A loophole of all ‘loophole-free’ Bell-type theorems

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I give a simple example of a local realistic problem with free will of observers and no detection loophole, but where the Bell inequality cannot be proved. The trick is based on random variables modeled by functions whose sum does not exist due to incompatibility of their domains.

I. INTRODUCTION

When one speaks of a physical system that violates the Bell inequality [1], what one really has in mind is a system that does not satisfy at least one assumption needed for its proof. The assumption that detectors are perfect enough is physically obvious [2]. There is also no problem with understanding the physical meaning of nonlocal influences, the point explicitly made by Bell. Yet, when it comes to postulating a joint probability measure for all random variables [3], the situation is much less clear. Is it just synonymous to realism, another explicit postulate of Bell? Does it mean that counterfactual probabilities are identical to the measurable ones, even in cases where the alternative measurements cannot be simultaneously performed in principle, because of purely classical logical inconsistencies?

The latter is especially visible in the proof of the CHSH inequality [4], which involves the following elementary step:

\[ a_0(x)b_0(y) + a_1(x)b_0(y) = (a_0(x) + a_1(x))b_0(y). \] (1)

The value \( b_0(y) \) of the random variable \( b_0 \), measured by Bob, does not depend on the choice of \( a_0 \) or \( a_1 \), measured by Alice. This is precisely the assumption of locality in the sense of Bell. In a nonlocal case we would have something like

\[ a_0(x, b_0)b_0(y, a_0) + a_1(x, b_0)b_0(y, a_1) \]

so we could not further simplify the expression and proceed with the proof. The problem is easy to understand and is not a source of controversies.

However, the occurrence of \( a_0(x) + a_1(x) \) in (1) means that the sum of the two functions is well defined, which is not always the case in mathematics (think of \( \ln x + \ln(-x) \)). The subtlety is, of course, of a purely local nature. If a measurement of \( a_0 \) can influence in some way the one of \( a_1 \), the result of \( a_0(x) + a_1(x) \) might in principle depend on the order in which the two measurements are performed, but addition is order independent and thus cannot feel it. On the other hand, it can be shown on explicit examples [5] that in cases where the measurements of \( a_0 \) and \( a_1 \) influence each other, the actual probabilities may differ from the counterfactual ones. One can prove a counterfactual inequality, but it may not apply to actually performed measurements. The attempts of finding a local counterexample to Bell’s theorem [6] are, in my opinion, based on this loophole.

The goal of the present note is to give a simple example that seems to satisfy all the implicit and explicit assumptions made in Bell-type theorems, but an attempt of proving a Bell-type inequality gets stuck already in the first line. The main trick I employ was published in 1988 [7], but since the paper is virtually unknown and unavailable on the arXiv, I have decided to rewrite it, simultaneously simplifying the core argument.

II. EXAMPLE

Alice and Bob are fans of two football teams, FC Aces and FC Bees, playing on Saturday nights in Acetown and Beetown. Whenever the Aces play in Acetown, Alice travels there and stays in a hotel. The hotels in Acetown are named \( A_\alpha \), where \( 0 \leq \alpha \leq 1 \). Alice is free to choose any of them, but typically stays at \( A_0 \) or \( A_1 \). Rooms in \( A_\alpha \) are numbered by \( x \), \( \alpha < x < \alpha + 1 \). An analogous system works in Beetown. Unfortunately, the couple cannot travel together so if Alice supports her Aces in Acetown, then Bob is with the Bees in Beetown. Bob typically stays in \( B_0 \) or \( B_1 \), but is also completely free to make his choice. A peculiarity of the hotel system is that the same room can belong to several hotels (in the same town), a fact related to the structure of the local tax system, the antitrust law, and the unusual architecture typical of the region.

While leaving a hotel a visitor is asked to fill out a short questionnaire, reducing to a single question: Was it a nice visit? The answer is ‘+’ or ‘−’. Experience shows that it is acceptable to stay in a room whose number is somewhere in the middle of the list: \( \alpha + 1/4 < x < \alpha + 3/4 \). Otherwise the noise made by fans who accompany the guest team becomes unbearable. Since decibels correspond to a logarithmic scale, the statistics of positive and negative answers in Acetown hotels is well described by the following random variable (Fig. 1 and Fig. 2):

\[ a_0(x) = \text{sign}(\ln(16x(1-x)/3)), \] (2)

\[ a_\alpha(x) = a_0(x - \alpha). \] (3)

The same model works in Beetown.

Each time Alice and Bob leave their hotels they fill out the forms and produce a pair of ‘results’ ± (each run of the experiment produces a pair of results, hence no ‘detection loophole’). During their travels Alice and Bob do not communicate with each other (‘locality’). Once one
knows the room number, the result ‘+’ or ‘−’ is uniquely defined (‘deterministic hidden variables’). Alice and Bob are free to choose the hotels (free will assumption). If Alice and Bob stayed in rooms with numbers \( x \) in \( A_\alpha \), and \( y \) in \( B_\beta \), then random variables \( a_\alpha(x) \), \( b_\beta(y) \), and \( a_\alpha(x)b_\beta(y) \) are simultaneously defined.

### III. CAN WE PROVE BELL’S INEQUALITY?

So, can we prove Bell or CHSH inequalities? Let us try. Locality is satisfied, so apparently

\[
a_0(x)b_0(y) + a_1(x)b_0(y) = (a_0(x) + a_1(x))b_0(y). \quad (4)
\]

However, a look at Fig. 2 and Fig. 3 shows that \( a_0(x) + a_1(x) \) is undefined. The domain of the function is empty. It is not a zero function, but a function that does not exist at all. The putative proof got stuck already in its first line. The same will happen if instead of CHSH or Bell inequality one will try to prove the GHZ theorem [8]. Then, instead of the sum, one arrives at a product of the form \( a_0(x)a_1(x) \), which does not exist either.

There is completely no problem with measuring experimentally an expectation value of \( a_0(x)b_0(y) \), so there exists a probability distribution \( \rho_{00} \) such that

\[
\langle a_0b_0 \rangle = \int_0^1 dx \int_0^1 dy \ a_0(x)b_0(y)\rho_{00}(x,y). \quad (5)
\]

The two functions, \( a_0(x)b_0(y) \) and \( \rho_{00}(x,y) \), are defined on the same domain \((0,1) \times (0,1)\). Similarly,

\[
\langle a_1b_0 \rangle = \int_1^2 dx \int_0^1 dy \ a_1(x)b_0(y)\rho_{10}(x,y), \quad (6)
\]

with \( a_1(x)b_0(y) \) and \( \rho_{10}(x,y) \) defined on \((1,2) \times (0,1)\).

Note that the domains are disjoint, but there are no undetected signals. In 1988, pressed by the referee while resubmitting [7], I wrote that disjoint domains imply undetected signals (if \( x \notin (\alpha,\alpha + 1) \) then \( a_\alpha \) ‘produces no result’), although already at that time I was not quite convinced that the referee was right. Of course, one way of modeling non-ideal detectors is to use the trick with non-identical domains, but the inverse implication is not true. Another argument of the referee was that my random variables were three-valued: \( \pm 1 \) and 0, where zero occurred when there ‘was no result’. This type of argument does not apply here: Logarithm is not ‘zero’ if its argument is negative. The same applies to \( a_\alpha(x) \) for \( x \notin (\alpha,\alpha + 1) \).

The only CHSH-type inequality one can find in our example is the trivial one,

\[
|\langle a_0b_0 \rangle + \langle a_1b_0 \rangle - \langle a_0b_1 \rangle - \langle a_1b_1 \rangle| \leq 4. \quad (7)
\]

Having four different \( \rho_{\alpha\beta} \), it is not difficult to give examples that saturate the right-hand side of (7), while maintaining

\[
\langle a_0 \rangle = \langle a_1 \rangle = \langle b_0 \rangle = \langle b_1 \rangle = 0. \quad (8)
\]
The ‘nonlocal boxes’ are then explicitly local, which agrees with the recent analysis of Brassard and Raymond-Robichaud [9].

An example of a theory that reproduces quantum probabilities, and which is implicitly based on the trick with disjoint domains of complementary observables, can be found in Pitowsky’s monograph [10]. When stripped of abstract details (decomposition of unit interval into infinitely many disjoint non-measurable sets), it turns out that what is essential for the Pitowsky construction is the disjointness and not the non-measurability. So, instead of decomposing $[0, 1]$ into non-measurable sets, one can replace $[0, 1]$ by $[0, 1] \times [0, 1]$ and the effect is the same.

What I have shown is a variant of the argument given by Fine [3], but the emphasis is shifted from states to observables. The proof of the inequality does not work, no matter which probability distributions (‘states’) one works with, and is entirely related to the structure of random variables (‘observables’). The example of the hotel system is a toy model with no ambition of reproducing quantum mechanics, but it shows that random variables with disjoint domains can occur in situations where the detector inefficiency loophole is absent. I am convinced that the difficulty can be found in any proof of a Bell-type theorem, so that various recent claims of loophole-free Bell tests are probably overoptimistic. Maybe we are not just bright enough to invent a convincing and practically working counterexample to the Bell theorem, but we have no proof it does not exist. Overoptimism is dangerous, especially in cryptography.

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