On Non-Commutative Orbifolds of K3 Surfaces

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Abstract

Using the algebraic geometry method of Berenstein and Leigh for the construction of the toroidal orbifold $\frac{T^2 \times T^2 \times T^2}{Z_2 \times Z_2}$ with discrete torsion and considering local K3 surfaces, we present non-commutative aspects of the orbifolds of product of K3 surfaces. In this way, the ordinary complex deformation of K3 can be identified with the resolution of stringy singularities by non-commutative algebras using crossed products. We give representations and make some comments regarding the fractionation of branes. Illustrating examples are presented.
1 Introduction

It has been known for a long time that non-commutative (NC) geometry plays an interesting role in the context of string theory \cite{1} and, more recently, in certain compactifications of the Matrix formulation of M-theory on NC torii \cite{2}, which has opened new lines of research devoted, for example, to the study of solitons in connection with NC quantum field theories \cite{3}.

In the context of superstring theory, NC geometry is involved whenever a $B$-field is turned on. For example, in the study of $D(p - 4)/Dp$ brane systems ($p > 3$) where, in particular, one can consider the ADHM construction of the $D0/D4$ system \cite{4}, the NC version of the Nahm construction for monopoles \cite{4, 5, 6}, the determination of the vacuum field solutions of the Higgs branch of supersymmetric gauge theories with eight supercharges \cite{8, 9, 10} or in the study of tachyon condensation using the so called GMS approach \cite{11}.

However, most of the NC spaces considered in all these studies involve mainly NC $\mathbb{R}^d_\theta$ \cite{11}, NC torii $T^d_\theta$ \cite{12}, few cases of orbifolds of NC torii and some generalizations to NC higher dimensional cycles such as the NC Hirzebruch complex surface $F_0$ used in \cite{13}.

Recently some efforts have been devoted to go beyond these geometric spaces. In particular, a special interest has been given to build NC Calabi-Yau (NCCY) manifolds containing the commutative ones as subalgebras and, in the case of orbifolds of Calabi-Yau (CY) threefolds, an explicit construction has been given by means of the so-called the NC algebraic geometric method \cite{14}. In that work, Berenstein and Leigh (BL) gave a realization of two NCCY 3-folds with discrete torsion:

1. The toroidal orbifolds $\frac{T^6}{\mathbb{Z}_2 \times \mathbb{Z}_2}$, where $T^6$ is viewed as the product of three elliptic curves as $T^2 \times T^2 \times T^2$. This construction involves non-commuting variables satisfying the 2-dimensional Clifford algebra.

2. The orbifold of the quintic in the $CP^4$ projective space,

$$P_5(z_j) = z_1^5 + z_2^5 + ... + z_5^5 + \lambda \prod_{i=1}^{5} z_i = 0,$$

(1)

by the $\mathbb{Z}_5^3$ discrete torsion symmetry group. The quintic algebra $A_5(5)$ reads as:

$$z_1 z_2 = \alpha z_2 z_1, \quad z_1 z_3 = \alpha^{-1} \beta z_3 z_1,$$
\[ z_1 z_4 = \beta^{-1} z_4 z_1, \quad z_2 z_3 = \alpha \gamma z_3 z_2, \]
\[ z_2 z_4 = \gamma^{-1} z_4 z_2, \quad z_3 z_4 = \beta \gamma z_4 z_3, \]
\[ z_i z_5 = z_5 z_i, \quad i = 1, 2, 3, 4, \tag{2} \]

where \( \alpha, \beta \) and \( \gamma \) are fifth roots of the unity generating the \( Z_5^3 \) discrete group and where the \( z_i \)'s are now the generators of the quintic algebra.

In this context, thinking of D-branes as coherent sheaves with support on a NC subvariety, they also explained the fractionation of branes by using a limit where the rank of the sheaf could jump at the singularity, leading to reducible matrix representations of the algebra.

Such formulation has been extended to higher dimensional orbifolds, understood as homogeneous hypersurfaces \( P_{n+2}(z_1, z_2, ..., z_{n+2}) \) in \( \mathbb{C}P^{n+1} \) with some discrete group of isometries \( Z_{n+2}^{(n+1)/2} \). In all these works, the CY algebra has a typical form which reminds quantum groups and the Yang-Baxter equations \([16, 17]\):

\[ z_i z_j = R_{ij}^{\alpha \beta} z_\alpha z_\beta, \tag{3} \]

where the four rank tensor \( R_{ij}^{\alpha \beta} \) was determined by the discrete torsion and the CY conditions. As we will see in this work, these \( R_{ij}^{\alpha \beta} \) can take the following form

\[ R_{ij}^{\alpha \beta} = \delta_i^{\beta} \delta_j^{\alpha} w^{\ell_{ij}}, \tag{4} \]

where \( w \) is an element of the discrete group \( G \), which leaves invariant the CY algebraic equation, and \( \ell_{ij} \) is an antisymmetric matrix satisfying the identity \( \sum_i \ell_{ij} = 0 \) which can be interpreted as the CY condition.

This analysis can also be adapted for lower dimensional CY manifolds \([18]\). In particular, we are interested in the case of the \( K3 \) surface. This is a very special surface because it is the only two-dimensional CY manifold. It can be represented in different ways depending on which property one is willing to study.

It can be easily seen to be related to the superconformal model corresponding to the polynomial constraint in \( WCP^3 \) plus deformations \([19, 20]\) and so can be seen as a complex surface in this space. This representation makes very clear the complex structure of the surface. Another description is a local one in terms of the ADE classification of singularities near the singular loci of the orbifold \( T^4/Z_2 \). A third description is in terms of an elliptic fibration, which means that locally the surface could be seen as a two torus times a complex plane.
All along the paper we will be dealing with the two first descriptions, although the last one could be used to find a proper interpretation of the results we will find, as will be explained in the conclusions.

The aim of this work is to extend the results found in the case of the orbifold \( \frac{T^6}{\mathbb{Z}_2 \times \mathbb{Z}_2} \) to higher CY manifolds in terms of product of K3 surfaces and, as we are considering \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) orbifolds, which have \( H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) \simeq \mathbb{Z}_2 \), we can include the effect of discrete torsion.

It is known that when the discrete torsion is considered, the twisted sector modes are in \( H^{2,1} \) and so act in the deformation of the complex structure of the orbifold \([20]\). However, there are not enough deformations available to resolve the singularities, because the discrete torsion is supported at them. In these cases, the only known way to resolve the singularities of the space is via NC geometry.

The outline of the paper is as follows. In section 2 we review the basic facts of the construction of the K3 surfaces in terms of the ADE classification of singularities and study the deformations which can be made to the equations which define them. In section 3 we will study how to construct the NC algebra associated to the orbifolds of CY manifolds. In section 4 we specialize to the case of orbifolding the product of three K3 and construct the realization of the associated algebra. In section 5 we extend the study to higher dimensional cases. We finish in section 6 with a discussion and some conclusions.

## 2 K3 surfaces, with ADE singularities, in string theory compactifications

In this section we give certain essential aspects of K3 surfaces as well as methods for the resolution of ADE singularities. This study is based on the results of the geometric engineering of \( D = 4 \) \( N = 2 \) quantum field theory embedded in superstring theory compactifications \([21, 22, 23]\).

Roughly speaking, K3 is a two complex dimensional compact Kähler CY manifold with \( SU(2) \) holonomy group. It has many types of realizations, the simplest one is to consider the orbifold \( \frac{T^4}{G} \), where \( T^4 \) is defined by the following complex identification equations

\[
\begin{align*}
  z_j &\equiv z_j + 1, \\
  z_j &\equiv z_j + i, \quad j = 1, 2, 
\end{align*}
\]

and where \( G \) is a discrete subgroup of \( SU(2) \). For instance, if we consider \( G = \mathbb{Z}_2 \), the K3 surface is obtained by imposing a extra constraint equations on \( T^4 \), namely
\[ z_j \equiv -z_j, \quad j = 1, 2. \] (6)

This symmetry has sixteen singular fixed points. Near such points \((z_1, z_2) \equiv (-z_1, -z_2)\), the K3 surface looks like \(\mathbb{C}^2/\mathbb{Z}_2\) and can be determined algebraically in terms of the \(\mathbb{Z}_2\) invariant coordinates on \(\mathbb{C}^2\), which are given by

\[
\begin{align*}
    x &= z_1^2 \\
    y &= z_2^2 \\
    z &= z_1 z_2,
\end{align*}
\] (7)

and give a map from \(\mathbb{C}^2/\mathbb{Z}_2\) to \(\mathbb{C}^3\).

Locally, K3 can be viewed as a hypersurface in \(\mathbb{C}^3\) defined by

\[ z^2 = xy \] (8)

The equation (8), which is known by \(A_1\) singularity, can be extended to the so called \(A_{n-1}\) singularity having the following form

\[ A_{n-1} : z^n = xy. \] (9)

Other singularities of local K3 surfaces are classified by the following equations

\[
\begin{align*}
    D_n : \quad & x^2 + y^2 z + z^{n-1} = 0, \\
    E_6 : \quad & x^2 + y^3 + z^4 = 0, \\
    E_7 : \quad & x^2 + y^3 + yz^3 = 0, \\
    E_8 : \quad & x^2 + y^3 + z^5 = 0.
\end{align*}
\] (10)

Basically there are two ways for smoothing out the ADE singularities, either by deforming its Kähler or its complex structure. For later use we shall focus our attention on the resolution of the \(A_{n-1}\) singularity, where the complex deformation deals with the left hand side of equation (8), while the Kähler deformation, which consists in blowing up the singular point with the help of \((n - 1)\) intersecting real \(S^2\), treats the right hand side of equation (8). In the case of K3 seen as an \(A_1\) singularity these two operations are related, because the \(A_1\) singularity can be seen as a vanishing 2-sphere, so either deforming the complex structure or making a blow-up consists in giving finite volume to it.

This method has a very nice interpretation in terms of the toric geometry realization of local K3 surfaces, where the Mori vectors are intimately related to the \(A_{n-1}\) Cartan
subalgebra charges of the gauge symmetry involved in the geometric engineering method. Moreover, the corresponding toric graph looks similar to the $A_{n-1}$ Dynkin diagram.

Since this method of doing is mirror to the complex deformation and, for latter use, we will only give the complex deformation of the $A_{n-1}$ singularity. Indeed, equation (9) admits a discrete $Z_n$ symmetry acting as follows

$$z \rightarrow wz, \quad w^n = 1$$
$$x \rightarrow x,$$
$$y \rightarrow y,$$

leaving $x$ and $y$ invariants. The deformation of the complex structure of the $A_{n-1}$ singularity introduces extra terms breaking the $Z_n$ symmetry as follows

$$xy = z^n + P(z).$$

In this equation, the extra polynomial is given by

$$P(z) = \sum_{i=1}^{n-1} a_i z^{n-i-1},$$

where the $a_i$'s are complex parameters carrying the complex deformation of the $A_{n-1}$ singularity. Their number is $(n - 1)$ which is the rank of the $A_{n-1}$ Lie algebra. These results have been used in many directions in string theory and F-theory compactifications, in particular, in the study of the quantum field theory using the geometric engineering method.

It should be interesting to note the following points for the $A_{n-1}$ geometry

1. Since K3 is a self mirror, equation (13) means that each monomial $z^k$ is associated to a divisor of K3 explaining the monomial/divisor map involved in the mirror symmetry application in the toric geometry framework.

2. The complex deformation acts only on the $z$ variable, by introducing terms breaking the $Z_n$ symmetry. The restoration of this symmetry leads to a limit where K3 develops the $A_{n-1}$ singularity.

3. The complex deformation of $A_{n-1}$ singularity is similar to the resolution of stringy singularities by a NC algebra involved in the study of the orbifold $C^2/\mathbb{Z}_2$ using the crossed product algebra $\mathbb{C}^2$. Indeed, identifying the role of the $Z_n$ symmetry involved in the complex deformation with the $Z_n$ discrete torsion of the crossed product of $\mathbb{C}^2$, one
can identify the complex deformation and the resolution of the stringy singularity of the orbifold $\mathbb{C}^2/\mathbb{Z}_2$.

This link can be understood by the fact the center of the algebras, being the singular geometry, is invariant under the $\mathbb{Z}_n$ symmetry corresponding to $P(z) = 0$ in the commutative deformation. Taking into account this fact, one can see that the terms of the deformation, in the NC sense, must not be in the center of the algebra. By this argument, one can see that the complex deformation of $\mathbb{C}^2/\mathbb{Z}_2$, in the commutative sense, is similar to the stringy singularities by NC algebra involved in the study of the orbifold $\mathbb{C}^2/\mathbb{Z}_2$ using NC algebraic geometry method and the crossed product algebra.

3 NC algebraic geometry method

In this section, we will briefly review the NC algebraic geometry approach, introduced first in [14], for treating the NC aspects of orbifolds of the CY manifolds. In this method, the (singular) orbifold with discrete torsion can be viewed as a NC algebra. In other words, the algebraic realization of a commutative orbifold space with discrete torsion has a nice interpretation using NC algebra.

In this method one proceeds following the next steps. Firstly, one takes a $d$-dimensional (singular) complex CY manifold $M^d$ defined by an equation of the form

$$f_j(u_i) = 0, \quad i - j = d, \quad (14)$$

where the $u_i$ are complex local coordinates. One looks for a discrete symmetry $G$

$$G : \quad u_i \rightarrow gu_i, \quad g \in G, \quad (15)$$

leaving $f_j(u_i)$ invariant

$$G : \quad f_j(u_i) \rightarrow f_j(u_i), \quad (16)$$

and preserving the CY condition. After that, one considers the orbifold $\frac{M^d}{G}$ which is constructed by identifying the points which are in the same orbit under the action of the group, i.e., $u_i \rightarrow gu_i$. The resulting space is smooth everywhere, except at the fixed points, which are invariant under non trivial group elements.
Following [14, 15, 18, 24, 25, 23] and using the discrete symmetry group \( G \), one can build the NC extensions of the above orbifold, \( \left( \mathcal{M}_G \right)_{nc} \). This procedure may be summarized as follows: the NC extension of this orbifold is obtained, as usual, by extending the commutative algebra \( \mathcal{A}_c \) of functions on \( \mathcal{M}_G \) to a NC one \( \mathcal{A}_{nc} \sim \left( \mathcal{M}_G \right)_{nc} \). In this algebra, the coordinate functions \( u_i \) on the deformed geometry will obey the following constraint equations

\[
 u_i u_j = \theta_{ij} u_j u_i, \tag{17}
\]

where \( \theta_{ij} \) are the NC parameters constrained by

\[
 \theta_{ij} \in G, \quad \theta_{ij} \theta_{ji} = 1. \tag{18}
\]

As we will see, the solution of these equations can take the following form

\[
 \theta_{ij} = g^\ell_{ij}, \tag{19}
\]

where \( g \) are the generators of \( G \) and \( \ell_{ij} \) is an antisymmetric tensor. An explicit solution is obtained with the help of extra constraints on the \( \theta_{ij} \)'s which can be easily specified once we know the elements of the center of the NC version of the orbifold, \( Z(\mathcal{A}_{nc}) \).

The elements of \( Z(\mathcal{A}_{nc}) \), which yield the commutative algebra, are the quantities invariant under the action of \( G \). In this way, the algebraic geometry of \( Z(\mathcal{A}_0) \) is identified with the algebraic realization [13], which may be singular, while the algebraic geometry of the NC algebra will resolve the singularities. In other words, the commutative singularity can be deformed in a NC algebraic realization sense.

Since the deformation part is not invariant under \( G \), one may say that this part resolving the singularity must be in \( \frac{\mathcal{A}_0}{Z(\mathcal{A}_0)} \) which may be a NC subspace algebra of \( \mathcal{A}_{nc} \). By this argument, we think that the same feature appears in the ordinary complex deformation of \( A_{n-1} \) singularity of \( K3 \) surfaces where the extra terms solving the singularity are not invariant under the \( Z_n \) symmetry.

This important link between the complex deformations and the resolution of stringy singularities by the NC algebras push us to think about the extension of the result of BL concerning the orbifold of the torus \( \frac{T_6}{\mathbb{Z}_2 \times \mathbb{Z}_2} \) in terms of \( K3 \) surfaces using NC algebraic geometry method. Before doing this, let us first recall the BL work for \( \frac{T_6}{\mathbb{Z}_2 \times \mathbb{Z}_2} \). In this work, \( T_6 \) is viewed as the product of three elliptic curves as \( T^2 \times T^2 \times T^2 \), each given in the Weierstrass form.
\[ y_i^2 = x_i(x_i - 1)(x_i - a_i), \quad i = 1, 2, 3, \]  
\[ i = 1, 2, 3, \text{ with a point added at infinity. The later can be brought to a finite point by} \]
\[ \text{a change of variables} \]
\[ y_i \rightarrow y_i' = \frac{y_i}{x_i^2}, \]
\[ x_i \rightarrow x_i' = \frac{1}{x_i} \]  
\[ i = 1, 2, 3, \]
\[ (20) \]
\[ \text{The } Z_2 \times Z_2 \text{ discrete symmetry acts by } y_i \rightarrow \pm y_i \text{ and } x_i \rightarrow x_i \text{ so that the holomorphic} \]
\[ \text{three form } dy_1 \wedge dy_2 \wedge dy_3 \text{ is invariant under the orbifold action satisfying the CY condition.} \]
\[ \text{After introducing the discrete torsion, the constraints of the NC reads} \]
\[ y_i y_j = -y_j y_i, \quad \text{for } i \neq j \]
\[ x_i x_j = x_j x_i, \quad \text{for } i, j = 1, 2, 3, \]
\[ x_i y_j = y_j x_i. \]
\[ i, j = 1, 2, 3, \]  
\[ (22) \]
\[ \text{and can be solved by} \]
\[ y_i = a_i \sigma_i, \]
\[ x_i = b_i I_2. \]
\[ (23) \]
\[ \text{By this approach, the orbifold } \frac{T^6}{Z_2 \times Z_4} \text{ with torsion defines a NCCY threefold, where} \]
\[ \text{the NC is carried by the discrete torsion phases and having a remarkable interpretation} \]
\[ \text{in terms of closed string states. On the fixed planes, the branes fractionate and local} \]
\[ \text{deformations are no more trivial. In what follows, we want to extend this result to higher} \]
\[ \text{dimensional CY manifolds. In particular we will consider CY’s realized as orbifolds of K3} \]
\[ \text{surfaces with discrete torsion. In other words, instead of having products of the } T^2 \text{ elliptic} \]
\[ \text{curves, we will have products of K3 surfaces.} \]

4 NC orbifolds of the K3 surfaces

In this section, we start by consider a general K3. The latter are given by the following
\[ z^2 = f(x, y), \]  
\[ (24) \]
where $f$ is obtained from a homogeneous function $F$ with total degree 6 in complex variables $u, v, w$ as follows

$$F(u, v, w) = F_6(u, v, w).$$

(25)

Note that a special form which has been used in [27] for studying $N$-point deformation of algebraic K3 surfaces is given by

$$F(u, v, w) = u^2v^3w + u^4v^2.$$  

(26)

However, in order to connect the algebraic geometry (24) to ones described in section 2, we will take here a special form of (25) as follows

$$F(u, v, w) = u^4vw.$$  

(27)

By this form, it is not difficult to see that (24) leads to the algebraic equation describing the $A_1$ singularity of K3 surfaces. Indeed, dividing equation (27) by $u^6$, one obtains

$$\frac{F(u, v, w)}{u^6} = \left(\frac{v}{u}\right) \left(\frac{w}{u}\right),$$

(28)

and so $f$ is given by

$$f(x, y) = xy,$$  

(29)

where

$$x = \frac{v}{u},$$

$$y = \frac{w}{u}.$$  

(30)

In this case, (24) looks like as the ALE space with $A_1$ singularity given by (7), and the analogue of the equations (21) reads now as

$$z \rightarrow z' = \frac{z}{x^3},$$

$$x \rightarrow x' = \frac{y}{x},$$

$$y \rightarrow y' = \frac{1}{x}.$$  

(31)
By these equations, now we are in position to extend the results of the orbifold $T^6_{Z_2 \times Z_2}$ with discrete torsion. To start, we consider the following geometric realization of $K_3^{\otimes 3}$ with discrete torsion. That is, $K_3^{\otimes 3}$ is represented by the product of three K3 as follows

$$z_i^2 = x_i y_i, \quad i = 1, 2, 3,$$  

(32)

with an orbifold group $G$ specified later on. A priori there are different symmetries leaving these equations invariant, but in order to keep the same analysis of [14], we will take $G$ as $Z_2^2$ acting only on the $z_i$ variables as follows

$$z_i \to \pm z_i,$$

$$x_i \to x_i,$$

$$y_i \to y_i.$$  

(33)

The reason behind choosing this symmetry is that the complex deformation of K3 surfaces acts only on each $z$ variable of K3 surfaces. The CY condition of this orbifold requires that the holomorphic six form

$$\Omega_6 = dz_1 \wedge dz_2 \wedge dz_3 \wedge \frac{dx_1}{y_1} \wedge \frac{dx_2}{y_2} \wedge \frac{dx_3}{y_3},$$  

(34)

should be invariant under (33). Furthermore, since the $Z_2 \times Z_2$ symmetry acts only on $z_i$, it follows that the invariance $\Omega_6$ is reduced to the invariance of $dz_1 \wedge dz_2 \wedge dz_3$. Having introduced these data, now we would like to introduce the discrete torsion. The orbifold $K_3^{\otimes 3} / Z_2^2$ with discrete torsion can be viewed as a NC hyper-Kahler CY manifold, where the $Z_2 \times Z_2$ invariant terms are elements of the center of the algebra. Using this feature and the CY condition, the NC version of the orbifold $K_3^{\otimes 3} / Z_2^2$ is obtained by taking the coordinates functions as follows

$$z_i z_j = -z_j z_i,$$

$$z_i x_j = x_j z_i,$$

$$y_i z_j = z_j y_i,$$

$$x_i x_j = x_j x_i,$$

$$y_i y_j = y_j y_i,$$

$$y_i x_j = x_j y_i,$$  

(35)

with
\[ z_i^2 z_j = z_j z_i^2, \]  
\[ [z_i, \prod_{i=1}^{3} z_i] = 0, \]  
(36)

which means that the \( x_i, y_i, z_i^2 \) and \( \prod_i z_i \) are all in the center of the algebra. To find the points of the NC geometry, the algebra (35-36) can be represented in terms of the Pauli matrices as follows

\[ z_i = a_i \sigma_i, \]
\[ x_i = b_i I_2, \]
\[ y_i = c_i I_2, \]

(37)

where the \( a_i, b_i, c_i \) are complex scalars and \( I_2 \) is the two dimensional identity matrix. Since the algebra (35-36) is very similar to the one describing the NC version of the orbifold torus, it follows that one should have the same interpretation in terms of resolution of singularities and reducibility of representations.

We can make the following remarks about the analysis made: the first one, which will be given in this section, is that we may find a Clifford algebra, in particular the Dirac algebra involved in the quantum field theory on Euclidean space. Another point is to consider the higher order of the discrete symmetries appearing in the geometry of ALE space, which will be treated in the next section.

Before going ahead, let us recall some useful properties of the Dirac algebra. The later, which is involved in the study of fermions, is given by

\[ \{\gamma_i, \gamma_j\} = \delta_{ij}, \]

(38)

where \( \gamma_i \) are complex matrices satisfying

\[ \gamma_0^\dagger = \gamma_0, \]
\[ \gamma_i^\dagger = -\gamma_i, \]
\[ \gamma_5^\dagger = \gamma_5. \]

(39)

Note that the minus sign in (39) can be absorbed by transforming \( \gamma_i \rightarrow i\gamma_i \), giving hermitian Dirac matrices which will be useful for discussing the brane fractionation in this context.

As an illustrating application, we can consider the product of five K3 surfaces with \( Z_2^5 \) discrete symmetry. The later acts on \( z_i, x_i \)'s and \( y_i \)'s as:
where \( \omega = \pm 1 \) and \( q_i^a \) are integer vectors satisfying the CY condition

\[
\sum_{i=1}^{5} q_i^a = 0, \quad \text{mod } 2; \quad a = 1, \ldots, 4.
\]  

Using the constraints on the \( \theta \) parameters and the CY condition, one can write

\[
\theta_{ij} = (-1)^{\ell_{ij}},
\]

\[
\sum_{i=1}^{5} \ell_{ij} = 0, \quad \text{mod } 2,
\]

where \( \ell_{ij} \) is an antisymmetric matrix of the following form

\[
\ell_{ij} = \Omega_{ab}q_i^aq_j^b;
\]

where \( \Omega_{ab} = -\Omega_{ba} \), and \( \Omega_{ab} = 1 \) for \( a < b \).

Now, if we take \( \ell_{ij} = 1 \) that is

\[
\theta_{ij} = -1 \quad \forall \ i \neq j
\]

the NC algebra reduces to

\[
z_i z_j = -z_j z_i, \quad \text{for } i \neq j, \quad \text{for } \ i, j = 1, \ldots, 5,
\]

with all others commutations relations. Using the Dirac matrices, a four dimensional realization of the algebra \([15]\) can be written as follows.

\[
z_i = a_i \gamma_i
\]

\[
x_i = b_i I_4
\]

\[
y_i = c_i I_4
\]

where now \( \gamma_i \) are given by
\[ \gamma_1 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \gamma_{i+1} = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad i = 1, 2, 3, \quad \gamma_5 = -i \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \] (47)

and where the \( \sigma_i \) are the Pauli matrices. At fixed locus, this representation becomes reducible as four out of the five variables \( z_i \) act by zero. Thus we get four distinct NC points, and so there are four different irreducible representations corresponding to the four eigenvalues of the non zero \( z_i \).

5 More on the Orbifolds of K3 surfaces

As we have mentioned, the above geometry can be extended to ones with higher dimensional discrete symmetries. In this case, the analogue of equation of (32) is

\[ z_i^n = x_i y_i, \quad i = 1, \ldots, m, \] (48)

where \( m \) is an integer, which will be fixed later on. As in the previous examples, equations (48) have a \( Z_n^{m-1} \) discrete group symmetry acting on the variables as follows

\[
\begin{align*}
z_i & \rightarrow z'_i = \omega^{q_i^a} z_i & \quad a = 1, \ldots, m-1, \\
x_i & \rightarrow x'_i = x_i, \\
y_i & \rightarrow y'_i = y_i,
\end{align*}
\] (49)

so that \( dz_1 \wedge dz_2 \wedge \ldots \wedge dz_m \) is invariant. This satisfies the CY condition on the quotient space. In equation (48), \( \omega \) is an element of the discrete group \( Z_n^{m-1} \) and where \( q_i^a \) are integers satisfying the following condition \( \sum_{i=1}^{n+1} a_i = 0, \quad \text{mod} \ m \) which is also interpreted as the CY condition. Using the previous analysis, the NC version of the orbifold \( K_{\mathbb{CP}^2}^{m} \) is obtained by substituting the usual commutative algebra of the functions by the NC one. In this way, the coordinate functions \( x_i, y_i \) and \( z_i \) on the deformed NC manifold obey the following identities

\[
\begin{align*}
z_i z_j &= \theta_{ij} z_j z_i, \\
z_i x_j &= x_j z_i, \\
y_i z_j &= z_j y_i, \\
x_i x_j &= x_j x_i, \\
y_i y_j &= y_j y_i, \\
y_i x_j &= x_j y_i,
\end{align*}
\] (50)
with

\[ z_i^n z_j = z_j z_i^n, \quad [z_i, \prod_{i=1}^m z_i] = 0, \]  

(51)

which means that \( z_i^n \) and \( \prod_{i=1}^m z_i \) belong to the center of the NC algebra. Using all these identities, one can easily see that the \( \theta_{ij} \) parameters must satisfy the following constraint equations

\[ \theta_{ij}^{n} = 1, \]  
\[ \prod_{i=1}^m \theta_{ij} = 1, \quad \forall i, \]  
\[ \theta_{ij} \theta_{ji} = 1. \]  

(52), (53), (54)

These constraints can be solved as follows: First, equations (52) show that

\[ \theta_{ij} = \omega^\ell_{ij}, \quad \omega = \exp^2 \frac{2i\pi}{n} \]  

(55)

where \( \ell_{ij} \) is a \( m \times m \) matrix. Second, putting this equation back into (52), one finds that \( \ell_{ij} \) must satisfy

\[ \ell_{ij} = -\ell_{ji}, \]  
\[ \sum_{i=1}^m \ell_{ij} = 0, \quad \text{mod } n. \]  

(56)

Next we will build the irreducible representations of the NCCY algebra for a regular representation. Then we will give the representation for the fixed points (where becomes reducible). It turns out that the \( d \) dimension of the finite matrix representations of the orbifolds geometry algebra is a multiple of \( n \). To see this property it is enough to take the determinant of both sides of NC variables namely

\[ \det (z_i z_j) = (\theta_{ij})^d \det (z_j z_i) = \det (z_j z_i) \]  

(57)

which constraint the dimension \( d \) of the representation to be such that:

\[ \theta_{ij}^d = 1. \]  

(58)

Using the identity (52), one discovers that \( d \) is a multiple of \( n \).

We return to equation (58), the change of variables (51) takes now the following form
\[ z_i \rightarrow \frac{z_i}{x_i^{6/n}} \]
\[ x_i \rightarrow x'_i = \frac{y_i}{x_i} \]  
\[ y_i \rightarrow y'_i = \frac{1}{x_i}. \]  
\[ (59) \]

If we require that \(6/n\) must be integer, therefore one has only \(n = 2, 3, 6\).

1. **Case of** \(n = 2\)

   we get the geometry related to \(A_1\) singularity, described in section 4.

2. **Case of** \(n = 3\)

   Instead of being general, we give a concrete example corresponding to \(m = 3\). In this case the equation \((48)\) reduces to

   \[ z_i^3 = x_i y_i, \quad i = 1, 2, 3. \]
   \[ (60) \]

   being the \(A_2\) singularity. Of course the NC version of this geometry is obtained from the one given in \((50)\). In this case, the equations \((52)\) can be solved as follows

   \[ \theta_{ij} = \omega^\ell_{ij}, \]
   \[ (61) \]

   where \(\omega\) is a phase so that \(\omega^3 = 1\) and \(\ell_{ij}\) is \(3 \times 3\) antisymmetric matrix

   \[ \ell_{ij} = \begin{pmatrix} 0 & k & -k \\ -k & 0 & k \\ k & -k & 0 \end{pmatrix} \]
   \[ (62) \]

   associated to the following commutations relations among \(z_i\)

   \[ z_1 z_2 = w^k z_2 z_1 \]
   \[ z_1 z_3 = w^{-k} z_3 z_1 \]
   \[ z_2 z_3 = w^k z_3 z_2 \]
   \[ (63) \]

   Note that for \(k = 1\), this algebra has the same structures of the non quartic K3 studied in \([24]\) but with \(\omega^4 = 1\). It is simple to see that there are 3-dimensional representations. Indeed, we introduce the two following matrices
\[ P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \] (64)

and so the algebra (53), for \( k = 1 \), can be solved by taking the \( z_i \) variables matrices as

\[
\begin{align*}
z_1 &= aP \\
z_2 &= bQ \\
z_3 &= cP^{-1}Q^{-1}
\end{align*}
\] (65)

Note that the only singularity in the commutative space happens when we take \( b = c = 0 \). The representation theory on the NC algebra becomes reducible at that point. Therefore, we obtain three distinct irreducible representations.

3. **Case of \( n = 6 \)**

Taking \( m = 6 \), we have the algebraic geometry corresponding to the \( A_5 \) singularity.

\[ z_i^6 = x_iy_i, \quad i = 1, \ldots, 6. \] (66)

This equation has \( Z_6^5 \) discrete symmetry, where in this case we have \( \sum_{i=1}^{6} q_i^a = 0, \quad a = 1, \ldots, 6 \). The NC extension (57) is given by the following algebra

\[ z_iz_j = \omega^{\ell_{ij}} z_jz_i, \] (67)

where \( \omega \) is a phase such that, \( \omega^6 = 1 \), and \( \ell_{ij} \) is \( 6 \times 6 \) antisymmetric matrix given by

\[
\ell_{ij} = \begin{pmatrix}
0 & \ell_{12} & \ell_{13} & \ell_{14} & \ell_{15} & \ell_{16} \\
-\ell_{12} & 0 & k_1 & k_2 & k_3 & k_4 \\
-\ell_{13} & -k_1 & 0 & k_5 & k_6 & k_7 \\
-\ell_{14} & -k_2 & -k_5 & 0 & k_8 & k_9 \\
-\ell_{15} & -k_3 & -k_6 & -k_8 & 0 & k_{10} \\
-\ell_{16} & -k_4 & -k_7 & -k_9 & -k_{10} & 0
\end{pmatrix} \] (68)

where
\( \ell_{12} = k_1 + k_2 + k_3 + k_4 \)
\( \ell_{13} = -k_1 + k_5 + k_6 + k_7 \)
\( \ell_{14} = -k_2 - k_5 + k_8 + k_9 \)
\( \ell_{15} = -k_3 - k_6 - k_8 + k_{10} \)
\( \ell_{16} = -k_4 - k_7 - k_9 - k_{10} \) \hspace{1cm} (69)

In what follows we consider the fundamental 6 \( \times \) 6 matrix representation obtained by using the following two matrices set \( Q, P_{\alpha\beta}; \alpha \beta = 1, ..., 6 \) as follows

\[
P_{\alpha\beta} = \text{diag}(1, \eta_{\alpha\beta}, \eta^2_{\alpha\beta}, ..., \eta^5_{\alpha\beta}), \quad Q = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 
\end{pmatrix} \hspace{1cm} (70)
\]

where \( \eta_{\alpha\beta} = w^{m_{\alpha\beta}} \) satisfying \( \eta^6_{\alpha\beta} = 1 \). From these expressions, it is not difficult to see that the above matrices satisfy:

\[
P_{\alpha} P_{\beta} = P_{\alpha\beta}, \quad P^6_{\alpha} = 1, \quad Q^6 = 1. \hspace{1cm} (71)
\]

Using these identities and the CY condition, one can check that the \( z_i \) variables can be presented as

\[
z_i = a_i \prod_{\alpha, \beta=1}^{6} \left( P_{\alpha\beta}^{a_i} Q_{i}^{\beta} \right), \quad x_i = b_i \mathbf{I}_6, \quad y_i = c_i \mathbf{I}_6 \hspace{1cm} (72)
\]

In the end of this section we would like to give a comment regarding the reducible representations for \( A_5 \) geometry. We will focus our attention herebelow on giving
a particular solution. In this solution we will consider an algebra described by $Z_6^2$ orbifold with $Z_3^6$ discrete torsions and more general solutions can be given using similar analysis; more details can be found in [13]. In this way, there exists situations where the representations are reducible. To see this, we start by recalling that the representation $(72)$ corresponds to regular points of NC orbifolds of K3 CY surfaces. These solutions are irreducibles. However similar solutions may be worked out as well for orbifold points with the $Z_3^6$ discrete torsions. Indeed, choosing matrix coordinates $z_5$ and $z_6$ in the centre of the algebra by setting

$$k_3 = k_4 = k_6 = k_7 = k_8 = k_9 = k_{10} = 0,$$  \hspace{1cm} (73)

the algebra reduces to

$$
\begin{align*}
  z_1 z_2 &= w_2^{k_1+k_2} z_2 z_1 \\
  z_1 z_3 &= w^{-k_1+k_5} z_3 z_1 \\
  z_1 z_4 &= w^{-k_2+k_5} z_4 z_1 \\
  z_2 z_3 &= w^{k_1} z_3 z_2 \\
  z_2 z_4 &= w^{k_2} z_4 z_2, \\
  z_3 z_4 &= w^{k_5} z_4 z_3
\end{align*}
$$

and all remaining other relations are commuting. In this equation, the $w$ are such that $w^6 = 1$; these are the phases of the $Z_3^6$ discrete torsions. In the singularity where the $z_1$, $z_2$, $z_3$, and $z_4$ moduli of equation (35) act by zero, the representation becomes reducible at $z_1 = z_2 = z_3 = z_4 = 0$.

6 Conclusion and Discussions

In this paper we have studied the NC version of orbifolds of product of K3 surfaces using the algebraic geometry approach of [14, 25]. In particular we have used a local description of K3 in terms of $A_{n-1}$ geometry to extend the analysis on the NC orbifold torus with discrete torsion initiated in [14] and exposed explicitly the relation between NC data and the CY charges. Among our results, we have worked out several representations of the corresponding NC algebra by using generic CY charges and given comments regarding the fractionation of branes.
In this context, the ordinary complex deformation of K3 surfaces near an $A_{n-1}$ singularity can be identified with the resolution of stringy singularities by NC algebras using crossed products in the $C_2^{Z_n}$ orbifold space. This analysis can be generalized to $D$ and $E$ geometries by replacing the $Z_n$ discrete symmetry by the corresponding ones.

On general grounds, it could be said that the appearance of NC geometry when considering discrete torsion is a natural thing. The first appearance of discrete torsion was related to some B-flux on a 2-cycle [28], and a relation between the discrete torsion and the torsion part of the homology of the target space was carried in [29].

The implementation in the presence of D-branes [30, 31] makes use of projective representations of the orbifold group, which are classified by $H^2(\Gamma, U(1))$, in perfect correspondence with the previous arguments.

So there is an intimate relation between discrete torsion and the B-field and, in this way, with NC geometry. Even more interesting is the fact that it is precisely the presence of this NC geometry which desingularizes the space.

This is important because could be applied to the resolution of singularities not only from a space-time point of view, but in the moduli space of certain theories. For example, a very close case to the ones studied in this paper is that of a $D2$ brane wrapped $n$ times over the fiber of an elliptic $K3$, which can be easily seen to have as moduli space the symmetric product [32]

$$\mathcal{M}_{1,n} = Sym(K3) = \frac{K3^{\otimes n}}{S_n}$$

(75)

where $\mathcal{M}_{1,n}$ denotes the moduli space of a D2-brane with charges $(1, n)$ and $S_n$ is the group of permutation of $n$ elements.

On the other hand, the fact that it can be found a reducibility property in the representations of the algebras have lead previously, as we have already mentioned, to an interpretation in terms of the fractionation of branes. However, as we would interpret this configuration as arising as the moduli space of certain configurations, the precise meaning of this result is still not clear for us. However, all these facts will be explored in a future work.

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