1. Introduction

An \(n\)-dimensional \((n \geq 3)\) Riemannian manifold \((M^n, g)\) is called a Ricci soliton if there exist a smooth vector field \(X\) and a constant \(\lambda \in \mathbb{R}\) on \(M^n\) such that

\[
Rc + \frac{1}{2} \mathcal{L}_X g = \lambda g,
\]

where \(Rc\) and \(\mathcal{L}_X g\) denote the Ricci tensor and the Lie derivative of \(g\) in the direction of \(X\), respectively, and \(\lambda\) is sometimes called the soliton constant. The soliton is shrinking, steady, or expanding if \(\lambda > 0\), \(\lambda = 0\), or \(\lambda < 0\), respectively. When \(X\) is a gradient of a smooth function \(f\) on \(M^n\), the soliton is called a gradient Ricci soliton and (1) becomes

\[
Rc + \text{Hess}\ f = \lambda g.
\]

Note that when \(X\) or \(\nabla f\) is a Killing vector field, equations (1) and (2) reduce to the Einstein equation. Thus, Ricci solitons are natural generalizations of Einstein manifolds. In particular, when \(X = 0\) or \(f\) is a constant, the soliton is trivial.

In recent decades, increasing investigations have been done to the rigidity of gradient shrinking Ricci solitons (gradient shrinker for short). In dimension 2, Hamilton [1] showed that a gradient shrinker is isometric to \(\mathbb{R}^2\) or to a quotient of \(S^2\). The first rigidity theorem in dimension 3 was proved by Ivey [2] saying that a 3-dimensional compact gradient shrinker is a quotient of \(S^3\). In the noncompact case, the relevant rigidity result was showed by Perelman [3] with noncollapsing assumption, which was removed by Naber [4] later. Adopting different arguments, Ni and Wallach [5] and Cao et al. [6] obtained the full classification; they proved that any 3-dimensional gradient shrinker must be isometric to \(\mathbb{R}^3\) or to a quotient of \(\mathbb{R} \times S^2\) or \(S^3\). Some relevant conclusions can be found in [4, 7, 8].

When \(n \geq 4\), under the assumption of nonnegative curvature operator or vanishing Weyl tensor, Naber [4], Ni and Wallach [5], Petersen and Wylie [8], and Zhang [9] proved corresponding rigidity results on gradient shrinkers, which were improved by Catino [10] using a general pointwise pinching condition on the Weyl tensor.

On the other hand, Munteanu and Wang [11] investigated the curvature behavior of 4-dimensional gradient shrinker and proved that there exists a constant \(C > 0\) for 4-dimensional gradient shrinkers with bounded scalar curvature \(R\) so that

\[
|Rm| \leq CR,
\]
which along with the fact \(|Rm|^2 = |R' m|^2 + (R^2/6)\) implies
\[
|\hat{R} m| \leq \left( C^2 - \frac{1}{6} \right) R^2,
\]
and \(C \geq (\sqrt{6}/6)\). Here, \(R' m\) is the trace-free curvature tensor.

In [12], the authors established the following rigidity theorem under pointwise pinching condition of \(R' m\):

**Theorem 1** (Theorem 1.1 in [12]). Let \((M^n, g, f)\) be an \(n\)-dimensional \((n \geq 3)\) complete gradient shrinker. If
\[
|\hat{R} m| \leq \frac{1}{C(n)} \left( \lambda - \frac{n - 2}{n(n-1)} R \right),
\]
then \((M^n, g)\) is isometric to \(\mathbb{R}^n\) or a finite quotient of \(S^n\).

In this paper, we will restrict our attention to the rigidity of gradient shrinkers with pointwise pinched conditions associated with \(R' m\) and the traceless Ricci tensor \(R' c = Rc - (R/n)g\). By establishing \(f\)-parabolic and algebraic curvature estimates, we prove two rigidity results for gradient shrinkers. More precisely, setting \(C(n) = ((n-2)/(\sqrt{n(n-1)})) + ((n^2-n-4)/(2\sqrt{(n^2-2)(n-1)(n+1)}))\), which is defined in Lemma 10, we have the following Theorem 2.

**Theorem 2.** Assume that \((M^n, g, f)\) is a complete gradient shrinker of dimension \(n \geq 3\). If
\[
|\hat{R} m|^2 \leq \frac{1}{C^2(n)} \left( \lambda - \frac{n - 2}{n(n-1)} R \right)^2 + \frac{4}{\lambda} R^2 c^2,
\]
then \((M^n, g)\) is isometric to \(\mathbb{R}^n\) or a finite quotient of \(S^n\). Moreover, when the pinching condition in the right hand of (6) is weakened to
\[
|\hat{R} m|^2 \leq \frac{2(n-1)}{n-2} \left( \lambda - \frac{n - 2}{n(n-1)} R \right)^2 + \frac{4}{\lambda} R^2 c^2 ,
\]
then \((M^n, g)\) is Einstein.

**Remark 3.** When
\[
|\hat{R} m| \leq \frac{1}{C(n)} \left( \lambda - \frac{n - 2}{n(n-1)} R \right),
\]
by equation (6) and Theorem 2, we see that \((M^n, g)\) is isometric to \(\mathbb{R}^n\) or a finite quotient of \(S^n\). Therefore, Theorem 2 can be seen as a generalization of Theorem 1.

**Theorem 4.** Let \((M^n, g, f)\) be a complete gradient shrinker of dimension \(n \geq 3\) with nonnegative Ricci curvature. If
\[
|\hat{R} m| \leq \frac{1}{C(n)} \left( R \frac{n}{n-1} + \frac{|R^c|^2}{R} \right),
\]
then \((M^n, g)\) is isometric to \(\mathbb{R}^n\) or a finite quotient of \(S^n\).

**Remark 5.** As is shown in the proof, the condition of non-negative Ricci curvature in Theorem 4 can be relaxed to that \(|Rc| \leq cR^{1+n}\) for some constants \(c > 0\) and \(\alpha \geq 0\) satisfying \(cR^\alpha \geq \sqrt{(n-2)/(n(n-1))}\).

**Remark 6.** Since any three-dimensional gradient shrinker must have nonnegative sectional curvature (cf. Corollary 2.4 of [13]), we see that the condition on Ricci curvature in Theorem 4 is not needed.

### 2. Preliminaries of Curvature Estimates

Let \((M^n, g)\) be a connected Riemannian manifold of dimension \(n \geq 3\). In local coordinates, denoting by \(R_{ijkl}, W_{ijkl}, R^c\), and \(R'_{jk} = R_{jk} - (R/n)g_{jk}\) the components of the curvature tensor \(Rm\), the Weyl tensor \(W\), and the traceless Ricci tensor \(R' c\), respectively, we have the well-known orthogonal decomposition of \(Rm\) (see e.g., [14]).

\[
R_{ijkl} = W_{ijkl} + \frac{R}{n(n-1)} \left( g_{il}g_{jk} - g_{ik}g_{jl} \right) + \frac{1}{n-2} \left( R_{ik}g_{jl} - R_{jl}g_{ik} \right). \tag{10}
\]

Correspondingly, the soliton equation (2) is rewritten as
\[
R_{ij} + V_i V_j f = \lambda g_{ij} . \tag{11}
\]

Taking the trace in equation (11) gives
\[
R + \Delta f = n\lambda . \tag{12}
\]

Writing \(R' m = \{R'_{ijkl}\} = \{R_{ijkl} - (R/n(n-1))(g_{il}g_{jk} - g_{ik}g_{jl})\}\) and using the properties of \(Rm\), one can easily derive the following equalities:

\[
g^{ij} \hat{R}_{ijkl} = \hat{R}_{kj} , \tag{13}
\]

\[
\hat{R}_{ijkl} + \hat{R}_{iklj} + \hat{R}_{ikjl} = 0 , \tag{14}
\]

\[
\hat{R}_{ijkl} = \hat{R}_{klij} = -\hat{R}_{ikjl} = -\hat{R}_{iklj} , \tag{15}
\]

\[
|\hat{R} m|^2 = |W|^2 + \frac{4}{n-2} |R^c|^2 , \tag{16}
\]
where the norm of a \((0,4)\)-type tensor \(T\) is defined by
\[
|T|^2 = |T^{|ijkl}|^2 = g^{im} g^{jn} g^{kw} g^{lq} T_{ijkl} T_{lmnt} = \delta^{|ijkl| T_{ijkl}}.
\] (17)

Here and subsequently, the notations \(u_\ast = \Delta\inf_u\) as well as \(u_\ast = \Delta\sup_u\) for a function \(u\) on \(M^n\) and Einstein summation convention are always adopted.

Recall the \(f\)-Laplacian \(\Delta_f\), which is sometimes called the drifted Laplacian or Witten-Laplacian and is defined on a function \(u\in\text{Lip}_{\text{loc}}(M^n)\) by
\[
\Delta_f u = \Delta u - g(\nabla f, \nabla u) = e^f \text{div} \left(e^{-f} \nabla u \right),
\] (18)
in the weak sense, which is a self-adjoint operator on the space of square integrable functions on \((M^n, g, f)\) with respect to weighted volume form \(e^f \text{d}V_g\). That is,
\[
\int_{M^n} \nabla \varphi \cdot \nabla e^{-f} \text{d}V_g = -\int_{M^n} (\Delta_f \varphi) e^{-f} \text{d}V_g,
\] (19)
for any \(\varphi, \psi \in C_0^\infty(M^n)\), where \(\text{d}V_g\) is the volume element induced by the metric \(g\).

First of all, we will compute the \(f\)-Laplacian of the norm square of \(R\) \(m\), by which we will establish the key estimate for any gradient Ricci soliton of dimension \(n \geq 3\) in Lemma 10. We start from Lemma 7.

**Lemma 7.** For any gradient Ricci soliton of dimension \(n \geq 3\), we have
\[
2R \nabla_l \nabla^p R_{ijkp} = 2R 
\]
\[
= 2R \nabla_l \nabla^p R_{ijkp} - \frac{2R}{n} |R| m^2 + 2R \nabla_l \nabla^p R_{ijkp}.
\] (20)

**Proof.** For convenience, we set
\[
A = 2R \nabla_l \nabla^p R_{ijkp} - \frac{1}{2} |R| m^2 \cdot \nabla f = 2R \nabla_l \nabla^p R_{ijkp} - 2R \nabla_l \nabla^p R_{ijkp}.
\] (21)

On the one hand, by the second Bianchi identity, we get
\[
-\frac{1}{2} (\nabla_l \nabla^p R_{ijkp}) \nabla^f f = -R (\nabla_l \nabla^p R_{ijkp}) \nabla^f f = -2R \nabla_l \nabla^p R_{ijkp} \nabla^f f.
\] (22)

On the other hand, by the Ricci identity and the equation (11), we deduce that
\[
\frac{1}{2} \Delta |R_{ijkl}|^2 = |R \cdot m|^2 + 2R \nabla_l \nabla^p R_{ijkp} + \frac{4R}{n(n-1)} \nabla_l \nabla^p R \cdot m.
\] (28)
Making use of the Ricci identity and (13), we have
\[ 2R^{ijkl} \nabla^p R_{jilk} = 2R^{ijkl} \left( \nabla^p R_{jilkp} - R^{\nu}_{\quad l} R_{jilk}^{\quad \nu p} + R^{\nu}_{\quad k} R_{jilk}^{\quad \nu p} - R^{\nu}_{\quad p} R_{jilk}^{\quad \nu p} \right) \]
\[ + 2R_{ilk} \left( R_{jilp} - R_{jlip} + R_{jilp} - R_{jilp} \right) - \frac{2R}{n(n-1)} R^{ijkl} \]
\[ = 2R^{ijkl} \left( \nabla^p R_{jilkp} - R^{\nu}_{\quad l} R_{jilk}^{\quad \nu p} + R^{\nu}_{\quad k} R_{jilk}^{\quad \nu p} - R^{\nu}_{\quad p} R_{jilk}^{\quad \nu p} \right) \]
\[ + 2R_{ilk} \left( R_{jilp} - R_{jlip} + R_{jilp} - R_{jilp} \right) - \frac{2R}{n(n-1)} R^{ijkl} \]
\[ = 2R^{ijkl} \left( \nabla^p R_{jilkp} - R^{\nu}_{\quad l} R_{jilk}^{\quad \nu p} + R^{\nu}_{\quad k} R_{jilk}^{\quad \nu p} - R^{\nu}_{\quad p} R_{jilk}^{\quad \nu p} \right) \]
\[ + 2R_{ilk} \left( R_{jilp} - R_{jlip} + R_{jilp} - R_{jilp} \right) - \frac{2R}{n(n-1)} R^{ijkl} \]
\[ = 2R^{ijkl} \left( \nabla^p g_{il} - g^p g_{il} \right) \]
\[ + \nabla^p R_{jilkp} \Delta R_{jilkp} = 2R^{ijkl} \left( \nabla^p g_{il} - g^p g_{il} \right) \]
\[ + \nabla^p R_{jilkp} \Delta R_{jilkp} \]
\[ = 2R^{ijkl} \left( \nabla^p g_{il} - g^p g_{il} \right) \]
\[ + \nabla^p R_{jilkp} \Delta R_{jilkp} \]
\[ = 2R^{ijkl} \left( \nabla^p g_{il} - g^p g_{il} \right) \]
\[ + \nabla^p R_{jilkp} \Delta R_{jilkp} \]
\[ \tag{29} \]

Combining (13) and (14) with (29), we get
\[ 2R^{ijkl} \nabla^p R_{jilkp} = -2R^{ijkl} \left( \nabla^p R_{jilkp} - R^{\nu}_{\quad l} R_{jilk}^{\quad \nu p} + R^{\nu}_{\quad k} R_{jilk}^{\quad \nu p} - R^{\nu}_{\quad p} R_{jilk}^{\quad \nu p} \right) \]
\[ + 2R_{ilk} \left( R_{jilp} - R_{jlip} + R_{jilp} - R_{jilp} \right) - \frac{2R}{n(n-1)} R^{ijkl} \]
\[ \tag{30} \]

where
\[ 2R^{ijkl} \nabla^p R_{jilkp} = -2R^{ijkl} \left( \nabla^p R_{jilkp} - R^{\nu}_{\quad l} R_{jilk}^{\quad \nu p} + R^{\nu}_{\quad k} R_{jilk}^{\quad \nu p} - R^{\nu}_{\quad p} R_{jilk}^{\quad \nu p} \right) \]
\[ + 2R_{ilk} \left( R_{jilp} - R_{jlip} + R_{jilp} - R_{jilp} \right) - \frac{2R}{n(n-1)} R^{ijkl} \]
\[ \tag{31} \]

Substituting (30) and (31) into (28), we obtain
\[ \frac{1}{2} \Delta \nabla^p R_{jilkp} = \left| \nabla^p R \right|^2 - 2R^{ijkl} \left( \nabla^p R_{jilkp} - R^{\nu}_{\quad l} R_{jilk}^{\quad \nu p} + R^{\nu}_{\quad k} R_{jilk}^{\quad \nu p} - R^{\nu}_{\quad p} R_{jilk}^{\quad \nu p} \right) \]
\[ + 2R^{ijkl} \left( \nabla^p R - \frac{4R}{n(n-1)} \right) \]
\[ \tag{32} \]

where the formula
\[ 2R^{ijkl} \nabla^p R_{jilkp} + \frac{2R}{n(n-1)} \nabla^p R = R^{ijkl} \nabla^p R \]
\[ \tag{33} \]
is used in (32).

By Lemmas 7 and 8 and the fact that
\[ \frac{1}{2} \Delta \nabla^p R_{jilkp} = \frac{1}{2} \Delta \nabla^p R_{jilkp} - R^{ijkl} \nabla^p R_{jilkp} \nabla^p f, \]
\[ \tag{34} \]
we now arrive at the f-Laplacian formula of \(|\nabla R|^2\) for all gradient Ricci solitons.

**Lemma 9.** Let \((M^n, g, f)\) be an \(n\)-dimensional \((n \geq 3)\) gradient Ricci soliton. Then,
\[ \frac{1}{2} \Delta |\nabla R|^2 = \left| \nabla R \right|^2 - 2R \left( \nabla^p R_{jilkp} - \frac{1}{2} R_{jilkp} \nabla^p f \right) \]
\[ + 2\lambda |\nabla R|^2 - \frac{8R}{n(n-1)} |\nabla c|^2. \]
\[ \tag{35} \]

Consequently, we conclude from (35) and (16) that
\[ \frac{1}{2} \Delta |\nabla R|^2 = \left| \nabla R \right|^2 - 2R \left( \nabla^p R_{jilkp} - \frac{1}{2} R_{jilkp} \nabla^p f \right) \]
\[ + 2\lambda |\nabla R|^2 - \frac{2(n-2)R}{n(n-1)} |\nabla R|^2 \]
\[ + \frac{2(n-2)R}{n(n-1)} |\nabla c|^2, \]
\[ \tag{36} \]
for any gradient Ricci soliton.

Utilizing the inequalities proved by Li and Zhao [15] and Huisken [16] (see also [17]), we have
\[ \left| \nabla R \right|^2 \leq \frac{n-2}{n(n-1)} |\nabla R|^3, \]
\[ \tag{37} \]
\[ \left| \nabla R \right|^2 \leq \frac{n^2 - n - 4}{(n-2)(n-1)n(n+1)} |\nabla R|^3. \]
\[ \tag{38} \]
Combining (37) and (38) gives

**Lemma 10.** Let \((M^n, g, f)\) be an \(n\)-dimensional \((n \geq 3)\) gradient Ricci soliton. Then,
\[ \frac{1}{2} \Delta |\nabla R|^2 \geq \left| \nabla R \right|^2 \]
\[ - 2 \left( \frac{n-2}{n(n-1)} + \frac{n^2 - n - 4}{2(n-2)(n-1)n(n+1)} \right) |\nabla R|^3 \]
\[ + \lambda |\nabla R|^2 - \frac{2(n-2)R}{n(n-1)} |\nabla R|^2 \]
\[ + \frac{2(n-2)R}{n(n-1)} |\nabla c|^2 - 2C(n) |\nabla R|^3 \]
\[ + \lambda |\nabla R|^2 + \frac{2(n-2)R}{n(n-1)} |\nabla c|^2. \]
\[ \tag{39} \]

**Remark 11.** Inequality (39) can also be derived by setting \(X = \nabla f\) in Lemma 2.5 of [12]; here, we give its proof for the sake of completeness.
Correspondingly, the f-Laplacian of $|Rc|^2$ is (see e.g. Lemma 2.1 of [18])

**Lemma 12.** Let $(M^n, g, f)$ be a gradient Ricci soliton of dimension $n \geq 3$. Then,

$$
\frac{1}{2} \Delta_f |Rc|^2 = |\nabla Rc|^2 + 2 \left( \lambda - \frac{(n-2)R}{n(n-1)} \right) |Rc|^2 + \frac{4}{n-2} R_{jk} R_k - 2 W_{ijk} R R_k.
$$

Employ the following curvature inequality.

**Lemma 13.** (Proposition 2.1 of [19]). Let $(M^n, g)$ be an n-dimensional ($n \geq 3$) Riemannian manifold. Then,

$$
-W_{ijk} R^i R^j R^k + 2 \frac{n-2}{n-2} W_{ij} R_{jk} R^k \leq \left( \frac{n-2}{2(n-1)} \left( |W|^2 + \frac{8}{n(n-2)} |Rc|^2 \right) \right) \frac{1}{2} |Rc|^2.
$$

We deduce from Lemma 12 the following.

**Lemma 14.** Let $(M^n, g)$ be an n-dimensional ($n \geq 3$) Riemannian manifold. Then,

$$
\frac{1}{2} \Delta_f |Rc|^2 \geq |\nabla Rc|^2 + 2 \left( \lambda - \frac{(n-2)R}{n(n-1)} \right) |Rc|^2 + \frac{4}{n-2} R_{jk} R_k - 2 W_{ijk} R R_k.
$$

Since the scalar curvature of nonflat Ricci shrinker is positive, by Proposition 2.7 of [12], we get the following curvature inequality.

**Lemma 15.** Let $(M^n, g, f)$ be an n-dimensional ($n \geq 3$) complete nonflat gradient shrinker. Then,

$$
\Delta_{f-2\log R} \left( \frac{|Rc|^2}{R^2} \right) \geq \frac{2}{R^4} \left( \sqrt{Rc} \frac{|\nabla Rc|}{\sqrt{Rc}} - \frac{|Rm|}{\sqrt{Rc}} \right)^2 + \frac{4(n-2)|W|^2}{n(n-1)R} + \frac{4 |Rm|^2}{R^3} + \frac{R^2}{n(n-1)} + |Rc|^2 - C(n)|Rm|,
$$

For the sake of the proofs of our main theorems, we recall the following results due to Catino [10], Pigola et al. [20], and Petersen and Wylie [8] (see also [4]).

**Lemma 16** (Proposition 1 of [10]). Let $(M^n, g, f)$ be a complete gradient nonflat shrinker of dimension $n \geq 3$. Then,

$$
\Delta_{f-2\log R} \left( \frac{Rc}{R^2} \right) \geq \frac{2}{R^4} \left( |Rc|^2 - \frac{R}{\sqrt{n(n-1)}} \right)^2 + \frac{4}{R^3} \left( \frac{|Rc|^2}{R} - \frac{R}{\sqrt{n(n-1)}} \right)^2 - RW_{ijkl} R R_k R_l.
$$

**Lemma 17** (Theorem 3 of [20]). Let $(M^n, g, f)$ be a complete gradient shrinker of dimension $n \geq 3$. Then, $0 \leq R_+ \leq n\lambda$. Moreover, $R_+ < n\lambda$ unless $(M, g)$ is Einstein and the soliton is trivial, and $R_+ > 0$ unless $R = 0$ and $(M, g)$ is isometric to $\mathbb{R}^n$.

**Lemma 18** (Theorem 22 of [20]). Any complete gradient shrinker $(M^n, g, f)$ is f-parabolic, namely, every solution of $\Delta_f u \geq 0$ satisfying $u^+ < +\infty$ must be a constant.

**Lemma 19** (Lemma 4.2 of [8]). Assume that $(M^n, g)$ is an n-dimensional manifold with finite -volume, i.e., $\int_M e^{-u} dV_g < +\infty$. If a smooth function $u \in L^2(e^{-u} dV_g)$ is bounded below such that $\Delta_f u \geq 0$, then $u$ is a constant.

## 3. Proofs of Main Theorems

We are now in a position to give the proofs of our main theorems.

**Proof of Theorem 2.** Using (16), we see that pinching conditions (6) and (7) in Theorem 2 are equivalent to the following inequality, respectively:

$$
|W|^2 + \frac{8}{n(n-2)} |Rc|^2 \leq \frac{1}{C^2(n)} \left( \lambda - \frac{n-2}{n(n-1)} R \right)^2,
$$

and

$$
|W|^2 + \frac{8}{n(n-2)} |Rc|^2 < \frac{2(n-1)}{n-2} \left( \lambda - \frac{n-2}{n(n-1)} R \right)^2.
$$
By Lemma 14 and (46), we have

\[
\frac{1}{2} \Delta_j |\dot{R} m|^2 \geq \left| \nabla^g R m \right|^2 + 2 \left( \lambda - \frac{(n-2)R}{n(n-1)} - \sqrt{\frac{n-2}{2(n-1)}} \right) \left| \dot{R} m \right|^2
\]

\[
\geq 2 \left( \lambda - \frac{(n-2)R}{n(n-1)} - \sqrt{\frac{n-2}{2(n-1)}} \right) \left| \dot{R} m \right|^2
\]

where the fact that \( R \geq 0 \) for shrinking solitons (see Lemma 17 or Corollary 2.5 of [13]) is used in the second inequality in (51). It follows from Lemma 18 and (50) that \( |\dot{R} m|^2 \) is a constant and therefore all equalities in (51) hold.

If there exists \( x_0 \in M^n \) such \( R(x_0) = 0 \), then, we see from Lemma 17 that \( M^n \) is isometric to \( \mathbb{R}^n \).

Otherwise, the facts \( R > 0, |\dot{R} c| = 0 \) and the equalities of (51) imply that \( \dot{R} m = W = 0 \). Hence, we know that \( (M^n, g) \) has constant sectional curvature when \( R > 0 \); it follows from the Myers theorem and the condition \( R > 0 \) that \( (M^n, g) \) is compact and therefore is a finite quotient of \( S^n \).

**Proof of Theorem 4.** It is well known that \( R \geq 0 \) for shrinking solitons. When \( R \) achieves its infimum 0, Lemma 17 says that \( (M^n, g, f) \) is flat and therefore is isometric to \( \mathbb{R}^n \).

In the rest of the proofs of Theorem 4, we assume that \( R > 0 \). By (43) and (9), we see that

\[
\Delta_{f,-2\log R} \left( \left| \frac{\dot{R} m}{R^2} \right|^2 \right) \geq \frac{2}{R^3} \left( \sqrt{\dot{R} \nabla \dot{R} m} - \frac{|\dot{R} m|}{\sqrt{R}} \right)^2
\]

\[
+ \frac{4(n-2)}{n(n-1)} R \left| \dot{R} m \right|^2
\]

\[
> \frac{2}{R^3} \left( \sqrt{\dot{R} \nabla \dot{R} m} - \frac{|\dot{R} m|}{\sqrt{R}} \right)^2 + \frac{4}{n(n-1)} R \left| \dot{R} m \right|^2
\]

\[
\geq \frac{4}{R^3} \left( \frac{R^2}{n(n-1)} + \left| \dot{R} \right|^2 - C(n) |\dot{R} m|^2 \right)
\]

\[
\geq 0.
\]

(52)

Set \( \omega = f - \nabla \log (R^2) \) and \( u = (|\dot{R} m|^2)/R^2 \). In order to apply Lemma 19 to (52), we need to verify

\[
\int_{M^n} e^{\omega} dV_g = \int_{M^n} R^2 e^{-f} dV_g < +\infty,
\]

\[
\int_{M^n} \frac{|\dot{R} m|^4}{R^4} e^{\omega} dV_g = \int_{M^n} \frac{|\dot{R} m|^4}{R^4} e^{-f} dV_g < +\infty.
\]

In fact, (53) follows from the result that \( R \in L^p(e^{-f} dV_g) \) for \( 1 \leq p < +\infty \) (see e.g., [10]) for all gradient shrinkers.

Under the assumption that \( (M^n, g, f) \) has nonnegative Ricci curvature and (9), we get \( |Rc| \leq \bar{R} \). Thus,

\[
|\dot{R} m|^2 \leq \frac{1}{C(n)} \left( R - \frac{(n-2)R}{n(n-1)} \right) = \frac{n^2 - 2n + 2}{n(n-1)C(n)} R,
\]

\[
|\dot{R} m|^4 \leq \left( \frac{n^2 - 2n + 2}{n(n-1)C(n)} \right)^2 R^2 e^{\omega} dV_g.
\]

Furthermore, as observed in Remark 5, if we relax the condition of negative Ricci curvature to that \( |Rc| \leq cR^{1+\alpha} \) for some constants \( c > 0 \) and \( \alpha \geq 0 \) satisfying \( cR^{\alpha} \geq \sqrt{(n-2)/(n(n-1))} \), then

\[
|\dot{R} m|^2 \leq \frac{R}{C(n)} \left( R^{2\alpha} - \frac{n-2}{n(n-1)} \right).
\]

(56)

It follows from the result \( R \in L^p(e^{-f} dV_g) \) for \( 1 \leq p < +\infty \) that...
$\frac{|\mathbf{R}|^4}{R^2} \in L(e^{-f}dV_\theta).$ \hfill (57)

These together with (52) and Lemma 19 yield $R^* m = 0$ or $W = 0$ and $|R^* m| = 1/C(n)((R/(n(n-1))) + (|R^* c|^2/R)).$

What we need to prove now is that $R^* c = 0$ in the latter case. In fact, by (16), the facts $W = 0$ and $|R^* m| = 1/C(n)((R/(n(n-1))) + (|R^* c|^2/R))$, we derive

$$|R^* c| = \frac{\sqrt{n-2}}{2} |R^* m| = \frac{\sqrt{n-2}}{2C(n)} \left( \frac{R}{n(n-1)} + \frac{|R^* c|^2}{R} \right). \hfill (58)$$

It is easy to check from the definition of $C(n)$ that the different two solutions of equation (58) satisfy

$$|R^* c|_1 = \frac{C(n)R}{\sqrt{n-2}} + R \frac{C^2(n)}{n-2} - \frac{1}{n(n-1)} > \frac{\sqrt{n-2}}{\sqrt{n(n-1)}} R,$$

$$|R^* c|_2 = \frac{(1/(n(n-1)))R^2}{(C(n)R/\sqrt{n-2}) + R[(C^2(n)/n-2) - (1/(n(n-1)))]} \quad < \frac{R}{\sqrt{(n-2)/(n-1)}} n.$$ \hfill (59)

Combining (59) and (60) and the fact $W = 0$ with Lemma 16 gives

$$\Delta_{\text{f-2vol}} R \left( \frac{|R^* c|^2}{R^2} \right) \geq \frac{2}{R^4} |R^* VR^* \theta - \nabla R^* \theta|^2 + \frac{4}{R^2} \left( \frac{|R^* c|^2}{R^2} \right)^2 \hfill (61)$$

$$\cdot \left( \frac{|R^* c| - \frac{R}{\sqrt{n(n-1)}}}{R} \right)^2 \geq \frac{4}{R^2} \left( \frac{|R^* c| - \frac{R}{\sqrt{n(n-1)}}}{R} \right)^2 \geq 0.$$

By a similar argument, we conclude from Lemma 19 and the assumption on the Ricci curvature that $|R^* c|^2/R$ is a constant and $R^* c = 0$ since $|R^* c|_1 \neq R/\sqrt{n(n-1)}$. This concludes the proof of Theorem 4.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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