omega((\Phi_1, \Pi_1, V_1, W_1), (\Phi_2, \Pi_2, V_2, W_2)) = \int_\Sigma d^3x (\Pi_1 \Phi_2 - \Phi_1 \Pi_2 + W_1 V_2^\mu - V_1^\mu W_2\mu).
\] (31)

It follows at once that (i) \(\Omega(v_1, v_2) = -\Omega(v_2, v_1)\); (ii) \(\Omega\) is weakly non-degenerate (see [13]); and (iii) \(\Omega\) is not only closed but also exact.

As \(\Omega\) is only a weak symplectic form, not every differentiable function on \(\Gamma\) will have a Hamiltonian vector field. The class of functions that do can be characterized as follows. If \(F : \Gamma \to \mathbb{R}\), we say that \(F\) has a gradient if the following two conditions are satisfied:

1. the Fréchet derivative, \(DF|_{(\varphi, \pi, X, P)} : T_{(\varphi, \pi, X, P)}\Gamma \to \mathbb{R}\) is a bounded linear map;

2. there exists \(\text{grad} F \in T^*_{(\varphi, \pi, X, P)}\Gamma\) such that \(\langle \text{grad} F, v \rangle = DF|_{(\varphi, \pi, X, P)}(v)\) for all \(v \in T_{(\varphi, \pi, X, P)}\Gamma\). The ‘components’ of this gradient will be denoted by the collection of functions \((\text{grad}_\varphi F, \text{grad}_\pi F, (\text{grad}_X F)_\mu, (\text{grad}_P F)^\nu)\).

Condition 2 means that \(DF\) must be regular (no distributions are accepted!), and hence we have to work with smeared objects. This will not lead to any real loss of generality.
The quantity \( \text{grad} F \) is calculated from \( DF \) as usual by integration by parts (if \( F \) contains derivatives).

For a differentiable function with a gradient, we can define an associated ‘Hamiltonian vector field’. Specifically, if \( F \) is such a function, then \( \xi_F \in T(\varphi,\pi,X,P)\Gamma \) is defined by the relation

\[
\langle \text{grad} F, v \rangle = \Omega(v, \xi_F), \text{ for all } v \in T(\varphi,\pi,X,P)\Gamma. \tag{32}
\]

Hence, because of Eq. (30), we see that \( \langle \text{grad} F, v \rangle = \langle J^{-1}\xi_F, v \rangle \) for all \( v \), and so \( \xi_F = J(\text{grad} F) \).

Finally, the Poisson bracket of a pair of differentiable functions \( F \) and \( G \) is defined as
\[
\{ F, G \} := -\Omega(\xi_F, \xi_G), \tag{33}
\]
This Poisson bracket is antisymmetric and, since \( \Omega \) is closed, it satisfies the Jacobi identity.

**IV. THE CONSTRAINT MANIFOLD**

The parametrized scalar field theory possesses a set of constraints \( \mathcal{H}_\mu = 0 \) on the Cauchy data \( (\varphi, \pi, X, P) \). These constraints are contained in the map

\[
C : \Gamma \rightarrow T^*\mathcal{E}, \tag{34}
\]

\[
(\varphi, \pi, X, P) \mapsto (X, \mathcal{H}(\varphi, \pi, X, P)) \tag{35}
\]

where \( \mathcal{H}(\varphi, \pi, X, P) \in T_X^*\mathcal{E} \), i.e., \( \mathcal{H}(\varphi, \pi, X, P)(x) \in T_{X(x)}^*(\mathcal{M}) \) for all \( x \in \Sigma \), so that \( \mathcal{H}(\varphi, \pi, X, P)(x) = \mathcal{H}(\varphi, \pi, X, P)_\mu(x) \, dy^\mu|_{X(x)} \) in a coordinate system \( y^\mu \) on \( \mathcal{M} \). The specific form of the constraints \( \mathcal{H}_\mu(x) \) is given in [7] as

\[
\mathcal{H}_\mu = P_\mu + \mathcal{H}_\mu^\phi, \tag{36}
\]
\[
\mathcal{H}_\mu^\phi = -\mathcal{H}_\perp n_\mu + \mathcal{H}_k^\phi X_\mu^k, \tag{37}
\]
where \( n_\mu(x) \) is the unit normal space-time vector at \( X(x) \in \mathcal{M} \), and where \( X_k^\mu(x) := g_{\mu\nu}(x) \gamma^{kl}(X(x)) X_{,l}^\nu(x) \). The components \( \mathcal{H}_\mu(\varphi, \pi, X) \) can be projected normal, and tangential, to the hypersurface \( X(\Sigma) \) to give

\[
\mathcal{H}_\mu = \frac{1}{2}(\det \gamma)^{1/2} \left( \frac{\pi^2}{\det \gamma} + \gamma^{kl} \varphi_k \varphi_l + m^2 \varphi^2 \right),
\]

\[
\mathcal{H}_k = \pi \varphi_{,k}.
\]

The components \( H_{\varphi}^{\phi, \pi}(\varphi, \pi, X) \) can be projected normal, and tangential, to the hypersurface \( X(\Sigma) \) to give

\[
H_{\varphi}^{\phi, \pi} = \frac{1}{2}(\det \gamma)^{1/2} \left( \frac{\pi^2}{\det \gamma} + \gamma^{kl} \varphi_k \varphi_l + m^2 \varphi^2 \right),
\]

\[
H_k = \pi \varphi_{,k}.
\]

The goal of this section is to explore the pre-symplectic structure of the manifold \( \tilde{\Gamma} \) of solutions to these constraints, i.e., \( \tilde{\Gamma} := \{(\varphi, \pi, X, P) \in \Gamma \mid C(\varphi, \pi, X, P) = (X, 0)\} \). In particular, we have the following theorem

**Theorem 2** Let the Sobolev class of all spaces involved be \( \infty \). Then, \( \tilde{\Gamma} \) is a \( C^\infty \)-submanifold of \( \Gamma \) in a neighbourhood of any of its points. This constraint manifold \( \tilde{\Gamma} \) is given by \( \tilde{\Gamma} = \tilde{C}(\Gamma \times \mathcal{E}) \), where \( \tilde{C} : \Gamma \phi \times \mathcal{E} \to \Gamma \) is defined by \( \tilde{C}(\varphi, \pi, X) := (\varphi, \pi, X, -\mathcal{H}^{\phi}(\varphi, \pi, X)) \).

**Proof** The proof is similar to that in [13], but requires less sophisticated functional analysis because our constraints are available in an explicit form; in particular—unlike the case in [13]—we do not need to use the Fredholm alternative theorem.

We start by assuming that \( T^*\Gamma \) is \( C^\infty \), and postulate that the Sobolev class of \( \Pi \) and \( W \) is \( s-1 \), where \( s \) is the class of \( X, \Phi \) and \( V \). Consider the map \( DC|_{(\varphi, \pi, X, \Pi)} : T_{(\varphi, \pi, X, \Pi)} \Gamma \to T_{H(\varphi, \pi, X, \Pi)}(T^*\mathcal{E}) \). To use the implicit function theorem, this map must be a surjection with a splitting kernel (see, e.g., [21]). However, \( DC|_{(\varphi, \pi, X, \Pi)} \) is trivially surjective. Moreover, its kernel in \( T_{\tilde{C}(X, \varphi, \pi)} \Gamma \) is given by \( D\tilde{C}|_{(\varphi, \pi, X)}(T_{(\varphi, \pi, X})(\Gamma \phi \times \mathcal{E})) \). We shall now show that \( DC|_{(\varphi, \pi, X)} \) is injective with a splitting image.

The map \( D\tilde{C}|_{(\varphi, \pi, X)} : T_{(\varphi, \pi, X)}(\Gamma \phi \times \mathcal{E}) \to T_{\tilde{C}(\varphi, \pi, X)} \Gamma \) between the Sobolev spaces \( T_{(\varphi, \pi, X)}(\Gamma \phi \times \mathcal{E}) \cong \Gamma^s \times H^s_0 \) and \( T_{\tilde{C}(\varphi, \pi, X)} \Gamma \cong \Gamma^s \times H^s_0 \times H^{s-1}_1 \) has a derivative given by

\[
D\tilde{C}|_{(\varphi, \pi, X)}(\Phi, \Pi, V) = (\Phi, \Pi, V, -DH^{\phi}|_{(\varphi, \pi, X)}(\Phi, \Pi, V)),
\]

where

\[
DH^{\phi}|_{(\varphi, \pi, X)}(\Phi, \Pi, V) = D_\Phi H^{\phi}|_{(\varphi, \pi, X)}(\Phi) + D_\Pi H^{\phi}|_{(\varphi, \pi, X)}(\Pi) + DX H^{\phi}|_{(\varphi, \pi, X)}(V).
\]
Using the results of [7] and Eqs. (37–39), we find after some calculation that

\[ D_{\phi} \mathcal{H}_{\mu}^{\phi}_{|x}(\Phi) = (\det \gamma)^{1/2}(L_{\mu}^k \Phi_k - n_\mu m^2 \varphi \Phi), \]  

(42)

\[ D_{\pi} \mathcal{H}_{\mu}^{\phi}_{|x}(\Pi) = L_{\mu}^\perp \Pi, \]  

(43)

\[ D_{X} \mathcal{H}_{\mu}^{\phi}_{|x}(V) = \frac{1}{2}(\det \gamma)^{1/2}K_{\mu\nu}^k V^\nu + \mathcal{H}_{\mu}^{\phi} \Gamma_{\mu\nu}^k V^\nu, \]  

(44)

where

\[ L_{\mu}^k = -\gamma^{kl} \varphi_{|l} n_\mu + \frac{\pi}{(\det \gamma)^{1/2}} X^k_{\mu}, \]  

(45)

\[ L_{\mu}^\perp = -\frac{\pi}{(\det \gamma)^{1/2}} n_\mu + \varphi_{|k} X^k_{\mu}, \]  

(46)

\[ K_{\mu\nu}^m = \left(\frac{\pi^2}{(\det \gamma)^{1/2}} - \gamma^{kl} \varphi_{|l} \varphi_{|l} - m^2 \varphi^2 \right) n_\mu X^m_{\nu} \]  

\[ + \left(\frac{\pi^2}{(\det \gamma)^{1/2}} + \gamma^{kl} \varphi_{|l} \varphi_{|l} + m^2 \varphi^2 \right) X^m_{\mu} n_\nu + 2 \varphi_{|l} \varphi_{|l} \gamma^{km} n_\mu X^k_{\nu} \]  

\[ -2 \frac{\pi}{(\det \gamma)^{1/2}} \varphi_{|k} (\gamma^{km} n_\mu n_\nu + X^k_{\nu} X^m_{\mu}), \]  

(47)

and ‘\( || \)’ denotes the bi-covariant derivative, for example \( Y_{k||l} = Y^{\mu}_{k,l} + \Gamma_{\rho\sigma}^{\mu} Y^\rho_{k,l} X^\sigma_{l} - \gamma_{kl}^m Y^m_{\mu} \),

where \( \Gamma_{\rho\sigma}^{\mu} \) is the Christoffel symbol of the metric \( g_{\mu\nu} \) on \( M \), and \( \gamma_{kl}^m \) is the Christoffel symbol of the pull-back of \( g \) to \( \Sigma \) by \( X \). By inspection, these formulas imply the crucial result that the map \( D_{\phi} \mathcal{H}_{\mu}^{\phi}_{|x} : H^s_0 \times E^s_0 \to H^{-1}_1 \) is continuous and bounded (recall that \( \varphi, \pi \) and \( X \) are \( C^\infty \)). Hence, for all \( (\Phi, \Pi, V) \in \Gamma_{\phi} \times H^s_0 \), the map \( D_{\phi} C_{|x}^{\phi} \) is continuous and bounded, and—by inspection—injective. It follows from the closed graph theorem that the image \( D_{\phi} \mathcal{C}_{|x}^{\phi} (\Gamma_{\phi} \times H^s_0) \) is therefore closed in \( \Gamma_{\phi} \times H^s_0 \times H^{-1}_1 \). However, a closed subspace of a Hilbert space splits, which proves the theorem. \( \text{QED} \)

By abuse of language, we will often call \( \Gamma_{\phi} \times E \) the ‘constraint manifold’.

For each fixed value of \( x \) and \( \mu \), the quantity \( \mathcal{H}_{\mu}(x) \) can be viewed as a function on the phase space, but it has no gradient; in particular, the Poisson bracket of a pair of such functions is not well-defined. Constraint functions with gradients can be constructed by ‘smearing’. Specifically, if \( N \in TXE \) (i.e., \( N(x) \in TX(x)M \)), then \( \mathcal{H}_N \) is defined as

\[ \mathcal{H}_N := \int_{\Sigma} d^3x N^\mu(x) \mathcal{H}_\mu(x). \]  

(48)
It is clear that the equations $\mathcal{H}_N = 0$ for all $N \in T_X E$ are equivalent to $\mathcal{H}_\mu(x) = 0$ for all $\mu$, $x$. In particular, let $U$ be a $C^\infty$ vectorfield on $M$. Then each embedding $X$ determines an element, $x \mapsto U^\mu(X(x)) (\partial/\partial y^\mu)_{X(x)}$, of $T_X E$. This is the type of smearing used in [7].

Let us calculate the gradient of $\mathcal{H}_N$. Starting from Eq. (37), using Eqs. (42), (43) and (44), and integrating by parts, we obtain

\[
\text{grad}_\varphi \mathcal{H}_N = (\det \gamma)^{1/2} [-(N^\mu L^k_\mu)_{||k} - m^2 \varphi n_\mu N^\mu],
\]

\[
\text{grad}_\pi \mathcal{H}_N = \frac{1}{\mu} N^\mu,
\]

\[
(\text{grad}_X \mathcal{H}_N)_\nu = N^\mu_{||\mu} \mathcal{H}_\mu - \frac{1}{2} (\det \gamma)^{1/2} (N^\mu K^k_{\mu\nu})_{||k} - P_\kappa \Gamma^\kappa_{\mu\nu} N^\mu,
\]

\[
(\text{grad}_P \mathcal{H}_N)_\nu = N^\nu,
\]

where $L^k_\mu$, $L^k_\mu$ and $K^k_{\mu\nu}$ are given by Eqs. (45), (46) and (47). In [7], the following theorem was shown:

**Theorem 3** Let $M$ and $N$ be two $C^\infty$ vector fields on $M$. Then

\[
\{\mathcal{H}_M, \mathcal{H}_N\} = -\mathcal{H}_{[M,N]},
\]

where $[M, N]$ is the Lie bracket of the fields $M^\mu$ and $N^\mu$.

Substituting the expressions Eqs. (49)–(52) for the gradients into the Poisson brackets (53) we obtain an identity that plays an important role in some proofs:

\[
\frac{1}{2} \int_S d^3x (\det \gamma)^{1/2} [(M^\mu K^k_{\mu\nu})_{||k} N^\nu - (N^\mu K^k_{\mu\nu})_{||k} M^\nu] = \int_S d^3x (\text{grad}_\varphi \mathcal{H}_M \text{grad}_\pi \mathcal{H}_N - \text{grad}_\pi \mathcal{H}_M \text{grad}_\varphi \mathcal{H}_N).
\]

The following theorem was essentially shown in [7]:

**Theorem 4** Let $\phi$ satisfy the Klein-Gordon equation (1) on $M$. Then for each curve $\lambda \mapsto X_\lambda$ with the tangent vector field $N$ on $E$, the initial data $(\varphi_\lambda, \pi_\lambda)$ for $\phi$ on $X_\lambda(\Sigma)$ satisfy the evolution equation

\[
(\dot{\varphi}, \dot{\pi}, \dot{X}, \dot{P}) = J(\text{grad}\mathcal{H}_N).
\]
The equation for $\dot{P}$ is a consequence of the first three equations and the constraints $\mathcal{H}_N = 0$ for all $N \in T_X \mathcal{E}$.

Conversely, if a curve $\lambda \mapsto (\varphi_\lambda, \pi_\lambda, X_\lambda)$ on $\Gamma \times \mathcal{E}$ satisfies the evolution equations \[57\], then it defines a unique solution $\phi$ of the Klein-Gordon equation.

Thus the Hamiltonian vector fields $J(\text{grad}\mathcal{H}_N)$ of the functions $\mathcal{H}_N$ are tangential to $\tilde{\Gamma}$, and hence the system is first class, according to the definition given in [10].

A simple consequence of theorem 4 is that the pull-back of the vector field \[53\] to $\Gamma \times \mathcal{E}$ is given by

\begin{align*}
\dot{\varphi} &= \text{grad}_\pi \mathcal{H}_N, \\
\dot{\pi} &= -\text{grad}_\varphi \mathcal{H}_N, \\
\dot{X} &= N.
\end{align*}

Let us denote the space of longitudinal vectors at $(\varphi, \pi, X) \in \Gamma \times \mathcal{E}$ by $\Xi(\varphi, \pi, X)$, i.e.,

$$\Xi(\varphi, \pi, X) := \left\{(\Phi, \Pi, V) \in T_{(\varphi, \pi, X)} \tilde{\Gamma} \mid \Phi = \text{grad}_\pi \mathcal{H}_N, \Pi = -\text{grad}_\varphi \mathcal{H}_N, V = N \in T_X \mathcal{E}\right\}.\quad \text{(59)}$$

The map $(\text{grad}_\pi \mathcal{H}_N, -\text{grad}_\varphi \mathcal{H}_N) : T_X \mathcal{E} \rightarrow T_{(\varphi, \pi)} \Gamma \phi$ is continuous and hence, by the closed graph theorem, $\Xi(\varphi, \pi, X)$ is a closed subspace of $T_{(\varphi, \pi, X)} \tilde{\Gamma}$.

Another consequence of theorem 4 is that the Fréchet derivative of the map $\rho_{XX'}$ with respect to $X'$ is given by

$$D_{X'} \rho_{XX'}(\varphi, \pi)(V) = (\text{grad}_\pi \mathcal{H}_V(\varphi', \pi', X'), -\text{grad}_\varphi \mathcal{H}_V(\varphi', \pi', X')),$$

where $(\varphi', \pi') := \rho_{XX'}(\varphi, \pi)$. \quad \text{(60)}

We shall need the pull-back of the symplectic form to the constraint manifold. This is given by the following theorem.

**Theorem 5** The pull-back $\tilde{\Omega}$ of the form $\Omega$ is given by the formula

$$\tilde{\Omega}((\Phi_1, \Pi_1, V_1), (\Phi_2, \Pi_2, V_2)) =$$

$$\int_{\Sigma} d^3x \left[ (\Pi_1 + \text{grad}_\varphi \mathcal{H}_{V_1})(\Phi_2 - \text{grad}_\pi \mathcal{H}_{V_2}) - (\Pi_2 + \text{grad}_\pi \mathcal{H}_{V_2})(\Phi_1 - \text{grad}_\varphi \mathcal{H}_{V_1}) \right]. \quad \text{(61)}$$
Proof The pull-back by the map $\tilde{C}$ of the form $\Omega$ is given by

$$\tilde{\Omega}((\Phi_1, \Pi_1, V_1), (\Phi_2, \Pi_2, V_2)) := \Omega(D\tilde{C}(\Phi_1, \Pi_1, V_1), D\tilde{C}(\Phi_2, \Pi_2, V_2)).$$

Substituting into this equation the expressions for $D\tilde{C}$ from Eqs. (40) and (41), and using Eq. (54), one easily arrives at Eq. (61).

QED

Thus, $\tilde{\Omega}$ is degenerate, and the degeneracy subspace at the point $(\varphi, \pi, X) \in \tilde{\Gamma}$ coincides with $\Xi_{(\varphi, \pi, X)}$.

The last important notion involving the constraint submanifold is that of a ‘c-orbit’, defined to be the set of points in $\tilde{\Gamma}$ that correspond to just one maximal classical solution (see [10]). Let $\tilde{\gamma}_{(\varphi, \pi, X)}$ be the map $\tilde{\gamma}_{(\varphi, \pi, X)} : \mathcal{E} \to \Gamma_{\varphi} \times \mathcal{E}$ defined by

$$\tilde{\gamma}_{(\varphi, \pi, X)}(X') := (\rho_{XX'}(\varphi, \pi), X')$$

Then the c-orbit $\gamma_{(\varphi, \pi, X)}$ through the point $(\varphi, \pi, X) \in \Gamma_{\varphi} \times \mathcal{E}$ is defined as

$$\gamma_{(\varphi, \pi, X)} := \tilde{\gamma}_{(\varphi, \pi, X)}(\mathcal{E})$$

i.e., $\gamma_{(\varphi, \pi, X)}$ is the collection of all embeddings, and Cauchy data on such, induced by the unique solution to the field equations whose Cauchy data on $X(\Sigma)$ is $(\varphi, \pi)$.

We shall show that the c-orbits are smooth submanifolds of $\tilde{\Gamma}$ and that their tangent spaces coincide with $\Xi_{(\varphi, \pi, X)}$; the proof is analogous to that of theorem 2.

The tangent space to $\gamma$ at $\tilde{\gamma}_{(\varphi, \pi, X)}(X')$ is the image of the map $D\tilde{\gamma}_{(\varphi, \pi, X)}|_{X'} : T_{X'}\mathcal{E} \to T_{(\varphi', \pi', X')}(\Gamma_{\varphi} \times \mathcal{E})$, where $(\varphi', \pi') := \rho_{XX'}(\varphi, \pi)$. Using Eq. (60) we obtain, for all $V \in T_{X'}\mathcal{E}$,

$$D\tilde{\gamma}_{(\varphi, \pi, X)}|_{X'}(V) = (D\rho_{XX'}(\varphi, \pi)|_{X'}(V), V)$$

$$= (\text{grad}_{\varphi'}\mathcal{H}_V(\varphi', \pi', X'), -\text{grad}_{\pi'}\mathcal{H}_V(\varphi', \pi', X'), V).$$

Hence, $D\tilde{\gamma}_{(\varphi, \pi, X)}|_{X'}$, is injective, and a comparison of Eq. (55) with Eqs. (56–58) shows that $D\tilde{\gamma}_{(\varphi, \pi, X)}|_{X'}(T_{X'}\mathcal{E}) = \Xi_{(\varphi, \pi, X)}$. As $\Xi$ is a closed subspace of a Hilbert space, it splits, and the claims above are proved.
V. CONCLUSIONS

We have shown that, with proper functional-analytical care, the geometrical structure of an infinite-dimensional parametrized system can be developed in a way that is analogous to that of a finite-dimensional system. In particular, for the model considered, the extended phase space is a (weak-)symplectic infinite-dimensional manifold, the constraint set is a submanifold of the phase space, and the $c$-orbits are submanifolds of the constraint set. The criteria for a constrained system to be first class are of the same form as those of a finite-dimensional system. Many constructions available for a finite-dimensional system can now be performed in the infinite-dimensional case. The only difference is that the symplectic form is only weakly non-degenerate. However, physicists usually work with a restricted class of functions so that the Poisson brackets are still well-defined.

We anticipate that our main results are broadly generalizable. For example, an extension to the case where the Cauchy hypersurfaces are asymptotically flat is likely to be relatively straightforward. We hope that our results will be useful for a number of purposes. In particular, our main goal was to apply the perennial formalism to the scalar field system; this is be done in the accompanying paper.

Acknowledgements

P.H. thanks the Theoretical Physics Group at Imperial College for their hospitality. Both authors gratefully acknowledge financial support from the European Network Physical and Mathematical Aspects of Fundamental Interactions.

APPENDIX A: CAUCHY PROBLEM

We collect together some well-known results about the Cauchy problem of linear hyperbolic systems and then use them to sketch a proof of the theorem 1 in section [ⅰ].

First, we state some lemmas about the space-time $(\mathcal{M}, g)$. 

24
Lemma 1 Let $X$ and $X'$ be two embeddings in $E$ such that $X' (\Sigma) \subset I^+ (X(\Sigma))$. Then, given $T > 0$ and $\epsilon > 0$, there is a one-dimensional family $\{X_t\}$, $t \in (-\epsilon, T + \epsilon)$, of embeddings such that:

a) $X_0 = X$, $X_T(\sigma) = X'(\Sigma)$;

b) if $(U, h)$ is a chart of $\Sigma$ ($h(U) \subset \mathbb{R}^3$), then $(\bigcup_{t \in (-\epsilon, T + \epsilon)} X_t(U), (X_t(h^{-1}(x)))^{-1})$ is a $C^{r+1}$ chart in $\mathcal{M}$, where $X_t(h^{-1}(x))$ is considered as a map

$$X_t(h^{-1}(x)) : (-\epsilon, T + \epsilon) \times h(U) \to \mathcal{M};$$

(A1)

c) the components $g_{\alpha\beta}(t, x)$ of the metric in any such chart satisfy the equations

$$g_{00}(t, x) < 0$$

in $(-\epsilon, T + \epsilon) \times h(U)$ (the $t$-curves are everywhere timelike), and

$$g_{00}(0, x) = g_{00}(T, x) = -1, \quad g_{0i}(0, x) = g_{0i}(T, x) = 0,$$

(A3)

for all $x \in h(U)$.

The proof is simple. Observe that $g^{00}$ is negative and $g_{kl}$ is positive-definite everywhere in $(-\epsilon, T + \epsilon) \times h(U)$; this is because any $X_t(U)$ hypersurface is a part of a Cauchy hypersurface, and hence spacelike. Then the condition that $g^{kl}$ is also positive-definite everywhere is equivalent to Eq. (A2). Hence, for any $t \in [0, T]$ and $x \in h(U)$, we have

$$|g^{00}| > c_1,$$

$$g^{kl}\xi_k\xi_l > c_2 e^{kl}\xi_k\xi_l,$$

(A4)

(A5)

where $c_1$ and $c_2$ are positive constants, $e_{kl}$ is a positive-definite $C^\infty$ metric on $\Sigma$, and $\xi_k$ is an arbitrary covector field on $\Sigma$.

The proof of the following lemma can be found in [10].
Lemma 2 If $\Sigma_1$ and $\Sigma_2$ are two Cauchy hypersurfaces in $(M, g)$ then there is a Cauchy hypersurface $\Sigma_3$ such that

$$\Sigma_3 \subset (I^+(\Sigma_1) \cap I^-(\Sigma_2)).$$  \hfill (A6)

Thus, $\Sigma_1 \cap \Sigma_3 = \Sigma_2 \cap \Sigma_3 = \emptyset$.

Next, consider the Klein-Gordon equation (1) for the field $\phi$, and the associated Cauchy problem. In the chart described in lemma 1, Eq. (1) has the form

$$-g^{00} \frac{\partial^2 \phi}{\partial t^2} = g^{kl} \frac{\partial^2 \phi}{\partial x^k \partial x^l} + 2g^{0k} \frac{\partial \phi}{\partial t \partial x^k} + (|\det g|^{-1/2} \partial_{\mu} (|\det g|^{1/2} g^{0\mu}) \frac{\partial \phi}{\partial t} + m^2 \phi. \hfill (A7)$$

An initial datum for $\phi$ at the time $t$ is the pair of scalar fields $(\varphi_t, \dot{\varphi}_t)$ on $\Sigma$ given by

$$\varphi_t(x) = \phi(t, x), \hfill (A8)$$
$$\dot{\varphi}_t(x) = \frac{\partial \phi}{\partial t}, \hfill (A9)$$

where $\varphi_t(x)$ coincides with the Cauchy datum for $X_t(\Sigma)$ as defined by Eq. (4). For $t = 0$ and $t = T$, we have also

$$(\det \gamma)^{1/2} \varphi_0(x) = \pi_0(x), \hfill (A10)$$
$$(\det \gamma)^{1/2} \varphi_T(x) = \pi_T(x), \hfill (A11)$$

where $\pi_0(x)$ and $\pi_T(x)$ are the pieces of Cauchy data defined by Eq. (3) for $X_0(\Sigma)$ and $X_T(\Sigma)$.

Consider the coefficient functions of Eq. (A7). They define tensor fields on $\Sigma$. Indeed,

$$a^{00}(t, x) = -g^{00}(t, x), \hfill (A12)$$
$$a^0(t, x) = |\det g|^{-1/2} \partial_{\mu} (|\det g|^{1/2} g^{0\mu})|_{t, x}, \hfill (A13)$$
$$a(t, x) = m^2, \hfill (A14)$$

are scalar fields on $\Sigma$ for each $t$, whereas
\begin{align}
a^{0k}(t, x) &= g^{0k}(t, x), \quad (A15) \\
a^k(t, x) &= |\det g|^{-1/2} \partial_\mu(\det g)^{1/2} \partial^{k\mu}|_{t,x}, \quad (A16)
\end{align}

are contravariant vector fields for each \( t \), and

\[ a^{kl}(t, x) = g^{kl}(t, x) \quad (A17) \]

is a contravariant tensor of second rank on \( \Sigma \) for each \( t \).

The \( C^{r+1} \)-differentiability of \( X \) implies the following properties of these tensor fields:

\[ a^{\alpha\beta} \in \text{Lip}([0, T]; H^{r-1}_0(\Sigma)) \subset L^\infty([0, T]; H^r_0(\Sigma)), \]
\[ a^\alpha \in \text{Lip}([0, T]; H^{r-2}_0(\Sigma)) \subset L^\infty([0, T]; H^{r-1}_0(\Sigma)), \]
\[ a \in \text{Lip}([0, T]; H^\infty_0(\Sigma)) = L^\infty([0, T]; H^r_0(\Sigma)). \quad (A18) \]

Moreover, according to Eqs. (A4) and (A5), we have \( a^{00}(t, x) \geq c_1 \), for all \( t \in [0, T] \) and \( x \in \Sigma \), and

\[ a^{kl}(t, x)\xi_k(x)\xi_l(x) \geq c'_2 e^{kl}(x)\xi_k(x)\xi_l(x), \quad (A19) \]

for all \( t \in [0, T] \) and \( x \in \Sigma \). The relations above enable us to use ‘localized’ forms of the theorems 4.15 and 4.13 in [15], and to apply them to construct unique local evolution systems \( F_{t,s} \) which can be ‘patched together’. By this means we are able to prove the following lemma (for further details see [15]):

**Lemma 3** Let the assumptions of Lemma 2 be satisfied, and let \( r \geq 4 \). Then, each initial datum \((\varphi_0(x), \dot{\varphi}_0(x)) \) \( \in H^{s+1}_0(\Sigma) \times H^s_0(\Sigma) \) on \( X_0(\Sigma) \), where \( 1 \leq s \leq r - 1 \), defines a unique solution \( \phi \) to the equation (4) in \( U \) whose initial datum \((\varphi_T(X), \dot{\varphi}_T(x)) \) on \( X_T(\Sigma) \) belongs to the Sobolev space \((\varphi_T(X), \dot{\varphi}_T(x)) \in H^{s+1}_0(\Sigma) \times H^s_0(\Sigma) \). Furthermore, the maps \( U(0,T) : H^{s+1}_0(\Sigma) \times H^s_0(\Sigma) \to H^{s+1}_0(\Sigma) \times H^s_0(\Sigma) \) are automorphisms of Banach spaces.

Note that it is a trivial matter to pass from the initial data \((\varphi, \dot{\varphi})\) to the Cauchy data \((\varphi, \pi)\): since \((\det \gamma)^{1/2} \) is \( C^\infty \) and bounded below by zero in \( U \), the definition \( \pi(x) := (\det \gamma)^{1/2} \dot{\varphi}(x) \) describes an isomorphism between the Banach spaces \( H^s_0(\Sigma) \) and \( H^s_1(\Sigma) \) for any \( s \leq r \).
Finally, given any pair of arbitrary embeddings $X_1$ and $X_2$, we can use lemma 2 to find an ‘intermediate’ embedding $X_3$. To be able to apply lemma 1 to the pairs $\{X_1, X_3\}$ and $\{X_3, X_2\}$, we must find two embeddings $X'_3$ and $X''_3$ such that the corresponding $t$-curves are timelike, and with $X'_3(\Sigma) = X''_3(\Sigma) = X_3(\Sigma)$. Then, we can use lemma 3 and the diffeomorphism invariance of $B$ to prove the theorem 1. QED
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