Spectral curves for the multi-phase solutions of Manakov system

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Abstract. The hierarchies of multi-component models of Manakov type are briefly outlined. Many of them have important applications to nonlinear optics. Next we develop a method for calculating the spectral curves for the multi-phase solutions of these equations. We discuss also other multi-component generalization of Manakov system to which our method can be extended.

1. Introduction
The ever growing traffic in data transmission system networks requires one to search for increase of the throughput (capacity) of the optical fibers. As a result the research units are actively working on new models in nonlinear optics. Many of these models have already been studied long enough but their practical use had started only recently. One of these models is Manakov system \cite{1}

\begin{equation}
\begin{aligned}
\partial_z p_1 &= i\sigma \partial^2_{z} p_1 - 2i\sigma (|p_1|^2 + |p_2|^2) p_1, \\
\partial_z p_2 &= i\sigma \partial^2_{z} p_2 - 2i\sigma (|p_1|^2 + |p_2|^2) p_2.
\end{aligned}
\end{equation}

Here \(\sigma = 1\) corresponds to defocusing regime (normal dispersion) while \(\sigma = -1\) is for focusing regime (anomalous dispersion). In particular Manakov system describes soliton signal propagation in optical fibers with randomly varying birefringence, see e.g. \cite{2-4} and the references therein.

It is well known that the inverse scattering method can be interpreted as nonlinear Fourier transform \cite{5,6}. These ideas have been applied to the analysis of signals whose propagation is described by the nonlinear Schrödinger equation (NLS)

\begin{equation}
\partial_z p = i\partial^2_{z} p - 2i\sigma |p|^2 p = 0
\end{equation}

see, e.g. \cite{7,8}. These ideas can be applied also not only to all equations from the AKNS hierarchy \cite{5}, but also to their multi-component generalizations related to the symmetric spaces \cite{6,9}. In practical applications using the direct nonlinear Fourier transform one could determine the continuous and the discrete spectrum of the signal, see e.g. \cite{10,11}. The reconstruction of the signal from its
spectrum is done using the inverse nonlinear Fourier transform. Important role for the recovery of the signal plays also the spectral curve [12,13].

Important difference between the nonlinear Fourier transforms for the system (1) and for the NLS equation (2) is in the dimension of the corresponding linear problem and the relevant spectral curves. The spectral curves for the multi-phase solutions of the NLS equation are hyperelliptic, see e.g., [14]. One of the methods for constructing these curves using the values of the function \( p \) and its derivatives has been proposed by one of us (AOS) in [15,16]. The spectral curves of the system (1) are not hyperelliptic, but the method for their construction proposed by Dubrovin in [17] is not so well known to practitioners. That is why in the present paper we use modification of this method used for constructing spectral curves for multi-phase solutions of Manakov hierarchy. We will see that this method can be used for constructing the spectral curves for related hierarchies. Naturally, this method can not be applied to non-integrable models in nonlinear optics.

In Section 2 we briefly describe the hierarchies of multi-component models, generalizing the nonlinear Schrödinger equation (NLS). The simplest non-trivial one of them is known as Manakov model [1] and has important applications to nonlinear optics. In Section 3 we develop a method for calculating the spectral curves for the multi-phase solutions of these equations. In the last Section 4 we discuss other multi-component generalization of Manakov system to which our method can be extended.

2. Hierarchies of integrable equations

As a rule the integrable nonlinear evolution equations (NLEE) are obtained as the compatibility condition of systems of linear differential equations. A large and important class of NLEE are obtained as the compatibility conditions of the following matrix differential equations:

\[
\begin{align*}
\partial_t \Psi &= U \Psi, \\
\partial_{z_k} \Psi &= V_k \Psi,
\end{align*}
\]

where

\[ U = \lambda J + U^0, \quad V_i = s \lambda U + V_i^0, \quad V_{k+1} = s \lambda V_k + V_{k+1}^0, \quad k \ldots 1, \]

\( J \) is a constant diagonal matrix, \( \text{Tr} J = 0 \), and \( s \) is a normalization factor, \( \text{Im}(s) = 0 \).

The compatibility conditions \((\Psi_i, \Psi_k) = (\Psi_k, \Psi_i)\), and \((\Psi_i, z_k) = (\Psi_k, z_i)\), give rise to the matrix equations:

\[
\begin{align*}
\partial_t V_n &= [U, V_n] + \partial_{z_k} U, \\
\partial_{z_k} V_n &= [V_k, V_n] + \partial_{\zeta_n} V_k,
\end{align*}
\]

which must hold true identically with respect to \( \lambda \). Equating in (4) the matrix coefficients of the same powers of \( \lambda \) we obtain the following relations between the matrices \( V_k^0 \)

\[
\partial_t U^0 = \frac{1}{s} [J, V_1^0], \quad \partial_k V_k^0 = \frac{1}{s} [J, V_{k+1}^0] - [V_k^0, U^0], \quad k \ldots 1,
\]

and finally derive the following integrable NLEE:

\[
\partial_{z_k} U^0 = \partial_k V_k^0 + [V_k^0, U^0] = \frac{1}{s} [J, V_{k+1}^0].
\]

The equations of the AKNS hierarchy [5,16,18] are the compatibility conditions of the equations (3) for \( s = 2 \).
\[ J = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad U^0 = \begin{pmatrix} 0 & ip \\ -iq & 0 \end{pmatrix}. \]

For the case \( s=1 \) and
\[
J = \frac{1}{3} \begin{pmatrix} -2i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{pmatrix}, \quad U^0 = \begin{pmatrix} 0 & ip_1 & ip_2 \\ -iq_1 & 0 & 0 \\ -iq_2 & 0 & 0 \end{pmatrix}
\]

the matrix \( V_i^0 \) equals
\[
V_i^0 = \begin{pmatrix} -F_{i1}^{11} - F_{i1}^{22} & H_i^1 & H_i^2 \\ G_i^1 & F_{i1}^{11} & F_{i1}^{12} \\ G_i^2 & F_{i1}^{21} & F_{i1}^{22} \end{pmatrix},
\]

where
\[
H_i^1 = -\partial_i p_1, \quad H_i^2 = -\partial_i p_2, \quad G_i^1 = -\partial_i q_1, \quad G_i^2 = -\partial_i q_2,
\]

and
\[
\partial_i F_{i1}^{11} = -iq_1 H_i^1 - ip_1 G_i^1, \quad \partial_i F_{i1}^{12} = -iq_1 H_i^2 - ip_2 G_i^1, \\
\partial_i F_{i1}^{21} = -iq_2 H_i^1 - ip_1 G_i^2, \quad \partial_i F_{i1}^{22} = -iq_2 H_i^2 - ip_2 G_i^2.
\]

Integrating we obtain:
\[
F_{i1}^{11} = ip_1 q_1 + i\alpha_i, \quad F_{i1}^{12} = ip_2 q_1, \quad F_{i1}^{21} = ip_1 q_2, \quad F_{i1}^{22} = ip_2 q_2 + i\alpha_i,
\]

where \( \alpha_i \) is an integration constant.

The integrable NLEE obtained from the compatibility condition take the form:
\[
\partial_{t_1} p_1 = -iq_1 H_i^1 - ip_1 F_{i1}^{21} - p_1 (2F_{i1}^{11} + F_{i1}^{22}), \\
\partial_{t_1} p_2 = -iq_1 H_i^2 - ip_2 F_{i1}^{12} - p_2 (2F_{i1}^{11} + F_{i1}^{12}), \\
\partial_{t_1} q_1 = i\partial_i G_i^1 + q_2 F_{i1}^{12} + q_1 (2F_{i1}^{11} + F_{i1}^{22}), \\
\partial_{t_1} q_2 = i\partial_i G_i^2 + q_1 F_{i1}^{21} + q_2 (2F_{i1}^{11} + F_{i1}^{22}),
\]

or (for \( \alpha_i \neq 0 \))
\[
\partial_{t_1} p_1 = i\partial_i^2 p_1 - 2i(p_1 q_1 + p_2 q_2) p_1 - 3i\alpha_i p_1, \\
\partial_{t_1} p_2 = i\partial_i^2 p_2 - 2i(p_1 q_1 + p_2 q_2) p_2 - 3i\alpha_i p_2, \\
\partial_{t_1} q_1 = -i\partial_i^2 q_1 + 2i(p_1 q_1 + p_2 q_2) q_1 + 3i\alpha_i q_1, \\
\partial_{t_1} q_2 = -i\partial_i^2 q_2 + 2i(p_1 q_1 + p_2 q_2) q_2 + 3i\alpha_i q_2.
\]

(8)

If \( q_j = \sigma_j p_j^* \), where \( \sigma_j = \pm 1 \), and \( \text{Im} \alpha_i = 0 \), then the four equations (8) reduce to the following two:
\begin{equation}
\partial_\tau p_1 = i\hat{\partial}_\tau^2 p_1 - 2i(\sigma_1 | p_1 |^2 + \sigma_2 | p_2 |^2) p_1 - 3i\alpha_1 p_1, \\
\partial_\tau p_2 = i\hat{\partial}_\tau^2 p_2 - 2i(\sigma_1 | p_1 |^2 + \sigma_2 | p_2 |^2) p_2 - 3i\alpha_1 p_2.
\end{equation} \tag{9}

For \(\sigma_1 = \sigma_2 = \sigma, \alpha_j = 0\) the system (9) takes the form (1).

The next equations of the hierarchy are obtained through the recurrent relations. Let:

\[
V_k^0 = \begin{pmatrix}
-F_k^{11} - F_k^{22} & H_k^1 & H_k^2 \\
F_k^1 & F_k^{11} & F_k^{12} \\
G_k^2 & F_k^{21} & F_k^{22}
\end{pmatrix}
\]

Then

\[
H_k^1 = i\partial_\tau H_{k-1}^1 + p_2 F_k^{21} + \rho_i (2F_k^{11} + F_k^{22}), \\
H_k^2 = i\partial_\tau H_{k-1}^2 + p_2 F_k^{12} + \rho_i (2F_k^{22} + F_k^{11}), \\
G_k^1 = -i\partial_\tau G_{k-1}^1 - q_2 F_k^{12} - q_2 (2F_k^{11} + F_k^{22}), \\
G_k^2 = -i\partial_\tau G_{k-1}^2 - q_2 F_k^{21} - q_2 (2F_k^{22} + F_k^{11})
\]

and

\[
\partial_\tau F_k^{11} = -iq_1 H_k^1 - ip_1 G_k^1, \quad \partial_\tau F_k^{12} = -iq_1 H_k^2 - ip_2 G_k^1, \\
\partial_\tau F_k^{21} = -iq_2 H_k^1 - ip_2 G_k^2, \quad \partial_\tau F_k^{22} = -iq_2 H_k^2 - ip_2 G_k^2.
\]

The integrable NLEE in the Manakov hierarchy acquire the form:

\[
\partial_\tau p_1 = -i\partial_\tau H_k^1 - p_2 F_k^{21} + \rho_i (2F_k^{11} + F_k^{22}) = -H_{k+1}^1, \\
\partial_\tau p_2 = -i\partial_\tau H_k^2 - p_2 F_k^{12} + \rho_i (2F_k^{22} + F_k^{11}) = -H_{k+1}^2, \\
\partial_\tau q_1 = i\partial_\tau G_k^1 + q_2 F_k^{12} + q_2 (2F_k^{11} + F_k^{22}) = -G_{k+1}^1, \\
\partial_\tau q_2 = i\partial_\tau G_k^2 + q_2 F_k^{21} + q_2 (2F_k^{22} + F_k^{11}) = -G_{k+1}^2.
\] \tag{10}

In particular

\[
H_k^1 = -i\hat{\partial}_\tau^2 p_1 + 2i(p_1 q_1 + p_2 q_2) p_1 + 3i\alpha_1 p_1, \\
H_k^2 = -i\hat{\partial}_\tau^2 p_2 + 2i(p_1 q_1 + p_2 q_2) p_2 + 3i\alpha_1 p_2, \\
G_k^1 = i\hat{\partial}_\tau^2 q_1 - 2i(p_1 q_1 + p_2 q_2) q_1 - 3i\alpha_1 q_1, \\
G_k^2 = i\hat{\partial}_\tau^2 q_2 - 2i(p_1 q_1 + p_2 q_2) q_2 - 3i\alpha_1 q_2
\]

and

\[
F_k^{11} = p_1 \partial_\tau q_1 - q_1 \partial_\tau p_1 + \alpha_2, \quad F_k^{12} = p_2 \partial_\tau q_1 - q_1 \partial_\tau p_2, \\
F_k^{21} = p_2 \partial_\tau q_2 - q_2 \partial_\tau p_2 + \alpha_2, \quad F_k^{22} = p_1 \partial_\tau q_2 - q_2 \partial_\tau p_1.
\]

The integrable NLEE for \(n=2\) are the vector mKdV equations:
\[
\begin{aligned}
\partial_{z_1} p_1 &= -\partial_z^3 p_1 + 3q_2 (p_2 \partial_z p_1 + p_1 \partial_z p_2) + 6 p_1 q_1 \partial_z p_1 + 3\alpha_1 \partial_z p_1 - 3\alpha_2 p_1, \\
\partial_{z_2} p_2 &= -\partial_z^3 p_2 + 3q_1 (p_2 \partial_z p_1 + p_1 \partial_z p_2) + 6 p_2 q_2 \partial_z p_2 + 3\alpha_1 \partial_z p_2 - 3\alpha_2 p_2, \\
\partial_{z_1} q_1 &= -\partial_z^3 q_1 + 3 p_2 (q_2 \partial_z q_1 + q_1 \partial_z q_2) + 6 p_1 q_1 \partial_z q_1 + 3\alpha_1 \partial_z q_1 + 3\alpha_2 q_1, \\
\partial_{z_2} q_2 &= -\partial_z^3 q_2 + 3 p_1 (q_2 \partial_z q_1 + q_1 \partial_z q_2) + 6 p_2 q_2 \partial_z q_2 + 3\alpha_1 \partial_z q_2 + 3\alpha_2 q_2.
\end{aligned}
\]

From the equations (10) there follows, that, like in the scalar case [16], using the multi-phase solutions

\[
p_j(t, z_1, z_2, \ldots), \quad q_j(t, z_1, z_2, \ldots)
\]

of the hierarchy, one can obtain solutions

\[
p_j(t, \gamma_1(z), \gamma_2(z), \ldots), \quad q_j(t, \gamma_1(z), \gamma_2(z), \ldots)
\]

of the mixed equations:

\[
\begin{aligned}
\partial_{z_1} p_1 &= -\sum_{k=1}^n \gamma_k'(z) H_{k+1}^1, & \partial_{z_2} p_2 &= -\sum_{k=1}^n \gamma_k'(z) H_{k+1}^2, \\
\partial_{z_1} q_1 &= -\sum_{k=1}^n \gamma_k'(z) G_{k+1}^1, & \partial_{z_2} q_2 &= -\sum_{k=1}^n \gamma_k'(z) G_{k+1}^2.
\end{aligned}
\]

3. Spectral curves equations

In the case of periodic potentials in eq. (3) the function

\[
\Psi(t + T, z_1, z_2, \ldots) = M(\lambda)\Psi(t, z_1, z_2, \ldots), \quad \text{ (11)}
\]

will also satisfy the system (3). Here \(T\) is the period and \(M(\lambda)\) is the monodromy matrix. In the case of finite-zone solutions the monodromy matrix is polynomial in \(\lambda\) and the equation for the spectral curve takes the form [17]

\[
det(M(\lambda) - \mu(\lambda)I) = 0,
\]

where \(I\) is the unit matrix.

Substituting (11) into (3) and after some simplifications we obtain:

\[
\begin{aligned}
\partial_t M &= [U, M], \\
\partial_{z_k} M &= [V_k, M].
\end{aligned} \quad \text{ (12)}
\]

The equations (12) put restrictions on the matrix elements of \(M\) and provides differential equations that they should satisfy. It is easy to check that any solution of the equations (12) can be written in the form:

\[
M(\lambda, t, z_1, z_2, \ldots) = \chi^\pm M_0(\lambda)(\chi^\mp)^{-1}, \quad \text{ (13)}
\]

where \(M_0(\lambda)\) is a matrix independent of \(t\) and \(z_k\) and for further convenience we have replaced \(\Psi\) by the fundamental analytic solutions \(\chi^\pm(\lambda, t, z_1, z_2, \ldots)\) of the Lax operator \(L = (\partial_z - U)\) given by (3) and (7). Here the superscripts \(\pm\) mean analyticity in \(\lambda\) for \(\text{Im } \lambda > 0\) or \(\text{Im } \lambda < 0\), respectively; for the methods of their construction see e.g. [6]. If we choose as dispersion law \(M_0(\lambda) = s \lambda^k J\) which is
associated with the variable \( z_k \), then the polynomial part of \( M(13) \) will coincide with \( V_k \) introduced in (3). In addition, the coefficients \( C_m \) in the expansion

\[
\text{Tr} \left( J \chi^\pm J(\chi^\pm)^{-1} \right) = -\frac{2}{3} + \sum_{m=1}^\infty \lambda^{-m} C_m, \tag{14}
\]

will be the densities of the local series of conserved quantities of the Manakov model, see [18,6].

Another possibility is to choose \( M_0 \) as linear combinations of \( E_{1,2} \), \( E_{1,3} \), \( E_{2,1} \) and \( E_{2,3} \), and restrict the right hand side of (13) to the image of \( \text{ad}_J \), see (7):

\[
M_{ab}^z = \text{ad}^{-1}_J \left( \chi^z E_{ab} (\chi^z)^{-1} \right), \tag{15}
\]

where the matrices \( E_{ab} \) are defined by \( (E_{ab})_{kn} = \delta_{ak} \delta_{bm}, \) and \( (a, b) \in \{(1, 2), (1, 3), (2, 1), (3, 1)\} \).

The sets \( M_{ab}^z \) are the well known sets of “squared solutions” which are complete sets of functions in the space of all allowed potentials \( U^0 \) like in (7), see [6].

The analysis of eqs. (12), (13) shows, that the matrix \( M(\lambda) \) for the \( n \)-phase finite-zone solution must have the form:

\[
M = V_n + \sum_{k=1}^{n-1} c_k V_{n-k} + c_n U + J_n, \tag{16}
\]

where \( J_n \) is a diagonal constant matrix with \( \text{Tr} J_n = 0 \).

Inserting (16) into (12) and simplifying by the use of (4), we obtain the following equations:

\[
\begin{align*}
\left( \partial_{z_k} + \sum_{k=1}^{n-1} c_k \partial_{z_{k+1}} + c_n \partial_{t} \right) U &= c_{n+1} [U, J_n], \\
\left( \partial_{z_k} + \sum_{k=1}^{n-1} c_k \partial_{z_{k+1}} + c_n \partial_{t} \right) V_k &= c_{n+1} [V_k, J_n].
\end{align*} \tag{17}
\]

The \( z_n \)-derivatives of the matrix elements of \( U^0 \) with the conditions (17) can be replaced by the matrix elements of \( U^0 \).

In particular from the equations (10) for the Manakov system we obtain the relations:

\[
\partial_{z_n} U = \begin{bmatrix}
0 & -iH_{n+1}^1 & -iH_{n+1}^2 \\
iG_{n+1}^1 & 0 & 0 \\
iG_{n+1}^2 & 0 & 0
\end{bmatrix}.
\]

Assuming the matrix \( J_n \) takes the form:

\[
J_n = \begin{bmatrix}
-i c_{n+1} - i c_{n+2} & 0 & 0 \\
0 & 2i c_{n+1} - i c_{n+2} & 0 \\
0 & 0 & 2i c_{n+2} - i c_{n+1}
\end{bmatrix},
\]

we obtain

\[
[U, J_n] = \begin{bmatrix}
0 & -3c_{n+1}q_1 & -3c_{n+2}p_2 \\
-3c_{n+1}q_1 & 0 & 0 \\
-3c_{n+2}q_2 & 0 & 0
\end{bmatrix}.
\]
The first of the equations in (17) of the Manakov hierarchy can be written down as the system:

\[-iH^1_{n+1} - i \sum_{k=1}^{n-1} c_k H^1_{n+1-k} + ic_n \partial_t p_1 = -3c_{n+1} p_1,\]

\[-iH^2_{n+1} - i \sum_{k=1}^{n-1} c_k H^2_{n+1-k} + ic_n \partial_t p_2 = -3c_{n+2} p_2,\]

\[iG^1_{n+1} + i \sum_{k=1}^{n-1} c_k G^1_{n+1-k} - ic_n \partial_t q_1 = -3c_{n+1} q_1,\]

\[iG^2_{n+1} + i \sum_{k=1}^{n-1} c_k G^2_{n+1-k} - ic_n \partial_t q_2 = -3c_{n+2} q_2.\]

From the algebraic-geometric integration theory [14,16,19] there follows that for any \(n\)-phase solutions the following relation holds:

\[\partial_{z_{n-k}} p_j = \sum_{m=1}^{n-1} c_{k,m} \partial_{z_{m-n}} p_j + c_{k,n} \partial_t p_j + c_{k,n+1} p_j, \quad k \geq 0,
\]

where \(c_{k,m}\) are some constants. Therefore the \(N\)-phase solutions of Manakov hierarchy satisfy the equations (18) where \(N \leq n\).

**Example 1.** The monodromy matrix of the 0-phase solution takes the form:

\[M_0 = U + J_0 = \begin{pmatrix} -2i\lambda /3 - ic_1 - ic_2 & ip_1 & ip_2 \\ -iq_1 & i\lambda /3 + 2ic_1 - ic_2 & 0 \\ -iq_2 & 0 & i\lambda /3 - ic_1 + 2ic_2 \end{pmatrix},\]

The equations (18), satisfied by the 0-phase solutions substantially simplify to:

\[i\partial_t p_1 = -3c_1 p_1, \quad -i\partial_t q_1 = -3c_1 q_1,\]

\[i\partial_t p_2 = -3c_2 p_2, \quad -i\partial_t q_2 = -3c_2 q_2.\]

Therefore the 0-phase solutions of Manakov system (1) become:

\[p_1 = a_1 e^{3ic_j t + i\phi_j(z)}, \quad q_1 = \sigma a_1^* e^{-3ic_j t - i\phi_j(z)},\]

\[p_2 = a_2 e^{3ic_j t + i\phi_j(z)}, \quad q_2 = \sigma a_2^* e^{-3ic_j t - i\phi_j(z)},\]

where \(\text{Im}(c_j) = 0\). The expression for the functions \(\phi_j(z)\) follow from the equations (1)

\[\phi_j(z) = -(9c_j^2 + 2\sigma(|a_1|^2 + |a_2|^2))z, \quad j = 1, 2.\]

The spectral curve of the solution (20) is given by the equation

\[\nu^3 + P(\lambda)\nu + Q(\lambda) = 0,\]

where \(\nu = iv /3,\)

\[P(\lambda) = -3\lambda^2 - 9(c_1 + c_2)\lambda + 9\sigma(|a_1|^2 + |a_2|^2) - 27(c_1^2 - c_1c_2 + c_2^2),\]
\[ Q(\lambda) = 2\lambda^3 + 9(c_1 + c_2)\lambda^2 - (9\sigma(|a_1|^2 + |a_2|^2) + 27(c_1^2 - 4c_1c_2 + c_2^2))\lambda \\
+ 27\sigma(c_1(|a_1|^2 - 2|a_2|^2) + c_2(|a_2|^2 - 2|a_1|^2)) - 27(c_1 + c_2)(c_1 - 2c_2)(2c_1 - c_2). \]

The discriminant of the eq. (21) is a polynomial of fourth order with respect to \( \lambda \). Therefore, the spectral curve has four branching points. Applying the Riemann-Hurwitz formula \[ n = N/2 - M + 1 \]
where \( N \) is the number of branching points and \( M \) is the number of sheets of the spectral curve we obtain that \( n=0 \).

**Example 2.** The monodromy matrix of the one-phase solution is given by:

\[ M_1 = V_1 + cU + J_1. \]

The equations for the one-phase solutions (18) have the form:

\[
\begin{align*}
\partial^2_t p_1 &= ic_1\partial_t p_1 + 3c_2 p_1 + 2p_1(p_1q_1 + p_2q_2), \\
\partial^2_t p_2 &= ic_1\partial_t p_2 + 3c_3 p_2 + 2p_2(p_1q_1 + p_2q_2), \\
\partial^2_t q_1 &= -ic_1\partial_t q_1 + 3c_2 q_1 + 2q_1(p_1q_1 + p_2q_2), \\
\partial^2_t q_2 &= -ic_1\partial_t q_2 + 3c_3 q_2 + 2q_2(p_1q_1 + p_2q_2).
\end{align*}
\]

(21)

It is well known, that the simplest solutions of the vector NSL equations are related to solutions of the scalar NLS. In particular, it is easy to see that for \( c_1 = -4b, c_2 = c_3 = 4(a^2 + b^2) \) the equations (22) satisfy the one-soliton solution of the form:

\[
\begin{align*}
p_1 &= \frac{2a \sin(\alpha) e^{-2ibt+4i(\alpha^2-b^2)\zeta}}{\cosh(2at+8ab\zeta)}, \\
q_1 &= \frac{-2a \sin(\alpha) e^{2ibt-4i(\alpha^2-b^2)\zeta}}{\cosh(2at+8ab\zeta)}, \\
p_2 &= \frac{2a \cos(\alpha) e^{-2ibt+4i(\alpha^2-b^2)\zeta}}{\cosh(2at+8ab\zeta)}, \\
q_2 &= \frac{-2a \cos(\alpha) e^{2ibt-4i(\alpha^2-b^2)\zeta}}{\cosh(2at+8ab\zeta)}.
\end{align*}
\]

(22)

The spectral curve of the solution (23) splits into components. Its equation is of the form:

\[ (v + 2\lambda^2 - 8b\lambda + 8a^2 + 8b^2)(v - \lambda^2 + 4b\lambda - 4b^2 - 4a^2)^2 = 0. \]

This is related to the fact that components of the solution \( p_1 \) and \( p_2 \) are linearly dependent. Indeed, transforming the matrix differential equation for the \( \Psi \)-function into a scalar third order equation for its first component we get:

\[ W\psi_1 - \partial_t W\psi_1 + A(\lambda)\psi_1 + B(\lambda)\psi_1 = 0, \]

(23)

where

\[
A(\lambda) = \frac{1}{3} W\lambda^2 - \frac{i}{3} \partial_t W\lambda + W_1 - (p_1q_1 + p_2q_2)W, \\
B(\lambda) = -\frac{2i}{27} W\lambda^3 \frac{2}{9} \partial_t W\lambda^2 + \frac{i}{3} (2W_1 + (p_1q_1 + p_2q_2)W)\lambda \\
+ (p_1q_1 + p_2q_2)\partial_t W - (p_1\partial_1 q_1 + 2q_1\partial_1 p_1 + p_2\partial_2 q_2 + 2q_2\partial_2 p_2)W, \\
W = \begin{vmatrix} p_1 & p_2 \\ \partial_1 p_1 & \partial_2 p_2 \end{vmatrix}, \\
W_1 = \begin{vmatrix} \partial_1 p_1 & \partial_2 p_2 \\ \partial_1^2 p_1 & \partial_2^2 p_2 \end{vmatrix}.
\]
It is easy to see that equation (24) becomes singular if the functions $p_1$ and $p_2$ are linearly dependent. Therefore, solutions of Manakov system constructing from solutions of the scalar NLS do not allow non-degenerate spectral curve.

Note that in the case of the solution (20) equation (24) takes the form:

$$\psi_1 - 3i(c_1 + c_2)\psi_1 + A_0(\lambda)\psi_1 + B_0(\lambda)\psi_1 = 0,$$

where

$$A_0(\lambda) = \frac{1}{3} \lambda^2 + (c_1 + c_2)\lambda - 9c_1c_2 - \sigma(|a_1|^2 + |a_2|^2),$$

$$B_0(\lambda) = -\frac{2i}{27} \lambda^3 - \frac{2i}{3}(c_1 + c_2)\lambda^2 - \frac{i}{3}(18c_1c_2 - \sigma(|a_1|^2 + |a_2|^2))\lambda + 3i\sigma(c_1|a_2|^2 + c_2|a_1|^2).$$

It is easy to check that equation (25) allows solutions of the form:

$$\psi_1 = \exp \left\{ i \left( \frac{1}{3} \nu(\lambda) + c_1 + c_2 \right) t \right\},$$

where $\nu(\lambda)$ satisfies the spectral curve equation (21).

4. Generalizations of Manakov system

It is well known that Manakov system is just one of a variety of multi-component NLS equations. Indeed, Fordy and Kulish [9] demonstrated that with each symmetric space [20] one can relate a generalization of Manakov system.

We will mention here $n$-component vector NLS equation:

$$\partial_z \tilde{p} = i\tilde{\sigma}^2 \tilde{p} - 2i\sum_{m=1}^{n} \sigma_m \left| p_m \right|^2 \tilde{p},$$

(25)

where $\sigma_m = \pm 1$ and $\tilde{p} = (p_1, \ldots, p_n)^T$. The potential $U = \lambda J + U_0$ of the corresponding Lax operator is of the form:

$$U = i \begin{pmatrix} n\lambda & p_1 & p_2 & \cdots & p_n \\ \sigma_1 p_1^* & -\lambda & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \sigma_n p_n^* & 0 & 0 & \cdots & -\lambda \end{pmatrix}.$$

The corresponding symmetric space is the factor group $SU(n+1)/S(U(1)\times U(n))$; it is of type A.III in Cartan classification [20]. Eq. (26) with $\sigma_m = 1$ finds also application in nonlinear optics and in Bose-Einstein condensates, see e.g. [21] and the references therein.

Another integrable form of the multi-component NLS equation was discovered by Kulish and Sklyanin [22,23]. The corresponding Lax operator is of the form (3) with

$$U = i \begin{pmatrix} -\lambda & \tilde{p}^T & 0 \\ \bar{q} & 0 & s_0 \tilde{p} \\ 0 & \bar{q}^T s_0 & \lambda \end{pmatrix}, \quad s_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \tilde{p}(x,t) = (p_1, p_2, p_3)^T, \quad \bar{q}(x,t) = (q_1, q_2, q_3)^T.$$

(26)
The generic NLEE's related to it are:

\[ i \frac{\partial \bar{p}}{\partial z} + \frac{\partial^2 \bar{p}}{\partial t^2} + 2(\bar{p}^T \bar{q}) \bar{p} - (\bar{p}^T s_0 \bar{p})s_0 \bar{q} = 0, \]

\[ -i \frac{\partial \bar{q}}{\partial z} + \frac{\partial^2 \bar{q}}{\partial t^2} + 2(\bar{q}^T \bar{p}) \bar{q} - (\bar{q}^T s_0 \bar{q})s_0 \bar{p} = 0. \]

This system of equations allows reduction

\[ q_1 = \sigma_1 p_1^*, \quad q_2 = \sigma_2 p_2^*, \quad q_3 = \sigma_3 p_3^*, \]

Where \( \sigma_i = \pm 1 \). The Kulish-Sklyanin model is obtained if \( \sigma_1 = \sigma_2 = 1 \):

\[ i \frac{\partial \bar{q}}{\partial z} + \frac{\partial^2 \bar{q}}{\partial t^2} + 2(\bar{q}^T \bar{q}) \bar{q} - (\bar{q}^T s_0 \bar{q})s_0 \bar{q}^* = 0. \]

This model describes Bose-Einstein condensate of spin 1. The direct and the inverse scattering problem for the Lax operator (3) with \( U \) given by (27) has been solved in [23]. The derivation of the soliton solutions and their interactions have been described in [24].

The list of multi-component NLS equations can be extended by models related to other symmetric spaces [20], e.g. to spaces \( Sp(2n)/U(n) \) of type C.I, to spaces \( Sp(2n+2m)/Sp(2n) \times Sp(2m) \) of type C.II, etc. For each of them the above method of calculating the spectral curves for the multi-phase solutions can also be extended. These solutions and their properties will be analyzed elsewhere.

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