MODERATE DEVIATIONS FOR MARTINGALES WITH BOUNDED JUMPS

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Abstract
We prove that the Moderate Deviation Principle (MDP) holds for the trajectory of a locally square integrable martingale with bounded jumps as soon as its quadratic covariation, properly scaled, converges in probability at an exponential rate. A consequence of this MDP is the tightness of the method of bounded martingale differences in the regime of moderate deviations.

1 Introduction
Suppose \( \{X_m, \mathcal{F}_m\}_{m=0}^{\infty} \) is a discrete-parameter real valued martingale with bounded jumps \( |X_m - X_{m-1}| \leq a, \ m \in \mathbb{N}, \) filtration \( \mathcal{F}_m \) and such that \( X_0 = 0. \) The basic inequality for the method of bounded martingale differences is Azuma-Hoeffding inequality (c.f. [1]):
\[
P\{X_k \geq x\} \leq e^{-x^2/2ka^2} \quad \forall x > 0. \tag{1}
\]
In the special case of i.i.d. differences \( P\{X_m - X_{m-1} = a\} = 1 - P\{X_m - X_{m-1} = -a/(1-\epsilon)\} = \epsilon \in (0,1), \) it is easy to see that \( P\{X_k \geq x\} \leq \exp[-kH(\epsilon + (1-\epsilon)x/(ak)\epsilon)], \) where
\[
H(q|p) = q \log(q/p) + (1-q) \log((1-q)/(1-p)).
\]
For \( \epsilon \to 0, \) the latter upper bound approaches 0, thus demonstrating that (1) may in general be a non-tight upper bound. Let \( B(u) = 2u^{-2}((1+u)\log(1+u) - u) \) and
\[
\langle X \rangle_m = \sum_{k=1}^{m} E[(X_k - X_{k-1})^2|\mathcal{F}_{k-1}]
\]
denote the quadratic variation of \( \{X_m, \mathcal{F}_m\}_{m=0}^{\infty}. \) Then,
\[
P\{X_k \geq x\} \leq P\{\langle X \rangle_k \geq y\} + e^{-x^2B(ax/y)/2y} \quad \forall x, y > 0 \tag{2}
\]

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(c.f. [4, Theorem (1.6)]). In particular, \( B(0,\epsilon) = 1 \), recovering (1) for the choice \( y = k a^2 \) and \( x/y \to 0 \). The inequality (2) holds also for the more general setting of locally square integrable (continuous-parameter) martingales with bounded jumps (c.f. [7, Theorem II.4.5]).

In this note we adopt the latter setting and demonstrate the tightness of (2) in the range of moderate deviations, corresponding to \( x/y \to 0 \) while \( x^2/y \to \infty \) (c.f. Remark 5 below). We note in passing that for \textit{continuous} martingales [6] studies the tightness of the inequality:

\[
\mathbb{P}\{ X_k \geq \frac{1}{2} x (1 + \langle X \rangle_t/y) \} \leq e^{-x^2/2y},
\]

using Girsanov transformations, whereas we apply large deviation theory and concentrate on martingales with (bounded) jumps, encompassing the case of discrete-parameter martingales.

Recall that a family of random variables \( \{ Z_k : k > 0 \} \) with values in a topological vector space \( \mathcal{X} \) equipped with \( \sigma \)-field \( \mathcal{B} \) satisfies the Large Deviation Principle (LDP) with speed \( a_k \downarrow 0 \) and good rate function \( I(\cdot) \) if the level sets \( \{ x; I(x) \leq \alpha \} \) are compact for all \( \alpha < \infty \) and for all \( \Gamma \in \mathcal{B} \)

\[
- \inf_{x \in \Gamma} I(x) \leq \liminf_{k \to \infty} a_k \log \mathbb{P}\{ Z_k \in \Gamma \} \leq \limsup_{k \to \infty} a_k \log \mathbb{P}\{ Z_k \in \Gamma \} \leq - \inf_{x \in \Gamma} I(x)
\]

(where \( \Gamma^0 \) and \( \GammaTriplet \) denote the interior and closure of \( \Gamma \), respectively). The family of random variables \( \{ Z_k : k > 0 \} \) satisfies the Moderate Deviation Principle with good rate function \( I(\cdot) \) and critical speed 1/\( h_k \) if for every speed \( a_k \downarrow 0 \) such that \( h_k a_k \to \infty \), the random variables \( \sqrt{a_k} Z_k \) satisfy the LDP with the good rate function \( I(\cdot) \).

Let \( D(\mathbb{R}^d) (= D(\mathbb{R}^+, \mathbb{R}^d)) \) denote the space of all \( \mathbb{R}^d \)-valued càdlàg (i.e. right-continuous with left-hand limits) functions on \( \mathbb{R}_+ \) equipped with the locally uniform topology. Also, \( C(\mathbb{R}^d) \) is the subspace of \( D(\mathbb{R}^d) \) consisting of continuous functions.

The process \( X \in D(\mathbb{R}^d) \) is defined on a complete stochastic basis \((\Omega, \mathcal{F}, \mathbb{F} = \mathcal{F}_t, \mathbb{P})\) (c.f. [5, Chapters I and II] or [7, Chapters 1-4] for this and the related definitions that follow). We equip \( D(\mathbb{R}^d) \) hereafter with a \( \sigma \)-field \( \mathcal{B} \) such that \( X : \Omega \to D(\mathbb{R}^d) \) is measurable (\( \mathcal{B} \) may well be strictly smaller than the Borel \( \sigma \)-field of \( D(\mathbb{R}^d) \)).

Suppose that \( X \in \mathcal{M}^2_{\text{loc}, 0} \) is a locally square integrable martingale with bounded jumps \( |\Delta X| \leq a \) (and \( X_0 = 0 \)). We denote by \((A, C, \nu)\) the triplet predictable characteristics of \( X \), where here \( A = 0, C = (C_t)_{t \geq 0} \) is the \( \mathbb{F} \)-predictable quadratic variation process of the continuous part of \( X \) and \( \nu = \nu(ds, dx) \) is the \( \mathbb{F} \)-compensator of the measure of jumps of \( X \). Without loss of generality we may assume that

\[
\nu(\{t\}, \mathbb{R}^d) = \int_{|x| \leq a} \nu(\{t\}, dx) \leq 1, \quad \int_{|x| \leq a} x \nu(\{t\}, dx) = 0, \quad t > 0
\]

and for all \( s < t, (C_t - C_s) \) is a symmetric positive-semi-definite \( d \times d \) matrix. The predictable quadratic characteristic (covariation) of \( X \) is the process

\[
\langle X \rangle_t = C_t + \int_0^t \int_{|x| \leq a} xx'd\nu,
\]

where \( x' \) denotes the transpose of \( x \in \mathbb{R}^d \), and \( \|A\| = \sup_{|\lambda| = 1} |\lambda' A \lambda| \) for any \( d \times d \) symmetric matrix \( A \).

Our main result is as follows.
Proposition 1 Suppose the symmetric positive-semi-definite $d \times d$ matrix $Q$ and the regularly varying function $h_t$ of index $\alpha > 0$ are such that for all $\delta > 0$:

\[
\limsup_{t \to \infty} h_t^{-1} \log P(\|h_t^{-1} \langle X \rangle_t - Q \| > \delta) < 0. \tag{5}
\]

Then $\{h_k^{-1/2} X_k\}$ satisfies the MDP in $(D[\mathbb{R}^d], \mathcal{B})$ (equipped with the locally uniform topology) with critical speed $1/h_k$ and the good rate function

\[
I(\phi) = \begin{cases} 
\int_0^\infty \Lambda^*(\phi(t)) \alpha^{-1} t^{1-\alpha} dt & \phi \in \mathcal{AC}_0 \\
\infty & \text{otherwise},
\end{cases}
\tag{6}
\]

where $\Lambda^*(v) = \sup_{t, \lambda \in I^d}(\lambda^t \frac{1}{2} \lambda Q \lambda)$, and $\mathcal{AC}_0 = \{\phi : I_R^+ \to I_R^d \text{ with } \phi(0) = 0 \text{ and absolutely continuous coordinates}\}$.

Remark 1 Note that both (5) and the MDP are invariant to replacing $h_t$ by $g_t$ such that $h_t g_t \to c \in (0, \infty)$ and taking $cQ$ instead of $Q$. Thus, if $Q \neq 0$ we may take $h_t = \text{median } \| \langle X \rangle_t \|$, and in general we may assume with no loss of generality that $h_t \in D(I^R_+)$ is strictly increasing of bounded jumps.

Remark 2 If $X$ is a locally square integrable martingale with independent increments, then $\langle X \rangle$ is a deterministic process, hence suffices that $h_t^{-1} \langle X \rangle_t \to Q$ for (5) to hold.

As stated in the next corollary, less is needed if only $X_k$ (or $\sup_{s \leq k} X_s$) is of interest.

Corollary 1

(a) Suppose that (5) holds for some unbounded $h_t$ (possibly not regularly varying). Then, $\{h_k^{-1/2} X_k\}$ satisfies the MDP in $I^d$ with critical speed $1/h_k$ and good rate function $\Lambda^*(\cdot)$.

(b) If also $d = 1$, then $\{h_k^{-1/2} \sup_{s \leq k} X_s\}$ satisfies the MDP with the good rate function $I(z) = z^2/(2Q)$ for $z \geq 0$ and $I(z) = \infty$ otherwise.

Remark 3 For $d = 1$, discrete-time martingales, and assuming that $h_k = \langle X \rangle_k$ is non-random, strong Normal approximation for the law of $h_k^{-1/2} X_k$ is proved in [9] for the range of values corresponding to $\alpha h_k \to \infty$.

Remark 4 The difference between Proposition 1 and Corollary 1 is best demonstrated when considering $X_t = B_{h_t}$, with $B_s$ the standard Brownian motion. The MDP for $h_t^{-1/2} B_{h_t}$ in $I^R$ then trivially holds, whereas the MDP for $h_k^{-1/2} B_{h_k}$ is equivalent to Schilder’s theorem (c.f. [3, Theorem 5.2.3]), and thus holds only when $h_t$ is regularly varying of index $\alpha > 0$.

Remark 5 When $d = 1$ and $Q \neq 0$, the rate function for the MDP of part (a) of Corollary 1 is $x^2/(2Q)$. For $y = h_k Q(1 + \delta)$, $\delta > 0$ and $x = x_k = o(y)$ such that $x^2/y \to \infty$, this MDP then implies that $P\{X_k \geq x\} = \exp\{-(1 + \delta + o(1))x^2/2y\}$ while $P\{(X)_k \geq y\} = o(\exp(-x^2/2y))$ by (5). Consequently, for such values of $x, y$ the inequality (2) is tight for $k \to \infty$ (see also Remark 9 below for non-asymptotic results).

Remark 6 In contrast with Corollary 1 we note that the LDP with speed $m^{-1}$ may fail for $m^{-1} X_m$ even when $X$ is a real valued discrete-parameter martingale with bounded independent increments such that $\langle X \rangle_m = m$. Specifically, let $b : I^N \to \{1, 2\}$ be a deterministic sequence such that $p_m = m^{-1} \sum_{k=1}^m 1_{\{b(k) = 1\}}$ fails to converge for $m \to \infty$ and let $\mu_i, i = 1, 2$ be two probability measures on $[-a, a]$ such that $\int x d\mu_i = 0, \int x^2 d\mu_i = 1, i = 1, 2$ while $c_1 \neq c_2$ for
\[ c_i = \log \int e^{x} \, d\mu_i. \] Then, \( \Delta X_k \) independent random variables of law \( \mu_{\ell(k)}, k \in \mathbb{N} \), result with \( X_m \) as above. Indeed, \( m^{-1} \log \mathbb{E}(\exp(X_m)) = p_m c_1 + (1 - p_m) c_2 \) fails to converge for \( m \to \infty \), hence by Varadhan’s lemma (c.f. [3, Theorem 4.3.1]), necessarily the LDP with speed \( m \) fails for \( m^{-1} X_m \).

**Remark 7** Corollary 1 may fail when \( X \) is a real valued discrete-parameter martingale with unbounded independent increments such that \( (X)_m = m \). Specifically, for \( m_j = 2^{j^2}, j \in \mathbb{N} \) let \( M(m_j) = 2(m_j \log m_j)^{1/2} \) and \( M(k) = 1 \) for all other \( k \in \mathbb{N} \). Let \( Z_k \) be independent Bernoulli \( (1/(M(k)^2 + 1)) \) random variables. Then, \( \Delta X_k = M(k)Z_k - M(k)^{-1}(1 - Z_k) \) result with \( X_m \) as above, with the LDP of speed 1 not holding for \( (m \log m)^{-1/2} X_m \) (c.f. Corollary 1), while \( (m \log m)^{-1/2} X_m \) converges to \( -\infty \) in probability. Consequently, the LDP bounds fail for \( (m \log m)^{-1/2} X_m \geq 2 \).

Proposition 1 is proved in the next section with the proof of Corollary 1 provided in Section 3. Both results build upon Lemma 1. Indeed, Proposition 1 is a direct consequence of Lemma 1 and [8]. Also, with Lemma 1 holding, it is not hard to prove part (a) of Corollary 1 as a consequence of the Gärtner–Ellis theorem (c.f. [3, Theorem 2.3.6]), without relying on [8].

## 2 Proof of Proposition 1

The cumulant \( G(\lambda) = (G_t(\lambda))_{t \geq 0} \) associated with \( X \) is

\[
G_t(\lambda) = \frac{1}{2} \lambda^t C_t \lambda + \int_0^t \int_{|x| \leq a} (e^{\lambda x} - 1 - \lambda x) \nu(ds, dx), \quad t > 0, \lambda \in \mathbb{R}^d. \tag{7}
\]

The stochastic (or the Doléans-Dade) exponential of \( G(\lambda) \), denoted \( \mathcal{E}(G(\lambda)) \) is given by

\[
\varphi_t(\lambda) = \log \mathcal{E}(G(\lambda)) = G_t(\lambda) + \sum_{s \leq t} \left[ \log(1 + \Delta G_s(\lambda)) - \Delta G_s(\lambda) \right], \tag{8}
\]

where

\[
\Delta G_s(\lambda) = \int_{|x| \leq a} (e^{\lambda x} - 1) \nu(ds, dx) = \int_{|x| \leq a} (e^{\lambda x} - 1 - \lambda x) \nu(ds, dx). \tag{9}
\]

The next lemma which is of independent interest, is key to the proof of Proposition 1.

**Lemma 1** For \( \epsilon > 0 \), let \( v(\epsilon) = 2(e^{\epsilon} - 1 - \epsilon) / \epsilon^2 \geq 1 \geq v(-\epsilon) - \epsilon^2 v(\epsilon)^2 / 4 = w(\epsilon) \). Then, for any \( 0 \leq u \leq t < \infty \), \( \lambda \in \mathbb{R}^d \)

\[
\frac{1}{2} w(|\lambda| \lambda) \lambda^t (X)_t - (X)_u \lambda \leq \varphi_t(\lambda) - \varphi_u(\lambda) \leq \frac{1}{2} w(|\lambda| \lambda) \lambda^t (X)_t - (X)_u \lambda. \tag{10}
\]

**Remark 8** Since \( \exp[\lambda X_t - \varphi_t(\lambda)] \) is a local martingale (c.f. [7, Section 4.13]), Lemma 1 implies that \( \exp[\lambda X_t - \varphi_t(\lambda)] \lambda^t (X)_t \lambda \) is a non-negative super-martingale while \( \exp[\lambda X_t - \frac{1}{2} w(|\lambda| \lambda) \lambda^t (X)_t \lambda] \) is a non-negative local sub-martingale. Noting that \( w(|\lambda| \lambda) \to 1 \) for \( |\lambda| \to 0 \), these are to be compared with the local martingale property of \( \exp[\lambda X_t - \frac{1}{2} \lambda^t (X)_t \lambda] \) when \( X \in \mathcal{M}_{loc,0} \) is a continuous local martingale (c.f. [7, Section 4.13]).
Remark 9 For $d = 1$ it follows that for every $\lambda \in \mathbb{R}$,
\[
\mathbb{E}\{\exp[\lambda X_m - \frac{1}{2}v(|\lambda|a)\lambda^2 \langle X \rangle_m]\} \leq 1
\] (11)
(c.f. Remark 8). The inequality (2) then follows by Chebycheff’s inequality and optimization over $\lambda \geq 0$. For the special case of a real-valued discrete-parameter martingale $X_m$ also
\[
\mathbb{E}\{\exp[\lambda X_m - \frac{1}{2}w(|\lambda|a)\lambda^2 \langle X \rangle_m]\} \geq 1,
\] (12)
and we can even replace $w(|\lambda|a)$ in (12) by $v(-|\lambda|a)$ (c.f. [4, (1.4)]) where the sub-martingale property of $\exp(\lambda X_m - \frac{1}{2}\nu(-|\lambda|a)\lambda^2 \langle X \rangle_m)$ is proved.

Proof: To prove the upper bound on $\varphi_t(\lambda) - \varphi_u(\lambda)$ note that $\log(1 + x) - x \leq 0$ implying by (8) that $\varphi_t(\lambda) - \varphi_u(\lambda) \leq G_t(\lambda) - G_u(\lambda)$. The required bound then follows from (7) since $e^{\lambda x} - 1 - \lambda'x \leq \frac{1}{2} v(|\lambda|a)\lambda'xx$ for $|x| \leq a$, and $\lambda'(C_t - C_u)\lambda \geq 0$ for $u \leq t$.

To establish the corresponding lower bound, note that since $\Delta G_s(\lambda) \geq 0$ (see (9)) and $\log(1 + x) - x \geq -x^2/2$ for all $x \geq 0$, we have that
\[
\varphi_t(\lambda) - \varphi_u(\lambda) \geq G_t(\lambda) - G_u(\lambda) - \frac{1}{2} \sum_{u < s \leq t} \Delta G_s(\lambda)^2.
\]
Moreover, again by (9) we see that
\[
0 \leq \Delta G_s(\lambda) \leq \frac{1}{2} v(|\lambda|a)\lambda' \left[ \int_{|x| \leq a} xx'\nu(\{s\}, dx) \right] \lambda \leq \frac{1}{2} v(|\lambda|a)^2(|\lambda|a)^2.
\]
Hence,
\[
\frac{1}{2} \sum_{u < s \leq t} \Delta G_s(\lambda)^2 \leq \frac{1}{8} v(|\lambda|a)^2(|\lambda|a)^2 \lambda' \left[ \sum_{u < s \leq t} \int_{|x| \leq a} xx'\nu(\{s\}, dx) \right] \lambda \\
\leq \frac{1}{8} v(|\lambda|a)^2(|\lambda|a)^2 \lambda' \left[ \langle X \rangle_t - \langle X \rangle_u \right] \lambda,
\]
and the required lower bound follows by noting that
\[
G_t(\lambda) - G_u(\lambda) \geq \frac{1}{2} v(-|\lambda|a)\lambda' \left[ \langle X \rangle_t - \langle X \rangle_u \right] \lambda.
\]

To prove Proposition 1 we need the following immediate consequence of Lemma 1.

Lemma 2 Suppose there exists $q \in C[0,\infty)$, a positive-semi-definite matrix $Q$ and an unbounded function $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $\delta > 0, T < \infty$
\[
\lim_{k \to \infty} \frac{1}{h_k} \log \mathbb{P} \left\{ \sup_{0 \leq u \leq T} \left| \frac{(X)_{u_k} - q(u)Q}{h_k} \right| > \delta \right\} < 0.
\] (13)

Then, for every $\lambda \in \mathbb{R}^d$ and $a_k \to 0$ such that $h_k a_k \to \infty$,
\[
\limsup_{k \to \infty} a_k \log \mathbb{P} \left\{ \sup_{0 \leq u \leq T} \left| a_k h_k (\lambda/\sqrt{h_k a_k}) - \frac{1}{2} q(u)\lambda^2 \right| > \delta \right\} = -\infty.
\] (14)
Hence, suffices to show that for every \( i \) \( q \) rate function follows from [8, (2.4)] taking there the LDP in Skorohod topology follows from [8, Theorem 2.2] and the explicit form (15) of the

Proof: Use (10), noting that \( a_k = \frac{1}{h_k}(a_k h_k) \) with \( a_k h_k \to \infty \), and that \( \lim_{k \to \infty} v(|\lambda|/\sqrt{a_k h_k}) = \lim_{k \to \infty} v(|\lambda|/\sqrt{a_k h_k}) = 1 \), while \( \sup_{u \in [0,T]} |q(u)| < \infty \). □

The next lemma is a simple application of the results of [8], relating (14) with the LDP (with speed \( a_k \)) of \( \sqrt{\frac{a_k}{h_k}} X_{k} \).

**Lemma 3** When (14) holds, the processes \( \left\{ \sqrt{\frac{a_k}{h_k}} X_{k}, k > 0 \right\} \) satisfy the LDP in \( (D(\mathbb{R}^d), \mathcal{B}) \) with speed \( a_k \) and the good rate function

\[
I(\phi) = \begin{cases} 
\int_0^\infty \Lambda^* \left( \frac{d\phi}{dq} \right) q(dt) & \phi \ll q, \quad \phi(0) = 0 \\
\text{otherwise} & 
\end{cases}
\]

(\( q \in M_+ (\mathbb{R}^d) \) is the continuous locally finite measure on \( (\mathbb{R}_+, \mathcal{B}_{\mathbb{R}_+}) \) such that \( q([0,t]) = q(t) \)).

Proof: For each sequence \( k_n \to \infty \) we shall apply [8, Theorem 2.2] for the local martingales \( \sqrt{a_{k_n}/h_{k_n}} X_{k_n,t} \) replacing \( h_{\alpha} \) throughout by \( a_{k_n} \). Cramer’s condition [8, (2.6)] is trivially holding in the current setting, while for \( G_t(\lambda) = \frac{1}{3} q(t)^2 Q \lambda \) the condition (sup \( \mathcal{E} \)) of [8, Theorem 2.2] is merely (14). Moreover, for this \( G_t(\lambda) \) the condition [8, (G)] is easily shown to hold (as \( H_{s,t}(\cdot) \) is then a positive-definite quadratic form on the linear subspace \( \text{dom} H_{s,t} \) for all \( s < t \)). Thus, the LDP in Skorohod topology follows from [8, Theorem 2.2] and the explicit form (15) of the rate function follows from [8, (2.4)] taking there \( g_t(\lambda) = \frac{1}{3} \lambda^2 Q \lambda \). Suppose \( I(\phi) < \infty \). Then, \( \phi \ll q \) and since \( q \in C(0, \infty) \) it follows that \( \phi \in C(\mathbb{R}^d) \). Hence, by [8, Theorem C] we may replace the Skorohod topology by the stronger locally uniform topology on \( D(\mathbb{R}^d) \). □

Proposition 4 follows by combining Lemmas 2 and 3 with the next lemma.

**Lemma 4** If \( h_t \) is regularly varying of index \( \alpha > 0 \) then (5) implies that (13) holds for \( q(u) = u^\alpha \).

Proof: Fix \( T < \infty \) and \( \delta > 0 \). Since \( h_t \) is regularly varying of index \( \alpha > 0 \), clearly \( h_{u_k}/h_k \to u^\alpha \) for all \( u \in (0, \infty) \) (c.f. [2, page 18]). Take \( \epsilon > 0 \) small enough for \( \sup_{0 \leq s \leq [T/\epsilon]} |q(s+\epsilon) - q(s)| \leq \delta/(3|Q|) \), and \( k_0 < \infty \) such that \( \sup_{0 \leq s \leq [T/\epsilon]} |h_{i s k}/h_k - q(s)| \leq \delta/(3|Q|) \) whenever \( k \geq k_0 \) (note that \( q(0) = 0 \)).

The monotonicity of \( \langle X \rangle_{i k} \) in \( t \) (and \( \langle X \rangle_0 = 0 \)) implies that for all \( k \geq k_0 \)

\[
\left\{ \sup_{u \in [0,T]} \left| \frac{\langle X \rangle_{i k}}{h_k} - q(u) Q \right| > \delta \right\} \leq \left\{ \sup_{1 \leq s \leq [T/\epsilon]} \left| \langle X \rangle_{i s k} - h_{i s k} Q \right| > \frac{1}{3} \delta h_k \right\}.
\]

Hence, suffices to show that for every \( i \in \mathbb{N} \), \( \epsilon > 0 \)

\[
\limsup_{k \to \infty} \frac{1}{h_k} \log \mathbb{P} \left\{ \left| \langle X \rangle_{i k} - h_{i k} Q \right| > \frac{1}{3} \delta h_k \right\} < 0.
\]

Since \( h_{i k} / h_k \to q(\epsilon) \in (0, \infty) \) this inequality follows from (5). □
3 Proof of Corollary 1

(a) Assume first that $h_t$ is regularly varying of index 1. Given Proposition 1, this case is easily settled by applying the contraction principle for the continuous mapping $\phi \mapsto \phi(1) : D[\mathbb{R}^d] \to \mathbb{R}^d$. In the general case, we take without loss of generality $h_t \in D(\mathbb{R}_+)$ strictly increasing of bounded jumps (see Remark 1). Let $\sigma_s = \inf\{t \geq 0 : h_t \geq s\}$ and $g_s = h_{\sigma_s}$. Note that $g_s - s$ is bounded, while (5) holds for the locally square integrable martingale $Y_s = X_{\sigma_s}$ of bounded jumps and the regularly varying function $g_s$ of index 1. Consequently, $\{g_s^{-1/2}Y_s\}$ satisfies the MDP with the critical speed $1/g_s$ and the good rate function $\Lambda^*(\cdot)$. Since $h_t$ is strictly increasing and unbounded it follows that $\sigma(\mathbb{R}_+) = \mathbb{R}_+$. Hence, this MDP is equivalent to the MDP for $\{h_k^{-1/2}X_k\}$.

(b) As in part (a) above suffices to prove the stated MDP for $h_t$ regularly varying of index 1. Applying the contraction principle for the continuous mapping $\phi \mapsto \sup_{s \leq 1} \phi(s)$ we deduce the stated MDP from Proposition 1. Since $\Lambda^*(v) = v^2/(2Q)$, the good rate function for this MDP is (c.f. (6))

$$I(z) = \frac{1}{2Q} \inf_{\phi \in AC_0 : \sup_{s \leq 1} \phi(s) = z} \int_0^\infty \dot{\phi}(s)^2 ds \geq \frac{z^2}{2Q}.$$  

Clearly, $\phi(0) = 0$ implies that $I(z) = \infty$ for $z < 0$, while taking $\phi(s) = (s \wedge 1)z$ we conclude that $I(z) = z^2/(2Q)$ for $z \geq 0$.

References

[1] K. Azuma (1967): Weighted sums of certain dependent random variables, *Tohoku Math. J.* 3, 357–367.

[2] N.H. Bingham, C.M. Goldie and J.L. Teugels (1987): *Regular Variation* Cambridge Univ. Press.

[3] A. Dembo and O. Zeitouni (1993): *Large Deviations Techniques and Applications* Jones and Bartlett, Boston.

[4] D. Freedman (1975): On tail probabilities for martingales, *Ann. Probab.* 3, 100–118.

[5] J. Jacod and A.N. Shiryaev (1987): *Limit theorems for stochastic processes* Springer-Verlag, Berlin.

[6] D. Khoshnevisan (1995): Deviation inequalities for continuous martingales, (preprint)

[7] R. Sh.Liptser and A.N. Shiryaev (1989): *Theory of Martingales* Kluwer, Dorndrecht.

[8] A. Puhalskii (1994): The method of stochastic exponentials for large deviations, *Stoch. Proc. Appl.* 54, 45–70.

[9] A. Rackauskas (1990): On probabilities of large deviations for martingales, *Liet. Matem. Rink.* 30, 784–794.