Lusin characterisation of Hardy spaces associated with Hermite operators

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Abstract

Let $d \in \{3, 4, 5, \ldots \}$ and $p \in (0, 1]$. We consider the Hermite operator $L = -\Delta + |x|^2$ on its maximal domain in $L^2(\mathbb{R}^d)$. Let $H^p_L(\mathbb{R}^d)$ be the completion of $\{f \in L^2(\mathbb{R}^d) : M_L f \in L^p(\mathbb{R}^d)\}$ with respect to the quasi-norm $\| \cdot \|_{H^p_L} = \| \mathcal{M} \cdot \|_{L^p}$, where $\mathcal{M}_L f(\cdot) = \sup_{t>0} |e^{-tL} f(\cdot)|$ for all $f \in L^2(\mathbb{R}^d)$. We characterise $H^p_L(\mathbb{R}^d)$ in terms of Lusin integrals associated with Hermite operator.

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1 Introduction

Hermite operators have been studied over the years due to their significant role in harmonic oscillator. Many of their interesting properties have been discovered. An in-depth study of Hermite operators can be found in the monograph [Tha93]. On the other hand, Hardy spaces with their rich structures are of fundamental interest in harmonic analysis (cf. [Ste95]). In this paper we will examine Hardy spaces with index $p \in (0, 1]$ associated with Hermite operators. We aim to characterise these spaces in terms of Lusin area integrals. Similar results are available in the literature (cf. [Ste95, Section III.4.4], [SY16], [JPY16], etc. and references therein). Our approach differs these in that the Lusin area integrals of our consideration are more general and are defined using derivatives of the form $\partial_j x_j$ which were first suggested by Thangavelu in [Tha90] and are specific to Hermite operators. Our work is motivated by [Jiz12] whose results are for Hardy spaces associated with Hermite operators with index $p = 1$.

Next we formulate our problem. Let $d \in \mathbb{N}$ with $d \geq 3$. Consider the sesquilinear form

$$a_0(u, v) = \int_{\mathbb{R}^d} D u \cdot D v + |x|^2 u v,$$

where $D = (\partial_1, \ldots, \partial_d)$, on the domain $D(a_0) = C_c^\infty(\mathbb{R}^d)$. Then $a_0$ is positive. Consequently it follows from [Kat80, Theorem VI.1.27] that $a_0$ is closable.

Let $L$ be the operator associated with the closure $\overline{a_0}$ in the sense of Kato’s First Representation Theorem [Kat80, Theorem VI.2.1]. Then $C_c^\infty(\mathbb{R}^d) \subset D(L)$ and

$$Lu = -\Delta u + |x|^2 u$$

for all $u \in C_c^\infty(\mathbb{R}^d)$. In the literature $L$ is known as Hermite operator. It is well-known that $L$ generates a contraction $C_0$-semigroup $T$ on $L^2(\mathbb{R}^d)$. If $f \in L^2(\mathbb{R}^d)$ then

$$T_t f = e^{-tL} f = \sum_{n=0}^{\infty} e^{-t(2n+d)} f_n,$$

where

$$f_n = \sum_{|\alpha|=n} (f, h_\alpha) h_\alpha.$$

The Poisson semigroup $P$ on $L^2(\mathbb{R}^d)$ associated with $L$ is given by

$$P_t = e^{-tL^{1/2}} f = \sum_{n=0}^{\infty} e^{-t(2n+d)^{1/2}} f_n.$$

It can be shown that $P$ is also a contraction $C_0$-semigroup on $L^2(\mathbb{R}^d)$.

Let $p \in (0, 1]$. We define $H^p_L(\mathbb{R}^d)$ as the completion of

$$\{ f \in L^2(\mathbb{R}^d) : M_L f \in L^p(\mathbb{R}^d) \}$$
under the quasi-norm
\[ \| \cdot \|_{H^p_L} = \| M_{L} \cdot \|_{L^p}, \]
where \( M_L f(\cdot) = \sup_{t > 0} |T_t f(\cdot)| \) for all \( f \in L^2(\mathbb{R}^d) \).

For each \( x \in \mathbb{R}^d \), we define the Lusin integral associated with Hermite operator as
\[
Sf(x) := \left( \int_{\Gamma(x)} t^{1-d} |\nabla_L P_t f(y)|^2 \, dy \, dt \right)^{1/2}
\] (1)
where \( \nabla_L = (\partial_t, \partial_1 + x_1, \ldots, \partial_d + x_d) \)
and
\[
\Gamma(x) := \{(y,t) \in \mathbb{R}^d \times (0, \infty) : |x - y| < t\}
\] (2)
Let \( H^p_S(\mathbb{R}^d) \) be the completion of
\[ \{ f \in L^2(\mathbb{R}^d) : Sf \in L^p(\mathbb{R}^d) \} \]
under the quasi-norm
\[ \| \cdot \|_{H^p_S(\mathbb{R}^d)} = \| S \cdot \|_{L^p(\mathbb{R}^d)}. \]

In this paper we aim to characterise the space \( H^p_L(\mathbb{R}^d) \) in terms of the Lusin integral defined above. Our main result is the following.

**Theorem 1.1.** We have
\[
\left( H^p_L(\mathbb{R}^d), \| \cdot \|_{H^p_L(\mathbb{R}^d)} \right) = \left( H^p_S(\mathbb{R}^d), \| \cdot \|_{H^p_S(\mathbb{R}^d)} \right).
\]

Following [Tha90], we also consider the Riesz transforms
\[
R^L_j := (\partial_j + x_j) L^{-1/2}, \quad j \in \{1, 2, 3, \ldots\}.
\]
We have the following.

**Theorem 1.2.** The operators \( R^L_j \) is bounded on \( H^p_L(\mathbb{R}^d) \) for all \( j \in \{1, 2, 3, \ldots\} \).

The outline of the paper is as follows. In proving Theorem 1.1 we will need two intermediate spaces \( H_{L,\text{max}}(\mathbb{R}^d) \) and \( H^p_A(\mathbb{R}^d) \) which will be defined in the corresponding sections. In Section 2 we provide some preliminaries for later use. In Section 3 we show that \( H^p_L(\mathbb{R}^d) \subset H_{L,\text{max}}(\mathbb{R}^d) \). In Section 4 we give a characterisation \( H^p_L(\mathbb{R}^d) = H^p_A(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \). In Section 5 we prove the main theorem. In Section 6 we prove Theorem 1.2.

Throughout the paper, we let \( C \) be a positive constant independent of the main parameters whose value varies from line to line. We also set \( \mathbb{N} = \{0, 1, 2, \ldots\} \).
2 Some prior estimates

Let \( \rho(x) = \frac{1}{1 + |x|} \).

**Proposition 2.1** ([She95, Lemma 1.4]). There exist \( M > 0 \) and \( k > 0 \) such that

\[
\frac{1}{M} \left( 1 + \frac{|x - y|}{\rho(x)} \right)^{-k} \leq \rho(y) \leq M \left( 1 + \frac{|x - y|}{\rho(x)} \right)^{-\frac{k+1}{M}}.
\]

In particular, \( \rho(x) \sim \rho(y) \) if \( |x - y| \lesssim \rho(x) \).

Next we consider some kernel estimates of the \( C_0 \)-semigroup \( T \) generated by \( L \).

**Proposition 2.2.** Let \( g_t \) be the heat kernel of \( T_t \) for each \( t > 0 \). Then following hold.

(i) There exists a \( C > 0 \) such that for every \( N > 0 \), there is a constant \( C_N > 0 \) that satisfies

\[
0 \leq g_t(x, y) \leq C_N t^{-d/2} e^{-C|x-y|^2/t} \left( 1 + \frac{\sqrt{t}}{\rho(x)} \right)^{-N} \left( 1 + \frac{\sqrt{t}}{\rho(y)} \right)^{-N}
\]

for all \( x, y \in \mathbb{R}^d \).

(ii) There exist a \( \delta \in (0, 1) \) and a \( C > 0 \) such that for every \( N > 0 \), there is a constant \( C_N \) such that

\[
|g_t(x + h, y) - g_t(x, y)| \leq C_N \left( \frac{|h|}{\sqrt{t}} \right)^{\delta} t^{-d/2} e^{-C|x-y|^2/t} \left( 1 + \frac{\sqrt{t}}{\rho(x)} \right)^{-N} \left( 1 + \frac{\sqrt{t}}{\rho(y)} \right)^{-N}
\]

for all \( x, y \in \mathbb{R}^d \) and \( |h| < \sqrt{t} \).

**Proof.** This follows from [JPY16, Lemma 2.2 and Proposition 3.1]. \( \square \)

**Proposition 2.3.** Let \( p_t \) be the heat kernel of \( P_t \) for each \( t > 0 \). Then following hold.

(i) For every \( N > 0 \), there is a constant \( C_N > 0 \) such that

\[
0 \leq p_t(x, y) \leq C_N \frac{t}{(t + 2|x-y|)^{d+1}} \left( 1 + \frac{\sqrt{t}}{\rho(x)} \right)^{-N} \left( 1 + \frac{\sqrt{t}}{\rho(y)} \right)^{-N}
\]

for all \( x, y \in \mathbb{R}^d \).

(ii) There exist a \( \delta \in (0, 1) \) and a \( C > 0 \) such that for every \( N > 0 \), there is a constant \( C_N \) such that

\[
|p_t(x + h, y) - p_t(x, y)| \leq C_N \left( \frac{|h|}{t} \right)^{\delta} \frac{t}{(t + 2|x-y|)^{d+1}} \left( 1 + \frac{\sqrt{t}}{\rho(x)} \right)^{-N} \left( 1 + \frac{\sqrt{t}}{\rho(y)} \right)^{-N}
\]

for all \( x, y \in \mathbb{R}^d \) and \( |h| < \sqrt{t} \).

**Proof.** This follows from [JPY16, Propositions 3.4 and 3.5]. \( \square \)
We end this preliminary section with an atom decomposition of the space $H^p_L(\mathbb{R}^d)$. Let $p \in (0,1]$. We first define $H^p_L$-atoms.

**Definition 2.4.** Let $x_0 \in \mathbb{R}^d$ and $r > 0$. A function $a$ is an $H^p_L$-atom associated with a ball $B(x_0,r)$ if

1. $\text{supp } a \subset B(x_0,r)$,
2. $\|a\|_{L^\infty} \leq |B(x_0,r)|^{-1/p}$ and
3. $\int_{\mathbb{R}^d} a = 0$ if $r < \rho(x_0)/4$ (moment condition).

It is of fundamental importance that each $H^p_L$-function can be written as the sum of $H^p_L$-atoms and that the $H^p_L$-quasi-norm is equivalent to that given by the atom decomposition.

**Proposition 2.5.** Let $\frac{d}{d+\delta} < p \leq 1$, where $\delta$ is given by (3). Then for all $f \in H^p_L(\mathbb{R}^d)$, there exist a sequence $\{c_j\} \subset l^p(\mathbb{R})$ and $H^p_L$-atoms $a_j$’s such that

$$f = \sum_{j \in \mathbb{N}} c_j a_j \quad (5)$$

in $H^p_L(\mathbb{R}^d)$.

For all $f \in H^p_L(\mathbb{R}^d)$, define the quasi-norm

$$\|f\|_{L,\text{at}} = \inf \left\{ \left( \sum_{j \in \mathbb{N}} |c_j|^p \right)^{1/p} \right\}$$

where the infimum is taken over all decompositions (5). Then

$$\left( H^p_L(\mathbb{R}^d), \| \cdot \|_{H^p_L} \right) = \left( H^p_L(\mathbb{R}^d), \| \cdot \|_{L,\text{at}} \right).$$

**Proof.** This follows from [BDL18, Theorem 2.15].

3  $H^p_L \subset H^p_{L,\text{max}}$

For each $x \in \mathbb{R}^d$ and $f \in L^2(\mathbb{R}^d)$, define

$$f^*_L(x) = \sup_{\{(t,y) \in \mathbb{R}^+ \times \mathbb{R}^d : |x-y|<t\}} |P_t f(y)|.$$

Let $p \in (0,1]$. We define $H^p_{L,\text{max}}(\mathbb{R}^d)$ be the completion of

$$\{ f \in L^2(\mathbb{R}^d) : f^*_L \in L^p(\mathbb{R}^d) \}$$

under the quasi-norm

$$\|f\|_{H^p_{L,\text{max}}(\mathbb{R}^d)} = \|f^*_L\|_{L^p(\mathbb{R}^d)}, \quad f \in H^p_{L,\text{max}}(\mathbb{R}^d).$$
Proposition 3.1. Let $\frac{d}{d+\delta} p \leq 1$, where $\delta$ is given by (1). Then

\[ H^p_L(\mathbb{R}^d) \subset H^p_{L,max}(\mathbb{R}^d). \]

Proof. Due to Proposition 2.3, it suffices to check that there exists a $C > 0$ such that \( \|a\|_{H^p_{L,max}} \leq C \) for all \( H^p_L \)-atom $a$.

Let $a$ be an $H^p_L$-atom. Let $y_0 \in \mathbb{R}^d$ and $r > 0$ such that $\text{supp } a \subset B(y_0, r) =: B$. Consider

\[ \|a\|_{H^p_{L,max}} = \int_{\mathbb{R}^d} |a^*_L|^p \int_{2B} |a^*_L|^p + \int_{(2B)^C} |a^*_L|^p =: I_1 + I_2, \]

where $2B := B(y_0, 2r)$ and $(2B)^C = \mathbb{R}^d \setminus (2B)$.

For $I_1$, we have

\[ |P_t a(y)| \leq \|a\|_{L^\infty} \int_B |p_t(y, z)| dz \leq C \|a\|_{L^\infty} \int_B \frac{t}{(t + |y - z|)^{d+1}} dz \leq C |2B|^{1/p} \]

for all $y \in B$, where we used Proposition 2.3 in the second step and [BDY12, Lemma 2.1] in the third step. It follows that $I_1 \leq C$.

To estimate $I_2$, we first show that there exists a $C > 0$ such that

\[ |a^*_L(x)| \leq C |B|^{1+1/d-1/p} \frac{1}{|x - y_0|^{d+1}} \quad (6) \]

for all $x \in (2B)^C$.

Let $x \in (2B)^C$. We consider the following two cases.

Case 1: Suppose that $r < \rho(y_0)/4$. Then $a$ satisfies the moment condition. For all $y \in \mathbb{R}^d$ and $t > 0$ such that $|x - y| < t$, we have

\[ |P_t a(y)| = \left| \int_B (p_t(y, z) - p_t(y, y_0)) a(z) dz \right| \leq \|a\|_{L^\infty} \int_B |p_t(y, z) - p_t(y, y_0)| dz \]

\[ \leq C \|a\|_{L^\infty} \int_B \left( \frac{|z - y_0|}{t} \right)^\delta \frac{t}{(t + |y - y_0|)^{d+1}} \left( \frac{\rho(y_0)}{t} \right)^{1-\delta} dz \]

\[ \leq C \|a\|_{L^\infty} \int_B \left( \frac{|z - y_0|}{t} \right)^\delta \frac{t}{(|x - y| + |y - y_0|)^{d+1}} \left( \frac{\rho(y_0)}{t} \right)^{1-\delta} dz \]

\[ \leq C \|a\|_{L^\infty} \int_B \frac{|z - y_0|^{\delta} \rho(y_0)^{1-\delta}}{|x - y_0|^{d+1}} dz, \]

where we used Proposition 2.3 in the third step. Note that $|z - y_0| \leq r$ as $z \in B$. Also $\rho(y_0) \leq Cr$ for some $C > 0$ by Proposition 2.1. These imply

\[ |P_t a(y)| \leq C \|a\|_{L^\infty} \int_B \frac{r}{|x - y_0|^{d+1}} dz \leq C |B|^{1+1/d-1/p} \frac{1}{|x - y_0|^{d+1}} \]

for all $y \in \mathbb{R}^d$ and $t > 0$ such that $|x - y| < t$. Hence (6) follows.
Case 2: Suppose that \( r \geq \rho(y_0)/4 \). So \( a \) need not satisfy the moment condition. For all \( y \) and \( z \) such that \(|x - y| < t\) and \(|y_0 - z| < r\), we have
\[
t + |y - z| \geq t + |x - y_0| - |x - y| - |y_0 - z| \geq |x - y_0| - r \geq \frac{|x - y_0|}{2},
\]
where we used the fact that \( x \in (2B)^C \) in the last step. By Proposition 2.1 there exists a \( C > 0 \) such that \( \rho(z) \leq Cr \) for all \( z \in B \). Therefore
\[
|P_a(y)| = \left| \int_B p_t(y, z) a(z) \, dz \right| \leq \|a\|_{L^\infty} \int_B |p_t(y, z)| \, dz \leq C \|a\|_{L^\infty} \int_B \frac{t}{(t + |y - z|)^{d+1}} \rho(z) \, dz
\]
\[
\leq C \|a\|_{L^\infty} \int_B \frac{r}{|x - y_0|^{d+1}} \, dz \leq C |B|^{1+1/d-1/p} \frac{1}{|x - y_0|^{d+1}}
\]
for all \( y \in \mathbb{R}^d \) and \( t > 0 \) such that \(|x - y| < t\), where we used Proposition 2.3 in the third step. Hence (6) follows.

Having proved (6), we now obtain
\[
I_2 = \int_{(2B)^C} |a_L|^p \leq C |B|^{(1+1/d-1/p)p} \int_{(2B)^C} \frac{1}{|x - y_0|^{(d+1)p}} \, dx \leq C |B|^{p+p/d-1} |B|^{-(d+1)p/d} = C.
\]
Thus the proposition follows.

4 \( H_L^p = H_A^p \cap L^2 \)

The work in this section is inspired by HLM+11 Chapter 4 whose results are for \( p = 1 \) on spaces of homogeneous type.

Define
\[
Af(x) := \left( \int_0^\infty \int_{|x-y|<t} |t (\partial_t P_t f)(y)|^2 \frac{dy \, dt}{t^{d+1}} \right)^{1/2} = \left( \int_0^\infty \int_{|x-y|<t} t^{1-d} |(\partial_t P_t f)(y)|^2 \, dy \, dt \right)^{1/2},
\]
where \( f \in L^2(\mathbb{R}^d) \) and \( x \in \mathbb{R}^d \).

Next we show that \( A \) is a bounded operator on \( L^2(\mathbb{R}^d) \). For this we need the following lemma.

Lemma 4.1. Let \( s > 0 \). Let \( \psi : (0, \infty) \to \mathbb{C} \) be such that
\[
|\psi(t)| \leq C \frac{t^s}{1 + t^{2s}}
\]
for some \( C > 0 \) and for all \( t > 0 \). Then there exists a \( C > 0 \) such that
\[
\left( \int_0^\infty \|\psi(t \sqrt{L})\|^2_{L^2(\mathbb{R}^d)} \frac{dt}{t} \right)^{1/2} = C \|f\|_{L^2(\mathbb{R}^d)}.
\]

Proof. Let \( C := \left( \int_0^\infty |\psi(t)|^2 \frac{dt}{t} \right)^{1/2} < \infty \). Then
\[
\int_0^\infty \|\psi(t \sqrt{L})\|^2_{L^2(\mathbb{R}^d)} \frac{dt}{t} = \int_0^\infty \left( \int_0^\infty \psi(t \sqrt{L}) f, \psi(t \sqrt{L}) f \right) \frac{dt}{t} = \left( \int_0^\infty \psi(t \sqrt{L}) f, f \right)\frac{dt}{t} = \left( \int_0^\infty \left( \int_0^\infty |\psi|^2(t \sqrt{L}) \, dE_{\sqrt{L}}(\lambda) \right) \frac{dt}{t} f, f \right)\frac{dt}{t} = \int_0^\infty \left( \int_0^\infty |\psi|^2(t \sqrt{L}) \, dE_{\sqrt{L}}(\lambda) \right) f, f \right)\frac{dt}{t} = C \|f\|^2_{L^2(\mathbb{R}^d)},
\]
where $E_{\sqrt{L}}(\lambda)$ is the spectral decomposition of $\sqrt{L}$. \hfill \Box

**Proposition 4.2.** The operator $A$ is bounded on $L^2(\mathbb{R}^d)$.

**Proof.** Let $f \in L^2(\mathbb{R}^d)$. Then

$$
\|Af\|_{L^2(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^d} \int_0^\infty \int_{|x-y|<t} t^{1-d} |(\partial_t P_t f)(y)|^2 \, dy \, dt \, dx
$$

$$
= \int_0^\infty \int_{\mathbb{R}^d} \int_{|x-y|<t} t^{1-d} |(\partial_t P_t f)(y)|^2 \, dx \, dy \, dt
$$

$$
= C \int_0^\infty \int_{\mathbb{R}^d} t |(\partial_t P_t f)(y)|^2 \, dy \, dt = C \int_{\mathbb{R}^d} \int_0^\infty \left( t \sqrt{L} e^{-t \sqrt{L}} f \right)(y) \frac{2}{t} \, dt \, dy
$$

$$
= C \|f\|_{L^2(\mathbb{R}^d)}^2
$$

for some $C > 0$, where we used Lemma 4.1 in the last step. \hfill \Box

Let $p \in (0, 1]$. We define $H^p_A(\mathbb{R}^d)$ as the completion of

$$\{ f \in L^2(\mathbb{R}^d) : Af \in L^p(\mathbb{R}^d) \}
$$

under the quasi-norm

$$
\| \cdot \|_{H^p_A(\mathbb{R}^d)} = \| A \cdot \|_{L^p(\mathbb{R}^d)}.
$$

**Definition 4.3.** Let $M$ be a positive integer. A function $a \in L^2(\mathbb{R}^d)$ is called a $(p, 2, M)$-atom associated to the operator $L$ if there exist a function $b \in D(L_M)$ and a ball $B$ such that

(i) $a = L^M b$,

(ii) $\text{supp } \lambda^k b \subset B$, where $k = 0, 1, \ldots, M$,

(iii) $\| (r_B^2 L)^k b \|_{L^2(\mathbb{R}^d)} \leq r_B^{2M} |B|^{1/2 - 1/p}$, where $k = 0, 1, \ldots, M$.

**Definition 4.4.** Let $f \in L^2(\mathbb{R}^d)$. If there exists a sequence $\{\lambda_j\} \in l^p$ such that

$$
f = \sum_{j \in \mathbb{N}} \lambda_j a_j
$$

in $L^2(\mathbb{R}^d)$, where each $a_j$ is a $(p, 2, M)$-atom, then we say that (7) is an atomic $(p, 2, M)$-representation of $f$.

Let $H^p_{L,at,M}(\mathbb{R}^d)$ be the completion of

$$\{ f \in L^2(\mathbb{R}^d) : f \text{ has an atomic } (p, 2, M) - \text{representation} \}
$$

with respect to the quasi-norm

$$
\| f \|_{H^p_{L,at,M}(\mathbb{R}^d)} = \inf \left\{ \left( \sum_{j \in \mathbb{N}} |\lambda_j|^p \right)^{1/p} : f = \sum_{j \in \mathbb{N}} \lambda_j a_j \text{ is an atomic } (p, 2, M) - \text{representation} \right\}.
$$

It turns out that certain functions in $H^p_A(\mathbb{R}^d)$ can be decomposed into $(p, 2, M)$-atoms. Specifically, we will prove that $H^p_A(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and $H^p_{L,at,M}(\mathbb{R}^d)$ equal as quasi-norm spaces.
Lemma 4.5. Let $M > \frac{d}{2} \left( \frac{1}{p} - \frac{1}{q} \right)$. Then
\[
\left( H^p_{L^q,M}(\mathbb{R}^d), \| \cdot \|_{H^q_{L^p,M}(\mathbb{R}^d)} \right) \subset \left( H^p_A(\mathbb{R}^d) \cap L^2(\mathbb{R}^d), \| \cdot \|_{H^p_A(\mathbb{R}^d)} \right).
\]

Proof. Let $f \in H^p_{L^q,M}(\mathbb{R}^d)$. We need $Af \in L^q(\mathbb{R}^d)$. But $f = \sum_{j \in \mathbb{N}} \lambda_j a_j$, where $\{\lambda_j\} \in l^p$ and each $a_j$ is a $(p, 2, M)$-atom. Therefore it suffices to show that there exists a $C > 0$ such that 
\[
\|Af\|_{L^q(\mathbb{R}^d)} \leq C \text{ for all } (p, 2, M)-\text{atom } a.
\]

Let $a$ be a $(p, 2, M)$-atom. Let $x_0 \in \mathbb{R}^d$ and $r > 0$ be such that $\text{supp } a \subset B(x_0, r) =: B$. By a generalized Holder’s inequality (cf. [AF03 Corollary 2.5]), we have
\[
\|Af\|_{L^q(\mathbb{R}^d)} \leq C \sum_{j \in \mathbb{N}} \|2^j B\|^{1/q} \|Af\|_{L^2(U_j)}
\]
for some $C > 0$, where $1/q = 1/p - 1/2$, $U_0 = B$ and $U_j = 2^j B \setminus 2^{j-1} B$ for $j \in \{1, 2, 3, \ldots \}$. Since $A$ is bounded on $L^2(\mathbb{R}^d)$ by Proposition 1.2 we deduce that
\[
\|Af\|_{L^2(U_j)} \leq C \|a\|_{L^2(B)} \leq C |B|^{1/2 - 1/p}
\]
for some $C > 0$ and for all $j = 0, 1, 2$, where the last step follows from the bounded property given in the definition of a $(p, 2, M)$-atom.

Let $j \geq 3$ and $b \in D(L^M)$ be such that $a = L^M b$. Then
\[
\|Af\|_{L^2(U_j)}^2 = \int_{U_j} \int_0^\infty \int_{|x-y|<t} \left| t \left( \sqrt{L} P_t a \right)(y) \right|^2 \frac{dy \, dt \, dx}{t^{d+1}}
\]
\[
= \int_{U_j} \int_0^\infty \int_{|x-y|<t} \left| \left( t \sqrt{L} \right)^{1+2M} P_t b \right|(y) \frac{dy \, dt \, dx}{t^{d+1} + 4M}
\]
\[
= \int_{U_j} \int_0^\infty \frac{|x-x_0|^4}{4} \int_{|x-y|<t} \left| \left( t \sqrt{L} \right)^{1+2M} P_t b \right|(y) \frac{dy \, dt}{t^{d+1} + 4M}
\]
\[
=: (I) + (II).
\]

We now estimate each term separately. For (I), set
\[
F_j = \{ y \in \mathbb{R}^d : |x - y| \leq \frac{|x - x_0|}{4} \text{ for some } x \in U_j \}.
\]
If $z \in B$, $y \in F_j$ and $x \in U_j$ is such that $|x - y| \leq |x - x_0|/4$, then
\[
|y - z| \geq |x - x_0| - |x - y| - |z - x_0| \geq \frac{3}{4}|x - x_0| - r \geq \frac{|x - x_0|}{2} \geq 2^{j-2}r.
\]

We deduce that $d(F_j, B) \geq 2^{j-2} r$. Therefore
\[
(I) \leq C \int_0^{2^{j-2} r} \int_{F_j} \left| \left( t \sqrt{L} \right)^{1+2M} P_t b \right|(y) \frac{dy \, dt}{t^{1+4M}}
\]
\[
\leq C \|b\|_{L^2(B)}^2 \int_0^{2^{j-2} r} \left( \frac{t}{d(F_j, B)} \right)^{4M+2} \frac{dt}{t^{1+4M}} \leq C r^{4M} |B|^{1-2/p} \int_0^{2^{j-2} r} \left( \frac{t}{2^{j} r} \right)^{4M+2} \frac{dt}{t^{1+4M}}
\]
\[
= C |B|^{1-2/p} 2^{-4Mj} = C |2^j B|^{1-2/p} 2^{-j(d(1-2/p)+4M)},
\]

(11)
where we used \[11\] Proposition 3.1 in the third step. For (II), we have
\[\langle II \rangle \leq C \int_{2^{j-3}r}^{\infty} \int_{R^d} \left| \left( t \sqrt{L} \right)^{1+2M} P_t b \right|^2 \frac{dy dt}{t^{1+4M}} \leq C \|b\|_{L^2(B)}^2 \int_{2^{j-3}r}^{\infty} \frac{dt}{t^{1+4M}} \]
\[\leq C r^{4M} |B|^{1-2p} \int_{2^{j-3}r}^{\infty} \frac{dt}{t^{1+4M}} \leq C |B|^{1-2p} 2^{-4Mj} = C |2^j B|^{1-2/p} 2^{-j(d(1-2/p)+4M)}. \] (12)
It follows from \[5, 9, 10, 11\] and \[12\] that \(\|Aa\|_{L^p(R^d)} \leq C\). Hence the claim follows. \[\square\]

Next we will show that the reverse inclusion \(H^p_{L, at, M}(R^d) \supset H^p_{A}(R^d) \cap L^2(R^d)\) holds. This requires some techniques from tent spaces. Therefore we will diverge a little to study tent spaces.

The aim is to make use of the atomic decomposition already available in tent spaces (cf. \[Rus07\]) to study the space \(H^p_{A}(R^d) \cap L^2(R^d)\) in terms of \((p, 2, M)\)-atoms defined above.

First we define tent spaces. Let \(\alpha > 0\). For any closed subset \(F \subset R^d\), let
\[R_\alpha(F) := \bigcup_{x \in F} \Gamma_\alpha(x),\]
where
\[\Gamma_\alpha(x) := \{(y, t) \in R^d \times (0, \infty) : |x - y| < \alpha t\}, \quad x \in R^d\] (13)
Note that \(\Gamma_1\) agrees with \(\Gamma\) defined in \[2\]. If \(O \subset R^d\) is open then we define
\[T_\alpha(O) := (R_\alpha(O^C))^C = \{(x, t) \in R^d \times (0, \infty) : d(x, O^C) \geq \alpha t\},\]
which is called the tent over \(O\) with aperture \(\alpha\). For short, we will write \(R(F)\) and \(T(O)\) in place of \(R_1(O)\) and \(T_1(O)\) respectively.

For each measurable function \(f\) on \(R^d \times (0, \infty)\) and \(x \in R^d\), define
\[(Sf)(x) = \left( \int_{\Gamma(x) \times (0, \infty)} |f(y, t)|^2 \frac{dy dt}{t^{d+1}} \right)^{1/2}\]
We say that \(f \in T^p_2(R^d)\) if
\[\|f\|_{T_2^p(R^d)} := \|Sf\|_{L^p(R^d)} < \infty.\]

**Definition 4.6.** A measurable function \(a\) on \(R^d(0, \infty)\) is said to be a \(T^p_2(R^d)\)-atom if there exists a ball \(B \subset R^d\) such that \(a\) is supported in \(T(B)\) and
\[\int_{R^d \times (0, \infty)} |a(y, t)|^2 \frac{dy dt}{t^d} \leq |B|^{1-2/p}.\]

In what follows, we let \(\phi \in C^\infty_c(R)\) be such that

(i) \(\text{supp } \phi \subset (-1, 1)\)
(ii) \(\phi\) is even,
(iii) \(\phi \geq 0\) on \((-1, 1)\) and \(\phi > 0\) on \((-1/2, 1/2)\).
Lemma 4.7. Let $M$ for each $a$ \[ \text{Let} \quad L \], for all $\Omega$. Observe that the functions $\psi, L \in L^2(\mathbb{R}^d)$ given by
\[
\pi \Omega \in \psi, \quad L \in \psi, \quad \overline{\int \psi, \quad L \in \psi,} \quad \overline{\int \psi, \quad L \in \psi,} \quad \overline{\int \psi, \quad L \in \psi,}
\]
It is known that the improper integral converges weakly in $L^2(\mathbb{R}^d)$ and
\[
\| \pi \Omega (F) \|_{L^2(\mathbb{R}^d)} \leq C_M \| F \|_{T_r^2(\mathbb{R}^d)}
\]
for each $M \geq 1$ (cf. [HLM+11, p.23]).

**Lemma 4.7.** Let $B$ be a ball in $\mathbb{R}^d$ and $F$ a $T^p_2(\mathbb{R}^d)$-atom associated with $B$. Let $M \geq 1$. Then there exists a $C_M > 0$ such that $C^{-1}_M \pi \Omega (F)$ is a $(p, 2, M)$-atom associated with $2B$.

**Proof.** By definition, we have
\[
\int_{\mathbb{R}^d \times (0, \infty)} |F(x, t)|^2 \frac{dx dt}{t} \leq |B|^{1-2/p}.
\]
Let $a := \pi \Omega (F) = L^M b$, where
\[
b := \int_0^\infty t^{2M} t^2 L \Phi(-t \sqrt{L}) (F(\cdot, t))(\frac{dt}{t}).
\]
Observe that the functions $L^k b$ are all supported on $2B$ for $k = 0, 1, \ldots, M$ as $F$ is supported in $T(B)$.

Next let $g \in L^2(2B)$ and $k \in \{0, 1, \ldots, M\}$. Let $r$ be the radius of $B$. Then
\[
\left| \int_{\mathbb{R}^d} (r^2 L)^k b \right| = \lim_{\delta \to 0} \int_{\mathbb{R}^d} \left( \int_{\delta}^{1/\delta} t^{2M} t^2 k^2 L \Phi(-t \sqrt{L}) (F(\cdot, t))(\frac{dt}{t}) \right) g(x) \frac{dx dt}{t}
\]
\[
= \int_{T(B)} F(x, t) t^{2M} r^{2k} L \Phi(-t \sqrt{L}) g(x) \frac{dx dt}{t}
\]
\[
\leq r^{2M} \left( \int_{\mathbb{R}^d \times (0, \infty)} \left| F(x, t) \right|^2 \frac{dx dt}{t} \right)^{1/2} \left( \int_{T(B)} \left| (t^2 L)^{k+1} \Phi(-t \sqrt{L}) g(x) \right|^2 \frac{dx dt}{t} \right)^{1/2}
\]
\[
\leq r^{2M} |B|^{1/2-1/p} \| g \|_{L^2(2B)}.
\]
where the last step follows from [12], Lemma [4.1] and the fact that $k \leq M$. Consequently
\[
\|(r^2 L)^k b\|_{L^2(2B)} \leq C r^2 |B|^{1/2-1/p}
\]
for all $k = 0, 1, \ldots, M$. The claim now follows.

**Lemma 4.8.** Let $M \geq 1$. Then
\[
\left( H^p_{L, \alpha, M}(\mathbb{R}^d), \| \cdot \|_{H^1_{L, \alpha, M}(\mathbb{R}^d)} \right) \supset \left( H^p_{A}(\mathbb{R}^d) \cap L^2(\mathbb{R}^d), \| \cdot \|_{H^p_{A}(\mathbb{R}^d)} \right).
\]
Proof. Let $f \in H^p_A(R^d)$. Set $F(\cdot,t) = t\sqrt{Le^{-t\sqrt{L}}f}$. Then $F \in T^2(R^d) \cap T^p_2(R^d)$ by Lemma 4.1 and the definition of $H^p_A(R^d)$. It follows from [Rus07, Theorem 1.1] that $F = \sum_{j \in \mathbb{N}} \lambda_j A_j$, where $A_j$’s are $T^p_2(R^d)$-atoms, $\{\lambda_j\} \in l^p$ and

$$
\left( \sum_{j \in \mathbb{N}} |\lambda_j|^p \right)^{1/p} \leq C \|F\|_{T^2_2(R^d)} = C \|f\|_{H^p_A(R^d)}.
$$

(15)

Using the Calderon reproducing formula, we obtain

$$
f = c_\Psi \int_0^\infty \Psi(t\sqrt{L})(t\sqrt{Le^{-t\sqrt{L}}f}) \frac{dt}{t} = c_\Psi \sum_{j \in \mathbb{N}} \lambda_j \pi_{\Psi,L}(A_j)
$$

(16)
in $L^2(R^d)$. Note that $a_j := \pi_{\Psi,L}(A_j)$ is a $(p,2,M)$-atom for $M \geq 1$ and $j \in \mathbb{N}$ by Lemma 4.7. Therefore (16) is an atomic $(p,2,M)$-representation and hence $f \in H^p_{L,at,M}(R^d)$. Moreover,

$$
\|f\|_{H^p_{L,at,M}(R^d)} = \inf \left( \sum_{j \in \mathbb{N}} |\lambda_j|^p \right)^{1/p} \leq C \|f\|_{H^p_A(R^d)}
$$

by (15). □

Proposition 4.9. Let $M > \frac{4(p_1)}{2} \left( 1 - \frac{1}{2} \right) \vee 1$. We have

$$
\left( H^p_A(R^d) \cap L^2(R^d), \| \cdot \|_{H^p_A(R^d)} \right) = \left( H^p_{L,at,M}(R^d), \| \cdot \|_{H^p_{L,at,M}(R^d)} \right).
$$

Proof. This follows from Lemmas 4.9 and 4.8. □

Proposition 4.10. Let $M \geq \frac{4}{2} \left( 1 - \frac{1}{p} \right)$. We have

$$
\left( H^p_L(R^d), \| \cdot \|_{H^p_L(R^d)} \right) = \left( H^p_{L,at,M}(R^d), \| \cdot \|_{H^p_{L,at,M}(R^d)} \right).
$$

Proof. This follows from [SY16, Theorem 1.4] and [BDL18, Theorem 2.15]. □

Proposition 4.11. We have

$$
\left( H^p_L(R^d), \| \cdot \|_{H^p_L(R^d)} \right) = \left( H^p_A(R^d) \cap L^2(R^d), \| \cdot \|_{H^p_A(R^d)} \right).
$$

Proof. The claim is a consequence of Propositions 4.9 and 4.10. □

5 $H^p_L = H^p_S$

Let $\alpha > 0$. For each $x \in R^d$, define

$$
\Gamma_\alpha^{\varepsilon,R}(x) := \{(y,t) \in R^d \times (\varepsilon,R) : |x-y| < \alpha t\},
$$

where $0 < \varepsilon < R < \infty$. 

11
For each \( x \in \mathbb{R}^d \), define
\[
S_\alpha f(x) = S_{\alpha,L} f(x) := \left( \int_{\Gamma_\alpha(x)} t^2 \left| \nabla L P_t f(y) \right|^2 \frac{dy dt}{t^{d+1}} \right)^{1/2} = \left( \int_{\Gamma_\alpha(x)} t^{1-d} \left| \nabla L P_t f(y) \right|^2 \frac{dy dt}{t^{d+1}} \right)^{1/2}
\]
and
\[
S_\varepsilon,R_\alpha f(x) = S_{\varepsilon,R_\alpha,L} f(x) := \left( \int_{\Gamma_\varepsilon,R_\alpha(x)} t^2 \left| \nabla L P_t f(y) \right|^2 \frac{dy dt}{t^{d+1}} \right)^{1/2} = \left( \int_{\Gamma_\varepsilon,R_\alpha(x)} t^{1-d} \left| \nabla L P_t f(y) \right|^2 \frac{dy dt}{t^{d+1}} \right)^{1/2},
\]
where \( \Gamma_\alpha(x) \) is defined by (13) and \( \nabla L = (\partial_t, \partial_1 + x_1, \ldots, \partial_d + x_d) \).

Note that \( S_1 \) coincides with \( S \) defined in (1). Let \( H^p_S(\mathbb{R}^d) \) be the completion of
\[
\{ f \in L^2(\mathbb{R}^d) : Sf \in L^p(\mathbb{R}^d) \}
\]
under the quasi-norm
\[
\| \cdot H^p_S(\mathbb{R}^d) = \| S \cdot \|_{L^p(\mathbb{R}^d)}.
\]

**Lemma 5.1** ([Jiz12, Lemma 3]). Let \( 0 \leq \alpha < 1 \). Then there is a \( C_\alpha \) such that
\[
S_\varepsilon,R_\alpha f(x) \leq C_\alpha \left( 1 + |\ln(R/\varepsilon)| \right)^{1/2} f_*^L(x)
\]
for all \( f \in L^2(\mathbb{R}^d) \).

Next we define
\[
\tilde{S}_\varepsilon,R_\alpha f(x) := \left( \int_{1}^{\varepsilon} \int_{\Gamma_{\alpha/a,R}(x)} t^{1-d} \left| \nabla L P_t f(y) \right|^2 \frac{dy dt da}{t^{d+1}} \right)^{1/2},
\]
Simple estimation gives
\[
S_{1/2}^{2\varepsilon,R} f(x) \leq \tilde{S}_\varepsilon,R_\alpha f(x) \leq S_{2\alpha}^{2\varepsilon,R} f(x). \tag{17}
\]

**Lemma 5.2.** There exists a \( C > 0 \) such that
\[
| \{ x \in \mathbb{R}^d : \tilde{S}_1^{2\varepsilon,R} f(x) > 2\lambda \ and \ f_*^L(x) \leq \gamma \lambda \} | \leq C \gamma^2 | \{ x \in \mathbb{R}^d : \tilde{S}_1^{1/2} f(x) > \lambda \} |
\]
for all \( 0 < \gamma < 1, \lambda > 0, 0 < \varepsilon < R < \infty \) and \( f \in H^p_{L_{\max}}(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \).

**Proof.** The proof follows verbatim to that of [Jiz12, Lemma 4]. \( \square \)

**Lemma 5.3.** Let \( \alpha, \beta > 0 \) and \( 0 < \varepsilon < R < \infty \). Then
\[
\| S_\varepsilon^{2\alpha,R} \|_{L^p(\mathbb{R}^d)} \sim \| S_\beta^{2\varepsilon,R} \|_{L^p(\mathbb{R}^d)},
\]
where the implicit constants are independent of \( \varepsilon, R \) and \( f \).

**Proof.** The proof follows from that of [CMS85, Proposition 4] with obvious modifications. \( \square \)
Proposition 5.4. Let $0 < \varepsilon < R < \infty$. Let $f \in H^p_{L,\max}(\mathbb{R}^d)$ be such that $\tilde{S}^{\varepsilon,R}_{1/20}f \in L^p(\mathbb{R}^d)$. Then there is a $C > 0$ such that

$$\|Sf\|_{L^p(\mathbb{R}^d)} \leq C \|f^*_s\|_{L^p(\mathbb{R}^d)}. \quad (18)$$

**Proof.** Since $f \in H^p_{L,\max}(\mathbb{R}^d)$, we have $f^*_L \in L^p(\mathbb{R}^d)$. This implies $P_s f \in L^p(\mathbb{R}^d)$ for all $s > 0$. We deduce from the definition of $f^*_L$ that $|P_s f(x)| \leq f^*_L(y)$ for all $s > 0$ and for all $x, y \in \mathbb{R}^d$ such that $y \in B(x, s)$. Therefore

$$|P_s f(x)| \leq \frac{1}{|B(x, s)|} \int_{B(x, s)} f^*_L \leq \frac{C}{s^d} \int_{\mathbb{R}^d} f^*_L < \infty$$

for some $C > 0$. Hence $P_s f \in L^\infty(\mathbb{R}^d)$ for all $s > 0$. Interpolation gives $P_s f \in L^2(\mathbb{R}^d)$ for all $s > 0$.

In what follows, we denote $f_s = P_s f$ for ease of notation. By Lemma 5.2 there is a $C > 0$ such that

$$\left| \{ x \in \mathbb{R}^d : \tilde{S}^{\varepsilon,R}_{1/20} f_s(x) > 2 \lambda \text{ and } (f_s)_1^*(x) \leq \gamma \lambda \} \right| \leq C \gamma^2 \left| \{ x \in \mathbb{R}^d : \tilde{S}^{\varepsilon,R}_{1/2} f_s(x) > \lambda \} \right|$$

for all $0 < \gamma < 1$ and $\lambda > 0$. Multiplying both sides by $\lambda^{p-1}$ and then integrating with respect to $\lambda$ give

$$\| \tilde{S}^{\varepsilon,R}_{1/2} f_s \|_{L^p(\mathbb{R}^d)} \leq C \left( \gamma^{-1} \|(f_s)_1^*\|_{L^p(\mathbb{R}^d)} + \gamma^2 \| \tilde{S}^{\varepsilon,R}_{1/2} f_s \|_{L^p(\mathbb{R}^d)} \right). \quad (19)$$

It follows from (17) and Lemma 5.3 that there exists a $C > 0$ such that

$$\| \tilde{S}^{\varepsilon,R}_{1/2} f_s \|_{L^p(\mathbb{R}^d)} \leq C \| \tilde{S}^{\varepsilon,R}_{1/20} f_s \|_{L^p(\mathbb{R}^d)} \leq C \| \tilde{S}^{\varepsilon,R}_{1/20} f_s \|_{L^p(\mathbb{R}^d)}. \quad (19)$$

Also notice that $\|u + v\|_{L^p(\mathbb{R}^d)} \leq 2^{(1-p)/p}(\|u\|_{L^p(\mathbb{R}^d)} + \|v\|_{L^p(\mathbb{R}^d)})$ when $0 < p < 1$, where $u, v \in \|u\|_{L^p(\mathbb{R}^d)}$. Consequently, there exists a $C > 0$ such that

$$\| \tilde{S}^{\varepsilon,R}_{1/2} f_s \|_{L^p(\mathbb{R}^d)} \leq C \| S^{\varepsilon/2,2R}_{1/2} f_s \|_{L^p(\mathbb{R}^d)}$$

$$\leq C \left( \| S^{\varepsilon/2, \varepsilon}_{1} f_s \|_{L^p(\mathbb{R}^d)} + \| S^{\varepsilon,2R}_{1} f_s \|_{L^p(\mathbb{R}^d)} + \| S^{R,2R}_{1} f_s \|_{L^p(\mathbb{R}^d)} \right)$$

$$\leq C \left( \| S^{\varepsilon/2R}_{1} f_s \|_{L^p(\mathbb{R}^d)} + \| (f_s)_L^\ast \|_{L^p(\mathbb{R}^d)} \right)$$

$$\leq C \left( \| S^{\varepsilon/2R}_{1/20} f_s \|_{L^p(\mathbb{R}^d)} + \| (f_s)_L^\ast \|_{L^p(\mathbb{R}^d)} \right), \quad (20)$$

where we used Lemma 5.4 in the third step. Substituting (20) into (19) and choosing an appropriate value for $\gamma$, we obtain (18) for $f_s$. In addition, we also have that

$$(f_s)_L^\ast(x) = \sup_{|x-y| < t} |P_t f_s(y)| = \sup_{|x-y| < t} |P_{t+s} f(y)| \leq \sup_{|x-y| < t+s} |P_{t+s} f(y)| = f^*_L(x)$$

for all $x \in \mathbb{R}^d$. Hence

$$\|Sf_s\|_{L^p(\mathbb{R}^d)} \leq C \|f^*_s\|_{L^p(\mathbb{R}^d)}.$$

Finally we use Lebesgue dominated convergence theorem and take limit $s \rightarrow 0$ to derive

$$\|Sf\|_{L^p(\mathbb{R}^d)} \leq C \|f^*_s\|_{L^p(\mathbb{R}^d)}.$$

This completes the proof. □
We are now ready to prove our main theorem.

**Proof of Theorem 1.1** (⊂) This follows from Propositions 3.1 and 5.4. (⊃) This is a consequence of Proposition 4.11 and the fact that \( \| \cdot \|_{\mathcal{H}_L^p(\mathbb{R}^d)} \leq \| \cdot \|_{\mathcal{H}_S^p(\mathbb{R}^d)} \).

6 Boundedness of Riesz transforms

In this section we prove Theorem 1.2. We first consider some auxiliary results.

Define \( \mathcal{H}_L^p(\mathbb{R}^d) \) as the completion of \( \{ f \in L^2(\mathbb{R}^d) : M_{L+2}f \in L^p(\mathbb{R}^d) \} \) under the quasi-norm \( \| \cdot \|_{\mathcal{H}_L^{p+2}} = \| M_{L+2} \cdot \|_{L^p} \), where \( M_{L+2}f(\cdot) = \sup_{t>0} | e^{-(L+2)t} T_t f(\cdot) | \) for all \( f \in L^2(\mathbb{R}^d) \).

Recall the two operators \( A \) and \( S \) associated with \( L \) considered in the previous sections. In this section we will also consider the operator \( L+2 \). Clearly the previous results applied to \( L+2 \).

To make notation clear, we will write \( A_L \) and \( S_L \) to emphasize \( A \) and \( S \) are associated with \( L \).

Similarly we can also define \( A_{L+2} \) and \( S_{L+2} \) associated with \( L+2 \).

In what follows, we let \( L^2_c(\mathbb{R}^d) \) be the space of functions in \( L^2(\mathbb{R}^d) \) with compact supports.

**Lemma 6.1.** The following inclusions hold:

\[
\left( L^2_c(\mathbb{R}^d), \| \cdot \|_{L^2} \right) \subset \left( \mathcal{H}_L^p(\mathbb{R}^d), \| \cdot \|_{\mathcal{H}_L^p} \right) \subset \left( \mathcal{H}_{L+2}^p(\mathbb{R}^d), \| \cdot \|_{\mathcal{H}_{L+2}^p} \right).
\]

Moreover, \( L^2_c(\mathbb{R}^d) \) is dense in both \( \mathcal{H}_L^p(\mathbb{R}^d) \) and \( \mathcal{H}_{L+2}^p(\mathbb{R}^d) \).

**Proof.** (First inclusion) Let \( f \in L^2_c(\mathbb{R}^d) \). Then by a generalised Holder’s inequality (cf. [AF03, Corollary 2.5])

\[
\| Af \|_{L^p(\mathbb{R}^d)} \leq \| \text{supp} f \|^{1/q} \| Af \|_{L^q(\mathbb{R}^d)} \leq C \| \text{supp} f \|^{1/q} \| f \|_{L^2(\mathbb{R}^d)} < \infty,
\]

where \( 1/q = 1/p - 1/2 \) and we used Lemma 4.2 in the second step. The first inclusion now follows from Proposition 4.11.

(Second inclusion) Let \( f \in \mathcal{H}_L^p(\mathbb{R}^d) \). Then \( \mathcal{M}_L f \in L^p(\mathbb{R}^d) \). However

\[
\mathcal{M}_{L+2} f(x) = \sup_{t>0} | e^{-2t} T_t f(x) | \leq \sup_{t>0} | T_t f(x) | = \mathcal{M}_L f
\]

for all \( x \in \mathbb{R}^d \). From this we deduce that \( f \in \mathcal{H}_{L+2}^p(\mathbb{R}^d) \) and the second inclusion holds.

(Dense) By the atomic characterisation in Proposition 2.5 each function in \( \mathcal{H}_L^p(\mathbb{R}^d) \) can be approximated by a finite linear combination \( \mathcal{H}_L^p \)-atoms. But each such finite linear combination is clearly in \( L^2_c(\mathbb{R}^d) \). Hence the claim follows. \qed
The following lemma is immediate from Lemma 6.1.

**Lemma 6.2.** We have

\[ \left( H^p_L(\mathbb{R}^d), \| \cdot \|_{H^p_L(\mathbb{R}^d)} \right) = \left( H^p_{L+2}(\mathbb{R}^d), \| \cdot \|_{H^p_{L+2}(\mathbb{R}^d)} \right). \]

**Proof of Theorem 1.2.** Let \( j \in \{1, 2, 3, \ldots \} \). We will show that \( R^L_j \) is bounded on \( H^p_L(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \). The claim then follows by density of \( H^p_L(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \) in \( H^p_L(\mathbb{R}^d) \).

Let \( f \in H^p_L(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \). Then \( f \in H^p_{L+2}(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \) by Lemma 6.2. There exists a \( C > 0 \) such that

\[
\| R^L_j f \|_{H^p_{L+2}(\mathbb{R}^d)} \leq C \| A_{L+2} R^L_j f \|_{L^p(\mathbb{R}^d)}
\]

\[
= C \left\| \left( \int_0^\infty \int_{|x-y|<t} t^{1-d} \left| (\partial_t e^{-t(L+2)^{1/2}} R^L_j f)(y) \right|^2 dy dt \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)}
\]

\[
= C \left\| \left( \int_0^\infty \int_{|x-y|<t} t^{1-d} \left| (t(\partial_j + x_j)e^{-tL^{1/2}} f)(y) \right|^2 dy dt \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)}
\]

\[
\leq C \| S_L f \|_{L^p(\mathbb{R}^d)} \leq C \| f \|_{L^p(\mathbb{R}^d)},
\]

where we used Proposition 4.11 in the first step, [Jiz12] Lemma 8 in the third step and Theorem 1.1 in the last step. \( \square \)

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