SYMmetric BRACKETS ON SKEW-SYMMETRIC ALGEBROIDS

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Abstract. In this note, we discuss the concept of symmetric brackets on skew-symmetric algebroids. Given a pseudo-Riemannian metric structure, we describe symmetric brackets induced by connections with totally skew-symmetric torsions in the language of Lie derivatives and differentials of functions. In particular, we obtain an explicit formula of the Levi-Civita connection. We present some symmetric brackets on almost Hermitian manifolds. In particular, we discuss the first canonical Hermitian connection and show the formula for this connection in the case of nearly Kähler manifold using properties of symmetric brackets.

1. Introduction

An anchored vector bundle $(A, \rho_A)$ over a manifold $M$ is a vector bundle $A$ over $M$ equipped with a homomorphism of vector bundles $\rho_A : A \rightarrow TM$ over the identity, which is called an anchor (cf. [13], [14]). If, additionally, in the space $\Gamma(A)$ of smooth sections of $A$ we have $\mathbb{R}$-bilinear skew-symmetric mapping $[\cdot, \cdot] : \Gamma(A) \times \Gamma(A) \rightarrow \Gamma(A)$ associated with the anchor with the following derivation law

\[(1.1) \quad [X, f \cdot Y] = f \cdot [X, Y] + (\rho_A \circ X)(f) \cdot Y \]

for $X, Y \in \Gamma(A)$, $f \in C^\infty(M)$, we say that $(A, \rho_A, [\cdot, \cdot])$ is a skew-symmetric algebroid over $M$. The terminology of skew-symmetric algebroid was introduced in [8]. If the anchor preserves $[\cdot, \cdot]$ and the Lie bracket $[\cdot, \cdot]_{TM}$ of vector fields on $M$, i.e. $\rho_A \circ [X, Y] = [\rho_A \circ X, \rho_A \circ Y]_{TM}$ for $X, Y \in \Gamma(A)$, a skew-symmetric algebroid is an almost Lie algebroid [15]. Any skew-symmetric algebroid in which $[\cdot, \cdot]$ satisfies the Jacobi identity is a Lie algebroid in the sense of Pradines [16]. However, we recall that the Leibniz type rule (1.1) and the Jacobi identity imply that the anchor preserves the brackets. For general theory of Lie algebroids, we refer to the monographs [11], [12].

Given an anchored vector bundle, we can associate a connection. Given a skew-symmetric algebroid, we can associate connection with a torsion.

An $A$-connection in a vector bundle $E \rightarrow M$ is an $\mathbb{R}$-bilinear map $\nabla : \Gamma(A) \times \Gamma(E) \rightarrow \Gamma(E)$ with the following properties:

\[
\begin{align*}
\nabla_{X+Y}(u) &= \nabla_X(u) + \nabla_Y(u), \\
\nabla_{X}(u+w) &= \nabla_X(u) + \nabla_X(w), \\
\nabla_{f \cdot X}(u) &= f \cdot \nabla_X(u), \\
\nabla_X(f \cdot u) &= f \cdot \nabla_X(u) + (\rho_A \circ X)(f) \cdot u
\end{align*}
\]

for any $X, Y \in \Gamma(A)$, $u, w \in \Gamma(E)$, $f \in C^\infty(M)$.

The torsion of an $A$-connection $\nabla$ in $A$ is the tensor $T^\nabla \in \Gamma(\wedge^2 A^* \otimes A)$ defined by

\[T^\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]\]

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for $X, Y \in \Gamma(A)$. We say that an $A$-connection is torsion-free if its torsion equals zero. Given an $A$-connection $\nabla$ in $A$, we associate a symmetric $\mathbb{R}$-bilinear mapping $\langle \cdot : \cdot \rangle : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$, defined for $X, Y \in \Gamma(A)$, by

$$\langle X : Y \rangle = \nabla X Y + \nabla Y X.$$  

We say that this mapping is a symmetric product induced by $\nabla$. The symmetric product in the case of tangent bundles was introduced by Crouch [5]. Observe that $\langle \cdot : \cdot \rangle$ satisfies the Leibniz rules. In this paper, we discuss about symmetric brackets defined as real bilinear symmetric mappings $\langle \cdot : \cdot \rangle : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ such that $\langle X : f Y \rangle = f \langle X : Y \rangle + (\varrho_A \circ X)(f)Y$ for $X, Y \in \Gamma(A)$. The identity (1.2) shows now that there is one-to-one correspondence between symmetric brackets on a given skew-symmetric algebroid $(A, \varrho_A, [\cdot, \cdot])$ and torsion-free $A$-connections in this algebroid.

Our first purpose is to determine the symmetric products for connections related to the (pseudo)metric structure. We give an explicit formula for a metric connection with totally skew-symmetric torsion, in particular, for the Levi-Civita connection. To describe these symmetric brackets, we use the Lie derivative and the exterior derivative with the metric is that the (alternating) Lie derivative of the metric is equal to the minus counter parts. We show that the condition for connections with torsions to be compatible with the metric is that the (alternating) Lie derivative of the metric is equal to the minus of the symmetric Lie derivative of the metric.

We also consider almost Hermitian structure and define some symmetric brackets associated with connections that are compatible with the metric structure and the almost complex structure. We consider two structures of the skew-symmetric algebroid in almost Hermitian manifold $(M, g, J)$. The first structure is the tangent bundle with the identity as an anchor and with Lie bracket of vector fields. The second skew-symmetric algebroid structure is induced by the almost complex structure $J$, where $J$ is the anchor and the bracket is associated with the Nijenhuis tensor.

## 2. Skew-symmetric and symmetric derivative operators

Let $(A, \varrho_A, [\cdot, \cdot])$ be a skew-symmetric algebroid over a manifold $M$. The substitution operator

$$i_X : \Gamma \left( \bigotimes^k A^* \right) \to \Gamma \left( \bigotimes^{k-1} A^* \right),$$

for $X \in \Gamma(A)$, is defined by

$$(i_X \zeta)(X_1, \ldots, X_{k-1}) = \zeta(X, X_1, \ldots, X_{k-1})$$

for $\zeta \in \Gamma(\bigotimes^k A^*)$, $X, X_1, \ldots, X_k \in \Gamma(A)$.

The (alternating) Lie derivative $\mathcal{L}_X^\omega : \Gamma(\bigotimes^k A^*) \to \Gamma(\bigotimes^k A^*)$ for $X \in \Gamma(A)$ is defined by

$$\mathcal{L}_X^\omega \Omega (X_1, \ldots, X_k) = (\varrho_A \circ X)(\Omega(X_1, \ldots, X_k)) - \sum_{i=1}^k \Omega(X_1, \ldots, [X, X_i], \ldots, X_k),$$

for $\Omega \in \Gamma(\bigotimes^k A^*)$, $X_1, \ldots, X_k \in \Gamma(A)$. Notice that $\mathcal{L}_X^\omega (\eta) \in \Gamma(\bigwedge^k A^*)$ if $\eta \in \Gamma(\bigwedge^k A^*)$.

Moreover, let

$$\nabla : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$$
Lemma 2.1. For any $A$-connection in $A$. We define in $A$-connection in a classical way the dual bundle by the following formula

$$(\nabla_X \omega)Y = \rho_A(X)(\omega(Y)) - \omega(\nabla_X Y)$$

for $\omega \in \Gamma(A^*)$, $X, Y \in \Gamma(A)$. And next, by the Leibniz rule, we extend this connection to the $A$-connection $\nabla$ in the whole tensor bundle $\bigotimes A^*$, which will also be denoted by $\nabla$. Then for $\zeta \in \Gamma\left(\bigotimes^k A^*\right)$, $X, X_1, \ldots, X_k \in \Gamma(A)$,

$$(\nabla_X \zeta)(X_1, \ldots, X_k) = \rho_A(X)(\zeta(X_1, \ldots, X_k)) - \sum_{j=1}^k \zeta(X_1, \ldots, \nabla_X X_j, \ldots, X_k).$$

Now, we define the operator

$$\nabla : \Gamma\left(\bigotimes^k A^*\right) \to \Gamma\left(\bigotimes^{k+1} A^*\right)$$

by

$$(\nabla \zeta)(X_1, X_2, \ldots, X_{k+1}) = (\nabla_{X_1} \zeta)(X_2, \ldots, X_{k+1}).$$

We recall that the exterior derivative operator on the skew-symmetric algebroid $(A, \rho_A, [\cdot, \cdot])$ is defined by

$$(d^\sigma \eta)(X_1, \ldots, X_{k+1}) = \sum_{j=1}^{k+1} (-1)^{j+1} (\rho_A \circ X_j)\left(\eta(X_1, \ldots, \hat{X}_j, \ldots, X_{k+1})\right)$$

$$+ \sum_{i<j} (-1)^{i+j} \eta([X_i, X_j], X_1, \ldots, \hat{X}_i \ldots \hat{X}_j, \ldots, X_{k+1})$$

for $\eta \in \Gamma(\wedge^k A^*)$, $X_1, \ldots, X_k \in \Gamma(A)$. If the bracket $[\cdot, \cdot]$ satisfies the Jacobi identity, $d^\sigma \circ d^\sigma = 0$. If $\nabla$ be torsion-free for $A$-connection in $A$, then $d^\sigma$ can be written as the alternation of the operator $\nabla$ (cf. [2]), i.e.,

$$d^\sigma = (k + 1) \cdot (\text{Alt} \circ \nabla) \quad \text{on} \quad \Gamma(\bigotimes^k A^*),$$

where Alt is the alternator given by

$$(\text{Alt} \zeta)(X_1, \ldots, X_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn} \sigma \, \zeta(X_{\sigma(1)}, \ldots, X_{\sigma(k)}) \quad \text{for} \quad \zeta \in \Gamma(\bigotimes^k A^*);$$

equivalently,

$$(d^\sigma \eta)(X_1, \ldots, X_{k+1}) = \sum_{j=1}^{k+1} (-1)^{j+1} (\nabla_{X_j} \eta)(X_1, \ldots, \hat{X}_j, \ldots, X_{k+1})$$

for $\eta \in \Gamma(\wedge^k A^*)$, $X_1, \ldots, X_k \in \Gamma(A)$.

Here, we recall the classical Cartan’s formulas:

**Lemma 2.1.** For any $X, Y \in \Gamma(A)$,

(a) $L^X_Y = i_X d^\sigma + d^\sigma i_X$,

(b) $L^X_Y i_Y - i_Y L^X_Y = i_{[X, Y]}$.

Now, let $S^k A^*$ denote the $k$-th symmetric power of the bundle $A^*$ and $S(A) = \bigoplus_{k \geq 0} S^k A^*$. In the theory of generalized gradients in the sense of Stein-Weiss (cf. [2]) appears in a natural way the symmetric derivative operator

$$d^\sigma : \Gamma(S^k A^*) \to \Gamma(S^{k+1} A^*)$$
being the symmetrization of $\nabla$ up to a constant on the symmetric power bundle,

$$d^s = (k + 1) \cdot (\text{Sym} \circ \nabla) \quad \text{on} \quad \Gamma(S^k A^*)$$

where $\text{Sym}$ is the symmetrizer defined by

$$(\text{Sym} \zeta) (X_1, \ldots, X_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \zeta(X_{\sigma(1)}, \ldots, X_{\sigma(k)}) \quad \text{for} \quad \zeta \in \Gamma(\bigotimes^k A^*).$$

Equivalently,

$$(2.1) \quad (d^s \eta)(X_1, \ldots, X_{k+1}) = \sum_{j=1}^{k+1} \left( \nabla_{X_j} \eta \right) \left( X_1, \ldots, \hat{X}_j, \ldots, X_{k+1} \right)$$

for $\eta \in \Gamma(S^k A^*)$, $X_1, \ldots, X_{k+1} \in \Gamma(A)$. The operator $d^s$ in the case of tangent bundles was first introduced by Sampson in [17], where a symmetric version of Chern’s theorem is proved. Next, this operator on tangent bundles was discussed in [7], where a Frölicher-Nijenhuis bracket for vector valued symmetric tensors is also discussed and in [3], where the Dirac type operator on symmetric tensors was considered.

One can check that for $\eta \in \Gamma(S^k A^*)$, $X_1, \ldots, X_{k+1} \in \Gamma(A)$ holds

$$(d^s \eta)(X_1, \ldots, X_{k+1}) = \sum_{j=1}^{k+1} \left( \rho_A \circ X_j \right) \left( \eta(X_1, \ldots, \hat{X}_j, \ldots, X_{k+1}) \right)$$

$$- \sum_{i<j} \eta \left( \langle X_i : X_j \rangle, X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1} \right),$$

where

$$\langle X : Y \rangle = \nabla_X Y + \nabla_Y X$$

for $X, Y \in \Gamma(A)$. So, $d^s$ can be described as the Koszul type formula. This shape of $d^s$ in the case $A = TM$ was considered in [7]. The symmetric $\mathbb{R}$-bilinear form

$$\langle \cdot : \cdot \rangle : \Gamma(A) \times \Gamma(A) \rightarrow \Gamma(A), \quad \langle X : Y \rangle = \nabla_X Y + \nabla_Y X$$

is called the symmetric product induced by the $A$-connection $\nabla$. The symmetric product in the case $A = TM$ was first introduced by Crouch in [5]. However, the symmetric product for Lie algebroids was first considered in the context of control systems by Cortés and Martínez in [4]. Observe that

$$\langle X : f \cdot Y \rangle = f \cdot \langle X : Y \rangle + (\varrho_A \circ X) (f) \cdot Y$$

for all $X, Y \in \Gamma(A)$ and $f \in C^\infty(M)$. Therefore, $\langle \cdot : \cdot \rangle$ satisfies the Leibniz kind rule. Lewis in [22] gives some interesting geometrical interpretation of the symmetric product associated with the geodesically invariant property of a distribution. We say that a smooth distribution $D$ on a manifold $M$ with an affine connection $\nabla^TM$ is geodesically invariant if for every geodesic $c : I \rightarrow M$ satisfying the property $c'(s) \in D_{c(s)}$ for some $s \in I$, we have $c'(s) \in D_{c(s)}$ for every $s \in I$. Lewis proved in [22] that a distribution $D$ on a manifold $M$ equipped with an affine connection $\nabla^TM$ is geodesically invariant if and only if the symmetric product induced by $\nabla^TM$ is closed under $D$.

3. Symmetric bracket. Symmetric Lie derivative

Let $(A, \varrho_A, [\cdot : \cdot])$ be a skew-symmetric algebroid over a manifold $M$. A symmetric bracket on the anchored vector bundle $(A, \varrho_A)$ is an $\mathbb{R}$-bilinear symmetric mapping

$$\langle \cdot : \cdot \rangle : \Gamma(A) \times \Gamma(A) \rightarrow \Gamma(A)$$

being the symmetrization of $\nabla$ up to a constant on the symmetric power bundle,
satisfying the following Leibniz kind rule:

\[ \langle X : fY \rangle = f \langle X : Y \rangle + (\varrho_A \circ X)(f)Y \]

for \( X, Y \in \Gamma(A) \), \( f \in C^\infty(M) \).

Let us assume that the skew-symmetric algebroid \((A, \varrho_A, [\cdot, \cdot])\) is equipped with a symmetric bracket \(\cdot : \Gamma(A) \times \Gamma(A) \to \Gamma(A)\).

The symmetric derivative operator \(d^s : \Gamma(S^kA^*) \to \Gamma(S^{k+1}A^*)\) on symmetric power bundle \(S(A)\) for \(\eta \in \Gamma(S^kA^*)\), \(X_1, \ldots, X_{k+1} \in \Gamma(A)\) is defined by

\[
(d^s\eta)(X_1, \ldots, X_{k+1}) = \sum_{j=1}^{k+1} (\rho_A \circ X_j)\left(\eta(X_1, \ldots, \widehat{X}_j, \ldots, X_{k+1})\right)
- \sum_{i<j} \eta(\langle X_i : X_j \rangle, X_1, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_{k+1}).
\]

Next, we extend this operator to the whole tensor algebra by the formula

\[
d^s : \Gamma(\bigotimes^k A^*) \to \Gamma(\bigotimes^{k+1} A^*),
\]

\[
(d^s\Omega)(X_1, \ldots, X_{k+1}) = \sum_{j=1}^{k+1} (\rho_A \circ X_j)\left(\Omega(X_1, \ldots, \widehat{X}_j, \ldots, X_{k+1})\right)
- \sum_{i<j} \Omega(X_1, \ldots, \widehat{X}_i, \ldots, \langle X_i : X_j \rangle, \ldots, X_{k+1}),
\]

for \(\Omega \in \Gamma(\bigotimes^k A^*)\), \(X_1, \ldots, X_k \in \Gamma(A)\).

The symmetric Lie derivative \(L_X^* : \Gamma(\bigotimes^k A^*) \to \Gamma(\bigotimes^k A^*)\) for \(X \in \Gamma(A)\), is defined by

\[
(L_X^*\Omega)(X_1, \ldots, X_k) = (\varrho_A \circ X)(\Omega(X_1, \ldots, X_k)) - \sum_{i=1}^k \Omega(X_1, \ldots, \langle X : X_i \rangle, \ldots, X_k)
for \(\Omega \in \Gamma(\bigotimes^k A^*)\), \(X_1, \ldots, X_k \in \Gamma(A)\). Notice that the image \(L_X^*(\varphi)\) of a symmetric tensor \(\varphi\) is also a symmetric tensor.

Next, observe that the symmetric Lie derivative satisfies the following Cartan’s identities.

**Lemma 3.1.** For any \(X, Y \in \Gamma(A)\),

(a) \(L_X = i_X d^s - d^s i_X\),

(b) \(L_X^* i_Y - i_Y L_X^* = i_{\langle X, Y \rangle}\).

**Proof.** Let \(X, Y, X_1, \ldots, X_k \in \Gamma(A)\) and \(\Omega \in \Gamma(\bigotimes^k A^*)\).

Observe that

\[
(i_X d^s\Omega)(X_1, \ldots, X_k)
= (\varrho_A \circ X)(\Omega(X_1, \ldots, X_k)) + \sum_{i=1}^k (\varrho_A \circ X_i)(\Omega(X, X_1, \ldots, \widehat{X}_i, \ldots, X_k))
- \sum_{i=1}^k \Omega(X_1, \ldots, \langle X : X_i \rangle, \ldots, X_k) - \sum_{i<j} \Omega(X, X_1, \ldots, \widehat{X}_i, \ldots, \langle X_i : X_j \rangle, \ldots, X_k)
\]

and

\[
(d^s i_X\Omega)(X_1, \ldots, X_k)
= \sum_{i=1}^k (\varrho_A \circ X_i)(\Omega(X, X_1, \ldots, \widehat{X}_i, \ldots, X_k)) - \sum_{i<j} \Omega(X, X_1, \ldots, \widehat{X}_i, \ldots, \langle X_i : X_j \rangle, \ldots, X_k).
\]
Hence, we obtain (a) in Lemma 3.1.

Moreover,

\[
(\mathcal{L}_X^s i_Y \Omega)(X_1, \ldots, X_{k-1}) = (\varrho_A \circ X)(\Omega(Y, X_1, \ldots, X_{k-1})) - \sum_{i=1}^{k-1} \Omega(Y, X_1, \ldots, \langle X : X_i \rangle, \ldots, X_{k-1})
\]

and

\[
(i_Y \mathcal{L}_X^s \Omega)(X_1, \ldots, X_{k-1}) = (\varrho_A \circ X)(\Omega(Y, X_1, \ldots, X_{k-1})) - \Omega((X : Y), X_1, \ldots, X_{k-1}) - \sum_{i=1}^{k-1} \Omega(Y, X_1, \ldots, \langle X : X_i \rangle, \ldots, X_{k-1}),
\]

which give immediately (b) in Lemma 3.1. □

By the definition of the symmetric Lie derivative and properties of differentiations, we have the following result:

**Lemma 3.2.** For \( f \in C^\infty(M) \), \( X \in \Gamma(A) \), \( \omega \in \Gamma(A^*) \), we have

(a) \( \mathcal{L}_f^s X \omega = f \cdot \mathcal{L}_X^s \omega - (i_X \omega) \cdot df \),

(b) \( \mathcal{L}_X^s (f \cdot \omega) = f \cdot \mathcal{L}_X^s \omega + (\varrho_A \circ X)(f) \cdot \omega \).

**4. The symmetric bracket of a skew-symmetric algebroid with a totally skew-symmetric torsion**

Let \((A, \varrho_A, [\cdot, \cdot])\) be a skew-symmetric algebroid over a manifold \( M \) equipped with a pseudo-Riemannian metric \( g \in \Gamma(S^2 A^*) \) in the vector bundle \( A \). The pseudo-Riemannian metric defines two homomorphism of vector bundles

\[ b : A \to A^*, \]
\[ g^\#: A^* \to A \]

by

\[ b(X) = i_X g \]

and

\[ g(\sharp(\omega), X) = \omega(X) \]

for \( X \in \Gamma(A), \omega \in \Gamma(A^*) \), respectively. For any \( X \in \Gamma(A) \), the 1-form \( i_X g = g(X, \cdot) \) will be denoted, briefly, by \( X^\flat \).

We say that \( \nabla \) is a connection with totally skew-symmetric torsion with respect to a pseudo-Riemannian metric \( g \) if the tensor \( T^g \in \Gamma\left( \bigotimes^3 A^* \right) \) given by

\[ T^g(X, Y, Z) = g(T^\nabla(X, Y), Z) \]

for \( X, Y, Z \in \Gamma(A) \), is a 3-form on \( A \), i.e., \( T^g \in \Gamma(\bigwedge^3 A^*) \) – cf. [1].

**Theorem 4.1.** Let \( X, Z \in \Gamma(A) \).

Then

\[
g(\nabla_X X, Z) = g(\sharp(\mathcal{L}_X^s X^\flat - \frac{1}{2} d^a g(X, X)), Z) - g(T^\nabla(X, Z), X) + (\nabla g)(Z, X, X) - \frac{1}{2} (d^a g)(X, X, Z).
\]
In particular, if $\nabla$ is a connection with totally skew-symmetric torsion compatible with $g$, then

$$\nabla_X X = \nabla^a (\mathcal{L}_X^a X^b - \frac{1}{2}d^a (g(X, X))).$$

\textbf{Proof.} Let $X, Z \in \Gamma(A)$. First, observe that

$$(d^a g)(X, X, Z) = 2(\nabla g)(X, X, Z) + (\nabla g)(Z, X, X).$$

Therefore, we have

$$(\nabla g)(Z, X, X) - \frac{1}{2}(d^a g)(X, X, Z) = \frac{1}{2}(\nabla g)(Z, X, X) - (\nabla g)(X, X, Z).$$

Then

$$\begin{align*}
\frac{1}{2} (\nabla g)(Z, X, X) - (\nabla g)(X, X, Z) \\
= (\nabla^a g)(X, X) - (\nabla^a g)(X, X, Z) \\
= \frac{1}{2} g_A(Z)(g(X, X), X) + g(\nabla^a g)(X, X, Z) + g(X, \nabla^a g)(X, X, Z) \\
= \frac{1}{2} g_A(Z)(g(X, X), X) + g(\nabla^a g)(X, X, Z) + g([X, Z] X) + g(\nabla^a g)(X, X, Z). \\
\end{align*}$$

Since

$$\mathcal{L}_X^a X^b (Z) = g_A (X) X^b (Z) - X^b ([X, Z])$$

and

$$g_A(Z)(g(X, X), X) = d^a (g(X, X))(Z) = g(\nabla^a g)(X, X, Z),$$

we have

$$\frac{1}{2} (\nabla g)(Z, X, X) - (\nabla g)(X, X, Z) = \frac{1}{2} d^a (g(X, X))(Z) + g(\nabla^a g)(X, X, Z) + g(\nabla^a g)(X, X, Z).$$

Moreover, if $\nabla$ is a metric connection with totally skew-symmetric torsion, then $\nabla g = 0$, $d^a g = 0$ and $g(T^a (X, X), X) = 0$ and, in consequence, we obtain (4.1). This completes the proof.\hfill \Box

\textbf{Theorem 4.2.} Let $X, Y, Z \in \Gamma(A)$ and let $\langle X : Y \rangle$ be the symmetric bracket of vector fields induced by $\nabla$, i.e., $\langle X : Y \rangle = \nabla_X Y + \nabla_Y X$. Then

$$\begin{align*}
g(\langle X : Y \rangle, Z) &= g(\nabla^a (\mathcal{L}_X^a Y^b + \mathcal{L}_Y^a X^b - d^a (g(X, Y))), Z) \\
&= -g(T^a (X, Z), Y) - g(T^a (Y, Z), X) \\
&+ 2(\nabla g)(Z, X, Y) - (d^a g)(X, Y, Z). \\
\end{align*}$$

\textbf{Proof.} Using the following polarization formula

$$\langle X : Y \rangle = \nabla_{X+Y} (X + Y) - \nabla_X X - \nabla_Y Y$$
and Theorem \[1.1\] we obtain

\[
\langle X : Y \rangle = g(\hat{\mathcal{L}}_{X+Y}^a (X + Y)^b - \frac{1}{2} d^a(g(X + Y, X + Y)), Z) - g(T^\nabla (X + Y, Z), X + Y) + (\nabla g)(Z, X + Y, X + Y) - \frac{1}{2} (d^a g)(X + Y, X + Y, Z) - g(\hat{\mathcal{L}}_{Y}^{a} Y^b - \frac{1}{2} d^a(g(Y, Y)) + g(T^\nabla (Y, Z), Y) - (\nabla g)(Z, Y, Y) + \frac{1}{2} (d^a g)(Y, Y, Z)
\]

First observe that

\[
\mathcal{L}_{X+Y}^a (X + Y)^b - \mathcal{L}_{X}^{a} X^b - \mathcal{L}_{Y}^{a} Y^b = \mathcal{L}_{X}^{a} Y^b + \mathcal{L}_{Y}^{a} X^b
\]

and

\[
-\frac{1}{2} d^a(g(X + Y, X + Y) + \frac{1}{2} d^a(g(Y, Y)) = -d^a(g(X, Y)).
\]

Since \( g \) is a symmetric tensor and \( T^\nabla \) is skew-symmetric, we conclude that

\[
-g(T^\nabla (X + Y, Z), X + Y) + g(T^\nabla (X, Z), X) + g(T^\nabla (Y, Z), Y)
\]

is equal to

\[
-g(T^\nabla (X, Z), Y) - g(T^\nabla (Y, Z), X).
\]

Moreover,

\[
(\nabla g)(Z, X + Y, X + Y) - (\nabla g)(Z, X, X) - (\nabla g)(Z, Y, Y) = 2(\nabla g)(Z, X, Y)
\]

and

\[
(d^a g)(X, Y, Z) = \frac{1}{2} (d^a g)(X, Y, Z) + \frac{1}{2} (d^a g)(Y, X, Z)
\]

\[
= \frac{1}{2} (d^a g)(X + Y, X + Y, Z) - \frac{1}{2} (d^a g)(X, X, Z) - \frac{1}{2} (d^a g)(Y, Y, Z).
\]

Hence, it is clear that some summands of \( \langle X : Y \rangle \) cancels. This establishes the formula \[1.2\].

The formula in Theorem \[1.2\] gives an explicit formula of symmetric bracket defined by any metric connection with totally skew-symmetric torsion.

**Corollary 4.3.** Let \( \nabla \) be any metric A-connection in \( A \) with totally skew-symmetric torsion with respect to a pseudo-Riemannian metric \( g \). Then

\[
\nabla_X Y + \nabla_Y X = \hat{\mathcal{L}}_{X}^{a} Y^b + \mathcal{L}_{Y}^{a} X^b - d^a(g(X, Y)).
\]

5. A general metric compatibility condition of connections having totally skew-symmetric torsion. The Levi-Civita connection

Let \((A, g_A, [\cdot, \cdot])\) be a skew-symmetric algebroid over a manifold \( M \) equipped with a pseudo-Riemannian metric \( g \in \Gamma(S^2 A^*) \) in the vector bundle \( A \) and a symmetric bracket \( \langle \cdot : \cdot \rangle : \Gamma(A) \times \Gamma(A) \to \Gamma(A) \). By definition we recall that the symmetric bracket is an \( \mathbb{R} \)-bilinear symmetric mapping which satisfies the following Leibniz kind rule:

\[
\langle X : fY \rangle = f \langle X : Y \rangle + (g_A \circ X)(f) \cdot Y,
\]

\[
\langle fX : Y \rangle = f \langle X : Y \rangle + (g_A \circ Y)(f) \cdot X
\]

for \( X, Y \in \Gamma(A), f \in C^\infty(M) \).

Given the bundle metric \( g \) on \( A \), there is a unique \( A \)-connection in \( A \) which is torsion-free and metric-compatible (i.e., \( T^\nabla = 0 \) and \( \nabla g = 0 \)). We call such the \( A \)-connection.
the Levi-Civita connection with respect to \( g \). Let \( \mathcal{L}^s \) and \( d^s \) denote the symmetric Lie derivative and the symmetric derivative operator, respectively, and both are induced by \( \langle \cdot, \cdot \rangle \).

**Theorem 5.1.** Let \( \nabla \) be any \( A \)-connection in \( A \) with totally skew-symmetric torsion with respect to a pseudo-Riemannian metric \( g \) on \( A \) given by

\[
(5.1) \quad \nabla_X Y = \frac{1}{2} \left( [X, Y] + \langle X : Y \rangle \right) + \frac{1}{2} T(X, Y)
\]

for \( X, Y \in \Gamma(A) \). Then

\[
(i_X \circ \nabla) g = \frac{1}{2} (\mathcal{L}_X^s + \mathcal{L}_X^a) g
\]

for \( X \in \Gamma(A) \).

**Proof.** Let \( X, Y, Z \in \Gamma(A) \). Since \( T \in \Gamma \left( \bigwedge^2 A^* \right) \) is a 2-skew-symmetric tensor with the property that

\[
g(Y, T(X, Z)) = g(T(X, Y), Z) = -g(T(X, Y), Z),
\]

we have,

\[
(\nabla_X g)(Y, Z) = \rho_A(X)(g(Y, Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = \frac{1}{2} \left( \rho_A(X)(g(Y, Z)) - g([X, Y], Z) - g(Y, [X, Z]) \right) + \frac{1}{2} \left( \rho_A(X)(g(Y, Z)) - g(\langle X : Y \rangle, Z) - g(Y, \langle X : Z \rangle) \right) - \frac{1}{2} g(T(X, Y), Z) - \frac{1}{2} g(Y, T(X, Z)) = \frac{1}{2} (\mathcal{L}_X^a g + \mathcal{L}_X^s g)(Y, Z) + 0.
\]

Hence, we can conclude the following condition on a connection with skew-symmetric torsion to be a metric connection:

**Corollary 5.2.** If \( \nabla \) is an \( A \)-connection with totally skew-symmetric torsion with respect to \( g \) given by \( (5.1) \), then \( \nabla \) is metric with respect to \( g \) if and only if

\[
\mathcal{L}_X^a g = -\mathcal{L}_X^s g
\]

for any \( X \in \Gamma(A) \).

Now, we recall some properties of the (skew-symmetric) Lie derivative.

**Lemma 5.3.** For \( f \in C^\infty(M) \), \( X \in \Gamma(A) \), \( \omega \in \Gamma(A^*) \), we have

(a) \( \mathcal{L}^s_{f \cdot X} \omega = f \cdot \mathcal{L}^a_X \omega + (i_X \omega) \cdot df \),

(b) \( \mathcal{L}^a_X (f \cdot \omega) = f \cdot \mathcal{L}^a_X \omega + (g_A \circ X)(f) \cdot \omega \).

**Theorem 5.4.** Given a skew-symmetric algebroid \( (A, g_A, [\cdot, \cdot]) \), we define

\[
\langle X : Y \rangle^* : \Gamma(A) \times \Gamma(A) \to \Gamma(A)
\]

by

\[
(5.2) \quad \langle X : Y \rangle^* = \frac{1}{2} (\mathcal{L}_X^a Y^a + \mathcal{L}_Y^a X^a - d^a(g(X, Y))
\]

for \( X, Y \in \Gamma(A) \). Then \( \langle \cdot, \cdot \rangle^* \) is a symmetric bracket which defines the symmetric Lie derivatives \( \mathcal{L}^s \) satisfying \( \mathcal{L}_X^s g = -\mathcal{L}_X^a g \).
Proof. It is evident that $\langle \cdot : \cdot \rangle^a$ is a symmetric and $\mathbb{R}$-bilinear mapping. Let $X, Y, Z \in \Gamma(A)$. Lemma [5.3] now gives

$$\mathcal{L}_X^a(Y^b) = f \mathcal{L}_X^a Y^b + X(f)Y^b$$

and

$$\mathcal{L}_Y^a X^b = f \mathcal{L}_Y^a X^b + g(X, Y)df.$$ 

Since

$$df(g(X, fY)) = df(g(X, Y)) + g(X, Y)df,$$

we conclude that $\langle \cdot : \cdot \rangle^a$ satisfies the Leibniz rule. In consequence, $\langle \cdot : \cdot \rangle^a$ is a symmetric bracket. Observe that

$$g(\langle X : Y \rangle^s, Z) = \langle (X : Y)^s \rangle (Z) = (\mathcal{L}_X^a Y^b + \mathcal{L}_Y^a X^b - df(g(X, Y)))(Z)$$

$$= (g_A \circ X)(g(Y, Z)) - g(Y, [X, Z]) + (g_A \circ Y)(g(X, Z)) - g(X, [Y, Z]) - (g_A \circ Z)(g(X, Y)).$$

Similarly,

$$g(Y, \langle X : Z \rangle^s) = (g_A \circ X)(g(Y, Z)) - g(Z, [X, Y]) + (g_A \circ Z)(g(X, Y)) - g(X, [Z, Y]) - (g_A \circ Y)(g(X, Z)).$$

Therefore,

$$(\mathcal{L}_X^a g)(Y, Z) = (g_A \circ X)(g(Y, Z)) - g(\{X, Y\}^s, Z) - g(Y, \langle X : Z \rangle^s)$$

$$= g(Y, [X, Z] + g(X, [Y, Z]) - (g_A \circ X)(g(Y, Z)) + g(Z, [X, Y]) + g(X, [Z, Y])$$

$$+ (g_A \circ Z)(g(X, Y)) + g([X, Y], Z) + g(Y, [X, Z])$$

$$+ g(X, [Y, Z] + [Z, Y])$$

$$= -(\mathcal{L}_X^a g)(Y, Z) + 0.$$

□

Theorem [5.1] now yields

Corollary 5.5. If $\nabla$ is an $A$-connection in the bundle $A$ with totally skew-symmetric torsion defined, for $X, Y \in \Gamma(A)$, by

$$\nabla_X Y = \frac{1}{2} ([X, Y] + \langle X : Y \rangle^s) + \frac{1}{2} T(X, Y),$$

where $\langle X : Y \rangle^s$ is given in (5.2), then $\nabla$ is compatible with the metric $g$.

In particular, we can write the Levi-Civita connection explicitly:

Corollary 5.6. The connection defined, for $X, Y \in \Gamma(A)$, by

$$\nabla_X Y = \frac{1}{2} ([X, Y] + \langle X : Y \rangle^s),$$

where

$$\langle X : Y \rangle^s = \frac{1}{2}(\mathcal{L}_X^a Y^b + \mathcal{L}_Y^a X^b - df(g(X, Y)),$$

is the Levi-Civita connection with respect to $g$.

Theorem 5.7. The mapping $\{\cdot, \cdot\}^s : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ defined by

$$\{X, Y\}^s = \frac{1}{2}(\mathcal{L}_X^a Y^b + \mathcal{L}_Y^a X^b + df(g(X, Y))$$

for $X, Y \in \Gamma(A)$, is a symmetric bracket in the skew-symmetric algebroid $(A, g_A, [\cdot, \cdot])$. 
Proof. It is obvious that \( \langle \cdot, \cdot \rangle^s \) is a symmetric bilinear mapping over \( \mathbb{R} \). Let \( X, Y \in \Gamma(A) \) and \( f \in C^\infty(M) \). On account of properties of the symmetric Lie derivatives written in Lemma 3.2
\[
\mathcal{L}_X(fY)^b = f\mathcal{L}_X(Y^b) + (g_A \circ X)(f)Y^b
\]
and
\[
\mathcal{L}_{fY}X^b = f\mathcal{L}_YX^b - (i_YX^b)df = f\mathcal{L}_YX^b - g(X, Y)df.
\]
Furthermore, since
\[
d^s(g(X, fY)) = fd^s(g(X, Y)) + g(X, Y)d^sf,
\]
we immediately conclude that
\[
\{X, fY\}^s = f\{X, Y\}^s + \#((g_A \circ X)(f)Y^b) = f\{X, Y\} + (g_A \circ X)(f)Y,
\]
and thus, the proof is complete. \( \square \)

To compare the symmetric brackets \( \langle \cdot, \cdot \rangle^s \) and \( \{\cdot, \cdot\}^s \) given in [5.3] and [5.4], respectively, we note that \( \langle \cdot, \cdot \rangle^s \) is a symmetric product induced by the Levi-Civita connection, and then, for any \( X, Y \in \Gamma(A) \), we have
\[
\mathcal{L}_X^sY^b = \mathcal{L}_X^sY^g = i_Y\mathcal{L}_X^sX^g + i_{\langle X, Y \rangle^g}g = -i_Y\mathcal{L}_X^sX^g + i_{\langle X, Y \rangle^g}g
\]
since Theorem 5.1 and the Cartan identities for Lie derivatives given in lemmas 2.1–3.1 hold. It follows that
\[
\{X, Y\}^s = 2\langle X : Y \rangle - \langle X : Y \rangle^s
\]
for \( X, Y \in \Gamma(A) \). Note that there is a more general property saying that the affine sum of symmetric brackets is again a symmetric bracket.

Lemma 5.8. If \( \{\cdot, \cdot\}^0, \{\cdot, \cdot\}^1, \ldots, \{\cdot, \cdot\}^n \) are symmetric brackets in anchored vector bundle \( (A, g_A), t_1, \ldots, t_n \in C^\infty(M) \), then the affine sum
\[
\{\cdot, \cdot\}^{0,1,\ldots,n} = \left(1 - \sum_{i=1}^n t_i\right)\{\cdot, \cdot\}^0 + \sum_{i=1}^n t_i\{\cdot, \cdot\}^i
\]
is a symmetric bracket in \( (A, g_A) \).

Theorem 5.9. If \( \{\cdot, \cdot\}^s \) is the symmetric bracket defined in [5.4], then
\[
\{X, Y\}^s = \langle X : Y \rangle + \#(i_Xi_Yd^sg)
\]
for \( X, Y \in \Gamma(A) \).

Proof. Let \( X, Y, Z \in \Gamma(A) \). Since
\[
\left(\mathcal{L}_X^sY^b + \mathcal{L}_Y^sX^b + d^s(g(X, Y))^b\right)(Z)
= (g_A \circ X)(g(Y, Z)) - g(Y, \langle X : Z \rangle) - (g_A \circ Y)(g(X, Z)) - g(X, \langle Y : Z \rangle)
+ (g_A \circ Z)(g(X, Y))
= (d^sg)(X, Y, Z) + g(Z, \langle X : Y \rangle)
= (i_Xi_Yd^sg)(Z)
\]
it follows that
\[
\{X, Y\}^s = \langle X : Y \rangle + \#(i_Xi_Yd^sg).
\]
\( \square \)
6. Symmetric brackets on almost Hermitian manifolds

In this chapter, we consider the symmetric brackets induced by structures of almost Hermitian manifolds. Let $(M, g, J)$ be an almost Hermitian manifold, that is $(M, g)$ is a 2n-dimensional Riemannian manifold admitting an orthogonal almost complex structure $J : TM \to TM$. Associated to the structures $g$ and $J$ are the Kähler form $\Omega \in \Gamma(\wedge^2 T^*M)$ given by

$$\Omega(X, Y) = g(JX, Y)$$

for $X, Y \in \Gamma(TM)$ and the Nijenhuis tensor $N_J \in \Gamma(\wedge^2 T^*M \otimes TM)$ of $J$, which is defined by

$$N_J(X, Y) = J [JX, Y] + J [X, JY] + [X, Y] - [JX, JY]$$

for $X, Y \in \Gamma(TM)$. One can observe that for any $X, Y \in \Gamma(TM)$ we have

$$N_J(X, Y) = J[X,Y]^J - [JX, JY]$$

where

$$[X, Y]^J = [JX, Y] + [X, JY] - J [X, Y].$$

**Theorem 6.1.** The tangent bundle together with the almost complex structure $J$ as an anchor and the mapping $[\cdot, \cdot]^J$ given in (6.1) as a skew-symmetric bracket is a skew-symmetric algebroid.

**Proof.** Let $X, Y \in \Gamma(TM)$ and $f \in C^\infty(M)$. It is sufficient to observe that

$$[X, fY]^J = [JX, fY] + [X, fJY] - J [X, fY]$$

$$= f[X,Y]^J + (JX)(f)Y - X(f)(JY) - X(f)(JY)$$

$$= f[X,Y]^J + (JX)(f)Y.$$

If $N_J = 0$, then

$$[[X,Y]^J, Z]^J = -J [[JX, JY], JZ]$$

for $X, Y, Z \in \Gamma(TM)$. This property gives

**Lemma 6.2.** If $N_J = 0$, then $\text{Jac}_{[\cdot, \cdot]^J}(X, Y, Z) = -J \text{Jac}_{[\cdot, \cdot]^J}(JX, JY, JZ) = 0$ for all $X, Y, Z \in \Gamma(TM)$.

In consequence, we have the following corollary.

**Corollary 6.3.** If the almost complex structure $J$ is integrable, then the skew-symmetric algebroid $(TM, J, [\cdot, \cdot]^J)$ is a Lie algebroid over $M$.

Let $d^a$ and $\mathcal{L}_X^a$ be the exterior derivative operator and the (alternating) Lie derivative for the Lie algebroid $(TM, \text{Id}_{TM}, [\cdot, \cdot])$, respectively, where $[\cdot, \cdot]$ is the Lie bracket of vector fields on $M$. However, let $d^J$ and $\mathcal{L}_X^J$ be the exterior derivative operator and the Lie derivative for the skew-symmetric algebroid $(TM, J, [\cdot, \cdot]^J)$, respectively.

**Theorem 6.4.** Let $\nabla^g$ be the Levi-Civita connection in the Lie algebroid $(TM, \text{Id}_{TM}, [\cdot, \cdot])$ and let $\nabla^{J,g}$ be the Levi-Civita connection in the skew-symmetric algebroid $(TM, J, [\cdot, \cdot]^J)$, both metric with respect to $g$. If $(X : Y) = \nabla^g_X Y + \nabla^a_X Y$ and $(X : Y)^J = \nabla^{J,g}_X Y + \nabla^{J,g}_Y X$ for $X, Y \in \Gamma(TM)$, then

$$\langle X : Y \rangle^J = \langle JX : Y \rangle + \langle X : JY \rangle + \sharp(\langle X : Y \rangle^b \circ J).$$
**Proof.** Let $X, Y \in \Gamma(TM)$. Corollary 5.6 now yields
\[
\langle X : Y \rangle^\rho = \frac{2}{n} (\mathcal{L}_X^P Y + \mathcal{L}_Y^P X - d^P(g(X, Y))).
\]
One can check that
\[
\mathcal{L}_X^P Y = \mathcal{L}_{[X, Y]}^P + \mathcal{L}_X^P (JY)^\rho + (\mathcal{L}_Y^P X)^\rho \circ J.
\]
Moreover, since
\[
d^P(g(X, Y)) = d^\rho(g(X, Y)) \circ J
\]
and
\[
d^\rho(g(JX, Y)) + d^\rho(g(X, JY)) = 0,
\]
we have [6.2].
\[
\square
\]
Now, we define some symmetric brackets on almost Hermitian manifolds.

Let $(TM, \rho, [\cdot, \cdot]^{\rho})$ be a structure of skew-symmetric algebroid and let $\langle \cdot : \cdot \rangle^\rho$ be a symmetric bracket in this algebroid. By definition,
\[
\langle X : fY \rangle^\rho = f \langle X : Y \rangle^\rho + (\rho \circ X)(f)Y
\]
for $X, Y \in \Gamma(TM)$.

We define two $\mathbb{R}$-bilinear symmetric operators $P^\rho, Q^\rho : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$,
\[
P^\rho(X, Y) = -J ([X, Y]^\rho + [Y, JX]^\rho)
\]
and
\[
Q^\rho(X, Y) = -J ((X : JY)^\rho + (Y : JX)^\rho)
\]
for $X, Y \in \Gamma(TM)$.

**Lemma 6.5.** For any $X, Y \in \Gamma(TM)$, $f \in C^\infty(M)$, we have
\[
(\text{a}) \quad P^\rho(X, f \cdot Y) = f \cdot P^\rho(X, Y) + (\rho \circ X)(f) \cdot Y + (\rho \circ JX)(f) \cdot JY,
\]
\[
(\text{b}) \quad Q^\rho(X, f \cdot Y) = f \cdot Q^\rho(X, Y) + (\rho \circ X)(f) \cdot Y - (\rho \circ JX)(f) \cdot JY.
\]

**Proof.** Compute directly,
\[
P^\rho(X, f \cdot Y) = -J ([X, f \cdot JY]^\rho + [f \cdot Y, JX]^\rho)
\]
\[
= -J (f \cdot [X, JY]^\rho + (\rho \circ X)(f) \cdot JY + f \cdot [Y, JX]^\rho - (\rho \circ JX)(f) \cdot Y)
\]
\[
= f \cdot P^\rho(X, Y) + (\rho \circ X)(f) \cdot Y + (\rho \circ JX)(f) \cdot JY
\]
and
\[
Q^\rho(X, f \cdot Y) = -J ((X : f \cdot JY)^\rho + (f \cdot JX)^\rho)
\]
\[
= -J ((f \cdot (X : JY)^\rho + (\rho \circ X)(f) \cdot JY + f \cdot (Y : JX)^\rho + (\rho \circ JX)(f) \cdot JY)
\]
\[
= f \cdot Q^\rho(X, Y) + (\rho \circ X)(f) \cdot Y - (\rho \circ JX)(f) \cdot JY.
\]
\[
\square
\]
In consequence of Lemma 6.5, we immediately get the following results.

**Theorem 6.6.** The mapping
\[
\frac{1}{2}(P^\rho + Q^\rho)
\]
is a symmetric bracket in the skew-symmetric algebroid $(TM, \rho, [\cdot, \cdot]^{\rho})$.

**Corollary 6.7.** The mapping $\langle \cdot : \cdot \rangle : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ given by
\[
\langle X : Y \rangle = -\frac{1}{2} J ([X, JY] + [Y, JX] + 2(\mathcal{L}_X^P(JY)^\rho + \mathcal{L}_Y^P(JX)^\rho + \mathcal{L}_{[X, Y]}^P)).
\]
is a symmetric bracket in the Lie algebroid $(TM, \text{Id}_TM, [\cdot, \cdot])$, where $[\cdot, \cdot]$ is the Lie bracket of vector fields on $M$ and $\mathcal{L}_{[X, Y]}^P$ is the Lie derivative on $M$. 

Proof. Let $d^a$ be the exterior derivative on manifold $M$. Taking $g = \text{Id}_{TM}$ in Theorem 6.6 and using Theorem 5.4, we deduce that the following formula

$$\langle X : Y \rangle = -\frac{1}{2} J ([X, JY] + [Y, JX]) - \frac{1}{2} (J \circ \sharp) (\mathcal{L}_X^a (JY)^b + \mathcal{L}_Y^a (JX)^b - d^a (g(X, JY)))$$

$$- \frac{1}{2} (J \circ \sharp) (\mathcal{L}_X^a Y^b + \mathcal{L}_Y^a (JX)^b - d^a (g(JX, Y))),$$

defines a symmetric bracket in the tangent bundle with $\text{Id}_{TM}$ as an anchor and with the classical Lie bracket. Since $\Omega$ is a skew-symmetric 2-form on $M$, it follows that

$$g(X, JY) + g(JX, Y) = \Omega(Y, X) + \Omega(Y, X) = 0.$$

Therefore

$$\langle X : Y \rangle = -\frac{1}{2} J ([X, JY] + [Y, JX])$$

$$- \frac{1}{2} (J \circ \sharp) (\mathcal{L}_X^a Y^b + \mathcal{L}_Y^a (JX)^b - d^a (g(JX, Y))).$$

□

It is obvious that the bracket in Corollary 6.7 is a totally symmetric part of the connection $\nabla^J : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ defined by

$$\nabla^J_X Y = -\frac{1}{2} J ([X, JY] + \langle X : JY \rangle).$$

Let $\nabla$ be the Levi-Civita connection with respect to $g$ given by

$$\nabla_X Y = \frac{1}{2} \left( [X, Y] + \langle X : Y \rangle \right).$$

Now let $\langle \cdot : \cdot \rangle = \langle \cdot : \cdot \rangle^\nabla$. Hence,

$$\nabla^J_X Y = -J \nabla_X (JY).$$

One can observe that the affine sum

$$\nabla = \frac{1}{2} (\nabla + \nabla^J)$$

of connections $\nabla$ and $\nabla^J$ is Lichnerowicz’s first canonical Hermitian connection [10], which is compatible with both the metric structure and the almost complex structure. This is a direct consequence of properties of $\nabla$ and $\nabla^J$ given in the following lemma.

Lemma 6.8.

(a) $(\nabla^J g)(X, Y, Z) = (\nabla g)(X, JY, JZ)$ for $X, Y, Z \in \Gamma(TM)$.

(b) $\nabla^J J = - \nabla J$.

Lemma 6.8 now yields

$$\nabla^J g = 0, \quad \nabla = \frac{1}{2} (\nabla + \nabla^J) g = \frac{1}{2} (\nabla g + \nabla^J g) = 0$$

and

$$\nabla J = \frac{1}{2} (\nabla J + \nabla^J J) = \frac{1}{2} (\nabla J - \nabla J) = 0.$$

We will now consider some further properties of $\nabla^J$ and $\nabla$.

For an $A$-connection $\nabla$ on $A$, we define the operators

$$d^\nabla_A, d^\nabla_T : \Gamma(\bigotimes^k T^* M) \to \Gamma(\bigotimes^{k+1} T^* M)$$

as the alternation and the symmetrization of $\nabla$, respectively, i.e., for $\zeta \in \Gamma(\bigotimes^k T^* M)$, $X_1, \ldots, X_{k+1} \in \Gamma(TM)$, we have

$$(d^\nabla_A \zeta)(X_1, \ldots, X_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} (\nabla X_i \zeta) (X_1, \ldots, \hat{X}_i, \ldots, X_{k+1})$$

and

$$(d^\nabla_T \zeta)(X_1, \ldots, X_{k+1}) = \sum_{i=1}^{k+1} \frac{1}{k+1} (\nabla X_i \zeta) (X_1, \ldots, X_{k+1}).$$
and
\[(d_\nabla^s\zeta)(X_1, \ldots, X_{k+1}) = \sum_{i=1}^{k+1} (\nabla_{X_i} \zeta)(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1}).\]

We say that an almost Hermitian manifold \((M, g, J)\) is nearly Kähler if \((\nabla_X J)Y = -(\nabla_Y J)X\) for \(X, Y \in \Gamma(TM)\); cf. [6]. We have the following theorem.

**Lemma 6.9.** An almost manifold \((M, g, J)\) is nearly Kähler if and only if \(d_\nabla^s \nabla J = 0\).

Moreover, if \((M, g, J)\) is nearly Kähler, \(\nabla\) is a Hermitian connection with totally skew-symmetric torsion (e.g., cf. [1]).

Now, we compare the symmetric brackets induced by \(\nabla\) and by \(\nabla^s\). We will denote by \(\langle \cdot : \cdot \rangle\) the symmetric product of \(\nabla\).

**Theorem 6.10.** For \(X, Y \in \Gamma(TM)\), we have
\[J((d_\nabla^s J)(X, Y)) = \langle X : Y \rangle^\nabla - \langle X : Y \rangle^{\nabla J}.\]

**Proof.** We first observe that
\[(d_\nabla^s J)(X, Y) = (\nabla_X J)Y + (\nabla_Y J)X = \nabla_X (JY) + \nabla_Y (JX) - J(\nabla_X Y + \nabla_Y X) = \nabla_X (JY) + \nabla_Y (JX) - J \langle X : Y \rangle^\nabla.\]

From this equality, we obtain
\[J((d_\nabla^s J)(X, Y)) = J\nabla_X (JY) + J\nabla_Y (JX) + \langle X : Y \rangle^\nabla = -\langle X : Y \rangle^{\nabla J} + \langle X : Y \rangle^\nabla.\]

□

**Theorem 6.11.** For \(X, Y \in \Gamma(TM)\), we have
\[\langle X : Y \rangle^\nabla = \langle X : Y \rangle^{\nabla} - \frac{1}{2} J((d_\nabla^s J)(X, Y)).\]

**Proof.** Since \(\nabla^s = \frac{1}{2} (\nabla + \nabla^J)\) is a affine sum of connections \(\nabla\) and \(\nabla^J\),
\[\langle X : Y \rangle^\nabla = \frac{1}{2} \langle X : Y \rangle^{\nabla} + \frac{1}{2} \langle X : Y \rangle^{\nabla J}.\]

From this result and Theorem [6.10], we see that
\[\langle X : Y \rangle^\nabla = \frac{1}{2} \langle X : Y \rangle^{\nabla} + \frac{1}{2} \left(\langle X : Y \rangle^{\nabla} - J((d_\nabla^s J)(X, Y))\right) = \langle X : Y \rangle^\nabla - \frac{1}{2} J((d_\nabla^s J)(X, Y)).\]

□

Since \(\nabla^s = \frac{1}{2} (\nabla + \nabla^J)\) and \(\nabla\) is torsion-free, we have
\[T^{\nabla^s} = \frac{1}{2} T^{\nabla} + \frac{1}{2} T^{\nabla^J} = \frac{1}{2} T^{\nabla^J}.\]

**Theorem 6.12.** \(T^{\nabla^J} = -J \circ (d_\nabla^s J).\)

**Proof.** Let \(X, Y \in \Gamma(TM)\). Then
\[(d_\nabla^s J)(X, Y) = (\nabla_X J)Y - (\nabla_Y J)X = \nabla_X (JY) - \nabla_Y (JX) - J [X, Y].\]
Moreover, it follows that Corollary 6.14. If $X, Y \in \Gamma(TM)$, we have
\[
2T^{\nabla J}(X, Y) = -N_J(X, Y) + (d^e_{\nabla J})(X, JY) - (d^e_{\nabla J})(JX, Y).
\]
In particular, if $(M, g, J)$ is nearly Kähler, then
\[
T^{\nabla J} = -\frac{1}{2}N_J.
\]

**Proof.** Let $X, Y \in \Gamma(TM)$. Then (e.g., see [1]) for the first equality:
\[
-N_J(X, Y) = (\nabla_X J)Y - (\nabla_Y J)X + (\nabla_{JX} J)Y - (\nabla_{JY} J)X
\]
\[
= (\nabla_X J)Y - (\nabla_Y J)X - (\nabla_Y J)JX + (d^e_{\nabla J})(JX, Y)
\]
\[
+ (\nabla_X J)JY - (d^e_{\nabla J})(X, JY)
\]
\[
= 2((\nabla_X J)JY - (\nabla_Y J)JX) + (d^e_{\nabla J})(JX, Y) - (d^e_{\nabla J})(X, JY).
\]
Moreover,
\[
(\nabla_X J)JY - (\nabla_Y J)JX = -\nabla_X Y - J(\nabla_X (JY)) + \nabla_Y X + J(\nabla_Y (JX))
\]
\[
= -J(\nabla_X (JY)) - (-J(\nabla_Y (JX))) - \nabla_X (Y) + \nabla_Y X
\]
\[
= \nabla^J_X Y - \nabla^J_Y X - [X, Y] = T^{\nabla J}(X, Y).
\]
It follows that
\[
-N_J(X, Y) = 2T^{\nabla J}(X, Y) + (d^e_{\nabla J})(JX, Y) - (d^e_{\nabla J})(X, JY).
\]

Since $\nabla$ is a totally skew-symmetric connection, Theorem 6.11 now leads to
\[
\nabla^J_X Y = \frac{1}{2} [X, Y] + \langle X : Y \rangle^\nabla + \frac{1}{2}T^{\nabla J}(X, Y)
\]
\[
= \frac{1}{2} [X, Y] + \langle X : Y \rangle^\nabla - J((d^e_{\nabla J})(X, Y)) + \frac{1}{2}T^{\nabla J}(X, Y)
\]
\[
= \nabla_X Y - \frac{1}{2}J((d^e_{\nabla J})(X, Y)) + \frac{1}{2}T^{\nabla J}(X, Y).
\]
Combining (6.3) with Lemma 6.9 and theorems 6.12, 6.13 we get the following result:

**Corollary 6.14.** If $(M, g, J)$ is nearly Kähler, then $d^e_{\nabla J} = 0$, and, in consequence,
\[
\nabla = \nabla - \frac{1}{2}J \circ (d^e_{\nabla J}) = \nabla - \frac{1}{2}N_J.
\]

**References**

[1] Agricola, I., *The Srn’ı lectures on non-integrable geometries with torsion*, Arch. Math. (Brno) **42** (suppl.) (2006), 5–84, [http://eudml.org/doc/188558](http://eudml.org/doc/188558).

[2] Balcerzak B., Pierzchalski A., *Generalized gradients on Lie algebroids*, Ann. Global. Anal. Geom. **44**, no. 3 (2013), 319–337; [doi: 10.1007/s10455-013-9368-y](https://doi.org/10.1007/s10455-013-9368-y).

[3] Balcerzak B., *On the Dirac Type Operators on Symmetric Tensors*, In: Kielanowski P., Odzijewicz A., Previtali E. (eds) Geometric Methods in Physics XXXVI. Trends in Mathematics. Birkhäuser (2019), 215–222; [doi: 10.1007/978-3-030-01156-7_23](https://doi.org/10.1007/978-3-030-01156-7_23).

[4] Cortés J., Martínez E., *Mechanical control systems on Lie algebroids*, IMA J. Math. Control Info. **21** (4) (2004), 457–492; [doi.org/10.1093/imamci/21.4.457](https://doi.org/10.1093/imamci/21.4.457).
[5] Crouch P., *Geometric structures in systems theory*. IEEE Proc. D Control Theory Appl. 128 (5) (1981), 242–252; [doi: 10.1049/ip-d.1981.0051](https://doi.org/10.1049/ip-d.1981.0051).

[6] Gray A., *Nearly Kähler manifolds*, J. Diff. Geometry 4 (1970), 283–309; [doi:10.4310/jdg/1214429504](https://doi.org/10.4310/jdg/1214429504).

[7] Heydari A., Boroojerdian N., Peyghan E., *A description of derivations of the algebra of symmetric tensors*, Arch. Math. (Brno) 42 (2) (2006), 175–184; [https://eudml.org/doc/249832](https://eudml.org/doc/249832).

[8] de León M., Marrero J. C., Martín de Diego D., *Linear almost Poisson structures and Hamilton-Jacobi equation. Applications to nonholonomic mechanics*, J. Geom. Mech. 2 (2), 159–198 (2010); [doi: 10.3934/jgm.2010.2.159](https://doi.org/10.3934/jgm.2010.2.159).

[9] Lewis A. D., *Affine connections and distributions with applications to nonholonomic mechanics*. Reports Math. Phys. 42, No. 1/2 (1998), 135–164; [doi: 10.1016/S0034-4877(98)80008-6](https://doi.org/10.1016/S0034-4877(98)80008-6).

[10] Lichnerowicz A., *Théorie globale des connexions et des groupes d'holonomie*, Edizioni Cremonese, Roma, 1962.

[11] Mackenzie K. C. H., *Lie Groupoids and Lie Algebroids in Differential Geometry*. London Math. Soc. Lecture Note Ser. 124, Cambridge Univ. Press, 1987.

[12] Mackenzie K. C. H., *General Theory of Lie Groupoids and Lie Algebroids*. London Math. Soc. Lecture Note Ser. 213, Cambridge Univ. Press, 2005.

[13] Popescu P., *Categories of Modules with Differentials*. J. Algebra 185 (1) (1996), 50–73; [doi: 10.1006/jabr.1996.0312](https://doi.org/10.1006/jabr.1996.0312).

[14] Popescu M., Popescu P., *Geometric objects defined by almost Lie structures*. In: Lie algebroids and related topics, Banach Centre Publ. 54 (2001), 217–233; [doi: 10.4064/bc54-0-5](https://doi.org/10.4064/bc54-0-5).

[15] Popescu M., Popescu P., *Almost Lie Algebroids and Characteristic Classes*, SIGMA Symmetry Integrability Geom. Methods Appl. 15 (2019), 021, 12 pages; [doi.org/10.3842/SIGMA.2019.021](https://doi.org/10.3842/SIGMA.2019.021).

[16] Pradines J., *Théorie de Lie pour les groupides différentiables. calcul différentiel dans la catégorie des groupides infinitésimaux*. C. R. Acad. Sci. Paris 264 (1967) 245–248.

[17] Sampson J. H., *On a theorem of Chern*, Trans. Amer. Math. Soc. 177 (1973), 141–153; [doi: 10.1090/S0002-9947-1973-0317221-7](https://doi.org/10.1090/S0002-9947-1973-0317221-7).

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