Dispersion Modes of Hot Plasma for Schwarzschild de-Sitter Horizon in a Veselago Medium

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Abstract

We analyze the dispersion modes of hot plasma around Schwarzschild de-Sitter event horizon in the presence of Veselago medium. For this purpose, we apply ADM 3 + 1 formalism to develop GRMHD equations for the Schwarzschild de-Sitter spacetime. Implementation of linear perturbation yields perturbed GRMHD equations that are used for the Fourier analysis of rotating (non-magnetized and magnetized) plasma. Wave analysis is described by the graphical representation of the wave vector, refractive index, change in refractive index, phase and group velocities. The outcome of this work supports the validity of Veselago medium.

Keywords: 3 + 1 formalism; SdS black hole; GRMHD equations; Veselago medium; Hot plasma; Dispersion relations.

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1 Introduction

Plasma is a distinct state of matter with a collection of free electric charge carriers which behave collectively and respond strongly to electromagnetic

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Debye shielding effect is the common example of collective effects of the plasma particles. The number density of positively charged particles remains almost the same as that of negatively charged particles to preserve the state of quasi-neutrality. Plasma is formed by the recombination of free electrons and ions, so it must have thermal energies to overcome the coupling of charged particles and can interact with electromagnetic fields. The nature of particle interactions distinguish between weakly and strongly ionized plasma. In weakly ionized plasma, charge-neutral interactions dominate the multiple Coulomb interactions, otherwise it is strongly ionized \[1\]. It is mentioned here that the most general plasma is hot and highly ionized \[2\].

Black holes are amongst the mysterious compact objects, depicting region of space with strong gravitational field that nothing not even light can escape from it. According to the predictions of relativists, black holes are formed when matter is compressed to a point. These are commonly created from deaths of massive stars or by collapse of a supergiant star. Black holes contain the energy flux which generates a relatively large magnetic field \[3\]. The theory of magnetohydrodynamics (MHD) is developed for the study of plasma flow having both positive and negative charge carriers. This describes the study of plasma flow in the presence of magnetic field. The MHD theory along with the effects of gravity is said to be general relativistic magnetohydrodynamics (GRMHD). The strong gravity of black hole perturbs the magnetospheric plasma. The theory of GRMHD is the most active field of study to examine the dynamics of magnetized plasma and impressions of black hole gravity.

The de-Sitter black hole is a vacuum solution of the Einstein field equations with a positive cosmological constant \[1\]. The Schwarzschild de-Sitter (SdS) spacetime expresses a black hole describing a patch of the de-Sitter spacetime. Inclusion of the positive cosmological constant in the Schwarzschild spacetime leads to SdS black hole which is non-rotating, thus plasma in magnetosphere falls along the radial direction. Current observations indicate that our universe is accelerating, inclusion of positive cosmological constant leads to the expanding universe \[5\]-\[7\]. Thus our universe approaches to de-Sitter universe in far future.

Regge and Wheeler \[8\] and Zerilli \[9\] investigated the stability and gravitational field of the Schwarzschild black hole with the help of non-spherical perturbations. Petterson \[10\] found that gravitational field is more strong near the surface of non-rotating black hole. Price \[11\] explored the dynamics of nearly spherical star by considering linear perturbation. Gleiser et al. \[12\]
discussed the stability of black holes by applying second order perturbations.

The more appropriate approach towards General Relativity is to decompose metric field into layers of three-dimensional spacelike hypersurfaces and one-dimensional time, called as ADM 3+1 formalism, presented by Arnowitt, Deser and Misner (ADM) [13]. Thorne and Macdonald [14] examined that 3 + 1 split is suitable approach for black hole theory. Smarr and York [15] used ADM formalism to explain spacetime kinematics numerically. Israel [16] studied event horizons in static vacuum and static electro-vacuum spacetimes. Zhang [17] reformulated the laws of perfect GRMHD for the general line element and discussed the significant features of stationary symmetric GRMHD solutions. The same author [18] studied the dynamics of rotating black hole. Buzzi et al. [19] investigated wave analysis of two fluid plasma model around Schwarzschild event horizon.

Rezolla et al. [20] considered the effects of cosmological constant and provided the dynamics of thick disks around SdS black hole. Myung [21] studied the entropy change for SdS black hole. Suneeta [22] presented the quasinormal modes for scalar field perturbations of SdS black hole. Ali and Rahman [23] investigated transverse wave propagation in two fluid plasma for SdS black hole. Sharif and his collaborators [24]-[27] found dispersion modes of cold and isothermal plasmas for non-rotating as well as rotating black holes in the usual medium. They have also investigated the wave properties around Schwarzschild magnetosphere for the hot plasma.

Artificial materials which exhibit unusual electromagnetic properties are termed as metamaterials. Veselago medium is the well-known metamaterial named after Russian physicist Veselago [28], which has negative electric permittivity and magnetic permeability simultaneously. It has various names, double negative medium (DNM), backward wave medium (BWM), left-handed medium (LHM), negative refractive index medium (NIM) or negative phase velocity medium (NPV). Many people [29]-[33] have explored the unusual properties of such a medium. Ziolkowski and Heyman [34] provided wave properties of NPV both analytically and numerically. Ramakrishna [35] discussed the importance of NIM for the perfect lensing. Sharif and Mukhtar [36] explored wave properties around Schwarzschild magnetosphere for isothermal as well as hot plasma in this unusual medium.

In a recent paper (Sharif and Noureen, submitted), we have extended this study to SDS black hole for isothermal plasma. Here we investigate wave analysis of hot plasma for SdS black hole in NPV. For this purpose, the 3 + 1 GRMHD equations are considered to determine dispersion relations
with the help of Fourier analysis after insertion of linear perturbation to the 
GRMHD equations for both non-magnetized and magnetized backgrounds. 
The three dimensional plot of wave vector, refractive index and change in 
refractive index describes the wave properties of hot plasma. The paper is 
arranged as follows: In section 2, linearly perturbed 3 + 1 GRMHD equa-
tions and their Fourier analysis is considered for hot plasma. The Fourier 
analyzed form of the GRMHD equations for rotating (non-magnetized and 
magnetized) plasmas are taken in sections 3 and 4 respectively to discuss the 
corresponding wave properties. Last section provides the conclusion.

## 2 GRMHD Equations for Hot Plasma in a 
Veselago Medium

The general line element in ADM 3 + 1 formalism is represented as follows 
\[ ds^2 = -\alpha^2 dt^2 + \eta_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt). \]  
(2.1)

Here \( \alpha \) denotes lapse function (ratio of FIDO proper time to universal time 
i.e., \( \frac{d\tau}{dt} \)), a natural observer linked with this spacetime is called fiducial ob-
server (FIDO), \( \beta^i \) describes three-dimensional shift vector and \( \eta_{ij} \) (\( i, j = 1, 2, 3 \)) denote the components of three-dimensional spacelike hypersurfaces. The SdS spacetime in Rindler coordinates is given by 
\[ ds^2 = -\alpha^2(z) dt^2 + dx^2 + dy^2 + dz^2, \]  
(2.2)

where the directions \( z, y \) and \( x \) are analogous to the Schwarzschild coordi-
nates \( r, \phi \) and \( \theta \) respectively. The SdS black hole is non-rotating, thus the 
shift vector vanishes. From the comparison of Eqs.(2.1) and (2.2), we have 
\[ \alpha = \alpha(z), \quad \beta = 0, \quad \eta_{ij} = 1 \ (i = j). \]  
(2.3)

Appendix provides the 3+1 GRMHD equations for the line element (2.2) 
in a Veselago medium, i.e. Eqs.(A5)-(A9). The equation of state for hot 
plasma is 
\[ \mu = \frac{\rho + p}{\rho_0}, \]  
(2.4)

where the rest mass density, moving mass density, pressure and specific en-
thalpy are denoted by \( \rho_0, \rho, p \) and \( \mu \) respectively. For hot plasma, the
specific enthalpy is not constant which shows that there is energy exchange between plasma and magnetic field of fluid. The $3+1$ GRMHD equations \((A5)-(A9)\) for hot plasma around the SdS black hole takes the form

\[
\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times (\alpha \mathbf{V} \times \mathbf{B}), \quad (2.5)
\]

\[
\nabla \cdot \mathbf{B} = 0, \quad (2.6)
\]

\[
\frac{\partial (\rho + p)}{\partial t} + (\rho + p)\gamma^2 \mathbf{V} \cdot \frac{\partial \mathbf{V}}{\partial t} + (\rho + p)\gamma^2 \mathbf{V} \cdot (\alpha \mathbf{V} \cdot \nabla) \mathbf{V}
\]

\[
+ (\rho + p) \nabla \cdot (\alpha \mathbf{V}) = 0, \quad (2.7)
\]

\[
\left\{ \left( (\rho + p)\gamma^2 + \frac{\mathbf{B}^2}{4\pi} \right) \delta_{ij} + (\rho + p)\gamma^4 \mathbf{V}_i \mathbf{V}_j - \frac{1}{4\pi} B_i B_j \right\} \left( \frac{1}{\alpha} \frac{\partial}{\partial t} \right)
\]

\[
+ \mathbf{V} \cdot \nabla \mathbf{V}^j + \gamma^2 \mathbf{V}_i (\mathbf{V} \cdot \nabla)(\rho + p) - \left( \frac{\mathbf{B}^2}{4\pi} \delta_{ij} - \frac{1}{4\pi} B_i B_j \right) \mathbf{V}_i^j \mathbf{V}^k
\]

\[
= -(\rho + p)\gamma^2 a_i - p_i + \frac{1}{4\pi} (\mathbf{V} \times \mathbf{B})_i \nabla \cdot (\mathbf{V} \times \mathbf{B}) - \frac{1}{8\pi \alpha^2} (\alpha \mathbf{B})_i^2
\]

\[
+ \frac{1}{4\pi \alpha} (\alpha \mathbf{B})_i \cdot \mathbf{B} - \frac{1}{4\pi \alpha} [\mathbf{B} \times \{ \mathbf{V} \times (\nabla \times (\alpha \mathbf{V} \times \mathbf{B}))) \}]_i, \quad (2.8)
\]

\[
\left( \frac{1}{\alpha} \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) (\rho + p)\gamma^2 - \frac{1}{\alpha} \frac{\partial p}{\partial t} + 2(\rho + p)\gamma^2 (\mathbf{V} \cdot \mathbf{a}) + (\rho + p)
\]

\[
\gamma^2 (\nabla \cdot \mathbf{V}) - \frac{1}{4\pi \alpha} (\mathbf{V} \times \mathbf{B}) \cdot (\mathbf{V} \times \frac{\partial \mathbf{B}}{\partial t}) - \frac{1}{4\pi \alpha} (\mathbf{V} \times \mathbf{B}) \cdot (\mathbf{B} \times \frac{\partial \mathbf{V}}{\partial t})
\]

\[
+ \frac{1}{4\pi \alpha} (\mathbf{V} \times \mathbf{B}) \cdot (\nabla \times \alpha \mathbf{B}) = 0. \quad (2.9)
\]

We assume that for rotating background plasma flow is in two dimensions, i.e., in \(xz\)-plane. Thus FIDO’s measured velocity \(\mathbf{V}\) and magnetic field \(\mathbf{B}\) become

\[
\mathbf{V} = V(z) \mathbf{e}_x + u(z) \mathbf{e}_z, \quad \mathbf{B} = B[\lambda(z) \mathbf{e}_x + \mathbf{e}_z], \quad (2.10)
\]

where \(B\) is an arbitrary constant. The following expression provides the relation between the quantities \(\lambda, u\) and \(V\) \([24]\)

\[
V = \frac{V^F}{\alpha} + \lambda u, \quad (2.11)
\]

where \(V^F\) is constant of integration. The Lorentz factor, \(\gamma = \frac{1}{\sqrt{1 - u^2 - V^2}}\), takes the form

\[
\gamma = \frac{1}{\sqrt{1 - u^2 - V^2}}. \quad (2.12)
\]
Strong gravity of black hole perturbs the plasma flow. To determine the effect of black hole gravity on plasma flow, we apply linear perturbation. The flow variables (mass density $\rho$, pressure $p$, velocity $V$ and magnetic field $B$) become

$$\rho = \rho^0 + \delta \rho = \rho^0 + \rho \tilde{\rho}, \quad p = p^0 + \delta p = p^0 + p \tilde{\rho},$$
$$V = V^0 + \delta V = V^0 + v, \quad B = B^0 + \delta B = B^0 + B \tilde{b},$$

(2.13)

where $\rho^0$, $p$, $V^0$ and $B^0$ represent the unperturbed quantities. The linearly perturbed quantities are denoted by $\delta \rho$, $\delta p$, $\delta V$ and $\delta B$. We introduce the following dimensionless quantities $\tilde{\rho}$, $\tilde{p}$, $v_x$, $v_z$, $b_x$ and $b_z$ which correspond to the perturbed quantities

$$\tilde{\rho} = \tilde{\rho}(t, z), \quad \tilde{p} = \tilde{p}(t, z), \quad \tilde{v} = \delta V = v_x(t, z)e_x + v_z(t, z)e_z,$$
$$\tilde{b} = \frac{\delta B}{B} = b_x(t, z)e_x + b_z(t, z)e_z.$$

(2.14)

After the insertion of linear perturbations in the perfect GRMHD equations (Eqs.(2.5)-(2.9)) and using Eq.(2.14), the component form of linearly perturbed GRMHD equations become [36]

$$\frac{1}{\alpha} \frac{\partial b_x}{\partial t} = \frac{1}{\alpha} \frac{\partial b_z}{\partial t} = 0,$$
$$b_{z,z} = 0,$$

(2.15)

(2.16)

(2.17)
\[
\begin{align*}
\rho \frac{1}{\alpha} \frac{\partial \tilde{p}}{\partial t} + p \frac{1}{\alpha} \frac{\partial \tilde{p}}{\partial t} + (\rho + p)\gamma^2 V \left( \frac{1}{\alpha} \frac{\partial v_x}{\partial t} + u v_{x,z} \right) + (\rho + p)\gamma^2 u \\
\times \frac{1}{\alpha} \frac{\partial v_x}{\partial t} + (\rho + p)(1 + \gamma^2 u^2) v_{x,z} = -\gamma^2 u (\rho + p) [(1 + 2\gamma^2 V^2) V' \\
+ 2\gamma^2 u V' u] v_x + (\rho + p) [(1 - 2\gamma^2 u^2)(1 + \gamma^2 u^2) \frac{u'}{u} - 2\gamma^4 u^2 V'^2] v_z, \\
= 0,
\end{align*}
\]
(2.18)

\[
\begin{align*}
&\left\{ (\rho + p)\gamma^2 (1 + \gamma^2 u^2) + \frac{B^2}{4\pi} \right\} \frac{1}{\alpha} \frac{\partial v_x}{\partial t} + \left\{ (\rho + p)\gamma^4 u V - \frac{\lambda B^2}{4\pi} \right\} \\
&\times \frac{1}{\alpha} \frac{\partial v_x}{\partial t} + \left\{ (\rho + p)\gamma^2 (1 + \gamma^2 u^2) + \frac{B^2}{4\pi} \right\} u v_{x,z} + \left\{ (\rho + p)\gamma^4 u V \\
&\quad - \frac{\lambda B^2}{4\pi} \right\} u v_{x,z} = \beta^2 (1 + 2\gamma^2 V^2)(\rho' + p') v_x + [(\rho + p)\gamma^2 [(1 + 2\gamma^2 u^2) \\
&\quad (1 + 2\gamma^2 V^2) V' - \gamma^2 V^2 V' + 2\gamma^2 (1 + 2\gamma^2 u^2) u V' u'] - \frac{B^2 u}{4\pi \alpha} (\lambda \alpha)' \\
&\quad + \gamma^2 V (1 + 2\gamma^2 u^2)(\rho' + p')] v_z = 0,
\end{align*}
\]
(2.19)

\[
\begin{align*}
&\left\{ (\rho + p)\gamma^2 (1 + \gamma^2 u^2) + \frac{\lambda^2 B^2}{4\pi} \right\} \frac{1}{\alpha} \frac{\partial v_x}{\partial t} + \left\{ (\rho + p)\gamma^4 u V - \frac{\lambda B^2}{4\pi} \right\} \\
&\times \frac{1}{\alpha} \frac{\partial v_x}{\partial t} + \left\{ (\rho + p)\gamma^2 (1 + \gamma^2 u^2) + \frac{\lambda^2 B^2}{4\pi} \right\} u v_{x,z} + \left\{ (\rho + p)\gamma^4 u V \\
&\quad - \frac{\lambda B^2}{4\pi} \right\} u v_{x,z} + \beta^2 (1 + 2\gamma^2 V^2)(\rho' + p') v_x + [(\rho + p)\gamma^2 [(1 + 2\gamma^2 u^2) \\
&\quad (1 + 2\gamma^2 V^2) V' - \gamma^2 V^2 V' + 2\gamma^2 (1 + 2\gamma^2 u^2) u V' u'] - \frac{B^2 u}{4\pi \alpha} (\lambda \alpha)' \\
&\quad + \gamma^2 V (1 + 2\gamma^2 u^2)(\rho' + p')] v_z = 0,
\end{align*}
\]
(2.20)
\[\frac{1}{\alpha} \gamma^2 \rho \frac{\partial \tilde{p}}{\partial t} + \frac{1}{\alpha} \gamma^2 p \frac{\partial \tilde{p}}{\partial t} + \gamma^2 (\rho' + p') v_z + u \gamma^2 (\rho \tilde{p}_z + p \tilde{p}_z + \rho' \tilde{p} + p' \tilde{p}) + 2 \gamma^2 u (\rho \tilde{p} + p \tilde{p}) a_z + \gamma^2 u'(\rho \tilde{p} + p \tilde{p}) + 2 (\rho + p) \gamma^4 (u V') + 2 u V a_z + u' V) v_z + 2 (\rho + p) \gamma^4 (1 + 2 \gamma^2 u^2) v_{z,z} - \frac{B^2}{4 \pi \alpha} \left[ (V^2 + u^2) \lambda \frac{\partial b_z}{\partial t} \right.
\left. + (V^2 + u^2) \frac{\partial b_z}{\partial t} - \lambda V (\lambda V + u) \frac{\partial b_z}{\partial t} - u (\lambda V + u) \frac{\partial b_z}{\partial t} \right] - \frac{B^2}{4 \pi \alpha} \times \left[ \lambda (\lambda V + u) v_{x,t} + (\lambda V + u) v_{z,t} - (\lambda^2 + 1) V v_{x,t} - (\lambda^2 + 1) u v_{z,t} \right] \right. + \left. \frac{B^2}{4 \pi} (\lambda' v_z - \lambda' v_x - \lambda' V b_z + \lambda' u b_x - V b_{x,z} + u \lambda b_{x,z}) = 0. \right]

For the Fourier analysis of the linearly perturbed GRMHD equations, we take the following harmonic spacetime dependence

\[\tilde{\rho}(t, z) = c_1 e^{-i(\omega t - k z)}, \quad \tilde{p}(t, z) = c_2 e^{-i(\omega t - k z)}, \quad v_z(t, z) = c_3 e^{-i(\omega t - k z)}, \quad v_x(t, z) = c_4 e^{-i(\omega t - k z)}, \quad b_z(t, z) = c_5 e^{-i(\omega t - k z)}, \quad b_x(t, z) = c_6 e^{-i(\omega t - k z)}, \]

(2.22)

where \(\omega\) represents angular frequency and \(k\) denotes the \(z\)-component of the wave vector \((0, 0, k)\). The wave vector is helpful to determine refractive index and the properties of plasma about the event horizon. Frequency dependence effects on the wave propagating in a medium are related to dispersion. Using Eq. (2.22) in Eqs. (2.15)-(2.21), we obtain their Fourier analyzed form

\[c_4 (\alpha' + ik \alpha) - c_3 \{(\alpha \lambda') + ik \alpha \lambda\} - c_5 (\alpha V') - c_6 \{(\alpha u') - \omega \]

+ ik \alpha) = 0, \quad (2.23)

\[c_5 \left( \frac{-i \omega}{\alpha} \right) = 0, \quad (2.24)
\]

\[c_5 ik = 0, \quad (2.25)
\]

\[c_1 \left( \frac{-i \omega}{\alpha} \right) \rho + c_2 \left( \frac{-i \omega}{\alpha} \right) p + c_3 (\rho + p) \left( \frac{-i \omega}{\alpha} \right) \gamma^2 u + (1 + \gamma^2 u^2) ik \]

\[- (1 - 2 \gamma^2 u^2) (1 + \gamma^2 u^2) u' \frac{u'}{u} + 2 \gamma^4 u^2 V V' + c_4 (\rho + p) \gamma^2 \left( \frac{-i \omega}{\alpha} \right) + ik u) V + u(1 + 2 \gamma^2 V^2) V' + 2 \gamma^2 u^2 V u' \right] = 0. \quad (2.26)

8
\[ c_1[\rho^2 u((1 + \gamma^2 V^2)V' + \gamma^2 V u') + \gamma^2 Vu(\rho' + ikp)] \\
+ c_2[\rho^2 u((1 + \gamma^2 V^2)V' + \gamma^2 V u') + \gamma^2 Vu(p' + ikp)] \\
+ c_3[(\rho + p)\gamma^2((1 + 2\gamma^2 u^2)(1 + 2\gamma^2 V^2)V' + (\frac{-i\omega}{\alpha} + iku)\gamma^2 V u \\
- \gamma^2 V^2 V' + 2\gamma^2(1 + 2\gamma^2 u^2)uv u' + \gamma^2 V(1 + 2\gamma^2 u^2)(\rho' + p') \\
- \frac{B^2 u}{4\pi\alpha} (\lambda\alpha)' + \frac{\lambda B^2}{4\pi}(\frac{\omega}{\alpha} - iku)] + c_4[(\rho + p)\gamma^4 u((1 + 4\gamma^2 V^2) \\
\times uu' + 4V V'(1 + \gamma^2 V^2)) + (\rho + p)\gamma^2(1 + \gamma^2 V^2)(\frac{-i\omega}{\alpha} + iku)] \\
+ \gamma^2 u(1 + 2\gamma^2 V^2)(\rho' + p') + \frac{B^2 u\alpha'}{4\pi\alpha} - \frac{B^2}{4\pi}(\frac{i\omega}{\alpha} - iku) \\
- c_6\frac{B^2}{4\pi\alpha}\left[\alpha uu' + \alpha' (1 + u^2) + (1 + u^2)ik\alpha\right] = 0, \quad (2.27) \\
c_1[\rho^2 a_z + (1 + \gamma^2 u^2)uu' + \gamma^2 u^2 V V'] + \gamma^2 u^2(\rho' + ikp)] \\
+ c_2[\rho^2 a_z + (1 + \gamma^2 u^2)uu' + \gamma^2 u^2 V V'] + (1 + \gamma^2 u^2) \\
\times (p' + ikp)] + c_3[(\rho + p)\gamma^2((1 + \gamma^2 u^2)((\frac{-i\omega}{\alpha} + iku) \\
+ u'(1 + \gamma^2 u^2)(1 + 4\gamma^2 u^2) + 2\gamma^2(a_z + (1 + 2\gamma^2 u^2)VV')) \\
+ 2\gamma^2 u(1 + \gamma^2 u^2)(\rho' + p') + \frac{\lambda B^2 u}{4\pi\alpha}(\lambda\alpha)' - \frac{\lambda^2 B^2}{4\pi}(\frac{i\omega}{\alpha} - iku)] \\
+ c_4[(\rho + p)\gamma^4((\frac{-i\omega}{\alpha} + iku)u V + u^2 V'(1 + 4\gamma^2 V^2) + 2V(a_z \\
+ (1 + 2\gamma^2 u^2)uu') + 2\gamma^4 u^2 V(\rho' + p') + \frac{\lambda B^2}{4\pi}(\frac{i\omega}{\alpha} - iku) \\
- \frac{\lambda B^2 u\alpha'}{4\pi\alpha}] + c_6\frac{B^2}{4\pi\alpha}\{-\lambda\alpha') + \alpha' \lambda - u\lambda(u\alpha' + u'\alpha)\} \\
+ \frac{\lambda B^2}{4\pi}(1 + u^2)ik = 0, \quad (2.28) \\
c_1[\frac{(-i\omega}{\alpha}\gamma^2 + iku\gamma^2 + 2\gamma^2 a_z + \gamma^2 u')\rho + u\rho' \gamma^2} + c_2[\frac{(-i\omega}{\alpha}(1 - \gamma^2) \\
+ iku\gamma^2 + 2\gamma^2 u a_z + \gamma^2 u')p + u\gamma^2 p'] + c_3\gamma^2 \{(\rho' + p') + 2 \\
\times (2\gamma^4 uu' + a_z + 2\gamma^2 u^2 a_z)(\rho + p) + (1 + 2\gamma^2 u^2)(\rho + p)ik + \frac{\lambda B^2}{4\pi\alpha} \\
\times (\lambda u - V)\omega + \alpha\lambda') + c_4[2(\rho + p)\gamma^2((u V' + 2Vu a_z + u' V) + u V i k) \\
+ \frac{B^2}{4\pi\alpha}(V - u\lambda)\omega - \alpha\lambda'] + c_6\frac{B^2}{4\pi\alpha}\{(\omega u + i\alpha k)(\lambda u - V) - \alpha\lambda' \\
+ u V i a k\} = 0. \quad (2.29)
3 Rotating Non-Magnetized Background

In the rotating non-magnetized plasma flow, we take $B = 0 = \lambda$ and $c_5 = 0 = c_6$ in the Fourier analyzed perturbed GRMHD equations ((2.26)-(2.29)) [36].

3.1 Numerical Solutions

To determine numerical solutions of the Fourier analyzed equations in rotating non-magnetized background, we assume

- Specific enthalpy: $\mu = \sqrt{1 + \alpha^2}$,
- Time lapse: $\alpha = \frac{z}{2r_h}$,
- Velocity components: $u = V, x$ and $z$-components of velocity yield $u = V = -\frac{1}{\sqrt{z^2 + 2}}$,
- Stiff fluid: $\rho = p = \frac{\mu}{2}$,
- Lorentz factor: $\gamma = \frac{1}{\sqrt{1 - u^2}} = \frac{\sqrt{z^2 + 2}}{z}$,

where $r_h$ is the SdS event horizon greater than that of the Schwarzschild event horizon and $r_h \approx 2M \left(1 + \frac{4M^2}{z^2} + \ldots\right) \approx \zeta 2.948 km, 1 \leq \zeta \leq 1.5$ [23]. It is known that strong gravity of black hole plays an important role to perturb plasma present in its surroundings which causes the plasma wave propagation around black holes. The frequency dependence effects in wave propagation leads to the dispersion, i.e., the phenomenon in which phase or group velocity of the wave depends upon the frequency [38].

We consider the region $4 \leq z \leq 10$ for wave analysis considering that event horizon is exactly at $z = 0$. The determinant of the coefficients of constants of the corresponding equations for the rotating non-magnetized plasma yields a complex dispersion relation. The real part of the determinant gives a quartic equation in $k$

$$A_1(z)k^4 + A_2(z, \omega)k^3 + A_3(z, \omega)k^2 + A_4(z, \omega)k + A_5(z, \omega) = 0 \quad (3.30)$$

which yields two real and two imaginary roots. A cubic equation in $k$ is obtained from the imaginary part of dispersion relation

$$B_1(z)k^3 + B_2(z, \omega)k^2 + B_3(z, \omega)k + B_4(z, \omega) = 0 \quad (3.31)$$
which provides three real roots. The wave vector, refractive index, its change with respect to angular frequency, group velocity and phase velocity lead to the wave properties of the SdS black hole in the presence of Veselago medium. These are shown by graphical representation in Figures 1-5 expressing real values of $k$ in Eqs.(3.30) and (3.31).

The dispersion is normal if phase velocity is greater than the group velocity, otherwise anomalous [38] or equivalently dispersion is normal if change in refractive index with respect to angular frequency is positive and anomalous otherwise. The information about energy exchange within the magnetosphere can be extracted only where the dispersion is normal. It is clear from figures that some waves move towards the event horizon and some move away from the horizon. The dispersion is normal in Figures 2, 3 and 5, while Figures 1 and 4 indicate normal as well as anomalous at random points of the region. The wave properties of non-magnetized hot plasma are illustrated in the tables I and II.

Table I. Direction and refractive index of waves

| Fig. | Direction of Waves | Refractive Index $(n)$ |
|------|--------------------|------------------------|
| 1    | Move towards the event horizon | $n < 1$ and decreases in the region $4 \leq z \leq 9, 0 \leq \omega \leq 10$ with the decrease in $z$ |
| 2    | Move away from the event horizon | $n < 1$ and increases in the region $4 \leq z \leq 9.8, 0 \leq \omega \leq 10$ with the decrease in $z$ |
| 3    | Move towards the event horizon | $n < 1$ and increases in the region $4 \leq z \leq 5.8, 0 \leq \omega \leq 10$ with the decrease in $z$ |
| 4    | Move towards the event horizon | $n < 1$ and decreases in the region $4 \leq z \leq 8.8, 0 \leq \omega \leq 10$ with the decrease in $z$ |
| 5    | Moves away from the event horizon | $n < 1$ and increases in the region $4 \leq z \leq 5.8, 0 \leq \omega \leq 10$ with the decrease in $z$ |

The regions for anomalous and normal dispersion for figures exhibiting random dispersion are separated as follows:

Table II. Regions of normal and anomalous dispersion
Here plasma is considered to be rotating and magnetized. The flow is supposed to be in two dimensions, so the magnetic field and velocity of fluid lie in $xz$-plane. Equations (2.23)-(2.29) represent the perturbed Fourier analyzed GRMHD equations for the rotating magnetized plasma.

### 4.1 Numerical Solutions

The assumptions for velocity, lapse function and specific enthalpy are the same as in the previous section. We put the following restrictions on magnetic field to find numerical solutions.

- $\frac{B^2}{4\pi} = 2$ with $u = V$
- putting $V^F = 1$ in Eq. (2.11) so that $\lambda = 1 + \frac{\sqrt{2 + z^2}}{z}$.

Again our region of consideration is $4 \leq z \leq 10$ and $0 \leq \omega \leq 10$. Equations (2.24)-(2.25) yield $c_5 = 0$. The dispersion relation, originated from the determinant of the Fourier analyzed form, gives the real part

$$A_1(z)k^4 + A_2(z, \omega)k^3 + A_3(z, \omega)k^2 + A_4(z, \omega)k + A_5(z, \omega) = 0 \quad (4.1)$$

which provides four roots, all are imaginary. The imaginary part of the dispersion relation

$$B_1(z)k^5 + B_2(z, \omega)k^4 + B_3(z, \omega)k^3 + B_4(z, \omega)k^2 + B_5(z, \omega)k$$
$$+ B_6(z, \omega) = 0 \quad (4.2)$$

yields five roots of $k$, three real and two complex. The real roots indicate wave propagation in the region $4 \leq z \leq 10$ and $0 \leq \omega \leq 10$, shown in Figures 6-8. The waves are directed towards the event horizon given in Figure 6, move away from the event horizon as shown in Figures 7 and 8. The dispersion
Figure 1: Dispersion is normal and anomalous in the region.

Figure 2: Whole region exhibits the normal dispersion.
Figure 3: Normal dispersion of waves is observed.

Figure 4: Random points of normal and anomalous dispersion are found in the region.
Figure 5: Whole region admits normal dispersion.

is normal as well as anomalous at random points in Figures 6 and 8, while Figure 7 indicates normal dispersion in the whole region. The tables III and IV summarize the results deduced from these figures.

Table III. Direction and refractive index of waves

| Fig. | Direction of Waves               | Refractive Index \((n)\)                                                   |
|------|----------------------------------|---------------------------------------------------------------------------|
| 6    | Move towards the event horizon   | \(n < 1\) and decreases in the region \(6.95 \leq z \leq 7.3, 0 \leq \omega \leq 10\) with the decrease in \(z\) |
| 7    | Move away from the event horizon | \(n < 1\) and increases in the region \(4.3 \leq z \leq 4.7, 0 \leq \omega \leq 10\) with the decrease in \(z\) |
| 8    | Move away from the event horizon | \(n < 1\) and increases in the region \(4.4 \leq z \leq 4.9, 0 \leq \omega \leq 10\) with the decrease in \(z\) |

The regions of normal and anomalous dispersion for the roots exhibiting random dispersion can be separated as follows:
Figure 6: Normal and anomalous dispersion of waves is observed.

Figure 7: Waves disperse normally in the whole region.
Figure 8: Waves exhibit normal as well as anomalous dispersion at random points.
Table IV. Regions of random dispersion

| Fig. | Normal dispersion | Anomalous dispersion |
|------|-------------------|----------------------|
| 6    | \(4 \leq z \leq 7, 2 \leq \omega \leq 10\) | \(7 \leq z \leq 7.8, 1 \leq \omega \leq 4\) |
|      | \(7.9 \leq z \leq 10, 3 \leq \omega \leq 10\) |                            |
| 8    | \(5.7 \leq z \leq 10, 0.5 \leq \omega \leq 10\) | \(4 \leq z \leq 4.6, 0.03 \leq \omega \leq 0.35\) |
|      |                                  | \(4.7 \leq z \leq 4.9, 0.45 \leq \omega \leq 1.6\) |

5 Summary

This paper is devoted to the study of dispersion modes for hot plasma surrounding SdS black hole in a Veselago medium. For the formulation of GRMHD equations in this unusual medium, we use the ADM 3 + 1 formalism. The strong gravitational field of black hole disturbs the flow of surrounding plasma. To determine the impression of black hole gravity on plasma flow, we have applied linear perturbations to the GRMHD equations. Further, we have assumed that plasma flow is in two dimensions and evaluated the component form of linearly perturbed GRMHD equations. Finally, we have established dispersion relations for the rotating (non-magnetized and magnetized) plasma with the help of Fourier analysis techniques.

In rotating non-magnetized plasma, waves move away from the event horizon shown in Figures 2 and 5 while waves are directed towards the event horizon given in Figures 1, 3 and 4. The dispersion is found to be normal in Figures 2, 3 and 5 in the whole region. It can be noticed that Figures 1 and 4 exhibit normal as well as anomalous dispersion at random points. For rotating magnetized background, waves are directed towards the event horizon in Figure 6 while Figures 7 and 8 indicate that waves moves away from the event horizon. The dispersion is randomly distributed (normal as well as anomalous) shown in Figures 6 and 8 while Figure 7 admits normal dispersion in the whole region.

In the usual medium, the refractive index is always greater than one but for Veselago medium its value must be less than one. The graphs of refractive index in all figures show that region has refractive index less than one and increases or decreases in small regions. Also, the phase velocity is greater than group velocity for both non-magnetized and magnetized backgrounds. These are the prominent features of the Veselago medium which demonstrate the validity of this unusual medium for both rotating (non-magnetized and magnetized) background around SdS event horizon.
We have seen that for hot plasma surrounding the Schwarzschild black hole, most of the waves have anomalous and normal dispersion at random points while some waves exhibit anomalous dispersion in the whole region \[36\]. Moreover, most of the waves are directed away from the event horizon of the Schwarzschild black hole. However, here we have found that dispersion is normal at most of the points. Thus it can be established that more information can be extracted from magnetosphere by the inclusion of positive cosmological constant in non-rotating black hole. Also, the outcome of SdS black hole for isothermal plasma in a Veselago medium \[37\] shows wave propagation in different regions, i.e., \(-5 \leq z \leq -1\), \(0 \leq \omega \leq 10\) and \(1 \leq z \leq 5\), \(0 \leq \omega \leq 10\). For the isothermal plasma, some waves move towards event horizon and other move towards outer end of magnetosphere. This comparison shows that the wave properties can be better understood for the highly ionized, i.e., hot plasma around SdS magnetosphere.

**Appendix A1**

The Maxwell equations for such a medium are

\[
\nabla \cdot \mathbf{B} = 0, \quad \text{(A1)}
\]

\[
\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \quad \text{(A2)}
\]

\[
\nabla \cdot \mathbf{E} = -\frac{\rho_e}{\epsilon}, \quad \text{(A3)}
\]

\[
\nabla \times \mathbf{B} = -\mu_0 \mathbf{j} + \frac{\partial \mathbf{E}}{\partial t} = 0. \quad \text{(A4)}
\]
The GRMHD equations for the SdS spacetime in Rindler coordinates become

\[
\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times (\alpha \mathbf{V} \times \mathbf{B}), \quad (A5)
\]

\[
\nabla \cdot \mathbf{B} = 0, \quad (A6)
\]

\[
\frac{\partial \rho_0}{\partial t} + (\alpha \mathbf{V} \cdot \nabla) \rho_0 + \rho_0 \gamma^2 \mathbf{V} \cdot \frac{\partial \mathbf{V}}{\partial t} + \rho_0 \gamma^2 \mathbf{V} \cdot (\alpha \mathbf{V} \cdot \nabla) \mathbf{V} + \rho_0 \nabla \cdot (\alpha \mathbf{V}) = 0, \quad (A7)
\]

\[
\left\{ \left[ \rho_0 \mu \gamma^2 + \frac{\mathbf{B}^2}{4\pi} \right] \delta_{ij} + \rho_0 \mu \gamma^4 \mathbf{V}_i \mathbf{V}_j - \frac{1}{4\pi} \mathbf{B}_i \mathbf{B}_j \right\} \left( \frac{1}{\alpha} \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \mathbf{V}^j - \left( \frac{\mathbf{B}^2}{4\pi} \delta_{ij} - \frac{1}{4\pi} \mathbf{B}_i \mathbf{B}_j \right) \mathbf{V}^j \cdot \mathbf{V}^k + \rho_0 \gamma^2 \mathbf{V}_i \left\{ \frac{1}{\alpha} \frac{\partial \mu}{\partial t} + \left( \mathbf{V} \cdot \nabla \right) \mu \right\}
\]

\[
= -\rho_0 \mu \gamma^2 a_i - p, a_i + \frac{1}{4\pi} \mathbf{V} \cdot \{ \mathbf{V} \times \mathbf{B} \} \mathbf{V} \cdot \{ \mathbf{V} \times (\alpha \mathbf{V} \times \mathbf{B}) \} - \frac{1}{8\pi \alpha^2} (\alpha \mathbf{B}^2)^2, a_i
\]

\[
+ \frac{1}{4\pi \alpha} (\alpha \mathbf{B}_i)_j B^j - \frac{1}{4\pi \alpha} [\mathbf{B} \times \{ \mathbf{V} \times (\nabla \times (\alpha \mathbf{V} \times \mathbf{B})) \}], \quad (A8)
\]

\[
\left( \frac{1}{\alpha} \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) (\mu \rho_0 \gamma^2) - \frac{1}{\alpha} \frac{\partial p}{\partial t} + 2 \mu \rho_0 \gamma^2 (\mathbf{V} \cdot \mathbf{a}) + \mu \rho_0 \gamma^2 (\nabla \cdot \mathbf{V})
\]

\[
- \frac{1}{4\pi} (\mathbf{V} \times \mathbf{B}) \cdot (\mathbf{V} \times \frac{1}{\alpha} \frac{\partial \mathbf{B}}{\partial t}) - \frac{1}{4\pi} (\mathbf{V} \times \mathbf{B}) \cdot (\mathbf{B} \times \frac{1}{\alpha} \frac{\partial \mathbf{V}}{\partial t})
\]

\[
+ \frac{1}{4\pi \alpha} (\mathbf{V} \times \mathbf{B}) \cdot (\nabla \times \alpha \mathbf{B}) = 0. \quad (A9)
\]

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