Abstract. A generalization of metric space is presented which is shown to admit a theory strongly related to that of ordinary metric spaces. To avoid the topological effects related to dropping any of the axioms of metric space, first a new, and equivalent, axiomatization of metric space is given which is then generalized from a fresh point of view. Naturally arising examples from metric geometry are presented.

1. Introduction

Symmetry, in the axiomatization of metric space, was under stress early on in the development of the theory, for various reasons. Simply neglecting the axiom of symmetry carries with it certain topological difficulties that are discomforting, thus dictating proceeding with caution. Below we argue, by presenting a completely equivalent axiomatization of metric space that makes no mention of symmetry, that symmetry as an axiom had received far too much attention. This new axiomatization enables us to develop a new extension of metric space whose study is the aim of this work. But first, some more background information and motivation regarding metric spaces, the symmetry axiom, and its weakening.

History. The idea that the abstract notion of distance is a symmetric one appears to be deeply ingrained and probably emanates from our basic intuition about distance in Euclidean spaces. However, the subtleties of the imposition of symmetry were noted early on in the development of metric spaces and General Topology. In 1918 Finsler introduced in [15], his Ph.D. dissertation, what was later coined by Élie Cartan Finsler geometry; a generalization of Riemannian manifold having an induced quasimetric instead of a metric. In 1931 Wilson, in an article titled “On quasi metric spaces” ([41]), comments that:

“In one sense a quasi-metric space is merely the result of suppressing the axiom [of symmetry] from the definition of metric space. Usually the result of such a limitation on a set of axioms is to diminish the number of theorems easily deducible, but in this case there is an embarrassing richness of material”.

Some 38 years later, Stoltenberg, in [38], an article bearing the exact same title, reviews the than recent developments in the study of the implications of removing the limitation of symmetry. More on the history of metric spaces can be found in [29].

Topological difficulties. The aim of this work is to develop a new generalization of metric space with little ill effect on the accompanying topological notions. Let us first recount the topological pathologies that arise when each of the axioms of metric space is neglected.
Zero distance and non-Hausdorff spaces. Allowing distinct points to have distance equal to 0 corresponds to the topological phenomenon of a non-Hausdorff topology. These were initially considered to be invalid spaces, but the rich structure of finite topologies shows such restrictions were not in place (see [28] for more on the applicability of non-Hausdorff spaces).

Positive self distance and ghostly neighborhoods. Allowing a point to have positive distance from itself (such spaces are called partial metrics or dislocated metrics) has the topological effect of a point possessing neighborhoods not containing that point. Topologically, such ghostly neighborhoods appear to be very pathological.

The triangle inequality and open balls. It is easily seen that the triangle inequality assures that open balls are open sets. Thus neglecting the triangle inequality will destroy the most fundamental example of an open set.

Symmetry and closed balls. It is equally easy to see that symmetry assures that closed balls are closed sets. Thus, neglecting the axiom of symmetry will destroy the most fundamental example of a closed set.

Remark 1.1. Viewed this way, it is a bit peculiar that the triangle inequality axiom is related to open balls being open sets while the symmetry axiom is related to closed balls being closed sets. The two axioms do not appear to be dual to each other and one may wonder if there is an equivalent axiomatization that better reflects the open/closed duality nature of topology. Indeed, an affirmative answer is provided incidentally by the axiomatization we present below.

Further motivation. The relevance of quasi-metrics to Physics and Biology goes back, respectively, at least to 1941 with Rander's [27] and to 1976 with Waterman, Smith, and Beyer's [40]. In [22] Lawvere shows that a significant portion of metric geometry that does not rely on symmetry can be seen as a form of extended logic and, at the same time, as a special case of the theory of enriched categories. More recently, Vickers continued this line of investigation in [39] and a unifying approach is given in [9]. Some general properties of quasimetrics are presented in [12] and [36]. An asymmetric Arzelà-Ascoli theorem is established in [10]. Generalized metrics in the theory of computation can be found in [31] and [35], and in computation semantics in [19] and [34]. For further applications in Computer Science we mention [4] and [6]. The recent encyclopedia of distances [11], including a wealth of generalizations of metric space, is yet another reflection of the importance of these structures.

It thus appears that the demand for symmetry to hold in any metric space might have been adopted prematurely. Aside from these motivating forces for considering non-symmetry, a very naive, yet illuminating, reason is given by Gromov in [17] when referring to how the symmetry axiom in the definition of a metric space unpleasantly limits many applications:

“the effort of climbing up to the top of a mountain, in real life as well as in mathematics, is not at all the same as descending back to the starting point”.

Another reason to consider various ways in which symmetry can be weakened is of a more theoretical nature. In order to understand the role of the axiom of symmetry in the general theory of metric spaces one is led to study the effects of weakening it. Perhaps the most studied instance is that of simply neglecting the axiom of
symmetry. The accompanying topological theory is that of bitopological spaces introduced in [21] with its rich structure expounded in [13].

However, as alluded to above, the accompanying bitopological theory does not seem to have the same intimate relation that topology has with metric spaces. Below, we present the theory of metric 1-spaces for which we are able to establish fundamental results so that both results and proofs echo the familiar theory of ordinary metric spaces quite strongly.

**Plan of the paper.** Section 2 after a very simple analysis of the axioms of metric space, presents a new axiomatization of metric space which does not mention symmetry at all. The importance of this reformulation is that it enables new possibilities for generalizations. Section 3 then develops such a generalization and introduces the main objects of study, namely metric 1-spaces. With most of the article devoted to the fine structure of metric 1-spaces, Section 4 is a detour providing a brief study of the coarse structure of metric 1-spaces. In it, coarse 1-spaces are defined and a metrizability theorem is proven. Section 5 is concerned with notions of convergence in metric 1-spaces. Forward sequences and series are defined, as well as their dual notions of backward sequences and series. A notion of limit for each of these entities is defined and is shown to extend the usual notion of limits of sequences in ordinary metric spaces. Section 6 then introduces appropriate notions of continuity and studies their interrelations. Section 7 pieces together the results of the previous sections to establish three fundamental results about metric 1-spaces. These results generalize the familiar results of ordinary metric space theory regarding metrizability of function spaces, the equivalence between continuity and uniform continuity on compact domains, and the Banach fixed point theorem for a contracting self-map. In Section 8 a hierarchy of symmetry is introduced by means of dagger structures with different properties. Finally, Section 9 provides more examples of metric 1-spaces, presents several possible applications in physics, and discusses metric n-spaces and their relation to other structures.

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### 2. Reformulating the Classical Definition

The definition of a metric space as formulated by Fréchet in 1906 is, of course, very well known, and yet we revisit it here briefly and in full detail. Our aim in this short section is to examine the axiomatization and arrive at an equivalent one that makes no mention of symmetry. While quite a simple result the author, surprisingly, could not find any trace of it in more than several books and articles. The significance of this reformulation of the axioms lies in the shift of focus it enables when generalizing the notion. Suddenly, there is no need to weaken symmetry since it is no longer demanded explicitly. The new axiomatization suggests a new possibility for extending the theory of metric spaces whose exploration is the aim of the sections that follow.

Consider a set $X$ and a function $w : X \times X \to \mathbb{R}_+$, where $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\} \cup \{\infty\}$ is the set of extended non-negative real numbers. We say that:

- $w$ is **locally finite** if for every $x, y \in X$ holds that $w(x, y) \neq \infty$.
- $w$ is **non-degenerate** if, for every $x, y \in X$, $w(x, y) = 0$ implies $x = y$.
- $w$ is **reflexive** if for all $x \in X$ holds that $w(x, x) = 0$.
• $w$ is symmetric if for all $x, y \in X$ holds that $w(x, y) = w(y, x)$.
• $w$ satisfies the full triangle inequality if the inequalities
  \[ |w(x, y) - w(y, z)| \leq w(x, z) \leq w(x, y) + w(y, z) \]
  hold for all $x, y, z \in X$.
• $w$ satisfies the restricted triangle inequality if the inequality
  \[ w(x, z) \leq w(x, y) + w(y, z) \]
  holds for all $x, y, z \in X$.

**Remark 2.1.** For the full and restricted triangle inequalities we agree that $x + \infty = \infty + x = \infty + \infty = \infty$, that $x - \infty = -\infty$, and that $\infty - x = \infty$ hold for every real $x \geq 0$ (here $-\infty$ is a new symbol satisfying $|-\infty| = \infty$). The quantity $\infty - \infty$ is left undefined and if $w(x, y) = \infty = w(y, z)$ then the full triangle inequality is to be interpreted as setting no particular lower bound on $w(x, z)$. We remark as well that while technically the full triangle inequality consists of two inequalities we still refer to it in the singular as a single entity to stress the point that it should be treated as one whole.

With this terminology in place we may now state the common-place definition of metric space.

**Definition 2.2.** A metric space is a set $X$ together with a locally finite, reflexive, non-degenerate and symmetric function $w : X \times X \to \mathbb{R}^+$ which satisfies the restricted triangle inequality.

**Lemma 2.3.** Let $X$ be a set and $w : X \times X \to \mathbb{R}^+$ a function. If $w$ is reflexive and satisfies the restricted triangle inequality then the following are equivalent

• $w$ is symmetric.
• $w$ satisfies the full triangle inequality.

**Proof.** Assuming symmetry we need to establish that $|w(x, y) - w(y, z)| \leq w(x, z)$ which, without loss of generality, would follow from showing that $w(x, y) \leq w(x, z) + w(y, z)$. Using symmetry this is equivalent to $w(x, y) \leq w(x, z) + w(z, y)$ which holds by the restricted triangle inequality. In the other direction, the full triangle inequality implies that $|w(x, y) - w(y, x)| \leq w(x, x) = 0$, and symmetry follows. \(\square\)

**Corollary 2.4.** (Equivalent Definition of Metric Space) A metric space is a set $X$ together with a locally finite, reflexive and non-degenerate function $w : X \times X \to \mathbb{R}^+$ which satisfies the full triangle inequality.

**Remark 2.5.** Our choice of using $w : X \times X \to \mathbb{R}^+$ instead of the more common $d : X \times X \to \mathbb{R}$ is meant to distinatiate the exposition from the deeply grained prejudices suggested by the word ’distance’. The reader may thus think instead of the word ’weight’.

With this result we achieved our preliminary goal of a symmetry-free axiomatization of metric spaces. As a by product notice that, as alluded to in Section [1] this axiomatization reflects the open/closed duality of topology in a rather straightforward manner: Each half of the full triangle inequality corresponds to either open balls being open sets or closed balls being closed sets.
Remark 2.6. As Michael Lockyer pointed out to the author, the axiom of symmetry can also be replaced, in the presence of reflexivity, by the strong triangle inequality: $d(x, z) \leq d(x, y) + d(z, y)$. Below we present a generalization of metric spaces based on the axiomatization in Corollary 2.4. It is also possible to develop such a generalization based on the strong triangle inequality. Such a generalization is already captured by the work below, as explained in Remark 8.8 below.

3. WEAKENING THE NEW DEFINITION

The main point of the previous section was that we arrived at an axiomatization of classical metric spaces which makes no mention of symmetry. We are now free to seek out a generalization from a fresh point of view on a very old subject.

Disposing of local finiteness is most easily justified. The immediate benefit of doing so is the existence of the coproduct of two metric spaces (and in fact any small colimit). In fact, the local finiteness axiom is already starting to disappear from textbooks on metric spaces, (see, e.g., the definition of metric space in [8] and the discussion following it).

Disposing of non-degeneracy is also easy to digest. In the literature the resulting structure is called a semi-metric space or a quasi-metric space. We prefer the more descriptive use of the term 'degenerate'. We refer again to page 2 of [8] for more details. In light of these facts we will use the term 'metric space' for spaces that might be degenerate or not locally finite.

We are thus left with reflexivity and the full triangle inequality and we feel most unwilling to depart with any of these due to the topological consequences of doing so. Reflexivity ensures that every neighborhood of a point contains that point, and the full triangle inequality guarantees that open balls are open sets and that closed balls are closed sets. Interestingly, though not the path we take below, it is useful to consider relaxing these axioms (see e.g., [20] for relevance to electronic engineering, [18] for applications in programming semantics, [2] in domain theory, and [7] for uses in theoretical computer science).

We are interested in developing a theory that retains as much of the character of topology as possible and thus insist on retaining both reflexivity and the full triangle inequality. As we saw above, these two axioms imply symmetry and thus there seems to be no escape from symmetry. However, we now identify a hidden axiom in the axiomatization, the weakening of which will allow us to proceed.

The axiom of univalence is the assumption that for every two points $x, y \in X$ there is a unique associated number $w(x, y)$. This assumption stems from the idea that $w(x, y)$ should signify the distance from $x$ to $y$ and that this distance is uniquely determined by the end points alone. However, when measuring a quantity from $x$ to $y$ one may consider measuring different aspects as signified by a parameter. It thus becomes natural to refine $w$ and allow it to become multivalued. In fact it is sensible to allow $w(x, y)$ to attain no value at all, for instance if it is not at all possible to measure anything from $x$ to $y$. If a measurement is possible then there could be a whole array of parameters indicating how one should measure. Thus, we wish to replace the underlying set of a metric space by a richer structure known as a category.

A category (definition follows) can be seen as a 1-dimensional analogue of a set. In more detail, a category is a set (or a class) of objects, thought of as 0-dimensional
point-like elements, together with 1-dimensional arrows between objects together with a notion of composition of such arrows.

**Definition 3.1.** A category $\mathcal{C}$ consists of a class of objects $\text{ob}(\mathcal{C})$ and, for every two objects $x, y \in \text{ob}(\mathcal{C})$, a set $\mathcal{C}(x, y)$. These sets are to be disjoint, in the sense that if $\mathcal{C}(x, y) \cap \mathcal{C}(x', y') \neq \emptyset$ then $x = x'$ and $y = y'$. An element $\psi \in \mathcal{C}(x, y)$ is also denoted by $\psi : x \to y$ and called an arrow or a morphism. The object $x$ is the domain of $\psi$ and the object $y$ is the codomain of $\psi$. For each object $x \in \text{ob}(\mathcal{C})$ there is a designated arrow $\text{id}_x : x \to x$ called the identity arrow at $x$. Lastly, there is a composition rule that assigns to arrows $\xrightarrow{\psi} y \xrightarrow{\varphi} z$ their composition $\varphi \circ \psi : x \to z$. With respect to the composition, the identity arrows are required to be neutral in the sense that if $\psi : x \to y$ is any arrow then $\psi \circ \text{id}_x = \psi$ and $\text{id}_y \circ \psi = \psi$. The final condition is that the composition be associative in the sense that if $x \xrightarrow{\psi} y \xrightarrow{\varphi} z \xrightarrow{\rho} w$ are any three arrows then $\rho \circ (\varphi \circ \psi) = (\rho \circ \varphi) \circ \psi$.

The class of all arrows in the category $\mathcal{C}$ is denoted by $\text{Arr}(\mathcal{C})$.

**Remark 3.2.** When considering general categories, size issues can become important. Namely, if the class of arrows in a category is a proper class then certain constructions are not guaranteed to exist. We adopt here the common solution due to Grothendieck of assuming implicitly a hierarchy of universes so that the class of arrows of a given category is small with respect to some universe. From this point on we tacitly ignore all size issues.

Examples of categories include the category $\text{Set}$ with objects all sets and arrows all functions, the category $\text{Top}$ of all topological spaces as objects and all continuous mappings as arrows, the category $\text{Grp}$ of all groups as objects and group homomorphisms as arrows and so on. Another class of examples of categories useful in what follows is the following one. Any set $S$ naturally gives rise to a category $I_S$, called an indiscrete category, where

- $\text{ob}(I_S) = S$
- $I_S(x, y) = \{\psi_{x,y}\}$

with $\text{id}_x = \psi_{x,x}$ and compositions determined uniquely. Via this construction categories can be seen to extend sets.

Structure preserving maps between categories are known as functors. The formal definition is as follows.

**Definition 3.3.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A functor $F : \mathcal{C} \to \mathcal{D}$ consists of an assignment of an object $F(c) \in \text{ob}(\mathcal{D})$ to any object $c \in \text{ob}(\mathcal{C})$ and to every pair $c, c' \in \text{ob}(\mathcal{C})$, a function $F : \mathcal{C}(c, c') \to \mathcal{D}(F(c), F(c'))$ such that for every $c \in \text{ob}(\mathcal{C})$ holds that $F(\text{id}_c) = \text{id}_{F(c)}$ and for every composable pair of arrows holds that $F(\varphi \circ \psi) = F(\varphi) \circ F(\psi)$.

**Remark 3.4.** Categories were introduced in [14] by Eilenberg and Mac Lane in 1945 not as generalizations of sets but rather in order to make precise the illusive exact meaning of the naturality of certain mathematical constructions (such as the natural isomorphism between a finite dimensional vector space and its double dual), an effort that proved crucial in advancing homology theory. Later, category theory, found uses in algebraic geometry, computer science, and logic just to mention a few areas. For more information on categories the reader is referred to [24].
We can now formulate the final step in the weakening of the axioms of a metric space by removing the assumption of univalence.

**Definition 3.5.** A metric 1-space is a category $X$ together with, for every two objects $x, y \in ob(X)$, a function $w : X(x, y) \to \mathbb{R}_+$ which satisfies reflexivity and the full triangle inequality in the following sense.

- For every $x \in ob(X)$ the equality $w(id_x) = 0$ holds.
- For every $x, y, z \in ob(X)$ and arrows $\psi : x \to y$ and $\varphi : y \to z$ the inequalities
  \[
  |w(\varphi) - w(\psi)| \leq w(\varphi \circ \psi) \leq w(\varphi) + w(\psi)
  \]
  hold.

It is assumed that we follow the same convention set out in Remark 2.1 above regarding computations involving $\infty$. In particular, if $w(\varphi) = w(\psi) = \infty$ then the full triangle inequality sets no lower bound on $w(\varphi \circ \psi)$.

We note immediately that every ordinary metric space $(S, d)$ gives rise to a metric 1-space structure on the indiscrete category $IS$ by defining $w(\psi_{x,y}) = d(x, y)$, for every arrow $\psi_{x,y}$ in $IS$. Moreover, any metric structure on $IS$ arises in this way as we now show.

**Proposition 3.6.** Let $X$ be a metric 1-space and $\psi : x \to y$ and $\varphi : y \to z$ arrows in $X$. If $w(\varphi \circ \psi) = 0$ then $w(\psi) = w(\varphi)$.

**Proof.** $|w(\varphi) - w(\psi)| \leq w(\varphi \circ \psi) = 0$. □

**Corollary 3.7.** If $\psi$ is an isomorphism (i.e., $\psi$ has an inverse) in a metric 1-space then $w(\psi) = w(\psi^{-1})$.

Since in an indiscrete category $IS$ every arrow is an isomorphism we obtain

**Corollary 3.8.** If $X$ is a metric 1-space with an indiscrete underlying category $IS$ then defining $d(x, y) = w(\psi_{x,y})$ defines a symmetric metric structure on $S$.

We thus see that metric spaces can be identified with metric 1-spaces having an indiscrete underlying category.

**Example 3.9.** In the context of ordinary metric spaces recall that a function $f : X \to Y$ between metric spaces is called bi-Lipschitz if there exists a constant $C$, called a bi-Lipschitz constant, such that

\[
\frac{1}{C}d(x, x') \leq d(f(x), f(x')) \leq Cd(x, x')
\]

holds for all $x, x' \in X$ (note that such a $C$, if it exists, satisfies $C \geq 1$). Consider now the category $\text{BiLip}$ of all metric spaces and bi-Lipschitz mappings between them. For each arrow $f : X \to Y$ in that category let $C_f$ be the infimum over all bi-Lipschitz constants for $f$ and let $w(f) = \ln(C_f)$. It is straightforward to verify that this choice of $w$ turns $\text{BiLip}$ into a metric 1-space.

Recall, from [22], that a Lawvere space is a set $X$ equipped with a reflexive function $d : X \times X \to \mathbb{R}_+$ satisfying the restricted triangle inequality. Given any metric 1-space $(X, w)$ one may define a Lawvere structure on $S = ob(X)$ by the formula

\[
d(x, y) = \inf_{\psi : x \to y} w(\psi)
\]
for any two objects $x, y \in \text{ob}(X)$. We denote this Lawvere space by $L(X)$ and note that it would usually fail to be a metric space since symmetry would not generally hold.

**Example 3.10.** Continuing Example 3.9 it is easily seen that in $L(\text{BiLip})$ the distance $d(X, Y)$ is the usual Lipschitz distance between $X$ and $Y$.

Before embarking on the study of metric 1-spaces we close this section by mentioning the concept of categorical duality. Given a category $\mathcal{C}$ one may construct a new category, denoted $\mathcal{C}^{op}$, and called the *opposite category*, by formally reversing the directions of all arrows in $\mathcal{C}$. More concretely, $\text{ob}(\mathcal{C}^{op}) = \text{ob}(\mathcal{C})$ and for every arrow $\psi : c \rightarrow d$ in $\mathcal{C}$ there is an arrow $\psi^{op} : d \rightarrow c$ in $\mathcal{C}^{op}$. It follows that every result about categories can be dualized to give another true result. This observation will be used repeatedly below and we refer the reader to [24] for more details on duality. We do comment that for most categories $\mathcal{C}$ the opposite category $\mathcal{C}^{op}$ is very different than the original category. For instance, the opposite of the category $\text{Set}$ of sets and functions is essentially the same as the category of complete atomic boolean algebras. Duality can also be used to define new objects of study. For instance, non-commutative geometry *defines* its objects of study to be the objects in the opposite of a category of algebras.

4. **THE COARSE STRUCTURE OF A METRIC 1-SPACE**

The main aim of this work is to investigate the fine structure of a metric 1-space. In this short section we take a detour to consider the coarse structure of metric 1-spaces as well. The further study of the coarse structure of metric 1-spaces is postponed to a future article.

If $E, E_1, E_2$ are relations on a fixed set $X$ then recall that $E_1 \circ E_2 = \{(x, z) \in X \times X \mid \exists y \in X \quad (x, y) \in E_1, (y, z) \in E_2\}$ and that $E^{-1} = \{(y, x) \in X \times X \mid (x, y) \in E\}$. Recall (30) that a *coarse structure* on a set $X$ is a collection $\mathcal{E} = \{E_i\}_{i \in I}$, whose elements are called *controlled sets*, where each $E_i$ is a relation $E_i \subseteq X \times X$, such that the following axioms are satisfied.

- **Reflexivity**: The diagonal $\Delta = \{(x, x) \mid x \in X\}$ is in $\mathcal{E}$.
- **Downward Saturation**: If $E_i \in \mathcal{E}$ and $F \subseteq E_i$ then $F \in \mathcal{E}$.
- **Upward Saturation**: $\mathcal{E}$ is closed under taking finite unions.
- **Composition Stability**: if $E_1, E_2 \in \mathcal{E}$ then $E_1 \circ E_2 \in \mathcal{E}$.
- **Symmetry**: if $E \in \mathcal{E}$ then $E^{-1} \in \mathcal{E}$.

A *coarse space* is then a set $X$ together with a coarse structure $\mathcal{E}$ on it. The archetypical example of a coarse space is the *bounded coarse structure* associated to an ordinary metric space $(X, d)$ where the controlled sets are all subsets $E \subseteq X \times X$ such that $\text{sup}\{d(x, y) \mid (x, y) \in E\}$ is finite. To adapt this definition to the setting of metric 1-spaces we first reformulate the definition of coarse space to obtain an equivalent axiomatization that does not mention symmetry. The steps we take are analogous to those taken above on the way to the symmetry-free axiomatization of metric space.

Given a relation $E \subseteq X \times X$, let $E^*$ be the union of the sets

$$\{(y, z) \in X \times X \mid \exists x \in X \quad (x, y), (x, z) \in E\}$$

and

$$\{(x, y) \in X \times X \mid \exists z \in X \quad (x, z), (y, z) \in E\}.$$
We now obtain the following coarse version of Lemma \[2,3\].

**Lemma 4.1.** If \( \mathcal{E} \) is a collection of relations on a set \( X \) that satisfies reflexivity, upward and downward saturation, and composition stability then \( \mathcal{E} \) is a coarse structure if, and only if, \( \mathcal{E} \) is \( * \) closed (i.e., if \( E \in \mathcal{E} \) then \( E^* \in \mathcal{E} \)).

**Proof.** We need to show that, under the given assumptions, symmetry is satisfied if, and only if, \( \mathcal{E} \) is \( * \) closed. If symmetry holds then noting that for every \( E \in \mathcal{E} \) holds that \( E^* \subseteq E \circ E^{-1} \cup E^{-1} \circ E \), shows that \( \mathcal{E} \) is \( * \) closed. Conversely, if \( \mathcal{E} \) is \( * \) closed then, given \( E \in \mathcal{E} \), form first \( E_0 = \Delta \cup E \) and then note that \( E^{-1} \subseteq E_0^* \), to finish the proof. \( \square \)

Thus, a coarse space can equivalently be defined as a collection \( \mathcal{E} \) of relations on a set \( X \) which satisfies reflexivity, upward and downward saturation, composition stability, and \( * \) stability. It is this formulation that is the appropriate one to generalize to metric 1-spaces. Given \( E,E_1,E_2 \), sets of arrows in a fixed category, we write

\[ E_1 \circ E_2 = \{ \psi_1 \circ \psi_2 \mid \psi_1 \in E_1, \psi_2 \in E_2 \} \]

and

\[ E^* = \{ \psi \in Arr(\mathcal{C}) \mid \exists \varphi \in E \ \psi \circ \varphi \in E \} \cup \{ \psi \in Arr(\mathcal{C}) \mid \exists \varphi \in E \ \varphi \circ \psi \in E \}. \]

**Definition 4.2.** A coarse structure on a category \( \mathcal{C} \) is a collection \( \mathcal{E} = \{ E_i \}_{i \in I} \), where each \( E_i \) is a set of arrows in \( \mathcal{C} \), called a controlled set, such that the following axioms hold.

- **Reflexivity:** The set \( \Delta = \{ id_x : x \to x \mid x \in ob(\mathcal{C}) \} \) is in \( \mathcal{E} \).
- **Downward Saturation:** If \( E \in \mathcal{E} \) and \( F \subseteq E \) then \( F \in \mathcal{E} \).
- **Upward Saturation:** \( \mathcal{E} \) is closed under taking finite unions.
- **Composition Stability:** If \( E_1,E_2 \in \mathcal{E} \) then \( E_1 \circ E_2 \in \mathcal{E} \).
- **\( * \) Stability:** If \( E \in \mathcal{E} \) then \( E^* \in \mathcal{E} \).

A coarse 1-space is a category \( X \) together with a coarse structure on it. One can easily show that given a metric 1-space \( X \), defining \( \mathcal{E} \) to consist of all sets \( E \) of arrows in \( X \) such that \( sup \{ w(\psi) \mid \psi \in E \} \) is finite endows the underlying category \( X \) with a coarse structure which is called the bounded coarse structure of the metric 1-space \( X \).

It is evident that coarse 1-spaces can be seen as an extension of coarse spaces via the construction of indiscrete categories analogously to the case of metric 1-spaces described above.

We close this section by proving a metrizability theorem for coarse 1-spaces. A coarse 1-space is metrizable if it is the bounded coarse structure of some metric 1-space. A generating set for a coarse 1-space \((X,\mathcal{E})\) is a collection \( \{ E_j \}_{j \in J} \subseteq \mathcal{E} \) such that every controlled set \( E \in \mathcal{E} \) is contained in some \( E_i \).

**Theorem 4.3.** A coarse 1-space \((X,\mathcal{E})\) is metrizable if, and only if, it admits a countable generating set.

**Proof.** If \( X \) is metrizable then defining, for each \( n \geq 0 \), the set \( E_n = \{ \psi \in Arr(X) \mid w(\psi) \leq n \} \) gives a countable generating set for the bounded coarse structure. To prove the converse assume a countable generating set \( \{ E_n \}_{n=0}^{\infty} \) is given, and define the sets \( F_0 = \Delta \) and \( F_{n+1} = F_n^* \cup F_n \circ F_n \cup E_n \cup E_n^* \), for each \( n \geq 0 \). Note that in general, if \( \Delta \subseteq E \) then \( E \subseteq E^* \) and thus it follows that \( F_n \subseteq F_{n+1} \) for every
Clearly, the set \( \{ F_n \}_{n=0}^{\infty} \) is a generating set for the coarse structure \( \mathcal{E} \). Now, for each arrow \( \psi \in \text{Arr}(X) \), define \( w(\psi) = \inf \{ n \in \mathbb{N} \mid \psi \in F_n \} \). Note that \( w(\psi) = 0 \) if, and only if, \( \psi \) is an identity arrow. To verify the full triangle inequality it suffices to only consider non-identity arrows \( \varphi \) and \( \psi \). To that end assume that \( w(\varphi) = n \leq m = w(\psi) \). Then \( \varphi, \psi \in F_m \) and so, by construction, \( \varphi \circ \psi \in F_{m+1} \) proving that \( w(\varphi \circ \psi) \leq m + 1 \leq m + n \). This, and a similar argument, establish the restricted triangle inequality. The full triangle inequality follows by considering cases such as: \( w(\varphi \circ \psi) = k \leq m = w(\varphi) \), from which follows that \( \varphi \circ \psi, \varphi \in F_m \) and so \( \psi \in F_{m+1} \), proving that \( w(\psi) \leq m + 1 \leq m + k \). We thus obtain the metric 1-space \( (X, w) \) and it is trivial to verify that the bounded coarse structure for this metric 1-space coincides with \( (X, \mathcal{E}) \). □

5. CONVERGENCE

The notions of convergence introduced in this section are a fusion of concepts from the theory of limits in ordinary metric spaces and in categories. Since a category is a more elaborate structure than a mere set there are different ways to proceed. We present here two notions of convergence in metric 1-spaces which will be used below to prove some fundamental results on metric 1-spaces. The comparison with the notion of topological limit will be quite self evident. It is also interesting to compare and contrast with the categorical notion of a (co)limit and thus we first consider categorical (co)limits of a special kind. Due to categorical duality all concepts of category theory come in pairs, commonly prefixed by ‘left’ and ‘right’ with the prefix ‘co’ used to signify an application of duality. As a result, the notions of sequence and series we introduce have duals as well. This is in accordance with current literature on quasi-metric spaces (e.g., [33]) where convergence and all other topological notions appear in two variants. These are usually called forward and backward (e.g., forward/backward convergence, forward/backward Cauchy etc.) and this is the terminology we adopt.

Remark. We point out that in [33] it is shown that limits in Lawvere spaces are related to the notion of weighted limits in enriched category theory. Our presentation below does not involve enrichment.

5.1. Sequences and series. Let \( \mathbb{N}_\bullet \) be the category whose objects are the natural numbers together with an object \( \bullet \) such that, besides the identity arrows, there is in the category, for every natural number \( n \), precisely one arrow from \( \bullet \) to \( n \). The category \( \mathbb{N}_\bullet \) can be depicted diagrammatically as

\[
\begin{array}{c}
\bullet \\
\downarrow \\
0 \quad 1 \quad \cdots \quad n \quad \cdots
\end{array}
\]

Dually, let \( \mathbb{N}^\bullet = \mathbb{N}_\circ \) be the opposite category whose diagram is

\[
\begin{array}{c}
\bullet \\
\downarrow \\
0 \quad 1 \quad \cdots \quad n \quad \cdots
\end{array}
\]

Definition 5.1. Let \( \mathcal{C} \) be a category. A forward sequence in the category \( \mathcal{C} \) is a functor \( \mathbb{N}_\bullet \to \mathcal{C} \). A backward sequence in the category \( \mathcal{C} \) is a functor \( \mathbb{N}^\bullet \to \mathcal{C} \).
Clearly a forward sequence \( \mathbb{N}_+ \to \mathcal{C} \) amounts to specifying a family \( \{ \psi_n : c \to c_n \}_{n=0}^\infty \) of arrows in \( \mathcal{C} \) with a common domain. If we wish to make the domain explicit we will speak of a forward sequence \textit{from an object} \( c \). Similarly, a backward sequence \( \mathbb{N}_- \to \mathcal{C} \) consists of a family \( \{ \psi_n : c_n \to c \}_{n=0}^\infty \) of arrows in \( \mathcal{C} \) with a common codomain which can be made explicit by speaking of a backward sequence \textit{to an object} \( c \).

Let \( \mathbb{N}_+ \) be the category whose objects are the natural numbers and, besides the identity arrows, there is an arrow from \( n \) to \( m \) if, and only if, \( n < m \). Dually, let \( \mathbb{N}_- = \mathbb{N}_+^{\text{op}} \) be the opposite category. The respective diagrams of these categories are

\[
\begin{align*}
0 &\to 1 &\cdots &\to n &\to \cdots \\
\cdots &\to n &\to \cdots &\to 1 &\to 0
\end{align*}
\]

(the identities and compositions are omitted from the diagrams).

**Definition 5.2.** Let \( \mathcal{C} \) be a category. A \textit{forward series} in the category \( \mathcal{C} \) is a functor \( \mathbb{N}_+ \to \mathcal{C} \). A \textit{backward series} in the category \( \mathcal{C} \) is a functor \( \mathbb{N}_- \to \mathcal{C} \).

Clearly, a forward series amounts to a sequence of arrows \( \{ \psi_n \}_{n=0}^\infty \) such that for each \( n \geq 0 \) the domain of \( \psi_{n+1} \) is equal to the codomain of \( \psi_n \). In other words, a sequence of arrows \( \{ \psi_n \}_{n=0}^\infty \) is a forward series if, and only if, for each \( n \geq 0 \) the composition \( \psi_n \circ \psi_{n-1} \circ \cdots \circ \psi_0 \) exists. Similarly, a sequence \( \{ \psi_n \}_{n=0}^\infty \) of arrows is a backward series precisely when for each \( n \geq 0 \) the composition \( \psi_0 \circ \cdots \circ \psi_{n-1} \circ \psi_n \) exists.

Our notation is meant to resonate with the familiar concepts of sequences and series in, e.g., \( \mathbb{R} \). However, there are marked differences which show up below. For instance, the two notions are generally not interchangeable. Series can be related to sequences by the evident construction of partial compositions as follows. Let \( \{ \psi_n \}_{n=0}^\infty \) be a forward series in \( \mathcal{C} \). Define for each \( n \geq 0 \) the arrow \( \varphi_n = \psi_n \circ \cdots \circ \psi_0 \). The resulting sequence \( \{ \varphi_n \}_{n=0}^\infty \) is called the forward sequence of \textit{partial compositions} associated to the forward series \( \{ \psi_n \}_{n=0}^\infty \). Similarly, one can associate to a backward series a backward sequence. However, it will be evident below that convergence of the forward (respectively backward) series is generally stronger than convergence of the associated forward (respectively backward) sequence of partial compositions. It is also evident that not every forward (respectively backward) sequence can so be obtained from a forward (respectively backward) series.

**5.2. Pushouts, Pullbacks, and transfinite compositions.** Before presenting the definitions of limits for the concepts introduced above we define, for the sake of completeness, categorical limits and colimits of sequences and series first.
Definition 5.3. Let \( \mathcal{C} \) be a category and consider a forward sequence \( \mathbb{N}_* \to \mathcal{C} \) represented by the solid arrows from the object \( c \) in the diagram

![Diagram of a forward sequence](image)

A cone over the forward sequence is a sequence of arrows to an object \( d \) such that the diagram of solid arrows commutes. Such a cone is called a weak pushout of the forward sequence if given any other cone (over the same forward sequence), represented by the dashed arrows to \( d' \), there exists a mediating arrow from \( d \) to \( d' \) that makes the entire diagram commute. A pushout is then a weak pushout such that each such mediating arrow is unique.

It can easily be shown that a pushout of a forward sequence, if it exists, is unique up to an isomorphism.

The dual notion is that of a pullback of a backward sequence, obtained by formally reversing all the arrows in the above definition.

Definition 5.4. Let \( \mathcal{C} \) be a category and \( \{ \psi_n : x_n \to x_{n+1} \}_{n=0}^{\infty} \) a series of arrows in it. A categorical weak transfinite composition of the series is an object \( x_\infty \in ob(\mathcal{C}) \) together with arrows \( \mu_n : x_n \to x_\infty \) forming the solid commutative diagram

![Diagram of a categorical weak transfinite composition](image)

with the property that if \( x \in ob(\mathcal{C}) \) is any object and the dotted arrows \( \iota_n : x_n \to x \) are any arrows that form a commutative diagram with the given series \( \{ \psi_n \}_{n=0}^{\infty} \), then there exists a mediating arrow \( \varphi : x_\infty \to x \) such that \( \iota_k = \varphi \circ \mu_k \) for all \( k \geq 0 \). A categorical transfinite composition is then a weak categorical transfinite composition for which each such mediating arrow is unique.

Again, the dual notion of (weak) categorical transfinite composition of a backward series is obtained by formally reversing all arrows and again we omit the details. We remark that the notions of transfinite composition, pushout, and pullback are special cases of general categorical colimits and limits (see, e.g., [24]).
5.3. Limits of sequences. Note that a (weak) pushout of a forward sequence \( \{ \psi_n : c \rightarrow c_n \}_{n=0}^\infty \) is highly sensitive to finite changes in the sequence. Indeed, changing just one of the arrows in the sequence can change a sequence that has a pushout to one that does not. Our definition of limits below removes this sensitivity by adapting the definition of cone so that any limit will essentially depend only on what happens ‘towards the end of the sequence’.

**Definition 5.5.** Let \( S : \mathbb{N}_\bullet \rightarrow X \) be a forward sequence in a metric 1-space \( X \), represented in the diagram

\[
\begin{array}{ccccccc}
& & x & & \\
& & \downarrow & & \downarrow & & \\
x_0 & x_1 & \cdots & x_m & x_{m+1} & \cdots & x_{m+t} & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
y & \rho_m & \rho_{m+1} & \rho_{m+t} & \cdots & & & \\
\end{array}
\]

by the arrows from the object \( x \). An *essential cone* over the forward sequence is given by arrows \( \{ \rho_k : x_k \rightarrow y \}_{k=m}^\infty \) to an object \( y \) such that the diagram commutes. Such an essential cone is called a *forward limiting cone* of the forward sequence if \( \lim_{k\rightarrow\infty} w(\rho_k) = 0 \). In that case, the arrow \( x \rightarrow y \) (the common value of all compositions in the diagram) is called the *forward limiting arrow* associated to the forward limit and is denoted, ambiguously, by \( \lim_{n\rightarrow\infty} S \).

Applying categorical duality we obtain the definition of a *backward limiting arrow* \( \lim_{n\rightarrow\infty} S \) of a backward sequence \( S : \mathbb{N}^\bullet \rightarrow X \). Once more, we omit the details.

Addressing uniqueness requires the following notion.

**Definition 5.6.** Consider two essential cones over the same forward sequence as in the diagram (drawn from some index \( m \) where both cones are defined)

\[
\begin{array}{ccccccc}
& & x & & \\
& & \downarrow & & \downarrow & & \\
x_0 & x_1 & \cdots & x_m & x_{m+1} & \cdots & x_{m+t} & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
y & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & y' \\
\end{array}
\]

A *factorization* of the cone to \( y' \) through the cone to \( y \) is a dashed arrow, called a *mediating arrow*, such that for infinitely many values \( l \in \mathbb{N} \) holds that the triangle
commutes. If a factorization between two cones exists then we say that they are compatible.

**Lemma 5.7.** Let $X$ be a metric 1-space, $S : N \to X$ a forward sequence from $x$, and $C_\mu$ and $C_\nu$ two forward limiting cones. If $y \to y'$ is a mediating arrow from $C_\mu$ to $C_\nu$ then $w(y \to y') = 0$. Dually, mediating arrows between backward limiting cones of a backward sequence have weight 0.

**Proof.** The proof is a straightforward application of the full triangle inequality. □

**Corollary 5.8.** If $X$ is non-degenerate then a forward limiting arrow, if it exists, is unique within compatible cones. To be more precise, if $X$ is non-degenerate then two forward limiting arrows (of the same forward sequence) with compatible cones are equal. Dually, backward limiting arrows in non-degenerate metric 1-spaces are similarly essentially unique.

**Example 5.9.** In any metric 1-space a constant forward (respectively backward) sequence $\{\psi_n\}_{n=0}^{\infty}$, $\psi_n = \psi$, has $\psi$ as a forward (respectively backward) limiting arrow. If $S$ is a set then we saw above that ordinary metric structures on $S$ correspond to metric structures on the indiscrete category $I_S$. Forward and backward sequences in $I_S$ are then essentially the same as sequences in $S$ and convergence in $S$ and in $I_S$ agree. More explicitly, if $\{\psi_n : y \to z_n\}_{n=0}^{\infty}$ is a forward sequence in $I_S$ then its limit exists in $I_S$ if, and only if, the sequence of points $\{z_n\}_{n=0}^{\infty}$ in $S$ converges in the ordinary sense. In that case, a limiting object of the sequence is the unique arrow from $y$ to the limit point of $\{z_n\}_{n=0}^{\infty}$ in $S$. A similar remark holds for backward limits. Moreover, starting with any convergent sequence of points in $S$ one may obtain its limit as a limiting object in $I_S$ of both a forward and a backward sequence in $I_S$.

**Remark 5.10.** All of the results that follow, when specialized to metric 1-spaces on an indiscrete category, relate to familiar notions in the ordinary theory of limits. One of the aims of this work is to show that the standard theory extends, in just this sense, to our more general setting. For the sake of keeping the presentation short we will not point out precisely how each result below extends familiar results. More often than not, it is self evident.

**Proposition 5.11.** Let $X$ be a metric 1-space. If the forward sequence $S : N \to X$ admits a limit then

$$\lim_{n \to \infty} w(S(\bullet \to n)) = w(\lim_{n \to \infty} S).$$

Dually, if the backward sequence $S : N^* \to X$ admits a limit then

$$\lim_{n \to \infty} w(S(n \to \bullet)) = w(\lim_{n \to \infty} S).$$

**Proof.** Since the assertions are dual it suffices to prove the first one. Let $\mu : x \to y$ be a forward limiting arrow with forward limiting cone given by arrows $\rho_m$ as in the diagram in Definition 5.5. Since for almost all $k$ the diagram

\[
\begin{array}{ccc}
 x & \xrightarrow{\mu} & y \\
 \downarrow{\psi_k} & & \downarrow{\rho_k} \\
 x_k & \xrightarrow{S(\bullet \to k)} & y_k
\end{array}
\]

commutes, the assertion follows.
commutes, the full triangle inequality implies that

\[ w(\psi_k) - w(\rho_k) \leq w(\mu) \leq w(\psi_k) + w(\rho_k) \]

and thus

\[ w(\mu) - w(\rho_k) \leq w(\psi_k) \leq w(\mu) + w(\rho_k) \]

The result now follows since \( \lim_{k \to \infty} w(\rho_k) = 0 \). □

Remark 5.12. For ordinary metric spaces the uniqueness of the limit is easily proven but depends in an essential way on the axiom of symmetry. Indeed, in a Lawvere space one may consider two kinds of convergence (see [33]) none of which exhibits uniqueness of the limiting point, not even after identifying points with distance 0. As shown above, in our approach the essential uniqueness of the limiting arrow of a sequence is a result of an interplay between the underlying categorical essential cones (i.e., their compatibility) and the metric structure defined on the category (with the full triangle inequality playing an essential role). Viewed this way, the uniqueness of the limit in ordinary metric spaces is a consequence of the fact that any two forward (respectively backward) limiting cones over the same forward (respectively backward) sequence in an indiscrete category \( I_S \) are compatible.

5.4. Limits of series. The ability to compose arrows in a category naturally gives rise to another limiting notion in metric 1-spaces.

**Definition 5.13.** Let \( X \) be a metric 1-space and \( S : \mathbb{N} \rightarrow X \) a forward series of arrows in it, depicted by the horizontal arrows in the following diagram.

\[ \begin{array}{cccc}
  x_0 & \xrightarrow{\psi_0} & x_1 & \xrightarrow{\psi_2} \cdots \xrightarrow{\psi_n} x_n & \xrightarrow{\psi} \cdots \\
  \mu_0 & \mu_1 & \cdots & \mu_n & \cdots \\
  x_\infty \\
\end{array} \]

A **forward limit** of the series is an object \( x_\infty \in \text{ob}(X) \) together with arrows \( \mu_n : x_n \rightarrow x_\infty \), forming a forward limiting cone in the sense that the diagram commutes and \( \lim_{n \to \infty} w(\mu_n) = 0 \). We then say that the series converges to \( \mu_0 : x_0 \rightarrow x_\infty \) which we denote by \( \bigcap_{n=0}^{\infty} \psi_n = \mu_0 \).

As usual, the dual notion of a limit of a backward series is obtained by reversing arrows.

Note that if a forward (respectively backward) series \( S : \mathbb{N} \rightarrow X \) converges to \( \mu_0 \) then its associated forward (respectively backward) sequence of partial compositions converges to \( \mu_0 \) (the reverse implication is not generally true). The following is thus an immediate consequence of Lemma 5.7.

**Lemma 5.14.** Given a forward (respectively backward) series that converges to \( \mu \) and \( \nu \), as witnessed by cones \( C_\mu \) and \( C_\nu \), if \( \varphi \) is a mediating arrow from \( C_\mu \) to \( C_\nu \) (as in Definition 5.4) then \( w(\varphi) = 0 \).

**Remark 5.15.** Note that the uniqueness within compatible cones of the limit of a forward (respectively backward) series follows from a less stringent condition than the existence of a mediating arrow as in the preceding lemma. The precise relation
between series and sequences can be elaborated much more but these subtleties
play no significant role in this work and are thus neglected.

The following simple result, whose proof is omitted, shows again the importance
of the full triangle inequality to obtain results that echo the fundamentals of the
theory of convergence in ordinary metric spaces.

**Proposition 5.16.** If a forward (respectively backward) series \( \{\psi_n\}_{n=0}^\infty \) converges
then \( \lim_{n \to \infty} w(\psi_n) = 0 \). Moreover, if the series forward converges then
\[
\lim_{n \to \infty} w(\psi_n \circ \psi_{n-1} \circ \cdots \circ \psi_0) = w(\bigcirc_{n=0}^\infty \psi_n)
\]
while if the series backward converges then
\[
\lim_{n \to \infty} w(\psi_0 \circ \psi_1 \circ \cdots \circ \psi_n) = w(\bigcirc_{n=0}^\infty \psi_n).
\]

The following lemma will be used in the proof of the Banach fixed point theorem
below. Again a proof is left for the reader.

**Lemma 5.17.** Let \( S : \mathbb{N} \to X \) be a forward series in a metric 1-space \( X \) with
forward limiting cone \( \{\psi_n\}_{n=0}^\infty \). If \( k \geq 0 \) is any natural number then the forward
series \( S^k : \mathbb{N} \to X \), given by \( S^k(n) = S(n+k) \) and extended uniquely to arrows,
has \( \{\psi_m\}_{m=k}^\infty \) as forward limiting cone.

The dual notion relates truncations of a backward limiting cone of a backward
series to truncations of the backward series.

6. Continuity

We present two notions of continuity at an arrow and some related concepts. As
we show below, the two concepts of continuity coincide when the domain is suitably
compact. We mention as well that there is a third natural notion of continuity at
an arrow which is obtained by considering decompositions of \( \psi \) as \( \rho \circ \tau \). The theory
of this kind of limit is similar to the other two and there are interrelations between
all notions. However, to keep the presentation more concise we do not consider this
third possibility here.

In what follows it will be useful to introduce the following convention. For
an arrow \( x \to y \) in a metric 1-space and a non-negative real number \( t \) we write
\( x \xrightarrow{t} y \) to indicate that \( w(x \to y) < t \).

**Definition 6.1.** \( F \) is said to be **forward continuous** at the arrow \( \psi : x \to z \) if for
every \( \epsilon > 0 \) there is a \( \delta > 0 \) such that \( F \) sends any commuting diagram of the form

\[
\begin{array}{ccc}
x & \xrightarrow{t} & y \\
\psi & \quad & \\
\downarrow & \quad & \downarrow \\
z & \to & \delta
\end{array}
\]
to a diagram of the form

\[
\begin{array}{ccc}
F_x & \xrightarrow{\psi} & F_y \\
\downarrow & & \downarrow \\
F_z & \xleftarrow{\epsilon} & \\
\end{array}
\]

We say that \(F\) is \textit{forward continuous} if it is forward continuous at every arrow in \(X\). We say that \(F\) is \textit{uniformly continuous} if for every \(\epsilon > 0\) there is a \(\delta > 0\) such that \(w(F(\psi)) < \epsilon\) holds for every \(\psi\) with \(w(\psi) < \delta\).

The notions of a functor being \textit{backward continuous} at an arrow and \textit{backward continuous} are defined by duality. Notice that the concept of uniform continuity is self-dual.

**Definition 6.2.** Let \(F : X \to Y\) be a functor between metric 1-spaces. We say that

- \(F\) is \textit{forward continuous at an object} \(x_0 \in \text{ob}(X)\) if for every \(\epsilon > 0\) there exists \(\delta > 0\) such that \(w(\psi) < \delta\) implies \(w(F(\psi)) < \epsilon\) for every \(\psi : x \to x_0\).
- \(F\) is \textit{backward continuous} at an object \(x_0 \in \text{ob}(X)\) if for every \(\epsilon > 0\) there exists \(\delta > 0\) such that \(w(\psi) < \delta\) implies \(w(F(\psi)) < \epsilon\) for every \(\psi : x_0 \to x\).
- \(F\) is \textit{object forward (respectively backward) continuous} if it is forward (respectively backward) continuous at every object \(x_0 \in \text{ob}(X)\).

**Proposition 6.3.** Let \(F : X \to Y\) be a functor between metric 1-spaces and \(\psi : x \to z\) an arrow in \(X\).

- Forward continuity at \(z\) implies forward continuity at \(\psi : x \to z\).
- Backward continuity at \(x\) implies backward continuity at \(\psi : x \to z\).

**Proof.** Straightforward. \(\square\)

**Proposition 6.4.** Let \(F : X \to Y\) be a functor between metric 1-spaces. If \(F\) is uniformly continuous then \(F\) is object forward and object backward continuous.

**Proof.** To prove the forward case (the backward case being obtained by duality) fix an object \(z \in \text{ob}(X)\). Given \(\epsilon > 0\) there is a \(\delta > 0\) such that any commuting square of the form

\[
\begin{array}{ccc}
x & \xrightarrow{\psi} & y \\
\downarrow & & \downarrow \\
z & \xleftarrow{\delta} & \\
\end{array}
\]
yields, upon applying $F$, a square of the form

\[
\begin{array}{c}
Fx \\
F\psi \\
Fz
\end{array}
\begin{array}{c}
\downarrow \\
\downarrow \\
\downarrow
\end{array}
\begin{array}{c}
FY \\
FY \\
FY
\end{array}
\begin{array}{c}
\epsilon \\
\epsilon \\
\epsilon
\end{array}
\]

In particular thus, given an arrow $y \xrightarrow{\phi} z$, with $w(\phi) < \delta$, consider the commuting diagram

\[
\begin{array}{c}
y \\
\phi \\
z
\end{array}
\begin{array}{c}
\downarrow \\
\downarrow \\
\downarrow
\end{array}
\begin{array}{c}
y \\
\phi \\
z
\end{array}
\begin{array}{c}
\downarrow \\
\downarrow \\
\downarrow
\end{array}
\begin{array}{c}
y \\
\phi \\
z
\end{array}
\]

and apply $F$ to it to obtain the diagram

\[
\begin{array}{c}
FY \\
F\phi \\
Fz \\
Fy \\
F\phi
\end{array}
\begin{array}{c}
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow
\end{array}
\begin{array}{c}
FY \\
FY \\
FY \\
FY \\
FY \\
FY
\end{array}
\begin{array}{c}
\epsilon \\
\epsilon \\
\epsilon \\
\epsilon \\
\epsilon
\end{array}
\]

proving that $w(F(\phi)) < \epsilon$ and so $F$ is forward continuous at $z$. \hfill \Box

Thus we see that forward object continuity lies between forward continuity and uniform continuity. A similar remark holds for backward continuity.

The next result is the metric analogue of the fact that a functor $F : \mathcal{C} \to \mathcal{D}$ must map an isomorphism to an isomorphism.

**Lemma 6.5.** Let $X$ and $Y$ be metric 1-spaces. If $F : X \to Y$ is either forward or backward continuous at the arrow $\psi$ and $w(\psi) = 0$ then $w(F(\psi)) = 0$.

**Proof.** Let us assume $F$ is forward continuous (the proof in the backward case is similar) and let $\epsilon > 0$ with a corresponding $\delta > 0$ obtained from forward continuity at $\psi$. Since we can always write

\[
\begin{array}{c}
x \\
\psi \\
y
\end{array}
\begin{array}{c}
\downarrow \\
\downarrow \\
\downarrow
\end{array}
\begin{array}{c}
x \\
\psi \\
y
\end{array}
\begin{array}{c}
\downarrow \\
\downarrow \\
\downarrow
\end{array}
\begin{array}{c}
x \\
\psi \\
y
\end{array}
\]

and since $w(\psi) < \delta$ we may conclude that $w(F(\psi)) < \epsilon$, and the result follows. \hfill \Box
Lastly, the following result is the expected claim that forward (respectively backward) continuous functor preserve forward (respectively backward) limits of both sequences and series. The proof is trivial and thus omitted.

**Theorem 6.6.** For a functor $F : X \to Y$ between metric 1-spaces the following hold.

- For every forward sequence $S : \mathbb{N} \to X$, a forward limiting arrow $\mu$, and a forward limiting cone $C$ holds that if $F$ is forward continuous at $\mu$ then $F(\mu)$ is a forward limiting arrow of the forward sequence $\mathbb{N} \to X \to Y$ with forward limiting cone $F(C)$.
- For every forward series $S : \mathbb{N} \to X$, a forward limiting arrow $\mu$, and a forward limiting cone $C$ holds that if $F$ is forward continuous at $\mu$ then $F(\mu)$ is a forward limiting arrow of the forward series $\mathbb{N} \to X \to Y$ with forward limiting cone $F(C)$.
- For every backward sequence $S : \mathbb{N}^\bullet \to X$, a backward limiting arrow $\mu$, and a backward limiting cone $C$ holds that if $F$ is backward continuous at $\mu$ then $F(\mu)$ is a backward limiting arrow of the backward sequence $\mathbb{N}^\bullet \to X \to Y$ with backward limiting cone $F(C)$.
- For every backward series $S : \mathbb{N}^\bullet \to X$, a backward limiting arrow $\mu$, and a backward limiting cone $C$ holds that if $F$ is backward continuous at $\mu$ then $F(\mu)$ is a backward limiting arrow of the backward series $\mathbb{N}^\bullet \to X \to Y$ with backward limiting cone $F(C)$.

7. **Fundamental results**

The aim of this section is to show that three fundamental results from the theory of ordinary metric spaces extend to metric 1-spaces. The results we generalize are that continuity implies uniform continuity when the domain is compact, that for compact spaces mapping spaces are metric spaces, and the Banach fixed point theorem for contractive mappings on complete spaces. We note immediately that due to the different notions of convergence (i.e., sequences vs. series) and the forward/backward duality there are several ways to interpret how compactness and completeness should be generalized. The definitions we present here are chosen to best fit the proofs of the three theorems we aim at. Other possibilities exist and are useful in different situations. We also mention that the proofs use most of the results recounted above and thus all depend essentially on the full triangle inequality. In particular, the proofs below make no use of symmetry and thus, upon specializing them to metric 1-spaces with indiscrete underlying categories, we obtain proofs of the classical versions of the mentioned results that do not make explicit use of symmetry. After presently stating the relevant definitions of compactness and completeness we proceed to prove the three results in the just stated order.

**Definition 7.1.** A metric 1-space $X$ is said to be **forward compact** (respectively **backward compact**) if every forward (respectively backward) sequence has a convergent subsequence.

Here the meaning of 'subsequence' is the obvious one.

**Definition 7.2.** A metric 1-space $X$ is said to be **object compact** if given any sequence $\{x_n\}_{n=1}^\infty$ of objects in $ob(X)$ there exists an object $x_0 \in ob(X)$ such that for every $\epsilon > 0$ and $N \in \mathbb{N}$ there exist $m, n > N$ and arrows $\psi : x_n \to x_0$ and $\varphi : x_0 \to x_m$ such that both $w(\psi) < \epsilon$ and $w(\varphi) < \epsilon$. 

Of course this definition could be refined to include backward and forward versions. For the proofs below such generality is not needed.

**Definition 7.3.** Let $X$ be a metric 1-space. A forward series $\{\psi_n : x_n \rightarrow x_{n+1}\}_{n=1}^\infty$ in $X$ is a *Cauchy forward series* if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for every $n > m > N$ holds that $w(\psi_n \circ \psi_{n-1} \circ \cdots \circ \psi_m) < \epsilon$. Dually, a backward series $\{\psi_n : x_{n+1} \rightarrow x_n\}_{n=0}^\infty$ is a *Cauchy backward series* if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for every $n > m > N$ holds that $w(\psi_m \circ \psi_{m+1} \circ \cdots \circ \psi_n) < \epsilon$.

We omit the details of the straightforward fact that any convergent forward or backward series is Cauchy.

**Definition 7.4.** A metric 1-space $X$ is said to be *forward complete* (respectively *backward complete*) if every forward (respectively backward) Cauchy series $\{\psi_n\}_{n=1}^\infty$ converges.

### 7.1. Uniform continuity

**Theorem 7.5.** Let $F : X \rightarrow Y$ be a forward (respectively backward) continuous functor between metric 1-spaces. If $X$ is forward (respectively backward) compact then $F$ is object forward (respectively backward) continuous. If moreover $X$ is object compact then $F$ is uniformly continuous.

**Proof.** Assume that $F$ is forward continuous at every arrow but not forward continuous at some object $x \in \text{ob}(X)$. Thus, there exists an $\epsilon_0 > 0$ and, for every $n \in \mathbb{N}$, an arrow $\psi_n : x \rightarrow x_n$ such that $w(\psi_n) < \frac{1}{n}$ and $w(F\psi_n) \geq \epsilon_0$. The arrows $\psi_n$ form a forward sequence which, by forward compactness of $X$, we may assume converges to some arrow $\mu$. Proposition 5.11 implies that $w(\mu) = 0$ and together with Theorem 6.6 we may also conclude that $w(F\mu) \geq \epsilon_0$. We now arrive at a contradiction with Lemma 6.5. It follows that $F$ is object forward continuous.

Now, under the extra assumption that $X$ is object compact, suppose that $F$ is not uniformly forward continuous. Then there exists an $\epsilon_0 > 0$ and for every $n \in \mathbb{N}$ an arrow $\phi_n : x_n \rightarrow z_n$ such that $w(\phi_n) < \frac{1}{n}$ and $w(F\phi_n) \geq \epsilon_0$. By object compactness there is an object $x$ and arrows $\psi_n : x \rightarrow x_n$ with $w(\psi_n) < \frac{1}{n}$. Since $F$ is forward continuous at $x$ we may assume without loss of generality that $w(F(\phi_n)) < \frac{\epsilon_0}{2}$ holds for all $n \geq 1$. The full triangle inequality now yields that $w(F(\phi_n \circ \psi_n)) > \frac{\epsilon_0}{2}$ while $w(\phi_n \circ \psi_n) < \frac{\epsilon_0}{2}$. Using forward compactness we again obtain a contradiction with Lemma 6.5, thus completing the proof. □

Since uniform continuity is a self dual property we obtain:

**Corollary 7.6.** If $X$ is object compact as well as forward and backward compact then a functor $F : X \rightarrow Y$ is forward continuous if, and only if, it is backward continuous.

### 7.2. Mapping spaces

To construct the mapping metric 1-space of metric 1-spaces we introduce the notion of natural transformations between functors.

**Definition 7.7.** Given categories $\mathcal{C}$ and $\mathcal{D}$ and functors $F, G : \mathcal{C} \rightarrow \mathcal{D}$, a *natural transformation* $\alpha : F \rightarrow G$ is a family $\{\alpha_c : F(c) \rightarrow G(c)\}_{c \in \text{ob}(\mathcal{C})}$ of arrows in $\mathcal{D}$.
such that for every arrow $\psi : c \to c'$ in $\mathcal{C}$ the diagram

\[
\begin{array}{ccc}
F(c) & \xrightarrow{\alpha_c} & G(c) \\
F(\psi) \downarrow & & \downarrow G(\psi) \\
F(c') & \xrightarrow{\alpha_{c'}} & G(c')
\end{array}
\]
commutes.

It is well known that given categories $\mathcal{C}$ and $\mathcal{D}$ the collection of all functors $F : \mathcal{C} \to \mathcal{D}$ as objects and natural transformations $\alpha : F \to G$ as arrows forms a category known as the functor category $\text{Cat}(\mathcal{C}, \mathcal{D})$. We refer to [24] for more details and just recall here that the composition (also known as vertical composition) of natural transformations that turns $\text{Cat}(\mathcal{C}, \mathcal{D})$ into a category is given, for natural transformations $\alpha : F \to G$ and $\beta : G \to H$, by the family of arrows

\[
\{ F(c) \xrightarrow{\alpha_c} G(c) \xrightarrow{\beta_c} H(c) \}_{c \in \text{ob}(\mathcal{C})}.
\]

In the presence of a metric structure on the categories $\mathcal{C}$ and $\mathcal{D}$, any natural transformation $\alpha$ in $\text{Cat}(\mathcal{C}, \mathcal{D})$ naturally acquires a weight as follows.

**Definition 7.8.** Let $X,Y$ be metric 1-spaces, $F,G : X \to Y$ functors, and $\alpha : F \to G$ a natural transformation. The weight of the natural transformation $\alpha$ is given by the formula

\[
w(\alpha) = \sup_{x \in \text{ob}(X)} \{ w(\alpha_x : F(x) \to G(x)) \}.
\]

**Remark 7.9.** Notice that it is only the metric structure on $Y$ that is used here. For this definition to make sense the functors $F$ and $G$ may be any functors at all and $X$ can be an arbitrary category. However, we will not need this extra generality.

Given metric 1-spaces $X$ and $Y$ it is natural to consider a subcategory of the category $\text{Cat}(\mathcal{C}, \mathcal{D})$, the one spanned by continuous functors $F : X \to Y$. We are most interested in the case where $X$ is object compact as well as forward and backward compact. Corollary [7.6] then shows that forward and backward continuity coincide. In that case we will simply say a functor is continuous.

**Definition 7.10.** Let $X$ and $Y$ be metric 1-spaces with $X$ object compact as well as forward and backward compact. The mapping 1-space $[X,Y]$ is the subcategory of $\text{Cat}(X,Y)$ whose objects are the continuous functors and whose arrows are all natural transformations between such functors.

We now establish that the weights defined above turn $[X,Y]$ into a metric 1-space, thus justifying its name.

**Lemma 7.11.** Let $X$ and $Y$ be metric 1-spaces with $X$ object compact as well as forward and backward compact. If $F,G : X \to Y$ are continuous and $\alpha : F \to G$ is a natural transformation then

\[
\max_{x \in \text{ob}(X)} \{ w(\alpha_x) \}
\]
exists.
Theorem 7.12. Let \( m > n \) so assume that \( S \) is a metric object compactness we may assume the existence of an object \( x_0 \) as in Definition 7.2. We claim that \( w(x_0) = S \), which will prove the result. Clearly \( w(x_0) \leq S \), so assume that \( w(x_0) < S \) and let \( \epsilon = S - w(x_0) \). Since \( F \) and \( G \) are uniformly continuous find \( \delta > 0 \) such that for every \( \psi : x \to x_0 \) with \( w(\psi) < \delta \) holds that \( w(F(\psi)) < \frac{S}{\delta} \) and \( w(G(\psi)) < \frac{S}{\delta} \). Now, let \( n_0 \in \mathbb{N} \) be such that for all \( n > n_0 \) holds that \( S - w(x_n) < \frac{S}{\delta} \). We may find an arrow \( \psi : x_m \to x_0 \) such that both \( w(\psi) < \delta \) and \( m > n_0 \) hold. In the corresponding naturality square

\[
\begin{array}{ccc}
F(x_m) & \xrightarrow{F(\psi)} & F(x_0) \\
\downarrow{\alpha_{x_m}} & & \downarrow{\alpha_{x_0}} \\
G(x_m) & \xrightarrow{G(\psi)} & G(x_0)
\end{array}
\]

let \( \varphi : F(x_m) \to G(x_0) \) be the common composition. Then, by the (restricted) triangle inequality, \( w(\varphi) \leq w(F(\psi)) + w(\alpha_{x_0}) < \frac{S}{\delta} + w(\alpha_{x_0}) = S - \frac{\delta}{\delta} \). However, the full triangle inequality yields that \( w(\varphi) \geq w(\alpha_{x_m}) - w(G(\psi)) > S - \frac{\delta}{\delta} = \frac{S}{\delta} \), a contradiction. The proof when \( S = \infty \) is similar.

Theorem 7.12. Let \( X \) and \( Y \) be metric 1-spaces with \( X \) object compact as well as forward and backward compact. The mapping 1-space \([X,Y]\), with the weights defined above, is then a metric 1-space.

Proof. It is trivial that the identity natural transformation \( id_F : F \to F \) has \( w(id_F) = 0 \) for every \( F : X \to Y \) in \([X,Y]\). Now, given natural transformations \( F \xrightarrow{\alpha} G \xrightarrow{\beta} H \), we need to verify that the inequalities

\[ |w(\alpha) - w(\beta)| \leq w(\beta \circ \alpha) \leq w(\alpha) + w(\beta) \]

hold. Let \( x_1, x_2, x_3 \in \text{ob}(X) \) be objects such that,

\[ w(\alpha) = w(\alpha_{x_1} : F(x_1) \to G(x_1)), \]

\[ w(\beta) = w(\beta_{x_2} : G(x_2) \to H(x_2)), \]

and

\[ w(\beta \circ \alpha) = w(\beta_{x_3} \circ \alpha_{x_3} : F(x_3) \to H(x_3)). \]

We now obtain

\[ w(\beta \circ \alpha) = w(\beta_{x_3} \circ \alpha_{x_3}) \leq w(\beta_{x_3}) + w(\alpha_{x_3}) \leq w(\alpha) + w(\beta) \]

establishing the restricted part of the triangle inequality. To obtain the full triangle inequality we need to show that

\[ w(\alpha) \leq w(\beta \circ \alpha) + w(\beta) \]

and that

\[ w(\beta) \leq w(\beta \circ \alpha) + w(\alpha). \]

Indeed,

\[ w(\beta \circ \alpha) + w(\beta) = w(\beta_{x_3} \circ \alpha_{x_3}) + w(\beta) \geq w(\beta_{x_3} \circ \alpha_{x_3}) + w(\beta) \geq w(\alpha_{x_1}) - w(\beta_{x_1}) + w(\beta). \]
and since \( w(\beta) \geq w(\beta_{x_1}) \), by definition of \( \beta \), it follows that

\[
w(\alpha_{x_1}) - w(\beta_{x_1}) + w(\beta) \geq w(\alpha_{x_1}) = w(\alpha).
\]

The other required inequality follows similarly. \( \square \)

Lemma 7.11 and Theorem 7.12 above point already to a significant difference between our metric \( 1 \)-spaces and the more classical Lawvere spaces, as one can easily construct a counter example showing that mapping spaces of Lawvere spaces are generally not Lawvere spaces, even under compactness conditions.

### 7.3. The Banach fixed point theorem.

Since categories are more involved than sets, the needed condition for the Banach fixed point result for a self map of a metric \( 1 \)-space, presented below, requires both a categorical component and a metric component. The metric condition of a contraction is the obvious one.

**Definition 7.13.** Let \( F : X \to X \) be a functor from a metric \( 1 \)-space \( X \) to itself. \( F \) is called a contraction if there exists a real number \( 0 \leq \alpha < 1 \) such that the inequality \( w(F(\psi)) \leq \alpha \cdot w(\psi) \) holds for every arrow \( \psi \) in \( X \).

The categorical condition for a contraction is the following one.

**Definition 7.14.** Let \( F : \mathcal{C} \to \mathcal{C} \) be a functor. A forward natural contraction of \( F \) is a natural transformation \( \alpha : id_{\mathcal{C}} \to F \) such that for every object \( c \in ob(\mathcal{C}) \) the equation \( F(\alpha_c) = \alpha_{Fc} \) holds. Dually, a backward natural contraction of \( F \) is a natural transformation \( \alpha : F \to id_{\mathcal{C}} \) such that for every object \( c \in ob(\mathcal{C}) \) the equality \( F(\alpha_c) = \alpha_{Fc} \) holds.

**Remark 7.15.** Recall that in a category an arrow is called an epimorphism if it is right cancelable and a monomorphism if it if left cancelable. To justify the terminology in the definition above note that a natural transformation \( \alpha : id_{\mathcal{C}} \to F \) which is an epimorphism is automatically a forward natural contraction. Dually any natural transformation \( \alpha : F \to id_{\mathcal{C}} \) which is a monomorphism is automatically a backward natural contraction. In many categories epimorphisms correspond to surjections and monomorphisms to injections. Thus an epimorphic \( \alpha : id_{\mathcal{C}} \to F \), being a family of epimorphisms \( c \to Fc \), shows that \( c \) ’surjects’ to \( Fc \). Similarly, a monomorphic \( \alpha : F \to id_{\mathcal{C}} \), being a family of monomorphisms \( Fc \to c \), shows that \( Fc \) injects into \( c \). In both cases, \( F \) is, in some sense, contracting.

Clearly, when viewing an ordinary metric space \( S \) as a metric \( 1 \)-space the notion of contraction just defined extends the classical notion. Moreover, any function \( f : S \to S \) gives rise to a corresponding functor \( F : IS \to IS \) which always admits a unique forward natural contraction and a unique backward natural contraction.

**Definition 7.16.** Let \( F : \mathcal{C} \to \mathcal{C} \) be a functor and \( \alpha : id_{\mathcal{C}} \to F \) a forward natural contraction. An arrow \( \psi : c \to d \) is an \( \alpha \)-fixed arrow if \( F(d) = d \) and the diagram

\[
c \xrightarrow{\alpha_c} Fc \\
\psi \downarrow \downarrow F\psi \\
F(d) = d
\]

commutes.
The dual notion is that of an $\alpha$-fixed arrow for a backward natural contraction $\alpha$. We may now state and prove the Banach fixed point theorem for metric 1-spaces. For simplicity we state it for a non-degenerate metric 1-space.

**Theorem 7.17.** Let $X$ be a forward (respectively backward) complete non-degenerate metric 1-space. If $F : X \to X$ is a forward (respectively backward) continuous contraction and $\alpha$ is a forward (respectively backward) natural contraction of $F$ then $F$ has an $\alpha$-fixed arrow. In particular $F$ has a fixed object.

**Proof.** The backward case is the dual of the forward case, which is the one we prove. Fix an object $x \in \text{ob}(X)$ and construct the forward series $\{\psi_n = \alpha F^n x\}_{n=1}^{\infty}$. Note that $F\psi_n = \psi_{n+1}$ holds for every $n \geq 0$, which thus clearly shows that the forward series is Cauchy. The following diagram depicts this forward series together with a forward limiting arrow $\mu_0 : x \to y$ and a forward limiting cone for it.

Applying $F$ to the diagram above yields

which, since $F$ is forward continuous, is a convergent forward cone. We may now form the following diagram

where the solid arrows are the arrows $\{\mu_k\}_{k=1}^{\infty}$ and the dashed arrows are $\{F\mu_k\}_{k=0}^{\infty}$, which we claim commutes. To establish that, we need to show that for each $k \geq 1$ the triangle

where the solid arrows are the arrows $\{\mu_k\}_{k=1}^{\infty}$ and the dashed arrows are $\{F\mu_k\}_{k=0}^{\infty}$, which we claim commutes. To establish that, we need to show that for each $k \geq 1$ the triangle
commutes. Since \( \mu_{k-1} = \mu_k \circ \psi_{k-1} \) and since \( F\psi_{k-1} = \psi_k = \alpha_{F^k x} \), the triangle becomes

\[
\begin{array}{c}
F^k x \\
\downarrow \mu_k \\
y \\
\end{array}
\begin{array}{c}
\downarrow \alpha_y \\
F y \\
\end{array}
\begin{array}{c}
\alpha_{F^k x} \\
\downarrow F\mu_k \\
F^{k+1} x \\
\end{array}
\]

which indeed commutes since \( \alpha \) is a natural transformation. We thus established, using Lemma 5.17 that the arrow \( \alpha_y : y \rightarrow F y \) is a mediating arrow for two forward limiting cones for the same forward series and thus conclude, by Lemma 6.14 that \( w(\alpha_y) = 0 \). Non-degeneracy of \( X \) now implies that \( \alpha_y = id \). We may thus conclude that the first diagram above can be rewritten as

\[
\begin{array}{c}
x \xrightarrow{\psi} Fx \\
\downarrow \mu_0 \\
y \\
\end{array}
\begin{array}{c}
\downarrow F\mu_0 \\
F^2 x \\
\downarrow \mu_2 \\
\end{array}
\begin{array}{c}
\downarrow F^2 \mu_0 \\
F^3 x \\
\downarrow \mu_2 \\
\end{array}
\ldots
\begin{array}{c}
\downarrow F^k \mu_0 \\
\vdots \\
\downarrow F^{k+1} \mu_0 \\
\end{array}
\begin{array}{c}
\ldots \\
F^k x \\
\end{array}
\]

and in particular the left most triangle exhibits \( \mu_0 \) as an \( \alpha \)-fixed arrow. \( \square \)

8. Dagger structures and a hierarchy of symmetry

We now turn to consider metric 1-spaces with extra structure, namely a dagger structure. We consider three conditions a dagger can satisfy and identify a four level hierarchy of symmetry for metric 1-spaces. Interestingly, the canonical embedding of ordinary metric spaces as metric 1-spaces with indiscrete underlying categories takes values in the lowest, most symmetric, level which shows again the inevitability of symmetry discussed in Section 2.

Recall, that a dagger category is a category equipped with an involution. More precisely:

**Definition 8.1.** A **dagger** structure on a category \( \mathcal{C} \) is a functor \( \dagger : \mathcal{C} \rightarrow \mathcal{C}^{op} \) which is the identity on objects and (writing \( f^\dagger \) instead of \( \dagger(f) \)) so that \( f^{\dagger\dagger} = f \) holds for every arrow \( f \) in \( \mathcal{C} \). A pair \( (\mathcal{C}, \dagger) \), with \( \dagger \) a dagger structure on \( \mathcal{C} \), is called a **dagger category**.

**Example 8.2.** This example will allow us later to interpret the Gromov-Hausdorff distance as a suitable weight function \( w \) on the category of cospans of a category of metric spaces. If a category \( \mathcal{C} \) admits pushouts then one can construct its category of cospans. A cospan in \( \mathcal{C} \) is a diagram of the form

\[
\begin{array}{c}
c \\
\downarrow f \\
d \\
\end{array}
\begin{array}{c}
g \\
\downarrow g \\
c' \\
\end{array}
\begin{array}{c}
c' \\
\downarrow c' \\
c \\
\end{array}
\begin{array}{c}
c \\
\end{array}
\]
and can be thought of as a generalized arrow from \( c \) to \( c' \). Given another cospan from \( c' \) to \( c'' \) one can construct a cospan from \( c \) to \( c'' \) by considering the diagram

\[
\begin{array}{ccc}
  c & \rightarrow & c' \\
  \downarrow & & \downarrow \\
  c'' & \rightarrow & c''
\end{array}
\]

where the bottom diamond is a pushout. The generally arbitrary choice of a particular object \( e \) for the pushout implies that this composition will only be associative up to an isomorphism. Thus, to obtain an honest category one needs to consider the evident equivalence classes of cospans. Once this is done one obtains the category \( \text{coSpan}(C) \) whose objects are the objects of \( C \) and an arrow \( c \rightarrow c' \) is an equivalence class \([([f, g]])\) of cospans (another possibility is to construct a weak 2-category of cospans). Note that \( \text{coSpan}(C) \) has an obvious dagger structure sending \([([f, g]])\) to \([([g, f]])\).

**Remark 8.3.** Dagger categories with some extra structure (compact closed) are studied in [1] (called there strongly compact closed categories) to allow for an abstract categorical presentation of quantum computations.

A **groupoid** is a category \( C \) in which all arrows are isomorphisms. A groupoid is the many objects version of a group in the sense that a groupoid with just one object is essentially a group. Any groupoid has a canonical dagger structure since one can always define \( \dagger : G \rightarrow G^{\text{op}} \) to be the identity on objects and to send an arrow \( f : a \rightarrow b \) in \( G \) to its (by assumption existing and necessarily unique) inverse \( f^{-1} : b \rightarrow a \).

If \( X \) is a metric 1-space then \( X^{\text{op}} \) is obviously a metric 1-space by defining \( w(\psi^{\text{op}} : y \rightarrow x) = w(\psi : x \rightarrow y) \) for every \( \psi^{\text{op}} \) in \( X^{\text{op}} \). The following definition distinguishes different ways that a dagger structure on \( X \) can be compatible with this metric.

**Definition 8.4.** Let \( X \) be a metric 1-space. A dagger structure \( \dagger : X \rightarrow X^{\text{op}} \) is called

- an **iso** dagger structure if \( w(\psi^{\dagger}) = w(\psi) \) holds for every arrow \( \psi \) in \( X \).
- a **uniform** dagger structure if the functor \( \dagger : X \rightarrow X^{\text{op}} \) is uniformly continuous.
- a **continuous** dagger structure if the functor \( \dagger : X \rightarrow X^{\text{op}} \) is both forward and backward continuous.

A few comments are in order. The properties listed above are clearly written in decreasing strength. Note, that the apparent weaker property of a dagger structure satisfying \( w(\psi^{\dagger}) \leq w(\psi) \) is in fact equivalent to the iso dagger property. Note as well that each of these properties is self dual in the sense that if \( \dagger : X \rightarrow X^{\text{op}} \) is iso, uniform, or continuous then so is \( \dagger^{\text{op}} : X^{\text{op}} \rightarrow X \). Lastly, it is possible to introduce another property in between iso and uniform, namely that \( \dagger \) satisfies a Lipschitz condition.

**Proposition 8.5.** If \( X \) is a metric 1-space with underlying groupoid then the canonical dagger structure on \( X \) is an iso dagger structure.
Proof. This is just Corollary 3.7.

Corollary 3.8 may now be restated as simply saying that an indiscrete category is trivially a groupoid.

Proposition 8.6. If $X$ is a metric 1-space that admits an iso dagger structure then the Lawvere space $L(X)$ is a metric space.

Proof. Immediate from the definition of $L(X)$.

The following lemma shows that the presence of a continuous dagger structure implies that forward and backward notions coincide. We leave the proof to the reader.

Lemma 8.7. Let $X$ and $Y$ be metric 1-spaces, each with a continuous dagger structure. The following then hold:

- A functor $F : X \to Y$ is forward continuous if, and only if, it is backward continuous.
- $X$ is forward compact if, and only if, it is backward compact.
- $X$ is forward complete if, and only if, it is backward complete.

Of course, these results can be refined to consider continuity at an arrow as well as other notions of compactness and completeness.

We can thus identify four types of symmetry a metric 1-space may possess. If the underlying category is a groupoid then the canonical dagger structure is automatically an iso dagger structure and we call such metric 1-spaces groupoidal. Then there are the iso dagger structures that are not necessarily canonical, followed by uniform dagger structures and lastly continuous ones. Metric structures in any of these classes exhibit various degrees of unification of the forward and backward topological notions and are thus, in a sense, symmetric.

Remark 8.8. Referring back to Remark 2.6 we note that a generalization, along the lines carried out above, of the axiomatization of metric spaces using the strong triangle inequality would lead to the concept of an iso dagger metric 1-space. It is in this sense that we justify our claim that the approach presented here subsumes the putative generalization of the strong triangle inequality.

9. Examples, related structures, and notes

In this final section we consider examples of metric 1-spaces, related structures, and we point out some of the potential applications of metric 1-spaces.

9.1. Lipschitz distance. Referring back to Example 3.9 above, clearly, every bi-Lipschitz function is invertible and the inverse function is again bi-Lipschitz. Thus the underlying category of the metric 1-space $\text{BiLip}$ is a groupoid and thus the induced dagger is automatically an iso dagger. The metric 1-space $\text{BiLip}$ is thus an example of a groupoidal metric 1-space and by Proposition 8.6, the associated Lawvere space $L(\text{BiLip})$ is a metric space. The distance in $L(\text{BiLip})$ is identical to the classical Lipschitz distance.
9.2. Gromov-Hausdorff distance. Given a subset \( Y \) of a metric space \( X \) and \( r > 0 \), let \( B_r(Y) = \bigcup_{y \in Y} B_r(y) \). Recall that \( d_H(C,D) \), the Hausdorff distance between subsets \( C,D \) of a metric space \( X \), is defined as the infimum over \( r > 0 \) such that both \( C \subseteq B_r(D) \) and \( D \subseteq B_r(C) \). Then, \( d_{GH}(X,Y) \), the Gromov-Hausdorff distance between metric spaces \( X \) and \( Y \), is defined as the infimum of \( d_H(i(X),j(Y)) \) as \( i : X \to Z \) and \( j : Y \to Z \) range over all possible isometric embeddings of \( X \) and \( Y \) into arbitrary metric spaces \( Z \). Ignoring set theoretic difficulties, one can construct the space \( S \) of all metric spaces and it is well-known that the Gromov-Hausdorff distance turns it into a metric space.

Consider now the category \( \text{Met}_e \) of all ordinary metric spaces as objects and isometric embedding as arrows. Note that the definition of the Gromov-Hausdorff distance \( d_{GH}(X,Y) \) can be restated as being the infimum over all cospans \((i,j)\)

\[
\begin{array}{ccc}
X & \searrow & Y \\
\downarrow & & \downarrow \\
Z & \nearrow & \\
\end{array}
\]

from \( X \) to \( Y \) of the quantity \( w(i,j) = d_H(i(X),j(Y)) \). In fact, it is not hard to show that this definition of \( w \) turns \( \text{CoSpan}(\text{Met}_e) \) into a metric 1-space and that the evident dagger structure on the cospan category is an iso dagger. It then follows immediately that we obtain the formula \( S = L(\text{CoSpan}(\text{Met}_e)) \) exhibiting the associated Lawvere space as the space of metric spaces.

Remark 9.1. In [3], the classical Gromov-Hausdorff distance is studied in the context of quantale valued metric 1-spaces and we suspect that the categorical insights obtained in [3] would be valid in that setting too.

9.3. Directed spaces. Recall from Section [8] that a groupoidal metric 1-space carries a canonical iso dagger structure and thus its associated Lawvere space is an ordinary metric space. We propose that groupoidal metric 1-spaces \( X \) can naturally serve as models for directed spaces. In any groupoidal metric 1-space \( X \) each set of arrows \( X(x,y) \) is a heap and if we assume \( X \) is a connected groupoid (i.e., every two objects are connected by some arrow) then all such heaps are isomorphic, let’s say to \( H \). It can be shown that the smaller \( H \) is the more symmetric the weight function must be. In more detail, let \( \sigma(x,y) = \sup |w(\psi) - w(\varphi)| \) where \( \psi : x \to y \) and \( \varphi : y \to x \) range over all possible such arrows. Then the smaller the size of \( H \) is with respect to the cardinality of \( \text{ob}(X) \) (we assume now that it is a set) the smaller the upper bound of \( \sigma(x,y) \) is. The extreme case where \( |H| = 1 \) forces \( \sigma(x,y) = 0 \) and thus the weight function is completely symmetric. The case where \( \text{ob}(X) = 2 \) and \( |H| = 2 \) can easily be analyzed to produce an exact upper bound for \( \sigma(x,y) \), which is different than 0. Further research is required to produce workable models of, for instance, the directed real line modeled by a groupoidal metric 1-space whose underlying groupoid has the real numbers as objects and, between any two real numbers, a continuum of arrows.

Remark 9.2. We wish to emphasize a conceptual difference between the directed extension of metric spaces we present here and Grandis’ extension of topology to directed topology (see [16]). Grandis defines a \( d \)-space to be a topological space together with a set \( D \) of allowed paths in \( D \). Mappings between \( d \)-spaces are then...
required to respect these chosen paths. Thus, underlying a directed space is an undirected space. In other words, directedness is more structure. In contrast, in our approach symmetry is seen as extra structure that might be present on an otherwise non-symmetric structure. In other words, undirectedness is a property.

9.4. Bi-metric spaces and phase space of interfering measurements. Consider a metric 1-space \( X \) such that \( X(x, y) \), for every two objects \( x, y \in ob(X) \), consists of precisely two arrows which we denote by \( \pm 1_{xy} : x \to y \), with composition given by multiplication. Let us assume that for every \( x \in ob(X) \) holds that \( w(-1_{xx}) = h \), a constant. If we interpret, for \( x \neq y \), \( a_1 = w(1_{xy}) \) as the accuracy in measuring quantity \( q_1 \) from \( x \) to \( y \) and \( a_2 = w(-1_{xy}) \) as the accuracy of measuring quantity \( q_2 \) from \( x \) to \( y \) then the full triangle inequality reads

\[
|a_1 - a_2| \leq h \leq a_1 + a_2
\]

and thus the constant \( h \) can be seen as setting lower and upper bounds the accuracy in simultaneous measurements of different quantities. The upper bound implies that the quantities measured are related and the upper bound implies that the measurements interfere.

The structure described above is certainly related to a set equipped with two metric structures. Such structures originated with Rosen’s \[32\] which gave rise to what is called bi-metric theories of gravitation. More recent ideas include \[37\] on spacetime as a pseudo-Finslerian bi-metric space and \[25\] on quantum Einstein gravity. Further research is needed to determine the applicability of metric 1-spaces to modeling phase space of quantum measurements and their relation to bi-metric spaces.

9.5. Metric \( n \)-spaces. The passage from metric structures based on sets to metric structures based on higher dimensional categories naturally raises the question as to metric structures based on higher dimensional categories. Very loosely speaking, an \( n \)-dimensional category has various cells of dimensions \( k \) for \( 0 \leq k \leq n \) such that each cell has a boundary composed of cells of lower dimensions ‘glued’ together. On top of this structure there is a composition rule that dictates how cells with matching boundaries can be composed. This composition is required to satisfy some notion of associativity and typically comes in two flavours: strict and weak. The combinatorial freedom allowed by higher dimensional cells makes turning this idea into a rigorous definition highly non-trivial. Indeed, especially for the weak flavour of higher categories, there are more than a dozen different definitions proposed and for most of them it is not yet known if they are equivalent or not. A survey of such definitions can be found in \[23\]. In any case, defining metric \( n \)-spaces carries with it all of the difficulties present in the theory of higher categories and thus we adopt a very informal approach to explain the following construction.

Assume that some notion of metric \( n \)-space is given where such an \( n \)-space is an \( n \)-category in which each cell has a weight together with some compatibilities. We describe, very non-rigorously, a recursive construction that, given a metric \( n \)-space, produces what we call its associated \( n \)-simplex metric. Thus, let \( X \) be a metric \( n \)-space and consider \( n + 1 \) objects \( x_0, \ldots, x_n \) in \( ob(X) \). If \( n = 1 \) then we set

\[
d_1(x_0, x_1) = \inf_{\psi : x_0 \to x_1} \{w(\psi)\}.
\]

This is the same formula defining the associated Lawvere space \( L(X) \). We now call it the associated 1-simplex metric on \( ob(X) \). Assume that for \( 1 \leq k \leq n \) we have
defined the notion of the $k$-simplex metric $d_k(x_0, \cdots, x_k)$ on $ob(X)$ associated to a metric $k$-space $X$. We now define the $(n+1)$-simplex metric associated to a metric $(n+1)$-space $X$. Given points $x_0, \cdots, x_n \in ob(X)$ consider for any $n$ of these points $x_0, \cdots, \hat{x}_j, \cdots, x_n$ the value of $d_n(x_0, \cdots, \hat{x}_j, \cdots, x_n)$. Let us assume for simplicity that it is obtained at a unique $n$-cell of $X$ which we denote by $\alpha_j$. Now, define

$$d_{n+1}(x_0, \cdots, x_n) = \inf_{\beta} \{ w(\beta) \}$$

where $\beta$ ranges over all $(n+1)$-cells having as boundary the composition of $\alpha_0, \cdots, \alpha_n$.

As stated, this is not a rigorous definition and further research is needed to make it precise and study its properties. We can make a few remarks we believe to be important since they relate this construction to existing notions.

The case $n = 1$ clearly gives back the construction $L(X)$ of Lawvere space associated to a metric 1-space. The case $n = 2$ produces a notion of metric structure which is related to structures called 2-metric spaces studied, for instance, in [26] and [5] with the latter source containing many references to other sources. In general, what we call the $n$-simplex metric is related to a notion called $n$-hemi-metric in [11].

We conclude this final section by responding to the following question posed, at the end of the introduction, in [5] by Aliouche and Simpson. In light of the relation between metric spaces and categories enriched in $\mathbb{R}_{\geq 0}$ (namely, that Lawvere spaces, seen as generalized metric spaces, are precisely categories enriched in $\mathbb{R}_{\geq 0}$ with a suitable monoidal structure) what is the category theoretical counterpart of 2-metric spaces (as they define in their article). In light of the work above we propose the following.

Consider the category $\text{Set}_w$ of weighted sets. A typical object in it is a set $S$ together with a function $S \to \mathbb{R}_+$ assigning a weight $s_w$ to each element $s \in S$. A typical arrow $f : S \to S'$ is then a function such that $f(s)_w \leq s_w$ for all $s \in S$. A category enriched in $\text{Set}_w$ amounts to a category with a weight for each arrow such that each identity arrow has weight 0 and the restricted triangle inequality holds. Thus, our notion of metric 1-spaces (with the more restricted class of non-expanding functors) can be identified with a proper subcategory of the category of categories enriched in $\text{Set}_w$. This point of view extends the fact that Lawvere spaces are equivalent to categories enriched in $\mathbb{R}_+$ and ordinary metric spaces are equivalent to a proper subcategory thereof.

Similarly, one may consider 2-categories weakly enriched in $\text{Set}_w$ where the notion of metric 2-spaces (with non-expanding functors) can again be identified with a proper subcategory of such enriched 2-categories. Just as Lawvere spaces arise as the 1-simplex metric associated to a metric 1-space we hypothesize that $n$-categories weakly enriched in $\mathbb{R}_+$ arise as the $n$-simplex metric associated to metric $n$-spaces.

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