Study of critical behavior of the supersymmetric spin chain that models plateau transitions in the integer quantum Hall effect

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Abstract. We use the density-matrix renormalization-group (DMRG) algorithm and finite-size scaling to study a supersymmetric (SUSY) spin chain that models plateau transitions in the integer quantum Hall effect. To illustrate the method, we first present results for the $S=1$ antiferromagnetic spin chain. Different scaling behavior of the dimerization operator is obtained for the Haldane phase, the dimerized phase, and at the critical point separating the two phases. We then investigate the supersymmetric spin chain. As the on-site Hilbert space has infinite dimension, it is non-compact. We present DMRG calculations for supersymmetric truncations of the on-site Hilbert space and also for non-supersymmetric truncations. These two cases scale differently, in accord with a Lieb-Schultz-Mattis type theorem.

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1 Introduction

Quantum critical points are characterized by fluctuations over all length and time scales and by the appearance of power law scaling. Transitions between plateaus in the integer quantum Hall effect (IQHE) provide the clearest experimental example of quantum critical behavior in a disordered system.

Critical behavior of the supersymmetric (SUSY) spin chain that models plateau transitions in the IQHE is studied here using the density-matrix renormalization-group (DMRG) algorithm and finite-size scaling. In Sec. 2, we briefly describe the method and show results for the rather well known $S=1$ antiferromagnetic spin chain, and then analyze the SUSY chain. In Sec. 3 we discuss the results, which accord with the predictions of a generalized LSM theorem.

2 DMRG results

We use the so-called “infinite-size” DMRG algorithm as it is particularly simple to implement. It is a systematic way of building up a one-dimensional chain to calculate its low energy properties. At each step, two sites with on-site Hilbert space $D$ are added at the center of the chain. The left and right halves of the chain are considered...
as separate blocks, and the $M$ most probable states (as determined by the eigenvalues of the reduced density matrix) are retained for each block.

2.1 $S = 1$ chain

Consider the general nearest-neighbor Hamiltonian for an isotropic $S = 1$ quantum antiferromagnetic Heisenberg spin chain:

$$H = \sum_{j=0}^{L-2} \left[ \cos \theta \left( \vec{S}_j \cdot \vec{S}_{j+1} \right) + \sin \theta \left( \vec{S}_j \cdot \vec{S}_{j+1} \right)^2 \right]. \quad (1)$$

Dimerization, defined here as $\Delta(j) = \left| \left\langle S_{j}^{x} S_{j+1}^{x} + S_{j}^{y} S_{j+1}^{y} \right\rangle - \left\langle S_{j-1}^{x} S_{j}^{x} + S_{j-1}^{y} S_{j}^{y} \right\rangle \right|$, is induced in the interior of the chain by the open boundary conditions. We monitor the decrease of the induced dimerization at the center of the chain $\Delta(L/2)$ as the chain length $L$ is enlarged via the DMRG algorithm. At a critical point, we expect power-law decay, with possible logarithmic and additive higher order corrections \cite{3, 4}. In contrast, exponential decay or constant dimerization is expected for gapped systems.

The phase diagram of the spin-1 chain (Eq. 1) can be represented on a circle where the parameter $\theta$ corresponds to the polar angle, as shown in Fig. 1 (a). Fig. 1 (b) is a log-log plot of the induced dimerization at the center of the chain for $\theta$ in the Haldane phase, at the critical point and in the dimerized phase.

(b) is a log-log plot of the induced dimerization at the center of the chain for three values of $\theta$. Power-law decay occurs at $\theta = -\pi/4$, the critical point. The point $\theta = \tan^{-1}(1/3) \approx 0.321751$ corresponds to the exactly solvable Affleck-Kennedy-Lieb-Tasaki valence bond solid \cite{5} which is in the Haldane phase \cite{6}. The system is gapped and the induced dimerization decays exponentially to zero. The $\theta = -\pi/2$ point is also gapped but in this case the induced dimerization decays to a constant since it is in the dimerized phase and translational symmetry is spontaneously broken (making $\sin \theta$ more negative favors the concentration of singlet correlations on isolated dimers).
2.2 Supersymmetric chain

Plateau transitions in the integer quantum Hall effect can be described by a quantum tunneling network model introduced by Chalker and Coddington [7]. In the anisotropic limit this model can be represented [8, 9] by an independent-particle Hamiltonian which describes a chain of edge states alternating in propagation forward and backward in imaginary time. The method of supersymmetry (SUSY) [10, 11, 12, 13, 14, 15] can then be used to perform the disorder average of the corresponding functional integral [16, 17]. Two spin species are introduced to permit the calculation of the disorder-averaged product of retarded and advanced Green’s functions which determines the conductivity. The resulting effective Hamiltonian [18] describes interacting spin-up and spin-down fermions, and can be written in terms of 16 spin operators.

At the transition point between plateaus, a generalization of the theorem of Lieb, Schultz and Mattis shows that the system is quantum critical [18].

The SUSY spin chain that models plateau transitions in the IQHE is non-compact [16, 17] since an arbitrarily large number of bosons on each site is permitted. To employ the DMRG method, a truncation of the on-site Hilbert space is required. Introducing the integer level index \( n = 0, 1, 2, \ldots \) we construct the on-site Hilbert space by adding to the vacuum state \( |0\rangle \) a tower of states built out of the quartets [18]:

\[
|4n + 1\rangle = \frac{1}{n!} (b^\dagger_\uparrow b^\dagger_\downarrow)^n c^\dagger_\uparrow c^\dagger_\downarrow |0\rangle , \\
|4n + 2\rangle = \frac{1}{\sqrt{n!(n+1)!}} (b^\dagger_\uparrow b^\dagger_\downarrow)^n c^\dagger_\uparrow c^\dagger_\downarrow |0\rangle , \\
|4n + 3\rangle = \frac{1}{\sqrt{n!(n+1)!}} (b^\dagger_\uparrow b^\dagger_\downarrow)^n c^\dagger_\uparrow b^\dagger_\downarrow |0\rangle , \\
|4n + 4\rangle = \frac{1}{(n+1)!} (b^\dagger_\uparrow b^\dagger_\downarrow)^n b^\dagger_\downarrow b^\dagger_\uparrow |0\rangle .
\] (2)

Truncations with \( D = 4n + 1 \) states contain full quartets and thus preserve SUSY. Also considered are non-SUSY truncations \( D = 4n + 2 \), with the state \( |4n+1\rangle \) selected as the final state at the top of the tower. Fig. 2 shows the induced dimerization at the center of the chain for SUSY truncations \( D = 5, 9, 13 \) and for non-SUSY truncations \( D = 2, 6, 10 \). Calculations were done at the transition point (the dimerization parameter \( R = 0 \), and the imaginary frequency, which defines advanced and retarded propagators, is small, \( \eta = 10^{-4} \)). As \( R \to 0 \), the correlation length diverges as \( \xi \sim R^{-\nu} \).

![Fig. 2](image-url) Log-log (a) and semi-log (b) plots of the induced dimerization at the center of the chain for SUSY truncations \( D = 5, 9, 13 \) and for non-SUSY truncations \( D = 2, 6, 10 \).
3 Discussion of results

In Ref. [18], we showed that the supersymmetric chain is quantum critical at $D \to \infty$ in the thermodynamic limit. Closely following the theorem of Lieb, Schultz and Mattis [19] for half-odd-integer spin antiferromagnets, we considered a slow-twist operator $U$ which satisfies $U^\dagger[H,U] = O(1/L)$. Now $U$ is invariant under global supersymmetry rotations and therefore does not create low-energy excitations for truncations which respect SUSY. However, chains with non-SUSY truncations $D = 4n + 2$ are gapless in the thermodynamic limit. Continuity in the $D \to \infty$ limit requires that both truncations converge to the gapless behavior. This result was checked by DMRG calculations of the gap for different truncations and system sizes [18]. The method used here, however, only requires the ground state as critical behavior can be extracted through the finite-size scaling analysis of the induced dimerization at the chain center.

DMRG results for the $S = 1$ chain show power-law behavior at the critical point. Power-law behavior is also obtained for the superspin chain with non-SUSY truncations $D = 4n + 2$, as expected from the generalized LSM theorem. The special case $D = 2$ corresponds to the ordinary $S = 1/2$ Heisenberg antiferromagnet. Fig. 2 (a) shows that non-SUSY truncations seem to be in the same universality class as the $S = 1/2$ antiferromagnet. SUSY truncations are gapped and show exponential decay of the induced dimerization, as can be seen from the semi-log plot of Fig. 2 (b). From the slopes in the semi-log plot, the relative sizes of the gap can be inferred for different SUSY $D = 4n + 1$ truncations. The gap decreases as the SUSY truncation size $D$ increases, and it must vanish in the physically relevant limit $D \to \infty$.

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