J-HOLOMORPHIC CURVES AND DIRAC-HARMONIC MAPS

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ABSTRACT. Dirac-harmonic maps are critical points of a fermionic action functional, generalizing the Dirichlet energy for harmonic maps. We consider the case where the source manifold is a closed Riemann surface with the canonical Spin^c-structure determined by the complex structure and the target space is a Kähler manifold. If the underlying map \( f \) is a \( J \)-holomorphic curve, we determine a space of spinors on the Riemann surface which form Dirac-harmonic maps together with \( f \). For suitable complex structures on the target manifold the tangent bundle to the moduli space of \( J \)-holomorphic curves consists of Dirac-harmonic maps. We also discuss the relation to the A-model of topological string theory.

1. INTRODUCTION

We briefly recall the definition of Dirac-harmonic maps (see [8, 9] and Section 2 for more details). Let \((\Sigma, h)\) and \((M, g)\) be Riemannian manifolds, where \(\Sigma\) is closed and oriented, and \(f: \Sigma \to M\) a smooth map. We assume that \(\Sigma\) is a spin manifold and choose a spin structure \(s\) with associated complex spinor bundle \(S\). We can then form the twisted spinor bundle \(S \otimes_{\mathbb{R}} f^*TM\) of spinors on \(\Sigma\) with values in the pullback \(f^*TM\) (also called spinors along the map \(f\)). The Dirac operator

\[
D^f: \Gamma(S \otimes_{\mathbb{R}} f^*TM) \to \Gamma(S \otimes_{\mathbb{R}} f^*TM)
\]

determined by the Levi–Civita connections on \(\Sigma\) and \(M\).

Dirac-harmonic maps \((f, \psi)\), where \(\psi \in \Gamma(S \otimes_{\mathbb{R}} f^*TM)\), are solutions of the following system of coupled equations [8, 9]:

\[
\tau(f) = \mathcal{R}(f, \psi) \\
D^f\psi = 0.
\] (1.1)

Here \(\tau(f) \in \Gamma(f^*TM)\) is the so-called tension field of \(f\) (cf. [12] and equation (2.2)). The curvature term \(\mathcal{R}(f, \psi)\) is determined by the curvature tensor \(\mathcal{R}\) of the Riemannian metric \(g\) on \(M\) and is an algebraic expression in the differential \(df\) and the spinor \(\psi\) (linear in \(df\) and quadratic in \(\psi\)); see Appendix B for a definition.

The system of equations (1.1) for Dirac-harmonic maps makes sense more generally if we replace the spin structure \(s\) by a Spin^c-structure \(s^c\) and consider twisted
spinors $\psi \in \Gamma(S^c \otimes_R f^*TM)$, where $S^c$ is the complex spinor bundle associated to $\mathfrak{s}^c$. For the definition of the Dirac operator

$$D^f : \Gamma(S^c \otimes_R f^*TM) \longrightarrow \Gamma(S^c \otimes_R f^*TM)$$

one has to choose (in addition to the Riemannian metrics $h$ and $g$) a Hermitian connection on the characteristic complex line bundle of $\mathfrak{s}^c$. We assume throughout that such a choice has been made and fixed (in the case we discuss there is a canonical choice of such a connection, determined by the Riemannian metric $h$).

Every almost Hermitian manifold $(\Sigma, j, h)$ has a canonical Spin$^c$-structure $\mathfrak{s}^c$ whose associated spinor bundle $S^c = S^{c+} \oplus S^{c-}$ is the direct sum of the positive and negative Weyl spinor bundles

$$S^{c+} = \Lambda^{0, \text{even}}$$
$$S^{c-} = \Lambda^{0, \text{odd}}.$$  

We focus on the special case where $(\Sigma, j, h)$ is a closed Riemann surface, so that

$$S^{c+} = \Lambda^{0,0} = \mathbb{C}$$
$$S^{c-} = \Lambda^{0,1} = K^{-1}.$$  

The Levi–Civita connection $\nabla^h$ induces a connection on $S^c$ with Dirac operator

$$D : \Gamma(S^{c\pm}) \longrightarrow \Gamma(S^{c\mp})$$

equal to the classical Dolbeault–Dirac operator

$$\sqrt{2}(\bar{\partial} + \partial^*).$$

Suppose that the target space $(M^{2n}, J, g, \omega)$ is an almost Hermitian manifold with a Hermitian connection $\nabla^M$ and $f : \Sigma \to M$ a smooth map. We first derive a formula for the twisted Dirac operator

$$D^f : \Gamma(S^{c\pm} \otimes_R f^*TM) \longrightarrow \Gamma(S^{c\mp} \otimes_R f^*TM).$$

**Proposition 1.1.** The spinor bundle $S^c \otimes_R f^*TM$ decomposes into two twisted complex spinor bundles

$$S^c \otimes_R f^*TM = (S^c \otimes_{\mathbb{C}} f^*T^{1,0}M) \oplus (S^c \otimes_{\mathbb{C}} f^*T^{0,1}M).$$

There is a corresponding decomposition $D^f = D^{\prime \prime} + D^{\prime \prime \prime}$ of the Dirac operator into two twisted Dolbeault–Dirac operators

$$D^{\prime \prime} = \sqrt{2}(\bar{\partial}^{\prime \prime} + \partial^{\prime \prime *})$$
$$D^{\prime \prime \prime} = \sqrt{2}(\bar{\partial}^{\prime \prime \prime} + \partial^{\prime \prime \prime *}).$$

The Hirzebruch–Riemann–Roch Theorem implies for the indices

$$\text{ind}_{\mathbb{C}} D^{\prime \prime} = n(1 - g_{\Sigma}) + c_1(A)$$
$$\text{ind}_{\mathbb{C}} D^{\prime \prime \prime} = n(1 - g_{\Sigma}) - c_1(A),$$

where $g_{\Sigma}$ is the genus of $\Sigma$, $A = f_*[\Sigma] \in H_2(M; \mathbb{Z})$ is the integral homology class represented by $\Sigma$ under $f$ and $c_1(A) = \langle c_1(TM, J), A \rangle$. 

We then restrict to the case where \((M, J, g, \omega)\) is Kähler, \(\nabla^M = \nabla^g\) the Levi-Civita connection and \(f: \Sigma \to M\) a \(J\)-holomorphic curve. In this case, the Dolbeault operator \(\bar{\partial}'\) is equal to the linearization \(L_f \bar{\partial}_J\) of the non-linear Cauchy–Riemann operator \(\bar{\partial}_J\) in \(f\). In particular, the kernel of \(D'f\) is given by the direct sum of the deformation and obstruction space for the \(J\)-holomorphic curve \(f\) (cf. Remark 4.4):
\[
\ker D'f = \ker \bar{\partial}' \oplus \ker \bar{\partial}'^\ast = \text{Def}_J(f) \oplus \text{Obs}_J(f).
\]
The pair \((f, J)\) is called regular if \(\text{Obs}_J(f) = 0\).

**Theorem 1.2.** Suppose that \((M, J, g, \omega)\) is a Kähler manifold of complex dimension \(n > 0\) and \(f: \Sigma \to M\) a \(J\)-holomorphic curve. If \(\psi \in \Gamma(S^c \otimes_C f^*T^C M)\) is an element of one of the following vector spaces, then \((f, \psi)\) is Dirac-harmonic:
\[
\ker \bar{\partial}' \oplus \ker \bar{\partial}'^\ast, \quad \ker \bar{\partial}'^\ast \oplus \ker \bar{\partial}'^\ast, \quad \ker \bar{\partial}' \oplus \ker \bar{\partial}'^\ast, \quad \ker \bar{\partial}'^\ast \oplus \ker \bar{\partial}'^\ast.
\]
At least one of these vector spaces is non-zero, except possibly in the case that \(g_\Sigma = 1\) and \(c_1(A) = 0\).

**Corollary 1.3.** Let \((\Sigma, j)\) be a Riemann surface, \(A \in H_2(M; \mathbb{Z})\) and denote by \(\mathcal{M}(A, J)\) the moduli space of all \(J\)-holomorphic curves \(f: \Sigma \to M\) with \(f_*[\Sigma] = A\). Suppose that \((f, J)\) is regular for all \(f \in \mathcal{M}(A, J)\). Then \(\mathcal{M}(A, J)\) is a smooth manifold (possibly empty) of dimension
\[
\dim \mathcal{M}(A, J) = 2n(1 - g_\Sigma) + 2c_1(A).
\]
Every element \((f, \psi) \in T \mathcal{M}(A, J)\) of the tangent bundle of the moduli space is a Dirac-harmonic map.

**Remark 1.4.** For the case of spin structures the vector spaces corresponding to the ones in (1.2) appear in the proof of [25, Theorem 1.1].

**Remark 1.5.** Dirac-harmonic maps for the canonical Spin\(^c\)-structure on Riemann surfaces are closely related to the A-model of topological string theory [27, 28] (with a fixed metric \(h\), i.e. without worldsheet gravity); see Section 5 for a short discussion. In particular, in the A-model path integrals of certain operators localize to integrals over the finite-dimensional moduli spaces \(\mathcal{M}(A, J)\) and the tangent bundle \(T \mathcal{M}(A, J)\) can be identified with the space of \(\chi\)-zero modes (in our notation \(\chi = \psi \in \ker \bar{\partial}'\)).

In the last section we consider a generalization of Theorem 1.2 to twisted Spin\(^c\)-structures \(S^c \otimes_C L\) with a holomorphic line bundle \(L \to \Sigma\); see Corollary 7.2. For \(L = K^\pm\) this includes the case of the spinor bundle \(S = S^c \otimes_C \mathbb{C} K^\pm\) of a spin structure \(s\).

Dirac-harmonic maps \((f, \psi)\) from surfaces \(\Sigma\) with a spin structure to Riemannian target manifolds \(M\) have been studied before. We summarize some of the results in [2, 3, 9, 10, 20, 24, 25, 29].
Examples of Dirac-harmonic maps for $\Sigma = M = S^2$ were constructed in [9] where $f$ is a conformal map and $\psi$ is defined using a twistor spinor on $S^2$. This method was generalized in [20] to arbitrary Riemann surfaces $\Sigma$ admitting twistor spinors and arbitrary Riemannian manifolds $M$, where the map $f$ is harmonic (among closed surfaces only $S^2$ and $T^2$ admit non-zero twistor spinors [14, A.2.2]). In [29] and [10] it was shown that all Dirac-harmonic maps with source $\Sigma$ of genus $g_\Sigma$ and target $M = S^2$, so that $|\deg(f)| + 1 > g_\Sigma$, can be obtained using the constructions from [9, 20], where $f$ is holomorphic or antiholomorphic and $\psi$ is defined using a twistor spinor on $\Sigma$, possibly with isolated singularities (see also [24]). Dirac-harmonic maps $(f, \psi)$ from spin Kähler manifolds to arbitrary Kähler manifolds were studied in [25]. In Example 7.5 below we consider the case where the source is a Riemann surface $\Sigma$ with a spin structure and the map $f$ is $J$-holomorphic.

Existence results for Dirac-harmonic maps related to the $\alpha$-genus $\alpha(\Sigma, s, f)$ for a spin structure $s$ on $\Sigma$ were discussed in [2]. Section 10.1 in [2] contains several results for Dirac-harmonic maps from surfaces to Riemannian manifolds $M$ of dimension $\geq 3$. In [3] Dirac-harmonic maps from surfaces to Riemannian manifolds were constructed with methods related to an ansatz in [20].

In [13, 16, 17] another fermionic generalization of $J$-holomorphic curves was studied (see Remark 7.3 for a brief discussion of the relation to Dirac-harmonic maps).

**Conventions.** In the following, all Riemann surfaces $\Sigma$ are closed (compact and without boundary), connected and oriented by the complex structure. For Riemannian metrics $h$ on $\Sigma$ and $g$ on $M$ we denote by $\nabla^h$ and $\nabla^g$ the Levi–Civita connections. Tensor products of vector spaces and vector bundles are over the complex numbers $\mathbb{C}$, unless indicated otherwise.

## 2. Some Background on Dirac-Harmonic Maps

Recall that harmonic maps $f: \Sigma \to M$ from a closed, oriented Riemannian manifold $(\Sigma, h)$ to a Riemannian manifold $(M, g)$ are smooth maps, defined as the critical points of the Dirichlet energy functional [12]

$$L[f] = \frac{1}{2} \int_{\Sigma} |df|^2 \, d\text{vol}_h,$$

where $df$ is the differential of $f$ and $|df|^2$ is the length-squared determined by the metrics $h$ and $g$. The Euler–Lagrange equation for stationary points of $L[f]$ under variations of $f$ is

$$\tau(f) = 0,$$

where $\tau(f)$ is the tension field

$$\tau(f) = \text{tr}_h(\nabla^h df) = \sum_\alpha (\nabla^h_{e_\alpha} df)(e_\alpha).$$
Here $df$ is considered as an element of $\Omega^1(f^*TM)$ and the connection $\nabla^f$ on the vector bundle $f^*TM \to \Sigma$ is induced from the Levi–Civita connection $\nabla^M = \nabla^g$. The basis $\{e_\alpha\}$ is a local orthonormal frame on $\Sigma$.

**Remark 2.1.** If the connection $\nabla^M$ on $M$ is compatible with $g$, but not torsion-free, then harmonic maps $f$ do not necessarily satisfy $\tau(f) = 0$.

Suppose that $\Sigma$ is a spin manifold and let $s$ be a spin structure on $\Sigma$ with associated complex spinor bundle $S$ and twisted spinor bundle $S \otimes_R f^*TM$. Note that if $V$ is a complex vector space and $W$ a real vector space, then $V \otimes_R W$ is a complex vector space isomorphic to $V \otimes_C W$.

It follows that there is a (canonical) isomorphism of complex vector bundles $S \otimes_R f^*TM \cong S \otimes_C f^*T^C M$, with $T^C M = TM \otimes_R \mathbb{C}$ (see [29, Section 2]).

The Levi–Civita connection on $\Sigma$ and the connection $\nabla^f$ on $f^*TM$ yield a Dirac operator $D^f : \Gamma(S \otimes_R f^*TM) \to \Gamma(S \otimes_R f^*TM)$.

Dirac-harmonic maps $(f, \psi)$ are defined as the critical points of the fermionic action functional [8, 9]

$$L[f, \psi] = \frac{1}{2} \int_\Sigma (|df|^2 + \langle \psi, D^f \psi \rangle) \, dvol_h. \quad (2.3)$$

A pair $(f, \psi)$ is Dirac-harmonic if and only if it is a solution of the system of coupled Euler–Lagrange equations (1.1) (see [9, Proposition 2.1] for a proof of the formulae below):

- If $f$ is fixed and $\psi_t$ a variation of $\psi$ with
  $$\psi_0 = \psi, \quad \frac{d\psi_t}{dt} \bigg|_{t=0} = \eta \in \Gamma(S^c \otimes_R f^*TM),$$

  then
  $$\frac{d}{dt} \bigg|_{t=0} \int_\Sigma \langle \psi_t, D^f \psi_t \rangle \, dvol_h = 2 \int_\Sigma \langle \eta, D^f \psi \rangle \, dvol_h.$$

- If $f_t$ is a variation of $f$ with
  $$f_0 = f, \quad \frac{df_t}{dt} \bigg|_{t=0} = f^*X \in \Gamma(f^*TM),$$

  then
  $$\frac{d}{dt} \bigg|_{t=0} \int_\Sigma |df_t|^2 \, dvol_h = -2 \int_\Sigma g(\tau(f), f^*X) \, dvol_h.$$

Suppose in addition that $\psi_t = \sum_\mu \psi_\mu \otimes f_t^* \partial_\mu$ is a twisted spinor with time-independent components $\psi_\mu$ with respect to local coordinates $\{x_\mu\}$ (or a local frame) of $M$. If $\psi = \psi_0$ satisfies $D^f \psi = 0$, then

$$\frac{d}{dt} \bigg|_{t=0} \int_\Sigma \langle \psi_t, D^f \psi_t \rangle \, dvol_h = 2 \int_\Sigma g(\mathcal{R}(f, \psi), f^*X) \, dvol_h. \quad (2.4)$$
More details on the calculation of this variation can be found in Appendix B.

Dirac-harmonic maps are generalizations of harmonic maps: For the trivial spinor $\psi \equiv 0$, the curvature term $R(f, \psi)$ vanishes identically and the system of equations (1.1) reduces to the equation

$$\tau(f) = 0,$$

i.e. $(f, 0)$ is Dirac-harmonic for any harmonic map $f$.

The fermionic action functional (2.3) is motivated by theoretical physics: Suppose that $\Sigma$ is 2-dimensional and $h, g$ Lorentzian metrics. The Dirichlet energy $L[X]$ for smooth maps $X: \Sigma \to M$ is (up to a normalization constant) the non-linear $\sigma$-model (Polyakov) action for bosonic strings propagating in $(M, g)$, cf. [7]. The functional $L[X, \psi]$ for Dirac-harmonic maps is part of the supersymmetric non-linear $\sigma$-model action [1]: Choosing coordinates $\{x_\mu\}$ on an open subset $U \subset M$ we can write every spinor $\psi \in \Gamma(S \otimes f^*TM)$ on $\tilde{U} = f^{-1}(U)$ as

$$\psi = \sum_\mu \psi_\mu \otimes f^*\partial_\mu, \quad \text{with } \psi_\mu \in \Gamma(\tilde{U}, S).$$

The spinors $\psi_\mu$ are the fermionic superpartners of the scalar fields $X_\mu \in C^\infty(\tilde{U}, \mathbb{R})$, i.e. the coordinate fields of the map $X$ (in physics, the spinors $\psi_\mu$ take values in a Grassmann algebra).

In the supersymmetric non-linear $\sigma$-model action in [1] there is an additional curvature term which is determined by the curvature tensor $R$ of $g$ and of order 4 in the spinor $\psi$ (cf. [11]). The full action for superstrings contains also a gravitino $\chi$, the superpartner of the metric $h$. This action was studied from a mathematical point of view in [19].

3. Spin$^c$-structures on Riemann surfaces

We discuss some background material concerning Spin$^c$-structures on Riemann surfaces (more details can be found e.g. in [18, 5, 13, 21]).

Let $(\Sigma, j, h)$ be a closed Riemann surface with complex structure $j$ and compatible Riemannian metric $h$. The canonical Spin$^c$-structure $s^c$ on $\Sigma$ has spinor bundles

$$S^{c+} = \Lambda^{0.0} = \mathbb{C}$$
$$S^{c-} = \Lambda^{0.1} = K^{-1},$$

where $\mathbb{C}$ is the trivial complex line bundle and $K^{-1} = K$ is the anticanonical line bundle. The spaces of smooth sections are

$$\Gamma(S^{c+}) = C^\infty(\Sigma, \mathbb{C})$$
$$\Gamma(S^{c-}) = \Omega^{0,1}(\Sigma).$$

Our notation for tangent vectors and 1-forms of type $(1, 0)$ and $(0, 1)$ can be found in Appendix A. The Riemannian metric $h$ extends to Hermitian bundle metrics on
$T^{1,0} \oplus T^{0,1}$ and $\Lambda^{1,0} \oplus \Lambda^{0,1}$ and the choice of a local $h$-orthonormal basis $(e_1, e_2)$ of $T\Sigma$ with $e_2 = je_1$ determines local unit basis vectors

\[ \epsilon \in T^{1,0}, \quad \bar{\epsilon} \in T^{0,1} \]

and dual unit basis $1$-forms

\[ \kappa \in \Lambda^{1,0}, \quad \bar{\kappa} \in \Lambda^{0,1}. \]

Any element $\beta \in \Lambda^{0,1}$ can be written as

\[ \beta = \sqrt{2} \beta(e_1) \bar{\kappa}. \]  

The spinor bundle $S$ has a Clifford multiplication

\[ \gamma : T\Sigma \times S^{c\pm} \longrightarrow S^{c\mp}, \quad (v, \psi) \mapsto \gamma(v)\psi = v \cdot \psi, \]

that satisfies the Clifford relation

\[ v \cdot w \cdot \psi + w \cdot v \cdot \psi = -2h(v, w)\psi. \]

Let $\alpha \in (T^c\Sigma)^*$. For $\phi \in \mathbb{C} = \Lambda^{0,0}$ Clifford multiplication is given by

\[ \alpha \cdot \phi = \sqrt{2} \alpha^{0,1} \phi, \]

which implies

\[ e_1 \cdot \phi = \phi \bar{\kappa} \]
\[ e_2 \cdot \phi = i \phi \bar{\kappa}. \]

For $\beta \in K^{-1} = \Lambda^{0,1}$ Clifford multiplication is given by contraction

\[ \alpha \cdot \beta = -\sqrt{2} i \alpha \beta, \]

implying

\[ e_1 \cdot \beta = -\beta(\bar{\epsilon}) \]
\[ e_2 \cdot \beta = i \beta(\bar{\epsilon}). \]

In particular, the volume form $d\text{vol}_h = e_1^* \wedge e_2^*$ acts as

\[ d\text{vol}_h = \pm(-i) \quad \text{on} \quad S^{c\pm}. \]

The decomposition of the differential

\[ d : \mathcal{C}^\infty(\Sigma, \mathbb{C}) \longrightarrow \Omega^1(\Sigma, \mathbb{C}) \]

into $(1, 0)$- and $(0, 1)$-components is denoted by

\[ d\phi = (d\phi)^{1,0} + (d\phi)^{0,1} = \partial \phi + \bar{\partial} \phi \]

and the Dolbeault operator is given by

\[ \bar{\partial} : \mathcal{C}^\infty(\Sigma, \mathbb{C}) \longrightarrow \Omega^{0,1}(\Sigma), \quad \bar{\partial} \phi = \frac{1}{2} (d\phi + id\phi \circ j) \]

with formal adjoint

\[ \bar{\partial}^* : \Omega^{0,1}(\Sigma) \longrightarrow \mathcal{C}^\infty(\Sigma, \mathbb{C}). \]
The Levi–Civita connection $\nabla^h$ of the Kähler metric $h$ satisfies $\nabla^h j = j\nabla^h$ and induces a connection on $K^{-1}$ and thus a Hermitian connection on $S^c$, compatible with Clifford multiplication. We consider the associated Dirac operator

$$D : \Gamma(S^{c\pm}) \longrightarrow \Gamma(S^{c\mp}).$$

**Lemma 3.1** (cf. [18]). The Dirac operator $D$ is equal to the Dolbeault–Dirac operator

$$\sqrt{2}(\bar{\partial} + \bar{\partial}^*).$$

The Riemann–Roch theorem implies for the index

$$\text{ind}_\Sigma D = 1 - g_\Sigma,$$

where $g_\Sigma$ is the genus of $\Sigma$.

**Proof.** Let $\phi \in C^\infty(\Sigma, \mathbb{C})$ be a positive spinor. On $C^\infty(\Sigma, \mathbb{C})$ the connection is just the differential $d$, hence

$$D\phi = e_1 \cdot d\phi(e_1) + e_2 \cdot d\phi(e_2) = (d\phi(e_1) + id\phi(e_2))\tilde{\kappa} = 2\bar{\partial}\phi(e_1)\tilde{\kappa} = \sqrt{2}\bar{\partial}\phi,$$

where the last step follows from equation (3.1). Thus

$$D : C^\infty(\Sigma, \mathbb{C}) \longrightarrow \Omega^{0,1}(\Sigma)$$

$$\phi \mapsto \sqrt{2}\bar{\partial}\phi.$$

Since the Dirac operator is formally self-adjoint, the claim follows. \qed

**Remark 3.2.** Riemann surfaces are spin, hence we can choose a spin structure $s$ on $\Sigma$, which is equivalent to the choice of a holomorphic square root $K^{1/2}$ of the canonical bundle $K$ (see [4, 18]). The spinor bundles of $s$ are

$$S^+ = K^{1/2}$$

$$S^- = K^{-1/2}$$

and the spinor bundle of the canonical Spin$^c$-structure is obtained by twisting

$$S^c = S \otimes K^{-1/2}.$$ 

There is another Spin$^c$-structure with spinor bundle

$$\bar{S}^c = S \otimes K^{1/2},$$

i.e.

$$\bar{S}^{c+} = K$$

$$\bar{S}^{c-} = \mathbb{C}.$$

**Remark 3.3.** Let $L \to \Sigma$ be a complex line bundle with a Hermitian bundle metric. Then there is a twisted Spin$^c$-structure $s^c \otimes L$ with spinor bundles

$$S^{c+} \otimes L = L$$

$$S^{c-} \otimes L = K^{-1} \otimes L.$$
A connection $\nabla^B$ on $L$, compatible with the Hermitian bundle metric, together with the Levi–Civita connection $\nabla^h$ yields a Hermitian connection on $S^c \otimes L$ and a Dirac operator

$$ DB : \Gamma(S^c \pm \otimes L) \longrightarrow \Gamma(S^c \mp \otimes L). $$

With the Dolbeault operator

$$ \bar{\partial}_B : \Gamma(L) \longrightarrow \Omega^{0,1}(L), \quad \bar{\partial}_B \phi = \frac{i}{2}(\nabla^B \phi + i\nabla^B \phi \circ j) $$

the Dirac operator $D_B$ is equal to the Dolbeault–Dirac operator

$$ \sqrt{2}(\bar{\partial}_B + \bar{\partial}_B^*). $$

4. Dirac Operator Along Maps and $J$-Holomorphic Curves

Let $(\Sigma, j, h)$ be a Riemann surface and $(M, J, g, \omega)$ an almost Hermitian manifold of real dimension $2n$ with almost complex structure $J$, Riemannian metric $g$ and non-degenerate 2-form $\omega$, related by

$$ g(Jx, Jy) = g(x, y) \quad \omega(x, y) = g(Jx, y) \quad \forall x, y \in TM. $$

We fix a Hermitian connection $\nabla^M$ on $TM$, i.e. an affine connection such that $\nabla^M g = 0$ and $\nabla^M J = 0$. For a general almost Hermitian manifold the connection $\nabla^M$ has non-zero torsion. The Hermitian connection $\nabla^M$ can be chosen torsion-free, hence equal to the Levi–Civita connection $\nabla^g$ of $g$, if and only if $(M, J, g, \omega)$ is Kähler.

Let $f : \Sigma \to M$ be a smooth map and consider the pullback $f^*TM \to \Sigma$ of the tangent bundle $TM$. If $X$ is vector field on $M$, then the pullback

$$ f^*X : \Sigma \longrightarrow f^*TM, \quad z \longmapsto X_{f(z)} $$

is a section of $f^*TM$. There is a unique Hermitian connection $\nabla^f$ on $f^*TM$ so that

$$ \nabla^f_V(f^*X) = f^*(\nabla^M_{df(V)}X) \quad \forall X \in \mathfrak{X}(M), V \in T\Sigma. $$

We consider the twisted spinor bundle

$$ S^c \otimes_R f^*TM \cong S^c \otimes f^*T^CM $$

on $\Sigma$. The Riemannian metric $g$ extends to a Hermitian bundle metric $\langle \cdot, \cdot \rangle$ on $T^CM$. There is a decomposition into orthogonal $ \pm i$-eigenspaces of the complex linear extension of $J$,

$$ T^CM = T^{1,0}M \oplus T^{0,1}M $$

and a corresponding decomposition of $S^c \otimes_R f^*TM$ into two twisted complex spinor bundles (cf. [29 Section 3])

$$ S^c \otimes_R f^*TM = (S^c \otimes f^*T^{1,0}M) \oplus (S^c \otimes f^*T^{0,1}M) $$

(4.1)

(the tensor products on the right are over $\mathbb{C}$). The connection $\nabla^M$ extends to a Hermitian connection on $T^CM$ which preserves both complex subbundles $T^{1,0}M$ and $T^{0,1}M$. The connections $\nabla^h$ and $\nabla^f$ thus define a Hermitian connection on
\[ S^c \otimes_{\mathbb{R}} f^*TM, \] also denoted by \( \nabla^f \), which preserves both complex spinor bundles on the right hand side of equation (4.1).

**Definition 4.1 (cf. [9]).** The associated twisted Dirac operator
\[ D^f : \Gamma(S^{c\pm} \otimes_{\mathbb{R}} f^*TM) \rightarrow \Gamma(S^{c\mp} \otimes_{\mathbb{R}} f^*TM) \]
\[ \psi \mapsto \sum_{\alpha=1}^2 e_{\alpha} \cdot \nabla^f_{e_{\alpha}} \psi \]
is called the \textbf{Dirac operator along the map} \( f \). Under the splitting in equation (4.1) the Dirac operator \( D^f \) decomposes into two twisted Dirac operators
\[ D'^f : \Gamma(S^{c\pm} \otimes f^*T^{1,0}M) \rightarrow \Gamma(S^{c\mp} \otimes f^*T^{1,0}M) \]
\[ D''^f : \Gamma(S^{c\pm} \otimes f^*T^{0,1}M) \rightarrow \Gamma(S^{c\mp} \otimes f^*T^{0,1}M). \]

Since the connection \( \nabla^f \) on the twisted spinor bundle is obtained from the Levi–Civita connection \( \nabla^h \) on \( \Sigma \), the Dirac operator \( D^f \) is formally self-adjoint. We consider the Dolbeault operators for the complex vector bundles \( f^*T^{1,0}M \) and \( f^*T^{0,1}M \),
\[ \bar{\partial}'^f : \Gamma(f^*T^{1,0}M) \rightarrow \Omega^{0,1}(f^*T^{1,0}M) \]
\[ \bar{\partial}''^f : \Gamma(f^*T^{0,1}M) \rightarrow \Omega^{0,1}(f^*T^{0,1}M) \]
defined by
\[ \bar{\partial}'^f \psi = \frac{1}{2}(\nabla^f \psi + J \circ \nabla^f \circ j) \]
\[ \bar{\partial}''^f \psi = \frac{1}{2}(\nabla^f \psi - J \circ \nabla^f \circ j). \]
The formal adjoints are denoted by \( \bar{\partial}'^f \) and \( \bar{\partial}''^f \).

**Proof of Proposition 4.1** Let \( \psi \in \Gamma(f^*T^{1,0}M) \). Then
\[ D'^f \psi = e_1 \cdot \nabla^f_{e_1} \psi + e_2 \cdot \nabla^f_{e_2} \psi = \bar{k} \otimes (\nabla^f_{e_1} \psi + i \nabla^f_{e_2} \psi) = \bar{k} \otimes (\nabla^f_{e_1} \psi + J \nabla^f_{je_{e_1}} \psi) \]
\[ = \sqrt{2} \bar{\partial}'^f \psi. \]
This implies the claim for the Dirac operator \( D'^f \), because it is self-adjoint. The claim for \( D''^f \) follows similarly. \( \square \)

Recall that a \textit{J}-holomorphic curve is a smooth map \( f : \Sigma \rightarrow M \) such that
\[ df \circ j = J \circ df, \]
where
\[ df : T\Sigma \rightarrow TM \]
is the differential. With the non-linear Cauchy–Riemann operator
\[ \bar{\partial}_J f = \frac{1}{2}(df + J \circ df \circ j), \]
the map \( f \) is a \textit{J}-holomorphic curve if and only if
\[ \bar{\partial}_J f = 0. \]
Corollary 4.2. Suppose that \((M, J, g, \omega)\) is Kähler, \(\nabla^M = \nabla^g\) the Levi–Civita connection and \(f : \Sigma \to M\) a \(J\)-holomorphic curve.

1. \(T^{1,0}M \cong (TM, J)\) and \(f^*T^{1,0}M\) is a holomorphic vector bundle over \(\Sigma\).
2. \(\bar{\partial}''\) is equal to the linearization \(L_f \partial_J\) of the non-linear Cauchy–Riemann operator \(\partial_J\) in \(f\).
3. The kernel of \(D''\) is given by
   \[
   \ker D'' = \ker \bar{\partial}'' \oplus \ker \bar{\partial}''^* \cong \ker L_f \partial_J \oplus \text{coker } L_f \partial_J \cong H^0(\Sigma, f^*T^{1,0}M) \oplus H^1(\Sigma, f^*T^{1,0}M).
   \]
4. The kernel of \(D'''\) is given by
   \[
   \ker D''' = \ker \bar{\partial}''' \oplus \ker \bar{\partial}'''^* \cong H^1(\Sigma, K_{\Sigma} \otimes f^*T^{1,0}M)^* \oplus H^0(\Sigma, K_{\Sigma} \otimes f^*T^{1,0}M)^*.
   \]

Proof. The claim in (2) follows from [22, p. 28]. For the formula in (3), note that
\[
\ker \bar{\partial}'' = H^0(\Sigma, f^*T^{1,0}M), \quad \text{coker } \bar{\partial}'' = H^0,1(\Sigma, f^*T^{1,0}M).
\]
The claim in (4) follows with Serre duality. 

Remark 4.3. For a non-integrable almost complex structure \(J\), the operators \(\bar{\partial}''\) and \(L_f \partial_J\) differ by an operator of order 0, cf. [22, p. 28].

Remark 4.4 (cf. [22, 23, 26]). For an arbitrary smooth map \(f : \Sigma \to M\), smooth sections of \(f^*TM\) correspond to infinitesimal deformations of \(f\). Suppose that \(f\) is \(J\)-holomorphic. Then elements of
\[
\text{Def}_J(f) = \ker L_f \partial_J
\]
correspond to infinitesimal deformations of \(f\) through \(J\)-holomorphic curves. The vector space
\[
\text{Obs}_J(f) = \text{coker } L_f \partial_J
\]
is called the obstruction space and the pair \((f, J)\) is called regular if \(\text{Obs}_J(f) = 0\), i.e. \(L_f \partial_J\) is surjective. If \((f, J)\) is regular, then \((f', J)\) is regular for all \(J\)-holomorphic curves \(f' : \Sigma \to M\) in a small neighbourhood of \(f\) (inside the space of all smooth maps \(\Sigma \to M\)). In this case, it follows that the local moduli space, i.e. the set of all \(J\)-holomorphic curves \(f'\) near \(f\), is a smooth manifold of real dimension \(2\text{ind}_C D^f\) with tangent space in \(f\) given by \(\text{Def}_J(f)\).

Remark 4.5. For a twisted \(\text{Spin}^c\)-structure \(S^c \otimes L\) with complex line bundle \(L \to \Sigma\), as in Remark 3.3, we can consider the spinor bundle \(S^c \otimes L \otimes_{\mathbb{R}} f^*TM\). The choice of a Hermitian connection \(B\) on \(L\) then defines a connection \(\nabla^{f \otimes B}\) on \(S^c \otimes L \otimes_{\mathbb{R}} f^*TM\) with Dirac operator
\[
D_B^f : \Gamma(S^{c\pm} \otimes L \otimes_{\mathbb{R}} f^*TM) \to \Gamma(S^{c\mp} \otimes L \otimes_{\mathbb{R}} f^*TM)
\]
given by a generalization of Proposition 1.1.
5. Relation to Topological String Theory

Dirac-harmonic maps on Riemann surfaces $\Sigma$ with the canonical Spin$^c$-structure are related to topological string theory, introduced by Edward Witten [27, 28]. We combine the Spin$^c$ spinor bundles

$$S^c = \mathbb{C} \oplus K^{-1}$$

$$\bar{S}^c = K \oplus \bar{\mathbb{C}}$$
on the Riemann surface to a twisted complex spinor bundle

$$\Delta = (S^c \oplus \bar{S}^c) \otimes f^*T^c M$$

with Weyl spinor bundles

$$\Delta^+ = T^{1,0}_f M \oplus (K \otimes T^{0,1}_f M) \oplus T^{0,1}_f M \oplus (K \otimes T^{0,1}_f M)$$

$$\Delta^- = (K^{-1} \otimes T^{1,0}_f M) \oplus T^{1,0}_f M \oplus (K^{-1} \otimes T^{0,1}_f M) \oplus T^{0,1}_f M.$$  

Here $(M, J, g, \omega)$ is a Kähler manifold of complex dimension $n$ and the pullback $f^*$ of $T^{1,0} M$ and $T^{0,1} M$ is abbreviated by an index $f$.

**Definition 5.1.** We define the following subbundles:

+ twist:

$$\Delta^+_{(+)} = T^{1,0}_f M \oplus (K \otimes T^{0,1}_f M)$$

$$\Delta^-_{(+)} = T^{1,0}_f M \oplus (K^{-1} \otimes T^{0,1}_f M).$$

− twist:

$$\Delta^+_{(-)} = (K \otimes T^{1,0}_f M) \oplus T^{0,1}_f M$$

$$\Delta^-_{(-)} = (K^{-1} \otimes T^{1,0}_f M) \oplus T^{0,1}_f M.$$  

We also define the following spinor bundles:

**A-model:**

$$\Delta_A = \Delta^+_{(+)} \oplus \Delta^-_{(-)}$$

with sections $$(\chi, \psi', \bar{\psi}, \chi')$$

**B-model:**

$$\Delta_B = \Delta^+_{(-)} \oplus \Delta^-_{(-)}$$

with sections $$(\rho_2, \frac{1}{2}(\eta' + \theta'), \rho_2, \frac{1}{2}(\eta' - \theta'))$$

To explain these definitions we consider the action functional

$$L[f, \psi] = \frac{1}{2} \int_{\Sigma} (|df|^2 + \langle \psi, D_1^\psi \rangle) \, dvol_h.$$  

The complete supersymmetric $\sigma$-model action functional also contains the quartic spinor term involving the Riemann curvature tensor of $g$, mentioned at the end of Section [2]. We ignore this term in the following discussion.

---

1. We follow the conventions in [28].
We first consider the case where \((M, g)\) is a Riemannian manifold and the spinor a section \(\psi \in \Gamma(S \otimes_R f^*TM)\) for the spinor bundle \(S\) of a spin structure on \(\Sigma\). One allows a slightly more general situation where the Weyl spinor bundles come from different spin structures: Let \(K^{\frac{j}{2}}\) and \(\bar{K}^{\frac{j}{2}}\) be holomorphic square roots of \(K\) and \(\bar{K}\), not necessarily related by \(\bar{K}^{\frac{j}{2}} = K^{\frac{j}{2}}\). Then
\[
\psi_+ \in \Gamma(K^{\frac{j}{2}} \otimes_R f^*TM), \quad \psi_- \in \Gamma(\bar{K}^{\frac{j}{2}} \otimes_R f^*TM).
\]
The non-linear \(\sigma\)-model has \(N = 2\) supersymmetry generated by spinors
\[
\epsilon_- \in \Gamma(K^{-\frac{j}{2}}), \quad \epsilon_+ \in \Gamma(\bar{K}^{-\frac{j}{2}}),
\]
which are holomorphic and antiholomorphic sections of \(K^{-\frac{j}{2}}\) and \(\bar{K}^{-\frac{j}{2}}\), respectively.

Suppose that \((M, J, g, \omega)\) is a Kähler manifold of complex dimension \(n\). We can decompose \(T^CM\) into the \((1,0)\)- and \((0,1)\)-part and denote the Weyl spinors by
\[
(\psi_+, \psi'_+) \in (K^{\frac{j}{2}} \otimes T^{1,0}_fM) \oplus (K^{\frac{j}{2}} \otimes T^{0,1}_fM)
\]
\[
(\psi_-, \psi'_-) \in (\bar{K}^{\frac{j}{2}} \otimes T^{1,0}_fM) \oplus (\bar{K}^{\frac{j}{2}} \otimes T^{0,1}_fM).
\]
The non-linear \(\sigma\)-model now has \(N = (2,2)\) supersymmetry generated by (anti)-holomorphic sections
\[
\alpha_-, \bar{\alpha}_- \in \Gamma(K^{-\frac{j}{2}}), \quad \alpha_+, \bar{\alpha}_+ \in \Gamma(\bar{K}^{-\frac{j}{2}}).
\]
(5.1) For a Riemann surface of genus \(g_{\Sigma} \neq 1\) the canonical and anticanonical bundle are non-trivial, hence the sections in (5.1) have zeroes. In particular, the only covariantly constant sections, corresponding to global (rigid) supersymmetries, are identically zero.

This can be remedied with the topological + and - twists, i.e. using the \(\text{Spin}^c\)-spinor bundle \(S^c\) instead of the spinor bundle \(S\). In the A-model the sections
\[
\alpha_-, \bar{\alpha}_+ \in \Gamma(\mathbb{C})
\]
and in the B-model the sections
\[
\bar{\alpha}_-, \bar{\alpha}_+ \in \Gamma(\mathbb{C})
\]
can be chosen covariantly constant. These sections yield a global fermionic symmetry \(Q\) of the non-linear \(\sigma\)-model for arbitrary genus \(g_{\Sigma}\), which implies that the A-model and B-model (for suitable target spaces) define topological quantum field theories (TQFTs).

We consider the A-model spinor bundle in more detail. The vector bundle \(\Delta_A\) can be decomposed as
\[
\Delta_A = (S^c \otimes T^{1,0}_fM) \oplus (\bar{S}^c \otimes T^{0,1}_fM)
\]
with sections
\[
(\Psi, \Psi'), \quad \Psi = (\chi, \psi_z), \quad \Psi' = (\psi'_z, \chi').
\]
The fermionic action (2.3) for the spinor bundle $\Delta_A$ can then be written as

$$L_A[f, \Psi, \Psi'] = \frac{1}{2} \int_\Sigma \left( |df|^2 + \langle \Psi, Df'\Psi \rangle + \langle \Psi', \bar{D}f''\Psi' \rangle \right) d\text{vol}_h.$$ 

There is a complex antilinear bundle isomorphism

$$S^c \otimes T^{1,0}_f M \xrightarrow{\cong} \bar{S}^c \otimes T^{0,1}_f M$$

given by complex conjugation and exchanging positive and negative Weyl spinors, which induces a corresponding isomorphism between $\ker Df'$ and $\ker \bar{D}f''$. Defining the numbers of zero modes

$$a = \dim_{\mathbb{C}} \{ (\chi, \chi') \mid Df'\chi = 0 = \bar{D}f''\chi' \}$$

$$b = \dim_{\mathbb{C}} \{ (\psi_z, \psi'_z) \mid Df'\psi_z = 0 = \bar{D}f''\psi'_z \},$$

the index of the Dirac operator $Df'$ is related to the so-called ghost number or $U(1)_A$-anomaly by

$$w = a - b = 2\text{ind}_{\mathbb{C}} Df' = 2n(1 - g_\Sigma) + 2c_1(A).$$

6. DIRAC-HARMONIC MAPS TO KÄHLER MANIFOLDS

Let $(\Sigma, j, h)$ be a Riemann surface and $(M, J, g, \omega)$ a Kähler manifold of complex dimension $n$ with Levi–Civita connection $\nabla^M = \nabla^g$.

Let $f: \Sigma \to M$ be a smooth map and $\psi \in \Gamma(S^c \otimes_{\mathbb{R}} f^*TM)$ a twisted spinor. Then $(f, \psi)$ is called a Dirac-harmonic map if it is a critical point of the fermionic action functional (2.3) (with the spinor bundle $S$ replaced by $S^c$). The same proof as in [9, Proposition 2.1] for spin structures shows that a pair $(f, \psi)$ is a Dirac-harmonic map if and only if it satisfies the Euler–Lagrange equations (1.1).

**Definition 6.1.** For $A \in H_2(M; \mathbb{Z})$ let

$$\mathcal{X}_A = \text{Map}(\Sigma, M; A)$$

be the set of all smooth maps $f: \Sigma \to M$ with $f_*[\Sigma] = A$, where $[\Sigma] \in H_2(\Sigma; \mathbb{Z})$ is the generator determined by the complex orientation of $\Sigma$.

**Proposition 6.2.** If $f: \Sigma \to M$ is $J$-holomorphic, then $f$ is harmonic and satisfies $\tau(f) = 0$. More precisely, the absolute minima of the Dirichlet energy $L[f]$ on $\mathcal{X}_A$ are given by the $J$-holomorphic curves $f$ with $f_*[\Sigma] = A$. The Dirichlet energy of a $J$-holomorphic curve $f$ has value

$$L[f] = \langle \omega, [A] \rangle,$$

where $\omega$ is the Kähler form on $M$.

**Proof.** The vanishing of the tension field $\tau(f)$ for $J$-holomorphic curves $f$ is well-known, cf. an example on [12, p. 118], and can be derived directly from formula (2.2) with respect to a local orthonormal frame $\{e_1, e_2 = je_1\}$, using that $df(e_2) = Jdf(e_1)$ and that the connection $\nabla^M = \nabla^g$ is torsion-free and Hermitian. The second part is proved in [23, Lemma 2.2.1] (note that deformations of $f$ do not change the integral homology class $f_*[\Sigma]$). \qed
Remark 6.3. More generally, if the target manifold is only almost Kähler, [23, Lemma 2.2.1] shows that $J$-holomorphic maps from closed Riemann surfaces are still absolute minima of the Dirichlet energy functional, hence harmonic maps. However, if $\nabla^M$ has torsion, the equation $\tau(f) = 0$ does not necessarily follow. Dirac-harmonic maps for connections $\nabla^M$ with torsion have been studied in [6].

The following statement appears in the proof of [25, Theorem 1.1] (more details on the definition of the curvature term $R(f, \psi)$ can be found in Appendix B).

**Proposition 6.4.** Let $f : \Sigma \to M$ be smooth map. Then

$$R(f, \psi) = 0$$

for all twisted spinors $\psi$ which are sections of one of the following subbundles of $S^c \otimes f^*T^C M$ (using the notation of Section 5):

$$S^{c+} \otimes (T^1_{f} M \oplus T^0_{f} M)$$

$$S^{c-} \otimes (T^1_{f} M \oplus T^0_{f} M)$$

$$\left( S^{c+} \otimes T^1_{f} M \right) \oplus \left( S^{c-} \otimes T^0_{f} M \right)$$

$$\left( S^{c+} \otimes T^0_{f} M \right) \oplus \left( S^{c-} \otimes T^1_{f} M \right).$$

**Proof.** This can be proved as in [25] by considering the expression (using the notation from Appendix B)

$$2g(R(f, \psi), f^*X) = \langle \psi, R^f(X, \psi) \rangle.$$

Alternatively, consider a smooth map $f : \Sigma \to M$ with variation $f_t$ given by a vector field $X \in \Gamma(f^*TM)$. Any spinor $\psi \in \Gamma(S^c \otimes f^*T^C M)$ defines a spinor $\psi_t = \sum \psi_\mu \otimes f_t^\mu \partial_\mu$ with time-independent components $\psi_\mu$ with respect to local coordinates on $M$. By equation (2.4)

$$\frac{d}{dt} \bigg|_{t=0} \int_\Sigma \langle \psi_t, D^{ft} \psi_t \rangle \dd vol_h = 2 \int_\Sigma g(R(f, \psi), f^*X) \dd vol_h.$$

For any variation $f_t$ the Dirac operator $D^{ft}$ maps positive (negative) to negative (positive) Weyl spinors and preserves the $(1, 0)$- and $(0, 1)$-type of twisted spinors. Furthermore, the bundles $S^{c+}$ and $S^{c-}$ as well as $T^1_{f} M$ and $T^0_{f} M$ are orthogonal with respect to the Hermitian bundle metric.

This implies for every section $\psi$ of the bundles in (6.1) that the corresponding spinor $\psi_t$ satisfies

$$\langle \psi_t, D^{ft} \psi_t \rangle = 0 \quad \forall t.$$

□

**Remark 6.5.** The first two bundles in (6.1) can be described as the $(\mp i)$-eigenspaces of the bundle automorphism $\dd vol_h = \dd vol_h \otimes \Id$ on $S^c \otimes f^*T^C M$ with $\dd vol_f = -\Id$ (cf. equation (3.2)). The other two bundles are the $(\pm 1)$-eigenspaces of the bundle automorphism $I = \dd vol_h \otimes J$ on $S^c \otimes f^*T^C M$ with $I^2 = \Id$. 

Proof of Theorem 1.2. The first claim is a direct consequence of the Euler–Lagrange equations (1.1) and Propositions 6.2, 6.4 and 1.1. The second claim follows because if all of the vector spaces are zero, then
\[ \text{ind}_C D'' = \text{ind}_C D''' = 0. \]

□

Remark 6.6. A Dirac-harmonic map \((f, \psi)\) as in Theorem 1.2, whose underlying map \(f\) is harmonic, is called uncoupled in [2]. The Dirac-harmonic maps \((f, \psi)\) in Theorem 1.2 have minimal bosonic action \(L[f]\) in their homology class \(A\).

Example 6.7. Suppose that \((M, J, g, \omega)\) is a Calabi–Yau manifold of complex dimension \(n\), hence \(c_1(TM) = 0\), and \(f : \mathbb{CP}^1 \to M\) is a \(J\)-holomorphic sphere. If \((f, J)\) is regular, then the vector space \(\ker \bar{\partial}''\) has complex dimension \(n\) and is the tangent space \(\text{Def}_f(f)\) in \(f\) of the local moduli space of \(J\)-holomorphic spheres (compare with [17, Remark 2.4]). For every \(\psi \in \ker \bar{\partial}''\), the pair \((f, \psi)\) is Dirac-harmonic.

Definition 6.8. Let \((\Sigma, j)\) be a fixed Riemann surface. For a class \(A \in H_2(M; \mathbb{Z})\) we denote by \(\mathcal{M}(A, J)\) the space of all \(J\)-holomorphic curves \(f : \Sigma \to M\) with \(f_*[\Sigma] = A\).

Proof of Corollary 1.3. This follows, because under the assumptions \(T_f \mathcal{M}(A, J) = \ker \bar{\partial}''\) for all \(f \in \mathcal{M}(A, J)\) (cf. Remark 4.4).

Example 6.9. Suppose that \((M, J, g, \omega)\) is a Kähler surface and \(f : \mathbb{CP}^1 \to M\) an embedded \(J\)-holomorphic sphere representing a class \(A\) of self-intersection \(A^2 = A \cdot A \geq -1\). Then every \(f' \in \mathcal{M}(A, J)\) is an embedding and \((f', J)\) is regular (see [22, Corollary 3.5.4]). By the adjunction formula
\[ -2 = A^2 - c_1(A), \]
hence \(\mathcal{M}(A, J)\) is a smooth manifold of real dimension \(8 + 2A^2 \geq 6\). The tangent bundle \(T\mathcal{M}(A, J)\) is a complex vector bundle and consists of Dirac-harmonic maps.

7. Generalization to twisted Spin\(\hat{c}\)-structures on \(\Sigma\)

We consider the following generalization for the same setup as in Section 6. Let \(L \to \Sigma\) be a holomorphic Hermitian line bundle with Chern connection \(\nabla\) and Dolbeault operator
\[ \bar{\partial} = \bar{\partial}_{\nabla} : \Gamma(L) \to \Omega^{0,1}(L). \]
Then \(s^c \otimes L\) is a Spin\(c\)-structure with holomorphic spinor bundles
\[ S^{c+} \otimes L = L \]
\[ S^{c-} \otimes L = K^{-1} \otimes L \]
and Dolbeault–Dirac operator
\[ D = \sqrt{2}(\bar{\partial} + \bar{\partial}^*): \Gamma(S^{c\pm} \otimes L) \to \Gamma(S^{c\mp} \otimes L). \]
Lemma 7.1. Let \( f : \Sigma \to M \) be a smooth map. The twisted Dirac operator
\[
D^f : \Gamma(S^c \otimes L \otimes f^*T^C M) \to \Gamma(S^c \otimes L \otimes f^*T^C M)
\]
decomposes into the sum \( D^f = D'' + D''' \) of two twisted Dolbeault–Dirac operators
\[
D'' = \sqrt{2} (\bar{\partial}'' + \bar{\partial}''^*) \\
D''' = \sqrt{2} (\bar{\partial}''' + \bar{\partial}'''^*).
\]

In this situation we can define Dirac-harmonic maps \((f, \psi)\) as solutions of the analogue of the system of equations (1.1).

Corollary 7.2. Let \( f : \Sigma \to M \) be a \( J \)-holomorphic curve with \( A = f_*[\Sigma] \). If \( \psi \in \Gamma(S^c \otimes L \otimes f^*T^C M) \) is an element of one of the following vector spaces, then \((f, \psi)\) is Dirac-harmonic:
\[
\ker \bar{\partial}'' \oplus \ker \bar{\partial}''^*, \quad \ker \bar{\partial}''' \oplus \ker \bar{\partial}'''^*.
\]

By the Hirzebruch–Riemann–Roch Theorem
\[
\text{ind}_C D'' = n(1 - g_\Sigma + c_1(L)) + c_1(A)
\]
\[
\text{ind}_C D''' = n(1 - g_\Sigma + c_1(L)) - c_1(A),
\]
where we write \( c_1(L) \) for \( \langle c_1(L), [\Sigma] \rangle \).

Remark 7.3. A Dirac-harmonic map \((f, \psi)\), where \( f \) is a \( J \)-holomorphic curve and \( \psi \in \ker \bar{\partial}'' \), is a \((\nabla^g, J)\)-holomorphic supercurve as studied in [17], cf. also [15].

Example 7.4. Consider again the situation in Example 6.9 of a Kähler surface \((M, J, g, \omega)\) with an embedded \( J \)-holomorphic sphere \( f : \mathbb{CP}^1 \to M \) of self-intersection \( A^2 = A \cdot A \geq -1 \) and smooth moduli space \( \mathcal{M}(A, J) \). Let \( L \to \Sigma \) be a holomorphic line bundle with \( c_1(L) > 0 \). Then
\[
c_1(L \otimes f^*T^{1,0} M) = 2c_1(L) + c_1(A) \geq 3
\]
and the arguments in [22] Section 3.5] using the Kodaira vanishing theorem show that \( \text{coker} \bar{\partial}'' = 0 \). Hence the complex vector space \( \ker \bar{\partial}'' \) has constant dimension
\[
\dim \ker \bar{\partial}'' = 4 + A^2 + 2c_1(L)
\]
for all \( f \in \mathcal{M}(A, J) \). There is a complex vector bundle over the infinite-dimensional manifold \( X_A \) from Definition 6.1 with fibre \( \Gamma(L \otimes f^*T^{1,0} M) \) over \( f \in X_A \). Since \( \mathcal{M}(A, J) \) is a submanifold of \( X_A \), it follows that the subset of Dirac-harmonic maps \((f, \psi)\) with
\[
f \in \mathcal{M}(A, J), \quad \psi \in \ker \bar{\partial}'' \subset \Gamma(L \otimes f^*T^{1,0} M)
\]
is a smooth complex vector bundle \( E \) over \( \mathcal{M}(A, J) \) of rank
\[
\text{rk}_C E = 4 + A^2 + 2c_1(L).
\]
In particular, for $L = K^\otimes(-q)$ with integers $q \geq 1$, we have $c_1(L) = 2q$ and the complex vector bundle $E$ over $\mathcal{M}(A, J)$ of Dirac-harmonic maps has rank

$$\text{rk}_C E = 4 + A^2 + 4q,$$

which becomes arbitrarily large for $q \gg 1$.

**Example 7.5.** Let $s$ be a spin structure on $\Sigma$ and $L = K^{\frac{1}{2}}$ the associated holomorphic square root of the canonical bundle $K$. Then $S \cong S^c \otimes K^{\frac{1}{2}}$ is the spinor bundle of $s$ with spin Dirac operator $D = \sqrt{2}(\bar{\partial} + \bar{\partial}^*)$ (cf. [18]) and

$$\text{ind}_C D'' = c_1(A)$$

$$\text{ind}_C D''' = -c_1(A).$$

The vector spaces in (7.1) are called $V^\pm_{\text{even}}$ and $V^\pm_{\text{odd}}$ in the proof of [25, Theorem 1.1].

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**APPENDIX A.**

Let $(\Sigma, j, h)$ be a closed Riemann surface with complex structure $j$ and compatible Riemannian metric $h$. We fix some notation for the decomposition of tangent vectors and 1-forms into those of type $(1, 0)$ and $(0, 1)$.

The almost complex structure $j$ on $T\Sigma$ extends canonically to a complex linear isomorphism on $T^C\Sigma = T\Sigma \otimes_{\mathbb{R}} \mathbb{C}$ and we decompose

$$T^C\Sigma = T^{1,0} \oplus T^{0,1}$$

(A.1)

into the complex $(+i)$- and $(-i)$-eigenspaces of $j$. The Riemannian metric $h$ extends to a Hermitian bundle metric on $T^C\Sigma$ and the decomposition in (A.1) is orthogonal.

The dual space $(T^C\Sigma)^*$ of complex linear 1-forms on $T^C\Sigma$ decomposes into

$$(T^C\Sigma)^* = \Lambda^{1,0} \oplus \Lambda^{0,1},$$

where $\Lambda^{1,0} = K$ and $\Lambda^{0,1} = K^{-1}$ are the bundles of complex linear 1-forms on $T^{1,0}$ and $T^{0,1}$. We have

$$\alpha \circ j = i\alpha \quad \forall \alpha \in \Lambda^{1,0}$$

$$\beta \circ j = -i\beta \quad \forall \beta \in \Lambda^{0,1}.$$

If $\tau \in (T^C\Sigma)^*$ is a 1-form, then its decomposition into $(1, 0)$- and $(0, 1)$-components is given by

$$\tau = \tau^{1,0} + \tau^{0,1}$$

with

$$\tau^{1,0} = \frac{1}{2}(\tau - i\tau \circ j), \quad \tau^{0,1} = \frac{1}{2}(\tau + i\tau \circ j).$$

Let $(e_1, e_2)$ with $e_2 = je_1$ be a local $h$-orthonormal basis of $T\Sigma$. Then

$$\epsilon = \frac{1}{\sqrt{2}}(e_1 - ie_2), \quad \bar{\epsilon} = \frac{1}{\sqrt{2}}(e_1 + ie_2)$$

and
are local unit basis vectors of $T^{1,0}$ and $T^{0,1}$. We extend the dual real basis $(e_1^*, e_2^*)$ of $T^*\Sigma$ to a basis of complex linear 1-forms of $(T^C\Sigma)^*$. Then

$$\kappa = \frac{1}{\sqrt{2}}(e_1^* + ie_2^*), \quad \bar{\kappa} = \frac{1}{\sqrt{2}}(e_1^* - ie_2^*)$$

are the dual local unit basis vectors of $K$ and $K^{-1}$.

**APPENDIX B.**

We summarize the definition of the curvature term $\mathcal{R}(f, \psi)$ that appears in the Euler–Lagrange equations (1.1) for Dirac-harmonic maps. Let $(\Sigma, j, h)$ be a Riemann surface, $(M^n, g)$ a Riemannian manifold and $f: \Sigma \to M$ a smooth map. We denote by

$$R: TM \times TM \times TM \longrightarrow TM$$

the curvature tensor, where we use the sign convention

$$R(X, Y)Z = [\nabla^g_X, \nabla^g_Y]Z - \nabla^g_{[X, Y]}Z.$$ 

There is an induced map

$$TM \times (S^c \otimes f^*T^C M) \longrightarrow T^*\Sigma \times (S^c \otimes f^*T^C M)$$

$$(X, \phi \otimes f^* Z) \longmapsto \phi \otimes f^*(R(X, df(\cdot)))Z.$$ 

Composing with Clifford multiplication

$$\gamma: T^*\Sigma \times S^c \longrightarrow S^c$$

we get the map

$$R^f: TM \times (S^c \otimes f^*T^C M) \longrightarrow S^c \otimes f^*T^C M$$

$$(X, \psi) \longmapsto R^f(X, \psi).$$

**Definition B.1.** We define

$$\mathcal{R}(f, \cdot): S^c \otimes f^*T^C M \longrightarrow f^*TM, \quad \psi \longmapsto \mathcal{R}(f, \psi)$$

by

$$g(\mathcal{R}(f, \psi), f^* X) = \frac{1}{2} \langle \psi, R^f(X, \psi) \rangle \quad \forall f^* X \in f^*TM.$$ 

With respect to a local orthonormal frame $e_1, e_2$ for $T\Sigma$ we can write

$$R^f(X, \phi \otimes f^* Z) = \sum_{\alpha=1}^2 e_\alpha \cdot \phi \otimes f^*(R(X, df(e_\alpha)))Z.)$$

With the components of the curvature tensor $R$ with respect to a local frame $\{y_k\}_{k=1}^n$

$$\sum_{i=1}^n R_{ijml}y_i = R(y_m, y_l)y_j$$

we obtain the original formula for the definition of the curvature term $\mathcal{R}$ in [9]:

$$\mathcal{R}(f, \psi) = \frac{1}{2} \sum_{i,j,m,l,\alpha} R_{ijml}df(e_\alpha)_i(\psi_l, e_\alpha \cdot \psi_j)f^* y_m.$$
The symmetries
\[ R_{ijml} = -R_{jiml}, \quad \langle \psi_i, e_\alpha \cdot \psi_j \rangle = -\langle \psi_j, e_\alpha \cdot \psi_i \rangle \]
 imply that \( \mathcal{R}(f, \psi) \) is indeed a real vector in \( f^*TM \).

Suppose that \( f_t \) a variation of the smooth map \( f : \Sigma \to M \) with
\[ f_0 = f, \quad \frac{df_t}{dt} \bigg|_{t=0} = f^*X \in \Gamma(f^*TM). \]

Let \( \phi, \phi' \in \Gamma(S^c) \) be time-independent spinors on \( \Sigma, Z, Z' \) time-independent vector fields on \( M \) and define spinors
\[ \psi_t = \phi \otimes f_t^* Z, \quad \psi'_t = \phi' \otimes f_t^* Z' \in \Gamma(S^c \otimes_R f^*TM). \]

**Definition B.2.** We set \( df_- (e_\alpha) \) for the vector field \( df_t(e_\alpha) \) along \( f_t \) and
\[ \nabla^g_X \psi = \phi \otimes f^* \nabla^g_X Z \]
\[ \nabla^g_X e_\alpha \psi = \nabla^h e_\alpha \phi \otimes f^* \nabla^g_X Z + \phi \otimes f^* \nabla^g_X \nabla^g_{df_- (e_\alpha)} Z \]
\[ = \nabla^h e_\alpha \phi \otimes f^* \nabla^g_X Z + \phi \otimes f^* (\nabla^g_{df_- (e_\alpha)} \nabla^g_X Z + R(X, df_- (e_\alpha)) Z). \]

In the last line we used that \( |X, df_- (e_\alpha)| = 0 \), since \( f_t \) is generated (to first order) by the flow of \( X \).

We calculate (cf. the proof of [9, Proposition 2.1])
\[ \frac{d}{dt} \bigg|_{t=0} (\psi'_t, D^t \psi_t) \]
\[ = \frac{d}{dt} \bigg|_{t=0} \sum_{\alpha = 1}^2 \langle \phi'_t \otimes f_t^* Z', e_\alpha \cdot (\nabla^h e_\alpha \phi) \otimes f_t^* Z + \phi \otimes f_t^* \nabla^g_{df_- (e_\alpha)} Z \rangle \]
\[ = \sum_{\alpha = 1}^2 \left( \langle \phi'_t, e_\alpha \cdot (\nabla^h e_\alpha \phi) \rangle L_X g(Z', Z) + \langle \phi'_t, e_\alpha \cdot \phi \rangle L_X g(Z', \nabla^g_{df_- (e_\alpha)} Z) \right) \]
\[ = \langle \nabla^g_X \psi', D^t \psi \rangle + \langle \psi', D^t \nabla^g_X \psi \rangle + \langle \psi', R^t (X, \psi) \rangle. \]

In particular, for \( \psi' = \psi \) and \( D^t \psi = 0 \) we get formula (2.4), using that \( D^t \) is formally self-adjoint.

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