Odd Chern-Simons Theory, Lie Algebra Cohomology and Characteristic Classes

Jian Qiu and Maxim Zabzine

Department of Physics and Astronomy, Uppsala university,
Box 516, SE-75120 Uppsala, Sweden

Abstract
We investigate the generic 3D topological field theory within AKSZ-BV framework. We use the Batalin-Vilkovisky (BV) formalism to construct explicitly cocycles of the Lie algebra of formal Hamiltonian vector fields and we argue that the perturbative partition function gives rise to secondary characteristic classes. We investigate a toy model which is an odd analogue of Chern-Simons theory, and we give some explicit computation of two point functions and show that its perturbation theory is identical to the Chern-Simons theory. We give concrete example of the homomorphism taking Lie algebra cocycles to $Q$-characteristic classes, and we reinterpreted the Rozansky-Witten model in this light.
1 Introduction

Topological field theory (TFT) is a well-developed subject spreading across physics and mathematics. TFT can be viewed as a very powerful machine for producing the topological invariants. If one looks at TFT from the point of view of path integral, then one should deal with the appropriate gauge symmetries and thus with BRST formalism. A usual way of constructing a topological field theory is that one proposes a set of BRST transformations for a set of fields, and then write down an action which is usually a BRST-exact term plus perhaps some additions of topological nature (e.g., such as the pull back of the Kähler form of the target manifold). Apart from the insight required to come up with a reasonable BRST rule, one is constantly faced with the problem that the BRST transformation closes only on-shell, and the problem of determination of observables etc. Thus dealing with all these issues is somewhat ad hoc.

The Alexandrov-Kontsevich-Schwarz-Zaboronsky (AKSZ) construction [1] is an elegant and powerful tool to engineer the topological field theories in various dimensions within the Batalin-Vilkovisky (BV) formalism. Many problem are all avoided with one single ingenious stroke of the AKSZ construction. Its beauty lies in that it converts the finding of the BRST transformation rules to a purely geometrical problem, namely, one seeks the so called $Q$-structure on a target manifold. The $Q$-structure is by definition an odd nilpotent vector field. This does not seem much improvement so far, but with the unifying language of graded manifolds (GrMfld), the possible $Q$-structures are well understood. For example, on a degree 1 GrMfld, a $Q$-structure encodes the data of a Lie algebroid. Thus the BRST rule will be related to the Lie algebroid differential for the target manifold, e.g. see [4] for the construction of a whole gamut of topological models. The AKSZ construction is done naturally within the BV formalism, which then clarifies the problem of on-shell closure of BRST transformation and at the same time gives geometrical interpretation to the otherwise unilluminating routine of gauge fixing.

On the other hand, in physics we are equipped with the handy tool of path integral which, albeit being totally formal, allows one to manipulate the formalisms conveniently. And it is no new phenomenon that one could use a topological field theory and path integral to produce non-trivial mathematical results. In this work we offer the systematic study of the perturbative AKSZ-BV topological theories. Moreover we suggest the interpretation of the perturbative correlators and partition function in these theories. In particular we concentrate our attention on 3-dimensional (3D) theories.

The present work is heavily influenced by several pieces of work along this direction. First the Chern-Simons perturbation theory [2], where the evaluation of the partition func-
tion led to the physical construction of invariants of 3-manifolds. Later Kontsevich [12] exposed the connection between the Feynman integral and graph (co)homology (namely the Feynman integral gives a cocycle in the graph complex); and thereby the construction of the low dimension topological invariants. Another piece of inspiration came from the works of Schwarz [22] and Lazarev and Hamilton [9], especially the latter, who used the tool of BV path integral to furnish a proof of the claim made by Kontsevich. Their proof made an excursion of first showing that the path integral is a cocycle in the cohomology of the Lie algebra of Hamiltonian vector fields. Since the latter is proven to be isomorphic to the graph cohomology, one can first send a graph chain to an element in the Lie algebra chain complex, then evaluate this chain in the path integral giving the desired cochain. We will show that all these ideas arise naturally within AKSZ-BV framework. Indeed BV path integral always give rise to a certain cocycles and the perturbative theory offers the concrete way of calculating them. Although we look mainly at 3D AKSZ models, many ideas can be extended to other AKSZ theories.

Being furnished with a cocycle coming from BV path integral one is led naturally to construct some characteristic classes using the Chern-Weil homomorphism. Now instead of plugging in the curvature two form to an invariant polynomial of Lie algebra, we plug in a flat connection into a cocycle. This is exactly what happens when we calculate the partition function of the AKSZ theory.

One purpose of this work is clarify what exactly the perturbation theory of these AKSZ models is computing. The partition function for such models turns out to be the (hopefully non-vanishing) characteristic classes of the relevant $Q$-(super)manifold. In the work by Lyakhovich, Mosman and Sharapov [14], they are able to use graph cohomology to find three infinite series of characteristic classes of any $Q$-manifold. Especially, their B,C series depend on the properties of the homological vector $Q$ alone and survive even for a flat manifold. In a nutshell, due to the observation $L_Q \partial_i \partial_j Q^k = 0$ where $Q^i \partial_i$ is a homological vector field, they show if one plugs the second Taylor coefficient of $Q^i$ into certain graphs made of 3-valent vertices, out comes some $Q$-characteristic classes. This version of the characteristic classes for the flat bundle is again tied to a second construction of graph cycle (except they are using it backwards) by Kontsevich. The construction is intuitive, one obtains a graph cocycle by plugging into the vertices the Taylor coefficient of the Hamiltonian lift of $Q$ and connecting edges using the symplectic form. We shall show that this is indeed what happens when one

---

1 Their graph complex is slightly different from what we consider and it is isomorphic to the cohomology of Lie algebra of formal vector fields vanishing at the origin.

2 This gives their B,C series of invariants, their A series come from two valent graphs and requires the vanishing of Pontryagin class.
evaluates the partition function for the AKSZ models. For such model the interaction term is just the Hamiltonian lift of \( Q \), and for anyone who knows anything about perturbation theory in physics, the evaluation of the Feynman diagrams are just about plugging the Taylor coefficients of the interaction terms.

The article is organized as follows, the BV formalism is reviewed in section 2. We also show that the quantum observables form a closed algebra and the path integral gives a cocycle in Lie algebra cohomology of formal Hamiltonian vector fields generated by these observables. In section 3 we review some relevant background material. The characteristic classes of flat bundles are recalled and we discuss the scenario in which they can arise in path integral. The isomorphism between Lie algebra (co)homology and graph (co)homology is sketched without any claim for rigor. We give the construction of the AKSZ model in section 4 in particular, the free theory gives a cocycle in Lie algebra cohomology of formal Hamiltonian vector fields of the target space. To do serious perturbation calculation, one needs to gauge fix the model; this is the topic of section 5. There we present the set of Feynman rules and we investigate the perturbative partition function. We claim that the partition function corresponds to a characteristic class of appropriate flat bundle. Sections 6-8 are dealing with different examples of 3D AKSZ models. In section 6 we consider the AKSZ model associated to the \( Q \)-equivariant vector bundle. In section 7 we examine 3D AKSZ model constructed on a flat symplectic space \( \mathbb{R}^{2m} \), and we show that it is a kind of odd analogue of Chern-Simons perturbation theory and the weight functions associated with the diagrams are identical to Chern-Simons and the Rozansky-Witten model. Finally as a grand finale section 8 we put all the ingredients together and reformulate the Rozansky-Witten model in the light of Lie algebra cohomology and the characteristic classes of flat bundles. At the end of the paper there are two appendices which contains some technical calculations relevant for the paper.

2 BV Formalism

We give the essential facts about BV formalism in this section and show that the standard manipulations in the BV framework allow us to interpret the path integral as a certain cocycle.

The original BV formalism was for the supermanifolds, namely manifolds with \( \mathbb{Z}_2 \) grading, yet the formalism may be carried to \( \mathbb{Z} \)-graded manifolds making some of the results stronger. A degree \( n \) graded manifold is by definition locally parameterized by coordinates of degrees 0 up to \( n \). And these coordinates are glued together through degree preserving transition
functions (for more details on the graded manifolds, see [23] and [17]). An example of such a manifold is: $T[1]M$; the notation being: $M$ is an ordinary manifold, $T[1]$ means that we take the total space of the tangent bundle of $M$ and we assign the fiber coordinate degree 1. This is an odd manifold since the highest fiber degree is 1. An example of even graded manifolds is $T^*[2]T[1]M$, locally, we have $x^\mu$ as the coordinate of $M$, the coordinate $v^\mu$ parameterizing the fiber of $T[1]M$ is given degree 1, the coordinates dual to $x^\mu$, $v^\mu$ are $P_\mu$ and $q_\mu$ of degree $-\deg(x)+2 = 2$ and $-\deg(v)+2 = 1$ respectively. The advantage of using graded manifolds instead of supermanifolds is that degrees of these coordinates eventually correspond to the ghost number in a physical theory.

The BV manifold is a manifold where the space of functions is equipped with the structure of BV algebra which is defined as the Gerstenhaber algebra (odd Poisson algebra) together with an odd Laplacian. Simply speaking the BV manifold is a manifold equipped with an odd symplectic form. The archetypical example of such spaces is of the form $T^*[−1]M$, where $M$ itself is allowed to be a graded manifold. The reason for the degree $−1$ shift is to make the BRST transformation of ghost number $+1$ in the end. For definiteness, let us take the coordinate of $M$ as $x$ and that of the fiber $x^+$, then the canonical symplectic form of the BV space is just $\omega = dx \wedge dx^+$. If $M$ has dimension $n$, then a Lagrangian submanifold (LagSubMfld) $L$ is a dimension $n$ submanifold of the BV space such that $\omega|_L = 0$. Suppose that a volume form $\mu(x)$ is given for $M$, then we have also a volume form for $T^*[−1]M$ which is $\mu^2(x) \wedge^n dx^+ \wedge^n dx$. With the density $\mu(x)$ we can define a Laplacian

$$\Delta \equiv \frac{1}{\mu^2(x)} \frac{\partial}{\partial x^+} \mu^2(x) \frac{\partial}{\partial x^+},$$

which can be checked to satisfy $\Delta^2 = 0$.

The key fact of the BV formalism [21] is the statement that the integral of a function $f$ over a LagSubMfld is invariant under continuous deformation of the LagSubMfld provided $f$ satisfies $\Delta f = 0$; and the integral of $\Delta$-exact functions gives zero. This statement is just the Stokes theorem in disguise [24]. By Fourier transforming the odd degree coordinates in $T^*[−1]M$ (namely, exchanging the coordinate and its dual momentum), the Laplacian $\Delta$ becomes the de Rham differential $d$ over the degree even submanifold of $T^*[−1]M$. And the integration of functions over LagSubMfld is reformulated as integration of forms along submanifolds. In contrast to $d$, $\Delta$ is not a derivation (does not obey the Leibnitz rule), in fact, when acting on a product of functions, it gives

$$\Delta(fg) = (\Delta f)g + (-)^{|f|}f(\Delta g) + (-)^{|f|}\{f, g\}, \quad (1)$$
where \(\{\cdot,\cdot\}\) is the odd Poisson bracket corresponding to the odd symplectic form \(\omega\).

We are going to explore the consequence of (1). The usual use of BV formalism is in the quantization of gauge system: suppose one has an action \(S\) satisfying \(\Delta e^{-S} = 0\), then one seeks a suitable \(L\) such that the restriction of \(S\) to \(L\) has a non-degenerate quadratic term. The choice of the LagSubMfld is the choice of the gauge fixing condition, and due to \(\Delta e^{-S} = 0\), the end result should not depend on the choice of gauge fixing. Having chosen \(L\), one then inserts operators \(O\) with \(\Delta (O e^{-S}) = 0\) into the path integral and obtain the expectation value of \(O\). It is usually stated that the path integral is a homomorphism sending elements of \(H(T^*[−1]M,\Delta_q)\) (\(\Delta_q \equiv e^S\Delta e^{-S} = \Delta - \{S,\cdot\}\)) to the number fields. Due to the fact that \(\Delta\) is not a derivation, there is no ring structure defined for the cohomology group \(H(T^*[−1]M,\Delta_q)\). This point of view is of course correct, however, it misses some rich structure innate in the BV formalism. In fact the cohomology group of \(\Delta\) is quite boring, as \(\Delta\) can always be Fourier transformed into a de Rham differential. One of the purposes of this paper is to elaborate some results in the paper by Schwartz [22] and by Hamilton and Lazarev [9]. The first crucial observation made by Schwartz is that the quantum observables (namely functions satisfying \(\Delta_q f = 0\)) form a closed algebra under the Poisson bracket, more concretely, by using (1)

\[
\{f, g\} = (-1)^{|f|}\Delta(fg) - (-1)^{|f|}(\Delta f)g - f(\Delta g) \\
= (-1)^{|f|}(\Delta_q(fg) + \{S, fg\}) - (-1)^{|f|}\{S, f\}g - f\{S, g\} = (-1)^{|f|}\Delta_q(fg) ,
\]

hence the bracket quantity \(\{f, g\}\) remains closed under \(\Delta_q\). But the bracket here does not yield a super Lie algebra structure for the quantum observables: the difference between the two is a shift in the assignment of the degree. More concretely, the Poisson bracket appearing here is odd and obeys \(\{f, g\} = -(-1)^{|f|+1)(|g|+1}\{g, f\}\), while for a super Lie algebra we would like to have graded anti-commutativity or \(\{f, g\} = -(-1)^{|f||g|}\{g, f\}\). So a shift of the degree by 1 solves the problem. This shift can be achieved by considering the Lie algebra of Hamiltonian vector fields generated by the observables instead.

If \(\omega\) is the symplectic form of the BV space, then the Hamiltonian vector field generated by a function is defined such that

\[
\mathcal{L}_X f \equiv \{f, g\} ,
\]

where \(g\) is any function on the BV space and \(X_f = \iota df \omega^{-1}\). Since \(\omega\) has degree \(-1\), \(\deg X_f = \deg f + 1\). We have the relation \([X_f, X_g] = X_{\{f, g\}}\), note the degree shift converts the Gerstenhaber algebra on the right hand side to the super Lie algebra on the left hand side. The Hamiltonian vector fields \(X_f\) are in one to one correspondence with Hamiltonians \(f\) modulo constants. Thus we can fix all functions to vanish at a given point to
remove this ambiguity. The Chevalley-Eilenberg (CE) complex of the Lie algebra of such Hamiltonian vector fields at degree $n$ is spanned by $n$-chain

$$c_n = X_{f_0} \wedge \cdots \wedge X_{f_n}.$$  

The boundary operator for such a chain is the conventional one

$$\partial (X_{f_0} \wedge X_{f_1}, \ldots, X_{f_n}) = \sum_{i<j} \text{sgn}_{ij} (-1)^{|f_i|} X_{\{f_i, f_j\}} \wedge X_{f_0} \wedge \cdots \wedge \hat{x}_{f_i} \wedge \cdots \wedge \hat{x}_{f_j} \wedge \cdots \wedge X_{f_n},$$

where the sgn is the Koszul sign factor $(-1)^{(|f_0| + \cdots + |f_i| - 1) |f_i| + (|f_0| + \cdots + |f_j| - 1) |f_j| - |f_i| |f_j|}$, which accounts for the minus’s caused by moving $X_{f_i}$ and $X_{f_j}$ to the front. Here we make a remark about the convention of graded (anti)-commutativity. One can either understand $X_{f} \wedge X_{g}$ as graded anti-commutative, i.e. $X_{f} \wedge X_{g} = -1 \times (-1)^{|X_{f}| |X_{g}|} X_{g} \wedge X_{f}$. Another point of view is to shift the degree $X_{f}$ up by 1 and call it graded commutative: $X_{f} \wedge X_{g} = (-1)^{|X_{f}| + 1} (-1)^{|X_{g}| + 1} X_{g} \wedge X_{f}$. The two views make no difference so long as $X_{f}$ has degree 0, yet in working with graded manifolds, the latter is more advantageous, for then all the commutation relations are controlled by the degree. In the above Koszul sign, we used the latter convention, therefore $\deg X_{f} = \deg f - 1 + 1$ ($-1$ because the symplectic form has degree $-1$) and $X_{f} \wedge X_{g} = (-1)^{|f| |g|} X_{g} \wedge X_{f}$.  

The cochains of the CE complex are just the dual of the chains $c^n : c_n \to \mathbb{R}$. The differential $\delta$ for the cochain is induced from $\partial$ through $\delta c^{n+1}(c_{n+1}) = c^n(\partial c_{n+1})$.

These definitions fit neatly into the BV framework as follows. Consider all functions $f_i$ which satisfy $\Delta q f_i = 0$ then the the corresponding Hamiltonian vector fields $X_{f_i}$ give rise to a closed Lie algebra $A_q$ since

$$[X_{f_i}, X_{f_j}] = (-1)^{|f_i|} X_{\Delta_q(f_i f_j)}.$$  

We can construct the $n$-chains and boundary operator for $A_q$ in the way described above. Using the property (2) we can prove the following identity

$$\Delta_q(f_0 f_1 \cdots f_n) = \sum_{i<j} \text{sgn}_{ij} (-1)^{|f_i|} \{f_i, f_j\} f_0 \cdots \hat{f}_i \cdots \hat{f}_j \cdots f_n.$$  

(3)

In BV context we have a naturally defined cochain, which evaluated on $X_{f_0} \wedge X_{f_1} \wedge \cdots \wedge X_{f_n}$ according to the following expression

$$c^n(X_{f_0} \wedge X_{f_1} \wedge \cdots \wedge X_{f_n}) \equiv \int \mathcal{L} f_0 f_1 \cdots f_n e^{-S} \in \mathbb{R}.$$  

(4)
One can check easily that it is a multilinear functional with the correct symmetry properties. This cochain defined through the path integral is in fact a \textit{cocycle}. This is shown by using the definition of the coboundary operator and the relation [3]

\[
\delta c^n(\mathbf{X} f_0 \wedge \mathbf{X} f_1 \wedge ... \wedge \mathbf{X} f_{n+1}) = c^n (\partial (\mathbf{X} f_0 \wedge \mathbf{X} f_1 \wedge ... \wedge \mathbf{X} f_{n+1}))
\]

\[
= \int_\mathcal{L} \Delta_q (f_0 f_1 ... f_{n+1}) e^{-S} = \int_\mathcal{L} \Delta (f_0 f_1 ... f_{n+1} e^{-S}) = 0,
\]

where in the last step we used the fact the integral of any \(\Delta\)-exact function is zero.

We would like to emphasize that the cochain thus defined \textit{does depend on the choice of the Lagrangian submanifold}. Although each \(f_i\) obeys \(\Delta_q(f_i) = 0\), \(\Delta_q(f_0 \cdots f_n) \neq 0\) in general, so the Stokes theorem does not apply. Hence we denote the cochain by \(c^n_L\) and we study the \(\mathcal{L}\) dependence next. By Schwarz’s explicit construction, every \(\mathcal{L}\) is locally embedded in the BV space as \(T^*[−1]M = T^*[−1]L\); the simplest \(\mathcal{L}\) namely \(M\) itself is such an example. If we denote the coordinates of \(\mathcal{L}\) as \(x^a\) and \(x^+_a\) that of the transverse direction to \(\mathcal{L}\) (\(\mathcal{L}\) is given by \(x^+ = 0\) locally). Then any small deformation is parameterized as

\[
x^+_a = \frac{\partial}{\partial x^a} \Psi(x).
\]

The function \(\Psi\) only depends on \(x\) and may be regarded as the generating function for the canonical transformation going from \(\mathcal{L}\) to \(\mathcal{L} + \delta \mathcal{L}\). Locally, the Laplacian is \(\Delta = \partial_{x^a} \partial_{x^+_a}\), so \(\Delta \Psi = 0\) trivially. Now

\[
\left( \int_{\mathcal{L} + \delta \mathcal{L}} - \int_{\mathcal{L}} \right) f_0 f_1 ... f_n e^{-S} = \int_{\mathcal{L}} \Psi \frac{\partial}{\partial x^a} \frac{\partial}{\partial x^+_a} (f_0 f_1 ... f_n e^{-S}) = \int_{\mathcal{L}} \{ \Psi, f_0 f_1 ... f_n e^{-S} \}
\]

\[
= - \int_{\mathcal{L}} (\Delta(\Psi f_0 f_1 ... f_n e^{-S}) + \Psi \Delta(f_0 f_1 ... f_n e^{-S}))
\]

\[
= - \int_{\mathcal{L}} \Psi \Delta_q(f_0 f_1 ... f_n) e^{-S}.
\]

If we define a new \((n−1)\)-cochain by

\[
\bar{c}^{n-1}(\mathbf{X} f_0 \wedge \mathbf{X} f_1 \wedge ... \wedge \mathbf{X} f_{n-1}) \equiv - \int_{\mathcal{L}} \Psi (f_0 f_1 ... f_{n-1}) e^{-S}.
\]

This cochain is \textit{not} closed, however we have

\[
(c^n_L - c^n_L)(\mathbf{X} f_0 \wedge \mathbf{X} f_1 \wedge ... \wedge \mathbf{X} f_n) = \bar{c}^{n-1}(\partial(\mathbf{X} f_0 \wedge \mathbf{X} f_1 \wedge ... \wedge \mathbf{X} f_n))
\]

\[
= \delta \bar{c}^{n-1}(\mathbf{X} f_0 \wedge \mathbf{X} f_1 \wedge ... \wedge \mathbf{X} f_n).
\]
Our observation is thus, the change of the LagSubMfld changes the cochain $c^n$ by a coboundary $\delta z^{n-1}$. Thus for any choice of $\mathcal{L}$, the path integral gives a representative of the class in the cohomology of the Lie algebra of the quantum observables. Yet two choices of $\mathcal{L}$ that are not homotopic to each other will produce different classes in the cohomology.

So far our discussion has been formal, and may only be applied properly to a finite dimensional BV manifold. While for most cases of interest to physics, the BV space is the space of mappings and hence of infinite dimension. One usually does not have a well defined Laplacian, and the condition $\Delta_q f = 0$ can at best be realized formally. Another drawback is that the relevant Lie algebra cohomology is on the space of mappings, while we quite often would like to ask questions about the properties of the target manifold alone, the formalism developed above becomes unwieldy. In the section 4 we will set up a 3D topological field theory that focuses on the Lie algebra cohomology of Hamiltonian vector fields of the target manifold. The discussion there is along the lines of [9].

But before we do so, we would have to digress a little for some other background material.

### 3 Background material

In this section, we review the relevant background material. We remind the idea behind the construction of characteristic classes of flat bundles. We hint on the application of this construction within BV formalism. We also review the necessary facts concerning Lie algebra homology of formal Hamiltonian vector fields and its relation to the graph homology.

#### 3.1 Characteristic Classes for Flat Bundles

Consider the principal bundle $P$ over base $M$ with structure group $G$,

$$
P \xleftarrow{\quad G \quad} \quad \downarrow \quad M \tag{7}
$$

If we choose the connection $A$ on $P$ with curvature $R$, then $R$ is a Lie algebra valued 2-form on $M$. The procedure we are familiar with is to take an invariant polynomial of the generators of the Lie algebra $g$ (usually a trace or a determinant), and plug in the curvature 2-form $R$. The Chern-Weil theorem guarantees that the resulting form is a closed form and so we have the mapping

$$
\mathbb{C}[g^*]^{Ad G} \rightarrow H^{2k}(M, \mathbb{R}) .
$$
This is the standard construction of the classical characteristic classes for the principle bundles.

A flat bundle is a principal bundle equipped with a connection whose curvature vanishes identically, flat connection. Thus, by the Chern-Weil theory all characteristic classes vanish and it may appear that the flat bundle is close to a trivial bundle. However, it is far from being true. Let us sketch the main idea behind the construction of the characteristic classes for flat bundles, which are also called secondary characteristic classes. Now we use the connection rather than the curvature. For the Lie algebra \( \mathfrak{g} \) there is the CE cochain complex \( c^\bullet = \bigwedge \mathfrak{g}^* \) with the standard CE differential. Instead of invariant polynomials, take any cocycle \( c^n \) in this complex and plug in the connection, resulting in a differential form on the bundle \( P \) given by

\[
c^n \xrightarrow{A} c^n(A, ..., A) \in \Omega^{n+1}(P) .
\]

This mapping from the cochain complex to the differential forms on \( P \) does not yet send cochain differential to de Rham differential. To mend this, one must require the connection to be flat, i.e. it satisfies the Maurer-Cartan equation \( dA + A \wedge A = 0 \). To make it look more familiar, we pick a basis \( t^a \) for the Lie algebra \( \mathfrak{g} \) and we can write the flatness condition as

\[
(dA_a)(t^a) + \frac{1}{2} (A_b \wedge A_c)[t^b, t^c] = 0 ,
\]

where \([ , , ]\) is Lie bracket for \( \mathfrak{g} \). The last identity makes it clear that the flatness condition qualifies the mapping \( (8) \) as a differential graded map, for

\[
dc^n(A, ..., A) = d(A_{a_0} \wedge ... \wedge A_{a_n}) \ c^n(t^{a_0}, ..., t^{a_n}) \\
= -\frac{1}{2} \sum_i (-1)^i A_{a_0} \wedge ... \wedge \underbrace{A_{b} \wedge A_{c} \wedge ... \wedge A_{a_n}}_{i} c^n(t^{a_0}, ..., [t^b, t^c], t^{a_n}) \\
= -\frac{1}{2} A_{a_0} \wedge ... \wedge A_{a_{n+1}}(\delta c^n)(t^{a_0}, ..., t^{a_{n+1}}) .
\]

Moreover, if \( c^n \) is a cocycle in the CE complex, the map \( (8) \) gives us a closed form on \( P \). Thus the flat connection induces the map of the cohomology groups

\[
H^n(\mathfrak{g}, \mathbb{R}) \xrightarrow{A} H^{n+1}(P, \mathbb{R}) \xrightarrow{s} H^{n+1}(M, \mathbb{R}) ,
\]

where the last step involves the choice of the section \( s \) (or trivialization of \( P \)). The above map does not change if we choose another trivialization of \( P \) in the same homotopy class of trivializations. This is the construction of the secondary characteristic classes. This
theory can be applied to the case of infinite dimensional algebras (groups) as well and it plays the central role in the characteristic classes of foliations. For further details about the characteristic classes of the flat bundles the reader may consult the book by Morita [15].

The flat connections appear a lot in physics. Let us discuss the relevant setup in which we generalize this slightly to include not just the Lie algebra valued differential forms but a general $Q$-structure. Recall a $Q$-structure is a degree one vector field satisfying $Q^2 = 0$. As a $Q$-structure is a natural generalization of the de Rham differential, the $Q$-equivariant fiber bundles are the generalization of flat bundles in the following way. Given any fiber bundle $\mathcal{E} \xrightarrow{\pi} \mathcal{M}$, suppose there is $Q$ structure over graded manifold $\mathcal{M}$ and a $\tilde{Q}$ over total space $\mathcal{E}$, which is also graded manifold. The $Q$-equivariantness says $\pi_* \tilde{Q} = Q$. In a local coordinate such $\tilde{Q}$ can be written as (taking $e^I$ as the coordinates of the fiber)

$$\tilde{Q}(x, e^I) = Q(x) + A^I(x, e^I) \frac{\partial}{\partial e^I},$$

where $A^I$ is a vector field along a fiber. $\tilde{Q}^2 = 0$ implies that $A$ satisfies the Cartan-Maurer equation

$$QA + \frac{1}{2}[A, A] = 0,$$

(11)

where $[\ , \ ]$ stands for the Lie bracket of vector fields along the fiber. Thus in this setup the Lie algebra $\mathfrak{g}$ can be identified with the algebra of formal vector fields along the fiber. By using the construction analogous to (9) one obtains $Q$-closed functions by evaluating the $A$ on the cocycle of this infinite dimensional algebra of $\mathfrak{g}$. These $Q$-closed functions are the characteristic classes for the $Q$-structure. As the $Q$-structure includes a wide variety of differentials such as the de Rham, Doubeault, Chevalley-Eilenberg, Poisson-Lichnerowicz etc we have a more uniform way of investigating the characteristic classes associated with these structures.

There is an immediate application of these ideas in the BV path integral framework. Recall from section 2 that

$$c^n(\mathcal{X}_{f_0} \wedge \ldots \wedge \mathcal{X}_{f_n}) = \int_{\mathcal{E}} f_0 \ldots f_n$$

(12)

defines the cocycle for the Lie algebra of divergenceless Hamiltonian vector fields (i.e., $\Delta f = 0$) on BV space. Consider the BV action $S$ which satisfies $\Delta S = 0$. Suppose that the action also depends on some extra parameters and that there exists another odd differential $Q$ acting on those parameters, such that

$$QS + \frac{1}{2}\{S, S\} = 0.$$
This is a quite typical setup in BV theory. Equation \([13]\) appears as a consequence of the classical master equation and the extra parameters can originate from the zero modes of the theory, for example. Now let us evaluate the partition function of this BV theory

\[
Z = \int_{\mathcal{L}} e^{-S} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} c^{n-1}(X_S \wedge \ldots \wedge X_S) ,
\]

where \(c^n(X_S \wedge \ldots \wedge X_S)\) is a cocycle since \(\Delta S = 0\) and it is now a function of the extra parameters. We can show easily that this function is annihilated by \(Q\)

\[
Q c^n(X_S \wedge \ldots \wedge X_S) = \frac{1}{2} c^n(\partial(X_S \wedge \ldots X_S)) = 0 ,
\]

where we used the property \([13]\). The most important example where this situation can arise is of course when we have a bundle structure whose fiber is equipped with an odd symplectic form and the extra parameter is the coordinate of the base. Then the relation \([13]\) is nothing but the \(Q\)-equivariantness condition \([11]\), namely the \(Q\)-structure on the base is lifted to \(\tilde{Q} = Q + \{S, \cdot\}\) in the total space. Within this picture the partition function \(Z\) gives rise to \(Q\) characteristic class (the concrete representative depends on the choice of \(\mathcal{L}\)). Although the present argument is formal, we will argue later that this is a generic feature of 3D TFTs.

3.2 Lie Algebra/Graph Cohomology

In this subsection we review briefly the algebra of formal Hamiltonian vector fields. We will use these material in the next sections.

Consider the vector space \(\mathbb{R}^{2m}\) equipped with the canonical symplectic structure. Let \(\text{Ham}^0_{2m}\) be the Lie algebra of formal (polynomial) Hamiltonian vector fields over \(\mathbb{R}^{2m}\) preserving the origin; let \(\text{Ham}^1_{2m}\) consist of those elements of \(\text{Ham}^0_{2m}\) whose Taylor expansion starts from the quadratic term and finally \(sp(2m, \mathbb{R})\) are those elements whose coefficients are linear. If one chooses to talk about the Hamiltonian function instead, then \(sp(2m, \mathbb{R})\) corresponds to quadratic polynomials, \(\text{Ham}^1_{2m}\) corresponds to cubic or higher polynomials. Let \(C_*(\text{Ham}^0_{2m})\) be the Chevalley-Eilenberg complex of \(\text{Ham}^0_{2m}\) and \(sp(2m, \mathbb{R})\) acts on this complex through the adjoint action. We shall consider the \(sp(2m, \mathbb{R})\) coinvariants\(^3\) of the complex \(C_*(\text{Ham}^1_{2m})\). If we denote such coinvariants as \(H_*(\text{Ham}^0_{2m}, sp(2m, \mathbb{R}))\), then we have the isomorphism due to Kontsevich \([11]\) that

\[
H_*(\text{Ham}^0_{2m}, sp(2m, \mathbb{R})) \sim H_*(\mathcal{G}) ,
\]

\(^3\)The coinvariants, in contrast to the invariants, are the largest quotient of \(C_*(\text{Ham}^0_{2m})\) on which \(sp(2m, \mathbb{R})\) acts trivially, or simply speaking, the orbits of the \(sp(2m, \mathbb{R})\) action.
where \( \mathcal{G} \) is the (undecorated) graph complex. The reason for `modding` out the \( sp(2m, \mathbb{R}) \) subgroup will become clear once we consider this isomorphism from the path integral point of view. The same isomorphism \( (16) \) can be generalized to the superspace \( \mathbb{R}^{2m|k} \) with the even symplectic structure, see \( [8] \).

We use here the same conventions as in the previous section. However we are interested in a different Lie algebra now. We use \( X_f \) to denote a Hamiltonian vector field generated by \( f \) over \( \mathbb{R}^{2m} \) with the canonical symplectic structure. The CE complex will be spanned by the exterior product of the form

\[
c_n = X_{f_0} \wedge \cdots \wedge X_{f_n}.
\]

The Chevalley-Eilenberg boundary operator is

\[
\partial c_n = \sum_{i<j} (-1)^{i+j+1} [X_{f_i}, X_{f_j}] \wedge X_{f_0} \cdots \wedge \hat{X}_{f_i} \cdots \wedge \hat{X}_{f_j} \cdots \wedge X_{f_n}.
\]

By using the relation \([X_f, X_g] = X_{\{f,g\}}\), we can abbreviate

\[
X_{f_0} \wedge \cdots \wedge X_{f_n} \text{ as } (f_0, \cdots, f_n),
\]

and the boundary operator by

\[
\partial (f_0, \cdots, f_n) = \sum_{i<j} (-1)^{i+j+1} (\{f_i, f_j\}, f_0, \cdots \hat{f}_i, \cdots \hat{f}_j, \cdots, f_n).
\]

Apart from the petitt details, the mapping in (16) is easy to understand. Take the Euclidean space \( \mathbb{R}^{2m} \) equipped with the standard symplectic structure \( \sum \Omega_{\mu\nu} dx^\mu \wedge dx^\nu \).

The function \( f \)'s are all polynomials on \( \mathbb{R}^{2m} \), so a given chain corresponds to a sum

\[
c_n = \sum (m_0, m_1, \cdots, m_n),
\]

where \( m_i \) are all monomials. An \( l \)-th order monomial will correspond to an \( l \)-valent vertex in the graph. For every propagator connecting leg \( \mu \) and \( \nu \) one incorporates a factor \( \Omega_{\mu\nu} \) into the coefficient of the graph

\[
(x^\mu x^\nu x^\rho) \sim \text{\small Graph}, \quad (x^\mu x^\nu x^\rho x^\lambda) \sim \text{\small Graph}, \quad \Omega_{\sigma\gamma} ((x^\mu x^\nu x^\sigma), (x^\rho x^\lambda x^\gamma)) \sim \text{\small Graph}.
\]

The Poisson bracket between two monomials of degree \( p \) and \( q \) produces a sum of monomials of order \( p + q - 2 \), so

\[
\partial (\Omega_{\sigma\gamma}(\cdots, (x^\mu x^\nu x^\sigma), (x^\rho x^\lambda), \cdots)) = \cdots + \Omega_{\sigma\gamma}(\cdots, (x^\mu x^\nu x^\sigma, x^\gamma x^\rho x^\lambda), \cdots) + \cdots = \cdots + (\cdots, x^\mu x^\nu x^\rho x^\lambda, \cdots) + \cdots,
\]

13
where we have only focused on the propagator $\sigma_\gamma$ while assuming the legs $\mu\nu\rho\lambda$ are connected to other parts of the graph in a certain way. In the graph language the boundary operator acts as

$$
\partial = \pm \begin{array}{c}
\mu \\
\lambda \\
\rho \\
\nu
\end{array}
; \quad \partial = \pm \begin{array}{c}
\mu \\
\lambda \\
\rho \\
\nu
\end{array}.
$$

So the boundary operator acts on a graph by deleting one propagator. This is exactly the differential for the graph complex. We have omitted lots of details, especially those concerning how to work out the sign factors and the orientation of the graph; the reader may see [8] for a full treatment. The similar construction can be applied to the superspace $\mathbb{R}^{2m|k}$ with even symplectic structure. In what follows we use the Greek letters for even case $\mathbb{R}^{2m}$, while upper case Latin letters for the supercase $\mathbb{R}^{2m|k}$.

## 4 3D AKSZ topological field theory

The Alexandrov-Kontsevich-Schwarz-Zaboronsky (AKSZ) construction [1] allows one to produce a variety of topological $\sigma$-models in a rather canonical fashion. The AKSZ approach uses mapping space between a source supermanifold (graded manifold) and target supermanifold (graded manifold). The source and target manifolds are equipped with additional structures. In this section we define 3D AKSZ model and explore the geometrical meaning of the correlation functions in AKSZ model on a 3D source manifold, the result turns out related to the Lie algebra cohomology of Hamiltonian vector fields on the target space. In particular we make some remarks on the perturbative theory. The correlators of the two 3D AKSZ models we know, namely the Chern-Simons theory and the Rozansky-Witten model, both fit this description.

### 4.1 Construction of AKSZ model

The general construction of the AKSZ models is standard by now. We shall be brief and concentrate our attention only on 3D models. For more details the reader may consult [1] [3] [18].

Consider an even symplectic supermanifold $\mathcal{M}$ with the local coordinates $X^A$ and $|A|$ denotes the degree $X^A$. Suppose the even symplectic form on the target manifold is $\Omega$, and it can be written locally as the differential of the Liouville form $\Omega = d\Xi$. Assume we are in
a Darboux coordinate then $\Xi = X^A \Omega_{AB} \, dX^B$. We will refer to $\mathcal{M}$ as a target. Also let us consider the three dimensional manifold $\Sigma_3$. We are interested in the odd tangent bundle $T[1] \Sigma_3$, where $\xi$ is the bosonic coordinate of $\Sigma_3$ and $\theta$ is the odd fiber coordinate of $T[1] \Sigma_3$. We will refer to $\Sigma_3$ as source. In the following discussion we have chosen the source manifold $\Sigma_3$ to be $S^3$ or more generally a rational homology sphere.

Consider the mapping space $\text{Maps}(T[1] \Sigma_3, \mathcal{M})$ and denote the mapping by $X^A(\xi, \theta)$. The even symplectic form on $\mathcal{M}$ induces an odd symplectic form in the space of mappings $\text{Maps}(T[1] \Sigma_3, \mathcal{M})$ according to

$$\omega = \frac{1}{2} \int_{T[1] \Sigma_3} d^6 z \left( \Omega_{AB} \delta X^A \delta X^B \right), \quad (20)$$

where we write $\xi, \theta$ collectively as $z$ and $d^6 z \equiv d^3 \theta \, d^3 \xi$. Note that each $X$ is a superfield, hence can be expanded into components

$$X(\xi, \theta) = X(\xi) + \theta^a X(\xi)_a + \frac{1}{2} \theta^b \theta^a X(\xi)_{ab} + \frac{1}{3!} \theta^c \theta^b \theta^a X(\xi)_{abc}. \quad (21)$$

The components correspond to forms of different degrees on $\Sigma_3$. When we do not want to spell out all the indices, we will write

$$X(0) = X(\xi) ; \quad X(1) = X(\xi)_a d\xi^a ; \quad X(2) = \sum_{a < b} X(\xi)_{ab} d\xi^a \wedge d\xi^b ; \quad \cdots$$

We may obtain the odd symplectic form written in components by integrating out $d^3 \theta$ in (20) (details left to the appendix)

$$\omega = - \int_{\Sigma_3} d^3 \xi \, \Omega_{AB} \left( \delta X^A_{(3)} \wedge \delta X^B_{(0)} + \delta X^A_{(1)} \wedge \delta X^B_{(2)} \right). \quad (21)$$

The BV space $\text{Maps}(T[1] \Sigma_3, \mathcal{M})$ is infinite dimensional and there is no well defined measure for the path integral, we shall use the naive one

$$\text{vol} = \wedge^{\text{top}} dX_{(0)} \wedge^{\text{top}} dX_{(1)} \wedge^{\text{top}} dX_{(2)} \wedge^{\text{top}} dX_{(3)}.$$

With this volume form we have a naive odd Laplacian

$$\Delta \equiv \int_{\Sigma_3} d^3 \xi \, \left( \Omega^{-1} \right)^A_{AB} (-1)^{|A||B|} \left( \frac{\delta}{\delta X^A_{(3)}(\xi)} \frac{\delta}{\delta X^B_{(0)}(\xi)} + \frac{\delta}{\delta X^A_{(1)}(\xi)} \frac{\delta}{\delta X^B_{(2)}(\xi)} \right).$$

Note that the odd Laplacian is the restriction of a distribution on $\Sigma_3 \times \Sigma_3$ to the diagonal, and is a singular object in this infinite dimensional context. Thus it should be understood
as the limit of a suitably regularized expression (see appendix). The odd Laplacian has a number of the formal properties

\[ \Delta \int_{T[1]\Sigma_3} d^6z \int_{T[1]\Sigma_3} d^6z_1 f(X(z)) = (-1)^f \int_{T[1]\Sigma_3} d^6z \{f(X(z)), g(X(z))\}, \]

\[ \{ \int_{T[1]\Sigma_3} d^6z f(X(z)), \int_{T[1]\Sigma_3} d^6z g(X(z)) \} = - \int_{T[1]\Sigma_3} d^6z \{f(X(z)), g(X(z))\}. \]  

(22)

We refer the reader to the appendix for further details.

Let us choose an odd function \( \Theta \) on \( M \) which satisfies \( \{\Theta, \Theta\} = 0 \) with respect to the even symplectic structure on \( M \). Then the AKSZ construction gives the standard BV action

\[ S = S_{kin} + S_{int} = \int_{T[1]\Sigma_3} d^6z X^A \Omega_{AB} DX^B + X^*\Theta, \]

(23)

where the first term involving the Liouville form is called the kinetic term. The kinetic term is independent of the concrete choice of the Liouville form. \( X^*\Theta \) is pullback of \( \Theta \) through \( X \) to the space of mappings. It serves as the interaction term. We often write the pull back \( X^*\Theta \) simply as \( \Theta \). One can check easily that \( S \) satisfies the classical master equation

\[ \{S, S\} = - \int_{T[1]\Sigma_3} d^6z D(\Xi_A DX^A + \Theta) + X^*\{\Theta, \Theta\} = - \int_{T[1]\Sigma_3} d^6z \{\Theta, \Theta\} = 0, \]

where the first term drops because it is a total derivative and \( \Sigma_3 \) has no boundary. So the only requirement is merely \( \{\Theta, \Theta\} = 0 \). In the expression above and in further discussion we use the same notations for the bracket on \( M \) and for the BV bracket on Maps(\( T[1]\Sigma_3, M \)). Hopefully it is not confusing since it can be understood from the context which bracket is used. Thus the action (23) formally satisfies the quantum master equation \( \Delta e^{-S} = 0 \), by using the first property of \( \Delta \) in (22) and \( \{S, S\} = 0 \). For further reference the component form of kinetic term and the interaction terms are

\[ S_{kin} = \int_{\Sigma_3} d^3\xi \Omega_{AB} \left( - X^A_{(1)} \wedge dX^B_{(1)} + X^A_{(2)} \wedge dX^B_{(0)} \right), \]

(24)

\[ S_{int} = \int_{\Sigma_3} d^3\xi X^A_3 (\partial_A \Theta) + X^A_{(2)} \wedge X^B_{(1)} (\partial_B \partial_A \Theta) + \frac{1}{6} X^A_{(1)} \wedge X^B_{(1)} \wedge X^C_{(1)} (\partial_C \partial_B \partial_A \Theta). \]
In the present discussion we keep in mind only $\mathbb{Z}_2$-grading. The construction can be refined to $\mathbb{Z}$-grading with the source and target being graded manifolds equipped with the extra structure [18], the main example is given by the Courant sigma model. The Chern-Simons theory is special case of the Courant sigma model with the target being $\mathcal{M} = g[1]$, where $g$ is a metric Lie algebra. In principle we will allow the BV theory to depend on extra free parameters, e.g. $\Theta$ may depend on the parameters other then the coordinates of $\mathcal{M}$.

4.2 Formal Properties of Correlators

In this subsection we apply the formal observation from section 2 to 3D AKSZ theory constructed in the previous subsection. The simple observation is that certain subalgebra of quantum observables can be mapped to specific subalgebra of Hamiltonian vector fields on $\mathcal{M}$. Thus the corresponding correlator can be interpreted entirely in term of target space geometry.

Consider the objects of the form $F \equiv \int d^6z \, f(X(z))$. According to the property (22) the BV bracket between $\{F_i, F_j\}$ gets mapped to the even bracket $\{f_i, f_j\}$ on $\mathcal{M}$. Moreover $F$’s are quantum observables if they satisfy

$$0 = \Delta (e^{-S} \int_{T[1]\Sigma_3} d^6z \, f(X(z))) = e^{-S} \int_{T[1]\Sigma_3} d^6z \, Df(X(z)) + X^\ast \{\Theta, f(X)\}.$$

Thus the only requirement on $F$ to be a quantum observable is that the corresponding $f$ commutes with $\Theta$ on $\mathcal{M}$, i.e. $\{\Theta, f\} = 0$. The observables of this type form a closed algebra and we define the correlator using the path integral

$$\langle F_0 F_1 \ldots F_n \rangle \equiv \int L \left( \int d^6z_0 f_0(z_0) \int d^6z_1 f_1(z_1) \cdots \int d^6z_n f_n(z_n) \right) e^{-S} , \quad (25)$$

where $f(z)$ is the short hand of $f(X(z))$. Repeating the formal argument from section 2 and using the fact that $F$’s algebra is mapped to $f$’s algebra on $\mathcal{M}$ we arrive at the conclusion that the correlator

$$\langle F_0 F_1 \ldots F_n \rangle = e^n (X_{f_0} \wedge X_{f_1} \wedge \ldots \wedge X_{f_n}) \quad (26)$$

corresponds to cocycle of the Lie algebra of Hamiltonian vector fields $X_f$ which commute with $X_\Theta$. We once again remark that this way of writing the correlation function (25) agrees with the graded commutativity on the left hand side, namely $X_f \wedge X_g = (-1)^{(f-2+1)(g-2+1)} X_g \wedge X_f$.

\footnote{To avoid ugly expressions we adopt the simple notation for degree, namely $\deg f = |f| = f$. Since the degree is essential only for signs, it appear in the expressions like $(-1)^f$ and thus there should be no confusion.}
\(X_f\) \((-2\) because the symplectic form has degree \(2\)). Because the integration measure \(d^6z\) carries \(-3\) degree, so

\[
\int_{T[1]\Sigma_3} d^6z_1 f(z_1) \int_{T[1]\Sigma_3} d^6z_2 g(z_2) = (-1)^{(f-3)(g-3)} \int_{T[1]\Sigma_3} d^6z_1 g(z_1) \int_{T[1]\Sigma_3} d^6z_2 f(z_2) .
\]

One may compare this to the situation of section 2 the degree shift is due to the odd symplectic form while here it is due to the degree of the measure for the source. Moreover using the standard BV manipulations and the correspondence of BV algebra of \(F\)'s with the algebra on \(M\) one can easily check that our correlator is a cocycle

\[
\delta c^{n-1}(X_{f_0} \wedge X_{f_1} \wedge ... \wedge X_{f_n}) = \int_\mathcal{L} \Delta_q(F_0 F_1 \cdots F_n) e^{-S} = \int_\mathcal{L} \sum_{i<j} (-1)^{1+(f_i+3)(f_0+\cdots+f_{i-1}+3i)+(f_j+3)(f_0+\cdots+f_{j-1}+3j)+(f_i+3)(f_j+3) \times

(-1)^n \{F_i,F_j\} F_0 \cdots \hat{F_i} \cdots \hat{F_j} \cdots F_n e^{-S} = c^{n-1}(\partial(X_{f_0} \wedge X_{f_1} \wedge ... \wedge X_{f_n})) ,
\]

where everything matches including the signs.

4.3 Formal Properties of Perturbation Theory

Our argument so far was quite formal and we would like to convert it into the concrete calculation with the precise properties. For this we will have to resolve to the perturbation theory. Before defining the precise Feynman rules, let us make a few comments about the expected properties of the correlators in the perturbative theory.

We can repeat the formal argument from the previous subsection with \(\Theta = 0\). The correlators are now

\[
\int_\mathcal{L} (\int d^6z_0 f_0(z_0) \cdots \int d^6z_n f_n(z_n)) e^{-S_{kin}} = c^n(X_{f_0} \wedge X_{f_1} \wedge ... \wedge X_{f_n}) ,
\]

which should be cocycles for the Lie algebra of Hamiltonian vector fields on \(M\). We can give some general remarks about the structure of the correlation function. Assuming that \(X^A\) are the Darboux coordinates of the target space, then the kinetic term is \(X^A \Omega_{AB} DX^B\). If this gives a non-degenerate quadratic term when restricted to \(\mathcal{L}\), then we can invert it and obtain the propagator. In the model we have the propagator will basically consist of \(\Omega^{-1}\) and the inversion of the de Rham operator \(G(z_1, z_2)\) (inverting the de Rham operator requires one to choose \(\mathcal{L}\) carefully, more of this later). Applying the Wick theorem the correlation
function (27) is represented by the Feynman diagrams (graphs), and the end result is written schematically as

\[ c^n(X_{f_0} \wedge X_{f_1} \wedge ... \wedge X_{f_n}) = \left\langle F_0 F_1 ... F_n \right\rangle = \sum_{\Gamma} b_{\Gamma} I_{\Gamma}, \]  

where we sum over the graphs \( \Gamma \). Concretely, any graph \( \Gamma \) gives a particular way of routing the propagators. Since every insertion \( F_i \) is integrated with the measure \( d^6 z_i \), we have an integration of these propagators over the configuration space \( T[1]\Sigma_3 \times \cdots \times T[1]\Sigma_3 \). This integral thus associates a graph with a number which is called the weight function \( b_{\Gamma} \). While \( I_{\Gamma} \) corresponds to the combination of derivatives of \( f^{'}s \) (vertices) contracted by \( \Omega^{-1} \) (edges) in the way prescribed by \( \Gamma \). The essential property of sum \( \sum_{\Gamma} b_{\Gamma} I_{\Gamma} \) is that it should give a cocycle for the Hamiltonian vector fields on \( \mathcal{M} \). In the next section we are going to discuss the concrete prescription behind the formula (28).

5 Perturbative expansion of the AKSZ Model

In this section we construct the perturbation theory for 3D AKSZ model constructed in the previous section. From now on we assume that \( \mathcal{M} \) is super(graded) vector space \( \mathbb{R}^{2m|k} \) equipped with the canonical even symplectic form. This assumption is not essential and it is done for the clarity of argument. For the general supermanifold \( \mathcal{M} \) we will have to apply the exponential map in order to map the problem to the vector space and keep the covariance. The example of this full covariant construction will be given when we discuss the Rozansky-Witten theory in section 8.

5.1 Gauge fixing

We continue to discuss the AKSZ model of the previous section. In general it may be tricky to pick a Lagrangian subspace \( \mathcal{L} \) such that the restriction of the action to \( \mathcal{L} \) has a non-degenerate quadratic term. In our case we expand out the kinetic term from the action (23)

\[ S_{kin} = \int_{\Sigma^3} d^3 \xi \ \Omega_{AB} \left( -X^A_{(1)} \wedge dX^B_{(1)} + X^A_{(2)} \wedge dX^B_{(0)} \right), \]

where we have assumed that we are in a Darboux coordinate so that \( \Omega \) is a constant. So the kinetic term to be inverted is the de Rham differential, which has infinitely many zero modes. It is an intricate game trying to find a set of constraints upon the component fields...
such that we are able to invert $d$. However, a brutal gauge fixing is possible and works for all model at the cost of explicit covariance on $\mathcal{M}$ and this is why we discuss the case when $\mathcal{M}$ is a vector space.

To this end, we introduce a metric on the source manifold $\Sigma_3$. Using this metric we can use the Hodge decomposition to break any differential form on $\Sigma_3$ into three parts

$$\omega = \omega^h + \omega^e + \omega^c = \omega^h + d\tau + d^\dagger \lambda,$$

where $h$, $e$, $c$ stand for harmonic, exact and co-exact respectively and $d^\dagger$ is the adjoint of $d$. The three parts are mutually orthogonal under the following non-degenerate pairing

$$\left(\omega_1, \omega_2\right) \equiv \int_{\Sigma_3} \omega_1 \wedge * \omega_2.$$

Since all the component of a superfield $X^A$ are some differential forms on $\Sigma_3$, we can decompose them likewise

$$X^A_{(p)} = (X^A_{(p)})^h + (X^A_{(p)})^e + (X^A_{(p)})^c.$$

The trouble maker is the exact part, since they are annihilated by $d$ and infinite in number. Our choice for $\text{LagSubMfld} \, \mathcal{L}$ will be to simply stay clear of these exact parts. More concretely, we first decompose the symplectic form (21) into

$$\omega \sim \int_{\Sigma_3} d^3x \, \Omega_{AB} \left( \delta(X^A_{(3)})^h \wedge \delta(X^B_{(0)})^h + \delta(X^A_{(3)})^e \wedge \delta(X^B_{(0)})^e \right. \left. + \delta(X^A_{(1)})^h \wedge \delta(X^B_{(2)})^h + \delta(X^A_{(1)})^e \wedge \delta(X^B_{(2)})^e + \delta(X^A_{(1)})^c \wedge \delta(X^B_{(2)})^c \right).$$

Moreover using the integration by parts there are no ee and cc combinations and the harmonic part is decoupled from the rest. Since we are on a rational homology 3-sphere there are no harmonic terms for the 1 and 2 forms, i.e. $(X^A_{(1)})^h = (X^A_{(2)})^h = 0$. We make the following gauge choice

$$(X^A_{(1)})^e = 0, \quad (X^A_{(2)})^e = 0, \quad X^A_{(3)} = 0,$$

where we put to zero both harmonic and exact parts of 3-forms. For further discussion we adopt the following notations $(X^A_{(0)})^h = \chi_0$ which does not appear in the kinetic term and thus corresponds to the zero modes. We will not perform the integral over zero modes! We will treat the zero modes as formal parameters and the integral will be performed only over co-exact fields

$$\int_{\mathcal{L}} F_0 \ldots F_n \, e^{-S_{\text{kin}}} = \int DX^c_{(0)} DX^c_{(1)} DX^c_{(2)} \, F_0 \ldots F_n \, e^{-S_{\text{kin}}}, \quad (29)$$
thus the correlator $\langle F_0 \ldots F_n \rangle$ is a function of $x_0 \in M$. The observable

$$F \equiv \int \Sigma_3 d^3 \xi d^3 \theta \ f (X (\xi, \theta))$$

$$= \int \Sigma_3 d^3 \xi \left( X^{(3)}_A \partial_A f (X(0)) + X^{(2)}_A \wedge X^{(1)}_B \partial_B \partial_A f (X(0)) + \frac{1}{6} X^{(1)}_A \wedge X^{(1)}_B \wedge X^{(1)}_C \partial_C \partial_B \partial_A f (X(0)) \right)$$

upon the gauge fixing and Taylor expansion becomes

$$F = \sum_{k=0}^{\infty} \frac{1}{k!} \int \Sigma_3 d^3 \xi \left( X^c_2 X^c_1 X^{(0)} f(x_0) + \frac{1}{6} (X^c_1)^3 X^{(0)} f(x_0) \right), \quad (30)$$

where we suppressed all indices and wedges. Now we have to contract the co-exact fields according to the Wick theorem. Using the propagator which is proportional to $\Omega^{-1}$ the fields $X^c_1$ are constructed to $X^{(1)}_1$ and the fields $X^c_2$ are contracted to $X^{(0)}$, namely

$$\langle X^A_2 (\xi_1) X^B_1 (\xi_2) \rangle = (\Omega^{-1})^{AB} G^0 (\xi_1, \xi_2), \quad (31)$$

$$\langle X^A_1 (\xi_1) X^B_1 (\xi_2) \rangle = (\Omega^{-1})^{AB} G^1 (\xi_1, \xi_2), \quad (32)$$

where we suppressed superscript $c$ on the fields. Since the observable (30) is composite in terms of elementary fields, we may have to use the point splitting procedure and study the tadpole contributions, if they are there. Equivalently we may work with the superfields and develop the perturbation theory entirely in terms of superfields, thus avoiding the components. For this we have to introduce the adjoint of $D$ which can be written as $D^\dagger = \nabla^a \partial_{\theta^a}$. The superfield admits the Hodge decomposition with respect to $D$ and $D^\dagger$. Thus the gauge fixing corresponds to setting to zero the exact part of superfield.

We leave the explicit formulas for the propogators for the later discussion in section 7.

Now we would like to concentrate on the general features of the Feynman rules.

### 5.2 Feynman rules

The current gauge fixing is very explicit, allowing us to sharpen some features of the perturbation expansion. The main consequence of this gauge fixing is that \emph{every diagram will have even number of vertices, all of which are 3-valent and there are no tadpoles.}

As discussed in the previous subsection the Feynman rules are defined by the $(k+2)$- and $(k+3)$-valent vertices (30) and the propagators (31 32). The first observation is that there will be no 2-valent vertices since

$$\int \Sigma_3 d^3 \xi \ X^c_2 X^c_1 \partial^2 f (x_0) = 0 \quad (33)$$
is identically zero. For the next observation it can be useful to think about the Feynman rules in terms of superfields. Let us look at the correlator

$$\int \mathcal{L} F_0 \ldots F_n e^{-S_{\text{kin}}} = \int \mathcal{L} \int d^6 z_0 f_0(z_0) \ldots \int d^6 z_n f_n(z_n) e^{-S_{\text{kin}}} ,$$

where $f_i(z_i) = f_i(X(z_i))$. We have a total of $(n + 1) \int d^3 \theta$. The propagator $\langle X(z_1) X(z_2) \rangle$ is quadratic in $\theta$’s. Since we have to saturate all $\theta$-integration there is the following relation between the number $\mathcal{V}$ of vertices and the number $\mathcal{P}$ of propagators

$$3\mathcal{V} = 2\mathcal{P} .$$

Equivalently we can make the similar argument within the component form of the perturbation theory. The integration over the configuration space $\Sigma_3 \times \ldots \times \Sigma_3$ requires a $3(n+1)$-form. The propagator is a 2-form on $\Sigma_3 \times \Sigma_3$ since according to (24) we have propagators between $X(1), X(1)$ and between $X(0), X(2)$. Thus to absorb all integration we have to require (34).

The property (34) says that in order for the diagram to be non-zero there should be 3-valent vertices on the average. For example, if there is 4-valent vertex then it should accompanied by 2-valent vertex. However we have argued that 2-valent vertices vanish identically. Therefore we can conclude that only 3-valent vertices contribute

$$\int_{\Sigma_3} d^3 \xi X^A_{(0)} X^B_{(2)} X^C_{(1)} \partial_C \partial_B \partial_A f(x_0) ,$$

$$\frac{1}{6} \int_{\Sigma_3} d^3 \xi X^A_{(1)} X^B_{(1)} X^C_{(1)} \partial_C \partial_B \partial_A f(x_0) .$$

Since only 3-valent graphs contribute, we need the even number of 3-valent vertices to contract all legs. Therefore only graphs with the even number of vertices give non-zero contribution. Now we have to discuss the tadpoles, the situation when the vertex leg is contracted with another leg from the same vertex. The tadpoles contain the following contribution

$$\partial_A \partial_B \partial_C f(x_0) (\Omega^{-1})^{AB} ,$$

One may wonder that this argument is too rough. At best we can say that 2-valent vertex $\int d^3 \xi X^c_{(2)} X^c_{(1)} \partial^2 f(x_0)$ is a surface term and due to the singularities in the propagators there may be non-trivial contributions of 2-valent vertices. However one may perform more careful analysis taking into account the possible singularities of the propagator and arrive at the same conclusion that 2-valent vertices do not contribute.

Strictly speaking the correlator is 2-form on $(\Sigma_3 \times \Sigma_3 - \text{diagonal})$. However the singularity of the propagator along the diagonal is not enough to spoil the argument.
which is identically zero due to the fact that we contract (graded) symmetric combination \( \partial_A \partial_B \) with (graded) antisymmetric \( (\Omega^{-1})^{AB} \). Thus all tadpoles are automatically zero in the theory.\footnote{It is important to stress that on \( \Sigma_3 \) we can do systematically the point-splitting regularization by picking nowhere vanishing vector field.}

Thus the correlator \( \langle F_0 F_1 ... F_n \rangle \) has the form \( \sum_{\Gamma} b_{\Gamma} I_{\Gamma} \), where \( \Gamma \)’s are all 3-valent graphs with \( (n + 1) \) vertices. The number \( b_{\Gamma} \) is an integral of the collection of propagators \( G \) over \( \Sigma_3 \times ... \times \Sigma_3 \) dictated by the graph \( \Gamma \). While \( I_{\Gamma} \) is collection of third derivatives \( \partial^3 f_i(x_0) \) contracted by \( \Omega^{-1} \) in the way dictated by the 3-valent graph \( \Gamma \). Thus \( I_{\Gamma} \) is a function of zero modes \( x_0 \). The explicit example of the calculation of the correlator is presented later.

5.3 The properties of correlators

In previous subsection we discussed the calculation of the correlators in the perturbative theory corresponding to 3D AKSZ models. Now we would like to go back to our formal BV arguments about the properties of the correlators, see subsections 4.2 and 4.3. We want to understand if those arguments are applicable to the perturbative theory, maybe with the possible refinements.

The perturbative correlator associated with the the collection of functions \( f_0, f_1, ..., f_n \) on target \( \mathcal{M} \) is defined as follows

\[
\mathcal{C}^n(\mathbb{X}_{f_0} \wedge \mathbb{X}_{f_1} \wedge ... \wedge \mathbb{X}_{f_n}) = \langle F_0 F_1 ... F_n \rangle = \sum_{\Gamma} b_{\Gamma} I_{\Gamma}(x_0) ,
\]

and it depends on zero modes parametrized by \( \mathcal{M} \) itself. We choose not to integrate over zero modes due to the fact that they do not enter the perturbative theory. Moreover quite often the integration over zero modes is either not well-defined or even when it is defined we may miss some interesting structures if we perform the integral right away. From the formal BV arguments we expect that (37) is cocycle of Lie algebra of Hamiltonian vector fields on \( \mathcal{M} \). However now \( \mathcal{C}^n \) is a function on \( \mathcal{M} \) itself. Thus we are dealing with the cochain \( \mathcal{C}^n \) taking values in the function on \( \mathcal{M} \) and one would expect that the differential \( \delta \) should be modified. The natural modification looks as follows

\[
(\delta \mathcal{C}^n)(\mathbb{X}_{f_0} \wedge ... \wedge \mathbb{X}_{f_{n+1}}) = \sum_{i < j} (-1)^{l_{ij}} \mathcal{C}^n(\mathbb{X}_{\{f_i, f_j\}} \wedge ... \mathbb{X}_{f_i} \wedge ... \mathbb{X}_{f_j} \wedge ... \mathbb{X}_{f_{n+1}}) \\
- \sum_i (-1)^{n_i} \mathcal{C}^n(\mathbb{X}_{f_0} \wedge ... \mathbb{X}_{f_i} \wedge ... \mathbb{X}_{f_{n+1}}) ,
\]
where we assume that \((n + 1)\) is even and the following sign conventions are valid

\[
\begin{align*}
    l_{ij} &= f_i + (f_i + 3)(f_0 + \cdots f_{i-1} + i3) + (f_j + 3)(f_0 + \cdots f_{j-1} + j3) + (f_i + 3)(f_j + 3), \\
p_i &= (f_i + 3)(f_0 + \cdots f_{i-1} + i3).
\end{align*}
\]

Indeed the formula (38) can be derived from the first principle with the BV framework. We have to treat carefully the contribution of ”zero modes” to the odd Laplacian operator and a regularization of the odd Laplacian is required in order to make some of the manipulations well-defined. It all can be done and we present the BV derivation of the formula (38) in the Appendix. The final claim is that the perturbative correlator is cocycle with the values in functions on \(\mathcal{M}\).

\[
(\delta c^n)(X_{f_0} \wedge \cdots X_{f_{n+1}}) = \delta \left( \sum \Gamma b_{\Gamma} I_{\Gamma}(x_0) \right) = 0, \tag{39}
\]

where \(\delta\) is defined by the formula (38).

In order to avoid the discussion of the covariance on \(\mathcal{M}\) we consider the case when \(\mathcal{M} = \mathbb{R}^{2m|k}\) with even constant symplectic structure \(\Omega\). In this case the correlator (37) is automatically \(osp(2m|k, \mathbb{R})\) invariant since our model has global \(osp(2m|k, \mathbb{R})\) symmetry. We believe that the property (39) is true for any functions \(f_i\) on \(\mathbb{R}^{2m|k}\) with the constant even symplectic structure (see subsection 7 for some simple explicit check). Now if we fix the point \(x_0\) (e.g, choose it to be an origin \(x_0 = 0\)) and consider the polynomial functions \(f_i\) with property \(f_i(x_0) = \partial f_i(x_0) = 0\), namely members of \(\text{Ham}^0_{2m|k}\), then the last term (38) disappears and we can regard \(c^n\) as a cocycle with the values in \(\mathbb{R}\). Since the construction of \(c^n\) as \(\sum b_{\Gamma} I_{\Gamma}\) is \(osp(2m|k, \mathbb{R})\)-invariant we get that \(c^n\) is the representative of the relative cohomology class

\[
H^\bullet(\text{Ham}^0_{2m|k}, osp(2m|k, \mathbb{R}), \mathbb{R}).
\]

Another observation by Kontsevich which is natural within the present BV context is that the following cochain in the graph complex

\[
\sum_{\Gamma} b_{\Gamma} \Gamma^\ast
\]

is a closed. The rough proof of this goes as follows. We take any graph chain in \(\mathcal{G}\) and find the corresponding chain in the CE complex \(\mathcal{C}(\text{Ham}^0_{2m|k}) \equiv \wedge \cdot \text{Ham}^0_{2m|k}\) according to section 3. Evaluating this CE chain in the path integral results in the number \(b_{\Gamma}\), which shows \(b_{\Gamma}\) is a graph cocycle. For a more careful proof of this statement see reference [9].
can illustrate this argument by the following example. Recall from the equation (19) the correspondence between the graph complex and the CE complex $C^*(\text{Ham}^{0}_{2m|k})$. Suppose we are given the following 3-valent graph with two vertices

$$\Gamma = \begin{array}{c}\end{array}$$

(40)

To construct a cochain mapping this graph to the number field, one can first map it to an element of CE complex

$$\Gamma \longrightarrow \Omega_{AB}\Omega_{CD}\Omega_{KL} x^A x^B \wedge x^C x^K \wedge x^L x^D \wedge x^B,$$

and then evaluate this Lie algebra chain in the path integral according to equation (25). If one defines the two point correlator as

$$\langle X^A(z_1)X^B(z_2) \rangle \equiv \Omega^{AB} G(z_1, z_2),$$

then the result of evaluating the Lie algebra chain is

$$\Omega_{AB}\Omega_{CD}\Omega_{KL} c^1 (x^A x^B \wedge x^C x^K \wedge x^L x^D \wedge x^B) \sim \int_{T[1]k_{-3}} d^6z_1 d^6z_2 G(z_1, z_2)^3 \equiv b_{\Gamma}.$$

Since the mapping between the graph complex and the CE chain complex is an isomorphism, and moreover the path integral is a cocycle in the CE cochain complex, we conclude that $\sum_{\Gamma} b_{\Gamma} \Gamma^*$ is a cocycle.

### 5.4 Partition function

In this subsection we study the perturbative partition function for the action (23) with the interaction term. We want to apply the general ideas about the characteristic classes of flat bundles reviewed in subsection 3.1 to the perturbative partition function of 3D AKSZ models.

Using the gauge fixing and Feynman rules from the previous subsections the partition function has the following expansion

$$Z(x_0) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} c^{n-1}(x^{\Theta} \wedge \ldots \wedge x^{\Theta}),$$

(41)
where $c^n$ is cocycle evaluated at $\Theta$

$$c^n(\Theta_0 \wedge ... \wedge \Theta_0) \equiv \int \mathcal{L} \int d^6z_0 \Theta(z_0) \cdots \int d^6z_n \Theta(z_n) \ e^{-S_{\text{kin}}} = \sum_{\Gamma} b_{\Gamma} c_{\Gamma}(x_0), \quad (42)$$

with $c_{\Gamma}(x_0)$ is constructed by contracting $\partial^3 \Theta(x_0)$ with $\Omega^{-1}$ according to the graph $\Gamma$. We stress that we do not perform the integral over zero modes which are parametrized by the target $\mathcal{M}$ and thus our partition function is function of $x_0$. However in the Chern-Simons theory $\Theta$ is cubic in fields and thus $\partial^3 \Theta(x_0)$ are constants and we end up with the constant partition function, in this particular example.

Since the function $\Theta$ on $\mathcal{M}$ satisfies $\{\Theta, \Theta\} = 0$ there is homological vector field $Q^A(x_0) = \Omega^{AB} \partial_B \Theta$, which squares to zero (i.e., the Lie bracket $Q^A \partial_A Q^B = 0$). Thus there is natural differential acting on the functions on $\mathcal{M}$ and we can define a cohomology group $H_Q(M)$. We would like to argue that $Q \cdot c^n = 0$ and as a result $Q \cdot Z(x_0) = 0$. Therefore the partition function $Z(x_0)$ can be understood as some sort of $Q$-characteristic class, i.e. the element of $H_Q(M)$.

Let us start from very elementary proof of this fact. As we argued before $c^n(x_0)$ is cocycle with respect to the differential $\delta$ defined by the formula (38). Thus we have the following chain of relations

$$Q \cdot c^n = \{\Theta(x_0), c^n\} \sim c^n(\Theta_0 \wedge ... \wedge \Theta_0) = 0 , \quad (43)$$

where we used $\delta c^n = 0$ and $\{\Theta, \Theta\} = 0$. Although this derivation is correct and simple, it misses some important geometrical aspects. Now we will resolve to more elaborate argument, but with clear geometrical meaning.

For simplicity, we take the target space $\mathcal{M}$ to be a vector space and identify its tangent space with the manifold itself $T\mathcal{M} = \mathcal{M} \times \mathcal{M}$, both are equipped with a symplectic structure and a bracket. Let us try to compute the partition function $Z(x_0)$ of the AKSZ model. The standard method is to split any field into the classical part and a fluctuation part $X = x_0 + \xi$. $x_0$ is treated as the background while the fluctuation $\xi$ is taken to parameterize the fiber of the tangent bundle of $\mathcal{M}$ at $x_0$. For a general curved manifold the simple splitting $x_0 + \xi$ does not make sense, one has to use the exponential map to identify a neighborhood of the tangent bundle with the neighborhood of $x_0$; this is what we do in section [8].

From the master equation $\{\Theta, \Theta\} = 0$, we want to derive some sort of Cartan-Maurer equation for the bundle $T\mathcal{M}$. We Taylor expand $\{\Theta, \Theta\} = 0$ around $x_0$ in powers of $\xi$, and
get a series of equations

\begin{align}
0 &= 2(\partial_C \partial_A \Theta)(\Omega^{-1})^{AB} \partial_B \Theta, \\
0 &= (\partial_A \Theta)(\Omega^{-1})^{AB} \partial_C \partial_B \Theta + (\partial_C \partial_A \Theta)(\Omega^{-1})^{AB} \partial_B \Theta, \\
&\quad \ldots \\
0 &= \frac{2}{n!} (\partial_A \Theta)(\Omega^{-1})^{AB} \partial_C \ldots \partial_C \partial_B \Theta \\
&\quad \quad + \frac{1}{n!} \sum_{p=1}^{n-1} \binom{n}{p} (\partial_C \ldots \partial_A \Theta)(\Omega^{-1})^{AB} \partial_C \ldots \partial_C \partial_B \Theta. 
\end{align}

(44)

If we define \( \Theta' \) as

\[
\Theta' = \sum_{n=2}^{\infty} \frac{1}{n!} (\partial_C \ldots \partial_C \Theta(x_0)) \xi^C_1 \ldots \xi^C_n,
\]

the series of equations except the first one can be packaged into the following compact form

\[
(\partial_A \Theta(x_0))(\Omega^{-1})^{AB} \frac{\partial}{\partial x^B} \Theta'(x_0) + \frac{1}{2} \{\Theta', \Theta'\} \xi = 0,
\]

where the bracket is written as \( \{.,.\} \xi \) to stress that it is the bracket of the fiber of \( T\mathcal{M} \). This is then our favorite Cartan-Maurer equation

\[
Q^B(x_0) \frac{\partial}{\partial x^B} \Theta'(x_0) + \frac{1}{2} \{\Theta', \Theta'\} \xi = 0. 
\]

(45)

Now we regard \( \Theta'(x_0, \xi) \) as a function on \( T\mathcal{M} = \mathcal{M} \times \mathcal{M} \). Since the zero term in expansion \( \Theta(x_0) \) will not be joined by the propagators, the connected diagrams in \( c^n \) from (41) will be given by

\[
\int \mathcal{L} \int d^6 z_0 \Theta'(z_0) \ldots \int d^6 z_n \Theta'(z_n) e^{-S_{\text{kin}}},
\]

(46)

which can be regarded as cocycle of Lie algebra formal Hamiltonian vector fields along the fiber \( T\mathcal{M} \) (i.e., \( \xi \)-direction). By construction \( \Theta'(x_0, 0) = 0 \) and \( \partial_\xi \Theta'(x_0, 0) = 0 \) and thus the expression (46) is annihilated by \( \delta \) as defined in (38), but without the last term. From (45) \( \Theta' \) can be thought of as flat connection on bundle \( \mathcal{M} \times \mathcal{M} \) with the Lie algebra being a Lie algebra of formal vector fields \( \text{Ham}^0_{2m|k} \) and the base differential being \( Q \). The perturbative calculation can be understood as plugging the flat connection \( \Theta' \) into a cocycle \( c^n \) and obtaining the characteristic class in \( H_Q(\mathcal{M}) \) of the flat bundle \( \mathcal{M} \times \mathcal{M} \). Thus according to the general discussion around the equation (15), the correlator \( \langle \int \Theta' \ldots \int \Theta' \rangle \) will be
annihilated by $Q^4(x_0)$. But it may be helpful to understand this in a concrete context. If we adopt the particular gauge fixing given in section 5, then $x_0$ is naturally taken to be the harmonic 0-form part of the fields $X^h_{(0)}$, $\xi$ is the non-harmonic part $\xi \sim X^e + X^c$. Recall from the previous discussion, the perturbation expansion picks up only the part of $\Theta'$ that is cubic in $\xi$, and the correlator $\langle \int \Theta' \cdots \int \Theta' \rangle$ is given by tri-valent graphs. Since the higher powers of $\Theta'$ does not enter the computation, we should be able to understand the invariance of the correlator under $Q$ in a direct way. Indeed, using the third equation of (44), we have

$$Q(\partial^3 \Theta(x_0) \xi^3) = -\frac{1}{6} \{ \partial^2 \Theta \xi^2, \partial^3 \Theta \xi^3 \} \xi,$$

and recall that the quadratic Hamiltonian functions generate the $osp$ rotations, so $Q$ will act on a correlator $\langle \int \partial^3 \Theta \xi^3 \cdots \int \partial^3 \Theta \xi^3 \rangle$ as a rotation in the $\xi$ space. But the correlator clearly has an invariance under such rotations. The homomorphism induced by (45) gives us a mapping

$$H^*(\text{Ham}^0_{2m|k}, osp(2m|k, \mathbb{R}), \mathbb{R}) \longrightarrow H_Q(\mathcal{M}),$$

which offers a better explanation for (43). We have already explained the modding out of $osp(2m|k, \mathbb{R})$. Taking the Lie algebra to be the formal Hamiltonian vector fields has two advantages, first it has more stable cohomology classes than, say, the Lie algebra of formal vector fields preserving zero (in which case, there is only three infinite series of cocycles, see [14, 6]). Larger number of cocycles means more characteristic classes after plugging the flat connection into cocycles. Secondly any foliation with a symplectic structure transverse to the leaf naturally gives rise to a flat connection taking values in the Lie algebra of Hamiltonian vector fields [7]. Indeed any symplectic graded(super)manifold $\mathcal{M}$ with the nilpotent Hamiltonian will give rise to the flat connection on $T\mathcal{M}$ upon using the exponential map.

To summarize, since the partition function of the AKSZ model will consist of series of (42), each of which is invariant under homological vector field $Q$, the AKSZ model calculates the characteristic class associated to $Q$. If the integration over zero modes is well-defined then one can calculate the characteristic number for the corresponding class. This a possible way of producing characteristic classes associated with a $Q$-structure using path integral for 3D AKSZ models and this agrees with the prescription given in [14] where the authors showed that basically all that matters are the 3-valent graphs.

Finally let us make comments about the partition function. There is a dual construction of graph cycles as follows. Suppose we have an odd Hamiltonian function $\Theta$ satisfying $\{\Theta, \Theta\} = 0$, and $\Theta$ is at least quadratic in its Taylor expansion. We take the cubic or higher
Taylor coefficients as vertices, then we follow the graph and connect the vertices together using $\Omega^{-1}$, take the graph as in (40) (denoting $\partial_A \cdots \partial_B \Theta|_{x=0}$ as $\Theta_{A \cdots B}$)

$$\Gamma \rightarrow \partial_A \Omega_{CD} \Omega_{KL} \Theta_{ACK} \Theta_{BDL} \equiv c_\Gamma,$$

and the chain $\sum c_\Gamma$ is a graph cycle. Now we have seen two dual constructions of the graph (co)cycle, when we compute the partition function of a general AKSZ model with the action (23), each term in the perturbation expansion is of the form

$$Z_{AKSZ} = \infty \sum_{n=0} (-1)^n \frac{1}{n!} \int \langle \sum b_\Gamma \Gamma^*, \sum c_\Gamma \Gamma \rangle = \sum b_\Gamma c_\Gamma,$$

which realizes the pairing of the two dual constructions.

6 Example 1: $Q$-Equivariant Bundle

In this subsection, we give a slightly more concrete example of the general discussion above, preparing the way for the Rozansky-Witten model. We consider an example of $Q$-equivariant vector bundle $E \rightarrow M$ (see the discussion around the equation (11)) and we set up the corresponding 3D AKSZ model which provides the realization of the characteristic classes described in section 3.

We assume that $M = T[1]M$ with $M$ being usual smooth manifold and the fiber of $E$ has a symplectic structure $\Omega_{AB}$, and we assign to it degree 2. The Lie algebra of the structure group of the bundle is the Lie algebra of formal Hamiltonian vector fields. Suppose for definiteness, the $Q$-structure of the base is the de Rham differential and it is lifted to a $Q$-structure $\tilde{Q}$ acting on the total space $E$. This action preserves $\Omega_{AB}$ so $\tilde{Q}$ can be written as

$$\tilde{Q} = v^\mu \frac{\partial}{\partial X^\mu} + v^\mu \{ A_\mu(X,e), \cdot \}_{\Omega},$$

for some function $A_\mu$ of the total space $E$. Of course, we need $\partial_{[\mu} A_{\nu]} + \{ A_\mu, A_\nu \} = 0$ to ensure $\tilde{Q}^2 = 0$

To set up an AKSZ model we need a Hamiltonian lift of $\tilde{Q}$. We do this within the minimal symplectic realization of $E$. Denote the local coordinate of this symplectic space to
be $X^\mu, P_\mu, v^\mu, q_\mu, e^A$, where $e^A$ is the fiber coordinate of $\mathcal{E}$ and $\nu^\mu$ is that of $T[1]M$ etc. The degree assignment is 0 2 1 1 0 respectively. This big space has the symplectic structure

$$\omega = \delta P_\mu \wedge \delta X^\mu + \delta q_\mu \wedge \delta v^\mu + \frac{1}{2} \Omega_{AB} \delta e^A \wedge \delta e^B,$$

with $\deg \Omega = 2$. Then we can lift $\tilde{Q}$ into a Hamiltonian function

$$\Theta = P_\mu v^\mu + v^\mu A_\mu.$$

Using $\Theta$ as the interaction term, we have the standard 3D AKSZ action

$$S = \int_{T[1]\Sigma^3} d^6z \left( P_\mu D X^\mu + q_\mu D v^\mu + \frac{1}{2} e^A \Omega_{AB} D e^B + P_\mu v^\mu + v^\mu A_\mu(X, e) \right).$$

We adopt the same gauge fixing by breaking every field into the harmonic and non-harmonic parts and setting to zero all the exact fields. Since $q^c$ only appears in the kinetic term $q^c D v^c$, we can integrate it out enforcing $D v^c = 0$ and hence $v^c = 0$, i.e. $v$ is harmonic. We are then left with

$$S = \int_{T[1]\Sigma^3} d^6z \left( (P_\mu)^c D(X^\mu)^c + \frac{1}{2} (e^A)^c \Omega_{AB} D(e^B)^c 
+ (v^\mu)^h (P_\mu)^h + (v^\mu)^h A_\mu(X^h + X^c, e^c + e^h) \right).$$

We integrate out $P^c$ enforcing $X^c = 0$:

$$S = \int_{T[1]\Sigma^3} d^6z \left( \frac{1}{2} e^c \Omega D e^c + v^h P^h + (v^\mu)^h A_\mu(X^h, e^c + e^h) \right).$$

Remember we are regarding the harmonic fields as parameters rather than dynamical variables, we can for example put $e^h, P^h$ to zero, and reduce the action down to its minimal ingredients

$$S = \int_{T[1]\Sigma^3} d^6z \left( \frac{1}{2} e^c \Omega D e^c + (v^\mu)^h A_\mu(X^h, e^c) \right).$$

It is then clear from the earlier discussion that the perturbation expansion will have the Taylor coefficients of $(v^\mu)^h A_\mu(X^h, e^c)$ as interaction vertices and the propagators will contract $e^c$’s together. The result of the path integral is a function of $X^h(0)$ and $v^h(0)$ only, which is just a differential form on $M$. This form is closed by construction, and is the secondary characteristic class.
7 Example 2: Odd Chern-Simons theory

Here we present a toy model on a vector space that reproduces the same weight function $b_T$ as in Chern-Simons and Rozansky-Witten theory. This model also provides us with cocycles in Lie algebra cohomology, setting the stage for section 8.

7.1 Chern-Simons theory

In this subsection we briefly remind some well-known perturbative aspects of the Chern-Simons theory. The 3D Chern-Simons theory is defined for any metric Lie algebra $g$ by the following classical action

$$S_{CS} = \frac{k}{4\pi} \int_{\Sigma_3} d^3\xi \left[ \eta_{\alpha\beta} A_{(1)}^\alpha \wedge dA_{(1)}^\beta + \frac{2}{3} f_{\alpha\beta\gamma} A_{(1)}^\alpha \wedge A_{(1)}^\beta \wedge A_{(1)}^\gamma \right],$$

(48)

where $A_{(1)}$ is a connection 1-form on $\Sigma_3$, $\eta$ is the metric and $f$ is the structure constant on $g$. The Chern-Simons theory can be embedded into BV-framework through the AKSZ action

$$\int_{T[1]\Sigma_3} d^6z \left[ \eta_{\alpha\beta} A^\alpha D^\beta + \frac{2}{3} f_{\alpha\beta\gamma} A A A \right],$$

(49)

where $A$ is a degree 1 superfield valued in a Lie algebra understood as

$$T[1]\Sigma_3 \rightarrow g[1]$$

and the odd symplectic structure is

$$\int_{T[1]\Sigma_3} d^6z \ \eta_{\alpha\beta} \delta A^\alpha \wedge \delta A^\beta.$$

The naive gauge fixing of (49) with $A_{(3)} = A_{(2)} = 0$ leads to the action (48) which is not suited for the perturbative theory. We have to resolve to the gauge fixing we have discussed by setting the exact parts of the fields to zero. Namely we get

$$S_{CS} = \int_{\Sigma_3} d^3\xi \ \text{Tr} \left[ A_{(1)}^c \wedge dA_{(1)}^c + A_{(2)}^c \wedge dA_{(0)}^c + \frac{2}{3} A_{(1)}^c A_{(1)}^c A_{(1)}^c + A_{(2)}^c [A_{(1)}^c, A_{(0)}^c] \right],$$

(50)

where for the sake of clarity we suppressed $\eta$ and $f$. One can recognize in this action $A_{(1)}$ as the connection 1-form, $A_{(0)}$ as the ghost $c$ and $A_{(2)}$ as the anti-ghost $d^c\bar{c}$. While the Lagrange multiplier appearing in the standard gauge fixing in [2] has been integrated out here forcing
every field to be co-exact. Therefore our gauge fixing is equivalent to the gauge fixing used in [2] for the Chern-Simons action (expanded around a trivial connection).

If we look at the correlators in the perturbation theory then according to the BV-argument we have to get a cocycle of Lie algebra of formal Hamiltonian vector fields on $g[1]$ with the even symplectic structure given by metric $\eta$. The functions on $g[1]$ are $\wedge^* g^*$ and thus the perturbation theory gives us map

$$c^n : \wedge^* g^* \otimes \wedge^* g^* \otimes \ldots \otimes \wedge^* g^* \rightarrow \mathbb{R},$$

which is cocycle with respect to differential $\delta$ defined previously. But of course calculating the Lie algebra cocycles is hardly the principle use of Chern-Simons theory.

### 7.2 Odd Chern-Simons theory

We take a $2m$ dimensional vector space $M = \mathbb{R}^{2m}$ equipped with the standard symplectic structure $\Omega$. The BV model will be $T[1]\Sigma_3 \rightarrow M$. The action is the free action

$$S = \frac{1}{2} \int_{T[1]\Sigma_3} d^6z \ X^\mu \Omega_{\mu\nu} DX^\nu,$$

where we assign formally the symplectic form grading 2 to match the degree as is done in [16]. The model is interesting because if we perform a naive gauge fixing by setting the 2- and 3-form components of the superfield $X^\mu$ to zero, we get a component action

$$S = \frac{1}{2} \int_{\Sigma_3} d^3\xi \ \Omega_{\mu\nu} \left( -X^\mu_{(2)} \wedge dX^\nu_{(0)} - (X^\mu_{(1)})^c \wedge d(X^\nu_{(1)})^c \right),$$

which can be compared to the free part of Chern-Simons action (48). The only difference between this model and the Chern-Simons is that the symmetric metric $\eta_{\alpha\beta}$ is replaced with the anti-symmetric symplectic form $\Omega_{\mu\nu}$, while the even 1-form $A^\alpha_{(1)}$ is replaced with the odd 1-forms $X^\mu_{(1)}$. The similarity does not stop here, as we go on to look at their perturbation expansion. We will refer to this new theory as odd Chern-Simons theory.

We use the gauge fixing of the previous section. The resulting action is

$$\int_{\Sigma_3} d^3\xi \ \Omega_{\mu\nu} \left( (X^\mu_{(2)})^c \wedge d(X^\nu_{(0)})^c - (X^\mu_{(1)})^c \wedge d(X^\nu_{(1)})^c \right),$$

which again can be compared to free part of the gauge fixed Chern-Simons action (50).

We want to discuss the perturbative theory for odd Chern-Simons model. From the above consideration, it is not surprising that, as far as the weight function $b_T$ is concerned,
these two models (odd and even Chern-Simons) are equivalent. And what is more, although
the Rozansky-Witten model, being an AKSZ model was gauge fixed slightly differently, also
produces the same weight function. Next we look at a two point function show the total
agreement between the odd CS model, the CS model and the Rozansky-Witten model.
The Green’s function may be worked out in a conventional manner, we first insert sources
for the fields and compute the partition function
\[ Z[J] = \int D\tilde{X} \exp \left\{ \int d^3\xi \left( X_2(\Omega)dX_0 - X_1(\Omega)dX_1 + J_1X_2 + J_2X_1 + J_3X_0 \right) \right\}. \]

Complete the square for the action (watch out \( J_1, J_2 \) are odd \( J_3 \) are even)
\[ S[J] = \int d^3\xi \left( (X_2 - \frac{1}{d}J_3\Omega^{-1})\Omega(dX_0 + \Omega^{-1}J_1) - J_1\Omega^{-1}\frac{1}{d}J_3 \right. \\
- \left. (X_1 + \frac{1}{2d}J_2\Omega^{-1})\Omega(dX_1 - \Omega^{-1}\frac{1}{2}J_2) - \frac{1}{4}J_2\Omega^{-1}\frac{1}{d}J_2 \right). \]

So the partition function is
\[ Z[J] = \frac{1}{\Delta} \exp \int d^3\xi \left( -J_1\Omega^{-1}\frac{1}{d}J_3 - \frac{1}{4}J_2\Omega^{-1}\frac{1}{d}J_2 \right), \]
where we used \( \Delta \) to denote the 1-loop determinant factor. Note that the absolute value of
it is the Ray-Singer torsion and it is independent of the metric on \( \Sigma_3 \). The phase of \( \Delta \) is
much more delicate, according to [25], a gravitational Chern-Simons term must be added to
the phase factor of \( \Delta \) to restore the metric independence.

The Green’s functions for \( J_2, J_3 \) are
\[ \frac{1}{d}J_3(u) = *d_u \int dv G(u,v) * J_3(v)g^{1/2}, \]
\[ \frac{1}{d}J_2(u) = \int dv H_{\alpha\beta}(u,v)J_{\alpha\beta}(v)g^{1/2}, \]
where \( G(u,v) \) is just the scalar Green’s function satisfying \( \nabla_u^2 G(u,v) = \delta(u,v)/\sqrt{g} \). We
do not have the Green’s function for \( J_2 \) over arbitrary \( \Sigma_3 \), so we write it as \( H_{\alpha\beta}(u,v) \) in
general. But in flat \( \Sigma_3 \), the Green’s function is \( \int G(u,v)d^1J_2 \).

The correlators are obtained by varying \( Z[J] \) with respect to the source
\[ (\Omega^{-1})G^0_{\alpha\beta}(u,v) = \langle X_{\alpha\beta}(u), X(v) \rangle = \epsilon_{\alpha\beta\delta} \frac{\partial}{\partial J_d(u)} \frac{\partial}{\partial J_3(v)} Z[J] \]
\[ = -g^{1/2}(u)\partial_u G(u,v)\epsilon_{\alpha\beta}, \]
\[ (\Omega^{-1})G^1_{\alpha\beta}(u,v) = \langle X_{\alpha}(u), X_{\beta}(v) \rangle = \frac{1}{4}\epsilon_{\alpha\beta\delta\epsilon} \frac{\partial}{\partial J_{\alpha\delta}(u)} \frac{\partial}{\partial J_{\epsilon\beta}(v)} Z[J] \]
\[ = H_{\alpha\beta}^{\gamma\delta}(u,v)g^{1/2}\epsilon_{\gamma\delta\epsilon}, \]
(51)
where we suppressed the target space indices. We can assemble the Green’s function into the superfield form

\[
\langle \mathbf{X}(u, \theta), \mathbf{X}(v, \eta) \rangle = \frac{1}{2} \theta^a \theta^b G_{ab}^0(u, v) - \theta^a \eta^b G_{ab}^1(u, v) + \frac{1}{2} \eta^b \eta^a G_{ab}^0(v, u)
\]

Note in the limit \( \Sigma_3 \) is flat, the \( G_{ab}^1 \) is given by \( \epsilon_{ab} \partial_\sigma \mathcal{G}(u, v) \), so the super Green’s function becomes

\[
\langle \mathbf{X}(u, \theta), \mathbf{X}(v, \eta) \rangle = -\frac{1}{2} \theta^a \theta^b \epsilon_{ab} \partial_\sigma \mathcal{G}(u, v) - \theta^a \eta^b \epsilon_{ab} \partial_\sigma \mathcal{G}(u, v) - \frac{1}{2} \eta^b \eta^a \epsilon_{ab} \partial_\sigma \mathcal{G}(u, v) \sim (\theta - \eta)^2
\]

hence there can not be two propagators between two vertices in the flat limit. This is important for this accounts for the improved short distance behavior. Since one propagator blows up with the square inverse of distance between two insertions, so there will be a naive UV divergence when three propagators connect two insertions. Yet, as two insertions become coincident, the flat space propagator dominates and we just saw the mitigation of such UV behavior. In fact ref. [2] proved the finiteness of the perturbation expansion.

Now we try to compute the two point function

\[
c^1(\mathbf{X}_{f_1} \wedge \mathbf{X}_{f_2}) = \int L \int d^6 z f_1 \int d^6 z f_2 e^{-S}.
\]

There is only one 3-valent diagram

\[
\begin{array}{c}
\includegraphics[width=1cm]{diagram.png}
\end{array}
\]

in order to have enough \( \theta \)'s to satisfy the Grassmann integral. The correlator is formally written as

\[
c^1(\mathbf{X}_{f_1} \wedge \mathbf{X}_{f_2}) = b_T \cdot (\Omega^{-1})^\mu_1^\nu_1 (\Omega^{-1})^\mu_2^\nu_2 (\Omega^{-1})^\mu_3^\nu_3 (\partial_\mu_1 \partial_\mu_2 \partial_\mu_3 f_1) (\partial_{\nu_1} \partial_{\nu_2} \partial_{\nu_3} f_2), \quad (52)
\]

where \( b_T \) is the weight function\(^8\)

\[
b_T = \int d^3 u d^3 \theta \int d^3 v d^3 \eta \left( \frac{1}{2} \theta^a \theta^b G_{ab}^0(u, v) - \theta^a \eta^b G_{ab}^1(u, v) + \frac{1}{2} \eta^b \eta^a G_{ab}^0(v, u) \right)^3
\]

\[
= - \int d^3 u d^3 \theta d^3 v d^3 \eta \left( \frac{6}{4} \theta^a \theta^b G_{ab}^0(u, v) \right) \left( \theta^a \eta^b G_{ab}^1(u, v) \right) \left( \theta^a \theta^b G_{ab}^0(u, v) \right) + \left( \theta^a \eta^b G_{ab}^1(u, v) \right)^3
\]

\[
= - \int d^3 u d^3 v \left( \frac{6}{4} G_{ab}^0(u, v) G_{cd}^1(u, v) G_{ef}^0(v, u) + G_{ab}^1(u, v) G_{cd}^1(u, v) G_{ef}^1(u, v) \right) e_{abc} e_{def}
\]

\[
= - \int d^3 u d^3 v \left( 6g^{1/2} \partial_\sigma G(u, v) G_{cd}^1(u, v) \partial_\sigma G(v, u) g^{1/2}(v) + G_{ab}^1(u, v) G_{cd}^1(u, v) G_{ef}^1(u, v) e_{abc} e_{def} \right).
\]

\(^8\)Our convention is that \( e_{abc}, \epsilon_{abc} = \epsilon_{abc} \) is the Levi-Civita symbol, while \( e^{abc} = 1/g e^{abc} \)

34
We observe that this is the same weight function appearing in the CS perturbation theory \[2\] and in the Rozansky-Witten model \[19\]. And we expect the agreement\[9\] to continue even for larger diagrams. The metric dependence of \(b_T\) can be addressed in the same way as for the Chern-Simons theory and thus we leave this issue aside.

For this simple correlator \(52\) we may check explicitly that it is cocycle

\[
(\delta c^1)(\mathbb{X}_{f_1} \wedge \mathbb{X}_{f_2} \wedge \mathbb{X}_{f_3}) = c^1(\mathbb{X}_{\{f_1,f_2\}} \wedge \mathbb{X}_{f_3}) - c^1(\mathbb{X}_{\{f_1,f_3\}} \wedge \mathbb{X}_{f_2}) + c^1(\mathbb{X}_{\{f_2,f_3\}} \wedge \mathbb{X}_{f_1})
- \{f_1, c^1(\mathbb{X}_{f_2} \wedge \mathbb{X}_{f_3})\} + \{f_2, c^1(\mathbb{X}_{f_1} \wedge \mathbb{X}_{f_3})\} + \{f_3, c^1(\mathbb{X}_{f_1} \wedge \mathbb{X}_{f_2})\} = 0,
\]

where magically all derivatives cancel out for any functions \(f_1, f_2, f_3\).

Some comments must be made regarding the subtlety of the (source) metric independence. The original model is written down without any metric and hence is classically invariant under any orientation preserving diffeomorphism of \(\Sigma_3\). The metric only comes in through the gauge fixing we use, and so the correlator should not depend on the gauge choice by the discussion of section \[2\] (known as the the Ward-identity in gauge theory). However, caution is needed when making such assertions, for the Ward-identity maybe spoiled by the quantum correction and we already saw that the 1-loop determinant has an anomalous dependence on the metric. Even when a correlator is finite accidentally, like the two point function above, one cannot conclude based on finiteness its gauge invariance, indeed \[2\] showed that this two loop diagram suffers a similar anomaly as the determinant factor. Another somewhat remote example for this is the 1-loop light by light scattering in QED. The diagrams when computed in 4D are finite accidentally, yet the result is not gauge invariant. The usual wisdom for the gauge theory is that the question hangs upon whether one possesses a gauge invariant regulator, if yes, then one can subtract the divergence in a manner preserving the symmetry in question and the symmetry is anomaly free. For the CS model, the common practice is that when one integrates over the configuration space, which is copies of \(T[1] \Sigma_3\), one carefully subtracts the diagonals because these correspond to the singular configuration where two insertions are coincident. As a result one is led to consider certain compactification of the configuration space, but this is out of the scope of this paper.

### 8 Example 3: Reinterpreting Rozansky-Witten Model

The Rozansky-Witten (RW) model was introduced in \[19\] and it gives rise to the Rozansky-Witten invariants, see \[20\] for the nice review of these invariants. The authors of \[19\] constructed the model by writing down a set of BRST rules associated to a hyperKähler

\[9\]Indeed we expect that the same agreement for \(b_T\) for the generic theory with the target \(\mathbb{R}^{2m|k}\).
manifold. They also pointed out the similarity between their model and the Chern-Simons model and went as far as calling it the odd Chern-Simons model. Yet one important difference between the two is that the perturbation expansion of the RW model stops at finite order while that of the CS does not. The reason is basically due to the need to saturate the zero modes. This feature did not fail to catch the attention of Kontsevich, who then pointed out that RW model is an AKSZ model with parameters and the model gives the characteristic classes of the holomorphic foliation. We can understand this from the discussion of section 6. In particular, the RW model is a special case of the model (47). The 'parameters' alluded to in [13] are the harmonic fields in (47). At the same time, Kapranov [10] interpreted the RW model from the point of view of Atiyah-class. In this section, we will investigate this model from the field theory perspective and try to endow physical embodiment to the works [13, 10].

RW model is also an AKSZ model. The space of fields is

\[ T[1]_{\Sigma} \rightarrow T^*[2]T^*[1]M , \]

where \( M \) is a HyperKähler manifold.\(^{10}\) The symplectic form for the target space \( T^*[2]T^*[1]M \) is given by

\[ \omega = \delta P_\mu \delta X^\mu + \delta v^\mu \delta q_\mu + \frac{1}{2} \Omega_{ij} \delta X^i \delta X^j , \]

where \( \Omega \) is the holomorphic symplectic 2-form for \( M \). We use labels \( \mu, \nu, ... \) for real coordinates and \( i, j, \bar{i}, \bar{j}, ... \) for complex coordinates. The kinetic term of the AKSZ action is the standard one determined by the symplectic form above

\[ S_{\text{kin}} = \int_{T[1]_{\Sigma}} d^6z \ P_\mu DX^\mu + \frac{1}{2} X^i \Omega_{ij} DX^j + q_\mu DV^\mu . \] (53)

The interaction term is the one corresponding to the Doubeault differential

\[ S_{\text{int}} = \int_{T[1]_{\Sigma}} d^6z \ P_i D\bar{v}^i . \] (54)

It is possible to find a LagSubMfld, and restriction of this action to it gives the RW model [16]

\[ S_{\text{RW}} = \int g_{ij} \ dX_{(0)}^i \wedge *dX_{(0)}^j + g_{ij} X_{(1)}^i \wedge *d\tilde{v}^j - \Omega_{ij} X_{(1)}^i \wedge d\bar{v} X_{(1)}^j - \frac{1}{3} R_{kk} X_{(1)}^k \wedge \Omega_{ij} X_{(1)}^l \wedge X_{(1)}^j v^k . \] (55)

\(^{10}\)The construction can be relaxed to the case of holomorphic symplectic manifold.
The first line of the action gives a non-degenerate kinetic term. This construction of the RW model is of course correct, but when we are only interested in the computation of invariants of holomorphic foliation, we can strip down the extraneous parts of the model and make the geometrical meaning more pronounced.

As far as the RW invariant is concerned, all we need is a cocycle in the cohomology of Lie algebra of formal Hamiltonian vector fields and a Hamiltonian function to plug in. At a given point in \( M \), the holomorphic tangent bundle is identified as \( \mathbb{C}^{2m} \) (\( \dim_{\mathbb{R}} M = 4m \)) and equipped with the symplectic form \( \Omega_{ij} \).

First recall that a tangent vector induces a flow; when we have a flat connection, we may fix the flow to be the geodesic flow and unambiguously identify a neighborhood of the origin in the tangent space of \( x_0 \) with a neighborhood of \( x_0 \) in \( M \). In this way we obtain the so-called normal coordinates. Any function in \( M \) defined in a neighborhood of \( x_0 \) may be pulled back to the normal coordinate system. We denote the geodesic flow induced by the vector \( \xi \) as \( \exp^* \), so

\[
X(x_0, \xi) = x_0 + \xi^\mu - \frac{1}{2} \Gamma^\mu_{\nu\rho} \xi^\nu \xi^\rho - \frac{1}{6} \xi^\nu \xi^\rho \xi^\sigma \partial_\nu \Gamma^\mu_{\rho\sigma} + \frac{1}{3} \xi^\nu \xi^\rho \xi^\sigma \Gamma^\mu_{n\mu} \Gamma^\kappa_{\rho\sigma} + \cdots \quad (56)
\]

\[
\exp^* \phi(X) = \phi(X(x_0, \xi)) = \phi(x_0) + \xi^\mu \partial_\mu \phi(x_0) + \frac{1}{2} \xi^\mu \xi^\nu \nabla_\nu \partial_\mu \phi(x_0) + \cdots,
\]

where we assumed that \( \Gamma \) is some flat connection. Of course, the exponential map requires just a connection, but for the sake of further discussion we assume in (56) that \( \Gamma \) is flat. As a mnemonic, the pull back of the function \( \phi \) by \( \exp \) is just the Taylor expansion of \( \phi(x_0 + \xi) \) around \( x_0 \), except that one uses covariant derivative rather than ordinary derivative. The same remark applies to the tensors as well, for example

\[
\exp^* \phi_\mu dX^\mu = \phi_\mu(x_0) d\xi^\mu + \xi^\nu \nabla_\nu \phi_\mu(x_0) d\xi^\mu + \frac{1}{2} \xi^\nu \xi^\rho \nabla_\nu \nabla_\rho \phi_\mu(x_0) d\xi^\mu + \cdots.
\]

Now we take the base coordinate \( x_0 \) and also \( v \) as fixed parameters. The target space for the AKSZ model is

\[
\mathcal{M} = T^{(1,0), \infty}_M(x_0),
\]

where \( T^{(1,0), \infty} \) denotes the formal neighborhood of the zero section of the holomorphic tangent bundle at \( x_0 \). Due to the Kähler property, the Levi-Civita connection has either totally holomorphic or totally anti-holomorphic indices, and the curvature \( R_{ij}^{\times} = 0 \), namely so \( \Gamma^i_{jk} \) can be regarded as the flat connection. Of course, we now only have the holomorphic half of the tangent bundle, so the geodesic flow is understood as the analytical continuation away from the real one. For all practical purposes, we just understand the geodesic flow as
given by formal Taylor expansion as in (56). Now the target space will be parameterized by the coordinates \( \xi^i \) in the formal neighborhood of \( x_0 \). We have originally a holomorphic symplectic form \( dX^i \Omega_{ij} dX^j \), which we pull back to the \( \xi \) coordinate

\[
\exp^* dX^i \Omega_{ij} dX^j = d\xi^i \Omega_{ij} d\xi^j + \xi^k (\nabla_k \Omega_{ij}) d\xi^i d\xi^j + \cdots = d\xi^i \Omega_{ij}(x_0) d\xi^j .
\]  

(57)

We can now set up a free AKSZ theory as in section [7]. The odd symplectic structure is given by

\[
\omega = \frac{1}{2} \int_{T[1]\Sigma_3} d\bar{6}z \, \delta \xi^i \Omega_{ij}(x_0) \, \delta \xi^j .
\]

We stress that now \( \Omega_{ij} \) is constant in the \( \xi \) space and equal \( \Omega_{ij}(x_0) \). The action is

\[
S = \frac{1}{2} \int_{T[1]\Sigma_3} d\bar{6}z \, \xi^i \Omega_{ij}(x_0) \, D\xi^j .
\]

(58)

The path integral provides us with the desired cocycle, and we will evaluate the correlator of the particular function

\[
\Theta = v^i \Theta_i = v^i \sum_{n=0}^{\infty} \frac{1}{(n+3)!} R^n_{i_1 \cdots i_{n+3}} \xi^{i_1} \cdots \xi^{i_{n+3}} ,
\]

where

\[
R^n_{i_1 \cdots i_{n+3}} = \nabla_{i_1} \cdots \nabla_{i_n} \nabla_{i_{n+1}} R_{i_{n+1} i_{n+3}} j_{i_{n+2}} \Omega_{j i_{n+2}} ,
\]

note that due to the Kähler property as well as the covariant constancy and holomorphy of \( \Omega \), all the holomorphic indices in \( R^n \) are symmetric. We show in the appendix that

\[
\bar{\partial} \Theta + \frac{1}{2} \{ \Theta, \Theta \} = 0 ,
\]

(59)

where the Poisson bracket is with respect to \( \Omega \) in \( \xi \)-direction.

Without specifying what gauge fixing, we still know that the path integral gives a cocycle with respect to \( \Omega \). The correlator is a function of extra parameters \( x_0, v \),

\[
c^\partial (\mathcal{X}_\Theta \wedge \cdots \wedge \mathcal{X}_\Theta) = f(x_0, v) ,
\]

which can be regarded as an anti-holomorphic form on \( M \). In fact this is an element in \( H^\partial_\partial(M) \), for the Dolbeault differential acts on \( c^\partial \) as the differential in the graph cohomology

\[
\bar{\partial} c^\partial (\mathcal{X}_\Theta \wedge \cdots \wedge \mathcal{X}_\Theta) = \frac{1}{2} \sum c^\partial (\mathcal{X}_\Theta \wedge \cdots \wedge \mathcal{X}_{\{\Theta, \Theta\}} \wedge \cdots \wedge \mathcal{X}_\Theta) \sim c^\partial (\bar{\partial}(\mathcal{X}_\Theta \wedge \cdots \wedge \mathcal{X}_\Theta)) = 0 .
\]

\[11\] Please notice our convention for \( R^n_{i_1 \cdots i_{n+3}} \), where the superscript \( n \) is not a holomorphic index!
Furthermore, since the space of $\xi$ now has a flat structure $C^2_m$, we can apply the naive gauge fixing by setting exact forms to zero as in section 7. This gauge fixing automatically keeps only tri-valent graphs. These graphs are given by contracting the $\xi$'s in $v^i R_{ik}^{\ell} \omega_{ij} \xi^i \xi^j \xi^k$ with $\Omega^{-1}$. This exactly reproduces the Rozansky-Witten invariants. The $\bar{\partial}$-closedness is automatic due to $\bar{\partial} R \Omega = 0$. We pointed out before that the cocycle given by the path integral does depend on the choice of the LagSubMfld, or the gauge fixing, it is reasonable to expect that a drastically different gauge choice shall produce a different class in $H^\bullet_\Omega$ due to the abundance of cocycles in the graph cohomology.

The above construction grasps the main feature of Rozansky Witten model, yet we would like to incorporate also the extra parameters $x_0$, $v$ into the theory and furthermore justify the definition of $\Theta$. In particular we show that RW model fits the general description of AKSZ model for flat bundles of section 6.

We first complexify $M$ by taking two copies $M \times M$, the second one is equipped with the opposite complex structure as the first one. So the diagonal embedding $M \Delta \rightarrow M \times M$ gives the real slice. The picture here can also be studied in the light of holomorphic foliation \[13\]. We try to motivate the analogy between our problem and the holomorphic foliation with a few words, though this analogy is not strictly necessary for the rest of the paper, so for the first reading, the reader may jump over the next five paragraphs.

We label the two copies of $M$ by the holomorphic and anti-holomorphic coordinate respectively, i.e, a point $(p, q) \subset M \times M$ is parameterized by

$$(X^i(p), \bar{X}^i(q)), \forall (p, q) \subset M \times M.$$  

We can take the second factor of the product as the leaf space of the foliation while the first factor as the transverse direction. The holomorphic geodesic exponential map amounts to the following change of coordinates

$$(X^i(p), \bar{X}^i(q)) \rightarrow (\exp_\xi X^i(q), \bar{X}^i(q)). \quad (60)$$

For a foliation with constant co-dimension, we have principle bundle structure. The fiber is (rather abstractly) all possible ways of identifying the transverse space at the neighborhood of a point with the Euclidean space $C^{2m}$. The structure group is by definition isomorphic to the fiber and can be taken as the group of formal diffeomorphism of the transverse direction. Now that there is a symplectic structure $\Omega_{ij}$ in the transverse direction, the relevant structure group should become the group of formal symplectomorphism. The previously defined exponential map offers one way of such identification, namely at point $q$, the point $p$ in the transverse space is identified with $\xi \in C^{2m}$ through $X^i(p) = \exp_\xi X^i(q)$. So the exponential map is a section of the principle bundle.
This principle has a flat connection. Here we follow the work of Fuks [7]. Let the foliation be determined by a system of $2n$ 1-forms $\theta^i$, which means the leaf is the null-space of these forms. By the Frobenius theorem for an integrable system of 1-forms, the differential $d\theta^i$ is

$$d\theta^i = \gamma^i_j \wedge \theta^j,$$

where $\gamma$ is again some 1-forms, and can be thought of as the connection of the principle bundle. This connection is flat only along the leaf: $(d\gamma^i_j - \gamma^i_k \gamma^k_j) \theta^j = 0$. But this in turn implies $d\gamma^i_j - \gamma^i_k \gamma^k_j = \gamma^i_{jk} \theta^k$ for some 1-form $\gamma^i_{jk}$. One can carry on this procedure and obtain a collection of such $\gamma$’s, and out of these one can construct a connection

$$\Gamma = \Gamma_i \frac{\partial}{\partial \eta^i} = (\theta^i + \gamma^i_j \eta^j + \gamma^i_{jk} \eta^j \eta^k + \cdots) \frac{\partial}{\partial \eta^i},$$

where $\eta$ is some formal variable. This connection now takes value in the Lie algebra of formal vector fields in the $\eta$-space. This connection is flat in all directions.

Next we apply the above machinery to bear upon our problem. Due to the mapping Eq.60, the previous $x_0(q)$ is identified as $X^i(q) = (X^i(q))^*$. The holomorphic foliation is determined by 2n 1-forms $dX^i$, because the leaf is clearly the null space of these 1-forms. It will be shown in the appendix that the pull back of these 1-forms $\exp_\xi dX^i$ is the linear combination of the system $\theta^i = d\xi^i - dX^i \{\Theta_i, \xi^i\}$. According to the above recipe, we differentiate the 1-forms and it turns out that

$$\gamma^i_j \sim dX^j \partial \xi^i \{\Theta_i, \xi^i\}; \quad \gamma^i_{jk} \sim dX^j \partial \xi^k \{\Theta_i, \xi^i\}; \quad \cdots.$$

And the flat connection for the foliation is given by

$$\Gamma = \left( d\xi^i - dX^i \sum_{n=0}^{n=\infty} \frac{1}{n!(\eta \partial \xi)^n} \{\Theta_i, \xi^i\} \right) \frac{\partial}{\partial \eta^i},$$

$$d\Gamma - \Gamma \Gamma = 0$$

This connection is flat in all (including the $\xi$) directions after applying Eq.59.

Now that we have a flat principle bundle defined over $M \times M$, but we can restrict it to the diagonal (given by $\xi = 0$). The pull back of the connection, for which we use the same symbol, is

$$\Gamma = -dX^i \sum_{n=0}^{n=\infty} \frac{1}{n!(\eta \partial \xi)^n} \{\Theta_i, \xi^i\} \partial \eta^i \bigg|_{\xi=0}$$

This is obviously is just $dX^i \{\Theta_i, \xi^i\}$ with all the $\xi$’s replaced with $\eta$. Finally, we see that it is in this way Eq.59 is interpreted as the flat connection of holomorphic foliation and the correlator of the RW model gives rise to the characteristic class of the holomorphic foliation.
Back from our digression, the key observation by Kapranov [10] is the following, under the mapping \( T^{(1,0),\infty} \exp M \times M, (\xi, X^i) \to (\exp_\xi (X^i)^*, X^i) \), the differential \( \bar{\partial} \) is pulled by as
\[
\left( \{ \Theta_i, \cdot \} , \bar{\partial}_i \right) \exp (0, \bar{\partial}_i).
\]
This motivates (59) because the Dolbeault differential on the rhs is nilpotent. It is also instructive to prove this relation explicitly which we do in the appendix.

On the space \( M \times M \), we can construct the GrMfld
\[
\mathcal{M} = M \times T^{(0,1)}[2]T^{(0,1)}[1]M
\]
parameterized by \((X^i, P_i, q_i, v^i, X^\bar{i})\). It has the even symplectic form
\[
\omega = \delta P_i \land \delta X^i + \delta q_i \land \delta v^i + \frac{1}{2} \Omega_{ij} \delta X^i \land \delta X^j.
\]
With this data one can set up the standard AKSZ model, whose homological vector field is \( \bar{\partial} \)–the rhs of Eq.(61)
\[
S = \int d^6 z \ P_i D X^\bar{i} + q_i D v^i + \frac{1}{2} X^i \Omega_{ij} D X^j + P_i v^i.
\]
This action is a truncation of the AKSZ action (53)+(54) and it still reduces to the gauge fixed RW action (55) along the lines presented in [16].

While for the manifold \( T_M^{(1,0),\infty} \) one has the GrMfld
\[
\mathcal{M} = T^{(1,0),\infty} M \oplus T^{(0,1)}[2]T^{(0,1)}[1]M
\]
parameterized by \((\xi^i, P_i, q_i, v^i, X^i)\). We want to pull the model (63) on \( M \times T^{(0,1)}[2]T^{(0,1)}[1]M \) back to this target space. The required change of variable is \((x_0 = (X^\bar{i})^*)\)
\[
\xi^i \exp \Rightarrow x^i_0 + \xi^i - \frac{1}{2} \Gamma_{jk}^i \xi^j \xi^k + \cdots = e^{\xi^i \partial_\xi^i - \xi^i \xi^j \Gamma^{ij}_{\xi^i}} x_0^i,
\]
\[
P_i, q_i, v^i, X^\bar{i} \Rightarrow P_i, q_i, v^i, X^i.
\]
The same calculation that led to (61) shows
\[
\delta X^i = \left( \delta \xi^j - \delta X^{\bar{i}} \{ \Theta_i, \xi^j \} \right) \frac{\partial X^i(\xi)}{\partial \xi^j}.
\]
The holomorphic symplectic form is pulled back according to (using the equation (57))
\[
\exp^* \delta X^i \Omega_{ij} \delta X^j = (\delta \xi^i - \delta X^{\bar{i}} \{ \Theta_i, \xi^i \})(\exp^* \xi^i \Omega_{ij})(\delta \xi^j - \delta X^{\bar{j}} \{ \Theta_j, \xi^j \})
\]
\[
= \delta \xi^i \Omega_{ij} \delta \xi^j - 2 \delta X^{\bar{i}} \{ \Theta_i, \xi^i \} \Omega_{ij} \delta \xi^j + \delta X^{\bar{j}} \{ \Theta_i, \xi^i \} \Omega_{ij} \delta X^j \{ \Theta_j, \xi^j \}
\]
\[
= \delta \xi^i \Omega_{ij} \delta \xi^j - 2 \delta X^{\bar{i}} \delta \Theta^j_i \delta \xi^j - \delta X^{\bar{j}} \delta X^j \{ \Theta_i, \Theta_j \}
\]
\[
= \delta \xi^i \Omega_{ij} \delta \xi^j - 2 \delta X^{\bar{i}} \delta \Theta^j_i.
\]
So the symplectic form Eq. 62 is pulled back to
\[
\exp^* \omega = \int d^6 z \, \delta P_i \wedge \delta X^i + \delta q_i \delta v^i + \frac{1}{2} \delta \xi^i \Omega_{ij} \delta \xi^j + \delta X^i \delta \Theta_i.
\]

The action Eq. 63 is pulled back as
\[
\exp^* S = \int d^6 z \, (P_i + \Theta_i) D X^i + \frac{1}{2} \xi^i \Omega_{ij} D \xi^j + q_i D v^i + P_i v^i.
\]

Since now the momentum dual to \( X^i \) is \( P_i + \Theta_i \), it is proper that we changed variable \( \tilde{P}_i := P_i + \Theta_i \)
\[
\exp^* S = \int d^6 z \, \tilde{P}_i D X^i + \frac{1}{2} \xi^i \Omega_{ij} D \xi^j + q_i D v^i + \tilde{P}_i v^i - \Theta_i v^i
\]
Note that \( \tilde{P}_i v^i - \Theta_i v^i \) generates the vector field \(- (v^i \partial_i + v^j \{ \Theta_i, \cdot \})\) on functions of \( X^i \) and \( \xi^i \), which is the lhs of Eq. 61. This model is totally in line with the general picture for the model given by (117).

We can perform the partial gauge fixing in the \( \tilde{P}_i, X^i, q_i, v^i \) sector by setting \( \tilde{P}_i = q_i = 0 \). Then we are left with the action with only \( \xi \) as dynamical variables, relegating \( X_i \) and \( v^i \) as extra parameters. This gives back the odd Chern-Simons model (58) with an interaction term \( \Theta \).

9 Summary

In this paper, we have explained the idea of using AKSZ-BV path integral as a construction of cocycles and that its relation to the graph cohomology is nothing but the Feynman integrals and the standard Wick’s theorem. We took the construction of [9] and put it into a concrete physical system. In particular, we discussed how to deal with zero modes which must exist in any realistic field theory. This leads to the embodiment of Kontsevich’s idea of applying homomorphism to a cocycle of Lie algebra cohomology to obtain secondary Chern-Simons type invariants (characteristic classes of flat bundles). Thus we conclude that the AKSZ construction of TFT is powerful not only at the classical level, it also offers very unified perturbative treatment of the corresponding TFTs.

We constructed the odd Chern-Simons theory over the target \( \mathbb{R}^{2n} \) and showed that its perturbation expansion is identical to that of the Chern-Simons theory, in particular, we obtained identical weight function for each given graph. We did this for the Rozansky-Witten model painstakingly, and showed from the field theory perspective that this model
fits the picture painted by Kontsevich, namely, it is a model associated with a flat bundle related to the holomorphic foliation.

The further issues include of course applying the presented ideas for the general AKSZ model for different algebroids and foliations and construct explicitly characteristic classes and invariants. One can naturally associate 3D AKSZ models to Courant algebroids and Lie algebroids. Thus applying the ideas presented in this work one may hope to obtain interesting characteristic classes for these algebroids. The main complication in the treatment of these models would be the application of the exponential map carefully, or in other words, performing the covariant Taylor expansions.

Another interesting issue would to apply the formal BV arguments from section 2 to a wide class of quantum observables. In this paper we concentrate our attention on observables which are written as full integral over source manifold. The BV algebra of these observables can be mapped to the algebra of functions on the target space. However we may look at the quantum observables which are integrals over cycles on the source (or even full Wilson loops). One has to embed these wide class of observables into the BV framework and calculate the corresponding BV algebra generated by them. The path integral evaluations of those observables should still give rise to a cocycle for some Lie algebra. We hope to return to this idea in the future.

Acknowledgement:

The research of M.Z. was supported by VR-grant 621-2008-4273. The authors would like to thank Francesco Bonechi for helpful discussions.

A Brackets of Even and Odd Type

In this section we fix the sign conventions of the symplectic form, Poisson bracket and odd Laplacian etc. These signs are important for the perturbation theory, it is worth the effort.

The degree $n$ symplectic form

$$\Omega = \sum_{A < B} \Omega_{AB} dX^A \wedge dX^B, \quad |A| + |B| = n,$$

where $|A| = \deg X^A$ and $|B| = \deg X^B$. We always assume that $\Omega_{AB}$ is a constant for simplicity, and it satisfies $\Omega_{AB} = (-1)^{|A|+1} (|B|+1) \Omega_{BA}$, matching the graded commutativity

$$dX^A dX^B = (-1)^{|A|+1} (|B|+1) dX^B dX^A.$$
We take the odd symplectic form as the starting point. This gives rise to an odd Poisson bracket, which can be induced from the odd Laplacian according to the formula (1). We fix the convention for this odd Laplacian first, and from there we derive the conventions of other brackets. The reason is that whether or not a Laplacian annihilates some function is crucial for the discussion of section 4. We assume that $\Omega_{AB}$ is a constant for simplicity, and fix the following

$$\sum_{A<B} \Omega_{AB} \, dX^A \wedge dX^B \Rightarrow \Delta = \sum_{A<B} (\Omega^{-1})^{AB} \frac{\partial}{\partial X^A} \frac{\partial}{\partial X^B}. \tag{64}$$

One may check that $\Delta$ satisfies (1), with the Poisson bracket given by

$$\{f, g\} = \sum_{A,B} (\Omega^{-1})^{AB} (f \partial_A) \partial_B g. \tag{65}$$

Note that our convention for the right derivative is $X^B \partial_A = (-1)^{|A|} \delta^B_A$.

The bracket (65) is derived for the odd case, but we take this as the definition of the Poisson bracket, both for even and odd. This bracket satisfies $\{f, g\} = -(1)^{(f+n)(g+n)} \{g, f\}$.

Suppose we have now a deg $n$ symplectic GrMfld $\mathcal{M}$, with symplectic form $\Omega$. Because we give degree 1 to $\delta$, as a result, $\delta \theta = -\theta \delta$ and we dispense with the $\wedge$. In the AKSZ construction, we build a TFT on a dimension $n + 1$ source manifold $\Sigma_{n+1}$ with $\mathcal{M}$ as the target. For such degree $n$ GrMfld, we can form the degree $-1$ symplectic form over $\text{Maps}(T[1] \Sigma_{n+1}, \mathcal{M})$ by

$$\omega = \int_{T[1] \Sigma_{n+1}} d^{2(n+1)} z \left( \sum_{A<B} \Omega_{AB} \delta X^A(z) \delta X^B(z) \right), \tag{66}$$

where $d^{2(n+1)} z = d^{n+1} \xi \, d^{n+1} \theta$ and $X$ stands for a map from $T[1] \Sigma_{n+1}$ to $\mathcal{M}$. This form has the desired degree $-1$ because the measure carries degree $-(n + 1)$.

We may obtain the odd symplectic form written in component fields by integrating out $d^{n+1} \theta$. Assume $\Omega_{AB}$ is a constant the symplectic form (66) can be rewritten as

$$\omega = \sum_p (-1)^{A(n+1-p)+p} \int d^{n+1} \xi \, \Omega_{AB} \delta X^A_{(p)} \delta X^B_{(n+1-p)}. \tag{66}$$

The Laplacian according to the rule (64) is

$$\Delta = \sum_{p,A<B} (-1)^{A(n+1-p)+p+(A+1)(B+1)} \int d^{n+1} \xi \left( \Omega^{-1})^{AB} \frac{\delta}{\delta X^A_{(p)}} \frac{\delta}{\delta X^B_{(n+1-p)}} \right).$$

44
This naive form of the Laplacian must be improved, otherwise, when it hits a local functional, it will produce $\delta(0)$. Nevertheless, if we proceed and investigate $\Delta \int d^{2(n+1)}z \ f(X(x)) = 0$, we find the following formal expression

$$
\Delta \int d^{2(n+1)}z \ f(X(z)) = 4 \sum_p \binom{n+1}{p} (-1)^p \sum_{A<B} (-1)^{|A|B|+n+1} (\Omega^{-1})^{AB} \int d^{n+1}\xi \frac{\partial}{\partial X^A_{(n+1)-p}} \frac{\partial}{\partial X^B_{I_p}} \delta(0).
$$

Note that this sum formally vanishes for any $n+1$,

$$
\sum_{p=0}^{n+1} \binom{n+1}{p} (-1)^p = (1 - 1)^{n+1} = 0.
$$

A better definition of $\Delta$ with regularization\(^\text{12}\), which is suited both for separating the subtlety from zero modes and for renormalization is the following. We expand any $p$-form on the source manifold into eigen-modes of the self-adjoint operator $\square = \{d^\dagger, d\}$, where $d$ is de Rham differential and $d^\dagger$ is its adjoint,

$$
X_{(p)} = X_{I_p} \psi^{I_p};
\square \psi^{I_p} = \lambda^2_{I_p} \psi^{I_p};
\sum_{I_p} \psi^{I_p}(x)_{i_1... i_p} (\psi^{I_p}(y))_{i_{p+1}...i_{n+1}} = \epsilon_{i_1...i_{n+1}} \delta(x - y).
$$

After changing variables from $X^A_{(p)}$ to $X^A_{I_p}$, the original Laplacian becomes

$$
\Delta = \sum_{A<B,p,I_p} (-1)^{A_p + p(n+1) + AB} (\Omega^{-1})^{AB} \frac{\partial}{\partial X^A_{(n+1)-p}} \frac{\partial}{\partial X^B_{I_p}}.
$$

To regularize this expression, one inserts the factor $\exp(-\epsilon^2 \lambda^2_{I_p})$ in the summation. This regularization is commonly known as the heat kernel regularization, we denote

$$
\Delta_\epsilon = \sum_{A<B,p,I_p} (-1)^{A_p + p(n+1) + AB} e^{-\epsilon^2 \lambda^2_{I_p}} (\Omega^{-1})^{AB} \frac{\partial}{\partial X^A_{(n+1)-p}} \frac{\partial}{\partial X^B_{I_p}}.
$$

What happens here is that we are effectively replacing the original Laplacian with

$$
\Delta_\epsilon = \sum_{p,A<B} (-1)^{A(n+1-p) + p + AB + n+1} \int d^{n+1}\xi_1 d^{n+1}\xi_2 \ K_p(\xi_1, \xi_2, \epsilon) \frac{\delta}{\delta \psi^{I_p}(\xi_1)} \frac{\delta}{\delta \psi^{I_{n+1-p}}(\xi_2)} ,
$$

$$
K_p(\xi_1, \xi_2, \epsilon) = \sum_{I_p} e^{-\epsilon^2 \lambda^2_{I_p}} \psi^{I_p}(\xi_1) \wedge \psi^{I_{n+1-p}}(\xi_2).
$$

\(^\text{12}\)We use the regularization of odd Laplacian which is similar to the one discussed in \[5\].
For small \( \epsilon \), the heat kernel \( K \) asymptotes to
\[
K_p(\xi_1, \xi_2, \epsilon) \sim \epsilon^{-(n+1)} e^{\frac{|\xi_1 - \xi_2|^2}{4\epsilon^2}} ,
\]
which is just the smeared delta function, and we recover the original definition of \( \Delta \) in the limit \( \epsilon \to 0 \).

It turns out that \( \Delta_\epsilon \) acts on a full integral as
\[
\Delta_\epsilon \int d^{2(n+1)} z \, f(X(z)) = \sum_{p,A < B} (-1)^p (\Omega^{-1})^{AB} \int d^{n+1} \xi \, K_p(\xi, \xi, \epsilon) \partial_A \partial_B f(X(\xi)) .
\]
Suppose that \( \partial_A \partial_B f(X(\xi)) \) is a constant, then the sum over \( p \) gives nothing but the index of the de Rham operator on \( \Sigma_{n+1} \)
\[
\sum_p (-1)^p \int d^{n+1} \xi \, K_p(\xi, \xi, \epsilon) = \text{Tr}[e^{-\epsilon^2 \Box}(-1)^p] = \chi(\Sigma_{n+1}) .
\]
While for odd \( (n+1) \) which is the main interest of this paper, the sum can be reshuffled into
\[
\Delta_\epsilon \int d^{2(n+1)} z \, f(X(z)) = \sum_{p \leq (n/2), A, B} (-1)^p (\Omega^{-1})^{AB} \int d^{n+1} \xi \, K_p(\xi, \xi, \epsilon) \partial_A \partial_B f(X(\xi)) , \quad (67)
\]
where now sum is taken over all \( A \) and \( B \). We used the fact that if \( \psi^I_p \) is eigenfunction of \( \Box \) with the eigenvalue \( \lambda^2_p \), then \( \ast \psi^I_p \) is eigenfunction of \( \Box \) with the same eigenvalue. The expression \((67)\) vanishes for nonzero \( \epsilon \) since \( (\Omega^{-1})^{AB} \partial_A \partial_B f = 0 \) due to contraction of (graded)symmetric with (graded)anti-symmetric and this happens only when \( n + 1 \) is even. Here we can see the crucial difference between even and odd dimension theory.

We would now like to investigate the relation between the Poisson bracket on the target space \( \mathcal{M} \) and the induced odd bracket in the mapping space, in particular whether \( \Delta(\int f \int g) = \int \{ f, g \} \) is true. After a lengthy but straightforward calculation, we obtain
\[
\Delta(\int d^{2(n+1)} z_1 \, f(X(z_1)) \int d^{2(n+1)} z_2 \, g(X(z_2))) = (-1)^f \int d^{2(n+1)} z \, \{ f(X(z)), g(X(z)) \} .
\]
From this we can get
\[
\{ \int d^{2(n+1)} z \, f(X(z)), \int d^{2(n+1)} z \, g(X(z)) \} = (-1)^{n+1} \int d^{2(n+1)} z \, \{ f(X(z)), g(X(z)) \} .
\]
Furthermore, in the text we quite often treat the harmonic modes (i.e., the eigenfunctions with \( \lambda^2_p = 0 \)) as parameters of the theory and the path integral is taken only for the non-harmonic fields. If we denote the Laplacian of the non-harmonic fields by \( \Delta' \), the Ward identity is given by
\[
\int \Delta'(\cdots) = 0 ,
\]
46
where \( \mathcal{L} \) is Lagrangian only in non-harmonic sector. Thus we should investigate what is the bracket induced by the \( \Delta' \).

To do this, we do not need the regularization above. We restrict ourselves to the case of rational homology sphere for simplicity. Then in the mode sum of (67), we need to exclude two modes

\[
\psi^h_0 = \frac{1}{\sqrt{\text{vol}}} \quad \psi^h_{n+1} = \sqrt{g} \sqrt{\text{vol}} \epsilon_{i_1 \ldots i_{n+1}} .
\]

As a result we obtain

\[
\Delta' \left( \int d^{2(n+1)} z_1 f(X(z_1)) \int d^{2(n+1)} z_2 g(X(z_2)) \right) = (-1)^f \int d^{2(n+1)} z \left\{ f(X(z)) , g(X(z)) \right\} \\
- \sum_{A,B} (\Omega^{-1})^{AB} (-1)^{n+f+A(n+1)} \frac{1}{\sqrt{\text{vol}}} \int \psi^h_{n+1}(\xi) f(X(\xi)) \frac{\partial}{\partial A} \int d^{2(n+1)} z \partial_B g(X(z)) \\
- \sum_{A,B} (\Omega^{-1})^{AB} (-1)^{f(n+1)+g+g f+A B+n+1} \frac{1}{\sqrt{\text{vol}}} \int \psi^h_{n+1}(\xi) g(X(\xi)) \frac{\partial}{\partial A} \int d^{2(n+1)} z \partial_B f(X(z)) .
\]

The first term is the usual one, while the last two are due to the exclusion of the zero modes.

To make sense of the formula, we have to resort to the explicit gauge fixing and the Feynmann rules we introduced in section [5]. Thus from now on \( n+1 = 3 \). Our claim is that we can do the following replacement in the path integral

\[
\int \mathcal{L} \left( \frac{1}{\sqrt{\text{vol}}} \int \psi^h_3(\xi) f(X(\xi)) \cdots \right. = f(x_0) \int \mathcal{L} \left. \cdots ,
\]

as long as all the other insertions are of the form \( \int d^6 z g(X) \). The reasoning is, suppose that \( \cdots \) consists of \( q \) insertions, then the number of propagators routed amongst themselves is \( \# = 3q/2 \). If the insertion \( \int d^3 \xi \psi^h_3(\xi) f(X(\xi)) \) was connected to the rest of the diagram with \( p \geq 2 \) propagators, then \( \# \) will be reduced to \( \# \leq (q - 2p)/3 \), forcing 2-valent vertex to appear somewhere and it vanishes within our rules.

With this consideration, the second term of the previous formula under the path integral becomes

\[
- (-1)^n f(x_0), \int_{\mathcal{L}} \cdots
\]

where the Poisson bracket is now taken over \( x_0 \). The path integral over the non-harmonic fields produces a function of \( x_0 \), accordingly the path integral should be interpreted now as a cochain of the CE complex of formal Hamiltonian vector fields of \( \mathcal{M} \), taking values in
\( C^\infty(\mathcal{M}) \). So the differential of such cochains must be modified correspondingly, and this new differential is induced by \( \Delta' \).

\[
(\delta c^q)(\mathbb{X}_{f_0} \wedge \cdots \mathbb{X}_{f_{q+1}}) = \sum_{i<j} (-1)^{j+1} \mathcal{L}_{\partial x^i} \omega ([\mathbb{X}_{s_i}, \mathbb{X}_{s_j}], \mathbb{X}_{s_0}, \cdots, \mathbb{X}_{s_i}, \cdots, \mathbb{X}_{s_j}, \cdots, \mathbb{X}_{s_q}).
\]

In fact, this formula is completely in accordance with the de Rham differential

\[
d\omega(X_0, \cdots, X_q) = \sum_{i<j} (-1)^{i+j+1} \omega ([X_i, X_j], X_0, \cdots, \dot{X}_i, \cdots, \dot{X}_j, \cdots, X_q)
- \sum_i (-1)^i X_i \omega (X_0, \cdots, \dot{X}_i, \cdots, X_q).
\]

Because of the formal relation \( \int_{\mathcal{L}} \Delta' \cdots = 0 \), the path integral is a cocycle for the modified differential. What is not expected in (68) is perhaps the sudden appearance of \( (f_i + n) \) deg \( c^q \). When one derives the formula of the differential using \( \Delta' \), one is restricted to \( n + 1 = 3 \) and \( q = 2k - 1 \), so the factor \( (f_i + n) \) deg \( c^q = (f_i + 2)6k \) is invisible. But for a general CE differential for general degree \( n \) bracket, this factor is needed in order \( \delta^2 = 0 \). If one wishes to check this point for himself, he will find the following useful

\[
\{f, g\} = -(-1)^{(f+n)(g+n)} \{g, f\},
\]

\[
\{\{f, g\}, h\} + \{\{g, h\}, f\}(-1)^{(g+h)(f+n)} + \{\{h, f\}, g\}(-1)^{(g+f)(h+n)} = 0.
\]

Lastly, we also check that for an odd symplectic form, the Hamiltonian vector fields satisfy

\[
[X_f, X_g] = (-1)^{(|f|+1)(|g|+1)} X_g X_f = X_{\{f, g\}}.
\]

We do this in Darboux coordinates for simplicity. The symplectic form is \( \omega = \delta x^+ \delta x^- \) where \( x^+ \) is odd \( x \) is even

\[
[X_f, X_g] = \{f, g\} \frac{\partial}{\partial x^+} \frac{\partial}{\partial x^-} + \{f, g\} \mathcal{L}_{\partial x^+} \mathcal{L}_{\partial x^-} - (-1)^{(|f|+1)(|g|+1)} (f \leftrightarrow g)
\]

\[
= X_{\{f, g\}} + (-1)^{|g|} \{f, g\} \frac{\partial}{\partial x^+} \frac{\partial}{\partial x^-} - \{f, g\} \frac{\partial}{\partial x^+} \frac{\partial}{\partial x^-}
\]

\[
-(-1)^{(|f|+1)(|g|+1)} \{g, f\} \frac{\partial}{\partial x^+} \frac{\partial}{\partial x^-} - (-1)^{(|f|+1)(|g|+1)} \{g, f\} \frac{\partial}{\partial x^+} \frac{\partial}{\partial x^-} = X_{\{f, g\}}.
\]

where we have used \( \{f, g\} = -(-1)^{(|f|+1)(|g|+1)} \{g, f\} \).
B \quad L_\infty \text{ Structure from HyperKähler Manifold}

In this Appendix we present some explicit formulas about $L_\infty$-structure for the hyperKähler manifold. The idea was presented in [10], but we could not read off the explicit numerical factors from this work. Therefore we present our own derivation of these relations. All expressions are written in complex coordinates.

For a hyperKähler manifold, the three indices $i, j, k$ are totally symmetric in $(R\Omega)_{\bar{i}ijk} = R_{\bar{i}i}^\ell \Omega_{\bar{j}\ell jk}$ and $\partial_{[\bar{j}}(R\Omega)_{\bar{i}ijk]} = 0$. If we define

$$R^n_{\bar{i}l_1\ldots l_{n+3}} \equiv \nabla_{l_1} \cdots \nabla_{l_n}(R\Omega)_{\bar{i}l_{n+1}l_{n+2}l_{n+3}},$$

then apply the covariant derivatives to $0 = \partial_{[\bar{j}}(R\Omega)_{\bar{i}ijk]}$

$$0 = \frac{1}{(n+3)!} \nabla_{l_1 \ldots l_n} \bar{\partial}_{[\bar{j}}(R\Omega)_{\bar{i}l_{n+1}l_{n+2}l_{n+3}] + \text{perm in } l_i$$

$$= \sum_{k=1}^{n-1} \frac{(n-k+2)}{(n+3)!} \nabla_{l_1 \ldots l_k} [(R_{\bar{j}[l_{k+1} \ldots l_{k+2}]}^m R_{\bar{i}m \ldots l_{n+3}]}^{n-k-1} + \frac{1}{(n+3)!} \bar{\partial}_{[\bar{j}} R^n_{\bar{i}l_1 \ldots l_{n+3}] + \text{perm in } l_i.$$ 

So we get

$$\bar{\partial}_{[\bar{j}} R^n_{\bar{i}l_1 \ldots l_{n+3}] = -\sum_{k=0}^{n-1} \frac{(n-k+2)}{(n+3)!} \nabla_{l_1 \ldots l_k} [(R_{\bar{j}[l_{k+1} \ldots l_{k+2}]}^m R_{\bar{i}m \ldots l_{n+3}]}^{n-k-1} + \text{perm in } l_i.$$ 

The rhs can be worked out explicitly

$$\text{rhs} = \sum_{k=0}^{n-1} \sum_{p=0}^{k} \frac{(k)!}{p!(n+3)!} (n-k+2) \nabla_{l_1 \ldots l_p} [(R_{\bar{j}[l_{k+1} \ldots l_{k+2}]}^m R_{\bar{i}m \ldots l_{n+3}]}^{n-k-1} + \text{perm in } l_i$$

$$= \sum_{k=0}^{n-1} \sum_{p=0}^{k} \frac{(k)!}{p!(n+3)!} R^{n-p-1}_{\bar{j}[l_{k+1} \ldots l_{k+2}]} \Omega^{mn} R^{m-p-1}_{\bar{i}[l_{p+1} \ldots l_{n+3}]} + \text{perm in } l_i$$

$$= \sum_{p=0}^{n-1} \sum_{k=p}^{n-1} \frac{(n-k+2)}{(n+3)!p!(k-p)!} R^{n-p-1}_{\bar{j}[l_{p+1} \ldots l_{p+2}]} \Omega^{mn} R^{m-p-1}_{\bar{i}[l_{p+3} \ldots l_{n+3}]} + \text{perm in } l_i.$$
The summation of factorials can be worked out as follows

\[
\sum_{k=p}^{n} \frac{k!}{(k-p)!} = \lim_{\epsilon \to 0} \sum_{k=p}^{n} \frac{k!}{(k-p+\epsilon)!} = \lim_{\epsilon \to 0} \sum_{k=p}^{n} \frac{\Gamma(k+1)\Gamma(\epsilon-p)}{\Gamma(k-p+\epsilon+1)\Gamma(\epsilon-p)}
\]

\[
= \lim_{\epsilon \to 0} \frac{1}{\Gamma(\epsilon-p)} \sum_{k=p}^{n} \int_{0}^{1} x^k (1-x)^{\epsilon-p-1}
\]

\[
= \lim_{\epsilon \to 0} \frac{1}{\Gamma(\epsilon-p)} \int_{0}^{1} (x^p - x^{n+1})(1-x)^{\epsilon-p-2}
\]

\[
= \lim_{\epsilon \to 0} \frac{1}{\Gamma(\epsilon-p)} \left[ \frac{\Gamma(p+1)\Gamma(\epsilon-p-1)}{\Gamma(\epsilon)} - \frac{\Gamma(n+2)\Gamma(\epsilon-p-1)}{\Gamma(\epsilon-p+n+1)} \right]
\]

\[
= \frac{(n+1)!}{(n-p)!(p+1)!}
\]

This leads to

\[
\sum_{p=0}^{n-1} \sum_{k=p}^{n-1} \frac{k!(n-k+2)}{(n+3)!p!(k-p)!} = \sum_{p=0}^{n-1} \frac{2p+n+5}{(p+2)!(n-p-1)!(n+1)(n+2)(n+3)}
\]

\[
= \frac{1}{2} \sum_{p=0}^{n-1} \frac{1}{(p+2)!(n-p+1)!},
\]

where we take the average between \(p \leftrightarrow n-p-1\) for the last step. The final result is

\[
\bar{\partial}_{\bar{j}} R_{\bar{i}\bar{l}_{1}...\bar{l}_{n+3}}^{m} = -\frac{1}{2} \sum_{p=0}^{n-1} \frac{1}{(p+2)!(n-p+1)!} R_{[\bar{j}\bar{l}_{1}...\bar{l}_{p+2m}}^{p} \Omega^{mn} R_{\bar{l}_{p+3}...\bar{l}_{n+3}]}^{n-p-1} + \text{perm in } l_{i}.
\]

Introducing formal variable \(\xi^i\) which transforms as a vector we define

\[
\Theta_l(x, \xi) \equiv \sum_{n=0}^{\infty} \frac{1}{(n+3)!} R_{[\bar{l}_{1}...\bar{l}_{n+3}]}^{n} (x) \xi^{l_{1}} \ldots \xi^{l_{n+3}},
\]

which satisfies the key identity

\[
\bar{\partial}_{\bar{j}} \Theta_{\bar{i}} = -\frac{1}{2} \{ \Theta_{\bar{j}}, \Theta_{\bar{i}} \},
\]

where \(\{ , \}\) stands for Poisson bracket in \(\xi\)-direction with respect to \(\Omega_{ij}(x)\). The equation (70) is flatness condition and hence there is an \(L_{\infty}\) structure defined for a hyperKähler manifold.

Next we show that

\[
\bar{\partial}_{\bar{i}} = \exp^{-1*}(\bar{\partial}_{\bar{i}} + \{ \Theta_{\bar{i}}, \cdot \}) \exp^{*}.
\]
It should be understood that the right $\bar{\partial}$ acts on the base of $T^{(1,0)}_M$ and the left one acts on the second factor of $M \times M$.

We need to show $(\bar{\partial}_i + \{\Theta_i, \cdot\}) \exp_\xi X^i = 0$ on the diagonal (where $X^i = (X^i)^*$), and this is equivalent to showing

$$e^{-\xi \nabla} (\bar{\partial}_i + \{\Theta_i, \cdot\}) e^{\xi \nabla} X^i = 0; \quad \nabla_i := \frac{\partial}{\partial X^i} - \xi^j \Gamma^k_{ij} \frac{\partial}{\partial \xi^k}$$

By using the formula $e^{-A} Be^A = e^{-[A, \cdot]} B$, and the flatness property $R^\times_{ij} = 0$, we can show for example

$$e^{-\xi \nabla} \partial_i e^{\xi \nabla} = \partial_i + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \xi_1 \cdots \xi_n \left( \xi_{n+1} \nabla_i \cdots \nabla_{i-n} R^k_{i_{n+1} i_n} \partial_{\xi^k} \right. \nabla_{i-n} \cdots \nabla_{i_1} R^k_{i_1 \cdots i_n} \nabla_i \right).$$

Let us agree to write $\xi^n \nabla^{n-2} R^k_i := \xi_1 \cdots \xi_n \nabla_i \cdots \nabla_{i-n} R^k_{i_1 \cdots i_n}$, then

$$e^{-\xi \nabla} \left( \xi^{m+2} \nabla^m R^k_i \partial_{\xi^k} \right) e^{\xi \nabla} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left( \xi^{n+2} \nabla^{n+2} R^k_i \partial_{\xi^k} - n \xi^{n+1} \nabla^{n+1} R^k_i \partial_{\xi^k} \right).$$

And combining the two,

$$e^{-\xi \nabla} \left( \bar{\partial}_i + \{\Theta_i, \cdot\} \right) e^{\xi \nabla} = e^{-\xi \nabla} \left( \bar{\partial}_i + \sum_{m=0}^{\infty} \frac{1}{(m+2)!} \xi^{m+2} \nabla^m R^k_i \partial_{\xi^k} \right) e^{\xi \nabla}$$

$$= \bar{\partial}_i + \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)!} \xi^{n+1} \nabla^{n+1} R^k_i \partial_{\xi^k},$$

This shows clearly that $(\bar{\partial}_i + \{\Theta_i, \cdot\}) \exp_\xi X^i = 0$.

References

[1] M. Alexandrov, M. Kontsevich, A. Schwartz and O. Zaboronsky, “The Geometry of the master equation and topological quantum field theory,” Int. J. Mod. Phys. A 12 (1997) 1405 [arXiv:hep-th/9502010].

[2] S. Axelrod and I. M. Singer, “Chern-Simons perturbation theory,” Proceedings of the XXth International Conference on Differential Geometric Methods in Theoretical Physics, Vol. 1, 2 (New York, 1991), 3–45, World Sci. Publ., River Edge, NJ, 1992. [arXiv:hep-th/9110056].

51
[3] A. S. Cattaneo and G. Felder, “On the AKSZ formulation of the Poisson sigma model,” Lett. Math. Phys. 56 (2001) 163 [arXiv:math.qa/0102108].

[4] A. S. Cattaneo, J. Qiu and M. Zabzine, “2D and 3D topological field theories for generalized complex geometry,” arXiv:0911.0993 [hep-th].

[5] K. J. Costello, “Renormalisation and the Batalin-Vilkovisky formalism,” arXiv:0706.1533 [math.QA].

[6] D. B. Fuks, ”Stable cohomologies of a Lie algebra of formal vector fields with tensor coefficients,” Funktsional. Anal. i Prilozhen. 17 (1983), no. 4, 62.

[7] D.B.Fuks, ”Cohomology of infinite-dimensional Lie algebras and characteristic classes of foliations,” Journal of Mathematical Sciences, Vol 11, 6 (1979).

[8] A. Hamilton and A. Lazarev, ”Characteristic classes of A∞ algebras,” J. Homotopy Relat. Struct. 3 (2008), no. 1, 65 [arXiv:math/0608395].

[9] A. Hamilton and A. Lazarev, ”Graph cohomology classes in the Batalin-Vilkovisky formalism”, J. Geom. Phys. 59, 555 (2009) [arXiv:math/0701825].

[10] M. Kapranov, ”Rozansky-Witten invariants via Atiyah classes,” Compositio Math. 115 (1999), no. 1, 71–113. [arXiv:alg-geom/9704009]

[11] M. Kontsevich, ”Formal (non)-commutative symplectic geometry,” The Gelfand Mathematical Seminars, 1990 - 1992, Birkhäuser (1993), 173 - 187.

[12] M. Kontsevich, ”Feynman diagrams and low-dimensional topology,” First European Congress of Mathematics, 1992, Paris, Volume II, Progress in Mathematics 120, Birkhäuser 1994, 97 - 121.

[13] M. Kontsevich, “Rozansky-Witten invariants via formal geometry,” Compositio Math. 115 (1999) 115 [arXiv:dg-ga/9704009].

[14] S. L. Lyakhovich, E. A. Mosman and A. A. Sharapov, “Characteristic classes of Q-manifolds: classification and applications,” arXiv:0906.0466 [math-ph].

[15] S. Morita, “Geometry of characteristic classes,” Iwanami Series in Modern Mathematics, American Mathematical Society, Providence, RI, 2001. xiv+185 pp.

[16] J. Qiu and M. Zabzine, “On the AKSZ formulation of the Rozansky-Witten theory and beyond,” JHEP 0909, 024 (2009) [arXiv:0906.3167 [hep-th]].
[17] D. Roytenberg, “On the structure of graded symplectic supermanifolds and Courant algebroids,” in: Quantization, Poisson Brackets and Beyond, Theodore Voronov (ed.), Contemp. Math, Vol. 315, Amer. Math. Soc., Providence, RI, 2002, arXiv:math/0203110.

[18] D. Roytenberg, “AKSZ-BV formalism and Courant algebroid-induced topological field theories,” Lett. Math. Phys. 79 (2007) 143 [arXiv:hep-th/0608150].

[19] L. Rozansky and E. Witten, “Hyper-Kähler geometry and invariants of three-manifolds,” Selecta Math. 3 (1997) 401 [arXiv:hep-th/9612216].

[20] J. Sawon, “Rozansky-Witten invariants of hyperkähler manifolds,” PhD thesis, Oxford 1999.

[21] A. S. Schwarz, “Geometry of Batalin-Vilkovisky quantization,” Commun. Math. Phys. 155, 249 (1993) arXiv:hep-th/9205088.

[22] A. S. Schwarz, “Quantum observables, Lie algebra homology and TQFT,” Lett. Math. Phys. 49, 115 (1999) arXiv:hep-th/9904168.

[23] T. Voronov, “Graded manifolds and Drinfeld doubles for Lie bialgebroids,” in: Quantization, Poisson Brackets and Beyond, Theodore Voronov (ed.), Contemp. Math, Vol. 315, Amer. Math. Soc., Providence, RI, 2002, arXiv:math/0105237.

[24] E. Witten, “A note on the antibracket formalism,” Mod. Phys. Lett. A 5, 487 (1990).

[25] E. Witten, “Quantum field theory and the Jones polynomial,” Commun. Math. Phys. 121, 351 (1989).