Vector Bundles over Multipullback Quantum Complex Projective Spaces

Albert Jeu-Liang Sheu†

Department of Mathematics, University of Kansas, Lawrence, KS 66045, U. S. A.

e-mail: asheu@ku.edu

Abstract

We work on the classification of isomorphism classes of finitely generated projective modules over the C*-algebras $C(\mathbb{P}^n(T))$ and $C(S^{2n+1}_H)$ of the quantum complex projective spaces $\mathbb{P}^n(T)$ and the quantum spheres $S^{2n+1}_H$, and the quantum line bundles $L_k$ over $\mathbb{P}^n(T)$, studied by Hajac and collaborators. Motivated by the groupoid approach of Curto, Muhly, and Renault to the study of C*-algebraic structure, we analyze $C(\mathbb{P}^n(T))$, $C(S^{2n+1}_H)$, and $L_k$ in the context of groupoid C*-algebras, and then apply Rieffel’s stable rank results to show that all finitely generated projective

\footnote{This work was partially supported by the grant H2020-MSCA-RISE-2015-691246-QUANTUM DYNAMICS and the Polish government grant 3542/H2020/2016/2.}

\footnote{The author would like to thank the Mathematics Institute of Academia Sinica for the warm hospitality and support during his visit in the summer of 2017.}
modules over $C(S_{H}^{2n+1})$ of rank higher than $\left\lfloor \frac{n}{2} \right\rfloor + 3$ are free modules. Furthermore, besides identifying a large portion of the positive cone of the $K_0$-group of $C(\mathbb{P}^n(T))$, we also explicitly identify $L_k$ with concrete representative elementary projections over $C(\mathbb{P}^n(T))$.

**Keywords:** multipullback quantum projective space; multipullback quantum sphere; quantum line bundle; finitely generated projective module; cancellation problem; Toeplitz algebra of polydisk; groupoid C*-algebra; stable rank; noncommutative vector bundle

**AMS 2010 Mathematics Subject Classification:** 46L80; 46L85
1 Introduction

Since the concept of noncommutative geometry first popularized by Connes [5], many interesting examples of a $C^*$-algebra $A$ viewed as the algebra $C(X_q)$ of continuous functions on a virtual quantum space $X_q$ have been constructed with a topological or geometrical motivation, and analyzed in comparison with their classical counterpart. For example, quantum odd-dimensional spheres and associated complex projective spaces have been introduced and studied by Soibelman, Vaksman, Meyer, and others [32, 14] as $S^{2n+1}_q$ and $CP^n_q$ via a quantum universal enveloping algebra approach, and by Hajac and his collaborators including Baum, Kaygun, Matthes, Nest, Pask, Sims, Szymański, Zieliński, and others [2, 10, 9, 12] as $S^{2n+1}_H$ and $P^n(T)$ via a multi-pullback and Toeplitz algebra approach. Actually $S^{2n+1}_H$ is the untwisted special case of the more general version of $\theta$-twisted spheres $S^{2n+1}_{H,\theta}$ introduced in [12].

Motivated by Swan’s work [30], the concept of a noncommutative vector bundle $E_q$ over a quantum space $X_q$ can be reformulated as a finitely generated projective (left) module $\Gamma(E_q)$ over $C(X_q)$. Based on the strong connection approach to quantum principal bundles [8] for compact quantum groups [34, 35], Hajac and his collaborators introduced quantum line bundles $L_k$ of degree $k$ over $P^n(T)$ as some rank-one projective modules realized as spectral subspaces $C(S^{2n+1}_H)_k$ of $C(S^{2n+1}_H)$ under a $U(1)$-action [12]. Besides having the $K_0$-group of $C(P^n(T))$ computed, they found that $L_k$ is not stably free unless $k = 0$, extending earlier results for the case of $n = 1$ [10] [11].

It has always been an interesting but challenging task to classify finitely generated pro-
jective modules over an algebra up to isomorphism, which goes beyond their classification up to stable isomorphism by $K_0$-group and appears in the form of so-called cancellation problem. Classically it is known that the cancellation law holds for complex vector bundles of rank no less than $\frac{d}{2}$ over a $d$-dimensional CW-complex, which implies that all complex vector bundles over $S^{2n+1}$ of rank $n+1$ or above are trivial.

The study of such classification problem for C*-algebras was popularized by Rieffel [21, 22] who introduced useful versions of stable ranks for C*-algebras to facilitate the analysis involved. Some successes have been achieved for certain quantum algebras [22, 23, 25, 1, 19]. In particular, Peterka showed that all finitely generated projective modules over the $\theta$-deformed 3-spheres $S^3_\theta$ are free, and constructed all those over $S^4_\theta$ up to isomorphism [19]. With more effort, the result of Bach [1] on the cancellation law for $S^{2n+1}_q$ and $\mathbb{C}P^n_q$ can be strengthened to a complete classification of finitely generated projective modules over them, which we will address elsewhere.

With the $K_0$-group of $C(\mathbb{P}^n(T))$ known [12], it is natural to try to classify finitely generated projective modules over $C(\mathbb{P}^n(T))$ and identify the line bundles $L_k$ among them. In [29], a complete solution was obtained for the special case of $n = 1$.

In this paper, we use the powerful groupoid approach to C*-algebras initiated by Renault [20] and popularized by Curto, Muhly, and Renault [6, 15] to study multi-variable Toeplitz C*-algebras $\mathcal{T}^\otimes n$, quantum spheres $C(S^2_{H+1})$, and quantum complex projective spaces $C(\mathbb{P}^n(T))$. Utilizing results on stable ranks of C*-algebras obtained by Rieffel [21], we analyze finitely generated projective modules over $\mathcal{T}^\otimes n+1$ and $C(S^2_{H+1})$, and get those
of rank higher than $\left\lceil \frac{4}{7} \right\rceil + 3$ and also a large class of “standard” modules classified up to isomorphism. Furthermore, besides identifying a large portion of the positive cone of the $K_0$-group $K_0(C(\mathbb{P}^n(T)))$, we explicitly identify the quantum line bundles $L_k$ with concrete representative elementary projections.

On the other hand, there are still a lot of questions to be further investigated, e.g. whether the cancellation law holds for low-ranked finitely generated projective modules, and whether the more general case of $\theta$-twisted multipullback quantum sphere $S^{2n+1}_{H,\theta}$ brings in new phenomena. Finally it is of interest to note the recent work of Farsi, Hajac, Maszczyk, and Zieliński [7] on $K_0(C(\mathbb{P}^2(T)))$, identifying its free generators arising from Milnor modules as sums of $L_k$, which are also expressed in terms of elementary projections, showing a perfect consistency with our result.

2 Notations

Taking the groupoid approach to C*-algebras initiated by Renault [20] and popularized by the work of Curto, Muhly, and Renault [6, 15], we give a description of the C*-algebras $C(S^{2n-1}_H)$ and $C(\mathbb{P}^{n-1}(T))$ of [12] as some concrete groupoid C*-algebras. We refer to [20, 15] for the concepts and theory of groupoid C*-algebras used freely in the following discussion.

By abuse of notation, for any C*-algebra homomorphism $\phi : A \to B$, we denote the C*-algebra homomorphism $M_k(\phi) : M_k(A) \to M_k(B)$ for $k \in \mathbb{N} \equiv \{1, 2, 3, \ldots\}$ also by $\phi$. We use $A^\times$ to denote the set of all invertible elements of an algebra $A$, and use $A^+$ to denote
the minimal unitization of $\mathcal{A}$. For any topological group $G$, we use $G^0$ to denote the identity component of $G$, i.e. the connected component that contains the identity element of $G$.

We denote by $M_\infty (\mathcal{A})$ the direct limit (or the union as sets) of the increasing sequence of matrix algebras $M_n (\mathcal{A})$ over $\mathcal{A}$ with the canonical inclusion $M_n (\mathcal{A}) \subset M_{n+1} (\mathcal{A})$ identifying $x \in M_n (\mathcal{A})$ with $x \oplus 0 \in M_{n+1} (\mathcal{A})$ for any algebra $\mathcal{A}$, where $\oplus$ denotes the standard diagonal concatenation (sum) of two matrices. So the size of an element in $M_\infty (\mathcal{A})$ can be taken arbitrarily large. We also use $GL_\infty (\mathcal{A})$ to denote the direct limit of the general linear groups $GL_n (\mathcal{A})$ over a unital C*-algebra $\mathcal{A}$ with $GL_n (\mathcal{A})$ embedded in $GL_{n+1} (\mathcal{A})$ by identifying $x \in GL_n (\mathcal{A})$ with $x \oplus 1 \in GL_{n+1} (\mathcal{A})$.

By an idempotent $P$ over a unital C*-algebra $\mathcal{A}$, we mean an element $P \in M_\infty (\mathcal{A})$ with $P^2 = P$, and a self-adjoint idempotent in $M_\infty (\mathcal{A})$ is called a projection over $\mathcal{A}$. Two idempotents $P, Q \in M_\infty (\mathcal{A})$ are called equivalent, denoted as $P \sim Q$, if there exists $U \in GL_\infty (\mathcal{A})$ such that $UPU^{-1} = Q$. Each idempotent $P \in M_n (\mathcal{A})$ over $\mathcal{A}$ defines a finitely generated left projective module $E := \mathcal{A}^n P$ over $\mathcal{A}$ where elements of $\mathcal{A}^n$ are viewed as row vectors. The mapping $P \mapsto \mathcal{A}^n P$ induces a bijective correspondence between the equivalence classes of idempotents over $\mathcal{A}$ and the isomorphism classes of finitely generated left projective modules over $\mathcal{A}$. From now on, by a module over $\mathcal{A}$, we mean a left $\mathcal{A}$-module, unless otherwise specified.

Two finitely generated projective modules $E, F$ over $\mathcal{A}$ are called stably isomorphic if they become isomorphic after being augmented by the same finitely generated free $\mathcal{A}$-module, i.e. $E \oplus \mathcal{A}^k \cong F \oplus \mathcal{A}^k$ for some $k \geq 0$. Correspondingly, two idempotents $P$ and $Q$ are
called stably equivalent if $P \boxplus I_k$ and $Q \boxplus I_k$ are equivalent for some identity matrix $I_k$. The $K_0$-group $K_0(\mathcal{A})$ classifies idempotents over $\mathcal{A}$ up to stable equivalence. The classification of idempotents over a $C^*$-algebra up to equivalence, appearing as the so-called cancellation problem, was popularized by Rieffel’s pioneering work [21, 22] and is in general an interesting but difficult question.

The set of all equivalence classes of idempotents over a $C^*$-algebra $\mathcal{A}$ is an abelian monoid $\mathfrak{P}(\mathcal{A})$ with its binary operation provided by the diagonal sum $\boxplus$. The image of the canonical homomorphism from $\mathfrak{P}(\mathcal{A})$ into $K_0(\mathcal{A})$ is the so-called positive cone of $K_0(\mathcal{A})$.

Furthermore, it is well-known [3] that in the above descriptions of $\mathfrak{P}(\mathcal{A})$ and $K_0(\mathcal{A})$, one can restrict to the self-adjoint idempotents, called projections over $\mathcal{A}$, and their unitary equivalence classes, which faithfully represent the elements of $\mathfrak{P}(\mathcal{A})$ and $K_0(\mathcal{A})$.

In this paper, we use freely the basic techniques and manipulations for $K$-theory found in [3, 31].

For a Hilbert space $\mathcal{H}$, we denote the $C^*$-algebra consisting of all compact linear operators on $\mathcal{H}$ by $\mathcal{K}(\mathcal{H})$, or simply by $\mathcal{K}$ if $\mathcal{H}$ is the essentially unique separable infinite-dimensional Hilbert space.

In the following, we use the notations $\mathbb{Z}_{\geq k} := \{ n \in \mathbb{Z} | n \geq k \}$ and $\mathbb{Z}_{\geq} := \mathbb{Z}_{\geq 0}$. In particular, $\mathbb{N} = \mathbb{Z}_{\geq 1}$. We use $I$ to denote the identity operator canonically contained in $\mathcal{K}^+ \subset \mathcal{B}(\ell^2(\mathbb{Z}_{\geq}))$, and

$$P_m := \sum_{i=1}^{m} e_{ii} \in M_m(\mathbb{C}) \subset \mathcal{K}$$

to denote the standard $m \times m$ identity matrix in $M_m(\mathbb{C}) \subset \mathcal{K}$ for any integer $m \geq 0$ (with
$M_0(C) = 0$ and $P_0 = 0$ understood). We also use the notation

$$P_{-m} := I - P_m \in \mathcal{K}^+$$

for integers $m > 0$, and take symbolically $P_{-0} \equiv I - P_0 = I \neq P_0$.

### 3 Quantum spaces as groupoid $C^*$-algebras

Let $\mathfrak{G}_n := \left( \mathbb{Z}^n \rtimes \mathbb{Z}^n \right) |_{\mathbb{Z}^n_{\geq n}}$ with $n \geq 1$ be the transformation group groupoid $\mathbb{Z}^n \rtimes \mathbb{Z}^n$ restricted to the positive “cone” $\mathbb{Z}^n_{\geq n}$ where $\mathbb{Z} \setminus \{ +\infty \}$ containing $\mathbb{Z}_{\geq} \equiv \{ n \in \mathbb{Z} | n \geq 0 \}$ carries the standard topology, and $\mathbb{Z}^n$ acts on $\mathbb{Z}^n$ componentwise in the canonical way. From the groupoid isomorphism

$$\left( \mathbb{Z}^n \rtimes \mathbb{Z}^n \right) |_{\mathbb{Z}^n_{\geq n}} \cong \times^n \left( \left( \mathbb{Z} \rtimes \mathbb{Z} \right) |_{\mathbb{Z}_2} \right)$$

and the well-known $C^*$-algebra isomorphism $C^* \left( \left( \mathbb{Z} \rtimes \mathbb{Z} \right) |_{\mathbb{Z}_2} \right) \cong \mathcal{T}$ for the Toeplitz $C^*$-algebra $\mathcal{T}$, we get the groupoid $C^*$-algebra

$$C^* (\mathfrak{G}_n) \equiv C^* \left( \left( \mathbb{Z}^n \rtimes \mathbb{Z}^n \right) |_{\mathbb{Z}^n_{\geq n}} \right) \cong \mathcal{T}^n \equiv \otimes^n \mathcal{T}.$$

We consider two important nontrivial invariant open subsets of the unit space $\mathbb{Z}^n_{\geq}$ of $\mathfrak{G}_n$, namely, $\mathbb{Z}_{\geq}^n$ the smallest one and $\mathbb{Z}_{\geq}^n \setminus \{ \infty^n \}$ the largest one, where $\infty^n := (\infty, \ldots, \infty) \in \mathbb{Z}^n_{\geq}$. By the theory of groupoid $C^*$-algebras developed in Renault’s book [20], they give rise to two short exact sequences of $C^*$-algebras

$$0 \to C^* \left( \left( \mathbb{Z}^n \rtimes \mathbb{Z}^n \right) |_{\mathbb{Z}^n_{\geq}} \right) \cong \mathcal{K} \left( \ell^2 \left( \mathbb{Z}^n_{\geq} \right) \right) \to C^* (\mathfrak{G}_n) \equiv \mathcal{T}^\otimes n \to C^* (\mathfrak{G}_n) \to 0$$
with $\mathcal{K} (\ell^2 (\mathbb{Z}_n)) \cong \otimes^n \mathcal{K} \equiv \mathcal{K}^{\otimes n}$ where

$$\mathfrak{G}_n := \left. \left( \mathbb{Z}^n \rtimes \mathbb{Z}^n \right) \right|_{\mathbb{Z}_n^\infty \setminus \mathbb{Z}_n^n}$$

is $\mathfrak{G}_n$ restricted to the “limit boundary” of its unit space, and

$$0 \to C^* \left( \left. \left( \mathbb{Z}^n \rtimes \mathbb{Z}^n \right) \right|_{\mathbb{Z}_n^\infty \setminus \{\infty\}^n} \right) \to C^* (\mathfrak{G}_n) \equiv \mathcal{T}^{\otimes n} \xrightarrow{\sigma_n} C^* \left( \left. \left( \mathbb{Z}^n \rtimes \mathbb{Z}^n \right) \right|_{\{\infty\}^n} \right) \cong C^* (\mathbb{Z}^n) \cong C (\mathbb{T}^n) \to 0$$

where the quotient map $\sigma_n$ extends the notion of the well-known symbol map $\sigma$ on $\mathcal{T}$ in the case of $n = 1$.

Note that the open invariant set $\mathbb{Z}_n^n$ being dense in the unit space $\mathbb{Z}_n^n$ of $\mathfrak{G}_n$ induces a faithful representation $\pi_n$ of $C^* (\mathfrak{G}_n)$ on $\ell^2 (\mathbb{Z}_n^n)$ that realizes the groupoid C*-algebra $C^* (\mathfrak{G}_n)$ and its closed ideal $C^* \left( \left. \left( \mathbb{Z}^n \rtimes \mathbb{Z}^n \right) \right|_{\mathbb{Z}_n^n} \right)$ respectively as a C*-subalgebra of $\mathcal{B} (\ell^2 (\mathbb{Z}_n^n))$ and the closed ideal $\mathcal{K} (\ell^2 (\mathbb{Z}_n^n))$ consisting of all compact operators on $\ell^2 (\mathbb{Z}_n^n)$.

In this paper, we freely identify elements of $C^* (\mathfrak{G}_n) \equiv \mathcal{T}^{\otimes n}$ with operators on $\ell^2 (\mathbb{Z}_n^n)$ via the faithful representation $\pi_n$ and use these two conceptually different notions interchangeably.

In [12], Hajac, Nest, Pask, Sims, and Zieliński defined the (untwisted) multipullback or Heegaard quantum odd-dimensional sphere $S_{H}^{2n-1}$ as the quantum space of the multipullback C*-algebra [18] determined by homomorphisms of the form $\text{id}^\otimes j \otimes \sigma \otimes \text{id}^{\otimes n-j-1}$ from $\mathcal{T}^{\otimes i} \otimes C (\mathbb{T}) \otimes \mathcal{T}^{\otimes n-i-1}$ with $i \neq j$ to some $\mathcal{T}^{\otimes m} \otimes C (\mathbb{T}) \otimes \mathcal{T}^{\otimes k} \otimes C (\mathbb{T}) \otimes \mathcal{T}^{\otimes n-m-k-2}$. (Actually more general $\theta$-twisted quantum spheres $S_{H, \theta}^{2n-1}$ are studied there.) They showed that

$$C \left( S_{H}^{2n-1} \right) \cong (\otimes^n \mathcal{T}) / (\otimes^n \mathcal{K})$$
and hence we have

\[ C^* \left( S_{H}^{2n-1} \right) \cong C^* (G_n) \]

identified as a groupoid C*-algebra.

With the ideal \( C^* \left( \left( \mathbb{Z}^n \rtimes \mathbb{Z}^n \right) \mid \mathbb{Z}_{\geq n} \setminus \{\infty\} \right) \) containing the ideal \( C^* \left( \left( \mathbb{Z}^n \rtimes \mathbb{Z}^n \right) \mid \mathbb{Z}_{\geq n} \right) \), the quotient map \( \sigma_n \) induces a well-defined quotient map \( \tau_n \) in the short exact sequence

\[ 0 \to C^* \left( \left( \mathbb{Z}^n \rtimes \mathbb{Z}^n \right) \mid \mathbb{Z}_{\geq n} \setminus \{\infty\} \right) \to C^* \left( S_{H}^{2n-1} \right) \cong \left( \otimes^n \mathcal{T} \right) / \left( \otimes^n K \right) \xrightarrow{\tau_n} C \left( \mathbb{T}^n \right) \to 0. \]

### 4 Stable ranks of quantum spaces

In his seminal paper [21], Rieffel introduced and popularized the notions of topological stable rank \( \text{tsr} (\mathcal{A}) \) and connected stable rank \( \text{csr} (\mathcal{A}) \) of a C*-algebra \( \mathcal{A} \), which are useful tools in the study of cancellation problems for finitely generated projective modules. Later, Herman and Vaserstein [13] showed that for C*-algebras \( \mathcal{A} \), Rieffel’s topological stable rank coincides with the Bass stable rank used in algebraic K-theory. So we will denote \( \text{tsr} (\mathcal{A}) \) simply as \( \text{sr} (\mathcal{A}) \) in our discussion.

In this section, we review an estimate of the stable ranks of the Toeplitz algebras \( \mathcal{T}^{\otimes n} \) and quantum spheres \( C^* (S_{H}^{2n-1}) \), which will be used in our study of their finitely generated projective modules. For the case of \( n = 1 \), it is known [21] that \( \text{sr} (\mathcal{T}) = \text{csr} (C (\mathbb{T})) = 2 \).

As an illustration of the groupoid approach to C*-algebras, we first establish some composition sequence structure for \( \mathcal{T}^{\otimes n} \) and \( C^* (S_{H}^{2n-1}) \), which leads to an easy estimate of their stable ranks.
**Proposition 1.** There is a finite composition sequence of closed ideals

\[ \mathcal{T}^\otimes_n \equiv C^*(\mathfrak{T}_n) \equiv \mathcal{I}_n \supset \mathcal{I}_{n-1} \supset \cdots \supset \mathcal{I}_1 \supset \mathcal{I}_0 \supset \mathcal{I}_0 \equiv \{0\} \]

such that \( \mathcal{T}^\otimes_n / \mathcal{I}_0 \cong C(\mathbb{S}^{2n-1}_H) \), and for \( 0 \leq j \leq n \),

\[ \mathcal{I}_j / \mathcal{I}_{j-1} \cong \bigoplus_{j!(n-j)!} C^* \left( \mathbb{P}^2 (\mathbb{Z}_n^{n-j}) \right) \otimes C(\mathbb{T}^j) , \]

where \( \mathbb{T}^0 \) and \( \mathbb{Z}_0^0 \) denote a singleton.

**Proof.** For \( 0 \leq j \leq n \), let \( X_j \) be the set consisting of \( z \in \mathbb{Z}_n^\otimes \) with exactly \( j \) of the components \( z_1, z_2, ..., z_n \) being equal to \( \infty \), and hence \( X_n = \{\infty^n\} \). Then the sets

\[ Y_j := X_0 \sqcup X_1 \sqcup \cdots \sqcup X_j \]

are open invariant subsets of the unit space \( \overline{\mathbb{Z}_n^\otimes} \) of \( \mathfrak{T}_n \) with

\[ \mathbb{Z}_n^\otimes = Y_0 \subset Y_1 \subset \cdots \subset Y_n = \overline{\mathbb{Z}_n^\otimes} \]

which determines an increasing chain of closed ideals \( \mathcal{I}_0 \triangleleft \mathcal{I}_1 \triangleleft \cdots \triangleleft \mathcal{I}_n \) of \( C^*(\mathfrak{T}_n) \) defined by

\[ \mathcal{I}_j := C^* \left( (\mathbb{Z}^n \ltimes \mathbb{Z}_n^\otimes) \big|_{Y_j} \right) \equiv C^* \left( \mathfrak{T}_n \big|_{Y_j} \right) . \]

Note that \( Y_j \setminus Y_{j-1} = X_j \) with \( Y_{-1} := \emptyset \) is a disjoint union of \( \frac{n!}{j!(n-j)!} \) copies of \( \mathbb{Z}_n^{n-j} \times \{\infty^j\} \) each of which is gotten from one of the \( \frac{n!}{j!(n-j)!} \) possible selections of exactly \( j \) of the \( n \) components of \( \mathbb{Z}_n^\otimes \). With each such copy of \( \mathbb{Z}_n^{n-j} \times \{\infty^j\} \) clearly a closed invariant subset of \( Y_j \setminus Y_{j-1} \), these \( \frac{n!}{j!(n-j)!} \) copies of \( \mathbb{Z}_n^{n-j} \times \{\infty^j\} \) are open invariant subsets of \( Y_j \setminus Y_{j-1} \), and hence

\[ C^* \left( \mathfrak{T}_n \big|_{Y_j \setminus Y_{j-1}} \right) = \bigoplus_{j!(n-j)!} C^* \left( (\mathbb{Z}^n \ltimes \mathbb{Z}_n^\otimes) \big|_{\mathbb{Z}_n^{n-j} \times \{\infty^j\}} \right) \]

11
Thus with $I_j = C^* \left( \mathfrak{T}_n|_Y \right)$ and $I_{j-1} = C^* \left( \mathfrak{T}_n|_{Y_{j-1}} \right)$, we get

$$I_j/I_{j-1} \cong C^* \left( \mathfrak{T}_n|_{Y_{j-1}} \right) \cong \oplus \frac{n!}{j! (n-j)!} \left( K \left( \ell^2 \left( \mathbb{Z}^n_{\geq j} \right) \right) \otimes C \left( T^j \right) \right).$$

□

Corollary 1. There is a finite composition sequence of closed ideals

$$C \left( S^{2n-1}_H \right) \equiv C^* \left( \mathfrak{S}_n \right) \equiv \mathcal{J}_n \triangleright \mathcal{J}_{n-1} \triangleright \cdots \triangleright \mathcal{J}_1 \triangleright \mathcal{J}_0 \equiv \{0\}$$

such that for $1 \leq j \leq n$,

$$\mathcal{J}_j/\mathcal{J}_{j-1} \cong \oplus \frac{n!}{j! (n-j)!} \left( K \left( \ell^2 \left( \mathbb{Z}^n_{\geq j} \right) \right) \otimes C \left( T^j \right) \right).$$

Proof. With $I_0 = K \left( \ell^2 \left( \mathbb{Z}^n_G \right) \right)$ and hence $C^* \left( \mathfrak{T}_n \right)/I_0 \cong C \left( S^{2n-1}_H \right)$, we simply take $\mathcal{J}_j := I_j/I_0$. □

The above composition sequences lead to the straightforward estimates

$$\left\lfloor \frac{n}{2} \right\rfloor + 1 \leq \sr \left( C \left( S^{2n-1}_H \right) \right) \leq \sr \left( T^{\otimes n} \right) \leq \left\lfloor \frac{n+1}{2} \right\rfloor + 1$$

and

$$\csr \left( T^{\otimes n} \right) \leq \csr \left( C \left( S^{2n-1}_H \right) \right) \leq \left\lfloor \frac{n+1}{2} \right\rfloor + 1$$

for all $n \geq 1$, based on the general rules established in [21] that (i) $\sr \left( \mathcal{A} \otimes K \right) = \min \{2, \sr \left( \mathcal{A} \right) \}$, (ii) for any closed ideal $\mathcal{I}$ of a $C^*$-algebra $\mathcal{A}$,

$$\max \{ \sr \left( \mathcal{A}/\mathcal{I} \right), \sr \left( \mathcal{I} \right) \} \leq \sr \left( \mathcal{A} \right) \leq \max \{ \sr \left( \mathcal{A}/\mathcal{I} \right), \sr \left( \mathcal{I} \right), \csr \left( \mathcal{A}/\mathcal{I} \right) \},$$

12
and (iii) \( \text{sr} (C (X)) = \left\lceil \frac{n}{2} \right\rceil + 1 \) for any \( n \)-dimensional CW-complex \( X \), and the rule \([25, 16, 17]\) that for any closed ideal \( \mathcal{I} \) of a C*-algebra \( A \), (iv) \( \text{csr} (A \otimes K) \leq 2 \) (with \( \text{csr} (K) = 1 \)) and (v)

\[
\text{csr} (A) \leq \max \{ \text{csr} (A/\mathcal{I}) , \text{csr} (\mathcal{I}) \}.
\]

Indeed, for \( n > 1 \), applying (i)-(ii) and (iv)-(v) to the short exact sequences

\[
0 \to \mathcal{I}_{j-1} \to \mathcal{I}_j \to \mathcal{I}_j/\mathcal{I}_{j-1} \cong \bigoplus_{n(n-j)} \left( \mathcal{K} (\ell^2 (\mathbb{Z}^{n-j}_2)) \otimes C (\mathbb{T}^j) \right) \to 0
\]

inductively for \( j \) increasing from 1 to \( n - 1 \), starting with the exact sequence

\[
0 \to \mathcal{K} (\ell^2 (\mathbb{Z}^n_2)) \cong \mathcal{I}_0 \to \mathcal{I}_1 \to \mathcal{I}_1/\mathcal{I}_0 \cong \bigoplus^n \left( \mathcal{K} (\ell^2 (\mathbb{Z}^n_2)) \otimes C (\mathbb{T}) \right) \to 0
\]

for \( j = 1 \), we get \( \text{csr} (\mathcal{I}_j) , \text{sr} (\mathcal{I}_j) \leq 2 \) for all \( 1 \leq j \leq n - 1 \). In particular, \( \text{csr} (\mathcal{I}_{n-1}) , \text{sr} (\mathcal{I}_{n-1}) \leq 2 \), which is also valid for \( n = 1 \) since \( \mathcal{I}_0 \cong \mathcal{K} (\ell^2 (\mathbb{Z}^n_2)) \). Then with \( \text{csr} (C (\mathbb{T}^n)) \leq \left\lfloor \frac{n+1}{2} \right\rfloor + 1 \) by homotopy theory \([33]\), we get \( \text{csr} (\mathcal{T}^\otimes n) \leq \left\lfloor \frac{n+1}{2} \right\rfloor + 1 \) and

\[
\left\lfloor \frac{n}{2} \right\rfloor + 1 \leq \text{sr} (\mathcal{T}^\otimes n) \leq \left\lfloor \frac{n+1}{2} \right\rfloor + 1
\]

by further applying (ii)-(iii) and (v) to the short exact sequence

\[
0 \to \mathcal{I}_{n-1} \to \mathcal{I}_n \equiv \mathcal{T}^\otimes n \to \mathcal{I}_n/\mathcal{I}_{n-1} \cong C (\mathbb{T}^n) \to 0.
\]

Similar argument yields \( \text{csr} (C (S^2n-1)_{\mathcal{H}}) \leq \left\lfloor \frac{n+1}{2} \right\rfloor + 1 \) and \( \left\lceil \frac{n}{2} \right\rceil + 1 \leq \text{sr} (C (S^2n-1)_{\mathcal{H}}) \leq \left\lfloor \frac{n+1}{2} \right\rfloor + 1 \), with the inequality \( \text{sr} (C (S^2n-1)_{\mathcal{H}}) \leq \text{sr} (\mathcal{T}^\otimes n) \) obviously valid by (ii). Also \( \text{csr} (\mathcal{T}^\otimes n) \leq \text{csr} (C (S^2n-1)_{\mathcal{H}}) \) by (iv)-(v).

Such an estimate determining \( \text{sr} (\mathcal{T}^\otimes n) \) sharply for even \( n \) and up to an error of 1 for odd \( n > 1 \) as stated above was first obtained by G. Nagy in \([16]\) and then sharpened to the exact
value
\[
\text{sr} \left( \mathcal{T}^{\otimes n} \right) = \left\lfloor \frac{n}{2} \right\rfloor + 1 \quad \text{(and hence } \text{sr} \left( C \left( S_{H}^{2n-1} \right) \right) = \left\lfloor \frac{n}{2} \right\rfloor + 1) \]

for general \( n > 1 \) by Nistor in [17] which also gives \( \text{csr} \left( \mathcal{T}^{\otimes n} \right) \leq \left\lfloor \frac{n+1}{2} \right\rfloor + 1 \). We summarize these results as follows.

**Proposition 2.** For all \( n > 1 \),
\[
\text{sr} \left( C \left( S_{H}^{2n-1} \right) \right) = \text{sr} \left( \mathcal{T}^{\otimes n} \right) = \left\lfloor \frac{n}{2} \right\rfloor + 1
\]
and
\[
\text{csr} \left( \mathcal{T}^{\otimes n} \right) \leq \text{csr} \left( C \left( S_{H}^{2n-1} \right) \right) \leq \left\lfloor \frac{n+1}{2} \right\rfloor + 1.
\]

**Corollary 2.** For any \( n > 1 \) and any \( k \geq \left\lfloor \frac{n}{2} \right\rfloor + 3 \), the topological group \( GL_{k} \left( \mathcal{T}^{\otimes n} \right) \) is connected.

Proof. By the K"unneth formula [3] for \( K \)-groups, we get \( K_{1} \left( \mathcal{T}^{\otimes n} \right) = 0 \) since \( K_{1} \left( \mathcal{T} \right) = 0 \) is well known. So by the theorem [21] that \( K_{1} \left( A \right) \cong GL_{k} \left( A \right)/GL_{k}^{0} \left( A \right) \) for any unital \( C^{*} \)-algebra \( A \) with \( k \geq \text{sr} \left( A \right) + 2 \), we get \( GL_{k} \left( \mathcal{T}^{\otimes n} \right) = GL_{k}^{0} \left( \mathcal{T}^{\otimes n} \right) \) for any \( k \geq \left\lfloor \frac{n}{2} \right\rfloor + 3 \geq \text{sr} \left( \mathcal{T}^{\otimes n} \right) + 2 \).

\[ \square \]

Note that the above statement holds for the case of \( n = 1 \), since \( GL_{k} \left( \mathcal{T} \right) \) is connected for all \( k \geq 1 \) in the case of \( n = 1 \) by the index theory of Toeplitz operators for the unit disk \( \mathbb{D} \).


5 Projective modules over $\mathcal{T}^\otimes n$

Before proceeding to study finitely generated projective modules over $\mathcal{T}^\otimes n$, we now point out a structure of $\mathcal{T}^\otimes n$ which facilitates some inductive procedures for the study of such modules.

For all $n \in \mathbb{N}$, the topological groupoid $\mathfrak{S}_n|_{\mathbb{Z}_>^{n-1} \times \{\infty\}}$ is isomorphic to the product topological groupoid $\mathfrak{S}_{n-1} \times \mathbb{Z}$, while the topological groupoid $\mathfrak{S}_n|_{\mathbb{Z}_>^{n-1} \times \mathbb{Z}_>}$ is isomorphic to the product topological groupoid $\mathfrak{S}_{n-1} \times (\mathbb{Z} \ltimes \mathbb{Z})|_{\mathbb{Z}_>}$, where the closed subset $\mathbb{Z}_>^{n-1} \times \{\infty\}$ and its open complement $\mathbb{Z}_>^{n-1} \times \mathbb{Z}_>$ in the unit space $\mathbb{Z}_>^n$ of $\mathfrak{S}_n$ are invariant. (Here it is understood that when $n-1 = 0$, the first factor $\mathbb{Z}_>^{n-1}$ is dropped.) Hence we get the short exact sequence of $C^*$-algebras

$$0 \to C^*(\mathfrak{S}_n|_{\mathbb{Z}_>^{n-1} \times \mathbb{Z}_>}) \cong \mathcal{T}^\otimes (n-1) \otimes K(\ell^2(\mathbb{Z}_>)) \to C^*(\mathfrak{S}_n) \equiv \mathcal{T}^\otimes n \otimes C(\mathbb{T}) \to 0$$

with $\mathcal{T}^\otimes 0 := \mathbb{C}$. Furthermore the quotient maps $\kappa_n$ for $n \in \mathbb{N}$ resulting from a groupoid restriction satisfy the commuting diagram

$$\begin{array}{ccc}
M_k(\mathcal{T}^\otimes n) & \xrightarrow{\kappa_n} & M_k(\mathcal{T}^\otimes (n-1) \otimes C(\mathbb{T})) \\
\downarrow \sigma_n & \circ & \downarrow \sigma_{n-1} \otimes \text{id} \\
M_k(C(\mathbb{T}^n)) & \equiv & M_k(C(\mathbb{T}^{n-1}) \otimes C(\mathbb{T}))
\end{array}$$

where $\equiv$ stands for a canonical isomorphism and $\sigma_0 := \text{id}_\mathbb{C}$.

To classify the isomorphism classes of finitely generated projective $\mathcal{T}^\otimes n$-modules $E$ or equivalently the equivalence classes of idempotents $P \in M_\infty(\mathcal{T}^\otimes n)$ over $\mathcal{T}^\otimes n$, we first define
the rank of (the class of) $E$ or $P$ as the classical rank of (the isomorphism class of) the vector bundle corresponding to (the class of) the $C(\mathbb{T}^n)$-module $C(\mathbb{T}^n) \otimes_T \mathcal{T}^n$ $E$ or the projection $\sigma_n(P)$ over $C(\mathbb{T}^n)$.

The set of equivalence classes of idempotents $P \in M_\infty(\mathcal{T}^\otimes n)$ equipped with the binary operation $\boxplus$ becomes an abelian graded monoid

$$\mathfrak{P}(\mathcal{T}^\otimes n) = \bigsqcup_{m=0}^\infty \mathfrak{P}_m(\mathcal{T}^\otimes n)$$

where $\mathfrak{P}_m(\mathcal{T}^\otimes n)$ is the set of all (equivalence classes of) idempotents over $\mathcal{T}^\otimes n$ of rank $m$, and

$$\mathfrak{P}_m(\mathcal{T}^\otimes n) \boxplus \mathfrak{P}_l(\mathcal{T}^\otimes n) \subset \mathfrak{P}_{m+l}(\mathcal{T}^\otimes n)$$

for $m, l \geq 0$. Clearly $\mathfrak{P}_0(\mathcal{T}^\otimes n)$ is a submonoid of $\mathfrak{P}(\mathcal{T}^\otimes n)$.

Next we define a submonoid of $\mathfrak{P}(\mathcal{T}^\otimes n)$ generated by “standard” type of idempotents, which turns out to contain (equivalence classes of) all idempotents of sufficiently high ranks, and then classify its elements.

Note that each permutation $\Theta$ on $\{1, 2, ..., n\}$ induces canonically a C*-algebra automorphism, still denoted as $\Theta$ by abuse of notation, on $\mathcal{T}^\otimes n$ by permuting the indices of the factors in $a_1 \otimes a_2 \otimes \cdots \otimes a_n \in \mathcal{T}^\otimes n$ for $a_i \in \mathcal{T}$. A permutation $\Theta$ on $\{1, 2, ..., n\}$ is called a $(j, n-j)$-shuffle on $\{1, 2, ..., n\}$ if $\Theta(1) < \Theta(2) < \cdots < \Theta(j)$ and $\Theta(j+1) < \Theta(j+2) < \cdots < \Theta(n)$.

Some basic projections over $\mathcal{T}^\otimes n$ are given by $\Theta(P_{j,l})$ where

$$P_{j,l} := \boxplus^l \left((\otimes^j I) \otimes (\otimes^{n-j} P_1)\right) \in M_l(\mathcal{T}^\otimes n)$$

for $l \geq 0$ and $0 \leq j \leq n$ (in particular, $P_{n,m} \equiv \boxplus^m (\otimes^n I) \equiv \boxplus^m \tilde{I}$ for the unit $\tilde{I}$ of
T^\otimes n), and Θ is (the automorphism defined by) a \((j, n-j)\)-shuffle on \(\{1, 2, ..., n\}\). Note that
\[
Θ(P_{j,t}) = Θ(⊞^tP_{j,1}) = ⊞^tΘ(P_{j,1}),
\]
and \((⊞^j I) ⊟ (⊞^{n-j-1} P_1) ⊟ P_t \sim P_{j,t} \otimes T^\otimes n\) since \(P_t \sim ⊞^t P_1 \otimes K^+ \subset T\). Furthermore
\[
σ_n(Θ(P_{j,t})) = \begin{cases} 0, & \text{if } 0 \leq j \leq n-1 \\ ⊞^t 1, & \text{if } j = n \end{cases},
\]
and hence \(Θ(P_{j,t}) \in \mathcal{P}_0 (T^\otimes n)\) if \(j < n\) and \(Θ(P_{n,t}) = P_{n,t} \in \mathcal{P}_1 (T^\otimes n)\), where \(1 \in C (\mathbb{T}^n)\) is the constant function 1 on \(\mathbb{T}^n\). So the set \(\mathcal{P}_0 (T^\otimes n) \subset \mathcal{P}_0 (T^\otimes n)\) consisting of (the equivalence classes of) all possible \(⊞\)-sums of \(Θ(P_{j,t})\) with \(l \geq 0\) and \(Θ\) a \((j, n-j)\)-shuffle on \(\{1, 2, ..., n\}\) for \(0 \leq j \leq n-1\) is a submonoid of \(\mathcal{P}_0 (T^\otimes n)\). For \(m \geq 1\), we define a singleton
\[
\mathcal{P}_m (T^\otimes n) := \{ P_{n,m} \equiv ⊞^n \tilde{I} \} \subset \mathcal{P}_m (T^\otimes n)
\]
where \(\tilde{I}\) denotes the identity element of \(T^\otimes n\). Clearly \(\sqcup_{m=1}^∞ \mathcal{P}_m (T^\otimes n)\) is also a submonoid of \(\mathcal{P} (T^\otimes n)\).

We define a partial ordering \(\prec\) on the collection
\[
Ω := \{(j, Θ) : 0 \leq j \leq n \text{ and } Θ \text{ is a } (j, n-j)\text{-shuffle}\}
\]
by the condition that \((j', Θ') \prec (j, Θ)\) if and only if \(Θ(\{1, 2, ..., j\}) \supseteq Θ'(\{1, 2, ..., j'\})\) (and hence \(j > j'\)). Here \(\{1, 2, ..., 0\} \equiv \emptyset\) is understood. Note that \(id_{\{1,2,...,n\}}\) is a \((j, n-j)\)-shuffle for every \(j\), and \((n, id_{\{1,2,...,n\}})\) is the greatest element while \((0, id_{\{1,2,...,n\}})\) is the smallest element in \(Ω\) with respect to \(\prec\).
Proposition 3. $Ψ'(T^{\otimes n}) = \bigsqcup_{m=0}^{\infty} Ψ_m'(T^{\otimes n})$ is a graded submonoid of $Ψ(T^{\otimes n})$ and its monoid structure is explicitly determined by that for any $l,l' > 0$ and any $(j',\Theta') \prec (j,\Theta)$ in $Ω$,

$$\Theta(P_{j,l}) \boxplus \Theta'(P_{j',l'}) \sim \Theta(P_{j,l}).$$

Proof. Note that since $Ψ'_0(T^{\otimes n})$ and $\sqcup_{m=1}^{\infty} Ψ_m'(T^{\otimes n})$ are submonoids of $Ψ(T^{\otimes n})$, the set $Ψ'(T^{\otimes n})$ is a submonoid if $\Theta(P_{n,m}) \boxplus \Theta'(P_{j',l'}) \sim \Theta(P_{n,m})$ holds for all $m > 0$ and all $\Theta'(P_{j',l'})$ with $j' \leq n - 1$. Since $(n,\text{id}_{\{1,2,...,n\}})$ is the greatest element in $Ω$, it remains to show that $\Theta(P_{j,l}) \boxplus \Theta'(P_{j',l'}) \sim \Theta(P_{j,l})$ for $n \geq j > j' \geq 0$ with $\Theta(\{1,2,...,j\}) \supset \Theta'(\{1,2,...,j'\})$ and $l,l' > 0$.

Note that for $\Theta(\{1,2,...,j\}) \supset \Theta'(\{1,2,...,j'\})$, there exists a permutation $\Theta''$ (not necessarily a shuffle) on $\{1,2,...,n\}$ such that $\Theta''(\Theta(P_{j,l})) = P_{j,l}$ and $\Theta''(\Theta'(P_{j',l'})) = P_{j',l'}$. (In fact, one can find a permutation $\Theta''$ such that $\Theta''\Theta$ fixes each of $j + 1,..,n$, and $\Theta''\Theta'$ is each of $1,2,...,j'$.) So it suffices to prove that

$$P_{j,l} \boxplus P_{j',l'} \sim P_{j,l}$$

whenever $j > j'$ and $l,l' > 0$. Furthermore since $P_{j,l} = \boxplus l P_{j,1}$, we only need to show that $P_{j,1} \boxplus P_{j',1} \sim P_{j,1}$ for $j > j'$.

Note that $U (P_1 \boxplus I) U^* = 0 \boxplus I$ in $M_2(Ω)$ for the unitary

$$U := e_{11} \otimes S^* + e_{22} \otimes S + e_{21} \otimes e_{11} \in M_2(C) \otimes Ω \equiv M_2(Ω)$$

where $S \in Ω$ is the (forward) unilateral shift on $ℓ^2(\mathbb{Z}_\geq)$. So

$$P_{j,1} \boxplus P_{j-1,1} = ((\otimes^j I) \otimes (\otimes^{n-j} P_1)) \boxplus ((\otimes^{j-1} I) \otimes (\otimes^{n-j+1} P_1))$$

18
\[
\left(\otimes^{j-1} I\right) \otimes (I \oplus P_1) \otimes \left(\otimes^{n-j} P_1\right) \sim \left(\otimes^{j-1} I\right) \otimes I \otimes \left(\otimes^{n-j} P_1\right) = P_{j,1}.
\]

Thus by iteration of this result, we can “expand” \(P_{j,1}\) to get for any \(0 \leq k < j\),

\[
P_{j,1} \sim P_{j,1} \oplus P_{j-1,1} \oplus \cdots \oplus P_{k,1},
\]

and hence

\[
P_{j,1} \oplus P_{j',1} \sim P_{j,1} \oplus P_{j-1,1} \oplus \cdots \oplus P_{j',1} \oplus P_{j'+1,1} \sim P_{j,1}.
\]

\(\square\)

For each \((j, \Theta) \in \Omega\), let \(X_\Theta \subset \mathbb{Z}_{\geq n}^n\) be the invariant closed subset of the unit space of \(\Sigma_n\) consisting of \(z \in \mathbb{Z}_{\geq n}^n\) with \(z_k = \infty\) for all \(k \in \Theta\) (\(\{1, 2, \ldots, j\}\)), and let

\[
\sigma_{(j, \Theta)} : C^* \left(\Sigma_n\right) \to C^* \left(\Sigma_n|_{X_\Theta}\right) \cong C \left(\mathbb{T}^j\right) \otimes \mathcal{T}^\otimes n-j \subset C \left(\mathbb{T}^j\right) \otimes \mathcal{B} \left(\ell^2 \left(\mathbb{Z}_{\geq n}^{n-j}\right)\right)
\]

be the canonical quotient map, where the isomorphism implicitly involves a rearrangement of factors by the inverse permutation \(\Theta^{-1}\). Here as before, \(\mathbb{T}^0\) is a singleton. Defining \(\rho_{(j, \Theta)}(P)\) for an idempotent \(P\) over \(C^* \left(\Sigma_n\right)\) as the rank of the projection operator \(\sigma_{(j, \Theta)}(P)(t) \in \mathcal{B} \left(\ell^2 \left(\mathbb{Z}_{\geq n}^{n-j}\right)\right)\) for any \(t \in \mathbb{T}^j\), which depends only on the equivalence class of \(P\), we get a well-defined monoid homomorphism

\[
\rho_{(j, \Theta)} : (\mathcal{P} \left(\mathcal{T}^\otimes n\right), \oplus) \to (\mathbb{Z}_{\geq} \cup \{\infty\}, +).
\]

A (finite) \(\oplus\)-sum of (the equivalence classes of) projections \(\Theta(P_{j,l})\) indexed by some \((j, \Theta) \in \Omega\) that are mutually unrelated by \(<\) with \(l \equiv l_{(j, \Theta)} > 0\) depending on \((j, \Theta)\) is called a reduced \(\oplus\)-sum of standard projections over \(\mathcal{T}^\otimes n\). It is understood that an “empty” \(\oplus\)-sum represents the zero projection and is a reduced \(\oplus\)-sum. Two reduced \(\oplus\)-sums are
called different when they have different sets of (mutually \(\prec\)-unrelated) indices \((j, \Theta) \in \Omega\) or have different weight functions \(l\) of \((j, \Theta)\). We are going to show that different reduced \(\boxplus\)-sums are inequivalent projections. Clearly each projection \(\Theta (P_{j,l})\) with \((j, \Theta) \in \Omega\) and \(l > 0\) is a reduced \(\boxplus\)-sum.

**Theorem 1.** The submonoid \(\mathfrak{P}' (\mathcal{T}^\otimes n) = \sqcup_{m=0}^\infty \mathfrak{P}'_m (\mathcal{T}^\otimes n)\) of \(\mathfrak{P} (\mathcal{T}^\otimes n)\) consists exactly of reduced \(\boxplus\)-sums of standard projections over \(\mathcal{T}^\otimes n\), and different reduced \(\boxplus\)-sums are mutually inequivalent projections. Furthermore the monoid homomorphism

\[
\rho : P \in \mathfrak{P}' (\mathcal{T}^\otimes n) \mapsto \prod_{(j, \Theta) \in \Omega} \rho_{(j, \Theta)} (P) \in \prod_{(j, \Theta) \in \Omega} \mathbb{Z}_\geq
\]

is injective, with \(\rho_{(j, \Theta)} (\Theta (P_{j,l})) = l \in \mathbb{N}\).

Proof. By definition, \(\mathfrak{P}' (\mathcal{T}^\otimes n)\) consists of \(\boxplus\)-sums of (the equivalence classes of) projections \(\Theta (P_{j,l})\) with \((j, \Theta) \in \Omega\) and \(l > 0\). Since \(\Theta (P_{j,l}) + \Theta (P_{j,l'}) \sim \Theta (P_{j,l+l'})\), we only need to consider in the following those \(\boxplus\)-sums, in which all summands \(\Theta (P_{j,l})\) are indexed by distinct \((j, \Theta) \in \Omega\) with \(l\) depending on \((j, \Theta)\). For any such a \(\boxplus\)-sum, using the property that \(\Theta (P_{j,l}) \boxplus \Theta' (P'_{j',l'}) \sim \Theta (P_{j,l})\) for any \((j', \Theta') \prec (j, \Theta)\), we can remove one by one those \(\boxplus\)-summands \(\Theta' (P'_{j',l'})\) with \((j', \Theta')\) dominated by the index of another summand, without changing the equivalence class, until we reach a \(\boxplus\)-sum of \(\Theta (P_{j,l})\) with \((j, \Theta) \in \Omega\) mutually unrelated by \(\prec\), i.e. a reduced \(\boxplus\)-sum. So \(\mathfrak{P}' (\mathcal{T}^\otimes n)\) consists of the reduced \(\boxplus\)-sums.

Note that for \((j, \Theta) \in \Omega\) and \(l > 0\),

\[
\sigma_{(j, \Theta)} (\Theta (P_{j,l})) = \sigma_{(j, \Theta)} (\boxplus \Theta ((\otimes^j I) \otimes (\otimes^{n-j} P_1)))
\]

\[
= 1 \otimes (\boxplus (\otimes^{n-j} P_1)) \in C (\mathbb{T}^j) \otimes (\boxplus \mathcal{B} (\ell^2 (\mathbb{Z}_{\geq}^{n-j})))
\]
and hence $\rho_{(j,\Theta)}(\Theta(P_{j,l})) = l \in \mathbb{N}$ the operator rank of $\mathbb{H}^l(\otimes^{n-j} P_1) \in \mathcal{B}(\oplus^l \ell^2(\mathbb{Z}_2^{n-j})).$ But for $(j',\Theta') \neq (j,\Theta),$

$$\rho_{(j,\Theta)}(\Theta'(P_{j',l'})) := \begin{cases} \infty, & \text{if } (j,\Theta) \prec (j',\Theta') \\ 0, & \text{if otherwise} \end{cases}$$

because either $\sigma_{(j,\Theta)}(\Theta'(P_{j',l'})) = 0$ when $\Theta(\{1, 2, \ldots, j\}) \setminus \Theta'(\{1, 2, \ldots, j'\}) \neq \emptyset,$ or $\sigma_{(j,\Theta)}(\Theta'(P_{j',l'}))$ is an infinite-dimensional projection when $\Theta'(\{1, 2, \ldots, j'\}) \supset \Theta(\{1, 2, \ldots, j\})$ (but $\Theta(\{1, 2, \ldots, j\}) \neq \Theta'(\{1, 2, \ldots, j'\})$ since $(j',\Theta') \neq (j,\Theta)$), i.e. when $(j,\Theta) \prec (j',\Theta').$

For a reduced $\boxplus$-sum $P$ of $\Theta'(P_{j',l'})$ indexed by $(j',\Theta')$ in some subset $A \subset \Omega,$ the $(j,\Theta)$-component of $\rho(P)$ is

$$\sum_{(j',\Theta') \in A} \rho_{(j,\Theta)}(\Theta'(P_{j',l'})) = \begin{cases} l \in \mathbb{N}, & \text{if } (j,\Theta) \in A \text{ with } \Theta(P_{j,l}) \text{ a summand of } P \\ \in \{0, \infty\}, & \text{if otherwise} \end{cases}$$

for any $(j,\Theta) \in \Omega,$ since if $(j,\Theta) \in A$ then $(j,\Theta)$ is $\prec$-unrelated to any other $(j',\Theta') \in A.$ So $\rho(P)$ completely determines the summands of a reduced $\boxplus$-sum $P,$ namely, $P$ is the $\boxplus$-sum of exactly those $\Theta(P_{j,l})$ with $l$ equal to the $(j,\Theta)$-component of $\rho(P)$ that is a strictly positive integer. Since $\Psi'(T^{\boxplus n})$ consists of reduced $\boxplus$-sums, this also shows that the clearly well-defined monoid homomorphism $\rho$ is injective.

Thus if $P \sim P'$ for two reduced $\boxplus$-sums $P$ and $P'$ and hence $\rho(P) = \rho(P'),$ then the summands of $P$ and $P'$ are exactly the same, i.e. $P$ and $P'$ are the same reduced $\boxplus$-sum. So different reduced $\boxplus$-sums are mutually inequivalent projections.
Proposition 4. \( \mathfrak{P}(\mathcal{T}) = \mathfrak{P}'(\mathcal{T}) \). More concretely,

\[
\mathfrak{P}(\mathcal{T}) \cong \{(0, l) : l \in \mathbb{Z}_{\geq}\} \cup \{(m, \infty) : m > 0\} \subset \mathbb{Z}_{\geq}^2
\]

where \( \mathbb{Z}_{\geq}^2 \) is equipped with the canonical monoid structure.

Proof. It suffices to show that any element of \( \mathfrak{P}_0(\mathcal{T}) \equiv \mathfrak{P}_0(\mathcal{T}^{\otimes 1}) \) is of the form \( P_{0,l} \) (realized as \( (0, l) \in \mathbb{Z}_{\geq}^2 \)) and any element of \( \mathfrak{P}_m(\mathcal{T}) \equiv \mathfrak{P}_m(\mathcal{T}^{\otimes 1}) \) for \( m \in \mathbb{N} \) is of the form \( P_{1,m} \) (realized as \( (m, \infty) \in \mathbb{Z}_{\geq}^2 \)).

The argument sketched below is similar to one used in [29].

Since any complex vector bundle over \( \mathbb{T} \) is trivial, any idempotent over \( C(\mathbb{T}) \) is equivalent to the standard projection \( 1 \otimes P_m \in C(\mathbb{T}) \otimes M_{\infty}(\mathbb{C}) \) for some \( m \in \mathbb{Z}_{\geq} \). So for any idempotent \( P \in M_{\infty}(\mathcal{T}) \) over \( \mathcal{T} \), there is some \( U \in GL_{\infty}(C(\mathbb{T})) \) such that

\[
U \sigma(P) U^{-1} = 1 \otimes P_m = \sigma(\boxplus m I)
\]

for some \( m \in \mathbb{Z}_{\geq} \) where \( I \) is the identity of \( \mathcal{K}^+ \subset \mathcal{T} \), and hence \( V P V^{-1} - \boxplus m I \in M_{\infty}(\mathcal{K}) \) for any lift \( V \in GL_{\infty}(\mathcal{T}) \) (which exists) of \( U \boxplus U^{-1} \in GL_0(\mathbb{T}) \) along \( \sigma \). Replacing \( P \) by the equivalent \( V P V^{-1} \), we may assume that \( P \in (\boxplus m I) + M_{k-1}(\mathcal{K}) \subset M_{k-1}(\mathcal{K}^+) \) for some large \( k \geq m + 1 \). Now since \( M_{\infty}(\mathbb{C}) \) is dense in \( \mathcal{K} \), there is an idempotent \( Q \in (\boxplus m I) + M_{k-1}(\mathcal{M}(\mathbb{C})) \) sufficiently close to and hence equivalent to \( P \) for some large \( N \). So replacing \( P \) by \( Q \), we may assume that \( K := P - \boxplus m I \in M_{k-1}(\mathcal{M}(\mathbb{C})) \).

Rearranging the entries of \( P \equiv K + \boxplus m I \in M_{k-1}(\mathcal{T}) \subset M_{k}(\mathcal{T}) \) via conjugation by the unitary

\[
U_{k,N} := \sum_{j=1}^{k-1} \left( e_{jj} \otimes (S^*)^N + e_{kj} \otimes (S^{(j-1)N}P_N) \right) + e_{kk} \otimes S^{(k-1)N} \in M_{k}(\mathbb{C}) \otimes \mathcal{T} \equiv M_{k}(\mathcal{T})
\]

22
we get
\[ U_{k,N} P U_{k,N}^{-1} \equiv U_{k,N} (P \boxplus 0) U_{k,N}^{-1} = \left( (\boxplus^m I) \boxplus (\boxplus^{k-1-m} 0) \right) \boxplus R \]
for some \( R \in M_{(k-1)N} (\mathbb{C}) \subset \mathcal{K} \subset \mathcal{T} \) which must be an idempotent. Since any idempotent in \( \mathcal{K} \) is equivalent over \( \mathcal{K}^+ \) to a standard projection \( P_l \), we get
\[ P \sim \left( (\boxplus^m I) \boxplus (\boxplus^{k-1-m} 0) \right) \boxplus P_l \]
for some \( l \in \mathbb{Z}_\geq \).

If \( m = 0 \), then clearly \( P \sim P_l \). Since it is well known that \( P_l \) and \( \boxplus^l P_1 \equiv P_{0,l} \) are equivalent over \( \mathcal{K}^+ \) and hence over \( \mathcal{T} \supset \mathcal{K}^+ \), we get \( P \sim P_{0,l} \).

If \( m \in \mathbb{N} \), then we can rearrange entries via conjugation by the unitary
\[ U_l := e_{11} \otimes I + e_{1k} \otimes P_1 + \sum_{j=2}^{k-1} e_{jj} \otimes I + e_{kk} \otimes (S^*)^l \in M_k (\mathbb{C}) \otimes \mathcal{T} \equiv M_k (\mathcal{T}) \]
to get
\[ U_l \left( \left( (\boxplus^m I) \boxplus (\boxplus^{k-1-m} 0) \right) \boxplus P_l \right) U_l^{-1} = (\boxplus^m I) \boxplus (\boxplus^{k-m} 0) \equiv \boxplus^m I \equiv P_{1,m} \cdot \]
□

**Theorem 2.** For \( n > 1 \) and \( m > 0 \), if \( \mathfrak{P}_m (\mathcal{T}^{\otimes n-1}) = \mathfrak{P}_m' (\mathcal{T}^{\otimes n-1}) \equiv \{ \boxplus^m (\otimes^{n-1} I) \} \) and \( GL_m (\mathcal{T}^{\otimes n-1}) \) is connected, then \( \mathfrak{P}_m (\mathcal{T}^{\otimes n}) = \mathfrak{P}_m' (\mathcal{T}^{\otimes n}) \).

Proof. In this proof, we use \( I \) and \( \bar{I} \) to denote respectively the identity elements of \( \mathcal{T}^{\otimes n-1} \) and \( \mathcal{T}^{\otimes n} \).

Let \( P \in \mathfrak{P}_m (\mathcal{T}^{\otimes n}) \). The idempotent \( \kappa_n (P) \) over \( \mathcal{T}^{\otimes n-1} \otimes C (\mathbb{T}) \) satisfies that for any \( z \in \mathbb{T} \),
\[ \sigma_{n-1} (\kappa_n (P) (z)) = \sigma_n (P) (\cdot, z) \in M_\infty \left( C \left( \mathbb{T}^{n-1} \right) \right) \]
23
which is of rank $m$ pointwise, and hence

$$\kappa_n (P) (z) \in \mathfrak{p}_m \left( T^{\otimes n-1} \right) = \mathfrak{p}_m' \left( T^{\otimes n-1} \right),$$

i.e. $\kappa_n (P) (z) \sim \boxplus^m I$ over $T^{\otimes n-1}$. In particular, there is a continuous idempotent-valued path $\gamma : [0, 1] \to M_k \left( T^{\otimes n-1} \right)$ for $k$ sufficiently large going from the idempotent $\kappa_n (P) (1)$ to $(\boxplus^m I) \boxplus (\boxplus^{k-m} 0)$. Clearly we may assume that $\gamma$ is locally constant at 1, say, $\gamma (t) = \boxplus^m I$ for $t \geq 1/2$. The concatenation of the path $\gamma^{-1}$, the loop $\kappa_n (P)$, and the path $\gamma$ defines an idempotent-valued continuous loop $\Gamma : \mathbb{T} \to M_k \left( T^{\otimes n-1} \right)$ starting and ending at $\boxplus^m I$ with $\Gamma (e^{i\theta}) = (\boxplus^m I) \boxplus (\boxplus^{k-m} 0)$, say, for all $\theta \in [3\pi/2, 2\pi]$ (and $[0, \pi/2]$), and is homotopic to the loop $\kappa_n (P)$ via idempotents, i.e. there is a path of idempotents in $M_k \left( T^{\otimes n-1} \otimes C (\mathbb{T}) \right)$ from $\kappa_n (P)$ to $\Gamma$. Consequently, there is a continuous path of invertibles $U_t \in GL_k \left( T^{\otimes n-1} \otimes C (\mathbb{T}) \right)$ with $U_0 = I_k$ such that $U_1 \kappa_n (P) U_1^{-1} = \Gamma$, which can be lifted along $\kappa_n$ to a continuous path of invertible $V_t \in GL_k \left( T^{\otimes n} \right)$ with $V_0 = I_k$ such that $\kappa_n \left( V_1 PV_1^{-1} \right) = \Gamma$.

Replacing $P$ by the equivalent idempotent $V_1 PV_1^{-1}$, we may now assume directly that the idempotent $\kappa_n (P)$ over $T^{\otimes n-1} \otimes C (\mathbb{T})$ is a continuous loop of idempotents in $M_k \left( T^{\otimes n-1} \right)$ such that $\kappa_n (P) \left( e^{i\theta} \right) = (\boxplus^m I) \boxplus (\boxplus^{k-m} 0)$ for all $\theta \in [3\pi/2, 2\pi]$. So there is a continuous path

$$\theta \in [0, 3\pi/2] \mapsto W_\theta \in GL_k \left( T^{\otimes n-1} \right)$$

with $W_0 = I_k$ such that

$$W_\theta \left( \kappa_n (P) \left( e^{i\theta} \right) \right) W_\theta^{-1} = \kappa_n (P) (1) = (\boxplus^m I) \boxplus (\boxplus^{k-m} 0)$$

24
for all $\theta \in [0, 3\pi/2]$. In particular,

$$W_{3\pi/2} \left( (\oplus^m I) \oplus (\oplus^{k-m} 0) \right) = \left( (\oplus^m I) \oplus (\oplus^{k-m} 0) \right) W_{3\pi/2}$$

and hence $W_{3\pi/2} = W' \oplus W''$ for some invertibles $W' \in GL_m (\mathcal{T}^{\otimes n-1})$ and $W'' \in GL_{k-m} (\mathcal{T}^{\otimes n-1})$.

By the connectedness assumption on $GL_m (\mathcal{T}^{\otimes n-1})$, there is a continuous path $\alpha : [3\pi/2, 2\pi] \to GL_m (\mathcal{T}^{\otimes n-1})$ with $\alpha (3\pi/2) = W'$ and $\alpha (2\pi) = I_m$. Since by Künneth formula, $K_1 (\mathcal{T}^{\otimes n-1}) = 0$ and hence $GL_N (\mathcal{T}^{\otimes n-1})$ is connected for $N$ sufficiently large, we may suitably increase the value of $k$ by adding diagonal $\oplus$-summands 0 to idempotents and diagonal $\boxplus$-summands $I$ to invertibles, so that $GL_{k-m} (\mathcal{T}^{\otimes n-1})$ is also connected and hence there is a continuous path $\beta : [3\pi/2, 2\pi] \to GL_{k-m} (\mathcal{T}^{\otimes n-1})$ with $\beta (3\pi/2) = W''$ and $\beta (2\pi) = I_{k-m}$.

Now the function $\theta \mapsto W_\theta$ can be continuously extended to the whole interval $[0, 2\pi]$ by setting

$$W_\theta := \alpha (\theta) \oplus \beta (\theta) \in GL_k (\mathcal{T}^{\otimes n-1})$$

for $\theta \in [3\pi/2, 2\pi]$, giving rise to a well-defined continuous loop

$$W : e^{i\theta} \in \mathbb{T} \mapsto W_\theta \in GL_k (\mathcal{T}^{\otimes n-1}) ,$$

i.e. $W \in GL_k (\mathcal{T}^{\otimes n-1} \otimes C (\mathbb{T}))$, satisfying

$$W (\kappa_n (P)) W^{-1} = (\oplus^m I) \oplus (\oplus^{k-m} 0) .$$

So the idempotent $\kappa_n (P)$ over $\mathcal{T}^{\otimes n-1} \otimes C (\mathbb{T})$ is equivalent to the idempotent $\oplus^m I$.

Replacing $P$ by the equivalent idempotent $\tilde{W} (P \oplus (\oplus^k 0)) \tilde{W}^{-1}$ for any fixed lifting $\tilde{W} \in GL_0^0 (\mathcal{T}^{\otimes n})$ of $W \oplus W^{-1} \in GL_{2k}^0 (\mathcal{T}^{\otimes n-1} \otimes C (\mathbb{T}))$ along $\kappa_n$, we may now assume that

$$\kappa_n (P) = (\oplus^m I) \oplus (\oplus^{2k-m} 0) = \kappa_n \left( (\oplus^m \tilde{I}) \oplus (\oplus^{2k-m} 0) \right) .$$
and proceed to show that $P \sim \bigoplus^m I$.

Note that $P - \left( \left( \bigoplus^m I \right) \bigoplus (\bigoplus^{2k-m} 0) \right) \in M_{2k} \left( \mathcal{T} \otimes_{n-1} \otimes \mathcal{K} \right)$. Since $M_\infty (\mathbb{C})$ is dense in $\mathcal{K}$, we may replace $P$ by a suitable equivalent idempotent and assume that

$$K := P - \left( \left( \bigoplus^m I \right) \bigoplus (\bigoplus^{2k-m} 0) \right) \in M_{2k} \left( \mathcal{T} \otimes_{n-1} \otimes M_N (\mathbb{C}) \right) \subset M_{2k} \left( \mathcal{T} \otimes^n \right)$$

for some $N \in \mathbb{N}$.

Rearranging the entries of $P \equiv P \bigoplus 0 \in M_{2k+1} \left( \mathcal{T} \otimes_{n-1} \otimes M_N (\mathbb{C}) \right)$ via conjugation by the unitary

$$U_{k,N} := \sum_{j=1}^{2k} \left( e_{jj} \otimes (I \otimes S^*)^N + e_{2k+1,j} \otimes (I \otimes S^{(j-1)N} P_N) \right) + e_{2k+1,2k+1} \otimes (I \otimes S^{2kN})$$

$$\in M_{2k+1} (\mathbb{C}) \otimes \mathcal{T} \otimes_{n-1} \otimes \mathcal{T} \equiv M_{2k+1} (\mathcal{T} \otimes^n)$$

we get

$$P \sim U_{k,N} P U_{k,N}^{-1} \equiv U_{k,N} (P \bigoplus 0) U_{k,N}^{-1} = \left( \left( \bigoplus^m I \right) \bigoplus (\bigoplus^{2k-m} 0) \right) \bigoplus R$$

for some

$$R \in M_{2kN} (\mathcal{T} \otimes_{n-1}) \equiv \mathcal{T} \otimes_{n-1} \otimes M_{2kN} (\mathbb{C}) \subset \mathcal{T} \otimes_{n-1} \otimes \mathcal{T} \equiv \mathcal{T} \otimes^n$$

which must be an idempotent over $\mathcal{T} \otimes_{n-1}$.

Since $K_0 (\mathcal{T} \otimes_{n-1}) = \mathbb{Z}$ by Künneth formula, $R \bigoplus (\bigoplus^r I) \sim (\bigoplus^{r+[R]} I)$ for a sufficiently large $r \in \mathbb{N}$ where $[R] \in \mathbb{Z}$ denotes the class of $R$ in $K_0 (\mathcal{T} \otimes_{n-1})$. So there is an invertible $U \in GL_d (\mathcal{T} \otimes_{n-1})$ for some large $d \geq \max \{ 2kN + r, r + [R] \}$ such that

$$U \left( R \bigoplus (\bigoplus^{d-2kN-r} 0) \bigoplus (\bigoplus^r I) \right) U^{-1} = (\bigoplus^{d-r-[R]} 0) \bigoplus (\bigoplus^{r+[R]} I) .$$
With \( m > 0 \), we can rearrange entries via conjugation by the unitary

\[
U_{d-r} := e_{11} \otimes (I \otimes S^{d-r}) + e_{1,2k+1} \otimes I \otimes P_{d-r} + \sum_{j=2}^{2k} e_{jj} \otimes \tilde{1} + e_{2k+1,2k+1} \otimes (I \otimes S^*)^{d-r}
\]

\[\in M_{2k+1}(\mathbb{C}) \otimes \mathcal{T}^{\otimes n-1} \otimes \mathcal{T} = M_{2k+1}(\mathcal{T}^{\otimes n})\]

to get

\[P \sim U_{d-r} \left( ((\oplus^m \tilde{1}) \oplus (\oplus^{2k-m}0) \oplus R) U_{d-r}^{-1} = R' \oplus (\oplus^{m-1} \tilde{1}) \oplus (\oplus^{2k+1} m0)\right)\]

where

\[R' = (R \oplus (\oplus^{d-2kN-r}0)) + (\tilde{1} - I \otimes P_{d-r}) \in \tilde{1} + (\mathcal{T}^{\otimes n-1} \otimes M_{d-r} (\mathbb{C}))\]

\[\subset \tilde{1} + (\mathcal{T}^{\otimes n-1} \otimes \mathcal{K}) \subset \mathcal{T}^{\otimes n-1} \otimes \mathcal{T} = \mathcal{T}^{\otimes n}.\]

Note that \( R' \) can be interpreted as \( R \oplus (\oplus^{d-2kN-r}0) \oplus (\oplus^\infty I) \in \mathcal{T}^{\otimes n-1} \otimes \mathcal{K}^+ \subset \mathcal{T}^{\otimes n} \), which when conjugated by the invertible \( U \equiv U \oplus (\oplus^\infty I) \in \mathcal{T}^{\otimes n-1} \otimes \mathcal{K}^+ \subset \mathcal{T}^{\otimes n} \) becomes

\[((\oplus^{d-r-[R]}0) \oplus (\oplus^{r+[R]} I) \oplus (\oplus^\infty I) = \tilde{1} - I \otimes P_{d-r-[R]} \in \tilde{1} + (\mathcal{T}^{\otimes n-1} \otimes \mathcal{K}) \subset \mathcal{T}^{\otimes n}.\]

So we get

\[P \sim (\tilde{1} - I \otimes P_{d-r-[R]} \oplus (\oplus^{m-1} \tilde{1}) \oplus (\oplus^{2k+1} m0),\]

the latter of which when conjugated by \( U_{d-r-[R]}^{-1} \) yields \( \tilde{1} \oplus (\oplus^{m-1} \tilde{1}) \oplus (\oplus^{2k+1} m0), \) where \( U_{d-r-[R]} \) is defined as \( U_{d-r} \) by replacing \( d-r \) by \( d-r-[R] \). Thus we get \( P \sim (\oplus^m \tilde{1}) \oplus (\oplus^{2k+1} m0) \equiv \oplus^m \tilde{I}.\)

\[\square\]

**Corollary 3.** \( \mathfrak{P}_m (\mathcal{T}^{\otimes n}) = \mathfrak{P}'_m (\mathcal{T}^{\otimes n}) \equiv \left\{ \oplus^m \tilde{1} \right\} \) for all \( m \geq \left\lceil \frac{n-1}{2} \right\rceil + 3 \) and any \( n \in \mathbb{N} \), where \( \tilde{1} \) is the identity element of \( \mathcal{T}^{\otimes n} \).
Proof. We prove by induction on \( n \in \mathbb{N} \). For \( n = 1 \), we already know that \( \mathcal{P}'_m (\mathcal{T}^{\otimes n}) \equiv \mathcal{P}_m (\mathcal{T}^{\otimes n}) \) for all \( m > 0 \).

Now assume as the induction hypothesis that \( \mathcal{P}'_m (\mathcal{T}^{\otimes n}) = \mathcal{P}_m (\mathcal{T}^{\otimes n}) \) for all \( m \geq \left\lfloor \frac{n-1}{2} \right\rfloor + 3 \) for an \( n \in \mathbb{N} \).

Since we know that \( GL_m (\mathcal{T}^{\otimes n}) \) is connected for all \( m \geq \left\lfloor \frac{n}{2} \right\rfloor + 3 \), the above theorem implies that \( \mathcal{P}'_m (\mathcal{T}^{\otimes n+1}) = \mathcal{P}_m (\mathcal{T}^{\otimes n+1}) \) for all \( m \geq \left\lfloor \frac{n}{2} \right\rfloor + 3 \).

\( \square \)

It remains open the problem of classification of low-rank idempotents over \( \mathcal{T}^{\otimes n} \). In particular, it is not clear whether there are idempotents of non-standard (equivalence) type.

6 Projective modules over \( C \left( S_H^{2n-1} \right) \)

Most of the arguments and results in the above study of projective modules over \( \mathcal{T}^{\otimes n} \) can be adapted to the case of the quantum spheres \( C \left( S_H^{2n-1} \right) \).

Let \( \partial_n : \mathcal{T}^{\otimes n} \to C \left( S_H^{2n-1} \right) \) be the canonical quotient map by restricting the groupoid \( \Xi_n \) to the closed invariant set \( \overline{Z_{\geq}^n} \setminus Z_{\geq}^{n} \) in its unit space.

First we note that there is a short exact sequence of C*-algebras

\[
0 \to C \left( S_H^{2n-3} \right) \otimes K (l^2 (\mathbb{Z}_\geq)) \to C \left( S_H^{2n-1} \right) \xrightarrow{\lambda_n} C^n \left( \Xi_n |_{\overline{Z_{\geq}^{n-1}} \times \{\infty\}} \right) \cong \mathcal{T}^{\otimes n-1} \otimes C (\mathbb{I}) \to 0
\]

for all \( n > 1 \). Indeed, since \( \left( \overline{Z_{\geq}^{n-1}} \setminus Z_{\geq}^{n-1} \right) \times Z_{\geq} \) is an open invariant subset of the unit space \( \overline{Z_{\geq}^n} \setminus Z_{\geq}^n \) of the groupoid \( \mathcal{G}_n \equiv (\mathbb{Z}^n \times \overline{Z_{\geq}^n}) |_{\overline{Z_{\geq}^n} \setminus Z_{\geq}^n} \) with the invariant complement

\[
(\overline{Z_{\geq}^n} \setminus Z_{\geq}^n) \setminus \left( \left( \overline{Z_{\geq}^{n-1}} \setminus Z_{\geq}^{n-1} \right) \times Z_{\geq} \right) = \overline{Z_{\geq}^{n-1}} \times \{\infty\},
\]

28
the groupoid C*-algebra

\[
C^* \left( \mathfrak{G}_n \big|_{\mathbb{Z}^{n-1} \times \mathbb{Z}^{n-1} \times \mathbb{Z}^2} \right) = C^* \left( \left( \mathbb{Z}^{n-1} \ltimes \mathbb{Z}^{n-1} \right) \big|_{\mathbb{Z}^{n-1} \times \mathbb{Z}^{n-1} \times \mathbb{Z}^2} \times (\mathbb{Z} \ltimes \mathbb{Z}) \big|_{\mathbb{Z}^2} \right)
\]

\[
\cong C^* \left( \left( \mathbb{Z}^{n-1} \ltimes \mathbb{Z}^{n-1} \right) \big|_{\mathbb{Z}^{n-1} \times \mathbb{Z}^2} \right) \otimes C^* \left( (\mathbb{Z} \ltimes \mathbb{Z}) \big|_{\mathbb{Z}^2} \right) = C^* \left( \mathbb{S}_H^{2n-3} \otimes K \left( \ell^2 (\mathbb{Z}_2) \right) \right)
\]

is a closed ideal of \( C^* (\mathfrak{G}_n) = C \left( \mathbb{S}_H^{2n-1} \right) \) with quotient

\[
C^* (\mathfrak{G}_n) / C^* \left( \mathfrak{G}_n \big|_{\mathbb{Z}^{n-1} \times \mathbb{Z}^{n-1} \times \mathbb{Z}^2} \right) \cong C^* \left( \mathfrak{G}_n \big|_{\mathbb{Z}^{n-1} \times \mathbb{Z}^{n-1} \times \{\infty\}} \right)
\]

\[
= C^* \left( \left( \mathbb{Z}^{n-1} \ltimes \mathbb{Z}^{n-1} \right) \big|_{\mathbb{Z}^{n-1} \times \mathbb{Z}} \right) \cong C^* \left( \left( \mathbb{Z}^{n-1} \ltimes \mathbb{Z}^{n-1} \right) \big|_{\mathbb{Z}^{n-1}} \right) \otimes C (\mathbb{T}) = T^\otimes n - 1 \otimes C (\mathbb{T})
\]

So we get the above short exact sequence with \( \lambda_n \) being the canonical map from \( C^* (\mathfrak{G}_n) \) to its quotient \( T^\otimes n - 1 \otimes C (\mathbb{T}) \) resulting from restricting the groupoid \( \mathfrak{G}_n \) to the closed invariant set \( \mathbb{Z}^{n-1} \times \{\infty\} \).

Clearly \( \kappa_n = \lambda_n \circ \partial_n \). Furthermore all the quotient maps \( \sigma_{(j, \Theta)} \) on \( T^\otimes n \) with \( j > 0 \) factors through \( \partial_n \) and induces a quotient map

\[
\tau_{(j, \Theta)} : C \left( \mathbb{S}_H^{2n-1} \right) \rightarrow C (\mathbb{T}^j) \otimes B \left( \ell^2 \left( \mathbb{Z}_2^{n-j} \right) \right)
\]

such that \( \sigma_{(j, \Theta)} = \tau_{(j, \Theta)} \circ \partial_n \).

Note that the quotient maps \( \lambda_n \) for \( n \in \mathbb{N} \) satisfy the commuting diagram

\[
M_k \left( C \left( \mathbb{S}_H^{2n-1} \right) \right) \xrightarrow{\lambda_n} M_k \left( T^\otimes n - 1 \otimes C (\mathbb{T}) \right) \equiv M_k \left( T^\otimes n - 1 \otimes C (\mathbb{T}) \right) \downarrow \tau_n \quad \oplus \quad \downarrow \sigma_{n-1} \otimes \text{id} \quad \downarrow \sigma_{n-1} \otimes \text{id}
\]

\[
M_k \left( C (\mathbb{T}^n) \right) \xrightarrow{\tau_n} M_k \left( C (\mathbb{T}^n - 1) \otimes C (\mathbb{T}) \right) \equiv M_k \left( C (\mathbb{T}^n - 1) \otimes C (\mathbb{T}) \right)
\]

We define the rank of (the equivalence class of) an idempotent \( Q \in M_{\infty} \left( C \left( \mathbb{S}_H^{2n-1} \right) \right) \) over \( C \left( \mathbb{S}_H^{2n-1} \right) \) as the rank of the matrix value \( \tau_n (Q) (z) \in M_{\infty} (C) \) at any \( z \in \mathbb{T}^n \) (independent of \( z \) since \( \mathbb{T}^n \) is connected). Then the set of equivalence classes of idempotents
\( Q \in M_\infty (C (S_{H}^{2n-1})) \) equipped with the binary operation \( \boxplus \) becomes an abelian graded monoid

\[
\mathfrak{P} (C (S_{H}^{2n-1})) = \bigsqcup_{m=0}^{\infty} \mathfrak{P}_m (C (S_{H}^{2n-1}))
\]

where \( \mathfrak{P}_m (C (S_{H}^{2n-1})) \) is the set of all (equivalence classes of) idempotents over \( C (S_{H}^{2n-1}) \) of rank \( m \), with clearly

\[
\mathfrak{P}_m (C (S_{H}^{2n-1})) \boxplus \mathfrak{P}_l (C (S_{H}^{2n-1})) \subset \mathfrak{P}_{m+l} (C (S_{H}^{2n-1}))
\]

for \( m, l \geq 0 \).

Since \( \sigma_n = \tau_n \circ \partial_n \), the rank of an idempotent \( P \) over \( C (T^{\otimes n}) \) equals the rank of the idempotent \( \partial_n P \) over \( C (S_{H}^{2n-1}) \). We now define

\[
\mathfrak{P}_m' (C (S_{H}^{2n-1})) := \partial_n (\mathfrak{P}_m (T^{\otimes n})) \subset \mathfrak{P}_m (C (S_{H}^{2n-1}))
\]

and the projections

\[
Q_{j, \Theta, l} := \partial_n (\Theta (P_{j,l}))
\]

over \( C (S_{H}^{2n-1}) \). Note that \( \mathfrak{P}_m' (C (S_{H}^{2n-1})) = \{ \boxplus^m \tilde{I} \} \) for \( m > 0 \), where \( \tilde{I} \) denotes the identity element of \( C (S_{H}^{2n-1}) \).

Also note that \( Q_{0,\text{id},l} = 0 \) for all \( l \), where \( \text{id} \equiv \text{id}_{(1,2,...,n)} \) is the only \((0,n)\)-shuffle. The monoid homomorphism

\[
\rho_0 : P \in \mathfrak{P}' (T^{\otimes n}) \mapsto \prod_{(j,\Theta) \in \Omega_0} \rho_{(j,\Theta)} (P) \in \prod_{(j,\Theta) \in \Omega_0} \mathbb{Z}_{\geq 0}
\]

with

\[
\Omega_0 := \Omega \setminus \{(0,\text{id})\} \equiv \{(j,\Theta) : 0 < j \leq n \text{ and } \Theta \text{ is a } (j, n-j) \text{-shuffle}\},
\]

30
“truncated” from \( \rho \) induces a well-defined monoid homomorphism

\[
\rho_0 : \mathfrak{P}'(C(S_{H}^{2n-1})) \to \prod_{(j, \Theta) \in \Omega_0} \mathbb{Z}_\geq,
\]

in the sense that \( \rho = \rho_0 \circ \partial_n \). Indeed for \((j, \Theta) \in \Omega_0\), i.e. with \(j > 0\), the quotient map

\[
\sigma_{(j, \Theta)} : T^\otimes_n \equiv C^*(\mathfrak{S}_n) \to C^*(\mathfrak{S}_n|_{X_\Theta})
\]
factors through \( \partial_n \) since the unit space \( \mathbb{Z}_\geq \setminus \mathbb{Z}_n^\otimes \) of \( \mathfrak{S}_n \) contains \( X_\Theta \), and hence the map \( \rho_{(j, \Theta)} \) factors through \( \partial_n \).

We call a \( \boxplus \)-sum of \( Q_{j, \Theta, l} \) indexed by \( \prec \)-unrelated \((j, \Theta) \in \Omega_0\) (i.e. \( 1 \leq j \leq n \)) and \( l \equiv l_{(j, \Theta)} > 0 \) depending on \((j, \Theta)\) to be a reduced \( \boxplus \)-sum of standard projections over \( C(S_{H}^{2n-1}) \). (The degenerate empty \( \boxplus \)-sum 0 is taken as a reduced \( \boxplus \)-sum.) Two such reduced \( \boxplus \)-sums are called different when they have different sets of (mutually \( \prec \)-unrelated) indices \((j, \Theta) \in \Omega_0\) or have different weight functions \( l \) of \((j, \Theta)\). Each \( Q_{j, \Theta, l} \) with \( j, l > 0 \) is a reduced \( \boxplus \)-sum of standard projections over \( C(S_{H}^{2n-1}) \).

**Proposition 5.** Different reduced \( \boxplus \)-sums of standard projections over \( C(S_{H}^{2n-1}) \) are mutually inequivalent projections over \( C(S_{H}^{2n-1}) \), and they form a graded submonoid

\[
\mathfrak{P}'(C(S_{H}^{2n-1})) = \bigsqcup_{m=0}^{\infty} \mathfrak{P}'_m(C(S_{H}^{2n-1}))
\]

of the monoid \( \mathfrak{P}(C(S_{H}^{2n-1})) \), with its monoid structure explicitly determined by \( Q_{j, \Theta, l} \boxplus Q_{j', \Theta', l'} \sim Q_{j, \Theta, l} \) for \((j', \Theta') \prec (j, \Theta)\) with \( j, j', l, l' > 0 \). Furthermore the monoid homomorphism

\[
\rho_0 : \mathfrak{P}'(C(S_{H}^{2n-1})) \to \prod_{(j, \Theta) \in \Omega_0} \mathbb{Z}_\geq
\]

31
is injective.

Proof. The submonoid $\Psi' (C (S_{H}^{2n-1})) = \partial_n (\Psi' (C (\mathcal{T}^\otimes n)))$ consists of reduced $\oplus$-sums of $Q_{j,\Theta,l} = \partial_n (\Theta (P_{j,l}))$ with $j > 0$, since $Q_{0,\text{id},l} = 0$.

Let $\mathcal{M}$ be the subset of $\Psi' (C (\mathcal{T}^\otimes n))$ consisting of all reduced $\oplus$-sums $P$ of $\Theta (P_{j,l})$ with $j > 0$. Then $\partial_n|_{\mathcal{M}} : \mathcal{M} \rightarrow \Psi' (C (S_{H}^{2n-1}))$ is still surjective, and $\rho_0|_{\mathcal{M}}$ still factors through $\rho_\Theta$, i.e. $\rho_0|_{\mathcal{M}} = \rho_\Theta \circ \partial_n|_{\mathcal{M}}$. These imply that $\rho_\Theta$ is injective if $\rho_0|_{\mathcal{M}}$ is injective.

For any reduced $\oplus$-sum $P \in \mathcal{M}$ of $\Theta (P_{j,l})$ with $j > 0$, the $(j, \Theta)$-component of $\rho (P)$ is the same as that of $\rho_0 (P)$ for all $(j, \Theta) \in \Omega_0$, while the only other component, namely, the $(0, \text{id})$-component of $\rho (P)$ is $\infty$ since $\rho_{(0,\text{id})} (\Theta (P_{j,l})) = \infty$ for any $j > 0$. Thus we get $\rho (P) = (\infty, \rho_0 (P))$ for all $P \in \mathcal{M}$. Hence the injectivity of $\rho|_{\mathcal{M}}$ implies the injectivity of $\rho_0|_{\mathcal{M}}$ on $\mathcal{M}$, and hence the injectivity of $\rho_\Theta$.

Since two different reduced $\oplus$-sums $Q, Q'$ over $C (S_{H}^{2n-1})$ are of the form $\partial_n (P), \partial_n (P')$ respectively for two different reduced $\oplus$-sums $P, P' \in \mathcal{M}$ over $C (\mathcal{T}^\otimes n)$ which are inequivalent over $C (\mathcal{T}^\otimes n)$ and hence $\rho_0 (P) \neq \rho_0 (P')$, we get $\rho_\Theta (Q) \neq \rho_\Theta (Q')$ showing that $Q, Q'$ are different equivalence classes in $\Psi' (C (S_{H}^{2n-1}))$.

The property that $\Theta (P_{j,l}) \oplus \Theta' (P'_{j',l'}) \sim \Theta (P_{j,l})$ over $\mathcal{T}^\otimes n$ for $(j', \Theta') \prec (j, \Theta)$ is clearly preserved under the quotient map $\partial_n$, i.e. $Q_{j,\Theta,l} \oplus Q_{j',\Theta',l'} \sim Q_{j,\Theta,l}$ over $C (S_{H}^{2n-1})$.

□

**Theorem 3.** For $n > 1$ and $m \in \mathbb{N}$, if $\Psi_m (\mathcal{T}^\otimes n-1) = \Psi'_m (\mathcal{T}^\otimes n-1)$ and $GL_m (\mathcal{T}^\otimes n-1)$ is connected, then $\Psi'_m (C (S_{H}^{2n-1})) = \Psi_m (C (S_{H}^{2n-1}))$.

Proof. Many arguments used to prove a similar theorem for $\mathcal{T}^\otimes n$ instead of $C (S_{H}^{2n-1})$...
can be used again here with minor modifications. In this proof, $I$ and $\tilde{I}$ denote respectively the identity element of $T^{\otimes n-1}$ and $C \left(S_H^{2n-1}\right)$.

Let $P \in \mathfrak{P}_m \left(C \left(S_H^{2n-1}\right)\right)$. The idempotent $\lambda_n (P)$ over $T^{\otimes n-1} \otimes C \left(\mathbb{T}\right)$ satisfies that for any $z \in \mathbb{T}$,

$$\sigma_{n-1} \left(\lambda_n (P) (z)\right) = \tau_n (P) \left(\cdot, z\right) \in M_\infty \left(C \left(\mathbb{T}^{m-1}\right)\right)$$

which is of rank $m$ pointwise, and hence

$$\lambda_n (P) (z) \in \mathfrak{P}_m \left(T^{\otimes n-1}\right) = \mathfrak{P}_m \left(T^{\otimes n-1}\right),$$

i.e. $\lambda_n (P) (z) \sim \bigoplus^m I$ over $T^{\otimes n-1}$. As before, for some large $k$, there is an idempotent-valued continuous loop $\Gamma : \mathbb{T} \to M_k \left(T^{\otimes n-1}\right)$ starting and ending at $\bigoplus^m I$ with $\Gamma \left(e^{i\theta}\right) = (\bigoplus^m I) \bigoplus \left(\bigoplus^{k-m} 0\right)$, say, for all $\theta \in [3\pi/2, 2\pi]$, and homotopic to the loop $\lambda_n (P)$ via idempotents. Consequently, there is a continuous path of invertibles $U_t \in GL_k \left(T^{\otimes n-1} \otimes C \left(\mathbb{T}\right)\right)$ with $U_0 = I_k$ such that $U_1 \lambda_n (P) U_1^{-1} = \Gamma$, which can be lifted along $\lambda_n$ to a continuous path of invertible $V_t \in GL_k \left(C \left(S_H^{2n-1}\right)\right)$ with $V_0 = I_k$ such that $\lambda_n \left(V_1 P V_1^{-1}\right) = \Gamma$.

Replacing $P$ by the equivalent idempotent $V_1 P V_1^{-1}$, we may now assume directly that the idempotent $\lambda_n (P)$ over $T^{\otimes n-1} \otimes C \left(\mathbb{T}\right)$ is a continuous loop of idempotents in $M_k \left(T^{\otimes n-1}\right)$ such that $\lambda_n (P) \left(e^{i\theta}\right) = (\bigoplus^m I) \bigoplus \left(\bigoplus^{k-m} 0\right)$ for all $\theta \in [3\pi/2, 2\pi]$. As before, by the connectedness assumption on $GL_m \left(T^{\otimes n-1}\right)$, after suitably increasing the size $k$, we can find a well-defined continuous loop

$$W : e^{i\theta} \in \mathbb{T} \mapsto W_\theta \in GL_k \left(T^{\otimes n-1}\right),$$

i.e. $W \in GL_k \left(T^{\otimes n-1} \otimes C \left(\mathbb{T}\right)\right)$, satisfying

$$W \left(\lambda_n (P)\right) W^{-1} = (\bigoplus^m I) \bigoplus \left(\bigoplus^{k-m} 0\right).$$
So the idempotent $\lambda_n (P)$ over $T^{\otimes n-1} \otimes C (\mathbb{T})$ is equivalent to the idempotent $\boxplus^m I$.

Replacing $P$ by the equivalent idempotent $\tilde{W} (P \boxplus (\boxplus^k 0)) \tilde{W}^{-1}$ for any fixed lifting $\tilde{W} \in GL^0_{2k} \left(C \left(S^2_n \otimes H^{-1}\right)\right)$ of $W \boxplus W^{-1} \in GL^0_{2k} \left(T^{\otimes n-1} \otimes C (\mathbb{T})\right)$ along $\lambda_n$, we may now assume that

$$\lambda_n (P) = (\boxplus^m I) \boxplus (\boxplus^{2k-m} 0) = \lambda_n \left(\left(\boxplus^m \tilde{I}\right) \boxplus (\boxplus^{2k-m} 0)\right)$$

and proceed to show that $P \sim \boxplus^m \tilde{I}$ over $C \left(S^2_n \otimes H^{-1}\right)$, where we use $\tilde{I}$ to denote the identity element in $C \left(S^2_n \otimes H^{-1}\right)$ so as to distinguish it from the identity element $I$ of $T^{\otimes n-1}$.

With $P - \left(\left(\boxplus^m \tilde{I}\right) \boxplus (\boxplus^{2k-m} 0)\right) \in M_{2k} \left(C \left(S^2_n \otimes H^{-1}\right) \otimes K\right)$ and $M_\infty (\mathbb{C})$ dense in $K$, we may replace $P$ by a suitable equivalent idempotent and assume that

$$P = K + \left(\left(\boxplus^m \tilde{I}\right) \boxplus (\boxplus^{2k-m} 0)\right) \in M_{2k} \left(C \left(S^2_n \otimes H^{-1}\right) \otimes K\right) \subset M_{2k} \left(C \left(S^2_n \otimes H^{-1}\right)\right)$$

for some $K \in M_{2k} \left(C \left(S^2_n \otimes H^{-1}\right) \otimes N (\mathbb{C})\right)$ and some $N \in \mathbb{N}$.

As before, by rearranging entries via conjugation, we get

$$P \sim \partial_n (U_{k,N}) P \partial_n (U_{k,N}^{-1}) \equiv \partial_n (U_{k,N}) (P \boxplus 0) \partial_n (U_{k,N}^{-1}) = \left(\left(\boxplus^m \tilde{I}\right) \boxplus (\boxplus^{2k-m} 0)\right) \boxplus R$$

for some

$$R \in M_{2kN} \left(C \left(S^2_n \otimes H^{-1}\right)\right) \equiv C \left(S^2_n \otimes H^{-1}\right) \otimes M_{2kN} (\mathbb{C}) \subset \left(C \left(S^2_n \otimes H^{-1}\right) \otimes K\right)^+ \subset C \left(S^2_n \otimes H^{-1}\right)$$

which must be an idempotent over $C \left(S^2_n \otimes H^{-1}\right)$. More precisely, we can lift $P$ to

$$\hat{P} = \hat{K} + \left(\left(\boxplus^m \hat{I}_{T^{\otimes n}}\right) \boxplus (\boxplus^{2k-m} 0)\right) \in M_{2k} \left(T^{\otimes n-1} \otimes K\right)^+ \subset M_{2k} \left(T^{\otimes n}\right)$$

for some $\hat{K} \in M_{2k} \left(T^{\otimes n-1} \otimes N (\mathbb{C})\right)$ and conjugate it by the unitary $U_{k,N}$ over $T^{\otimes n}$ to get the form $\left(\left(\boxplus^m \hat{I}_{T^{\otimes n}}\right) \boxplus (\boxplus^{2k-m} 0)\right) \boxplus \hat{R}$ with $\hat{R} \in M_{2kN} (T^{\otimes n-1})$ as we did for the case of
Then the above $R$ is $\partial_n(\hat{R})$. Note that even though $\hat{P}$ and $\hat{R}$ are not necessarily idempotents, $R$ is since it is the idempotent $P$ conjugated by the unitary $\partial_n(U_{k,N})$ over $C(S_{H}^{2n-1})$.

Since $K_0(C(S_{H}^{2n-3})) = \mathbb{Z}$ [12], $R \boxplus (\boxplus^r \hat{I}) \sim (\boxplus^r [R] \hat{I})$ for a sufficiently large $r \in \mathbb{N}$ where $[R] \in \mathbb{Z}$ denotes the class of $R$ in $K_0(C(S_{H}^{2n-3}))$ and $\hat{I}$ is the identity element of $C(S_{H}^{2n-3})$. So there is an invertible $U \in GL_d(C(S_{H}^{2n-3}))$ for some large $d \geq \max\{2kN + r, r + [R]\}$ such that

$$U \left( R \boxplus (\boxplus^{d-2kN-r} 0) \boxplus (\boxplus^r \hat{I}) \right) U^{-1} = (\boxplus^{d-r-[R]} 0) \boxplus (\boxplus^{r+[R]} \hat{I}).$$

As before, with $m > 0$, by rearranging entries via conjugation, we can get

$$P \sim R' \boxplus (\boxplus^{m-1} \hat{I}) \boxplus (\boxplus^{2k+1-m} 0)$$

where the idempotent

$$R' = (R \boxplus (\boxplus^{d-2kN-r} 0)) + (\hat{I} - \hat{I} \otimes P_{d-r}) \in (C(S_{H}^{2n-3}) \otimes \mathcal{K})^+ \subset C(S_{H}^{2n-3})$$

when conjugated by the invertible $U \equiv U \boxplus (\hat{I} - \hat{I} \otimes P_{d}) \in (C(S_{H}^{2n-3}) \otimes \mathcal{K})^+$ becomes

$$(\boxplus^{d-r-[R]} 0) \boxplus (\hat{I} - \hat{I} \otimes P_{d-r-[R]}) \in (C(S_{H}^{2n-3}) \otimes \mathcal{K})^+ \subset C(S_{H}^{2n-3}).$$

So we get

$$P \sim \left( (\boxplus^{d-r-[R]} 0) \boxplus (\hat{I} - \hat{I} \otimes P_{d-r-[R]}) \right) \boxplus (\boxplus^{m-1} \hat{I}) \boxplus (\boxplus^{2k+1-m} 0),$$

the latter of which as before is equivalent to $\hat{I} \boxplus (\boxplus^{m-1} \hat{I}) \boxplus (\boxplus^{2k+1-m} 0)$ by a further conjugation by $U_{d-r-[R]}^{-1}$. Thus $P \sim (\boxplus^{m} \hat{I}) \boxplus (\boxplus^{2k+1-m} 0) \equiv \boxplus^{m} \hat{I}$. 

35
\[ \text{Corollary 4. } \mathcal{P}_m \left( C \left( S_H^{2n-1} \right) \right) = \mathcal{P}_m \left( C \left( S_H^{2n-1} \right) \right) \equiv \left\{ \mathbb{H}^n \tilde{I} \right\} \text{ for all } m \geq \left\lfloor \frac{n-1}{2} \right\rfloor + 3 \text{ and any } n \in \mathbb{N}, \text{ where } \tilde{I} \text{ is the identity element of } C \left( S_H^{2n-1} \right). \]

Proof. The case of \( n = 1 \) is well known. For \( n > 1 \), since \( \mathcal{P}_m \left( \mathcal{T}^{\otimes n-1} \right) = \mathcal{P}_m \left( \mathcal{T}^{\otimes n-1} \right) \) for all \( m \geq \left\lfloor \frac{n-2}{2} \right\rfloor + 3 \) and \( GL_m \left( \mathcal{T}^{\otimes n-1} \right) \) is connected for all \( m \geq \left\lfloor \frac{n-1}{2} \right\rfloor + 3 \), the above theorem implies that \( \mathcal{P}_m \left( C \left( S_H^{2n-1} \right) \right) = \mathcal{P}_m \left( C \left( S_H^{2n-1} \right) \right) \) for all \( m \geq \left\lfloor \frac{n-1}{2} \right\rfloor + 3 \).

\[ \square \]

It is not clear whether there are (low-rank) idempotents over \( C \left( S_H^{2n-1} \right) \) of non-standard (equivalence) type and whether the cancellation law holds for them.

\section{Projective modules over \( C \left( \mathbb{P}^{n-1} \left( \mathcal{T} \right) \right) \)}

In this section we study the problem of classification of finitely generated projective modules over the multipullback quantum complex projective space \( \mathbb{P}^{n-1} \left( \mathcal{T} \right) \) that was introduced and studied by Hajac, Kaygun, Ziębiński in \[9\].

In \[12\], \( K_0 \left( C \left( \mathbb{P}^{n-1} \left( \mathcal{T} \right) \right) \right) = \mathbb{Z}^n \) and \( K_1 \left( C \left( \mathbb{P}^{n-1} \left( \mathcal{T} \right) \right) \right) = 0 \) are computed, and \( \mathbb{P}^{n-1} \left( \mathcal{T} \right) \) is shown to be a quantum quotient space of \( S_H^{2n-1} \). More precisely, the C*-algebra \( C \left( \mathbb{P}^{n-1} \left( \mathcal{T} \right) \right) \) is isomorphic to the invariant C*-subalgebra \( \left( C \left( S_H^{2n-1} \right) \right)^{U(1)} \) of \( C \left( S_H^{2n-1} \right) \) under the canonical diagonal \( U(1) \)-action on \( C \left( S_H^{2n-1} \right) \cong \mathcal{T}^{\otimes n} / \mathcal{K}^{\otimes n} \), which in the groupoid context can be implemented by the multiplication operator

\[ U_\zeta : f \in C_c \left( \mathfrak{G}_n \right) \mapsto h_\zeta f \in C_c \left( \mathfrak{G}_n \right) \]
for $\zeta \in U(1) \equiv \mathbb{T}$ where

$$h_\zeta : (m, p) \in \mathcal{G}_n \subset \mathbb{Z}^n \times \mathbb{Z}^n \mapsto \zeta^{\Sigma m} \in \mathbb{T}$$

with $\Sigma m := \sum_{i=1}^{n} m_i$ is a groupoid character. Then $C(\mathbb{P}^{n-1}(\mathcal{T}))$ is realized as the groupoid C*-algebra $C^* ((\mathcal{G}_n)_0)$ of the subgroupoid $((\mathcal{G}_n)_0)$ of $\mathcal{G}_n$, where

$$(\mathcal{G}_n)_k := \{(m, p) \in \mathcal{G}_n : \Sigma m = k\}$$

for $k \in \mathbb{Z}$. Furthermore, $C^* (\mathcal{G}_n)$ becomes a (completion of the) graded algebra $\bigoplus_{k \in \mathbb{Z}} C^* (\mathcal{G}_n)_k$ with the component $C^* (\mathcal{G}_n)_k$ being the quantum line bundle $C (S_{H}^{2n-1})_k$ of degree $k$ over the quantum space $\mathbb{P}^{n-1}(\mathcal{T})$.

It is easy to see that the standard projections $Q_{j,\Theta,l} \equiv \partial_n (\Theta (P_{j,l}))$ over $C (S_{H}^{2n-1})$ with $j, l > 0$ found in the previous section lie in $M_\infty (C^* ((\mathcal{G}_n)_0))$ since $P_{j,l} = \bigoplus l (\bigotimes^j I) \otimes (\bigotimes^{n-j} P_1)$ is in $C^* ((\mathcal{G}_n)_0)$, and hence are also projections over $C^* ((\mathcal{G}_n)_0) \equiv C(\mathbb{P}^{n-1}(\mathcal{T}))$. Furthermore with $C(\mathbb{P}^{n-1}(\mathcal{T})) \subset C (S_{H}^{2n-1})$, inequivalent $\bigoplus$-sums of standard projections over $C (S_{H}^{2n-1})$ must be inequivalent over $C(\mathbb{P}^{n-1}(\mathcal{T}))$ as well.

**Proposition 6.** Different reduced $\bigoplus$-sums of standard projections $Q_{j,\Theta,l}$ over $C (S_{H}^{2n-1})$ with $j, l > 0$ when viewed as projections over $C(\mathbb{P}^{n-1}(\mathcal{T}))$ are mutually inequivalent over $C(\mathbb{P}^{n-1}(\mathcal{T}))$, and they form a graded submonoid

$$\mathcal{P}' (C(\mathbb{P}^{n-1}(\mathcal{T}))) = \bigsqcup_{m=0}^{\infty} \mathcal{P}'_m (C(\mathbb{P}^{n-1}(\mathcal{T})))$$

of the monoid $\mathcal{P} (C(\mathbb{P}^{n-1}(\mathcal{T})))$. Furthermore the monoid homomorphism

$$\mathcal{P}' (C(\mathbb{P}^{n-1}(\mathcal{T}))) \to \prod_{(j,\Theta) \in \Omega_0} \mathbb{Z}_{\geq}$$

37
inherited from $\rho_\theta$ is injective.

However, for $(j', \Theta') \prec (j, \Theta)$ with $j, j', l, l' > 0$, it is no longer true in general that $Q_{j, \Theta, l} \boxplus Q_{j', \Theta', l'} \sim Q_{j, \Theta, l}$ over $C(\mathbb{P}^{n-1}(T))$, even though $Q_{j, \Theta, l} \boxplus Q_{j', \Theta', l'} \sim Q_{j, \Theta, l}$ over $C(S_H^{2n-1})$ since the invertible matrix over $C(S_H^{2n-1})$ intertwining $Q_{j, \Theta, l} \boxplus Q_{j', \Theta', l'}$ and $Q_{j, \Theta, l}$ may not be replaced by one over the subalgebra $C(\mathbb{P}^{n-1}(T))$ of $C(S_H^{2n-1})$.

In the following, we show that the standard projections $Q_{j, \text{id}, 1}$ with $j > 0$ provide a set of representatives of $K_0$-classes that freely generate the abelian $K_0$-group of $C(\mathbb{P}^{n-1}(T))$.

The subgroupoid $\mathfrak{H}_j := \mathfrak{G}_j \times (\mathbb{Z}^{n-j} \times \mathbb{Z}^{n-j}_>)$ of $\mathfrak{G}_n$ for $1 \leq j \leq n$ is the groupoid $\mathfrak{G}_n$ restricted to the open invariant subset $\big(\mathbb{Z}_>^j \times \mathbb{Z}_>^{n-j}\big) \setminus \mathbb{Z}_>^n$ and inherits the grading of $\mathfrak{G}_n$. The grade-0 part $(\mathfrak{H}_j)_0$ of $\mathfrak{H}_j$ is the groupoid $(\mathfrak{G}_n)_0$ restricted to $\big(\mathbb{Z}_>^j \times \mathbb{Z}_>^{n-j}\big) \setminus \mathbb{Z}_>^n$, and from the increasing chain of $(\mathfrak{H}_j)_0$, we get an increasing composition sequence of closed ideals of $C^*((\mathfrak{G}_n)_0)$ as

$$0 =: C^*((\mathfrak{H}_0)_0) \triangleleft C^*((\mathfrak{H}_1)_0) \triangleleft \cdots \triangleleft C^*((\mathfrak{H}_{n-1})_0) \triangleleft C^*((\mathfrak{H}_n)_0) = C^*((\mathfrak{G}_n)_0)$$

such that with $\big(\mathbb{Z}_>^j \times \mathbb{Z}_>^{n-j}\big) \setminus \big(\mathbb{Z}_>^j \times \mathbb{Z}_>^{n-j+1}\big) = \mathbb{Z}_>^{j-1} \times \{\infty\} \times \mathbb{Z}_>^{n-j}$,

$$C^*((\mathfrak{H}_j)_0)/C^*((\mathfrak{H}_{j-1})_0) \cong C^*\left(\mathfrak{G}_n|_{\mathbb{Z}_>^j \times \{\infty\} \times \mathbb{Z}_>^{n-j}}\right)_0 \cong C^*\left(\mathfrak{T}_{n-1}|_{\mathbb{Z}_>^{j-1} \times \mathbb{Z}_>^{n-j}}\right) \cong T^*_{\otimes j^{-1}} \otimes K(\mathbb{Z}^{n-j}_>)$$

because the groupoid $\left(\mathfrak{G}_n|_{\mathbb{Z}_>^{j-1} \times \{\infty\} \times \mathbb{Z}_>^{n-j}}\right)_0$ is isomorphic to the groupoid $\mathfrak{T}_{n-1}|_{\mathbb{Z}_>^{j-1} \times \mathbb{Z}_>^{n-j}}$ via the groupoid isomorphism

$$(m, k, l, p, \infty, q) \mapsto (m, l, p, q)$$

where

$$(m, k, l, p, \infty, q) \in \mathfrak{G}_n|_{\mathbb{Z}_>^{j-1} \times \{\infty\} \times \mathbb{Z}_>^{n-j}} \subset \mathbb{Z}_>^{j-1} \times \mathbb{Z} \times \mathbb{Z}^{n-j} \times \mathbb{Z}_>^{j-1} \times \{\infty\} \times \mathbb{Z}_>^{n-j}$$

38
with $\sum_{i=1}^{j-1} m_i + k + \sum_{i=1}^{n-j} l_i = 0$ and hence $k = -\sum m - \sum l$ determined by $m, l$.

Since $K_1 \left( T^{\otimes j-1} \otimes K \left( \mathbb{Z}_\geq^{n-j} \right) \right) = 0$ and $K_0 \left( T^{\otimes j-1} \otimes K \left( \mathbb{Z}_\geq^{n-j} \right) \right) = \mathbb{Z}$, it is easy to conclude from the cyclic six-term exact sequence of $K$-groups for the pair $C^* \left( (\mathcal{H}_{j-1})_0 \right) \lhd C^* \left( (\mathcal{H}_j)_0 \right)$ that the following sequence is exact and splits

$$0 \to K_0 \left( C^* \left( (\mathcal{H}_{j-1})_0 \right) \right) \to K_0 \left( C^* \left( (\mathcal{H}_j)_0 \right) \right) \to K_0 \left( T^{\otimes j-1} \otimes K \left( \mathbb{Z}_\geq^{n-j} \right) \right) \cong \mathbb{Z} \to 0$$

where the projection $(\otimes^j - I) \otimes (\otimes^{n-j} P_1)$ is a generator of $K_0 \left( T^{\otimes j-1} \otimes K \left( \mathbb{Z}_\geq^{n-j} \right) \right)$. Note that this $(\otimes^j - I) \otimes (\otimes^{n-j} P_1)$ lifts to the projection element $\chi_{A_j} \in C_c \left( (\mathcal{H}_j)_0 \right) \subset C^* \left( (\mathcal{H}_j)_0 \right)$ given by the characteristic function of the set

$$A_j := \{0\} \times \{0\} \times \left( \mathbb{Z}_\geq^j \setminus \mathbb{Z}_\geq^j \right) \times \{0\} \subset \mathcal{H}_j \subset \mathbb{Z}^j \times \mathbb{Z}^{n-j} \times \mathbb{Z}_\geq^j \times \mathbb{Z}_\geq^{n-j}.$$

Furthermore $\chi_{A_j} = Q_{j,\text{id},1}$ in $C \left( \mathbb{P}^{n-1} (T) \right) \subset C \left( S_H^{2n-1} \right)$. So we get

$$K_0 \left( C^* \left( (\mathcal{H}_j)_0 \right) \right) \cong K_0 \left( C^* \left( (\mathcal{H}_{j-1})_0 \right) \right) \oplus \mathbb{Z} [Q_{j,\Theta,1}]$$

with $K_0 \left( C^* \left( (\mathcal{H}_{j-1})_0 \right) \right)$ canonically embedded in $K_0 \left( C^* \left( (\mathcal{H}_j)_0 \right) \right)$.

Putting together these results for all $j$, we get

$$K_0 \left( C \left( \mathbb{P}^{n-1} (T) \right) \right) \equiv K_0 \left( C^* \left( (\mathcal{H}_n)_0 \right) \right) \cong \bigoplus_{j=1}^n \mathbb{Z} [Q_{j,\text{id},1}] \cong \mathbb{Z}^n$$

and hence $Q_{j,\text{id},1}$ freely generate the abelian group $K_0 \left( C \left( \mathbb{P}^{n-1} (T) \right) \right)$. Note that $Q_{j,\text{id},l} = \mathbb{H}^l Q_{j,\text{id},1}$ and $[Q_{j,\text{id},l}] = l [Q_{j,\text{id},1}]$ in $K_0 \left( C \left( \mathbb{P}^{n-1} (T) \right) \right)$ for any $l \in \mathbb{N}$.

We now summarize the above discussion.

**Theorem 4.** The projections $Q_{j,\Theta,l}$ over $C \left( \mathbb{P}^{n-1} (T) \right)$ with $l \in \mathbb{N}$ and $\Theta$ a $(j, n-j)$-shuffle for $0 < j \leq n$ are mutually inequivalent, and the projections $Q_{j,\text{id},1}$ with $0 < j \leq n$
freely generate the abelian group $K_0 \left( C(\mathbb{P}^{n-1}(T)) \right)$, such that if $[p] = \sum_{j=1}^{n} m_j [Q_{j,\text{id},1}]$ for a projection $p$ over $C(\mathbb{P}^{n-1}(T))$, then the coefficient $m_n$ of $[Q_{n,\text{id},1}]$ is the rank of $p$.

Proof. We only need to note that the rank of $Q_{n,\text{id},1}$ is 1 and the rank of any other $Q_{j,\text{id},1}$ is 0. □

Remark. Since any permutation $\Theta$ on $\{1, 2, ..., n\}$ canonically induces a $U(1)$-equivariant (outer) C*-algebra automorphism of $\mathcal{T}^\otimes n$ permuting its tensor factors and preserving its ideal $\mathcal{K}^\otimes n$, the above expression of free generators $[\partial_n (\otimes^j I \otimes \otimes^{n-j} P_1)]$ with $0 < j \leq n$ of $K_0 \left( C(\mathbb{P}^{n-1}(T)) \right)$ can be changed by a permutation to yield some other free generators. For example, both $\{[\partial_3 (1 \otimes P_1 \otimes P_1)], [\partial_3 (1 \otimes 1 \otimes P_1)], [\partial_3 (1 \otimes 1 \otimes 1)]\}$ and $\{[\partial_3 (P_1 \otimes P_1 \otimes 1)], [\partial_3 (P_1 \otimes 1 \otimes 1)], [\partial_3 (1 \otimes 1 \otimes 1)]\}$ are sets of free generators of $K_0 \left( C(\mathbb{P}^{2}(T)) \right)$.

The above theorem shows that for $(j',\text{id}) < (j,\text{id})$ in $\Omega_0$, i.e. $0 < j' < j$, it is not true that $Q_{j,\text{id},1} \boxplus Q_{j',\text{id},1} \sim Q_{j,\text{id},1}$ over $C(\mathbb{P}^{n-1}(T))$ because

$$[Q_{j,\text{id},1} \boxplus Q_{j',\text{id},1}] = [Q_{j,\text{id},1}] + [Q_{j',\text{id},1}] \neq [Q_{j,\text{id},1}]$$

in $K_0 \left( C(\mathbb{P}^{n-1}(T)) \right)$.

Next we consider the positive cone of $K_0 \left( C(\mathbb{P}^{n-1}(T)) \right)$. In the following, we use $\hat{I}$ and $\tilde{I}$ to denote the identity elements of $\mathcal{T}^\otimes n$ and $\mathcal{T}^\otimes n$ respectively.

First, it is easy to see that for $k > 0$, the projection $\hat{I} \otimes P_k$ is a sum of $k$ mutually orthogonal projections $\hat{I} \otimes e_{jj}$, each equivalent to $\hat{I} \otimes P_1$ over $(\mathcal{T}^\otimes n \otimes \mathcal{K})^+ \subset \mathcal{T}^\otimes n$, and hence the projection $\partial_n \left( \hat{I} \otimes P_k \right)$ is a sum of $k$ mutually orthogonal projections $\partial_n \left( \hat{I} \otimes e_{jj} \right)$, each equivalent to $\partial_n \left( \hat{I} \otimes P_1 \right)$ over $C(\mathbb{S}_{\mathcal{H}}^{2n-1})$. So

$$\hat{I} \otimes P_k \sim \bigoplus_{j=1}^{k} (\hat{I} \otimes P_1) \equiv \bigoplus_{j=1}^{k} P_{n-1,1} \equiv P_{n-1,1,k} \text{ over } (\mathcal{T}^\otimes n \otimes \mathcal{K})^+ \subset \mathcal{T}^\otimes n,$$
and \( \partial_n (\hat{I} \otimes P_k) \sim \oplus^k Q_{n-1,\text{id},1} \equiv Q_{n-1,\text{id},k} \) over \( C(S_H^{2n-1}) \). Similarly, by rearranging entries via conjugation by shifts, the projection \( \hat{I} \otimes P_{-k} \) is equivalent to \( \hat{I} \) over \( T^{\otimes n} \), and hence \( \partial_n (\hat{I} \otimes P_{-k}) \sim \partial_n (\hat{I}) \) over \( C(S_H^{2n-1}) \). However, such equivalences no longer hold over the algebra \( C(\mathbb{P}^{n-1}(T)) \subset C(S_H^{2n-1}) \). For example, \( \partial_n (\hat{I} \otimes P_{-k}) \oplus \partial_n (\hat{I} \otimes P_{k}) \sim \partial_n (\hat{I}) \) over \( C(\mathbb{P}^{n-1}(T)) \) since \( \partial_n (\hat{I} \otimes P_{-k}) \) and \( \partial_n (\hat{I} \otimes P_{k}) \) are orthogonal projections in \( C(\mathbb{P}^{n-1}(T)) \) which add up to \( \hat{I} \). So

\[
\left[ \partial_n (\hat{I} \otimes P_{-1}) \right] = \left[ \partial_n (\hat{I}) \right] - \left[ \partial_n (\hat{I} \otimes P_1) \right] = [Q_{n,\text{id},1}] - [Q_{n-1,\text{id},1}] \quad \text{in} \ K_0(C(\mathbb{P}^{n-1}(T)))
\]

showing that

\[
\left[ \partial_n (\hat{I} \otimes P_{-1}) \right] \in \mathbb{Z}^{n-2} \times \{-1\} \times \{1\} \subset \mathbb{Z}^n \cong K_0(C(\mathbb{P}^{n-1}(T)))
\]

and \( \partial_n (\hat{I} \otimes P_{-1}) \) is not even stably equivalent over \( C(\mathbb{P}^{n-1}(T)) \) to any \( \oplus \)-sum of the \( K_0 \)-generating projections \( Q_{j,\text{id},1} \) with \( 0 < j \leq n \).

From now on, we include all projections of the form \( \partial_n ((\otimes^{j-1}I) \otimes P_k \otimes (\otimes^{n-j}P_1)) \) with \( k \in \mathbb{Z} \) as elementary projections over \( C(\mathbb{P}^{n-1}(T)) \), where it is understood that for \( k = 0 \), we take \( P_k := P_{-0} \equiv I \) instead of \( P_0 \equiv 0 \).

**Theorem 5.** The positive cone of \( K_0(C(\mathbb{P}^{n-1}(T))) \cong \mathbb{Z}^n \equiv \bigoplus_{j=1}^n \mathbb{Z} [Q_{j,\text{id},1}] \) contains

\[
\mathbb{Z}^n \setminus \{ z \in \mathbb{Z}^n : z_j < 0 = z_{j+1} = \cdots = z_n \text{ for some } 1 \leq j \leq n \}
\]

which is the part of the cone generated/spanned by the equivalence classes of the elementary projections \( \partial_n ((\otimes^{j-1}I) \otimes P_k \otimes (\otimes^{n-j}P_1)) \) with \( k \in \mathbb{Z} \) and \( 1 \leq j \leq n \), where for \( k = 0 \), we take \( P_k := P_{-0} \equiv I \).
Proof. In [29], it has been established that in the case of
\( n = 2 \), the positive cone of \( K_0 ( C ( P^1 ( T ) ) ) \) consists of \((k, m) \in \mathbb{Z}^2 \) with either \( k \geq 0 \) or the rank \( m > 0 \), such that \([\partial_2 (I \otimes P_k)] = k [\partial_2 (I \otimes P_1)] = (k, 0)\) and
\([\partial_2 (I \otimes P_{-k})] = [\partial_2 (I \otimes I)] - k [\partial_2 (I \otimes P_1)] = (-k, 1)\)
in \( K_0 ( C ( P^1 ( T ) ) ) \) for all \( k > 0 \).

By induction on \( n \), we can show that the positive cone of \( K_0 ( C ( P^n-1 ( T ) ) ) \) contains the set \((\mathbb{Z}^n_{\geq 1} \times \{0\}) \cup (\mathbb{Z}^n_{\geq 1} \times \mathbb{N})\) consisting of \((k_1, ..., k_{n-1}, m) \in \mathbb{Z}^n \) with either \( k_j \geq 0 \) for all \( j \) or the rank \( m > 0 \).

Indeed, under the canonical embedding
\[ \iota : C ( P^{n-2} ( T ) ) \equiv C^* ((\mathfrak{g}_{n-1})_0) \to C ( P^{n-1} ( T ) ) \equiv C^* ((\mathfrak{g}_n)_0) \]
due to the degree-preserving groupoid embedding of \((\mathbb{Z}^{n-1_{\geq 1}} \times \mathbb{Z}^{n-1_{\geq 1}}) \big|_{\mathbb{Z}_{\geq 1}} \) in \((\mathbb{Z}^n \times \mathbb{Z}^n) \big|_{\mathbb{Z}_{\geq 1}}\) as \((\mathbb{Z}^{n-1_{\geq 1}} \times \{0\}) \times (\mathbb{Z}^{n-1_{\geq 1}} \times \{0\}) \big|_{\mathbb{Z}^{n-1_{\geq 1}} \times \{0\}}\), a projection \( p \) (for example, \( \partial_{n-1} (P_{k_1} \otimes \cdots \otimes P_{k_{n-1}}) \)) over \( C ( P^{n-2} ( T ) ) \) becomes the projection \( p \otimes P_1 \) (for example, \( \partial_n (P_{k_1} \otimes \cdots \otimes P_{k_{n-1}} \otimes P_1) \)) over \( C ( P^{n-1} ( T ) ) \). Furthermore if \( p \sim q \) over \( C ( P^{n-2} ( T ) ) \), say, \( upu^{-1} = q \) for some \( u \in GL_\infty ( C ( P^{n-2} ( T ) ) ) \) then the equivalence \( p \otimes P_1 \sim q \otimes P_1 \) over \( C ( P^{n-1} ( T ) ) \) can be explicitly constructed as
\[ ((u \otimes P_1) + \partial_n (I \otimes P_{-1})) (p \otimes P_1) ((u \otimes P_1) + \partial_n (I \otimes P_{-1}))^{-1} = q \otimes P_1 \]
with \((u \otimes P_1) + \partial_n (I \otimes P_{-1}) \in GL_\infty (C (\mathbb{P}^{n-1} (\mathcal{T})))\). Now consider the well-defined group homomorphism

\[
K_0 (\iota) : K_0 (C (\mathbb{P}^{n-2} (\mathcal{T}))) \cong \mathbb{Z}^{n-1} \to K_0 (C (\mathbb{P}^{n-1} (\mathcal{T}))) \cong \mathbb{Z}^n
\]
mapping the positive cone of \(K_0 (C (\mathbb{P}^{n-2} (\mathcal{T})))\) into that of \(K_0 (C (\mathbb{P}^{n-1} (\mathcal{T})))\). Since under \(\iota\), the projection \(Q_{j, \text{id}, 1}\) over \(C (\mathbb{P}^{n-2} (\mathcal{T}))\) for \(0 < j \leq n - 1\) is sent to the projection \(Q_{j, \text{id}, 1}\) over \(C (\mathbb{P}^{n-1} (\mathcal{T}))\), by induction hypothesis, we get that the positive cone of \(K_0 (C (\mathbb{P}^{n-2} (\mathcal{T})))\) contains \((\mathbb{Z}^{n-2} \times \{0\} \times \{0\}) \cup (\mathbb{Z}^{n-2} \times \mathbb{N} \times \{0\})\), and hence \((\mathbb{Z}^{n-1} \times \{0\}) \cup (\mathbb{Z}^{n-2} \times \mathbb{N} \times \mathbb{Z}_\geq)\).

On the other hand, for \(k > 0\),

\[
\hat{I} \otimes P_{-k} = \left( \hat{I} \otimes P_{-(k+1)} \right) \boxplus \left( \hat{I} \otimes e_{kk} \right) \sim \left( \hat{I} \otimes P_{-(k+1)} \right) \boxplus \left( I' \otimes P_{-k} \otimes P_1 \right) \quad \text{over} \quad \mathcal{T}^{\otimes n}
\]

where \(I'\) denotes the identity element of \(\mathcal{T}^{\otimes n-2}\) and \(e_{ij}\) with \(i, j \in \mathbb{Z}_\geq\) represents a matrix unit projection, because \(\hat{I} \otimes P_{-k}\) is the sum of orthogonal projections \(\left( \hat{I} \otimes P_{-(k+1)} \right)\) and \(\left( \hat{I} \otimes e_{kk} \right)\), and \(\left( \hat{I} \otimes e_{kk} \right) \boxplus 0\) when conjugated by

\[
u_k := \begin{pmatrix}
I' \otimes I \otimes P_k & I' \otimes (S^k)^* \otimes S^k \\
I' \otimes S^k \otimes (S^k)^* & I' \otimes P_k \otimes I
\end{pmatrix} \in GL_2 (\mathcal{T}^{\otimes n})
\]

becomes \(0 \boxplus (I' \otimes P_{-k} \otimes P_1)\). Since \(\partial_n (u_k)\) of total degree 0 is in \(M_2 (C (\mathbb{P}^{n-1} (\mathcal{T})))\), we get

\[
\partial_n \left( \hat{I} \otimes P_{-k} \right) \sim \partial_n \left( \left( \hat{I} \otimes P_{-(k+1)} \right) \right) \boxplus \iota (\partial_{n-1} (I' \otimes P_{-k})) \quad \text{over} \quad C (\mathbb{P}^{n-1} (\mathcal{T}))
\]

and hence

\[
[\partial_n \left( \hat{I} \otimes P_{-k} \right) - \partial_n \left( \left( \hat{I} \otimes P_{-(k+1)} \right) \right)] \in \mathbb{Z}^{n-2} \times \{1\} \times \{0\} \quad \text{in} \quad \mathbb{Z}^n
\]
because \( [\partial_{n-1} (I' \otimes P_k)] \in \mathbb{Z}^{n-2} \times \{1\} \) for the rank-one projection \( I' \otimes P_k \) over \( \mathcal{T}^{\otimes n-1} \).

With

\[
\left[ \partial_n \left( \tilde{I} \otimes P_{-k} \right) \right] = \left[ \partial_n \left( \tilde{I} \right) \right] - \left[ \partial_n \left( \tilde{I} \otimes P_{1} \right) \right] = (0, ..., 0, -1, 1) \in \mathbb{Z}^n,
\]

we get inductively

\[
\left[ \partial_n \left( \tilde{I} \otimes P_{-k} \right) \right] \in \mathbb{Z}^{n-2} \times \{-k\} \times \{1\} \subset \mathbb{Z}^n \cong K_0 \left( C \left( \mathbb{P}^{n-1} \left( \mathcal{T} \right) \right) \right)
\]

for all \( k > 0 \). Thus the positive cone of \( K_0 \left( C \left( \mathbb{P}^{n-1} \left( \mathcal{T} \right) \right) \right) \cong \mathbb{Z}^n \) contains \((\mathbb{Z}_{\geq 1}^{n-1} \times \{0\}) \cup (\mathbb{Z}^{n-1} \times \{1\})\) and hence \((\mathbb{Z}_{\geq 1}^{n-1} \times \{0\}) \cup (\mathbb{Z}^{n-1} \times \mathbb{N})\). On the other hand, the positive cone of \( K_0 \left( C \left( \mathbb{P}^{n-2} \left( \mathcal{T} \right) \right) \right) \) is mapped into the positive cone of \( K_0 \left( C \left( \mathbb{P}^{n-1} \left( \mathcal{T} \right) \right) \right) \) by the homomorphism \( \cdot \times \{0\} \equiv K_0 (i) \). So it is easy to get inductively the conclusion. \( \square \)

We note that for the case of \( n = 2 \), the finitely generated projective modules over \( C \left( \mathbb{P}^{1} \left( \mathcal{T} \right) \right) \) are completely classified with the positive cone of \( K_0 \left( C \left( \mathbb{P}^{1} \left( \mathcal{T} \right) \right) \right) \) explicitly identified in [29].

### 8 Quantum line bundles

In this section, we identify the quantum line bundles \( L_k := C \left( \mathbb{S}^{2n-1}_H \right) \) of degree \( k \) over \( C \left( \mathbb{P}^{n-1} \left( \mathcal{T} \right) \right) \) with a concrete (equivalence class of) projection described in terms of the elementary projections defined in the previous section. We continue to use \( \hat{I} \) and \( \bar{I} \) to denote the identity elements of \( \mathcal{T}^{\otimes n-1} \) and \( \mathcal{T}^{\otimes n} \) respectively, and we start to use \( 0^{(l)} \) to denote the zero of \( \mathbb{Z}^l \).

To distinguish between ordinary function product and convolution product, we denote
the groupoid C*-algebraic (convolution) multiplication of elements in $C_c(\mathcal{G}) \subset C^*(\mathcal{G})$ by $\ast$, while omitting $\ast$ when the elements are presented as operators or when they are multiplied together pointwise as functions on $\mathcal{G}$. We also view $C_c(\mathfrak{S}_n)$ or $C_c ((\mathfrak{S}_n)_k)$ (also abbreviated as $C_c(\mathfrak{S}_n)_k$) as left $C_c((\mathfrak{S}_n)_0)$-modules with $C_c(\mathfrak{S}_n)$ carrying the convolution algebra structure as a subalgebra of the groupoid C*-algebra $C^*(\mathfrak{S}_n)$. Similarly, for a closed subset $X$ of the unit space of $\mathfrak{S}_n$, the inverse image $\mathfrak{S}_n |_X$ of $X$ under the source map of $\mathfrak{S}_n$ or its grade-$k$ component $(\mathfrak{S}_n |_X)_k$ gives rise to a left $C_c((\mathfrak{S}_n)_0)$-module $C_c(\mathfrak{S}_n |_X)$ or $C_c((\mathfrak{S}_n |_X)_k)$.

We define a partial isometry in $C(\mathbb{S}^{2n-1}_H) \equiv C^*(\mathfrak{S}_n)$ for each $k \in \mathbb{Z}$ as the characteristic function $\chi_{B_k}$ of the compact open set

$$B_k := \left\{ (0, k, p, q) \in \mathfrak{S}_n \subset \mathbb{Z}^{n-1} \times \mathbb{Z} \times \frac{\mathbb{Z}^{n-1}}{\mathbb{Z}} \times \frac{\mathbb{Z}}{\mathbb{Z}^2} : q + k \geq 0 \right\} \subset \mathfrak{S}_n.$$  

It is easy to verify that $\chi_{B_k} \in C_c(\mathfrak{S}_n)_k$ and $(\chi_{B_k})^\ast \in C_c((\mathfrak{S}_n)_{-k})$ such that

$$(\chi_{B_k})^\ast \chi_{B_k} = \begin{cases} \chi_{\{0^{(n)}\} \times (\mathbb{Z}_2 \times \mathbb{Z}_2^n)} = 1_{C^*(\mathfrak{S}_n)} \equiv 1_{C^*(\mathfrak{S}_n)_0} & \text{if } k \geq 0 \\ \chi_{\{0^{(n)}\} \times ((\mathbb{Z}_2^{n-1} \times \mathbb{Z}_2) \setminus \mathbb{Z}_2^n)} = \partial_n (\hat{I} \otimes P_{-|k|}) & \text{if } k < 0 \end{cases}$$

and

$$\chi_{B_k} (\chi_{B_k})^\ast = \begin{cases} \chi_{\{0^{(n)}\} \times ((\mathbb{Z}_2^{n-1} \times \mathbb{Z}_2) \setminus \mathbb{Z}_2^n)} = \partial_n (\hat{I} \otimes P_{-k}) & \text{if } k \geq 0 \\ \chi_{\{0^{(n)}\} \times (\mathbb{Z}_2^n \setminus \mathbb{Z}_2^n)} = 1_{C^*(\mathfrak{S}_n)} \equiv 1_{C^*(\mathfrak{S}_n)_0} & \text{if } k < 0 \end{cases}.$$  

For $k \geq 0$, we have $C_c((\mathfrak{S}_n)_k) \ast (\chi_{B_k})^\ast \subset C_c(\mathfrak{S}_n)_0$ and

$$(C_c((\mathfrak{S}_n)_k) \ast (\chi_{B_k})^\ast) \ast \chi_{B_k} = C_c((\mathfrak{S}_n)_k)$$

which implies that the convolution operator $\cdot \ast \chi_{B_k}$ maps $C_c((\mathfrak{S}_n)_0)$ onto $C_c(\mathfrak{S}_n)_k$. Since $\chi_{B_k} (\chi_{B_k})^\ast = \chi_{\{0^{(n)}\} \times ((\mathbb{Z}_2^{n-1} \times \mathbb{Z}_2) \setminus \mathbb{Z}_2^n)}$, we get $\cdot \ast \chi_{B_k}$ mapping $C_c((\mathfrak{S}_n)_0) \ast \chi_{\{0^{(n)}\} \times ((\mathbb{Z}_2^{n-1} \times \mathbb{Z}_2) \setminus \mathbb{Z}_2^n)}$.
continuously to provide an isomorphism between the $C_c(\mathfrak{g}_n)_k$ with $\cdot \ast (\chi_{B_k})^*$ as the inverse. Furthermore $\cdot \ast \chi_{B_k}$ is a left $C_c(\mathfrak{g}_n)_0$-module homomorphism. With $\chi_{B_k}$ being a partial isometry, $\cdot \ast \chi_{B_k}$ and $\cdot \ast (\chi_{B_k})^*$ extend continuously to provide an isomorphism between the $C^*(\mathfrak{g}_n)_0$-modules

$$C^*(\mathfrak{g}_n)_0 \chi_{\{0^{(n)}\}} \times ((\mathbb{Z}_2^{n-1} \times \mathbb{N}) \setminus \mathbb{Z}_2^n) \equiv C^*(\mathfrak{g}_n)_0 \partial_n \left( \hat{I} \otimes P_{\cdot -k} \right)$$

and $C^*(\mathfrak{g}_n)_k \equiv C_c(\mathfrak{g}_n)_k$. So the quantum line bundle $C^*(\mathfrak{g}_n)_k$ is identified with the projection $\partial_n \left( \hat{I} \otimes P_{\cdot -k} \right)$.

For $k < 0$, we consider the direct sum decomposition as left $C_c(\mathfrak{g}_n)_0$-modules

$$C_c(\mathfrak{g}_n)_k = \left( C_c(\mathfrak{g}_n)_k \ast \chi_{\{0^{(n)}\}} \times ((\mathbb{Z}_2^{n-1} \times \mathbb{N}) \setminus \{0,1,\ldots,k\}) \right) \oplus \left( C_c(\mathfrak{g}_n)_k \ast \chi_{\{0^{(n)}\}} \times ((\mathbb{Z}_2^{n-1} \times \mathbb{N}) \setminus \mathbb{Z}_2^n) \right)$$

$$= C_c\left( \mathfrak{g}_n \uparrow (\mathbb{Z}_2^{n-1} \setminus \mathbb{Z}_2) \times \{0,1,\ldots,k\} \right) \oplus \left( C_c(\mathfrak{g}_n)_k \ast \chi_{\{0^{(n)}\}} \times ((\mathbb{Z}_2^{n-1} \times \mathbb{N}) \setminus \mathbb{Z}_2^n) \right).$$

From

$$C_c(\mathfrak{g}_n)_0 \ast \chi_{B_k} \ast (\chi_{B_k})^* \equiv C_c(\mathfrak{g}_n)_0 \ast 1_{C^*(\mathfrak{g}_n)} = C_c(\mathfrak{g}_n)_0$$

and

$$C_c(\mathfrak{g}_n)_k \ast (\chi_{B_k})^* \ast \chi_{B_k} = C_c(\mathfrak{g}_n)_k \ast \chi_{\{0^{(n)}\}} \times ((\mathbb{Z}_2^{n-1} \times \mathbb{Z}_2) \setminus \mathbb{Z}_2^n)$$

we see that $\cdot \ast \chi_{B_k}$ is a left $C_c(\mathfrak{g}_n)_0$-module isomorphism between $C_c(\mathfrak{g}_n)_0$ and $C_c(\mathfrak{g}_n)_k \ast \chi_{\{0^{(n)}\}} \times ((\mathbb{Z}_2^{n-1} \times \mathbb{Z}_2) \setminus \mathbb{Z}_2^n)$ with $\cdot \ast (\chi_{B_k})^*$ as its inverse.

On the other hand, in the $C_c(\mathfrak{g}_n)_0$-module decomposition

$$C_c\left( \mathfrak{g}_n \uparrow (\mathbb{Z}_2^{n-1} \setminus \mathbb{Z}_2) \times \{0,1,\ldots,k\} \right) \kappa = \bigoplus_{j=0}^{\lfloor k \rfloor - 1} C_c\left( \mathfrak{g}_n \uparrow (\mathbb{Z}_2^{n-1} \setminus \mathbb{Z}_2) \times \{j\} \right) \kappa,$$

each $C_c\left( \mathfrak{g}_n \uparrow (\mathbb{Z}_2^{n-1} \setminus \mathbb{Z}_2) \times \{j\} \right) \kappa$ is isomorphic to the $C_c(\mathfrak{g}_n)_0$-module $C_c\left( \mathfrak{g}_n \uparrow (\mathbb{Z}_2^{n-1} \setminus \mathbb{Z}_2) \times \{0\} \right) \kappa + \jmath$. 

46
with \( k + j < 0 \) via the homeomorphism

\[
(m, l, p, j) \in \left( \mathfrak{S}_n \upharpoonright (\mathbb{Z}_2^{n-1} \times \mathbb{Z}_2^{n-1}) \times \{j\} \right)_k \subset \mathbb{Z}^{n-1} \times \mathbb{Z} \times \mathbb{Z}_2^{n-1} \times \mathbb{Z}_2 \mapsto (m, l + j, p, 0) \in \left( \mathfrak{S}_n \upharpoonright (\mathbb{Z}_2^{n-1} \times \mathbb{Z}_2^{n-1}) \times \{0\} \right)_{k+j}
\]

where the implicit condition \( l + j \geq 0 \) is equivalent to \( l \geq -j \). So we focus on analyzing \( C_c(\mathfrak{S}_n)_0 \)-modules of the form

\[
C_c\left( \left( \mathfrak{S}_n \upharpoonright (\mathbb{Z}_2^{n-1} \times \mathbb{Z}_2^{n-1}) \times \{0\} \right)_{-r} \right) = C_c(\mathfrak{S}_n)_{-r} \ast \chi_{\{0^{(n)}\}} \times (\mathbb{Z}_2^{n-1} \times \mathbb{Z}_2^{n-1} \times \{0\}) = C_c(\mathfrak{S}_n)_{-r} \partial_n \left( I \otimes P_1 \right)
\]

with \( r \geq 0 \). Note that the \( C^* (\mathfrak{S}_n)_0 \)-module

\[
\mathfrak{C}_c(\mathfrak{S}_n)_0 \partial_n \left( I \otimes P_1 \right) \equiv Q_{n-1,\text{id},1}
\]

For \( r > 0 \), similar to the argument used above, it can be checked that the compact open subset

\[
B'_r := \left\{ (0, -r, 0, p, q, 0) \in \mathfrak{S}_n \subset \mathbb{Z}^{n-2} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_2^{n-2} \times \mathbb{Z}_2 \times \mathbb{Z}_2 : q \geq r \right\} \subset \mathfrak{S}_n
\]

defines a partial isometry \( \chi_{B'_r} \in C_c(\mathfrak{S}_n)_{-r} \) with \( \left( \chi_{B'_r} \right)^* \in C_c(\mathfrak{S}_n)_r \) such that

\[
\left( \chi_{B'_r} \right)^* \chi_{B'_r} = \chi_{\{0^{(n)}\}} \times (\mathbb{Z}_2^{n-2} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) = \partial_n \left( I^{n-2} \otimes P_{-r} \otimes I \right)
\]

and

\[
\chi_{B'_r} \ast \left( \chi_{B'_r} \right)^* = \chi_{\{0^{(n)}\}} \times (\mathbb{Z}_2^{n-1} \times \mathbb{Z}_2^{n-1} \times \{0\})
\]

In the decomposition

\[
C_c(\mathfrak{S}_n)_{-r} \ast \chi_{\{0^{(n)}\}} \times (\mathbb{Z}_2^{n-1} \times \mathbb{Z}_2^{n-1} \times \{0\})
\]
the second summand is isomorphic, via the right convolution * \( C \mapsto r \)
for \( r \) is isomorphic to via the homeomorphism

\[
A(c,r,l) \sim (C \mapsto r,c,l) \sim (C \mapsto r,c,l)
\]

Applying the same kind of arguments as shown above, we get the isomorphism of

\[
C_c(\mathfrak{G}_n)_0 \otimes (\hat{1} \otimes P_1).
\]

Now we introduce the notation of a \( C_c(\mathfrak{G}_n)_0 \)-module

\[
A_{n,l} := C_c \left( \left( \left( \mathfrak{G}_n \mid_{\geq l} \times \{0 \} \right) \right) \right) \subset C_c (\mathfrak{G}_n) \subset C_c (\mathfrak{G}_n)
\]
for \( r \geq 0 \) and \( 1 \leq l \leq n - 1 \). We note that the \( C_c(\mathfrak{G}_n)_0 \)-module

\[
A_{n,1} = C_c \left( \left( \left( \mathfrak{G}_n \mid_{\geq 1} \times \{0 \} \right) \right) \right)
\]
is isomorphic to

\[
C_c \left( \left( \left( \mathfrak{G}_n \mid_{\geq 1} \times \{0 \} \right) \right) \right) = C_c \left( \left( \left( \mathfrak{G}_n \mid_{\geq 1} \times \{0 \} \right) \right) \right)
\]

\[
\equiv C_c (\mathfrak{G}_n)_0 \otimes \chi(0) \times ((\mathfrak{G}_n \mid_{\geq 1} \times \{0 \} \}) \times \mathbb{Z}_2^n
\]

via the homeomorphism

\[
(s,t,\infty,0^{n-1}) \in \left( \mathfrak{G}_n \mid_{\geq 1} \times \{0 \} \right) \mapsto \mathbb{Z} \times \mathbb{Z}^{n-1} \times \{\infty\} \times \mathbb{Z}_2^{n-1}
\]

Applying the same kind of arguments as shown above, we get the isomorphism of

\[
C_c(\mathfrak{G}_n)_0 \text{-modules}
\]

\[
A_{n,l} \approx C_c \left( \left( \left( \mathfrak{G}_n \mid_{\geq l} \times \{0 \} \right) \right) \right) \oplus \left( \left( \left( \mathfrak{G}_n \mid_{\geq l} \times \{0 \} \right) \right) \right).
\]

48
\[
\bigoplus_{j=0}^{r-1} C_c \left( \mathfrak{G}_n \upharpoonright \left( \frac{Z_{l-1}}{2} \backslash \frac{Z_{l-1}}{2} \right) \times \{j\} \times \{0^{(n-i)}\} \right)_{-r} \oplus \left( C_c (\mathfrak{G}_n)_0 \partial_n \left( I^{\otimes l} \otimes P_1^{\otimes n-l} \right) \right)
\]

\[
\bigoplus_{j=0}^{r-1} C_c \left( \mathfrak{G}_n \upharpoonright \left( \frac{Z_{l-1}}{2} \backslash \frac{Z_{l-1}}{2} \right) \times \{0\} \times \{0^{(n-l)}\} \right)_{-r+j} \oplus \left( C_c (\mathfrak{G}_n)_0 \partial_n \left( I^{\otimes l} \otimes P_1^{\otimes n-l} \right) \right)
\]

\[
= \bigoplus_{j=0}^{r-1} A_{r-j,l-1} \oplus \left( C_c (\mathfrak{G}_n)_0 \partial_n \left( I^{\otimes l} \otimes P_1^{\otimes n-l} \right) \right)
\]

for \(2 \leq l \leq n - 1\). This provides a recursive formula to reduce the index \(l\) of the module \(A_{r,l}\).

For \(n > 2\), we define a combinatorial number \(\nu_n (m, l)\) recursively by

\[
\nu_n (m, l) := \sum_{s=0}^{m} \nu_n (s, l - 1)
\]

and \(\nu_n (m, 1) := 1\), for \(m \geq 0\) and \(2 \leq l \leq n - 1\), to be used in the following theorem.

Thanks to Thomas Timmermann, as he pointed out to the author, \(\nu_n (m, l)\) can be identified with a familiar combinatorial number, namely,

\[
\nu_n (m, l) = C_{m+l-1}^{m+l-1}
\]

for all \(m \geq 0\) and \(l \geq 1\). Indeed, if either \(l = 1\) or \(m = 0\) (e.g. when \(m + l \leq 2\)), we get easily from the definition, \(\nu_n (m, l) = 1 = C_{m+l-1}^{m+l-1}\). On the other hand, for \(l \geq 2\) and \(m \geq 1\), since

\[
\nu_n (m, l) = \sum_{s=0}^{m-1} \nu_n (s, l - 1) + \nu_n (m, l - 1) = \nu_n (m - 1, l) + \nu_n (m, l - 1),
\]

the identification can be proved by an induction on \(m + l \geq 3\) as shown in

\[
\nu_n (m - 1, l) + \nu_n (m, l - 1) = C_{m-1}^{m+l-2} + C_m^{m+l-2} = C_m^{m+l-1}
\]

which is valid due to either the induction hypothesis for \(m+l-1\) (in the case of \(m+l-1 > 2\)) or the already established identification (for the case of \(m + l - 1 = 2\)).
Theorem 6. For $n > 2$, the quantum line bundle $L_k \equiv C(S_{k}^{n-1})_k$ of degree $k \in \mathbb{Z}$ over $C(\mathbb{P}^{n-1}(T))$ is isomorphic to the finitely generated projective left module over $C(\mathbb{P}^{n-1}(T))$ determined by the projection $\partial_n (\otimes^{n-1} I \otimes P_{-k})$ if $k \geq 0$, and the projection

$$
\left(\bigoplus_{m=0}^{|k|-1} \nu_{n}(|k|,m) \otimes C_{c}(G_{n}^{-1} \mathbb{P}^{n-1} \mathbb{P}) \cup \bigoplus_{l=1}^{|k|-1} \nu_{n}(|k|,l) \otimes C_{c}(G_{n}^{-1} \mathbb{P}^{n-1} \mathbb{P}) \right) \otimes C_{c}(G_{n}^{-1} \mathbb{P}^{n-1} \mathbb{P})$$

if $k < 0$.

Proof. Only the case of $k < 0$ remains to be proved as follows.

For $k < 0$, starting with the established isomorphism

$$
C_{c}(G_{n})_k = C_{c}(G_{n}^{-1} \mathbb{P}^{n-1} \mathbb{P})_k \oplus C_{c}(G_{n})_0
$$

we apply repeatedly the recursive formula

$$
A_{r,l} = \bigoplus_{j=0}^{r-1} A_{r-j,l-1} \oplus \left( C_{c}(G_{n})_0 \otimes C_{c}(G_{n})_0 \otimes P_{-k} \right)
$$

reducing $l$ for $A_{r,l}$ with $2 \leq l \leq n$ until $l$ reaches 2 with

$$
A_{r,2} \cong \bigoplus_{j=0}^{r-1} \left( C_{c}(G_{n})_0 \otimes C_{c}(G_{n})_0 \otimes P_{-k} \right) \oplus \left( C_{c}(G_{n})_0 \otimes P_{-k} \right),
$$

in order to convert all terms to $C_{c}(G_{n})_0$-modules of the form $C_{c}(G_{n})_0 \otimes P_{-k}$ for some $0 < j \leq n$.

In fact, we check inductively on $1 \leq j \leq n - 2$ that

$$
\left( C_{c}(G_{n})_k \cong \bigoplus_{m=0}^{|k|-1} \nu_{n}(m,j) A_{|k|-m,n} \oplus \bigoplus_{l=1}^{|k|-1} \nu_{n}(l,j) C_{c}(G_{n})_0 \otimes P_{-k} \right).
$$
The case of \( j = 1 \) is our starting point already proved. Now assuming that it holds for \( j \), we get by the above recursive formula

\[
C_c ( \mathfrak{G}_n )_k \cong \bigoplus_{m=0}^{\lfloor k \rfloor - 1} \bigoplus_{s=0}^{\lfloor k \rfloor - m - 1} A_{k - m - s, n - j - 1} \bigoplus \bigoplus_{l=0}^{j} \left( \bigoplus_{m=0}^{\lfloor k \rfloor - l - 1} \nu_n (m, j) C_c ( \mathfrak{G}_n )_0 \partial_n \left( I^{\otimes n - l - 1} \otimes P_1^{\otimes l - 1} \right) \right)
\]

\[
= \bigoplus_{m=0}^{\lfloor k \rfloor - 1} \bigoplus_{s=0}^{\lfloor k \rfloor - m - 1} A_{k - m - s, n - j - 1} \bigoplus \bigoplus_{l=0}^{j} \left( \bigoplus_{m=0}^{\lfloor k \rfloor - l - 1} \nu_n (m, j) C_c ( \mathfrak{G}_n )_0 \partial_n \left( I^{\otimes n - l - 1} \otimes P_1^{\otimes l - 1} \right) \right)
\]

which verifies (*) for \( j + 1 \).

For \( j = n - 2 \), (*) says

\[
C_c ( \mathfrak{G}_n )_k \cong \bigoplus_{m=0}^{\lfloor k \rfloor - 1} \left( \bigoplus_{s=0}^{\lfloor k \rfloor - m - 2} A_{k - m - 2, n - j} \right) \bigoplus \bigoplus_{l=1}^{n - 2} \left( \bigoplus_{m=0}^{\lfloor k \rfloor - l - 1} \nu_n (m, l) C_c ( \mathfrak{G}_n )_0 \partial_n \left( I^{\otimes n - l - 1} \otimes P_1^{\otimes l - 1} \right) \right)
\]

\[
= \bigoplus_{m=0}^{\lfloor k \rfloor - 1} \left( \bigoplus_{s=0}^{\lfloor k \rfloor - m - 2} A_{k - m - 2, n - j} \right) \bigoplus \bigoplus_{l=1}^{n - 2} \left( \bigoplus_{m=0}^{\lfloor k \rfloor - l - 1} \nu_n (m, l) C_c ( \mathfrak{G}_n )_0 \partial_n \left( I^{\otimes n - l - 1} \otimes P_1^{\otimes l - 1} \right) \right)
\]

51
\[
\oplus \left( \sum_{m=0}^{\lfloor k \rfloor - 1} \nu_n(m, n-2) C_c \left( \mathfrak{S}_n \right)_0 \partial_n \left( I \otimes P_1^{\oplus n-1} \right) \right) \\
\oplus \left( \sum_{m=0}^{\lfloor k \rfloor - 1} \nu_n(m, n-2) \left( C_c \left( \mathfrak{S}_n \right)_0 \partial_n \left( I^\otimes 2 \otimes P_1^\otimes n-2 \right) \right) \right) \\
\oplus \bigoplus_{l=1}^{n-2} \left( \nu_n(\lfloor k \rfloor - 1,l) C_c \left( \mathfrak{S}_n \right)_0 \partial_n \left( I^\otimes n-l+1 \otimes P_1^\otimes l-1 \right) \right)
\]

\[
\oplus \left( \sum_{m=0}^{\lfloor k \rfloor - 1} \nu_n(m, n-2) \left( C_c \left( \mathfrak{S}_n \right)_0 \partial_n \left( I^\otimes n-l+1 \otimes P_1^\otimes l-1 \right) \right) \right) \\
\oplus \bigoplus_{l=1}^{n-1} \left( \nu_n(\lfloor k \rfloor - 1,l) C_c \left( \mathfrak{S}_n \right)_0 \partial_n \left( I^\otimes n-l+1 \otimes P_1^\otimes l-1 \right) \right)
\]

After completion, we get the \( C^* \left( \left( \mathfrak{S}_n \right)_0 \right) \)-module \( L_k \) isomorphic to

\[
\left( \sum_{m=0}^{\lfloor k \rfloor - 1} \nu_n(m, n-2) \left( C_c \left( \mathfrak{S}_n \right)_0 \partial_n \left( I^\otimes n-l+1 \otimes P_1^\otimes l-1 \right) \right) \right)
\]

which corresponds to the projection

\[
\left( \sum_{m=0}^{\lfloor k \rfloor - 1} \nu_n(m, n-2) \partial_n \left( I \otimes P_1^\otimes n-1 \right) \right) \boxplus \left( \sum_{l=1}^{n-1} \nu_n(\lfloor k \rfloor - 1,l) \partial_n \left( I^\otimes n-l+1 \otimes P_1^\otimes l-1 \right) \right)
\]

\[
\left( \sum_{m=0}^{\lfloor k \rfloor - 1} \nu_n(m, n-2) \partial_n \left( I \otimes P_1^\otimes n-1 \right) \right) \boxplus \left( \sum_{l=1}^{n-1} \nu_n(\lfloor k \rfloor - 1,l) \partial_n \left( I^\otimes n-l+1 \otimes P_1^\otimes l-1 \right) \right)
\]

Little is known about the cancellation problem and hence the classification problem for finitely generated projective modules over \( C \left( \mathbb{P}^{n-1} (\mathcal{T}) \right) \). We expect that these problems will be far more complicated than those for over \( C \left( S^2_{\mathcal{H}} \right) \) and \( C \left( \mathcal{T}^\otimes n \right) \).

The recent work of Farsi, Hajac, Maszczyk, and Zieliński \[7\] identifies one of three free generators of \( K_0 \left( C \left( \mathbb{P}^2 (\mathcal{T}) \right) \right) \) as \([L_1]+[L_{-1}]-2[I]\) (in addition to \([L_1]-[I]\) and \([I]\) constructed from a Milnor module and then expresses it in terms of elementary projections, showing a perfect consistency with our result.
References

[1] K. A. Bach, *A cancellation problem for quantum spheres*, Thesis, U. of Kansas, Lawrence, 2003.

[2] P. Baum, P. Hajac, R. Matthes, and W. Szymański, *The K-theory of Heegaard-type quantum 3-sphere*, K-Theory 35 (2005), no. 1-2, 159–186.

[3] B. Blackadar, “*K-theory for Operator Algebras*”, MSRI Publications Vol. 5,. Cambridge University Press, Cambridge, 2nd ed., 1998.

[4] T. Brzeziński and S. A. Fairfax, *Quantum teardrops*, Comm. Math. Phys. 316 (2012), 151-170.

[5] A. Connes, “*Noncommutative Geometry*”, Academic Press, New York, 1994.

[6] R. E. Curto and P. S. Muhly, *C*-algebras of multiplication operators on Bergman spaces, J. Func. Anal. 64 (1985), 315-329.

[7] C. Farsi, P. M. Hajac, T. Maszczyk, and B. Zieliński, *Rank-two Milnor idempotents for the multipullback quantum complex projective plane*, arXiv:1708.04426

[8] P. M. Hajac, *Strong connections on quantum principal bundles*, Comm. Math. Phys. 182 (1996), 579–617.

[9] P. M. Hajac, A. Kaygun, and B. Zieliński, *Quantum complex projective spaces from Toeplitz cubes*, J. Noncomm. Geom. 6 (2012), 603-621.
[10] P. M. Hajac, R. Matthes, and W. Szymański, Chern numbers for two families of non-commutative Hopf fibrations, C. R. Acad. Sci. Paris, Ser. I 336 (2003), 925-930.

[11] P. M. Hajac, R. Matthes, and W. Szymański, Noncommutative index theory for mirror quantum spheres, C. R. Acad. Sci. Paris, Ser. I 343 (2006), 731-736.

[12] P. Hajac, R. Nest, D. Pask, A. Sims, and B. Zieliński, The K-theory of twisted multipullback quantum odd spheres and complex projective spaces, arXiv:1512.08816v2.

[13] R. H. Herman and L. N. Vaserstein, The stable range of $C^*$-algebras, Invent. Math. 77 (1984), 553-555.

[14] U. Meyer, Projective quantum spaces, Lett. Math. Phys. 35 (1995), 91–97.

[15] P. S. Muhly and J. N. Renault, $C^*$-algebras of multivariable Wiener-Hopf operators, Trans. Amer. Math. Soc. 274 (1982), 1-44.

[16] G. Nagy, Stable rank of $C^*$-algebras of Toeplitz operators on polydisks, Operators in Indefinite Metric Spaces, Scattering Theory and Other Topics (Bucharest, 1985), 227–235, Oper. Theory Adv. Appl., 24, Birkhäuser, Basel, 1987.

[17] V. Nistor, Stable range for tensor products of extensions of $K$ by $C(X)$, J. Operator Theory, 16 (1986), no. 2, 387–396.

[18] G. K. Pedersen, Pullback and pushout constructions in $C^*$-algebra theory, J. Func. Anal. 267 (1999), 243-344.
[19] M. A. Peterka, *Finitely-generated projective modules over the \( \theta \)-deformed 4-sphere*, Comm. Math. Phys. 321, 577–603 (2013).

[20] J. Renault, “*A Groupoid Approach to C*-algebras*”, Lecture Notes in Mathematics, Vol. 793, Springer-Verlag, New York, 1980.

[21] M. A. Rieffel, *Dimension and stable rank in the K-theory of C*-algebras*, Proc. London Math. Soc. 46 (1983), 301-333.

[22] ______, *The cancellation theorem for projective modules over irrational rotation C*-algebras*, Proc. London Math. Soc. 47 (1983), 285-302.

[23] ______, *Projective modules over higher-dimensional noncommutative tori*, Canad. J. Math., 40 (1988), 257–338.

[24] N. Salinas, A. J. L. Sheu, and H. Upmeier, *Toeplitz operators on pseudoconvex domains and foliation C*-algebras*, Ann. Math., 130 (1989), 531-565.

[25] A. J. L. Sheu, *A cancellation theorem for modules over the group C*-algebras of certain nilpotent Lie groups*, Canad. J. Math., 39 (1987), 365-427.

[26] ______, *Compact quantum groups and groupoid C*-algebras*, J. Func. Anal. 144 (1997), 371-393.

[27] ______, *Quantization of the Poisson SU(2) and its Poisson homogeneous space - the 2-sphere*, Comm. Math. Phys. 135 (1991), 217-232.

[28] ______, *The structure of line bundles over quantum teardrops*, SIGMA 10 (2014), 027.
[29] _____, *Projective modules over quantum projective line*, International J. Math., 28 (2017), 1750022.

[30] R. W. Swan, *Vector bundles and projective modules*, Trans. Amer. Math. Soc. 705 (1962), 264-277.

[31] J. Taylor, *Banach algebras and topology*, in “Algebras in Analysis”, Academic Press, New York, 1975.

[32] L. L. Vaksman and Ya. S. Soibelman, *The algebra of functions on the quantum group SU(n + 1), and odd-dimensional quantum spheres*, Leningrad Math. J. 2 (1991), 1023-1042.

[33] G. W. Whitehead, “*Elements of Homotopy Theory*”, Springer-Verlag, New York, 1978.

[34] S. L. Woronowicz, *Compact quantum groups*, in “Les Houches, Session LXIV, 1995, Quantum Symmetries”, Elsevier, 1998, pp. 845-884.

[35] _____, *Compact matrix pseudogroups*, Comm. Math. Phys. 111 (1987), 613-665.