A concept of Dirac-type tensor equations

N.G.Marchuk *

October 27, 2018

PACS: 04.20Cv, 04.62, 11.15, 12.10

Steklov Mathematical Institute, Gubkina st.8, Moscow 119991, Russia
nmarchuk@mi.ras.ru, www.orc.ru/~nmarchuk

Abstract

Considering a four dimensional parallelisable manifold, we develop a concept of Dirac-type tensor equations with wave functions that belong to left ideals of the set of nonhomogeneous complex valued differential forms.

Contents

1 Two pictures of the Dirac equation. 3

2 A Differentiable manifold \( X \). Tensors. 5

3 A Parallelisable manifold with a tetrad. 5

4 Differential forms. 7

5 A central product of differential forms. 8

*Research supported by the Russian Foundation for Basic Research grants 00-01-00224, 00-15-96073.
6 Tensors from $\Lambda^r_k$.  

7 Dirac-type tensor equations. A general case.  

8 Unitary and Spin gauge symmetries.  

9 A connection between the Dirac-type tensor equation and the Dirac equation.  

10 Idempotents and bases of left ideals.  

11 Special cases  

In the previous papers [2], [3], [4], developing ideas of [6]-[12], we consider Dirac-type tensor equations with non-Abelian gauge symmetries on a four dimensional parallelisable manifold with wave functions that belong to the algebras $\Lambda^c, \Lambda, \Lambda^c_{\text{even}}, \Lambda_{\text{even}}$ of real dimensions 32, 16, 16, 8 respectively, where $\Lambda$ is the set of nonhomogeneous differential forms, $\Lambda_{\text{even}}$ is the set of even differential forms, and $\Lambda^c, \Lambda^c_{\text{even}}$ are the corresponding sets of complex valued differential forms. Now we develop a concept of Dirac-type tensor equations with wave functions that belong to left ideals of $\Lambda^c$. This concept gives us possibility to find an SU(3) invariant Dirac-type tensor equation with 12 complex valued components of wave function. In addition to unitary gauge symmetries Dirac-type tensor equations have a gauge symmetry with respect to the spinor group Spin($\mathcal{W}$). In Section 9 we investigate in detail a connection between the (spinor) Dirac equation and the Dirac-type tensor equation. Of cause, Dirac spinors and tensors are different mathematical objects and, generally speaking, we can’t establish an equivalence between them. But it nevertheless, if we take Dirac spinors that are solutions of the Dirac equation and tensors (differential forms) that are solutions of the corresponding Dirac-type tensor equation, then, with the aid of above mentioned spinor group symmetry, we can establish a one-to-one correspondence between these spinors and tensors (Section 9).
1 Two pictures of the Dirac equation.

Let $\mathcal{R}^{1,3}$ be the Minkowski space with coordinates $x^\mu (\mu = 0, 1, 2, 3)$, with a metric tensor $\eta_{\mu\nu} = \eta^{\mu\nu}$ defined by the Minkowski matrix

$$\eta = \|\eta_{\mu\nu}\| = \|\eta^{\mu\nu}\| = \text{diag}(1, -1, -1, -1).$$

Consider the Dirac equation for an electron

$$\gamma^\mu (\partial_\mu \psi + ia_\mu \psi) + im\psi = 0, \quad (1)$$

where $\partial_\mu = \partial/\partial x^\mu$, $\psi = \psi(x)$ is a column of four complex valued smooth functions (an electron wave function), $a_\mu = a_\mu(x)$ is a real valued covector field (a potential of electromagnetic field), $m$ is a real constant (the electron mass), $i = \sqrt{-1}$, and $\gamma^\mu$ are (Dirac) matrices that satisfy the identities ($1$ is the identity matrix)

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} 1.$$

In particular we may take

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad (2)$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Consider a change of coordinates

$$x^\mu \rightarrow \hat{x}^\mu = p_\nu^\mu x^\nu \quad (3)$$

from the proper orthochronous Lorentz group $\text{SO}^+(1, 3)$, i.e.,

$$P^T \eta P = \eta, \quad \det P = 1, \quad p^0_0 > 0, \quad (4)$$

where $P = \|p^\mu_\nu\|$. Then

$$\partial_\mu \rightarrow \hat{\partial}_\mu = q^\nu_\mu \partial_\nu, \quad a_\mu \rightarrow \hat{a}_\mu = q^\nu_\mu a_\nu.$$
where $q^\nu_\mu$ are elements of the inverse matrix $P^{-1}$, i.e.,

$$q^\nu_\mu P^\mu_\lambda = \delta^\nu_\lambda, \quad q^\mu_\nu p^\lambda_\mu = \delta^\lambda_\nu$$

and $\delta^\nu_\mu$ is the Kronecker tensor. P. Dirac in [1] postulate that $\gamma^\mu$ is a vector with values in $\mathcal{M}^C(4)$ ($\mathcal{M}^C(4)$ is the algebra of complex valued $4 \times 4$-matrices). That means

$$\gamma^\mu \rightarrow \dot{\gamma}^\mu = p^\nu_\mu \gamma^\nu$$

and eq. (1) in coordinates $(\dot{x})$ has the form

$$\dot{\gamma}^\mu (\dot{\partial}^\mu \psi + i \dot{a}^\mu \psi) + im\psi = 0,$$

where $\dot{\partial}^\mu = \partial/\partial \dot{x}^\mu$, $\psi = \psi(x(\dot{x}))$. It is proved in the theory of representations of Lorentz group that there exists a pair of nondegenerate matrices $\pm R \in \mathcal{M}^C(4)$ such that

$$R^{-1} \gamma^\mu R = p^\nu_\mu \gamma^\nu = \dot{\gamma}^\mu. \quad (6)$$

Substituting $R^{-1} \gamma^\mu R$ for $\dot{\gamma}^\mu$ in (3), we get

$$R^{-1} \gamma^\mu R(\dot{\partial}^\mu \psi + i \dot{a}^\mu \psi) + im\psi = 0, \quad (7)$$

or, equivalently,

$$\gamma^\mu (\dot{\partial}^\mu (R \psi) + i \dot{a}^\mu (R \psi)) + im(R \psi) = 0, \quad (8)$$

We see that an invariance of the Dirac equation can be proved in two ways.

If we suppose the following transformation rules for $\gamma^\mu$ and $\psi$ under every change of coordinates (3,4):

$$\gamma^\mu \rightarrow R^{-1} \gamma^\mu R, \quad \psi \rightarrow \psi,$$

then we get a spinorless picture of the Dirac equation.

In the case of transformation rules

$$\gamma^\mu \rightarrow \gamma^\mu, \quad \psi \rightarrow R \psi,$$

we get a conventional spinor picture of the Dirac equation. In sect.9, considering a connection between the Dirac equation and the corresponding Dirac-type tensor equation, we shall see that the two pictures of the Dirac equation are the consequence of a gauge symmetry of the Dirac-type tensor equation with respect to the spinor group.
2 A Differentiable manifold $X$. Tensors.

Let $X$ be a four-dimensional orientable differentiable manifold with local coordinates $x^\mu$ ($\mu = 0, 1, 2, 3$). Summation convention over repeating indices is assumed. Let $\mathcal{T}^q_p$ be the set of smooth type $(p, q)$ real valued tensors (tensor fields) on $X$ 

$$\mathcal{T}^q_p = \{ u^{\nu_1 \ldots \nu_q}_{\mu_1 \ldots \mu_p}(x) \}.$$ 

Under a change of coordinates $(x) \rightarrow (\hat{x})$ a tensor $u^{\nu_1 \ldots \nu_q}_{\mu_1 \ldots \mu_p}$ transforms as follows:

$$u^{\nu_1 \ldots \nu_q}_{\mu_1 \ldots \mu_p} \rightarrow \hat{u}^{\alpha_1 \ldots \alpha_q}_{\beta_1 \ldots \beta_p} = u^{\nu_1 \ldots \nu_q}_{\mu_1 \ldots \mu_p} \frac{\partial \hat{x}^\alpha_1}{\partial x^\nu_1} \cdots \frac{\partial \hat{x}^\alpha_p}{\partial x^\mu_p} \frac{\partial x^{\mu_p}}{\partial \hat{x}^{\alpha_p}} \frac{\partial x^{\nu_p}}{\partial \hat{x}^{\alpha_p}} \cdots \frac{\partial x^{\nu_1}}{\partial \hat{x}^{\alpha_1}}.$$

Operations of addition and tensor product are defined in the usual way

$$(u + v)^{\nu_1 \ldots \nu_q}_{\mu_1 \ldots \mu_p} = u^{\nu_1 \ldots \nu_q}_{\mu_1 \ldots \mu_p} + v^{\nu_1 \ldots \nu_q}_{\mu_1 \ldots \mu_p} \in \mathcal{T}^q_p, \quad u, v \in \mathcal{T}^q_p,$$

$$(u \otimes v)^{\nu_1 \ldots \nu_q}_{\mu_1 \ldots \mu_p + r} = u^{\nu_1 \ldots \nu_q}_{\mu_1 \ldots \mu_p} v^{\nu_{p+1} \ldots \nu_{p+r}} \in \mathcal{T}^{q+s}_{p+r}, \quad u \in \mathcal{T}^q_p, v \in \mathcal{T}^s_r.$$ 

3 A Parallelisable manifold with a tetrad.

An $n$-dimensional differentiable manifold is called parallelisable if there exist $n$ linear independent vector or covector fields on it. Let $X$ be a four dimensional parallelisable manifold with local coordinates $x = (x^\mu)$ and $e_\mu^a = e_\mu^a(x)$, $a = 0, 1, 2, 3$ be four covector fields on $X$. This set of four covectors is called a tetrad. The full set $\{X, e_\mu^a, \eta\}$, where $\eta$ is the Minkowski matrix, is denoted by $W$. Here and in what follows we use Greek indices as tensorial indices and Latin indices as nontensorial (tetrad) indices.

We can define a metric tensor on $W$

$$g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}$$

such that

$$g_{\mu\nu} = g_{\nu\mu}, \quad g_{00} > 0, \quad g = \det\|g_{\mu\nu}\| < 0,$$

and the signature of the matrix $\|g_{\mu\nu}\|$ is equal to $-2$. Hence we may consider $W$ as pseudo-Riemannian space with the metric tensor $g_{\mu\nu}$. We raise and
lower Latin indices with the aid of the Minkowski matrix $\eta^{ab} = \eta_{ab}$ and Greek indices with the aid of the metric tensor $g_{\mu\nu}$

$$e^{\nu a} = g^{\mu \nu} e_\mu^a, \quad e_{\mu a} = \eta_{ab} e_\mu^b.$$  

With the aid of the tetrad we can replace all or part of Greek indices in a tensor $u_{\mu_1 \ldots \mu_p}^{\nu_1 \ldots \nu_q} \in \mathcal{T}^q_p$ by Latin indices

$$u_{a_1 \ldots a_p}^{b_1 \ldots b_q} = u_{\mu_1 \ldots \mu_p}^{\nu_1 \ldots \nu_q} e_{\mu_1}^{a_1} \ldots e_{\mu_p}^{a_p} e_{\nu_1}^{b_1} \ldots e_{\nu_q}^{b_q}.$$  

As a result we get a set of invariants $u_{b_1 \ldots b_q}^{a_1 \ldots a_p}$ enumerated by Latin indices. On the contrary we can replace Latin indices by Greek indices

$$u_{\nu_1 \ldots \nu_q}^{\mu_1 \ldots \mu_p} = u_{a_1 \ldots a_p}^{b_1 \ldots b_q} e_{\mu_1}^{a_1} \ldots e_{\mu_p}^{a_p} e_{\nu_1}^{b_1} \ldots e_{\nu_q}^{b_q}.$$  

Note that

$$e^\mu_a e^b_\mu = \delta^b_a, \quad e^\mu_a e^a_\nu = \delta^\mu_\nu.$$  

The metric tensor $g_{\mu\nu}$ defines the Levi-Civita connection, the curvature tensor, the Ricci tensor, and the scalar curvature

$$\begin{align*}
\Gamma_{\mu\nu}^\lambda &= \frac{1}{2} g_{\kappa\lambda}(\partial_{\mu} g_{\nu\kappa} + \partial_{\nu} g_{\mu\kappa} - \partial_{\kappa} g_{\mu\nu}), \\
R_{\lambda\mu\nu\kappa} &= \partial_{\mu} \Gamma_{\nu\kappa}^{\lambda} - \partial_{\nu} \Gamma_{\mu\kappa}^{\lambda} + \Gamma_{\mu\eta}^{\lambda} \Gamma_{\nu\kappa}^{\eta} - \Gamma_{\nu\eta}^{\lambda} \Gamma_{\mu\kappa}^{\eta}, \\
R_{\nu\rho} &= R_{\lambda\mu\nu\rho}, \\
R &= g^{\mu\nu} R_{\mu\nu}
\end{align*}$$

with symmetries

$$\Gamma_{\mu\nu}^\lambda = \Gamma_{\nu\mu}^\lambda, \quad R_{\mu\nu\lambda\rho} = R_{\lambda\mu\nu\rho} = R_{[\mu\nu]\lambda\rho}, \quad R_{\mu[p\lambda\rho]} = 0, \quad R_{\nu\rho} = R_{\rho\nu}$$

Covariant derivatives $\nabla_\mu : \mathcal{T}^q_p \to \mathcal{T}^q_{p+1}$ are defined via the Levi-Civita connection by the usual rules

1. If $u = u(x), \ x \in X$ is a scalar function, then

$$\nabla_\mu u = \partial_\mu u.$$  

2. If $u^\nu \in \mathcal{T}^1$, then

$$\nabla_\mu u^\nu \equiv u^\nu_{;\mu} = \partial_\mu u^\nu + \Gamma_{\lambda\mu}^\nu u^\lambda.$$  

6
3. If $u_\nu \in \mathcal{T}_1$, then

$$\nabla_\mu u_\nu \equiv u_{\nu;\mu} = \partial_\mu u_\nu - \Gamma^\lambda_{\nu\mu} u_\lambda.$$ 

4. If $u = (u_{\nu_1...\nu_k}^{\lambda_1...\lambda_3}) \in \mathcal{T}_k$, $v = (v_{\rho_1...\rho_s}^\tau) \in \mathcal{T}_s$, then

$$\nabla_\mu (u \otimes v) = (\nabla_\mu u) \otimes v + u \otimes \nabla_\mu v.$$ 

With the aid of these rules it is easy to calculate covariant derivatives of arbitrary type tensors. Also we can check the formulas

$$\nabla_\mu g^\alpha_\beta = 0, \quad \nabla_\mu g_\alpha^\beta = 0, \quad \nabla_\mu \delta^\beta_\alpha = 0,$$

$$\nabla_\alpha (R^\alpha_\beta - \frac{1}{2} R g^\alpha_\beta) = 0,$$

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) a_\rho = R^\lambda_{\rho\mu\nu} a_\lambda,$$

for any $a_\rho \in \mathcal{T}_1$.

4 Differential forms.

Let $\Lambda_k$ be the set of all exterior differential forms of rank $k = 0, 1, 2, 3, 4$ ($k$-forms) on $\mathcal{W}$ and

$$\Lambda = \Lambda^0 \oplus \ldots \oplus \Lambda^4 = \Lambda^{\text{even}} \oplus \Lambda^{\text{odd}},$$

$$\Lambda^{\text{even}} = \Lambda^0 \oplus \Lambda^2 \oplus \Lambda^4, \quad \Lambda^{\text{odd}} = \Lambda^1 \oplus \Lambda^3.$$ 

The set of smooth real valued functions on $\mathcal{W}$ is identified with the set of 0-forms $\Lambda_0$. A $k$-form $U \in \Lambda_k$ can be written as

$$U = \frac{1}{k!} u_{\nu_1...\nu_k} dx^{\nu_1} \wedge \ldots \wedge dx^{\nu_k} = \sum_{\mu_1<...<\mu_k} u_{\mu_1...\mu_k} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_k}, \quad \text{(13)}$$

where $u_{\nu_1...\nu_k} = u_{\nu_1...\nu_k}(x)$ are real valued components of a covariant antisymmetric $(u_{\nu_1...\nu_k} = u_{[\nu_1...\nu_k]})$ tensor. Differential forms from $\Lambda$ can be written as linear combinations of the 16 basis elements

$$1, dx^\mu, dx^{\mu_1} \wedge dx^{\mu_2}, \ldots, dx^0 \wedge \ldots \wedge dx^3, \quad \mu_1 < \mu_2 < \ldots. \quad \text{(14)}$$
The exterior product of differential forms is defined in the usual way. If $U \in \Lambda^r, V \in \Lambda^s$, then

$$U \wedge V = (-1)^{rs}V \wedge U \in \Lambda^{r+s}.$$ 

In the sequel we also use complex valued differential forms from $\Lambda^C_k, \Lambda^C, \Lambda^C_{\text{even}}, \Lambda^C_{\text{odd}}$.

The tetrad $e^a_\mu$ can be written with the aid of four 1-forms

$$e^a = e^a_\mu dx^\mu \quad \text{or} \quad e_a = \eta_{ab}e^b.$$ 

The $k$-form $U$ from (13) can be written as

$$U = \frac{1}{k!} u_{a_1 \ldots a_k} e^{a_1} \wedge \ldots \wedge e^{a_k},$$ 

where invariants $u_{a_1 \ldots a_k} = u_{[a_1 \ldots a_k]}$ connected with tensor components $u_{\mu_1 \ldots \mu_k}$ by the formula

$$u_{a_1 \ldots a_k} = u_{\mu_1 \ldots \mu_k} e^{a_1}_{\mu_1} \cdots e^{a_k}_{\mu_k}.$$ 

Differential forms from $\Lambda$ can be written as

$$\sum_{k=0}^4 \frac{1}{k!} u_{a_1 \ldots a_k} e^{a_1} \wedge \ldots \wedge e^{a_k}, \quad u_{a_1 \ldots a_k} = u_{[a_1 \ldots a_k]}$$

and the 16 differential forms

$$1, e^a, e^{a_1} \wedge e^{a_2}, \ldots e^0 \wedge \ldots \wedge e^3, \quad a_1 < a_2 < \ldots$$

(15)

can be considered as basis forms of $\Lambda$.

5 A central product of differential forms.

Let us define a central product of differential forms $U, V \rightarrow UV$ by the following rules:

1. For $U, V, W \in \Lambda, \alpha \in \Lambda_0$

$$1U = U1 = U, \quad (\alpha U)V = U(\alpha V) = \alpha(UV),$$

$$U(VW) = (UV)W, \quad (U + V)W = UW + VW.$$ 

2. $e^a e^b = e^a \wedge e^b + \eta^{ab}$. 

3. $e^{a_1} \ldots e^{a_k} = e^{a_1} \wedge \ldots \wedge e^{a_k}$ for $a_1 < \ldots < a_k$. 


Note that from the second rule we get the equality
\[ e^a e^b + e^b e^a = 2\eta^{ab}, \] (16)
which appear in the Clifford algebra\(^1\). Substituting into (16)
\[ e^a = e_\mu^a dx^\mu, \quad e^b = e_\nu^b dx^\nu, \quad \eta^{ab} = g^{\mu\nu} e_\mu^a e_\nu^b, \]
we get
\[ e_\mu^a e_\nu^b (dx^\mu dx^\nu + dx^\nu dx^\mu - 2g^{\mu\nu}) = 0. \]
Thus,
\[ dx^\mu dx^\nu + dx^\nu dx^\mu = 2g^{\mu\nu}. \]
Evidently, the operation of central product maps \( \Lambda \times \Lambda \to \Lambda \). From this
definition of the central product it is evident that for \( u_{a_1...a_k} = u_{[a_1...a_k]} \)
\[ \frac{1}{k!} u_{a_1...a_k} e^{a_1} \wedge ... \wedge e^{a_k} = \frac{1}{k!} u_{a_1...a_k} e^{a_1} ... e^{a_k}. \]
In the sequel we use notation
\[ e^{a_1...a_k} = e^{a_1} ... e^{a_k} = e^{a_1} \wedge ... \wedge e^{a_k} \quad \text{for} \quad a_1 < \cdots < a_k. \]
Let us define an operation of conjugation, which maps \( \Lambda_k \to \Lambda_k \) or \( \Lambda^C_k \to \Lambda^C_k \). For \( U \in \Lambda^C_k \)
\[ U^* = (-1)^{\frac{k(k-1)}{2}} \bar{U}, \]
where \( \bar{U} \) is the differential form with complex conjugated components (if \( U \in \Lambda \), then \( \bar{U} = U \)). We see that
\[ (UV)^* = V^* U^*, \quad U^{**} = U, \quad (e^{a_1} ... e^{a_k})^* = e^{a_k} ... e^{a_1} \]
and
\[ U^* = U \quad \text{for} \quad U \in \Lambda_0 \oplus \Lambda_1 \oplus \Lambda_4, \]
\[ U^* = -U \quad \text{for} \quad U \in \Lambda_2 \oplus \Lambda_3. \]
\(^1\)In a special case the central product was invented by H. Grassmann\([14]\) in 1877 as
an attempt to unify the exterior calculus (the Grassmann algebra) with the quaternion
calculus. A discussion on that matter see in\([15]\). In some papers the central product is
called a Clifford product.
Let us define a trace of differential form as linear operation $\text{Tr} : \Lambda \to \Lambda_0$ such that

$$\text{Tr}(1) = 1, \quad \text{Tr}(e^{a_1} \wedge \ldots \wedge e^{a_k}) = 0 \quad \text{for} \quad k = 1, 2, 3, 4.$$ 

It is easy to prove that

$$\text{Tr}(UV - VU) = 0 \quad \text{for} \quad U, V \in \Lambda.$$ 

Let us take the set of forms

$$\text{Spin}(\mathcal{W}) = \{ S \in \Lambda_{\text{even}} : S^* S = 1 \},$$

which can be considered as a Lie group with respect to the central product. It can be shown that the real Lie algebra of the Lie group Spin($\mathcal{W}$) is coincide with the set of 2-forms $\Lambda_2$ with the commutator $[U, V] = UV - VU$, which maps $\Lambda_2 \times \Lambda_2 \to \Lambda_2$. If $U \in \Lambda_2$, then $\pm \exp(U) \in \text{Spin}(\mathcal{W})$, where

$$\exp(U) = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} U^k.$$ 

In some cases it is suitable to use for calculations an exterior exponent $\exp(U)$.

If $u_{a_1a_2} = u_{[a_1a_2]}$ and $U = \frac{1}{2} u_{a_1a_2} e^{a_1} \wedge e^{a_2} \in \Lambda_2$ is such that

$$\lambda = 1 - u_{01}^2 - u_{02}^2 - u_{03}^2 + u_{12}^2 - u_{03}^2 u_{12}^2 + 2u_{02}u_{03}u_{12}u_{13}$$

$$+ u_{13}^2 - u_{02}^2 u_{13}^2 - 2u_{01}u_{03}u_{12}u_{23} + 2u_{01}u_{02}u_{13}u_{23} + u_{23}^2 - u_{01}^2 u_{23}^2 > 0,$$

then

$$\pm \frac{1}{\sqrt{\lambda}}(1 + U + \frac{1}{2} U \wedge U) \in \text{Spin}(\mathcal{W}).$$

It is not hard to prove that if $U \in \Lambda_k$ and $S \in \text{Spin}(\mathcal{W})$, then

$$S^{-1}US \in \Lambda_k,$$

i.e., $S^{-1}\Lambda_k S \subseteq \Lambda_k$. In particular,

$$S^{-1} e^a S = p_b^a e^b,$$

where the matrix $P = |p_b^a|$ has the properties

$$P^T \eta P = \eta, \quad \det P = 1, \quad p_0^0 > 0.$$
The transformation
\[ e^a \rightarrow \tilde{e}^a = S^{-1}e^a S, \quad (17) \]
where \( S \in \text{Spin}(W) \), is called a Lorentz rotation of the tetrad. Obviously,
\[
S^{-1}\left(\frac{1}{k!} u_{a_1\ldots a_k} e^{a_1} \wedge \ldots \wedge e^{a_k}\right) S
= \frac{1}{k!} u_{a_1\ldots a_k} (S^{-1}e^{a_1}S) \ldots (S^{-1}e^{a_k}S)
= \frac{1}{k!} u_{a_1\ldots a_k} \tilde{e}^{a_1} \ldots \tilde{e}^{a_k} = \frac{1}{k!} u_{a_1\ldots a_k} \tilde{e}^{a_1} \wedge \ldots \wedge \tilde{e}^{a_k}.\]

From the identities \( e^a e^b + e^b e^a = 2\eta^{ab} \) it follows that
\[ \tilde{e}^a \tilde{e}^a + \tilde{e}^b \tilde{e}^b = 2\eta^{ab}. \]

6 Tensors from \( \Lambda_k \mathcal{T}_s^r \).

Let us take a tensor
\[ u_{\lambda_1^1 \ldots \lambda_r^1 \nu_1^1 \ldots \nu_s^1 \mu_1^1 \ldots \mu_k^1} = u_{\lambda_1^1 \ldots \lambda_r^1 \nu_1^1 \ldots \nu_s^1 \mu_1^1 \ldots \mu_k^1} \in \mathcal{T}_s^r, \]
antisymmetric with respect to \( k \) covariant indices. One may consider the following objects:
\[ U^{\lambda_1^1 \ldots \lambda_r^1}_{\nu_1^1 \ldots \nu_s^1} = \frac{1}{k!} u^{\lambda_1^1 \ldots \lambda_r^1}_{\nu_1^1 \ldots \nu_s^1 \mu_1^1 \ldots \mu_k^1} dx^{\mu_1^1} \wedge \ldots \wedge dx^{\mu_k^1}, \quad (18) \]
which are formally written as \( k \)-forms. Under a change of coordinates \((x) \rightarrow (\tilde{x})\) values \( (18) \) transform as components of tensor of type \((s, r)\), i.e.,
\[ \tilde{U}^{\alpha_1 \ldots \alpha_r}_{\beta_1 \ldots \beta_s} = q_{\beta_1}^{\alpha_1} \ldots q_{\beta_s}^{\alpha_s} p_{\lambda_1}^{\alpha_1} \ldots p_{\lambda_r}^{\alpha_r} U^{\lambda_1 \ldots \lambda_r}_{\nu_1 \ldots \nu_s}, \quad q_{\beta}^{\nu} = \frac{\partial \tilde{x}^{\nu}}{\partial x^{\beta}} \quad \text{and} \quad p_{\lambda}^{\alpha} = \frac{\partial x^{\lambda}}{\partial \tilde{x}^{\alpha}}. \]
The objects \( (18) \) are called tensors of type \((s, r)\) with values in \( \Lambda_k \). We write this as
\[ U^{\lambda_1 \ldots \lambda_r}_{\nu_1 \ldots \nu_s} \in \Lambda_k \mathcal{T}_s^r. \]

We take
\[ \Lambda \mathcal{T}_s^r = \Lambda_0 \mathcal{T}_s^r \oplus \ldots \oplus \Lambda_4 \mathcal{T}_s^r. \]

Note that \( dx^{\mu} = \delta^{\mu}_{\nu} dx^{\nu} \), where \( \delta^{\mu}_{\nu} \) is the Kronecker tensor \( (\delta^{\mu}_{\nu} = 0 \text{ for } \mu \neq \nu \text{ and } \delta^{\mu}_{\mu} = 1) \). Hence, \( dx^{\mu} \in \Lambda_1 \mathcal{T}_1^1 \).
Elements of $\Lambda_0 T^r_s$ are identified with tensors from $T^r_s$. For the sequel it is suitable to write tensors (18) as

$$U_{\nu_1...\nu_s}^{\lambda_1...\lambda_r} = \frac{1}{k!} u_{\nu_1...\nu_s a_1...a_k}^{\lambda_1...\lambda_r} e^{a_1} \wedge \ldots \wedge e^{a_k} \in \Lambda_k T^r_s.$$  

Let us define a central product of elements $U_{\nu_1...\nu_s}^{\mu_1...\mu_r} \in \Lambda T^r_s$ and $V_{\beta_1...\beta_q}^{\alpha_1...\alpha_p} \in \Lambda T^p_q$ as a tensor from $\Lambda T^{r+p}_{s+q}$ of the form

$$W_{\nu_1...\nu_s \beta_1...\beta_q}^{\mu_1...\mu_r \alpha_1...\alpha_p} = U_{\nu_1...\nu_s}^{\mu_1...\mu_r} V_{\beta_1...\beta_q}^{\alpha_1...\alpha_p},$$

where on the right-hand side there is the central product of differential forms (the indices $\mu_1, \ldots, \mu_r, \alpha_1, \ldots, \alpha_p, \nu_1, \ldots, \nu_s, \beta_1, \ldots, \beta_q$ are fixed).

If $U_{\nu_1...\nu_s}^{\mu_1...\mu_r} \in \Lambda_0 T^r_s$ and $V_{\beta_1...\beta_q}^{\alpha_1...\alpha_p} \in \Lambda_0 T^p_q$, then the central product of these elements is identified with the tensor product.

Let us define operators (Upsilon) $\Upsilon_{\mu}$, which act on tensors from $\Lambda T^r_s$ by the following rules:

a) If $u = (u_{\nu_1...\nu_s}^{\mu_1...\mu_r}) \in \Lambda_0 T^r_s$, then

$$\Upsilon_{\mu} u = \partial_{\mu} u.$$

b) $\Upsilon_{\mu} dx^\nu = -\Gamma^\nu_{\mu \lambda} dx^\lambda$.

c) If $U \in \Lambda T^r_s$, $V \in \Lambda T^p_q$, then

$$\Upsilon_{\lambda}(UV) = (\Upsilon_{\lambda} U)V + U \Upsilon_{\lambda} V.$$

d) If $U, V \in \Lambda T^r_s$, then

$$\Upsilon_{\lambda}(U + V) = \Upsilon_{\lambda} U + \Upsilon_{\lambda} V.$$

With the aid of these rules it is easy to calculate how operators $\Upsilon_{\mu}$ act on arbitrary tensor from $\Lambda T^r_s$.

**Theorem 1.** If $U \in \Lambda_k$ has the form (13), then

$$\Upsilon_{\nu} U = \frac{1}{k!} u_{\mu_1...\mu_k \nu}^a dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_k} \in \Lambda_k T^1_1.$$  

The proof in straightforward.
From the rule b) we get
\[(\Upsilon_\mu \Upsilon_\nu - \Upsilon_\nu \Upsilon_\mu)dx^\lambda = -R^\lambda_{\rho\mu\nu}dx^\rho.\] (19)

**Theorem 2.** Under a change of coordinates \((x) \rightarrow (\hat{x})\) operators \(\Upsilon_\mu\) transform as components of a covector, i.e.,
\[\hat{\Upsilon}_\mu = q^\nu_\mu \Upsilon_\nu, \quad q^\nu_\mu = \frac{\partial x^\nu}{\partial \hat{x}^\mu}.\]

**Proof.** This fact follows from the transformation rule of Christoffel symbols \(\Gamma^\lambda_{\mu\nu}\).

**Theorem 3.**
\[e^a U e_a = \begin{cases} 
4U & \text{for } U \in \Lambda_0 \top_q^p \\
-2U & \text{for } U \in \Lambda_1 \top_q^p \\
0 & \text{for } U \in \Lambda_2 \top_q^p \\
2U & \text{for } U \in \Lambda_3 \top_q^p \\
-4U & \text{for } U \in \Lambda_4 \top_q^p 
\end{cases}\]

The proof is by direct calculations.
Let us take the tensor
\[B_\mu = -\frac{1}{4} e^a \wedge \Upsilon_\mu e_a = -\frac{1}{4} e^a \Upsilon_\mu e_a \in \Lambda_2 \top_1.\] (20)

**Theorem 4.** Under the Lorentz rotation of tetrad \([17]\) the tensor \(B_\mu\) transforms as a connection
\[B_\mu \rightarrow \hat{B}_\mu = S^{-1}B_\mu S - S^{-1}\Upsilon_\mu S.\]

**Proof.** We have
\[-4\hat{B}_\mu = \hat{e}^a \Upsilon_\mu \hat{e}_a = S^{-1}e^a SY_\mu (S^{-1}e_a S) \\
= S^{-1}e^a S (\Upsilon_\mu S^{-1}) e_a S + S^{-1}e^a \Upsilon_\mu (e_a S) + S^{-1}e_a \Upsilon_\mu S \\
= -4S^{-1}B_\mu S + 4S^{-1} \Upsilon_\mu S + S^{-1}e^a S (\Upsilon_\mu S^{-1}) e_a S.\]

Here we use the formula \(e^a e_a = 4\) from Theorem 3. It can be checked that \(S \Upsilon_\mu S^{-1} \in \Lambda_2 \top_1\). Consequently from Theorem 3 we have
\[e^a S \Upsilon_\mu S^{-1} e_a S = 0.\]
These completes the proof.

From the formula (20) it is easily shown that
\[ \Upsilon_{\mu} e^a = [B_\mu, e^a], \ \ \ \ U_{\mu} e_a = [B_\mu, e_a]. \]

Hence, if we take the operators
\[ \mathcal{D}_\mu = \Upsilon_\mu - [B_\mu, \cdot], \]
then
\[ \mathcal{D}_\mu e^a = 0, \ \ \mathcal{D}_\mu e_a = 0. \]

**Theorem 5.** The operators \( \mathcal{D}_\mu \) satisfy the Leibniz rule
\[ \mathcal{D}_\mu(UV) = (\mathcal{D}_\mu U)V + U\mathcal{D}_\mu V \quad \text{for} \quad U \in \Lambda^{\top^r}, \ V \in \Lambda^{\top^p} \]
and
\[ \mathcal{D}_\mu \mathcal{D}_\nu - \mathcal{D}_\nu \mathcal{D}_\mu = 0. \]

**Proof** is by direct calculations.

Note that the volume form
\[ \ell = e^0 \wedge e^1 \wedge e^2 \wedge e^3 = \sqrt{-g} \, dx^0 \wedge \ldots \wedge dx^3 \in \Lambda_4 \]
is constant with respect to these operators, i.e.,
\[ \mathcal{D}_\mu \ell = 0. \]

We have the following consequences from the theorems 4,5. Under the Lorentz rotation of tetrad (17) \( B_\mu, \mathcal{D}_\mu \) transform as follows
\[ B_\mu \rightarrow \tilde{B}_\mu = B_\mu - S^{-1} \mathcal{D}_\mu S, \]
\[ \mathcal{D}_\mu \rightarrow \tilde{\mathcal{D}}_\mu = \mathcal{D}_\mu + [S^{-1} \mathcal{D}_\mu S, \cdot]. \] (21)

Let us denote 1-form
\[ H := e^0. \]

We define the operator of *Hermitian conjugation* of tensors \( \dagger : \Lambda^{\top^p} \rightarrow \Lambda^{\top^p} \)
\[ U^\dagger = HU^*H \quad \text{for} \quad U \in \Lambda^{\top^p}. \]
Evidently,
\[(UV)^\dagger = V^\dagger U^\dagger, \quad U^{\dagger\dagger} = U, \quad i^\dagger = -i.\]

We shall see in Section 10 that this operator is connected to the operator of Hermitian conjugation of matrices.

We say that a tensor \( U \in \Lambda^p \) is **Hermitian** if \( U^\dagger = U \) and **anti-Hermitian** if \( U^\dagger = -U \). Every tensor \( U \) can be decomposed into Hermitian and anti-Hermitian parts
\[U = \frac{1}{2}(U + U^\dagger) + \frac{1}{2}(U - U^\dagger).\]

Note that all discussed constructions, which were defined in this paper for tensors from \( \Lambda^p \), are also valid for complex valued tensors from \( \Lambda^C \).

Now we may define the operation
\[(\cdot, \cdot) : \Lambda^C \times \Lambda^C \to \Lambda^C_0\]
by the formula
\[(U, V) = \text{Tr}(U^\dagger V).\]

This operation has all the properties of Hermitian scalar product at every point \( x \in X \)
\[
\begin{align*}
\alpha(U, V) &= (\bar{\alpha}U, V) = (U, \alpha V), \\
(U, V) &= (\overline{V}, U), \\
(U + W, V) &= (U, V) + (W, V), \\
(U, U) > 0 \quad \text{for} \quad U \neq 0,
\end{align*}
\]
where \( U, V, W \in \Lambda^C, \alpha \in \Lambda^C_0 \), and a bar means complex conjugation. The operation \((\cdot, \cdot)\) converts \( \Lambda^C \) into the unitary space at every point \( x \in X \).

Let us denote by \( T_0, \ldots T_{15} \) the following differential forms:
\[
i, ie^0, e^1, e^2, e^3, ie^01, ie^02, ie^03, e^{12}, e^{13}, e^{23}, e^{012}, e^{013}, e^{023}, ie^{123}.\tag{22}
\]
which form an orthonormal basis of \( \Lambda^C \)
\[
(T_k, T^N) = \delta^N_k, \quad k, n = 0, \ldots 15
\]
and
\[
T_k = -T^\dagger_k, \quad D_\mu T_k = 0,
\]
where \( T^N = T_N \). This basis is said to be the **anti-Hermitian basis of** \( \Lambda^C \).
A differential form $t \in \Lambda^C$ such that
\[ t^2 = t, \quad D_\mu t = 0, \quad t^\dagger = t \] (23)
is called an idempotent. We suppose that under a Lorentz rotation of tetrad $e^a \rightarrow \tilde{e}^a = S^{-1}e^aS$, $S \in \text{Spin}(\mathbb{W})$ an idempotent $t$ transforms as
\[ t \rightarrow \tilde{t} = S^{-1}tS. \]
In this case
\[
\tilde{t}^2 = \tilde{t}, \\
\tilde{t}^\dagger = \tilde{H}^* \tilde{H} = \tilde{t}, \\
\tilde{D}_\mu \tilde{t} = 0.
\]
We may consider the left ideal generated by the idempotent $t$
\[ \mathcal{I}(t) = \{ Ut : U \in \Lambda^C \} \subseteq \Lambda^C. \] (24)
Let us define the set of differential forms
\[ L(t) = \{ U \in \mathcal{I}(t) : U^\dagger = -U, [U, t] = 0 \}. \]
This set is closed with respect to the commutator (if $U, V \in L(t)$, then $[U, V] \in L(t)$) and can be considered as a real Lie algebra. With the aid of the real Lie algebra $L(t)$ we define the corresponding Lie group
\[ G(t) = \{ \exp(U) : U \in L(t) \}. \]
In Section 10 we consider $t, \mathcal{I}(t), L(t), G(t)$ in details.
Finally let us summarize properties of the operators $D_\mu$
\[
D_\mu (UV) = (D_\mu U)V + UD_\mu V, \\
D_\mu D_\nu - D_\nu D_\mu = 0, \\
D_\mu (U + V) = D_\mu U + D_\mu V, \\
D_\mu e^a = 0, \quad D_\mu e_a = 0, \\
D_\mu \ell = 0, \\
D_\mu (U^*) = (D_\mu U)^*, \\
D_\mu (U^\dagger) = (D_\mu U)^\dagger, \\
D_\mu (\text{Tr}(U)) = \partial_\mu (\text{Tr}(U)) = \text{Tr}(D_\mu U), \\
D_\mu (U, V) = \partial_\mu (U, V) = (D_\mu U, V) + (U, D_\mu V).
\]
Under a change of coordinates \((x) \rightarrow (\dot{x})\) operators \(\mathcal{D}_\mu\) transform as components of a covector, i.e.,

\[ \mathcal{D}_\mu \rightarrow \dot{\mathcal{D}}_\mu = \frac{\partial x^\nu}{\partial \dot{x}^\mu} \mathcal{D}_\nu. \]

Under a Lorentz rotation of tetrad \(e^a \rightarrow \tilde{e}^a = S^{-1} e^a S\) \((S \in \text{Spin}(\mathcal{W}))\) operators \(\mathcal{D}_\mu\) transform as \(\mathcal{D}_\mu \rightarrow \tilde{\mathcal{D}}_\mu = \mathcal{D}_\mu + [S^{-1} \mathcal{D}_\mu S, \cdot].\)

7 Dirac-type tensor equations. A general case.

We begin with the following equation in \(\mathcal{W}\) (a tetrad \(e^a\) is given and, consequently, the tensor \(B_\mu \in \Lambda_2 \mathbb{T}_1\) and the operators \(\mathcal{D}_\mu\) are defined):

\[ dx^\mu (\mathcal{D}_\mu \Phi + \Phi A_\mu + B_\mu \Phi) + im \Phi = 0, \tag{25} \]

where \(\Phi \in \Lambda^C\), \(i = \sqrt{-1}\), \(m\) is a given real constant, and \(A_\mu \in \Lambda^C \mathbb{T}_1\) is such that \(A_\mu^\dagger = -A_\mu\). We consider the differential form \(\Phi\) as unknown (16 complex valued components) and \(A_\mu\) as known. Writing eq. (25) as a system of equations for components of \(\Phi\), we see that the number of equations is equal to the number of unknown values.

An equation is said to be a tensor equation if all values in it are tensors and all operations in it take tensors to tensors.

In eq. (25) we have \(dx^\mu \mathcal{D}_\mu \Phi, dx^\mu \Phi A_\mu, dx^\mu B_\mu \Phi \in \Lambda^C\). Hence eq. (25) is a tensor equation.

Let \(t \in \Lambda^C\) be an idempotent and \(A_\mu\) be a tensor from \(L(t) \mathbb{T}_1\). Then we may consider the equation in \(\mathcal{W}\)

\[ (dx^\mu (\mathcal{D}_\mu \Phi + \Phi A_\mu + B_\mu \Phi) + im \Phi)t = 0, \tag{26} \]

From the identities \(\mathcal{D}_\mu t = 0, [A_\mu, t] = 0\) it follows that eq. (26) can be written as the equation for \(\Psi = \Phi t \in \mathcal{I}(t)\)

\[ dx^\mu (\mathcal{D}_\mu \Psi + \Psi A_\mu + B_\mu \Psi) + im \Psi = 0, \tag{27} \]

where the idempotent \(t\), the real constant \(m\), and the tensor \(A_\mu \in L(t) \mathbb{T}_1\) are considered as known. The differential form \(\Psi \in \mathcal{I}(t)\) is considered as unknown.
In Section 10 we shall see that there are four types of idempotents $t$. Consequently, there are four types of equations (27). These equations are called *Dirac-type tensor equations*. A connection of eqs. (27) with the Dirac equation will be discussed in Section 9.

Denoting $\alpha^\mu = H dx^\mu \in (\Lambda_0 \oplus \Lambda_2)^\top$, we see that $(\alpha^\mu)^\dagger = \alpha^\mu$.

**Theorem 6.** If $\Psi \in \mathcal{I}(t)$ satisfies eq. (27), then the tensor

$$J^\mu = i\Psi^\dagger \alpha^\mu \Psi$$

satisfies the equality

$$\frac{1}{\sqrt{-g}}D_\mu(\sqrt{-g} J^\mu) - [A_\mu, J^\mu] = 0,$$

which is called a (non-Abelian) charge conservation law.

Note that $\Psi = \Psi^t$ and $J^\mu = t J^\mu t$. Therefore,

$$[J^\mu, t] = [t J^\mu t, t] = 0,$$

i.e., $J^\mu \in L(t)^\top$.

**Proof** of Theorem 6. Let us multiply eq. (26) from the left by $H$ and denote the left-hand side of resulting equation by

$$Q = \alpha^\mu(D_\mu \Psi + \Psi A_\mu + B_\mu \Psi) + imH \Psi. \quad (29)$$

Then

$$Q^\dagger = (D_\mu \Psi^\dagger A_\mu \Psi^\dagger + \Psi^\dagger B_\mu^\dagger) \alpha^\mu - im \Psi^\dagger H.$$

Consider the expression

$$i(\Psi^\dagger Q + Q^\dagger \Psi) = i(\Psi^\dagger \alpha^\mu D_\mu \Psi + D_\mu \Psi^\dagger \alpha^\mu \Psi + \Psi^\dagger (D_\mu \alpha^\mu) \Psi) - [A_\mu, i \Psi^\dagger \alpha^\mu \Psi] + i \Psi^\dagger (-D_\mu \alpha^\mu + \alpha^\mu B_\mu + B_\mu^\dagger \alpha^\mu) \Psi$$

$$= \frac{1}{\sqrt{-g}}D_\mu(\sqrt{-g} J^\mu) - [A_\mu, J^\mu]. \quad (30)$$

Here we use the formulae

$$D_\mu \alpha^\mu = -\Gamma_{\mu\nu}^\nu \alpha^\nu + \alpha^\mu B_\mu + B_\mu^\dagger \alpha^\mu,$$

$$D_\mu J^\mu + \Gamma_{\mu\nu} J^\nu = \frac{1}{\sqrt{-g}}D_\mu(\sqrt{-g} J^\mu).$$
which can be easily checked. In (30) equality $Q = 0$ leads to the equality (28). These completes the proof.

Now we may write eq. (27) together with Yang-Mills equations (32, 33)

$$dx^\mu(D_\mu \Psi + \Psi A_\mu + B_\mu \Psi) + im\Psi = 0,$$

$$D_\mu A_\nu - D_\nu A_\mu - [A_\mu, A_\nu] = F_{\mu\nu},$$

$$\frac{1}{\sqrt{-g}}D_\mu(\sqrt{-g} F^{\mu\nu}) - [A_\mu, F_{\mu\nu}] = J^\nu,$$

$$J^\nu = i\Psi^\dagger \alpha^\nu \Psi,$$

where $\Psi \in \mathcal{I}(t)$, $A_\mu \in L(t)^\top_1$, $F_{\mu\nu} \in L(t)^\top_2$, $J^\nu \in L(t)^\top_1$. In this system of equations we consider $\Psi, A_\mu, F_{\mu\nu}, J^\nu$ as unknown values and $m, t$ as known values.

**Theorem 7.** Let us denote the left-hand side of eq. (33) by $R^\nu$

$$R^\nu := \frac{1}{\sqrt{-g}}D_\mu(\sqrt{-g} F^{\mu\nu}) - [A_\mu, F_{\mu\nu}],$$

where $F_{\mu\nu}$ satisfy (32). Then

$$\frac{1}{\sqrt{-g}}D_\mu(\sqrt{-g} R^\mu) - [A_\mu, R^\mu] = 0.$$

The proof is by direct calculations.

This theorem means that eq. (33) is consistent with the charge conservation law (28).

8 Unitary and Spin gauge symmetries.

**Theorem 8.** Let $\Psi, A_\mu, F_{\mu\nu}, J^\nu$ satisfy eqs. (31-34) with a given idempotent $t$ and constant $m$. And let $U \in G(t)$, where the Lie group $G(t)$ is defined in Section 6. Then the following values with tilde:

$$\tilde{\Psi} = \Psi U, \quad \tilde{A}_\mu = U^{-1} A_\mu U - U^{-1} D_\mu U, \quad \tilde{F}_{\mu\nu} = U^{-1} F_{\mu\nu} U,$$

$$\tilde{J}^\nu = U^{-1} J^\nu U, \quad \{\tilde{i}, \tilde{B}_\mu, \tilde{D}_\mu\} = \{t, B_\mu, D_\mu\}$$

satisfy the same equations (31-34).

The proof is straightforward.

This theorem means that eqs. (31-34) are invariant under gauge transformations with the symmetry Lie group $G(t)$. 

19
Theorem 9. Let $\Psi, A_\mu, F_{\mu\nu}, J^\nu$ satisfy eqs. (31-34) with a given idempotent $t$ and constant $m$. And let $S \in \text{Spin}(W)$. Then the following values with check:

$$
\breve{\Psi} = \Psi S, \quad \breve{A}_\mu = S^{-1}A_\mu S, \quad \breve{F}_{\mu\nu} = S^{-1}F_{\mu\nu} S, \quad \breve{J}^\nu = S^{-1}J^\nu S, \quad (35)
$$

satisfy the same equations (31-34).

The proof is by direct calculations.

Note that for values with check the operation of Hermitian conjugation is defined by $\breve{U}^\dagger = \breve{H} \breve{U}^\ast \breve{H}$, where $\breve{H} = S^{-1}HS$.

This theorem means that eqs. (31-34) are invariant under gauge transformations with the symmetry Lie group Spin($W$).

Eqs. (31-34) can be derived from the following Lagrangian:

$$
L = \frac{1}{4}i\sqrt{-g} \text{Tr}(\Psi^\dagger HQ - Q^\dagger H \Psi) + C \frac{1}{4}\sqrt{-g} \text{Tr}(\frac{1}{8}F_{\mu\nu}F^{\mu\nu}), \quad (36)
$$

where $Q$ is from (29), $F_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu - [A_\mu, A_\nu] \in L(t) \up_2$, and $C$ is a real constant. If we have a basis $\{t_1, \ldots, t_D\}$ of $I(t)$ and a basis $\{\tau_1, \ldots, \tau_d\}$ of $L(t)$ that satisfy (37), (42), then we may substitute $\Psi = \psi_k t_k, A_\mu = a_\mu^\alpha \tau_\alpha$ into the Lagrangian $L$. Variating $L$ with respect to $\psi_\alpha$ and $a_\mu^\alpha$, we arrive at eqs. (31-34). We discuss bases of $I(t)$ and $L(t)$ in Section 10.

In [4] we discuss a gravitational term for the Lagrangian $L$.

9 A connection between the Dirac-type tensor equation and the Dirac equation.

Let $t \in \Lambda^c$ be an idempotent with the properties (23) and $I(t)$ be the left ideal (24) of complex dimension $D$. In the next section we shall see that $D$ may take one of four possible values $D = 4, 8, 12, 16$. We use an orthonormal basis $t_k = t^k, k = 1, \ldots, D$ of $I(t)$ such that

$$
D_\mu t_k = 0, \quad (t_n, t_k) = \delta_k^\alpha. \quad (37)
$$

Small caps font indices run from 1 to $D$. Consider a linear operator $\|\|$ that maps $I(t)$ to $C^D$. If

$$
\Omega = \omega_k t_k \in I(t),
$$

20
then

\[ |\Omega\rangle = (\omega^1 \ldots \omega^D)^T. \]

In particular,

\[ |t_k\rangle = (0 \ldots 1 \ldots 0)^T \]

with only 1 on the \(k\)-th place of the column.

By \( M^c(D)^{T^q_p} \) denote the set of type \((p,q)\) tensors with values in \( D \times D \) complex matrices. Let \( \gamma : \Lambda^c \rightarrow M^c(D)^{T^q_p} \) be a map such that for \( U = (U_{\mu_1 \ldots \mu_p}) \in \Lambda^c \)

\[ Ut_n = \gamma(U)^k_n t_k. \]  

\hbox{(38)}

Hence,

\[ \gamma(U)^k_n = (t^k, Ut_n), \]

where \( \gamma(U)^k_n \) are elements of the matrix \( \gamma(U) \) (an upper index enumerates rows and a lower index enumerates columns).

If \( U \in \Lambda^c \) and \( \Omega \in I(t) \), then

\[ U\Omega = U\omega^nt_n = \omega^s \gamma(U)^k_n t_k. \]

That means

\[ |U\Omega\rangle = \gamma(U)|\Omega\rangle. \]

If \( U, V \in \Lambda^c, \Omega \in I(t) \), then

\[ |UV\Omega\rangle = \gamma(U)\gamma(V)|\Omega\rangle = \gamma(UV)|\Omega\rangle. \]

Consequently,

\[ \gamma(UV) = \gamma(U)\gamma(V), \]

i.e., \( \gamma \) is a matrix representation of \( \Lambda^c \). For example, if we take \( dx^\mu = \delta^\mu_\nu dx^\nu \in \Lambda_1 \), then we get

\[ dx^\mu t_n = \gamma(dx^\mu)^k_n t_k. \]

Denoting \( \gamma^\mu = \gamma(dx^\mu) \), we see that the equality \( dx^\mu dx^\nu + dx^\nu dx^\mu = 2g^{\mu\nu} \)

leads to the equality

\[ \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \mathbf{1}, \]

where \( \mathbf{1} \) is the identity matrix of dimension \( D \).

Let us take the set of differential forms

\[ \mathcal{K}(t) = \{ V \in \Lambda^c : [V, t] = 0 \}, \]
which can be considered as an algebra (at any point $x \in X$). Now we define a map

$$\theta : \mathcal{K}(t) \rightarrow M_c^q(D) \rightarrow$$

such that for $V = (V_{\mu_1,..,\mu_q}) \in \mathcal{K}(t)$

$$t_n V = \theta(V)^{k_n}_n t_k.$$

Therefore,

$$\theta(V)^{k_n}_n = (t^k_n, t_n V).$$

If $V \in \mathcal{K}(t)$ and $\Omega \in \mathcal{I}(t)$, then

$$\Omega V = \omega^n t_n V = \omega^n \theta^{k_n}_n t_k.$$

Than means

$$|\Omega V\rangle = \theta(V)|\Omega\rangle.$$

If $U, V \in \mathcal{K}(t), \Omega \in \mathcal{I}(t)$, then

$$|\Omega U V\rangle = \theta(V) \theta(U)|\Omega\rangle = \theta(UV)|\Omega\rangle.$$

Consequently,

$$\theta(UV) = \theta(V) \theta(U).$$

If $U \in \Lambda^c, V \in \mathcal{K}(t), \Omega \in \mathcal{I}(t)$, then $U \Omega \in \mathcal{I}(t), \Omega V \in \mathcal{I}(t)$ and

$$|U \Omega V\rangle = \gamma(U)|\Omega V\rangle = \gamma(U) \theta(V)|\Omega\rangle,$$

$$|U \Omega V\rangle = \theta(V)|U \Omega\rangle = \theta(V) \gamma(U)|\Omega\rangle.$$

Consequently,

$$[\gamma(U), \theta(V)] = 0.$$

Denoting the left-hand side of eq. (27) by

$$\Omega = dx^\mu(D_\mu \Psi + \Psi A_\mu + B_\mu \Psi) + im\Psi$$

(39)

and using formulas

$$\psi := |\Psi\rangle,$$

$$|D_\mu \Psi\rangle = |D_\mu (t_k \psi^K)\rangle = |t_k \partial_\mu \psi^K\rangle = \partial_\mu \psi,$$

$$|dx^\mu \Psi A_\mu\rangle = \gamma^\mu |\Psi A_\mu\rangle = \gamma^\mu \theta(A_\mu) \psi,$$

$$|dx^\mu B_\mu \Psi\rangle = \gamma^\mu |B_\mu \Psi\rangle = \gamma^\mu \gamma(B_\mu) \psi.$$
we see that
\[ |\Omega\rangle = \gamma^\mu (\partial_\mu + \theta(A_\mu) + \gamma(B_\mu))\psi + im\psi. \] (40)

Note that
\[ B_\mu = \frac{1}{2} b_{\mu ab} e^a \wedge e^b = \frac{1}{4} b_{\mu ab} (e^a e^b - e^b e^a). \]

Therefore,
\[ \gamma(B_\mu) = \frac{1}{4} b_{\mu ab} [\gamma^a, \gamma^b], \]

where \( \gamma^a = \gamma(e^a) \) and the equalities \( e^a e^b + e^b e^a = 2\eta^{ab} \) leads to the equalities
\[ \gamma^a \gamma^b + \gamma^b \gamma^a = 2\eta^{ab}. \]

If we take \( V \in G(t) \), then
\[ \Omega V = (dx^\mu(D_\mu \Psi + \Psi A_\mu + B_\mu \Psi) + im\Psi)V \]
\[ = dx^\mu(D_\mu(\Psi V) + \Psi V(V^{-1}A_\mu V - V^{-1}D_\mu V) + B_\mu(\Psi V)) + im(\Psi V) \]

and
\[ |\Omega V\rangle = \theta(V)|\Omega\rangle = P(\gamma^\mu (\partial_\mu + \theta(A_\mu) + \gamma(B_\mu))\psi + im\psi) \]
\[ = \gamma^\mu (\partial_\mu + (P\theta(A_\mu)P^{-1} - (\partial_\mu P)P^{-1}) + \gamma(B_\mu))(P\psi) + imP\psi, \]

where \( P = \theta(V) \) and we use formulae \([P, \gamma^\mu] = 0, [P, \gamma(B_\mu)] = 0\). Thus the invariance of eq. (27) under the gauge transformation \((V \in G(t))\)
\[ \Psi \rightarrow \Psi V, \quad A_\mu \rightarrow V^{-1}A_\mu V - V^{-1}D_\mu V \]
leads to the invariance of the equation
\[ \gamma^\mu (\partial_\mu + \theta(A_\mu) + \gamma(B_\mu))\psi + im\psi = 0 \] (41)
under the gauge transformation \((P = \theta(V))\)
\[ \psi \rightarrow P\psi, \quad \theta(A_\mu) \rightarrow P\theta(A_\mu)P^{-1} - (\partial_\mu P)P^{-1}. \]

Let \( S \in \text{Spin}(W) \). Consider the Lorentz rotation of the tetrad \( e^a \rightarrow \tilde{e}^a = S^{-1}e^aS \), which leads to the transformation
\[ t \rightarrow \tilde{t} = S^{-1}tS, \quad I(t) \rightarrow I(\tilde{t}), \quad t_\nu \rightarrow \tilde{t}_\nu = S^{-1}t_\nu S. \]
Let us define a map $|\tilde{\Phi}| : \mathcal{I}(\tilde{t}) \to \mathcal{C}^D$. If $\Phi = \phi^k \tilde{t}_k \in \mathcal{I}(\tilde{t})$, then
\[ |\Phi| = (\phi^1 \ldots \phi^D)^T. \]
Evidently for $\Omega \in \mathcal{I}(t)$
\[ |S^{-1}\Omega S\rangle = |\Omega\rangle. \]
Also we may define a map $\tilde{\gamma} : \Lambda^c \mathcal{T}^q_p \to M^c(D) \mathcal{T}^q_p$ such that
\[ U \tilde{t}_k = \tilde{\gamma}(U)_{n}^{k} \tilde{t}_k \]
and
\[ \tilde{\gamma}(UV) = \tilde{\gamma}(U) \tilde{\gamma}(V). \]
It is easily seen that
\[ \tilde{\gamma}(U) = \gamma(S) \gamma(U) \gamma(S^{-1}). \]
In particular,
\[ \tilde{\gamma}(S) = \gamma(S). \]
For $\Omega \in \mathcal{I}(t)$ we have
\[ |\Omega S\rangle = |S(S^{-1}\Omega S)\rangle = \tilde{\gamma}(S)|S^{-1}\Omega S\rangle = \gamma(S)|\Omega\rangle. \]
If we apply these identities to
\[ \Omega S = (dx^\mu(\mathcal{D}_\mu \psi + \Psi A_\mu + B_\mu \Psi) + im\Psi)S \]
\[ = dx^\mu(\tilde{\mathcal{D}}_\mu \tilde{\psi} + \tilde{\Psi} \tilde{A}_\mu + \tilde{B}_\mu \tilde{\Psi}) + im\tilde{\psi}, \]
then we get
\[ |\Omega S\rangle = \gamma(S)|\Omega\rangle = R(\gamma^\mu(\partial_\mu \psi + \theta(A_\mu)\psi + \gamma(B_\mu)\psi) + im\psi) \]
\[ = R\gamma^\mu R^{-1}(\partial_\mu (R\psi) + \theta(A_\mu)R\psi + (R\gamma(B_\mu)R^{-1} - (\partial_\mu R)R^{-1})R\psi) + imR\psi, \]
where $R = \gamma(S)$. Note that $[R, \theta(A_\mu)] = 0$.

This implies that the invariance of eq. (27) under the gauge transformation (33) leads to the invariance of eq. (41) under the gauge transformation ($R = \gamma(S)$)
\[ \psi \to R\psi, \quad \theta(A_\mu) \to \theta(A_\mu) \quad \gamma^\mu \to R\gamma^\mu R^{-1}, \]
\[ \gamma(B_\mu) \to R\gamma(B_\mu)R^{-1} - (\partial_\mu R)R^{-1}. \]
Now consider a transformation of eqs. (27),(41) under a change of coordinates \((x) \rightarrow (\hat{x})\). Coordinates \((\hat{x})\) we denote with the aid of primed indices \(x^{\mu'}\). In coordinates \((\hat{x})\) eq. (27) has the form

\[
\hat{\Omega} \equiv dx^{\mu'}(\mathcal{D}_{\mu'}\Psi + \Psi A_{\mu'} + B_{\mu'}\Psi) + im\Psi = 0,
\]

where

\[
\mathcal{D}_{\mu'} = \frac{\partial x^\nu}{\partial x^{\mu'}}\mathcal{D}_\nu, \quad dx^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\nu}dx^\nu, \quad A_{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu''}}A_{\nu'}, \quad B_{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu''}}B_{\nu'},
\]

and eq. (41) has the form

\[
|\hat{\Omega}\rangle = \gamma^{\mu'}(\partial_{\mu'} + \theta(A_{\mu'}) + \gamma(B_{\mu'}))\psi + im\psi = 0,
\]

where

\[
\gamma^{\mu'} = \frac{\partial}{\partial x^{\mu'}}, \quad \gamma^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\nu}}\gamma^{\nu}.
\]

Note that

\[
\gamma^{\mu} = \gamma(dx^{\mu}) = \gamma^a e^{\mu}_a, \quad \gamma^a = \gamma(e^a)
\]

and

\[
\gamma^{\mu'} = \gamma^a e^{\mu'}_a = \gamma^a \frac{\partial x^{\mu'}}{\partial x^{\nu}}e^{\nu}_a.
\]

Finally, let us note that in the Lagrangian (36)

\[
\frac{1}{4}\sqrt{-g} \text{Tr}(i(\Psi^\dagger H\Omega - \Omega^\dagger H\Psi)) = \frac{1}{4}\sqrt{-g} i(\langle\Psi|\gamma^0|\Omega\rangle - \langle\Omega|\gamma^0|\Psi\rangle),
\]

where \(\langle\Psi\rangle = |\Psi\rangle^\dagger\).

10 \hspace{1cm} \textbf{Idempotents and bases of left ideals.}

Let us take the idempotent

\[
t_{(1)} = \frac{1}{4}(1 + e^0)(1 + ie^{12}) = \frac{1}{4}(1 + e^0 + ie^{12} + ie^{012}),
\]

which satisfies conditions (23), and consider the left ideal \(I(t_{(1)})\). It can be shown that the complex dimension of \(I(t_{(1)})\) is equal to four (this \(I(t_{(1)})\) is a
minimal left ideal of $\Lambda^C$ and $t_{(1)}$ is a primitive idempotent) and the following differential forms

\[
\begin{align*}
t^1 &= 2t_{(1)} = \frac{1}{2}(1 + e^0 + ie^{12} + ie^{012}), \\
t^2 &= -2e^{13}t_{(1)} = \frac{1}{2}(-e^{13} + ie^{23} - e^{013} + ie^{023}), \\
t^3 &= -2e^{03}t_{(1)} = \frac{1}{2}(-e^3 + e^{03} - ie^{123} + ie^{0123}), \\
t^4 &= 2e^{01}t_{(1)} = \frac{1}{2}(-e^1 + ie^2 + e^{01} - ie^{02})
\end{align*}
\]

can be taken as basis forms of $\mathcal{I}(t_{(1)})$, which satisfy (37). This basis, according to the formula (38), defines the matrix representation of $\Lambda^C$ ($\gamma_{(1)}$ is a one-to-one map)

\[
\gamma_{(1)} : \Lambda^C\mathcal{T}_p^q \rightarrow M^C\mathcal{T}_p^q.
\]

In particular we get matrices $\gamma_{(1)}^\mu = \gamma_{(1)}(dx^\mu)$ identical to (3). Denote

\[
\mathbf{U} := \gamma_{(1)}(U) \quad \text{for} \quad U \in \Lambda^C\mathcal{T}_p^q.
\]

Let $Y^N_k \in \Lambda^C$, $(k, N = 1, 2, 3, 4)$ be differential forms such that $Y^N_k$ are $4 \times 4$ matrices with only nonzero element that equal to 1 on the intersection of $n$-th row and $k$-th column. We can calculate that

\[
\begin{align*}
Y^1_1 &= (1 + e^0 + e^{012}i + e^{12}i)/4, \\
Y^2_2 &= (e^{013} + e^{13} + e^{023}i + e^{23}i)/4, \\
Y^3_3 &= (e^{03} + e^3 + e^{0123}i + e^{123}i)/4, \\
Y^4_4 &= (e^{01} + e^1 + e^{02}i + e^{2}i)/4, \\
Y^2_1 &= (-e^{013} - e^{13} + e^{023}i + e^{23}i)/4, \\
Y^1_2 &= (1 + e^0 - e^{012}i - e^{12}i)/4, \\
Y^3_2 &= (e^{01} + e^1 - e^{02}i - e^{2}i)/4, \\
Y^4_3 &= (-e^{03} - e^{3} + e^{0123}i + e^{123}i)/4, \\
Y^1_3 &= (e^{03} - e^{3} + e^{0123}i - e^{123}i)/4, \\
Y^2_4 &= (e^{01} - e^{1} + e^{02}i - e^{2}i)/4, \\
Y^3_4 &= (e^{01} + e^{1} - e^{02}i + e^{2}i)/4, \\
Y^1_4 &= (1 - e^{0} - e^{012}i + e^{12}i)/4, \\
Y^2_3 &= (-e^{013} + e^{13} - e^{023}i + e^{23}i)/4, \\
Y^3_1 &= (e^{01} - e^{1} - e^{02}i + e^{2}i)/4.
\end{align*}
\]
\[ Y_2^4 = (-e^{03} + e^3 + e^{0123}i - e^{123}i)/4, \]
\[ Y_3^4 = (e^{013} - e^{13} - e^{023}i + e^{23}i)/4, \]
\[ Y_4^4 = (1 - e^0 + e^{012}i - e^{12}i)/4. \]

We see that
\[ t_{(1)} = Y_1^1, \quad \mathcal{L}_{(1)} = \text{diag}(1, 0, 0, 0) \]
and
\[ t^1 = 2Y_1^1, \quad t^2 = 2Y_1^2, \quad t^3 = 2Y_1^3, \quad t^4 = 2Y_1^4. \]

Now we may define the idempotents
\[ t_{(2)} = Y_1^1 + Y_2^2 = \frac{1}{2}(1 + e^0), \]
\[ t_{(3)} = Y_1^1 + Y_2^2 + Y_3^3 = \frac{1}{4}(3 + e^0 + ie^{12} - ie^{012}), \]
\[ t_{(4)} = Y_1^1 + Y_2^2 + Y_3^3 + Y_4^4 = 1 \]
such that
\[ \mathcal{L}_{(2)} = \text{diag}(1, 1, 0, 0), \quad \mathcal{L}_{(3)} = \text{diag}(1, 1, 1, 0), \quad \mathcal{L}_{(4)} = \text{diag}(1, 1, 1, 1). \]

Also we can take the following differential forms \( t^1, \ldots, t^{16} : \)
\[ t^{4(n-1)+k} = 2Y_n^k, \quad k, n = 1, 2, 3, 4, \]
which satisfy conditions \( D_{\mu}t^k = 0, \ (t_k, t^N) = \delta_N^k. \) Evidently, \( \{t_1, \ldots, t_8\} \) is a basis of \( \mathcal{I}(t_{(2)}), \) \( \{t_1, \ldots, t_4\} \) is a basis of \( \mathcal{I}(t_{(3)}), \) and \( \{t_1, \ldots, t_8\} \) is a basis of \( \mathcal{I}(t_{(4)}) = \Lambda^C. \) In accordance with the formula (38), these bases define the maps
\[ \gamma_{(k)} : \Lambda^C \top^q_p \rightarrow M^C(4k) \top^q_p, \quad k = 1, 2, 3, 4 \]
such that
\[ \gamma_{(k)}(UV) = \gamma_{(k)}(U)\gamma_{(k)}(V) \quad \text{for} \quad U \in \Lambda^C \top^q_p, \ V \in \Lambda^C \top^s_r. \]

Also the maps \( \gamma_{(k)} \) have the important property
\[ \gamma_{(k)}(U^\dagger) = (\gamma_{(k)}(U))^\dagger \quad \text{for} \quad U \in \Lambda^C \top^q_p, \]
where \( U^\dagger = HU^*H \) is the Hermitian-conjugated differential form and \( (\gamma_{(k)}(U))^\dagger \) is the Hermitian conjugated matrix (transposed matrix with complex conjugated elements). Consider the set of differential forms
\[ \mathcal{K}_0(t) = \{U \in \mathcal{I}(t) : [U, t] = 0\} = \mathcal{K}(t) \cap \mathcal{I}(t) \]

27
and the corresponding set of $4 \times 4$-matrices

$$K_0(t) = \{ U : U \in K_0(t) \}.$$ 

Evidently $K_0(t_k)$, $(k = 1, 2, 3, 4)$ are sets of matrices with all zero elements except elements in the left upper $k \times k$-block.

Considering the sets of differential forms

$$L(t_k) = \{ U \in K_0(t_k) : U^\dagger = -U \}$$

and the corresponding sets of matrices $L(t_k)$ as real Lie algebras, we see that

$$L(t_k) \simeq L(t_k) \simeq u(k) \simeq u(1) \oplus su(k),$$

where $u(k)$, $(k = 1, 2, 3, 4)$ are Lie algebras of anti-Hermitian $k \times k$-matrices, $su(k)$ are the Lie algebras of traceless anti-Hermitian matrices, and the sign $\simeq$ denote isomorphism. We have the Lie groups

$$G(t_k) \simeq G(t_k) \simeq U(1) \oplus SU(k),$$

where $SU(k)$ are the Lie groups of unitary $k \times k$-matrices with determinants equal to 1.

For elements of $L(t_k)$ we define the normalized scalar product

$$(u, v)(k) := \frac{4}{k} (u, v)$$

such that

$$(it_k, it_k)(k) = 1, \quad k = 1, 2, 3, 4.$$ 

Now we show that for every $k = 1, 2, 3, 4$ we may take generators $\tau$ of $L(t_k)$ such that

$$D_\mu \tau_n = 0, \quad (\tau_n, \tau^m)(k) = \delta^m_n, \quad \tau_n^\dagger = -\tau_n, \quad [\tau_n, \tau_l] = c^m_{nl} \tau_m,$$ (42)

where $\tau_n = \tau^n$ and $c^m_{nl}$ are real structure constants of the Lie algebra $L(t_k)$.

We use differential forms

$$\begin{align*}
\lambda_1 &= Y_2^1 + Y_1^2 \\
\lambda_2 &= -iY_2^1 + iY_1^2 \\
\lambda_3 &= Y_1^1 - Y_2^2
\end{align*}$$

28
\[ \lambda_4 = Y_3^1 + Y_1^3 \]
\[ \lambda_5 = -iY_3^1 + iY_1^3 \]
\[ \lambda_6 = Y_3^2 + Y_2^3 \]
\[ \lambda_7 = -iY_3^2 + iY_2^3 \]
\[ \lambda_8 = \frac{1}{\sqrt{3}}(Y_1^1 + Y_2^2 - 2Y_3^3) \]

such that \( \{\lambda_1, \ldots, \lambda_8\} \) is equivalent to the Gell-Mann basis of the real Lie algebra \( \text{su}(3) \). We take the following generators of \( L(t_{(k)}) \):

1. For \( L(t_{(1)}) \simeq \text{u}(1) \)
   \[ \tau_0 = it_{(1)}. \]
2. For \( L(t_{(2)}) \simeq \text{u}(1) \oplus \text{su}(2) \)
   \[ \tau_0 = it_{(2)}, \quad \tau_n = i\lambda_n, \quad n = 1, 2, 3. \]
3. For \( L(t_{(3)}) \simeq \text{u}(1) \oplus \text{su}(3) \)
   \[ \tau_0 = it_{(3)}, \quad \tau_n = i\sqrt{\frac{3}{2}}\lambda_n, \quad n = 1, \ldots, 8. \]
4. For \( L(t_{(4)}) \simeq \text{u}(1) \oplus \text{su}(4) \) we take as a basis \( \tau_0, \ldots, \tau_{15} \) the anti-Hermitian basis (22)

Therefore,

\[ L(t_{(k)}) = \{f_n\tau^n\}, \]

where \( f_n = f_n(x) \) are real valued scalar functions.

Let us define the set of differential forms

\[ L(t) = \{ U \in L(t) : \text{Tr}U = 0 \}. \]

We see that

\[ L(t_{(k)}) \simeq \text{su}(k), \quad k = 2, 3, 4. \]

If we replace \( L(t) \) by \( L(t_{(k)}) \) in above considerations, then we get that all results are valid (we must take \( J^\mu = i\Psi\dagger\alpha^\mu\Psi - \text{Tr}(i\Psi\dagger\alpha^\mu\Psi) \) in Theorem 6 and in eq. (34)).
11 Special cases

Let us denote
\[ I = -i e^{12}. \]
Then
\[ t_{(1)} = \frac{1}{4}(1 + H)(1 - i I) \]
and
\[ it_{(1)} = It_{(1)}. \]

**Theorem 10.** For a given \( \Phi \in \mathcal{I}(t_{(1)}) \) the equation
\[ \Psi t_{(1)} = \Phi, \] (43)
has a unique solution \( \Psi \in \Lambda_{\text{even}}. \)

**Proof.** We have the orthonormal basis of \( \mathcal{I}(t_{(1)}) \)
\[ t_{k} = F_{k}t_{(1)}, \quad k = 1, 2, 3, 4, \]
where \( F_{k} \in \Lambda_{\text{even}}. \) Decomposing \( \Phi \in \mathcal{I}(t_{(1)}) \) with respect to the basis \( t_{k} \)
\[ \Phi = (\alpha^{k} + i \beta^{k})t_{k}, \quad \alpha^{k}, \beta^{k} \in \Lambda_{0}, \] (44)
and using the relation \( it_{(1)} = It_{(1)}, \) we see that the differential form
\[ \Psi = F_{k}(\alpha^{k} + I \beta^{k}) \in \Lambda_{\text{even}} \]
is a solution of eq. (43).

We claim that if
\[ U = u + u_{01}e^{01} + u_{02}e^{02} + u_{03}e^{03} + u_{12}e^{12} + u_{13}e^{13} + u_{23}e^{23} + u_{0123}e^{0123} \in \Lambda_{\text{even}} \]
is a solution of the homogeneous equation
\[ Ut_{(1)} = 0, \]
then \( U = 0. \) Indeed, the differential form \( Ut_{(1)} \in \mathcal{I}(t_{(1)}) \) can be decomposed into the basis \( t_{k} \)
\[ Ut_{(1)} = \frac{1}{2}(u - iu_{12})t_{1} + \frac{1}{2}(-u_{13} - iu_{23})t_{2} + \frac{1}{2}(u_{03} - iu_{0123})t_{3} + \frac{1}{2}(u_{01} + iu_{02})t_{4}, \]
Thus the identity \( Ut_{(1)} = 0 \) implies \( U = 0. \) So the solution (44) of eq. (43) is unique. These completes the proof.
Theorem 11. For a given $\Phi \in \mathcal{I}(t^{(2)})$ the equation
\[ \Psi t^{(2)} = \Phi, \] (45)
has a unique solution $\Psi \in \Lambda_{\text{even}}^C$.

Proof. We have the orthonormal basis of $\mathcal{I}(t^{(2)})$
\[ t_k = F_k t^{(2)}, \quad k = 1, \ldots, 8, \]
where $F_k \in \Lambda_{\text{even}}^C$. Decomposing $\Phi \in \mathcal{I}(t^{(2)})$ with respect to the basis $t_k$
\[ \Psi = (\alpha^k + i\beta^k) t_k, \quad \alpha^k, \beta^k \in \Lambda_0, \] (46)
we see that the differential form
\[ \Psi = F_k (\alpha^k + i\beta^k) \in \Lambda_{\text{even}}^C \]
is a solution of eq. (19).

We claim that if
\[ U = (v + iw) + \sum_{0 \leq a < b \leq 3} (v_{ab} + iw_{ab}) e^{ab} + (v_{0123} + iw_{0123}) e^{0123} \]
is a solution of the homogeneous equation
\[ Ut^{(2)} = 0, \]
then $U = 0$. Indeed, the differential form $Ut^{(2)} \in \mathcal{I}(t^{(2)})$ can be decomposed into the basis $t_k$
\[ Ut^{(2)} = \phi_k t^k, \quad \phi_k = (t_k, Ut^{(2)}). \]
We get
\begin{align*}
\phi_1 &= (v - iv_{12} + iw + w_{12})/2 \\
\phi_2 &= (-v_{13} - iv_{23} - iw_{13} + w_{23})/2 \\
\phi_3 &= (v_{03} - iv_{0123} + iw_{03} + w_{0123})/2 \\
\phi_4 &= (v_{01} + iv_{02} + iw_{01} - w_{02})/2 \\
\phi_5 &= (v_{13} - iv_{23} + iw_{13} + w_{23})/2 \\
\phi_6 &= (v + iv_{12} + iw - w_{12})/2 \\
\phi_7 &= (v_{01} - iv_{02} + iw_{01} + w_{02})/2 \\
\phi_8 &= (v_{03} - iv_{0123} - iw_{03} + w_{0123})/2.
\end{align*}
Evidently the identity $Ut(2) = 0$ implies $U = 0$. So the solution (46) of eq. (45) is unique. These completes the proof.

Denote
$$\frac{1}{L} (t) = \{ U \in \Lambda_{\text{even}} : U^\dagger = -U, [U, t(1)] = 0 \}.$$ The Lie algebra $\frac{1}{L} (t)$ is isomorphic to the Lie algebra $u(1)$ and as a generator of $\frac{1}{L} (t)$ we may take $\tau_0 = I$.

**Theorem 12.** Differential forms $\frac{1}{\Psi} \in \Lambda_{\text{even}}, \frac{1}{A_\mu} \in L (t) \uparrow_1$ satisfy the equation
$$dx^\mu (D_\mu \frac{1}{\Psi} + \frac{1}{\Psi} A_\mu + B_\mu \frac{1}{\Psi})I + m \frac{1}{\Psi} I = 0 \quad (47)$$
iff differential forms $\Psi = \frac{1}{\Psi} t(1) \in \mathcal{I}(t(1)), A_\mu = \frac{1}{A_\mu} t(1) \in L(t) \uparrow_1$ satisfy eq. (27).

**Proof.** Multiplying (17) from the right by $t(1)$ and using relations $Ht(1) = t(1), It(1) = it(1)$, we obtain that $\Psi = \frac{1}{\Psi} t(1), A_\mu = \frac{1}{A_\mu} t(1)$ satisfy (27).

Conversely, let $\Psi \in \mathcal{I}(t), A_\mu \in L(t) \uparrow_1$ satisfy (27). By Theorem 10 there exists a unique solution $\frac{1}{\Psi} \in \Lambda_{\text{even}}, \frac{1}{A_\mu} \in \Lambda_{\text{even}} \uparrow_1$ of the system of equations $\frac{1}{\Psi} t(1) = \Psi, \frac{1}{A_\mu} t(1) = A_\mu$. It can be shown that $\frac{1}{A_\mu} \in \frac{1}{L} (t) \uparrow_1$. Substituting $\frac{1}{\Psi} t(1), \frac{1}{A_\mu} t(1)$ for $\Psi, A_\mu$ in (27), we arrive at the equality
$$(dx^\mu (D_\mu \frac{1}{\Psi} + \frac{1}{\Psi} A_\mu + B_\mu \frac{1}{\Psi})I + m \frac{1}{\Psi} I)t(1) = 0,$$
which we rewrite as
$$\frac{1}{\Omega} t(1) = 0.$$

We see that $\frac{1}{\Omega} \in \Lambda_{\text{even}}$ and, according to Theorem 10, we get
$$\frac{1}{\Omega} = 0.$$

These completes the proof.

Denote
$$\frac{2}{L} (t) = \{ U \in \Lambda_{\text{even}}^c : U^\dagger = -U, [U, t(2)] = 0 \}.$$
The Lie algebra \( \mathcal{L}(t) \) is isomorphic to the Lie algebra \( u(1) \oplus su(2) \) and we may take the following generators of \( \mathcal{L}(t) \):

\[
\tau_0 = i, \quad \tau_1 = e^{23}, \quad \tau_2 = -e^{13}, \quad \tau_3 = e^{12}.
\]

**Theorem 13.** Differential forms \( \Psi \in \Lambda^C_{\text{even}}, A_\mu \in \mathcal{L}(t) \mathcal{T}_1 \) satisfy the equation

\[
dx(\mathcal{D}_\mu \frac{\Psi}{2} + \frac{\Psi}{2} A_\mu + B_\mu \frac{\Psi}{2})H + im \, \frac{\Psi}{2} = 0 \tag{48}
\]

iff differential forms \( \Psi = \Psi(t_2) \in \mathcal{I}(t_2), A_\mu = \mathcal{A}_\mu(t_2) \in L(t) \mathcal{T}_1 \) satisfy eq. (27).

**Proof** is word for word identical to the proof of the preceding theorem.

Equations (47), (48) together with correspondent Yang-Mills equations were considered in [2]-[4].

**References**

[1] Dirac P.A.M., Proc. Roy. Soc. Lond. A117 (1928) 610.

[2] Marchuk N.G., Nuovo Cimento, 117B, 01, (2002) 95.

[3] Marchuk N.G., Nuovo Cimento, 117B, 05, (2002) 613.

[4] Marchuk N.G., Dirac-type tensor equations on a parallelisable manifold, to appear in Nuovo Cimento B.

[5] Marchuk N.G., Nuovo Cimento, 115B, N.11, (2000) 1267.

[6] Ivanenko D., Landau L., Z. Phys., 48 (1928)340.

[7] Kähler E., Randiconti di Mat. (Roma) ser. 5, 21, (1962) 425.

[8] Riesz M., pp.123-148 in C.R. 10 Congres Math. Scandinaves, Copenhagen, 1946. Jul. Gjellerups Forlag, Copenhagen, 1947. Reprinted in L.Gårding, L.Hörmander (eds.): *Marcel Reisz, Collected Papers*, Springer, Berlin, 1988, pp.814-832.

[9] Gürsey F., Nuovo Cimento, 3, (1956) 988.
[10] Hestenes D., *Space-Time Algebra*, Gordon and Breach, New York, 1966.

[11] Hestenes D., J. Math. Phys., 8, (1967) 798-808.

[12] Benn I.M., Tucker R.W., *An introduction to spinors and geometry with applications to physics*, Bristol, 1987.

[13] Lounesto P., *Clifford Algebras and Spinors*, Cambridge Univ. Press (1997, 2001)

[14] Grassmann H., Math. Commun. 12, 375 (1877).

[15] Doran C., Hestenes D., Sommen F., Van Acker N., J.Math.Phys. 34(8), (1993) 3642.