An inverse problem for semilinear equations involving the fractional Laplacian

Pu-Zhao Kow§, Shiqi Ma‡,∗ and Suman Kumar Sahoo§

1 Department of Mathematical Sciences, National Chengchi University, Taipei 16302, Taiwan
2 School of Mathematics, Jilin University, Changchun, People’s Republic of China
3 Department of Mathematics and Statistics, University of Jyväskylä, Jyväskylä, Finland

E-mail: mashiqi01@gmail.com and mashiqi@jlu.edu.cn

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Abstract

Our work concerns the study of inverse problems of heat and wave equations involving the fractional Laplacian operator with zeroth order nonlinear perturbations. We recover nonlinear terms in the semilinear equations from the knowledge of the fractional Dirichlet-to-Neumann type map combined with the Runge approximation and the unique continuation property of the fractional Laplacian.

Keywords: fractional Laplacian, fractional Calderón problem, nonlocal semilinear equations, fractional diffusion equation, fractional wave equation, Runge approximation

1. Introduction and main results

We investigate inverse problems for heat and wave equations involving the fractional Laplacian operator with zeroth order nonlinear perturbations. In [GSU20], Ghosh, Salo, and Uhlmann proposed and proved a Calderón type inverse problem for a linear fractional Schrödinger equation. The Calderón problem was initiated by Calderón in his work [Cal80] for nonfractional Laplace equations. There is ample amount of literature available on the nonfractional Calderón problem and we refer the readers to the survey [Uhl09]. The key tool for studying fractional type of inverse problems is the Runge approximation property, which is a consequence of the fractional unique continuation property, i.e. if $u = (-\Delta)^s u = 0$ in certain open set, then $u = 0$ everywhere. Utilizing these tools, inverse problems involving fractional operators have been greatly investigated by numerous authors in recent years. We refer...
Here, the fractional Laplacian denotes the space of smooth compactly supported functions on their domain of definition. The following fractional diffusion equation with nonlinear term nonlinear parabolic equations have been well studied. We refer to references therein for more results.

Motivated by the works mentioned above, in this article we consider an inverse problem for nonlinear fractional parabolic equations. Fractional parabolic equations have applications in random processes. We study the fractional type heat equations as well as the fractional type wave equations, and we start with the heat equation first.

Let $n \geq 1$ be a non-negative integer and $0 < s < 1$. Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$ and $\Omega^c := \mathbb{R}^n \setminus \overline{\Omega}$. Let $W$ be any bounded Lipschitz domain in $\Omega^c$. Let $u = u(t,x)$ satisfy the following fractional diffusion equation with nonlinear term $q = q(t,x,z)$:

\[
\begin{cases}
\partial_t u(t,x) + (-\Delta)^s u(t,x) + q(t,x,u(t,x)) = 0 & \text{in } \Omega_T := (0,T) \times \Omega, \\
u(t,x) = f(t,x) & \text{in } \Omega_T^c := (0,T) \times \Omega^c, \\
0 & \forall x \in \Omega,
\end{cases}
\]  

(1.1)

for certain appropriate exterior data $f = f(t,x) \in C^\infty_c(W_T)$, where $W_T := (0,T) \times W$ and $C^\infty_c(\cdot)$ denotes the space of smooth compactly supported functions on their domain of definition. Here, the fractional Laplacian $(-\Delta)^s$ is defined via the Fourier transform: $\mathcal{F}((-\Delta)^s)v(\xi) := |\xi|^{2s} \hat{v}(\xi)$ for all $\xi \in \mathbb{R}^n$, where $\hat{v} = \mathcal{F}v$ is the Fourier transform of distribution $v$. Given any open sets $V$ and $W$ in $\Omega^c$, we define the DN-map corresponding to (1.1) as follows:

\[
\Lambda^\text{heat}_q(f) := (-\Delta)^s u|_{\partial \Omega_T} \quad \text{for all} \ 'sufficiently small' \ f \in C^\infty_c(W_T),
\]  

(1.2)
where \( \nu \) is the unique solution of (1.1), see proposition 2.10. We now state the assumptions on the coefficient under which we state and prove our main results. Denote \( \Omega_T := (0, T) \times \Omega \). Let \( C^k(\cdot) \) be the space of \( k \)-times continuously differentiable functions for all integers \( k \geq 0 \).

**Assumption A.** Fix an integer \( m > 0 \) and \( \delta > 0 \). We assume that \( q(t,x,z) \) satisfies the following properties:

1. **Q.1** For each \( (t,x) \in \Omega_T \), the mapping \( z \mapsto q(t,x,z) \) is in \( C^{m+1}((\delta,\delta)) \).
2. **Q.2** \( q(t,0,0) = 0 \) for all \( (t,x) \in \Omega_T \).
3. **Q.3** There exists a non-decreasing function \( \Phi : (-\delta, \delta) \to \mathbb{R}_+ \) such that
   \[
   \| \partial_0 q(t,x,z) \| \leq \Phi(\epsilon)
   \]
   for all \( 0 < \epsilon < \delta \) and \( \lim_{\epsilon \to 0} \Phi(\epsilon) = 0 \).
4. **Q.4** Given any \( k = 2, 3, \ldots, m+1 \), there exists \( M_k \) (depending on \( k \)) such that
   \[
   \sup_{(t,x) \in \Omega_T, |z| \leq \delta} |\partial^k_x q(t,x,z)| \leq M_k.
   \]

With these assumptions on the coefficient, the following is our first main result:

**Theorem 1.1 (Global uniqueness from DN-map).** Let \( n \in \mathbb{N} \) and \( 0 < s < 1 \). Let \( \Omega \subset \mathbb{R}^n \) be a bounded Lipschitz domain. Let \( W, V \subset \Omega^e \) be any open sets, both with Lipschitz boundary, satisfying \( \nabla \cap \Omega = \emptyset \) and \( \nabla \cap \Omega = \emptyset \). Fix an integer \( m \geq 0 \) and a number \( \delta > 0 \). Assume that both \( q_j (j = 1, 2) \) satisfy assumption A for the same \( m \) and \( \delta \). Then there exists a constant \( \tilde{e}_0 = \tilde{e}_0(n, s, \Omega, T, \delta) \) such that the following holds: If
   \[
   \Lambda_{q_1}^{heat}(f) = \Lambda_{q_2}^{heat}(f) \quad \forall f \in C^\infty_c(W_T) \text{ satisfying } ||f||_{ext} \leq \tilde{e}_0,
   \]
   where the norm \( || \cdot ||_{ext} \) is defined in (2.2) below, then we conclude
   \[
   \partial^k_x q_1(t,x,0) = \partial^k_x q_2(t,x,0) \quad \forall (t,x) \in \Omega_T, \quad k = 0, 1, 2, \ldots, m.
   \]

If we additionally assume that \( z \mapsto q(t,x,z) \) is analytic for \( (t,x) \in \Omega_T \), then we conclude
   \[
   q_1(t,x,z) = q_2(t,x,z) \quad \forall (t,x) \in \Omega_T, \quad \forall z \in (-\delta, \delta).
   \]

We mention that in [CLL19] the authors established the uniqueness of the obstacle and the linear potential without using the full information of the DN-map. Furthermore, in [GRSU20, L123], only finite dimensional data set are used to establish the unique determination. All these results benefit from the nonlocal property of fractional operators. Through a careful examination of the proof of theorem 1.1, along with the ideas from [GRSU20], it is possible to determine the unknown \( q \) with fewer measurements when \( q \) is \( t \)-independent. This leads us to the following theorem.

**Theorem 1.2 (Recovery of \( m \)-jet from \( m \)-dimensional measurements).** Suppose the assumptions in theorem 1.1 hold. We further assume that for \( j = 1, 2 \)
   \[
   \partial^k_x q_j(\cdot,0) \in C^0(\overline{\Omega}) \text{is independent of time variable } t.
   \]

Let \( g_1, \ldots, g_m \in C^\infty(W_T) \) such that \( g_1, \ldots, g_m \neq 0 \). If \( \Lambda_{q_{m+1}}^{heat}(\epsilon_1 g_1 + \cdots + \epsilon_m g_m) = \Lambda_{q_{m+1}}^{heat}(\epsilon_1 g_1 + \cdots + \epsilon_m g_m) \) for all sufficiently small \( \epsilon_j > 0 (j = 1, \ldots, m) \), we conclude (1.4).
In this article, we also take into consideration a nonlinear inverse problem for fractional wave equations in one spatial dimension. Let \( u = u(t, x) \) satisfy

\[
\begin{align*}
\partial_t^\alpha u(t, x) + (\Delta)^\beta u(t, x) + q(t, x, u(t, x)) &= 0 \quad \text{in } \Omega_T, \\
u(t, x) &= f(t, x) \quad \text{in } \Omega_T, \\
u(0, x) &= \partial_x \nu(0, x) = 0 \quad \text{for all } x \in \Omega,
\end{align*}
\]

for certain appropriate exterior data. We can define the following hyperbolic DN-map corresponding to (1.6) as follows:

\[
\Lambda^\text{wave}_q(f) := (-\Delta)^\beta u \bigg|_{V_T},
\]

for all ‘sufficiently small’ \( f \in C_0^\infty(W_T) \), where \( u \) is the unique solution of (1.6), see proposition 4.4 for the well-posedness. The following result can be proved adapting the similar ideas:

**Theorem 1.3** (Global uniqueness from DN-map). Let \( n = 1 \) and \( 1/2 < s < 1 \). Let \( \Omega \subset \mathbb{R}^n \) be a bounded Lipschitz domain. Let \( W, V \subset \Omega^c \) be any open sets, both with Lipschitz boundary, satisfying \( V \cap \Omega = \emptyset \) and \( W \cap \Omega = \emptyset \). Fix an integer \( m \geq 2 \) and a number \( \delta > 0 \). Assume that both \( q_j (j = 1, 2) \) satisfy assumption A for the same \( m \) and \( \delta \). Then there exists a constant \( \tilde{\epsilon}_0 = \tilde{\epsilon}_0(n, s, \Omega, \varnothing, T, \delta) \) such that the following holds: If

\[
\Lambda^\text{wave}_{q_j}(f) = \Lambda^\text{wave}_{q_j}(f) \quad \forall f \in C_0^\infty(W_T) \text{ satisfying } \| f \|_{\text{ext}} \leq \tilde{\epsilon}_0,
\]

where the norm \( \| \cdot \|_{\text{ext}} \) is defined in (2.2) below, then we conclude (1.4). If we additionally assume that \( z \rightarrow q(t, x, z) \) is analytic for \( (t, x) \in \Omega_T \), then we conclude (1.5).

The next theorem is analogous to theorem 1.2.

**Theorem 1.4.** [Recovery of \( m \)-jet from \( m \)-dimensional measurements]. Suppose the assumptions in theorem 1.3 hold. We further assume that

\[
\partial_t^\beta q_j(\cdot, 0) \in C^0(\Omega^c) \text{ is independent of time variable } t.
\]

Let \( g_1, \ldots, g_m \in C_0^\infty(W_T) \) such that \( g_1, \ldots, g_m \neq 0 \). If \( \Lambda^\text{wave}_{q_j}(\epsilon_1 g_1 + \cdots + \epsilon_m g_m) = \Lambda^\text{wave}_{q_j}(\epsilon_1 g_1 + \cdots + \epsilon_m g_m) \) for all sufficiently small \( \epsilon_j > 0 \) \( (j = 1, \ldots, m) \), we conclude (1.4).

There are only a few works available in the literature about the inverse problems for fractional heat equations as well as fractional wave equations. To motivate our work, we mention several closely related ones. In [KW23, Li22, Li23b], the authors considered inverse problems for generalized Kerr-type nonlinearity, which is different with ours. Then in [KLL22], the authors studied an inverse problem involving fractional wave equation, while in [LLL21], the authors solved an inverse problem for hyperbolic systems.

The rest of the paper is organized as follows. We discuss the forward problem of the fractional diffusion equation in section 2, then we prove theorems 1.1 and 1.2 in section 3. We next prove theorems 1.3 and 1.4 in section 4. To make our paper self-contained, we also present the proof of the well-posedness of the linear fractional diffusion equation (proposition 2.2) in appendix A. Finally, we discuss the issue of considering theorem 1.3 in one spatial dimension in appendix B.

2. The forward problem for the fractional diffusion equation

In this section, we introduce several preliminaries that will be useful in this work.
2.1. Fractional sobolev spaces

We use notations for fractional Sobolev spaces as in [KLW22]. To make the paper self-contained, we give brief introductions to them. For \( \alpha \in \mathbb{R} \), denote as \( H^\alpha (\mathbb{R}^n) \) the standard \( L^2 \)-based fractional Sobolev spaces, which is defined via Fourier transform [DNPV12, Kwa17, Ste16]. For \( s \in (0,1) \), in fact

\[
H^s (\mathbb{R}^n) = \left\{ u \in L^2 (\mathbb{R}^n) \mid \frac{|u(x) - u(y)|}{|x-y|^{\frac{n}{2} + s}} \in L^2 (\mathbb{R}^n \times \mathbb{R}^n) \right\} \quad \text{(as sets)}
\]

with equivalent norm: \( \|u\|_{H^s (\mathbb{R}^n)}^2 = \|u\|_{L^2 (\mathbb{R}^n)}^2 + \|u\|_{H^s (\mathbb{R}^n)}^2 \), where

\[
[u]^2_{H^s (\mathbb{R}^n)} = \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \, dx \, dy. \tag{2.1}
\]

Here, (2.1) is called the Aronszajn–Gagliardo–Slobodeckij seminorm, see [DNPV12, equation (2.2)] for reference.

Let \( \mathcal{O} \) be any open set in \( \mathbb{R}^n \), and let \( \alpha \in \mathbb{R} \). We define the following Sobolev spaces:

\[
\begin{align*}
H^\alpha (\mathcal{O}) &:= \left\{ u_{|\mathcal{O}} \mid u \in H^\alpha (\mathbb{R}^n) \right\}, \\
\tilde{H}^\alpha (\mathcal{O}) &:= \text{closure of } C^\infty_0 (\mathcal{O}) \text{ in } H^\alpha (\mathbb{R}^n), \\
H^\alpha_0 (\mathcal{O}) &:= \text{closure of } C^\infty_0 (\mathcal{O}) \text{ in } H^\alpha (\mathcal{O}), \\
H^\alpha_{\mathcal{O}} &:= \left\{ u \in H^\alpha (\mathbb{R}^n) \mid \text{supp}(u) \subset \mathcal{O} \right\}.
\end{align*}
\]

The Sobolev space \( H^\alpha (\mathcal{O}) \) is complete under the quotient norm

\[
\|u\|_{H^\alpha (\mathcal{O})} := \inf \left\{ \|v\|_{H^\alpha (\mathbb{R}^n)} \mid v \in H^\alpha (\mathbb{R}^n) \text{ and } v_{|\mathcal{O}} = u \right\}.
\]

It is easy to see that \( \tilde{H}^\alpha (\mathcal{O}) \subset H^\alpha_0 (\mathcal{O}) \), and that \( H^\alpha_{\mathcal{O}} \) is a closed subspace of \( H^\alpha (\mathbb{R}^n) \). If \( \Omega \) is a bounded Lipschitz domain, then we also have following identifications (with equivalent norms):

\[
\begin{align*}
\tilde{H}^\alpha (\Omega) &= H^\alpha_{\Omega}, \\
\tilde{H}^\alpha_0 (\Omega) &= H^{-\alpha} (\Omega) \text{ and } (H^\alpha (\Omega))' = H^{-\alpha}_{\Omega} \forall \alpha \in \mathbb{R}, \\
H^\alpha (\Omega) &= H^\alpha_{\Omega} = H^\alpha_0 (\Omega) \forall -1/2 < s < 1/2,
\end{align*}
\]

see e.g. [GSU20, section 2A], [McL00, chapter 3], and [Tri02]. Next following [Eva10, chapter 5], we define time dependent fractional Sobolev space for all integers \( p \geq 1 \) denoted by \( L^p ((0,T); H^\alpha) \). Then the exterior norm of \( f \in C^\infty_0 (W_T) \) is given by

\[
\|f\|_{W_T}^2 := \|f\|_{L^2 ((0,T);L^2 (\mathbb{R}^n))}^2 + \|(-\Delta)^{\frac{s}{2}} f\|_{L^2 (\Omega_T)}^2. \tag{2.2}
\]

Moreover, for any measurable set \( A \subset \mathbb{R}^n \) we use the following notations:

\[
(f,g)_{L^2 (A)} := \int_A fg \, dx, \quad (F,G)_{L^2 (A)} := \int_0^T \int_A FG \, dx \, dt.
\]

2.2. Well-posedness for the linear equation

We state the well-posedness of the linear fractional diffusion equation. Let \( T > 0, s \in (0,1) \), and \( a = a(t,x) \in L^\infty (\Omega_T) \), and we consider the following initial-exterior value problem:

\[
\begin{align*}
\left\{ \begin{array}{ll}
(\partial_t + (-\Delta)^s + a)u &= F & \text{in } \Omega_T, \\
u &= f & \text{in } \Omega_T, \\
u &= \varphi & \text{in } \{0\} \times \mathbb{R}^n,
\end{array} \right.
\]
where $f \in C^\infty_0(W_T)$ for some open set with Lipschitz boundary $W \subset \Omega$, satisfying $\mathcal{W} \cap \overline{\Omega} = \emptyset$, and $\varphi \in H^1(\Omega) = \{ \varphi \in L^2(\mathbb{R}^n) \mid \text{supp} \varphi \subset \overline{\Omega} \}$. Setting $v := u - f$, we then consider the following linear equation with zero exterior data:

\[
\begin{cases}
(\partial_t + (-\Delta)^s + a)v = \tilde{F} & \text{in } \Omega_T, \\
v = 0 & \text{in } \Omega_T^c, \\
v = \varphi & \text{in } \{0\} \times \mathbb{R}^n,
\end{cases}
\tag{2.4}
\]

where $\tilde{F} = F - (-\Delta)^s f$. Here $W_T = (0, T) \times W$, therefore $f \in C^\infty_0(W_T)$ implies that $f = 0$ on $\{0\} \times \mathbb{R}^n$. Now it suffices to study the well-posedness of (2.4).

Define functions $v : [0, T] \to \dot{H}^s(\Omega)$ and $F : [0, T] \to L^2(\Omega)$ by

\[
[v(t)](x) := v(t, x), \quad [\tilde{F}(t)](x) := \tilde{F}(t, x) \quad \text{for } (t, x) \in [0, T] \times \mathbb{R}^n.
\tag{2.5}
\]

Let $\langle \cdot, \cdot \rangle$ be the duality pairing on $H^{-s}(\Omega) \oplus \dot{H}^s(\Omega)$. Multiplying (2.4) by any $\phi \in \dot{H}^s(\Omega)$ gives

\[
\langle v'(t), \phi \rangle + B[v, \phi; t] = \langle \tilde{F}(t), \phi \rangle_{L^2(\Omega)} \quad \text{for } 0 \leq t \leq T,
\]

where $B[v, \phi; t]$ is the bilinear form given by

\[
B[v, \phi; t] := \int_{\mathbb{R}^n} (-\Delta)^{s/2}v(t) (-\Delta)^{s/2}\phi \, dx + \int_{\Omega} a(t, \cdot)v(t)\phi \, dx.
\]

**Definition 2.1 (Weak solutions).** We say that $v$ is a weak solution of (2.4), if

(a) $v \in L^2((0, T); \dot{H}^s(\Omega))$ and $v' \in L^2(0, T; H^{-s}(\Omega));$

(b) $\langle v'(t), \phi \rangle + B[v, \phi; t] = \langle \tilde{F}(t), \phi \rangle_{L^2(\Omega)}$ for all $\phi \in \dot{H}^s(\Omega)$ for (almost) all $0 \leq t \leq T$;

(c) $v(0) = \varphi$,

where $v$ and $\tilde{F}$ are defined according to (2.5).

**Proposition 2.2 (Well-posedness).** Given any $n \in \mathbb{N}$ and $0 < s < 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain in $\mathbb{R}^n$. Let $a \in L^2(\Omega_T)$ and $\varphi \in H^1(\Omega)$, there exists a unique weak solution $v$ of (2.4) and satisfies the following estimate:

\[
\|v\|^2_{L^2((0, T); L^2(\Omega))} + \|v\|^2_{L^2(0, T; \dot{H}^s(\Omega))} + \|\partial_t v\|^2_{L^2(0, T; H^{-s}(\Omega))} \leq C(\|\varphi\|^2_{L^2(\Omega)} + \|\tilde{F}\|^2_{L^2(\Omega_T)})
\tag{2.6}
\]

for some constant $C = C(n, s, T, \|a\|_{L^\infty(\Omega_T)})$. If we further assume $\varphi \in \dot{H}^s(\Omega)$, then $v \in L^2(0, T; \dot{H}^s(\Omega))$ and $\partial_t v \in L^2(\Omega_T)$. In this case, the unique weak solution also satisfies the following estimate:

\[
\|v\|^2_{L^\infty((0, T); L^2(\Omega))} + \|\partial_t v\|^2_{L^2(\Omega_T)} \leq C(\|\varphi\|^2_{\dot{H}^s(\Omega)} + \|\tilde{F}\|^2_{L^2(\Omega_T)})
\tag{2.7}
\]

for some constant $C = C(n, s, T, \|a\|_{L^\infty(\Omega_T)})$.

The proof of proposition 2.2 is analogous to the standard well-posedness proof of the classical diffusion equation. However, for completeness, we present a proof in appendix A.

**Corollary 2.3.** Given any $n \in \mathbb{N}$ and $0 < s < 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain in $\mathbb{R}^n$, and $W \subset \Omega^c$ be any open set with Lipschitz boundary satisfying $\mathcal{W} \cap \overline{\Omega} = \emptyset$. Let $a \in L^\infty(\Omega_T)$, then for any $\tilde{F} \in L^2(\Omega_T)$, $\varphi \in \dot{H}^s(\Omega)$, and $f \in C^\infty_0(W_T)$, there exists a unique weak solution $u = v + f$ of (2.3) satisfying

\[
\|u - f\|^2_{L^2((0, T); L^2(\Omega))} + \|\partial_t (u - f)\|^2_{L^2(0, T; \dot{H}^s(\Omega))} \leq C(\|\varphi\|^2_{L^2(\Omega)} + \|\tilde{F}\|^2_{L^2(\Omega_T)} + \|f\|^2_{L^2((0, T); \dot{H}^s(\Omega))} + \|\partial_t f\|^2_{L^2(0, T; H^{-s}(\Omega))})
\]
for some constant $C = C(n, s, T, \|a\|_{L^\infty(\Omega_T)})$. If we further assume $\varphi \in \dot{H}^1(\Omega)$, then the unique weak solution $u$ also satisfies the following estimate:

$$\|u - f\|_{L^2(0,T;L^2(\Omega))}^2 + \|\partial_t u\|_{L^2(\Omega_T)}^2 \leq C((\|\varphi\|_{\dot{H}^1(\Omega_T)}^2 + \|f - (-\Delta)^{s/2} f\|_{L^2(\Omega_T)}^2)$$

(2.8)

for some constant $C = C(n, s, T, \|a\|_{L^\infty(\Omega_T)})$.

We skip the proof of corollary 2.3 as it is a straightforward consequence of proposition 2.2.

2.3. Maximum principle for the linear equation

Modifying the ideas in [LL19, proposition 3.1] or [RO16, proposition 4.1], we can obtain the following proposition:

**Proposition 2.4 (Maximum principle).** Given any $n \in \mathbb{N}$ and $0 < s < 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain in $\mathbb{R}^n$. Let $a \in L^\infty(\Omega_T)$. Suppose that $u \in L^2(0,T;H^s(\mathbb{R}^n)) \cap H^1(0,T;L^2(\Omega))$ is a weak solution of (2.3). If $F \geq 0$ in $\Omega_T$, $f \geq 0$ in $\Omega_T$, $\varphi \geq 0$ in $\mathbb{R}^n$, then $u \geq 0$ in $\Omega_T$.

**Proof.** Let $M$ be a real number which shall be determined later. We define

$$u_M(t,x) := e^{-Mt}u(t,x), \quad a_M(t,x) := a(t,x) + M, \quad F_M(t,x) := e^{-Mt}F(t,x) \quad \text{in} \quad \Omega_T,$$

$$f_M(t,x) := e^{-Mt}f(t,x) \quad \text{in} \quad \Omega_T.$$  

(2.9)

We see that $u_M$ satisfies

$$\begin{cases}
(\partial_t + (-\Delta)^s + a_M)u_M = F_M & \text{in} \quad \Omega_T, \\
u_M = f_M & \text{in} \quad \Omega_T, \\
u_M = \varphi & \text{on} \quad \{0\} \times \mathbb{R}^n.
\end{cases}$$

(2.10)

We choose $M = \|a\|_{L^\infty(\Omega_T)}$, then $a_M \geq 0$ in $\Omega_T$. Next we write $u_M = u_M^+ - u_M^-$, where $u_M^+ = \max\{u_M,0\}$ and $u_M^- = \max\{-u_M,0\}$. Since $u_M \in L^2(0,T;H^s(\mathbb{R}^n)) \cap H^1(0,T;L^2(\Omega))$, then $u_M^+ \in L^2(0,T;H^s(\mathbb{R}^n)) \cap H^1(0,T;L^2(\Omega))$ and that

$$\partial_t (u_M^-) = \begin{cases}
-\partial_t u_M^- & \text{in} \quad \{u_M < 0\}, \\
0 & \text{in} \quad \{u_M \geq 0\}.
\end{cases}$$

Since $u_M = f_M \geq 0$ in $\Omega_T$, hence $u_M^- = 0$ in $\Omega_T$, which implies $u_M^- \in L^2(0,T;H^s(\Omega)) \cap H^1(0,T;L^2(\Omega))$. Testing the first equation of (2.10) by $u_M$, we have

$$0 \leq (F_M(t),u_M(t))_{L^2(\Omega)} \quad \text{(because $F_M \geq 0$ and $u_M \geq 0$ in $\Omega_T$)},$$

$$= \int_{\Omega} (\partial_t u_M(t)u_M(t) \, dx) + \int_{\mathbb{R}^n} (-\Delta)^{s/2} u_M(t) (-\Delta)^{s/2} u_M(t) \, dx + \int_{\Omega} a_M(t,\cdot) u_M u_M \, dx$$

$$= -\frac{d}{dt} \left( \frac{1}{2} \int_{\Omega} |u_M(t)|^2 \, dx \right) + \int_{\mathbb{R}^n} (-\Delta)^{s/2} u_M(t) (-\Delta)^{s/2} u_M(t) \, dx - \int_{\Omega} a_M(t,\cdot) |u_M|^2 \, dx$$

for all $0 < t < T$. In [LL19, proposition 3.1] or [RO16, proposition 4.1], they showed that

$$\int_{\mathbb{R}^n} (-\Delta)^{s/2} u_M(t) (-\Delta)^{s/2} u_M(t) \, dx \leq 0 \quad \text{for all} \quad 0 < t < T.$$

Combining the preceding two inequalities, we then conclude that $\frac{d}{dt} \left( \frac{1}{2} \int_{\Omega} |u_M(t)|^2 \, dx \right) \leq 0$ holds true for all $0 < t < T$. Since $u_M = 0$ on $\mathbb{R}^n \times \{0\}$ (because $\varphi \geq 0$ in $\mathbb{R}^n$), then we conclude $\int_{\Omega} |u_M|^2 \, dx = 0$ for all $0 < t < T$, which completes our proof. \qed
Corollary 2.5 (Comparison principle). Given any \( n \in \mathbb{N} \) and \( 0 < s < 1 \). Let \( \Omega \subset \mathbb{R}^n \) be a bounded Lipschitz domain in \( \mathbb{R}^n \), and let \( a \in L^\infty(\Omega_T) \). Let \( u_1 \) and \( u_2 \) be weak solutions of

\[
\begin{align*}
(\partial_t + (-\Delta)^s + a)u_j &= F_j & \text{in } \Omega_T, \\
u_j &= f_j & \text{in } \Omega_T^c, \\
u_j &= \varphi_j & \text{on } \{0\} \times \mathbb{R}^n,
\end{align*}
\]

for \( j = 1, 2 \). If \( F_1 \geq F_2 \) in \( \Omega_T \), \( f_1 \geq f_2 \) in \( \Omega_T^c \), \( \varphi_1 \geq \varphi_2 \) in \( \mathbb{R}^n \), then \( u_1 \geq u_2 \) in \( \Omega_T \).

**Proof.** By applying proposition 2.4 with \( u = u_1 - u_2 \), this can be proved immediately. \( \square \)

**Remark 2.1.** Proposition 2.4 as well as corollary 2.5 also imply the uniqueness part of proposition 2.2 and corollary 2.3.

2.4. \( L^\infty \)-bounds of solutions of the linear equation

For our purposes, we require the following \( L^\infty \)-bound estimate, which can be found in [Li22, proposition 3.3]:

**Proposition 2.6.** Given any \( n \in \mathbb{N} \) and \( 0 < s < 1 \). Let \( \Omega \subset \mathbb{R}^n \) be a bounded Lipschitz domain in \( \mathbb{R}^n \), and let \( a \in L^\infty(\Omega_T) \). Suppose that \( u \in L^2(0,T;H^s(\mathbb{R}^n)) \cap H^1(0,T;L^2(\Omega)) \) is a weak solution of

\[
\begin{align*}
(\partial_t + (-\Delta)^s + a)u &= F & \text{in } \Omega_T, \\
u &= f & \text{in } \Omega_T^c, \\
u &= 0 & \text{on } \{0\} \times \mathbb{R}^n,
\end{align*}
\]

with \( F \in L^\infty(\Omega_T) \) and \( f \in L^\infty(\Omega_T^c) \). Then \( \|u\|_{L^\infty(\Omega_T)} \leq C(\|f\|_{L^\infty(\Omega_T^c)} + \|F\|_{L^\infty(\Omega_T)}), \) for some constant \( C = C(n,s,T,\Omega,\|a\|_{L^\infty(\Omega_T)}). \)

To make our paper more self-contained, here we sketch the proof of proposition 2.6. The following lemma can be found in [LL19, lemma 3.4] (with \( a \equiv 0 \)) or [RO16, lemma 5.1].

**Lemma 2.7 (Elliptic barrier).** Given any \( n \in \mathbb{N} \) and \( 0 < s < 1 \). Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^n \). There exists a function \( \phi = \phi(x) \in C^\infty(\mathbb{R}^n) \) such that

\[
(-\Delta)^s \phi \geq 1 \quad \text{in } \Omega, \quad \phi \geq 0 \quad \text{in } \mathbb{R}^n, \quad \phi \leq C \quad \text{in } \Omega, \quad \text{for some constant } C = C(n,s,\Omega).
\]

If we define \( \Phi(t,x) := e^t \phi(x) \), we immediately obtain the following corollary:

**Corollary 2.8 (Parabolic barrier).** Given any \( n \in \mathbb{N} \) and \( 0 < s < 1 \). Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^n \). There exists a function \( \Phi = \Phi(x) \in C^\infty([0,T] \times \mathbb{R}^n) \) such that

\[
(\partial_t + (-\Delta)^s)^t \Phi \geq 1 \quad \text{in } \Omega_T, \quad \Phi \geq 0 \quad \text{in } [0,T] \times \mathbb{R}^n, \quad \Phi \leq C \quad \text{in } \Omega_T, \quad \text{for some constant } C = C(n,s,T,\Omega).
\]

Using the barrier in corollary 2.8, we now can obtain the following \( L^\infty \)-bound for the solution of (2.3).

**Proof of proposition 2.6.** Using the functions given in (2.9) with \( M = \|a\|_{L^\infty(\Omega_T)} \), we know that

\[
\begin{align*}
(\partial_t + (-\Delta)^s + a_M)u_M &= F_M & \text{in } \Omega_T, \\
u_M &= f_M & \text{in } \Omega_T^c, \\
u_M &= 0 & \text{on } \{0\} \times \mathbb{R}^n,
\end{align*}
\]

...
with \( a_M \geq 0 \). Let \( v(t,x) := \| F_M \|_{L^\infty(\Omega_T)} + \| F_M \|_{L^\infty(\Omega_T)} \Phi(t,x) \geq 0 \) in \( [0,T) \times \Omega \), where \( \Phi \) is the barrier given in corollary 2.8. We see that 
\[
(\partial_t + (-\Delta)^s + a_M)v \geq (\partial_t + (-\Delta)^s)v = \| F_M \|_{L^\infty(\Omega_T)} (\partial_t + (-\Delta)^s) \Phi \geq \| F_M \|_{L^\infty(\Omega_T)}
\]
\[
\Rightarrow \Phi F_M = \Phi (\partial_t + (-\Delta)^s + a_M) u_M \text{ in } \Omega_T.
\]
Moreover, we also have 
\[
(\partial_t + (-\Delta)^s + a_M)(v \pm u_M) \geq 0 \quad \text{in } \Omega_T, \\
v \pm u_M = v \pm u_M \geq 0 \quad \text{on } \{0\} \times \mathbb{R}^n.
\]
Combining relations in (2.11), and proposition 2.4, we see that \( v \geq \pm u_M \) in \( \Omega_T \), which further implies that \( \| u_M \|_{L^\infty(\Omega_T)} \lesssim \| v \|_{L^\infty(\Omega_T)} \lesssim \| f_M \|_{L^\infty(\Omega_T)} + C \| F_M \|_{L^\infty(\Omega_T)} \), where \( C = C(n,s,T,\Omega) \) is the constant given in the corollary 2.8. Finally, utilizing 
\[
|u(t,x)| = e^{Mt} |u_M|(t,x)| = e^{T\|u\|_{L^\infty(\Omega_T)} \|u_M\|_{L^\infty(\Omega_T)}} \quad \text{in } \Omega_T, \\
|F_M(t,x)| = e^{-Mt} |F(t,x)| \lesssim \| F \|_{L^\infty(\Omega_T)} \quad \text{in } \Omega_T, \\
|f_M(t,x)| = e^{-Mt} |f(t,x)| \lesssim \| f \|_{L^\infty(\Omega_T)} \quad \text{in } \Omega_T, \\
\]
we conclude the proof.

We skip the proof of the following well-posedness result as it follows from combining corollary 2.3 and proposition 2.6.

**Proposition 2.9.** Given any \( n \in \mathbb{N} \) and \( 0 < s < 1 \). Let \( \Omega \subset \mathbb{R}^n \) be a bounded Lipschitz domain in \( \mathbb{R}^n \), let \( W \subset \Omega^c \) be any open set with Lipschitz boundary satisfying \( \overline{W} \cap \overline{\Omega} = \emptyset \). Then for any \( F \in L^\infty(\Omega_T) \) and \( f \in C_c^\infty(W_T) \), there exists a unique weak solution \( u \) of 
\[
(\partial_t + (-\Delta)^s + a)u = F \quad \text{in } \Omega_T, \\
u = f \quad \text{in } \Omega_T, \\
u = 0 \quad \text{in } \{0\} \times \mathbb{R}^n,
\]
satisfying 
\[
\|u\|_{L^\infty(0,T; H^s(\mathbb{R}^n))} + \|u\|_{L^\infty(\Omega_T)}^2 \\
\lesssim C(\| F \|_{L^\infty(\Omega_T)} + \| f \|_{L^\infty(0,T; H^s(\mathbb{R}^n)))} + \| (-\Delta)^s F \|_{L^\infty(\Omega_T)})
\]
for some constant \( C = C(n,s,T,\Omega) \).

### 2.5. Well-posedness for the nonlinear equation

We now state the well-posedness of (1.1) for small exterior data:

**Proposition 2.10.** Given any \( n \in \mathbb{N} \) and \( 0 < s < 1 \). Let \( \Omega \subset \mathbb{R}^n \) be a bounded Lipschitz domain in \( \mathbb{R}^n \), and \( W \subset \Omega^c \) be any open set with Lipschitz boundary satisfying \( \overline{W} \cap \overline{\Omega} = \emptyset \). Fixing any parameter \( \delta > 0 \). Assume that \( q \) satisfies (Q.1)–(Q.3). Then there exists a sufficiently small parameter \( \epsilon_0 = \epsilon_0(n,s,\Omega,T,\delta) > 0 \) such that the following statement holds: given any \( f \in C_c^\infty(W_T) \) with \( \| f \|_{\text{ext}} \leq \epsilon_0 \), there exists a unique solution \( u \in L^\infty(0,T; H^s(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n_T) \) of (1.1) with 
\[
\|u\|_{L^\infty(0,T; H^s(\mathbb{R}^n))} + \|u\|_{L^\infty(\Omega_T)}^2 \lesssim C \| f \|_{\text{ext}}
\]
for certain constant \( C = C(n,s,T,\Omega) \).
Remark 2.2. In order to prove proposition 2.10, we only need \( q \) to be \( C^1 \)-smooth in \( z \) variable. However to recover \( m \)th jet of \( q \) we need to assume (Q.1).

Remark 2.3. In [MBRS16, theorem 11.2], they showed that there exist infinitely many solutions \( w_j \) to

\[
\begin{cases}
(-\Delta)^j w_j + q(x, w_j) + h(x) = 0 & \text{in } \Omega, \\
w_j = 0 & \text{in } \Omega^F,
\end{cases}
\]

such that \( \|w_j\|_{H^3(\mathbb{R}^n)} \to \infty \) as \( j \to \infty \). Therefore, the smallness assumption on \( f \) seems to be necessary to ensure the uniqueness of the solution to (1.1).

Proof of proposition 2.10. Step 1: initialization. Given any \( f \in C^\infty_0(\Omega_T) \), from proposition 2.9, there exists a unique solution \( u_0 = u_0(t, x) \) of

\[
\begin{cases}
(\partial_t + (-\Delta)^j)u = 0 & \text{in } \Omega_T, \\
u_0 = f & \text{in } \Omega^*_T, \\
u_0 = 0 & \text{in } \{0\} \times \mathbb{R}^n,
\end{cases}
\]

with

\[
\|u_0\|_{L^\infty(0, T; H^0(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^3_T)} \leq C\|f\|_{\text{ext}}.
\]

for some constant \( C = C(n, s, T, \Omega) \). We use the notation for simplicity:

\( \mathcal{F} : v(t, x) \mapsto -q(t, x, (v + u_0)(t, x)) \).

If \( u \) is a solution of (1.1), then the remainder function \( v \equiv u - u_0 \) satisfies

\[
\begin{cases}
(\partial_t + (-\Delta)^j)v = \mathcal{F}(v) & \text{in } \Omega_T, \\
v = 0 & \text{in } \Omega^*_T, \\
v = 0 & \text{in } \{0\} \times \mathbb{R}^n.
\end{cases}
\]

Again, using proposition 2.9, given any \( F = F(t, x) \in L^\infty(\Omega_T) \), there exists a unique solution \( SF \in L^\infty(0, T; H^0(\Omega)) \cap L^\infty(\mathbb{R}^3_T) \) of

\[
\begin{cases}
(\partial_t + (-\Delta)^j)SF = F & \text{in } \Omega_T, \\
SF = 0 & \text{in } \Omega^*_T, \\
SF = 0 & \text{in } \{0\} \times \mathbb{R}^n,
\end{cases}
\]

with \( \|SF\|_{L^\infty(0, T; H^0(\Omega)) \cap L^\infty(\mathbb{R}^3_T)} \leq C\|F\|_{L^\infty(\Omega_T)} \) for some constant \( C = C(n, s, T, \Omega) \). In other words, the solution operator

\[
S : L^\infty(\Omega_T) \to L^\infty(0, T; H^0(\Omega)) \cap L^\infty(\mathbb{R}^3_T)
\]

of (2.15) is a bounded linear operator.

Step 2: contraction. Let \( \epsilon = \|f\|_{\text{ext}} \), and we define

\[
X_\epsilon := \left\{ v \in L^\infty(0, T; H^0(\Omega)) \cap L^\infty(\mathbb{R}^3_T) \mid \|v\|_{L^\infty(0, T; H^0(\Omega)) \cap L^\infty(\mathbb{R}^3_T)} \leq \epsilon \right\}.
\]

We denote the composition of operators \( S \) and \( \mathcal{F} \) as \( S \circ \mathcal{F} \). We first show that \( S \circ \mathcal{F}(v) \in X_\epsilon \) for all \( v \in X_\epsilon \).

Given any \( v \in X_\epsilon \), using (2.13), we know

\[
\|u_0 + v\|_{L^\infty(0, T; H^0(\mathbb{R}^3_T)) \cap L^\infty(\mathbb{R}^3_T)} \leq C\epsilon.
\]
By choosing $\tilde{\epsilon}_0 = \tilde{\epsilon}_0(n, s, \Omega, T, \delta) > 0$ to be sufficiently small, we can guarantee that $2C\epsilon \leq 2C\tilde{\epsilon}_0 < \delta$. From (Q.1) and (Q.2), by using the mean value theorem, we can find a function $0 \leq \zeta(t, x) \leq 1$ such that

$$F(v)(t, x) = q(t, x, (u_0 + v)(t, x)) - q(t, x, 0)$$

$$= \partial_t q(t, x, (\zeta(u_0 + v))(t, x))(u_0 + v)(t, x), \quad \text{for all } x \in \Omega.$$  \hspace{1cm} (2.19)

Therefore, using (Q.3), combining (2.18) and (2.19), we obtain $\|F(v)\|_{L^\infty(\Omega_2)} \leq \Phi(C\epsilon)\|u_0 + v\|_{L^\infty(\mathbb{R}^2)} \leq C\Phi(C\epsilon)\epsilon$. Using (2.16), we then obtain $\|S \circ F(v)\|_{L^\infty(0, T; H^1(\Omega)) \cap L^\infty(\mathbb{R}^2)} \leq \tilde{C}\Phi(C\epsilon)\epsilon$. Since $\Phi$ is non-decreasing, using the assumption (Q.3), and by choosing a smaller $\tilde{\epsilon}_0 = \tilde{\epsilon}_0(n, s, \Omega, T, \delta) > 0$, we can assure that $\Phi(C\epsilon) \leq \Phi(C\tilde{\epsilon}_0) \leq C^{-1}$, and can obtain

$$\|S \circ F(v)\|_{L^\infty(0, T; H^1(\Omega)) \cap L^\infty(\mathbb{R}^2)} \leq \epsilon,$$  \hspace{1cm} (2.20)

which concludes (2.17).

We next show that

$$S \circ F$$ is a contraction on $X_c$.  \hspace{1cm} (2.21)

Let $v_1, v_2 \in X_c$, similar to (2.18), we have

$$\|u_0 + v_j\|_{L^\infty(0, T; H^1(\Omega)) \cap L^\infty(\mathbb{R}^2)} \leq C\epsilon \quad \text{for all } j = 1, 2.$$  \hspace{1cm} (2.22)

From (Q.1), by using the mean value theorem, we can find a function $0 \leq \zeta(t, x) \leq 1$ such that

$$F(v_1)(t, x) - F(v_2)(t, x) = q(t, x, (u_0 + v_1)(t, x)) - q(t, x, (u_0 + v_2)(t, x))$$

$$= \partial_t q(t, x, (\zeta(u_0 + v_1) + (1 - \zeta)(u_0 + v_2))(t, x))(v_1 - v_2)(t, x).$$  \hspace{1cm} (2.23)

Using (2.22), we know that

$$\|\zeta(u_0 + v_1) + (1 - \zeta)(u_0 + v_2)\|_{L^\infty(\mathbb{R}^2)} \leq C\epsilon \leq C\tilde{\epsilon}_0.$$  \hspace{1cm} (2.24)

Since $C\tilde{\epsilon}_0 < \delta$, using (Q.3), combining (2.23), we have

$$\|F(v_1) - F(v_2)\|_{L^\infty(\Omega_2)} \leq \Phi(C\tilde{\epsilon}_0)\|v_1 - v_2\|_{L^\infty(\Omega_2)} \leq \Phi(C\tilde{\epsilon}_0)\|v_1 - v_2\|_{L^\infty(0, T; H^1(\Omega)) \cap L^\infty(\mathbb{R}^2)}.$$  \hspace{1cm} (2.25)

Using (2.16), we then obtain

$$\|S \circ F(v_1) - S \circ F(v_2)\|_{L^\infty(0, T; H^1(\Omega)) \cap L^\infty(\mathbb{R}^2)} \leq \tilde{C}\Phi(C\tilde{\epsilon}_0)\|v_1 - v_2\|_{L^\infty(0, T; H^1(\Omega)) \cap L^\infty(\mathbb{R}^2)}.$$  \hspace{1cm} (2.26)

Since $\Phi$ is non-decreasing, using (Q.3), possibly choosing a smaller $\tilde{\epsilon}_0 = \tilde{\epsilon}_0(n, s, \Omega, T, \delta) > 0$, we can assure that $\Phi(C\epsilon) \leq \frac{1}{2}C^{-1}$, and we obtain

$$\|S \circ F(v_1) - S \circ F(v_2)\|_{L^\infty(0, T; H^1(\Omega)) \cap L^\infty(\mathbb{R}^2)} \leq \frac{1}{2}\|v_1 - v_2\|_{L^\infty(0, T; H^1(\Omega)) \cap L^\infty(\mathbb{R}^2)}.$$  \hspace{1cm} (2.27)

which concludes (2.21).

**Step 3: conclusion.** From (2.17) and (2.21), by using the Banach fixed point theorem, there exists a unique $v \in X_c$ such that $v = S \circ F(v)$, that is, there exists a unique $v \in X_c$ satisfying (2.14). Hence we know that $u \equiv v + u_0 \in L^\infty(0, T; H^1(\Omega)) \cap L^\infty(\mathbb{R}^2)$ is the unique solution of (1.1). Moreover, from (2.13) and (2.20), we can conclude (2.12).
3. The inverse problems for the fractional diffusion equation

In this section we perform higher-order linearizations to the nonlinear fractional diffusion equation (1.1) as well as the DN map (1.2), which is also nonlinear. For each linearization step we derive certain identities and combine them with the Runge approximation to recover partial derivatives of \( q \). We first recall the following fundamental properties of \( (-\Delta)^{\alpha} \):

**Lemma 3.1.** ([GSU20, theorem 1.2]) Suppose \( u = (-\Delta)^{\alpha} u = 0 \) in \( \Omega \), for some open set \( \Omega \subset \mathbb{R}^n \), then \( u \equiv 0 \) in \( \mathbb{R}^n \).

By using proposition 2.9, we see that there exists a unique solution \( u \) satisfying

\[
\begin{aligned}
(\partial_t + (-\Delta)^{\alpha} + \alpha t(x))u &= 0 \quad \text{in} \quad \Omega_T, \\
\epsilon u &= f \quad \text{in} \quad \Omega_T^\epsilon, \\
\epsilon u &= 0 \quad \text{on} \quad \{0\} \times \mathbb{R}^n.
\end{aligned}
\]

(3.1)

We define the solution operator \( P_a : f \mapsto P_a f := u \). As a consequence of lemma 3.1, we also have the following Runge approximation for the solution operator \( P_a \):

**Proposition 3.2.** ([LI22, proposition 2.4]) Assume that all assumptions in proposition 2.9 hold, let \( a \in L^\infty(\Omega_T) \) and let \( f \in L^\infty(W_T) \). Then the set \( \{ (P_a f)|_{\Omega_T^\epsilon} | f \in C_c^\infty(W_T) \} \) is dense in \( L^2(\Omega_T) \).

With these tools at hand, we are able to begin our discussions. We start with the zeroth order linearization.

3.1. Zeroth order linearization

Let \( u^\epsilon \) be the unique solution of

\[
\begin{aligned}
\partial_t u^\epsilon + (-\Delta)^{\alpha} u^\epsilon + q_j(\cdot, u^\epsilon) &= 0 \quad \text{in} \quad \Omega_T, \\
u^\epsilon = \epsilon \cdot g &= \epsilon_1 g_1 + \cdots + \epsilon_m g_m \quad \text{in} \quad \Omega_T^\epsilon, \\
u^\epsilon &= 0 \quad \text{on} \quad \{0\} \times \mathbb{R}^n,
\end{aligned}
\]

(3.2)

where \( g = (g_1, \ldots, g_m) \in (C_c^\infty(W_T))^m \). Since \( q_j \) \((j = 1, 2)\) satisfies (Q.1)–(Q.3), there exists a constant \( c_0 = c_0(n, s, \Omega, T, \delta, g) > 0 \) with \( c_0 \leq c_0 \), where \( c_0 \) is the constant given in proposition 2.10, such that the following statement holds: Given any \( \epsilon \) with \( |\epsilon| = \max_{1 \leq k \leq m} |\epsilon_k| < c_0 \), there exists a unique solution \( u^\epsilon \in L^\infty(0, T; H^s(\Omega)) \cap L^\infty(\mathbb{R}^n_T) \) of (3.2) with

\[
\|u^\epsilon\|_{L^\infty(0, T; H^s(\mathbb{R}^n))} \leq C(n, s, \Omega, T, g, m)|\epsilon|.
\]

(3.3)

Therefore, the corresponding DN-map is \( \Lambda_0(\cdot, g) = (-\Delta)^{\alpha} u^\epsilon |_{\partial\Omega} \) for all \( 0 \leq |\epsilon| < c_0 \). We now show that \( \epsilon \to u^\epsilon \) is continuous in the following sense:

**Lemma 3.3.** The mapping \( \epsilon \to u^\epsilon \) is continuous in \( L^\infty(0, T; H^s(\Omega)) \), that is,

\[
\lim_{|\epsilon| \to 0} \|u^{\epsilon + \theta} - u^\epsilon\|_{L^\infty(0, T; H^s(\mathbb{R}^n_T))} = 0 \quad \text{for each} \quad |\epsilon| < c_0.
\]

(3.4)

**Proof.** Let \( \theta = (\theta_1, \ldots, \theta_m) \in \mathbb{R}^m \) with \( |\theta| \leq |\epsilon| \) and \( |\epsilon| + |\theta| < c_0 \). We define \( \delta_\theta u^\epsilon = u^{\epsilon + \theta} - u^\epsilon \), and observe that

\[
\begin{aligned}
(\partial_t + (-\Delta)^{\alpha})\delta_\theta u^\epsilon &= G \quad \text{in} \quad \Omega_T, \\
\delta_\theta u^\epsilon &= \theta \cdot g \quad \text{in} \quad \Omega_T^\epsilon, \\
\delta_\theta u^\epsilon &= 0 \quad \text{on} \quad \{0\} \times \mathbb{R}^n,
\end{aligned}
\]
where \( G = -g_j \mathcal{L}_j u^{\mathcal{L}_j} + q_j \), From proposition 2.9, we know that
\[
\| \delta \mathcal{Q} u^f \|_{L^\infty(0, T; H^1(\mathbb{R}^n))} \leq C \| G \|_{L^\infty(\Omega_T)} + |\theta|.(3.5)
\]
Using mean value theorem on the \( z \) variable of \( q \), there exists \( 0 \leq \zeta(t, x) \leq 1 \) such that \( G = -\partial_z q_j \mathcal{L}_j u^{\mathcal{L}_j} + (1 - \zeta) u^f \) in \( \Omega_T \). From (3.3), we know that
\[
\| \mathcal{L}_j u^{\mathcal{L}_j} \|_{L^\infty(0, T; H^1(\mathbb{R}^n))} \leq C \| \epsilon \| (\text{because } |\theta| \leq |\epsilon|).
\]
Using (Q.3), we see that
\[
\| G \|_{L^\infty(\Omega_T)} \leq \Phi(C|\epsilon|)\| \delta \mathcal{Q} u^f \|_{L^\infty(\Omega_T)} \leq \Phi(C|\epsilon|)\| \delta \mathcal{Q} u^f \|_{L^\infty(0, T; H^1(\Omega))} + C|\theta|.
\]
Substituting this inequality into (3.5), we obtain
\[
\| \delta \mathcal{Q} u^f \|_{L^\infty(0, T; H^1(\mathbb{R}^n))} \leq \Phi(C|\epsilon|)\| \delta \mathcal{Q} u^f \|_{L^\infty(0, T; H^1(\mathbb{R}^n))} + C|\theta|.
\]
Since \( \Phi \) is non-decreasing, using the assumption (Q.3), and possibly by choosing a smaller constant \( \epsilon_0 = \epsilon_0(n, s, \Omega, T, \delta, g, m) > 0 \), we can assure \( \Phi(C|\epsilon|) \leq \Phi(C\epsilon_0) \leq \frac{1}{2} C^{-1} \), thus \( \| \delta \mathcal{Q} u^f \|_{L^\infty(0, T; H^1(\mathbb{R}^n))} \leq 2C|\theta| \), which implies (3.4). This completes the proof. \( \square \)

By setting \( \epsilon = 0 \) in (3.2), we obtain
\[
\begin{align*}
\partial_t u^0 + (-\Delta) u^0_j + g_j (\cdot, u^0_j) &= 0 &\text{in } \Omega_T, \\
u^0_j &= 0 &\text{in } \Omega_T^c, \\
\partial_t u^0_j &= 0 &\text{on } \{0\} \times \mathbb{R}^n.
\end{align*}
\]
From (3.3), we know that \( u^0_j = 0 \) in \( \mathbb{R}^n_T \).

3.2. First order linearization

Assuming the derivative \( \partial_c \) to (3.2) is well-defined, we obtain
\[
\begin{cases}
\left( \partial_t + (-\Delta)^c + \partial_c g_j (\cdot, u^f_j) \right)(\partial_c u^f_j) = 0 &\text{in } \Omega_T, \\
\partial_c u^f_j &= g_1 &\text{in } \Omega_T^c, \\
\partial_t u^f_j &= 0 &\text{on } \{0\} \times \mathbb{R}^n.
\end{cases}
\]
Using (Q.3), we know that \( \| \partial_c g_j (\cdot, u^f_j) \|_{L^\infty(\Omega_T)} \leq \Phi(C\epsilon_0) \leq 1 \). Therefore, using proposition 2.9, given any \( \epsilon \) with \( |\epsilon| < \epsilon_0 \), there exists a unique solution \( v^f_j \in L^\infty(0, T; H^1(\Omega)) \cap L^\infty(\mathbb{R}^n_T) \) to (3.6) with
\[
\| v^f_j \|_{L^\infty(0, T; H^1(\mathbb{R}^n))} \leq C\| g_1 \|_{\text{ext}}.
\]
Here, \( v^f_j \) is just a intermediate function which will be dropped after showing that \( \partial_c u^f_j \) is well-defined.

**Lemma 3.4.** There exists a constant \( \epsilon_0 = \epsilon_0(n, s, \Omega, T, \delta, g, m) > 0 \) with \( 0 < \epsilon_0 < \tilde{\epsilon}_0 \), where \( \tilde{\epsilon}_0 \) is given in proposition 2.10, such that for each \( \epsilon \) with \( |\epsilon| < \epsilon_0 \), we have
\[
\lim_{\epsilon_1 \to 0} \| v^f_j - \delta_{\epsilon_1} u^f_j \|_{L^\infty(0, T; H^1(\mathbb{R}^n))} = 0,
\]
where \( \delta_{\epsilon_1} u^f_j = \frac{u^f_j + \epsilon_1 u^* - u^*}{\epsilon_1} \) for all \( (t, x) \in \Omega_T \), provided that \( |\epsilon| + |\epsilon_1| < \epsilon_0 \).

**Proof.** Let \( \epsilon_1 \) satisfies \( |\epsilon_1| \leq |\epsilon| \) and \( |\epsilon| + |\epsilon_1| < \epsilon_0 \). Note that
\[
\begin{cases}
\left( \partial_t + (-\Delta)^c \right)(v^f_j - \delta_{\epsilon_1} u^f_j) = G_1 &\text{in } \Omega_T, \\
(v^f_j - \delta_{\epsilon_1} u^f_j) = 0 &\text{in } \Omega_T^c \text{ and on } \{0\} \times \mathbb{R}^n.
\end{cases}
\]
with \(-G_1 = \partial_t q(\cdot, u^f_j) v^f_j - q(\cdot, u^{f+\epsilon}_{j+1}) - \partial_t u^{f+\epsilon}_{j+1}\). From proposition 2.9, we know that

$$
\|v^f_j - \delta_t u^f_j\|_{L^\infty(0,T;B^1(R^n))} \leq C\|g_1\|_{L^\infty(\Omega_T)}.
$$

Using the mean value theorem on the \(z\) variable of \(q\), there exists \(0 \leq \zeta(t, x) \leq 1\) such that

\[-G_1 = [\partial_t q(\cdot, u^f_j) - \partial_t q(\cdot, \zeta u^{f+\epsilon}_{j+1}) + (1 - \zeta)u^f_j] v^f_j + \partial_t q(\cdot, \zeta u^{f+\epsilon}_{j+1}) + (1 - \zeta)u^f_j [v^f_j - \delta_t u^f_j].\]

Using the mean value theorem on the \(z\) variable of \(\partial_t q\), there exists \(0 \leq \eta(t, x) \leq 1\) such that

\[-G_1 = -\zeta \partial^2_t q(\cdot, u^f_j) - (1 - \eta)(\zeta u^{f+\epsilon}_{j+1} + (1 - \zeta)u^f_j) (u^{f+\epsilon}_{j+1} - u^f_j) v^f_j + \partial_t q(\cdot, \zeta u^{f+\epsilon}_{j+1} + (1 - \zeta)u^f_j) [v^f_j - \delta_t u^f_j].\]

From (3.3) and \(|\epsilon_1| \leq |\epsilon|\), we have

\[
\|\eta u^f_j - (1 - \eta)(\zeta u^{f+\epsilon}_{j+1} + (1 - \zeta)u^f_j)\|_{L^\infty(0,T;B^1(R^n))} \leq C|\epsilon|,
\]

\[
\|\zeta u^{f+\epsilon}_{j+1} + (1 - \zeta)u^f_j\|_{L^\infty(0,T;B^1(R^n))} \leq C|\epsilon|.
\]

Hence, by (Q.3) and (Q.4), we know that

\[
\|\partial_t q(\cdot, \zeta u^{f+\epsilon}_{j+1} + (1 - \zeta)u^f_j)\|_{L^\infty(\Omega_T)} \leq \Phi(C|\epsilon|).
\]

Hence, by using (3.7) we know that

\[
\|G_1\|_{L^\infty(\Omega_T)} \leq CM_2\|g_1\|_{B^1(\Omega_T)}\|\|u^{f+\epsilon}_{j+1} - u^f_j\|_{L^\infty(\Omega_T)} + \Phi(C\|\epsilon\|)\|v^f_j - \delta_t u^f_j\|_{L^\infty(\Omega_T)}.
\]

Combining this with (3.8), we have

\[
\|v^f_j - \delta_t u^f_j\|_{L^\infty(0,T;B^1(R^n))} \leq \tilde{C}M_2\|g_1\|_{B^1(\Omega_T)}\|\|u^{f+\epsilon}_{j+1} - u^f_j\|_{L^\infty(\Omega_T)} + \Phi(C\|\epsilon\|)\|v^f_j - \delta_t u^f_j\|_{L^\infty(0,T;B^1(R^n))}.
\]

Since \(\Phi\) is non-decreasing, using the limiting assumption of \(\Phi\) in (Q.3), possibly choosing a smaller \(\epsilon_0 = \epsilon_0(m, n, \Omega, T, \delta, g, m) > 0\), we can assure that \(\Phi(C\|\epsilon\|) \leq \frac{1}{2}\), and hence

\[
\|v^f_j - \delta_t u^f_j\|_{L^\infty(0,T;B^1(R^n))} \leq \tilde{C}M_2\|g_1\|_{B^1(\Omega_T)}\|\|u^{f+\epsilon}_{j+1} - u^f_j\|_{L^\infty(\Omega_T)}.
\]

Finally, using lemma 3.3, we conclude lemma 3.4. \(\square\)

Using (Q.4), we also see that \(\partial_t u^f_j|_{\epsilon=0}\) satisfies

\[
\begin{cases}
(\partial_t + (\Delta)^\gamma)(\partial_t u^f_j|_{\epsilon=0}) = 0 & \text{in } \Omega_T, \\
\partial_t u^f_j|_{\epsilon=0} = g_1 & \text{in } \Omega_T^\gamma, \\
\partial_t u^f_j|_{\epsilon=0} = 0 & \text{on } \{0\} \times \mathbb{R}^n.
\end{cases}
\]

By uniqueness of solutions (see proposition 2.9), we know that \(\partial_t u^f_j|_{\epsilon=0} = \partial_t u^f_j|_{\epsilon=0} = \partial_t u^f_j|_{\epsilon=0}\) in \(\Omega_T\).

For later convenience, we simply denote

\[
\partial_t u^f_j|_{\epsilon=0} = \partial_t u^f_j|_{\epsilon=0} = \partial_t u^f_j|_{\epsilon=0} \text{ in } \Omega_T.
\]

In the next lemma we show that the information from DN-map can be passed to the first-order linearized DN-map.
Lemma 3.5. If $\Lambda_{q_1}(f) = \Lambda_{q_2}(f)$ for all $f \in C^\infty_c(W_T)$ with $\|f\|_{\text{ext}} \leq \tilde{\epsilon}_0$, where $\tilde{\epsilon}_0$ is the constant given in proposition 2.10, then there exists a constant $\epsilon_0 = \epsilon_0(n, s, \Omega, \delta, g, m) > 0$ with $0 < \epsilon_0 < \tilde{\epsilon}_0$ such that

$$(-\Delta)^{j} \partial_{t_1} u_{f}^{e} \bigg|_{V_f} = (-\Delta)^{j} \partial_{t_1} u_{f}^{e} \bigg|_{V_f} \quad \text{for all} \quad \epsilon \quad \text{with} \quad |\epsilon| \leq \epsilon_0. \quad (3.11)$$

Proof. We have

$$\left\| (-\Delta)^{j} \partial_{t_1} u_{f}^{e} - \frac{\Lambda_{q_1}(\epsilon + \epsilon e_1) \cdot g - \Lambda_{q_2}(\epsilon \cdot g)}{\epsilon_1} \right\|_{L^\infty(0, T; H^{-j}(V))} \leq C \left\| \left( \partial_{t_1} u_{f}^{e} - \frac{u_{f}^{e+\epsilon e_1} - u_{f}^{e}}{\epsilon_1} \right) \right\|_{L^\infty(0, T; H^{-j}(\mathbb{R}^m))} \cdot$$

From lemma 3.4, we have

$$\lim_{\epsilon_1 \to 0} \left\| (-\Delta)^{j} \partial_{t_1} u_{f}^{e} - \frac{\Lambda_{q_1}(\epsilon + \epsilon e_1) \cdot g - \Lambda_{q_2}(\epsilon \cdot g)}{\epsilon_1} \right\|_{L^\infty(0, T; H^{-j}(V))} = 0.$$

Combining this equality with the assumption $\Lambda_{q_1} = \Lambda_{q_2}$, we conclude (3.11). \qed

3.3. Second order linearization

First of all, we recall (3.7):

$$\left\| \partial_{t_1} u_{f}^{e} \right\|_{L^\infty(0, T; H^{-1}(\mathbb{R}^m)) \cap L^\infty(\mathbb{R}^m_{x_2})} \leq C \|g_f\|_{\text{ext}} \quad \text{for} \quad p = 1, 2. \quad (3.12)$$

see lemma 3.4. Acting a formal differential operator $\partial_{x_2}$ on (3.6), we obtain

$$\begin{cases} 
(\partial_{t} + (-\Delta)^{j} + \partial_{t} q_{j}(\cdot, u_{f}^{e})) (\partial_{t_1} \partial_{x_2} u_{f}^{e}) = 0 & \text{in} \ \Omega_T, \\
+ \partial_{x_2} q_{j}(\cdot, u_{f}^{e}) (\partial_{t_1} u_{f}^{e}) (\partial_{x_2} u_{f}^{e}) = 0 & \text{in} \ \Omega_T^c \text{ and on} \ \{0\} \times \mathbb{R}^m. 
\end{cases} \quad (3.13)$$

Since the term $\partial_{x_2} q_{j}(\cdot, u_{f}^{e}) (\partial_{t_1} u_{f}^{e}) (\partial_{x_2} u_{f}^{e})$ is bounded in $\Omega_T$, using proposition 2.9, there exists a unique solution $u_{f}^{e} \in L^\infty(0, T; \mathcal{H}^{p}(\Omega)) \cap L^\infty(\mathbb{R}^m_{x_2})$ to (3.13) with
Using (Q.3) and (Q.4)

\[ \|v^\varepsilon\|_{L^\infty(0,T;H^1(\Omega))} \leq C \|\partial_t^2 q_j(\cdot, u^\varepsilon)(\partial_{t_1} u^\varepsilon)\|_{L^\infty(\Omega_T)} \]

\[ \leq CM_2 \|g_1\|_{L^2(\Omega)} \|g_2\|_{L^2(\Omega)} \]  \quad \text{(using (Q.4) and (3.12))} \tag{3.14}

Again, \(v^\varepsilon\) is temporary notation, which will be dropped after showing \(\partial_{t_1} u^\varepsilon\) is well-defined. We emphasize that we have already dropped \(v^\varepsilon\), so this will not conflict with the section with section 3.2.

**Lemma 3.6.** There exists a constant \(\varepsilon_0 = \varepsilon_0(n, s, \Omega, T, \delta, g, m) > 0\) with \(0 < \varepsilon_0 < \varepsilon_0\), where \(\varepsilon_0\) is given in proposition 2.10, such that for each \(\varepsilon\) with \(|\varepsilon| < \varepsilon_0\), we have

\[ \lim_{\varepsilon \to 0} \|v^\varepsilon - \varepsilon_0 \partial_{t_1} u^\varepsilon\|_{L^\infty(0,T;H^1(\Omega))} = 0, \] \tag{3.15}

where \(\varepsilon_0 \partial_{t_1} u^\varepsilon = \frac{\partial_{t_1} (u^\varepsilon + \varepsilon \varepsilon_0)}{\partial_{t_1} u^\varepsilon}\) in \(\Omega_T\), provided \(|\varepsilon| + |\varepsilon_0| < \varepsilon_0\).

**Proof.** Let \(\varepsilon_2\) satisfies \(|\varepsilon_2| < |\varepsilon|\) and \(|\varepsilon| + |\varepsilon_2| < \varepsilon_0\). Note that

\[ \begin{cases} (\partial_t + (-\Delta)^s)(v^\varepsilon - \varepsilon_0 \partial_{t_1} u^\varepsilon) = G_2 \quad \text{in } \Omega_T, \\ v^\varepsilon - \varepsilon_0 \partial_{t_1} u^\varepsilon = 0 \quad \text{in } \Omega_T^r \text{ and on } \{0\} \times \mathbb{R}^n, \end{cases} \]

where

\[ G_2 = \partial_t q_j(\cdot, u^\varepsilon) v^\varepsilon + \partial_{t_1}^2 q_j(\cdot, u^\varepsilon)(\partial_{t_1} u^\varepsilon)(\partial_{t_2} u^\varepsilon) - \frac{\partial_{t_1} q_j(\cdot, u^\varepsilon + \varepsilon \varepsilon_2)}{\varepsilon_2} \partial_{t_1} u^\varepsilon \partial_{t_2} u^\varepsilon \cdot \varepsilon_2 \]

After some computation we can write \(-G_2 = G_{21} + G_{22} + G_{23}\), where

\[ \begin{cases} G_{21} = \partial_t q_j(\cdot, u^\varepsilon)[v^\varepsilon - \varepsilon_0 \partial_{t_1} u^\varepsilon], \\ G_{22} = \left[ \partial_{t_1}^2 q_j(\cdot, u^\varepsilon)(\partial_{t_2} u^\varepsilon) - \frac{\partial_{t_1} q_j(\cdot, u^\varepsilon)}{\varepsilon_2} \right] \partial_{t_1} u^\varepsilon \partial_{t_2} u^\varepsilon, \\ G_{23} = \partial_{t_1}^2 q_j(\cdot, u^\varepsilon) \partial_{t_2} u^\varepsilon \partial_{t_1} u^\varepsilon + \partial_{t_1} q_j(\cdot, u^\varepsilon) \partial_{t_2} u^\varepsilon. \end{cases} \]

Note that \(\|v^\varepsilon - \varepsilon_0 \partial_{t_1} u^\varepsilon\|_{L^\infty(0,T;H^1(\Omega))} \leq C \|G_2\|_{L^\infty(\Omega)}\). Possibly choosing a smaller \(\varepsilon_0\), we have \(\|G_2\|_{L^\infty(\Omega)} \leq \frac{1}{4} \|v^\varepsilon - \varepsilon_0 \partial_{t_1} u^\varepsilon\|_{L^\infty(0,T;H^1(\Omega))} \leq \frac{1}{2} \|v^\varepsilon - \varepsilon_0 \partial_{t_1} u^\varepsilon\|_{L^\infty(0,T;H^1(\Omega))} \leq \frac{1}{2} \|v^\varepsilon - \varepsilon_0 \partial_{t_1} u^\varepsilon\|_{L^\infty(\Omega_T)} \).

Using the mean value theorem and lemma 3.4, we know that \(\lim_{\varepsilon \to 0}(\frac{\|G_2\|_{L^\infty(\Omega_T)} + \|G_2\|_{L^\infty(\Omega_T)}) = 0\). We then conclude (3.15) by using arguments similar to lemma 3.4. \(\square\)

Akin to lemma 3.5, we next demonstrate the following lemma.

**Lemma 3.7.** If \(\Lambda_q(f) = \Lambda_q(f) \text{ for all } f \in C_0^\infty(\mathcal{W}_T)\) with \(\|f\|_{\mathcal{W}_T} \leq \varepsilon_0\), where \(\varepsilon_0\) is the constant given in proposition 2.10, then there exists a constant \(\varepsilon_0 = \varepsilon_0(n, s, \Omega, T, \delta, g, m) > 0\) with \(0 < \varepsilon_0 < \varepsilon_0\) such that

\[ (-\Delta)^s \partial_{t_1} u^\varepsilon \big|_{\Omega_T^r} = (-\Delta)^s \partial_{t_1} u^\varepsilon \big|_{\Omega_T^r} \quad \forall \varepsilon \text{ with } |\varepsilon| \leq \varepsilon_0. \] \tag{3.16}

**Proof.** Using similar arguments as in lemma 3.5 (with lemma 3.6), we can show that (3.11) implies (3.16). Then we conclude the lemma by lemma 3.5. \(\square\)

**Proof of theorem 1.1 for \(m = 2\).** Using (Q.3) and (3.10), we know that \(\partial_{t_1} u^\varepsilon\big|_{\varepsilon = 0}\) satisfies

\[ \begin{cases} (\partial_t + (-\Delta)^s)(\partial_{t_1} u^\varepsilon) = 0 \quad \text{in } \Omega_T, \\ + \partial_{t_1}^2 q_j(\cdot, 0)(\partial_{t_2} u^\varepsilon)(\partial_{t_2} u^\varepsilon) = 0 \quad \text{in } \Omega_T, \\ \partial_{t_1} u^\varepsilon \big|_{\varepsilon = 0} = 0 \quad \text{in } \Omega_T^r \text{ and on } \{0\} \times \mathbb{R}^n. \end{cases} \]

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Hence, we know that $v := \partial_{\epsilon \ell} u^{\epsilon}_{1}|_{\epsilon=0} - \partial_{\epsilon \ell} u^{\epsilon}_{2}|_{\epsilon=0}$ satisfies
\[
\begin{cases}
(\partial_t + (-\Delta)^{\alpha}) v + (\partial^2 q_1(\cdot,0) - \partial^2 q_2(\cdot,0))(\partial_{\epsilon \ell} u^{\epsilon}_{1}|_{\epsilon=0})(\partial_{\epsilon \ell} u^{\epsilon}_{1}|_{\epsilon=0}) = 0 & \text{in } \Omega_T, \\
v = 0 & \text{in } \Omega_T^\infty \text{ and on } \{0\} \times \mathbb{R}^n.
\end{cases}
\]
From lemma 3.7, we know that \((-\Delta)^{\alpha} v|_{\Gamma_T} = 0\). Since $v = 0$ in $V_T$, using the unique continuation property of the fractional Laplacian in lemma 3.1, we conclude that $v \equiv 0$. Therefore, we know that
\[
(\partial^2 q_1(\cdot,0) - \partial^2 q_2(\cdot,0))(\partial_{\epsilon \ell} u^{\epsilon}_{1}|_{\epsilon=0})(\partial_{\epsilon \ell} u^{\epsilon}_{1}|_{\epsilon=0}) = 0. \tag{3.17}
\]
Since $g_1, g_2 \in C^\infty_c(W_T)$ are arbitrary, using (3.9) and the Runge approximation for fractional diffusion equation in proposition 3.2, we conclude $\partial^2 q_1(\cdot,0) - \partial^2 q_2(\cdot,0) = 0$ in $\Omega_T$, which proves theorem 1.1 for $m = 2$.

For the case when $q$ is independent of $t$, one can recover $\partial^2 q_1(\cdot,0)$ using lesser measurements.

**Proof of theorem 1.2 for $m = 2$.** We fix $g_1, g_2 \in C^\infty_c(W_T)$. Using the same argument as above, we reach (3.17):
\[
(\partial^2 g_1(\cdot,0) - \partial^2 g_2(\cdot,0))(\partial_{\epsilon \ell} u^{\epsilon}_{1}|_{\epsilon=0})(\partial_{\epsilon \ell} u^{\epsilon}_{1}|_{\epsilon=0}) = 0. \tag{3.18}
\]
Using (3.9), the unique continuation property of the fractional Laplacian in lemma 3.1 and a simple contradiction argument, for each $x_0 \in \Omega$, we can find a sequence $\{x_k\}_{k \in \mathbb{N}} \subset \Omega$ with $x_k \to x_0$ and a sequence $\{t_k\}_{k \in \mathbb{N}} \subset (0, T)$ such that
\[
\partial_{\epsilon \ell} u^{\epsilon}_{1}|_{\epsilon=0}(t_k, x_k) \neq 0 \quad \text{and} \quad \partial_{\epsilon \ell} u^{\epsilon}_{2}|_{\epsilon=0}(t_k, x_k) \neq 0, \tag{3.19}
\]
see also [Li23a, page 10] for a similar argument. Therefore from (3.18) we know that $\partial^2 q_1(x_0,0) = \partial^2 q_2(x_0,0)$. Note that here we have assumed both $\partial^2 q_1(\cdot,0)$ and $\partial^2 q_2(\cdot,0)$ are independent of $t$. Hence by continuity of $\partial^2 q_1(\cdot,0), \partial^2 q_2(\cdot,0)$ and the arbitrariness of $x_0 \in \Omega$, we conclude $\partial^2 q_1(\cdot,0) = \partial^2 q_2(\cdot,0)$ in $\Omega$, which proves theorem 1.2 for $m = 2$.

### 3.4. Higher order linearization

For each $2 \leq p \leq m$, we denote $\partial_{(p)} = \partial_{\epsilon_1} \ldots \partial_{\epsilon_p}$. By repeating formal differentiations to the equation (3.13), we obtain the following $p$th order linearization
\[
\begin{cases}
(\partial_t + (-\Delta)^{\alpha}) \partial_{(p)} u^{\epsilon}_{p} + \partial_{(p)} q_1(\cdot, u^{\epsilon}_{p}) = 0 & \text{in } \Omega_T, \\
\partial_{(p)} u^{\epsilon}_{p} = 0 & \text{in } \Omega_T^\infty \text{ and on } \{0\} \times \mathbb{R}^n.
\end{cases}
\]
where we simply denote $\partial_{(p)} = \partial_{\epsilon_1} \ldots \partial_{\epsilon_p}$. By induction, we can verify
\[
\partial_{(p)} q_1(\cdot, u^{\epsilon}_{p}) = \partial_{(p)} q_1(\cdot, u^{\epsilon}_{p}) \partial_{(p)} u^{\epsilon}_{p} + \sum_{\ell=2}^{p-1} \partial_{(p)} q_1(\cdot, u^{\epsilon}_{p}) T_{\ell}^p(u^{\epsilon}_{p}) + \partial_{(p)} q_1(\cdot, u^{\epsilon}_{p}) \prod_{\ell=1}^{p} \partial_{\epsilon_\ell} u^{\epsilon}_{\ell},
\]
where $T_{\ell}^p(u^{\epsilon}_{p})$ is a generic notation (in order $p$ linearization) signifying a combination of the terms $\partial_{(\alpha)} u^{\epsilon}_{\ell}$ with multi-index $\alpha$ satisfying $1 \leq |\alpha| \leq p - 1$. The following facts can be proved using strong induction on $m$:

1. Functions $\partial_{(p)} u^{\epsilon}_{p} \in L^\infty(0, T; H^p(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n_T)$ are well-defined for each $2 \leq p \leq m$. 

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(2) There exists \( \epsilon_0 = \epsilon_0(n,s,\Omega,T,\delta,g,m) > 0 \) with \( 0 < \epsilon_0 < \bar{\epsilon}_0 \), where \( \bar{\epsilon}_0 \) is the constant given in proposition 2.10, such that

\[
\lim_{\epsilon \to 0} \| \partial_x \epsilon f^\epsilon - \partial_x \epsilon g^\epsilon \|_{L^\infty(\Omega;H^T(\mathbb{R}^n))} = 0 \quad \text{for all} \quad 2 \leq p \leq m,
\]

where \( \epsilon \), \( \epsilon g^\epsilon \), and \( \epsilon = g^\epsilon \) denote the solution of the fractional diffusion equation proved in proposition 2.10.

Moreover, if \( \epsilon \) is the solution of the fractional diffusion equation proved in proposition 2.10, provided \( |\epsilon| + |\epsilon_p| < \epsilon_0 \), we have

\[
(-\Delta)^s \partial_x \epsilon f^\epsilon |_{V_T} = (-\Delta)^s \partial_x \epsilon g^\epsilon |_{V_T} \quad \text{for all} \quad 2 \leq p \leq m. \tag{3.20}
\]

Using (3.10), we see that

\[
\partial_x \epsilon f^\epsilon |_{e=0} = \partial_x \epsilon g^\epsilon |_{e=0} + \sum_{\ell=2}^{m-1} \partial_x \epsilon f^\epsilon |_{e=0} + \partial_x \epsilon g^\epsilon |_{e=0} + \partial_x \epsilon u^\epsilon |_{e=0}.
\]

Using (3.21), we see that \( \sum_{\ell=2}^{m-1} \partial_x \epsilon f^\epsilon |_{e=0} + \partial_x \epsilon g^\epsilon |_{e=0} + \partial_x \epsilon u^\epsilon |_{e=0} \) satisfies

\[
\left\{ \begin{array}{l}
(-\Delta)^s v + (\partial_x \epsilon f^\epsilon |_{e=0} - \partial_x \epsilon g^\epsilon |_{e=0}) \prod_{\ell=1}^{m} \partial_x \epsilon u^\epsilon |_{e=0} = 0 \\
\epsilon = 0
\end{array} \right. \quad \text{in} \quad \Omega_T \quad \text{and on} \quad \{0\} \times \mathbb{R}^n.
\]

Using (3.20), we know that \( (-\Delta)^s v |_{V_T} = 0 \). Since \( \epsilon = 0 \) in \( V_T \), by using the unique continuation principle of the fractional Laplacian (see lemma 3.1), we conclude that \( \epsilon = 0 \). Hence, we know that

\[
(\partial_x \epsilon f^\epsilon |_{e=0} - \partial_x \epsilon g^\epsilon |_{e=0}) \prod_{\ell=1}^{m} \partial_x \epsilon u^\epsilon |_{e=0} = 0 \quad \text{in} \quad \Omega_T. \tag{3.22}
\]

Since \( g_1, \ldots, g_m \in C^\infty_c(W_T) \) are arbitrary, using (3.9) and the Runge approximation for the fractional diffusion equation proved in proposition 3.2, we conclude \( \partial_x \epsilon f^\epsilon |_{e=0} = \partial_x \epsilon g^\epsilon |_{e=0} \) in \( \Omega_T \). This completes the proof of theorem 1.1.

**Proof of theorem 1.2.** We fix \( g_1, g_2, \ldots, g_m \in C^\infty_c(W_T) \) and assume the hypothesis (3.21). Under this hypothesis, by using the same arguments as above, we reach (3.22). Similar to (3.19), one reaches

\[
\partial_x \epsilon u^\epsilon |_{e=0}(u,x) \neq 0 \quad \text{for all} \quad j = 1, 2, \ldots, m.
\]

Since both \( \partial_x \epsilon f^\epsilon |_{e=0} \) and \( \partial_x \epsilon g^\epsilon |_{e=0} \) are both independent of \( \epsilon \), then we conclude theorem by the continuity of \( \partial_x \epsilon f^\epsilon |_{e=0} \) and \( \partial_x \epsilon g^\epsilon |_{e=0} \).
4. Analogous result for the fractional wave equation

The following results can be proved by modifying the ideas in [KLW22, corollary 2.2] (or [Eva10, chapter 7]), which we use later to prove the well-posedness of (1.6) with small exterior data and to solve inverse problem as well.

**Lemma 4.1.** Given any \( n \in \mathbb{N} \) and \( 0 < s < 1 \). Let \( \Omega \subset \mathbb{R}^n \) be a bounded Lipschitz domain in \( \mathbb{R}^n \), let \( W \subset \Omega' \) be any open set with Lipschitz boundary satisfying \( W \cap \Omega = \emptyset \). Let \( a \in L^\infty(\Omega') \). Then for any \( F \in L^2(\Omega') \), \( f \in C_c(\mathbb{R}) \), \( \psi \in \dot{H}^0(\Omega) \), \( \phi \in \dot{H}^1(\Omega) \), there exists a unique solution \( u \) of

\[
\begin{aligned}
(\partial_t^2 + (-\Delta)^s + a)u &= F \quad \text{in } \Omega_T, \\
u &= f \quad \text{in } \Omega_T', \\
u &= \phi, \quad \partial_t u = \psi \quad \text{on } \{0\} \times \mathbb{R}^n,
\end{aligned}
\]

(4.1)

satisfying

\[
\|u - f\|_{L^\infty(0,T;H^s(\mathbb{R}^n))} + \|\partial_t u\|_{L^\infty(0,T;L^2(\Omega))} \\
\leq C(\|\phi\|_{L^2(\Omega')} + \|\psi\|_{L^2(\Omega')} + \|\psi\|_{L^2(\Omega)} + \|F - (-\Delta)^s f\|_{L^2(\Omega)})
\]

(4.2)

for some constant \( C = C(n,s,T,\|a\|_{L^\infty(\Omega')} ) \).

**Remark 4.1.** It is interesting to compare (4.2) with (2.8): both solutions (wave and diffusion) have regularity \( L^\infty(0,T;H^s(\mathbb{R}^n)) \). In [KLW22, corollary 2.2], they only consider the case when \( a \) is independent of time \( t \). The existence of solutions can be proved using exactly the same argument, but the proof of uniqueness result need extra care. In contrast to [Eva10, chapter 7], here we do not assume the \( W^{1,\infty}(0,T;L^2(\Omega)) \) regularity for the coefficient \( a \).

**Proof of uniqueness result of lemma 4.1.** Let \( u \in L^2(0,T;H^s(\mathbb{R}^n)) \cap H^1(0,T;L^2(\Omega)) \) be the solution of

\[
\begin{aligned}
(\partial_t^2 + (-\Delta)^s + a)u &= 0 \quad \text{in } \Omega_T, \\
u &= 0 \quad \text{in } \Omega_T', \\
u &= \partial_t u = 0 \quad \text{on } \{0\} \times \mathbb{R}^n.
\end{aligned}
\]

(4.3)

We want to show that \( u \equiv 0 \). Fix \( 0 \leq \eta \leq T \) and set

\[
v(t,\cdot) := \begin{cases}
\int_0^\eta u(t,\cdot) \, d\tau & \text{if } 0 \leq t \leq \eta, \\
0 & \text{if } \eta \leq t \leq T.
\end{cases}
\]

Then \( v(t,\cdot) \in \dot{H}^s(\Omega) \) for each \( 0 \leq t \leq T \), and so

\[
\int_{\Omega} \int_0^\eta (\partial_t^2 u)v \, d\tau \, dx + \int_{\mathbb{R}^n} \int_0^\eta (-\Delta)^s u(-\Delta)^s v \, d\tau \, dx + \int_{\Omega} \int_0^\eta anu \, d\tau \, dx = 0.
\]

(4.4)

Since \( \partial_t u(0) = v(0) = 0 \) and \( \partial_t v = -u \) for all \( 0 \leq t \leq \eta \), we see that

\[
\int_{\Omega} \int_0^\eta (\partial_t^2 u)v \, d\tau \, dx = -\int_{\Omega} \int_0^\eta (\partial_t u)(\partial_t v) \, d\tau \, dx = \int_{\Omega} \int_0^\eta (\partial_t u)u \, d\tau \, dx
\]

\[
= \frac{1}{2} \frac{d}{dt} \int_{\Omega} \int_0^\eta |u(\eta,\cdot)|^2 \, d\tau \, dx = \frac{1}{2} \|u(\eta,\cdot)\|_{L^2(\Omega)}^2.
\]

(4.5)
Using the fact $\partial_t v = -u$ for all $0 \leq t \leq \eta$, we also have
\[
\int_{\mathbb{R}^2} \int_0^\eta (-\Delta)^{\frac{1}{2}} u (-\Delta)^{\frac{1}{2}} v \, dx \, dt = - \int_{\mathbb{R}^2} \int_0^\eta \partial_t((-\Delta)^{\frac{1}{2}} v) (-\Delta)^{\frac{1}{2}} v \, dx \, dt \\
= - \frac{1}{2} \int_{\mathbb{R}^2} \int_0^\eta \frac{d}{dt} ((-\Delta)^{\frac{1}{2}} v)^2 \, dx \, dt \leq \frac{1}{2} \|(-\Delta)^{\frac{1}{2}} v(0, \cdot)\|_{H^1(\mathbb{R}^2)}^2.
\] (4.6)

Since $a \in L^\infty(\Omega_T)$, combining (4.4)–(4.6), together with the Hardy–Littlewood–Sobolev inequality (A.5), we obtain
\[
\|v(0, \cdot)\|_{H^1(\Omega)}^2 + \|u(t, \cdot)\|_{L^2(\Omega)}^2 \leq C \int_0^\eta \left( \|v(t, \cdot)\|_{L^2(\Omega)}^2 + \|u(t, \cdot)\|_{L^2(\Omega)}^2 \right) \, dt.
\] (4.7)

Let us write $w(t, \cdot) := \int_0^t u(t, \cdot) \, dt$. Since $v(0, \cdot) = w(\eta, \cdot)$ and $v(t, \cdot) = w(\eta, \cdot) - w(t, \cdot)$, from (4.7) we know that
\[
\|w(\eta, \cdot)\|_{H^1(\Omega)} + \|u(t, \cdot)\|_{L^2(\Omega)} \leq \int_0^\eta \left( \|w(\eta, \cdot) - w(t, \cdot)\|_{L^2(\Omega)}^2 + \|u(t, \cdot)\|_{L^2(\Omega)}^2 \right) \, dt \\
\leq \eta \|w(\eta, \cdot)\|_{L^2(\Omega)} + \int_0^\eta \left( \|w(t, \cdot)\|_{L^2(\Omega)}^2 + \|u(t, \cdot)\|_{L^2(\Omega)}^2 \right) \, dt.
\]

Therefore, we can choose $T_1$, which is independent of $\eta$, such that
\[
\|w(\eta, \cdot)\|_{H^1(\Omega)} + \|u(t, \cdot)\|_{L^2(\Omega)} \leq C \int_0^\eta \left( \|w(t, \cdot)\|_{L^2(\Omega)}^2 + \|u(t, \cdot)\|_{L^2(\Omega)}^2 \right) \, dt,
\]
for all $0 \leq \eta \leq T_1$. Using the Grönwall’s inequality in [Eva10, section B.2], we know that $u(t, \cdot) = 0$ for all $t \in [0, T_1]$. Applying the same argument on the intervals $[T_1, 2T_1], [2T_1, 3T_1], \ldots$ we conclude that $u \equiv 0$. \hfill $\square$

We need the following Sobolev embedding to obtain $L^\infty(\Omega_T)$-regularity of the solution:

**Lemma 4.2.** (A special case of [DNPV12, theorem 8.2]) Let $n = 1$ and $1/2 < s < 1$. There exists a constant $C = C(s, \Omega)$ such that
\[
\|f\|_{L^2(\Omega)} \leq C(\|\partial_t f\|_{L^2(\Omega)}^2 + \|f\|_{H^1(\Omega)}^2) = C\|f\|_{H^1(\mathbb{R}^3)}^2,
\]
for any $f \in L^2(\Omega)$ with $\alpha = (2s - 1)/2$.

Therefore, lemma 4.1 implies the following result.

**Proposition 4.3.** Let $n = 1$ and $1/2 < s < 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set in $\mathbb{R}^n$, let $W \subset \Omega$ be any open set satisfying $\overline{W} \cap \overline{\Omega} = \emptyset$. Let $a \in L^\infty(\Omega_T)$. Then for any $F \in L^2(\Omega_T)$ and $f \in C_c^\infty(W_T)$, there exists a unique weak solution $u$ of (4.1) satisfying
\[
\|u\|_{L^\infty(0, T; H^s(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3))} + \|\partial_t u\|_{L^\infty(0, T; L^2(\Omega_T))} \\
\leq C(\|\partial_t \varphi\|_{H^s(\mathbb{R}^3)} + \|\varphi\|_{L^2(\Omega_T)} + \|F\|_{L^2(\Omega_T)} \\
+ \|f\|_{L^\infty(0, T; H^s(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3))} + \|(-\Delta)^{\frac{1}{2}} f\|_{L^2(\Omega_T)})
\]
for certain constant $C = C(s, T, \|a\|_{L^\infty(\Omega_T)}, \Omega)$.

Using the same argument as in proposition 2.10, we also can prove the well-posedness of (1.6) for small exterior data:
Proposition 4.4. Let \( n = 1 \) and \( \frac{1}{2} < s < 1 \). Let \( \Omega \subset \mathbb{R} \) be a bounded open set, let \( W \subset \Omega^c \) be any open set satisfying \( W \cap \Omega = \emptyset \). Fix any parameter \( \delta > 0 \). Assume \( q \) satisfies (Q.1)–(Q.3). There exists a sufficiently small parameter \( \bar{c}_0 = \bar{c}_0(s, \Omega, \delta) > 0 \) such that the following statement holds: Given any \( f \in C^\infty_0(W_T) \) with \( \|f\|_{C^0} \leq \bar{c}_0 \), there exists a unique solution \( u \in L^\infty(0, T; H^s) \cap L^\infty(W_T) \) of (1.6) with
\[
\|u\|_{L^\infty(0, T; H^s)} \leq C\|f\|_{C^0}
\]
for certain constant \( C = C(s, T, \Omega) \).

Finally, the inverse problem for the nonlinear fractional wave equation (1.6), i.e. theorem 1.3 and theorem 1.4, can be proved using exactly the same idea as in theorem 1.1 and theorem 1.2, respectively, see section 3.

Data availability statement
No new data were created or analysed in this study.

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Appendix A. Well-posedness of the linear fractional diffusion equation

A.1. Uniqueness of weak solution
We first prove the uniqueness of weak solution of (2.3) as well as (2.4). It suffices to prove the following statement: if \( u \) a weak solution of
\[
\begin{cases}
(\partial_t + (-\Delta)^s + a)v = 0 & \text{in } \Omega_T, \\
v = 0 & \text{in } \Omega^c_T \text{ and on } \{0\} \times \mathbb{R}^n.
\end{cases}
\] (A.1)
then \( u \equiv 0 \). Multiplying the first equation of (A.1) by \( v \), we obtain
\[
0 = \langle \dot{v}, v \rangle + \mathcal{B}[v, v; l] = \frac{d}{dt} \left( \frac{1}{2} \|v(t)\|_{L^2(\Omega)}^2 \right) + \mathcal{B}[v, v; l] \geq \frac{d}{dt} \left( \frac{1}{2} \|v(t)\|_{L^2(\Omega)}^2 \right) - \|a\|_{L^\infty(\Omega)} \|v(t)\|_{L^2(\Omega)}^2,
\]
that is, \( \frac{d}{dt}(\|v(t)\|_{L^2(\Omega)}^2) \leq 2\|a\|_{L^\infty(\Omega)} \|v(t)\|_{L^2(\Omega)}^2 \). Using the Grönwall’s inequality in [Eva10, section B.2], we conclude \( \|v(t)\|_{L^2(\Omega)}^2 = 0 \) for all \( 0 \leq t \leq T \), hence \( u \equiv 0 \). The uniqueness is proved.

A.2. Existence of weak solution
Now it suffices prove that there exists a weak solution of (2.4).

Step 1: Galerkin approximation. We now set up the Galerkin approximation for (2.4). Similar to [KLW22, appendix A], we consider an eigenbasis \( \{w_k\}_{k \in \mathbb{N}} \) associated with the
Dirichlet fractional Laplacian in a bounded domain $\Omega$. We normalize these eigenfunctions so that
\[
\{w_k\}_{k \in \mathbb{N}} \text{ be an orthogonal basis in } \tilde{H}^s(\Omega),
\]
\[
\{w_k\}_{k \in \mathbb{N}} \text{ be an orthonormal basis in } L^2(\Omega).
\]

Given any fixed integer $m \in \mathbb{N}$, we consider the following ansatz:
\[
v_m(t) := \sum_{k=1}^{m} d_k^m(t)w_k. \quad (A.2)
\]

Plugging the ansatz (A.2) into definition 2.1(b), we obtain
\[
\begin{cases}
(v'_m(t), w_k)_{L^2(\Omega)} + B[v_m, w_k; \tilde{t}] = (\tilde{F}(t), w_k)_{L^2(\Omega)} & \text{for all } 0 \leq t \leq T,

d_k^m(0) = (\tilde{\varphi}, w_k)_{L^2(\Omega)}.
\end{cases} \quad (A.3)
\]

Note that $(v'_m(t), w_k)_{L^2(\Omega)} = (d_k^m)'(t)$, $B[v_m, w_k; \tilde{t}] = \sum_{j=1}^{m} v^j(t)d_k^m(t)$ with the coefficients $e^j(t) := B[w_i, w_k; \tilde{t}]$. This shows that $d_k^m(t)$ satisfies the following linear system of ordinary differential equation (ODE):
\[
\begin{cases}
(d_k^m)'(t) + \sum_{j=1}^{m} e^j(t)d_k^m(t) = (\tilde{F}(t), w_k)_{L^2(\Omega)} & \text{for all } 0 \leq t \leq T,

d_k^m(0) = (\tilde{\varphi}, w_k)_{L^2(\Omega)}.
\end{cases}
\]

Therefore, the standard ODE theory guarantees the existence and uniqueness of such $d_k^m(t)$, and thus (A.2) is a valid discretization of (2.4).

**Step 2: energy estimate.** Multiplying (A.3) by $d_k^m(t)$, and summing over index $k = 1, \ldots, m$, we have
\[
(v'_m, v_m)_{L^2(\Omega)} + B[v_m, v_m; \tilde{t}] = (\tilde{F}, v_m)_{L^2(\Omega)}. \quad (A.4)
\]

The following Hardy–Littlewood–Sobolev inequality can be found in [Pon16, proposition 15.5] or in [KLW22, equation (A.11)]:
\[
\|v_m\|_{L^p(\mathbb{R}^n)} \leq \|v_m\|_{L^2(\Omega)} \leq C(n, s)\|\phi\|_{L^\infty(\mathbb{R}^n)} \leq C(n, s)\|(-\Delta)^{s}\phi\|_{L^2(\mathbb{R}^n)} \quad (A.5)
\]
for $n = 1$ and for all $\phi \in \tilde{H}^s(\Omega)$. On the other hand, we observe that $(v'_m, v_m)_{L^2(\Omega)} = \int \frac{d}{dt} \left( \frac{1}{2} \|v_m\|^2_{L^2(\Omega)} \right)$. Hence, from (A.4) we have
\[
\frac{d}{dt} \left( \frac{1}{2} \|v_m\|^2_{L^2(\Omega)} \right) + \|v_m\|^2_{L^2(\Omega)} \leq C(n, s, x)\|a\|_{L^\infty(\Omega^s)} \left( \|v_m\|^2_{L^2(\Omega)} + \|\tilde{F}\|^2_{L^2(\Omega)} \right) \quad (A.6)
\]
for all $0 \leq t \leq T$. Using the Grönwall’s inequality in [Eva10, section B.2], we have
\[
\|v_m(t)\|^2_{L^2(\Omega)} \leq e^{Ct} \left( \|v_m(0)\|^2_{L^2(\Omega)} + C \int_0^T \|\tilde{F}(s)\|^2_{L^2(\Omega)} \, ds \right) \quad \text{for all } 0 \leq t \leq T.
\]

Since $\|v_m(0)\|^2_{L^2(\Omega)} = \sum_{k=1}^{m} \|\tilde{\varphi}, w_k\|^2_{L^2(\Omega)} \leq \sum_{k=1}^{\infty} \|\tilde{\varphi}, w_k\|^2_{L^2(\Omega)} = \|\varphi\|^2_{L^2(\Omega)}$, then we have
\[
\sup_{0 \leq t \leq T} \|v_m(t)\|^2_{L^2(\Omega)} \leq C_{x, T, \|a\|_{L^\infty(\Omega^s)}} \left( \|\varphi\|^2_{L^2(\Omega)} + \|\tilde{F}\|^2_{L^2(\Omega)} \right). \quad (A.7)
\]

Integrating (A.6) on $t \in [0, T]$, we obtain
\[
\|v_m\|^2_{L^2(0, T; L^2(\Omega))} \leq C_{x, T, \|a\|_{L^\infty(\Omega^s)}} \left( \|v_m\|^2_{L^2(\Omega^s)} + \|\tilde{F}\|^2_{L^2(\Omega^s)} \right). \quad (A.8)
\]
Combining (A.7) and (A.8), we obtain the following energy estimate:
\[
\sup_{0 \leq t \leq T} \|v_m(t)\|_{L^2(\Omega)}^2 + \|v'_m(t)\|_{L^2(0,T;H^1(\Omega))}^2 \leq C(n,s,T,\|a\|_{L^\infty(\Omega)}) (\|\varphi\|_{L^2(\Omega)}^2 + \|\tilde{F}\|_{L^2(\Omega)}^2)
\] (A.9)

Fixing any \(\phi \in \tilde{H}'(\Omega)\) with \(\|\phi\|_{\tilde{H}'(\Omega)} \leq 1\), we write \(\phi = \phi_1 + \phi_2\), where \(\phi_1 \in \text{span} \{w_k\}_{k=1}^m\) and \((\phi_2, w_k)_{L^2(\Omega)} = 0\) for \(k = 1, \ldots, m\). Using (A.3), we see that
\[
(v'_m(t), \phi)_{L^2(\Omega)} = (v'_m(t), \phi_1)_{L^2(\Omega)} = (\tilde{F}, \phi_1) - B[v_m, \phi_1; t].
\]
Since \(\|\phi_1\|_{\tilde{H}'(\Omega)} \leq 1\), this implies \(|(v'_m(t), \phi)_{L^2(\Omega)}| \leq C(\|\tilde{F}(t)\|_{L^2(\Omega)}^2 + \|v'_m\|_{P(\Omega)}^2)\). Hence we know that
\[
\|v'_m(t)\|_{H^{-1}(\Omega)}^2 := \sup_{\|\phi\|_{\tilde{H}'(\Omega)} \leq 1} |(v'_m(t), \phi)_{L^2(\Omega)}| \leq C_{\|u\|_{\infty}} (\|\tilde{F}(t)\|_{L^2(\Omega)}^2 + \|v'_m\|_{P(\Omega)}^2).
\]

Integrating the inequality above on \(t \in [0,T]\), and combining the result with (A.9), we obtain
\[
\sup_{0 \leq t \leq T} \|v'_m(t)\|_{L^2(\Omega)}^2 + \|v'_m\|_{L^2(0,T;H^1(\Omega))}^2 + \|v'_m\|_{L^2(0,T;H^{-1}(\Omega))}^2
\]
\[
\leq C_{s,T,\|u\|_{\infty}} (\|\varphi\|_{L^2(\Omega)}^2 + \|\tilde{F}\|_{L^2(\Omega)}^2).
\] (A.10)

**Step 3: passing to the limit.** By (A.10), we can extract a subsequence of \(\{v_m\}_{m \in \mathbb{N}}\) (for simplicity), such that
\[
\begin{align*}
v_m &\to v \text{ weakly in } L^2(0,T;\tilde{H}'(\Omega)), \\
v'_m &\to v' \text{ weakly in } L^2(0,T;H^{-1}(\Omega)).
\end{align*}
\] (A.11)

Given any fixed integer \(N\), we write \(\tilde{v}(t) := \sum_{k=1}^N d^k(t)w_k\), where \(d^k(t) (k = 1, \ldots, N)\) are arbitrary smooth functions (not the one in (A.2)). Choosing \(m \geq N\), multiplying (A.3) by \(d^k(t)\), and summing over \(k = 1, \ldots, N\), we obtain
\[
\int_0^T ((v'_m(t), \tilde{v}(t))_{L^2(\Omega)} + B[v_m, \tilde{v}; t]) \, dt = \int_0^T (\tilde{F}(t), \tilde{v}(t))_{L^2(\Omega)} \, dt.
\] (A.12)

Taking \(m \to +\infty\) in (A.12), and from (A.11), we know that
\[
\int_0^T ((v'(t), \tilde{v}(t)) + B[v, \tilde{v}; t]) \, dt = \int_0^T (\tilde{F}(t), \tilde{v}(t))_{L^2(\Omega)} \, dt.
\] (A.13)

Due to the arbitrariness of \(N\) and \(d^k\) for \(k = 1, \ldots, N\), we have
\[
\langle v', \phi \rangle + B[v, \phi; t] = (\tilde{F}(t), \phi)_{L^2(\Omega)} \text{ for all } \phi \in \tilde{H}'(\Omega).
\]
This together with (A.10) verifies definition 2.1(a) and (b).

It remains to show \(v\) verifies definition 2.1(c). To that end, let us choose any \(\tilde{v} \in C^1(0,T;\tilde{H}'(\Omega))\) with \(\tilde{v}(T) = 0\). From (A.13), we have
\[
\int_0^T ((\tilde{v}'(t), v(t))_{L^2(\Omega)} + B[v, \tilde{v}; t]) \, dt = \int_0^T (\tilde{F}(t), \tilde{v}(t))_{L^2(\Omega)} \, dt + (\tilde{v}'(0), v(0))_{L^2(\Omega)}.
\] (A.14)

Similarly, from (A.12), we have
\[
\int_0^T ((\tilde{v}'(t), v_m(t))_{L^2(\Omega)} + B[v_m, \tilde{v}; t]) \, dt = \int_0^T (\tilde{F}(t), \tilde{v}(t))_{L^2(\Omega)} \, dt + (\tilde{v}'(0), v_m(0))_{L^2(\Omega)}.
\] (A.15)

Combining (A.11) and (A.15), we obtain
\[
\int_0^T ((v'(t), v(t))_{L^2(\Omega)} + B[v, \tilde{v}; t]) \, dt = \int_0^T (\tilde{F}(t), \tilde{v}(t))_{L^2(\Omega)} \, dt + (v'(0), \varphi)_{L^2(\Omega)}.
\]
Comparing this with (A.14), we see that \((\tilde{v}'(0), v(0))_{\mathcal{L}^2(\Omega)} = (\tilde{v}'(0), \varphi)_{\mathcal{L}^2(\Omega)}\). Due to the arbitrariness of \(\tilde{v}\), we conclude that \(v\) verifies definition 2.1(c).

**Step 4. Higher regularity.** We now further assume \(\varphi \in \mathcal{H}'(\Omega)\). Multiplying (A.3) by \((d_m^k)'(t)\), and summing over \(k = 1, \ldots, m\), we have

\[
(v_m', v_m')_{\mathcal{L}^2(\Omega)} + B[v_m', v_m'] = (\tilde{F}, v_m')_{\mathcal{L}^2(\Omega)}. \tag{A.16}
\]

Note that we have

\[
B[v_m', v_m'] = \int_{\mathbb{R}^n} (-\Delta)^\frac{j}{2} v_m(t)(-\Delta)^\frac{j}{2} v_m'(t) \, dx + \int_{\Omega} a(t, x)v_m(t, x)v_m'(t, x) \, dx
\]

and

\[
\int_{\Omega} a(t, x)v_m(t)v_m'(t) \, dx \leq \varepsilon \|v_m(t)\|^2_{\mathcal{L}^2(\Omega)} + C\varepsilon^{-1} \|v_m(t)\|^2_{\mathcal{L}^2(\Omega)} \quad \text{and} \quad |(\tilde{F}, v_m')_{\mathcal{L}^2(\Omega)}| \leq \varepsilon \|v_m(t)\|^2_{\mathcal{L}^2(\Omega)} + C\varepsilon^{-1} \|\tilde{F}\|^2_{\mathcal{L}^2(\Omega)}.
\]

These along with (A.16) imply

\[
\|v_m'(t)\|^2_{\mathcal{L}^2(\Omega)} + \frac{d}{dt} \left( \frac{1}{2} \int_{\mathbb{R}^n} |(-\Delta)^\frac{j}{2} v_m(t)|^2 \, dx \right) \leq 2\varepsilon \|v_m(t)\|^2_{\mathcal{L}^2(\Omega)} + C\varepsilon^{-1} \|v_m(t)\|^2_{\mathcal{L}^2(\Omega)} + C\varepsilon^{-1} \|\tilde{F}(t)\|^2_{\mathcal{L}^2(\Omega)}.
\]

Choosing \(\varepsilon = 1/4\), we obtain

\[
\|v_m'(t)\|^2_{\mathcal{L}^2(\Omega)} + \frac{d}{dt} \left( \frac{1}{2} \int_{\mathbb{R}^n} |(-\Delta)^\frac{j}{2} v_m(t)|^2 \, dx \right) \leq C(\|v_m(t)\|^2_{\mathcal{L}^2(\Omega)} + \|\tilde{F}(t)\|^2_{\mathcal{L}^2(\Omega)}). \tag{A.17}
\]

Given any \(0 \leq \tilde{t} \leq T\), we integrate (A.17) on \(t \in [0, \tilde{t}]\),

\[
\int_0^{\tilde{t}} \|v_m(t)\|^2_{\mathcal{L}^2(\Omega)} \, dt + \frac{1}{2} \int_{\mathbb{R}^n} |(-\Delta)^\frac{j}{2} v_m(\tilde{t})|^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^n} |(-\Delta)^\frac{j}{2} v_m(0)|^2 \, dx \leq C \left( \int_0^{\tilde{t}} \|v_m(t)\|^2_{\mathcal{L}^2(\Omega)} \, dt + \int_0^{\tilde{t}} \|\tilde{F}(t)\|^2_{\mathcal{L}^2(\Omega)} \right) \leq C \left( \|v_m(\tilde{t})\|^2_{\mathcal{L}^2(\Omega)} + \|\tilde{F}(\tilde{t})\|^2_{\mathcal{L}^2(\Omega)} \right).
\]

Combining this inequality with (A.5), we obtain

\[
\|v_m(\tilde{t})\|^2_{\mathcal{L}^2(\Omega)} + \|v_m\|_{L^\infty(0, T; \mathcal{H}(\Omega))}^2 \leq C \left( \|v_m'(0)\|^2_{\mathcal{L}^2(\Omega)} + \sup_{0 \leq t \leq T} \int_{\mathbb{R}^n} |(-\Delta)^\frac{j}{2} v_m(t)|^2 \, dx \right)
\]

\[
\leq C \left( \int_{\mathbb{R}^n} |(-\Delta)^\frac{j}{2} v_m(0)|^2 \, dx + \|v_m(\tilde{t})\|^2_{\mathcal{L}^2(\Omega)} + \|\tilde{F}(\tilde{t})\|^2_{\mathcal{L}^2(\Omega)} \right) \leq C(\|v_m(0)\|^2_{\mathcal{H}(\Omega)} + \|v_m\|^2_{L^\infty(0, T; \mathcal{H}(\Omega))} + \|\tilde{F}\|^2_{\mathcal{L}^2(\Omega)}). \tag{A.18}
\]

Since \(\|v_m(0)\|^2_{\mathcal{H}(\Omega)} \leq \sum_{k=1}^{\infty} \|(g, w_k)_{\mathcal{L}^2(\Omega)}\|^2_{\mathcal{H}(\Omega)} = \|\varphi\|^2_{\mathcal{H}(\Omega)}\), (A.18) implies

\[
\|v_m'(\tilde{t})\|^2_{\mathcal{L}^2(\Omega)} + \|v_m\|^2_{L^\infty(0, T; \mathcal{H}(\Omega))} \leq C(\|\varphi\|^2_{\mathcal{H}(\Omega)} + \|v_m\|^2_{L^2(\Omega)} + \|\tilde{F}\|^2_{\mathcal{L}^2(\Omega)}).
\]

Therefore, combining this inequality with (A.10), we obtain

\[
\|v_m'(\tilde{t})\|^2_{\mathcal{L}^2(\Omega)} + \|v_m\|^2_{L^\infty(0, T; \mathcal{H}(\Omega))} \leq C(\|\varphi\|^2_{\mathcal{H}(\Omega)} + \|\tilde{F}\|^2_{\mathcal{L}^2(\Omega)}).
\]

Finally, taking the limit \(m \to \infty\), we complete our proof.
Appendix B. Some discussions

The main difficulty in proving theorem 1.3 is the regularity of the solutions. Due to this difficulty, we are only able to prove theorem 1.3 in one dimension. The method we used requires the $L^\infty(\Omega_T)$-regularity for the linear fractional wave equation. The $L^\infty(\Omega_T)$-regularity is required to guarantee the well-posedness of (1.6), and it is essential to prove that the linearization is well-defined as we see in section 2. However, we are only able to obtain this regularity in the case when $n=1$ and $\frac{1}{2} < s < 1$. If one can prove the well-posedness of (1.6) for general $n \in \mathbb{N}$ and $0 < s < 1$, then theorem 1.3 immediately extends for general $n \in \mathbb{N}$ and $0 < s < 1$.

In view of standard elliptic regularity results (see [GT01] or [JLS17, proposition A.1] in terms of other norms) as well as Sobolev embedding, an attempt to improve the result in theorem 1.3 is to try to obtain the $L^\infty(0; T; H^{2s}(\mathbb{R}^n))$ regularity for the solution. However, this idea is less likely to be feasible. Using [GSU20, lemma 2.3], we know there exists a unique solution $w_{\tilde{H}} \in C^s(\Omega)$ of

\[
\left\{
\begin{array}{ll}
(-\Delta)^s w = F & \text{in } \Omega, \\
w = 0 & \text{in } \Omega^c,
\end{array}
\right.
\]  

(B.1)

for $F \in H^{-s}(\Omega)$. Choose $\Omega$ to be the unit disk and $F$ to be a positive constant in $\Omega$. Then the best regularity result of (B.1) we know is $C^s(\mathbb{R}^n)$ [RO16, proposition 7.2]. In fact, [RO16, lemma 5.4] gives an explicit solution $w(x) = (1 - |x|^2)^s \in C^s(\mathbb{R}^n)$, and such $w$ does not belong to $C^{s'}(\mathbb{R}^n)$ for any $s' > s$. When $n = 1$, we have the continuous embedding $H^{2s}(\mathbb{R}) \hookrightarrow C^{2s-1}(\mathbb{R})$. Therefore, at least when $n = 1$ and $s > \frac{1}{2}$, such a solution $w$ of (B.1) cannot be in $H^{2s}(\mathbb{R})$.

ORCID iDs

Pu-Zhao Kow https://orcid.org/0000-0002-2990-3591
Shiqi Ma https://orcid.org/0000-0002-0192-493X
Suman Kumar Sahoo https://orcid.org/0000-0002-6459-1597

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