Brownian motion is a well-known phenomenon, and since its theoretical foundations were laid, more than one hundred years ago [1,2], diffusion processes and random walk models have been attracting the attention of researchers. The importance of random walk (RW) resides in the fact that it is the simplest realisation of Brownian motion, with applications in almost every field of science where stochastic dynamics play a role [3]. It is worth to remark that, even though a random walker evolves according to simple rules, a considerable effort may be needed to solve the dynamic problem in detail and unexpected behaviours may emerge. Thus, for example, our understanding of the mechanisms responsible for anomalous diffusion has been strongly influenced by the large amount of work devoted to the study of RW in non-Euclidean media, during the last three decades [4–6].

In the last years, it has been often reported that, sometimes, the time behaviour of a RW is modulated by logarithmic-periodic oscillations. These fluctuations have been rigorously studied by mathematicians on special kinds of graphs. A proof of the fluctuating behaviour of the n-step probabilities for a simple RW on a Sierpiński graph was given in ref. [7] and a generalisation to the broad class of symmetrically self-similar graphs can be found in ref. [8]. Within the physical community, it has been shown that, on Sierpiński gaskets, the mean number of distinct sites visited at time \( t \) by \( N \) noninteracting random walkers presents an oscillatory behaviour [9] and, more recently, detailed studies of the log-periodic modulations on fractals with finite ramification order, were presented in refs. [10,11].

Log-periodic modulations were also observed for biased diffusion of tracers on random systems [12–15] but are not restricted to random walks. It is in general believed that they appear because of an inherent self-similarity [16], responsible for a discrete scale invariance (DSI) [17]. Examples of these oscillations have been detected in earthquakes [18,19], escape probabilities in chaotic maps close to crisis [20], kinetic and dynamic processes on random quenched and fractal media [21–24], diffusion-limited aggregates [25], growth models [26], and stock markets near a financial crash [27–30].

In this work we introduce a minimal model of RW with a discrete hierarchy of hopping rates, which results in log-periodic modulations of some observables. The main objective is to show that the origin of these modulations is due to the dependence of the diffusion coefficient on the length scale, which emerge from the hierarchical structure mentioned above. Sometimes, physical phenomena can be more easily grasped with the help of simple models. Thus, the present study may be useful to provide a good insight into the mechanisms involved in this kind of oscillations.

For the sake of simplicity, we consider the problem of a single particle moving on a one-dimensional lattice. At every time step, the particle can hop to a nearest-neighbour (NN) lattice site with a probability per unit time, which depends on the involved (initial and final) sites only. An additional condition is that the forward and backward hopping rates between a given pair of NN sites must be identical. One of the basic ingredients of
Fig. 1: A one-dimensional periodic lattice with $M = 5$ sites per unit cell. A schematic barrier of height $c/k(i)$ is drawn between every pair of NN sites ($i$ and $i + 1$), where $k(i)$ is the hopping rate from site $i$ to site $i + 1$ (and from site $i + 1$ to site $i$). The periodicity conditions $k(i + M) = k(i)$ are satisfied.

The model is a self-similar distribution of hopping rates. As we will show, this leads to a length-scale dependence of the diffusion coefficient that serves to explain both, the anomalous overall behaviour and the modulations observed in higher resolution measurements.

Before going into the details of the self-similar model, let us consider a one-dimensional periodic lattice with $M$ sites per unit cell. The hopping rules stated above can be schematically represented by a set of barriers of height $h(i) = c/k(i)$ ($i \in Z$), where $c$ is an arbitrary constant and $k(i)$ the probability of a hop from site $j$ to site $j + 1$ (and from site $j + 1$ to site $j$) per unit time. A sketch of this structure, with $M = 5$, is shown in fig. 1.

The effective diffusion constant $D_{\text{eff}}$ for a periodic model can be computed following the steady-state method introduced in ref. [31]. If $a$ is the distance between NN sites, it is found that

$$D_{\text{eff}} = \left( \frac{1}{a^2 M} \sum_{j \text{ in a unit cell}} \frac{1}{k(j)} \right)^{-1}, \quad (1)$$

(in what follows we use $a = 1$). The meaning of eq. (1) is that, for times $t \gg t_{\text{esc}}$, where $t_{\text{esc}}$ is the average time for the particle to escape from the initial unit cell, the particle mean-square displacement $\langle \Delta^2 x \rangle$ satisfies the normal diffusion equation

$$\langle \Delta^2 x(t) \rangle = \langle |x(t) - x(0)|^2 \rangle = 2D_{\text{eff}} t. \quad (2)$$

This is a very simple example of a length-scale dependence of the diffusion behaviour. There is an asymptotic regime, described by eq. (2), for $\sqrt{\langle \Delta^2 x \rangle}$ greater than $M$, and a transitory one, not described here, for $\sqrt{\langle \Delta^2 x \rangle}$ smaller than $M$.

In order to obtain the oscillations we are interested in, we proceed now to modify the periodic lattice to allow for a greater number of kinetic regimes. Let us first introduce a parameter $L$, which is an odd natural number greater than 1. The model is built in stages and the result of every stage is called a generation. The building process is illustrated in fig. 2, for $L = 5$.

The zeroth-generation lattice corresponds to the situation in which all the hopping rates are identical ($k(i) = q_0$, $\forall i \in Z$). In this stage, eq. (2), is valid for $\sqrt{\langle \Delta^2 x \rangle}$ greater than 1 with $D_{\text{eff}} = D(0) = q_0$, see eq. (1)).

In the first generation, the hopping rate $k(i)$ is set to $q_1$ ($< q_0$) for every $j = pL - (L + 1)/2$, with $p$ integer. All the other hopping rates remain as in the generation zero. Equation (2) is also valid in first-generation model but for $\sqrt{\langle \Delta^2 x \rangle}$ greater than $L$ and a new value of $D_{\text{eff}} = D(1)$ (also given by eq. (1)).

This process is iterated indefinitely and, in general, the generation $n$ is obtained from the generation $n - 1$ after replacing by $q_n$ ($< q_{n-1}$) the values of $k(i)$, for every $j = pL^n - (L^n + 1)/2$, with $p$ integer.

In the limit of an infinite number of iterations we get the model, with a self-similar distribution of hopping rates and an infinite set of diffusion constants $\{D(n), n = 0, 1, 2, \ldots \}$, discussed in the rest of the work.
It is not hard to convince oneself that, in this model, eq. (2), with $D_{\text{eff}} = D^{(0)}$, should hold for $\sqrt{\langle \Delta^2 x \rangle}$ in some range between $L^n$ and $L^{n+1}$, and this for every non-negative integer $n$. If the RW starts at the site $j = 0$ (the centre of symmetry of the lattice), for short enough times (though longer than $t_{\text{esc}}^{(0)} \sim 1/q_1$) the particle behaves as being in the zeroth-generation lattice. It only feels the action of the lowest barriers and normal diffusion, with a constant $D^{(0)}$, should be observed. However, when $\sqrt{\langle \Delta^2 x \rangle}$ is of the order of $L \langle t \sim t_{\text{esc}}^{(1)} \rangle$, the particle starts to interact with the barriers of height $c/q_1$. For even longer length scales, though shorter than $L^2 \langle t < t_{\text{esc}}^{(2)} \rangle$, everything happens as in the first-generation lattice, and one should then observe normal diffusion with a constant $D^{(1)}$. Because of the self-similar properties of the lattice, this sequence of changes continues indefinitely, and the effective constant $D^{(n)}$, which corresponds to the normal diffusion in the $n$-th-generation lattice, should appear at a scale $L^n$.

If, in addition, we impose that

$$D^{(n)} = \frac{q_0}{(1 + \lambda)^n}, \quad \text{for} \quad n = 0, 1, 2, \ldots,$$

(3)

where $\lambda > 0$ is another parameter of the model, both the diffusion coefficients and the hopping rates are determined, up to a multiplicative constant, (see eqs. (1) and (3)) through

$$D^{(n)} = \frac{q_0}{(1 + \lambda)^n}, \quad \text{for} \quad n = 0, 1, 2, 3, \ldots$$

(4)

and the iterative relation

$$\frac{q_0}{q_i} = \frac{q_0}{q_{i-1}} + (1 + \lambda)^{i-1} \lambda L, \quad \text{for} \quad i = 1, 2, 3 \ldots$$

(5)

From the discussion in the paragraph immediately preceding eq. (3), we can anticipate that the mean-square displacement behaves qualitatively as in fig. 3. This is a sketch of $\langle \Delta^2 x \rangle$ (thick curve), which, as a function of $t$, has a power law form modulated by a log-periodic amplitude. That is

$$\langle \Delta^2 x \rangle(t) = C t^{2\nu} f(t), \quad \text{for} \quad t > t_{\text{esc}}^{(0)},$$

(6)

where $\nu$ is the RW exponent, and $f(t)$ a log-periodic function, which satisfies $f(\lambda t) = f(t)$, with the logarithmic period $\log(\tau)$. The value of the constant $C$ is obtained by asking that the log-time average of $\log(f)$ over one period be zero (see, for more details, fig. 5). We also observe, in fig. 3, two groups of inclined straight lines. On the one hand, the dashed line, which corresponds to the $\langle \Delta^2 x \rangle$ global power law trend, and has a slope of $2\nu$. On the other hand, the solid lines, which represent normal diffusion in each of the different generation lattices, and have slopes of 1. From these slopes, it is clear that $2\nu = \log(L^2)/\log(\tau)$ (dashed line), and that $\log(\tau) = \log(L^2) + \log(1 + \lambda)$ (solid line), which gives both $\tau$ and $\nu$ expressed in terms of the parameters $L$ and $\lambda$,

$$\tau = L^{1/\nu} = (1 + \lambda)L^2,$$

(7)

$$\nu = \frac{1}{2 + \frac{\log(1 + \lambda)}{\log L}}.$$  

(8)

Note that, since $\lambda > 0$, an anomalous diffusion appears ($\nu < 1/2$, see eq. (8)), and that, from the sketch in fig. 3, we can predict that the amplitude of the modulation increases with the increase of $\lambda$ or $L$.

Let us remark that the self-similarity in the mean-square displacement, schematically shown in fig. 3 and mathematically described by eq. (6), is a direct consequence of the set of constraints (3). Because of these relations, the distance between any pair of nearest solid straight lines is a constant and any pair of nearest equivalent points in the graph (like a and b) are related by the transformation $(t \rightarrow rt, \langle \Delta^2 x \rangle \rightarrow r^{2\nu} \langle \Delta^2 x \rangle)$. Even though we have focused on the properties of the mean-square displacement, a similar analysis applies to the average number $S(t)$ of distinct sites visited by the particle, after a time $t$. As we are working with a one-dimensional lattice, $S \sim \sqrt{\langle \Delta^2 x \rangle}$, and it is thus expected that

$$S = C' t^{\nu} g(t),$$

(9)

where the exponent $\eta$ is equal to $\nu$, $g(t)$ is a log-periodic function, $g(\tau t) = g(t)$ (with $\tau$ given by eq. (7)) and the constant $C'$ is obtained by asking that the log-time average of $\log(g)$ over one period be zero.

Fig. 3: (Color online) Schematic of the mean-square displacement as a function of the time, shown by the thick red curve. The length of the segment bc is $\log(2^2D^{(1)}) = \log(2D^{(1)} \langle t \rangle)$, because of eq. (3). From the slopes ($= 1$) of the full straight lines (representing the normal diffusion behaviours, $\langle \Delta^2 x \rangle = 2D^{(0)} \langle t \rangle$), one gets that the segments ad and cd have the same length or, equivalently, that $\log(\tau) = \log(L^2) + \log(1 + \lambda)$. The dashed straight line represents the global power law $\langle \Delta^2 x \rangle \sim t^{2\nu}$, with $2\nu = \log(L^2)/\log(\tau)$. More details in the text.
To check the validity of the analytical predictions stated above, we have performed Monte Carlo (MC) simulations, with $q_0 = 1/2$ and a time step $\Delta t = 1$. Every simulation begins with the particle at the center of a sixth-generation lattice and stops after a given number of MC steps, always chosen to avoid that the particle reaches the highest (6th-order) barriers.

The numerical results of the mean-square displacement as a function of the time is plotted in fig. 4 for two sets of parameters, $L = 5, \lambda = 0.2$ and $L = 5, \lambda = 5$. It is apparent in this figure that $\langle \Delta^2 x \rangle(t)$ satisfies a modulated power law.

The modulations can however be better observed in fig. 5. The logarithm of the scaled mean-square displacement $\langle \Delta^2 x \rangle/(C t^{2\nu})$ as a function of the logarithm of the time, for $L = 5, \lambda = 0.2$ (squares), and $L = 5, \lambda = 5$ (circles). The values of $\nu$ were obtained from eq. (8) and $C$ are appropriately chosen constants. The curvilinear lines represent first-harmonic approximations of the data, $A \sin((2\pi \log t)/(\log \tau) + \alpha)$. The period $\tau$ is given by eq. (7). $A$ and $\alpha$ are fitted constants.

In general, it can be said that for $\nu = 0.4732$ and $\nu = 0.3212$, respectively, given by eq. (8). These lines are drawn to guide the eyes.

In fig. 5 and in the insets of fig. 6 how the former increases with the increase of the latter, which confirm our predictions. We would like to mention that, as shown in fig. 6, the agreement between numerical and theoretical results is as good for the mean number of distinct visited sites as it is for the mean-square displacement.

So far, we have considered RWs that always start at the centre of symmetry (central site in fig. 2). To examine what happens when the RW starts at another site, the evolution of the mean-square displacement on a lattice with $L = 5$ and $\lambda = 5$ is plotted, in fig. 7, for two different initial conditions. In one sample (squares), the particle was initially located at $j = 0$ (the center of symmetry), in the other (stars), at $j = 12$ (the rightmost site in fig. 2, bottom). As can be expected, two clearly different behaviours are present at short times. However, at long times, a superposition of the data is observed. This can be understood as follows. For times much longer than the time for the particle to escape the second-generation unit cell (i.e., when $\sqrt{\langle \Delta^2 x \rangle} \gg L^n$) the diffusion is governed by the coefficients $D^{(n)}$, with $n \geq 2$, and becomes independent of the initial position. Similar results were obtained for several initial positions and different values of $L$ and $\lambda$. In general, it can be said that for $\sqrt{\langle \Delta^2 x \rangle} \gg L^n$, the behaviour of the RW does not depend on the initial position, provided that the latter is at a distance less than $(L^n - 1)/2$ from the centre of symmetry.

We want to emphasise that, although the system seems to become super diffusive for $\lambda < 0$ (provided that the denominator of eq. (8) is positive), the extension of the...
model to include negative values of $\lambda$ is not straightforward. In fact, for any $\lambda < 0$, the right-hand side of eq. (5) becomes negative after a finite number of iterations, giving a negative hopping rate (a nonsense result). This limitation results from the ansatz (3) we impose to obtain log-periodic modulations. It would be interesting but beyond the scope of this letter to investigate the possibility of obtaining a super-diffusive dynamics when the constraints (3) are removed.

To summarise, we have presented a one-dimensional self-similar model of RW, which yields a log-periodic modulated power law for the most commonly measured quantities ($\langle \Delta^2 x(t) \rangle$ and $S(t)$). The goal of this paper is to introduce a minimal model that provides a simple way to gain insight into the effect of self-similarity on this kind of oscillations. As was mentioned above, log-periodic modulations appear in the time behaviour of RWs on fractals with finite ramification order [7–11]. In these objects, at a given length scale, a random walker cannot easily pass from one array to the next, due to the small number of connections between equivalent structures; a difficulty that increases with the structure linear size (because of the larger ratio of internal to connecting sites). In the model here, a similar effect occurs at a length $L^n$, as a consequence of the decrease of the hopping rates at the edges of the $n$-th-generation unit cell, when $n$ changes to $n+1$. This may be considered as a rough qualitative explanation of the log-periodic modulations in fractals with finite connexion order. Even when, for these objects, we have not calculated how the effective diffusion constant depends on a characteristic length, we expect that the knowledge of this dependence would allow us to obtain the values of $\nu$ and $\tau$, in the same way as for the one-dimensional model. Let us also stress that, by tuning the values of the two parameters $L$ and $\lambda$, it is possible to design a structure that leads to an oscillatory power law behaviour with predetermined values of $\nu$ and $\tau$ (whenever $\nu < 1/2$ and $\tau = L^{1/\nu}$). So far, the parameter $L$ has been restricted to odd values, because in these cases the lattice has a centre of symmetry. However, as fig. 7 shows, the initial position of the random walker is not relevant for its long-time behaviour and thus the model can be easily generalised to include even values of $L$ greater than 2.
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