Linear functional state bounding for positive singular systems with unbounded delay and disturbances varying within a bounded set

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Abstract
In this paper, the linear functional state bounding problem, which is considered in [25], is extended to the singular system with unbounded delay. Firstly, some conditions are presented to guarantee positivity, regularity, impulse-free, and the component-wise bound for the state vector of the singular system without disturbance. Then, based on the results obtained and by using state transformations, the smallest component-wise ultimate bound of the state vector of the singular system with bounded disturbances is derived. By using the new technique, some sufficient conditions were proposed given in terms of the linear programming/Hurwitz matrix/spectral abscissa for linear functional state bounding problems of the singular system with unbounded delay. Finally, a numerical example is given to illustrate the obtained results.

1 | INTRODUCTION

Singular systems (also called implicit systems, differential-algebraic equations) have an essential position in system control theory and becoming increasingly important in various technical areas such as biological systems, economic systems, power systems, aircraft control systems etc. (see, [1–3] and the references therein). Moreover, disturbances often occur and are not eliminated in practical engineering systems for many reasons, such as measurement errors, external noises, linear approximation. In general, it is challenging to get asymptotic stability for the dynamic systems in which noise occurs. The objective of the state bounding problem is to find an ultimate bound, which is a set such that the state vector converges within it when the time tends to infinity [4] or the time tends to prespecified time [5, 6]. Recently the problem of finding the smallest ultimate bound set for perturbed systems has been increasingly concerned with research and has become an essential issue in practical system control theory (see, [5, 7–9]).

To solve this problem, two commonly used methods. The first method is based on the Lyapunov way combined with linear matrix inequality, and the second for the positive system is based on the Hurwitz/Schur matrices combined with the solution comparison method [5, 6, 10–12]. The first method is widely used for classes of linear systems whose matrices are constant, while, the second method is very useful for classes of positive linear systems and classes of nonlinear/time-varying systems which are bounded by positive linear systems.

Note that researching the singular system is much more complicated than the standard system because one needs to consider not only stability but also regularity and causality (discrete-time systems) or non-impulsiveness (continuous-time systems), and due to the singularity of the derivative matrix and the positive restriction on variables, much of the developed theory for positive descriptor systems is still not up to a quantitative level [13, 14, 16, 17]. Recently, the problem of state bounding for positive regular systems with time-varying delays and bounded disturbances has been considered in [6, 10, 15]. However, it is very difficult to extend the results of this work to positive singular systems because in order to deal with positive singular systems it requires more new techniques. Although there have been many results on the state bounding for linear systems or nonlinear systems with time-delay, little attention has been paid to this problem for positive singular systems with time-varying bounded/unbounded delay.

Very recently, in [16] studied the state bounding for positive singular systems problem, but the results only apply to discrete-time systems with delay bounded. It is worth noting that in all of the papers cited above, the delays are assumed to be bounded, while time delays in many real-life dynamical systems such as
neural networks systems are often unbounded. Therefore, many researchers have paid much attention to problem of stability of systems with unbounded delays and have obtained many results [18–22]. Recently, the linear functional state bounding problem for the positive system has been very interested in research and obtained many exciting results [23–25]. Besides, the approach used in [23–25] seems challenging to apply to the singular system with unbounded delay. Moreover, to the best of the authors’ knowledge, there are no results for the linear functional state bounding problem of the singular system with unbounded delay. Note that, if we find component-wise state bounds of the system then manifest, we also obtain linear functional bound of the system. However, the results obtained by such a method do not give an optimal linear functional bound. Note that the problem of finding state bounds in previous studies, the disturbance vector is bounded (or bounded component-wise) by known constant limits [10, 15, 16, 26–28]. In this paper, we consider a more general case where the disturbance vector is assumed to vary within a known bounded set. Therefore, we can study the state bounding problem for broader classes of perturbed dynamical systems. Inspired by the work in [25], in this study, we consider the linear state bounding problem of the singular system with unbounded delay. Different from the techniques used in [25], by introducing new techniques and Lemma, we proposed some sufficient conditions given in terms of the linear programming/Hurwit matrix/spectral abscissa for linear functional state bounding problems of the singular system with unbounded delay. The main results of the paper are summarised as follows:

i. We first derive some sufficient conditions to guarantee the considered singular system is regular, impulse-free, and positive. Then, we obtain component-wise bound for the singular system without disturbances with unbounded delay. Based on this result, we also derive sufficient conditions to guaranteed asymptotic stability for the singular system with unbounded delay.

ii. We present a method for establishing the smallest component-wise ultimate bound of the state vector of the singular system with bounded disturbances.

iii. Finally, based on the results obtained, by some new techniques, we derive linear functional state bounding for positive singular systems with unbounded delay and disturbances varying within a bounded set.

The remaining of this paper is organised as follows. In Section 2, we provide the problem statement and preliminaries. The main results are given in Section 3. A numerical example is given in Section 4. A conclusion is presented in Section 5.

2 PROBLEM FORMULATION AND PRELIMINARIES

Notation: \( \mathbb{R}^n_+ \) (\( \mathbb{R}^n_{0+} \)) denotes the set of all positive (nonnegative) vectors in \( \mathbb{R}^n \); \( \mathbb{R}^{m \times p} \) denotes the space of all real \( p \times m \) matrices. \( I_n \) is the \( n \)-dimensional identity matrix. For \( n = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k, x \geq 0 (> 0) \) means that \( x_j \geq 0 (> 0) \) for all \( j = 1, \ldots, k \). A matrix \( \mathcal{A} \in \mathbb{R}^{m \times n} \) is Metzler if all its off diagonal elements are non-negative. \( C([-b_0], \mathbb{R}^n) \) denotes the set of all \( \mathbb{R}^n \)- valued continuous functions on \([-b_0], \mathbb{R}^n \). \( \| \cdot \|_\infty \) denotes the infinity norm of \( \cdot \). For a polynomial \( M(x) = \sum_{\nu=0}^{2n} \sum_{\sigma=0}^{2n} c_{\nu\sigma} x_\nu x_\sigma \), we denote its degree as \( \text{deg}(M(x)) \). For real valued continuous functions on \([-b_0], \mathbb{R}^n \), \( \| \cdot \|_\infty \) denotes the infinity norm of \( \cdot \) and \( \sup_{x \in [-b_0]} | \cdot | \) denotes the maximum value of \( \cdot \) on \([-b_0], \mathbb{R}^n \).
Consider the linear function vector of the state system of the following form:
\[ L(t) = R^{T}X(t), \]
where \( R \in \mathbb{R}^{n \times d} \) is a given matrix.

In this paper, we find the smallest component-wise upper bound of the \( L(t) \).

**Definition 1** [29].
(i) If the pair \((E, A_{0})\) is regular, that is, \( \text{det}(sE - A_{0}) \neq 0 \), then the singular system \((1)\) is regular.
(ii) If the pair \((E, A_{0})\) is impulse-free, that is, \( \text{deg}(\text{det}(sE - A_{0})) = \text{rank}(E) \), then the singular system \((1)\) is impulse free.

**Definition 2** [29]. System \((1)\) is positive if for all initial value \( \psi \geq 0 \), and for any nonnegative input \( \vartheta(\cdot) \geq 0 \) implies the corresponding trajectory \( X(t, \psi, \vartheta) \geq 0 \) for all \( t \geq 0 \).

**Lemma 3** [29]. Let \( D \in \mathbb{R}^{n \times d} \) be a Metzler matrix. Then the following conditions are equivalent.

(i) \( \exists \lambda \in \mathbb{R}^{n} : \lambda > 0 \) and \( D\lambda < 0 \).
(ii) \( \text{det}(D) \neq 0 \) and \( -D^{-1} \succeq 0 \).
(iii) \( D \) is Hurwitz matrix.

We see that, if \( A_{j} \geq 0 \), \( A_{0} \) is a Metzler matrix, then \( A_{0} + A_{j} \) is also a Metzler matrix. Based on Lemma 1, we get the following lemma:

**Lemma 2.** Let \( A_{0} \) be a matrix Metzler matrix and \( A_{j} \geq 0 \). Then, the following conditions are equivalent:

(i) There exist \( \lambda = (\lambda_{1}, \lambda_{2}) > 0 \), \( \lambda_{1} \in \mathbb{R}^{n} \), \( \lambda_{2} \in \mathbb{R}^{n\times r} \) such that
\[ (A_{0} + A_{j})\lambda = \begin{pmatrix} A + M & B + N \\ C + P & D + Q \end{pmatrix} \begin{pmatrix} \lambda_{1} \\ \lambda_{2} \end{pmatrix} < 0. \]

(ii) \( s(D + Q) < 0 \) and
\[ s(A + M - (B + N)(D + Q)^{-1}(C + P)) < 0. \]

(iii) \( s(A + M) < 0 \) and
\[ s(D + Q - (C + P)(A + M)^{-1}(B + N)) < 0. \]

We obtain
\[ (A + M)\bar{\lambda}_{1} + (B + N)\bar{\lambda}_{2} < 0. \]

**Proof.** (i) \( \Rightarrow \) (ii). From (i) we get
\[ (C + P)\bar{\lambda}_{1} + (D + Q)\bar{\lambda}_{2} < 0. \]

Since, \( (C + P)\bar{\lambda}_{1} \geq 0 \), it follows that \( (D + Q)\bar{\lambda}_{2} < 0 \). Using Lemma 1 we have \( s(D + Q) < 0 \), and \(-(D + Q)^{-1} \succeq 0 \). Combine this with the inequality (5) we get
\[ -(B + N)(D + Q)^{-1}(C + P)\bar{\lambda}_{1} \leq (B + N)\bar{\lambda}_{2}. \]

We use the condition (i) and inequality (6) we obtain
\[ (A + M)\bar{\lambda}_{1} - (B + N)(D + Q)^{-1}(C + P)\bar{\lambda}_{1} \leq (A + M)\bar{\lambda}_{1} + (B + N)\bar{\lambda}_{2} < 0. \]

This implies \( (A + M - (B + N)(D + Q)^{-1}(C + P)) \bar{\lambda}_{1} \leq 0 \). On the other hand \( (A + M - (B + N)(D + Q)^{-1}(C + P)) \) is Metzler, using Lemma 1 we get \( s(A + M - (B + N)(D + Q)^{-1}(C + P)) < 0 \).

(\(ii\) \( \Rightarrow \) (i). Combining Lemma 1 and \( s(D + Q) < 0 \), \( D + Q \) is Metzler it follows that \( (D + Q)^{-1} \succeq 0 \), and there exist \( \lambda_{0} > 0 \) such that \( (D + Q)\lambda_{0} < 0 \). It is easy to check that:
\[ A + M - (B + N)(D + Q)^{-1}(C + P), \]

is a Metzler matrix, combine this with \( s(A + M - (B + N)(D + Q)^{-1}(C + P)) < 0 \) and Lemma 1 implies there exist \( \lambda_{1} > 0 \) such that
\[ (A + M - (B + N)(D + Q)^{-1}(C + P))\lambda_{1} < 0. \]

This implies that
\[ (A + M - (B + N)(D + Q)^{-1}(C + P))\lambda_{1} + \epsilon(B + N)\lambda_{0} < 0, \]

for sufficiently small \( \epsilon \). Setting
\[ \lambda_{2} := -(D + Q)^{-1}(C + P)\lambda_{1} + \epsilon\lambda_{0} > 0. \]

We obtain
\[ (A + M)\lambda_{1} + (B + N)\lambda_{2} \]
\[ = (A + M)\lambda_{1} + (B + N)\epsilon\lambda_{0} < 0. \]

Pre - multiplying both sides of equation
\[ \lambda_{2} := -(D + Q)^{-1}(C + P)\lambda_{1} + \epsilon\lambda_{0}, \]

with the matrix \( (D + Q) \) we get
\[ (C + P)\lambda_{1} + (D + Q)\lambda_{2} = \epsilon ((A + M)\lambda_{0}) < 0. \]
Therefore, (i) holds.

(i) $\Rightarrow$ (ii). We have obtained from (i) the following: $(B + N)\lambda_2 \geq 0$, and

$$(A + M)\lambda_1 + (B + N)\lambda_2 < 0.$$ 

This implies $(A + M)\lambda_1 < 0$. Using Lemma 1 we have $s(A + M) < 0$ and $-(A + M)^{-1} \geq 0$. Combine this with the following inequality

$$(A + M)\lambda_1 + (B + N)\lambda_2 < 0,$$

we get

$$-(C + P)(A + M)^{-1}(B + N)\lambda_2 \leq (C + P)\lambda_1. \quad (7)$$

Using inequality (7) with the condition (ii), we derive the following:

$$(D + Q)\lambda_2 - (C + P)(A + M)^{-1}(B + N)\lambda_2 \leq (D + Q)\lambda_2 + (C + P)\lambda_1 < 0.$$ 

This implies that

$$(D + Q - (C + P)(A + M)^{-1}(B + N))\lambda_2 < 0.$$ 

Moreover, $(D + Q - (C + P)(A + M)^{-1}(B + N))$ is Metzler, using Lemma 1 we get

$$s(D + Q - (C + P)(A + M)^{-1}(B + N)) < 0,$$

(ii) $\Rightarrow$ (i). Quickly obtain $A + M$ as a Metzler matrix. Using Lemma 1 and $s(A + M) < 0$, it follows that $-(A + M)^{-1} \geq 0$, and $\exists v > 0$ such that $(A + M)v < 0$. We have $D + Q - (C + P)(A + M)^{-1}(B + N)$ is Metzler, and $s(D + Q - (C + P)(A + M)^{-1}(B + N)) < 0$ and Lemma 1 implies there exist $\beta_1 > 0$ such that

$$(D + Q - (C + P)(A + M)^{-1}(B + N))\beta_1 < 0.$$ 

This implies

$$(D + Q - (C + P)(A + M)^{-1}(B + N))\beta_1 + \varepsilon(C + P)v < 0,$$

for sufficiently small $\varepsilon$. Setting

$$\beta_2 := -(A + M)^{-1}(B + N)\beta_1 + \varepsilon v > 0.$$ 

Then, we obtain

$$-\beta_2 := -(A + M)^{-1}(B + N)\beta_1 + \varepsilon v = 0.$$ 

Pre - multiplying both sides of equation

$$\beta_2 := -(A + M)^{-1}(B + N)\beta_1 + \varepsilon v,$$

with the matrix $A + M$ we get

$$(B + N)\beta_1 + (A + M)\beta_2 = \varepsilon (A + M)v < 0.$$ 

Therefore, (i) holds.

(i) $\Leftrightarrow$ (iv). Using Lemma 1, we have (i) and (iv) are equivalent.

**Lemma 3.** Assume that $A_1 \geq 0$ and $A_0$ is a Metzler matrix. If $A_0 + A_1$ is Hurwitz matrix, then

i. For all $q_1(t) \geq 0$, $q_2(t) \geq 0$, $t \geq 0$, the following system is positive

$$\begin{cases}
\dot{x}(t) = Ax(t) + By(t) + Mx(t - r(t)) + Ny(t - r(t)) \\
+ q_1(t)
\end{cases}$$

$$\begin{cases}
\dot{y}(t) = -D^{-1}(Cx(t) + P\gamma(t - r(t)) + Q\phi(t - r(t))) \\
+ q_2(t).
\end{cases} \quad (8)$$

ii. For $\phi_1(s) \leq \phi_2(s)$, $i = 1, 2$, $s \in [-n_0, 0]$ we have

$$\dot{x}(t, \phi_1) \leq \dot{x}(t, \phi_2), \forall t \geq 0,$$ 

$$\dot{y}(t, \phi_1) \leq \dot{y}(t, \phi_2), \forall t \geq 0.$$ 

iii. There exists $\eta \in (0, 1)$ such that

$$-D^{-1}(C + P)\beta - D^{-1}Q\mu < (1 - \eta)\mu.$$ 

$$-(D + Q)^{-1}(C + P)\beta < (1 - \eta)\mu.$$ 

**Proof.** (i) It follows from $A_0$ is Metzler matrix and $A_1 \geq 0$ we obtain that $A_1$ and $D$ are Metzler matrices and $B, C, M, N, P, Q$ are nonnegative matrices. It is easy to see that $A_0 + A_1$ is also Metzler matrix. Combining this with Lemma 1 and $A_0 + A_1$ is Hurwitz matrix, implies there exists $\gamma = (\beta, \mu) \in R^r_+$ where

$$\beta \in R^{r}\_+, \mu \in R^{n-r}_+$$

such that

$$(A_0 + A_1)\lambda = \left((A + M)B + N\right)(\beta) \times 0,$$

which is equivalent to

$$(A + M)\beta + (B + N)\mu < 0,$$

$$(C + P)\beta + (D + Q)\mu < 0.$$ 

It follows from $(C + P)\beta \geq 0$, and (14) that

$$(D + Q)\mu < 0.$$ 

(15)
By $Q \succeq 0$ and (15) we get $D\mu < 0$ and from $D$ Metzler matrix, using Lemma 1, it can be deduced that $\det(D) \neq 0$ and $-D^{-1} \succeq 0$. Note that the matrix $A$ is Metzler, $B, M, N \succeq -D^{-1} C$, $-D^{-1} P - D^{-1} Q$ are nonnegative matrices. Then, the proof of (i) is similar to [22]. Therefore, we omit it here.

(ii) Based on the linearity and positivity of the system (8) (with $q_i(t) = 0, i = 1, 2$), we get that $\dot{x}(t, \phi_1, \phi_2) - x(t, \phi_1, \phi_2) = x(t, \phi_1 - \psi_1, \phi_2 - \psi_2) \geq 0$ for $t \geq 0$, which gives the inequality (9). The condition (10) can be proved by similar arguments.

(iii) Taking Lemma 1 and $D + Q$ is Metzler matrix and (15) into account, we obtain $-D^{-1} (D + Q)^{-1} \succeq 0$. It follows from $-D^{-1} \succeq 0$, and (14), we get that

$$-D^{-1} (C + P) \beta + (-D^{-1}) Q \mu < \mu.$$  \hfill (16)

Similarly, it follows from $-D^{-1} (D + Q)^{-1} \succeq 0$ and (14), we have that

$$-D^{-1} (D + Q)^{-1} (C + P) \beta < \mu.$$ \hfill (17)

This inequality and (16) are precisely, so exist $\delta \in (0, 1)$ so that estimates (11) and (12) are satisfied.

\section{MAIN RESULTS}

From now on, we always assume that $A_0$ is a Metzler matrix, $A_\phi \succeq 0, F \succeq 0$.

\subsection{State bounding for positive singular systems with unbounded time-variable delay}

Firstly, we consider the systems as follows

$$\begin{cases} E\dot{X}(t) = A_0 X(t) + A_\phi X(t - r(t)) \geq 0, \\ X(t) = \psi(t), \ t \in [-\tau, 0]. \end{cases} \hfill (18)$$

We get the following result for the system (18).

\textbf{Theorem 1.} Assume that $A_0, A_\phi$ satisfy one of the conditions in Lemma 2. Then system (18) is regular, impulse-free, positive, and $\exists \delta \in (0, 1), A \in \mathbb{R}^n_+$ and a sequence $0 = T_0 < T_1 < T_2 < \cdots < T_n < \cdots + \infty$ such that

$$X(t) \leq (1 - \delta)^n \lambda, \ \forall t \in [T_n, T_{n+1}].$$ \hfill (19)

\textbf{Proof.} By $A_0 + A_\phi$ is Metzler and Hurwitz matrix, it follows from Lemma 1, $\exists \lambda \in \mathbb{R}^n_+$ such that:

$$(A_0 + A_\phi) \lambda < 0.$$ \hfill (20)

Let us denotes: $\kappa = (\beta_0, \mu_0)$, where $\beta_0 \in \mathbb{R}^n_+, \mu_0 \in \mathbb{R}^{m-n}$. From (20), we obtain

$$(A_0 + A_\phi) \kappa = \begin{pmatrix} A + M B + N \\ C + P \end{pmatrix} \begin{pmatrix} \beta_0 \\ \mu_0 \end{pmatrix} < 0,$$ which is equivalent to

$$\begin{cases} (A + M) \beta_0 + (B + N) \mu_0 < 0, \\ (C + P) \beta_0 + (D + Q) \mu_0 < 0. \end{cases} \hfill (21)$$

Using the second inequality of (21), combined with the condition $(C + P) \beta_0 \geq 0$ we derive $(D + Q) \mu_0 < 0$. This together with Lemma 1 with the note that $D + Q$ is a Metzler matrix we have $D + Q$ is a Hurwitz matrix. This together with $D$ is a Metzler matrix, $Q \succeq 0$ we get $D$ is also Hurwitz matrix and $-D^{-1} \succeq 0$ by Lemma 1. This implies that the system (18) is regular and impulse-free. Besides, the system (18) can rewrite in vector form as follows

$$\begin{cases} \dot{x}(t) = Ax(t) + By(t) + Mx(t - r(t)) + Ny(t - r(t)), \\ y(t) = -D^{-1} (Cx(t) + Bx(t - r(t)) + Qy(t - r(t))). \end{cases} \hfill (22)$$

Applying Lemma 3 to the system (22), we derive the system (22) is a positive system, which implies the system (18) is also a positive system.

Let $\delta^\ast = \max \left\{ \frac{\psi_1}{\kappa_1}, \cdots, \frac{\psi_a}{\kappa_a} \right\}$, where $x = (x_1, x_2, \cdots, x_a) \in \mathbb{R}^n_+^a, \overline{\psi} = (\overline{\psi}_1, \overline{\psi}_2, \cdots, \overline{\psi}_a) \in \mathbb{R}^n_+^a$ and choose $\lambda = \delta^\ast x$. Then, we get $\lambda \geq \overline{\psi}$ and $(A_0 + A_\phi) \lambda < 0$. Let $\overline{\phi}(t) = \lambda, \ t \in [-\tau, 0]$. Using Lemma 3 and $\psi(t) \leq \overline{\psi} \leq \lambda, \ t \in [-\tau, 0]$, we obtain that

$$x(t, \psi) \leq x(t, \overline{\phi}), \ t \geq 0.$$ \hfill (23)

Let us denotes: $\lambda = (\beta, \mu)$, where $\beta \in \mathbb{R}^n_+, \mu \in \mathbb{R}^{m-n}$. Note that, from the inequality (21) and $\lambda = \delta^\ast x$ we get

$$(A + M) \beta + (B + N) \mu < 0,$$ \hfill (24)

Now, our proof will be divided into two main parts, as follows:

\textbf{Part 1:} We first show that $\exists \delta \in (0, \eta]$ and $t_1 > 0$ such that

$$x(t, \beta, \mu) \leq (1 - \delta) \beta, \ \forall t \geq t_1,$$ \hfill (25)

$$y(t, \beta, \mu) \leq (1 - \delta) \mu, \ \forall t \geq t_1,$$ \hfill (26)

where $\eta$ is defined as in Lemma 3. Define $e_1(t) := \beta - x(t, \beta, \mu), e_2(t) := \mu - y(t, \beta, \mu).$ Then, we have

$$e_1(t) = A_0 e_1(t) + B_2 e_2(t) + M_1 (r(t) - r(t)) + N_2 (t - r(t))$$ \hfill (27)
\[ e_2(t) = -D^{-1}C_1(t) - D^{-1}P_1(t - r(t)) - D^{-1}Q_2(t - r(t)) + D^{-1}((C + \beta \mu + Q) + \mu). \] (28)

It immediately follows that \( e_1(t) \geq 0, e_2(t) \geq 0 \) for all \( t \geq 0 \), by Lemma 3 and regarding \(-((1 + M)\beta + (B + N)\mu)\) and \( D^{-1}((C + \beta \mu + Q) + \mu)\) as a nonnegative input. This implies that
\[ x(t, \beta, \mu) \leq \beta, \quad \forall t \geq 0, \] (29)
\[ y(t, \beta, \mu) \leq \mu, \quad \forall t \geq 0. \] (30)

Further, considering the second equation of (22), we derive that
\[ y(t) = \begin{cases} -D^{-1}(C\xi(t) + P\xi(t - r(t)) + Qy(t - r(t))), & \text{if } r(t) > 0 \\ -(D + Q)^{-1}(C + P)x(t), & \text{if } r(t) = 0. \end{cases} \] (31)

From (11), (12), (29), (30) and (31) it follows that
\[ y(t, \beta, \mu) < (1 - \eta)\mu, \quad \forall t \geq 0. \] (32)

The condition (24) and \( M\beta + (B + N)\mu \geq 0 \) imply \( A\beta < 0 \). According to Lemma 1, we get \( A \) is invertible and \(-A^{-1} \geq 0\). Moreover, using the first inequality of (24) we obtain that
\[ -A^{-1}M\beta - A^{-1}(B + N)\mu < \beta. \] (33)

We now show that
\[ \lim_{t \to \infty} x(t, \beta, \mu) \leq -A^{-1}M\beta - A^{-1}(B + N)\mu, \] (34)
and \( x(t, \beta, \mu) \) is decreasing. Indeed, combining the conditions (24), (29), (30) with the first inequality of (22), we derive \( x(t, \beta, \mu) < 0, \quad \forall t \geq 0. \) Therefore \( x(t, \beta, \mu) \) is a decreasing on \([0, +\infty)\). We consider the systems as follows:
\[ Z(t) = AZ(t) + M\beta + (B + N)\mu, \quad t \geq 0. \] (35)

By virtue of (29), (30), we have
\[ x(t, \beta, \mu) \leq Z(t, \beta), \quad t \geq 0. \] (36)

We easily point out that \( \dot{w}(t) = Aw(t) \), where \( w(t) = Z(t) + A^{-1}M\beta + A^{-1}(B + N)\mu \). On the other hand, since \( A \) is Metzler and Hurwitz matrix, system \( \dot{w}(t) = Aw(t) \) is positive and asymptotically stable. From (33), we derive that
\[ \beta + A^{-1}M\beta + A^{-1}(B + N)\mu > 0, \]
and so we obtain
\[ w(t, \beta + A^{-1}M\beta + A^{-1}(B + N)\mu) \geq 0, \quad t \geq 0, \]
and
\[ \lim_{t \to \infty} w(t, \beta + A^{-1}M\beta + A^{-1}(B + N)\mu) = 0, \]
which implies that
\[ \lim_{t \to \infty} Z(t, \beta) = -A^{-1}M\beta - A^{-1}(B + N)\mu. \] (37)

Combining (36) and (37) we obtain (34). Note that, inequality (33) is strict, then implies \( \exists \delta \in (0, \eta) \) such that
\[ -A^{-1}M\beta - A^{-1}(B + N)\mu < (1 - \delta)\beta < \beta. \] (38)

It follows from (34) and (38) we get that \( \exists t_1 > 0 \) such that (25) holds. It follows from (32) and \( \delta \eta \), we obtain (26).

**Part 2:** We show that there is a sequence \( 0 = T_0 < T_1 < T_2 < \cdots < T_n < +\infty \) such that
\[ x(t, \beta, \mu) \leq (1 - \delta)^n\beta, \quad \forall t \in [T_n, T_{n+1}], \] (39)
\[ y(t, \beta, \mu) \leq (1 - \delta)^n\mu, \quad \forall t \in [T_n, T_{n+1}]. \] (40)

By Assumption 1, putting \( b_0 = 0, b_1 = T, b_{n+1} = [b_n(1 - \delta)^{-1}], m = 1, 2, \ldots, \) we get that: (i) \( (b_i) \) is a strict increasing sequence, \( b_i \to +\infty \); (ii) \( \forall t > 0 \) we have \( t - r(t) \geq b_i, \quad \forall t \geq b_i+1. \)

For \( n = 0 \), putting \( t_1 = \min\{t \in \mathbb{N} : t_1 \leq b_1\} \) and choose \( T_1 = b_1. \) Then, from (29) and (30) implies the estimation (39) and (40) hold for \( n = 0. \) For \( n = 1, t \geq b_{n+1} \) consider the system as follows:
\[ \begin{cases} x_1(t) = Ax_1(t) + B\phi_1(t) + M\phi_1(t - r(t)) + N\phi_1(t - r(t)), \\ y_1(t) = -D^{-1}(C\phi_1(t) + P\phi_1(t - r(t)) + Q\phi_1(t - r(t))). \end{cases} \] (41)

It is easy to check that, for \( \psi_i(i) \leq \phi_i(i), i = 1, 2, \) then
\[ x_1(t, \psi_1, \phi_2) \leq x_1(t, \phi_1, \phi_2), \quad \forall t \geq b_{1+1}, \] (42)
\[ y_1(t, \psi_1, \phi_2) \leq y_1(t, \phi_1, \phi_2), \quad \forall t \geq b_{1+1}. \] (43)

Let us set
\[ \beta_1 = (1 - \delta)\beta, \quad \mu_1 = (1 - \delta)\mu. \] (44)
Combining (25), (26), (42) and (43) we have
\[ x(t, \beta, \mu) \leq x_1(t, \beta_1, \mu_1), \forall t \geq b_{h+1}, \quad (45) \]
\[ y(t, \beta, \mu) \leq y_1(t, \beta_1, \mu_1), \forall t \geq b_{h+1}. \quad (46) \]

On the other hand, we can verify that (11)-(12) also hold for \( \beta = \beta_1 \) and \( \mu = \mu_1 \). By the same way as in the proof of the Part 1 for (41), we get a time \( t_2 > b_{h+1} \) such that
\[ x_1(t, \beta_1, \mu_1) \leq (1 - \delta) \beta_1, \forall t \geq t_2, \quad (47) \]
\[ y_1(t, \beta_1, \mu_1) \leq (1 - \delta) \mu_1, \forall t \geq t_2. \quad (48) \]

By virtue of (44)-(48), we obtain
\[ x(t, \beta, \mu) \leq (1 - \delta)^2 \beta, \forall t \geq t_2, \quad (49) \]
\[ y(t, \beta, \mu) \leq (1 - \delta)^2 \mu, \forall t \geq t_2. \quad (50) \]

Denote \( t_2 = \min\{t \in \mathbb{N} : t \leq b_1 \} \) and choose \( T_2 = b_2 \). Then, from (49) and (50) imply the estimation (39) and (40) hold for \( n = 1 \). Similarly, we also find \( T_2 < T_2 < ... \) such that (39) and (40) hold for \( n = 2, 3, ... \). Combine this with (23) deduce (19) holds. The proof is completed. \( \square \)

**Remark 1.** Note that the vector \( \lambda \) in (19) computed by \( \lambda = \delta^* \xi = \max_{1 \leq i \leq n} x_i \) is not minimised.

Finding the “smallest” vector \( \lambda_s \).

We present a method to find the “smallest” upper bound vector \( \lambda_s \geq \bar{\psi} \) satisfies condition \( \langle A_0 + A_d \rangle \lambda_s \leq 0 \). Here, the meaning of “smallest” is understood as that the 1-norm distance between two these vectors is smallest. Setting \( t = [1 \ 1 \ \cdots \ 1]^T \), for \( \lambda \geq \bar{\psi} \), then we get \( \|\lambda - \bar{\psi}\|_1 = t^T (\lambda - \bar{\psi}) \).

Denote \( g(\lambda) = t^T (\lambda - \bar{\psi}) \). Then, \( \lambda_s \) is an optimal solution of a linear optimisation problem: \( \min g(\lambda) \) subject to \( \lambda \geq \bar{\psi} \) and \( \langle A_0 + A_d \rangle \lambda \leq 0 \). Based on algorithms/software for solving linear programming problems, for example, “linprog” in the Optimisation toolbox in MATLAB, we can find the optimal solution \( \lambda_s \).

The following corollary of Theorem 1 gives a sufficient condition for the stability of the system (18) with any time-variable delay function satisfying (2).

**Corollary 1.** Assume that \( A_0, A_d \) satisfy one of the conditions in Lemma 2. Then, the system (18) is regular, impulse-free, positive and asymptotically stable.

**Proof.** Prove the same as in Theorem 1; we show that the systems (18) are regular, impulse-free, and positive. Moreover, \( \forall \epsilon > 0 \), choose \( \delta = \epsilon \frac{\lambda}{\|\lambda\|_\infty} \), where \( \lambda_{\text{min}} = \min_{1 \leq i \leq n} \lambda_i, \lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{R}^n_+ \). For any initial function \( \psi(\cdot) \) satisfying \( \|\psi\|_\infty < \delta \), we obtain \( \psi(t) < \frac{\epsilon}{\|\lambda\|_\infty}, t \in [0] \). Since system (18) is linear and using (29),(30), we get
\[ X(t, \psi) \leq \frac{\epsilon}{\|\lambda\|_\infty} X(t, \lambda) \leq \frac{\epsilon}{\|\lambda\|_\infty} \lambda, \forall t \geq 0, \quad (51) \]
this implies that \( \|X(t, \psi)\|_\infty \leq \epsilon \). Moreover, using (39), (40) imply \( \lim_{t \to \infty} x(t, \beta, \mu) = 0 \) and \( \lim_{t \to \infty} y(t, \beta, \mu) = 0 \) which implies that \( \lim_{t \to \infty} X(t, \lambda) = 0 \). Combining this with (51), we get that \( \lim_{t \to \infty} X(t, \psi) = 0 \). The proof is completed. \( \square \)

**Remark 2.** We know that, for a standard positive system of the form \( \dot{x}(t) = A_0 x(t) + A_d x(t - r(t)), \ t \geq 0 \), with the delay \( r(t) \) is bounded or unbounded, the necessary and sufficient condition to ensure the asymptotic stability of the system is \( A_0 + A_d \) is the Hurwitz matrix (see in [30, 31]). For the singular system (18) with unbounded delay considered in this paper, as Corollary 1 shows that, when the \( E \) matrix is of the form \( E = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \), the condition to guarantee that the system is asymptotically stability is also the \( A_0 + A_d \) is the Hurwitz matrix. Note that this condition is independent of the matrix \( E \).

Now, we investigate the state bounding problem for the singular system (1). To do that, we first consider the following system:
\[ \begin{cases} \dot{E}z(t) = A_0 z(t) + A_d z(t - r(t)) + \bar{\sigma}, \ t \geq 0, \\ Z(t) = \phi(s), s \in [-\tau_0, 0]. \end{cases} \quad (52) \]
The following lemma provides a relationship between the state trajectory of the system (1) and the state trajectory of the system (52).

**Lemma 4.** Assume that \( D \) is Hurwitz matrix. The following statements hold:

i. If \( \psi(t) \leq \phi(t), \forall s \in [-\tau_0, 0] \), then
\[ X(t, \psi, \bar{\sigma}) \leq Z(t, \phi, \bar{\sigma}), \forall t \geq 0. \]

ii. If \( \phi'(t) \leq \phi(t), \forall t \in [-\tau_0, 0] \), then
\[ Z(t, \phi, \bar{\sigma}) \leq Z(t, \phi, \bar{\sigma}), \forall t \geq 0. \]

**Proof.** (i) We consider the system
\[ \dot{E}w(t) = A_0 w(t) + A_d w(t - r(t)) + p(t), \ t \geq 0, \]
\[ w(s) = \phi(s) - \psi(s), s \in [-\tau_0, 0], \quad (53) \]
where \( w(t) := Z(t) - X(t), p(t) := \bar{\sigma} - F \sigma(t) \). In virtue of Lemma 1, and \( D \) is Hurwitz, we obtain \( \det(D) \neq 0 \) and \( -D^{-1} \geq \]
0. So system (53) is rewritten as follows:

\[
\begin{align*}
\dot{w}_1(t) &= Aw_1(t) + Bw_2(t) + Mw_1(t - r(t)) + N\bar{y}(t) + p_1(t) \\
\dot{w}_2(t) &= -D^{-1}Cw_1(t) - D^{-1}Bw_2(t - r(t)) - D^{-1}M\bar{y}(t)
\end{align*}
\]

where \( w(t) = (w_1(t), w_2(t)) \), \( p(t) = (p_1(t), p_2(t)) \). Apply Lemma 3; we can immediately deduce that \( w(t, \Phi_1 - \Psi, p(t)) \geq 0, \forall t \geq 0 \), it follows that \( Z(t, \Phi_1, \Theta) - X(t, \Psi, \Theta) \geq 0, \forall t \geq 0 \).

(ii) By the same method as in the proof of part (i) and Lemma 3, we get that (ii).

The following theorem provides a condition sufficient to ensure that system (1) is regular, impulse-free, and the existence of an ultimate component-wise bound for the system.

Let us denote: \( \xi = -(A_0 + A_d)^{-1} \xi \).

**Theorem 2.** Assume that \( A_0, A_d \) satisfy one of the conditions in Lemma 2. Then, the system (1) is regular, impulse-free, positive and bounded such that

\[ X(t, \Psi, \Theta) \leq \xi + (1 - \delta) \eta, \forall t \in [T_n, T_{n+1}] \].

(i) There exist \( \delta \in (0, 1) \), \( \eta \in \mathbb{R}_+^n \), and a sequence \( 0 = T_0 < T_1 < T_2 < \cdots < T_n < \cdots < +\infty \) and \( \xi \) such that

\[ \limsup_{t \to \infty} X(t, \Psi, \Theta) \leq \xi \].

**Proof.** In case (i), similar to (i) of Lemma 3, implies that \( \exists \lambda \in \mathbb{R}_+^n \) such that

\[ (A_0 + A_d) \lambda \leq 0. \]  

Similar to Theorem 1, from the condition (56) implies that \( D \) is Hurwitz matrix and \( -D^{-1} \geq 0 \). By \( \det(D) \neq 0 \), we can conclude that system (1) is regular, impulse-free. Moreover, the system (1) can be rewritten as follows

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + By(t) + Mx(t - r(t)) + N\bar{y}(t) + p(t) \\
\dot{y}(t) &= -D^{-1}(Cx(t) + By(t) - r(t)) + D^{-1}M\bar{y}(t)
\end{align*}
\]

It follows from this and Lemma 3 that the system (1) is positive. Let \( \delta_1 = \max\{\frac{1}{\lambda_1}, \cdots, \frac{1}{\lambda_n}\} \), where \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}_+^n \), \( \xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}_+^n \) and choose \( \xi = \delta_1 \lambda \). Then, we get \( \xi \geq \psi \) and

\[ (A_0 + A_d) \xi \leq 0. \]

Let \( \Phi_1 = \xi, \ s \in [-n, 0] \). Using Lemma 3 and \( \psi(t) \leq \xi \), \( s \in [-n, 0] \), we obtain that

\[ X(t, \Psi, \Theta) \leq Z(t, \Phi_1, \Theta) \], \( t \geq 0 \).

By \( \xi > 0 \), it is easy to choose \( \rho > 1 : \rho \xi \geq \xi \). Set \( \eta := \rho \xi \) this together with (57) we have that

\[ (A_0 + A_d) \eta < 0. \]

Putting \( \Phi_1 = \eta, \ s \in [-n, 0] \). Since \( \rho > 1 \) we have \( \xi < \eta \), it follows from Lemma 3, we obtain that

\[ Z(t, \Phi_1, \Theta) \leq Z(t, \Psi, \Theta) \], \( t \geq 0 \).

We can easily get \( \xi \geq 0 \). Let \( \Phi_1 = \xi, \ s \in [-n, 0] \), it follows from \( \xi \geq \eta \) then, we obtain that

\[ \Phi_1 \preceq \eta \], \( s \in [-n, 0] \).

Setting \( \Phi_1 = \xi, \ s \in [-n, 0] \), using (61) we get \( \Phi_1 \preceq \eta \). Under coordinate transformation

\[ Z(t) = \psi(t) + \xi \]

then, from system (52), we get

\[ E \psi(t) = A \psi(t) + A_d \psi(t - r(t)) \]

and

\[ Z(t, \Phi_1, \Theta) = \xi + \psi(t, \Phi_1, \Theta) \]

where \( \psi(t, \Phi_1, \Theta) \) is the solution of system (63) with the initial function \( \Phi_1 \preceq \eta \). It follows from \( \eta - \xi \leq \xi \), implies that \( \Phi_1 \preceq \eta \), \( s \in [-n, 0] \), then using Lemma 3, we get

\[ \Phi_1 \preceq \eta \]

It follows from Theorem 1 for system (63) and (59) imply \( \exists \delta \in (0, 1) \) and a sequence \( 0 = T_0 < T_1 < T_2 < \cdots < T_n < \cdots < +\infty \) such that

\[ \psi(t, \Phi_1) \preceq (1 - \delta) \eta, \forall t \in [T_n, T_{n+1}] \]

Combining (58), (60), (64), (65) and (66), we get that (54).
(ii) For $t \to \infty$, it follows from (54), we get (55). Then, $\xi$ is a component-wise ultimate bound of system (1). We now show that $\lim_{t \to \infty} Z(t, \phi_0, \overline{\theta}) = \xi$, where $\phi_0(i) = 0, i \in [-n_0, 0]$. Using coordinate transformation

$$v(t) = \xi - \overline{Z}(t). \quad (67)$$

This together with (52), imply that

$$Ei(t) = A_0 v(t) + A_1 v(t - r(t)), \quad t \geq 0, \quad (68)$$

and

$$v(t, \phi_0(\xi)) = \xi - \overline{Z}(t, \phi_0, \overline{\theta}), \quad \forall t \geq 0, \quad (69)$$

where $\phi_0(\xi) = \xi, i \in [-n_0, 0]$. It follows from Lemma 3 that $v(t, \phi_0(\xi)) \geq 0, t \geq 0$. Applying Theorem 2 of [10], the author considered the problem of state bounding for positive coupled differential-difference equations with bounded disturbances of the form

$$\dot{x}(t) = Ax(t) + By(t - \tau(t)) + \theta_i(t), \quad y(t) = Cx(t) + D(y(t - \tau(t)) + \theta_2(t)), \quad (71)$$

the disturbance vector $\theta_i(t), i = 1, 2$ is an unknown continuous vector-valued function satisfying $0 \leq \theta_i(t) \leq \overline{\theta}, \forall t \geq 0, i = 1, 2$. It has been showed in Theorem 3 of [10] that the vector $\xi = -\left[\begin{array}{cc}A & B \\ C & D - I\end{array}\right]^{-1}\overline{\theta}$ is the smallest component-wise ultimate bound of system (71) if the following conditions hold

i) $A$ is a Metzler matrix, $B, C, D$ are nonnegative, $D$ is a Schur matrix,

ii) $\xi(A + B(I - D)^{-1}C) < 0$.

It is not hard to see that this result is equivalent to Theorem 2 of this paper. However, to obtain this result, the authors of the work [10] have to assume that the system satisfies regular condition and impulse-free condition. In contrast, Theorem 2 of this paper can be applied to a general time-delay system which does not need to be satisfied regular condition and impulse-free condition.

### 3.2 Linear functional state bounding

Let us denote:

$$R = [R_1 \ldots R_d] \in \mathbb{R}^{\alpha \times d}, \quad (72)$$

$$R_j = [f_{1j} \ldots f_{nj}]^T \in \mathbb{R}^{\alpha \times 1}, j = 1, \ldots, d, \quad (73)$$

$$\overline{\theta} := \max_{\theta \in \Sigma} F \theta; \quad \overline{\theta} = \max_{\theta \in \Sigma} R^T F \theta, \quad \overline{V} = \left[\begin{array}{c} \overline{\theta}^T \\ \overline{\theta}^T \end{array}\right]^T, \quad (74)$$

$$U_+ = \text{diag}(-\theta_i^+, \ldots, -\theta_j^+) \in \mathbb{R}^{d \times d}, \quad (75)$$

where $A_{ij}$ the jth column vector of matrix $A_0, j = 1, \ldots, d$.

**Lemma 5.** Let $A_0 \in \mathbb{R}^{\alpha \times \alpha}$ be a Metzler and Hurwitz matrix. For any $q = [q_1 \ldots q_\alpha]^T \in \mathbb{R}^{\alpha \times 1}$ then we have

i. The set $\Sigma = \{\theta \in \mathbb{R} : A_0^T q + \theta q \geq 0\} \neq \emptyset$.

ii. $\min \Psi = -\min_{1 \leq i \leq \alpha, q_i > 0} \frac{\psi_{\Psi, i}}{q_i} > 0$, where $\Psi_{i}$ the jth column vector of matrix $A_j$.

iii. For each $\theta_j \in \mathbb{R}_+$ satisfies $A_0^T R_j + \theta_j R_j \geq 0, j = 1, \ldots, d$, we obtain

$$R^T A_0 = W + U R^T, \quad (76)$$

where $W = (w_{ij}) \in \mathbb{R}^{d \times \alpha}, \quad w_{ij} = A_{ij}^T R_j + \theta_j f_{ij}, \quad i = 1, \ldots, d, \quad j = 1, \ldots, n$ and $U = \text{diag}(-\theta_i^+, \ldots, -\theta_j^+) \in \mathbb{R}^{d \times d}$.

**Proof.** i) We denote $A_0 = (a_{ij}) \in \mathbb{R}^{n \times n}, 1 \leq i, j \leq n$, there always exist $\theta^* = \max_{1 \leq i \leq n} |a_{ij}|$ such that $a_{ij} + \theta^* 1 \geq 0$, for all $1 \leq i \leq n$. Moreover, since $A_0$ is a Metzler matrix, $a_{ij} \geq 0$ for all $i \neq j$. So there always exists $\overline{\theta}^* \in \mathbb{R}$ satisfying

$$A_0^T q + \theta^* q = (A_0^T + \theta^* I_n) q \geq 0,$$

for all $q \geq 0$.

ii) We set $A_0^T q = (a_1 a_2 \ldots q_n)^T, \quad I = \{1, 2, \ldots, n\}, \quad I^- = \{i : a_i < 0, i \in I\}, \quad I^* = \{i : a_i \geq 0, i \in I\}$. Since matrix $A_0$ is Metzler and Hurwitz. According to part (iv) of Lemma 1, we have the set $I^\perp$ is nonempty. It is easy to see that if $\theta \in \mathbb{R}$ satisfies $A_0^T q + \theta q \geq 0$ then $\theta > 0$. Indeed, the opposite assumes that $\theta \leq 0$, since the set $I^\perp$ is nonempty, there exists an index $j \in I^\perp$ such that $a_j < 0$, so $a_j + \theta q_j < 0$. This contradicts the condition $A_0^T q + \theta q \geq 0$. Note that, for all $i \in I^\perp$, then $a_i + \theta q_i \geq 0$.
for all $\theta \in \mathbb{R}_+$. Therefore, we get

$$
\min \Xi = \min \{ \theta \in \mathbb{R} : a_j + \theta q_j \geq 0, j \in I^+ \cup I^- \}
$$

$$
= \min \{ \theta \in \mathbb{R}_+ : a_j + \theta q_j \geq 0, j \in I^+ \cup I^- \}
$$

$$
= \min \{ \theta \in \mathbb{R}_+ : a_j + \theta q_j \geq 0, j \in I^- \}.
$$

(72)

We set $A_{ij}^T = (c_{ij} c_{i2} \cdots c_{ij} \cdots c_{i\mu})$ as the $j$th row vector of matrix $A_{ij}$. Then we get

$$
a_j = A_{ij}^T q = \sum_{k=1}^n c_{jk} q_k.
$$

Because $A_{ij}^T$ is a Metzler matrix, we have $c_{ij} < 0$, and $c_{jk} \geq 0, k \neq j$. From this, we infer that for all $j \in I^+, a_j < 0$ deduces that $q_j > 0$. Combining this with (72) we get the following result:

$$
\min \Xi = \min \{ \theta \in \mathbb{R}_+ : a_j + \theta q_j \geq 0, j \in I^- \}
$$

$$
= \min \{ \theta \in \mathbb{R}_+ : \theta \geq -\frac{a_j}{q_j}, j \in I^+, q_j > 0 \}
$$

$$
= \max_{j \in I^+, q_j > 0} \left(-\frac{a_j}{q_j}\right) > 0.
$$

(73)

Note that $\max_{j \in I^+, q_j > 0} \left(-\frac{a_j}{q_j}\right) \leq 0$. Combining this with (73) we get

$$
\min \Xi = \max_{j \in I^+, q_j > 0} \left(-\frac{a_j}{q_j}\right) = -\min_{1 \leq i \leq n, q_j > 0} \left(\frac{a_j}{q_j}\right)
$$

$$
= -\min_{1 \leq i \leq n, q_j > 0} \left\{ A_{ij}^T q \right\} > 0.
$$

iii) Based on Remark 7 in [25], we obtain iii).

**Theorem 3.** Assume that $A_0, A_d$ satisfy one of the conditions in Lemma 2. Then

$$
\bar{L}(t) = -U^{-1} \bar{g}(t) - R^T (A_0 + A_d)^{-1} \bar{g}.
$$

is the smallest linear functional bound of (4).

**Proof.** Using (4) and Lemma 5 implies that

$$
\bar{L}(t) = R^T \bar{x}(t) = R^T A_0 x(t) + R^T A_d x(t - r(t)) + R^T F \vartheta(t)
$$

$$
= (W + UR^T)x(t) + R^T A_d x(t - r(t)) + R^T F \vartheta(t)
$$

$$
= W x(t) + UL(t) + R^T A_d x(t - r(t)) + R^T F \vartheta(t),
$$

(75)

Combining this with (1) we obtain

$$
\bar{E} \hat{x}(t) = A_0 \hat{x}(t) + A_d \hat{x}(t - r(t)) + F \vartheta(t), t \geq 0,
$$

(76)

where

$$
\bar{E} = \text{diag}(I_d, L, 0_{n-1}), \bar{A}_0 = \begin{bmatrix} U & W \end{bmatrix}.
$$

$$
\bar{A}_d = \begin{bmatrix} 0 & R^T A_d \\ 0 & A_d \end{bmatrix}, \quad \bar{F} = \begin{bmatrix} R^T F \\ F \end{bmatrix}, \quad \hat{x}(t) = \begin{bmatrix} L(t) \\ \bar{x}(t) \end{bmatrix}.
$$

We have

$$
\bar{A}_0 + \bar{A}_d = \begin{bmatrix} U & W + R^T A_d \\ 0 & A_0 + A_d \end{bmatrix}, \quad U = \text{diag}(-\vartheta_1, \ldots, -\vartheta_d) \in \mathbb{R}^{d \times d}.
$$

We can immediately deduce that $\bar{A}_0 + \bar{A}_d$ is Hurwitz and Metzler matrix. Applying Theorem 2 to the system (76), we get the following estimation of $\bar{x}(t)$:

$$
\bar{x}(t) \leq -\left(\bar{A}_0 + \bar{A}_d\right)^{-1} \bar{F}, \forall t \geq 0.
$$

(77)

Setting

$$
\bar{L}(t) = -\left( I_d \ 0_{d \times n} \right) \left( \bar{A}_0 + \bar{A}_d \right)^{-1} \bar{F}.
$$

(78)

So we get that:

$$
\bar{L}(t) \leq \bar{L}, \forall t \geq 0.
$$

(79)

By some simple calculation, we get

$$
\left( \bar{A}_0 + \bar{A}_d \right)^{-1} = \begin{bmatrix} U^{-1} - U^{-1}(W + R^T A_d)(A_0 + A_d)^{-1} \\ 0 \end{bmatrix}.
$$

(80)

Combining (78), (80) and $W = R^T A_0 - UR^T$, and $\bar{F} = \begin{bmatrix} \bar{F} \\ \bar{F} \end{bmatrix}$ we obtain

$$
\bar{L} = -U^{-1}(\bar{F} - R^T \bar{g} - R^T (A_0 + A_d)^{-1} \bar{g}.
$$

(81)

It is note that $\max(f + \vartheta) \leq \max(f) + \max(\vartheta)$, then we have $\bar{F} \leq R^T \bar{g}$. Combining this with (81) implies that $\bar{L}$ is smallest if and only if

$$
-\vartheta_i = \min_{1 \leq i \leq n, q_j > 0} \left\{ A_{ij}^T q \right\}, i = 1, \ldots, d.
$$

(82)

**Remark 4.** Recently, the linear functional state bounding problem for positive system

$$
\dot{x}(t) = A_0 x(t) + A_d x(t - r(t)) + F \vartheta(t), t \geq 0,
$$

(83)
was studied in [25]. It is obvious that the system (1) is more general than the system (83), when the matrix $E = I$, applying Theorem 3, we obtain the smallest linear functional bound of (83) is

$$\mathcal{T} = -U^{-1}_s\left(\overline{\vartheta}_R - R^T\overline{\vartheta}\right) - R^T(A_0 + A_d)^{-1}\overline{\vartheta}. $$

Then, Theorem 9 in [25] is a particular case of Theorem 3. We know that studying the singular system is much more complicated than the standard system because one needs to consider not only stability but also regularity and causality (discrete-time systems) or non-impulsiveness (continuous-time systems). In this paper, we provide sufficient conditions to ensure that the singular system is regular, causal, positive and finding a linear functional bound for a class of positive singular systems with unbounded delay.

4 | NUMERICAL EXAMPLE

Example 1. Consider singular system (1) with

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_0 = \begin{bmatrix} -6 & 1 & 0 \\ 2 & -6 & 0 \\ 3 & 2 & -6 \end{bmatrix}, \quad A_d = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix},$$

$$\vartheta(t) \in \mathcal{S} = \left\{ \begin{pmatrix} 2 & 2 & 2 \\ 1.8 & 2.2 & 2.2 \\ 2.6 & 1.9 & 2.4 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \leq \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \right\}.$$  

$F = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}, R = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. The time-varying unbounded delay function is chosen as $r(t) = \begin{cases} 1, & t \in [0,1] \\ 0.3t, & t > 1. \end{cases}$. By some simple calculation in Matlab, we show that the set of eigenvalues of matrix $A_0 + A_d$ is: $\sigma(A_0 + A_d) = \{-6, -2, -8\}$, which implies that $A_0 + A_d$ is a Hurwitz matrix. Moreover, by simple calculation, we get that $U_s = -\frac{13}{2}$. On the other hand, by using linear programming, we calculate that:

$$\overline{\vartheta} := \max_{\vartheta \in \mathcal{S}} F\vartheta = \begin{bmatrix} 22 \\ 25 \\ 20 \\ 13 \\ 20 \end{bmatrix}^T,$$

and

$$\overline{\vartheta}_R = \max_{\vartheta \in \mathcal{S}} R^T F\vartheta = 116.$$  

So by using the formula (74) in Theorem 3, we show that the smallest linear functional bound of the system:

$$\mathcal{T} = -U_s^{-1}\left(\overline{\vartheta}_R - R^T\overline{\vartheta}\right) - R^T(A_0 + A_d)^{-1}\overline{\vartheta} = 895.$$  

Note that $A_0$ and $A_d$ satisfy the conditions in Corollary 1, therefore, system (1) with $\vartheta(t) = 0, t \geq 0$, is positive and asymptotically stable. For a visual simulation, we choose initial values $\psi(\varphi) = (\frac{a_1}{8}, \frac{a_2}{8}, \frac{a_3}{8})$, $\varphi \in [-1, 0]$, $a_1 \in \{0, 2, 4, 6, 8\}$, $a_2 \in \{0, 2, 4, 6, 8\}$, $a_3 \in \{1, 2, 3\}$. Figure 1 show the trajectories of the system (1) with $\vartheta(t) = 0, t \geq 0$. Figure 2 show the
FIGURE 2 Trajectories of $L(t)$ and its bound

trajectories of $L(t) = R^T X(t)$ and its bound with

$\Theta(t) = \left\{ (0.3k \cdot \sin^2(2t), 0.3k \cdot |\sin(8t)|, 0.2k \cdot |\sin(4t)|) \right\}, t \geq 0,$

$k \in \{0, 1/4, 1/2, 3/4, 1\}.$

5 CONCLUSIONS

We have considered the problem of linear functional state bounding for positive singular systems with unbounded delay and disturbances varying within a bounded set. (i) we have given a condition for the existence of component-wise bounds and a condition for the asymptotic stability of the singular systems without disturbances, (ii) we have provided a sufficient condition for the existence of component-wise ultimate bounds of the singular systems with bounded disturbances, (iii) we proposed some sufficient conditions given in terms of the linear programming/Hurwitz matrix/spectral abscissa for linear functional state bounding problems of the singular system with unbounded delay. A numerical example is given to illustrate the effectiveness of the proposed results. Besides, extending the results of this paper to solve the controller design problem for the singular positive systems would be an interesting problem for future research. Furthermore, future works can apply the techniques used in this paper to study positive observer synthesis for positive singular systems with unbounded delay and interval parameter uncertainties.

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