A NOTE ON ANOSOV HOMEOMORPHISMS

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ABSTRACT. We give an elementary proof that a sufficient condition for an expansive homeomorphism acting on a compact space has the shadowing property is that it has the $\alpha$-shadowing property for one-jump pseudo orbits, $\alpha$ being an expansivity constant. The proof is based on a reformulation of the property of expansivity of a homeomorphism of a compact space in terms of a property of the pseudo orbits of the system.

1. INTRODUCTION

In [1, Theorem 1.2.1] it is proved, among other things, that Anosov diffeomorphisms has the shadowing property, called pseudo orbit tracing property there. In the proof, on [1, p. 23], the authors only uses the fact that $W^s_\varepsilon(x) \cap W^u_\varepsilon(y) \neq \emptyset$ if $\delta > 0$ is chosen small enough for a given $\varepsilon > 0$, and the special (hyperbolic) properties of the metric $d$ coming from the Riemannian structure of the manifold supporting the system given in [1, (B), p. 20].

As can be easily checked the first of these two conditions is equivalent to the shadowing property for pseudo orbits with one jump, that is, for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every bi-sequence of points of the form

$$\ldots, z_{-2} = T^{-2}y, z_{-1} = T^{-1}y, z_0 = x, z_1 = Tx, z_2 = T^2x, \ldots$$

with $d(x, y) < \delta$, where $T$ denotes the diffeomorphism, there exists a point $z$ such that $d(T^nz, z_n) < \varepsilon$ for all $n \in \mathbb{Z}$.

On the other hand, in [3, Theorem 5.1] it is shown that for every expansive homeomorphism on a compact space there exists a compatible metric (which we call hyperbolic metric) with similar properties to those of the metric $d$ in the case of Anosov diffeomorphisms. Then we realize that the proof of the shadowing property for Anosov diffeomorphisms given in [1] carries over the more general case of expansive systems.

In this note we give an alternative and elementary proof of this shadowing condition (Proposition 4.1) not making use of the hyperbolic metric of Fathi. Instead we use a reformulation of the condition of expansivity of a system (Proposition 3.1) which seems interesting on its own right.

2. TERMINOLOGY AND NOTATION

Throughout this note $X$ will be a compact metric space with metric $d$ and $T: X \to X$ a homeomorphism. The orbit under $T$ of a point $x \in X$ is the bi-sequence $O(x) = (T^n x)_{n \in \mathbb{Z}}$. 

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For terminology and notation in the Introduction we refer [1, §1.2]
Definition 2.1. $T$ is said to be expansive if there is $\alpha > 0$ such that if $x, y \in X$ and $d(T^n x, T^n y) \leq \alpha$ for all $n \in \mathbb{Z}$ then $x = y$. Such an $\alpha$ is called expansivity constant for $T$.

Definition 2.2. Let $\xi = (x_n)_{n \in \mathbb{Z}} \in X^\mathbb{Z}$ be a bi-sequence of elements of $X$.

If $\delta > 0$ and $d(T x_n, x_{n+1}) < \delta$ for all $n \in \mathbb{Z}$ then $\xi$ is called $\delta$-pseudo orbit of $T$.

We say that $\xi$ has a jump at the $n$-th step if $T x_{n-1} \neq x_n$.

Given $\varepsilon > 0$ a bi-sequence $\eta = (y_n)_{n \in \mathbb{Z}} \in X^\mathbb{Z}$ is said to $\varepsilon$-shadow $\xi$ if $d(x_n, y_n) < \varepsilon$ for all $n \in \mathbb{Z}$. If in the previous situation $\eta = O(y)$ is the orbit of an element $y \in X$ we simply say that $y$ $\varepsilon$-shadows $\xi$. We say that $\xi$ is $\varepsilon$-shadowed if is $\varepsilon$-shadowed by some point $y \in X$.

Definition 2.3. Given $\varepsilon > 0$ we say that $T$ has the $\varepsilon$-shadowing property if there exists $\delta > 0$ such that for each $\delta$-pseudo orbit $\xi$ there exists $x \in X$ that $\varepsilon$-shadows $\xi$. We say that $T$ has the shadowing property if it has the $\varepsilon$-shadowing property for all $\varepsilon > 0$. If $T$ is expansive and has the shadowing property then it is called Anosov homeomorphism.

3. Rephrasing expansivity

The following simple result states an equivalent condition for the expansivity of the system $(X, T)$. This alternative characterization of expansivity will allow us to give an elementary proof of the shadowing condition in Proposition 4.1.

Proposition 3.1. Let $\alpha > 0$. The following conditions are equivalent.

1. $T$ is expansive with expansivity constant $\alpha$.
2. For every $\varepsilon > 0$ there exists $\delta > 0$ such that

   if $d(x_n, y_n) \leq \alpha$ for all $n \in \mathbb{Z}$ then $d(x_n, y_n) < \varepsilon$ for all $n \in \mathbb{Z}$,

   for every pair of $\delta$-pseudo orbits $(x_n)_{n \in \mathbb{Z}}$ and $(y_n)_{n \in \mathbb{Z}}$ of $T$.

Proof. (1 $\Rightarrow$ 2) Suppose that the thesis is not true. Then, there exists $\varepsilon > 0$ such that for every $k \in \mathbb{N}$ one can find $1/k$-pseudo orbits $(x_n^k)_{n \in \mathbb{Z}}$ and $(y_n^k)_{n \in \mathbb{Z}}$ of $T$ satisfying $d(x_n^k, y_n^k) \leq \alpha$ for all $n \in \mathbb{Z}$ but $d(x_n^k, y_n^k) \geq \varepsilon$ for a suitable $n_k \in \mathbb{Z}$. Changing the indexing of the pseudo orbits if necessary it can be assumed that $n_k = 0$ for all $k \in \mathbb{N}$. As $X$ is compact it can be also assumed that $x_0^k \to x$ and $y_0^k \to y$ for some $x, y \in X$. It is easy to see that then $x_n^k \to T^n x$ and $y_n^k \to T^n y$ for all $n \in \mathbb{Z}$ (the pseudo orbits converge pointwise to actual orbits). But now, as $d(x_n^k, y_n^k) \leq \alpha$ for all $k \in \mathbb{N}$ and $n \in \mathbb{Z}$ we have $d(T^n x, T^n y) \leq \alpha$ for all $n \in \mathbb{Z}$, and as $d(x_0^k, y_0^k) \geq \varepsilon$ for all $k \in \mathbb{N}$ we get $d(x, y) \geq \varepsilon$, so that $x \neq y$. This contradicts that $\alpha$ in an expansivity constant and the proof finishes.

(2 $\Rightarrow$ 1) Suppose $x, y \in X$, $d(T^n x, T^n y) \leq \alpha$ for all $n \in \mathbb{Z}$ and note that $(T^n x)_{n \in \mathbb{Z}}$ and $(T^n y)_{n \in \mathbb{Z}}$ are $\delta$-pseudo orbits of $T$ for every $\delta > 0$. Then, by hypothesis, for every $\varepsilon > 0$ we have $d(T^n x, T^n y) < \varepsilon$ for all $n \in \mathbb{Z}$. That is $(T^n x)_{n \in \mathbb{Z}} = (T^n y)_{n \in \mathbb{Z}}$ and in particular $x = y$. This show that $\alpha$ is an expansivity constant for $T$. 

For later reference we recall the following basic property of expansive homeomorphisms on compact spaces known as uniform expansivity.
Proposition 3.2 ([2, Theorem 5]). Let $\alpha > 0$. The following conditions are equivalent.

1. $T$ is expansive with expansivity constant $\alpha$.
2. For every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that
   \[ d(T^n x, T^n y) \leq \alpha \text{ for all } |n| \leq N \text{ then } d(x, y) < \varepsilon, \]
   for every $x, y \in X$.

4. The shadowing condition

As pointed out in the Introduction the following is a known result that can be proved with the techniques in [1, p. 23] replacing the metric coming from the Riemannian structure in that argument by the hyperbolic metric introduced by Fathi in [3, Theorem 5.1].

Proposition 4.1. If $T$ is expansive with expansivity constant $\alpha > 0$ then the following conditions are equivalent.

1. $T$ has the shadowing property.
2. There exists $\delta > 0$ such that every one-jump $\delta$-pseudo orbit is $\alpha$-shadowed.

Proof. Clearly we only need to prove that the last statement implies the first one. By [4, Lemma 8] it is enough to show that for every $\varepsilon > 0$ there exists $\rho > 0$ such that all $\rho$-pseudo orbits with a finite number of jumps are $\varepsilon$-shadowed. To do that it is sufficient to find a $\rho > 0$ corresponding only to $\varepsilon = \alpha$, because by Proposition 3.1 for any $\varepsilon > 0$ taking a smaller value of $\rho$, more precisely choosing $\rho \leq \delta$ where $\delta$ is given by the cited proposition, we have that to $\alpha$-shadow a $\rho$-pseudo orbit is equivalent to $\varepsilon$-shadow it.

To find $\rho > 0$ such that every $\rho$-pseudo orbit with a finite number of jumps is $\alpha$-shadowed, let $\delta > 0$ be as in the statement of this proposition, that is, such that
\[ \text{every } \delta \text{-pseudo orbit with one jump is } \alpha \text{-shadowed.} \]

By Proposition 3.1 (with $\varepsilon = \alpha/2$) we can take a smaller value of $\delta$ if necessary to also guarantee that
\[ \text{if a } \delta \text{-pseudo orbit } \xi \alpha \text{-shadows a } \delta \text{-pseudo orbit } \eta \text{ then } \xi \alpha/2 \text{-shadows } \eta. \]

Obviously we can also require that $\delta \leq \alpha$. For this $\delta > 0$ according to Proposition 3.2 there exists $N \in \mathbb{N}$ such that
\[ \text{if } d(T^n x, T^n y) \leq \alpha \text{ for all } |n| \leq N \text{ then } d(x, y) < \delta, \]
for all $x, y \in X$. Finally, as $T$ is uniformly continuous we can take $\rho > 0$ such that any segment of length $2N + 1$ of a $\rho$-pseudo orbit, say $x_0, \ldots, x_{2N+1}$, is $\delta$-shadowed by its first element $x_0$, that is,
\[ \text{if } d(Tx_n, x_{n+1}) < \rho, \text{ } 0 \leq n \leq 2N, \text{ then } d(T^n x_0, x_n) < \delta, \text{ } 0 \leq n \leq 2N + 1. \]

Clearly we can also specify that $\rho \leq \delta$.

We will prove that this $\rho$ works by induction in the number of jumps in the $\rho$-pseudo orbits. If a $\rho$-pseudo orbit has only one jump, as $\rho \leq \delta$ we know by condition (1) that it can be $\alpha$-shadowed. Assume now that $\xi = (x_n)_{n \in \mathbb{Z}}$ is a $\rho$-pseudo orbit with $k \geq 2$ jumps. Indices can be arranged so that the last jump takes place in
the step from $x_{2N}$ to $x_{2N+1}$, so that $(x_n)_{n>2N}$ is a segment of a true orbit. By condition (4) we can replace $x_0, \ldots, x_{2N}$ in $\xi$ getting a $\delta$-pseudo orbit
\[ \xi' = (x_n)_{n<0} \sqcup (T^nx_0)_{0\leq n \leq 2N} \sqcup (x_n)_{n>2N} \]
which $\alpha$-shadows $\xi$ because $\delta \leq \alpha$. Note that
\[ \eta = (x_n)_{n<0} \sqcup (T^nx_0)_{n \geq 0} \]
is a $\rho$-pseudo orbit with less than $k$ jumps, then by the inductive hypothesis there exists $y \in X$ that $\alpha$-shadows $\eta$. By condition (2) we know that in fact $y/2$-shadows $\eta$. On the other hand consider
\[ \zeta = (T^nx_0)_{n \leq 2N} \sqcup (x_n)_{n>2N} \]
which is a $\delta$-pseudo orbit with one jump. Then by condition (1) there exists $z \in X$ that $\alpha$-shadows $\zeta$. Again condition (2) implies that $z/2$-shadows $\zeta$.

Now, as the segment of orbit $(T^n x_0)_{0 \leq n \leq 2N}$ is in both sequences $\eta$ and $\zeta$ we have that the corresponding segments of the orbits of $y$ and $z$ verifies $d(T^n y, T^n z) < \alpha$ for $0 \leq n \leq 2N$ Hence, by condition (3) we have that $d(T^N y, T^N z) < \delta$. Consequently $\tau = (T^ny)_{n \leq N} \sqcup (T^nz)_{n>2N}$ is a $\delta$-pseudo orbit with one jump that $\alpha/2$-shadows $\xi'$. A new application of condition (1) gives an element $w \in X$ that $\alpha$-shadows $\tau$. Finally, as $w/2$-shadows $\tau$, $\tau/2$-shadows $\xi'$ and $\xi'$-shadows $\xi$, we obtain by repeated application of condition (2) that $w$ $\alpha$-shadows $\xi$, and we are done. $\square$

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2We denote $(x_n)_{n \in I} \sqcup (x_n)_{n \in J} = (x_n)_{n \in I \cup J}$ if $I, J \subseteq \mathbb{Z}$ are disjoint sets of indices.