On numerical density approximations of solutions of SDEs with unbounded coefficients

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Abstract We study a numerical method to compute probability density functions of solutions of stochastic differential equations. The method is sometimes called the numerical path integration method and has been shown to be fast and accurate in application oriented fields. In this paper we provide a rigorous analysis of the method that covers systems of equations with unbounded coefficients. Working in a natural space for densities, $L^1$, we obtain stability, consistency, and new convergence results for the method, new well-posedness and semigroup generation results for the related Fokker-Planck-Kolmogorov equation, and a new and rigorous connection to the corresponding probability density functions for both the approximate and the exact problems. To prove the results we combine semigroup and PDE arguments in a new way that should be of independent interest.

Keywords Stochastic differential equations · Numerical method · Path integration · Density tracking · Probability density · Semigroup generation · Convergence

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1 Introduction

Over the past decades there has been a large number of publications in the field of stochastic dynamics and its various application areas – including physics, biology, engineering, and finance [2, 12, 35]. In this field, the response of dynamical systems to stochastic excitation is studied, and the typical model is (a system of) stochastic differential equations (SDEs) of the form

\[
\begin{align*}
  dY_t &= b(t, Y_t) \, dt + \sigma(t, Y_t) \, dB_t, \\
  Y_0 &= Z,
\end{align*}
\]

where \( b: \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d \), \( \sigma: \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d \times n} \), \( B_t \) is an \( n \)-dimensional Brownian motion, and the initial data \( Z \) is a random variable in \( \mathbb{R}^d \) independent of \( B_t \). The solution \( Y_t \) of the SDE is a state space process in \( \mathbb{R}^d \).

In most cases the solutions of such problems must be computed numerically, and various discrete approximation methods are widely used in many application areas [25]. There are two main approaches: (i) Path-wise approximations of the SDE based on stochastic simulation, and (ii) approximations of the statistics or distributions of the SDE. The first approach is more efficient in high dimensions and the second in low dimensions. For path-wise approximations we refer to [25] for the classical literature and e.g. to [32] and references therein for some promising recent developments. Approach (ii) is deterministic and based on approximating the forward or backward Kolmogorov partial differential equations. In this paper we study a method of the second type, called the numerical path integration method [11, 30, 37, 42–44] or density tracking method [5]. This method is an explicit deterministic iteration scheme that produces approximate probability density functions (PDFs) for the solution of the SDE. The iteration step is based on a short time approximation of the SDE. For simplicity we use here the strong Euler-Maruyama method, the most basic numerical method for SDEs. The convergence of our path integration method is therefore equivalent to the convergence of the PDFs of the Euler-Maruyama method.

The path integration/density tracking approach (i.e. simulating the PDFs) enjoys several favorable properties. First, it introduces an extra perspective to the system, which enables deeper insights and invites broader mathematical tools. Secondly, as an explicit method, one can formally implement the path integration algorithm on a vast number of scenarios. Since the formulation is deterministic, it is also free from perturbation by extreme outcomes during stochastic simulation. Finally, the result of the method is an explicit density function rather than bundle of random paths. This means that many characteristics of the system become more transparent, and can be captured and displayed by e.g. visualisation methods.

The numerical path integration method has been applied in many fields, including financial mathematics [11, 30, 42, 43]. Many of these studies show that it can provide highly accurate numerical solutions [5, 36, 37, 44]. Even though convergence problems have been reported in some cases, cf. Section 7.1 in [36], little or no emphasis has been devoted to conditions for convergence of the method in the path integration literature. A very natural and relevant mode of convergence for this method is strong \( L^1 \)-convergence of the resulting densities. Such convergence seems not to be a direct consequence of either strong or weak convergence of the Euler-Maruyama scheme.