Beyond the Horizon

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Abstract

Cosmic horizons arise in general relativity in the context of black holes and in certain cosmologies. Classically, regions beyond a horizon are inaccessible to causal observers. However, quantum mechanical correlations may exist across horizons that may influence local observations. For the case of de Sitter space, we show how a single particle excitation behind the horizon changes the density matrix governing local observables. As compared to the vacuum state, we calculate the change in the average energy and entropy per unit volume. This illustrates what may be a generic property allowing some features of spacetime beyond a horizon to be inferred.

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1 Introduction

The analogies between thermodynamic concepts and black hole parameters have posed numerous puzzles for quantum field theory (QFT) in curved spacetime. In particular, one would like to understand whether there is a statistical mechanical origin of the Beckenstein-Hawking entropy associated with the surface gravity or horizon of a black hole \[1\]. This issue is complicated further by the fact that the apparent horizon is frame dependent, so that the associated entropy is also observer dependent. Usually, discussions implicitly make reference to stationary observers at infinity, and this frame dependence is ignored. Similarly, Hawking's black hole information paradox\[2\] deals with the changes experienced by external observers who can measure black hole radiation. The challenges posed have led some to speculate that, ultimately, local QFT as we know, must break down even at distances large compared to the Planck length \[3\]. Indeed, the holographic principle suggests that quantum field theory in general has many redundant degrees of freedom. Similar paradoxes beset special reference frames or cosmological situations in which there exist horizons, for which different observers may describe things quite differently \[4\]. Such an observer dependence was recently discussed by comparing observers in Rindler and Minkowski reference frames \[5\].

Several approaches to these problems have illuminated the issues, while not resolving them completely. For eternal black holes, where there is a delicately arranged balance between incoming and outgoing radiation, a microscopic explanation is available, at least in some cases, in terms of Israel’s thermofield formulation\[6\]. Israel’s construction, which may seem a bit artificial for ordinary systems in flat spacetime, becomes natural for systems with horizons. This is conceptually quite satisfying, inasmuch as the global structure of the QFT is that of a unique state, whereas the entropy ascribed to an observer is associated with the inability to access states behind the horizon.

The case of de Sitter space in static coordinates can be put into one-to-one correspondence with Israel’s discussion of the Schwarzschild black hole \[7\], \[8\]. An observer in the southern diamond is classically blind to events occurring in the northern diamond. In this brief note, we will show that quantum correlations provide a window for this observer to peek into the northern diamond. In light of this result, we will reflect on some of the so-called failures of QFT in this context.

In the next section, we review the construction of the vacuum state for a scalar field in de Sitter background. We then consider the effects of a single particle excitation in the northern diamond on the mean energy (Section 3) and entropy (Section 4) associated with an observer in the southern diamond. We summarize our results and discuss some of their implications in Section 5.

2 Preliminaries

To be self-contained, we will review in this section the description of a scalar field in de Sitter background. We shall mostly follow the notation in ref. \[8\]. Four dimensional de Sitter space can be thought of an embedding in (4+1) dimensional Minkowskian space by a hyperboloid

\[ z^2_0 - z^2_1 - z^2_2 - z^2_3 - z^2_4 = -\alpha^2. \] (1)
\( \alpha \) is the de Sitter radius, which henceforth will be set to unity. To define a vacuum state for a free massless scalar field, one takes a mode expansion

\[ \phi = \sum_n [a_n \phi_n(x) + a_n^\dagger \phi_n^*(x)] \quad (2) \]

in terms of creation and annihilation operators with canonical commutation relations (CCR). The vacuum is then defined as a state destroyed by all the annihilation operators.

\[ a_n |\Omega\rangle = 0 \quad (3) \]

The two-point function is defined as

\[ G_{\Omega}(x, x') = \langle \Omega | \phi(x) \phi(x') | \Omega \rangle = \sum_n \phi_n(x) \phi_n^*(x) \quad (4) \]

The Euclidean counterpart of de Sitter space is a four-dimensional sphere \( S^4 \). There is a unique choice of wave-function \( \phi_n \) that yields a Green’s function nonsingular on \( S^4 \). Correspondingly, the two-point function obtained by the analytic continuation from the Euclidean sphere is associated with a unique state called the Euclidean vacuum \( |E\rangle \). But one can define different vacua with respect to different coordinate systems which appears equally ‘natural’ in them. For example, consider static coordinates in de Sitter space. Regarded as an embedding in five-dimensional Minkowskian space, they are defined in the southern diamond \( (0 < r < 1) \) as

\[
\begin{align*}
    z_0 &= (1 - r^2)^{1/2} \sinh(t) \\
    z_1 &= (1 - r^2)^{1/2} \cosh(t) \\
    z_2 &= r \sin \theta \cos \phi \\
    z_3 &= r \cos \theta \cos \phi \\
    z_4 &= r \cos \theta
\end{align*}
\]

The line element takes the form

\[ ds^2 = [1 - r^2]dt^2 - [1 - r^2]^{-1}dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2), \quad (6) \]

with the timelike Killing vector \( \partial_t \) defining time-evolution in the southern diamond. The Penrose diagram for de Sitter space is shown in Fig. 1 \[.\] These coordinates may be extended outside this causal diamond; for \( r > 1 \), the roles of \( t \) and \( r \) are reversed, and \( \partial_t \) becomes spacelike. These coordinates describe only half of de Sitter space, with the remaining half described by replacing \( t \) by \(-t\) in the formulas above. The Klein-Gordon equation in these coordinates is

\[ \left\{ -\frac{1}{1 - r^2} \partial_t^2 + \frac{1}{r^2} \partial_r (1 - r^2) r^2 \partial_r + \frac{1}{r^2 \sin \theta} \partial_\theta (\sin \theta \partial_\theta) + \frac{1}{r^2 \sin^2 \theta} \partial_\phi^2 - m^2 \right\} \Phi = 0 \quad (7) \]
It has a solution for a given energy $\omega$ and angular momentum quantum numbers $l, m$,

$$
\Phi_{S,\omega,lm} = f_{\omega,lm}(r)e^{-i\omega t}Y_{l,m}(\theta, \phi) \quad (8)
$$

$$
f_{\omega,lm}(r) = r^l(1-r^2)F(a, b, c, r^2)
$$

$$
a = \frac{1}{2}(l + i\omega + h_+),
$$

$$
b = \frac{1}{2}(l + i\omega + h_-),
$$

$$
c = l + \frac{3}{2}.
$$

$Y_{l,m}(\theta, \phi)$ are the spherical harmonics and the function $F(a, b, c, r^2)$ is a hypergeometric function. These form a complete set of states for solutions regular at $r = 0$. But they cannot be extended to the whole of de Sitter space. One can similarly define modes in the northern diamond as

$$
\Phi_{N,\omega,lm} = f_{\omega,lm}(r)e^{-i\omega t}Y_{l,m}(\theta, \phi). \quad (9)
$$

One can consider the mode expansions

$$
\Phi_S = \int_0^\infty d\omega \sum_{l,m} a_{S,\omega,lm} \Phi_{S,\omega,lm} + a_{S,\omega,lm}^\dagger \Phi_{S,\omega,lm}^* \quad \text{in } S \quad (10)
$$

$$
= 0 \quad \text{in } N
$$

$$
\Phi_N = \int_0^\infty d\omega \sum_{l,m} a_{N,\omega,lm} \Phi_{N,\omega,lm} + a_{N,\omega,lm}^\dagger \Phi_{N,\omega,lm}^* \quad \text{in } N \quad (11)
$$

$$
= 0 \quad \text{in } S.
$$

On any spacelike slice through the origin, the $\Phi_S$ and $\Phi_N$ together form a complete set of functions for all of de Sitter space. Therefore, we may take $\Phi(x) = \Phi_S(x) + \Phi_N(x)$. The creation and annihilation operators obey canonical commutation relations. Since modes in
one diamond have no support in the other, one can take the corresponding operators to commute,
\[
[a^\dagger_{N,\omega,l,m}, a^\dagger_{S,\nu,l'm'}] = 0.
\]
(12)

As indicated in Fig. 1, the coordinate \( t \) representing time runs in forward direction in the southern diamond and backwards in the northern one. So the modes in eq. (9) are of negative frequency, so that we must take \( a^\dagger_{N,\omega,l,m} \) as the annihilation operator rather than \( a_{N,\omega,l,m} \). The full Hamiltonian is therefore difference
\[
H = H_S - H_N \text{ (or vice versa.)}
\]
One may then define a vacuum state as
\[
|0\rangle_N = |0\rangle_{N,\omega,l,m}, |0\rangle_S = |0\rangle_{S,\omega,l,m}
\]
(13)

Henceforth, we shall simply abbreviate this vacuum, which differs from the Euclidean vacuum, as \( |0\rangle \) when reference to the quantum numbers is not required. Note that the associated Fock space is the direct product of the southern and northern Fock spaces, \( \mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_N \).

## 3 Energy change by particle excitation

Components of \( \Phi_N \) and \( \Phi_S \) in equations (10) and (11) together account for a complete set of modes in de Sitter space. The Euclidean vacuum, when expressed in terms of these states is
\[
|E\rangle = \prod_{\omega,l,m} (1 - e^{-2\pi\omega})^{1/2} \exp[-\pi\omega(a_{N,\omega,l,m})(a^\dagger_{S,\omega,l,m})] |0\rangle
\]
(14)

Or, if we define \( |m_{N,\omega,l,m}, n_{S,\omega,l,m}\rangle \) to represent \( m \) excitations of \((\omega, l, m)\) kind of modes in northern diamond and \( n \) excitations in southern diamond.
\[
|m_{N,\omega,l,m}, n_{S,\omega,l,m}\rangle = \frac{(a_{N,\omega,l,m})^m (a^\dagger_{S,\omega,l,m})^n}{\sqrt{m!} \sqrt{n!}} |0\rangle
\]
(15)

Then it can be shown that
\[
|E\rangle = \prod_{\omega,l,m} (1 - e^{-2\pi\omega})^{1/2} \sum_{n_{\omega,l,m}} e^{-\pi\omega n_{\omega,l,m}} |n_{N,\omega,l,m}, n_{S,\omega,l,m}\rangle
\]
(16)

Measurements are classical, so an observer in the southern diamond cannot directly probe states of the northern diamond. From the point of view of a “southern” observer, all observables can be obtained from the density matrix formed by tracing over all the states corresponding to the northern diamond modes.[9] Thus, a particular observer actually appears to see a mixed state, even though the vacuum is a unique state globally.

\[
\rho^0_S = \text{Tr}_N |E\rangle \langle E| = \prod_{\omega,l,m} \sum_{k=0}^\infty \langle k_{N,\omega,l,m}| (|E\rangle \langle E|) |k_{N,\omega,l,m}\rangle
\]
Calculating the contribution for a given mode (say, $\omega = \nu$, $\{l, m\} = \{l', m'\}$),

$$
\langle k_N | (1 - e^{-2\pi \nu}) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} e^{-\pi \nu (m+n)} | m_N, m_S \rangle \langle n_N, n_S | k_N \rangle
= (1 - e^{-2\pi \nu}) e^{-2\pi \nu k} | k_S \rangle \langle k_S |
$$

This gives

$$
\rho^0_S = (\rho^0_S)' \sum_{k=0}^{\infty} (1 - e^{-2\pi \nu}) e^{-2\pi \nu k} | k_S \rangle \langle k_S |
$$

where

$$
(\rho^0_S)' = \prod_{\omega, l, m, k_{\omega, l, m}=0}^{\infty} (1 - e^{-2\pi \omega}) e^{-2\pi \omega k_{\omega, l, m}} | k_{S, \omega, l, m} \rangle \langle k_{S, \omega, l, m} |.
$$

Here, the prime over the product denotes that it is evaluated over all modes except the particular mode ($\omega = \nu$, $\{l, m\} = \{l', m'\}$). One can check that the density matrix is properly normalised, i.e., $\text{Tr} \rho^0_S = 1$. Now suppose a single excitation of ($\nu$, $l'$, $m'$) mode of northern type is created in Euclidean vacuum. Since its wavefunction vanishes in southern diamond, one might naively expect the southern diamond observer to remain blind to this excitation. The corresponding density matrix would be

$$
\rho^1_S = \mathcal{N} \text{Tr}_N \{ a_N | E \rangle \langle E | a_N^\dagger \} = \mathcal{N} (\rho^0_S)' (1 - e^{-2\pi \nu}) \sum_{m=0}^{\infty} e^{-2\pi \nu m} (m+1) | m_S \rangle \langle m_S |.
$$

To keep the density matrix normalised properly, the normalization factor must be taken to be $\mathcal{N} = (1 - e^{-2\pi \nu})$. Therefore

$$
\rho^1_S = (\rho^0_S)' (1 - e^{-2\pi \nu})^2 \sum_{m=0}^{\infty} e^{-2\pi \nu m} (m+1) | m_S \rangle \langle m_S |.
$$

The change in average energy in the southern diamond observer due to the excitation of a northern diamond mode in Euclidean vacuum is therefore

$$
\Delta E_S = \nu \text{Tr} \{ a^\dagger a (\rho^1_S - \rho^0_S) \}/\nu = (e^{2\pi \nu} - 1)
$$

which is precisely the energy $\nu$ of a single particle in a mixed state corresponding to a Bose-Einstein distribution at the temperature $(T = 1/2\pi)$.

The energy change corresponding to an observer in the northern diamond can be calculated in a similar fashion. The corresponding density matrix for the case without any particles excited is obtained by tracing over all the states corresponding to the southern diamond.

$$
\rho^0_N = \text{Tr}_S (| E \rangle \langle E |)
$$

(20)
The normalised northern diamond density matrix for one particle being created would be

\[ \rho_N^1 = (\rho_N^0)'(1 - e^{-2\pi\nu})^2 \sum_{m=0}^{\infty} e^{-2\pi\nu(m-1)} m |m_N\rangle \langle m_N| \]  

(21)

The change in energy corresponding to the northern diamond observer due to the excitation of a northern diamond mode in Euclidean vacuum is

\[ \Delta E_N = \nu \text{Tr}[a_{N,\nu l}a_{N,\nu l}^\dagger (\rho_N^1 - \rho_N^0)] = \frac{\nu}{1 - e^{-2\pi\nu}} \]  

(22)

One can check that \( \Delta E_N - \Delta E_S = \nu \), the total change in energy in de Sitter space due to a single particle excitation in the northern diamond.

The lesson is that, because the Euclidean vacuum involves states in which the southern and northern excitations are correlated, an excitation in the northern diamond does have observable consequences in the southern diamond, even though a southern observer cannot directly probe northern states. Of course, to interpret the density matrix, a southern observer would need to know much about the global state, but symmetry principles and the properties of quantum field theory go a long way toward determining them.[10]

4 Change in Entropy

One can also calculate the entropy change observed by static observers due to a northern diamond mode excitation. The entropy is simply \( S_S = -\text{Tr}\rho_S \ln(\rho_S) \). To evaluate this, it is convenient to introduce some additional notation. Let \( x \equiv \exp(-2\pi\nu) \), and let \( A_S \) and \( B_S \) denote the matrices corresponding to \( (\nu, l', m') \) submatrices of \( \rho_S^0 \) and \( \rho_S^1 \) in the southern diamond. Then

\[ A_S = (1 - x) \sum_{n=0}^{\infty} x^n |n_S\rangle \langle n_S| \]  

(23)

\[ B_S = (1 - x)^2 \sum_{n=0}^{\infty} x^n(n + 1) |n_S\rangle \langle n_S| \]  

(24)

The change in entropy noted by southern diamond observer turns out to be

\[ \Delta S_S = -\text{Tr}(B_S \ln B_S - A_S \ln A_S) \]

\[ = -\ln(1 - x) - \frac{x}{1 - x} \ln x - (1 - x)^2 \sum_{n=1}^{\infty} (n \ln n)x^{n-1} \]  

(25)

The change in the entropy from the point of view of a northern diamond observer is the same, since globally the system is in a pure state. This is a special case of an important general theorem about the entropy of factorized subsystems. When a system is in a pure state,
each subsystem has the same set of nonzero eigenvalues, even if their dimension is different.\textsuperscript{3} Thus, even though the excitations appear very different in the two causal diamonds, and the change in energy in each is very different, the change in entropy is the same.

\section{Conclusion}

The Euclidean vacuum is a pure state with respect to the global de Sitter space. Any observation can at best determine correlation functions within a causal domain. The presence of cosmological horizons in spacetime renders it impossible to measure correlations everywhere, so all observables can be obtained from a density matrix formed by tracing over states outside of the causal domain. Thus, any observer perceives a mixed state. In the case of the Euclidean vacuum, an observer in static coordinates perceives a density matrix corresponding to a thermal state.

Naively, one might think that an observer in the southern diamond, for example, would be insensitive to an excitation in the northern diamond because its past support and influence in the future lie entirely outside the southern diamond. However, because of correlations in the Euclidean vacuum, such excitations change the density matrix for the southern diamond and, therefore, can influence measurements there. We have illustrated this for the case of a single particle excitation in the northern diamond, but the lesson is quite general.

These seemingly acausal results are reminiscent of other nonintuitive correlations in quantum mechanics, as with the EPR paradox \textsuperscript{12}. While we have discussed only the case of a particular cosmological horizon, the same reasoning may help explain some of the properties of black holes. Certainly, “eternal” black holes, such as the classical Schwarzschild geometry, will have their immediate counterparts to our discussion. It is perhaps not so incredible that a similar situation is obtained for the case of gravitational collapse from a pure state to a black hole followed by its subsequent evaporation due to Hawking radiation. These correlations across classical horizons may help resolve the so-called black hole information paradox \textsuperscript{2}.

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\textsuperscript{3}See for example, \textsuperscript{11}.
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