LOCAL SATURATION AND SQUARE EVERYWHERE

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Abstract. We show that it is consistent relative to a huge cardinal that for all infinite cardinals \( \kappa \), \( \square_\kappa \) holds and there is a stationary \( S \subseteq \kappa^+ \) such that \( \text{NS}_{\kappa+} \upharpoonright S \) is \( \kappa^+ \)-saturated.

1. Introduction

In his work on Suslin’s problem, Jensen introduced the principles square \( \square \) and diamond \( \Diamond \) and proved that they hold everywhere in Gödel’s constructible universe \( L \) \[12\]. More specifically, in \( L \), \( \square_\kappa \) holds for every infinite cardinal \( \kappa \), and \( \Diamond_\kappa(S) \) holds for all regular uncountable \( \kappa \) and all stationary \( S \subseteq \kappa \). A natural opposite of \( \Diamond_\kappa(S) \) is the statement that the nonstationary ideal on \( \kappa \) restricted to \( S \), denoted \( \text{NS}_\kappa \upharpoonright S \), is \( \kappa^+ \)-saturated. If such a stationary set \( S \subseteq \kappa \) exists, we will say that \( \text{NS}_\kappa \) is locally saturated. While it is consistent relative to large cardinals that \( \text{NS}_\kappa \) is locally saturated for a variety of cardinals \( \kappa \), Gitik and Shelah \[10\] proved that the unrestricted \( \text{NS}_\kappa \) can never be \( \kappa^+ \)-saturated, except in the case \( \kappa = \omega_1 \). For background on saturated ideals and related topics, see \[7\].

Forcing \( \text{NS}_\kappa \) to be locally saturated typically results in the failure of \( \square \) in the vicinity of \( \kappa \) as a side-effect. Moreover, Foreman \[7\] observed that certain structural properties of Boolean algebras of the form \( \mathcal{P}(\kappa^+)/I \) for \( \kappa^+ \)-complete ideals \( I \) can imply the failure of \( \square_\kappa \). A similar result was observed by Zeman in unpublished work, which we reproduce here with his permission. Furthermore, square principles are generally opposed to very large cardinals, while saturation properties of ideals can carry significant large cardinal strength \[15\]. It is thus natural to ask what is the extent of the tension between these kinds of principles. We address the situation with the following result:

Theorem 1. It is consistent relative to a huge cardinal that for all infinite cardinals \( \kappa \), \( \square_\kappa \) holds and \( \text{NS}_{\kappa+} \) is locally saturated.

This improves a result of Foreman \[6\], who proved that it is consistent relative to a huge cardinal that every successor cardinal \( \kappa \) carries a \( \kappa^+ \)-saturated ideal.

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The saturation of $\text{NS}_{\omega_1}$ is equiconsistent with a Woodin cardinal; this is due to Shelah and Jensen-Steel [13]. For general successors of regular cardinals, Woodin showed in unpublished work how to force local saturation from an almost-huge cardinal, and details were given by Foreman and Komjath [9]. Because of the particulars of their construction, it was not immediately clear how to extend the result to get the nonstationary ideal to be locally saturated on several successive cardinals at once. We overcome this technical challenge by using a crucial observation of Usuba and by building on the alternative approach to saturated ideals forged by Shioya [17]. In order to achieve the global result, we use Cummings’ method of interleaving posets into Radin forcing [2]. An earlier version of this manuscript used a supercompact-based Radin forcing. The advantage of Cummings’ method is that since it is based on a degree of strongness rather than supercompactness, it is possible to carry out in a universe in which square holds everywhere.

The paper is organized as follows. In Section 2, we present the essential background material on forcing and large cardinals that we will need. In Section 3, we present a “modular” version of the Foreman-Komjath construction that allows us to transform a saturated ideal on a successor cardinal into a localization of the nonstationary ideal while retaining saturation, given that the original ideal satisfies certain combinatorial properties. In Section 4, we define a type of collapse forcing that, when combined with certain large cardinals, forces the existence of saturated ideals on successor cardinals that possess the desired combinatorial properties in a rather indestructible way. In Section 5, we construct a preparatory model in which square holds everywhere, local saturation holds at the first few successors of every Mahlo cardinal, and there exists a superstrong cardinal. Finally, in Section 6, we complete the proof of Theorem 1 using a version of Radin forcing that achieves the desired property at all successors of limits by ensuring that every such cardinal was a successor of a large cardinal in the preparatory model, while interleaving posets that recreate the desired situation at double successors.

2. Preliminaries

2.1. General forcing facts. We start by recalling some general notions and folklore results about forcing, most of which we state without proof.

A partial order $P$ is said to be separative when $p \not\leq q \Rightarrow (\exists r \leq p) r \perp q$. Every partial order $P$ has a canonically associated equivalence relation $\sim_s$ and a separative quotient $P_s$, which is isomorphic to $P$ if $P$ is already separative. For every separative partial order $P$, there is a canonical complete Boolean algebra $B(P)$ with a dense set isomorphic to $P$.

A map $e : P \to Q$ is an embedding when it preserves order and incompatibility. An embedding is said to be regular when it preserves the maximality of antichains. If $P \subseteq Q$, we say $P$ is a regular suborder if the identity map from $P$ to $Q$ is a regular embedding. A order-preserving map $\pi : Q \to P$ is called a projection when $\pi(1_Q) = 1_P$, and $p \leq \pi(q) \Rightarrow (\exists q' \leq q) \pi(q') \leq p$.

Lemma 2.1. Suppose $P$ and $Q$ are partial orders.

1. $G$ is a generic filter for $P$ if and only if $\{[p]_s : p \in G\}$ is a generic filter for $P_s$.

2. $e : P \to Q$ is a regular embedding if and only if for all $q \in Q$, there is $p \in P$ such that for all $r \leq p$, $e(r)$ is compatible with $q$.

3. The following are equivalent:
(a) There is a regular embedding \( e : P_s \rightarrow B(Q_s) \).
(b) There is a projection \( \pi : Q_s \rightarrow B(P_s) \).
(c) There is a \( Q \)-name \( \dot{g} \) for a \( P \)-generic filter such that for all \( p \in P \), there is \( q \in Q \) such that \( q \Vdash p \in \dot{g} \).

4. Suppose \( \pi : Q \rightarrow P \) is a projection. If \( G \) is a filter on \( P \), let \( Q/G = \pi^{-1}[G] \). The following are equivalent:
   (a) \( H \) is \( Q \)-generic over \( V \).
   (b) \( G = \pi[H] \) is \( P \)-generic over \( V \), and \( H \) is \( Q/G \)-generic over \( V[G] \).

**Lemma 2.2.** Suppose \( P \) and \( Q \) are partial orders. \( B(P_s) \cong B(Q_s) \) if and only if the following holds. Letting \( G, H \) be the canonical names for the generic filters for \( P, Q \) respectively, there is a \( P \)-name for a function \( f_0 \) and a \( Q \)-name for a function \( \dot{f}_1 \) such that:

1. \( \Vdash_P f_0(G) \) is a \( Q \)-generic filter,
2. \( \Vdash_Q \dot{f}_1(H) \) is a \( P \)-generic filter,
3. \( \Vdash_P G = f_0^P(f_0(G)) \), and \( \Vdash_Q \dot{H} = \dot{f}_1^Q(\dot{f}_1(H)) \).

An isomorphism is given by \( p \mapsto ||\dot{p} \in \dot{f}_1(\dot{H})||_{B(Q_s)} \).

A partial order \( P \) is said to be \( \kappa \)-closed when any descending sequence of elements of length less than \( \kappa \) has a lower bound. A weaker property is being \( \kappa \)-strategically closed, which is when the “good” player has a winning strategy in the following game: \( \text{Bad} \) starts by playing some element \( p_0 \in P \), and \( \text{Good} \) must play some \( p_1 \leq p_0 \). The players alternate in choosing elements of a descending sequence, with \( \text{Good} \) playing at limit stages. \( \text{Good} \) wins if sequence of length \( \kappa \) is produced, and \( \text{Bad} \) wins if at some stage \( \alpha < \kappa \), a sequence has been produced with no lower bound. A still weaker property is being \( \kappa \)-distributive, which means that the intersection of fewer than \( \kappa \) dense open sets is dense.

If \( \kappa < \lambda \) are ordinals, \( \text{Col}(\kappa, \lambda) \) is the collection of function whose domain is a bounded subset of \( \kappa \) and whose range is contained in \( \lambda \), ordered by \( p \leq q \) if \( p \supseteq q \).

We will use the following well-known lemma about \( \kappa \)-closed forcing:

**Lemma 2.3.** If \( P \) is a \( \kappa \)-closed partial order that forces \( |P| = \kappa \), then \( B(P) \cong B(\text{Col}(\kappa, |P|)) \).

**Lemma 2.4.** If \( \kappa^+ = \kappa \) and \( P \) is \( (\kappa + 1) \)-strategically closed, then \( P \) preserves stationary subsets of \( \kappa^+ \).

**Proof.** Let \( \sigma \) be a strategy witnessing that \( P \) is \( (\kappa + 1) \)-strategically closed, and let \( p_0 \in P \). Let \( S \subseteq \kappa^+ \) be stationary and \( \dot{C} \) be a \( P \)-name for a club. Let \( \theta \) be a large regular cardinal, and let \( \langle M_\alpha : \alpha < \kappa^+ \rangle \) be an increasing continuous sequence of elementary submodels of \( \langle H_\theta, \in, P, \sigma, S \rangle \), each of size \( \kappa \), having transitive intersection with \( \kappa^+ \), and such that \( M_\alpha \cap \kappa^+ = \alpha^* \in S \). Let \( \lambda = \text{cf}(\alpha^*) \), and let \( \langle \beta_i : i < \lambda \rangle \) be an increasing sequence converging to \( \alpha^* \). Build a descending chain \( \langle p_i : i < \lambda \rangle \subseteq M_{\alpha^*} \) below \( p_0 \) such that at odd \( i \), \( p_i \) decides some ordinal \( \geq \beta_i \) to be in \( \dot{C} \), and even stages are chosen by following \( \sigma \). Since \( M_{\alpha^*} \subseteq M_{\alpha^*} \), the construction continues, and there is a condition \( p_\lambda \) below all conditions chosen. \( p_\lambda \Vdash \alpha^* \in \dot{C} \cap S \). □

A partial order \( P \) has the \( \kappa \)-chain condition (\( \kappa \)-c.c.) if every antichain \( A \subseteq P \) has size \( < \kappa \).
Lemma 2.5 (Easton). Suppose \( P, Q \) are partial orders, \( Q \) is \( \kappa \)-distributive, and \( \Vdash_Q P \) is \( \kappa \)-c.c. Then \( \Vdash_P Q \) is \( \kappa \)-distributive.

Proof. Suppose \( G \times H \) is \( P \times Q \)-generic, and \( X \) is a sequence of ordinals of length \( < \kappa \) in \( V[G][H] \). Then in \( V[H] \), \( X \) has a \( P \)-name \( \tau \), and by the \( \kappa \)-c.c., \( \tau \) can be assumed to be a subset of \( V \) of size \( < \kappa \). By the distributivity of \( Q \), \( \tau \in V \), so \( \tau^G = X \in V[G] \).

The above lemma was crucial in Easton’s proof \(^4\) that the continuum function can be “anything reasonable.” There, he introduced the notion of an Easton-support product, which we will use several times. A set of ordinals \( X \) is **Easton** for every regular cardinal \( \kappa \), \( |X \cap \kappa| < \kappa \). A collection of partial functions has **Easton support** if the domain of each function in the collection is an Easton set of ordinals.

We will use several times a stronger version of the \( \kappa \)-c.c. introduced by Shelah \(^{10}\) which is easier to preserve under other forcings:

**Definition.** Let \( \kappa \) be a regular cardinal and \( S \subseteq \kappa \). A partial order \( P \) is \( S \)-layered if there is a \( \subseteq \)-increasing sequence of regular suborders \( \langle Q_\alpha : \alpha < \kappa \rangle \) such that \( P = \bigcup_{\alpha < \kappa} Q_\alpha \), each \( |Q_\alpha| < \kappa \), and for some club \( C \) and all \( \alpha \in S \cap C \), \( Q_\alpha = \bigcup_{\beta < \alpha} Q_\beta \).

**Lemma 2.6.** If \( S \) is a stationary subset of \( \kappa \) and \( P \) is \( S \)-layered, then \( P \) is \( \kappa \)-c.c.

**Lemma 2.7.** Suppose \( S \subseteq \kappa \) is stationary, \( P \) is \( S \)-layered, \( \Vdash_P Q \) is \( S \)-layered, and \( R \) is regular suborder of \( P \) of size \( < \kappa \). Then:

1. \( P \ast Q \) is \( S \)-layered.
2. \( \Vdash_R P / G \) is \( S \)-layered.

Proof. For (1), let \( \langle P_\alpha : \alpha < \kappa \rangle \) witness that \( P \) is \( S \)-layered, and let \( \langle Q_\alpha : \alpha < \kappa \rangle \) be a sequence of \( P \)-names for a witness to the \( S \)-layeredness of \( Q \). By the \( \kappa \)-c.c., we may assume that we have an increasing sequence \( \langle \beta_\alpha : \alpha < \kappa \rangle \) of ordinals below \( \kappa \) such that for each \( \alpha \), \( Q_\alpha \) is a \( P_{\beta_\alpha} \)-name. Let \( C \subseteq \kappa \) be a club such that each \( \gamma \) in \( C \) is closed under \( \alpha \mapsto \beta_\alpha \). Now assume \( \gamma \in C \cap S \). \( P_\gamma = \bigcup_{\alpha < \gamma} P_\alpha \), and \( \Vdash \bigcup_{\alpha < \gamma} Q_\alpha \subseteq P_\gamma \). Hence we may form a \( P_\gamma \)-name for \( Q_\gamma \). It is routine to show that \( P_\gamma \ast Q_\gamma \) is a regular suborder of \( P \ast Q \).

For (2), let \( \gamma \) be such that \( R \subseteq P_\gamma \). Then \( R \) is regular in \( P_\alpha \) for \( \alpha \geq \gamma \). If \( G \subseteq R \) is generic, then \( P_\alpha / G = \bigcup_{\gamma \leq \beta < \alpha} P_{\beta} / G \) for all \( \alpha \in S \setminus \gamma \).

2.2. Large cardinals and generic embeddings. A cardinal \( \kappa \) is called **huge** if there is an elementary embedding \( j : V \rightarrow M \) with critical point \( \kappa \), where \( M \) is a transitive class such that \( M^{j(\kappa)} \subseteq M \). \( \kappa \) is called **almost-huge** if the closure requirement of \( M \) is weakened to \( M^{<j(\kappa)} \subseteq M \). \( \kappa \) is called **superstrong** if the requirement is weakened further to just \( V_{j(\kappa)} \subseteq M \). The value of \( j(\kappa) \) in each case will be called the **target**.

It is straightforward to show that \( \kappa \) is huge with target \( \lambda \) iff there is a normal \( \kappa \)-complete ultrafilter on \( |\lambda|^\kappa := \{ z \subseteq \lambda : \text{ot}(z) = \kappa \} \). The first-order characterizations of almost-hugeness and superstrongness are more complicated, and we refer the reader to \(^{14}\) for details. We will just need the following facts:

**Lemma 2.8.** Suppose \( \kappa \) is almost-huge with target \( \lambda \). Then there is an elementary \( j : V \rightarrow M \) with the following properties:

1. \( \text{crit}(j) = \kappa \), \( j(\kappa) = \lambda \), and \( M^{<\lambda} \subseteq M \).
Lemma 2.9. Suppose \( H \) Then by the directed closure, \( q \cap H \). Let \( \hat{\alpha} \) can find a filter \( \hat{\alpha} \) of size \( \kappa \) and \( M^\kappa \subseteq M \).

(4) The embedding is generated by a tower of measures \( T \subseteq V_\lambda \), which we will call a \((\kappa, \lambda)\)-tower. The fact that \( T \) generates an embedding with the above properties is equivalent to a first-order property in \( \langle V_\lambda, \in, T \rangle \).

Silver observed that if \( j : M \to N \) is an elementary embedding between models of set theory, \( P \in M \) is a partial order, and \( G \) is \( P \)-generic over \( M \), then \( j \) can be extended to an elementary embedding with domain \( M[G] \) if and only if we can find a filter \( \hat{G} \) that is \( j(P) \)-generic over \( N \), with \( j(G) \subseteq \hat{G} \). We now describe some general situations in which almost-huge and superstrong embeddings can be generically extended.

Definition. A partial order \( Q \) is \((\kappa, \lambda)\)-nice when there is a sequence \( \langle Q_\alpha : \alpha \leq \lambda \rangle \) of regular suborders such that:

1. The sequence is \( \subseteq \)-increasing, \( \bigcup_{\alpha \leq \lambda} Q_\alpha = Q_\lambda = Q \), and for all \( \alpha < \lambda \), \( |Q_\alpha| < \lambda \).
2. For each \( \alpha \leq \lambda \), any two compatible elements of \( Q_\alpha \) have an infimum in \( Q_\alpha \), and every directed subset of \( Q_\alpha \) of size \( \kappa \) has an infimum in \( Q_\alpha \).

Lemma 2.10. Suppose the following:

1. \( j : V \to M \) is an almost-huge embedding derived from a \((\kappa, \lambda)\)-tower.
2. \( P \subseteq V_\kappa \) is a partial order, and \( j(P) \) is \( \lambda \)-c.c. in \( V \).
3. \( \hat{Q} \) is a \( P \)-name for a \((\kappa, \lambda)\)-nice partial order.
4. There is a projection \( \pi : j(P) \to P * \hat{Q} \) such that for \( p \in P \), \( \pi(p) = (p, 1) \).

If \( \hat{G} \) is \( j(P) \)-generic over \( V \) and \( G * H = \pi[\hat{G}] \), then in \( V[\hat{G}] \) we can extend \( j \) to \( j[V[G * H]] \to M[\hat{G} * \hat{H}] \), such that \( M[\hat{G} * \hat{H}]^{<\lambda} \cap V[\hat{G}] \subseteq M[\hat{G} * \hat{H}] \).

Proof. Let \( \hat{G} \subseteq j(P) \) be generic, and let \( G * H = \pi[\hat{G}] \). Since \( \pi \) and \( j \) are the identity on \( P \), we can extend to \( j : V[G] \to M[\hat{G}] \). By the \( \lambda \)-c.c. and the closure of \( M \), \( \text{Ord}^{<\lambda} \cap V[\hat{G}] \subseteq M[\hat{G}] \). We will build the desired \( \hat{H} \) in \( V[\hat{G}] \), so we will get \( \text{Ord}^{<\lambda} \cap V[\hat{G}] \subseteq M[\hat{G} * \hat{H}] \) as well.

Let \( \langle Q_\alpha : \alpha \leq \lambda \rangle \) witness the \((\kappa, \lambda)\)-niceness of \( Q \) in \( V[G] \). For each \( \alpha < \lambda \), let \( H_\alpha = H \cap Q_\alpha \), and let \( m_\alpha = \inf j[H_\alpha] \), which exists because \( j[H_\alpha] \) is an element of \( M[\hat{G}] \) and a directed subset of \( j(Q_\alpha) \) of size \( < j(\kappa) \). Let us observe the following: If \( \alpha < \beta < \lambda \) and \( q \leq m_\alpha \) is in \( j(Q_\alpha) \), then \( q \) is compatible with \( m_\beta \). To show this, note that for any \( r \in Q_\beta \), set \( D_r = \{ p \in Q_\alpha : p \perp r \text{ or } (\forall p' \leq p) p' \notin r \} \) is dense in \( Q_\alpha \). Suppose towards a contradiction that \( q \leq m_\alpha \) is in \( j(Q_\alpha) \) and \( q \perp m_\beta \). Then by the directed closure, \( q \perp j(r) \) for some \( r \in H_\beta \). But there is \( p \in D_r \cap H_\alpha \),
and $p \not\leq r$. Since $q \leq m_\alpha \leq j(p)$, by elementarity $q$ is compatible with $j(r)$, a contradiction.

Let $(A_\alpha : \alpha < \lambda)$ list in $V(\hat{G})$ the maximal antichains of $j(Q)$ that live in $M[\hat{G}]$. We inductively build a filter $\hat{H}$ that is $j(Q)$-generic over $M[\hat{G}]$, and contains each $m_\alpha$. This will guarantee $j[H] \subseteq \hat{H}$, and thus by Silver’s criterion we will be done. Choose an increasing sequence of ordinals $\langle \beta_\alpha : \alpha < \lambda \rangle$ such that $A_\alpha \subseteq j(Q_{\beta_\alpha})$. Find some $a_0 \in A_0$ such that $m_{\beta_0}$ is compatible with $a_0$, and let $q_0 = m_{\beta_0} \wedge a_0$. Suppose inductively that for some $\gamma < \lambda$, we have constructed a descending sequence $\langle q_\alpha : \alpha < \gamma \rangle$ such that for each $\alpha$, $q_\alpha \in j(Q_{\beta_\alpha})$ and $q_\alpha \leq a_\alpha \wedge m_{\beta_\alpha}$ for some $a_\alpha \in A_\alpha$. By the observation of the previous paragraph, $m_{\beta_\alpha}$ is compatible with $q_\alpha$ for all $\alpha < \gamma$. Hence the directed set $\{m_{\beta_\alpha} \wedge q_\alpha : \alpha < \gamma\}$ has an infimum $q'_\gamma$. Let $q_\gamma = q'_\gamma \wedge a_\gamma$ for some $a_\gamma \in A_\gamma$. This completes the induction. \(\square\)

Lemma 2.11. Suppose the following:

1. $j : V \to M$ is an superstrong embedding derived from a $(\kappa, \lambda)$-extender with $\lambda$ inaccessible.
2. $P \subseteq V_\kappa$ is a partial order, and $j(P)$ is $\lambda$-c.c. in $V$.
3. $P$ forces that the quotient $j(P)/G$ is $\kappa^+$-distributive.

If $\hat{G}$ is $j(P)$-generic over $V$, then $\kappa$ is superstrong with target $\lambda$ in $V[\hat{G}]$.

Proof. Since $P$ is $\kappa$-c.c., $j(A) = A$ for every maximal antichain $A \subseteq P$, so $P$ is a regular suborder of $j(P)$. Let $\alpha < \kappa$. Since every subset of $\alpha$ added by $j(P)$ is added by $P$, reflection gives that there is a regular suborder $Q \subseteq P$ of size $< \kappa$ that adds all subsets of $\alpha$. Thus $M$ satisfies that for every $\alpha < \lambda$, there is a regular suborder $Q \subseteq j(P)$ of size $< \lambda$ that adds all subsets of $\alpha$. This is true in $V$ as well, since by the $\lambda$-c.c., any $j(P)$-name $\tau$ for a subset of $\alpha$ is equivalent to an $Q'$-name $\tau'$ for some regular suborder $Q' \subseteq V_\lambda \subseteq M$, and $\tau'$ must be equivalent to a $Q'$-name by what $M$ thinks of $Q$. Thus $\lambda$ remains inaccessible after forcing with $j(P)$.

Let $G$ be $j(P)$-generic over $V$, and let $G = \hat{G} \cap P$. We can extend $j$ to $j : V[G] \to M[\hat{G}]$. By the $\lambda$-c.c., every element of $(V_\lambda)^{V[\hat{G}]}$ is $\tau^{\hat{G}}$ for some $j(P)$-name $\tau$ in $(V_\lambda)^V$. Since $V_\lambda \subseteq M$, $(V_\lambda)^{M[G]} = (V_\lambda)^{V[\hat{G}]}$.

For every $x \in M[\hat{G}]$, there is a $j(P)$-name $\tau$ such that $x = \tau^{\hat{G}}$, and there is an $a \in [\lambda]^{<\omega}$ and a function $f$ with domain $[\kappa]^{<\omega}$ in $V$ such that $\tau = j(f)(a)$. We may assume that for every $b \in \text{dom } f$, $f(b)$ is a $P$-name. If we define a function $g$ in $V[G]$ by $g(b) = f(b)^{\hat{G}}$, then we have $x = j(g)(a)$.

Let $Q$ be the quotient $j(P)/G$ in $V[G]$, and let us write $\hat{G}$ as $G * H \subseteq P * \hat{Q}$. Let $D \in M[\hat{G}]$ be dense open subset of $j(Q)$. Let $a, g$ be a such that $a \in [\lambda]^{<\omega}$, $g \in V[G]$ is a function with domain $[\kappa]^{<\omega}$, and $D = j(g)(a)$. We may assume that $g(b)$ is a dense open subset of $P$ for all $b \in \text{dom } g$. Let $E = \bigcap_{b \in \text{dom } g} g(b)$. By the distributivity of $Q$, $E$ is dense. Thus there is $q \in E \cap H$. Since $E \subseteq g(b)$ for all $b \in \text{dom } g$, $j(q) \in j(E) \subseteq D$. Therefore the image $j[H]$ generates a filter $\hat{H}$ which is $j(Q)$-generic over $M[G]$. We may extend the embedding to $j : V[G * H] \to M[\hat{G} * H]$. Since $(V_\lambda)^{V[G]} = (V_\lambda)^{M[G]} \subseteq (V_\lambda)^{M[\hat{G} * H]} \subseteq (V_\lambda)^{V[\hat{G}]}, \kappa$ is superstrong with target $\lambda$ in $V[\hat{G}]$. \(\square\)

2.3. Ideals and duality. We will be interested in extending embeddings that arise from forcing with Boolean algebras of the form $P(Z)/I$, where $I$ is an ideal over $Z$, and in computing what happens to this algebra in generic extensions. To this
end, we present an optimal generalization of a result of Foreman [8] on this topic from the author’s thesis [3]. Let us first review some basic facts concerning ideals, which can be found in [2]. An ideal $I$ over a set $Z$, gives a notion of smallness or “$I$-measure-zero” for subsets of $Z$. We will sometimes refer to the family $\mathcal{P}(Z) \setminus I$ as $I^+$ or the “$I$-positive sets,” and refer to the family $\{Z \setminus A : A \in I\}$ as $I^*$ or the “$I$-measure-one sets.” Recall that an ideal $I$ over a set $Z$ is called precipitous if whenever $G \subseteq \mathcal{P}(Z)/I$ is generic, then the ultrapower $V^Z/G$ is well-founded.

**Fact 2.12.** Let $I$ be an ideal over $Z \subseteq \mathcal{P}(\lambda)$. Suppose that $I$ is normal and $\lambda^+$-saturated. Then $I$ is precipitous, and whenever $j : V \rightarrow M \subseteq V[G]$ is a generic ultrapower embedding arising from $I$, then $M^\lambda \cap V[G] \subseteq M$. Furthermore, if $\kappa = \mu^+$, then $\{z \in Z : cf(\sup z) = cf(\mu)\} \in I^*$.

**Theorem 2.13.** Suppose $I$ is a precipitous ideal on $Z$ and $\mathbb{P}$ is a Boolean algebra. Let $j : V \rightarrow M \subseteq V[\mathbb{P}]$ denote a generic ultrapower embedding arising from $I$. Suppose $\hat{K}$ is a $\mathcal{P}(Z)/I$-name for an ideal on $j(\mathbb{P})$ such that whenever $G * h$ is $\mathcal{P}(Z)/I * j(\mathbb{P})/\hat{K}$-generic and $H = \{p : [p]_K \in h\}$, we have:

1. $1 \forces_{\mathcal{P}(Z)/I * j(\mathbb{P})/\hat{K}} \hat{H}$ is $j(\mathbb{P})$-generic over $M$,
2. $1 \forces_{\mathcal{P}(Z)/I * j(\mathbb{P})/\hat{K}} j^{-1}[\hat{H}]$ is $\mathbb{P}$-generic over $V$, and
3. for all $p \in \mathbb{P}$, $1 \not\forces_{\mathcal{P}(Z)/I} j(p) \in \hat{K}$.

Then there is $\mathbb{P}$-name $\hat{J}$ for an ideal on $Z$ and a canonical isomorphism

$$\iota : B(\mathbb{P} * \mathcal{P}(Z)/\hat{J}) \cong B(\mathcal{P}(Z)/I * j(\mathbb{P})/\hat{K}).$$

Proof. Let $e : \mathbb{P} \rightarrow B(\mathcal{P}(Z)/I * j(\mathbb{P})/\hat{K})$ be defined by $p \mapsto ||j(p)\in \hat{H}||$. By (3), this map has trivial kernel. By elementarity, it is an order and antichain preserving map. If $A \subseteq \mathbb{P}$ is a maximal antichain, then it is forced that $j^{-1}[\hat{H}] \cap A \neq \emptyset$. Thus $e$ is regular.

Whenever $H \subseteq \mathbb{P}$ is generic, there is a further forcing yielding a generic $G * h \subseteq \mathcal{P}(Z)/I * j(\mathbb{P})/\hat{K}$ such that $j[H] \subseteq \hat{H}$. Thus there is an embedding $j : V[H] \rightarrow M[\hat{H}]$ extending $j$. In $V[H]$, let $J = \{A \subseteq Z : 1 \forces_{\mathcal{P}(Z)/I * j(\mathbb{P})/\hat{K}/[id]_M} j(A)\}$. In $V$, define a map $\iota : \mathbb{P} * \mathcal{P}(Z)/\hat{J} \rightarrow B(\mathcal{P}(Z)/I * j(\mathbb{P})/\hat{K})$ by $(p, A) \mapsto e(p) \& ||[id]_M \in j(A)||$. It is easy to check that $\iota$ is order and antichain preserving.

We want to show the range of $\iota$ is dense. Let $(B, \hat{q}) \in \mathcal{P}(Z)/I * j(\mathbb{P})/\hat{K}$. Without loss of generality, there is some $f : Z \rightarrow V$ in $V$ such that $B \models \hat{q} = [[f]_M]_K$. By the regularity of $e$, let $p \in \mathbb{P}$ be such that for all $p' \leq p$, $e(p') \& (B, \hat{q}) \neq 0$. Let $\hat{A}$ be a $\mathbb{P}$-name such that $p \forces \hat{A} = \{z \in B : f(z) \in \hat{H}\}$, and $-p \forces \hat{A} = Z$. $1 \forces p \hat{A} \in j^+$. Because for any $p' \leq p$, we can take a generic $G * h$ such that $e(p') \& (B, \hat{q}) \in G * h$, here we have $[id]_M \in j(B)$ and $[f]_M \in \hat{H}$, so $[id]_M \in \hat{J}(A)$. Furthermore, $\iota(p, \hat{A})$ forces $B \in G$ and $q \in h$, showing $\iota$ is a dense embedding.

**Proposition 2.14.** If $Z, I, \mathbb{P}, \hat{J}, \hat{K}, \iota$ are as in Theorem 2.13, then whenever $H \subseteq \mathbb{P}$ is generic, $J$ is precipitous and has the same completeness and normality that $I$ has in $V$. Also, if $G \subseteq \mathcal{P}(Z)/\hat{J}$ is generic and $G * h = \iota[H * \hat{G}]$, then if $\hat{j} : V[H] \rightarrow M[\hat{H}]$ is as above, $M[\hat{H}] = V[H]^Z/\hat{G}$ and $\hat{j}$ is the canonical ultrapower embedding.

Proof. Suppose $H * G \subseteq \mathcal{P} * \mathcal{P}(Z)/\hat{J}$ is generic, and let $G * h = \iota[H * \hat{G}]$ and $\hat{H} = \{p : [p]_K \in h\}$. For $A \in j^+, A \in \hat{G}$ if and only if $[id]_M \in j(A)$. If $i : V[H] \rightarrow N = V[H]^Z/\hat{G}$ is the canonical ultrapower embedding, then there is
an elementary embedding \(k : N \rightarrow M[\hat{H}]\) given by \(k([f]_N) = \hat{j}(f)([\text{id}]_M)\), and \(\hat{j} = k \circ i\). Thus \(N\) is well-founded, so \(J\) is precipitous. If \(f : Z \rightarrow \text{Ord}\) is a function in \(V\), then \(k([f]_N) = \hat{j}(f)([\text{id}]_M) = [f]_M\). Thus \(k\) is surjective on ordinals, so it must be the identity, and \(N = M[\hat{H}]\). Since \(i = \hat{j}\) and \(\hat{j}\) extends \(j\), \(i\) and \(j\) have the same critical point, so the completeness of \(J\) is the same as that of \(I\). Finally, since \([\text{id}]_N = [\text{id}]_M\), \(J\) is normal in \(V\) if and only if \(J\) is normal in \(V[H]\), because \(j \upharpoonright \bigcup Z = j \upharpoonright \bigcup Z\), and normality is equivalent to \([\text{id}] = j(\bigcup Z)\).

Theorem 2.13 is optimal in the sense that it characterizes exactly when an elementary embedding coming from a precipitous ideal can have its domain enlarged via forcing:

**Proposition 2.15.** Let \(I\) be a precipitous ideal on \(Z\) and \(P\) a Boolean algebra. The following are equivalent.

1. In some generic extension of a \(P(Z)/I\)-generic extension, there is an elementary embedding \(j : V[H] \rightarrow M[\hat{H}]\), where \(j : V \rightarrow M\) is the elementary embedding arising from \(I\) and \(H\) is \(P\)-generic over \(V\).
2. There are \(p \in P\), \(A \in I^+\), and a \(P(A)/I\)-name \(\hat{K}\) for an ideal on \(j(P \upharpoonright p)\) such that \(P(A)/I \ast j(P \upharpoonright p)/\hat{K}\) satisfies the hypothesis of Theorem 2.13.

**Proof.** (2) \(\Rightarrow\) (1) is trivial. To show (1) \(\Rightarrow\) (2), let \(\hat{Q}\) be a \(P(Z)/I\)-name for a partial order, and suppose \(A \in I^+\) and \(\hat{H}_0\) are such that \(\Vdash_{P(A)/I \ast \hat{Q}} "\hat{H}_0\) is \(j(P)\)-generic over \(M\) and \(j^{-1}[\hat{H}_0]\) is \(P\)-generic over \(V\).” By the genericity of \(j^{-1}[\hat{H}_0]\), the set of \(p \in P\) such that \(\Vdash_{P(A)/I \ast j(P \upharpoonright p)}/\hat{K}\) has a \(P\)-generic over \(V\). Let \(G\) be \(P(A)/I \ast j(P \upharpoonright p)\)-generic. In \(V[G \ast h]\), let \(\hat{H} = \{ p \in j(P \upharpoonright p_0) : [p]_K \in h \}\).

1. If \(D \in M\) is open and dense in \(j(P \upharpoonright p_0)\), then \(\{ [d]_K : d \in D \text{ and } d \notin K \}\) is dense in \(j(P \upharpoonright p_0)/K\). For otherwise, there is \(p \in j(P \upharpoonright p_0) \setminus K\) such that \(p \land d \in K\) for all \(d \in D\). By the definition of \(K\), we can force with \(Q\) over \(V[G]\) to obtain a \(M\)-generic filter \(H_0 \subseteq j(P)\) with \(p \in H_0\). But \(H_0\) cannot contain any elements of \(D\), so it is not generic over \(M\), a contradiction. Thus if \(h \subseteq j(P \upharpoonright p_0)/K\) is generic over \(V[G]\), then \(\hat{H}\) is \(j(P \upharpoonright p_0)\)-generic over \(M\).
2. If \(A \in V\) is a maximal antichain in \(P \upharpoonright p_0\), then \(\{ [j(a)]_K : a \in A \text{ and } j(a) \notin K\}\) is a maximal antichain in \(j(P \upharpoonright p_0)/K\). For otherwise, there is \(p \in j(P \upharpoonright p_0) \setminus K\) such that \(p \land j(a) \in K\) for all \(a \in A\). We can force with \(Q\) over \(V[G]\) to obtain a filter \(H_0 \subseteq j(P)\) with \(p \in H_0\). But \(H_0\) cannot contain any elements of \(j[A]\), so \(j^{-1}[\hat{H}_0]\) is not generic over \(V\), a contradiction.
3. If \(p \in P \upharpoonright p_0\), \(\hat{H}_0\) is not generic over \(V\), a contradiction.

**Lemma 2.16.** Suppose the ideal \(K\) in Theorem 2.13 is forced to be principal. Let \(\hat{m}\) be such that \(\Vdash_{P(Z)/I} K = \{ p \in j(P) : p \leq \neg \hat{m}\}\). Suppose \(f\) and \(A\) are such that \(A \Vdash \hat{m} = [f]\), and \(\hat{B}\) is a \(P\)-name for \(\{ z \in A : f(z) \in H\}\). Let \(\hat{I}\) be the ideal generated by \(I\) in \(V[H]\). Then \(\hat{I} \upharpoonright B = J \upharpoonright B\), where \(J\) is given by Theorem 2.13.
For $\alpha < \mu$ and $\lambda$ a cardinal, and let $\langle p, q \rangle$ be a name for a function from $\mathbb{P}$ to $\lambda$. Without loss of generality, $\mathbb{P}$ is a complete Boolean algebra. For each $z \in Z$, let $b_z = \{z \in \mathcal{C}|z\}$. In $V$, define $C' = \{z : p_1 \land b_z \land f(z) \neq 0\}$. Let $G$ be a generic over $\mathbb{P}$ with $C' \subseteq I^+$. If $G \subseteq \mathcal{P}(\mathbb{P})/\mathcal{I}$ is generic with $C' \subseteq G$, then $j(p_1) \land b_{[\mu]} \land m \neq 0$. Take $\hat{H} \subseteq j(\mathbb{P})$ generic over $V[G]$ with $j(p_1) \land b_{[\mu]} \land m \in \hat{H}$. Since $b_{[\mu]} \parallel M, [\mu] \in j(C)$, $p_1 \not\in \mathcal{C} \in \hat{J}$ as $p_1 \in H = j^{-1}[\hat{H}]$. Thus $p_0 \parallel \mathcal{C} \in \hat{J}^+$.

**Corollary 2.17.** If $I$ is a $\kappa$-complete precipitous ideal on $Z$ and $\mathbb{P}$ is $\kappa$-c.c., then there is a canonical isomorphism $\iota : \mathbb{P} \ast \mathcal{P}(\mathbb{P})/\mathcal{I} \cong \mathcal{P}(\mathbb{P})/\mathcal{I} * j(\mathbb{P})$.

**Proof.** If $G \ast \hat{H} \subseteq \mathcal{P}(\mathbb{P})/\mathcal{I} * j(\mathbb{P})$ is generic, then for any maximal antichain $A \subseteq \mathbb{P}$ in $V$, $j[A] = j(A)$, and $M \models j(A)$ is a maximal antichain in $j(\mathbb{P})$. Thus $j^{-1}[\hat{I}]$ is $\mathbb{P}$-generic over $V$, and clearly for each $p \in \mathbb{P}$, we can take $\hat{H}$ with $j(p) \in \hat{H}$. Taking a $\mathcal{P}(\mathbb{P})/\mathcal{I}$-name $\hat{K}$ for the trivial ideal on $j(\mathbb{P})$, Theorem 2.13 implies that there is a $\mathbb{P}$-name $\hat{J}$ for an ideal on $Z$ and an isomorphism $\iota : B(\mathbb{P} \ast \mathcal{P}(\mathbb{P})/\hat{J}) \rightarrow B(\mathcal{P}(\mathbb{P})/\mathcal{I} * j(\mathbb{P}))$, and Lemma 2.10 implies that $\parallel \mathcal{P} \hat{J} = \hat{I}$.

**3. A local saturation module**

In this section, we show how to transform saturated ideals with certain properties into a restriction of the nonstationary ideal, while retaining saturation. Some key ideas are taken from [9]. Given a set of ordinals $S$, let $C(S)$ denote the forcing for shooting a club through $\text{sup}(S)$ by initial segments.

**Lemma 3.1.** Assume GCH, $\mu$ is regular, $\kappa = \mu^+$, and $S \subseteq \kappa \cap \text{cof}(\mu)$ is stationary. Let $\langle \mathbb{P}_\alpha, \check{Q}_\beta : \alpha \leq \lambda, \beta < \lambda \rangle$ be an iteration with $<\kappa$-supports such that for each $\alpha$, there is a $\mathbb{P}_\alpha$-name $\check{S}_\alpha$ for a subset of $\kappa$ such that $\parallel \mathbb{P}_\alpha \check{Q}_\alpha = C(\check{S}_\alpha \cup S \cup \text{cof}(<\mu))$. Then:

1. $\mathbb{P}_\lambda$ is $\mu$-closed.
2. $\mathbb{P}_\lambda$ is $\kappa$-distributive.
3. $\mathbb{P}_\lambda$ preserves every stationary $T \subseteq S$.
4. The set $\overline{\mathbb{P}}_\lambda = \{ p \in \mathbb{P}_\lambda : (\forall \alpha < \lambda)(\exists \beta \leq \kappa) p|\alpha \parallel \alpha p(\alpha) = \check{r}\}$, is dense in $\mathbb{P}_\lambda$.

**Proof.** (1) is easy. For (2) and (3), fix a stationary $T \subseteq S$, let $p \in \mathbb{P}_\lambda$, and let $\check{f}$ be a name for a function from $\mu$ to the ordinals. Let $\theta$ be a large enough regular cardinal, and let $N \prec H_\theta$ be elementary such that $N^{<\mu} \subseteq N$, $N \cap \kappa \in T$, $|N| = \mu$, and $p, \check{f}, \mathbb{P}_\lambda \in N$. List the dense open subsets of $\mathbb{P}_\lambda$ in $N$ as $\langle D_\alpha : \alpha < \mu \rangle$. Note that for all $\alpha < \kappa$, the set of $q$ such that for all $\beta \in \text{sprt}(q)$, $q \parallel \beta \parallel \text{sup}(q) > \check{\alpha}$ is dense. Note also that for all $q \in \mathbb{P}_\lambda \cap N$, $\text{sprt}(q) \subseteq N$. Using $\mu$-closure, build a descending chain $\langle q_\alpha : \alpha < \mu \rangle \subseteq N$ below $p$ such that each $q_\alpha \in D_\alpha$. Let $q$ be a function with domain $N \cap \kappa$ such that for all $\beta$, $q(\beta)$ is the canonical $\mathbb{P}_\beta$-name for $\bigcup_{\alpha < \mu} q_\alpha(\beta) \cup \{N \cap \kappa\}$. By induction we see that $q$ is a condition in $\mathbb{P}_\lambda$ below each $q_\alpha$.

If $q \parallel \beta$ is a condition below each $q_\alpha \parallel \beta$, then $q \parallel \beta \parallel \check{\beta}$ "$\langle q_\alpha(\beta) : \alpha < \mu \rangle$ is a chain of bounded closed subsets of $(\check{S}_\beta \cup S \cup \text{cof}(<\mu)) \cap N \cap \kappa$ ordered by end-extension, and $\{\text{sup} q_\alpha(\beta) : \alpha < \mu \}$ is unbounded in $N \cap \kappa$.” Hence $q|\beta \parallel q(\beta) \leq q_\alpha(\beta)$ for all $\alpha < \mu$. Limit steps are trivial. Thus $q$ decides $\check{f}$ and forces $T \cap \check{C} \neq \emptyset$.

For (4), we proceed by induction on $\lambda$. Suppose $\lambda = \beta + 1$ and the result holds for $\mathbb{P}_\beta$. If $p \in \mathbb{P}_\lambda$, then by $\kappa$-distributivity we may extend $p \parallel \beta$ to some $q \in \mathbb{P}_\beta$ such that $q \parallel p(\beta) = \check{r}$ for some $r \subseteq \kappa$. If $\text{cf}(\lambda) = \delta < \mu$, choose an increasing sequence $\langle \lambda_i : i < \delta \rangle$ cofinal in $\lambda$. Let $p \in \mathbb{P}_\lambda$, and build a descending chain
Note that in $V$ and every $X \to \pi$ induction hypothesis, we build a descending chain of length $\mu$ below this chain as in the previous claims, which will be in $\mathbb{P}_{\lambda}$. Finally, if $\text{cf}(\lambda) = \mu$, then given a $p \in \mathbb{P}_{\lambda}$, we take an elementary substructure $N$ with $p \in N$ as in the previous claims. Using the induction hypothesis, we build a descending chain of length $\mu$ of elements below $p$ contained in $\mathbb{P}_{\lambda}$, as in the case $\text{cf}(\lambda) < \mu$. We then construct a master condition $q$ below this chain as in the previous claims, which will be in $\mathbb{P}_{\lambda}$.

In a context like above where the fixed objects $\kappa, \mu, S$ are clear, we will abbreviate the forcing $\mathbb{C}(T \cup S \cup \text{cof}(\mu))$ by $\mathbb{C}(T)$. An iteration of such forcings with $\langle \kappa \rangle$-support will be called an $S$-iteration.

Lemma 3.2 (Foreman-Komjath). Assume GCH, $\mu$ is regular, $\kappa = \mu^+$, and $S \subseteq \kappa \cap \text{cof}(\mu)$ is stationary. Let $\langle \mathbb{P}_\alpha, \dot{Q}_\alpha : \alpha < \lambda \rangle$ be an $S$-iteration. There is a sequence $\langle \pi_0 : \alpha \leq \lambda \rangle$ such that:

1. $\pi_0 : B(\mathbb{C}(\emptyset) \times \text{Add}(\kappa, \alpha)) \to B(\mathbb{C}(\emptyset) \times \mathbb{P}_\alpha)$ is a projection.
2. For $\alpha < \beta < \lambda$, $\pi_0 = \pi_\beta \upharpoonright B(\mathbb{C}(\emptyset) \times \text{Add}(\kappa, \alpha))$.

Proof. It suffices to show that, after forcing with $\mathbb{C}(\emptyset)$, which makes $S$ contain almost all ordinals of cofinality $\mu$, there exists a system of projections that commutes as desired, defined on a dense subset of $\text{Add}(\kappa, \lambda)$ and mapping into $\mathbb{P}_\lambda$. Note that $\mathbb{C}(\emptyset)$ adds no new $< \kappa$-sequences, if $G \subseteq \mathbb{C}(\emptyset)$ is generic, then $\text{Add}(\kappa, \lambda)^G = \text{Add}(\kappa, \lambda)^{\mathbb{C}(\emptyset)}$, and if $\mathbb{P}^G_\lambda$ is $S$-iteration defined in $\mathbb{C}(\emptyset)$ using the same sequence of names for subsets of $\kappa$, then we see inductively that $\mathbb{P}^G_\lambda = \mathbb{P}_\lambda^\mathbb{C}(\emptyset)$.

Let us work in a generic extension by $\mathbb{C}(\emptyset)$, and let $C \subseteq \kappa$ be the generic club. Note that in $\mathbb{C}(\emptyset)$, there is a $\kappa$-closed dense subset of $\mathbb{P}_\lambda$; namely the set of those $p$ such that for all $\alpha \in \text{sprt}(p)$, $\max p(\alpha) \in C$.

Say a condition $p \in \text{Add}(\kappa, \lambda)$ is flat if $\text{dom} p = X \times \xi$ for some $X \subseteq \lambda$ and some $\xi < \kappa$. Clearly, the set of flat conditions is dense. Fix some bijection $f : \kappa \to [\kappa]^{<\kappa}$ such that $f(\emptyset) = \emptyset$. We will define the map $\pi_\lambda$, and it will be clear from the construction that the same definition can be run at a different $\lambda'$, and the desired coherence condition (2) above will hold.

Let $X \in [\lambda]^{<\kappa}$, and let $p : X \to [\kappa]^{<\kappa}$. Let $q \in \mathbb{P}_\lambda$ with $\text{sprt}(q) = X$. Inductively define $q \land p \in \mathbb{P}_\lambda$ on $\alpha \in X$ by putting $(q \land p)(\alpha) = p(\alpha)$ if:

1. $p(\alpha)$ is a closed bounded set end-extending $q(\alpha)$.
2. $\max p(\alpha) \in C$.
3. $(q \land p) \upharpoonright \alpha \Vdash \varphi(\alpha) \in \dot{Q}_\alpha$.

Define $(q \land p)(\alpha) = q(\alpha)$ otherwise. Let $p \in \text{Add}(\kappa, \lambda)$ be a flat condition with domain $X \times \xi$. For $i < \xi$, let $p_i = p \upharpoonright (X \times i)$; we will define $\pi(p)$ as a limit of the $\pi(p_i)$. Let $\pi(p_0) = 1_{\mathbb{P}_\alpha}$. Given $\pi(p_i)$, consider the sequence $q_i = \langle f(p(\beta, i)) : \beta \in X \rangle$. Let $\pi(p_{i+1}) = \pi(p_i) \land q_i$. At limit $j$, we take $\pi(p_j) = \inf_{i < j} \pi(p_i)$.

For $p, q \in \mathbb{P}_\lambda$, let us say that $q$ is a horizontal extension of $p$ if $q(\alpha) = p(\alpha)$ for $\alpha \in \text{sprt}(p)$. Call a flat condition $p \in \text{Add}(\kappa, \lambda)$ with domain $X \times \xi$ good if for every $i < \xi$ and every $\alpha \in X$, if $\pi(p_{i+1}) \upharpoonright \alpha$ has a horizontal extension in $\mathbb{P}_\alpha$ deciding whether $f(p(\alpha, i)) \in \dot{Q}_\alpha$, then already $\pi(p_{i+1}) \upharpoonright \alpha$ decides this. We claim that the set of good conditions is dense, and that $\pi$ restricted to this set is a projection.

Let $p \in \text{Add}(\kappa, \lambda)$ be flat with domain $X_0 \times \xi$. We can recursively add points to $X_0$ until we have a good condition $p_1' \supseteq p_1$. To see this, take an increasing
for ideals. Suppose the following:

Let $\pi \in \mathcal{P}(\kappa)$ be a generic such that $\pi(\kappa) = 0$ when a new point $\alpha$ is introduced. This will ensure $p_1 \restr \kappa$ is flat, and also not alter the fact that $p_2 \restr X_2 \times 1$ is good. We continue in this way up to $\xi$, adding zeros to the lower positions whenever necessary and simply taking unions at limits. The resulting condition will extend $p$, be flat with domain $X_\xi \times \xi$, and be good.

Suppose $q \leq p$ are good flat conditions, where dom $p = X \times \zeta$, and dom $q = Y \times \eta$. We show by induction on $i < \zeta$, and in each case by induction on $\alpha < \lambda$, that $\pi(q_i) \not\models p_i$. Suppose this is true for $j < i$ and $\beta < \alpha$. If $i$ is a limit, the induction proceeds trivially. Suppose $i$ is a successor and $\alpha \in Y \setminus X$. Then $\pi(p_i)$ is trivial at $\alpha$, so $\pi(q_i) \restr (\alpha + 1)$ is a horizontal extension. Otherwise, we have $\pi(p_i) \restr \alpha$ and $\pi(q_i) \restr \alpha$ must either both not decide, or both decide whether the set $f(p(\alpha, i - 1)) = f(q(\alpha, i - 1))$ is in $\mathcal{P}_\alpha$. Since by induction $\pi(p_{i-1})(\alpha) = \pi(q_{i-1})(\alpha)$, we must have $\pi(p_{i})(\alpha) = \pi(q_{i})(\alpha)$, showing that we may continue the induction along $\lambda$ in the case of successor $i$. We conclude in particular that $\pi(q) \leq \pi(p)$.

Now suppose $p \in \text{Add}(\kappa, \lambda)$, dom $p = X \times \xi$, and $q \leq p$ is in $\mathcal{P}_\lambda$. For each $\alpha \in \text{sprt}(q)$, there is $\eta_\alpha \subseteq [\kappa]^{<\kappa}$ such that $q \restr \alpha \not\models \eta(\alpha) = \r_\alpha$. If $\alpha \in X$, define $p'(\alpha, \xi) = f^{-1}(\r_\alpha)$. If $\alpha \in \text{dom} q \setminus X$, let $p'(\alpha, i) = 0$ for all $i < \xi$, and $p'(\alpha, \xi) = f^{-1}(\r_\alpha)$. Since $\pi(p' \restr (\text{sprt}(q) \times \xi)) = \pi(p)$, we have $\pi(p') = q$. □

**Remark 3.3.** Suppose we are in the situation as in the previous lemma. Suppose $C \times G_\lambda$ is $(\mathcal{C}(\emptyset) \times \mathcal{P}_\lambda)$-generic, and let $G_\alpha$ be the generic for the subforcing $\mathcal{P}_\alpha$ for $\alpha < \lambda$. Then for all $p \in \text{Add}(\kappa, \lambda)$, $\pi_\lambda(p) \in G_\lambda$ iff $\pi_\alpha(p \restr \alpha) \in G_\alpha$ for all $\alpha < \lambda$. It follows that for all $\alpha < \lambda$, the identity map is a regular embedding from the quotient $\text{Add}(\kappa, \alpha)/G_\alpha$ into $\text{Add}(\kappa, \lambda)/G_\lambda$. Therefore, if $\lambda$ is regular and $\alpha^{<\kappa} \leq \lambda$ for all $\alpha < \lambda$, then $\text{Add}(\kappa, \lambda)/G_\lambda$ is $(\lambda \cap \text{cof}(\kappa))$-layered.

**Theorem 3.4.** Suppose the following:

1. GCH, $\mu$ is regular, and $\kappa = \mu^+$.
2. $I$ is a normal $\kappa^+$-saturated ideal on $\kappa$.
3. There is a stationary $A \in I^+$ and a nonstationary $B \subseteq \kappa^+$ such that $\models \mathcal{P}(\kappa)/I\vdash (\mathcal{A}) = \mathcal{B}$.
4. There is a projection $\pi : \mathcal{P}(\kappa)/I \to \text{Col}(\mu, \kappa) \times \text{Add}(\kappa, \kappa^+)$. Let $S = \kappa \cap \text{cof}(\mu) \setminus A$. Then $S$ is stationary, and there is an $S$-iteration $\mathcal{P}$ of length $\kappa^+$ such that $\mathcal{B}[\mathcal{P}^* \mathcal{P}(A)/\text{NS}_\kappa] \cong \mathcal{P}(\kappa)/I$.

**Proof.** If $j : V \to M \subseteq V[G]$ is any generic ultrapower arising from $I$, then $\text{cof}(\mu)^M = \text{cof}(\mu)^{V[G]}$. Thus since $B$ not almost all of $\text{cof}(\mu) \cap j(\kappa)$, $S$ is stationary. In $V$, we will construct an $S$-iteration $\mathcal{P}$ of length $\kappa^+$, and simultaneously construct a sequence of projections from $\mathcal{P}(\kappa)/I$ to $\mathcal{P}$ and a sequence of $\mathcal{P}$-names for ideals.
Suppose $G \subseteq P(\kappa)/I$ is generic, and $j : V \to M \subseteq V[G]$ is the generic ultrapower embedding. Since $B$ is nonstationary, the forcing $C(j(S) \cup \text{cof}(<\mu))^M$ has a $\kappa^+$-closed dense set in $V[G]$, as does $\text{Add}(j(\kappa), j(\kappa^+))^M$. Since $C(j(S) \cup \text{cof}(<\mu)) \times \text{Add}(\kappa, 1)$ has the $j(\kappa^+)$-c.c. in $M$, and $j(\kappa^+) < (\kappa^+)^V$, there are only $j(\kappa)$-many dense open subsets that live in $M$. Using this and the closure of $M$, we can build in $V[G]$ a filter $H$ that is generic over $M$. Fix a $P(\kappa)/I$-name for such an object.

In $V$, let $\langle S_\beta^\alpha : \gamma < \kappa^+ \rangle$ enumerate $P(\kappa)$. We begin our $S$-iteration by first forcing with $C(S_0^\alpha)$ if $S_0^\alpha \in I^*$, and forcing with $C(\kappa)$ otherwise. Since $\text{Col}(\mu, \kappa)$ absorbs $C(S \cup \text{cof}(<\mu))$, Lemma 3.2 gives a projection $\pi_1 : \text{Col}(\mu, \kappa) \times \text{Add}(\kappa, 1)$, so that a generic for this first step is absorbed by $P(\kappa)/I$. Suppose $C_0$ is a generic club for $P_1$, and $G \subseteq P(\kappa)/I$ is a generic absorbing $C_0$ via $\pi_1$. Note that $C_0 \cup \{\kappa\}$ is a condition in $j(P_1)$. The generic filter $H$ yields, via the projection of Lemma 3.2, a club $C_\beta \subseteq j(\kappa)$ that is $j(P_1) \upharpoonright C_0 \cup \{\kappa\}$-generic over $M$. Thus we can extend the embedding to $j_1 : V[C_0] \to M[C_0]$. By Theorem 2.13 we have in $V$ a $P_1$-name $I_1$ for a normal ideal extending $I$ such that $B(P_1 \ast P(\kappa)/I_1) \equiv P(\kappa)/I$, where $I_1$ is the collection of $X \subseteq \kappa$ for which it is forced that $\kappa \notin j_1(X)$.

Let $\alpha \mapsto \langle \alpha_0, \alpha_1 \rangle$ denote the Gödel pairing function; note that $\alpha_0, \alpha_1 \leq \alpha$. Let $\alpha < \kappa^+$ and assume inductively that:

1. In $V$, we have an $S$-iteration $\langle P_\beta, Q_\beta : \beta < \alpha \rangle$.
2. At each stage $\beta < \alpha$, we have chosen an enumeration $\langle S_\gamma^\beta : \gamma < \kappa^+ \rangle$ of $P(\kappa)^{V^\beta}$.
3. We have a commuting system of projections $\pi_\beta : \text{Col}(\mu, \kappa) \times \text{Add}(\kappa, \beta) \to P_\beta$, for $\beta \leq \alpha$, as given by Lemma 3.2. This implies that $P(\kappa)/I$ absorbs $P_\alpha$. Let $C_\beta$ denote the generic club added at stage $\beta$.
4. For $\beta < \alpha$, it is forced that $m_\beta = \langle (j(\gamma), C_\gamma \cup \{\kappa\}) : \gamma < \beta \rangle$ is a condition in $j(P_\beta)$.
5. For $\beta < \alpha$, $H$ yields a $j(P_\beta) \upharpoonright m_\beta$-generic sequence $\langle \hat{C}_\gamma : \gamma < j(\beta) \rangle$. By the coherence of the projections from Lemma 3.2 we have that for $\beta' < \beta$, the sequence associated to $\beta$ is an end-extension of that associated to $\beta'$.

Since we use $<\kappa$-supports, for all $\beta < \alpha$ and all $p$ in the generic filter corresponding to $(C_\gamma : \gamma < \beta)$, $m_\beta \leq j(p)$. Thus for all $\beta < \alpha$, we have elementary embeddings $j_\beta : V[C_\gamma : \gamma < \beta] \to M[(\hat{C}_\gamma : \gamma < j(\beta))] \subseteq V[G]$, which are forced to extend one another. Theorem 2.13 gives that for each $\beta < \alpha$, there is a $P_\beta$-name $I_\beta$ for a normal ideal on $\kappa$ equal to the set of $X \subseteq \kappa$ for which it is forced that $\kappa \notin j_\beta(X)$, and $B(P_\beta \ast P(\kappa)/I_\beta) \equiv P(\kappa)/I$.

If $\alpha = \beta + 1$, then we continue the construction by letting $Q_\beta$ be a $P_\beta$-name for $C(S_0^\beta)$ if $\mathbb{P}_\beta S_0^\beta \in I_\beta$ and $C(\kappa)$ otherwise. We then choose an enumeration of $P(\kappa)^{V_{\alpha+1}}$. The first three induction hypotheses are easily preserved. For (4), first note that it is preserved at successors because $P_\alpha$ is dense in $P_\alpha$, and because $k$ is forced to be in $j(S_0^\beta)$. At limits, it is preserved by the closure of $M$ and because $j(P_\alpha)$ uses $<j(\kappa)$-supports. Thus (5) makes sense and follows from Lemma 3.2.

Regarding the final stage $\kappa^+$, note that if $A \subseteq j(P_{\kappa^+})$ is a maximal antichain in $M$, then $A \subseteq j(P_\beta)$ for some $\beta < \kappa^+$. Thus the filter induced by $(C_\gamma : \gamma < \beta)$ meets $A$. Thus $(C_\gamma : \gamma < j(\kappa^+))$ is $j(P_{\kappa^+})$-generic over $M$, and we may extend the embedding to $j_{\kappa^+} : V[(C_\gamma : \gamma < \kappa^+)] \to M[(\hat{C}_\gamma : \gamma < j(\kappa^+))] \subseteq V[G]$. 

Suppose $\check{X}$ is a $P_{κ^+}$-name for a set in $I_κ^+$. Then $\check{X}$ is a $P_κ$-name for some $β < κ^+$ by the $κ^+$-c.c. If $\check{X} \in I_κ^+$, then we can take a generic $G \subseteq P(κ)/I$ such that $κ \notin j_β(X)$. But $G$ also determines an embedding $j_{κ^+}$ extending $j_β$, and by assumption it is forced that $κ \in j_{κ^+}(X)$, a contradiction. Now in $V^{P_κ}$, $X = S^β_κ$ for some $γ$, and there is $α ≥ β$ such that $⟨α_0, α_1⟩ = ⟨β, γ⟩$. Thus $Qα$ shoots a club $C$ through $X ≺ (κ \setminus A)$, so that in the final model, $C ∩ A ⊆ X$. □

4. Obtaining saturated ideals

Let $μ < κ$ be regular cardinals. Deviating slightly from convention, we define:

$$\text{Col}(μ, <κ) := \prod_{μ ≤ α < κ} \text{Col}(μ, α).$$

Now define:

$$\mathbb{P}(μ, κ) := \prod_{μ ≤ α < κ} \text{Col}(α, <κ).$$

It is convenient to identify $\mathbb{P}(μ, κ)$ with a collection of partial functions on $κ^3$. If $μ < δ < κ$ and $δ$ is regular, then $\mathbb{P}(μ, δ) = \mathbb{P}(μ, κ) ∩ V_δ$. Thus if $κ$ is Mahlo, then $\mathbb{P}(μ, κ)$ is $S$-layered, where $S$ is the stationary set of regular cardinals below $κ$.

The key to our construction is Shioya’s argument [17], which shows that $\mathbb{P}(μ, κ)$ can absorb future versions of itself.

Lemma 4.1. Assume GCH. Suppose $κ < λ$ are regular cardinals, and $Q$ is a $κ$-c.c. partial order of size $≤ κ$. There is a projection $π : Q × P(κ, λ) → Q × P(κ, λ)$.

Proof. Using GCH and the chain condition, choose an enumeration $⟨τ_α : α < λ⟩$ of $Q$-names for ordinals such that whenever $η ∈ [κ, λ]$ is regular and $\forces{σ < η}$ there is $α < η$ such that $\forces{σ = τ_α}$. Let $π$ be defined by $⟨q, p⟩ ↦ ⟨q, p⟩$, where $p$ is the canonical $Q$-name for the function with the same domain as $p$, and such that for all $α ∈ \text{dom } p$, $\forces{p(α)} = τ_π(α)$.

Suppose $⟨q_1, p_1⟩ ≤ p(γ_0, p_0)$. By the $κ$-c.c., there is a set $X ⊆ λ^3$ such that $\models \text{dom } p_1 ⊆ X$, where:

- $\{α : 3βγ⟨α, β, γ⟩ ∈ X\} ⊆ [κ, λ]$ and is Easton.
- $∀α(⟨β, γ⟩ : 〈α, β, γ⟩ ∈ X) ⊆ [α, λ] × α$ and has size $< α$.

Define $p_2$ such that $p_2 ↾ \text{dom } p_0 = p_0$, and if $α ∈ X \setminus \text{dom } p_0$, then $p_2(α) = δ$, where $δ < α(1)$ is such that the following is forced about the $Q$-name $τ_δ$: “If $α ∈ \text{dom } p_1$, then $τ_δ = τ_1(α)$.” If $α ∈ \text{dom } p_0$, then $q_1 ↾ p_0(α) = p_1(α) = τ_2(α)$; if $α ∈ \text{dom } p_2 \setminus \text{dom } p_0$, then $q_1 ↾ α ∈ \text{dom } p_1 → τ_1(α) = τ_2(α)$. Thus $q_1 ↾ p_2 ≤ p_1$. □

Lemma 4.2. Suppose the following:

1. $μ < κ ≤ δ < λ$ are regular cardinals, and $κ$ and $λ$ are Mahlo.
2. $j : V → M$ is an almost-huge embedding derived from a $(κ, λ)$-tower.
3. $Q$ is a $P(μ, κ)$-name $κ$-closed, $δ$-c.c. poset of size $≤ δ$.

Let $G * h * H$ be $P(μ, κ) * Q * P(δ, λ)$-generic. In $V[G * h * H]$, there is a normal $κ$-complete ideal $I$ on $[δ]^{<κ}$ such that $P([δ]^{<κ})/I$ projects to $\text{Col}(μ, κ) × \text{Col}(κ, δ) × \text{Add}(κ, λ)$, and is $S$-layered, where $S$ is the set of $V$-regular $α$, $δ ≤ α < λ$. Any generic embedding arising from forcing with this ideal extends $j$.
Proof. Let us first claim that there is a projection from $\mathbb{P}(\mu, \lambda)$ to
$$
\mathbb{P}(\mu, \kappa) \ast \left( [\hat{Q} \ast \dot{\mathbb{P}}(\delta, \lambda)] \times \text{Col}(\kappa, \delta) \times \text{Add}(\kappa, \lambda) \right) \times \text{Col}(\mu, \kappa),
$$
which is the identity on $\mathbb{P}(\mu, \kappa)$. Note that there is a natural projection from $\mathbb{P}(\mu, \lambda)$ to $\mathbb{P}(\mu, \kappa) \times \mathbb{P}(\kappa, \lambda) \times \text{Col}(\mu, \kappa)$, given by
$$
p \mapsto \langle p \upharpoonright \kappa^3, p \upharpoonright [\kappa, \lambda], \lambda^2, p(\mu, \kappa, \cdot) \rangle.
$$
By Lemma 4.4, the first two factors project to $\mathbb{P}(\mu, \kappa) \ast \dot{\mathbb{P}}(\delta, \lambda)$. Since by Lemma 2.3, $\text{Col}(\alpha, \beta) \cong \text{Col}(\alpha, \beta) \times \mathbb{R}$, whenever $\alpha$ is regular and $\mathbb{R}$ is $\alpha$-closed and of size $\leq \beta$, we see that in $V^{\mathbb{P}(\mu, \kappa)}$, $\mathbb{P}(\kappa, \lambda)$ is forcing-equivalent to $\mathbb{P}(\kappa, \lambda) \times \text{Col}(\kappa, \delta) \times \text{Add}(\kappa, \lambda)$. The first factor projects to $\mathbb{Q}$ by Lemma 2.3 again. By Lemma 4.4, we get a projection from the first two factors to $\mathbb{Q} \ast \dot{\mathbb{P}}(\delta, \lambda)$. This finishes the argument for the claim.

Now let $G \ast h \ast H$ be as hypothesized, and let $\hat{G} \subseteq \mathbb{P}(\mu, \lambda)$ be a generic projecting to $G \ast h \ast H$. By the above argument, $\hat{G}$ also projects to $G \ast K \subseteq \mathbb{P}(\mu, \kappa) \ast \dot{\mathbb{P}}(\kappa, \lambda)$, which in turn projects to $G \ast h \ast H$. Since $\mathbb{P}(\kappa, \lambda)$ is clearly $(\kappa, \lambda)$-nice, we get by Lemma 2.10 an extended elementary embedding $j : V[G \ast K] \to M[\hat{G} \ast K]$, such that $\text{Ord}^G \cap V[\hat{G}] \subseteq M[\hat{G} \ast K]$. By elementarity, there is some $\hat{h} \ast \hat{H}$ such that the embedding restricts to $j : V[G \ast h \ast H] \to M[\hat{G} \ast \hat{H}]$. By the closure of the relevant forcings, the latter model also has the same $\text{Ord}^G$ as $V[G]$. In $V[G \ast h \ast H]$, let $\mathbb{R}$ be the quotient forcing $\mathbb{P}(\mu, \lambda)/(G \ast h \ast H)$. We define the ideal $I$ as $\{ X \subseteq [\delta]^{<\kappa} : 1 \not\Vdash_R j[\delta] \notin j(X) \}$. Let $e : \mathcal{P}([\delta]^{<\kappa})/I \to \mathcal{B}(\mathbb{R})$ be defined by $e([X]) = [j[\delta] \in j(X)]$. It is routine to check that $I$ is normal and $\kappa$-complete and that $e$ is a Boolean embedding. Since $\mathbb{P}(\mu, \lambda)$ is $\lambda$-c.c., $I$ is $\lambda$-saturated. It follows from this that $e$ is a complete embedding. For let $\{ [X_\alpha]_I : \alpha < \delta \}$ be a maximal antichain in $\mathcal{P}([\delta]^{<\kappa})/I$. Then $[\cap_{\alpha<\delta} X_\alpha]_I = [[\delta]^{<\kappa}]_I$, so $1 \not\Vdash_R j[\delta] \in j(\cap_{\alpha<\delta} X_\alpha)$. Thus it is forced that $j[\delta] \in j(X_\alpha)$ for some $\alpha < \delta$.

To show the isomorphism, let $U \subseteq \mathcal{P}(\kappa)/I$ be generic over $V[G \ast h \ast H]$, and let $j_U : V[G \ast h \ast H] \to N$ be the generic ultrapower embedding. Forcing further with $\mathbb{R}/e[U]$ yields an embedding $j : V[G \ast h \ast H] \to M[G \ast h \ast H]$ as above. We have that $X \in U$ if and only if $j[\delta] \in j(X)$, so we can define an elementary embedding $k : N \to M[G \ast h \ast H]$ by $k([f]_U) = j(f)(j[\delta])$, and we have $j = k \circ j_U$. Note that for $\alpha \leq \delta$, $k(\alpha) = k(\text{ot}(j_U(\alpha) \cap [\text{id}]_U)) = \text{ot}(j(\alpha) \cap j[\delta]) = \alpha$. Thus $\text{crit}(k) \geq \lambda$.

Let $\beta$ be any ordinal. Since $j : V \to M$ was derived from a $(\kappa, \lambda)$-tower, there is some $\alpha < \delta$ and some $f \in V$ such that $\beta = j(f)(j[\alpha])$. Let $b : \delta \to \alpha$ be a surjection in $V[G \ast h \ast H]$. Then
$$
\beta = j(f)(j(b)(j[\delta])) = k(j_U(f)(j_U(b)([\text{id}]_U))).
$$
Thus $\beta \in \text{ran}(k)$, so $k$ does not have a critical point and $N = M[\hat{G} \ast \hat{H}]$. For any generic $\hat{G}$, if $U = e^{-1}[\hat{G}]$, then $\hat{G} = j_U(G)$. For any generic $U$, if $\hat{G} = j_U(G)$, then $U = \{ X \subseteq \kappa : \kappa \in j_U(X) \} = e^{-1}[\hat{G}]$. Thus Lemma 2.2 implies that $\mathcal{P}([\delta]^{<\kappa})/I \cong \mathcal{B}(\mathbb{R})$. It follows that $\mathcal{P}([\delta]^{<\kappa})/I$ projects to $\text{Col}(\mu, \kappa) \times \text{Col}(\kappa, \delta) \times \text{Add}(\kappa, \lambda)$ as desired.

It remains to show that $\mathcal{P}([\delta]^{<\kappa})/I$ is $S$-layered, where $S$ is the stationary set of $V$-regular cardinals between $\delta$ and $\lambda$. This follows because the projection $\pi$ of Lemma 4.4 has the following property: For any $p$ in the quotient $\mathbb{R}$, any $\alpha < \lambda$, and any $q \leq p$ with $\alpha$ of rank $\leq \alpha$ which is also in the quotient, we have that $p \cup q$ is also in the quotient. Thus $[\mathbb{R} \cap V_\alpha : \alpha < \lambda]$ witnesses that $\mathbb{R}$ is $S$-layered. \qed
Although we are ultimately interested in saturated ideals on regular cardinals, rather than sets of the form $[\delta]^{<\kappa}$, the ideals on such sets will be useful for us because of their resilience under collapses:

**Lemma 4.3.** Assume GCH. Suppose $\kappa < \delta$ are regular, and $I$ is a normal ideal on $[\delta]^{<\kappa}$ such that $P([\delta]^{<\kappa})/I$ projects to $Col(\kappa, \delta)$, and is $S$-layered for some stationary subset $S \subseteq \delta^+$. If $g \subseteq Col(\kappa, \delta)$ is generic, then in $V[G]$, there is an $S$-layered ideal $J$ on $\kappa$ such that $P(\kappa)/J \cong (P([\delta]^{<\kappa})/I)/g$. Furthermore, any generic ultrapower arising from forcing with $J$ over $V[g]$ extends one arising from forcing with $I$ over $V$.

**Proof.** Let $g \subseteq Col(\kappa, \delta)$ be generic. Further forcing yields a generic $G \subseteq P([\delta]^{<\kappa})/I$ and an ultrapower embedding $j : V \to M \subseteq V[G]$. The quotient $(P([\delta]^{<\kappa})/I)/g$ is $S$-layered by Lemma 2.14. Since $j(Col(\kappa, \delta))$ is $j(\kappa)$-directed-closed and $j[g] \in M$ is a directed set of size $< j(\kappa)$, there is a condition $m \in j(Col(\kappa, \delta))$ below $j[g]$.

A counting argument shows that $j(\delta^+) < \delta^{++}$, so there are only $\delta^+$-many dense subsets of $j(Col(\kappa, \delta))$ in $M$. Since $j(\kappa) > \delta$ and $M^\delta \cap V[G] \subseteq M$, we can build a filter $g_j \subseteq j(Col(\kappa, \delta))$ in $V[G]$ that is generic over $M$, with $m \in g_j$. Thus we may extend the embedding to $j : V[g_j] \to M[g]$. By Theorem 2.13 we get a normal $\kappa$-complete ideal $J'$ on $[\delta]^{<\kappa}$ in $V[g]$ such that $P([\delta]^{<\kappa})/J' \cong (P([\delta]^{<\kappa})/I)/g$. By Proposition 2.14 any generic embedding coming from $J'$ extends one coming from $I$.

Now since $|\delta| = \kappa$ in $V[g]$, a bijection $f : \kappa \to \delta$ yields an ideal $J$ on $[\kappa]^{<\kappa}$ given by $J = \{z : f[z] \in J'\}$, and clearly $P([\kappa]^{<\kappa})/J \cong P([\delta]^{<\kappa})/J'$. By normality, $\kappa$ is a $J$-measure-one set, so $J$ is essentially an ideal on $\kappa$. \qed

5. The preparatory model

We build the preparatory model towards Theorem 1 in three rounds. We warn the reader that we will continually change the reference of “$V$” to mean whatever ground model we on which are currently focused. Let $\theta$ be a huge cardinal. A standard reflection argument shows that there is a large set $X \subseteq \theta$ such that for all $\alpha < \beta$ in $X$, $\alpha$ is almost-huge with target $\beta$. In the first round, we arrange that for all such $\alpha < \beta$, there is an $A \subseteq \alpha$ and an $(\alpha, \beta)$-tower $T$ such that $\kappa \in j_T(A)$ and $j_T(A)$ is nonstationary. We then collapse many cardinals so that $\theta$ is still very large, the set of cardinals below $\theta$ is almost equal to $X$, and there are many saturated ideals with the properties that make Theorem 3.4 applicable. In the second round, we introduce square at every cardinal without collapsing, while preserving many superstrong cardinals, and preserving the desired saturated ideals. In the third round, we arrange local saturation on the first few successors of Mahlo cardinals, while still preserving many superstrongs.

5.1. Many saturated ideals that get stationarity wrong. In [9], local saturation was obtained at a single successor cardinal by exploiting a precise degree of almost-hugeness: $\Lambda (\kappa, \lambda)$-tower was chosen with $\lambda$ non-Mahlo. Such towers always exist whenever almost-huge cardinals exist, but they will not serve our purposes here since we want the target to also be almost-huge. In order to achieve this, we use a forcing argument supplied by Toshimichi Usuba.

**Lemma 5.1 (Usuba).** There is a forcing $\mathbb{P}$ such that whenever $\kappa$ is almost-huge in $V$ with Mahlo target $\lambda$, then in $V^\mathbb{P}$, there is a $(\kappa, \lambda)$-tower $T$ and an $A \subseteq \kappa$ such that $\kappa \in j_T(A)$ and $j_T(A)$ is nonstationary.
Proof. Let $\mathbb{P}$ be the Easton-support iteration of adding a Cohen subset of $\alpha$ whenever $\alpha$ is inaccessible. Let $j : V \to M$ be an almost-huge embedding generated by a $(\kappa, \lambda)$-tower in $V$, where $\lambda$ is Mahlo. Let $G$ be generic for $\mathbb{P}$, and let $G_\alpha = G \cap \mathbb{P}_\alpha$. Then $j$ can be extended to $j : V[G_\alpha] \to M[G_\lambda]$.

It is easy to show that the Cohen-generic function $g : \kappa \to 2$ added at stage $\kappa$ has the property that if $A = \{ \alpha : g(\alpha) = 1 \}$ and $S \in V$ is a stationary subset of $\kappa$, then $S \cap A$ and $S \setminus A$ are both stationary. We now build a subset of $\lambda$ that is Add($\lambda$)-generic over $M[G_\lambda]$ with some specific properties. Since $j(\lambda) < \lambda^+$, we can list all dense open subsets of $\text{Add}(\lambda)^{M[G_\lambda]}$ that live in $M[G_\lambda]$ as $\langle D_\alpha : \alpha < \lambda \rangle$. We construct an extension $\hat{g}$ of $g$ as $\bigcup_{\alpha < \lambda} \check{g}_\alpha$ and along the way choose a continuous, increasing, cofinal sequence of ordinals $\langle \beta_\alpha : \alpha < \lambda \rangle \subseteq \lambda$ with the following properties:

1. $\text{dom}(\hat{g}_0) = \kappa + 1$, $\check{g}_0 \upharpoonright \kappa = g$ and $\hat{g}_0(\kappa) = 1$.
2. For $\alpha > 0$, $\text{dom}(\check{g}_\alpha) = \beta_\alpha + 1$, and $\hat{g}_\alpha(\beta_\alpha) = 0$.
3. For all $\alpha$, $\check{g}_{\alpha + 1} \in D_\gamma$.
4. For limit $\alpha$, $\beta_\alpha = \sup_{\gamma < \alpha} \beta_\gamma$.

Clearly, $\hat{g}$ is generic over $M[G_\lambda]$, and $\{ \alpha : \hat{g}(\alpha) = 1 \}$ is disjoint from the club $\{ \beta_\alpha : \alpha < \lambda \}$. So we extend the embedding to $j : V[G_{\kappa + 1}] \to M[G_\lambda \ast \hat{g}]$.

The method of the proof of Theorem 2.11 lets us build a filter $H \subseteq j(\mathbb{P}_\lambda)/(\mathbb{P}_\lambda \ast \hat{g})$-generic over $M[G_\lambda \ast \hat{g}]$, with $j[G_\lambda] \subseteq G_\lambda \ast \hat{g} \ast H$, so we can extend the embedding to $j : V[G_\lambda] \to M[G_\lambda \ast \hat{g} \ast H]$. By the $\lambda$-c.c. of $\mathbb{P}_\lambda$ and the $\lambda$-closure of $j(\mathbb{P}_\lambda)/(\mathbb{P}_\lambda \ast \hat{g})$, we have that $\text{Ord}^{< \lambda} \cap V[G_\lambda] \subseteq M[G_\lambda \ast \hat{g} \ast H]$. The appropriate $(\kappa, \lambda)$-tower inducing $j$ exists in $V[G_\lambda]$, and it is preserved by the $\lambda$-closed forcing $\mathbb{P}/G_\lambda$. \hfill $\square$

It is not hard to show that the forcing $\mathbb{P}$ of the previous lemma preserves huge cardinals as well. Let us therefore work in a model satisfying the conclusion of the previous lemma, and in which there is a huge cardinal $\theta$. Let $\mathcal{U}$ be an ultrafilter on $\theta$ derived from an embedding witnessing $\theta$ is huge. $X \in \mathcal{U}$ be such that for $\alpha < \beta$ in $X$, $\alpha$ is almost-huge with target $\beta$. Let $\langle \alpha_i : i < \theta \rangle$ enumerate the closure of $X \cup \{ \omega \}$. Let $\mathbb{P}(\kappa, \lambda)$ denote the product forcing defined in Section 4. Let us force with the following Easton-support iteration $\langle \mathbb{P}_i, \check{\mathbb{Q}}_i : i < \theta \rangle$:

- If $i$ is 0 or a successor, let $\Vdash_i \check{\mathbb{Q}}_i = \mathbb{P}(\alpha_i, \alpha_{i+1})$.
- If $\alpha_i$ is a non-Mahlo limit of $X$, let $\Vdash_i \check{\mathbb{Q}}_i = \mathbb{P}(\alpha_i^+, \alpha_{i+1})$.
- If $\alpha_i$ is a Mahlo limit of $X$, let $\Vdash_i \check{\mathbb{Q}}_i = \mathbb{P}(\alpha_i, \alpha_{i+1})$.

It is routine to check that after forcing with this iteration, the set of cardinals below $\theta$ are the ordinals $\alpha_i$ and those of the form $(\alpha_i^+)^V$ for $\alpha_i$ a non-Mahlo limit of $X$.

Let $\mu < \delta$ be either successor cardinals or a Mahlo cardinals after forcing with $\mathbb{P}_\theta$. Let $i < \theta$ be such that either $\mu = \alpha_i$ or $\mu = (\alpha_i^+)^V$, and define $i'$ similarly with respect to $\delta$. Let $G_i \subseteq \mathbb{P}_i$ be generic, and let $\kappa = \alpha_{i+1}$. Since $|\mathbb{P}_i| \leq \mu$, any almost-huge embedding with critical point $\kappa$ in $V$ extends to one in $V[G_i]$. Consider the forcing $\mathbb{P}_{i'+1}/G_i$. It takes the form $\mathbb{P}(\mu, \kappa) \ast \check{\mathbb{Q}} \ast \check{\mathbb{P}}(\delta, \lambda)$, where $\lambda = \alpha_{i'+1} \in X$ and $\check{\mathbb{Q}}$ is forced to be $\kappa$-closed, $\delta$-c.c., and of size $\leq \delta$. The hypotheses of Lemma 4.2 are satisfied, so there exists an ideal as in the conclusion after forcing with $\mathbb{P}_{i'+1}$. As the tail-end is $\lambda$-closed, this still holds in the extension by $\mathbb{P}_\theta$.

Furthermore, many almost-huge cardinals below $\theta$ are preserved. For let $j : V \to M$ witness the hugeness of $\theta$. If $T$ is the almost-huge tower derived from $j$, then $T \in M$. We have $j(\mathbb{P}_\theta) \cap V_\theta = \mathbb{P}_\theta$, so reflection gives us that, if $\mathcal{U}$ is the ultrafilter on $\theta$ derived from $j$, then there are $\mathcal{U}$-many $\alpha < \theta$ that are almost-huge
with embedding \( j_\alpha \), with \( j_\alpha(\mathcal{P}_\alpha) = \mathcal{P}_\alpha \). Reflecting again yields a set \( Y \in \mathcal{U} \) such that for all \( \alpha < \beta \) in \( Y \), there is an \( (\alpha, \beta) \)-tower \( T \) with \( j_T(\mathcal{P}_\alpha) = \mathcal{P}_\beta \). The proof of Lemma 2.11 shows that such embeddings can be extended through the forcing. Let us record what we have as:

**Lemma 5.2.** It is consistent relative to a huge cardinal that there is an inaccessible \( \theta \) and a sequence \( \langle S_\alpha : \alpha < \theta \rangle \) such that:

1. \( V_\theta \) satisfies GCH and that there is a proper class of almost-huge cardinals with Mahlo targets.
2. Whenever \( \mu \) is regular and \( \kappa = \mu^+ \), \( S_\kappa \) is a stationary subset of \( \kappa \cap \text{cof}(\mu) \).
3. For every pair of cardinals \( \mu < \delta < \theta \) which are either successor or Mahlo, if \( \kappa = \mu^+ \) and \( \lambda = \delta^+ \), then there is \( \kappa \)-complete normal ideal \( I \) on \([\delta]^{< \kappa}\) such that:
   a. \( \mathcal{P}([\delta]^{< \kappa})/I \) is \( S_\lambda \)-layered.
   b. There is a stationary \( A \subseteq \kappa \cap \text{cof}(\mu) \) and a nonstationary \( B \subseteq \lambda \) such that for any generic embedding \( j \) arising from \( I \), \( \kappa \in j(A) = B \).
   c. \( \mathcal{P}([\delta]^{< \kappa})/I \) projects to \( \text{Col}(\mu, \kappa) \times \text{Col}(\kappa, \delta) \times \text{Add}(\kappa, \lambda) \), in a way such that the quotient is forced to be \( S_\lambda \)-layered.

5.2. **Squares.** For a cardinal \( \kappa \), \( \square_\kappa \) holds if there is a sequence \( \langle C_\alpha : \alpha < \kappa^+ \rangle \) such that if \( \alpha \) is a limit ordinal,

1. \( C_\alpha \) is a club subset of \( \alpha \).
2. If \( \beta \in \text{lim} C_\alpha \), then \( C_\beta = C_\alpha \cap \beta \).
3. \( \alpha \in C_\alpha \).

We will refer to a sequence satisfying (1) and (2) as a **coherent sequence of clubs** and a sequence satisfying all three as a \( \square_\kappa \)-sequence. A weaker notion, \( \square(\kappa^+) \), holds when there is a coherent sequence of clubs with the property that there is no “thread” \( C \subseteq \kappa^+ \), a club such that if \( \alpha \in \text{lim} C \), then \( C_\alpha = C \cap \alpha \).

There is some tension between squares and saturated ideals. The following two propositions show that if \( \mu \) has uncountable cofinality and \( \square_\mu \) holds, then there cannot be a saturated ideal on \( \mu^+ \) whose associated forcing is either weakly homogeneous or proper.

**Proposition 5.3 (Zeman).** Suppose \( I \) is a normal \( \kappa \)-complete ideal on \( Z \), \( \mathcal{P}(Z)/I \) is weakly homogeneous, and \( \mathcal{P}(Z)/I \) preserves that \( \kappa \) has uncountable cofinality. Then \( \square(\kappa) \) fails.

**Proof.** Suppose \( \langle C_\alpha : \alpha < \kappa \rangle \) is a coherent sequence of clubs. Let \( G \subseteq \mathcal{P}(Z)/I \) be generic, and let \( j : V \to M \) be the associated embedding. By [7, Section 2.6], \( M \) is well-founded up to \( \kappa^+ \). \( C^* = j(\mathcal{C}) \) is a thread of \( \langle C_\alpha : \alpha < \kappa \rangle \). Suppose \( C'' \) is another thread. Then \( C'' = C'' \cap C^* \) is a club in \( \kappa \), and whenever \( \alpha \in \text{lim} C'' \), \( C'' \cap \alpha = C' \cap \alpha = C^* \cap \alpha = C_\alpha \). Thus \( C'' = C^* \), hence \( C^* \) is definable from parameters in the ground model. By weak homogeneity, \( C^* \in V \). \( \square \)

**Proposition 5.4.** Suppose \( I \) is a normal \( \kappa \)-complete ideal on \( Z \), and \( \mathcal{P}(Z)/I \) is a proper forcing. Then every stationary subset of \( \kappa \cap \text{cof}(\omega) \) reflects.

**Proof.** Let \( S \subseteq \kappa \cap \text{cof}(\omega) \) be stationary. Let \( G \subseteq \mathcal{P}(Z)/I \) be generic and let \( j : V \to M \) be the associated embedding. Then \( j(S) \cap \kappa = S \), and \( S \) is still stationary in \( V[G] \). By elementarity, \( S \cap \alpha \) is stationary in \( \alpha \) for some \( \alpha < \kappa \). \( \square \)
We will need the following to show the preservation of squares under some cardinal collapses:

**Lemma 5.5.** Let $\kappa$ be a cardinal and $\zeta < \kappa^+$. Suppose there is a coherent sequence of clubs $\langle C_\alpha : \alpha < \kappa^+ \rangle$ such that for all $\alpha$, $\ot C_\alpha \leq \zeta$. Then $\square_\kappa$ holds.

**Proof.** It is easy to show by induction that for each $\xi < \kappa^+$, there is a short square sequence $\langle D_\alpha : \alpha \leq \xi \rangle$, i.e. a sequence satisfying all requirements for $\square_\kappa$ except that its length is $< \kappa^+$. Fix one for $\xi = \zeta$. For $\alpha < \kappa^+$, let $C'_\alpha = \{ \beta \in C_\alpha : \ot(C_\alpha \cap \beta) \in D_{\ot(C_\alpha)} \}$. For each $\alpha$, $\ot(C'_\alpha) = \ot(D_{\ot(C_\alpha)}) \leq \kappa$. Suppose $\beta$ is a limit point of $C'_\alpha$. Then $\beta$ is a limit point of $C_\alpha$, so $C_\beta = C_\alpha \cap \beta$. Also, $\ot(C_\beta)$ is a limit point of $D_{\ot(C_\alpha)}$, so $D_{\ot(C_\alpha)} \cap \ot(C_\beta) = D_{\ot(C_\beta)}$, and therefore $C'_\beta = C'_\alpha \cap \beta$.

For a cardinal $\delta$, let $S_\delta$ be the collection of bounded approximations to a $\square_\delta$ sequence. That is, a condition is a sequence $\langle C_\alpha : \alpha \in \eta \rangle$ such that $\eta < \delta^+$ is a successor ordinal, each $C_\alpha$ is a club subset of $\alpha$ of order type $\leq \delta$, and whenever $\beta$ is a limit point of $C_\alpha$, $C_\alpha \cap \beta = C_\beta$. An induction argument shows that conditions can be extended to arbitrary length, so the forcing introduces a $\square_\delta$-sequence. The first and third claims of the following lemma are well-known, and the second follows from a general theorem of Ishiu and Yoshinobu [11]. We give a proof for the reader’s convenience.

**Lemma 5.6.** Let $\delta$ be a cardinal.

1. $S_\delta$ is $(\delta + 1)$-strategically closed.
2. If $\square_\delta$ holds, then $S_\delta$ is $\delta^+$-strategically closed.
3. For every regular $\lambda \leq \delta$, there is a $S_\delta$-name for a “threading” partial order $T_\delta^\lambda$ that adds a club $C \subseteq (\delta^+)^V$ of order type $\lambda$ and such that whenever $\alpha$ is a limit point of $C$, $C \cap \alpha = C_\alpha$. Furthermore, $S_\delta \star T_\delta^\lambda$ has a $\lambda$-closed dense subset of size $2^\delta$.

**Proof.** For (1), let us pit the players $\text{Good}$ and $\text{Bad}$ against each other. Let $\text{Bad}$ play any condition $p_\alpha$. If $\text{Bad}$ plays $p_\alpha$, let $\text{Good}$ play any condition $p_{\beta+1}$ strictly longer than $p_\beta$, where $\max(\text{dom} p_{\beta+1})$ is a limit ordinal. At limit stages $\lambda$, $\text{Good}$ plays $\bigcup_{\gamma < \lambda} p_\gamma \cup \langle \lambda : (\exists \beta < \lambda) \max(\text{dom} p_\beta) = \alpha \rangle$. The fact that $\text{Good}$ plays at all limit stages ensures coherence. This strategy succeeds in producing conditions in $S_\delta$ for $(\delta + 1)$-many turns, as the order types never get too large.

For (2), assume there is a square sequence $\langle D_\alpha : \alpha < \delta^+ \rangle$. $\text{Good}$ plays a similar strategy, except at limit $\lambda$, she plays $\bigcup_{\gamma < \lambda} p_\gamma \cup \langle \lambda : (\exists \beta < \lambda) \max(\text{dom} p_\beta) = \alpha \rangle$. This strategy allows the game to continue for $\delta^+$-many turns.

For (3), define $T_\delta^\lambda$ as the collection of bounded approximations to the desired set. By the strategic closure of $S_\delta$, the collection of $\langle p, \dot q \rangle \in S_\delta \star T_\delta^\lambda$ such that for some $x \in V$, $p \Vdash \dot q = \dot x$ is dense. If $\langle (p_\alpha, \dot x_\alpha) : \alpha < \beta < \lambda \rangle$ is a decreasing sequence of such conditions, let $x_\beta = \bigcup_{\alpha < \beta} x_\alpha$, and let $p_\beta = \bigcup_{\alpha < \beta} p_\beta \cup \langle \sup_{\alpha < \beta}(\text{dom} p_\alpha), \dot x_\beta \rangle$. This is a condition because for all limit points $\gamma$ of $x_\beta$, $x_\beta \cap \gamma = p_\beta(\gamma)$.

We can now perform our second round of forcing. We simply force with the Easton-support product of $S_\delta$, over all infinite cardinals $\delta < \theta$. First we check that this preserves superstrong cardinals with Mahlo target $\lambda$. For a set of ordinals $X$, let $P_X$ denote the sub-product where we restrict to indices in $X$. Let $\kappa$ be superstrong with target $\lambda$, $j(P_\kappa) = P_\lambda$, and $P_\lambda$ is $\lambda$-c.c. Since $P_{\lambda \setminus \kappa}$ is $(\kappa + 1)$-strategically-closed and $|P_\kappa| = \kappa$, Easton’s Lemma implies that $P_{\lambda \setminus \kappa}$ is still $\kappa^+$-distributive after forcing.
with $\mathbb{P}_\kappa$. By Lemma 2.11, $\kappa$ is still superstrong with target $\lambda$ after forcing with $\mathbb{P}_\lambda$. $\mathbb{P}_{\theta\setminus\lambda}$ does not add sets of rank $< \lambda$, so the superstrongness is preserved.

Now we argue that the conclusion of Lemma 5.2 still holds after forcing square everywhere below $\theta$, but with the proper class of almost-huge cardinals replaced with a proper class of superstrong cardinals. It will be important for the argument that we force square to hold everywhere with a product rather than an iteration.

Suppose $\mu < \delta < \theta$ are successor cardinals or Mahlo, and let $\kappa = \mu^+$ and $\lambda = \kappa^+$. Let $J$ be the ideal on $[\delta]^{<\kappa}$ as in Lemma 5.2 The forcing $\mathbb{P}_{\theta\setminus\delta}$ is $(\delta + 1)$-strategically closed. Thus it preserves the stationarity of $S_\lambda$, adds no subsets of $[\delta]^{<\kappa}$, and preserves that $J$ has all the properties as in Lemma 5.2. If $\kappa < \delta$, consider the forcing $\mathbb{P}_{[\kappa, \delta)}$. It is a coordinate-wise projection of

$$\prod_{\kappa < \delta} S_\mu * \bar{T}_\mu^{\kappa}.$$ 

By Lemma 5.6, this poset is $\kappa$-closed and has size $\delta$. Therefore, it is absorbed by $\text{Col}(\kappa, \delta)$, and thus by the Boolean algebra $\mathbb{P}([\delta]^{<\kappa})/I$. If $g \subseteq \text{Col}(\kappa, \lambda)$ is generic, then as in Lemma 4.3 forcing with the quotient $(\mathbb{P}([\delta]^{<\kappa})/I)/g$ yields an embedding $j : V[g] \to M[\check{g}]$. If $h$ is the projected generic for $\mathbb{P}_{[\kappa, \delta)}$, then the embedding restricts to $j : V[h] \to M[\check{h}].$ By Theorem 2.4.3, there is a normal $\kappa$-complete ideal $J$ on $[\delta]^{<\kappa}$ such that $\mathbb{P}([\delta]^{<\kappa})/I$ is isomorphic to the quotient $(\mathbb{P}([\delta]^{<\kappa})/I)/h.$ This Boolean algebra is still $S_\lambda$-layered by Lemma 2.7. If we make sure to use a projection that leaves a copy of $\text{Col}(\kappa, \delta)$ behind in the quotient (which can always be done as it is isomorphic to its square), we have that $\mathbb{P}([\delta]^{<\kappa})/I$ still projects to $\text{Col}(\mu, \kappa) \times \text{Col}(\kappa, \delta) \times \text{Add}(\kappa, \lambda)$. Since generic embeddings via $J$ extend those via $I$, the desired property of mapping a stationary $A \subseteq \kappa$ to a nonstationary $B \subseteq \lambda$ still holds.

Now we move to the extension by $S_\mu$, and it is here that we use the fact we have already forced with $\mathbb{P}_{\theta\setminus\kappa}$. The forcing $S_\mu * \bar{T}_\mu^{\kappa}$ is $\mu$-closed and of size $\kappa$, so it is absorbed by $\text{Col}(\mu, \kappa)$ and thus by the ideal $\mathbb{P}([\delta]^{<\kappa})/J$. Again, let us use a projection that leaves a copy of $\text{Col}(\mu, \kappa)$ behind. $G$ be generic for $\mathbb{P}([\delta]^{<\kappa})/J$ and let $j : V \to M$ be the generic ultrapower. Notice that since $\Box_\delta$ holds in $V$ and $\lambda = j(\kappa) = (\mu^+)^{V[\check{G}]}$, $\Box_\mu$ holds in $V[\check{G}]$ by Lemma 5.8. The projected generic $g * h \subseteq S_\mu * \bar{T}_\mu^{\kappa}$ yields a condition $m \in j(S_\mu) = S_\mu^{V[\check{G}]}$ that is below all conditions in $g$. Because $\Box_\mu$ already holds, $S_\mu^{V[\check{G}]}$ is in fact $\lambda$-strategically closed in $V[G]$ (but not necessarily in $M$). Since $(2^\lambda)^M = j(\kappa^+) \leq j(\lambda) < (\lambda^+)^V$, we can use this strategic closure and the $<\lambda$-closure of $M$ to build an $M$-generic $\check{g}$ with $m \in \check{g}$. By Theorem 2.13, there is an ideal $J'$ on $[\delta]^{<\kappa}$ in $V[g]$ such that $\mathbb{P}([\delta]^{<\kappa})/J' \cong (\mathbb{P}([\delta]^{<\kappa})/J)/g$. Since $S_\mu$ is $\kappa$-distributive, $\mathbb{P}([\delta]^{<\kappa})/J'$ still projects to $\text{Col}(\mu, \kappa) \times \text{Col}(\kappa, \delta) \times \text{Add}(\kappa, \lambda)$, and it is $S_\lambda$-layered.

Finally, consider the remaining forcing $\mathbb{P}_\mu$. Since $\mu$ is either Mahlo or a successor, it is either $\mu$-c.c., or of the form $\mathbb{P}_\mu \times S_\mu$, where $\mu = \nu^+$. Since $\mathbb{P}_\mu$ has size $\mu$ and $J'$ is $\kappa$-complete, if $J'$ is the ideal generated by $J'$ after forcing with $\mathbb{P}_\mu$, then every $J'$-positive set contains a $J'$-positive set from the ground model. Thus $\mathbb{P}([\delta]^{<\kappa})/J'$ remains $S_\lambda$-layered, and it projects to the version of $\text{Col}(\mu, \kappa) \times \text{Col}(\kappa, \delta) \times \text{Add}(\kappa, \lambda)$ from the ground model. If $\mu$ is Mahlo, then by Lemma 4.1 this projects to version of the same forcing as defined in the extension. If $\mu$ is a successor, then it projects to the version as defined in the extension by $S_\mu$, for the latter two factors because
|S_ν| = µ, and for Col(µ, κ) because S_ν adds no ν-sequences. Then this version projects to the version as defined in the further extension by P_ν, because |P_ν| = ν. Let us record what we have done:

**Lemma 5.7.** It is consistent relative to a huge cardinal that there is an inaccessible θ and a sequence ⟨S_α : α < θ⟩ such that:

1. V_θ satisfies GCH, □_κ for every infinite cardinal κ, and that there is a proper class of superstrong cardinals with Mahlo targets.
2. Clauses (4) and (5) of Lemma 3.2 hold.

5.3. **Frequent local saturation.** In order to provide the necessary set-up for the application of Radin forcing, we begin to introduce local saturation in the neighborhood of Mahlo cardinals, many of which will become singular in the end, leave room for some collapsing in between them, and retain superstrongness.

**Lemma 5.8.** Over a model satisfying the conclusion of Lemma 5.7, there is a cofinality-preserving forcing extension in which there is an inaccessible θ and a sequence ⟨S_α : α < θ⟩ such that:

1. V_θ satisfies GCH, □_κ for every infinite cardinal κ, and that there is a proper class of superstrong cardinals with Mahlo targets.
2. Whenever µ is Mahlo and κ = µ+n for 1 ≤ n ≤ 4, S_κ is a stationary subset of κ \cof(µ+n-1).
3. If µ < θ is a Mahlo cardinal and 1 ≤ n ≤ 3, there is a stationary A ⊆ µ+n such that P(A)/NS is S_{µ+n+1}-layered.
4. For every two Mahlo cardinals µ < δ < θ, if κ = µ+4, and λ = δ+, then there is κ-complete normal ideal I on [δ]^{κ} such that:
   a. P([δ]^{κ})/I is S_λ-layered.
   b. There is a stationary A ⊆ κ \cof(µ+3) and a nonstationary B ⊆ λ such that for any generic embedding j arising from I, κ ∈ j(A) = B.
   c. P([δ]^{κ})/I projects to Col(µ+3, κ) × Col(κ, δ) × Add(κ, λ).

*Proof.* Let ⟨S_α : α < θ⟩ be the sequence given by Lemma 5.2. Let µ < θ be Mahlo and let κ = µ+n for 1 ≤ n ≤ 3. Substituting δ = κ in clause (3) of Lemma 5.2, we have a normal ideal I on κ satisfying the hypotheses of Theorem 3.4. Let A ⊆ κ be the stationary set that is forced to be mapped to a nonstationary set B. By shrinking A if necessary, we may assume that S_κ \ A is stationary. Let P_κ be the (κ \cof(µ+n-1) \ A)-iteration as in the conclusion of Theorem 3.4. Let X = {κ < θ : κ = µ+n for µ Mahlo and 1 ≤ n ≤ 3}. We force with the product:

Q := \prod_{κ ∈ X} P_κ.

For clause (1), for the preservation of cofinalities (and thus cardinals and squares) and the GCH, the key is to note that for any regular cardinal µ, (a) Q \upharpoonright µ^+ is (µ^+ \cof(µ))-layered, and (b) Q \upharpoonright [µ^+, θ) is µ^+-distributive since it is of the form (µ^+-distributive) × (µ^+-closed). Thus by Easton’s Lemma, the latter retains its distributivity after forcing with Q \upharpoonright µ^+. The desired superstrongness is preserved by the same argument as in previous subsection.

For clause (2), we just need to check the preservation of S_{µ+n} for µ Mahlo and 1 ≤ n ≤ 4. If µ is Mahlo and 1 ≤ n ≤ 3, then P_{µ+n} preserves the stationarity of S_{µ+n} by Lemma 3.1 and thus so does Q \upharpoonright [µ^{n+1}, θ). Since the tail Q \upharpoonright [µ^{n+1}, θ)
remains \(\mu^{+n+1}\)-distributive, \(S_{\mu+n}\) remains stationary. If \(n = 4\), then \(Q\) factors as 
\((\mu^{+n}\text{-c.c.}) \times (\mu^{+n}\text{-closed})\).

For clause (3), let \(\mu < \theta\) be Mahlo and \(1 \leq n \leq 3\). After forcing with \(P_{\mu+n}\), the desired conclusion holds for some stationary \(A \subseteq \mu^{+n}\), and thus it holds after forcing with \(Q \upharpoonright [\mu^{+n}, \theta)\) by the distributivity of the tail. Temporarily let \(V\) denote an extension by \(Q \upharpoonright [\mu^{+n}, \theta)\). In this model, the forcing \(Q \upharpoonright \mu^{+n}\) is still \((\mu^{+n} \cap \text{cof}(\mu^{+n-1}))\)-layered. If \(j : V \to M \subseteq V[G]\) is a generic embedding arising from forcing with \(P(A)/\text{NS}\), then \(M \models \langle j(Q \upharpoonright \mu^{+n}) = j(\mu^{+n}) \cap \text{cof}(\mu^{+n-1})\rangle\)-layered,
and this holds in \(V[G]\) as well since \(M\) is closed under \(\mu^{+n-1}\)-sequences from \(V[G]\). Since \(Q \upharpoonright \mu^{+n}\) is \(\mu^{+n}\text{-c.c.}, the ideal generated by \(\text{NS}\) of the ground is \(\text{NS}\) of the extension. By Corollary 2.17

\[ Q \upharpoonright \mu^{+n} * P(A)/\text{NS} \cong P(A)/\text{NS} * j(Q \upharpoonright \mu^{+n}). \]

By Lemma 2.7, the right-hand side is \(S_{\mu+n+1}\)-layered, and since \(|Q \upharpoonright \mu^{+n}| = \mu^{+n}\), it is forced that the quotient \(P(A)/\text{NS}\) is \(S_{\mu+n+1}\)-layered.

For clause (1), let \(\mu < \delta\) Mahlo cardinals below \(\theta\), and let \(\kappa = \mu^{+4}\). The subforcing \(Q \upharpoonright [\kappa, \delta)\) is \(\kappa\)-closed and of size \(\delta\). By the same arguments as in the previous subsection, there is an ideal \(J\) on \([\delta]^{<\kappa}\) with the desired properties after forcing with \(Q \upharpoonright [\kappa, \delta)\), and this holds after forcing further with the tail \(Q \upharpoonright [\delta, \theta)\) by distributivity. Now consider adjoining a generic for the forcing \(Q \upharpoonright \kappa\). By precisely the same argument as for (3), the Boolean algebra associated to the generated ideal \(J\) is still \(S_{\delta+}\)-layered. Subclause (b) holds by the fact that a generic embedding arising from \(J\) will extend one arising from \(J\), per Proposition 2.14.

For subclause (c), we have that

\[ Q \upharpoonright \kappa * P([\delta]^{<\kappa})/\bar{J} \cong P([\delta]^{<\kappa})/\bar{J} \times j(Q \upharpoonright \kappa). \]

Let \(H * \bar{G}\) be generic for the left-hand side. If we transfer this generic to one for the right-hand side \(G * \bar{H}\), via the isomorphism \(\iota\) of Theorem 2.13, we get that 
\(G = \bar{G} \cap P([\delta]^{<\kappa})^V\). Furthermore, since \(j\) is the identity of \(Q \upharpoonright \kappa\), \(H = H \cap (Q \upharpoonright \kappa)\). By the layeredness of \(j(Q \upharpoonright \kappa), Q \upharpoonright \kappa\) is a regular suborder, and \(H\) is generic over \(V[G]\). Thus the map \(\langle q, Y \rangle \mapsto \langle q, \bar{Y} \rangle\) is a regular embedding of \((Q \upharpoonright \kappa) \times P([\delta]^{<\kappa})/\bar{J}\)
into \(Q \upharpoonright \kappa * P([\delta]^{<\kappa})/\bar{J}\). Hence, if \(H \subseteq Q \upharpoonright \kappa\) is generic over \(V[H]\), then in \(V[H]\) there is a projection from \(P([\delta]^{<\kappa})/\bar{J}\) to \(\text{Col}(\mu^{+3}, \kappa) \times \text{Col}(\kappa, \delta) \times \text{Add}(\kappa, \lambda)^V\). This is equal to \(\text{Col}(\mu^{+3}, \kappa)^{V[H]} \times \text{Col}(\kappa, \delta) \times \text{Add}(\kappa, \lambda)^V\). Since \(Q \upharpoonright \kappa\) is \(\kappa\text{-c.c.}\) and of size \(\kappa\), Lemma 4.4 gives that the latter factor projects to \(\text{Col}(\kappa, \delta) \times \text{Add}(\kappa, \lambda)^V\). □

We would like to point out that in the argument for (1) above, the layeredness of \(Q \upharpoonright \kappa\) played a substantial role. In general, the \(\kappa\text{-c.c.}\) alone is not enough to carry out such arguments. See [1] Theorems 7.3 and 7.4 for further discussion.

6. The final model

In this section, we introduce the Radin forcing with interleaved posets as presented by Cummings [cite], and show how it forces a model of Theorem 1 over a model satisfying the conclusion of Lemma 5.8. The details of the forcing are a slight generalization of those presented by Cummings. However, no new ideas are needed to prove its basic properties, so we defer to his article for some key lemmas. As the reader may check, each step in Cummings’ detailed proof of the Prikry Property can be carried out using the slightly more general hypotheses that we write here.
The construction of the Radin forcing and the proof its important properties depends crucially on the following:

**Background Hypothesis.** There is a class of partial orders $\mathbb{P}(\alpha, \beta)$ indexed by Mahlo $\alpha < \beta$ such that:

1. For $\alpha < \beta$ Mahlo, $\mathbb{P}(\alpha, \beta)$ is a partial order that is $(2^\alpha)^{++}$-closed and of size $\leq 2^\beta$.
2. For all measurable $\kappa$, if $j : V \to M$ is an embedding derivied from a normal measure on $\kappa$, then there is an $M$-generic $G \subseteq \mathbb{P}(\kappa, j(\kappa))^M$.

Suppose $j : V \to M$ is an elementary embedding with critical point $\kappa$ derived from an extender $E$. Let $\mathcal{U}$ be the normal measure on $\kappa$ derived from $j$, and let $i_{0,1} : V \to N_1$ be the ultrapower embedding by $\mathcal{U}$. Let $i_{1,2} : N_1 \to N_2$ be the ultrapower embedding of $N_1$ by $i_{0,1}(\mathcal{U})$, and let $i_{0,2} = i_{1,2} \circ i_{0,1}$. Then $\mathbb{P}(\kappa, i_{0,1}(\kappa))^{N_1} = \mathbb{P}(\kappa, i_{0,1}(\kappa))^{N_2}$. Let $G^* = \{ f : [\kappa]^2 \to V_\kappa : \text{dom } f \in \mathcal{U}^2 \land \forall \alpha \forall \beta f(\alpha, \beta) \in \mathbb{P}(\alpha, \beta) \}$.

Suppose $G \subseteq \mathbb{P}(\kappa, i_{0,1}(\kappa))^{N_1}$ is generic over $N_1$. Let $G^* = \{ f \in G^* : i_{0,2}(f)(\kappa, i_{0,1}(\kappa)) \in G \}$.

Notice that $\mathcal{U}$ can be read off from $G^*$. In a situation where $G \in M$, we define measure sequences $u$ as follows. Let $u(0) = \kappa$ and $u(1) = G^*$. For $\alpha > 1$, inductively let $u(\alpha) = \{ X \subseteq V_\kappa : u \upharpoonright \alpha \in j(X) \}$, in case $u \upharpoonright \alpha \in M$. We will say that the pair $(E, G)$ constructs $u$. We will say that $(E, G)$ is an acceptable pair if in addition, $E$ is a $(\kappa, \lambda)$-extender, with $[\lambda]^\kappa \subseteq M$ and $|\lambda| \leq (2^\kappa)^+$. This implies that $M^* \subseteq M$ and that if $k : N_1 \to M$ is the factor embedding, then $k[G]$ generates an $M$-generic filter for $\mathbb{P}(\kappa, j(\kappa))^M$.

Given a measure sequence $u$ constructed by an acceptable pair $(E, G)$, we say that a set $X \subseteq V_{u(0)}$ is $u$-measure-one if there is $f \in u(1)$ such that $\text{dom } f = A^2$ and $A \subseteq X$, and for all $\alpha$ such that $1 < \alpha < \text{len } u, X \in u(\alpha)$. We inductively define some well-behaved classes of measure sequences: Let $U_0$ be the class of measure sequences $u$ constructed by an acceptable pair. Given $U_n$, let $U_{n+1} = \{ w \in U_n : u \cap V_{u(0)} \text{ is } u\text{-measure-one} \}$. Let $U_\infty = \bigcap_{n<\omega} U_n$. If $u \in U_\infty$, then by countable completeness, $U_\infty \cap V_{u(0)}$ is $u$-measure-one, and $u \upharpoonright \alpha \in U_\infty$ for $1 \leq \alpha < \text{len } u$. It is worth noting at this point:

**Lemma 6.1** (Cummings). Suppose $E$ is a $(\kappa, (2^\kappa)^+)$ extender witnessing that $\kappa$ is $(\kappa + 2)$-strong, i.e. if $j : V \to M$ is the embedding by $E$, then $V_{\kappa+2} \subseteq M$.

Let $i : V \to N$ be the embedding by the normal measure $\mathcal{U}$ on $\kappa$ derived from $E$, and suppose $G$ is $\mathbb{P}(\kappa, i_\mathcal{U}(\kappa))^{N_1}$-generic over $N$. Then $(E, G)$ constructs a measure sequence $u \in U_\infty$ of length $(2^\kappa)^+$. 

**Proof.** See [2] Section 3.1.

Now we are ready to define the forcing $Q_u$ relative to a $u \in U_\infty$. Let $\kappa = u(0)$. $p$ is a condition in $Q_u$ if $p = \langle X_i : i \leq n \rangle$, where $n \geq 1$ and there exists an increasing sequence of Mahlo cardinals $\kappa_0 < \cdots < \kappa_n = \kappa$ such that:

1. $X_0 = (\kappa_0)$.
2. For $0 < i < n$, $X_i$ is either a pair $\langle \kappa_i, p_i \rangle$ with $p_i \in \mathbb{P}(\kappa_{i-1}, \kappa_i)$, or a quadruple $\langle w_i, A_i, H_i, h_i \rangle$, where:
   (a) $w_i \in U_\infty$, $\text{len } w_i > 1$, and $\kappa_i = w_i(0)$.
Proof. See [2, Section 3.4].

(b) \( A_i \) is \( w_i \)-measure-one and contained in \( V_{\kappa_i} \setminus V_{\kappa_{i-1}} \).

(c) If \((E_i,G_i)\) is an acceptable pair that constructs \( w_i, \) then \( H_i \in G_i^* \).

(d) \( h_i \) is a function with domain \( A_i \cap \kappa_i, \) and \( (\forall \alpha) h_i(\alpha) \in P(\kappa_{i-1}, \alpha) \).

(e) \( \text{dom} H_i = [\text{dom} h_i]^2 \).

(3) \( X_n \) is a quadruple \( \langle w_n, A_n, H_n, h_n \rangle \) with the same properties as above, and \( w_n = u \).

If \( \langle X_i : i \leq n \rangle \) is a condition with associated sequence of cardinals \( \langle \kappa_i : i \leq n \rangle, \) put \( \kappa_{X_i} = \kappa_i \). Let \( p = \langle X_i : i \leq m \rangle \) and \( q = \langle Y_i : i \leq n \rangle \). We put \( p \leq q \) when:

(1) \( m \geq n \) and \( X_0 = Y_0 \).

(2) \( \{ \kappa_{X_i} : i \leq m \} \supseteq \{ \kappa_{Y_j} : i \leq n \} \).

(3) For \( 0 < i < m, \) if \( X_i = \langle \kappa_{i}, p_i \rangle \), then one of the following occurs:

(a) There is \( j < n \) such that \( \kappa_{X_i} = \kappa_{Y_j} \). In this case, \( \kappa_{X_{i-1}} = \kappa_{Y_{j-1}} \) also, and \( p_i \subseteq q_j \), where \( Y_j = \langle \kappa_j, q_j \rangle \).

(b) There is no \( j < n \) such that \( \kappa_{X_i} = \kappa_{Y_j} \). For the least \( j \leq n \) such that \( \kappa_{X_i} < \kappa_{Y_j} \), \( Y_j \) is a quadruple \( \langle w, A, H, h \rangle \) with \( \kappa_{X_i} \in A \). If \( \kappa_{X_{i-1}} = \kappa_{Y_{j-1}} \), then \( p_i \leq h(\kappa_{X_i}) \), and if \( \kappa_{X_{i-1}} > \kappa_{Y_{j-1}} \), then \( p_i \leq H(\kappa_{X_{i-1}}, \kappa_{X_i}) \).

(4) For \( 0 < i \leq m, \) if \( X_i \) is a quadruple \( \langle w, A, H, h \rangle \), then one of the following occurs:

(a) There is \( j \leq n \) such that \( Y_j = \langle w, A', H', h' \rangle \). In this case, \( A \subseteq A' \) and for all \( (\alpha, \beta) \in \text{dom} H, H(\alpha, \beta) \leq H'(\alpha, \beta) \). If \( \kappa_{X_{i-1}} = \kappa_{Y_{j-1}} \), then for all \( \alpha \in \text{dom} h, h(\alpha) \leq h'(\alpha) \). If \( \kappa_{X_{i-1}} > \kappa_{Y_{j-1}} \), then for all \( \alpha \in \text{dom} h, h(\alpha) \leq H'(\kappa_{X_{i-1}}, \alpha) \).

We say \( p \leq^* q \) when \( p \leq q \) and \( \text{len} p = \text{len} q \). The following three lemmas are straightforward to check:

Lemma 6.2. Suppose \( u \in U_\infty, p = \langle X_i : i \leq n \rangle \in Q_u, \) and \( X_0 = \langle \kappa_0 \rangle \). Then \( \langle Q_u \upharpoonright p \rangle \leq^* q \) is \( (2^{\omega_2})^{++} \)-closed.

Lemma 6.3. Suppose \( u \in U_\infty \) and \( p = \langle X_i : i \leq n \rangle \in Q_u \). Suppose \( m_0 < m_1 < n \) are such that \( X_{m_0} \) is a quadruple \( \langle w, A, H, h \rangle \), and \( X_i \) is a pair \( \langle \kappa_i, p_i \rangle \) for \( m_0 < i \leq m_1 \). Then \( Q_u \upharpoonright p \) is isomorphic to

\[
Q_u \upharpoonright \langle X_i : i \leq m_0 \rangle \times P(\kappa_{m_0}, \kappa_{m_0+1}) \times p_{m_0+1} \times \cdots \times P(\kappa_{m_1-1}, \kappa_{m_1}) \times p_{m_1} \\
\times Q_u \upharpoonright \langle \langle \kappa_{m_1}, X_{m_1+1}, \ldots, X_n \rangle \rangle.
\]

Lemma 6.4. Suppose \( u \in U_\infty \) and \( G \subseteq Q_u \) is generic. Let \( \kappa = u(0) \) and let \( C = \{ \alpha : (\exists p = \langle X_i : i \leq n \rangle \in G)(\exists i < n) \alpha = \kappa_{X_i} \} \). Then \( C \) is club in \( \kappa \).

Now for the more substantial results concerning this forcing:

Theorem 6.5 (Cummings). Suppose \( u \in U_\infty, p \in Q_u, \) and \( \sigma \) is a sentence in the forcing language of \( Q_u \). Then there is \( q \leq^* p \) deciding \( \sigma \).

Proof. See [2, Section 3.4].
Corollary 6.6. Suppose \( u \in U_\infty, \ p = (X_i : i \leq n) \in Q_u, \) and \( X_0 = \langle \kappa_0 \rangle. \) Then \( Q_u \upharpoonright p \) adds no subsets of \( (2^{\kappa_0})^+. \) Thus if \( P \) is a forcing of size \( (2^{\kappa_0})^+, \) then \( Q_u \) adds no subsets of \( (2^{\kappa_0})^+ \) over \( V^P. \)

Now we specify the partial orders \( P(\alpha, \beta) \) and show that the Background Hypothesis holds. In a model satisfying the conclusion of Lemma 5.8, we have by Lemma 6.7 that whenever \( \alpha < \beta \) are Mahlo, \( \text{Col}(\alpha^{++}, \beta) \) forces that there is an ideal on \( \alpha^{++} \) satisfying the hypotheses of Theorem 3.4. Thus in \( V^{\text{Col}(\alpha^{++}, \beta)} \), for some stationary \( S \subseteq S_{\alpha^{++}} \) (the latter being the one that witnesses layeredness of the ideal on \( \alpha^{++} \)), there is an \( S \)-iteration \( C_{\alpha^{++}} \) of length \( \beta^+ \) that forces \( \text{NS}_{\alpha^{++}} \) is locally saturated. We define \( P(\alpha, \beta) \) to be \( \text{Col}(\alpha^{++}, \beta) * C_{\alpha^{++}}. \) Then \( P(\alpha, \beta) \) is \( \alpha^{++}\)-closed and of size \( \beta^+. \) By GCH, whenever \( \kappa \) is measurable as witnessed by \( j : V \to M, \ \kappa^+ < j(\kappa^+) < \kappa^{++}. \) Thus by the \( \kappa^+ \)-closure and \( j(\kappa^+) \)-c.c. in \( M \) of \( P(\kappa, j(\kappa))^M \), we can build an \( M \)-generic filter \( G \in V. \) Therefore the Background Hypothesis holds for these \( P(\alpha, \beta). \)

From now on, when we refer to the Radin forcing \( Q_u \) as described above, we mean the one defined in our preparatory model using the above specification of the \( P(\alpha, \beta). \) The important feature of our version of Radin forcing, which is not shared by Cummings' version, is the chain condition:

Lemma 6.7. Suppose \( u \in U_\infty \) and \( \kappa = u(0). \) Then \( Q_u \) is \( \kappa^+\)-c.c. Moreover, it preserves \( \kappa^{++}\)-saturated ideals on \( \kappa^+. \)

Proof. Suppose \( (p_{\alpha} : \alpha < \kappa^+) \subseteq Q_u. \) Let \( p_{\alpha} = \vec{x}_{\alpha} = \langle u, \alpha, H_\alpha, h_\alpha \rangle. \) We can assume there is a fixed \( \vec{x} = \langle X_0, \ldots, X_{n-1} \rangle \) such that \( \vec{x}_{\alpha} = \vec{x} \) for all \( \alpha. \) Let \( U \) be the ultrafilter associated to \( u(1) \) and let \( \vec{j}_U : V \to M \) be the embedding by \( U. \) Each \( h_\alpha \) represents an element of \( P(\kappa_{n-1}, \kappa)^M, \) which is \( (\kappa^+ \cap \text{cof}(\kappa))-\text{layered}. \) Let \( \alpha < \beta \) be such that \( h_\alpha, h_\beta \) represent compatible conditions. Since \( H_\alpha, H_\beta \) represent conditions in a filter, \( p_\alpha \) and \( p_\beta \) are compatible.

To show that \( Q_u \) preserves saturated ideals on \( \kappa^+, \) suppose \( I \) is such an ideal and \( i : V \to N \subseteq V[G] \) is a generic ultrapower via \( I. \) Then the above argument can be carried out in \( N. \) In particular, if \( M' \) is the ultrapower of \( N \) by \( i(U), \) then \( N \) satisfies that \( P(\kappa_{n-1}, \kappa) \) is \( (i(\kappa^+) \cap \text{cof}(\kappa))-\text{layered.} \) This is true in \( V[G] \) as well since \( N^\kappa \cap V[G] \subseteq N. \) Thus by Corollary 2.17, \( Q_u \) forces that the ideal generated by \( I \) is \( \kappa^{++}\)-saturated.

With this information, we can give a proof of the following lemma for our case that is easier than Cummings' proof for the general case (see [2 Section 3.8]):

Lemma 6.8. Suppose \( u \in U_\infty, \) \( u(0) = \kappa, \) and \( \text{len} u \geq (2^\kappa)^+. \) Then \( Q_u \) preserves the measurability of \( \kappa. \)

Proof. Let \( u \) be as hypothesized. For every \( X \subseteq V_\kappa, \) let \( \alpha_X \) be the least \( \alpha \) such that \( X \in u(\alpha), \) if there is such an \( \alpha. \) Let \( \gamma^* = \text{sup}\{\alpha_X + 1 : X \subseteq V_\kappa\}. \) If \( \gamma^* \leq \beta < \text{len} u, \) then every \( X \in u(\beta) \) is \( u(\alpha) \) for some \( \alpha < \gamma^*. \) Let \( u^* = u \upharpoonright \gamma^* \) and consider the forcing \( Q_{u^*}. \) If \( \vec{x}^* \langle u, A, H, h \rangle \in Q_{u^*}, \) then \( \vec{x}^* \langle u^*, A, H, h \rangle \in Q_{u^*}, \) since every \( u\)-measure-one set is \( u^*\)-measure-one. Conversely, if \( \vec{x}^* \langle u, A, H, h \rangle \in Q_{u^*}, \) then we claim \( A \) is \( u\)-measure-one. For otherwise there is \( \beta \geq \gamma^* \) such that \( V_\kappa \setminus A \in u(\beta). \) But then \( V_\kappa \setminus A \in u(\alpha) \) for some \( \alpha < \gamma^* \), a contradiction. Thus \( \vec{x}^* \langle u, A, H, h \rangle \in Q_{u^*}. \) Therefore \( Q_{u^*} \models Q_{u^*}. \)

Since \( Q_{u^*} \) is \( \kappa^+\)-c.c. and of size \( \kappa^+, \) there are only \( \kappa^+ \)-many inequivalent \( Q_{u^*}\)-names for subsets of \( \kappa. \) Let \( \langle (q, \tau)_{\alpha} : \alpha < \kappa^+ \rangle \) enumerate all pairs of conditions
and names for subsets of $\kappa$. For every $p \in \mathbb{Q}_u^*$, the sequence $p^\frown \langle j(u), B, F, f \rangle$ is a member of $j(\mathbb{Q}_u)$, for any $B$ which is $j(u)$-measure-one, and $F, f$ appropriate functions defined on $|B|^2$, $B$ respectively.

Suppose $(E, G)$ is an acceptable pair that constructs $u$, and let $j : V \to M$ be the embedding via $E$. Note that $M^\kappa \subseteq M$ by hypothesis. Inductively build a descending $\leq^*$-descending sequence in $j(\mathbb{Q}_u)$ as follows. Start with the trivial condition. Given $(B_\alpha, F_\alpha, f_\alpha)$, let $(p, \bar{X}) = (q, \tau)_\alpha$. If there is a $j(u)$-measure-one $C \subseteq B_\alpha$, an $F \in j(G^*)$ coordinate-wise below $F_\alpha$, and an $f$ defined on $C$ that is coordinate-wise below $f_\alpha$, such that $p^\frown \langle j(u), C, F, f \rangle$ decides whether $\kappa \in j(\bar{X})$, let $B_{\alpha+1}, F_{\alpha+1}, f_{\alpha+1}$ be such objects. Otherwise let $B_{\alpha+1} = B_\alpha, F_{\alpha+1} = F_\alpha, f_{\alpha+1} = f_\alpha$. At limit stages $\lambda < \kappa^+$, let $B_\lambda = \bigcap_{\alpha < \lambda} B_\alpha$, and let $F_\lambda, f_\lambda$ be coordinate-wise lower bounds using the closure of the posets.

Let $H \subseteq \mathbb{Q}_u^*$ be generic. In $V[H]$, let $\mathcal{F}$ be the collection of $X \subseteq \kappa$ such that for some $\alpha < \kappa^+$, $(q, \tau)_\alpha$ is such that $\tau^H = X$, $p \in H$, and $p^\frown \langle j(u), B_\alpha, F_\alpha, f_\alpha \rangle \models \kappa \in j(\tau)$. It is easy to see that $\mathcal{F}$ is a filter, because $(p^\frown \langle j(u), B_\alpha, F_\alpha, f_\alpha \rangle : p \in H \land \alpha < \kappa^+)$ is a filter of conditions in $j(\mathbb{Q}_u)$. Next we claim $\mathcal{F}$ is an ultrafilter. Let $p \in \mathbb{Q}_u^*$ and suppose $p^\frown \langle j(u), B, F, f \rangle \in j(\mathbb{Q}_u)$. For any sentence $\sigma$ in the forcing language, there is a direct extension deciding $\sigma$ by Theorem 6.3. Thus for every sentence $\sigma$ and every appropriate sequence $\langle j(u), B, F, f \rangle$, there is $p \in H$ and a stronger $\langle j(u), B', F', f' \rangle$ such that $p^\frown \langle j(u), B', F', f' \rangle$ decides $\sigma$. So if $\bar{X}$ is a name for a subset of $\kappa$, let $p \in H$ and $\alpha < \kappa^+$ be such that $(p, \bar{X}) = (q, \tau)_\alpha$. There are $B, F, f$ below $B_\alpha, F_\alpha, f_\alpha$ such that $p^\frown \langle j(u), B, F, f \rangle$ decides $\kappa \in j(\bar{X})$, so $p^\frown \langle j(u), B_{\alpha+1}, F_{\alpha+1}, f_{\alpha+1} \rangle$ decides $\kappa \in j(\bar{X})$. Thus either $X$ or $\kappa \setminus X$ is in $\mathcal{F}$.

Finally, we argue that $\mathcal{F}$ is $\kappa$-complete. Suppose $\delta < \kappa$ and $\{X_\alpha : \alpha < \delta \} \subseteq \mathcal{F}$, and choose names $\bar{X}_\alpha$, $\bar{X}_\alpha'$ such that $\bar{X}_\alpha^H = X_\alpha$. Let $\beta < \kappa^+$ be large enough that for every $\alpha < \delta$, there is $p \in H$ such that $p^\frown \langle j(u), B_\beta, F_\beta, f_\beta \rangle \models \kappa \in j(\bar{X}_\alpha)$, and there is $p \in H$ such that $p^\frown \langle j(u), B_\beta, F_\beta, f_\beta \rangle$ decides whether $\kappa$ is in the intersection of all sets in $j(\langle X_\alpha \cup j(\bar{X}_\alpha) : \alpha < \delta \rangle) = \langle j(\bar{X}_\alpha) : \alpha < \delta \rangle$. If $K$ is generic over $V[H]$ for the forcing $j(\mathbb{Q}_u) \upharpoonright \langle \langle \kappa, \langle j(u), B_\beta, F_\beta, f_\beta \rangle \rangle \rangle$, then we have a generic $\hat{H}$ for $j(\mathbb{Q}_u)$, such that in $M[\hat{H}], \kappa \in \bigcap_{\alpha < \delta} j(X_\alpha)$. Therefore, $\bigcap_{\alpha < \delta} X_\alpha$ must be in $\mathcal{F}$.

Now fix a cardinal $\kappa$ which is $(\kappa + 2)$-strong as witnessed by an extender $(\kappa, \kappa^{++})$-extender $E$, let $G$ be such that $(E, G)$ is an acceptable pair, and let $u$ be a measure sequence of length $\kappa^{++}$ constructed by $(E, G)$. Let $H \subseteq \mathbb{Q}_u$ be generic. By Theorem 6.8, $V_{\kappa}^{\mathcal{V}[H]}$ is a model of ZFC. Let $C \subseteq \kappa$ be the Radin club introduced by $H$, and let $\kappa_0 = \min C$. Because of Theorem 6.3 and the interleaved collapses, the set of limit cardinals in $\kappa \setminus \kappa_0$ in $V[H]$ is simply the set of limit points of $C$. Let $h \subseteq \text{Col}(\omega, \kappa_0)$ be generic over $V[H]$. We claim that $V_{\kappa}^{\mathcal{V}[H][h]}$ is a model of Theorem 1.

First we check that $\Box_\delta$ holds for all $\delta < \kappa$. If $\mu$ is a successor cardinal of $V[H][h]$, then $\mu = \nu^+$ for some cardinal $\nu$ of $V$. $\Box_\nu$ holds in $V$. Although $\nu$ may be collapsed, Lemma 5.3 implies that $\Box_\eta$ holds in $V[H][h]$, where $\eta$ is the predecessor of $\mu$ in the final model.

Now let $\mu_0 < \mu_1$ be two successive points of $C$. Suppose first that $\mu_0$ is a limit point. Let $p \in H$ force this, so that $p = \langle X_i : i \leq n \rangle$ such that for some $m < n$, $X_m$ is a quadruple $\langle w, A, H, h \rangle$ with $w(0) = \mu_0$ and $X_{m+1}$ is a pair $\langle \mu_1, p_{m+1} \rangle$. By Lemma 6.3.

$\mathbb{Q}_u \upharpoonright p \cong \mathbb{Q}_u \upharpoonright \langle X_0, \ldots, X_m \rangle \times \mathbb{P}(\mu_0, \mu_1) \upharpoonright p_{m+1} \times \mathbb{Q}_u \upharpoonright \langle \langle \mu_1, X_{m+2}, \ldots, X_n \rangle$. 

\[ \mathbb{Q}_u \upharpoonright p \cong \mathbb{Q}_u \upharpoonright \langle X_0, \ldots, X_m \rangle \times \mathbb{P}(\mu_0, \mu_1) \upharpoonright p_{m+1} \times \mathbb{Q}_u \upharpoonright \langle \langle \mu_1, X_{m+2}, \ldots, X_n \rangle. \]
In $V$, $\text{NS}_{\mu_0^+}$ is locally saturated, and this is preserved by $Q_w$ by Lemma 6.7. The local saturation of $\text{NS}_{\mu_0^+}$ is preserved since $|Q_w| = \mu_0^+$. The upper factor $\mathbb{P}(\mu_0, \mu_1) \times Q_u \upharpoonright \langle \langle \mu_1, X_{m+2}, \ldots, X_n \rangle \rangle$ preserves this, since it adds no further subsets to $\mu_0^+\times\mu_0^+$.

For $\mu_0^+\times\mu_0^+$, in $V$ there is some stationary $A \subseteq \mu_0^+\times\mu_0^+$ such that $\mathbb{P}(A)/\text{NS}$ is $S_{\mu_0^+\times\mu_0^+}$-layered. The factor $\mathbb{P}(\mu_0, \mu_1)$ preserves the stationarity of $S_{\mu_0^+\times\mu_0^+}$ and adds no subsets of $\mu_0^+\times\mu_0^+$, and thus preserves that $\text{NS}_{\mu_0^+\times\mu_0^+}$ is locally saturated. This is preserved by the small lower factor $Q_w$, and by the upper factor $Q_u \upharpoonright \langle \langle \mu_1, X_{m+2}, \ldots, X_n \rangle \rangle$, which adds no further subsets of $\mu_0^+\times\mu_0^+$.

For $\mu_0^+\times\mu_0^+$, the local saturation of $\text{NS}_{\mu_0^+\times\mu_0^+}$ is explicitly forced by $\mathbb{P}(\mu_0, \mu_1)$. This is preserved by the small lower factor $Q_w$, and then by the upper factor $Q_u \upharpoonright \langle \langle \mu_1, X_{m+2}, \ldots, X_n \rangle \rangle$, which adds no further subsets of $\mu_0^+$.

Now suppose that $\mu_0$ is a successor point or the least point of $C$. If $p \in H$ forces this, then we may assume that $p = \langle X_i : i \leq n \rangle$ and $Q_u \upharpoonright p \cong \mathbb{R} \times \mathbb{P}(\mu_0, \mu_1) \upharpoonright p_{m+1} \times Q_u \upharpoonright \langle \langle \mu_1, X_{m+2}, \ldots, X_n \rangle \rangle$, where either $\mathbb{R}$ is trivial or is $(\mu_0^+ \cap \text{cof}(\mu_0))$-layered. $\mathbb{R}$ preserves the local saturation of $\text{NS}_{\mu_0^+\times\mu_0^+}$, and this is preserved by the upper factor, which adds no subsets of $\mu_0^+\times\mu_0^+$. The local saturation of $\text{NS}_{\mu_0^+\times\mu_0^+}$ for $2 \leq k \leq 4$ is forced for the same reasons as in the case that $\mu_0$ is a limit point.

Finally, all of these saturation properties are preserved by the small forcing $\text{Col}(\omega, \kappa_0)$, which makes the class of successor cardinals in $V[H]$ above $\kappa_0$ equal to the class of all successor cardinals. This concludes the proof of Theorem 4.

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