Green’s function for longitudinal shear of a periodic laminate

John Willis
DAMTP, Cambridge

The reasoning to follow is a slight extension of that employed in my article [1] and its further development [2]. A formal solution to the title problem is presented for a general periodic laminate and its application for the construction of “effective constitutive relations” is developed. Its implementation in detail will be reported separately.

1 Exact Green’s function

The medium occupies all of space and its properties are defined by the periodic functions $\mu_{13}(x_1)$, $\mu_{23}(x_1)$ and $\rho(x_1)$ which have period $h$. The medium is considered to be random, in the sense that its shear moduli and density at position $x = (x_1, x_2, x_3)$ take the values

$$
\mu_{13}(x, y) = \mu_{13}^0(x_1 + y), \quad \mu_{23}(x, y) = \mu_{23}^0(x_1 + y), \quad \rho(x, y) = \rho^0(x_1 + y),
$$

(1)

where the sample parameter $y$ is uniformly distributed on the interval $(0, h)$.

Now consider the application of a body force $f(x)$, with $i$-component

$$
f_i(x, t) = \delta_{33} \delta(x_1) \delta(x_2) \delta(t).
$$

(2)

It generates just a 3-component of displacement, $u_3(x_1, x_2, t, y)$, which satisfies the equation of motion

$$
\sigma_{13,1} + \sigma_{23,2} + \delta(x_1) \delta(x_2) = \dot{p},
$$

(3)

where

$$
\sigma_{13} = \mu_{13} w_1, \quad \sigma_{23} = \mu_{23} w_2 \quad \text{and} \quad p = \rho \dot{u}_3.
$$

(4)

Henceforth, equations will be considered in which Laplace transforms have been taken with respect to $t$ and $x_2$, which is equivalent to considering dependence $e^{(st+kw_2)}$. The complex variable $s$ is taken to have a small positive real part, to ensure causality.
Expressed in terms of \( u_3 \), the equation of motion \([4]\) becomes
\[
\{ \mu_{13}^0(x_1 + y)u_3(x_1, y) \}' + k^2 \mu_{23}^0(x_1 + y)u_3(x_1, y) + \delta(x_1) = \rho^0(x_1 + y)s^2u_3(x_1, y),
\]
where the prime represents differentiation with respect to \( x_1 \). Equivalently, with
\[
\overline{u}_3(x_1, y) = u_3(x_1 - y, y) \text{ so that } u_3(x_1, y) = \overline{u}_3(x_1 + y, y),
\]
the equation of motion can be written
\[
\{ \mu_{13}^0(x_1)\overline{u}_3(x_1, y) \}' + k^2 \mu_{23}^0(x_1)\overline{u}_3(x_1, y) + \delta(x_1 - y) = \rho^0(x_1)s^2\overline{u}_3(x_1, y).
\]
The solution of equation \([7]\) can be expressed in terms of the two “Floquet” solutions,
\[
\phi_+(x_1) = \psi_+(x_1)e^{\kappa_1 x_1}, \quad \phi_-(x_1) = \psi_-(x_1)e^{-\kappa_1 x_1},
\]
where \( \psi_\pm \) are periodic with period \( h \) and \( \kappa \) has negative real part.\(^1\) Correspondingly, \( \phi_+ \to 0 \) as \( x_1 \to +\infty \) and \( \phi_- \to 0 \) as \( x_1 \to -\infty \).

Then,
\[
\overline{u}_3(x_1, y) = \begin{cases} D\phi_+(x_1)\phi_-(y), & x_1 \geq y, \\ D\phi_+(y)\phi_-(x_1), & x_1 \leq y, \end{cases}
\]
where
\[
D = \{ \mu_{13}^0(x_1)\phi_+(y)\phi_-(y) - \phi_+(y)\phi_-(y) \}^{-1}
\]
(which, in fact, is independent of \( y \)). The desired Green’s function \( G(x_1, y) \) is just \( u_3(x_1, y) \). Thus, it follows from \([6]\) that
\[
G(x_1, y) = \begin{cases} D\phi_+(x_1 + y)\phi_-(y) \equiv D\psi_+(x_1 + y)\psi_-(y)e^{\kappa_1 x_1}, & x_1 \geq 0, \\ D\phi_+(y)\phi_-(x_1 + y) \equiv D\psi_+(y)\psi_-(x_1 + y)e^{-\kappa_1 x_1}, & x_1 \leq 0. \end{cases}
\]

Since they are periodic, the functions \( \psi_\pm \) can be expressed as Fourier series:
\[
\psi_+(x_1) = \sum_{m=-\infty}^{\infty} a_m(\kappa)e^{-2\pi imx_1/h}, \quad m \in \mathbb{Z},
\]
\[
\psi_-(x_1) = \sum_{m=-\infty}^{\infty} a_m(-\kappa)e^{-2\pi imx_1/h}, \quad m \in \mathbb{Z},
\]
where
\[
a_m(\kappa) = \frac{1}{h} \int_0^h \psi_+(x_1)e^{2\pi imx_1/h}dx_1 \equiv \frac{1}{h} \int_0^h \phi_+(x_1)e^{(2\pi im/h-\kappa)x_1}dx_1, 
\]
\[
a_m(-\kappa) = \frac{1}{h} \int_0^h \psi_-(x_1)e^{2\pi imx_1/h}dx_1 \equiv \frac{1}{h} \int_0^h \phi_-(x_1)e^{(2\pi im/h+\kappa)x_1}dx_1.
\]

\(^1\)This convention is a little different from that used in [1], in which a parameter \( \mu \) is used for \( \kappa \), and \( \mu \) was defined to have positive imaginary part. There are corresponding minor differences in subsequent formulae. The present convention is what I used in a program written for oblique waves.
2 Effective properties

Green’s function for the effective medium is the ensemble average:

\[
\langle G \rangle (x_1) = \frac{1}{h} \int_{0}^{h} G(x_1, y) dy
\]

\[
= \begin{cases} 
D e^{\kappa x_1} \sum_m a_m(\kappa) a_{-m}(\kappa) e^{-2\pi i m x_1/h}, & x_1 \geq 0 \\
D e^{-\kappa x_1} \sum_m a_m(\kappa) a_{-m}(\kappa) e^{2\pi i m x_1/h}, & x_1 \leq 0 \\
D e^{\kappa |x_1|} \sum_m a_m(\kappa) a_{-m}(\kappa) e^{-2\pi i m |x_1|/h}.
\end{cases}
\] (16)

Stresses \( \sigma_{13} \) and momentum \( p \) can be treated similarly. First, the exact stress component \( \sigma_{13} \) is

\[
\sigma_{13}(x_1, y) = \begin{cases} 
D \mu_{13}^0(x_1 + y) \phi'_+(x_1 + y) \phi_-(y), & x_1 > 0, \\
D \mu_{13}^0(x_1 + y) \phi'_-(x_1 + y), & x_1 < 0.
\end{cases}
\] (17)

Now with

\[
\mu_{13}^0(x_1) \phi'_+(x_1) e^{-\kappa x_1} = \sum_{m=-\infty}^{\infty} b_m(\kappa) e^{-2\pi i m x_1/h}
\] (18)

and

\[
\mu_{13}^0(x_1) \phi'_-(x_1) e^{\kappa x_1} = \sum_{m=-\infty}^{\infty} b_m(-\kappa) e^{-2\pi i m x_1/h},
\] (19)

the ensemble averaged stress component can be expressed

\[
\langle \sigma_{13} \rangle (x_1) = \begin{cases} 
D e^{\kappa x_1} \sum_m b_m(\kappa) a_{-m}(\kappa) e^{-2\pi i m x_1/h}, & x_1 > 0, \\
D e^{-\kappa x_1} \sum_m a_m(\kappa) b_{-m}(\kappa) e^{2\pi i m x_1/h}, & x_1 < 0.
\end{cases}
\] (20)

Similarly,

\[
\langle \sigma_{23} \rangle (x_1) = \begin{cases} 
D k e^{\kappa x_1} \sum_m c_m(\kappa) a_{-m}(\kappa) e^{-2\pi i m x_1/h}, & x_1 > 0, \\
D k e^{-\kappa x_1} \sum_m a_m(\kappa) c_{-m}(\kappa) e^{2\pi i m x_1/h}, & x_1 < 0
\end{cases}
\] (21)

and

\[
\langle p \rangle (x_1) = \begin{cases} 
s D e^{\kappa x_1} \sum_m d_m(\kappa) a_{-m}(\kappa) e^{-2\pi i m x_1/h}, & x_1 > 0, \\
s D e^{-\kappa x_1} \sum_m a_m(\kappa) d_{-m}(\kappa) e^{2\pi i m x_1/h}, & x_1 < 0.
\end{cases}
\] (22)

where

\[
\mu_{23}^0(x_1) \phi'_+(x_1) e^{-\kappa x_1} = \sum_{m=-\infty}^{\infty} c_m(\kappa) e^{-2\pi i m x_1/h},
\] (23)
\[
\mu_{23}(x_1)\phi_-(x_1)e^{\kappa x_1} = \sum_{m=-\infty}^{\infty} c_m(-\kappa)e^{-2\pi imx_1/h},
\]
(24)
\[
\rho^0(x_1)\phi_+(x_1)e^{-\kappa x_1} = \sum_{m=-\infty}^{\infty} d_m(\kappa)e^{-2\pi imx_1/h}
\]
(25)
and
\[
\rho^0(x_1)\phi_-(x_1)e^{\kappa x_1} = \sum_{m=-\infty}^{\infty} d_m(-\kappa)e^{-2\pi imx_1/h}.
\]
(26)

**Fourier transforms**

Now take
\[
s = \epsilon - i\omega \quad \text{and} \quad k = -i\xi_2,
\]
(27)
where \(\epsilon\) is small and positive, and Fourier transform the preceding ensemble averages with respect to \(x_1\). This gives
\[
\langle \tilde{G} \rangle(\xi_1, \xi_2, \omega) = -2D \sum_{m=-\infty}^{\infty} \frac{a_m(\kappa)a_{-m}(-\kappa)(\kappa - 2\pi im/h)}{(\kappa - 2\pi im/h)^2 + \xi_1^2},
\]
(28)
\[
\langle \tilde{\Sigma}_{13} \rangle(\xi_1, \xi_2, \omega) = -D \sum_m \left\{ \frac{a_m(\kappa)b_{-m}(-\kappa)}{\kappa - 2\pi im/h - i\xi_1} + \frac{b_m(\kappa)a_{-m}(-\kappa)}{\kappa - 2\pi im/h - i\xi_1} \right\},
\]
(29)
\[
\langle \tilde{\Sigma}_{23} \rangle(\xi_1, \xi_2, \omega) = i\xi_2 D \sum_m \left\{ \frac{a_m(\kappa)c_{-m}(-\kappa)}{\kappa - 2\pi im/h - i\xi_1} + \frac{c_m(\kappa)a_{-m}(-\kappa)}{\kappa - 2\pi im/h + i\xi_1} \right\},
\]
(30)
\[
\langle \tilde{P} \rangle(\xi_1, \xi_2, \omega) = i\omega D \sum_m \left\{ \frac{a_m(\kappa)d_{-m}(-\kappa)}{\kappa - 2\pi im/h - i\xi_1} + \frac{d_m(\kappa)a_{-m}(-\kappa)}{\kappa - 2\pi im/h + i\xi_1} \right\}.
\]
(31)

Upper-case \(\Sigma\) and \(P\) are employed, to identify these quantities as components of stress and momentum density associated with the Green’s function. Note that, in these expressions, \(\kappa\) is a function of \(\xi_2\) and \(\omega\).

**Effective constitutive relations**

Following [\(\Pi\)], equation (29) can be written
\[
\langle \tilde{\Sigma}_{13} \rangle = \tilde{C}_{1313}^{\text{eff}}(-i\xi_1\langle \tilde{G} \rangle) + \tilde{S}_{133}^{\text{eff}}(-i\omega\langle \tilde{G} \rangle),
\]
(32)
where
\[
\tilde{C}_{1313}^{eff} = D \sum_m \frac{a_m(\kappa)b_m(-\kappa) - a_{-m}(-\kappa)b_m(\kappa)}{(\kappa - 2\pi im/h)^2 + \xi_1^2} \langle G \rangle^{-1},
\]
\[
\tilde{S}_{133}^{eff} = -D \sum_m \frac{a_m(\kappa)b_m(-\kappa) + a_{-m}(-\kappa)b_m(\kappa)}{(\kappa - 2\pi im/h)^2 + \xi_2^2} \langle G \rangle^{-1}.
\]

There is a correspondingly “obvious” split for equation (31), whereas a similar split for (30) gives a less “intuitive” result. There is, however, the general formula for effective properties
\[
\mathcal{L}^{eff} = \langle \mathcal{L} \rangle - \langle \mathcal{L} \mathcal{E}(\mathcal{E}^\dagger)^\dagger \rangle + \langle \mathcal{L} \mathcal{E}\mathcal{G} \rangle \langle G \rangle^{-1} \langle (\mathcal{E}^\dagger)^\dagger \mathcal{L} \rangle
\]
(35)
derived in [3], which applies also in the presence of any non-random inelastic strain.\footnote{The formula as given in [3] was more general because it allowed also for a “weighted average”, not considered at present.}

In the present context, \( \mathcal{L} \) is the 3 \( \times \) 3 array of parameters\footnote{\( \mathcal{L} \) is diagonal but \( \mathcal{L}^{eff} \) is not.} that relates
\[
s = \begin{pmatrix} \sigma_{13} \\ \sigma_{23} \\ p \end{pmatrix} \quad \text{to} \quad \mathcal{E}u_3 = \begin{pmatrix} u_{3,1} \\ u_{3,2} \\ \dot{u}_3 \end{pmatrix}.
\]

Implementing (35) in the Fourier transform domain gives
\[
\tilde{\mathcal{L}}^{eff} = \begin{pmatrix} \tilde{C}^{eff}_{1313} & \tilde{C}^{eff}_{1323} & \tilde{S}^{eff}_{133} \\ \tilde{C}^{eff}_{2313} & \tilde{C}^{eff}_{2323} & \tilde{S}^{eff}_{233} \\ \tilde{S}^{eff}_{313} & \tilde{S}^{eff}_{123} & \tilde{\rho}^{eff}_{33} \end{pmatrix}.
\]
(36)

The term \( \langle \mathcal{L} \rangle - \langle \mathcal{L} \mathcal{E}(\mathcal{E}^\dagger)^\dagger \mathcal{L} \rangle \) is significant in the presence of inelastic deformation but makes zero contribution to the stress when strain and velocity are derived from a displacement. However, for the record, the required formula is
\[
\langle \tilde{\mathcal{L}} \rangle - \langle \mathcal{L} \mathcal{E}(\mathcal{E}^\dagger)^\dagger \mathcal{L} \rangle = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \langle C_{2323} \rangle & 0 \\ 0 & 0 & \langle \tilde{\rho} \rangle \end{pmatrix}
+ D \sum_m \frac{1}{\kappa - 2\pi im/h - i\xi_1} \begin{pmatrix} b_m(-\kappa) \\ -i\xi_2c_{-m}(-\kappa) \\ -i\omega d_{-m}(-\kappa) \end{pmatrix} \begin{pmatrix} b_m(\kappa) & i\xi_2c_m(\kappa) & -i\omega d_m(\kappa) \end{pmatrix}
\]
\[ + D \sum_{m} \frac{1}{\kappa - 2\pi i m / h + i \xi_1} \begin{pmatrix} b_m(\kappa) \\ -i \xi_2 c_m(\kappa) \\ -i \omega d_m(\kappa) \end{pmatrix} \begin{pmatrix} b_{-m}(\kappa) & i \xi_2 c_{-m}(\kappa) & -i \omega d_{-m}(\kappa) \end{pmatrix}. \]

(37)

The reason for the absence of a term \( \tilde{C}_{1313} \) in the \((1,1)\) place in the first matrix is that it is cancelled by a delta-function contribution from \( E(E^\dagger)^\dagger \) in that position.

The remaining term in (35) can be built up by noting that

\[
\langle \mathcal{L} \mathcal{E} G \rangle(\xi_1, \xi_2, \omega) = \begin{pmatrix} \langle \tilde{\Sigma}_{13} \rangle(\xi_1, \xi_2, \omega) \\ \langle \tilde{\Sigma}_{23} \rangle(\xi_1, \xi_2, \omega) \\ \langle \tilde{P} \rangle(\xi_1, \xi_2, \omega) \end{pmatrix} \]

(38)

where \( \langle \tilde{\Sigma}_{13} \rangle \) etc. are given respectively by (29), (30) and (31), and that

\[
\langle (E G^\dagger)^\dagger \mathcal{L} \rangle(\xi_1, \xi_2, \omega) = \langle \mathcal{L} \mathcal{E} G \rangle^\dagger(\xi_1, \xi_2, \omega) = \{ \langle \mathcal{L} \mathcal{E} G \rangle(-\xi_1, -\xi_2, \omega) \}^T, \]

(39)

while \( \langle \tilde{G} \rangle \) is given by (28). Note that \( \kappa \) depends on \( \xi_2 \) and \( \omega \) and is an even function of \( \xi_2 \).

Some elementary observations

Note first that equation (16) represents \( \langle G \rangle \) as a superposition of plane waves, with space and time-dependence \( e^{st + (\kappa - 2\pi i m / h)x_1 + kx_2} \) when \( x_1 > 0 \) and a corresponding dependence with \( \kappa \) replaced by \(-\kappa\) when \( x_1 < 0 \). Each of these of course corresponds to a single “Floquet–Bloch” mode (depending only on the sign before \( \kappa \)). Another view of the same result is obtained from considering the Fourier transform of \( \langle G \rangle \). Poles of the transform correspond to plane waves, and the poles are at points \( (\xi_1, \xi_2, \omega) \) at which

\[
i\xi_1 \pm (\kappa(-\xi_2, -i\omega) - 2\pi i m / h) = 0.
\]

(40)

Furthermore, the expression for \( \langle \tilde{\mathcal{L}} \rangle \) as \( (\xi_1, \xi_2, \omega) \) approaches one of the poles involves only the contributions from that pole. Thus, for instance,

\[
\langle \tilde{\Sigma}_{13} \rangle \sim -D \frac{b_m(\kappa)a_{-m}(\kappa)}{\kappa - 2\pi i m / h + i \xi_1}
\]

(41)

as \( i\xi_1 + (\kappa(-\xi_2, -i\omega) - 2\pi i m / h) \to 0 \). The expression for \( \langle \tilde{\mathcal{L}} \rangle \) simplifies correspondingly, and clearly depends on \( m \).
Next, a comment on Green’s function in physical space. In equation (16), the dependence on $x_1$ is explicit but (even assuming that harmonic time-dependence is required) it is necessary to invert the Laplace transform with respect to $k$. Alternatively, it is perhaps simpler to start from the Fourier transform (28) and invert first with respect to $\xi_2$. Poles in (say) the upper half of the complex $\xi_2$-plane are certain to appear wherever

$$\kappa(-i\xi_2, -i(\omega + 0i)) = 2\pi im/h - i\xi_1. \quad (42)$$

I am inclined to expect that there will be one solution for each $m$, but have not properly investigated; nor do I know if there are other (branch-cut) singularities. I do intend to investigate this further, in order to calculate the influence of a line of body-force, proportional to $e^{i(\omega t + \xi_1x_1)}\delta(x_2)$. This will be of interest in relation to considering transmission and reflection from a medium occupying a half-space $x_2 > 0$.

3 A note on implementation

Consider an $n$-phase laminate, in which material of type $j$ has elastic constants $C_{13}^j, C_{23}^j$ and density $\rho^j$, and occupies the region $z_{j-1} < x_1 < z_j$, where $z_0 = 0$ and $z_j = z_{j-1} + h_j$. This structure is repeated periodically, with period $h = \sum_{j=1}^n h_j$. To obtain the basic Floquet waves $\phi_{\pm}$, $A_j$ is defined to be

$$A_j = \begin{pmatrix} \sigma_{13}(z_{j-1}) \\ u_3(z_{j-1}) \end{pmatrix}. \quad (43)$$

Then, for $z_{j-1} < x_1 < z_j$,

$$\begin{pmatrix} \sigma_{13}(x_1) \\ u_3(x_1) \end{pmatrix} = M_j(x_1 - z_{j-1})A_j, \quad (44)$$

where the propagator matrix $M_j(x_1 - z_{j-1})$ is given as

$$M_j(x_1 - z_{j-1}) = \begin{pmatrix} \cosh(\kappa_j(x_1 - z_{j-1})) & \mu_{13}^j\kappa_j\sinh(\kappa_j(x_1 - z_{j-1})) \\ (\mu_{13}^j\kappa_j)^{-1}\sinh(\kappa_j(x_1 - z_{j-1})) & \cosh(\kappa_j(x_1 - z_{j-1})) \end{pmatrix}, \quad (45)$$

with

$$\kappa_j = \sqrt{\frac{\rho^j s^2 - \mu_{23}^j k^2}{\mu_{13}^j}}. \quad (46)$$

In particular,

$$A_{j+1} = M_j(h_j)A_j, \quad (47)$$

7
and the Floquet condition that fixes $\kappa$ is defined so that

$$MA_1 = e^{\kappa h} A_1,$$

where

$$M = M_1(h_1)M_2(h_2) \cdots M_n(h_n).$$

It may be noted, in passing, that $\det(M) = 1$, because each $M_j(h_j)$ has determinant 1, and hence the Floquet condition can be written as

$$\cosh(\kappa h) = \frac{1}{2} \text{trace}(M).$$

The stress component $\sigma_{23}$ will also be needed. When $z_{j-1} < x_1 < z_j$,

$$\sigma_{23}(x_1) = \mu_{23}^j k \left[ (\mu_{13}^j \kappa_j)^{-1} \sinh(\kappa_j(x_1 - z_{j-1})) \cosh(\kappa_j(x_1 - z_{j-1})) \right] A_j.$$

Now all that remains is the tedium of actually doing the calculations...

References

[1] J.R. Willis. Exact effective relations for dynamics of a laminated body. Mechanics of Materials 41 (2009), 385-393.

[2] J.R. Willis. A comparison of two formulations for effective relations for waves in a composite. Mechanics of Materials 47 (2012), 51-60.

[3] J.R. Willis. The construction of effective relations for waves in a composite. Comptes Rendus Mécanique 340 (2012), 181-192.