Admissible embedding of L-groups and essentially tame local Langlands correspondence

Geo Kam-Fai Tam
Department of Mathematics, University of Toronto
geo.tam@utoronto.ca

November 22, 2011

Abstract
Let $F$ be a non-Archimedean local field and $G$ be the general linear group $\text{GL}_n$ over $F$. Based on the previous results of the author, we can describe the Langlands parameter of an essentially tame supercuspidal representation of $G(F)$ by those admissible embeddings of L-groups constructed by Langlands and Shelstad. We therefore provide a different interpretation on Bushnell and Henniart’s essentially tame local Langlands correspondence.

Contents

1 Introduction

1.1 Historical background ........................................... 2
1.2 Main results of the author ...................................... 3
1.3 Notations and conventions ..................................... 5

2 Previous results

2.1 Admissible characters ........................................... 6
2.2 Induced representations and admissible embeddings .......... 6
2.3 Finite symplectic modules ..................................... 8
2.4 Complementary modules ....................................... 9
2.5 Table of values .................................................. 9
2.5.1 Galois groups ............................................... 9
2.5.2 Symplectic signs .......................................... 10
2.5.3 Quadratic Gauss sums ..................................... 11
2.5.4 Rectifiers .................................................. 11

3 Main results

3.1 The asymmetric case ......................................... 12
3.2 The symmetric ramified case ................................. 13
3.2.1 The case $[\sigma^k] \in D_{l+1}$ .............................. 13
3.2.2 The case $[\sigma^k] \in D_l$ ................................ 14
3.2.3 The case $[\sigma^k] \in D_{l-1} \sqcup \cdots \sqcup D_1$ .......... 15
3.3 The symmetric unramified case ............................... 15
3.3.1 The case $\lambda_{k,l/2}(\varpi_E) = 1$ ....................... 16
3.3.2 The case $\lambda_{k,l/2}(\varpi_E) = -1$ ..................... 18
3.3.3 The case $\lambda_{k,l/2}(\varpi_E) \neq \pm 1$ .................. 19
3.4 Proof of Proposition 3.5 .................................... 21

4 Related results (in preparation) .................................. 21
1 Introduction

1.1 Historical background

Let \( F \) be a non-Archimedean local field of characteristic 0, with residue field \( k_F \) of order \( q \) a power of \( p \). Let \( G \) be \( GL_n \), the general linear group of invertible \( n \times n \) matrices over \( F \). We know that the supercuspidal spectrum of \( G(F) \) is bijective to the collection of \( n \)-dimensional irreducible semi-simple complex representations of the Weil group \( W_F \) of \( F \). This is a major result of the local Langlands correspondence for \( GL_n \) proved in \( [HT01] \).

Bushnell and Henniart described in \( [BH05a] \) such correspondence when restricted to the essentially tame case, which is briefly summarized as follows. For each irreducible semi-simple \( n \)-dimensional complex representation \( \sigma \) of \( W_F \), let \( f(\sigma) \) be the number of unramified characters \( \chi \) of \( W_F \) such that \( \chi \otimes \sigma \cong \sigma \). We call \( \sigma \) essentially tame if \( p \) does not divide \( f(\sigma) \). Let \( \mathcal{G}_n^\text{et}(F) \) be the set of equivalence classes of essentially tame representations of degree \( n \). Similarly, for each irreducible supercuspidal representation (or supercuspidal for short) \( \pi \) of \( G(F) \) let \( f(\pi) \) be the number of unramified characters \( \chi \) of \( F^\times \) that \( \chi \otimes \pi \cong \pi \). Here \( \chi \) is regarded as a representation of \( G(F) \) by factoring through the determinant map. We call \( \pi \) essentially tame if \( p \) does not divide \( n/f(\pi) \). Let \( \mathcal{A}_n^\text{et}(F) \) be the set of isomorphism classes of essentially tame irreducible supercuspidals. Bushnell and Henniart proved that there exists a unique bijection

\[
\mathcal{L} = P\mathcal{L}_n^\text{et} : \mathcal{G}_n^\text{et}(F) \rightarrow \mathcal{A}_n^\text{et}(F),
\]

which satisfies certain canonical conditions together with the compatibilities of automorphic induction \( [HH95] \) and base-change \( [AC89] \) for cyclic extensions on both sides (see the precise statement in Proposition 3.2 of \( [BH05a] \)). We call the map \( \mathcal{L} \) the essentially tame local Langlands Correspondence.

To describe \( \mathcal{L} \) we first recall the following setup. A character \( \xi \) of \( E^\times \) is called admissible over \( F \) if

(i) it does not factor through any proper norm, and

(ii) if its restriction on \( U_E^1 \) factors through some proper norm \( N_{E/K} \) for some \( K/F \) then \( E/K \) is unramified.

We introduce the third set \( P_n(F) \) of equivalence classes \((E/F, \xi)\) of \( F \)-admissible characters \( \xi \) of \( E^\times \) where \( E \) goes through tame extensions over \( F \) of degree \( n \). By \( [BH05a] \) we know that \( P_n(F) \) bijectively parameterizes both \( \mathcal{G}_n^\text{et}(F) \) and \( \mathcal{A}_n^\text{et}(F) \) simultaneously. We denote the bijections by

\[
\sigma : P_n(F) \rightarrow \mathcal{G}_n^\text{et}(F), \quad (E/F, \xi) \mapsto \sigma \xi
\]

and

\[
\pi : P_n(F) \rightarrow \mathcal{A}_n^\text{et}(F), \quad (E/F, \xi) \mapsto \pi \xi.
\]

It is easy to describe the correspondence \( \sigma \), which is simply the induction of representation \( \sigma \xi = \text{Ind}_{W_E}^{W_F} \xi \) if we regard \( \xi \) as a character of \( W_E \) by class field theory \( [Tat79] \). The correspondence \( \pi \) is a compact induction of the form

\[
\pi \xi = c\text{Ind}_{J_\xi}^{G(F)} \Lambda \xi
\]

for certain representation \( \Lambda \xi \) on a compact-mod-center subgroup \( J_\xi \) of \( G(F) \). This representation is an extended maximal type associated to \( \xi \) in the sense of \( [BK93] \). In the case \( p \nmid n \), any irreducible supercuspidals of \( GL_n(F) \) and any \( n \)-dimensional irreducible complex representations of \( W_F \) are essentially tame, and \( P_n(F) \) consists of all \((E, \xi)\) for \( E \) goes through separable extensions over \( F \) of degree \( n \). In this case the correspondence of \( \mathcal{G}_n^\text{et}(F) \) and \( \mathcal{A}_n^\text{et}(F) \) using \( P_n(F) \) as parameters was studied in \( [Moy86] \) and \( [Rei91] \).

With the setup above, we can describe \( \mathcal{L} \) as follows. The composition of the bijections \( \sigma \), \( \pi \) and the inverse of \( \sigma \)

\[
\mu : P_n(F) \xrightarrow{\sigma} \mathcal{G}_n^\text{et}(F) \xrightarrow{\mathcal{L}} \mathcal{A}_n^\text{et}(F) \xrightarrow{\pi^{-1}} P_n(F)
\]
does not give the identity map on $P_n(F)$. Bushnell and Henniart proved in [BH10] that for any admissible character $\xi$ of $E^\times$, there is a unique tamely ramified character $\mu(\xi)$ of $E^\times$, depending on the wild part $\xi|_{U_{l\mu}}$ of $\xi$, so that $\mu(\xi)$ is also admissible and

$$\mu(E, \xi) = (E, \mu(E, \xi)).$$

We call $\mu(\xi)$ the rectifier of $\xi$. In the series [BH05a], [BH05b], [BH10], they express $\mu(\xi)$ explicitly, and hence also the correspondence $\mathcal{L}$.

We briefly explain how to deduce the values of $\mu(\xi)$. We construct a sequence of subfields $F \subseteq K \subseteq K_1 \subseteq \cdots \subseteq K_l \subseteq E,$

which satisfies the following conditions.

(i) $K/F$ is the maximal unramified sub-extension of $E/F$;

(ii) $K_l/K_{l-1}, \ldots, K_1/K$ are quadratic totally ramified;

(iii) $E/K_l$ is totally ramified of odd degree $d$.

Each rectifier $\mu(\xi)$ admits a factorization

$$\mu(E, \xi) = (K/F, \xi)(K_1/K, \xi) \cdots (K_{l-1}/K_l, \xi)(E/K_l, \xi)$$

with each factor being tamely ramified. The approach is to deduce each $L/M, \mu(\xi)$ on the right hand side through an induction process. In the initial case $(L, M) = (E, K_l)$, the rectifier $E/K_l, \mu(\xi)$ can be computed after establishing a base-change argument with the help of automorphic induction (see Theorem 4.4 and 4.6 of [BH05a]). Suppose that $K_{j+1}/K_j, \xi, \ldots, E, \xi, \mu(\xi)$ are all known. Since $K_l/K_{j-1}$ is cyclic by construction, we can make use of the automorphic induction formula in the sense of [HH95], and then compare it with the Mackey induction formula from the compact induction in [3]. After extensive arguments of elaborate representation theoretic technique as in [BH05b], [BH10], we boil down both formulae in terms of $\xi$ only. By a careful comparison we arrive in expressing the values of $K_j, \xi$.

1.2 Main results of the author

The author’s main result is to express the Bushnell-Henniart’s rectifier, and hence the essentially tame local Langlands correspondence, in terms of admissible embeddings of $L$-groups. Recall in [LS7] that Langlands and Shelstad introduced a collection of characters, called $\chi$-data, to construct certain admissible embedding of $L$-groups $L^G \rightarrow G$ for a pair $(G, T)$ of a quasi-split reductive group $G$ containing $T$ as a maximal torus. In the case $(G, T) = (GL_n, Res_{E/F}(G_m))$, a collection $\chi$-data is a set of characters $\{\chi(\lambda)\}_{\lambda \in W_{F}\Phi}$. Here $\lambda$ runs through a suitable subset of representatives of the $W_{F}$-orbits of the root system $\Phi = \Phi(G, T)$. We fix such a subset of $\Phi$ once and for all and denote it by $W_{F}\Phi$. For each $\lambda \in W_{F}\Phi$, the character $\chi(\lambda)$ is defined on the multiplicative group of a field extension $E_\lambda$ of $E$. The precise description

$$\{\chi(\lambda)\}_{\lambda \in W_{F}\Phi} \mapsto (\chi(\lambda) : L^G \rightarrow G)$$

of admissible character constructed from $\chi$-data, $\chi$-admissible character for short, can be found in (2.5) of [LS7].

The main result of this article is that particular $\chi$-admissible embeddings establish the essentially tame local Langlands correspondence.

**Theorem 1.1.** For any admissible $\xi$, the rectifier $\mu(\xi)$ has a factorization of the form

$$\mu(\xi) = \prod_{\lambda \in W_{F}\Phi} \chi(\lambda, \xi)|_{E^{\times}}$$

for canonical choices of tamely ramified $\chi$-data $\{\chi(\lambda, \xi)\}$ depending on $\xi$. 

3
Therefore we can interpret the essentially tame local Langlands correspondence in a reversed way as follows. Recall the local Langlands correspondence for the tori $T$

$$\text{Hom}(E^\times, \mathbb{C}^\times) \cong H^1(W_F, \hat{T}).$$

Let $\xi : W_F \to \mathbb{C}^\times$ be a 1-cocycle corresponding to the admissible character $\xi : E^\times \to \mathbb{C}^\times$. Moreover given the $\chi$-data as in Theorem 1 we consider the inverse collection $\{\lambda_{\xi}^{-1}\}_\lambda$ which is also a $\chi$-data by definition. Let $\chi_{\{\lambda_{\xi}^{-1}\}} : L^T \to L^G$ be the admissible embedding defined by the $\chi$-data $\{\lambda_{\xi}^{-1}\}_\lambda$.

**Corollary 1.2.** The natural projection of the Langlands parameter $\chi_{\{\lambda_{\xi}^{-1}\}} \circ \tilde{\xi} : W_F \to L^T \to L^G$ onto $GL_n(\mathbb{C})$ is isomorphic to $\sigma_{F, \mu_\xi}^{-1, \xi} = \text{Ind}_{W_F}^{E_F}(F \mu_\xi^{-1} \xi)$ as representations of $W_F$.

Indeed Corollary 1.2 is a straightforward consequence of Theorem 1 and a result Proposition 4.2 in [Tamb].

We give some idea in establishing Theorem 1. We briefly introduce these factors as follows. We give some idea in establishing Theorem 1. The choices of $\chi$-data $\{\chi_{\lambda, \xi}\}_\lambda$ are related to certain finite symplectic modules defined as follows. Denote $\Psi_{E/F} = E^*/F^*U_{E^0}^1$. For each admissible character $\xi$ there is a finite symplectic $k_F \Psi_{E/F}$-module $V = V_\xi$ emerges from the bijection $\mathfrak{B}$ with respect to an alternating bilinear form on $V$ naturally defined by $\xi$. The precise details of the construction of $V$ can be found in [Bak93, BH10]. As in [Tamb] we embed each $V$ into a fixed finite $k_F \Psi_{E/F}$-module $U$ which is independent of the admissible character $\xi$. Such module admits a complete decomposition

$$U = \bigoplus_{\lambda \in W_F \setminus \Phi} U_{\lambda}.$$ 

This decomposition is called the residual root space decomposition, which is analogous to the root space decomposition of Lie algebra in the absolute case. Therefore by restriction the submodule $V$ admits a decomposition

$$V = \bigoplus_{\lambda \in W_F \setminus \Phi} V_{\lambda}.$$ 

By construction we can show that $V_{\lambda}$ is either trivial or isomorphic to $U_{\lambda}$. In [Tamb] we proved that such decomposition on $V$ is orthogonal, with symplectic components

$$V_{\lambda} = \begin{cases} V_{\lambda} \oplus V_{-\lambda}, & \text{if } W_F \lambda \neq W_F (-\lambda), \text{ i.e. asymmetric} \\ V_{\lambda}, & \text{if } W_F \lambda = W_F (-\lambda), \text{ i.e. symmetric} \end{cases}$$

Let $W_{\lambda}$ be the complementary module of $V_{\lambda}$ so that

$$V_{\lambda} \oplus W_{\lambda} = U_{\lambda}.$$ 

Let $\mu$ and $\varpi$ be the subgroups of $\Psi_{E/F}$ generated by the image of a primitive root of unity $\zeta \in \mu_E$ and a chosen prime element $\varpi_E$ of $E$ respectively. What we are interested are the $k_F \mu$-module and the $k_F \mathfrak{w}$-module structures of these $V_{\lambda}$.

For each $\lambda \in W_F \setminus \Phi$ the values of our desired character $\chi_{\lambda, \xi}$ will depend on certain invariants called $t$-factors. We briefly introduce these factors as follows.

(i) By regarding each symplectic component $V_{\lambda} = V_{\lambda} \oplus V_{-\lambda}$ or $V_{\lambda}$ (depending on the symmetry of $\lambda$) as $k_F \mu$-module and $k_F \mathfrak{w}$-module respectively, we have the symplectic signs

$$t^0_{\mu}(V_{\lambda}), t^1_{\mu}(V_{\lambda}), t^0_{\mathfrak{w}}(V_{\lambda}), \text{ and } t^1_{\mathfrak{w}}(V_{\lambda}).$$

Here $t^0_{\mu}(V_{\lambda})$ and $t^0_{\mathfrak{w}}(V_{\lambda})$ are signs $\pm 1$, while $t^1_{\mu}(V_{\lambda}) : \mu \to \{\pm 1\}$ and $t^1_{\mathfrak{w}}(V_{\lambda}) : \mathfrak{w} \to \{\pm 1\}$ are quadratic characters. They are all defined by the algorithm in section 3 of [BH10] and are computed in [Tamb] for each $\lambda \in W_F \setminus \Phi$.

(ii) For each $W_{\lambda}$ we attach a Gauss sum $t(W_{\lambda})$ with respect to certain quadratic form on $W_{\lambda}$. They are computed in section 4 of [BH05b].
We then assign a collection of tamely ramified characters \( \{ \chi_{\lambda, \xi} \}_{\lambda \in W_F \setminus \Phi} \) in terms of the \( t \)-factors above. The precise values are given in Theorem 3.1. To prove Theorem 1.1 it remains to show that

**Proposition 1.3.** If \( L/M \mu_\xi \) is one of the factors in (3), then

\[
L/M \mu_\xi = \prod_{\lambda \in W_F \setminus \Phi, \lambda|L \equiv 1, \lambda|M \equiv 1} \chi_{\lambda, \xi}|E^\times.
\]

The proof of Proposition 1.3 and hence that of Theorem 1.1 occupies the whole section 3 of this article.

With such chosen \( \chi \)-data we have the following consequence on arbitrary rectifier. Let \( L \) be an intermediate subfield between \( E/F \). We regard an \( F \)-admissible character \( \xi \) as being admissible over \( L \) and compute the corresponding the rectifier \( L\mu_\xi \). Then we have (Corollary 5.7)

\[
L\mu_\xi = \prod_{\lambda \in W_F \setminus \Phi, \lambda|L \equiv 1} \chi_{\lambda, \xi}|E^\times.
\]

Notice that the index set \( \{ \lambda \in W_F \setminus \Phi, \lambda|L \equiv 1 \} \) above can be identified with \( W_L \Phi_L \GL_{E/L} \Res_{E/L} \Gal_m \), the \( W_L \)-orbits of the root system of the maximal torus \( \Res_{E/L} \Gal_m \) in the group \( \GL_{E/L} \) over \( L \).

During the proofs of the above statements, we need to have an explicit expression of each orbit \( \lambda \in W_F \Phi \). As in [Tama] and [Tamb] we once again identify \( W_F \Phi \) with the non-trivial double cosets \( (W_F/W_F/W_E) - \{ W_E \} \). In subsequent sections our indexes \( \lambda \in W_F \Phi \) will be replaced by group elements \( g \in W_F \) that represent the corresponding double cosets.

### 1.3 Notations and conventions

Let \( E/F \) be a tame extension of degree \( n \), with ramification index \( e \) and residue degree \( f \). The group \( E^\times \) decomposes into subgroups

\[
(E^\times)^{\times} \times \mu_F \times U_F^n,
\]

namely the group generated by a prime element, the group of roots of unity, and the 1-unit group. We have similar decomposition for \( F^\times \). By [Lan94] II, Sec 5 we can always assume our choices of \( \varpi_E \) and \( \varpi_F \) satisfy

\[
\varpi_E^e = \zeta_{E/F} \varpi_F \text{ for some } \zeta_{E/F} \in \mu_E.
\]

Let \( k_F \) be the residual field of \( F \). We may identify \( \mu_F \) with \( k_F^\times \) in the canonical way. Write \( \Psi_{E/F} = E^\times/F^\times \times U_F^n \). It has two cyclic subgroups \( \mu = \mu_E \mu_F \) and \( \varpi = (\varpi_E) \) the subgroup in \( \Psi_{E/F} \) generated by \( \varpi_E \).

Let \( W_E \subseteq W_F \) be the corresponding Weil groups of the fields \( E/F \). We denote induction \( \Ind_{W_E}^{W_F} \) by \( \Ind_{E/F} \) and restriction \( \Res_{W_E}^{W_F} \) by \( \Res_{E/F} \). We write \( \delta_{E/F} = \det \Ind_{E/F} 1_{W_E} \). This character of \( W_F \) is computed in Chapter 2 of [Moy80].

Suppose \( H \) acts on a set \( X \). For \( h \in H \) and \( x \in X \), we write \( h \cdot x \) for the action of \( h \) on \( x \). The \( H \)-orbit of \( x \in X \) is denoted by \( H \cdot x \). The collection of all \( H \)-orbits of \( X \) is denoted by \( H \backslash X \). The set of fixed points is denoted by \( X^H \). If \( f \) is a map whose domain is \( X \), we write \( h \cdot f(x) = f^{h^{-1}}(x) = f^{(h^{-1})} \).

Given a finite field \( V \) and a multiplicative subgroup \( H \) of \( V \), we write \( \text{sgn}_H(V) \) to be the sign character of the multiplicative action of \( H \) on \( V \), and for \( h \in H \) we write \( \text{sgn}_h(V) = \text{sgn}_H(V)(h) \). For a group morphism \( f : H \to V^\times \), we write \( \text{sgn}_{f(H)}(V) \) as a character of \( H \). For any finite cyclic group \( K \), we write

\[
\left( \frac{x}{K} \right) = \begin{cases} 
1, & \text{if } x \in K^2 \\
-1, & \text{otherwise}
\end{cases}
\]

For any integer \( N \) and odd integer \( M \), we write \( \left( \frac{N}{M} \right) \) to be the Jacobi symbol, \( \mu_n \) to be the group of \( n \)-th roots of unity, and \( \zeta_n \) to be a primitive \( n \)-th root of unity.

Throughout we fix an additive character \( \psi_F : F \to \mathbb{C}^\times \) with level 0, i.e. \( \psi_F \) is trivial on \( \mathfrak{p}_F \) but not on \( \mathfrak{o}_F \).
2 Previous results

2.1 Admissible characters

Let $\xi$ be a character of $E^\times$, i.e. a smooth morphism $E^\times \to \mathbb{C}^\times$.

**Definition 2.1.**
1. The $E$-level $r_E(\xi)$ of $\xi$ is the minimum integer $r$ so that $\xi|_{U_{E}^{r+1}} = 1$.
2. We say that $\xi$ is tamely ramified or just tame if $r_E(\xi) = 0$.

Suppose $\xi$ is admissible over $F$ as defined in section [I]. From [Moy86] we know that $\xi$ admits a Howe-factorization of the form

$$\xi = (\xi_{d+1} \circ N_{E/F}) (\xi_d \circ N_{E/E_d}) \cdots (\xi_0 \circ N_{E/E_0}) \xi_{-1}.$$  

We need to specify the notations appeared above.

1. We have a decreasing sequence of fields

$$E = E_{-1} \supseteq E_0 \supseteq E_1 \supseteq \cdots \supseteq E_d \supseteq E_{d+1} = F.$$  

Each $\xi_i$ is a character of $E_i^\times$, and $\xi_d$ is a character of $F^\times$.

2. Let $r_i$ be the $E$-level of $\xi_i$, i.e. the $E$-level of $\xi_i \circ N_{E/E_i}$, and $r_{d+1}$ be the $E$-level of $\xi$. We assume that $\xi_{d+1}$ is trivial if $r_{d+1} = r_d$. We call the $E$-levels $r_0 < \cdots < r_d$ the jumps of $\xi$.

3. If $E_0 = E$, then we replace $(\xi_0 \circ N_{E/E_0}) \xi_{-1}$ by $\xi_0$. If $E_0 \subseteq E$ we have that $\xi_{-1}$ is tame and $E/E_0$ is unramified.

There is also a generic property [Moy86] for which those $\xi_i$ canonically satisfy. The genericity implies that

$$\text{gcd}(r_j, e(E/E_{j+1})) = e(E/E_j).$$  \hspace{1cm} (5)

Let $P(E/F)$ be the set of admissible characters of $E^\times$ over $F$. Two admissible characters $\xi \in P(E/F)$ and $\xi' \in P(E'/F)$ are called equivalent if there is $g \in W_F$ that $^gE = E'$ and $^g\xi = \xi'$. Following [BH05a] we denote the equivalence class of $\xi$ by $(E/F, \xi)$. Let $P_n(F)$ be the set of equivalence classes of admissible characters, i.e.

$$P_n(F) = \text{equivalence classes of } \left( \bigcup_E P(E/F) \right)$$

for $E$ goes through tame extensions over $F$ of degree $n$.

**Definition 2.2.** the jump data of $\xi$ are the sequences of subfields $E \supseteq E_0 \supseteq E_1 \supseteq \cdots \supseteq E_d \supseteq F$ and the jumps $r_0, \ldots, r_d$.

We can define a $W_F$-action on the sequence

$$g: \{E \supseteq E_0 \supseteq E_1 \supseteq \cdots \supseteq E_d \supseteq F\} \mapsto \{^gE \supseteq ^gE_0 \supseteq ^gE_1 \supseteq \cdots \supseteq ^gE_d \supseteq F\}$$

for all $g \in W_F$. The jumps of $^g\xi$ is clearly the same as those of $\xi$. Hence we can define a jump data of the equivalence class $(E/F, \xi)$ in the obvious sense.

2.2 Induced representations and admissible embeddings

We write $G = \text{GL}_n$ and $T = \text{Res}_{E/F}(\mathbb{G}_m)$, both regarded as algebraic groups over $F$. We choose an $F$-embedding of $T$ into $G$. Attached to the root system $\Phi = \Phi(G, T)$ is a collection of characters, called $\chi$-data, defined as follows. Let $E_\lambda$ and $E_{\pm \lambda}$ be the respective fixed fields of the following stabilizers

$$\{w \in W_F|w\lambda = \lambda\} \text{ and } \{w \in W_F|w\lambda = \pm\lambda\}. \hspace{1cm} (6)$$

We call $\lambda \in \Phi$ symmetric if $-\lambda \in W_F \lambda$, and asymmetric otherwise.
Definition 2.3. We call a collection \( \{ \chi_{\lambda} : E_{+\lambda}^\times \to \mathbb{C}^\times \}_{\lambda \in \Phi} \) of characters \( \chi \)-data if it satisfies

(i) \( \chi_{-\lambda} = \chi_{\lambda}^{-1} \) and \( \chi_{w\lambda} = \chi_{\lambda}^{w^{-1}} \) for all \( w \in W_F \);

(ii) if \( \lambda \) is symmetric, then \( \chi|_{E_{+\lambda}^\times} \) equals the quadratic character \( \chi_{E_{+\lambda}/E_{-\lambda}} \).

Notice that by condition (i) it suffices to attach characters to the symmetrized orbits of \( \Phi \), i.e. if \( R \) is a subset of \( \Phi \) representing the following orbits

\[ W_F \{ \lambda \in \Phi \text{ symmetric} \} \bigsqcup W_F \{ \lambda \in \Phi \text{ asymmetric} \} / \{ \lambda \sim -\lambda \} \]

then the collection \( \{ \chi_{\lambda} \}_{\lambda \in \Phi} \) is determined by its subset \( \{ \chi_{\lambda} \}_{\lambda \in R} \).

For computational convenience, we make use of the following fact.

Proposition 2.4. The collection of \( W_F \)-orbits of \( \Phi \) is bijective to the non-trivial double cosets \( (W_E \backslash W_F/W_E)' \) by

\[ W_F \Phi \to (W_E \backslash W_F/W_E)', \quad W_F \lambda = W_F [\lambda] \mapsto [g] := E_EgW_E \]

where \( [\lambda] \) is the root such that \( [\lambda] t = gtt^{-1} \) for any \( t \in T(F) = E^\times \).

We write \( \chi_g = \chi_{[\lambda]} \), \( E_g = E_{[\lambda]} = gEE \), and \( E_{\pm g} = E_{\pm [\lambda]} \). Analogous to the description of the roots, we call \( g \in W_F \) symmetric if the double cosets \( [g] \) and \( [g^{-1}] \) are equal, and asymmetric otherwise. Notice that by condition (i) of a collection \( \chi \)-data \( \{ \chi_{\lambda} \} \) is determined by its subset \( \{ \chi_g \} \) for \( g \in W_F/W_E \) runs through a set of representatives of non-trivial double cosets \( (W_E \backslash W_F/W_E)' \) modulo symmetry, i.e. if \( D \) is a subset of \( W_F \) representing the following double cosets

\[ \{ [g] \in (W_E \backslash W_F/W_E)' \text{ symmetric} \} \bigsqcup \{ [g] \in (W_E \backslash W_F/W_E)' \text{ asymmetric} \} / \{ [g] \sim [g^{-1}] \} \]

then the collection \( \{ \chi_{\lambda} \}_{\lambda \in \Phi} \) is determined by \( \{ \chi_g \}_{g \in D} \).

The purpose of introducing \( \chi \)-data is to construct certain embedding of the \( \mathbb{L} \)-groups \( \mathbb{L}T \hookrightarrow \mathbb{L}G \).

Definition 2.5. An admissible embedding from \( \mathbb{L}T \) to \( \mathbb{L}G \) is a morphism of groups \( \chi : \mathbb{L}T \to \mathbb{L}G \) of the form

\[ \chi(t \times w) = \iota(t) \tilde{\chi}(w) \times w \]

for some injective morphism \( \iota : \hat{T} \to \hat{G} \) and some map \( \tilde{\chi} : W_F \to \hat{G} \).

The precise construction of an admissible embedding is in \cite{LSS} and is summarized in \cite{Tama}. We can interpret any admissible embeddings by induced representations of Weil groups as follows. Take a character \( \xi \) of \( E^\times \). An application of Shapiro’s Lemma gives the local Langlands correspondence for the torus \( T \)

\[ \text{Hom}(W_F, C^\times) = H^X(W_F, \hat{T}) = \text{Hom}(W_F, \mathbb{L}T)/\text{Int}\hat{T}. \]

If \( \tilde{\xi} \) is a 1-cohomology class corresponding to \( \xi \) by above, let \( \hat{\xi} \in \text{Hom}_{W_F}(W_F, \mathbb{L}T) \) be a representative of \( [\xi] \). Suppose that \( \chi \) is an admissible embedding \( \mathbb{L}T \to \mathbb{L}G \) constructed by \( \chi \)-data \( \{ \chi_g \} \). Let \( \mu \) to be the character

\[ \mu(\chi_g) := \prod_{[g] \in (W_E \backslash W_F/W_E)'} \text{Res}_{E_g}^E \chi_g. \]

A remark here is that if \( [g] = [h] \), then \( \chi_g|_{E^\times} = \chi_h|_{E^\times} \). Hence \( \mu \) is a well-defined character, uniquely determined by the \( \chi \)-data \( \{ \chi_g \} \). Finally let proj : \( \mathbb{L}G \to \mathbb{GL}_n(\mathbb{C}) \) be the canonical projection.

Proposition 2.6 (Proposition 4.2 of \cite{Tama}). The representation

\[ W_F \tilde{\xi} \otimes \mathbb{L}T \xrightarrow{\chi} \mathbb{L}G \xrightarrow{\text{proj}} \mathbb{GL}_n(\mathbb{C}) \]

is isomorphic to \( \text{Ind}_{E/F}(\xi \mu) \).
2.3 Finite symplectic modules

In [Tamb], we briefly studied the correspondence \( P_n(F) \to A^\xi_t(F) \), whose detail is referred to [Moy86], [BK93], [BH05a]. We are interested in the finite symplectic \( k_F\Psi_{E/F}-\text{module} V = \zeta \) emerged during the process.

The module \( V \) is a quotient of two compact subgroups \( H^F_k \subseteq J^F_k \) in \( G(F) \) of which the maximal torus \( E^\times \) normalizes both of them. It is known that the \( E^\times \)-action on \( V \) factors through \( \Psi_{E/F} = E^\times/F^\times U^F_k \). Moreover, we can choose a character \( \theta \) of \( H^F_k \) among those simple characters of \( \xi \). (There is a canonical one among the poll given in chapter 3 of [Moy86], which is defined using the fixed additive character \( \psi_F \) in section 12.3.) Such simple character gives rise to an alternating \( F_p \)-bilinear form \( h_\theta \) on \( V \). Hence we obtain a finite symplectic \( k_F\Psi_{E/F}-\text{module} \) structure on \( V \). In [BH10] we have the following structural property of \( V \).

**Proposition 2.7.** The \( k_FE^\times\)-module \( V \) decomposes into an orthogonal sum
\[
V = \bigoplus_{1 \leq j \leq d} V_{E_{j-1}/E_j}
\]
with respect to the alternating form \( h_\theta \) such that for each \( j = 0, \ldots, d \),

(i) \( V_{E_{j-1}/E_j} \) is non-trivial if and only if the jump \( r_j \) is even,

(ii) the \( E^\times \)-action on \( V_{E_{j-1}/E_j} \) is trivial.

The \( k_F\Psi_{E/F}-\text{module} V \) is a submodule module of \( U = \mathfrak{A}/\mathfrak{A}' \), where \( \mathfrak{A} \) is the hereditary order in \( \text{End}_F(E) \) defined by the \( \sigma_F \)-lattice \( \{ \mathfrak{p}_k \}_{k \in \mathbb{Z}} \) in \( E \) and \( \mathfrak{A}' \) is the radical of \( \mathfrak{A} \). It is direct from its construction that \( U \) has a decomposition (called the residual root space decomposition in the sense of [Tamb])
\[
U = \bigoplus_{[g] \in (W_E/W_E')/W_E'} U_{[g]}.
\]
The index of each submodule indicates the action of \( \Psi_{E/F} \), which is \( tv = gtt^{-1}v \) for all \( t \in \Psi_{E/F} \) and \( v \in U_{[g]} \). Notice that if \( [g] = [h] \) then \( U_{[g]} \cong U_{[h]} \) as \( \Psi_{E/F}-\text{modules} \). Hence the index notation \( U_{[g]} \) for the submodules are well-defined.

The subspace \( V \) of \( U \) is \( \Psi_{E/F} \)-invariant. Such action on \( V \) is equivalent to the \( \Psi_{E/F} \)-module structure induced by the normalization of \( E^\times \subseteq G(F) \). Hence \( V \) decomposes accordingly. We therefore obtain a finer decomposition of \( V \) as follows.

**Proposition 2.8 (Complete decomposition of \( V \)).** The \( k_F\Psi_{E/F}-\text{module} V \) decomposes into an orthogonal sum
\[
V = \bigoplus_{[g] \in (W_E/W_E')/W_E} V_{[g]}
\]
with respect to the alternating form \( h_\theta \) such that

(i) each \( V_{[g]} \) is either trivial or isomorphic to the field extension of \( k_{E/F} \) and

(ii) \( V_{[g]} \subseteq V_{E_{j-1}/E_j} \) if and only if \( [g] \in W_E/W_{E_{j-1}}/W_{E_{j-1}/E_j} \).

Each non-trivial symplectic components of \( V \) is either of the form \( U_{[g]} \oplus U_{[g]} \) for \( [g] \) asymmetric, or \( U_{[g]} \) for \( [g] \) symmetric. We write \( V_{[g]} \) to be the corresponding symplectic component. We attach to each cyclic subgroup \( \Gamma = \mu \) in \( \Psi_{E/F} \) and each symplectic component \( V_{[g]} \) of \( V \) the symplectic signs, which are called \( t \)-factors in [Tamb]. They are of the form
\[
t^G_{\gamma}(V_{[g]}) = \pm 1 \text{ and a character } t^G_{\gamma}(V_{[g]}): \Gamma \to \{ \pm 1 \}.
\]

We also denote
\[
t^G_{\gamma}(V_{[g]}) = t^G(\zeta_1(V_{[g]}))(\gamma)
\]
for any generator \( \gamma \in \Gamma \). The \( t \)-factors are certain invariants of a symplectic module which reflect its structural (hyperbolic/anisotropic) properties. The factors also appear in the automorphic induction formula of the supercuspidal representation, as formulated in [BH10]. Since the essentially tame local Langlands correspondence is based on these formulae, the \( t \)-factors also appear in the description of the rectifiers. We summarize those values in a tables in section [2.5.2].
2.4 Complementary modules

Let $W$ be the complementary module of $V$, in the sense that

$$W = \bigoplus_{[g] \in (W_E \backslash W_F / W_E)'} W_{[g]}$$

where $W_{[g]}$ is either trivial or isomorphic to $U_{[g]}$ and

$$W \oplus V = \bigoplus_{[g] \in (W_E \backslash W_F / W_E)'} U_{[g]}.$$ 

For those $[g]$ such that $W_{[g]}$ is non-trivial, we write $W_{[g]}$ the component $U_{[g]} \oplus U_{[g-1]}$ for $[g]$ asymmetric or $U_{[g]}$ for $[g]$ symmetric. Let $K$ be the maximal unramified extension in $E/F$. We introduce the extra $t$-factor $t(W_{[g]})$ which appear in the explicit expression of the rectifier.

We regard $\xi$ as an admissible character over $K$. Let $E = E_0 \supseteq E_1 \supseteq \cdots \supseteq E_t \supseteq E_{t+1} = K$ and \{r_0, \ldots, r_t\} be the jump data of $\xi$. Write $W^{K}_{t,-1} = W_{E_0/E_1} \oplus \cdots \oplus W_{E_t/E_{t+1}}$. As in [BH05b], there is a quadratic form $Q_K$ on $W^K$ emerged when analyzing the Mackey induction formula of the supercuspidal $\pi_\xi$ as a compactly induced representation. The form $Q_K$ decomposes into a product of $Q_{ij}^k$ on each $W_{E_i/E_{j+1}}$, which is expressed in Proposition 5.7 of [BH05b]. We can further decompose the module

$$W^K = \bigoplus_{[g] \in (W_E \backslash W_K / W_E)'} W_{[g]}.$$ 

Let $Q_{[g]}$ be the restriction of $Q_{ij}^k$ on $W_{[g]}$ if $[g] \in W_E \backslash W_E / W_E - W_E \backslash W_{E_{j+1}} / W_E$. We therefore define in all cases

$$t(W_{[g]}) = \begin{cases} n(Q_{[g]}, \psi_F), & \text{if } W_{[g]} \text{ is non-trivial} \\ 1, & \text{if } W_{[g]} \text{ is trivial} \end{cases}.$$ 

Here $n(Q_{[g]}, \psi_F)$ is a quadratic Gauss sum. In section 2.5.3 we will deal with quadratic Gauss sums in general and compute the values of $t(W_{[g]})$ for each $[g] \in (W_E \backslash W_K / W_E)'$.

**Remark 2.9.** The quadratic form $Q_K$ is defined by a chosen simple character $\theta$ of $\xi$. This means that each factor $Q_{ij}^k$ depends on the Howe factor $\xi_j$ and the character $\psi_K = \psi_F \circ \text{tr}_{K/F}$ fixed already in section 1.3.

We give a final remark in this section about the remaining $[g] \in W_E \backslash W_F / W_E - W_E \backslash W_K / W_E$. The comparison between the Mackey formula and the automorphic induction formula over the cyclic extension $K/F$ is more complicated than that in the totally ramified case. (For example, compare Theorem C in the Introduction of [BH05b] with (6.5.3) of [BH11].) We do not have a quadratic form in this case and therefore we do not have the extra $t$-factor. We then circumvent this problem by a fact Proposition 5.4 about the parity of the set $W_E \backslash W_F / W_E - W_E \backslash W_K / W_E$. Then we assign an extra $-1$, no matter $V_{[g]}$ is trivial or not, to each $\chi_{[g]}(w_F)$ in addition to the $t$-factors.

2.5 Table of values

2.5.1 Galois groups

We first recall [Tamb] a description of the elements in a Weil group, or more precisely a finite Galois group. Let $L$ be the Galois closure of $E/F$. Define the $F$ operators

$$\phi : \zeta \mapsto \zeta^q \text{ for all } \zeta \in \mu_L \text{ and } w_E \mapsto \zeta_\phi w_E$$

$$\sigma : \zeta \mapsto \zeta^{q^i} \text{ for all } \zeta \in \mu_L \text{ and } w_E \mapsto \zeta^{q^i}_\sigma w_E$$

for some $\zeta_\phi \in \mu_F$ and an eth root of $\zeta^{-1}_{\phi'^{-1}}$, since $\phi \circ \sigma \circ \phi'^{-1} = \sigma^q$. Therefore

(i) we can write our Galois group as $\Gamma_{L/F} = W_F / W_E = \langle \sigma \rangle \rtimes \langle \phi \rangle$ and

(ii) we can choose representatives of $W_E \backslash W_F / W_E$ of the form $\sigma^k \phi^i$ where $i \in \mathbb{Z}/f$ and $k \in q^f \setminus (\mathbb{Z}/e)$ the set of orbits in $\mathbb{Z}/e$ under the action $x \mapsto q^f x$.

Let $\lambda_{ki}$ be a root in $\Phi(G, T)$ whose $W_F$-orbit corresponds to $[\sigma^k \phi^i]$, i.e. $\lambda_{ki}(\zeta) = \zeta^{q^i-1}$ for all $\zeta \in \mu_E$ and $\lambda_{ki}(w_E) = \zeta^{q^i}_k \zeta^{q^i}_\phi$, for some fixed $\zeta^{q^i}_\phi \in \mu_E$. 
2.5.2 Symplectic signs

Write $\Psi_{E/F} = E^\times/F^\times U_{E/F}$. Recall that a non-trivial symplectic component in $V$ is either of the form $V = U_{[g]} = U_{[g]} \oplus U_{[g^{-1}]}$ for $[g]$ asymmetric or $V = U_{[g]}$ for $[g]$ symmetric. In [Tamb] we compute the following $t$-factors.

(i) If $V$ is trivial, then $t_0^0(V) = 1$ and $t_1^1(V)$ is a trivial character for all cyclic subgroup $\Gamma$ of $\Psi_{E/F}$.

(ii) If $[g] = [\sigma^k \phi^t]$ is asymmetric, then

$$t_0^0(U_{[\sigma^k \phi^t]}) = 1 \text{ and } t_1^1(U_{[\sigma^k \phi^t]}) = \text{sgn}_{\alpha^{\phi^{-1}}}(U_{[\sigma^k \phi^t]}).$$

and

$$t_0^0(U_{[\sigma^k \phi^t]}) = 1 \text{ and } t_1^1(U_{[\sigma^k \phi^t]})(\varpi_E) = \text{sgn}_{\alpha^\phi}(U_{[\sigma^k \phi^t]}).$$

(iii) If $[g]$ is symmetric, then $[g] = [\sigma^k] \text{ or } [g] = [\sigma^k \phi^{f/2}]$. By slightly abusing language, we call those symmetric $[\sigma^k]$ totally ramified and those symmetric $[\sigma^k \phi^{f/2}]$ unramified. If $U_{[g]}$ constitute a component in $V$ then we have shown in [Tamb] that $[g] \neq [1]$ or $[\sigma^{e/2}]$.

(a) If $[g]$ is symmetric totally ramified, i.e. $[g] = [\sigma^k]$, $k \neq 0$ or $e/2$, then

$$t_0^0(U_{[\sigma^k]}) = 1 \text{ and } t_1^1(U_{[\sigma^k]}) \equiv 1. \text{ and}$$

$$t_0^0(U_{[\sigma^k]}) = -1 \text{ and } t_1^1(U_{[\sigma^k]}) : \varpi_E \mapsto \left(\frac{\zeta^k}{\mu_p^{e+1}}\right),$$

where $|\mathbb{F}_p[\zeta^k]/\mathbb{F}_p| = 2s$ and $\mu_p^{e+1} = \text{ker}(N_{\mathbb{F}_p[\zeta^k]/\mathbb{F}_p[\zeta^k]} \cdot \text{ for the quadratic extension } \mathbb{F}_p[\zeta^k]/\mathbb{F}_p[\zeta^k].$

(b) If $[g]$ is symmetric unramified, i.e. $[g] = [\sigma^k \phi^{f/2}]$, then

$$t_0^0(U_{[\sigma^k \phi^{f/2}]}) = -1 \text{ and } t_1^1(U_{[\sigma^k \phi^{f/2}]}) : \zeta \mapsto \left(\frac{\zeta^{f/2-1}}{\mu_p^{e+1}}\right)$$

and $t_0^0(U_{[\sigma^k \phi^{f/2}]})$ by the following cases. Write $\lambda_{k,f/2}$ the corresponding root $x \mapsto \sigma^k \phi^{f/2} x x^{-1}$.

(I) If $\lambda_{k,f/2}(\varpi_E) = 1$, then

$$t_0^0(U_{[\phi^{f/2}]}) = 1 \text{ and } t_1^1(U_{[\phi^{f/2}]}) \equiv 1.$$

(II) If $\lambda_{k,f/2}(\varpi_E) = -1$, then

$$t_0^0(U_{[\phi^{f/2}]}) = 1 \text{ and } t_1^1(U_{[\phi^{f/2}]})(\varpi_E) = (-1)^{(f/2-1)/2}.$$

(III) If $\lambda_{k,f/2}(\varpi_E) \neq \pm 1$, then

$$t_0^0(U_{[\phi^{f/2}]}) = -1 \text{ and } t_1^1(U_{[\phi^{f/2}]}) : \varpi_E \mapsto \left(\frac{\lambda_{k,f/2}(\varpi_E)}{\mu_p^{e+1}}\right),$$

where $|\mathbb{F}_p[\lambda_{k,f/2}(\varpi_E)]/\mathbb{F}_p| = 2s$ and $\mu_p^{e+1} = \text{ker}(N_{\mathbb{F}_p[\lambda_{k,f/2}(\varpi_E)]/\mathbb{F}_p[\lambda_{k,f/2}(\varpi_E)]}).$ for the quadratic extension $\mathbb{F}_p[\lambda_{k,f/2}(\varpi_E)]/\mathbb{F}_p[\lambda_{k,f/2}(\varpi_E)].$
2.5.3 Quadratic Gauss sums

Let $W$ be a finite dimensional $k_F$-vector space, $Q : W \to k_F$ be a quadratic form and $\psi_F$ be a non-trivial character of $k_F$. We define the Gauss sum $g(Q, \psi_F) = \sum_{v \in W} \psi_F(Q(v))$. The simplest example is when $W = k_F$ and $Q(x) = x^2$. In this case we denote the Gauss sum by $g(\psi_F)$. Suppose $Q$ is non-degenerate, i.e. $\det Q \neq 0$, then we can show that

$$g(Q, \psi_F) = \left( \frac{\det Q}{k_F^\times} \right) g(\psi_F)^{\dim k_F} W.$$  \hspace{1cm} (7)

We also define the normalized Gauss sum $n(Q, \psi_F) = (\#W)^{-1/2}g(Q, \psi_F)$ and $n(\psi_F) = q^{-1/2}g(\psi_F)$. The equation (7) is still true if we replace $g$ by $n$. We can easily show that the Gauss sum is a convoluted sum

$$g(\psi_F) = \sum_{x \in k_F} \left( \frac{x}{k_F^\times} \right) \psi_F(x).$$

From this point we can deduce that $n(Q, \psi_F)^2 = \left( \frac{-1}{q} \right)$, i.e. the normalize sum $n(Q, \psi_F)$ is a 4th root of unity.

We consider those $[g]$ such that $U_{[g]}$ is a component of $W$, and write $W_{[g]}$ to be the component $U_{[g]} \oplus U_{[g]^{-1}}$ for $[g]$ asymmetric or $U_{[g]}$ for $[g]$ symmetric. Write $Q_{[g]}$ be the restriction of $Q$ on $W_{[g]}$. Suppose $Q_{[g]}$ is non-degenerate. We first consider $[g] \neq [\sigma^c/2]$, i.e. the corresponding root $\lambda$ satisfies $\lambda^2 \neq 1$. By Proposition 4.4 of [BH05b] we have

$$\left( \frac{\det Q_{[g]}}{k_F^\times} \right) = \left( \frac{-1}{q} \right)^{\dim k_F W_{[g]}} = \begin{cases} \left( \frac{-1}{q} \right)^{\dim k_F U_{[g]}}, & \text{if } [g] \text{ is asymmetric} \\ \left( \frac{-1}{q} \right)^{\dim k_F U_{[g]/2}}, & \text{if } [g] \text{ is symmetric} \end{cases}$$  \hspace{1cm} (8)

In particular the discriminant of $Q_{[g]} \mod k_F^\times$ is determined by the underlying $k_F\Psi_E/F$-module structure. Therefore using (7) we have

$$n(Q_{[g]}, \psi_F) = \left( \frac{\det Q_{[g]}}{k_F^\times} \right) n(\psi_F)^{\dim k_F W_{[g]}} = \pm \left( \frac{-1}{q} \right)^{\dim k_F W_{[g]/2}} \left( \frac{-1}{q} \right)^{\dim k_F W_{[g]/2}} = \pm 1,$$

where the sign is determined by the symmetry of $[g]$ as in (8). In other words, the Gauss sum depends only on the symmetry of $[g]$ and is independent to the quadratic form $Q$. In the exceptional case $[g] = [\sigma^c/2]$, the sum $n(Q_{[g]}, \psi_F)$ is an arbitrary 4th root of unity.

2.5.4 Rectifiers

For each character $\xi$ we give the values of the rectifier $\mu_E$ following [BH05a], [BH05b] and [BH10]. Recall that each rectifier $\mu_E$ admits a factorization as in (4). Since each factor being tamely ramified, it is enough to give the values of each factor on $\mu_E$ and at $\varpi_E$.

We first give some notations. We write

(i) the submodule

$$V_{L/M} = \prod_{\lambda \in \Psi_E \setminus \Phi} V_{\lambda}$$

for a field extension $L/M$ lying between $E/F$,

(ii) the submodule $V^\equiv$ as the subspace of fixed points of the subgroup $\varpi$ of $\Psi_E/F$, and

(iii) for any field extension $K$, the notations $\mathfrak{D}_K = \mathfrak{D}_K(\varpi_K, \psi_K)$, $R_K = R_K(\alpha_0, \varpi_K, \psi_K)$ and $\mathfrak{R}_K^\equiv = R_K^\equiv(\alpha_0, \varpi_K, \psi_K)$ for various Gauss sums and Kloostermann sums in [BH05b]. The exact meanings of them will be specified in the proof of Theorem 3.1.
We give the values of those $L/M\mu_\xi$ appearing in the right side of [4].

(i) $E/K_i\mu_{\xi|E} \equiv 1$ and $E/K_i\mu_{\xi}(\bar{\omega}_E) = \left(\frac{q_i}{\omega_{E/K_i}}\right)$,

(ii) $K_i/K_{i-1}\mu_{\xi|E} = \left(\frac{\mu_i}{\mu_{E/K_i}}\right)$ and $K_i/K_{i-1}\mu_{\xi}(\bar{\omega}_E) = t_\omega(V_{K_i/K_{i-1}})(\bar{\Theta}_{K_{i-1}}/\bar{\Theta}_{K_i})(\bar{R}_{K_{i-1}}^\circ/\bar{R}_{K_i})$,

(iii) $K_i/K_{i-1}\mu_{\xi|E} \equiv 1$ and $K_i/K_{i-1}\mu_{\xi}(\bar{\omega}_E) = t_\omega(V_{K_i/K_{i-1}})(\bar{\Theta}_{K_{i-1}}/\bar{\Theta}_{K_i})(\bar{R}_{K_{i-1}}^\circ/\bar{R}_{K_i})$ for $j = l-1, \ldots, 1$, and

(iv) $F/K\mu_{\xi|E} \equiv t_\mu^0(V_{K/F})$ and $F/K\mu_{\xi}(\bar{\omega}_E) = (-1)^{c(f-1)}t_\mu^0(\bar{V}/\bar{V})t_\omega(V_{K/F})$.

3 Main results

In this section we prove the main result Theorem 3.1 that the rectifier $F\mu_\xi$ for each $\xi \in P(E/F)$ can be constructed by $\chi$-data, in the sense that $F\mu_\xi$ is a product of canonically chosen characters in $\chi$-data $\{\chi_\lambda\}$. Let's recall the preliminary setup. For any $g = [\sigma^k\phi] \in (W_E\backslash W_E)^\prime$, let $U_{g_{\xi}}$ be the standard $k_F(\Psi_{E/F})$-module defined in Section 2.3. The action of $\Psi_{E/F}$ is given in Section 2.3. For $g \in P(E/F)$ let $V = V_{\xi}$ be the $k_F(\Psi_{E/F})$-module defined by $\xi$. Let $V_{[g]}$ be the $U_{g_{\xi}}$-isotypic part of $V$ so that $V_{[g]}$ is either trivial or isomorphic to $U_{g_{\xi}}$. Let $W_{g}$ be the isotypic part of $V_{[g]}$ of $U_{g_{\xi}}$. We write $V_{[g]} = V_{[g]} \otimes V_{[g-1]}$ and $W_{g} = W_{[g]} \otimes W_{[g-1]}$. The t-factors $t_{\mu}^i(V_{[g]})$ and $t_{\mu}^i(W_{[g]})$, $i = 0, 1$, and $t(W_{[g]})$ are computed in section 2.3.

Theorem 3.1. (i) Let $V$ be the $k_F(\Psi_{E/F})$-module defined by $\xi \in P(E/F)$. The following characters define a collection of $\chi$-data $\{\chi_{g}\}$,

(a) All $\chi_{g}$ are tamely ramified.
(b) If $g$ is asymmetric, then we assign $\chi_{g}|_{E/\mu_g} = \text{sgn}_{\mu_g}(V_{[g]})$ and $\chi_{g}(\bar{\omega}_E)$ to be anything appropriate.
(c) If $g$ is symmetric, then we assign $\chi_{g}|_{E/\mu_g} = \left\{ \begin{array}{ll} \left(\frac{\mu}{\mu_{E/K_i}}\right), & \text{if } [g] = [\sigma^{e/2}], \\
\left(t_{\mu}^0(V_{[g]})\right), & \text{otherwise}, \end{array} \right.$

and $\chi_{g}(\bar{\omega}_E) = \left\{ \begin{array}{ll} t_\omega(V_{[g]})t(W_{[g]}), & \text{if } [g] \text{ is ramified}, \\
t_{\mu}^0(V_{[g]})t(W_{[g]}), & \text{if } [g] \text{ is unramified}. \end{array} \right.$

(ii) Let $F\mu_\xi$ be the rectifier of $\xi \in P(E/F)$ and $\{\chi_{g}\}$ be the $\chi$-data defined above, then $F\mu_\xi = \prod_{[g] \in (W_E\backslash W_E)^\prime} \chi_{g}|_{E^\times}$.

The proof occupies the remaining of this section. The structure of the proof is as follows. Recall the factorization in [4] $F\mu_\xi = (K/F\mu_\xi)(K_{i}/K\mu_\xi) \cdots (K_{1}/K_{i-1}\mu_\xi)(E/K_{i}\mu_\xi)$ where each factor is explicitly expressed in [BH05a], [BH05b] and [BH10]. We then decompose the double cosets $(W_E\backslash W_E)^\prime$ into subsets

$D_{l+1} = D_{E/K_i} = W_E\backslash W_{K_i}/W_E - [1]$,

$D_{l} = D_{K_i/K_{i-1}} = W_E\backslash W_{K_{i-1}}/W_E - W_E\backslash W_{K_i}/W_E$,

$\vdots$

$D_{1} = D_{K_1/K} = W_E\backslash W_{K_1}/W_E - W_E\backslash W_K/W_E$,

$D_{0} = D_{K/F} = W_E\backslash W_F/W_E - W_E\backslash W_K/W_E$. 
Let's call $D_{t+1} \sqcup \cdots \sqcup D_1$ the totally ramified part and $D_0$ the unramified part. We will prove the following

1. For each $[g] \in D_j$, prove that if we define $\chi_g$ by the values in Theorem 3.1, then it satisfies the conditions of $\chi$-data.

2. Show that

$$
\chi_{D_j}|E^x = \prod_{[g] \in D_j} \chi_g|E^x = k_j/k_{j-1} \mu_\xi.
$$

We start proving Theorem 3.1 in case when $\text{char}(k_F) = p \neq 2$. We first consider those asymmetric $[g]$ and then those symmetric $[g] \in D_j$ for each $j$.

### 3.1 The asymmetric case

For $[g] = [\sigma^k \phi^i]$ asymmetric, we denote the modules $V_{[\sigma^k \phi^i]} = V_{[\sigma^k \phi^i]} \oplus V_{[(\sigma^k \phi^i)^{-1}]}$, which is non-trivial. Since $\chi_{(\sigma^k \phi^i)^{-1}} = (\chi_{\sigma^k \phi^i})^{-1}$, we are going to compute the restriction to $E^x$ of the product

$$
\chi_{\sigma^k \phi^i} (\chi_{\sigma^k \phi^i})^{-1}|E^x = \chi_{\sigma^k \phi^i} \circ \lambda_{ki}.
$$

Our assignment gives

$$
(\chi_{\sigma^k \phi^i} \circ \lambda_{ki})|_{\mu_E} = \text{sgn}_{\mu_E, \sigma^k \phi^i} (V_{[\sigma^k \phi^i]}) = t_\mu^i (V_{[\sigma^k \phi^i]})
$$

and

$$
(\chi_{\sigma^k \phi^i} \circ \lambda_{ki})(\sigma E) = \chi_{[\sigma^k \phi^i]} (\lambda_{ki}(\sigma E)) = \text{sgn}_{\lambda_{ki}(\sigma E)} (V_{[\sigma^k \phi^i]}) = t^i_{\sigma^k}(V_{[\sigma^k \phi^i]})(\sigma E).
$$

Notice that the relation in (10) does not depend on the exact value of $\chi_{\sigma^k \phi^i}(\sigma E)$ and $\chi_{(\sigma^k \phi^i)^{-1}}(\sigma E)$, therefore we can assign them to anything appropriate. We will use the $t$-factors on both right sides of (9) and (10) later in the symmetric cases when we compare the $\chi$-data with the rectifier.

### 3.2 The symmetric ramified case

We now consider those $[g] \in (W_E \setminus W_F \setminus W_E)_{\text{sym-ram}}$. They are of the form $[g] = [\sigma^k]$.

#### 3.2.1 The case $[\sigma^k] \in D_{t+1}$

We first consider those symmetric $[\sigma^k] \in D_{t+1} = W_E \setminus W_{K^s}/W_E - [1]$, i.e. $k$ has odd order in $\mathbb{Z}/e$. Let $s$ be a positive integer and minimum so that $e|(1 + q^s)k$ and $e|(1 + p^s)k$. We recall that we have assigned

$$
\chi_{\sigma^k}|_{\mu_E} = t^i_{\mu} (V_{[\sigma^k]})
$$

which is 1 since $\mu_E$ acts on each $V_{[\sigma^k]}$ trivially, and

$$
\chi_{\sigma^k} (\sigma E) = t^i_{\sigma} (V_{[\sigma^k]}) t(W_{[\sigma^k]}).
$$

If $V_{[\sigma^k]}$ is non-trivial, then we have

$$
t^0_{\sigma} (V_{[\sigma^k]}) = -1 \quad \text{and} \quad t^1_{\sigma} (V_{[\sigma^k]})(\sigma E) = \left( \frac{c_k^k}{\mu_{p^s+1}} \right) = 1
$$

since $c_k^k$ has odd order and $p^s + 1$ is even. Therefore the right side of (12) is $-1$. If $V_{[\sigma^k]}$ is trivial, so that $W_{[\sigma^k]}$ is non-trivial, then

$$
t_{\sigma} (V_{[\sigma^k]}) = 1 \quad \text{and} \quad t(W_{[\sigma^k]}) = -1.
$$

Hence we always have $\chi_{\sigma^k}(\sigma E) = t_{\sigma} (V_{[\sigma^k]}) = -1$.

We have to check that such $\chi_{\sigma^k}$ in (11) and (12) satisfies the condition of $\chi$-data

$$
\chi_{\sigma^k}|_{E^x_{\pm \sigma^k}} = \delta_{E^x_{\pm \sigma^k}}.
$$
Since $E_{\sigma^k}/E_{\pm\sigma^k}$ is quadratic unramified, the norm group $N_{E_{\sigma^k}/E_{\pm\sigma^k}}(E_{\sigma^k}^\times)$ has a decomposition

$$\mu_{E_{\pm\sigma^k}} \times \langle \zeta_{E_{\sigma^k}}^k \rangle \times U_{E_{\pm\sigma^k}}^1$$

and let $\zeta_0 \in \mu_{E_{\sigma^k}}$ so that

$$\zeta_0 \omega_E \in E_{\pm\sigma^k} - N_{E_{\sigma^k}/E_{\pm\sigma^k}}(E_{\sigma^k}^\times).$$

Therefore the condition (13) is explicitly

$$\chi_{\sigma^k}|_{\mu_{E_{\pm\sigma^k}}} \equiv 1, \chi_{\sigma^k}(\zeta_{E_{\sigma^k}}^k) = 1$$

and

$$\chi_{\sigma^k}(\zeta_0)\chi_{\sigma^k}(\omega_E) = -1.$$  \hspace{1cm} (14)

Since $\mu_E \subseteq \mu_{E_{\pm\sigma^k}}$, the first condition of (14) implies that the quadratic character $\chi_{E_{\sigma^k}/E_{\pm\sigma^k}}$ extends $\chi_{\sigma^k}$ from $\mu_E$ to $\mu_{E_{\pm\sigma^k}}$. Therefore our assignment (11) satisfies the first condition in (14). If we assume $\chi_{\sigma^k}(\omega_E) = \pm 1$, then the second condition of (14) gives $\chi(\zeta_{\sigma^k}) = 1$. Finally we use the third condition of (14) that $\chi_{\sigma^k}(\omega_E) = -\chi_{\sigma^k}(\zeta_0)$. Since $\zeta_0^{-1}q^{t_1} = \zeta^k = \zeta_N$ for $N$ being odd (indeed if $k = 2^sk'$ for some odd $k'$ and $e = 2^s$ for some odd $d$, then $N = d/gcd(k',d)$ is odd), we have $\zeta_0 \in \mu_{(1-q^{t_1})N}$. Hence we have $\chi_{\sigma^k}(\zeta_0^{-1}) = 1$ as $\zeta_0^{-1} \in \mu_{q^{t_1} - 1} = \mu_{E_{\pm\sigma^k}}$. We assumed that $\chi_{\sigma^k}(\zeta_0) = \pm 1$, and so it must be 1. Therefore we have shown that our assignment (12) satisfies the conditions in (14). In other words, $\chi_{\sigma^k}$ is a well-defined character in $\chi$-data.

Finally we have to show that the product

$$\chi_{D_{l+1}} = \prod_{[\sigma^k] \in D_{l+1}} \chi_{\sigma^k} = \prod_{[\sigma^k] \in (D_{l+1})_{sym}} \chi_{\sigma^k} \prod_{[\sigma^k] \in (D_{l+1})_{asym/\pm}} \chi_{\sigma^k}^{-1} \circ \lambda_{\sigma^k}$$

equals $K_i/E\mu_\xi$. We first compute its value on $\mu_E$. For asymmetric $[\sigma^k]$ we have right side of (19) being 1 since $\mu_E$ acts trivially on $U_{[\sigma^k]}$. Combining these with the symmetric ones in (11) and using Theorem 4.4 of [BH05a] we have

$$\chi_{D_{l+1}}|_{\mu_E} = 1 = \delta_{E/K_i}|_{\mu_{K_i}} = K_i/E\mu_\xi|_{\mu_E}. \hspace{1cm}$$

Next we compute the value on $\omega_E$. Combining (10) and (12) we have

$$\chi_{D_{l+1}}(\omega_E) = t_{\omega}(U_{D_{l+1}}).$$

By Lemma 4.2(ii) of [Rei91] we have the product

$$t_{\omega}(U_{D_{l+1}}) = \left( \frac{q^{t_1}}{e_{E/K_i}} \right)$$

which equals $E/K_i\mu_\xi(\omega_E)$ by Theorem 4.4 of [BH05a]. We have proved Theorem 3.1 for symmetric $[\sigma^k] \in D_{l+1}$.

3.2.2 The case $[\sigma^k] \in D_l$

We next study those $[\sigma^k] \in D_l = D_{K_i/K_{i-1}}$. We first consider the distinguished element $[g] = [\sigma^{s/2}]$. By Proposition 5.6 of [Lam] the module $V_{[\sigma^{s/2}]}$ is always trivial. Recall that $E_{\sigma^{s/2}} = E$ and $E/E_{\pm\sigma^{s/2}}$ is quadratic totally ramified. We assign

$$\chi_{\sigma^{s/2}}|_{\mu_E} : \zeta \mapsto \left( \frac{\zeta}{\mu_E} \right). \hspace{1cm}$$

(15)

Since $N_{E/E_{\pm\sigma^{s/2}}}(\omega_E) = -\omega_{E_{\pm\sigma^{s/2}}} = -\omega_{E}$, it remains to assign $\chi_{\sigma^{s/2}}(\omega_E)$ which satisfies

$$\chi_{\sigma^{s/2}}(\omega_E)^2 = \chi_{\sigma^{s/2}}(\omega_{E_{\pm\sigma^{s/2}}}) = \chi_{\sigma^{s/2}}(-1) = \left( \frac{1}{\mu_E} \right)$$

so that it satisfies the condition of $\chi$-data

$$\chi_{\sigma^{s/2}}|_{E_{\pm\sigma^{s/2}}^\times} = \delta_{E/E_{\pm\sigma^{s/2}}^\times}.$$
If we assign $\chi_{\sigma^{\nu}/2}(\varpi_E) = t(W_{\sigma^{\nu}/2})$ then $\chi_{\sigma^{\nu}/2}(\varpi_E)^2 = n(\psi_{K_{l+1}})^2 = \left(\frac{-1}{\mu_E}\right)$. Therefore we have shown that $\chi_{\sigma^{\nu}/2}$ is a character in $\chi$-data.

For $[\sigma^k] \in \mathcal{D}$ symmetric, $k \neq e/2$, since $\mu_E$ acts trivially on $V_{[\sigma]}$, we can assign

$$\chi_{\sigma^k}|_{\mu_E} = t_\mu(V_{[\sigma^k]}) = 1.$$  \hspace{1cm} (16)

To obtain the value of $\chi_{\sigma^k}(\varpi_E)$ so that $\chi_{\sigma^k}$ defines a character in a $\chi$-data, i.e.

$$\chi_{\sigma^k}|_{E_{\pm, k}} = \delta_{E_{\pm, k}} = \delta_E_{\pm, k},$$

we recall the explicit $\chi$-data conditions in (14) that

$$\chi_{\sigma^k}|_{E_{\pm, k}} \equiv 1, \chi_{\sigma^k}(\zeta^k) = 1 \text{ and } \chi_{\sigma^k}(\zeta_0) \chi_{\sigma^k}(\varpi_E) = -1.$$  \hspace{1cm} (17)

Our assignment (16) satisfies the first condition since $\mu_E \subseteq E_{\pm, k}$ and

$$\chi_{\sigma^k}(\mu_{E_{\pm, k}}) = \delta_{E_{\pm, k}} = \delta_E_{\pm, k} = 1.$$  \hspace{1cm} (18)

If we assume $\chi_{\sigma^k}(\varpi_E) = \pm 1$, then the second condition of (17) implies that $\chi_{[\sigma^k]}(\zeta^k) = 1$. Let $\zeta_{\pm} = \zeta_N$ for $N = 2M$ and $M$ being odd. (Indeed for $[\sigma^k] \in \mathcal{D}$ we have

$$k \in q^f \setminus (2^{s-1}Z/eZ - 2^sZ/eZ).$$

So if $k = 2^{s-1}l', e = 2^{s}d$, then $\zeta^k = \zeta^k_{\pm} = \zeta_N$ for $N = 2d/\gcd(l', d.)$. Recall that $\zeta_0^{1-q_{-}^{q_{+}}} = \zeta^k$, hence $\zeta_0^{1-q_{-}^{q_{+}}} = \mu_{E_{\pm, k}}$. Since $N$ is even and $\chi_{\sigma^k}|_{E_{\pm, k}} = 1$, we can assign $\chi_{\sigma^k}(\zeta_0)$ to be either $\pm 1$, and from the third condition of (17) we can assign $\chi_{\sigma^k}(\varpi_E) = -\chi_{\sigma^k}(\zeta_0)$ to be either $\pm 1$. As in Theorem 3.1 we assign

$$\chi_{\sigma^k}(\varpi_E) = t_E(V_{[\sigma^k]})t(W_{[\sigma^k]})$$

and $\chi_{\sigma^k}$ is hence a character in $\chi$-data.

We are ready to compute $\chi_{\mathcal{D}}$. On $\mu_E$ we combine (15), (16) and (11) to obtain

$$\chi_{\mathcal{D}|_{\mu_E}} = \left(\frac{\mu_{E_{\pm, k}}}{\mu_E}\right) = \left(\frac{\mu_{K_{l+1}}}{\mu_{K_{l-1}}}\right) = \delta_{K_{l-1}}/\delta_{K_{l+1}},$$

which equals $K_{l-1}/\mu_{E_{\pm, k}}$ by (3.1.1) of [BH05b].

To compare $\chi_{\mathcal{D}}(\varpi_E)$ with $K_{l-1}/\mu_{E_{\pm, k}}(\varpi_E)$, we recall from Theorem 6.6 [BH05b] that

$$K_{l-1}/\mu_{E_{\pm, k}}(\varpi_E) = t_E(V_{\mathcal{D}})(\mathfrak{G}_{K_{l-1}}/\mathfrak{G}_{K_{l}})(\mathfrak{R}_{K_{l-1}}/\mathfrak{R}_{K_{l}})$$

where $\mathfrak{G}_{K_{l-1}}$ and $\mathfrak{G}_{K_{l}}$ are products of certain Gauss sum and $\mathfrak{R}_{K_{l-1}}$ and $\mathfrak{R}_{K_{l}}$ are certain Kloostermann sum. Their values were be specified in the subsequent calculations. We need to proceed case by case by the virtue of [BH05b] and adopt the notations there. Regard $\xi$ as a admissible character over $K_{l-1}$ and let $E = E_0 \supseteq E_1 \supseteq \cdots \supseteq E_t \supseteq E_{t+1} = K_{l-1}$ and $\{r_0, \ldots, r_t\}$ be the jump data of $\xi$. We let (see section 1.2 of [BH05b])

$$i^+$$ be the largest odd jump of $\xi$,

$$d^+ [E_{j+1}/K_{l-1}]$$ if $r_j = i^+$, and

$$i_+$$ be the least jump of $\xi$ so that $[E_{j+1}/K_{l-1}]$ is odd.

(Notice that there is a shift of index between ours and those in [BH05b], e.g. our $r_0$ is denoted by $r_1$ in [BH05b].) By genericity of $\xi$ we always have $i_+ \geq i^+$. We first distinguish between $r_0 > 1$ and $r_0 = 1$. If $r_0 > 1$, then $\chi_{\mathcal{D}_t} = t_E(V_{\mathcal{D}_t})t(W_{\mathcal{D}_t})$ where

$$t(W_{\mathcal{D}_t}) = \prod_{j=0}^t n(\zeta_j(\varpi_E)Q_{K_j}^j, \psi_{K_j}).$$
This equals $\mathfrak{S}_{K_{i-1}/\mathfrak{S}_K}$ by (8.2.1) of [BH05b]. Also by convention $\mathcal{R}^e_{K_{i-1}}/\mathcal{R}_{K_{i-1}} = 1$ in the case $r_0 > 1$. Therefore we have checked that
\[
\chi_{D_i}(\varpi_E) = t_{\varpi}(V_{D_i})t(W_{D_i}) = t_{\varpi}(V_{D_i})/(\mathfrak{S}_{K_{i-1}/\mathfrak{S}_K})(\mathcal{R}^e_{K_{i-1}}/\mathcal{R}_{K_{i-1}}) = K_i/K_{i-1} \mu_\zeta(\varpi_E).
\]

If $r_0 = 1$, we separate into three subcases, namely
\[
i_+ = i^+ = r_0 = 1, \quad i_+ > i^+ = r_0 = 1 \quad \text{and} \quad i_+ \geq i^+ > r_0 = 1.
\]
If $i_+ = i^+ = r_0 = 1$, then the jumps are of the form $\{1, 2s_1, ..., 2s_t\}$ and $d^+ = |E_1/K_{i-1}|$ is odd, we have
\[
\bigoplus_{j \geq 1} W_{E_j/E_{j+1}}^{K_{i-1}} = 0 \quad \text{and} \quad W_{K_{i-1}}^{E_1} = W_{K_{i-1}}^{E_1/E_1}.
\]

The first equation gives
\[
t\left(\bigoplus_{j \geq 1} W_{E_j/E_{j+1}}^{K_{i-1}}\right) = 1
\]
which equals $\mathfrak{S}_{K_{i-1}/\mathfrak{S}_K}$ by Lemma 8.1(1) of [BH05b]. The second equation is indeed
\[
W_{E_1/E_1}^{K_{i-1}} = \bigoplus_{k=1}^{\lceil E_1/K_1 \rceil} W_{[\sigma^k E_1/K_1]}^{K_1}.
\]

Here $K$ is the maximal unramified extension of $E/F$. Therefore we have
\[
(W_{E_1/E_1}^{K_{i-1}})_{D_i} = \bigoplus_{k \text{ is odd}} W_{[\sigma^k E_1/K_1]}^{K_1}.
\]

This module is denoted by $\mathcal{Y}$ in [BH05b] section 8.2. Notice that $\dim_{K_1} \mathcal{Y} = |E_1/K_1|/2$ is odd. It is enough to check the Kloosterman sum equals to our normalized Gauss sum $n(\zeta_0(\varpi_E)Q_{K_{i-1}/1}, \psi_{K_{i-1}})$. Indeed if we define the bilinear form $\zeta_0(\varpi_E)Q_{K_{i-1}/1}$ on $W_0$, then by the proof of [BH05b] Proposition 8.3 (still goes for $s = 0$ verbatim) we have \((Q_{K_{i-1}/1})^{\frac{1}{2}} = \left(\frac{|E_1/K_1|}{q^{l'}}\right)^{\dim \mathcal{Y}}\) and hence
\[
t(\mathcal{Y}) = n(\zeta_0(\varpi_E)Q_{K_{i-1}/1}, \psi_{K_{i-1}}) = \left(\frac{\zeta_0(\varpi_E)}{q^{l'}}\right)^{\dim \mathcal{Y}} \left(\frac{|E_1/K_1|}{q^{l'}}\right)^{\dim \mathcal{Y}} = \left(\frac{\zeta_0(\varpi_E)}{q^{l'}}\right)^{\dim \mathcal{Y}} \left(\frac{|E_1/K_1|}{q^{l'}}\right)^{\dim \mathcal{Y}}.
\]

This equals $\mathcal{R}^e_{K_{i-1}/\mathcal{R}_{K_{i-1}}}$ by Proposition 9.3 of [BH05b]. Therefore
\[
\chi_{D_i}(\varpi_E) = t_{\varpi}(V_{D_i})t(\mathcal{Y}) = \chi_{D_i}(\varpi_E) = t_{\varpi}(V_{D_i})/(\mathfrak{S}_{K_{i-1}/\mathfrak{S}_K})(\mathcal{R}^e_{K_{i-1}}/\mathcal{R}_{K_{i-1}}) = K_i/K_{i-1} \mu_\zeta(\varpi_E).
\]

For $i_+ > i^+ = r_0 = 1$, we show that such case does not exist. Here $i^+ = 1$ implies that the set $D_{E_i/K_{i-1}}$ of jumps of $\xi$ equals $\{1, r_1, \ldots, r_t\}$, where $r_i$ are all even for $i > 0$. Let $E \supseteq E_1 \supseteq \cdots \supseteq E_t \supseteq E_{i+1} = K_{i-1}$ be the corresponding subfields. The condition $\mathfrak{a}^2$ implies $e(E_i/E_1)$ and $e(E/E_{i+1})$ have the same parity for all $i$. Since $e(E_i/E_{i+1}) = e(E/K_{i-1}) = 2d$ is even, we have that $e(E_i/E_1)$ is also even and $e(E_1/K_{i-1})$ divides $d$. So $e(E_1/K_{i-1})$ is odd. This implies $i_+ = 1$.

If $i_+ \geq i^+ > r_0 = 1$, let $r_i$ be the smallest jump such that $e(E_i/E_1)$ is odd and $e(E/E_{i+1})$ is even. Hence $r_i$ must be odd by $\mathfrak{a}^2$ and $r_i \leq i^+$. We have that $e(E_{i+1}/K_{i-1})$ is odd and so $i_+ = i_{i+1}$. Therefore $r_i = i^+$. Hence
\[
W_{K_{i-1}}^{E_1} = \bigoplus_{r_j \leq i^+, r_j \text{ is odd}} W_j
\]
which contains $W_{E_1}$ and $W_{E_i/E_{i+1}}$ with $W_{[\sigma^k/2]} \subseteq W_{E_i/E_{i+1}}$. Since $|E_{i+1}/E_{i+1}|$ is even, We can check $W_{D_i}^{K_{i-1}} = W_{E_{i+1}/E_{i+1}}$. In particular $\mathcal{Y} = (W_{E_1/E_1})_{D_i} = 0$ and so $t(\mathcal{Y}) = 1$. This is $\mathcal{R}^e_{K_{i-1}}/\mathcal{R}_{K_{i-1}}$ by Lemma 8.1(3) of [BH05b]. Therefore $t(W_{D_i}) = \mathfrak{S}_{K_{i-1}/\mathfrak{S}_K}$ and $\chi_{D_i}(\varpi_E) = K_i/K_{i-1} \mu_\zeta(\varpi_E)$.

Combining all cases, we have proved Theorem 8.1 for $[\sigma^k] \in D_i \cup D_{i-1}$.
3.2.3 The case $[\sigma^k] \in D_{l-1} \cup \cdots \cup D_1$

For $[\sigma^k] \in D_{l-1}$ we first consider another distinguished element $[g] = [\sigma^{e/4}]$, in case it is symmetric.

**Lemma 3.2.** The following are equivalent.

(i) $[\sigma^{e/4}]$ is symmetric.

(ii) $(1 + qf^t)/2$ is even for some $t$.

(iii) $q f \equiv 3 \mod 4$.

(iv) The degree 4 totally ramified extension $K_t/K_{l-2}$ is non-Galois.

**Proof.** By the definition of symmetry, (i) is equivalent to the statement that $e$ divides $(1 + q f^t) e/4$ for some $t$, which is clearly equivalent to (ii). That (ii) $\iff$ (iii) and (iii) $\iff$ (iv) are clear. A totally ramified extension $E/F$ of degree 4 is non-Galois if and only if 4 does not divide $q f - 1$. Hence (iv) is equivalent to (iii). □

Consider $\xi$ as a admissible character of $E^\times$ over $K_{l-2}$ and let $E = E_0 \supset E_1 \supset \cdots \supset E_t \supset K_{l-1} = K_{l-2}$ and $\{r_0, \ldots, r_t\}$ be the corresponding jump data. Also define $i^+$ and $i_+$ as in [18] and recall that $i_+ \geq i^+$ always. We can characterize $V_{[\sigma^{e/4}]}$ by the following.

**Lemma 3.3.** $V_{[\sigma^{e/4}]}$ is non-trivial if and only if $i_+ > i^+$.

**Proof.** Let $r_j$ be the jump that $V_{[\sigma^{e/4}]} \subseteq V_j$. The sufficient condition is equivalent to that $r_j$ is even, and by Lemma 1.2 of [BH05b] the necessary condition is equivalent to that $i_+ > i^+$ is even. Hence $4|e(E_j/K_{l-2})$ is odd. We have $r_j = i_+$.

We assign

$$\chi_{\sigma^{e/4}}|_{\mu_E} = t_{i_+}^{i_+}(V_{[\sigma^{e/4}]}),$$

and

$$\chi_{\sigma^{e/4}}(w_E) = t_{w}(V_{[\sigma^{e/4}]}) t(W_{[\sigma^{e/4}]}) = \begin{cases} t_{w}(V_{[\sigma^{e/4}]}) & \text{if } V_{[\sigma^{e/4}]} \text{ is non-trivial} \\ t(W_{[\sigma^{e/4}]}) = -1 & \text{if } V_{[\sigma^{e/4}]} \text{ is trivial} \end{cases}.$$  

The verification of $\chi_{\sigma^{e/4}}$ being a $\chi$-data, no matter $V_{[\sigma^{e/4}]}$ is trivial or not, is similar to the statement following [17].

**Remark 3.4.** We can compute the value of $t_{w}(V_{[\sigma^{e/4}]})$ if $V_{[\sigma^{e/4}]}$ is non-trivial, i.e. $V_{[\sigma^{e/4}]} = k_E(\zeta_4)$. By Lemma 3.2 $q f \equiv 3 \mod 4$, which implies also $p \equiv 3 \mod 4$. We have $|F_p(\zeta_4)/F_p| = 2$ and that $r = |k_E(\zeta_4)/F_p(\zeta_4)| = |k_E/F_p|$ must be odd by Lemma 4.5 of [Tamb]. Therefore

$$t_{w}(V_{[\sigma^{e/4}]}) = \begin{pmatrix} \zeta_4 \\ \mu_{p+1} \end{pmatrix}^r = - \begin{pmatrix} \zeta_4 \\ \mu_{p+1} \end{pmatrix} = \begin{cases} 1 & \text{if } p \equiv 3 \mod 8 \\ -1 & \text{if } p \equiv 7 \mod 8 \end{cases}.$$ □

For other symmetric $[\sigma^k]$, $k \neq e/4$, we proceed as in the case $[\sigma^k] \in D_t$, $k \neq e/2$, and obtain

$$\chi_{\sigma^k}|_{\mu_E} = t_{i_+}^{i_+}(V_{[\sigma^k]}) = 1$$

and

$$\chi_{\sigma^k}(w_E) = t_{w}(V_{[\sigma^k]}) t(W_{[\sigma^k]}).$$

Again we can use similar statements following [17] to verify that $\chi_{\sigma^k}$ is a character in a $\chi$-data.

On $\mu_E$ we combine [20], [21] and [22] to obtain $\chi_{D_{l-1}}|_{\mu_E} = 1$, which equals $K_{l-1}/K_{l-2} \mu_E|_{\mu_E}$ by Theorem 3.2 of [BH05b]. To compare $\chi_{D_{l-1}}(w_E)$ with $K_{l-1}/K_{l-2} \mu_E|_{\mu_E}(w_E)$, again we distinguish between cases. When $r_0 > 1$, the situation is similar to the case $[g] \in D_{K_l/K_{l-1}}$. When $r_0 = 1$ we distinguish between $i_+ = i^+ = r_0 = 1$, $i_+ > i^+ = r_0 = 1$ and $i^+ > r_0 = 1$. 

17
If $i_+ = i^+ = r_0 = 1$, then $|E_1/K_{i-2}|$ is odd. Let
\[
\mathcal{Y} = (W_{E_1/E_1})_{D_{i-1}} = \bigoplus_{k \in \mathbb{Z}/|E/E_1|, \ k \text{ is odd}} W_{\sigma_k E_1/K_1}
\]
defined similarly as in [19]. Then $\dim \mathcal{Y} = |E/E_1|/2$ is even. As in the proof of [BH05b] Proposition 8.4 (which is still true for $r_0 = 1$), we have \(\left(\frac{\det Q_{K_{i-1}}}{q^f}\right) = 1\) and so
\[
t(\mathcal{Y}) = \left(\frac{\zeta_0(\varpi_E)}{q^f}\right)^{\dim \mathcal{Y}} \left(\frac{\det Q_{K_{i-1}}}{q^f}\right) \phi(\psi_{K_{i-2}})^{\dim \mathcal{Y}} = \left(\frac{-1}{q^f}\right)^{|E/E_1|/4}.
\]
This equals $R_{K_{i-2}}^E / R_{K_{i-2}}$ by 9.3 of [BH05b]. Moreover $i^+ = 1$ implies that $\bigoplus_{j \geq 1} (W_{E_j/E_{j+1}})_{D_{j-1}} = 0$. Hence
\[
t \left(\bigoplus_{j \geq 1} (W_{E_j/E_{j+1}})_{D_{j-1}}\right) = 1 = \Phi_{K_{i-1}} / \Phi_{K_{i-1}}
\]
by Lemma 8.1(1) of [BH05b]. Therefore $\chi_{K_{i-1}/K_{i-2}}(\varpi_E) = K_{i-1}/K_{i-2} \mu_{\xi}(\varpi_E)$.

For $i_+ > i^+ = r_0 = 1$, we must have that $|E_1/K_{i-2}|$ is even, otherwise $i_+ = 1$. If $[\sigma^k K_{i-2}/K_1] \in D_{i-1} \cap W_E \setminus W_{E_1}/W_E$, then $k$ is odd and also a multiple of $|E_1/K_{i-2}|$. No such $k$ exists. Therefore $\mathcal{Y}$ is trivial and $t(\mathcal{Y}) = 1 = R_{K_{i-2}}^E / R_{K_{i-2}}$ by Proposition 8.1 of [BH05b]. Hence again $\chi_{K_{i-1}/K_{i-2}}(\varpi_E) = K_{i-1}/K_{i-2} \mu_{\xi}(\varpi_E)$. The case $i^+ > r_0 = 1$ is similar. In all case we have proved that $\chi_{K_{i-1}/K_{i-2}}(\varpi_E) = K_{i-1}/K_{i-2} \mu_{\xi}(\varpi_E)$.

For $[\sigma^k] \in D_{i-2} \sqcup \cdots \sqcup D_1$, the proof that $\chi_{\sigma^k}$ defines a character in $\chi$-data and that $\chi_{\sigma^k} = K_{i-1}/K_{i-2} \mu_{\xi}$ are very similar to the previous cases. Notice in this case we have, by Proposition 8.4 and 9.3 of [BH05b], that
\[
\Phi_{K_{i-1}} / \Phi_{K_{i-1}} = R_{K_{i-2}}^E / R_{K_{i-2}} = 1.
\]
We have proved Theorem 3.1 for all symmetric $[g] = [\sigma^k]$.

### 3.3 The symmetric unramified case

We now consider those $[g] \in (W_E \setminus W_{E_1}/W_E)_{\text{sum-unr}}$. They are of the form $[g] = [\sigma^k \phi/f/2]$. If $\lambda_{k,f/2}$ is the corresponding root $x \mapsto \sigma^k \phi/f/2 \cdot x^{-1}$, then we distinguish the following three cases

\[
\lambda_{k,f/2}(\varpi_E) = 1, \quad \lambda_{k,f/2}(\varpi_E) = -1, \quad \text{and} \quad \lambda_{k,f/2}(\varpi_E) \neq \pm 1.
\]

#### 3.3.1 The case $\lambda_{k,f/2}(\varpi_E) = 1$

When $\lambda_{k,f/2}(\varpi_E) = 1$ we have $E_0 = E$. If we denote $E_{\pm g} = E_\pm$, then the $\chi$-data conditions are
\[
\chi_g |_{\mu_{E_{\pm}}(\varpi_E)} \equiv 1, \quad \chi_g(\varpi_E^2) = 1 \quad \text{and} \quad \chi_g(\zeta_0 \varpi_E) = -1,
\]
where $\zeta_0 \in \mu_E$ so that $\zeta_0^{q^{f/2} - 1} = 1$, i.e. $\zeta_0 \in \mu_{E_{\pm}} \subseteq \mu_E$. If $V_{[g]}$ is trivial, then our assignments are
\[
\chi_g |_{\mu_E} = t_1^0(V_{[g]}) = 1 \quad \text{and} \quad \chi_g(\varpi_E) = -t_0^0(V_{[g]})^2 t_\infty(V_{[g]}) = -1,
\]
which satisfies the conditions in (22). If $V_{[g]}$ is non-trivial, then the assignments are
\[
\chi_g |_{\mu_E} = t_1^0(V_{[g]}) : \zeta \mapsto \left(\frac{\zeta^{q^{f/2} - 1}}{\mu_{q^{f/2} + 1}}\right)
\]
and
\[
\chi_g(\varpi_E) = -t_0^0(V_{[g]})^2 t_\infty(V_{[g]}) = -(-1)^2(1) = -1.
\]
Since $\mu_{E_{\pm}} = \mu_{q^{f/2} - 1}$, we have $t_1^0(V_{[g]}) |_{\mu_{E_{\pm}}} = 1$. The conditions in (22) are readily satisfied. Therefore $\chi_g$ is a character of $\chi$-data.
3.3.2 The case $\lambda_{k,f/2}(\varpi_E) = -1$

For $\lambda_{k,f/2}(\varpi_E) = -1$, again $E_g = E$ and is quadratic unramified over $E_\pm = E_{\pm g}$. The $\chi$-data conditions are

$$\chi_g|_{\mu_{E_\pm}} \equiv 1, \quad \chi_g(-\varpi_E^2) = 1 \quad \text{and} \quad \chi_g(\zeta_0 \varpi_E) = -1.$$  \hfill (23)

Here $\zeta_0 \in \mu_E$ satisfies $\zeta_0^{q^{l/2} - 1} = -1$, i.e. $\zeta_0$ is a primitive root in $\mu_{2(q^{l/2} - 1)}$. If $V_{[g]}$ is trivial, we assign

$$\chi_g|_{\mu_E} = t^1_\mu(V_{[g]}) \equiv 1 \quad \text{and} \quad \chi_g(\varpi_E) = -t^0_\mu(V_{[g]})t_\varpi(V_{[g]}) = -1.$$  

By noticing that $\zeta_0 \in \mu_{2(q^{l/2} - 1)} \subset \mu_{q^{l/2} - 1} = \mu_E$, the verification is an easy analogue of the previous case.

When $V_{[g]}$ is non-trivial, our assignments are

$$\chi_g|_{\mu_E} = t^1_\mu(V_{[g]}); \quad \zeta \mapsto \left( \frac{\zeta^{q^{l/2} - 1}}{\mu_{q^{l/2} + 1}} \right)$$  \hfill (24)

and

$$\chi_g(\varpi_E) = -t^0_\mu(V_{[g]})t_\varpi(V_{[g]}) = (-1)(1)(-1)\frac{1}{2(q^{l/2} - 1)} = (-1)\frac{1}{2(q^{l/2} - 1)}.$$  \hfill (25)

Since $\mu_{E_\pm} = \mu_{q^{l/2} - 1}$, our assignment (24) satisfies the first condition of (23). Also the second condition of (23) is satisfied if we assume $\chi_g(\varpi_E) = \pm 1$. Since

$$\chi_g(\zeta_0) = t^1_\mu(V_{[g]})(\zeta_0) = \left( \frac{\zeta_0^{q^{l/2} - 1}}{\mu_{q^{l/2} + 1}} \right) = \left( \frac{1}{\mu_{q^{l/2} + 1}} \right) = (-1)\frac{1}{2(q^{l/2} - 1)},$$

with (25) we have

$$\chi_g(\zeta_0 \varpi_E) = (-1)\frac{1}{2(q^{l/2} - 1)}(-1)\frac{1}{2(q^{l/2} - 1)} = -1.$$  

Hence the third condition of (23) is satisfied, and $\chi_g$ is a character of $\chi$-data.

3.3.3 The case $\lambda_{k,f/2}(\varpi_E) \neq \pm 1$

For other $[g] = [\sigma^k \phi^{l/2}]$ symmetric with $\lambda_{k,f/2}(\varpi_E) = \zeta_0^{k} \zeta_{\phi^{l/2}} \neq \pm 1$, we assign

$$\chi_g|_{\mu_E} = t^1_\mu(V_{[g]})$$  \hfill (26)

and

$$\chi_g(\varpi_E) = -t^0_\mu(V_{[g]})t_\varpi(V_{[g]}).$$  \hfill (27)

We have to show that it satisfies the condition of $\chi$-data

$$\chi_g(\mu_{E_\pm}) = 1, \quad \chi_g(\lambda_{k,f/2}(\varpi_E)\varpi_E^2) = 1 \quad \text{and} \quad \chi_g(\zeta_0 \varpi_E) = -1,$$  \hfill (28)

where $\zeta_0 \in \mu_{E_g}$ so that $\zeta_0 \varpi_E \in E_{\pm g}$. Since $\mu_E \cap \mu_{E_\pm} = \mu_{q^{l-1}} \cap \mu_{q^{(2l+1)/2} - 1} = \mu_{q^{l/2} - 1}$, which is mapped by

$$t^1_\mu(V_{[g]}); \quad \zeta \mapsto \begin{cases} 1, & \text{if } V_{[g]} \text{ is trivial} \\ \left( \frac{z^{q^{l/2} - 1}}{\mu_{q^{l/2} + 1}} \right), & \text{if } V_{[g]} \text{ is non-trivial} \end{cases}$$

into $1$, we can extend $\chi_g$ so that $\chi_g|_{E_{\pm g}^2} \equiv 1$ and hence (26) satisfies the first condition of (28). Now notice that

$$\lambda_{k,f/2}(\varpi_E)^{1+q^{(2l+1)/2}} = \sigma^k \phi^{(2l+1)/2} (\lambda_{k,f/2}(\varpi_E))^1 = 1$$

and so

$$\lambda_{k,f/2}(\varpi_E) \in \mu_{1+q^{(2l+1)/2}} = \ker(N_{E_g/E_{\pm g}}).$$

Since

$$\langle \mu_E, \mu_{E_\pm} \rangle \cap \ker(N_{E_g/E_{\pm g}}) = \mu_{1+q^{(2l+1)/2}} (\mu_{1+q^{(2l+1)/2} - 1} \cap \mu_{1+q^{(2l+1)/2}}) = \mu_{q^{l/2} + 1}$$
and $t_\mu^0(V_{[g]})|\mu_{g^2+1} = 1$, we can extend $\chi_g$ so that $\chi_g|\ker(N_{E_g/E_g}) = 1$ and hence (20) satisfies the second condition of (28). It remains to show that (20) satisfies the last condition of (28) and hence determines $\chi_g(W_\varpi)$ as in (27). The proof is the same as that in the cases $[g] = [\sigma^k] \in (W_E/W_F/W_E)$sym-unram. In case $\lambda_{k,f/2}(W_\varpi)$ has odd order, we can show that $\chi_g(\zeta_0)$ must be 1 and also $\left(\frac{\lambda_{k,f/2}(W_\varpi)}{\mu_{g^2+1}}\right) = 1$. We have that

$$-t_\mu^0(V_{[g]}) t_{\varpi}(V_{[g]}) = -t_\mu^0(V_{[g]}) t_{\varpi}(V_{[g]}) t_{\varpi}^1(V_{[g]})(W_\varpi)$$

$$= \begin{cases} -(1)(1)(1) = -1 & \text{if } V_{[g]} \text{ is trivial} \\ -(-1)(-1) \left(\frac{\lambda_{k,f/2}(W_\varpi)}{\mu_{g^2+1}}\right) = -1 & \text{if } V_{[g]} \text{ is non-trivial} \end{cases}$$

Hence we should assign $\chi_g(W_\varpi) = -1$. In case $\lambda_{k,f/2}(W_\varpi)$ has even order, we can assign $\chi_g(\zeta_0)$, hence $\chi_g(W_\varpi)$, to be either $\pm 1$. We then assign $\chi_g(W_\varpi)$ as in (27).

We have checked that each $\chi_g$ is a character in $\chi$-data for symmetric unramified $[g]$. Together with the asymmetric $[g]$ and those $[\sigma^k \phi^f] \in D_0$ which are all automatically asymmetric, we can check whether

$$\chi_{D_0}(W_\varpi) = K/F \mu_{\xi}(W_\varpi). \tag{20}$$

By Theorem 5.2 of [BH10] the value of the right side is

$$(1)^{e(f-1)} t_{\mu}^0(V_{D_0}) t_{\varpi}^1(V_{D_0}) t_{\varpi}(V_{D_0}).$$

The $t$-factors on both sides of (20) match by an easy comparison. Canceling the $t$-factors, the remaining sign on the left side of (20) is the parity of $\#(W_E/W_F/W_E)$sym-unram. To show that this sign equals the remaining one on the right side, namely $(-1)^{e(f-1)}$, we apply the following fact.

**Proposition 3.5.** The parity of $\#(W_E/W_F/W_E)$sym-unram = $\#\{[\sigma^k \phi^f] \in (W_E/W_F/W_E)$sym \} is equal to that of $e(f-1)$.

The proof will be in the last subsection. Granting this fact, we have proved Theorem 3.1 for $[g] \in D_0$. Hence we have finished the proof of Theorem 3.1 for the residual characteristic $p$ being odd.

The case when the residual characteristic $p = 2$ is much simpler. When $E/F$ is tamely ramified, we have the sequence of subfields $F \subseteq K \subseteq E$ where $K/F$ is unramified and $E/K$ is totally ramified and has odd degree. Since the order of $\Psi_{E/F}$ is odd, all sign characters and Jacobi symbols are trivial. Hence Theorem 3.1 reduces to the following.

1. If $[g] \in (W_E/W_F/W_E)$sym, then

$$\chi_g|\mu_{E_g} = 1 \text{ and } \chi_g(W_\varpi) = \text{ anything appropriate.}$$

2. If $[g] \in (W_E/W_F/W_E)$sym, then

$$\chi_g|\mu_{E_g} = 1 \text{ and } \chi_g(W_\varpi) = t_{\varpi}^1(V_{[g]}) t(W_{[g]}) = -1.$$
Proof. We first compute $\prod_{[g]\in(W_E\setminus W_L)/W_E'} \chi_g|\mu_E$ if $e(E/L)$ is odd, then

$$\prod_{[g]\in(W_E\setminus W_L)/W_E'} \chi_g|\mu_E = \prod_{[g]\in(W_E\setminus W_L)/W_E'} t_{\mu_E}^1(V_g) = t_{\mu_E}^1(V_L).$$

which equals $L\mu_\xi|\mu_E$ in this case. If $e(E/L)$ is even, then

$$\prod_{[g]\in(W_E\setminus W_L)/W_E'} \chi_g|\mu_E = \left(\prod_{[g]\in(W_E\setminus W_L)/W_E'} t_{\mu_E}^1(V_g)\right) \left(\prod_{[g]\in(W_E\setminus W_L)/W_E'} t_{\mu_E}^0(V_g)\right) \left(\prod_{[g]\in(W_E\setminus W_L)/W_E'} t_{\mu_E}^0(V_g)\right).$$

which also equals $L\mu_\xi|\mu_E$ in this case. We then compute $\prod_{[g]\in(W_E\setminus W_L)/W_E'} \chi_g|\mu_E|\varpi_E$. Let $M$ be the maximal unramified extension in $E/L$ and $V_{M/L}$ be the complementary module of $V^M$ in $V^L$, then

$$\prod_{[g]\in(W_E\setminus W_L)/W_E'} \chi_g|\varpi_E = \left(\prod_{[g]\in(W_E\setminus W_L)/W_E'} t_{\varpi_E}^1(V_g)\right) \left(\prod_{[g]\in(W_E\setminus W_L)/W_E'} t_{\varpi_E}^0(V_g)\right).$$

We re-group the $t$-factors and obtain

$$(t_{\varpi_E}^1(V^M)t(W^M)) \left((-1)^{e(E/L)(f(E/L)-1)}t_{\mu_E}^1(V_{M/L})t_{\mu_E}^0(V_{M/L})t_{\varpi_E}(V_{M/L})\right).$$

The first factor is $M\mu_\xi(\varpi_E)$, which the second factor is $M\mu_\xi(\varpi_E)$. Therefore the product is $L\mu_\xi(\varpi_E)$.

3.4 Proof of Proposition 3.5

The proof of Proposition 3.5 relies on knowing the parity of the number of double cosets $W_E\setminus W_K/W_E$ when $E/K$ is totally ramified. If $|E/K| = e$ and $q = q_K$, then by section 2.5 it is equivalent to check the number of $q$-orbits of $\mathbb{Z}/e$. Let $o(q,d)$ be the multiplicative order of $q$ in $(\mathbb{Z}/d)^\times$.

Lemma 3.8. (i) The number of double cosets $W_E\setminus W_F/W_E$ equals $\sum_{d|e} \phi(d)/o(q,d)$.

(ii) If $d = 1$ or 2, then $\phi(d)/o(q,d) = 1$. If $d \geq 3$ and $q$ is a square, then $\phi(d)/o(q,d)$ is even.

Proof. We can partition $\mathbb{Z}/e$ into subsets $\bigcup_{d|e} (e/d)^\times$. Since $(q,e) = 1$, each subset is $q$-invariant and has $q$-orbits of the same cardinality $o(q,d)$. Hence (i) is proved. (ii) is an easy calculation.

Proof. (of Proposition 3.5) If $f$ is odd, then the set $\{[\sigma^k\phi^{f/2}]\}$ is empty, so the statement is true. Now we assume that $f$ is even. Since $\phi^{f/2}$ normalizes $W_E$, we have indeed a bijection

$$\{[\sigma^k\phi^{f/2}] \in W_E\setminus W_F/W_E\} \leftrightarrow \{[\sigma^k] \in W_E\setminus W_K/W_E\}.$$ 

Since those asymmetric $[\sigma^k\phi^{f/2}]$ pair up, it suffices to show that the parity of $W_E\setminus W_K/W_E$ is the same as the parity of $e$. It is then clear by Proposition 3.8(1).

4 Related results (in preparation)

To appear
5 Appendix (in preparation)

To appear

5.1 Transfer factors

5.2 Archimedean case

References

[AC89] James Arthur and Laurent Clozel. *Simple algebras, base change, and the advanced theory of the trace formula*, volume 120 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1989.

[BH05a] Colin J. Bushnell and Guy Henniart. The essentially tame local Langlands correspondence. I. *J. Amer. Math. Soc.*, 18(3):685–710 (electronic), 2005.

[BH05b] Colin J. Bushnell and Guy Henniart. The essentially tame local Langlands correspondence. II. Totally ramified representations. *Compos. Math.*, 141(4):979–1011, 2005.

[BH10] Colin J. Bushnell and Guy Henniart. The essentially tame local Langlands correspondence, III: the general case. *Proc. Lond. Math. Soc. (3)*, 101(2):497–553, 2010.

[BK93] C.J. Bushnell and P.C. Kutzko. *The admissible dual of GL(N) via compact open subgroups*. Annals of mathematics studies. Princeton University Press, 1993.

[HH95] Guy Henniart and Rebecca Herb. Automorphic induction for GL(n) (over local non-Archimedean fields). *Duke Math. J.*, 78(1):131–192, 1995.

[HT01] Michael Harris and Richard Taylor. *The geometry and cohomology of some simple Shimura varieties*, volume 151 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2001. With an appendix by Vladimir G. Berkovich.

[Lan94] Serge Lang. *Algebraic number theory*, volume 110 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1994.

[LS87] R. P. Langlands and D. Shelstad. On the definition of transfer factors. *Math. Ann.*, 278(1-4):219–271, 1987.

[Moy86] Allen Moy. Local constants and the tame Langlands correspondence. *Amer. J. Math.*, 108(4):863–930, 1986.

[Rei91] Harry Reimann. Representations of tamely ramified $p$-adic division and matrix algebras. *J. Number Theory*, 38(1):58–105, 1991.

[Tama] Geo Kam-Fai Tam. On Certain Admissible Embeddings of L-groups. [arXiv:1109.4529v1 [math.RT]].

[Tamb] Geo Kam-Fai Tam. On finite symplectic modules arising from supercuspidal representations. (preprint).

[Tat79] J. Tate. Number theoretic background. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977)*, Part 2, Proc. Sympos. Pure Math., XXXIII, pages 3–26. Amer. Math. Soc., Providence, R.I., 1979.