Relational evolution with oscillating clocks

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Abstract

A fundamental description of time can be consistent not only with the usual monotonic behavior but also with a periodic physical clock variable, coupled to the degrees of freedom of a system evolving in time. Generically, one would in fact expect some kind of oscillating motion of a system that is dynamical and interacts with its surroundings, as required for a fundamental clock that can be noticed by any other system. Unitary evolution does not require a monotonic clock variable and can be achieved more generally by formally unwinding the periodic clock movement, keeping track not only of the value of the clock variable but also of the number of cycles it has gone through at any moment. As a result, the clock is generically in a quantum state with a superposition of different clock cycles, a key feature that distinguishes oscillating clocks from monotonic time. Because the clock and an evolving system have a common conserved energy, the clock is in different cycles for different energy eigenstates of the system state. Coherence could therefore be lost faster than observed, for instance if a system that would be harmonic in isolation is made anharmonic by interactions with a fundamental clock, implying observational bounds on fundamental clocks. Numerical computations show that coherence is maintained over long time scales provided the clock period is much smaller than the system period. A small loss of coherence nevertheless remains and, measured in terms of the relative standard deviation of the system period, is proportional to the ratio of the system period and the clock period. Since the precision of atomic clocks could not be achieved if atomic frequencies would be subject to additional variations from coupling to a fundamental clock, an upper bound on the clock period can be obtained that turns out to be much smaller than currently available direct or indirect measurements of time.

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1 Introduction

Quantum mechanics does not treat space and time on the same footing, even in its relativistic version. For instance, a meaningful probabilistic interpretation of quantum states requires that time evolution is unitary such that an evolving state is always normalized. Unitarity implies that time will keep going on forever. Positions in space, by contrast, can easily be limited to finite regions or periodic boundary conditions, as in the basic examples of an infinite square well and of a particle required to move on a circle.

Such a distinction between space and time appears to violate relativity, in particular the general covariance of arbitrary transformations of time and space coordinates realized in general relativity. In this context, attempts to combine quantum properties with general relativity in a theory of quantum gravity have indeed encountered several obstacles collectively referred to as the problem of time [1, 2, 3]. For instance, the perpetual nature of time, encoded mathematically in the condition of unitarity in quantum mechanics, is incompatible with the possibility of the universe (and therefore time) having a beginning at the big bang, or an end if the universe happens to collapse in some distant future. Moreover, time in general relativity is a local coordinate that, in general, need not be defined in the same way everywhere in space-time even if the universe does not encounter a physical boundary.

The picture of unitary evolution in which monotonic time seems to be required to never cease increasing is also add odds with our physical measurements of time, which are based on periodic phenomena such as planetary orbits, the moving hands on a clock, or the vibrations of a quartz. (For a detailed discussion of physical clocks, see for instance [4].) Time is measured by periodic processes, but represented mathematically in a linear, monotonic fashion. There is an interesting dichotomy between periodic and monotonic phenomena in the context of time, which is hard to resolve because we have a very intuitive understanding of how we experience time but do not know well what time is on a fundamental level. We experience time as pointing from the past to the future without being able to move back, as perhaps indicated by the second law of thermodynamics. The temporal labels we attach to events are accordingly based on conventions that imply monotonic behavior: While the numbers we conventionally assigned to what we call time in a strict sense (second, minute, hour) reflect the periodic nature of how we measure brief intervals, the complete date (day, month, year) of an event renders the assignment monotonic.

The appearance of the traditional periodic processes in measurements of time is not fundamental but determined by their utility. It is easier to recognize change by observing a periodic system returning to a fixed state multiple times, compared with the gradual motion of a monotonic process. In addition, the compactness of a periodic scale makes it easier to construct portable or wearable clocks. The prevalence of periodic processes in time measurements therefore does not imply that time must fundamentally be based on a periodic process. Then again, the possibility of constructing a monotonic label of time (and date) by counting the cycles that a hierarchy of periodic processes goes through, as in our actual time-and-date measurements, shows that the monotonicity of our experienced time does not imply either that time fundamentally must be monotonic.
In this situation, any statement about fundamental clocks requires fundamental physics. Here, we use two main ingredients to argue that time should fundamentally be based on a periodic process. At the same time, we define what we mean by a “periodic process” in a formulation that does not use a monotonic background time. First, in order to bring space and time on a more equal footing in quantum mechanics, a decades-old treatment postulates that time $t$ should be represented by an operator (or a phase-space coordinate in a classical theory) just like the spatial position $x$. Such a formulation of evolution is called relational because it describes how one physical degree of freedom, $x$, evolves with respect to another physical degree of freedom, $t$, rather than how a single physical degree of freedom evolves with respect to some external time parameter. (Of course, in a relational formulation we could equally well describe how $t$ evolves with respect to $x$, but in keeping with conventions, we choose to call the reference degree of freedom $t$.) This idea goes back to Dirac [5] and Bergmann [6] and has seen much recent interest in the context of combining quantum physics with relativity and gravity [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22].

Secondly, if time as a physical degree of freedom is on an equal footing with space, or with matter degrees of freedom, it may in general be expected to have interactions with itself or with other degrees of freedom. In a fundamental theory, the reference degree of freedom we use to describe time could be selected from one of the fundamental fields of the standard model of particle physics, or from a physical degree of freedom that determines the structure of space-time. If we require that the degree of freedom we call time is such that it does not have any self-interactions, as in traditional formulations of relational evolution, this degree of freedom takes on a special form devised just for the purpose of being able to play the role of time as we think we know it. Such an assumption would forgo any possibility of determining fundamental properties of time. More generally, if time is based on a fundamental clock, it should generically be expected to have self-interactions, perhaps described by a potential. The reference clock degree of freedom could then have “turning points” which confine its values to a certain finite range. At this point we also have to address a common language problem: Our standard experience of a monotonic background time is so common that it is built into several physics concepts, such as “turning points,” that we still have to refer to even when we try to formulate an oscillating fundamental clock. The conventional term “turning points” assumes motion with respect to some background time, but here we are not interested in this motion. We merely refer to the confinement to a finite range that may be implied by a potential and, for lack of an alternative term, use the established dynamical term. Any process with turning points in this sense will be referred to as “periodic.”

In a fundamental description, we should therefore be able to make sense of relational evolution with respect to a confined or periodic reference degree of freedom. For the sake of clarity, we will reserve the word “time” for a monotonic label in accordance with our common experience of time. A non-monotonic reference system on which the measure of time is based will be called a “clock.” The distinction between “time” and “clock” does not appear in traditional relational evolution, but it is relevant for a discussion on a fundamental level as presented in this paper.
We will describe and evaluate relational evolution with respect to a periodic clock degree of freedom in what follows, demonstrating that it is not only consistent with standard requirements on quantum mechanics but also implies new and potentially observable effects, as announced in [23].

2 Relational evolution

Relational evolution as a formal device, used often in quantum gravity and quantum cosmology, helps to bring time conceptually closer to space by accompanying the canonical pair of position and momentum, $x$ and $p$, with a second canonical pair of energy and time, $E$ and $t$. For consistent signs, note that time $t$ is the momentum of $E$, or $-E$ is the momentum of $t$, as a consequence of the usual negative sign in the time components of the Minkowski metric $\eta_{\mu\nu}$. In 4-dimensional notation, one can write these canonical relationships through a Poisson bracket, $\{x_\mu, p_\nu\} = \eta_{\mu\nu}$.

The extended description by canonical variables implies that the corresponding quantum theory should include a time operator $\hat{t}$, in addition to an energy operator $\hat{E}$, such that $[\hat{t}, \hat{E}] = -i\hbar$. The classical energy equation $E = H(x, p; t)$ with the Hamiltonian $H(x, p; t)$ of the system, possibly having an explicit time dependence, is then quantized to the Schrödinger equation by representing the energy as a derivative operator $\hat{E}\psi = i\hbar \partial\psi/\partial t$, acting on wave functions depending on $x$ and $t$.

In order to make the relationship between time and space more apparent, we can formulate the energy equation as a constraint,

$$C_1 = -E + H(x, p; t) = 0,$$

moving time and space to the same side of the equation. We initially introduced a new degree of freedom into the usual formulation, $t$ with momentum $-E$, and now impose a constraint to make sure that the correct number of independent parameters is maintained.

In spite of a certain formal semblance between $x$ and $t$ in this formulation, it does not manage to put space and time on an equal footing. One remaining difference between these two variables is that the latter’s canonical momentum, $E$, appears linearly in the constraint, while the former’s canonical momentum, $p$, usually appears in a quadratic form. A relativistic energy equation, such as

$$C_2 = -E^2 + p^2 + m^2 = 0$$

for a free particle with mass $m$, helps to reduce this difference.

2.1 Classical formulation

The reformulation of standard Hamiltonian evolution as a constraint linear in $E$ does not change the assumption that time is monotonic. A constraint generates equations of motion
with respect to an auxiliary parameter $\epsilon$ in the same form as a Hamilton function generates Hamilton’s equations in time. For instance, the equations generated by $C_1$ are

\[
\begin{align*}
\frac{dx}{d\epsilon} &= \frac{\partial C_1}{\partial p} = \frac{\partial H}{\partial p}, \\
\frac{dp}{d\epsilon} &= -\frac{\partial C_1}{\partial x} = -\frac{\partial H}{\partial x}, \quad (4) \\
\frac{dE}{d\epsilon} &= \frac{\partial C_1}{\partial t} = \frac{\partial H}{\partial t}, \\
\frac{dt}{d\epsilon} &= -\frac{\partial C_1}{\partial E} = 1. \quad (6)
\end{align*}
\]

The last equation implies that $t$ is monotonic with respect to $\epsilon$; in fact, it can be identified with $\epsilon$ up to a constant shift. The remaining equations then obtain their usual Hamiltonian form, demonstrating the equivalence of the Hamiltonian and constrained formulations. Equation (6), derived from the non-relativistic constraint, means that time in non-relativistic mechanics is, unlike space, inevitably given by a function that is monotonic in $\epsilon$, and therefore perpetual.

The relativistic constraint introduced so far, $C_2$ in equation (2), implies a similar monotonic behavior of time. Initially,

\[
\frac{dt}{d\epsilon} = -\frac{\partial C_2}{\partial E} = 2E
\]  

is not just a numerical constant. However, the constraint $C_2$ for a free particle implies, via $dE/d\epsilon = \partial C_2/\partial t = 0$, that $E$ is constant. The time variable $t$ is therefore still monotonic in $\epsilon$, although its rate of change, given by $2E$, is no longer universal but depends on the energy.

This observation shows how we can move closer to a local notion of time based on a periodic clock: If we find a relativistic model in which there is a time-dependent potential added to $C_2$, the energy will no longer be constant. If its value can move through zero and change sign, $dt/d\epsilon = 2E$ would change sign, and $t$ might oscillate for a suitable potential. These oscillations would be with respect to an external parameter $\epsilon$, but this parameter is only auxiliary because it can locally be eliminated from solutions. Yet, in spite of this auxiliary nature of the parameter in which oscillations may unfold, dynamics with respect to an oscillating clock degree of freedom representing time would be markedly different from a monotonic time $t$ because of the presence of a potential, which we take as the defining feature of an oscillating clock. Recall that this definition only refers to the form of the constraint and does not require a background time.

Systems with time-dependent potentials in a relational interpretation indeed exist in fundamental physics. For instance, the cosmological dynamics of an expanding universe on large scales is determined by the Friedmann equation

\[
\left(\frac{1}{a} \frac{da}{dt}\right)^2 = \frac{8\pi G}{3c^2} \rho ,
\]

\( (8) \)
for the scale factor $a$, whose time derivative is related to the energy density $\rho$ of matter, Newton’s constant $G$, and the speed of light $c$. It can be formulated as a constraint

$$C_3 = -H(\phi, p_\phi; V) + \frac{6\pi G}{c^2} V p_V^2 = 0$$

(9)

in canonical variables [24] given by the expanding volume, $V$, and its momentum, $p_V = -c^2 H/(4\pi G)$ related to the Hubble parameter $H = a^{-1} da/dt$. The matter variables are often described by another canonical pair, $\phi$ and $p_\phi$, which appear in the matter Hamiltonian $H = V \rho$. A common example of an isotropic matter degree of freedom $\phi$ is a scalar field with mass $m$, in which case the energy density equals

$$\rho = \frac{c^2}{2} \frac{p_\phi^2}{V^2} + \frac{1}{2} m^2 \phi^2.$$ (10)

For $m = 0$, equations of motion imply that $p_\phi$ is conserved and $\phi$ is monotonic, much like $E$ and $t$ as determined by the previous constraint, $C_2$. But the more generic case of $m \neq 0$ leads to a representation of time through a clock degree of freedom $\phi$ that evolves in a non-monotonic, periodic fashion.

Simplifying some coefficients in the cosmological example, we will now work with a constraint of the form

$$C_4 = -p_\phi^2 - \lambda^2 \phi^2 + H(x, p)^2$$

(11)

with some system Hamiltonian $H(x, p)$, keeping the quadratic dependence on $\phi$ and $p_\phi$ but applying it to non-cosmological models. For $\lambda = 0$, in which case $\phi$ is not confined or periodic, we can factorize the quadratic constraint into two factors linear in $p_\phi$ such that solutions to the constraint equation $C_4 = 0$ are equivalent to solutions of the non-relativistic constraint $C_1 = 0$ with $p_\phi = \pm E$. For $\lambda \neq 0$, $\phi$ is confined for a given system energy, allowing us to generalize monotonic time behavior to an oscillating clock, $\phi$. Imposing the constraint $C_4 = 0$ then couples the system degrees of freedom, $x$ and $p$, to the clock degrees of freedom, $\phi$ and $p_\phi$. Even though there is no force between system and clock for a constraint of the form (11), their dynamics are related by the energy-balance constraint $C_4 = 0$. The absence of a coupling force relieves us from the obligation to justify any specific form from fundamental physics. If there were such a coupling force in addition to the energy-balance constraint, it would only strengthen the effects of clock-system interactions that we will observe in what follows.

2.2 Quantum formulation

Quantum cosmology aims to quantize constraints such as (9) or the simpler (11) by solving and interpreting the quantum constraint equation $\hat{C}_4 \psi(x, \phi) = 0$. However, solutions of this equation for the wave function $\psi$ do not evolve in an obvious way because the imposed quantum constraint implies that the evolution operator associated with $\hat{C}_4$, given by $\exp(-i\hat{C}_4 \epsilon / \hbar)$ where $\epsilon$ is analogous to the auxiliary parameter of the same letter used in the classical formulation, acts trivially on solutions $\psi(x, \phi)$ of $\hat{C}_4 \psi(x, \phi) = 0$. 

6
2.2.1 Ordering questions

A common method to address this problem goes back to Dirac [5], called deparameterization and implemented in detail in [25] for quantum cosmology. This method, which assumes $\lambda = 0$ in (11) or $m = 0$ in (10), amounts to an inversion of the process that led us from Hamiltonians to constraints: We factorize the quantum constraint equation as

$$\hat{C}_4^{(\lambda=0)} \psi = (-\hat{p}_\phi^2 + H(\hat{x}, \hat{p}))\psi = (\hat{p}_\phi + H(\hat{x}, \hat{p})) \psi = 0,$$  

(12)

such that it can be solved by either of the parentheses being zero when acting on $\psi$: $\hat{p}_\phi \psi = \mp H(\hat{x}, \hat{p}) \psi$, or

$$i\hbar \frac{\partial \psi}{\partial \phi} = \pm H(\hat{x}, \hat{p}) \psi.$$  

(13)

We have obtained Schrödinger evolution (for both choices of the orientation of time) from the quantum constraint.

For $\lambda \neq 0$, however, the procedure suggested by Dirac does not go through precisely because time is no longer monotonic. Deparameterized evolution with respect to $\phi$ then cannot be unitary because the classical $\phi$ oscillates. A further difficulty appears at a formal level, noting that the factorization (12) is not correct if a Hamiltonian $\hat{\bar{H}}$, such as a quantization of $\bar{H} = \sqrt{H(x, p) - \lambda^2 \phi^2}$ which classically solves $C_4 = 0$ for $p_\phi$, depends on $\phi$. Because $[\hat{\rho}, \hat{\bar{H}}] \neq 0$,

$$\hat{C}'_4 := (-\hat{p}_\phi + \hat{\bar{H}})(\hat{p}_\phi + \hat{\bar{H}}) = -\hat{p}_\phi^2 - [\hat{p}_\phi, \hat{\bar{H}}] + \hat{\bar{H}}^2 = -\hat{p}_\phi^2 + \hat{\bar{H}}^2 + i\hbar \frac{\partial \bar{H}}{\partial \phi}$$  

(14)

does not agree with the constraint $-\hat{p}_\phi^2 + \hat{\bar{H}}^2$; see also [11] [11].

The commutator term, being proportional to $\hbar$, could be interpreted as a quantum correction (although a complex-valued one), modifying the classical constraint $C_4$. The modified constraint equation $\hat{C}'_4 \psi = 0$ can then be solved by $-\hat{p}_\phi \psi = \hat{\bar{H}} \psi$, using the rightmost factor in (14) next to the wave function $\psi$ in $\hat{C}'_4 \psi = 0$. However, if $\phi$ is a local oscillating clock, both factors in a version of (12) are required for forward and backward evolution with respect to $\phi$. But exchanging the factors in (14), such that $-\hat{p}_\phi + \hat{\bar{H}}$ now acts directly on a wave function $\psi$, modifies the constraint:

$$\hat{C}''_4 := (\hat{p}_\phi + \hat{\bar{H}})(-\hat{p}_\phi + \hat{\bar{H}}) = -\hat{p}_\phi^2 + [\hat{p}_\phi, \hat{\bar{H}}] + \hat{\bar{H}}^2 = -\hat{p}_\phi^2 + \hat{\bar{H}}^2 - i\hbar \frac{\partial \bar{H}}{\partial \phi} \neq \hat{C}'_4.$$  

(15)

It therefore seems impossible to include both signs in a quantized $p_\phi = \pm \bar{H}$ for a unique quantum model, based on a single quantum constraint.

2.2.2 Gribov horizons

In a classical treatment, as $\phi$ evolves through its turning points in the quadratic potential $\lambda^2 \phi^2$, the sign of $p_\phi$ alternates. Different half-cycles of this periodic evolution are therefore
governed by not just one but both factors in the classical version of (12). The factorization of quadratic quantum constraints appears to be in conflict with this elementary behavior. However, quantum mechanics is more subtle. As shown in [26], the problem of time is a special case of the Gribov problem of gauge theories [27, 28], where the $\epsilon$-flow generated by the constraint plays the role of the gauge flow, and selecting a variable such as $\phi$ as time fixes the gauge as long as $p_{\phi} \neq 0$: Setting $\phi = \tau$ to a constant value $\tau$ of a global time parameter then gives a cross-section of the flow. When $p_{\phi} = 0$ at a turning point of $\phi$, $\phi$ is at an extremum and the condition $\phi = \tau$ is not transversal to the flow. Moreover, because $\phi$ is not monotonic, the condition $\phi = \tau$ evaluated on the full evolution does not have a unique solution.

These issues are common to all gauge theories with Gribov problems, in which transition surfaces in phase space, such as $p_{\phi} = 0$, are called Gribov horizons. The usual solution to this problem in quantized gauge theories is to ensure that Gribov horizons are never crossed on a single gauge orbit in order to avoid double-counting gauge-fixed solutions in a path integral. In the present case, this means that quantum mechanics cannot allow $p_{\phi}$ to change sign on a single gauge orbit. However, since our gauge fixing, $\phi = \tau$, is time-dependent, we may choose a different Gribov region at different times, such that we choose the region with $p_{\phi} < 0$ when $\phi$ moves forward and the region with $p_{\phi} > 0$ for backward motion. (According to (7), forward motion with respect to $\epsilon$ implies $E > 0$, which corresponds to $p_{\phi} < 0$.)

In a canonical treatment and with positive $H$, these two regions correspond to solutions of the constraint annihilated by the left and right factors, respectively, in (14). Acting on wave functions, we therefore seem subject to the ordering problem, $\hat{C}_4' \neq \hat{C}_4''$, when $H$ is $\phi$-dependent. However, when $\phi$ runs backwards ($p_{\phi} > 0$), evolution with respect to $\phi$ is reversed compared with forward motion ($p_{\phi} < 0$). Since time reversal in quantum mechanics is associated with complex or Hermitian conjugation, we may impose the constraint on wave functions by acting to the left in this case, $\psi \hat{C}_4' = 0$, while using the standard action to the right for forward motion. A single constraint in a fixed ordering, $\hat{C}_4'$, can then be used to describe both forward and backward motion of $\phi$. Implementing this concept formally on a Hilbert space is subtle because, as we will discuss in more detail, turning points where $p_{\phi} = 0$ are energy dependent according to the constraint equation. (Algebraic formulations of quantum mechanics [29] that generalize Hilbert-space treatments may be useful in this context, as they turned out to be in other questions about time as well [17].) Time reversals therefore happen at different times for different energy eigenstates that are superimposed in an evolving wave function. Our specific constructions will demonstrate that such a formulation is meaningful and feasible, but we postpone a detailed general discussion of such time-reversal states to later work.

2.2.3 Clock and time

The specific implementation of cycles in which a local clock such as $\phi$ may move forward or backward, solving the problem of oscillating clocks, is perhaps obvious, with hindsight, but it has been noticed only recently [26]: We should distinguish carefully not only between
background time $\epsilon$ and a clock variable $\phi$, as formalized by deparameterization which would also identify $\phi$ with time, but rather between three conceptually different notions: background time $\epsilon$, a clock variable $\phi$, and (as a new ingredient) global monotonic time $\tau$. The roles of $\phi$ and $\tau$ are indistinguishable in the usual treatment of deparameterization in which $\phi$ is monotonic and can be assumed to be identical with $\tau$. If $\phi$ has turning points, however, it can be identified with a linear function of $\tau$ only locally, for periods of evolution that do not contain a turning point of $\phi$. After a turning point, we should realign the relationship between $\phi$ and $\tau$ such that $\tau$ keeps on going forward while $\phi$ moves back. For instance, if $\phi(\tau) = \tau + A$ before a turning point at $\tau_t$ (with some constant $A$), $\phi(\tau) = -\tau + A + 2\tau_t$ rewinds $\phi$ in a way that is connected continuously to $\phi(\tau)$ before the turning point.

The new distinction between three types of variables related to time has conceptual implications that will not be the focus of this paper, and which we mention only briefly in this paragraph. For instance, there may be no time operator because time $\tau$, as outlined below, is a constructed, effective parameter and not fundamental. There would only be a clock operator $\hat{\phi}$ which can measure the direction in which $\phi$ points in its cycle, but not which cycle it is in. The number of cycles, and therefore time, would have to be determined by keeping track of a suitable succession of clock measurements. It would not be possible to determine time by a single measurement because, unlike the clock, it is not represented by a fundamental degree of freedom. As we will also see, a quantum clock is generically in a superposition of different cycles, such that the cycle is not a sharply defined observable.

Since classical physics, not restricted by the condition of unitarity, can easily be formulated with local times, we usually do not have to introduce a time parameter such as $\tau$. In this context, the transition from a background parameter $\epsilon$ to a clock variable $\phi$ is often motivated as moving a step closer toward a fundamental description of time, no longer given by a mathematical coordinate but rather by a physical measurement by means of a clock. The value of $\phi$ is then the position of a periodic phenomenon used as a clock, modeled in our example with regular periods by the harmonic-oscillator Hamiltonian $p^2_{\phi} + \lambda^2 \phi^2$ added to our constraints. However, observing $\phi$ no longer corresponds to a physical measurement of time. It is merely the reading of a clock instant without putting it into the context of a constructed time-and-date label of events. The variable $\phi$, like a Cartesian coordinate used to determine the position of the hands on a clock, oscillates back and forth, but the time we infer from this motion always increases. This perpetually increasing time is the global time $\tau$ introduced here.

Because $d\phi/d\tau$ changes sign at the turning points of $\phi$, a parameterization with respect to global time is consistent with having alternating signs of $p_\phi$ (related to our specification of Gribov regions) even while the energy (of a stable system) should always be positive: Building on (13), the Schrödinger equation

\[
\frac{i\hbar}{\partial\tau} \psi = i\hbar \frac{d\phi}{d\tau} \frac{\partial\psi}{\partial\phi} = -\frac{d\phi}{d\tau} \text{sgn}(\hat{p}_\phi) H(\hat{x}, \hat{p}, \phi(\tau)) \psi = H(\hat{x}, \hat{p}, \phi(\tau)) \psi
\]

with respect to global time $\tau$ contains both branches of wave-function evolution with $\hat{p}_\phi = \mp H(\hat{x}, \hat{p})$, even if the Hamiltonian $\hat{H}$ is always positive. Since we have forward
motion (now with respect to our new parameter $\tau$) for $p_\phi < 0$ and backward motion for $p_\phi > 0$, we always obey the condition $-(d\phi/d\tau)\text{sgn}(p_\phi) = 1$. Equation (16) is therefore the standard Schrödinger equation with a positive Hamiltonian. Global, unitary time evolution with respect to $\tau$ is then defined by taking into account the sign changes of $p_\phi$ in strict correspondence with the sign changes of $d\phi/d\tau$ for forward and backward motion of $\phi(\tau)$.

Still, even though (16) looks like a standard Schrödinger equation, it depends on a piecewise linear but not strictly linear parameterization $\phi(\tau)$ and therefore implies new features. At turning points of $\phi$, the dependence of $\phi$ on $\tau$ changes abruptly, which can be implemented in solutions by concatenating evolution operators $\exp(-i \int \hat{H} d\tau/\hbar)$ derived for strictly linear branches of $\phi(\tau)$, or their transition amplitudes. The sudden changes in a wave function constructed from such concatenated evolution operators imply that the time-dependent phase lacks smoothness, but continuity and unitarity are never compromised.

While this procedure introduced in [26] presents a solution to some aspects of the problem of time, the first to make sense of a local notion of an oscillating clock variable $\phi$, the question of its physical viability remained open. For instance, one might worry that the rather sudden changes of $\tau$-evolution operators at turning points, and correspondingly of the phase of the wave function, could destroy coherence faster than in standard quantum mechanics with a background time. They might then be in conflict with sensitive experimental observations, for instance in atomic clocks. In particular, the turning-point condition $p_\phi = 0$ is met at different clock values $\phi$ for different system energies because of the constraint, and therefore at different global times $\tau$. The independent energy contributions in a coherent state would therefore be affected differently by turning points, endangering their delicate balance required for long-term coherence.

In the next section we will show that this concern is unwarranted: Coherence remains intact over long time scales, provided the fundamental clock is sufficiently fast, with durations of cycles much shorter than the typical rate of change of the non-time observable, $x$. The procedure of oscillating clocks is therefore physically viable, and it is testable by measurements of quantum coherence. Detailed calculations presented in what follows impose a tight upper bound on the possible fundamental period of time.

3 Global evolution

Our formal results are valid for a constraint of the form (11) with an arbitrary Hamiltonian $H(x, p)$ of a bound-state system. Since any initial state can be written as a superposition of eigenstates of $\hat{H} = H(\hat{x}, \hat{p})$, it is sufficient to compute evolving wave functions or transition amplitudes by solving the ordinary differential equations

$$i\hbar \frac{d\psi_k(\phi)}{d\phi} = \pm \sqrt{E_k^2 - \lambda^2 \phi^2} \psi_k(\phi),$$

(17)
in the energy representation, where $E_k$ is one of the energy eigenvalues of $\hat{H}$ and $\psi_k$ the corresponding eigenfunction. This differential equation is straightforward to solve, giving

$$\psi_k(\phi) = \psi_k(0) \exp \left( \pm \frac{i}{2\hbar} \left( \phi \sqrt{E_k^2 - \lambda^2 \phi^2} + \frac{E_k^2}{\lambda} \arcsin \left( \frac{\lambda\phi}{E_k} \right) \right) \right).$$

(18)

(Without loss of generality, we assume that the initial phase of $\psi_k(\phi)$ with respect to the energy eigenstate $\psi_k$ vanishes.) As a function of the energy eigenvalues, the phase function

$$\Theta_k(\phi) = -\frac{1}{2\hbar} \left( \phi \sqrt{E_k^2 - \lambda^2 \phi^2} + \frac{E_k^2}{\lambda} \arcsin \left( \frac{\lambda\phi}{E_k} \right) \right)$$

(19)
can be used for any bound-state system, provided the clock Hamiltonian is given by $p_\phi^2 + \lambda^2 \phi^2$.

When $E_k^2 < \lambda^2 \phi^2$, Eq. (18), taken at face value, produces a wave function that is not normalized, highlighting the unitarity problem of deparameterization with an oscillating clock. Unitarity starts being violated precisely when $\phi = \pm \phi_t$, where

$$\phi_t = \frac{E_k}{\lambda},$$

(20)

reaches a turning point corresponding to the energy $E_k$. For later reference, we illustrate the phase-space trajectory of the clock degree of freedom in phase space in Fig. 1. For a complete cycle, we clearly need both positive and negative $p_\phi$. 

Figure 1: Harmonic clock degree of freedom in phase space.
Figure 2: A periodic clock degree of freedom $\phi$ as a function of global monotonic time, as constructed in Eq. (21). The parameter $n$ counts the clock cycle according to Eq. (22), starting at $\tau = 0$, while $\pm$ indicate the sign of $d\phi/d\tau$, determining which one of the two options (25) and (26) should be used at a given $\tau$.

3.1 Unwinding time

In order to solve the unitarity and sign problems, following [26], we introduce a monotonic global time $\tau$ related to the clock variable $\phi$ in a continuous and piecewise linear fashion:

$$\phi(\tau) = \begin{cases} 
\tau - 4n\phi_t & \text{if } 4n - 1 \leq \tau/\phi_t \leq 4n + 1 \\
(4n + 2)\phi_t - \tau & \text{if } 4n + 1 \leq \tau/\phi_t \leq 4n + 3 
\end{cases}$$  \hspace{1cm} (21)

Here, the integer

$$n = \left\lfloor \frac{1 + \tau/\phi_t}{4} \right\rfloor$$  \hspace{1cm} (22)

equals the number of clock cycles, starting with $n = 0$ at $\tau = 0$. This parameterization, illustrated in Fig. 2, is constructed such that $\phi$ (i) is related to $\tau$ in a piecewise linear fashion with equal rates for $\phi$ and $\tau$ ($d\phi/d\tau = \pm 1$), and (ii) never takes values outside of the range delimited by its turning points, $\pm \phi_t$. Time $\tau$ therefore progresses at the same rate as the clock, and it keeps track of the number of clock cycles that have passed while it unwinds the periodic behavior of the clock.

Implicitly, each energy eigenstate contained in a system state dictates its own clock period $4\phi_t$ through the $\phi_t$-dependence in $\phi(\tau)$, where $\phi_t$ depends on $E_k$ according to (20). For simplicity, we dropped the subscript $k$ in the more complete notation $\phi_k(\tau)$ because we will for some time be working with individual energy eigenstates. However, when we bring different energy eigenstates back in superposition, the $E_k$-dependence of $\phi_t$ implies that a unique global time $\tau$ for the entire state requires different energy eigenstates to be at different clock values $\phi$ and in different cycles. The combined clock-system state therefore evolves into a superposition of different clock cycles whenever the system is in a superposition of different energy eigenstates.

Inserting $\phi(\tau)$ in $\psi_k(\phi) = \psi_k(\phi_0) \exp(\pm i(\Theta_k(\phi) - \Theta_k(\phi_0)))$, with some initial $\phi_0$ in a given half-cycle of the clock, results in the local solutions

$$\psi_k(\tau) = \psi_k(\phi_0) \exp(\pm i(\Theta_k(\phi(\tau)) - \Theta_k(\phi_0)))$$  \hspace{1cm} (23)
of (16). In any given half-cycle of the clock, the sign in the exponent has to be chosen such that it cancels the sign \( \frac{d\phi}{d\tau} \) produced by acting on \( \Theta_k(\phi(\tau)) \) with a \( \tau \)-derivative in the Schrödinger equation, using the chain rule. This condition, as introduced before, ensures that \( \tau \)-evolution is generated by a positive Hamiltonian for a stable system.

Therefore,

\[
\psi_k(\tau) = \psi_k(\tau_0) \exp(i\text{sgn}(\frac{d\phi}{d\tau})(\Theta_k(\phi(\tau)) - \Theta_k(\phi(\tau_0))))
\]

or, in a piecewise description,

\[
\psi_k(\tau) = \psi_k(\tau_1) \exp(i(\Theta_k(\phi(\tau)) - \Theta_k(\phi(\tau_1))))
\]

during any half-cycle with \( \frac{d\phi}{d\tau} > 0 \) (starting at some \( \tau_1 \)), while

\[
\psi_k(\tau) = \psi_k(\tau_2) \exp(i(\Theta_k(\phi(\tau)) - \Theta_k(\phi(\tau_2))))
\]

when \( \frac{d\phi}{d\tau} < 0 \) (ending at some \( \tau_2 \)). The latter equation implies that

\[
\psi_k(\tau_2) = \psi_k(\tau) \exp(-i(\Theta_k(\phi(\tau)) - \Theta_k(\phi(\tau_2))))
\]

with the opposite sign in the phase for forward evolution in a backward half-cycle, as required. This equation may be used whenever \( \tau_2 > \tau \) in the same backward half-cycle.

The concatenated solution then has a continuous phase because the phases of the two wave functions (25) and (26) indeed meet in the middle: At a turning point, if \( \phi_k(\tau_1) \) marks the beginning of a forward half-cycle and \( \phi_k(\tau_2) \) marks the end of the next backward half-cycle, we may choose \( \tau \) to be in both half-cycles, interpreted either as the end of the first one or the beginning of the second one. Correspondingly, the phase \( \Theta_k(\phi(\tau)) - \Theta_k(\phi(\tau_1)) \) added to \( \psi_k(\tau_1) \) and the phase \( \Theta_k(\phi(\tau)) - \Theta_k(\phi(\tau_2)) \) added to \( \psi_k(\tau_2) \) produce the same state. This solution is globally valid because \( |\phi(\tau)| \) never surpasses \( \phi_k \), demonstrating unitarity. While phase continuity is guaranteed by construction, smoothness or even differentiability is not. The system maintains unitary evolution as the wavefunction itself is smooth during any half-cycle, whereas different evolution operators are concatenated (rather than extended by solving a single differential equation) precisely where the phase is not differentiable.

In our specific example, according to the phase \( \Theta_k(\phi(\tau)) \) in (18), each half-clock cycle of \( \phi \), changing monotonically from \( -\phi_t \) to \( \phi_t \) or back, adds an amount of

\[
\frac{1}{2} \Delta \Theta_k = \Theta_k(\phi_t) - \Theta_k(-\phi_t) = -\frac{E_k^2}{2\lambda \hbar} (\arcsin(1) - \arcsin(-1)) = -\frac{\pi E_k^2}{2\lambda \hbar}
\]

to the phase. A forward half-cycle starts at \( -\phi_t \) and ends at \( \phi_t \) and has a phase changing according to \( \Theta_k(\phi) \), while a backward half-cycle starts at \( \phi_t \) and ends at \( -\phi_t \) but has a phase changing according to \( -\Theta_k(\phi) \). Therefore, in both cases the phase added per half-cycle equals \( \Theta_k(\phi_t) - \Theta_k(-\phi_t) \). A full clock cycle, going from \( -\phi_t \) to \( \phi_t \) and back, adds twice this phase. Going back in \( \phi \) does not cancel out the phase of the previous half cycle because of our specific construction in which we flip the sign of \( \Theta_k \) according to the sign of \( \frac{d\phi}{d\tau} \), dictated by positivity of the Hamiltonian for a stable system.
As an example, starting at the beginning of the zeroth cycle according to Fig. 2, such that \( \tau_0 = -\phi_t \), \( \tau \) increases from \(-\phi_t\) to \(\phi_t\) during the first monotonic phase of \( \phi \), where the latter also increases from \(-\phi_t\) to \(\phi_t\). The next monotonic phase then completes the zeroth cycle and has \( \tau \) increasing from \(\phi_t\) to \(3\phi_t\) while \( \phi \) decreases back to its initial value, \(-\phi_t\). The next clock cycles \((n = 1, 2, \ldots)\) repeat this process. Since a full clock cycle adds \(-\pi E_k^2/(\lambda\hbar)\) to the phase, according to (28), at the end of the \(n\)th cycle the wave function equals

\[
\psi_k(\tau_1) = \psi_k(\tau_0) \exp \left( -\frac{in\pi E_k^2}{\lambda\hbar} \right) \tag{29}
\]

where \( \phi(\tau_1) = -\phi_t = \tau_0 \). It then proceeds by

\[
\psi_k(\tau) = \psi_k(\tau_0) \exp \left( -\frac{in\pi E_k^2}{\lambda\hbar} \right) \exp (i(\Theta_k(\phi(\tau)) - \Theta_k(-\phi_t)))
\]

\[
= \psi_k(\tau_0) \exp \left( -\frac{i(n + 1/4)\pi E_k^2}{\lambda\hbar} \right) \exp (i\Theta_k(\phi(\tau))) \tag{30}
\]

as long as \(4n - 1 \leq \tau/\phi_t \leq 4n + 1\) (\( \phi(\tau) \) increasing). This half-cycle, ending at a time \( \tau_2' \) when

\[
\psi_k(\tau_2') = \psi_k(\tau_0) \exp \left( -\frac{i(n + 1/4)\pi E_k^2}{\lambda\hbar} \right) \exp (i\Theta_k(\phi_t)) = \psi_k(\tau_0) \exp \left( -\frac{i(n + 3/4)\pi E_k^2}{\lambda\hbar} \right) \exp (i\Theta_k(\phi(\tau))) \tag{31}
\]

adds an additional \(-\pi E_k^2/(2\lambda\hbar)\) to the phase. According to (27), the wave function in the next half-cycle is then given by

\[
\psi_k(\tau) = \psi_k(\tau_0) \exp \left( -\frac{i(n + 1/2)\pi E_k^2}{\lambda\hbar} \right) \exp \left( -i(\Theta_k(\phi(\tau)) - \Theta_k(\phi_t)) \right)
\]

\[
= \psi_k(\tau_0) \exp \left( -\frac{i(n + 3/4)\pi E_k^2}{\lambda\hbar} \right) \exp (i\Theta_k(\phi(\tau))) \tag{32}
\]

\((4n + 1 \leq \tau/\phi_t \leq 4n + 3, \phi(\tau) \) decreasing). Because this half-cycle ends when \( \phi(\tau) = -\phi_t \), the final phase increase during the combination of two half-cycles equals \(-\pi E_k^2/(\lambda\hbar)\), in agreement with (28).

### 3.2 Harmonic oscillators

In order to highlight features of coherence, we now consider the example of a harmonic system Hamiltonian. Since there is perfect coherence in suitable states of the standard harmonic oscillator, any loss of coherence implied by our treatment of local clocks can easily be discerned. Section 3.3 will show analogous properties in non-harmonic examples in order to confirm the general qualitative features.

The parameter \( \lambda \) in the harmonic clock Hamiltonian \( p_\phi^2 + \lambda^2 \phi^2 \) acts as a scaling factor for the frequency of the clock variable. It does not have the units of frequency but rather
Figure 3: Density plot of the wave function for intermediate (top) and large \( \lambda \) (bottom), using a harmonic-oscillator Hamiltonian \( \hat{H} = \frac{1}{2}(\hat{p}^2 + \hat{x}^2) \) and a standard coherent initial state. Coherence is quickly lost in \( \tau \)-evolution for intermediate \( \lambda \) as soon as the first turning points are reached (around \( \tau \sim 10 \) for \( \lambda = 10^{-1} \), top). By contrast, coherence is maintained for long times for large \( \lambda \) even though billions of turning points are crossed during a single system period for \( \lambda = 10^{10} \) (bottom). Strong coherence is maintained in this case even though the actual \( \phi \)-Hamiltonian \( (\hat{H}^2 - \lambda^2 \phi^2)^{1/2} \) is not harmonic if \( \lambda \neq 0 \).
Figure 4: Phase-space trajectory and quantum uncertainty (bottom bar) for a harmonic system Hamiltonian with intermediate $\lambda = 0.1$

Figure 4: Phase-space trajectory and quantum uncertainty (bottom bar) for a harmonic system Hamiltonian with intermediate $\lambda = 0.1$.

of frequency times energy. In fact, the number of clock cycles in any range of global time depends, through the value $\phi_t = E_k/\lambda$, on the energy of the system, or rather on the contributions of energy eigenstates to an evolving system state. Since $\lambda$ determines how often the clock turns around, it must therefore refer to the energy.

The parameter $\lambda$ directly affects how often an energy eigenstate (of the system Hamiltonian) moves through turning points of $\phi$ in a given $\tau$-interval. As usual, any initial state can be expanded as a superposition of energy eigenstates, but since different eigenstates imply different times for the turning points of a clock, an oscillating clock makes the superposition evolve in a modified way compared with absolute time in standard quantum mechanics. Figures 3 and 4 show how complicated this new dynamics can be, even for a coherent initial state of a standard harmonic system Hamiltonian. However, the behavior simplifies not only when $\lambda \to 0$, in which case we have standard quantum mechanics with a monotonic time, but also, surprisingly, when $\lambda \to \infty$ as evidenced by Fig. 3. The different behaviors of quantum fluctuations and deviations of the expectation values from classical sinusoidal behavior are also illustrated in Figs. 4 and 5.
Figure 5: Phase-space trajectory and quantum uncertainty (bottom bars) for a harmonic system Hamiltonian. Compared with Fig. 4, coherence and semiclassical behavior are regained as $\lambda$ is increased, here showing the examples of $\lambda = 1$ and $\lambda = 5$. 

17
3.2.1 Small-λ approximation

As \( \lambda \to 0 \), the state never approaches a turning point in any finite range of \( \tau \). Far away from these turning points, the phase of the state resembles that of an unconstrained system, solving (17). Indeed, rewriting the constraint equation (11) by substituting \( \lambda \approx 0 \) yields the relation \( \hat{p_\phi} \approx \pm \hat{H} \), belonging to a system with global time \( \phi \).

Utilizing this approximation, the equation of motion is equivalent to the familiar Schrödinger equation

\[
i\hbar \frac{\partial \psi_k(\phi)}{\partial \phi} = \hat{H}_k(\phi) = E_k \psi_k(\phi)
\]

(choosing a sign suitable for stability from a positive Hamiltonian). This equation generates the expected global time-dependent phase for a stationary state,

\[
\psi_k(\tau) = \psi_k(0) \exp \left( \frac{-i\tau E_k}{\hbar} \right) \tag{33}
\]

if we identify clock \( \phi \) and time \( \tau \) in this case.

3.2.2 Large-λ approximation

As \( \lambda \to \infty \), the stationary state passes through turning points at a very high rate. The evolving state is obtained by concatenating many branches of wave functions (30) and (32). The fact that concatenations happen at different times for different energy eigenstates in a superposition makes it hard to understand the long-term behavior of an evolving state by analytic means. Nevertheless, numerical features, seen in several model systems, show surprising simplifications, as demonstrated by the example of a harmonic-oscillator system Hamiltonian in Figs. 3 and 5. According to numerical results, each stationary state as a function of \( \tau \) is again governed by a sinusoidally varying phase, except for small intervals around turning points. The main visible difference with standard evolution is that the sinusoidal frequency is a multiple of the expected frequency at \( \lambda = 0 \) by a factor of \( \frac{1}{4} \pi \). An example is shown in Fig. 6, where movements of the clock through turning points (about two per system cycle) are clearly visible for intermediate \( \lambda \), while the many turning points the clock goes through per system cycle for large \( \lambda \) merely rescale the system period but do not lead to noticeable deviations from sinusoidal behavior.

The same features are shown by numerical evolution of expectation values of \( \hat{x} \) and \( \hat{p} \) and their second-order moments (fluctuations and the covariance), seen in Fig. 7 for small \( \lambda \) (close to the standard harmonic oscillator), Fig. 8 for intermediate \( \lambda \), and Fig. 9 for large \( \lambda \). A single stationary state does not show time-dependent expectation values and moments. For this analysis, we have therefore chosen an initial state of minimum uncertainty which would be a dynamical coherent state of the harmonic oscillator (\( \lambda = 0 \) or small).

For intermediate \( \lambda \), the system rapidly loses coherence, as expected because the \( \phi \)-Hamiltonian \( \sqrt{H^2 - \lambda^2 \phi^2} \) is no longer harmonic. As noted before, the phase of a wave function, obtained from concatenated evolutions for each half-cycle of the clock, is not smooth. For large \( \lambda \), the high frequency at which the system crosses turning points smoothes out...
Figure 6: Phases of the ground state of the harmonic potential for different values of $\lambda$. Small plateaus caused by individual turning points of $\phi$ are clearly visible for intermediate $\lambda = 10^{-1}$. These plateaus stretch out the curve in the time direction. As a consequence, the system period for a large $\lambda = 10^3$ is greater than the period for a small $\lambda = 10^{-5}$. Plateaus are no longer visible for large $\lambda$ because the clock period is much smaller in this case compared with intermediate $\lambda$. Accordingly, the phases behave sinusoidally for large $\lambda$, just as for the standard harmonic oscillator approached at small $\lambda$. 

![Diagram showing evolution of Im(Phase) of the Ground State](image)

\[ \tau \text{ Evolution of Im(Phase) of the Ground State} \]
the phase on scales larger than the clock period, which may have been expected. More surprisingly, the same non-harmonic Hamiltonian $\sqrt{H^2 - \lambda^2 \phi^2}$ that implies rapid loss of coherence at intermediate $\lambda$ leads to strongly coherent behavior at large $\lambda$. This feature, as well as its explanation and applications below, are the main results of [23], for which the present paper provides a detailed discussion.

To explain this behavior, the large-$\lambda$ limit of Eqs. (30) and (32) can be simplified by ignoring the last exponential factors as they approach zero in this limit: The first term in each of the last exponentials is reduced to zero as $\lambda \to \infty$ since the amplitude of $\phi(\tau)$ approaches zero as $\lambda \to \infty$. The second term in each of the last exponentials is an arcsine function divided by $\lambda$, which also approaches zero as $\lambda \to \infty$. The reduced equations therefore become

$$
\psi_k(\tau) \approx \psi_k(0) \exp \left( -\frac{i(n + 1/4)\pi E_k^2}{\lambda \hbar} \right) \tag{34}
$$

if $4n - 1 \leq \tau / \phi_t \leq 4n + 1$, and

$$
\psi_k(\tau) \approx \psi_k(0) \exp \left( -\frac{i(n + 3/4)\pi E_k^2}{\lambda \hbar} \right) \tag{35}
$$

otherwise. Since $\phi_t \to 0$ in this limit for fixed $E_k$, even small changes in $\tau$ imply transitions between different clock cycles. Any extended range of $\tau$ therefore leads to large numbers of clock cycles, $n$, such that the remaining exponentials in (34) and (35) are non-trivial even for large $\lambda$. However, there is a negligible difference between $n + 3/4$ and $n + 1/4$ in the exponents, which therefore are nearly identical in this limit.

As $\lambda \to \infty$ it is possible to approximate the floor function in (22), defining $n$, in a continuous form:

$$
\frac{n}{\lambda} = \left\lfloor \frac{1/4 + \lambda \tau / (4E_k)}{\lambda} \right\rfloor \approx \frac{\tau}{4E_k} \tag{36}
$$

Using this result in Eq. (34) yields

$$
\psi_k(\tau) \approx \psi_k(0) \exp \left( -\frac{i\pi E_k}{4\hbar} \right) \tag{37}
$$

Comparing this time-dependent phase with the usual solution

$$
\psi_k(t) = \psi_k(0) \exp(-iE_k t / \hbar) \tag{38}
$$

of the time-independent Schrödinger equation shows that the frequency of the phase in the large-lambda limit is $\frac{1}{4\pi}$ times the frequency of the system for $\lambda \to 0$, in agreement with our numerical plots for the harmonic oscillator with Hamiltonian $\hat{H} = \frac{1}{2}(\hat{p}^2 + x^2)$: When $\lambda = 10^{-5}$ (approximating the limit $\lambda \to 0$), a sinusoidal function with a period of $2\pi$ is returned, as expected for the ground state of the harmonic oscillator of frequency parameter $\omega = 1$. For large values of lambda, $\lambda = 10^3$ (approximating $\lambda \to \infty$), the period of the oscillations is multiplied by a factor of $4/\pi$ resulting in a value of 8. With intermediate values, turning points are spaced at easily visible intervals, for instance located at $\tau = 5 + 10j$ with integer $j$ in the case of $\lambda = 10^{1}$ as shown in Fig. 6.
Figure 7: Basic expectation values (top) and second-order moments (bottom) for small $\lambda$, using a harmonic system Hamiltonian.
Figure 8: Basic expectation values (top) and second-order moments (bottom) for intermediate $\lambda$, using a harmonic system Hamiltonian.
Figure 9: Basic expectation values (top) and second-order moments (bottom) for large $\lambda$, using a harmonic system Hamiltonian.
3.3 Non-harmonic examples

Our specific equations for the $\tau$-dependent phase can be used for any system Hamiltonian. The harmonic example enjoys the most coherent dynamics and therefore highlights any loss of coherence implied by an oscillating fundamental clock. Non-harmonic systems cannot regain strong coherence for large $\lambda$ simply because their dynamics is not coherent, but it is nevertheless possible to see an approach to standard quantum mechanics for a periodic clock with large $\lambda$.

Figure 10 shows the example of a basic hydrogen Hamiltonian, based on the Coulomb potential, with a non-coherent initial state, chosen as a certain superposition of finitely many energy eigenstates. Quantum fluctuations vary rather strongly for any $\lambda$, as they would also do in standard quantum mechanics in this case. There is a notable difference in the expectation values for different $\lambda$, shown by a high-frequency signal visibly superimposed for intermediate $\lambda$ that disappears for large $\lambda$ where the expectation values approach the standard behavior.

Similar features are obtained for a non-harmonic clock Hamiltonian. The main difference of a non-harmonic system compared with a harmonic clock is that its period depends on the energy. However, in our coupled system the clock period $T_C = 4\phi_t = 4E_k/\lambda$ already depends on the system energy, which equals the clock energy by the energy-balance constraint. Non-harmonic behavior of the clock therefore does not introduce crucial new features, although it makes explicit calculations more complicated.

4 Observational bounds

So far, we have found one quantitative difference between the evolution of a harmonic system with respect to an oscillating clock one one hand, and with respect to an absolute time on the other: A rescaling of the system period by a factor of $4/\pi$ for the clock Hamiltonian used here, as described in Section 3.2.2. However, this difference is not observable because it depends only on the clock Hamiltonian and therefore rescales all system frequencies or their characteristic time scales in the same way. The rescaling factor can therefore be removed by absorbing it in a bare frequency $\omega' = 4\omega/\pi$ that appears in the mathematical expression of the system Hamiltonian, and therefore gives rise to the observed frequency $\omega$ after our rescaling. The scaling factor depends neither on the energy or initial state for a given system, nor on the system itself. It depends only on the clock dynamics, which is the same for all systems if the clock is fundamental.

The second implication of an oscillating clock is its effect on the decoherence time, depending on the clock parameter $\lambda$. Since a coherent state of the harmonic oscillator has an infinite decoherence time in standard quantum mechanics, a finite decoherence time as shown by our numerical results cannot be eliminated by a simple rescaling. Moreover, the time of how long a system can maintain coherence is under good observational control. For instance, the current relative precision of $10^{-19}$ of atomic clocks [30] could not be obtained if nature had provided us with an intermediate $\lambda$ in our fundamental clock that would
Figure 10: Expectation value and quantum uncertainty of the orbital radius with a standard hydrogen Hamiltonian. The quantum uncertainty is similar for all $\lambda$, while the time-dependent expectation value has an additional high-frequency contribution for intermediate $\lambda$ (top) compared with large $\lambda$ (bottom) or standard quantum mechanics.
Figure 11: Relative standard deviation of the system period over many system cycles as a function of $\lambda$. The analytical approximation (40) agrees well with a numerical computation of many system periods, both confirming a $1/\lambda$-behavior. The analytical result is rescaled by a factor of $2/\pi$ to take into account the average of a sine function over a quarter-cycle.

destroy coherence or a stable system period over many system cycles.

Taking (36) to the next order in $1/\lambda$ indicates that relative deviations from strict sinusoidal behavior should be of this order. More precisely, for given finite $\lambda$, we can compute the variance of the phase $\Theta_k(\phi(\tau))$, given in (19) around

$$\Theta_k^\infty(\tau) := \lim_{\lambda \to \infty} \Theta_k(\phi(\tau)) = \frac{\pi}{4} E_k \tau$$

averaged over a quarter-cycle of $\phi$:

$$\sigma^2 = \frac{1}{\phi_t} \int_0^{\phi_t} (\Theta_k(\phi(\tau)) - \Theta_k^\infty(\tau))^2 \, d\tau$$

$$= \frac{E_k^4 (21\pi^2 - 1024/5)}{24^2 \lambda^2 \hbar^2}.$$  \hfill (40)

Introducing the clock period $T_C = 4\phi_t = 4E_k/\lambda$ and the system period $T_S = 2\pi \hbar / E_k$ allows us to eliminate the parameters $\lambda$ and $E_k$ in favor of more general clock characteristics. Equation (40) then takes the form

$$T_C = \frac{48\sigma T_S}{\pi \sqrt{21\pi^2 - 1024/5}} \approx 9.7\sigma T_S.$$  \hfill (41)
The dependence on $\sigma$ implies a strong magnification factor that can make precision measurements sensitive to a small period of the fundamental clock even if they operate on a larger system period. Indirect bounds on the clock period therefore come within reach.

Even though (41) no longer refers directly to an energy eigenvalue, its derivation remains strictly valid only for a system period of an eigenstate. It is more difficult to obtain analytical control over the variance of the system period of a superposition of several energy eigenstates, such as a coherent state. A natural expectation is that $E_k$ in (40) should then be replaced with a certain expectation value determined by the Hamilton operator $\hat{H}$, for which there are different options. As shown by Fig. 11, replacing $E_k^2$ in (40) with the expectation value $\langle \hat{H}^2 \rangle$ is in good agreement with numerical computations of the relative standard deviation, $\sigma$, of the system period over many system cycles, shown as a function of $\lambda$ in the figure. While the close agreement is encouraging, it is also surprising and remains incompletely understood: The quarter-cycle calculation in (40) does not directly refer to turning points of the clock and is therefore insensitive to flipping the signs of the phase according to (25) and (26), while the latter is important for global evolution over many cycles during which we notice the restoration of coherence. The quarter-cycle calculation does, however, depend on the non-linearity of the phase implied by an oscillating clock, which is most prominent in the approach to a turning point.

For very large $\lambda$, one should evolve through many system cycles in order to determine the standard deviation accurately, considering the collection of cycles as a statistical ensemble for the observable period. Such long-term evolution was beyond the numerical capacity available to us. However, since the upper bound on the standard deviation shown in the figure follows a simple $1/\lambda$-behavior, we are justified in extrapolating it to values of $\lambda$ even larger than those shown in the plot. According to this extrapolation to large $\lambda$, a relative accuracy of $10^{-19}$, as reported for recent atomic clocks in [30], requires ten orders of magnitude less than the smallest relative standard deviations shown in Fig. 11. Since one order of magnitude in $\lambda$ corresponds to about one order of magnitude in $\sigma$, taking into account the numerical factor of about ten in (41), we need $\lambda$ at least as large as $10^{18}$ times the system frequency, or a fundamental clock period of at most $10^{-18}$ times the system period.

The system period (or atomic clock period) used in [30] is based on the transition from the $^3P_0$ state to the $^1S_0$ ground state of Strontium, with a wavelength of 698 nm, amounting to a system period of about 2 fs. The upper bound on the fundamental clock frequency is therefore

$$T_C < 10^{-18} \cdot 2 \text{ fs} = 2 \times 10^{-33} \text{ s}.$$  \hspace{1cm} (42)

Although this upper bound is still several orders of magnitude larger than the Planck time $t_P = 5 \times 10^{-44} \text{ s}$, which is often suggested as a fundamental period of time based on dimensional arguments, it is much smaller than any value that could at present be obtained from a direct time measurement. For instance, the current value of the shortest time interval measured directly, given by the photon travel time $247 \cdot 10^{-21} \text{ s}$ across a hydrogen molecule [31], is more than ten orders of magnitude larger than our upper bound.

Our new upper bound is also stronger than previous indirect measurements of short
time scales. In particular, the shortest length measurements currently possible are of the order $10^{-19}$ m, achieved at high-energy particle accelerators. Using the speed of light, this value translates into an upper bound of $10^{-19}$ m/c $\approx 3 \cdot 10^{-28}$ s on the time scale, about five orders of magnitude above our new upper bound. This value, like ours, is based on an indirect measurement because it translates a direct measurement of a scattering cross section into a length, and then into a time parameter. By exploiting the dephasing time, our indirect measurement is much more sensitive even than indirect measurements at high energy.

5 Conclusions

We have analyzed a combination of an oscillating clock variable $\phi$ and an evolving system degree of freedom $x$, coupled minimally through an energy-balance constraint (11). Expressed as relational evolution of $x$ with respect to $\phi$, the dynamics is governed by a standard Schrödinger equation (17) with time-dependent Hamiltonian $\sqrt{\hat{H}^2 - \lambda^2 \phi^2}$ if $\hat{H}$ is the Hamiltonian that determines the energy of the system. Therefore, an oscillating clock implies that the usual equality between the energy operator $i\hbar \partial/\partial t$ and the system Hamiltonian no longer holds. The operator $i\hbar \partial/\partial \phi$ instead determines the momentum of the clock variable $\phi$ and therefore its kinetic energy, but not the full clock energy because an oscillating clock also has potential energy. The energy-balance constraint makes sure that the combined energy of the system and the clock is conserved. But the clock momentum, measured in quantum mechanics by $i\hbar \partial/\partial \phi$, no longer equals the system Hamiltonian.

Experience with standard quantum mechanics would suggest that evolution with a Hamiltonian $\sqrt{\hat{H}^2 - \lambda^2 \phi^2}$ is rather complicated for any $\lambda \neq 0$, even if $\hat{H}$ belongs to the harmonic oscillator as in our main example. (Fractional powers of Hamiltonians can imply additional subtleties; see for instance [32].) This expectation is confirmed by Fig. 3 for small and intermediate values of $\lambda$, defined such that $\lambda T_s / \langle \hat{H} \rangle$ is not very large, where $T_s$ is the system period and the expectation value $\langle \hat{H} \rangle$ of the system energy is taken in an initial system state. For this range of $\lambda$, the coherence of an initial standard coherent state is quickly lost as soon as the term $\lambda^2 \phi^2$ becomes relevant in the action of $\sqrt{\hat{H}^2 - \lambda^2 \phi^2}$ on an evolving state. For large $\lambda$, one would then expect that coherence is lost even faster, well before the system can complete a single period or just move in a noticeable way. Surprisingly, however, coherence is restored for very large $\lambda$, as also shown in Fig. 3.

The unexpected restoration of coherence for small periods of the fundamental clock demonstrates that an oscillating fundamental clock is consistent not only conceptually, as already shown in [26], but also physically: Even though the potential required for a periodic clock affects the coupling between system and clock through the energy-balance constraint, leading to a non-harmonic system Hamiltonian of the form $\sqrt{\hat{H}^2 - \lambda^2 \phi^2}$, coherence can be maintained for surprisingly long times for large $\lambda$. Here, it is important to note that the dynamics is governed not only by the linear Schrödinger equation (17), but also by

28
a discrete process given by flipping the sign of the phase according to (16). The precise mathematical origin of restored coherence remains to be understood. This restoration of coherence is not simply a perturbation of the standard harmonic oscillator because it occurs for large $\lambda$, where perturbation theory cannot be used in $\sqrt{\hat{H}^2 - \lambda^2 \phi^2}$. While intermediate $\lambda$ are ruled out by observations of long coherence in isolated quantum systems, large $\lambda$ and therefore sufficiently small fundamental periods are consistent with current observations. A slight dephasing persists, however, even at large $\lambda$, giving rise to our upper bound (12).

The origin of the coherence effect lies in quantum mechanics with an oscillating fundamental clock. It does not have a complete classical analog. A fundamental period of time would imply that a system period that is not an integer multiple of the fundamental period cannot be sampled precisely. Successive system periods may therefore appear slightly longer or shorter depending on which system cycle an incomplete clock period is attributed to in a measurement. This classical model would also lead to a certain variance in system periods, but any such effect would quickly average out over a few system cycles. Moreover, two systems starting at the same time would be affected by the over/undercounting of complete clock cycles in the same way. If they are synchronized, like an atom and a photon of the right energy to generate a transition of energy levels or like two atomic clocks, they would therefore not get out of tune by classical variations of the system period. Observable implications of such a classical model would be insignificant.

Our new quantum effect, which also implies variations of the system frequency, is of a different nature. It acts on subtle coherence properties in a superposition of energy eigenstates of the system. If, again, we have two synchronized systems starting at the same time, they do not have identical quantum states, and therefore are affected in different ways by the new coherence and dephasing effects. Their system periods still vary, and subsequent cycles of one system present a statistical ensemble independent of the cycles of the other system because the initial states are largely independent except in certain macroscopic properties that have been arranged to agree in the synchronization procedure. Therefore, detuning can in principle be observed by comparing the two systems. Similarly, if the two systems are an atom and a photon which are “synchronized” in the sense that the energy of the photon matches a transition energy of the atom, the atom and photon states are necessarily different and therefore react differently to the new coherence effect.

Unfortunately, it seems difficult to derive the new coherence effect at large $\lambda$ in a controlled analytical approximation, even though it is clearly presented by numerical simulations. We have been able to compute the variance of the system period (40) in good agreement with a numerical analysis of the statistical ensemble given by the periods of a simulated wave function, as demonstrated by Fig. 11. However, the agreement remains somewhat mysterious because the calculation in (40) is based on a quarter cycle of the fundamental clock, which is a much shorter time than spanned by the large number of system periods that are used in the numerical analysis. The agreement is encouraging and supports the relevance of the new effect, but for a detailed analysis and further predictions, as well as stricter upper bounds, it would be desirable to have an analytical approximation that could accurately describe the statistical and coherence properties of the system over
many periods. Developing such an approximation is challenging because the dynamics over many clock cycles is governed not only by the linear partial differential equation (17), but also by the phase “reflections” in Eqs. (25) and (26).

Our results have several conceptual implications for the quantum nature of clocks and time. The fact that evolution for large $\lambda$ agrees with $\lambda = 0$ to a good degree demonstrates that deparameterization, a procedure going back to Dirac and now widely used to evade the problem of time in quantum gravity and quantum cosmology, can be considered a controlled approximation of quantum dynamics at least as long as system periods are not Planckian. The deparameterization procedure may therefore be applied at low curvature, but it remains questionable at Planckian curvature where the system period (or any rate of change if the system is not periodic, like the expanding universe) itself is Planckian and comparable with the period of a fundamental clock. New, unexpected effects may therefore be implied by a fundamental clock at the big bang, which remain to be evaluated.

Finally, our analysis has shown that a periodic fundamental clock is, in general, in a superposition of different clock cycles. We reach this conclusion because the turning points of the clock variable, minimally coupled to the system through energy balance, depend on the system energy as shown by (20). Since a system is generically in a superposition of different energy eigenstates, a fundamental period will quickly evolve into a superposition of different clock cycles even if it is assumed to start in a state that is sharply peaked around a given clock value. In particular, an application of equation (21) shows that the requirement of having a unique value of global time $\tau$ for all energy eigenstates in superposition implies different clock values $\phi_k(\tau)$ for different energy eigenvalues $E_k$. The more time $\tau$ progresses, the more the various $\phi_k$ for a given system state differ, potentially stretching over many clock cycles. Our procedure of concatenating half-cycle evolutions for each eigenstate and then bringing them back into superposition allows us to avoid dealing directly with a complicated clock state, but such a state is indirectly realized. This effect does not appear in non-periodic clocks such as those used in deparameterization. To the best of our knowledge, our model is the first in which the clock is truly quantum, understood in the sense that it is by necessity in a superposition of different cycles.

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A Ingredients of the code

We present crucial parts of the MATLAB program used for our results as pseudocode:

- Calculating $\phi(\tau)$:
if ((tau/Phi_t) <= 4*n+1)
Phi_tau = t-4*n*Phi_t;
elseif ((tau/Phi_t) > (4*n+1))
Phi_tau = (4.*n+2)*Phi_t-t;

• Calculating $n$:

$$n = \text{floor}((1+(tau/Phi_t))/4);$$

• Calculating $\Theta(\phi(\tau))$:

if ((tau/Phi_t)>(4*n+1))
Theta = \exp(pi*i*(n+1/2)*((k+1/2)^2)/\lambda/\hbar)*
\exp(-i/2/\lambda/\hbar*(\lambda*Phi_tau*((k+1/2)^2-\lambda^2*Phi_tau^2)^.5))*
\exp(-i/2/\lambda/\hbar*(k+1/2)^2*asin(\lambda*Phi_tau/(k+1/2)));
elseif ((tau/Phi_t(k,\lambda))<=(4*n(k,\lambda,\tau)+1))
Theta = \exp(pi*i*n*(k+1/2)^2/\lambda/\hbar)*
\exp(i/2/\lambda/\hbar*(\lambda*Phi_tau*((k+1/2)^2-\lambda^2*Phi_tau^2)^.5))*
\exp(i/2/\lambda/\hbar*(k+1/2)^2*asin(\lambda*Phi_tau/(k+1/2)));
end

• Calculating zero crossing statistics:

for (a given range of $\lambda$)
$$Qbar(t)=\text{Integral}(|\Psi(q,t)|^2*q)$$
for (a given number of $q$ intervals)
$$\text{distance from exact} = (\lambda \text{ is infinity}) - \text{findzero}(Qbar,q \text{ interval})$$
$$\text{std dev}(\text{distance from exact})$$
$$\text{average}(\text{distance from exact})$$

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