Redshift - distance relation in inhomogeneous cosmology

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Abstract

We continue to study a cosmological model with large-scale inhomogeneity. Working in the cosmic rest frame we determine null geodesics, redshift and area and luminosity distances. Combining the result with Hubble’s law enables us to calculate the distance of the local group of galaxies from the origin \( r = 0 \) where the Big Bang has taken place. We obtain a surprisingly small value of about 2 million lightyears.

Keyword: Cosmology

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1 Introduction

In this paper we continue the study of inhomogeneous cosmology in the cosmic rest frame. The perturbative solution of Einstein’s equations investigated in [1] is now crucially tested by determining the corresponding redshift - distance relation. We work in the cosmic rest frame throughout. The big advantage of this frame over comoving coordinates is that it is really a frame, that means a global coordinate system for the whole universe. As a consequence its origin $r = 0$ is the point where the Big Bang has taken place, and $t = 0$ is the time 14 billion years ago when this has happened. A natural and meaningful question which now occurs is: Where is $r = 0$ with respect to our place in the Milky Way? In [1] this question remained open. Here we will answer it: Surprisingly enough $r = 0$ is not further away than the Andromeda galaxy! This follows by taken Hubble’s law into account in the inhomogeneous cosmological model.

The paper is organized as follows. In the next section we recall the main results of [1] and use it to determine null geodesics; these are needed to find the light rays in the expanding cosmic gravitational field. In section 3 we consider Maxwell’s equations in the eikonal approximation and calculate the redshift. In section 4 we compute the area and luminosity distances for an observer near the origin $r = 0$. Combining the result with Hubble’s law enables us to determine the distance $R$ of the observer from the origin. We obtain a surprisingly small value for $R$ of about 2 million lightyears.

2 Null geodesics

We choose spherical coordinates $t, r, \vartheta, \phi$ in the cosmic rest frame and assume the inhomogeneous metric of the form

$$ds^2 = dt^2 + 2b(t, r)dt dr - a^2(t, r)[dr^2 + r^2(\vartheta^2 + \sin^2 \vartheta d\phi^2)].$$  \hspace{1cm} (2.1)

In [3] we have constructed a perturbative solution of Einstein’s equations given by

$$a(t, r) = \alpha t^{2/3} + tg_1(r) + h_1(r) + O(\alpha^{-1})$$  \hspace{1cm} (2.2)

$$b(t, r) = \alpha t^{2/3}Lr + tg_2(r) + h_2(r) + O(\alpha^{-1}).$$  \hspace{1cm} (2.3)

Here $\alpha$ and $L$ are universal constants of integration and the functions $g_1, g_2$ and $h_1, h_2$ are explicitly known. For later use we also list the components of the metric tensor corresponding to the line element (2.1)

$$g_{00} = 1, \quad g_{01} = b(t, r), \quad g_{11} = -a^2(t, r).$$
\[ g_{22} = -r^2a^2(t, r), \quad g_{33} = -r^2a^2(t, r) \sin^2 \vartheta, \quad (2.4) \]

and zero otherwise. The components of the inverse metric are equal to

\[ g^{00} = \frac{a^2}{D}, \quad g^{01} = \frac{b}{D}, \quad g^{11} = -\frac{1}{D} \]

\[ g^{22} = -\frac{1}{a^2r^2}, \quad g^{33} = -\frac{1}{a^2r^2 \sin^2 \vartheta} \]

where

\[ D = a^2 + b^2 \]

is the determinant of the \( 2 \times 2 \) matrix of the \( t, r \) components. The non-vanishing Christoffel symbols are given by

\[ \Gamma^{00}_{00} = \frac{bb'}{D}, \quad \Gamma^{00}_{01} = -\frac{ab}{D} \dot{a}, \quad \Gamma^{00}_{11} = \frac{1}{D} (a^3 \dot{a} - aba' + a^2 b') \]

\[ \Gamma^{02}_{22} = \frac{r}{D} (ra^3 \dot{a} + ba^2 + rba'), \quad \Gamma^{03}_{33} = \frac{r \sin^2 \vartheta}{D} (ra^3 \dot{a} + ba^2 + rba') \]

\[ \Gamma^{10}_{00} = -\frac{\dot{b}}{D}, \quad \Gamma^{10}_{01} = \frac{a \dot{a}}{D}, \quad \Gamma^{10}_{11} = \frac{1}{D} (ab \dot{a} + aa' + bb') \]

\[ \Gamma^{12}_{22} = \frac{r}{D} (rab \dot{a} - a^2 - raa'), \quad \Gamma^{13}_{33} = \frac{r}{D} (rab \dot{a} - a^2 - raa') \sin^2 \vartheta, \]

\[ \Gamma^{22}_{22} = \frac{\dot{a}}{a}, \quad \Gamma^{22}_{12} = \frac{a'}{a} + \frac{1}{r}, \quad \Gamma^{23}_{33} = -\sin \vartheta \cos \vartheta \]

\[ \Gamma^{32}_{03} = \frac{\dot{a}}{a}, \quad \Gamma^{32}_{13} = \frac{a'}{a} + \frac{1}{r}, \quad \Gamma^{33}_{23} = \frac{\cos \vartheta}{\sin \vartheta}. \]

Here the dot means \( \partial / \partial t \) and the prime \( \partial / \partial r \).

The propagation of radiation in the cosmic gravitational field is described by the geodesic equation

\[ \frac{dk^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta} k^\alpha k^\beta = 0 \quad (2.12) \]

where

\[ k^\mu = \frac{dx^\mu}{d\tau} \]

is the wave vector and \( \tau \) is the affine parameter along the ray. We want to determine the null geodesics which starts from a point on the \( z \)-axis under
a certain angle and lies in a plane $\phi = \text{const}$. Then only $k^0, k^1$ and $k^2$ are different from zero and the geodesic equation in leading order read as follows

$$\frac{dk^0}{d\tau} + \frac{L^2 r^2}{\beta} \frac{\dot{a}}{a} (k^0)^2 - 2 \frac{L r}{\beta} \dot{a} k^0 k^1 + \frac{a \dot{a}}{\beta} (k^1)^2 + \frac{r}{\beta} (ra \ddot{a} + L r) (k^2)^2 = 0 \quad (2.15)$$

$$\frac{dk^1}{d\tau} - \frac{L r}{\beta} \frac{\dot{a}}{a^2} (k^0)^2 + 2 \frac{\dot{a}}{\beta a} k^0 k^1 + \frac{1}{\beta} L r \ddot{a} + L^2 r (k^1)^2 + \frac{r}{\beta} (L r^2 a \ddot{a} - 1) (k^2)^2 = 0 \quad (2.16)$$

$$\frac{dk^2}{d\tau} + 2 \frac{\dot{a}}{a} k^0 k^2 + \frac{2}{r} k^1 k^2 = 0 \quad (2.17)$$

where

$$\beta = L^2 r^2 + 1. \quad (2.18)$$

Equation (2.17) can immediately be integrated

$$k^2 = \frac{C}{(ar)^2} = \frac{d\theta}{d\tau} \quad (2.19)$$

where $C$ is a constant of integration which fixes the angle under which the trajectory starts at the $z$-axis. In the following we are interested in small $C$, that means the trajectory is nearly radial. To solve the two remaining equations (2.15-16) we use the ansatz

$$k^0 = K \frac{a}{a} + f_0(t, r), \quad k^1 = K \frac{L r - \sqrt{\beta}}{a^2} + f_1(t, r). \quad (2.20)$$

Here $K$ is another integration constant which multiplies the radial null geodesics. The latter follows easily from

$$ds^2 = 0, \quad k_\mu k^\mu = 0, \quad (2.21)$$

it corresponds to $C = 0$. The sign of the square root has been chosen in such a way that the light ray moves towards $r = 0$. For small $C$ we neglect quadratic term in $f_j$, then we obtain from (2.15) the following linear equation

$$\frac{df_0}{d\tau} + 2 f_0 \frac{K L r}{\sqrt{\beta} a^2} \frac{\dot{a}}{a^2} - 2 f_1 \frac{K}{\sqrt{\beta}} \frac{\dot{a}}{a^2} = - \frac{C^2 \ddot{a}}{a^3 \beta r^2} \quad (2.22)$$
The normalization (2.21) gives another linear equation between \( f_0 \) and \( f_1 \):

\[
2Kf_1 \sqrt{\beta} = \frac{C^2}{a^2 r^2} - 2f_0 \frac{K}{a}(\beta - Lr \sqrt{\beta}).
\]  

(2.23)

Substituting this into (2.22) we find the simple equation

\[
\frac{df_0}{d\tau} + 2f_0 \frac{K \dot{a}}{a^2} = 0
\]

(2.24)

with the solution

\[
f_0 = \frac{K_1}{a^2}
\]

(2.25)

where \( K_1 \) is another constant of integration.

Now by (2.23) we get the second function \( f_1 \)

\[
f_1 = \frac{C^2}{2K a^2 r^2 \sqrt{\beta}} - \frac{K_1}{a^3}(\sqrt{\beta} - Lr).
\]

(2.26)

Since

\[
k^0 = \frac{dt}{d\tau} = \frac{K}{a} + f_0
\]

(2.27)

the constant \( K_1 \) in (2.25) only enters in the dependence \( t(\tau) \). Therefore, without loss of generality we can put \( K_1 = 0 \) by a redefinition of the affine parameter. Finally we have to check that the second geodesic equation (2.16) is satisfied by (2.25-26). This is indeed true in leading order \( O(a^{-3}) \), higher orders do not interest us because we have neglected them in the Christoffel symbols already.

Summing up the wave vector has the following components

\[
k^0 = \frac{K}{a} = \frac{dt}{d\tau}
\]

(2.28)

\[
k^1 = K \frac{Lr - \sqrt{\beta}}{a^2} + \frac{C^2}{2K a^2 r^2 \sqrt{\beta}} = \frac{dr}{d\tau}
\]

(2.29)

\[
k^2 = \frac{C}{a^2 r^2} = \frac{d\vartheta}{d\tau}.
\]

We put

\[
r_0 = \frac{C}{K}
\]

(2.30)
and assume this to be positive; its physical meaning will soon become clear. By dividing (2.31) and (2.28) we get

$$\frac{d\vartheta}{dt} = \frac{r_0}{ar^2}. \tag{2.31}$$

Since this is positive the polar angle $\vartheta(t)$ is monotonously increasing with time. We consider a trajectory starting on the z-axis at a point $r_1$ at time $t_1$ so that $\vartheta(t_1) = 0$. This is the place of our radiating galaxy. To find the light path we divide (2.29) by (2.28)

$$\frac{k^1}{k^0} = \frac{dr}{dt} = \frac{1}{a} \left( Lr - \sqrt{\beta} + \frac{r_0^2}{2r^2\sqrt{\beta}} \right) \tag{2.32}$$

and separate the variables:

$$\frac{dt}{a(t)} = \frac{r^2\sqrt{L^2r^2 + 1}}{Lr^3\sqrt{L^2r^2 + 1 - r^2(L^2r^2 + 1) + r_0^2/2}} \, dr. \tag{2.33}$$

We consider the light rays towards the origin $r = 0$, that means

$$\frac{dr}{dt} < 0.$$ 

Setting this equal to 0 we find from (2.32) the minimal distance of the path from the origin

$$r^2_{\text{min}} = \frac{r_0^2}{2} + \sqrt{\frac{r_1^4}{4} + \frac{r_0^2}{L^2}}, \tag{2.34}$$

so $r_0$ determines this minimal distance.

To find the path we must integrate (2.33). To make this integration simple we assume

$$L^2r^2 \gg 1 \tag{2.35}$$

and expand

$$\sqrt{\beta} = Lr + \frac{1}{2Lr}.$$ 

The usefulness of this approximation was discussed in [1]. We integrate (2.33) from an emission time $t_1$ to the present time $T$:

$$\int_{t_1}^{T} \frac{dt}{\alpha t^{2/3}} = \frac{1}{3\alpha} (T^{1/3} - t_1^{1/3}) = -2L \int_{r_1}^{R} \frac{r^3}{r^2 - r_0^2} \, dr.$$
Here \( r_1 \) and \( R \) are the radial coordinates of the emitter and observer, respectively. Solving for \( t_1 \) we obtain the equation \( t = t(r) \) of the light path

\[
t = \left[ T^{1/3} - 3\alpha L(r^2 - R^2 + r_0^2 \log \left| \frac{r_2^2 - r_0^2}{R^2 - r_0^2} \right| \right]^3.
\]  (2.37)

3 The redshift

The propagation of electromagnetic radiation in the expanding universe is described by Maxwell’s equations in a gravitational field. The homogeneous equations are

\[
\nabla_\lambda F_{\mu\nu} + \nabla_\mu F_{\nu\lambda} + \nabla_\nu F_{\lambda\mu} = 0
\]

(3.1)

where \( \nabla \) stands for the covariant derivative with respect to our inhomogeneous metric. In addition the field tensor \( F_{\mu\nu} \) satisfies the inhomogeneous equations

\[
\nabla_\mu F^{\mu\nu} = 0
\]

away from the radiating galaxy. As in Minkowski vacuum, equation (3.1) implies the existence of a vector potential

\[
F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu
\]

and similarly for the upper indices. Here we can impose the “Lorentz” gauge condition

\[
\nabla^\mu A_\mu = 0.
\]

(3.4)

As in classical electrodynamics we write (3.2) in terms of the vector potential

\[
\nabla_\mu \nabla^\mu A_\nu - \nabla_\mu \nabla^\nu A_\mu = 0.
\]

(3.5)

Now we want to commute the derivatives in the second term and use (3.4). Since the covariant derivatives do not commute we obtain an additional term with the Ricci tensor

\[
\nabla_\mu \nabla^\mu A_\nu - R^\nu_\alpha A^\alpha = 0.
\]

In cosmology it is sufficient to solve this equation in the eikonal approximation by considering a solution of the form

\[
A_\nu(x) = f_\nu(x) \sin[\varphi(x)].
\]

(3.6)
Here \( f^\nu \) is a slowly varying amplitude, \( \varphi \) is the phase which varies from 0 to 2\( \pi \) over the wave length of the radiation. Since the latter is extremely small compared to cosmic distances, \( \varphi \) is big so that we shall consider only the first two orders of \( \varphi \) in (3.6). The leading order \( O(\varphi^2) \) gives

\[
\nabla^\mu \varphi \nabla_\mu \varphi = 0
\]

(3.7)

and in order \( O(\varphi) \) we obtain

\[
2 \nabla^\mu f^\nu \nabla_\mu \varphi + f^\nu \nabla_\mu \nabla^\mu \varphi = 0.
\]

(3.8)

As in optics one introduces the wave vector

\[
k^\mu = \nabla^\mu \varphi.
\]

(3.9)

Then the two equations to be solved read

\[
k^\mu k_\mu = 0
\]

(3.10)

\[
k^\mu \nabla_\mu f^\nu = -\frac{1}{2} f^\nu \nabla_\mu k^\mu.
\]

(3.11)

From equation (3.10) we have

\[
k^\mu \nabla_\nu k_\mu = 0,
\]

(3.12)

and since (3.9) implies

\[
\nabla_\nu k_\mu = \nabla_\mu k_\nu
\]

we arrive at

\[
k^\mu \nabla_\mu k^\nu = k^\mu \left( \frac{\partial k^\nu}{\partial x^\mu} + \Gamma^\nu_{\mu \lambda} k^\lambda \right) = 0.
\]

(3.13)

Let now \( x^\nu(\tau) \) be the curve

\[
\frac{dx^\mu}{d\tau} = k^\mu,
\]

(3.14)

then we have

\[
\frac{\partial k^\nu}{\partial x^\mu} = \frac{d^2 x^\nu}{d\tau^2} \frac{d\tau}{dx^\mu} = \frac{d^2 x^\nu}{d\tau^2} \frac{1}{k^\mu}
\]

and (3.13) becomes

\[
\frac{d^2 x^\nu}{d\tau^2} + \Gamma^\nu_{\mu \lambda} \frac{dx^\mu}{d\tau} \frac{dx^\lambda}{d\tau} = 0.
\]

(3.15)

This is the geodesic equation (2.12) for \( x^\nu(\tau) \) which we have solved in the last section.
Following G.F.R. Ellis [2] (see also [3]) the redshift is given by
\[
\nu_{\text{em}} = 1 + z = \frac{(u_{\mu}k^{\mu})_{\text{em}}}{(u_{\mu}k^{\mu})_{\text{obs}}}
\]  
(3.16)
where \(u_{\mu}\) is the 4-velocity of the emitting galaxy and the observer, respectively. The radial velocity of the galaxy we know from [1] equations (6.8)-(6.9):
\[
u_0 = \sqrt{\beta}, \quad u_1 = \frac{L^2rt}{\sqrt{\beta}}.
\]  
(3.17)
Then, since
\[
\omega \equiv u_{\mu}k^{\mu} = K \sqrt{\beta} \left[ 1 + \frac{L^2rt}{\beta a(t)} \left( Lr - \sqrt{\beta} + \frac{r_0^2}{2r^2\sqrt{\beta}} \right) \right]
\]  
(3.18)
the redshift becomes
\[
1 + z = \frac{\sqrt{\beta} a(T)}{\sqrt{\beta R} a(t)} \left[ 1 + \frac{L^2rt}{\beta a(t)} \left( Lr - \sqrt{\beta} + \frac{r_0^2}{2r^2\sqrt{\beta}} \right) \right] \times
\]
\[
\times \left[ 1 + \frac{L^2RT}{\beta Ra(T)} \left( LR - \sqrt{\beta R} + \frac{r_0^2}{2R^2\sqrt{\beta R}} \right) \right]^{-1},
\]  
(3.19)
where \(T\) is the present age of the Universe and \(R > 0\) the radial coordinate of the observer. The emission time \(t\) follows from the coordinate distance \(r = r_1\) of the galaxy by means of (2.37). For later use we introduce the expression
\[
K_r = 1 + \frac{L^2rt}{\beta} \left( Lr - \sqrt{\beta} + \frac{r_0^2}{2r^2\sqrt{\beta}} \right).
\]  
(3.20)
Then the redshift is equal to
\[
1 + z = \frac{\sqrt{\beta} a(T)}{\sqrt{\beta R} a(t)} K_r.
\]  
(3.21)
where
\[
\beta_R = L^2R^2 + 1
\]  
(3.22)
according to (2.18).
4 Area and luminosity distances

In this section we must specify the radial coordinate $R$ of the observer. We choose

$$R = r_{\text{min}} \quad (4.1)$$

equal to the minimal distance (2.34) of the light path from the origin $r = 0$ where the Big Bang has taken place. This greatly simplifies the following calculations because we have

$$\frac{dr}{dt} \bigg|_R = 0 \quad (4.2)$$

$$K_R = 1 \quad (4.3)$$

so that the redshift (3.21) becomes

$$1 + z = \sqrt{\beta} \frac{a(T)}{\sqrt{\beta_R a(t)}} K_r. \quad (4.4)$$

Theoretically the simplest radial distance is the area or angular-diameter distance [4]

$$D_A = a(t) r(t) \quad (4.5)$$

where $r(t)$ is the radial coordinate of the galaxy at the time of emission $t$ which can be found by inverting (2.37). Assuming reciprocity [3] $D_A$ is related to the luminosity distance $D_L$ which is measured by the astronomers by

$$D_L = (1 + z)^2 D_A = a(t) r(t) (1 + z)^2. \quad (4.6)$$

Considering $t$ as a parameter the equations (4.4) and (4.6) determine the distance redshift relation $D_L = D_L(z)$ in parametric form.

We want to calculate the derivative

$$\frac{dD_L}{dz} \bigg|_{z=0} = \frac{c}{H_0} \quad (4.7)$$

which gives the Hubble constant $H_0$ at present time $T$ (we have put the velocity of light $c$ equal to 1). From (4.6) we obtain

$$\frac{dD_L}{dz} \bigg|_{z=0} = \dot{a}(T) \frac{dt}{dz} \bigg|_T R + 2a(T)R \quad (4.8)$$

where (4.2) has been taken into account. Again using (4.2) we get from (4.4)

$$\frac{dz}{dt} \bigg|_T = \frac{\partial z}{\partial t} = -\frac{\dot{a}}{a} \bigg|_T + \partial_t K_r \bigg|_T. \quad (4.9)$$
Since
\[ \partial_t K_r |_{T} = 0 \]
because \( r(T) = R \) is the minimal distance (4.1) we get in leading order \( O(\alpha) \) (2.2)
\[ \frac{dz}{dt} |_{T} = -\frac{2}{3T} \tag{4.10} \]
Using
\[ \dot{a}(T) = \frac{2}{3T} a \tag{4.11} \]
we finally find
\[ \frac{dD_L}{dz} \bigg|_{z=0} = a(T) R. \tag{4.12} \]

Now it is time to put some simple numbers in. Let
\[ \rho_{\text{crit}} = 1.878 \times 10^{-29} h^2 \text{g/cm}^3 \tag{4.13} \]
be the critical density [4] and we assume a Hubble constant
\[ h = 0.7, \quad H_0 = 70 \text{ km s}^{-1} \text{Mpc}^{-1}. \tag{4.14} \]
We use the empirical fact that the corresponding Hubble time
\[ \frac{1}{H_0} = 14 \times 10^9 \text{years} = T \tag{4.15} \]
coincides with the age \( T \) of the universe. A realistic density of normal matter at present time \( T \) is
\[ \rho_m = 0.01 \times \rho_{\text{crit}} = \frac{1}{6\pi G \beta_R T^2} \tag{4.16} \]
where the last equality is the lowest order result [1] eq.(6.7). This allows to determine
\[ \beta_R = L^2 R^2 + 1 = 44.3 \tag{4.17} \]
which gives
\[ LR = 6.6. \tag{4.18} \]
Next we assume a radial velocity of our Galaxy of 300 km/sec that means
\[ v_m = 0.001 = \frac{LR}{a(T)} \tag{4.19} \]
where the last equality comes from [1] eq.(7.6). Using (4.18) this allows to determine
\[ a(T) = 6.6 \times 10^3. \]  
(4.20)

Now we are able to calculate \( R \) from (4.12) using (4.7) (4.12) and the Hubble time (4.15)
\[ R = \frac{T}{a(T)} = 2.1 \times 10^6 \text{ly} = 640 \text{kpc}. \]  
(4.21)

This is the distance between our Galaxy and the origin \( r = 0 \) where the Big Bang has taken place. It is even smaller than the distance of the Andromeda galaxy which is \( 2.5 \times 10^6 \) lightyears away. From the dipole anisotropy of the CMB one has derived a net velocity of the local group of galaxies of \( 627 \pm 22 \) km/sec in a direction between the Hydra and Centaurus clusters of galaxies [4] [6]. We expect that the origin \( r = 0 \) lies in the opposite direction. The velocity about 600 km/sec seems to be rather high. But this velocity is the sum of the cosmic radial velocity and a local velocity due to the attraction of the Hydra and Centaurus clusters. Since the latter has not been estimated we have made the simple assumption that 300 km/sec is the cosmic velocity (4.19).

If we calculate the second derivative \( \frac{d^2 D_L}{dz^2} \) in the same way we get zero. This shows that a lowest order calculation is not good enough to determine the acceleration parameter \( q_0 \).

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