EXTINCTION AND UNIFORM STRONG PERSISTENCE OF A SIZE-STRUCTURED POPULATION MODEL

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Dedicated to Steve Cantrell in honor of his 60th birthday

Abstract. In this paper, we study the long-time behavior of a size-structured population model. We define a basic reproduction number $R$ and show that the population dies out in the long run if $R < 1$. If $R > 1$, the model has a unique positive equilibrium, and the total population is uniformly strongly persistent. Most importantly, we show that there exists a subsequence of the total population converging to the positive equilibrium.

1. Introduction. In this paper, we consider the following size-structured population model

$$
\begin{cases}
    u_t + (g(x)u)_x + m(x, P(t))u = 0, & x \in (0, l), \ t \in (0, \infty), \\
    g(0)u(0, t) = \int_0^l \beta(x, P(t))u(x, t)dx, & t \in (0, \infty), \\
    u(x, 0) = u_0(x), & x \in [0, l].
\end{cases}
$$

(1.1)

Here, the function $u(x, t)$ represents the density of individuals of size $x$ at time $t$, and $P(t) = \int_0^l u(x, t)dx$ is the total population at time $t$. The parameter $g(x)$ is the size-dependent growth rate, and $m(x, P)$ and $\beta(x, P)$ are the mortality rate and the reproduction rate of an individual of size $x$, respectively, which depend on the total population. The function $u_0(x)$ denotes the initial population density. Moreover, the following assumptions on the parameters in model (1.1) are imposed:

(H1) $g \in C^1[0, l]$, $g > 0$ on $[0, l)$ and $g(l) = 0$;

(H2) $m, \beta \in C([0, l] \times [0, \infty))$, $m, \beta \geq 0$, and $\frac{\partial m}{\partial P} \geq 0$, $\frac{\partial \beta}{\partial P} \leq 0$ with $\left(\frac{\partial m}{\partial P}\right)^2 + \left(\frac{\partial \beta}{\partial P}\right)^2 \neq 0$;

(H3) $u_0 \in C[0, l]$, and $u_0 \geq 0$ is nontrivial.

Linear and nonlinear problems similar to (1.1) have been discussed extensively in the literature (e.g., see [13]). Many works focus on the well-posedness of the models. And certain articles focus on qualitative behavior of the models. For instance, in [5] it is shown that for a nonlinear model, there exists a compact global attractor, and if the total population $P(t)$ converges to a steady state $P^*$, then the solution $u(x, t)$ converges to a corresponding steady state solution $u^*(x)$ in $L^1(0, l)$. Later, stability

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of the steady state solution of nonlinear models has been studied in [6, 8, 9]. Very recently, sensitivity analysis for linear and nonlinear versions of model (1.1) has been conducted in [2] and [4], respectively.

Our main objective here is to establish extinction and persistence results for model (1.1) with a basic reproduction number \( \mathcal{R} \) serving as a threshold value. Our method to study the persistence of the solution bears a resemblance to that in [10, 11], where the authors considered certain age-structured population models. However, we improve our persistence result by proving that there is a subsequence of the total population converging to the positive equilibrium (this yields the uniform weak persistence and hence implies the uniform strong persistence of the total population). The paper is organized as follows. In Section 2, we introduce the definition of upper and lower solutions of problem (1.1) and present a comparison principle and revisit the existence of solutions of (1.1). In Section 3, we introduce a basic reproduction number \( \mathcal{R} \) and show the uniform convergence of the solution to the trivial steady state if \( \mathcal{R} < 1 \). In Section 4, we establish a uniform strong persistence result for (1.1) if \( \mathcal{R} > 1 \).

2. **Comparison principle and existence of solutions.** The existence of solutions of (1.1) has been established using the characteristic method with fixed point argument or the semigroup of linear operators theoretic approach (e.g., [3, 4, 5]). However, in order to study the long-time behavior of the solution, especially the extinction of the population in \( L^\infty \) norm, we introduce the upper/lower solution method based on the comparison principle and revisits the existence of solutions of (1.1) here.

**Definition 2.1.** A pair of nonnegative functions \( \bar{u}(x,t) \) and \( u(x,t) \) are called an upper solution and a lower solution of (1.1) in \((0,l)\times(0,T)\), respectively, if all the following hold:

1. \( \bar{u} \) and \( u \) are bounded measurable functions, defined everywhere on \([0,l] \times [0,T]\) with \( \bar{u}(\cdot,t) \) and \( u(\cdot,t) \) integrable;
2. \( \bar{u}(x,0) \geq u_0(x) \geq u(x,0) \) a.e. in \((0,l)\);
3. For each \( t \in (0,T) \) and every nonnegative \( \psi \in C^1((0,l) \times (0,T)) \),

\[
\begin{align*}
\int_0^l \bar{u}(x,t)\psi(x,t)dx &\geq \int_0^l \bar{u}(x,0)\psi(x,0)dx + \int_0^t \int_0^l \beta(x,P(s))\bar{u}(x,s)\psi(x,s)dxd\psi + \int_0^t \int_0^l m(x,P(s))\bar{u}(x,s)\psi(x,s)dxd\psi \\
&\quad + \int_0^t \int_0^l (\psi_s(x,s) + g(x)\psi_x(x,s))\bar{u}(x,s)dxd\psi,
\end{align*}
\tag{2.1}
\]

\[
\begin{align*}
\int_0^l u(x,t)\psi(x,t)dx &\leq \int_0^l u(x,0)\psi(x,0)dx + \int_0^t \int_0^l \beta(x,P(s))u(x,s)\psi(0,s)dxd\psi + \int_0^t \int_0^l m(x,P(s))u(x,s)\psi(x,s)dxd\psi \\
&\quad - \int_0^t \int_0^l (\psi_s(x,s) + g(x)\psi_x(x,s))u(x,s)dxd\psi,
\end{align*}
\tag{2.2}
\]

where

\[
P(t) = \int_0^l \bar{u}(x,t)dx \quad \text{and} \quad P(t) = \int_0^l u(x,t)dx.
\]

A function \( u(x,t) \) is called a solution of (1.1) in \((0,l) \times (0,T)\) if \( u \) satisfies (2.1) with \( \geq \) replaced by \( = \) and \( P(s) \) by \( P(s) \).

Based on such a definition, the following comparison principle and existence result can be established using similar techniques as those in [1].
Theorem 2.2. Suppose that $\overline{u}(x,t)$ and $\underline{u}(x,t)$ are a pair of upper and lower solutions of (1.1) in $(0, l) \times (0, T)$. Then $\overline{u}(x,t) \geq \underline{u}(x,t)$ a.e. in $(0, l) \times (0, T)$. Moreover, there is a unique solution $u(x,t)$ of (1.1) such that
\[ \overline{u}(x,t) \geq u(x,t) \geq \underline{u}(x,t) \quad \text{in} \quad (0, l) \times (0, T). \]

It is easy to check that the functions $\overline{u}(x,t) = M e^{rt} e^{-qx}$ and $\underline{u}(x,t) = 0$ are a pair of upper and lower solutions of (1.1), where $M, r$ and $q$ are positive constants appropriately chosen, and thus we have the following global existence result.

Corollary 2.3. Problem (1.1) has a unique solution in $(0, l) \times (0, \infty)$.

We then introduce the solution representation for problem (1.1) via the method of characteristics. For the equation in (1.1), the characteristic curves are given by
\[ \begin{cases} \frac{d}{ds} s(t) = 1, \\ \frac{d}{ds} x(s) = g(x(s)). \end{cases} \tag{2.3} \]

Under the assumption (H1), equation (2.3) has a unique solution for any initial point $(x(s_0), t(s_0))$. Parameterizing the characteristic curves with the variable $t$, then a characteristic curve passing through $(\hat{x}, \hat{t})$ is given by $(X(t; \hat{x}, \hat{t}), t)$, where $X$ satisfies
\[ \frac{d}{dt} X(t; \hat{x}, \hat{t}) = g(X(t; \hat{x}, \hat{t})) \]
and $X(\hat{t}; \hat{x}, \hat{t}) = \hat{x}$. By (H1) the function $X$ is strictly increasing, and therefore a unique inverse function $\Gamma(x; \hat{x}, \hat{t})$ exists. Let $z(t) = X(t; 0, 0)$, then $(z(t), t)$ represents the characteristic curve passing through $(0, 0)$, and this curve divides the $(x, t)$-plane into two parts. Proceeding analogously as in [1], one can see that the solution of (1.1) can be represented as follows:
\[ u(x, t) = \begin{cases} \frac{B(t) e^{-\int_0^t m(X(s;0,\tau), P(s)) d\tau}}{\int_0^t e^{-\int_0^{\tau} m(X(s;0,\tau), P(s)) d\tau} d\tau} u_0(X(0; x, t)) e^{-\int_0^t \int_0^\xi m(x, s, \xi, 0, P(s)) ds d\xi}, & \text{if} \ x < z(t), \\ \int_0^t e^{-\int_0^{\tau} m(X(s;0,\tau), P(s)) d\tau} d\tau \int_0^t \int_{0}^{\xi} e^{-\int_0^{\tau} m(X(s;0,\tau), P(s)) d\tau} ds d\xi, & \text{if} \ x \geq z(t), \end{cases} \tag{2.4} \]
where $B(t)$ is given by
\[ B(t) = \int_0^t \beta(x, P(t)) u(x, t) dx, \tag{2.5} \]
the inflow of newborns in the population at time $t$.

Integrating (2.4) with respect to $x$, we obtain an integral equation for $P(t)$:
\[ P(t) = \int_0^t B(\tau) e^{-\int_0^\tau m(X(s;0,\tau), P(s)) ds d\tau} d\tau + \int_0^t u_0(\xi) e^{-\int_0^\tau m(X(s;0,\tau), P(s)) ds d\tau} d\xi, \tag{2.6} \]
where we have changed the variable $x$ in the first integral on the right-hand side by $\tau = \Gamma(0; x, t)$ and in the second integral by $\xi = X(0; x, t)$.

Substituting (2.6) into (2.4) and integrating with respect to $x$, we also obtain an integral equation for $B(t)$:
\[ B(t) = \int_0^t \int_0^\xi e^{-\int_0^\tau m(X(s;0,\tau), P(s)) ds d\tau} B(\tau) e^{-\int_0^\tau m(X(s;0,\tau), P(s)) ds d\tau} d\tau + \int_0^t \int_0^\xi \beta(X(t; \xi, 0), P(t)) u_0(\xi) e^{-\int_0^\tau m(X(s;0,\tau), P(s)) ds d\tau} d\xi. \tag{2.7} \]
3. Basic reproduction number, equilibrium, and population extinction.

An equilibrium of problem (1.1) is a solution of the following problem:

\[ g(x)u'(x) = -(g'(x) + m(x, P))u(x), \]
\[ g(0)u(0) = \int_0^l \beta(x, P)u(x)dx, \]
\[ P = \int_0^l u(x)dx. \]

By (3.1), one has
\[ u(x) = u(0)\pi(x, P), \]
where
\[ \pi(x, P) = e^{-\int_0^x g'(y)+m(y, P)\frac{dy}{g(y)}} \frac{g(0)}{g(x)}e^{-\int_0^x m(y, P)\frac{dy}{g(y)}}. \]

Then by (3.2), either
\[ u(0) = 0 \]
or
\[ \int_0^l \beta(x, P)\pi(x, P)dx = 1. \]

Denote the left-hand side of (3.4) by \( R(P) \). Then \( R \) is a decreasing function in \( P \) by the monotonicity assumptions on \( m \) and \( \beta \). Define the basic reproduction number of (1.1) as
\[ R = R(0), \]
where
\[ R = \int_0^l \frac{\beta(x, 0)}{g(0)}\pi(x, P)dx. \]

If \( R \leq 1 \), then (3.4) has no positive solution and so the unique nonnegative equilibrium of (1.1) is the trivial equilibrium \( u \equiv 0 \). If \( R > 1 \), we assume (H4) \( \lim_{P \to \infty} R(P) < 1 \), and then (3.4) has a unique positive solution \( P^* \). It then follows from \( u(x) = u(0)\pi(x, P) \) that \( u(0) = P^*/\int_0^l \pi(x, P^*)dx \), and so the positive equilibrium of (1.1) is
\[ u^*(x) = \frac{\pi(x, P^*)}{\int_0^l \pi(x, P^*)dx}P^*. \]

We then impose an additional assumption as follows:

(H5) There exist a positive number \( \mu \) and \( x_0 \in [0, l] \) such that
\[ \min_{[x_0,l]} m(x, 0) \geq \mu. \]

Applying the comparison principle, we now show that the population dies out if \( R < 1 \).

**Theorem 3.1.** Suppose that (H1)-(H3) and (H5) hold. If \( R < 1 \), then the solution \( u(x, t) \) of (1.1) converges to zero uniformly as \( t \to \infty \).
which holds if we let
\[
K(x) = e^{-\int_0^x \frac{g(y)}{g(x)} \, dy} = \frac{g(0)}{g(x)} e^{-\int_0^x \frac{m(y,y) - r}{g(y)} \, dy}.
\]
The second inequality in (3.5) is equivalent to the following
\[
1 \geq \int_0^l \frac{\beta(x,0)}{g(x)} e^{-\int_0^x \frac{m(y,y) - r}{g(y)} \, dy} \, dx,
\]
which holds if \( r \) is small enough by \( R < 1 \) and (H5). Finally, the third inequality in (3.5) holds if \( M \) is large. Hence, \( \tilde{u} \) and \( u \) are a pair of upper and lower solution of (1.1), respectively. It then follows from \( \lim_{t \to \infty} u(x,t) = 0 \) and Theorem 2.2 that
\[
\lim_{t \to \infty} u(x,t) = 0 \text{ uniformly in } x.
\]

4. Population persistence. In this section, we prove that the population persists if \( R > 1 \). To this end, we need one more assumption on \( \beta \) and \( m \):
\[ (H6) \quad \beta \in C^1([0,l] \times [0,\infty)). \]
Moreover, there exist two positive numbers \( P_0 \) and \( N \) such that
\[
\sup_{(x,P) \in [0,l] \times [P_0,\infty)} \{ \beta(x,P) - m(x,P) \} \leq -N.
\]

**Remark 1.** We note that (H6) implies (H4). To see this, suppose that (H6) holds. Then for \( P > P_0 \), we have
\[
R(P) = \int_0^l \frac{\beta(x,P)}{g(x)} e^{-\int_0^x \frac{m(y,y) - r}{g(y)} \, dy} \, dx
\]
\[
< \int_0^l \frac{m(x,P)}{g(x)} e^{-\int_0^x \frac{m(y,y) - r}{g(y)} \, dy} \, dx
\]
\[
= -e^{-\int_0^l \frac{m(y,y) - r}{g(y)} \, dy} \bigg|_0^l = 1 - e^{-\int_0^l \frac{m(y,y) - r}{g(y)} \, dy} \leq 1.
\]
Hence, (H4) is valid.

By [5], the semiflow induced by (1.1) has a compact global attractor in \( L^1(0,l) \), if (H1)-(H3) and (H5)-(H6) hold.

To prove the persistence of the population, we will focus on the equation (2.7). We first show that this equation is a Volterra integral equation of the second type if \( P(t) \) is a constant. To see this, for some fixed positive constant \( \tilde{P} \), we define
\[
f_{\tilde{P}}(t,\tau) = \beta(X(t;0,\tau), \tilde{P}) e^{-\int_0^t m(X(s;0,\tau), \tilde{P}) \, ds}, \quad t \geq \tau \geq 0
\]
and
\[
h_{\tilde{P}}(t) = \int_0^l \beta(X(t;\xi,0), \tilde{P}) u_0(\xi) e^{-\int_0^t m(X(s;\xi,0), \tilde{P}) \, ds} \, d\xi. \quad (4.1)
\]
We then prove the following result.

**Lemma 4.1.** For any fixed positive constant \( \tilde{P} \), we have that \( f_{\tilde{P}}(t,\tau) = \tilde{f}_{\tilde{P}}(t - \tau) \) for some function \( \tilde{f}_{\tilde{P}} \).

**Proof.** Since the right-hand side of the characteristic equation depends only on \( x \), it is an autonomous equation, and so we have
\[
X(t;0,\tau) = X(t - \tau;0,0) = z(t - \tau).
\]
Then we have
\[
\begin{align*}
\hat{f}_p(t, \tau) &= \beta(z(t - \tau), \hat{P}) e^{-\int_0^t m(z(s - \tau), \hat{P}) ds} \\
&= \beta(z(t - \tau), \hat{P}) e^{-\int_0^{\tau} m(z(s), \hat{P}) ds},
\end{align*}
\]
which is a function of \(t - \tau\).

By the lemma, if we set \(P(t) = \hat{P}\) in (2.7), it becomes
\[
B(t) = \int_0^t \hat{f}_p(t - \tau) B(\tau) d\tau + h_p(t),
\]
which is a Volterra integral equation of the second type.

The following lemma is important for us to study the asymptotic behavior of the solution of (1.1).

**Lemma 4.2.** Suppose that (H1)-(H3) and (H5)-(H6) hold. Then there exists a positive constant \(C\) such that
\[
\hat{f}_p(t) \leq Ce^{-\mu t} \quad \text{and} \quad h_p(t) \leq Ce^{-\mu t}.
\]

**Proof.** Since \(z(s) \to l\) as \(s \to \infty\) by (H1), there exists \(s_0 > 0\) such that \(z(s) \geq x_0\) for all \(s \geq s_0\), where \(x_0\) is specified in (H5). Hence, we have
\[
\begin{align*}
\hat{f}_p(t) &= \beta(z(t), \hat{P}) e^{-\int_0^t m(z(s), \hat{P}) ds} e^{-\int_0^t m(z(s), \hat{P}) ds} \\
&\leq \max_{x \in [0, l]} \beta(x, 0) e^{-\int_0^t m(z(s), \hat{P}) ds} e^{-\int_0^t \mu ds} \\
&\leq \left( \max_{x \in [0, l]} \beta(x, 0) e^{\mu s_0} \right) e^{-\mu t}.
\end{align*}
\]

On the other hand, since \(X(s; \xi, 0) \geq z(s)\) for all \(s \geq 0\) and \(\xi \in [0, l]\), \(X(s; \xi, 0) \geq x_0\) for all \(s \geq s_0\). Therefore, we have
\[
\begin{align*}
h_p(t) &= \int_0^t \beta(X(t; \xi, 0), \hat{P}) u_0(\xi) e^{-\int_0^t m(X(s; \xi, 0), \hat{P}) ds} e^{-\int_0^t m(X(s; \xi, 0), \hat{P}) ds} d\xi \\
&\leq \max_{x \in [0, l]} \beta(x, 0) \int_0^l u_0(\xi) e^{-\int_0^t \mu ds} d\xi \\
&\leq \left( \max_{x \in [0, l]} \beta(x, 0) \int_0^l u(\xi) d\xi e^{\mu s_0} \right) e^{-\mu t}.
\end{align*}
\]

We now prove the following result.

**Lemma 4.3.** Suppose that \(f\) and \(h\) are bounded, continuous nonnegative functions on \(\mathbb{R}^+\) such that they are positive in a subset of positive measure and satisfy
\[
f(t) \leq ce^{-\alpha t} \quad \text{and} \quad h(t) \leq ce^{-\alpha t}
\]
for some positive constants \(\alpha\) and \(c\).

(a) Suppose that \(\int_0^\infty f(t) dt < 1\), then the solution \(y(t)\) of the following equation
\[
y(t) = \int_0^t f(t - s) y(s) ds + h(t) \quad (4.2)
\]
satisfies
\[
y(t) \leq y_0 e^{-\alpha_1 t}
\]
for some \(y_0 > 0\) and \(\alpha_1 \in (0, \alpha]\).
(b) Suppose that \( \int_0^\infty e^{\alpha_2 t} f(t) dt < 1 \) for some \( \alpha_2 \in (0, \alpha) \), then the solution \( y(t) \) of (4.2) satisfies
\[
y(t) \leq y_0 e^{-\alpha_2 t}
\]
for some \( y_0 > 0 \).

Proof. The assumptions on \( f \) imply that \( f \) is absolutely Laplace transformable in the half plane \( \mathcal{R} \lambda > -\alpha \). Define a real valued function \( F(x) = \int_0^\infty e^{-\alpha t} f(t) dt \).

Since \( f \) is positive in a nonzero measurable set, \( F \) is a strictly decreasing function. Moreover, \( F \) is continuous on \((-\alpha, \infty)\). To see this, for any \( [a_1, a_2] \subseteq (-\alpha, \infty) \), we have \( e^{-\alpha t} f(t) \leq c_0 e^{-\alpha_1 t} \) for some \( c_0 > 0 \), and so \( F(x) \) is absolutely convergent on \([a_1, a_2]\). Thus, \( F \) is continuous on \([a_1, a_2]\). Since \( [a_1, a_2] \subseteq (-\alpha, \infty) \) is arbitrary, \( F \) is continuous on \((-\alpha, \infty)\).

We now prove part (a). Since \( \int_0^\infty f(t) dt < 1 \) and \( F \) is continuous on \((-\alpha, \infty)\), there exists \( \alpha_1 \in (0, \alpha) \) such that \( F(-\alpha_1) < 1 \). Let \( R(t) \) be the resolvent kernel of (4.2) (e.g., see [12]), that is,
\[
R(t) = -\sum_{n=1}^\infty (f * f * n \times \cdots * f)(t),
\]
where * denotes the convolution. To obtain a bound on \( R(t) \), we have that
\[
f * f(t) = \int_0^t f(t-s) f(s) ds = e^{-\alpha_1 t} \int_0^t f(t-s) e^{\alpha_1 (t-s)} f(s) ds \leq c_1 e^{-\alpha_1 t},
\]
where \( c_1 = F(-\alpha_1) < 1 \), and
\[
f * f * f(t) \leq \int_0^t c_1 e^{-\alpha_1 (t-s)} f(s) ds \leq c_2 e^{-\alpha_1 t}.
\]
By induction, we have
\[
R(t) \geq -\sum_{n=1}^\infty c_1^{n-1} e^{-\alpha_1 t} \geq -\frac{c}{1-c_1} e^{-\alpha_1 t}.
\]
It then follows that
\[
y(t) = h(t) - \int_0^t R(t-s) h(s) ds \leq c e^{-\alpha_1 t} + \int_0^t \frac{c}{1-c_1} e^{-\alpha_1 (t-s)} e^{-\alpha s} ds \leq y_0 e^{-\alpha_1 t}
\]
for some \( y_0 > 0 \).

The proof of part (b) is similar to that of part (a), and hence is omitted. \( \square \)

Let \( Lf \) denote the Laplace transform of \( f \). Considering the equation
\[
Lf(\lambda) = 1, \tag{4.3}
\]
we present the following result which is similar to Theorem 5.1 of [12].

Lemma 4.4. Suppose that \( f \) is a continuous nonnegative function on \( \mathbb{R}^+ \) such that it is positive in a subset of positive measure and satisfies
\[
\limsup_{t \to \infty} f(t) e^{\alpha t} \leq c
\]
for some positive constants \( \alpha \) and \( c \). Suppose that \( \int_0^\infty e^{\alpha_0 t} f(t) dt \geq 1 \) for some \( x_0 \in [0, \alpha) \), then there exists a unique real root \( \alpha^* \) of (4.3). Moreover, \( \alpha^* > -\alpha \) is
simple, and any other root $\alpha$ of (4.3) satisfies that $\Re \alpha < \alpha^*$, where $\alpha^* < 0$ (or $> 0$) if and only if $\int_0^\infty f(t)dt < 1$ (or $> 1$). Furthermore, within any strip $\sigma_1 < \Re \lambda < \sigma_2$, there are at most a finite number of roots of (4.3).

Proof. The proof is essentially the same as that for Theorem 5.1 of [12]. The assumptions on $f$ imply that $f$ is absolutely Laplace transformable in the half plane $\Re \lambda > -\alpha$. Define a real valued function $F(x) = \int_0^\infty e^{-xt}f(t)dt$. Since $f$ is positive in a nonzero measurable set, $F$ is a strictly decreasing function with $F(-x_0) \geq 1$ and $F(\infty) = 0$. Hence, there exists a unique real number $\alpha^*$ with $\alpha^* > -\alpha$ such that $F(\alpha^*) = 1$. The rest of the proof is the same as that for Theorem 5.1 of [12].

We now present a result on the asymptotic behavior of the Volterra integral equation of the second type.

Lemma 4.5. Suppose that $f$ and $h$ are continuous nonnegative functions on $\mathbb{R}^+$ such that they are positive in a subset of positive measure and satisfy

$$\limsup_{t \to \infty} f(t)e^{\alpha t} \leq c$$

and

$$\limsup_{t \to \infty} h(t)e^{\alpha t} \leq c$$

for some positive constants $\alpha$ and $c$. Suppose that $\int_0^\infty e^{x_0 t}f(t)dt \geq 1$ for some $x_0 \in [0, \alpha)$. Let $\alpha^*$ be given as in Lemma 4.4. Then the solution $y(t)$ of the following equation

$$y(t) = \int_0^t f(t-s)y(s)ds + h(t)$$

satisfies

$$y(t) = y_0 e^{\alpha^* t}(1 + \Omega(t)),$$

where

$$y_0 > 0 \quad \text{and} \quad \lim_{t \to \infty} \Omega(t) = 0.$$

Proof. The assumptions on $f$ and $h$ imply that they are absolutely Laplace transformable in the half plane $\Re \lambda > -\alpha$. Then the rest of the proof is essentially the same as that for Theorem 5.2 of [12] except that we work in the half plane $\Re \lambda > -\alpha$ here.

Remark 2. If $\int_0^\infty f(t)dt \geq 1$, then we have $\alpha^* \geq 0$.

By the result of [5], if $P(t) \to P^*$, then $u(x, t) \to u^*(x)$ in $L^1$ as $t \to \infty$. We now prove that there is a subsequence of $P(t)$ converging to $P^*$ and the total population is uniformly strongly persistent.

Theorem 4.6. Suppose that (H1)-(H3) and (H5)-(H6) hold and $\Re > 1$. Let $u(x, t)$ be the solution of (1.1). Then there exists a sequence $\{t_n\}_{n=1}^\infty$ with $t_n \to \infty$ such that

$$\lim_{n \to \infty} P(t_n) = \lim_{n \to \infty} \int_0^1 u(x, t_n)dx = P^*.$$

Moreover, (1.1) is uniformly strongly persistent in the sense that there exists a $\epsilon > 0$ independent of the initial data such that

$$\liminf_{t \to \infty} P(t) \geq \epsilon.$$
Proof. Assume to the contrary that the sequence satisfying the above conditions does not exist. Then there exists either a $\hat{P} > P^*$ and $t_1 > 0$ such that $P(t) \geq \hat{P}$ for all $t > t_1$ or a positive $\tilde{P} < P^*$ and $t_2 > 0$ such that $P(t) \leq \tilde{P}$ for all $t > t_2$.

We first consider the case that there exists $\hat{P} > P^*$. Without loss of generality, we may assume that $t_1 = 0$. Then noticing (H2) and $P(t) \geq \hat{P}$, we have

$$B(t) \leq \int_0^t \dot{\hat{P}}(t - \tau) B(\tau) d\tau + h_{\hat{P}}(t).$$

Then by the comparison principle for Volterra equations, we find that $B(t) \leq y(t)$, where $y(t)$ satisfies

$$y(t) = \int_0^t \dot{\hat{P}}(t - \tau) y(\tau) d\tau + h_{\hat{P}}(t).$$

Note that

$$1 = \int_0^t \frac{\beta(x, P^*)}{g(x)} e^{-\int_0^t \frac{m(s, P^*)}{g(s)} ds} dx = \int_0^\infty \beta(z(t), P^*) e^{-\int_0^t \frac{m(z(s), P^*)}{g(s)} ds} dt,$$

Since $\hat{P} > P^*$, we have

$$\int_0^\infty \dot{\hat{P}}(t) dt < 1.$$

Then by Lemma 4.3, we have that $y(t) \to 0$ as $t \to \infty$. Hence $B(t) \to 0$, which is a contradiction. We then consider the case that there exists a positive $\tilde{P} < P^*$. Without loss of generality, we may also assume that $t_2 = 0$. We find that

$$B(t) \geq \int_0^t \dot{\tilde{P}}(t - \tau) B(\tau) d\tau + h_{\tilde{P}}(t).$$

Then by the comparison principle for Volterra equations, we have that $B(t) \geq \tilde{y}(t)$, where $\tilde{y}(t)$ satisfies

$$\tilde{y}(t) = \int_0^t \dot{\tilde{P}}(t - \tau) \tilde{y}(\tau) d\tau + h_{\tilde{P}}(t).$$

Note that $\tilde{y}(0) = h_{\tilde{P}}(0) > 0$ by (H3). It then follows from $\tilde{P} < P^*$ that

$$\int_0^\infty \dot{\tilde{P}}(t) dt > 1,$$

which implies that $\alpha^* > 0$. Hence, by Lemmas 4.2 and 4.5, we have that $\tilde{y}(t) \to \infty$ as $t \to \infty$, and consequently $B(t) \to \infty$, which is a contradiction.

From the above discussion, it follows that there exists a sequence $\{t_n\}$ such that $\lim_{n \to \infty} P(t_n) = P^*$, which implies the uniform weak persistence of $P(t)$. It is well known that uniform weak persistence implies uniform strong persistence (e.g., see [14]), and so the proof is completed.

For the reader’s convenience, we include certain details of using the well-known result that the uniform weak persistence implies the uniform strong persistence. To use the abstract result in [14], we adopt the definition of a solution in [5] (which has weaker requirements than those assumed here), and let $Y = \{u \in L^1(0, l) : u \geq 0\}$. Then $Y = Y_0 \cup \partial Y_0$ with $\partial Y_0 = \{0\}$ and $Y_0 = Y/\partial Y_0$. By [5], the solution of (1.1) generates a continuous semiflow $\mathcal{T} : [0, \infty) \times Y \to Y$, which has a compact global attractor $A$. Let $E = A \cap Y_0$, which is relatively closed in $Y_0$. It is not hard to see that $Y_0$ is invariant with respect to $\mathcal{T}$, and so we consider the restriction of $\mathcal{T}$ onto $Y_0$, that is, $\Psi : [0, \infty) \times Y_0 \to Y_0$. Define a continuous, strictly positive functional $\rho : Y \to (0, \infty)$ by $\rho(u) = \int_0^l u(x) dx$ for all $u \in Y_0$. Since $A$ is the compact global
attractor of $T$ on $Y$, $E$ satisfies the property (CA) in [14]. Moreover, we have shown that $\lim\sup_{t\to\infty} \rho(u(\cdot, t)) = \lim\sup_{t\to\infty} P(t) \geq P^*$ (this holds for the initial data $u_0 \in Y_0$), i.e., $\Psi$ is uniformly weakly $\rho$–persistent. Hence, Theorem 2.2 of [14] implies that $\Psi$ is uniformly strongly $\rho$–persistent (note that the semiflow $\Psi$ is autonomous, and so the assumptions on $\sigma$ in Theorem 2.2 of [14] are not needed here), that is, $\lim\inf_{t\to\infty} P(t) \geq \epsilon$ for some $\epsilon > 0$.

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