NON-SYMPLECTIC AUTOMORPHISMS OF K3 SURFACES WITH ONE-DIMENSIONAL MODULI SPACE

MICHELA ARTEBANI, PAOLA COMPARIN, AND MARÍA ELISA VALDÉS

Abstract. The moduli space of K3 surfaces $X$ with a purely nonsymplectic automorphism $\sigma$ of order $n \geq 2$ is one dimensional exactly when $\varphi(n) = 8$ or 10. In this paper we classify and give explicit equations for the very general members $(X, \sigma)$ of the irreducible components of maximal dimension of such moduli spaces. In particular we show that there is a unique one-dimensional component for $n = 20, 22, 24$, three irreducible components for $n = 15$ and two components in the remaining cases.

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1. Introduction

An automorphism $\sigma$ of finite order $n \geq 2$ of a complex K3 surface $X$ is purely non-symplectic if $\sigma^*(\omega_X) = \zeta_n \omega_X$, where $\omega_X$ is a nowhere vanishing holomorphic 2-form of $X$ and $\zeta_n$ is a primitive $n$th root of unity. By [MO98, Main Theorem 3] there exists one such pair $(X, \sigma)$ if and only if $n$ belongs to the set $TV_{K3} = \{n \in \mathbb{N} - \{60\} | \varphi(n) \leq 20\}$.

The structure of the moduli space of such K3 surfaces can be described by means of the Global Torelli theorem and the surjectivity theorem for periods of K3 surfaces (see [DK07, §11]). In particular it is known that an irreducible component of the moduli space of pairs $(X, \sigma)$ for $n \geq 3$ is an arithmetic quotient of a Zariski open subset of a complex ball of dimension $\dim(V^\sigma) - 1$, where $V^\sigma$ is the $\zeta_n$-eigenspace of $\sigma^*$ in $H^2(X, \mathbb{C})$.

In this paper we consider the orders $n$ such that the moduli space of K3 surfaces carrying a purely non-symplectic automorphism of order $n$ is one dimensional. We show that the orders $n$ with such property, as expected, are exactly those $n \in TV_{K3}$ with $\varphi(n) = 8$ or 10, i.e. $11, 15, 16, 20, 22, 24$ and $30$ (see [MO98]). For all these values of $n$ we classify pairs $(X, \sigma)$ such that $\dim(V^\sigma) = 2$, i.e. we identify the fixed locus of $\sigma$ and of its powers, determine the dimensions of the eigenspaces of $\sigma^*$ in $H^2(X, \mathbb{C})$ and compute the Néron-Severi lattice of a very general pair. The orders $n = 11$ and $n = 16$ had been previously studied in [OZ11, AST11] and [ATST16] respectively. We collect these results in the following Theorem.

**Theorem 1.1.** Let $X$ be a complex K3 surface with a purely non-symplectic automorphism $\sigma$ of order $n \geq 2$ such that $\varphi(n) = 8$ or $10$ and $\dim(V^\sigma) = 2$. Then Table 1 provides all possible values for the vector $d$ describing the dimensions of the eigenspaces of $\sigma^*$ in $H^2(X, \mathbb{C})$, the topological invariants describing the fixed locus of powers of $\sigma$ (see Section 2 for the notation) and the Néron-Severi lattice of a very general K3 surface in each case. Moreover, all cases in the table exist.

This classification allows us to prove the following result, which provides explicit birational models for a very general pair $(X, \sigma)$ under the previous conditions (see Remark 2.4 about the generality assumption in the statement).

**Theorem 1.2.** Let $X$ be a very general complex K3 surface with a purely non-symplectic automorphism $\sigma$ of order $n \geq 2$ such that $\varphi(n) = 8$ or $10$ and $\dim(V^\sigma) = 2$, then up to a birational isomorphism $(X, \sigma)$ belongs to the families described in Table 2, where $a \in \mathbb{C}$ is a parameter, $\zeta_n$ denotes a primitive $n$th root of unity and $(\ast)$ means: minimal resolution of a degree 11 covering of a principal homogeneous space of order 11 of the rational elliptic surface $y^2 = x^3 + x + t$ (see Example 3.3).
Corollary 1.3. The moduli space of $K3$ surfaces carrying a purely non-symplectic automorphism of order $n$ has a unique one-dimensional component for $n = 20, 22, 24$, three irreducible components for $n = 15$ and two irreducible components for $n = 11, 16, 30$.

For orders 22, 15, 30 and 20 we actually prove a stronger version of Theorem 1.2, since we provide projective models without assuming $X$ to be very general.
Finally, in case $n = 22$ and $n = 15$ we classify purely non-symplectic automorphisms of order $n$, that is we provide the same type of information contained in Table 1, without assuming $\dim(V^\sigma) = 2$, see Theorem 4.2 and Theorem 5.1.

The structure of the paper is the following. In the first section we give preliminaries on non-symplectic automorphisms of K3 surfaces and we fix the corresponding notation: fixed loci, invariant lattices and eigenspaces in cohomology, moduli spaces. In the second section, for each order $n \in \{11, 22, 15, 30, 16, 20, 24\}$, we prove Theorem 1.1 (see Theorem 3.1 and Propositions 3.4, 3.6, 3.11, 3.14, 3.19, 3.24) and Theorem 1.2. In the third and fourth section we prove Theorem 4.2 and Theorem 5.1 respectively.

2. Background and preliminary results

We will work over the complex numbers and we will denote by $\zeta_i$ a primitive $i$th root of unity. Let $X$ be a K3 surface over $\mathbb{C}$ and let $\sigma$ be a purely non-symplectic automorphism of $X$ of order $n \geq 3$, i.e. $\sigma^*(\omega_X) = \zeta_n \omega_X$, where $\omega_X$ is a generator of the complex vector space $H^{2,0}(X)$. 

| $n$ | $X$ | $\sigma$ |
|-----|-----|---------|
| 11  | $y^2 = x^3 + ax + (t^{11} - 1)$ | $(x, y, \zeta_{11}t)$ |
|     | (* ) |         |
| 15  | $a) \ y^2 = x^3 + (t^5 - 1)(t^5 - a)$ | $(\zeta_3 x, y, \zeta_t)$ |
|     | $b) \ y^2 = x_0^6 + x_0 x_1^5 + x_2^6 + a x_0^3 x_2^3$ | $(x_0, \zeta_5 x_1, \zeta_4 x_2, y)$ |
|     | $c) \ y^3 = x_0^3 x_1 + x_1^2 x_2^2 + x_1^4 x_2 + a x_1^6$ | $(\zeta_5 x_0, x_1, x_2, \zeta_3 y)$ |
| 16  | $a) \ y^2 = x^3 + t^2 x + a t^3(t^6 + 1)$ | $(\zeta_{16}^2 x, \zeta_{16}^3 y, \zeta_{16}^2 t)$ |
|     | $b) \ y^2 = x_0(x_0^4 x_2 + x_1^5 + x_1 x_2^4 + a x_1^3 x_2^3)$ | $(x_0, \zeta_6^7 x_1, \zeta_6^4 x_2, \zeta_{16}^3 y)$ |
| 20  | $y^2 = x_0(x_1^5 + x_2^5 + x_0^2 x_2^3 + a x_0^4 x_2)$ | $(-x_0, \zeta_5 x_1, x_2, iy)$ |
| 22  | $y^2 = x^3 + ax + (t^{11} - 1)$ | $(x, -y, \zeta_{11} t)$ |
| 24  | $y^2 = x^3 + t(t^4 - 1)(t^4 - a)$ | $(\zeta_{12} x, \zeta_8 y, it)$ |
| 30  | $a) \ y^2 = x^3 + (t^5 - 1)(t^5 - a)$ | $(\zeta_3 x, -y, \zeta_5 t)$ |
|     | $b) \ y^2 = x_0^6 + x_0 x_1^5 + x_2^6 + a x_0^3 x_2^3$ | $(x_0, \zeta_5 x_1, \zeta_4 x_2, -y)$ |

Table 2. One dimensional families of K3 surfaces with non-symplectic automorphisms
Notation 2.1. In what follows we will denote by $\sigma_k$ an element of $\langle \sigma \rangle$ whose order is $k$.

2.1. Fixed locus. We start describing the fixed locus of $\sigma$. The local action of $\sigma$ in a neighborhood of one of its fixed points can be linearized and can be described by a matrix of the form

$$A_{i,n} = \begin{pmatrix} \zeta_i^{i+1} & 0 \\ 0 & \zeta_n^{n-i} \end{pmatrix}, \quad i = 0, 1, 2, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor,$$

see [Nik79a, §5]. When $i = 0$ the fixed point belongs to a fixed curve, otherwise it is an isolated fixed point. This description implies that the fixed locus of $\sigma$ is the union of isolated points and disjoint smooth curves. Moreover, by the Hodge index theorem, the fixed locus contains at most one curve of genus $g \geq 2$. In what follows we will use the following notation for the fixed locus of $\sigma$:

$$\text{Fix}(\sigma) = C_g \sqcup R_1 \sqcup \ldots \sqcup R_k \sqcup \{p_1, \ldots, p_N\},$$

where $C_g$ is a smooth curve of genus $g$, $R_1, \ldots, R_k$ are smooth rational curves and $p_1, \ldots, p_N$ are isolated fixed points. The fixed points such that the local action is given by the matrix $A_{i,n}$ will be called points of type $A_{i,n}$ and the number of such points will be denoted by $a_{i,n}$.

We now recall the holomorphic Lefschetz formula [AS68], which relates these numbers with the action of $\sigma^*$ on the cohomology groups $H^j(X, \mathcal{O}_X)$:

$$\sum_{j=0}^{2} \text{tr} \left( \sigma^* | H^j(X, \mathcal{O}_X) \right) = \sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{a_{i,n}}{(1 - \zeta_n^{i+1})(1 - \zeta_n^{n-i})} + \alpha \frac{1 + \zeta_n}{(1 - \zeta_n)^2},$$

where $\alpha := \sum_{C \subset \text{Fix}(\sigma)} (1 - g(C))$. Observe that

$$H^1(X, \mathcal{O}_X) = H^3(X, \mathcal{O}_X) = \{0\}$$

since $X$ is a K3 surface, $\sigma^* = \text{id}$ on $H^0(X, \mathcal{O}_X) \cong \mathbb{C}$ and $\sigma^*$ acts as multiplication by $\zeta_n$ on $H^2(X, \mathcal{O}_X) \cong H^{0,2}(X) = \mathbb{C}\omega_X$. Thus the left hand side of the formula is equal to $1 + \zeta_n$.

Finally, we recall Hurwitz formula for a ramified covering $f : X \to Y$ of degree $d$ between smooth complex projective varieties, which will be used several times in the paper for both curves and surfaces:

$$K_X \sim f^* K_Y + \sum_i (e_i - 1) C_i,$$

where the $C_i$’s are the irreducible components of the ramification locus and $e_i$ is the associated ramification index (see for example [BHPV04, §16, §17]).

2.2. Eigenspaces and invariant lattices. We now consider the action of $\sigma^*$ in $H^2(X, \mathbb{Z})$ and $H^2(X, \mathbb{C})$. We will denote by $S(\sigma^i) \subset H^2(X, \mathbb{Z})$ the invariant lattice of $\sigma^i$ for $i = 0, \ldots, n - 1$. Moreover, for any divisor $k$ of $n$ let

$$H^2(X, \mathbb{C})^\sigma_k := \{ x \in H^2(X, \mathbb{C}) : \sigma^i x = \zeta_k x \}$$
and let \( d_k \) be its dimension. In particular \( d_1 \) is the rank of \( S(\sigma) \) and \( d_n \) is the dimension of \( V^\sigma = H^2(X, \mathbb{C})^\sigma \). In what follows we will denote by \( d \) the vector whose entries are the numbers \( d_k \), as \( k \) varies in the set of divisors of \( n \) in decreasing order:

\[
d = (d_n, \ldots, d_k, \ldots, d_1), \quad k|n.
\]

**Remark 2.2.** Observe that, since \( \sigma \) is purely non symplectic, then \( S(\sigma^i) \) is contained in the Néron-Severi lattice of \( X \) for any \( i = 0, \ldots, n-1 \). In fact, given \( x \in S(\sigma^i) \) we have

\[
(x, \omega_X) = ((\sigma^i)^* x, (\sigma^i)^* \omega_X) = (x, \zeta_n^i \omega_X) = \zeta_n^i (x, \omega_X),
\]

which implies \( (x, \omega_X) = 0 \) and thus \( x \in H^2(X, \mathbb{Z}) \cap \omega_X^1 = \text{NS}(X) \).

We also recall the *topological Lefschetz formula* [AS68, Theorem 4.6], for simplicity we state it only for \( \sigma \):

\[
\chi(\text{Fix}(\sigma)) = \sum_{i=0}^4 (-1)^i \text{tr}\left( \sigma^* \big|_{H^i(X, \mathbb{R})} \right),
\]

where the right side is equal to \( 2 + \text{tr}(\sigma^* \big|_{H^2(X, \mathbb{R})}) \) since \( H^i(X, \mathbb{R}) = \{0\} \) for \( i = 1, 3 \) and \( \sigma^* = \text{id} \) on \( H^i(X, \mathbb{R}) \) for \( i = 0, 4 \).

Finally, we recall some notation for lattices which will appear in the paper: \( A_\ell \ (\ell \geq 1) \), \( D_m \ (m \geq 4) \) and \( E_n \ (n = 6, 7, 8) \) denote the negative definite even lattices associated to the Dynkin diagrams of the corresponding types, \( U \) and \( H_5 \) denote the lattices with the following Gram matrices

\[
U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad H_5 = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix},
\]

and \( U(r) \) with \( r \geq 2 \) denotes the lattice whose Gram matrix is the one of \( U \) multiplied by \( r \).

### 2.3. Moduli spaces.

Let \( X \) be a K3 surface with an order \( n \) automorphism \( \sigma \) such that \( \sigma^4(\omega_X) = \zeta_n^4 \omega_X \). The period line \( \mathbb{C} \omega_X \) belongs to the domain

\[
\mathcal{D}^\sigma = \{ \mathbb{C}z \in \mathbb{P}(V^\sigma) : (z, \bar{z}) > 0, (z, z) = 0 \},
\]

where \( V^\sigma \) is the \( \zeta_n \)-eigenspace of \( \sigma^* \) in \( H^2(X, \mathbb{C}) \). Observe that, for \( n \geq 3 \), we have

\[
(z, z) = (\sigma^* z, \sigma^* z) = \zeta_n^2 (z, z),
\]

thus the condition \( (z, z) = 0 \) is not necessary and \( \mathcal{D}^\sigma \) can be easily proved to be isomorphic to a complex ball. On the other hand, if \( n = 2 \), then \( \mathcal{D}^\sigma \) is a type IV Hermitian symmetric space. By [DK07, Theorem 11.3] an arithmetic quotient of a Zariski open subset of \( \mathcal{D}^\sigma \) parametrizes isomorphism classes of \((\rho, M)\)-polarized K3 surfaces, where \( \rho : C_n \rightarrow O(L_{K3}) \) is a representation induced by the isometry \( \sigma^* \) of \( H^2(X, \mathbb{C}) \) and the choice of an isometry \( H^2(X, \mathbb{Z}) \rightarrow L_{K3} \), and \( M \subseteq L_{K3} \) is the invariant lattice of \( \text{Im}(\rho) \).

In particular such moduli space has dimension \( \dim(\mathcal{D}^\sigma) = \dim(V^\sigma) - 1 \) if \( n \geq 3 \) and \( \dim(V^\sigma) - 2 \) if \( n = 2 \).
On the other hand, if $T_X$ is the transcendental lattice of $X$, it is known that the eigenvalues of $\sigma^*$ in $T_X \otimes \mathbb{C}$ are the primitive $n$th roots of unity [Nik79a, Section 3], thus $\text{rank}(T_X) = \dim(V^\sigma) \varphi(n)$. Since $\text{rank}(T_X) \leq 21$, this implies that

$$\dim(D^{\sigma}) \leq \gamma(n) : = \left\lfloor \frac{21}{\varphi(n)} \right\rfloor - 1.$$  

In particular the dimension of $D^{\sigma}$ is at most one if $\gamma(n) = 1$. We show that the converse also holds.

**Lemma 2.3.** Let $n \neq 60$ be a positive integer with $\varphi(n) \leq 20$ and $\gamma(n) > 1$, then there exist a K3 surface $X$ and a purely non-symplectic automorphism $\sigma$ of $X$ of order $n$ such that $\dim(D^{\sigma}) > 1$.

**Proof.** We will denote by $d(n)$ the dimension of the moduli space of K3 surface carrying a purely non-symplectic automorphism of order $n$. The orders $n \geq 2$ with $\varphi(n) \leq 20$ and $\gamma(n) > 1$ are $n = 7, 9, 14, 18$, $n = 5, 8, 10, 12$ with $\gamma(n) = 4$, $n = 3, 4, 6$ with $\gamma(n) = 9$, and $n = 2$.

For prime orders $n = 3, 5, 7$ it is known by [AST11] that $d(n) = \gamma(n)$. Moreover, the same is true for orders $n = 6, 10, 14$ by Proposition 2.5 and [AST11].

For order $n = 9$ it is known by [ACV20] that $d(n) = 2$. Moreover, the general member of one of its components of maximal dimension is an elliptic K3 surface with Weierstrass equation

$$y^2 = x^3 + t(t^3 - a)(t^3 - b)(t^3 - c), \quad a, b, c \in \mathbb{C},$$

which carries the order nine automorphism $\sigma(x, y, t) = (\zeta_9^4 x, \zeta_9^6 y, \zeta_3 t)$. This surface also admits the non-symplectic involution $\tau(x, y, t) = (x, -y, t)$ which commutes with $\sigma$, so it carries the non-symplectic automorphism $\sigma \tau$ of order 18. This shows that $d(18) = 2$ as well.

When $n = 4$, [AS15, Example 6.3] is a 9-dimensional family of K3 surfaces with a purely non-symplectic automorphism of order 4.

When $n = 8$, [ATS18, Example 4.1] is a 2-dimensional family of K3 surfaces with a purely non-symplectic automorphism of order 8.

When $n = 12$ the family of elliptic K3 surfaces defined by the Weierstrass equation

$$y^2 = x^3 + t \prod_{i=1}^{5}(t^2 - a_i), \quad a_i \in \mathbb{C}$$

is 4-dimensional and has an order 12 automorphism, $\sigma(x, y, t) = (-\zeta_3 x, iy, -t)$ which can be easily checked to be purely non-symplectic.

When $n = 2$ it is well known that $d(2) = 19$ and there is a unique component of maximal dimension whose general element is a double cover of $\mathbb{P}^2$ branched along a smooth plane sextic. \qed

**Remark 2.4.** Under the hypotheses of Theorem 1.2, since $T_X$ has the structure of a $\mathbb{Z}[\zeta_n]$-module by [Nik79a, Section 3] and $\dim(V^\sigma) = 2$ we have that
rk NS(X) ≥ 22 − 2ϕ(n). The generality assumption in the statement of the theorem means that the Néron-Severi lattice of X has the minimal rank.

Finally, we recall a result contained in [GS13, Theorem 1.4, Theorem 1.5] and in [Dil12].

**Proposition 2.5.** Let X be a K3 surface with a non-symplectic automorphism σ of order n. If either

i. \( n = 5, 13, 17, 19 \),

ii. \( n = 7, 11 \) and the fixed locus of σ contains a curve,

iii. \( n = 3 \) and the fixed locus of σ contains at least two curves,

iv. \( n = 3 \) and the fixed locus of σ contains a curve and two points,

then X admits a non-symplectic automorphism τ of order 2n with \( \tau^2 = \sigma \).

Moreover, if \( n = 11 \) and the fixed locus of σ consists of only isolated fixed points, then X does not admit a non-symplectic automorphism τ of order 22 with \( \tau^2 = \sigma \).

3. Proof of Theorem 1.1 and Theorem 1.2

In this section we prove the two main theorems for each order.

3.1. Order 11. Non-symplectic automorphisms of order 11 have been classified in [OZ11] and [AST11, Sec. 7]. In particular the proof of Theorem 1.2 for order 11 follows from the following result.

**Theorem 3.1.** Let X be a K3 surface with a non-symplectic automorphism σ of order 11 such that rank \( S(\sigma) = 2 \) (or equivalently \( \dim(V^\sigma) = 2 \)). Then two cases can occur:

a) \( \text{Fix}(\sigma) = C_1 \sqcup \{p_1, p_2\} \) and \( S(\sigma) = \text{NS}(X) \cong U \),

b) \( \text{Fix}(\sigma) = \{p_1, p_2\} \) and \( S(\sigma) = \text{NS}(X) \cong U(11) \).

where \( C_1 \) is a smooth curve of genus one. In both cases \( d = (2, 2) \). Moreover, up to birational isomorphisms, \((X, \sigma)\) belongs to the family in Example 3.2 in case a) and to the family in Example 3.3 in case b).

**Example 3.2.** Given \( a \in \mathbb{C} \), let \( X_{11a} \) be the elliptic fibration with Weierstrass equation

\[
y^2 = x^3 + ax + (t^{11} - 1).
\]

For general \( a \in \mathbb{C} \) the fibration has one fiber of Kodaira type II over \( t = \infty \) and 22 fibers of type I₁. Observe that \( X_{11a} \) carries the order 11 automorphism

\[
\sigma_{11a}(x, y, t) = (x, y, \zeta_{11} t),
\]

which fixes the smooth fiber over \( t = 0 \) and two points in the fiber over \( t = \infty \).

**Example 3.3.** Consider the extremal rational elliptic surface \( \phi : Y \to \mathbb{P}^1 \) with Weierstrass equation

\[
y^2 = x^3 + x + t.
\]
The fibration has a fiber of type $II^*$ over $t = \infty$ and two fibers of type $I_1$ over the zeroes of $\Delta = 4 + 27t^2$ thus it is extremal. Given $\alpha \in \mathbb{P}^1$ such that $\phi^{-1}(\alpha)$ is smooth, let $\phi_{\alpha,e} : Y_{\alpha,e} \to \mathbb{P}^1$ be the principal homogeneous space of $\phi$ associated to a non-trivial 11-torsion element $e$ in $\phi^{-1}(\alpha)$. We recall that $\phi_{\alpha,e}$ has the same configuration of singular fibers as $\phi$ and it has a fiber $F = 11F_0$ of multiplicity 11 over $\alpha$ such that $(F_0)|_{F_0} = e \in \text{Pic}^0(F_0)$ (see [CD89, §4, Chapter V] as a reference for principal homogeneous spaces of rational jacobian elliptic fibrations). Let $\psi_{\alpha,e} : Z_{\alpha,e} \to \mathbb{P}^1$ be the degree 11 base change of $\phi_{\alpha,e}$ branched along $t = \infty$ and $t = \alpha$. A minimal resolution of $Z_{\alpha,e}$ is a K3 surface $X_{\alpha,e}$ carrying an elliptic fibration $\pi_{\alpha,e}$ induced by $\psi_{\alpha,e}$ which has 22 fibers of type $I_1$ over the two fibers of type $I_1$ of $\psi_{\alpha,e}$ and a fiber of type $II$ over $t = \infty$. The covering automorphism of $Z_{\alpha,e} \to Y_{\alpha,e}$ induces an order 11 automorphism $\sigma_{11b}$ of $X_{\alpha,e}$.

$$
\begin{array}{ccc}
X_{\alpha,e} & \longrightarrow & Z_{\alpha,e} \\
\pi_{\alpha,e} \downarrow & & \psi_{\alpha,e} \downarrow & & \phi_{\alpha,e} \\
\mathbb{P}^1 & \longrightarrow & \mathbb{P}^1 & \longrightarrow & \mathbb{P}^1.
\end{array}
$$

We will denote by $(X_{11b}, \sigma_{11b})$ the family of K3 surfaces with automorphism obtained with this construction. The automorphism $\sigma_{11b}$ fixes exactly two points in the fiber of $\pi_{\alpha,e}$ of type $II$.

3.2. **Order 22.** In this section we will give the classification of purely nonsymplectic automorphisms of order 22 with $\text{dim}(V^\sigma) = 2$. The full classification, including the cases with $\text{dim}(V^\sigma) = 1$, will be given in Section 4.

**Proposition 3.4.** Let $X$ be a K3 surface with a purely nonsymplectic automorphism $\sigma$ of order 22 such that $\text{dim}(V^\sigma) = 2$. Then the fixed loci of $\sigma = \sigma_{22}$ and of its powers $\sigma_{11} = \sigma^2$ and $\sigma_2 = \sigma^{11}$ are as follows:

$$
\begin{array}{ccc}
\text{Fix}(\sigma_{22}) & | & \text{Fix}(\sigma_{11}) & | & \text{Fix}(\sigma_2) \\
\{p_1, \ldots, p_6\} & | & C_1 \sqcup \{p_5, p_6\} & | & C_{10} \sqcup R
\end{array}
$$

where $g(C_1) = 1$, $g(C_{10}) = 10$ and $g(R) = 0$. Moreover $d = (2, 0, 2)$ and $\text{NS}(X) \cong U$ for a very general K3 surface with such property.

**Proof.** Decomposing $H^2(X, \mathbb{C})$ as the direct sum of the eigenspaces of $\sigma^+$ we obtain, with the notation in Section 2:

$$
\text{dim} H^2(X, \mathbb{C}) = 22 = 10d_{22} + 10d_{11} + d_2 + d_1 = 20 + 10d_{11} + d_2 + d_1.
$$

Since $d_{22} = 2$, thus $d_1 + d_2 = 2$ and $d_{11} = 0$, so either $d = (2, 0, 1, 1)$ or $(2, 0, 0, 2)$. Let $\chi_i := \chi(\text{Fix}(\sigma_i)), i \in \{2, 11, 22\}$. By the topological Lefschetz formulas we have

$$
\begin{cases}
\chi_{22} = d_{22} - d_{11} - d_2 + d_1 + 2 \\
\chi_{11} = -d_{22} - d_{11} + d_2 + d_1 + 2 \\
\chi_2 = -10d_{22} + 10d_{11} - d_2 + d_1 + 2.
\end{cases}
$$
This implies $\chi_{11} = 2$. By Proposition 2.5, if a K3 surface admits a non-symplectic automorphism of order 11 without fixed curves, it does not admit a non-symplectic automorphism of order 22. This result and Theorem 3.1 imply that $\text{Fix}(\sigma_{11})$ is the union of a smooth genus one curve $C$ and two points $p,q$. On the other hand the same equations give that $\chi_{22} = 4$ if $d = (2,0,1,1)$ and $\chi_{22} = 6$ if $d = (2,0,0,2)$. This implies that $\sigma_{22}$ is not the identity on $C$, thus it acts on it as an involution with 4 fixed points, and it either exchanges or fixes $p$ and $q$.

We will now show that $\sigma_{22}$ must fix $p$ and $q$, i.e. that $\chi_{22} = 6$. Observe that the fixed points of $\sigma_{22}$ on $C$ are of type $A_{10}$, since they are contained in a fixed curve of $\sigma_{22}^2$. If these were the only fixed points of $\sigma_{22}$, an easy computation shows that holomorphic Lefschetz formula does not hold, giving a contradiction.

Finally $\chi_{2} = -16$. By [Nik79a] this implies that the fixed locus of $\sigma_{2}$ is either a genus 9 curve or the union of a genus 10 curve and a rational curve. The first case is not possible since a curve of genus 9 has no order 11 automorphisms by the Riemann-Hurwitz formula.

Observe that for a very general K3 surface as in the statement $\text{rk} \, \text{NS}(X) = 22 - 2\varphi(22) = 2$ (see Remark 2.4) and $S(\sigma_{11}) \subseteq \text{NS}(X)$ by Remark 2.2, thus $\text{NS}(X) = S(\sigma_{11}) \cong U$ by Theorem 3.1.

**Example 3.5.** The elliptic K3 surface in Example 3.2

$$y^2 = x^3 + ax + (t^{11} - 1), \ a \in \mathbb{C}$$

admits the order 22 automorphism

$$\sigma_{22}(x,y,t) = (x,-y,\zeta_{11}t),$$

which fixes four points in the smooth fiber over $t = 0$ and two points in the fiber of type $II$ over $t = \infty$. The involution $\sigma_2 = \sigma_{22}^{11}$ fixes the curve $y = 0$, which has genus 10, and the sections at infinity. Since $\sigma_2$ has fixed curves and since there exist no symplectic automorphism of a K3 surface of order 11 [Nik79a], then $\sigma$ is purely non-symplectic.

**Proof of Theorem 1.2, order 22.** Let $X$ be a K3 surface with a purely non-symplectic automorphism $\sigma = \sigma_{22}$ of order 22. By Proposition 3.4 $\text{Fix}(\sigma_{11})$ contains an elliptic curve $C_1$ and two points. Thus by Theorem 3.1 $(X,\sigma_{11})$ belongs to the family in Example 3.2 up to isomorphism, i.e. it carries an elliptic fibration $\pi : X \to \mathbb{P}^1$ with Weierstrass equation

$$y^2 = x^3 + ax + (t^{11} - 1), \ a \in \mathbb{C}$$

and $\sigma_{11}(x,y,t) = (x,y,\zeta_{11}t)$. The lattice generated by the class of a fiber and the class of a section of $\pi$ is isometric to the lattice $U$ and is fixed by the automorphism $\sigma_{11}^*$, thus it coincides with $S(\sigma_{11})$ by Theorem 3.1. Since $\sigma_{22}^*$ preserves the lattice $S(\sigma_{11})$ and this contains a unique class of elliptic fibration and a unique class of smooth rational curve, then $\sigma_{22}^*$ preserves both. By Proposition 3.4 the fixed locus of the involution $\sigma_2$ is the disjoint union of a smooth curve $C_{10}$ of genus 10 and a smooth rational curve $R$. 
The curve $C_{10}$ is clearly transverse to the fibers of $\pi$, thus each fiber of $\pi$ contains fixed points of $\sigma_2$. This implies that the action induced by $\sigma_2$ on $\mathbb{P}^1$ is the identity, i.e. each fiber of $\pi$ is preserved by $\sigma_2$. Moreover, the unique section $S$ of $\pi$ must be pointwise fixed by $\sigma_2$, so that $R = S$. Since $\sigma_2$ is an involution which preserves each fiber of $\pi$ and fixes $S$, then it is defined by $(x, y, t) \mapsto (x, -y, t)$. This shows that the action of $\sigma_2 = \sigma_1 \circ \sigma_2$ on $\pi$ is the one described in the statement of Theorem 1.2, concluding the proof. □

3.3. Order 15. In this section we will give the classification of purely non-symplectic automorphisms of order 15 with $\dim(V^\sigma) = 2$. The full classification, including the cases with $\dim(V^\sigma) = 1$, will be given in Section 5.

Proposition 3.6. Let $X$ be a K3 surface with a purely non-symplectic automorphism $\sigma$ of order 15 such that $\dim(V^\sigma) = 2$. Then the fixed loci of $\sigma = \sigma_{15}$ and its powers $\sigma_i = \sigma_{15}^i$ are as follows:

|   | $\text{Fix}(\sigma_{15})$ | $\text{Fix}(\sigma_5)$ | $\text{Fix}(\sigma_3)$ |
|---|---------------------------|-----------------------|-----------------------|
| a | $\{p_1, \ldots, p_5\}$   | $C_2 \cup \{p_1\}$   | $C'_2 \cap \{p_2, p_3\}$ |
| b | $\{p_1, \ldots, p_7\}$   | $C_1 \cup \{p_1, \ldots, p_4\}$ | $C_4 \cap R \cup \{p_1\}$ |
| c | $\{p_1, \ldots, p_4\}$   | $C_1 \cup \{p_1, q_1, q_2, q_3\}$ | $C_4$ |

where $g(C_1) = 1, g(C_2) = g(C'_2) = 2, g(C_4) = 4$ and $g(R) = 0$. Moreover, $d = (2, 1, 0, 2)$ in case a), $d = (2, 0, 1, 4)$ in case b) and $d = (2, 0, 2, 2)$ in case c). Finally $\text{NS}(X) \cong U(3) \oplus A_2 \oplus A_2$ for a very general K3 surface $X$ in case a) and $\text{NS}(X) \cong H_5 \oplus A_4$ for a very general K3 surface $X$ in cases b) and c), where $H_5$ is the lattice defined in [AST11, Section 1].

Proof. Decomposing $H^2(X, \mathbb{C})$ as the direct sum of the eigenspaces of $\sigma^*$ we obtain, with the notation in Section 2:

$$22 = 8d_{15} + 4d_5 + 2d_3 + d_1 = 16 + 4d_5 + 2d_3 + d_1,$$

thus $d \in \{(2, 1, 0, 2), (2, 0, 2, 2), (2, 0, 1, 4), (2, 0, 0, 6)\}$. Let $\chi_i := \chi(\text{Fix}(\sigma_i)), i \in \{3, 5, 15\}$. By the topological Lefschetz fixed point formulas:

$$\begin{align*}
\chi_{15} &= d_{15} - d_5 - d_3 + d_1 + 2 \\
\chi_5 &= -2d_{15} - d_5 + 2d_3 + d_1 + 2
\end{align*}$$

We will show that $d = (2, 1, 0, 2), d = (2, 0, 1, 4)$ and $d = (2, 0, 2, 2)$ are the only possible cases.

Assume that $d = (2, 1, 0, 2)$. Thus $(\chi_{15}, \chi_5, \chi_3) = (5, -1, 0)$. By [AST11] we have that $\text{Fix}(\sigma_5)$ is the union of a curve $C_2$ of genus 2 and one point. Since $\chi_{15} = 5$ the action of $\sigma$ on $C_2$ has order 3 with 4 fixed points by the Riemann-Hurwitz formula. In particular $\text{Fix}(\sigma)$ is the union of 5 points. Finally by [AS08] $\text{Fix}(\sigma_3)$ is either the union of a genus 2 curve and 2 points or contains a curve of genus three. The second case is not possible since there is no genus 3 curve with an order five automorphism by [Bro91, Table 5].
If $d \neq (2, 1, 2)$, then $\chi_5 = 4$ and $\chi_3 = -6, -3, 0$ if $d = (2, 0, 2, 2), (2, 0, 1, 4)$ or $(2, 0, 0, 6)$ respectively. By [AST11] $\text{Fix}(\sigma_5)$ is either the union of an elliptic curve $C_1$ and 4 points, or the union of 4 points. Observe that $C_1$ can not be contained in $\text{Fix}(\sigma_3)$ since by [AS08] this would imply $\chi_3 \geq 3$. Thus, looking at the possible actions of $\sigma$ on $C_1$ and the 4 points we find that $\chi_{15}$ is either 1, 4 or 7.

If $d = (2, 0, 0, 6)$, then $\chi_{15} = 10$ by (2), giving a contradiction.

If $d = (2, 0, 1, 4)$, then $\chi_{15} = 7$ by (2). Thus $\text{Fix}(\sigma_5)$ is the union of an elliptic curve $C_1$ and 4 points and $\text{Fix}(\sigma)$ consists of 7 points, 3 of them on $C_1$. Moreover $\chi_3 = -3$, thus by [AS08] $\text{Fix}(\sigma_3)$ is either the union of a genus 4 curve, a rational curve and one point, or it contains a curve of genus 3. The last case is not possible by [Bro91, Table 5].

If $d = (2, 0, 2, 2)$, then $\chi_{15} = 4$ and $\chi_3 = -6$ by (2). Observe that $\sigma$ has 7 types of isolated fixed points. The fixed points of type $A_{1,15}, A_{4,15}, A_{7,15}$ are isolated fixed points for $\sigma_3$ too, while points of type $A_{2,15}, A_{3,15}, A_{5,15}, A_{6,15}$ lie on a curve fixed by $\sigma_3$. Observe that $\sigma_{15}$ acts on the set of isolated fixed points of $\sigma_3$ with orbits of length either 1 or 5. Thus we have

$$a_{1,15} + a_{4,15} + a_{7,15} \leq a_{1,3}, \quad a_{1,15} + a_{4,15} + a_{7,15} \equiv a_{1,3} \mod 5.$$

Moreover, points of type $A_{4,15}, A_{5,15}$ lie on a curve fixed by $\sigma_5$, while points of type $A_{1,15}, A_{2,15}, A_{3,15}, A_{6,15}, A_{7,15}$ are isolated fixed points for $\sigma_5$ too. Checking types one has

$$a_{1,15} + a_{3,15} + a_{6,15} \leq a_{1,5}, \quad a_{2,15} + a_{7,15} \leq a_{2,5}. \quad (3)$$

Since $\chi_3 = -6$, then $a_{1,3} = 0$ by [AS08]. Applying the holomorphic Lefschetz formula to $\sigma$ with this condition and using the fact that $\alpha = 0$, we find that $(a_{1,15}, a_{2,15}, \ldots, a_{7,15}) = (0, 1, 0, 0, 3, 0, 0)$. Since $a_{5,15} = 3$, then we find that $\text{Fix}(\sigma_5)$ contains an elliptic curve $C_1$ and $\sigma$ fixes three points on it.

For $\text{Fix}(\sigma_3)$ there are two possibilities by [AS08]: it is either a curve of genus four or the union of a genus five curve and a rational curve. The second case is excluded by Lemma 3.7, where $C'$ is the elliptic curve $C_1$.

We now compute the Néron-Severi lattice of a very general $X$ in each case. Observe that since $d_{15} = 2$ and $\varphi(15) = 8$, the Néron-Severi lattice of $X$ has rank $22 - 2 \cdot 8 = 6$. In case a) the invariant lattice $S(\sigma^5) = S(\sigma_3)$ has rank $d_1 + 4d_5 = 6$, thus $\text{NS}(X) = S(\sigma_3) \cong U(3) \oplus A_2 \oplus A_2$, where the last isomorphism is by [AST11]. In cases b) and c) the invariant lattice $S(\sigma^3) = S(\sigma_5)$ has rank $d_1 + 2d_3 = 6$, thus we conclude as before that $\text{NS}(X) = S(\sigma_5)$.

In both cases $S(\sigma_5)$ is isomorphic to $H_5 \oplus A_4$ by [AST11].

\textbf{Lemma 3.7.} Let $X$ be a $K3$ surface and $\tau$ be a non-symplectic automorphism of order three of $X$ whose fixed locus is the disjoint union of a smooth curve $C$ of genus five and a smooth rational curve $R$. Then $X$ has no purely non-symplectic automorphism $\sigma$ of order 15 such that $\sigma^5 = \tau$ and such that the fixed locus of $\sigma^3$ contains a curve $C'$ distinct from $C$ and $R$. \hfill \square
Proof. Let $\pi : X \to Y$ be the quotient morphism by $\tau$. Since the fixed locus of $\tau$ is a smooth curve and the automorphism is non-symplectic of order 3, then $Y$ is a smooth rational surface. Moreover, the invariant lattice $S(\tau)$ has rank two by [AS08]. Since $\pi^*$ is injective and $\pi^*NS(Y)$ is a sublattice of $S(\tau)$, then rank $\text{NS}(Y) \leq 2$. Thus $Y$ is isomorphic to either $\mathbb{P}^2$ or a Hirzebruch surface $\mathbb{F}_r$, $r \geq 0$. The covering $\pi$ is branched along a smooth curve $B$ whose class $[B] \in \text{NS}(Y)$ satisfies $-3K_Y = 2[B]$ by the Hurwitz formula. This excludes the case $Y \cong \mathbb{P}^2$. We recall that if $Y = \mathbb{F}_r$, then $-K_Y = (r + 2)f + 2e$, where $f^2 = 0, e^2 = -r$ and $f \cdot e = 1$. Thus $r$ must be even and 

$$[B] = \frac{3(r + 2)}{2}f + 3e.$$ 

Since $R^2 < 0$, then its image in $Y$ has the same property and is the unique curve of negative self-intersection in $Y$, i.e. $[\pi(R)] = e$. Moreover, since $B$ is the disjoint union of $\pi(R)$ and $\pi(C)$ then 

$$([B] - e) \cdot e = \frac{3(r + 2)}{2} - 2r = \frac{6 - r}{2} = 0.$$ 

Thus $Y \cong \mathbb{F}_6$ and the class of $\pi(C)$ is $12f + 2e$. Let $p : \mathbb{F}_6 \to \mathbb{P}^1$ be the natural fibration. Observe that the restriction of $p$ to $\pi(C)$ is a double cover of $\mathbb{P}^1$ since $(12f + 2e) \cdot f = 2$, thus $\pi(C)$ is hyperelliptic. This implies that there are 12 fibers of $p$ which are tangent to $\pi(C)$.

Assume now that $X$ has an automorphism $\sigma$ of order 15 with $\sigma^5 = \tau$. Then $\sigma$ induces an automorphism $\bar{\sigma}$ of $Y$ of order 5 which preserves both $\pi(R)$ and $\pi(C)$. Since $\sigma^5$ is not the identity on $R$, then $\bar{\sigma}$ is not the identity on $\pi(R)$, thus we can assume that it acts on the basis of the fibration $p$ as $(x, y) \mapsto (\zeta_5 x, y)$, where $\zeta_5$ is a primitive 5-th root of unity. In particular there are exactly two fibers of $p$ which are invariant for $\bar{\sigma}$.

This implies that the image of the curve $C'$ in $\text{Fix}(\sigma^3)$ is a fiber of $p$, which is invariant for $\bar{\sigma}$. On the other hand $\bar{\sigma}$ preserves $\pi(C)$ and thus permutes the 12 fibers of $p$ which are tangent to $\pi(C)$. Thus it should preserve at least two of them. The curve $\pi(C)$ can not be tangent to $\pi(C')$, since otherwise $C'$ would be singular, thus $\bar{\sigma}$ should leave invariant three fibers of $p$, a contradiction. \qed

Example 3.8. Let $B$ be the plane sextic defined by 

$$F_6(x_0, x_1, x_2) = a_1x_0^6 + a_2x_0x_1^5 + a_3x_0^6 + a_4x_0^3x_2^3,$$ 

with general $a_1, a_2, a_3, a_4 \in \mathbb{C}$. Let $X$ be a double cover of $\mathbb{P}^2$ branched along $B$, which can be defined by $x_3^2 - F_6(x_0, x_1, x_2) = 0$ in $\mathbb{P}(1, 1, 1, 3)$. Then $X$ is a K3 surface carrying an order 15 automorphism 

$$\sigma_{15}(x_0, x_1, x_2, x_3) = (x_0, \zeta_5 x_1, \zeta_3 x_2, x_3)$$ 

whose fixed locus is the union of 5 points, which project to the points $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ of $\mathbb{P}^2$. Observe that $\sigma_5$ fixes the genus two curve defined by $x_1 = 0$ and the point $(0, 1, 0, 0)$, while $\sigma_3$ fixes the genus two curve $x_2 = 0$. 

}\end{proof}
and the points \((0,0,1,\pm 1)\). Since both \(\sigma_3\) and \(\sigma_5\) fix curves, then none of them is symplectic by [Nik79a], thus \(\sigma_{15}\) is purely non-symplectic. This is an example of case a) in Proposition 3.6.

**Example 3.9.** Consider the elliptic surface with Weierstrass equation

\[
y^2 = x^3 + (t^5 - 1)(t^5 - a),
\]

with general \(a \in \mathbb{C}\). Then \(X\) is a K3 surface with the automorphism of order 15:

\[
\sigma_{15}(x, y, t) = (\zeta_3 x, y, \zeta_5 t).
\]

The elliptic fibration has one fiber of type IV over \(t = \infty\) and 10 fibers of type II. The automorphism \(\sigma_3\) fixes the genus 4 curve defined by \(x = 0\), the section at infinity and the center of the fiber of type IV. The automorphism \(\sigma_5\) fixes the smooth fiber over \(t = 0\) and four points in the fiber of \(t = \infty\). The automorphism \(\sigma_{15}\) fixes 3 points in the fiber over \(t = 0\) and 4 points in the fiber over \(t = \infty\). As in the previous Example, \(\sigma_3\) and \(\sigma_5\) fix curves, thus they are non-symplectic and \(\sigma_{15}\) is purely non-symplectic. This is an example of case b) in Proposition 3.6.

**Example 3.10.** Consider \(P = \mathbb{P}(1, 1, 2)\) with coordinates \(x_0, x_1, x_2\) and let \(D\) be a curve of degree 6 in \(P\) of equation

\[
G_6(x_0, x_1, x_2) = x_0^2 x_1 + a_1 x_0^2 x_2 + a_2 x_1^4 x_2 + a_3 x_0^6 + a_4 x_2^3 = 0,
\]

where \(a_1, a_2, a_3, a_4 \in \mathbb{C}\) are general. Observe that \(D\) is smooth, since it does not pass through the singular point \((0, 0, 1)\) and its partial derivatives only vanish at the origin. Let \(Y \cong \mathbb{F}_2\) be the blow up of the singular point of \(P\) and let \(B\) be the preimage of \(D\) in \(Y\). Since \(2|D| \sim -3K_P\) and the resolution \(Y \to P\) is crepant, then \(2|B| \sim -3K_Y\). Let \(X\) be the triple cover of \(Y\) branched along \(B\). By the Hurwitz formula \(X\) is a K3 surface. Observe that the curve \(D\) has the order 5 automorphism

\[
(x_0, x_1, x_2) \mapsto (\zeta_5 x_0, x_1, x_2),
\]

which lifts to an order five automorphism \(\varphi\) of \(X\). The composition of \(\varphi\) with the covering automorphism of \(X \to Y\) is an order 15 automorphism \(\sigma\) of \(X\). A birational model of \((X, \sigma)\) in \(\mathbb{P}(1, 1, 2, 2)\) is

\[
x_3^3 + G_6(x_0, x_1, x_2) = 0, \quad \sigma(x_0, x_1, x_2, x_3) = (\zeta_5 x_0, x_1, x_2, \zeta_3 x_3).
\]

Embedding \(\mathbb{P}(1, 1, 2, 2)\) in \(\mathbb{P}^4\) via the map \((x_0, x_1, x_2, x_3) \mapsto (x_0^3, x_0 x_1, x_1^2, x_2, x_3)\), we also obtain a birational model of \((X, \sigma)\) as complete intersection in \(\mathbb{P}^4\) with three \(A_1\) singularities at \(q_i = (0, 0, 0, \zeta_3^i, 1), i = 1, 2, 3:\)

\[
\begin{cases}
y_1^2 - y_0 y_2 = 0, \\
y_3^3 + y_1^3 y_3 + a_1 y_0^2 y_1^2 + a_2 y_0^2 y_3 + a_3 y_0^3 + a_4 y_3^3 = 0,
\end{cases}
\]

\[
\sigma(y_0, y_1, y_2, y_3, y_4) = (\zeta_5^i y_0, \zeta_5 y_1, y_2, y_3, \zeta_3 y_4).
\]

The fixed locus of \(\sigma_3\) is the genus four curve \(y_4 = 0\). The fixed locus of \(\sigma_5\) is the union of the curve \(C_1\) of genus one defined by \(y_0 = y_1 = 0\), the point
would contain a smooth rational curve. This implies that we can assume that Table contains more than three fixed points of . An equation of the form contains a line fixed by either \( \bar{\sigma} \). Moreover, by the previous description, the two lines must be distinct. Thus we can assume that \( \bar{\sigma} \) is invariant for \( \bar{\sigma} \). Let \( \bar{\sigma} = \bar{\sigma}_1 \) and \( \bar{\sigma}_5 = \bar{\sigma}_3 \). Up to a projectivity we can assume that \( \bar{\sigma}_5 \), and thus \( \bar{\sigma}_3 \) and \( \bar{\sigma}_5 \) are diagonal. Observe that both \( \bar{\sigma}_3 \) and \( \bar{\sigma}_5 \) must fix pointwise a line and a point in \( \mathbb{P}^2 \), since both \( \bar{\sigma}_3 \) and \( \bar{\sigma}_5 \) fix pointwise a curve of positive genus. Moreover, by the previous description, the two lines must be distinct. Thus we can assume that

\[
\bar{\sigma}_3(x_0 : x_1 : x_2) = (x_0 : x_1 : \zeta_3 x_2) \quad \bar{\sigma}_5(x_0 : x_1 : x_2) = (x_0 : \zeta_5 x_1 : x_2).
\]

The branch sextic \( B \) of \( \varphi \) is invariant for \( \bar{\sigma}_{15} \). Observe that \( B \) can not contain a line fixed by either \( \bar{\sigma}_3 \) or \( \bar{\sigma}_5 \) since otherwise \( \text{Fix}(\bar{\sigma}_3) \) and \( \text{Fix}(\bar{\sigma}_5) \) would contain a smooth rational curve. This implies that \( B \) is defined by an equation of the form \( F_0(x_0, x_1, x_2) = 0 \) as given in Example 3.8. If either \( a_2 \) or \( a_3 \) vanishes, then \( B \) would contain a line. If \( a_1 = 0 \), then \( B \) would contain a singular point of type \( E_8 \), whose central component would be fixed by both \( \bar{\sigma}_3 \) and \( \bar{\sigma}_5 \), a contradiction. Thus \( a_1 a_2 a_3 \neq 0 \), in particular \( B \) is smooth. Up to rescaling the variables, an equation for \( X \) is the one given in Table 2 with \( a \in \mathbb{C} \).

**Case b).** We now consider the case when \( \text{Fix}(\bar{\sigma}_{15}) \) consists of 7 points, \( \text{Fix}(\bar{\sigma}_5) \) is the union of an elliptic curve \( C_1 \) and 4 points and \( \text{Fix}(\bar{\sigma}_3) \) is the union of a genus 4 curve \( C_4 \), a rational curve \( R \) and one point. Let \( \pi : X \to \mathbb{P}^1 \) be the morphism associated to the linear system \( |C_1| \), which is an elliptic fibration. Since \( C_1 \) is invariant for \( \bar{\sigma}_{15} \), then the elliptic fibration is invariant for \( \bar{\sigma}_{15} \). On the other hand, since \( \bar{\sigma}_3 \) fixes the curve \( C_4 \) of genus \( g > 1 \), then it induces the identity on \( \mathbb{P}^1 \). The automorphism \( \bar{\sigma}_3 \) acts on \( C_1 \) and has fixed points in \( C_1 \cap C_4 \), thus it has exactly 3 fixed points by the Riemann-Hurwitz formula. This implies that \( C_1 \cdot C_4 \leq 3 \). Moreover, \( C_1 \cdot C_4 = 1 \) since otherwise the restriction of \( \pi \) to \( C_4 \) would be an isomorphism onto \( \mathbb{P}^1 \). Thus \( C_1 \cdot C_4 \) is either 2 or 3. We now show that the second case does not appear.

If \( C_1 \cdot C_4 = 3 \), then the curve \( R \) can not intersect \( C_1 \), since otherwise \( C_1 \) would contain more than three fixed points of \( \bar{\sigma}_5 \). Thus \( R \) must be contained in a reducible fiber \( F \) of \( \pi \). The fiber \( F \) is invariant for \( \bar{\sigma}_3 \), it can only contain an isolated fixed point of \( \bar{\sigma}_3 \) and \( C_4 \cdot F = 3 \). By [AS08, Lemma 4.1] \( F \) should

\[ p_1 = (1, 0, 0, 0, 0) \]

and three points over \( q_1, q_2, q_3 \). Finally, \( \sigma \) fixes \( p_1 \) and three points in \( C_1 \cap C_4 \). This is an example of case c) in Proposition 3.6.

**Proof of Theorem 1.2, order 15.** Let \( X \) be a K3 surface with a purely nonsymplectic automorphism \( \sigma_{15} \) of order 15. By Proposition 3.6, \( \text{Fix}(\sigma_{15}) \) contains either 4, 5 or 7 isolated fixed points.

**Case a).** We first assume that \( \text{Fix}(\sigma_{15}) \) consists of 5 fixed points, \( \text{Fix}(\sigma_5) \) is the union of a curve \( C_2 \) of genus 2 and one point and \( \text{Fix}(\sigma_3) \) is the union of a genus 2 curve \( C_2' \) and 2 points. Let \( \varphi : X \to \mathbb{P}^2 \) be the morphism associated to the linear system \( |C_2'| \), which is a degree two morphism branched along a plane sextic \( B \) which possibly contracts the smooth rational curves disjoint from \( C_2' \) to simple singular points of \( B \) [SD74]. Since \( |C_2'| \) is fixed by \( \sigma_{15} \), the automorphism \( \sigma_{15} \) descends to an automorphism \( \bar{\sigma}_{15} \) of \( \mathbb{P}^2 \). Let \( \bar{\sigma}_3 = \bar{\sigma}_{15}^3 \) and \( \bar{\sigma}_5 = \bar{\sigma}_{15}^2 \). Up to a projectivity we can assume that \( \bar{\sigma}_{15} \), and thus \( \bar{\sigma}_3 \) and \( \bar{\sigma}_5 \) are diagonal. Observe that both \( \bar{\sigma}_3 \) and \( \bar{\sigma}_5 \) fix pointwise a line and a point in \( \mathbb{P}^2 \), since both \( \bar{\sigma}_3 \) and \( \bar{\sigma}_5 \) fix pointwise a curve of positive genus. Moreover, by the previous description, the two lines must be distinct. Thus we can assume that

\[
\bar{\sigma}_3(x_0 : x_1 : x_2) = (x_0 : x_1 : \zeta_3 x_2) \quad \bar{\sigma}_5(x_0 : x_1 : x_2) = (x_0 : \zeta_5 x_1 : x_2).
\]
be of type $I^*_1$, but this contradicts the fact that the rank of the invariant lattice of $\sigma_3$ is 4 by [AS08, Theorem 2.2].

Thus $C_1 \cdot C_4 = 2$. Since the general fiber of $\pi$ must contain three fixed points of $\sigma_3$ by the Riemann-Hurwitz formula, then the curve $R$ must be a section of $\pi$. Thus $\pi$ is a jacobian elliptic fibration invariant for an order three automorphism and with a fixed section. This implies that up to a coordinate change $\pi$ has Weierstrass equation

$$y^2 = x^3 + p(t),$$

with $\sigma_3(x, y, t) = (\zeta_3 x, y, t)$, where $\deg(p) \leq 12$. In these coordinates $C_1$ is the fiber over $t = 0$ and $C_4$ is the curve $x = 0$. Since $\sigma_5$ has order 5 on $R$, then it induces an order five automorphism on $\mathbb{P}^1$ which can be assumed to be $\bar{\sigma}_5(t) = \zeta_5 t$. Since $\sigma_5$ is the identity when $t = 0$, then $\sigma_5(x, y, t) = (x, y, \zeta_5 t)$. Thus up to a coordinate change $\pi$ has Weierstrass equation of the form

$$y^2 = x^3 + (t^5 - 1)(t^5 - a),$$

with $a \in \mathbb{C}$ and $\sigma_{15}(x, y, t) = (\zeta_3 x, y, \zeta_5 t)$, as in Example 3.9.

**Case c.** Finally, assume that $\text{Fix}(\sigma_{15})$ consists of 4 points. $\text{Fix}(\sigma_5)$ is the union of an elliptic curve $C_1$ and 4 points and $\text{Fix}(\sigma_3)$ is a curve of genus 4. Following the same argument in the proof of Lemma 3.7 we obtain that the quotient of $X$ by $\sigma_3$ is a smooth rational surface $Y$ isomorphic to a Hirzebruch surface $\mathbb{F}_r$ for some even $r \geq 0$. This quotient is a cyclic degree three covering branched along a smooth curve $B$ of genus four whose class is $[B] = \frac{3(r+2)}{2} f + 3e$, where $f^2 = 0, e^2 = -r, f \cdot e = 1$. Since $B$ is smooth, then $[B] \cdot e = \frac{6-3r}{2} \geq 0$, thus $r \leq 2$.

The case when $r = 0$, i.e. $Y \cong \mathbb{P}^1 \times \mathbb{P}^1$ can be excluded as follows. In this case $B$ is a curve of type $(3,3)$. The automorphism $\sigma_5$ descends to an automorphism $\overline{\sigma}_5$ of $\mathbb{P}^1 \times \mathbb{P}^1$ which has a fixed curve, thus up to a coordinate change we can assume

$$\overline{\sigma}_5 : (x_0 : x_1), (y_0 : y_1) \mapsto (x_0 : x_1), (\zeta_5 y_0 : y_1).$$

However there is no curve of type $(3,3)$ which is invariant for this automorphism, giving a contradiction.

Thus $Y \cong \mathbb{F}_2$, $[B] = 6f + 3e$ and $[B] \cdot e = 0$. After contracting the $(-2)$-curve of $Y$ we obtain the surface $P \cong \mathbb{P}(1,1,2)$. Since the contraction is a crepant morphism the image of $B$ is a smooth curve $D$ with $2[D] \sim -3K_P$, i.e. of degree 6. The automorphism $\sigma_5$ descends to an automorphism $\overline{\sigma}_5$ of $P$ which preserves $D$ and fixes the image of the curve $C_1$. This implies that, after a coordinate change, we can assume $\overline{\sigma}_5(x_0, x_1, x_2) = (\zeta_5 x_0, x_1, x_2)$. The equation of $D$ must be invariant for $\overline{\sigma}_5$, thus it is of the form given in Example 3.10. If all the coefficients of $G_6$ are non-zero, then one obtains the equation in Table 2 up to rescaling the variables. \qed

3.4. Order 30.
**Proposition 3.11.** Let $X$ be a K3 surface with a purely non-symplectic automorphism $\sigma_{30}$ of order 30 such that $\dim(V^\sigma) = 2$. Then there are two possibilities for the fixed locus of $\sigma_{30}$ and of its powers:

|   | $\text{Fix}(\sigma_{30})$ | $\text{Fix}(\sigma_{15})$ | $\text{Fix}(\sigma_5)$ | $\text{Fix}(\sigma_3)$ | $\text{Fix}(\sigma_2)$ |
|---|----------------|----------------|----------------|----------------|----------------|
| a) | $\{p_1\}$ | $\{p_1, \ldots, p_5\}$ | $C_2 \sqcup \{p_1\}$ | $C_2' \sqcup \{p_2, p_3\}$ | $C_{10}$ |
| b) | $\{p_1, p_2, p_3\}$ | $\{p_1, \ldots, p_7\}$ | $C_1 \sqcup \{p_1, \ldots, p_4\}$ | $C_4 \sqcup R \sqcup \{p_1\}$ | $C_9 \sqcup R$ |

where $C_g, C'_g$ have genus $g$ and $g(R) = 0$. Moreover, $d = (2, 0, 1, 0, 0, 0, 1, 1)$ in case a) and $d = (2, 0, 0, 1, 0, 0, 1, 3)$ in case b).

Finally $\text{NS}(X) \cong U(3) \oplus A_2 \oplus A_2$ for a very general K3 surface $X$ in case a) and $\text{NS}(X) \cong H_5 \oplus A_4$ for a very general K3 surface $X$ in case b).

**Proof.** Let $\chi_i = \chi(\text{Fix}(\sigma_i)), i = 30, 15, 5, 3, 2$. First observe that given a one dimensional family of K3 surfaces admitting a purely non-symplectic automorphism of order 30, every element in the family admits a purely non-symplectic automorphism of order 15. Thus this corresponds to one of the three families in Proposition 3.6 and the vector $(\chi_{15}, \chi_5, \chi_3)$ is either $(5, -1, 0)$, $(7, 4, -3)$ or $(4, 4, -6)$.

Decomposing $H^2(X, \mathbb{C})$ as the direct sum of the eigenspaces of $\sigma^*$ we obtain:

\[
22 = 8d_{30} + 8d_{15} + 4d_{10} + 2d_6 + 4d_5 + 2d_3 + d_2 + d_1.
\]

Assuming $d_{30} = 2$, this gives $d_{15} = 0$. Using the topological Lefschetz fixed point formulas we compute the topological Euler characteristic of the fixed loci of $\sigma_{30}$ by:

\[
\begin{align*}
\chi_{30} &= d_{10} + d_6 - d_5 - d_3 - d_2 + d_1 \\
\chi_{15} &= -d_{10} - d_6 - d_5 - d_3 + d_2 + d_1 + 4 \\
\chi_5 &= -d_{10} + 2d_6 - d_5 + 2d_3 + d_2 + d_1 - 2 \\
\chi_3 &= 4d_{10} - d_6 + 4d_5 - d_3 + d_2 + d_1 - 6 \\
\chi_2 &= -4d_{10} - 2d_6 + 4d_5 + 2d_3 - d_2 + d_1 - 14. 
\end{align*}
\]

We first assume to be in case a) of Proposition 3.6, i.e. $\text{Fix}(\sigma_5)$ is the union of a smooth curve $C_2$ of genus 2 and a point $p_1$. $\text{Fix}(\sigma_3)$ is the union of a smooth curve $C'_2$ of genus 2 and two isolated points and $\text{Fix}(\sigma_{15})$ consists of 5 isolated points $p_1, \ldots, p_5$. In particular $(\chi_{15}, \chi_5, \chi_3) = (5, -1, 0)$. Moreover, since the fixed locus of $\sigma_{15}$ only contains isolated points, the same holds for $\sigma_{30}$. Thus $\chi_{30} \geq 0$. By (4) and (5) we get the possibilities in Table 3.

| $d_{30}$ | $d_{15}$ | $d_{10}$ | $d_6$ | $d_5$ | $d_3$ | $d_2$ | $d_1$ | $\chi_{30}$ | $\chi_{15}$ | $\chi_5$ | $\chi_3$ | $\chi_2$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 5 | -1 | 0 | -18 |
| 2 | 0 | 1 | 0 | 0 | 0 | 0 | 2 | 3 | 5 | -1 | 0 | -16 |
| 2 | 0 | 0 | 0 | 1 | 0 | 0 | 2 | 1 | 5 | -1 | 0 | -8 |

**Table 3.**
In particular $\chi_{30}$ is either 3 or 1, thus $\text{Fix}(\sigma_{30})$ is either the union of $p_1$ and two of the $p_i$'s with $i \geq 2$ (and the other two are exchanged) or $\text{Fix}(\sigma_{30}) = \{p_1\}$ and $\sigma_{30}$ has no fixed points on $C_2$. By the proof of Theorem 1.2 in the case $n = 15$, the linear system associated to $C_2$ defines a double cover $\varphi : X \to \mathbb{P}^2$ which can be defined in $\mathbb{P}(1,1,1,3)$ by an equation of the form

$$y^2 = x_0^6 + x_0x_1^5 + x_2^6 + ax_0^3x_2^3,$$

where $a \in \mathbb{C}$ and in these coordinates $\sigma_{15}(x_0, x_1, x_2, y) = (x_0, \zeta_5 x_1, \zeta_3 x_2, y)$. Since $\sigma_{30}$ preserves $C_2$, then it induces an automorphism $\overline{\sigma}_{30}$ of $\mathbb{P}^2$. The involution $\sigma_2$ either induces the identity or an involution of $\mathbb{P}^2$. The latter is not possible since the fixed locus of $\sigma_2$ would contain a curve of genus at most 2, while $\chi_2 \leq -8$ by Table 3. Thus $\sigma_2$ coincides with the (automorphism induced by) the covering involution of $\varphi$, which fixes a smooth genus 10 curve, so that $\chi_2 = -18$ and $\chi_{30} = 1$ by Table 3. Thus $\sigma_{30}$ fixes a unique point. Since $C_2$ is invariant for $\sigma_{30}$, then $\varphi(C_2)$ is a line which contains two fixed points for $\sigma_{30}$. Since $\chi_{30} = 1$, their preimages by $\varphi$ are four points exchanged in pairs by $\sigma_2$.

Assume now to be in case b) of Proposition 3.6, i.e. $\text{Fix}(\sigma_3)$ is the union of a curve $C_4$ of genus 4, a rational curve $R$ and a point, $\text{Fix}(\sigma_5)$ is union of an elliptic curve $C_1$ and four points, and $\text{Fix}(\sigma_{15})$ is the union of 7 points (3 on $C_1$). In particular $(\chi_{15}, \chi_5, \chi_3) = (7, 4, -3)$. Moreover, $\chi_{30} \geq 0$ since $\text{Fix}(\sigma_{15})$ only contains isolated points, and thus the same holds for $\sigma_{30}$. There are five possible vectors $d$ such that $(\chi_{15}, \chi_5, \chi_3) = (7, 4, -3)$ (see Table 4). By the proof of Theorem 1.2 in the case $n = 15$, $X$ admits an elliptic fibration $\pi : X \to \mathbb{P}^1$ with Weierstrass equation

$$y^2 = x^3 + (t^5 - 1)(t^5 - a),$$

with $a \in \mathbb{C}$ and $\sigma_{15}(x, y, t) = (\zeta_3 x, y, \zeta_5 t)$. By the same argument in the proof of Theorem 1.2 in the case $n = 15$, using [AS15, Lemma 5], one concludes that the elliptic fibration is invariant for $\sigma_{30}$. Since $\chi_2 \leq -8$, then $\sigma_2$ fixes a curve of genus $> 1$. Such curve is clearly transverse to all fibers of $\pi$, thus $\sigma_2$ induces the identity on the basis of the fibration. Moreover, $\sigma_2$ must fix the section at infinity $R$ of the fibration, since it preserves $R$ and each fiber of $\pi$. This implies that $\sigma_2(x, y, t) = (x, -y, t)$. In particular $\sigma_2$ fixes $R$ and the curve defined by $y = 0$, which has genus 9, so that $\chi_2 = -14$.

| $d_{30}$ | $d_{15}$ | $d_{10}$ | $d_6$ | $d_5$ | $d_4$ | $d_2$ | $d_1$ | $\chi_{30}$ | $\chi_{15}$ | $\chi_5$ | $\chi_3$ | $\chi_2$ |
|---------|---------|---------|-------|-------|-------|-------|-------|-------------|-------------|----------|----------|----------|
| 2       | 0       | 0       | 1     | 0     | 0     | 2     | 2     | 1           | 7           | 4        | -3        | -16       |
| 2       | 0       | 0       | 1     | 0     | 0     | 1     | 3     | 3           | 7           | 4        | -3        | -14       |
| 2       | 0       | 0       | 0     | 0     | 0     | 1     | 1     | 1           | 7           | 4        | -3        | -10       |
| 2       | 0       | 0       | 1     | 0     | 0     | 0     | 4     | 5           | 7           | 4        | -3        | -12       |
| 2       | 0       | 0       | 0     | 0     | 0     | 1     | 0     | 3           | 7           | 4        | -3        | -8        |

Table 4.
Moreover $\sigma_{30}$ fixes three points: two points on $R$ and the center of the fiber of type $IV$ over $t = \infty$.

Assume now to be in case c) of Proposition 3.6, i.e. $\text{Fix}(\sigma_3)$ is a curve $C_4$ of genus 4, $\text{Fix}(\sigma_5)$ is union of an elliptic curve $C_1$ and four points, and $\text{Fix}(\sigma_{15})$ is the union of 4 points. In particular $(\chi_{15}, \chi_5, \chi_3) = (4, 4, -6)$. Moreover, $\chi_{30} \geq 0$ since $\text{Fix}(\sigma_{15})$ only contains isolated points, and thus the same holds for $\sigma_{30}$. By the proof of Theorem 1.2 in the case $n = 15$, $X$ is the minimal resolution of the double cover of $P = \mathbb{P}(1, 1, 2)$ branched along a smooth curve $D$ of degree 6 not passing through the singular point of $P$. The automorphism induced by $\sigma_5$ in $P$ can be assumed to be $(x_0, x_1, x_2) \mapsto (\zeta_5 x_0, x_1, x_2)$. The automorphism $\sigma_2$, since it commutes with $\sigma_3$, induces an involution $\tilde{\sigma}_2$ of $P$ which preserves the curve $D$. Moreover it can be also diagonalized. However, no diagonal involution leaves invariant the general equation as in Example 3.10, thus this case is not possible.

The Néron-Severi lattice of a very general $X$ in cases a) and b) is clearly the same as in Proposition 3.6. $\square$

Example 3.12. The double cover $X$ of $\mathbb{P}^2$ in Example 3.8 carries the order 30 automorphism $\sigma_{30}(x_0, x_1, x_2, x_3) = (x_0, \zeta_5 x_1, \zeta_3 x_2, -x_3)$. Observe that for a general choice of the coefficients the fixed locus of $\sigma_2$ is the smooth plane sextic defined by $x_3 = 0$, which has genus 10. Moreover, the fixed locus of $\sigma_{30}$ consists of the point $(0, 1, 0, 0)$. This is an example of case a) in Proposition 3.11.

Example 3.13. The elliptic K3 surface in Example 3.9 carries the order 30 automorphism $\sigma_{30}(x, y, t) = (\zeta_3 x, -y, \zeta_5 t)$. Observe that for general $a \in \mathbb{C}$ the fixed locus of $\sigma_2$ is the curve $y = 0$, which has genus 9. Moreover, as observed in the proof of Proposition 3.11, the fixed locus of $\sigma_{30}$ consists of two points in the section at infinity (over $t = 0$ and $t = \infty$) and the center of the fiber of type $IV$ over $t = \infty$. This is an example of case b) in Proposition 3.11.

Proof of Theorem 1.2, order 30. Let $X$ be a K3 surface with a purely non-symplectic automorphism $\sigma$ of order 30 such that $\dim(V^\sigma) = 2$. It is straightforward from the proof of Proposition 3.11 that, up to isomorphism, $(X, \sigma)$ belongs to one of the families in Examples 3.12 and 3.13. $\square$

3.5. Order 16. Purely non-symplectic automorphisms of order 16 on K3 surfaces have been classified in [ATST16]. The following result has the same statement as [ATST16, Theorem 4.1], but we provide a slightly different proof since we use the weaker hypothesis $\dim(V^\sigma) = 2$.

Proposition 3.14. Let $\sigma_{16}$ be a purely non-symplectic automorphism of order 16 of a K3 surface $X$ and assume that $\dim(V^\sigma) = 2$ (or equivalently
$S(\sigma_2)$ has rank 6). Then there are two possibilities for the fixed locus of $\sigma_{16}$ and of its powers:

| $a)$ | Fix($\sigma_{16}$) | Fix($\sigma_8$) | Fix($\sigma_4$) | Fix($\sigma_2$) |
|------|------------------|-----------------|-----------------|-----------------|
| $b)$ | $\{p_1, \ldots, p_6\} \sqcup R$ | $\{p_1, \ldots, p_6\} \sqcup R$ | $\{p_1, \ldots, p_6\} \sqcup R$ | $C_7 \sqcup R \sqcup R$ |
|      | $\{p_1, p_2, p_7, p_8\}$ | $\{p_1, \ldots, p_6\} \sqcup R$ | $\{p_1, \ldots, p_6\} \sqcup R$ | $C_6 \sqcup R$ |

where $g(C_6) = 6$, $g(C_7) = 7$ and $g(R) = g(R') = 0$. Moreover, $d = (2, 0, 0, 0, 6)$ in case $a)$ and $d = (2, 0, 0, 2, 4)$ in case $b)$.

Finally $\text{NS}(X) \cong U \oplus D_4$ for a very general $X$ in case $a)$ and $\text{NS}(X) \cong U(2) \oplus D_4$ for a very general $X$ in case $b)$.

**Proof.** Decomposing $H^2(X, \mathbb{C})$ as the direct sum of the eigenspaces of $\sigma_{16}^*$ we obtain:

$22 = 8d_{16} + 4d_8 + 2d_4 + d_2 + d_1$.

Since $d_{16} = 2$, this implies that $d_8$ is either 0 or 1 and gives the 14 possibilities for the vector $d$ in Table 5.

| $d_{16}$ | $d_8$ | $d_4$ | $d_2$ | $d_1$ | $\chi_{16}$ | $\chi_8$ | $\chi_4$ | $\chi_2$ |
|---------|-------|-------|-------|-------|-------------|---------|---------|---------|
| 2       | 1     | 0     | 1     | 1     | 2           | 4       | 0       | -8      |
| 2       | 0     | 2     | 1     | 1     | 2           | 2       | 8       | -8      |
| 2       | 0     | 1     | 3     | 1     | 0           | 5       | 8       | -8      |
| 2       | 0     | 0     | 5     | 1     | -2          | 8       | 8       | -8      |
| 2       | 1     | 0     | 0     | 2     | 4           | 4       | 0       | -8      |
| 2       | 0     | 2     | 0     | 2     | 4           | 2       | 8       | -8      |
| 2       | 0     | 1     | 2     | 2     | 2           | 5       | 8       | -8      |
| 2       | 0     | 0     | 4     | 2     | 0           | 8       | 8       | -8      |
| 2       | 0     | 1     | 1     | 3     | 4           | 5       | 8       | -8      |
| 2       | 0     | 0     | 3     | 3     | 2           | 8       | 8       | -8      |
| 2       | 0     | 1     | 0     | 4     | 6           | 5       | 8       | -8      |
| 2       | 0     | 0     | 2     | 4     | 4           | 8       | 8       | -8      |
| 2       | 0     | 0     | 1     | 5     | 6           | 8       | 8       | -8      |
| 2       | 0     | 0     | 0     | 6     | 8           | 8       | 8       | -8      |

Table 5.

Let $N_i$ be the number of isolated fixed points of $\sigma_i$, $\chi_i = \chi(\text{Fix}(\sigma_i))$ and

$$\alpha_i = \sum_{C \subseteq \text{Fix}(\sigma_i)} (1 - g(C))$$

for $i \in \{2, 4, 8, 16\}$. By the topological Lefschetz fixed point formula we get

$$\begin{align*}
\chi_{16} &= -d_2 + d_1 + 2 \\
\chi_8 &= -d_4 + d_2 + d_1 + 2 \\
\chi_4 &= -4d_8 + 2d_4 + d_2 + d_1 + 2 \\
\chi_2 &= -8d_{16} + 4d_8 + 2d_4 + d_2 + d_1 + 2.
\end{align*}$$

Table 5 shows the values of $(\chi_{16}, \chi_8, \chi_4, \chi_2)$ for each possible vector $d$. 
Observe that \( \chi_2 = -8 \). By [Nik79a], \( N_2 = 0 \) and \( \text{Fix}(\sigma_2) \) is the union of a curve of genus \( g \) and \( k \) rational curves with \((g, k) = (5, 0), (6, 1) \) or \((7, 2)\).

Moreover, \( \chi_4 = 0 \) or 8. By [AS15, Proposition 1] we have that \( N_4 = 2\alpha_4 + 4 \). Since \( \chi_4 = 2\alpha_4 + N_4 \) one has

\[
\chi_4 = 4\alpha_4 + 4.
\]

If \( \chi_4 = 0 \), then \( \alpha_4 = -1 \), but this is not possible since \( \text{Fix}(\sigma_4) \subseteq \text{Fix}(\sigma_2) \) and it is not compatible with the aforementioned possibilities for \( \text{Fix}(\sigma_2) \). Thus \( \chi_4 = 8 \), \( \alpha_4 = 1 \) and \( \text{Fix}(\sigma_4) \) contains a rational curve (and no more curves) and 6 points. This implies that the case \((g, k) = (5, 0)\) is impossible.

The cases \((g, k) = (6, 1) \) and \((7, 2)\) are treated in Lemma 3.15 and Lemma 3.16. We conclude that the only admissible cases are the ones in Proposition 3.14. Observe that both in case a) and b) we have that \( d_4 = d_8 = 0 \) and \( d_1 + d_2 = 6 \). This implies that \( S(\sigma_8) = S(\sigma_4) = S(\sigma_2) \) has rank 6. If \( X \) is very general, then rank \( \text{NS}(X) = 22 - 2\varphi(20) = 6 \) and thus by Remark 2.2 \( \text{NS}(X) = S(\sigma_2) \). Moreover by [Nik79b, Theorem 4.2.2] or [AST11, Figure 1] the invariant lattice of \( \sigma_2 \) is isometric to \( U\oplus D_4 \) in case a) and \( U(2)\oplus D_4 \) in case b), see [ATST16].

**Lemma 3.15.** If \( \text{Fix}(\sigma_2) \) is the union of a curve of genus 6 and a rational curve, then the fixed loci of \( \sigma_{16}, \sigma_8 \) and \( \sigma_4 \) are as follows:

\[
\text{Fix}(\sigma_{16}) = \{p_1, p_2, p_7, p_8\}, \quad \text{Fix}(\sigma_8) = \{p_1, \ldots, p_6\} \sqcup R,
\]

\[
\text{Fix}(\sigma_4) = \{p_1, \ldots, p_6\} \sqcup R.
\]

**Proof.** Let \( C_6 \) (resp. \( R \)) be the smooth curve of genus 6 (resp. rational curve) in \( \text{Fix}(\sigma_2) \). By the previous analysis we know that \( \sigma_4 \) fixes pointwise \( R \) and has 6 isolated fixed points \( p_1, \ldots, p_6 \) on \( C_6 \).

By the Riemann-Hurwitz formula for \( \sigma_8 \) on \( C_6 \), we observe that either a) 2 of the \( p_i \)'s are fixed and the other four are permuted in pairs by \( \sigma_8 \) or b) the points \( p_1, \ldots, p_6 \) are fixed points for \( \sigma_8 \). Observe that case a) is not possible since \( \chi_4 = 8 \) and \( \chi_8 = 4 \) does not appear in Table 5.

By the Riemann-Hurwitz formula for \( \sigma_{16} \) on \( C_6 \), we obtain that \( \sigma_{16} \) fixes 2 of the \( p_i \)'s and exchanges the other four in pairs. Thus \((\chi_{16}, \chi_8, \chi_4, \chi_2) = (4, 8, 8, -8)\).

Observe that six of the fixed points of \( \sigma_8 \) lie on a curve fixed pointwise by \( \sigma_2 \) and not by \( \sigma_4 \), thus the local action of \( \sigma_8 \) at such points is either of type \( A_{2,8} \) or \( A_{3,8} \). By [ATS18, Proposition 2.2] we have that \( 6 = 2 + 4\alpha_8 \), thus \( \alpha_8 = 1 \). This implies that \( N_8 = 6 \) and the curve \( R \) is pointwise fixed by \( \sigma_8 \). On the other hand, by [ATST16, Proposition 2], \( N_{16} \) is bigger or equal to \( 2\alpha_{16} + 1 \). This implies that \( \alpha_{16} = 0 \), i.e. \( R \) is not pointwise fixed by \( \sigma_{16} \).

**Lemma 3.16.** If \( \text{Fix}(\sigma_2) \) is the union of a curve of genus 7 and 2 rational curves, then the fixed loci of \( \sigma_{16}, \sigma_8 \) and \( \sigma_4 \) are as follows:

\[
\text{Fix}(\sigma_{16}) = \{p_1, \ldots, p_4, q_1, q_2\} \sqcup R, \quad \text{Fix}(\sigma_8) = \{p_1, \ldots, p_4, q_1, q_2\} \sqcup R,
\]

\[
\text{Fix}(\sigma_4) = \{p_1, \ldots, p_4, q_1, q_2\} \sqcup R.
\]
**Proof.** Let $C_7$ (resp. $R, R'$) be the smooth curve of genus 7 (resp. rational curves) in $\text{Fix}(\sigma_2)$. We already know that one rational curve is fixed by $\sigma_4$, say $R$. Thus $\sigma_4$ fixes 2 points $q_1, q_2$ on $R'$ and 4 points $p_1, \ldots, p_4$ on $C_7$. This implies that the curves $R$ and $R'$ cannot be exchanged by $\sigma_{16}$ nor by $\sigma_8$ and that $\chi_{16} \geq 4$ and $\chi_8 \geq 4$.

By the Riemann-Hurwitz formula for $\sigma_8$ on $C_7$, either the four $p_i$’s are fixed by $\sigma_8$ or none of them is fixed by $\sigma_8$. This implies that either $\chi_8 = 4$ or $\chi_8 = 8$. Looking at Table 5, we find that we are left with the three possibilities of Table 6.

| $d_{16}$ | $d_8$ | $d_4$ | $d_2$ | $d_1$ | $\chi_{16}$ | $\chi_8$ | $\chi_4$ | $\chi_2$ |
|-------|-------|-------|-------|-------|-------------|----------|----------|----------|
| 2     | 0     | 0     | 2     | 4     | 4           | 8        | 8        | -8       |
| 2     | 0     | 0     | 1     | 5     | 6           | 8        | 8        | -8       |
| 2     | 0     | 0     | 0     | 6     | 8           | 8        | 8        | -8       |

**Table 6.**

In particular $\chi_8 = 8$ and $\{p_1, \ldots, p_4, q_1, q_2\} \subset \text{Fix}(\sigma_8)$. Moreover, by [ATS18, Proposition 2.2] we obtain that $2 + 4\alpha_8 = 6$, thus $\alpha_8 = 1$. This implies that $\sigma_8$ fixes pointwise the curve $R$.

By the Riemann-Hurwitz formula for $\sigma_{16}$ on $C_7$, either a) $\sigma_{16}$ fixes the four $p_i$’s and thus $\chi_{16} = 8$, or b) it does not fix any of them and $\chi_{16} = 4$. By [ATST16, Proposition 2, Remark 1.3] the cases $(N_{16}, \alpha_{16}) = (2, 1)$ and $(N_{16}, \alpha_{16}) = (8, 0)$ are impossible. Thus in case a) $\alpha_{16} = 1$ and $N_{16} = 4$, i.e. the fixed locus of $\sigma_{16}$ contains $p_1, \ldots, p_4, q_1, q_2$ and the curve $R$. On the other hand in case b) we have that $\alpha_{16} = 0$, i.e. $\sigma_{16}$ fixes exactly $q_1, q_2$ and two points on $R$. We now show that this case can not appear. By [ATST16, Remark 1.3] if $N_{16} = 4$, then $n_{3,16} = n_{7,16} = 1$ and $n_{8,16} = 2$. Observe that the points of type $A_{8,16}$ lie on a curve fixed by $\sigma_8$, thus they must be the two points on $R$. This implies that the points of type $A_{3,16}$ and $A_{7,16}$ are $q_1, q_2$. However, two isolated fixed points of $\sigma_{16}$ lying on an invariant smooth rational curve can not be of these types by the proof of [AS15, Lemma 4].

For the following examples, see [ATST16, Example 4.2].

**Example 3.17.** Consider the elliptic fibration defined by

$$y^2 = x^3 + t^2x + at^3 + t^{11}, \ a \in \mathbb{C}$$

with the order 16 automorphism $\sigma_{16}(x, y, t) = (\zeta_8 x, \zeta_{16}^2 y, \zeta_8 t)$. The action of $\sigma_{16}^*$ on the holomorphic two form $\omega_X = (dx \wedge dt)/2y$ is the multiplication by $\zeta_8$, thus $\sigma_{16}$ is purely non-symplectic. The fibration has a fiber of type $I_0^*$ over $t = 0$ and a fiber of type $II$ over $t = \infty$. The automorphism $\sigma_{16}$ fixes the central component of the fiber of type $I_0^*$, four points in the other components of the same fiber and two more fixed points in the fiber over $t = \infty$. This is an example of case a) in Proposition 3.14.
Example 3.18. Consider the plane sextic $B$ defined by
\[ F_6(x_0, x_1, x_2) = x_0(x_1^4x_2 + a_1x_1^5 + a_2x_1x_2^4 + a_3x_1^3x_2^2) = 0 \]
for general $a_1, a_2, a_3 \in \mathbb{C}$. Observe that $B$ is the union of a smooth plane quintic $C$ and a line $L$. Let $Y$ be the double cover of $\mathbb{P}^2$ branched along $B$, which can be defined by the equation $x_3^3 = F_6(x_0, x_1, x_2)$ in $\mathbb{P}(1, 1, 1, 3)$. The surface $Y$ has the order 16 automorphism
\[ \sigma_{16}(x_0, x_1, x_2, y) = (x_0, \zeta_8^7 x_1, \zeta_8^3 x_2, \zeta_16 y). \]
The surface $Y$ has 5 singular points of type $A_1$ over the intersection points of $C$ and $L$. Its minimal resolution $X$ is a K3 surface and $\sigma_{16}$ lifts to an automorphism $\tilde{\sigma}_{16}$ of $X$. The automorphism $\tilde{\sigma}_{16}$ has 4 fixed points: two of them over the points $(1,0,0,0)$ and $(0,1,0,0)$ and the other two in the exceptional divisor over $(0,0,1,0)$ (which is a singular point of $Y$). Thus this is an example of case b) in Proposition 3.14.

Proof of Theorem 1.2, order 16. Let $X$ be a K3 surface with a purely nonsymplectic automorphism $\sigma$ of order 16 such that $\dim(V^*) = 2$. By Proposition 3.14, $\text{Fix}(\sigma_{16})$ is either the union of a rational curve and 6 points or the union of 4 isolated points.

Case a). By Proposition 3.14 $\text{NS}(X) = S(\sigma_2) \cong U \oplus D_4$ for a very general K3 surface. In what follows we assume $X$ to be very general. By [Kon92, Lemma 2.1] or [PSS71, §3, proof of Corollary 3], $X$ has an elliptic fibration $\pi : X \rightarrow \mathbb{P}^1$ with a section $S$ and a reducible fiber of type $\tilde{D}_4 = I_0^6$. The curve $C_7$ fixed by the involution $\sigma_2$ has to be transverse to the fibers of $\pi$, since its genus is bigger than 1. Thus $\sigma_2$ induces the identity on the basis of the fibration. Since $\text{NS}(X) = S(\sigma_2)$, $\sigma_2^2$ is the identity on $\text{NS}(X)$, hence each smooth rational curve is invariant for $\sigma_2$. This implies that the section $S$ and the central component of the fiber of type $I_0^6$ are pointwise fixed by $\sigma_2$. Since a smooth fiber of $\pi$ must contain four fixed points for $\sigma_2$ and one of them is on $S$, then $C_7$ intersects it in 3 points. Applying [AS15, Lemma 5] with $x = [C_7]$, one concludes that the elliptic fibration $\pi$ is invariant under $\sigma_{16}$. The section $S$ corresponds to the curve $R'$ (see notation of Proposition 3.14) i.e. it is not fixed pointwise by $\sigma_{16}$, otherwise each fiber of $\pi$, including the smooth ones, would have an order 16 automorphism with a fixed point, which is impossible for an elliptic curve [Har77]. Thus $\sigma_{16}$ induces an automorphism of order 8 on the basis of $\pi$. This implies that $\sigma_8$ preserves each fiber of $\pi$ and acts on it as an involution with a fixed point.

Consider a Weierstrass equation for $\pi$ with respect to the section $S$:
\[ y^2 = x^3 + A(t)x + B(t), \quad t \in \mathbb{P}^1. \]
We can assume that the two invariant fibers of $\sigma_{16}$ are over $t = 0$ and $t = \infty$, and that the fiber of type $I_0^6$ is over $t = 0$. Since $\text{NS}(X) \cong U \oplus D_4$ the fiber $F_0$ over 0 is the only reducible fiber of $\pi$, moreover $24 - e(F_0)$ is divisible by 8. This implies that the fiber over $t = \infty$ is of type $II$. By [Mir89, Table IV.3.1], this implies that the vanishing order $v(\Delta)$ of $\Delta(t)$ at $t = 0$ is 6 and
at \( t = \infty \) is 2. Thus \( \Delta(t) = t^6 P(t) \) with \( P(0) \neq 0, \deg(P(t)) = 16 \). Moreover \( v(B(\infty)) = 1 \), thus \( B(t) = t^3 Q(t) \) with \( \deg(Q(t)) = 8 \). Since the action of \( \sigma_{16} \) on the basis of \( \pi \) has order 8 and the fibers over \( t = 0, \infty \) are preserved by \( \sigma_{16} \), then \( Q(t) = t^8 + a \) with \( a \in \mathbb{C} \). By [Mir89, Table IV.3.1], we have that \( A(t) = t^2 \). Moreover, \( \sigma_{16}(x,y,t) = (\zeta_8 x, \zeta_8^6 y, \zeta_8 t) \) and \( X \) belongs to the family in Example 3.17.

**Case b).** In this case \( \text{Fix}(\sigma_{16}) \) is the union of 4 isolated points, and \( S(\sigma_2) \cong U(2) \oplus D_4 \) by Proposition 3.14. As before we assume \( X \) to be very general, i.e. that \( \text{NS}(X) = S(\sigma_2) \). It is known that the surface \( X \) has a degree two morphism \( \pi : X \to \mathbb{P}^2 \) which is the minimal resolution of a double cover ramified along the union of a line \( \ell \) and a quintic \( C \) [ATST16, Section 4]. In particular, \( X \) has six \((-2)\)-curves, i.e. 5 exceptional divisors \( E_1, \ldots, E_5 \) over the points \( \ell \cap C \) and the proper transform \( E \) of (the double cover) of \( \ell \). It follows from Vinberg’s algorithm (see [Rou]) that these are the only \((-2)\)-curves of \( X \). This implies that the linear system of the divisor \( 2E + \sum_{i=1}^{5} E_i \), which is the one defining the morphism \( \pi \), is invariant for \( \sigma^* \). Thus the automorphism \( \sigma \) induces an automorphism \( \overline{\sigma} \) of \( \mathbb{P}^2 \) preserving the branch curve \( \ell \cup C \).

The involution \( \sigma_2 \) fixes a genus 6 curve and a rational curve \( R \). If the induced automorphism \( \overline{\sigma}_4 \) were an involution, it would fix a line and a point. This would imply that in \( \text{Fix}(\sigma_2) \) the maximum possible genus is 2, giving a contradiction. Thus \( \overline{\sigma}_2 \) is the identity on \( \mathbb{P}^2 \), \( \sigma_2 \) is the covering involution and \( R \) is the transform of the line \( \ell \) in the branch locus. Moreover, since \( R \) is contained in \( \text{Fix}(\sigma_4) \) and \( \text{Fix}(\sigma_8) \), \( \ell \) is fixed by \( \overline{\sigma}_4 \) and \( \overline{\sigma}_8 \). In addition, \( \overline{\sigma}_{16} \) has order 8 on \( \mathbb{P}^2 \) and it only fixes points, \( \overline{\sigma}_8 \) has order 4 and \( \overline{\sigma}_4 \) has order 2.

Assume that \( \ell \) is defined by \( x_0 = 0 \), thus

\[
\overline{\sigma}_8(x_0, x_1, x_2) = (ix_0, x_1, x_2), \quad \overline{\sigma}_4(x_0, x_1, x_2) = (-x_0, x_1, x_2).
\]

Since \( \overline{\sigma}_{16} \) only fixes points in \( \mathbb{P}^2 \), it is \( \overline{\sigma}_{16}(x_0, x_1, x_2) = (\zeta_8 x_0, x_1, -x_2) \) and the equation of \( X \) is obtained taking invariant monomials and recalling that we need the quintic to be smooth, otherwise \( \text{Fix}(\sigma_2) \) would not contain a genus 6 curve. Thus the equation of \( X \) is as in Example 3.18 and

\[
\sigma_{16}(x_0, x_1, x_2, y) = (\zeta_8 x_0, x_1, -x_2, \zeta_8^6 y) = (x_0, \zeta_8^4 x_1, \zeta_8^3 x_2, \zeta_8^1 y),
\]

where \( \zeta_{16} \) is a primitive 16-th root of unity with \( \zeta_{16}^6 = \zeta_8 \). Observe that \( \sigma_{16}^9 \) is the automorphism in Example 3.18. If all the coefficients of \( F_6 \) are non-zero, then one obtains the equation in Table 2 up to rescaling the variables.

### 3.6. Order 20.

**Proposition 3.19.** Let \( X \) be a K3 surface with a purely non-symplectic automorphism \( \sigma_{20} \) of order 20 such that \( \dim(V^\sigma) = 2 \). Then the fixed loci of \( \sigma_{20} \) and of its powers are as follows:

| \( \text{Fix}(\sigma_{20}) \) | \( \text{Fix}(\sigma_{10}) \) | \( \text{Fix}(\sigma_5) \) | \( \text{Fix}(\sigma_4) \) | \( \text{Fix}(\sigma_2) \) |
| \( \{p_1, p_2, p_3\} \) | \( \{p_1, \ldots, p_7\} \) | \( C_2 \cup \{p_1\} \) | \( \{p_1, \ldots, p_6\} \cup \{p_7\} \) | \( C_6 \cup \{p_7\} \) |
where \( g(C_i) = i \) for \( i = 2, 6 \) and \( g(R) = 0 \). Moreover \( d = (2, 0, 1, 0, 0, 2) \) and \( \text{NS}(X) = S(\sigma_2) \) for a very general such K3 surface \( X \).

**Proof.** Decomposing \( H^2(X, \mathbb{C}) \) as the direct sum of the eigenspaces of \( \sigma_{20} \) we obtain:

\[
22 = 8d_{20} + 4d_{10} + 4d_5 + 2d_4 + d_2 + d_1.
\]

Since \( d_{20} = 2 \), then \( d_{10} \) is either 0 or 1 and this gives 16 possibilities for the vector \( d \). Let \( \chi_i = \chi(\text{Fix}(\sigma_i)), i \in \{2, 4, 5, 10, 20\} \). By the topological Lefschetz fixed point formula we get

\[
\begin{align*}
\chi_{20} &= d_{10} - d_5 - d_2 + d_1 + 2, \\
\chi_{10} &= 2d_{10} - d_5 - 2d_4 + d_2 + d_1 + 2, \\
\chi_{5} &= -2d_{20} - d_{10} - d_5 + 2d_4 + d_2 + d_1 + 2, \\
\chi_{4} &= -4d_{10} + 4d_5 - d_2 + d_1 + 2, \\
\chi_{2} &= -8d_{20} + 4d_{10} + 4d_5 - 2d_4 + d_2 + d_1 + 2.
\end{align*}
\]

(8)

By (8) we compute \( \chi_5 \) for the 16 possible \( d \)'s and find that it is either \(-1 \) or \( 4 \). Lemma 3.20 and Lemma 3.21 study these two cases separately. Observe that, since \( d_{20} = 2 \) and \( \varphi(20) = 8 \), the Néron-Severi lattice of a very general \( X \) has rank \( 22 - 2 \cdot 8 = 6 \). Moreover \( S(\sigma_2) \subseteq \text{NS}(X) \) by Remark 2.2. On the other hand, since the fixed locus of \( \sigma_2 \) is the union of a curve of genus 6 and a rational curve, then \( \text{rk} S(\sigma_2) = 6 \) by [Nik79a], thus \( S(\sigma_2) = \text{NS}(X) \).

**Lemma 3.20.** If \( \chi_5 = -1 \), then the fixed loci of \( \sigma_{20}, \sigma_{10}, \sigma_5, \sigma_4, \sigma_2 \) are

\[
\begin{align*}
\text{Fix}(\sigma_{20}) &= \{p_1, p_2, p_3\}, \quad \text{Fix}(\sigma_{10}) = \{p_1, \ldots, p_6, p\}, \\
\text{Fix}(\sigma_5) &= C_2 \sqcup \{p\}, \quad \text{Fix}(\sigma_4) = \{p_1, \ldots, p_6\} \sqcup R, \quad \text{Fix}(\sigma_2) = C_6 \sqcup R.
\end{align*}
\]

**Proof.** By [AST11], if \( \chi_5 = -1 \), then \( \text{Fix}(\sigma_5) \) is the union of a smooth curve \( C \) of genus two and one point \( p \). This corresponds to the cases in Table 7.

In all these cases \( \chi_{10} = 7 \), so that \( \text{Fix}(\sigma_{10}) \) is the union of \( p \) and 6 points on \( C \). By the Riemann-Hurwitz formula, this implies that \( \sigma_{20} \) has two fixed points on \( C \), so that \( \text{Fix}(\sigma_{20}) \) consist of the union of three points. Moreover, in all these cases \( \chi_2 = -8 \) by Table 7, so that by [Nik79a] \( \text{Fix}(\sigma_2) \) is a) the union of a curve of genus 7 and two rational curves, or b) the union of a curve of genus six and a rational curve, or c) a genus 5 curve. In all cases \( \sigma_{20} \) acts with order 10 on the curve of positive genus, since otherwise either \( \sigma_{10} \) or \( \sigma_4 \) should contain such curve in its fixed locus, contradicting the previous remarks for \( \sigma_{10} \) and [AS15, Theorem 0.1].
In case a), $\sigma_{20}$ must fix exactly three points on the curve $C$ of genus 7 and exchange the two rational curves. By the Riemann-Hurwitz formula this implies that $\sigma_5$ fixes the same points on $C$ and $\sigma_4$ has exactly 8 fixed points on $C$ and exchanges the two rational curves. This is not possible since by [AS15, Theorem 0.1] the number of fixed points equals $2\alpha + 4$, where $\alpha = \sum_{C \in \text{Fix}(\sigma_i)} (1 - g(C))$.

In case b), $\sigma_{20}$ has exactly one fixed point on the genus six curve and two points on the rational curve $R$, while $\sigma_5$ has exactly five fixed points on the curve by the Riemann-Hurwitz formula. By the same formula $\sigma_4$ has 6 fixed points on $C$. By [AS15, Theorem 0.1] $R$ is fixed by $\sigma_4$. This case corresponds to the statement.

Case c) is impossible since, by the Riemann-Hurwitz formula, a curve of genus 5 can not have an order five automorphism with more than two fixed points (and $\sigma_5$ would have this property).

Lemma 3.21. If $\chi_5 = 4$, there are no admissible cases.

Proof. By [AST11], if $\chi_5 = 4$, then $\text{Fix}(\sigma_5)$ contains either four isolated points or an elliptic curve and four isolated points. In both cases $a_{1,5} = 3, a_{2,5} = 1$. Observe that points of type $A_{4,20}, A_{5,20}, A_{9,20}$ lie on a curve fixed by $\sigma_5$ while points of type $A_{i,20}$ with $i \in \{1, 2, 3, 6, 7, 8\}$ are isolated fixed points for $\sigma_5$. Since the action of $\sigma_{20}$ on $\text{Fix}(\sigma_5)$ has order 2 or 4, in both cases the point of type $A_{2,5}$ is fixed by $\sigma_{20}$ and $a_{2,20} + a_{7,20} = 1$ and $a_{1,20} + a_{3,20} + a_{6,20} + a_{8,20}$ is either 1 or 3.

If $\text{Fix}(\sigma_5)$ consists of four isolated points, $a_{4,20} + a_{5,20} + a_{9,20} = 0$ since there are no curves in $\text{Fix}(\sigma_5)$. A Magma computation shows that the holomorphic Lefschetz formula has no solutions satisfying these conditions. If $\text{Fix}(\sigma_5)$ consists of four isolated points and an elliptic curve $E$, by the Riemann-Hurwitz formula $E$ contains 0, 2 or 4 isolated points for $\sigma_{20}$ thus $a_{4,20} + a_{5,20} + a_{9,20} \in \{0, 2, 4\}$. The holomorphic Lefschetz formula has no solutions with these restrictions.

Example 3.22. Let $B$ be the plane sextic defined by

$$F_6(x_0, x_1, x_2) = x_0(x_1^5 + a_1x_2^5 + a_2x_0^2x_2^3 + a_3x_0^4x_2) = 0,$$

where $a_1, a_2, a_3 \in \mathbb{C}$ are general. Observe that $B$ is the union of a smooth plane quintic $C$ and a line $L$. Let $Y$ be the double cover of $\mathbb{P}^2$ branched along $B$, which can be defined by the equation $x_3^2 - F_6(x_0, x_1, x_2) = 0$ in $\mathbb{P}(1, 1, 1, 3)$. The surface $Y$ has the order 20 automorphism

$$\sigma_{20}(x_0, x_1, x_2, y) = (-x_0, \zeta_5 x_1, x_2, iy).$$

The surface $Y$ has 5 singular points of type $A_1$ over the intersection points of $C$ and $L$. Its minimal resolution $X$ is a K3 surface and $\sigma_{20}$ lifts to an automorphism $\tilde{\sigma}_{20}$ of $X$. The automorphism $\tilde{\sigma}_{20}$ has 3 fixed points over $(1, 0, 0), (0, 1, 0), (0, 0, 1)$. 
**Proof of Theorem 1.2, order 20.** Let $X$ be a K3 surface with a purely non-symplectic automorphism $\sigma_{20}$ of order 20. By Proposition 3.19 $\text{Fix}(\sigma_5)$ contains a curve $C_2$ of genus 2 and one point. The linear system $|C_2|$ defines a morphism $\varphi : X \to \mathbb{P}^2$ of degree two which contracts all smooth rational curves orthogonal to $C_2$. Since $\sigma$ leaves $C_2$ invariant, then it induces an automorphism $\bar{\sigma}$ of $\mathbb{P}^2$ which can be assumed to be diagonal.

Let $\sigma_2 = \sigma^{10}$ and assume it has order two. Thus its fixed locus is the union of a line and one point, so that $\text{Fix}(\sigma_2)$ contains a fixed curve of genus at most 2, contradicting the fact that $\sigma_2$ fixes a curve of genus 6. Thus $\sigma_2$ coincides with the covering involution of $\varphi$.

Now consider the automorphism $\sigma_4 = \sigma^5$, whose order is equal to two. Its fixed locus contains a line, we can assume it to be $L = \{x_0 = 0\}$ up to projectivities. By Proposition 3.19 the line $L$ must be a component of the branch curve $B$ of $\varphi$.

Finally, let $\sigma_5 = \sigma^4$. Since $\sigma_5$ has a fixed curve, then $\sigma_5$ must fix a line $L'$ which is not equal to the line fixed by $\sigma_4$, thus up to projectivities we can assume $L' = \{x_1 = 0\}$. In these coordinates

$$\sigma(x_0, x_1, x_2) = (-x_0, \zeta_5 x_1, x_2).$$

The branch curve $B$ is reduced, invariant for $\sigma$ and must contain the line $L$ as a component. This implies that its equation is as in Example 3.22. If all the coefficients of $F_6$ are non-zero, then one obtains the equation in Table 2 up to rescaling the variables. \[\square\]

**Remark 3.23.** It follows from the proof of Theorem 1.2, $n = 20$, that there are five smooth rational curves $R_1, \ldots, R_5$ in $X$, each intersecting at one point the two fixed curves $C_6$ and $R$ of $\sigma_2$. The classes of the curves $R, R_1, \ldots, R_5$ all belong to the invariant lattice $S(\sigma_2)$. Observe that the classes of

$$2R + R_1 + R_2 + R_3 + R_4, \quad 2R + R_1 + R_2 + R_3 + R_5, \quad R, \quad R_1, \quad R_2, \quad R_3,$$

generate a lattice $S$ isometric to $U(2) \oplus D_4$. Since $S$ is contained in $S(\sigma_2)$ and $\det(S(\sigma_2)) = \det(S) = -2^4$ by [AST11, Theorem 0.1], then $S = S(\sigma_2)$.

**3.7. Order 24.**

**Proposition 3.24.** Let $X$ be a K3 surface with a purely non-symplectic automorphism $\sigma = \sigma_{24}$ of order 24 such that $\dim(V^\sigma) = 2$. Then the fixed loci of $\sigma_{24}$ and some of its powers are as follows:

| Fix($\sigma_{24}$) | Fix($\sigma_{12}$) | Fix($\sigma_6$) |
|-----------------|-----------------|-----------------|
| $\{p_1, p_2, p_3, p_{12}, p_{13}\}$ | $\{p_1, p_2, p_3, p_{12}, p_{13}\}$ | $R_1 \cup \{p_1, \ldots, p_{11}\}$ |
| Fix($\sigma_3$) | Fix($\sigma_2$) |
| $C_4 \sqcup R_1 \sqcup \{p_1\}$ | $C_7 \sqcup R_1 \sqcup R_2$ |

where $g(C_i) = i$ for $i = 4, 7$ and $g(R_1) = g(R_2) = 0$. Moreover $\chi(\text{Fix}(\sigma_4)) = \chi(\text{Fix}(\sigma_8)) = 8$, $d = (2, 0, 0, 0, 0, 1, 0, 4)$ and $\text{NS}(X) = S(\sigma_2) \cong U \oplus D_4$ for a very general such K3 surface $X$. 
We obtain:

\[
\begin{align*}
22 &= 8d_{24} + 4d_{12} + 4d_8 + 2d_6 + 2d_4 + 2d_3 + d_2 + d_1.
\end{align*}
\]

Let \( \chi_i = \chi(\text{Fix}(\sigma_i)), i \in \{2, 3, 4, 6, 8, 12, 24\} \). By the topological Lefschetz fixed point formula we get

\[
\begin{align*}
\chi_{24} &= d_6 - d_3 - d_2 + d_1 + 2 \\
\chi_{12} &= 2d_{12} - d_6 - 2d_4 - d_3 + 2d_2 + d_1 + 2 \\
\chi_8 &= -2d_6 + 2d_3 - d_2 + d_1 + 2 \\
\chi_{6} &= 4d_{24} - 2d_{12} - 4d_8 - d_6 + 2d_4 - d_3 + 2d_2 + d_1 + 2 \\
\chi_4 &= -4d_{12} + 2d_6 - 2d_4 + 2d_3 + 2d_2 + d_1 + 2 \\
\chi_3 &= -4d_{24} - 2d_{12} + 4d_8 - d_6 + 2d_4 - 2d_3 + 2d_2 + d_1 + 2 \\
\chi_2 &= -8d_{24} + 4d_{12} - 4d_8 + 2d_6 + 2d_4 + 2d_3 + 2d_2 + d_1 + 2
\end{align*}
\]

Computing all possible values of the vector \( d \) one can see that \( \chi_3 \in \{0, -3, -6\} \).

Assume \( \chi_3 = 0 \). By [AS08] \( \text{Fix}(\sigma_3) \) is either the union of genus two curve and two isolated points or the union of a genus three curve, a smooth rational curve and two isolated points. Clearly \( \text{Fix}(\sigma_6) \subseteq \text{Fix}(\sigma_3) \) and in this case \( \chi_6 = 16 \) or 8. The first case is incompatible with the structure of the fixed locus of \( \sigma_3 \). If \( \chi_6 = 8 \), then the fixed locus of \( \sigma_3 \) must be the union of a genus three curve \( C \), a smooth rational curve \( R \) and two isolated points \( p, q \). The automorphism \( \sigma_6 \) fixes 4 points on \( C \) and \( p, q \). Moreover, it either fixes pointwise \( R \) or it has two isolated fixed points on it. Both cases are incompatible with [Dil12, Theorem 4.1] since the fixed points of \( \sigma_6 \) contained in the fixed curve of \( \sigma_3 \) are those of type \( A_{2,6} \) (of type \( \frac{1}{6}(3,4) \) in [Dil12]).

If \( \chi_3 = -6 \) we have \( \chi_6 = 10 \) and this can be seen to be incompatible with [Dil12, Theorem 4.1] with an argument similar to the previous one.

If \( \chi_3 = -3 \) and by [AS08] \( \text{Fix}(\sigma_3) \) is either the union of a curve of genus 3 and one point, or the union of a curve of genus 4, a smooth rational curve and one point. In these cases we have \( \chi_6 = 13 \), which excludes the first possibility for \( \text{Fix}(\sigma_3) \). Thus \( \text{Fix}(\sigma_3) \) is the union of a curve \( C \) of genus 4, a smooth rational curve \( R \) and one point \( p \). Using the Riemann-Hurwitz formula for \( \sigma_6 \) and the fact that \( \chi_6 = 13 \) we obtain that \( \sigma_6 \) fixes \( p \) and 10 points on \( C \). Moreover, by [Dil12, Theorem 4.1] the curve \( R \) is pointwise fixed by \( \sigma_6 \). In this case one computes that \( \chi_{12} \) is either 5 or 1, but the second case is not possible since \( \sigma_{12} \) either fixes pointwise or has two fixed points on \( R \). Thus \( \text{Fix}(\sigma_{12}) \) fixed \( p \), two points on \( C \) and it either fixes pointwise or has two fixed points on \( R \). A computation using the holomorphic Lefschetz formula shows that the first case does not occur. In this case one computes that \( \chi_{24} \in \{-1, 1, 3, 5, 7\} \). The only cases compatible with the structure
of $\text{Fix}(\sigma_{12})$ are $\chi_{24} = 3$ or 5. The first case is impossible by the Riemann-Hurwitz formula.

Assuming $\chi_3 = -3$, $\chi_6 = 13$, $\chi_{12} = 5$ and $\chi_{24} = 5$ we find two possible vectors $d = (2, 0, 0, 0, 1, 0, 4), (2, 0, 0, 1, 0, 0, 1, 3)$. For these cases $\chi_2 = -8$, $\chi_4 = 8$. Moreover $\chi_8 = 8$ in the first case and $2$ in the second case.

By [Nik79a] the fixed locus of $\sigma_2$ is either the union of a curve $C_7$ of genus 7 and two smooth rational curves $(R_1$ and $R_2)$, or the union of a curve $C_6$ of genus 6 and $R_1$. The latter is not possible by the Riemann-Hurwitz formula applied to $\sigma_6$ restricted to $C_6$. Since $\chi_4 = 8$, $\sigma_4$ must fix 4 points on $C_7$, two points on $R_1$ and it either fixes pointwise $R_2$ or it has two fixed points on it. This implies that $\text{Fix}(\sigma_8)$ contains isolated points and, at most, a smooth rational curve. Thus $\chi_8 \geq \chi_{24} = 5$, which excludes the case $d = (2, 0, 0, 1, 0, 0, 1, 3)$.

Finally, by [Nik79b, Theorem 4.2.2] or [AST11, Figure 1] the invariant lattice of $\sigma_2$ is isometric to $U \oplus D_4$. For a very general K3 surface we have $\text{rk} \text{NS}(X) = 22 - 2\varphi(24) = 6$. Moreover $S(\sigma_2) \subseteq \text{NS}(X)$ by Remark 2.2, thus $S(\sigma_2) = \text{NS}(X)$.

**Example 3.25.** Consider the elliptic surface with equation

$$y^2 = x^3 + t(t^4 - 1)(t^4 - a), \quad a \in \mathbb{C}.$$ 

For general $a \in \mathbb{C}$ it is a K3 surface and carries the order 24 automorphism

$$\sigma_{24}(x, y, t) = (\zeta_{12}x, \zeta_8y, it).$$

The action of $\sigma^*$ on the holomorphic two form $\omega_X = (dx \wedge dt)/2y$ is the multiplication by $\zeta_{12} \zeta_4 \zeta_8^{-1}$, thus $\sigma_{24}$ is purely non-symplectic. For general $a \in \mathbb{C}$ the elliptic fibration has a singular fiber $F_\infty$ of type $I_0^*$ over $t = \infty$ and 9 fibers of type $II$. The automorphism $\sigma_2$ fixes the section at infinity $R_1$, the genus 7 curve defined by $y = 0$ and the central component $R_2$ of the fiber $F_\infty$. The automorphism $\sigma_3$ fixes $R_1$, the curve of genus 4 defined by $x = 0$ and the intersection point $p_1$ between $R_2$ and the component of $F_\infty$ intersecting $R_1$. Observe that the remaining three components of $F_\infty$ are permuted by $\sigma_3$. The automorphism $\sigma_6$ fixes the 9 singular points $p_1, \ldots, p_{11}$ of the fibers of type $II$, the point $p_1$ and the intersection point $p_2$ between the fiber $F_\infty$ and the curve $x = 0$. Finally, the automorphisms $\sigma_{12}$ and $\sigma_{24}$ fix the singular point $p_3$ of the fiber $F_0$ of type $II$ over $t = 0$, the intersection points of $R_1$ with the fibers $F_0, F_\infty, p_1$ and $p_2$.

**Proof of Theorem 1.2, order 24.** By Proposition 3.24 the fixed locus of $\sigma_2$ is the union of a curve $C_7$ of genus 7 and two rational curves $R_1, R_2$. Moreover $\text{NS}(X) = S(\sigma_2) \cong U \oplus D_4$ for a very general $X$. Following the first part of the proof of Theorem 1.2 for order 16, we find that a very general $X$ has a jacobian elliptic fibration $\pi : X \to \mathbb{P}^1$ with a fiber of type $I_0^*$ such that $R_1$ can be assumed to be a section of $\pi$, $R_2$ is the central component of the reducible fiber, and $C_7$ intersects a general fiber in three points. It follows from [AS15, Lemma 5] with $x = [C_7]$ that $\pi$ is invariant for $\sigma_{24}$. Since the
fixed locus of $\sigma_3$ contains a curve $C_4$ of genus $> 1$, then each fiber of $\pi$ is invariant for $\sigma_3$. Moreover $\sigma_3$ fixes pointwise the curve $R_1$. Thus up to a coordinate change $\pi$ has Weierstrass equation of the form
\[ y^2 = x^3 + p(t), \]
where $\deg(p) \leq 12$, $\sigma_2(x,y,t) = (x,-y,t)$ and $\sigma_3(x,y,t) = (\zeta_3 x, y, t)$. Observe that $\sigma_8$ preserves $R_1$ but $\sigma_4 = \sigma_8^2$ does not fix it pointwisely, since otherwise $R_1$ would be contained in the fixed locus of $\sigma_{12}$, contradicting Proposition 3.24. Thus $\sigma_8$ induces an automorphism $\bar{\sigma}_8$ of order 4 on $\mathbb{P}^1$.

Up to a coordinate change, we can assume that $\bar{\sigma}_8(t) = it$. Since the reducible fiber of type $I^*_0$ must be preserved by $\sigma_8$, then we can assume it to be over $t = \infty$. By [Mir89] this implies that the $\deg(p) = 9$ (so that it has a triple root at infinity). Moreover, since its zero set is invariant for $\bar{\sigma}_8$, then $p(t) = t(t^4-a)(t^4-b)$ for some $a, b \in \mathbb{C}$. Finally, since $\sigma_8^4 = \sigma_2$ we can assume that $\sigma_8(x,y,t) = (-ix, \zeta_8 y, it)$. This implies that up to a coordinate change $X$ belongs to the family in Example 3.25.

4. Classification for order 22

We now provide a classification theorem of purely non-symplectic automorphisms $\sigma$ of order 22 on a K3 surface according to their fixed locus. Observe that, since $\varphi(22) = 10$, then $\dim(V_\sigma) \in \{1, 2\}$. The case when $\dim(V_\sigma) = 2$ has been studied in Section 3.

We recall that the fixed locus of any power $\sigma_i := \sigma^{22_i}$ of $\sigma$ is of the form
\[ C_i \sqcup R_1 \sqcup \ldots \sqcup R_{k_i} \sqcup \{p_1, \ldots, p_{N_i}\}, \]
where $g(C_i) = g_i$, $g(R_\ell) = 0$ for $\ell = 1, \ldots, k_i$.

Remark 4.1. As in [ACV20, Lemma 1.3], a straightforward computation using the holomorphic Lefschetz formula shows that a non-symplectic automorphism of order 22 is purely non-symplectic.

Theorem 4.2. Let $\sigma$ be a purely non-symplectic automorphism of order 22 of a complex K3 surface $X$. Then the invariants $(g_i, k_i, N_i)$ of the fixed locus of $\sigma_i := \sigma^{22_i}$, the vector $d = (d_2, d_{11}, d_2, d_{11})$ giving the dimensions of the eigenspaces of $\sigma^* \in H^2(X, \mathbb{C})$ and the Néron-Severi lattice of a very general K3 surface carrying an automorphism with such invariants are given by one of the rows of Table 8. Moreover, all cases in the table exist.

| $N_{22}$ | $g_{22}$ | $k_{22}$ | $N_{11}$ | $g_{11}$ | $k_{11}$ | $g_2$ | $k_2$ | $d$ | $\text{NS}$ |
|-------|--------|--------|--------|--------|--------|------|------|----|--------|
| A1    | 6      | 0      | 2      | 1      | 0      | 10   | 1    | (2,0,0,2) | $U$ |
| B1    | 11     | 0      | 0      | 11     | 0      | 0    | 5    | (1,0,1,11) | $U \oplus A_{10}$ |
| B2    | 9      | 0      | 0      | 11     | 0      | 0    | 5    | (1,0,2,10) | $U \oplus A_{10}$ |
| B3    | 5      | -      | 0      | 11     | 0      | 0    | 5    | (1,0,5,7) | $U \oplus A_{10}$ |

Table 8. Order 22
Proof. Let \( \sigma_{11} \) be the square of \( \sigma_{22} \). According to [AST11, Table 4], the fixed locus of \( \sigma_{11} \) is either a) the union of a smooth elliptic curve and 2 points, or b) the union of a rational curve and 11 points. In the first case \( m := \frac{1}{10}(22 - \text{rank} S(\sigma_{11})) = 2 \), while in the second case \( m = 1 \).

Recall that fixed points of type \( A_{10,22} \) lie on a curve in \( \text{Fix}(\sigma_{11}) \), while points of type \( A_{i,22}, A_{10-i,22} \) correspond to isolated points for \( \sigma_{11} \) of type \( A_{i,11}, i = 1, \ldots, 4 \). The Lefschetz holomorphic formula with the restrictions

\[
a_{5,22} \leq a_{5,11}, \quad a_{i,22} + a_{10-i,22} \leq a_{i,11}, \quad i = 1, 2, 3, 4
\]
gives the solutions as in Table 9, where we compute \( \chi_{22} \) and \( \chi_2 \) by (1).

| \( (a_{1,22}, a_{2,22}, \ldots, a_{10,22}) \) | \( \alpha \) | \( \chi_{22} \) | \( \chi_2 \) |
|---|---|---|---|
| A1 | (0, 0, 0, 1, 0, 0, 0, 0, 1, 4) | 0 | 6 | -16 |
| B1 | (3, 2, 1, 1, 1, 2, 1, 0, 0, 0) | 1 | 13 | 2 |
| B2 | (3, 2, 2, 1, 1, 0, 0, 0, 0, 0) | 1 | 11 | 0 |
| B3 | (0, 0, 0, 1, 1, 0, 0, 0, 1, 2) | 0 | 5 | -6 |

Table 9.

In case A1 we have \( d_1 + d_2 = 2 \) and \( d_{11} + d_{22} = 2 \) by [AST11, Table 4]. Since \( \chi_{22} = 6 \), then \( (d_{22}, d_{11}, d_2, d_1) = (2, 0, 0, 2) \) by (1). The description of the fixed locus of \( \sigma_2 \) is thus obtained as in the proof of Proposition 3.4.

We now study the possibilities for \( \text{Fix}(\sigma_{22}) \) when \( \text{Fix}(\sigma_{11}) \) is the union of a rational curve and 11 points. By [Nik79a] the fixed locus of the involution \( \sigma_2 \) is the union of a curve of genus \( g_2 \) and \( k_2 \) rational curves and \( \chi_2 = 2(1 - g_2 + k_2) \). Thus in case B1 one has \( (g_2, k_2) \in \{(0, 0), (1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\} \). The only admissible one is \( (g_2, k_2) = (5, 5) \), since otherwise, recalling that isolated points of \( \sigma_{22} \) lie on fixed curves for \( \sigma_2 \), one gets a contradiction with the Riemann-Hurwitz formula.

As for case B2, one has \( (g_2, k_2) \in \{(1, 0), (2, 1), (3, 2), (4, 3), (5, 4), (6, 5)\} \). The first four cases give a contradiction to the Riemann Hurwitz formula. Case \( (g_2, k_2) = (6, 5) \) is not admissible since by [FK92, Proposition V.2.14] a curve of genus 6 does not admit an automorphism of order 11 acting on it.

Similarly, in case B3 the possibilities are \( (g_2, k_2) \in \{(4, 0), (5, 1), (6, 2)\} \) and the only admissible one is \( (g_2, k_2) = (5, 1) \). The vector \( d = (d_{22}, d_{11}, d_2, d_1) \) is obtained in all cases by means of (1).

The Néron-Severi lattice of a very general K3 surface in case A1 has been given in Section 3.2. In the remaining cases, which have \( d_{22} = 1 \), the rank of the Néron-Severi lattice in the very general case is \( 22 - \varphi(22) = 12 \). Since the lattice \( S(\sigma_{11}) \) is a primitive sublattice of \( NS(X) \) by Remark 2.2 and has rank \( d_1 + d_2 = 12 \) in each case, then \( NS(X) = S(\sigma_{11}) \). By [OZ11] or [AST11, Sec. 7] the lattice \( S(\sigma_{11}) \) is isometric to \( U \oplus A_{10} \).

An example for case A1 has been given in Section 3.2. We now provide examples for the cases B1, B2, B3. \( \square \)
Example 4.3. (Case B1) Let \( X \) be the elliptic K3 surface whose elliptic fibration is given by
\[
y^2 = x^3 + t^7 x + t^5.
\]
The singular fibers of the fibration are \( II^* \) over \( t = 0 \), \( III \) over \( t = \infty \) and 11 fibers of type \( I_1 \). The automorphism
\[
\sigma_{22} : (x, y, t) \mapsto (\zeta_{22}^2 x, \zeta_{22}^3 y, \zeta_{22}^{10} t)
\]
is purely non-symplectic of order 22 since its action on the two form \( \frac{dx \wedge dt}{2y} \) is the multiplication by \(-\zeta_{11}\). The automorphism \( \sigma_{22} \) preserves the fibers over \( t = 0 \) and \( t = \infty \). In the fiber over \( t = 0 \), which is of type \( II^* \), it must fix the component of multiplicity 6 and has 8 isolated fixed points in the other components. In the fiber over \( t = \infty \) it fixes three isolated points. The involution \( \sigma_2 \) preserves each fiber of the elliptic fibration, thus it must fix \( R \), three more components of the fiber over \( t = 0 \), the section at infinity and the 3-section \( y = 0 \), which has genus 5. This corresponds to case B1.

Example 4.4. (Case B2) Let us consider the elliptic fibration
\[
y^2 = x^3 + t^5 x + t^2,
\]
The fibration has a fiber of type \( IV \) over \( t = 0 \), a fiber of type \( III^* \) over \( t = \infty \) and 11 fibers of type \( I_1 \). The automorphism
\[
\sigma_{22}(x, y, t) = (\zeta_{11}^8 x, -\zeta_{11} y, \zeta_{11} t)
\]
is purely non-symplectic of order 22 since its action on the two form \( \frac{dx \wedge dt}{2y} \) is the multiplication by \(-\zeta_{11}^8\). By [AST11, Example 7.4], \( \sigma_{11} \) has fixed locus \( R \cup \{p_1, \ldots, p_{11}\} \), where \( R \) is the central component of the fiber of type \( III^* \). The involution \( \sigma_2 \) maps \((x, y, t)\) to \((x, -y, t)\), thus it preserves each fiber. This implies that it fixes \( R \) and two more rational components of the fiber of type \( III^* \), as well as the section at infinity and the 3-section \( y = 0 \), whose genus is 5. This corresponds to case B2.

Example 4.5. (Case B3) We already observed in Section 3.2 that the elliptic K3 surface defined by
\[
y^2 = x^3 + ax + (t^{11} - 1), \quad a \in \mathbb{C}^*
\]
with the automorphism \( \sigma_{22} : (x, y, t) \mapsto (x : -y : \zeta_{11} t) \), is an example of case A. If \( a^3 = -\frac{27}{4} \), thus the fibration admits a singular fiber of type \( II \) over \( t = 0 \), \( I_{11} \) over \( t = \infty \) and 11 fibers of type \( I_1 \). The fixed locus of the automorphism \( \sigma_{11} \) is contained in the fibers over \( t = 0 \) and \( t = \infty \). Since it fixes 11 isolated points and one rational curve, then it must fix one of the components of the fibre of type \( I_{11} \), say \( R \), has 9 fixed points in the other components of the same fibers and two more fixed points in the fiber of type \( II \). The involution \( \sigma_2 \) fixes the section at infinity and the curve \( y = 0 \), which has genus 5. Moreover, \( \sigma_2 \) can not preserve each component of the fiber of type \( I_{11} \) by [AS15, Lemma 4]. Thus \( \sigma_2 \) acts on the fiber of type \( I_{11} \) as a reflection, without fixed components and with a unique invariant component. This corresponds to case B3.
Remark 4.6. By Section 2.3 the moduli space of K3 surfaces having a purely non-symplectic automorphism of order 22 whose invariants are as in cases B1, B2 or B3 is 0-dimensional, since \( \dim(V^\sigma) = 1 \). In fact, since \( \text{rk} T(X) = 10 = \varphi(22) \) and \( f^* \) has order 11 on \( \text{NS}(X) \), then it follows from [Bra19, Theorem 5.9] that there is a unique K3 surface \( X \) which carries the three types of non-symplectic automorphisms of order 22. Thus the K3 surfaces given in Examples 4.3, 5.4 and 4.5 are isomorphic.

5. Classification for order 15

We now provide a classification theorem of purely non-symplectic automorphisms \( \sigma \) of order 15 on a K3 surface according to their fixed locus. Observe that, since \( \varphi(15) = 8 \), then \( \dim(V^\sigma) \in \{1, 2\} \). The case when \( \dim(V^\sigma) = 2 \) has been studied in Section 3.

Theorem 5.1. Let \( \sigma \) be a purely non-symplectic automorphism of order 15 of a complex K3 surface \( X \). Then the invariants \((g_i, k_i, N_i)\) of the fixed locus of \( \sigma_i := \sigma^{15} \), the vector \( d = (d_{15}, d_5, d_3, d_1) \) giving the dimensions of the eigenspaces of \( \sigma^* \) in \( H^2(X, \mathbb{C}) \) and the Néron-Severi lattice of a very general K3 surface carrying an automorphism with such invariants are given by one of the rows of Table 10. Moreover, all cases in the table exist.

| \( N_{15} \) | \( g_{15} \) | \( k_{15} \) | \( N_5 \) | \( g_5 \) | \( k_5 \) | \( N_3 \) | \( g_3 \) | \( k_3 \) | \( d \) | \( \text{NS} \) |
|---|---|---|---|---|---|---|---|---|---|---|
| A1 | 5 | - | 0 | 1 | 2 | 0 | 2 | 2 | 0 | (2, 1, 0, 2) | \( U(3) \oplus A_2 \oplus A_2 \) |
| B1 | 7 | - | 0 | 4 | 1 | 0 | 1 | 4 | 1 | (2, 0, 1, 4) | \( H_5 \oplus A_4 \) |
| B2 | 7 | - | 0 | 4 | 1 | 0 | 6 | 0 | 2 | (1, 2, 0, 6) | \( U \oplus E_6 \oplus A_2^{\oplus 3} \) |
| B3 | 4 | - | 0 | 4 | 1 | 0 | 0 | 4 | 0 | (2, 0, 2, 2) | \( H_5 \oplus A_4 \) |
| D1 | 10 | 0 | 0 | 7 | 1 | 1 | 6 | 0 | 2 | (1, 1, 0, 10) | \( U \oplus E_6 \oplus A_2^{\oplus 3} \) |
| F3 | 9 | 0 | 0 | 10 | 0 | 1 | 4 | 2 | 2 | (1, 0, 2, 10) | \( H_5 \oplus A_4 \oplus E_8 \) |
| F7 | 12 | 0 | 0 | 10 | 0 | 1 | 5 | 2 | 3 | (1, 0, 1, 12) | \( H_5 \oplus A_4 \oplus E_8 \) |
| F8 | 5 | - | 0 | 10 | 0 | 1 | 2 | 2 | 0 | (1, 0, 4, 6) | \( H_5 \oplus A_4 \oplus E_8 \) |

Table 10. Order 15

Proof. According to [AST11], the fixed locus of the cube of \( \sigma_{15} \), i.e. \( \sigma_5 \), is the union of a smooth curve of genus \( g_5 \), \( k_5 \) rational curves and \( a_{1,5} + a_{2,5} \) isolated points, with \( g_5, k_5, a_{1,5}, a_{2,5} \) as in one of the lines of Table 11.

We recall that \( \alpha = \sum_{C \in \text{Fix}(\sigma_{15})} (1 - g(C)) \). In order to find all possibilities for \( \text{Fix}(\sigma_{15}) \), we will look for a solution \( a := (a_{1,15}, a_{2,15}, \ldots, a_{7,15}, \alpha) \) of the holomorphic Lefschetz formula compatible with the system of equations (2).

Remark 5.2. We recall that points of type \( A_{4,15}, A_{5,15} \) lie on a curve fixed by \( \sigma_5 \) and not by \( \sigma_{15} \). Thus if \( a_{4,15} + a_{5,15} > 0 \), there is at least a curve in \( \text{Fix}(\sigma_5) \backslash \text{Fix}(\sigma_{15}) \).
Remark 5.3. Observe that by [Bro91], a curve of genus 3 does not admit an automorphism of order 5. Thus if Fix(σ₃) contains a curve of genus 3, such curve is also fixed by σ₁₅.

We now analyze each line of the previous table separately.

- **Case A**: corresponds to χ(Fix(σ₅)) = −1. By equations (3) it follows that a₂₁₅ = a₇₁₅ = 0. The only solution of the holomorphic Lefschetz formula with this property is a = (0, 0, 1, 2, 2, 0, 0, 0). In particular χ(Fix(σ₁₅)) = 5. It follows from equations (2) that d = (2, 1, 0, 2). The proof thus follows as in the proof of Proposition 3.6.

- **Case B**: corresponds to χ(Fix(σ₅)) = 4, i.e. Fix(σ₅) is the disjoint union of a smooth curve of genus one and 4 points. By [AST11, Example 5.6] X has an elliptic fibration π : X → ℙ¹ which can be defined by a Weierstrass equation of the form

  \[ y^2 = x^3 + (t^5 + \alpha)x + (t^{10} + \beta t^5 + \gamma), \quad \alpha, \beta, \gamma \in \mathbb{C}, \]

where σ₅(x, y, t) = (x, y, ζ₅t). The automorphism σ₅ fixes pointwise the smooth fiber F₀ over t = 0 and leaves invariant the fiber F∞ over t = ∞, which contains 4 fixed points. This property and the fact that 24 − e(F∞) must be divisible by 5, imply that F∞ is of Kodaira type IV, i.e. the union of three smooth rational curves intersecting transversally at one point. Observe that the elliptic fibration π is invariant for σ₃, since the smooth fiber over t = 0 is invariant for σ₃, and thus the same holds for the associated linear system. Moreover σ₃ must preserve all fibers of π, since otherwise 15 should divide 24 − e(F∞) = 20, a contradiction. The remaining singular fibers of π, considering the fact that they are preserved by σ₃ (thus J = 0) and that 24 − e(F∞) = 20, are either 5 fibers of type IV or 10 fibers of type II.

By the holomorphic Lefschetz formula and equations (3) we find that either a = (0, 1, 0, 0, 3, 0, 0, 0) or a = (0, 0, 0, 0, 3, 3, 1, 0).

If a = (0, 1, 0, 0, 3, 0, 0, 0), then it follows from equations (2) that either d = (2, 0, 2, 2) or d = (1, 2, 1, 4). The first case has been considered in the proof of Proposition 3.6 (case B₃). In the second case by (2) and

|   | a₁₅ | a₂₁₅ | g₅ | k₅ | m   |
|---|-----|------|----|----|-----|
| A | 1   | 0    | 2  | 0  | 5   |
| B | 3   | 1    | 1  | 0  | 4   |
| C | 3   | 1    | -  | -  | 4   |
| D | 5   | 2    | 1  | 1  | 3   |
| E | 5   | 2    | 0  | 0  | 3   |
| F | 7   | 3    | 0  | 1  | 2   |
| G | 9   | 4    | 0  | 2  | 1   |

Table 11. Fixed locus of σ₅
[AS08, Table 1], \( \chi_3 = 9 \) and the fixed locus of \( \sigma_3 \) contains at least two curves. We now exclude this case.

The automorphism \( \sigma_{15} \) fixes four points: three of them lie on the unique curve \( F_0 \) fixed by \( \sigma_5 \) and the other one is an isolated fixed point for \( \sigma_5 \). By the previous description, it follows that \( \sigma_3 \) must fix the center of the fiber \( F_\infty \) and permutes the other three fixed points of \( \sigma_5 \) on it (and thus the three components of the fiber \( F_\infty \)). Moreover, being of types \( A_{2,15} \) and \( A_{5,15} \), the fixed points of \( \sigma_{15} \) are all contained in a curve \( C \) fixed by \( \sigma_3 \). Since \( C \) passes through the center of the fiber \( F_\infty \), then it is connected and by the Riemann-Hurwitz formula it is the unique fixed curve of \( \sigma_3 \) which is transversal to the fibers of \( \pi \). On the other hand, \( \sigma_3 \) can not fix a curve \( R \) contained in a fiber of \( \pi \), since the other singular fibers are either of type \( II \), or of type \( IV \), and in both cases \( R \) would intersect \( C \), a contradiction. Thus \( \sigma_3 \) fixes at most one (connected) curve, so that the case \( d = (1,2,1,4) \) is not possible.

If \( a = (0,0,0,0,3,3,1,0) \), then it follows from equations (2) that either \( d = (2,0,1,4) \) or \( d = (1,2,0,6) \). If \( d = (2,0,1,4) \), then by (2) and [AS08, Table 1], \( \chi_3 = -3 \) and the fixed locus of \( \sigma_3 \) consists either of the disjoint union of a genus three curve and one point or the disjoint union of a curve of genus four, a rational curve and one point. The first case is not possible by Remark 5.3.

If \( d = (1,2,0,6) \), then by (2) and [AS08, Table 1], \( \chi_3 = 12 \) and the fixed locus of \( \sigma_3 \) consists either of the union of three disjoint rational curves and 6 points or the disjoint union of a curve of genus one, three rational curves and six points. We now exclude the second case. Observe that in this case \( \sigma_{15} \) fixes three points on \( F_0 \) and four isolated points in the fiber \( F_\infty \). Six of these points are contained in a curve fixed by \( \sigma_3 \), which will intersect each fiber of \( \pi \) at three points counting multiplicity. The same argument as before shows that \( \sigma_3 \) cannot fix a curve contained in a fiber of \( \pi \). Thus \( \sigma_3 \) fixes at most three (connected) curves.

To conclude, the only possible cases have \( a = (0,0,0,0,3,3,1,0) \) and either \( d = (2,0,1,4) \) with \( \sigma_3 \) fixing a genus four curve, a rational curve and one point (case \( B_1 \)), or \( d = (1,2,0,6) \) with \( \sigma_3 \) fixing three smooth rational curves and six points (case \( B_2 \)).

- **Case C**: in this case \( \sigma_5 \) fixes exactly four points, more precisely \( a_{1,5} = 3 \) and \( a_{2,5} = 1 \). As before, by the holomorphic Lefschetz formula one obtains that either

\[
   a = (0,1,0,0,3,0,0,0) \quad \text{or} \quad a = (0,0,0,0,3,3,1,0).
\]

In both cases \( a_{4,15} + a_{5,15} > 0 \), thus this case is not possible by Remark 5.2.

- **Case D**: in this case the fixed locus of \( \sigma_5 \) contains an elliptic curve, a smooth rational curve \( R \) and 7 isolated fixed points, with \( a_{1,5} = 5, a_{2,5} = 2 \). The holomorphic Lefschetz formula with the restrictions of (3) gives four
solutions for the vector $a$:

\[(10) \quad (0, 0, 0, 3, 3, 1, 0), (0, 0, 1, 2, 2, 0, 0, 0), (0, 1, 0, 0, 3, 0, 0, 0), (3, 2, 2, 3, 0, 0, 0, 1)\]

The only one compatible with equations in (2) is $a = (3, 2, 2, 3, 0, 0, 0, 1)$. By Remark 5.2 a solution with $\alpha = 1$ means that only $R$ is fixed by $\sigma_{15}$. By (2), this gives $\chi_3 = 9$. According to [AS08, Table 1], there are two possibilities for $\text{Fix}(\sigma_3)$:

**D1** disjoint union of 3 smooth rational curves and 6 points;

**D2** disjoint union of an elliptic curve, 3 smooth rational curves and 6 points.

We now show that case D2 is not possible. Let

$$\text{Fix}(\sigma_3) = E \cup R_1 \cup R_2 \cup R_3 \cup \{p_1, p_2, \ldots, p_6\}$$

and consider the elliptic fibration $\pi : X \to \mathbb{P}^1$ defined by the linear system $|E|$. The automorphism $\bar{\sigma}_3$ induced by $\sigma_3$ on $\mathbb{P}^1$ is not the identity, since otherwise $\sigma_3$ should act on the general fiber of $\pi$ either as a translation (which is impossible since $\sigma_3$ is non-symplectic) or with fixed points (impossible, since otherwise $\sigma_3$ should fix a curve which is transverse to all fibers, and thus intersecting $E$). Thus $\bar{\sigma}_3$ has order three and fixes two points in $\mathbb{P}^1$, one of them corresponding to the fiber $E$. The smooth rational curves and the isolated points fixed by $\sigma_3$ must be components of the other invariant fiber. This implies that such fiber is of type $I_6^{\ast} = \tilde{D}_{10}$.

Since the curve $E$ is preserved by $\sigma_5$, thus the fibration $\pi$ is preserved too. The fixed locus of $\sigma_5$ contains a curve of genus one $E'$. The curve $E'$ can not be transverse to the fibers of $\pi$, since otherwise the general fiber of $\pi$ would have an order five automorphism with a fixed point, which is impossible by [Har77, Corollary 4.7, IV]. Thus $E'$ is one of the fibers of $\pi$. A similar reasoning to the one used for $\sigma_3$ implies that $\sigma_5$ induces an order 5 automorphism of $\mathbb{P}^1$, thus it preserves exactly two fibers of $\pi$. Observe that $\sigma_5$ must preserve both $E$, since it commutes with $\sigma_3$, and the fiber of type $I_6^{\ast} = \tilde{D}_{10}$, since an elliptic fibration of a K3 surface can not have five fibers of this type (the Euler number of the fiber is 12). This implies that $E = E'$, thus $E$ would be a fixed curve of $\sigma_{15}$, a contradiction.

- **Case E**: as in the previous case, $a_{1,5} = 5, a_{2,5} = 2$ and the holomorphic Lefschetz formula with the restrictions of (3) has the four solutions of (10). Since in each case $a_{4,15} + a_{5,15} > 0$, then by Remark 5.2 the only curve fixed by $\sigma_5$ is not fixed by $\sigma_{15}$ and $\alpha = 0$. For each one of the three possibles $a$'s with $\alpha = 0$, the system (2) has no solutions. Thus there are no $\sigma_{15}$ such that $\sigma_5$ has invariants as in case E.

- **Case F**: in this case $\text{Fix}(\sigma_5)$ contains two rational curves $R_1, R_2$ and ten points with $a_{1,5} = 7, a_{2,5} = 3$. The holomorphic Lefschetz formula with the restrictions of (3) gives nine solutions, all of them with $\alpha = 0$ or 1. Thus at most one of the two curves $R_i$ is contained in $\text{Fix}(\sigma_{15})$. 
If \( \text{Fix}(\sigma_{15}) \) contains a rational curve, then \( \alpha = 1 \) and combining the nine solutions of the Lefschetz formula with (2) one gets the possibilities F1-F7 of Table 12. If \( \text{Fix}(\sigma_{15}) \) only contains points, then \( \alpha = 0 \) and by (2) we get possibilities F8 and F9.

| \( a_{1,5} \) | \( a_{2,5} \) | \( a_{1,3} \) | \( g_3 \) | \( k_3 \) | \( a_{1,15} \) | \( a_{2,15} \) | \( a_{3,15} \) | \( a_{4,15} \) | \( a_{5,15} \) | \( a_{6,15} \) | \( a_{7,15} \) | \( \alpha \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| F1 | 7 | 3 | 4 | 0 | 0 | 3 | 3 | 1 | 1 | 1 | 0 | 0 | 1 |
| F2 | 7 | 3 | 4 | 1 | 1 | 3 | 3 | 1 | 1 | 1 | 0 | 0 | 1 |
| F3 | 7 | 3 | 4 | 2 | 2 | 3 | 3 | 1 | 1 | 1 | 0 | 0 | 1 |
| F4 | 7 | 3 | 4 | 3 | 3 | 3 | 3 | 1 | 1 | 1 | 0 | 0 | 1 |
| F5 | 7 | 3 | 5 | 0 | 1 | 3 | 2 | 1 | 1 | 1 | 3 | 1 | 1 |
| F6 | 7 | 3 | 5 | 1 | 2 | 3 | 2 | 1 | 1 | 1 | 3 | 1 | 1 |
| F7 | 7 | 3 | 5 | 2 | 3 | 3 | 2 | 1 | 1 | 1 | 3 | 1 | 1 |
| F8 | 7 | 3 | 2 | 2 | 0 | 0 | 0 | 1 | 2 | 2 | 0 | 0 | 0 |
| F9 | 7 | 3 | 2 | 3 | 1 | 0 | 0 | 1 | 2 | 2 | 0 | 0 | 0 |

Table 12. Case F

By Remark 5.3 we exclude cases F4 and F9.

Case F1 has to be excluded for the following reason: the total number of fixed points for \( \sigma_{15} \) is 9 and \( \sigma_{15} \) fixes a rational curve. Thus, \( a_{2,15} + a_{3,15} + a_{5,15} + a_{6,15} = 5 \) of the isolated fixed points for \( \sigma_{15} \) lie on curves fixed by \( \sigma_3 \). However, \( \text{Fix}(\sigma_3) \) contains just one rational curve, which is fixed by \( \sigma_{15} \), giving a contradiction.

Case F2 has to be excluded for the following reason: \( \sigma_{15} \) acts as an automorphism of order 5 on the elliptic curve in \( \text{Fix}(\sigma_3) \) and it contains fixed points, which is not possible by [Har77, Corollary 4.7, IV]. Case F6 is analogous.

In case F5, the total number of fixed points for \( \sigma_{15} \) is 12: 5 of them are isolated for \( \sigma_3 \), thus 7 points should lie on the rational curve in \( \text{Fix}(\sigma_3) \setminus \text{Fix}(\sigma_{15}) \). This is not possible by the Riemann-Hurwitz formula.

- **Case G:** in this case \( \text{Fix}(\sigma_5) \) contains three rational curves and all solutions of the holomorphic Lefschetz formula with the restrictions of (3) have \( \alpha = 0 \) or 1. Thus at most one of the three rational curves in \( \text{Fix}(\sigma_5) \) is contained in \( \text{Fix}(\sigma_{15}) \). Checking (2) for all solutions in both cases \( \alpha = 0,1 \) we find no solutions. Thus there are no possible \( \sigma_{15} \) such that \( \text{Fix}(\sigma_5) \) is as in case G.

The Néron-Severi lattice of a very general K3 surface in cases A1, B1 and B3 has been given in Section 3.3. In the remaining cases, which have \( d_{15} = 1 \), the rank of the Néron-Severi lattice in the very general case is \( 22 - \varphi(8) = 14 \). In the cases B2 and D1 we have that the rank of \( S(\sigma_3) \) is \( d_1 + 4d_5 = 14 \). Since \( S(\sigma_3) \) is a primitive sublattice of \( \text{NS}(X) \) by Remark 2.2, we conclude that \( \text{NS}(X) = S(\sigma_3) \). By [AS08] the lattice \( S(\sigma_3) \) in the
two cases is isometric to $U \oplus E_6 \oplus A_2 \oplus 3$. A similar argument in the cases $E_3, F_7, F_8$ shows that for a very general $X$ we have $\text{NS}(X) = S(\sigma_5)$ and that it is isometric to $H_5 \oplus A_4 \oplus E_8$ by [AST11]. □

In the following we will provide Examples for all cases collected in Table 10, thus completing the proof of Theorem 5.1. Examples of cases A1, B1 and B3 can be found in Section 3.3.

Example 5.4. (Case B2) The elliptic K3 surface with Weierstrass equation
\[ y^2 = x^3 + (t^5 - 1)^2 \]
has six fibers of type IV, over $t = \infty$ and over the zeroes of $t^5 - 1$. It carries the order 15 automorphism
\[ \sigma_{15} : (x, y, t) \mapsto (\zeta_3 x, y, \zeta_5 t). \]
The fixed locus of $\sigma_5$ is contained in the union of the smooth fiber over $t = 0$ and in the fiber over $t = \infty$. The fixed locus of $\sigma_3$ contains the section at infinity, the two sections defined by $x = y \pm (t^5 - 1) = 0$ and the six centers of the fibers of type IV.

Example 5.5. (Case D1) This surface appears in [Bra19]. Let $X$ be the elliptic K3 surface with Weierstrass equation
\[ y^2 = x^3 + t^5 x + 1, \]
The fibration has one fiber of type $III^* = \tilde{E}_7$ over $t = \infty$ and 15 fibers of type $I_1$. It carries the order 15 automorphism
\[ \sigma_{15} : (x, y, t) \mapsto (\zeta_{15}^{10} x, y, \zeta_{15} t). \]
The automorphism $\sigma_5 = \sigma_{15}^3$ fixes the smooth fiber $E$ over $t = 0$, the smooth rational curve of multiplicity 4 of the fiber over $t = \infty$ and 7 isolated points in the same reducible fiber. Thus the invariants of $\sigma_5$ are $(g_5, k_5) = (1, 1)$, which corresponds to case D. The elliptic curve $E$ is not fixed by $\sigma_3 = \sigma_5^3 : (x, y, t) \mapsto (\zeta_3 x, y, \zeta_3 t)$. The automorphism $\sigma_3$ fixes three smooth rational curves and 3 isolated points in the fiber over $t = \infty$, and 3 points in the curve $E$.

Example 5.6. (Case F3) Let $Y$ be the double cover of $\mathbb{P}^2$ defined by the following equation in $\mathbb{P}(1, 1, 1, 3)$:
\[ y^2 = x_2(x_0^2x_1^3 + x_2^5 + x_0^5). \]
The branch sextic $B$ is the union of a line $L$ and a quintic curve $Q$. The surface $Y$ has four rational double points: one point of type $D_7$ at $(0, 1, 0, 0)$ and three points of type $A_1$ at $(-\zeta_3^i, 1, 0, 0)$, for $i = 0, 1, 2$. The minimal resolution of $Y$ is a K3 surface $X$. The surface has the order 15 automorphism
\[ \sigma_{15} : (x_0, x_1, x_2, y) \mapsto (x_0, \zeta_3 x_1, \zeta_5 x_2, \zeta_3^2 y). \]
We will denote by $\tilde{\sigma}_{15}$ the lifting of $\sigma_{15}$ to $X$. The automorphism $\sigma_3$ fixes the genus two curve $C_2$ defined by $x_1 = 0$ and the singular point $(0, 1, 0, 0)$. Thus $\tilde{\sigma}_3$ fixes the proper transform of $C_2$ and the union of two components
and four isolated points in the exceptional divisor of type $D_7$. Thus we are in case F3.

**Example 5.7.** (Case F7) Let $Y$ be the double cover of $\mathbb{P}^2$ defined by the following equation in $\mathbb{P}(1,1,1,3)$:

$$y^2 = x_2(2x_2^5 + x_1^5 + x_0^3x_1x_2).$$

The branch sextic $B$ is the union of a line $L$ and a quintic curve $Q$. The surface $Y$ has a rational double point of type $D_{10}$ at $(1,0,0,0)$. The minimal resolution of $Y$ is a K3 surface $X$. The surface has the order 15 automorphism

$$\sigma_{15}: (x_0, x_1, x_2, y) \mapsto (\zeta_5^2x_0, \zeta_3^7x_1, \zeta_3^2x_2, y).$$

We will denote by $\tilde{\sigma}_{15}$ the lifting of $\sigma_{15}$ to $X$. The automorphism $\sigma_3$ fixes the genus two curve $C_2$ defined by $x_0 = 0$ and the point $(1,0,0,0)$. Thus $\tilde{\sigma}_3$ fixes the proper transform of $C_2$ and the union of three components and five isolated points in the exceptional divisor of type $D_{10}$. Thus we are in case F7.

**Example 5.8.** (Case F8) Let $Y$ be the double cover of $\mathbb{P}^2$ defined by the following equation in $\mathbb{P}(1,1,1,3)$:

$$y^2 = x_0^5x_1 + (x_1^3 - x_2^3)^2.$$ 

The surface $Y$ has three rational double points of type $A_4$ at $(0,1,\zeta_3^i,0)$, with $i = 0,1,2$. The minimal resolution of $Y$ is a K3 surface $X$. The surface has the order 15 automorphism

$$\sigma_{15}: (x_0, x_1, x_2, y) \mapsto (\zeta_3x_0, x_1, \zeta_3x_2, y).$$

We will denote by $\tilde{\sigma}_{15}$ the lifting of $\sigma_{15}$ to $X$. The automorphism $\sigma_3$ fixes the genus two curve $C_2$ defined by $x_2 = 0$ and the smooth points $(0,0,1,\pm 1)$. Thus we are either in case F8 or in case A1. The automorphism $\sigma_5$ fixes the two smooth rational curves defined by $x_0 = y \pm (x_1^3 - x_2^3) = 0$ and the point $(1,0,0,0)$. Thus its lifting $\tilde{\sigma}_5$ fixes two smooth rational curves, so we are in case F8.

**Remark 5.9.** By Section 2.3 the moduli space of K3 surfaces having a purely non-symplectic automorphism of order 15 whose invariants are as in cases B2, D1, F3, F7 or F8 is 0-dimensional, since $\dim(V^\sigma) = 1$. In cases B2 and D1 the isometry $f^*$ has order 5 on $NS(X)$, while in cases F3, F7, F8 it has order 3. Moreover, in all cases $\operatorname{rk} T(X) = 8 = \varphi(15)$. It follows from [Bra19, Theorem 5.9] that there is a unique K3 surface $X$ which carries purely non-symplectic automorphisms of order 22 of types B2 and D1, and a unique K3 surface carrying automorphisms of types F3, F7, F8. Thus the K3 surfaces given in Examples 5.4 and 5.5 are isomorphic, and the same is true for the K3 surfaces given in Examples 5.6, 5.7, 5.8.
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Departamento de Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile

Email address: martebani@udec.cl

Departamento de Matemática y Estadística, Universidad de la Frontera, Temuco, Chile

Email address: paola.comparin@ufrontera.cl

Departamento de Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile

Email address: mariaevaldes@udec.cl