SUPREMA OF LÉVY PROCESSES

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In this paper we study the supremum functional \( M_t = \sup_{0 \leq s \leq t} X_s \), where \( X_t, t \geq 0 \), is a one-dimensional Lévy process. Under very mild assumptions we provide a simple uniform estimate of the cumulative distribution function of \( M_t \). In the symmetric case we find an integral representation of the Laplace transform of the distribution of \( M_t \) if the Lévy-Khintchin exponent of the process increases on \((0, \infty)\).

1. Introduction. By a classical reflection argument, the supremum functional \( M_t = \sup_{0 \leq s \leq t} X_s \) of the Brownian motion \( X_t \) has truncated normal distribution, \( \mathbf{P}(M_t \geq x) = 2\mathbf{P}(X_t \geq x) \) \( (x \geq 0) \). A similar question for symmetric \( \alpha \)-stable processes was first studied by Darling [11], and the case of general Lévy processes \( X_t \) was addressed by Baxter and Donsker [3]. Theorem 1 therein gives a formula for the double Laplace transform of the distribution of \( M_t \), which for a symmetric Lévy process \( X_t \) with Lévy-Khintchin exponent \( \Psi(\xi) \) reads

\[
\int_0^{\infty} \int_0^{\infty} e^{-\xi x - z t} \mathbf{P}(M_t \in dx) dt = \frac{1}{\sqrt{z}} \exp \left( -\frac{1}{\pi} \int_0^{\infty} \frac{\xi \log(z + \Psi(\xi))}{\xi^2 + \xi^2} d\xi \right). 
\]

Inversion of the double Laplace transform is typically a very difficult task. Apart from the Brownian motion case, an explicit formula for the distribution of \( M_t \) was found for the Cauchy process (the symmetric 1-stable process) by Darling [11], for a compound Poisson process with \( \Psi(\xi) = 1 - \cos \xi \) by Baxter and Donsker [3] and for the Poisson process with drift by Pyke [32].

The development of the fluctuation theory for Lévy processes resulted in many new identities involving the supremum functional \( M_t \), see, for example,
There are numerous other representations for the distribution of $M_t$, at least in the stable case, see \cite{4, 7, 11, 12, 15, 16, 19, 20, 27, 28, 30, 36}. The main goal of this article is to give a more explicit formula for $\mathbb{P}(M_t < x)$ and simple sharp bounds for $\mathbb{P}(M_t < x)$ in terms of the Lévy-Khintchin exponent $\Psi(\xi)$ for a class of Lévy processes. Most estimates of the cumulative distribution function of $M_t$ are proved for very general Lévy processes, without symmetry assumptions.

Let $\tau_x$ denote the first passage time through a barrier at the level $x$ for the process $X_t$,

$$\tau_x = \inf \{ t \geq 0 : X_t \geq x \}, \quad x \geq 0,$$

with the infimum understood to be infinity when the set is empty. We always assume that $X_0 = 0$. Since $\mathbb{P}(M_t < x) = \mathbb{P}(\tau_x > t)$, the problems of finding the cumulative distribution functions of $M_t$ and $\tau_x$ are the same. The supremum functional and first passage time statistics are important in various areas of applied probability (\cite{1, 2}), as well as in mathematical physics (\cite{21, 26}). The recent progress in the potential theory of Lévy processes is, in part, due to the application of fluctuation theory, see \cite{9, 10, 18, 22, 23, 24, 25}.

The paper is organized as follows. Section 2 contains some preliminary material related to Bernstein functions, Stieltjes functions and estimates for the Laplace transform. In Section 3 (Theorem 3.1 and Corollary 3.2) we prove, under mild assumptions, the estimate

$$\mathbb{P}(M_t < x) \approx \min \left(1, \kappa(1/t, 0)V(x)\right), \quad t, x > 0,$$

where $V(x)$ and $\kappa(z, 0)$ are the renewal function for the ascending ladder-height process, and the Laplace exponent of the the ascending ladder-time process corresponding to $X_t$, respectively. Here $f(x) \approx g(x)$ means that there are constants $c_1, c_2 > 0$ such that $c_1 g(x) \leq f(x) \leq c_2 g(x)$. In Section 4 we show that in the symmetric case, given some regularity of $\Psi(\xi)$, we have

$$V(x) \approx \frac{1}{\sqrt{\Psi(1/x)}}, \quad x > 0,$$

see Theorem 4.4. Therefore the estimate of the above cumulative distribution function of $M_t$ takes a very explicit form

$$\mathbb{P}(M_t < x) \approx \min \left(1, \frac{1}{\sqrt{t\Psi(1/x)}}\right), \quad t, x > 0.$$
The other main result of Section 4 is an explicit formula for the (single, in the space variable) Laplace transform of the distribution of $M_t$ (Theorem 4.1), under the assumption that $X_t$ is symmetric and $\Psi(\xi)$ is increasing on $[0, \infty)$.

When $\Psi(\xi) = \psi(\xi^2)$ for a complete Bernstein function $\psi(\xi)$, the above results can be significantly improved. Following the approach of [30], a (rather complicated) explicit formula for $P(M_t < x)$ can be given, and estimates and asymptotic formulae for $P(M_t < x)$ extend to $(d/dt)^nP(M_t < x)$ when $x$ is small or $t$ is large. These results will be covered in a forthcoming paper.

**Notation.** We denote by $C$, $C_1$, $C_2$ etc. constants in theorems, and by $c$, $c_1$, $c_2$ etc. temporary constants in proofs. Any dependence of a constant on some parameters is always indicated by writing, for example, $c(n, \varepsilon)$. We write $f(x) \sim g(x)$ when $f(x)/g(x) \to 1$. We use the terms increasing, decreasing, concave, convex function etc. in the weak sense.

**2. Preliminaries.**

**2.1. Complete Bernstein and Stieltjes functions.** A function $\psi(\xi)$ is said to be a complete Bernstein function (CBF) if

\[
\psi(\xi) = c_1 + c_2\xi + \frac{1}{\pi} \int_{0^+}^{\infty} \frac{\xi}{\xi + \zeta} \frac{\mu(d\zeta)}{\zeta}, \quad \xi \in C \setminus (-\infty, 0),
\]

where $c_1, c_2 \geq 0$, and $\mu$ is a measure on $(0, \infty)$ such that the integral $\int_0^{\infty} \min(\zeta^{-1}, \zeta^{-2})\mu(d\zeta)$ is finite. A function $\tilde{\psi}(\xi)$ is said to be a Stieltjes functions if

\[
\tilde{\psi}(\xi) = \tilde{c}_1 + \tilde{c}_2 + \frac{1}{\pi} \int_{0^+}^{\infty} \frac{1}{\xi + \zeta} \tilde{\mu}(d\zeta), \quad \xi \in C \setminus (-\infty, 0],
\]

for some $\tilde{c}_1, \tilde{c}_2 \geq 0$ and some measure $\tilde{\mu}$ on $(0, \infty)$ such that the integral $\int_0^{\infty} \min(1, \zeta^{-1})\tilde{\mu}(d\zeta)$ is finite. See [34] for a general account on complete Bernstein functions, Stieltjes functions and related notions.

It is known that $\psi(\xi)$ is a CBF if and only if $\tilde{\psi}(\xi)$ is nonnegative and increasing on $(0, \infty)$, holomorphic in $C \setminus (-\infty, 0]$, and $\text{Im} \psi(\xi) > 0$ when $\text{Im} \xi > 0$. Furthermore, if $\psi(\xi)$ is a CBF, then $\xi/\psi(\xi)$ is a CBF, and $1/\psi(\xi)$ and $\psi(\xi)/\xi$ are Stieltjes functions.

The function $\tilde{\psi}(\xi)$ given by (2.2) is the Laplace transform of $\tilde{c}_2\delta_0(dx) + (\tilde{c}_1 + L\tilde{\mu}(x))dx$ ([34], Theorem 2.2). Furthermore, $\pi\tilde{c}_1\delta_0(d\zeta) + \tilde{\mu}(d\zeta)$ is the limit of measures $-\text{Im}(\tilde{\psi}(\zeta + i\varepsilon))d\zeta$ as $\varepsilon \to 0^+$ ([34], Corollary 6.3 and Comments 6.12), so, in a sense, it is the boundary value of $\tilde{\psi}$. Therefore, we use a shorthand notation $-\text{Im}(\tilde{\psi}^+(\zeta))d\zeta$ for $\tilde{\mu}(d\zeta)$. Furthermore, we have $\tilde{c}_1 = \lim_{\xi \to 0}(\xi \psi(\xi))$ and $\tilde{c}_2 = \lim_{\xi \to \infty} \psi(\xi)$.  
Following [30], we define
\begin{equation}
\psi^\dagger(\xi) = \exp \left( \frac{1}{\pi} \int_0^\infty \frac{\xi \log \psi(\zeta^2)}{\xi^2 + \zeta^2} d\zeta \right), \quad \Re \xi > 0,
\end{equation}
for any function \( \psi(\xi) \) such that \( \min(1, \zeta^{-2}) \log \psi(\zeta^2) \) is integrable in \( \zeta > 0 \).

By a simple substitution,
\begin{equation}
\psi^\dagger(\xi) = \exp \left( \frac{1}{\pi} \int_0^\infty \log \frac{\psi(\xi^2 \zeta^2)}{1 + \zeta^2} d\zeta \right), \quad \xi > 0.
\end{equation}

By [30], Lemma 4, if \( \psi(\xi) \) is a CBF, then also \( \psi^\dagger(\xi) \) is a CBF (this was independently proved in [24], Proposition 2.4), and
\begin{equation}
\psi^\dagger(\xi) \psi^\dagger(-\xi) = \psi(-\xi^2), \quad \xi \in \mathbb{C} \setminus \mathbb{R}.
\end{equation}

**Proposition 2.1.** If \( \psi(\xi) \) is nonnegative on \( (0, \infty) \) and both \( \psi(\xi) \) and \( \xi/\psi(\xi) \) are increasing on \( (0, \infty) \), then
\begin{equation}
e^{-2C/\pi} \sqrt{\psi(\xi^2)} \leq \psi^\dagger(\xi) \leq e^{2C/\pi} \sqrt{\psi(\xi^2)},
\end{equation}
where \( C \approx 0.916 \) is the Catalan constant. Note that \( e^{2C/\pi} \leq 2 \).

If, in addition, \( \psi(\xi) \) is regularly varying at \( \infty \), then
\begin{equation}
\psi^\dagger(\xi) \sim \sqrt{\psi(\xi^2)}, \quad \xi \to \infty.
\end{equation}

An analogous statement for \( \xi \to 0 \) holds for \( \psi(\xi) \) regularly varying at \( 0 \).

In particular, (2.6) holds for any CBF. Likewise, (2.7) holds for any regularly varying CBF.

A result similar to (2.6) was obtained independently in [25], Proposition 3.7, while (2.7) for CBFs was derived in [22], Proposition 2.2.

**Proof.** By the assumptions, we have
\begin{equation}
\psi(\xi^2) \min(1, \zeta^{-2}) \leq \psi(\xi^2 \zeta^2) \leq \psi(\xi^2) \max(1, \zeta^{-2}), \quad \xi, \zeta > 0.
\end{equation}

It follows that
\begin{align*}
\psi^\dagger(\xi) &= \exp \left( \frac{1}{\pi} \int_0^\infty \log \frac{\psi(\xi^2 \zeta^2)}{1 + \zeta^2} d\zeta \right) \\
&\leq \sqrt{\psi(\xi^2)} \exp \left( \frac{1}{\pi} \int_1^\infty \log \frac{\zeta^2}{1 + \zeta^2} d\zeta \right) = e^{2C/\pi} \sqrt{\psi(\xi^2)}.
\end{align*}
The lower bound is obtained in a similar manner.

The second statement of the proposition is proved in a very similar manner to Lemma 15 in [30]. Define an auxiliary function $h(\xi, \zeta) = \psi(\xi^2 \zeta^2)/\psi(\xi^2)$. By (2.8) we have $|\log h(\xi, \zeta)| \leq 2|\log \zeta|, \xi, \zeta > 0$. Since $\psi$ is regularly varying at infinity, for some $\alpha$, $\lim_{\xi \to \infty} h(\xi, \zeta) = \zeta^{2\alpha}$ for each $\zeta > 0$. Hence, by dominated convergence,

$$
\lim_{\xi \to \infty} \int_{0}^{\infty} \frac{\log h(\xi, \zeta)}{1 + \zeta^2} d\zeta = \int_{0}^{\infty} \frac{\log \zeta^{2\alpha}}{1 + \zeta^2} d\zeta = 0.
$$

It follows that

$$
\lim_{\xi \to \infty} \left( \int_{0}^{\infty} \frac{\log(\xi^2 \zeta^2)}{1 + \zeta^2} d\zeta - \frac{\pi}{2} \log \psi(\zeta^2) \right) = 0,
$$

and so finally $\lim_{\xi \to \infty} \psi^\dagger(\xi)/\sqrt{\psi(\xi^2)} = 1$, as desired. Regular variation at 0 is proved in a similar way.

As in [30], for differentiable functions $\psi(\xi)$ with positive derivative we define

$$
\psi^\lambda(\xi) = \frac{1 - \xi/\lambda^2}{1 - \psi(\xi)/\psi(\lambda^2)}, \quad \lambda > 0, \xi \in \mathbb{C} \setminus (-\infty, 0).
$$

This definition is extended continuously by $\psi^\lambda(\lambda^2) = \psi(\lambda^2)/(\lambda^2 \psi'(\lambda^2))$. Note that if $\psi(0) = 0$, then $\psi^\lambda(0) = 1$. For simplicity, we denote $\psi^\dagger(\xi) = (\psi^\lambda)^\dagger(\xi)$. By [30], Lemma 2, if $\psi(\xi)$ is a CBF, then $\psi^\lambda(\xi)$ is a CBF for any $\lambda > 0$.

2.2. Estimates for the Laplace transform. This short section contains some rather standard estimates for the inverse Laplace transform.

**Proposition 2.2.** Let $a > 0$, $c \geq 1$. If $f$ is nonnegative and $f(x) \leq cf(a)\max(1, x/a)$ ($x > 0$), then for any $\xi > 0$,

$$
f(a) \geq \frac{\xi \mathcal{L} f(\xi)}{c(1 + (a \xi)^{-1} e^{-a \xi})}.
$$

**Proof.** We have

$$
\xi \mathcal{L} f(\xi) = \int_{0}^{a} \xi e^{-\xi x} f(x) dx + \int_{a}^{\infty} \xi e^{-\xi x} f(x) dx
$$

$$
\leq cf(a) \int_{0}^{a} \xi e^{-\xi x} dx + \frac{cf(a)}{a} \int_{a}^{\infty} \xi e^{-\xi x} dx
$$

$$
= cf(a)(1 - e^{-a \xi}) + \frac{cf(a)}{a \xi} (1 + a \xi) e^{-a \xi} = cf(a)(1 + (a \xi)^{-1} e^{-a \xi}),
$$

as desired. \hfill \Box
Proposition 2.3. If $f$ is nonnegative and increasing, then for $a, \xi > 0$,

$$f(a) \leq e^{\alpha \xi} \xi \mathcal{L}f(\xi).$$

Proof. As before,

$$\xi \mathcal{L}f(\xi) = \int_0^a \xi e^{-\xi x} f(x) dx + \int_a^\infty \xi e^{-\xi x} f(x) dx \geq f(a) \int_a^\infty \xi e^{-\xi x} dx = f(a) e^{-\alpha \xi},$$

as claimed.

Proposition 2.4. If $f$ is nonnegative and decreasing, then for $a, \xi > 0$,

$$f(a) \leq \frac{\xi \mathcal{L}f(\xi)}{1 - e^{-\alpha \xi}}.$$

Proof. Again,

$$\xi \mathcal{L}f(\xi) = \int_0^a \xi e^{-\xi x} f(x) dx + \int_a^\infty \xi e^{-\xi x} f(x) dx \geq f(a) \int_0^a \xi e^{-\xi x} dx = f(a)(1 - e^{-\alpha \xi}),$$

as claimed.

3. Suprema of general Lévy processes. We briefly recall the basic notions of the fluctuation theory for Lévy processes. Let $L_t$ be the local time of the process $X_t$ reflected at its supremum $M_t$, and denote by $L_{s^{-1}}$ the right-continuous inverse of $L_t$, the ascending ladder-time process for $X_t$. This is a (possibly killed) subordinator, and $H_s = X(L_{s^{-1}}) = M(L_{s^{-1}})$ is another (possibly killed) subordinator, called the ascending ladder-height process. The Laplace exponent of the ascending ladder process, that is, the (possibly killed) bivariate subordinator $(L_{s^{-1}}, H_s)$ $(s < L(\infty))$, is denoted by $\kappa(z, \xi)$. By, e.g., [5], Corollary VI.10,

$$\kappa(z, \xi) = c \exp \left( \int_0^\infty \int_0^\infty (e^{-t} - e^{-z t - \xi x}) t^{-1} P(X_t \in dx) dt \right),$$

where $c$ is a normalization constant of the local time. Since our results are not affected by the choice of $c$, we assume that $c = 1$. We note that $\kappa(z, 0)$ is a Bernstein function of $z$, and also $z/\kappa(z, 0)$ is a Bernstein function (this follows from (3.1) by Frullani’s integral; see [5], formula (VI.3) for the case...
when \(X_t\) is not a compound Poisson process). For more account on the fluctuation theory we refer the reader to [5, 13, 31]. In general, there is no closed-form formula for \(\kappa(z, \xi)\). For a list of special cases, see [29] and the references therein. For a symmetric process which is not a compound Poisson process, we have \(\kappa(z, 0) = \sqrt{z}\).

As usual, \(\tau_x\) denotes the first passage time through a barrier at \(x \geq 0\) for \(X_t\) (or for \(M_t\)). Following [5], for \(x, z \geq 0\) we define

\[
V^z(x) = \mathbb{E} \left( \int_0^\infty \exp(-z L_s^{-1}) 1_{[0,x)}(H_s) ds \right) = \mathbb{E} \left( \int_0^\infty e^{-zt} 1_{[0,x)}(M_t) dL_t \right).
\]

For \(z = 0\), we simply have \(V^0(x) = \int_0^\infty \mathbb{P}(H_s < x) ds\), so that \(V^0(x) = V(x)\) is the renewal function of the process \(H_s\), studied in more detail for symmetric Lévy processes in Section 4. By [5], formula (VI.8),

\[
\int_0^\infty e^{-zt} \mathbb{P}(M_t < x) dt = \frac{\kappa(z, 0)V^z(x)}{z}, \quad x, z \geq 0.
\]

(Note that in [5] a weak inequality \(M_t \leq x\) is used in the definition of \(V^z(x)\).) Hence, for a symmetric process \(X_t\) which is not a compound Poisson process, we have

\[
\int_0^\infty e^{-zt} \mathbb{P}(M_t < x) dt = \frac{V^z(x)}{\sqrt{z}}, \quad x, z \geq 0.
\]

This is a partial inverse of the double Laplace transform in (1.1); however, there is no known explicit formula for \(V^z(x)\). For a different and, in a sense, more explicit partial inverse, see (4.2) below.

By [5], Section VI.4, the Laplace transform of \(V^z(x)\) is \(1/(\xi \kappa(z, \xi))\). Hence, when \(X_t\) is symmetric and it is not a compound Poisson process, the right hand side of the Baxter-Donsker formula (1.1) can be written as \(\sqrt{z}/(z \kappa(z, \xi))\) (see [14], Corollary 9.7).

**Theorem 3.1.** Let \(X_t\) be a Lévy process, \(M_t = \sup_{0 \leq s \leq t} X_s\), and let \(\kappa(z, \xi)\) be the bivariate Laplace exponent of its ascending ladder process. Suppose that

\[
K(s) = \int_s^\infty \frac{\kappa(z, 0)}{z^2} dz < \infty, \quad s > 0,
\]

and that \(\kappa(z, 0)/z\) is unbounded (near 0). For \(t, x > 0\), we have

\[
\min \left( C_1, C_2(\kappa, t)\kappa(1/t, 0)V(x) \right) \leq \mathbb{P}(M_t < x) \leq \min \left( 1, \frac{e}{e - 1} \kappa(1/t, 0)V(x) \right).
\]
Here

\[ C_1 = \frac{e - 1}{8e^2} \quad \text{and} \quad C_2(\kappa, t) = \frac{zt}{2e}, \]

where \( z \in (0, 1/t) \) solves

\[ \frac{\kappa(z, 0)}{z} = \frac{4e^2}{e - 1} K(1/t). \]

**Proof.** The upper bound in (3.5) is a direct consequence of (3.2) and Proposition 2.4 with \( \xi = 1/t \).

Following [5], Lemma VI.21, we find a lower bound for \( V^z(x) \). We have

\[
V(x) = \mathbb{E} \left( \int_0^\infty 1_{[0,x)}(M_t) dL_t \right)
\leq e \mathbb{E} \left( \int_0^{1/z} e^{-zt} 1_{[0,x)}(M_t) dL_t \right) + \mathbb{E} \left( \int_{1/z}^\infty 1_{[0,x)}(M_t) dL_t \right),
\]

which implies

(3.6) \[ eV^z(x) \geq V(x) - \mathbb{E} \left( \int_{1/z}^\infty 1_{[0,x)}(M_t) dL_t \right). \]

Let \( \sigma_z = \inf\{ t \geq 1/z : X_t = M_t \} = L^{-1}(L_{1/z}); \) \( \sigma_z \) is a stopping time. Since the support of the measure \( dL_t \) is contained in the set \( \{ t : X_t = M_t \} \) of zeros of the reflected process, we have

\[
\mathbb{E} \left( \int_{1/z}^\infty 1_{[0,x)}(M_t) dL_t \right) = \mathbb{E} \left( \int_{\sigma_z}^\infty 1_{[0,x)}(M_t) dL_t; M_{1/z} < x \right)
\leq \mathbb{E} \left( \int_{\sigma_z}^\infty 1_{[0,x)}(M_t - M_{\sigma_z}) dL_t; M_{1/z} < x \right).
\]

Next, observe that \( M_{\sigma_z} = X_{\sigma_z}, \) so that

\[ M_t - M_{\sigma_z} = \sup_{s \leq t - \sigma_z} (X_{\sigma_z+s} - X_{\sigma_z}), \quad t \geq \sigma_z. \]

Hence,

\[
\mathbb{E} \left( \int_{1/z}^\infty 1_{[0,x)}(M_t) dL_t \right)
\leq \mathbb{E} \left( \int_{\sigma_z}^\infty 1_{[0,x)} \left( \sup_{s \leq t - \sigma_z} (X_{\sigma_z+s} - X_{\sigma_z}) \right) dL_t; M_{1/z} < x \right)
= \mathbb{E} \left( \int_0^{\infty} 1_{[0,x)} \left( \sup_{s \leq u} (X_{\sigma_z+s} - X_{\sigma_z}) \right) d(L_{\sigma_z+u} - L_{\sigma_z}); M_{1/z} < x \right).
\]
Since $\sigma_z \geq 1/z$, by the strong Markov property,

$$E \left( \int_{1/z}^{\infty} \mathbf{1}_{[0,x)}(M_t)dL_t \right) \leq P(M_{1/z} < x)E \left( \int_{0}^{\infty} \mathbf{1}_{[0,x)}(M_u)dL_u \right) = P(M_{1/z} < x)V(x),$$

which, by (3.6), yields

$$V^z(x) \geq \frac{(1 - P(M_{1/z} < x))V(x)}{e} = \frac{P(M_{1/z} \geq x)V(x)}{e}.$$

Let $k > 0$. By (3.2) and the already proved upper bound of (3.5),

$$V^z(x)\kappa(z,0) = z \int_0^{k/z} e^{-zt}P(M_t < x)dt + z \int_{k/z}^{\infty} e^{-zt}P(M_t < x)dt \leq \frac{e}{e - 1} V(x)z \int_0^{k/z} e^{-zt}\kappa(1/t,0)dt + P(M_{k/z} < x).$$

The last two estimates give

$$P(M_{k/z} < x) \geq \frac{\kappa(z,0)P(M_{1/z} \geq x)V(x)}{e} - \frac{e}{e - 1} V(x)z \int_0^{k/z} \kappa(1/t,0)dt$$

$$= \frac{V(x)\kappa(z,0)}{e} \left( P(M_{1/z} \geq x) - \frac{e^2}{e - 1} \frac{zK(z/k)}{\kappa(z,0)} \right).$$

Fix $\varepsilon \in (0,1)$ (later we choose $\varepsilon = 1/4$). Note that the function $\kappa(z,0)/z$ is continuous, decreasing and unbounded. Hence, it maps the interval $(0,1/t]$ onto the interval $[t\kappa(1/t,0), \infty)$. Furthermore, $\kappa(z,0)$ is increasing, so that $K(z) \geq \kappa(z,0)/z$. In particular, $\frac{e^2}{e - 1} K(1/t) > K(1/t) \geq t\kappa(1/t,0)$. It follows that we can choose $z = z(t) < 1/t$ such that

$$\frac{\kappa(z,0)}{z} = \frac{e^2}{\varepsilon(e - 1)} K(1/t).$$

Setting $k = zt < 1$, the above equality can be rewritten as

$$\frac{e^2}{e - 1} \frac{zK(z/k)}{\kappa(z,0)} = \varepsilon.$$

Suppose now that $V(x)\kappa(z,0) \leq \varepsilon(e - 1)/e$. Then, by the upper bound of (3.5), we have $P(M_{1/z} \geq x) = 1 - P(M_{1/z} < x) \geq 1 - \varepsilon$. This, (3.7) and (3.8) give

$$P(M_t < x) = P(M_{k/z} < x) \geq \frac{V(x)\kappa(z,0)}{e} (1 - 2\varepsilon).$$
This estimate holds for $t \geq t_0$, where $V(x)\kappa(z(t_0), 0) = \varepsilon(e - 1)/e$ (here we use continuity of $\kappa(z(t), 0)$ as a function of $t$). Hence, by monotonicity of $P(M_t < x)$ in $t$,

$$P(M_t < x) \geq \min \left( \frac{\varepsilon(1 - 2\varepsilon)(e - 1)}{e^2}, \frac{(1 - 2\varepsilon)V(x)\kappa(z(0))}{e} \right).$$

The lower bound in (3.5) follows by taking $\varepsilon = 1/4$ and using the inequality $\kappa(z, 0) = \kappa(k/t, 0) \geq k\kappa(1/t, 0)$.

To formulate the next result we define upper scaling conditions:

\begin{align}
\text{(3.9)} & \quad \text{for some } \varrho \in (0, 1) \text{ and } c > 0, \quad \frac{\kappa(z_2, 0)}{\kappa(z_1, 0)} \leq c \frac{z_2^\varrho}{z_1^\varrho} \quad \text{when } 0 < z_1 < z_2 < 1, \\
\text{(3.10)} & \quad \text{for some } \varrho \in (0, 1) \text{ and } c > 0, \quad \frac{\kappa(z_2, 0)}{\kappa(z_1, 0)} \leq c \frac{z_2^\varrho}{z_1^\varrho} \quad \text{when } 1 < z_1 < z_2.
\end{align}

Observe that the condition (3.10) implies that for any $z^* > 0$ there is $c^*$ such that

$$\frac{\kappa(z_2, 0)}{\kappa(z_1, 0)} \leq c^* \frac{z_2^\varrho}{z_1^\varrho} \quad \text{when } z^* < z_1 < z_2.$$

Corollary 3.2. Let $X_t$ be a Lévy process, $M_t = \sup_{0 \leq s \leq t} X_s$, and let $\kappa(z, \xi)$ be the bivariate Laplace exponent of its ascending ladder process. If $\kappa(z, 0)$ satisfies condition (3.9) with $0 < \varrho < 1$ and the integral $\int_1^\infty \kappa(z, 0)z^{-2}dz$ is finite then

$$C(\kappa) \min(1, \kappa(1/t, 0)V(x)) \leq P(M_t < x) \leq \min(1, 2\kappa(1/t, 0)V(x)),$$

for every $x > 0$ and $t \geq 1$. If $\kappa(z, 0)$ satisfies (3.10) with $0 < \varrho < 1$ and $\lim_{z \to 0} z/\kappa(z, 0) = 0$ then (3.12) holds for $x > 0$ and $t \leq 1$.

In particular, if $\kappa(z, 0)$ satisfies both (3.9) and (3.10), that is, there are $c > 0$ and $\varrho \in (0, 1)$ such that $\kappa(\lambda z, 0) \leq c\lambda^\varrho\kappa(z, 0)$ for $\lambda \geq 1$ and $z > 0$, then (3.12) is true for every $x > 0$ and $t > 0$.

Proof. We begin with the first part of the statement. By the condition (3.9),

$$\kappa(z, 0) \leq c_1(\kappa) \left( \frac{z}{s} \right)^\varrho \kappa(s, 0), \quad s \leq z \leq 1.$$
In particular, $\kappa(s,0)/s$ is unbounded. Furthermore, using also finiteness of the integral $\int_1^\infty \kappa(z,0)z^{-2}dz$, we obtain

$$K(s) \leq c_2(\kappa)\frac{\kappa(s,0)}{s}, \quad s \leq 1. \quad (3.13)$$

This implies that the assumptions of Theorem 3.1 are satisfied.

Let $t \geq 1$ and define $z = z(t) \in (0,1/t)$ as in Theorem 3.1. By the condition (3.9) we have

$$\frac{\kappa(1/t,0)}{\kappa(z,0)} \leq \frac{c_3(\kappa)}{(zt)^{\varrho}}.$$ 

By definition of $z$ and (3.13) (with $s = 1/t$), we have

$$\frac{1}{z} = \frac{4e^2}{e - 1} \frac{K(1/t)}{\kappa(z,0)} \leq \frac{4e^2c_2(\kappa)c_3(\kappa)}{e - 1} \frac{t}{(zt)^{\varrho}},$$

which gives $zt \geq c_4(\kappa)$. Hence, the constant $C_2$ in Theorem 3.1 satisfies $C_2 = zt/(2e) \geq c_4(\kappa)/(2e)$. This ends the proof of the first part.

The second part can be justified in a similar way, since the condition (3.10) implies that

$$K(s) \leq c_5(\kappa)\frac{\kappa(s,0)}{s}, \quad s \geq 1.$$ 

Moreover, for $t < 1$ and $z = z(t)$ selected according to Theorem 3.1 we have $z(1) \leq z(t) < 1/t$. Applying (3.10) (with $z^* = z(1)$) we obtain

$$\frac{\kappa(1/t,0)}{\kappa(z,0)} \leq \frac{c_6(\kappa)}{(zt)^{\varrho}}, \quad z \leq \frac{1}{t}.$$

Finally, the last statement is a direct consequence of the previous ones. 

**Remark 3.3.** Due to Potter’s theorem ([8], Theorem 1.5.6) the condition (3.9) is implied by regular variation of $\kappa(z,0)$ at zero with index $0 < \varrho^* < 1$. Likewise, the condition (3.10) is implied by regular variation of $\kappa(z,0)$ at $\infty$ with index $0 < \varrho^* < 1$.

In the second part of the above corollary the assumption $\lim_{z \to 0} z/\kappa(z,0) = 0$ can be removed at the expense that the lower bound holds for $t \leq t_0$, where $t_0 = t_0(\kappa)$ is sufficiently small. This is due to the fact that since $\lim_{t \to 0} K(1/t) = 0$, $z = z(t)$ in Theorem 3.1 is well defined for $t$ small enough.
By the results of [5], Theorem VI.14, and [6], the regular variation of order \( g \in (0, 1) \) of \( \kappa(z, 0) \) at 0 or at \( \infty \) is equivalent to the existence of the limit of \( P(X_t \geq 0) \) as \( t \to \infty \) or \( t \to 0^+ \), respectively. Hence, Corollary 3.2 implies the following result.

**Corollary 3.4.** Let \( X_t \) be a Lévy process and \( M_t = \sup_{0 \leq s \leq t} X_s \). If

\[
\lim_{t \to \infty} P(X_t \geq 0) \in (0, 1) \quad \text{and} \quad \limsup_{t \to 0^+} P(X_t \geq 0) < 1,
\]

then (3.12) holds for \( x > 0 \) and \( t \geq 1 \). If

\[
\lim_{t \to 0^+} P(X_t \geq 0) \in (0, 1) \quad \text{and} \quad \limsup_{t \to \infty} P(X_t \geq 0) < 1,
\]

then (3.12) is true for \( x > 0 \) and \( t \leq 1 \). Finally, if

\[
\lim_{t \to \infty} P(X_t \geq 0) \in (0, 1) \quad \text{and} \quad \lim_{t \to 0^+} P(X_t \geq 0) \in (0, 1),
\]

then (3.12) holds for every \( x > 0 \) and \( t > 0 \).

**Proof.** We only need to verify that \( \kappa(z, 0)/z^2 \) is integrable at infinity, and that \( \lim_{z \to 0^+} (z/\kappa(z, 0)) = 0 \). In each of the cases, there is \( \varepsilon > 0 \) such that \( P(X_t \geq 0) \leq 1 - \varepsilon \) for all \( t > 0 \). Therefore, by (3.1) and the Frullani integral, \( \kappa(z, 0) \leq z^{1-\varepsilon} \) for \( z \geq 1 \), and \( \kappa(z, 0) \geq z^{1-\varepsilon} \) when \( 0 < z < 1 \). The result follows.

**Remark 3.5.** The uniform estimates of Corollary 3.4 complement the existing results from [17] about the asymptotic behavior of \( P(M_t < x) \), where it was shown that

\[
\lim_{t \to \infty} \sqrt{\pi} \kappa(1/t, 0) P(M_t < x) = V(x),
\]

under the assumption that \( \kappa(z, 0) \) is regularly varying at zero with index \( g \in (0, 1) \).

4. **Suprema of symmetric Lévy processes.** In this section we assume that \( X_t \) is a symmetric Lévy process with Lévy-Khintchin exponent \( \Psi(\xi) \). In a rather general setting, we can invert the Laplace transform in time variable in (1.1).
Theorem 4.1. Suppose that $X_t$ is a symmetric Lévy process with Lévy-Khintchin exponent $\Psi(\xi)$. Suppose that $\Psi(\xi)$ is increasing in $\xi > 0$. If $M_t = \sup_{0 \leq s \leq t} X_s$, then

\begin{equation}
E e^{-\xi M_t} = \frac{1}{\pi} \int_0^\infty \frac{\xi \Psi'(\lambda)}{(\lambda^2 + \xi^2)^{1/2} \sqrt{\Psi(\lambda)}} \times \\
\times \exp \left( \frac{1}{\pi} \int_0^\infty \frac{\xi \log \frac{\lambda^2 - \xi^2}{\Psi(\lambda) - \Psi(\zeta)}}{\lambda^2 + \zeta^2} d\zeta \right) e^{-t \Psi(\lambda)} d\lambda.
\end{equation}

(4.1)

Since $P(M_t < x) = P(\tau_x > t)$, the following integrated form of (4.1) is sometimes more convenient.

Corollary 4.2. With the notation and assumptions of Theorem 4.1,

\begin{equation}
\int_0^\infty e^{-\xi x} P(\tau_x > t) dx = \frac{E e^{-\xi M_t}}{\xi} = \frac{1}{\pi} \int_0^\infty \frac{\Psi'(\lambda)}{(\lambda^2 + \xi^2)^{1/2} \sqrt{\Psi(\lambda)}} \times \\
\times \exp \left( \frac{1}{\pi} \int_0^\infty \frac{\xi \log \frac{\lambda^2 - \xi^2}{\Psi(\lambda) - \Psi(\zeta)}}{\lambda^2 + \zeta^2} d\zeta \right) e^{-t \Psi(\lambda)} d\lambda.
\end{equation}

(4.2)

Proof of Theorem 4.1. Let $\psi(\xi) = \Psi(\sqrt{\xi})$ for $\xi > 0$. For any $z \in \mathbb{C} \setminus (-\infty, 0]$ and $\xi > 0$, we define (see (1.1) and (2.3))

\[
\varphi(\xi, z) = \sqrt{z} \exp \left( -\frac{1}{\pi} \int_0^\infty \frac{\xi \log(z + \Psi(\zeta))}{\xi^2 + \zeta^2} d\zeta \right) \\
= \exp \left( -\frac{1}{\pi} \int_0^\infty \frac{\xi \log(1 + \psi(\zeta)/z)}{\xi^2 + \zeta^2} d\zeta \right).
\]

For any $\xi > 0$, the function $\varphi(\xi, z)$ is positive and increasing in $z \in (0, \infty)$. As $z \to 0$ or $z \to \infty$, $\varphi(\xi, z)$ converges to 0 and 1, respectively. Furthermore, if $\text{Im} \ z > 0$, then $\arg(1 + \psi(\zeta^2)/z) \in (-\pi, 0)$ for all $\zeta > 0$, and therefore

\[
\arg \varphi(\xi, z) = -\frac{1}{\pi} \int_0^\infty \frac{\xi \arg(1 + \psi(\zeta^2)/z)}{\xi^2 + \zeta^2} d\zeta \in (0, \pi/2).
\]

Hence, for any $\xi > 0$, $\varphi(\xi, z)$ (and even $(\varphi(\xi, z))^2$) is a complete Bernstein function of $z$. Note that the continuous boundary limit $\varphi^+(\xi, -z)$ exists for
\[ z > 0: \text{if } z = \psi(\lambda^2), \text{ or } \lambda = \sqrt{\psi^{-1}(z)}, \text{ then} \]

\[
\varphi^+(\xi, -z) = \exp \left( -\frac{1}{\pi} \int_0^{\infty} \frac{\xi \log (1 - \frac{\psi(\zeta^2)}{\psi(\lambda^2)})}{\xi^2 + \zeta^2} \, d\zeta \right)
\]

\[
= \exp \left( -\frac{1}{\pi} \int_0^{\infty} \frac{\xi \log \left[1 - \frac{1}{\psi(\lambda^2)}\right]}{\xi^2 + \zeta^2} \, d\zeta + i \int_{\lambda}^{\infty} \frac{\zeta}{\xi^2 + \zeta^2} \, d\zeta \right)
\]

\[
= \exp \left( \frac{1}{\pi} \int_0^{\infty} \frac{\xi \log \psi(\zeta^2) - \log \left[1 - \frac{\xi^2}{\lambda^2}\right]}{\xi^2 + \zeta^2} \, d\zeta + i \arctan \frac{\xi}{\lambda} \right);
\]

see (2.9) for the notation. Here \( \log^- \) denotes the boundary limit on \(( -\infty, 0)\) approached from below, \( \log^- (-\zeta) = -i\pi/2 + \log \zeta \) for \( \zeta > 0 \). The function \( \log |1 - \zeta^2/\lambda^2| \) is harmonic in the upper half-plane \( \text{Im} \zeta > 0 \), so that

\[
\frac{1}{\pi} \int_0^{\infty} \frac{\xi \log \left[1 - \frac{\xi^2}{\lambda^2}\right]}{\xi^2 + \zeta^2} \, d\zeta = \frac{1}{2} \log \left(1 + \frac{\xi^2}{\lambda^2}\right).
\]

Furthermore, \( \exp(i \arctan(\xi/\lambda)) = (\lambda + i\xi)/\sqrt{\lambda^2 + \xi^2} \). Therefore, with \( z = \psi(\lambda^2) \),

\[
\varphi^+(\xi, -z) = \frac{\lambda(\lambda + i\xi)}{\lambda^2 + \xi^2} \exp \left( \frac{1}{\pi} \int_0^{\infty} \frac{\xi \log \psi(\zeta^2)}{\xi^2 + \zeta^2} \, d\zeta \right)
\]

(4.3)

\[
= \frac{\lambda(\lambda + i\xi)\psi^!(\xi)}{\lambda^2 + \xi^2};
\]

see (2.3) for the notation. Note that if \( \psi(\xi) \) is bounded on \((0, \infty)\) and \( z \geq \sup_{\xi > 0} \psi(\xi) \), then \( \varphi^+(\xi, -z) \) is real.

By (1.1), \( \varphi(\xi, z) \) is the double Laplace transform of the distribution of \( M_t \). But for all \( \xi > 0 \), \( \varphi(\xi, z)/z \) is a Stieltjes function of \( z \). Therefore, by (2.2),

\[
\frac{\varphi(\xi, z)}{z} = \frac{1}{\pi} \int_0^{\infty} \frac{\text{Im} \varphi^+(\xi, -\zeta)}{z + \zeta} \, d\zeta
\]

\[
= \frac{1}{\pi} \int_0^{\infty} 2\lambda \psi'(\lambda^2) \text{Im} \frac{\varphi^+(\xi, -\psi(\lambda^2))}{\psi(\lambda^2)} \frac{1}{z + \psi(\lambda^2)} \, d\lambda
\]

\[
= \frac{2}{\pi} \int_0^{\infty} \frac{\lambda \xi \psi^!(\xi)}{\psi(\lambda^2)} \frac{1}{\lambda^2 + \xi^2} \frac{1}{z + \psi(\lambda^2)} \, d\lambda.
\]
Note that the second equality holds true also when $\psi(\xi)$ is bounded. Since
\[
\frac{\varphi(\xi, z)}{z} = \int_0^\infty \left( \frac{2}{\pi} \int_0^\infty \frac{\lambda \psi'(\lambda^2)}{\psi(\lambda^2)} \frac{\lambda \xi \psi_\lambda^+(\xi)}{\lambda^2 + \xi^2} e^{-t\psi(\lambda^2)} d\lambda \right) e^{-zt} dt.
\]
The theorem follows by the uniqueness of the Laplace transform.

Let $V(x) = V^0(x)$ be the renewal function for the ascending ladder-height process $H_s$ corresponding to $X_t$; see Section 3 for the definition. When $X_t$ satisfies the absolute continuity condition (for example, if $1/(1 + \Psi(\xi))$ is integrable in $\xi$), then $V(x)$ is the (unique up to a multiplicative constant) increasing harmonic function for $X_t$ on $(0, \infty)$, and $V'(x)$ is the decreasing harmonic function for $X_t$ on $(0, \infty)$, cf. [35]. It is known ([5], formula (VI.6)) that for $\xi > 0$,
\[
\mathcal{L}V(\xi) = \frac{1}{\xi \kappa(0, \xi)}.
\]
Moreover, if $X_t$ is not a compound Poisson process, then by [14], Corollary 9.7,
\[
\kappa(0, \xi) = \exp \left( \frac{1}{\pi} \int_0^\infty \frac{\xi \log \Psi(\zeta)}{\xi^2 + \zeta^2} d\zeta \right) = \psi^+(\xi),
\]
where $\Psi(\xi) = \psi(\xi^2)$, see (2.3) for the notation. Clearly, we have $\mathcal{L}V'(\xi) = \xi \mathcal{L}V(\xi) = 1/\psi^+(\xi)$; here $V'$ is the distributional derivative of $V$ on $[0, \infty)$. We remark that when $X_t$ is a compound Poisson process, then, also by [14], Corollary 9.7,
\[
(4.4) \quad \kappa(0, \xi) = c\psi^+(\xi), \quad \text{with} \quad c = \exp \left( -\frac{1}{2} \int_0^\infty \frac{1 - e^{-t}}{t} \mathbb{P}(X_t = 0) dt \right).
\]
For simplicity, we state the next three results only for the case when $X_t$ is not a compound Poisson process. However, extensions for compound Poisson processes are straightforward due to (4.4).

As an immediate consequence of Proposition 2.1 and Karamata’s Tauberian theorem ([8], Theorem 1.7.1), we obtain the following result, which in the case of complete Bernstein functions was derived in Proposition 2.7 of [22].
Proposition 4.3. Let $\Psi(\xi)$ be the Lévy-Khintchin exponent of a symmetric Lévy process $X_t$, which is not a compound Poisson process, and suppose that both $\Psi(\xi)$ and $\xi^2/\Psi(\xi)$ are increasing in $\xi > 0$. If $\Psi(\xi)$ is regularly varying at $\infty$, then $V$ is regularly varying at 0 and $\Gamma(1 + \alpha)V(x) \sim 1/\sqrt{\Psi(1/x)}$ as $x \to 0$. Similarly, if $\Psi(\xi)$ is regularly varying at 0, then $\Gamma(1 + \alpha)V(x) \sim 1/\sqrt{\Psi(1/x)}$ as $x \to \infty$.

Another consequence of Proposition 2.1 is a uniform estimate of the renewal function (see also Proposition 3.9 of [25]).

Theorem 4.4. Let $\Psi(\xi)$ be the Lévy-Khintchin exponent of a symmetric Lévy process $X_t$, which is not a compound Poisson process, and suppose that both $\Psi(\xi)$ and $\xi^2/\Psi(\xi)$ are increasing in $\xi > 0$. Then

$$\frac{1}{5} \frac{1}{\sqrt{\Psi(1/x)}} \leq V(x) \leq \frac{5}{\sqrt{\Psi(1/x)}}, \quad x > 0.$$

Proof. Let $\psi(\xi) = \Psi(\sqrt{\xi})$ for $\xi > 0$. By Proposition 2.1, we obtain $e^{-2C/\pi}/\sqrt{x^2\psi(x^2)} \leq LV(\xi) \leq e^{2C/\pi}/\sqrt{x^2\psi(x^2)}$, $\xi > 0$. Since $V$ is increasing, Proposition 2.3 gives

$$V(x) \leq \frac{eLV(1/x)}{x} \leq \frac{e^{1+2C/\pi}}{\sqrt{\psi(1/x^2)}} \leq \frac{5}{\sqrt{\psi(1/x^2)}}.$$

Furthermore, using subadditivity and monotonicity of $V$ (see [5], Section III.1), for $x = ka + r$ ($k \geq 0$, $r \in [0,a]$) we obtain $V(x) \leq kV(a) + V(r) \leq (k + 1)V(a)$. It follows that $V(x) \leq 2V(a)\max(1,x/a)$ for all $a, x > 0$, and so, by Proposition 2.2,

$$V(x) \geq \frac{LV(1/x)}{2x(1+e^{-1})} \geq \frac{1}{2(1+e^{-1})e^{2C/\pi}\sqrt{\psi(1/x^2)}} \geq \frac{1}{5\sqrt{\psi(1/x^2)}},$$

as desired. \qed

We remark that when $V$ is a concave function on $(0, \infty)$ (for example, when $\psi$ is a complete Bernstein function, see below), then clearly $V(x) \leq \max(1,x/a)V(a)$, so that the lower bound in Theorem 4.4 holds with constant $2/5$ instead of $1/5$.

If $\psi(\xi)$ is a complete Bernstein function (CBF, see (2.1)), then $\psi^\dagger(\xi)$ and $\xi/\psi^\dagger(\xi)$ are CBFs, and hence $1/\psi^\dagger(\xi)$ is a Stieltjes function (see (2.2)). Therefore, $V'(x)$ is a completely monotone function on $(0, \infty)$, and $V(x)$ is a Bernstein function (see [34] for the relation between completely monotone, Bernstein, complete Bernstein and Stieltjes functions). More precisely, we have the following result.
**Proposition 4.5.** Let $\Psi(\xi)$ be the Lévy-Khintchine exponent of a symmetric Lévy process $X_t$, which is not a compound Poisson process, and suppose that $\Psi(\xi) = \psi(\xi^2)$ for a complete Bernstein function $\psi$. Then $V$ is a Bernstein function, and

\begin{align}
V(x) &= bx + \frac{1}{\pi} \int_{0^+}^{\infty} \text{Im} \left( -\frac{1}{\psi^+(\xi^2)} \right) \psi^\dagger(\xi) \left( 1 - e^{-x\xi} \right) d\xi, \quad x > 0, \\
V'(x) &= b + \frac{1}{\pi} \int_{0^+}^{\infty} \text{Im} \left( -\frac{1}{\psi^+(\xi^2)} \right) \psi^\dagger(\xi) e^{-x\xi} d\xi, \quad x > 0,
\end{align}

where $b = \lim_{\xi \to 0^+} (\xi/\sqrt{\psi(\xi^2)})$.

As explained after formula (2.2), the expression $\text{Im}(1/\psi^+(\xi))d\xi$ in (4.5) and (4.6) should be understood in the distributional sense, as a weak limit of measures $\text{Im}(1/\psi(-\xi^2 + i\varepsilon))d\xi$ on $(0, \infty)$ as $\varepsilon \to 0^+$. The measure $\text{Im}(1/\psi^+(\xi))d\xi$ has an atom of mass $\pi b$ at 0, and this atom is not included in the integrals from $0^+$ to $\infty$ in (4.5) and (4.6).

**Proof.** Since $1/\psi^\dagger(\xi)$ is a Stieltjes function, it has the form (2.2),

\[ L V'(\xi) = \frac{1}{\psi^\dagger(\xi)} = a + b \frac{1}{\xi} + \frac{1}{\pi} \int_{0^+}^{\infty} \frac{1}{\xi + \zeta} \tilde{\mu}(d\zeta), \quad \xi \in \mathbb{C} \setminus (-\infty, 0], \]

where, using (2.5),

\[ \tilde{\mu}(d\xi) = -\text{Im} \left( \frac{1}{(\psi^\dagger)^2(-\xi)} \right) d\xi = -\text{Im} \left( \frac{\psi^\dagger(\xi)}{\psi^+(\xi^2)} \right) d\xi, \]

and

\[ a = \lim_{\xi \to \infty} \frac{1}{\psi^\dagger(\xi)}, \quad b = \lim_{\xi \to 0^+} \frac{\xi}{\psi^\dagger(\xi)}. \]

Using Proposition 2.1, we can express $a$ and $b$ in terms of $\psi$. Since $\psi$ is unbounded, also $\psi^\dagger$ is unbounded (by (2.6)), and so in fact $a = 0$. In a similar way, if $\xi/\psi(\xi)$ converges to 0 as $\xi \to 0^+$, then (2.6) gives $\xi/\psi^\dagger(\xi) \to 0$, so that $b = 0$. When the limit of $\xi/\psi(\xi)$ is positive (since $\xi/\psi(\xi)$ is a CBF, the limit always exists), then $\psi$ is regularly varying at 0, and so $b = \lim_{\xi \to 0^+} (\xi/\sqrt{\psi(\xi^2)})$, as desired. By the uniqueness of the Laplace transform,

\[ V'(x) = b + \frac{1}{\pi} \int_{0^+}^{\infty} e^{-x\xi} \tilde{\mu}(d\xi), \quad x > 0. \]

The result follows by integration in $x$. \qed
Note that for a compound Poisson process, we have $a > 0$, so there is an extra positive constant in (4.5).

As a combination of Theorem 3.1 and Theorem 4.4, we obtain the following result.

**Theorem 4.6.** Let $\Psi(\xi)$ be the Lévy-Khintchin exponent of a symmetric Lévy process $X_t$. Suppose that both $\Psi(\xi)$ and $\xi^2/\Psi(\xi)$ are increasing in $\xi > 0$. If $M_t = \sup_{0 \leq s \leq t} X_s$, then for all $t, x > 0$,

$$\frac{1}{100} \min \left(1, \frac{1}{200 \sqrt{t \Psi(1/x)}} \right) \leq \mathbb{P}(M_t < x) \leq \min \left(1, \frac{10}{\sqrt{t \Psi(1/x)}} \right).$$

**Proof.** When $X_t$ is not a compound Poisson process, then the result follows from Theorems 3.1 and 4.4, and from $\kappa(z, 0) = \sqrt{z}$. Suppose that $X_t$ is a compound Poisson process. For $\varepsilon > 0$ consider $X_t^\varepsilon = \varepsilon B_t + X_t$, where the Brownian motion $B_t$ is independent of $X_t$. Then the Lévy-Khintchin exponent of $X_t^\varepsilon$ equals to $\Psi_\varepsilon(\xi) = (\varepsilon \xi)^2 + \Psi(\xi)$. It is easy to check that $\xi^2/\Psi_\varepsilon(\xi)$ is increasing. Moreover, $M_t^\varepsilon$ converges in distribution to $M_t$ as $\varepsilon \to 0$. The result follows by the continuity of $\Psi(\xi)$. \hfill $\square$

**Remark 4.7.** Clearly, the condition $\Psi(\xi)$ and $\xi^2/\Psi(\xi)$ are increasing in $\xi > 0$ in Theorem 4.4, Proposition 4.3 and Theorem 4.6 can be replaced with

$$0 < \Psi'(\xi) < \frac{2\Psi(\xi)}{\xi}, \quad \xi > 0. \tag{4.7}$$

If $\Psi(\xi) = \psi(\xi^2)$, then (4.7) reads

$$0 < \psi'(\xi) < \frac{\psi(\xi)}{\xi}, \quad \xi > 0. \tag{4.8}$$

Using the standard representation of Bernstein functions, it is easy to check that any Bernstein function $\psi(\xi)$ (not necessarily a complete one) satisfies (4.8). Hence, Theorem 4.6 applies to any subordinate Brownian motion: a process $X_t = B_{\eta_t}$, where $B(s)$ is the standard Brownian motion (with $\mathbb{E}(B_s) = 0$ and $\text{Var}(B_s) = 2s$), $\eta_t$ is a subordinator (with $\mathbb{E}(e^{-\xi \eta_t}) = e^{-t\psi(\xi)}$), and $B_s$ and $\eta_t$ are independent processes.

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