Thermodynamic interpretation of the quantum error correcting criterion

Vladimir Korepin
Yang Institute of Theoretical Physics, State University of New York, Stony Brook, NY 11794-3840.

John Terilla
Department of Mathematics, State University of New York, Stony Brook, NY 11794-3651.

(Dated: June 18, 2021)

Shannon’s fundamental coding theorems relate classical information theory to thermodynamics. More recent theoretical work has been successful in relating quantum information theory to thermodynamics. For example, Schumacher proved a quantum version of Shannon’s 1948 classical noiseless coding theorem. In this note, we extend the connection between quantum information theory and thermodynamics to include quantum error correction.

There is a standard mechanism for describing errors that may occur during the transmission, storage, and manipulation of quantum information. One can formulate a criterion of necessary and sufficient conditions for the errors to be detectable and correctable. We show that this criterion has a thermodynamical interpretation.

PACS numbers: 03.67, 05.30, 63.10

I. INTRODUCTION

Modern information theory was invented more than fifty years ago by Claude Shannon [33]. In his seminal paper, he gave a mathematical definition of information and proved his theoretical noiseless coding and noisy coding theorems. The noiseless theorem describes the physical resources necessary to store and transmit the information contained in a message. The noisy theorem describes the informational capacity of a noisy channel. The basic concept behind both theorems is that of typical messages. As one considers messages of increasing length, some messages emerge as likely to appear and some messages recede and become unlikely to appear.

In statistical mechanics, one studies systems of a very large number of particles. The business of thermodynamics is to study the properties of the entire system that materialize as the number of particles grows. Information theory as conceived by Shannon can be compared to the physical science of one-dimensional statistical mechanics and the concept of typical sequences is like a thermodynamic equilibrium in this statistical mechanical system.

The success of treating information theory as a physical science is by now unquestioned and the theoretical bridge between information theory and thermodynamics has brought a profitable trade route between the subjects. Presently, quantum information theory is being hotly pursued and investigators are discovering how quantum information and thermodynamics are connected. One direct descendant of Shannon’s noiseless coding theorem is a quantum version proved by Schumacher [32], which shows that entropy is important in quantum information theory. Also, connections between information and thermodynamics have been built using entanglement in quantum systems [17, 25, 31]. The bridge between information science and statistical mechanics lengthens to include the quantum branches of both fields and in this tradition we present a thermodynamic interpretation of quantum error correction.

The concept of quantum error correcting codes has developed rapidly. First quantum error correcting codes were discovered in [8, 34, 35]. Error avoiding codes were discovered in [17]; a systematic way of building nontrivial models in which dynamical symmetry allows unitary evolution of a subspace (decoherence-dissipation free) while the remaining part of the Hilbert space gets strongly entangled with the environment. Necessary and sufficient conditions for the
ability of quantum error correcting codes to correct errors appeared [4, 6, 15, 22, 30]. Let us remind the reader about these conditions.

A quantum code $C$ is a subspace of a Hilbert space $W$ equipped with an inner product $\langle \cdot | \cdot \rangle$. $W$ is sometimes called an encoding space. Errors are represented by a collection $E = \{ E_a : W \rightarrow W \}$ of linear operators. One imagines the diagram $C \hookrightarrow W \xrightarrow{E} W \rightarrow C$ where the middle arrow represents deterioration of the medium while either during storage in a quantum device, or during transmission down a noisy channel. The arrow on the left represents encoding the information into the encoding space and the arrow on the right represents some kind of recovery procedure. The goal of quantum error correction is to control the code space $C$, the encoding procedure, and the decoding procedure, so that the composition of all three arrows acts as the identity on $C$. Here we highlight the following important result:

**Quantum Error Correction (QEC) Condition.** The necessary and sufficient condition for the errors $E$ to be correctable is that $\langle \psi | E_a^\dagger E_b | \psi \rangle$ be the same for all unit vectors $\psi \in C$ and for every $E_a, E_b \in E$.

In the appendix, we briefly review quantum error correction and various criteria equivalent to the QEC stated above. In particular, we show that the QEC criterion is equivalent to the widely known condition that for any orthonormal basis $\{|\psi_j\rangle\}$ of the code space $C$ and for all $E_a, E_b \in E$, there exists a constant $c_{ab}$ so that $\langle \psi_j | E_a^\dagger E_b | \psi_k \rangle = c_{ab} \delta_{jk}$. Since 1996, several good quantum codes have been developed. Many are adapted from the classical theory of error correction codes, the most famous being the CSS codes [3, 33] and their generalization—the stabilizer codes [4, 7, 13, 14]. See the textbook [3] and the report [36] for a summary, and also the references within.

Let us now turn to statistical mechanics. A qubit of quantum information will be identified with a spin $\frac{1}{2}$ and we consider a one dimensional system of interacting spins. The state of the system is a unit vector in a Hilbert space, which we will again call $W$. The dynamics of the system is determined by a Hermitian operator $\mathcal{H} : W \rightarrow W$ called the Hamiltonian. Akin to the emergence of typical sequences in Shannon’s theory, or the typical subspace of thermodynamic equilibrium space will serve, equally well, to represent the macroscopic physical properties of the entire equilibrium. Physical properties are properties such as energy, scattering matrix, and most importantly for this paper, local operators and the TCF conditions hold in the thermodynamic limit, thus asymptotically delivering the quantum error correcting conditions. The reader may wonder how one can manipulate encoded information, say for the purposes of computation, if no local operators act non-trivially on the code space. The TCF condition is strong, but does not preclude using thermodynamic systems as quantum processing devices since nonlocal operators are available. Two different approaches come immediately to mind. One is to use long products of Pauli operators affecting a number of qubits comparable to the size of the entire system. Another is to use adiabatic changes of boundary conditions to obtain encoded gates. We expand the adiabatic approach in the final remarks of the conclusion.

We remark that the subspace of thermo-equilibrium is somewhat like a positive temperature, dynamic analog of the zero temperature ground state degeneracy that appears in topological quantum computing—a subject that is also being investigated for its asymptotic ability to correct errors [12, 21].
II. SHANNON’S NOISELESS CODING THEOREM

We briefly review Shannon’s noiseless coding theorem. Consider a random binary variable \( X \) where the probability that \( X = 0 \) is \( p \) and the probability that \( X = 1 \) is \( 1 - p \). Consider a sequence consisting of \( n \) values of \( X \). Then, the expected number of 0’s in the sequence is \( np \) and the number of sequences with this expected number of 0’s is

\[
\binom{n}{np} = \frac{n!}{(np)!(n(1-p))!}.
\]

For large \( n \), we have the asymptotic result

\[
\binom{n}{np} \sim 2^{nS(p)} \text{ where } S(p) = -p \log p - (1-p) \log(1-p).
\]

Here, and throughout this paper, \( \log \) means \( \log \) base 2. Note that \( 0 \leq S(p) \leq 1 \) and \( S(p) = 1 \) only if \( p = \frac{1}{2} \). A sequence \( b_1b_2 \cdots b_n \) of values of \( X \) that contains the expected number of number of zeros (assuming \( np \) is an integer) is called a typical sequence. The probability that \( b_1b_2 \cdots b_n \) is typical approaches 1 as \( n \) approaches infinity. So, in order to communicate a given sequence of length \( n \), one needs only communicate which one of \( 2^{nS(p)} \) typical sequences is at hand. In this way \( n \) bits can be encoded in \( nS(p) \) bits. Translated into a coding theorem, one encodes blocks of \( n \) bits by using \( nS(p) \) bits and the probability of being able to successfully decode a block approaches one as \( n \) tends to infinity.

One easily generalizes to the case that the random variable \( X \) takes values in some finite set \( \{x\} \). If \( X \) takes the value \( x \) with probability \( p(x) \), then one defines the Shannon entropy of \( X \) to be

\[
S(X) = -\sum_x p(x) \log p(x)
\]

and the same analysis and conclusions hold. The probability that a sequence \( x_1x_2 \cdots x_n \) of values of \( X \) is typical approaches one as \( n \) approaches infinity.

III. SCHUMACHER’S CODING THEOREM

A quantum bit, or a qubit, can be represented by a vector in a two dimensional complex linear space with an inner product. One should consider two vectors that differ by a nonzero scaler factor to represent the same qubit. An ensemble of \( n \) qubits is represented by an element of the \( n \)-fold tensor product of qubit spaces. Let us fix notation. Let \( W = \otimes_{j=1}^n W_j \), where each \( W_j \) is a two dimensional vector space with orthonormal basis \( \{ |0_j \rangle, |1_j \rangle \} \). Each \( W_j = \mathbb{C}|0_j \rangle \oplus \mathbb{C}|1_j \rangle \simeq \mathbb{C}^2 \) is identified with a space of qubits. The following convenient notation is suggested for a basis of \( W \). Every basis vector \( |b_1 \rangle_1 \otimes |b_2 \rangle_2 \otimes \cdots \otimes |b_n \rangle_n \) can be referred to by the shortened name \( |b_1b_2 \cdots b_n \rangle \). Bit strings of length \( n \) index the basis vectors of \( W \) and, in this way, a general quantum state consisting of \( n \) qubits can be thought of as a linear combination of bit strings of length \( n \).

Let us define the density matrix associated to a state \( |\psi \rangle \in W \). One has the linear functional \( \langle \psi | \) in the dual space \( W^* \) whose value on \( |\phi \rangle \in W \) is given by \( \langle \psi | \phi \rangle \). A matrix \( \rho \) can then be identified with \( |\psi \rangle \otimes \langle \psi | \) via an isomorphism \( W \otimes W^* \simeq \text{Hom}(W,W) \simeq 2^n \times 2^n \) matrices. It is common to drop the tensor sign and write \( \rho = |\psi \rangle \langle \psi | \). Of course, the information contained in \( \rho \) is no different than the information contained in \( |\psi \rangle \). If \( |\psi \rangle \) is a unit vector and \( \{ |b \rangle \} \) is an orthonormal basis of \( W \), then \( \rho \) has an expansion

\[
\rho = \sum_b p(b) |b \rangle \langle b |
\]

and the numbers \( p(b) \) are nonnegative real numbers satisfying \( \sum_b p(b) = 1 \). The numbers \( \{ p(b) \} \) define a probability distribution on the collection \( \{ |b \rangle \} \). One can extend the definition of \( p \) to an arbitrary \( |\phi \rangle \in W \) by \( p(|\phi \rangle) = \langle \phi | \rho | \phi \rangle \).

Now, we can view the density matrix \( \rho \) as a quantum random variable which produces the state \( |b \rangle \) with a probability of \( p(b) \). Or, one imagines a “quantum signal source” which encodes the classical bit string \( b \), which is the output
of a random variable with probability \( p(b) \), as the quantum state \(| b \rangle \). One can then define a state \(| b_1 b_2 \cdots b_n \rangle \) to be a typical state provided \( b_1 b_2 \cdots b_n \) is a typical sequence à la Shannon. Define the typical subspace of \( W \) to be the subspace \( T \) spanned by all typical states. Then the subspace \( T \) serves to compress \( W \) as the probability that a random vector from \( W \) will lie in the subspace \( T \) approaches 1, as \( n \to \infty \). The dimension of \( T \) is \( 2^{nS(\rho)} \), where \( S(\rho) \) is the Shannon entropy defined in equation (3). In the present context, it is natural to note that \( S(\rho) = \text{tr}(\rho \log(\rho)) \), which is called the von Neumann entropy of \( \rho \). However, in the case that \( \{| b \rangle \} \) is a collection of arbitrary, not necessarily orthogonal, states then it is the von Neumann entropy, not the Shannon entropy, which computes the dimension of \( T \). Further refinements to the role von Neumann entropy plays in quantum compression are still being developed [10].

IV. QUANTUM SPIN CHAINS

Now let us show how to formulate the quantum error correcting condition via thermodynamics. The subject of quantum statistical mechanics and thermodynamics is treated in several textbooks [23, 37]. There are various models determined by different Hamiltonians describing different interactions, and many, such as the spin chain models, have relevance to quantum information theory. For example, programs are underway to investigate various aspects of the XYZ family of quantum spin chain models with relevance to quantum computing, aspects like the implementation of gates [1], decoherence free subspaces [3], entanglement [27], and encoded universality [2, 25, 39]. Also, some two dimensional spin models arise in topological quantum computing [12, 21].

A. Notation

Let \( \sigma^x, \sigma^y, \) and \( \sigma^z \) be the Pauli matrices, which act on \( \mathbb{C}^2 = \mathbb{C}|0 \rangle \oplus \mathbb{C}|1 \rangle \) as the matrices

\[
\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

It is common in quantum information theory to use the notation \( X, Y, \) and \( Z \) instead of \( \sigma_x, \sigma_y, \) and \( \sigma_z \). Set, as before, \( W = \otimes_{j=1}^n W_j \), where \( W_j = \mathbb{C}|0 \rangle_j \oplus \mathbb{C}|1 \rangle_j \simeq \mathbb{C}^2 \). Sometimes the integer \( n \) is called the length of the lattice or the size of the model, and the reader may compare it to the length of the message in information theory. For any \( j = 1, \ldots, n \) and superscripts \( \alpha = x, y, z \) define an operator \( \sigma^\alpha_j : W \to W \) by

\[
\sigma^\alpha_j |W_m \rangle = \begin{cases} \text{Id} & \text{if } j \neq m, \\ \sigma^\alpha & \text{if } j = m. \end{cases}
\]

(6)

The operator \( \sigma^\alpha_j \) acts non trivially on the \( j \)-th qubit as the Pauli matrix \( \sigma^\alpha \) and leaves all other qubits unchanged.

One also has the matrix

\[
\sigma^- := \frac{1}{2} (\sigma^x - i\sigma^y) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},
\]

(7)
called a creation operator. By setting \( \sigma^-_j = \frac{1}{2} (\sigma^x_j - i\sigma^y_j) \), one can obtain every basis vector in \( W \) by acting on \( |0 \rangle := |00 \cdots 0 \rangle \) by products of the \( \sigma^-_j \).

A local operator \( \mathcal{O} : W \to W \) is defined to be a linear combination of products of the operators \( \sigma^\alpha_j \). If \( \mathcal{O} \) is a product of \( t \) such operators

\[
\mathcal{O} = \sigma^{\alpha_{m_1}} \sigma^{\alpha_{m_2}} \cdots \sigma^{\alpha_{m_t}}
\]

with distinct \( m_j \), we say that \( \mathcal{O} \) is an operator of weight \( t \).
B. The partition function and thermodynamic equilibrium

We now wish to study the model as the spins interact. The Hilbert space of the model is $W \cong (\mathbb{C}^2)^\otimes n$ and the interacting spins are governed by a Hamiltonian $H : W \rightarrow W$. What follows is quite general, though later we illustrate more details with the XX0 model. So the reader may have the XX0 Hamiltonian in mind:

$$H = -\sum_{j=1}^{n} \left( \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + h \sigma_j^z \right)$$  \hspace{1cm} (9)$$

The real parameter $h$ is called the magnetic field. The XX0 model with periodic boundary conditions was originally solved in 1961 by E. Lieb, T. Schultz, and D. Mattis [26]. The XX0 model is sometimes called the “isotropic XY model” and is also known (in quantum information theory) as the “XY model with Zeeman splitting.”

The thermodynamic limit of the model is mathematical idealization of a very large system defined by a controlled limit $n \rightarrow \infty$. Quantities of interest are computed for finite $n$ and then studied as $n$ tends to infinity. Often these quantities are proportional to $n$ and the proportionality factor has a finite value in the thermodynamic limit. These asymptotic are studied much the same way that they are in information theory when the length of the message grows to infinity.

The central object of thermodynamics is the partition function $Z$, which is defined at a temperature $T \geq 0$, by

$$Z = \text{tr} \left( 2^{-\frac{H}{T}} \right).$$  \hspace{1cm} (10)$$

In the thermodynamic limit, the partition function can be computed by the method of steepest descent. One has

$$Z = \text{tr} 2^{-\frac{H}{T}} = \sum_{\text{eigenvectors } v} 2^{-\frac{E(v)}{T}} = \sum_{\text{eigenvalues } E} 2^{n S} 2^{-\frac{E}{T}}$$  \hspace{1cm} (11)$$

where the factor $2^{nS}$ is the degeneracy of the energy level $E$ and $S$ is entropy. Both energy and entropy increase linearly with $n$. So evaluating

$$\lim_{n \rightarrow \infty} Z = \lim_{n \rightarrow \infty} \sum_{E} 2^{n S} 2^{-\frac{E}{T}}$$  \hspace{1cm} (12)$$

by the method of steepest descent leads to the variational equaiton

$$\delta \left( S - \frac{E}{nT} \right) = 0.$$  \hspace{1cm} (13)$$

This brings us to the key definition:

**Definition 1.** The subspace of thermodynamic equilibrium is defined to be the span of the set of eigenvectors that solve equation (12).

This definition of the thermo-equilibrium subspace makes sense for all solvable models, including XX0, XXZ, XYZ, etc... For the model we’ve chosen to work with, we can be more specific. For the XX0 model, equation (12) becomes

$$Z = \text{tr}_C \left( 2^{-\frac{H}{T}} \right) \sim 2^{\frac{nS}{T}} \int_{-\pi}^{\pi} dp \log \left( 1 + 2^{-\frac{E(p)}{T}} \right)$$  \hspace{1cm} (14)$$

where $\epsilon$ is given by

$$\epsilon(p) = -4 \cos(p) + 2h.$$  \hspace{1cm} (15)$$

The symbol $\text{tr}_C$ means the trace over the subspace $C$. It is defined for any operator $A : W \rightarrow W$ by $\text{tr}_C(A) = \sum_{j=1}^{n} a_{jj}$ where $a_{jj}$ are the diagonal entries of $A$ when expressed as a matrix using a basis $\{|\psi_j\rangle\}_{j=1}^{n}$ for $W$ extending a basis $\{|\psi_j\rangle\}_{j=1}^{n}$ for $C$. 


The precise meaning of the right hand side of equation (14) is that \( \lim_{n \to \infty} \frac{1}{n} \log Z \) exists and is given by
\[
\lim_{n \to \infty} \frac{1}{n} \log Z = \frac{1}{2\pi} \int_{-\pi}^{\pi} dp \log \left( 1 + 2^{-i \epsilon(p)} \right).
\]
(16)
The quantity in equation (16) is called bulk free energy.
From equations (34) and (41), in section IV E, we determine, asymptotically, the dimension of \( C \). The dimension of \( C \sim 2^{nS} \) where \( S \) is the entropy:
\[
S = -\frac{1}{2\pi} \int_{-\pi}^{\pi} dp \left[ \theta(p) \log \theta(p) + (1 - \theta(p)) \log(1 - \theta(p)) \right],
\]
(17)
where \( \theta \), called the Fermi weight, is defined by
\[
\theta(p) = \left( 1 + 2^{-i \epsilon(p)} \right)^{-1}.
\]
(18)
Physicists call \( \epsilon \) the energy of the spin wave and call \( p \) the momentum of the spin wave.

C. Correlation functions

Let us define correlation functions. Let \( \mathcal{O} \) be a linear combination of products of operators \( \sigma_j^\alpha \). The correlation function \( \langle \mathcal{O} \rangle_T \) is defined by
\[
\langle \mathcal{O} \rangle_T = \frac{\text{tr} \left( 2^{-\frac{\mathcal{H}}{T}} \mathcal{O} \right)}{Z}.
\]
(19)
If \( \mathcal{O} \) has weight \( t \), then \( \langle \mathcal{O} \rangle_T \) can be related to \( t \)-point correlation function. In the thermodynamic limit, the correlation functions also can be calculated explicitly by the method of steepest descent, just as for the partition function \( Z = \text{tr}(2^{-\frac{\mathcal{H}}{T}}) \). Only the space of thermo equilibrium \( C \) contributes to the trace (23, page 25). So instead of taking the trace over all of \( W \), one has
\[
\langle \mathcal{O} \rangle_T = \frac{\text{tr}_C \left( 2^{-\frac{\mathcal{H}}{T}} \mathcal{O} \right)}{Z}.
\]
(20)
An even stronger statement is true. In 11 13 15 23 it was shown that each term of the trace in equation (20) contributes equally, and so equation (20) simplifies further:
\[
\langle \mathcal{O} \rangle_T = \frac{\text{tr}_C \left( 2^{-\frac{\mathcal{H}}{T}} \mathcal{O} \right)}{Z} = \frac{\langle \psi | \mathcal{O} | \psi \rangle}{\langle \psi | \psi \rangle} \text{ for any } \psi \in C,
\]
(21)
which gives the thermodynamic correlation function condition stated in the introduction.
The TCF condition is a quite general feature of quantum statistical mechanics. It holds not just for XX0, but also for other integrable models (XY, XXZ, XYZ, nonlinear Schrödinger, Hubbard model, etc...). We conjecture that the TCF condition is valid for a wide class of physically interesting models, including non-integrable models in some vicinity of an integrable one.

Let us comment on the the special case \( T = 0 \). The correlation functions simplify dramatically. There is a unique vector \( |G\rangle \), first identified in 20, called the ground state. It corresponds to the lowest eigenvalue of \( \mathcal{H} \). In the case of zero temperature
\[
\langle \mathcal{O} \rangle_{T=0} = \langle G | \mathcal{O} | G \rangle.
\]
(22)
The TCF condition is precisely a generalization the equation (22) to \( T > 0 \).
D. Eigenvectors of the XX0 Hamiltonian

Except for equations (14–18), the discussion above applies to a quantum statistical model governed by most any Hamiltonian. Now, to further illustrate the thermodynamics, we work specifically with XX0. The Hamiltonian

\[ H = -\sum_{j=1}^{n} (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + h \sigma_j^z) \]

can be written as \( H = H_0 - 2hS^z \) where

\[ H_0 = -\sum_{j=1}^{n} (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y) \quad \text{and} \quad S^z = \frac{1}{2} \sum_{j=1}^{n} \sigma_j^z. \]  

Note that \( [H_0, S^z] = 0 \). The problem of finding the eigenvectors of \( H \) can be reduced to finding the common eigenvectors of \( H_0 \) and \( S^z \). We now describe a complete set of eigenvectors of \( H \) for any positive integer \( n \). Recall that \( \sigma_j^z = \frac{1}{2}(\sigma_j^x - i \sigma_j^y) \).

The eigenvectors of \( H \) are determined by a positive integer \( m \leq n \) and a collection of real numbers \( \{p_i\}_{i=1}^{m} \), with each \( -\pi < p_i < \pi \), called momenta. For short, the collection \( \{p_i\}_{i=1}^{m} \) may be denoted simply by \( \{p\} \). Define a vector \( |\{p\}\rangle_m \in W \) by

\[ |\{p\}\rangle_m = \frac{1}{\sqrt{m!}} \sum_{x_1,\ldots,x_m} \chi_m(\{x\}|\{p\}) \sigma_{x_m}^- \cdots \sigma_{x_1}^- |0\rangle. \]  

The complex valued function \( \chi \) is defined by

\[ \chi_m(\{x\}|\{p\}) = \frac{1}{\sqrt{m!}} \left( \prod_{1 \leq a < b \leq m} \text{sign}(x_b - x_a) \right) \det(A), \]  

where \( A \) is the \( m \times m \) matrix with \((j,k)\) entry \( A_{jk} = \exp(ip_j p_k) \) and the sign function is given by

\[ \text{sign}(x) = \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0. \end{cases} \]  

The function \( \chi(\{x\}|\{p\}) \) is called a wave function. It is symmetric in \( x \) and antisymmetric in \( p \), so we assume that \( x_1 \leq x_2 \leq \cdots \leq x_m \) and that \( p_1 < p_2 < \cdots < p_m \). We may drop the subscript \( m \) from \( |\{p\}\rangle_m \in W \) if the size of \( \{p\} \) does not need emphasis and just write \( |\{p\}\rangle \).

The periodic boundary conditions of the model lead to the following equation for each \( p_j \):

\[ \exp(ip_j n) = (-1)^{m+1} \]  

Note that \( 24 \) has \( n \) solutions and a collection \( \{p\} \) amounts to a choice of \( m \) of these solutions. So, for each \( m = 0, \ldots, n \), there are \( \binom{n}{m} \) choices of momenta each identifying one eigenvector of \( H \). In total, there are \( 2^n \) eigenvectors \( 24 \) as \( m \) varies from 0 to \( n \). A straightforward calculation of their scalar products shows that these eigenvectors form an orthogonal basis of \( W \):

\[ \langle \{p\}|\{p\}\rangle = n^m \quad \text{and} \quad \langle \{p\}|\{q\}\rangle = 0 \quad \text{if} \quad \{p\} \neq \{q\}. \]

The eigenvalue \( E(\{p\}) \) of \( |\{p\}\rangle_m \) is given by

\[ E(\{p\}) = \sum_{j=1}^{m} \epsilon(p_j), \quad \epsilon(p) = -4 \cos p + 2h. \]
E. Thermodynamics of the XX0 model

One may think of each of the eigenvectors of $\mathcal{H}$ as being obtained from the ferromagnetic state $|0\rangle$ by adding $m$ particles (flipping $m$ spins) via the creation operators $\sigma^+_j$, with momenta $\{p_j\}$ and energies $\{\epsilon(p_j)\}$. In this section we consider the situation as the number of particles occupying positions in the interval $[-\pi, \pi]$ tends to infinity (see [23]). Divide the interval $[-\pi, \pi]$ into $n$ subintervals, each of length $\frac{2\pi}{n}$. Each subdivision point will be called a position and should be thought of as a possible location for a particle. They represent the $n$ solutions of $\frac{2\pi}{n}$, which the momenta satisfy. When a wave function has been selected, and a choice $\{p\}$ of momenta has been made, one should think that each of the $m$ positions appearing in $\{p\}$ is occupied by a particle and the remaining $n-m$ positions are empty (see figure [1]).

Once a collection $\{p\} = \{p_1 < p_2 < \cdots < p_n\}$ has been chosen, one may look at the numbers $\rho(p_j)$:

$$\rho(p_j) = \frac{1}{n(p_j+1 - p_j)}. \quad \text{(29)}$$

Now we consider the situation where the number of positions becomes large, the number of particles becomes large, and the density of particles remains fixed. The thermodynamic limit is defined by

$$m \to \infty, \quad n \to \infty, \quad \text{and} \quad d := \frac{m}{n} \text{ is constant.} \quad \text{(30)}$$

The reader may compare it to the double-scaling limit in matrix models. For now, consider $m$ and $n$ as very large, but still finite. As the number of particles grows, it becomes difficult to keep track of the individual momenta $p_j$ since there are so many of them. Instead, the growing collections $\{p\}$ give way to a finite function $\rho : [-\pi, \pi] \to \mathbb{R}$, which exists as a limit of expressions [24]. It has the interpretation that for large $n$

the number of particles in the interval $[p, p + \Delta p]$ $\approx n\rho(p)\Delta p$,

provided the scale for $\Delta p$ is chosen properly. One should have

$$\frac{2\pi}{n} << \Delta p, \quad \text{(31)}$$

but $\Delta p$ should still be small enough to be able to approximate $\rho$ by a constant on the interval $[p, p + \Delta p]$.

One should think that $\rho$ is part of a macroscopic description of the model, and that a collection $\{p\}$ is part of a microscopic description. Be aware that each $\rho$ may be the limit of many different sequences of microscopic states $\{p\}$, the number of which can be called the degeneracy of the macroscopic state $\rho$. For a fixed $\rho$, we can compute the number of microscopic states corresponding to $\rho$. First, partition the interval $[-\pi, \pi]$ into subintervals of length $\Delta p$. It is important that the scale of $\Delta p$ is such that $\rho$ is well approximated by a constant on the interval $[p, \Delta p]$. The number of ways of selecting $n\rho(p)\Delta p$ positions from $\frac{2\pi}{n}\Delta p$ vacancies in which to place particles is equal to the binomial coefficient:

$$\binom{n\rho(p)\Delta p}{\frac{2\pi}{n}\Delta p} = \frac{(n\rho(p)\Delta p)!}{(n \rho(p)\Delta p)! (n (\frac{1}{2\pi} - \rho(p)) \Delta p)!}. \quad \text{(32)}$$

Using Stirling’s formula to approximate a factorial, one arrives at equation [32] asymptotically becomes $2^{n\Delta S}$ where

$$\Delta S = \left[ \frac{1}{2\pi} \log \left( \frac{1}{2\pi} - \rho(p) \log(\rho(p)) \right) - \left( \frac{1}{2\pi} - \rho(p) \right) \log \left( \frac{1}{2\pi} - \rho(p) \right) \right] \Delta p. \quad \text{(33)}$$
Now, as $\Delta p \to 0$, we find that asymptotically, $\rho$ has a degeneracy of $2^n S$ where $S$, called the entropy, has the form

$$S = \int_{-\pi}^{\pi} dp \left[ \frac{1}{2\pi} \log \left( \frac{1}{2\pi} \right) - \rho(p) \log(\rho(p)) - \left( \frac{1}{2\pi} - \rho(p) \right) \log \left( \frac{1}{2\pi} - \rho(p) \right) \right]$$

(34)

The following picture emerges. For a finite system, each collection $\{p\}$ corresponds to precisely one $|\{p\}\rangle \in W$, which is an eigenvector of $\mathcal{H}$. In the thermodynamic limit when the number of spins becomes infinite, we replace the momenta $\{p\}$ by the function $\rho$. But unlike the finite system, each $\rho$ corresponds to many eigenvectors in the now infinite dimensional Hilbert space—the different microscopic states corresponding to a single macroscopic state. Moreover, each of these eigenvectors have the same energy density (see equation (36) below). The fact that every eigenvector corresponding to a single $\rho$ has the same energy density is an instance of a crucial principle. Namely, all of the local observables in the thermodynamic limit depend only on the macroscopic variable $\rho$.

Consider, for instance, the energy $E(\{p\})$:

$$E(\{p\}) = \sum_{j=1}^{m} \epsilon(p_j) = n \sum_{j=1}^{m} \frac{p_{j+1} - p_j}{n(p_{j+1} - p_j)}.$$  

(35)

In the thermodynamic limit, one finds that energy density is equal to

$$\lim_{n \to \infty} \frac{E}{n} = \int_{-\pi}^{\pi} \epsilon(p) \rho(p) dp.$$  

(36)

The thermodynamic correlation function condition is a consequence of the principle stated above. The matrix elements

$$\lim_{n \to \infty} \frac{\langle \{p\}|\mathcal{O}|\{p\} \rangle}{\langle \{p\}|\{p\} \rangle}$$

(37)

depend only on $\rho$, not on the set $\{p\}$. Therefore, in the thermodynamic limit,

$$\langle \mathcal{O} \rangle_T = \frac{\text{tr} \left( 2 - \frac{\mathcal{H}}{Z} \right)}{Z} = \lim_{n \to \infty} \frac{\langle \{p\}|\mathcal{O}|\{p\} \rangle}{\langle \{p\}|\{p\} \rangle}$$

(38)

In order to determine which function $\rho(p)$ defines the space of thermodynamic equilibrium, we return to the variational equation, which arose from the steepest descent approximation. Using variational calculus, one has for XX0

$$\delta S = \int_{-\pi}^{\pi} dp \left[ -\delta \rho(p) \log(\rho(p)) + \delta \rho(p) \log \left( \frac{1}{2\pi} - \rho(p) \right) \right]$$

(39)

and

$$\delta E = \int_{-\pi}^{\pi} dp [-\delta \rho(p) \epsilon(p)].$$

(40)

Setting $\delta \left( S - \frac{E}{nT} \right) = 0$ gives the solution

$$\rho(p) = \frac{1}{2\pi} \theta(p) = \frac{1}{2\pi \left( 1 + 2^{-\epsilon(p)} \right)}.$$  

(41)

Therefore, the span of the eigenvectors that correspond to the function $\rho = \frac{1}{2\pi} \theta$ comprise the space $C$ of thermodynamic equilibrium. Since every eigenvector in the thermo-equilibrium space corresponds to the same $\rho$, the correlation functions $\langle \psi|\mathcal{O}|\psi \rangle$ do not depend on which $\psi \in C$ is chosen. It can be a challenge to compute these correlation functions explicitly. In [11, 18, 23], they are computed for several models including XX0 by direct calculations.
V. CONCLUDING REMARKS

In statistical mechanics, the probability that an eigenstate $|\psi\rangle \in W$ will appear is given by $\frac{1}{2} 2^{-E(|\psi\rangle)}$ where the energy $E(|\psi\rangle)$ is the eigenvalue of the eigenvector $|\psi\rangle$. This leads to a definition of a typical state and the subspace of thermo-equilibrium becomes comparable to Schumacher’s typical subspace. We have shown that this subspace of thermo-equilibrium satisfies, asymptotically, the quantum error correction criterion for all errors of finite weight, thus strengthening the bond between information science and statistical mechanics. The reader may imagine that for finite $n$, an approximation of the thermo-equilibrium space may serve as quantum code. In order to develop this idea, one needs ways of producing gates and making measurements. Gates in quantum spin chain models have already been implemented in several circumstances [5, 20, 38]. Presumably, gates operating in the thermo-equilibrium subspace (or some finite dimensional approximation of it) can be obtained by similar means. Short products of Pauli matrices cannot be employed to map states in the thermo-equilibrium space into one another since such products are topological. We wrote this paper assuming periodic boundary conditions. That is, the wave function $\phi$ satisfies boundary conditions become twisted by a real phase $\phi$: 

$$\chi_m(x_1 + n, x_2, \ldots, x_n|\{p\}) = e^{i\phi} \chi_m(x_1, x_2, \ldots, x_n|\{p\}).$$

Thus, one can imagine the lattice forming a circle. Now, by introducing a magnetic flux threading this circle, the boundary conditions become twisted by a real phase $\phi$: 

$$\chi_m(x_1 + n, x_2, \ldots, x_n|\{p\}) = e^{i\phi} \chi_m(x_1, x_2, \ldots, x_n|\{p\}).$$

By adiabatically changing $\phi$ from 0 to $2\pi$ the subspace of thermodynamic equilibrium, i.e. the codes space, will be mapped into itself. Even at zero temperature, this map is nontrivial—Berry’s phase for such an adiabatic process was calculated in [27].

APPENDIX A: QUANTUM ERROR CORRECTION CRITERIA

The occurrence of errors during storage or transmission of quantum data is governed by a quantum operation, also called a super-operator. Given a finite set of linear transformations $\mathcal{E} = \{E_a : W \to W\}_{a=1}^m$ satisfying $\sum_{a=1}^m E_a^\dagger E_a = \text{Id}$, one defines a super-operator $S_\mathcal{E}$ acting on density matrices describing states in $W$. The action of $S_\mathcal{E}$ is defined on a density matrix $P$ describing states in $W$ by

$$S_\mathcal{E}(P) = \sum_{a=1}^m E_a P E_a^\dagger. \quad (A1)$$

Let $C \subseteq W$ be a quantum code. One says that the code $C$ can correct the errors $\mathcal{E}$, or that the errors $\mathcal{E}$ are correctable, provided there exists another super-operator $S_\mathcal{R}$ (expressed by a collection $\mathcal{R} = \{R_b\}_{b=1}^s$ satisfying $\sum_{b=1}^s R_b^\dagger R_b = \text{Id}$) such that

$$S_\mathcal{R}(S_\mathcal{E}(P)) = P \quad \text{for all density matrices } P \text{ describing states in } C \subseteq W. \quad (A2)$$

We now recall (theorem 10.1 from [2])

**Theorem.** Let $\pi : W \to C$ be the orthogonal projector onto the code subspace and $\mathcal{E} = \{E_a : W \to W\}_{a=1}^m$ be a collection of linear operators with $\sum_{a=1}^m E_a^\dagger E_a = \text{Id}$. A recovery super-operator $S_\mathcal{R}$ inverting $S_\mathcal{E}$ on density matrices from $C$ exists if and only if for every $E_a, E_b \in \mathcal{E}$, there exists a constant $c_{ab}$ satisfying $\pi E_b^\dagger E_a \pi = c_{ab} \pi$.

This theorem is evidently equivalent to
QEC II condition. In order for the errors $E$ to be correctable, it is necessary and sufficient that for any orthonormal basis $\{ |ψ_j⟩\}_{j=1}^l$ of the code space $C$ and for each $E_a, E_b ∈ E$, there exists a constant $c_{ab}$ so that $⟨ψ_j | E_a^† E_b | ψ_k⟩ = c_{ab} δ_{jk}$.

The condition QEC II as stated above seems to be familiar to those working in the field and can be found in many places (for example, section 3 of [22], in chapter 7 (pages 9 and 86) of [29], and section 6.4 of [4].) In the introduction, we stated the

QEC condition. The necessary and sufficient condition for the errors $E$ to be correctable is that $⟨ψ | E_a^† E_b | ψ⟩$ be the same for all unit vectors $ψ ∈ C$ and for every $E_a, E_b ∈ E$.

We now prove that the conditions QEC and QEC II are equivalent.

Proof. Suppose that $⟨ψ | E_a^† E_b | ψ⟩ = c_{ab}$ for every unit vector $|ψ⟩ ∈ C$. Note that the matrix $A = [c_{ab}]$ is Hermitian, hence there exists a unitary $U = [u_{ra}]$ such that $UAU^† = D = [d_{ab}]$ is a real diagonal matrix. Now let us define $F_r = ∑_{a=1}^m u_{ra} E_a$. Note that for any unit vector $ψ ∈ C$, we have

$$⟨ψ | F_r^† F_s | ψ⟩ = d_{rs} δ_{rs}.$$  \hspace{1cm} (A3)

The operator $F_r^† F_s$ is Hermitian so we can find an orthonormal basis $\{ |φ_j⟩\}_{j=1}^l$ of eigenvectors of $F_r$ for $E_r$. Hence $F_r^† F_s = d_{rs} I$. By substituting $|φ_j⟩$ for $|ψ⟩$ in equation (A3) one finds that the eigenvalue corresponding to $|φ_j⟩$ is $d_{rr}$ for every $j$. Thus

$$⟨ψ_j | F_r^† F_s | ψ_k⟩ = d_{rs} δ_{jk}.$$  \hspace{1cm} (A4)

By changing back from the $\{F_r\}$ to the $\{E_a\}$ by using $E_a = ∑_{r=1}^m u_{ra} F_r$, we find that

$$⟨ψ_j | E_a^† E_b | ψ_k⟩ = c_{ab} δ_{jk}.$$  \hspace{1cm} (A5)

This proves that QEC $⇒$ QECII.

Now, suppose that $\{ |ψ_j⟩\}_{j=1}^l$ be an orthonormal basis for $C$ and that $⟨ψ_j | E_a^† E_b | ψ⟩ = c_{ab} δ_{jk}$. For any unit vector $ψ ∈ C$ we have $ψ = ∑_{j=1}^l a_j ψ_j$ for some $a_j$ with $∑_{j=1}^l a_j^2 = 1$. We compute

$$⟨ψ | E_a^† E_b | ψ⟩ = ∑_{j=1}^l ∑_{k=1}^l a_j a_k ⟨ψ_j | E_a^† E_b | ψ_k⟩$$

\hspace{1cm} (A6)

$$= ∑_{j=1}^l a_j^2 c_{ab}$$

\hspace{1cm} (A7)

$$= c_{ab}.$$  \hspace{1cm} (A8)

Thus QECII $⇒$ QEC. In such a way we proved the equivalence of QEC II and QEC.

[1] D. Bacon, G. Burkard, D. DiVincenzo, J. Kempe, and K.B. Whaley. Universal quantum computation with the exchange interaction. quant-ph/0005116, 2000.

[2] D. Bacon, J. Kempe, D.P. DiVincenzo, D.A. Lidar, and K.B. Whaley. Encoded universality in physical implementations of a quantum computer. quant-ph/0102140, 2001.

[3] D. Bacon, J. Kempe, D.A. Lidar, and K.B. Whaley. Theory of decoherence-free fault-tolerant universal quantum computation. Phys. Rev. A., 63:042307, 2001.

[4] C. Bennett, D. DiVincenzo, J. Smolin, and W. Wootters. Mixed state entanglement and quantum error correction. Phys. Rev. A, 54:3824, 1996.

[5] N. E. Bonesteel, D. P. DiVincenzo, and D. Stepanenko. Anisotropic spin exchange in pulsed quantum gates. quant-ph/0106161, 2001.
[6] A. R. Calderbank, E. M. Rains, P. W. Shor, and N. J. Sloane. Quantum error correction and orthogonal geometry. *Phys. Rev. Lett.*, 78:405–408, 1997.

[7] A. R. Calderbank, E. M. Rains, P. W. Shor, and N. J. Sloane. Quantum error correction via codes over $gf(4)$. *IEEE Transactions on Information Theory*, IT-44:1369 – 1387, 1998.

[8] A. R. Calderbank and P. W. Shor. Good quantum error-correcting codes exist. *Phys. Rev. A*, 54(2):1098–1106, 1996.

[9] I. L. Chuang and M. A. Nielsen. *Quantum Computation and Quantum Information*. Cambridge Univ. Press, 2000.

[10] P. Zanardi and M. Rasetti. Noiseless Quantum Codes. *Phys. Rev. Lett*. 79(17):3306-3309, 1997.

[11] F. Colomo, A. Izergin, V. Korepin, and V. Tognetti. Fredholm determinant representation for correlation functions in XXO Heisenberg chain. *Phys. Lett. A*, 169:243, 1992.

[12] M. Freedman, A. Kitaev, M. Larsen, and Z. Wang. Topological quantum computation. *quant-ph/0101025*, 2001.

[13] D. Gottesman. A class of quantum error-correcting codes saturating the quantum Hamming bound. *Phys. Rev. A*, 54:1862, 1996.

[14] D. Gottesman. *Stabilizer Codes and Quantum Error Correction*. PhD thesis, Caltech, 1997. *quant-ph/9705052*.

[15] D. Gottesman. Theory of fault-tolerant quantum computation. *Phys. Rev. A*, 57:127–137, 1998.

[16] P. Hayden, R. Jozsa, and A. Winter. Trading quantum for classical resources in quantum data compression. *quant-ph/0204038*, 2002.

[17] R. Horodecki, M. Horodecki, and P. Horodecki. Balance of information in bipartite quantum-communication systems: Entanglement-energy analogy. *Phys. Rev. A.*, 63:022310, 2001.

[18] I. L. Chuang, M. A. Nielsen. *Quantum Computation and Quantum Information*. Cambridge Univ. Press, 2000.

[19] A. G. Izergin, A. G. Pron’ko, and M. B. Zvonarev. Long-wave asymptotics of correlation functions of the third component of spins in XX0 model with quartic interaction. *Zap. Nauch Semin. POMI*, 268(16):207, 2000.

[20] J. Kempe and B. Whaley. Exact gate-sequences for universal quantum computation using the XY-interaction alone. *quant-ph/0112014*, 2001.

[21] A. Kitaev. *quant-ph/9707021*, 1997.

[22] E. Knill and R. Laflamme. A theory of quantum error-correcting codes. *Phys. Rev. A.*, 55:900, 1997.

[23] V. E. Korepin, N. M. Boguliubov, and A. G. Izergin. *Quantum Inverse Scattering Method and Correlation Functions*. Cambridge Univ. Press, 1993.

[24] V. E. Korepin and A. C. T. Wu. Adiabatic transport properties and Berry’s phase in Heisenberg-Ising ring. *Int. J. of Mod. Phys.*, 5(3):497, 1991.

[25] D. A. Lidar and L.-A. Wu. Adiabatic transport properties and Berry’s phase in Heisenberg-Ising ring. *Int. J. of Mod. Phys.*, 5(3):497, 1991.

[26] E. Lieb, T. Schultz, and D. Mattis. Two soluble models of an antiferromagnetic chain. *Annals of Physics*, 16:461, 1961.

[27] A. Osterloh, L. Amico, G. Falci, and R. Fazio. Scaling of entanglement close to a quantum phase transition. *Nature*, 416:608, April 2002.

[28] S. Popescu and D. Rohrlich. Thermodynamics and the measure of entanglement. *Phys. Rev. A*, 56:R3319, 1997.

[29] J. Preskill. Lectures notes. *http://www.theory.caltech.edu/people/preskill/ph219/*, 1997-1999.

[30] J. Preskill. Reliable quantum computers. *Proc. Roy. Soc. Lond.*, A454:385–410, 1998.

[31] D. Rohrlich. Thermodynamical analogues in quantum information theory. *quant-ph/0107026*, 2001.

[32] B. Schumacher. A mathematical theory of communication. *Bell Syst. Tech. J.*, 27:379, 1948.

[33] P. W. Shor. Schemes for reducing decoherence in quantum computer memory. *Phys. Rev. A*, 52:2493–2496, 1995.

[34] A. M. Steane. Error correcting codes in quantum theory. *Phys. Rev. Lett.*, 77(5):793–797, 1996a.

[35] A. M. Steane. Quantum computing. *Rept. Prog. Phys.*, 61:117–173, 1998.

[36] M. Takahashi. *Thermodynamics of One-Dimensional Solvable Models*. Cambridge Univ. Press, 1999.

[37] B. Terhal and D. P. DiVincenzo. Classical simulation of noninteracting-fermion quantum circuits. *quant-ph/0108010*, 2001.

[38] L.-A. Wu and D. A. Lidar. Power of anisotropic exchange interactions: Universality and efficient codes for quantum computing. *Phys. Rev. A.*, 65:042318, 2002.