THE $i$-QUANTUM GROUP $U^i(n)$

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Abstract. This paper reveals some new structural property for the $i$-quantum group $U^i(n)$ and constructs a certain hyperalgebra from the new structure which has connections to finite symplectic groups at the modular representation level. This work is built on certain finite dimensional $Q(\nu)$-algebras $S^i(n, r)$ whose integral form $S^i_Z(n, r)$ is investigated as a convolution algebra arising from the geometry of type $C$ in [BKLW18]. We investigate $S^i_Z(n, r)$ as an endomorphism algebra of a certain $q$-permutation module over the Hecke algebra of type $C_r$ and interpret the convolution product as a composition of module homomorphisms. We then prove that the action of $U^i(n)$ on the $r$-fold tensor space of the natural representation of $U^i(n)$ (via an embedding $U^i(n) \hookrightarrow U(gl_{2n})$) coincides with an action given by multiplications in $S^i(n, r)$. In this way, we re-establish the surjective homomorphism from $U^i(n)$ to $S^i(n, r)$ due to Bao–Wang [BW18]. We then embed $U^i(n)$ into the direct product of $S^i(n, r)$ and completely determine its image. This gives a new realisation for $U^i(n)$ and, as an application, the aforementioned hyperalgebra is an easy consequence of this new construction.

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1. Introduction

Building on two fundamental results—some “short” multiplication formulas in [BKLW18, Lem. 3.2] and a triangular relation between two bases in [BKLW18 Thm. 3.10]—for the $q$-Schur algebras of type $B$, the authors successfully established certain “long” multiplication formulas which are used to develop a new presentation for the $i$-quantum group $U^i(n)$. This new realisation has an important application to a partial integral version of the Bao–Wang’s Schur duality [BW18 Thm. 6.27]. Thus, this type of $q$-Schur algebras can now play a bridging role between the modular representation theory of the $i$-quantum groups $U^i(n)$ and that of $U^i(n)$.

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finite orthogonal groups, a well-known connection in the type $A$ case established almost thirty years ago; see [DJ89] and [Du95].

It is natural to expect that the $q$-Schur algebra $S^t_\mathbb{Z}(n, r)$ of type $C$ should play a similar bridging role between the $i$-quantum groups $U^i(n)$ and finite symplectic groups. Since $S^t_\mathbb{Z}(n, r)$ is isomorphic to a centralizer subalgebra $eS^t_\mathbb{Z}(n, r)e$ for a certain idempotent $e$, some of the structure of $S^t_\mathbb{Z}(n, r)$ such as the aforementioned two fundamental results should be transferred from that of $eS^t_\mathbb{Z}(n, r)e$. However, the “twin products” used in a triangular relation described in [BKLW18] §5.4 cannot be transferred. So, we must modify the triangular relation in $eS^t_\mathbb{Z}(n, r)e$ so that it can be transferred to $S^t_\mathbb{Z}(n, r)$. Also, in order to develop the link between representations of $U^i(n)$, regarded as a coideal subalgebra of $U(\mathfrak{gl}_{2n})$, and the finite symplectic groups, no transferring can apply; they all need to be developed independently.

This paper also follows an approach different from that used in [DW]. After the definition of $U^i(n)$, we identify $U^i(n)$ as a coideal subalgebra of the quantum linear group $U(\mathfrak{gl}_{2n})$ and compute its actions on the tensor space $\Omega^\otimes r$, where $\Omega$ is the natural representation of $U(\mathfrak{gl}_{2n})$ (see Proposition 2.3). Once we introduce the $q$-Schur algebra $S^t_\mathbb{Z}(n, r)$ of type $C$, we introduce an action on $\Omega^\otimes r$ by the Hecke algebra $H(r)$ via the vector space isomorphism from $\Omega^\otimes r$ to a $S^t(n, r)$-$H(r)$-bimodule. We then identify the $U^i(n)$-action on $\Omega^\otimes r$ with the $S^t(n, r)$-action (Theorem 4.5). In this way, we prove that the $U^i(n)$-action commutes with the $H(r)$-action and establish the partial Schur–Weyl duality in Corollary 6.5. This approach is different from the original proof given in [BW18] Thm. 5.4 and is convenient to lift the partial duality to the integral level.

We organise the paper as follows. In §2, we review the definition of $i$-quantum groups $U^i(n)$ and their realisation as a coideal subalgebra of the quantum linear group $U(\mathfrak{gl}_{2n})$. Then we compute the action of $U^i(n)$ on the tensor product $\Omega^\otimes r$ via the coideal subalgebra embedding into $U(\mathfrak{gl}_{2n})$. In §3, we introduce the $q$-Schur algebra $S^t_\mathbb{Z}(n, r)$ of type $C$ through the (Iwahori–)Hecke algebras of type $C_r$ and identify $\Omega^\otimes r$ as an $S^t(n, r)$-$H(r)$-bimodule in (3.1.8). This allows us to transfer the $H(r)$-action on the bimodule to an action on $\Omega^\otimes r$. In §4, short multiplication formulas in $S^t_\mathbb{Z}(n, r)$ (Lemma 4.3) are introduced, following [BKLW18] Thm 3.7, Lem. A.13, and are used to prove that the $U^i(n)$-action commutes with the $H(r)$-action via multiplication formulas. This results in a map $\rho^r_t$ sending $U^i(n)$ into $\text{End}_{H(r)}(\Omega^\otimes r)$ (Theorem 4.5). In §5, we develop a triangular relation between two bases of $S^t_\mathbb{Z}(n, r)$. This improves a similar relation via twin products in [BKLW18] §5.4. Long multiplication formulas in $S^t(n, r)$ are derived in §6 (Theorem 6.2). We further use them together with the triangular relation to prove that the map $\rho^r_t$ is surjective. As an application, we immediately get in §7 a monomorphism from $U^i(n)$ to the direct product $S^t(n)$ of $S^t(n, r)$. The triangular relation further allows us to determine the image of the embedding. This gives us a new realisation of $U^i(n)$ (Theorem 7.3). Finally, in the last section, we introduce a $\mathbb{Z}$-subalgebra $U^t_\mathbb{Z}(n)$ so that $\rho^r_t$ induces an epimorphism from $U^t_\mathbb{Z}(n)$ onto $S^t_\mathbb{Z}(n, r)$ and, at the same time, use the interpretation of $S^t_\mathbb{Z}(n, r)$ as the endomorphism algebra of a permutation module over the finite symplectic group $G = Sp_{2n}(q)$ to establish a link between representations of $U^t_\mathbb{R}(n)$ and $RG$ via $S^t_\mathbb{R}(n, r)$ for any field $R$.

Some notations. For a positive integer $a$, let

$$[1, a] = \{1, 2, \ldots, a\}, \quad [1, a] = \{1, 2, \ldots, a-1\}.$$

Let $\mathbb{Z} = \mathbb{Z}[v, v^{-1}]$ be the integral Laurent polynomial ring and let $\mathcal{A} := \mathbb{Z}[q]$ ($q = v^2$). For any integers $n, m$ with $m > 0$, we set

$$\binom{n}{m} = \frac{\prod_{i=0}^{m-1}(q^{n-i} - 1)}{\prod_{i=1}^{m}(q^i - 1)} \in \mathcal{A}, \text{ where } n \geq 0,$$
\[ [n] = \frac{v^n - v^{-n}}{v - v^{-1}} \quad \text{and} \quad \left[ \begin{array}{c} n \\ m \end{array} \right] = \frac{[n][n-1]\ldots[n-m+1]}{[1][2]\ldots[m]} = \prod_{i=1}^{m} \frac{v^{n-i+1} - v^{-(n-i+1)}}{v^i - v^{-i}}. \]

Denote \([n]_1\) as \([n]\) and set \([0]_0 = 1 = [0]_0\). Note that
\[
\left[ \begin{array}{c} n \\ m \end{array} \right] = v^{m(m-n)} \left[ \begin{array}{c} n \\ m \end{array} \right]. \quad (1.0.1)
\]

We also define, for \(s, t \in \mathbb{Z}\) with \(t > 0\) and an element \(K\) in a \(\mathbb{Q}(v)\)-algebra,
\[
\left[ K; s \atop t \right] = \prod_{i=1}^{t} \frac{Kv^{s-i+1} - K^{-1}v^{-(s-i+1)}}{v^i - v^{-i}}. \quad (1.0.2)
\]

2. The \(i\)-Quantum Group \(U^i(n)\) and Its Associated Tensor Modules

In Bao-Wang’s study [BW18] of canonical bases for the quantum symmetric pairs introduced in [Le99, Le03], they investigated two classes of such quantum symmetric pairs whose associated coideal subalgebras or \(i\)-quantum groups are denoted by \(U^i(n)\) and \(U^i(n)\), where \(n\) indicates the rank of the \(i\)-quantum group. We now follow the definition of \(U^i(n)\) given in [BKLW18, §A.4].

**Definition 2.1.** The algebra \(U^i(n)\) is defined to be the associative algebra over \(\mathbb{Q}(v)\) with generators \(e_i, f_i, d_a, d_a^{-1}, t\), for \(i = 1, 2, \ldots, n-1\), for \(a = 1, 2, \ldots, n\) and the following relations:

- (iQG1) \(d_a^{-1} = d_a^{-1}d_a = 1, d_ad_b = d_bd_a\);
- (iQG2) \(d_ae_jd_a^{-1} = v^{\delta_{a,j}+\delta_{a,j+1}}e_j, d_af_jd_a^{-1} = v^{-\delta_{a,j}+\delta_{a,j+1}}f_j, d_atd_a^{-1} = t;\)
- (iQG3) \(e_if_j - f_je_i = \delta_{ij}d_{d_a^{-1}d_{a+1}}v^{-|i-j|};\)
- (iQG4) \(e_ie_j - e_je_i = e_if_j - f_je_i\), if \(|i - j| > 1;\)
- (iQG5) \(e_i^2e_j + e_je_i^2 = [2]e_i^2e_j, f_i^2f_j + f_jf_i^2 = [2]f_if_j, f_i^2, f_j^2, f_i; f_j, \) if \(|i - j| = 1;\)
- (iQG6) \(a) e_it = te_i, for \(i \neq n - 1,\)
  \(b) t^2e_{n-1} + e_{n-1}t^2 = [2]t^2e_{n-1} + e_{n-1};\)
  \(c) e_i^2 + t^2 = [2]e_i^2 + t^2;\)
- (iQG7) \(a) f_jt = tf_j, for \(j \neq n - 1,\)
  \(b) t^2f_{n-1} + f_{n-1}t^2 = [2]t^2f_{n-1} + f_{n-1};\)
  \(c) f_{n-1}t + t^2f_{n-1} = [2]t^2f_{n-1}f_{n-1}.\)

We use the following diagram to indicate the relations of the generators.

\[
\begin{align*}
1 & \quad \cdots \quad 2 \quad \cdots \quad 3 \quad \cdots \quad n-1 \quad \cdots \quad t
\end{align*}
\]

Figure 1.

Here the dashed line represents some unusual relations between \(t\) and \(e_{n-1}\) (resp., \(f_{n-1}\)) in (iQG6) (resp., (iQG7)).

The algebra \(U^i(n)\) admits an involution (i.e., algebra automorphism of order 2)
\[
\omega : U^i(n) \longrightarrow U^i(n), \quad e_i \mapsto f_i, \quad f_i \mapsto e_i, \quad d_a \mapsto d_a^{-1}, \quad t \mapsto t, \quad (2.1.1)
\]

and an anti-involution
\[
\tau : U^i(n) \longrightarrow U^i(n), \quad e_i \mapsto e_i, \quad f_i \mapsto f_i, \quad d_a \mapsto d_a^{-1}, \quad t \mapsto t; \quad (2.1.2)
\]

see [BW18] Lem. 2.1).

1Roughly speaking, an \(i\)-quantum group is a quantum analogue of the universal enveloping algebras of a fixed-point Lie subalgebra \(g^\theta\) of a semisimple Lie algebra \(g\) with an involution \(\theta\). Here \(i\) stands for involution.
2The relation \(d_atd_a^{-1} = t\) is missing in [BKLW18, (A.11)].
Consider the quantum linear group $U(gl_{2n})$, a (Hopf) $\mathbb{Q}(v)$-algebra defined by generators

$$E_h, F_h, K_j^\pm, h \in [1, 2n], j \in [1, 2n].$$

and relations similar to $(iQG1)-(iQG5)$ with $E_h, F_h, K_j^\pm$ replacing $e_h, f_h, d_j^\pm$, respectively. Its comultiplication is defined by

$$\Delta : U(gl_N) \rightarrow U(gl_N) \otimes U(gl_N), \quad F_h \mapsto F_h \otimes 1 + \tilde{K}_h^{-1} \otimes F_h, \quad K_j \mapsto K_j \otimes K_j.$$ (2.1.3)

(We omit the counit and antipode maps as they are not used.)

The algebra $U(gl_{2n})$ admits involution $\tilde{\omega}$ similar to (2.1.1) with $t$ omitted:

$$\tilde{\omega} : U(gl_{2n}) \rightarrow U(gl_{2n}), \quad E_h \mapsto E_h, \quad F_h \mapsto E_h, \quad K_j \mapsto K_j^{-1},$$

and an anti-involution

$$\tilde{\tau} : U(gl_{2n}) \rightarrow U(gl_{2n}), \quad E_i \mapsto E_i, \quad F_i \mapsto F_i, \quad K_j \mapsto K_j^{-1}\text{ similarly to (2.1.2) with } t \text{ omitted. (See, e.g., DDPW08 Lem. 6.5.)}$$

We also need the “graph automorphism”:

$$\tilde{\gamma} : U(gl_{2n}) \rightarrow U(gl_{2n}), \quad E_h \mapsto E_{2n-h}, \quad F_h \mapsto F_{2n-h}, \quad K_j \mapsto K_{2n+1-j}^{-1}.$$ (2.1.3)

Note that

$$\tilde{\gamma}(\tilde{K}_i) = \tilde{K}_{2n-i}, \quad \text{and} \quad \tilde{\tau}(\tilde{K}_i) = \tilde{K}_i^{-1} = \tilde{\omega}(\tilde{K}_i),$$

where $\tilde{K}_i = K_iK_{i+1}^{-1}$.

The following realisation of $U'(n)$ is modified from [BW18 Prop. 2.2]. We intentionally make the embedding to agree with the one given in [BKLW18 Prop. 4.5], thus, the embedding was chosen as below.

**Lemma 2.2.** There is an injective $\mathbb{Q}(v)$-algebra homomorphism $\iota : U'(n) \rightarrow U(gl_{2n})$ defined, for $i \in [1, n], j \in [1, n]$, by

$$d_j \mapsto K_j^{-1}K_{2n+1-j}^{-1}, \quad e_i \mapsto F_i + \tilde{K}_i^{-1}E_{2n-i}, \quad f_i \mapsto E_i\tilde{K}_{2n-i}^{-1} + F_{2n-i},$$

$$t \mapsto F_n + v^{-1}E_n\tilde{K}_n^{-1} + \tilde{K}_n^{-1}.$$ (2.1.3)

Moreover, relative to the coalgebra structure, $\iota(U'(n))$ is a coideal of $U(gl_{2n})$ so that the comultiplication $\Delta$ in (2.1.3) restricts to an algebra homomorphism

$$\Delta : \iota(U'(n)) \rightarrow \iota(U'(n)) \otimes U(gl_{2n}).$$

**Proof.** Let $U'(n)'$ be the $i$-quantum group generated by $t, e_{n+i}, f_{n+i}, d_{n+j}^\pm$ for all $i \in [1, n], j \in [1, n]$ obtained by an index shift $[-n+1, n-1] \rightarrow [1, 2n], i \mapsto n+i$ from the $U^i$ in [BW18 p.24] (which is extended similarly from $U(sl_{2n})$ to $U(gl_{2n})$ as in [BKLW18]). Note that the associated diagram of $U'(n)'$ has the form

```
\begin{verbatim}
\textbullet \rightarrow \textbullet \rightarrow \textbullet \rightarrow \textbullet \rightarrow \textbullet \rightarrow \textbullet \rightarrow \textbullet \rightarrow \textbullet \rightarrow \textbullet

\end{verbatim}
```

Figure 2.

Thus, after index shifting, the injective $\mathbb{Q}(v)$-algebra homomorphism $\iota : U^i \rightarrow U$ in [BW18 Prop. 2.2] takes the form

$$\iota : U'(n)' \rightarrow U(gl_{2n}), \quad e_{n+i} \mapsto E_{n+i} + \tilde{K}_n^{-1}E_{n-i}, \quad f_{n+i} \mapsto F_{n+i}\tilde{K}_n^{-1} + E_{n-i},$$

$$d_{n+j} \mapsto K_{n+j}K_{n+1-j}, \quad t \mapsto E_n + vF_n\tilde{K}_n^{-1} + \tilde{K}_n^{-1},$$
On the other hand, relabelling gives an algebra isomorphism
\[ \gamma : U^r(n) \longrightarrow U^r(n), \quad t \longmapsto t, e_{n-i} \longmapsto e_{n+i}, f_{n-i} \longmapsto f_{n+i}, d_{n-j} \longmapsto d_{n+1+j}. \]
Now, one checks easily that \( \iota = \tilde{\omega} \circ \tilde{\tau} \circ \tilde{\gamma} \circ \tau \circ \gamma \circ \iota \). For example, for \( i \in \{1, n\} \),
\[ e_{n-i} \xrightarrow{\tau} e_{n-i} \xrightarrow{\gamma} e_{n+i} \xmapsto{\iota} E_{n-i} + \tilde{K}^{-1}_{i_n} E_{n-i}, \]
and, for \( j \in \{1, n\} \), \( \iota(d_{n+1-j}) = \tilde{\omega} \circ \tilde{\tau} \circ \tilde{\gamma} \circ \iota(d_{n+j}) = \tilde{\omega} \circ \tilde{\tau} \circ \tilde{\gamma}(K_{n+j}K_{n+1-j}) = K_{n+j}K_{n+1-j}^{-1} \), as desired.

**Remark 2.3.** Dropping \( \tilde{\omega} \) or \( \tilde{\omega} \) together with \( \tilde{\gamma} \) and \( \gamma \) in \( \iota \) results in other embeddings. However, as shown in [DW1, Thm. 7.1], the embedding \( \iota : U^r(n) \to U(gl_{2n}) \) with the resulting action on the tensor space \( \Omega^{\otimes r} \) is compatible with the action induced from \( q \)-Schur algebra at level \( r \); see Theorem 4.3 below.

Let \( \Omega = \Omega_{2n} \) be the natural representation of \( U(gl_{2n}) \) with a \( \mathbb{Q}(v) \)-basis \( \{\omega_1, \omega_2, \ldots, \omega_{2n}\} \) via the following actions:
\[ E_h.\omega_i = \delta_{i,h+1} \omega_h, \quad F_h.\omega_i = \delta_{i,h} \omega_{h+1}, \quad K_j.\omega_i = v^{\delta_{i,j}} \omega_i. \] (2.3.1)
Let
\[ I(2n, r) := \{\mathbf{i} = (i_1, \ldots, i_r) | i_j \in \{1, 2n\}\}. \] (2.3.2)
For \( \mathbf{i} = (i_1, \ldots, i_r) \in I(2n, r) \), let
\[ \hat{\mathbf{i}} = (i_1, \ldots, i_{r+1}, i_{r+2}, \ldots, i_{2r}) \in I(2n, 2r) \] (2.3.3)
be defined by setting \( i_{2r+1-j} = 2n+1-i_j \) for all \( j \in \{1, r\} \). Define
\[ wt(\hat{\mathbf{i}}) = (\lambda_1, \lambda_2, \ldots, \lambda_{2n}), \quad \text{where} \quad \lambda_j = \# \{a \in \{1, 2r\} | i_a = j\}, \] (2.3.4)
and becomes a \( U(gl_{2n}) \)-module via the actions:
\[ E_h.\omega_{\mathbf{i}} = \Delta^{(r-1)}(E_h)\omega_{\mathbf{i}}, \quad F_h.\omega_{\mathbf{i}} = \Delta^{(r-1)}(F_h)\omega_{\mathbf{i}}, \quad K_j.\omega_{\mathbf{i}} = \Delta^{(r-1)}(K_j)\omega_{\mathbf{i}}, \]
where
\[ \Delta^{(r-1)} = (\Delta \otimes 1 \otimes \cdots \otimes 1) \circ \cdots \circ (\Delta \otimes 1) \circ \Delta : U(gl_{2n}) \longrightarrow U(gl_{2n})^{\otimes r}. \]
Thus, we obtain a \( \mathbb{Q}(v) \)-algebra homomorphism
\[ \rho_r = \rho_{n,r} : U(gl_{2n}) \longrightarrow \text{End}(\Omega^{\otimes r}_{2n}). \] (2.3.5)
For any \( \mathbf{i} = (i_1, \ldots, i_r) \in I(2n, r) \), we use the abbreviation \( \omega_{\mathbf{i}} := \omega_{i_1} \otimes \omega_{i_2} \otimes \cdots \otimes \omega_{i_r} \), below for \( \omega_{\mathbf{i}} = \omega_{i_1} \otimes \omega_{i_2} \otimes \cdots \otimes \omega_{i_r} \) and call \( wt(\mathbf{i}) \) the weight of \( \omega_{\mathbf{i}} \).

**Proposition 2.4.** The \( \mathbb{Q}(v) \)-algebra homomorphism
\[ \rho_r^* := \rho_r \circ \iota : U^r(n) \longrightarrow U(gl_{2n}) \longrightarrow \text{End}(\Omega^{\otimes r}_{2n}) \] (2.4.1)
defines a \( U^r(n) \)-module structure on \( \Omega^{\otimes r}_{2n} \) which is given by the following action formulas: for all \( \mathbf{i} = (i_1, \ldots, i_r) \in I(2n, r) \),
\[ \delta(j, i) = |\{k | 1 \leq k \leq r, i_k = j\}| + |\{k | 1 \leq k \leq r, i_k = 2n+1-j\}|. \]
(2) $\rho_i^r(e_h).\omega_i = \sum_{1 \leq l \leq r} v^\varepsilon_1(l) \omega_{i_1} \cdots \omega_{i_{l-1}} \omega_{h+1} \omega_{i_{l+1}} \cdots \omega_{i_r}$
+ \sum_{1 \leq l \leq r} v^\varepsilon_2(l) \omega_{i_1} \cdots \omega_{i_{l-1}} \cdot \omega_{2n-h} \cdot \omega_{i_{l+1}} \cdots \omega_{i_r},$

where
\begin{align*}
\varepsilon_1(l) &= -|\{k \mid 1 \leq k < l, i_k = h\}| + |\{k \mid 1 \leq k < l, i_k = h + 1\}|
\varepsilon_2(l) &= -|\{k \mid l < k \leq r, i_k = h\}| + |\{k \mid l < k \leq r, i_k = h + 1\}|
+ |\{k \mid l < k \leq r, i_k = 2n - h\}| - |\{k \mid l < k \leq r, i_k = 2n - h + 1\}|.
\end{align*}

(3) $\rho_i^r(f_h).\omega_i = \sum_{1 \leq l \leq r} v^\varepsilon_1(l) + \varepsilon_2(l) \omega_{i_1} \cdots \omega_{i_{l-1}} \cdot \omega_{h} \cdot \omega_{i_{l+1}} \cdots \omega_{i_r},$
+ \sum_{1 \leq l \leq r} v^\varepsilon_2(l) \omega_{i_1} \cdots \omega_{i_{l-1}} \omega_{2n-h} \omega_{i_{l+1}} \cdots \omega_{i_r},$

where
\begin{align*}
\varepsilon_1'(l) &= -|\{k \mid 1 \leq k < l, i_k = 2n - h\}| + |\{k \mid 1 \leq k < l, i_k = 2n - h + 1\}|
\varepsilon_2'(l) &= |\{k \mid l < k \leq r, i_k = h\}| - |\{k \mid l < k \leq r, i_k = h + 1\}|
- |\{k \mid l < k \leq r, i_k = 2n - h\}| + |\{k \mid l < k \leq r, i_k = 2n - h + 1\}|.
\end{align*}

(4) $\rho_i^r(t).\omega_i = v^{\tau_0} \omega_{i_1} \cdots \omega_{i_{n-1}} \omega_{i_n} \omega_{i_{n+1}} \cdots \omega_{i_r} + \sum_{1 \leq l \leq r} v^{\tau_1(l)} \omega_{i_1} \cdots \omega_{i_{l-1}} \omega_{i_{l+1}} \cdots \omega_{i_r},$
+ \sum_{l=1}^{r} v^{\tau_1(l)} \omega_{i_1} \cdots \omega_{i_{l-1}} \omega_n \omega_{i_{l+1}} \cdots \omega_{i_r},$

where
\begin{align*}
\tau_0 &= |\{k \mid 1 \leq k \leq r, i_k = n + 1\}| - |\{k \mid 1 \leq k \leq r, i_k = n\}|
\tau_1(l) &= |\{k \mid 1 \leq k < l, i_k = n + 1\}| - |\{k \mid 1 \leq k < l, i_k = n\}|.
\end{align*}

Proof. We first compute $\Delta^{(r-1)}(E_h)$, $\Delta^{(r-1)}(F_h)$, and $\Delta^{(r-1)}(K_j)$ as in [DW] (2.3.2) and then apply the algebra homomorphism $\Delta^{(r-1)}$ to $t(d_j) = K_j^{-1}K_{2n-j}^{-1}$, $t(e_i) = F_i + \tilde{K}_n^{-1}E_{2n-i}$, $t(f_i) = E_n \tilde{K}_{2n-i}^{-1} + F_{2n-i}$, and $t(t) = F_n + v^{-1}E_n \tilde{K}_n^{-1} + \tilde{K}_n^{-1}$. The remaining calculation via (2.3.1) is straightforward. See the proof of [DW] Thm. 7.1 for action formulas (1)–(3), excluding the $h = n$ case there. For (4), we have
\begin{align*}
\rho_i^r(t).\omega_i &= \Delta^{(r-1)}(F_n + v^{-1}E_n \tilde{K}_n^{-1} + \tilde{K}_n^{-1})(\omega_{i_1} \omega_{i_2} \cdots \omega_{i_r})
= \sum_{l=1}^{r} \tilde{K}_n^{-1} \omega_{i_1} \cdots \tilde{K}_n^{-1} \omega_{i_{l-1}} F_n \omega_{i_l} \omega_{i_{l+1}} \cdots \omega_{i_r} + \tilde{K}_n^{-1} \omega_{i_1} \cdots \tilde{K}_n^{-1} \omega_{i_r}
+ v^{-1} \sum_{l=1}^{r} \tilde{K}_n^{-1} \omega_{i_1} \cdots \tilde{K}_n^{-1} \omega_{i_{l-1}} E_n \tilde{K}_n^{-1} \omega_{i_l} \omega_{i_{l+1}} \cdots \omega_{i_r}.
\end{align*}
Now applying (2.3.1) gives the desired formula. \qed

It is proved in [BW13] Thm. 5.4] that, for the Hecke algebra $\mathcal{H}(r)$ of type $B_r$, there is a $U^\ast(n)\cdot \mathcal{H}(r)$-bimodule structure on $\Omega^\circ_{2n}$ such that $\text{im}(\rho_i^r) = \text{End}_{\mathcal{H}(r)}(\Omega^\circ_{2n})$. We will provide a different proof later in §6 as a by-product of the multiplication formulas developed in §4 (see Theorem 4.5 and Corollary 6.5).
3. The $q$-Schur algebra of type $C$

The Weyl group of type $C_r$ is isomorphic to the Weyl group of type $B_r$ which is the Coxeter system $(W, S)$, where $S = \{s_1, \ldots, s_{r-1}, s_r\}$ with the subgroup $W' := \langle s_1, \ldots, s_{r-1} \rangle \cong S_r$, the symmetric group on $r$ letters, and $s_{r-1}s_r$ has order 4. In this case, $W$ is regarded as a fixed-point subgroup of $S_2$, under the graph automorphism. More precisely, there is a type $C$ embedding (cf. the type B embedding in [DW, §3]):

$$
\sigma : W \rightarrow S_{2r}, \ s_1 \mapsto (1, 2)(2r, 2r-1), \ldots, s_{r-1} \mapsto (r-1, r)(r+2, r+1), s_r \mapsto (r, r+1).
$$

Then $i, j$ satisfying $i + j = 2r + 1$.

If $W$ is regarded as a fixed-point subgroup of $S_{2r+1}$ (see Remark 4.1 below), $W$ is called the Weyl group of type $B_r$. See [DW] for more details in this case and also [LW] in general.

Let $\mathcal{H}_A(r) = \mathcal{H}_A(C_r)$ be the Hecke algebra over $A := \mathbb{Z}[q]$ ($q = v^2$) associated with $(W, S)$. Then it is generated by $T_i = T_{s_i}$ for $1 \leq i \leq r$ subject to the relations:

$$
T_i^2 = (q - 1)T_i + q, \forall i; \ T_iT_j = T_jT_i, |i - j| \geq 2,
$$

$$
T_iT_{j+1}T_j = T_{j+1}T_jT_{j+1}, 1 \leq j < r - 1;
$$

$$
T_{r-1}T_rT_{r-1} = T_rT_{r-1}T_r.
$$

Let $\mathcal{Z} = \mathbb{Z}[v, v^{-1}]$. We will mainly use the $\mathcal{Z}$-algebra $\mathcal{H}_Z(r) = \mathcal{H}_A(r) \otimes \mathcal{Z}$ in the sequel. Both $\mathcal{H}_A(r)$ and $\mathcal{H}_Z(r)$ have basis $\{T_w\}_{w \in W}$. The subalgebra generated by $T_1, \ldots, T_{r-1}$ is the Hecke algebra $\mathcal{H}_A(S_r)$ over $A$ or $\mathcal{H}_Z(S_r)$ over $\mathcal{Z}$ associated with the symmetric group $S_r$.

Let $\ell : W \rightarrow \mathbb{N}$ be the length function relative to $S$. Then, for $s \in S, w \in W$, we have

$$
T_sT_w = \begin{cases} 
T_{sw}, & \text{if } \ell(sw) = \ell(w) + 1; \\
(q - 1)T_w + qT_{sw}, & \text{if } \ell(sw) = \ell(w) - 1.
\end{cases}
$$

Let

$$
\Lambda(n, r) = \{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{N}^n \ | \lambda_1 + \cdots + \lambda_n = r\}.
$$

For $\lambda \in \Lambda(n, r)$, let $W_\lambda$ be the parabolic subgroup of $W$ generated by

$$
S \setminus \{s_{\lambda_1 + \cdots + \lambda_i} \ | \ i \in [1, n]\}.
$$

Note that $W_\lambda$ is a subgroup of $W'$. Let

$$
x_\lambda = \sum_{w \in W_\lambda} T_w, \ \mathcal{T}_Z(n, r) = \bigoplus_{\lambda \in \Lambda(n, r)} x_\lambda \mathcal{H}_Z(r).
$$

(3.0.2)

The endomorphism $\mathcal{Z}$-algebra:

$$
S_\mathcal{Z}(n, r) = \text{End}_{\mathcal{H}_Z(r)}(\mathcal{T}_Z(n, r))
$$

(3.0.3)

is called the (generic) $q$-Schur algebra of type $C$. Note that $S_\mathcal{Z}(n, r)$ has also an $\mathcal{A}$-form $S_\mathcal{A}(n, r)$ defined by using $\mathcal{H}_A(r)$ as above.

For $\lambda \in \Lambda(n, r)$, let $\mathcal{D}_\lambda$ be the set of shortest representatives of right cosets of $W_\lambda$ in $W$, and let $\mathcal{D}_\lambda \mu = \mathcal{D}_\lambda \cap \mathcal{D}_\mu^{-1}$. Then $\mathcal{D}_\lambda \mu$ is the set of shortest representatives of $W_\lambda$-$W_\mu$ double cosets.

Define

$$
\gamma : \Lambda(n, r) \rightarrow \Lambda(2n, 2r), \ \lambda = (\lambda_1, \ldots, \lambda_n) \mapsto \hat{\lambda} = (\lambda_1, \ldots, \lambda_n, \lambda_n, \ldots, \lambda_1).
$$

(3.0.4)
Note that, for any \( i \in I(2n, r) \) with \( \hat{i} \) defined in (2.3.3), wt(\( \hat{i} \)) = \( \hat{\lambda} \) for some \( \lambda \in \Lambda(n, r) \). In other words, wt(\( \hat{i} \)) = wt(\( \hat{\lambda} \)).

For a positive integer \( N \) and integer \( r \geq 0 \), let

\[
\Xi_N = \{A = (a_{i,j}) \in \text{Mat}_N(N) \mid a_{i,j} = a_{N+1-i,N+1-j}, \forall i, j \in [1, N]\},
\]

\[
\Xi^0_{N,r} = \{A - \text{diag}(a_{1,1}, a_{2,2}, \ldots, a_{N,N}) \mid A = (a_{i,j}) \in \Xi_N\},
\]

\[
\Xi_{N,r} = \{A = (a_{i,j}) \in \Xi_N \mid |A| = \sum_{i,j} a_{i,j} = r\}, \quad \text{and} \quad \Xi^0_{N,r} = \Xi^0_{N} \cap \Xi_{N,r}.
\]

Note that if we represent \( w \in \mathcal{S}_{2r} \) by a \( 2r \times 2r \) permutation matrix \( P(w) = (p_{k,l}) \), where \( p_{k,l} = \delta_{k,l} \), then \( w \in W = \mathcal{S}_{2r}^\theta \) if and only if \( P(w) \in \Xi_{2r,2r} \).

For an \( N \times N \) matrix \( A = (a_{i,j}) \), let

\[
\text{ro}(A) := (\sum_j a_{1,j}, \sum_j a_{2,j}, \ldots, \sum_j a_{N,j})
\]

\[
\text{co}(A) := (\sum_i a_{i,1}, \sum_i a_{i,2}, \ldots, \sum_i a_{i,N}).
\]

Clearly, we have

\[
\{\text{ro}(A) \mid A \in \Xi_{N,2r}\} = \{\text{co}(A) \mid A \in \Xi_{N,2r}\} = \hat{\Lambda}(n, r) := \{\hat{\lambda} \mid \lambda \in \Lambda(n, r)\}.
\]

**Lemma 3.1.** (1) There is a bijection

\[
m : \{(\lambda, d, \mu) \mid \lambda, \mu \in \Lambda(n, r), d \in \mathcal{D}_{\lambda, \mu}\} \longrightarrow \Xi_{2n,2r}.
\]

(2) The \( \mathbb{Z} \)-algebra \( S^\mathbb{Z}_{\lambda}(n, r) \) is a free \( \mathbb{Z} \)-module with basis

\[
\{\phi_A = \phi^d_{\lambda, \mu} \mid \lambda, \mu \in \Lambda(n, r), d \in \mathcal{D}_{\lambda, \mu}\},
\]

where \( \phi^d_{\lambda, \mu} \) is defined by \( \phi^d_{\lambda, \mu}(x_\nu) = \delta_{\mu, \nu} T_{w_\lambda dW_\mu} \) and \( A = m(\lambda, d, \mu) \).

**Proof.** For assertion (1), note that the matrix \( A = m(\lambda, d, \mu) \) is the matrix associated with the double coset \( \mathcal{S}_{\lambda} \mathcal{S}_{\mathcal{D}_{\lambda}} \) in \( \mathcal{S}_{2r} \), where \( \sigma(W_\nu) = (\mathcal{S}_{\nu})^\theta \) for \( \nu = \lambda \) or \( \mu \), and \( \hat{\lambda} = \sigma(d) \). For more details, see, e.g., [11 Lem. 2.2.1]. Assertion (2) follows from (1) and [Du94 1.4]. \( \square \)

For any \( A = m(\lambda, d, \mu) \), let

\[
[A] = \nu^{-\ell(d^+) + \ell(w_{\mu, d})} \phi^d_{\lambda, \mu}, \quad [\lambda] = \phi^1_{\lambda, \lambda}
\]

where \( d^+ \) (resp. \( w_{\mu, d} \)) is the longest element in the double coset \( W_\lambda dW_\mu \) (resp. \( W_\mu \)). Here we follow the definition given in [Du94 1.4] or [11 (3.22)] (cf. also [DDPW08 (9.3.1)]).

Note that, for \( A, B \in \Xi_{2n,2r} \),

\[
[A][B] \neq 0 \implies \text{co}(A) = \text{ro}(B) \quad \text{and} \quad [\text{ro}(A)][A] = [A] = [A][\text{co}(A)].
\]

If \( n \geq r \), then the basis element \( e_\emptyset := [\text{diag}(\emptyset)] \in S^\mathbb{Z}_{\lambda}(n, r) \) is an idempotent, where

\[
\emptyset = \left(\begin{array}{cccc}
1, & 1, & 0, & \ldots, 0, 0, & \ldots, 0, & 1, & \ldots, 1
\end{array}\right) \in \Xi_{2n,2r},
\]

and \( e_\emptyset S^\mathbb{Z}_{\lambda}(n, r) e_\emptyset \cong \mathcal{H}_Z(r) \) via the evaluation map \( \phi^w_{n,0} \mapsto \phi^w_{n,0}(1) = T_w \) for all \( w \in W \). Via this isomorphism, \( S^\mathbb{Z}_{\lambda}(n, r) e_\emptyset \) becomes an \( S^\mathbb{Z}_{\lambda}(n, r)-\mathcal{H}_Z(r) \)-bimodule.

\[^3\text{The Hecke algebra there is the Hecke algebra here with } v \text{ replaced by } v^{-1}.\]
For $i = (i_1, i_2, \ldots, i_r) \in I(2n, r)$ and $A_i = (a_{k,l}) \in \Xi_{2n,2r}$ defined by
\[
    a_{k,l} = \begin{cases} 
        \delta_{k,i_l}, & \text{if } l \in [1,r]; \\
        0, & \text{if } l \in [r+1, 2n-r]; \\
        \delta_{2n+1-k,i_{2n+1-l}}, & \text{if } l \in [2n+1-r, 2n]. 
    \end{cases} \tag{3.1.2}
\]
Note that $co(A_i) = (1^r, 0^{n-r}, 0^{n-r}, 1^r) = \emptyset$ and $ro(A_i) = \text{wt}(\hat{i}^*)$, where $\hat{i}^* = (i, 0^{n-r})$ and $\hat{i}$ is similarly defined as in \(2.3.3\) with $i_{2n+1-j} = 2n+1-i_j$. If we write $\hat{i}^* = (i_1, \ldots, i_r, i_{r+1}, \ldots, i_{2n})$, then the entry $a_{k,l}$ of $A_i$ has the form $a_{k,l} = \delta_{k,i_1}$ for all $k, l \in [1,2n]$. By Lemma 3.1(1), there exist $\lambda = \lambda_i \in \Lambda(n, r)$, $d_i = d_i \in \mathcal{D}_\lambda$ such that $A_i = m(\lambda, d_i, 0)$. Thus, under the assumption $n \geq r$, the evaluation map
\[
    \text{ev} : S^r_{\Z}(n, r)e_0 \longrightarrow \mathcal{I}_\Z(n, r), \quad [A_i] \longmapsto v^{-\ell(d_i^*)}x_{\lambda_i}T_{d_i} (= v^{-\ell(d_i^*)}\phi_{\lambda_i,0}(1)) \tag{3.1.3}
\]
defines an $S^r_{\Z}(n, r)$-$\mathcal{H}_\Z(r)$-bimodule isomorphism.

If $n < r$, then we may identify $\Xi_{2n,2r}$ as a subset of $\Xi_{2r,2r}$ via the following embedding:
\[
    \Xi_{2n,2r} \longrightarrow \Xi_{2r,2r}, \quad A \longmapsto A^0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.1.4}
\]
where each 0 at a corner position of $A^0$ is a square zero matrix of size $r-n$ and other zeros represent zero matrices of appropriate sizes. Thus, if $n < r$, we may regard $S^r_{\Z}(n, r)$ as a centraliser subalgebra of $S^r_{\Z}(r, r)$ via the induced embedding $[A] \mapsto [A^0]$.

Note also that the embedding $A \mapsto A^0$ induces an embedding
\[
    \Lambda(n, r) \longrightarrow \Lambda(r, r), \quad \lambda \longmapsto \lambda^0 := (0^{r-n}, \lambda). \tag{3.1.5}
\]
In $S^r_{\Z}(r, r)$, the idempotent $f = \sum_{\lambda \in \Lambda(n, r)}[\hat{\lambda}^0]$ induces an algebra isomorphism
\[
    fS^r_{\Z}(r, r)f \cong S^r_{\Z}(n, r). \tag{3.1.6}
\]
Thus, for $n < r$, the evaluation map is in fact an $S^r_{\Z}(n, r)$-$\mathcal{H}_\Z(r)$-bimodule isomorphism
\[
    \text{ev} : fS^r_{\Z}(r, r)e_0 \longrightarrow \mathcal{I}_\Z(n, r), \quad [A_i] \longmapsto [A_i](1). \tag{3.1.6}
\]
We record the right $\mathcal{H}_\Z(r)$-action from \(3.1.3\) and \(3.1.6\) as follows:
\[
    [A_i]T_{s_j} = \begin{cases} 
        v[A_{i,s_j}], & \text{if } \ell(d_is_j) = \ell(d_i), d_is_j \in \mathcal{D}_{\lambda_i} \text{ or } i_j < i_{j+1}; \\
        v^2[A_i], & \text{if } \ell(d_is_j) > \ell(d_i), d_is_j \not\in \mathcal{D}_{\lambda_i} \text{ or } i_j = i_{j+1}; \\
        (v^2-1)[A_i] + v[A_{i,s_j}], & \text{if } \ell(d_is_j) < \ell(d_i), \text{ or } i_j > i_{j+1}. 
    \end{cases} \tag{3.1.7}
\]
Here, for $j = r, i_{r+1} = 2n + 1 - i_r$ (cf. \(2.3.3\)).

Let $\mathcal{H}(r) = \mathcal{H}_\Z(r) \otimes \Q(v)$ and $S^r(n, r) = S^r_{\Z}(n, r) \otimes \Q(v)$. In both cases, we obtain the vector space isomorphisms
\[
    \eta_r = \eta_{n,r} : \begin{pmatrix} \Omega_{2n}^r \longrightarrow S^r(n, r)e_0, & \omega_i \longmapsto [A_i], & \text{if } n \geq r; \\
        \Omega_{2r}^r \longrightarrow fS^r(r, r)e_0, & \omega_i \longmapsto [A_i], & \text{if } n < r. \end{pmatrix} \tag{3.1.8}
\]

The $\mathcal{H}(r)$-action defined in \(3.1.7\) is transferred to a right $\mathcal{H}(r)$-action on $\Omega_{2n}^r$ so that $\eta_r$ is an $\mathcal{H}(r)$-module isomorphism. Note that the transferred action here coincides with those given in \(BKLW18\) (6.3),(6.4). \footnote{The place permutation $i_{s,j}$ is “truncated” from the place permutation of $G_{2r}$ on all $\hat{i}$ defined in \(2.3.3\) via the embedding $\sigma$ in \(3.0.1\).}
Remark 3.2. (1) If $\lambda \in A(n + 1, r)$, then $S \setminus \{s_{\lambda_1 + \ldots + \lambda_i} \mid i \in [1, n]\}$ also generates a parabolic subgroup $W_\lambda$ of $W$ that is not necessarily a subgroup of $W'$. Using this $W_\lambda$, define $x_\lambda$ as in (3.0.2). The endomorphism algebra

$$S^1_\lambda(n, r) := \text{End}_{H_\lambda(r)} \left( \bigoplus_{\lambda \in A(n+1, r)} x_\lambda H_\lambda(r) \right)$$

is called the $q$-Schur algebra of type $B$. This algebra has a basis $\{[A] \mid A \in \Xi_{2n+1,2r+1}\}$ indexed by the matrix set $\Xi_{2n+1,2r+1}$.

As observed in [BKLW18, §5], this algebra contains a centraliser subalgebra $eS^1_\lambda(n, r)e$, where $e^2 = e \in S^1_\lambda(n, r)$, that is isomorphic to $S^1_\lambda(n, r)$; see §4 for more details.

(2) There is a quotient coordinate algebra approach to both $S^1_\lambda(n, r)$ and $S^1_\lambda(n, r)$ developed by Lai, Nakano and Xiang in [LNX] where they realise such an algebra as the dual of the $r$-th homogeneous component of the “coordinate algebra” of the corresponding $i$-quantum group.

4. Short multiplication formulas in $S^1_\lambda(n, r)$

We first embed $S^1_\lambda(n, r)$ into $S^1_\lambda(n, r)$ as a centralizer subalgebra of the form $eS^1_\lambda(n, r)e$ for an idempotent $e$ and then derive some “short” multiplication formulas via their counterpart in $eS^1_\lambda(n, r)e$ extracted from [BKLW18]. As a byproduct, we prove that the image of the map $\rho'$ in (2.4.1) is isomorphic to a subalgebra of $S^i(n, r) := S^1_\lambda(n, r) \otimes \mathbb{Q}(v)$.

We start with the canonical embedding of $S_{2r}$ into $S_{2r+1}$. Let $\iota : [1, 2r] \to [1, 2r+1]$ be the embedding defined by $\iota(x) = \begin{cases} x, & \text{if } x \leq r; \\ x + 1, & \text{if } x > r. \end{cases}$ Then, $\iota$ induces an injective group homomorphism

$$\iota : S_{2r} \to S_{2r+1}, \ w \mapsto \iota(w),$$

where $\iota(w)$ is the permutation that fixes $r + 1$ and equals $\iota \circ w \circ \iota^{-1}$ on $[1, 2r+1] \setminus \{r + 1\}$. In other words, we may identify $S_{2r}$ as a subgroup $\iota(S_{2r})$ of $S_{2r+1}$ consisting of permutations that fix $r + 1$.

Remark 4.1. If $w_0$ is the longest element of $S_{2r+1}$ sending $i$ to $2r+2-i$, then $w_0(r+1) = r+1$ and so $w_0 \in \iota(S_{2r})$. Thus, $w_0$ induces an (inner) automorphism $\tilde{\theta}$ sending $w$ to $w_0wv_0$ on $S_{2r+1}$ which restricts to the automorphism $\theta = \iota^{-1} \circ \tilde{\theta} \circ \iota$ on $S_{2r}$. Thus, we have the type $B$ identification $W = (S_{2r+1})^\theta$. Compare the type $C$ identification $W = (S_{2r})^\theta$ given in (3.0.1).

Now consider the following embeddings

$$\begin{aligned}
(\ )^\dagger : \Xi_{2n} &\to \Xi_{2n+1}, \ A = \begin{pmatrix} X & Y \\ Y' & X' \end{pmatrix} \mapsto A^\dagger = \begin{pmatrix} X & Y \\ -Y & -X' \end{pmatrix}, \\
(\ )^\dagger : \Lambda(n, r) &\to \Lambda(2n + 1, 2r + 1), \ \lambda \mapsto \hat{\lambda}^\dagger := (\lambda_1, \lambda_2, \ldots, \lambda_n, 1, \lambda_n, \ldots, \lambda_2, \lambda_1),
\end{aligned}$$

(4.1.1)

where $-\cdot$ and $\cdot$ represent a zero row and column, respectively.

Let $\Xi_{2n}$ be the image of $\Xi_{2n}$ in $\Xi_{2n+1}$. Then $\Xi_{2n}$ consists of matrices such that all entries in the $n + 1$st row or the $n + 1$st column are 0 except the $(n + 1, n + 1)$ entry which is 1. Let

$$\Xi_{2n, 2r}^\dagger = \Xi_{2n}^\dagger \cap \Xi_{2n+1,2r+1}^\dagger, \quad \Xi_{2n}^{1\text{\,diag}} = \Xi_{2n}^\dagger \cap \Xi_{2n+1}^{1\text{\,diag}}.$$

Then, $\Xi_{2n, 2r}^\dagger = \{A^\dagger \mid A \in \Xi_{2n, 2r}\}$.

We also set

$$\hat{\lambda}^\dagger(n, r) = \{\hat{\lambda}^\dagger \mid \lambda \in \Lambda(n, r)\} \subset \Lambda(2n + 1, 2r + 1).$$
Clearly, $\tilde{\Lambda} = \{ \text{ro}(A) \mid A \in \Xi_{2n,2r}^1 \} = \{ \lambda \mid \lambda \in \Lambda(n+1,r), \lambda_{n+1} = 0 \}$ under the notation of [DW (3.0.1)] (or Remark 3.2 above).

Let $e = \sum_{\lambda \in \Lambda(n,r)}[\text{diag}(\lambda)]$. Then $e$ is an idempotent in $S^2_{\pm}(n,r)$. Define the centraliser subalgebra

$$S^n(n,r) := eS^2_{\pm}(n,r)e. \quad (4.1.2)$$

**Lemma 4.2.** There is an algebra embedding $\hat{\iota} : S^2_{\pm}(n,r) \longrightarrow S^2_{\pm}(n,r)$ sending $[A]$ to $[A^\dagger]$, which induces an algebra isomorphism

$$\hat{\iota} : S^2_{\pm}(n,r) \longrightarrow S^n(n,r); \quad [A] \longmapsto [A^\dagger]. \quad (4.2.1)$$

For $i,j \in [1,2n]$, let $e_i = (0,\ldots,0,1,0,\ldots,0) \in \mathbb{Z}^{2n}$ and $E_{i,j} \in \text{Mat}_{2n}(\mathbb{N})$ the standard matrix units. Define

$$e_i^\theta = e_i + e_{2n+1-i}, \quad E_{i,j}^\theta := E_{i,j} + E_{2n+1-i,2n+1-j} \in \Xi_{2n}. \quad (4.2.2)$$

Then $e_i^\theta = e_i^\theta E_{i,j}^\theta = \text{co}(E_{i,j}^\theta)$ for all $i,j \in [1,2n]$ and $E_{i,j}^\theta = E_{2n+1-i,2n+1-j}^\theta$. In particular, $E_{n,n+1}^\theta = E_{n+1,n}^\theta$.

Similarly, for the standard basis elements $e_i^\theta \in \mathbb{Z}^{2n}+1$ and $E_{i,j}^\theta \in \text{Mat}_{2n+1}(\mathbb{N})$, define

$$e_i^\theta = e_i^\theta + e_{2n+2-i}, \quad E_{i,j}^\theta := E_{i,j}^\prime + E_{2n+2-i,2n+2-j} \in \Xi_{2n+1} \quad (4.2.3)$$

For $A \in \Xi_{2n}$, $p \in [1,2n]$, let

$$\beta_p(A,h) = \sum_{j \geq p} a_{h,j} - \sum_{j > p} a_{h+1,j}, \quad \beta'_p(A,h) = \sum_{j \leq p} a_{h,j} - \sum_{j < p} a_{h,j}. \quad (4.2.4)$$

Note that $\beta_p(A,h)$ is slightly different from the one defined in [DW (4.0.1)] for matrices in $\Xi_{2n+1}$. Moreover, we have the symmetry property $\beta'_p(A,n) = \beta_{2n+1-p}(A,n)$.

As set in [BKLW18 §5.1], the multiplication formulas given in [BKLW18 Lem. 3.2] or [DW Lem. 4.1] continue to hold in $S^n(n,r)$ whenever $h \neq n$. An extra formula related to the generator $t$, replacing $e_n, f_n$, is given in [BKLW18 Lem. A.13]. We now write these formulas in $S^2_{\pm}(n,r)$.

**Lemma 4.3.** The $\mathbb{Z}$-algebra $S^2_{\pm}(n,r)$ has a basis $\{ [A] \mid A \in \Xi_{2n,2r} \}$. If $A = (a_{i,j}) \in \Xi_{2n,2r}$, $\lambda \in \Lambda(n,r-1)$ and $1 \leq h < n$, the following multiplication formulas hold in $S^2_{\pm}(n,r)$:

1. $[E_{h,h+1}^\theta + \tilde{\lambda}] \cdot [A] = \delta_{e_{h+1},i} \cdot \text{ro}(A) \sum_{p \in [1,2n], a_{h,p} \geq 1} v^{\beta_p(A,h)} [a_{h,p} + 1][A + E_{h,p}^\theta - E_{h+1,p}^\theta];$
2. $[E_{h+1,h}^\theta + \tilde{\lambda}] \cdot [A] = \delta_{e_{h+1},i} \cdot \text{ro}(A) \sum_{p \in [1,2n], a_{h+1,p} \geq 1} v^{\beta'_p(A,h)} [a_{h+1,p} + 1][A - E_{h,p}^\theta + E_{h+1,p}^\theta];$
3. $[E_{n+1,n}^\theta + \tilde{\lambda}] \cdot [A] = \delta_{e_{n+1},i} \cdot \text{ro}(A) \left( c_A [A] + \sum_{p \in [1,2n], a_{n+1,p} \geq 1} v^{\beta'_p(A,n)} [a_{n+1,p} + 1][A - E_{n,p}^\theta + E_{n+1,p}^\theta] \right),$

where $c_A$ and $\epsilon = \delta_{n+1}^\epsilon$ are defined by

$$c_A = v^{-\sum_{j \leq n} a_{n,j}} \left( v^{\sum_{j \leq n} a_{n+1,j} - \sum_{j \leq n} a_{n,j}} \right) \text{ and } \delta_{i,j}^\epsilon = \begin{cases} 1, & \text{if } i \leq j; \\ 0, & \text{if } i > j. \end{cases} \quad (4.3.1)$$

\footnote{There is no such a symmetry property for the similarly named functions in [DW (4.0.1)].}
Proof. If we choose $A := A^1$, $\lambda := \lambda^1$ in [DW, Lem. 4.1] for some $A \in \Xi_{2n,2r}$, $\lambda \in \Lambda(n,r)$, then both formulas in [DW, Lem. 4.1(1)&(2)] are closed in $S^p(n, r)$ for $h \in [1,n)$. So, (1) and (2) are the $f$-inverse images of the two.

To see (3), we use $A^1 = (a_{i,j}^1)$ to replace $A$ in the displayed formula in [BKLW18, Lem. A.13], which holds in $S^n(n,r)$. Thus, by using the notation in [4.2.3], each term on the RHS of the formula has the form $[A^1 - E^\theta_{n,p'} + E^\theta_{n+1,p'}] = [(A - E^\theta_{n,p} + E^\theta_{n+1,p})]'$ if $p' \neq n + 1$ ($p = p'$ for $p' \leq n$ and $p = p' - 1$ for $p' \geq n + 2$) or $[A^1]$ if $p' = n + 1$ which has coefficient

$$v\sum_{j \leq p'} a^1_{n+2,j} - \sum_{j < p'} a^1_{n,j} - \sum_{j > n + 1} \delta_{j,j} [a^1_{n+2,p'} + 1] = \begin{cases} 
v^{\theta}(A,n) - \delta_{n+1,p}[a_{n+1,p} + 1], & \text{if } p' \neq n + 1; \\
v\sum_{j \leq n} a_{n,j} - \sum_{j > n} a_{n,j}, & \text{if } p' = n + 1.
\end{cases}$$

Here, in the $p' = n + 1$ case, we used the fact that column $n + 1$ or row $n + 1$ of $A^1$ has zero entries except the $(n + 1, n + 1)$ entry.

On the other hand, by [BKLW18, (A.9)], the left hand side of the formula in [BKLW18, Lem. A.13] contains a summand $v^{-\text{row}(A^1)n}[A^1]$ which is moved to the right hand side. Thus, the coefficient of $[A^1]$ is $v\sum_{j \leq a_{n+1,j}} - \sum_{j \leq n} a_{n,j} - v\sum_{i \in [1,2n]} v_{i,n}$, which equals $c_A$ since $\sum_{n+1 \leq i} a_{n,i} = \sum_{j \leq n} a_{n+1,j}$. Hence, (3) follows.

The following values will be used in the sequel: for $h \in [1,n]$, $a \in \mathbb{N}$,

$$c_{aE_{h,b+1}}^a = c_{aE_{h+1,b+h}}^a = \begin{cases} 0, & \text{if } h \neq n; \\
v^a - v^{-a}, & \text{if } h = n. \end{cases} \quad (4.3.2)$$

For $A = (a_{i,j}) \in \Xi_{2n,2r}$ and $\nu = (\nu_i) \in \Lambda(2n,m)$ ($m > 0$), we set

$$\nu \leq \text{row}_h(A) \iff \nu_i \leq a_{n,i} \forall i \in [1,2n], \text{ where } \text{row}_h(A) = (a_{h,1}, \ldots, a_{h,2n}). \quad (4.3.3)$$

When $h < n$, multiplication formulas in [BKLW18, Thm. 3.7(1)&(2)] are closed in $S^n(n,r)$. For later use, we record their counterpart in $S^p(n,r)$ as follows.

**Proposition 4.4.** If $A = (a_{i,j}) \in \Xi_{2n,2r}$, $m > 0$, $\lambda \in \Lambda(n,r - m)$, and $1 \leq h < n$, the following multiplication formulas hold in $S^p_p(n,r)$:

$$[mE^\theta_{h,b+1} + \lambda][A] = \varepsilon \sum_{\nu \in \Lambda(2n,m), \nu \leq \text{row}^\lambda(A)} \prod_{u=1}^{2n} \frac{[a_{h,u} + \nu_u]}{v_u} [A + \sum_{u=1}^{2n} \nu_u(E^\theta_{h,u} - E^\theta_{h+1,u})],$$

where $\varepsilon = \delta_{mE_{h+1} + \lambda, \text{ro}(A)}$ and $\beta^\lambda(A,h) = \sum_{j \geq p} a_{h,j} \nu_j - \sum_{j > p} a_{h+1,j} \nu_p + \sum_{j < p} \nu_j \nu_p$.

$$[E^\theta_{h+1,b} + \lambda][A] = \varepsilon' \sum_{\nu \in \Lambda(2n,m), \nu \leq \text{row}^\lambda(A)} \prod_{u=1}^{2n} \frac{[a_{h+1,u} + p_u]}{v_u} [A + \sum_{u=1}^{2n} \nu_u(E^\theta_{h,u} - E^\theta_{h+1,u})],$$

where $\varepsilon' = \delta_{mE_{h+1} + \lambda, \text{ro}(A)}$ and $\beta^\lambda(A,h) = \sum_{j \leq p} a_{h+1,j} \nu_p - \sum_{j < p} a_{h,j} \nu_p + \sum_{j > p} \nu_j \nu_p$.

It seems too complicated to write down a formula for the product $[mE^\theta_{n+1,n} + \lambda][A]$.

We now give an application of the multiplication formulas given in Lemma 4.3. The following result allows us to transfer the $S^p_p(n,r)$-bimodule structure on $\mathcal{T}_Z(n,r)$, after base change, to a $U^1(n)$-bimodule structure on $\Omega^r_{2n}$ via (3.1.3), (3.1.6), and (3.1.8).

**Theorem 4.5.** The right $\mathfrak{H}_Z(r)$-module structure on $\Omega^r_{2n}$ defined by (3.1.7) and (3.1.8) commutes with the action of $U^1(n)$. In other words, the map $\rho^r : U^1(n) \rightarrow \text{End}(\Omega^r_{2n})$ defined in (2.4.1) can be refined to

$$\rho^r = \rho^r_{n,r} : U^1(n) \rightarrow \text{End}_{\mathfrak{H}_Z(r)}(\Omega^r_{2n}). \quad (4.5.1)$$
Proof. Recall the $\mathcal{H}(r)$-action on $\Omega_{2n}^{\otimes r}$ via (3.1.7) and the $\mathcal{H}(r)$-module isomorphisms $\eta_r$ in (3.1.8). By restriction to the subalgebra $\mathcal{H}(\mathfrak{g}_r)$, $\Omega_{2n}^{\otimes r}$ becomes a right $\mathcal{H}(\mathfrak{g}_r)$-module (cf. [DDPW08] (14.6.4)) and the map $\rho_r$ in (2.3.5) induces an algebra homomorphism, denoted by $\rho_r$ again,

$$\rho_r : \mathbf{U}(\mathfrak{g}_l^{2n}) \rightarrow \text{End}_{\mathcal{H}(\mathfrak{g}_r)}(\Omega_{2n}^{\otimes r}).$$

We now prove that the restriction map $\rho_r$ sends $\mathbf{U}^*(n)$ into $\text{End}_{\mathcal{H}(r)}(\Omega_{2n}^{\otimes r})$.

We first assume $n \geq r$. In this case, $\eta_r$ induces an algebra isomorphism

$$\tilde{\eta}_r := \tilde{\eta}_{n,r} : \text{End}_{\mathcal{H}(r)}(\Omega_{2n}^{\otimes r}) \rightarrow S^r(n,r).$$

We claim that, under the linear isomorphism $\eta_r$ in (3.1.8), the action formulas on the basis $\{\omega_i \mid i \in I(2n,r)\}$ given in Proposition 2.4(1)–(4) coincide with the action formulas of certain elements in $S^r(n,r)$ on the basis $\{[A_i] \mid i \in I(2n,r)\}$. More precisely, we claim that, for all $j \in [1,n], h \in [1,n]$, and $i \in I(2n,r)$ with $\text{wt}(i) = \lambda$ (so that $\hat{\lambda} = \text{ro}(A_i)$),

1. $\eta_r(\rho_r(d_j) \cdot \omega_i) = v^{-\lambda_j} \tilde{\lambda} \cdot [A_i]$;
2. $\eta_r(\rho_r(e_h) \cdot \omega_i) = [E_{h+1}^{\theta} + \hat{\lambda} - e_{\theta,h}^{\theta}] \cdot [A_i]$;
3. $\eta_r(\rho_r(\eta_i) \cdot \omega_i) = [E_{h+1}^{\theta} + \hat{\lambda} - e_{\theta,h}^{\theta}] \cdot [A_i]$;
4. $\eta_r(\rho_r(t) \cdot \omega_i) = (E_{h+1}^{\theta} + \hat{\lambda} - e_{\theta,h}^{\theta}) + v^{-\lambda_h} \tilde{\lambda} \cdot [A_i]$.

Note that, if $\lambda_h = \text{ro}(A_i)_h = 0$ in (2), or $\lambda_{h+1} = \text{ro}(A_i)_{h+1} = 0$ in (3), then both sides are zeros since there are no components in $i$ equal $h$ or $h+1$ in these cases.

By the claim, we see from (3.1.8) that the $\mathbf{U}^*(n)$ action on $\Omega_{2n}^{\otimes r}$ commutes with the $\mathcal{H}(r)$ action transferred above. Hence, we have $\text{im}(\rho_r) \subseteq \text{End}_{\mathcal{H}(r)}(\Omega_{2n}^{\otimes r})$ in this case.

We now prove (1)–(4) in the claim. Recall the definition of $A_i$ in (3.1.2).

For Part (1), it suffices to prove $\text{ro}(A_i j) = \delta(j,i)$. This is clear since

$$\text{ro}(A_i) j = |\{l \mid l \in [1,r], a_{j,l} = 1 = \delta_{j,i}\} \cup \{l \mid l \in [2n-r+1,2n], a_{j,l} = 1\}| = |\{k \mid 1 \leq k \leq r, i_k = j\}| + |\{k \mid 1 \leq k \leq r, i_k = 2n+1-j\}| = \delta(j,i).$$

Parts (2) and (3) can be easily checked by the short multiplication formulas in Lemma 4.3(1)&(2) by mimicking part of the proof of [DW, Thm. 7.1].

Finally, we prove Part (4). By Lemma 4.3(3), we have

$$\cdot[A_i] = c_{A_i}[A_i] + \sum_{l \in [1,2n], a_{n,l} \geq 1} v^{\theta,l}[A_i - E_{n,l}^{\theta} + E_{n+1,l}^{\theta}]$$

$$= c_{A_i}[A_i] + \sum_{l \in [1,r], a_{n,l} = \delta_{n,i,l}} v^{\theta,l}[A_i - E_{n,l}^{\theta} + E_{n+1,l}^{\theta}]$$

$$+ \sum_{l \in [2n-r+1,2n]} v^{\theta,l}[A_i - E_{n,l}^{\theta} + E_{n+1,l}^{\theta}]$$

$$= c_{A_i}[A_i] + \sum_{l \in [1,r], i_l = n} v^{\theta,l}[A_i - E_{n,l}^{\theta} + E_{n+1,l}^{\theta}]$$

$$+ \sum_{l \in [1,r]} v^{\theta,l}(A_i n)_{i_l = n+1}[A_i - E_{n,l}^{\theta} + E_{n+1,l}^{\theta}]$$

$$+ \sum_{l \in [1,r]} v^{\theta,l}(A_i n)_{i_l = n+1}[A_i - E_{n+1,l}^{\theta} + E_{n+1,2n+1-l}^{\theta}].$$
We now compare this action with the action formula in Proposition 2.4(4). First, we have
\[
\eta_r(\omega_i \cdots \omega_{i_{n-1}} \omega_{n+1} \omega_{i_{n+1}} \cdots \omega_i) = [A_i - E_{n,l} + E_{n+1,l}], \quad \text{and} \\
\eta_r(\omega_i \cdots \omega_{i_{n-1}} \omega_{n+1} \omega_{i_{n+1}} \cdots \omega_i) = [A_i - E_{n,l} + E_{n+1,l}] = [A_i - E_{n+2l} + E_{n+2l+1}].
\]
It remains to verify that the corresponding coefficients are equal, i.e., to prove that
\[
\begin{align*}
(a) & \ c_{A_i} + v^{-\rho_0(A_i)n} = v^{\tau_0}; \\
(b) & \beta'_l(A_i, n) = \tau_1(l), \quad \text{if } i_l = n; \\
(c) & \beta'_{2n+1-l}(A_i, n) = \tau_1(l) + 1 \quad \text{if } i_l = n + 1.
\end{align*}
\]
Since \( c_{A_i} = v^{-\sum_{j \leq n} a_{n,j}}(v^{\sum_{j \leq n} a_{n+1,j}} - v^{-\sum_{j \leq n} a_{n,j}}) = \sum_{j \leq n} a_{n+1,j} - \sum_{j \leq n} a_{n,j} = v^{-\rho_0(A_i)n} \), it follows that \( c_{A_i} + v^{-\rho_0(A_i)n} = v^{\sum_{j \leq n} a_{n+1,j} - \sum_{j \leq n} a_{n,j}} = v^{(\sum_{k|1 \leq k \leq r,i_k=n+1} 1 - \sum_{k|1 \leq k \leq r,i_k=n})} = v^{\tau_0}, \) proving (a).

For (b), this is the \( i_l = n \) case:
\[
\beta'_l(A_i, n) = \sum_{k \leq l} a_{n+1,k} - \sum_{k < l} a_{n,k} = |\{k \mid 1 \leq k \leq l, a_{n+1,k} = \delta_{n+1,i_k} = 1\}| - |\{k \mid 1 \leq k < l, a_{n,k} = \delta_{n,i_k} = 1\}| = |\{k \mid 1 \leq k < l, i_k = n + 1\}| - |\{k \mid 1 \leq k < l, i_k = n\}| = \tau_1(l).
\]
Finally, for the \( i_l = n + 1 \) case, we have
\[
\beta'_{2n+1-l}(A_i, n) = \sum_{k \leq 2n+1-l} a_{n+1,k} - \sum_{k < 2n+1-l} a_{n,k} = |\{k \mid 1 \leq k \leq r, a_{n+1,k} = 1\}| + |\{k \mid 2n+1 - r \leq k \leq 2n+1 - l, a_{n+1,k} = 1\}| - |\{k \mid 1 \leq k \leq r, a_{n,k} = 1\}| - |\{k \mid 2n+1 - r \leq k \leq 2n+1 - l, a_{n,k} = 1\}| = |\{k \mid 1 \leq k \leq r, i_k = n + 1\}| - |\{k \mid 1 \leq k \leq r, i_k = n\}| + 1 = \tau_1(l) + 1,
\]
proving (c) and hence (4). This completes the proof for the \( n \geq r \) case.

Assume now \( n < r \). We have proved that \( \rho^r_{i,r}(U^r(n)) \subseteq \End_{\mathcal{H}(r)}(\Omega^\otimes_{2r}) \). Consider the algebra embedding \( \iota_1 : U^r(n) \rightarrow U^r(r) \) (resp., the space embedding \( \Omega_{2n} \rightarrow \Omega_{2r} \)) induced by the index embedding \( [1, n] \rightarrow [1, r] \) (resp., \( [1, 2n] \rightarrow [1, 2r] \)), \( i \mapsto r - n + i, \forall i \).

The latter induces an \( \mathcal{H}(r) \)-module embedding \( \Omega^\otimes_{2n} \rightarrow \Omega^\otimes_{2r} \) so that its image is a direct summand of the \( \mathcal{H}(r) \)-module \( \Omega^\otimes_{2r} \). Thus, we have a centraliser subalgebra embedding
\[
\iota_2 : \End_{\mathcal{H}(r)}(\Omega^\otimes_{2n}) \rightarrow \End_{\mathcal{H}(r)}(\Omega^\otimes_{2r})
\]
whose image \( \text{im}(\iota_2) = (\tilde{\eta}_{r,r})^{-1}(fS^r(r,f)) \) (see (4.5.3)), where \( f = \sum_{\lambda \in \Lambda(n,r)} [\lambda^\circ] \). Now, one sees easily the inclusion \( \rho^r_{i,r}(U^r(n)) \subseteq \iota_2(\End_{\mathcal{H}(r)}(\Omega^\otimes_{2n})) \). Hence, \( \rho^r_{i,r}(U^r(n)) \subseteq \End_{\mathcal{H}(r)}(\Omega^\otimes_{2n}) \) and \( \rho^r_{i,r} = \iota_2^{-1} \rho^r_{i,r} \iota_1 \). \( \square \)

We will see in §6 that the map \( \rho^r_{i,r} \) given in (4.5.1) is surjective.
In the next result, we identify \( \text{End}_{\mathcal{H}(r)}(\Omega_2^{\otimes r}) \) with \( S^r(n, r) \) via (4.5.3), For \( j \in [1, n] \) and \( A \in \{ E_{j,j+1}^\theta, E_{j+1,j}^\theta \} \), let
\[
O(-e_j, r) = \sum_{\lambda \in \Lambda(n, r)} v^{-\lambda_j} \lceil \lambda \rceil, \quad A(0, r) = \sum_{\mu \in \Lambda(n, r-1)} [A + \mu], \quad (4.5.4)
\]

**Corollary 4.6.** The \( \mathbb{Q}(v) \)-algebra homomorphism \( \tilde{\eta}_r \circ \rho_r : U^t(n) \to \text{End}_{\mathcal{H}(r)}(\Omega_2^{\otimes r}) \) \( \overset{\sim}{\to} \) \( S^r(n, r) \) has the following images on generators: for all \( h \in [1, n], j \in [1, n] \),
\[
d_j \mapsto O(-e_j, r), \quad e_h \mapsto E_{h+1,h}^\theta(0, r), \quad f_h \mapsto E_{h,h+1}^\theta(0, r), \quad t \mapsto E_{n+1,n}^\theta(0, r) + O(-e_n, r).
\]

**Proof.** This follows easily from the equations (1)–(4) in the proof of the theorem since the right hand sides of (1)–(4) are equal to \( O(-e_j, r)[A_i], E_{h+1,h}^\theta(0, r)[A_i], E_{h,h+1}^\theta(0, r)[A_i] \), and \( (E_{n+1,n}^\theta(0, r) + O(-e_n, r))[A_i] \), respectively. \( \Box \)

5. A Triangular Relation in \( S^r_+(n, r) \)

In this section, we develop a triangular relation for two bases of \( S^r_+(n, r) \). This is built on the determination of leading terms in certain multiplication formulas with respect to a preorder. Recall the preorder \( \preceq \) in [BKLW18] (3.22) defined on \( \Xi_N \) \( (N = 2n \text{ or } 2n+1) \): for \( A = (a_{i,j}), B = (b_{i,j}) \in \Xi_N \),
\[
A \preceq B \iff \sum_{i \leq u, j \geq v} a_{i,j} \leq \sum_{i \leq u, j \geq v} b_{i,j}, \quad \text{for all } 1 \leq u < v \leq N. \quad (5.0.1)
\]

Equivalently, for \( N = 2n \), we have
\[
A \preceq B \iff \begin{cases} 
\sum_{i \leq u, j \geq v} a_{i,j} \leq \sum_{i \leq u, j \geq v} b_{i,j}, & \text{for all } u \in [1, n], v \in [1, 2n], u < v; \\
\sum_{i \leq u, j \leq v} a_{i,j} \leq \sum_{i \leq u, j \leq v} b_{i,j}, & \text{for all } u, v \in [1, n], u > v.
\end{cases} \quad (5.0.2)
\]

We write \( A < B \) if \( A \preceq B \) and \( B \not\preceq A \). Note that, with (4.1.1), we have \( A \preceq B \iff A^\dagger \preceq B^\dagger \).

We first derive some multiplication formulas in \( S^r_+(n, r) \) (or more precisely in its centraliser subalgebra \( S^\mu(n, r) \)). With the notations \( e_n^\theta, E_n^\theta \) given in (4.2.3) and by the display [BKLW18] (5.4), the following formula holds in \( S^\mu(n, r) \): for \( m \geq 0, \lambda \in \Lambda(n, r) \), and \( \beta(i) : = i \lambda_n - (i+1)_2 \),
\[
[mE_{n,n+1}^\theta + \hat{\lambda}] \cdot [mE_{n+1,n}^\theta + \hat{\lambda}] = \sum_{i=0}^m v^{\frac{\lambda_n + i}{i}} \left[ \binom{\lambda_n + i}{i} \right] [(m - i)E_{n+2,n}^\theta + \hat{\lambda}^\dagger + i e_n^\theta]
\]
\[
= [mE_{n+2,n}^\theta + \hat{\lambda}^\dagger] + \sum_{i=1}^m v^{-\frac{(i+1)_2}{i}} \left[ \binom{\lambda_n + i}{i} \right] [(m - i)E_{n+2,n}^\theta + \hat{\lambda}^\dagger + i e_n^\theta],
\]
where we use (1.0.1) to get the second equality. Thus, for \( i \in [1, m] \), we have
\[
[(m - i)E_{n,n+1}^\theta + \hat{\lambda}^\dagger + i e_n^\theta] \cdot [(m - i)E_{n+1,n+1}^\theta + \hat{\lambda}^\dagger + i e_n^\theta] =
\]
\[
[(m - i)E_{n+2,n+2}^\theta + \hat{\lambda}^\dagger + i e_n^\theta] + \sum_{j=1}^{m-i} v^{-\frac{(i+j+1)_2}{j}} \left[ \binom{\lambda_n + i + j}{j} \right] [(m - i - j)E_{n+2,n+2}^\theta + \hat{\lambda}^\dagger + (i + j) e_n^\theta].
\]

Hence, for \( 0 \leq i \leq m \), there exist \( p_i \in \mathbb{Z} \) with \( p_0 = 1 \) such that
\[
[mE_{n+2,n}^\theta + \hat{\lambda}] = \sum_{i=0}^m p_i [(m - i)E_{n,n+1}^\theta + \hat{\lambda}^\dagger + i e_n^\theta] \cdot [(m - i)E_{n+1,n}^\theta + \hat{\lambda}^\dagger + i e_n^\theta]]. \quad (5.0.3)
\]

We first derive a transferable leading term formula in \( S^\mu(n, r) \). The “lower terms” in an expression of the form \( [M]+\text{(lower terms)} \) represents a linear combination of \([B]\) with \( B < M \).
Lemma 5.1. Suppose that $A \in \mathbb{Z}_{2n+2}^{n \times n}$ satisfies that: $a_{n,k} \geq m > 0$ for some $k \in [1, n]$ and $a_{n,j} = 0, \forall \ j < k$ and $j \geq 2n + 2 - k$. Then, for any $\tilde{\lambda} = \text{ro}(A) - m e_{n}^{\theta}$, we have in $\mathbb{S}^{n}(n,r)$

$$[mE_{n+2,n}^{\theta} + \tilde{\lambda}] \cdot [A] = [A - mE_{n,k}^{\theta} + mE_{n+2,k}^{\theta}] + \text{(lower terms)}_{\leq 4}. \quad (5.1.1)$$

**Proof.** Since $A - (m-i)E_{n,k}^{\theta} + (m-i)E_{n+2,k}^{\theta} < A - mE_{n,k}^{\theta} + mE_{n+2,k}^{\theta}$ for all $i \in [1, m]$, by (5.0.3), it suffices to prove that for every $m$ with $a_{n,k} \geq m \geq 1$, the leading term in the product $[mE_{n,n+1}^{\theta} + \tilde{\lambda}][mE_{n+1,n}^{\theta} + \tilde{\lambda}] \cdot [A]$ is $[A - mE_{n,k}^{\theta} + mE_{n+2,k}^{\theta}]$.

Let $N = 2n + 1$ and let $t^{(0)} = (t_{i}) = (t_{1}, t_{2}, \ldots, t_{N}) \in \Lambda(N, m)$, where $t_{k} = m$ and $t_{i} = 0$ for $i \neq k$. Let $\Lambda(N, m)_{0} = \Lambda(N, m) - \{t^{(0)}\}$. By [BKLW18] Thm. 3.7(b), we have

$$[mE_{n+1,n}^{\theta} + \tilde{\lambda}] \star [A] = [A - mE_{n,k}^{\theta} + mE_{n+2,k}^{\theta}] + \sum_{\substack{t \in \Lambda(N, m)_{0} \ \ t \leq \text{row}_n(A) \ \ 1 \leq u \leq N}} f_{t}[A - \sum_{u \leq N} t_{u}(E_{n,u}^{\theta} - E_{n+1,u}^{\theta})], \quad (5.1.2)$$

where $f_{t} = v^{\beta''(u)} \prod_{j < n+1} \left[ t_{j} + t_{N+1-j} \right]$ with $\beta''(t)$ as defined in [BKLW18], (3.21), and $t \leq \text{row}_n(A)$ means $t_{u} \leq a_{n,u}$ for all $u$. Note that we have $t_{n+1} = 0$, since $a_{n,n+1} = 0$. (So, the missing factor $\prod_{i=0}^{n+1} [2+1]^{i}$ in $f_{t}$ is 1.) Note also that, since $a_{n,j} = 0, \forall \ j < k$ and $j \geq N + 1 - k$, it follows that that $t_{u} = 0$ for all $u < k$ and $u \geq N + 1 - k$.

Moreover, one checks easily by the definition that $A - mE_{n,k}^{\theta} + mE_{n+1,k}^{\theta}$ is the leading term on the right hand side of (5.1.2).

Let $t^{(0)} = (t_{i}) = (t_{1}, t_{2}, \ldots, t_{N})$, where $t_{N+1-k} = m$ and $t_{i} = 0$ for $i \neq N + 1 - k$. By [BKLW18] Thm. 3.7(a), we have, for $A' = A - mE_{n,k}^{\theta} + mE_{n+1,k}^{\theta} = (a'_{ij}), \Lambda(N, m)_{0} := \Lambda(N, m) - \{t^{(0)}\}$ and some $g_{v} \in \mathbb{Z}$,

$$[mE_{n,n+1}^{\theta} + \tilde{\lambda}] \cdot [A'] = [mE_{n,n+1}^{\theta} + \tilde{\lambda}] \cdot [A - mE_{n+2,N+1-k}^{\theta} + mE_{n+1,N+1-k}^{\theta}]$$

$$= [A - mE_{n+2,N+1-k}^{\theta} + mE_{n+1,N+1-k}^{\theta}] + \sum_{\substack{t' \in \Lambda(N, m)_{0} \ \ t' \leq \text{row}_n(A') \ \ t'+(t')^{T} \leq \text{row}_n(A')}} g_{v'}[A - mE_{n,k}^{\theta} + mE_{n+1,k}^{\theta} + \sum_{\mu \leq N+1} t'_{\mu}(E_{n,\mu}^{\theta} - E_{n+1,\mu}^{\theta})]$$

$$= [A - mE_{n,k}^{\theta} + mE_{n+1,k}^{\theta}] + \sum_{\substack{t' \in \Lambda(N, m)_{0} \ \ t' \leq \text{row}_n(A') \ \ t'+(t')^{T} \leq \text{row}_n(A')}} g_{v'}[A - mE_{n,k}^{\theta} + mE_{n+1,k}^{\theta} + \sum_{\mu \leq N+1} t'_{\mu}(E_{n,\mu}^{\theta} - E_{n+1,\mu}^{\theta})],$$

where $t'+(t')^{T} = (t'_{1} + t'_{N}, t'_{2} + t'_{N-1}, \ldots, t'_{N} + t'_{1})$, and $t'+(t')^{T} \leq \text{row}_n(A')$ means $t'_{u} + t'_{N+1-u} \leq a'_{n+1,u}$ for all $u$.

Since $a'_{n+1,u} = 0$ for all $u \neq k, N + 1 - k, n + 1$, it follows that $t'_{u} = 0$ for all $u \neq k, N + 1 - k$. Thus, $m = t'_{k} + t'_{N+1-k} \leq a'_{n+1,k} = m$ and

$$A - mE_{n,k}^{\theta} + mE_{n+1,k}^{\theta} + \sum_{u} t'_{u}(E_{n,u}^{\theta} - E_{n+1,u}^{\theta})$$

$$= A - mE_{n,k}^{\theta} + mE_{n+1,k}^{\theta} + t'_{k}(E_{n,k}^{\theta} - E_{n+1,k}^{\theta}) + t'_{N+1-k}(E_{n,N+1-k}^{\theta} - E_{n+1,N+1-k}^{\theta})$$

$$= A - mE_{n,k}^{\theta} + mE_{n+1,k}^{\theta} + t'_{k}E_{n,k}^{\theta} + t'_{N+1-k}E_{n,N+1-k}^{\theta} - (t'_{k} + t'_{N+1-k})E_{n+1,k}^{\theta}$$

$$= A - (m-t'_{k})E_{n,k}^{\theta} + (m-t'_{k})E_{n,N+1-k}^{\theta} < A - mE_{n,k}^{\theta} + mE_{n+2,k}^{\theta}. \quad \text{6}$$

The condition $a_{n,j} = 0, \forall \ j \geq 2n + 2 - k$ is equivalent to $a_{n+2,j} = 0, \forall \ j \leq k$. 


Now consider a lower term in (5.1.2) of the form

\[ A'' = (a''_{ij}) : = A - \sum_{u=1}^{N} t_u(E^{\theta}_{n,u} - E^{\theta}_{n+1,u}) = A - \sum_{u=k}^{N+1-k} t_u(E^{\theta}_{n,u} - E^{\theta}_{n+1,u}). \]

The product \([mE^{\theta}_{n,n+1} + \hat{\lambda}] \cdot [A']\) is a linear combination of basis elements of the form

\[ [A - \sum_{u=1}^{N} t_u(E^{\theta}_{n,u} - E^{\theta}_{n+1,u}) + \sum_{v=1}^{N} l_v(E^{\theta}_{n,v} - E^{\theta}_{n+1,v})], \]

where all \(l_v \in \mathbb{N}\) satisfy that \(\sum l_v = m\) and \(l_v + l_{N+1-v} = a''_{n+1,v} = t_v + t_{N+1-v}\) for all \(v\). These, together with the fact that \(\sum_v t_v = m\), force \(l_v + l_{N+1-v} = t_v + t_{N+1-v}\) (forcing \(l_v = 0\) for all \(v < k\) or \(v > N + 1 - k\)). Thus, \(t_{n+1} = 0\) implies \(l_{n+1} = 0\), and \(\sum_{u=1}^{N} t_uE^{\theta}_{n+1,u} = \sum_{u=1}^{N} l_vE^{\theta}_{n+1,v}\). Hence, the matrix in (5.1.3) becomes

\[ [A - \sum_{u=k}^{N+1-k} t_uE^{\theta}_{u,n} + \sum_{v=k}^{N+1-k} l_vE^{\theta}_{v,n+1}] \in S^\mu(n, r), \]

which is clearly a term lower than \([A - mE^{\theta}_{n,k} + mE^{\theta}_{n+2,k}]\) (as \(t \neq t^{(0)}\) or \(t_k < m\)).

This completes the proof of the lemma. \(\square\)

We now return to the setup for \(S^\mu_z(n, r)\). Recall the dominance order on \(\Lambda(N, m)\): for \(\mu, \nu \in \Lambda(N, m)\),

\[ \mu \preceq \nu \iff \mu_1 \leq \nu_1, \mu_1 + \mu_2 \leq \nu_1 + \nu_2, \ldots, \mu_1 + \cdots + \mu_N \leq \nu_1 + \cdots + \nu_N, \quad (5.1.4) \]

Observe that, if \(\mu \preceq \nu\), then \(\mu_N = m - (\mu_1 + \cdots + \mu_{N-1}) \geq m - (\nu_1 + \cdots + \nu_{N-1}) = \nu_N, \mu_N + \mu_{N-1} \geq \nu_N + \nu_{N-1}, \ldots, \mu_N + \cdots + \mu_1 \geq \nu_N + \cdots + \nu_1\),Thus, if we set \(\lambda^\tau := (\lambda_1, \ldots, \lambda_N)\) for every \(\lambda = (\lambda_1, \ldots, \lambda_N) \in \Lambda(N, m)\), then we have

\[ \mu \preceq \nu \iff \mu^\tau \succeq \nu^\tau. \]

For \(h \in [1, n]\), recall \(4.3.3\) and let

\[ \Lambda(2n, m)_h := \{ \nu \in \Lambda(2n, m) \mid \nu \leq \text{row}_h(A) \} \neq \emptyset, \]

Parts (1) and (2) of the following result is a generalisation of the leading term formulas in \cite[Lem. 3.9]{BKLW18}.

**Proposition 5.2.** Let \(A = (a_{ij}) \in \Xi_{2n,2r}\), \(h \in [1, n]\), \(\lambda \in \Lambda(n, r)\) and \(m\) a positive integer. Then the following formulas with leading terms hold in \(S^\mu_z(n, r)\):

1. If \(h \neq n\) and \(A\) satisfies \(a_{h,j} = 0\), \(a_{h+1,k} > 0\), and \(a_{h+1,j'} = 0\) for all \(j \geq k, j' > k\), then, for \(\hat{\lambda} = \text{ro}(A) - mE^{\theta}_{h+1}\),

\[ [mE^{\theta}_{h,h+1} + \hat{\lambda}] \cdot [A] = [A + \sum_{j=1}^{k} \nu^{(0)}(E^{\theta}_{h,j} - E^{\theta}_{h+1,j})] + (\text{lower terms}) \preceq, \quad (5.2.1) \]

where \(\nu^{(0)}\) is the least element of \((\Lambda(2n, m)_{h+1}, \preceq)\). In particular, if \(a_{h+1,k} \geq m\), then \(\nu^{(0)} = (0, \ldots, 0, m, 0, \ldots, 0)\) and (5.2.1) becomes

\[ [mE^{\theta}_{h,h+1} + \hat{\lambda}] \cdot [A] = [A + m(E^{\theta}_{h,k} - E^{\theta}_{h+1,k})] + (\text{lower terms}) \preceq. \quad (5.2.2) \]
(2) If \( h \neq n \) and \( A \) satisfies \( a_{h,k} > 0 \), \( a_{h,j} = 0 = a_{h+1,j} \) for all \( j < k \), \( j' \leq k \), then, for \( \hat{\lambda} = \text{ro}(A) - me^0_h \),
\[
[mE^0_{h+1,h} + \hat{\lambda}] \cdot [A] = [A + \sum_{j=k}^{2n} \nu_j^{(0)}(E^0_{h+1,j} - E^0_{h,j})] + \text{(lower terms)} \leq . \tag{5.2.3}
\]
where \( \nu^{(0)} \) is the largest element of \( \Lambda(2n,m)_h, \leq \). In particular, if \( a_{h,k} \geq m \), then \( \nu^{(0)} = (0, \ldots, 0, m, 0, \ldots, 0) \) and (5.2.3) becomes
\[
[mE^0_{h+1,h} + \hat{\lambda}] \cdot [A] = [A + m(E^0_{h+1,k} - E^0_{h,k})] + \text{(lower terms)} \leq . \tag{5.2.4}
\]
(3) If \( A \) satisfies \( a_{n,k} \geq m > 0 \) for some \( k \in [1,n] \) and \( a_{n,j} = 0, \forall j < k \) and \( j \geq 2n+1-k \), then, for any \( \hat{\lambda} = \text{ro}(A) - me^0_n \), we have in \( S^1_n(n,r) \)
\[
[mE^0_{n+1,n} + \hat{\lambda}] \cdot [A] = [A + m(E^0_{n+1,k} - E^0_{n,k})] + \text{(lower terms)} \leq .
\]
Proof. Part (3) is the \( \hat{f} \)-pullback of (5.1.1). We now prove (1). The proof of (2) is symmetric.

By the hypothesis, each \( \nu \in \Lambda(2n,m)_{h+1} \) has the form \( \nu = (\nu_1, \ldots, \nu_k, 0, \ldots, 0) \). Thus, by Proposition 4.4(1), the left hand side of (5.2.1) is a linear combination of \( [A_\nu] \), where \( A_\nu = A + \sum_{j=1}^k \nu_j(E^0_{h,j} - E^0_{h+1,j}) \). Putting \( A_\nu = (m_{i,j}) \) and \( A_{\nu^{(0)}} = (m^{(0)}_{i,j}) \), we have for all \( i < j \),
\[
m_{i,j} = \begin{cases} 
a_{i,j} + \nu_j(\delta_{h,i} - \delta_{h+1,i}), & \text{if } i \in [1,n]; \\
a_{i,j} + \nu_{2n+1-j}(\delta_{2n+1-h,i} - \delta_{2h-h,i}), & \text{if } i \in [n+1,2n]. 
\end{cases}
\]
We now check (5.0.1). For all \( u < v \) with \( u \in [1,n] \), since \( \sum_{i \leq u}(\delta_{h,i} - \delta_{h+1,i}) = \begin{cases} 1, & \text{if } u = h; \\
0, & \text{otherwise.} \end{cases} \)
\[
\sum_{i \leq u < v \leq j} (m^{(0)}_{i,j} - m_{i,j}) = \sum_{j \geq v} \left( \sum_{i \leq u} (\nu^{(0)}_j - \nu_j) \sum_{i \leq u} (\delta_{h,i} - \delta_{h+1,i}) \right) = \begin{cases} \sum_{j=v}^k (\nu^{(0)}_j - \nu_j), & \text{if } u = h < v \leq k; \\
0, & \text{otherwise}, \end{cases}
\]
which is non-negative since \( (\nu^{(0)})^r \geq \nu^r \).

For all \( u < v \) with \( u \in [n+1,2n] \), since \( \sum_{n < i \leq u} (\delta_{2n+1-h,i} - \delta_{2h-h,i}) = \begin{cases} -1, & \text{if } u = 2n - h; \\
0, & \text{otherwise} \end{cases} \),
\[
\sum_{i \leq u < v \leq j} (m^{(0)}_{i,j} - m_{i,j}) = \sum_{n < i \leq u} (m^{(0)}_{i,j} - m_{i,j}) = \sum_{j \geq v} \left( \sum_{n < i \leq u} (\nu^{(0)}_{2n+1-j} - \nu_{2n+1-j}) \right) \\
\cdot \sum_{n < i \leq u} (\delta_{2n+1-h,i} - \delta_{2h-h,i}) = \begin{cases} -\sum_{j \leq 2n+1-u} (\nu^{(0)}_j - \nu_j'), & \text{if } u = 2n - h; \\
0, & \text{otherwise}, \end{cases}
\]
which is non-negative as \( \nu^{(0)} \leq \nu \). This completes the proof of (1). \( \square \)

Note that the two special cases in (5.2.2) and (5.2.4) are extracted from [BKLW18 Lem. 3.9] via the isomorphism \( \hat{f} \) in (1.2.1).\(^7\) Note also that the least/largest elements \( \nu^{(0)} \) and \( \nu^{(0')} \) in \( \Lambda(2n,m)_{h+1} \) and \( \Lambda(2n,m)_h \), respectively, have the form
\[
\nu^{(0)} = (0, \ldots, 0, a_{h+1,j} - a, a_{h+1,j+1}, \ldots, a_{h+1,k}, 0, \ldots, 0),
\nu^{(0')} = (0, \ldots, 0, a_{h,k}, a_{h,k+1}, \ldots, a_{h,j' + 1}, a_{h,j'} - a', 0, \ldots, 0),
\]
\(\square\)

\(^7\)We modified their “\( \cdots = R \)” condition to a “\( \cdots \geq m \)” condition.
where \( j \) (resp. \( j' \)) is the index such that \( \sum_{i=j+1}^{k} a_{h+1,i} < m \leq \sum_{i=j}^{k} a_{h+1,i} \) (resp., \( \sum_{i=k}^{j'+1} a_{h,i} < m \leq \sum_{i=k}^{j} a_{h,i} \)) and \( a = \sum_{i=j}^{k} a_{h+1,i} - m \) (resp., \( a' = \sum_{i=k}^{j'} a_{h,i} - m \)).

\[
\mathcal{T}_N = \{ (i, h, j) \mid 1 \leq j \leq h < i \leq N \}. \tag{5.2.5}
\]

We order the set as in [BKLW18, Thm. 3.10]:

\[
(i, h, j) \leq (i', h', j') \iff i < i' \text{ or } i = i', j < j' \text{ or } i = i', j = j', h > h'. \tag{5.2.6}
\]

For example, \( \mathcal{T}_3 \) has the following order:

\[
\mathcal{T}_3 = \{ (2, 1, 1), (3, 2, 1), (3, 1, 1), (3, 2, 2), (4, 3, 1), (4, 2, 1), (4, 1, 1), (4, 3, 2), (4, 2, 2), (4, 3, 3) \}.
\]

Like the construction in [BKLW18, Thm. 3.10], for \( A \in \Xi_{2n,2r} \) and the largest element \((N, N-1, N-1)\) in \( \mathcal{F}_{2n} \), where \( N = 2n \), let \( D_{N,N-1,N-1} = \operatorname{diag}(\alpha(A) - a_{N,N-1}E_{N-1}) \) so that \( \operatorname{co}(D_{N,N-1,N-1} + a_{N,N-1}E_{N-1}) = \alpha(A) \). Inductively, suppose \( D_{i',h',j'} \) is defined and \((i, h, j)\) is an immediate predecessor of \((i', h', j')\), then the diagonal matrix \( D_{i,h,j} \) is uniquely defined by the equation

\[
\operatorname{co}(D_{i,h,j} + a_{i,j}E_{h+1,h}) = \operatorname{ro}(D_{i',h',j'} + a_{i',j'}E_{h'+1,h'}). 
\]

Note that, for the least element \((2, 1, 1)\) in \( \mathcal{F}_{2n} \), \( D_{2,1,1} \) is the diagonal matrix uniquely defined by \( \operatorname{co}(D_{2,1,1} + a_{2,1}E_{2,1}) = \operatorname{ro}(D_{3,2,1} + a_{3,1}E_{3,2}) \). In particular, \( \operatorname{ro}(D_{2,1,1} + a_{2,1}E_{2,1}) = \alpha(A) \).

The following result is the type \( C \) counterpart of similar results for type \( B \) in [BKLW18, Thm. 3.10] and for type \( A \) in [BLM90, 3.5]. In [BKLW18, §5.4], a similar triangular relation is developed via the twin products [\( D_{i+n+1,j} + a_{i,j}E_{n+2,n+1} \), \( D_{i,n+j} + a_{i,j}E_{n+1+n} \)] within the centralizer subalgebra \( S^\bullet(n, r) \) of \( S^\bullet_{2n}(n, r) \). Our version is independent of \( S^\bullet(n, r) \) and simplify their version by using only the leading term of a twin product, i.e., by dropping a long tail. This becomes possible thanks to the new leading term formulas in Proposition 5.2.3.

**Theorem 5.3.** For any \( A = (a_{i,j}) \in \Xi_{2n,2r} \), the following triangular relation holds in \( S^\bullet_{2n}(n, r) \):

\[
\mathbf{m}(A) := \prod_{(i,j) \in (\mathcal{F}_{2n}, \leq)} [D_{i,h,j} + a_{i,j}E_{h+1,h}] = [A] + (\text{lower terms}), \tag{5.3.1}
\]

**Proof.** Similar to that of the type \( A/B \) case, the proof is standard by repeatedly using the leading term formulas in Proposition 5.2. We use the example for \( n = 2 \) above to illustrate it. From the set \( \mathcal{T}_4 \) above, there are 10 factors in the product:

\[
[D_{2,1,1} + a_{2,1}E_{2,1}] \cdot 9 [D_{3,2,1} + a_{3,1}E_{3,2}] \cdot 8 [D_{3,1,1} + a_{3,1}E_{3,2}] \cdot 7 [D_{3,2,2} + a_{3,2}E_{3,2}] \\
\cdot 6 [D_{4,3,1} + a_{4,1}E_{4,2}] \cdot 5 [D_{4,2,1} + a_{4,1}E_{4,2}] \cdot 4 [D_{4,1,1} + a_{4,1}E_{4,2}] \\
\cdot 3 [D_{4,3,2} + a_{4,2}E_{4,2}] \cdot 2 [D_{4,2,2} + a_{4,2}E_{4,2}] \cdot 1 [D_{4,3,3} + a_{4,3}E_{4,2}] \cdot 0 [\operatorname{co}(A)]
\]

Here we have used the fact \( E_{h+1,h}^g = E_{h+1,h}^g \). We make the product in the order as indicated. Each step gives a leading term by the lemma above. The following 10 matrices is the leading term of the 10 multiplications ordered from 0 to 9. Recall \( a_{i,j} = a_{5-1,5-j} \).

\[
\begin{pmatrix}
0 & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & a_{12} & a_{13} & \bullet \\
\bullet & a_{13} & a_{14} & \bullet \\
\bullet & a_{41} & a_{42} & \bullet \\
\bullet & a_{42} & a_{43} & \bullet \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & a_{12} & a_{13} & a_{14} \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & a_{41} & \bullet & \bullet \\
\bullet & a_{42} & \bullet & \bullet \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & a_{12} & a_{13} & a_{14} \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & a_{41} & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & a_{12} & a_{13} & a_{14} \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & a_{41} & \bullet & \bullet \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & a_{12} & a_{13} & a_{14} \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & a_{12} & a_{13} & a_{14} \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & a_{12} & a_{13} & a_{14} \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & a_{12} & a_{13} & a_{14} \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & a_{12} & a_{13} & a_{14} \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & a_{12} & a_{13} & a_{14} \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & a_{12} & a_{13} & a_{14} \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & a_{12} & a_{13} & a_{14} \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & a_{12} & a_{13} & a_{14} \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\end{pmatrix}
\]
Here the \( j \)th \( \bullet \) on the diagonal of the first matrix is the sum of \( j \)th column of \( A \) which decreases to \( \bullet = a_{i,j} \) in the last matrix, an arrow \( \uparrow_i \) or \( \downarrow_j \) below or above an entry tells how the entry is moved up or down, and \( i \) indicates the \( j \)th multiplication. Note that steps 1, 4, 6, and 8 used the leading term formula in Proposition 5.2(3).

**Corollary 5.4.** The \( \mathbb{Z} \)-algebra \( S^t_\mathbb{Z}(n, r) \) is generated by the elements \( \hat{\Lambda}(\lambda) \) \((\lambda \in \Lambda(n, r))\), \( [E^\theta_{h,j} + \hat{p}_i] \) \((h \in [1, 2n], \mu \in \Lambda(n, r - 1))\), and has a new basis \( \{m(A) | A \in \Xi_{2n, 2r}\} \) which is triangularly related to the basis \( \{[A] | A \in \Xi_{2n, 2r}\} \).

The following result requires the generalised leading term in Proposition 5.2(1)&(2) and will be needed in §7.

**Corollary 5.5.** Maintain the notations in Theorem 5.3. If several factors of the form \([D_{i,n+j} + a_{i,j}E^\theta_{n+1,n}]\) in the product (5.3.1) are replaced by \([D'_{i,n+j} + (a_{i,j} - s)E^\theta_{n+1,n}]\) with \(0 < s \leq a_{i,j}\), then the resulting product is either 0 or a linear combination of \([B]\) with \(B \prec A\).

*Proof.* The multiplication by \([D'_{i,n+j} + (a_{i,j} - s)E^\theta_{n+1,n}]\), if nonzero, moves \(a_{i,j} - s\) from \((n, j)\) position to \((n+1, j)\) position. Thus, the resulting leading term is (strictly) \(\prec\) the corresponding leading term when multiplied by \([D_{i,n+j} + a_{i,j}E^\theta_{n+1,n}]\). Now applying (5.2.1) and (5.2.3) to the multiplication by the next factor(s) produces leading terms that are \(\prec\) the corresponding leading term in the original product.

### 6. Long Multiplication Formulas in \( S^t(n, r) \)

Recall the elements in (4.5.4) that define the actions of the generators of \( U^t(n) \) on the tensor space \( \Omega^\otimes_{2n} \). We now introduce the general version of such elements and extend the short multiplication formulas given in §4 to these “long” elements defined below.

Recall from (3.0.4) the map \( \lambda \mapsto \hat{\lambda} \) from \( \Lambda(n, r) \) to \( \Lambda(2n, 2r) \) and, for \( j, j' \in \mathbb{Z}^n \), let

\[
j \cdot j' = j_1j'_{1} + j_2j'_{2} + \cdots + j_Nj'_{N}.
\]

Associated with \( A \in \Xi^\text{diag}_{2n} \) and \( j \in \mathbb{Z}^{2n} \), we define the following elements in \( S^t_\mathbb{Z}(n, r) \) (cf. [DW (4.1.1)]):

\[
A(j, r) = \begin{cases} 
\sum_{\lambda \in \Lambda(n, r-\frac{|j|}{2})} v^{\lambda j}[A + \hat{\lambda}], & \text{if } |A| \leq 2r, \\
0, & \text{if } |A| > 2r,
\end{cases}
\]

(6.0.1)

For any \( j = (j_1, \ldots, j_n, j_{n+1}, \ldots, j_{2n}) \in \mathbb{Z}^{2n} \), let

\[
j^* = (j_1 + j_{2n}, \ldots, j_n + j_{n+1}) \in \mathbb{Z}^n, \quad j^0 = (j_1, \ldots, j_n, 0, j_{n+1}, \ldots, j_{2n}) \in \mathbb{Z}^{2n+1},
\]

(6.0.2)

Then \( \hat{\lambda} \cdot j = \lambda \cdot j^* \). Thus, \( A(j, r) = A(j^*, r) \) where \( j^* \) is regarded as an element of \( \mathbb{Z}^n \) by adding \( n \) zeros at the end. In particular, if \( O \) denotes the zero matrix in \( \Xi_{2n} \), then \( O(e_i, r) = O(e_{2n+1-i}, r) \) for all \( i \in \{1, n\} \).

Recall the idempotent \( e = \sum_{\lambda \in \Lambda(n,r)}[\text{diag}(\hat{\lambda})] \in S^t_\mathbb{Z}(n, r) \) and the algebra isomorphism \( \tilde{f} : S^t_\mathbb{Z}(n, r) \rightarrow eS^t_\mathbb{Z}(n, r)e \)

**Lemma 6.1.** For \( A \in \Xi^\text{diag}_{2n, 2r}, j \in \mathbb{Z}^{2n}, \) let \( A^t \) be defined as in (4.1.1) and \( A^t(j^0, r) \in S^t_\mathbb{Z}(n, r) \) defined in [DW (4.1.1)]. Then \( A(j, r) = \tilde{f}^{-1}(eA^t(j^0, r)e) \).
Proof. By definition, we may assume $|A| \leq 2r$ and so $A^\dagger(j^o, r) = \sum_{\mu \in \Lambda(n+1, r - \frac{|A|}{2})} \nu^\beta \tilde{\mu}^\rho [A^\dagger + \tilde{\mu}]$, where $\tilde{\mu} = (\mu_1, \ldots, \mu_n, 2\mu_{n+1} + 1, \mu_n, \ldots, \mu_1)$, then

$$e[A^\dagger + \tilde{\mu}] e = \begin{cases} 0, & \text{if } \mu_{n+1} > 0; \\
[A^\dagger + \lambda^\dagger], & \text{if } \mu_{n+1} = 0,
\end{cases}$$

where $\lambda = (\mu_1, \ldots, \mu_n)$. The assertion now follows from the fact that $\tilde{\mu} \cdot j^o = \lambda \cdot j$.

For $1 \leq h \leq n$, put

$$\alpha_h = e_h - e_{h+1}, \quad \alpha_h^- = -e_h + e_{h+1} \in \mathbb{Z}^{2n}.$$ (6.1.1)

**Theorem 6.2.** Maintain the notations introduced above. For $A = (a_{i,j}) \in \mathbb{Z}^{2n}_{\text{diag}}$, $h \in [1, n)$, and $j = (j_1, j_2, \ldots, j_{2n}) \in \mathbb{Z}^{2n}$, let $\beta_p(A, h), \beta'_p(A, h)$ be defined as in (4.2.4). Then the following multiplication formulas hold in $S^\prime(n, r)$ for all $r \geq \frac{|A|}{2}$:

1. $O(j, r)A(j', r) = \nu^{\text{ro}(A)}jA(j + j', r), \quad A(j', r)O(j, r) = \nu^{\text{co}(A)}jA(j + j', r);

2. $E_{h,h+1}^\theta(0, r) \cdot A(j, r) = \sum_{1 \leq p < h \atop a_{h,p} \geq 1} v^{\beta_p(A,h)}[a_{h,p} + 1](A + E^\theta_{h,p} - E^\theta_{h+1,p})(j + \alpha_h, r) + \varepsilon \cdot \frac{v^{\beta_h(A,h) - j_h - j_{h+1} - h}}{v - v^{-1}} \left( (A - E^\theta_{h+1,h})(j + \alpha_h, r) - (A - E^\theta_{h,h+1})(j + \alpha_h^-, r) \right) + \nu^{\beta_h(A,h) + j_h + j_{h+1} + j_{2n - h}}[a_{h,h+1} + 1](A + E^\theta_{h,h+1})(j, r) + \sum_{h+1 < p \leq 2n \atop a_{h,p} \geq 1} v^{\beta_p(A,h)}[a_{h,p} + 1](A + E^\theta_{h,p} - E^\theta_{h+1,p})(j, r),$

where $\varepsilon = \delta^\leq_{\alpha, h+1}$ is given in (4.3.1).

3. $E_{h+1,h}^\theta(0, r) \cdot A(j, r) = \sum_{1 \leq p < h \atop a_{h,p} \geq 1} v^{\beta_p(A,h)}[a_{h+1,p} + 1](A - E^\theta_{h,p} + E^\theta_{h+1,p})(j, r) + \varepsilon' \cdot \frac{v^{\beta_h(A,h) + j_h + j_{2n - h}}[a_{h+1,h} + 1](A + E^\theta_{h,h+1})(j, r)}{v - v^{-1}} + \sum_{h+1 < p \leq 2n \atop a_{h,p} \geq 1} v^{\beta_p(A,h)}[a_{h+1,p} + 1](A - E^\theta_{h,p} + E^\theta_{h+1,p})(j - \alpha_h, r),$

where $\varepsilon' = \delta^\leq_{\alpha, h,h+1}$.

4. $E_{n+1,n}^\theta(0, r) \cdot A(j, r) = \sum_{1 \leq p < n+1 \atop a_{n+1,i} \geq 1} v^{\delta^\leq_{n+1}}[a_{n+1,i} + 1](A - E^\theta_{n,i} + E^\theta_{n+1,i})(j, r) + \delta^\leq_{\alpha, n+1} \frac{v^{\nu_i}}{v - v^{-1}} \left( (A - E^\theta_{n,n+1})(j, r) - (A - E^\theta_{n+1,n})(j - e_n^\theta, r) \right) + c_A A(j - e_n, r)$

where $c_A$ is defined in (4.3.1).
Proof. For the $h < n$ case, by using Lemma 4.3(1) and (2), the formulas (1)–(3) can be proved similarly to that of [DWW] Thm. 4.2. Alternatively, they can also be obtained by applying Lemma 6.1 to the formulas (1)–(3) in [DWW] Thm. 4.2 using $O^1(j^0, r)$, $E^\theta_{h,h+1}(0^0, r)$, $E^\theta_{h+1,h}(0^0, r)$, and $A^1(j^0, r)$.

We now use Lemma 4.3(3) to prove (4), the $h = n$ case. By the definition of $A(j, r)$,

$$E^\theta_{n,n+1}(0, r) \cdot A(j, r) = \left( \sum_{\nu \in \Lambda(n,r-1)} [E^\theta_{n,n+1} + \nu]\right) \left( \sum_{\mu \in \Lambda(n,r-|A|)} \nu^\mu [A + \mu] \right)$$

$$= \sum_{\mu \in \Lambda(n,r-|A|)} \nu^\mu \left( \sum_{\nu \in \Lambda(n,r-1)} [E^\theta_{n,n+1} + \nu]\right) [A + \mu] \quad (6.2.1)$$

$$= \sum_{\mu \in \Lambda(n,r-|A|)} \nu^\mu [E^\theta_{n,n+1} - E^\theta_{n,n} + ro(A) + \mu] \cdot [A + \mu].$$

Here, by (3.1.1), the only nonzero terms satisfy $co(E^\theta_{n,n+1}) + \nu = ro(A) + \mu$. Write $A + \mu = (\hat{a}_{i,j}) \in \Xi_{2n,2r}$. By Lemma 4.3(3),

$$[E^\theta_{n,n+1} - E^\theta_{n,n} + ro(A) + \mu] \cdot [A + \mu] = c_A + \mu \sum_{i \in [1,2n], a_{i,j}^\mu \geq 1} \nu^\mu [A + \mu]$$

Since $a_{i,j}^\mu = \hat{\mu}_i$, $a_{i,j}^\mu = a_{i,j}^\nu$ for $i \neq j$, and $\hat{\mu}_{n+1} = \hat{\mu}_n$, it follows that $\beta_p(A + \mu, n) = \beta_p(A, n)$ for all $p \in [1,2n]$, and $c_A + \mu = v^\mu c_A$. Hence, substituting into (6.2.1) gives the last term in (4):

$$\sum_{\mu \in \Lambda(n,r-|A|)} \nu^\mu [A + \mu] = c_A \sum_{\mu \in \Lambda(n,r-|A|)} \nu^\mu [A + \mu] = c_A A(j - e_n, r).$$

On the other hand, substituting the summation into (6.2.1) yields

$$\sum_{\mu \in \Lambda(n,r-|A|)} \nu^\mu \left( \sum_{i \in [1,2n], a_{i,j}^\mu \geq 1} \nu^\mu [A + \mu - E^\theta_{n,i} + E^\theta_{n+1,i}] \right)$$

$$= \sum_{i \neq n,n+1, a_{i,j}^\mu \geq 1} \nu^\mu [A + \mu - E^\theta_{n,i} + E^\theta_{n+1,i}]$$

$$+ \sum_{\mu \in \Lambda(n,r-|A|), \hat{\mu}_i \geq 1} \nu^\mu [A + \mu - E^\theta_{n,i} + E^\theta_{n+1,i}]$$

$$+ \sum_{\mu \in \Lambda(n,r-|A|), \hat{\mu}_i \geq 1} \nu^\mu [A + \mu - E^\theta_{n,i} + E^\theta_{n+1,i}]$$

Since $|A| = |A - E^\theta_{n,i} + E^\theta_{n+1,i}|$, the inner summation of the double sum term in (6.2.2) gives, for $i \neq n, n+1$, $(A - E^\theta_{n,i} + E^\theta_{n+1,i})(j, r)$. So, the double sum gives the summation term in (4).
For the second summation term of (6.2.2), since \( E_{n,n}^\theta = \text{diag}(e_n^\theta) \), we have

\[
\sum_{\mu \in \Lambda(n,r-\lfloor \frac{\vert A \vert}{2} \rfloor), \mu \geq 1} u_{\beta_n^\theta(A,n)+j_n+j_n+1}^{\beta_n^\theta(A,n)+j_n+j_n+1}[a_{n+1,n}+1][A + E_{n+1,n}^\theta + \mu - e_n^\theta]
\]

\[
= u_{\beta_n^\theta(A,n)+j_n+j_n+1}^{\beta_n^\theta(A,n)+j_n+j_n+1}[a_{n+1,n}+1] \sum_{\mu \in \Lambda(n,r-\lfloor \frac{\vert A \vert}{2} \rfloor), \mu \geq 1} u^{(\tilde{\mu} - e_n^\theta)} [A + E_{n+1,n}^\theta + \mu - e_n^\theta]
\]

\[
= u_{\beta_n^\theta(A,n)+j_n+j_n+1}^{\beta_n^\theta(A,n)+j_n+j_n+1}[a_{n+1,n}+1](A + E_{n+1,n}^\theta)(j, r),
\]

giving the second term in (4).

Finally, if \( a_{n+1,n} \geq 1 \), then, by noting \( E_{n+1,n}^\theta = E_{n,n}^\theta = \text{diag}(e_n^\theta) \) and \( \tilde{\mu}_n = \tilde{\mu}_{n+1} \), the last summation in (6.2.2) has the form

\[
\sum_{\mu \in \Lambda(n,r-\lfloor \frac{\vert A \vert}{2} \rfloor)} u_{\beta_n^\theta(A,n)+j_n+j_n}^{\beta_n^\theta(A,n)+j_n} [a_{n+1,n}+1][A + \mu - E_{n,n+1}^\theta + e_n^\theta]
\]

\[
= \frac{u_{\beta_n^\theta(A,n)+j_n+j_n}^{\beta_n^\theta(A,n)+j_n} [a_{n+1,n}+1]}{1 - v^{-2}} \sum_{\mu \in \Lambda(n,r-\lfloor \frac{\vert A \vert}{2} \rfloor)} u^{(\tilde{\mu} + e_n^\theta)} (1 - v^{-2(\tilde{\mu}_{n+1}+1)}) [A - E_{n,n+1}^\theta + \mu + e_n^\theta]
\]

\[
= \frac{u_{\beta_n^\theta(A,n)+j_n+j_n}^{\beta_n^\theta(A,n)+j_n} [a_{n+1,n}+1]}{v - v^{-1}} \sum_{\mu \in \Lambda(n,r-\lfloor \frac{\vert A \vert}{2} \rfloor)} (u^{(\tilde{\mu} + e_n^\theta)} - u^{(\tilde{\mu} + e_n^\theta)} (j - e_n^\theta)) [A - E_{n,n+1}^\theta + \mu + e_n^\theta]
\]

\[
= \frac{u_{\beta_n^\theta(A,n)+j_n+j_n}^{\beta_n^\theta(A,n)+j_n} [a_{n+1,n}+1]}{v - v^{-1}} \sum_{\lambda \in \Lambda(n,r-\lfloor \frac{\vert A \vert}{2} \rfloor + 1)} (v^{\tilde{\lambda} - j - e_n^\theta}) [A - E_{n,n+1}^\theta + \tilde{\lambda}],
\]

giving the third term in (4). This completes the proof of the theorem. \( \square \)

We may now compute certain divided powers in \( S'(n,r) \).

**Corollary 6.3.** Let \( m \) be a positive integer.

1. If \( h \in [1, n) \), then we have, for all \( r \geq m \),

\[
\frac{E_{h,h+1}^\theta(0,r)^m}{[m]} = (mE_{h,h+1}^\theta(0,r))^m = (mE_{h+1,h}^\theta(0,r))^m = (mE_{h+1,h}^\theta(0,r)).
\]

2. If \( h = n \), then there exist \( f_{s,t} \in \mathbb{Q}(v) \), \( j_{s,t} \in \mathbb{Z}^{2n} \) for \( s \in [0, m), 1 \leq t \leq n_a \), independent of \( r \geq m \), such that

\[
\frac{E_{n,n+1}^\theta(0,r)^m}{[m]} = (mE_{n,n+1}^\theta(0,r)) + \sum_{s=0}^{m-1} \sum_{t=1}^{n_a} f_{s,t}(sE_{n+1,n}^\theta)(j_{s,t}, r).
\]

3. For \( A = (a_{i,j}) \in \mathbb{Z}_{2n}^{\text{diag}} \) and \( j = (j_1, j_2, \ldots, j_{2n}) \in \mathbb{Z}^{2n} \), there exist finitely many \( B_a \in \mathbb{Z}_{2n}^{\text{diag}}, j^{(b)} \in \mathbb{Z}^{2n}, \) and \( g_{B_a,j^{(b)}} \in \mathbb{Q}(v) \) such that, for all \( r \geq \frac{|A|}{2} \),

\[
(mE_{n+1,n}^\theta(0,r)) \cdot A(j, r) = \sum_{a,b} g_{B_a,j^{(b)}} B_a(j^{(b)}, r).
\]
Proof. The first assertion is clear; see the proof of [DW Cor. 4.5]. We now prove the $h = n$ case. For $A = E^\theta_{n,n+1}$, we first observe that, for a positive integer $a$,

$$
\beta'_p(aE^\theta_{n,n+1}, n) = \sum_{l \leq p} a_{n+1,l} - \sum_{l < p} a_{n,l} = \begin{cases} 
0, & \text{if } p < n; \\
 a, & \text{if } p = n, n + 1; \\
0, & \text{if } p > n + 1.
\end{cases}
$$

Thus, by Theorem 6.2(4) and (4.3.2), we have, for any positive integer $a$,

$$
E^\theta_{n,n+1}(0, r)(aE^\theta_{n,n+1})(j, r)
= v^{a+j_n+j_{n+1}}((a + 1)E^\theta_{n,n+1})(j, r)
+ \frac{v^{a-j_n-j_{n+1}}}{v-1}(((a - 1)E^\theta_{n,n+1})(j, r) - ((a - 1)E^\theta_{n,n+1})(j - e_n^\theta, r))
+ (v^a - v^{-a})(aE^\theta_{n,n+1})(j - e_n, r).
$$

Hence, for $j = 0$, we have

$$
E^\theta_{n,n+1}(0, r)(aE^\theta_{n,n+1})(0, r) = [a + 1][(a + 1)E^\theta_{n,n+1}(0, r)] + (v^a - v^{-a})(aE^\theta_{n,n+1})(-e_n, r)
= [a + 1][(a + 1)E^\theta_{n,n+1}(0, r)] + (v^a - v^{-a})(aE^\theta_{n,n+1})(-e_n, r).
$$

Now assertion (2) follows easily from an induction on $m$.

Finally, for assertion (3), let $E^\theta_{n,n+1}(0, r)(m) := E^\theta_{n,n+1}(0, r)^{m}$. By (2), we write $(mE^\theta_{n,n+1})(0, r)$ as a linear combination of $E^\theta_{n,n+1}(0, r)(m-i)O(j, a, r)$ ($i \in [0, m], a \in [1, p_i]$) and so (3) follows from Theorem 6.2(1), (4).

Corollary 6.4. For $h \in [1, n-1]$ and $j \in \mathbb{Z}^{2n}$, we have

\begin{enumerate}
  \item $O(j, r)E^\theta_{n,n+1}(0, r) = E^\theta_{n,n+1}(0, r)O(j, r)$,
  \item $E_{h,h+1}(0, r)E^\theta_{n,n+1}(0, r) = E^\theta_{n,n+1}(0, r)E_{h,h+1}(0, r)$,
  \item $E_{h,h+1}(0, r)E^\theta_{n,n+1}(0, r) = E^\theta_{n,n+1}(0, r)E_{h,h+1}(0, r)$.
\end{enumerate}

Proof. By Theorem 5.2(1), $O(j, r)E^\theta_{n,n+1}(0, r) = v^{E^\theta_{n,n+1}(0, r)}E^\theta_{n,n+1}(j, r) = E^\theta_{n,n+1}(0, r)O(j, r)$, proving (1). For (2), since $h < n - 1$, all $\beta_{h+1}(E^\theta_{n,n+1}, h), \beta'_h(E^\theta_{n,n+1}, n)$, and $\beta_{h+1}(E^\theta_{n,n+1}, n)$ are 0. Thus, by Theorem 5.2(2)&(4),

$$
E^\theta_{h,h+1}(0, r)E^\theta_{n,n+1}(0, r) = v^{E^\theta_{n,n+1}(h)}(E^\theta_{h,h+1} + E^\theta_{n,n+1})(0, r) = (E^\theta_{n,n+1} + E^\theta_{h,h+1})(0, r)
= v^{E^\theta_{n,n+1}(h)}(E^\theta_{h,h+1} + E^\theta_{n,n+1})(0, r) + cE^\theta_{h,h+1}E^\theta_{h,h+1}(-e_n, r)
= E^\theta_{n,n+1}(0, r)E^\theta_{h,h+1}(0, r),
$$

as desired. The proof of (3) is similar. \hfill \Box

We end this section with an application by showing the map in (4.5.1) is surjective. For $i \in [1, n], m \in \mathbb{N}$, let $k_i,r := O(e_i, r)$, and $\begin{bmatrix} k_i,r \end{bmatrix}_m$ as defined in (1.0.2). Then, for any $\lambda \in \Lambda(n, r)$, we have in $S^*_n(n, r)$

$$
\begin{bmatrix} k \end{bmatrix} := \prod_{i=1}^{n} \begin{bmatrix} k_i,r \end{bmatrix} = [\lambda].
$$

See [DW, Lemma 6.4] for a proof.
Corollary 6.5. The $\rho'_r$ in \ref{4.5.1} is surjective.

Proof. By Corollaries 6.3 and 4.6, elements $[\tilde{\lambda}]$, $(mE^q_{h+1,h})(0,r)$, $(mE^q_{h,h+1})(0,r)$ are all in the image of $\tilde{\eta}_r \circ \rho'_r$, where $\tilde{\eta}_r$ is given in \ref{4.5.3}. Thus, for any $A \in \mathbb{Z}_{2n,2r}$, Theorem 5.3 implies that

$$\prod_{(i,j) \in \mathbb{T}_{2n,\leq}} (a_{i,j} E^q_{h+1,h})(0,r) \cdot [\co(A)] = [A] + (\text{lower terms})_\leq,$$

where the product is taken over the order $\leq$ on the set $\mathbb{T}_{2n}$ defined in \ref{5.2.5} and \ref{5.2.6}. Hence, all $[A] \in \text{im}(\tilde{\eta}_r \circ \rho'_r)$. Consequently, $\rho'_r$ is surjective. \hfill $\square$

We remark that the epimorphism was first obtained by Bao–Wang in \cite[Thm. 5.4]{BW18}.

7. A BLM type construction for $U^i(n)$

We are ready to give a new presentation for $U^i(n)$ via a basis and multiplication formulas of basis elements by generators.

Consider the $\mathbb{Q}(v)$-algebra of the direct product of $S^i(n, r)$:

$$S^i(n) := \prod_{r \geq 0} S^i(n, r).$$

For all $A \in \mathbb{Z}^{0\text{diag}}_{2n}$ and $j \in \mathbb{Z}^{2n}$, write

$$A(j) = (A(j, r))_{r \in \mathbb{N}}.$$

The algebra homomorphisms in \ref{4.5.1} and isomorphisms in \ref{4.5.3} induce a homomorphism and isomorphism, respectively,

$$\rho^i := (\rho'_r)_{r \in \mathbb{N}} : U^i(n) \rightarrow \prod_{r \geq 0} \text{End}_{\mathfrak{g}(r)}(\Omega^\otimes_{2n})$$

$$\tilde{\eta} := \Pi_{r \geq 0} \tilde{\eta}_r : \prod_{r \geq 0} \text{End}_{\mathfrak{g}(r)}(\Omega^\otimes_{2n}) \rightarrow S^i(n).$$

Recall also the involution $\omega$ given in \ref{2.1.1}. Corollary 4.6 gives immediately the following result.

Proposition 7.1. There is a $\mathbb{Q}(v)$-algebra injective homomorphism

$$\phi^i := \tilde{\eta} \circ \rho^i \circ \omega : U^i(n) \rightarrow S^i(n)$$

such that $e_h \mapsto E^q_{h,h+1}(0)$, $f_h \mapsto E^q_{h+1,h}(0)$, $a_i^{\pm 1} \mapsto O(\pm e_i)$, and $t \mapsto E^q_{n,n+1}(0) + O(-e_n)$ for all $1 \leq h \leq n - 1$ and $1 \leq i \leq n$.

Proof. Clearly, $\phi^i$ is an algebra homomorphism. Since $\rho^i$ is the restriction to $i(U^i(n))$ of the injective algebra homomorphism

$$\rho = (\rho_r)_{r \in \mathbb{N}} : U(\mathfrak{g}_{2n}) \rightarrow \prod_{r \geq 0} \text{End}_{\mathfrak{g}(r)}(\Omega^\otimes_{2n}),$$

where $\rho_r$ are given in \ref{4.5.2}, it follows that $\phi^i$ is injective. \hfill $\square$

We now determine the image of $\phi^i$. Let $\mathfrak{A}^i(n)$ be the subspace of $S^i(n)$ spanned by the linear independent set

$$\mathcal{B} = \{A(j) \mid A \in \mathbb{Z}^{0\text{diag}}_{2n}, j \in \mathbb{Z}^{2n}\}.$$

\footnote{See, e.g., \cite{DF10} Prop. 4.1 for one proof.}
Theorem 7.2. We have $\mathfrak{A}(n) = \text{im}(\phi^i)$. Moreover, the algebra homomorphism $\phi^i$ induces an algebra isomorphism $\phi^i : U^i(n) \rightarrow \mathfrak{A}(n)$.

Proof. The proof for the first assertion is somewhat standard (see that of [DW, Thm. 5.2] or [BLM90, Lem. 5.5]). We outline it as follows.

By Proposition 7.1, $\text{im}(\phi^i)$ is generated by

$$E_{i,i+1}^\theta(0), E_{i+1,i}^\theta(0), O(\pm e_i)$$

for all $1 \leq i \leq n$ (Note that $E_{n,n+1}^\theta = E_{n+1,n}^\theta$). By Theorem 6.2, $\text{im}(\phi^i) \subseteq \mathfrak{A}(n)$. We now prove the reverse inclusion.

For $A \in \Xi_2^{\text{0diag}}$, $(i, h, j) \in S_{2n}$ as in (5.2.5), $(\phi^i)^{-1}((a_{i,j}E_{i+1}^\theta_{h+1})(0)) = \left\{ \begin{array}{ll} f_{h}^{(a_{i,j})}, & \text{if } h < n; \\ e_{2n-h}^{(a_{i,j})}, & \text{if } h > n, \end{array} \right.$ by

Corollary 6.3

Let

$$(a_{i,j})g := (\phi^i)^{-1}((a_{i,j}E_{n+1}^\theta_{n})(0)).$$

Note that $(a_{i,j})g$ is not a divided power and $g := (1)g = t - d_n^{-1}$ by Proposition 7.1.

For $A \in \Xi_2^{\text{0diag}}$, define

$$m^{A,0} := \prod_{(i, h, j) \in (S_{2n}, \leq)} (\phi^i)^{-1}((a_{i,j}E_{i+1}^\theta_{h+1})(0)) \in U^i(n),$$

(7.2.1)

where the product is taken with respect to the order $\leq$ (cf. [DW, (5.0.3)]). We claim that

$$M^{A,0} := \phi^i(m^{A,0}) = \prod_{(i, h, j) \in (S_{2n}, \leq)} (a_{i,j}E_{i+1}^\theta_{h+1})(0) = A(0) + \text{a linear comb. of } B(j) \text{ with } B < A.$$

By Theorem 6.2 and Corollary 6.3, $M^{A,0} \in \text{im}(\phi^i)$ is a linear combination of $B(j)$. We need to determine its leading term.

Let

$$\pi_r : S^r(n) \rightarrow S^r(n, r)$$

(7.2.2)

be the canonical projection on $r$th component. Then

$$\pi_r(M^{A,0}) = \prod_{(i, h, j) \in (S_{2n}, \leq)} (a_{i,j}E_{i+1}^\theta_{h+1})(0, r),$$

Now a proof similar to that of [DW, Lemma 5.1] via Theorem 5.3 together Corollary 6.3 shows that

$$\prod_{(i, h, j) \in (S_{2n}, \leq)} (a_{i,j}E_{i+1}^\theta_{h})(0, r) = A(0, r) + \text{a linear comb. of } C(j, r) \text{ with } C < A,$$

proving the claim.

Finally, by induction on $\|A\| := \sum_{i=1}^{2n} (j_i - n)(a_{i,j} + a_{j,i})$, a proof similar to that of [DW, Thm. 5.2] shows that every $A(0) \in \text{im}(\phi^i)$, and hence, all $A(j) \in \text{im}(\phi^i)$. \hfill $\square$

By identifying $U^i(n)$ with $\mathfrak{A}(n)$ via $\phi^i$, we now summarise our discovery so far in the following result.

Let $m^{A,J} = O(j)M^{A,0} = O(e_1)^{j_1} \cdots O(e_n)^{j_n}M^{A,0}$, where $(j_1^*, \ldots, j_n^*) = j^*$ is defined in (6.0.2). Let $Z^{n*}$ denote the subset of $Z^{2n}$ consisting of $(j, 0^n)$ for all $j \in Z^n$.

Theorem 7.3. The $i$-quantum group $U^i(n)$ has two bases

$$B = \{ A(j) \mid A \in \Xi_2^{\text{0diag}}, j \in Z^{2n} \} = \{ A(j) \mid A \in \Xi_2^{\text{0diag}}, j \in Z^{n*} \} \text{ and}$$

$$M = \{ m^{A,J} \mid A \in \Xi_2^{\text{0diag}}, j \in Z^{2n} \} = \{ m^{A,J} \mid A \in \Xi_2^{\text{0diag}}, j \in Z^{n*} \}.$$
Furthermore, with the notation in (6.11), we may present \( U^1(n) \) by the basis \( \mathcal{B} \) and the following multiplication formulas by generators \( d_j, e_h, f_h, g = t - d_n^{-1} \):

1. \( d_j \cdot A(j) = d^{\sum_{i=1}^{2n} a_{j,i}} A(j + e_j), \quad A(j) \cdot d_j = d^{\sum_{i=1}^{2n} a_{j,i}} A(j + e_j) \);

2. \( e_h \cdot A(j) = \sum_{1 \leq p < h, a_{h+1,p} = 1} v^\beta_p(A,h) [a_{h,p} + 1] (A + E_{h,p}^\theta - E_{h+1,p}^\theta)(j + \alpha_h) \\
+ \varepsilon \sum_{h+1 < p \leq 2n, a_{h+1,p} = 1} v^\beta_p(A,h) (A + E_{h,p}^\theta - E_{h+1,p}^\theta)(j + \alpha_h) \\
+ \sum_{h+1 < p \leq 2n, a_{h+1,p} = 1} v^\beta_p(A,h) [a_{h,p} + 1] (A + E_{h,p}^\theta - E_{h+1,p}^\theta)(j) \)

where \( \varepsilon = \delta_1^{a_{h+1,h}} \).

3. \( f_h \cdot A(j) = \sum_{1 \leq p < h, a_{h,p} = 1} v^\beta_p(A,h) [a_{h+1,p} + 1] (A + E_{h,p}^\theta - E_{h+1,p}^\theta)(j) \\
+ v^\beta(A,h) [a_{h+1,h} + 1] (A + E_{h+1,h}^\theta)(j) \\
+ \varepsilon' \sum_{h+1 < p \leq 2n, a_{h+1,p} = 1} v^\beta_p(A,h) [a_{h+1,p} + 1] (A + E_{h,p}^\theta - E_{h+1,p}^\theta)(j) - \alpha_h) \\
+ \sum_{h+1 < p \leq 2n, a_{h+1,p} = 1} v^\beta_p(A,h) [a_{h+1,p} + 1] (A + E_{h,p}^\theta - E_{h+1,p}^\theta)(j) - \alpha_h) \)

where \( \varepsilon' = \delta_1^{a_{h+1,h+1}} \) is given in (4.3.1).

4. \( g \cdot A(j) = \sum_{i = 1}^{2n} v^\gamma(A,n) [a_{n+1,i} + 1] (A + E_{n,i}^\theta - E_{n+1,i}^\theta)(j) \\
+ v^\gamma(A,n) [a_{n+1,n} + 1] (A + E_{n+1,n}^\theta)(j) \\
+ \delta_1^{a_{n+1,n}} \frac{v^\gamma(A,n) - j_n + j_{n+1}}{v - 1} (A - E_{n,n+1}^\theta)(j) - \alpha_n) \\
+ c_A A(j - e_n) \)

where \( c_A \) is defined in (4.3.1).

\textbf{Proof.} We only make some comments on the first assertion. In the two descriptions for each basis, the first ones have duplications since \( A(j) = A(j^*) \) and \( m^{4j} = m^{4j^*} \) where \( j^* \) is defined in (6.0.2) and regarded as an element of \( \mathbb{Z}^{n*} \). The second ones are a more accurate description.\(^9\) The two descriptions giving the same basis set follows from the fact that the map \( \mathbb{Z}^{2n} \to \mathbb{Z}^{n*}, j \to j^* \) is surjective. One then uses a standard argument (see, e.g., the proof of [DF10 Prop. 4.1]) to prove that the (second) set for \( \mathcal{B} \) is linearly independent. Hence, the assertion for \( M \) follows.

\textbf{Remark 7.4.} The presentation given in the theorem defines the regular representation of \( U^1(n) \). It would be interesting to know if the regular representation can be constructed directly via the \( v \)-differential operator approach developed in [DZ20].

\(^9\)A similar description for [DW] Cor. 5.3 is also needed to avoid possible confusions.
8. Finite symplectic groups and quantum hyperalgebras of $\mathbf{U}^i(n)$

We now bring finite symplectic groups into the game. As seen from [BKLW18], the $q$-Schur algebra $S^*_q(n, r)$ or rather its $\mathcal{A}$-form $S^*_A(n, r)$, where $\mathcal{A} = \mathbb{Z}[q]$ ($q = \nu^2$), has a convolution algebra specialisation via a certain flag variety of type $C$. This specialisation links such algebras to representations of finite symplectic groups in certain Harish-Chandra series at the cross-characteristic level. Now, the new construction of $\mathbf{U}^i(n)$ developed in §7 allows us to extend further the link to representations of a certain hyperalgebra of $\mathbf{U}^i(n)$.

For any field $k$, let $\text{GL}_{2n}(k)$ be the general linear group over $k$ and consider the group isomorphism

$$\vartheta : \text{GL}_{2n}(k) \longrightarrow \text{GL}_{2n}(k), \ x \longmapsto J^{-1}(x^t)^{-1}J,$$

where $J = \left( \begin{array}{cc} 0 & I_n \\ -I_n & 0 \end{array} \right)$ with the $n \times n$ identity matrix $I_n$. The symplectic group

$$\text{Sp}_{2n}(k) := \{ x \in \text{GL}_{2n}(k) \mid J = x^t J x \}$$

is the fixed-point group of $\vartheta$.

Let $G(q) := \text{Sp}_{2n}(k)$ for $k = \mathbb{F}_q$, the finite field of $q$ elements.

For $\lambda \in \Lambda(n, r)$, let $P_\lambda(q)$ be the standard parabolic subalgebra of $\text{GL}_{2n}(\mathbb{F}_q)$ associated with $\widehat{\lambda}$, consisting of upper quasi-triangular matrices with blocks of sizes $\widehat{\lambda}_i$ on the diagonal. Let

$$P_\lambda(q) = P_{\widehat{\lambda}}(q) \cap G(q).$$

Then $G(q)$ acts on the set $G(q)/P_\lambda(q)$ of left cosets $gP_\lambda(q)$ in $G(q)$. For any commutative ring $R$, this action induces a permutation representation over $R$ which is isomorphic to the induced representations $\text{Ind}^{G(q)}_{P_\lambda(q)} 1_R$ of the trivial representation $1_R$ of $P_\lambda(q)$ to $G(q)$ and define

$$\mathcal{E}_{q, R}(n, r) = \text{End}_{R G(q)} \left( \bigoplus_{\lambda \in \Lambda(n, r)} \text{Ind}^{G(q)}_{P_\lambda(q)} 1_R \right)^{\text{op}}. \quad (8.0.1)$$

For any integral domain $R$ with $q \in R$, base change via the specialisation $A \rightarrow R, \nu^2 \mapsto q$ induces an isomorphism

$$S^*_R(n, r) := S^*_A(n, r) \otimes_A R \cong \mathcal{E}_{q, R}(n, r).$$

See [BKLW18] Prop. 6.6[10], or [ LW Thm. 4.2].

**Remark 8.1.** By Remark 3.2, we may also introduce the $q$-Schur algebra $S^*_q(n, r)$ of type $B$ over $A = \mathbb{Z}[q]$ which has an interpretation similar to (8.0.1) via finite orthogonal groups $G(q) = O_{2r+1}(q)$ with $P_\lambda$ replaced by $P_{\lambda}(q)$ and $\Lambda(n, r)$ replaced by $\Lambda(n + 1, r)$; see [DW] (3.0.2)]. Here, for $\lambda = (\lambda_1, \ldots, \lambda_n, \lambda_{n+1}) \in \Lambda(n + 1, r), \widehat{\lambda} = (\lambda_1, \ldots, \lambda_n, 2\lambda_{n+1} + 1, \lambda_n, \ldots, \lambda_1)$.

As seen above, representations of $S^*_R(n, r)$ is related to those of finite symplectic groups $G(q)$. If we can lift the epimorphism in Corollary 6.3 to the integral level (i.e., a homomorphism from some quantum hyperalgebra $U_q^i(n)$ to $S^*_R(n, r)$), then the representation category of $S^*_R(n, r)$ is a full subcategory of that of $U_q^i(n)$. In this way, we establish a link between representations of $i$-quantum groups and finite symplectic groups in cross-characteristics.

To define $U_q^i(n)$, we need a candidate Lusztig type form $U^*_q(n)$ of $\mathbf{U}^i(n)$. Traditionally, $U^*_q(n)$ is the $\mathbb{Z}$-subalgebra of $\mathbf{U}^i(n)$ generated by divided powers $e^{(m)}_h, f^{(m)}_h, \ell^{(m)}$ and $d_i, \left[ \begin{array}{c} d_i \end{array} \right; 0_s]$.\footnote{The algebra $S^*_i$ in [BKLW18] is the centraliser subalgebra $S^\nu(n, r)$ given in [11.1.2].}
for all \( m, s \in \mathbb{N} \), \( 1 \leq h \leq n-1 \) and \( 1 \leq i \leq n \). However, by Theorem 7.3 \( E^\theta_{n,n+1}(0)O(-\mathbf{e}_n) = O(-\mathbf{e}_n)E^\theta_{n,n+1}(0) \) and, by identifying \( \mathbf{U}^i(n) \) as \( 2\mathbf{U}^i(n) \) under \( \phi^i \),

\[
t^{(m)} = \frac{1}{[m]!}(E^\theta_{n,n+1}(0) + O(-\mathbf{e}_n))^m = \frac{1}{[m]^j \sum_{j=0}^m \binom{m}{j} E^\theta_{n,n+1}(0)^{m-j}O(-\mathbf{e}_n)^j.
\]

Hence,

\[
t^{(m)} = E^\theta_{n,n+1}(0)^{(m)} + f_1 E^\theta_{n,n+1}(0)^{(m-1)}O(-\mathbf{e}_n) + \cdots + f_{m-1} E^\theta_{n,n+1}(0)^{(0)}O(-\mathbf{e}_n)^{m-1} + f_m O(-\mathbf{e}_n)^m
\]

for some \( f_1, \ldots, f_m \in \mathbb{Q}(v) \). These rational function coefficients in the display above and in Corollary 6.3 show that this form cannot be used as the image \( \pi_r \circ \phi^i(U^i_\mathbb{Z}(n)) \) cannot be inside \( S^r_\mathbb{Z}(n,r) \).

Motivated by the proof of Corollary 4.6, we make the following definition.

**Definition 8.2.** Let \( U^i_\mathbb{Z}(n) \) be the \( \mathbb{Z} \)-subalgebra of \( \mathbf{U}^i(n) \) generated by

\[
ed_h^{(m)}, f_h^{(m)}, d_i, \begin{bmatrix} d_i; 0 \end{bmatrix}^s, (m) g,
\]

for all \( h, i \in [1, n] (h \neq n) \) and \( m, s \in \mathbb{N} \).

Note that, if we identify \( \mathbf{U}^i(n) \) with \( 2\mathbf{U}^i(n) \) as in Theorem 7.3 then, by Corollary 6.3 the generators \( e_h^{(m)}, f_h^{(m)}, (m) g \) can be unified as the generators \( (mE^\theta_{h,h+1})(0), (mE^\theta_{h+1,h})(0) \).

For \( \lambda \in \mathbb{N}^n \) and \( \tau \in \mathbb{N}_2 \), where \( \mathbb{N}_2 = \{0, 1\} \), let

\[
\begin{bmatrix} d_i \end{bmatrix}^\lambda = \prod_{i=1}^n \begin{bmatrix} d_i; 0 \end{bmatrix}^\lambda_i, \quad d^\tau = d_1^{\tau_1} d_2^{\tau_2} \cdots d_n^{\tau_n},
\]

where \([d_i; 0]^\lambda_i\) is defined in (1.0.2). Recall the elements \( \mathbf{m}^{A,0} \) defined in (7.2.1) and the canonical projection \( \pi_r \) in (7.2.2).

**Theorem 8.3.** The \( \mathbb{Z} \)-algebra \( U^i_\mathbb{Z}(n) \) contains the basis

\[
\mathcal{M}_\mathbb{Z} = \left\{ d^\tau \begin{bmatrix} d \end{bmatrix}^\lambda \mathbf{m}^{A,0} \mid A \in \Xi_{2n}^{\text{diag}}, \tau \in \mathbb{N}_2^n, \lambda \in \mathbb{N}^n \right\}
\]

for \( \mathbf{U}^i(n) \). Hence, restricting the map \( \phi^i \) in Proposition 7.1 to \( U^i_\mathbb{Z}(n) \) induces a surjective homomorphism

\[
\phi^i_{r,\mathbb{Z}} := \pi_r \circ \phi^i : U^i_\mathbb{Z}(n) \to S^r_\mathbb{Z}(n,r).
\]

**Proof.** By definition, we have \( \mathcal{M}_\mathbb{Z} \subset U^i_\mathbb{Z}(n) \). The basis claim follows from Theorem 7.3 and the fact that both \( \{d^\theta \mid j \in \mathbb{N}^n\} \) and \( \{d^\tau \begin{bmatrix} d \end{bmatrix}^\lambda \mid \tau \in \mathbb{N}_2^n, \lambda \in \mathbb{N}^n\} \) form bases for \( \mathbb{Q}(v)[d_1^{\pm 1}, \ldots, d_n^{\pm 1}] \) (see, e.g., the proof for [DDPW08, Thm. 14.20]).

The surjectivity assertion is seen from the proof of Corollary 6.5 since, for any \( A \in \Xi_{2n,2r} \) and \( \lambda = \text{ro}(A) \), we have, by (6.4.1), \( \phi^i_{r,\mathbb{Z}}(\begin{bmatrix} d \end{bmatrix}^\lambda) = \begin{bmatrix} \lambda \end{bmatrix}^\lambda = \text{ro}(A) \) and, by Theorem 5.3

\[
\phi^i_{r,\mathbb{Z}}(\begin{bmatrix} d \end{bmatrix}^\lambda \mathbf{m}^{A,0}) = \text{ro}(A) \cdot \prod_{(i,h,j) \in (B_{2n}, \leq)} (a_{i,j} E^\theta_{h+1,h})(0, r) = [A] + (\text{a \( \mathbb{Z} \)-linear comb. of } [B] \text{ with } B < A),
\]

where \( A' \) is the matrix obtained from \( A \) by replacing its diagonal with zeros. Thus, the set \( \{\phi^i_{r,\mathbb{Z}}(\begin{bmatrix} d \end{bmatrix}^\lambda \mathbf{m}^{A,0}) \mid A \in \Xi_{2n,2r} \} \) forms a basis for \( S^r_\mathbb{Z}(n,r) \). Hence, the theorem is proved. \( \square \)
Note that, if we identify $S^I_z(n,r)$ as a $\mathcal{Z}$-form of $\text{End}_{\mathcal{H}(r)}(\Omega^{\otimes r}_{2n})$, then $\phi_{r,z}^I$ may be identified as the restriction of the map $\rho^I_r$ given in (4.5.1) to $U^I_z(n)$.

**Corollary 8.4.** For any commutative ring $R$ which is a $\mathcal{Z}$-algebra via $\nu \mapsto \sqrt{q} \in R$, the $q$-Schur algebra $S^I_R(n,r)$ is a homomorphic image of $U^I_R(n)$. In particular, the category $S^I_R(n,r)$-mod of $S^I_R(n,r)$-modules is a full subcategory of the category $U^I_R(n)$-mod of $U^I_R(n)$-modules.

**Remark 8.5.** (1) If $R$ takes a member from a modular system $(0,K,k)$, where $0$ is a local DVR with fraction field $K$ and residue field $k$, and $q$ is a prime power and non-zero in $k$, representations of $S^I_R(n,r)$ are closely related to the representations of finite symplectic group $G(q)$ in cross-characteristics, especially those in the unipotent principal series. For example, the decomposition matrix for $S^I_R(n,r)$ is unitriangular and is part of the decomposition matrix $RG(q)$. See [DPS] Chap. 5 for more details.

(2) The $\mathcal{Z}$-subalgebra $U^I_z(n)$ of $U^I(n)$ contains the $\mathcal{Z}$-subalgebra $U^I_z(\mathfrak{g}_n)$ having the same generators with all $(mE_{i,n+1})\{0\}$ removed. This subalgebra is the Lusztig form of $U^I(\mathfrak{g}_n)$ which is $\mathcal{Z}$-free. It would be interesting to know if the set $\mathcal{M}_z$ spans $U^I_z(n)$ and hence, forms a basis for $U^I_z(n)$.

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**References**

[BLM90] A. A. Beilinson, G. Lusztig, and R. MacPherson, *A geometric setting for the quantum deformation of $GL_n$*, Duke Math. J. **61** (1990), 655-677.

[BW18] H. Bao and W. Wang, *A new approach to Kazhdan-Lusztig theory of type $B$ via quantum symmetric pairs*, Astérisque **402**, 2018, vii+134pp.

[BKLW18] H. Bao, J. Kujawa, Y. Li and W. Wang, *Geometric Schur Duality of Classical Type*, Transf. Groups **23** (2018), 329-389.

[BLMW91] A. A. Beilinson, G. Lusztig, and R. MacPherson, *A geometric setting for the quantum deformation of $GL_n$*, Duke Math. J. **61** (1990), 655-677.

[DDPW08] B. Deng, J. Du, B. Parshall, and J. Wang, *Finite Dimensional Algebras and Quantum Groups*, Math. Surveys and Monographs **150**, Amer. Math. Soc. (2008).

[DJ89] R. Dipper and G. James, *The $q$-Schur algebra*, Proc. London Math. Soc. **59** (1989), 23-50.

[Du94] J. Du, *IC bases and quantum linear groups*, In: Algebraic groups and their generalizations: quantum and infinite-dimensional methods (University Park, PA 1991), 135–148. Proc. Sympos. Pure Math. **56**, Part 2, AMS, Providence, RI, 1994.

[Du95] J. Du, *A note on quantised Weyl reciprocity at roots of unity*, Algebra Colloquium, 2 (1995), 363–372.

[DF10] J. Du and Q. Fu, *A modified BLM approach to quantum affine $\mathfrak{gl}_n$*, Math. Z. **266** (2010), 747–781.

[DPS] J. Du, B. Parshall, and L. Scott, *An exact category approach to Hecke endomorphism algebras*, in preparation.

[DW] J. Du and Y. Wu, *A new realisation of the $i$-quantum groups $U^I(n)$*, J. Pure App. Algebra, available online at https://doi.org/10.1016/j.jpaa.2021.106793.

[DZ20] J. Du and Z. Zhou, *The regular representation of $U^I(\mathfrak{g}_m|\mathfrak{n})$*, Proc. Amer. Math. Soc., **148** (2020), 111–124.

[LL] C.-J. Lai and L. Luo, *Schur algebras and quantum symmetric pairs with unequal parameters*, Int. Math. Res. Not. (2020), https://doi.org/10.1093/imrn/rnz110.

[LNX] C.-J. Lai, D. Nakano, and Z. Xiang, *On $q$-Schur algebras corresponding to Hecke algebras of type $B$*, Transformation Groups, (2020), DOI 10.1007/s00031-020-09628-7.

[LW] L. Luo and W. Wang, *The $q$-Schur algebras and $q$-Schur dualities of finite type*, J. Inst. Math. Jussieu, to appear.

[Le99] G. Letzter, *Symmetric pairs for quantized enveloping algebras*, J. Algebra **220** (1999), 729–767.

[Le03] G. Letzter, *Quantum symmetric pairs and their zonal spherical functions*, Tranf. Groups **8** (2003), 261–292.
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