On higher spin $U_q(sl_2)$–invariant $R$–matrices

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Dedicated to Professor L.D. Faddeev on the occasion of his 70th birthday

Abstract

The spectral decomposition of regular $U_q(sl_2)$–invariant solutions of the Yang–Baxter equation is studied. An algorithm for finding all possible solutions of spin $s$ is developed. It also allows to reconstruct the $R$–matrix from a given nearest neighbour spin chain Hamiltonian. The algorithm is based on reduction of the Yang–Baxter equation to certain subspaces. As an application, the complete list of inequivalent regular $U_q(sl_2)$–invariant $R$–matrices is obtained for generic $q$ and spins $s \leq \frac{3}{2}$. Some further results about spectral decompositions for higher spins are also obtained. In particular, it is proved that certain types of regular $sl_2$–invariant $R$–matrices have no $U_q(sl_2)$–invariant counterparts.

1 Preliminaries

The quantum Lie algebra $U_q(sl_2)$ is defined as a universal enveloping algebra over $\mathbb{C}$ with the identity element $e$ and generators $S^\pm, S^z$ obeying the following defining relations [1]

$$[S^+, S^-] = [2S^z]_q, \quad [S^z, S^\pm] = \pm S^\pm,$$

where $[t]_q = (q^t - q^{-t})/(q - q^{-1})$. $U_q(sl_2)$ can be equipped with a Hopf algebra structure [2]. In particular, the co–multiplication (a co–associative linear homomorphism) is defined as follows

$$\Delta(S^\pm) = S^\pm \otimes q^{-S^z} + q^{S^z} \otimes S^\pm, \quad \Delta(S^z) = S^z \otimes e + e \otimes S^z. \quad (2)$$

For generic $q$, the algebra [1–2] has the same structure of representations as the undeformed algebra $sl_2$ [3]. In particular, irreducible highest weight representations $V_s$ are parameterized by a non–negative integer or half–integer number $s$ (referred to as spin) and are $(2s+1)$–dimensional. We will use the standard notation, $|k\rangle$, $k = -s, \ldots, s$, for the basis vectors of $V_s$ such that $S^z|k\rangle = k|k\rangle$, $\langle k'|k\rangle = \delta_{kk'}$.

Let $E$ denote the identity operator on $V_s \otimes V_s$. Consider operator valued functions, $R(\lambda) : \mathbb{C} \rightarrow \text{End } V_s \otimes V_s$, that have the following properties:

- **regularity**: $R(0) = E$, \hspace{1cm} (3)
- **unitarity**: $R(\lambda) R(-\lambda) = E$, \hspace{1cm} (4)
- **spectral decomposition**: $R(\lambda) = \sum_{j=0}^{2s} r_j(\lambda) P_j$, \hspace{1cm} (5)
- **normalization**: $r_{2s}(\lambda) = 1$. \hspace{1cm} (6)
Here $P^j$ is the projector onto the spin $j$ subspace $V_j$ in $V_s^{\otimes 2}$ and $r_j(\lambda)$ is a scalar function. Property (5) is equivalent to the requirement of $U_q(sl_2)$–invariance, i.e.,

$$\left[R(\lambda), \Delta(\xi)\right] = 0 \quad \forall \xi \in U_q(sl_2).$$

(7)

In order to fulfill (3)–(4), the coefficients $r_j(\lambda)$ must satisfy the relations

$$r_j(0) = 1, \quad r_j(\lambda)r_j(-\lambda) = 1.$$  

(8)

In what follows we will assume that $r_j(\lambda)$ are analytic in some neighbourhood of $\lambda = 0$. The normalization condition (6) is imposed in order to eliminate inessential freedom of rescaling $R(\lambda)$ by an arbitrary analytic function preserving the conditions (8).

For a given $R(\lambda)$ satisfying (3)–(4), let us define the Yang–Baxter (YB) operator, $Y(\lambda, \mu) : C^2 \rightarrow \text{End} V_s^{\otimes 3}$, as follows

$$Y(\lambda, \mu) = R_{12}(\lambda) R_{23}(\lambda + \mu) R_{12}(\mu) - R_{23}(\lambda + \mu) R_{12}(\mu) R_{23}(\lambda).$$

(9)

Here and below we use the standard notations – the lower indices specify the components of the tensor product $V_s^{\otimes 3}$. We will say that $R(\lambda)$ is an ($U_q(sl_2)$–invariant) $R$–matrix if the corresponding YB operator vanishes on $V_s^{\otimes 3}$,

$$Y(\lambda, \mu) = 0.$$  

(10)

An advantage of treating the YB equation (10) as the condition of vanishing of the YB operator is that, as we will show below, conditions of vanishing of the YB operator on some subspaces of $V_s^{\otimes 3}$ involve fewer coefficients $r_j(\lambda)$. Moreover, $r_j(\lambda)$ found by resolving such condition for a given subspace can be further used in order to write down and solve conditions of vanishing of the YB operator on other subspaces. A recursive procedure of this type will be presented in the next section.

Remark 1. Eqs. (8) and (10) are preserved under rescaling of the spectral parameter,

$$\lambda \rightarrow \gamma \lambda,$$

(11)

by an arbitrary finite constant $\gamma$. We will regard R–matrices related by such transformation with a finite nonzero $\gamma$ as equivalent.

Remark 2. Conditions (8) and (10) along with the YB equation ensure unitarity of an R–matrix (see Appendix A).

For $q = 1$, there are known [4, 5, 6, 7] four different types of $sl_2$–invariant R–matrices:

$$R(\lambda) = (1 - \lambda)^{-1}\left(E - \lambda P\right),$$

(12)

$$R(\lambda) = P^{2s} + \sum_{j=0}^{2s-1} \left( \prod_{k=j+1}^{2s} \frac{k + \lambda}{k - \lambda} \right) P^j,$$

(13)

$$R(\lambda) = (1 - \lambda)^{-1}\left(E - \lambda P + \frac{\beta \lambda}{\lambda - \alpha} P^0\right),$$

(14)

$$\alpha = s + \frac{1}{2} + (-1)^{2s+1}, \quad \beta = (2s + 1)(-1)^{2s+1},$$

and

$$R(\lambda) = E + (b^2 + 1)\frac{1 - e^\lambda}{e^\lambda - b^2} P^0, \quad b + b^{-1} = 2s + 1,$$

(15)

where

$$P = \sum_{j=0}^{2s} (-1)^{2s-j} P^j$$

(16)
is the permutation in $V_s \otimes V_s$. Observe that for all but the last type of solutions we have

$$R(\pm \infty) = P.$$  

(17)

For $s = \frac{1}{2}$, R–matrices [13] and [14] degenerate into [12] and the fourth solution, [15], is absent. For $s = 1$, R–matrices [13] and [14] are equivalent. For $q = 1$ and $s = 3$, there is known an additional solution which is not of the form [12]– [15]. It is given by

$$R(\lambda) = P^6 + \frac{1 + \lambda}{1 - \lambda} P^5 + \frac{4 + \lambda}{4 - \lambda} P^3 + P^2 + \frac{1 + \lambda}{1 - \lambda} P^1 + \frac{1 + \lambda}{1 - \lambda} 6 + \lambda P^0.$$  

(18)

Numerical, computer–based checks [8] suggest that eqs. (12)–(15) and (18) exhaust the list of $sl_2$–invariant R–matrices, but no corresponding theorem has been proved yet.

The $q \neq 1$ counterparts of (13) and (15) are given by [9, 7]

$$R(\lambda) = \sum_{j=0}^{2s-1} \prod_{k=j+1}^{2s} \left( \frac{[k + \lambda]_q}{[k - \lambda]_q} \right) P^j ;$$  

(19)

$$R(\lambda) = E + (b^2 + 1) \frac{1 - e^\lambda}{e^\lambda - b^2} P^0 , \quad b + b^{-1} = [2s + 1]_q.$$  

(20)

The aim of the present paper is to study $U_q(sl_2)$–invariant solutions of the Yang–Baxter equation for a generic $q$ (i.e., $q$ is not a root of unity and $q \neq 0, \infty$) and to develop a systematic method of finding all possible sets of $r_j(\lambda)$ for a given spin $s$. In particular, we will prove that [12], [14], and [18] have no regular $U_q(sl_2)$–invariant counterparts. Our approach will be based on the fact that $U_q(sl_2)$–invariance of an R–matrix implies that the corresponding YB operator commutes with the action of $U_q(sl_2)$ on $V_s \otimes^3$. This action is defined as

$$S^z_{123} = (\Delta \otimes id) \Delta S^z = S^z_1 + S^z_2 + S^z_3 ,$$

$$S^\pm_{123} = (\Delta \otimes id) \Delta S^\pm = S^\pm_{12} q^{-S^z_3} + q^{S^z_{12}} S^\pm_3 = q S^z_2 S^\pm_3 + S^\pm_1 q^{-S^z_3}.$$  

(21)

(22)

The assertion

$$[Y(\lambda, \mu), S^z_{123}] = [Y(\lambda, \mu), S^\pm_{123}] = 0$$  

(23)

follows from the fact that $P^j$ are functions of $\Delta C$, where $C$ is the Casimir element of $U_q(sl_2)$. It is obvious from (21)–(22) that $P^j_l , l = \{12\}, \{23\}$ commute with $S^z_{123}$ and $S^\pm_{123}$.

2 Reduced Yang–Baxter equations

2.1 Hecke–Temperley–Lieb algebra in YB

Interrelations between Hecke algebras, braid groups, and constant (independent on the spectral parameter $\lambda$) solutions of the YB equation are well known. In the case of nontrivial spectral parameter dependence, a construction of an R–matrix employing the Temperley–Lieb algebra [10] generators was introduced by Baxter in [7]. For the purposes of the present article, we will need the following, slightly generalized, version of this construction.
Lemma 1 Consider an associative algebra over $\mathbb{C}$ with the unit element $E$ and the generators $U_l$ labeled by $l = \{12\}, \{23\}$ obeying the following Hecke-type relations (with $\eta_0$ and $\eta_1$ being scalar constants, $\Re \eta_0 \geq 0$)

\begin{align*}
U_l^2 &= \eta_0 U_l + \eta_1 E, \quad (24) \\
U_{12} U_{23} U_{12} - U_{23} U_{12} U_{23} &= U_{12} - U_{23}. \quad (25)
\end{align*}

Let $g(\lambda)$ be a function analytic in a neighbourhood of $\lambda = 0$ and satisfying the condition $g(0) = 0$. Then $R_l(\lambda) = E + g(\lambda) U_l$ satisfy the YB equation (10) if and only if

$$g(\lambda) = \begin{cases} 
\frac{2\gamma \lambda}{1 - \gamma \lambda}, & \text{if } \eta_0 = 2; \\
\frac{b}{e^{\gamma \lambda} - e^{-\gamma \lambda}}, & b + b^{-1} = \eta_0 \text{ if } \eta_0 \neq 2.
\end{cases} \quad (26)$$

Here $\gamma$ is an arbitrary finite constant.

Remark 3. R–matrices (12) and (15) provide two examples where $U_l$ are elements of $\text{End} V_s \otimes \mathbb{C}$ given by

$$U_{12} = U \otimes e, \quad U_{23} = e \otimes U \quad (27)$$

with $U = E + P$, $\eta_0 = 2$ and $U = P^0$, $\eta_0 = 2s+1$, respectively, and $e$ being the unit element on $V_s$. Let us however stress that, in general, the hypotheses of Lemma 1 do not require that $U_l$ be of the form (27) with $U \in \text{End} V^{\otimes 2}$. In fact, below we will apply Lemma 1 in the cases where the underlying linear space is not of the form $V^{\otimes 3}$.

Proof. For the sake of completeness of the exposition, let us give a proof of the lemma. Substituting $R_l(\lambda)$ into (10) and employing (24)–(25), one reduces the YB equation to the form $(\ldots)(U_{12} - U_{23}) = 0$, where $(\ldots)$ is a scalar factor. Therefore, the YB equation holds if and only if this factor vanishes, which is equivalent to the requirement that $g(\lambda)$ satisfy the following functional relation

$$g(\lambda - \mu) g(\lambda) g(\mu) + \eta_0 g(\lambda - \mu) g(\mu) + g(\lambda - \mu) - g(\lambda) + g(\mu) = 0. \quad (28)$$

By differentiating (28) w.r.t. $\mu$, setting $\mu = \lambda$, and taking into account the condition $g(0) = 0$, one derives the differential equation

$$g'(\lambda) = g'(0) \left((g(\lambda))^2 + \eta_0 g(\lambda) + 1\right). \quad (29)$$

Its solution is given by (26) and it is easily verified that this solution does satisfy (28). $\square$

Remark 4. Analysis of (28) differs if we relax the condition that $g(0) = 0$. In this case, by setting $\mu = \lambda$, eq. (28) is reduced to an algebraic equation,

$$(g(\lambda))^2 + \eta_0 g(\lambda) + 1 = 0, \quad (30)$$

which implies that $g(\lambda)$ is a constant function. The possible values of the constant, i.e., the roots of (30), are the values of (26) in the limit $\gamma \lambda \to \pm \infty$.

Now, we will apply Lemma 1 in order to obtain some information about the spectral decomposition of a regular $U_q(sl_2)$–invariant R–matrix.
Proposition 1 Let \( R(\lambda) \) be a \( U_q(sl_2) \)-invariant solution of the YB equation (10) on \( V_s^{\otimes 3} \) satisfying (7) and (8). Then the second highest coefficient in its spectral decomposition is given by

\[
r_{2s-1}(\lambda) = \begin{cases} 
\frac{1+\gamma \lambda}{1-\gamma \lambda} & \text{if } q = 1; \\
\frac{[2s+\gamma \lambda]}{[2s-\gamma \lambda]} & \text{if } q \neq 1,
\end{cases}
\]

(31)

where \( \gamma \) is an arbitrary finite constant.

Proof. Let \( \tilde{W}_1 \) denote the subspace of \( V_s^{\otimes 3} \) which is the linear span of the vectors

\[
|1\rangle_{123} = |s-1\rangle_1 |s\rangle_2 |s\rangle_3, \quad |2\rangle_{123} = |s\rangle_1 |s-1\rangle_2 |s\rangle_3, \quad |3\rangle_{123} = |s\rangle_1 |s\rangle_2 |s-1\rangle_3.
\]

(32)

From (26) and the Clebsch–Gordan (CG) decomposition of \( V_s^{\otimes 2} \) (see [11] for an explicit form of the CG coefficients),

\[
|2s, 2s-1\rangle = \alpha_s |s\rangle |s-1\rangle + \beta_s |s-1\rangle |s\rangle,
\]

(33)

\[
|s, s\rangle = \beta_s |s\rangle |s-1\rangle - \alpha_s |s-1\rangle |s\rangle,
\]

(34)

\[
\alpha_s = q^s (q^{2s} + q^{-2s})^{-\frac{1}{2}}, \quad \beta_s = q^{-s} (q^{2s} + q^{-2s})^{-\frac{1}{2}},
\]

(35)

we infer that \( \tilde{W}_1 \) is an invariant subspace of the YB operator for the R–matrix under consideration. Notice that the restrictions of \( P^j_l \), \( l = \{12, 13\} \) onto \( \tilde{W}_1 \) vanish if \( j < 2s-1 \). Thus,

\[
R_l(\lambda) \Big|_{\tilde{W}_1} = P^{2s}_l + r_{2s-1}(\lambda) P^{2s-1}_l.
\]

(36)

Furthermore, taking into account (33)–(34) and introducing \( \tilde{g}(\lambda) = r_{2s-1}(\lambda) - 1 \), we observe that (36) can be rewritten in the following form

\[
R_l(\lambda) \Big|_{\tilde{W}_1} = \mathbb{E} + \tilde{g}(\lambda) \pi_l,
\]

(37)

where \( \pi_l \) are projectors, \( \pi^2_l = \pi_l \), given in the basis (22) by

\[
\pi_{12} = \alpha_s^2 |1\rangle \langle 1| - \alpha_s \beta_s |1\rangle \langle 2| - \alpha_s \beta_s |2\rangle \langle 1| + \beta_s^2 |2\rangle \langle 2|,
\]

(38)

\[
\pi_{23} = \alpha_s^2 |2\rangle \langle 2| - \alpha_s \beta_s |2\rangle \langle 3| - \alpha_s \beta_s |3\rangle \langle 2| + \beta_s^2 |3\rangle \langle 3|.
\]

(39)

Now, noticing that

\[
\pi_{12} \pi_{23} \pi_{12} = (\alpha_s \beta_s)^2 \pi_{12}, \quad \pi_{23} \pi_{12} \pi_{23} = (\alpha_s \beta_s)^2 \pi_{23},
\]

(40)

we see that fulfill the conditions of Lemma 1 upon identification \( U_1 = (\alpha_s \beta_s)^{-1} \pi_1 \), \( g(\lambda) = \alpha_s \beta_s \tilde{g}(\lambda) \), and \( \eta_0 = (\alpha_s \beta_s)^{-1} \). Substituting this value of \( \eta_0 \) into (20) and recalling that \( \tilde{g}(\lambda) = r_{2s-1}(\lambda) - 1 \), we obtain (31), where we replaced \( e^{\gamma \lambda} \) with \( q^{2\gamma \lambda} \) for the sake of convenience of comparison with the \( q = 1 \) limit. Since we require that \( R(\lambda) \) be regular, the constant \( \gamma \) must be finite. \( \square \)
2.2 Invariant subspaces

The proof of Proposition 1 demonstrates that reduction of the YB operator to some invariant subspace facilitates finding the coefficients \( r_j(\lambda) \) of an R–matrix. In what follows we will develop this approach further exploiting available knowledge about the CG decomposition of tensor products of representations of \( U_q(sl_2) \). On this way, we will derive systems of coupled functional equations similar to eq. (28) and show that the corresponding necessary conditions are provided by a set of coupled algebraic equations.

In Proposition 1 we used that the YB operator \( \Psi \) commutes with \( S^\pm_{123} \). Now we are going to use that \( Y(\lambda, \mu) \) commutes with \( S^\pm_{123} \) as well.

Let \([t]\) denote the entire part of \( t \). Let us define the subspace \( W_n^{(s)} \subset V_s^\otimes 3 \) for \( n = 0, 1, \ldots, [3s] \) as the linear span of highest weight vectors of spin \((3s−n)\), i.e.,

\[
W_n^{(s)} = \{ \psi \in V_s^\otimes 3 \ | \ S^+_{123} \psi = 0, \ S^z_{123} \psi = (3s−n)\psi \}.
\]

Consider the following two orthonormal bases in \( W_n^{(s)} \) (here and below \([::]\) and \([::]\) stand, respectively, for the CG coefficients and 6–j symbols of \( U_q(sl_2) \))

\[
|n;k\rangle_{123} = \sum_m |m\rangle_1 |2s−k,3s−n−m\rangle_23 \begin{bmatrix} s & 2s−k \\ m & 3s−n−m \end{bmatrix}_{q,3s−n}, \quad (42)
\]

\[
|n;k\rangle'_{123} = \sum_m |2s−k,3s−n−m\rangle_1 |m\rangle_3 \begin{bmatrix} 2s−k & s \\ 3s−n−m & m \end{bmatrix}_{q,3s−n}. \quad (43)
\]

The basis vectors of \( W_n^{(s)} \) are enumerated by integer \( k \in I_n^{(s)} \), where

\[
I_n^{(s)} = \{ 0 \leq k \leq n \quad \text{for} \quad 0 \leq n \leq 2s;
\]

\[
\quad n−2s \leq k \leq 4s−n \quad \text{for} \quad 2s \leq n \leq [3s]. \quad (44)
\]

The sum in (42)–(43) runs over those \( m \) for which the CG coefficients on the r.h.s. of (42), (43) do not vanish, i.e., \((s−n+k) \leq m \leq \min(s,5s−n−k)\).

Consider the transition matrix, \( A^{(s,n)} \), from the basis (42) to the basis (43), i.e., the orthogonal matrix with entries being the following scalar products

\[
A^{(s,n)}_{kk'} = \langle n;k|n;k'\rangle'. \quad (45)
\]

The transition matrix is \( q \)–dependent but for compactness of notations we will not write the argument \( q \) explicitly unless required by the context.

**Proposition 2**

i) entries of \( A^{(s,n)} \) are expressed in terms of 6–j symbols of \( U_q(sl_2) \) as follows

\[
A^{(s,n)}_{kk'} = (-1)^{2s−n} \sqrt{[4s−2k+1]_q [4s−2k'+1]_q} \ \begin{bmatrix} s & 2s−k \\ s & 3s−n \end{bmatrix}_{q,2s−k'} \quad (46)
\]

ii) \( A^{(s,n)} \) is self–dual in \( q \),

\[
A^{(s,n)}_{q} = A^{(s,n)}_{q^{-1}}. \quad (47)
\]

iii) \( A^{(s,n)} \) is orthogonal, symmetric, and coincides with its inverse (\( t \) denotes the matrix transposition),

\[
A^{(s,n)} = (A^{(s,n)})^t = (A^{(s,n)})^{-1}. \quad (48)
\]
As a consequence, the only eigenvalues of $A^{(s,n)}$ are $\pm 1$.

iv) Transition matrices enjoy the following “spin–level duality” relations

$$A^{(s,1)}_q = A^{(\frac{1}{2}+1)}_{q^{2s}},$$
$$A^{(s,n)}_q = A^{(2s-\frac{1}{2},2s-2n)}_q,$$

where $n \leq 2s$.

Explicit formulae for entries of matrix $A^{(s,n)}$ and a proof of its properties listed above are given in Appendices B and C.

Remark 5. It follows from iii) and iv) that $\frac{1}{2}(E \pm A^{(s,n)})$ are projectors of ranks $n_{\pm}$. In particular, for $n \leq 2s$, we have $n_+ = \lfloor \frac{s}{2}+1 \rfloor$, $n_- = \lfloor \frac{s+1}{2} \rfloor$.

Properties of the transition matrix given in Proposition 2 make it an efficient tool for dealing with restrictions of $U_q(sl_2)$–invariant operators to subspaces $W^{(s)}_n$. As a simple example, let us prove the following well–known statement.

Lemma 2 The following identity holds on $V_{s}^{\otimes 3}$

$$P^0_{23} P^j_{12} P^0_{23} = \frac{[2j+1]}{2j} P^0_{23}.$$  

Proof. Observe that $P^0_{12}|_{W^{(s)}_n}$ and $P^0_{23}|_{W^{(s)}_n}$ vanish for all $n$ except $n = 2s$. Thus it suffices to prove (51) when it is restricted to $W^{(s)}_{2s}$. Denote $p^j = P^j_{23}|_{W^{(s)}_2}$. In the basis (12) we have $p^j_{ab} = \delta_a,2s-j \delta_b,2s-j$. Therefore, in this basis, the l.h.s. of (51) acquires the following form

$$p^0 A^{(s,2s)} p^j A^{(s,2s)} p^0 = \left(A^{(s,2s)}_{2s-j,2s}\right)^2 p^0.$$  

The value of $A^{(s,2s)}_{2s-j,2s}$ is easily computable (see eq. (97) in Appendix B) and its square yields the scalar coefficient on the r.h.s. of (51).

2.3 Reduced Yang–Baxter equations

Eqs. (23) imply that $W^{(s)}_n$ is an invariant subspace for the YB operator (9). Let us introduce the reduced YB operator: $Y_n(\lambda, \mu) = Y(\lambda, \mu)|_{W^{(s)}_n}$ (the restriction of $Y(\lambda, \mu)$ onto $W^{(s)}_n$). Notice that restrictions of $P^j_l$ to $W^{(s)}_n$ are diagonal in the bases (43) and (42) for $l = (12)$ and $l = (23)$, respectively. Moreover, they vanish unless

$$|2s-n| \leq j \leq \min(2s, 4s-n).$$

Therefore, in the basis (43), $R_l(\lambda)|_{W^{(s)}_n}$ are represented as

$$R_{12}(\lambda)|_{W^{(s)}_n} = A^{(s,n)} D(\lambda) \left(A^{(s,n)}\right)^{-1}, \quad R_{23}(\lambda)|_{W^{(s)}_n} = D(\lambda),$$

$$D_{kk'}(\lambda) = \delta_{kk'} r_{2s-k}(\lambda),$$

where $k \in I^{(s)}_n$ as specified in (44). Whence, taking the property (48) into account, we conclude that $Y_n(\lambda, \mu)$ acquires the following form in the basis (43)

$$Y_n(\lambda, \mu) = A^{(s,n)} D(\lambda - \mu) A^{(s,n)} D(\lambda) A^{(s,n)} D(\mu)$$

$$- D(\mu) A^{(s,n)} D(\lambda) A^{(s,n)} D(\lambda - \mu) A^{(s,n)}. $$

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The corresponding reduced YB equation reads

\[ A^{(s,n)} D(\lambda - \mu) A^{(s,n)} D(\lambda) A^{(s,n)} D(\mu) = D(\mu) A^{(s,n)} D(\lambda) A^{(s,n)} D(\lambda - \mu) A^{(s,n)}. \]  

(57)

It is the condition of vanishing of the YB operator \( W \) on \( W \). Observe that \( \mathbf{48} \) implies that the reduced YB operator is antisymmetric, \( (Y_n(\lambda, \mu))^\dagger = -Y_n(\lambda, \mu) \). Therefore independent relations contained in \( \mathbf{57} \) are

\[
\sum_{i,j \in I_n^{(s)}} r_{2s-i}(\lambda - \mu) r_{2s-j}(\lambda) \ A_{ij}^{(s,n)} \left( r_{2s-a}(\mu) \ A_{a}^{(s,n)} A_{b}^{(s,n)} \right) - r_{2s-b}(\mu) \ A_{a}^{(s,n)} A_{b}^{(s,n)} = 0, \quad a < b, \quad a, b \in I_n^{(s)}. \]  

(58)

Let us emphasize that eqs. \( \mathbf{58} \) at the level \( n \) ensure, thanks to commutativity of the YB operator with \( S_{123}^{-} \), that the YB operator vanishes not only on the subspace \( W_n^{(s)} \) but also on the larger subspace that is spanned by all vectors obtained by acting on \( W_n^{(s)} \) with \( (S_{123}^{-})^m \), \( m = 0, \ldots, 6s - 2n \). (This picture resembles closely the structure of eigenvectors in the algebraic Bethe ansatz, see \( \mathbf{13} \) for a review). Thus, the set of reduced YB equations \( \mathbf{55} \), \( n = 1, \ldots, [3s] \), is less overdetermined than the initial YB equation \( \mathbf{10} \) containing \( \dim V_n^{(s)} = (2s+1)^3 \) functional equations (although some of them are in general not independent). However, even this set of equations is still overdetermined. Indeed, \( \mathbf{58} \) at level \( n \leq 2s \) involves \( r_j(\lambda) \) with \( j = 2s-n, \ldots, 2s \). Therefore, it suffices to solve \( \mathbf{58} \) for \( n = 1, \ldots, 2s \) to determine all coefficients \( r_j(\lambda) \) of an R–matrix. But these coefficients also have to satisfy the remaining reduced YB equations for \( n = 2s+1, \ldots, [3s] \).

Remark 6. For \( s = \frac{1}{2} \) we have \( 2s = [3s] = 1 \) and therefore the corresponding set of reduced YB equations is not overdetermined. Indeed, in this case \( \mathbf{58} \) contains only one independent equation. A slightly less trivial remark is that the set of reduced YB equations is not overdetermined for \( s = 1 \) as well (see the proof of Proposition 5).

Now, as an immediate application of the reduced YB equation technique, let us prove the following statement.

Proposition 3 Let \( R(\lambda) \) be a \( U_q(sl_2) \)--invariant solution of the YB equation \( \mathbf{17} \) on \( V_n^{(s)} \) for a spin \( s \geq \frac{1}{2} \), \( n \in \mathbb{Z}_+ \) satisfying \( \mathbf{2} \) and \( \mathbf{3} \). Suppose that \( n \) highest coefficients in its spectral decomposition coincide, \( r_{2s}(\lambda) = r_{2s-1}(\lambda) = \ldots = r_{2s-n+1}(\lambda) = 1 \). Then

\[ r_{2s-n}(\lambda) = 1 + \eta_0 \ g(\lambda), \]  

(59)

where \( g(\lambda) \) is given by \( \mathbf{29} \) with

\[ \eta_0 = \frac{[2s-n]! \ [4s-n+1]!}{[2s]! \ [4s-2n+1]!}. \]  

(60)

Here the \( q \)–factorial is defined as \( [n]! = \prod_{k=1}^{n} [k]_q \) for \( n \in \mathbb{Z}_+ \) and \( [0]! = 1 \).

Proof. The corresponding reduced YB equation (with \( n \) in \( \mathbf{57} \) being the same as in \( \mathbf{59} \)) multiplied from the left by \( A^{(s,n)} \) can be regarded as the YB equation \( \mathbf{10} \) for \( R_{12}(\lambda) = D(\lambda) \) and \( R_{23}(\lambda) = A^{(s,n)} D(\lambda) A^{(s,n)}. \) Further, we notice that

\[ D(\lambda) = \mathbb{E} + \tilde{g}(\lambda) \pi, \quad A^{(s,n)} D(\lambda) A^{(s,n)} = \mathbb{E} + \tilde{g}(\lambda) \pi'. \]  

(61)
where $\hat{g}(\lambda) = r_{2s-n}(\lambda) - 1$, $\pi$ is a matrix such that $\pi_{ab} = \delta_{an}\delta_{bn}$, $a, b = 0, \ldots, n$, and $\pi' = A^{(s,n)}\pi A^{(s,n)}$. Obviously, $\pi$ and $\pi'$ are projectors of rank one. Moreover, a computation similar to (52) shows that

$$\pi\pi' = \pi'\pi = \eta_0^{-2}\pi', \quad \pi'\pi' = \eta_0^{-2}\pi', \quad \eta_0 = |A^{(s,n)}|^{-1}. \quad (62)$$

Hence (59) follows by invoking Lemma 1 upon the identification $U_{12} = \eta_0\pi$, $U_{23} = \eta_0\pi'$, and $\hat{g}(\lambda) = \eta_0 g(\lambda)$. Explicit form of $\eta_0$ given in (60) is easily obtained from (57).

This Proposition generalizes both Lemma 1 and Proposition 1. For $n = 1$, eq. (60) yields $\eta_0 = q^{2s}+q^{-2s}$ and we recover the case of Proposition 1. For $n = 2s$, eq. (60) yields $\eta_0 = [2s+1]q$; the corresponding R–matrix is given by (20) which is a particular example covered by Lemma 1 (cf. Remark 3). It is not clear whether an example of R–matrix with such coefficients $r_j(\lambda)$ as described in Proposition 3 exists for $n \neq 1$ and $n \neq 2s$. Nevertheless this proposition is useful for the analysis of solutions of the YB equation (see the next section). Another statement useful for this analysis reads as follows.

**Proposition 4** Let $R(\lambda)$ be a $U_q(sl_2)$–invariant solution of the YB equation (66) on $V^{\otimes 3}$ for a half–integer spin $s \geq \frac{3}{2}$ satisfying (62) and (63). Then the coefficients $r_{s-\frac{1}{2}}(\lambda)$ and $r_{s+\frac{1}{2}}(\lambda)$ in its spectral decomposition are related as follows

$$r_{s-\frac{1}{2}}(\lambda) = \left\{ \begin{array}{ll}
\frac{1+\gamma\lambda}{1-\gamma\lambda} & \text{if } q = 1; \\
\frac{|s+\frac{1}{2}+\gamma\lambda|_q}{|s+\frac{1}{2}-\gamma\lambda|_q} & \text{if } q \neq 1,
\end{array} \right. \quad (63)$$

where $\gamma$ is an arbitrary finite constant.

**Proof.** Matrix $D(\lambda)$ in the reduced YB equation can be multiplied by an arbitrary function $\varphi(\lambda)$ analytic in a neighbourhood of $\lambda = 0$ and satisfying $\varphi(\lambda)\varphi(-\lambda) = 1$. Therefore, in (57) for $n = 3s-\frac{1}{2}$, we can choose $D(\lambda) = \text{diag}(1, g(\lambda))$, where $g(\lambda) = \frac{r_{s-\frac{1}{2}}(\lambda)}{r_{s+\frac{1}{2}}(\lambda)}$. Further, by the duality relation (20), we have $A^{(s,3s-\frac{1}{2})} = A^{(\frac{1}{2}+\frac{1}{2})}$ for half–integer spins $s \geq \frac{3}{2}$. Whence, applying Proposition 1 we conclude that $g(\lambda) = r_{2s'}-1(\lambda)$, where $s' = \frac{s}{2} + \frac{1}{4}$. 

### 2.4 Necessary conditions

Differentiating (58) w.r.t. $\mu$, setting $\mu = \lambda$, and taking into account the regularity condition $D(0) = E$, we derive the following system of equations (prime denotes derivative w.r.t. the spectral parameter):

$$\sum_{i,j \in I_n^{(s)}} r'_{2s-i}(0) r_{2s-j}(\lambda) A^{(s,n)}_{i,j} \left( r_{2s-a}(\lambda) A^{(s,n)}_{a,j} A^{(s,n)}_{a,i} - r_{2s-b}(\lambda) A^{(s,n)}_{a,i} A^{(s,n)}_{j,b} \right) = A^{(s,n)}_{a,b} \left( r'_{2s-a}(\lambda) r_{2s-b}(\lambda) - r'_{2s-b}(\lambda) r_{2s-a}(\lambda) \right), \quad a < b, \quad a, b \in I_n^{(s)}. \quad (64)$$

Here we have carried out the summation on the r.h.s. by using that $(A^{(s,n)} A^{(s,n)})_{ab} = \delta_{ab}$. It is important to emphasize here that although (64) contain derivatives, they are actually linear algebraic equations on $r_j(\lambda)$ for $j \neq a, b$.

It is easy to check that (64) is satisfied trivially for $\lambda = 0$. Therefore, let us look at higher order terms in the expansion of (64) about $\lambda = 0$. Denote $r'_{2s-a}(0) \equiv \xi_a$. In the first order in $\lambda$, the summation over $i, j$ can be carried out, and we obtain the conditions

$$A^{(s,n)}_{a,b} \left( r''_{2s-a}(0) - r''_{2s-b}(0) \right) = A^{(s,n)}_{a,b} \left( \xi_a^2 - \xi_b^2 \right), \quad (65)$$
that are always satisfied, because the unitarity \(^4\) implies that
\[
r_{2s-a}'(0) = \xi_a^2.
\]
In the second order in \(\lambda\), eqs. \((64)\) turn into a system of algebraic equations
\[
\sum_{i,j \in I_n^{(s)}} \xi_i \xi_j A_{ij}^{(s,n)} \left( A_{aj}^{(s,n)} A_{ib}^{(s,n)} - A_{ai}^{(s,n)} A_{jb}^{(s,n)} \right) \\
+ (\xi_a - \xi_b) \sum_{i,j \in I_n^{(s)}} \xi_i \xi_j A_{ij}^{(s,n)} \left( A_{aj}^{(s,n)} A_{ib}^{(s,n)} + A_{ai}^{(s,n)} A_{jb}^{(s,n)} \right) \\
= A_{ab}^{(s,n)} \left( r_{2s-a}'(0) - r_{2s-b}'(0) - \xi_a^3 + \xi_b^3 + \xi_a^2 \xi_b - \xi_b^2 \xi_a \right), \quad a < b, \quad a, b \in I_n^{(s)}.
\]
Remarkably, equations \((64)\) and \((67)\) can be solved in a recursive way. Let us provide the corresponding algorithm. We start with the level \(n = 1\), where we have \(r_{2s}(\lambda) = 1\) and \(r_{2s-1}(\lambda)\) is given by \((31)\). Now, suppose we have found \(r_j(\lambda), \; j = 2s-n, \ldots, 2s\) that solve eqs. \((63)\) for a given level \(n < 2s\). Then \((63)\) and \((67)\) at the level \((n+1)\) allow us to express \(r_{2s-n-1}(\lambda)\) algebraically in terms of the previously found \(r_j(\lambda)\). Indeed, since we already know \(\xi_j\) for \(j = 0, \ldots, n\), eqs. \((67)\) for \(2s-n \leq a < b \leq n\) turn into quadratic equations w.r.t. \(\xi_{n+1}\). Solving them and substituting the found values of \(\xi_{n+1}\) into \((64)\) for \(2s-n \leq a < b \leq n\), we obtain a system of linear equations on \(r_{2s-n-1}(\lambda)\). Finding all possible solutions to this system completes the \((n+1)\)-th step of recursion. Continuing this procedure up to \(n = 2s\), we will obtain all possible solutions for all \(r_j(\lambda)\) and thus construct all possible ansätze for regular \(U_q(sl_2)\)-invariant R–matrices of spin \(s\). Next, since \((64)\) provide necessary but not sufficient conditions, we have to check which of these ansätze indeed satisfy the YB equation \((10)\) or, alternatively, the reduced YB equations \((58)\) for all \(n\) up to \([3s]\).

2.5 Spin chain Hamiltonians and reconstruction of R–matrices

The utmost importance of the YB equation in the quantum inverse scattering method (see \([12,13]\) for a review) is due to the fact that its solutions can be used to construct families of quantum integrals of motion in involution. In particular, regular solutions of the YB equation allow to construct local integrals of motion for lattice models. For the R–matrix of type \((5)\), the first of these integrals,
\[
\mathcal{H} = \sum_k H_{k,k+1}, \quad H = \partial_\lambda R(\lambda) \bigg|_{\lambda=0} = \sum_{j=0}^{2s-1} \xi_{2s-j} P^j,
\]
is usually regarded as a Hamiltonian of a spin \(s\) magnetic chain with the nearest neighbour interaction. Here \(H \in \text{End} \; V_{s}^\otimes 2\) and \(\mathcal{H} \in \text{End} \; V_{s}^\otimes L\), where \(L\) is the number of lattice sites. Notice that in \((58)\) we took into account the normalization condition \((6)\), which implies \(\xi_0 = 0\) (this fixes the choice of the additive constant in the Hamiltonian). R–matrices equivalent in the sense of transformation \((11)\) yield Hamiltonians related simply by rescaling \(H \rightarrow \gamma H\); we will regard such Hamiltonians as equivalent.

It is important to remark that, as it was observed in \([5]\), for regular solutions of the YB equation different Hamiltonians correspond to inequivalent R–matrices. In the present context this statement can be formulated as follows.
Lemma 3 Let $R^{(1)}(\lambda)$ and $R^{(2)}(\lambda)$ be two solutions of the YB equation \((10)\) on $V_s \otimes V_s$ satisfying \((3), (7), \) and \((8)\). The corresponding Hamiltonians given by \((68)\) coincide, $H^{(1)} = H^{(2)}$, if and only if $R^{(1)}(\lambda) = R^{(2)}(\lambda)$.

Proof. The “if” part is obvious. Further, Theorem 3 in [8] asserts that if the Hamiltonians corresponding to two regular $R$–matrices analytic in a neighbourhood of $\lambda = 0$ coincide, then $R^{(1)}(\lambda) = \varphi(\lambda) R^{(2)}(\lambda)$, where the scalar function $\varphi(\lambda)$ is analytic in a neighbourhood of $\lambda = 0$ and satisfies the condition $\varphi(0) = 1$. In the case under consideration, analyticity of $r^j(\lambda)$ along with condition \((6)\) imply that $\varphi(\lambda) = 1$. □

Remark 7. The algorithm described at the end of the previous subsection complements this Lemma with a constructive procedure that allows us to reconstruct the $R$–matrix from a given Hamiltonian. Indeed, if we know a Hamiltonian in the form \((68)\), i.e., we know all $\xi_j$, then we can solve \((64)\) recursively starting with $r^2 \equiv 1$ thus recovering all the coefficient $r^j(\lambda)$ of the corresponding regular $U_q(sl_2)$–invariant $R$–matrix. In contrast to the general situation, Lemma 3 guarantees that the resulting set of $r^j(\lambda)$ will be unique.

3 Analysis of reduced Yang–Baxter equations

3.1 Asymptotic solutions

Let us remark that the technique described above applies in the limit $\lambda \rightarrow \pm \infty$ as well. In this limit, assuming that $R^{\pm 1} = \lim_{\lambda \rightarrow \pm \infty} R(\lambda)$ exist, the YB equation \((10)\) turns into

$$
\hat{R}_{12} \hat{R}_{23} \hat{R}_{12} = \hat{R}_{23} \hat{R}_{12} \hat{R}_{23}.
$$

Denote $d_j = \lim_{\lambda \rightarrow \pm \infty} r^j(\lambda)$, so that $\hat{R} = \sum_{j=0}^{2s} d_j P^j$. Condition \((6)\) implies that $d_{2s} = 1$. Taking the limit $\lambda \rightarrow \infty$ in \((57)\), we obtain a set of algebraic equations on the coefficients $d_j$,

$$
A^{(s,n)} D A^{(s,n)} D A^{(s,n)} D = D A^{(s,n)} D A^{(s,n)} D A^{(s,n)} D.
$$

Here $n = 1, \ldots, [3s]$, and $D_{kk'} = \delta_{kk'} d_{2s-k}$, where $k \in I_{n}^{(s)}$. Analogously to the spectral dependent case, independent equations contained in \((70)\) are

$$
(d_a - d_b) \sum_{i,j \in I_{n}^{(s)}} d_i d_j A^{(s,n)}_{ij} A^{(s,n)}_{ai} A^{(s,n)}_{jb} = 0, \quad a < b, \quad a, b \in I_{n}^{(s)}.
$$

This system of equations can be solved in a recursive way with the help of the algorithm described at the end of subsection 2.4. In this context, it is worth noticing that one particular solution is known a–priory, namely

$$
d_j = (-1)^{2s-j} q^{2s(2s+1)-j(j+1)},
$$

which corresponds to \((14)\) in the limit $q^\lambda \rightarrow \infty$.

3.2 Reduced YB for $n = 1$ and $n = 2$

The explicit form of $n = 1$ transition matrix is

$$
A^{(s,1)} = \frac{1}{\{2s\}_q} \left( \frac{1}{\sqrt{1+\{4s\}_q}} \begin{pmatrix} \sqrt{1+\{4s\}_q} \\ 1 \end{pmatrix} \right),
$$

(73)
where we denoted \( \{t\}_q = q^t + q^{-t} \). In this case eq. \( \text{(54)} \) contains only one equation. Taking into account that \( r_{2s}(\lambda) = 1 \) and making substitution \( r_{2s-1}(\lambda) = 1 + \{2s\}_q g(\lambda) \), it is easy to see that this equation coincides with \( (29) \). Hence we recover the same expression for \( r_{2s-1}(\lambda) \) as in Proposition \( \text{[3]} \). Accordingly, eq. \( (71) \) is either satisfied trivially if \( d_{2s-1} = 1 \) or it represents a quadratic equation with roots \( d_{2s-1} = -q^{\pm 4s} \).

The explicit form of \( n = 2 \) transition matrix is

\[
A^{(s,2)} = \begin{pmatrix}
\frac{[2s-1]_q}{\{2s\}_q [4s-1]_q} & \rho_s \sqrt{\frac{\{2q\} [6s-1]_q}{\{2s\}_q [4s-1]_q}} & \rho_s \sqrt{\frac{[6s-1]_q [6s-2]_q}{[4s-1]_q}} \\
\rho_s \sqrt{\frac{\{2s\}_q [6s-1]_q}{[4s-1]_q}} & \{4s-1\}_q (\rho_s)^2 & -\rho_s \sqrt{\frac{\{2q\} [6s-2]_q}{[2s-1]_q [4s-1]_q}} \\
\rho_s \sqrt{\frac{[6s-1]_q [6s-2]_q}{[4s-1]_q}} & -\rho_s \sqrt{\frac{\{2s\}_q [6s-2]_q}{[2s-1]_q [4s-1]_q}} & \frac{[2s]_q}{[4s-1]_q}
\end{pmatrix}, \tag{74}
\]

where we introduced \( \rho_s = \{(2s-1)_q \{2s\}_q\}^{-\frac{1}{2}} \). For computations, it is useful to observe the following identities:

\[
A_{01}^{(s,2)} A_{12}^{(s,2)} = A_{02}^{(s,2)} (A_{11}^{(s,2)} - 1), \quad A_{11}^{(s,2)} = 1 - \{2\}_q (\rho_s)^2. \tag{75}
\]

Analysis of \( \text{(58)} \) splits into two cases: \( r_{2s-1}(\lambda) = 1 \) and \( r_{2s-1}(\lambda) \neq 1 \). The former case is covered by Proposition \( \text{[3]} \) which yields

\[
r_{2s-2}(\lambda) = \frac{b^2 e^\lambda - 1}{b^2 - e^\lambda}, \quad b + b^{-1} = \frac{[4s-1]_q [2s-1]_q}{[2s]_q}. \tag{76}
\]

In the latter case, without loss of generality we can choose \( \gamma = 1 \) in \( \text{(31)} \), which corresponds to

\[
r_{2s-1}(\lambda) = \frac{[2s+\lambda]_q}{[2s-\lambda]_q}, \quad d_{2s-1} = -q^{4s}, \quad \xi_1 = \kappa_q \frac{\{2s\}_q}{[2s]_q}, \quad \xi_q \equiv \frac{2 \log q}{q - q^{-1}}. \tag{77}
\]

In this case, one finds easily that the three equations contained in \( \text{(71)} \) for \( n = 2 \) have only one common root, namely

\[
d_{2s-2} = q^{8s-2}. \tag{78}
\]

Substituting \( \xi_0 = 0 \) and \( \xi_1 \) given by \( \text{(77)} \) into \( \text{(67)} \), we obtain a quadratic equation on \( \xi_2 \) with roots given by

\[
\xi_2 = \frac{2 \kappa_q [4s-1]_q}{[2s-1]_q [2s]_q}, \quad \xi_2 = \kappa_q (q - q^{-1})^2 \frac{[4s-1]_q}{\{4s-1\}_q}. \tag{79}
\]

The corresponding solutions of \( \text{(64)} \) are given by

\[
r_{2s-2}(\lambda) = \frac{[2s+\lambda]_q [2s-1+\lambda]_q}{[2s-\lambda]_q [2s-1-\lambda]_q}, \quad r_{2s-2}(\lambda) = \frac{[4s-1+\lambda]_q}{\{4s-1-\lambda\}_q}. \tag{80}
\]

It is straightforward to verify that both these solutions satisfy the \( n = 2 \) level reduced YB equation \( \text{(58)} \).
Proposition 5 For a generic $q$ and spin $s = 1$, inequivalent regular $U_q(sl_2)$–invariant solutions of the YB equation (17) satisfying the condition (7) are exhausted by the following three types

$$R(\lambda) = P^2 + P^1 + \frac{b^2 e^{\lambda} - 1}{b^2 - e^{\lambda}} P^0, \quad b + b^{-1} = [3]_q,$$

(81)

$$R(\lambda) = P^2 + \frac{[2+\lambda]_q}{[2-\lambda]_q} P^1 + \frac{[2+\lambda]_q [1+\lambda]_q}{[2-\lambda]_q [1-\lambda]_q} P^0,$$

(82)

$$R(\lambda) = P^2 + \frac{[2+\lambda]_q}{[2-\lambda]_q} P^1 + \frac{[3+\lambda]_q}{[3-\lambda]_q} P^0.$$

(83)

Proof. For $n \leq 2s$ we have $\dim W_n^{(s)} = (n+1)$. As it has already been mentioned, reduced YB equation (57) ensures vanishing of the YB operator on all vectors of the form $(S_{123})^m W_n^{(s)}$, i.e., on a subspace of dimension $\Delta_n^{(s)} = (6s-2n+1) \dim W_n^{(s)}$. In particular, we have $\Delta_0^{(1)} + \Delta_1^{(1)} + \Delta_2^{(1)} = 26$, which means that the level $n = 3$ reduced YB equation is satisfied automatically, since the corresponding subspace $W_3^{(1)}$ is one dimensional. Therefore, for $s = 1$, the $n = 1, 2$ reduced YB equations provide not only necessary but also sufficient conditions. As we have shown above in this subsection, solutions to these equations are exhausted by (76) and (80) which for $s = 1$ yield (81)–(83).

Thus, for spin $s = 1$, all three inequivalent $sl_2$–invariant R–matrices (15), (13), and (12) have $U_q(sl_2)$–invariant counterparts. Two of them belong to the well–known types (19) and (20). The last one, (83), appears to be rather an exceptional case; it was found previously (14) by means of Baxterization of the Birman–Wenzl–Murakami algebra.

3.3 Reduced YB for $n = 3$

For $n = 3$ the transition matrix has 12 entries given by (97)–(98) and the remaining four entries are

$$A_{11}^{(s,3)} = \frac{[2]_q [2s-1]_q [6s-2]_q - ([2s-2]_q)^2}{[2s-1]_q [4s-3]_q [4s]_q},$$

(84)

$$A_{12}^{(s,3)} = A_{21}^{(s,3)} = \frac{[2s-2]_q}{[4s-2]_q} \sqrt{\frac{[2s-1]_q [6s-3]_q}{[4s-4]_q [4s-3]_q [4s-1]_q [4s]_q}},$$

(85)

$$A_{22}^{(s,3)} = \frac{[2s-2]_q - [2]_q [6s-3]_q}{[2s-2]_q [2s-1]_q [4s-1]_q}.$$  

(86)

Proposition 6 For a generic $q$ and spin $s = \frac{3}{2}$, inequivalent regular $U_q(sl_2)$–invariant solutions of the YB equation (11) satisfying the condition (6) are exhausted by the two types given by (19) and (20).

Proof. Let us analyse the spectral decomposition of possible spin $\frac{3}{2}$ solutions to the reduced YB equations for $n = 1, 2, 3$. The first possibility is $r_2(\lambda) = r_1(\lambda) = 1$, in which case $r_0(\lambda)$ is determined by Proposition 4; the corresponding R–matrix is given by (20).

Next, the case $r_2(\lambda) = 1, r_1(\lambda) \neq 1$ is covered by the same proposition for $n = 2$ and $r_1(\lambda)$ is given by (76). However, this case is ruled out, because (76) for $s = 1$ is incompatible with the statement of Proposition 4 which requires $b = q^2$. In the remaining case, $r_2(\lambda) \neq 1,$
without loss of generality we can choose \( \gamma = 1 \) in (61), which yields \( r_2(\lambda) = \frac{[3+\lambda q][2+\lambda q]}{[3-\lambda q][2-\lambda q]} \), and, according to the analysis carried out in the previous subsection, \( r_1(\lambda) \) is given by one of the expressions in (80). However, the second form in (80) is ruled out again as incompatible with Proposition 4. Thus, we are left with expressions in (80). However, the second form in (80) is ruled out again as incompatible with Proposition 4. Thus, we are left with expressions in (80). However, the second form in (80) is ruled out again as incompatible with Proposition 4. Thus, we are left with expressions in (80). However, the second form in (80) is ruled out again as incompatible with Proposition 4. Thus, we are left with expressions in (80). However, the second form in (80) is ruled out again as incompatible with Proposition 4. Thus, we are left with

\[
\xi_0 = 0, \quad \xi_1 = \kappa_q \frac{[3]\lambda_q}{[2][3]q}, \quad \xi_2 = 2\kappa_q \frac{[5]q}{[2][3]q}. \tag{87}
\]

Substituting these values into (67) for \( n = 3 \) and \( s = \frac{3}{2} \), we obtain a system of three quadratic equations on \( \xi_3 \). A direct computation using (37)–(38) and (84)–(85) shows that these equations have only one common root given by

\[
\xi_3 = \kappa_q \frac{[2][5]+3[2]\lambda_q}{[2][3]q}, \tag{88}
\]

which is the value corresponding to (19) for \( s = \frac{3}{2} \). By Lemma 3, an \( R \)-matrix determined by (87) and (88) is unique and therefore it is the one given by (19).

The proven proposition shows that, unlike the case of spin \( s = 1 \), only two out of four \( sl_2 \)-invariant \( R \)-matrices (12)–(15) have \( U_q(sl_2) \)-counterparts for spin \( s = \frac{3}{2} \). Actually, analysing the \( n = 3 \) reduced YB equations, we can extend this observation to higher spins as well.

**Proposition 7** Let \( R(\lambda) \) be a \( U_q(sl_2) \)-invariant solution of the YB equation (77) on \( V_s^\otimes 3 \) for a spin \( s \geq 2 \) satisfying (7) and (8). If \( r_{2s-1}(\lambda) = \frac{[2s+\lambda q]}{[2s-\lambda q]} \), then

\[
r_{2s-2}(\lambda) = \frac{[2s+\lambda q][2s-1+\lambda q]}{[2s-\lambda q][2s-1-\lambda q]}. \tag{89}
\]

As a consequence, for \( s \geq 2 \), there exist no \( U_q(sl_2) \)-invariant regular \( R \)-matrices whose \( q \to 1 \) limit coincides with (12) or (14).

**Proof.** Let \( q = 1 + h, h \ll 1 \), so that \( [t]_q = t + (t-1)h^2/3 + O(h^3) \). Since \( A_q^{(s,n)} \) depends on \( q \) smoothly, we have \( A_q^{(s,n)} = A_{q=1}^{(s,n)} + O(h^2) \). Consider the \( n = 3 \) reduced YB equations (67) where \( \xi_0 = 0, \xi_1 \) is as in (77), and \( \xi_2 \) is given by the second expression in (79). Using (87)–(88) and (84)–(85), we find the following \( h \)-expansions of these equations for \( (a,b) = (0,1), (0,2), \) and \( (1,3) \), respectively

\[
0 = (5s^2-3s)\xi_3^2 + (3-6s)\xi_3 + 1 \tag{90}
\]

\[
+ \frac{2}{3}h^2 \left( (25s^4-32s^3+9s^2)\xi_3^2 + (78s^3-81s^2+21s)s\xi_3 - 47s^2+35s-3 \right) + O(h^3),
\]

\[
0 = h^2 \left( (7s^2-3s)\xi_3^2 + (3-10s)\xi_3 + 3 \right) + O(h^3), \tag{91}
\]

\[
0 = (5s^2-3s)\xi_3^2 + (3-6s)\xi_3 + 1 \tag{92}
\]

\[
+ \frac{2}{3}h^2 \left( (19s^4-35s^3+17s^2-3s)\xi_3^2 + (138s^3-201s^2+96s-15)\xi_3 - 85s^2+90s-20 \right) + O(h^3).
\]

We see that, in the zeroth order in \( h \), (91) is satisfied trivially, whilst (90) and (92) yield the same quadratic equation which has the following roots

\[
\xi_3 = \frac{1}{s}, \quad \xi_3 = \frac{1}{5s-3}. \tag{93}
\]
Thus, for \( q = 1 \), eqs. (90)–(92) are compatible (in particular, the first value in (93) corresponds to solutions of the type (12) and (13)).

In the second order in \( h \), (91) has roots \( \xi_3 = \frac{1}{3} \) and \( \xi_3 = \frac{1}{6} \). But, for (90) and (92), the \( h^2 \) corrections to the first value in (93) are

\[
\xi_3 = \frac{1}{3} + h^2 \left( \frac{28}{3} s - 6 \right) + O(h^3), \quad \xi_3 = \frac{1}{6} + h^2 \left( \frac{46}{3} s - 12s \right) + O(h^3).
\]

Therefore already in the second order in \( h \) compatibility of (90)–(92) is lost. Which implies that the second expression in (80) for \( r_{2s-2}(\lambda) \) is ruled out. And, as follows from the analysis of the previous subsection, the only possible form of \( r_{2s-2}(\lambda) \) is the first expression in (80). An R–matrix with such spectral coefficient cannot be a \( q \)--deformation of (12) or (13) for \( s \geq 2 \), because the corresponding value of \( \xi_2 \) does not vanish in the limit \( q \rightarrow 1 \).

Remark 8. Notice that the coefficient \( r_3(\lambda) \) of the exceptional solution (18) corresponds (after rescaling \( \lambda \rightarrow \lambda/6 \)) to the second value in (93). As we have seen in the proof of Proposition 7 this value is not a root of (91) for \( h \neq 0 \). Therefore, we conclude that (18) has no regular \( U_q(sl_2) \)--invariant counterpart.

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Appendix A

Lemma 4 Let \( R(\lambda) \) be a \( U_q(sl_2) \)--invariant solution of the YB equation (16) on \( V^\otimes 3_s \) satisfying conditions (3) and (4). Then \( R(\lambda) \) is unitary, i.e., it satisfies (7) as well.

Proof. Eq. (5) ensures that \( R(\lambda) \) commutes with \( R(\mu) \). Introduce \( X^\lambda = R(\lambda)R(-\lambda) \). Then the YB equation for \( \mu = -\lambda \) implies that \( X^\lambda_{12} = X^\lambda_{23} \). Applying \( \text{tr}_{23} \) and \( \text{tr}_3 \) to this equality (along the lines of (8), where a less trivial equation \( X_{12} - X_{23} = Z_{123} \) was considered), one infers that \( X^\lambda = cE \), \( c \) being a scalar constant. On the other hand, we have \( X^\lambda = P^{2s} + \ldots \), according to (9). Hence \( c = 1 \) and \( X^\lambda = E \).

Appendix B

The co-multiplication (2) determines the structure of the Clebsch–Gordan (CG) decomposition of tensor products of irreducible representations. The corresponding CG coefficients and 6–j symbols were derived and studied in (11). The particular 6–j symbol which appeared in (16) is given by

\[
\begin{pmatrix} s & s & 2s-k \\ s & 3s-n & 2s-k' \end{pmatrix}_q = F_k^s F_k'^s \sum_l (-1)^l [l+1]! \left([l-4s+k]![l-4s+k']! \times [l-6s+n+k]![l-6s+n+k']![6s-n-l]![6s-k-k'-l]![8s-n-k-k'-l]!\right)^{-1},
\]

where

\[
F_k^s = [2s-k]! \left(\frac{[k]! [n-k]! [2s-n+k]! [4s-n-k]!}{[4s-k+1]! [6s-n-k+1]!}\right)^{\frac{1}{2}}.
\]

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and the q–factorial is defined as \([n]! = \prod_{k=1}^{n}[k]_q\) for \(n \in \mathbb{Z}_+\) and \([0]! = 1\). The sum in (95) runs over those \(l\) for which the arguments of the q–factorials are non–negative.

For \(n \leq 2s\) and \(k' = 0\) or \(k' = n\), the sum on the r.h.s. of (95) contains only one term \((l = 6s – n)\) of \((l = 6s – n/k)\), respectively, and we obtain

\[
A_{k,n}^{(s,n)} = \frac{(-1)^k \sqrt{[4s-2k+1]_q}}{[2s-n]!} \times \left( \frac{[n]! [2s]! [2s-n+k]! [4s-n-k]! [4s-2n+1]! [6s-n-k+1]!}{[k]! [n-k]! [4s-k+1]! [4s-n+1]! [6s-2n+1]!} \right)^{1/2},
\]

\[
A_{0,k}^{(s,n)} = \frac{[2s]! \sqrt{[4s-2k+1]_q}}{[2s-n]!} \times \left( \frac{[n]! [4s-n]! [4s-n-k]! [6s-n+1]!}{[k]! [n-k]! [2s-n+k]! [2s-n]! [4s-k+1]! [6s-2n-k+1]!} \right)^{1/2}.
\]

**Appendix C**

**Proof of Proposition 2**

i) Applying the CG decomposition to the \(\{23\}\) and \(\{12\}\) components in (12) and (13), respectively, and using the orthonormality of the basis of \(V_s\), \(\langle p|p'\rangle = \delta_{pp'}\), it is straightforward to find that the scalar product in (15) is given by

\[
A_{kk'}^{(s,n)} = \sum_{m,m'} \left[ \begin{array}{ll} s & 2s-k \\ m & 3s-n-m \end{array} \right]_q \left[ \begin{array}{ll} 2s-k' & s \\ m' & 3s-n-m' \end{array} \right]_q \times \left[ \begin{array}{ll} 2s-k & 2s-n \\ 3s-n-m & 3s-n \end{array} \right]_q,
\]

(99)

In order to carry out the summation over \(m\) we invoke the following identity [11]

\[
\sum_m \left[ \begin{array}{ll} a & b \\ m & m' \\ m'' & m''+m' \end{array} \right]_q \left[ \begin{array}{ll} b & d \\ m' & m''+m' \end{array} \right]_q \left[ \begin{array}{ll} f & c \\ m'' & m''+m' \end{array} \right]_q \left[ \begin{array}{ll} a & e \\ m & m''+m' \end{array} \right]_q = (-1)^{a+b+c+d} \left[ \begin{array}{ll} 2e+1 \\ q \end{array} \right]_q \left[ 2f+1 \right]_q \left[ \begin{array}{ll} e & d \\ m' & m''+m' \end{array} \right]_q \left[ \begin{array}{ll} a & e \\ m & m''+m' \end{array} \right]_q \left[ \begin{array}{ll} d & e \\ c & f \end{array} \right]_q.
\]

(100)

After this the summation over \(m'\) reduces to

\[
\sum_{m'} \left[ \begin{array}{ll} 2s-k' & s \\ 3s-n-m' & 3s-n \end{array} \right]_q^2 = \langle n; k' | n; k \rangle = 1.
\]

(101)

The remaining factors in (100) yield the r.h.s. of (16).

ii) The self–duality of the transition matrix with respect to \(q \rightarrow q^{-1}\) follows from the fact that 6–j symbols are invariant with respect to this operation (because, unlike the CG coefficients, they are expressed entirely in terms of q–numbers [11]).

iii) The obvious invariance of (95) with respect to \(k \leftrightarrow k'\) implies that the transition matrix is symmetric. Since \(A_{q(k,n)}^{(s,n)}\) is orthogonal by construction, we conclude that \(A_{q(k,n)}^{(s,n)}\) coincides with its inverse.

iv) Formula (19) is obvious from (73). The duality relation (30) in terms of matrix entries looks as follows

\[
A_{k,k'}^{(s,n)} = A_{k,k'}^{(\tilde{s},\tilde{n})},
\]

(102)

\[
\tilde{s} = 2s - \frac{n}{2}, \quad \tilde{n} = 6s - 2n, \quad \tilde{k} = k - n + 2s, \quad \tilde{k}' = k' - n + 2s,
\]

(103)
where $0 \leq k, k' \leq n$. The shifts in $\tilde{k}, \tilde{k}'$ are necessary in order to satisfy (44) (notice that $2s-n = \tilde{n} - 2\tilde{s} \geq 0$). Eq. (102) is checked straightforwardly by making the change of variables (103) in (105) and using the explicit expressions (95)–(96). This completes the proof. □

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