Lorentzian symmetric spaces in supergravity

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Abstract. I will discuss the emergence of lorentzian symmetric spaces as supersymmetric supergravity backgrounds. I will focus on supergravity theories in dimension 11, 10, and 6, and will concentrate on the determination of the so-called maximally supersymmetric backgrounds, for which a classification exists up to local isometry. A special class of lorentzian symmetric spaces also plays a rôle in the determination of parallelisable supergravity backgrounds in type II supergravity, which I will also summarise.

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The purpose of this short review is to highlight the rôle of lorentzian geometry in the supergravity limit of string theories. That lorentzian geometry plays a rôle in such theories should not come as a surprise, given that the supergravity theories in question are lorentzian theories; that is, some (if not all) of the gravitational degrees of freedom are encoded in the form of a (local) lorentzian metric. What may be a little surprising is the fact that, with notable exceptions, until relatively recently the lorentzian nature of the solutions had not been fully exploited. Indeed, most early papers studying solutions to the supergravity field equations concentrated on decomposable geometries $L \times R$, where $L$ is a lorentzian space-form (e.g., Minkowski or (anti) de Sitter spacetime) and $R$ a riemannian manifold, which would invariably become the focus of the ensuing analysis. This is not to say that such solutions are geometrically or physically uninteresting. In fact, they have motivated the study of a large class of riemannian geometries which otherwise might have remained largely in obscurity: Calabi–Yau manifolds, manifolds with $G_2$ and Spin(7) holonomy, as well as manifolds whose metric cones have such holonomies, e.g., Sasaki–Einstein and nearly Kähler manifolds, among others. However they do miss interesting solutions, as we will try to illustrate in this review.

This review is organised as follows. In Section 2 we introduce the geometries of interest, namely lorentzian symmetric spaces and in particular those which admit an absolute parallelism, which translates into the question of which of these spaces are Lie groups admitting bi-invariant lorentzian metrics. In Section 3 we discuss the geometrical aspects of supergravity theories. Much more could and should eventually be written about this, but for the purposes of this review we will limit ourselves to treat supergravity theories as collections of geometric PDEs whose form is highly constrained, despite at first seeming ad hoc. For reasons explained in the body of the review, we will consider only the following supergravity theories: eleven-dimensional supergravity, ten-dimensional type IIB and the chiral six-dimensional $(1,0)$ and $(2,0)$ supergravities. In Section 4 we discuss the classification of maximally supersymmetric solutions of the above theories and also of ten-dimensional type IIA supergravity, which can be obtained from eleven-dimensional supergravity via Kaluza–Klein reduction, a technique we review in Section 4.1. Finally in Section 5 we discuss parallelisable backgrounds in the common sector of type II supergravity.

### 5 Parallelisable type II backgrounds

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2. The geometries of interest

In this section we will quickly review the geometries of interest: lorentzian symmetric spaces and, in particular, those which are parallelisable.

2.1. Lorentzian symmetric spaces. We start by reviewing the classification of lorentzian symmetric spaces.

The classification of symmetric spaces in indefinite signature is hindered by the fact that there is no splitting theorem saying that if the holonomy representation is reducible, the space is locally isometric to a product. In fact, local splitting implies both reducibility and a nondegeneracy condition on the factors [1]. This means that one has to take into account reducible yet indecomposable holonomy representations. The general semi-riemannian case is still open, but indecomposable lorentzian symmetric spaces were classified by Cahen and Wallach [2] almost four decades ago. Indeed, they stated the following theorem

Theorem 2.1 (Cahen–Wallach [2]). Let \((M, g)\) be a simply-connected lorentzian symmetric space. Then \(M\) is isometric to the product of a simply-connected riemannian symmetric space and one of the following:

- \(\mathbb{R}\) with metric \(-dt^2\);
- the universal cover of \(n\)-dimensional de Sitter or anti de Sitter spaces, where \(n \geq 2\); or
- a Cahen–Wallach space \(\text{CW}_n(A)\) with \(n \geq 3\) and metric given by (2.2) below.

If we drop the hypothesis of simply-connectedness then this theorem holds up to local isometry, which is the version of the theorem of greater relevance in supergravity.

The \(n\)-dimensional Cahen–Wallach spaces \(\text{CW}_n(A)\) are constructed as follows. Let \(V\) be a real vector space of dimension \(n−2\) endowed with a euclidean structure \((-,-)\). Let \(V^*\) denote its dual. Let \(Z\) be a real one-dimensional vector space and
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$Z^*$ its dual. We will identify $Z$ and $Z^*$ with $\mathbb{R}$ via canonical dual bases $\{e_+\}$ and $\{e_-\}$, respectively. Let $A \in S^2 V^*$ be a symmetric bilinear form on $V$. Using the euclidean structure on $V$ we can associate with $A$ an endomorphism of $V$ also denoted $A$:

$$\langle A(v), w \rangle = A(v, w) \quad \text{for all } v, w \in V.$$  

We will also let $\flat : V \to V^*$ and $\sharp : V^* \to V$ denote the musical isomorphisms associated to the euclidean structure on $V$.

Let $k^A$ be the Lie algebra with underlying vector space $V \oplus V^* \oplus Z \oplus Z^*$ and with Lie brackets

$$\begin{align*}
[e_-, v] &= v^b \\
[e_-, \alpha] &= A(\alpha^i) \\
[\alpha, v] &= A(v, \alpha^i) e_+ ,
\end{align*}$$

for all $v \in V$ and $\alpha \in V^*$. All other brackets not following from these are zero. The Jacobi identity is satisfied by virtue of $A$ being symmetric. Notice that since its second derived ideal is central, $g_A$ is (three-step) solvable.

Notice that $k^A = V^*$ is an abelian Lie subalgebra, and its complementary subspace $p^A = V \oplus Z \oplus Z^*$ is acted on by $k^A$. Indeed, it follows easily from (2.1) that

$$[k^A, p^A] \subset p^A \quad \text{and} \quad [p^A, p^A] \subset k^A ,$$

whence $g_A = k^A \oplus p^A$ is a symmetric split. Lastly, let $B \in (S^2 p^*_A)^{k^A}$ denote the invariant symmetric bilinear form on $p^A$ defined by

$$B(v, w) = \langle v, w \rangle \quad \text{and} \quad B(e_+, e_-) = 1 ,$$

for all $v, w \in V$. This defines on $p^A$ a $k^A$-invariant lorentzian scalar product of signature $(1, n - 1)$.

We now have the required ingredients to construct a (lorentzian) symmetric space. Let $G_A$ denote the connected, simply-connected Lie group with Lie algebra $g_A$ and let $K_A$ denote the Lie subgroup corresponding to the subalgebra $k^A$. The lorentzian scalar product $B$ on $p^A$ induces a lorentzian metric $g$ on the space of cosets

$$M_A = G_A / K_A ,$$

turning it into a symmetric space.

Introducing coordinates $x^\pm, x^i$ naturally associated to $e_\pm, e_i$, where $e_i$ is an orthonormal frame for $V$, we can write the Cahen–Wallach metric explicitly as

$$g = 2dx^+ dx^- + \left( \sum_{i,j=1}^{n-2} A_{ij} x^i x^j \right) (dx^-)^2 + \sum_{i=1}^{n-2} (dx^i)^2 .$$

(2.2)

**Proposition 2.2** (Cahen–Wallach [2]). The metric on $M_A$ defined above is indecomposable if and only if $A$ is nondegenerate. Moreover, $M_A$ and $M_{A'}$ are isometric if and only if $A$ and $A'$ are related in the following way:

$$A'(v, w) = c A(Ov, Ow) \quad \text{for all } v, w \in V,$$
for some orthogonal transformation \( O : V \to V \) and a positive scale \( c > 0 \).

From this result one sees that the moduli space \( M_n \) of indecomposable such metrics in \( n \) dimensions is given by

\[
M_n = \left( S^{n-3} - \Delta \right) / S_{n-2}
\]

where

\[
\Delta = \{ (\lambda_1, \ldots, \lambda_{n-2}) \in S^{n-3} \subset \mathbb{R}^{n-2} \mid \lambda_1 \cdots \lambda_{n-2} = 0 \}
\]

is the singular locus consisting of eigenvalues of degenerate \( A \)'s, and \( S_{n-2} \) is the symmetric group in \( n - 2 \) symbols, acting by permutations on \( S^{n-3} \subset \mathbb{R}^{n-2} \).

**Local isometric embeddings.** Indecomposable lorentzian symmetric spaces in \( d \geq 2 \) are locally isometric to algebraic varieties in pseudo-euclidean spaces. This is well-known for both de Sitter and anti de Sitter spaces. Indeed, let \( \kappa > 0 \). Then the quadric in \( E^{n,1} \) consisting of points \((x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \) such that

\[
-x_0^2 + x_1^2 + x_2^2 + \cdots + x_n^2 = 1/\kappa^2
\]

has constant sectional curvature \( \kappa \) and hence is locally isometric to a de Sitter space, whereas the quadric in \( E^{n-1,2} \) consisting of points \((x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \) such that

\[
-x_0^2 + x_1^2 + x_2^2 + \cdots + x_{n-1}^2 - x_n^2 = -1/\kappa^2
\]

has constant section curvature \(-\kappa \) and hence is locally isometric to an anti de Sitter space.

Similarly, the \( n \)-dimensional Cahen–Wallach spaces are locally isometric to the intersection of two quadrics in \( E^{n,2} \). Indeed, let \((u^1, v^1, u^2, v^2, x^i)\), for \( i = 1, \ldots, n-2 \), be flat coordinates in \( E^{n,2} \) relative to which the metric takes the form

\[
2du^1dv^1 + 2du^2dv^2 + \sum_{i=1}^{9} dx^idx^i \ . \quad (2.3)
\]

Then the Cahen–Wallach space with matrix \( A \) is locally isometric to the induced metric on the intersection of the two quadrics

\[
(u^1)^2 + (u^2)^2 = 1 \quad \text{and} \quad 2u^1v^1 + 2u^2v^2 = \sum_{i,j=1}^{9} A_{ij}x^ix^j \ . \quad (2.4)
\]

This was proven in [3].

**2.2. Lorentzian parallelisable manifolds.** A subclass of the lorentzian symmetric spaces are the parallelisable manifolds.

Recall that a differentiable manifold \( M \) is said to admit an absolute parallelism if it admits a smooth trivialisation of the frame bundle. Such a trivialisation consists of a smooth global frame and hence also trivialises the tangent bundle;
whence manifolds admitting absolute parallelisms are parallelisable in the topological sense. The reduction theorem for connections on principal bundles (see, for example, [4, Section II.7]) allows us to think of absolute parallelisms in terms of holonomy groups of connections. Indeed, an absolute parallelism is equivalent to a smooth connection on the frame bundle with trivial holonomy. This implies, in particular, that the connection is flat and if the manifold is simply-connected then flatness is also sufficient.

So far these notions are purely (differential) topological and make no mention of metrics or any other structure on the manifold. The question arises whether there is a metric on $M$ which is consistent with a given absolute parallelism, so that parallel transport is an isometry; or turning the question around, whether a given pseudo-riemannian manifold $(M, g)$ admits an absolute parallelism consistent with it. In terms of connections, a consistent absolute parallelism is equivalent to a metric connection with torsion with trivial holonomy; or, locally, to a flat metric connection with torsion.

Cartan and Schouten [5, 6] essentially solved the riemannian case by generalising Clifford’s parallelism on the 3-sphere in two different ways. The three-sphere can be understood both as the unit-norm quaternions and also as the Lie group $SU(2) = Sp(1)$. The latter characterisation generalises to other (semi)simple Lie groups, whereas the former gives rise to the parallelism of the 7-sphere thought of as the unit-norm octonions. It follows from the results of Cartan and Schouten that a simply-connected irreducible riemannian manifold admitting a consistent absolute parallelism (equivalently a flat metric connection) is isometric to one of the following: the real line, a simple Lie group with the bi-invariant metric induced from a multiple of the Killing form, or the round 7-sphere.

Their proofs might have had gaps which were addressed by Wolf [7, 8], who also generalised these results to arbitrary signature, subject to an algebraic curvature condition saying that the pseudo-riemannian manifold $(M, g)$ is of “reductive type,” a condition which is automatically satisfied in the riemannian case. (See Wolf’s paper for the precise condition.) In the case of lorentzian signature, Cahen and Parker [9] showed that one can relax the “reductive type” condition; completing the classification of absolute parallelisms consistent with a lorentzian metric.

Wolf also showed that if one also assumes that the torsion is parallel, then, in any signature, $(M, g)$ is locally isometric to a Lie group with a bi-invariant metric. In fact, as we will show below in Section 2.3, one obtains the same result starting with the weaker hypothesis that the torsion three-form is closed, which will be the case needed in supergravity.

The results of Cahen and Parker [9] actually show that in lorentzian signature one gets for free that the torsion is parallel. Therefore it follows that an indecomposable lorentzian manifold $(M, g)$ admits a consistent absolute parallelism if and only if it is locally isometric to a lorentzian Lie group with bi-invariant metric.

2.3. Flat metric connections with closed torsion. We will now show that a pseudo-riemannian manifold $(M, g)$ with a flat metric connection with closed torsion three-form is locally isometric to a Lie group admitting a bi-invariant
metric.

Let \((M, g)\) be a pseudo-riemannian manifold and let \(D\) be a metric connection with torsion \(T\). In other words, \(Dg = 0\) and for all vector fields \(X, Y\) on \(M\), \(T : \wedge^2 TM \to TM\) is defined by

\[
T(X, Y) = D_X Y - D_Y X - [X, Y].
\]

In terms of the torsion-free Levi-Civita connection \(\nabla\), we have

\[
D_X Y = \nabla_X Y + \frac{1}{2}T(X, Y).
\]

Since both \(Dg = 0\) and \(\nabla g = 0\), \(T\) is skew-symmetric:

\[
g(T(X, Y), Z) = -g(T(X, Z), Y),
\]

for all vector fields \(X, Y, Z\) and gives rise to a torsion three-form \(H \in \Omega^3(M)\), defined by

\[
H(X, Y, Z) = g(T(X, Y), Z).
\]

We will assume that \(H\) is closed and in this section we will characterise those manifolds for which \(D\) is flat.

Let \(R^D\) denote the curvature tensor of \(D\), defined by

\[
R^D(X, Y)Z = D_{[X,Y]}Z - D_X D_Y Z + D_Y D_X Z.
\]

Our strategy will be to consider the equation \(R^D = 0\), decompose it into types and solve the corresponding equations. We will find that \(T\) is parallel with respect to both \(\nabla\) and \(D\), and this will imply that \((M, g)\) is locally a Lie group with a bi-invariant metric and \(D\) the parallelising connection of Cartan and Schouten [5].

The curvature \(R^D\) is given by

\[
R^D(X, Y)Z = R(X, Y)Z - \frac{1}{2}(\nabla_X T)(Y, Z) + \frac{1}{2}(\nabla_Y T)(X, Z) - \frac{1}{4}T(X, T(Y, Z)) + \frac{1}{4}T(Y, T(X, Z)),
\]

where \(R = R^\nabla\) is the curvature of the Levi-Civita connection. The tensor

\[
R^D(X, Y, Z, W) := g(R^D(X, Y)Z, W)
\]

takes the following form

\[
R^D(X, Y, Z, W) = R(X, Y, Z, W) - \frac{1}{2}g((\nabla_X T)(Y, Z), W) + \frac{1}{2}g((\nabla_Y T)(X, Z), W) - \frac{1}{4}g(T(X, T(Y, Z)), W) + \frac{1}{4}g(T(Y, T(X, Z)), W),
\]

where we have defined the Riemann tensor as usual:

\[
R(X, Y, Z, W) := g(R(X, Y)Z, W).
\]
Using equation (2.5) we can rewrite $R^D$ as

$$R^D(X,Y,Z,W) = R(X,Y,Z,W) - \frac{1}{2} g((\nabla_X T)(Y,Z), W) + \frac{1}{2} g((\nabla_Y T)(X,Z), W) + \frac{1}{4} g(T(X,W), T(Y,Z)) - \frac{1}{4} g(T(Y,W), T(X,Z)),$$

which is manifestly skew-symmetric in $X,Y$ and in $Z,W$. Observe that unlike $R$, the torsion terms in $R^D$ do not satisfy the first Bianchi identity. Therefore breaking $R^D$ into algebraic types will give rise to more equations and will eventually allow us to characterise the data $(M,g,T)$ for which $R^D = 0$.

Indeed, let $R^D = 0$ and consider the identity

$$\sum_{X,Y,Z} R^D(X,Y,Z,W) = 0,$$

where $\sum$ denotes signed permutations. Since $R$ does obey the Bianchi identity

$$\sum_{X,Y,Z} R(X,Y,Z,W) = 0,$$

we obtain the following identity

$$\sum_{X,Y,Z} g((\nabla_X T)(Y,Z), W) = -\frac{1}{4} \sum_{X,Y,Z} g(T(W,X), T(Y,Z)). \quad (2.6)$$

Now we use the fact that the torsion three-form $H$ is closed, which can be written as

$$g((\nabla_X T)(Y,Z), W) - g((\nabla_Y T)(X,Z), W) + g((\nabla_Z T)(X,Y), W) - g((\nabla_W T)(X,Y), Z) = 0,$$

or equivalently,

$$g((\nabla_W T)(X,Y), Z) = \frac{1}{4} \sum_{X,Y,Z} g((\nabla_X T)(Y,Z), W).$$

This turns equation (2.6) into

$$g((\nabla_W T)(X,Y), Z) = -\frac{1}{4} \sum_{X,Y,Z} g(T(W,X), T(Y,Z)). \quad (2.7)$$

From this equation it follows that

$$g((\nabla_W T)(X,Y), Z) = -g((\nabla_X T)(W,Y), Z),$$

so that $g((\nabla_W T)(X,Y), Z)$ is totally skew-symmetric. This means that $\nabla H = dH = 0$, whence $H$ and hence $T$ are parallel. Therefore equation (2.6) simplifies to

$$\sum_{X,Y,Z} g(T(W,X), T(Y,Z)) = 0. \quad (2.8)$$
Equation (2.9) is the Jacobi identity for $T$. Indeed, notice that
\[
g(T(W, X), T(Y, Z)) = H(W, X, T(Y, Z)) \\
= H(X, T(Y, Z), W) = g(T(X, T(Y, Z)), W) ,
\]
whence equation (2.9) is satisfied if and only if
\[
S_{XYZ} T(X, T(Y, Z)) = 0 .
\] (2.9)

This means that the tangent space $T_p M$ of $M$ at every point $p$ becomes a Lie algebra where the Lie bracket is given by the restriction of $T$ to $T_p M$. More is true and the restriction to $T_p M$ of the metric $g$ gives rise to an (ad-)invariant scalar product:
\[
g(T(X, Y), Z) = g(X, T(Y, Z)) .
\]

By a theorem of Wolf [7, 8] (based on the earlier work of Cartan and Schouten [5, 6]) if $(M, g)$ is complete then it is a discrete quotient of a Lie group with a bi-invariant metric. In general, we can say that $(M, g)$ is locally isometric to a Lie group with a bi-invariant metric.

Indeed, since $D$ is flat, there exists locally a parallel frame $\{\xi_i\}$ for $TM$. Since $\xi_i$ is parallel, from the definition of the torsion,
\[
T(\xi_i, \xi_j) = -[\xi_i, \xi_j] .
\]
Moreover, since $T$ is parallel relative to $D$, we see that $[\xi_i, \xi_j]$ is also parallel with respect to $D$, whence it can be written as a linear combination of the $\xi_i$ with constant coefficients. In other words, they span a real Lie algebra $g$. The homomorphism $g \to C^\infty(M, TM)$ whose image is the subalgebra spanned by the $\{\xi_i\}$ integrates, once we choose a point in $M$, to a local diffeomorphism $G \to M$. This is also an isometry if we use on $G$ the metric induced from the one on the Lie algebra, whence we conclude that $(M, g)$ is locally isometric to a Lie group with a bi-invariant metric.

2.4. Bi-invariant lorentzian metrics on Lie groups. In this section we will briefly review the structure of Lie groups admitting a bi-invariant lorentzian metric. It is well-known that bi-invariant metrics on a Lie group are in bijective correspondence with (ad-)invariant scalar products on the Lie algebra. Therefore it is enough to study those Lie algebras possessing an invariant lorentzian scalar product. We shall call them Lorentzian Lie algebras in this review.

It is well-known that reductive Lie algebras admit invariant scalar products: Cartan’s criterion allows us to use the Killing forms on the simple factors and any scalar product on an abelian Lie algebra is trivially invariant. Another well-known example of Lie algebras admitting an invariant scalar product are the classical doubles. Let $\mathfrak{h}$ be any Lie algebra and let $\mathfrak{h}^*$ denote the dual space on which $\mathfrak{h}$ acts via the coadjoint representation. The definition of the coadjoint representation is such that the dual pairing $\mathfrak{h} \otimes \mathfrak{h}^* \to \mathbb{R}$ is an invariant scalar product on the semidirect product $\mathfrak{h} \ltimes \mathfrak{h}^*$ with $\mathfrak{h}^*$ an abelian ideal. The Lie algebra $\mathfrak{h} \ltimes \mathfrak{h}^*$ is called
the classical double of \( h \) and the invariant metric has split signature \((r, r)\) where \( \dim h = r \).

It turns out that all Lie algebras admitting an invariant scalar product can be obtained by a mixture of these constructions. Let \( g \) be a Lie algebra with an invariant scalar product \( \langle -, - \rangle_g \). Now let \( h \) act on \( g \) as skew-symmetric derivations; that is, preserving both the Lie bracket and the scalar product. First of all, since \( h \) acts on \( g \) preserving the scalar product, we have a linear map

\[
h \to so(g) \cong \Lambda^2 g,
\]

with dual map

\[
c : \Lambda^2 g \to h^*,
\]

where we have used the invariant scalar product to identify \( g \) and \( g^* \) equivariantly. Since \( h \) preserves the Lie bracket in \( g \), this map is a cocycle, whence it defines a class \([c] \in H^2(g; h^*)\) in the second Lie algebra cohomology of \( g \) with coefficients in the trivial module \( h^* \). Let \( g \times_c h^* \) denote the corresponding central extension. The Lie bracket of \( g \times_c h^* \) is such that \( h^* \) is central and if \( X, Y \in g \), then

\[
[X, Y] = [X, Y]_g + c(X, Y),
\]

where \([-, -]_g\) is the original Lie bracket on \( g \). Now \( h \) acts naturally on \( g \times_c h^* \) preserving the Lie bracket; the action on \( h^* \) being given by the coadjoint representation. This then allows us to define the double extension of \( g \) by \( h \),

\[
\mathfrak{d}(g, h) = h \ltimes (g \times_c h^*)
\]

as a semidirect product. Details of this construction can be found in [10, 11]. The remarkable fact is that \( \mathfrak{d}(g, h) \) admits an invariant scalar product:

\[
\langle (X, h, \alpha), (Y, k, \beta) \rangle = \langle X, Y \rangle_g + \alpha(k) + \beta(h) + B(h, k),
\]

for all \( X, Y \in g, h, k \in h, \alpha, \beta \in h^* \) and where \( B \) is any invariant symmetric bilinear form on \( h \).

We say that a Lie algebra with an invariant scalar product is indecomposable if it cannot be written as the direct product of two orthogonal ideals. A theorem of Medina and Revoy [10] (see also [12] for a refinement) says that an indecomposable (finite-dimensional) Lie algebra with an invariant scalar product is either one-dimensional, simple, or a double extension \( \mathfrak{d}(g, h) \) where \( h \) is either simple or one-dimensional and \( g \) is a (possibly trivial) Lie algebra with an invariant scalar product. Any (finite-dimensional) Lie algebra with an invariant scalar product is then a direct sum of indecomposables.

If the scalar product on \( g \) has signature \((s, t)\), then the scalar product on the double extension \( \mathfrak{d}(g, h) \) has signature \((s + r, t + r)\), where \( r = \dim h \). This means that if we are interested in lorentzian signature, we can double extend at most once and by a one-dimensional \( h \).

Therefore indecomposable lorentzian Lie algebras are either reductive or double extensions \( \mathfrak{d}(g, h) \) where \( g \) has a positive-definite invariant scalar product and \( h \) is
one-dimensional. In the reductive case, indecomposability means that it has to be simple, whereas in the latter case, since the scalar product on $g$ is positive-definite, $g$ must be reductive. A result of [11] (see also [12]) then says that any semisimple factor in $g$ splits off resulting in a decomposable Lie algebra. Thus if the double extension is to be indecomposable, $g$ must be abelian. In summary, an indecomposable lorentzian Lie algebra is either simple or a double extension of an abelian Lie algebra by a one-dimensional Lie algebra and hence solvable (see, e.g., [10]).

In summary, an indecomposable lorentzian Lie algebra is either isomorphic to $\mathfrak{so}(1, 2)$ with (a multiple of) the Killing form, or else is solvable and can be described as a double extension $\mathfrak{d}_{2n+2} := \mathfrak{d}(E^{2n}, \mathbb{R})$ of the abelian Lie algebra $E^{2n}$ with the (trivially invariant) euclidean “dot” product by a one-dimensional Lie algebra acting on $E^{2n}$ via a non-degenerate skew-symmetric linear map $J : E^{2n} \to E^{2n}$. Let $\omega \in \Lambda^2(E^{2n})^*$ denote the associated 2-form: $\omega(v, w) = \langle v, Jw \rangle$.

More concretely, the double extension $\mathfrak{d}_{2n+2}$ has underlying vector space $V = E^{2(n-1)} \oplus \mathbb{R} \oplus \mathbb{R}$, and if $(v, v^-, v^+), (w, w^-, w^+) \in V$, then their Lie bracket is given by

$$[(v, v^-, v^+), (w, w^-, w^+)] = (v^- J(w) - w^- J(v), 0, v \cdot J(w))$$

and their inner product follows by polarisation from

$$|(v, v^-, v^+)|^2 = v \cdot v + 2v^+v^- + b(v^-)^2,$$

where $b \in \mathbb{R}$ is arbitrary. One can however always set $b = 0$ via a Lie algebra automorphism and we will do so here; although there are situations when one may wish to retain this freedom.

The unique simply-connected Lie group with Lie algebra $\mathfrak{d}_{2n+2}$ is a solvable $(2n + 2)$-dimensional Lie group admitting a bi-invariant metric

$$ds^2 = 2dx^+dx^- - \langle Jx, Jx \rangle (dx^-)^2 + \langle dx, dx \rangle,$$

relative to natural coordinates $(x, x^-, x^+)$. The parallelising torsion has 3-form

$$H = dx^- \wedge \omega.$$  

(2.12)

The non-degenerate skew-symmetric endomorphism $J$ can be brought to a Jordan normal form consisting of nonzero $2 \times 2$ blocks via an orthogonal transformation. The skew-eigenvalues $\lambda_1, \ldots, \lambda_n$, which are different from zero, can be arranged so that they obey: $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Finally a positive rescaling of $J$ can be absorbed into reciprocal rescalings of $x^\pm$, so that we can set $\lambda_n$, say, equal to 1 without loss of generality. Therefore we see that the moduli space of metrics (2.11) is given by an $(n - 1)$-tuple $\lambda = (\lambda_1, \ldots, \lambda_{n-1})$ where $0 < \lambda_1 \leq \cdots \leq \lambda_{n-1} \leq 1$. It is clear that they are particular cases of the Cahen–Wallach spaces discussed in Section 2.1.
3. Supergravity

Supergravity is one of the later jewels of 20th century theoretical physics. It started out as an attempt to ‘gauge’ the supersymmetry of certain quantum field theories, but it was quickly realised that it provides a nontrivial extension of Einstein gravity. Supergravity theories are fairly rigid—their structure dictated largely by the representation theory of the spin groups. A good modern review of the structure of supergravity theories is [13]. It is fair to say, however, that supergravity theories are still somewhat mysterious to most mathematicians and much remains to be done to make this beautiful chapter of modern mathematical physics accessible to a larger mathematical audience. That, however, is a task for a different occasion. For our present purposes, each supergravity theory will be a collection of geometric PDEs and our interest will be in finding special types of solutions. We shall be interested uniquely in lorentzian supergravity theories in dimension \(d \geq 4\). There are supergravity theories in lower dimensions and in other metric signatures, but we will not discuss them here. Neither will we discuss other types of supergravity theories: heterotic, gauged, conformal, massive,... The two-volume set [14] reprints many of the foundational supergravity papers.

For reasons which are well-known, namely the otherwise non-existence of non-trivial interacting theories, the dimension of the spacetime will be bounded above by 11. Apart from the dimension of the spacetime, the other important invariant is the “number of supercharges”, denoted \(n\), which is an integer multiple of the dimension of the smallest irreducible real spinor representation in that spacetime dimension. For dimension \(\geq 4\) the number of supercharges ranges from 4 to 32.

In Table 3, which is borrowed from [13], we tabulate the different supergravity theories in \(d \geq 4\). The seemingly baroque notation is not too important: \(M\) refers to the unique eleven-dimensional supergravity theory [15, 16] which is a low-energy limit of M-theory (hence the name), types I [17, 18], IIA [19, 20, 21] and IIB [22, 23, 24] supergravities are the low-energy limits of the similarly named string theories, whereas the notation \(N = n\) or \((p,q)\) is historical and denotes the multiplicity of the spinor (or half-spinor) representations in the corresponding supersymmetry algebra. The original supergravity theory [25, 26] is the four-dimensional \(N = 1\) theory. The top entry in each column has been highlighted to indicate that upon dimensional reduction it gives rise to all the theories below it in the same column. As we will explain below, this means that for many purposes, especially the classification of solutions, it is generally enough to understand the ‘top’ theories and, indeed, we will concentrate on those.

Indeed, supergravity theories in different dimensions may be related by a procedure known as Kaluža–Klein reduction. This can be read off from Table 3: any supergravity theory in the table can be obtained by Kaluža–Klein reduction from any theory sitting above it in the same column. In practice this means that a solution to any of the supergravity theories in the table can be lifted to a solution of any theory above it in the same column, should there be any. Conversely, any solution of a supergravity theory which is invariant under a one-dimensional Lie group gives rise to a local solution (and indeed global if the action is free and proper) of the supergravity theory immediately below it in the same column. We
shall be particularly interested in the reduction from $d = 11$ supergravity to $d = 10$ type IIA supergravity, and in the reduction and subsequent truncation from the $d = 6$ $(1,0)$ supergravity to minimal $d = 5$ $N = 2$ supergravity.

We shall be interested in solutions of the field equations coming from these supergravity theories. Such a solution is described in geometric terms by the following data:

- a $d$-dimensional lorentzian spin manifold $(M,g)$ with a (possibly twisted) real rank $n$ spinor bundle $S \rightarrow M$, and
- certain additional geometric data, which will be different in each supergravity theory, consisting of differential forms or, more generally, sections of certain fibre bundles over $M$,

all subject to field equations which generalise the coupled Einstein–Maxwell equations familiar from four-dimensional Physics.

The above geometric data defines a connection $D$ on the spinor bundle $S$ as well as a (possibly empty) set of endomorphisms of $S$. Together they define a class of sections of $S$, parallel with respect to $D$ and in the kernel of the endomorphisms, which are called Killing spinors.

We will be particularly interested in the cases where the connection $D$ is flat, so that it admits the maximum number of parallel sections. In this case, the field equations are automatically satisfied. In general, the field equations are intimately related to integrability conditions for the existence of parallel sections of $D$.

| $d \downarrow n \rightarrow$ | 32 | 24 | 20 |
|-----------------|----|----|----|
| 11 M            |    |    |    |
| 10 IIA          | IIB|    |    |
| 9 N=2           |    |    |    |
| 8 N=2           |    |    |    |
| 7 N=4           |    |    |    |
| 6 (2,2)         | (3,1) | (4,0) | (2,1) | (3,0) |
| 5 N=8           |    | N=6|    |
| 4 N=8           |    | N=6| N=5|

Table 1. Lorentzian supergravity theories in $d \geq 4$

| $d \downarrow n \rightarrow$ | 16 | 12 | 8 | 4 |
|-----------------|----|----|---|---|
| 10 I            |    |    |   |   |
| 9 N = 1         |    |    |   |   |
| 8 N = 1         |    |    |   |   |
| 7 N = 2         |    |    |   |   |
| 6 (1,1)         | (2,0) | (1,0) | N = 3 | N = 2 | N = 1 |
| 5 N=4           |    | N=2|    |    |
| 4 N=4           | N = 3 | N = 2 | N = 1 |
3.1. Eleven-dimensional supergravity. Eleven-dimensional supergravity was predicted by Nahm [15] and constructed soon thereafter by Cremmer, Julia and Scherk [16]. We will only be concerned with the bosonic equations of motion. The geometrical data consists of \((M, g, F)\) where \((M, g)\) is an eleven-dimensional lorentzian manifold with a spin structure and \(F \in \Omega^4(M)\) is a closed 4-form. The equations of motion generalise the Einstein–Maxwell equations in four dimensions. The Einstein equation relates the Ricci curvature to the energy momentum tensor of \(F\). More precisely, the equation is

\[
\text{Ric}(g) = T(g, F) \tag{3.1}
\]

where the symmetric tensor

\[
T(X, Y) = \frac{1}{2} \langle i_X F, i_Y F \rangle - \frac{1}{6} g(X, Y)|F|^2,
\]

is related to the energy-momentum tensor of the (generalised) Maxwell field \(F\). In the above formula, \(\langle -, - \rangle\) denotes the scalar product on forms, which depends on \(g\), and \(|F|^2 = \langle F, F \rangle\) is the associated (indefinite) norm. The generalised Maxwell equations are now nonlinear:

\[
d \star F = -\frac{1}{2} F \wedge F. \tag{3.2}
\]

**Definition 3.1.** A triple \((M, g, F)\) satisfying the equations (3.1) and (3.2) is called a (bosonic) background of eleven-dimensional supergravity.

Let \(S\) denote the bundle of spinors on \(M\). It is a real vector bundle of rank 32 with a spin-invariant symplectic form \((-, -)\). A differential form on \(M\) gives rise to an endomorphism of the spinor bundle via the composition

\[
c : \Lambda T^* M \xrightarrow{\cong} \text{Cl}(T^* M) \rightarrow \text{End} S,
\]

where the first map is the bundle isomorphism induced by the vector space isomorphism between the exterior and Clifford algebras, and the second map is induced from the action of the Clifford algebra \(\text{Cl}(1, 10)\) on the spinor representation \(S\) of \(\text{Spin}(1, 10)\). In signature \((1, 10)\) one has the algebra isomorphism

\[
\text{Cl}(1, 10) \cong \text{Mat}(32, \mathbb{R}) \oplus \text{Mat}(32, \mathbb{R}),
\]

hence the map \(\text{Cl}(1, 10) \rightarrow \text{End} S\) has kernel. In other words, the map \(c\) defined above involves a choice. This comes down to choosing whether the (normalised) volume element in \(\text{Cl}(1, 10)\) acts as \(\pm\) the identity. In our conventions, the volume element acts as minus the identity.

**Definition 3.2.** We say that a background \((M, g, F)\) is supersymmetric if there exists a nonzero spinor \(\varepsilon \in \Gamma(S)\) which is parallel with respect to the supersymmetric connection

\[
D : \Gamma(S) \rightarrow \Gamma(T^* M \otimes S)
\]
Lorentzian symmetric spaces in supergravity

defined, for all vector fields $X$, by

$$D_X \epsilon = \nabla_X \epsilon + \Omega_X(F)\epsilon,$$

(3.3)

where $\nabla$ is the spin connection and $\Omega(F) : TM \to \text{End} S$ is defined by

$$\Omega_X(F) = \frac{1}{12} c(X^\flat \wedge F) - \frac{1}{6} c(\iota_X F),$$

with $X^\flat$ the one-form dual to $X$.

A nonzero spinor $\epsilon$ which is parallel with respect to $D$ is called a Killing spinor. This is a generalisation of the usual geometrical notion of Killing spinor (see, for example, [27]). The name is apt because Killing spinors are “square roots” of Killing vectors. Indeed, one has the following

**Proposition 3.3.** Let $\epsilon_i$, $i = 1, 2$ be Killing spinors: $D\epsilon_i = 0$. Then the vector field $V$ defined, for all vector fields $X$, by

$$g(V, X) = (\epsilon_1, X \cdot \epsilon_2)$$

is a Killing vector and moreover [28] preserves $F$.

The fundamental object in eleven-dimensional supergravity is the connection $D$, whose curvature encodes the field equations. Indeed, the field equations are equivalent [28, 29] to

$$e^i \cdot R^D_X, e_i = 0$$

for every vector field $X$,

where $(e_i)$ is an orthonormal frame and $(e^i)$ is dual coframe and $\cdot$ is Clifford multiplication.

Alas, $D$ is not induced from a connection on the tangent bundle and in fact, it does not even preserve the symplectic structure. Nevertheless one has the following

**Proposition 3.4.** [30] The holonomy of $D$ is contained in $\text{SL}(32, \mathbb{R})$.

An important open problem is to determine the possible holonomy groups of $D$ subject to the field equations. In a way, the field equations play the rôle of the torsion-free condition in the holonomy problem for affine connections. Except for the above result there are no other results of a general nature and although the infinitesimal holonomy of a number of solutions are known [31, 32], a general pattern has yet to emerge.

A coarser invariant than the holonomy of $D$ is the dimension of its kernel; that is, the dimension of the space of Killing spinors. It is customary to write this as a fraction

$$\nu = \frac{\dim \{ \text{Killing spinors} \}}{\text{rank } S}$$

which in this case is of the form $k/32$ for some integer $k = 0, 1, \ldots, 32$.

In Section 4.2 we will review the classification of those backgrounds with $\nu = 1$; that is, those backgrounds where $D$ is flat. We will see that they are all given by
lorentzian symmetric spaces. In fact, it was shown in [33] that if \( \nu > \frac{3}{4} \), then \( M \) is locally homogeneous and moreover it was conjectured that there exist backgrounds with \( \nu = \frac{3}{4} \) which are not locally homogeneous; although at present none have been constructed. At the other end of the spectrum, the general form of \((g, F)\) which admit (at least) one Killing spinor is known [28, 34].

3.2. Ten-dimensional IIB supergravity. Ten-dimensional IIB supergravity [22, 23, 24] is somewhat more complicated than eleven-dimensional supergravity due to the proliferation of dynamical fields and the fact that it cannot be obtained by Kaluža–Klein reduction from any higher-dimensional supergravity theory.

A type IIB supergravity background is described by the geometric data we describe presently. First of all, we have a ten-dimensional lorentzian spin manifold \((M, g)\) together with a self-dual 5-form \(F\). Now let \(\mathcal{H}\) be the complex upper half-plane, thought of as the riemannian symmetric space \(SU(1,1)/U(1)\) and let \(\tau : M \rightarrow \mathcal{H}\) be a smooth map. We may think of \(SU(1,1)\) as the total space of a principle circle bundle over \(\mathcal{H}\) and we let \(\mathcal{L} \rightarrow \mathcal{H}\) denote the associated complex line bundle. Let \(L_\tau = \tau^* \mathcal{L}\) denote the pull-back bundle over \(M\). Choosing a section \(\sigma : \mathcal{H} \rightarrow SU(1,1)\), we may pull back to \(M\) the left-invariant Maurer–Cartan form on \(SU(1,1)\): its component along \(u(1)\) defines a connection \(A\) on \(L_\tau\), whereas the component perpendicular to \(u(1)\), relative to the invariant lorentzian scalar product on \(su(1,1)\), defines a one-form \(B\) on \(M\) with values in \(L^2_{\tau}\). Both \(A\) and \(B\) can be written explicitly in terms of \(\tau\). Indeed, if we let \(z = (\tau - i)/(\tau + i)\) be the Cayley transform of \(\tau\), so that \(|z| < 1\), then there is a choice of section \(\sigma\) for which

\[
A = \frac{\text{Im}(zd\overline{z})}{1 - |z|^2} \quad \text{and} \quad B = \frac{dz}{1 - |z|^2}.
\]

Finally, let \(G\) be a 3-form on \(M\) with values in \(L^2_\tau\). On the bundles \(\Lambda^p T^* M \otimes L^2_\tau\) we have connections \(\nabla^{p,q}\) obtained from the Levi-Civitá connection on \(TM\) (and hence the tensor bundles) and the connection \(A\) on \(L_\tau\) (and hence its powers). We will let

\[
d^{\nabla^{p,q}} : \Omega^p(M; L^2_\tau) \rightarrow \Omega^{p+1}(M; L^2_\tau) \quad \text{and} \quad \delta^{\nabla^{p,q}} : \Omega^p(M; L^2_\tau) \rightarrow \Omega^{p-1}(M; L^2_\tau)
\]

denote the associated differential and co-differential on \(L^2_\tau\)-valued differential forms. We will let \((\cdot, \cdot)\) denote the natural pairing

\[
\Omega^p(M; L^2_\tau) \otimes \Omega^q(M; L^2_\tau) \rightarrow \Omega^{p+q}(M; L^2_\tau)
\]

induced from the metric \(g\). With these notational remarks behind us, we can finally define a IIB supergravity background.

**Definition 3.5.** The data \((M, g, \tau, F, G)\) described above defines a IIB supergrav-
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In order to provide that the following equations are satisfied:

\[ \delta \nabla^{1/2} B = \frac{1}{4}|G|^2 \]
\[ \delta \nabla^{3/2} G(X, Y) = \langle B, \iota_X \iota_Y G \rangle - \frac{2i}{3} \langle \iota_X \iota_Y F, G \rangle \]
\[ d\nabla^{1/2} G = -B \wedge G \]
\[ d\nabla^{3/2} F = \frac{i}{8} G \wedge G \]
\[ \text{Ric}(X, Y) = B(X) \overline{B}(Y) + B(Y) \overline{B}(X) + 4 \langle \iota_X F, \iota_Y F \rangle \]
\[ + \frac{1}{4} (\langle \iota_X G, \iota_Y G \rangle + \langle \iota_Y G, \iota_X G \rangle) - \frac{1}{8} \langle G, G \rangle g(X, Y). \]

Let \( S_\pm \) denote the half-spinor bundles over \( M \). They are real, symplectic and have rank 16. Let \( S := S_- \otimes L^{1/2}_1, \) where \( L^{1/2}_1 \) is the square-root bundle of \( L_\tau \). Let \( \overline{S} := S_- \otimes L^{-1/2}_\tau. \) Notice that if \( \varepsilon \in \Omega^0(M; S), \) then \( \overline{\varepsilon} \in \Omega^0(\overline{S}). \) Furthermore the Clifford action of differential forms on spinors extends to an action

\[ c : \Omega^p(M; L^\xi_\tau) \rightarrow \text{Hom}(\Omega^p(M; S_\pm \otimes L^r_\tau), \Omega^0(M; S_{(-1)p} \otimes L^{r+q}_\tau)) \].

Similarly we have a connection \( \nabla^s \) acting on \( \Omega^0(M; S_\pm \otimes L^s_\tau) \) which is defined using the spin connection and the connection \( A \) on \( L_\tau. \) We are now in a position to define a type IIB supergravity Killing spinor.

**Definition 3.6.** A IIB supergravity Killing spinor is a nonzero section \( \varepsilon \) of \( S \) satisfying the following two conditions

\[ c(B)\overline{\varepsilon} = \frac{1}{4} c(G)\varepsilon \]
\[ \nabla^{1/2} \varepsilon = -\frac{1}{4} c(F)c(X^\flat)\varepsilon - \frac{1}{16} \left( c(\iota_X G) - 2c(X^\flat \wedge G) \right) \overline{\varepsilon}. \]

Just like in eleven-dimensional supergravity, IIB Killing spinors are square roots of Killing vectors. Indeed, the image of the natural map

\[ \Omega^0(M; S) \otimes \Omega^0(M; \overline{S}) \rightarrow \text{vector fields} \]

consists of Killing vectors which in addition preserve the geometric data of a background. Again it is possible to show that if the space of Killing spinors of a supersymmetric background of type IIB supergravity has (real) dimension > 24, then the background is locally homogeneous [35].

Type IIB supergravity backgrounds are acted upon by SU(1, 1), which is the duality group of type IIB supergravity. The metric \( g \) and the five-form \( F \) are SU(1, 1)-invariant, whereas SU(1, 1) acts on \( z \) (hence on \( \tau \)) via fractional linear transformations:

\[ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \cdot z = \frac{az + b}{bz + a}. \]

Moreover, the bundle \( L \rightarrow \mathcal{H} \) is a homogeneous bundle of SU(1, 1) hence there is an action of SU(1, 1) on sections of \( L \) and its powers. Putting these two actions together we see that \( \gamma \in \text{SU}(1, 1) \) sends sections of \( L^p_\tau \) (and also differential forms and spinors with values in such a bundle) to sections of \( L^p_\tau. \)
The classification of the maximally supersymmetric background will be presented in Section 4.4, which is based on the papers [36] and [37]. In the opposite extreme, there has been steady progress recently on the determination of the general form of the backgrounds admitting some supersymmetry [38, 39, 40].

3.3. Six-dimensional (2, 0) and (1, 0) supergravities. We start by describing the field content and Killing spinor equations of (1, 0) [41] and (2, 0) [42, 43] chiral supergravities in six dimensions. We start as usual by describing the relevant spinorial representations. The spin group \( \text{Spin}(1, 5) \cong \text{SL}(2, \mathbb{H}) \), whence the irreducible spinorial representations are quaternionic of complex dimension 4. There are two inequivalent representations \( S_\pm \) which are distinguished by their chirality. Let \( S_1 \) denote the fundamental representation of \( \text{Sp}(1) \): it is a quaternionic representation of complex dimension 2, and similarly let \( S_2 \) denote the fundamental representation of \( \text{Sp}(2) \), which is a quaternionic representation of complex dimension 4. The tensor products \( S_+ \otimes S_1 \) and \( S_+ \otimes S_2 \) are complex representations of \( \text{Spin}(1, 5) \times \text{Sp}(1) \) and \( \text{Spin}(1, 5) \times \text{Sp}(2) \), respectively, with a real structure. We will let \( S = [S_+ \otimes S_1] \) and \( \mathcal{S} = [S_+ \otimes S_2] \) denote the underlying real representations. Clearly \( S \) is a real representation of dimension 8 and \( \mathcal{S} \) is a real representation of dimension 16. If \( (M, g) \) is a six-dimensional lorentzian spin manifold, then we will let \( S \) and \( \mathcal{S} \) denote the bundles of spinors associated with the representations \( S \) and \( \mathcal{S} \), respectively. The groups \( \text{Sp}(1) \) and \( \text{Sp}(2) \) are the R-symmetry groups of these supergravity theories.

**Definition 3.7.** A (1, 0) supergravity background consists of a six-dimensional lorentzian spin manifold \( (M, g) \) together with a closed antiself-dual 3-form \( H \) subject to the Einstein equation

\[
\text{Ric}(X, Y) = -\frac{1}{4} \langle \mathbf{i}_X H, \mathbf{i}_Y H \rangle.
\]

Such a background is said to be supersymmetric if there are nonzero sections \( \varepsilon \) of \( S \) obeying

\[
D_X \varepsilon := \nabla_X \varepsilon - \frac{1}{4} c(\mathbf{i}_X H) \varepsilon = 0,
\]

for all vector fields \( X \), where \( c : \Omega(M) \to \text{Cl}(T^*M) \to \text{End}(S) \) is the action of forms on sections of \( S \), and \( \nabla \) is induced from the Levi-Civita connection.

We remark that the connection \( D \) in equation (3.6) is induced from a spin connection with torsion three-form \( H \).

Similarly for (2, 0) supergravity, we have the following

**Definition 3.8.** A (2, 0) supergravity background consists of a six-dimensional lorentzian spin manifold \( (M, g) \), a \( V \)-valued closed antiself-dual 3-form \( H \), where \( V \) is the five-dimensional real representation of the R-symmetry group \( \text{Sp}(2) \cong \text{Spin}(5) \) together with a \( \text{Sp}(2) \)-invariant scalar product, subject to the Einstein equation

\[
\text{Ric}(X, Y) = -\frac{1}{4} \langle \mathbf{i}_X H, \mathbf{i}_Y H \rangle,
\]
where \(\langle -, - \rangle\) now also includes the \(\text{Sp}(2)\)-invariant inner product on \(V\). Such a background is said to be \textit{supersymmetric} if there are nonzero sections \(\varepsilon\) of \(\mathcal{S}\) obeying

\[
D_X \varepsilon = \nabla_X \varepsilon - \frac{1}{2} c(\iota_X H) \varepsilon = 0,
\]

(3.7)

for all vector fields \(X\) and where \(c : \Omega(M; V) \to \text{Cl}(T^*M) \otimes \text{Cl}(V) \to \text{End}(\mathcal{S})\) is the action of \(V\)-valued forms on sections of \(\mathcal{S}\).

Notice that in \((2, 0)\) supergravity, the anti-selfduality of \(H\) imply that \(H \wedge H = 0\) in \(\Omega^6(M; \Lambda^2 V)\).

Maximal supersymmetry implies that the connections \(D\) acting on \(\mathcal{S}\) and \(D\) on \(\mathcal{S}\) are flat. In the case of \((1, 0)\) supergravity, \(D\) is a spin connection with torsion and maximally supersymmetric solutions correspond to six-dimensional lorentzian manifolds admitting a flat metric connection with anti-selfdual closed torsion three-form. We saw in Section 2.3 that \((M, g)\) is locally isometric to a Lie group with a bi-invariant lorentzian metric. In the case of \((2, 0)\) supergravity, \(D\) does not have such an obvious geometrical interpretation, but it is proven in [44] that, up to the natural action of the R-symmetry group, the \((2, 0)\) maximally supersymmetric backgrounds are in one-to-one correspondence with those of \((1, 0)\) supergravity.

The general form of a supersymmetric background in \((1, 0)\) supergravity has been obtained in [45], who in particular also determine the maximally supersymmetric backgrounds by a different method, closely related to the one in [36, 37].

4. \textbf{Maximally supersymmetric backgrounds}

In this section we review the known results about maximally supersymmetric backgrounds in a number of the more interesting supergravity theories. Several of the theories under the consideration will be tackled directly: \(d = 11\) supergravity, \(d = 10\) IIB supergravity and the \(d = 6\) supergravities, whereas the maximally supersymmetric backgrounds of \(d = 10\) IIA and \(d = 5\) \(N = 2\) supergravities will be obtained from those \(d = 11\) and \(d = 6\) supergravities by the technique of Kaluža–Klein reduction. As this technique is very useful, we will review it briefly now.

4.1. \textbf{Kaluža–Klein reduction.} In this section we will briefly review the geometric underpinning of Kaluža–Klein reduction. We start with a supergravity background \((M, g, F, \ldots)\) which is invariant under a one-dimensional Lie group \(\Gamma\), acting freely and properly on \(M\) by isometries which in addition preserve any other geometric data \(F, \ldots\). We shall let \(\xi\) denote a Killing vector field for the \(\Gamma\)-action. Since the action is free, \(\xi\) is nowhere vanishing. We will also assume that \(\xi\) is spacelike; although this is not strictly necessary and indeed time-like reductions can be quite useful, especially in the context of topological field theories.

The original spacetime \(M\) is to be thought of as the total space of a principal \(\Gamma\)-bundle \(\pi : M \to N = M/\Gamma\), where \(\pi\) the map taking a point in \(M\) to the \(\Gamma\)-orbit on which it lies. At every point \(p\) in \(M\), the tangent space \(T_p M\) of \(M\) at \(p\)
decomposes into two orthogonal subspaces: $T_pM = V_p \oplus H_p$, where the **vertical subspace** $V_p = \ker \pi^*$ consists of those vectors tangent to the $\Gamma$-orbit through $p$, and the **horizontal subspace** $H_p = V_p^\perp$ is its orthogonal complement relative to the metric $g$. The resulting decomposition is indeed a direct sum by virtue of the nowhere-vanishing of the norm of $\xi$, whose value at $p$ spans $V_p$ for all $p$. The derivative map $\pi^*$ sets up an isomorphism between $T_pM$ and $T_qN$, where $\pi(p) = q$.

As is well-known, there is a unique metric on $N$ for which this isomorphism is also an isometry and for which the map $\pi$ is a riemannian submersion. We will call this metric $h$.

The horizontal sub-bundle $H$ gives rise to a connection one-form $\alpha$ on $M$ such that $H = \ker \alpha$ and such that $\alpha(\xi) = 1$. We remark that $\alpha$ is invariant, so that $L_\xi \alpha = 0$. This means that the curvature 2-form $\alpha \wedge \alpha$ is both invariant and horizontal—that is, $\iota_\xi \alpha \wedge \alpha = 0$. Such forms are called **basic** and it is a basic fact that they define forms on $N$. Hence $\alpha \wedge \alpha$ defines a 2-form on $N$.

Finally the norm $|\xi|$ of the Killing vector is itself $\Gamma$-invariant and hence defines a function on $N$. Since $\xi$ is spacelike, this function is positive and hence it is convenient to write it as the exponential of a real valued function $\phi : N \to \mathbb{R}$ which is (up to a constant multiple) called the **dilaton**.

In summary, and omitting the pull-backs on $h$ and $\phi$, we can write the metric $g$ as

$$g = h + e^{2\phi} \alpha^2.$$  

The other geometric data also reduces. For example, if $F$ is an invariant differential form on $M$, it gives rise to two differential forms on $N$ simply by decomposing

$$F = G - \alpha \wedge H,$$

where $\alpha$ is the connection one-form defined above. The forms $G$ and $H$ are basic and hence define differential forms on $N$. Indeed, it is clear from the above expression that $H = -\iota_\xi F$, so that it is manifestly horizontal. Invariance of $F$ means that $H$ is closed, whence it is also invariant. Finally, we observe that $G$ is also basic. It is manifestly horizontal, and invariance follows by a simple calculation using that $G$, $H$ and $\alpha \wedge \alpha$ are horizontal.

### 4.2. Eleven-dimensional supergravity.

Maximal supersymmetry implies the flatness of the supersymmetric connection (3.3). Calculating the curvature of this connection and separating into types one arrives at the following conditions:

- $\nabla F = 0$;
- the Riemann curvature tensor is given by

$$\text{Riem}(g) = \frac{1}{12} T^{[4]} + \frac{1}{36} (g \odot T^{[2]}) - \frac{1}{72} |F|^2 (g \odot g),$$

where $\odot$ is the Kulkarni–Nomizu product (see, e.g., [46, §1.G, 1.110]), and the tensors $T^{[2k]} \in \mathcal{C}^\infty(M, S^{2k} \Lambda^k T^* M)$, for $k = 1, 2$ are defined by

$$T^{[2]}(X, Y) := \langle \iota_X F, \iota_Y F \rangle$$

$$T^{[4]}(X, Y, W, Z) := \langle \iota_X \iota_Y F, \iota_W \iota_Z F \rangle.$$


for all vector fields $X,Y,W,Z$; and

- $F$ obeys the Plücker identity:

$$t_Xt_Yt_ZF \wedge F = 0,$$

for all vector fields $X,Y,Z$. The first two conditions imply that the Riemann tensor is parallel, whence $(M,g)$ is locally symmetric, whence locally isometric to one of the spaces in Theorem 2.1. Every such space is acted on transitively by a Lie group $G$ (the group of transvections), whence if we fix a point in $M$ (the origin) with isotropy $H$, $M$ is isomorphic to the space of cosets $G/H$. Let $\mathfrak{g}$ denote the Lie algebra of $G$ and $\mathfrak{h}$ the Lie subalgebra corresponding to $H$. Then $\mathfrak{g}$ admits a vector space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{m}$ is isomorphic to the tangent space of $M$ at the origin. The Lie brackets are such that $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$, $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$, $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$.

The metric $g$ on $M$ is determined by an $\mathfrak{h}$-invariant inner product $B$ on $\mathfrak{m}$.

Since $F$ is parallel, it is $G$-invariant. This means that it is uniquely specified by its value at the origin, which defines an $\mathfrak{h}$-invariant four-form on $\mathfrak{m}$. For $F \neq 0$, the right-hand side of equation (4.1) vanishes, and hence $g$ is flat. We will therefore assume that $F \neq 0$. The Plücker identity says that it is then decomposable, whence it determines a four-dimensional vector subspace $n \subset \mathfrak{m}$ as follows: if at the origin $F = \theta_1 \wedge \theta_2 \wedge \theta_3 \wedge \theta_4$, then $\mathfrak{m}$ is the span of (the dual vectors to) the $\theta_i$. Furthermore, because $F$ is invariant, we have that $H$ leaves the space $\mathfrak{n}$ invariant, whence $[\mathfrak{h}, \mathfrak{n}] \subset \mathfrak{n}$, which means that the holonomy group of $M$ (which is isomorphic to $H$) acts reducibly. In lorentzian signature this does not imply that the space is locally isometric to a product, since the metric may be degenerate when restricted to $\mathfrak{n}$. Therefore we must distinguish between two cases, depending on whether or not the restriction $B|_n$ of $B$ to $\mathfrak{n}$ is or is not degenerate.

If $B|_n$ is non-degenerate, then it follows from the de Rham–Wu decomposition theorem [1] that the space is locally isometric to a product $N \times P$, with $N$ and $P$ locally symmetric spaces of dimensions four and seven, respectively. Explicitly, we can see this as follows: there exists a $B$-orthogonal decomposition $\mathfrak{m} = \mathfrak{n} \oplus \mathfrak{p}$, with $\mathfrak{p} := \mathfrak{m}^\perp$, where $[\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{p}$ because of the invariance of the inner product. Let $\mathfrak{g}_N = \mathfrak{h} \oplus \mathfrak{n}$ and $\mathfrak{g}_P = \mathfrak{h} \oplus \mathfrak{p}$. They are clearly both Lie subalgebras of $\mathfrak{g}$. Let $G_N$ and $G_P$ denote the respective (connected, simply-connected) Lie groups. Then $N$ will be locally isometric to $G_N/H$ and $P$ will be locally isometric to $G_P/H$, and $M$ will be locally isometric to the product. The metrics on $N$ and $P$ are induced by the restrictions of $\mathfrak{n}$ and $\mathfrak{p}$ respectively of the inner product $B$ on $\mathfrak{n} \oplus \mathfrak{p}$, denoted

$$B_n = B|_n,$$

$$B_p = B|_p.$$  

(4.2)

We shall denote the metrics on $N$ and $P$ induced from the above inner products by $h$ and $m$, respectively.
On the other hand if the restriction $B|_{n}$ is degenerate, so that $n$ is a null four-dimensional subspace of $m$, the four-form $F$ is also null. From Theorem 2.1 one sees (see, e.g., [47]) that the only lorentzian symmetric spaces admitting parallel null forms are those which are locally isometric to a product $M = CW_d(A) \times Q_{11-d}$, where $CW_d(A)$ is a $d$-dimensional Cahen-Wallach space and $Q_{11-d}$ is an $(11-d)$-dimensional riemannian symmetric space.

In summary, there are two separate cases to consider:

1. $(M,g) = (N_4 \times P_7, h \oplus m)$ (locally), where $(N,h)$ and $(P,m)$ are symmetric spaces and where $F$ is proportional to (the pull-back of) the volume form on $(N,h)$; or

2. $M = CW_d(A) \times Q_{11-d}$ (locally) and $d \geq 3$, where $Q_{11-d}$ is a riemannian symmetric space.

In [36] these cases are analysed further, resulting in the following theorem.

**Theorem 4.1 (FO-Papadopoulos [36]).** Let $(M,g,F)$ be a maximally supersymmetric solution of eleven-dimensional supergravity. Then it is locally isometric to one of the following:

- $\text{AdS}_7(-7R) \times S^4(8R)$ and $F = \sqrt{6R} \text{dvol}(S^4)$, where $R > 0$ is the constant scalar curvature of $M$;
- $\text{AdS}_4(8R) \times S^7(-7R)$ and $F = \sqrt{-6R} \text{dvol}(\text{AdS}_4)$, where $R < 0$ is again the constant scalar curvature of $M$; or
- $\text{CW}_{11}(A)$ with $A = -\frac{\mu^2}{36} \text{diag}(4,4,1,1,1,1,1,1,1,1)$ and $F = \mu dx^- \wedge dx^1 \wedge dx^2 \wedge dx^3$. One must distinguish between two cases:
  - $\mu = 0$: which recovers the flat space solution $\mathbb{P}^{1,10}$ with $F = 0$; and
  - $\mu \neq 0$: all these are isometric and describe a symmetric plane wave.

The first two solutions are the well-known Freund–Rubin backgrounds [48] and [49], whereas the plane wave was originally discovered by Kowalski-Glikman [50] and rediscovered subsequently in [47]. All of these solutions are locally isometric to the intersection of two quadrics in $E^{11,2}$. Moreover, as shown in [51, 3] they are related by “plane-wave limits” [52, 53].

### 4.3. Ten-dimensional IIA supergravity

Type IIA supergravity [21, 19, 20] is obtained by dimensional reduction from eleven-dimensional supergravity. From the discussion in Section 4.1 and the fact that a $d = 11$ background is characterised by a metric $g$ and a 4-form $F$, it follows that a IIA supergravity background is characterised by a quintuplet $(h, \phi, \Omega, G, H)$ where $h$ is a lorentzian metric on a ten-dimensional spacetime $N$, $\phi$ is a real function on $N$, $\Omega$ a closed 2-form which is the curvature of a principal $\Gamma$-bundle over $N$, $G$ a 4-form on $N$ and $H$ a 3-form on $N$. The PDEs satisfied by these fields are obtained by reducing those in $d = 11$ supergravity by the action of the group $\Gamma$. 
It is a fundamental property of the Kaluza–Klein reduction, that any IIA supergravity background can be lifted (or “oxidised”) to a background of eleven-dimensional supergravity possessing a one-parameter group symmetries. If the IIA supergravity solution preserves some supersymmetry, its lift to eleven dimensions will preserve at least the same amount of supersymmetry. This means that a maximally supersymmetric solution of IIA supergravity will uplift to one of the maximally supersymmetric solutions of eleven-dimensional supergravity determined in the previous section. Therefore the determination of the maximally supersymmetric IIA backgrounds reduces to classifying those dimensional reductions of the maximally supersymmetric eleven-dimensional backgrounds which preserve all supersymmetry.

As explained already in [47], the only such reductions are the reductions of the flat eleven-dimensional background by a translation subgroup of the Poincaré group. In summary, one has

\textbf{Corollary 4.2} (FO-Papadopoulos [36]). Any maximally supersymmetric solution of type IIA supergravity is locally isometric to $\mathbb{E}^{1,9}$ with zero fluxes and constant dilaton.

\section{4.4. Ten-dimensional IIB supergravity.} A maximally supersymmetric background of IIB supergravity admits a (real) 32-dimensional space of Killing spinors. Since this is the (real) rank of the spinor bundle $S$ defined in Section 3.2, it means that at any given point, there is a basis for the spinor bundle consisting of Killing spinors. These spinors satisfy equation (3.4), whence $c(B)$ and $c(G)$ must vanish separately, which in turn imply the vanishing of $G$ and $B$. In particular, this has a consequence that $z$ and hence $\tau$ are constant, whence the connection $A$ on $L_\tau$ also vanishes. Maximally supersymmetric backgrounds have the form $(M, g, F)$ and are parametrised by the upper half-plane via the constant parameter $\tau$. Maximally supersymmetry now implies the flatness of the connection $D$ defined by equation (3.5) and which takes the simplified form

$$D_X \varepsilon = \nabla_X \varepsilon + \frac{i}{4} c(F)c(X)\varepsilon,$$

where $\nabla$ is the spin connection. Notice that the equations of motion now say that $F$ is closed.

Computing the curvature of this connection, and separating into types, we arrive at the following conditions:

- $\nabla F = 0$;
- the Riemann curvature tensor is given by

$$R(X, Y, Z, W) = \langle t_X t_Z F, t_Y t_W F \rangle - \langle t_X t_W F, t_Y t_Z F \rangle. \quad (4.3)$$

Since $F$ is parallel, this means that so is the Riemann tensor, whence $(M, g)$ is locally symmetric; and
• \( F \) obeys an identity reminiscent of both the Plücker and Jacobi identities:

\[
\lambda(t_X t_Y t_Z F) F = 0 \quad \text{for all vector fields } X, Y, Z,
\]

where \( \lambda : \Omega^2(M) \to \operatorname{End}(\Lambda^5 T^* M) \) is the composition of the (metric-induced) isomorphism \( \Omega^2(TM) \cong \mathfrak{so}(TM) \) between 2-forms and skew-symmetric endomorphisms of the tangent bundle and the action of such endomorphisms on the 5-forms.

It was proved in [37] that equation (4.4) implies that \( F = G + \star G \), where \( G = \theta_1 \wedge \theta_2 \wedge \theta_3 \wedge \theta_4 \wedge \theta_5 \) is a parallel decomposable form.

The ensuing analysis follows closely the case of eleven-dimensional supergravity and will not be repeated here. We must distinguish between two cases, depending on whether or not the five-form \( G \) is null. First suppose that \( G \) (and hence \( F \)) is not null. Then the five-form \( G \) induces a local decomposition of \((M,g)\) into a product \( N_5 \times P_5 \) of two five-dimensional symmetric spaces \((N,h)\) and \((P,m)\), where \( G \propto \text{dvol}(N) \) and hence \( \star G \propto \text{dvol}(P) \). Since \((M,g)\) is lorentzian, one of the spaces \((N,h)\) and \((P,m)\) is lorentzian and the other riemannian. By interchanging \( G \) with \( \star G \) if necessary, we can assume that \( G \) has positive norm and hence that \( N \) is riemannian.

In summary, there are two separate cases to consider:

1. \((M,g) = (N_5 \times P_5, h \oplus m)\) (locally), where \((N,h)\) and \((P,m)\) are symmetric spaces and where \( F = G + \star G \) and \( G \) is proportional to (the pull-back of) the volume form on \((N,h)\); or
2. \( M = CW_d(A) \times Q_{10-d} \) (locally) and \( d \geq 3 \), where \( Q_{10-d} \) is a riemannian symmetric space.

In [36] these cases are analysed further, resulting in the following theorem.

**Theorem 4.3** (FO-Papadopoulos [36]). Let \((M,g,F_5^+, \ldots)\) be a maximally supersymmetric solution of ten-dimensional type IIB supergravity. Then it has constant axi-dilaton (normalised so that \( z = 0 \) in the formulas below), all fluxes vanish except for the one corresponding to the self-dual five-form, and is locally isometric to one of the following:

- \( \text{AdS}_5(-R) \times S^5(R) \) and \( F = 2 \sqrt{\frac{R}{5}} \left( \text{dvol(AdS}_5) + \text{dvol(S}^5) \right) \), where \( \pm R \) are the scalar curvatures of \text{AdS}_5 and \text{S}^5, respectively; or
- \( \text{CW}_{10}(A) \) with \( A = -\mu^2 \mathbf{1} \) and \( F = \frac{1}{2} \mu \operatorname{d}x^- \wedge (\operatorname{d}x^1 \wedge \operatorname{d}x^2 \wedge \operatorname{d}x^3 \wedge \operatorname{d}x^4 + \operatorname{d}x^5 \wedge \operatorname{d}x^6 \wedge \operatorname{d}x^7 \wedge \operatorname{d}x^8) \). One must distinguish between two cases:
  - \( \mu = 0 \): which yields the flat space solution \( \mathbb{E}^{1,9} \) with zero fluxes; and
  - \( \mu \neq 0 \): all these are isometric and describe a symmetric plane wave.

The first solution is the well-known Freund–Rubin background mentioned originally in [22]. The plane wave solution was discovered in [54]. As in eleven-dimensional supergravity, the solutions above are locally isometric to the intersection of two quadrics in \( \mathbb{E}^{10,2} \) and as shown in [51, 3] they are related by plane-wave limits.
4.5. Six-dimensional $(2, 0)$ and $(1, 0)$ supergravities. In this case, maximal supersymmetry implies the flatness of the supersymmetric connection $D$ in (3.6) which, as explained in Section 3.3, is induced from a metric connection with closed torsion 3-form $H$. In Section 2.3 we showed that $(M, g)$ is locally isometric to a six-dimensional Lie group with a bi-invariant lorentzian metric. The only extra condition is that $H$, the canonical bi-invariant 3-form associated to such a Lie group, should be self-dual.

We therefore look for Lie algebras with invariant lorentzian scalar products relative to which the canonical invariant 3-form is anti-self dual. As explained in Section 2.4, such Lie algebra is a direct sum of indecomposables. Furthermore, if the Lie algebra is indecomposable then it must be the double extension of an abelian Lie algebra by a one-dimensional Lie algebra and hence solvable (see, e.g., [10]).

These considerations make possible the following enumeration of six-dimensional lorentzian Lie algebras:

1. $E^{1,5}$
2. $E^{1,2} \oplus \mathfrak{so}(3)$
3. $E^{3} \oplus \mathfrak{so}(1, 2)$
4. $\mathfrak{so}(1, 2) \oplus \mathfrak{so}(3)$
5. $\mathfrak{d}(E^{4}, \mathbb{R})$

where the last case actually corresponds to a family of Lie algebras, depending on the action of $\mathbb{R}$ on $E^{4}$, which is given by a homomorphism $\mathbb{R} \to \mathfrak{so}(4)$.

Imposing the condition of anti-selfduality trivially discards cases (2) and (3) above. Case (1) is the abelian Lie algebra with Minkowski metric. The remaining two cases were investigated in [44] (see also [45]) in detail and we review this below.

4.5.1. A six-dimensional Cahen–Wallach space. Let $e_i$, $i = 1, 2, 3, 4$, be an orthonormal basis for $E^{4}$, and let $e_- \in \mathbb{R}$ and $e_+ \in \mathbb{R}^*$, so that together they span $\mathfrak{d}(E^{4}, \mathbb{R})$. The action of $\mathbb{R}$ on $E^{4}$ defines a map $\rho : \mathbb{R} \to \Lambda^2 E^{4}$, which can be brought to the form $\rho(e_-) = \alpha e_1 \wedge e_2 + \beta e_3 \wedge e_4$ via an orthogonal change of basis in $E^{4}$ which moreover preserves the orientation. The Lie brackets of $\mathfrak{d}(E^{4}, \mathbb{R})$ are given by

\[
\begin{align*}
[e_-, e_1] &= \alpha e_2 & [e_-, e_3] &= \beta e_4 \\
[e_-, e_2] &= -\alpha e_1 & [e_-, e_4] &= -\beta e_3 \\
[e_1, e_2] &= \alpha e_+ & [e_3, e_4] &= \beta e_+
\end{align*}
\]

and the scalar product is given (up to scale) by

\[
\langle e_-, e_- \rangle = b \quad \langle e_+, e_- \rangle = 1 \quad \langle e_i, e_j \rangle = \delta_{ij}.
\]

The first thing we notice is that we can set $b = 0$ without loss of generality by the automorphism fixing all $e_i, e_+$ and mapping $e_- \mapsto e_- - \frac{1}{2}be_+$. We will assume that
this has been done and that $\langle e_-, e_- \rangle = 0$. A straightforward calculation shows that the three-form $H$ is anti-selfdual if and only if $\beta = \alpha$. Let us put $\beta = \alpha$ from now on. We must distinguish between two cases: if $\alpha = 0$, then the resulting algebra is abelian and is precisely $\mathbb{E}^{1,5}$. On the other hand if $\alpha \neq 0$, then rescaling $e_+ \mapsto \alpha^{-1} e_+$ we can effectively set $\alpha = 1$ without changing the scalar product. Finally we notice that a constant rescaling of the scalar product can be undone by an automorphism of the algebra. As a result we have two cases: $\mathbb{E}^{1,5}$ (obtained from $\alpha = 0$) and the algebra

\[
\begin{align*}
[e_-, e_1] &= e_2 & [e_-, e_3] &= e_4 \\
[e_-, e_2] &= -e_1 & [e_-, e_4] &= -e_3 \\
[e_1, e_2] &= e_+ & [e_3, e_4] &= e_+
\end{align*}
\] (4.5)

with scalar product given by

\[
\langle e_+, e_- \rangle = 1 \quad \text{and} \quad \langle e_i, e_j \rangle = \delta_{ij}.
\] (4.6)

There is a unique simply-connected Lie group with the above Lie algebra which inherits a bi-invariant lorentzian metric. This Lie group is a six-dimensional analogue of the Nappi–Witten group [55], which is based on the double extension $\mathfrak{d}(\mathbb{E}^2, \mathbb{R})$ [11]. This was denoted NW$_6$ in [56], where one can find a derivation of the metric on this six-dimensional group. The supergravity solution was discovered by Meessen [57] who called it KG6 by analogy with the maximally supersymmetric plane wave of eleven-dimensional supergravity discovered by Kowalski-Glikman [50] and rediscovered in [47].

The metric is easy to write down once we choose a parametrisation for the group. The calculation is routine (see, for example, [56]) and the result is

\[
g = 2dx^+ dx^- - \frac{4}{3} \sum_i (x^i)^2 (dx^-)^2 + \sum_i (dx^i)^2.
\] (4.7)

In these coordinates the three-form $H$ is given by

\[
H = \frac{2}{3} dx^- \wedge (dx^1 \wedge dx^2 + dx^3 + dx^4).
\]

4.5.2. The Freund–Rubin backgrounds. Finally we discuss case (4), with Lie algebra $\mathfrak{so}(1,2) \oplus \mathfrak{so}(3)$. Let $e_0, e_1, e_2$ be a pseudo-orthonormal basis for $\mathfrak{so}(1,2)$. The Lie brackets are given by

\[
[e_0, e_1] = -e_2 \quad [e_0, e_2] = e_1 \quad [e_1, e_2] = e_0.
\]

Similarly let $e_3, e_4, e_5$ denote an orthonormal basis for $\mathfrak{so}(3)$, with Lie brackets

\[
[e_5, e_3] = -e_4 \quad [e_5, e_4] = e_3 \quad [e_3, e_4] = -e_5.
\]
The most general invariant lorentzian scalar product on \( \mathfrak{so}(1,2) \oplus \mathfrak{so}(3) \) is labelled by two positive numbers \( \alpha \) and \( \beta \) and is given by

\[
\begin{pmatrix}
  e_0 & e_1 & e_2 & e_3 & e_4 & e_5 \\
  e_0 & -\alpha & 0 & 0 & 0 & 0 \\
  e_1 & 0 & \alpha & 0 & 0 & 0 \\
  e_2 & 0 & 0 & \alpha & 0 & 0 \\
  e_3 & 0 & 0 & 0 & \beta & 0 \\
  e_4 & 0 & 0 & 0 & 0 & \beta \\
  e_5 & 0 & 0 & 0 & 0 & 0 & \beta 
\end{pmatrix}.
\]

Anti-selfduality of the canonical three-form implies that \( \beta = \alpha \). There is a unique simply-connected Lie group with Lie algebra \( \widetilde{\mathfrak{so}}(1,2) \oplus \mathfrak{so}(3) \), namely \( \widetilde{\mathrm{SL}}(2,\mathbb{R}) \times \mathrm{SU}(2) \), where \( \widetilde{\mathrm{SL}}(2,\mathbb{R}) \) denotes the universal covering group of \( \mathrm{SL}(2,\mathbb{R}) \). This group inherits a one-parameter family of bi-invariant metrics. This solution is none other than the standard Freund–Rubin solution \( \mathrm{AdS}_3 \times S^3 \), with equal radii of curvature, where strictly speaking we should take the universal covering space of \( \mathrm{AdS}_3 \).

In summary, the following are the possible maximally supersymmetric backgrounds of \((1,0)\) supergravity, and of \((2,0)\) supergravity up to the action of the R-symmetry group. First of all we have a one-parameter family of Freund-Rubin backgrounds locally isometric to \( \mathrm{AdS}_3 \times S^3 \), with equal radii of curvature. The anti-selfdual three-form \( H \) is then proportional to the difference of the volume forms of the two spaces. Then we have a six-dimensional analogue \( \mathrm{NW}_6 \) of the Nappi–Witten group, locally isometric to a Cahen–Wallach symmetric space. Finally there is flat Minkowski spacetime \( \mathbb{E}^{1,5} \). These backgrounds are related by Penrose limits which can be interpreted in this case as group contractions. The details appear in [56].

### 4.6. Five-dimensional \( N = 2 \) supergravity

In this section we will review the dimensional reductions of the six-dimensional backgrounds just found. Dimensional reduction usually breaks some supersymmetry: in the ten- and eleven-dimensional supergravity theories, only the flat background remains maximally supersymmetric after dimensional reduction and then only by a translation. However for the six-dimensional backgrounds the situation is different. Indeed, in [58] it was shown that the thereto known maximally supersymmetric backgrounds with eight supercharges in six, five and four dimensions are related by dimensional reduction and oxidation. As we will see presently, this perhaps surprising phenomenon stems from the fact that the six-dimensional backgrounds are parallelised Lie groups. Our results will also give an \textit{a priori} explanation to the empirical fact that these backgrounds are homogeneous [59].

We now explain the technical result which underlies this result. Let \( D \) be a metric connection with torsion \( T \). We observe that if a vector field \( \xi \) is \( D \)-parallel then it is Killing. Now let \( \psi \) be a Killing spinor; that is, \( D\psi = 0 \). Then the Lie derivative of \( \psi \) along \( \xi \) is well-defined (see, for example, [60]) and, furthermore, it vanishes identically. Moreover, if \( L_\xi \psi = 0 \) for \textit{all} Killing spinors then \( D\xi = 0 \).
For a parallelised Lie group $G$, the $D$-parallel vectors are either the left- or right-invariant vector fields, depending on the choice of parallelising connection. For definiteness, we will choose the connection whose parallel sections are the left-invariant vector fields. Left-invariant vector fields generate right translations and are in one-to-one correspondence with elements of the Lie algebra $\mathfrak{g}$. Therefore every left-invariant vector field $\xi$ determines a one-parameter subgroup $K$, say, of $G$ and the orbits of such a vector field in $G$ are the right $K$-cosets. The dimensional reduction along this vector field is smooth and diffeomorphic to the space of cosets $G/K$. We will be interested in subgroups $K$ such that $G/K$ is a five-dimensional lorentzian spacetime, which requires that the right $K$-cosets are spacelike. In other words, we require that the Killing vector $\xi$ be spacelike. Bi-invariance of the metric guarantees that this is the case provided that the Lie algebra element $\xi(e) \in \mathfrak{g}$ is spacelike relative to the invariant scalar product. Further notice that a constant rescaling of $\xi$ does not change its causal property nor the subgroup $K$ it generates: it is simply reparameterised. Therefore, in order to classify all possible reductions we need to classify all spacelike elements of $\mathfrak{g}$ up to scale. Moreover elements of $\mathfrak{g}$ which are related by isometric automorphisms (e.g., which are in the same adjoint orbit of $G$) give rise to isometric quotients. Thus, to summarise, we want to classify spacelike elements of $\mathfrak{g}$ up to scale and up to automorphisms.

As discussed in Section 4.1, the reduction of the six-dimensional metric to five dimensions gives rise to a metric $h$, a dilaton $\phi$ and a curvature 2-form $F$. The dilaton $\phi$ is a logarithmic measure of the fibre metric $\|\xi_X\|$ which in our case is constant, and $F = d\alpha$ (omitting pullbacks). We can give an explicit formula for $\Omega$ using the Maurer–Cartan structure equations. Indeed,

$$F = d\alpha = \langle X, d\theta \rangle = -\frac{1}{2} \langle X, [\theta, \theta] \rangle.$$ (4.8)

In terms of this data, the metric on the $G$ is given by the usual Kaluža–Klein ansatz

$$ds^2 = h + \alpha^2,$$

where we have set the dilaton to zero in agreement with the choice of normalisation for $\xi_X$. More explicitly the metric on the five-dimensional quotient is given by

$$h = \langle \theta, \theta \rangle - \langle X, \theta \rangle^2.$$

To reduce the anti-selfdual three-form $H$ we first decompose it as

$$H = G_3 + \alpha \wedge G_2,$$

where $G_2 = \iota_{\xi_X} H$ and $G_3$ are basic. Because $dH = 0$ it follows that $dG_2 = 0$ and that $dG_3 + F \wedge G_2 = 0$ where $F = d\alpha$ was defined above. Finally because $H$ is anti-selfdual, it follows that $G_3$ and $G_2$ are related by Hodge duality in five dimensions: $G_3 = *_h G_2$. In other words, we have that

$$H = *_h G_2 + \alpha \wedge G_2,$$

where $dG_2 = 0$ and $d* \, G_2 = -F \wedge G_2$. 

In fact, in this case we have $F = G_2$. Indeed, using that $H = -\frac{1}{6} \langle \theta, [\theta, \theta] \rangle$, we compute

\[ G_2 = \iota_\xi H = -\frac{1}{2} \langle X, [\theta, \theta] \rangle , \]

which agrees with the expression for $F$ derived in (4.8).

In summary, for the reductions under consideration, we obtain a maximally supersymmetric background of the minimal $N=2$ supergravity with bosonic fields $(h, F)$ given by the reduction of $(g, H)$ where $F = d\alpha$, $h = g - \alpha^2$ and $H = \star_\xi F + \alpha \wedge F$.

The different reductions were classified in [44], to where we send the reader for details, hence obtaining all the maximally supersymmetric backgrounds of the minimal $N=2$ supergravity and thus completing the classification of supersymmetric backgrounds in [61]. Among the maximally supersymmetric backgrounds one finds the near-horizon geometries [62] of the rotating black holes of [63, 64], the symmetric plane-wave of [57] and the Gödel-like background discovered in [61].

5. Parallelisable type II backgrounds

In this section we will present a classification of parallelisable type II backgrounds, by which we mean backgrounds of both type IIA and type IIB supergravity. Since these theories contain different dynamical degrees of freedom, common backgrounds are necessarily very special.

**Definition 5.1.** A type II supergravity background consists of a ten-dimensional lorentzian spin manifold $(M, g)$ together with a closed 3-form $H$ and a smooth function $\phi : M \to \mathbb{R}$ subject to the equations of motion obtained by varying the (formal) action functional

\[ \int_M e^{-2\phi} \left( R + 4|d\phi|^2 - \frac{1}{4}|H|^2 \right) d\text{vol}_g , \quad (5.1) \]

where $R$ and $d\text{vol}_g$ are the scalar curvature and the volume form associated to $g$.

We are interested in parallelisable backgrounds, for which the metric connection $D$ with torsion 3-form $H$ is flat. In that case, the equations of motion simplify to the following three conditions:

\[ \nabla d\phi = 0 \]
\[ d\phi \wedge \star H = 0 \]
\[ |d\phi|^2 - \frac{1}{4}|H|^2 = 0 . \]

(5.2)

To discuss supersymmetry, we need to distinguish whether we are in type IIA or type IIB supergravity, since the spinor bundles are different. Let $S_{\pm}$ denote the real 16-dimensional half-spin representations of Spin(1,9) and let $S_A = S_+ \oplus S_-$ and $S_B = S_+ \oplus \bar{S}_+$. Let $S_A$ and $S_B$ denote the spinor bundles on $M$ associated to $S_A$ and $S_B$, respectively. We will let $S$ denote either $S_A$ or $S_B$, depending on which type II theory we are considering.
Definition 5.2. A type II background is **supersymmetric** if there are nonzero sections $\varepsilon$ of $\mathcal{S}$ satisfying the two conditions:

$$D\varepsilon = 0 \quad \text{and} \quad c\left(d\phi + \frac{1}{2} H\right)\varepsilon = 0,$$

where $c : \Omega(M) \to \mathcal{C}(TM) \to \text{End}(\mathcal{S})$ is the Clifford action of forms on spinors.

The supersymmetric parallelisable type II backgrounds were classified in [65, 66] and revisited in the context of heterotic supergravity in [67], whose treatment we follow.

5.1. Ten-dimensional parallelisable geometries. As explained in Section 2.2, it is possible to list all the simply-connected parallelisable lorentzian manifolds in any dimension. The ingredients out of which we can make them are given in Table 2, whose last column follows from equation (5.2).

Indeed, in the case of a Lie group, that is, when $dH = 0$, equation (5.2) says that $d\phi$ must be central, when thought of as an element in the Lie algebra. Since AdS$_3$, $S^3$ and SU(3) are simple, their Lie algebras have no centre, whence $d\phi = 0$. In the case of an abelian group there are no conditions, and in the case of CW$_{2n}(A)$, the Lie algebra has a one-dimensional centre corresponding to $\partial_+$, whose dual one-form is $dx^-$. This means that $d\phi$ must be proportional to $dx^-$, whence $\phi$ can only depend on $x^-$. Finally for $S^7$, the equation of motion $\ast H \wedge d\phi = 0$ implies that $d\phi = 0$. To see this, notice that the parallelised $S^7$ possesses a nearly parallel $G_2$ structure and the differential forms decompose into irreducible types under $G_2$. For example, the one-forms corresponding to the irreducible seven-dimensional irreducible representation $m$ of $G_2$ coming from the embedding $G_2 \subset SO(7)$, whereas the two-forms decompose into $g_2 \oplus m$, where $g_2$ is the adjoint representation which is irreducible since $G_2$ is simple. Now, $H$ and $\ast H$ both are $G_2$-invariant and hence the map $\Omega^1(S^7) \to \Omega^2(S^7)$ defined by $\theta \mapsto \ast(\ast H \wedge \theta)$ is $G_2$-equivariant. Since it is not identically zero, it must be an isomorphism onto its image. Hence if $\ast H \wedge d\phi = 0$, then also in this case $d\phi = 0$.

It is now a simple matter to put these ingredients together to make up all possible ten-dimensional combinations with lorentzian signature. Doing so, we arrive at Table 3 (see also [65], where the entry corresponding to $E_{1,0} \times S^3 \times S^3 \times S^3$ had been omitted inadvertently and where the entries with $S^7$ had also been omitted due to the fact that in type II string theory $\ast H = 0$).

5.2. Type II backgrounds. First of all we notice that $S^7$ cannot appear because $dH = 0$. Therefore the allowed backgrounds follow *mutatis mutandis* from the analysis of [65, 66]. We start by listing the possible backgrounds and then counting the amount of supersymmetry that each preserves. The results are summarised in Table 4 and Table 5, which also contains the analysis of the supersymmetry preserved by the background.

**AdS$_3 \times S^3 \times S^3 \times E$.** Here $d\phi$ can only have nonzero components along the flat direction, which is spacelike, whence $|d\phi|^2 \geq 0$. Equation (5.2) says that $|H|^2 \geq 0$, etc.
so that if we call \( R_0, R_1 \) and \( R_2 \) the radii of curvature of \( \text{AdS}_3 \) and of the two 3-spheres, respectively, then

\[
\frac{1}{R_1} + \frac{1}{R_2} \geq \frac{1}{R_0}.
\]

This bound is saturated if and only if the dilaton is constant.

**\( \text{AdS}_3 \times S^3 \times \mathbb{E}^4 \).** This is the limit \( R_2 \to \infty \) of the above case.

**\( \text{AdS}_3 \times \mathbb{E}^7 \).** This would be the limit \( R_1 \to \infty \) of the above case, but then the inequality \( R_0^{-2} \leq 0 \) cannot be satisfied. Hence this geometry is not a background
(with or without supersymmetry).

$E^{1,0}$. In this case $H = 0$, so $|d\phi|^2 = 0$. So we can take a linear dilaton along a null direction: $\phi = a + bx^-$, for some constants $a, b$ say.

$E^{1,0} \times S^3 \times S^3 \times S^3$. The dilaton can only depend on the flat coordinate, which is timelike, so $|d\phi|^2 \leq 0$. However $|H|^2 > 0$, whence this geometry is never a background (with or without supersymmetry).

$E^{1,1} \times SU(3)$. Here $|H|^2 > 0$, and $d\phi$ can have components along $E^{1,1}$. Letting $(x^0, x^1)$ be flat coordinates for $E^{1,1}$, we can take $\phi = a + \frac{1}{2} |H| x^1$, for some constant $a$, without loss of generality.

$E^{1,3} \times S^3 \times S^3$. Here $|H|^2 > 0$ and $d\phi$ can have components along $E^{1,3}$ [68]. With $(x^0, x^1, x^2, x^3)$ being flat coordinates for $E^{1,3}$, we take $\phi = a + \frac{1}{2} |H| x^1$, for some constant $a$.

$E^{1,6} \times S^3$. This is the limit $R_2 \to \infty$ of the above case, where $R_2$ is the radius of curvature of one of the spheres [69, 70].

$CW_{2n}(A) \times E^{10-2n}$, $n = 2, 3, 4, 5$. In these cases $|H|^2 = 0$ and hence $|d\phi|^2 = 0$, so that it cannot have components along the flat directions (if any). This means $\phi = a + bx^-$, for constants $a, b$.

$CW_4(A) \times S^3 \times S^3$. Here $|d\phi|^2 = 0$, whereas $|H|^2 > 0$, hence there are no backgrounds with this geometry.

$CW_{2n}(A) \times S^3 \times E^{7-2n}$, $n = 2, 3$. Here $|H|^2 > 0$, whence $|d\phi|^2 > 0$. This means that we can take $\phi = a + bx^- + \frac{1}{2} |H| y$, where $y$ is any flat coordinate in $E^{7-2n}$ and $a, b$ are constants.

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| Geometry          | Dilaton                                           |
|-------------------|--------------------------------------------------|
| $\text{AdS}_3 \times S^3 \times S^3 \times E$ | $\phi = a + \frac{1}{4} |H| y$          |
| $\text{AdS}_3 \times S^3 \times E^4$            | $\phi = a + \frac{1}{4} |H| y$          |
| $\text{E}^{1,1} \times \text{SU}(3)$            | $\phi = a + \frac{1}{4} |H| y$          |
| $\text{E}^{1,3} \times S^3 \times S^3$          | $\phi = a + \frac{1}{4} |H| y$          |
| $\text{E}^{1,6} \times S^3$                      | $\phi = a + \frac{1}{4} |H| y$          |
| $\text{E}^{1,9}$                                   | $\phi = a + bx^-$     |
| $\text{CW}_{10}(A)$                               | $\phi = a + bx^-$     |
| $\text{CW}_6(A) \times E^2$                       | $\phi = a + bx^-$     |
| $\text{CW}_6(A) \times S^3 \times E$             | $\phi = a + bx^- + \frac{1}{2} |H| y$    |
| $\text{CW}_6(A) \times E^4$                       | $\phi = a + bx^-$     |
| $\text{CW}_4(A) \times S^3 \times E^3$           | $\phi = a + bx^- + \frac{1}{2} |H| y$    |
| $\text{CW}_4(A) \times E^6$                       | $\phi = a + bx^-$     |

Table 4. Parallelisable backgrounds with a linear dilaton. The notation is such that $y$ is a spacelike flat coordinate.

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| Parallelisable geometry | Supersymmetries with dilaton being constant | nonconstant |
|-------------------------|------------------------------------------|------------|
| AdS$_3 \times S^3 \times S^3 \times E$ | 16 | 16 |
| AdS$_3 \times S^3 \times E^4$ | 16 | 16 |
| $E^{1,1} \times SU(3)$ | $\times$ | 16 |
| $E^{1,3} \times S^3 \times S^3$ | $\times$ | 16 |
| $E^{1,6} \times S^3$ | $\times$ | 16 |
| $E^{1,9}$ | 32 | 16 |
| CW$_{10}(A)$ | 16, 18(A), 20, 22(A), 24(B), 28(B) | 16 |
| CW$_{8}(A) \times E^2$ | 16, 20 | 16 |
| CW$_{6}(A) \times S^3 \times E$ | $\times$ | 16 |
| CW$_{6}(A) \times E^4$ | 16, 24 | 16 |
| CW$_{4}(A) \times S^3 \times E^3$ | $\times$ | 16 |
| CW$_{4}(A) \times E^6$ | 16 | 16 |

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