A SIMPLIFIED STRUCTURE FOR THE SECOND ORDER COSMOLOGICAL PERTURBATION EQUATIONS

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May 5, 2014

Abstract

Increasingly accurate observations of the cosmic microwave background and the large scale distribution of galaxies necessitate the study of nonlinear perturbations of Friedmann-Lemaître cosmologies, whose equations are notoriously complicated. In this paper we present a new derivation of the governing equations for second order perturbations within the framework of the metric-based approach that is minimal, as regards amount of calculation and length of expressions, and flexible, as regards choice of gauge and stress-energy tensor. Because of their generality and the simplicity of their structure our equations provide a convenient starting point for determining the behaviour of nonlinear perturbations of FL cosmologies with any given stress-energy content, using either the Poisson gauge or the uniform curvature gauge.

PACS numbers: 04.20.-q, 98.80.-k, 98.80.Bp, 98.80.Jk

1 Introduction

Over the past ten years there have been a number of significant developments in the theory of nonlinear cosmological perturbations and its applications that have
transformed the area from a quiet backwater into one that is at the forefront of research in cosmology. The main impetus for this resurgence has been the availability of increasingly accurate observations of the cosmic microwave background (CMB) and of the large scale distributions of galaxies. These observations necessitate the study of possible deviations from linearity, for example, non-Gaussianity in the CMB anisotropies, using nonlinear perturbations of FL cosmologies\(^1\) (see for example, Bartolo \textit{et al} (2004a), Bartolo \textit{et al} (2010b), and Pitrou \textit{et al} (2010)).

In order to motivate our work we give a brief overview of recent developments. First, Bartolo and collaborators have written a series of papers that apply second order perturbation theory to physical problems relating to the early universe and to the CMB. Bartolo and collaborators restrict their considerations \textit{ab initio} to purely scalar metric perturbations at linear order and to a flat FL background. Their main theoretical tool is the set of expressions for the perturbed Einstein tensor at second order for this class of perturbed metrics, derived without restricting the gauge. They then introduce the Poisson gauge via gauge-fixing. We refer to Acquaviva \textit{et al} (2003) (equation (4) for the perturbed metric and Appendix A5 for the perturbed Einstein tensor\(^2\)). As regards physical applications they consider the case of a scalar field (Acquaviva \textit{et al} (2003)), a perfect fluid with linear equation of state (Bartolo \textit{et al} (2004b)), and dust and a cosmological constant (Bartolo \textit{et al} (2010a)). They use the expressions for the perturbed Einstein tensor referred to above to obtain the governing equations for the second order perturbations in these cases.

Second, Malik and collaborators (see, Malik (2007), Malik \textit{et al} (2008) and Huston and Malik (2011)) have written a series of papers that apply second order perturbation theory to FL cosmologies with one or more scalar fields. They make the same simplifying assumptions as Bartolo and collaborators, but instead use the uniform curvature gauge. They formulate the governing equations in a way that is suitable for numerical computation. Third, Nakamura (2003) introduced a geometrical method for constructing gauge invariants for linear and nonlinear (second order) perturbations which he later applied to derive the governing equations (see Nakamura (2006) and Nakamura (2007)) effectively using the Poisson gauge. Finally, Noh and Hwang (2004) have given a comprehensive treatment of second order perturbations, with arbitrary spatial curvature and arbitrary stress-energy tensor, including a scalar field and a perfect fluid as special cases. They use the 3 + 1-formulation of the Einstein equations and write the governing equations in a so-called gauge-ready form\(^3\).

Despite the impressive progress that has been made to date the theory of nonlinear cosmological perturbations nevertheless presents challenges. Indeed, the theory is notorious for tedious calculations leading to lengthy quadratic expressions, the so-called \textit{source terms}, that can obscure the overall structure of the equations. There is thus a need for a formulation of the governing equations that is both general and concise and that will hence provide a suitable starting point for future investigations.

\(^1\)We follow the nomenclature of Wainwright and Ellis (1997): a Friedmann-Lemaître (FL) cosmology is a Robertson-Walker (RW) geometry that satisfies Einstein’s field equations.

\(^2\)See also Bartolo \textit{et al} (2004a), equations (104) and (A.36–(A.43).

\(^3\)We refer to Hwang and Noh (2007) for further details and to Hwang \textit{et al} (2012) for an application of the formalism to determining second order perturbations of dust cosmologies.
Motivated by this state of affairs we present a new approach to the derivation of the governing equations for second order perturbations within the framework of the metric-based approach\(^4\). We have designed our approach to be minimal in the sense that we calculate the least number of objects and keep the number of lengthy expressions to a minimum, thereby revealing useful mathematical structure. We rely extensively on the notation and formalism for describing linear perturbations introduced in two recent papers (Uggla and Wainwright (2011) and (2012), hereafter referred to as UW1 and UW2, respectively). In UW1 we gave the linearized Einstein equations with an arbitrary stress-energy tensor in two different but complementary gauge-invariant forms, which we referred to as the Poisson form, associated with the work of Bardeen (1980), and the uniform curvature form, associated with the work of Kodama and Sasaki (1984). In the present paper we derive and cast the perturbed Einstein equations at second order into a convenient form analogous to the linearized equations in UW1 but differing by the addition of source terms.

We begin our derivation by introducing so-called geometric perturbation operators, which provide the first level of structure by decomposing the perturbed Riemann and Einstein tensors into a linear leading order term and a quadratic source term, the latter being present only at second order. A second level of structure is provided by the use of certain linear combinations of the perturbed Einstein tensor components and their derivatives, leading directly to a simple minimal set of governing equations that is convenient for analysis. The use of a shorthand notation for differential operators that occur frequently and reveal key mathematical structure makes the equations much more tractable. A third level of structure is provided by our strategy of decomposing the general source terms into simpler pieces and identifying common expressions, without expanding them fully. Special cases of the source terms that appear in the literature can easily be extracted in a convenient form from our general expressions.

The outline of the paper is as follows. In Section 2 we present 'geometric perturbation operators' for the Riemann and Einstein tensors up to second order, and specialize the key operators to the Poisson gauge and the uniform curvature gauge. Section 3 gives the perturbed Einstein field equations to second order in both the Poisson and uniform curvature forms. We conclude the main part of the paper with a discussion in Section 4. Finally Appendix A contains the definitions and equations needed to provide the background for the results in the main part of the paper, as well as the so-called Replacement Principle to second order, while Appendix B contains the detailed expressions for the Einstein source terms assuming a purely scalar perturbation at linear order, using the uniform curvature gauge.

\(^4\)By this we mean the standard approach to cosmological perturbations in which one formulates the governing equations in terms of gauge-invariant variables associated with the perturbed metric tensor and the perturbed stress-energy tensor, using local coordinates.
# Geometric perturbation operators

## 2.1 Background

Following standard cosmological perturbation theory, we consider a 1-parameter family of spacetimes $g_{ab}(\epsilon)$, where $g_{ab}(0)$, the unperturbed metric, is a RW metric, and $\epsilon$ is referred to as the *perturbation parameter*. We assign physical dimension *length* to the scale factor $a$ of the RW metric and $(\text{length})^2$ to $g_{ab}(\epsilon)$. Then the conformal transformation

$$g_{ab}(\epsilon) = a^2 \bar{g}_{ab}(\epsilon),$$

yields a dimensionless metric $\bar{g}_{ab}(\epsilon)$.

The Riemann tensor associated with the metric $g_{ab}(\epsilon)$ is a function of $\epsilon$, denoted $R_{abcd}(\epsilon)$, as is the Einstein tensor, $G_{ab}(\epsilon)$. The stress-energy tensor of the matter distribution is also assumed to be a function of $\epsilon$, denoted $T_{ab}(\epsilon)$. We include all these possibilities by considering a 1-parameter family of tensor fields $A(\epsilon)$, which we assume can be expanded in powers of $\epsilon$, i.e. as a Taylor series:

$$A(\epsilon) = (0)A + \epsilon (1)A + \frac{1}{2} \epsilon^2 (2)A + \ldots .$$

The coefficients are given by

$$(0)A = A(0), \quad (1)A = \frac{\partial A}{\partial \epsilon} \bigg|_{\epsilon=0}, \quad (2)A = \frac{\partial^2 A}{\partial \epsilon^2} \bigg|_{\epsilon=0}, \quad \ldots ,$$

where $(0)A$ is called the *unperturbed value*, $(1)A$ is called the *first order (linear) perturbation* and $(2)A$ is called the *second order perturbation* of $A(\epsilon)$.

In particular we assume that we can expand the conformal metric $\bar{g}_{ab}(\epsilon)$ in (1) in powers of $\epsilon$,

$$\bar{g}_{ab}(\epsilon) = (0)\bar{g}_{ab} + \epsilon (1)\bar{g}_{ab} + \frac{1}{2} \epsilon^2 (2)\bar{g}_{ab} + \ldots .$$

We label the coefficients as

$$\gamma_{ab} := (0)\bar{g}_{ab} = \bar{g}_{ab}(0), \quad (1)f_{ab} := (1)\bar{g}_{ab} = \frac{\partial \bar{g}_{ab}}{\partial \epsilon} \bigg|_{\epsilon=0}, \quad (2)f_{ab} := (2)\bar{g}_{ab} = \frac{\partial^2 \bar{g}_{ab}}{\partial \epsilon^2} \bigg|_{\epsilon=0},$$

which is consistent with (2b). We refer to $(1)f_{ab}$ as the *first order metric perturbation* and $(2)f_{ab}$ as the *second order metric perturbation*. To simplify the notation, we will denote $(1)f_{ab}$ by $f_{ab}$ when there is no risk of confusion. The conformal background metric $\gamma_{ab}$ and its inverse $\gamma^{ab}$ will play an important role in that they are used to lower and raise indices on perturbed objects.

## 2.2 Perturbation operators for the Riemann and Einstein tensors

We next expand the Riemann and Einstein tensors in the form

$$R_{abcd}(\epsilon) = (0)R_{abcd} + \epsilon (1)R_{abcd} + \frac{1}{2} \epsilon^2 (2)R_{abcd} + \ldots ,$$

$$G_{ab}^c(\epsilon) = (0)G_{ab}^c + \epsilon (1)G_{ab}^c + \frac{1}{2} \epsilon^2 (2)G_{ab}^c + \ldots .$$

We use Latin letters $a, b, \ldots , f$ to denote abstract spacetime indices.
To describe the perturbations of the spacetime geometry up to second order we introduce dimensionless leading order linear operators and quadratic source term operators for the Riemann tensor $R_{abcd}$ and the Einstein tensor $G_{ab} = R_{ac}^{\ cd} bc - \frac{1}{2} \delta_{ab} R_{cd}^{\ cd}$. (5)

We refer to these operators as the geometric perturbation operators and use the notation $R_{abcd}^{\ (\, f)}$ and $G_{ab}^{\ (\, f)}$ for the leading order operators, and $R_{abcd}^{\ (1), (1)} f$, $G_{ab}^{\ (1), (1)} f$ for the source term operators. These operators determine the dependence of the linear and quadratic terms in the Taylor series (4) on the perturbations of the metric, through equations of the following form:

\begin{align}
a^{2(1)} R_{cd}^{\ ab} &= R_{cd}^{\ ab\ (1) f}, \quad a^{2(2)} R_{cd}^{\ ab} = R_{cd}^{\ ab\ (2) f} + R_{cd}^{\ ab\ (1) f, (1) f}, \tag{6a} \\
a^{2(1)} G_{b}^{\ a} &= G_{b}^{\ a\ (1) f}, \quad a^{2(2)} G_{b}^{\ a} = G_{b}^{\ a\ (2) f} + G_{b}^{\ a\ (1) f, (1) f}. \tag{6b}
\end{align}

It is important that there is only one leading order operator for each tensor: the same operator acts on both $(1) f$ and $(2) f$. In Appendix A.2 we derive concise expressions for $R_{cd}^{\ ab\ (r) f}, r = 1, 2$, and $R_{cd}^{\ ab\ (1) f, (1) f}$ (see equations (71) and (72)).

### 2.2.1 Local coordinates and differential operators

To proceed further we need to work in a coordinate frame so that we can calculate time and spatial components separately. We thus introduce local coordinates $x^\mu = (\eta, x^i)$, with $\eta$ being the usual conformal time coordinate for the RW metric $g_{ab}(0)$, and such that the unperturbed conformal metric $\gamma_{ab} := \bar{g}_{ab}(0)$ has components

\begin{align} 
\gamma_{00} = -1, \quad \gamma_{0i} = 0, \quad \gamma_{ij},
\end{align}

where $\gamma_{ij}$ is the metric of a spatial geometry of constant curvature. The function $a = a(\eta)$ is the background cosmological scale-factor, which determines the dimensionless background Hubble scalar $\mathcal{H}$ according to

\begin{align} 
\mathcal{H} = \frac{a'}{a} = aH,
\end{align}

where $H$ is the true background Hubble scalar. Here and elsewhere in this paper $'$ denotes the derivative with respect to $\eta$ of a background function that depends only on $\eta$. As in UW1 we will use the geometric background scalars $\mathcal{A}_G$ and $\mathcal{C}_G^2$ that can be defined in terms of $\mathcal{H}$ by the following equations:

\begin{align} 
\mathcal{A}_G := 2(-\mathcal{H}' + \mathcal{H}^2 + K), \quad \mathcal{A}'_G = -(1 + 3\mathcal{C}_G^2)\mathcal{H} \mathcal{A}_G
\end{align}

(see UW1, equation (42)).

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6Here and elsewhere we use the shorthand notation $(r) f$ for $(r) f_{ab}, r = 1, 2$.

7We use Greek letters to denote spacetime coordinate indices on the few occasions that they occur, and we use Latin letters $i, j, k, m$ to denote spatial coordinate indices, which are lowered and raised using $\gamma_{ij}$ and its inverse $\gamma^{ij}$, respectively.

8Since we assigned $a$ to have physical dimension length, the conformal time $\eta$ and the conformal spatial line-element $\gamma_{ij} dx^i dx^j$ are dimensionless. We choose the $x^i$ to be dimensionless, which implies that the $\gamma_{ij}$ are also dimensionless.
In order to formulate the perturbation equations concisely we have found it helpful to introduce a shorthand notation for certain differential operators that occur frequently and clarify the structure of the equations. The first operator determines the evolution of scalar perturbations when using the Poisson gauge and is defined by

\[ L := \partial^2 + 3 (1 + C^2) \mathcal{H} \partial_\eta + \mathcal{H}^2 \mathcal{B} - (1 + 3 C^2) K, \]  

(10a)

where

\[ \mathcal{B} := \frac{2 H'}{H^2} + 1 + 3 C^2, \]  

(10b)

(see UW1, equation (56)). This operator has the important property that it can be written as the product of two first order differential operators:

\[ L(\bullet) = H L_A L_B (\frac{\bullet}{H}), \]  

(11a)

where

\[ L_A := \partial_\eta + H \mathcal{B}, \quad L_B := \partial_\eta + 2 H, \]  

(11b)

(see UW1, equation (55) and UW2, equation (39)). These operators play a central role, in particular when using the uniform curvature gauge.

We will make extensive use of the second order spatial differential operators defined by

\[ D^2 := \gamma^{ij} D_i D_j, \quad D_{ij} := D(i D_j) - \frac{1}{3} \gamma_{ij} D^2, \]  

(12)

where \( D_i \) denotes covariant differentiation with respect to the spatial metric \( \gamma_{ij} \). We will also use the following shorthand notation

\[ (DA)^2 := (D^k A)(D_k A), \]  

(13)

where \( A \) is a scalar field.

### 2.2.2 Minimal representation of the perturbed Einstein tensor

In deriving the governing equations for linear and second order perturbations in section 3 we follow the approach of UW1 and consider four linear combinations of the components of the perturbed Einstein tensor and their derivatives, which we denote by

\[ \langle r \rangle G_{ij}, \langle r \rangle G, \langle r \rangle G_i, \langle r \rangle G^0_i \rangle, \quad r = 1, 2, \]  

(14)

where

\[ \langle r \rangle G_{ij} := \gamma_{ki} \langle r \rangle G^k_j - \frac{1}{3} \gamma_{ij} \langle r \rangle G^k, \]  

\[ \langle r \rangle G := \langle r \rangle G^0_0 + \frac{1}{3} \langle r \rangle G^k_k, \]  

\[ \langle r \rangle G_i := -D_i \langle r \rangle G^0_0 - 3 \mathcal{H} \langle r \rangle G^0_i, \]  

(15a-c)

\footnote{See the final paragraph of section 3.2 in UW1 for references to the literature where related forms of this operator appear.}

\footnote{Note that \( \langle r \rangle G_{ij} \equiv \gamma_{ki} \langle r \rangle G^k_j \) is not symmetric when \( r = 2 \). We thus compensate by symmetrizing in \( \langle 15a \rangle \), so that \( \langle 2 \rangle G_{ij} \), as defined by that equation, is symmetric.}
and $C^2_G$ is defined by (9). The use of these combinations leads to a convenient form of the governing equations. In particular, in the linear case ($r = 1$) when using the Poisson gauge the expression (15b) leads directly to the evolution equation for the Bardeen potential in terms of the differential operator $L$ in (10), while (15c) leads to the generalized Poisson equation that determines the density perturbation in terms of the Bardeen potential (see UW1 equations (54)). We will see that this pattern is repeated in the nonlinear case.

In order to calculate the Einstein combinations (15) we express them in terms of the perturbations of the Riemann tensor using (5) and (4):

\[
\hat{\hat{G}}_{ij}^{(r)} = \gamma_{k(i}^{(r)} R_{\alpha k|\alpha j)}^{(r)}, \quad (16a)
\]
\[
\hat{G}^{(r)} = -\frac{1}{6}(1 + 3C^2_G) R_{pq}^{pq} - \frac{2}{3}(r) R_{0k}^{0k}, \quad (16b)
\]
\[
\hat{G}^i_i = \frac{1}{2} D_i^{(r)} R_{pq}^{pq} - 3H(r) R_{0k}^{0k}, \quad (16c)
\]
\[
\hat{G}^0_i = (r) R_{0k}^{0k}. \quad (16d)
\]

It follows from (6b) that the dependence of the Einstein combinations (15) on the metric perturbations is given by equations of the form

\[
a^{(1)}G_{ij} = \hat{G}_{ij}^{(1)}(f), \quad a^{(2)}G_{ij} = \hat{G}_{ij}^{(2)}(f) + G_{ij}(f, f), \quad (17a)
\]
\[
a^{(1)}G = G^{(1)}(f), \quad a^{(2)}G = G^{(2)}(f) + G(f, f), \quad (17b)
\]
\[
a^{(1)}G_i = G_i^{(1)}(f), \quad a^{(2)}G_i = G_i^{(2)}(f) + G_i(f, f), \quad (17c)
\]
\[
a^{(1)}G^0_i = G^0_i^{(1)}(f), \quad a^{(2)}G^0_i = G^0_i^{(2)}(f) + G^0_i(f, f), \quad (17d)
\]

where for brevity we have denoted $(^{(1)}f$ by $f$ in the source terms.

### 2.2.3 The leading term operators

We can express the leading order operators in (17) in terms of $R^{ab}_{cd}(f)$ using equations (16):

\[
\hat{G}_{ij}(f) = \gamma_{k(i}^{(r)} R_{0k|\alpha j)}^{(r)}, \quad (18a)
\]
\[
G(f) = -\frac{1}{6}(1 + 3C^2_G) R_{pq}^{pq} - \frac{2}{3} R_{0k}^{0k}(f), \quad (18b)
\]
\[
G_i(f) = \frac{1}{2} D_i^{(r)} R_{pq}^{pq} - 3H R_{0k}^{0k}(f), \quad (18c)
\]
\[
G^0_i(f) = R_{0k}^{0k}(f), \quad (18d)
\]
where \( f \) denotes \( (1)f \) or \( (2)f \). On using equations (71) for \( R_{cd}^{ab}(f) \), in conjunction with (74), we obtain after some manipulation\(^{12}\)

\[
\hat{G}_{ij}(f) = \frac{1}{2}D_{ij}(f_{00} - \frac{1}{3}f_k^k) + \mathcal{L}_B \hat{Y}_{ij}(f) + D^k_{ij}\hat{f}_{j0} - \frac{1}{6}(D^2 + 3K)\hat{f}_{ij},
\]

(19a)

\[
G(f) = -\frac{1}{3}[(\mathcal{L} - C^2_G D^2) f_k^k + (3\mathcal{H}C_A + D^2)(f_{00} - \frac{1}{3}f_k^k)] - \frac{1}{6}(1 + 3C^2_G)D^k_i f_m^m + \frac{2}{3}(\mathcal{L}_B + 3\mathcal{H}C^2_G) D^k f_{k0},
\]

(19b)

\[
G_i(f) = -D_i\left[\frac{3}{2}(D^2 + 3K)f_k^k - \frac{1}{2}D^k m f_m^m\right] + 3\mathcal{H}D^k \hat{Y}_{ik}(f),
\]

(19c)

\[
G^0_i(f) = \frac{2}{3}D_i Y(f) - D^k \hat{Y}_{ik}(f),
\]

(19d)

where

\[
Y(f) := \frac{3}{2}\mathcal{H}f_{00} + \frac{1}{2}D^k f_k - D^k f_{k0}, \quad \hat{Y}_{ij}(f) := \frac{1}{2}D^l f_{ij} - D_{(ij)} f_{00},
\]

(20)

and \( f \) denotes \( (1)f \) or \( (2)f \). These equations are the main result of this subsection and play a central role in our derivation of the perturbation equations.

### 2.2.4 The source term operators

Calculating the Einstein source terms \( \mathcal{G}(f, f) \) in (17) presents a major challenge. Our strategy is to first express them in terms of \( R_{cd}^{ab}(f, f) \) using equations (16):

\[
\hat{G}_{ij}(f, f) = \gamma_{ij} R_{cd}^{ab}(f, f),
\]

(21a)

\[
G(f, f) = -\frac{1}{6}(1 + 3C^2_G) R_{pq}^{pq}(f, f) - \frac{2}{3} R_{0k}^{0k}(f, f),
\]

(21b)

\[
G_i(f, f) = \frac{1}{2}D_i R_{pq}^{pq}(f, f) - 3\mathcal{H} R_{0k}^{0k}(f, f),
\]

(21c)

\[
G^0_i(f, f) = R_{0k}^{0k}(f, f),
\]

(21d)

We then substitute the expression for \( R_{cd}^{ab}(f, f) \) given by equations (72) in the Appendix into (21). This leads to the expressions (73) for the Einstein source terms, each as a sum of simpler terms that can be calculated separately. The constituent terms are given by equations (76)–(78).

### 2.3 Gauge invariants for the Einstein tensor

We associate gauge invariants with the linear and second order perturbations of any tensor using a method pioneered by Nakamura\(^{13}\), which we modify to ensure that the gauge invariants are dimensionless. The process of construction, which involves introducing so-called compensating gauge fields denoted by \( (r)X \), \( r = 1, 2 \), is described briefly in Appendix [A, 3] and is specified by equation (81) for an arbitrary tensor and by (87) for the conformal metric tensor. The gauge invariants associated with the perturbations of the conformal metric tensor and the Einstein tensor by this process, which we refer to as X-compensation, are denoted by \( (r)f_{ab}[X] \) and \( (r)\mathcal{G}_{\hat{Y}_i}[X] \), respectively.

In equation (65) we expressed the perturbations \( (r)G^0_{ab}, r = 1, 2 \), of the Einstein tensor in terms of the perturbations \( (r)f_{ab} \) of the metric tensor in gauge-variant form,

\(^{12}\)We refer to footnote [11] for the (,) and hat notation, and to equations (10)–(12) for the definitions of the differential operators.

\(^{13}\)See for example Nakamura (2007), section 2.3.
using the Einstein geometric operators $G^a_b$ and $G'_b$. By applying the Replacement Principle\footnote{Make the replacements $(r)f \rightarrow (r)f[X], r = 1, 2$, in (6b), in analogy with (83).} to equation (6b) we obtain the following expressions for the Einstein gauge invariants in terms of the metric gauge invariants:

\begin{equation}
(1)G^a_b[X] = G^a_b(1)f, \quad (2)G^a_b[X] = G^a_b(2)f + G^a_b(1)f(1)f, \tag{22}
\end{equation}

where we use the shorthand notation $(r)f \equiv (r)f_{ab}[X], r = 1, 2$.

In this section we derive explicit expressions for the gauge invariants on the right side of (22) for two specific choices of the gauge fields $X$. The starting point is to consider the gauge invariants associated with the first and second order metric perturbations.

### 2.3.1 Gauge invariants for the metric tensor perturbations

To construct dimensionless gauge invariants associated with the linear perturbation $f_{ab}$ of the conformal metric tensor we define \footnote{See equation (87a) in this paper and equation (16) in UW1.}

\[ f_{ab}[X] := f_{ab} - a^{-2}L_{(1)}X(a^2\gamma_{ab}), \tag{23} \]

where the gauge field $(1)X^a$ has to be chosen appropriately. In order to construct a metric gauge field one has to decompose $f_{ab}$ into scalar, vector and tensor modes, which we label as follows (see UW1, equation (18)):

\begin{align}
 f_{00} &= -2\varphi, \tag{24a} \\
 f_{0i} &= D_i B + B_i, \tag{24b} \\
 f_{ij} &= -2\psi_{ij} + 2D_iD_jC + 2D_{(i}C_{j)} + 2C_{ij}, \tag{24c}
\end{align}

where the vectors $B_i$ and $C_i$ and the tensor $C_{ij}$ satisfy:

\[ D^iB_i = 0, \quad D^iC_i = 0, \quad C^i_i = 0, \quad D^iC_{ij} = 0. \tag{24d} \]

We use equations (24) as a model for doing a mode decomposition of $f_{ab}[X]$, using an obvious notation\footnote{For example, $\varphi \rightarrow \Phi[X], B \rightarrow B[X]$, as in equation (88).}.

As shown in UW1 there are two ways to choose $X$ uniquely in terms of $f_{ab}$, leading to the Poisson gauge field $X_p$ and the uniform curvature gauge field $X_c$. The corresponding expressions for $f_{ab}[X]$ are as follows (see UW1, equations (28)–(31)):

\begin{align}
 f_{00}[X_p] &= -2\Phi, \quad f_{0i}[X_p] = B_i, \quad f_{ij}[X_p] = -2\Psi_{ij} + 2C_{ij}, \tag{25a} \\
 \Phi &= \Phi[X_p], \quad \Psi := \Psi[X_p], \tag{25b} \\
 \text{and} \quad f_{00}[X_c] &= -2A, \quad f_{0i}[X_c] = D_i B + B_i, \quad f_{ij}[X_c] = 2C_{ij}, \tag{26a}
\end{align}
where\(^\text{17}\)

\[ A := \Phi[X_c], \quad B := B[X_c]. \] (26b)

In both case the vector and tensor modes satisfy

\[ D^i B_i = 0, \quad C^i_i = 0, \quad D^i C_{ij} = 0. \] (27)

To construct gauge invariants associated with the second order perturbation \(^{(2)}f_{ab}\) we introduce a second gauge field \(^{(2)}X\) and define\(^\text{18}\)

\[ (2)f_{ab}[X] := (2)f_{ab} - a^{-2} L_{(2)X} (a^2 \gamma_{ab}) + \mathcal{F}_{ab}[X], \] (28a)

where

\[ \mathcal{F}_{ab}[X] := -a^{-2} L_{(1)X} (2a^2 f_{ab} - L_{(1)X} (a^2 \gamma_{ab})). \] (28b)

The key point is that one can construct a gauge field \(^{(2)}X_p\) such that \(^{(2)}f_{ab}[X_p]\) has the same form as \(f_{ab}[X_p]\) in (25). In other words, \(^{(2)}f_{ab}[X_p]\) can be obtained by making the substitutions

\[ \Phi \to (2)\Phi, \quad \Psi \to (2)\Psi, \quad B_i \to (2)B_i[X_p], \quad C_{ij} \to (2)C_{ij}[X_p], \] (29)

in (25). Similarly, one can construct a gauge field \(^{(2)}X_c\) such that \(^{(2)}f_{ab}[X_c]\) is obtained by making the substitutions

\[ A \to (2)A, \quad B \to (2)B, \quad B_i \to (2)B_i[X_c], \quad C_{ij} \to (2)C_{ij}[X_c], \] (30)

in (26). Details about the construction of these gauge fields and the expressions for the metric gauge invariants in equations (25), (26), (29) and (30) in terms of the gauge-variant metric perturbations are given in Appendix \(A.4\). It is important to note, however, that these explicit expressions are not required in what follows. All that is required is the general form of \(f_{ab}[X]\) and \(^{(2)}f_{ab}[X]\) in the Poisson gauge and in the uniform curvature gauge, as given by equations (25), (26), (29) and (30).

### 2.3.2 The leading order terms

To obtain gauge-invariant expressions for the leading order terms we simply make the substitutions \(f_{ab} \to f_{ab}[X]\), \(^{(2)}f_{ab} \to (2)f_{ab}[X]\) in equations (19). For the Poisson gauge we use (25) which gives the leading order Einstein operator \(G\) acting on the first order metric perturbation \(f_p \equiv f_{ab}[X_p]\). After some manipulation we obtain\(^\text{19}\)

\[
\begin{align*}
\hat{G}_{ij}(f_p) & = D_{ij} (\Psi - \Phi) - D_i (\mathcal{L}_B B_j) + (\mathcal{L}_B \partial \eta + 2K - D^2) C_{ij}, \\
G(f_p) & = 2 \left[ (\mathcal{L} - C^2 A) D^2 \right] \Psi - \left( \mathcal{H} \mathcal{L}_A + \frac{1}{3} D^2 \right) (\Psi - \Phi), \\
G_i (f_p) & = 2 D_i (D^2 + 3K) \Psi - \frac{3}{2} \mathcal{H} (D^2 + 2K) B_i, \\
G^i_j (f_p) & = -2 D_i (\partial \eta \Psi + \mathcal{H} \Phi) + \frac{1}{2} (D^2 + 2K) B_i.
\end{align*}
\] (31a-31d)

\(^{17}\)In UW1 we introduced the symbols \(A\) and \(B\) for these gauge invariants, following the notation of Kodama and Sasaki (1984), equations (3.4) and (3.5).

\(^{18}\)See equation (27) in Appendix \(A.4\).

\(^{19}\)The identities (B.39b) and (B.39f) in UW1 are needed.
To obtain $\mathbf{G}$ acting on the second order metric perturbation $(^2f)_p \equiv (^2f_{ab}[X_p])$ we simply make the replacements $^{(29)}$ in $^{(31)}$.

Similarly for the uniform curvature gauge we use $(^{26})$, which gives the leading order Einstein operator $\mathbf{G}$ acting on the first order metric perturbation $f_\iota \equiv f_{ab}[X_\iota]$. After some manipulation we obtain $^{20}$

$$
\hat{G}_{ij}(f_\iota) = -D_{ij}(\mathcal{L}_B B + A) - D_{(i} \mathcal{L}_B B_{j)} + (\mathcal{L}_B \partial_\iota + 2K - D^2) C_{ij},
$$

(32a)

$$
G(f_\iota) = 2\mathcal{H}[\mathcal{L}_A A + C_G^2 D^2 B] + \frac{2}{3} D^2(\mathcal{L}_B B + A),
$$

(32b)

$$
G_i(f_\iota) = -2\mathcal{H} D_i (D^2 + 3K) B - \frac{2}{3} \mathcal{H} (D^2 + 2K) B_i,
$$

(32c)

$$
G^0_i(f_\iota) = -2D_i (HA - KB) + \frac{1}{2} (D^2 + 2K) B_i.
$$

(32d)

To obtain $\mathbf{G}$ acting on the second order metric perturbation $(^2f)_\iota \equiv (^2f_{ab}[X_\iota])$ we simply make the replacements $^{(30)}$ in $^{(32)}$.

Equations $(^{31})$ and $(^{32})$ provide the leading order terms in the perturbed Einstein equations $(^{10})$ at linear and second order using the Poisson gauge and the uniform curvature gauge, respectively. For the reader’s convenience we note that the various differential operators are defined by equations $(^{10})$–$(^{12})$.

2.3.3 The source terms

The source terms are given in general in gauge-variant form by $(^{75})$ in conjunction with $(^{76})$–$(^{78})$. They are obtained in gauge-invariant form by simply making the replacement $f_{ab} \rightarrow f_{ab}[X]$ in these equations, in particular with $f_{ab}[X_p]$ for the Poisson gauge and $f_{ab}[X_\iota]$ for the uniform curvature gauge (see equations $(^{25})$ and $(^{26})$). To illustrate our approach we consider a popular special case, namely the Poisson gauge with the metric perturbation restricted as follows:

**Metric assumptions:** The vector and tensor modes of the metric perturbation are zero at first order, i.e.

$$
B_i = 0, \quad C_{ij} = 0,
$$

(33a)

and in addition the scalar mode at first order satisfies

$$
\Phi = \Psi.
$$

(33b)

Subject to these assumptions $(^{25})$ reduces to

$$
f_{00}[X_p] = -2\Psi, \quad f_{0\iota}[X_p] = 0, \quad f_{ij}[X_p] = -2\Psi \gamma_{ij},
$$

(34)

where $\Psi$ is the Bardeen potential. We now make the replacement $f_{ab} \rightarrow f_{ab}[X_p]$ in the expressions $(^{75})$ for the Einstein source terms using the special metric perturbation $(^{34})$. The constituent terms, as given by equations $(^{76})$–$(^{78})$, can be evaluated separately and one finds that many terms are zero. This calculation yields the following simple expressions for the source terms:

$$
\hat{G}_{ij}(f_p, f_p) = 4(D_{ij}\Psi^2 - (D_i\Psi)(D_j\Psi)),
$$

(35a)

$$
G(f_p, f_p) = -\frac{1}{3}(1 + 3C_G^2) \mathcal{R}(\Psi, \Psi) - \frac{8}{3} (D\Psi)^2 - 8\mathcal{H}\mathcal{L}_A \Psi^2,
$$

(35b)

$$
G_i(f_p, f_p) = D_i \mathcal{R}(\Psi, \Psi) + 12\mathcal{H}(\partial_\iota\Psi) D_i\Psi,
$$

(35c)

$$
G^0_i(f_p, f_p) = 8\mathcal{H}D_i\Psi^2 - 4(\partial_\iota\Psi)(D_i\Psi),
$$

(35d)

$^{20}$The terms involving the vector and tensor modes are the same as in the Poisson case.
where
\[ R(\Psi, \Psi) := 2 [3(\partial_\eta \Psi)^2 - 5(D\Psi)^2 + 4(D^2 + 3K)\Psi^2] . \] (35e)

Note that the source terms are quadratic expressions in the Bardeen potential \( \Psi \) and its derivatives. Nakamura (2007) has given the source terms in this case in the form \( G^a_b(f_p, f_p) \). We find complete agreement with his equations (6.13)–(6.16) when they are transformed into the form (35).

In summary, equations (31) and (35) give the constituent geometrical parts in the second order field equations (40) in section 3. They thus provide the foundation for determining second order metric perturbations in the Poisson gauge, subject to the simplifying metric assumption (33).

Finally we note that one can similarly derive expressions for the source terms using the uniform curvature gauge, subject to the simplifying assumption (33a). The resulting expressions are more complicated than (35) and so we give them in Appendix B.

3 Perturbed Einstein equations

3.1 General structure of the governing equations

At zeroth order the non-zero components of Einstein’s field equations are given by
\[ a^2 (0) G_0^0 = -3(\mathcal{H}^2 + K) = -a^2(0) \rho = a^2(0) T_0^0, \] (36a)
\[ a^2 (0) G^i_j = -(2\mathcal{H}' + \mathcal{H}^2 + K) \delta^i_j = a^2(0) p \delta^i_j = a^2(0) T^i_j, \] (36b)

where \( \mathcal{H} \) is given by (8) and \( K \) is the curvature index, defined in equation (67) in Appendix A.1.

The perturbed Einstein equations at linear and second order are given by
\[ (1) G^a_b = (1) T^a_b, \quad (2) G^a_b = (2) T^a_b. \] (37)

Assuming that the background Einstein equations are satisfied we can write these equations in terms of the gauge invariants \( ^{(r)} G^a_b[X] \) and \( ^{(r)} T^a_b[X] \):
\[ (1) G^a_b[X] = (1) T^a_b[X], \quad (2) G^a_b[X] = (2) T^a_b[X], \] (38)

as follows from the definition (81) in Appendix A.3. We can now use (22) to express the left side of these equations in terms of the Einstein operators:
\[ G^a_b(\text{f}^{(1)}) = (1) T^a_b[X], \] (39a)
\[ G^a_b(\text{f}^{(2)}) + G^a_b(\text{f}^{(1)}, \text{f}) = (2) T^a_b[X]. \] (39b)

At this stage we are considering an arbitrary stress-energy tensor, whose components are regarded as primary objects, i.e. they are not constructed from other quantities.

\footnote{See, for example, Mukhanov et al (1992), equation (4.2), noting the difference in signature. We use units \( c = 1 \) and \( 8\pi G = 1 \), where \( c \) is the speed of light and \( G \) is the gravitational constant.}
as in the case of the Einstein tensor. Before continuing we note that equations (39) correspond to equations (2.49) and (2.50) in Nakamura (2007). \[\text{22}\]

We next consider the combinations of equations (39) corresponding to the combinations of the Einstein components defined in (15):

\[\begin{align*}
\hat{G}_{ij}^{(1)f} &= (1)\hat{T}_{ij}[X], \\
\hat{G}_{ij}^{(2)f} + \hat{G}_{ij}(f, f) &= (2)\hat{T}_{ij}[X], \\
G^{(1)f} &= (1)T[X], \\
G^{(2)f} + G(f, f) &= (2)T[X], \\
G_i^{(1)f} &= (1)T_i[X], \\
G_i^{(2)f} + G_i(f, f) &= (2)T_i[X], \\
G_0^{(1)f} &= (1)T_0[X], \\
G_0^{(2)f} + G_0(f, f) &= (2)T_0[X].
\end{align*}\]

(40a)

(40b)

(40c)

(40d)

The linear combinations of \((r)T^a_b[X]\) in (40) are defined in analogy with (15) by (see UW1, \[\text{23}\]

\[\begin{align*}
(r)\hat{T}_{ij}[X] &= \gamma_{k(i}^{(r)}T_{j)k}[X] - \frac{1}{3}\gamma_{ij}^{(r)}T^k_k[X], \\
(r)T[X] &= C^2(r)T_0^0[X] + \frac{1}{3}(r)T^k_k[X], \\
(r)T_i[X] &= -D_i^{(r)}T_0^0[X] - 3\mathcal{H}^{(r)}T^0_i[X],
\end{align*}\]

(41a)

(41b)

(41c)

where

\[C^2 = \frac{(0)p'}{(0)\rho'}, \quad (0)p = \frac{1}{3}(0)T^k_k, \quad (0)\rho = -(0)T_0^0.\]

(42)

Equations (40) give a convenient minimal form of the governing equations for linear and second order perturbations for any choice of gauge field \(X\) and any stress-energy tensor. The leading order terms \(G(f)\), where \(f = (1)f\) or \((2)f\), are given by (31) for the Poisson gauge \(X = X_p\) and by (32) for the uniform curvature gauge \(X = X_c\). The source terms \(G(f, f)\) are obtained in general by making the substitution \(f_{ab} \rightarrow f_{ab}[X]\) in equations (75)–(78) for an arbitrary \(X\), and are given directly by (33) for the special metric perturbation (33) when using the Poisson gauge.

The final step is to decompose the governing equations (40) into equations for the scalar mode, the vector mode and the tensor mode. As with the metric we perform a mode decomposition of the stress-energy gauge invariants (see UW1, \[\text{24}\]

\[\begin{align*}
(r)\hat{T}_{ij}[X] &= D_{ij}^{(r)}\Pi + 2D_{ij}^{(r)\Pi_j} + (r)\Pi_{ij}, \\
(r)T_i[X] &= D_i^{(r)}\Delta + (r)\Delta_i, \\
(r)T_0^0[X] &= D_i^{(r)}\bar{V} + (r)\bar{V}_i, \\
(r)T[X] &= (r)\bar{T},
\end{align*}\]

(43a)

(43b)

(43c)

(43d)

where

\[D_i^{(r)\Pi_i} = 0, \quad (r)\Pi_k^k = 0, \quad D_i^{(r)\Pi_j^i} = 0, \quad D_i^{(r)\Delta_i} = 0, \quad D_i^{(r)\bar{V}_i} = 0.\]

\[\text{22}\text{See also, equations (38) and (39) in Nakamura (2006). Nakamura’s metric gauge invariants are related to ours according to } \mathcal{L}_{ab} = a^2(1)\hat{f}_{ab}[X], \text{ and } \mathcal{H}_{ab} = a^2(2)f_{ab}[X].\]

\[\text{23}\text{At first order } (1)\hat{T}_{ij}[X], (1)T[X], (1)T_i[X] \text{ are intrinsic gauge invariants since they do not depend on the choice of gauge field and hence can be written as } (1)\hat{T}_{ij}, (1)T, (1)T_i \text{ (see UW1, section 2.3). At second order this is no longer the case.}\]

\[\text{24}\text{For brevity we drop the argument } [X] \text{ for the various mode terms.}\]
A difficulty arises that is not present at the linear level. The leading order terms $G^{(2)}(f_p)$ and the stress-energy terms $(2)T[X]$ in (10) are expressed explicitly as a sum of a scalar term, a vector term and a tensor term. On the other hand, the source terms $G(f, f)$ do not have this form, as can be seen, for example, from (35). One thus has to apply what we call mode extraction operators to (40) in order to separate the modes in the source terms. These operators are defined in (85), using the letters $S, V$ and $T$ to denote the scalar, vector and tensor modes, respectively.

For later use we note that applying the mode extraction operators to (43) yields

\[
\begin{align*}
^{(r)}\Pi &= S^{ij(r)}T_{ij}[X], &^{(r)}\Pi_i &= V_i^{jk(r)}T_{jk}[X], &^{(r)}\Pi_{ij} &= T_{ij}^{pq(r)}T_{pq}[X], \\
^{(r)}\Delta &= S^{i(r)}T_i[X], &^{(r)}\Delta_i &= V_i^{j(r)}T_j[X], \\
^{(r)}V &= S^{i(r)}T^0_i[X], &^{(r)}\tilde{V}_i &= V_i^{j(r)}T^0_j[X].
\end{align*}
\]  

3.2 The mode-decomposed governing equations

In this section we give the mode-decomposed form of the governing Einstein field equations at second order for perturbations of an FL cosmology with arbitrary matter content, first using Poisson gauge invariants, and then using uniform curvature gauge invariants. The source terms, identified by the kernel $G$, are obtained by making the substitution $f \to f_p$ or $f \to f$ in equations (75)–(78), or directly by (35) for the special metric perturbation (33) when using the Poisson gauge. We note that, in accordance with (40), the governing equations at first order can be simply obtained from the equations at second order by dropping the source terms, indicated by the kernel $G$, and dropping the exponent $^{(2)}$ (or replacing it by $^{(1)}$).

The Poisson form

To obtain the Poisson form we substitute (31) and (43) into (10) and then apply the mode extraction operators (85), which leads to:

**Scalar mode**

\[
\begin{align*}
^{(2)}\Psi - ^{(2)}\Phi &= ^{(2)}\Pi_p - ^{(2)}S^{ij}\hat{G}_{ij}(f_p, f_p), \\
(\mathcal{L} - C^2_0 D^2)\, ^{(2)}\Psi &= \frac{1}{2}^{(2)}\Gamma_p + \\
&\quad \left(\mathcal{H}\mathcal{L}_A + \frac{i}{3}D^2\right)\left(^{(2)}\Pi_p - ^{(2)}S^{ij}\hat{G}_{ij}(f_p, f_p)\right) - \frac{1}{2}G(f_p, f_p), \\
^{(2)}\Delta_p &= \frac{1}{2}^{(2)}\Delta_p - \frac{1}{2}S^i\hat{G}_i(f_p, f_p), \\
\partial^\eta \Psi + \mathcal{H}^{(2)}\Phi &= \frac{1}{2}^{(2)}V_p + \frac{1}{2}S^i\hat{G}_i^0(f_p, f_p),
\end{align*}
\]  

**Vector mode**

\[
\begin{align*}
\mathcal{L}_B^{(2)}B_i[X_p] &= -2^{(2)}\Pi_i[X_p] + 2V_i^{jk}\hat{G}_{jk}(f_p, f_p), \\
(D^2 + 2K)^{(2)}B_i[X_p] &= 2^{(2)}\tilde{V}_i[X_p] - 2V_i\hat{G}_i^0(f_p, f_p),
\end{align*}
\]  

**Tensor mode**

\[
\begin{align*}
(\mathcal{L}_B \partial^\eta + 2K - D^2)^{(2)}C_{ij}[X_p] &= ^{(2)}\Pi_{ij}[X_p] - T_{ij}^{km}\hat{G}_{km}(f_p, f_p).
\end{align*}
\]
Here \( (2)\Psi, (2)\Phi, (2)B_i[X_p] \) and \( (2)C_{ij}[X_p] \) are the second order Poisson metric gauge invariants and \( f_p \) is shorthand for the first order metric perturbation \( f_{ab}[X_p] \) as given by (25).

The evolution of the scalar perturbations is governed by equation (46b), a second order partial differential equation for \( (2)\Psi \). In order to obtain a solution one first has to solve the linearized field equations for the first order gauge-invariant metric perturbation \( f_{ab}[X_p] \), which then determines the Einstein source terms. Once the second order matter terms \( (2)\Gamma_p \) and \( (2)\Pi_p \) have been specified, one can solve (46b) for \( (2)\Psi \) and then successively use (46a), (46c) and (46d) to calculate \( (2)\Phi, (2)\Delta_p \) and \( (2)V_p \), respectively.

**The uniform curvature form**

To obtain the uniform curvature form we substitute (32) and (43) into (40) and then apply the mode extraction operators. For the sake of brevity we give only the scalar mode since the vector and tensor modes have essentially the Poisson form.

**Scalar mode**

\[
\mathcal{L}_B (2)B + (2)A = - (2)\Pi_c + S^{ij} \hat{\Gamma}_{ij}(f_c, f_c), \tag{49a}
\]

\[
\mathcal{H}(\mathcal{L}_A (2)A + C_G^2 D^2 (2)B) = \frac{1}{2} (2)\Gamma_c + \frac{1}{2} D^2 (2)\Pi_c - S^{ij} \hat{\Gamma}_{ij}(f_c, f_c) - \frac{1}{2} \hat{\mathcal{G}}(f_c, f_c), \tag{49b}
\]

\[
\mathcal{H}(D^2 A - 3K)^2 (2)B = - \frac{1}{2} (2)\Delta_c + \frac{1}{2} S^i \hat{\Gamma}_i(f_c, f_c), \tag{49c}
\]

\[
\mathcal{H}(2)A - K (2)B = - \frac{1}{2} (2)V_c + \frac{1}{2} S^i \hat{\Gamma}_i^0(f_c, f_c). \tag{49d}
\]

Here \( (2)A \) and \( (2)B \) are the second order uniform curvature metric gauge invariants and \( f_c \) is shorthand for the first order (in time) metric metric perturbation \( f_{ab}[X_c] \) as given by (26).

These equations differ from the Poisson form (46) in that the evolution of the scalar potentials \( A \) and \( B \) is governed by two coupled first order partial differential equations (49a) and (49b). In order to obtain a solution one has to follow a two step procedure, as with the Poisson form.

**Commentary**

The systems of equations (46) and (49), together with the expressions (75)–(78) for the source terms, are new and constitute one of the main results of this paper. Either set of equations determines the behaviour of second order scalar perturbations of an FL cosmology with arbitrary stress-energy content. We emphasize that the specific form of the evolution equations (46) and (49) depends on the matter terms, specifically, the \( \Gamma \)-terms and the \( \Pi \)-terms, both at first order and at second order. These quantities are determined by the stress-energy tensor using equation (84b) in conjunction with (43d) and (45a). In the next section we illustrate how to calculate these quantities for the simple case of a perfect fluid.

Equations (46) and (49) illustrate a fundamental difference between second order and first order perturbations. The analysis of linear perturbations is simplified by the fact that the three modes, namely scalar, vector and tensor, decouple and hence can...
be analyzed separately. At second order each mode involves leading order terms, described by the same operators that occur at first order, but also complicated quadratic source terms that, in general, contain all linear modes, which leads to a phenomenon one may refer to as source mode coupling. For example, this means that the first order metric perturbation $f_0$ that determines the Einstein source terms in (46) contains scalar, vector and tensor modes in general (see (25)). In other words the vector and tensor perturbations at linear order contribute to the scalar perturbation at second order. On the other hand equations (46) show that a purely scalar linear perturbation (i.e. if the vector and tensor modes at the linear level are assumed to be zero, as is often done) generate all three modes at second order.

### 3.3 Matter gauge invariants for a perfect fluid and $\Lambda$

In this section we determine the second order stress-energy perturbations $(2)\Gamma[X], (2)\Pi[X], (2)\Pi_i[X]$ and $(2)\Pi_{ij}[X]$ that appear in the governing equations in Poisson form, as given by equations (46), (47) and (48), when the stress-energy tensor describes a perfect fluid and a cosmological constant $\Lambda$:

$$
T^a_b = (\rho + p)u^a u_b + p\delta^a_b - \Lambda\delta^a_b. 
$$

(50)

Here $u^a$, $\rho$, $p$ are the fluid’s 4-velocity, energy-density and pressure, respectively. Since $\Lambda$ is $\epsilon$-independent it follows that it does not appear in $(1)T^a_b$ and $(2)T^a_b$, and it only affects the perturbed field equations indirectly via background quantities determined by the zeroth order field equations. For simplicity we assume $p = w\rho$ with $w = constant$, which implies that

$$(r)p = C^2_T (r)\rho, \quad \text{with} \quad C^2_T = w, \quad \text{for} \quad r = 1, 2. \quad (51)$$

In order to find the desired quantities we need to calculate $(2)\Gamma[X]$ and $(2)\Pi_{ij}[X]$ for the stress-energy tensor (50). We begin by expanding (50) to second order, which yields

$$
a^2 (2)T^0_0 = -a^2 (2)\rho - 2A_T \gamma^{ij}(1)v_i (1)v_j - f_{0j}), \quad (52a)$$

$$
a^2 (2)T^k_k = 3a^2 (2)p + 2A_T \gamma^{ij}(1)v_i (1)v_j - f_{0j}), \quad (52b)$$

$$
a^2 (2)\Pi_{ij} = 2A_T (1)v_i - f_{0(i)(1)v_j), \quad (52c)$$

where $(1)v_i := a^{-1(1)}u_i$. Applying the Replacement Principle\(^\text{26}\) gives

$$
^{(2)}T^0_0[X] = -(2)\rho[X] - U^k_k[X], \quad (53a)$$

$$
^{(2)}T^k_k[X] = -3C^2_T(2)\rho[X] + U^k_k[X], \quad (53b)$$

$$
^{(2)}\Pi_{ij}[X] = \hat{U}_{ij}[X], \quad (53c)$$

where

$$
U_{ij}[X] := 2A_T^{-1(1)}V_i[X] (1)V_j[X] - A_T f_{0j}[X], \quad (54)
$$

\(^\text{25}\)See, for example, Bartolo et al (2004a), section 3.1.

\(^\text{26}\)To obtain $(2)T^k_k[X]$ replace each perturbation variable on the right side by its gauge invariant formed by $X$-compensation i.e. $(2)\rho \rightarrow (2)\rho[X]$ and $A_T(1)v_i \rightarrow (1)V_i[X]$. See equation (34).
We then choose $X$ invariants (2), i.e., and hence at second order, denoted (2), the required quantities plying the mode extraction operators as in (45a).

Equations (53a) and (53b), together with (41b), result in

$$\Gamma[X] = \frac{2}{3}(1 - 3C_T^2)U_k^k[X].$$

As expected, we see that $\Gamma[X]$ and $\hat{T}_{ij}[X]$ are purely source terms.

For simplicity we now assume that the linear vector modes $\tilde{V}_i[X]$ and $B_i[X]$ are zero, i.e.,

$$V_i[X] = D_iV[X], \quad f_{ij}[X] = D_iB_j[X].$$

We then choose $X = X_p$, noting that $B[X_p] = 0$ and $V[X_p] \equiv V$. It follows that

$$U_{ij}[X_p] = 2A_T^{-1}(D_iV)(D_jV),$$

and hence

$$\Gamma[X] = \frac{2}{3}(1 - 3C_T^2)A_T^{-1}(DV)^2, \quad \hat{T}_{ij}[X_p] = 2A_T^{-1}D_iV D_jV.$$

The required quantities $\Pi[X_p]_j$, $\Pi[X_p]$ and $\hat{T}_{ij}[X_p]$ are then obtained by applying the mode extraction operators as in (45a).

In order to facilitate comparison with the literature we relate the gauge invariants associated with the matter density and with the matter velocity in the Poisson gauge at second order, denoted (2)$\rho_p$, and (2)$\nu_p$, respectively, to our stress-energy gauge invariants (2)$\Delta_p$ and (2)$\nu_p$:

$$(2)\rho_p = (2)\Delta_p + 3\mathcal{H}(2)V_p - 2A_T^{-1}(DV_p)^2, \quad (2)\nu_p = A_T^{-1} \left( \frac{(2)\nu_p + 2\mathcal{S}^i}{\Psi - \rho_p/(3\mathcal{H}^2)} D_iV_p \right),$$

where

$$\rho_p = \Delta + 3\mathcal{H}V_p, \quad \Omega_m = a^2 (0)/(3\mathcal{H}^2).$$

### 3.4 The governing equations in a simple example

We now specialize the general governing equations (46), (47) and (48) at second order in the Poisson gauge to a perfect fluid and a cosmological constant with only scalar first order contributions and $\Phi = \Psi$, i.e. we impose the metric assumptions (33). We hence substitute for the metric source terms $G(f_p, f_p)$ from (33) and for the stress-energy perturbations (2)$\Gamma$ and the three (2)$\Pi$-terms from (15a) and (59). This leads to the following governing equations:

**Scalar mode**

$$(2)\Psi - (2)\Phi = -4\Psi^2 + 2S^{ij}M_{ij}(\Psi, \Psi),$$

$$(\mathcal{L} - C_T^2 D^2)^2(2)\Psi = \frac{1}{3}(1 - 3C_T^2)A_T^{-1}(DV)^2 + \frac{1}{5}(1 + 3C_T^2)\mathcal{R}(\Psi, \Psi) + \frac{\mathcal{L}}{3}(DV)^2 - \frac{4}{3}DV^2 \Psi^2 + 2(\mathcal{H}\mathcal{L}_A + \frac{1}{3}D^2)S^{ij}M_{ij}(\Psi, \Psi),$$

$$(D^2 + 3K)(2)\Psi = \frac{1}{2}(2)\Delta - \frac{1}{7}\mathcal{R}(\Psi, \Psi) - 6\mathcal{H}S^i((\partial_i\Psi)(D_i\Psi)),$$

$$(\partial_\eta + \mathcal{H})(2)\Psi = -\frac{1}{2}(2)\mathcal{V} - 2S^i((\partial_i\Psi)(D_i\Psi)) + 2\mathcal{H}S^iM_{ij}(\Psi, \Psi),$$

Where

27Since $(1)\Gamma[X] = 0$ and $(1)\hat{T}_{ij}[X] = 0$ for a perfect fluid it follows that the leading order term in $(2)\Gamma[X]$ and in $(2)\hat{T}_{ij}[X]$ is zero.
where
\[ R(\Psi, \Psi) := 2 \left[ 3(\partial_\eta \Psi)^2 - 5(D^2 \Psi)^2 + 4(D^2 + 3K)\Psi^2 \right], \] (61e)
\[ M_{ij}(\Psi, \Psi) := 2(D_{i} \Psi)(D_{j} \Psi) + A^{-1}_T(D_{i} V)(D_{j} V), \] (61f)
and \( V \) is given by\(^{28}\)
\[ V = -2(\partial_\eta + \mathcal{H})\Psi. \] (61g)

Vector mode
\[ \mathcal{L}_B (2) B_i = -4V^{jk}M_{jk}(\Psi, \Psi), \] (62a)
\[ (D^2 + 2K)(2) B_i = 2(2)V_i + 8V^j_i((\partial_\eta \Psi)(D_j \Psi)). \] (62b)

Tensor mode
\[ (\mathcal{L}_B \partial_\eta + 2K - D^2) (2) C_{ij} = 2T_{ij}^{km}M_{km}(\Psi, \Psi). \] (63)

Commentary
An attractive feature of the Poisson gauge at first order is that if the anisotropic stress is zero (for example, for a perfect fluid or a scalar field) then \( \Phi = \Psi \), i.e. there is only one metric gauge invariant for the scalar mode. Equation (61a) shows that this feature is not preserved at second order due to the presence of the source terms. There are thus two second order metric gauge invariants, \( (2) \Phi \neq (2) \Psi \). We shall refer to \( (2) \Phi \) as the second order Bardeen potential and to \( (2) \Psi \) as the second order Bardeen curvature, to distinguish their roles.\(^{29}\)

The system of equations (61)-(63) is closely related to but simpler than equations that appear in the literature. Nakamura (2007) has given a system of equations that can be transformed into\(^{30}\) our equations (61)-(63). There are, however, two important differences. First, Nakamura chooses the Bardeen potential \( (2) \Phi \) rather than the Bardeen curvature \( (2) \Psi \) to be the primary metric gauge invariant for the scalar mode, which has a major drawback: the source terms contain second order time derivatives and are significantly more complicated. The second difference is in the treatment of the density perturbation. We use \( (2) \Delta_p \), which satisfies a generalized Poisson equation and can be viewed as being analogous at second order to the well known Bardeen gauge invariant \( \epsilon_m \). In contrast, Nakamura uses the gauge invariant \( (2) \rho_p \) associated with the density perturbation at second order in the Poisson gauge, as given by\(^{31}\)

Equations (62) and (63) show that a purely scalar linear perturbation gives rise to a vector and a tensor perturbation at second order, with the link provided by the tensor \( M_{ij} \). The gravitational waves described by this tensor perturbation have been investigated in detail by Ananda et al (2007) and Baumann et al (2007).\(^{28}\)

\(^{28}\)This is obtained from the governing equations for the first order perturbations.

\(^{29}\)Strictly speaking one should also make this distinction at first order, but since \( \Phi = \Psi \) in many applications we simply refer to \( \Psi \) as the Bardeen potential.

\(^{30}\)His equations (6.38), (6.44), (6.41) and (6.42) can be used to obtain the first four of our equations (61), his equations (6.39) and (6.33) can be transformed into our equations (62), and finally his equation (6.40) yields our equation (63). We also refer to Nakamura (2006) for a brief summary.

\(^{31}\)Note that \( (2) \rho [X_p] \equiv a^2 (2) \epsilon \) in Nakamura’s notation.
4 Discussion

The systems of equations (46) and (49) which govern nonlinear perturbations, together with the expressions (75)–(78) for the source terms, are new and constitute one of the main results of this paper. Because of their generality these equations provide a starting point for determining the behaviour of nonlinear perturbations of FL cosmologies with any given stress-energy content, using either the Poisson gauge or the uniform curvature gauge. These equations exhibit the same concise structure as the governing equations for linear perturbations given in UW1 (see equations (52) and (54)), in which the evolution of the metric perturbations is determined by the second order factored differential operator $\mathcal{L}$ in the Poisson form and by the pair of first order differential operators $\mathcal{L}_A$ and $\mathcal{L}_B$ in the uniform curvature form. This structure arises directly from our use of specific linear combinations of the components of the perturbed Einstein tensor and their derivatives, as given by (15) (see UW1, equations (39) and (40) for the motivation). Indeed the three operators are visible at an early stage in the derivation in the leading order expressions (19) for the perturbed Einstein tensor. This is in contrast to the literature where it is customary to simply calculate all the components $(2)G_{ab}$ of the perturbed Einstein tensor, and then form linear combinations of the perturbed Einstein equations (see for example, Acquaviva et al (2003), equation (4) for the perturbed metric and Appendix A5 for the perturbed Einstein equations, Bartolo et al (2004a), equations (104) and (A.36)–(A.43).). This process involves more extensive calculations than in our approach and may lead to expressions that are not optimally simplified while hiding important mathematical structures.

There is one issue that deserves particular attention, namely the fact that there is no unique choice of gauge invariant associated with the perturbations of the matter density. At the linear level there are three commonly used choices, the Poisson gauge invariant, the uniform curvature gauge invariant that is related to one of the so-called conserved quantities, and the total matter (or comoving) gauge invariant. The last-mentioned is the well-known Bardeen gauge invariant, and is related to the spatial gradient of the matter density orthogonal to the fluid flow. At the second order level the situation is more complicated and requires further investigation. The appropriate choice may depend on the physical situation under consideration.

In this paper we have focussed exclusively on using the perturbed Einstein field equations to describe the dynamics of nonlinear perturbations. There are, however, two alternatives to the direct use of the Einstein equations. First, one can use the perturbed conservation equations for the stress-energy tensor and second, one

\[\text{For example, by specializing these equations we can derive in an efficient manner the various equations for second order perturbations that appear in the papers by Bartolo and collaborators (Poisson form) and by Malik and collaborators (uniform curvature form), referred to in the introduction.}\]

\[\text{See, for example, Malik and Wands (2004), equations (4.17) and (4.27).}\]

\[\text{See Bartolo et al (2010a), section 3 and in particular equation (29) for $(2)\rho_p$ in a $\Lambda CDM$ model, Christopherson and Malik (2009), equation (4.8) for $(2)\rho_c$ and Noh and Huang (2004), equation (273) for $(2)\rho_v$.}\]

\[\text{See, for example Bartolo et al (2004b), equation (4.3), and Noh and Hwang (2004), equations (104) and (200).}\]
can use the $1 + 3$ formalism\footnote{See Bruni et al (1992) for a comprehensive treatment of linear perturbations using this formalism.} expanding the exact equations to second order and making them gauge-invariant. More work needs to be done in this regard. An additional aspect of the dynamics of scalar perturbations that we have likewise not touched on is that under certain conditions (i.e. in the long wavelength regime) the governing equations admit so-called conserved quantities, i.e. quantities that remain approximately constant during a restricted epoch. These quantities, which were initially introduced for linear perturbations (see, for example UW2 section 4 for a unified overview) have now been generalized to second order perturbations\footnote{Malik and Wands (2004), equations (4.17) and (4.18) and Christopherson and Malik (2009), equations (4.11)--(4.13).

\section*{Acknowledgments}
CU is supported by the Swedish Research Council (VR grant 621-2009-4163). CU also thanks the Department of Applied Mathematics at the University of Waterloo for kind hospitality. JW acknowledges financial support from the University of Waterloo.

\appendix
\section{A Derivation of the perturbation equations}
\subsection*{A.1 Exact curvature expressions}
In UW1 we derived an exact expression for the Riemann curvature tensor $R^{ab}_{cd}(\epsilon)$ of $g_{ab}(\epsilon) = a^2 \bar{g}_{ab}(\epsilon)$ by replacing the covariant derivative $\nabla_a$ of $g_{ab}(\epsilon)$ with the covariant derivative $\nabla_a$ of $\bar{g}_{ab}(\epsilon) = \gamma_{ab}$. We first make the following definitions:

\begin{align}
    r_a & := \nabla_a (\ln a), \\
    \tilde{Q}^a_{bc}(\epsilon) & := \bar{g}^{ad}(\epsilon) \left( \nabla_b \bar{g}_{c}d(\epsilon) - \frac{1}{2} \nabla_d \bar{g}_{bc}(\epsilon) \right). 
\end{align}

Since $\nabla_a \gamma_{bc} = 0$ it follows that

$$\tilde{Q}^a_{bc}(0) = 0.$$ \hfill (65)

The desired expression is as follows (see UW1, (B.8), (B.10) and (B.12b)):

$$a^2 R^{ab}_{cd}(\epsilon) = \bar{R}^{ab}_{cd}(\epsilon) + 4 \delta^{[a}[c} \bar{U}^{b]}_{df}(\epsilon),$$ \hfill (66a)

where

\begin{align}
    \bar{R}^{ab}_{cd}(\epsilon) & = \bar{g}^{be}(\epsilon) \left( \nabla_{[c} \bar{Q}_{d]e}(\epsilon) + 2 \bar{Q}^{a}_{f|c}(\epsilon) \tilde{Q}^{f}_{d]e}(\epsilon) \right), \\
    \bar{U}^{b}_{d}(\epsilon) & = - \left[ \bar{g}^{be}(\epsilon) \left( \nabla_d - r_d \right) + \frac{1}{2} \delta^{b}_{d} \bar{g}^{ef}(\epsilon) r_f - \bar{g}^{bf}(\epsilon) \tilde{Q}^{e}_{d}(\epsilon) \right] r_e. 
\end{align}

Here $\bar{R}^{ab}_{cd}(\epsilon)$ is the curvature tensor of the metric $\bar{g}_{ab}(\epsilon)$, and $\nabla_{[c} \bar{Q}_{d]e}(\epsilon)$ is the curvature tensor of the metric $\gamma_{ab}$. Note that $\nabla_{[c} \bar{Q}_{d]e}(\epsilon)$ is zero if at least one index is temporal, while if all indices are spatial

$$0 R^{ij}_{km} = 2 K \delta^{[i}_{[k} \delta^{j]}_{m]}. $$ \hfill (67)
A.2 First and second order gauge-variant perturbations

Our first goal is to derive expressions for the geometric operators $\mathcal{R}_{ab}(r)f$ and $\mathcal{R}_{ab}(f, f)$ that determine the perturbed Riemann tensor through equations (6a). To accomplish this we need the Taylor expansion of $R_{ab}(\epsilon)$, given by (66).

We begin by deriving expressions for $(1)\tilde{g}^{ab}$ and $(2)\tilde{g}^{ab}$. We Taylor expand the relation $\delta^a_b = \tilde{g}^{0a}(\epsilon) \tilde{g}_{db}(\epsilon)$, which gives

$$
(1)\tilde{g}^{ab} = -f^{ab}, \quad (2)\tilde{g}^{ab} = -f^{ab} + 2f^{ac}f^{b}_c,
$$

where we used the following relations

$$
(1)(AB) = (0)A^{(1)}B + (1)A^{(0)}B, \quad (69a)
$$

$$
(2)(AB) = (0)A^{(2)}B + 2(1)A^{(1)}B + (2)A^{(0)}B. \quad (69b)
$$

We next Taylor expand $\tilde{Q}^a_{bc}(\epsilon)$, as given by (64b), and use (69). This leads to

$$
(1)\tilde{Q}^a_{bc} = \tilde{Q}^a_{bc}(f), \quad (2)\tilde{Q}^a_{bc} = \tilde{Q}^a_{bc}(2f) - 2f^{ad}\tilde{Q}_{dc}(f). \quad (70a)
$$

where the operator $\tilde{Q}^a_{bc}$ is defined by

$$
\tilde{Q}^a_{bc}(f) := \gamma^{ad}(\nabla_b f_c)d - \frac{1}{2}\partial_d f_{bc}. \quad (70b)
$$

Finally Taylor expanding $\mathcal{R}_{ab}(\epsilon)$ as given by (66) and using (6a), in conjunction with (68), (65) and (70a), gives the leading order term

$$
\mathcal{R}_{ab}(r)f = \bar{\mathcal{R}}_{ab}(r)f + 4\delta^a_c\tilde{U}^b_d(f), \quad r = 1, 2, \quad (71a)
$$

where

$$
\bar{\mathcal{R}}_{ab}(r)f = -2\partial_c^a(\nabla^b d f_b) + (r)f^{[a}\epsilon_0 \tilde{R}^{b]c}d, \quad (71b)
$$

$$
\tilde{U}^b_d(r)f = \tilde{U}^{bc}(r)f, \quad (71c)
$$

and the source term

$$
\mathcal{R}_{ab}(f, f) = \bar{\mathcal{R}}_{ab}(f, f) + 4\delta^a_c\tilde{U}^b_d(f, f), \quad (72a)
$$

where

$$
\bar{\mathcal{R}}_{ab}(f, f) = 2f^{[a}_e(2\tilde{R}^{b]e}(f) - f^{[a}_e\tilde{R}^{b]e}f) - 4\tilde{Q}^f_{[a}(f)\tilde{Q}^b_{d]}(f), \quad (72b)
$$

$$
\tilde{U}^b_d(f, f) = -2f^{[a}_e(\tilde{U}^{bc}(f) + \delta^b_f\tilde{Q}^{ce}(f)r_g). \quad (72c)
$$

In the above equations we have also defined

$$
\tilde{U}^{bc}(r)f := \left[r^{[a}f^{bc}(\nabla_d f - r_d) + \frac{1}{2}\delta^b_a(r) f^{ch} f_{gh} + \gamma^b_{gh} \tilde{Q}^{ag}_{d}(r)f\right]r_f. \quad (73)
$$

To obtain (71b) we used $\partial^a\nabla^b c (r)f_{ab} = (r) f^{(a}\tilde{R}^{b) c}f$, while $\partial^c(r)f_{ab} = 2\tilde{Q}^{(ab}_{c}(r)f$ was used to obtain (72b).
Equations (71) and (72) constitute one of the main results of this paper. They express the first and second order Riemann tensor perturbations, as given by (6a), in terms of the metric perturbations as given by equations (3), with the second order perturbation written as the sum of a leading order term and a source term.

To proceed further we need to introduce local coordinates as in section 2.2.1 which implies that the covariant derivative of a tensor, \( \nabla_a A \), and the gradient \( r_a \) defined by (63a), assume the form

\[
0\nabla_0 A = \partial_\eta A, \quad 0\nabla_i A = D_i A, \quad r_0 = \mathcal{H}, \quad r_i = 0. \tag{74}
\]

where \( \partial_\eta \) denotes partial differentiation with respect to \( \eta \) and \( D_i \) is the spatial covariant derivative of \( \gamma_{ij} \).

We have already used equations (71) to write the Einstein leading terms in the form (19) in the main text. We now use equations (72) to write the Einstein source terms, as given by (21), in the form

\[
\begin{align*}
\hat{G}_{ij}(f, f) &= \hat{S}_{ij} + \mathcal{W}_{ij} + 2\hat{Q}_{\alpha\beta(i} \hat{Q}^{\alpha\beta j)} - 2\hat{Q}^{\alpha\beta}_{\beta(i} \hat{Q}_{\alpha(j)}, \\
\mathcal{G}(f, f) &= -\frac{1}{2} \left[ (1 + 3\mathcal{C}_0^2)\mathcal{R} + 2\mathcal{R}_0 + \mathcal{W}^k - 6\mathcal{H}L_A(f_0 f_0^\alpha) \right], \\
\mathcal{G}_i(f, f) &= D_i \mathcal{R} - 3\mathcal{H} \tilde{G}_i, \\
\mathcal{G}^{0i}(f, f) &= -2\mathcal{H}D_i(f_0 f_0^\alpha) + \bar{\mathcal{G}}^{0i},
\end{align*}
\tag{75}
\]

where

\[
\begin{align*}
\mathcal{W}_{ij}(f, f) &= 4\mathcal{H} \left[ f_{00} \hat{Q}^{0\alpha}_{ij} - \frac{1}{2} f_{0i} D_j f_{00} - f_{ik} \hat{Q}^{0k}_{ij} \right], \\
\hat{S}_{ij}(f, f) &= -2 \left[ R^{\alpha k}_{\beta(\gamma f)k} f^{\beta}_{ij} + R^{\alpha\beta(\alpha f)j} - K(f^k_{(i} f^j_{j)k} - \frac{1}{3} f^k_{ij} f^k_{ij}) \right], \\
\mathcal{R}(f, f) &= \mathcal{W}^k - 2f^k_{\alpha} R^{\alpha m}_{k m} + K(f^m_{k} f^k_{m} - \frac{2}{3} (f^k_{k})^2) - 2\hat{Q}^{\alpha k} \hat{Q}^m_{\alpha m}, \\
\mathcal{G}^{0i}(f, f) &= \mathcal{G}_i + 2K(f^k_{0i} f^k_{0m} - f^m_{0k} f^k_{0i}), \\
\mathcal{R}_\alpha(f, f) &= 2 \left[ (f_{00} - \frac{1}{2} f^k_{k}) R^{0m}_{k m} f_{0k} \hat{R}^{0k}_{k m} - \hat{f}^m_{k} \hat{R}^{0k}_{k m} - 4\hat{Q}^{\beta [0}_\alpha \hat{Q}^m_{\beta m]} \right].
\end{align*}
\tag{76}
\]

For notational brevity we have dropped the arguments \( f, f \) of the quantities (76) and the argument \( f \) of \( \hat{Q}_{bc}(f) \) and \( \hat{Q}_{ab}(f) \) when they appear on the right side of (75) and (76). It remains to give the components of \( \hat{R}^{ab}_{cd}(f) \) and \( \hat{Q}^a_{bc}(f) \), that are defined by (71b) and (70b):

\[
\begin{align*}
\hat{R}^i_{0m} &= \frac{1}{2} (D^i_{m} + \frac{1}{3} \delta^i_{m} D^2) f_{00} + \partial_\eta \hat{Q}^0_{m}, \\
\hat{R}^i_{km} &= 2D^i_{[k \hat{Q}^0_{m}]}, \\
\hat{R}^i_{0m} &= -2D^i_{[j} \hat{Q}^{0][j]}\hat{m} + 2K f^i_{0[|j}\delta_{m]^|j], \\
\hat{R}^i_{km} &= -2(D^i_{[k} D^i_{[k} + K \delta_{[i}^{[i} f^i_{m]}), \\
\hat{Q}^0_{00} &= -\frac{1}{2} \partial_\eta f_{00}, \quad \hat{Q}^0_{0i} = -\frac{1}{2} D_i f_{00}, \\
\hat{Q}^0_{ij} &= \frac{1}{2} \partial_\eta f_{ij} - D_i f_{j0}, \quad \hat{Q}^i_{00} = 0, \\
\hat{Q}^i_{0j} &= \frac{1}{2} \partial_\eta f_{ij} - D_i f_{j0}, \quad \hat{Q}^i_{0i} = 0.
\end{align*}
\tag{77}
\]

In summary, equations (75), in conjunction with (70)-(78), give the general expressions for the Einstein source terms.
A.3 Gauge invariance and the Replacement Principle

Gauge fields and gauge invariants

In cosmological perturbation theory a second order gauge transformation can be represented in coordinates as follows:

\[ \tilde{x}^a = x^a + e^{(1)}\xi^a + \frac{1}{2} \epsilon^a (^{(2)}\xi^a + (1)\xi^a_{\ b} (1)\xi^b), \]  

(79)

where \( (1)\xi^a \) and \( (2)\xi^a \) are independent dimensionless background vector fields. Each transformation induces a change in the first and second order perturbations of a tensor field \( A \) according to

\[
\begin{align*}
(1)\tilde{A}[\xi] &= (1)A + \mathcal{L}_{(1)\xi}^{(0)}A, \\
(2)\tilde{A}[\xi] &= (2)A + \mathcal{L}_{(2)\xi}^{(0)}A + \mathcal{L}_{(1)\xi}^{(0)} (2^{(1)}A + \mathcal{L}_{(1)\xi}^{(0)}A),
\end{align*}
\]

(80a)

(80b)

where \( \mathcal{L} \) is the Lie derivative (see, e.g., Bruni et al (1997), equations (1.1)–(1.3)).

One can impose restrictions on the tensor perturbations \( ^{(r)}\tilde{A}[\xi] \) by letting \( ^{(r)}\xi^a \) depend suitably on the dimensionless perturbations \( a^n (^{(r)}A \ (r = 1, 2) \), a procedure that can be referred to as perturbative gauge fixing. If the vector fields \( ^{(r)}\xi^a \) up to order \( r \) are fully determined via the assumed perturbative restrictions, we say that the gauge is fully fixed to order \( r \). An important special case of gauge fixing is order by order gauge fixing, in which the same conditions are imposed on the first and second (or higher) order perturbations. When the gauge has been fully fixed to order \( r \), all remaining perturbative quantities to order \( r \) are rendered gauge-invariant.

The key features of our dimensionless version of Nakamura’s method for constructing gauge invariants up to second order are as follows (see Nakamura (2007), and UW1, equations (5)–(8), for the linear case and further discussion and references). Given a family of tensor fields \( A(\epsilon) \) with \( a^n A(\epsilon) \) dimensionless we define\(^{39}\)

\[
\begin{align*}
(1)A[X] := a^n (^{(1)}A - \mathcal{L}_{^{(1)}X}^{(0)}A), \\
(2)A[X] := a^n (^{(2)}A - \mathcal{L}_{^{(2)}X}^{(0)}A - \mathcal{L}_{^{(1)}X}^{(0)} (2^{(1)}A - \mathcal{L}_{^{(1)}X}^{(0)}A)).
\end{align*}
\]

(81a)

(81b)

Comparing \((81)\) and \((80)\) multiplied by \( a^n \), reveals that the equations have precisely the same form if we identify \( ^{(r)}X^a \) with \( ^{(r)}\xi^a \). Hence given a fully specified gauge choice \( ^{(r)}\xi^a \), if we choose \( ^{(r)}X^a = -^{(r)}\xi^a \) then the \( ^{(r)}A[X] \) will be gauge invariants that coincide with the \( a^n (^{(r)}A[X]) \). In other words, imposing conditions on the \( ^{(r)}A[X] \) that fully determine the \( ^{(r)}X^a \) corresponds precisely to full gauge fixing. Due to the close relation between the roles of \( ^{(r)}X^a \) and \( ^{(r)}\xi^a \) we refer to \( ^{(r)}X^a \) as gauge fields, and we say that \( ^{(1)}A[X] \) and \( ^{(2)}A[X] \) are the first and second order dimensionless gauge invariants associated with \( ^{(1)}A \) and \( ^{(2)}A \), respectively, by \( X \)-compensation.

\(^{39}\)Compare with equations (2.26)–(2.27) in Nakamura (2007) and (2.34)–(2.35) in Nakamura (2010). The factor of \( a^n \) in our equations ensures that our expressions \( ^{(r)}A[X], \ r = 1, 2, \) are dimensionless.
We note in passing that another way of ensuring that the \((r)A[X]\) are gauge-invariant expressions is to choose dimensionless fields \((r)X^a\) that satisfy:

\[
\begin{align*}
\Delta^{(1)}X^a &= (1)X^a - (1)X^a = (1)\xi^a, \\
\Delta^{(2)}X^a &= (2)X^a - (2)X^a = (2)\xi^a + [(1)\xi, (1)X]^a. 
\end{align*}
\] (82a, 82b)

These conditions are obtained by applying (80) to (81) and demanding that \(\Delta^{(1)}A[X] = 0\) and \(\Delta^{(2)}A[X] = 0\). In UW1 we made use of (82a) in constructing gauge invariants (see section 2.2). At second order, however, the length of the computation makes it impractical to verify (82b) directly.

The one-to-one correspondence between full gauge fixing and the Nakamura approach makes the choice between them a matter of aesthetics and personal preference. We choose to work in the formally ‘dimensionless gauge-invariant’ picture, but everything we do has a ‘fully gauge fixed picture’ analogue (replace \(X\) with \(-\xi\) and \((r)A[X]\) with \(a^n(r)\tilde{A}[\xi]\)).

In Appendix A.4 we apply the Nakamura approach to the metric tensor to construct gauge fields and gauge invariants.

The Replacement Principle

The equivalence of gauge fixing and the Nakamura approach is expressed in the Replacement Principle. We present two versions, one for the Riemann and Einstein tensors, and one for the stress-energy tensor.

The dependence of the perturbations of the Riemann tensor on the metric perturbations can be written symbolically in the form:

\[
\begin{align*}
a^{2(1)}R_{cd} &= R_{cd}(1)f, \\
a^{2(2)}R_{cd} &= R_{cd}(2)f + \mathcal{R}_{cd}(1)f, (1)f, 
\end{align*}
\] (83a)

where \((r)f\) is shorthand for \((r)f_{ab}\), with \(r = 1, 2\). The Replacement Principle for the Riemann curvature states that the gauge invariants associated with \((r)R_{cd}\) and with \((r)f_{ab}\) by \(X\)-compensation are related by the same operators:

\[
\begin{align*}
(1)R_{cd}[X] &= R_{cd}(1)f, \\
(2)R_{cd}[X] &= R_{cd}(2)f + \mathcal{R}_{cd}(1)f, (1)f, 
\end{align*}
\] (83b)

where \((r)f\) is shorthand for \((r)f_{ab}[X]\). A similar result for the Einstein tensor can be derived from the above using (5).

This Replacement Principle has its origins in the work of Nakamura, although he does not state it explicitly in the above form. See, for example, Nakamura (2010), equations (B9)–(B13), in conjunction with equations (2.36) and (2.37).

The stress-energy tensor of a perfect fluid, as given by (50), can be viewed as a function of the variables \(F = (\rho, p, u_a, g_{ab})\). The perturbations of the stress-energy tensor can be written symbolically in the form:

\[
\begin{align*}
a^{2(1)}T^a_{\ b} &= T^a_{\ b}(1)F, \\
a^{2(2)}T^a_{\ b} &= T^a_{\ b}(2)F + \mathcal{T}^a_{\ b}(1)F, (1)F, 
\end{align*}
\] (84a)

These conditions arise in Nakamura’s work. See for example Nakamura (2007), equations (2.23) and (2.25).

The relation between gauge fixing and Nakamura’s method has also been discussed by Christopherson et al (2011).
where $T^a_b$ is the linear leading order operator and $T^a_b$ is the quadratic source term operator, and $(\tau)F = (\tau)_{\rho}^{(\rho)}p_{(\tau)u_{(\tau)g_{ab}}}$, with $r = 1, 2$. The Replacement Principle for the stress-energy tensor states that the gauge invariants associated with $(\tau)T^a_b$ and with $(\tau)F$ by $X$-compensation are related by the same operators:

$$(1)T^a_b[X] = T^a_b(1)F, \quad (2)T^a_b[X] = T^a_b(2)F + T^a_b(1)F, \quad (84)$$

where $(2)F$ is shorthand for $(2)F[X]$. This result can be deduced from Nakamura (2007).

### A.4 Construction of the gauge fields $X^a$

We begin with the metric mode decomposition (24). This decomposition assumes that the inverse operators (Green’s functions) $D^{-2}$, $(D^2 + 2K)^{-1}$ and $(D^2 + 3K)^{-1}$ exist (see UW1, and Nakamura (2007), page 19, for further discussion), as seen explicitly when one extract the various modes. In the present context this is accomplished by means of the following mode extraction operators:

$$S^i = D^{-2}D^i, \quad S^{ij} = \frac{3}{2}D^{-2}(D^2 + 3K)^{-1}D^{ij},$$

$$(85a)$$

$$\nabla^i = \delta^i_j - D_i S^j, \quad \nabla^i j^k = (D^2 + 2K)^{-1} \nabla^i (D^k),$$

$$(85b)$$

$$T_{ij}^{km} = \delta_{ij} (D^{km} - D^{(i} \nabla_{j)} km).$$

$$(85c)$$

Applying these operators to (24) gives

$$\varphi = -\frac{1}{2} f_{00}, \quad B = S^i f_{0i}, \quad B_i = \nabla^i f_{0j},$$

$$C = \frac{1}{2} S^{ij} f_{ij}, \quad C_i = \nabla^i \nabla^j f_{ij}, \quad C_{ij} = \frac{1}{2} T_{ij}^{km} f_{km},$$

$$\psi = -\frac{1}{6} (f - D^2 S^{ij} f_{ij}).$$

$$(86)$$

Similar relations hold for the modes of $(2)f_{ab}$. Explicitly extracting modes by means of the mode extraction operators (85) becomes essential at second order, and these operators are therefore used frequently in this paper.

In order to construct gauge invariants associated with the metric perturbations we apply (81) with $n = -2$ to $A_{ab} = g_{ab} = a^2 \tilde{g}_{ab}$ and define

$$f_{ab}[X] := f_{ab} - a^{-2} \mathcal{L}_{(1)X} \left(a^2 \gamma_{ab}\right),$$

$$(2)f_{ab}[X] := (2)f_{ab} - a^{-2} \mathcal{L}_{(2)X} \left(a^2 \gamma_{ab}\right) + \mathcal{F}_{ab}[X],$$

$$(87a)$$

$$(87b)$$

where

$$\mathcal{F}_{ab}[X] := -a^{-2} \mathcal{L}_{(1)X} \left(2a^2 f_{ab} - \mathcal{L}_{(1)X} \left(a^2 \gamma_{ab}\right) \right).$$

$$(87c)$$

We perform a mode decomposition of $f_{ab}[X]$ and $(2)f_{ab}[X]$ using equations (24) as a model, and we denote the dimensionless gauge invariant associated with $(\tau)g_{ab}$ by $(\tau)f_{ab}[X]$, $r = 1, 2$. For example, $(2)\varphi \rightarrow (2)\Phi[X], (2)B \rightarrow (2)B[X]$, as in equations (85).
We begin by expressing \( \Phi^{(r)} \) and \( \Psi^{(r)} \) in terms of the modes as defined by (24), and the mode decomposed spatial vectors \( (r)X_i = D_i^{(r)}X + (r)\tilde{X}_i \), which gives

\[
\begin{align*}
(r)\Phi[X] &= (r)\varphi - (\partial_\eta + H)(r)X^0 - \frac{1}{2}F_{00}[X], \\
(r)B[X] &= (r)B + (r)X^0 - \partial_\eta (r)X + S^iF_{0i}[X], \\
(r)B_i[X] &= (r)B_i - \partial_\eta (r)\tilde{X}_i + \psi_i F_{0j}[X], \\
(r)C[X] &= (r)C - (r)X + \frac{1}{2}S^{ij}\tilde{F}_{ij}[X], \\
(r)C_i[X] &= (r)C_i - (r)\tilde{X}_i + \psi_{jk}\tilde{F}_{jk}[X], \\
(r)C_{ij}[X] &= (r)C_{ij} + \frac{1}{2}T_{ij}^{km}\tilde{F}_{km}[X], \\
(r)\Psi[X] &= (r)\psi + H(r)X^0 - \frac{1}{6}(F^k_k[X] - D^2S^{ij}\tilde{F}_{ij}[X]),
\end{align*}
\]

where \( r = 1, 2 \), and the source terms \( F_{ab}[X] \) do not appear when \( r = 1 \).

We next determine \( (r)X \) and \( (r)\tilde{X}_i \) uniquely by imposing the conditions \( (r)C[X] = 0 \) and \( (r)C_i[X] = 0 \) in (88d) and (88e), respectively, which lead to

\[
\begin{align*}
(1)X &= (1)C, \\
(2)X &= (2)C + \frac{1}{2}S^{ij}\tilde{F}_{ij}[X], \\
(1)\tilde{X}_i &= (1)C_i, \\
(2)\tilde{X}_i &= (2)C_i + \psi_{jk}\tilde{F}_{jk}[X].
\end{align*}
\]

At this stage the mode decomposition for \( (r)f_{ab}[X], r = 1, 2 \) assumes the form

\[
\begin{align*}
(r)f_{00}[X] &= -2(r)\Phi[X], \\
(r)f_{0i}[X] &= D_i^{(r)}B[X] + (r)B_i[X], \\
(r)f_{ij}[X] &= -2(r)\Psi[X]_{ij} + 2(r)C_{ij}[X].
\end{align*}
\]

It remains to determine \( (r)X^0 \), where \( (1)X^0 \) together with \( (1)X = (1)C \) and \( (1)\tilde{X}_i = (1)C_i \) form \( (1)X^a \) which is to be inserted in \( \tilde{F}_{ij}[X] \) in (89i). There are two ways to determine \( (r)X^0 \) algebraically in a unique manner. The first way is to set \( (r)B[X] = 0 \), which via (88d) with (89) inserted yields

\[
\begin{align*}
(1)X^0 &= (1)X^0_c := -(1)B + \partial_\eta (1)C, \\
(2)X^0 &= (2)X^0_c := -(2)B + \partial_\eta (2)C - S^iF_{0i}[X_p].
\end{align*}
\]

Note that \( (1)X^0_p \) together with (89a) yields \( (1)X^a_p \). Substituting \( (1)X^a_p \) into \( \tilde{F}_{ij}[X] \) and \( F_{0i}[X_p] \) in (89b) and (91b), respectively, gives \( (2)X^a_p \). We refer to the gauge fields \( (r)X^a_p \) as the Poisson gauge fields since they result in the Poisson gauge invariants:

\[
\begin{align*}
(r)\Phi := (r)\Phi[X_p], \\
(r)\Psi := (r)\Psi[X_p], \\
(r)B_i[X_p], \\
(r)C_{ij}[X_p],
\end{align*}
\]

when (89) and (91) are inserted into (88).

The second way is to set \( (r)\Psi[X] = 0 \), which via (88g) with (89a) inserted gives

\[
\begin{align*}
(1)X^0 &= (1)X^0_c := -H^{-1}(1)\psi, \\
(2)X^0 &= (2)X^0_c := -H^{-1}(2)\psi + \frac{1}{6}(F^k_k[X_c] - D^2S^{ij}\tilde{F}_{ij}[X_c]),
\end{align*}
\]

This choice of the spatial gauge is essentially that made by Noh and Hwang (2004). See their equation (259) and the discussion on page 37.
where \((1)X^0\) together with \((89a)\) yields \((1)X^0_c\). Inserting \((1)X^0_c\) into \(\mathcal{F}_{ij}[X]\) in \((89b)\) and \((93b)\), and into \(\mathcal{F}^k_{ij}[X_c] \) in \((93b)\), gives \((2)X^0_c\). We refer to the gauge fields \((r)X^0_c\) as the uniform curvature gauge fields since they result in the uniform curvature gauge invariants.\(^{47}\)

\[
\begin{align*}
(r)A &:= (r)\Phi[X_c], \\
(r)B &:= (r)B[X_c], \\
(r)B_i[X_c], \\
(r)C_{ij}[X_c],
\end{align*}
\]

when \((89)\) and \((93)\) are inserted into \((88)\).

Note that at first order \((1)C_{ij}[X] = (1)C_{ij}, \) i.e., \((1)C_{ij}\) is independent of \((1)X^a,\) while at second order \((2)C_{ij}[X]\) depends on \(X\), so that \((2)C_{ij}[X_p]\) and \((2)C_{ij}[X_c]\) are unequal.

### B Source terms in the uniform curvature gauge

We make the replacement \(f_{ab} \rightarrow f_{ab}[X_c]\) in the expressions \((75)\) for the Einstein source terms using the metric perturbation \((26)\) in the uniform curvature gauge subject to the restriction \((33a)\), i.e.

\[
\begin{align*}
f_{00}[X_c] &= -2A, \\
f_{0i}[X_c] &= D_iB, \\
f_{ij}[X_c] &= 0.
\end{align*}
\]

The constituent terms, as given by equations \((61)\)–\((68)\), can be evaluated separately. This calculation yields the following expressions for the source terms in the uniform curvature gauge, assuming a purely scalar linear perturbation:

\[
\begin{align*}
\tilde{G}_{ij}(f_c, f_c) &= 2\left[2AD_{ij}(A + L_B B) + (\partial_\eta A + \frac{1}{3}D^2B)D_{ij}B\right] + (D_iA)D_{ij}B + (2\mathcal{H})B^2 - (D_kD_jB) - (D_kD_jB), \\
\mathcal{G}(f_c, f_c) &= -\frac{1}{3}(1 + 3C_0)R - \frac{1}{3}W^k + 2\mathcal{H}L_A(-4A^2 + (DB)^2) \\
&\quad - \frac{4}{3}[2AD^2(A + L_B B) + (\partial_\eta A - 4\mathcal{H}A)D^2B + (DA)^2], \\
\mathcal{G}_i(f_c, f_c) &= D_iR - 3\mathcal{G}_0, \\
\mathcal{G}^0_i(f_c, f_c) &= -2\mathcal{H}D_i(-4A^2 + (DB)^2) + \mathcal{G}^0, \\
\end{align*}
\]

where

\[
\begin{align*}
W^k &= 4\mathcal{H}[2AD^2B + (D^2A)D_kB], \\
R &= W^k - (D_kB)D_m^mB + \frac{2}{3}(D^2B)^2 - 4K(DB)^2, \\
\mathcal{G}^0_i &= -2(D_jA)D^iB + \frac{2}{3}(D_jA)D^2B - 8KA DB.
\end{align*}
\]

We note that the source terms in the Poisson gauge as given by \((55)\) were derived by imposing the restriction \(\Phi = \Psi,\) which led to significant simplification. If we impose the corresponding restriction in the uniform curvature case, namely \(A = -\mathcal{L}_B B,\) (see equations \((38a)\) and \((40a)\) in UW2), then the above expressions simplify somewhat, as can be seen by inspection.

\(^{47}\)We note in passing that that Christopherson et al (2011) have given the relation between the uniform curvature metric gauge invariants \((2)A, (2)B\) and the Poisson gauge invariants \((2)\Phi, (2)\Psi\) (see their equations \((4.56)\) and \((4.58)\)).
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