Drinfel’d twist and Noncommutative oscillators

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Abstract. Using the Drinfel’d twist of Hopf algebras, noncommutative coordinates are derived for quantum mechanics. Two cases are presented, the constant noncommutativity and a Snyder-like one. This approach brings naturally new features, such as non-additivity in the energy and pseudo-hermiticity.

1. Introduction
In recent works [1, 2, 3], noncommutative coordinates in quantum mechanics were achieved by applying the Drinfel’d twist technique [4], using a suitably chosen dynamical Lie algebra, and its universal envelopment. This allows the noncommutativity to be derived from first principles - the unfolded and hybrid formalisms - rather than as an ansatz.

The main idea (as discussed in further details below) is that the deformed generators of the Lie algebra, as defined by Woronowicz [5], will be a deformed algebra, even though the deformed adjoint action leads to the same structure constants. In order to use this derivation we need to work in the setting of Hopf algebras [6, 7, 8], with the coproduct defining multiparticle states.

Because of this derivation some new features arise naturally, like a non-additivity in the energy of multiparticle states, and a non-hermiticity on the Hamiltonian. This leads to the expectation that noncommutative effects could be intrinsically tied to these features and any experimental verification should take that into account.

Two different noncommutativities will be derived here, following [2, 3]. In the first section, the general aspects of the formalism will be discussed, using the specific dynamical Lie algebra for the harmonic oscillator. In the second section the abelian twist will be presented, leading to a constant noncommutativity. The deformed Hamiltonian will have the same spectrum as derived in the literature for noncommutative quantum mechanics [2], and (in 2 dimensions) the same rotation symmetry operator. In the third section the non-abelian case is shown to lead to Snyder-like commutation relations [10], and to a non-hermitian Hamiltonian - which turns out to be pseudo-hermitian [11, 12].

2. The Unfolded Formalism
When dealing with Lie algebras, the Drinfel’d twist provides a solid approach for deformations. The first step is to use the universal enveloping algebra $U(g)$ - meaning we have now all polynomials in the generators. This is a Hopf algebra, when endowed with the appropriate
structures. For a primitive element - a generator of the Lie algebra - the coproduct is given by
\[ \Delta(u) = u \otimes 1 + 1 \otimes u, \]
while for other elements we use \( \Delta(uv) = \Delta(u)\Delta(v) \). With this expression, we have a physical interpretation of primitive coproducts: it lifts an additive quantity to a two-particle state, e.g.

For a primitive element - a generator of the Lie algebra - the coproduct is given by
\[ \Delta(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha, \]
with \( \alpha \) being a multi-index) by

This is extended naturally to multi-particle states, with the property that \( (\Delta \otimes \text{Id})\Delta = (\text{Id} \otimes \Delta)\Delta \) (the coproduct is coassociative). This means we will have an associative construction of multiparticle states.

The commutator is recovered by using the antipode for primitive elements, \( S(u) = -u \) (and \( S(uv) = S(v)S(u) \)). If the coproduct is written as \( \Delta(u) = (u_1)_{i \otimes (u_2)} \equiv u_1 \otimes u_2 \) - summation understood - then we have the adjoint action given by
\[ ad_u(v) = [u, v] = u_1vS(u_2). \]

The Drinfel’d twist deforms both the coproduct and the antipode, with the outcome still being a Hopf algebra. This means the coassociativity is preserved, but not the additivity, as will be shown below. With a deformed coproduct and antipode, we have a deformed adjoint action \( ad^F \) and Woronowicz [5] proved that there is a subspace \( g^F \subset U(g) \) closed under this deformed commutator (also recovering the same structure constants). Whenever dealing with a spectral theory however, it is the regular commutator of the deformed generators \( g^F \in g^F \) that will play a significant role, therefore leading to a hybrid formalism. In this framework, the deformed generators will lead to a noncommutative deformation of some operator algebra.

2.1. Dynamical Lie Algebra
For this construction, we will use a formal Lie algebra, with the following construction. Take the Heisenberg algebra
\[ [x_i, p_j] = i\hbar \delta_{ij}, \quad i, j = 1, 2, \ldots, d \]
together with \( H = \frac{1}{2M}p_i p_i, \quad K = \frac{1}{2M}x_i x_i \) and \( D = \frac{1}{2M}(x_i p_i + p_i x_i) \) \( (H + K) \) is the Hamiltonian of the harmonic oscillator. This gives the following commutation relations
\[ [D, H] = iH, \quad [D, K] = -iK, \quad [K, H] = 2iD \]
\[ [x_i, H] = ip_i, \quad [x_i, K] = 0, \quad [x_i, D] = \frac{i}{\hbar}x_i \]
\[ [p_i, H] = 0, \quad [p_i, K] = -ix_i, \quad [p_i, D] = \frac{\hbar}{2}p_i \]
(3)

with \( [\hbar, \cdot] = 0 \) and we may still add the angular momentum in a similar way. Considering \( \hbar \) as a central extension, this is a Lie algebra. Notice that the definitions of \( H, K \) and \( D \) were essential so we wouldn’t have quadratic terms in the generators, and that they no longer make sense after we lift \( \hbar \) to the role of a central extension. Also they may no longer be thought of as quadratic polynomials. This all reflected by their their antipode and their coproduct, which should be written as \[ 1 \]

2.2. Twisting and Deformed Generators
The Drinfel’d twist is performed by a invertible counitary 2-cocycle \( F = f^\alpha \otimes f_\alpha \in U(g) \otimes U(g) \) \( (\alpha \) being a multi-index) by
\[ \Delta^F(u) = F \Delta(u) F^{-1} \]
\[ S^F(u) = \chi S(u) \chi^{-1}, \]
(4)
with \( \chi = f^\alpha S(f_\alpha) \). With these, the deformed generators can be written as \((\mathcal{F}^{-1} = \bar{f}^\alpha \otimes \bar{f}_\alpha) \)

\[
g^\mathcal{F} = \bar{f}^\alpha[g] \bar{f}_\alpha,
\]

(5)

where \( \bar{f}^\alpha[g] \) denotes the adjoint action (e.g. if \( f^2 = D \cdot D \) then \( f^2[g] = [D, [D, g]] \)). With this, the deformed generators are power series in the old ones, and finding their commutators becomes possible.

2.3. Multi-particle states
For the two twists considered below, \( \mathcal{F} \) is a unitary transformation in \( U(\mathfrak{g}) \otimes U(\mathfrak{g}) \) and we may consider the undeformed coproduct as defining the multiparticle states - the twist will be encoded in the deformed generators. The advantage is that the undeformed coproduct will be symmetric under particle exchange while the deformed one requires a deformed flip operator.

In the undeformed case the coproduct implied in additivity for the energy spectrum, but now this is no longer the case. With the undeformed coproduct of the deformed generators we now have a non-additive energy for two-particle states, compatible with the noncommutative quantum mechanics. We should notice however that the coassociative property remains, and thus we may still use this construction for multiparticles. Explicit examples for this non-additivity will be given below.

3. An Abelian Twist
The Abelian twist is given by

\[
\mathcal{F} = \exp (i \alpha_{ij} p_i \otimes p_j), \quad \alpha_{ij} = -\alpha_{ji}.
\]

(6)

For this twist the antipode is undeformed. It is also easy to see that \( \mathcal{F} \) is a unitary transformation in \( U(\mathfrak{g}) \otimes U(\mathfrak{g}) \), under the conjugation \( \dagger : u \otimes v \mapsto u^\dagger \otimes v^\dagger \).

In \( d = 2 \) there is only one parameter for the deformation, \( \alpha = \alpha_{12} \), and the angular momentum is a scalar. In \( d = 3 \) both are vectors: \( L_i \) and \( \alpha_i = \frac{1}{2} \epsilon_{ijk} \alpha_{jk} \). Nevertheless we may choose a system of coordinates where \( \alpha_i = (0, 0, \alpha) \).

3.1. Deformed Generators and Noncommutativity
The expression for the deformed generators \(^5\) becomes

\[
g^\mathcal{F} = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \alpha_{i_1 j_1} \alpha_{i_2 j_2} \cdots \alpha_{i_n j_n} [p_{i_n}, \ldots [p_{i_1}, g] \ldots] p_{j_1} \cdots p_{j_n}.
\]

(7)

There are only 2 generator that are deformed (not counting the angular momentum),

\[
x^F_i = x_i - \alpha_{ik} p_k \hbar, \\
K^F = K - \alpha_{jk} x_j p_k + \frac{\alpha_{jk} \alpha_{il}}{2l!} p_k p_l \hbar.
\]

(8)

With this the deformed coordinates now have a constant noncommutativity,

\[
[x^F_i, x^F_j] = i \Theta_{ij},
\]

(9)

where \( \Theta_{ij} = 2\alpha_{ij} h^2 \).
3.2. Spectrum
The Hamiltonian of the harmonic oscillator is deformed to $H^F = (H + K)^F = H + K^F$, and with the choice of coordinates introduced above, both $d = 2$ and $d = 3$ deformed Hamiltonian take the same form, i.e.,

$$H^F = H + K^F = H + K - \alpha xp_y + \alpha yp_x + \frac{\alpha^2}{2} \hbar (p_z^2 + p_y^2). \tag{10}$$

Notice that for $d = 3$ the $p_z$ component seems to be “missing” from the deformation.

We can also find the two-particle Hamiltonian,

$$\Delta(H^F) = H^F \otimes 1 + 1 \otimes H^F + \alpha (y \otimes p_x + p_x \otimes y - x \otimes p_y - p_y \otimes x)$$

$$+ \frac{\alpha^2}{2} \sum_{i=1}^{2} (2p_i \hbar \otimes p_i + 2p_i \otimes p_i \hbar + p_i^2 \otimes \hbar + \hbar \otimes p_i^2). \tag{11}$$

We should keep in mind that, as far as this point, $\hbar$ may not be taken across any tensor product, since it is considered a generator, not a multiple of the identity. As discussed before, we have a non-additivity on the two-particle spectrum. Also, the $d = 3$ again has no $p_z$ term in the deformation.

The spectrum for a single particle can be easily found in both cases. For $d = 2$ the eigenstates can be labeled by the two integers $n \in \mathbb{N}$ and $m = -n, -n + 2, ..., 0, ..., n - 2, n$, defined by $H|n m\rangle = (n + 1)|n m\rangle$ and $L|n m\rangle = m|n m\rangle$, $L$ being the angular momentum. The eigenvalues of the deformed Hamiltonian are then

$$H^F|n m\rangle = \left(\sqrt{1 + \alpha^2} (n + 1) - \alpha m\right)|n m\rangle, \tag{12}$$

and so the degeneracy of the spectrum is lost. Introducing the appropriate constants, this can be matched to results in the literature [9].

For $d = 3$ the picture is fairly similar, and the eigenstates are labeled by $n_z, n_{xy} \in \mathbb{N}$, and $m = -n_{xy}, -n_{xy} + 2, ..., 0, ..., n_{xy} - 2, n_{xy}$, with the spectrum

$$H^F|n_{xy} n_z m\rangle = \left(\sqrt{1 + \alpha^2} (n_{xy} + 1) - \alpha m + \left( n_z + \frac{1}{2} \right) \right)|n_{xy} n_z m\rangle. \tag{13}$$

Again the degeneracy is lost.

3.3. Rotation Symmetry
It should be obvious from [10] [12] and [13] that the rotation symmetry is only maintained in $d = 2$. In fact, for $d = 3$ the $so(3)$ symmetry is broken to a $so(2)$. This can be seen by the presence of a term proportional to $xp_y - yp_x$ in the Hamiltonian, which obviously commutes with the angular momentum in $d = 2$, but only with the $z$ component in $d = 3$.

If we add the angular momentum to the dynamical algebra, it will also be deformed to $L^F = L - \alpha (p_x^2 + p_y^2)$, or $L_i^F = L_i + \alpha p_i p_j - \alpha_i p_j p_i$, respectively in the $d = 2$ and $d = 3$ cases. But these will not give the correct structure constants, nor will they commute with the Hamiltonian. So we see that it is the undeformed angular momentum that will play the role of symmetry operator in both cases.
4. Non Abelian Case

In [3] we have an $sl_2$ subalgebra. Therefore we may now perform the non-abelian Jordanian twist,

$$F = \exp (-iD \otimes \sigma)$$

with $\sigma = \ln (1 + \xi H)$. The deformed generators are

$$\begin{align*}
x^F_i &= x_i e^{\sigma} \\
p^F_i &= p_i e^{\sigma} \\
H^F &= He^{-\sigma} \\
K^F &= Ke^{\sigma}
\end{align*}$$

Again the coordinates become noncommutative under the usual commutators, this time we obtain the non constant commutator

$$\left[ x^F_i , x^F_j \right] = \frac{-i\xi}{2} (x^F_i p^F_j - x^F_j p^F_i ).$$

This is exactly the spatial part of the algebra defined by Snyder [10], together with

$$\left[ x^F_i , p^F_j \right] = i\hbar \delta_{ij} + \frac{i\xi}{2} p^F_i p^F_j ,$$

as required from the Jacobi identity. With this algebra we have full $so(3)$ symmetry. In fact, unlike the abelian case, the Jordanian deformation may be used (non trivially) in $d = 1$, deforming the $[x, p]$ commutator.

The Hamiltonian is now $H^F = He^{-\sigma} + Ke^{\sigma}$. For two-particle states the non-additivity is given by

$$\Delta (H^F) = Ke^{\sigma} \otimes e^{\sigma} + e^{\sigma} \otimes Ke^{\sigma} - \xi^2 (KH \otimes H + H \otimes KH) + \sum_{n=1}^{\infty} (-\xi)^{n-1} \sum_{k=1}^{n} \binom{n}{k} H^k \otimes H^{n-k}.$$  \hspace{1cm} (18)

4.1. Non-hermiticity

The deformed harmonic oscillator derived here, will have a particular difficulty in the case of the Jordanian twist. The Hamiltonian is no longer hermitian. In fact

$$\left( H^F \right)^\dagger \neq \eta H^F \eta^{-1},$$

with $\eta = e^{\sigma} = 1 + \xi H$. This happens because $e^{-\sigma} K \neq Ke^{-\sigma}$.

The Hamiltonians that are not hermitian, but its conjugate is a similarity transformation like the one given above, are called pseudo-hermitian, and they still have a real spectrum. Following [11 12], we define a deformed inner product

$$\langle \langle \psi, \phi \rangle \rangle = \langle \psi, \eta \phi \rangle,$$

under which the deformed Hamiltonian is now self-adjoint. As vector spaces, the two Hilbert spaces are the same, but as inner product spaces they differ. Also, $\rho = e^{\sigma}$ is a hermitian operator in the usual Hilbert space, but plays the role of a unitary transformation between both inner
products. This means we may define an equivalent Hamiltonian in the usual Hilbert space, given by

$$H^F = \rho H^F \rho^{-1} = \left( 1 - \frac{\xi^2}{4} \right) H^F + K^F + i\xi D,$$

which is explicitly hermitian.

5. Conclusions

Considering the Heisenberg algebra as a Lie algebra with a central extension allows the use of a Hopf algebraic approach in the study of noncommutative quantum mechanics. The introduction of the concept of a coproduct brings naturally a multiparticle construction, consistent with the adjoint action. The use of a Hopf algebra allows as well the use of the Drinfel’d twist for its deformation, preserving the coassociativity of the multiparticle states, while leading to a non additive energy spectrum. Together with the pseudo-hermiticity, this raises the question on whether we can have noncommutative quantum mechanics without these extra features. The approach given here also leads to a natural representation of noncommutative quantum mechanics in terms of its commutative version, making it easy to see all important aspects of the theory.

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