Some improved bounds in sampling discretization of integral norms

F. Dai, E. Kosov, V. Temlyakov *

Abstract

The paper addresses a problem of sampling discretization of integral norms of elements of finite-dimensional subspaces satisfying some conditions. We prove sampling discretization results under a standard assumption formulated in terms of the Nikol’skii-type inequality. In particular, we obtain some upper bounds on the number of sample points sufficient for good discretization of the integral $L_p$ norms, $1 \leq p < 2$, of functions from finite-dimensional subspaces of continuous functions. Our new results improve upon the known results in this direction. We use a new technique based on deep results of Talagrand from functional analysis.

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1 Introduction

Let $\Omega$ be a compact subset of $\mathbb{R}^d$ with the probability measure $\mu$. By $L_p$ norm, $1 \leq p < \infty$, we understand

$$
\|f\|_p := \|f\|_{L_p(\Omega)} := \|f\|_{L_p(\Omega, \mu)} := \left( \int_{\Omega} |f|^p d\mu \right)^{1/p}.
$$

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By $L_\infty$ norm we understand the uniform norm of continuous functions
\[ \|f\|_\infty := \max_{x \in \Omega} |f(x)| \]
and with some abuse of notation we occasionally write $L_\infty(\Omega)$ for the space $C(\Omega)$ of continuous functions on $\Omega$.

By discretization of the $L_p$ norm we understand a replacement of the measure $\mu$ by a discrete measure $\mu_m$ with support on a set $\xi = \{\xi_j\}_{j=1}^m \subset \Omega$ in such a way that the error $\|f\|_{L_p(\Omega,\mu)}^p - \|f\|_{L_p(\Omega,\mu_m)}^p$ is small for functions from a given class. In this paper we focus on discretization of the $L_p$ norms of elements of finite-dimensional subspaces. Namely, we work on the following problem.

The Marcinkiewicz discretization problem. Let $\Omega$ be a subset of $\mathbb{R}^d$ with the probability measure $\mu$. We say that a linear subspace $X_n$ (the index $n$ here, usually, stands for the dimension of $X_n$) of $L_p(\Omega,\mu)$, $1 \leq p < \infty$, admits the Marcinkiewicz-type discretization theorem with parameters $m \in \mathbb{N}$ and $p$ and positive constants $C_1 \leq C_2$ if there exists a set
\[ \{\xi_j \in \Omega : j = 1, \ldots, m\} \]
such that for any $f \in X_n$ we have
\[ C_1 \|f\|_p^p \leq \frac{1}{m} \sum_{j=1}^m |f(\xi_j)|^p \leq C_2 \|f\|_p^p. \tag{1.1} \]

The Marcinkiewicz discretization problem with $\epsilon \in (0,1)$. We write $X_n \in \mathcal{M}(m,p,\epsilon)$ if (1.1) holds with $C_1 = 1 - \epsilon$ and $C_2 = 1 + \epsilon$. There are known results on the Marcinkiewicz discretization problem proved for subspaces $X_n$ satisfying some conditions. There are two types of conditions used in the literature: (I) Conditions on the entropy numbers and (II) Conditions in terms of the Nikol’skii-type inequalities. The reader can find a detailed discussion of known results in the very recent survey [14]. In this paper we only prove some discretization results under conditions (II). We now describe these conditions in detail.

Nikol’skii inequalities. Let $q \in [1, \infty)$ and $X_n \subset L_\infty(\Omega)$. The inequality
\[ \|f\|_\infty \leq C\|f\|_q, \quad \forall f \in X_n \tag{1.2} \]
is called the Nikol’skii inequality for the pair $(q, \infty)$ with the constant $C$. In this paper it is convenient for us to write the constant $C = \]
$(Kn)^{1/q}$. We obtain here discretization results under the Nikol’skii inequality for the pair $(2, \infty)$. In Section 3 we prove the following Theorem 1.1 which is one of the main results of the paper.

**Theorem 1.1.** There exists a positive absolute constant $C$ such that for any subspace $X_n$ of $C(\Omega)$ of dimension at most $n$ satisfying the Nikol’skii inequality

$$\|f\|_\infty \leq \sqrt{Kn}\|f\|_{L_2(\Omega, \mu)}, \quad \forall f \in X_n$$

for some probability measure $\mu$, and for any $\epsilon \in (0, 1)$, there is a finite set of points $\{\xi^1, \ldots, \xi^m\} \subset \Omega$ with

$$m \leq C\epsilon^{-2}Kn\log n,$$

which provides the following discretization inequalities for any $f \in X_n$

$$(1 - \epsilon)\|f\|_{L_1(\Omega, \mu)} \leq \frac{1}{m} \sum_{j=1}^{m} |f(\xi^j)| \leq (1 + \epsilon)\|f\|_{L_1(\Omega, \mu)},$$

$$(1 - \epsilon)\|f\|_{L_2(\Omega, \mu)}^2 \leq \frac{1}{m} \sum_{j=1}^{m} |f(\xi^j)|^2 \leq (1 + \epsilon)\|f\|_{L_2(\Omega, \mu)}^2.$$ 

Theorem 1.1 guarantees good discretization with equal weights under the Nikol’skii inequality for the pair $(2, \infty)$ with the bound on the number of points $m \leq C_1\epsilon^{-2}Kn\log n$. This covers the case of simultaneous discretization of the $L_1$ and $L_2$ norms. In the case of simultaneous discretization of the $L_p$, $1 < p < 2$, and $L_2$ norms the following Theorem 1.2, which is proved in Section 4, provides a little worse guarantees on the number of points for good discretization. Theorem 1.2 is the second main result of the paper.

**Theorem 1.2.** Let $1 < p < 2$. There exists a positive constant $C(p)$ such that for any subspace $X_n$ of $C(\Omega)$ of dimension at most $n$ satisfying the Nikol’skii inequality

$$\|f\|_\infty \leq \sqrt{Kn}\|f\|_{L_2(\Omega, \mu)}, \quad \forall f \in X_n$$

for some probability measure $\mu$, and for any $\epsilon \in (0, 1)$ there is a finite set of points $\{\xi^1, \cdots, \xi^m\} \subset \Omega$ with

$$m \leq C(p)\epsilon^{-2}Kn(\log(Kn) + \log(1/\epsilon))(\log(1/\epsilon) + \log \log(Kn))^2,$$
which provides the following discretization inequalities for any $f \in X_n$

\begin{align*}
(1 - \epsilon)\|f\|^p_{L_p(\Omega,\mu)} &\leq \frac{1}{m} \sum_{j=1}^{m} |f(\xi_j)|^p \leq (1 + \epsilon)\|f\|^p_{L_p(\Omega,\mu)} \quad \text{(1.7)} \\
(1 - \epsilon)\|f\|^2_{L_2(\Omega,\mu)} &\leq \frac{1}{m} \sum_{j=1}^{m} |f(\xi_j)|^2 \leq (1 + \epsilon)\|f\|^2_{L_2(\Omega,\mu)}. \quad \text{(1.8)}
\end{align*}

Several comments are in order.

**Comment 1.1.** We point out that one always has $K \geq 1$ in the Nikol’skii inequality assumed above.

**Comment 1.2.** Historical discussions on sampling discretization for the cases $p = 2$ and $1 \leq p < 2$ can be found in Subsections D.15 and D.16 of [14] respectively. Here we only mention that the best previously known results are the following. It was proved in [6] that under the Nikol’skii inequality for the pair $(2, \infty)$ we have $X_n \in M(m, p, \epsilon)$ for $1 \leq p < 2$ provided $m \geq C(p, K, \epsilon) n (\log n)^{3/2}$. This last estimate on $m$ was further improved to $m \geq C(p, K, \epsilon) n (\log n)^2$ for $1 < p < 2$ in [17]. We also point out that sampling discretization for $p > 2$ under the Nikol’skii inequality for the pair $(p, \infty)$ was studied in [17] as well, where the results were further improved for $p > 3$ in [8].

**Comment 1.3.** The Banach-Mazur distance between two finite dimensional spaces $X$ and $Y$ of the same dimension is defined to be

\[ d(X, Y) := \inf \{\|T\|\|T^{-1}\| : T \text{ linear isomorphism from } X \text{ to } Y\}. \]

In Section 5 of [3], the authors discuss the following important problem from functional analysis. Given an $n$ dimensional subspace $X_n$ of $L_q(0,1)$ and $\epsilon > 0$, what is the smallest positive integer $N = N(X_n, q, \epsilon)$ such that there is a subspace $Y_n$ of $\ell^q$ with $d(X_n, Y_n) \leq 1 + \epsilon$? Clearly, if $X_n \in M(m, q, \epsilon)$ then $N(X_n, q, C(q)\epsilon) \leq m$. On the other hand, however, results on the behavior of $N(X_n, q, \epsilon)$ do not seem to imply bounds on $m$ for $X_n \in M(m, q, \epsilon)$. Nevertheless, techniques developed for studying behavior of $N(X_n, q, \epsilon)$ turn out to be very useful for the Marcinkiewicz-type discretization. A detailed discussion of this connection can be found in Section 4.2 of [14]. Also, we refer to [10] for the history of this problem.

**Comment 1.4.** We give a brief description of the scheme of our proofs of Theorems 1.1 and 1.2. Let $X_n$ be a subspace of $L_\infty$ of dimension at
most \( n \). First, we establish results on simultaneous discretization of the \( L_2 \) and \( L_p \) norms. At this step we use \( M \leq C_p \epsilon^{-r_1} n (\log n)^{r_2} \) points for good discretization. This step allows us to reduce the original problem of discretization to a problem in \( \mathbb{R}^M \). Second, we use deep known results of Talagrand and Rudelson on the expectations of random vectors to reduce the number of points for good discretization by half (approximately). Third, we iterate the second step with an appropriate stopping time and finish the proof.

**Comment 1.5.** Clearly, a good discretization set \( \{\xi^j \in \Omega, j = 1, \ldots, m\} \) depends on the set \( \Omega \) and on the probability measure \( \mu \). For convenience, we will not indicate this fact in our further formulations.

**Comment 1.6.** We note that Theorems 1.1 and 1.2 combined with Lewis’ change of density theorem (see [19] or [31]) imply the following two statements concerning weighted discretization (see the proof of Theorem 2.3 of [6] for the detailed argument).

**Corollary 1.1.** There exists a positive absolute constant \( C \) such that for any subspace \( X_n \) of \( C(\Omega) \) of dimension at most \( n \), for any \( \epsilon \in (0,1) \) and any probability measure \( \mu \), there are a finite set of points \( \{\xi^1, \ldots, \xi^m\} \subset \Omega \) with

\[
m \leq C \epsilon^{-2} n \log n,
\]

and a set of nonnegative weights \( \{\lambda_j\}_{j=1}^m \) which provide the following discretization inequalities for any \( f \in X_n \):

\[
(1 - \epsilon) \|f\|_{L_1(\Omega,\mu)} \leq \sum_{j=1}^m \lambda_j |f(\xi^j)| \leq (1 + \epsilon) \|f\|_{L_1(\Omega,\mu)}. \quad (1.9)
\]

**Corollary 1.2.** Let \( 1 < p < 2 \). There exists a positive constant \( C(p) \) such that for any subspace \( X_n \) of \( C(\Omega) \) of dimension at most \( n \), for any \( \epsilon \in (0,1) \) and any probability measure \( \mu \), there are a finite set of points \( \{\xi^1, \ldots, \xi^m\} \subset \Omega \) with

\[
m \leq C(p) \epsilon^{-2} n (\log(n) + \log(1/\epsilon))(\log(1/\epsilon) + \log \log(n))^2,
\]

and a set of nonnegative weights \( \{\lambda_j\}_{j=1}^m \) which provide the following discretization inequalities for any \( f \in X_n \):

\[
(1 - \epsilon) \|f\|_{L_p(\Omega,\mu)}^p \leq \sum_{j=1}^m \lambda_j |f(\xi^j)|^p \leq (1 + \epsilon) \|f\|_{L_p(\Omega,\mu)}^p. \quad (1.10)
\]
The paper is organized as follows. Certain iteration techniques play an important role in our proofs of sampling discretization results. In Section 2, we prove several technical lemmas that will be needed in our later applications of these iteration techniques.

After that, in Section 3, we prove Theorem 1.1, the discretization theorem of the $L_1$ norm for the finite dimensional subspaces satisfying the Nikol’skii inequality between $L_\infty$ and $L_2$ norms. The proof combines an iteration technique with a deep result of Talagrand [32] stated in Theorem 3.1. The proof also relies on a technical lemma, Lemma 3.1, on preliminary simultaneous discretizations. While both Theorem 3.1 and Lemma 3.1 were essentially known previously, they were not clearly stated. As a result, we include a proof of Lemma 3.1 in Section 6 and a proof of Theorem 3.1 in Appendix I in Section 7 for the sake of completeness.

Section 4 is devoted to the proof of Theorem 1.2, the discretization result of the $L_p$ norm for $1 < p < 2$. Indeed, we prove a slight improvement of Theorem 1.2 in this section. The proof uses a similar iteration technique, however, the key ingredient to the iteration is Theorem 4.2, a version of Proposition 2.3 from Talagrand’s paper [33] for $1 < p < 2$ with uniform probability measure $\mu$ but weaker assumption on the $n$-dimensional space $X_n$ (see also Theorem 16.8.2 in [34]). Theorem 4.2 can be deduced by modifying the proof of Theorem 16.8.2 in [34]. Since the proof in [34] is very complicated and difficult, to make the paper relatively self-contained, we give a detailed proof of Theorem 4.2 in Appendix II in Section 8. Our intent there is to present both the result and the proof in their simplest possible form, with a slightly weaker assumption on the space $X_n$, so that readers who are not familiar with all the involved technicalities (e.g., the majorizing measure theorem of Talagrand) can follow the details easily.

In Section 5 we discuss a connection between sampling discretization of integral norms and frames. We formulate there (see Comment 5.2) a simple observation that the properties of a point set $\{\xi^\nu\}_{\nu=1}^m$ to provide the sampling discretization inequalities and to provide a subsystem, which is a frame, of the dictionary consisting of the Dirichlet kernels are equivalent. We formulate some direct corollaries of our main results on sampling discretization for construction of finite frames out of the infinite dictionary consisting of the Dirichlet kernels. In Subsection 5.2 we comment on the known results on sampling discretization of the uniform norm, which is closely connected with a special bilinear approximation of the Dirichlet kernels.
2 Iteration lemmas

Iteration techniques play an important role in our proofs. In this section we present the corresponding results. The following Lemma 2.1 from [26] was used in proving sampling discretization results of the $L_2$ norm.

**Lemma 2.1 ([26 Lemma 1]).** Let $0 < \delta < 1/100$, and let $\alpha_j, \beta_j, j = 0, 1, \ldots$, be defined inductively

\[
\alpha_0 = \beta_0 = 1, \quad \alpha_{j+1} := \alpha_j \frac{1 - 5\sqrt{\delta/\alpha_j}}{2}, \quad \beta_{j+1} := \beta_j \frac{1 + 5\sqrt{\delta/\alpha_j}}{2}.
\]

Then there exist a positive absolute constant $C$ and a number $L \in \mathbb{N}$ such that

\[
\alpha_j \geq 100\delta \quad \text{for} \ j \leq L, \quad 25\delta \leq \alpha_{L+1} < 100\delta, \quad \beta_{L+1} < C\alpha_{L+1}.
\]

We prove a version of Lemma 2.1, which is convenient for us.

**Lemma 2.2.** Let $\delta \in (0, 1/4)$ and $\theta \in (0, 1/2]$ be such that $\delta < \theta^2$. Consider the sequence

\[
\alpha_0 = 1, \quad \alpha_{j+1} = \frac{1}{2} \alpha_j (1 - (\delta/\alpha_j)^{1/2}),
\]

for $j = 0, \ldots, s$, where $s := s(\delta, \theta) \in \mathbb{N}_0$ is determined by the condition

\[
\alpha_s \geq \frac{\delta}{\theta^2}, \quad \alpha_{s+1} < \frac{\delta}{\theta^2}. \quad (2.1)
\]

Then we have the following inequalities

\[
2^{-s-1} \geq \alpha_{s+1} \geq \frac{\delta}{4\theta^2}, \quad (2.2)
\]

\[
\prod_{j=0}^{t} ((1 + \kappa(\delta/\alpha_j)^{1/2}) \leq \exp(c_1 2^{-(s-t)/2} \kappa \theta) \quad (2.3)
\]

and

\[
\prod_{j=0}^{t} ((1 - \kappa(\delta/\alpha_j)^{1/2}) \geq \exp(-c_2 2^{-(s-t)/2} \kappa \theta) \quad (2.4)
\]

for every $0 < \kappa < \frac{1}{2}\theta^{-1}$, every $t \in \{0, 1, \ldots, s\}$ and some positive absolute constants $c_1$ and $c_2$. 

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Proof. Consider the function \( g(x) := x - (\delta x)^{1/2}, x \in (0, 1) \). Then \( g'(x) = 1 - (\delta/x)^{1/2} > 0 \) for \( x > \delta/4 \). Therefore, using our assumption \( \alpha_s \geq \frac{\delta}{\theta} \geq 4\delta > \delta/4 \) we obtain

\[
\alpha_{s+1} = \frac{1}{2} \alpha_s (1 - (\delta/\alpha_s)^{1/2}) = \frac{1}{2} g(\alpha_s) \geq \frac{1}{2} g\left(\frac{\delta}{\theta}\right)
\]

\[
= \frac{1}{2} \left( \frac{\delta}{\theta^2} - \frac{\delta}{\theta} \right) = \frac{1}{2} \cdot \frac{\delta}{\theta^2} (1 - \theta) \geq \frac{\delta}{4\theta^2},
\]

which proves (2.2).

Using the trivial inequality \( \alpha_{j+1} \leq \alpha_j/2 \), we obtain from (2.1) that

\[
\alpha_j \geq \frac{\delta}{\theta^2} 2^{s-j}, \quad j = 0, \ldots, s.
\] (2.5)

This implies for \( j = 0, \ldots, s \)

\[
1 + \kappa (\delta/\alpha_j)^{1/2} \leq 1 + \kappa \theta 2^{-(s-j)/2} \leq \exp(\kappa \theta 2^{-(s-j)/2}).
\] (2.6)

Next,

\[
\sum_{j=0}^{t} 2^{-(s-j)/2} = \sum_{k=s-t}^{s} 2^{-k/2} \leq \sum_{k=s-t}^{\infty} 2^{-k/2}
\]

\[
= \frac{\sqrt{2}}{\sqrt{2} - 1} \cdot 2^{-(s-t)/2} = c_1 \cdot 2^{-(s-t)/2}, \quad c_1 := \frac{\sqrt{2}}{\sqrt{2} - 1},
\] (2.7)

and

\[
\prod_{j=0}^{t} ((1 + \kappa (\delta/\alpha_j)^{1/2}) \leq \exp(c_1 2^{-(s-t)/2} \kappa \theta),
\]

which proves (2.3).

Since \( 1 - x = (1 + \frac{x}{1-x})^{-1} \geq \exp(-\frac{x}{1-x}) \geq \exp(-2x) \) for every \( x \in (0, 1/2] \) then using (2.5), we obtain for \( 0 \leq j \leq s \)

\[
1 - \kappa (\delta/\alpha_j)^{1/2} \geq 1 - \kappa \theta 2^{-(s-j)/2} \geq \exp(-\kappa \theta 2^{-(s-2-j)/2}).
\]

Therefore,

\[
\prod_{j=0}^{t} ((1 - \kappa (\delta/\alpha_j)^{1/2}) \geq \exp(-c_2 2^{-(s-t)/2} \kappa \theta),
\]

which proves (2.4). \( \square \)
Lemma 2.3. Let $\{\alpha_j\}_{j=0}^{s+1}$ be from Lemma 2.2 and let $0 < \kappa < \frac{1}{2} \theta^{-1}$. Consider the sequences

$$a_0 = b_0 = 1, \quad a_{j+1} = \frac{1}{2} a_j (1 - \kappa (\delta/\alpha_j)^{1/2}), \quad b_{j+1} = \frac{1}{2} b_j (1 + \kappa (\delta/\alpha_j)^{1/2}),$$

$j = 0, \ldots, s$. Then

$$b_{t+1} \leq \exp(c_3 2^{-(s-t)/2} \kappa \theta) a_{t+1}$$

(2.8)

for every $t \in \{0, 1, \ldots, s\}$ for some positive absolute constant $c_3$. In particular, for $\kappa = 1$ and for the sequence

$$\beta_0 = 1, \quad \beta_{j+1} = \frac{1}{2} \beta_j (1 + (\delta/\alpha_j)^{1/2}), \quad j = 0, \ldots, s$$

one has

$$\beta_{t+1} \leq \exp(c_3 2^{-(s-t)/2} \theta) \alpha_{t+1}$$

(2.9)

for every $t \in \{0, 1, \ldots, s\}$ for some positive absolute constant $c_3$.

Proof. By the definition of $b_j$ we obtain from (2.3)

$$b_{t+1} = 2^{-t} \prod_{j=0}^{t} ((1 + \kappa (\delta/\alpha_j)^{1/2}) \leq 2^{-t} \exp(c_1 2^{-(s-t)/2} \kappa \theta).$$

Further, using the definition of the $a_j$ and (2.4) we get

$$a_{t+1} = 2^{-t} \prod_{j=0}^{t} ((1 - \kappa (\delta/\alpha_j)^{1/2}) \geq 2^{-t} \exp(-c_2 2^{-(s-t)/2} \kappa \theta).$$

(2.10)

Combining the above two inequalities, we complete the proof of (2.8). \qed

Finally, we prove one simple inequality for a recurrent sequence.

Lemma 2.4. Let the sequence $\{m_j\}_{j=0}^\infty$ of positive numbers satisfy the following conditions:

$$m_0 = M, \quad (m_j - (m_j)^{1/2})/2 \leq m_{j+1} \leq m_j/2, \quad j = 0, 1, \ldots$$

where $M > 0$ is a constant. Then for every integer $k \geq 0$ we have

$$M - 2^{k/2} M^{1/2} (\sqrt{2} - 1)^{-1} \leq 2^k m_k \leq M.$$
Proof. The right inequality is obvious. For the left inequality we have
\[ 2m_k \geq m_{k-1} - (m_{k-1})^{1/2} \geq m_{k-1} - (2^{-k+1}M)^{1/2}, \]
which implies
\[ 2^km_k \geq 2^{k-1}m_{k-1} - 2^{(k-1)/2}\sqrt{M} \quad \text{for } k = 1, 2, \ldots, \]
and hence
\[ 2^km_k \geq M - M^{1/2}2^{k/2} \sum_{j=1}^{k}2^{-j/2} \geq M - 2^{k/2}M^{1/2}(\sqrt{2} - 1)^{-1}. \]
The lemma is proved.

3 The case $p = 1$: proof of Theorem 1.1

In this section we prove the discretization theorem of the $L_1$ norm for the finite dimensional subspaces satisfying the Nikol’skii inequality between $L_\infty$ and $L_2$ norms. We now proceed to the detailed argument.

Proof of Theorem 1.1. Let $\epsilon \in (0, 1/4)$ be a fixed number and let $\epsilon_0 = \alpha_1\epsilon$ and $\theta = \alpha_2\epsilon$, where $\alpha_1, \alpha_2 \in (0, 1)$ are positive absolute constants, which will be specified later.

Step 1. Preliminary discretization. We need the following lemma, which can be deduced by combining certain estimates from [5, 6, 7].

Lemma 3.1. Let $1 \leq p < 2$ and $0 < \epsilon_0 < 1/4$. Let $X_n$ be a subspace of $L_\infty(\Omega)$ of dimension at most $n$ satisfying
\[ \|f\|_\infty \leq \sqrt{Kn}\|f\|_{L_2(\Omega)}, \quad \forall f \in X_n \quad (3.1) \]
for some constant $K \geq 1$. Then there exists a finite set of points $x_1, \cdots, x_m \in \Omega$ with $m \leq C_p\epsilon_0^{-8}Kn(\log(Kn))^3$ such that for any $f \in X_n$, we have
\[ (1 - \epsilon_0)\|f\|_{L_p(\Omega)}^p \leq \frac{1}{m} \sum_{j=1}^{m} |f(x_j)|^p \leq (1 + \epsilon_0)\|f\|_{L_p(\Omega)}^p, \quad (3.2) \]
and
\[ (1 - \epsilon_0)\|f\|_{L_2(\Omega)}^2 \leq \frac{1}{m} \sum_{j=1}^{m} |f(x_j)|^2 \leq (1 + \epsilon_0)\|f\|_{L_2(\Omega)}^2. \quad (3.3) \]
For the reader's convenience, we present a detailed proof of Lemma 3.1 in Section 6.

By Lemma 3.1, we can find a finite set \( \Lambda_M := \{x_1, \ldots, x_M\} \subset \Omega \) such that both (3.2) and (3.3) hold with \( m = M \) for all \( f \in X_n \). Then (3.1) implies

\[
\sup_{f \in X_n} \|f\|_\infty \|f\|_{2,\Lambda_M} \leq \sqrt{Kn} \sqrt{1 - \epsilon_0} \leq \sqrt{2Kn},
\]

where \( \|f\|_{2,\Lambda_M} := \left( \frac{1}{M} \sum_{i=1}^{M} |f(x_i)|^2 \right)^{1/2} \). Thus, without loss of generality, we may assume that \( \Omega = \Omega_M := \{1, 2, \ldots, M\} \) and \( \mu \) is the probability measure on \( \Omega_M \) given by \( \mu\{j\} = M^{-1} \) for \( 1 \leq j \leq M \).

**Step 2.** We identify each vector in \( \mathbb{R}^M \) with a function on the set \( \Omega_M := \{1, 2, \ldots, M\} \). For \( I \subset \Omega_M \) and \( f \in \mathbb{R}^M \), we define

\[
\|f\|_{p,I} := \left( \frac{1}{|I|} \sum_{j \in I} |f(j)|^p \right)^{1/p}, \quad 1 \leq p < \infty, \quad \text{and} \quad \|f\|_{\infty,I} := \max_{j \in I} |f(j)|,
\]

where \( |I| \) denotes the cardinality of the set \( I \). Let \( \|\cdot\|_p \) denote the usual norm of \( \ell^M_p \), that is, \( \|f\|_p := M^{1/p} \|f\|_{p,\Omega_M} \). Let \( B^M_p := \{ f \in \mathbb{R}^M : \|f\|_p \leq 1 \} \). For each \( I \subset \Omega_M \), we denote by \( R_I \) the orthogonal projection onto the space spanned by \( e_i, i \in I \), where \( e_1 = (1, 0, \ldots, 0), \ldots, e_M = (0, \ldots, 0, 1) \) is a canonical basis of \( \mathbb{R}^M \). Thus, for each \( f \in \mathbb{R}^M \), \( (R_I f)(j) = f(j) \) for \( j \in I \), and \( (R_I f)(j) = 0 \) for \( j \in \Omega_M \setminus I \). Throughout this note, \( \{\varepsilon_i : i = 1, 2, \ldots\} \) denotes a sequence of independent Bernoulli random variables taking values \( \pm 1 \) with probability 1/2.

**Theorem 3.1.** Let \( X_n \) be a subspace of \( \mathbb{R}^M \) of dimension at most \( n \) satisfying

\[
\|f\|_\infty \leq \sqrt{Kn} \|f\|_{2,\Omega_M} = \sqrt{\frac{Kn}{M}} \|f\|_2, \quad \forall f \in X_n. \tag{3.4}
\]

Then for \( p = 1 \) and \( p = 2 \) we have

\[
\mathbb{E} \left( \sup_{f \in X_n \cap B^M_p} \left( \sum_{i=1}^{M} |f(i)|^p \right) \right) \leq C_4 \sqrt{\frac{Kn \log n}{M}}, \tag{3.5}
\]

where \( C_4 \) is a positive absolute constant.

Several remarks on Theorem 3.1 are in order:

(i) Theorem 3.1 for \( p = 2 \) was proved by Rudelson [28, Lemma 1].
(ii) For $p = 1$, Theorem 3.1 was essentially proved by Talagrand [32], while not explicitly stated there (see also [18, Proposition 15.16] and [10, Theorem 13]).

For completeness, we will include a proof of Theorem 3.1 for $p = 1$ in Appendix I.

The following lemma, which is a consequence of Theorem 3.1, plays an important role in the proof of Theorem 1.1.

**Lemma 3.2.** Let $X_n$ be a subspace of $\mathbb{R}^M$ of dimension at most $n$ satisfying (3.4) for some constant $K \geq 1$. Let $J \subset \Omega_M := \{1, 2, \cdots, M\}$. Assume that there exist positive constants $\alpha_J, \beta_J$ such that for any $f \in X_n$ we have for both $p = 1$ and $p = 2$

$$\alpha_J \|f\|^p_p \leq \|R_J f\|^p_p \leq \beta_J \|f\|^p_p.$$  \hspace{1cm} (3.6)

Then there exists a subset $I \subset J$ with

$$\frac{|J|}{2} \left(1 - \frac{1}{\sqrt{|J|}}\right) \leq |I| \leq \frac{|J|}{2}$$ \hspace{1cm} (3.7)

such that for any $f \in X_n$ we have for both $p = 1$ and $p = 2$

$$\alpha_I \|f\|^p_p \leq \|R_I f\|^p_p \leq \beta_I \|f\|^p_p$$ \hspace{1cm} (3.8)

where

$$\alpha_I := \frac{(1 - \sigma_1)\alpha_J}{2}, \quad \beta_I := \frac{(1 + \sigma_1)\beta_J}{2}, \quad \sigma_1 := C_5 \sqrt{\frac{K n \log n}{\alpha_J M}},$$ \hspace{1cm} (3.9)

and $C_5$ is an absolute constant.

**Proof.** Without loss of generality, we may assume that $J = \{1, 2, \cdots, M_1\}$. By (3.4) and (3.6), we have

$$\sup_{f \in X_n} \|R_J f\|_\infty \leq \sup_{f \in X_n} \|R_J f\|_2 \leq K \sqrt{\frac{n \log n}{|J|}},$$ \hspace{1cm} (3.10)

where $K_J := K \frac{|J|}{\alpha_J M}$. Set $X_n(I) := \{R_I(f) : f \in X_n\}$. By Theorem 3.1 applied to $\Omega_{M_1} = J$ and $K = K_J$, we obtain for $p = 1$ and $p = 2$

$$\mathbb{E}\left(\sup_{f \in X_n(I) \cap \mathcal{B}_{p}^{M_1}} \left|\sum_{j=1}^{M_1} \varepsilon_j |f(j)|^p\right|\right) \leq C_4 \sqrt{\frac{K n \log n}{|J|}},$$

$$= C_4 \sqrt{\frac{K n \log n}{\alpha_J M}} := \frac{1}{8} \sigma_1.$$ \hspace{1cm} (3.11)
Next, consider the random set \( I := \{ i \in J : \varepsilon_i = 1 \} \). Clearly,
\[
|I| = \sum_{i \in J} \frac{\varepsilon_i + 1}{2} = \frac{|J|}{2} + \frac{1}{2} \sum_{i \in J} \varepsilon_i.
\]
We use the following result from [16].

**Lemma 3.3.** [16] If \((a_1, \cdots, a_m) \in \mathbb{R}^m\) and \(\sum_{j=1}^m a_j^2 = 1\), then
\[
\mathbb{P}\left(\left|\sum_{j=1}^m a_j \varepsilon_j\right| \leq 1\right) \geq \frac{1}{2}.
\]
(3.12)

Using (3.12), we have
\[
\mathbb{P}\left(-\frac{\sqrt{|J|}}{2} \leq |I| - \frac{|J|}{2} \leq 0\right) = \mathbb{P}\left(-\sqrt{M_1} \leq \sum_{i=1}^{M_1} \varepsilon_i \leq 0\right)
\]
\[
= \frac{1}{2} \mathbb{P}\left(|\sum_{i=1}^{M_1} \varepsilon_i| \leq \sqrt{M_1}\right) \geq \frac{1}{4}.
\]
This means that with probability \(\geq \frac{1}{4}\), we have
\[
\frac{|J|}{2} \left(1 - \frac{1}{\sqrt{|J|}}\right) \leq |I| \leq \frac{|J|}{2}.
\]
(3.13)

Now combining (3.11) with (3.13), we can find a finite sequence \(\{\varepsilon_j : 1 \leq j \leq M_1\} \subset \{\pm 1\}\) such that (3.13) with \(I := \{ i \in J : \varepsilon_i = 1 \}\) is satisfied, and such that for every \(f \in X_n\),
\[
\left|\sum_{i=1}^{M_1} \varepsilon_i |f(i)|^2\right| \leq \sigma_1 \|R_I f\|_2^2 \quad \text{and} \quad \sum_{i=1}^{M_1} \varepsilon_i |f(i)|^p \leq \sigma_1 \|R_I f\|_1.
\]
(3.14)

On the other hand, note that for any \(x \in \mathbb{R}^M\) we have for \(1 \leq q < \infty\),
\[
\left|\sum_{i=1}^{M_1} \varepsilon_i |x(i)|^q\right| = \left|\sum_{i=1}^{M_1} (1 + \varepsilon_i) |x(i)|^q - \|R_J x\|_q^q\right| = \left|2\|R_I x\|_q^q - \|R_J x\|_q^q\right|.
\]
Thus, we obtain from (3.14) that for \(p = 1, 2\)
\[
\frac{1 - \sigma_1}{2} \sum_{i \in J} |f(i)|^p \leq \|R_I f\|_p^p \leq \frac{1 + \sigma_1}{2} \sum_{i \in J} |f(i)|^p.
\]
(3.15)

Combining (3.15) with (3.6), we obtain the desired estimate (3.8). \(\square\)
Step 3. Iteration. For given numbers $K$, $n$, and $M$ define

$$\delta := \delta(K, n, M) := C_5^2 \frac{Kn \log n}{M},$$

where $C_5$ is from (3.9). Recall that $\theta = \varkappa_2 \varepsilon$. Without loss of generality we may assume that $\delta \leq \theta^2$, since otherwise we already have $M \leq C_5^2 \varkappa_2^{-2} \varepsilon^{-2} Kn \log n$ and we have discretization with $\epsilon_0 \leq \epsilon$ on the first step. Consider the sequence $\{\alpha_j\}_{j=0}^{s+1}$ from Lemma 2.2 with $\theta = \varkappa_2 \varepsilon$. We now iterate applications of Lemma 3.2. We begin with $I_0 := \{1, 2, \ldots, M\}$ and obtain the sequence $\{I_j\}_{j=0}^{s+1}$. Let the sequence $\{\beta_j\}_{j=0}^{s+1}$ be from Lemma 2.3 (i.e. we take $\varkappa = 1$). Then by Lemma 3.2 we obtain for any $f \in X_n$ and $p = 1, 2$

$$\alpha_j \|f\|_p \leq \|R_{I_j} f\|_p \leq \beta_j \|f\|_p, \quad j = 0, \ldots, s + 1. \tag{3.16}$$

By Lemma 2.3 we obtain

$$\alpha_{s+1} \|f\|_p \leq \|R_{I_{s+1}} f\|_p \leq e^{c_3 \theta} \alpha_{s+1} \|f\|_p. \tag{3.17}$$

Note that Lemma 2.3 and the trivial inequalities $\alpha_j \leq \alpha_j / 2$, $\beta_j \geq \beta_j / 2$ imply

$$\alpha_{s+1} \leq 2^{-s-1} \leq \beta_{s+1} \leq e^{c_3 \theta} \alpha_{s+1}. \tag{3.18}$$

Set $m_j := |I_j|, j = 0, \ldots, s + 1$. Then, by Lemma 2.4 we obtain

$$M - 2^{(s+1)/2} M^{1/2}(\sqrt{2} - 1)^{-1} \leq 2^{s+1} m_{s+1} \leq M. \tag{3.19}$$

Set $m := m_{s+1}$. Under assumption $\delta \geq 4(\sqrt{2} - 1)^{-2} M^{-1}$, which we certainly can impose without loss of generality, we obtain from (3.18) and (2.2) that

$$4\theta^2 \cdot 2^{-s-1} \geq 4\theta^2 \cdot \alpha_{s+1} \geq 4\theta^2 \cdot \frac{\delta}{4\theta^2} = \delta \geq 4(\sqrt{2} - 1)^{-2} M^{-1}.$$

Thus, $2^{(s+1)/2} M^{-1/2} \leq \theta(\sqrt{2} - 1)$ and, combining this bound with (3.19) and with (3.18), we obtain

$$\alpha_{s+1} (1 - \theta) M \leq 2^{-s-1} (1 - \theta) M \leq m \leq 2^{-s-1} M \leq e^{c_3 \theta} \alpha_{s+1} M. \tag{3.20}$$

Finally, (3.17) and (3.20) imply

$$e^{-c_3 \theta} \frac{1}{M} \|f\|_p \leq \frac{1}{m} \sum_{k \in I_{s+1}} |f(k)|^p \leq (1 - \theta)^{-1} e^{c_3 \theta} \frac{1}{M} \|f\|_p.$$
with
\[ m = |I_{s+1}| \leq 2^{-s-1}M \leq e^{c_3\theta^2} \alpha_{s+1} M \leq e^{c_3\theta^2 - 2\delta M} = C_5 \kappa_2 e^{c_3\kappa_2 \epsilon} e^{-2K_n \log n} \leq C_8 \epsilon^{-2} K_n \log(n). \]

We now choose \( \kappa_1 \) and \( \kappa_2 \) such that
\[
1 - \epsilon \leq (1 - \epsilon_0) e^{-c_3\theta} \text{ and } (1 + \epsilon_0)(1 - \theta)^{-1} e^{c_3\theta} \leq 1 + \epsilon.
\]

Theorem is proved.

4 The case \( 1 < p < 2 \): proof of Theorem 1.2

The main purpose of this section is to prove Theorem 1.2. The argument follows along the lines of Section 3. Indeed, we will prove a slight improvement of Theorem 1.2.

**Theorem 4.1.** Let \( X_n \) be a subspace of \( C(\Omega) \) of dimension at most \( n \) satisfying the Nikol’skii inequality
\[
\| f \|_{\infty} \leq \sqrt{K_n} \| f \|_{L^2(\Omega)}, \quad \forall f \in X_n \tag{4.1}
\]
for some constant \( K \geq 1 \). Then for any \( 1 < p < 2 \), there exists a positive constant \( C_1(p) \) depending only on \( p \) such that for any \( \epsilon \in (0, 1/4) \), there is a finite set of points \( \{ \xi_1, \ldots, \xi_m \} \subset \Omega \) with
\[
m \leq C_1(p) \epsilon^{-2} K_n \left( \log(K_n) + \log(1/\epsilon) \right) \left( \log(1/\epsilon) + \log \log(K_n) \right)^2,
\]
which provides the discretization inequalities for both the \( L_p \) norm and the \( L_2 \) norm:
\[
(1 - \epsilon) \| f \|_{L^p(\Omega)}^p \leq \frac{1}{m} \sum_{j=1}^m |f(x_j)|^p \leq (1 + \epsilon) \| f \|_{L^p(\Omega)}^p, \quad \forall f \in X_n,
\]
\[
(1 - \epsilon_n) \| f \|_{L^2(\Omega)}^2 \leq \frac{1}{m} \sum_{j=1}^m |f(x_j)|^2 \leq (1 + \epsilon_n) \| f \|_{L^2(\Omega)}^2, \quad \forall f \in X_n,
\]
with \( \epsilon_n = \epsilon \cdot (\log \log(2K_n))^{-1} \).
Proof. Let $\varepsilon \in (0, 1/4)$ be a fixed number, and set

$$\varepsilon_0 := \frac{\kappa_1 \varepsilon}{\log \log (4Kn)},$$

where $\kappa_1 \in (0, 1)$ is an absolute constant to be specified later.

Step 1p. This step is the same as in the proof of Theorem 1.1. We use Lemma 3.1 to obtain a finite set $\Lambda_M := \{x_1, \ldots, x_M\} \subset \Omega$ with $M \leq C_p \varepsilon_0^8 Kn (\log (Kn))^3$ such that both (3.2) and (3.3) hold with $m = M$ for all $f \in X_n$. Without loss of generality, we may also assume that

$$M \geq C_9 \varepsilon^{-2} Kn (\log (Kn) + \log (1/\varepsilon)) (\log (1/\varepsilon) + \log \log (Kn))^2,$$

(4.2)
since otherwise there’s nothing to prove.

Step 2p. It is similar to Step 2 of the proof of Theorem 1.1. Instead of Theorem 3.1 we use the following result.

Theorem 4.2. Let $X_n$ be a subspace of $\mathbb{R}^M$ of dimension at most $n$ satisfying

$$\|f\|_\infty \leq \sqrt{Kn} \|f\|_{2, \Omega_M} = \sqrt{\frac{Kn}{M}} \|f\|_2, \quad \forall f \in X_n$$

(4.3)

for some constant $K \geq 1$. Then for $p \in (1, 2)$, we have

$$\mathbb{E} \left( \sup_{f \in X_n \cap B_{p}^M} \left| \sum_{i=1}^{M} |f(i)|^p \varepsilon_i \right| \right) \leq C(p) \sqrt{\frac{Kn \log M}{M}} \log \left( \frac{M}{Kn} + 2 \right).$$

(4.4)

For $1 < p < 2$, Talagrand [34, Theorem 16.8.2] proved a more general result for a probability measure $\mu$ on $\mathbb{R}^M$ satisfying $\mu \{j\} \leq \frac{2}{M}$ for $1 \leq j \leq M$ under a stronger assumption on the space $X_n$:

$$\frac{1}{n} \sum_{j=1}^{n} \varphi_j(i)^2 = 1, \quad i = 1, 2, \ldots, M,$$

where $\{\varphi_j\}_{j=1}^{n}$ is an orthonormal basis of $(X_n, \| \cdot \|_{L_2(\mu)})$. The proof of Talagrand [34, Theorem 16.8.2] is very difficult, but can be modified and slightly simplified to obtain Theorem 4.2. For completeness, we will present a relatively self-contained proof of Theorem 4.2 in Appendix II.

In the same way as Lemma 3.2 has been proven in Section 3, we can prove the following result.
Lemma 4.1. Let $X_n$ be a subspace of $\mathbb{R}^M$ of dimension at most $n$ satisfying (4.3) for some constant $K \geq 1$. Let $1 < p < 2$ and $J \subset \Omega_M := \{1, 2, \cdots, M\}$. Assume that there exist positive constants $\alpha_J, \beta_J, a_J, b_J$ such that for any $f \in X_n$,

$$\alpha_J \|f\|_2^2 \leq \|R_J f\|_2^2 \leq \beta_J \|f\|_2^2 \quad \text{and} \quad a_J \|f\|_p^p \leq \|R_J f\|_p^p \leq b_J \|f\|_p^p. \quad (4.5)$$

Then there exists a subset $I \subset J$ with

$$\frac{|J|}{2} \left(1 - \frac{1}{\sqrt{|J|}}\right) \leq |I| \leq \frac{|J|}{2} \quad (4.6)$$

such that

$$\alpha_I \|f\|_2^2 \leq \|R_I f\|_2^2 \leq \beta_I \|f\|_2^2 \quad \text{and} \quad a_I \|f\|_p^p \leq \|R_I f\|_p^p \leq b_I \|f\|_p^p, \quad (4.7)$$

where

$$\alpha_I := \frac{(1 - \sigma_1)\alpha_J}{2}, \quad \beta_I := \frac{(1 + \sigma_1)\beta_J}{2}, \quad \sigma_1 := C_5 \sqrt{\frac{Kn \log n}{\alpha_J M}},$$

$$a_I := \frac{(1 - \sigma_2)a_J}{2}, \quad b_I := \frac{(1 + \sigma_2)b_J}{2}, \quad \sigma_2 := C_p \sqrt{\frac{Kn \log |J|}{\alpha_J M} \log \left(2 + \frac{M}{Kn}\right)},$$

and $C_p$ is a constant depending only on $p$.

We point out that

$$\sigma_1 \leq C \sqrt{\frac{Kn \log M}{\alpha_J M}} := \sigma'_1, \quad \sigma_2 \leq C \sqrt{\frac{Kn \log M}{\alpha_J M} \log \left(2 + \frac{M}{Kn}\right)} := \sigma'_2,$$

where $C := \max\{C_5, C_p\}$. Thus, we can use $\sigma'_1$ and $\sigma'_2$ in place of $\sigma_1$ and $\sigma_2$ in the lemma above.

**Step 3p.** As in the proof of Theorem 1.1, we iterate applications of Lemma 4.1 and obtain sequences $\{\alpha_j\}_{j=0}^{s+1}, \{\beta_j\}_{j=0}^{s+1}, \{a_j\}_{j=0}^{s+1}, \{b_j\}_{j=0}^{s+1}$ as defined in Lemma 2.2 and Lemma 2.3 with

$$\delta := C^2 \frac{Kn \log M}{M}, \quad \kappa := \log \left(2 + \frac{M}{Kn}\right) \quad \text{and} \quad \theta := \frac{\kappa \epsilon}{\kappa},$$
where $\varkappa_2 \in (0, 1)$ is an absolute constant to be specified later. Recalling that

$$M \leq C_p \epsilon_0^{-8} K n (\log(Kn))^3 \quad \text{and} \quad \epsilon_0 = \frac{\varkappa_1 \epsilon}{\log \log(4Kn)},$$

we may choose the constant $C_9$ in (4.2) sufficiently large so that the parameters $\delta$, $\theta$ and $\varkappa$ satisfy all the conditions of Lemma 2.2 and Lemma 2.3; that is,

$$\delta \in (0, \frac{1}{4}), \quad \theta \in (0, \frac{1}{2}), \quad \delta < \theta^2, \quad 0 < \varkappa < \frac{1}{2}\theta^{-1}.$$ 

Thus, we have

$$a_{s+1} \| f \|^p_p \leq \| R_{I_{s+1}} f \|^p_p \leq b_{s+1} \| f \|^p_p \leq \exp(c_3 \varkappa \theta) a_{s+1} \| f \|^p_p;$$

$$\alpha_{s+1} \| f \|^2_2 \leq \| R_{I_{s+1}} f \|^2_2 \leq \beta_{s+1} \| f \|^2_2 \leq \exp(c_3 \theta) \alpha_{s+1} \| f \|^2_2.$$ 

As in the proof of Theorem 1.1, for $m = m_{s+1} = |I_{s+1}|$ we have

$$2^{-s-1}(1 - \varkappa \theta) M \leq 2^{-s-1}(1 - \theta) M \leq m \leq 2^{-s-1} M.$$ 

Since

$$\alpha_{s+1} \leq 2^{-s-1} \leq \beta_{s+1} \leq e^{c_3 \theta} \alpha_{s+1}; \quad a_{s+1} \leq 2^{-s-1} \leq b_{s+1} \leq e^{c_3 \varkappa \theta} a_{s+1},$$

we get

$$e^{-c_3 \varkappa \theta} \frac{1}{M} \| f \|^p_p \leq \frac{1}{m} \sum_{k \in I_{s+1}} |f(k)|^p \leq (1 - \varkappa \theta)^{-1} e^{-c_3 \varkappa \theta} \frac{1}{M} \| f \|^p_p$$

and

$$e^{-c_3 \theta} \frac{1}{M} \| f \|^2_2 \leq \frac{1}{m} \sum_{k \in I_{s+1}} |f(k)|^2 \leq (1 - \theta)^{-1} e^{-c_3 \theta} \frac{1}{M} \| f \|^2_2$$

with

$$m = |I_{s+1}| \leq 2^{-s-1} M \leq e^{c_3 \theta} \alpha_{s+1} M \leq e^{c_3 \theta} \theta^{-2} \delta M.$$ 

We now choose $\varkappa_1$ and $\varkappa_2$ such that

$$1 - \epsilon \leq (1 - \varkappa_1 \epsilon) e^{-c_3 \varkappa_2 \epsilon} \quad \text{and} \quad (1 + \varkappa_1 \epsilon)(1 - \varkappa_2 \epsilon) e^{-c_3 \varkappa_2 \epsilon} \leq 1 + \epsilon.$$ 

Since $\log M \leq c_1(\log \epsilon^{-1} + \log(Kn))$ and $\varkappa \leq c_2(\log \epsilon^{-1} + \log \log(Kn))$ we get the desired result. Theorem is proved. \qed
In this section we discuss finite-dimensional subspaces $X_n$ of the space $C(\Omega)$ defined on a compact set $\Omega \subset \mathbb{R}^d$ equipped with a probability measure $\mu$. For convenience we only consider the case of real functions.

**Dirichlet kernel.** For an orthonormal system $U_n := \{u_j(x)\}_{j=1}^n$ on $(\Omega, \mu)$ we define the Dirichlet kernel as follows

$$D_n(U_n, x, y) := \sum_{j=1}^n u_j(x)u_j(y).$$

Here is a very simple claim that the Dirichlet kernel $D_n(U_n, x, y)$ does not depend on the orthonormal basis of a given subspace $X_n$.

**Proposition 5.1.** For any two orthonormal bases $U_n$ and $V_n$ of a given subspace $X_n$ we have

$$D_n(U_n, x, y) = D_n(V_n, x, y).$$

**Proof.** For $x \in \Omega$ denote the column vectors $u(x) := (u_1(x), \ldots, u_n(x))^T$, $v(x) := (v_1(x), \ldots, v_n(x))^T$. Then there exists an orthogonal matrix $O$ such that for all $x \in \Omega$ we have $v(x) = Ou(x)$ and, therefore, $v(x)^T = u(x)^TO^T$. This implies that

$$D_n(V_n, x, y) = v(x)^Tv(y) = u(x)^TO^Tu(y) = u(x)^Tu(y) = D_n(U_n, x, y).$$

Proposition 5.1 shows that the Dirichlet kernel $D_n(U_n, x, y)$ with $U_n$ being an orthonormal basis of $X_n$ can be seen as a characteristic of the subspace $X_n$. Denote

$$D(X_n, x, y) := D_n(U_n, x, y).$$

Consider the system $D := \{D(X_n, x, y)\}_{x \in \Omega}$ as a dictionary (not normalized) of functions on $y$ in the subspace $X_n$.

We recall the definition of the $p$-frame of the subspace $X_n$, $1 \leq p < \infty$ (see [1]).

**Definition 5.1.** The system $\Psi := \{\psi_j\}_{j=1}^m$ is said to be a $p$-frame of $X_n$ with positive constants $A$ and $B$ if for any $f \in X_n$ we have

$$A\|f\|_{L_p(\Omega, \mu)}^p \leq \sum_{j=1}^m |\langle f, \psi_j \rangle|^p \leq B\|f\|_{L_p(\Omega, \mu)}^p.$$
Using a well known property of the Dirichlet kernel: For any \( f \in X_n \) we have

\[
f(x) = \int_{\Omega} D(X_n, x, y) f(y)d\mu(y),
\]
we derive from results of Sections 3 and 4 the following corollaries.

**Theorem 5.1.** There exist three positive absolute constants \( C_i, i = 1, 2, 3 \) such that for any \( n \)-dimensional subspace \( X_n \) of \( C(\Omega) \) satisfying the Nikol’skii inequality

\[
\|f\|_{\infty} \leq \sqrt{K_n}\|f\|_{L_2(\Omega)}, \quad \forall f \in X_n \tag{5.1}
\]
for some constant \( K \geq 1 \), there is a finite set of points \( \xi^1, \ldots, \xi^m \in \Omega \) with \( m \leq C_1 Kn \log n \) such that the finite subdictionary \( \Psi := \{D(X_n, \xi^\nu, y)\}_{\nu=1}^m \) of dictionary \( D \) forms \( p \)-frames of the \( X_n \) with constants \( C_2m \) and \( C_3m \) for \( p = 1 \) and \( p = 2 \).

**Theorem 5.2.** Let \( 1 < p < 2 \). There exist three positive constants \( C'_i, i = 1, 2, 3 \) (\( C'_1 \) may depend on \( p \) and \( C'_2 \) and \( C'_3 \) are absolute constants) such that for any \( n \)-dimensional subspace \( X_n \) of \( C(\Omega) \) satisfying the Nikol’skii inequality

\[
\|f\|_{\infty} \leq \sqrt{K_n}\|f\|_{L_2(\Omega)}, \quad \forall f \in X_n \tag{5.2}
\]
there is a finite set of points \( \xi^1, \ldots, \xi^m \in \Omega \) with

\[
m \leq C'_1 Kn \log(Kn)(\log \log(Kn))^2
\]
such that the finite subdictionary \( \Psi := \{D(X_n, \xi^\nu, y)\}_{\nu=1}^m \) of dictionary \( D \) forms a \( p \)-frame of the \( X_n \) with constants \( C'_2m \) and \( C'_3m \) and forms a 2-frame of the \( X_n \) with constants \( (1 - \epsilon_n)m \) and \( (1 + \epsilon_n)m \), where

\[
\epsilon_n \asymp (\log \log(Kn))^{-1}.
\]

**Comment 5.1.** Theorem 5.1 is a corollary of Theorem 1.1. If instead of Theorem 1.1 we use the known results from [21] on the sampling discretization of the \( L_2 \) norm, then we obtain a version of Theorem 5.1 about a 2-frame: Under condition 5.1 there exists a finite subdictionary \( \Psi := \{D(X_n, \xi^\nu, y)\}_{\nu=1}^m \) of dictionary \( D \), which forms a 2-frame of the \( X_n \) with constants \( C_2m \) and \( C_3m \) with \( m \leq C_1 Kn \).

**Comment 5.2.** We pointed out that Theorems 5.1 and 5.2 about frames are corollaries of Theorems 1.1 and 4.1 on the sampling discretization. Actually, these problems are equivalent. Namely, the following two statements are equivalent.
Statement 1. The set of points \( \{\xi^\nu\}_{\nu=1}^m \) is such that for all \( f \in X_n \) we have the sampling discretization inequalities
\[
A \|f\|^p \leq \frac{1}{m} \sum_{\nu=1}^m |f(\xi^\nu)|^p \leq B \|f\|^p.
\]

Statement 2. The set of points \( \{\xi^\nu\}_{\nu=1}^m \) is such that the subsystem \( \Psi := \{D(X_n, \xi^\nu, y)\}_{\nu=1}^m \) of the system \( D \) forms a \( p \)-frame of the \( X_n \) with constants \( A_m \) and \( B_m \).

Comment 5.3. We have discussed above a connection between sampling discretization with equal weights of the \( L_p \) norm of functions from a subspace \( X_n \) and the \( p \)-frame properties of a subsystem of the system generated by the corresponding Dirichlet kernel. In the sampling discretization theory we study weighted discretization along with discretization with equal weights. This motivates us to introduce a concept of \( (\Lambda, p) \)-frame. Here is a rigorous definition.

**Definition 5.2.** Let \( \Lambda := \{\lambda_j\}_{j=1}^m \) and \( 1 \leq p < \infty \). The system \( \Psi := \{\psi_j\}_{j=1}^m \) is said to be a \( (\Lambda, p) \)-frame of \( X_n \) with positive constants \( A \) and \( B \) if for any \( f \in X_n \) we have
\[
A \|f\|_{L_p(\Omega, \mu)}^p \leq \sum_{j=1}^m \lambda_j |\langle f, \psi_j \rangle|^p \leq B \|f\|_{L_p(\Omega, \mu)}^p.
\]

Then the following two statements are equivalent.

**Statement 1w.** The sets of points \( \{\xi^\nu\}_{\nu=1}^m \) and weights \( \Lambda := \{\lambda^\nu\}_{\nu=1}^m \) are such that for all \( f \in X_n \) we have the weighted sampling discretization inequalities
\[
A \|f\|^p \leq \sum_{\nu=1}^m \lambda^\nu |f(\xi^\nu)|^p \leq B \|f\|^p.
\]

**Statement 2w.** The sets of points \( \{\xi^\nu\}_{\nu=1}^m \) and weights \( \Lambda := \{\lambda^\nu\}_{\nu=1}^m \) are such that the subsystem \( \Psi := \{D(X_n, \xi^\nu, y)\}_{\nu=1}^m \) of the system \( D \) forms a \( (\Lambda, p) \)-frame of the \( X_n \) with constants \( A \) and \( B \).

Comment 5.4. Let \( U_n := \{u_j(x)\}_{j=1}^N \) be an orthonormal basis of \( X_n \) on \( (\Omega, \mu) \). It is well known and easy to see that
\[
\sup_{f \in X_n, \|f\|_2 \leq 1} |f(x)| = \left( \sum_{j=1}^n u_j(x)^2 \right)^{1/2} =: w(x).
\]
The function $w(x)^{-1}$ is known as the Christoffel function of the subspace $X_n$. Clearly,

$$w(x)^2 = D(X_n, x, x).$$

Thus, the conditions (5.1) and (5.2) in Theorems 5.1 and 5.2 can be formulated as

$$D(X_n, x, x) \leq Kn, \quad x \in \Omega.$$

Note, that the condition $w(x)^2 \leq nt^2, \quad x \in \Omega,$ is known in the sampling discretization theory under the name Condition E (see, for instance, [14]).

Comment 5.5. It is known (see [13]) that sampling discretization of the uniform norm of functions from $X_n$ is connected to a special type of bilinear approximation of the Dirichlet kernel $D(X_n, x, y)$ of this subspace. We refer the reader to [13] for a detailed discussion of the corresponding results.

6 Proof of Lemma 3.1

For the proof of Lemma 3.1 we need several technical lemmas from [5, 6, 7]. First, we need a conditional result from [7], which allows us to estimate the number of points needed for the sampling discretization in terms of an integral of $\varepsilon$-entropy. Recall that given a positive number $\varepsilon$, the $\varepsilon$-entropy $H_\varepsilon(A, X)$ of the compact set $A$ in a Banach space $(X, \| \cdot \|_X)$ is defined as

$$N_\varepsilon(A, X) := \min \left\{ n \in \mathbb{N} : \exists g^1, \ldots, g^n \in A, \sup_{f \in A} \min_{1 \leq j \leq n} \|f - g^j\|_X \leq \varepsilon \right\}.$$

**Lemma 6.1.** [7, Theorem 5.1] Let $W$ be a set of uniformly bounded functions on $\Omega$ with

$$1 \leq R := \sup_{f \in W} \sup_{x \in \Omega} |f(x)| < \infty.$$

Assume that $H_\varepsilon(W, L_\infty) < \infty$ for every $t > 0$, and

$$(\lambda \cdot W) \cap B_{L_p} \subset W \subset B_{L_p}, \quad \forall \lambda > 0$$

for some $1 \leq p < \infty$, where $B_{L_p} := \{ f \in L_p(\Omega) : \|f\|_p \leq 1 \}$. Then there exist positive constants $C_p, c_p$ depending only on $p$ such that for any $\varepsilon \in (0, 1)$ and any integer

$$m \geq C_p \varepsilon^{-5} \left( \int_{10^{-1} \varepsilon^{1/p}}^R u^{\frac{2}{p} - 1} \left( \int_u^R \frac{H_{c_p \varepsilon t}(W, L\infty)}{t} dt \right)^{\frac{1}{2}} du \right)^2,$$

(6.2)
there exist \( m \) points \( x_1, \ldots, x_m \in \Omega \) such that for all \( f \in W \),

\[
(1 - \varepsilon) \|f\|^p_p \leq \frac{1}{m} \sum_{j=1}^{m} |f(x_j)|^p \leq (1 + \varepsilon) \|f\|^p_p.
\] (6.3)

**Remark 6.1.** The proof in [7] actually yields the following result. Let \( x_1, \ldots, x_m \) be independent random points identically distributed according to the probability measure \( \mu \) on \( \Omega \). Then under the conditions of Lemma 6.1, the inequalities (6.3) hold for all \( f \in W \) with probability > \( \frac{3}{4} \).

We will also need the following estimates of entropy numbers from [6].

**Lemma 6.2.** [6, Theorem 2.1] Assume that \( X_n \) is an \( n \)-dimensional subspace of \( L_\infty(\Omega) \) satisfying the following two conditions:

\begin{enumerate}[(i)]
  \item There exists a constant \( K_1 > 1 \) such that
  \[
  \|f\|_\infty \leq (K_1 n)^\frac{1}{p} \|f\|_2, \quad \forall f \in X_n.
  \] (6.4)
  \item There exists a constant \( K_2 > 1 \) such that for \( q_n := \log n \),
  \[
  \|f\|_\infty \leq K_2 \|f\|_{q_n}, \quad \forall f \in X_n.
  \] (6.5)
\end{enumerate}

Then for each \( 1 \leq p \leq 2 \), there exists a constant \( C_p > 0 \) depending only on \( p \) such that

\[
\mathcal{H}_t(X^p_n; L_\infty) \leq C_p K_1 K_2^{\frac{1}{p}} \frac{n \log n}{t^p}, \quad \forall t > 0,
\] (6.6)

where \( X^p_n := \{ f \in X_n : \|f\|_p \leq 1 \} \).

Note that Lemma 3.1 can be deduced directly from Lemma 6.1, Remark 6.1 and Lemma 6.2 under the additional assumption (6.5). To drop the extra condition (6.5), we need the following lemma proved in [5].

**Lemma 6.3.** [5, Lemma 4.3] Let \( 1 \leq p < \infty \) and \( 0 < \varepsilon < 1/4 \). Let \( X_n \) be an \( n \)-dimensional subspace of \( L_\infty(\Omega) \) satisfying

\[
\|f\|_\infty \leq (Kn)^\frac{1}{p} \|f\|_{L_p(\Omega)}, \quad \forall f \in X_n
\]

for some constant \( K \geq 1 \). Let \( \{\xi_j\}_{j=1}^\infty \) be a sequence of independent random points identically distributed in accordance with \( \mu \) on the set \( \Omega \). Then there exists an absolute constant \( C > 0 \) such that for any integer

\[
m \geq CK \varepsilon^{-2} \left( \log \frac{2}{\varepsilon} \right) n^2 \log n,
\] (6.7)
with probability \( \geq 1 - m^{-n/\log K} \), one has

\[
(1 - \varepsilon)\|f\|_p^p \leq \frac{1}{m} \sum_{j=1}^{m} |f(\xi_j)|^p \leq (1 + \varepsilon)\|f\|_p^p, \quad \forall f \in X_n. \tag{6.8}
\]

**Proof of Lemma 3.1.** Without loss of generality, we may assume that \( \log K \leq C \log n \) for some absolute constant \( C > 1 \) since otherwise we may use \( Kn \) to replace \( n \), considering \( X_n \) as a subspace of dimension at most \( Kn \). We may also assume that \( 0 < \varepsilon < \varepsilon_0 \) and \( n \geq n_0 \), where \( \varepsilon_0 \in (0, 1) \) is a sufficiently small absolute constant, and \( n_0 > 1 \) is a sufficiently large absolute constant. For \( z = (z_1, \ldots, z_m) \in \Omega^m \), define the operator \( S_{m,z} : X_n \to \mathbb{R}^m \) by \( S_{m,z}f = (f(z_1), \ldots, f(z_m)) \). Note that (3.1) implies that (see, for instance, [6])

\[
\|f\|_\infty \leq (Kn)^{\frac{1}{p}}\|f\|_p, \quad \forall f \in X_n, \quad 1 \leq p \leq 2.
\]

By Lemma 6.3 there exists a vector \( z = (z_1, \ldots, z_m) \in \Omega^m \) with

\[
C\varepsilon^{-2}(\log \frac{1}{\varepsilon})Kn^2 \log n \leq m_1 \leq C^2\varepsilon^{-2}(\log \frac{1}{\varepsilon})Kn^2 \log n
\]

such that

\[
\left| \|f\|_{L_p(\Omega)} - \|S_{m_1,z}f\|_{p,\Omega_{m_1}} \right| \leq \frac{\varepsilon}{4} \|f\|_p, \quad \forall f \in X_n^p, \tag{6.9}
\]

and

\[
\left| \|f\|_{L_2(\Omega)} - \|S_{m_1,z}f\|_{2,\Omega_{m_1}} \right| \leq \frac{\varepsilon}{4} \|f\|_2, \quad \forall f \in X_n^2, \tag{6.10}
\]

where \( \Omega_{m_1} := \{z_1, \ldots, z_{m_1}\} \).

Now consider the \( n \)-dimensional subspace \( \tilde{X}_n := S_{m_1,z}(X_n) \) of \( L_\infty(\Omega_{m_1}) \). Using (6.4) and (6.10), we have that for any \( f \in X_n \),

\[
\|S_{m_1,z}f\|_{\infty,\Omega_{m_1}} \leq \sup_{x \in \Omega} |f(x)| \leq (Kn)^{\frac{1}{2}}\|f\|_2 \leq (2Kn)^{\frac{1}{2}}\|S_{m_1,z}f\|_{2,\Omega_{m_1}}, \tag{6.11}
\]

Furthermore, since \( \log K \leq C \log n \), we have

\[
\log m_1 \leq 3 \log \varepsilon^{-1} + C \log(\sqrt{Kn}) \leq 3 \log \varepsilon^{-1} + C \log n,
\]

which in turn implies that for any \( f : \Omega_{m_1} \to \mathbb{R} \),

\[
\|f\|_{\infty,\Omega_{m_1}} \leq m_1^{\frac{1}{m_1}}\|f\|_{q_n,\Omega_{m_1}} \leq C\varepsilon^{-\frac{1}{2}}\|f\|_{q_n,\Omega_{m_1}}.
\]

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Thus, the $n$-dimensional subspace $\tilde{X}_n$ of $L_\infty(\Omega_{m_1})$ satisfies both the conditions (6.4) and (6.5) with $K_1 = 2K$ and $K_2 = C\varepsilon^{-\frac{1}{4}}$. By Lemma 6.2 (applied with a discrete measure), we then obtain

$$\mathcal{H}_t(\tilde{X}_p^n; \| \cdot \|_\infty, \Omega_{m_1}) \leq C_p K\varepsilon^{-\frac{1}{2}} n \log n t^p, \quad \forall t > 0, \quad (6.12)$$

where

$$\tilde{X}_p^n := \{ f \in \tilde{X}_n : \| f \|_{p,\Omega_{m_1}} \leq 1 \}.$$

This implies that for $R = (2Kn)^{1/p}$ and any $1 \leq p \leq 2$,

$$\varepsilon^{-5} \left( \int_u^{R} u^{-\frac{1}{2} - 1} \left( \int_u^R \mathcal{H}_{c,p,t}(\tilde{X}_p^n; \| \cdot \|_\infty, \Omega_{m_1}) dt \right) \frac{1}{t} du \right)^2 \leq C\varepsilon^{-8} Kn \log^3 n.$$

Thus, applying Lemma 6.1 and Remark 6.1 to the subspace $\tilde{X}_n$ of $L_p(\Omega_{m_1})$, we get a subset $\Lambda \subset \{1, 2, \ldots, m_1\}$ with $|\Lambda| \leq C_p \varepsilon^{-8} Kn \log^3 n$ such that the inequalities

$$\left| \frac{1}{|\Lambda|} \sum_{j \in \Lambda} |S_{m_1,\mathbf{z}}f(j)|^q - \|S_{m_1,\mathbf{z}}f\|_{q,\Omega_{m_1}}^q \right| \leq \varepsilon \|S_{m_1,\mathbf{z}}f\|_{q,\Omega_{m_1}}^q, \quad f \in X_n$$

hold simultaneously for $q = p$ and $q = 2$. This together with (6.9) and (6.10) implies that for $q = p$ and $q = 2$,

$$(1 - \varepsilon)\|f\|_q \leq \frac{1}{|\Lambda|} \sum_{j \in \Lambda} |f(z_j)|^q \leq (1 + \varepsilon)\|f\|_q, \quad \forall f \in X_n.$$

The theorem is proved. \(\square\)

**Remark 6.2.** We point out that Lemma 3.1 for $1 < p < 2$ also follows directly from a result of Rudelson (Theorem 6.1) and a result from [17] (see Theorem 6.2).

**Theorem 6.1** ([28], see also Corollary 4.1 of [14]). Let $\mu$ be a probability Borel measure on a compact set $\Omega$. There is a constant $C$ such that, if $X_n$ is an $n$-dimensional subspace of $L_\infty(\mu)$ such that

$$\|f\|_\infty \leq \sqrt{Kn} \|f\|_2 \quad \forall f \in X_n,$$
then
\[ \mathbb{E} \sup_{f \in B_2(X_n)} \left| \frac{1}{m} \sum_{j=1}^{m} |f(x_j)|^2 - \|f\|^2 \right| \leq C \left( \frac{Kn \log n}{m} + \sqrt{\frac{Kn \log n}{m}} \right), \]

where \( B_2(X_n) := \{ f \in X_n : \|f\|_2 \leq 1 \} \).

**Theorem 6.2** (see Corollary 4.11 in [17]). Let \( p \in (1, 2) \) and let \( \mu \) be a probability Borel measure on a compact set \( \Omega \). There is a constant \( C := C(p) \) such that, if \( X_n \) is an \( n \)-dimensional subspace of \( L_\infty(\mu) \) such that
\[
\|f\|_\infty \leq \sqrt{Kn} \|f\|_2 \quad \forall f \in X_n,
\]
then
\[
\mathbb{E} \sup_{f \in B_p(X_n)} \left| \frac{1}{m} \sum_{j=1}^{m} |f(x_j)|^p - \|f\|_p^p \right|
\leq C \left( \frac{[\log m]^{1+p} [\log 4Kn]^{-\frac{p}{2}}}{m} Kn + \left[ \frac{[\log m]^{1+p} [\log 4Kn]^{-\frac{p}{2}}}{m} Kn \right]^{1/2} \right),
\]
where \( B_p(X_n) := \{ f \in X_n : \|f\|_p \leq 1 \} \).

**7 Appendix I. Proof of Theorem 3.1**

Recall the notion of \( K \)-convexity constant. Let \((L, \| \cdot \|)\) be a Banach space. The \( K \)-convexity constant of the space \( L \) is defined as
\[
K(L, \| \cdot \|) := \sup \left( \mathbb{E} \left\| \sum_{j=1}^{k} \varepsilon_j \mathbb{E}[f(\varepsilon)] \right\|^2 \right)^{1/2}
\]
with the supremum being taken over all integers \( k \in \mathbb{N} \) and all functions \( f : \{-1, 1\}^k \to L \) such that \( \mathbb{E} \| f(\varepsilon) \|^2 = 1 \).

**Remark 7.1.** It is known (see e.g. Theorem in paragraph 14.6 of [24]) that there is an absolute constant \( C > 0 \) such that \( K(L, \| \cdot \|) \leq C \log N \) for any \( N \)-dimensional space \((L, \| \cdot \|)\). Moreover (see [10, Lemma 17]), there is an absolute constant \( C > 0 \) such that for an \( N \)-dimensional subspace \( L \subset L_1(\mu) \) one has \( K(L, \| \cdot \|_1) \leq C \sqrt{\log N} \).
In [32] the following theorem was proved (see also [18, Proposition 15.16] and [10, Theorem 13]).

**Theorem 7.1.** Let $X_n$ be an $n$-dimensional subspace of $\mathbb{R}^M$ equipped with the counting measure on $\Omega_M := \{1, \ldots, M\}$. Assume that there is a number $\theta > 0$ such that for each $f \in X_n$ one has $\|f\|_{\infty} \leq \theta \|f\|_2$. Then

$$
\mathbb{E}_\varepsilon \sup_{f \in X_n \cap B^M_1} \left| \sum_{j=1}^M \varepsilon_j |f(j)| \right| \leq 2\sqrt{\pi} \theta K(X_n, \| \cdot \|_1).
$$

The proof of the above theorem relies on the following lemma (see [10, Proposition 16]).

**Lemma 7.1.** Let $X_n$ be an $n$-dimensional subspace of $L_2(\mu)$, where $\mu$ is a positive but not necessarily a probability measure. Assume that there is a constant $\theta > 0$ such that $\|f\|_{\infty} \leq \theta \|f\|_{L_2(\mu)}$ for each $f \in X_n$. Then

$$
\mathbb{E}_g \sup_{f \in X_n, \|f\|_{L_1(\mu)} \leq 1} \left| \sum_{k=1}^n g_k \langle u_k, f \rangle_{L_2(\mu)} \right| \leq \sqrt{2} \theta K(X_n, \| \cdot \|_{L_1(\mu)}),
$$

where $\{u_1, \ldots, u_n\}$ is any orthonormal basis of $X_n$, and $g = (g_1, \ldots, g_n) \sim \mathcal{N}(0, I_n)_{\mathbb{R}^n}$ is the standard Gaussian vector, i.e. its components are i.i.d. copies of $Z \sim \mathcal{N}(0, 1)$.

Since in [32], [18], and [10] the statement of Theorem 7.1 is presented in slightly different terms, we now show how one can prove Lemma 7.1 and how to deduce Theorem 7.1 from Lemma 7.1. The proof follows the argument from the proof of [10, Proposition 16] almost verbatim.

**Proof of Lemma 7.1.** Let $B_1 := \{ f \in X_n : \|f\|_{L_1(\mu)} \leq 1 \}$. Let $\{\varepsilon_i\}, i \in \{1, \ldots, N\}, k \in \{1, \ldots, n\}$ — be i.i.d. symmetric Bernoulli random variables. Then by the multivariate central limit theorem, we have

$$
\mathbb{E}_g \sup_{f \in B_1} \left| \sum_{k=1}^n g_k \langle u_k, f \rangle \right| = \lim_{N \to \infty} N^{-1/2} \mathbb{E}_\varepsilon \sup_{f \in B_1} \left| \sum_{k=1}^n \sum_{i=1}^N \varepsilon_{ik} \langle u_k, f \rangle \right|.
$$

For any fixed vector $\varepsilon = (\varepsilon_{ik})$ there is a vector $f_{\varepsilon} \in X_n$ such that $\|f_{\varepsilon}\|_1 \leq 1$ and

$$
\sup_{f \in B_1} \left| \sum_{k=1}^n \sum_{i=1}^N \varepsilon_{ik} \langle u_k, f \rangle \right| = \sum_{k=1}^n \sum_{i=1}^N \varepsilon_{ik} \langle u_k, f_{\varepsilon} \rangle.
$$
Thus, using Fubini’s theorem, we get

\[
\mathbb{E}_\varepsilon \sup_{f \in B_1} \left| \sum_{k=1}^n \sum_{i=1}^N \varepsilon_{ik} \langle u_k, f \rangle \right| = \mathbb{E}_\varepsilon \left[ \sum_{k=1}^n \sum_{i=1}^N \varepsilon_{ik} \langle u_k, f_\varepsilon \rangle \right] \\
= \sum_{k=1}^n \sum_{i=1}^N \langle u_k, f_{ik} \rangle =: S_{N,n},
\]

where \( f_{ik} = \mathbb{E}_\varepsilon [\varepsilon_{ik} f_\varepsilon] \). By Hölder’s inequality, we have

\[
S_{N,n} = \int_\Omega \left[ \sum_{k=1}^n \sum_{i=1}^N u_k(x) f_{ik}(x) \right] d\mu(x) \\
\leq \int_\Omega \left( \sum_{k=1}^n \sum_{i=1}^N |u_k(x)|^2 \right)^{1/2} \left( \sum_{k=1}^n \sum_{i=1}^N |f_{ik}(x)|^2 \right)^{1/2} \ d\mu(x) \\
\leq N^{1/2} \left\| \left( \sum_{k=1}^n |u_k|^2 \right)^{1/2} \right\|_\infty \int_\Omega \left( \sum_{k=1}^n \sum_{i=1}^N |f_{ik}(x)|^2 \right)^{1/2} \ d\mu(x).
\]

Since \( \{u_1, \ldots, u_n\} \) is an orthonormal basis of \( X_n \), we have

\[
\left\| \left( \sum_{k=1}^n |u_k|^2 \right)^{1/2} \right\|_\infty = \sup_{f \in X_n, \|f\|_2 = 1} \|f\|_\infty \leq \theta.
\]

It then follows by Khintchine’s inequality that

\[
S_{N,n} \leq \sqrt{2} N^{1/2} \theta \int_\Omega \left[ \mathbb{E}_\varepsilon \left| \sum_{k=1}^n \sum_{i=1}^N \varepsilon_{ik} f_{ik}(x) \right| \right] \ d\mu(x) \\
= \sqrt{2} N^{1/2} \theta \mathbb{E}_\varepsilon \left\| \sum_{k=1}^n \sum_{i=1}^N \varepsilon_{ik} f_{ik} \right\|_1 \leq \sqrt{2} N^{1/2} \theta K(X_n, \| \cdot \|_1).
\]

This leads to the desired upper bound:

\[
\mathbb{E}_g \sup_{f \in B_1} \left| \sum_{k=1}^n g_k \langle u_k, f \rangle \right| = \lim_{N \to \infty} N^{-1/2} S_{N,n} \leq \sqrt{2} \theta K(X_n, \| \cdot \|_1).
\]

The lemma is proved. \( \square \)
Proof of Theorem 7.1 It is known (see [18, Theorem 4.12 and Estimate (4.8)]) that
\[
\mathbb{E}_\varepsilon \sup_{f \in X_n \cap B_1^M} \left| \sum_{j=1}^M \varepsilon_j f(j) \right| \leq \sqrt{2\pi} \mathbb{E}_g \sup_{f \in X_n \cap B_1^M} \left| \sum_{j=1}^M g_j f(j) \right| = \\
= \sqrt{2\pi} M^{-1} \mathbb{E}_g \sup_{f \in X_n, \|f\|_{L_1(\mu)} \leq 1} \left| \sum_{j=1}^M g_j f(j) \right|
\]
where \( g_1, \ldots, g_m \) are i.i.d. \( \mathcal{N}(0, 1) \) random variables and \( \mu \) is a uniform probability distribution on the set \( \Omega_M = \{1, \ldots, M\} \). For any orthonormal basis \( \{u_1, \ldots, u_n\} \) of \( X_n \) with respect to the norm \( \|f\|_{L_2(\mu)} = \frac{1}{\sqrt{M}} \|f\|_2 \) we have
\[
\mathbb{E}_g \sup_{f \in X_n, \|f\|_{L_1(\mu)} \leq 1} \left| \sum_{j=1}^M g_j f(j) \right| = \mathbb{E}_g \sup_{f \in X_n, \|f\|_{L_1(\mu)} \leq 1} \left| \sum_{k=1}^n \langle f, u_k \rangle_{L_2(\mu)} \sum_{j=1}^M g_j u_k(j) \right| = \\
= \sqrt{M} \mathbb{E}_g \sup_{f \in X_n, \|f\|_{L_1(\mu)} \leq 1} \left| \sum_{k=1}^n G_k \langle f, u_k \rangle_{L_2(\mu)} \right|
\]
where \( G_k := \frac{1}{\sqrt{M}} \sum_{j=1}^M g_j u_k(j) \). Since \( \{u_1, \cdots, u_n\} \) is an orthonormal basis of \( (X_n, \| \cdot \|_{L_2(\mu)}) \), it follows by the rotation invariance of the Gaussian random vector that \( (G_1, \cdots, G_n) \sim \mathcal{N}(0, I_n)_{\mathbb{R}^n} \). In our case we have
\[
\|f\|_{L_\infty(\mu)} \leq \theta \|f\|_2 = \theta \sqrt{M} \|f\|_{L_2(\mu)}, \quad \forall f \in X_n.
\]
Theorem 7.1 then follows by Lemma 7.1. \( \square \)

8 Appendix II. Proof of Theorem 4.2

The proof of Theorem 4.2 is very close to that of Theorem 16.8.2 of [34]. We first recall some notations. We identify a vector from \( \mathbb{R}^M \) with a function on the set \( \Omega_M := \{1, \cdots, M\} \), and denote by \( \ell_p^M \), \( 0 < p \leq \infty \) the space \( \mathbb{R}^M \)
equipped with the norm
\[ \|f\|_p := \begin{cases} 
\left( \sum_{i=1}^{M} |f(i)|^p \right)^{1/p}, & \text{if } 0 < p < \infty, \\
\max_{1 \leq i \leq M} |f(i)|, & \text{if } p = \infty.
\end{cases} \]

Let \( B_p^M := \{ f \in \mathbb{R}^M : \|f\|_p \leq 1 \} \) denote the unit ball of \( \ell_p^M \). For each \( I \subset \Omega_M \), we denote by \( R_I \) the orthogonal projection onto the space spanned by \( e_i, i \in I \), where \( e_1 = (1, 0, \cdots, 0) \), \( \cdots \), \( e_M = (0, \cdots, 0, 1) \in \mathbb{R}^M \). Finally, we set \( N_0 = 1 \) and \( N_k = 2^k \) for \( k = 1, 2, \cdots \).

The proof of Theorem 4.2 uses a majorizing measure theorem of Talagrand [34], which we now recall. Let \((T, \rho)\) be a metric space, and let \( B_{\rho}(s, r) := \{ t \in T : \rho(s, t) \leq r \} \) denote the ball with center \( s \in T \) and radius \( r > 0 \). Let \( H_1, \cdots, H_N \) be subsets of \( T \). We say \( H_1, \cdots, H_N \) are \((a, \delta, \rho)\)-separated for some constants \( a > 0 \) and \( 0 < \delta < 1/2 \) if there exist points \( t_0 \in T \) and \( t_1, \cdots, t_N \in B_{\rho}(t_0, 4a) \) such that
\[
\min_{1 \leq j \neq k \leq N} \rho(t_j, t_k) \geq a \quad \text{and} \quad H_j \subset B_{\rho}(t_j, \delta a) \quad \text{for all } 1 \leq j \leq N.
\]

A functional \( F \) on the metric space \( T \) is a nonnegative function on the collection of all subsets of \( T \) satisfying \( F(H) \leq F(H') \) whenever \( H \subset H' \subset T \). A sequence \((F_k)_{k=0}^\infty\) of functionals on \( T \) is called decreasing if \( F_k(H) \geq F_{k+1}(H) \) for all \( k \geq 0 \) and \( H \subset T \).

The following version of the majorizing measure theorem of Talagrand [34] can be obtained as a combination of [34, Theorem 2.7.6, p. 68] and [34, Theorem 2.2.18, p. 25].

**Theorem 8.1 (Majorizing measure theorem).** [34] Let \( T = (T, \rho) \) be a metric space with \( \text{diam}(T) := \sup_{s, t \in T} \rho(s, t) < \infty \), and let \((V_t)_{t \in T}\) be a random process satisfying that
\[
\mathbb{P}(|V_t - V_s| > u) \leq 2 \exp\left( -\frac{c u^2}{\rho^2(s, t)} \right), \quad \forall u > 0, \quad \forall s, t \in T,
\]
where \( c > 0 \) is an absolute constant. Assume that there exists a decreasing sequence of functionals \((F_k)_{k=0}^\infty\) on \( T \) satisfying the following growth condition: there exist a constant \( \sigma > 0 \) and an integer \( k_0 \geq 2 \) such that for each integer \( k \geq 0 \),
\[
F_k\left( \bigcup_{\ell=1}^{N_{k+1}} H_{\ell} \right) \geq \sigma a \sqrt{\log N_k} + \min_{1 \leq \ell \leq N_{k+1}} F_{k+k_0}(H_{\ell}) \quad (8.1)
\]
whenever $H_1, \cdots, H_{N_k+1}$ are $(a, 4^{-k}, \rho)$-separated for some constant $a > 0$. Then there exists an absolute constant $C > 0$ such that

$$
\mathbb{E} \sup_{t \in T} V_t \leq C k_0 \left( \frac{F_0(T)}{\sigma} + \text{diam}(T) \right).
$$

To prove Theorem 4.2, we also need some known estimates of entropy numbers. Let $(X, \| \cdot \|_X)$ be a Banach space. Let $B_X(g, r)$ denote the closed ball $\{ f \in X : \| f - g \| \leq r \}$ with center $g \in X$ and radius $r > 0$. The entropy numbers $e_k(A, X)$ of a set $A$ in $X$ are defined as

$$
e_k(A, \| \cdot \|_X) := \inf \left\{ \epsilon > 0 : \exists g^1, \cdots, g^{2^k} \in A \text{ such that } A \subset \bigcup_{j=1}^{2^k} B_X(g^j, \epsilon) \right\},
$$

where $k = 0, 1, \cdots$. The following estimates can be found in [6, Theorem 2.1 and (2.5)].

**Lemma 8.1.** [6, Theorem 2.1 and (2.5)] Let $X_n$ be an $n$-dimensional subspace of $\mathbb{R}^M$ satisfying

$$
\| f \|_\infty \leq \sqrt{\frac{Kn}{M}} \| f \|_2, \quad \forall f \in X_n
$$

for some constant $K \geq 1$. Then for any $1 \leq p \leq 2 < q \leq \infty$,

$$
e_k(X_n \cap B^M_p, \| \cdot \|_q) \leq C_{p,q} \left( \frac{\log M}{M} \right)^{\frac{1}{p} - \frac{1}{q}} \left( \frac{Kn}{k} \right)^{\frac{1}{p} - \frac{1}{q}}, \quad k = 1, 2, \cdots. \quad (8.2)
$$

Now we turn to the proof of Theorem 4.2. Throughout the proof, we use the letter $C_1$ to denote a sufficiently large positive constant depending only on $p$, and use the letter $C$ to denote a general positive constant which depends only on $p$ but may vary at each appearance.

Without loss of generality, we may assume that $K = 1$ since otherwise we may replace $Kn$ by $n$ and consider $X_n$ as a space of dimension at most $Kn$. We may also assume that $M \geq C_1 n \log(2M)$ since otherwise (8.3) holds trivially. In particular, this implies that $\log \log M \leq \log \frac{M}{n}$.

Let

$$
T := \left\{ \| f \|_p : f \in X_n \cap B^M_p \right\} \subset B^M_1.
$$

For $s \in T$, define $V_s := \sum_{i=1}^M s(i) \varepsilon_i$. Then $\{V_s\}_{s \in T}$ is a symmetric random process satisfying (see [18, Lemma 4.3]) that

$$
\mathbb{P}\{|V_s - V_t| > u\} = \mathbb{P}\{|V_{s-t}| > u\} \leq 2 \exp \left( -\frac{u^2}{2\|s - t\|_2^2} \right), \quad \forall s, t \in T, \ \forall u > 0.
$$
Our aim is to show that
\[ \mathbb{E}\left( \sup_{t \in T} |V_t| \right) \leq C \sqrt{\frac{n \log M}{M} \log \frac{M}{n}}. \quad (8.3) \]
To prove (8.3), we will apply Theorem 8.1 to the random process \( \{V_s\}_{s \in T} \) on the compact metric space \((T, \| \cdot \|_2)\). Since the condition (4.3) implies
\[ \|f\|_\infty \leq \left( \frac{n}{M} \right)^{1/p} \|f\|_p, \quad \forall f \in X_n, \quad (8.4) \]
we have that for \( t = |f|^p \in T \) with \( f \in X_n \cap B^M_p \),
\[ \|t\|_2^2 = \sum_{i=1}^{M} |f(i)|^{2p} \leq \|f\|_\infty \sum_{i=1}^{M} |f(i)|^p \leq \frac{n}{M}, \]
implying
\[ \text{diam}(T) = \sup_{s, t \in T} \|s - t\|_2 \leq 2 \sqrt{\frac{n}{M}}. \quad (8.5) \]

To apply Theorem 8.1, we need to construct a decreasing sequence of functionals \((F_k)_{k \geq 0}\) on the metric space \( T := (T, \| \cdot \|_2) \) satisfying \( F_0(T) \leq 2 \) and a growth condition. To this end, for \( k = 0, 1, \ldots \), we let \( A_k := \frac{2^k}{(\log M)^T} \), and let \( \Sigma_k \) denote the collection of all subsets \( I \) of \( \Omega_M := \{1, 2, \ldots, M\} \) with \( |I| \leq A_k \). For a subset \( H \subset T \) and an integer \( k \geq 0 \), we define \( \varphi_k(H) = 0 \) if \( A_k < 1 \), and
\[ \varphi_k(H) := \max_{I \in \Sigma_k} \inf_{x \in H} \|R_I x\|_1 \quad (8.6) \]
if \( A_k \geq 1 \). Clearly, \( 0 \leq \varphi_k(H) \leq \varphi_{k+1}(H) \leq 1 \), and \( \varphi_k(H_1) \geq \varphi_k(H_2) \geq \varphi_k(T) = 0 \) whenever \( H_1 \subset H_2 \subset T \). Now we define the sequence of functionals on \( T \) as follows:
\[ F_k(H) := 1 - \varphi_k(H) + \frac{1}{n_1} \max\{n_1 - k, 0\}, \quad H \subset T, \quad k = 0, 1, \ldots, \quad (8.7) \]
where \( n_1 \) is a positive integer satisfying
\[ C_1 \log M \leq n_1 < 2C_1 \log M. \quad (8.8) \]
Clearly, \((F_k)_{k \geq 0}\) is a decreasing sequence of functionals on \( T \), and \( F_0(T) \leq 2 \).
We need to prove \((F_k)_{k \geq 0}\) satisfies a growth condition. Let \(k_0 \in \mathbb{N}\) be such that
\[
2^{k_0 - 1} < \frac{C_1 M (\log M)^{6 + \frac{1}{2p}}}{n} \leq 2^k,
\]
where we choose the constant \(C_1\) large enough so that
\[
\log_2 \frac{M}{n} \leq k_0 \leq \frac{C_1}{8} \log_2 \frac{M}{n}.
\]
Let
\[
\sigma := c_1 \sqrt{\frac{M}{n \log M}}, \quad \text{where } c_1 = \frac{1}{4C_1}.
\]
Our aim is to show that for each integer \(k \geq 0\),
\[
F_k \left( \bigcup_{t=1}^{N_{k+1}} H_t \right) \geq \sigma a 2^{k/2} + \min_{1 \leq \ell \leq N_{k+1}} F_{k+k_0}(H_\ell)
\]
whenever \(H_1, \ldots, H_{N_{k+1}} \subset T\) are \((a, 4^{-k_0}, \| \cdot \|_2)\)-separated for some constant \(a > 0\). Once (8.12) is proved, the desired estimate (8.3) will follow immediately from Theorem 8.1 and (8.10) since according to (8.5), we have \(\text{diam}(T) \leq C \sigma^{-1}\).

For the proof of (8.12), we fix the integer \(k \geq 0\) so that \(A_k \geq 1\), set \(N := N_{k+1}\), and assume that \(H_1, \ldots, H_N \subset T\) are \((a, 4^{-k_0}, \| \cdot \|_2)\)-separated. Denote by \(B(x, r)\) the Euclidean ball \(\{z \in \mathbb{R}^M : \|z - x\|_2 \leq r\}\) with center \(x \in \mathbb{R}^M\) and radius \(r > 0\), and define
\[
B^T(x, r) := B(x, r) \cap T = \{z \in T : \|z - x\|_2 \leq r\} \quad \text{for } x \in T \text{ and } r > 0.
\]
Then there exist \(t_0 \in T\) and \(t_1, \ldots, t_N \subset B^T(t_0, 4a)\) such that \(\min_{1 \leq i \neq j \leq N} \|t_i - t_j\|_2 \geq a\) and \(H_j \subset B^T(t_j, 4^{-k_0}a)\) for \(j = 1, \ldots, N\).

We will reduce the inequality (8.12) to a relatively simpler one in several steps. First, we claim that it suffices to prove (8.12) for \(k < \log_2(3M)\). Indeed, since \(t_1, \ldots, t_N \subset B(t_0, 4a)\) are \(a\)-separated with respect to the Euclidean distance (i.e., \(\|t_i - t_j\|_2 \geq a\) for any \(1 \leq i \neq j \leq N\)), it follows that \(\log_2 N = 2^{k+1} \leq 2M \log_2 3\), which implies that \(k + 1 = \log_2 \log_2 N \leq \log_2(5M)\).

Second, we claim that it suffices to show the inequality
\[
\varphi_k \left( \bigcup_{j=1}^{N} H_j \right) + \sigma a 2^{k/2} \leq \max_{1 \leq j \leq N} \varphi_{k+k_0}(H_j)
\]
(8.13)
under the conditions
\[ 0 \leq k < \log_2(5M) \quad \text{and} \quad a^2 2^k \geq \frac{n}{M \log M}. \tag{8.14} \]

Indeed, by the first claim, we may assume that \( 0 \leq k < \log_2(5M) \). By (8.10) and (8.8), we then have \( k + k_0 < n_1 \). Thus, by (8.7), the desired inequality (8.12) is equivalent to the inequality,
\[ \varphi_k \left( \bigcup_{j=1}^{N} H_j \right) + \sigma a^{2k/2} \leq \max_{1 \leq j \leq N} \varphi_{k+k_0} (H_j) + \frac{k_0}{n_1}. \tag{8.15} \]

This last inequality holds trivially if \( \sigma a^{2k/2} \leq \frac{k_0}{n_1} \) because
\[ \varphi_k \left( \bigcup_{j=1}^{N} H_j \right) \leq \min_{1 \leq j \leq N} \varphi_k (H_j) \leq \min_{1 \leq j \leq N} \varphi_{k+k_0} (H_j). \]

Thus, we may always assume that \( \sigma a^{2k/2} > \frac{k_0}{n_1} \), which, using (8.11) and (8.8), implies \( a^2 2^k \geq \frac{n}{M \log M} \). Thus, it is enough to prove (8.15) under the conditions (8.14). Since (8.13) implies (8.15), the claim then follows.

Third, we claim that for the proof of (8.13), it is enough to show that for each set \( I \in \Sigma_k \),
\[ \max_{1 \leq \ell \leq N} \max_{J \in \Sigma_{k+k_0-1}} \inf_{x \in H_\ell} \| R_I x \|_1 \geq \sigma a^{2k/2}. \tag{8.16} \]

Indeed, by the definition (8.6), the inequality (8.13) is equivalent to the assertion that for each set \( I \in \Sigma_k \), there exist an integer \( 1 \leq \ell \leq N \) and a set \( J \in \Sigma_{k+k_0} \) such that
\[ \inf_{x \in \bigcup_{j=1}^{N} H_j} \| R_I x \|_1 + \sigma a^{2k/2} \leq \inf_{x \in H_\ell} \| R_J x \|_1. \tag{8.17} \]

Once (8.16) is proven, then for each \( I \in \Sigma_k \), there exist an integer \( 1 \leq \ell \leq N \) and a set \( J_\ell \subset \Omega_M \setminus I \) such that \( |J_\ell| \leq A_{k+k_0-1} \) and \( \inf_{x \in H_\ell} \| R_{J_\ell} x \|_1 \geq \sigma a^{2k/2} \). Setting \( J := I \cup J_\ell \), we have
\[ |J| = |I| + |J_\ell| \leq A_{k+k_0-1} + A_k \leq A_{k+k_0}, \]
and
\[
\inf_{x \in \bigcup_{j=1}^{N} H_j} \|R_J x\|_1 + \sigma a 2^{k/2} \leq \inf_{x \in H_t} \|R_I x\|_1 + \inf_{x \in H_t} \|R_J x\|_1
\]
\[
\leq \inf_{x \in H_t} \left[ \|R_I x\|_1 + \|R_J x\|_1 \right] = \inf_{x \in H_t} \|R_J x\|_1,
\]
proving (8.17).

Finally, we claim that it is enough to prove that for each integer \(k\) satisfying (8.14), and each \(I \in \Sigma_k\),
\[
\max_{1 \leq \ell \leq N} \max_{J \in \Sigma_{k+k_0-1}} \|R_{J \ell} t\|_1 \geq 2\sigma a 2^{k/2} = 2c_1 \sqrt{\frac{M}{n \log M}} 2^{k/2} a. \tag{8.18}
\]
According to the second and the third claims that have already been proven, we need to prove that (8.18) implies (8.17), assuming \(k\) satisfies (8.14). Indeed, by (8.18), there exist an integer \(1 \leq \ell \leq N\) and a set \(J_{\ell} \subset \Omega_M \setminus I\) such that
\[
|J_{\ell}| \leq A_{k+k_0-1}
\]
and
\[
\|R_{J_{\ell}} t\|_1 \geq 2\sigma a 2^{k/2} \geq \sigma 2^{k/2}. \tag{8.19}
\]
Since \(H_t \subset B(t_{\ell}, 4^{-k_0} a) \cap T\), it follows that for any \(x \in H_t\),
\[
\|R_{J_{\ell}} t\|_1 - \|R_{J_{\ell}} x\|_1 \leq \|R_{J_{\ell}} (x - t_{\ell})\|_1 \leq \sqrt{|J_{\ell}|} \|t_{\ell} - x\|_2
\]
\[
\leq 2^{(k+k_0-1)/2} 4^{-k_0} a \leq a 2^{k/2} 2^{-k_0} (\log M)^{-2} \leq \sigma a 2^{k/2}.
\]
Thus, using (8.19), we have
\[
\inf_{x \in H_t} \|R_{J_{\ell}} x\|_1 \geq \|R_{J_{\ell}} t\|_1 - \sigma a 2^{k/2} \geq \sigma 2^{k/2} a
\]
implying (8.16).

In summary, we reduce to showing that (8.18) holds for each \(I \in \Sigma_k\) with \(k\) satisfying (8.14). This is an easy consequence of the following proposition.

**Proposition 8.1.** Assume that \(1 \leq p < 2\), \(M \geq Cn(\log M)\) for some large constant \(C = C_p > 1\), and \(k \geq 0\) is an integer satisfying \(a 2^{k} \geq \frac{n}{M \log M}\) for some constant \(a > 0\). Assume in addition that \(N = N_{k+1}\) and \(t_1, \ldots, t_N\) are
points in a ball $B^T(t_0, 4a)$ with $t_0 \in T$ satisfying $\min_{1 \leq i \neq j \leq N} \|t_i - t_j\|_2 \geq a$. Let
\[
\tau := \frac{1}{A_{k+k_0-1}} = 2^{-k-k_0+1}(\log M)^4
\] (8.20)
with $k_0$ being the integer given in (8.9) with $C_1 = C$. Let $I$ be a subset of $\Omega_M$ satisfying $|I| \leq A_k = 2^k(\log M)^{-4}$. For $1 \leq \ell \leq N$, let $J_\ell := \{\Omega_M \setminus I : t_\ell(i) \geq \tau\}$. Then there exists a positive constant $c$ depending only on $p$ such that
\[
\max_{1 \leq \ell \leq N} \|R_{J_\ell} t_\ell\|_1 \geq c \cdot a \frac{2^k}{\sqrt{n \log M}}. \tag{8.21}
\]

The proof of Proposition 8.1 is long and will be given in the next subsection. For the moment, we take it for granted and proceed with the proof of (8.18) for each $I \in \Sigma_k$ with $k$ satisfying (8.14). By (8.21), there exists an integer $1 \leq \ell \leq N$ such that
\[
\|R_{J_\ell} t_\ell\|_1 \geq c a \frac{2^{k/2}}{\sqrt{n \log M}}, \tag{8.22}
\]
where $J_\ell := \{\Omega_M \setminus I : t_\ell(i) \geq \tau\}$, and $\tau$ is given in (8.20). Since $\|t_\ell\|_1 \leq 1$, we have
\[
|J_\ell| \leq \frac{1}{\tau} = \frac{2^{k+k_0-1}}{(\log M)^4} = A_{k+k_0-1}.
\]
(8.18) then follows with $c_1 = \frac{1}{2}c$.

### 8.1 Proof of Proposition 8.1

Recall that $N = N_{k+1} = 2^{k+1}$, $1 \leq p < 2$, and $a \frac{2^k}{M \log M} \geq \frac{n}{M \log M}$. For each $1 \leq \ell \leq N$, let $f_\ell \in X_n \cap B_p^M$ be such that $t_\ell = |f_\ell|^p$. Let $I \subset \Omega_M$ be a fixed set such that $|I| \leq A_k := \frac{2^k}{(\log M)^{4}}$, and let $I^c = \Omega_M \setminus I$. Set
\[
S := \max_{1 \leq \ell \leq N} \sum_{i \in I^c} t_\ell(i) \chi_{\{t_\ell(i) \geq \tau\}}(i). \tag{8.23}
\]

Our aim is to prove
\[
S \geq c \cdot a \frac{2^{k/2}}{\sqrt{n \log M}}. \tag{8.24}
\]

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For the proof of (8.24), we need several lemmas. The first lemma allows us to replace \( \{ t_\ell \}_{\ell=1}^N \) with a subset of points \( t_\ell, \ell \in V \) that are “well” distributed.

**Lemma 8.2.** We can find a subset \( V \) of \( \{ 1, 2, \cdots, N_{k+1} \} \) with \( |V| > N_{k-2} \) and the following properties:

\[
\| R_I(t_\ell - t_{\ell'}) \|_2 \geq \frac{a}{\sqrt{2}} \quad \text{whenever} \quad \ell, \ell' \in V \quad \text{and} \quad \ell \neq \ell', \tag{8.25}
\]

\[
\| f_\ell - f_{\ell'} \|_\infty \leq C_p \left( \frac{n}{2^k} \cdot \frac{\log M}{M} \right)^{1/p}, \quad \forall \ell, \ell' \in V, \tag{8.26}
\]

\[
\| R_I(f_\ell) - R_I(f_{\ell'}) \|_p \leq C_p \left( \frac{n}{M(\log M)^3} \right)^{1/p}, \quad \forall \ell, \ell' \in V. \tag{8.27}
\]

**Proof.** First, by Lemma 8.1

\[
e_{2k-2}(X_n \cap B_p^M; \| \cdot \|_\infty) \leq C_p \left( \frac{\log M}{M} \right)^{\frac{1}{p}} \left( \frac{n}{2^k} \right)^{\frac{1}{p}}.
\]

Since \( \{ f_1, \cdots, f_N \} \subset X_n \cap B_p^M \), we can partition \( \{ 1, 2, \cdots, N \} \) into at most \( N_{k-2} \) sets \( V_{\alpha,1}, \alpha \in A_1 \) so that (8.26) with \( V = V_{\alpha,1} \) is satisfied for each \( \alpha \in A_1 \).

Second, since \( \| R_I f \|_p \leq |I|^{1/p} \| f \|_\infty \leq A_k^{1/p} \| f \|_\infty \), we have

\[
e_{2k-2} \left( R_I X_n \cap B_p^M, \| \cdot \|_p \right) \leq A_k^{1/p} \cdot e_{2k} \left( R_I X_n \cap B_p^M, \| \cdot \|_\infty \right)
\]

\[
\leq A_k^{1/p} \cdot e_{2k} \left( X_n \cap B_p^M, \| \cdot \|_\infty \right) \leq C_p \left( A_k \cdot \frac{\log M}{M} \cdot \frac{n}{2^k} \right)^{\frac{1}{p}} = C_p \left( \frac{n}{M(\log M)^3} \right)^{\frac{1}{p}}.
\]

Thus, we can partition \( \{ 1, 2, \cdots, N \} \) into at most \( N_{k-2} \) sets \( V_{\alpha,2}, \alpha \in A_2 \) so that (8.27) with \( V = V_{\alpha,2} \) is satisfied for each \( \alpha \in A_2 \).

Third, setting \( E := \{ R_I t_\ell : \ell = 1, 2, \cdots, N \} \), we have

\[
E \subset \{ u \in \mathbb{R}^{|I|} : \| u - R_I t_0 \|_2 \leq 4a \}.
\]

Since \( |I| \leq A_k \), \( E \) can be covered by at most \( 32|I| \leq N_{k-2} \) Euclidean balls of radius \( \frac{a}{4} \) in \( \mathbb{R}^{|I|} \). Thus, we may partition the set \( \{ 1, 2, \cdots, N \} \) into at most \( N_{k-2} \) sets \( V_{\alpha,3}, \alpha \in A_3 \) such that

\[
\max_{\ell, \ell' \in V_{\alpha,3}} \| R_I(t_\ell) - R_I(t_{\ell'}) \|_2 \leq \frac{a}{4}, \quad \alpha \in A_3.
\]
Since $\|t_\ell - t_{\ell'}\|_2 \geq a$ for any two distinct $\ell, \ell' \in \{1, \ldots, N\}$, this in turn implies that (8.23) with $V = V_{\alpha,3}$ is satisfied for each $\alpha \in \mathcal{A}_3$.

Finally, let $\{V_\alpha\}_{\alpha \in \mathcal{A}}$ denote the partition of $\{1, \ldots, N\}$ generated by the above three partitions. Then $\{V_\alpha\}_{\alpha \in \mathcal{A}}$ contains at most $N_{k-2}^3 < N_k$ sets for which (8.25), (8.26), and (8.27) with $V = V_\alpha$ are satisfied for all $\alpha \in \mathcal{A}$. Since $N_{k-2}^4 = N_k < N = N_{k+1}$, we can find a set $V$ from this last partition $\{V_\alpha\}_{\alpha \in \mathcal{A}}$ with $|V| > N_{k-2}$.

For the remainder of the proof, we will always use the letter $V$ to denote a subset of $\{1, \ldots, N\}$ with the properties stated in Lemma 8.2. Therefore, we will work on the set of points $f_\ell, \ell \in V$ instead.

Our second lemma can be used to obtain useful lower estimates of the quantity $S$.

**Lemma 8.3.** There exists a constant $c_p > 0$ depending only on $p$ such that

$$\min_{\ell, \ell' \in V, \ell \neq \ell'} \left( \|f_\ell - f_{\ell'}\|_\infty^p \right) \cdot S \geq c_p \cdot a^2.$$  (8.28)

In particular, this implies

$$S \geq c_p a^{2k/2} \sqrt{\frac{M}{n \log M \log M}} \geq \frac{c_p}{\log^2 M}.$$  (8.29)

**Proof.** We will use the following inequality. If $1 \leq p \leq 2$, $a, b > 0$ and $\eta > 0$, then

$$|a^p - b^p|^2 \leq 2^p p^2 (a^p_\eta + b^p_\eta)|b - a|^p + 2 \eta^p (a^p + b^p),$$  (8.30)

where $a_\eta = a \cdot \chi_{[\eta, \infty)}(a)$ and $b_\eta = b \cdot \chi_{[\eta, \infty)}(b)$. To show (8.30), without loss of generality, we may assume that $a \geq b$. If $a < \eta$, then

$$|a^p - b^p|^2 \leq 2a^p(a^p + b^p) \leq 2 \eta^p (a^p + b^p).$$

If $a \geq \eta$, then $a = a_\eta$ and

$$|a^p - b^p|^2 \leq (pa^{p-1}|b - a|)^2 = p^2 a^{2p-2} |b - a|^p |b - a|^{2-p} \leq p^2 (2a)^p |b - a|^p \leq 2^p p^2 (a_\eta^p + b_\eta^p)|b - a|^p.$$  (8.30)

In either case, we have (8.30).
Next, note that (8.30) implies that for each $J \subset \Omega_M$, $x, y \in B_p^M$ and $\eta \geq 0$,
\[
\left\| R_J x^p - R_J y^p \right\|^2 \leq 4\eta^p + C_p \| x - y \|_\infty^p A,
\]
where
\[
A := \max\left\{ \sum_{i \in J} |x(i)|^p \chi\{|x(i)| \geq \eta\}(i), \sum_{i \in J} |y(i)|^p \chi\{|y(i)| \geq \eta\}(i) \right\}.
\]
Thus, setting $\eta = \tau^{1/p}$, and using (8.25), we have that for any $\ell, \ell' \in V$ with $\ell \neq \ell'$,
\[
\frac{a^2}{2} \leq \sum_{i \in \Omega_M \setminus I} \left\| f_\ell(i) - f_{\ell'}(i) \right\|^p \leq 4\tau + C_p \| f_\ell - f_{\ell'} \|_\infty^p S.
\]
Since $a^{2^k} \geq \frac{n}{M \log M}$, we obtain from (8.9) with $C = C_1$ that
\[
4\tau = 2^{-k-k_0+3}(\log M)^4 \leq \frac{2^{-k-k_0+4n}}{C M (\log M)^{2+\frac{4}{2-p}}} \leq \frac{1}{4} a^2.
\]
The estimate (8.28) then follows from (8.32).

Finally, using (8.26), (8.28) and the assumption $a^{2^k} \geq \frac{n}{M \log M}$, we obtain
\[
S \geq c_p \frac{a^{2^k} m}{n \log M} \geq c_p a^{2^k/2} \sqrt{\frac{m}{n \log M \log M}} \geq \frac{c_p}{(\log M)^2},
\]
proving (8.29).

Our aim is to show (8.24), which is an improvement of the first inequality in (8.29). According to (8.28), for the proof of (8.24), it will suffice to show that there exist two distinct $\ell, \ell' \in V$ such that
\[
\| f_\ell - f_{\ell'} \|_\infty \leq C_p S \frac{n}{2^k M} \log M.
\]
The idea is to construct a set $U = U_S$ with the following two properties:

(i) $\{f_\ell - f_{\ell_0} : \ell \in V\} \subset U$ for some $\ell_0 \in V$;
(ii) there exists a partition \( \{ E_\alpha \}_{\alpha \in \mathcal{A}} \) of \( U \) such that \( |\mathcal{A}| < |V| \) and

\[
\operatorname{diam}(E_\alpha, \| \cdot \|_\infty) \leq C_p \left( \frac{S_n \log M}{2^k} \right)^{1/p}, \quad \forall \alpha \in \mathcal{A}.
\]

Since \( |V| > N_{k-2} \), to ensure (ii), it suffices to prove that

\[
e_{2^k-2}(U, \| \cdot \|_\infty) \leq C_p \left( \frac{S_n \log M}{2^k} \right)^{1/p}.
\]

To show (8.33), we need two additional lemmas. For \( \xi \geq 0 \), we define

\[
U(\xi) = \xi B_p^M + \tau \frac{1}{p} \cdot B_2^M \quad \text{and} \quad X_n(\xi) = U(\xi) \cap X_n.
\]

**Lemma 8.4.** Let \( \xi = (3S)^{1/p} \). Then

\[
\{ f_\ell - f_{\ell'} : \ell, \ell' \in V \} \subset 2X_n(\xi).
\]  

**Proof.** Fix \( \ell, \ell' \in V \) and set \( g := g_{\ell, \ell'} = f_\ell - f_{\ell'} \in X_n \). Our aim is to show that \( g \in 2U(\xi) \). First, by (8.27) and (8.29), we have

\[
\sum_{i \in I} |g(i)|^p = \sum_{i \in I} |f_\ell(i) - f_{\ell'}(i)|^p \leq \frac{C_p n \log M}{M \log M} \leq S,
\]

implying \( g \chi_I \in S^{1/p} \cdot B_p^M \). Second, setting \( \eta = \tau^{1/p} \), and using (8.23), we have

\[
2S \geq \sum_{i \in \Omega_M \setminus I} |g(i)|^p \chi_{\{|g(i)| \geq \eta\}}(i) + \sum_{i \in \Omega_M \setminus I} |f_\ell(i) - f_{\ell'}(i)|^p \chi_{\{|f_\ell(i)| \geq \eta\}}(i)
\]

\[
\geq 2^{-p} \sum_{i \in \Omega_M \setminus I} |g(i)|^p \chi_{\{|g(i)| \geq 2\eta\}}(i),
\]

where the last step uses the inequality,

\[
|u - v|^p \chi_{|u-v| \geq 2\eta} \leq 2^p \left( |u|^p \chi_{|u| \geq \eta} + |v|^p \chi_{|v| \geq \eta} \right), \quad u, v \in \mathbb{R}.
\]

This implies that

\[
g \chi_{\{|i| \geq 2\eta\}} \in 2 \cdot (2S)^{1/p} B_p^M.
\]

Finally, we write \( g = u + v \), where

\[
v := g \chi_{\{i : |g(i)| \geq 2\eta\}} \quad \text{and} \quad u := g \chi_{\{i : |g(i)| < 2\eta\}}.
\]
Clearly, \( \frac{1}{2}v \in (3S)^{1/p} \cdot B^M_p \) and
\[
\frac{u}{2} \in (\eta B^M_\infty) \cap B^M_p \subset \eta^{1-\frac{p}{2}} \cdot B^M_2.
\]
Thus, \( g \in 2U(\xi) \). \( \square \)

**Lemma 8.5.** Let \( \xi > 0 \) be such that
\[
\tau \leq \xi \frac{2p}{3 - p} \left( \frac{n \log M}{M} \right)^{1/p}.
\] (8.35)

Then
\[
e_j \left( X_n(\xi), \| \cdot \|_\infty \right) \leq C\xi \cdot \left( \frac{n \log M}{M} \right)^{1/p}, \quad j = 1, 2, \ldots, 2^k.
\] (8.36)

**Proof.** Note first that \( U(\xi) = \xi B^M_p + \tau \cdot \frac{1}{p-\frac{1}{2}} B^M_2 \) is a symmetric convex body in \( \mathbb{R}^M \). Let \( W \) denote the polar of \( U(\xi) \); that is,
\[
W := \left\{ x \in \mathbb{R}^M : \| x \|_W := \max_{y \in U(\xi)} x \cdot y \leq 1 \right\}.
\]
Then for each \( x \in W \),
\[
\| x \|_W = \sup_{\| u \|_p \leq \xi} \sup_{\| v \|_2 \leq \tau \cdot \frac{1}{p-\frac{1}{2}}} (x \cdot u + x \cdot v) \leq \| x \|_{p'} + \tau \cdot \frac{1}{p-\frac{1}{2}} \| x \|_2.
\]

By Lemma [8.1] this implies that for \( 1 \leq j \leq 2^k \),
\[
\begin{align*}
e_j (B^M_2 \cap X_n, \| \cdot \|_W) & \leq \xi \cdot e_j (B^M_2 \cap X_n, \| \cdot \|_{p'}) + 2\tau \cdot \frac{1}{p-\frac{1}{2}} \\
& \leq C\xi \cdot \left( \frac{n \log M}{M} \right)^{\frac{1}{p}-\frac{1}{2}} + 2\tau \cdot \frac{1}{p-\frac{1}{2}} \leq C\xi \cdot \left( \frac{n \log M}{M} \right)^{\frac{1}{p}-\frac{1}{2}},
\end{align*}
\]
where the last step uses (8.35). By duality (see [34, Theorem 16.8.10]), we deduce
\[
e_j (X_n(\xi), \| \cdot \|_2) \leq C\xi \cdot \left( \frac{n \log M}{M} \right)^{\frac{1}{p}-\frac{1}{2}}, \quad j = 1, 2, \ldots, 2^k.
\] (8.37)

Finally, using (8.37) and Lemma [8.1] we have that for \( 1 \leq j \leq 2^{k-1} \),
\[
e_{2j} (X_n(\xi), \| \cdot \|_\infty) \leq 2e_j (X_n(\xi), \| \cdot \|_2)e_j (X_n \cap B^M_2, \| \cdot \|_\infty) \leq C\xi \cdot \left( \frac{n \log M}{M} \right)^{\frac{1}{p}}.
\]

The stated estimate then follows by monotonicity. \( \square \)
Now we are in a position to prove Proposition 8.1.

Proof of Proposition 8.1. Let $\xi = (3S)^{1/p}$. A straightforward calculation using (8.29) and (8.9) shows that the condition (8.35) is satisfied. Consequently, using Lemma 8.5, we obtain

$$e_{2k-2} \left( X_n(\xi), \| \cdot \|_\infty \right) \leq C\xi \cdot \left( \frac{n \log M}{2^k} \right)^{1/p}.$$  \hfill (8.38)

Thus, we may partition the set $2X_n(\xi)$ into $N_{k-2}$ sets $E_\gamma$, $\gamma \in \Gamma$ such that

$$\text{diam}(E_\gamma, \| \cdot \|_\infty) \leq C\xi \cdot \left( \frac{n \log M}{2^k} \right)^{1/p}, \quad \forall \gamma \in \Gamma.$$  \hfill (8.39)

On the other hand, Lemma 8.4 implies that for each fixed $\ell_0 \in V$,

$$\{ f_\ell - f_{\ell_0} : \ell \in V \} \subset 2X_n(\xi).$$

Since $|V| > N_{k-2}$ and $|\Gamma| = N_{k-2}$, we can find two distinct $\ell, \ell' \in V$ such that $f_\ell - f_{\ell_0}, f_{\ell'} - f_{\ell_0}$ lie in a same set of the partition $\{ E_\gamma \}_{\gamma \in \Gamma}$, which, using (8.39), implies

$$\| f_\ell - f_{\ell'} \|_\infty \leq C\xi \cdot \left( \frac{n \log M}{2^k} \right)^{1/p}.$$  \hfill (8.40)

Thus, there exist two distinct $\ell, \ell' \in V$ such that the estimate (8.33) holds, which, using (8.28), in turn implies the desired estimate (8.24). \hfill \Box

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F. Dai  
DEPARTMENT OF MATHEMATICAL AND STATISTICAL SCIENCES  
UNIVERSITY OF ALBERTA, EDMONTON, ALBERTA T6G 2G1, CANADA  
E-MAIL: fdai@ualberta.ca

E. Kosov  
STEKLOV MATHEMATICAL INSTITUTE OF RUSSIAN ACADEMY OF SCIENCES, MOSCOW, RUSSIA; NATIONAL RESEARCH UNIVERSITY HIGHER SCHOOL OF ECONOMICS, RUSSIAN FEDERATION  
E-MAIL: ked−2006@mail.ru

V.N. Temlyakov  
UNIVERSITY OF SOUTH CAROLINA, 1523 GREENE ST., COLUMBIA SC, 29208, USA; STEKLOV INSTITUTE OF MATHEMATICS; LOMONOSOV MOSCOW STATE UNIVERSITY, AND MOSCOW CENTER FOR FUNDAMENTAL AND APPLIED MATHEMATICS.  
E-MAIL: temlyak@math.sc.edu