COMMUTATORS OF CERTAIN FRACTIONAL TYPE OPERATORS
WITH HÖRMANDER CONDITIONS, ONE-WEIGHTED AND
TWO-WEIGHTED INEQUALITIES

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Abstract. In this paper we study the commutators of fractional type integral operators. This operators are given by kernels of the form

\[ K(x, y) = k_1(x - A_1y)k_2(x - A_2y) \ldots k_m(x - A_my), \]

where \( A_i \) are invertibles matrices and each \( k_i \) satisfies a fractional size condition and generalized fractional Hörmander condition. We obtain weighted Coifman estimates, weighted \( L^p(w^p) \)-\( L^q(w^q) \) estimates and weighted BMO estimates. We also give a two weight strong estimate for pair of weights of the form \((u, Su)\) where \( u \) is an arbitrary non-negative function and \( S \) is a maximal operator depending on the smoothness of the kernel \( K \). For the singular case we also give a two-weighted endpoint estimate.

1. Introduction

In [22], Ricci and Sjögren obtained the \( L^p(\mathbb{R}, dx) \) boundedness, \( p > 1 \), for a family of maximal operators on the three dimensional Heisenberg group. Some of these operators arise in the study of the boundary behavior of Poisson integrals on the symmetric space \( SLR^3/SO(3) \). To get the principal result, they studied the boundedness on \( L^2(\mathbb{R}) \) of the operator

\[ T_\alpha f(x) = \int_\mathbb{R} |x - y|^{-\alpha} |x + y|^\alpha f(y)dy, \]
for \( 0 < \alpha < 1 \). Later, in [13], Godoy and Urciuolo studied a generalization of (1.1) for \( \mathbb{R}^n \).

During the last years, several authors studied operators such that are generalizations of (1.1) of the following form, let \( 0 \leq \alpha < n \) and \( m \in \mathbb{N} \). For \( 1 \leq i \leq m \), let \( A_i \) be matrices such that satisfy

\( (H) \quad A_i \) is invertible and \( A_i - A_j \) is invertible for \( i \neq j, 1 \leq i, j \leq m \).

For any locally integrable bounded function \( f, f \in L_\text{loc}^\infty(\mathbb{R}^n) \), we define

\[ T_{\alpha,m} f(x) = \int_{\mathbb{R}^n} K(x, y)f(y)dy, \]
where

\[ K(x, y) = k_1(x - A_1y)k_2(x - A_2y) \ldots k_m(x - A_my). \]

They consider particular functions \( k_i \) and studied the operator in different context: weighted Lebesgue and Hardy spaces with constant and variable exponent, also the endpoint estimates and boundedness in BMO and weighted BMO. See for example [10, 12, 14, 15, 23, 24, 26].

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These operators generalized classical operators as $I_\alpha$, the fractional integral operator, and the rough fractional and singular operators. In several cases these type of operators are not bounded in $H^p$, but instead are bounded from $H^p$ into $L^q$, $0 < p < 1$ and some $q$ (see [27, 28]). In the case of $\alpha = 0$, $T_{0,m}$ behaves like a singular integral operator. If $0 < \alpha < n$, $m = 1$, $A_1 = I$ and $k_1(x - A_1y) = \frac{1}{|x - y|^n}$ then $T_{0,1} = I_\alpha$.

In [23, 24] and [25], Urciuolo and the second author consider each $k_i$ as a rough fractional kernel, then each $k_i$ satisfies a $L^{n,r_\alpha}$-Hörmander regular condition, $k_i \in H_{\alpha,r_i}$, that is, for all $x \in \mathbb{R}^n$ and $R > |x|$

$$\sum_{m=1}^{\infty} (2^m R)^{n-\alpha}\|(K_\alpha(\cdot - x) - K_\alpha(\cdot))\chi_{B(x,2^m+1 R)\setminus B(x,2^m R)}\|_{r_i,B(x,2^m R)} < \infty.$$

More recently, in [16], we analyzed operators of the form (1.2) with conditions of regularity more generals that the $L^{n,r_\alpha}$-Hörmander condition and a fractional size condition. For the definitions of this conditions recall that a function $\Psi : [0, \infty) \to [0, \infty)$ is said to be a Young function if $\Psi$ is continuous, convex, no decreasing and satisfies $\Psi(0) = 0$ and $\lim_{t \to \infty} \Psi(t) = \infty$.

For each Young function $\Psi$ we can induce an average of the Luxemburg norm of a function $f$ in the ball $B$ defined by

$$\|f\|_{\Psi,B} := \inf \left\{ \lambda > 0 : \frac{1}{|B|} \int_B \Psi \left( \frac{|f|}{\lambda} \right) \leq 1 \right\},$$

and a fractional maximal operator $M_{\alpha,\Psi}$ defined by, given $f \in L^{1}_{\text{loc}}(\mathbb{R}^n)$ and $0 \leq \alpha < n$,

$$M_{\alpha,\Psi}f(x) := \sup_{B \ni x} |B|^{\alpha/n} \|f\|_{\Psi,B}.$$

Now, we present the fractional size condition and a generalized fractional Hörmander condition. For more details see [3] or [11].

Let $\Psi$ be a Young function and let $0 \leq \alpha < n$. Let us introduce some notation: $|x| \sim s$ means $s < |x| \leq 2s$ and we write

$$\|f\|_{\Psi,|x| \sim s} = \|f\chi_{|x| \sim s}\|_{\Psi, B(0,2s)}.$$

The function $K_\alpha$ is said to satisfies the fractional size condition, if there exists a constant $C > 0$ such that

$$\|K_\alpha\|_{\Psi,|x| \sim s} \leq Cs^{\alpha-n}.$$

In this case we denote $K_\alpha \in S_{\alpha,\Psi}$. When $\Psi(t) = t$ we write $S_{\alpha,\Psi} = S_\alpha$. Observe that if $K_\alpha \in S_\alpha$, then there exists a constant $c > 0$ such that

$$\int_{|x| \sim s} |K_\alpha(x)| dx \leq cs^\alpha.$$

The function $K_\alpha$ satisfies the $L^{n,\Psi,k}$-Hörmander condition ($K \in H_{\alpha,\Psi,k}$), if there exist constants $c_\Psi > 1$ and $C_{\Psi} > 0$ such that for all $x$ and $R > c_\Psi|x|$,

$$\sum_{m=1}^{\infty} (2^m R)^{n-\alpha} m^k \|K_\alpha(\cdot - x) - K_\alpha(\cdot)\|_{\Psi,|y| \sim 2^m R} \leq C_{\Psi}.$$

We say that $K_\alpha \in H_{\alpha,\infty,k}$ if $K_\alpha$ satisfies the previous condition with $\| \cdot \|_{L^{\infty},|x| \sim 2^m R}$ in place of $\| \cdot \|_{\Psi,|x| \sim 2^m R}$. When $l = 0$, we write $H_{\alpha,\Psi} = H_{\alpha,\Psi,0}$. 

[29, 30, 31].
When $\Psi(t) = t^r$, $1 \le r < \infty$, we simply write $H_{\alpha,r,k}$ instead of $H_{\alpha,\Psi,k}$.

In this paper, we study the $k$-order commutators of operator of the form (1.2) where $k_i \in S_{n-\alpha_i,\Psi_i} \cap H_{n-\alpha_i,\Psi_i,k_i}$.

Recall that given a locally integrable function $b$ and an operator $T_\alpha$ defined as (1.2), we define the $k$-order commutator, $k \in \mathbb{N} \cup \{0\}$, by

$$T_{\alpha,b}^k(f) = [b, T_{\alpha,b}^{k-1}]f = \int (b(x) - b(y))^k K(x,y) f(y) dy$$

where we assume that $T_{0,b}^0 = T_\alpha$.

We also consider the following condition for the weights, there exists $c > 0$ such that

$$w(A;x) \le cw(x),$$

$a.e. x \in \mathbb{R}^n$ and for all $1 \le i \le m$.

The following is an example of a weight $w$ that satisfies condition (1.4). Observe that also power weights satisfy this condition.

**Example 1.** Let $w(x) = \begin{cases} \log \left( \frac{1}{|x|} \right) & \text{if } |x| \le \frac{A}{r} \\ 1 & \text{if } |x| > \frac{A}{r} \end{cases}$. Then $w \in A_1$ and satisfies (1.4).

The main results in this paper is the following Coifman type estimate:

**Theorem 1.1.** Let $b \in BMO$, $0 \le \alpha < n$, $k \in \mathbb{N} \cup \{0\}$, $m \in \mathbb{N}$ and $1 \le i \le m$. Let $\Psi_i$ be Young functions and $0 \le \alpha_i < n$ such that $\alpha_1 + \cdots + \alpha_n = n - \alpha$. Let $T_{\alpha,m}$ be the integral operator defined by (1.2) and $T_{\alpha,m,k}^{\Psi}$ be the $k$-order commutator of $T_{\alpha,m}$. Suppose that the matrices $A_i$ satisfy the hypothesis (H) and $k_i \in S_{n-\alpha_i,\Psi_i} \cap H_{n-\alpha_i,\Psi_i,k_i}$.

If $\alpha = 0$, let $T_{0,m}$ be of strong type $(p_0, p_0)$ for some $1 < p_0 < \infty$.

Let $\varphi_k(t) = t \log(c+t)^k$ and let $\phi$ be a Young function such that $\Psi^{-1}(t) \cdots \Psi^{-1}(t) \varphi^{-1}(t) \lesssim t$ for $t \ge t_0$, some $t_0 > 0$.

Let $0 < p < \infty$. Then there exists $C > 0$ such that, for $f \in L_c^\infty(\mathbb{R}^n)$ and $w \in A_\infty$,

$$\int_{\mathbb{R}^n} |T_{\alpha,m,b}^k f(x)|^p w(x) dx \le C \|b\|_{BMO}^k \sum_{i=1}^{m} \int_{\mathbb{R}^n} |M_{\alpha,\phi} f(x)|^p w(A_i;x) dx.$$

whenever the left-hand side is finite.

Furthermore, if $w \in A_\infty$ satisfying (1.4), then

$$\int_{\mathbb{R}^n} |T_{\alpha,m,b}^k f(x)|^p w(x) dx \le C \|b\|_{BMO}^k \int_{\mathbb{R}^n} |M_{\alpha,\phi} f(x)|^p w(x) dx.$$

To prove this estimate, we need a pointwise estimate that relates de sharp delta maximal of the commutator with a sum of generalized fractional maximal function of $f$. As a consequence of the Coifman estimate we get strong weighted estimates for the operator $T_{\alpha,m,b}^k$ and weighted BMO estimates. We also obtain strong weighted estimates of the form

$$\|T_{\alpha,m,b}^k\|_{L^p(u)} \le c \|f\|_{L^p(Su)},$$

where $1 < p < n/\alpha$, $u$ is any weight and where $S$ is appropriate maximal operator. For $T_{0,m,b}^k$ we also give a two pair $(u, Su)$ endpoint estimate. that is,

$$u \{x \in \mathbb{R}^n : |T_{0,m,b}^k(x)| > \lambda \} \le c \int_{\mathbb{R}^n} \varphi_k \left( \frac{|f(x)|}{\lambda} \right) Su(x) dx.$$

The plan of the paper is the following, the next section contains some preliminaries, definitions and previous results that are needed to state the others results which appear in
section 3. The proof of the Coifman Theorem \[1\] is in the section 4. In section 5 we prove strong one weighted inequalities and in section 6 the two-weighted inequalities.

## 2. Preliminaries and previous results

In this section we present some notions about Young function, Luxemburg norm and weights that will be fundamental throughout all this paper. Also we present some previous results.

### 2.1. Young Function and Luxemburg norm

Now, we present some extra definitions and properties for Young functions. Also we given examples. For more details of these topics see \[19\] or \[21\].

Each Young function \(\Psi\) has an associated complementary Young function \(\overline{\Psi}\) satisfying the generalized Hölder inequality

\[
\frac{1}{|B|} \int_B |fg| \leq 2 \|f\|_{\Psi,B} \|g\|_{\overline{\Psi},B}.
\]

If \(\Psi_1, \ldots, \Psi_m, \phi\) are Young functions satisfying \(\Psi_1^{-1}(t) \cdots \Psi_m^{-1}(t) \phi^{-1}(t) \leq ct\), for all \(t \geq t_0\), some \(t_0 > 0\) then

\[
\|f_1 \cdots f_m\|_{L^1,B} \leq c \|f_1\|_{\Psi_1,B} \cdots \|f_m\|_{\Psi_m,B} \|g\|_{\phi,B},
\]

the function \(\phi\) is called the complementary of the functions \(\Psi_1, \ldots, \Psi_m\).

Here are some examples of maximal operators related to certain Young functions.

- \(\Psi(t) = t\), then \(\|f\|_{\Psi,Q} = f_Q := \frac{1}{|Q|} \int_Q |f|\) and \(M_{\alpha,\Psi} = M_\alpha\), the fractional maximal operator.
- \(\Psi(t) = t^r\) with \(1 < r < \infty\). In that case \(\|f\|_{\Psi,Q} = \|f\|_{r,Q} := \left(\frac{1}{|Q|} \int_Q |f|^r\right)^{1/r}\) and \(M_{\alpha,\Psi} = M_{\alpha,r}\), where \(M_{0,r}f = M_r f := M(f^r)^{1/r}\).
- \(\Psi(t) = \exp(t) - 1\). Then, \(M_{\alpha,\Psi} = M_{\alpha,\exp(L)}\).
- If \(\beta > 0\) and \(1 < r < \infty\), \(\Psi(t) = t^r \log(e + t)^\beta\) is a Young function then \(M_{\alpha,\Psi} = M_{\alpha,L^r(\log L)^\beta}\).
- If \(\alpha = 0\) and \(k \in \mathbb{N}\), \(\Psi(t) = t \log(e + t)^k\) it can be proved that \(M_\Psi \approx M^{k+1}\), where \(M^{k+1}\) is \(k + 1\) iterated \(k+1\) times.

**Remark 2.1.** Observe that if \(\Psi(t) = t^r\) then a simple computation shows that

\[
M_{\alpha,r}f = (M_{\alpha r}|f|^r)^{1/r}.
\]

**Proposition 2.2.** Let \(D\) be a Young function and \(A\) be a invertible matrix. Let \(w_A(x) = w(Ax)\), then

\[
M_{\alpha,D}(w_A)(A^{-1}x) \leq c_{A,n} M_{\alpha,D}(w)(x)
\]

for almost every \(x \in \mathbb{R}^n\).

**Proof.** Fix \(x \in \mathbb{R}^n\) and let \(B = B(A^{-1}x, r)\) be a ball

\[
\frac{1}{|B|} \int_B D \left(\frac{w(Ay)}{\lambda}\right) dy = \frac{1}{|AB|} \int_{AB} D \left(\frac{w(z)}{\lambda}\right) dz.
\]

Then, \(x \in AB\) and

\[
\|w_A\|_{D,B} = \|w\|_{D,AB}
\]
Let \( \|A\|_\infty = \sup_{x:|x|=1} |Ax| \). There exist balls \( B_1 = B(x, \|A^{-1}\|_\infty) \) and \( B_2 = B(x, \|A\|_\infty r) \) such that \( B_1 \subset AB \subset B_2 \), then

\[
\|w\|_{D,AB} \leq \|A^{-1}\|_\infty^n \|A\|_\infty^n \|w\|_{D,B_2}
\]

Hence,

\[
M_{\alpha,D}^c(w_A)(A^{-1}x) \leq \|A^{-1}\|_\infty^n \|A\|_\infty^n M_{\alpha,D}^c w(x)
\]

\[\square\]

2.2. **Weights.** A weight is a non negative locally integrable function in \( \mathbb{R}^n \) that takes values in \((0, \infty)\) almost every where. Let \( 0 \leq \alpha < n, 1 \leq p, q \leq \infty \), we say that a weight \( w \in A_{p,q} \) if

\[
[w]_{A_{p,q}} = \sup_B \|w\|_{q,B} \|w^{-1}\|_{p',B} < \infty,
\]

where the supremum is taken over all balls \( B \subset \mathbb{R}^n \).

If \( 1 \leq p < \infty \), \( A_p \) denotes the classical Muckenhoupt classes of weights and \( A_\infty = \cup_{p\geq 1} A_p \). Observe that \( w \in A_{p,p} \) if and only if \( w^p \in A_p \) and \( w \in A_{\infty,\infty} \) if, and only if \( w^{-1} \in A_1 \).

The fractional \( B_p \) condition, \( B_p^0 \), was introduced by Cruz-Uribe and Moen in [7]: Let \( 1 < p < n/\alpha \) and \( \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} \). A Young function \( \phi \in B_p^0 \) if

\[
\int_1^\infty \frac{\phi(t)^{q/p}}{t^{\frac{q}{p'}}} \frac{dt}{t} < \infty.
\]

They proved that if \( \phi \in B_p^0 \) then \( M_{\alpha,\phi} : L^p(dx) \to L^q(dx) \) and

\[
\|M_{\alpha,\phi}\|_{L^p \to L^q} \leq c \left( \int_1^\infty \frac{\phi(t)^{q/p}}{t^{\frac{q}{p'}}} \frac{dt}{t} \right)^{1/q}.
\]

We will consider the following bump conditions: let \( 1 < q < \infty \) and \( \Psi \) be a Young function, then a weight \( w \in A_{q,\Psi} \) if

\[
[w]_{A_{q,\Psi}} = \sup_Q \|w\|_{q,Q} \|w^{-1}\|_{\Psi,Q} < \infty
\]

where the supremum is over all balls \( B \subset \mathbb{R}^n \).

Let \( f \) be locally integrable function in \( \mathbb{R}^n \). The sharp maximal function is defined by

\[
M^# f(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B f(y) - \frac{1}{|B|} \int_B f(z)dz \ dy.
\]

A locally integrable function \( f \) has bounded mean oscillation (\( f \in BMO \)) if \( M^# f \in L^\infty \) and the norm \( \|f\|_{BMO} = \|M^# f\|_{\infty} \).

Observe that the \( BMO \) norm is equivalent to

\[
\|f\|_{BMO} = \|M^# f\|_{\infty} \sim \sup_{B \ni x} \frac{1}{|B|} \int_B |f(x) - a|dx.
\]

There is also a weighted version of \( BMO \), this is denoted by \( BMO(w) \), and it is described by the seminorm

\[
\|f\|_w = \sup_B \|w\chi_B\|_{\infty} \left( \int_B |f(y) - \frac{1}{|B|} \int_B f(z)dz| \ dy \right).
\]

It is easy to check that

\[
\|f\|_w \simeq \|wM^# f\|_{\infty}.
\]
2.3. Previous results. Here we enounce some known results for the operator $T_{a,m}$. See [16].

**Theorem 2.3.** [16] Let $0 \leq \alpha < n$, $m \in \mathbb{N}$ and let $T_{a,m}$ be the integral operator defined by $(L.3)$. For $1 \leq i \leq m$, let $\Psi_i$ be Young functions, $0 \leq \alpha_i < n$ such that $\alpha_1 + \cdots + \alpha_m = n - \alpha$. Also suppose $k_i \in S_{n-\alpha_i,\Psi_i} \cap H_{n-\alpha_i,\Psi_i}$ and let the matrices $A_i$ satisfy the hypothesis (H).

If $\alpha = 0$, suppose $T_{0,m}$ is of strong type $(p_0, p_0)$ for some $1 < p_0 < \infty$.

If $\phi$ is the complementary of the functions $\Psi_1, \ldots, \Psi_m$, then there exists $C > 0$ such that, for $0 < \delta < 1$ and $f \in L^\infty(\mathbb{R}^n)$

\[
(2.2) \quad M^2_\delta|T_{a,m}f|(x) := M^2 \left(|T_{a,m}f|^\delta\right)(x)^{1/\delta} \leq C \sum_{i=1}^{m} M_{\alpha,\phi}f(A_i^{-1}x).
\]

**Theorem 2.4.** [16] Let $0 \leq \alpha < n$ and $m \in \mathbb{N}$ and let $T_{a,m}$ be the integral operator defined by $(L.3)$. For $1 \leq i \leq m$, let $\Psi_i$ be Young functions, $0 \leq \alpha_i < n$ such that $\alpha_1 + \cdots + \alpha_m = n - \alpha$. Also suppose $k_i \in S_{n-\alpha_i,\Psi_i} \cap H_{n-\alpha_i,\Psi_i}$ and that matrices $A_i$ satisfy the hypothesis (H).

If $\alpha = 0$, suppose $T_{0,m}$ is of strong type $(p_0, p_0)$ for some $1 < p_0 < \infty$.

Let $0 < p < \infty$. If $\phi$ is the complementary of the functions $\Psi_1, \ldots, \Psi_m$, then there exists $C > 0$ such that, for $f \in L^\infty(\mathbb{R}^n)$ and $w \in A_\infty$,

\[
\int_{\mathbb{R}^n} |T_{a,m}f(x)|^p w(x)dx \leq C \sum_{i=1}^{m} \int_{\mathbb{R}^n} |M_{\alpha,\phi}f(x)|^p w(A_{i}x)dx,
\]

whenever the left-hand side is finite.

3. Main results

In this section we present the mains results

3.1. Pointwise estimate. To obtain an appropriate maximal operator which controls in weighted $L^p$ norms the operator $T^k_{a,m,b}$ we need the following result:

**Theorem 3.1.** Let $b \in BMO$, $0 \leq \alpha < n$, $k \in \mathbb{N} \cup \{0\}$, $m \in \mathbb{N}$ and $1 \leq i \leq m$. Let $\Psi_i$ be Young functions and $0 \leq \alpha_i < n$ such that $\alpha_1 + \cdots + \alpha_m = n - \alpha$. Let $T_{a,m}$ be the integral operator defined by $(L.2)$ and $T^k_{a,m,b}$ be the $k$-order commutator of $T_{a,m}$. Suppose that the matrices $A_i$ satisfy the hypothesis (H) and $k_i \in S_{n-\alpha_i,\Psi_i} \cap H_{n-\alpha_i,\Psi_i,k_i}$.

If $\alpha = 0$, let $T_{0,m}$ be of strong type $(p_0, p_0)$ for some $1 < p_0 < \infty$.

Let $\varphi_k(t) = t \log(\epsilon t)^k$ and let $\phi$ be a Young function such that $\Psi_1^{-1}(t) \cdots \Psi_m^{-1}(t) \varphi_k^{-1}(t) \lesssim t$ for $t \geq t_0$, some $t_0 > 0$.

Then, there exists $0 < C = C(n,\alpha,A_1,\ldots,A_m)$ such that, for $0 < \delta < \epsilon \leq 1$ and $f \in L^\infty(\mathbb{R}^n)$

\[
(3.1) \quad M^2_\delta|T^k_{a,m,b}f|(x) \leq C \sum_{i=1}^{m} \|b\|^k_{BMO} M_i(T^l_{a,m,b}) + C\|b\|^l_{BMO} \sum_{i=1}^{m} M_{\alpha,\phi}f(A_{i}^{-1}x).
\]

This Theorem is a generalization of several known results. The table illustrate some example of this results: the example (i) with $m = 1$ is a classical example proved in [3], (ii) with $k = 0$ is the example of fractional rough kernel proved in [25] and the last example (iii) is the commutator of the example given in [16].
3.2. One weight inequalities. In this subsection, we prove the boundedness of the operator, $T_{\alpha,m,b}^k$ in two different ways, using the Coifman inequality and using a Cauchy integral formula. Also we give a weighted $BMO$ estimate for weights in the class $A(\frac{n}{\alpha}, \infty)$.

**Theorem 3.2.** Let $0 \leq \alpha < n$, $1 < p < n/\alpha$ and $\frac{1}{p} = \frac{1}{q} - \frac{\alpha}{n}$. Let $T_{\alpha,m}$ be the integral operator defined by (1.2) under the hypothesis of Theorem 1.1 and $T_{\alpha,m,b}^k$ be the $k$-order commutator of $T_{\alpha,m}$. Suppose that one of the following hypothesis holds,

(a) If there exists $1 < r < p$ such that $\kappa_r < \infty$. Let $\eta$ be a Young function such that $\eta^{-1}(t)t_\alpha^{\frac{\alpha}{n}} \lesssim \phi^{-1}(t)$ for every $t > 0$. If $\phi^{\frac{1}{n}} = B_{\frac{n}{\alpha}}$ for every $s > r(n-\alpha)/(n-\alpha r)$ and $w^r \in A(\frac{n}{\alpha}, \frac{2}{r})$.

(b) Suppose that exist $B$ and $C$ be Young functions such that $B^{-1}(t)C^{-1}(t) \leq c\phi^{-1}(t)$, $t > t_0 > 0$, $C \in B_p^\alpha$ and $w \in A_{q,B}$.

(c) Suppose that the operator $T_{\alpha,m}$ is bounded from $L^p(w^p)$ into $L^q(w^q)$ for all $w \in A_{p,q}$.

If $w$ satisfies the condition (1.4) then there exists $c > 0$ such that, for every $f \in L^p(w^p)$,

$$\|T_{\alpha,m,b}^k f\|_{L^q(w^q)} \leq c\|b\|_{BMO}^k \| f \|_{L^p(w^p)}.$$

**Theorem 3.3.** Let $T_{\alpha,m}$ be the integral operator defined by (1.2) under the hypothesis of Theorem 1.1 and $T_{\alpha,m,b}^k$ be the $k$-order commutator of $T_{\alpha,m}$. Suppose that there exist $r > 1$ such that $\kappa_r < \infty$. If $w^r \in A(\frac{n}{\alpha}, \infty)$ and satisfies (1.4), then there exists $C > 0$ such that for $f \in L_n^\infty(\mathbb{R}^n)$,

$$\|T_{\alpha,m,b}^k f\|_{L^q(w^q)} \leq C\|b\|_{BMO}^k \| f \|_{L^\infty(\mathbb{R}^n)}.$$

3.3. Two weights inequalities. If $T = T_{\alpha,m}$ is defined by (1.2), then its adjoint $T^*$ is

$$T^* g(x) = \int \tilde{k}_1(x-A_1^{-1}y) \cdots \tilde{k}_m(x-A_m^{-1}y)g(y)dy,$$

where $\tilde{k}_i(x) = k_i(-A_i x)$. If $T^*$ satisfies hypothesis of Theorem 1.1 then

$$\int_{\mathbb{R}^n} |T^* f(x)|^q w(x)dx \leq c \int_{\mathbb{R}^n} \sum_{i=1}^m (M_{\alpha,\phi}(A_i x))^q w(x)dx.$$

for all $0 < q < \infty$ and $w \in A_\infty$

**Theorem 3.4.** Let $\phi$ be a Young function, $0 \leq \alpha < n$ and $1 < p < \infty$. Suppose that there exist Young functions $\mathcal{E}, \mathcal{F}$ such that $\mathcal{E} \in B_p^\alpha$ and $\mathcal{F}^{-1}(t) \leq \phi^{-1}(t)$. Set $\mathcal{D}(t) = \mathcal{F}(t^{1/p})$.

Let $T$ be a linear operator such that its adjoint $T^*$ satisfies

$$\int_{\mathbb{R}^n} |T^* f(x)|^q w(x)dx \leq c \int_{\mathbb{R}^n} \sum_{i=1}^m (M_{\alpha,\phi}(A_i x))^q w(x)dx,$$

for all $0 < q < \infty$ and $w \in A_\infty$. 

### Table 1. Examples

| $\Psi_i$ | $1 \leq i \leq m$ | $\phi$ | $M_{\alpha,\phi}$ |
|-----------|--------------------|--------|------------------|
| (i)       | $\infty$           | $t \log(e + t)^k$ | $M_{\alpha,L\log L^k}$ |
| (ii)      | $t^{r_i}, 1 < r_i < \infty$ | $t^s \log(e + t)^{s_k}, \sum_{i=1}^m \frac{1}{r_i} + \frac{1}{s} = 1$ | $M_{\alpha,L^s \log L^s}$ |
| (iii)     | $\psi_1 = t^r, \psi_2(t) = \exp(t) - 1, 1 < r < \infty$ | $t^{r'} \log(e + t)^{(k+1)r'}$ | $M_{\alpha,L^{r'} \log L^{r'(k+1)}}$ |
If there exists $r > 0$ such that $t' \leq c\phi(t)$ for $t \geq t_0 > 0$ then \eqref{1.6} holds for the pairs of weights 
\[
\left( u, \sum_{i=1}^{m} (M_{\phi}u(A_i)) \right).
\]
(b) If there exist Young functions $E, F$ such that $E \in B_{p'}$ and $E^{-1}(t)F^{-1}(t) \leq \phi^{-1}(t)$. Set
\[
\mathcal{D}(t) = F(t^{1/p}),
\]
then \eqref{1.6} holds for the pairs of weights 
\[
\left( u, \sum_{i=1}^{m} (M_{\mathcal{D}u(A_i)}) \right).
\]

\textbf{Remark 3.6.} Observe that the pairs of weights given in (a) are better than the one in (b).

(see Remark 3.3 in \cite{18})

4. PROOF OF SHARP THEOREM AND COIFMAN INEQUALITY

Recall some classical results concern to functions in BMO, we do not give the proof.

\textbf{Lemma 4.1.} Let $b \in \text{BMO}$.

(1) For any measurable subsets $A \subset B \subset \mathbb{R}^n$ such that $|A|, |B| > 0$, we have
\[
|b_A - b_B| \leq \frac{|B|}{|A|} \|b\|_{\text{BMO}}.
\]

In particular, if $\tilde{B}$ is a measurable set and $\tilde{B}_i = A_i^{-1}\tilde{B}$, $1 \leq i \leq m$ then
\[
|b_{\tilde{B}} - b_{\bigcup_{i=1}^{m} \tilde{B}_i}| \leq (1 + \sum_{i=1}^{m} |\text{det}(A_i^{-1})|) \|b\|_{\text{BMO}}.
\]

(2) Let $B = B(c_B, R)$ be a ball, centered at $c_B$ with radius $R$, and $B^j = B(c_B, 2^j R)$. Then,
\[
|b_B - b_{B^j}| \leq c_j \|b\|_{\text{BMO}}.
\]

In the proof of Theorem 3.1 we follow the idea of the proof of Theorem 2.2 in \cite{25}.

\textbf{Proof of Theorem 4.1.} We just consider the case $m = 2$ and $k = 1$, i.e. $T_{0,2,b}^1 = [b, T_{a,2}]$, and we will just write $[b, T_0]$. The general case is proved in an analogous way.

Let $f$ be a bounded function with compact support, $b \in \text{BMO}$ and $0 < \delta < \epsilon \leq 1$. Let $x \in \mathbb{R}^n$ and let $B = B(c_B, R)$ be a ball that contains $x$, centered at $c_B$ with radius $R$. We write $\tilde{B} = B(c_B, 2R)$ and for $1 \leq i \leq 2$, set $\tilde{B}_i = A_i^{-1}\tilde{B}$. Let $f_1 = f \chi_{\bigcup_{i=1}^{2} \tilde{B}_i}$ and $f_2 = f - f_1$. Suppose that $a := T_a((b - b_{\bigcup_{i=1}^{2} \tilde{B}_i})f_2)(c_B) < \infty$. 

We can write
\[ [b, T_\alpha f](x) = (b(x) - b_{B \cup B_1 \cup B_2}) T_\alpha f(x) - T_\alpha((b - b_{B \cup B_1 \cup B_2}) f)(x). \]
Now, we have
\[
\left( \frac{1}{|B|} \int_B |[b, T_\alpha f](y) - a|^\delta \, dy \right)^{1/\delta} \\
\leq \left( \frac{1}{|B|} \int_B |(b(y) - b_{B \cup B_1 \cup B_2}) T_\alpha f(y)|^\delta \, dy \right)^{1/\delta} \\
+ \left( \frac{1}{|B|} \int_B |T_\alpha((b - b_{B \cup B_1 \cup B_2}) f_1)(y)|^\delta \, dy \right)^{1/\delta} \\
+ \left( \frac{1}{|B|} \int_B |T_\alpha((b - b_{B \cup B_1 \cup B_2}) f_2)(y) - T_\alpha((b - b_{B \cup B_1 \cup B_2}) f)(c_B)|^\delta \, dy \right)^{1/\delta} \\
= I + II + III.
\]
To estimate I, let \( q = \epsilon/\delta > 1 \), by Hölder’s inequality and Lemma 4.1,
\[
I \leq \left( \frac{1}{|B|} \int_B |(b(y) - b_{B \cup B_1 \cup B_2}) T_\alpha f(y)|^\delta \, dy \right)^{1/\delta} + |b_{B \cup B_1 \cup B_2}| \left( \frac{1}{|B|} \int_B |T_\alpha f(y)|^\delta \, dy \right)^{1/\delta} \\
\leq \left( \frac{1}{|B|} \int_B |(b(y) - b_{B_2})|^\delta \, dy \right)^{1/q} \left( \frac{1}{|B|} \int_B |T_\alpha f(y)|^\delta \, dy \right)^{1/q} + C\|b\|_{BMO} M_\delta(T_\alpha f)(x) \\
\leq C\|b\|_{BMO} M_\epsilon(T_\alpha f)(x) + C\|b\|_{BMO} M_\delta(T_\alpha f)(x) \\
\leq C\|b\|_{BMO} M_\epsilon(T_\alpha f)(x).
\]
For II, by Jensen inequality
\[
II \leq \frac{1}{|B|} \int_B |T_\alpha((b - b_{B \cup B_1 \cup B_2}) f_1)(y)| \, dy \\
\leq \frac{1}{|B|} \int_B \int_{B_1 \cup B_2} |K(y, z)||b(z) - b_{B \cup B_1 \cup B_2}| |f_1(z)| \, dz \, dy \\
\leq \sum_{i=1}^2 \frac{1}{|B|} \int_{B_i} |b(z) - b_{B \cup B_1 \cup B_2}| |f_1(z)| \int_B |K(y, z)| \, dy \, dz.
\]
We estimate the first summand, that is \( z \in \bar{B}_1 \), the case \( z \in \bar{B}_2 \) is analogous. Observe that (4.3)
\[
\int_B |K(y, z)| \, dy \leq \int_{\{y \in B : |y - A_1 z| \leq |y - A_2 z|\}} |K(y, z)| \, dy + \int_{\{y \in B : |y - A_2 z| \leq |y - A_1 z|\}} |K(y, z)| \, dy.
\]
For \( j \in \mathbb{N} \), let us consider the set
\[
C_j^1 := \{ y \in B : |y - A_1 z| \leq |y - A_2 z|, |y - A_1 z| \sim 2^{-j-1}R \}.
\]
Observe that if \( y \in B \) and \( z \in B_1 \) then \( |y - A_1 z| \leq 3R < 4R \).
Thus,
\[
\int_{\{y \in B : |y - A_1 z| \leq |y - A_2 z|\}} |K(y, z)| \, dy \leq \sum_{j=-2}^{\infty} \int_{C_j^1} |K(y, z)| \, dy \\
\leq \sum_{j=-2}^{\infty} \frac{|A_1^{-1} B(c_B, 2^{-j}R)|}{|A_1^{-1} B(c_B, 2^{-j}R)|} \int_{A_1^{-1} B(c_B, 2^{-j}R)} |K(y, z)| \chi_{C_j^1} \, dy.
\]
\[
\leq C \sum_{j=-2}^{\infty} |A_1^{-1} B(c_B, 2^{-j} R)||k_1(\cdot - A_1 z)||_{\Psi_1, \|y-A_1 z\|^{-2-j-1} R}||k_2(\cdot - A_2 z)||_{\Psi_2, \|y-A_1 z\|^{-2-j-1} R}.
\]

Observe that if \( y \in C_j \) then \( |y - A_2 z| \geq |y - A_1 z| > 2^{j-1} R \) and since \( k_2 \in S_{n-\alpha_2, \Psi_2} \) we get
\[
\|k_2(\cdot - A_2 z)\|_{\Psi_2, \|y-A_1 z\|^{-2-j-1} R} \leq \sum_{k \geq 0} \|k_2(\cdot)\|_{\Psi_2, \|y-A_1 z\|^{-2-j+k-1} R} \leq \sum_{k \geq 0} \|k_2(\cdot)\|_{\Psi_2, \|y\|^{-2-j+k-1} R} = \sum_{k \geq 0} (2^{-j+k} R)^{-\alpha_2}.
\]

As \( k_1 \in S_{n-\alpha_1, \Psi_1} \) and using inequality (1.4) we get
\[
\int_{\{y \in B: |y - A_1 z| \leq |y - A_2 z|\}} |K(y, z)| dy \leq C \sum_{k \geq 0} (2^{-\alpha_2})^k \sum_{j=-2}^{\infty} (2^{-j} R)^{n-\alpha_1 - \alpha_2} = C R^\alpha.
\]

In an analogous way, we get
\[
\int_{\{y \in B: |y - A_2 z| \leq |y - A_1 z|\}} |K(y, z)| dy \leq CR^\alpha.
\]

Then, by (1.2) and (1.5), we obtain
\[
II \leq C R^\alpha \sum_{i=1}^{2} \frac{1}{|B_i|} \int_{B_i} |b(z) - b_{\tilde{B}_i \cup \tilde{B}_2}| |f(z)| dz
\]
\[
\leq C R^\alpha \sum_{i=1}^{2} \frac{1}{|B_i|} \int_{\tilde{B}_i} (|b(z) - b_{\tilde{B}_i}| + |b_{\tilde{B}_i} - b_{\tilde{B}_i \cup \tilde{B}_2}|) |f(z)| dz
\]
\[
\leq C \sum_{i=1}^{2} R^\alpha \left[ \|b - b_{\tilde{B}_i}\|_{\exp L, \tilde{B}_i} \|f\|_{\phi, \tilde{B}_i} + \|b\|_{BMO} M_{\alpha} f(A_i^{-1} x) \right]
\]
\[
\leq C \|b\|_{BMO} \sum_{i=1}^{2} M_{\alpha, \phi} f(A_i^{-1} x).
\]

For \( III \), by Jensen inequality we get
\[
III \leq \frac{1}{|B|} \int_B |T_{\alpha, 2}((b - b_{\tilde{B}_1 \cup \tilde{B}_2}) f_2)(y) - T_{\alpha, 2}((b - b_{\tilde{B}_1 \cup \tilde{B}_2}) f_2)(c_B)| dy
\]
\[
\leq \frac{1}{|B|} \int_B \int_{\tilde{B}_1 \cup \tilde{B}_2} |K(y, z) - K(c_B, z)||b(z) - b_{\tilde{B}_1 \cup \tilde{B}_2}| |f_2(z)| dz dy
\]
\[
\leq \frac{1}{|B|} \int_B \int_{Z^l} |K(y, z) - K(c_B, z)||b(z) - b_{\tilde{B}_1 \cup \tilde{B}_2}| |f_2(z)| dz dy,
\]
where
\[
Z^l = (\tilde{B}_1 \cup \tilde{B}_2)^c \cap \{z : |c_B - A_1 z| \leq |c_B - A_r z|, r \neq l, 1 \leq r \leq 2\}.
\]

Let us estimate \( |K(y, z) - K(c_B, z)| \) for \( y \in B \) and \( z \in Z^l \),
\[
|K(y, z) - K(c_B, z)| \leq |k_1(y - A_1 z) - k_1(c_B - A_1 z)||k_2(y - A_2 z)|
\]
(4.6) \[ + |k_1(c_B - A_1 z)||k_2(y - A_2 z) - k_2(c_B - A_2 z)|. \]

For simplicity we estimate the first summand of (4.6), the other one follows in an analogous way. For \( j \in \mathbb{N} \), let \( D_j^l = \{ z \in Z^l : |c_B - A_1 z| \sim 2^{j+1} R \} \).

Observe that \( D_j^l \subset \{ z : |c_B - A_1 z| \sim 2^{j+1} R \} \subset A_1^{-1} B(c_B, 2^{j+2} R) =: \tilde{B}_{l,j} \). Using generalized Hölder’s inequality

\[
\int_{Z^l} |k_1(y - A_1 z) - k_1(c_B - A_1 z)||k_2(y - A_2 z)||b(z) - b_{\tilde{B}_{l,j} \cup \tilde{B}_2}||f(z)||dz
\]

\[
\leq \sum_{j=1}^{\infty} \int_{D_j^l} |k_1(y - A_1 z) - k_1(c_B - A_1 z)||k_2(y - A_2 z)||b(z) - b_{\tilde{B}_{l,j} \cup \tilde{B}_2}||f(z)||dz
\]

\[
\leq \sum_{j=1}^{\infty} \frac{\tilde{B}_{l,j}}{|B_{l,j}|} \int_{\tilde{B}_{l,j}} \chi_{\{z : |c_B - A_2 z| \sim 2^{j+1} R\}} |k_1(y - A_1 z) - k_1(c_B - A_1 z)||k_2(y - A_2 z)|| |f(z)||dz
\]

\[
\leq \sum_{j=1}^{\infty} \frac{\tilde{B}_{l,j}}{|B_{l,j}|} \int_{\tilde{B}_{l,j}} \chi_{\{z : |c_B - A_2 z| \sim 2^{j+1} R\}} \sum_{k=1}^{\infty} |k_1(y - A_1 z) - k_1(c_B - A_1 z)||k_2(y - A_2 z)|| |f(z)||dz
\]

\[
\leq \sum_{j=1}^{\infty} \frac{\tilde{B}_{l,j}}{|B_{l,j}|} \int_{\tilde{B}_{l,j}} \chi_{\{z : |c_B - A_2 z| \sim 2^{j+1} R\}} \sum_{k=1}^{\infty} \left( |b(z) - b_{\tilde{B}_{l,j}}| + |b_{\tilde{B}_{l,j}} - b_{\tilde{B}_{l,j} \cup \tilde{B}_2}||f(z)|| \right) dz
\]

\[
\leq \sum_{j=1}^{\infty} \frac{\tilde{B}_{l,j}}{|B_{l,j}|} \int_{\tilde{B}_{l,j}} \chi_{\{z : |c_B - A_2 z| \sim 2^{j+1} R\}} \sum_{k=1}^{\infty} \left( |b(z) - b_{\tilde{B}_{l,j}}| + |b_{\tilde{B}_{l,j}} - b_{\tilde{B}_{l,j} \cup \tilde{B}_2}||f(z)|| \right) dz
\]

Observe that \(|c_B - A_1 z|/2 \leq |y - A_1 z| < 2|c_B - A_1 z| \) and if \(|c_B - A_1 z| \sim 2^{j+1} R \) then \(2^j R \leq |y - A_1 z| \leq 2^{j+2} R \). Thus, we have

\[
||k_1(y - A_1 \cdot)\chi_{D_j}||_{\Psi_1, |c_B - A_2 z| \sim 2^{j+1} R} \leq ||k_1(y - A_1 \cdot)||_{\Psi_1, |c_B - A_2 z| \sim 2^{j+1} R} + ||k_1(y - A_1 \cdot)||_{\Psi_1, |c_B - A_2 z| \sim 2^{j+1} R}
\]

\[
\leq ||k_1(\cdot)||_{\Psi_1, |\cdot| \sim 2^{j+1} R} + ||k_1(\cdot)||_{\Psi_1, |\cdot| \sim 2^{j+1} R}
\]

\[
\leq 2^{j+1} R - \alpha_1,
\]

where the last inequality holds since \(k_1 \in S_{n_\alpha, \Psi_1} \). Also, by hypothesis

\[
||k_1(c_B - A_1 \cdot)\chi_{D_j}||_{\Psi_1, |c_B - A_2 z| \sim 2^{j+1} R} \leq c(2^{j+1} R)^{-\alpha_1}.
\]

For \( r \neq l \), observe that if \( z \in D_j^l \) then \(|c_B - A_r z| \geq |c_B - A_1 z| \geq 2^{j+1} R \), so we decompose \( D_j^l = \cup_{k \geq j} (D_j^l)_{k,r} \) where

\[
(D_j^l)_{k,r} = \{ z \in D_j^l : |c_B - A_r z| \sim 2^{j+1} R \}.
\]

Note that \((D_j^l)_{k,r} \subset \{ z : |c_B - A_r z| \sim 2^{k+1} R \} \). Then, as \(k_r \in S_{n-\alpha, \Psi_r} \),

\[
||k_r(y - A_r \cdot)\chi_{D_j}||_{\Psi_r, |c_B - A_2 z| \sim 2^{j+1} R} \leq \sum_{k \geq j} ||k_r(y - A_r \cdot)\chi_{(D_j^l)_{k,r}}||_{\Psi_r, |c_B - A_2 z| \sim 2^{j+1} R}
\]

\[
\leq \sum_{k \geq j} ||k_r(y - A_r \cdot)\chi_{(D_j^l)_{k,r}}||_{\Psi_r, |c_B - A_2 z| \sim 2^{k+1} R}
\]
Then, we obtain
\[ \sum_{k \geq j} k_r(y - A_r) \| |_{\psi_{r,|_{c_B - A_r}}|^{2^k+1} R} \leq c \sum_{k \geq j} (2^k R)^{-\alpha_r} = c (2^j R)^{-\alpha_r}. \]

Also, using again that \( k_r \in S_{n-\alpha_r, \Psi_{r,\cdot}} \), we get
\[ \| k_r(c_B - A_r)^c \|_{\psi_{r,|_{c_B - A_r}}|^{2^j+1} R} \leq \sum_{k \geq j} (2^k R)^{-\alpha_r} = c \sum_{k \geq j} (2^k R)^{-\alpha_r}. \]

Now for \( l = 1 \),
\[ \int_{Z^1} \left| k_1(y - A_1 z) - k_1(c_B - A_1 z) \right| b(z) d z \leq c \| b \|_{BMO} \sum_{j=1}^{\infty} (2^j R)^{n-\alpha_2} \| (k_1(y - A_1) - k_1(c_B - A_1)) \chi_{D_j} \|_{\psi_{1,|_{c_B - A_1}}|^{2^j+1} R} \| f \|_{\varphi_{B_1}} \]
\[ \leq c \| b \|_{BMO} M_{\alpha_2} f (A_1^{-1} x) \sum_{j=1}^{\infty} (2^j R)^{n-\alpha_2} \| (k_1(y - A_1) - k_1(c_B - A_1)) \chi_{D_j} \|_{\psi_{1,|_{c_B - A_1}}|^{2^j+1} R} \]
\[ \leq c \| b \|_{BMO} M_{\alpha_2} f (A_1^{-1} x), \]
where the last inequality follows since \( k_1 \in H_{n-\alpha_1, \Psi_{1,\cdot}} \).

For \( l = 2 \) we observe that
\[ \| (k_1(y - A_1) - k_1(c_B - A_1)) \chi_{D_j} \|_{\psi_{1,|_{c_B - A_1}}|^{2^{j+1} R}} \]
\[ \leq \sum_{k \geq j} \| (k_1(y - A_1) - k_1(c_B - A_1))^c \|_{\psi_{1,|_{c_B - A_1}}|^{2^{k} R}}. \]

Then, we obtain
\[ \sum_{j=1}^{\infty} (2^j R)^{\alpha_1} \| (k_1(y - A_1) - k_1(c_B - A_1)) \chi_{D_j} \|_{\psi_{1,|_{c_B - A_1}}|^{2^{j+1} R}} \]
\[ \leq \sum_{j=1}^{\infty} (2^j R)^{\alpha_1} \sum_{k \geq j} \| (k_1(y - A_1) - k_1(c_B - A_1)) \chi_{D_j} \|_{\psi_{1,|_{c_B - A_1}}|^{2^{k+1} R}} \]
\[ \leq \sum_{j=1}^{\infty} (2^j R)^{\alpha_1} \sum_{k \geq j} (2^k R)^{\alpha_1} \| (k_1(y - A_1) - k_1(c_B - A_1)) \chi_{D_j} \|_{\psi_{1,|_{c_B - A_1}}|^{2^{k+1} R}} \]
\[ \leq \sum_{k=1}^{\infty} \left( \sum_{j=1}^{k} (2^{-\alpha_1} k)^{k-j} \right) (2^k R)^{\alpha_1} \| (k_1(y - A_1) - k_1(c_B - A_1)) \chi_{D_j} \|_{\psi_{1,|_{c_B - A_1}}|^{2^{k+1} R}} \]
\[ \leq c \sum_{k=1}^{\infty} (2^k R)^{\alpha_1} \| (k_1(y - A_1) - k_1(c_B - A_1)) \chi_{D_j} \|_{\psi_{1,|_{c_B - A_1}}|^{2^{k+1} R}} \leq c, \]
where the last inequality follows since $k_1 \in H_{n-\alpha_1,\Psi_{1,1}}$.

So as in the case $l = 1$, we obtain

$$
\left| \int_{\mathbb{R}^n} |k_1(y - A_1z) - k_1(c_2 - A_1z)||k_2(y - A_2z)||b(z) - b_{B_1 \cup B_2}||f(z)|dz \right|
\leq C_{B} ||b||_{BMO} M_{\alpha,\varphi} f(A_1^{-1} x).
$$

Then

$$
III \leq C_{B} ||b||_{BMO} \sum_{l=1}^{2} M_{\alpha,\varphi} f(A_1^{-1} x).
$$

For the case $\alpha = 0$, we repeat the same argument to the inequality (4.1). The terms $I$ and $III$ are analogous to the ones in the case $0 < \alpha < n$. For $II$, observe that $T_0$ is of weak-type $(1,1)$ with respect to the Lebesgue measure (see Lemma 5.3 in [9]), as $0 < \delta < 1$ and using Kolmogorov’s inequality (see Lemma 5.16 in [9]) we get

$$
II \leq \frac{C}{|B|} \int_{\mathbb{R}^n} |f_1(y)|dy \sum_{l=1}^{2} C \int_{B_i} |f_1(y)|dy \leq C \sum_{l=1}^{2} M f(A_1^{-1} f(x)),
$$

and the theorem follows in this case.

Proof of Theorem 1.1. By the extrapolation result Theorem 1.1 in [6], estimate (1.5) holds for all $0 < p < \infty$ and all $w \in A_{\infty}$ if, and only if, it holds for some $0 < p_0 < \infty$ and all $w \in A_{\infty}$. Therefore, we will show that (1.5) is true for $p_0$, which is taken such that $\frac{\alpha}{\alpha_1} < p_0 < \infty$. By homogeneity, we assume that $||b||_{BMO} = 1$. We proceed by induction.

When $k = 0$, then $T_{0,m,b}^0 = T_{a,m}$. As $k_i \in H_{n-\alpha_i,\Psi_{i,0}} = H_{n-\alpha_i,\Psi_{i}}$, Theorem 3.3 in [16] implies

$$
\int_{\mathbb{R}^n} |T_{a,m} f(x)|^p w(x)dx \leq C \sum_{i=1}^{m} \int_{\mathbb{R}^n} |M_{a,\varphi} f(x)|^p w(A_i x)dx.
$$

Next, we assume that the results holds for all $0 \leq j \leq k - 1$ and let us see how to derive the case $k$. Let $w \in A_{\infty}$, then there exists $r > 1$ such that $w \in A_r$. Let $0 < \delta < 1$ such that $1 < r < p_0/\delta$, thus $w \in A_{p_0/\delta}$. Then, by Lemma (5.1) in [16], we have $||T_{a,m} f||_{L^{p_0}(w)} < \infty$ and $||(T_{a,m} f)^\delta||_{L^{p_0/\delta}(w)} < \infty$.

For prove this, we consider $w, b \in L^\infty$,

$$
||T_{a,m,b}^k f||_{L^{p_0}(w)} = \| \sum_{j=1}^{k} c_{k,j} b^{k-j} T_{a,m}(b^j f) \|_{L^{p_0}(w)} \leq \| w \|_\infty \| \sum_{j=1}^{k} c_{k,j} b^{k-j} T_{a,m} (b^j f) \|_{L^{p_0}},
$$

and $||(T_{a,m,b}^k f)^\delta||_{L^{p_0/\delta}(w)} < \infty$. Then, we get

$$
\int_{\mathbb{R}^n} |T_{a,m,b}^k f(x)|^{p_0} w(x)dx \leq \int_{\mathbb{R}^n} |M (T_{a,m,b}^k f)^\delta(x)|^{p_0/\delta} w(x)dx \leq \int_{\mathbb{R}^n} (M_{b}^k (T_{a,m,b}^k f)(x))^{p_0} w(x)dx \leq C \sum_{l=0}^{k-1} ||(M_{a} (T_{a,m,b}^l f)||_{L^{p_0}(w)} + C \sum_{l=1}^{m} \int_{\mathbb{R}^n} (M_{a,\varphi} f(A_1^{-1} x))^{p_0} w(x)dx.
$$
Since $\delta < q/r < 1$, we can take $\epsilon > 0$ such that $\delta < \epsilon < p_0/r < 1$, and so $w \in A_{p_0/\epsilon}$.

Hence,
\[
\|(M_\epsilon(T^d_{a,m,b,f})w)\|_{L^{p_0}(w)} = \|(M_\epsilon(T^d_{a,m,b,f}f)\|_{L^{p_0}/\epsilon(w)}^{1/\epsilon} \leq c\|T^d_{a,m,b,f}\|_{L^{p_0}(w)}.
\]

Thus, the induction hypothesis implies that, for any $0 \leq l \leq k - 1$,
\[
\|(M_\epsilon(T^l_{a,m,b,f})w)\|_{L^{p_0}(w)} \leq c\|T^l_{a,m,b,f}\|_{L^{p_0}(w)} \leq c\sum_{i=1}^m \int_{\mathbb{R}^n} (M_{a,\phi} f(A_i^{-1}x))^p_0 w(x)dx.
\]

Hence, for $w$ and $b \in L^\infty$, (1.5) holds, that is
\[
\int_{\mathbb{R}^n} |T_{a,m,b,f}(x)|^p_0 w(x)dx \leq C \sum_{i=1}^m \int_{\mathbb{R}^n} (M_{a,\phi} f(x))^p_0 w(A_i x)dx.
\]

For the general case, if $b \in BMO$, for any $N \in \mathbb{N}$ we define $b_N = b\chi(-N,N] + N\chi(N,\infty) - N\chi(-\infty,-N)$, then $\|b_N\|_{BMO} = \|b\|_{BMO} \leq 2\|b\|_{BMO}$. For the weight $w \in A_\infty$, we define $w_N = \min\{w, N\}$, then $w_N \in A_\infty$ and $[w_N]_{A_\infty} \leq [w]_{A_\infty}$. Now, using convergence theorems, for details see [17], we conclude that (1.5) holds for any $b \in BMO$ and $w \in A_\infty$.

Thus, as mentioned, using the extrapolation results obtained in [6], (1.5) holds for all $0 < p < \infty$, $b \in BMO$ and $w \in A_\infty$.

If $w$ satisfies (1.4), we have
\[
\int_{\mathbb{R}^n} |T^l_{a,m,b,f}(x)|^p w(x)dx \leq C \sum_{i=1}^m \int_{\mathbb{R}^n} (M_{a,\phi} f(x))^p w(A_i x)dx
\]
\[
\leq C \sum_{i=1}^m \int_{\mathbb{R}^n} (M_{a,\phi} f(x))^p w(x)dx.
\]

5. PROOF OF ONE WEIGHTED INEQUALITIES

For the proof of Theorem 3.2 (a) and (b), we need the Coifman inequality (1.5) and the boundedness of the maximal operator, given in [2] (see Theorem 2.6). In the case of the classical Lebesgue spaces the theorem is the following

**Theorem 5.1.** [2] Let $0 \leq \alpha < n$, $w$ be a weight, $1 \leq \beta < n/\alpha$ and $1/q = 1/p - \alpha/n$. Let $\eta$ be a Young function such that $\eta^{1+(\alpha - 1)/\alpha} \in B_{\frac{\alpha}{\alpha - 1}}$ for every $\rho > \beta(n - \alpha)/(n - \alpha\beta)$, and let $\phi$ be a Young function such that $\phi^{-1}(t)t^{\alpha/n} \leq \eta^{-1}(t)$ for every $t > 0$. If $w^\beta \in A_{\frac{\beta}{n}\frac{q}{q}}$, then $M_{a,\phi}$ is bounded in $L^p(w^\beta)$ into $L^q(w^\beta)$.

The boundedness of the $M_{a,\phi}$ from $L^p(w^\beta)$ into $L^q(w^\beta)$ with bump conditions, given in [8] (see Theorem 5.37), is the following,

**Theorem 5.2.** [8] Let $0 \leq \alpha < n$, $1 < p < n/\alpha$, let $\frac{1}{q} = \frac{1}{p} - \frac{n}{\alpha}$. Let $\phi, B$ and $C$ be Young functions such that $B^{-1}(t)C^{-1}(t) \leq c\phi^{-1}(t)$, $t \geq t_0 > 0$. If $C \in B_{\alpha}^p$ and $w \in A_{q,B}$, then for every $f \in L^p(w^\beta),
\[
\int (M_{a,\phi} f)^p w^\beta \leq C \int |f|^p w^\beta.
\]

Now we prove the part (a) and (b) of Theorem 3.2
Lemma 5.3. Using the boundedness of the conjugation of the operator.

Proof of Theorem 3.2 (c). From the previous Theorems, hypothesis (a) or (b) implies that $M_{\alpha,\phi}$ is bounded from $L^p(w^p)$ into $L^q(w^q)$.

Then, by Theorem 1.1 and $w$ satisfies (1.3),

$$\|T^k_{\alpha,m,b}f\|_{L^q(w^q)} \leq c\|b\|_{\text{BMO}}\|M_{\alpha,\phi}f\|_{L^q(w^q)} \leq c\|b\|_{\text{BMO}}\|f\|_{L^p(w^p)}.$$  \hfill $\blacksquare$

For the proof of Theorem 4.1 (c) we use a Cauchy integral formula technique, see [3] and [1]. This technique is as follows, let $T$ be a linear operator, we can write $T^k_b$ as a complex integral operator

$$T^k_b f = \frac{d^k}{dz^k}e^{zb}T(f e^{-zb}) \bigg|_{z=0} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{T_z(f)}{z^2} dz,$$

where $\epsilon > 0$ and $T_z(f) = e^{zb}T(f e^{-zb})$, $z \in \mathbb{C}$. This is called the “conjugation” of $T$ by $e^{zb}$.

Now, if $\| \cdot \|$ is a norm we can apply Minkowski inequality,

$$\|T^k_b f\| \leq \frac{1}{2\pi \epsilon^k} \sup_{|z| = \epsilon} \|T_z(f)\| \quad \epsilon > 0.$$

Observe that using this technique we can obtain the boundedness of the commutator using the boundedness of the conjugation of the operator.

Lemma 5.3. [1] Fix $1 < r, \eta < \infty$. If $w^\eta \in A_r$ and $b \in \text{BMO}$. Then $w^{\lambda b} \in A_r$ for every $\lambda \in \mathbb{R}$ verifying

$$|\lambda| \leq \frac{\min\{1, p - 1\}}{\eta' \|b\|_{\text{BMO}}}.$$

Proof of Theorem 3.3 (c). Let $T = T_{\alpha,m}$. Let $w \in A_{p,q}$ and $\nu = w e^{Re(z)k}$, where $Re(z)$ is the real part of the complex number $z$. If $\nu \in A_{p,q}$, then

$$\|T_z f\|_{L^q(w^\eta)} = \|T(f e^{-zb})\|_{L^q(\nu^\eta)} \leq c\|f e^{-zb}\|_{L^p(\nu^\eta)} = c\|f\|_{L^p(w^\eta)},$$

since $T$ is boundedness from $L^p(\nu^\eta)$ into $L^q(\nu^\eta)$.

Let us prove that $\nu \in A_{p,q}$. If $w \in A_{p,q}$ then $w^\eta \in A_{1+\frac{q}{p}}$ and exists $r > 1$ such that $w^{qr} \in A_{1+\frac{q}{p}}$. Let $\epsilon_0 = \frac{\min\{1, \frac{q}{r'}\}}{q'\|b\|_{\text{BMO}}}$, if $|z| = \epsilon_0$ then

$$|qRe(z)| \leq q|z| = \frac{\min\{1, \frac{q}{r'}\}}{r'\|b\|_{\text{BMO}}}.$$

By Lemma 5.3 $\nu^\eta \in A_{1+\frac{q}{p}}$ and $\nu \in A_{p,q}$.

Hence,

$$\|T^k_b f\|_{L^p(w^\eta)} \leq \frac{1}{2\pi \epsilon_0^k} \sup_{|z|=\epsilon_0} \|T_z(f)\|_{L^p(w^\eta)} \leq \frac{1}{2\pi \epsilon_p^k} \|b\|_{\text{BMO}}\|f\|_{L^q(w^\eta)}.$$  \hfill $\blacksquare$

Now, we prove the weighted $\text{BMO}$ inequality
Proof of Theorem 3.3 Follows the ideas in [17], the authors prove that

\[(5.1) \quad w^r \in A \left(\frac{n}{\alpha r}, \infty\right) \Rightarrow \|w M_{\alpha,r} f\|_{\infty} \leq C \|fw\|_{n/\alpha} .\]

Now, by Lemma 4.1 in [4], Theorem 3.1 and (5.1), we get

\[ \|T_{\alpha,m,b,f}^k\|_w \simeq \|w^{\frac{1}{\alpha}} T_{\alpha,m,b}^k\|_w \leq C \|b\|_{BMO} \sum_{i=1}^m \|w M_{\alpha,\phi} f(A_i^{-1})\|_{\infty} \]

\[ \leq C \kappa_r \|b\|_{BMO} \sum_{i=1}^m \|w M_{\alpha,r} f(A_i^{-1})\|_{\infty} \]

\[ \leq C \kappa_r \|b\|_{BMO} \sum_{i=1}^m \|w f(A_i^{-1})\|_{n/\alpha} \]

\[ \leq C \kappa_r \|b\|_{BMO} \|wf\|_{n/\alpha} . \]

\[ \square \]

6. Proof of Two Weights Norm Inequalities

For the proof of the two weights norm inequality we need the following auxiliary results.

Lemma 6.1. (a) [20] Let \( \Phi \) be a Young function. If \( \Phi \in B_p \) then for every weight \( \nu \)

\[ \int |M_{\Phi} f(x)|^p \nu(x) dx \leq c \int |f(x)|^p \nu(x) dx. \]

(b) [18] If \( r > 1 \), then \( M(M_r) \approx M_r \).

Proof of Theorem 3.4 Let \( u \) a weight and \( \nu(x) = M_{\alpha,p,D} u(x) \). By duality, (3.3) turns out to be equivalent to

\[ \int_{\mathbb{R}^n} |T^* f(x)|^p \nu(x)^{1-p'} dx \leq c \int_{\mathbb{R}^n} \sum_{i=1}^m |f(A_i x)|^{p'} u(x)^{1-p'} dx. \]

Since \( \nu = M_{\alpha,p,D} u^{1-p'} \in A_{\infty} \), see [9], then by Remark 2.2 and the fact that \( E \in B_{p'} \) we get

\[ \int_{\mathbb{R}^n} |T^* f(x)|^{p'} \nu(x)^{1-p'} dx \leq c \int_{\mathbb{R}^n} M_{\alpha,\phi} f(A_i x)^{p'} \nu(x)^{1-p'} dx \]

\[ \leq c \int_{\mathbb{R}^n} M_{\xi}(f w^{-1/p}(A_i x)^{p'} M_{\alpha,p}(w^{1/p})(A_i x)^{p'} \nu(x)^{1-p'} dx \]

\[ = c \int_{\mathbb{R}^n} M_{\xi}(f w^{-1/p}(A_i x)^{p'} M_{\alpha,p,D}(w^{-1/p})(A_i x)^{p'} \nu(x)^{1-p'} dx \]

\[ \leq c \int_{\mathbb{R}^n} M_{\xi}(f w^{-1/p}(A_i x)^{p'} M_{\alpha,p,D}(w)^{1/p}(x)^{p'} \nu(x)^{1-p'} dx \]

\[ \leq c \int_{\mathbb{R}^n} M_{\xi}(f w^{-1/p}(A_i x)^{p'} dx \]

\[ \leq c \int_{\mathbb{R}^n} |f(A_i x)| w^{-1/p}(A_i x)^{p'} dx = c \int_{\mathbb{R}^n} |f(A_i x)|^p w(x)^{1-p'} dx. \]
Proof of Theorem 3.5. We consider \( m = 2, T = T_{0,2} \). The general case is analogous. Let \( u \) be a weight, suppose that \( u \in L^\infty_c \) (otherwise consider \( u_N = \min\{u, N\} \chi_{B(0,N)} \) and use monotone converge). Let \( 0 \leq f \in L^\infty_c \). By the standard Calderón-Zygmund decomposition of \( f \) at height \( \lambda \), there exists \( \{Q_j\}_j \) dyadic cubes such that

\[
\lambda < \frac{1}{|Q_j|} \int_{Q_j} f \leq 2^n \lambda,
\]

and write \( f = g + h \) where

\[
g = f \chi_{\mathbb{R}^n \setminus \cup_j Q_j} + \sum_j f_{Q_j} \chi_{Q_j}, \quad h = \sum_j h_j = \sum_j (f - f_{Q_j}) \chi_{Q_j},
\]

where \( f_{Q_j} \) denotes the average of \( f \) over \( Q_j \). Let us recall that \( 0 \leq g \leq 2^n \lambda \) a.e. and also that each \( h_j \) has vanishing integral. We set \( \tilde{Q}_{j,i} \), the cube with center \( \tilde{A}_j \) with length \( 2\sqrt{n} M |Q_j| \), where \( M = \max_{1 \leq i \leq 2} \| A_i \|_\infty \), \( \tilde{\Omega} = \bigcup_j (\tilde{Q}_{j,1} \cup \tilde{Q}_{j,2}) \) and \( \tilde{u} = u \chi_{\mathbb{R}^n \setminus \tilde{\Omega}} \). Then

\[
u \{ x \in \mathbb{R}^n : |Tf(x)| > \lambda \} \leq u(\tilde{\Omega}) + u \{ x \in \mathbb{R}^n \setminus \tilde{\Omega} : |Th(x)| > \lambda/2 \} + u \{ x \in \mathbb{R}^n \setminus \tilde{\Omega} : |Tg(x)| > \lambda/2 \} = I + II + III.
\]

For \( I \), observe that \( |\tilde{Q}_{j,i}| = (42\sqrt{n} M)^n |Q_j| \). Then, we have

\[
I = u \left( \bigcup_j (\tilde{Q}_{j,1} \cup \tilde{Q}_{j,2}) \right) \leq \sum_j \left[ u(\tilde{Q}_{j,1}) + u(\tilde{Q}_{j,2}) \right] \leq \frac{C_n}{\lambda} \sum_j \left[ \frac{u(\tilde{Q}_{j,1})}{|Q_{j,1}|} + \frac{u(\tilde{Q}_{j,2})}{|Q_{j,2}|} \right] \int_{Q_j} f \leq \frac{C_n}{\lambda} \sum_j \int_{Q_j} \left[ Mu(A_1 x) + Mu(A_2 x) \right] f(x) dx.
\]

where the last inequality follows since \( x \in Q_j \) then \( A_i x \in \tilde{Q}_{j,i} \).

To estimate \( II \), recall that the function \( h_j \) has vanishing integral, then

\[
II = u \{ x \in \mathbb{R}^n \setminus \tilde{\Omega} : |Th(x)| > \lambda/2 \} \leq \frac{2}{\lambda} \sum_j \int_{\mathbb{R}^n \setminus \tilde{\Omega}} |Th_j(x)| u(x) dx \leq \frac{2}{\lambda} \sum_j \int_{\mathbb{R}^n \setminus \tilde{\Omega}} \left( \int_{Q_j} (K(x, y) - K(x, c_j)) h_j(y) dy \right) u(x) dx \leq \frac{2}{\lambda} \sum_j \int_{Q_j} |h_j(y)| \int_{\mathbb{R}^n \setminus (\tilde{Q}_{j,1} \cup \tilde{Q}_{j,2})} \left| (K(x, y) - K(x, c_j)) \right| u(x) dx dy.
\]

We claim that for every \( y \in Q_j \) we have

\[
\int_{\mathbb{R}^n \setminus (\tilde{Q}_{j,1} \cup \tilde{Q}_{j,2})} \left| (K(x, y) - K(x, c_j)) \right| u(x) dx \leq c \text{ ess inf}_{x \in Q_j} \left[ M_\Phi u(A_1 x) + M_\Phi u(A_2 x) \right].
\]

This estimate drives us to
\[ II \leq \frac{c}{\lambda} \sum_{j} \essinf_{Q_j} [M_{q\Phi}(A_1) + M_{q\Phi}(A_2)] \int_{Q_j} |h_j(y)|dy \]
\[ \leq \frac{c}{\lambda} \sum_{j} \essinf_{Q_j} [M_{q\Phi}(A_1) + M_{q\Phi}(A_2)] \int_{Q_j} f(y)dy \]
\[ \leq \frac{c}{\lambda} \sum_{j} \int_{Q_j} f(y) [M_{q\Phi}(A_1y) + M_{q\Phi}(A_2y)] dy. \]

Let us proof (5.1). Using (4.10), we obtain
\[
\int_{\mathbb{R}^n \setminus (\tilde{Q}, \cup \tilde{Q}_2)} |(K(x, y) - K(x, c_j))| u(x)dx \\
\leq \int_{Z^1 \cup Z^2} |k_1(x - A_1y) - k_1(x - A_1c_j)| |k_2(x - A_2y)| u(x)dx \\
+ \int_{Z^1 \cup Z^2} |k_1(x - A_1c_j)| |k_2(x - A_2y) - k_2(x - A_2c_j)| u(x)dx,
\]
where \( Z^i = \mathbb{R}^n \setminus (\tilde{Q}_i, \cup \tilde{Q}_j, \cup \tilde{Q}_k) \cap \{x : |x - A_iw| \leq |x - A_vw|, r \neq i\}. \)

We only estimate the first summand, the other follows in an analogous way. Using generalized Hölder inequality and observing that \( |Q_j| = (42\sqrt{M})^n|Q_j| \), we have
\[
\int_{Z^1} |k_1(x - A_1y) - k_1(x - A_1c_j)| |k_2(x - A_2y)| u(x)dx \\
\leq c \sum_{l=1}^{\infty} |Q^l|^{\alpha} |k_1(- A_1y) - k_1(- A_1c_j)\chi_{Q^{l+1} \setminus Q^l}| \psi_{l, Q^{l+1}} |k_2(- A_2y)\chi_{Q^{l+1} \setminus Q^l}| \psi_{l, Q^{l+1}} u_{Q, Q^{l+1}}.
\]

where \( Q^l \) is the cube with center \( A_1c_j \) and length \( 2^\frac{l}{n} \sqrt{M} |Q_j| \). Observe \( Q^1 = \tilde{Q}_j,1 \).

Since, \( k_2 \in S_{n-\alpha, \psi^2} \), we get
\[ \|k_2(- A_2y)\chi_{Q^{l+1} \setminus Q^l}| \psi_{l, Q^{l+1}} \leq c |Q^l|^{-\alpha/2}. \]

Also, if \( x \in Q_j \) then for all \( t \in \mathbb{N} \) we get \( A_1x \in \tilde{Q}_j,1 \subset Q^t \) and
\[ |Q^t|^{\alpha} u_{Q, Q^{l+1}} \leq c \essinf_{Q_j} M_{q\Phi}(A_1). \]

Then,
\[
\int_{Z^1} |k_1(x - A_1y) - k_1(x - A_1c_j)| |k_2(x - A_2y)| u(x)dx \\
\leq c \essinf_{Q_j} M_{q\Phi}(A_1) \sum_{l=1}^{\infty} |Q^l|^{\alpha} |k_1(- A_1y) - k_1(- A_1c_j)\chi_{Q^{l+1} \setminus Q^l}| \psi_{l, Q^{l+1}} \\
\leq c \essinf_{Q_j} M_{q\Phi}(A_1),
\]
where the last inequality holds since \( k_1 \in H_{n-\alpha, \psi^1} \).

In an analogous way, we obtain
\[
\int_{Z^2} |k_1(x - A_1y) - k_1(x - A_1c_j)| |k_2(x - A_2y)| u(x)dx \leq c \essinf_{Q_j} M_{q\Phi}(A_2). \]
The estimate III is different in each case. We start with (a). For $p > 1$, using Theorem 1.1 the fact that $M_r u \in A_1$ and Lemma 6.1 we get

$$III = u \{ x \in \mathbb{R}^n \setminus \hat{\Omega} : |Tg(x)| > \lambda/2 \} \leq \frac{2^p}{\lambda p} \int_{\mathbb{R}^n} |Tg(x)|^p \tilde{u}(x)dx$$

$$\leq \frac{2^p}{\lambda p} \int_{\mathbb{R}^n} |Tg(x)|^p M_r \tilde{u}(x)dx \leq \frac{c}{\lambda p} \sum_{i=1}^2 \int_{\mathbb{R}^n} |M_\Phi g(A_i^{-1} x)|^p M_r \tilde{u}(x)dx$$

$$\leq \frac{c}{\lambda p} \sum_{i=1}^2 \int_{\mathbb{R}^n} |g(A_i^{-1} x)|^p M_r \tilde{u}(x)dx \leq \frac{c}{\lambda p} \sum_{i=1}^2 \int_{\mathbb{R}^n} |g(A_i^{-1} x)|^p M_r \tilde{u}(x)dx$$

$$\leq \frac{c}{\lambda p} \int_{\mathbb{R}^n} |g(x)|^p \sum_{i=1}^2 M_r \tilde{u}(A_i x)dx \leq \frac{c}{\lambda p} \int_{\mathbb{R}^n} |g(x)|^p \sum_{i=1}^2 M_\Phi \tilde{u}(A_i x)dx,$$

where the last inequality holds since $t'' \leq \Phi(t)$ for $t \geq t_0 > 0$. Since $g \leq 2^n \lambda$, then

$$III \leq \frac{c}{\lambda p} \int_{\mathbb{R}^n} |g(x)|^p \sum_{i=1}^2 M_\Phi \tilde{u}(A_i x)dx$$

$$\leq \frac{c}{\lambda} \int_{\mathbb{R}^n} |g(x)| \sum_{i=1}^2 M_\Phi \tilde{u}(A_i x)dx$$

$$\leq \frac{c}{\lambda} \int_{\mathbb{R}^n} f(x) \sum_{i=1}^2 M_\Phi \tilde{u}(A_i x)dx.$$
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