On convex hull of $d$-dimensional fractional Brownian motion

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Abstract:

It is well known that for standard Brownian motion $\{B(t), t \geq 0\}$ with values in $\mathbb{R}^d$ its convex hull $V(t) = \text{conv}\{B(s), s \leq t\}$ with probability 1 contains 0 as an interior point for each $t > 0$ (see [1]). The aim of this note is to state the analogous property for $d$-dimensional fractional Brownian motion.

Key-words: Brownian motion, Multi-dimensional fractional Brownian motion, convex hull.

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1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a basic probability space. Consider a $d$-dimensional centered Gaussian process $X = \{X(t), t \geq 0\}$ defined on $\Omega$ which is self-similar of index $H > 0$. It means that for each constant $c > 0$ the process $\{X(ct), t \geq 0\}$ has the same distribution as $\{c^H X(t), t \geq 0\}$.

We call $X$ fractional Brownian motion (FBM) if for each $e \in \mathbb{R}^d$ the scalar process $t \rightarrow \langle X(t), e \rangle$ is a standard one-dimensional FBM up to a constant $c(e)$.

It is easy to see that in this case $c^2(e) = \langle Qe, e \rangle$, where $Q$ is the covariance matrix of $X(1)$, and hence

$$\mathbb{E}\langle X(t), e \rangle\langle X(s), e \rangle = \langle Qe, e \rangle \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}), \quad t, s \geq 0; \ e \in \mathbb{R}^d$$

(see [2], [3] and references therein for more general definitions of operator self-similar FBM).

The next properties follow from the definition without difficulties.

1) Continuity. The process $X$ has a continuous version.

Below we always suppose $X$ to be continuous.

2) Reversibility. Define the process $Y$ by

$$Y(t) = X(1) - X(1 - t), \quad t \in [0, 1].$$

Then $\{Y(t), \ t \in [0, 1]\} \overset{\mathcal{L}}{=} \{X(t), \ t \in [0, 1]\}$, where $\overset{\mathcal{L}}{=}$ means equality in law.

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3) **Ergodicity.** Let \( L = \{L(u), \ u \in \mathbb{R}^1\} \) be the strictly stationary Gaussian process obtained from \( X \) by Lamperti transformation:

\[
L(u) = e^{-Hu}X(e^u), \quad u \in \mathbb{R}^1.
\]

(1)

Then \( L \) is ergodic (see [2], Ch. 14, §2, Th.1, Th.2).

2 **Results**

For Borel set \( A \subset \mathbb{R}^d \) we denote by \( \text{conv}(A) \) the closed convex hull generated by \( A \).

The object of our interest is the convex hull process related to \( X \):

\[
V(t) = \text{conv}\{ X(s), \ s \leq t \}.
\]

It is supposed below that the law of \( X(1) \) is non degenerate, that is the rank of the matrix \( Q \) is equal to \( d \).

**Theorem 1** With probability 1 for all \( t > 0 \) the point 0 is an interior point of \( V(t) \).

As a corollary we immediately deduce the following fact.

**Theorem 2** For each \( t > 0 \) with probability 1 the point \( X(t) \) is an interior point of \( V(t) \).

**Proof of Th. 2.** Denote by \( A^\circ \) the interior of \( A \). By self-similarity of the process \( X \) it is sufficient to state this property for \( t = 1 \). Then, due to the reversibility of \( X \) by Th. 1., a.s.

\[
0 \in [\text{conv}\{ X(1) - X(1 - t), \ t \in [0,1]\}]^\circ.
\]

(2)

As

\[
\text{conv}\{ X(1) - X(1 - t), \ t \in [0,1]\} = X(1) - \text{conv}\{ X(s), \ s \in [0,1]\},
\]

the relation (2) is equivalent to

\[
X(1) \in [\text{conv}\{ X(s), \ s \in [0,1]\}]^\circ,
\]

which concludes the proof.

Let \( K_d \) be the family of all compact convex subsets of \( \mathbb{R}^d \). It is well known that \( K_d \) equipped with Hausdorff metric is a Polish space.

We say that a function \( f : [0,1] \to K_d \) is increasing, if \( f(t) \subset f(s) \) for \( 0 \leq t < s \leq 1 \).

We say that a function \( f : [0,1] \to K_d \) is Cantor - staircase (C-S), if \( f \) is continuous, increasing and such that for almost every \( t \in [0,1] \) there exists an interval \( (t - \varepsilon, t + \varepsilon) \) where \( f \) is constant.

The next statement is an easy corollary of Th.2.
Theorem 3  With probability 1 the paths of the process $t \to V(t)$ are C-S functions.

Proof of Th. 3. We use the notation $X(s, \omega)$ for $X(s)$ to emphasize the dependance of $\omega \in \Omega$. By Th.2 for each $t \in (0, 1)$ with probability 1 $X(t) \in V(t)^\circ$. By continuity of $X$, for $\mathbb{P}$-almost each $\omega$ there exists $\varepsilon > 0$ such that $X(s, \omega) \in V(t)^\circ$ for all $s \in (t - \varepsilon, t + \varepsilon)$ which gives $V(s) = V(t)$, $\forall s \in (t - \varepsilon, t + \varepsilon)$. 

Remark 1 Let $h : \mathcal{K} \to \mathbb{R}^1$ be an increasing continuous function. Then almost all paths of the process $t \to h(V(t))$ are C-S real functions. This obvious fact may be applied to all reasonable geometrical caracteristics of $V(t)$, such as volume, surface area, diameter,...

3 Proof of Theorem 1

Let $\Theta = \{0, 1\}^d$ be the set of all diadic sequences of length $d$. Denote by $D_\theta$, $\theta \in \Theta$, the quadrant

$$D_\theta = \prod_{i=1}^d \mathbb{R}_{\theta_i},$$

where $\mathbb{R}_{\theta_i} = [0, \infty)$ if $\theta_i = 1$, and $\mathbb{R}_{\theta_i} = (-\infty, 0]$ if $\theta_i = 0$.

The positive quadrant $D_{(1,1,...,1)}$ for simplicity is denoted by $D$.

We first show that

$$p \overset{\text{def}}{=} \mathbb{P}\{ \exists t \in (0, 1) \mid X(t) \in D^\circ \} = 1. \quad (3)$$

Remark that $p$ is strictly positive:

$$p \geq \mathbb{P}\{X(1) \in D^\circ\} > 0 \quad (4)$$

due to the hypothesis that the law of $X(1)$ is non degenerate.

By self similarity

$$\mathbb{P}\{D^\circ \cap \{X(t), t \in [0, T]\} = \emptyset\} = 1 - p$$

for every $T > 0$.

The sequence of events $(A_n)_{n \in \mathbb{N}}$,

$$A_n = \{D^\circ \cap \{X(t), t \in [0, n]\} = \emptyset\},$$

being decreasing, it follows that

$$1 - p = \lim \mathbb{P}(A_n) = \mathbb{P}(\cap_n A_n) = \mathbb{P}\{X(t) \notin D^\circ, \forall t \geq 0\}.$$

In terms of the stationary process $L$ from Lamperti representation it means that

$$\mathbb{P}\{L(s) \notin D^\circ, \forall s \in \mathbb{R}^1\} = 1 - p.$$
As this event is invariant, by ergodicity of $L$ and due to $(4)$ we see that the value $p = 1$ is the only one possible.

Applying the analogous arguments to another quadrants $D_{\theta}$, we get that with probability 1 there exists points $t_{\theta} \in (0, 1]$, such that $X(t_{\theta}) \in D_{\theta}$, $\forall \theta \in \Theta$. Now, to end the proof it is sufficient to remark that

$$ V(1) = \operatorname{conv}\{X(t), t \in [0, 1]\} \supset \operatorname{conv}\{X(t_{\theta}), \theta \in \Theta\} $$

and that the last set evidently contains 0. ■

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