Coupled Tensor Completion via Low-rank Tensor Ring

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Abstract—The coupled tensor decomposition aims to reveal the latent data structure which may share common factors. Using the recently proposed tensor ring decomposition, in this paper we propose a non-convex method by alternately optimizing the latent factors. We provide an excess risk bound for the proposed alternating minimization model, which shows the improvement in completion performance. The proposed algorithm is validated on synthetic data.

Index Terms—tensor ring, coupled tensor completion, alternating least squares, excess risk bound, permutational Rademacher complexity

I. INTRODUCTION

Tensor is a multi-dimensional array and able to model the interaction between different modes in high-dimensional data. Analogous to singular value decomposition (SVD), tensor decomposition seeks an optimal form of tensor representation which results in a set of smaller and simpler components. Tensor completion recovers the missing entries based on low-rank assumptions that are induced by different form of tensor decompositions. The completion methods are mainly divided into two categories. One is the convex method which is based on optimizing the low-rank inducing norms, another is the non-convex method which is based on optimizing the latent factors given pre-defined tensor ranks. Tensor completion is applicable in many fields, such as signal processing [1]–[5], link prediction [6]–[8], recommendation system [9]–[14], bioinformatics [15]–[19], chemometrics [20], [21] and computer vision [22]–[30].

Fig. 1: Illustration of information sharing between three coupled tensors, through mode 1.

The recently proposed tensor ring (TR) decomposition represents a D-order tensor with cyclically contracted 3-order tensor factors of size $R \times I \times R$ by using the matrix product state expression (see Fig. 2(a)), resulting in $DIR^2$ parameters, where $[R, \ldots, R]$ is the TR-rank. The TR decomposition allows a cyclical shift of TR-factors due to the nature of trace operator, thus the reordering tensor’s dimensions make no difference to the result. As a quantum-inspired decomposition, the TR representation is shown to perform better than CP and TK representations due to its powerful representation ability [35], [36]. Though the TR-rank is a vector, assuming all ranks to be the same is validated to be effective [36], which alleviates the burden of tuning many parameters.

Fig. 2: Illustration of coupled TR decomposition.

In this paper, we focus on utilizing coupled TR decomposition for coupled tensor completion, and to the best of our knowledge, this is the first attempt to use TR decomposition for coupled tensor completion. The different novelty of this paper compared with [36] is the derivation of closed form of

tensor completion are based on CANDECOMP/PARAFAC (CP) decomposition, a generalization of SVD, which factorizes a D-order tensor into a linear combination of D rank-1 tensors, resulting in $DIR$ parameters, where $I$ is each dimensional size and $R$ is the CP-rank.

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The $d$-shifting $K$-unfolding yields a matrix $X_{\{d,K\}} \in \mathbb{R}^{I_d \times J_d}$ by permuting $X$ with order $[d, \ldots, D, 1, \ldots, d-1]$ and unfolding along its first $K$ dimensions, where $J_d = \prod_{n=1, n\neq d}^{D} I_n$.

### B. Preliminaries of tensor ring decomposition

This section introduces the TR decomposition. Suppose the tensor $X$ has size $I_1 \times \cdots \times I_D$. The TR decomposition factorizes a $D$-order into $D$ cyclically contracted 3-order tensors, and the formulation is

$$X(i_1, \ldots, i_D) = \text{tr} \left( U^{(1)}(:, i_1) \cdots U^{(D)}(:, i_D, :) \right),$$

where $U^{(d)} \in \mathbb{R}^{R_d \times I_d \times R_{d+1}}$.

Reference [35] mentions two methods for TR decomposition. The first method is based on the density matrix renormalization group [39]. It first reshapes $X$ into $X_{(1,1)}$ and applies SVD to derive $X_{(1,1)} = U \Sigma V$. Then it reshapes $U$ into the first TR-factor and applies SVD to $\Sigma V$. The algorithm is accomplished by performing $D-1$ SVDs. This method does not need the per-defined TR-rank and performs fast. The second method alternatively optimizes the TR-factor while keeping the others fixed. Repeatedly performing the optimization until the relative change $\|X^k - X^{k-1}\|/\|X^{k-1}\|$ or relative error $\|X^k - X_0\|/\|X_0\|$ drops below some pre-defined thresholds.

This method requires the pre-defined TR-rank which affects the performance and performs slowly compared with the first method.

### III. COUPLED TENSOR RING COMPLETION ALGORITHM

We use $\mathcal{R}$ to represent the tensor ring and assume the first $L$ TR-factors of $\mathcal{R}_1$ and $\mathcal{R}_2$ are coupled. Operator $\mathcal{R}(\cdot)$ means the TR contraction which yields a tensor of size $I_1 \times \cdots \times I_D$. Suppose $\{\mathcal{U}\} = \{U^{(1)}, \ldots, U^{(D)}\}$ are the TR-factors of $\mathcal{R}_1$, and $\{\mathcal{V}\} = \{V^{(1)}, \ldots, V^{(D)}\}$ are the TR-factors of $\mathcal{R}_2$. Assume $\mathcal{R}(\{\mathcal{U}\}) \in \mathbb{R}^{I_1 \times \cdots \times I_D}$ and $\mathcal{R}(\{\mathcal{V}\}) \in \mathbb{R}^{I'_1 \times \cdots \times I'_D}$. Then the model for coupled TR completion is

$$\min_{\{\mathcal{U}, \mathcal{V}\}} \frac{1}{2} \|P_{\Omega_1}(\mathcal{R}(\{\mathcal{U}\})) - P_{\Omega_1}(T_1)\|^2 + \frac{1}{2} \|P_{\Omega_2}(\mathcal{R}(\{\mathcal{V}\})) - P_{\Omega_2}(T_2)\|^2$$

s. t. $U^{(l)}(1: \Gamma_1, \ldots, 1: \Gamma_{l-1}, 1: \Gamma_{l+1}) = V^{(l)}(1: \Gamma_1, \ldots, 1: \Gamma_{l-1}, 1: \Gamma_{l+1})$,

$$l = 1, \ldots, L,$$

where $\Gamma_1 \in [1, \min \{R_l, R'_l\}]$, $l = 1, \ldots, L$ are the coupled distances, in which $[R_1, \ldots, R_{D_1}]$ and $[R'_1, \ldots, R'_{D_2}]$ are the TR-ranks of $T_1$ and $T_2$, respectively.

### A. Algorithm

To solve problem (6), we use the block coordinate descent method. Specifically, this method alternately optimizes the block variable $U^{(d_l)}$ (or $V^{(d'_l)}$) while keeping others fixed, thus the original problem is decomposed into $D_1 + D_2$ subproblems.
1) **Update of the uncoupled TR-factors of \( \mathfrak{R}_1 \):** we rewrite problem (6) as

\[
\min_{d = L + 1, \ldots, D_1} \frac{1}{2} \| P_{\Omega_1}(\mathfrak{R}(\{U\})) - P_{\Omega_1}(T_1) \|_2^2.
\]

We substitute \( W_U \) of (P) for \( \mathfrak{R}_1 \). Let \( A_d \in \mathbb{R}^{I_d \times R_d R_{d+1}} \), \( B_d \in \mathbb{R}^{R_d R_{d+1} \times J_d} \) and \( C_d \in \mathbb{R}^{I_d \times J_d} \) be the unfoldings of \( U_i \), \( B_d \) and \( T_1 \), respectively, where \( B_d \) is computed by contracting all the \( D_1 \) TR-factors of \( \mathfrak{R}_1 \) except the \( d \)-th factor. Then problem (7) is converted into

\[
\min_{A_d, d = L + 1, \ldots, D_1} \frac{1}{2} \| W_{\{d, 1\}} \circ A_d B_d - W_{\{1, d\}} \circ C_d \|_F^2.
\]

(8)

Define \( w_{d,(i,:)} = W_{\{d, 1\}}(i,d,:) \) and a permutation matrix \( P_{i,d} = e_{i,d}\) with a vector of length \( I_d \) whose values are all zero but one in the \( k \)-th entry, \( k = \mathbb{S}^d(1) \) and \( \mathbb{S}^d(1) = \{ j_d \in \mathbb{S}^d | (j_d) = 1 \}\).

Note that the \( d \)-th sub-problem in (8) can be divided into \( I_d \) sub-sub-problems, in which the row vectors \( a_{d,i} = A_d(i,d,:) \) are treated as the block variables. Reformulating the \( d \)-th sub-sub-problem in the quadratic form and calculating its first-order derivative, we have

\[
a_{d,i} = -g_{d,i} H_{d,i}^T
\]

(9)

where \( \dagger \) is the Moore-Penrose pseudoinverse and

\[
\begin{align*}
H_{d,i}^T &= B_d \circ B_{d,i}^T, & g_{d,i} &= -e_{d,i} B_{d,i}^T, \\
A_{d,i}^T &= c_{d,i} P_{i,d}, & B_{d,i} &= B^T P_{i,d}.
\end{align*}
\]

The TR-factor \( U_i \) is optimized by performing (9) \( I_d \) times to solve the \( d \)-th sub-problem of (8). Then the uncoupled TR-factors of \( \mathfrak{R}_1 \) are updated by optimizing all \( D_1 - L \) factors.

2) **Update of the uncoupled TR-factors of \( \mathfrak{R}_2 \):** this optimization is similar to the update of uncoupled TR-factors of \( \mathfrak{R}_1 \) and can refer to (9), hence ignore the deduction and just give the solution as follows.

\[
a_{d,i}^T = -g_{d,i} H_{d,i},
\]

(10)

where \( H_{d,i} = B_{d,i}^T B_d^T \), \( g_{d,i} = -e_{d,i} B_d^T \), \( z_{d,i} = c_{d,i} P_{d,i} \), \( c_{d,i} = c_{d,i}^T P_{d,i} \), and the symbols with superscript \( \circ \) means the corresponding terms derived from computation of \( \mathfrak{R}_2 \).

3) **Update of the coupled TR-factors of \( \mathfrak{R}_1 \) and \( \mathfrak{R}_2 \):** we rewrite problem (6) as

\[
\min_{U^{(1)}, \ldots, U^{(L)}} \frac{1}{2} \| \Omega_1(\mathfrak{R}(\{U\})) - \Omega_1(\mathfrak{R} \{U\}) \|_2^2 + \frac{1}{2} \| \Omega_2(\mathfrak{R}(\{V\})) - \Omega_2(\mathfrak{R} \{V\}) \|_2^2
\]

s. t. \( U^{(1)}(1, \Gamma_d, \ldots, 1, \Gamma_{d+1}) = \gamma^{(1)}(1, \Gamma_d, \ldots, 1, \Gamma_{d+1}), \ldots, \Gamma_d = 1, \ldots, L \).

Let \( A_{d}' \in \mathbb{R}^{I' \times R_d R_{d+1}} \) be the unfolding of \( \gamma^{(d)} \), \( C_{d}' \in \mathbb{R}^{I' \times J_d} \) be the \( \{d, 1\} \)-unfolding of \( T_2 \) and \( W' \) be the tensor form of \( P_{\Omega_d} \). Let \( C_{d} = \{1, \ldots, \Gamma_d + 1, R_{d+1} + 1, \ldots, R_{d+1} + \Gamma_{d+1}, \ldots, \Gamma_d R_{d+1} + 1, \ldots, \Gamma_d R_{d+1} + \Gamma_{d+1} \}, \Gamma_d = 1, \ldots, L, \Gamma_{d+1} = 1, \ldots, L, \Gamma_d R_{d+1} + 1, \Gamma_d R_{d+1} + \Gamma_{d+1} + 1, \ldots, \Gamma_d R_{d+1} + \Gamma_{d+1} + \Gamma_{d+1} \), \( d = 1, \ldots, L \).

We reformulate (11) as

\[
\min_{d = 1, \ldots, L} \frac{1}{2} \| W_{\{d, 1\}} \circ A_d B_d - W_{\{1, d\}} \circ C_d \|_F^2 + \frac{1}{2} \| W'_{\{d, 1\}} \circ A_d' B_d' - W'_{\{1, d\}} \circ C_d' \|_F^2
\]

s. t. \( A_d(1,d) = A_d'(1,d), C_d, C_d' \), \( d = 1, \ldots, L \), where the index sets \( C_d \in \mathbb{R}^{U_d T_d} \) and \( C_d' \in \mathbb{R}^{U_{d+1} T_{d+1}} \) indicate which columns are coupled in \( A_d \) and \( A_d' \), respectively.

Follow the analysis in optimization (8), we consider the \( i_d \)-th sub-problem of the \( d \)-th sub-problem of (12). Defining

\[
\begin{align*}
\alpha_{d,i} &\triangleq A_d(i,d) \\
\beta_{d,i} &\triangleq A_d(i,d) \\
\gamma_{d,i} &\triangleq A_d(i,d)
\end{align*}
\]

then \( A_d(i,d) = [\alpha_{d,i}, \beta_{d,i}] P_{d} \) and \( A_d'(i,d) = [\alpha_{d,i}, \beta_{d,i}] P_{d}' \), where \( P_{d} = [e_{C_d}, e\{1, \ldots, R_d R_{d+1}\} C_d] \) and \( P_{d}' = \left[e_{C_d'}, e\{1, \ldots, R_{d+1} R_{d+1}\} C_d'\right] \) are permutation matrices. Accordingly, we have the problem

\[
\min_{\alpha_{d,i}, \beta_{d,i}, \gamma_{d,i}} \frac{1}{2} \| w_{d,i} \circ \alpha_{d,i} P_{d} - w_{d,i} \circ \gamma_{d,i} \|_2^2 + \frac{1}{2} \| w'_{d,i} \circ \alpha_{d,i} P_{d}' - w'_{d,i} \circ \gamma_{d,i} \|_2^2
\]

(13)

Let \( \overline{H}'_{d,1} = P_{d,1} H_{d,1} P_{d} \) and \( \overline{H}'_{d,2} = P_{d,2} H_{d,2} P_{d} \). Defining

\[
\overline{H}'_{d,1} \triangleq \left[ \overline{H}'_{d,1}, \overline{H}'_{d,12}, \overline{H}'_{d,12}, \overline{H}'_{d,2}, \overline{H}'_{d,21}, \overline{H}'_{d,21} \right]
\]

such that

\[
\begin{align*}
\overline{H}'_{d,11} &\in \mathbb{R}^{U_{d+1} T_{d+1} \times U_{d+1} T_{d+1}} \\
\overline{H}'_{d,12} &\in \mathbb{R}^{U_{d+1} T_{d+1} \times U_{d+1} T_{d+1}} \\
\overline{H}'_{d,21} &\in \mathbb{R}^{U_{d+1} T_{d+1} \times U_{d+1} T_{d+1}} \\
\overline{H}'_{d,22} &\in \mathbb{R}^{U_{d+1} T_{d+1} \times U_{d+1} T_{d+1}}
\end{align*}
\]

and the similar sizes hold for \( \overline{H}'_{d,11}, \overline{H}'_{d,12}, \overline{H}'_{d,21} \) and \( \overline{H}'_{d,22} \).
such that $\xi_{id}^{(d)}, \xi'^{(d)}_{id} \in \mathbb{R}^{1 \times \Gamma_d \Gamma_{d+1}}, \eta_{id}^{(d)} \in \mathbb{R}^{R_d R_{d+1} \Gamma_d \Gamma_{d+1}}$, and $\eta'^{(d)}_{id} \in \mathbb{R}^{R_d R_{d+1} \Gamma_d \Gamma_{d+1} - \Gamma_d \Gamma_{d+1}}$.

We then deduce the solution (see Appendix for detail) as

$$\left[\begin{array}{c}
\alpha_{id}^{(d)}, \beta_{id}^{(d)}, \gamma_{id}^{(d)}
\end{array}\right] * = \arg \min_{\alpha_{id}^{(d)}, \beta_{id}^{(d)}, \gamma_{id}^{(d)}} \frac{1}{2} \left[ \alpha_{id}^{(d)}, \beta_{id}^{(d)}, \gamma_{id}^{(d)} \right] H_{id}^{(d)} \left[ \alpha_{id}^{(d)}, \beta_{id}^{(d)}, \gamma_{id}^{(d)} \right]^T$$

where $\left[\begin{array}{c}
\hat{\alpha}_{id}^{(d)}, \hat{\beta}_{id}^{(d)}, \hat{\gamma}_{id}^{(d)}
\end{array}\right]$ and the Hessian matrix is

$$\hat{H}_{id}^{(d)} = \left[\begin{array}{ccc}
\hat{H}_{id}^{(d)11} & \hat{H}_{id}^{(d)12} & \hat{H}_{id}^{(d)21}
\hat{H}_{id}^{(d)12} & \hat{H}_{id}^{(d)22} & 0
\hat{H}_{id}^{(d)21} & 0 & \hat{H}_{id}^{(d)22}
\end{array}\right]$$

The algorithm for coupled tensor completion via low-rank tensor ring is outlined in Algorithm 1.

**Algorithm 1** Alternating least squares for coupled tensor ring completion (CTRC)

**Input**: Two zero-filled tensors $T_1$ and $T_2$, two binary tensors $W_1$ and $W_2$, the maximal # iterations $K$

**Output**: Two recovered tensors $\mathcal{X}$ and $\mathcal{Y}$, two sets of TR-factors $\{U\}$ and $\{V\}$

1. Apply Algorithm 1 to initialize $\{U\}$ and $\{V\}$
2. for $k = 1$ to $K$
3. Update the uncoupled TR-factors of $\mathcal{X}_1$ according to
4. Update the uncoupled TR-factors of $\mathcal{X}_2$ according to
5. Update the coupled TR-factors of $\mathcal{X}_1$ and $\mathcal{X}_2$ according to
6. Update $\mathcal{X} = \mathfrak{R}(\{U\}), \mathcal{Y} = \mathfrak{R}(\{V\})$
7. if converged then
8. break
9. end if
10. end for
11. return $\mathcal{X}, \mathcal{Y}, \{U\}, \{V\}$

Note this algorithm can be easily extend to the case where more than two tensor rings are coupled, in which only the scheme for updating the coupled components is changed. We have the Hessian matrix defined in the form of block matrix

$$\left[\begin{array}{c}
\hat{H}_{id}^{(d)11} & \hat{H}_{id}^{(d)12} & \hat{H}_{id}^{(d)21}
\hat{H}_{id}^{(d)12} & \hat{H}_{id}^{(d)22} & 0
\hat{H}_{id}^{(d)21} & 0 & \hat{H}_{id}^{(d)22}
\end{array}\right]$$

B. Computational Complexity

Assume the tensors $X_1, \ldots, X_N$ are of size $I_1 \times \ldots \times I_D$ with all TR-ranks being $[R, \ldots, R]$. The computation of Hessian matrix $H_{id}^{(d)}$ costs $O \left( R^4 \prod_{k=1, k \neq d}^{D} I_d \right) = O \left( mR^4 / I_d \right)$, where $R$ is the sampling ratio. Thus updating the $d$-th TR-factor costs $O \left( mR^4 \right)$ and one iteration of CTRC costs $O \left( mN R^4 \right)$.

The computation of $H_{id}^{(d)}$ costs $O \left( R^6 \right)$ and updating the $d$-th TR-factor costs $O \left( I_d R^6 \right)$. Hence one iteration of CTRC costs $O \left( N R^6 \sum_{d=1}^{D} I_d \right)$.

The total computational cost of one iteration of CTRC is $O \left( N R^6 \sum_{d=1}^{D} I_d \right) = O \left( mN D R^4 \right)$.

C. Excess Risk Bound

We define $l_T (\cdot, \cdot)$ as the average of the perfect square trinomial $l (\cdot, \cdot)$ computed on a finite training set $T$. For concise expression of average test error, we use notation $l_T (\{X, Y\}, \{T_1, T_2\})$ to denote the average training error over $T$, where we simply refer to $T \subseteq \Omega$ as the union of $T_1 \subseteq \Omega_1$ and $T_2 \subseteq \Omega_2$. Similarly, we can define $I_T (\cdot, \cdot)$ as the average test error measured by $l (\cdot, \cdot)$ over $S \subseteq \Omega^2$. As in [40], we assume that $|S| = |T|$ for any $i \in \{1, 2\}$.

Given an assumption that $X = \mathfrak{R}(\{U\})$ with TR rank $[R, \ldots, R]$ and each TR factor is an independent Gaussian random tensor with zero mean and variance of $\sigma^2$, we can define a hypothesis class $\mathcal{H} \triangleq \{X, Y | U^{(d)} \sim \mathcal{N} (0, \sigma^2), Y^{(d)} \sim \mathcal{N} (0, \sigma^2)\}$. Without loss of generality, we assume $l (\cdot, \cdot)$ is $L$-Lipschitz continuous since the $F$-norms of two tensors are centralized with overwhelming probability.

By leveraging the recently proposed permutational Rademacher complexity [40], the following theorem characterizes the excess risk of coupled TR completion.
Theorem 1. Under the hypothesis $H$ mentioned before, the excess risk of the coupled TR completion (6) is bounded as

$$\tilde{I}_2 (\{X, Y\}, \{T_1, T_2\}) - \tilde{I}_2 (\{X, Y\}, \{T_1, T_2\}) \leq \Lambda \left( 1 + \frac{2}{\sqrt{2\pi |T_n| - 2}} \right) \frac{\sigma D_1 2^{D_1}}{\sqrt{|T_1|}} \frac{\Gamma D_1 (\frac{k + 1}{2})}{\Gamma D_1 (\frac{k}{2})} + D_2 + 1 - L F_{D_2 - L} \left( \left[ \frac{1}{1}, \frac{2}{1}, \ldots, \frac{L}{1} \right] \right) \left( -1 \right)^{D_1 + 1 - L} 2^{D_2 - D_1} \frac{|T_2|}{|T_1|^2}$$

for $D_1 \geq D_2$ and

$$\tilde{I}_2 (\{X, Y\}, \{T_1, T_2\}) - \tilde{I}_2 (\{X, Y\}, \{T_1, T_2\}) \leq \Lambda \left( 1 + \frac{2}{\sqrt{2\pi |T_n| - 2}} \right) \frac{\sigma D_2 2^{D_2}}{\sqrt{|T_2|}} \frac{\Gamma D_2 (\frac{k + 1}{2})}{\Gamma D_2 (\frac{k}{2})} + D_1 + 1 - L F_{D_2 - L} \left( \left[ \frac{1}{1}, \frac{2}{1}, \ldots, \frac{L}{1} \right] \right) \left( -1 \right)^{D_1 + 1 - L} 2^{D_1 - D_2} \frac{|T_1|}{|T_2|^2}$$

for $D_2 \geq D_1$ respectively with probability at least $1 - \delta$.

Moreover, with the same probability, the excess risk of each individual TR completion is bounded by

$$\Lambda \left( 1 + \frac{2}{\sqrt{2\pi |T_n| - 2}} \right) \frac{\sigma D_n 2^{D_n}}{\sqrt{|T_n|}} \frac{\Gamma D_n (\frac{L R^2 + 1}{2})}{\Gamma D_n (\frac{R^2}{2})} + \frac{2 |T_n \cup S_n| \ln (1/\delta)}{(|T_n \cup S_n| - 1/2)^2} \frac{|T_n|}{|T_n \cup S_n|}, \quad n = 1, 2.$$

Note that the hypergeometric series is well-poised, it is easy to illustrate that the risk bounds (15) and (16) are less than the sum of the bound (17) by transformation identity. Besides, the value of the risk bound decreases as the number of the coupled dimensions $L$ increases, which implies the coupled tensor rings benefit each other’s completion performance. This phenomenon can also be comprehended from the viewpoint of mutual information $I (X, Y) = \int p (X, Y) \ln \frac{p (X, Y)}{p (X)p (Y)} dX dY$. The two tensor rings have no mutual information if they are not coupled, thus they cannot help each other’s recovery. On the other hand, this term becomes the differential entropy $H (X)$ if they are totally coupled, meanwhile, the amount of information reaches the maximum which results in the best recovery performance. The information transfer consists in the summation of the Hessian matrix in Algorithm 1. In this case, the sampling bound for $N$ totally coupled tensors can be reduced to $1/N$ of an individual sampling bound.

IV. NUMERICAL EXPERIMENT

In this section, we test our algorithm on randomly generated completion problem. We generate two tensors of size $20 \times 20 \times 20 \times 20$ using the TR decomposition (5), in which the TR-factors are randomly sampled from the standard normal distribution, i.e., $U (d) (r_d, i_d, r_{d+1}) \sim N (0, 1)$, $\gamma (d) (r_d, i_d, r_{d+1}) \sim N (0, 1)$, $d = 1, \ldots, 4$. Then we couple two tensor rings by setting $U (d) = \gamma (d)$, $d = 1, \ldots, 3$. Next, we compute the tensors $T_1$ and $T_2$ according to these factors. We run our algorithm to plot the phase transition on TR-rank versus sampling ratio of tensor $T_1$ under different settings of sampling ratio of tensor $T_2$ and the number of coupled TR-factors. The sampling ratio of $T_1$ ranges from 0.005 to 0.1 with interval 0.005, and the sampling ratio of $T_2$ ranges from 0.05 to 0.2 with interval 0.05. The TR-rank varies from 2 to 8, and the number of TR-factors is 1, 2 and 3.

Fig. 3 reports the result. The Dim$_c$ in the figure represents the number of the coupled TR-factors, and SR$_1$ and SR$_2$ represent the sampling ratios of $T_1$ and $T_2$, respectively. In phase transition, the white patch means a successful recovery whose relative error is less than $1 \times 10^{-6}$, and the black patch means a failed recovery whose relative error is greater than $1 \times 10^{-6}$. The successful area increases when sampling ratio of $T_2$ increases and the number of the coupled TR-factors is fixed. The successful area also increases when the number of the coupled TR-factors increases and sampling ratio of $T_2$ is fixed. This is because the first tensor ring can learn information from the second one with increasing sampling ratio or the number of coupled factors, though the recovery of $T_1$ is beyond its sampling limit. Mathematically, the magnitude of the singular values of Hessian matrix $H_{T_1}$ is SR$_2$/SR$_1$ times the magnitude of the singular values of $H_{T_2}$, hence $H_{T_1}$ is dominated in the updating scheme.

V. CONCLUSION

This paper investigates the coupled tensor completion via tensor ring decomposition and propose a non-convex algorithm by alternating minimization. We also provides a excess risk bound which implies the sampling complexity can be reduced.
to below the theoretical bound. However, the more precise characterization of this reduction is needed as a future work.

**APPENDIX A**

**OPTIMIZATION ON COUPLED TR-FACTORS OF $\mathcal{R}_1$ AND $\mathcal{R}_2$**

To solve problem [13], we calculate the second-order partial derivatives of the objective function with respect to $\alpha_{i_d}$, $\beta_{i_d}$, $\gamma_{i_d}$, respectively. We write the objective function as

$$f_{i_d} = \frac{1}{2} \left[ \alpha_{i_d}^{(d)} , \beta_{i_d}^{(d)} , \gamma_{i_d}^{(d)} \right]^T \left[ \begin{array}{ccc} \alpha_{i_d}^{(d)} & \beta_{i_d}^{(d)} & \gamma_{i_d}^{(d)} \\ \alpha_{i_d}^{(d)} & \beta_{i_d}^{(d)} & \gamma_{i_d}^{(d)} \\ \alpha_{i_d}^{(d)} & \beta_{i_d}^{(d)} & \gamma_{i_d}^{(d)} \end{array} \right] \left[ \begin{array}{c} \alpha_{i_d}^{(d)} \\ \beta_{i_d}^{(d)} \\ \gamma_{i_d}^{(d)} \end{array} \right] - \frac{1}{2} \left[ \alpha_{i_d}^{(d)} , \beta_{i_d}^{(d)} , \gamma_{i_d}^{(d)} \right]^T \left[ \begin{array}{ccc} P_d & H_d & C_d \\ P_d & H_d & C_d \\ P_d & H_d & C_d \end{array} \right] \left[ \begin{array}{c} \alpha_{i_d}^{(d)} \\ \beta_{i_d}^{(d)} \\ \gamma_{i_d}^{(d)} \end{array} \right] + \frac{1}{2} \left[ \alpha_{i_d}^{(d)} , \gamma_{i_d}^{(d)} \right] \tilde{H}_d \left[ \alpha_{i_d}^{(d)} , \gamma_{i_d}^{(d)} \right]^T - \frac{1}{2} \left[ \alpha_{i_d}^{(d)} , \gamma_{i_d}^{(d)} \right] \tilde{P}_d \left[ \alpha_{i_d}^{(d)} , \gamma_{i_d}^{(d)} \right]^T + \frac{1}{2} \left[ \alpha_{i_d}^{(d)} , \gamma_{i_d}^{(d)} \right] \tilde{H}_d \left[ \alpha_{i_d}^{(d)} , \gamma_{i_d}^{(d)} \right]^T$$

then there is

$$\frac{\partial f_{i_d}}{\partial \alpha_{i_d}} = \frac{\partial}{\partial \alpha_{i_d}} \left( \frac{1}{2} \left[ \alpha_{i_d}^{(d)} , \beta_{i_d}^{(d)} , \gamma_{i_d}^{(d)} \right]^T \left[ \begin{array}{ccc} \alpha_{i_d}^{(d)} & \beta_{i_d}^{(d)} & \gamma_{i_d}^{(d)} \\ \alpha_{i_d}^{(d)} & \beta_{i_d}^{(d)} & \gamma_{i_d}^{(d)} \\ \alpha_{i_d}^{(d)} & \beta_{i_d}^{(d)} & \gamma_{i_d}^{(d)} \end{array} \right] \left[ \begin{array}{c} \alpha_{i_d}^{(d)} \\ \beta_{i_d}^{(d)} \\ \gamma_{i_d}^{(d)} \end{array} \right] - \frac{1}{2} \left[ \alpha_{i_d}^{(d)} , \beta_{i_d}^{(d)} , \gamma_{i_d}^{(d)} \right]^T \left[ \begin{array}{ccc} P_d & H_d & C_d \\ P_d & H_d & C_d \\ P_d & H_d & C_d \end{array} \right] \left[ \begin{array}{c} \alpha_{i_d}^{(d)} \\ \beta_{i_d}^{(d)} \\ \gamma_{i_d}^{(d)} \end{array} \right] + \frac{1}{2} \left[ \alpha_{i_d}^{(d)} , \gamma_{i_d}^{(d)} \right] \tilde{H}_d \left[ \alpha_{i_d}^{(d)} , \gamma_{i_d}^{(d)} \right]^T - \frac{1}{2} \left[ \alpha_{i_d}^{(d)} , \gamma_{i_d}^{(d)} \right] \tilde{P}_d \left[ \alpha_{i_d}^{(d)} , \gamma_{i_d}^{(d)} \right]^T + \frac{1}{2} \left[ \alpha_{i_d}^{(d)} , \gamma_{i_d}^{(d)} \right] \tilde{H}_d \left[ \alpha_{i_d}^{(d)} , \gamma_{i_d}^{(d)} \right]^T \right)$$

$$= \alpha_{i_d}^{(d)} \left( \tilde{H}_d^{(d)11} + \tilde{H}_d^{(d)11} \right) + \tilde{g}_{i_d}^{(d)} ,$$

where

$$\tilde{g}_{i_d}^{(d)} = \beta_{i_d}^{(d)} \left( \tilde{H}_d^{(d)21} + \tilde{H}_d^{(d)21} \right) + \gamma_{i_d}^{(d)} \left( \tilde{H}_d^{(d)21} + \tilde{H}_d^{(d)21} \right)$$

Thus we have

$$\frac{\partial^2 f_{i_d}}{\partial \alpha_{i_d} \partial \alpha_{i_d}} = \tilde{H}_d^{(d)11} + \tilde{H}_d^{(d)11}$$

$$\frac{\partial^2 f_{i_d}}{\partial \alpha_{i_d} \partial \beta_{i_d}} = \tilde{H}_d^{(d)12} + \tilde{H}_d^{(d)12}$$

$$\frac{\partial^2 f_{i_d}}{\partial \alpha_{i_d} \partial \gamma_{i_d}} = \tilde{H}_d^{(d)12} + \tilde{H}_d^{(d)12}$$

$$\frac{\partial^2 f_{i_d}}{\partial \alpha_{i_d} \partial \beta_{i_d}} = \tilde{H}_d^{(d)12} + \tilde{H}_d^{(d)12}$$

Then we deduce

$$\frac{\partial f_{i_d}}{\partial \beta_{i_d}} = \frac{\partial}{\partial \beta_{i_d}} \left( \frac{1}{2} \beta_{i_d}^{(d)} \tilde{H}_d^{(d)22} \left( \tilde{H}_d^{(d)12} + \tilde{H}_d^{(d)12} \right) + \beta_{i_d}^{(d)} \left( \tilde{H}_d^{(d)12} + \tilde{H}_d^{(d)12} \right) \right)$$

Similarly, we derive

$$\frac{\partial^2 f_{i_d}}{\partial \beta_{i_d} \partial \gamma_{i_d}} = \tilde{H}_d^{(d)12} + \tilde{H}_d^{(d)12}$$

Incorporating [18] – [20], we derive the Hessian matrix

$$\tilde{H}_d = \begin{bmatrix} \tilde{H}_d^{(d)11} & \tilde{H}_d^{(d)11} & \tilde{H}_d^{(d)11} & \tilde{H}_d^{(d)11} \\ \tilde{H}_d^{(d)12} & \tilde{H}_d^{(d)12} & \tilde{H}_d^{(d)12} & \tilde{H}_d^{(d)12} \\ \tilde{H}_d^{(d)21} & \tilde{H}_d^{(d)21} & \tilde{H}_d^{(d)21} & \tilde{H}_d^{(d)21} \\ \tilde{H}_d^{(d)22} & \tilde{H}_d^{(d)22} & \tilde{H}_d^{(d)22} & \tilde{H}_d^{(d)22} \end{bmatrix}.$$

**APPENDIX B**

**PROOF OF THEOREM 1**

A. The expectation of a linear combination of products of independent variables

Supposing $X_i$, $i = 1, \ldots, m$ are the independent Chi-square variables with a same degree of freedom, say $k$, and $Y_j$, $j = 1, \ldots, n$ are independent variables with the same distribution as $X_i$. It follows that the density function of $X_i$ is $p(x_i) = \frac{1}{\Gamma(k/2)} (x/2)^k \exp(-x/2)$, where $\Gamma(k)$ is the Gamma function. The expectation of $\sqrt{\alpha} \prod_{i=1}^{m} X_i + \beta \prod_{j=1}^{n} Y_j$ is given by the multiple integral

$$\int_0^{+\infty} \cdots \int_0^{+\infty} \frac{\alpha}{\sqrt{\Gamma(k/2)}} \prod_{i=1}^{m} x_i \cdot \beta \prod_{j=1}^{n} y_j \prod_{i=1}^{m} \prod_{j=1}^{n} (x_i + y_j) \, dx_1 \cdots dx_m \, dy_1 \cdots dy_n.$$
brackets. For example, the notation \( \binom{a}{b} \) stands for the divergent integral \( \int_{0}^{+\infty} x^{a-1} dx \). The indicator \( \phi_{n} \triangleq (-1)^{n} / \Gamma (n + 1) \) will be used in the series expressions when applying the method of brackets. The Pochhammer symbols defined as \( (b)_{n} \triangleq \Gamma (n + b) / \Gamma (b) \) is a systematic procedure in the simplification of the series. An exponential function \( \exp (-x) \) can be represented as \( \sum_{n} \phi_{n} x^{n} \) in the framework of the method of brackets. Another useful rule is that a multinomial \( (x_{1} + \cdots + x_{m})^{n} \) is expanded as \( \sum_{(n)} \phi_{n} x_{1}^{n_{1}} \cdots x_{m}^{n_{m}} \).

We start with the two rules, slinging out the terms that do not contain the integral variables, merging the remained terms and substituting the integral with brackets, the integral is transformed into

\[
\frac{1}{2} \Gamma \left( \frac{1}{2} \right) \Gamma \left( n \right) \sum_{w_{1}, w_{2}=0}^{+\infty} \phi_{w_{1}, w_{2}} \alpha^{w_{1}} \beta^{w_{2}} \langle w_{1} + w_{2} - 1/2 \rangle,
\]

To continue, we choose \( w_{1} \) and \( w_{2} \) as free variables and eliminate the other brackets. The result shown below follows from the rule that the value assigned to \( \sum_{n} \phi_{n} f (n) \langle cn+d \rangle \) is \( f (n^*) \Gamma (-n^*) / |c| \), where \( n^* \) is obtained from the vanishing of the bracket.

\[
\Gamma \left( \frac{1}{2} \right) \Gamma \left( n \right) \sum_{w_{1}, w_{2}=0}^{+\infty} \phi_{w_{1}, w_{2}} \alpha^{2w_{1}} \beta^{2w_{2}} \langle w_{1} + w_{2} - 1/2 \rangle.
\]

The matrix of coefficients left has rank 1, thus it produces two series as candidates for the values of the integral, one per free variable. The simplified formulation derives from the Pochhammer symbols and the transformation \( -b_{n} = (-1)^{n} / (1 - b_{n}) \), which can be proved by the Euler’s reflection formula. The final result is obtained by introducing the hypergeometric function \( \phi_{F_{2}} (\ldots) \).

1) Case 1: The variable \( w_{1} \) is free. Thus Plugging \( w_{1}^* = 1/2 - w_{2} \) into the rule gives

\[
T_{1} = \sqrt{2} \frac{\Gamma \left( n \right) \Gamma \left( \frac{1}{2} \right)}{\Gamma \left( n + \frac{1}{2} \right)} \sum_{w=0}^{+\infty} \left( \frac{1}{2} \right)^{w} \frac{\Gamma \left( n \right) \Gamma \left( \frac{1}{2} \right)}{\Gamma \left( n + \frac{1}{2} \right)} \langle (w_{1} + w_{2} - 1/2) \rangle.
\]

2) Case 2: The variable \( w_{2} \) is free. Then Plugging \( w_{2}^* = 1/2 - w_{1} \) into the rule yields

\[
T_{2} = \sqrt{2} \frac{\Gamma \left( n \right) \Gamma \left( \frac{1}{2} \right)}{\Gamma \left( n + \frac{1}{2} \right)} \sum_{w=0}^{+\infty} \left( \frac{1}{2} \right)^{w} \langle w_{1} + w_{2} - 1/2 \rangle.
\]

Which of the two expressions is used depends on the convergence condition of the hypergeometric function. The first one is employed if \( n \geq m \), otherwise the second one is considered.

B. Bounding the expectation of the \( F \)-norm of two coupled tensors

Without loss of generality, suppose \( X \) and \( Y \) are 3-order tensors coupled on their first \( L \) modes, with a same TR-rank and a dimensional size. To calculate \( \mathbb{E} \sqrt{\alpha \|X\|_{F}^{2} + \beta \|Y\|_{F}^{2}} \), we first note that \( \|X\|_{F} \) is submultiplicative, thus

\[
\mathbb{E} \sqrt{\alpha \|X\|_{F}^{2} + \beta \|Y\|_{F}^{2}} \leq \|L \sum_{l=1}^{\alpha} \|U^{(l)}\|_{F} \|D_{1} \|_{F} \|U^{(d_{1})}\|_{F} + \beta \|D_{2} \|_{F} \|D_{2} \|_{F} \leq \|D_{1} \|_{F} \|D_{2} \|_{F} \|Y^{(d_{2})}\|_{F}^{2}
\]

holds for \( D_{1} \geq D_{2} \) and

\[
\mathbb{E} \sqrt{\alpha \|X\|_{F}^{2} + \beta \|Y\|_{F}^{2}} \leq \sqrt{\frac{\alpha}{\beta}} \frac{\|D_{1} \|_{F} \|D_{2} \|_{F}}{\|D_{2} \|_{F}} \|Y^{(d_{2})}\|_{F}^{2}
\]

holds for \( D_{2} \geq D_{1} \).

C. Bounding the excess risk

A subset \( x_{m_{1}} \), containing \( m_{1} = |S_{1} \cup T_{1}| \) elements is sampled uniformly without replacement from \( \text{vec} (X) \). We concatenate \( x_{m_{1}} \) and \( y_{m_{2}} \) as a vector \( z_{m} \triangleq [x_{m_{1}}; y_{m_{2}}] \) where \( m = |S \cup T| \).

where

\[
\mathcal{Q}_{m,n} (l_{T}, z_{m}) = \mathbb{E}_{z_{m}} \left[ \sup_{y_{T} \in \mathcal{H}} l_{T} (z_{k}, t_{k}) - l_{T} (z_{n}, t_{n}) \right],
\]

where \( z_{n}, n \in \{1, \ldots, m-1\} \) is a random subset of \( z_{m} \) containing \( n \) elements sampled uniformly without replacement and \( z_{k} \triangleq z_{m} \backslash z_{n} \).

Under the hypothesis \( \mathcal{H} \) mentioned before, let \( m = 2n = |T_{1} \cup T_{2}| \), then the expectation of the permutilational
Rademacher complexity is bounded as

\[
\mathbb{E}_{z_m} \left[ \hat{Q}_{m,m/2}(I_T, z_m) \right] \\
 \leq \mathbb{E}_{z_m} \left\{ \left( 1 + \frac{2}{\sqrt{2\pi |T|}} - 2 \right) \mathbb{E} \left[ \sup_{x, \gamma \in \mathcal{H}} \left\{ \frac{1}{|T|} \mathbf{e}^T I_T (z_m, t_m) \right\} \right] ^2 \right\} \\
 \leq \Lambda \left( 1 + \frac{2}{\sqrt{2\pi |T|}} - 2 \right) \mathbb{E} \left[ \sup_{x, \gamma \in \mathcal{H}} \left\{ \frac{1}{|T|} \mathbf{e}^T m \right\} \right] \\
 \leq \Lambda \left( 1 + \frac{2}{\sqrt{2\pi |T|}} - 2 \right) \mathbb{E} \left[ \sup_{x, \gamma \in \mathcal{H}} \left\{ \|m\|_F \right\} \right] \\
 \leq \Lambda \left( 1 + \frac{2}{\sqrt{2\pi |T|}} - 2 \right) \mathbb{E} \left[ \sup_{x, \gamma \in \mathcal{H}} \left\{ \|m\|_F \right\} \right] \\
 \leq \Lambda \left( 1 + \frac{2}{\sqrt{2\pi |T|}} - 2 \right) \mathbb{E} \left[ \sum_{i=0}^{m} \left( \frac{(|T_{1,i}| - 1)}{m - i} \right) \left( \frac{(|T_{2,i}| - 1)}{m - i} \right) \left\{ \left\{ |x_i| \right\}^2 + \sum_{i \neq j \in z_m} \left\{ |\bar{x}_{i} \right\} \right\} \right] \\
 \leq \Lambda \left( 1 + \frac{2}{\sqrt{2\pi |T|}} - 2 \right) \mathbb{E} \left[ \sum_{i=0}^{m} \left( \frac{|T_{1,i}|}{m - i} \right) \left( \frac{|T_{2,i}|}{m - i} \right) \left\{ \left\{ |x_i| \right\}^2 + \sum_{i \neq j \in z_m} \left\{ |\bar{x}_{i} \right\} \right\} \right] \\
 = \sqrt{\Lambda} \left( 1 + \frac{2}{\sqrt{2\pi |T|}} - 2 \right) \mathbb{E} \left[ \left\{ |x_i| \right\}^2 + \left\{ |\bar{x}_{i} \right\} \right] \\
 \leq \Lambda \left( 1 + \frac{2}{\sqrt{2\pi |T|}} - 2 \right) \mathbb{E} \left[ \left\{ |x_i| \right\}^2 + \left\{ |\bar{x}_{i} \right\} \right]
\]

where the first inequality follows from the Theorem 3 in [40], the second inequality is a result of Rademacher contraction, the third inequality comes from the Hlder’s inequality, the forth inequality is a consequence of arithmetic-mean-quadratic mean inequality. Due to the hypothesis \( \mathcal{H} \), the final bounds can be derived by plugging (21) and (22) with \( \alpha = 1/|T_1| \) and \( \beta = 1/|T_2| \).

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