HIGHER LOCALISED $\hat{A}$-GENERA FOR PROPER ACTIONS AND APPLICATIONS

HAO GUO AND VARGHESE MATHAI

Abstract. For a finitely generated discrete group $\Gamma$ acting properly on a spin manifold $M$, we formulate new topological obstructions to $\Gamma$-invariant metrics of positive scalar curvature on $M$ that take into account the cohomology of the classifying space $B\Gamma$ for proper actions.

In the cocompact case, this leads to a natural generalisation of Gromov-Lawson’s notion of higher $\hat{A}$-genera to the setting of proper actions by groups with torsion. It is conjectured that these invariants obstruct the existence of $\Gamma$-invariant positive scalar curvature on $M$. For classes arising from the subring of $H^\ast(B\Gamma, \mathbb{R})$ generated by elements of degree at most 2, we are able to prove this, under suitable assumptions, using index-theoretic methods for projectively invariant Dirac operators and a twisted $L^2$-Lefschetz fixed-point theorem involving a weighted trace on conjugacy classes. The latter generalises a result of Wang-Wang [21] to the projective setting. In the special case of free actions and the trivial conjugacy class, this reduces to a theorem of Mathai [15], which provided a partial answer to a conjecture of Gromov-Lawson on higher $\hat{A}$-genera.

If $M$ is non-cocompact, we obtain obstructions to $M$ being a partitioning hypersurface inside a non-cocompact $\Gamma$-manifold with non-negative scalar curvature that is positive in a neighbourhood of the hypersurface. Finally, we define a quantitative version of the twisted higher index, as first introduced in [10], and use it to prove a parameterised vanishing theorem in terms of the lower bound of the total curvature term in the square of the twisted Dirac operator.

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1. Introduction

In this paper we develop the theory of higher indices of projectively equivariant Dirac operators with respect to proper actions of discrete groups, and relate this to numerical invariants that are computable in terms of characteristic classes. In the spin setting, these results can be applied to give new obstructions to metrics of positive scalar curvature that are invariant under the proper action of a discrete group. This generalises the obstructions provided by the higher $\hat{A}$-genera of Gromov and Lawson [7] to the setting of proper actions.

On the operator-algebraic side, where the higher index resides, we consider appropriately weighted traces on twisted group $C^*$-algebras. These traces generalise the traces associated to conjugacy classes in the setting of ordinary group $C^*$-algebras by taking into account a U(1)-valued group 2-cocycle. On the geometric side, there is a corresponding algebra of projectively invariant operators of an appropriate trace class, in which the heat operator associated to a projectively invariant Dirac operator lies. The traces on these two sides are related through a weighted $L^2$-Lefschetz fixed-point theorem that generalises a result of Wang-Wang [21, Theorem 6.1].

The special case where the group $\Gamma$ is torsion free and was considered in [5, 15], and led to a proof that the higher $\hat{A}$-genera of $M$ arising from the subring of $H^*(B\Gamma, \mathbb{R})$ generated by elements of degree at most 2 are obstructions to positive scalar curvature on $M/\Gamma$. This gave a partial answer to the following conjecture of Gromov-Lawson [7, Conjecture]; see also [19, Conjecture 2.1].

Now let us turn to the general case where an arbitrary discrete group $\Gamma$, possibly with torsion, acts properly on a manifold $M$. Denote by $B\Gamma$ the classifying space for proper actions [1], and let $f: M/\Gamma \to B\Gamma$ be the classifying map for the action of $\Gamma$ on $M$. For any integer $m \geq 0$, let $\alpha \in Z^m(B\Gamma, \mathbb{R})$. Let $\omega \in \Omega^m(M)$ (1.1) be the $\Gamma$-invariant lift of a differential form on the orbifold $M/\Gamma$ belonging to the class $f^*[\beta]$ in $H^{*\Gamma}(M/\Gamma)$.

If the quotient $M/\Gamma$ is compact, then for each $g \in \Gamma$, the centraliser $Z^g$ of $g$ acts properly and cocompactly on the fixed-point submanifold $M^g$. Let $c^g$ be a cut-off function for this action, so that for any $x \in M^g$, we have

$$\sum_{k \in Z^g} c^g(k^{-1}x) = 1.$$  

We define the higher localised $\hat{A}$-genus of $M$ with respect to $\omega$ to be

$$\hat{A}_g(M, \omega) := \int_{M^g} c^g \cdot \frac{\hat{A}(M^g) \cdot \omega|_{M^g}}{\det(1 - ge^{-R^N/2\pi i})^{1/2}},$$  

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where $R_N$ is the curvature of the Levi–Civita connection restricted to the normal bundle $N$ of $M^g$ in $M$ with respect to a $\Gamma$-invariant Riemannian metric on $M$.

This leads to the following conjecture:

**Conjecture 1.1.** If a proper, cocompact $\Gamma$-spin connected manifold $M$ admits a $\Gamma$-invariant Riemannian metric of positive scalar curvature, then all of the higher localised $\hat{A}$-genera of $M$ vanish. That is, for any closed $m$-form $\omega$ constructed as above, and for all $g \in \Gamma$, one has

$$\hat{A}_g(M, \omega) = 0.$$ 

**Remark 1.2.** We give evidence for the validity of this conjecture in Theorem 1.3 and Corollary 1.4, by proving it for all $\omega$ arising from the cohomology ring generated by $f^*(H^1(B\Gamma) \cup f^*(H^2(B\Gamma))$, under the assumption that $M^g$ is connected and $g$ is a regular element for the relevant group multiplier on $\Gamma$ (see subsection 2.2). In view of Remark 4.11, it is likely that Conjecture 1.1 is implied by the Baum-Connes conjecture.

We give an index-theoretic approach to a special case of Conjecture 1.1 as follows. From the data above, we construct a twisted Dirac operator that is invariant under a projective representation of $\Gamma$. We show that the integral (1.2) arises naturally as certain weighted traces of the associated heat operator, and relate this to the higher index of the twisted Dirac operator via a weighted trace map on the twisted group algebra. We prove:

**Theorem 1.3.** Let $\Gamma$ be a finitely generated group, and let $M$ be a connected, $\Gamma$-equivariantly spin manifold such that $M/\Gamma$ is compact. Let $\omega \in \Omega^2(M)$ be as in (2.1), and let $D$ be the associated projectively invariant Dirac operator on $M$ from Definition 2.5. Let $\alpha$ and $\sigma$ be the multipliers constructed in subsection 2.2, and suppose that $g \in \Gamma$ is an $\alpha$-regular element, in the sense of (3.3), whose conjugacy class $(g)$ has polynomial growth.

(i) Suppose $M$ is even-dimensional. Then

$$(\tau^{(g)}_\sigma)_* \text{Ind}_{\Gamma, \sigma}(D) = \sum_{j=1}^m \int_{M^g_j} e^{i\phi_g(x_j)} c^g \cdot \frac{\hat{A}(M^g_j) \cdot e^{-\omega/2\pi i} |M^g_j|}{\det(1 - ge^{-R^g_{\pi i}/2\pi i})^{1/2}}$$

where $\text{Ind}_{\Gamma, \sigma}(D) \in K_0(C^*_r(\Gamma, \sigma))$ be the $(\Gamma, \sigma)$-invariant higher index of $D$ from Definition 3.30, $(\tau^{(g)}_\sigma)_*$ is the homomorphism (4.8), $M^g_1, \ldots, M^g_m$ are the connected components of $M^g$ that intersect the support of a cut-off function $c^g$ for the action of $\mathbb{Z}^g$ on $M^g$, $\phi_g$ is as in (2.2), $x_j$ is an arbitrary point in $M^g_j$, and $R^g_{\pi i}$ is the curvature of the Levi-Civita connection restricted to the normal bundle $N_j$ of $M^g_j$ in $M$ and with respect to a $\Gamma$-invariant Riemannian metric on $M$.

(ii) If $M$ admits a $\Gamma$-invariant Riemannian metric of positive scalar curvature, then

$$\sum_{j=1}^m \int_{M^g_j} c^g \cdot \frac{\hat{A}(M^g_j) \cdot e^{(\phi_g(x_j) + \omega)/2\pi i} |M^g_j|}{\det(1 - ge^{-R^g_{\pi i}/2\pi i})^{1/2}} = 0,$$

where the notation is as explained in part (i). If $M^g$ is connected, then

$$\hat{A}_g(M, \omega^k) = 0$$

for each $k \geq 0$. 

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Corollary 1.4. Let $\Gamma$, $M$, and $g$ be as in the statement of Theorem 1.3. If $M^g$ is connected, then Conjecture 1.1 holds for $\omega$ arising from the subring of $H^*(B\Gamma, \mathbb{R})$ generated by $f^*H^1(B\Gamma, \mathbb{R}) \cup f^*H^2(B\Gamma, \mathbb{R})$.

Next, we prove two obstruction results when $M/\Gamma$ is non-compact, again via index theory of projectively invariant Dirac operators. First, we show that the higher localised $\hat{A}$-genera are obstructions to $\Gamma$-invariant Riemannian metrics with positive scalar curvature in a neighbourhood of $M$. More precisely, we prove:

Theorem 1.5. Let $M$ be a connected spin manifold on which a discrete group $\Gamma$ acts properly, preserving the spin structure. Let $H$ be a connected $\Gamma$-cocompact hypersurface in $M$ with trivial normal bundle. Let the multipliers $\alpha$ and $\sigma$ be as in subsection 5.2, and suppose that $g \in \Gamma$ is an $\alpha$-regular element, in the sense of (3.3), whose conjugacy class $(g)$ has polynomial growth.

Suppose $M$ admits a complete $\Gamma$-invariant Riemannian metric whose scalar curvature is non-negative everywhere on $M$ and positive in a neighbourhood of $H$. Let $\omega$ be as above in (1.2). Then if $H^g$ is connected,

$$\hat{A}_g(H, \omega^k|_H) = 0$$

for each integer $k \geq 0$. In particular,

$$\sum_{j=1}^{m} \int_{H_j^g} e^{-i\phi_j(x_j)\cdot g} \cdot \frac{\hat{A}(H_j^g) \cdot e^{-\omega/2\pi|_{H_j^g}}}{\det(1 - ge^{-R_{N_j}^g/2\pi i})^{1/2}} = 0,$$

where $g \in \Gamma$ is a $\sigma$-regular element, $H_1^g, \ldots, H_m^g$ are the connected components of the fixed-point set $H^g$ that intersect the support of a cut-off function $c_{H_j}^g$ for the $Z^g$-action on $H^g$, and $N_j$ is the normal bundle of $H_j^g$ in $H$.

The proof of this theorem uses a Callias-type index theorem for projectively invariant Dirac operators, which is discussed in section 5.

Remark 1.6. Instead of the assumption that $(g)$ has polynomial growth, the conclusions of Theorem 1.3 and Theorem 1.5 also hold whenever the trace $\tau_{\sigma^s}: C^*\Gamma \to \mathbb{C}$ extends continuously to $C^*_r(\Gamma, \sigma^s)$ for all $s$ in a small interval $(0, \delta)$.

Second, we show that the projectively invariant higher index is compatible with the framework of quantitative $K$-theory [18], which is a refinement of operator $K$-theory in the context of geometric $C^*$-algebras. As shown in subsection 6.1, this allows us to formulate a quantitative notion of projectively invariant higher index, generalising that defined in [10]. With respect to this index, we obtain a parameterised vanishing theorem where the vanishing propagation depends on the lower bound of the twisting curvature of the projective Dirac operator as well as the scalar curvature. More precisely, we prove:

Theorem 1.7. Fix $0 < \varepsilon < \frac{1}{29}$ and $N \geq 7$. There exists a constant $\lambda_0$ such that the following holds. Let $M$ be a smooth spin Riemannian manifold equipped with a proper isometric action by a discrete group $\Gamma$. Let $p: M \to M/\Gamma$ be the projection, $\kappa$ the scalar curvature on $M$, and $S \to M$ the spinor bundle.

Let $\omega \in \Omega^2(M)$ be as in (2.1), and let $D^s$ be the associated projectively invariant Dirac operator on $M$ from Definition 2.5, acting on sections of the bundle $S_{X_\omega}$. Let $\kappa$ denote the scalar curvature on $M$, and let $c$ denote Clifford multiplication. If the estimate

$$\kappa + 4isc(\omega) \geq C_s$$
holds for some positive constant $C_s$, then the quantitative $(\Gamma, \sigma^s)$-equivariant higher index of $D^s$ on $L^2(S_{\varphi})$ at scale $r$ vanishes for all $r$ ≥ \( \frac{k_0}{\sqrt{C_s}} \):

\[
\text{Ind}_{\Gamma, \sigma^s, L^2}(D^s) = 0 \in K^{\varepsilon, r, N}_{\sigma^s}(C^*(M; L^2(S_{\varphi}))^{\Gamma, \sigma^s}),
\]

where the algebra $C^*(M; L^2(S_{\varphi}))^{\Gamma, \sigma^s}$ is as in Definition 3.25, and $\text{Ind}_{\Gamma, \sigma^s, L^2}(D^s)$ is defined as in subsection 6.2.

**Remark 1.8.** When $s = 0$, Theorem 1.7 reduces to [10, Theorem 1.1].

The paper is organised as follows. In section 2, we formulate the preliminary definitions and properties of the relevant operator algebras and projectively invariant operators. In section 4, we prove Theorem 1.3 and Corollary 1.4. In section 5, we develop some Callias-type index theory in the projective setting and use it to prove Theorem 1.5. In section 6, we formulate the quantitative twisted higher index and prove Theorem 1.7.

## 2. Preliminaries

We first fix some notation and recall the necessary operator-algebraic and geometric terminology we will need.

### 2.1. Notation.

For $X$ a Riemannian manifold, we write $B(X)$, $C_b(X)$, $C_0(X)$, and $C_c(X)$ to denote the $C^*$-algebras of complex-valued functions on $X$ that are, respectively: bounded Borel, bounded continuous, continuous and vanishing at infinity, and continuous with compact support. If $S \subseteq X$ is a Borel subset, we write $1_S$ for the associated characteristic function.

For any $C^*$-algebra $A$, denote its unitization by $A^+$. If $E$ is a Hilbert module over $A$, let $B(E)$ and $K(E)$ denote the $C^*$-algebras of bounded adjointable and compact operators on $E$ respectively.

For an element $g$ of a group $G$, we use $Z_g$ to denote the centraliser of $g$ in $G$.

### 2.2. Multipliers and projective representations.

Let $G$ be a discrete group.

**Definition 2.1.** A multiplier on $G$ is a map $\sigma : \Gamma \times \Gamma \to U(1)$ satisfying

1. \( \sigma(\gamma, \mu)\sigma(\gamma \mu, \delta) = \sigma(\gamma, \mu \delta)\sigma(\mu, \delta) \);
2. \( \sigma(\gamma, e) = \sigma(e, \gamma) = 1 \),

for all $\gamma, \mu, \delta \in \Gamma$, where $e$ is the identity element in $\Gamma$.

**Remark 2.2.** Condition (ii) implies that $\sigma(\gamma, \gamma^{-1}) = \sigma(\gamma^{-1}, \gamma) = 1$ for all $\gamma \in \Gamma$.

In other words, a multiplier on $G$ is an element of $Z^2(G, U(1))$ satisfying the normalisation condition (ii). We will be concerned specifically with multipliers that arise from proper $\Gamma$-actions on manifolds in the following way.

Let $M$ be a smooth, connected manifold equipped with a proper action by a discrete group $\Gamma$, preserving the spin structure. Suppose $M/\Gamma$ is compact and that $H^1(M) = 0$. Let $B\Gamma$ be the classifying space for proper $\Gamma$-actions [1], and $f : M/\Gamma \to B\Gamma$ the classifying map for $M$.

Let $f[\beta] \in H^2(B\Gamma, \mathbb{R})$ be a 2-cocycle on $B\Gamma$, and $\omega_0$ a closed 2-form on $M$ representing the de Rham cohomology class $f^*[\beta] \in H^2_{\text{dR}}(M/\Gamma)$ on the orbifold $M/\Gamma$. Since $[\beta]$ lifts trivially to $\tilde{E}\Gamma$,
the lift $\omega$ of $\omega_0$ to $M$ is exact, so there exists a one-form $\eta$ (not necessarily $\Gamma$-invariant) such that

$$\omega = d\eta.$$  \hfill (2.1)

Since $\omega$ is $\Gamma$-invariant, $d(\gamma^*\eta - \eta) = \gamma^*\omega - \omega = 0$ for all $\gamma \in \Gamma$. Thus $\gamma^*\eta - \eta$ is a closed 1-form on $M$, and in fact exact. It follows that there exists a family

$$\phi := \{\phi_\gamma : \gamma \in \Gamma\}$$

of smooth functions on $M$ such that

$$\gamma^*\eta - \eta = d\phi_\gamma.$$  \hfill (2.2)

This implies that for any $\gamma, \gamma' \in \Gamma$,

$$d(\phi_\gamma + \gamma^{-1}\phi_{\gamma'} - \phi_{\gamma'\gamma}) = 0.$$  \hfill (2.3)

By way of normalisation, we may assume that there exists some $x_0$ such that

$$\phi_\gamma(x_0) = 0$$  \hfill (2.4)

for all $\gamma \in \Gamma$. Using (2.3), one easily verifies that $\phi_e \equiv 0$ and that for each $s \in \mathbb{R}$, the formula

$$\alpha_\phi(\gamma, \gamma') = \frac{1}{2\pi}\phi_\gamma(\gamma'x_0)$$  \hfill (2.5)

defines an element of $Z^2(\Gamma, \mathbb{R})$. The associated $U(1)$-valued 2-cocycle

$$\sigma_\phi^s(\gamma, \gamma') = e^{2\pi is\alpha_\phi(\gamma, \gamma')} = e^{is\phi_\gamma(\gamma'x_0)}$$  \hfill (2.6)

is a multiplier on $\Gamma$. When it is clear from context, we will simply write

$$\alpha = \alpha_\phi, \quad \sigma^s = \sigma_\phi^s, \quad \sigma = \sigma^1.$$

It can be shown that for a given class $[\beta]$, different choices of $\omega, \eta$, and $\phi$ all lead to cohomologous $\sigma$. Nevertheless, the numerical obstructions we compute in Theorems 1.3 and 1.5 depend not only on the class of $\sigma$ but also on the family $\phi$.

**Remark 2.3.** Restricting (2.2) to the fixed-point submanifold $M^\gamma$, one sees that

$$\gamma^*\eta|_{M^\gamma} - \eta|_{M^\gamma} = d_{M^\gamma}\phi_\gamma = 0,$$

hence on any connected component $M^\gamma$, the function $\phi_\gamma$ is constant.

**Definition 2.4.** Let $E \to M$ be a $\Gamma$-equivariant $\mathbb{C}$-vector bundle. For each $\gamma \in \Gamma$ and $s \in \mathbb{R}$, define the unitary operators $U_\gamma$, $S_\gamma^s$, and $T_\gamma^s$ on $L^2(E)$ by:

- $U_\gamma u(x) = \gamma u(\gamma^{-1}x)$;
- $S_\gamma^s u = e^{is\phi_\gamma} u$;
- $T_\gamma^s = U_\gamma \circ S_\gamma^s$,

for $u \in L^2(E)$ and $x \in M$. We refer to $T$ as a *projection action* on $L^2(E)$.

Note that for any $\gamma, \gamma' \in \Gamma$ and $s \in \mathbb{R}$, we have

$$T_\gamma^s T_{\gamma'}^s = \sigma^s(\gamma, \gamma') T_{\gamma\gamma'}^s.$$  \hfill (2.7)

Thus the map $T: \Gamma \to U(L^2(E))$ given by $T(\gamma) = T_\gamma$ defines a projective representation of $\Gamma$ in the sense of subsection 2.3 below. An operator on $L^2(E)$ that commutes with $T$ is said to be $(\Gamma, \sigma^s)$-invariant or simply *projectively invariant* if no confusion arises.

Now suppose $M$ is $\Gamma$-equivariantly spin, equipped with a $\Gamma$-invariant Riemannian metric. Let $E = \mathcal{S}$ be the spinor bundle and $\slashed{D}$ the spin-Dirac operator. We can obtain a $(\Gamma, \sigma^s)$-invariant Dirac
operator as follows. Let \( \mathcal{L} \to M \) be a \( \Gamma \)-equivariantly trivial line bundle. For each \( s \in \mathbb{R} \), consider the Hermitian connection

\[
\nabla^{L,s} := d + is\eta
\]
on \( \mathcal{L} \). Equip \( S_{\mathcal{L}} = S \otimes \mathcal{L} \) with the obvious \( \mathbb{Z}_2 \)-grading. Then we have:

**Definition 2.5.** For each \( s \in \mathbb{R} \), we will refer to the operator

\[
D^s := \partial \otimes \nabla^{L,s} : L^2(S_{\mathcal{L}}) \to L^2(S_{\mathcal{L}})
\]

as the \( (\Gamma, \sigma^s) \)-invariant Dirac operator, or simply a projectively invariant Dirac operator if no confusion arises. We write \( D = D^1 \).

It is easy to verify that \( D^s \) commutes with the projective \( (\Gamma, \sigma^s) \)-action defined by applying Definition 2.4 to \( E = S_{\mathcal{L}} \).

2.3. **Twisted group \( C^* \)-algebras.** Given a multiplier \( \sigma \) on a discrete group \( \Gamma \), a unitary \( (\Gamma, \sigma) \)-representation, or projective representation, of \( \Gamma \) is a map

\[
T : \Gamma \to U(H), \quad \gamma \mapsto T_\gamma,
\]

for some Hilbert space \( H \), such that

\[
T_e = \text{Id}_H; \\
T_{\gamma_1} \circ T_{\gamma_2} = \sigma(\gamma_1, \gamma_2)T_{\gamma_1 \gamma_2}
\]

for all \( \gamma_1, \gamma_2 \in \Gamma \). The twisted group algebra \( C^0 \Gamma \) is the associative \( * \)-algebra over \( \mathbb{C} \) with basis \( \{ \bar{\gamma} : \gamma \in \Gamma \} \), where the multiplication and involution are given on basis elements by

\[
\bar{\gamma}_1 \cdot \bar{\gamma}_2 = \sigma(\gamma_1, \gamma_2)\bar{\gamma}_1 \bar{\gamma}_2, \quad \bar{\gamma}^* = \gamma^{-1},
\]

and extended linearly and conjugate-linearly respectively.

The reduced twisted group \( C^* \)-algebra is constructed as follows. Consider \( l^2(\Gamma) \) with its usual basis \( \{ \delta_\gamma \}_{\gamma \in \Gamma} \), and define a projective representation

\[
T^L : (\Gamma, \sigma) \to \mathcal{B}(l^2(\Gamma))
\]

by the following action on basis elements:

\[
T_{\gamma_1}^L \delta_{\gamma_2} := \sigma(\gamma_1, \gamma_1^{-1} \gamma_2)\delta_{\gamma_1 \gamma_2}.
\]

Then \( T^L \) extends naturally to an injective \(*\)-representation on \( \mathcal{B}(l^2(\Gamma)) \), called the left regular representation of \( C^0 \Gamma \). The reduced twisted group \( C^* \)-algebra \( C^+_r(\Gamma, \sigma) \) is the completion of \( C^0 \Gamma \) with respect to the induced norm. When convenient, we will use simply abbreviate \( T^L_\gamma \) to \( \bar{\gamma} \).

When \( \sigma \equiv 1 \) is the trivial multiplier, \( C^0 \Gamma \) and \( C^+_r(\Gamma, \sigma) \) are the ordinary group algebra and reduced group \( C^* \)-algebra respectively.

**Remark 2.6.** It follows from Remark 2.2 that for any projective representation \( T \), we have \( T_{\gamma}^{-1} = T_{\gamma^{-1}} \) for all \( \gamma \in \Gamma \).
3. Weighted traces and projective indices

We now define the traces associated to conjugacy classes, in an appropriate sense, on the algebraic part of the twisted group algebra that we will use in this paper.

When \( \sigma \equiv 1 \) is the trivial multiplier, we recover from \( \mathbb{C}^w \Gamma \) the group algebra \( \mathbb{C} \Gamma \). In this case, for any conjugacy class \( (g) \subseteq \Gamma \), the map

\[
\tau^g: \mathbb{C} \Gamma \to \mathbb{C}, \quad \sum_{\gamma \in \Gamma} a_{\gamma} \gamma \mapsto \sum_{\gamma \in (g)} a_{\gamma} (3.1)
\]

is a trace. When \( \sigma \) is non-trivial, this formula ceases in general to define a trace on \( \mathbb{C}^w \Gamma \). However, for conjugacy classes of certain \( \sigma \)-regular elements, one can define a trace via a weighted sum determined by \( \sigma \).

**Definition 3.1.** Let \( \sigma \) be a multiplier on \( \Gamma \). An element \( g \in \Gamma \) is \( \sigma \)-regular if

\[
\sigma(g, z) = \sigma(z, g)
\]

for all \( z \in Z^g \).

The property of being \( \sigma \)-regular is invariant under conjugation in \( \Gamma \), hence we may speak of \( \sigma \)-regular conjugacy classes. The next lemma implies that in order to consider any traces at all on \( \mathbb{C}^w \Gamma \), it is necessary to deal with \( \sigma \)-regular elements 

**Lemma 3.2.** If \( g \) is not \( \sigma \)-regular, then for any trace map \( t: \mathbb{C}^w \Gamma \to \mathbb{C} \), \( t(g) = 0 \).

**Proof.** If \( g \) is not \( \sigma \)-regular, then there exists \( z \in Z_G(g) \) such that \( \bar{z}^{-1}g^{-1} = \lambda \bar{g} \) for \( \lambda \neq 1 \). Now

\[
\bar{z}^{-1}g^{-1}z - z^{-1}g^{-1} = \bar{g} - z^{-1}g = \bar{g} - (1 - \lambda)g.
\]

Since \( t \) is a trace, \( t(1 - \lambda)\bar{g} = 0 \) and hence \( t(\bar{g}) = 0 \). \( \square \)

To define traces for \( \sigma \)-regular conjugacy classes, we will use the following weighting function. Define a function \( \theta: \Gamma \to U(1) \) by

\[
\theta(\gamma) = \begin{cases} 
\sigma(g^{-1}, k)\sigma(k^{-1}, g^{-1}) & \text{if } \gamma = k^{-1}gk \text{ for some } k \in \Gamma, \\
0 & \text{otherwise.}
\end{cases} \quad (3.2)
\]

We claim that \( \theta \) is well-defined. Indeed, if \( \gamma = k^{-1}a^{-1}gak \) for some \( a \in Z^g \), then

\[
\sigma(k^{-1}a^{-1}, g)\sigma(k^{-1}a^{-1}g, ak)k^{-1}gk = \sigma(k^{-1}a^{-1}, g)\sigma(k^{-1}a^{-1}g, ak)k^{-1}a^{-1}gak
\]

\[
= k^{-1}a^{-1}g \cdot \bar{a} \cdot \bar{g}
\]

\[
= k^{-1}a^{-1}\sigma(k^{-1}, a^{-1})\bar{g}\sigma(a, k)\bar{a}k
\]

\[
= k^{-1}\bar{g}k
\]

\[
= \sigma(k^{-1}, g)\sigma(k^{-1}, k)k^{-1}gk,
\]

where in the second-last equation we used that \( (g) \) is \( \sigma \)-regular. Equating coefficients and taking complex conjugates then shows that \( \theta \) is well-defined.

**Remark 3.3.** Note that \( \theta(g) = \sigma(g^{-1}, e)\sigma(e, g^{-1}) = 1 \).

**Remark 3.4.** When \( \Gamma \) is finite, the set of \( \sigma \)-regular conjugacy classes in \( \Gamma \) is in bijection with the set of distinct irreducible \((\Gamma, \sigma)\)-representations.
Definition 3.5. Suppose that \((g)\) is \(\sigma\)-regular for some multiplier \(\sigma\) on \(\Gamma\). Define the \(\sigma\)-weighted \((g)\)-trace \(\tau_{\sigma}^{(g)} : \mathbb{C}\sigma\Gamma \to \mathbb{C}\) by

\[
\sum_{\gamma \in \Gamma} a_{\gamma} \bar{\gamma} \mapsto \sum_{\gamma \in (g)} \theta(\gamma) a_{\gamma}.
\]

We will show that \(\tau_{\sigma}^{(g)}\) is a trace in two steps. Define the multiplier

\[
\sigma' = \sigma d\theta,
\]

where we recall that \(d\theta(\gamma_1, \gamma_2) = \theta(\gamma_1 \gamma_2) \bar{\theta}(\gamma_1) \bar{\theta}(\gamma_2)\). Then \(\sigma'\) is a multiplier cohomologous to \(\sigma\). Let \(\mathbb{C}\sigma'\Gamma\) be the twisted group algebra defined by \(\sigma'\), with basis \(\{\gamma : \gamma \in \Gamma\}\). We first claim that:

Proposition 3.6. The map \(\tau^{(g)} : \mathbb{C}\sigma'\Gamma \to \mathbb{C}\) defined by

\[
\sum_{\gamma \in \Gamma} a_{\gamma} \gamma \mapsto \sum_{\gamma \in (g)} a_{\gamma}
\]

is a trace.

Proof. First note that \(\mathbb{C}\sigma'\Gamma\) has the property that for any \(k \in \Gamma\),

\[
k^{-1} g k = k^{-1} g k.
\]

By extension, this holds if \(g\) is replaced by any \(h \in (g)\): if \(h = m^{-1} g m\), then

\[
k^{-1} h k = k^{-1} m^{-1} g m k = m k^{-1} g m = (m k)^{-1} g m = k^{-1} h k.
\]

To see that \(\tau^{(g)}\) is a trace, it suffices to show that for any \(k, m \in \Gamma\), we have \(\text{tr}^{(g)} km = \text{tr}^{(g)} km\).

We may assume that \(km = h \in (g)\), so that \(km = \lambda h\) for some \(\lambda \in \mathbb{U}(1)\). Then clearly \(\text{tr}^{(g)} km = \lambda\). On the other hand,

\[
\lambda h = \lambda k^{-1} h k = \lambda (m k)^{-1} g m k = \lambda k^{-1} h k.
\]

By the preceding discussion, this is equal to

\[
\lambda k^{-1} h k
\]

hence \(\text{tr}^{(g)} km = \lambda\). \(\square\)

We now use Proposition 3.6 to deduce:

Proposition 3.7. \(\tau_{\sigma}^{(g)} : \mathbb{C}\sigma\Gamma \to \mathbb{C}\) is a trace.

Proof. Using that \(\sigma'\) is cohomologous to \(\sigma\) via the coboundary \(d\theta\), one verifies that

\[
f_\theta : \mathbb{C}\sigma\Gamma \to \mathbb{C}\sigma'\Gamma,
\]

\[
\gamma \mapsto \theta(\gamma) \gamma
\]

is an isomorphism of \(*\)-algebras. We have a commutative diagram

\[
\begin{array}{ccc}
\mathbb{C}\sigma'\Gamma & \xrightarrow{\tau_{\sigma}^{(g)}} & \mathbb{C} \\
| f_\theta \downarrow & & | f_\theta \downarrow \\
\mathbb{C}\sigma\Gamma & \xrightarrow{\tau_{\sigma}^{(g)}} & \mathbb{C},
\end{array}
\]
from which it easily follows that \( \tau^{(g)}_{\sigma} \) is a trace.

For real-valued group cocycles, the notion of regularity also applies: for any \( \alpha \in Z^2(\Gamma, \mathbb{R}) \), we say that \( g \in \Gamma \) is \( \alpha \)-regular if

\[
\alpha(g, z) = \alpha(z, g)
\]

for all \( z \in Z^g \). For such \( g \), let us define a function \( \psi: \Gamma \to \mathbb{R} \) by

\[
\psi(\gamma) = \begin{cases} 
\alpha(g^{-1}, k) + \alpha(k^{-1}, g^{-1}k) & \text{if } \gamma = k^{-1}gk \text{ for some } k \in \Gamma, \\
0 & \text{otherwise}.
\end{cases}
\]

(3.4)

As with the function \( \theta \) in (3.2), one checks that \( \psi \) is well-defined.

**Corollary 3.8.** Let \( \alpha \in Z^2(\Gamma, \mathbb{R}) \) be defined as in (2.5), and let \( \{\sigma^s\}_{s \in \mathbb{R}} \) be the associated family multipliers as in (2.6). If \( g \in \Gamma \) is \( \alpha \)-regular, then for every \( s \in \mathbb{R} \),

(i) the conjugacy class \( (g) \) is \( \sigma^s \)-regular;

(ii) the formula

\[
\sum_{\gamma \in \Gamma} a_{\gamma} \bar{\gamma} \mapsto \sum_{\gamma \in (g)} e^{2\pi is\psi(\gamma)}a_{\gamma}.
\]

defines a trace \( \tau^{(g)}_{\sigma^s} : C^{\sigma^s} \Gamma \to \mathbb{C} \), where \( \psi \) is as in (3.4).

**Proof.** By (2.6) and the fact that \( g \) is \( \alpha \)-regular,

\[
\sigma^s(g, z) = e^{2\pi is \alpha(g, z)} = e^{2\pi is \alpha(z, g)} = \sigma^s(z, g)
\]

for any \( z \in Z^g \), which proves (i). For (ii), note that the proofs of Propositions 3.6 and 3.7, with \( \theta \) replaced by

\[
\theta^s := e^{2\pi is\psi}
\]

(3.5)

and \( \sigma \)' replaced by \( (\sigma^s)' = \sigma^s d\theta^s \), imply that \( \tau^{(g)}_{\sigma^s} \) is a trace for each \( s \).

3.1. **Traces on operators.**

We now define weighted traces on projectively invariant operators of an appropriate class. Let \( M \), \( E \), \( \phi \), and \( \sigma \) be as in subsection 2.2.

**Definition 3.9.** A continuous function \( c: M \to [0, \infty) \) is called a cut-off function for the \( \Gamma \)-action on \( M \) if for any \( x \in M \) we have

\[
\sum_{g \in \Gamma} c(g^{-1}x) = 1.
\]

**Remark 3.10.** For any proper action, a cut-off function always exists. If the action is cocompact, the cut-off function can be chosen to be compactly supported.

We have the following useful lemma:

**Lemma 3.11.** Suppose \( M/\Gamma \) is compact, and let \( c \) be any cut-off function.

(i) Let \( f \) be a smooth function on \( M/\Gamma \) and \( \tilde{f} \) its lift to \( M \). Then

\[
\int_M c(x)\tilde{f}(x) \, dx = \int_{M/\Gamma} f(z) \, dz.
\]
(ii) Let $h$ be a continuous function on $M \times M$ that is invariant under the diagonal action of $\Gamma$. If $c(x)h(x,y), c(y)h(x,y)$ are integrable on $M \times M$, then

$$\int_{M \times M} c(x)h(x,y) \, dx \, dy = \int_{M \times M} c(y)h(x,y) \, dx \, dy.$$ 

**Proof.** Following [21, section 3], let us define the following special cut-off function. First write $M$ as

$$M = \bigcup_{i \in \mathbb{N}} \Gamma \times F_i U_i,$$

for finite subgroups $F_i$ of $\Gamma$ and $F_i$-invariant, relatively compact subsets $U_i \subseteq M$. Then $M/\Gamma$ admits the open cover $\{U_i/F_i\}_{i \in \mathbb{N}}$, along with a subordinate partition of unity $\{\phi_i\}_{i \in \mathbb{N}}$. Write $\tilde{\phi_i}$ for the $\Gamma$-invariant lift of $\phi_i$ to $M$. For each $i$, define the function $\varphi_i: G \times F_i U_i \rightarrow [0, \infty)$ by

$$\varphi_i([g, u]) := \begin{cases} \tilde{\phi_i}(u) & \text{for } g \in F_i, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\varphi_i$ extends by zero to an $F_i$-invariant function $\psi_i$ on all of $M$. Define a smooth function $c: M \rightarrow [0, \infty)$ by

$$c(x) := \sum_{i \in \mathbb{N}} \frac{1}{|F_i|} \psi_i(x). \quad (3.6)$$

By [21, Lemma 3.9], $c$ is a compactly supported, smooth cut-off function on $M$ whenever the $\Gamma$-action is cocompact.

The case of $c = c$, where $c$ is defined as in (3.6), is a special case of [21, Lemma 3.10]. Now observe that for any $\Gamma$-invariant function $r$ on $M$, the integral $\int_M c(x)r(x) \, dx$ is independent of the choice of cut-off function $c$. Applying this to $r = \tilde{f}$ yields the general case of (i), while letting

$$r(x) = \int_M h(x,y) \, dy, \quad r(y) = \int_M h(x,y) \, dx$$

yields the general case of (ii). \qed

**Definition 3.12.** Let $\sigma$ be a multiplier on $\Gamma$, and let $(g) \subseteq \Gamma$ be a $\sigma$-regular conjugacy class.

- A bounded $(\Gamma, \sigma)$-invariant operator $S$ on $L^2(S^*_\mathcal{F})$ is said to be of $(\sigma, g)$-trace class if for all $\phi_1, \phi_2 \in C_c(M),$

(i) the operator $\phi_1 T_{h^{-1}} S \phi_2$ is of trace class for any $h \in (g)$;

(ii) the sum

$$\sum_{h \in (g)} \text{tr}(\phi_1 T_{h^{-1}} S \phi_2) \quad (3.7)$$

converges absolutely.

- If $S$ is of $(\sigma, g)$-trace class, we define its $\sigma$-weighted $(g)$-trace to be

$$\text{tr}^{(g)}(S)_\sigma := \sum_{h \in (g)} \theta(h) \cdot \text{tr}(c_1 T_{h^{-1}} S c_2),$$

for some $c_1, c_2 \in C_c(M)$ such that $c_1c_2$ is a cut-off function on $M$.

**Remark 3.13.** If $S$ has finite propagation, then (3.7) is equivalent to the requirement that the operator

$$\sum_{h \in (g)} \phi T_{h^{-1}} S$$

...
is trace class for all $\phi \in C_c(M)$. Indeed, in this case there exists a compactly supported continuous function $f$ such that $\phi S = \phi Sf$. The $\sigma$-weighted $(g)$-trace of $S$ can then be equivalently defined to be
\[ \sum_{h \in (g)} \theta(h) \cdot \text{tr}(cT_{h^{-1}}S), \]
for any cut-off function $c$. Compare [21, Definition 3.13] and [22, Definition 3.1].

**Lemma 3.14.** If $S$ is a $(\Gamma, \sigma)$-invariant $(\sigma, g)$-trace class operator, then
\[ \text{tr}_g(S) = \sum_{h \in (g)} \theta(h) \int_M e^{-i\phi_h(x)}c(x)\text{Tr}(h^{-1}K_S(hx, x)) \, dx, \]
where $c$ is a cut-off function as in Definition 3.12, and $K_S$ denotes the Schwartz kernel of $S$.

**Proof.** Let $c_1, c_2 \in C_c(M)$ such that $c_1c_2 = c$. Note that for any $u \in L^2(S^\mathbb{R})$ and $h \in \Gamma$, we have
\[ (U_h^{-1}Sc_2)u(x) = \int_M h^{-1}K_S(hx, y)c_2(y)u(y) \, dy. \]
Since $T_h^{-1} = T_{h^{-1}} = S_h^{-1}U_h^{-1}$, this implies that
\[ (c_1T_h^{-1}Sc_2)u(x) = \int_M e^{-i\phi_h(x)}c_1(x)h^{-1}K_S(hx, y)c_2(y)u(y) \, dy. \]
Taking the trace gives
\[ \text{tr}(c_1T_h^{-1}Sc_2) = \int_M e^{-i\phi_h(x)}c_1(x)\text{Tr}(h^{-1}K_S(hx, x))c_2(x) \, dx = \int_M e^{-i\phi_h(x)}c(x)\text{Tr}(h^{-1}K_S(hx, x)) \, dx. \]
Multiplying by $\theta(h)$ and taking a sum finishes the proof. \hfill \Box

Our task now is to show that $\text{tr}_g(S)$ satisfies the tracial property and that it is independent of the choices of $c_1$, $c_2$, and $c$ made in Definition 3.12. This will be carried out in Lemma 3.18, after we collect some necessary observations in the form of Lemmas 3.14 – 3.27.

**Lemma 3.15.** For any multiplier $\sigma$ on $\Gamma$, we have
\[ \sigma(g, h)\sigma(h^{-1}, g^{-1}) = 1. \]

**Proof.** Since $\sigma(gh, h^{-1}g^{-1}) = 1$, we have
\[ \sigma(g, h)\sigma(h^{-1}, g^{-1}) = \sigma(g, h)\sigma(gh, h^{-1}g^{-1})\sigma(h^{-1}g^{-1}, gh)\sigma(h^{-1}, g^{-1}). \quad (3.8) \]
The cocycle identity and normalisation imply that
\[ \sigma(g, h)\sigma(gh, h^{-1}g^{-1}) = \sigma(g, g^{-1})\sigma(h, h^{-1}g^{-1}) = \sigma(h, h^{-1}g^{-1}), \]
\[ \sigma(h^{-1}g^{-1}, gh)\sigma(h^{-1}, g^{-1}) = \sigma(h^{-1}, h)\sigma(g^{-1}, gh) = \sigma(g^{-1}, gh). \]
Hence the right-hand side of equation (3.8) equals
\[ \sigma(h, h^{-1}g^{-1})\sigma(g^{-1}, gh) = \sigma(h, e)\sigma(h^{-1}g^{-1}, gh) = 1. \quad \Box \]

**Lemma 3.16.** Suppose $h = x^{-1}gx$. Then
\[ \sigma(\gamma^{-1}h, \gamma)\sigma(\gamma^{-1}, h) = \sigma(\gamma^{-1}x^{-1}, g)\sigma(\gamma^{-1}x^{-1}g, x\gamma)\sigma(x^{-1}, g)\sigma(x^{-1}g, x). \]
Proof. Observe that \( \tilde{\gamma}^{-1} h \tilde{\gamma} \) is equal to
\[
\tilde{\gamma} x^{-1} g x \tilde{\gamma} = \tilde{\gamma} x^{-1} \tilde{g} \tilde{x} \tilde{\gamma} \sigma(x^{-1}, g) \sigma(x^{-1} g, x).
\]
On the other hand, it is also equal to \( \sigma(\gamma^{-1}, h) \sigma(\gamma^{-1} h, \gamma) \gamma^{-1} h \gamma \), which can be written as
\[
\sigma(\gamma^{-1}, h) \sigma(\gamma^{-1} h, \gamma) \gamma^{-1} x^{-1} \tilde{g} \tilde{x} \tilde{\gamma} \sigma(x^{-1} g, x) \sigma(x^{-1}, g) \sigma(\gamma^{-1} x^{-1} g, x \gamma).
\]
We conclude by noting that \( \tilde{\gamma}^{-1} x^{-1} \tilde{g} \tilde{x} \tilde{\gamma} = \gamma^{-1} x^{-1} g \tilde{x} \tilde{\gamma} \) and equating coefficients. \( \Box \)

Lemma 3.17. Let \((g) \subseteq \Gamma\) be a \(\sigma\)-regular conjugacy class for some multiplier \(\sigma\). Then for any \(h \in (g)\) and \(\gamma \in \Gamma\), we have
\[
\theta(\gamma^{-1} h \gamma) \cdot \tilde{\sigma}(\gamma^{-1} h \gamma, \gamma^{-1}) = \theta(h) \cdot \tilde{\sigma}(\gamma^{-1}, h).
\]

Proof. We have
\[
\tilde{\sigma}(\gamma^{-1} h \gamma, \gamma^{-1}) = \sigma(\gamma, \gamma^{-1} h^{-1} \gamma)
= \tilde{\sigma}(\gamma^{-1}, h^{-1} \gamma)
= \sigma(\gamma^{-1} h, \gamma).
\]
The first and third equalities follow from Lemma 3.15; the second follows from the fact that \(\sigma(\gamma, \gamma^{-1} h^{-1} \gamma) \sigma(\gamma^{-1}, h^{-1} \gamma) = 1\). Thus it suffices to show that
\[
\theta(\gamma^{-1} h \gamma) \tilde{\theta}(h) \sigma(\gamma^{-1} h, \gamma) \sigma(\gamma^{-1}, h) = 1. \tag{3.9}
\]
For this, let \(h = k^{-1} g k\). Then by Lemma 3.16,
\[
\sigma(\gamma^{-1} h, \gamma) \sigma(\gamma^{-1}, h) = \sigma(\gamma^{-1} k^{-1} g, k) \sigma(\gamma^{-1} k^{-1} g, k) \sigma(k^{-1} g, k).
\]
It follows from the definition of \(\theta\) that the left-hand side of (3.9) equals
\[
\sigma(g^{-1}, k) \sigma(\gamma^{-1} k^{-1}, g^{-1} k) \sigma(g^{-1}, k) \sigma(k^{-1}, g^{-1} k)
\cdot \sigma(\gamma^{-1} k^{-1}, g) \sigma(\gamma^{-1} k^{-1} g, k) \sigma(k^{-1} g, k),
\]
which equals 1 by repeated applications of Lemma 3.15. \( \Box \)

With these preparations, we are now ready to prove that \(\text{tr}_{\sigma}^{(g)}\) is tracial and well-defined independently of the choice of functions used in Definition 3.12.

Proposition 3.18. Suppose \((g)\) is a \(\sigma\)-regular conjugacy class with respect to the multiplier \(\sigma\) defined by (2.6). Then:

(i) \(\text{tr}_{\sigma}^{(g)}\) does not depend on the choices of \(c_1\), \(c_2\), and \(c\) in Definition 3.12;

(ii) if \(S\) and \(T\) are bounded \(G\)-invariant operators on \(L^2(S_c)\) such that \(ST\) and \(TS\) are \((\sigma, g)\)-trace class, then \(\text{tr}_{\sigma}^{(g)}(ST) = \text{tr}_{\sigma}^{(g)}(TS)\).

Proof. We begin with (i). Let \(S\) be a \((\sigma, g)\)-trace class operator. By the formula in Lemma 3.14, it is clear that for any cut-off function \(c\), \(\text{tr}_{\sigma}^{(g)}(S)\) is independent of the choice of \(c_1\) and \(c_2\) such that \(c_1 c_2 = c\). Define the function \(m_1: M \to \mathbb{C}\) by
\[
m_1(x) := \sum_{h \in (g)} \theta(h) \cdot e^{-i\phi_h(x)} \text{Tr}(h^{-1} K_S(h x, x)),
\]
so that
\[
\text{tr}_{\sigma}^{(g)}(S) = \int_M c_1(x) m_1(x) c_2 \, dx,
\]
where \( c_1, c_2 \in C_c(M) \) such that \( c_1c_2 = c \) for some cut-off function \( c \). We will show that \( m_1 \) is \( \Gamma \)-invariant, whence by Lemma 3.11, \( \text{tr}^{(g)} \) is independent of \( c \).

By Lemma 3.27, we see that for any \( \gamma \in \Gamma \),
\[
\begin{align*}
m_1(\gamma x) &= \sum_{h \in (g)} \theta(h) \cdot e^{-i\phi_h(\gamma x)} \text{Tr}(h^{-1}K_S(h\gamma x, \gamma x)) \\
&= \sum_{h \in (g)} \theta(h) \cdot e^{-i\phi_h(\gamma x)} \text{Tr}(h^{-1}e^{-i\phi_{\gamma^{-1}}(h\gamma x)}\gamma K_S(\gamma^{-1}h\gamma x, x)\gamma^{-1}e^{i\phi_{\gamma^{-1}}(\gamma x)}) \\
&= \sum_{h \in (g)} \theta(h) \cdot e^{-i\phi_h(\gamma x)}e^{-i\phi_{\gamma^{-1}}(h\gamma x)} \text{Tr}(\gamma^{-1}h\gamma^{-1}K_S(\gamma^{-1}h\gamma x, x)e^{i\phi_{\gamma^{-1}}(\gamma x)}).
\end{align*}
\]
We claim that the \( \gamma^{-1}h\gamma \)-summand of \( m_1(x) \) equals the \( h \)-summand of \( m_1(\gamma x) \). To prove this, it suffices to show that
\[
\theta(\gamma^{-1}h\gamma) \cdot e^{-i(\phi_{\gamma^{-1}h\gamma}(x) + \phi_{\gamma^{-1}}(\gamma x))} = \theta(h) \cdot e^{-i(\phi_h(\gamma x) + \phi_{\gamma^{-1}}(h\gamma x))}.
\]
Applying identity (2.3) with \( \gamma \to \gamma^{-1}, \gamma' \to \gamma^{-1}h\gamma \), and \( x \to \gamma x \), one sees that the function
\[
x \mapsto \phi_{\gamma^{-1}h\gamma}(x) + \phi_{\gamma^{-1}}(\gamma x) - \phi_{\gamma^{-1}h}(\gamma x)
\]
is constant on \( M \). Letting \( x = x_0 \), and using the definition of \( \sigma \) in terms of \( \phi \) given by (2.6) together with the fact that \( \phi_k(x_0) = 0 \) for all \( k \in \Gamma \), we see that
\[
\theta(\gamma^{-1}h\gamma) \cdot e^{-i(\phi_{\gamma^{-1}h\gamma}(x_0) + \phi_{\gamma^{-1}}(\gamma x_0) - \phi_{\gamma^{-1}h}(\gamma x_0))} = \theta(h) \cdot e^{-i(h\gamma x_0)}\sigma(\gamma^{-1}h, \gamma).
\]
We then have
\[
\begin{align*}
\theta(\gamma^{-1}h\gamma) \cdot \sigma(\gamma^{-1}, \gamma)\sigma(\gamma^{-1}h, \gamma) &= \theta(\gamma^{-1}h\gamma) \cdot \sigma(\gamma^{-1}h, \gamma^{-1}) \\
&= \theta(h)\sigma(\gamma^{-1}, h) \\
&= \theta(h)\sigma(h, \gamma)\sigma(\gamma^{-1}, h\gamma)\sigma(\gamma^{-1}h, \gamma) \\
&= \theta(h) \cdot e^{-i(\phi_h(\gamma x_0) + \phi_{\gamma^{-1}}(h\gamma x_0) - \phi_{\gamma^{-1}h}(\gamma x_0))}.
\end{align*}
\]
The first equality follows from the fact that \( \sigma(\gamma^{-1}h, \gamma)\sigma(\gamma^{-1}h, \gamma^{-1}) = 1 \), which can be verified using the cocycle condition. The second follows from Lemma 3.17 and the third is again by the cocycle condition. The final equality is by (2.6).

Again by (2.3), this equals \( \theta(h) \cdot e^{-i(\phi_h(\gamma x) + \phi_{\gamma^{-1}}(h\gamma x) - \phi_{\gamma^{-1}h}(\gamma x))} \). From this (3.11) follows. Summing over elements of \( (g) \) then shows that \( m_1 \) is \( \Gamma \)-invariant, which completes the proof of (i).

For (ii), note that by Lemma 3.14,
\[
\text{tr}^{(g)}_\sigma(ST) = \sum_{h \in (g)} \theta(h) \cdot \int_{M \times M} e^{-i\phi_h(x)}c(x)\text{Tr}(h^{-1}K_S(hx, y)K_T(y, x)) \, dx \, dy,
\]
while
\[
\begin{align*}
\text{tr}^{(g)}_\sigma(TS) &= \sum_{h \in (g)} \theta(h) \cdot \int_{M \times M} e^{-i\phi_h(y)}c(y)\text{Tr}(h^{-1}K_T(hy, x)K_S(x, y)) \, dx \, dy \\
&= \sum_{h \in (g)} \theta(h) \cdot \int_{M \times M} e^{-i\phi_h(h^{-1}y)}c(y)\text{Tr}(h^{-1}K_T(y, x)K_S(x, h^{-1}y)) \, dx \, dy,
\end{align*}
\]
where we have used the change of variable \( y \mapsto h^{-1}y \) and the fact that \( y \mapsto c(h^{-1}y) \) is a cut-off function for any \( h \in \Gamma \).
Now define $m_2: M \times M \to \mathbb{C}$ by
\[ m_2(x, y) = \sum_{h \in (g)} \theta(h) \cdot e^{-i\phi_h(x)} \text{Tr}(h^{-1}K_S(hx, y)K_T(y, x)) \]
so that
\[ \text{tr}_\sigma^T(ST) = \int_{M \times M} c(x)m_2(x, y) \, dx \, dy. \]

We claim that
(1) $m_2$ is $\Gamma$-equivariant for the diagonal action on $M \times M$;
(2) $\text{tr}_\sigma^T(TS) = \int_{M \times M} c(y)m_2(x, y) \, dx \, dy$.

To prove claim (1), fix some $\gamma \in \Gamma$. By the change of variable $h \to \gamma^{-1}h\gamma$, we have
\[ m_2(x, y) = \sum_{\gamma^{-1}h\gamma \in (g)} \theta(\gamma^{-1}h\gamma) \cdot e^{-i\phi_{\gamma^{-1}h\gamma}(x)} \text{Tr}((\gamma^{-1}h\gamma)^{-1}K_S(\gamma^{-1}h\gamma x, y)K_T(y, x)). \]

Since $S$ and $T$ are $(\Gamma, \sigma)$-invariant, Lemma 3.27 implies that $m_2(\gamma x, \gamma y)$ equals
\[ \sum_{h \in (g)} \theta(h) \cdot e^{-i\phi_{\gamma h}(\gamma x)} \text{Tr}(h^{-1}K_S(h\gamma x, \gamma y)K_T(\gamma y, \gamma x)) \]
\[ = \sum_{h \in (g)} \theta(h) \cdot e^{-i\phi_h(\gamma x)} \text{Tr}(e^{-i\phi_{\gamma^{-1}h\gamma}(x)}h^{-1}\gamma K_S(\gamma^{-1}h\gamma x, y)K_T(y, x)\gamma^{-1}e^{i\phi_{\gamma^{-1}h\gamma}(x)}) \]
\[ = \sum_{h \in (g)} \theta(h) \cdot e^{-i(\phi_h(\gamma x) + \phi_{\gamma^{-1}h\gamma}(x) - \phi_{\gamma^{-1}h\gamma}(x))} \text{Tr}((\gamma^{-1}h\gamma)^{-1}K_S(\gamma^{-1}h\gamma x, y)K_T(y, x)). \]

We claim that the $\gamma^{-1}h\gamma$-summand of $m_2(x, y)$ equals the $h$-summand of $m_2(\gamma x, \gamma y)$, that is:
\[ \theta(\gamma^{-1}h\gamma) \cdot e^{-i(\phi_{\gamma^{-1}h\gamma}(x))} = \theta(h) \cdot e^{-i(\phi_h(\gamma x) + \phi_{\gamma^{-1}h\gamma}(x) - \phi_{\gamma^{-1}h\gamma}(x))}; \tag{3.13} \]
but this is precisely what we have already established in (3.11). Now summing over elements of $(g)$ proves claim (1).

Now for claim (2), let us denote the $h$-summand of $m_2(x, y)$ by $[m_2(x, y)]_h$. Then by (3.12), it suffices to show that for each $h \in (g)$, we have
\[ [m_2(x, y)]_h = \theta(h) \cdot e^{-i\phi_h(h^{-1}y)} \text{Tr}(h^{-1}K_T(y, x)K_S(x, h^{-1}y)). \tag{3.14} \]

To see this, note that by Lemma 3.27, we have
\[ \text{Tr}(h^{-1}K_S(hx, y)K_T(y, x)) = \text{Tr}(K_T(y, x)h^{-1}K_S(hx, y)) \]
\[ = \text{Tr}(K_T(y, x)h^{-1}e^{-i\phi_{h^{-1}}(hx)}hK_S(x, h^{-1}y)h^{-1}e^{i\phi_{h^{-1}}(y)}) \]
\[ = \text{Tr}(h^{-1}K_T(y, x)K_S(x, h^{-1}y))e^{-i(\phi_{h^{-1}}(hx) - \phi_{h^{-1}}(y))}. \]

Using the definition of $m_2(x, y)$, this means that
\[ [m_2(x, y)]_h = \theta(h) \cdot e^{-i(\phi_h(x) + \phi_{h^{-1}}(hx) - \phi_{h^{-1}}(y))} \text{Tr}(h^{-1}K_T(y, x)K_S(x, h^{-1}y)). \]

Thus to establish (3.14), it remains to show that
\[ \phi_h(h^{-1}y) + \phi_{h^{-1}}(y) = \phi_h(x) + \phi_{h^{-1}}(hx). \]

By the identity (2.3), the left-hand side equals $\phi_{hh^{-1}}(y) = \phi_e(y)$, while the right-hand side equals $\phi_{h^{-1}h}(hx) = \phi_e(hx)$. Thus equality follows from the normalisation condition $\phi_e \equiv 0$. This completes the proof of claim (2).
Finally, since $m_2$ is invariant under the diagonal $\Gamma$-action, we may apply Lemma 3.11 (ii) to $m_2$, and use claim (2), to obtain
\[
\text{tr}_\sigma^{(g)}(ST) = \int_{M \times M} c(x)m_2(x,y) \, dx \, dy \\
= \int_{M \times M} c(y)m_2(x,y) \, dx \, dy \\
= \text{tr}_\sigma^{(g)}(TS).
\]

To summarise the notation, we now have:
- the trace $\tau_\sigma^{(g)}$ on $C^\sigma \Gamma$,
- the trace $\text{Tr}$ on $\text{End}(S_{\mathcal{F}})$;
- the traces $\text{tr}$ and $\text{tr}_\sigma^{(g)}$ on operators on $L^2(S_{\mathcal{F}})$.

Recall that for any trace map $\mathcal{T}$ on operators on a Hilbert space $\mathcal{H}$, the associated supertrace applied to an operator $S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$ (3.15) on the $\mathbb{Z}_2$-graded space $\mathcal{H} \oplus \mathcal{H}$ is
\[
\mathcal{T}(S_{11}) - \mathcal{T}(S_{22}).
\]
The supercommutator of homogeneous elements $S, T \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ is
\[
[S, T]_s := ST - (-1)^{\text{deg}S \cdot \text{deg}T} TS.
\]
This extends linearly to arbitrarily elements of $\mathcal{B}(\mathcal{H} \oplus \mathcal{H})$. It follows that the supertrace vanishes on supercommutators.

**Definition 3.19.** Denote by $\text{Str}$, $\text{str}$, and $\text{str}_\sigma^{(g)}$ the supertraces associated to $\text{Tr}$, $\text{tr}$, and $\text{tr}_\sigma^{(g)}$ respectively.

Then Definition 3.12 generalises easily to:

**Definition 3.20.** Let $S$ be a $\mathbb{Z}_2$-graded operator on $L^2(S_{\mathcal{F}})$ written in the form (3.15). If $S_{11}$ and $S_{22}$ are of $(\sigma, g)$-trace class, then we define the $\sigma$-weighted $(g)$-supertrace of $S$ to be
\[
\text{str}_\sigma^{(g)}(S) := \sum_{h \in (g)} \theta(h) \cdot \text{str}(c_1 T_{h^{-1}} S c_2).
\]

**Remark 3.21.** If $S$ has finite propagation, then (3.16) equals
\[
\sum_{h \in (g)} \theta(h) \cdot \text{str}(c T_{h^{-1}} S),
\]
where $c$ is any cut-off function.

Proposition 3.18 implies:

**Corollary 3.22.** If $S$ and $T$ are $\mathbb{Z}_2$-graded $(\Gamma, \sigma)$-invariant operators on $L^2(S_{\mathcal{F}})$ such that the diagonal entries of $ST$ and $TS$ are of $(\sigma, g)$-trace class, then
\[
\text{str}_\sigma^{(g)}[S, T]_s = 0.
\]

**Remark 3.23.** Suppose $g$ is an $\alpha$-regular element, with $\alpha$ as in (2.5). Then the discussion in this subsection generalises easily to the family of multipliers $\{\sigma_s^*\}_{s \in \mathbb{R}}$ from (2.6), once we replace the projective representation $T$ by $T^s$ from Definition 2.4, the family $\phi$ by $s\phi$, and $\theta$ by $\theta^s$ from (3.5) in the proof of Corollary 3.8.
3.2. Twisted Roe algebras.

To formulate the higher index of projectively invariant operators, we will work with certain geometric $C^*$-algebras. The analogous construction in the non-twisted case is well-known; see for instance [23]. Since the discussion applies to $\sigma^a$ for any $s$, let us fix $s = 1$.

**Definition 3.24.** Let $A$ be an operator on $L^2(S_\phi)$.

- $A$ is $(\Gamma, \sigma)$-equivariant if
  \[ T_g A(T_g)^* = A \]
  for all $g \in \Gamma$, where $T_g$ is as in Definition 2.4;
- The support of $A$, denoted $\text{supp}(A)$, is the complement of all $(x, y) \in M \times M$ for which there exist $f_1, f_2 \in C_0(M)$ such that $f_1(x) \neq 0$, $f_2(y) \neq 0$, and
  \[ f_1 A f_2 = 0; \]
- The propagation of $A$ is the extended real number
  \[ \text{prop}(A) = \sup \{ d(x, y) \mid (x, y) \in \text{supp}(A) \}, \]
  where $d$ denotes the Riemannian distance on $M$;
- $A$ is locally compact if $A f$ and $A f \in K(L^2(S_\phi))$ for all $f \in C_0(M)$.

**Definition 3.25.** The $(\Gamma, \sigma)$-equivariant algebraic Roe algebra of $M$, denoted by $\mathbb{C}[M; L^2(S_\phi)]^\Gamma,\sigma$, is the $*$-subalgebra of $\mathcal{B}(L^2(S_\phi))$ consisting of $(\Gamma, \sigma)$-equivariant locally compact operators with finite propagation.

The $(\Gamma, \sigma)$-equivariant Roe algebra of $M$, denoted by $C^*(M; L^2(S_\phi))^\Gamma,\sigma$, is the completion of $\mathbb{C}[M; L^2(S_\phi)]^\Gamma,\sigma$ in $\mathcal{B}(L^2(S_\phi))$.

**Definition 3.26.** Consider the vector bundle $\text{Hom}(S_\phi) = S_\phi \boxtimes S_\phi \to M \times M$. Let $\mathcal{S}(M)^\Gamma,\sigma$ denote the convolution algebra of smooth sections $k$ of $\text{Hom}(S_\phi)$ such that

(i) $k$ is $(\Gamma, \sigma)$-invariant, in the sense that
  \[ e^{-i\phi}(x) \gamma^{-1} k_A(\gamma x, \gamma y) \gamma e^{i\phi}(y) = k_A(x, y) \]
  for all $\gamma \in \Gamma$ and $x, y \in M$;
(ii) $k$ has finite propagation, in the sense that there exists an $R > 0$ such that $k(x, y) = 0$ whenever $d(x, y) > R$.

An element $k \in \mathcal{S}(M)^\Gamma,\sigma$ acts on a section $u \in L^2(S_\phi)$ by
\[ (ku)(x) = \int_M k(x, y) u(y) \, dy. \quad (3.17) \]

**Lemma 3.27.** Suppose a $(\Gamma, \sigma)$-invariant operator $A$ on $L^2(S_\phi)$ with finite propagation has a smooth Schwartz kernel $k_A$. Then $k_A \in \mathcal{S}(M)^\Gamma,\sigma$.

**Proof.** The finite-propagation property for $k_A$ follows from the fact that $A$ has finite propagation. Now by assumption, $T_{\gamma^{-1}} A T_{\gamma} = A$. Since the cocycle $\sigma$ is normalised, it follows from 2.6 that $T_{\gamma^{-1}} = T_{\gamma}^{-1}$, whence
\[ k_A(x, y) = k_{T_{\gamma^{-1}} A T_{\gamma}}(x, y) = k_{S_{\gamma}^{-1} U_{\gamma}^{-1} A U_{\gamma} S_{\gamma}}(x, y) = e^{-i\phi}(x) k_{U_{\gamma}^{-1} A U_{\gamma}}(x, y) e^{i\phi}(y) = e^{-i\phi}(x) \gamma^{-1} k_A(\gamma x, \gamma y) \gamma e^{i\phi}(y). \]
\[ \square \]
Conversely, an element of $\mathcal{S}(M)^{\Gamma,\sigma}$ defines an element of $\mathbb{C}[M; L^2(\mathcal{S}_\mathcal{L})]^{\Gamma,\sigma}$ via the action (3.17). Let $\mathcal{M} \subseteq \mathcal{B}(L^2(\mathcal{S}_\mathcal{L}))$ denote the multiplier algebra of the $(\Gamma,\sigma)$-equivariant Roe algebra, and write $Q = \mathcal{M}/C^*(M; L^2(\mathcal{S}_\mathcal{L}))^{\Gamma,\sigma}$. We have a short exact sequence of $C^*$-algebras

$$0 \to C^*(M; L^2(\mathcal{S}_\mathcal{L}))^{\Gamma,\sigma} \to \mathcal{M} \to Q \to 0.$$  
(3.18)

### 3.3. The twisted higher index.

We discuss the index map first in a more general context. Recall that associated to any short exact sequence of $C^*$-algebras

$$0 \to I \to A \to A/I \to 0,$$

is a cyclic exact sequence in $K$-theory:

$$K_0(I) \xrightarrow{\partial_0} K_0(A) \xrightarrow{\partial_1} K_0(A/I) \xrightarrow{\partial_1} K_0(I),$$

where the connecting maps $\partial_0$ and $\partial_1$ are defined as follows.

**Definition 3.28.**

(i) $\partial_0$: let $u$ be an invertible matrix with entries in $A/I$ representing a class in $K_1(A/I)$. Write

$$w = \begin{pmatrix} 0 & -u^{-1} \\ u & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \begin{pmatrix} 1 & -u^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}.$$  

Then $w$ lifts to an invertible matrix $W$ with entries in $A$. Then

$$P = W \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} W^{-1}$$

is an idempotent, and we define

$$\partial_0[u] := [P] - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in K_0(I).$$  
(3.19)

(ii) $\partial_1$: let $q$ be an idempotent matrix with entries in $A/I$ representing a class in $K_0(A/I)$. Let $Q$ be a lift of $q$ to a matrix algebra over $A$. Then we define

$$\partial_1[q] := [e^{2\pi i Q}] \in K_1(I).$$  
(3.20)

Now let $M$ be a Riemannian spin manifold on which $\Gamma$ acts properly and isometrically, respecting the spin structure. Let $\sigma$ be a multiplier on $\Gamma$. Let $\mathcal{L} \to M$ be a trivial line bundle, and let $D$ be the twisted Dirac operator as in (2.7) acting on smooth sections of $\mathcal{S}_\mathcal{L}$. Pick any normalizing function $\chi : \mathbb{R} \to \mathbb{R}$, i.e. a continuous, odd function such that

$$\lim_{x \to \pm \infty} \chi(x) = 1,$$

and form the bounded self-adjoint operator $\chi(D)$ on $L^2(\mathcal{S}_\mathcal{L})$. When $\dim M$ is even, $\mathcal{S}_\mathcal{L}$ is naturally a direct sum $(S^+ \otimes \mathcal{L}) \oplus (S^- \otimes \mathcal{L})$, and $D$ and $\chi(D)$ are odd-graded:

$$D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}, \quad \chi(D) = \begin{pmatrix} 0 & \chi(D)_- \\ \chi(D)_+ & 0 \end{pmatrix}.$$  

**Proposition 3.29.** The class of $\chi(D)$ in $\mathcal{M}/C^*(M; L^2(\mathcal{S}_\mathcal{L}))^{\Gamma,\sigma}$ is invertible and independent of the choice of $\chi$, and $\frac{\chi(D)+1}{2}$ is an idempotent modulo $C^*(M; L^2(\mathcal{S}_\mathcal{L}))^{\Gamma,\sigma}$. 

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Proof. Since $\chi^2 - 1 \in C_0(\mathbb{R})$, it suffices to show that for any $f \in C_0(\mathbb{R})$, we have $f(D) \in C^*(M; L^2(S_\mathcal{D}))^{\Gamma,\sigma}$. In fact, functions in the Schwartz algebra $S(\mathbb{R})$ with compactly supported Fourier transform form a dense subset of $C_0(\mathbb{R})$, we may assume that $f$ is such a function. In that case, the operator $f(D)$ is given by a smooth $(\Gamma, \sigma)$-invariant Schwartz kernel, and hence an element of $\mathcal{S}(M)^{\Gamma,\sigma}$ (see Definition 3.26). It follows that $f(D) \in C^*(M; L^2(S_\mathcal{D}))^{\Gamma,\sigma}$. Finally, since the difference of any two normalizing functions lies in $C_0(\mathbb{R})$, the class of $\chi(D)$ does not depend on the choice of normalising function $\chi$. \hfill \square

Applying Definition 3.28 to the short exact sequence (3.18) leads to the following:

**Definition 3.30.** For $i = 0, 1$, let $\partial_i$ be the connecting maps from Definition 3.28. The $(\Gamma, \sigma)$-invariant higher index of $D$ on $L^2(S_\mathcal{D})$ is the element

$$\text{Ind}_{\Gamma, \sigma, L^2}(D) := \begin{cases} 
\partial_0 [\chi(D)_+] \in K_0(C^*(M; L^2(S_\mathcal{D}))^{\Gamma,\sigma}) & \text{if dim } M \text{ is even}, \\
\partial_1 \left[ \frac{\chi(D)+1}{2} \right] \in K_1(C^*(M; L^2(S_\mathcal{D}))^{\Gamma,\sigma}) & \text{if dim } M \text{ is odd}.
\end{cases}$$

From this, we can obtain an index in the $K$-theory of $C^*_r(\Gamma, \sigma)$ as follows. Define

$$j_\sigma : C_c(S_\mathcal{D}) \hookrightarrow C^*[\Gamma] \otimes C_c(S_\mathcal{D})$$

$$(j_\sigma e)_\gamma = c \cdot (T_{g^{-1}} e),$$

where $T$ is the $(\Gamma, \sigma)$-representation on $L^2(S_\mathcal{D})$. Then $j_\sigma$ extends to an isometry

$$j_\sigma : L^2(S_\mathcal{D}) \hookrightarrow l^2(\Gamma) \otimes L^2(S_\mathcal{D}).$$

The following is easily proved:

**Lemma 3.31.** The map $j_\sigma$ is equivariant with respect to the projective representation $T$ on $L^2(S_\mathcal{D})$ and $T^L \otimes 1$, where $T^L$ is the left-regular $(\Gamma, \sigma)$-representation on $l^2(\Gamma)$, as in subsection 2.3.

At the level of operators, the map $j_\sigma$ induces two inclusion maps

$$\oplus 0, \oplus 1 : \mathcal{B}(L^2(S_\mathcal{D})) \hookrightarrow \mathcal{B}(l^2(\Gamma) \otimes L^2(S_\mathcal{D})),$$

defined by first identifying operators on $L^2(S_\mathcal{D})$ with operators on $j(L^2(S_\mathcal{D}))$ via conjugation by $j$ and then extending them by the zero operator and identity operator, respectively, on the orthogonal complement of $j(L^2(S_\mathcal{D}))$. These maps restrict to inclusions

$$\oplus 0 : C^*(M; L^2(S_\mathcal{D}))^{\Gamma,\sigma} \to C^*_r(\Gamma, \sigma) \otimes \mathcal{K}(L^2(S_\mathcal{D})),$$

$$\oplus 1 : (C^*(M; L^2(S_\mathcal{D}))^{\Gamma,\sigma})^+ \to (C^*_r(\Gamma, \sigma) \otimes \mathcal{K}(L^2(S_\mathcal{D})))^+.$$ Let us write $T \oplus i := (\oplus i)(T)$, $i = 0, 1$. In this section we prove Theorem 1.3. Thus let $\Gamma, M, \alpha, \sigma$, and $g$ be given as in the statement of the theorem.

Then $\oplus i$ extends in the obvious way to maps between matrix algebras over $C^*(M; L^2(S_\mathcal{D}))^{\Gamma,\sigma}$ and its unitisation, preserving idempotence and invertibility when $i = 0$ and 1 respectively. We get induced maps

$$(\oplus i)_* : K_i(C^*(M; L^2(S_\mathcal{D}))^{\Gamma,\sigma}) \to K_i(C^*_r(\Gamma, \sigma) \otimes \mathcal{K}(L^2(S_\mathcal{D}))) \cong K_i(C^*_r(\Gamma, \sigma)).$$

This gives an index

$$\text{Ind}_{\Gamma, \sigma}(D) := (\oplus i)_* \text{Ind}_{\Gamma, \sigma, L^2}(D) \in K_i(C^*_r(\Gamma, \sigma)), \quad (3.22)$$

where $i = \text{dim } M \pmod{2}$.

**Remark 3.32.** When $\sigma \equiv 1$, (3.22) recovers the usual $\Gamma$-equivariant higher index of Dirac operators on cocompact manifolds.
4. Projective PSC obstructions on cocompact manifolds

In this section we prove Theorem 1.3. Thus let $\Gamma$, $M$, $\alpha$, $\sigma$, and $g$ be given as in the statement of the theorem.

We begin by describing how the trace $\tau^{(g)}_\sigma$ from Definition 3.5 can be extended to a trace on a smooth dense subalgebra of $C^*(\Gamma,\sigma) \otimes K(L^2(S_\sigma))$. Our construction is based on [6, section 6], [24, subsection 2.2], and [14, section 3], but some extra arguments are needed to make the adaptation to the proper action case, having to do with the form of the embedding $j_{\sigma}$ in (3.21).

Let $D$ and $S_\sigma$ be as in Definition 2.5, with the underlying manifold $M$ being cocompact with respect to the action of a finitely generated group $\Gamma$. We will work with the following choice of orthonormal basis of $L^2(S_\sigma)$ to work with. Let $c$ be the cut-off function for the $\Gamma$-action on $M$ used in the embedding $j$ from (3.21). Since the support of $c$ is compact, there exists an orthonormal basis $B_0 = \{u_i\}_{i \in \mathbb{N}}$ of $L^2(S_\sigma)_{|N}$, for some relatively compact neighbourhood $N$ of supp($c$) consisting of eigenfunctions of $D^2|_N$. Asymptotically, the eigenvalues $\lambda_i$ satisfy $\lambda_i \sim i^q$ for some constant $q > 0$. This allows us to complete $B_0$ to an orthonormal basis $B = \{v_i\}_{i \in \mathbb{N}}$ of $L^2(S_\sigma)$ in such a way that, upon identifying $L^2(S_\sigma)$ with $l^2(\mathbb{N})$, any operator of the form $e^{\frac{i}{2}S}\text{c}^2$, where $S \in \mathfrak{S}_{\Gamma,\sigma}$ (see Definition 3.26), corresponds to an element of the algebra

$$\mathcal{A} = \{(a_{ij})_{i,j \in \mathbb{N}}: \sup_{i,j} |i^k j^l a_{ij}| < \infty \text{ for all } k, l \in \mathbb{N}\}$$

of matrices with rapidly decreasing entries, which are dense inside $K(l^2(\mathbb{N}))$. Let

$$I = \{i \in \mathbb{N}: v_i \in B_0\}. \quad (4.1)$$

Fix a set of generators of $\Gamma$, and let $\ell(\gamma) = d_\Gamma(\gamma, e)$ denote the associated word length function, where $d_\Gamma$ is the word metric. Let $\mathcal{D}$ be the unbounded self-adjoint operator on $l^2(\mathbb{N})$ given by $\mathcal{D} = \ell(\gamma) \cdot \gamma$, where the notation $\gamma$ is as explained in subsection 2.3. Let $\Delta$ be the unbounded self-adjoint operator on $l^2(\mathbb{N})$ given by

$$\Delta(\delta_i) = \begin{cases} \lambda_i \delta_i & \text{if } i \in I, \\ i^q \delta_i & \text{otherwise}, \end{cases} \quad (4.2)$$

for $i \in \mathbb{N}$, where $\lambda_i$ is the $i$th eigenvalue of $D^2|_N$, and the set $I$ is as in (4.1).

Consider also the unbounded operators $\partial = [\mathcal{D}, \cdot]$ on $\mathcal{B}(l^2(\mathbb{N}))$ and $\partial = [\mathcal{D} \otimes 1, \cdot]$ on $\mathcal{B}(l^2(\Gamma) \otimes l^2(\mathbb{N}))$. Note that $\partial$ is a closed derivation with domain $\text{Dom}(\partial)$, which consists of all elements $a \in \mathcal{B}(l^2(\Gamma) \otimes l^2(\mathbb{N}))$ such that $a$ maps $\text{Dom}(\mathcal{D} \otimes 1)$ to itself, and the operator $\partial(a) = (\mathcal{D} \otimes 1) \circ a - a \circ (\mathcal{D} \otimes 1)$, defined initially on $\text{Dom}(\mathcal{D} \otimes 1)$, extends to a bounded operator on $l^2(\Gamma) \otimes l^2(\mathbb{N})$. Define

$$\mathcal{B}_\infty(\Gamma, \sigma) = \bigcap_{k \in \mathbb{N}} \text{Dom}(\partial^k) \cap (C_r^*(\Gamma, \sigma) \otimes K),$$

and

$$\mathcal{B}(\Gamma, \sigma) = \{a \in \mathcal{B}_\infty(\Gamma, \sigma): \partial^k(a) \circ (1 \otimes \Delta)^2 \text{ is bounded } \forall k \in \mathbb{N}\}. \quad (4.3)$$

Note that $\mathcal{B}(\Gamma, \sigma)$ is a left ideal as well as a Fréchet subalgebra of $\mathcal{B}_\infty(\Gamma, \sigma)$, with respect to the Fréchet topology given by the family of norms

$$\|a\|_n = \sum_{k=0}^{\infty} \frac{1}{k!} \|\partial^k(a) \circ (1 \otimes \Delta)^2\|,$$

for $n \in \mathbb{N}$, where the norm on the right-hand side is the operator norm.

Define the von Neumann algebra

$$\mathcal{A}(\Gamma, \sigma) = \{a \in \mathcal{B}(l^2(\Gamma) \otimes l^2(\mathbb{N})): [\gamma \otimes 1, a] = 0 \ \forall \gamma \in \Gamma\}. \quad (4.4)$$
By [14, Lemmas 1.1, 3.2], we have the following useful fact:

**Lemma 4.1.** Any \( a \in \mathcal{A}(\Gamma, \sigma) \) can be written as a strongly convergent sum

\[
a = \sum_{\gamma \in \Gamma} \tilde{\gamma} \otimes a_{\gamma},
\]

where \( a_{\gamma} \in \mathcal{B}(l^2(\mathbb{N})) \) for each \( \gamma \in \Gamma \). If \( a_{\gamma} \in \mathcal{R} \) for each \( \gamma \in \Gamma \) and

\[
\sum_{\gamma} \ell(\gamma)^k \| a_{\gamma} \circ \Delta \| < \infty,
\]

for all \( k \geq 0 \), then \( a \in \mathcal{B}(\Gamma, \sigma) \).

**Lemma 4.2.** The \( * \)-algebra \( \mathcal{B}(\Gamma, \sigma) \) is dense in \( C^*_r(\Gamma, \sigma) \otimes \mathcal{K}(l^2(\mathbb{N})) \) and stable under the holomorphic functional calculus.

**Proof.** First note that if \( a = \tilde{g} \otimes V \in \mathbb{C}^\sigma \Gamma \otimes \mathcal{R} \), then

\[
\partial^k(\mathcal{A}) \circ (1 \otimes \Delta)^2 = \partial^k(\tilde{g}) \otimes V \Delta^2,
\]

which is bounded since both \( \partial^k(\tilde{g}) \) and \( V \Delta^2 \) are bounded. It follows that \( \mathcal{B}(\Gamma, \sigma) \) is dense in \( C^*_r(\Gamma, \sigma) \otimes \mathcal{K}(l^2(\mathbb{N})) \).

Now recall that a subalgebra \( A \) of a Banach algebra \( B \) is called spectral invariant in \( B \) if the invertible elements of the unitisation \( A^+ \) are precisely those which are invertible in \( B^+ \). In the case that \( A \) is a Fréchet subalgebra of of \( B \), \( A \) is spectral invariant if and only if \( A \) is stable under the holomorphic functional calculus in \( B \), by [20]. Further, it is a purely algebraic fact that if \( I \) is a left ideal in \( A \), then \( I \) is itself a spectral invariant subalgebra of \( B \).

By [13, Theorem 1.2], \( \mathcal{B}_\infty(\Gamma, \sigma) \) is dense in \( C^*_r(\Gamma, \sigma) \otimes \mathcal{K}(l^2(\mathbb{N})) \) and closed under the holomorphic functional calculus, and hence a spectral invariant subalgebra of \( C^*_r(\Gamma, \sigma) \otimes \mathcal{K}(l^2(\mathbb{N})) \). Since \( \mathcal{B}(\Gamma, \sigma) \) is a left ideal in \( \mathcal{B}_\infty(\Gamma, \sigma) \) as well as a Fréchet subalgebra, it follows from the above discussion that \( \mathcal{B}(\Gamma, \sigma) \) is holomorphically closed. \( \square \)

Let \( \text{Tr} \) denote the operator trace on \( \mathcal{R} \), and denote by \( \tau^{(g)}_\sigma \otimes \text{Tr} \) the amplified trace on \( \mathbb{C}^\sigma \Gamma \otimes \mathcal{R} \). Recall that a conjugacy class \( (g) \) is said to have polynomial growth if there exist constants \( C \) and \( d \) such that the number of elements \( h \in (g) \) such that \( \ell(h) \leq l \) is at most \( Cl^d \).

**Lemma 4.3.** Let \( g \in \Gamma \) be a \( \sigma \)-regular element with respect to a multiplier \( \sigma \). If the conjugacy class \( (g) \) has polynomial growth, then \( \tau^{(g)}_\sigma \otimes \text{Tr} \) extends to a continuous trace on \( \mathcal{B}(\Gamma, \sigma) \).

**Proof.** This follows from an adaptation of the proof of [24, Lemma 2.7]. Indeed, let \( \mathcal{A} = (\mathcal{A}_{ij})_{i,j \in \mathbb{N}} \) be an arbitrary element of \( \mathcal{B}(\Gamma, \sigma) \), where \( \mathcal{A}_{ij} \in C^*_r(\Gamma, \sigma) \) can be written as

\[
\mathcal{A}_{ij} = \sum_{\gamma \in \Gamma} \mathcal{A}_{ij,\gamma} \tilde{\gamma}.
\]

Define the function \( t: \mathcal{B}(\Gamma, \sigma) \to \mathbb{C} \) by

\[
t(\mathcal{A}) = \sum_{i \in \mathbb{N}} \sum_{h \in (g)} \theta(h) \mathcal{A}_{ii,h}.
\]

It follows from Definition 3.5 that \( t \) agrees with \( \tau^{(g)}_\sigma \otimes \text{Tr} \) on \( \mathbb{C}^\sigma \Gamma \otimes \mathcal{R} \). Now since \( \theta(\gamma) \in \mathbb{U}(1) \), the right-hand side of (4.7) converges absolutely by the same estimates as in the proof of [24, Lemma 2.7], hence \( t \) extends to a continuous trace on \( \mathcal{B}(\Gamma, \sigma) \). \( \square \)

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We will continue to write \( \tau^{(g)}_\sigma \otimes \text{Tr} \) for the extended trace \( t \) on \( \mathcal{B}(\Gamma, \sigma) \) from the proof of Lemma 4.3. It follows from and Lemmas 4.2 and 4.3 that we have an induced map
\[
(\tau^{(g)}_\sigma)_*: K_0(C^*_r(\Gamma, \sigma) \otimes K(L^2(S_\mathcal{F}))) \cong K_0(\mathcal{B}(\Gamma, \sigma)) \rightarrow \mathbb{C}.
\]  
(4.8)
We are now ready to establish the connection between \( \tau^{(g)}_\sigma \) and the trace \( \text{tr}^{(g)}_\sigma \) from section 3.

**Proposition 4.4.** Let \( A \) be a \((\Gamma, \sigma)\)-invariant operator on \( L^2(S_\mathcal{F}) \) and \( g \in \Gamma \) a \( \sigma \)-regular element with respect to \( \sigma \). If \( A \) is of \((\sigma, g)\)-trace class and \( A \otimes 0 \in \mathcal{B}(\Gamma, \sigma) \), then
\[
(\tau^{(g)}_\sigma \otimes \text{Tr})(A \otimes 0) = \text{tr}^{(g)}_\sigma(A).
\]

**Proof.** For any \( \gamma \in \Gamma \), denote by
\[
q_\gamma: l^2(\Gamma) \otimes L^2(S_\mathcal{F}) \rightarrow \mathbb{C} \delta_\gamma \otimes L^2(S_\mathcal{F})
\]
the canonical projection. Let \( A \) be as given, and write \( A \otimes 0 \in \mathcal{B}(\Gamma, \sigma) \) as a matrix \((A_{ij})\), where
\[
A_{ij} = \sum_{\gamma \in \Gamma} A_{ij, \gamma \bar{\gamma}},
\]
(4.9)
as in (4.6). Letting \( A_\gamma \) be the element of \( \mathcal{K} \) given by the matrix \((A_{ij, \gamma \bar{\gamma}})_{i,j \in \mathbb{N}}\), one observes that
\[
q_\gamma(A \otimes 0)q_\gamma = \gamma \otimes A_\gamma.
\]
(4.10)
Now by (4.7), together with the fact that \( \tau^{(g)}_\sigma \otimes \text{Tr} \) converges absolutely on elements of \( \mathcal{B}(\Gamma, \sigma) \), we have
\[
(\tau^{(g)}_\sigma \otimes \text{Tr})(A \otimes 0) = \sum_{i \in \mathbb{N}} \sum_{h \in (g)} \theta(h) A_{ii, h}
\]
\[
= \sum_{h \in (g)} \theta(h) \text{Tr}(A_h),
\]
Thus by (3.12) it suffices to show that for each \( h \in (g) \),
\[
\text{Tr}(A_h) = \text{Tr}(c^{1/2} T_{\gamma^{-1}} A c^{1/2}).
\]
(4.11)
To this end, let \( B = \{u_i\}_{i \in \mathbb{N}} \) be the orthonormal basis of \( L^2(S_\mathcal{F}) \) from the start of this section. By definition,
\[
A \otimes 0 = j_\sigma A_{j_\sigma^*},
\]
where \( j_\sigma \) is as in (3.21). Hence by (4.10), we have
\[
(\hat{h} \otimes A_h)(\delta_e \otimes u_i) = q_h \circ j_\sigma \circ A \circ j_\sigma^*(\delta_e \otimes u_i)
\]
(4.12)
for each \( i \in \mathbb{N} \) and \( \gamma \in \Gamma \). The map \( j_\sigma^* \) can be written explicitly as follows: given \( v \in l^2(\Gamma) \otimes L^2(S_\mathcal{F}) \) and \( x \in M \),
\[
(j_\sigma^* v)(x) = \sum_{\gamma \in \Gamma} c(\gamma^{-1} x) S_{\gamma^{-1}}(\gamma^{-1} x) \cdot \gamma \cdot (v(\gamma, \gamma^{-1} x)).
\]
Applying this to \( \delta_e \otimes u_i \) and using that \( S_e \equiv 1 \) gives
\[
j_\sigma^*(\delta_e \otimes u_i)(x) = \sum_{\gamma \in \Gamma} c^{1/2}(\gamma^{-1} x) S_{\gamma^{-1}}(\gamma^{-1} x) \cdot \gamma \cdot ((\delta_e(\gamma) u_i(\gamma^{-1} x)))
\]
\[
= c^{1/2}(x) u_i(x).
\]
Thus (4.12) equals \( q_h \circ j_\sigma \circ A(c u_i) \). By (3.21), \( j_\sigma(A(c u_i))(\gamma) = c^{1/2} T_{\gamma^{-1}}(A(c^{1/2} u_i)) \), hence
\[
A_{\gamma}(u_i) = c^{1/2} T_{\gamma^{-1}} A(c^{1/2} u_i).
\]
(4.13)
It follows that for any \( h \in (g) \),
\[
\text{Tr}(A_h) = \sum_i \langle c^{1/2}T_{\gamma_i}A(c^{1/2}u_i), u_i \rangle_{L^2(S_\mathcal{F})} = \text{Tr}(c^{1/2}T_{\gamma_i}Ac^{1/2}),
\]
which establishes (4.11).

\[ \square \]

**Theorem 4.5.** Let \( M, \Gamma, \text{ and } D \) be as in the statement of Theorem 1.3 and Definition 2.5. Let \( g \in \Gamma \) be a \( \sigma \)-regular element for a multiplier \( \sigma \). Then

(i) \( e^{-tD^2} \oplus 0 \in \mathcal{B}(\Gamma, \sigma) \), where \( \mathcal{B}(\Gamma, \sigma) \) is as in (4.3);

(ii) we have
\[
(\tau^{(g)}_\sigma)^\ast \text{Ind}_{\Gamma, \sigma}(D) = \text{str}^{(g)}_\sigma(e^{-tD^2}).
\]

**Proof.** For (i), let us write the operator \( e^{-tD^2} \oplus 0 \) note that by (4.10) and (4.13), we can write
\[
e^{-tD^2} \oplus 0 = \sum_{\gamma \in \Gamma} \tilde{\gamma} \otimes c^{1/2}T_{\gamma}e^{-tD^2}c^{1/2}.
\]
Under the identification of \( L^2(S_\mathcal{F}) \) with \( \ell^2(\mathbb{N}) \) given at the start of this section, the operator \( c^{1/2}T_{\gamma}e^{-tD^2}c^{1/2} \) is an element of \( \mathcal{B} \). Hence by Lemma 4.1, it suffices to prove that
\[
\sum_{\gamma} \langle (\gamma) \rangle^k \|c^{1/2}T_{\gamma}e^{-tD^2}c^{1/2} \circ \Delta \| < \infty \tag{4.14}
\]
for any \( k \geq 0 \), where the norm is the operator norm on \( \ell^2(\mathbb{N}) \cong L^2(S_\mathcal{F}) \).

Note that by construction, the operator \( \Delta \) is local in the sense that for any \( u \in L^2(S_\mathcal{F}) \), we have \( \text{supp}(\Delta u) \subseteq \text{supp}(u) \), while if \( \text{supp}(u) \subseteq \text{supp}(c) \), then \( \Delta \) acts on \( u \) as \( D^2 \). Since the operator \( c^{1/2}T_{\gamma}e^{-tD^2}c^{1/2} \) is supported only on the subset \( \text{supp}(c) \times \text{supp}(c) \subseteq M \times M \), and \( c^{1/2}D^2 = D^2c^{1/2} + DA_1 + A_2 \) for some operators \( A_1, A_2 \) of order zero, it follows from the Gaussian decay of the heat kernel that there exist constants \( C_1, C_2, \text{ and } C_3 \) such that for all \( \gamma \) with \( \ell(\gamma) > C \), we have
\[
\|c^{1/2}T_{\gamma}e^{-tD^2}c^{1/2}D^2\| \leq C_2e^{-C_3\ell(\gamma)^2}.
\]
Since \( \Gamma \) is finitely generated, there exist constants \( C_4, C_5 \) such that the number of group elements \( \gamma \) with \( \ell(\gamma) \leq l \) is at most \( C_4e^{C_5 l} \). Setting
\[
C_6 = \sum_{\ell(\gamma) < C_1} \ell(\gamma)^k \|c^{1/2}T_{\gamma}e^{-tD^2}c^{1/2}D^2\|
\]
and combining the above observations shows that
\[
\sum_{\gamma} \ell(\gamma)^k \|c^{1/2}T_{\gamma}e^{-tD^2}c^{1/2}D^2\| = C_6 + \sum_{\ell(\gamma) \geq C_1} \ell(\gamma)^k \|c^{1/2}T_{\gamma}e^{-tD^2}c^{1/2}D^2\| \
\leq C_6 + \sum_{\ell(\gamma) \geq C_1} \ell(\gamma)^k C_4e^{C_5 l(\gamma)}C_2e^{-C_3l(\gamma)^2},
\]
which is finite. This establishes (4.14).

For (ii), note that it follows from [11, Exercise 12.7.3] that the element
\[
(\oplus 0), \text{Ind}_{\Gamma, \sigma, L^2}(D) \in K_0(C^*_r(\Gamma, \sigma) \otimes \mathcal{K}(L^2(S_\mathcal{F})))
\]
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can be represented explicitly by the following difference of idempotents constructed from the heat operator:

\[
\begin{pmatrix}
  e^{-tD^{-}D^{+}} \oplus 0 & e^{-\frac{t}{2}D^{-}D^{+}} \frac{1 - e^{-tD^{-}D^{+}}}{D^{-}D^{+}} \oplus 0 \\
  e^{-\frac{t}{2}D^{+}D^{-}} \frac{1 - e^{-tD^{+}D^{-}}}{D^{+}D^{-}} \oplus 0 & 1 - e^{-tD^{+}D^{-}} \oplus 0
\end{pmatrix} - \begin{pmatrix}
  0 & 0 \\
  0 & 1
\end{pmatrix}.
\]

(4.15)

Now, we have a commutative diagram

\[
\begin{array}{c}
K_0(C^*_r(\Gamma, \sigma) \otimes K(L^2(S_\mathcal{L}))) \\
\xrightarrow{\text{Tr}_s} K_0(C^*_r(\Gamma, \sigma)) \end{array}
\]

where the map \(\text{Tr}_s\) is induced by the operator trace on \(L^2(S_\mathcal{L})\). Thus

\[
(\tau^{(g)}_\sigma)_*(\text{Ind}_{\Gamma, \sigma}(D)) = (\tau^{(g)}_\sigma \otimes \text{Tr})_*(\oplus 0)_s \text{Ind}_{\Gamma, \sigma, L^2}(D)
\]

\[
= (\tau^{(g)}_\sigma \otimes \text{Tr})(e^{-tD^{-}D^{+}} \oplus 0) + (\tau^{(g)}_\sigma \otimes \text{Tr})(e^{-tD^{+}D^{-}} \oplus 0)
\]

\[
= \text{str}^{(g)}(e^{-tD^2}).
\]

where we have used Proposition 4.4 for the last equality.

We now turn our attention to Theorem 1.3. To begin, we have:

**Proposition 4.6.** Let \(M, \Gamma, D, \) and \(\sigma\) be as in Theorem 1.3. Let \(\omega\) be as in (2.1). If the \(\Gamma\)-invariant Riemannian metric on \(M\) satisfies

\[
\inf_{x \in M} (\kappa(x) - 4\|c(\omega)\|_x) > 0,
\]

where the norm is taken fibrewise in \(\text{End}(S_\mathcal{L})\), then

\[
\text{Ind}_{\Gamma, \sigma, L^2}(D) = 0 \in K_0(C^*(M; L^2(S_\mathcal{L}))^{\Gamma, \sigma}).
\]

In particular \(\text{Ind}_{\Gamma, \sigma}(D) = 0 \in K_0(C^*_r(\Gamma, \sigma))\).

The proof of this proposition uses the Bochner-Lichnerowicz formula.

**Lemma 4.7** (Bochner-Lichnerowicz). Let \(N\) be a spin Riemannian manifold, and let \(S_N \to N\) be the spinor bundle with connection \(\nabla^{S_N}\). Let \(\nabla^E\) be a Hermitian connection on a Hermitian vector bundle \(E \to M\). Then

\[
D^2_E = \nabla^* \nabla + \frac{\kappa}{4} + c(R^E),
\]

where \(D_E\) is the Dirac operator associated to the connection \(\nabla = \nabla^{S_N} \otimes 1 + 1 \otimes \nabla^E\) on \(S_N \otimes E\), \(\kappa\) is the scalar curvature on \(N\), and \(c(R^E)\) denotes Clifford multiplication by the curvature of \(\nabla^E\).

**Proof of Proposition 4.6.** Let \(\nabla^\mathcal{L}\) be the Hermitian connection on the trivial bundle \(\mathcal{L} \to M\) defined by \(i\eta\), where \(\eta\) is any one-form satisfying \(d\eta = \omega\). Applying Lemma 4.7 with \(D_E = \hat{\theta} \otimes \nabla^\mathcal{L}\) gives

\[
(\hat{\theta} \otimes \nabla^\mathcal{L})^2 = \nabla^S \nabla + \frac{\kappa}{4} + ic(\omega),
\]

where \(\nabla = \nabla^S \otimes 1 + 1 \otimes \nabla^\mathcal{L}\). Thus for all \(u \in L^2(M, S_\mathcal{L})\), we have

\[
\langle D^2 u, u \rangle_{L^2} = \langle \nabla u, \nabla u \rangle_{L^2} + \frac{1}{4} \langle \kappa u, u \rangle_{L^2} + \langle ic(\omega) u, u \rangle_{L^2} \geq 0
\]

by our assumption. Then \(\text{Ind}_{\Gamma, \sigma, L^2}(D)\) can be defined by choosing the normalising function \(\chi\) equal to the sign function, and a routine computation then shows that the index representative (6.4) is
is a cut-off function for the action of $\Gamma$ on $M$ using the Riemannian metric. Since the action of $\Gamma$ on $M$ is cocompact, $\kappa(x)$ is uniformly bounded below by some $\kappa_0 > 0$, while there exists a constant $C$ such that

$$\|c(\omega)\|_x \leq C \cdot \sum_{j=1}^n |\lambda_j(x)|$$

for each $x$. Since $\omega$ is $\Gamma$-invariant, the right-hand side is again uniformly bounded over $M$. It follows that (4.16) holds for $s$ sufficiently small.

**Proof.** The curvature of the connection $\nabla^{Z,\omega}$ is $s\omega$. Thus by Proposition 4.6, it suffices to show that

$$\inf_{x \in M} (\kappa(x) - 4\|sc(\omega)\|_x) > 0$$

(4.16)

for $s$ sufficiently small. For $x \in M$, let $i\lambda_1(x), \ldots, i\lambda_n(x)$ be the pointwise eigenvalues of $\omega$, which we view as a skew-symmetric endomorphism of $TM$ using the Riemannian metric. Since the action of $\Gamma$ on $M$ is cocompact, $\kappa(x)$ is uniformly bounded below by some $\kappa_0 > 0$, while there exists a constant $C$ such that

$$\|c(\omega)\|_x \leq C \cdot \sum_{j=1}^n |\lambda_j(x)|$$

for each $x$. Since $\omega$ is $\Gamma$-invariant, the right-hand side is again uniformly bounded over $M$. It follows that (4.16) holds for $s$ sufficiently small.

**Proposition 4.9.** Let $(g) \subseteq \Gamma$ be a $\sigma^*$-regular conjugacy class, with $\sigma^*$ as in (2.6). Then the operators $e^{-tD^*D^*_+}$ and $e^{-tD^*_+D^*_+}$ are of $(\sigma^*, g)$-trace class.

**Proof.** Fix $s \in \mathbb{R}$ and $\phi_1, \phi_2 \in C_c(M)$. Let $K_{s,t}$ be the Schwartz kernel of $e^{-t(D^*)^2}$. By standard estimates for the heat kernel on cocompact manifolds (see for example [21, Corollary 3.5]), there exists a constant $C_2$, depending on $t$, such that

$$\sum_{h \in (g)} \|h^{-1}K_{s,t}(hx,x)\|_{(S^{Z,\omega})_x} < C_2$$

for all $x \in M$. Then

$$\sum_{h \in (g)} |\theta(h) \cdot \text{Str}(\phi_1 T_{h^{-1}} e^{-t(D^*)^2} \phi_2)| \leq \sum_{h \in (g)} |\text{Str}(\phi_1 T_{h^{-1}} e^{-t(D^*)^2} \phi_2)|$$

$$\leq C_1 \sum_{h \in (g)} \int_M \phi_1(x) \phi_2(x) \|h^{-1}K_{s,t}(hx,x)\| dx$$

$$\leq C_1 C_2 \|\phi_1 \phi_2\|_{L^1}.$$
Proposition 4.10. Let $\alpha$ and $\sigma$ be as in (2.5) and (2.6). Let $(g) \subseteq \Gamma$ be the conjugacy class of an $\alpha$-regular element and $c^{\theta}$ as in (4.17). Then
\[
\text{str}_{\alpha}^{(g)} e^{-t(D^{\nu})^2} = \int_M e^{-is\phi_g(x)} c^{\theta}(x) \text{Str}(g^{-1}K_{s,t}(gx,x)) \, dx,
\]
where $K_{s,t}$ is the Schwartz kernel of $e^{-t(D^{\nu})^2}$.

Proof. We begin by establishing a useful algebraic identity:
\[
\theta^s(k^{-1}gk)e^{-is(\phi_{k^{-1}gk}(k^{-1}x)+\phi_k(k^{-1}gx)-\phi_k(k^{-1}x))} = e^{-is\phi_g(x)},
\]
where $\theta^s$ is as in (3.5).

It suffices to establish this identity for $s = 1$. The case of general $s$ follows by replacing $\theta$ and $\phi$ by $\theta^s$ and $s\phi$ respectively, similar to the proof of Corollary 3.8. To proceed, note that by (2.3) applied with $x \rightarrow k^{-1}x$, $\gamma \rightarrow k^{-1}gk$, and $\gamma' \rightarrow k$, the function
\[
x \mapsto -\phi_{k^{-1}gk}(k^{-1}x) - \phi_k(k^{-1}gx) + \phi_{gk}(k^{-1}x)
\]
is constant on $M$, hence
\[
e^{-i(\phi_{k^{-1}gk}(k^{-1}x)+\phi_k(k^{-1}gx))} = e^{-i\phi_{gk}(k^{-1}x)} \cdot C_1
\]
for some $C_1 \in U(1)$. Letting $x = kx_0$ and using that $\phi_1 \equiv 0$ gives
\[
C_1 = e^{-i\phi_{gk}(k^{-1}gx_0)} = \bar{\sigma}(k,k^{-1}gk).
\]
Now by the cocycle identity and Lemma 3.15,
\[
\theta(k^{-1}gk) = \sigma(g^{-1},k)\sigma(k^{-1},g^{-1}k)
= \sigma(k^{-1},g^{-1})\sigma(k^{-1}g^{-1}k)
= \bar{\sigma}(g,k)\sigma(k^{-1}g^{-1},k).
\]
Using this and (4.21), the claimed equality (4.18) becomes
\[
\bar{\sigma}(g,k)\sigma(k^{-1}g^{-1},k)\bar{\sigma}(k,k^{-1}gk)e^{-i(\phi_{gk}(k^{-1}x)+\phi_k(k^{-1}x))} = e^{-i\phi_g(x)}.
\]
Applying (2.3) with $x \rightarrow k^{-1}x$, $\gamma \rightarrow k$, and $\gamma' \rightarrow g$ now shows that the function
\[
x \mapsto -\phi_k(k^{-1}x) - \phi_g(x) + \phi_{gk}(k^{-1}x)
\]
is constant on $M$, hence
\[
e^{-i(\phi_{gk}(k^{-1}x)+\phi_k(k^{-1}x))} = e^{-i\phi_g(x)} \cdot C_2
\]
for some $C_2 \in U(1)$. Letting $x = kx_0$ shows that
\[
C_2 = e^{i\phi_g(kx_0)} = \sigma(g,k).
\]
This, together with the computation
\[
\sigma(k^{-1}g^{-1},k)\bar{\sigma}(k,k^{-1}gk) = \sigma(k^{-1}g^{-1},k)\sigma(k^{-1}g^{-1}k,k^{-1}) = 1,
\]
establishes (4.20) and therefore (4.18).

Next, by Lemma 3.14 and a computation similar to (3.10), we can use the set $\Omega$ defined above to write $\text{str}_{\alpha}^{(g)} e^{-t(D^{\nu})^2}$ as
\[
\sum_{k \in \Omega} \theta^s(k^{-1}gk) \int_M e^{-is(\phi_{k^{-1}gk}(k^{-1}x)+\phi_k(k^{-1}gx)-\phi_k(k^{-1}x))} c(k^{-1}x) \text{Str}(g^{-1}K_{s,t}(gx,x)) \, dx.
\]
Similarly to the proof of Proposition 4.9, we can take the sum inside the integral. Then by (4.18) and the definition of $c^g$ given by (4.17), this is equal to

$$
\int_M e^{-is\phi_\sigma(x)} c^g(x) \text{Str}(g^{-1}K_{s,t}(gx,x)) \, dx.
$$

We can now finish the proof of Theorem 1.3.

**Proof of Theorem 1.3.** For (i), first note that

$$(D^s)^2 e^{-t(D^s)^2} = \frac{1}{2}[D^s, D^s e^{-t(D^s)^2}]_s.$$  

It follows from Corollary 3.22 that

$$\frac{d(\text{str}_{\sigma^s}(g) e^{-t(D^s)^2})}{dt} = -\text{str}_{\sigma^s}((D^s)^2 e^{-t(D^s)^2}) = 0,$$

hence the function $t \mapsto \text{str}_{\sigma^s}(g) e^{-t(D^s)^2}$ is constant in $t > 0$. Let $K_{s,t}$ be the Schwartz kernel of the operator $e^{-t(D^s)^2}$. By standard heat kernel estimates, in the limit $t \to 0$ the integral

$$
\int_M e^{-is\phi_\sigma(x)} c^g(x) \text{Str}(g^{-1}K_{s,t}(gx,x)) \, dx
$$

localises to arbitrarily small neighbourhoods of the fixed-point submanifold $M^9$; see for example [12, Lemma 4.10]. Pick a sufficiently small tubular neighbourhood and identify it with the normal bundle $\mathcal{N}$ of $M^9$ in $M$. Since the curvature of the twisting bundle $\mathcal{L} \to M$ is $\Gamma$-invariant, the standard asymptotic expansion of $K_{s,t}$ (see [2, Theorem 6.11]) applies. The same argument as in the compact case (see [2, Theorem 6.16]) then shows that the above integral equals

$$
\int_{M^9} e^{-is\phi_\sigma} c^g \cdot \frac{\widehat{A}(M^9) \cdot e^{-s\omega/2\pi i}}{\det(1 - ge^{-R^N/2\pi i})^{1/2}},
$$

where we have used that $c^g|_{M^9}$ is a cut-off function for the action of $Z^9$ on $M^9$.

By Remark 2.3, $e^{-is\phi_\sigma(x)}$ is constant on each connected component of $M^9$. Further, the support of $c^g$ is compact and thus intersects only finitely many of these connected components, $M^9_1, \ldots, M^9_N$.

For each $j = 1, \ldots, N$, pick a point $x_j \in M^9_j$ and let $\mathcal{N}_j$ be the restriction of $\mathcal{N}$ to $M^9_j$. By Theorem 4.5 (ii), together with the above discussion, we have

$$
(t^{(g)}_{\sigma^s})_* \text{Ind}_{\Gamma,\sigma^s}(D^s) = \text{str}_{\sigma^s}(e^{-t(D^s)^2})
$$

\begin{align*}
&= \sum_{j=1}^m \int_{M^9_j} e^{-is\phi_\sigma(x_j)} c^g \cdot \frac{\widehat{A}(M^9_j) \cdot e^{-s\omega/2\pi i}}{\det(1 - ge^{-R^N/2\pi i})^{1/2}} \\
&= \sum_{k=0}^{\dim M/2} \frac{s^k}{k!} \sum_{j=1}^m \int_{M^9_j} c^g \cdot \frac{\widehat{A}(M^9_j) \cdot (\phi_\sigma(x_j) + \omega)^k}{\det(1 - ge^{-R^N/2\pi i})^{1/2}}.
\end{align*}

For (ii), observe that we can, by a straightforward suspension argument, reduce to the case where $M$ is even-dimensional. If $M$ admits a $\Gamma$-invariant metric of positive scalar curvature, then $\text{Ind}_{\Gamma,\sigma^s} D^s$ vanishes for all $s \in (0, \delta)$ for some $0 < \delta$, by Proposition 4.8. Since the map

$$s \mapsto (t^{(g)}_{\sigma^s})_* (\text{Ind}_{\Gamma,\sigma^s} D^s)$$

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is a polynomial in $s$, it must vanish identically on $\mathbb{R}$, hence

$$
\sum_{j=1}^{m} \int_{M^g_j} c^g \cdot \frac{\hat{A}(M^g) \cdot \left(\frac{\phi_j(x)}{2\pi i}\right)^k}{\det(1 - ge^{-Rjk/2\pi i})^{1/2}} |_{M^g_j} = 0
$$

(4.24)

for each $k \geq 0$, where we may also assume that each $M^g_j$ is even-dimensional.

In the case that $M^g$ is connected, we may choose the point $x_0$ from (2.4) to lie in $M^g$, whence Remark 2.3 implies that $e^{-i\omega \phi_j} \equiv 1$ on $M^g$. The formula (4.24) now simplifies to

$$
\hat{A}_g(M, \omega^k) = \int_{M^g} c^g \cdot \frac{\hat{A}(M^g) \cdot \omega^k |_{M^g}}{\det(1 - ge^{-R^g/2\pi i})^{1/2}} = 0
$$

for each $k \geq 0$. □

Remark 4.11. It should be possible to derive part (ii) of Theorem 1.3 independently of any growth assumptions on $(g)$ by using methods similar to the proof of [16, Theorem 3.4]; see also [24, Remark A.2]. In particular, this would imply that if the Baum-Connes conjecture holds for $\Gamma$, then the map $(\tau_{\sigma}^{(g)})_*$ from (4.8) can be defined via the $\sigma$-weighted $(g)$-supertrace str$^{(g)}_{\sigma}$ from (3.16).

4.1. Special cases.

We discuss two special cases of Theorem 1.3 that have already appeared elsewhere in the literature. These correspond to the extreme cases when either the conjugacy class or the multiplier is trivial.

4.1.1. Free action and $(g) = (e)$.

In this case, the canonical trace

$$
\tau_{\sigma}^{(e)}: C^\sigma \Gamma \to \mathbb{C}
$$

$$
\sum_{\gamma \in \Gamma} a_{\gamma} \gamma \mapsto a_e
$$

(4.25)

extends continuously to a trace $C^*_r(\Gamma) \to \mathbb{C}$ and induces a linear map

$$(\tau_{\sigma}^{(e)})_*: K_0(C^*_r(\Gamma, \sigma)) \to \mathbb{C}.$$ 

Since the action of $\Gamma$ on $M$ is free, we may work with the classifying space $B\Gamma$ instead of $B\Gamma$, as done in [15]. Let $f: M/\Gamma \to B\Gamma$ be the classifying map for $M$, and let $|\beta| \in H^2(B\Gamma, \mathbb{R})$. Let $\omega_0$ be a differential form on the quotient manifold $M/\Gamma$ such that $f^*[\beta] = [\omega_0]$, and let $\omega$ be the $\Gamma$-invariant lift of $\omega_0$ to $M$.

Since $\phi_e \equiv 0$, Theorem 1.3 (i) reduces to the twisted $L^2$-index theorem in of Mathai [15, Theorem 3.6] for the case of the spin-Dirac operator, namely that

$$(\tau_{\sigma}^{(e)})_* \text{Ind}_{\Gamma, \sigma}(D) = \int_M c \cdot \hat{A}(M) \cdot e^{-\omega/2\pi i} = \int_{M/\Gamma} \hat{A}(M/\Gamma) \cdot e^{-\omega_0/2\pi i},$$

where $c$ is a cut-off function for the $\Gamma$-action on $M$. Theorem 1.3 (ii) recovers the result if the $M/\Gamma$ admits a metric of positive scalar curvature, then for each non-negative integer $k$,

$$
\int_M \hat{A}(M/\Gamma) \cdot \omega_0^k = 0.
$$

This is [15, Corollary 1].
4.1.2. **Trivial multiplier** $\sigma \equiv 1$.

In this case, the projectively invariant Dirac operator $D$ is simply the $\Gamma$-invariant Dirac operator acting on sections of the spinor bundle $\mathcal{S} \to M$. The element $\text{Ind}_{\Gamma,\sigma}(D)$ reduces to the usual $\Gamma$-equivariant higher index $\text{Ind}_{\Gamma}(D) \in K_0(C^*_r(\Gamma))$ for cocompact actions [1]. The trace $\tau_{\sigma}^{(g)}$ is then the unweighted trace $\tau^{(g)} : \mathbb{C}\Gamma \to \mathbb{C}$ from (3.1).

Theorem 1.3 (i) then states that if $(g)$ has polynomial growth, then

$$\tau^{(g)}_{\sigma} \text{Ind}_{\Gamma}(D) = \int_{M^g} c^g \cdot \frac{\hat{A}(M^g)|_{M^g}}{\det(1 - g e^{-R^N/2\pi i})^{1/2}},$$

where we have implicitly taken $\omega$ to be zero. This recovers the formula [21, Theorem 6.1] in the case of the spin-Dirac operator.

**Proof of Corollary 1.4.** By a suspension argument, we may assume without loss of generality that both $M$ and $M^g$ are even-dimensional. Further, since the vanishing property is preserved under sums of forms, we may to restrict our attention to the case where

$$\omega = \omega_1 \omega_2 \ldots \omega_m,$$

for some $m \leq \dim M/2$, where each $\omega_i$ is the lift of a differential form on $M/\Gamma$ representing a class in $f^*H^2(B\Gamma, \mathbb{R})$. For each $i = 1, \ldots, m$, let $s_i \in \mathbb{R}$. Then applying to argument from the proof of Theorem 1.3 to $\sum_{i=1}^m s_i \omega_i$ instead of $s \omega$, with the twisted Dirac operator defined accordingly, shows that if $M$ admits a $\Gamma$-invariant metric of positive scalar curvature, then there exists a $\delta > 0$ such that

$$\sum_{k=0}^{\dim M/2} \frac{1}{k!} \int_{M^g} c^g \cdot \frac{\hat{A}(M^g) \cdot (s_1 \omega_1 + \ldots + s_m \omega_m)}{\det(1 - g e^{-R^N/2\pi i})^{1/2}} = 0$$

whenever $s_i \in (0, \delta)$ for all $i$. Since the left-hand side is a polynomial in the variables $s_1, \ldots, s_m$, it vanishes identically on $\mathbb{R}^m$. In particular, the coefficient of $s_1 s_2 \ldots s_m$ is zero, and this is equal to

$$\frac{(2\pi i)^{-m}}{m!} \int_{M^g} c^g \cdot \frac{\hat{A}(M^g) \cdot \omega}{\det(1 - g e^{-R^N/2\pi i})^{1/2}}.$$

This concludes the proof. \qed

5. A Neighbourhood PSC Obstruction

In this section we prove Theorem 1.5. We will make use of a Callias-type index theorem for projectively invariant operators.

5.1. **Projectively invariant Callias-type operators.**

Let us begin with a general definition and discussion of projectively invariant Callias-type operators, before specialising to our geometric situation of interest.

Let $\sigma$ be a multiplier on a discrete group $\Gamma$. Let $M$ be a complete Riemannian manifold on which $\Gamma$ acts properly and isometrically. Let $E^\sigma \to M$ be a $\mathbb{Z}_2$-graded $\Gamma$-equivariant Clifford bundle over $M$ and $D$ an odd-graded Dirac operator acting on smooth sections of $E^\sigma$ that commutes with a given $(\Gamma, \sigma)$-representation $T$ on $L^2(E^\sigma)$ of the form given in Definition 2.4.
**Definition 5.1.** An odd-graded, $\Gamma$-equivariant fibrewise Hermitian bundle endomorphism $\Phi$ of $E^\sigma$ is admissible for $D$ if $D\Phi + \Phi D$ is an endomorphism of $E^\sigma$ such that there exists a cocompact subset $Z \subseteq M$ and a constant $C > 0$ such that the pointwise estimate
\[ \Phi^2 \geq \|D\Phi + \Phi D\| + C \] (5.1)
holds over $M \setminus Z$. In this setting, $D + \Phi$ is called a ($\Gamma, \sigma$)-invariant Callias-type operator.

**Remark 5.2.** A bundle endomorphism on $L^2(E^\sigma)$ commutes with the projective action $T$ from Definition 2.4 if and only if it commutes with the unitary action $U$ of the group.

Two key properties of these operators are that they have an index in $K^*_C(C^*_r(\Gamma, \sigma))$ and that this index can be computed from the data on a cocompact hypersurface in $M$. The indices of such operators can be defined either via Roe algebras, similar to what was done in subsection 3.3, or using Hilbert $C^*_r(\Gamma, \sigma)$-modules. We will take the latter approach in order to frame the discussion in parallel with that in [9] for the untwisted case.

**Definition 5.3.** Let $E^\sigma$ be the Hilbert $C^*_r(\Gamma, \sigma)$-module completion of $C_c(E^\sigma)$ with respect to (5.2).

The admissibility condition (5.1) implies that the operator $D + \Phi$ is projectively Fredholm in the following sense:

**Proposition 5.4.** There exists a cocompactly supported, $G$-invariant continuous function $f$ on $M$ such that
\[ F := (D + \Phi)(D + \Phi)^{-1/2} \in \mathcal{B}(E^\sigma), \] (5.3)
such that the pair $(E^\sigma, F)$ is a cycle in $KK(C, C^*_r(\Gamma, \sigma))$. The class $[E^\sigma, F]$ is independent of the choice of $f$.

**Proof.** The proof is analogous to that of [8, Theorem 4.19]. Instead of the Hilbert module $C^*(G)$-module $E$ used there, we work with $E^\sigma$. □

**Definition 5.5.** The ($\Gamma, \sigma$)-index of $D + \Phi$ is the class
\[ Ind_{\Gamma, \sigma}(D + \Phi) := [E^\sigma, F] \in K_0(C^*_r(\Gamma, \sigma)) \cong KK(C, C^*_r(\Gamma, \sigma)). \]

**5.2. Localisation of projective Callias-type indices.**

One of the key properties of Callias-type operators in the equivariant setting is that their indices can be calculated by localising to a cocompact subset of the manifold [9, Theorem 3.4]. In the projective setting, a similar result holds. For this, we will assume that the ($\Gamma, \sigma$)-invariant Dirac operator $D$ from Definition 5.1 takes the form of a Dirac operator twisted by a line bundle, as in Definition 2.5.

To begin, let $M$ and $\Gamma$ be as in subsection 5.1. Let
\[ f: M/\Gamma \to B\Gamma \]
be the classifying map of $M$. Let $[\beta] \in H^2(B\Gamma, \mathbb{R})$, and let $[\omega_0] = f^*[\beta]$ in the de Rham cohomology of the orbifold $M/\Gamma$. Let $\omega$ be the $\Gamma$-invariant lift of $\omega_0$ to $M$. As in subsection 2.2, we obtain a
one-form $\eta$, a family $\phi$ of functions on $M$, and a family of multipliers $\sigma^s$ on $\Gamma$ parameterised by $s \in \mathbb{R}$.

**Remark 5.6.** The cocycles $\alpha$ and $\sigma^s$ from (2.5) and (2.6) can be defined equivalently in terms of the above data restricted to any submanifold $P \subseteq M$ preserved by the action of $\Gamma$. Indeed, by working with the de Rham differential on $P$ instead of $M$, (2.3) implies that the family

$$\phi|_P := \{\phi_\gamma|_P : \gamma \in \Gamma\}$$

satisfies

$$dp(\phi_\gamma|_P + \gamma^{-1}\phi_{\gamma'}|_P - \phi_{\gamma'\gamma}|_P) = 0.$$  

It follows that $\alpha$ and $\sigma^s$ can be defined equivalently as

$$\alpha(\gamma, \gamma') = \frac{1}{2\pi}(\phi_\gamma|_P(x) + \phi_{\gamma'}|_P(\gamma'x) - \phi_{\gamma'\gamma}|_P(x)),$$

$$\sigma^s(\gamma, \gamma') = e^{2\pi is\alpha(\gamma, \sigma)}.$$

Since the discussion in rest of this subsection applies uniformly to $\sigma^s$ for any $s$ with only minor and obvious adjustments, let us now fix $s = 1$ and write $\sigma = \sigma^1$.

Let $E_0$ be an ungraded $\Gamma$-equivariant Clifford bundle over $M$ equipped with a $\Gamma$-invariant Hermitian connection $\nabla E_0$. Define the Hermitian connection

$$\nabla^\mathcal{L} = d + i\eta$$

on a $\Gamma$-equivariantly trivial Hermitian line bundle $\mathcal{L} \to M$, and form the connection $\nabla^{E_0,\mathcal{L}} = \nabla E_0 \otimes 1 + 1 \otimes \nabla^\mathcal{L}$ on the bundle

$$E_0,\mathcal{L} := E_0 \otimes \mathcal{L}.$$  

In the notation of subsection 5.1, we will take

$$E^\sigma = E_\mathcal{L} := E_0,\mathcal{L} \oplus E_0,\mathcal{L}$$

where the first copy of $E_0,\mathcal{L}$ is given the even grading, and the second copy the odd grading. Let $D_0$ be the Dirac operator on $E_0,\mathcal{L}$ associated to $\nabla^{E_0,\mathcal{L}}$, and define

$$D = \begin{pmatrix} 0 & D_0 \\ D_0 & 0 \end{pmatrix}$$

on $E_\mathcal{L}$. Let $\Phi_0$ be $\Gamma$-invariant a Hermitian endomorphism of $E_0$ such that $\Phi^2 \geq \|D\Phi + \Phi D\| + C$ holds outside a cocompact subset $Z \subseteq M$ for some $C > 0$. Let

$$\Phi = \begin{pmatrix} 0 & i\Phi_0 \otimes 1 \\ -i\Phi_0 \otimes 1 & 0 \end{pmatrix}.$$  

Then $D + \Phi$ is a $(\Gamma, \sigma)$-invariant Callias-type operator acting on sections of $E_\mathcal{L}$, in the sense of Definition 5.1.

Let $M_- \subseteq M$ be a $\Gamma$-invariant, cocompact subset containing $Z$ in its interior, such that $N := \partial M_-$ is a (not necessary connected) smooth submanifold of $M$. Let $M_+$ be the closure of the complement of $M_-$, so that $N = M_- \cap M_+$ and $M = M_- \cup M_+$. We will use the notation

$$M = M_- \cup_N M_+.$$  

By (5.1), the restriction of $\Phi_0$ to $N$ is fibrewise invertible. Let

$$E^N_{0,\mathcal{L}},_\pm \subseteq E^N_{0,\mathcal{L}}$$
be the positive and negative eigenbundles of \(\Phi_0\). Clifford multiplication by \(i\) times the unit normal vector field \(\tilde{n}\) to \(N\) pointing into \(M_+\) defines \(\Gamma\)-invariant gradings on both \(E^{N,+,0}_{\mathcal{L}}\) and \(E^{N,0,-}_{\mathcal{L}}\).

Define the connections
\[
\nabla^{E^{N,\pm} \mathcal{L}} := p_+ \nabla^{E^{N,0}_{\mathcal{L}}} p_+ \tag{5.8}
\]
on \(E^{N,\pm}_{\mathcal{L}},\) where \(p_+: E^{N,0}_{\mathcal{L}} \to E^{N,\pm}_{\mathcal{L}}\) are the orthogonal projections. Along with the Clifford action of \(TM|_N\) on \(E^{N,\pm}_{\mathcal{L}}\), these connections give rise to two Dirac operators
\[
D^{E^{N,\pm}_{\mathcal{L}}} = D^{E^{N,0}_{\mathcal{L}}} \quad \text{and} \quad D^{E^{N,\pm}_{\mathcal{L}}},
\]
both odd-graded, acting on sections of \(E^{N,\pm}_{\mathcal{L}}\) respectively.

For each group element \(\gamma \in \Gamma\), define unitary operators \(U^{N,\pm}_{\gamma}, S^{N,\pm}_{\gamma},\) and \(T^{N,\pm}_{\gamma}\) on \(L^2(E^{N,\pm}_{\mathcal{L}})\) by:
\[
\begin{align*}
U^{N,\pm}_{\gamma} u(x) &= \gamma u(\gamma^{-1} x); \\
S^{N,\pm}_{\gamma} u &= e^{i\phi|_N} u; \\
T^{N,\pm}_{\gamma} &= U^{N}_{\gamma} \circ S^{N,\pm}_{\gamma},
\end{align*}
\]
where \(u \in L^2(E^{N,\pm}_{\mathcal{L}})\) and \(x \in N\). Note that (2.1) and (2.2) continue to hold if we restrict \(\omega, \eta,\) and \(\phi\) to \(N\) and work with the de Rham differential on \(N\) instead of \(M\). It follows that the operator \(D^{E^{N,\pm}_{\mathcal{L}}}\) is equivariant with respect the projective action \(T^{N,+}\). Since \(\Gamma\) acts on \(N\) cocompactly, \(D^{E^{N,\pm}_{\mathcal{L}}}\) has a \((\Gamma, \sigma)\)-invariant higher index
\[
\text{Ind}_{\Gamma, \sigma}(D^{E^{N,\pm}_{\mathcal{L}}} \mathcal{L}) \in K_0(C^*_r(\Gamma, \sigma))
\]
by Definition 3.30 and (3.22). Equivalently, this index can be formulated as in (5.3), using a bounded transform on a Hilbert module. As mentioned previously, we will adopt the latter in order to follow more closely the exposition of [9].

With these preparations, we have the following:

**Theorem 5.7** \((\Gamma, \sigma)\)-Callias-type index theorem.
\[
\text{Ind}_{\Gamma, \sigma}(D + \Phi) = \text{Ind}_{\Gamma, \sigma}(D^{E^{N,\pm}_{\mathcal{L}}}) \in K_0(C^*_r(\Gamma, \sigma)). \tag{5.10}
\]

This is the projective analogue of the equivariant Callias-type index theorem [9, Theorem 3.4]. The proof is analogous to that in the untwisted setting, once we make the following modifications:

1. instead of the Hilbert \(C^*(G)\)-modules used in [9], we work with Hilbert \(C^*_r(\Gamma, \sigma)\)-modules;
2. instead of the \(G\)-equivariant bundle \(S\) used in [9] (resp. bundles derived from \(S\)), we work with the bundle \(E^\sigma\), or specifically \(E\) from subsection 5.2 (resp. bundles derived from \(E^\sigma\) or \(E\));
3. instead of \(G\)-invariant differential operators, we work with \((\Gamma, \sigma)\)-invariant operators, as constructed above;
4. instead of the index \(\text{index}_G\) from [9, section 3], we work with \(\text{Ind}_{\Gamma, \sigma}\).

Given these similarities, we will for the most part only sketch the proof of Theorem 5.7, and invite the reader who is interested in a more detailed discussion to [9, section 5]. Nevertheless, let us give a detailed example of how one of the key technical tools used in the proof of [9, Theorem 3.4] can be adapted to the projective setting, namely the relative index theorem for Callias-type operators [9, Theorem 4.13]. The twisted analogue of that theorem is as follows.

For \(j = 1, 2\), let \(M_j, E^\sigma_j, D_j\) and \(\Phi_j\) be as \(M, E^\sigma, D,\) and \(\Phi\) were in Definition 5.1. Suppose there exist \(\Gamma\)-invariant, cocompact hypersurfaces \(N_j \subseteq M_j\), \(\Gamma\)-invariant tubular neighbourhoods \(U_j \supseteq N_j\), and a \(\Gamma\)-equivariant isometry \(\psi: U_1 \to U_2\) such that
\[
\bullet \ \psi(N_1) = N_2;
\]
• \( \psi^*(E^\sigma_2|_{U_2}) \cong E^\sigma_1|_{U_1} \);
• \( \psi^*(\nabla_2|_{U_2}) = \nabla_1|_{U_1} \), where \( \nabla_j \) is the Clifford connection used to define \( D_j \);
• \( \Phi_1|_{U_1} \) corresponds to \( \Phi_2|_{U_2} \) via \( \psi \).

Suppose that \( M_j = X_j \cup N_j \) for closed, \( \Gamma \)-invariant subsets \( X_j, Y_j \subseteq M_j \). We identify \( N_1 \) and \( N_2 \) via \( \psi \) and simply write \( N \) for this manifold. Construct

\[
M_3 := X_1 \cup N \subseteq U_2; \quad M_4 := X_2 \cup N \subseteq U_1.
\]

For \( j = 3, 4 \), let \( E^\sigma_j, D_j \) and \( \Phi_j \) be obtained from the corresponding data on \( M_1 \) and \( M_2 \) by cutting and gluing along \( U_1 \cong \equiv U_2 \) via \( \psi \). For \( j = 1, 2, 3, 4 \), form the Hilbert \( C^*_r(\Gamma, \sigma) \)-modules \( E^\sigma_j \) as in Definition 5.3.

**Proposition 5.8.** *In the above situation,*

\[
\text{Ind}_{r, \sigma}(D_1 + \Phi_1) + \text{Ind}_{r, \sigma}(D_2 + \Phi_2) = \text{Ind}_{r, \sigma}(D_3 + \Phi_3) + \text{Ind}_{r, \sigma}(D_4 + \Phi_4) \in K_0(C^*_r(\Gamma, \sigma)).
\]

**Proof.** (Compare the proof of [9, Theorem 4.13].) Define

\[
E^\sigma := E^\sigma_1 \oplus E^\sigma_2 \oplus E^\sigma_3 \oplus E^\sigma_4 \oplus E^\sigma_{\text{op}},
\]

where a superscript \( \text{op} \) indicates reversal of the \( \mathbb{Z}_2 \)-grading on the given module. Similar to (5.3), define

\[
F_j := (D_j + \Phi_j)((D_j + \Phi_j)^2 + f_j)^{-1/2},
\]

for \( j = 1, 2, 3, 4 \), and

\[
F := F_1 \oplus F_2 \oplus F_3 \oplus F_4.
\]

For \( j = 1, 2 \), let \( \chi_{X_j}, \chi_{Y_j} \in C^\infty(M_j) \) be real-valued functions such that:

(i) \( \text{supp}(\chi_{X_j}) \subseteq X_j \cup U_j \) and \( \text{supp}(\chi_{Y_j}) \subseteq Y_j \cup U_j \);

(ii) \( \psi^*(\chi_{X_j}|_{U_2}) = \chi_{X_1}|_{U_1} \) and \( \psi^*(\chi_{Y_j}|_{U_2}) = \chi_{Y_1}|_{U_1} \) and

(iii) \( \chi_{X_j}^2 + \chi_{Y_j}^2 = 1 \).

We view pointwise multiplication by these functions as operators

\[
\chi_{X_1} : E^\sigma_1 \to E^\sigma_3; \quad \chi_{Y_2} : E^\sigma_2 \to E^\sigma_3;
\]

\[
\chi_{X_1} : E^\sigma_1 \to E^\sigma_4; \quad \chi_{X_2} : E^\sigma_2 \to E^\sigma_4.
\]

(5.11)

Define the operator

\[
X := \gamma \begin{pmatrix}
0 & 0 & -\chi_{X_1}^* & -\chi_{Y_1}^* \\
0 & 0 & -\chi_{Y_2}^* & \chi_{X_2}^* \\
\chi_{X_1} & \chi_{Y_2} & 0 & 0 \\
\chi_{Y_1} & -\chi_{X_2} & 0 & 0
\end{pmatrix} \in \mathcal{B}(E^\sigma),
\]

where \( \gamma \) is the grading operator on \( E^\sigma \). Then \( X \) is an odd, self-adjoint operator on \( E^\sigma \). Further, using properties (ii) and (iii), one verifies directly that \( X^2 = 1 \). Let \( \mathcal{C}l \) denote the Clifford algebra generated by \( X \). It follows from a discussion analogous to [9, section 4] that

\[
XF + FX \in \mathcal{K}(E^\sigma).
\]

Since \( X \) generates \( \mathcal{C}l \) and anticommutes with \( F \) modulo \( \mathcal{K}(E^\sigma) \), the pair \( (E^\sigma, F) \) is a Kasparov \( (\mathcal{C}l, C^*_r(\Gamma, \sigma)) \)-cycle. Its class is mapped to \( [E^\sigma, F] \in K_*(C^*_r(\Gamma, \sigma)) \) by the homomorphism induced by the pullback along the inclusion \( \mathbb{C} \hookrightarrow \mathcal{C}l \). By [3, Lemma 1.15], that homomorphism is zero. Hence

\[
[E^\sigma, F] = 0 \in K_0(C^*_r(\Gamma, \sigma)),
\]

which is equivalent to the theorem. \( \square \)
Proof of Theorem 5.7. The first and most important step is to use Proposition 5.8 to reduce the computation of \( \text{Ind}_{\Gamma, \sigma}(D + \Phi) \) to the index of a \((\Gamma, \sigma)\)-invariant Callias-type operator on the cylinder \( N \times \mathbb{R} \). This is done by following the same geometric steps as in [9, subsection 5.3], only applied to the bundle \( E^\sigma \) instead of the bundle \( S \) used there. Where [9, Theorem 4.13] was used, we now apply Proposition 5.8. In [9], a homotopy invariance property of equivariant Callias-type operators [9, Proposition 4.9] was proved, together with the fact that the index of a Callias-type operator does not change if one modifies the potential \( \Phi \) on a cocompact subset [9, Corollary 4.10]. The proofs of both of these properties carry over to the projective setting after making the modifications (1) – (4) from above.

More precisely, the above discussion reduces the computation of \( \text{Ind}_{\Gamma, \sigma}(D + \Phi) \) to the \((\Gamma, \sigma)\)-index of the operator (5.12) below. To define this operator, let \( E_{N, \mathcal{L}} \) be as in (5.5), and denote by \( E_{0, \mathcal{L}}^N \) its restriction to \( N \). Let

\[
E_{N, \mathcal{L}, \pm}^N \rightarrow N \times \mathbb{R}
\]

be the pullbacks of \( E_{0, \mathcal{L}, \pm}^N \rightarrow N \) along the canonical projection \( N \times \mathbb{R} \rightarrow N \). Then \( E_{0, \mathcal{L}, \pm}^N \) are Clifford bundles over \( T(N \times \mathbb{R}) \), with Clifford action

\[
\tilde{c}(v, t) = c(v + tf),
\]

where \( v \in TN, t \in \mathbb{R}, \hat{n} \) is the normal vector field to \( N \) pointing into \( M_+ \), and \( c \) is Clifford multiplication on \( E_{0, \mathcal{L}} \). Let \( \nabla_{E_{0, \mathcal{L}, \pm}}^N \) and \( D_{E_{0, \mathcal{L}, \pm}}^N \) be as in (5.8) and (5.9). By pulling back \( \nabla_{E_{0, \mathcal{L}, \pm}}^N \) along \( N \times \mathbb{R} \rightarrow \mathbb{R} \) and composing with \( \tilde{c} \), we obtain Dirac operators \( D_{E_{0, \mathcal{L}, \pm}}^N \) acting on sections of \( E_{0, \mathcal{L}, \pm}^N \). In particular, operator \( D_{E_{0, \mathcal{L}, +}}^N \) is equivariant with respect to the pull-back of the projective action \( T^{N,+} \) to \( L^2(E_{0, \mathcal{L}, +}^N) \). Let \( \chi \in C^\infty(\mathbb{R}) \) be an odd function such that \( \chi(t) = t \) for all \( t \geq 2 \). Let \( \chi_{N \times \mathbb{R}} \) be its pullback along the projection \( N \times \mathbb{R} \rightarrow \mathbb{R} \). Define

\[
D_{E_{0, \mathcal{L}, +}}^N = \begin{pmatrix} 0 & D_{E_{0, \mathcal{L}, +}}^N \\ D_{E_{0, \mathcal{L}, +}}^N & 0 \end{pmatrix}
\]

on smooth sections of \( E_{0, \mathcal{L}, +}^N \). Then the endomorphism

\[
\chi_{N \times \mathbb{R}} = \begin{pmatrix} 0 & \ i\chi_{N \times \mathbb{R}} \\ -i\chi_{N \times \mathbb{R}} & 0 \end{pmatrix}
\]

is admissible for \( D_{E_{0, \mathcal{L}, +}}^N \) in the sense of Definition 5.1, and

\[
D_{E_{0, \mathcal{L}, +}}^N + \chi_{N \times \mathbb{R}} \tag{5.12}
\]

is a \((\Gamma, \sigma)\)-invariant Callias-type operator on \( N \times \mathbb{R} \). By the discussion in the first paragraph of this proof, we have

\[
\text{Ind}_{\Gamma, \sigma}(D + \Phi) = \text{Ind}_{\Gamma, \sigma}(D_{E_{0, \mathcal{L}, +}^N}^N + \chi_{N \times \mathbb{R}}). \tag{5.13}
\]

It then suffices to prove that

\[
\text{Ind}_{\Gamma, \sigma}(D_{E_{0, \mathcal{L}, +}^N}^N + \chi_{N \times \mathbb{R}}) = \text{Ind}_{\Gamma, \sigma}(D_{E_{0, \mathcal{L}, +}^N}^N), \tag{5.14}
\]

which is the projective analogue of [9, Proposition 5.7]. For this, note that the operator \( D_{E_{0, \mathcal{L}, +}^N}^N + \chi_{N \times \mathbb{R}} \) can be written explicitly as

\[
\begin{pmatrix} 0 & D_{E_{0, \mathcal{L}, +}^N} \\ D_{E_{0, \mathcal{L}, +}^N} & 0 \end{pmatrix} \otimes 1_{C^\infty(\mathbb{R})} + \gamma_{E_{0, \mathcal{L}, +}^N} \otimes \begin{pmatrix} 0 & i\frac{d}{dt} \\ i\frac{d}{dt} & 0 \end{pmatrix} + 1_{C^\infty(E_{0, \mathcal{L}, +}^N)} \otimes \begin{pmatrix} 0 & -i\chi \\ i\chi & 0 \end{pmatrix},
\]

for all...
where $\gamma_{E^N_{0,\mathcal{L},+}}$ is a grading on $E^N_{0,\mathcal{L},+}$ defined as $-i$ times Clifford multiplication by the unit normal vector field on $N$ pointing into $M_+$. The equality (5.14) then follows from the fact that the kernel of $i\frac{d}{dt} \pm i\chi$ in $C^\infty(\mathbb{R})$ is one-dimensional. Combining (5.13) and (5.14) concludes the proof. \[\square\]

**Remark 5.9.** Theorem 5.7 continues to hold if we replace $\sigma$ by $\sigma^s$ for any $s \in \mathbb{R}$. In this case, the connection $\nabla^\mathcal{L}$ from (5.4) would be replaced by $\nabla^\mathcal{L},s = d + is\eta$.

### 5.3. Proof of Theorem 1.5.

**Proof of Theorem 1.5.** This proof is similar to, but more subtle than, that of [9, Theorem 2.1], as it involves an additional scaling argument along with the use of an appropriate partition of unity. Hence we will give the full details.

First note that, by a suspension argument, we need only consider the case of odd-dimensional $M$. In this case, let $\mathcal{S}$ be the spinor bundle over $M$, let $\mathcal{D}$ be the spin-Dirac operator, and let $\mathcal{S}_\mathcal{L} = \mathcal{S} \otimes \mathcal{L}$ for a $\Gamma$-trivial line bundle $\mathcal{L}$. For $s \in \mathbb{R}$, let $\nabla^\mathcal{L},s$ be the Hermitian connection on $\mathcal{L}$ defined by the one-form $is\eta$. In the notation of subsection 5.2, take $E_0,\mathcal{L} = \mathcal{S}_\mathcal{L}$ and $D_0^s$ be the Dirac operator associated to the connection $\nabla^\mathcal{S} \otimes 1 + 1 \otimes \nabla^\mathcal{L},s$. Let

$$D^s = \begin{pmatrix} 0 & D_0^s \\ D_0^s & 0 \end{pmatrix}$$

act on sections of $\mathcal{S}_\mathcal{L} \oplus \mathcal{S}_\mathcal{L}$. We now construct a potential $\Phi$ that is admissible for $D^s$, for all $s$, in the sense of Definition 5.1.

Let $H$ be as in the statement of the theorem. Then $M \setminus H = X \cup Y$ for disjoint open subsets $X$ and $Y$. Pick a cocompact subset $K$ of $M$ such that $H \subseteq K \subseteq \overline{X}$ and $\kappa > 0$ on $K$, and the distance from $X \setminus K$ to $Y$ is positive. Pick a $\Gamma$-invariant function $\chi \in C^\infty(M)$ such that $\chi$ equals 1 on $Y$ and $-1$ on $X \setminus K$. Let $\Phi_0$ be the endomorphism of $\mathcal{S}_\mathcal{L}$ given by pointwise multiplication by $\chi$. Now define $D^s$ and $\Phi$ according to (5.6) and (5.7) respectively. Define the endomorphism

$$\Phi = \begin{pmatrix} 0 & i\chi \\ -i\chi & 0 \end{pmatrix}.$$  

One finds that $D_0^s = \mathcal{D} + is\eta$. Together with the fact that $[isc(\eta),i\chi] = 0$, this implies that

$$\{D^s,\Phi\} = -i \begin{pmatrix} sc(d\chi) & 0 \\ 0 & -sc(d\chi) \end{pmatrix}.$$  

By construction, the estimate $\Phi^2 \geq \|\{D^s,\Phi\}\| + 1$ holds pointwise on $M \setminus K$, hence $\Phi$ is admissible for $D^s$, for all $s \in \mathbb{R}$, in the sense of Definition 5.1.

Next, in the notation of subsection 5.1, take $M_- = K$. Then $N = \partial M_-$ is a disjoint union $N_- \cup H$ for a cocompact subset $N_-$ such that $f|_{N_-} = -1$. In this case,

$$E^N_+ = \mathcal{S}_\mathcal{L}|_H.$$  

(5.15)

By Theorem 5.7 and the proof of Theorem 1.3 (see also Remark 5.9), it suffices to show that $\text{Ind}_{\Gamma,\sigma}(D^s + \Phi) = 0$ for all sufficiently small $s$. For convenience, let us write $B^s_\lambda = D^s + \lambda\Phi$ for $\lambda > 0$. By a homotopy argument, $\text{Ind}_{\Gamma,\sigma}(B^s) = \text{Ind}_{\Gamma,\sigma}(B^s_\lambda)$ for any positive $\lambda$, so it suffices to show that

$$\text{Ind}_{\Gamma,\sigma}(B^s_\lambda) = 0$$

for some $\lambda > 0$ and all sufficiently small $s$. Note that the endomorphism $\lambda\Phi$ is still admissible for $D^s$, and (5.15) continues to hold. Let $K'$ be an arbitrary cocompact neighbourhood of $K$. By construction,

$$\left(B^s_\lambda\right)^2 \geq \lambda^2$$  

(5.16)
on $M \setminus K$. On the set $K'$, we can obtain an estimate as follows. Letting $\nabla^s$ be the connection used to define $D^s$, we have
\[
(B^s_\lambda)^2 = (D^s)^2 + \{D^s, \lambda \Phi\} + \Phi^2
= \nabla^s \nabla^s + \frac{\kappa}{4} + is\lambda c(\omega) + \{D^s, \lambda \Phi\} + \lambda^2 \Phi^2
\geq \frac{\kappa}{4} + is\lambda c(\omega) + \{D^s, \lambda \Phi\} + \lambda^2 \Phi^2.
\] (5.17)
Let $\kappa_0 = \inf_{x \in K} \kappa(x) > 0$ by cocompactness of $K$. Then
- on $K$: there exist $s_0, \lambda_0 > 0$ such for all $s < s_0$ and $\lambda < \lambda_0$, the endomorphism (5.17) is bounded below by $\frac{\kappa_0}{8}$;
- on $K' \setminus K$: since $\kappa \geq 0$ and $\{D^s, \lambda \Phi\} = 0$, the endomorphism (5.17) is bounded below by $is\lambda c(\omega) + \lambda^2 \Phi^2$. By cocompactness of $K'$, there exist $s_1, \lambda_1$ such that $is\lambda c(\omega) + \lambda^2 \Phi^2 \geq \frac{\kappa_0}{8}$ for all $s < s_1$ and $\lambda < \lambda_1$.
Combining this with (5.16), we see that the estimate
\[
(B^s_\lambda)^2 \geq \frac{\kappa_0}{8}
\]
holds on both $K'$ and $M \setminus K$ for all $s < \inf\{s_0, s_1\}$ and $\lambda < \inf\{\lambda_0, \lambda_1, \frac{\kappa_0}{8}\}$. To combine these these estimates, let $\phi_1, \phi_2$ smooth functions $M \to [0, 1]$ such that
- $\{\phi_1^2, \phi_2^2\}$ is a partition of unity on $M$;
- $\text{supp}(\phi_1) \subseteq K'$ and $\text{supp}(\phi_2) \subseteq M \setminus K$.
For any $\epsilon > 0$, we may take $K'$ to be sufficiently large so that $\|d\phi_1\|_\infty, \|d\phi_2\|_\infty < \epsilon$. Since $B^s$ is a perturbation of $\partial$ by an endomorphism,
\[
[\phi_i, B^s] = [\phi_i, \partial] = c(d\phi_i),
\] (5.18)
for $i = 1, 2$, hence $\phi_i B^s_\lambda = B^s_\lambda \phi_i + [\phi_i, \partial]$. For any $u \in L^2(S\varphi)$, one computes that
\[
\langle \phi_i B^s_\lambda u, \phi_i B^s_\lambda u \rangle = \langle [\phi_i, \partial] u, \phi_i B^s_\lambda u \rangle + \langle B^s_\lambda \phi_i u, \phi_i B^s_\lambda u \rangle
= \langle [\phi_i, \partial] u, \phi_i B^s_\lambda u \rangle + \langle \phi_i B^s_\lambda u, [\phi_i, \partial] u \rangle
+ \langle B^s_\lambda \phi_i u, B^s_\lambda \phi_i u \rangle
\geq \|B^s_\lambda \phi_i u\|^2 - 2\epsilon \|B^s_\lambda u\| \|u\| - \epsilon^2 \|u\|^2,
\]
where the norms and inner products are taken in $L^2(S\varphi)$ and we have used that $\|\phi_i\|_\infty \leq 1$. It follows that
\[
\langle (B^s_\lambda)^2 u, u \rangle = \langle \phi_1^2 B^s_\lambda u, B^s_\lambda u \rangle + \langle \phi_2^2 B^s_\lambda u, B^s_\lambda u \rangle
\geq 2(\|B^s_\lambda \phi_i u\|^2 - 2\epsilon \|B^s_\lambda u\| \|u\| - \epsilon^2 \|u\|^2)
\geq (\frac{\kappa_0}{4} - \epsilon^2) \|u\|^2 - 4\epsilon \|B^s_\lambda u\| \|u\|.
\]
Hence $(\|B^s_\lambda u\| + 2\epsilon \|u\|)^2 \geq (\frac{\kappa_0}{4} + 3\epsilon^2) \|u\|^2$, so that
\[
\|B^s_\lambda u\| \geq \left(\sqrt{\frac{\kappa_0}{4} + 3\epsilon^2} - 2\epsilon\right) \|u\|.
\]
Taking $\epsilon$ small enough, and hence $K'$ large enough, we see that $(B^s_\lambda)^2$ is strictly positive. Thus $B^s_\lambda$ is invertible for $\lambda < \inf\{\lambda_0, \lambda_1, \frac{\kappa_0}{8}\}$ and $s < \inf\{s_0, s_1\}$, whence $\text{Ind}_{\Gamma \sigma}(B^s_\lambda) = 0$. \hfill \square
When $M/\Gamma$ is non-compact, we can use quantitative $K$-theory to give obstructions to the existence of $\Gamma$-invariant metrics of positive scalar curvature on $M$. This uses the fact that the twisted Roe algebra is naturally filtered by propagation, making it an example of a geometric C*-algebra. We now review these concepts.

6.1. Geometric C*-algebras and quantitative K-theory.

Definition 6.1. A unital C*-algebra $A$ is geometric if it admits a filtration $\{A_r\}_{r>0}$ satisfying the following properties:

(i) $A_r \subseteq A_{r'}$ if $r \leq r'$;
(ii) $A_r A_{r'} \subseteq A_{r+r'}$;
(iii) $\bigcup_{r=0}^{\infty} A_r$ is dense in $A$.

If $A$ is non-unital, then its unitization $A^+$, viewed as $A \oplus \mathbb{C}$ as a vector space, is a geometric C*-algebra with filtration $\{A_r \oplus \mathbb{C}\}_{r>0}$. In addition, for each $n$, the matrix algebra $M_n(A)$ is a geometric C*-algebra with filtration $\{M_n(A_r)\}_{r>0}$.

Definition 6.2 ([4, Definition 2.15]). Let $A$ be a geometric C*-algebra. For $0 < \varepsilon < \frac{1}{20}$, $r > 0$, and $N \geq 1$,

- an element $e \in A$ is called an $(\varepsilon, r, N)$-quasiidempotent if
  \[ \|e^2 - e\| < \varepsilon, \quad e \in A_r, \quad \max(\|e\|, \|1_A - e\|) \leq N; \]
- if $A$ is unital, an element $u \in A$ is called an $(\varepsilon, r, N)$-quasiinvertible if $u \in A_r$, $\|u\| \leq N$, and there exists $v \in A_r$ with
  \[ \|v\| \leq N, \quad \max(\|uv - 1\|, \|vu - 1\|) < \varepsilon. \]

The pair $(u, v)$ is called an $(\varepsilon, r, N)$-quasinverse pair.

The quantitative $K$-groups $K^r_{0, \varepsilon} \geq (A)$ and $K_{1, \varepsilon}(A)$ are defined by collecting together all quasiidempotents and quasiinvertibles over all matrix algebras, quotienting by an equivalence relation, and taking the Grothendieck completion.

Definition 6.3 ([4, subsection 3.1]). Let $A$ be a unital geometric C*-algebra. Let $r > 0$, $0 < \varepsilon < \frac{1}{20}$, and $N > 0$.

(i) Denote by $\text{Idem}^r_{\varepsilon,r,N}(A)$ the set of $(\varepsilon, r, N)$-quasiidempotents in $A$. For each positive integer $n$, let
  \[ \text{Idem}^r_{\varepsilon,r,N}(A) = \text{Idem}^r_{\varepsilon,r,N}(M_n(A)). \]

We have inclusions $\text{Idem}^r_{\varepsilon,r,N}(A) \hookrightarrow \text{Idem}^r_{\varepsilon,r,N}(A)$ given by $e \mapsto \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}$. Set
  \[ \text{Idem}^{\infty}_{\varepsilon,r,N}(A) = \bigcup_{n=1}^{\infty} \text{Idem}_{\varepsilon,r,N}(A). \]

Define an equivalence relation $\sim$ on $\text{Idem}^{\infty}_{\varepsilon,r,N}(A)$ by $e \sim f$ if $e$ and $f$ are $(4\varepsilon, r, 4N)$-homotopic in $M_\infty(A)$. Denote the equivalence class of an element $e \in \text{Idem}^{\varepsilon}_{\varepsilon,r,N}(A)$ by $[e]$. Define addition on $\text{Idem}^{\varepsilon}_{\varepsilon,r,N}(A)/\sim$ by
  \[ [e] + [f] = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}. \]

With this operation, $\text{Idem}^{\varepsilon}_{\varepsilon,r,N}(A)/\sim$ is an abelian monoid with identity $[0]$. Let $K^{r}_{0, \varepsilon} \geq (A)$ denote its Grothendieck completion.
(ii) Denote by $GL^{\varepsilon,r,N}(A)$ the set of $(\varepsilon, r, N)$-quasiinvertibles in $A$. For each positive integer $n$, let

$$GL^{\varepsilon,r,N}_n(A) = GL^{\varepsilon,r,N}(M_n(A)).$$

We have inclusions $GL^{\varepsilon,r,N}_n(A) \hookrightarrow GL^{\varepsilon,r,N}_{n+1}(A)$ given by $u \mapsto \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$. Set

$$GL^{\varepsilon,r,N}_\infty(A) = \bigcup_{n=1}^\infty GL^{\varepsilon,r,N}_n(A).$$

Define an equivalence relation $\sim$ on $GL^{\varepsilon,r,N}_\infty(A)$ by $e \sim f$ if $u$ and $v$ are $(4\varepsilon, 2r, 4N)$-homotopic in $M_\infty(A)$. Denote the equivalence class of an element $u \in GL^{\varepsilon,r,N}_\infty(A)$ by $[u]$. Define addition on $GL^{\varepsilon,r,N}_\infty(A)/\sim$ by

$$[u] + [v] = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}.$$

With this operation, $GL^{\varepsilon,r,N}_\infty(A)/\sim$ is an abelian group identity $[1]$.

**Remark 6.4.** If $A$ is a non-unital geometric $C^*$-algebra, then we have a canonical $*$-homomorphism $\pi: A^+ \to \mathbb{C}$. Using contractivity of $\pi$, we have homomorphisms

$$\pi_*: K^{\varepsilon,r,N}_i(A^+) \to K^{\varepsilon,r,N}_i(\mathbb{C}),$$

where $i = 0$ or $1$. Define $K^{\varepsilon,r,N}_i(A) = \ker(\pi_*)$.

The following result on quasiidempotents and quasiinvertibles is useful.

**Lemma 6.5 ([10, Lemma 3.4]).** Let $A$ be a geometric $C^*$-algebra. If $e$ is an $(\varepsilon, r, N)$-idempotent in $A$, and $f \in A_r$ satisfies

$$\|f\| \leq N, \quad \|e - f\| < \frac{\varepsilon - \|e^2 - e\|}{2N + 1},$$

then $f$ is a quasiidempotent that is $(\varepsilon, r, N)$-homotopic to $e$. In particular, if

$$\|f\| < \frac{\varepsilon}{2N + 1},$$

then the class of $f$ is zero in $K^{\varepsilon,r,N}_0(A)$.

Suppose that $A$ is unital and $(u, v)$ is an $(\varepsilon, r, N)$-quasiinverse pair in $A$. If $a \in A_r$ satisfies

$$\|a\| \leq N, \quad \|u - a\| < \frac{\varepsilon - \max(||uv - 1||, ||vu - 1||)}{N},$$

then $a$ is a quasiinvertible that is $(\varepsilon, r, N)$-homotopic to $u$. In particular, if

$$\|1 - a\| < \frac{\varepsilon}{N},$$

then the class of $a$ is zero in $K^{\varepsilon,r,N}_1(A)$.

There is a homomorphism of abelian groups

$$\Psi: K^{\varepsilon,r,N}_*(A) \to K_*(A),$$

preserving the $\mathbb{Z}_2$-grading, where $K^{\varepsilon,r,N}_*(A)$ (resp. $K_*(A)$) denotes the direct sum of the quantitative (resp. operator) $K_0$ and $K_1$-groups.
6.2. The quantitative higher index.

Fix $0 < \varepsilon < \frac{1}{20}$ and $N \geq 7$. For each $s \in \mathbb{R}$, let the algebras $\mathbb{C}[M; L^2(S_{\mathcal{F}})]^{\Gamma,\sigma^s}$ and $C^*(M; L^2(S_{\mathcal{F}}))^{\Gamma,\sigma^s}$ be as in Definition 3.25, with $\sigma$ and the projective representation $T$ replaced by $\sigma^s$ and $T^s$, as in Definition 2.4.

For each $r > 0$, define the $\ast$-subalgebra
\[
\mathbb{C}[M; L^2(S_{\mathcal{F}})]^{\Gamma,\sigma^s}_r := \{ T \in \mathbb{C}[M; L^2(S_{\mathcal{F}})]^{\Gamma,\sigma^s} : \text{prop}(T) \leq r \}
\] (6.2)
of $C^*(M; L^2(S_{\mathcal{F}}))^{\Gamma,\sigma^s}$. Then with respect to the filtration
\[
\{ \mathbb{C}[M; L^2(S_{\mathcal{F}})]^{\Gamma,\sigma^s}_r \}_{r > 0},
\] $C^*(M; L^2(S_{\mathcal{F}}))^{\Gamma,\sigma^s}$ is a geometric $C^*$-algebra in the sense of Definition 6.1. As in [10], this structure allows us to define a refinement of the higher index that takes values in the quantitative $\ast$-invariant higher index $\text{Ind}^{\sigma^s}(D)$ given in Definition 3.30 can be represented explicitly as follows. Let $\chi$ be a normalising function. If dim $M$ is even, define the idempotent
\[
p_{\chi}(D) = \begin{pmatrix}
[(1 - \chi(D))^2]_{1,1} & [\chi(D)(1 - \chi(D)^2)]_{1,2} \\
[\chi(D)(2 - \chi(D)^2)(1 - \chi(D)^2)]_{2,1} & [\chi(D)^2(2 - \chi(D)^2)]_{2,2}
\end{pmatrix},
\] (6.3)
where the notation $[X]_{i,j}$ means the $(i, j)$-th entry of the matrix $X$. Then $\text{Ind}^{\sigma^s}(D)$ is represented by the difference of idempotents
\[
A_{\chi}(D) = p_{\chi}(D) - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\] (6.4)

For dim $M$ odd, $\text{Ind}^{\sigma^s}(D)$ can be represented by the unitary
\[
A_{\chi}(D) = e^{\pi i(\chi + 1)}(D).
\] (6.5)

Even-dimensional $M$.

Choose a normalizing function $\chi$ such that
\[
\text{supp} \tilde{\chi} \subseteq \left[ -\frac{r}{5}, \frac{r}{5} \right].
\] (6.6)

Let $A_{\chi}(D^s) = p_{\chi}(D^s) - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ as in (6.4). Then the $(\varepsilon, r, N)$-quantitative maximal higher index of $D$ is the class
\[
\text{Ind}_{\Gamma,\sigma^s,L^2}^{\varepsilon,r,N}(D^s) = [p_{\chi}(D^s)] - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in K_0^{\varepsilon,r,N}(C^*(M; L^2(S_{\mathcal{F}}))^{\Gamma,\sigma^s}).
\]

Odd-dimensional $M$.

For each integer $n \geq 0$ define polynomials
\[
f_n(x) = \sum_{k=0}^{n} \frac{(2\pi ix)^k}{k!}, \quad g_n(x) = f_n(x) - \left( \sum_{k=1}^{n} \frac{(2\pi i)^k}{k!} \right) x^2.
\] (6.7)

One finds that as $n \to \infty$, the difference $e^{2\pi ix} - g_n(x)$ converges uniformly to 0 for $x$ in the interval $[-2, 2]$. Let $m = m(\varepsilon, N)$ be the smallest number such that
\[
|g_m(x)g_m(-x) - 1| < \varepsilon,
\]
\[
|e^{2\pi ix} - g_m(x)| < 1,
\] (6.8)
for all $x \in [-2, 2]$. Pick a normalizing function $\chi$ satisfying
\[
\text{supp } \hat{\chi} \subseteq \left[ -\frac{r}{\deg g_m}, \frac{r}{\deg g_m} \right]. \tag{6.9}
\]

Then the operator
\[
S_\chi = \frac{\chi(D^s) + 1}{2}
\]
has propagation at most $\frac{r}{\deg g_m}$ and spectrum contained in $[-\frac{1}{2}, \frac{3}{2}]$. Then $g_m(S_\chi(D^s))$ is an $(\varepsilon, r, N)$-quasiinertial in the unitisation of $C^*(M; L^2(S_\mathcal{F}))^{\Gamma, \sigma^s}$. It was shown in [10, subsection 3.2.2] that
\[
\text{Ind}_{\Gamma, \sigma^s}(D^s) = [A_\chi(D^s)] = [g_m(S_\chi)] \in K_1(C^*(M; L^2(S_\mathcal{F}))^{\Gamma, \sigma^s}).
\]

The $(\varepsilon, r, N)$-quantitative higher index of $D^s$ is the class
\[
\text{Ind}_{\Gamma, \sigma^s}^{\varepsilon, r, N}(D^s) = [g_m(S_\chi)] \in K_1(C^*(M; L^2(S_\mathcal{F}))^{\Gamma, \sigma^s}).
\]

**Remark 6.6.** First, although in the above constructions we needed to make a choice of $\chi$, the quantitative higher index obtained is independent of this choice.

Second, the $(\Gamma, \sigma)$-higher index of $D^s$ relates to its quantitative refinement by $\text{Ind}_{\Gamma}(D^s) = \Psi(\text{Ind}_{\Gamma}(D^s))$, where $\Psi$ is the homomorphism from (6.1).

### 6.3. A quantitative obstruction

We now prove Theorem 1.7. This uses the construction of the twisted higher index from subsection 2.3 in terms of twisted Roe algebras, which are geometric $C^*$-algebras in the sense of [18]. The result we obtain generalises in [10, Theorem 1.1].

**Proof.** The technique of the proof is as in [10, section 4]. The differences are that we now work with the reduced rather than the maximal version of the twisted Roe algebra, and that bounds on $\kappa$ used in that paper are now replaced by bounds on the endomorphism $\kappa + 4isc(\omega)$. By Lemma 4.7, we have
\[
(D^s)^2 = \nabla^{ss^s} + \frac{\kappa}{4} + isc(\omega).
\]

Suppose that $\kappa + 4isc(\omega) \geq C_s$ holds as an estimate on operators on $L^2(S_\mathcal{F})$. Let $\chi$ be a normalizing function whose distributional Fourier transform $\hat{\chi}$ is supported on some finite interval $[-s, s]$ for $s > 0$. For each $t > 0$, let $\chi_t$ be the normalizing function defined by
\[
\chi_t(u) = \chi(tu), \tag{6.10}
\]

$u \in \mathbb{R}$. Let $A_\chi(D^s)$ be the index representative defined using $\chi$.

If $M$ is even-dimensional, let
\[
A_\chi(u) := \begin{pmatrix}
(1 - \chi(u)^2) & \chi(t)(1 - \chi(u)^2) \\
\chi(u)(2 - \chi(u)^2)(1 - \chi(u)^2) & \chi(u)^2(2 - \chi(u)^2) - 1
\end{pmatrix}, \quad u \in \mathbb{R}.
\]

Let $u_0 > 0$ and a function $\alpha$ be such that
\[
\|A_\chi(u)\| < \frac{\varepsilon}{2N + 1} \tag{6.11}
\]
whenever $|1 - \chi(u)^2| < \alpha(\varepsilon)$ for all $u$ such that $|u| > u_0$, where the norm of $A_\chi(u)$ is taken in $M_2(\mathbb{C})$. Note that for $N \geq 7$, (6.11) also implies that $\|A_\chi^2(u) - A_\chi(u)\| < \varepsilon$ if $|u| > u_0$. By (6.10), we have
\[
\left| 1 - \chi \frac{2u_0}{\sqrt{\varepsilon}} \right| = \left| 1 - \chi \left( \frac{2u_0}{\sqrt{\varepsilon}} \right)^2 \right| < \alpha(\varepsilon) \tag{6.12}
\]
whenever \( u \in \mathbb{R} \setminus (-\sqrt{\frac{\sqrt{s}}{C} }, \sqrt{\frac{\sqrt{s}}{C} }) \), while

\[
\text{supp} \left( \chi_{\frac{2u_0}{\sqrt{C}}}(D^s) \right) \subseteq \left[ -\frac{2u_0}{\sqrt{C}} s, \frac{2u_0}{\sqrt{C}} s \right].
\]

It follows that \( A_{\chi_{\frac{2u_0}{\sqrt{C}}}}(D^s) \) is an \((\varepsilon, \frac{10u_0}{\sqrt{C}} s, N)\)-quasidiempotent in \( 2 \times 2 \)-matrices over the unitisation of \( C^*(M; L^2(S_\mathcal{X}))^{\Gamma, \sigma^s} \) with norm strictly less than \( \frac{2}{2N+1} \). By Lemma 6.5,

\[
\text{Ind}^\varepsilon_{\Gamma, \sigma^s, L^2}(D^s) = 0 \in K^\varepsilon_{10u_0 \sqrt{C}} s, N(C^*(M; L^2(S_\mathcal{X})))^{\Gamma, \sigma^s}.
\]

Letting \( \lambda_0 = 10u_0 s \), we obtain \( \text{Ind}^\varepsilon_{\Gamma, \sigma^s, L^2}(D^s) = 0 \). Note that for any \( r \geq \frac{\lambda_0}{\sqrt{C}} \), \( \text{Ind}^\varepsilon_{\Gamma, \sigma^s, L^2}(D^s) \) can also be represented by \( A_{\chi_{\frac{2u_0}{\sqrt{C}}}}(D^s) \). The homomorphism

\[
K^\varepsilon_{\lambda_0 \sqrt{C}} (C^*(M; L^2(S_\mathcal{X})))^{\Gamma, \sigma^s} \rightarrow K^\varepsilon_{\lambda_0 \sqrt{C}} (C^*(M; L^2(S_\mathcal{X})))^{\Gamma, \sigma^s}
\]

induced by the inclusion

\[
\text{Idem}^\varepsilon_{\lambda_0 \sqrt{C}} (C^*(M; L^2(S_\mathcal{X})))^{\Gamma, \sigma^s} \rightarrow \text{Idem}^\varepsilon_{\lambda_0 \sqrt{C}} (C^*(M; L^2(S_\mathcal{X})))^{\Gamma, \sigma^s}
\]

takes \( \text{Ind}^\varepsilon_{\Gamma, \sigma^s, L^2}(D^s) \) to \( \text{Ind}^\varepsilon_{\Gamma, \sigma^s, L^2}(D^s) \), which therefore vanishes.

If \( M \) is odd-dimensional, let \( m = m(\varepsilon, N) \) and the polynomial \( g_m \) be as (6.8). Let \( \chi \) be a normalizing function satisfying (6.9), and let \( s = \frac{r}{\deg g_m} \). Let \( u_0 > 0 \) be such that

\[
\| 1 - g_m(P_{\chi}(u)) \| < \frac{\varepsilon}{N}
\]

whenever \( |1 - \chi(u)^2| < \alpha(\varepsilon) \) holds for all \( u \) such that \( |u| > u_0 \) or, equivalently, whenever

\[
|1 - \chi_{\frac{2u_0}{\sqrt{C}}}(u)^2| = |1 - \chi \left( \frac{2u_0 u}{\sqrt{C}} \right)^2| < \alpha(\varepsilon)
\]

for all \( u \in \mathbb{R} \setminus (-\sqrt{\frac{\sqrt{s}}{C} }, \sqrt{\frac{\sqrt{s}}{C} }) \). Meanwhile,

\[
\text{supp} \left( \chi_{\frac{2u_0}{\sqrt{C}}}(D^s) \right) \subseteq \left[ -\frac{2u_0}{\sqrt{C}} s, \frac{2u_0}{\sqrt{C}} s \right].
\]

Thus \( g_m(P_{\chi_{\frac{2u_0}{\sqrt{C}}}}(D^s)) \) is an \((\varepsilon, \frac{2mu_0 s}{\sqrt{C}} s, N)\)-quasidiempotent in \( 2 \times 2 \)-matrices over the unitisation of \( C^*(M; L^2(S_\mathcal{X})))^{\Gamma, \sigma^s} \) satisfying

\[
\| 1 - g_m(P_{\chi_{\frac{2u_0}{\sqrt{C}}}}(D^s)) \| < \frac{\varepsilon}{N}.
\]

By Lemma 6.5,

\[
\text{Ind}^\varepsilon_{\Gamma, \sigma^s, L^2}(D^s) = 0 \in K^\varepsilon_{\lambda_0 \sqrt{C}} s, N(C^*(M; L^2(S_\mathcal{X})))^{\Gamma, \sigma^s}.
\]

Letting \( \lambda_0 = 2mu_0 s \), we obtain \( \text{Ind}^\varepsilon_{\Gamma, \sigma^s, L^2}(D^s) = 0 \). For any \( r \geq \frac{\lambda_0}{\sqrt{C}} \), the element \( \text{Ind}^\varepsilon_{\Gamma, \sigma^s, L^2}(D^s) \) can also be represented by \( g_m(P_{\chi_{\frac{2u_0}{\sqrt{C}}}}(D^s)) \). The homomorphism

\[
K^\varepsilon_{\lambda_0 \sqrt{C}} (C^*(M; L^2(S_\mathcal{X})))^{\Gamma, \sigma^s} \rightarrow K^\varepsilon_{\lambda_0 \sqrt{C}} (C^*(M; L^2(S_\mathcal{X})))^{\Gamma, \sigma^s}
\]
induced by the inclusion

\[ GL_{\infty}^{\epsilon, r} \rightarrow GL_{\infty}^{\epsilon, r, N} \]

\[ ((C^* (M; L^2 (S_L)))^{\Gamma, \sigma_s})^+ \]

\[ \rightarrow GL_{\infty}^{\epsilon, r, N} \]

\[ ((C^* (M; L^2 (S_L)))^{\Gamma, \sigma_s})^+ \]

takes \( \text{Ind}^{\epsilon, r, N} \) to \( \text{Ind}^{\epsilon, r, N} \), which therefore vanishes.

\[ \square \]

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