Regularizing Effect and Local Existence for the Non-Cutoff Boltzmann Equation

Radjesvarane Alexandre, Yoshinori Morimoto, Seiji Ukai, Chao-Jiang Xu & Tong Yang

Communicated by T.-P. Liu

Abstract

The Boltzmann equation without Grad’s angular cutoff assumption is believed to have a regularizing effect on the solutions because of the non-integrable angular singularity of the cross-section. However, even though this has been justified satisfactorily for the spatially homogeneous Boltzmann equation, it is still basically unsolved for the spatially inhomogeneous Boltzmann equation. In this paper, by sharpening the coercivity and upper bound estimates for the collision operator, establishing the hypo-ellipticity of the Boltzmann operator based on a generalized version of the uncertainty principle, and analyzing the commutators between the collision operator and some weighted pseudo-differential operators, we prove the regularizing effect in all (time, space and velocity) variables on the solutions when some mild regularity is imposed on these solutions. For completeness, we also show that when the initial data has this mild regularity and a Maxwellian type decay in the velocity variable, there exists a unique local solution with the same regularity, so that this solution acquires the $C^\infty$ regularity for any positive time.

Contents

1. Introduction .................................... 40
2. Pseudo-differential calculus ............................ 47
   2.1. Upper bound estimates ............................ 47
   2.2. Coercivity estimates .............................. 64
   2.3. Commutator estimates ............................. 67
3. Regularizing effect ................................. 78
   3.1. Initialization .................................. 78
   3.2. Gain of regularity in $v$ .......................... 80
   3.3. Gain of regularity in $(t, x)$ ....................... 86
   3.4. Proof of Theorem 1.1 ............................. 94
4. Existence and uniqueness of local solutions .................... 96
   4.1. Modified Cauchy Problem .......................... 97
1. Introduction

Consider the Boltzmann equation,

$$f_t + v \cdot \nabla_x f = Q(f, f), \quad (1.1)$$

where \( f = f(t, x, v) \) is the density distribution function of particles with position \( x \in \mathbb{R}^3 \) and velocity \( v \in \mathbb{R}^3 \) at time \( t \). The right-hand side of (1.1) is given by the Boltzmann bilinear collision operator

$$Q(g, f) = \int_{\mathbb{R}^3} \int_{S^2} B(v - v_*, \sigma) \left\{ g(v'_*) f(v') - g(v_*) f(v) \right\} \, d\sigma \, dv_*,$$

which is well-defined for suitable functions \( f \) and \( g \) specified later. Notice that the collision operator \( Q(\cdot, \cdot) \) acts only on the velocity variable \( v \in \mathbb{R}^3 \). In the following discussion, we will use the \( \sigma \)-representation, that is, for \( \sigma \in S^2 \),

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma,$$

which give the relations between the post- and pre-collisional velocities.

It is well known that the Boltzmann equation is a fundamental equation in statistical physics. For the mathematical theories on this equation, we refer the readers to \([16,17,27,31,46]\), and the references therein, also, for the physics background.

In addition to the special bilinear structure of the collision operator, the cross-section \( B(v - v_*, \sigma) \) varies with different physical assumptions on the particle interactions, and it plays an important role in the well-posedness theory for the Boltzmann equation. In fact, except for the hard sphere model, for most of the other molecular interaction potentials, such as the inverse power laws, the cross section \( B(v - v_*, \sigma) \) has a non-integrable angular singularity. For example, if the interaction potential obeys the inverse power law \( r^{-(p-1)} \) for \( 2 < p < \infty \), where \( r \) denotes the distance between two interacting molecules, the cross-section behaves like

$$B(|v - v_*|, \cos \theta) \sim |v - v_*|^\gamma \theta^{-2-2s}, \quad \cos \theta = \left( \frac{v - v_*}{|v - v_*|}, \sigma \right), \quad 0 \leq \theta \leq \frac{\pi}{2},$$

with

$$-3 < \gamma = \frac{p - 5}{p - 1} < 1, \quad 0 < s = \frac{1}{p - 1} < 1.$$

As usual, the hard and soft potentials correspond to \( 2 < p < 5 \) and \( p > 5 \), respectively, and the Maxwellian potential corresponds to \( p = 5 \). The fact that the singularity \( \theta^{-2-2s} \) is not integrable on the unit sphere leads to the conjecture that