Nonlinear pseudo-bosons versus hidden Hermiticity

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Abstract

The increasingly popular concept of a ‘hidden’ Hermiticity of operators is compared with the recently introduced notion of nonlinear pseudo-bosons. The formal equivalence between these two notions is deduced under very general assumptions. Examples of their applicability in quantum mechanics are discussed.

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1. Introduction

The strong formal limitations imposed upon an observable operator $U$ by the requirement of its Hermiticity in a suitable Hilbert space $\mathcal{H}$, $U = U^\dagger$, have long been perceived as a challenge. In mathematics, for example, Dieudonné [1] introduced the notion of the so-called quasi-Hermiticity of $U$ based on the weakened requirement $TU = U^\dagger T$ with a suitable $T > 0$. He emphasized that without additional assumptions, unexpectedly, the adjoint $U^\dagger$ is not quasi-Hermitian in general, so the spectrum of $U$ is not necessarily real.

In the context of physics (and, most typically, in quantum mechanics), one usually tries to avoid similar paradoxes by accepting additional assumptions. In this context, one of the most successful attempted generalizations of the Hermiticity has been proposed by Scholtz \textit{et al} [2] who restricted their attention only to bounded observables $U \rightarrow A \in \mathcal{B}(\mathcal{H})$ and to bounded ‘metric’ operators $T$, $T^{-1} \in \mathcal{B}(\mathcal{H})$.

Unfortunately, being inspired and guided just by the well-known terminology used in linear algebra of the $N$ by $N$ matrices $A$ acting in the finite-dimensional Hilbert space $\mathcal{H}^{(N)} \equiv \mathbb{C}^N$, the authors of [2] gave their very narrow compact-operator subset of Dieudonné’s quasi-Hermitian set the same name. In our recent summary and completion of their proposal [3], therefore, we slightly modified the notation (replacing the symbol for ‘metric’, $T \rightarrow \Theta$) and, having recalled Smilga’s innovative terminology [4], we recommended to call the corresponding and very nicely behaved quasi-Hermitian operators $A_{\text{cryptohermitian}}$. 
Originally, the concept of cryptohermiticity (meaning, in essence, just a hidden form of Hermiticity [5]) proved successful just in the area of physics of heavy nuclei [2]. About 14 years ago a ‘new life’ of this concept has been initiated by Bender with coauthors. In a way inspired by the needs of quantum field theory [6, 7] and along the path independent of [2], they proposed an innovation of textbooks on quantum theory. Under the nickname of $\mathcal{PT}$-symmetric quantum theory, their formalism may be found described, e.g., in reviews [5, 8, 9].

Briefly, this formalism may be characterized by the heuristically fruitful postulate of the so-called $\mathcal{PT}$-symmetry of observables (here, $\mathcal{P}$ means parity while $\mathcal{T}$ denotes the antilinear operator of time reversal [10]). Secondly, the formalism replaces the most common physical assumption of the reality (i.e. observability) of the argument $x$ of the wavefunction $\psi(x)$ by the observability of the so-called charge $\mathcal{C}$. Ultimately, $\mathcal{PT}$-symmetric quantum theory eliminates the well-known interpretation ambiguities of the general cryptohermitian quantum theory [2, 11] by the recommended selection of the physical metric in the unique, factorized, $\mathcal{CPT}$-symmetry-minicking form of the product $\Theta^{(\mathcal{CPT})} = \mathcal{PC}$.

It is worth mentioning that the ideas coming from $\mathcal{PT}$-symmetric quantum theory proved inspiring and influential even far beyond quantum physics [12–14]. At the same time, the modified and extended forms of the cryptohermiticity with $\Theta \neq \mathcal{PC}$ had to be used for the description of the manifestly time-dependent quantum systems [15] and/or in the context of the scattering dynamical regime [16, 17]. In this paper, we intend to pay attention to another alternative to the formalism represented by the recent independent and parallel introduction and studies of the concept of the so-called pseudo-bosons (PB).

The most compact presentation of the latter PB concept is due to Trifonov [18]. A deeper understanding of its mathematics has been provided by the very recent series of papers [19–26] where one of us (FB) revealed the necessity as well as the key importance of the fully rigorous treatment of some of the underlying formal questions. We intend to continue these studies in what follows.

The open and interesting formal questions arise from the bosonic form of the canonical commutation relation $[a, a^\dagger] = 1$ upon replacing $a^\dagger$ by another (unbounded) operator $b$ not (in general) related to $a$: $[a, b] = 1$. More recently, FB has extended the general settings to what has been called nonlinear pseudo-bosons (NLPB, [27]) where the role of the commutation relation is replaced by a different requirement (see below). This extension is motivated by the attempt to include, in this general settings, Hamiltonian-like operators which have a rich spectrum, and in particular eigenvalues, labeled by a set of quantum numbers $n_1, n_2, \ldots$, which are not linear functions of $n_j$'s. An interesting aspect of this construction is the possibility of getting operators ($M$ and $M^\dagger$) which are not self-adjoint but still have real eigenvalues. This peculiarity is well explained by the presence of an intertwining operator (IO) between, say, $M$ and a third self-adjoint operator, $\tilde{M}$. General results on IO show that, in this case, $M$ and $\tilde{M}$ are isospectral, and their eigenvectors are also related by the IO itself.

On the other hand, the use of the above-mentioned notion of cryptohermiticity (CH) of a given operator opens the possibility of the parallel work with a given operator using its parallel representations in several Hilbert spaces, mutually not necessarily related by a unitary transformation. In this setting, it is obvious that the PB and CH concepts may be related, meaning that the pseudo-bosonic settings provide examples of the general statement introduced for cryptohermitian operators or vice versa.

In this paper, we intend to proceed with this analysis, showing a sort of equivalence between NLPB and CH. In particular, we will show that under very reasonable assumptions, any cryptohermitian operator gives rise to a family of NLPB which are regular (NLRPB), see
below, and vice-versa, each family of NLRPB produces in a natural way a cryptohermitian operator.

The paper is organized as follows: in the next section, after a short introduction to NLRPB, we prove the equivalence outlined above between these excitations and a certain cryptohermitian operator. Sections 3 and 4 are devoted to examples, while our conclusions are given in section 5.

2. NLRPB versus cryptohermiticity

We begin this section with a short review of NLPB, giving some details in particular on the role of bounded or unbounded operators. We refer to [27] for some preliminary examples of this construction. Other examples will be discussed in sections 3 and 4.

2.1. Nonlinear RPB

In [27], FB has used the main ideas which produce, out of coherent states, the so-called nonlinear coherent states, to extend the original framework proposed for pseudo-bosons to what he has called nonlinear pseudobosons. The starting point is a strictly increasing sequence \( \{ \epsilon_n \} \) such that \( \epsilon_0 = 0 \), \( 0 = \epsilon_0 < \epsilon_1 < \cdots < \epsilon_n < \cdots \). Then, given two operators \( a \) and \( b \) on the Hilbert space \( \mathcal{H} \),

**Definition 1.** We will say that the triple \((a, b, \{\epsilon_n\})\) is a family of NLRPB if the following properties hold:

- **p1:** a nonzero vector \( \Phi_0 \) exists in \( \mathcal{H} \) such that \( a \Phi_0 = 0 \) and \( \Phi_0 \in D(\mathcal{M}(a)) \).
- **p2:** a nonzero vector \( \eta_0 \) exists in \( \mathcal{H} \) such that \( b^\dagger \eta_0 = 0 \) and \( \eta_0 \in D(\mathcal{M}(b)) \).
- **p3:** calling \( \Phi_n := \frac{1}{\sqrt{\epsilon_n}} b^\dagger \Phi_0 \), \( \eta_n := \frac{1}{\sqrt{\epsilon_n}} a^\dagger \eta_0 \), (2.1)
we have, for all \( n \geq 0 \),

\[
a \Phi_n = \sqrt{\epsilon_n} \Phi_{n-1}, \quad b^\dagger \eta_n = \sqrt{\epsilon_n} \eta_{n-1}.
\] (2.2)

- **p4:** the sets \( \mathcal{F}_\Phi = \{ \Phi_n, n \geq 0 \} \) and \( \mathcal{F}_\eta = \{ \eta_n, n \geq 0 \} \) are bases of \( \mathcal{H} \).
- **p5:** \( \mathcal{F}_\Phi \) and \( \mathcal{F}_\eta \) are Riesz bases of \( \mathcal{H} \).

As noticed in [27], the definitions in (2.1) are well posed in the sense that, because of p1 and p2, the vectors \( \Phi_n \) and \( \eta_n \) are well-defined vectors of \( \mathcal{H} \) for all \( n \geq 0 \). Moreover, for p3, the other conditions above coincide exactly with those of RPB. In fact, we can show that p3 replaces (and extends) the commutation rule \([a, b] = 1\), which is recovered if \( \epsilon_n = n \). Moreover, if all but p5 are satisfied, then we have called our particles NLPB.

Let us introduce the following (not self-adjoint) operators:

\[
M = ba, \quad \mathcal{M} = M^\dagger = a^\dagger b^\dagger.
\] (2.3)
Then, we can check that \( \Phi_n \in D(M) \cap D(b) \), \( \eta_n \in D(\mathcal{M}) \cap D(a^\dagger) \), and, more than this, that

\[
b \Phi_n = \sqrt{\epsilon_n+1} \Phi_{n+1}, \quad a^\dagger \eta_n = \sqrt{\epsilon_n+1} \eta_{n+1}.
\] (2.4)
which is a consequence of definitions (2.1), as well as

\[
M \Phi_n = \epsilon_n \Phi_n, \quad \mathcal{M} \eta_n = \epsilon_n \eta_n.
\] (2.5)
These eigenvalue equations imply that the vectors in \( \mathcal{F}_\Phi \) and \( \mathcal{F}_\eta \) are mutually orthogonal. More explicitly,

\[
\langle \Phi_n, \eta_m \rangle = \delta_{n,m},
\]

(2.6)

where we have fixed the normalization of \( \Phi_0 \) and \( \eta_0 \) in such a way that \( \langle \Phi_0, \eta_0 \rangle = 1 \).

In [27], we have also proved that conditions \{p1, p2, p3, p4\} are equivalent to \{p1, p2, p3’, p4\}, where

p3’: the vectors \( \Phi_n \) and \( \eta_n \) defined in (2.1) satisfy (2.6).

In the following, therefore, we can use p3 or p3’ depending of which is more convenient for us.

Carrying on our analysis on the consequences of the definition on NLRPB, and in particular of p4, we rewrite this assumption in bra-ket formalism as

\[
\sum_n \langle \Phi_n | \eta_n \rangle = \sum_n |\eta_n \rangle \langle \Phi_n| = 1,
\]

(2.7)

while p5 implies that the operators \( S_\Phi := \sum_n |\Phi_n \rangle \langle \Phi_n| \) and \( S_\eta := \sum_n |\eta_n \rangle \langle \eta_n| \) are positive, bounded, invertible and that \( S_\Phi = S_\eta^{-1} \). The new fact is that the operators \( a \) and \( b \) do not, in general, satisfy any simple commutation rule. Indeed, we can check that, for all \( n \geq 0 \),

\[
[a, b] \Phi_n = (\epsilon_{n+1} - \epsilon_n) \Phi_n,
\]

(2.8)

which is different from \([a, b] = 1\) in general. In [27] it has also been proved that, not surprisingly, if \( \sup_n \epsilon_n = \infty \), then the operators \( a \) and \( b \) are unbounded. We end this overview mentioning also that \( M \) and \( \mathfrak{M} \) are connected by an intertwining operator related to \( S_\Phi \). We will use this property in what follows.

2.2. Connection with cryptohermiticity

The introduction of the above-mentioned notion of cryptohermiticity of a given operator \( H \) [3] enables us to distinguish between the ‘first’ Hilbert space \( \mathcal{H}^{(F)} \) (in which the operator in question is not self-adjoint, \( H \neq H^\dagger \)) and the ‘second’ Hilbert space \( \mathcal{H}^{(S)} \) in which the same operator is self-adjoint (one may write, e.g. [3], \( H = H^\dagger \)). The idea behind such an apparent paradox is that one can say that \( H \) can only be declared self-adjoint with respect to a definite scalar product. In this sense, one usually starts from the ‘friendly’ definition of the so-called Dirac’s (i.e. roughly speaking, ‘transposition plus complex conjugation’) definition of \( H^\dagger \) in \( \mathcal{H}^{(F)} \) and complements it by the mere modification of the inner product in \( \mathcal{H}^{(F)} \) yielding the explicit definition of \( H^\dagger = \Theta^{-1} H^\dagger \Theta \) written in terms of the positive metric operator \( \Theta = \Theta^{(S)} \neq I \) which should be, together with its inverse [2], bounded and self-adjoint in the ‘friendly’ space \( \mathcal{H}^{(F)} \).

In the models where \( \Theta \) as well as \( \Theta^{-1} \) are bounded, one can comparatively easily deal with mathematical questions. Otherwise, there emerge several subtle points related to the domain of the operators in question. Due to the relevance of the metric operator \( \Theta \), let us make now the standard notation conventions less ambiguous.

**Definition 2.** Let us consider two (not necessarily bounded) operators \( H \) and \( \Theta \) acting on the Hilbert space \( \mathcal{H} \), with \( \Theta \) positive and invertible. Let us call \( H^\dagger \) the adjoint of \( H \) in \( \mathcal{H} \) with respect to its scalar product and \( H^\dagger = \Theta^{-1} H^\dagger \Theta \), when this exists. We will say that \( H \) is cryptohermitian with respect to \( \Theta \) (CHwt(\Theta)) if \( H = H^\dagger \).

Using standard facts on the functional calculus, it is obvious that the operators \( \Theta^{\pm 1/2} \) are well defined. Hence, we can introduce an operator \( h := \Theta^{1/2} H \Theta^{-1/2} \) at least if the domains of the operators allow us to do so. More explicitly, \( h \) is well defined if taken \( f \in D(\Theta^{-1/2}) \),
\( \Theta^{-1/2} f \in D(H) \) and if \( H \Theta^{-1/2} f \in D(\Theta^{1/2}) \). Of course, these requirements are surely satisfied if \( H \) and \( \Theta^{1/2} \) are bounded. Otherwise some care is required. It is easy to check that \( h = h^\dagger \). Hence, the following definition appears natural:

**Definition 3.** Assume that \( H \) is \( CH\)Wrt \( \Theta \) for \( H \) and \( \Theta \) as above. \( H \) is well behaved w.r.t. \( \Theta \) if \( h \) has only discrete eigenvalues \( \epsilon_n, n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \), with eigenvectors \( e_n; \) \( he_n = \epsilon_n e_n, n \in \mathbb{N}_0 \).

It is convenient, but not really necessary, to restrict ourselves to the case in which the multiplicity of each eigenvalue \( \epsilon_n \), \( m(\epsilon_n) \), is 1. To fix the ideas, we also assume that \( 0 = \epsilon_0 < \epsilon_1 < \epsilon_2 < \cdots \). The above definition implies that the set \( \mathcal{E} = \{e_n, n \in \mathbb{N}_0\} \) is an orthonormal basis of \( \mathcal{H} \), so that it produces a resolution of the identity which we write in the bra-ket language as \( \sum_{n=0}^{\infty} |e_n\rangle \langle e_n| = 1 \). The following theorem can be proved:

**Theorem 1.** Let \( H \) be well behaved w.r.t. \( \Theta \), where \( \Theta, \Theta^{-1} \in B(\mathcal{H}) \). Then, it is possible to introduce two operators \( a \) and \( b \) on \( \mathcal{H} \), and a sequence of real numbers \( \{\epsilon_n, n \in \mathbb{N}_0\} \), such that the triple \( (a, b, \{\epsilon_n\}) \) is a family of NLRPB.

Vice versa, if \( (a, b, \{\epsilon_n\}) \) is a family of NLRPB, two operators can be introduced, \( H \) and \( \Theta \), such that \( \Theta, \Theta^{-1} \in B(\mathcal{H}) \), and \( H \) is well behaved w.r.t. \( \Theta \).

**Proof.** To prove the first part of the theorem we introduce the following families of vectors of \( \mathcal{H} \):

\[
\mathcal{F}_\Theta = \{\Phi_n := \Theta^{-1/2} e_n, n \in \mathbb{N}_0\}, \quad \mathcal{F}_\eta = \{\eta_n := \Theta^{1/2} e_n, n \in \mathbb{N}_0\}.
\]

Because of our assumptions, \( \mathcal{F}_\Theta \) and \( \mathcal{F}_\eta \) are Riesz bases of \( \mathcal{H} \) which are also biorthonormal: \( \langle \Phi_n, \eta_k \rangle = \delta_{n,k} \). Hence, conditions p4 and p5 are satisfied. On \( \mathcal{F}_\Theta \) we define two operators \( a \) and \( b \) as follows:

\[
a \Phi_n = \sqrt{\epsilon_n} \Phi_{n-1}, \quad b \Phi_n = \sqrt{\epsilon_{n+1}} \Phi_{n+1}, \tag{2.9}
\]

for all \( n \geq 0 \). By definition, p1 is satisfied: \( a \Phi_0 = 0 \) since \( \epsilon_0 = 0 \) and \( \Phi_0 \in D^\infty(b) \) since, iterating the second equation in (2.9) we deduce that \( b^n \Phi_0 = \sqrt{\epsilon_n!} \Phi_n \), which shows that \( b^n \Phi_0 \) is well defined for all \( n \), being \( \Phi_n = \Theta^{-1/2} e_n \in \mathcal{H} \). To check condition p2 we first have to compute the action of \( a^\dagger \) and \( b^\dagger \) on a suitable dense set in \( \mathcal{H} \). It is easy to check that (2.9), together with the fact that \( \langle \Phi_n, \eta_k \rangle = \delta_{n,k} \), imply that

\[
a^\dagger \eta_n = \sqrt{\epsilon_{n+1}} \eta_{n+1}, \quad b^\dagger \eta_n = \sqrt{\epsilon_n} \eta_{n-1}, \tag{2.10}
\]

for all \( n \geq 0 \). Equations (2.9) and (2.10) imply that, for instance, \( b \) is a raising operator for \( \mathcal{F}_\Theta \) but \( b^\dagger \) is a lowering operator for \( \mathcal{F}_\eta \). From (2.10), we see that \( b^\dagger \eta_0 = 0 \). Iterating the first equation, we also find that \( (a^\dagger)^n \eta_0 = \sqrt{\epsilon_n}! \eta_n = \sqrt{\epsilon_n!} \Theta^{1/2} e_n \), which is a well-defined vector in \( \mathcal{H} \) for all \( n \geq 0 \). Hence, p2 is satisfied. We end this part of the proof noting that p3' surely holds and, as a consequence, also p3 is verified. Hence, \( (a, b, \{\epsilon_n\}) \) is a family of NLRPB, as expected.

Let us now prove the converse: we assume that \( (a, b, \{\epsilon_n\}) \) is a family of NLRPB and we show how to construct two operators, \( H \) and \( \Theta \), such that \( H \) is cryptohermitian and well behaved w.r.t. \( \Theta \).

This proof is based on the fact that since \( \mathcal{F}_\Theta \) and \( \mathcal{F}_\eta \) are Riesz bases, the operators

\[
S_\Theta := \sum_{n=0}^{\infty} |\Phi_n\rangle \langle \Phi_n|, \quad S_\eta := \sum_{n=0}^{\infty} |\eta_n\rangle \langle \eta_n| \tag{2.11}
\]

are both positive and bounded. Assuming that \( \langle \Phi_0, \eta_0 \rangle = 1 \), they satisfy \( S_\Theta = S_\eta^{-1} \). The operator \( H := ba \) is well (and densely) defined since, because of p3, \( \Phi_n \in D(a) \) and \( a \Phi_n \in D(b) \). More than this, we deduce that \( H \Phi_n = \epsilon_n \Phi_n \). Analogously we find that
\( \eta_n \in D(H^\dagger) \), and that \( H^\dagger \eta_n = \epsilon_n \eta_n \) for all \( n \geq 0 \). Since \( S_{\Phi} \eta_n = \Phi_n \) and \( S_\eta \Phi_n = \eta_n \), we can rewrite this last eigenvalue equation as \( S_{\Phi}^{-1} H^\dagger S_\eta \Phi_n = \epsilon_n \Phi_n \), which together with the first eigenvalue equation and using the completeness of \( \mathcal{F}_\Phi \) implies that \( H = S_{\eta}^{-1} H^\dagger S_\eta \). Hence, \( H \) is CHwrt \( S_\eta \). Due to the properties of intertwining operators \( H, H^\dagger \) and \( h := S_{\eta}^{1/2} H S_{\eta}^{-1/2} \) all have the same eigenvalues and related eigenvectors. This concludes the proof. \( \square \)

We want to briefly consider few consequences and remarks of this theorem.

1. The formal expressions of the operators introduced so far can be easily deduced. For instance, we have

\[
a = \sum_{n=0}^\infty \sqrt{\epsilon_n} |\Phi_{n-1}\rangle \langle \eta_n|, \quad b = \sum_{n=0}^\infty \sqrt{\epsilon_n} |\Phi_{n+1}\rangle \langle \eta_n|.
\]

From these we can also deduce the formal expansions for \( a^\dagger \) and \( b^\dagger \). Moreover, \( h = \sum_{n=0}^\infty \epsilon_n |\epsilon_n\rangle \langle \epsilon_n|, H = \sum_{n=0}^\infty \epsilon_n |\Phi_n\rangle \langle \eta_n| \) and \( H^\dagger = \sum_{n=0}^\infty \epsilon_n |\eta_n\rangle \langle \Phi_n| \). These formulas show, among other features, that \( h, H \) and \( H^\dagger \) are isospectrals.

2. A straightforward computation shows that \( S_{\Phi} = \Theta^{-1} \) and \( S_\eta = \Theta \), as we have also deduced in the proof of the second part of the theorem. This fact, together with our previous results, shows that the frame operators \( S_{\Phi} \) and \( S_\eta \), as well as their square roots, behave as intertwining operators. This is exactly the same kind of results we can deduce for ordinary pseudo-bosons, where biorthogonal Riesz bases and intertwining operators are recovered.

3. Even if \( h \) is not required to be factorizable, because of our construction it turns out that it can be written as \( h = b_\theta a_\theta \), where \( a_\theta = \Theta^{1/2} a \Theta^{-1/2} \) and \( b_\theta = \Theta^{1/2} b \Theta^{-1/2} \). Incidentally, in general \( [a_\theta, b_\theta] = \Theta^{1/2} [a, b] \Theta^{-1/2} \neq [a, b] \), but if \( [[a, b], \Theta^{1/2}] = 0 \), which is the case for pseudo-bosons. Therefore, at least at a formal level, our construction shows that the Hamiltonian \( h \) can be written in a factorized form.

4. The reasons for the attention paid to the role of Riesz bases may be traced back to Mostafazadeh’ results. In chapter 2 of \([5]\) (cf also references therein, or \([11] \) and \([27]\)) he emphasized that in the methodical analyses of the formalism of pseudo-Hermitian quantum mechanics, it makes sense to pay particular attention to the finite-dimensional Hilbert spaces for simplicity. This inspired not only the present proof but also the popular constructions of metric operators using Riesz bases formed by eigenstates of non-Hermitian Hamiltonians. The same idea also helped to clarify the essence of the problem of the ambiguity of the metric as formulated by Scholtz et al \([2]\).

5. Although we deal here with an infinite-dimensional Hilbert space in general, it makes good sense to contemplate a reduction of our observations to a finite-dimensional Hilbert space. In such a simplified scenario, one reveals several interesting connections with the recent \( n \)-level coherent-state constructions by Najarbashi et al \([28]\).

3. Illustrative matrix models with ascending spectra

3.1. A two-by-two example with two free parameters

We consider first a two-dimensional illustrative schematic matrix example, originally introduced in \([27]\). Let \( \mathcal{H} = \mathbb{C}^2 \) be our Hilbert space and let us consider the following matrices on \( \mathcal{H} \):

\[
A = \begin{pmatrix} -1 & \beta \\ \frac{1}{\beta} & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & \delta \\ \frac{1}{\delta} & 1 \end{pmatrix},
\]

(3.1)
where \( \beta \neq \delta \) are real quantities. The vectors \( \Phi_0 = y(\beta) \) and \( \eta_0 = w(1/\beta) \) satisfy \( A\Phi_0 = B^*\eta_0 = 0 \) and contain normalization constants \( y \) and \( w \) which we take real and constrained, \( yw(\beta - \delta) = 1 \). Putting \( \epsilon_0 = 0 \) and \( \epsilon_1 = -\frac{1}{\beta^2}(\beta - \delta)^2 \), we define

\[
\Phi_1 = \frac{1}{\sqrt{\epsilon_1}} B\Phi_0 = \frac{y}{\sqrt{\epsilon_1}} \left( \frac{\delta - \beta}{\beta} + 1 \right) \quad \text{and} \quad \eta_1 = \frac{1}{\sqrt{\epsilon_1}} A^*\eta_0 = \frac{w}{\sqrt{\epsilon_1}} \left( \frac{\delta - \beta}{\beta} - 1 \right).
\]

Hence, both \( \mathcal{F}_\Phi = \{\Phi_0, \Phi_1\} \) and \( \mathcal{F}_\eta = \{\eta_0, \eta_1\} \) are (biorthogonal) Riesz bases of \( \mathcal{H} \), satisfying \( A\Phi_0 = B^*\eta_0 = 0, A\Phi_1 = \sqrt{\epsilon_1}\Phi_0 \) and \( B^*\eta_1 = \sqrt{\epsilon_1}\eta_0 \). With this choice, calling

\[
M = BA = \begin{pmatrix}
1 - \frac{\delta}{\beta} & \delta - \beta \\
\frac{1}{\beta} - \frac{\delta}{\beta} & -\frac{\delta}{\beta} + 1
\end{pmatrix} \quad \mathfrak{M} = A^*B^* = \begin{pmatrix}
1 - \frac{\delta}{\beta} & \frac{1}{\beta} - \frac{1}{\beta} \\
\delta - \beta & -\frac{\delta}{\beta} + 1
\end{pmatrix},
\]

(3.2)

we can check that \( M\Phi_k = \epsilon_k\Phi_k \) and \( \mathfrak{M}\eta_k = \epsilon_k\eta_k \), \( k = 0, 1 \). It is also easy to compute \( [A, B] \), which is different from zero if \( \delta \neq \beta \) and it is never equal to the identity operator. Also, we have \( \sum_{k=0}^{1} |\Phi_k| |\eta_k| = 1 \) and

\[
S_\Phi = y^2 \begin{pmatrix}
(\beta(\beta - \delta)) & 0 \\
0 & 1 - \frac{\beta}{\delta}
\end{pmatrix}, \quad S_\eta = w^2 \begin{pmatrix}
1 - \frac{\delta}{\beta} & 0 \\
0 & \delta(\delta - \beta)
\end{pmatrix}.
\]

(3.3)

A direct computation finally shows that \( S_\Phi = S_\eta^{-1} \) and that \( MS_\Phi = S_\Phi\mathfrak{M} \). This can be written as \( M = S_\eta^{-1}\mathfrak{M}S_\eta \), which shows that \( M \) is CHWrt\( S_\eta \). Moreover, as is clear, \( S_\eta \) and \( S_\eta^{-1} = S_\Phi \) are bounded operators. Hence, the first part of the second statement of theorem 1 is recovered. To check that \( M \) is also well behaved wrt\( S_\eta \), it is sufficient to compute \( h = S_h^{1/2}MS_h^{-1/2} \), and to compute the two eigenvalues which must have multiplicity 1. This is a simple exercise in linear algebra and will not be done here.

3.2. An \( N \) by \( N \) matrix example without free parameters

Whenever one tries to apply the principles of cryptohermitian quantum mechanics in phenomenology, say, of solid-state physics [29], one must contemplate matrices (2.3) of perceptibly larger dimensions \( N \gg 2 \). In such a realistic setting, one is usually forced to employ a suitable purely numerical method. Typically, it is practically impossible to employ the finite-dimensional version

\[
a = \sum_{n=1}^{N-1} \sqrt{\epsilon_n} |\Phi_{n-1}\rangle \langle \eta_n|, \quad b = \sum_{n=0}^{N-2} \sqrt{\epsilon_{n+1}} |\Phi_{n+1}\rangle \langle \eta_n|
\]

(3.4)

of the spectral-like expansion formula (2.12) because its components themselves are only available, generically, in a purely numerical representation. The situation further worsens if one tries to render the ‘phenomenological input’ matrices (2.3) varying with a suitable coupling-simulating parameter.

Fortunately, several arbitrary-\( N \) benchmark examples have been recently found in the context of a cryptohermitian reinterpretation of certain properties of the classical orthogonal polynomials [30–32]. For our present illustrative purposes, the latter reference proves particularly suitable since it renders both the underlying \( N \) by \( N \) Schrödinger equations, \( H\Phi_n = \epsilon_n\Phi_n \) and \( H^*\eta_n = \epsilon_n\eta_n \), exactly solvable.

At the general matrix dimension \( N \gg 2 \), the main message delivered by [32] may be read as the discovery of the feasibility of the construction of the \( N \)-parametric metrics \( \Theta \) of which the above-defined matrices \( S_\eta \) represent just the NLPB-related special cases of present interest.
In the opposite direction, the above-mentioned exact solvability of the pair of Schrödinger equations will make it easy, for us, to feel guided by our previous benchmark example of paragraph 3.1.

Our present extension of the above illustrative $N = 2$ considerations to all the finite integers $N = 2, 3, \ldots$ will be based on the results of [32] where the Hamiltonian-simulating matrices were chosen in the form which we will denote by the tilded symbol

$$\tilde{H} = \begin{bmatrix} 0 & 2 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}. \quad (3.5)$$

The standard Hermiticity condition is obviously violated here. In the notation of [32] and via the underlying conjugate pair of the linear algebraic Schrödinger eigenvalue problems

$$\tilde{H} \Phi_n = E_n \Phi_n, \quad [\tilde{H}]^\dagger \eta_n = E_n \eta_n \quad (3.6)$$

we get, in particular, the site-indexed components $|\alpha \rangle \Phi_n, \alpha = 1, 2, \ldots, N$ of the $n$th eigenstate $\Phi_n$ of our Hamiltonian (where, conventionally, $n = 0, 1, \ldots, N - 1$) in the following closed and arbitrarily normalized form:

$$\{1 \vert \Phi \rangle = T(0, x) = 1, \quad \{2 \vert \Phi \rangle = T(1, x) = x, \quad \{3 \vert \Phi \rangle = T(2, x) = 2x^2 - 1, \ldots, \{N \vert \Phi \rangle = T(N - 1, x), \quad (3.7)$$

where the letter $T$ denotes the classical orthogonal Chebyshev polynomials of the first kind.

One can easily deduce [32] that

$$\Phi_n = \begin{pmatrix} T(0, x) \\ T(1, x) \\ \vdots \\ T(N - 1, x) \end{pmatrix}.$$ 

The argument $x = x^{(N)}_n$ of these polynomials is fixed by the secular equation $T(N, x_n) = 0$ which is exactly solvable:

$$E_n = 2x^{(N)}_n = -2 \cos \frac{(n + 1/2)\pi}{N}, \quad n = 0, 1, \ldots, N - 1. \quad (3.8)$$

This formula defines the necessary $N$-plet of energies at every dimension $N$ (note that in comparison with [32] a more natural and convenient choice of the minus sign is being used here and in what follows).

For our present purposes, we still need to replace the tilded, auxiliary Hamiltonians $\tilde{H}$ of equation (3.5) (which do not exhibit the above-required positive-semidefiniteness of the spectrum) by our following untilded, constant-shifted ultimate matrices $M$ which are real and manifestly non-Hermitian in the conventional sense:

$$M = \begin{bmatrix} Z & 2 & 0 & \cdots & 0 \\ 1 & Z & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & Z & 1 \\ 0 & \cdots & 0 & 1 & Z \end{bmatrix} \quad (3.9)$$
and where we choose $Z = E_{N-1} > 0$. In terms of these matrices with the property $M \neq M^T := M^T$ (the superscript $T$ marks transposition), we may write down our final, untilded pair of toy-model Schrödinger equations

\[ M \Phi_n = \epsilon_n \Phi_n, \quad M^T \eta_n = \epsilon_n \eta_n \tag{3.10} \]

with the sharply ascending spectrum $\epsilon_n = E_n + Z \geq 0$.

4. Closed-form constructions at $N \leq 5$

One of the most important merits of our toy-model matrix (3.9) has been found to lie in the closed form of the solution of the second, conjugate Schrödinger equation in (3.10). For the corresponding lattice-site unnormalized components $\{\alpha|\eta\}$ one obtains the following, almost identical prescription:

\[ \{\alpha|\eta\} = T(n, x), \quad \alpha = 2, 3, \ldots, N \tag{4.1} \]

which does not differ from its predecessor (3.7) but which must be complemented by the single different missing item $\{1|\eta\} = T(0, x)/2 = 1/2$. The latter feature makes the resulting biorthogonal system of vectors deceptively similar to an orthogonal system. Thus, for many purposes it proves useful to separate the whole set of sites into the ‘exceptional’ item $\alpha = 1$ accompanied by the $(N - 1)$-dimensional rest. The more detailed description of several technical consequences of this split may be found in [32]. A particularly important question of the appropriate choice of normalization has been analyzed in [11].

4.1. The choice of $N = 2$

For our present purposes, it is particularly useful to recall formula (3.4) in its utterly elementary $N = 2$ version

\[ a = \sqrt{\epsilon_1}\Phi_0(\eta_1), \quad b = \sqrt{\epsilon_1}\Phi_1(\eta_0), \tag{4.2} \]

where we have to insert the eigenvalues $\epsilon_0 = 0$ and $\epsilon_1 = 2\sqrt{2}$ of the matrix

\[ M = \begin{bmatrix} \sqrt{2} & 2 \\ 1 & \sqrt{2} \end{bmatrix} \]

and of its transpose. The four properly re-normalized real eigenvectors of these matrices may be easily found and written, for typographical reasons, in the transposed form

\[ \Phi_0^T = c_{R0}[1/\sqrt{2}, -1/2], \quad \Phi_1^T = c_{R1}[1, 1/\sqrt{2}], \]

\[ \eta_0^T = c_{L0}[1/\sqrt{2}, -1], \quad \eta_1^T = c_{L1}[1/2, 1/\sqrt{2}]. \]

By the Schrödinger equations themselves, the quadruplet of the normalization constants $c$ is left arbitrary. In the present NLRPB context, it is assumed that we choose their values is such a way that our vectors form a biorthonormalized system. At $N = 2$, it is then the matter of elementary algebra to specify, say, $c_{R0} = c_{R1} = c_{L0} = c_{L1} = 1$.

At this point it is necessary to realize [11] that at any $j = 0, 1, \ldots, N - 1$, the simultaneous multiplication of $c_{Lj}$ and division of $c_{Rj}$ by the same constant $v_j$ will keep the biorthonormality and bicompleteness relations unchanged. In this sense, formulas (2.11) define just a very specific, $v_j = 1$ metric which is unique. At $N = 2$, in particular, our choice of the normalization leads to the metric

\[ S^{(2)}_n = |\eta_0\rangle\langle \eta_0| + |\eta_1\rangle\langle \eta_1| = \frac{1}{4} \begin{bmatrix} 3 & -\sqrt{2} \\ -\sqrt{2} & 6 \end{bmatrix}. \tag{4.3} \]
Its eigenvalues \( s_k = (9 \pm \sqrt{17})/8 \) are both positive. The same observation can also be made at the higher dimensions \( N \). The most important conclusion may be already drawn from our first nontrivial illustration (4.3) which shows that in contrast to the previous \( N = 2 \) model (3.3) of section 3.1, the matrix elements of the generic metric \( S^{(N)}_\eta \) (as well as of its inverse \( S^{(N)}_\eta^{-1} \)) will be all non-zero.

Another choice of the above-mentioned scaling parameters \( v_j \neq 1 \) could be employed converting the metric \( S^{(N)}_\eta \) into a diagonal or sparse matrix. The price to be paid is that the necessary proof of the positivity of this matrix becomes nontrivial and dimension dependent in general. More details (as well as a few elementary sample constructions of the families of metrics assigned to our present illustrative zero-parametric Hamiltonians) may be found in [32].

4.2. The next special case with \( N = 3 \)

The required insertion in the explicit \( N = 3 \) recipe
\[
\begin{align*}
    a &= \sqrt{\epsilon_1} |\Phi_0\rangle \langle \eta_1| + \sqrt{\epsilon_2} |\Phi_1\rangle \langle \eta_2|, \\
    b &= \sqrt{\epsilon_1} |\Phi_1\rangle \langle \eta_0| + \sqrt{\epsilon_2} |\Phi_2\rangle \langle \eta_1|
\end{align*}
\]  
may be based on the not too tedious evaluation of the eigenvalues \( \epsilon_0 = 0, \epsilon_1 = \sqrt{3} \) and \( \epsilon_2 = 2\sqrt{3} \) and of the respective eigenvectors
\[
\begin{align*}
    \Phi_0^T &= c_{R0}[1, \sqrt{-\sqrt{3}}, 1/2], \\
    \Phi_1^T &= c_{R1}[1, -1, 0, 1], \\
    \Phi_2^T &= c_{R2}[1, \sqrt{3}, 1], \\
    \eta_0^T &= c_{L0}[1, \sqrt{-\sqrt{3}}, 1], \\
    \eta_1^T &= c_{L1}[1, 0, -2], \\
    \eta_2^T &= c_{L2}[1, \sqrt{3}, 1]
\end{align*}
\]

of the matrix
\[
M = \begin{bmatrix}
\sqrt{3} & 2 & 0 \\
1 & \sqrt{3} & 1 \\
0 & 1 & \sqrt{3}
\end{bmatrix}
\]

and of its transpose. One can again proceed in full analogy with the above \( N = 2 \) example. It is perhaps interesting to add that the NLRPB-related special metric matrices
\[
S^{(3)}_\eta = |\eta_0\rangle \langle \eta_0| + |\eta_1\rangle \langle \eta_1| + |\eta_2\rangle \langle \eta_2|
\]  
need not necessarily remain non-sparse. For example, the judicious choice
\[
c_{R0} = 1/3, \quad c_{R1} = -1/3, \quad c_{R2} = 1/6, \quad c_{L0} = 1, \quad c_{L1} = 1, \quad c_{L2} = 1
\]
of the normalization parameters generates the elementary diagonal metric (4.5),
\[
S^{(3)}_\eta = \begin{bmatrix}
3 & 0 & 0 \\
0 & 6 & 0 \\
0 & 0 & 6
\end{bmatrix}
\]  

(4.6)

The main merit of such a diagonal special case (non-numerically accessible, in our present model, at any \( N \)) may be seen in the facilitated feasibility of the evaluation of the manifestly Hermitian isospectral Hamiltonian
\[
h = h^{(3)} = S^{1/2}
\]  

possessing the following illustrative set of orthonormal eigenvectors:
\[
\begin{align*}
    \epsilon_0^T &= c_{R0}[1, -\sqrt{3}, 1/2], \\
    \epsilon_1^T &= c_{R1}[1, 0, -2], \\
    \epsilon_2^T &= c_{R2}[1, \sqrt{3}, 1].
\end{align*}
\]

Obviously, all of the alternative normalizations and analogous insertions in the above-listed formulas remain routine. They will lead to non-numerical, fully exact formulas. The only remaining challenge may be seen in the extension of the spectral-like representation of the operators \( a \) and \( b \) beyond the ‘trivial’ cases, i.e. to \( N \geq 4 \).
4.3. The choice of $N = 4$

The $N = 4$ version of our main definition (3.4) requires, again, the evaluation of the trigonometric-function eigenvalues $\epsilon_j$ of the matrix

$$M = \begin{bmatrix}
\sqrt{2} + \sqrt{2} & 2 & 0 & 0 \\
1 & \sqrt{2} + \sqrt{2} & 1 & 0 \\
0 & 1 & \sqrt{2} + \sqrt{2} & 1 \\
0 & 0 & 1 & \sqrt{2} + \sqrt{2}
\end{bmatrix}. $$

It is easy to verify that in terms of the auxiliary constants $\alpha_0 = 0$, $\alpha_1 = 2 - \sqrt{2}$, $\alpha_2 = \sqrt{2}$ and $\alpha_3 = 2$, we can write $\epsilon_j = \alpha_j \sqrt{2} + \sqrt{2}$. What is less routine is the evaluation of the respective eigenvectors

$$\Phi_0^T = c_{R0} [-\sqrt{2} \sqrt{2} + \sqrt{2}, \sqrt{2} + 1, -\sqrt{2} + \sqrt{2}, 1]$$

$$\Phi_1^T = c_{R1} [1, -1/2 \sqrt{2} \sqrt{2} + \sqrt{2}, 1/2 \sqrt{2} + \sqrt{2}, -1/2 \sqrt{2}, 1/2 \sqrt{2} + \sqrt{2}]$$

$$\Phi_2^T = c_{R2} [1, 1/2 \sqrt{2} \sqrt{2} + \sqrt{2} - 1/2 \sqrt{2} + \sqrt{2}, -1/2 \sqrt{2}, -1/2 \sqrt{2} + \sqrt{2}]$$

$$\Phi_3^T = c_{R3} [\sqrt{2} \sqrt{2} + \sqrt{2}, \sqrt{2} + 1, \sqrt{2} + \sqrt{2}, 1]$$

$$\eta_0^T = c_{L0} [-1/2 \sqrt{2} \sqrt{2} + \sqrt{2}, \sqrt{2} + 1, -\sqrt{2} + \sqrt{2}, 1]$$

$$\eta_1^T = c_{L1} [1, -\sqrt{2} \sqrt{2} + \sqrt{2} + \sqrt{2}, -\sqrt{2}, \sqrt{2} + \sqrt{2}]$$

$$\eta_2^T = c_{L2} [1, \sqrt{2} \sqrt{2} + \sqrt{2} - \sqrt{2} + \sqrt{2}, -\sqrt{2}, -\sqrt{2} + \sqrt{2}]$$

$$\eta_3^T = c_{L3} [1/2 \sqrt{2} \sqrt{2} + \sqrt{2} + 1, \sqrt{2} + \sqrt{2}, 1].$$

Although these formulas still remain extremely elementary, their generation has been based on the computer-assisted symbolic manipulations.

4.4. The last, $N = 5$ illustration

Our next (and also last) Hamiltonian matrix

$$M = \begin{bmatrix}
\frac{1}{2} \sqrt{10 + 2 \sqrt{5}} & 2 & 0 & 0 & 0 \\
1 & \frac{1}{2} \sqrt{10 + 2 \sqrt{5}} & 1 & 0 & 0 \\
0 & 1 & \frac{1}{2} \sqrt{10 + 2 \sqrt{5}} & 1 & 0 \\
0 & 0 & 1 & \frac{1}{2} \sqrt{10 + 2 \sqrt{5}} & 1 \\
0 & 0 & 0 & 1 & \frac{1}{2} \sqrt{10 + 2 \sqrt{5}}
\end{bmatrix}$$

forces us to conclude that in spite of the purely non-numerical character of the recipe (based just on insertions), the $N \geq 5$ explicit formulas become rather lengthy. The typographical
considerations start to represent, in fact, the main limiting factor of the presentation of the $N \geq 5$ continuation of the series. For example, in spite of the existence of closed non-trigonometric formulas at $N = 5$, the pentaplet of energies $0.0, 0.726542529, 1.902113032, 3.077683536, 3.804226065$ is already better represented numerically. The same comment applies also to the closed-form eigenvectors, with
\[
\Phi_0^T = c_{R0} \left[ 1, -\frac{1}{4} \sqrt{10 + 2\sqrt{5}}, \frac{1}{4} + \frac{1}{4} \sqrt{5}, \frac{1}{8} \sqrt{10 + 2\sqrt{5}} - \frac{1}{8} \sqrt{10 + 2\sqrt{5}\sqrt{5}}, -\frac{1}{4} + \frac{1}{4} \sqrt{5} \right],
\]
etc, and with
\[
\eta_0^T = c_{L0} \left[ \frac{1}{2} + \frac{1}{2} \sqrt{5}, -\frac{1}{2} \sqrt{10 + 2\sqrt{5}} - \frac{1}{2} \sqrt{10 + 2\sqrt{5}\sqrt{5}}, \frac{1}{2} + \frac{1}{2} \sqrt{5}, -\frac{1}{2} \sqrt{10 + 2\sqrt{5}}, 1 \right],
\]
etc.

5. Conclusions

We have shown that two apparently different concepts previously introduced in the context of quantum mechanics with a non-self-adjoint Hamiltonian are strongly related, the one producing the other under very natural assumptions. We have also analyzed a few examples to show how the construction works. The analysis of non-regular NLPB, where unbounded metric operators play a crucial role, will be considered in the nearest future.

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