Relativistic dissipative hydrodynamics with extended matching conditions for ultra-relativistic heavy-ion collisions

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Abstract. Recently we proposed a novel approach to the formulation of relativistic dissipative hydrodynamics by extending the so-called matching conditions in the Eckart frame [Phys. Rev. C 85, (2012) 14906]. We extend this formalism further to the arbitrary Lorentz frame. We discuss the stability and causality of solutions of fluid equations which are obtained by applying this formulation to the Landau frame, which is more relevant to treat the fluid produced in ultra-relativistic heavy-ion collisions. We derive equations of motion for a relativistic dissipative fluid with zero baryon chemical potential and show that linearized solutions obtained from them are stable against small perturbations. It is found that conditions for a fluid to be stable against infinitesimal perturbations are equivalent to imposing restrictions that the sound wave, \( c_s \), propagating in the fluid, must not exceed the speed of light \( c \), i.e., \( c_s < c \). This conclusion is equivalent to that obtained in the previous paper using the Eckart frame [Phys. Rev. C 85, (2012) 14906].

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1 Introduction

It is known that one faces problems of instability and violation of causality in solutions obtained from the relativistic Naiver-Stokes (NS) equations (observed in the first-order theories [12] which naively extend the non-relativistic NS equation). Israel and Stewart (IS) provide a phenomenological framework for a relativistic dissipative fluid [44] accounting for these problems. In their model, a possible general form of the non-equilibrium entropy current is described by the dissipative part of the energy-momentum tensor and by the particle current up to second order in the deviation from the equilibrium state. After the stability and causality of the IS theory have been shown by Hiscock and Lindblom [5,6,7], the causal dissipative hydrodynamical model was adapted to study the dynamics of hot matter produced in ultra-relativistic heavy-ion collisions by Muronga [8-9]. In recent years, the IS model plays an important role in the analysis of the experimental data obtained by RHIC and LHC (see, for example, [10]).

In parallel to the application of IS theory, investigations of the basis of relativistic dissipative hydrodynamic theory was continued [11,12,13,14,15,16,17]. This is because the IS theory, in its present form, is too general and complex from the point of view of quantum chromodynamics (QCD), which is believed to be the fundamental theory for strongly interacting systems [11]. Another reason is that the theory of relativistic dissipative hydrodynamics is not yet fully understood because, for example, the equation of motion of the fluid it uses depends on the choice of the Lorentz frame [16] (or on the definition of the hydrodynamical flow). Since the dissipative part of the energy-momentum tensor \( \delta T^{\mu\nu} \) and the particle current \( \delta N^\mu \) cannot be determined uniquely by the second law of thermodynamics, one usually introduces some constraints to fix them. These constraints are known as matching conditions.

The other reason to introduce these matching (or fitting) conditions [4] is the necessity of matching the energy density and baryonic charge density, \( (\varepsilon, n) \), in a non-equilibrium state to the corresponding equilibrium densities: \( (\varepsilon_{eq}, n_{eq}) \), \( \varepsilon = \varepsilon_{eq}, n = n_{eq} \), or, equivalently,

\[
    u_\mu u_\nu \delta T^{\mu\nu} = 0, \quad u_\mu \delta N^\mu = 0. \tag{1}
\]

Matching conditions allow to determine the thermodynamical pressure, \( P_{eq}(\varepsilon_{eq}, n_{eq}) \), (defined as work done in isentropic expansion) via the equation of state for the equilibrium state. Here \( P_{eq} \) should be distinguished from the bulk viscous contribution, \( \Pi = -\frac{1}{3} \Delta_{\mu\nu} \delta T^{\mu\nu} \), present in the energy-momentum tensor [4]. Finally, matching conditions are also needed because they are necessary for the thermodynamical stability of the entropy current (see Appendix A in Ref. [20]). However, the matching conditions given by eq. (1) are not unique. So far, except of some recent works [10,17], they were not investigated in detail.

A state of a relativistic dissipative fluid is described by the energy-momentum tensor, \( T^{\mu\nu}(x) \), and by the baryon number current, \( N^\mu \), which obey the conservation laws

\[
    T^{\mu\nu}_{\mu} = 0, \tag{2}
\]
\[ N_\mu^\alpha = 0, \] (3)

and the second law of the thermodynamics
\[ S^\mu_\mu \geq 0. \] (4)

Because of the uncertainty in definition of the flow velocity \( u^\mu(x) \) for a non-equilibrium fluid, one needs, unlike in the case of perfect fluid, to fix the frame for the fluid. Two special frames can be defined: Landau and Eckart.

The Landau frame \([2]\) is defined by the vanishing of the energy flow, which consists of heat flow \( q^\mu \) and baryon number flow \( W^\mu \);
\[ W^\mu = u^\mu T^{\mu\lambda} \Delta^\lambda_\mu = 0, \quad q^\mu = -\frac{\delta_{\text{eq}} + P_{\text{eq}}}{n_{\text{eq}}} V^\mu, \] (5)

with \( \Delta^\mu_\mu = g^\mu_\mu - u^\mu u^\mu \), being the projection operator orthogonal to the four vector \( u^\mu \). In the Eckart frame, the hydrodynamic flow velocity \( u^\mu \) (with normalization \( u^\mu u^\mu = 1 \)) is defined by using baryon charge current \( \mu^\alpha \equiv N^\alpha/\sqrt{\text{Vol}} \). In this frame one always has
\[ V^\mu = \Delta^\mu_\mu = 0, \quad q^\mu = W^\mu. \] (6)

Equations of motion for the fluid should not depend on the choice of Lorentz frame. Since the relativistic dissipative fluid dynamics with extended matching conditions has been already formulated in the Eckart frame \([21]\), it is interesting to check it in another Lorentz frame, for example in the Landau frame. Also, the stability and causality conditions for a relativistic dissipative fluid should not depend on the Landau frame used. The purpose of this article is therefore to investigate how the stability and causality conditions should be imposed on a fluid depending on the Lorentz frame used.

This paper is organized as follows: In Sec.2 we re-formulate relativistic dissipative hydrodynamics with extended matching condition (as originally introduced in ref. [21]) in an arbitrary Lorentz frame with \( V^\mu \neq 0 \) and \( W^\mu \neq 0 \), the Eckart frame with \( V^\mu = 0 \) nor the Landau frame with \( W^\mu = 0 \). In such a general frame, the flow velocity field \( u^\mu(x) \) may be determined by using coexisting \( W^\mu \) and \( V^\mu \). We then choose the Landau frame and consider a fluid with zero baryon chemical potential \( \mu_b = 0 \), appearing in the central rapidity region of the ultra-relativistic heavy-ion collisions. In Sec.3, we check the stability and causality of the fluid obtained from our model in this frame. We close with Sec.4 containing a summary and some further discussion.

2 Extended matching condition in arbitrary frame

2.1 General form of off-equilibrium entropy current

The general off-equilibrium entropy current can be written in the following simple form using the vector \( \phi^\mu \) and 2-rank symmetric tensor \( \Phi^\mu_\nu \),
\[ S^\mu_\nu(x) = -\alpha \phi^\mu + \beta_\lambda \Phi^\lambda_\nu, \] (7)

where \( \alpha \equiv \mu_b/T, \beta_\epsilon = \beta \epsilon^\mu \) with \( \beta \equiv 1/T, T \) and \( \mu_b \) are, respectively, the temperature and baryon chemical potential. In the local equilibrium case it is given by
\[ S^\mu_\nu = -\alpha \phi^\mu + \beta_\epsilon \Phi^\lambda_\nu, \]
\[ \phi^\nu_\mu = N^\nu_\epsilon \Phi^\lambda_\nu, \]
\[ \Phi^\lambda_\nu = T^{\lambda \nu} - \frac{\delta^{\lambda \nu}}{3} \Delta^{\alpha \beta} T^{\alpha \beta}. \] (8)

In this case, the energy-momentum conservation, \( T^{\mu \nu}_{\text{eq}} = 0 \), and the baryon number conservation, \( \mu^\nu_{\text{eq}} = 0 \), together with thermodynamic relations result in the locally conserved entropy current:
\[ S^{\mu \nu}_{\text{eq}} = -\alpha N^{\nu}_{\epsilon} \Phi^{\lambda}_{\nu}, \]
\[ \mu^{\nu}_{\text{eq}} = n_{\text{eq}} u^{\nu}. \] (11)

To this current we now introduce dissipative corrections by adding corresponding dissipative corrections \( \delta T^{\mu \nu} \) and \( \delta N^{\mu} \) to the energy-momentum tensor and to the baryon number current appearing in \( \phi^\mu_\nu \) and \( \Phi^\mu_\nu \). In this way one extends the expression of equilibrium entropy current towards the off-equilibrium entropy current:
\[ S^{\mu \nu} = -\alpha N^{\nu}_{\epsilon} \Phi^{\lambda}_{\nu} + \beta_\lambda T^{\lambda \nu} + [\beta^\mu P_{\text{eq},\mu}], \]
\[ \delta T^{\mu \nu} = \delta N^{\mu}, \]
\[ \delta \phi^\mu_\nu = -\alpha \phi^\mu - \phi^\nu_\mu + \beta_\lambda \phi^{\lambda \nu} + \delta \Phi^{\lambda \nu}. \] (12)

Notice that term proportional to \( \Delta^{\alpha \beta} \delta T^{\alpha \beta} \) is stable and causality conditions for a relativistic dissipative fluid should not depend on the Landau frame used. Therefore the term \( \Delta^{\alpha \beta} \delta T^{\alpha \beta} \) is usually dropped (i.e., \( \chi = 0 \)). However, the problem of thermodynamic instability can be avoided by simultaneously demanding the natural extension of the form of the entropy current (eq. 13) with the following general extension of matching conditions:
\[ \chi \neq 0, \quad u^{\nu}_{\mu} \delta T^{\mu \nu} \neq 0, \quad u^{\mu}_{\nu} \delta N^{\nu} \neq 0. \] (16)

2.2 Extended matching conditions

To restore thermodynamical stability as discussed above, we propose to impose the following extended matching conditions on the dissipative correction of the energy-momentum tensor and baryon charge current, \( \delta T^{\mu \nu} \) and \( \delta N^{\mu} \), respectively:
\[ u^{\nu}_{\mu} \delta T^{\mu \nu} u^{\rho} = \Lambda, \quad \delta N^{\mu} u^{\mu} = \delta n. \] (17)
With these conditions, off-equilibrium contributions for the energy momentum tensor and baryon charge vector in general Lorentz frame are
\[
\delta T^{\mu\nu} = A u^\mu u^\nu - \Pi \Delta^{\mu\nu} + W^\mu u^\nu + W^\nu u^\mu + \pi^{\mu\nu},
\]
(18)
\[
\delta N^\mu = \delta n u^\mu + \nu^\mu \Delta_\nu^\mu,
\]
(19)
where \(\Pi\) is bulk pressure, \(\pi^{\mu\nu}\) is shear tensor and \(V^\mu\) is net flow of the baryonic charge. In this case, the off-equilibrium entropy current eq. (13) is given by
\[
S^\mu = S^\mu_{\text{eq}} - \alpha V^\lambda \Delta^\mu_\lambda + \beta W^\mu - \beta [\mu_b \delta n - A + \chi \Pi] u^\mu.
\]
(20)
To ensure thermodynamic stability, one may impose a condition on the entropy current \(S^\mu\), eq. (20), demanding that
\[
\frac{d}{d\Pi}(u_\mu S^\mu) = 0.
\]
(21)
This requirement can be satisfied by the following unique condition,
\[
\chi \Pi = -\mu_b \delta n + A.
\]
(22)
Note that, when both \(A\) and \(\delta n\) are set equal to zero, \(\chi\) should also be zero, as so far considered widely in the literature. However, once one assumes that \(A \neq 0\) and/or \(\delta n \neq 0\), there exists a term proportional to \(\chi\). By the extended thermodynamical stability condition, eq. (22), the entropy current in the arbitrary Lorentz frame is
\[
S^\mu = S^\mu_{\text{eq}} - \alpha V^\lambda \Delta^\mu_\lambda + \beta W^\mu.
\]
(23)
One can obtain the entropy current corresponding to Eckart or Landau’s formulation in the limit \(V^\mu \to 0\) and \(W^\mu \to 0\), respectively. The entropy current eq. (23) can be rewritten using eqs. (18), (19) and thermodynamical stability condition eq. (22),
\[
S^\mu = S^\mu_{\text{eq}} - \alpha \delta N^\mu + \beta \chi \delta T^{\lambda\mu} - \chi \Pi u^\mu.
\]
(24)
The entropy production in this case is given by
\[
S^\mu_{\text{pr}} = -\alpha \mu \delta N^\mu + \beta \lambda \chi \delta T^{\lambda\mu} - [\chi \Pi \beta^{\mu\nu}]_{\mu\nu}.
\]
(25)
It should be noted here that eq. (25) is exactly the same as eq. (19) in Ref. [21] obtained in the Eckart frame when using extended matching condition. This is because \(S^\mu_{\text{pr}}\) is a scalar and so it does not depend on Lorentz frame. This can be seen in the form of the entropy current, eq. (22). The expression for \(S^\mu\) has the same form both in Eckart frame \((V^\mu = 0\) but \(\delta N^\mu \neq 0\)) and in Landau frame \((W^\mu = 0\) but \(\beta \delta T^{\lambda\mu} \neq 0\)).

The entropy production is explicitly given by
\[
S^\mu_{\text{pr}} = -(\nabla \chi \lambda) V^\lambda + (\nabla \chi \beta) W^\lambda + \beta \frac{d u^\lambda}{d\tau} W^\lambda
- \beta \Pi \theta + \beta \left[\nabla_{(\mu u_\lambda)} \chi \pi^{\lambda\mu}\right]
+ [\alpha \delta n - \beta A] \theta,
\]
(26)
where \(\theta\) is the divergence of the flow velocity field, \(\theta \equiv u^\mu_{\mu}\).

Using the definition that \(\alpha = \mu_b \beta\) one can also write the entropy production as
\[
TS^\mu_{\text{pr}} = \left[\frac{\nabla_{\mu} \mu_b}{\mu_b} - \frac{\nabla_{\mu} T}{T}\right][-\mu_b V^\mu]
+ \left[\frac{d u^\mu}{d\tau} - \frac{\nabla_{\mu} T}{T}\right] W^\mu
- \Pi \theta + \nabla_{(\mu u_\lambda)} \chi \pi^{\lambda\mu}
+ [\mu_b \frac{d \delta n}{d\tau} - \frac{d \Lambda}{d\tau}] + [\mu_b \delta n - A] \theta.
\]
(27)
Note that the above equation can also be expressed in the following form:
\[
TS^\mu_{\text{pr}} = \left[\frac{d u^\mu}{d\tau} - \frac{\nabla_{\mu} T}{T}\right] \tilde{W}^\mu
+ \left[\frac{d u^\mu}{d\tau} - \frac{\nabla_{\mu} \mu_b}{\mu_b}\right] \tilde{V}^\mu
- \Pi \theta + \nabla_{(\mu u_\lambda)} \chi \pi^{\lambda\mu}
+ [\mu_b \frac{d \delta n}{d\tau} - \frac{d \Lambda}{d\tau}] + [\mu_b \delta n - A] \theta,
\]
(28)
where \(\tilde{W}^\mu \equiv W^\mu - \mu_b V^\mu\) and \(\tilde{V}^\mu \equiv \mu_b V^\mu\).

2.3 Constitutive equations for a dissipative fluid in the Landau frame

We shall now consider the Landau frame \((W^\mu \equiv 0)\) which is more relevant in the context of ultra-relativistic heavy-ion collisions. In particular, we consider a fluid in the central rapidity region where it is expected that the baryon chemical potential \(\mu_b\) is small. In this paper, we assume that, for simplicity, \(\mu_b = 0\) (this implies that \(n_{\text{eq}} = 0\).) In this case, the entropy production takes the simple form;
\[
TS^\mu_{\text{pr}} = -\Pi \theta + \nabla_{(\mu u_\lambda)} \chi \pi^{\lambda\mu} - \frac{d \Lambda}{d\tau} + A \theta
= -\Pi \theta + \nabla_{(\mu u_\lambda)} \chi \pi^{\lambda\mu} - \frac{d (\chi \Pi)}{d\tau} + (\chi \Pi) \theta.
\]
(29)
In the second line of eq. (29) we have used eq. (22) with \(\mu_b = 0\). For the \(\chi\) in eq. (29) we then use
\[
\chi = \kappa + \xi \Pi + \eta \pi^{\mu\nu} \pi_{\mu\nu} \Pi^{-1},
\]
(30)
i.e., the form of \(\chi\) used in eq. (21) in the ref. [21] with \(W^\mu \equiv 0\). The second law of thermodynamics is guaranteed (with \(\zeta\) and \(\eta\) being, respectively, the bulk pressure and shear viscosity, which are all positive constants) if the entropy production is given by
\[
TS^\mu_{\text{pr}} = \frac{\Pi^2}{\zeta} + \frac{\pi^{\mu\nu} \pi_{\mu\nu}}{2 \eta}.
\]
(31)
This requirement determines the following constitutive equation for, respectively, bulk and shear pressure:

\[
\begin{align*}
\Pi &= -(1 + \chi) \theta - (\kappa / \Pi + 2\chi) \frac{d\Pi}{dt}, \\
\frac{\pi_{\mu\nu}}{2\eta} &= \nabla_{(\mu}u_{\nu)} - \xi \frac{d\pi_{\mu\nu}}{dt},
\end{align*}
\]

(32) (33)

Because eq. (32) includes term proportional to \(1/\Pi d\Pi/dt\), one can introduce into the bulk pressure \(\Pi\) an arbitrary constant \(z\) (with dimension \([GeV]^4\)) and write

\[
\frac{\Pi}{z} + 2\xi \frac{d\Pi}{dt} z + \kappa \xi \frac{d}{dt} \ln\frac{\Pi}{z} = -\xi(1 + \chi) \theta.
\]

(34)

We shall now consider small perturbations of \(\Pi\) fields. The bulk pressure \(\Pi\) can be written as

\[
\Pi = \Pi_0 + \delta\Pi
\]

(35)

with the background reference field \(\Pi_0\) and its perturbation field \(\delta\Pi\). One can also regard \(\Pi_0\) as the value of \(\Pi\) at initial proper time \(\tau_0\), \(\Pi_0 = \Pi(\tau_0)\). Correspondingly, one can write \(\theta_0 = u_{0\mu}\) obtained from the initial flow vector field \(u_0^\mu\) at \(\tau_0\). In this case, the perturbation field \(\delta\Pi\) can be interpreted as \(\delta\Pi = \Pi(\tau) - \Pi(\tau_0)\). In this sense, \(\Pi_0\) is a kind of parameter showing the degree of non-equilibrium at initial stage. Identifying the arbitrary constant \(z\) with \(\Pi_0\) and noticing that \(\frac{d}{dt} \ln(1 + \frac{\Pi}{\Pi_0}) \approx \frac{1}{\Pi_0} \delta\Pi\), one can rewrite the above equation as:

\[
\begin{align*}
\Pi_0 &= -(1 + \chi) \xi \theta_0, \\
\frac{\tau_0 d\Pi}{dt} + \delta\Pi &= -\xi(1 + \chi) \xi \theta_0,
\end{align*}
\]

(36) (37)

where the relaxation time \(\tau_\Pi\) is given by

\[
\tau_\Pi = \frac{\xi(1 + \chi)}{\Pi_0}
\]

(38)

and \(\theta_0 = \theta - \theta_0\). It is interesting to note that \(\Pi_0\) contributes to the relaxation time \(\tau_\Pi\). This means that relaxation processes may depend on the initial condition. Note also that if \(\kappa = 0\) then contribution \(\kappa/\Pi_0\) in the relaxation time would disappear.

A similar approach can also be applied to the \(\pi_{\mu\nu}\) field, with perturbation of the shear viscosity around \(\pi_{\mu\nu} = 0\), leading to

\[
\frac{\tau_\pi d\pi_{\mu\nu}}{dt} + \delta\pi_{\mu\nu} = 2\eta \nabla(\mu u^\nu),
\]

(39)

where the relaxation time \(\tau_\pi\) is given by

\[
\tau_\pi = 2\xi\eta.
\]

(40)

3 Stability and causality of the fluid

The stability of a general class of dissipative relativistic fluid theories was investigated by Hiscock and Lindblom [5,6,7]. Denoting by \(\delta V(x)\) the difference between the actual non-equilibrium value of a field \(V(x)\) and the value in the background reference state, \(V_0(x)\), we assume that variations \(\delta V\) are small enough so that their evolution is adequately described by the linearized equations of motion describing the background state. We shall now investigate the stability of the fluid in our model following a prescription proposed in Ref. [5,6,7]. In what follows:

1. The background reference state is assumed to be homogeneous in space. Notice that, unlike in Ref. [5,6,7], in our case it is not an equilibrium state but rather a non-equilibrium one with \(\Pi = \Pi_0\) and with \(\pi_{\mu\nu} = 0\).

Further, the background space-time is assumed to be flat Minkowski space, so that all background field variables have vanishing gradients.

2. We consider following plane wave form of perturbation propagating in \(x\) direction

\[
\delta V = \delta V_0 \exp(i k x + \Gamma \tau).
\]

(41)

Linearized equations for dissipative fluid dynamical model are given by

\[
\delta[T^{\mu\nu}]_{\mu} = 0,
\]

(42)

with the perturbed energy-momentum tensor:

\[
\begin{align*}
\delta[T^{\mu\nu}] &= (\varepsilon^\star + P^\star)(\delta u^\mu u^\nu + u^\mu \delta u^\nu) + (\delta \varepsilon^\star + \kappa \delta \Pi) u^\mu u^\nu \\
&\quad - (\delta P^\star + \delta \Pi) \Delta^{\mu\nu} + \delta \pi^{\mu\nu}.
\end{align*}
\]

(43)

Here \(\varepsilon^\star\) and \(P^\star\) are energy density and pressure in the background non-equilibrium state

\[
\varepsilon^\star \equiv \varepsilon_\Pi + \kappa \Pi_0, \quad P^\star \equiv P_\Pi + \Pi_0.
\]

However, since \(\delta[\Pi(0)] = 0\) (it has vanishing gradient and is constant in \(\tau\)), terms proportional to \(\Pi_0\) do not contribute to the linearized equations eq. (43) (we ignore terms like \(\Pi_0 \delta u^\mu\) in the linearized equation). Hence, one can replace in the eq. (43) \(\varepsilon_{\Pi\Pi}\) and \(P_{\Pi\Pi}\) by the, respectively, \(\varepsilon_{\Pi\Pi}\) and \(P_{\Pi\Pi}\).

For baryon charge current, since we deal with a fluid in the region where the baryon chemical potential can be considered \(\mu_b = 0\) and the net baryon density \(n_{eq} = 0\), one has \(\delta[N^\mu] = 0\).

The perturbed fluid dynamical fields must satisfy constraints

\[
\begin{align*}
\varepsilon_{\Pi\Pi} \delta u^\mu &= 0, \\
\delta \pi^{\mu\nu} u_\mu &= 0.
\end{align*}
\]

Hence, in the rest frame of fluid, \(u^\mu = (1, 0, 0, 0)\), the proper time \(\tau\) component of the flow velocity field vanishes, \(\delta u^\tau \equiv 0\). We therefore obtain the following linearized equations for the energy-momentum tensor and baryon number current:

\[
\begin{align*}
\delta[T^{\mu\nu}]_{\mu} &= (\varepsilon_{\Pi\Pi} + P_{\Pi\Pi})(ik\delta u^\tau u^\nu + \Gamma \delta u^\nu) \\
&\quad + (\Gamma \delta \varepsilon_{\Pi\Pi} + \kappa \Gamma \delta \Pi) u^\nu - (\nabla^\nu \delta \Pi_{\Pi\Pi} + \nabla^\nu \delta \Pi) \\
&\quad + (ik) \delta \pi^{\tau\tau} = 0.
\end{align*}
\]

(44)

The linearized constitutive equations for \(\Pi\) and \(\pi^{\mu\nu}\) have the form (with \(\vec{k} \equiv 1 + \kappa ) :\)

\[
\begin{align*}
\frac{1 + \tau_\Pi \Gamma}{\xi} \delta \Pi &= -\vec{k} (ik) \delta u^\tau, \\
\frac{1 + \tau_\pi \Gamma}{2\eta} \delta \pi^{\mu\nu} &= -\frac{1}{2} (ik) [\delta \pi^{\mu\nu} + \delta \pi^{\nu\mu} - \frac{2}{3} \delta \pi^{\tau\tau}] \\
&\quad - \frac{1}{2} (ik) [\delta \pi^{\mu\nu} + \delta \pi^{\nu\mu} - \frac{2}{3} \delta \pi^{\tau\tau}]
\end{align*}
\]

(45) (46)
The parameters $\xi$ and $\xi'$ introduced above have been absorbed in the expressions for relaxation time, $\tau_\eta$ and $\tau_\pi$, respectively. On the other hand, the parameter $\kappa$ in the expression of the entropy production is kept and not absorbed in $\tau_\Pi$. Its role will be discussed later.

All perturbation equations can be expressed in concise matrix form:

$$M^B_2 \delta Y^B = 0, \quad (47)$$

where $\delta Y^B$ represents the list of fields. The system matrix $M^B_2$ can be expressed in a block-diagonal form when one chooses the following set of perturbation variables [6]

$$\delta Y^B = \{ \delta \varepsilon_{eq}, \delta u^x, \delta \Pi, \delta \pi^{xx}, \delta u^y, \delta \pi^{xy}, \delta u^z, \delta \pi^{xz}, \delta \pi^{yz}, \delta \pi^{yy} - \delta \pi^{zz} \}. \quad (48)$$

In this case,

$$M = \left( \begin{array}{cc} Q & R \\ R & I \end{array} \right), \quad (49)$$

where the matrices $Q$ and $R$ are given by

$$Q = \left( \begin{array}{ccc} \Gamma & ik \hbar_{eq} \kappa \Gamma & 0 \\ ik \partial P \partial \varepsilon_{eq} \Gamma \hbar_{eq} & ik & ik \\ 0 & ik \kappa \frac{1 + \xi \gamma \epsilon_{eq}}{\eta} & 0 \\ 0 & 0 & \frac{1 + \xi \gamma \epsilon_{eq}}{4\eta^2} \end{array} \right),$$

$$R = \left( \begin{array}{ccc} \hbar_{eq} \Gamma & ik \frac{1 + \xi \gamma \epsilon_{eq}}{\eta} \\ ik \kappa \frac{1 + \xi \gamma \epsilon_{eq}}{\eta} & 0 \end{array} \right), \quad (50)$$

respectively, and $I$ is the $2 \times 2$ unit matrix. The $\hbar_{eq}$ denotes the enthalpy density which is defined by $\hbar_{eq} \equiv \epsilon_{eq} + P_{eq}$. For $\Gamma$ and $\kappa$ satisfying dispersion relation

$$[\det M] = [\det R]^2 [\det Q] = 0, \quad (51)$$

one has plane-wave solution such as eq. (41) for the linearized equations of the system eq. (17).

In what follows we shall discuss in detail the stability of transverse and longitudinal modes separately.

### 3.1 Propagation of the transverse mode

The dispersion relation obtained by setting

$$\eta \det R = (\tau_\eta \hbar_{eq}) \Gamma^2 + \hbar_{eq} \Gamma + \eta k^2 = 0 \quad (52)$$

corresponds to the solution of the perturbation equation which is referred to as the so-called transverse mode. The solution of the above equation (52) is given by

$$\Gamma = -\hbar_{eq} - \sqrt{\hbar_{eq}^2 - 4\eta(\tau_\eta \hbar_{eq})k^2} \quad (53)$$

Note that we have always $\text{Re}[\Gamma] < 0$ independent of the value of $k$, which means that any small perturbation propagating in the transverse direction (perpendicular to the $x$ axis, direction which the perturbation wave propagates) will be damped with time $\tau$. Since the general solution is given by a linear combination of those solutions of the transverse mode, one can say that the plane-wave solution of the mode is stable against small perturbation.

Note also that when wave number $k \geq k_c$, where

$$k_c = \sqrt{\frac{4\eta(\tau_\eta \hbar_{eq})}{3\eta^2 \tau_\Pi}}, \quad (54)$$

the linear perturbation wave propagates towards the transverse direction, but waves with wave number $k < k_c$ are damped.

### 3.2 Propagation of the longitudinal mode

Frequencies of the so-called longitudinal mode (propagating parallel to the $x$ direction) are given by the roots of the following dispersion relation:

$$\sqrt{\frac{4\eta(\tau_\Pi \hbar_{eq})}{3\eta^2 \tau_\Pi \tau_\pi}} \det(Q) \equiv \sum_{n=0}^{n=4} q_n P_n = 0, \quad (55)$$

where the coefficients $q_n$ are given by

$$q_1 = \hbar_{eq}, \quad q_2 = \hbar_{eq} \left( \frac{1}{\tau_\eta \tau_\pi} + \frac{1}{\tau_\pi} \right), \quad (56)$$

$$q_2 = \hbar_{eq} \left( \frac{1}{\tau_\eta \tau_\pi} + \frac{\partial P_{eq}}{\partial \epsilon_{eq}} k^2 \right) + \left( \frac{4\eta}{3\eta^2 \tau_\eta \tau_\pi} + \kappa \left( 1 - \frac{\partial P_{eq}}{\partial \epsilon_{eq}} \frac{\lambda}{\tau_\pi} \right) \right) k^2, \quad (58)$$

$$q_1 = \hbar_{eq} \left( \frac{1}{\tau_\eta \tau_\pi} + \frac{1}{\tau_\pi} \right) \frac{\partial P_{eq}}{\partial \epsilon_{eq}} k^2 + \left( \frac{4\eta}{3\eta^2 \tau_\eta \tau_\pi} + \kappa \left( 1 - \frac{\partial P_{eq}}{\partial \epsilon_{eq}} \frac{\lambda}{\tau_\pi} \right) \right) k^2, \quad (59)$$

$$q_0 = \hbar_{eq} \frac{\partial P_{eq}}{\partial \epsilon_{eq}} \frac{1}{\tau_\eta \tau_\pi} k^2. \quad (60)$$

When all coefficients $q_n$ of the fourth-order equation (55) have the same sign, the four (complex or real) solutions of the real part is definitely negative. In this case, the general solution which is a linear combination of those four solutions, is stable. Since $q_4$, $q_3$, and $q_0$ are positive defined, then the stability condition sought after is that $q_2$ and $q_1$ must be simultaneously positive. The condition is

$$1 - \kappa \frac{\partial P_{eq}}{\partial \epsilon_{eq}} \bigg|_{\epsilon_{eq}=0} > 0. \quad (61)$$

Using thermodynamical relation $\frac{\partial P_{eq}}{\partial \epsilon_{eq}} \bigg|_{\epsilon_{eq}} = c_s^2 + \frac{1}{\partial \epsilon_{eq}} \bigg|_{\epsilon_{eq}}$, where $c_s^2 \equiv \frac{\partial P_{eq}}{\partial \epsilon_{eq}} \bigg|_{\epsilon_{eq}}$ is the adiabatic velocity of sound, one
can rewrite the stability condition eq. (61) with the following
\[ c_s^2 + \frac{1}{T} \frac{\partial P_{eq}}{\partial s_{eq}} \bigg|_{\varepsilon_{eq}} < \frac{1}{\kappa}, \] (62)
which is exactly the same condition found in previous work \cite{21} in the Eckart frame. Note that, when \( \kappa \to 0 \), one finds that the speed of sound can exceed unity violating eq. (21). In the linearized field equations, the speed of sound \( c_s \) is actually restricted so that \( 0 \leq c_s^2 \leq 1 \) when \( \kappa \) is chosen in the following range:
\[ \frac{1}{T} \frac{\partial P_{eq}}{\partial s_{eq}} \bigg|_{\varepsilon_{eq}} \leq \frac{1}{\kappa} \leq 1 + \frac{1}{T} \frac{\partial P_{eq}}{\partial s_{eq}} \bigg|_{\varepsilon_{eq}} , \] (63)
then the velocity of sound satisfies \( 0 \leq c_s \leq 1 \). Thus, when the condition for \( \kappa \), eq. (63) is satisfied, the fluid is stable against small perturbations and evolves without violating causality.

4 Summary and concluding remarks

We have proposed a novel formulation of the relativistic dissipative hydrodynamical model in an arbitrary frame by using extended matching conditions
\[ u_{\mu} u_{\nu} \delta T^{\mu\nu} = \Lambda, \quad u_{\mu} \delta N^\mu = \delta n. \]
To apply the above extended matching conditions, we have also generalized the form of the entropy current for non-equilibrium state [cf. eq. (20)]:
\[ S^\mu = S_{eq}^\mu - \alpha V^\lambda \Delta^\mu_\lambda + \beta W^\mu - \beta [\mu_0 \delta n + \Lambda - \chi \Pi] u^\mu. \]
The phenomenological parameter \( \chi \) introduced in the generalization of the entropy current can be fixed by the extended thermodynamic stability condition,
\[ \mu_0 \delta n + \Lambda - \chi \Pi = 0. \]
(Note that in the usual formulation \( \chi \equiv 0 \), because of \( \Lambda = 0 \) and \( \delta n = 0 \)). Taking the thermodynamical stability condition into account, the entropy current is given by
\[ S^\mu = S_{eq}^\mu - \alpha V^\lambda \Delta^\mu_\lambda + \beta W^\mu = S_{eq}^\mu - \alpha \delta N^\mu + \beta \delta T^{\lambda\mu} - \chi \Pi u^\mu. \]
As seen in the above equation, the last term \( \chi \Pi u^\mu \) is the new correction term. The corresponding entropy production evidently does not depend on the Lorentz frame considered. It is given by
\[ S^\mu_{ij} = -\alpha_{ij} \delta N^\mu + \beta_{ij\mu} \delta T^{\lambda\mu} - [\chi \Pi \beta_{ij\mu}]. \]
In this paper, we chose the Landau frame and considered a dissipative fluid with zero chemical potential \( \mu_0 \equiv 0 \), for simplicity. For this case, the \( \chi \) was assumed as
\[ \chi = \kappa + \xi \Pi + \xi'' \pi^{\mu\nu} \pi_{\mu\nu} / \Pi. \]
The physical meaning of \( \kappa, \xi \) and \( \xi'' \) are revealed in the discussion on the stability and causality of the fluid in Sec. 4. The \( \kappa \) is related to the bound for the speed of sound wave of the fluid and \( \xi \) and \( \xi'' \) are related to the relaxation of the off-equilibrium system. In the linearized field equations, the speed of sound \( c_s \) is actually restricted so that \( 0 \leq c_s^2 \leq 1 \) when \( \kappa \) is chosen in the following range:
\[ \frac{1}{T} \frac{\partial P_{eq}}{\partial s_{eq}} \bigg|_{\varepsilon_{eq}} \leq \frac{1}{\kappa} \leq 1 + \frac{1}{T} \frac{\partial P_{eq}}{\partial s_{eq}} \bigg|_{\varepsilon_{eq}}. \]
On the other hand, the relaxation time of the perturbed \( \Pi \) and \( \pi^{\mu\nu} \) fields are respectively given by
\[ \tau_\Pi = (2 \kappa + \Pi_0) \xi, \quad \tau_\pi = 2 \xi'' \eta. \]
The parameter \( \kappa \) also contributes to the relaxation time and may bring in a contribution to the initial condition characterized by \( \Pi_0 \).
We therefore conclude that, when the matching conditions
\[ \Lambda = \chi \Pi = \kappa \Pi + \xi \Pi^2 + \xi'' \pi^{\mu\nu} \pi_{\mu\nu} \]
and \( \delta n = 0 \) (also \( \mu_0 = 0 \) ) are imposed, the relativistic dissipative fluid can be applied to an analysis of the phenomena observed in the ultra-relativistic heavy-ion collisions not only in the Eckart frame, as discussed in \cite{21}, but also in the Landau frame. The conditions that should be imposed seem to be independent of the Lorentz frame used, i.e.,
\[ \frac{\lambda T}{\tau_w} \leq \hbar_{eq} \quad \text{and} \quad c_s^2 \leq \frac{1}{\kappa} - \frac{1}{T} \frac{\partial P_{eq}}{\partial s_{eq}} \bigg|_{\varepsilon_{eq}}, \] (64)
where \( \lambda \) and \( \tau_w \) are thermal energy conductivity and relaxation time of the thermal energy conduction, respectively \cite{21}. In the Landau frame, \( \Delta^\mu_{\gamma_\mu} \) should be regarded as \( 0 \) because of the definition of the frame, \( W^\mu = 0 \).

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