Nonlinearity of the field induced by a rotating superconducting shell

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For a thin superconducting shell with cylindrical symmetry, the magnetic field generated by its rotation is easily evaluated in the Ginzburg–Landau framework. We compare this field with the result that is obtained by using the London theory.

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I. INTRODUCTION

Almost a century ago Barnett 1 found that “any magnetic substance becomes magnetized when set into rotation” “by a sort of molecular gyroscopic action.” In 1933 Becker et al. 2 found that the magnetic field induced by a “perfect conductor” is

\[ B = -(2m^* c^e) \omega, \tag{1} \]

where \( m^* \) and \( e^* \) are the mass and charge of a free charge carrier, \( c \) is the speed of light and \( \omega \) is the angular velocity at which the body rotates. This is precisely the field that according to Barnett would be generated by a “perfectly diamagnetic” material. Since all the vectors in this article will be directed along the axis of rotation, we will disregard their vectorial character.

The “perfect conductor” presents a conceptual difficulty, since the currents in it depend on the initial conditions. On the other hand, for a superconductor, a thermodynamically significant result can be obtained. Using the London theory 3 and in the absence of an external field, Eq. (1) is recovered. \( m^* \) and \( e^* = -2e \) are now the mass and charge of a Cooper pair. The rotating superconductor was analyzed in the Ginzburg–Landau (GL) framework by Verkin and Kulik 4 in this kind of analysis has been recently revised by Capellmann 5.

Equation (1) has been verified by several experiments 6,7,8,9,10,11 and they have actually become a means for measuring the effective mass of a Cooper pair. The possibility of measuring the charge to mass ratio from the magnetic field generated by a rotating superconductor has been regarded as an example that stands on equal footing with “quantum protection” and symmetries, i.e., a case in which the result is insensitive to microscopics. 12 Denoting by \( m \) the mass of an electron, the most precise result obtained \(^{12} \) to my knowledge is \( m^*/2m = 1.000084(21) \). The relationship between the effective and the bare mass of the charge carriers has been analyzed from the microscopic 13,14,15,16 and the thermodynamic 17,18 points of view, and none of the results is in full agreement with experiment.

The possibility of an electric field generated by a rotating \(^{19} \) or even static \(^{20} \) superconductor has also been considered.

In spite of the fundamental character that has been attributed to Eq. (1), this article raises the point that it is just an approximation, valid for low angular velocities. For high rotation speeds \( B \) will not be a linear—and not even a monotonic—function of \( \omega \). This will be shown in the following section, by considering a superconducting sample with a shape such that the GL theory is very easily applied.

II. OUR MODEL

We consider a long superconducting cylindrical shell with radius \( R \) and thickness \( d \) \((d \ll R)\) that rotates around its axis with angular velocity \( \omega \). The entire analysis will be conducted from an inertial frame of reference. Let \( B_0 \) (resp. \( B_1 \)) denote the component of the magnetic field in the direction of the axis outside (resp. inside) the cylinder. \( (B_0 \text{ and } B_1 \text{ are assumed to be uniform.}) \) Let \( N \) be the number of superconducting pairs per unit area and \( v' \) their velocity relative to the ions of the shell. It follows that the current per unit length is \( Ne^*v' \) and, by Ampère’s law, the inner and outer fields are related by

\[ B_1 = B_0 + \frac{4\pi}{c} Ne^*v'. \tag{2} \]

Quantization of the canonical momentum requires

\[ 2\pi R m^* v + \frac{e^*}{c} \pi R^2 B_1 = Lh \tag{3} \]

where \( v \) is the velocity of the pairs relative to the laboratory, \( L \) is an integer that determines the trapped flux and \( h \) is Planck’s constant. \( v \) and \( v' \) are related by

\[ v = v' + \omega R \tag{4} \]

Solving the system of equations (2–4) we obtain

\[ B_1 - B_0 = \frac{2m^* e^* (\omega_\Phi - \omega)}{e^* (1 + \gamma)} \tag{5} \]

\[ v = R (\gamma_\omega + \omega_\Phi)/(1 + \gamma) \tag{6} \]

where we have defined

\[ \gamma = \frac{2\pi (e^*)^2 RN}{m^* c^2}; \quad \omega_\Phi = \frac{Lh}{2\pi R^2 m^* - \frac{e^* B_0}{2m^* c}} \tag{7} \]

Expressions analogous to (5) have been found in Refs. 1, 21.
If \( \gamma \gg 1 \) (for \( R \sim 1\text{cm} \) this means \( N \gg 10^{12}\text{cm}^{-2} \)), Eq. (1) reduces to
\[
B_1 - B_o = -\frac{2m^*c}{e^*} (\omega - \omega_\phi)
\]
which for \( L = 0 \) and in the absence of external field is just Becker’s result.

The condition \( \gamma \gg 1 \) is easily fulfilled, and this seems to be the reason that Eq. (1) is so widely accepted. In London’s theory, \( N \) is effectively a constant. Even in Ref. 3, where \( N \) could in principle be obtained from the GL theory, in practice \( B_1 \) was evaluated only in the case that “stiffness of the wave function” can be assumed. However, if the shell is very thin and the temperature close to critical, \( N \) can be noticeably dependent on \( \omega \) and this dependence will be the source of nonlinearity of \( B_1(\omega) \).

The density of pairs \( N \) is obtained by minimizing the free energy. If the thickness \( d \) of the shell is small compared with the coherence length \( \xi \), the free energy per unit length is
\[
G = 2\pi R N \left( \frac{1}{2} m^* v^2 + \alpha + \frac{\beta}{2}\right) + \frac{R^2}{8} (B_1 - B_o)^2
- \pi R (N_T - 2N) m (\omega R)^2
\]
where \( \alpha \) and \( \beta \) are the GL coefficients and \( N_T \) is the total number of electrons per unit area of the shell. The first term consists of the kinetic and the condensation energy of the pairs and the second is the contribution of \( B_1 \) to the free energy for given \( B_o \). The last term is due to the normal electrons, which have density \( N_T - 2N \) and energy per unit length \( \pi R (N_T - 2N) m (\omega R)^2 \). Noting that \( \omega \) acts as a Lagrange multiplier of the angular momentum of the normal electrons, we obtain the opposite sign.

Substituting the expressions (4) and (6), defining the frequencies
\[
\omega_\xi = \left( \frac{-2\alpha}{m^* R^2} \right)^{1/2} = \frac{\hbar}{m^* R \xi}

\omega_\kappa = \left( \frac{\beta \gamma c^2}{2\pi (e^*)^2 d R^3} \right)^{1/2} = \frac{\kappa \hbar}{m^* d^{1/2} R \gamma^{3/2}}
\]
where \( \kappa \) is the GL parameter, and dropping the constant \( N_T \), the free energy can be rewritten as
\[
G = \left( \frac{m^* c R}{e^*} \right)^2 \frac{\gamma}{2} \left( \gamma \omega_\xi^2 - \omega_\kappa^2 + \omega_\kappa^2 + \omega_\xi^2 \right) + \frac{2m^* \omega_\kappa^2}{\omega_\xi^2}
\]
(11)

III. RESULTS

Let us first assume that the fluxoid number \( L \) stays unchanged. For given \( \omega \), \( B_o \), \( L \), and \( m^*/m \), \( N \) can be obtained by numerical minimization of \( G \) in Eq. (11). A fair approximation, that fits \( N \) to order \( O(\omega_\xi^4, m^*-2m) \)

![Graph](image_url)

**FIG. 1:** Magnetic field generated by the rotating shell as a function of its angular velocity. Each curve is marked by the parameters \((\omega_\phi/\omega_\xi, \omega_\phi/\omega_\kappa)\). The effective mass of a pair was taken as exactly equal to the mass of two electrons. The curves were obtained by minimization of \( G \) in Eq. (11) and the dots from the approximation (12). From Eq. (11) one would obtain a straight line through the origin with slope 2.

at \( \omega = \omega_\xi \) and to order \( O(\omega^4, \omega^2 (m^*-2m), \omega^4) \) for \( \omega \ll \omega_\xi \) is
\[
\gamma = \frac{\omega_\xi^2 - \omega_\kappa^2}{2\omega_\kappa^2} \left( 1 - \frac{4\omega_\phi^2 \omega_\xi^2/\omega_\kappa^2 + \omega_\phi^2 (\omega_\xi^2 + 4\omega_\kappa^2)}{(\omega_\kappa^2 + 2\omega_\phi^2)^2} \right)
(12)

Representative results are shown in Fig. 1.

Let us now revise the assumption that \( L \) stays unchanged. \( L \) is related to \( \omega_\phi \) by definition (7). If \( L \) could change continuously, \( \omega_\phi \) would assume the value that minimizes \( G \) in Eq. (11), i.e., \( \omega_\phi = 0 \). Since \( L \) has to be integer, the most stable value of \( L \) will be the one that gives the lowest possible value of \( \omega_\phi \). Experiment (10) shows that several values of \( L \) (within the range of about five units from the one of maximal stability) are also possible; they give rise to metastable states. In order to evaluate the potential barriers between different values of \( L \) we would have to consider situations that deviate from cylindrical symmetry; however, the form of Eq. (11) suggests that there is no coupling between \( \omega \) and \( \omega_\phi \) and a state that is metastable for the sample at rest will remain metastable when the sample is rotating and fluxoid jumps will be rare events. The conditions for metastability of different fluxoid states were studied in detail in Ref. 4.

IV. DISCUSSION

We have found that the magnetic field generated by a rotating superconducting shell is not a linear function of the angular velocity; rather, it reaches a maximum for angular velocities of the order of \( \omega_\xi \). \( \omega_\kappa \) is a decreasing function of the coherence length and is smallest close to the critical temperature. For \( R \sim 1\text{cm} \) and \( \xi \sim 10^{-4}\text{cm} \),
\( \omega \xi \sim 10^4 \text{s}^{-1} \). Even if the angular velocity is smaller than \( \omega \xi \) by a few orders of magnitude, nonlinearity ought to be taken into account for the purpose of precision measurements.

We have only considered the case of a very thin shell, with perfect cylindric symmetry. The thinner the shell, the larger the value of \( \omega \xi \), and Fig. 1 shows that this implies an initial slope that is smaller than in Becker’s result. However, nonlinearity shows up also when the shell is less thin, suggesting that this qualitative feature will be present for every rotating superconductor, regardless of its shape.

The induced magnetic field depends also on \( \omega \Phi \). \( \omega \Phi \) depends on the mismatch between the applied magnetic flux and the angular momentum per Cooper pair that would lead to a vanishing velocity. This mismatch is usually of the order of a quantum. In this case we have \( \omega \Phi /\omega \xi \sim \xi/R \). Since this ratio is usually very small, the \( \omega \Phi \)-dependence is expected to be a minor effect when regarded in the entire scale \( -\omega \xi \leq \omega \leq \omega \xi \). Jumps between different possible values of \( \omega \Phi \) are expected to be rare, but when \( \omega \) approaches \( \omega \xi \) the density of superconducting pairs becomes small and the potential barriers between different states have to be small, too. In addition, if the thickness of the shell is not exactly uniform, continuous passage between different states may become possible.

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