Experimental demonstration of robust quantum steering

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We analyse and experimentally demonstrate quantum steering using criteria based on generalised entropies and criteria with minimal assumptions based on so-called dimension-bounded steering. Further, we investigate and compare their robustness against experimental imperfections such as misalignment in the shared measurement reference frame. Whilst entropy based criteria are robust against imperfections in state preparation, we demonstrate an advantage in dimension-bounded steering in the presence of measurement imprecision. As steering with such minimal assumptions is easier to reach than fully non-local correlations, and as our setting requires very little trust in the measurement devices, the results provide a candidate for the costly Bell tests while remaining highly device-independent.

Introduction. — Communication protocols based on quantum information have come a long way from abstract theoretical models to everyday technological applications. Some of the most celebrated achievements are undoubtedly the randomness generators \textsuperscript{1} and quantum key distribution \textsuperscript{2,3}. Such protocols demonstrate a quantum advantage compared to their classical counterparts by utilizing non-classical resources such as coherence, entanglement, and measurement incompatibility. Their verification is typically performed in a device-dependent manner, which implies trust in the measurement devices in the laboratory to perform precisely as their manufacturer promises. Such assumption naturally opens up a loophole.

The most rigorous way to verify non-classical resources without such loophole is a Bell test \textsuperscript{4,5}. Bell tests are fully-device independent in that they treat measurement devices as black boxes. However, the realization of such tests is experimentally challenging and extremely resource intensive, despite today’s technology. To overcome these difficulties, a relaxation of Bell tests entitled steering in the presence of measurement imprecision. As steering with such minimal assumptions is easier to reach than fully non-local correlations, and as our setting requires very little trust in the measurement devices, the results provide a candidate for the costly Bell tests while remaining highly device-independent.

and dimension-bounded steering inequalities \textsuperscript{15}. Entropic steering criteria allow for the detection of a large class of two-qubit states, can be extended to high dimensional systems, and have been reported to have a detection advantage over linear steering inequalities in terms of noise robustness \textsuperscript{12}. Albeit these advantages, all protocols for quantum steering are based on one party being trusted whilst the other is untrusted. Establishing trust can be very resource intensive, hence protocols making such assumptions redundant are advantageous. An example of such a protocol is dimension-bounded steering which allows for detection of steering from correlations with minimal assumptions about Bob’s measurement device \textsuperscript{15}. Whilst assumptions about the Hilbert space dimension that Bob’s devices act on remain, none are made about the exact form of Bob’s measurements. This brings steering protocols much closer to Bell tests.

As the source of imperfections in our protocols, we consider misalignment of the shared reference frame, a nontrivial experimental challenge that has to be addressed in quantum communication protocols and quantum fiber networks. Our analysis and experimental results show that the generalised entropic criteria are less robust against misalignments in the common reference frame than dimension-bounded steering. Surprisingly, the more device-independent criteria, that require only trust in Bob’s measurement to act on a qubit, turn out to be highly robust. We perform the analysis and experiments in a two-qubit system. We believe that the results are encouraging for theoretical as well as practical developments in entanglement-based quantum communication protocols beyond scenarios considered here and especially beyond the standard semi-device independent paradigm.

Steering inequalities from general entropic uncertainty relations. — In a general steering scenario, one assumes that two parties, Alice and Bob, share a quantum state

In this manuscript, we focus on two promising classes of steering criteria: generalised entropic steering criteria based on Shannon, Tsallis, and Renyi entropies \textsuperscript{11,14}
\( g_{AB} \). In each round, Bob receives his part of the shared state and announces randomly chosen measurement settings \( x \in \{1, \ldots, n\} \). Then Alice declares her corresponding measurement outcome \( a \) on her system which could be either a fabricated result or a genuine measurement outcome. Over many runs, Bob can obtain the correlation matrix, which is the joint probability distribution of the measurement outcomes, and test if it can be explained by a local hidden state (LHS) model \([3]\). To define a local hidden state model we consider a state assemblage of Bob’s unnormalised states conditioned on Alice’s measurement \( x \) and outcome \( a \) as given by \( g_{a|x} := \text{tr}_A [A_{a|x} \otimes \mathds{1} g_{AB}] \), where \( \{A_{a|x}\}_a \) is a positive-operator valued measure for each \( x \), i.e. \( \sum_a A_{a|x} = \mathds{1} \), and \( A_{a|x} \geq 0 \) for each \( a \), \( \mathds{1} \) is the identity, and \( g_{AB} \) is the state shared between Alice and Bob. The state assemblage allows a local hidden state model whenever

\[
\eta_{a|x} = \sum_{\lambda} p(\lambda) D(a|x, \lambda) \sigma_\lambda, \tag{1}
\]

where \( \{\rho(\lambda)\sigma_\lambda\}_\lambda \) is a state ensemble on Bob’s side and \( D(a|x, \lambda) \) is a deterministic probability distribution for each \( x \) and \( \lambda \). If such an LHS model does not exist, Bob can conclude that the shared state \( \rho_{AB} \) is entangled. In this way, Alice can steer his system via her measurements. Local hidden state models can be also defined on the level of correlations in which case we say that Alice can steer Bob if the following decomposition of the correlation table \( \{p(a, b|x, y)\} \) is not possible

\[
p(a, b|x, y) = \sum_{\lambda} p(\lambda)p(a|x, \lambda)p^O(b|y, \lambda), \tag{2}
\]

where \( p(\cdot|x, \lambda) \) are classical probability distribution and \( p^O(\cdot|y, \lambda) \) refers to a distribution that originates from Bob’s measurements on a local state \( \sigma_\lambda \). It is straightforward to check that whenever Bob can perform local tomography, the definitions are equivalent. Here, our criteria are based on correlation tables, but in order to introduce dimension-bounded steering, we use the assemblage based definition. It should be mentioned that despite using the assemblages in our theoretical considerations, the criteria can be nevertheless evaluated from correlations. General entropic uncertainty relations provide a state-independent tool to construct steering criteria. Two independent groups \([12, 13]\) proposed such criteria based on the Tsallis entropy \([16, 17]\) and Rényi entropy \([18]\), respectively. The former is parametrised by \( q > 1 \), and is given by

\[
S_q^e(\mathcal{P}) = -\sum_i p^\delta_i \ln_q(p_i), \tag{3}
\]

for a general probability distribution \( \mathcal{P} = (p_1, \ldots, p_n) \), where the \( q \)-logarithm is defined as \( \ln_q(x) = (x^{1-q} - 1)/(1 - q) \). In the limit of \( q \to 1 \), this entropy converges to the well-known Shannon entropy \([19]\). In the following we restrict our attention to the case where all outcomes are labelled by \( \pm 1 \), and Bob’s measurements correspond to a set of orthogonal spin directions on the Bloch sphere such that \( B_m = b_m \otimes \tilde{\sigma} \) with \( b_m \cdot b_{m'} = \delta_{m m'} \). Here \( \tilde{\sigma} = (\tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3) \) is the vector of Pauli operators in some fixed basis.

If the entropy given by Bob’s \( m \) measurement settings and its \( B_m \) outcomes can be bounded by the Tsallis entropic uncertainty (EUR) bound, \( \sum_m S_q^e(B_m) \geq C_B(q, m) \) (for more discussions about these bounds, see Ref. \([20]\)), it is possible to construct steering inequalities in the form of

\[
S^e_m = C_B(q, m) - \frac{1}{q - 1} \left[ \sum_{m} \left( 1 - \frac{(p_{ab}^{(m)})^q}{p_{a}^{(m)} p_{b}^{(m)}} \right) \right] \leq 0, \tag{4}
\]

Here, \( p_{a}^{(m)} \) is the probability of Alice and Bob for outcome \((a, b)\) when measuring \( A_m \otimes B_m \), and \( p_{a}^{(m)} \) are the marginal outcome probabilities of Alice’s measurement \( A_m \).

If the quantity \( S^e_m \) is positive then the system is steerable. This form of the steering criteria is not restricted to the case of two-level systems \([12]\) and allows for evaluation of any set of measurements, as long as they have a valid entropic uncertainty bound.

Alternatively, generalized entropic steering criteria can be constructed using the Rényi entropy \([13, 18]\) which is given by

\[
H_r(\mathcal{P}) = \frac{1}{1 - r} \ln \left( \sum_i p^\delta_i \right). \tag{5}
\]

In the limit of \( q \to 1 \), Rényi entropy converges to the Shannon entropy. The entropies of conditional measurement outcomes of Alice’s and Bob’s measurements, \( A_m \) and \( B_m \), can be bounded by a LHS model as \([13]\)

\[
H_r(B|A) + H_s(B|A2) \geq R_B(2), \tag{6}
\]

with the entropy of order \( r, s \geq 1/2 \), such that \( r^{-1} + s^{-1} = 2 \). If \( r \) and \( s \) fulfill these conditions, the bound \( R_B(2) \) with \( m = 2 \), independent of the order, trivialises to the EUR bound for the Shannon entropy \([21]\).

Similar to Eq. \([4]\), the Rényi entropic steering parameter is

\[
\mathcal{H}_2^{(r,s)} = R_B(2) - H_r(B|A) - H_s(B|A) \leq 0. \tag{7}
\]

Note that Eq. \([7]\) only holds for the two-measurement settings scenario, whereas the Tsallis entropic steering criteria do not have such restriction.

**Dimension-bounded steering.**— Bringing practical semi-device independent protocols closer to fully-device independent protocols imposes serious experimental efforts. Promising candidates to overcome these limitations are dimension-bounded steering inequalities \([15]\).
Instead of trusting Bob’s measurement devices, one can simply make the assumption that they act on a qubit system. Interestingly, one of the most basic steering protocols – three orthogonal qubit measurements acting on a singlet state – was shown to have an unaffected noise tolerance when decreasing the trust on Bob’s side. Demonstration of steering in such scenarios is referred to as dimension-bounded steering.

The technique for building dimension-bounded steering tests is a three-step process: First, one notices that any unsteerable state assembly can be prepared with a separable state. For example, consider an unsteerable assemblage \( g_{\alpha \beta} \) with two inputs, two outputs, and a local hidden state model given by the operators \( \{\omega_{ij}\} \). A separable state that can be used to prepare such assemblage is \( \Sigma_{AB} := \sum_{i,j} Z_{ij} \otimes \omega_{ij} \), where \( \Sigma_{AB} \) is a separable quantum state [15]. However, operators corresponding to an entangled state \( \Sigma_{AB} \) exist whilst simultaneously satisfying the elimination criterion. Importantly, this allows for detection of entanglement via extra constraints for the operators \( Z_{ij} \). In our example

\[
\Sigma_{AB} := \sum_{i,j} Z_{ij} \otimes \omega_{ij} = Z_{++} \otimes \varrho_{+1} + Z_{--} \otimes \varrho_{+2} + Z_{-+} \otimes \varrho_{-} + (Z_{++} - Z_{--} - Z_{-+} + Z_{+-}) \otimes \omega_{++}, \tag{8}
\]

with \( \varrho_{-} = \varrho_{+1} - \varrho_{+2} \) and the elimination of \( \omega_{++} \), because of \( Z_{++} - Z_{--} - Z_{-+} + Z_{+-} = 0 \).

For any unsteerable state assembly and any set of operators satisfying this elimination criterion, the operator \( \Sigma_{AB} \) is a separable quantum state [15]. However, operators corresponding to an entangled state \( \Sigma_{AB} \) exist whilst simultaneously satisfying the elimination criterion. Importantly, this allows for detection of entanglement via the swap entanglement witness. This provides a method of mapping steering problems into problems of entanglement detection, for which there exist dimension-bounded techniques.

Although the state \( \Sigma_{AB} \) is rather abstract, its correlations can be connected to those of the original steering experiment [6]. The entanglement of such states can be witnessed from the steering data in a dimension-bounded manner. In our experiments the relevant criterion is evaluated through the data matrix \( D_{xy} = \operatorname{tr}[G_y \otimes B_x \Sigma_{AB}] \), where \( B_y = M_{+1y} - M_{-1y} \) with \( M_{+1y} \) being Bob’s measurement operators and \( G_y \) are orthonormal Hermitian operators. The determinant of the data matrix can be used to lower bound the trace norm of a correlation matrix, i.e., a quantity for which an upper bound is known for separable states. This leads to the dimension-bounded steering inequality (for details see [15])

\[
|\det D| \leq \frac{1}{\sqrt{d_A}} \left( \frac{\sqrt{2d_A} - 1}{m\sqrt{d_A}} \right)^m, \tag{9}
\]

where \( m \) is the number of Bob’s measurements and \( d_A \) is the dimension of the chosen operators \( Z_{ij} \). Importantly, in our scenarios the data matrix is proportional to the correlation matrix (with factor \( 1/\sqrt{2} \)) with an additional factor of \( 1/\sqrt{m} \) for all terms including Alice’s measurements [15].

In the scenario of two-qubits systems (\( d_A = d_B = 2 \)), we can define a steering parameter for the dimension-bounded steering, where Eq. (9) reduces to

\[
DB_m = |\det D| - \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{m}} \right)^m \leq 0. \tag{10}
\]

**Entropic and dimension-bounded steering using mutually unbiased bases.** — In general, protocols for testing or exploiting quantum correlations assume mutually unbiased (MUB) measurements and a common reference frame between two parties. Their role has recently been investigated using a steering inequality that allows for deterministic violation for a larger class of states [10]. Here we implement the same framework for steering criteria based on EURs and dimension-bounded steering, and investigate the robustness of MUB to noise and the role of the number of measurement settings using a subset of data from [10].

In this framework, we consider that Alice and Bob share a two-qubit state, i.e., a Werner state — \( \varrho = \mu|\psi>_S\langle\psi| + \frac{1-\mu}{4} I_4 \) — which is a probabilistic mixture of a maximally entangled singlet state \( |\psi>_S \) with a symmetric noise state parametrized by the mixing probability \( \mu \in [0; 1] \) [22]. Firstly, we limit Bob to fixed MUBs for his measurements (Fig. 1), whilst Alice chooses MUBs that can be rotated with respect to the shared reference direction with Bob. Although Alice’s and Bob’s measurement directions will lie in the same plane, their relative orientation (which we denote as \( \alpha \)) within this plane may be unknown (Fig. 1).

For a maximally aligned shared reference direction, the measurements have to lie in the plane, e.g., spanned up by \( \sigma_x - \sigma_z \), corresponding to an angle of \( \Phi = 0^\circ \) between Alice’s and Bob’s measurement planes. Further, we consider the case when Alice and Bob do not share the same reference direction and Alice’s reference plane spanned up by her measurements is tilted by \( \Phi \neq 0^\circ \) (Fig. 1).

In order to verify how these rotations of Alice’s MUBs affect the detection of steering, we apply her measurement settings \( A_m = \frac{1}{2}(1 \pm \tilde{u}_m \cdot \vec{\sigma}) \) on the shared state, where \( \tilde{u}_m \in \mathbb{R}^3 \) depends on Alice’s measurements orientations (\( \alpha \) and \( \Phi \)) on the Bloch sphere.

We test our steering protocol considering the case of minimal set size \( m = 2 \) MUBs on Alice’s side and Bob’s side, where we have the bounds \( C_B(q, 2) = \ln_q(2) \) [20].
and \( R_B(2) = \ln(2) \) \(^{23}\) for the Tsallis and Rényi steering criteria, respectively. Then Eq. (4) simplifies to

\[
S_2^{(q)} = \frac{1}{(1-q)} \left[ 1 + 2(1-q) - f_q(\mu \cos \alpha) \right] - f_q(\mu \cos \Phi \cos \alpha),
\]

with \( f_q(x) := \left( \frac{1-x}{2} \right)^q + \left( \frac{1+x}{2} \right)^q \), and Eq. (7) results in

\[
H_2^{(r,s)} = \ln(2) - \frac{r}{1-r} \ln[f_r(\mu \cos \alpha)]^{1/r} - \frac{s}{1-s} \ln[f_s(\mu \cos \Phi \cos \alpha)]^{1/s}.
\]

The most interesting scenario for the Rényi entropic steering criteria is the case where \( r = 1/2 \) and \( s = \infty \) \(^{24}\), which applied in this example gives the following Rényi steering parameter

\[
H_2^{(1/2,\infty)} = - \ln[1 + \sqrt{1 - \mu^2 \cos^2 \alpha}] + \ln \left[ 1 + \mu |\cos \Phi \cos \alpha| \right].
\]

Interestingly, for this class of states and set of measurements, the Tsallis steering parameter, \( S_2^{(2)} \), detects steerability for the same range of parameters of \( H_2^{(1/2,\infty)} \), i.e. both steering parameters are positive if \( \mu > 1/\cos \alpha \sqrt{1 + \cos^2 \Phi} \). This shows the equivalence of both criteria for our case, given that the optimal parameters, \( q = 2 \), \( r = 1/2 \), and \( s = \infty \), are used. Furthermore, the constraints on Bob’s side reduce Eq. (10) to

\[
DB_2 = \frac{1}{8\sqrt{2}} (2\mu^2 |\cos \Phi| - 1).
\]

The steering protocol based on Eqs. (11)–(14) is dependent on \( \Phi \) and therefore rotationally variant in the case of two MUBs per site. Further, the entropic criteria are limited to specific misalignment (\( \alpha \)) within the measurement plane. Whilst deviations of Alice’s measurement directions will affect the detection of steering, we will show the robustness of entropic criteria for some specific cases and compare it to the dimension-bounded criterion.

Experimental details and results.— We implemented the steering protocols using a high-efficiency spontaneous parametric down-conversion (SPDC) source (Fig. 1d). This source, mounted in a Sagnac ring interferometer \(^{25, 26}\), consists of a 10 mm-long periodically poled potassium titanyl phosphate (ppKTP) crystal pumped bidirectionally by a 410 nm fiber-coupled continuous-wave laser with an output power (after fiber) of 2.5 mW. The generated state is verified using quantum state tomography \(^{27}\) at several stages throughout the experiment—in each case, we achieved a fidelity of ca. 98% with the singlet state \( |\Psi^-\rangle \). Alice and Bob’s measurement directions and projective measurements are implemented by rotating the QWPs and HWPs in front of polarising elements together with coincidence detections. The steering parameter and its error are calculated from the observed correlations. The error \( \Delta S^{(q)} = \sqrt{\langle \Delta S_{\text{obs}} \rangle^2 + \langle \Delta S_{\text{stat}} \rangle^2} \) consists of a systematic and a statistical error due to small imperfections in Bob’s measurement settings and Poissonian statistics in photon counting, respectively. An adversarial scenario and therefore time ordering of events were not considered in this protocol. The experimental data used here were collected in a previous experiment and some of it was used, in the context of different steering results, in Ref. \(^{10}\). We investigated the steering protocol using two MUBs aligned along \( \sigma_z \) and \( \sigma_x \) on Alice’s and Bob’s side (Fig. 2a,c) (blue and red). We successfully violated the steering inequalities (Eqs. (11), (13) and (14) with

![Figure 1: The Poincaré (Bloch) spheres (a-c) contain vectors showing one of the eigenstates of the two relevant directions (blue and red) in the experiments we performed. (a) Alice’s directions, in the case where Alice and Bob share a reference direction. We test the robustness of our inequalities to rotations in the plane (yellow), as the blue and red settings are rotated through 90° in steps (blue and red dots). \( \Phi = 0° \) denotes the fact that the plane is not tilted with respect to Bob. (b) Alice’s directions for \( m = 2 \) (blue and red dots) when her plane of measurement directions is tilted by \( \Phi = 30° \) and the settings are rotated in that plane, whilst maintaining local orthogonality. (c) Bob uses the same two measurement directions in each experiment. (d) The experiment consisted of entangled photon pair generation at 820 nm via SPDC in a Sagnac interferometer constructed of a polarizing beam splitter (PBS), two mirrors (M), a dual-coated half-wave plate (HWP), and a periodically poled KTP (ppKTP) crystal. Different measurement settings are performed by rotating HWP and quarter-wave plates (QWP) relative to the PBS. Long pass (LP) filters and an additional bandpass filter in Bob’s line, remove 410 nm pump photons copropagating with the 820 nm photons before photons are coupled into single-mode fibers and detected by single photon counting modules and counting electronics.](image-url)
$S_2^{(1)} \exp = 0.524 \pm 0.008$ (criterion based on the Shannon entropy), $S_2^{(2)} \exp = 0.433 \pm 0.004$, $H_2^{(1/2, \infty)} \exp = 0.486 \pm 0.008$ and $DB_2 = 0.076 \pm 0.002$. It should be noted that although the entropic steering criteria allow for stronger violation of the classical bound than the dimension-bounded criteria, the amounts of violation are, of course, not comparable with one another.

Next we increased the misalignment by tilting to angle $\Phi = 30^\circ$ whilst maintaining MUBs (Fig. 1b). Our experimental results demonstrate steering for $S_2^{(1)}$, $S_2^{(2)}$, and $H_2^{(1/2, \infty)}$ for $\alpha < 31.2^\circ$ and $\alpha < 39^\circ$, respectively (there is no difference between $S_2^{(2)}$ and $H_2^{(1/2, \infty)}$ in this scenario).

Finally, we investigated the case of no shared reference direction between Alice and Bob which corresponds to $\Phi = 90^\circ$. Although they maintain their MUBs for measurement on each side, the lack of reference makes it impossible for Alice to steer Bob’s state even for the dimension-bounded steering criterion. Our investigation shows that in the presence of misalignment entropic steering criteria lose their advantage over dimension-bounded steering. Thus, a detection method with fewer assumptions performs better.

Further, we extend the protocol to three MUBs per site and discuss the details in the Supplemental Material. Here, entropic criteria based on Rényi entropy are not valid as they are restricted to two MUBs. The robustness of the Tsallis entropic criteria is improved when considering a triad of measurements for each party, while the dimension-bounded steering becomes completely rotationally invariant.

Conclusions. — We have experimentally demonstrated steering protocols based on generalized entropic criteria and dimension-bounded steering and discussed their robustness to reference-frame misalignment. For two measurement settings per side, we showed that these steering inequalities can be violated using a sufficiently entangled state. Further, they showed robustness to misalignment of their measurement directions. The robustness with respect to such reference frame rotations makes dimension-bounded steering appealing for future semi-device independent communication protocols. Further, we demonstrate the equivalence of entropic criteria based on Tsallis and Rényi entropies with optimized parameters, i.e. $q = 2$, $r = 1/2$, and $s = \infty$ respectively. The Tsallis criteria are preferable as it allows the protocol to be extended to three measurement settings per side, providing greater noise robustness for the protocol. This work utilizes the advantages of quantum steering and brings it closer to desired but experimentally resource intensive Bell tests. Another interesting avenue to experimentally investigate is the steerability of higher dimensional states via generalized entropic criteria and dimension-bounded steering. Moreover, experiments involving multipartite entropic steering [20–22] seems to be a promising case of interest.

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SUPPLEMENTAL MATERIAL

Three-measurement settings for each party

Here, we consider the case where Alice and Bob each choose a triad of mutually orthogonal directions, i.e., $m = 3$. In this scenario, we focus on the Tsallis entropic steering criteria and the dimension-bounded steering as the Rényi entropic steering criteria cannot be implemented for more than two measurements per site. Then the entropic steering inequality is given by

$$S_3^{(q)} = \frac{1}{1-q} \left[ 1 + 2^{(1-q)} f_q(\mu \cos \Phi) - f_q(\mu \cos \alpha) \right] \leq 0,$$

with Tsallis EUR bounds given by $C_B(q,3) = 2 \ln_q(2)$, while the dimension-bounded steering is

$$DB_3 = \frac{1}{12} \left( \mu^3 - 1 \right) \leq 0.$$

Further, we considered the same misalignment scenarios by $\alpha$ and $\Phi$ (Fig. 3). First, Alice’s and Bob’s MUBs lie within the same measurement plane ($\Phi = 0^\circ$) and are rotated by angle $\alpha$. Whilst the Shannon inequality for $q \rightarrow 1$ (Eq. [15]) provides a greater violation of the bound in such scenario at $\alpha = 0^\circ$, the inequality for $q = 2$ allows for demonstration of steering for higher values of $\alpha$ than the Shannon inequality, with $\alpha < 80^\circ$ and $\alpha < 75^\circ$, respectively. For $\Phi = 30^\circ$ the inequality for $q = 2$ can be violated for $\alpha < 66^\circ$ whilst the $q \rightarrow 1$ inequality can only tolerate $\alpha < 56^\circ$. A complete loss of reference direction ($\Phi = 90^\circ$) does not allow for any demonstration of steering using three MUBs (Fig. 3). Interestingly, dimension-bounded steering for three-measurements per site is rotationally invariant, as one can see in Eq. (16), and detects steering in all scenarios.

Loss of MUBs

Further, we investigated the scenario where Alice chooses to perform non-orthogonal measurements (NOM), while Bob’s measurements remain orthogonal. First, let’s consider the case of $m = 2$ NOM measurement settings. Bob measures along $\sigma_z$ and $\sigma_x$ and Alice

\[\ldots\]
FIG. 3: Entropic steering parameter as function of the rotation angle $\alpha$ (degrees) in Alice’s measurement plane, for experiments with $m = 3$ measurement directions. Angle $\Phi$ denotes the angle of tilt between Alice’s and Bob’s measurement plane. Error bars are too small to be seen.

measures along $\vec{u}_1 = (0, 0, 1)$ and $\vec{u}_2 = (\sqrt{3}/2, 0, 1/2)$. This leads to the steering criteria

$$S_{2-NOM}^{(q)} = \frac{1}{1-q} \left[ 1 + 2^{(1-q)} - f_q(\mu) - f_q\left(\frac{\sqrt{3}}{3}\mu\right) \right],$$

$$H_{2-NOM}^{(1/2,\infty)} = \ln \left[ 1 + \frac{\sqrt{3}}{2} \mu \right] - \ln \left[ 1 + \sqrt{1 - \mu^2} \right],$$

$$DB_{2-NOM} = \frac{1}{8\sqrt{2}} (\sqrt{3}\mu^2 - 1).$$

The engineered state had a fidelity of 97.2% with the singlet state, corresponding to the Werner state with $\mu = 0.963$, which allows to demonstrate steering with $S_{2-NOM}^{(1)} \exp = 0.304 \pm 0.016$ and $S_{2-NOM}^{(2)} \exp = 0.316 \pm 0.01$, which strongly agrees with the theoretically determined steering values of $S_{2-NOM}^{(1)} \theo = 0.314$ and $S_{2-NOM}^{(2)} \theo = 0.311$, respectively. Concerning the Rényi entropic steering criteria and dimension-bounded criteria, the same agreement between the analytical and experimental results can be verified, where we have $H_{2-NOM}^{(1/2,\infty)} \exp = 0.365 \pm 0.01$ and $H_{2-NOM}^{(1/2,\infty)} \theo = 0.367$, and $DB_{2-NOM} = 0.051 \pm 0.004$ and $DB_{2-NOM} \theo = 0.053$.

Finally, we extend the analysis to the case of three measurements. While Bob measures along $\sigma_x$, $\sigma_y$, and $\sigma_z$, Alice measures along $\vec{u}_1 = (0, 0, 1)$, $\vec{u}_2 = (\sqrt{3}/2, 0, 1/2)$, and $\vec{u}_3 = (1/(2\sqrt{3}), \sqrt{2/3}, 1/2)$. This leads to the steering parameters

$$S_{3-NOM}^{(q)} = \frac{1}{1-q} \left[ 1 + 2^{(2-q)} - f_q(\mu) - f_q\left(\sqrt{2} \mu\right) \right],$$

$$S_{3-NOM}^{(1)} = 0.667, S_{3-NOM}^{(2)} = 0.620, \text{ and } DB_{3-NOM} = 0.021 \text{ for the Shannon and Tsallis entropic steering inequality and the dimension-bounded steering, respectively.}$$

The proposed steering criteria allow for demonstration of quantum steering when using NOM, the most rigorous form of misalignment. Whilst the violation of the bounds is not very strong, it still emphasizes the suitability of the investigated criteria for future applications in e.g. quantum networks.