Topological hypermultiplet on $N=2$ twisted superspace in four dimensions

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Abstract

We propose a $N=2$ twisted superspace formalism with a central charge in four dimensions by introducing a Dirac-Kähler twist. Using this formalism, we construct a twisted hypermultiplet action and find an explicit form of fermionic scalar, vector and tensor transformations. We construct an off-shell Donaldson-Witten theory coupled to the twisted hypermultiplet. We show that this action possesses $N=4$ twisted supersymmetry at the on-shell level. It turns out that the four-dimensional Dirac-Kähler twist is equivalent to Marcus’ twist.

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1 Introduction

In 1988 Witten pointed out that the $N=2$ super-Yang-Mills theory corresponds to the Donaldson-Witten theory [1] which is a kind of topological field theories (TFT). This correspondence is called topological twist or simply twist. This topological theory was soon after derived from quantizing a four dimensional topological Yang-Mills theory using instanton gauge fixing [2–5]. This suggest that a topological field theory can be constructed by twisting an extended supersymmetric gauge theory or by quantizing a topological invariant with a suitable gauge fixing condition in gauge theories. From the twisting procedure a BRST charge appearing in TFT was identified with a part of the supercharges appearing in supersymmetry (SUSY). Thus in the quantized TFT there should be other BRST-like fermionic symmetries. A vector supersymmetry [6, 7], that is, a BRST-like fermionic symmetry with a vector index was actually discovered in various models and dimensions [8–18]. It was recognized that the BRST symmetry and the vector SUSY belong to a twisted version of the $N=2$ or $N=4$ extended SUSY. For example in two dimensional $N=2$ case the twisted supercharges consist of a scalar (BRST), a vector and a pseudo-scalar (the second rank tensor) charge [19, 20]. One of the important characteristics of a quantization of TFT is that the twisted supersymmetry (TSUSY) spontaneously appears after a quantization. This suggests that the origin of the SUSY may be connected with quantization.

Kawamoto and Tsukioka proposed a new twisting procedure called Dirac-Kähler twist in two dimensional quantized topological Yang-Mills theory with instanton gauge [21] constructed from the generalized gauge theory [22–25]. It was found that the twisting procedure between the spinors (gaugino and matter field) and the tensor fermions (ghost, anti-ghost) is essentially the Dirac-Kähler fermion mechanism [26–33] and the flavor degrees of freedom of Dirac-Kähler fermion can be interpreted as that of the extended SUSY [34, 35]. One of the authors (J.K.) with Kawamoto and Uchida pointed out that the twisted superspace formalism is hidden behind the formulation [36]. It became clear that the two dimensional quantized topological Yang-Mills theory and the quantized BF theory were successfully derived from the twisted superspace formalism.

In the previous paper [37] the authors (J.K. and A.M.) with Kawamoto proposed a $N=4$ Dirac-Kähler twisting procedure and a twisted superspace formalism in four dimensions. It turned out that a $N=4$ off-shell twisted supersymmetric action which corresponds to the two dimensional counterpart of BF theory was constructed from this formalism, but the action has higher derivative terms and many auxiliary fields and is not actually the four dimensional BF theory. In the $N=2$ case the super Yang-Mills theory whose component fields belong to the vector multiplet was constructed by using a superconnection formalism [38, 39]. However a matter multiplet or equivalently hypermultiplet [40, 41] has not been found because of the necessity for a central charge.

In this paper we concentrate on studying $N=2$ twisted SUSY with a central charge. We propose a $N=2$ twisted superspace formalism with the central charge
using the Dirac-Kähler twist. Related works in a similar context to that of our formulation was given by Alvarez and Labastida for $N=2$ twisted SUSY by spinor formulation in four dimensions [42], while our formulation is based on the tensor formulation coming from the Dirac-Kähler twisting procedure. We then propose a new twisted hypermultiplet action and give a gauge covariant version of this action. We claim that this action plus Donaldson-Witten action has the on-shell $N=4$ TSUSY and the four-dimensional Dirac-Kähler twist is equivalent to the Marcus’s twist [43].

There is another off-shell formulation which is called harmonic superspace formulation [44]. It is interesting that this formulation gives all the off-shell $N=2$ supersymmetric theories. It is characterized by a presence of an infinite number of auxiliary fields and unconstrained superfields. It is interesting to construct a twisted version of this formulation and to establish a relation to the superspace formulation presented here.

Another important motivation of this work comes from the recent study of lattice SUSY. It is well known that the Dirac-Kähler fermion mechanism is fundamentally related to a lattice formulation [45–47]. In the two dimensional case $N=2$ exact supersymmetry is realized on the lattice by using the superspace formalism based on the Dirac-Kähler twist [48]. These types of lattice SUSY models are considered in three and four dimensions [49]. Other lattice SUSY models using the Dirac-Kähler mechanism were investigated in [50]. There are other models preserving part of SUSY on the lattice based on the TSUSY [51, 52].

The paper is organized as follows. In section 2 we give a general property of the $N=2$ twisted superspace formalism with a central charge coming from the Dirac-Kähler twisting procedure. Then we propose a new topological hypermultiplet action without interactions. In section 3 we introduce a Donaldson-Witten theory coupling to the hypermultiplet. We show that this action possess a $N=4$ TSUSY at on-shell level and the four-dimensional Dirac-Kähler twist is equivalent to the Marcus’s twist. In section 4 we establish a connection to the ordinary $N=4$ supersymmetric theory. We summarize the result in section 5. We provide several appendixes to summarize the notations and show the full transformation of on-shell $N=4$ TSUSY.

## 2 Twisted SUSY with Central Charge

We have investigated the properties of the four dimensional $N=4$ and $N=2$ twisted superspace formalism without a central charge in a previous paper [37]. In the $N=2$ case the Donaldson-Witten theory was constructed in our superspace formulation. It is well known that another multiplet exists in ordinary $N=2$ supersymmetric theories. Since this multiplet, that is, a hypermultiplet accompanies the central charge at off-shell level, it is difficult to deal with such a vector multiplet.

We consider a topological version of the hypermultiplet. The hypermultiplet satisfies the following $N=2$ SUSY algebra with the central charge,

$$\{Q_{\alpha i}, \overline{Q}^j_{\dot{\beta}}\} = 2\delta^j_i (\gamma^\mu)_{\alpha\dot{\beta}}P_\mu + 2\delta^j_i \delta_{\alpha\dot{\beta}}Z,$$  \hspace{1cm} (2.1)
where the indices \( \{ \alpha, \beta \} \) and the indices \( \{ i, j \} \) are Lorentz spinor and internal R-symmetry indices of the extended SUSY, respectively. In this case the internal symmetry group has the form \( SU(2)_I \), where the indices are taken to be \( i, j \in \{1, 2\} \), and the supercharge satisfies a \( SU(2) \) Majorana condition given by (B.7) and a two components spinor representation of the algebra (2.1) is discussed in Appendix C. Throughout this paper we consider the Euclidean flat spacetime.

We consider a twist of the algebra (2.1) by using the Dirac-Kähler twisting procedure. This procedure means that a \( SO(4)_I \) R-symmetry is identified with \( SO(4) \) Euclidean rotation symmetry. Thus this procedure is not available for \( SU(2)_I \) R-symmetry directly. We consider a twist of \( N=4 \) SUSY algebra with \( SO(4)_I \) R-symmetry. Since \( SO(4) \) group satisfies the following relation,

\[
SO(4) \simeq SU(2) \otimes SU(2),
\]

we can extract the corresponding \( SU(2)_I \) R-symmetry.

We define \( N=4 \) supercharge as \( Q_{\alpha i} \) to distinguish \( N=4 \) supercharge from \( N=2 \) supercharge. It should be noted that index \( \{ i \} \) takes \( i \in \{1 \ldots 4\} \), because the supercharge \( Q_{\alpha i} \) has index of the \( SO(4)_I \) R-symmetry. We apply the Dirac-Kähler mechanism to the \( N=4 \) supercharge:

\[
Q_{\alpha i} = \frac{1}{\sqrt{2}}(s + \gamma^\mu s_\mu + \frac{1}{2}\gamma^{\mu\nu} s_{\mu\nu} + \bar{s}_5 s_5 + \gamma_5 \bar{s}),
\]

where the index \( i \) and \( \alpha \) are the Euclidean rotation \( SO(4) \) symmetry indices and \( \{s, s_\mu, s_{\mu\nu}, \bar{s}_5, \bar{s}\} \) stand for the twisted supercharges. We redefine the twisted supercharges as the following form to extract the \( N=2 \) sector,

\[
\begin{aligned}
&\begin{cases}
  s^\pm &\equiv \frac{1}{\sqrt{2}}(s \pm \bar{s}) \\
s_\mu^\pm &\equiv \frac{1}{\sqrt{2}}(s_\mu \pm \bar{s}_\mu) \\
s_{\mu\nu}^\pm &\equiv \frac{1}{\sqrt{2}}(s_{\mu\nu} \mp \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} s^{\rho\sigma}).
\end{cases}
\end{aligned}
\]

The equation (2.3) is rewritten as the following,

\[
Q_{\alpha i} = (s^+_P + s^+_\mu \gamma^\mu P_+ + \frac{1}{4}s^+_A \gamma^A P_+) + s^-_P + s^-_\mu \gamma^\mu P_- + \frac{1}{4}s^-_A \gamma^A P_-)_{\alpha i},
\]

where the capital \( \{A\} \) means anti-symmetric tensor indices \( \{\mu\nu\} \), projection matrices \( (P_\pm)_{ij} \) of the internal symmetry are defined by \( (P_\pm)_{ij} = \frac{1}{2}(1 \pm \gamma_5)_{ij} \) and the second rank tensors satisfy (anti-)self-dual conditions: \( s^\pm_{\mu\nu} = \mp \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} s^{\pm\rho\sigma}. \)

As we are considering the \( N=2 \) case, we extract the corresponding supercharges from the equation (2.3). We exclude the supercharges \( \{s^-, s^-_\mu, s^-_{\mu\nu}\} \), applying \( P_+ \) to the equation (2.5),

\[
(QP_+)^{\alpha i} = (s^+_P + s^+_\mu \gamma^\mu P_+ + \frac{1}{4}s^+_A \gamma^A P_+)_{\alpha i}.
\]
This means the chiral projection with respect to the internal $SO(4)$ symmetry. It should be noted that the indices $i, j \in \{1, 2, 3, 4\}$ but $i \in \{3, 4\}$ components are zero because of the form of the matrix $P_+ = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$. We then identify the chiral supercharges as the $N=2$ sector of the supercharges,

$$Q_{\alpha i} \equiv (Q P_+)_{\alpha i}. \quad (2.7)$$

We define the conjugate supercharge as follows:

$$\overline{Q}_{i\alpha} \equiv (P_+ C^{-1} Q C)_{i\alpha}, \quad (2.8)$$

where $C$ is the charge conjugation matrix and satisfies

$$\gamma^T = C\gamma C^{-1}, \quad C^T = -C. \quad (2.9)$$

The twisted supercharges satisfy the following relations,

$$s^+ = \frac{1}{2} \text{Tr}(Q P_+), \quad s^+_\mu = \frac{1}{2} \text{Tr}(\gamma^\mu Q P_+), \quad s^+_A = -\frac{1}{2} \text{Tr}(\gamma_A Q P_+). \quad (2.10)$$

The algebra (2.1) is equivalently rewritten as

$$\{Q_{\alpha i}, (Q P_+)_{\beta j}\} = 2(P_+ C P_+)^{ji}(\gamma^\mu C^{-1})_{\alpha i} + 2(P_+)^{ij}\delta_{\alpha \beta} Z. \quad (2.11)$$

We calculate the anti-commutation relations of these twisted supercharges,

$$\{s^+, s^+_\mu\} = P_\mu, \quad \{s^+, s^+_A\} = -\delta^+_\alpha \mu P^\nu, \quad \{s^+, s^+_Z\} = 0, \quad \{s^+_\mu, s^+_A\} = \delta^{+}_{\mu \nu} P^\nu, \quad \{s^+_\mu, s^+_Z\} = \delta^{+}_{A, \mu} Z, \quad (2.12)$$

where $\delta^{+}_{\mu \nu, \rho} = \delta_{\mu \nu} \delta_{\nu \rho} - \delta_{\mu \rho} \delta_{\nu \rho} - \epsilon_{\mu \nu \rho}$.

This algebra is $N=2$ twisted supersymmetry algebra with the central charge.

We construct a $N=2$ twisted superspace formalism based on the algebra (2.12). Thus the $N=2$ twisted superspace with the central charge consists of $\{x^\mu, \theta^+, \theta^+_\mu, \theta^+_A, z\}$ coordinates, where $x^\mu$, and $z$ are bosonic and $\theta^+, \theta^+_\mu$ and $\theta^+_A$ are fermionic. We define differential operators corresponding with supercharges (2.10) as follows:

$$Q^+ = \frac{\partial}{\partial \theta^+} + \frac{i}{2} \theta^+ \partial_\mu + \frac{i}{2} \theta^+ \frac{\partial}{\partial z},$$

$$Q^+_\mu = \frac{\partial}{\partial \theta^+_\mu} + \frac{i}{2} \theta^+ \partial_\mu - \frac{i}{2} \theta^+ \partial^\nu + \frac{i}{2} \theta^+ \frac{\partial}{\partial z},$$

$$Q^+_A = -\frac{\partial}{\partial \theta^+_A} - \frac{i}{2} \delta^{+}_{A, \rho \sigma} \theta^+ \partial^\rho + \frac{i}{2} \theta^+ \frac{\partial}{\partial z},$$

$$Z = -i \frac{\partial}{\partial z}. \quad (2.13)$$

where $z$ is a bosonic parameter corresponding to the central charge $Z$. These differential operators satisfy the following relations,

$$\{Q^+, Q^+_\mu\} = i \partial_\mu, \quad \{Q^+_\mu, Q^+_A\} = -i \delta^+_\mu \nu \partial^\nu, \quad \{Q^+, Q^+_A\} = 0,$$

$$\{Q^+, Q^+\} = -Z, \quad \{Q^+_\mu, Q^+_\nu\} = -\delta^+_{\mu \nu} Z, \quad \{Q^+_A, Q^+_B\} = -\delta^+_A B Z. \quad (2.14)$$
We introduce another set of differential operators \( \{ D^+_I \} \) which anticommute with the differential operators \( \{ Q^+_I \} \),

\[
D^+ = \frac{\partial}{\partial \theta^+} - \frac{i}{2} \theta^{+\mu} \partial_{\mu} - \frac{i}{2} \theta^+ \frac{\partial}{\partial z},
\]

\[
D^+_\mu = \frac{\partial}{\partial \theta^{+\mu}} - \frac{i}{2} \theta^{+\nu} \partial_{\nu} - \frac{i}{2} \theta^+ \frac{\partial}{\partial z},
\]

\[
D^+_A = \frac{\partial}{\partial \theta^{+A}} + \frac{i}{2} \delta^+_{A,\rho\sigma} \theta^{+\rho} \partial^\sigma - \frac{i}{2} \theta^+ \frac{\partial}{\partial z}. \tag{2.15}
\]

Next we consider the general characteristics of a superfield with the central charge. If we know the TSUSY transformations of component fields, we define the general superfield \( \Phi \) as follows,

\[
\Phi(x^\mu, \theta^I, z) = e^{\delta_{\theta^+} z} \phi(x^\mu), \tag{2.16}
\]

where \( \phi(x^\mu) \) is any field and \( \delta_{\theta^+} = \theta^+ s^+ + \theta^+ s^+_\mu + \frac{1}{4} \theta^{+A} s^+_A + izZ = \delta_{\theta^+} + izZ \).

Since the operator \( Z \) commutes all the operator \( \{ s^+_I \} \), the superfield is given by

\[
\Phi(x^\mu, \theta^I, z) = e^{izZ} e^{\delta_{\theta^+} \phi(x^\mu)}. \tag{2.17}
\]

We can define the following operator,

\[
\cosh(z\partial) \equiv 1 + \frac{1}{2} z^2 \partial^2 + \frac{1}{4!} z^4 \partial^4 + \frac{1}{6!} z^6 \partial^6 + \cdots,
\]

\[
\frac{\sinh(z\partial)}{\partial} \equiv z + \frac{1}{3!} z^3 \partial^2 + \frac{1}{5!} z^5 \partial^4 + \frac{1}{7!} z^7 \partial^6 + \cdots. \tag{2.18}
\]

Using these operators and the relation \( Z^2 = -\partial^2 \), \( e^{izZ} \) is expressed as follows:

\[
e^{izZ} = \cosh(z\partial) + i \frac{\sinh(z\partial)}{\partial} Z. \tag{2.19}
\]

Thus the general superfield is given by

\[
\Phi(x^\mu, \theta^I, z) = \cosh(z\partial) e^{\delta_{\theta^+} \phi(x^\mu)} + i \frac{\sinh(z\partial)}{\partial} Z e^{\delta_{\theta^+} \phi(x^\mu)}
\]

\[
= \cosh(z\partial) \Psi(x^\mu, \theta^I) + i \frac{\sinh(z\partial)}{\partial} \bar{\Psi}(x^\mu, \theta^I), \tag{2.20}
\]

where \( \Psi(x^\mu, \theta^I) \equiv e^{\delta_{\theta^+} \phi(x^\mu)} \), \( \bar{\Psi}(x^\mu, \theta^I) \equiv Z \Psi(x^\mu, \theta^I) \). Since the general superfield is an infinite series with respect to \( z \), one may think that there are infinite number of component fields. The superfield, however, has a finite number of component fields because \( \Psi \) and \( \bar{\Psi} \) depend only on \( \theta^+ \), \( \theta^+ \) and \( \theta^+_{\mu\nu} \).

We consider a superfield which represents the hypermultiplet. Since the general superfield has many component fields as stated above, we need to eliminate superfluous fields. We then use the R-symmetry in order to impose a condition on the
superfield. We consider a superfield $V_\mu$ with a vector index $; V_{\mu}$. We also introduce a R-transformations for the superfield $V_{\mu}$ and supercharges:

$$R_A^+ V_{\mu} = -\frac{i}{2} \delta_{A,\mu\nu} V^\nu,$$

$$R_A^+ s^+ = -\frac{i}{2} s_A^+,$$

$$R_A^+ s_B^+ = \frac{i}{2} \delta_{A,B}^+ s^+ - \frac{i}{8} \Gamma_{ABC}^+ s^{+C},$$

$$R_A^+ s_\mu^+ = -\frac{i}{2} \delta_{A,\mu\nu} s^{+\nu}.$$  \hfill (2.21)

where $\Gamma_{ABC}^+$ is an anti-symmetric tensor defined in the Appendix. The $\{D\}$ operators transform in the same manner with respect to the supercharges. We can find the following R-invariant terms,

$$R_A^+ (D_\mu^+ V_\nu + D_\mu^+ V^\nu) = 0, \quad R_A^+ (D_\mu^+ V^\mu) = 0, \quad R_A^+ (\delta^{\mu,\rho,\sigma} D_\mu^+ V^\rho) = 0.$$  \hfill (2.22)

We impose the following condition,

$$R_A^+ (D_\mu^+ V_\nu) = 0.$$  \hfill (2.23)

We expand $R_A^+ (D_\mu^+ V_\nu)$ as follows:

$$R_A^+ (D_\mu^+ V_\nu) = R_A^+ \left( \frac{1}{4} \delta_{\mu\rho} D^{+\rho} V_\nu + \frac{1}{2} (D_\mu^+ V_\nu + D_\nu^+ V_\mu) - \frac{1}{2} \delta_{\mu\nu} D^{+\rho} V_\rho \right)$$

$$+ \frac{1}{4} \delta_{\mu,\rho,\sigma} D^{+\rho} V_\sigma + \frac{1}{4} \delta_{\mu,\rho,\sigma} D^{+\rho} V_\sigma \right),$$

$$= R_A^+ \left( \frac{1}{2} (D_\mu^+ V_\nu + D_\nu^+ V_\mu) - \frac{1}{2} \delta_{\mu\nu} D^{+\rho} V_\rho \right) + \frac{1}{4} \delta_{\mu,\rho,\sigma} D^{+\rho} V_\sigma \right),$$

where we use the equation (2.22). The right hand side of equation (2.24) must be zero because of the constraint (2.23). We can similarly find a constraint by using the first equation of (2.22). The superfield satisfies the following constraints,

$$D_\mu^+ V_\mu + \delta_{A,\mu\nu} D^{+\nu} = 0,$$

$$D_\mu^+ V_\nu + D_\nu^+ V_\mu = \frac{1}{2} \delta_{\mu\nu} D^{+\rho} V_\rho,$$

$$\delta_{A,\mu\nu} D^{+\rho} V_\rho = 0.$$  \hfill (2.25)

$^1$We can think of another construction of the hypermultiplet. For example, we can introduce $V$ and $V_A$ which transforms $R_A^+ V = -\frac{i}{2} V_A$ and $R_A^+ V_B = \frac{i}{2} \delta_{A,B}^+ V - \frac{i}{8} \Gamma_{A,B,C}^+ V^{+C}$, respectively. We can further introduce a superfield with spinor index $\{42\}$, but it is difficult to deal with this type of superfield in our tensor formalism.
We then derive the following relations from the constraints (2.25),
\[ D^+ D^+ \psi^\mu = -2i \partial^\mu \psi^\mu, \] (2.26)
\[ \delta_{\lambda^\mu, \nu} D^+ D^+ \psi^\nu = -2i \delta_{\lambda^\mu, \nu} \partial^\nu \psi^\nu, \] (2.27)
\[ D^+ D^+ \psi^\nu = 2Z \psi^\nu, \] (2.28)
\[ \delta_{\lambda^\mu, \nu} D^+ D^+ \psi^\nu = 2 \delta_{\lambda^\mu, \nu} Z \psi^\nu, \] (2.29)
\[ Z D^+ \psi^\nu = -i \delta_{\lambda^\mu, \nu} \partial^\nu D^+ \psi^\rho - i \delta^\perp_{\mu, \nu} \partial^\rho D^+ \psi^\sigma, \] (2.30)
\[ Z D^+ \psi^\mu = -\frac{i}{4} \partial^\rho D^+ \psi^\nu + \frac{i}{4} \delta_{\nu, \rho} \partial^\rho D^+ \psi^\sigma, \] (2.31)
\[ Z D^+ \psi^\nu = -\frac{i}{4} \delta_{\mu, \nu} \partial^\rho D^+ \psi^\rho - \frac{i}{4} \delta^\perp_{\mu, \nu} \partial^\rho D^+ \psi^\sigma, \] (2.32)
\[ Z^2 \psi^\mu = -\partial^2 \psi^\mu. \] (2.33)

It is difficult to generally solve the constraints (2.25). We find transformation laws of TSUSY with respect to component fields by the following method. We check whether these transformations satisfy the algebras (2.12).

We define component fields of the hypermultiplet as:
\[ V^\mu| = V^\mu, \]
\[ D^+ V^\mu| = \tilde{\psi}^\mu, \]
\[ \frac{1}{4} D^+ V^\mu| = \tilde{\eta}, \]
\[ \frac{1}{4} \delta_{\lambda^\mu, \nu} D^+ V^\nu| = -\chi^\perp_{\lambda^\mu}, \]
\[ Z V^\mu| = K^\mu, \] (2.33)

where | means the lowest component of \( \theta \)'s. One can derive transformation laws of the hypermultiplet as follows:
\[ s^+ V^\mu = Q^+ V^\mu = D^+ V^\mu| = \tilde{\psi}^\mu, \] (2.34)

where the first equality is a definition of the SUSY transformation, we use the fact that the operator \( Q^+ \) is equivalent to \( D^+ \) at the lowest component of \( \theta \)'s at the second one and we use the definition of the field \( \tilde{\psi}^\mu \) at the last one.

\[ s^+ V^\nu = Q^+ V^\nu = D^+ V^\nu| \]
\[ = \frac{1}{2} (D^+ V^\nu + D^+ V^\mu|) + \frac{1}{2} (D^+ V^\nu - D^+ V^\mu|) \]
\[ = \frac{1}{4} \delta_{\lambda^\mu, \nu} D^+ \psi^\nu| + \frac{1}{4} \delta_{\mu, \nu} D^+ \psi^\sigma| + \frac{1}{4} \delta^\perp_{\mu, \nu} D^+ \psi^\sigma| \]
\[ = \delta_{\mu, \nu} \tilde{\eta} - \chi^\perp_{\mu, \nu}, \] (2.35)

where we use constraints (2.26) and definitions of \( \tilde{\eta} \) and \( \chi^\perp_{\mu, \nu} \). We find the other transformations of component fields. We show the \( N=2 \) TSUSY transformations of the hypermultiplet in Table 1. Since these transformations satisfy the algebra (2.1), this multiplet is one of the solutions of the constraint (2.25). We can construct the superfield \( V^\mu \) by applying the equation (2.16).

We consider a TSUSY invariant action. Taking the highest component of \( \theta \)'s out of a superfield does not lead to a TSUSY invariant action. The reason is that the
derivative with respect to $z$ is added to these supercharges $\{Q\}$. For example, $Q^+$ is the following form,

$$Q^+ = \frac{\partial}{\partial \theta^+} + \frac{\imath}{2} \theta^{+\mu} \partial_\mu + \frac{\imath}{2} \theta^+ \frac{\partial}{\partial z}. \quad (2.36)$$

The first term does not contribute to the highest component of a superfield, the second one is a total derivative term and the last one is not a total derivative term. We recognize that the superfield to which we apply $Q^+$ is non-total derivative term. The other charges have the same structure. Thus the highest component of the superfield is not TSUSY invariant. We find a TSUSY invariant action to take the lowest component of $\theta$’s. The explicit form of the action is the following form,

$$S = \frac{1}{2} \int d^4x \left( -\frac{1}{6} \delta^{\mu\nu}_{\rho\sigma} D^{\mu\rho} D^{\nu\sigma}(\nabla^\rho Z V^\sigma) - \frac{1}{12} \delta^{\mu\nu}_{\rho\sigma} D^{\mu\rho} D^{\nu\sigma}(\nabla^\rho Z V^\sigma) \right)$$

$$= \int d^4x \left( V^\mu \partial^2 V_\mu + 4i \tilde{\psi}^\mu (\partial_\mu \tilde{\eta} + \partial^\nu \chi^-_{\mu\nu}) + K^\mu K_\mu \right). \quad (2.37)$$

This action possesses the TSUSY invariance of Table 1 and the following R-symmetry,

$$R^+_A V_\mu = -\frac{\imath}{2} \delta^{+\mu}_{A,\mu\nu} V^\nu, \quad R^+_A K_\mu = -\frac{\imath}{2} \delta^{+\mu}_{A,\mu\nu} K^\nu, \quad R^+_A \tilde{\psi} = R^+_A \tilde{\eta} = R^+_A \chi_B^- = 0. \quad (2.38)$$

### 3 Hypermultiplet coupling to Donaldson-Witten theory

We will construct the Donaldson-Witten theory coupled to the hypermultiplet. The constraints (2.25) must be covariantized because of the coupling to the gauge field. The equations (3.1) are:

$$\nabla^+_A V_\mu + \delta^{+\mu}_{A,\mu\nu} \nabla^+ V^\nu = 0,$$

$$\nabla^+_\mu V_\nu + \nabla^+_\nu V_\mu = \frac{\imath}{2} \delta^{\mu\nu}_{\rho\sigma} \nabla^+ V^\rho,$$

$$\delta^{+\nu}_{A,\mu\nu} \nabla^+ V^\nu = 0. \quad (3.1)$$
where $\nabla^+ \equiv D^+ - i\Gamma$, $\nabla^\mu_\mu \equiv D^{\mu\mu} - i\Gamma_\mu$, $\nabla^+_A \equiv D^+_A - i\Gamma_A$ and $\{\Gamma, \Gamma_\mu, \Gamma_A\}$ are connection superfields. We derive the following relations from the constraint (3.1) and the Jacobi identities,

$$\nabla^+ \nabla^+ \nu_\mu = -2i\nabla^\mu \nu_\mu,$$

(3.2)

$$\delta_{\mu,\nu} \nabla^+ \nabla^+ \nu^{\nu} = -2i\delta_{\mu,\nu} \nabla^\mu \nu^{\nu},$$

(3.3)

$$\nabla^+ \nabla^+ \nu_\nu = 2\Xi \nu_\mu - 2i[\mathcal{W}, \nu_\mu],$$

(3.4)

$$\mathcal{Z} \nabla^+ \nu_\nu = -i\delta_{\mu,\nu} \nabla^+ \nu_\rho - i\delta_{\nu,\rho} \nabla^+ \nu^{\nu} + i[\mathcal{F}, \nabla^+ \nu_\nu]$$

$$+ i\delta_{\mu,\nu} \nabla^+ \nu_\rho + \frac{i}{2} \delta_{\mu,\nu} \nabla^+ \nu^{\nu} + i[\mathcal{W}, \nu_\mu],$$

(3.6)

$$\mathcal{Z} \nabla^+ \nu_\mu = -\nabla^\nu \nabla^\nu \nu_\mu - \frac{i}{4} \left( \nabla^+ \mathcal{F}, \nabla^+ \nu_\nu \right) - \frac{i}{4} \delta_{\mu,\nu} \nabla^+ \nu_\rho$$

$$+ i\{[\nabla^+ \mathcal{W}, \nabla^+ \nu_\mu] - i\{[\nabla^+ \mathcal{W}, \nabla^+ \nu^{\nu}] + \frac{i}{2} \delta_{\mu,\nu} \nabla^+ \nu^{\nu} \mathcal{F}, \nu^{\nu} \right]$$

$$+ [\mathcal{W}, \nu_\mu] + i[\mathcal{F}, \nu_\mu],$$

(3.7)

$$\mathcal{Z}^2 \nu_\mu = -\nabla^\nu \nabla^\nu \nu_\mu + \frac{i}{4} \left( \nabla^+ \mathcal{F}, \nabla^+ \nu_\nu \right) - \frac{i}{4} \delta_{\mu,\nu} \nabla^+ \nu_\rho$$

$$+ i\{[\nabla^+ \mathcal{W}, \nabla^+ \nu_\mu] - i\{[\nabla^+ \mathcal{W}, \nabla^+ \nu^{\nu}] + \frac{i}{2} \delta_{\mu,\nu} \nabla^+ \nu^{\nu} \mathcal{F}, \nu^{\nu} \right]$$

$$+ [\mathcal{W}, \nu_\mu] + i[\mathcal{F}, \nu_\mu],$$

(3.8)

where $\mathcal{F}$ and $\mathcal{W}$ are bosonic curvature superfields with respect to the vector multiplet [37] and we use the following modified commutation relations,

$$\{\nabla^+, \nabla^+ \} = -i\mathcal{F} + \mathcal{Z}, \quad \{\nabla^+_A, \nabla^+_B \} = -i\delta^+_A \mathcal{F} + \delta^+_A \mathcal{Z},$$

$$\{\nabla^+_\mu, \nabla^+ \nu \} = -i\delta_{\mu,\nu} \mathcal{W} + \delta_{\mu,\nu} \mathcal{Z}, \quad \{\nabla^+ \nu_\mu, \nabla^+ \nu_\nu \} = i\delta_{\mu,\nu} \nabla^\nu,$$

$$\{\nabla^+_A, \nabla^+ \nu_\mu \} = i\delta_{A,\mu} \nabla^\nu,$$

$$[\nabla^+_A, \nabla^+_\mu] = -\frac{i}{2} \delta_{A,\mu} \nabla^+ \mathcal{F},$$

$$[\nabla^+_A, \nabla^+ \nu_\mu] = -\frac{i}{2} \delta_{A,\mu} \nabla^+ \nu_\mu + \frac{i}{2} \nabla^+ \mathcal{W}. \quad \tag{3.9}$$

The curvature superfields $\mathcal{F}$ and $\mathcal{W}$ commute with the central charge $\mathcal{Z}$. We define component fields of superfields $\mathcal{F}$, $\mathcal{W}$ and $\nu_\mu$ as follows:

$$\mathcal{F} = \phi, \quad \nabla^\mu \mathcal{F} = C_\mu, \quad \frac{1}{4} \delta_{A,\mu} \nabla^\mu \nabla^+ \mathcal{F} = -\phi^+_A,$$

$$\mathcal{W} = \bar{\phi}, \quad \nabla_A \mathcal{W} = \chi^+_A, \quad \nabla \mathcal{W} = \chi,$$

$$\nu_\mu = V_\mu, \quad \nabla^+ \nu_\mu = \psi_\mu, \quad \frac{1}{4} \nabla^+ \nu_\mu = \eta,$$

$$\frac{1}{4} \delta_{A,\mu} \nabla^+ \nu_\mu = -\chi^-_A, \quad \mathcal{Z} \nu_\mu = K_\mu. \quad \tag{3.10}$$
One can similarly derive transformation laws of the hypermultiplet as follows:

\[
\begin{align*}
    s^+ V_\mu &= \nabla^+ V_\mu = \bar{\psi}_\mu, \\
    s^+ \psi_\nu &= \nabla^+ \psi_\nu = \frac{1}{2} \left( \nabla^+ \psi_\nu + \nabla^+ \psi_\mu \right) + \frac{1}{2} \left( \nabla^+ \psi_\nu - \nabla^+ \psi_\mu \right) \\
    &= \frac{1}{4} \delta_{\mu\nu} \nabla^+ \psi_\rho + \frac{1}{4} \delta_{\mu\nu,\rho} \nabla^+ \psi_\sigma + \frac{1}{4} \delta_{\mu\nu,\rho} \nabla^+ \psi_\sigma \\
    &= \delta_{\mu\nu} \bar{\eta} - \chi_{\mu\nu},
\end{align*}
\]  

(3.11)

where we use the constraints (3.1) and the definition of \( \bar{\eta} \) and \( \chi_{\mu\nu} \). It should be noted that the lowest components of the fermionic superconnections vanish because we impose the Wess-Zumino gauge. We show the \( N=2 \) TSUSY transformations of the hypermultiplets in Table 2. The commutation relations of the twisted supercharges are given in the following form,

\[
\begin{align*}
    \{s^+, s^+\} \varphi &= Z \varphi - i[\phi, \varphi], \\
    \{s^+, s^+\} \varphi &= -i D_\mu \varphi, \\
    \{s^+, s^+\} \varphi &= i \delta^\mu_{\nu,\rho} D^\rho \varphi, \\
    \{s^+, s^+\} \varphi &= \delta_{\mu\nu} \varphi - i \delta_{\mu\nu} [\bar{\phi}, \varphi], \\
    \{s^+, s^+\} \varphi &= \delta_{\mu\nu} \varphi - i \delta_{\mu\nu} [\bar{\phi}, \varphi], \\
    \{s^+, s^+\} \varphi &= \delta_{\mu\nu} \varphi - i \delta_{\mu\nu} [\bar{\phi}, \varphi], \\
    \{s^+, s^+\} \varphi &= \delta_{\mu\nu} \varphi - i \delta_{\mu\nu} [\bar{\phi}, \varphi].
\end{align*}
\]

(3.12)

where \( \varphi = V_\mu, \bar{\psi}_\mu, \bar{\eta}, \chi_A, K_\mu \) and \( D_\mu \varphi = \partial_\mu \varphi - i [\omega_\mu, \varphi] \). Thus these algebras are closed at the off-shell level up to the gauge transformation.

| \( s^+ \) | \( s^+ \) | \( s^+ \) |
|---|---|---|
| \( V_\mu \) | \( \bar{\psi}_\mu \) | \( \frac{1}{2} K_\nu - \frac{i}{2} [\bar{\phi}, V_\nu] \) |
| \( \bar{\eta} \) | \( -\frac{i}{2} D^\mu V_\mu \) | \( \frac{i}{2} (\delta_{\mu\nu,\rho} D^\rho V_\sigma + \delta_{\mu\nu} D^\rho V_\rho - 2 D_\mu V_\nu) \) |
| \( \chi_B \) | \( \frac{i}{2} \delta^\mu_{\nu,\rho} D^\rho V_\sigma \) | \( \frac{i}{2} \delta^\mu_{\nu,\rho} D^\rho V_\sigma \) |
| \( K_\nu \) | \( -i (D_\nu \bar{\eta} + D_\nu \chi_{\nu\rho}) - \frac{i}{2} [\bar{\chi}, V_\nu] \) | \( -i (\delta_{\mu\nu} D^\rho V_\sigma + \delta_{\mu\nu} D^\rho V_\rho - 2 D_\mu V_\nu) \) |

Table 2: TSUSY transformation of the hypermultiplet coupling to the vector multiplet.
A covariantized action has the following form,

\[ S_H = \frac{1}{4} \int d^4x \text{Tr} \left( -\frac{1}{6} \delta_{\mu\nu,\rho\sigma} \nabla^\mu \nabla^\nu (\nabla^\rho \nabla^\sigma) - \frac{1}{12} \delta_{\mu\nu,\rho\sigma} \nabla^\mu \nabla^\nu \phi^\rho \phi^\sigma \right) \left| \phi \right|^2 \]

\[ = \int d^4x \text{Tr} \left( \frac{1}{2} V^\mu D^\nu D_\nu V_\mu + 2i \tilde{\psi}^\mu (D_\mu \bar{\eta} + D^\nu \chi^\mu_\nu) + \frac{1}{2} K^\mu K_\mu \right. \]

\[ + i \phi \{ \tilde{\psi}_\mu, \tilde{\psi}^\mu \} + i \phi \{ \bar{\eta}, \bar{\eta} \} + i \phi \{ \chi_A, \chi^{-A} \} \]

\[ - i V^\mu \{ C^\mu, \delta_{\mu\nu} \tilde{\eta} + \chi^\mu_\nu \} - i V^\mu \{ \delta_{\mu\nu} \chi - \chi^\mu_\nu, \tilde{\psi}^\mu \} \]

\[ - \frac{i}{2} \phi^+_\mu [V^\mu, V^\nu] - \frac{1}{4} V^\mu [\phi, [\bar{\Phi}, V_\mu]] - \frac{1}{4} V^\mu [\bar{\Phi}, [\phi, V_\mu]]. \]

(3.14)

The Donaldson-Witten theory is given by the following form:

\[ S_{DW} = \frac{1}{2} \int d^4x d^4 \theta \text{ Tr} F^2 \]

\[ = \frac{1}{2} \int d^4x \text{Tr} \left( -\phi D^\mu D_\mu \phi \right. \]

\[ \left. + i \phi \{ \chi, \chi \} + i \phi \{ C^A, C_A \} + i \phi \{ C^\mu, C_\mu \} \right) \]

\[ + \frac{1}{4} [\phi, \bar{\Phi}]^2 - \frac{1}{4} (\phi^+_\mu)^2. \]

(3.15)

The superspace description of this action have been studied in [42], [37].

We then obtain a new action,

\[ S_{total} = S_H + S_{DW} \]

\[ = \int d^4x \text{Tr} \left( -\frac{1}{2} \phi D^\mu D_\mu \phi \right. \]

\[ \left. + i \phi \{ \chi, \chi \} + i \phi \{ C^A, C_A \} + i \phi \{ C^\mu, C_\mu \} \right) \]

\[ + \frac{1}{4} [\phi, \bar{\Phi}]^2 - \frac{1}{4} (\phi^+_\mu)^2. \]

(3.16)

In the ordinary supersymmetric theory \( N=4 \) vector multiplet consists of \( N=2 \) vector multiplet and hypermultiplet at on-shell level. By integrating out the auxiliary fields; \( \phi^+_A \) and \( K_\mu \), we can construct an on-shell invariant action. From the equation of motion auxiliary fields are given by,

\[ \phi^+_A = -\frac{i}{2 \delta_{A,\rho\sigma}} [V^\rho, V^\sigma], \quad K_\mu = 0. \]

(3.17)

We redefine the field as

\[ C \to 2C, \quad \chi \to 2\chi, \quad \chi^+_A \to 2\chi^+_A, \quad \phi \to i\phi, \quad \bar{\Phi} \to i\bar{\Phi}. \]

(3.18)
and obtain

$$S_{total} = S_H + S_{DW}$$

$$= \int d^4x \text{Tr} \left( \frac{1}{2} \phi D^\mu D_\mu \phi + \frac{1}{2} V^\mu D^\nu D_\nu V_\mu + \frac{1}{2} (F^\mu_\nu)^2 + \frac{1}{8} [\phi, \bar{\phi}]^2 - \frac{1}{4} [V^\mu, V_\nu]^2 ight.$$  

$$+ 2i \bar{\psi}^\mu (D_\mu \bar{\eta} + D^\nu X^\nu) - \frac{1}{4} \phi \{ \bar{\eta}, \bar{\eta} \} - \frac{1}{4} \phi \{ \chi^+_A, \chi^{-A} \} - \bar{\phi} \{ \psi^\mu, \bar{\psi}^\mu \}$$

$$+ 2i C^\mu (D_\mu \chi - D^\nu \chi^+_\nu) + \phi \{ \chi, \chi \} + \frac{1}{4} \phi \{ \chi^+_A, \chi^{-A} \} + \bar{\phi} \{ C^\mu, C_\mu \}$$

$$- 2V^\mu \{ C^\mu, \delta_{\mu\nu} \bar{\eta} + \chi_{\mu\nu} \} - 2V^\mu \{ \delta_{\mu\nu} \chi - \chi_{\mu\nu}, \bar{\psi}^\nu \}$$

$$+ \frac{1}{4} V^\mu [\phi, [\bar{\phi}, V_\mu]] + \frac{1}{4} V^\mu [\bar{\phi}, [\phi, V_\mu]] \right).$$ 

(3.19)

This action possess the following discrete symmetry,

$$\phi \rightarrow -\phi, \quad \bar{\phi} \rightarrow -\bar{\phi}, \quad \chi \leftrightarrow \bar{\eta}, \quad \psi^\mu \leftrightarrow C^\mu, \quad \chi^{\pm}_A \rightarrow -\chi^{\mp}_A. \quad (3.20)$$

Applying the symmetry (3.20) to the TSUSY transformation, we get a new fermionic symmetry which is shown in Appendix A. Since the new symmetry satisfies the TSUSY algebra of $\{ s^- \}$ part, the on-shell action (3.19) has the $N=4$ TSUSY and $SO(4)$ R-symmetry and it is equal to Marcus’s $N=4$ twisted action [43] when we redefine the fields.

We summarize the $N=4$ twisting procedure here [53]. Yamron firstly pointed out the existence of three twisting manners with respect to $N=4$ SUSY [54]. This was analyzed by Vafa and Witten [55] and Marcus [43, 56] in detail. Let us focus on the Marcus’s twist. The Marcus’s twist has two BRST charges having the same ghost number. The Dirac-Kähler twist has the same characteristic as the Marcus’s twist in the sense that there are the scalar and the pseudo-scalar BRST charges. Therefore we have thought that the Dirac-Kähler twist has a close relation to the one of Marcus. We now claim that the four-dimensional Dirac-Kähler twist is equivalent to the Marcus twist because we have derived the Marcus action from our formalism.

## 4 Euclidean $N=4$ SUSY Action

In this section, we derive an ordinary spinor type $N=4$ SUSY action using Dirac-Kähler twist. The tensor fermions appearing in the twisted theory $\{ C^\mu, \psi^\mu, \chi^+_A, \chi^{-A}, \chi, \bar{\eta} \}$ are easily transformed into a spinor field with the internal symmetry. We define a Dirac-Kähler field as follows:

$$\Psi_{\alpha i} = (1_{\psi} + \gamma^\mu \psi^\mu + \frac{1}{2} \gamma^{\mu\nu} \psi_{\mu\nu} + \bar{\gamma}^\mu \psi'^\mu + \gamma_5 \bar{\psi})_{\alpha i}. \quad (4.1)$$

where $\chi = \psi + \bar{\psi}$, $\bar{\eta} = \psi - \bar{\psi}$, $C^\mu = \psi_\mu + \psi'^\mu$, $\tilde{\psi}^\mu = \psi_\mu - \psi'^\mu$, $\chi^{\pm\mu\nu} = \psi^{\mu\nu} - \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \psi_{\rho\sigma}$, $\chi^{-\mu\nu} = -\psi^{\mu\nu} + \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \psi_{\rho\sigma}$. Then the action (3.19) is rewritten as
follows:

\[
S = \int d^4x \text{Tr} \left( \frac{1}{2} \bar{\Psi} \gamma^{\mu} D_{\mu} \Psi + \frac{1}{2} \phi D^{\mu} D_{\mu} \bar{\phi} + \frac{1}{2} V^{\mu} D_{\nu} D_{\sigma} V_{\mu} + \frac{1}{2} (F_{\mu\nu})^2 \
+ \frac{1}{2} \phi (\bar{\Psi} \gamma^5 \Psi^i + \bar{\Psi} \gamma^5 \Psi^j (\hat{\gamma}_5)^{ij}) - \frac{1}{2} \bar{\phi} (\bar{\Psi} \gamma^5 \Psi^i - \bar{\Psi} \gamma^5 \Psi^j (\hat{\gamma}_5)^{ij}) \
- V^{\mu} \bar{\Psi} \gamma^{\mu} \Psi^i + \frac{1}{8} V^{\mu} [\phi, [\bar{\phi}, V_{\mu}]] + \frac{1}{4} V^{\mu} [\phi, V_{\mu}] \
- \frac{1}{4} [V_{\mu}, V_i]^2 + \frac{1}{8} [\phi, \bar{\phi}]^2 \right). \tag{4.2}
\]

This is an untwisted action. It should be noted that $V_{\mu}$ is not a spacetime vector field since it couples to the internal space $\gamma$-matrix $\{ (\hat{\gamma}_\mu)^{ij} \}$. The action (4.2) is easily rewritten by using $\phi^{ij}$, $\tilde{\phi}^{ij}$ as follows:

\[
S = \int d^4x \text{Tr} \left( \frac{1}{2} \bar{\Psi} \gamma^{\mu} D_{\mu} \Psi^i + \frac{1}{8} \bar{\tilde{\phi}}^{ij} D_{\mu} \phi^{ij} + \frac{1}{2} (F_{\mu\nu})^2 \
+ \bar{\Psi} P_+ \Psi^i \phi^{ij} + \bar{\Psi} P_- \Psi^j \tilde{\phi}^{ij} - \frac{1}{64} [\phi^{ij}, \tilde{\phi}^{lm}] [\phi^{ji}, \phi^{ml}] \right), \tag{4.3}
\]

where $\phi$ and $\tilde{\phi}$ are defined as follows:

\[
\phi^{ij} = (\phi P_+ - \bar{\phi} P_- - V_{\mu} \hat{\gamma}_\mu)^{ij}, \quad \tilde{\phi}^{ij} = (\bar{\phi} P_+ - \phi P_- - V_{\mu} \hat{\gamma}_\mu)^{ij}. \tag{4.4}
\]

$\phi^{ij}$ and $\tilde{\phi}^{ij}$ satisfy the following relations respectively,

\[
\phi^\dagger = C\phi^T C^{-1}, \quad \phi^{\dagger} = \phi, \quad \tilde{\phi}^\dagger = C\tilde{\phi}^T C^{-1}, \quad \tilde{\phi}^{\dagger} = \tilde{\phi}. \tag{4.5}
\]

Using (4.5), we find that $C\phi$ and $C\tilde{\phi}$ are anti-symmetric matrices. They are equivalently written in the following form

\[
(C\tilde{\phi})^{*}_{ij} = -\frac{1}{2} \epsilon_{ijkl} (C\phi)^{kl}, \tag{4.6}
\]

where $\epsilon_{1234} = 1$. They are the second rank self-dual tensor of the representation 6 of SU(4) group. This untwisted theory has the internal SU(4) R-symmetry, but the R-symmetry is reduced to SO(4) due to the Dirac-Kähler twist in the twisted theory. This action (4.3) is a $N=4$ SUSY action at on-shell level.

### 5 Conclusions and Discussions

We have proposed a twisted $N=2$ superspace formalism with the central charge based on the Dirac-Kähler twisting procedure. We have examined a general property of a superfield in this formalism. In this case the superfield has many superfluous
fields. We have introduced a superfield $V_\mu$ and found its constraints by means of R-symmetry. Using this superfield, we have found the off-shell action of the twisted hypermultiplet on superspace. It turned out that the R-symmetry played an important role in this formalism.

We have then extended this model to the covariantized theory. We obtained a Donaldson-Witten theory coupled to the twisted hypermultiplet. This theory is off-shell $N=2$ TSUSY. When the auxiliary fields are integrated out, the symmetry of the theory is enhanced to the $N=4$ TSUSY at on-shell level. This action is equal to that of Marcus after some fields are redefined.

Since the Dirac-Kähler twisted algebra and Marcus’s one include two scalar twisted supercharges which are assigned to the same ghost number $+1$ [58] and both twists lead to the same action, we then claim that the four-dimensional Dirac-Kähler twisting procedure is equivalent to the Marcus’s twist. It should be noted that the Dirac-Kähler twist can be defined in other dimensions [37, 58]. The advantage of the Dirac-Kähler twist is that since all the fermions are related to spinor by the Dirac-Kähler mechanism, we can easily construct a corresponding untwisted theory. This equivalence suggest that these $N=2$ and $N=4$ TSUSY models can be applied to a lattice SUSY theory, because the Dirac-Kähler mechanism is compatible with a lattice theory and corresponding two-dimensional models are realized on a lattice based on the twisted superspace formalism [48, 49].

In this paper, we have considered the $N=2$ twisted superspace formalism with a central charge. There is an alternative approach where a theory has the $N=4$ TSUSY with a central charge and the SO(4) R-symmetry. A corresponding untwisted theory has the maximal R-symmetry; USp(4) $\simeq$ SO(5) which can be reduced to SO(4) by Dirac-Kähler twist in twisted theory. In this case the off-shell $N=4$ TSUSY is preserved [59].

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## Appendix A

We show the full list of the on-shell $N=4$ twisted SUSY transformations,

|     | $s_\mu^+$ | $s_\mu^-$ |
|-----|-----------|-----------|
| $\phi$ | $-2C_\mu$ | $2\psi_\mu$ |
| $\bar{\phi}$ | 0 | 0 |
| $C_\nu$ | $\frac{i}{2}D_\nu\phi$ | $\frac{i}{2}D_\nu\phi$ |
| $\tilde{\psi}_\nu$ | $-\frac{1}{2}D_{\mu\nu}\phi$ | $-\frac{1}{2}D_{\mu\nu}\phi$ |
| $\chi_B^+$ | $-\frac{1}{2}\delta_{\mu,\rho\sigma}[V^\rho, V^\sigma] + iF_\mu^+$ | $\frac{i}{4}\delta_{\mu,\rho\sigma}[V^\rho, V^\sigma] + iF_\mu^+$ |
| $\chi_B^-$ | $\frac{1}{2}\delta_{\mu,\rho\sigma}D^\rho V^\sigma$ | $\frac{i}{4}\delta_{\mu,\rho\sigma}[V^\rho, V^\sigma] + iF_\mu^+$ |
| $\chi$ | $-\frac{1}{4}[\phi, \tilde{\phi}]$ | $-\frac{1}{4}[\phi, \tilde{\phi}]$ |
| $\tilde{\eta}$ | $-\frac{i}{2}D^\nu V_\mu$ | $-\frac{i}{2}D^\nu V_\mu$ |
| $\omega_\nu$ | $-\frac{1}{2}[\phi, V_\mu]$ | $-\frac{1}{2}[\phi, V_\mu]$ |
| $V_\nu$ | $\delta_{\mu,\nu}\tilde{\eta} - \chi_{\mu\nu}^+$ | $\delta_{\mu,\nu}\tilde{\eta} - \chi_{\mu\nu}^+$ |

|     | $s_A^+$ | $s_A^-$ |
|-----|--------|--------|
| $\phi$ | 0 | 0 |
| $\bar{\phi}$ | $-2\chi_A^+$ | $-2\chi_A^+$ |
| $C_\nu$ | $-\frac{1}{4}\delta_{\mu,\nu}[\phi, V^\nu]$ | $-\frac{1}{4}\delta_{\mu,\nu}[\phi, V^\nu]$ |
| $\tilde{\psi}_\nu$ | $-\frac{1}{2}\delta_{\mu,\nu\rho\sigma}D^\rho \phi$ | $-\frac{1}{2}\delta_{\mu,\nu\rho\sigma}D^\rho \phi$ |
| $\chi_B^+$ | $\frac{i}{8}\Gamma_{ABC}(\frac{i}{2}\delta_{C,\rho\sigma}[V_\rho, V_\sigma] + 2F^C - C) - \frac{1}{4}\delta_{\mu,\nu}\phi$ | $\frac{i}{8}\Gamma_{ABC}(\frac{i}{2}\delta_{C,\rho\sigma}[V_\rho, V_\sigma] + 2F^C - C) + \frac{1}{4}\delta_{\mu,\nu}\phi$ |
| $\chi_B^-$ | $\frac{1}{4}\delta_{\mu,\nu\rho\sigma}[V^\rho, V^\sigma] + iF_A^+$ | $\frac{1}{4}\delta_{\mu,\nu\rho\sigma}[V^\rho, V^\sigma] + iF_A^+$ |
| $\chi$ | $-\frac{1}{2}\delta_{\mu,\nu}\rho V^\sigma$ | $-\frac{1}{2}\delta_{\mu,\nu}\rho V^\sigma$ |
| $\tilde{\eta}$ | $-\delta_{\mu,\nu}\tilde{\eta}$ | $-\delta_{\mu,\nu}\tilde{\eta}$ |
| $\omega_\nu$ | $-\delta_{\mu,\nu}\psi_\mu$ | $-\delta_{\mu,\nu}\psi_\mu$ |
| $V_\nu$ | $-\delta_{\mu,\nu}\tilde{\eta}$ | $-\delta_{\mu,\nu}\tilde{\eta}$ |
The $N=4$ twisted supercharges $\{s^\pm, s^\pm_A, s^\pm_B\}$ satisfy the following commutation relations up to gauge transformation at on-shell level.

\[
\begin{align*}
\{s^\pm, s^\pm\} \varphi &= \delta_{\text{gauge}(\mp i\partial)} \varphi, \\
\{s^\pm_A, s^\pm_B\} \varphi &= \delta_{\mu\nu} \delta_{\text{gauge}(\mp i\partial)} \varphi, \\
\{s^\pm_A, s^\pm\} \varphi &= \delta_{A,B} \delta_{\text{gauge}(\mp i\partial)} \varphi, \\
\{s^\pm, s^\pm_B\} \varphi &= \delta_{\text{gauge}(iV)} \varphi, \\
\{s^\pm_A, s^\pm\} \varphi &= \delta_{\text{gauge}(iV)} \varphi, \\
\{s^\pm_A, s^\pm_B\} \varphi &= 0,
\end{align*}
\]

where $\delta_{\text{gauge}(\pm i\partial)\omega} = D_{\mu} \xi$, $\delta_{\text{gauge}(\pm i\partial)} \varphi' = -i[\varphi', \xi]$, $\varphi' = \{\phi, \bar{\phi}, C_{\mu}, \bar{\psi}_{\nu}, \chi_{B}^A, \chi_{B}^{-A}, \bar{\eta}, V_{\nu}\}$.

**Appendix B**

We define an Euclidean four dimensional $\gamma$-matrices:

\[
\{\gamma_{\mu}, \gamma_{\nu}\} = 2\delta_{\mu\nu}, \quad \gamma^{\mu\dagger} = \gamma^\mu,
\]

where $\gamma_{\mu}$ satisfies the Clifford algebra. We introduce the following notations:

\[
\gamma_{\mu\nu} \equiv \frac{1}{2}[\gamma_{\mu}, \gamma_{\nu}], \quad \bar{\gamma}_{\mu} \equiv \gamma_{\mu} \gamma_5.
\]

We use the following representation of $\gamma$-matrix:

\[
\gamma^\mu = \begin{pmatrix} 0 & i\sigma^\mu \\ i\sigma^\mu & 0 \end{pmatrix},
\]

where $\sigma^n = (\sigma^1, \sigma^2, \sigma^3, \sigma^4)$, $\bar{\sigma}^n = (-\sigma^1, -\sigma^2, -\sigma^3, \sigma^4)$ with $\sigma^4 = -i1_{2 \times 2}$ and $\sigma^i$ are Pauli matrices, $i \in \{1, 2, 3\}$. A charge conjugation matrix $C$ and $B$ matrix are in general defined as follows:

\[
\gamma_{\mu} = \eta B^{-1} \gamma_{\mu}^* B, \quad B^* B = \epsilon 1, \\
\bar{\gamma}_{\mu} = \eta C^{-1} \bar{\gamma}_{\mu}^T C, \quad C^T = \epsilon C,
\]

\[
\begin{array}{|c|c|}
\hline
\varphi & 0 \\
\psi_{\mu} & 0 \\
\chi_{B}^+ & -i\delta_{A,\nu\rho} \bar{\psi}_{\rho} \\
\chi_{B}^- & 0 \\
\chi & 0 \\
\bar{\eta} & i\frac{1}{2} \chi_{A} \\
\omega_{\mu} & 0 \\
V_{\nu} & -i\delta_{A,\nu\rho} V^\rho \\
\hline
R_{+}^A & R_{-}^A \\
\hline
\end{array}
\]

\[
\begin{align*}
\{s^\pm, s^\pm\} \varphi &= \delta_{\text{gauge}(\mp i\partial)} \varphi, \\
\{s^\pm_A, s^\pm_B\} \varphi &= \delta_{\text{gauge}(\mp i\partial)} \varphi, \\
\{s^\pm_A, s^\pm\} \varphi &= \delta_{A,B} \delta_{\text{gauge}(\mp i\partial)} \varphi, \\
\{s^\pm, s^\pm_B\} \varphi &= \delta_{\text{gauge}(iV)} \varphi, \\
\{s^\pm_A, s^\pm\} \varphi &= \delta_{\text{gauge}(iV)} \varphi, \\
\{s^\pm_A, s^\pm_B\} \varphi &= 0,
\end{align*}
\]

\[
\begin{align*}
\gamma_{\mu\nu} \equiv \frac{1}{2}[\gamma_{\mu}, \gamma_{\nu}], \quad \bar{\gamma}_{\mu} \equiv \gamma_{\mu} \gamma_5.
\end{align*}
\]
where \((\eta, \epsilon) = (\pm 1, -1)\).

In a four dimensional Euclidean space Majorana fermions do not exist because the factor \(\epsilon\) should be equal to \(-1\) [60]. A Majorana spinor satisfies the following condition,

\[
\psi^* = B \psi, \tag{B.5}
\]

which leads

\[
\psi = B^* \psi = B^* B \psi. \tag{B.6}
\]

Thus the existence of Majorana fermions requires \(B^* B = 1\). This condition can not be taken in four dimensional Euclidean space. We can, however, take a \(\text{SU}(2) \simeq \text{USp}(2)\) Majorana fermion and a USp(4) Majorana fermion which satisfies the following condition, respectively,

\[
\psi^{i*} = \epsilon^{ij} B \psi^j, \tag{B.7}
\]

\[
\psi^{l*} = C^{lm} B \psi^m, \tag{B.8}
\]

where \(i, j \in \{1, 2\}, \ l, m \in \{1, 2, 3, 4\}\) and these fermions correspond to the fermions which appear in \(N=2\) and \(N=4\) supersymmetric theory, respectively. In this paper we choose \((\eta, \epsilon) = (1, -1)\) and \(C = B = -\gamma_1 \gamma_3\).

\section*{Appendix C}

In this appendix we explain SUSY algebra with respect to two-component spinors. For simplicity we consider Minkowski space. We represent the supercharge appearing in algebra as the following form,

\[
Q^i_{\alpha} = \left( -i \epsilon^{ij} Q^j_{\alpha} \right), \tag{C.1}
\]

\[
\bar{Q}_{\dot{\alpha} i} = (Q^i_{\alpha}, i \epsilon^{ij} \bar{Q}_{\dot{\alpha} j}). \tag{C.2}
\]

In this notation the SU(2) Majorana condition is the following form,

\[
\bar{Q}_{\dot{\alpha}} = (Q_{\alpha i})^\dagger. \tag{C.3}
\]

We describe the algebra with respect to these two-component supercharges,

\[
\{Q_{\alpha i}, \bar{Q}_{\dot{\alpha}}^{\dot{\beta}}\} = 2 \delta^{\dot{\beta}}_{\dot{\alpha}} (\sigma^\mu)_{\alpha \dot{\beta}} P_\mu, \quad \{Q_{\alpha i}, Q_{\beta j}\} = 2 i \epsilon_{ij} \epsilon_{\alpha \beta} Z, \tag{C.4}
\]

where supercharges with upper and lower indices are related through the \(\epsilon\)-tensor.
Appendix D

We introduce the definition of $\delta_{A,B}^\pm$ and $\Gamma_{ABC}^\pm$. The suffix $A$ is the second rank tensor which denotes the suffix $\mu, \nu \in \{1, \cdots, 4\}$. The definition of $\delta_{A,B}^\pm$ is

$$\delta_{A,B}^\pm = \delta_{\mu\nu, \rho\sigma}^\pm = \delta_{\mu\rho}^\pm \delta_{\nu\sigma}^\pm - \delta_{\mu\sigma}^\pm \delta_{\nu\rho}^\pm \mp \epsilon_{\mu\nu\rho\sigma},$$

(D.1)

where $\delta_{A,B}^\pm \delta_{A,B}^\mp = 0$. (Anti-)self-dual tensors $\chi_A^\pm$ satisfy

$$\chi_A^\pm = \frac{1}{4} \delta_{A,B}^\pm \chi_B,$$

(D.2)

where $\chi_B = \chi_B^+ + \chi_B^-$. Variants of the definition of $\Gamma_{ABC}^\pm$ which stand for the third anti-symmetric tensor for $ABC$ is

$$\Gamma_{\mu\nu, \rho\gamma}^{\pm \alpha, \beta, \gamma} = \delta_{\alpha\nu}^{\pm \beta\rho} \delta_{\gamma}^{\pm \mu\gamma} + \delta_{\mu\nu}^{\pm \beta\gamma} \delta_{\rho\alpha}^{\pm \gamma\alpha} + \delta_{\alpha\beta}^{\pm \gamma\gamma} \delta_{\rho\nu}^{\pm \gamma\gamma} + \delta_{\alpha\gamma}^{\pm \beta\rho} \delta_{\nu\alpha}^{\pm \gamma\alpha},$$

$$\mp \epsilon_{\mu\alpha\beta\gamma}^\pm \delta_{\rho\nu}^{\pm \beta\gamma} \mp \epsilon_{\mu\alpha\nu\gamma}^\pm \delta_{\beta\rho}^{\pm \gamma\alpha} \mp \epsilon_{\mu\alpha\beta\rho}^\pm \delta_{\gamma\nu}^{\pm \gamma\alpha} + \delta_{\mu\alpha, \beta\gamma}^\pm \delta_{\rho\nu}^\pm - \delta_{\mu\alpha, \nu\gamma}^\pm \delta_{\beta\rho}^\pm - \delta_{\mu\alpha, \beta\rho}^\pm \delta_{\nu\gamma}^\pm,$$

(D.3)

$$\Gamma^{\pm AB\rho\sigma} = \frac{1}{2} (\delta_{A,\nu\rho}^{\pm B, \sigma} \sigma - \delta_{A,\nu\sigma}^{\pm B, \rho} \sigma),$$

(D.4)

$$\Gamma^{\pm ABC} = \frac{1}{4} \delta_{A,\nu\rho}^{\pm B, \sigma} \delta_{\sigma}^{\pm C, \rho\sigma}.$$  

(D.5)

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