Security of entanglement-based quantum key distribution with practical detectors

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We prove the unconditional security of an entanglement-based quantum-key-distribution protocol using detectors that respond to multiple modes of light and cannot distinguish between one from two or more photons. Even with such practical detectors, any defect in the source is automatically detected as an increase in the error rate or in the rate of double clicks.

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The idea of using quantum entanglement for absolutely secure secret communication was first proposed by Ekert [1], followed by a proposal of a modified quantum-key-distribution (QKD) protocol (BBM92) by Bennett et al. [2]. When ideal apparatuses are used and the source is possessed by a legitimate user, the BBM92 protocol is equivalent to the BB84 protocol [3], which does not use an entangled source. On one hand, this property has lead to a powerful security proof [4] based on entanglement, which is applicable to prepare-measure protocols such as the BB84 protocol [4] and the B92 protocol [5, 6, 7]. But on the other, the equivalence may have discouraged the use of an entangled source in an actual setup if the same function is available without the trouble of generating entanglement. In fact, a huge advantage of actually using an entangled source shows up when we take defects in the source into account. Defects may arise from limitation on technology, and in the BB84 protocol they raise new threats on the security such as the photon-number-splitting attack [8]. Even worse, in long-distance communication a source must be placed at an insecure relay station and hence its property cannot be trusted anymore. The entanglement-based protocol such as the BBM92 protocol provides a unique property in this situation. Since the protocol is based on testing a strong correlation unique to the entanglement, we may expect that any defect in the source will be revealed as a degradation of the correlation.

An important question at this point is what kind of detection apparatus is required to realize such a built-in mechanism for detecting the defects in the source. It would surely be a disappointment if we were forced to use an ideal detector for such a purpose. So far, it has been shown [9] that it is sufficient if one of the two parties have a detection apparatus with a so-called squashing property [10], that is, equivalence to a noisy quantum channel followed by an ideal BB84 measurement on a qubit. It is expected that a practical detection apparatus (as in Fig. 1) with two threshold (on/off) detectors will satisfy the squashing property if we assign a random bit whenever both detectors have clicked. Based on this conjecture, practical benefits of the BBM92 protocol, such as placing the source in the middle to achieve a larger communication distance, were discussed quantitatively. But the proof of the conjecture remains open, leaving an unsatisfactory situation that the BBM92 protocol, being one of the basic QKD protocols with many experimental demonstrations [11, 12, 13], still requires an assumption in the detectors for its security.

In this paper, we prove the unconditional security of the BBM92 protocol with practical threshold detectors which cannot distinguish between one photon from two or more, and cannot single out a single optical spatio-temporal mode either. Instead of proving the squashing property, we adopt a protocol in which the double-click events are simply discarded. The proof is based on a simple inner-product formula for the basis states, which shows that the parity of the number of incident photons has an important role. Eve can carry out a powerful attack by distributing odd and even numbers of photons to the two receivers. The security is essentially obtained by monitoring the bit-error rate and the double-click rate to watch out for the possibility of such an attack.

The protocol considered here is the BBM92 protocol with the detection apparatuses shown in Fig. 1. For each event, Alice randomly chooses between the Z basis and the X basis using a wave plate placed before the polarizing beam splitter (PBS). In the Z-basis measurement, horizontal (H) and vertical (V) polarization components are split at the PBS and sent toward two threshold detectors corresponding to bit values 0 and 1. In the X-basis measurement, the ±45° polarizations (D±) are split in-

![FIG. 1: Schematic of the setup for the BBM92 protocol.](image-url)
stead. Alice publicly announces whether she detected photons, and if so, she also announces whether both of the detectors clicked (double clicks). Bob follows the same protocol as Alice. The bit values are registered only when both parties have detected photons, but neither party has seen double clicks.

As usual, we assume that non-unit efficiency and dark counting of the detectors can be equivalently described by a noise source in front of the detection apparatus. This is satisfied, for example, if two detectors with matched efficiency are used and their roles are switched randomly. Hence, here and henceforth, we treat each detector as an ideal threshold detector that clicks when it receives one or more photons.

Let $N$ be the number of events where both Alice and Bob detected photons and their basis choices were the same. In principle, the number of photons ($n_B \geq 1$) incident on Alice’s apparatus can be determined for each event, since this observable commutes with Alice’s actual measurement. The same goes with Bob’s photon number $n_B \geq 1$. Accordingly, the $N$ events are classified into $N\xi$ multi-photon events satisfying $n_A+n_B \geq 3$ and $N(1-\xi)$ single-photon events with $n_A=n_B=1$. Among the $N\xi$ multi-photon events, suppose that $N\xi\delta_m$ events were discarded due to double clicks, and $N\xi\epsilon_m$ events showed bit errors, namely, different bit values were registered by Alice and Bob. The single-photon events should have no double clicks, and suppose that they include $N(1-\xi)\epsilon_1$ bit-error events. Whereas the parameters $(\xi, \delta_m, \epsilon_m, \epsilon_1)$ are all measurable in principle, the actual setup does not reveal $(n_A, n_B)$ and hence only tells us the overall double-click fraction $\delta$ and a good estimate of the overall error fraction $\epsilon$, which are related to $(\xi, \delta_m, \epsilon_m, \epsilon_1)$ as

$$\delta = \xi\delta_m,$$

$$\epsilon = (1-\xi)\epsilon_1 + \xi\epsilon_m.$$  \hspace{1cm} (1)

From the $N$ events, Alice and Bob produce sifted key of length $N(1-\delta)$ with a quantum bit error rate (QBER) $\epsilon/(1-\delta)$. For simplicity, we assume that the error correction is done by encrypted one-way communication from Alice to Bob by consuming the previously shared secret key of length $N(1-\delta)fHH(\epsilon/(1-\delta))$, where $H(x) \equiv -x\log_2 x - (1-x)\log_2(1-x)$ and $f \geq 1$ represents the inefficiency in the practical error correction schemes. The reconciled key is further shortened by $N\tau$ to amplify the privacy, where $\tau$ is determined from the observed values $(\delta, \epsilon)$. The fraction $R_{\text{key}}$ of the final key (normalized by $N$) is thus written as follows,

$$R_{\text{key}} = (1-\delta)[1-fH(\epsilon/(1-\delta))] - \tau(\delta, \epsilon).$$  \hspace{1cm} (3)

In the limit of large $N$, the final key is secure if

$$\tau(\delta, \epsilon) \geq (1-\xi)H(\epsilon_1) + \xi(1-\delta_m)$$  \hspace{1cm} (4)

holds for any attack by Eve, because the right-hand side is given by the argument by Gottesman et al. [10] with a pessimistic assumption that Eve perfectly knows Alice’s bit value in multi-photon events. One might expect that the use of multi-photon events inevitably leads to bit errors $\epsilon_m > 0$ and double clicks $\delta_m > 0$, but it turns out that either value can be zero by choosing a suitable state. But Eve cannot make both of the values to be zero at the same time. In what follows, we determine this trade-off relation and determine $\tau(\delta, \epsilon)$ satisfying Eq. (4).

Let us first suppose that Alice (or Bob) receives $n$ photons in a single spatio-temporal mode. Let $|Q, n\rangle$ be the state with $n$ photons in the same polarization $Q$, namely, $|Q, n\rangle \equiv (n!)^{-1/2}(a_Q^n)^\dagger|\text{vac}\rangle$ with $a_Q$ being the photon annihilation operator for polarization $Q$. On the $Z$-basis, the outcome 0 corresponds to the projection to the state $|0_Z^{(n)}\rangle = |H, n\rangle$, and 1 to the state $|1_Z^{(n)}\rangle = |V, n\rangle$. The other $n-1$ orthogonal states correspond to the double clicks. On the $X$-basis, the outcomes 0 and 1 correspond to the states $|0_X^{(n)}\rangle = |D+, n\rangle$ and $|1_X^{(n)}\rangle = |D-, n\rangle$. Using $(a_D \pm 2^{-1/2}(a_H \pm a_V))$, we obtain a relation vital to our discussion,

$$\langle b_X^{(n)}|b_Z^{(n)}\rangle = (-1)^{m-n}2^{-n/2}.$$  \hspace{1cm} (5)

We can show that this relation is unaltered even if $n$ photons are distributed over multiple modes. In such a case, the photon numbers $n_1, n_2, \ldots$ in each mode can be measured in principle. For fixed values of $\{n_j\}$, the state $|0_Z^{(n)}\rangle$ is given by $|H, n_1\rangle|H, n_2\rangle \cdots$, and so are the other three states. Noting that $\sum_j n_j = n$, one can see that the inner products are still given by Eq. (5). The only difference is the dimension $d = \prod(n_j + 1)$ of the state space, but it does not affect the argument below, in which only Eq. (5) is used.

When $n = 2l + 1$ ($l = 1, 2, \ldots$), Eq. (5) reads

$$\langle b_X^{(2l+1)}|b_Z^{(2l+1)}\rangle = \langle b_X^{(1)}|b_Z^{(1)}\rangle 2^{-l},$$

which leads to a clear physical interpretation. Since the dimension $d$ is even, the state space $\mathcal{H}_A$ can be identified with a combined system $\mathcal{H}_A \otimes \mathcal{H}_{A'}$ of a single photon (qubit) $A'$ and an ancilla $A''$, with the relations

$$|b_{W, A}^{(2l+1)}\rangle = |b_{W, A}^{(1)}\rangle|\phi_W^{(l)}\rangle_{A'},$$  \hspace{1cm} (6)

$$|\phi^{(l)}_{Z, A'}\rangle_{A''} = 2^{-l},$$  \hspace{1cm} (7)

which preserve the inner products (5). Hence for an odd number of incident photons, Alice’s measurement can be regarded as an ideal BB84 measurement on a qubit $A'$, except that the outcome is overridden by the occurrence of double clicks that is determined by a basis-dependent measurement on the ancilla $A''$. On the other hand, for even numbers we have a constant inner product

$$\langle b_X^{(2l)}|b_Z^{(2l)}\rangle = 2^{-l},$$  \hspace{1cm} (8)

which has no simple connection to a qubit.

Now let us derive a trade-off relation between $(\delta_m, \epsilon_m)$. For the moment, we consider the attacks using only a
where $P(|\psi\rangle) \equiv |\psi\rangle\langle\psi|$. $1 - F_{\text{cor}} - F_{\text{err}}$ corresponds to double clicks. Let us write an expectation value of observable $O$ as \( \langle O \rangle \equiv \text{Tr}(O \rho) \). If \( \rho \) is a state, the probability of \( \langle \delta_m, \epsilon_m \rangle \) to be deviated from the region \( r(1 - \delta_m - \epsilon_m, \epsilon_m) \leq 0 \) is exponentially small for large $N$. In what follows, we consider the limit $N \to \infty$ and ignore such rare possibilities. We divide the argument according to the parities of \((n_A, n_B)\).

i) Odd-odd, $n_A = 2l_A + 1$ and $n_B = 2l_B + 1$ with $l_A + l_B \geq 1$. From Eqs. (5), (9) and (10) with $P_W \equiv P(\phi_W^{l_A} | \phi_W^{l_B})$, we have

\[
F_{\text{cor}} + F_{\text{err}} = \frac{1}{2} \sum_{W=X,Z} P_{\text{err}}(W) / 2.
\]

Eq. (7) shows that the largest eigenvalue of $P_Z + P_X$ is $1 + 2^{-l_A - l_B}$, and we have

\[
\delta_m \geq (1 - 2^{-l_A - l_B}) / 2 \geq 1 / 4.
\]

ii) Odd-even, $n_A = 2l_A + 1 \geq 1$ and $n_B = 2l_B \geq 2$. According to Eq. (5), there exists a unitary $V$ satisfying

\[
V |\phi_W^{l_A} \rangle_A |\phi_W^{l_B} \rangle_B = |\phi_W^{l_A} \rangle_A (b + a \text{ mod } 2) |\phi_W^{l_B} \rangle_B.
\]

for $W = X, Z$. The operation of $V$ is regarded as a basis-independent controlled-NOT gate, which is possible because the target system $B$ is not a qubit but has a larger dimension. Since Eve is allowed to prepare any state, it makes no difference if we assume that she applies $V$ just before she sends systems $A$ and $B$ to Alice and Bob. Then the relevant observables take simple forms as follows:

\[
V^\dagger F_{\text{err}} V = 1_A \otimes (P_Z^+ + P_X^+)/2,
\]

\[
V^\dagger F_{\text{cor}} V = 1_A \otimes (P_Z^+ + P_X^+)/2,
\]

where $P_W^\pm \equiv P(\phi_W^{l_A} A \phi_W^{l_B})$. This leads to

\[
\epsilon_m \geq g(\delta_m) \text{ for } \delta_m \leq 1/3,
\]

where $g(\delta) \equiv [(1 - \delta)/2] - \sqrt{(1 - 2\delta)}$. The boundary is achievable with $n_A = 1$ and $n_B = 2$. Of course, the case with $n_A = 2l_A \geq 2$ and $n_B = 2l_B + 1 \geq 1$ follows the same condition.

Incidentally, the existence of the operation $V$ leads to an interesting attack by Eve with $n_A = 1$ and $n_B = 2$. Suppose that Eve prepares a maximally entangled state $|\phi^+ \rangle_{AE}$ and a pure state $|\chi \rangle_B$, and then applies unitary $V$ before she distributes the photons to Alice and Bob. As is seen from Eqs. (14) and (15), $(\delta_m, \epsilon_m)$ is determined solely by the state $|\chi \rangle_B$, and hence Eve can realize any point on the boundary $\epsilon_m = g(\delta_m)$ by choosing $|\chi \rangle_B$ to be $\sum W \alpha(0_{W^+}) + \beta(1_{W^+})$. On the other hand, Eq. (18) shows that Alice’s outcome can be regarded as obtained from the direct measurement on $|\phi^+ \rangle_{AE}$. Hence after the basis is announced, Eve precisely learns Alice’s bit. This particular attack constitutes a lower bound $\tau_{\text{low}}$ on $\tau(\delta, \epsilon)$ to have a secure key:

\[
\tau_{\text{low}}(\delta, \epsilon) \geq \max_{\xi} \left[ \xi - \delta - \left(1 - \xi\right)H\left(\frac{\epsilon - \xi g(\delta, \xi)}{1 - \xi}\right) \right].
\]

iii) Even-even, $n_A = 2l_A \geq 2$ and $n_B = 2l_B \geq 2$. Let $|\phi_W^\pm \rangle_{AB} \equiv 2^{-1/2} |0_{W^+} \rangle_A |0_{W^+} \rangle_B \pm |1_{W^+} \rangle_A |1_{W^+} \rangle_B$ and define the projection $P_W^\pm \equiv |\phi_W^\pm \rangle \langle \phi_W^\pm |$. Further, let $|\psi_{X}^\pm \rangle_{AB} \equiv 2^{-1/2} |0_{W^+} \rangle_A |1_{W^+} \rangle_B \pm |1_{W^+} \rangle_A |0_{W^+} \rangle_B$ and define $P_W^\pm$ accordingly. We see from Eq. (8) that the state with the minus sign (such as $|\psi_{X}^\pm \rangle_{AB}$) is orthogonal to any of the other seven states. Hence we can write

\[
F_{\text{err}} = 2^{-1} \left(|P_Z^{+} + P_X^{+}\rangle \langle P_Z^{+} + P_X^{+}| + 0\right),
\]

\[
F_{\text{cor}} = 2^{-1} \left(|P_Z^{+} + P_X^{+}\rangle \langle P_Z^{+} + P_X^{+}| + 0 + |P_Z^{-} + P_X^{-}\rangle \langle P_Z^{-} + P_X^{-}|\right).
\]

This leads to the same condition as Eq. (18).

Since the general attack is a mixture of attacks to various $(n_A, n_B)$, we conclude that $(\delta_m, \epsilon_m)$ must be in the shaded region of Fig. 2, obtained by taking convex combination of Eqs. (12) and (16). $\tau(\delta, \epsilon)$ is then determined as the maximum of the right-hand side of Eq. (11) under the constraints Eqs. (1) and (2). The optimization is reduced to a standard problem of determining the convex hull of the points $(0, \epsilon_1, H(\epsilon_1))$ (0 $\leq \epsilon_1 \leq 1/2$, $(\delta_m, g(\delta_m), 1 - \delta_m)$ (0 $\leq \delta_m \leq 1/3$, and (1/4, 0, 3/4). We classify the results into the three different regions (a)–(c) shown in Fig. 2. Let $\epsilon_1 \equiv 0.080$ be the root of $16\epsilon_1^3 (1 - \epsilon_1^3) = 1$.

(a) For $\epsilon \leq \epsilon_1^3 (1 - 4\delta)$,

\[
\tau(\delta, \epsilon) = 3\delta + (1 - 4\delta)H(\epsilon/(1 - 4\delta)).
\]

FIG. 2: The observed fractions $(\delta, \epsilon)$ of double clicks and of bit errors are a mixture of the multi-photon contribution (the shaded region) and the single photon contribution ($\delta = 0$).
FIG. 3: Dependence of key fraction $R_{\text{key}}$ on the double-click fraction $\delta$. Solid curves are proved to be secure in the discarding protocol. Dash-dotted curves are the upper bounds for the discarding protocol. Broken curves are the key fraction conjectured to be secure in the random-bit-assignment protocol.

(b) For $c_1^*(1 - 4\delta) \leq \epsilon \leq \min\{(1 - 6\delta)e_1^* + (\delta/2), 1/4 - \delta\}$,

$$\tau(\delta, \epsilon) = [c_1\delta + c_2\epsilon + c_3]/(1 - 4e_1^*)$$

with constants $c_1 \equiv 3 - 4H(e_1^*) + 4e^*_1$, $c_2 \equiv 4(1 - H(e_1^*))$, and $c_3 \equiv H(e_1^*) - 4e^*_1$.

c) For $(1 - 6\delta)e_1^* + (\delta/2) \leq \epsilon \leq g(\delta)$,

$$\tau(\delta, \epsilon) = \tau_{\text{low}}(\delta, \epsilon).$$

To conclude, we have proved the unconditional security of an entanglement-sharing QKD protocol (the BBM92 protocol) with the use of practical detection apparatuses and with no assumption on the source, which establishes the prominent feature of the protocol — the built-in mechanism for detecting defects in the source. We chose to discard the double-click events, which enabled us to build up the proof from a very simple nonorthogonality relation [Eq. (19)] that holds regardless of the mode structure of incident photons. The proved secure key rate is higher than or almost the same as the rate conjectured for the random-bit-assignment protocol, and hence practical benefits of the BBM92 protocol discussed by Ma et al. [11] are now confirmed with unconditional security. The security proof is also applicable to a long-distance QKD using quantum repeaters [19].

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