Ramanujan’s master theorem for sturm liouville operator

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Abstract
In this paper we prove an analogue of the Ramanujan’s master theorem in the setting of Sturm Liouville operator

\[ \mathcal{L} = \frac{d^2}{dt^2} + \frac{A'(t)}{A(t)} \frac{d}{dt}, \]

on \((0, \infty)\), where \(A(t) = (\sinh t)^{2\alpha+1}(\cosh t)^{2\beta+1}B(t); \alpha, \beta > -\frac{1}{2} \) with suitable conditions on \(B\). When \(B \equiv 1\) we get back the Ramanujan’s Master theorem for the Jacobi operator.

Keywords Ramanujan’s master theorem · Compact dual · Sturm Liouville operator

Mathematics Subject Classification Primary 43A62 · 43A85; Secondary 43E32 · 34L10

1 Introduction

Ramanujan’s Master theorem ([10]) states that if a function \(f\) can be expanded around 0 in a power series of the form

\[ f(t) = \sum_{n=0}^{\infty} a_n t^n. \]
\[ f(x) = \sum_{k=0}^{\infty} (-1)^k a(k)x^k, \]

then

\[ \int_0^\infty f(x)x^{-\lambda-1} \, dx = -\frac{\pi}{\sin \pi \lambda} a(\lambda). \]

One needs some assumptions on the function \( a \), as the theorem is not true for \( a(\lambda) = \sin \pi \lambda \). Hardy provides a rigorous statement of the theorem above as:

Let \( D, p, \delta \) be real constants such that \( D < \pi \) and \( 0 < \delta \leq 1 \). Let \( \mathcal{H}(\delta) = \{ \lambda \in \mathbb{C} \mid \Re \lambda > -\delta \} \). Also let \( \mathcal{H}(D, p, \delta) \) be the collection of all holomorphic functions \( a : \mathcal{H}(\delta) \to \mathbb{C} \) such that

\[ |a(\lambda)| \leq Ce^{-p(\Re \lambda)+D|\Im \lambda|} \text{ for all } \lambda \in \mathcal{H}(\delta). \]

**Theorem 1.1** (Ramanujan’s Master theorem, Hardy [10]) Suppose \( a \in \mathcal{H}(D, p, \delta) \).

Then the following holds:

1. The power series

\[ f(x) = \sum_{k=0}^{\infty} (-1)^k a(k)x^k, \]

converges for \( 0 < x < e^p \) and defines a real analytic function on that domain.

2. Let \( 0 < \sigma < \delta \). Then for \( 0 < x < e^p \) we have

\[ f(x) = \frac{1}{2\pi i} \int_{-\sigma-i\infty}^{-\sigma+i\infty} \frac{-\pi}{\sin \pi \lambda} a(\lambda)x^\lambda \, d\lambda. \]

The integral on the right side of the equation above converges uniformly on compact subsets of \([0, \infty)\) and is independent of \( \sigma \).

3. Also

\[ \int_0^\infty f(x)x^{-\lambda-1} \, dx = -\frac{\pi}{\sin \pi \lambda} a(\lambda), \]

holds for the extension of \( f \) to \([0, \infty)\) and for all \( \lambda \in \mathbb{C} \) with \( 0 < \Re \lambda < \delta \).

This theorem can be thought of as an interpolation theorem, which reconstructs the values of \( a(\lambda) \) from its given values at \( a(k), k \in \mathbb{N} \cup \{0\} \). In particular if \( a(k) = 0 \) for all \( k \in \mathbb{N} \cup \{0\} \), then \( a \) is identically 0. This theorem is used to find several definite and indefinite integrals involving hypergeometric functions, etc (see [1, 5]). This explains the name “master theorem”. We can rewrite the theorem above in terms of Fourier series and Fourier transform as follows:
Theorem 1.2 Suppose \( a \in \mathcal{H}(D, p, \delta) \). Then the following holds:

1. The Fourier series

\[
f(z) = \sum_{k=0}^{\infty} (-1)^k a(k) e^{-ikz},
\]

converges for \( \Im z < p \) and defines a holomorphic function on that domain.

2. Let \( 0 < \sigma < \delta \). Then for \( 0 < t < p \) we have

\[
f(it) = \frac{1}{2\pi i} \int_{-\sigma-i\infty}^{\sigma+i\infty} \frac{-\pi}{\sin \pi \lambda} a(\lambda) e^{-it\lambda} d\lambda.
\]

The integrals defined above are independent of \( \sigma \) and \( f \) extends as a holomorphic function to a neighbourhood \( \{ z \in \mathbb{C} \mid |\Re z| < \pi - D \} \) of \( i \mathbb{R} \).

3. Also

\[
\int_{\mathbb{R}} f(ix) e^{i\lambda x} dx = -\frac{\pi}{\sin \pi \lambda} a(\lambda),
\]

holds for the extension of \( f \) to \( i \mathbb{R} \) and for all \( \lambda \in \mathbb{C} \) with \( 0 < \Re \lambda < \delta \).

In summary, if a function \( f \) can be written in terms of Fourier series on \( \Im z < p \) whose Fourier coefficients lie in \( \mathcal{H}(D, p, \delta) \) defined above, then it has an integral representation at \( it, t \in (0, p) \). Furthermore the integral representation of \( f \) has a holomorphic extension around a neighbourhood of \( i \mathbb{R} \). Also the Fourier coefficients of \( f \) can be obtained as a restriction on discrete points of the Fourier transform of the extended \( f \). Bertram (in [2]) provides a group theoretical interpretation of the theorem in the following way: Consider \( x \mapsto x^k, \lambda \in \mathbb{C} \) and \( x \mapsto x^k, k \in \mathbb{Z} \) as the spherical functions on \( X_G = \mathbb{R}^+ \) and \( X_U = U(1) \) respectively. Both \( X_G \) and \( X_U \) can be realized as the real forms of their complexification \( X_C = \mathbb{C}^* \). Let \( \tilde{f} \) and \( \hat{f} \) denote the spherical transformation of \( f \) on \( X_G \) and on \( X_U \) respectively. Then from Theorem 1.1 it follows that,

\[
\tilde{f}(\lambda) = -\frac{\pi}{\sin \pi \lambda} a(\lambda), \quad \hat{f}(k) = (-1)^k a(k).
\]

Using the duality between \( X_U \) and \( X_G \) inside their complexification \( X_C \), Bertram proved an analogue of the Ramanujan’s Master theorem for semisimple Riemannian symmetric spaces of rank one. This theorem was further extended to arbitrary rank semisimple Riemannian symmetric spaces by Ólafsson and Pasquale (see [16]). It was also extended for the hypergeometric Fourier transform associated to root systems by Ólafsson and Pasquale (see [17]) and also to the radial sections of line bundles over Poincaré upper half plane by Pusti and Ray ([19]).

In this paper we prove an analogue of this theorem in the setting of Sturm-Liouville operator. Let

\[
A(z) = (\sinh z)^{2\alpha+1} (\cosh z)^{2\beta+1} B(z), z \in \Omega,
\]
where $\Omega = \{z : |\Re z| < \frac{\pi}{2} \}$. The function $B$ is assumed to be an even, non-zero holomorphic function on $\Omega$ and positive on real and imaginary axis in $\Omega$ along with some more conditions stated below. We define $L$ on $(0, \infty)$ as:

$$L = \frac{d^2}{dt^2} + \frac{A'(t)}{A(t)} \frac{d}{dt}.$$  

Then we consider the eigenfunctions $\varphi_\lambda$ of the initial value problem

$$L f + (\lambda^2 + \rho^2) f = 0 \quad \text{on} \quad (0, \infty), \quad f(0) = 1, \quad f'(0) = 0.$$  

We think of the eigenfunctions $\varphi_\lambda$ as an analogue of the spherical function $x \mapsto x^\lambda$ on $\mathbb{R}^+$. When $B \equiv 1$, $L$ is the Jacobi operator and the solution of the above initial value problem is given by the hypergeometric functions (see [12]). We can also replace $t \equiv (0, \infty)$ in $L$ by the variable $z \in \Omega \setminus \{0\}$ and apply it to holomorphic functions on $\Omega \setminus \{0\}$. With abuse of notation we call the extended operator as $L$.

For proving Ramanujan’s Master theorem corresponding to the Sturm Liouville operator $L$ we need it’s Fourier analysis on $(0, \infty)$ and on $z = it, t \in (0, \pi/2)$ as well. In ([4]) Brandolini and Gigante developed the Fourier analysis analogue of $L$ on $(0, \infty)$ for proving the equiconvergence of the eigenfunction expansion for the Sturm-Liouville operator $L$ on $(0, \infty)$ when $A(t)$ satisfies (C1) to (C6) conditions below. For dealing with the case when $L$ is restricted to $z = it, t \in (0, \pi/2)$, we define a positive, symmetric densely defined differential operator $-L$ on $(0, \pi/2)$ on a suitable Hilbert space such that

$$-L w(t) = L u(z)_{z=it},$$  

where $w(t) = u(it)$ for holomorphic function $u$ on $\Omega$. We consider the operator

$$L = \frac{d^2}{dt^2} + \frac{\tilde{A}'(t)}{\tilde{A}(t)} \frac{d}{dt},$$  

on $(0, \frac{\pi}{2})$, where $\tilde{A}(t) = (-i)^{2x+1} A(it)$ and $A$ as defined above. A simple computation gives that $L$ satisfies (1.1). We apply the theory developed in [15] for the self adjoint extension of $-L$ to obtain a spectral decomposition of $-L$ on $(0, \pi/2)$ under suitable boundary conditions such that the eigenfunctions of $L$ are the restriction of the eigenfunctions of $\tilde{L}$ at $z = it$. More explicitly, we get functions $\Psi_j$’s (defined in section 4) as (countable) eigenfunctions of $-L$ with the corresponding eigenvalue $\nu_j$ (see Theorem 4.1). The functions $\Psi_j$’s, $j = 0, 1, 2 \cdots$ are orthonormal basis of $L^2((0, \frac{\pi}{2}), \tilde{A}(t)dt)$. We think of these $\Psi_j$’s as an analogue of $x \mapsto x^k$ on $U(1)$. These $\varphi_\lambda$ and $\Psi_j$ are related by

$$\Psi_j(t) = c_j \varphi_i \sqrt{\nu_j + \rho^2} (it), \quad \text{on} \quad \left(0, \frac{\pi}{2}\right).$$  

In the non-structure case, that is the case when $B \equiv 1$, the functions $\Psi_j$’s are nothing but the Jacobi polynomials. In [8] the authors also study the spectral decomposition
of \(-L\) on \((0, \frac{\pi}{2})\) under suitable boundary conditions at 0 and \(\pi/2\). They obtained the eigenfunctions of \(-L\) as the perturbation of the Jacobi polynomials along with their corresponding eigenvalues in the asymptotic sense. Since \(\Psi_j\)'s are orthonormal basis of \(L^2 ((0, \frac{\pi}{2}), \tilde{A(t)}dt)\), we can think of the series (1.8) as the Fourier series corresponding to the operator \(-L\).

**Main results**

We now state Ramanujan’s Master theorem for the Sturm-Liouville operator \(L\) here. We postpone it’s proof to section 6. Before stating it let us introduce some conditions on the function \(A\) in the definition of \(L\).

Throughout this paper we always assume that, \(\alpha, \beta > -\frac{1}{2}\). Let \(\Omega = \{t + is \mid |s| < \frac{\pi}{2}\}\). We define \(A : \Omega \to \mathbb{C}\) by

\[
A(z) = (\sinh z)^{2\alpha + 1}(\cosh z)^{2\beta + 1}B(z),
\]

where \(B : \Omega \to \mathbb{C} \setminus \{0\}\) is holomorphic. In this paper we assume the following (C1) to (C6) conditions (cf. [4]):

(C1) The function \(B\) is even on \(\Omega\), positive on \(\mathbb{R}\) and \(B\left|_{ix; |s| < \frac{\pi}{2}}\right. > 0\).

(C2) The function \(B\) has an even (with respect to \(i\frac{\pi}{2}\), i.e. \(B(i\frac{\pi}{2} + z) = B(i\frac{\pi}{2} - z)\)) holomorphic extension to a neighbourhood of \(i\frac{\pi}{2}\).

(C3) The function \(\frac{A'(t)}{A(t)}\) is non-negative decreasing function, for large \(t\) and \(\lim_{t\to\infty} A(t) = \infty\). Then we define

\[
2\rho = \lim_{t\to\infty} \frac{A'(t)}{A(t)}.
\]

(C4) The function \(G\) defined in equation (2.8) is integrable along any straight line in \(\Omega\) and \(\sup_{\Im(z)\leq \theta} \left| \int_{\{0,z\}} G(s)ds \right| < \infty\) for any \(\theta < \pi/2\) and \([0, z]\) is the oriented line segment joining 0 and \(z\).

(C5) Let

\[
q(t) = \frac{1}{4} \left( \frac{A'(t)}{A(t)} \right)^2 + \frac{1}{2} \left( \frac{A'(t)}{A(t)} \right)' - \rho^2.
\]

We assume that there exists \(a \geq 0\) such that

\[
q(t) = \frac{a^2 - \frac{1}{4}}{t^2} + \zeta(t),
\]

for some \(\zeta \in L^1 ((1, \infty), t\,dt)\) for \(a > 0\) or \(\zeta \in L^1 ((1, \infty), t\log t\,dt)\) for \(a = 0\).
Remark 1.3 Note that the above properties (C1) to (C5) imply that as \( t \to \infty \), there is \( b \) with \( b > -\frac{1}{2} \) and \( |b| = a \) such that

\[
\sqrt{A(t)\varphi_0(t)} \propto t^{b + \frac{1}{2}},
\]

when \( b \neq 0 \) and

\[
\sqrt{A(t)\varphi_0(t)} \propto t^{\frac{1}{2}} \log t
\]

when \( b = 0 \).

(C6) We assume further that for \( -\frac{1}{2} < b < 0 \) and

\[
\sqrt{A(t)\varphi_0(t)} \propto t^{\frac{1}{2} + b},
\]

the function \( \zeta \in L^1 \left( (1, \infty), t^{2|b|+1} \, dt \right) \). When \( b = 0 \) and

\[
\sqrt{A(t)\varphi_0(t)} \propto t^{\frac{1}{2}} \log t,
\]

we assume that \( \zeta \in L^1 \left( (1, \infty), t \log^2 t \, dt \right) \).

(C5)* (cf. [20]): There exists \( \varepsilon > 0 \) such that for all \( t \) in \([t_0, \infty)\) (for some \( t_0 > 0 \))

\[
\frac{A'(t)}{A(t)} = \begin{cases} 
2\rho + e^{-\varepsilon t} E(t) & \text{if } \rho > 0, \\
\frac{2\alpha+1}{t} + e^{-\varepsilon t} E(t) & \text{if } \rho = 0,
\end{cases}
\]

for some smooth bounded function \( E \) whose derivatives are also bounded.

Let \( \alpha, \beta \geq 0 \). We assume the conditions (C1), (C2), (C3), (C4), (C5)* on function \( A \) and let \( \rho > 0 \). In this case \( \nu_0 \geq 0 \) and \( \nu_j > 0 \) for all \( j = 1, 2, \cdots \) (see section 4 for details). Let \( D, p, \delta \) be real numbers such that \( D < \pi/2, \ p > 0 \) and \( 0 < \delta \leq 1 \).

Define

\[
\mathcal{H}(\delta) = \{ \lambda \in \mathbb{C} \mid \Re \lambda > -\delta \rho \},
\]

and

\[
\mathcal{H}(D, p, \delta) = \left\{ a : \mathcal{H}(\delta) \to \mathbb{C} \text{ holomorphic} \mid |a(\lambda)| \leq C e^{-p(\Im \lambda) + D|\Re \lambda|} \text{ for all } \lambda \in \mathcal{H}(\delta) \right\}.
\]

We define the following functions:

\[
S(z) = \pi z \prod_{j=0}^{\infty} \left( 1 + \frac{z^2}{\nu_j + \rho^2} \right), \ z \in \mathbb{C},
\]
and
\[ S_1(z) = \frac{z^2}{S(z)}, \quad z \in \mathbb{C}. \] (1.7)

The function \( S_1 \) has simple poles at \( \lambda = i\sqrt{\nu_j + \rho^2}, \ j = 0, 1, 2, \ldots \). Let \( d_j \) be the residue of \( S_1 \) at \( \lambda = i\sqrt{\nu_j + \rho^2} \) for \( j = 0, 1, 2, \ldots \) and \( c_j \)'s are constants given in (1.3). Then the statement of our theorem is as follows:

**Theorem 1.4** Let \( a \in \mathcal{H}(D, p, \delta) \). Then the following holds:

1. The Fourier series
   \[
   f(t) = 2\pi i \sum_{j=0}^{\infty} \frac{d_j}{c_j} a\left(i\sqrt{\nu_j + \rho^2}\right) \Psi_j(t),
   \] (1.8)
   converges uniformly on compact subsets of \( \Omega_p := \{ t \in \mathbb{C} \mid |\Re t| < \frac{\pi}{2}, |\Im t| < p \} \) and hence holomorphic there.

2. Let \( 0 \leq \sigma < \delta \rho \). Then the function \( f \) can also be expressed in the integral form for \( t \in \mathbb{R} \) with \( |t| < p \) as
   \[
   f(it) = \frac{1}{2} \int_{-\infty-i\sigma}^{\infty-i\sigma} (a(\lambda)b(\lambda) + a(-\lambda)b(-\lambda)) \varphi_\lambda(t) \frac{d\lambda}{c(\lambda)c(-\lambda)},
   \] (1.9)
   where the function \( b \) is defined by
   \[
   \frac{b(\lambda)}{c(\lambda)c(-\lambda)} = S_1(\lambda).
   \] (1.10)

The integrals defined above are independent of \( \sigma \) and extends as a holomorphic function to a neighbourhood \( \{ z \in \mathbb{C} \mid |\Re z| < \frac{\pi}{2} - D \} \) of \( i\mathbb{R} \).

3. The extension of \( f \) to \( i\mathbb{R} \) satisfies, for all \( \lambda \in \mathbb{R} \)
   \[
   \int_{\mathbb{R}} f(it) \varphi_\lambda(t) A(t) \, dt = \frac{1}{2} \left( a(\lambda)b(\lambda) + a(-\lambda)b(-\lambda) \right). \] (1.11)

In the above theorem \( c(\lambda) \) is the Harish Chandra \( c \)-function defined in (2.10). For the case when either \( \alpha \) or \( \beta \) is less than 0 we refer to Section 6.

The main crux of the proof of the theorem is to find a function \( b \) for which (1.9) holds. In the Euclidean case and in the non perturbed case the function \( b \) is related to the reciprocal of sine function but here in the perturbed case reciprocal of sine function will not work. Instead, here the function \( b \) is related to inverse of some sine type function. In order to obtain estimates on the function \( b \) we use sharp asymptotics of the eigenvalues \( \nu_j \) (see [8, Lemma 17]) as \( j \to \infty \). We get the desired properties of the function \( b \) by combining these sharp estimates and the characterisation of sine type functions by Levin in [13].
Theorem 1.4 is the standard analogue of Ramanujan’s Master Theorem as proved in [2, 16] and [17]. However it requires more explicit condition on A than the one stated in [4] so that we get more information on the Harish Chandra c-function (see section 3 for more details). If we assume conditions (C1) to (C6) on A as in [4], we prove an $L^2$ version of the Ramanujan’s Master Theorem 7.2 (see section 7 for details).

**Plan of the paper:** In section 2 we define the necessary terminology and state some known results for $L$ and its eigenfunctions mainly from [4]. In section 3 under modified condition on the function A, we prove more properties of Harish-Chandra $c$-function. Section 4 is devoted to develop the Fourier series analogue of the operator $L$ (given in (1.2)) and we prove the relation (1.3). In section 5 we define and prove few important properties of the function $b$. After developing all the machinery we prove our main Theorem 1.4 in section 6. In section 7 we state the $L^2$ version of the Ramanujan’s Master theorem. For the sake of completeness, in section 8 we state and sketch outline of proof of some standard theorems in our context.

## 2 Preliminaries

For two non negative functions $f$ and $g$, we write $f \asymp g$ if there exists constants $d_1, d_2 > 0$ such that $d_1 f(x) \leq g(x) \leq d_2 f(x)$ for all $x$. Throughout this paper we always assume that, $\alpha, \beta > -\frac{1}{2}$.

Now we consider the following Sturm-Liouville operator

$$\mathcal{L} = \frac{d^2}{dt^2} + \frac{A'(t)}{A(t)} \frac{d}{dt}.$$  

(2.1)

For each $\lambda \in \mathbb{C}$, we define $\varphi_\lambda$ as the unique solution of

$$\mathcal{L} f + (\lambda^2 + \rho^2) f = 0, \text{ with } f(0) = 1, f'(0) = 0.$$  

(2.2)

For the case when $B(t) = 1$ for all $t$ (with certain values of $\alpha, \beta$) the Sturm-Liouville operator $\mathcal{L}$ becomes the radial part of the Laplace-Beltrami operator on the rank one symmetric spaces of noncompact type $X = G/K$ and in this case the function $\varphi_\lambda$ (defined in (2.2)) becomes the elementary spherical function on $X$. We call the case $B \equiv 1$ as the non perturbed case and otherwise as perturbed case.

We have the following properties of $\varphi_\lambda$ ([3, 4]):

1. For each $t \in \mathbb{R}$, the function $\lambda \mapsto \varphi_\lambda(t)$ is even, entire.
2. For each $\lambda \in \mathbb{C}$, the function $t \mapsto \varphi_\lambda(t)$ is even.
3. For $\lambda \in \mathbb{C}$ with $|\Re \lambda| \leq \rho, |\varphi_\lambda(t)| \leq 1$ for all $t \in \mathbb{R}$.
4. For $\lambda \in \mathbb{C}, t \in \mathbb{R}$, $|\varphi_\lambda(t)| \leq C(1 + |t|) e^{-\rho |t|} e^{\left|\Re \lambda \right||t||}$.

For a function $f \in L^1(\mathbb{R}, A(t) dt)$, the (Sturm-Liouville) Fourier transform of $f$ is defined by

$$\hat{f}(\lambda) = \int_{\mathbb{R}} f(t) \varphi_\lambda(t) A(t) \, dt.$$  

(2.3)
Also for a suitable function $f$ the inverse Fourier transform is given by
\[ f(t) = \frac{1}{4\pi} \int_{\mathbb{R}} \hat{f}(\lambda) \varphi_{\lambda}(t) |c(\lambda)|^{-2} d\lambda, \] (2.4)
where $c(\lambda)$ is the Harish-Chandra $c$-function associated with the Sturm-Liouville operator. Let $C^\infty_{c,R}(\mathbb{R})$ be the space of all compactly supported smooth functions on $\mathbb{R}$ with support in $[-R, R]$. Also, let $PW_R(\mathbb{C})$ be the space of all entire functions $F$ such that for each $N \in \mathbb{N},$
\[ \sup_{\lambda \in \mathbb{C}} (1 + |\lambda|)^N |F(\lambda)| e^{-R|\Im \lambda|} < \infty. \] (2.5)
We also denote $PW_R(\mathbb{C})_e$ as the space of even functions in $PW_R(\mathbb{C})$. Then we have the following Paley-Wiener theorem:

**Theorem 2.1** ([6, Theorem 3]) The (Sturm-Liouville) Fourier transform $f \mapsto \hat{f}$ is a topological isomorphism between $C^\infty_{c,R}(\mathbb{R})$ and $PW_R(\mathbb{C})_e$.

We already know that the function $x \mapsto \varphi_{\lambda}(x)$ is $C^\infty$ on $\mathbb{R}$. But the following lemma states that the function has a holomorphic extension to the “crown domain” $\Omega := \{ z \in \mathbb{C} \mid |\Im z| < \frac{\pi}{2} \}$.

**Lemma 2.2** The function $x \mapsto \varphi_{\lambda}(x)$ has holomorphic extension to $\Omega$.

**Proof** We recall that $\varphi_{\lambda}$ is the unique solution of
\[ \frac{d^2 f}{dt^2} + \frac{A'(t)}{A(t)} \frac{df}{dt} + (\lambda^2 + \rho^2) f = 0, \] (2.6)
with $f(0) = 1$, $f'(0) = 0$. In [7, Theorem 2] it is proved that $\varphi_{\lambda}$ has a real analytic extension on the real line around zero. The same proof also works in our case to show that $\varphi_{\lambda}$ has a holomorphic extension to a neighbourhood of 0 in $\mathbb{C}$, call it $\Omega_0$. Let $\Omega_1 = \Omega \setminus (-\infty, 0]$, $\Omega_2 = \Omega \setminus [0, \infty)$ and let $y_0 \in \Omega_1 \cap \Omega_0 \cap \Omega_2$. Then from Theorem 8.3 (in Appendix), there exists a unique holomorphic solution $f$ on $\Omega_1$ of (2.6) with initial condition $f(y_0) = \varphi_{\lambda}(y_0)$, $f'(y_0) = \varphi'_{\lambda}(y_0)$. Similarly there exists a unique holomorphic solution $f$ on $\Omega_2$ of (2.6) with initial condition $f(y_0) = \varphi_{\lambda}(y_0)$, $f'(y_0) = \varphi'_{\lambda}(y_0)$. Therefore by analytic continuation it follows that $\varphi_{\lambda}$ has a holomorphic extension to $\Omega$. \[ \square \]

Before going further we will rewrite $L$ as a perturbation of the Bessel equation to deduce some more properties of $\varphi_{\lambda}$. After applying the classical Liouville transformation i.e. $v(t) = \sqrt{A(t)} u(t)$, equation (2.6) reduces to
\[ \frac{d^2 v}{dt^2} - \frac{\alpha^2 - \frac{1}{4}}{t^2} v - G(t)v + \lambda^2 v = 0 \] (2.7)
where

\[ G(t) = \frac{1}{4} \left( \frac{A'(t)}{A(t)} \right)^2 + \frac{1}{2} \left( \frac{A'(t)}{A(t)} \right)' - \rho^2 - \frac{\alpha^2 - \frac{1}{4}}{t^2} = q(t) - \frac{\alpha^2 - \frac{1}{4}}{t^2}. \]

Let

\[ G_0(t) := \left(\left(\alpha + \frac{1}{2}\right)^2 - \left(\alpha + \frac{1}{2}\right)\right) \coth^2 t + \left(\left(\beta + \frac{1}{2}\right)^2 - \left(\beta + \frac{1}{2}\right)\right) \tanh^2 t + \left(\alpha + \frac{1}{2}\right) + \left(\beta + \frac{1}{2}\right) + 2 \left(\alpha + \frac{1}{2}\right) \left(\beta + \frac{1}{2}\right) - \rho^2 - \frac{\alpha^2 - \frac{1}{4}}{t^2}. \]

It is easy to check that \( \coth^2 t - \frac{1}{t^2} \) is a holomorphic function on \( \Omega \), hence \( G_0 \) is a holomorphic function on \( \Omega \).

A simple computation shows that

\[ G(t) = G_0(t) + \left(\beta + \frac{1}{2}\right) \tanh t \frac{B'(t)}{B(t)} + \left(\alpha + \frac{1}{2}\right) \coth t \frac{B'(t)}{B(t)} + \frac{1}{4} \left( \frac{B'(t)}{B(t)} \right)^2 + \frac{B''(t)}{2B(t)}. \] (2.8)

The assumptions on \( B \) implies that \( B'(0) = 0 \), which assure that \((\alpha + \frac{1}{2}) \coth t \frac{B'(t)}{B(t)}\) is a holomorphic function on \( \Omega \) and therefore \( G \) is a holomorphic function on \( \Omega \). The following theorem gives an estimate of \( \varphi_\lambda \) on the domain \( \Omega \) away from the boundary of \( \Omega \).

**Theorem 2.3** Let \( G \) satisfy the condition (C4). Then for any nonzero complex number \( \lambda \), there exists a polynomial \( P \) and a constant \( C > 0 \) such that

\[ |\varphi_\lambda(\xi)| \leq C |P(\xi)| e^{\frac{1}{2} |\xi|}, \]

for all \( \xi \in \Omega \) and \( C \) is uniform in \( |\Im \xi| \leq \theta \) for any \( \theta < \pi/2 \).

The proof of this theorem is similar to [4, Theorem 1.2] but for the sake of completeness we give the proof of the theorem above, in the Appendix.

**Remark 2.4** In the nonperturbed case, we have

\[ A(t) = (\sinh t)^{2\alpha+1} (\cosh t)^{2\beta+1}, \]

and hence

\[ G(t) = \left(\left(\alpha + \frac{1}{2}\right)^2 - \left(\alpha + \frac{1}{2}\right)\right) \coth^2 t + \left(\left(\beta + \frac{1}{2}\right)^2 - \left(\beta + \frac{1}{2}\right)\right) \tanh^2 t. \]
+ \left( \alpha + \frac{1}{2} \right) \left( \beta + \frac{1}{2} \right) + 2 \left( \alpha + \frac{1}{2} \right) \left( \beta + \frac{1}{2} \right) - \rho^2 - \frac{\alpha^2 - \frac{1}{4}}{t^2} = \left( \alpha^2 - \frac{1}{4} \right) \left( \text{cosech}^2 t - \frac{1}{t^2} \right) - \left( \beta^2 - \frac{1}{4} \right) \text{sech}^2 t.

This shows that $G$ is in $L^1$ in every direction in the domain $\Omega$ and $\left| \int_{[0,z]} G(s) ds \right|$ is uniformly bounded in $\{z : |\Im z| \leq \theta \}$ where $\theta < \pi / 2$. Therefore the condition (C4) on $G$ is satisfied automatically for the non-perturbed case. Also it can be verified that all other conditions $(C1) - (C6)$ are satisfied in the non-perturbed case. One can find $q(t)$ explicitly and check that $a = b = 1/2$. For $\rho > 0$, $(C6)$ does not occur as $b > 0$. For $\rho = 0$ it corresponds to the Euclidean case.

In [4, Theorem 1.17], the authors proved the existence of a solution $\Phi_\lambda$ of (2.6) under some conditions on the functions $A$ and $G$ defined above. It is easy to see that the conditions (C1) to (C4) on $A$ and $G$ stated earlier imply the conditions under which Theorem 1.17 in [4] holds. Also it follows that $\Phi_{-\lambda}$ is a solution of (2.6) on $(0, \infty)$. The Wronskian of $\Phi_\lambda, \Phi_{-\lambda}$ is given by ([4, Corollary 1.18])

$$W(\Phi_\lambda, \Phi_{-\lambda}) = -2i \lambda A(t)^{-1}.$$ 

Therefore, for non zero $\lambda \in \mathbb{C}$, the solutions $\Phi_\lambda, \Phi_{-\lambda}$ are linearly independent. Hence there is a function $c$ called Harish chandra $c$-function such that for $\lambda \neq 0, t > 0$,

$$\varphi_\lambda = c(\lambda) \Phi_\lambda + c(-\lambda) \Phi_{-\lambda}. \quad (2.9)$$

Therefore, for $t > 0$

$$\varphi'_\lambda(t) = c(\lambda) \Phi'_\lambda(t) + c(-\lambda) \Phi'_{-\lambda}(t).$$

Hence,

$$c(\lambda) = \frac{\varphi_\lambda(t) \Phi'_{-\lambda}(t) - \varphi'_{\lambda}(t) \Phi_{-\lambda}(t)}{W(\Phi_\lambda, \Phi_{-\lambda}) (t)} = \frac{i A(t)}{2 \lambda} \left( \varphi_\lambda(t) \Phi'_{-\lambda}(t) - \varphi'_{\lambda}(t) \Phi_{-\lambda}(t) \right). \quad (2.10)$$

We note that $c(\lambda)$ is independent of $t > 0$. Then we have the following properties of the Harish Chandra $c$-function ([4, Corollary 2.3, Theorem 2.4]):

**Theorem 2.5** Assume conditions (C1) to (C6) on $A$ and $G$.

1. If $|\lambda| \to +\infty, \Im \lambda \leq 0$, then

$$\frac{1}{|c(\lambda)|} \asymp |\lambda|^\alpha \cdot \frac{1}{2}.$$
(2) If \( \lambda \to 0, \Im \lambda \leq 0 \), then

\[
\frac{1}{|c(\lambda)|} \asymp |\lambda|^{b+\frac{1}{2}},
\]

and \( \lambda^{-b-\frac{1}{2}}c(\lambda)^{-1} \) is analytic in \( \Im \lambda < 0 \) and continuous in \( \Im \lambda \leq 0 \).

3 More information on Harish Chandra \( c \) function

In Theorem 2.5 above we only get information for \( c(\lambda) \) when \( \Im(\lambda) \leq 0 \). For the standard version of the Ramanujan’s Master theorem we need more information on \( c(\lambda) \) for \( \Im(\lambda) < \mu \) for some \( \mu > 0 \). In what follows we always assume the stronger conditions \( (C_1), (C_2), (C_3), (C_4), (C_5)^* \) instead of conditions \( (C_1) \) to \( (C_6) \) unless otherwise stated. We also assume that \( \rho > 0 \). We note that by [22, Lemma 6.1, p. 148-149], when \( \rho > 0 \), \( (C_5)^* \) implies that

\[
q(t) = O(e^{-\varepsilon t}),
\]

as \( t \to \infty \). The condition \( (C_5)^* \) implies \( (C_5) \) for \( a = \frac{1}{2} \), when \( \rho > 0 \).

Also the condition \( (C_5)^* \) assures that for large \( t \),

\[
A(t) = O(e^{2\rho t}) \text{ if } \rho > 0.
\]

Definition 3.1 For \( 1 \leq q \leq 2 \), the \( L^q \)-Schwartz space \( C^q(\mathbb{R}) \) is the collection of all \( C^\infty \) functions \( f \) on \( \mathbb{R} \) such that for each \( N, m \in \mathbb{N} \cup \{0\} \),

\[
\sup_{x \in \mathbb{R}} (1 + |x|)^N \left| \mathcal{L}^m f(x) \right| e^{\frac{2}{q} \rho |x|} < \infty.
\]

Using (3.2) it follows that \( C^q(\mathbb{R}) \subseteq L^q(\mathbb{R}, A(t)dt) \).

For \( 1 \leq q \leq 2 \), let \( S_q = \{ \lambda \in \mathbb{C} \mid |\Im \lambda| \leq \left( \frac{2}{q} - 1 \right) \rho \} \). Also let \( S(S_q)_{\mathfrak{c}} \) be the collection of all even \( C^\infty \) functions on \( S_q \) such that for each \( N, m \in \mathbb{N} \cup \{0\} \),

\[
\sup_{\xi \in S_q} (1 + |\xi|)^N \left| \frac{d^m}{d\xi^m} f(\xi) \right| < \infty.
\]

Theorem 3.2 ([20]) The map \( f \mapsto \hat{f} \) is a topological isomorphism between \( C^q(\mathbb{R}) \) and \( S(S_q)_{\mathfrak{c}} \).

In this case (that is, when \( A, G \) satisfies \( (C_1), (C_2), (C_3), (C_4), (C_5)^* \) and \( \rho > 0 \)) we prove that the \( c \)-function has simple pole at \( \lambda = 0 \) and get improved estimate of \( c(\lambda) \) in a larger domain (Corollary 3.6). The next two theorems are improved versions
of Theorem 1.17 and Theorem 1.19 in [4] respectively. These theorems are needed to prove the Corollary 3.6. We get the improved results (in Theorem 3.3 and Theorem 3.4) because of the stronger assumption \((C5)^*\).

**Theorem 3.3** Let \(A\) satisfies the conditions \((C1), (C2), (C3), (C4), (C5)^*\). Assume \(\rho > 0\). Then for \(t > 0\) and \(\lambda \in \mathbb{C} \setminus \{0\}\) there exists a unique solution of (2.7) such that

\[
\sqrt{A(t)} \Phi_{-\lambda}(t) = e^{-i\lambda t} + e^{-i\lambda t} \mathcal{R}(\lambda, t),
\]

with \(\mathcal{R}(\lambda, t) \to 0\) and \(\frac{\partial}{\partial t} \mathcal{R}(\lambda, t) \to 0\) as \(t \to \infty\). Let \(0 < \varepsilon' < \varepsilon\). When \(\Im \lambda \leq \varepsilon' / 2\) we have

\[
|\mathcal{R}(\lambda, t)| \leq e^{\frac{1}{|\lambda|} \int_t^\infty e^{\varepsilon' v} |q(v)| dv} \int_t^\infty e^{\varepsilon' v} |q(v)| dv,
\]

and

\[
\left| \frac{\partial}{\partial t} \mathcal{R}(\lambda, t) \right| \leq |\lambda| \left( e^{\frac{1}{|\lambda|} \int_t^\infty e^{\varepsilon' v} |q(v)| dv} \int_t^\infty e^{\varepsilon' v} |q(v)| dv \right).
\]

**Proof** The proof follows along the same lines as in [4, Theorem 1.17]. Note that (3.3) follows directly from equation (1.14) in [4, Theorem 1.17] just by replacing \(\lambda\) by \(-\lambda\). Due to the assumption on \(q\) in \((C5)^*\) we get exponential decay of \(q\) in terms of \(\varepsilon\) when \(\rho > 0\) (see (3.1)). As \(\Phi_{-\lambda}\) satisfies (2.7), using (3.3) one can check that \(\mathcal{R}(\lambda, t)\) satisfies

\[
\frac{d^2}{dt^2} \mathcal{R}(\lambda, t) - 2i\lambda \frac{d}{dt} \mathcal{R}(\lambda, t) = q(t) \mathcal{R}(\lambda, t) + q(t).
\]

When \(\Im \lambda \leq \varepsilon' / 2\), by [4, Lemma A.1], a solution of the integral equation

\[
\mathcal{R}(\lambda, t) = \int_t^\infty \frac{e^{2i\lambda(t-v)} - 1}{-2i\lambda} (q(v) \mathcal{R}(\lambda, v) + q(v)) dv,
\]

satisfies (3.4) as well. We now follow the same method as in [4, Theorem A.2]. Define \(k(t, v) = e^{-i\varepsilon' t} e^{2i\lambda(t-v) - 1} e^{\varepsilon' (t-v)}\), \(\phi(v) = \psi_0(v) = e^{i\varepsilon' v} q(v)\) and \(J(v) \equiv 1\). When \(\Im \lambda \leq \varepsilon' / 2, \lambda \neq 0\) we get that \(P_0(t) = \frac{1}{|\lambda|}, P_1(t) = 1\) and \(Q(v) = 1\). Therefore by [4, theorem A.2], there is a unique solution of (3.5) satisfying

\[
\mathcal{R}(\lambda, t) \to 0, \quad \frac{\partial}{\partial t} \mathcal{R}(\lambda, t) \to 0,
\]

as \(t \to \infty\). In fact we also get the estimates

\[
|\mathcal{R}(\lambda, t)| \leq e^{\frac{1}{|\lambda|} \int_t^\infty e^{\varepsilon' v} |q(v)| dv} \int_t^\infty e^{\varepsilon' v} |q(v)| dv,
\]

and

\[
\left| \frac{\partial}{\partial t} \mathcal{R}(\lambda, t) \right| \leq |\lambda| \left( e^{\frac{1}{|\lambda|} \int_t^\infty e^{\varepsilon' v} |q(v)| dv} \int_t^\infty e^{\varepsilon' v} |q(v)| dv \right).
\]

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To get more information on the $c$-function we also need the behaviour of $\Phi_1 - \lambda$ for $\lambda$ near zero.

We prove the following analogue of [4, Theorem 1.19].

**Theorem 3.4** Let $A$ satisfies the conditions $(C1), (C2), (C3), (C4), (C5)^*$. Assume $\rho > 0$. Then for $t > 1$

$$\sqrt{A(t)}\Phi_1 - \lambda(t) = W_1(t) + Z(\lambda, t), \quad (3.6)$$

such that $\lambda \to Z(\lambda, t)$ and $\lambda \to \frac{\partial}{\partial t} Z(\lambda, t)$ is analytic in $\Im \lambda < \epsilon/2$. We also have $\lim_{\lambda \to 0} Z(\lambda, t) = 0$ and $\lim_{\lambda \to 0} \frac{\partial}{\partial t} Z(\lambda, t) = 0$.

Let us first define $W_1$ in (3.6). Let $k(t, s) = (s - t)q(s)$, then from [4, Theorem 1.8] (for $a = 1/2$) $W_1(t)$ is the unique solution of the integral equation $W_1(t) = 1 + \int_t^\infty k(t, s)W_1(s)ds$ such that $W_1(t) - 1$ satisfies

$$|W_1(t) - 1| \leq \left( e^{c\int_t^\infty |q(s)|ds} - 1 \right)$$

and

$$\left| \frac{d}{dt} (W_1(t) - 1) \right| \leq t^{-1} \left( e^{c\int_t^\infty |q(s)|ds} - 1 \right)$$

for $t \to \infty$. Note that (3.6) follows from [4, Theorem 1.19] for $\Im(\lambda) < 0$ when $a = 1/2$. Therefore we only need to deal with the case when $0 \leq \Im(\lambda) < \epsilon/2$ and $a = 1/2$.

**Proof** We can rewrite (2.7) as

$$v''(t) + \lambda^2 v(t) = q(t)v(t). \quad (3.7)$$

Fix $0 < \epsilon' < \epsilon$. Also assume $0 \leq \Im(\lambda) \leq \epsilon'/2$. If we compare the above equation with [4, equation (1.22), p. 235] we have $a = \frac{1}{2}$ and $\zeta(t) = q(t)$. By (3.1) we further have $t e^{\epsilon' t} \zeta(t) \in L^1((1, \infty), dt)$. The two linearly independent solutions of the corresponding homogeneous equation of (3.7) are $\sin \lambda t$ and $e^{-i\lambda t}$. A simple computation shows that $W(\sin \lambda t, e^{-i\lambda t}) = \lambda$. As done in [4] we see that a solution of the integral equation

$$v(t) = c_1 e^{-i\lambda t} + \int_t^\infty k_1(t, s, \lambda)q(s)e^{i\lambda(s-t)}v(s)ds, \quad (3.8)$$
is a solution of (3.7). We will choose $c_1$ so that $\sqrt{A(t)}\Phi_{-\lambda}(t)$ is a solution of (3.8). Define $\tilde{v}_\lambda(t) = e^{i\lambda t}v(t)$ and $k_1(t, s, \lambda) = e^{i\lambda(t-s)}e^{-i\lambda s}sin\lambda s$. Then (3.8) can be rewritten as

$$\tilde{v}_\lambda(t) = c_1 + \int_t^\infty \frac{k_1(t, s, \lambda)}{se^{\varepsilon s}}se^{\varepsilon s}q(s)\tilde{v}_\lambda(s)ds. \quad (3.9)$$

When $t \leq s$ and $0 \leq \Re \lambda \leq \varepsilon'/2$ we have

$$|k_1(t, s, \lambda)| \leq c|\lambda|^{-1}\langle \lambda s \rangle e^{\varepsilon s},$$

and

$$\left| \frac{\partial}{\partial t} k_1(t, s, \lambda) \right| \leq c'\langle \lambda t \rangle^{-1}\langle \lambda s \rangle e^{\varepsilon s},$$

where $\langle x \rangle = \frac{|x|}{1+|x|}$. Note that when $\Re \lambda \leq 0$ we don’t get extra exponential growth $e^{\varepsilon s}$ in the above inequalities. See proof of [4, Theorem 1.19] for more details. To compensate this growth we need the integrability of $q(s)$ w.r.t. the weight $se^{\varepsilon s}$ ($0 < \varepsilon' < \varepsilon$) which is guaranteed in our case because of condition (C5)* when $\rho > 0$.

In the notation of [4, Theorem A.3] for the integral equation (3.9) involving $\tilde{v}_\lambda$, put $\phi(s) = se^{\varepsilon s}q(s)$. $J(s) = c_1$, $Q(s) = cs^{-1}\langle \lambda s \rangle$, $P_0(t) = |\lambda|^{-1}$ and $P_1(t) = \langle \lambda t \rangle^{-1}$. Also let $\kappa = c|\lambda|$ and $\kappa_0 = c$. From [4, Theorem A.3] we get there exists a solution $\tilde{v}_\lambda(t)$ satisfying the (3.9) such that

$$\tilde{v}_\lambda(t) - c_1 \to 0,$$

and

$$\frac{\partial}{\partial t} (\tilde{v}_\lambda(t) - c_1) \to 0,$$

as $t \to \infty$. We put $c_1 = 1$ which implies that $\tilde{v}_\lambda(t) = \sqrt{A(t)}\Phi_{-\lambda}(t)e^{i\lambda t}$ by the uniqueness of the solution of (2.7) in Theorem 3.3. Moreover from [4, Theorem A.3] we also get the estimates

$$|\tilde{v}_\lambda(t) - 1| \leq e^{c\int_t^\infty se^{\varepsilon s}|q(s)|ds - 1}, \quad (3.10)$$

and

$$\left| \frac{\partial}{\partial t} (\tilde{v}_\lambda(t) - 1) \right| \leq \left( \frac{t}{1+|\lambda|} \right)^{-1} \left( e^{c\int_t^\infty se^{\varepsilon s}|q(s)|ds - 1} \right). \quad (3.11)$$

Recall that $Z(\lambda, t) = \sqrt{A(t)}\Phi_{-\lambda}(t) - W_1(t)$ for $\Re \lambda < \varepsilon / 2$. Therefore the analyticity of $\lambda \to Z(\lambda, t)$ and $\lambda \to \frac{\partial}{\partial t} Z(\lambda, t)$ will follow from the analyticity of $\Phi_{-\lambda}(t)$ and $\frac{\partial}{\partial t} \Phi_{-\lambda}(t)$ respectively.
Next we prove that $\Phi_\lambda(t)$ and $\frac{\partial}{\partial t}\Phi_\lambda(t)$ when $\Im(\lambda) < \varepsilon'/2$ in a similar manner as in [4, Theorem 1.19]. We give a brief sketch to emphasise that because of the condition that $se^{\varepsilon's} q(s) \in L^1((1, \infty), ds)$ we get the analyticity of the said functions for $\Im(\lambda) < \varepsilon/2$ instead of just $\Im(\lambda) < 0$ in [4, Theorem 1.19]. From [4, Theorem A.3] we can write

$$\tilde{v}_\lambda(t) = \sum_{j=0}^{\infty} \left( v_{j+1}(t, \lambda) - v_j(t, \lambda) \right),$$  

(3.12)

where

$$v_{j+1}(t, \lambda) = 1 + \int_t^\infty \frac{k_1(t, s, \lambda)}{se^{\varepsilon's}}se^{\varepsilon's}q(s)v_j(s, \lambda)ds$$

for $j \geq 0$ and $v_0(t, \lambda) \equiv 0$. Furthermore when $\Im(\lambda) < \varepsilon'/2$,

$$|v_{j+1}(t, \lambda) - v_j(t, \lambda)| \leq c \left( \frac{\int_t^\infty se^{\varepsilon's}|q(s)|ds}{j!} \right)^{j+1}$$

(3.13)

and

$$\left| \frac{\partial}{\partial t}(v_{j+1}(t, \lambda) - v_j(t, \lambda)) \right| \leq c \left( \frac{t}{1+|\lambda t|} \right)^{-1} \left( \frac{\int_t^\infty se^{\varepsilon's}|q(s)|ds}{j!} \right)^{j+1}$$

(3.14)

hold.

For proving the holomorphicity of $\tilde{v}_\lambda(t)$ and $\frac{\partial}{\partial t}\tilde{v}_\lambda(t)$, we first observe that $k_1(t, s, \lambda)$ and $\frac{\partial}{\partial t}k_1(t, s, \lambda)$ are entire functions on $\mathbb{C}$ and $\sup_{t>1} \int_t^\infty se^{\varepsilon's}|q(s)|ds < \infty$ as $se^{\varepsilon's}q(s) \in L^1((1, \infty), ds)$. We see that by applying induction argument we can prove that $v_{j+1}(t, \lambda)$ is holomorphic for $\Im(\lambda) < \varepsilon'/2$. Moreover the sum in equation (3.12) converges uniformly on compact subsets of $\Im(\lambda) < \varepsilon'/2$ for a fixed $t > 1$. We can use similar arguments to prove that $\frac{\partial}{\partial t}\tilde{v}_\lambda(t), t > 1$ is a holomorphic function in $\Im(\lambda) < \varepsilon'/2$ as well. As $\varepsilon' < \varepsilon$ is arbitrary, we get analyticity for $\Im(\lambda) < \varepsilon/2$. Letting $\lambda \to 0$ in (3.9) we get

$$\tilde{v}_0(t) = 1 + \int_t^\infty k_1(t, s)q(s)\tilde{v}_0(s)ds,$$

(3.15)

where $k_1(t, s) = -(t-s)$. Taking the limit $\lambda \to 0$ in (3.10) and (3.11) we get

$$|\tilde{v}_0(t) - 1| \leq e^{\varepsilon \int_t^\infty s|q(s)|ds} - 1,$$

(3.16)

and

$$|\tilde{v}_0'(t)| \leq t^{-1} \left( e^{\varepsilon \int_t^\infty s|q(s)|ds} - 1 \right).$$

(3.17)
By [4, Theorem 1.8], $\tilde{v}_0(t)$ and $W_1(t)$ satisfy the same integral equation and the estimates (3.16) and (3.17) imply by uniqueness of such solution of (3.15) that $\tilde{v}_0(t) = W_1(t)$. Therefore we get that $\lim_{\lambda \to 0} Z(\lambda, t) = 0$. Similarly we get $\lim_{\lambda \to 0} \frac{\partial}{\partial t} Z(\lambda, t) = 0$ as in [4].

**Definition 3.5** [4] (def. 1.11) If the conditions (C1) to (C6) hold for some $a \geq 0$ and if $\sqrt{A} \varphi_0 = c'W_1(t)$ for some $c' \neq 0$ we say $A$ is recessive. Otherwise we say $A(t)$ is dominant.

Before stating the next corollary, we recall from [4, Lemma 1.12] that under the conditions (C1) to (C4) and (C5)* and $\rho > 0$ the function $A(t)$ is dominant as (C5)* implies (C5) for $a = \frac{1}{2}$. As $a > 0$ and $A(t)$ is dominant we have that $b = a$ in condition (C5). (See [4, remark 1.15]). Therefore we have $\sqrt{A(t)} \varphi_0(t) \approx t$ as $t \to \infty$. As a consequence of Theorem 3.4 we have the following corollary.

**Corollary 3.6** Let the conditions (C1), (C2), (C3), (C4), (C5)* and $\rho > 0$ hold. Then the $c$-function defined in (2.10), as $\lambda \to 0$ in $\Im \lambda < \epsilon / 2$

$$|c(\lambda)| \asymp |\lambda|^{-1}, \quad (3.18)$$

and also $(\lambda c(\lambda))^{-1}$ is holomorphic in $\Im \lambda < \epsilon / 2$.

**Proof** The proof is along the similar lines as in Theorem 2.4 in [4]. We extend the holomorphicity of $(\lambda c(\lambda))^{-1}$ in a larger domain here. Note that we have $a = b = 1/2$ in the notation of [4] in Theorem 2.4. Recall that $\lambda \mapsto \varphi_\lambda$ is an entire function. From (2.10) we know that

$$\lambda c(\lambda) = \frac{i}{2} \mathcal{W}\left(\sqrt{A} \varphi_\lambda, \sqrt{A} \Phi_{-\lambda}\right)(t).$$

From Theorem 3.4 we know that $\lambda c(\lambda)$ is holomorphic in $\Im \lambda < \epsilon / 2$. Using the notation in [4], let $c_1(\lambda) = \lambda c(\lambda)$. From Theorem 3.4 we know that $\lim_{\lambda \to 0, \Im \lambda < \epsilon / 2} c_1(\lambda)$ exists. In fact $c_1(0) = \frac{i}{2} \mathcal{W}(\sqrt{A} \varphi_0, W_1)(t)$. In our case $A(t)$ is dominant as explained before the corollary, so $\sqrt{A} \varphi_0$ and $W_1$ are linearly independent. Therefore $c_1(0)$ is a non zero quantity. Hence proved. □

### 4 Compact case

As mentioned in the introduction, to obtain an analogue of Ramanujan’s Master Theorem 1.2 for the Sturm Liouville operator we also need the corresponding Fourier series for $L$ when restricted to $z = it$, $t \in (0, \pi/2)$.

More precisely, we need a positive, symmetric densely defined differential operator $-L$ on $(0, \pi/2)$ on a suitable Hilbert space such that the following holds

$$-Lw(t) = Lu(z = it), \quad (4.1)$$
where \( w(t) = u(it) \) for \( u \) holomorphic in \( \Omega \). The spectral decomposition of \(-L\) on \((0, \pi/2)\) under suitable boundary conditions should be such that the eigenfunctions of \(-L\) are the restriction of the eigenfunctions of \(L\) on \( z = it \). This will be in direct analogy with the functions \( \{e^{iz}\} \) as discussed in the introduction. In this section we show the existence of such operator \( L \) with the desired properties mentioned above. Let us recall that

\[
A(t) = (\sinh t)^{2\alpha+1} (\cosh t)^{2\beta+1} B(t),
\]

where \( B \) is as defined in section 2. We know that \( B(it) > 0 \) when \( t \in (-\pi/2, \pi/2) \).

We define

\[
\tilde{A}(t) = (-i)^{2\alpha+1} A(it).
\]

Let \( \tilde{B}(t) = B(it) \). Indeed

\[
\tilde{A}(t) = (\sin t)^{2\alpha+1} (\cos t)^{2\beta+1} \tilde{B}(t).
\]

Clearly \( \tilde{A} > 0 \) on \((0, \pi/2)\). We define the Sturm Liouville operator corresponding to \( \tilde{A} \) on \((0, \pi/2)\) as

\[
L = \frac{d^2}{dt^2} + \frac{\tilde{A}'(t)}{\tilde{A}(t)} \frac{d}{dt}.
\] (4.2)

It is easy to verify that this choice of \( L \) satisfies (4.1) above. We define \( \mathcal{D}(L) \) to be the space of \( f \in L^2((0, \pi/2), \tilde{A}(t)dt) \) such that \( f \) and \( \tilde{A}(\cdot)f' \) are absolutely continuous on any compact subinterval of \((0, \pi/2)\) and \( L(f) \in L^2((0, \pi/2), \tilde{A}(t)dt) \).

The operator \(-L\) is a densely defined operator from \( \mathcal{D}(L) \) to \( L^2((0, \pi/2), \tilde{A}(t)dt) \).

Let

\[
\mathcal{D}(L)_0 := \{ f \in \mathcal{D}(L) : f \text{ is compactly supported on } (0, \pi/2) \}.
\]

Let us denote \(-L\) restricted on \( \mathcal{D}(L)_0 \) as \(-L_0\). An integration by parts argument shows that \(-L_0\) is a positive and symmetric operator on \( \mathcal{D}(L)_0 \). We need to obtain a selfadjoint extension of \(-L_0\) on \( L^2((0, \pi/2), \tilde{A}(t)dt) \) with suitable boundary conditions so that the eigenfunctions are restriction of \( \varphi_\lambda \) to \( z = it, t \in (0, \pi/2) \) for \( \lambda \) related to the spectrum of the selfadjoint extension of \(-L_0\). We get the following theorem:

**Theorem 4.1** The operator \(-L_0\) has a selfadjoint extension (with abuse of notation call it \(-L\)) on \( \tilde{D} := \left( \sqrt{\tilde{A}} \right)^{-1} \mathcal{D} \) (where \( \mathcal{D} \subset L^2(0, \pi/2) \) is as defined in Proposition 4.2). The spectrum of \(-L\) is purely discrete, bounded below and all eigenvalues are simple. The eigenvalues can be ordered by

\[
0 < \nu_1 < \ldots < \nu_n < \ldots
\]
with \( \nu_n \to +\infty \). The eigenfunctions \( \{\Psi_j\} \) corresponding to the eigenvalues \( \{\nu_j\} \) form an orthonormal basis of \( L^2((0, \pi/2), \tilde{A}(t)dt) \). Let \( j_0 \) is the least natural number for which \( \nu_j + \rho^2 \geq 0 \). Define \( \theta_j = i \sqrt{\nu_j + \rho^2} \) for \( j \geq j_0 \) and \( \theta_j = i \sqrt{-\nu_j - \rho^2} \); \( j = 0, \ldots, j_0 - 1 \). There exist constants \( c_j \) with a polynomial growth in \( j \) such that

\[
\Psi_j(t) = c_j \varphi_{\theta_j}(it)
\]  

(4.3)

for all \( j \geq 0 \). When \( \alpha, \beta \geq 0 \), the operator \(-L\) is the Friedrichs extension of \(-L_0\) and \( \nu_j \)'s are all nonnegative.

We postpone the proof of the above theorem to the end of this section. We will first set the ground for its proof.

Let \( u \in D(L) \) be an eigenfunction of \(-L\) with eigenvalue \( \mu \). After applying the classical Liouville transformation i.e. \( v(t) = \sqrt{\tilde{A}(t)}u(t) \) we get another differential operator \(-l\) such that \( v \) is an eigenfunction of \(-l\) with eigenvalue \( \mu \). One can explicitly write the expression of \(-l\) as follows:

\[
-l = -\frac{d^2}{dt^2} + p(t),
\]

where

\[
p(t) = \frac{1}{4} \left( \frac{\tilde{A}'(t)}{\tilde{A}(t)} \right)^2 + \frac{1}{2} \left( \frac{\tilde{A}'(t)}{\tilde{A}(t)} \right)'.
\]

In fact

\[
p(t) = \left( \left( \alpha^2 - \frac{1}{4} \right) \cot^2 t + \left( \beta^2 - \frac{1}{4} \right) \tan^2 t - \chi(t) \right),
\]

(4.4)

where

\[
\chi(t) = \left( \beta + \frac{1}{2} \right) \frac{\tilde{B}'(t)}{\tilde{B}(t)} \tan t - \left( \alpha + \frac{1}{2} \right) \frac{\tilde{B}'(t)}{\tilde{B}(t)} \cot t + \frac{1}{4} \left( \frac{\tilde{B}'(t)}{\tilde{B}(t)} \right)^2 - \frac{1}{2} \frac{\tilde{B}''(t)}{\tilde{B}(t)}.
\]

It follows from the assumptions on \( \tilde{B} \) that \( \chi \) is a smooth function on \([-\pi/2, \pi/2]\). Clearly \( p \in L^1(I) \) for any compact subset of \((0, \pi/2)\). Therefore \( p \) has singularities only at 0 and \( \pi/2 \). See [15] for more details. A simple evaluation gives that

\[
\sqrt{\tilde{A}(t)}Lu(t) = lv(t),
\]

(4.5)
where $v(t) = \sqrt{\tilde{A}(t)} u(t)$. The unbounded operator $l : \mathcal{D}(l) \subset L^2(0, \pi/2) \to L^2(0, \pi/2)$ is defined on

$$\mathcal{D}(l) = \{ f \text{ and } f' \text{ AC on any compact subinterval of } (0, \pi/2), f, l(f) \in L^2(0, \pi/2) \}.$$  

Here AC stands for absolutely continuous. We observe that $\tilde{A} > 0$ and $\sqrt{\tilde{A}}$ is an AC function on any compact sub interval of $(0, \pi/2)$ and therefore it follows that

$$\mathcal{D}(l) = \sqrt{\tilde{A}(\cdot)} \mathcal{D}(L). \quad (4.6)$$

Let

$$\mathcal{D}(l)_0 := \{ f \in \mathcal{D}(l) : f \text{ is compactly supported on } (0, \pi/2) \}.$$  

A similar identity $(4.6)$ also holds for $\mathcal{D}(l)_0$ and $\mathcal{D}(L)_0$. We denote $-l$ restricted on $\mathcal{D}(l)_0$ as $-l_0$. Let $v_1, v_2 \in \mathcal{D}(l)_0$. The following holds:

$$(-L_0 u_1, u_2)_{L^2((0,\pi/2),\tilde{A}(t)dt)} = (-l_0 v_1, v_2)_{L^2(0,\pi/2)},$$

where $v_i(t) = \sqrt{\tilde{A}(t)} u_i(t)$ for $i = 1, 2$. The last identity just uses the relation between $u_i$ and $v_i$, $i = 1, 2$ and equation $(4.5)$. This shows that $-l_0$ is a positive and symmetric operator on $\mathcal{D}(l)_0$. In fact if $u$ is an eigenfunction of $-L$ with eigenvalue $\mu$, the equation $(4.5)$ gives that $\sqrt{\tilde{A}(t)} u(t)$ is an eigenfunction of $-l$ with the same eigenvalue and vice versa too. Therefore the eigenfunctions of $-L$ and $-l$ are in one to one correspondence by Liouville transformation. A simple computation gives

$$\|v\|_{L^2(0,\pi/2)} = \|u\|_{L^2((0,\pi/2),\tilde{A}(t)dt)}.$$  

In fact the map $u \to \sqrt{\tilde{A}(\cdot)} u(\cdot)$ is an isometry from $L^2((0, \pi/2), \tilde{A}(t)dt)$ onto $L^2(0, \pi/2)$. In view of the above relation between $-L$ and $-l$, in order to obtain self adjoint extension of $-L_0$ on $L^2((0, \pi/2), \tilde{A}(t)dt)$, it is enough to obtain a self adjoint extension of $-l_0$ on $L^2(0, \pi/2)$.

**Relation between $L$ and $-L$:** Let $u$ be a holomorphic function in $\Omega$. Define $w(t) := u(it)$. By relation $(4.1)$ we get

$$Lu(z)_{z=it} = -Lw(t), \ t \in (0, \pi/2).$$

We define $w_{\mu}(t) = \varphi_i \sqrt{\mu + \rho^2}(it)$, where $\mu + \rho^2 \in \mathbb{C} \setminus (-\infty, 0]$ and $\sqrt{z}$ is the complex square root for $z \in \mathbb{C} \setminus \{(-\infty, 0]\}$.

We know that

$$L \varphi_i \sqrt{\mu + \rho^2}(z) = \mu \varphi_i \sqrt{\mu + \rho^2}(z).$$
for all $z \in \Omega \setminus \{0\}$ (in particular when $z = it, t \in (0, \pi/2)$). Therefore we have

$$-Lw_\mu(t) = \mu w_\mu(t), t \in (0, \pi/2).$$

In the case when $\mu + \rho^2 \leq 0$, define $\eta = \sqrt{-(\mu + \rho^2)}$. By the same principle as above we can check that $-Lu_\eta(t) = \mu u_\eta(t)$, where $u_\eta(t) = \varphi_\eta(it)$. We define

$$\tilde{\varphi}_i(\mu + \rho^2)(t) = \sqrt{A(t)} \varphi_i(\mu + \rho^2)(it)$$

for $t \in (0, \pi/2)$. By the correspondence between $-L$ and $-l$ (4.5), we also have

$$-l\tilde{\varphi}_i(\mu + \rho^2)(t) = \mu \tilde{\varphi}_i(\mu + \rho^2)(t), t \in (0, \pi/2).$$

Similarly when $\mu + \rho^2 \leq 0$, define $\eta = \sqrt{-(\mu + \rho^2)}$. We get that $-l\tilde{\varphi}_\eta = \mu \tilde{\varphi}_\eta$.

**Spectral Decomposition of $-l$**: Let us recall that

$$-l = -\frac{d^2}{dt^2} + p(t),$$

(4.7)

where $p(t)$ is defined as in Eq. (4.4). As mentioned before $p$ has singularities at 0 and $\pi/2$. Without loss of generality we assume $\beta \leq \alpha$, the case when $\alpha < \beta$ follows similarly. Let $\mu \in \mathbb{C}$. From [8, theorem 8, page 15] we know that ODE $-lu = \mu u$ on $(0, \pi/2)$ has a unique twice differentiable solution (call it $V_{\mu,\alpha}$) such that $V_{\mu,\alpha}(t) \sim t^{\alpha + 1/2}$ and $V'_{\mu,\alpha}(t) \sim t^{\alpha - 1/2}$ near 0+. Similarly there exists a unique twice differentiable solution of $-lu = \mu u$ on $(0, \pi/2)$ (call it $W_{\mu,\beta}$) such that $W_{\mu,\beta}(t) \sim (\pi/2 - t)^{\beta + 1/2}$ and $W'_{\mu,\beta}(t) \sim (\pi/2 - t)^{\beta - 1/2}$ near $\pi/2^-$. By the definition of $\tilde{\varphi}_i(\mu + \rho^2)$, it is clear that $\tilde{\varphi}_i(\mu + \rho^2)$ is a constant multiple of $V_{\mu,\alpha}$ when $\mu + \rho^2 \geq 0$. Similarly when $\mu + \rho^2 < 0$ we have $\tilde{\varphi}_\eta$ is a scalar multiple of $V_{\mu,\alpha}$ for $\eta = \sqrt{-(\mu + \rho^2)}$.

We state the following crucial proposition from [8, Proposition 10]:

**Proposition 4.2** The operator $-l_0$ has a self adjoint extension (call it $-l$) on

$$\mathcal{D} = \begin{cases} \mathcal{D}(l) : [y, V_{\mu,\alpha}](0) = [y, W_{\mu,\beta}](\pi/2) = 0 \text{ for } -1/2 < \beta \leq \alpha < 1 \\
[y, V_{\mu,\alpha}](0) = [y, W_{\mu,\beta}](\pi/2) = 0 \text{ for } -1/2 < \beta < 1 \leq \alpha \\
\mathcal{D}(l) \text{ for } 1 \leq \beta \leq \alpha. \end{cases}$$

Here

$$[y, u](0) = \lim_{t \to 0^+} \left( y(t) \overline{u'(t)} - y'(t) \overline{u(t)} \right)$$

$$[y, u](\pi/2) = \lim_{t \to \pi/2^-} \left( y(t) \overline{u'(t)} - y'(t) \overline{u(t)} \right).$$

\(\square\) Springer
The operator $-l$ is bounded from below in $L^2(0, \pi/2)$ and the domain is independent of the choice of $\mu$ and $\tilde{\mu}$. The spectrum of $-l$ is purely discrete, bounded below and all eigenvalues are simple. The eigenvalues can be ordered by

$$v_0 < v_1 < \ldots < v_n < \ldots$$

with $v_n \to +\infty$.

We are now in a position to prove Theorem 4.1.

**Proof of Theorem 4.1** The existence of a self adjoint extension of $-L_0$ on $\tilde{D}$ follows from the equation (4.5) and the relation between $D(L)_0$ and $D(l)_0$. In fact the eigenvalues of $-L$ are same as that of $-l$ as explained earlier and the eigenfunctions are related by the classical Liouville transformation. Let $\{u_j\}_{j \geq 0}$ be the eigenfunctions of $-\tilde{l}$ such that $-\tilde{l}u_j = v_j u_j$. We further assume that $\|u_j\|_{L^2(0,\pi/2)} = 1$ for all $j \geq 0$. As $\{v_j\}_{j \geq 0}$ is the spectrum of $-L$ and $\{u_j\}_{j \geq 0}$ are the corresponding eigenfunctions, therefore $\{u_j\}_{j \geq 0}$ form an orthonormal basis of $L^2(0, \pi/2)$.

**When $\alpha, \beta \geq 0$:** It is clear from the above estimates that $\tilde{\phi}_\sqrt{\mu+\rho^2}$ and $W_{\tilde{\mu}, \beta}$ are principal solutions respectively at 0 and $\pi/2$ (see Appendix). In fact the boundary conditions in the above proposition are same as in [15, Theorem 4.2] due to the uniqueness of the principal solutions upto constant multiples. In [15, Theorem 4.2] it is proved that such a self adjoint extension is Freidrichs extension. Hence when $\alpha, \beta \geq 0$ we obtain the Freidrichs extension of $-\tilde{L}$ on $\tilde{D}$. The lower bound of the Freidrichs extension is same as that of $-L_0$. We have already seen that $(L_0 f, f) \geq 0$ when $f \in D(L)_0$ using integration by parts. Therefore the self adjoint extension considered above is also non negative when $\alpha, \beta \geq 0$. This implies that $v_j$’s are non negative.

Let $\theta_j = \sqrt{v_j + \rho^2}$ when $v_j + \rho^2 \geq 0$ and $\theta_j = \sqrt{-v_j + \rho^2}$ if $v_j + \rho^2 < 0$. By the above proposition $v_j \to \infty$ as $j \to \infty$, therefore there exist only finitely many $j$’s such that $v_j + \rho^2 \leq 0$. We will now show that there exist non zero constants $c_j$ and $c_j'$ such that for all $t \in (0, \pi/2)$,

$$u_j(t) = c_j \tilde{\phi}_{\theta_j}(t) = c_j' W_{v_j, \beta}(t).$$

for all $-1/2 < \beta \leq \alpha$. We know that

$$-l u = v_j u$$

is a second order ODE on $(0, \pi/2)$. Let $V_{v_j, |\alpha|}$ be as defined earlier [8, Theorem 8]. It is a principle solution of (4.9). One can construct another solution of (4.9) for $t \in (0, \epsilon)$ defined as

$$\tilde{V}_{v_j, \alpha}(t) = V_{v_j, |\alpha|}(t) \int_0^\epsilon \frac{1}{V_{v_j, |\alpha|}(u)^2} du.$$
In fact $\tilde{V}_{v_j,|\alpha|}(t)$ and $V_{v_j,|\alpha|}(t)$ are linearly independent. As $t \to 0^+$ one can check from the asymptotics of $V_{v_j,|\alpha|}(t)$ that $\tilde{V}_{v_j,|\alpha|}(t) \sim t^{-|\alpha|+1/2}$ if $\alpha \neq 0$ and $-t^{1/2} \log t$ for $\alpha = 0$. See [15, Theorem 2.2] for more details. Similarly one can construct a linearly independent solution of (4.9) near $\pi/2^-$ from $W_{v_j,|\beta|}$. From the above explanation, equation (4.8) is clearly true for $\beta < 1$ and $\alpha < 1$ due to the boundary conditions of proposition 4.2.

When $\alpha \geq 1$, from the asymptotic estimates one can check that $\tilde{V}_{v_j,|\alpha|}$ does not lie in $L^2((0, \varepsilon))$. Similarly $\tilde{W}_{v_j,|\beta|} \notin L^2(\pi/2-\varepsilon, \pi/2)$ when $\beta \geq 1$. Therefore any solution of (4.9) which also lies in $L^2((0, \pi/2))$ is a scalar multiple of $\tilde{\varphi}_{\theta_j}$ when $\alpha \geq 1$.

We define

$$\Psi_j(t) = \left(\sqrt{\tilde{A}(t)}\right)^{-1} u_j(t), \ t \in (0, \pi/2).$$

Clearly $\{\Psi_j\}_{j \geq 0}$ forms an orthonormal basis of $L^2((0, \pi/2), \tilde{A}(t)dt)$ satisfying $-L\Psi_j(t) = v_j \Psi_j(t), \ t \in (0, \pi/2)$. Therefore we have

$$\Psi_j(t) = c_j \varphi_{\theta_j}(it),$$

for all $j \geq 0$.

As $\Psi_j$ coincides with a holomorphic function in $\Omega$ on $\{it : t \in (0, \pi/2)\}$, so it has a holomorphic extension on $\Omega$. We define $\Psi_j(-iz) = c_j \varphi_{\theta_j}(z)$ for all $z \in \Omega$.

**Corollary 4.3** For each $j \geq 0$, the function $\Psi_j$ satisfies the following inequality:

$$|\Psi_j(t+is)| \leq CQ \left(\sqrt{|v_j + \rho^2|}\right) |P(t+is)| e^{\left|\Im(z)\right| \sqrt{|v_j + \rho^2|}},$$

for some polynomials $P$, $Q$ and $|s| < \pi/2$.

**Proof** For $v_j + \rho^2 \geq 0$, we have

$$\Psi_j(z) = c_j \varphi_{i\sqrt{v_j + \rho^2}}(iz).$$

Also from Theorem 2.3, we get

$$|\varphi_{i\sqrt{v_j + \rho^2}}(iz)| \leq CQ \left(\sqrt{|v_j + \rho^2|}\right) |P(i_z)| e^{3|z| \sqrt{|v_j + \rho^2|}}.$$

Then using the polynomial growth of $c_j$ (Theorem 4.1) the required inequality will follow. The estimate for the other case that is when $v_j + \rho^2 < 0$, will follow similarly. □
5 Sine type function

In this section we consider the case when \( \alpha, \beta \geq 0 \) and \( \rho > 0 \). We recall that \( v_j \)'s \((j = 0, 1, 2, \ldots)\) are eigenvalues of \(-L\) with eigenfunctions \( \Psi_j \) respectively. In this case (i.e. for \( \alpha, \beta \geq 0 \)), \( \nu_0 \geq 0 \) and \( \nu_j > 0 \) for \( j = 1, 2, \ldots \). We recall:

\[
S(z) = \pi z \prod_{j=0}^{\infty} \left( 1 + \frac{z^2}{v_j + \rho^2} \right).
\] (5.1)

In what follows, we also need the asymptotic expansion of \( \nu_j \)'s in our analysis of \( S(z) \) defined above. In fact from [8, Lemma 17, p. 23] we have

\[
\sqrt{v_j} = 2j + 1 + \alpha + \beta - \frac{\Theta}{4j} + O(j^{-2}),
\] (5.2)

where \( \Theta = \alpha^2 + \beta^2 - 1/2 + \frac{2}{\pi} \int_{0}^{\pi} \chi(t) dt \) for \( j \to \infty \). The function \( S(z) \) has zeros exactly at \( \{ \pm i \sqrt{v_j + \rho^2} \}_{j \geq 0} \) and \( 0 \). Clearly \( \sqrt{v_j + \rho^2} \) is a perturbation of \( 2j + 1 + \alpha + \beta \) for \( n \) large enough. The function \( S(iz) \) is a function of sine type (see [13]).

In order to obtain Ramanujan’s master Theorem we need to find a uniform bound on the residue of \( S(z)^{-1} \) at \( \{ i \sqrt{v_j + \rho^2} \}_{j \geq 0} \). We prove the following theorem:

**Theorem 5.1** Let

\[
S_1(z) = \frac{z^2}{S(z)}.
\] (5.3)

Let \( d_j \) be the residue of \( S_1 \) at \( i \sqrt{v_j + \rho^2} \). Then the following holds:

1. \[
|S_1(z)| \asymp |z| e^{-\frac{\pi}{2}|\Re z|},
\] (5.4)

   as \( |z| \to \infty \).

2. \[
|d_j| \leq c|v_j + \rho^2| \text{ for all } j \text{ large}.
\] (5.5)

**Proof** Let us put \( \rho_0 = 1 + \alpha + \beta \). By using equation (5.2) we can write

\[
\sqrt{v_j + \rho^2} = 2j + \rho_0 + \frac{2\rho^2 - \Theta}{4j} + O(j^{-2}),
\]
for \( j \geq N, N \) large enough. Let \( \mu_j := \sqrt{v_j + \rho^2} \). We define

\[
M(z) = \frac{S(iz)}{\pi z}.
\]

Then \( M \) has zeros of order one precisely on the set \( \{ \pm \mu_j \}_{j \geq 0} \). The residue of \( S_1 \) at \( i\mu_j \) is given by

\[
d_j := \lim_{z \to i\mu_j} (z - i\mu_j) S_1(z).
\]

Since \( S_1(z) = z^2(-i\pi z M(-iz))^{-1} \), it is easy to see that

\[
d_j = -\mu_j^2 \lim_{-iz \to \mu_j} \frac{(-iz - \mu_j)}{-\pi z M(-iz)}.
\]

As \( \mu_j \) is a zero of \( M(z) \),

\[
d_j = \frac{\mu_j^2}{i\pi \mu_j M'(\mu_j)}.
\]

In order to show \( d_j \times \mu_j^{-2} \) is bounded for large \( j \), it is enough to show that \( (z M'(z))_{z=\mu_j} \) is bounded below away from zero as \( j \to \infty \). We define

\[
M_0(z) = \frac{2 \sin(\pi(z - \rho_0)/2)}{\pi(z - \rho_0)}.
\]

The function \( M_0 \) is a sine type function and exponential of type \( \pi/2 \). Clearly the zeros of \( M \) are the perturbation of the zeros of \( M_0 \). Indeed from (5.2) it is clear that the perturbation is of order 2. Using the notation of [13, p.86], let \( \lambda_k = 2k + \rho_0 \) and \( \psi_k = -\frac{2\rho^2 - \Theta}{4k} + O(k^{-2}) \). Note that \( O(k^{-2}) \) in the last expression corresponds to the order of perturbation which is 2 in this case. By [13, Theorem 2, p.86] with \( n = 2 \), i.e the order of perturbation, we have the following asymptotic expansion of \( M \) in terms of \( M_0 \)

\[
M(z) = a_0M_0(z) + z^{-1}(a_1 M_0(z) + a_2 M'_0(z)) + z^{-2}(b_0 M_0(z) + b_1 M'_0(z) + M''_0(z)) + z^{-2} f_2(z),
\]

(5.6)

where \( f_2 \) is a holomorphic function of exponential type \( \leq \frac{\pi}{2} \) and \( a_i, b_i, i = 0, 1, 2 \) are constants. For getting lower bound on \( z M'(z) \) at \( \mu_j \)'s for \( j \) large we use the above expression of \( M \) in terms of \( M_0 \). On calculating we get that

\[
M'_0(z) = \frac{\cos(\pi(z - \rho_0)/2)}{z - \rho_0} - \frac{2 \sin(\pi(z - \rho_0)/2)}{\pi(z - \rho_0)^2}.
\]
Using the fact that \( \mu_j \) is a perturbation of \( 2j + \rho_0 \) (see (5.2)) it is clear that \((z^\prime M_0(z))_{z=\mu_j}\) is bounded from below away from zero for \( j \) large enough. On computing the derivative of \( M \) from equation (5.6) it is easy to see that the most dominating term of \( z^\prime M_0(z) \) at \( z = \mu_j, j \geq N \) is \( \mu_j M_0^\prime(\mu_j) \) and the rest of the terms are of the order \( \mu_j^{-1} \) which can be made as small as possible for large \( N \). More precisely

\[ |\mu_j M_0^\prime(\mu_j)| \propto |\mu_j M_0^\prime(\mu_j)| + O(\mu_j^{-1}). \]

As \( \mu_j M_0^\prime(\mu_j) \) is bounded below for \( j \to \infty \), it implies that \((z^\prime M(z))_{z=\mu_j}\) is bounded from below as \( j \to \infty \).

The pointwise estimate (5.4) of \( S_1 \) is clear from the asymptotic expansion (5.6) of \( M \) above.

\[ \square \]

6 Main theorem

In this section we prove Ramanujan’s Master Theorem 1.4 for the case when \( \alpha, \beta \geq 0 \). We recall that, for this theorem we assumed the function \( A \) satisfies the conditions \((C1), (C2), (C3), (C4), (C5)* \) and \( \rho > 0 \) as well.

Proof of Theorem 1.4  We first prove (1). Using Corollary 4.3 and Eq. (5.5) we get that

\[
\sum_{j=0}^{\infty} \frac{|d_j|}{|c_j|} \left| a \left( i \sqrt{v_j + \rho^2} \right) \right| \left| \Psi_j \left( (t + is) \right) \right|
\leq C \sum_{m=0}^{\infty} (v_j + \rho^2) Q \left( \sqrt{v_j + \rho^2} \right) e^{-p \sqrt{v_j + \rho^2}} e^{\left| s \right| \sqrt{v_j + \rho^2}} |P(t + is)|
< \infty \text{ if } |s| < p.
\]

Here we have used the fact that \( c_j \to \infty \) as \( j \to \infty \).

Now we shall prove (2). Let \( R > 0 \) and \( 0 < \sigma < \delta \rho \). Let \( \Gamma_1 \) be the straight line joining \((-R, -\sigma)\) and \((R, -\sigma)\), \( \Gamma_2 \) be a straight line joining \((R, -\sigma)\) and \((R, R)\), \( \Gamma_3 \) be a straight line joining \((R, R)\) and \((-R, R)\), \( \Gamma_4 \) be a straight line joining \((-R, R)\) and \((-R, -\sigma)\). Let \( \Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \) be the rectangle with anticlockwise direction.

Let

\[
I = \int_{\Gamma} a(\lambda) b(\lambda) \varphi_\lambda(t) \frac{1}{c(\lambda) c(-\lambda)} d\lambda
= \int_{\Gamma_1} a(\lambda) \varphi_\lambda(t) \frac{b(\lambda)}{c(\lambda) c(-\lambda)} d\lambda + \int_{\Gamma_2} a(\lambda) \varphi_\lambda(t) \frac{b(\lambda)}{c(\lambda) c(-\lambda)} d\lambda
+ \int_{\Gamma_3} a(\lambda) \varphi_\lambda(t) \frac{b(\lambda)}{c(\lambda) c(-\lambda)} d\lambda + \int_{\Gamma_4} a(\lambda) \varphi_\lambda(t) \frac{b(\lambda)}{c(\lambda) c(-\lambda)} d\lambda.
\]
We claim that
\[ \int_{\Gamma_i} a(\lambda)\varphi_\lambda(t) \frac{b(\lambda)}{c(\lambda)c(-\lambda)} \, d\lambda \to 0, \quad (6.1) \]
as $R \to \infty$, for $i = 2, 3, 4$. Suppose that the claim is true. We observe that the function
\[ \frac{b(\lambda)}{c(\lambda)c(-\lambda)} = S_1(\lambda), \quad (6.2) \]
has simple poles at $\lambda = i\sqrt{v_j + \rho^2}$ for $j = 0, 1, 2, \cdots$ inside the rectangle $\Gamma$. Therefore from Cauchy’s theorem, (4.3) and (6.1) it follows that
\[
\int_{-\infty - i\sigma}^{\infty - i\sigma} a(\lambda) b(\lambda) \varphi_\lambda(t) \frac{1}{c(\lambda)c(-\lambda)} \, d\lambda = 2\pi i \sum_{j=0}^\infty a \left( i\sqrt{v_j + \rho^2} \right) \varphi \left( i\sqrt{v_j + \rho^2} \right) \text{Res_{\lambda=i\sqrt{v_j + \rho^2}} S_1(\lambda)}
\]
This proves (1.9). Now we shall prove the claim (6.1). Using Theorem 2.3 and (5.4) we have for $|t| < p$,
\[
\int_{\Gamma_2} \left| a(\lambda)\varphi_\lambda(t) \frac{b(\lambda)}{c(\lambda)c(-\lambda)} \right| \, d\lambda = \int_{-\sigma}^{R} |a(R + is)| \, |\varphi_{R+is}(t)| \, |S_1(R + is)| \, ds
\][\[
\leq \int_{-\sigma}^{R} e^{-ps + DR} e^{\frac{\pi}{2} R} (R^2 + s^2)^{\frac{1}{2}} \, ds
\][\[
= e^{(D-\pi/2)R} \int_{-\sigma}^{R} e^{-sp + |s| |t|} (R^2 + s^2)^{\frac{1}{2}} \, ds
\][\[
\to 0 \text{ as } R \to \infty.
\]
The last line follows as $D < \pi/2$. On $\Gamma_3$ for $|t| < p$,
\[
\int_{\Gamma_3} \left| a(\lambda)\varphi_\lambda(t) \frac{b(\lambda)}{c(\lambda)c(-\lambda)} \right| \, d\lambda = \int_{-\sigma}^{R} |a(s + iR)| \, |\varphi_{s+iR}(t)| \, |S_1(s + iR)| \, ds
\][\[
\leq \int_{-R}^{R} e^{-pR + D|s|} e^{R|t|} e^{-\pi/2 |s|} (R^2 + s^2)^{\frac{1}{2}} \, ds
\][\[
= e^{-R(p-|t|)} \int_{-R}^{R} e^{(D-\pi/2)|s|} (R^2 + s^2)^{\frac{1}{2}} \, ds
\][\[
\to 0 \text{ as } R \to \infty.
\]
For $\Gamma_4$, for $|t| < p$,
\[
\int_{\Gamma_4} \left| a(\lambda)\varphi_\lambda(t) \frac{b(\lambda)}{c(\lambda)c(-\lambda)} \right| \, d\lambda = \int_{-\sigma}^{R} |a(-R + is)| \, |\varphi_{-R+is}(t)| \, |S_1(-R + is)| \, ds
\]
Using the same method we can show that the right hand side of (1.9) is independent of $0 < \sigma < \delta \rho$.

Again using Theorem 2.3 and (5.4) we have

$$
\int_{\infty-i\sigma}^{\infty-i\sigma} \left| a(\lambda) b(\lambda) \varphi_\lambda(t + is) \frac{d\lambda}{c(\lambda)c(-\lambda)} \right| = \int_{\infty-i\sigma}^{\infty-i\sigma} \left| a(\lambda) \varphi_\lambda(t + is) S_1(\lambda) d\lambda \right|
\leq \int_{\mathbb{R}} \left| a(y - i\sigma) \varphi_{y - i\sigma}(t + i\sigma) S_1(y - i\sigma) (y^2 + \sigma^2)^{\frac{1}{2}} d\lambda \right|
\leq \int_{\mathbb{R}} e^{\rho \sigma + D |y|} e^{|y + i\sigma|} e^{-\pi/2 |y|} (y^2 + \sigma^2)^{\frac{1}{2}} d\lambda.
$$

This shows that the integral defined above is finite if $|s| < \pi/2 - D$ and hence holomorphic when $|s| < \pi/2 - D$.

We observe that the Equation (1.9) is true for every $0 < \sigma < \delta \rho$ and the right hand side of (1.9) is independent of $\sigma$. Hence using the fact that $c(-\lambda) = c(\lambda)$, $\lambda \in \mathbb{R}$ we have

$$
f(it) = \int_{\mathbb{R}} (a(\lambda) b(\lambda) + a(-\lambda) b(-\lambda)) \varphi_\lambda(t) |c(\lambda)|^{-2} d\lambda.
$$

To prove (3) if we prove that the map

$$
\lambda \mapsto a(\lambda) b(\lambda) + a(-\lambda) b(-\lambda),
$$

is in $\mathcal{S}(\mathbb{R})_c$, then using the inversion formula we will have (1.11). To show that the map

$$
\lambda \mapsto a(\lambda) b(\lambda) + a(-\lambda) b(-\lambda),
$$

is in $\mathcal{S}(\mathbb{R})_c$, it is enough to show (using Cauchy’s theorem) that the map is holomorphic on $\mathbb{R} + i[-\eta, \eta]$ and for each $N \in \mathbb{N}$,

$$
\sup_{\lambda \in \mathbb{R} + i[-\eta, \eta]} (1 + |\lambda|)^N |a(\lambda) b(\lambda)| < \infty,
$$

for some $\eta > 0$. We have

$$
b(\lambda) = c(\lambda) c(-\lambda) S_1(\lambda).
$$
From the definition of $S_1$ and Corollary 3.6 it follows that $b$ has simple pole at $\lambda = 0$. Also we have $b(-\lambda) = -b(\lambda)$. Therefore the function

$$a(\lambda)b(\lambda) + a(-\lambda)b(-\lambda) = b(\lambda)\left(a(\lambda) - a(-\lambda)\right),$$

is holomorphic around origin, as $b$ has simple pole at $\lambda = 0$ and $a(\lambda) - a(-\lambda)$ has zero at $\lambda = 0$. Hence it follows from Corollary 3.6 that the map

$$\lambda \mapsto a(\lambda)b(\lambda) + a(-\lambda)b(-\lambda),$$

is holomorphic on $\mathbb{R}^+ + i[-\delta/2, \delta/2], (\delta > 0$ as in condition $(C5)^*$). Using Corollary 3.6 and (5.4) we get that

$$\sup_{\lambda \in \mathbb{R}^+ + i[-\delta/2, \delta/2]\setminus\text{a nbd. of origin}} (1 + |\lambda|)^N |a(\lambda)b(\lambda)| < \infty,$$

for some $\delta > 0$. This completes the proof. \qed

**Remark 6.1** We now consider the non-perturbed case, i.e. the case when $B(t) \equiv 1$. In this case $\hat{A}(t) = (\sin t)^{2\alpha+1}(\cos t)^{2\beta+1}$ and $A(t) = (\sinh t)^{2\alpha+1}(\cosh t)^{2\beta+1}$. The corresponding Sturm Liouville operators for $A$ and $\hat{A}$ are well studied on $\mathbb{R}^+$ and $(0, \pi/2)$ respectively. Indeed the full spectral decomposition of $\mathcal{L}$ and $-L$ is known. Let $P_n^{\alpha,\beta}$ be a Jacobi polynomial of order $(\alpha, \beta)$ of degree $n$.

The operator $-l$ can be explicitly written as

$$-l = -\frac{d^2}{dt^2} + \left(\alpha^2 - \frac{1}{4}\right)\cot^2 t + \left(\beta^2 - \frac{1}{4}\right)\tan^2 t.$$

It is known (see [21, p.67, sec 4.24]) that $u_n(t) = \sqrt{\hat{A}(t)} P_n^{\alpha,\beta}(\cos 2t)$ are the eigenfunctions of $-l$ with eigenvalues $\nu_n = (2n + \rho)^2 - \alpha^2 - \beta^2 + \frac{1}{2}$, where $\rho = \alpha + \beta + 1$ and $n \geq 0$. Therefore in this case the eigenfunction $\Psi_n$ of $-L$ reduces to a normalising factor times the Jacobi polynomial $P_n^{\alpha,\beta}(\cos 2t)$ with eigenvalue $\nu_n = (2n + \rho)^2 - \alpha^2 - \beta^2 + \frac{1}{2}$. Hence the relation (1.3) becomes

$$P_n^{\alpha,\beta}(\cos 2t) = c_n \varphi_i \sqrt{\nu_n + \rho^2}(it), \quad \text{on} \quad \left(0, \frac{\pi}{2}\right),$$

with $\rho = \alpha + \beta + 1$. Then we define the function $b$ as in (1.10) and state the Ramanujan’s master theorem as in Theorem 1.4. We conclude here the function $b$ does not come out to be exactly $P(\lambda) \left(\sin \frac{\pi}{2}(\lambda - \rho)\right)^{-1}$ (where $P$ is a polynomial) as in [17, Theorem 5.1] (when restricted to rank one case) is because $\mathcal{L}$ differs from the Laplace Beltrami operator considered in [17] by a constant dependent on $\alpha, \beta$ times $I$ (Identity operator), which makes $\nu_n + \rho^2$ a complete square for all $n$ in their case.
6.1 The case when at least one of \( \alpha \) or \( \beta \) is not positive

Let us consider the case when \( \alpha \) or \( \beta \in (\frac{-1}{2}, 0) \). We also assume the conditions \((C1), (C2), (C3), (C4), (C5)^* \) and \( \rho > 0 \). We know from Theorem 4.1 that the eigenvalues \( v_n \) of \(-L\) goes to \(+\infty\) as \( n \to \infty \). In the case of \( \alpha, \beta \geq 0 \), all the eigenvalues \( v_n \)'s are non-negative. But in general (i.e. for \( \alpha \) or \( \beta < 0 \)) all of the eigenvalues may not be non-negative. Let \( n_0 \) be the smallest non-negative integer such that \( v_j + \rho^2 \) is positive for all \( j \geq n_0 \). It is easy to see that we can write \( \sqrt{- (v_j + \rho^2)} = \eta_j, \eta_j \geq 0 \) for \( j < n_0 \).

We define the following functions:

\[
\tilde{S}(z) = \pi z \prod_{j=n_0}^{\infty} \left( 1 + \frac{z^2}{v_j + \rho^2} \right),
\]

and

\[
S_2(z) = \frac{z^2}{\tilde{S}(z)}.
\]

Then the conclusion of the Theorem 5.3 will remain same for \( S_2 \). Now we state Ramanujan’s master theorem in this case as follows and its proof is similar to the proof of Theorem 1.4.

**Theorem 6.2** Let at least one of \( \alpha \) or \( \beta \in (\frac{-1}{2}, 0), \rho > 0 \) and let \( a \in H(D, p, \delta) \).

Then the following holds:

1. The Fourier series

\[
f(t) = 2\pi i \sum_{j=0}^{\infty} \frac{d_j}{c_j} a \left( i \sqrt{v_j + \rho^2} \right) \Psi_j(t),
\]

converges uniformly on compact subsets of \( \Omega_\rho := \{ t \in \mathbb{C} | |\Re t| < \frac{\pi}{2}, |\Im t| < p \} \) and hence holomorphic there.

2. Let \( 0 \leq \sigma < \delta \rho \). Then the function \( f \) can also be expressed in the integral form for \( t \in \mathbb{R} \) with \( |t| < p \) as

\[
f(it) = \frac{1}{2} \int_{-\infty-i\sigma}^{\infty-i\sigma} (a(\lambda)b(\lambda) + a(-\lambda)b(-\lambda)) \varphi_\lambda(t) \frac{d\lambda}{c(\lambda)c(-\lambda)}
\]

\[
+ 2\pi i \sum_{j=0}^{n_0-1} \frac{d_j}{c_j} a(\eta_j) \Psi_j(it),
\]

where the function \( b \) is defined by

\[
\frac{b(\lambda)}{c(\lambda)c(-\lambda)} = S_2(\lambda).
\]
The integrals defined above are independent of $\sigma$ and extends as a holomorphic function to a neighbourhood $\{z \in \mathbb{C} \mid |\Re z| < \frac{\pi}{2} - D\}$ of $i\mathbb{R}$.

(3) The extension of $f$ to $i\mathbb{R}$ satisfies, for all $\lambda \in \mathbb{R}$

$$\int_{\mathbb{R}} \left( f(it) - 2\pi i \sum_{j=0}^{n_0-1} \frac{d_j}{c_j} a(\eta_j) \Psi_j(it) \right) \varphi_\lambda(t) A(t) \, dt = \frac{1}{2} \left( a(\lambda)b(\lambda) + a(-\lambda)b(-\lambda) \right).$$

### 7 Ramanujan’s master theorem when $A$ satisfies (C1) to (C6)

In this section we prove Ramanujan’s Master theorem for the case when $\alpha, \beta \geq 0$. We assume the weaker conditions (C1) – (C6) and $\rho > 0$. In this case we get the Plancherel formula (7.2) instead of inversion formula (1.11) in the analogue of Ramanujan’s master theorem, Theorem 7.2. In this case we do not get the inversion formula, as we do not know the behaviour of Harish-Chandra $c$ function on (some portion of) the upper half plane.

**Theorem 7.1** Let

$$S_3(z) = \frac{z^{b+\frac{1}{2}}+2}{S(z)},$$

where $S(z)$ is defined in (5.1). Let $d_n$ be the residue of $S_3$ at $i\sqrt{\nu_n + \rho^2}$. Then the following holds:

(1)

$$|S_3(z)| \leq |z|^{b+\frac{1}{2}+1} e^{-\frac{\pi}{2} |\Re z|},$$

(7.1)

as $|z| \to \infty$.

(2)

$$|d_n| \leq c|\nu_n + \rho^2|$$

for all $n$ large.

**Theorem 7.2** Let $\alpha, \beta \geq 0$ and $\rho > 0$. Let $a \in \mathcal{H}(D, p, \delta)$. Then the following holds:

(1) The Fourier series

$$f(t) = 2\pi i \sum_{m=0}^{\infty} \frac{d_m}{c_m} a\left(i\sqrt{\nu_m + \rho^2}\right) \Psi_m(t),$$

converges uniformly on compact subsets of $\Omega_p := \{t \in \mathbb{C} \mid |\Re t| < \frac{\pi}{2}, |\Im t| < p\}$ and hence holomorphic there.
(2) Let $0 \leq \sigma < \delta \rho$. Then the function $f$ can also be expressed in the integral form for $t \in \mathbb{R}$ with $|t| < p$ as

$$f(it) = \frac{1}{2} \int_{-\infty-i\sigma}^{\infty-i\sigma} (a(\lambda)b(\lambda) + a(-\lambda)b(-\lambda)) \varphi_\lambda(t) \frac{d\lambda}{c(\lambda)c(-\lambda)},$$

where the function $b$ is defined by

$$\frac{b(\lambda)}{c(\lambda)c(-\lambda)} = S_3(\lambda).$$

The integrals defined above are independent of $\sigma$ and extends as a holomorphic function to a neighbourhood $\{z \in \mathbb{C} \mid |\Re z| < \frac{\pi}{2} - D\}$ of $i\mathbb{R}$.

(3) The extension of $f$ to $i\mathbb{R}$ satisfies,

$$\int_{\mathbb{R}} |f(it)|^2 A(t) \, dt = \frac{1}{2} \int_{\mathbb{R}} |a(\lambda)b(\lambda) + a(-\lambda)b(-\lambda)|^2 |c(\lambda)|^{-2} \, d\lambda. \quad (7.2)$$

**Proof** Proof of (1), (2) are similar. To prove (3) if we prove that the map

$$\lambda \mapsto a(\lambda)b(\lambda) + a(-\lambda)b(-\lambda),$$

is in $L^2(\mathbb{R}, |c(\lambda)|^{-2})_e$, then using the Plancherel formula we will have (7.2). Using Theorem 2.5 and (5.4) it is easy to check that $\lambda \mapsto a(\lambda)b(\lambda) + a(-\lambda)b(-\lambda)$ is in $L^2(\mathbb{R}, |c(\lambda)|^{-2})_e$. In deed, using the definition of $b$, Theorem 2.5 and (7.1) and we have

$$\int_{\mathbb{R}} |a(\lambda)b(\lambda) + a(-\lambda)b(-\lambda)|^2 |c(\lambda)|^{-2} \, d\lambda$$

$$\leq \int_{\mathbb{R}} e^{2D|\lambda|} |S_3(\lambda)|^2 |c(-\lambda)|^2 \, d\lambda$$

$$= \int_{0}^{1} e^{2D|\lambda|} |S_3(\lambda)|^2 |c(-\lambda)|^2 \, d\lambda + \int_{1}^{\infty} e^{2D|\lambda|} |S_3(\lambda)|^2 |c(-\lambda)|^2 \, d\lambda$$

$$= \int_{0}^{1} e^{2D|\lambda|} |\lambda|^{2(b+\frac{1}{2})+2} e^{-\pi |\lambda|} |\lambda|^{-2b-1} \, d\lambda + \int_{1}^{\infty} e^{2D|\lambda|} |\lambda|^{2(b+\frac{1}{2})+2} e^{-\pi |\lambda|} |\lambda|^{-2a-1} \, d\lambda$$

$$< \infty. \quad \Box$$

### 8. Appendix

In this section we prove Theorem 2.3 and state some well known preliminaries. The proof of Theorem 2.3 is similar to [4, Theorem 1.2]. To prove the theorem we need the following preliminaries:
Let $J_\alpha, Y_\alpha$ be the Bessel functions of first and second kind respectively. Also let $H^{(1)}_\alpha$ and $H^{(2)}_\alpha$ be the Hankel functions defined by

$$H^{(1)}_\alpha(x) = J_\alpha(x) + i Y_\alpha(x), \quad H^{(2)}_\alpha(x) = J_\alpha(x) - i Y_\alpha(x).$$

Also let

$$J_\alpha(x) = \sqrt{x} J_\alpha(x), \quad Y_\alpha(x) = \sqrt{x} Y_\alpha(x), \quad H^{(1)}_\alpha(x) = \sqrt{x} H^{(1)}_\alpha(x), \quad H^{(2)}_\alpha(x) = \sqrt{x} H^{(2)}_\alpha(x).$$

(8.1)

Let $\Omega = \{ t + is \mid |s| < \pi/2 \}$. Also let

$$\langle x \rangle = \frac{|x|}{1 + |x|}.$$

To prove the Theorem 2.3, the following two theorems are needed:

**Theorem 8.1** ([18, Theorem 10.1, p. 219]) Let $\alpha, \beta \in \mathbb{C}$ and let $\mathcal{P}$ be a finite chain of $R_2$ arcs in complex plane joining $\alpha$ and $\beta$. Let

$$h(\xi) = \int_\alpha^\xi K(\xi, v) (\psi_0(v) h(v) + \varphi(v) J(v)) \, dv,$$

(8.2)

where

1. the path of integration lies on $\mathcal{P}$,
2. the real/complex valued functions $J(v), \varphi(v), \psi_0(v)$ are continuous except for a finite number of discontinuity,
3. the real/complex valued kernel $K(\xi, v)$ and its first two partial $\xi$ derivatives are continuous function on two variables $\xi, v \in \mathcal{P}$, (here all differentiation with respect to $\xi$ are performed along $\mathcal{P}$).
4. The kernel satisfies
   
   (a) $K(\xi, \xi) = 0$,
   (b) $|K(\xi, v)| \leq p_0(\xi) q(v),$
   (c) $\left| \frac{\partial K}{\partial \xi}(\xi, v) \right| \leq p_1(\xi) q(v),$
   (d) $\left| \frac{\partial^2 K}{\partial \xi^2}(\xi, v) \right| \leq p_2(\xi) q(v),$

   for all $\xi \in \mathcal{P} \text{ and } v \in (\alpha, \xi)_{\mathcal{P}}$, for some continuous functions $p_0, p_1, p_2, q$. Here $(\alpha, \xi)_{\mathcal{P}}$ denotes the part of $\mathcal{P}$ lying between $\alpha$ and $\xi$.
5. Also let the functions

$$\Phi(\xi) = \int_\alpha^\xi |\varphi(v)\, dv|, \quad \Psi(\xi) = \int_\alpha^\xi |\psi_0(v)\, dv|$$

converges.
(6) Let
\[ \kappa = \sup_{\xi \in \mathcal{P}} \{ q(\xi) | J(\xi) | \} \] and \( \kappa_0 = \sup_{\xi \in \mathcal{P}} \{ p_0(\xi) q(\xi) \} \)
are finite.

Then the integral equation (8.2) has a unique solution \( h \) which is continuously differentiable in \( \mathcal{P} \) and satisfies
\[ |h(\xi)|, |h'(\xi)| \leq \kappa/\Phi_1(\xi) \exp(\kappa_0/\Psi_1(\xi)). \] (8.3)

**Theorem 8.2** ([4, Appendix, Lemma A.1]) Let \( u_1 \) and \( u_2 \) be two linearly independent solutions of the equation
\[ u'' + p_1 u' + p_2 u = 0, \]
on \( \mathbb{R} \). If \( \varphi \) is a \( C^2 \) solution of the integral equation
\[ u(t) = -\int_0^t u_1(t)u_2(s) - u_2(t)u_1(s) \frac{\psi_0(s)u(s) + J(s)\varphi(s)}{u_1(s)u'_2(s) - u_2(s)u'_1(s)} \, ds \]
then \( \varphi \) is a solution of
\[ u'' + p_1 u' + p_2 u = \psi_0 u + J \varphi. \]

**Proof of Theorem 2.3** We first assume that \( \lambda \neq 0 \). From the asymptotic expansion of \( \varphi_\lambda \) (see [4, p. 219]) we have
\[ \varphi_\lambda(t) = \sum_{m=0}^M a_m(t) \frac{J_{\alpha+m}(\lambda t)}{\sqrt{A(t)}\lambda^{m+\alpha+\frac{1}{2}}} + \frac{R_M(\lambda, t)}{\sqrt{A(t)}}, \] (8.4)
where \( a_m \) are holomorphic and \( M \geq 0 \).

The function \( R_M(\lambda, t) \) satisfies (see [4, (1.3)])
\[ \frac{d^2 R_M}{dt^2} + \left( \lambda^2 - \frac{\alpha^2 - 1}{t^2} \right) R_M = G(t) R_M + 2a'_{M+1}(t) \frac{J_{\alpha+M}(\lambda t)}{\lambda^{M+\alpha+\frac{1}{2}}}. \] (8.5)

Let \( \Omega_1 = \Omega \setminus (-\infty, 0] \). From Theorem 8.2, it follows that a solution of the following integral equation
\[ R_M(\lambda, t) = -\pi \int_0^t \frac{J_\alpha(\lambda t) Y_\alpha(\lambda s) - J_\alpha(\lambda s) Y_\alpha(\lambda t)}{2\lambda} \left( G(s) R_M(\lambda, s) + 2a'_{M+1}(s) \frac{J_{\alpha+M}(\lambda s)}{\lambda^{M+\alpha+\frac{1}{2}}} \right) ds, \] (8.6)
also satisfies (8.5). As shown in [4, p. 222] $R_M(\lambda, t)$ is the solution of the above integral equation (8.6), which satisfies the required Cauchy Condition. Let $t_0 \in (0, \infty)$. Then

$$R_M(\lambda, t) = -\pi \int_{t_0}^{t} \frac{\mathcal{J}_\alpha(\lambda t) \mathcal{Y}_\alpha(\lambda s) - \mathcal{J}_\alpha(\lambda s) \mathcal{Y}_\alpha(\lambda t)}{2\lambda} ds$$

$$+ \left( G(s) R_M(\lambda, s) + 2a_{\alpha+1} M(s) \right) ds$$

$$+ R_M(\lambda, t_0).$$

Since both sides of the equation above is holomorphic in $\Omega_1$, we have for all $\xi \in \Omega_1$,

$$R_M(\lambda, \xi) = -\pi \int_{t_0}^{\xi} \frac{\mathcal{J}_\alpha(\lambda \xi) \mathcal{Y}_\alpha(\lambda s) - \mathcal{J}_\alpha(\lambda s) \mathcal{Y}_\alpha(\lambda \xi)}{2\lambda} ds$$

$$+ \left( G(s) R_M(\lambda, s) + 2a_{\alpha+1} M(s) \right) ds$$

$$+ R_M(\lambda, t_0).$$

Let

$$K(\xi, s) = -\pi \frac{\mathcal{J}_\alpha(\lambda \xi) \mathcal{Y}_\alpha(\lambda s) - \mathcal{J}_\alpha(\lambda s) \mathcal{Y}_\alpha(\lambda \xi)}{2\lambda}$$

$$- \pi i \frac{\mathcal{H}_\alpha^{(1)}(\lambda \xi) - \mathcal{H}_\alpha^{(2)}(\lambda s) - \mathcal{H}_\alpha^{(1)}(\lambda s) \mathcal{H}_\alpha^{(2)}(\lambda \xi)}{4\lambda}.$$

Using the estimates of Bessel and Hankel functions (as in [4]) we get,

$$|K(\xi, s)| \leq \begin{cases} C |\lambda| |\lambda \xi|^{|\alpha|+\frac{1}{2}} |\lambda s|^{-|\alpha|+\frac{1}{2}} e^{\Im(\lambda \xi) - \Im(\lambda s)} & \text{for } \alpha \neq 0, \\ C |\lambda \xi|^{|\alpha|+\frac{1}{2}} |\lambda s|^{-|\alpha|+\frac{1}{2}} \log \left( \frac{2}{|\lambda s|} \right) e^{\Im(\lambda \xi) - \Im(\lambda s)} & \text{for } \alpha = 0. \end{cases}$$

Also we have

$$\left| \frac{\partial}{\partial \xi} K(\xi, s) \right| \leq \begin{cases} C |\lambda \xi|^{|\alpha|+\frac{1}{2}} |\lambda s|^{-|\alpha|+\frac{1}{2}} e^{\Im(\lambda \xi) - \Im(\lambda s)} & \text{for } \alpha \neq 0, \\ C |\lambda \xi|^{-\frac{1}{2}} |\lambda s|^{|\alpha|+\frac{1}{2}} \log \left( \frac{2}{|\lambda s|} \right) e^{\Im(\lambda \xi) - \Im(\lambda s)} & \text{for } \alpha = 0. \end{cases}$$

For $\Im(\lambda \xi) > \Im(\lambda s) > 0$, we let

(1) $p_0(\xi) = \frac{C}{|\lambda|} |\lambda \xi|^{|\alpha|+\frac{1}{2}} e^{\Im(\lambda \xi)}$

(2) $q(s) = \begin{cases} |\lambda s|^{-|\alpha|+\frac{1}{2}} e^{-\Im(\lambda s)} & \text{for } \alpha \neq 0 \\ |\lambda s|^{|\alpha|+\frac{1}{2}} \log \left( \frac{2}{|\lambda s|} \right) e^{-\Im(\lambda s)} & \text{for } \alpha = 0. \end{cases}$

(3) $p_1(\xi) = |\lambda \xi|^{|\alpha|+\frac{1}{2}} e^{-\Im(\lambda \xi)}$. 

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\( \psi_0(s) = G(s), J(s) = \frac{1}{q(s)} \) and \( \varphi(s) = 2a_{M+1}'(s) \frac{f_2(s)}{\lambda^{M+\alpha+\frac{1}{2}}} q(s) \).

Therefore

1. \( \kappa_0 := \sup \{ p_0(\xi) q(\xi) \} = \frac{C}{|\lambda|} \),

2. \( \kappa := \sup \{ q(\xi) J(\xi) \} = 1 \).

We have

\[
|\varphi(s)| \leq \begin{cases} 
C|a_{M+1}'(s)| |\lambda|^{-M-\alpha-\frac{1}{2}} \langle \lambda, s \rangle^\alpha |\lambda|^{M+1} & \text{for } \alpha \neq 0 \\
C|a_{M+1}'(s)| |\lambda|^{-M-\frac{1}{2}} \langle \lambda, s \rangle^M \log \left( \frac{2}{\langle \lambda, s \rangle} \right) & \text{for } \alpha = 0 
\end{cases}
\]

We know that \( |a_{M+1}'(s)| \leq Cs^M \) for \( s < 1 \) and \( a_{M+1}' \in L^1((1, \infty)) \). Therefore, we have

\[
|\Phi(\xi)| \leq \begin{cases} 
C|\lambda|^{-M-\alpha-\frac{1}{2}} \langle \lambda, \xi \rangle^\alpha |\lambda|^{M+1} \langle \xi \rangle^M & \text{for } \alpha \neq 0 \\
C|\lambda|^{-M-\frac{1}{2}} \langle \lambda, \xi \rangle^M \log \left( \frac{2}{\langle \lambda, \xi \rangle} \right) \langle \xi \rangle^M & \text{for } \alpha = 0 
\end{cases}
\]

Hence by Theorem 8.1 we have

\[
|R_M(\lambda, \xi)| \leq C|\lambda|^{-M-\alpha-\frac{3}{2}} \langle \lambda, \xi \rangle^M \langle \lambda, \xi \rangle^{\alpha+\frac{3}{2}} e^{\lambda(\xi)} \exp \left( \left( \frac{C}{|\lambda|} \mid \int_0^\xi G(s) \, ds \right) \right)
\]

\[
\leq C|P(\xi)| e^{\lambda(\xi)} \exp \left( \frac{C}{|\lambda|} \mid \int_0^\xi G(s) \, ds \right).
\]

for some polynomial \( P \) for the case \( \alpha \neq 0 \). Then as in the argument [4, Remark 1.3] we can improve the inequality above as

\[
|R_M(\lambda, \xi)| \leq C|P(\xi)| e^{\lambda(\xi)} \exp \left( \frac{C}{1+|\lambda|} \left| \int_0^\xi G(s) \, ds \right| \right).
\]

(8.7)

Similar estimate also holds for \( \alpha = 0 \). Therefore from the given condition (C4), it follows that for all \( \xi \in \Omega_1 \) with \( |\xi| \leq \theta, \theta < \frac{\pi}{2} \),

\[
|R_M(\lambda, \xi)| \leq C|P(\xi)| e^{\lambda(\xi)}.
\]

We can do the similar technique to the domain \( \Omega_2 = \Omega \setminus [0, \infty) \) and get the similar estimate for the domain \( \Omega_2 \) away from the boundary. Hence we have from (8.4) that

\[
|\varphi_\lambda(\xi)| \leq C |P(\xi)| e^{\lambda(\xi)},
\]

for all \( \xi \in \Omega \), away from the boundary. \( \Box \)

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Holomorphic ODE: We state the following theorem about the holomorphicity of solutions of a differential equation ([14, Theorem 1.4]). This is used in the proof of the Lemma 2.2.

Theorem 8.3 Let Ω be a simply connected region in ℂ and z₀ ∈ ℂ. Also let a₁, a₂, · · · , aₙ be holomorphic functions on Ω. For any complex numbers y₀, y₁, · · · , yₙ, there exists a unique holomorphic function y on Ω such that

\[ \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = 0, \]  

(8.8)

and

\[ y(z₀) = y₀, y^{(1)}(z₀) = y₁, \cdots , y^{(n-1)}(z₀) = y_{n-1}. \]  

(8.9)

Singular Sturm Liouville operator: In this subsection we give few well known preliminaries of Sturm-Liouville’s operator for the sake of completeness.

Let us define a singular Sturm Liouville operator on an interval I = (a, b) by

\[ My := \frac{1}{w} \left( - (py')' + qy \right), \]  

(8.10)

where p, q, w : I → ℝ, p, w > 0 a.e. and \( \frac{1}{p}, q, w \in L^1_{\text{loc}}(I) \). The operator M is called non-oscillatory at a if there exists a solution My = λy such that y ≠ 0 in the (a, a + δ) for some δ > 0 and some λ ∈ ℝ. Similar definition is for the other end point b.

For non oscillatory end points, Niessen and Zettl (in [15]) have completely characterised all the self adjoint extensions of the Sturm Liouville operator M on \( L^2((a, b), w(t) dt) \) with explicit boundary conditions at a and b.

We say \( u_a \) is a principal solution at a if \( u_a \) is non zero in a right neighbourhood of a and for any other solution y of My = λy on (a, b), \( u_a(t) = o(y(t)) \) as \( t \to a^+ \). It is known that a principle solution at a of the equation My = λy is unique upto multiplicative constant. When M is non-oscillatory at a and b, principal solutions do exist at a and b respectively.

If M is a limit point case at a i.e. only one solution of Mu = λu lies in \( L^2(a, a + \epsilon) \) for some \( \epsilon > 0 \) then we don’t require any boundary condition at a. This classification is independent of λ. For further details see [15].

In section 3 we consider \( -l = -\frac{d^2}{dt^2} + q(t) \), where p(t) is defined as in equation (4.4). If we compare it with (8.10), p and w ≡ 1 for −l. We have the following observations about −l.

1. The end points 0 and π/2 are non-oscillatory end points of −l.
2. It is known that for a fixed \( \mu \geq 0 \) there exist two linearly independent eigenfunctions \( u_1, u_2 \) of −l with eigenvalue \( \mu \) such that near \( 0^+ \) \( u_1(t) \sim t^{\alpha + 1/2}, u_1'(t) \sim t^{\alpha - 1/2} \) and \( u_2(t) \sim t^{-\alpha + 1/2}, u_2'(t) \sim t^{-\alpha - 1/2}, \alpha \neq 0 \). When \( \alpha = 0 \) \( u_2(t) \sim t^{1/2} \log t \) and \( u_2'(t) \sim t^{-1/2} \log t \) near 0+. Similarly, given \( \tilde{\mu} \geq 0 \), we can also
find two linearly independent eigenfunctions $W_{\tilde{\mu}, \pm \beta}$ of $-l$ with eigenvalue $\tilde{\mu}$ defined on $(0, \frac{\pi}{2})$ satisfying

$$-lW_{\tilde{\mu}, \pm \beta}(t) = \tilde{\mu}W_{\tilde{\mu}, \pm \beta}(t), \ t \in \left(0, \frac{\pi}{2}\right)$$

such that $W_{\tilde{\mu}, \pm \beta}(t) \sim (\frac{\pi}{2} - t)^{\pm \beta + \frac{1}{2}}$ and $W'_{\tilde{\mu}, \pm \beta}(t) \sim (\frac{\pi}{2} - t)^{\pm \beta - \frac{1}{2}}$ near $\frac{\pi}{2}$, $\beta \neq 0$. When $\beta = 0$, $W_{\tilde{\mu}, -\beta}(t) \sim (\frac{\pi}{2} - t)^{1/2} \log (\frac{\pi}{2} - t)$ and $W'_{\tilde{\mu}, -\beta}(t) \sim (\frac{\pi}{2} - t)^{-1/2} \log (\frac{\pi}{2} - t)$ near $\frac{\pi}{2}$. (See theorem 8, page 15 in [8] and Theorem 2.2, page 548 in [15] for further details)

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