CONSTRUCTION OF SIMPLE QUOTIENTS OF
BERNSTEIN-ZELEVINSKY DERIVATIVES AND HIGHEST
DERIVATIVE MULTISEGMENTS

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ABSTRACT. Let $F$ be a non-Archimedean field. A sequence of derivatives of generalized Steinberg representations can be used to construct simple quotients of Bernstein-Zelevinsky derivatives of irreducible representations of $\text{GL}_n(F)$. We introduce a notion of a highest derivative multisegment and use it to give a combinatorial criteria when two sequences of derivatives of generalized Steinberg representations give rise isomorphic quotients of Bernstein-Zelevinsky derivatives of an irreducible representation. Those results can be translated to affine Hecke algebra of type A, by using our previous formulation of Bernstein-Zelevinsky derivatives in affine Hecke algebra of type A setting.

1. Introduction

Let $F$ be a non-Archimedean local field. The Bernstein-Zelevinsky derivative is a twisted Jacquet functor, originally introduced in classifying the irreducible representations of $\text{GL}_{n+1}(F)$ in [BZ76, BZ77, Ze80]. On the other hand, there is a notion of $\rho$-derivatives introduced and studied by C. Jantzen [Ja07] and independently by Minguez [Mi09]. This article is to study more precise relations between two notions of derivatives. Our goal is applications towards branching laws for general linear groups or even other classical groups, in view of the recent derivative approach in studying branching laws e.g. [MW12, SV17, Pr18, CS21, Gn21, Ch21, GGP20, Ch21b, Ch21c]. Those applications will appear elsewhere.

To be more precise, the notion of derivatives in this article is the one using generalized Steinberg representations to replace cuspidal representations in [Ja07, Mi09], which we shall simply call St-derivatives. Such derivative is also used for other classical groups in a recent work of Atobe-Minguez [AM20]. A certain sequence of St-derivatives can be used to obtain some submodules of Bernstein-Zelevinsky derivatives (see Section 3). This is based on the observation that any standard module in GL case has unique submodule and such submodule is generic [JS83] (also see [CaSh98, Ch21]).

The highest derivative of an arbitrary irreducible representation is obtained by Zelevinsky [Ze80]. A complete description for all the derivatives has been previously obtained for Steinberg representations by Zelevinsky [Ze80], and ladder representations (including Speh representations) by Lapid-Minguez [LM16] (also see [Ta87, CS19]). It could be hard to obtain a nice explicit description for the general case, and so it may be desirable to study derivatives in terms of some properties and invariants.
1.1. Two notions of derivatives. We introduce the two notions of derivatives and more notations will be given in Sections 2 and 3. Let \( G_n = \text{GL}_n(F) \), the general linear group over a non-Archimedean field \( F \). For \( a, b \in \mathbb{Z} \) with \( b - a \geq 0 \) and a cuspidal representation \( \rho \) of \( G_m \), we shall call \([a, b]_\rho \) to be a segment and define \( l_a([a, b]_\rho) = (b - a + 1)m \). Zelevinsky [Ze80] showed that essentially square-integrable representations of \( G_n \) can be parametrized by those segments. For each segment \( \Delta \), we shall denote by \( \text{St}(\Delta) \) the corresponding essentially square-integrable representation (see Section 2.3).

Let \( \text{Irr}(G_n) \) be the set of (isomorphism classes of) irreducible smooth complex representations of \( G_n \). Let \( \text{Irr} = \sqcup_n \text{Irr}(G_n) \). Let \( N_i \subset G_n \) be the unipotent radical containing matrices of the form \( \begin{pmatrix} I_{n-i} & u \\ I_i \end{pmatrix} \), where \( u \) is a \((n - i) \times i\) matrix. There exists at most one irreducible module \( \tau \in \text{Irr}(G_{n-i}) \) such that
\[
\tau \boxtimes \text{St}(\Delta) \hookrightarrow \pi_N,
\]
where \( \pi_N \) is a Jacquet module of \( \pi \). If such \( \tau \) exists, we denote such \( \tau \) by \( D_\Delta(\pi) \). Otherwise, we set \( D_\Delta(\pi) = 0 \). We shall refer \( D_\Delta \) to be a \( \text{St-derivative} \).

Let
\[
R_i = \left\{ \begin{pmatrix} g & m \\ u & I_i \end{pmatrix} : g \in G_{n-i}, m \in \text{Mat}_{(n-i) \times i}, u \in U_i \right\}.
\]

The right \( i \)-th Bernstein-Zelevinsky derivative \( \pi^{(i)} \) of \( \pi \) is defined as the \( G_{n-i} \)-module
\[
\delta_{R_i}^{-1/2} \frac{\pi}{(x, v - \psi(x))v : x \in R_i, v \in \pi},
\]
where \( \delta_{R_i} \) is the modular character of \( R_{n-i} \), and \( \psi \) is a non-degenerate character on \( U_i \) extended trivially to \( R_i \). Here \( U_i \) is viewed as a subgroup of \( R_i \) via the embedding
\[
u \mapsto \begin{pmatrix} I_{n-i} & 0 \\ u \end{pmatrix}.
\]

The level of \( \pi \) is the largest integer \( i^* \) such that \( \pi^{(i^*)} \neq 0 \) and, for any \( i > i^* \), \( \pi^{(i)} = 0 \). For the level \( i^* \) of \( \pi \), let \( \pi^- = \pi^{(i^*)} \), which is known to be irreducible [Ze80]. We shall call \( \pi^- \) the highest derivative of \( \pi \).

1.2. Main results. Fix a cuspidal representation \( \rho \in \text{Irr} \). Let \( \text{Irr}_\rho \) be the subset of \( \text{Irr} \) containing irreducible representations which are an irreducible quotient of \( \nu^{a_1}\rho \times \ldots \times \nu^{a_k}\rho \), for some integers \( a_1, \ldots, a_k \in \mathbb{Z} \). Representations in \( \text{Irr}_\rho \) are the most interesting case, and the general case can be deduced from that.

A multsegment is a multiset of segments. Let Mult be the set of all multisegments and let Mult\( \rho \) be the subset of Mult containing all multisegments whose segments are of the form \([a, b]_\rho \) for some \( a, b \in \mathbb{Z} \). For two multisegments \( \mu_1 \) and \( \mu_2 \), we write \( \mu_2 \leq_Z \mu_1 \) if \( \mu_2 \) can be obtained by a sequence of elementary intersection-union operations from \( \mu_1 \) (see Section 2.3). In particular, if any pair of segments in \( \mu \) is unlinked, then \( \mu \) is a minimal element under \( \leq_Z \).

A sequence of segments \([a_1, b_1]_\rho, \ldots, [a_k, b_k]_\rho \) (all \( a_j, b_j \in \mathbb{Z} \)) is said to be in an ascending order if
\[
a_1 \leq a_2 \leq \ldots \leq a_k
\]
Theorem 1.1. (c.f. Theorem 5.3) Let $\pi \in \text{Irr}_\rho$. There exists a minimal multisegment $m$ such that when the segments $\{\Delta_1, \ldots, \Delta_k\}$ in $m$ are written in an ascending order, we have that:

$$D_{\Delta_k} \circ \ldots \circ D_{\Delta_1}(\pi) \cong \pi^-. $$

Here the minimality is with respect to $\leq_Z$.

The existence part of Theorem 1.1 will be given in Theorem 5.3 and the minimality is dealt in Section 9.2.

The multisegment in the above theorem is explicitly constructed in Section 7.1, which we shall call $hd(\pi)$ to be the highest derivative multisegment of $\pi$. The highest derivative multisegments of some special classes of representations are given in Section 10.

For a multisegment $n \in \text{Mult}_\rho$, which we write the segments in $n$ in an ascending order $\Delta_1, \ldots, \Delta_k$. Define

$$D_n(\pi) := D_{\Delta_k} \circ \ldots \circ D_{\Delta_1}(\pi). $$

We will show in Lemma 4.7 that the derivative is independent of the ordering of an ascending sequence. In general, we have the following connection between two notions of derivatives:

Proposition 1.2. Let $\pi \in \text{Irr}_\rho$. Let $n \in \text{Mult}_\rho$ such that

$$D_n(\pi) \neq 0. $$

Then $D_n(\pi)$ is a simple quotient of $\pi(i)$, where $i = l_a(\Delta_1) + \ldots + l_a(\Delta_k)$.

The above proposition motivates to study multiple St-derivatives. In general, there are plenty of examples that two different sequences can give isomorphic St-derivatives. Hence it is interesting to study the combinatorial structure of the following set: for $\pi \in \text{Irr}_\rho$ and for a simple quotient $\tau$ of $\pi(i)$ for some $i$, define

$$S(\pi, \tau) := \{n \in \text{Mult}_\rho : D_n(\pi) \cong \tau\}. $$

For that purpose, in Section 7 we define a combinatorial algorithm on a pair $(\Delta, h)$, where $\Delta$ takes the form $[a, b]_\rho$ and $h \in \text{Mult}_\rho$. The algorithm results a multisegment, denoted $r(\Delta, h)$. The case that we are interested in is that when $h = hd(\pi)$. We also develop some rules and properties for computing $r(\Delta, h)$ in Section 7, and the relation to $D_{\Delta}(\pi)$ is given in Theorem 7.18.

For a multisegment $n \in \text{Mult}_\rho$, which we write the segments in $n$ in an ascending order $\Delta_1, \ldots, \Delta_k$, we define

$$r(n, \pi) := r(\Delta_k, r(\Delta_{k-1}, \ldots r(\Delta_1, hd(\pi))) \ldots). $$

One remarkable property of the multisegment $r(n, \pi)$ is the following, which roughly says that $r(n, \pi)$ measures the difference between the derivative $D_n(\pi)$ and the highest derivative $\pi^-$:

Theorem 1.3. (=Theorem 8.7) Let $\pi \in \text{Irr}_\rho$. Let $n \in \text{Mult}_\rho$ such that $D_n(\pi) \neq 0$. Let $\tau = D_n(\pi)$. Then

$$D_{r(n, \pi)}(\tau) \cong \pi^-. $$
Indeed, $r(n, \pi)$ determines when two ascending sequences determine the same derivative:

**Theorem 1.4.** (=Theorem 8.2) Let $\pi \in \text{Irr}_\rho$. Let $n_1, n_2$ be multisegments such that $D_{n_1}(\pi) \neq 0$ and $D_{n_2}(\pi) \neq 0$. Then

$$D_{n_1}(\pi) \cong D_{n_2}(\pi) \iff r(n_1, \pi) = r(n_2, \pi).$$

We comments on the proof of Theorems 1.3 and 1.4. The multisegment associated to the derivative $D_{n}(\pi)$ can be in general computed via explicit algorithms, but our proof does not directly use that. Our proof is more combinatorially soft in the sense that we use some commutation relations between derivatives studied in Section 4 and we also provide more conceptual explanations on some construction of derivatives in Section 6.

An application of Theorem 1.4 is to show that $S(\pi, \tau)$ is closed under $\leq_Z$ in the following sense:

**Corollary 1.5.** (=Corollary 9.11) Let $\pi \in \text{Irr}_\rho$. Let $\tau$ be a simple quotient of $\pi^{(i)}$ for some $i$. If $n_1, n_2 \in S(\pi, \tau)$ and $n_1 \leq_Z n_2$, then any $n_3 \in \text{Mult}_\rho$ satisfying $n_1 \leq n_3 \leq n_2$ is also in $S(\pi, \tau)$.

1.3. **Remarks.** As mentioned in the beginning, our primary interest is on the branching law. The close relation of derivatives and branching law underlies in the Bernstein-Zelevinsky theory. Indeed the branching law also has applications on the derivative theory, and perhaps an interesting one is the multiplicity at-most-one phenomenon [AGRS10] giving the multiplicity-freeness on socles and cosocles of the Bernstein-Zelevinsky derivatives of an irreducible representation (see Sections 3.6 and 3.8).

Apart from branching laws, there are many other applications for those derivatives. For example, it is important for theta correspondence [M108, M109], $L$-functions [CPS17], other distinction problems such as [O118], and Zelevinsky-Aubert duals [MW80, AM20], and Arthur packets [Xu17] and many others.

We finally give some background of our study. In [CS19] joint with Savin, we formulated the analogous Bernstein-Zelevinsky derivative functor for affine Hecke algebras of type A and so one could also formulate the analogous results in such setting. Some parts in this article are originally inspired by the work of [GV01] Gronowski-Vazirani in Hecke algebra setting few years ago, in which they used $\rho$-derivatives to study branching problems.

Using $\rho$-derivatives instead of Bernstein-Zelevinsky derivatives also explicitly appears before, for example [De16] studying with orbital varieties and a more recent work [Gu21] studying with RSK model. However, we emphasis that $St$-derivatives are important in the study of simple quotients of classical Bernstein-Zelevinsky derivatives, while $\rho$-derivatives seem to be not enough for such purpose. In particular, using machinery in Sections 7 and 8 one can find some simple quotients of Bernstein-Zelevinsky cannot be constructed from a sequence of $\rho$-derivatives.
2. Preliminaries

2.1. Notations. All the representations are smooth and over \( \mathbb{C} \). We shall usually drop those descriptions. We sometimes do not distinguish representations in the same isomorphism class. We also use the notations in Section 1.1.

For \( n_1, \ldots, n_k = n \), we write \( P_{n_1, \ldots, n_k} \) to be the parabolic subgroup generated by the matrices \( \text{diag}(g_1, \ldots, g_k) \), where \( g_j \in G_{n_j} \), and upper triangular matrices. We shall say that \( P_{n_1, \ldots, n_k} \) be the standard parabolic subgroup associated to the partition \( (n_1, \ldots, n_k) \). For \( i \leq n \), \( N_i \) defined in Section 1.1 is the unipotent radical of the parabolic subgroup \( P_{n-i,i} \).

For \( \pi \in \text{Irr}(G) \), \( n(\pi) \) is defined to be the number that \( \pi \in \text{Irr}(G_{n(\pi)}) \). Let \( \text{Irr}^c(G_n) \) be the set of (irreducible) cuspidal representations of \( G_n \). Similarly, let \( \text{Irr}^e = \cup_n \text{Irr}^c(G_n) \).

For any \( \pi_1 \in \text{Irr}(G_{n_1}) \) and \( \pi_2 \in \text{Irr}(G_{n_2}) \), define

\[
\pi_1 \times \pi_2 = \text{Ind}_{P_{n_1,n_2}}^{G_{n_1+n_2}} \pi_1 \boxtimes \pi_2,
\]

where we inflate \( \pi_1 \boxtimes \pi_2 \) to a \( P_{n_1,n_2} \)-representation. For \( \pi_j \in \text{Irr}(G_{n_j}) \) \((j = 1, \ldots, k)\),

\[
\pi_1 \times \ldots \times \pi_k = \pi_1 \times (\pi_2 \times (\pi_3 \times \ldots \times \pi_k))
\]

The product is indeed an associative operation.

For \( b - a \in \mathbb{Z} \) and a cuspidal representation \( \rho \), we call

\[
[a, b]_\rho := \{ \nu^a \rho, \ldots, \nu^b \rho \}
\]

be a segment. We also write:

\[
[a]_\rho := [a, a]_\rho.
\]

We may also write \([\nu^a \rho, \nu^b \rho] \) for \([a, b]_\rho \) and write \([\nu^a \rho] \) for \([a]_\rho \). The relative length of a segment \([a, b]_\rho \) is defined as \( b - a + 1 \), and we shall denote by \( l_r([a, b]_\rho) \). The absolute length of a segment \([a, b]_\rho \) is defined as \( (b-a+1)n(\rho) \), and we shall denote by \( l_a([a, b]_\rho) \) as before. Two segments \([a, b]_\rho \) and \([a', b']_\rho' \) are said to be equal if \( \nu^{a + b} \equiv \nu^{a'} \) and

\[
b - a + 1 = b' - a' + 1.
\]

For any \( \pi \in \text{Irr} \), there exists cuspidal representations \( \rho_1, \ldots, \rho_r \) such that \( \pi \) is a simple composition factor for \( \rho_1 \times \ldots \times \rho_r \), and we shall denote such multiset to be \( \text{csupp}(\pi) \), which is called the cuspidal support of \( \pi \).

2.2. More notations for multisegments. Let \( \text{Seg} \) be the set of all segments. A multisegment is a multiset of segments. Let \( \text{Mult} \) be the set of all multisegments.

For \( \rho_1, \rho_2 \in \text{Irr}^c \), we write \( \rho_2 < \rho_1 \) if \( \rho_1 \cong \nu^a \rho_2 \) for some integer \( a > 0 \). For two segments \( \Delta_1, \Delta_2 \), we write \( \Delta_1 < \Delta_2 \) if \( \Delta_1 \) and \( \Delta_2 \) are linked and \( b(\Delta_1) < b(\Delta_2) \).

For a fixed cuspidal support \( \rho \), let \( \text{Irr}_\rho \) be the subset of \( \text{Irr} \) containing all irreducible representations \( \pi \) such that for any \( \rho' \in \text{csupp}(\pi) \), \( \rho' \cong \nu^a \rho \) for some \( a \in \mathbb{Z} \). Let \( \text{Seg}_\rho \) be the subset of segments in \( \text{Seg} \) containing all segments of the form \([a, b]_\rho \) for some \( a, b \in \mathbb{Z} \). Let \( \text{Mult}_\rho \) be the subset of \( \text{Mult} \) containing multisegments \( \mathfrak{m} \) such that for any segment \( \Delta \) in \( \mathfrak{m} \) is in \( \text{Seg}_\rho \). For an integer \( c \), let \( \text{Mult}_{\rho,c} \) be the subset of \( \text{Mult}_\rho \) containing all multisegments \( \mathfrak{m} \) such that any segment \( \Delta \) in \( \mathfrak{m} \) satisfies \( a(\Delta) \cong \nu^c \rho \). Similarly, define \( \text{Mult}_{\rho,c}^R \) to be the subset of \( \text{Mult}_\rho \) containing all
multisegments \( m \) such that any segment \( \Delta \) in \( m \) satisfies \( b(\Delta) \cong \nu^k \rho \). We shall also say that a multisegment \( m \in \text{Mult}_{\rho,c} \) is at a point \( \nu^k \rho \).

For a multisegment \( m \) in \( \text{Mult}_\rho \), let \( m[c] \) be the submultisegment of \( m \) containing all the segments \( \Delta \) satisfying \( a(\Delta) \cong \nu^k \rho \); and let \( m(c) \) be the submultisegment of \( m \) containing all the segments \( \Delta \) satisfying \( b(\Delta) \cong \nu^k \rho \).

For a multisegment \( m = \{ \Delta_1, \ldots, \Delta_k \} \), we also set:

\[
l_a(m) = l_a(\Delta_1) + \ldots + l_a(\Delta_k), \quad l_r(m) = l_r(\Delta_1) + \ldots + l_r(\Delta_k).
\]

2.3. **Ordering on segments.** For two segments \([a', b']_\rho \) and \([a'', b'']_\rho \) in \( \text{Mult}_\rho \), we write

\[
[a', b']_\rho \prec_R [a'', b'']_\rho
\]

if either \( a' < a'' \); or \( a' = a'' \) and \( b' < b'' \). We write

\[
[a', b']_\rho \prec_L [a'', b'']_\rho
\]

if either \( b' < b'' \); or \( b' = b'' \) and \( a' < a'' \). We also write \([a', b']_\rho \preceq_L [a'', b'']_\rho \) if \([a', b']_\rho \prec_L [a'', b'']_\rho \) or \([a', b']_\rho = [a'', b'']_\rho \) and similarly define \([a', b']_\rho \preceq_R [a'', b'']_\rho \).

2.4. **Ordering on \( \text{Mult}_{\rho,c} \).** We say that a multisegment \( m_2 \) is obtained from \( m_1 \) by an elementary intersection-union operation if for two segments \( \Delta_1, \Delta_2 \) in \( m_1 \),

\[
m_2 = m_1 - \{ \Delta_1, \Delta_2 \} + \{ \Delta_1 \cup \Delta_2, \Delta_1 \cap \Delta_2 \}.
\]

The ordering \( \leq_Z \) is defined in Section 1.2.

Fix an integer \( c \). Let \( \Delta_1 = [c, b_1]_\rho, \Delta_2 = [c, b_2]_\rho \) be two segments. We write \( \Delta_1 \leq_Z \Delta_2 \) if \( b_1 \leq b_2 \), and write \( \Delta_1 < Z \Delta_2 \) if \( b_1 < b_2 \).

For \( m_1, m_2 \) in \( \text{Mult}_{\rho,c} \), label the segments in \( m_1 \) as: \( \Delta_1, \ldots, \Delta_k \) and label the segments in \( m_2 \) as: \( \Delta_{1', \ldots, \Delta_{k'}} \). We define the lexicographical ordering: \( m_1 \leq_Z m_2 \) if \( k \leq r \) and, for any \( i \leq k \), \( \Delta_{1,i} \leq_Z \Delta_{2,i} \).

We also need a ‘right’ ordering. One can define \([a_1, c]_\rho \preceq_R [a_2, c]_\rho \) if \( a_1 < a_2 \), and similarly define \([a_1, c]_\rho \preceq_R [a_2, c]_\rho \). One similarly define \( \leq_R \) on \( \text{Mult}_{\rho,c} \).

2.5. **Zelevinsky and Langlands classification.** For a segment \( \Delta = [a, b]_\rho \in \text{Mult} \), define \( \langle \Delta \rangle \) to be the the unique submodule of

\[
\nu^a \rho \times \ldots \times \nu^b \rho
\]

and define \( \text{St}(\Delta) \) to be the unique quotient of

\[
\nu^a \rho \times \ldots \times \nu^b \rho.
\]

For any multisegment \( m = \{ \Delta_1, \ldots, \Delta_k \} \) with the labeling satisfying that, for \( i < j \), \( \Delta_j \not\leq \Delta_i \). Define, as in [Ze80, Theorem 6.1], \( \langle m \rangle \) to be the unique submodule of

\[
\zeta(m) := \langle \Delta_1 \rangle \times \ldots \times \langle \Delta_k \rangle.
\]

Define \( \text{St}(m) \) to be the unique quotient of

\[
\lambda(m) := \text{St}(\Delta_1) \times \ldots \times \text{St}(\Delta_k).
\]

We frequently use the following standard fact (see [Ze80, Theorems 4.2, 6.1]): for two unlinked segments \( \Delta_1 \) and \( \Delta_2 \),

\[
(\Delta_1) \times (\Delta_2) \cong (\Delta_2) \times (\Delta_1),
\]

\[
(\Delta_1) \cup (\Delta_2) \cong (\Delta_1) \cup (\Delta_2).
\]
\[(2.3) \quad \text{St}(\Delta_1) \times \text{St}(\Delta_2) \cong \text{St}(\Delta_2) \times \text{St}(\Delta_1).\]

2.6. Quotients and submodules of Jacquet functors. Let \(\theta = \theta_n : G_n \to G_n\) be given by \(\theta(g) = g^{-t}\), the inverse transpose of \(g\). This induces a self-equivalence exact functor on \(\text{Alg}(G_n)\), still denoted by \(\theta\). We shall call it the Gelfand-Kazhdan involution.

**Proposition 2.1.** Let \(\pi \in \text{Irr}(G_n)\). Let \(n_1 + \ldots + n_r = n\). Let \(\mathcal{N}\) be the unipotent radical of the parabolic subgroup \(P_{n_1,\ldots,n_r}\). Let \(\theta'\) be the involution on \(\text{Alg}(G_{n_1} \times \ldots \times G_{n_r})\) arisen from \(\theta'(g_1,\ldots,g_r) = (\theta(g_1),\ldots,\theta(g_r))\). Then \(\theta'([\pi_N]) \cong \pi_N\). In particular, for an irreducible representation \(\omega\) of \(G_{n_1} \times \ldots \times G_{n_r}\), \(\omega\) is a simple submodule of \(\pi_N\) if and only if \(\omega\) is a simple quotient of \(\pi_N\).

**Proof.** Recall that \(\theta(\pi)\) and \(\pi\) have the same underlying space, which we refer to \(V\). Note that
\[
W := \{\theta(n).v - v : n \in \mathcal{N}^-, v \in V\} = \{n.v - v : v \in \mathcal{N} \in V\}.
\]
Then it induces a natural identification as vector space:
\[
\theta_n(\pi)_{\mathcal{N}^-} = \theta'(\pi_N) = V/W.
\]
Now one checks the isomorphism lifting to a \(G_{n_1} \times \ldots \times G_{n_r}\)-morphism. This proves that:

\[
(\ast) \quad \theta(\pi)_{\mathcal{N}^-} \cong \theta'(\pi_N).
\]

On the other hand, by a result of Casselman,

\[
(\ast\ast) \quad (\theta(\pi)_{\mathcal{N}^-})^\vee \cong (\theta(\pi)^\vee)_N \cong \pi_N
\]

The proposition follows by combining (\ast) and (\ast\ast).

2.7. Jacquet functors on Steinberg representations. We shall frequently use the following formula [Ze80]:
\[
\langle [a,b]_{\rho}\rangle_{\text{in}(\rho)} = \langle [a,b-i]_{\rho} \rangle \boxtimes \langle [b-i+1,b]_{\rho} \rangle
\]
and
\[
\text{St}([a,b]_{\rho})_{\text{in}(\rho)} = \text{St}([a+i,b]_{\rho}) \boxtimes \text{St}([a,a+i-1]_{\rho}).
\]

3. Two notions of derivatives

3.1. \(\rho\)-derivative.

**Lemma 3.1.** [GV01, Ja07, Mi09] Let \(\rho \in \text{Irr}^c(G_k)\) and let \(\pi \in \text{Irr}\). For any positive integer \(k\),
\[
\pi \times \rho \times \ldots \times \rho
\]
has unique irreducible submodule and unique irreducible quotient.

We introduce the following notations [Ja07, Mi09]:

**Notation 3.2.** (1) We shall also write \(\pi \times \rho^k\) for \(\pi \times \rho \times \ldots \times \rho\) for representations \(\pi\) and \(\rho\).
Lemma 3.5. $D_\pi$ corollary 3.6. Let $\tau$ for some $D_\pi$. Then $\tau$ is an admissible $G$ submodule of $\pi_{n(\rho)}$.

For $k \geq 0$, we shall write

\[
\underbrace{D_\pi \circ \ldots \circ D_\pi}_k(\pi).
\]

(One may also replace the submodule condition by the corresponding quotient condition, see Proposition 2.4.) When $k = 0$, $D_0(\pi) = \pi$. We shall call $D_\pi(\pi)$ to be a $\rho$-derivative of $\pi$ (depending on $c$).

(3) We shall denote the largest non-negative $k$ such that $D_\pi^k(\pi) \neq 0$ by $\epsilon_c(\pi)$.

3.2. Some more results on socle and cosocle. For a representation $\pi$ of finite length, we denote by $\text{soc}(\pi)$ and $\text{cosoc}(\pi)$ the socle and cosocle of $\pi$ respectively. We need the following result later (see e.g. LM16, c.f. Lemma 3.1) and we refer to LM16 for a definition of a ladder representation:

Lemma 3.3. LM16 Let $\pi \in \text{Irr}_\rho(G_n)$ be a ladder representation or a generic representation. Let $\tau_1 \in \text{Irr}_\rho(G_k)$ and let $\tau_2 \in \text{Irr}_\rho(G_k)$. Then

- $\text{soc}(\pi \times \tau_1)$ and $\text{cosoc}(\pi \times \tau_1)$ are irreducible;
- $\text{soc}(\pi \times \tau_1) \cong \text{soc}(\pi \times \tau_2)$ if and only if $\tau_1 \cong \tau_2$;
- $\text{cosoc}(\pi \times \tau_1) \cong \text{cosoc}(\pi \times \tau_2)$ if and only if $\tau_1 \cong \tau_2$.

3.3. Properties of $\rho$-derivatives. For the following lemma, see GV01 Lemma 3.5], Ja07 Corollary 2.3.2], Mi09 Corollaire 6.5.:

Lemma 3.4. GV01, Ja07, Mi09 Let $\pi \in \text{Irr}_\rho$. Let $c$ be an integer. Let $\tilde{\pi}$ be the unique submodule of $\pi \times (\nu^c \rho)^\times k$. Then

1. $\epsilon_c(\tilde{\pi}) = \epsilon_c(\pi) + k$;
2. $\tilde{\pi}$ appears with multiplicity one in $\pi \times (\nu^c \rho)^\times k$;
3. For any irreducible composition factor $\tau$ of $(\pi \times (\nu^c \rho)^\times k)$ which is not isomorphic to $\tilde{\pi}$, $\epsilon_c(\tau) < \epsilon_c(\pi) + k$.

The following result follows from an application on geometric lemma:

Lemma 3.5. GV01, Ja07, Mi09 Let $\pi \in \text{Irr}_\rho$. Let $c$ be an integer such that $D_c(\pi) \neq 0$. Let $k = \epsilon_c(\pi)$. Then there is only one simple composition factor in $\pi_{N_{k\nu}(\rho)}$ of the form

$\tau \boxtimes (\nu^c \rho)^\times k$

for some $\tau \in \text{Irr}$. Moreover, such $\tau \cong D_c^k(\pi)$.

As a consequence, we have the following:

Corollary 3.6. Let $\pi \in \text{Irr}_\rho(G_n)$. Let $k = \epsilon_c(\pi)$. Suppose $k > 0$. Let $\omega$ be an admissible $G_n - kn(\rho)$-representation such that

$\pi \hookrightarrow \omega \times (\nu^c \rho)^\times k$

Then $D_c^k(\pi) \hookrightarrow \omega$. 

Proof. By Frobenius reciprocity, we have a non-zero map:
\[ \pi_{kn(\rho)} \to \omega \boxtimes (\nu^e\rho)^{\times k}. \]

By Lemma 3.5, the only composition factor of the form \( \tau \boxtimes (\nu^e\rho)^{\times k} \) and hence is mapped to the submodule of \( \omega \boxtimes (\nu^e\rho)^{\times k} \). It follows from Künneth formula (see [Ra07]) that \( D^k_c(\pi) \) is a submodule of \( \omega \). \( \square \)

3.4. Highest derivative using \( \rho \)-derivatives.

**Proposition 3.7.** Let \( m \in \text{Mult}_\rho \). Let \( c \) (resp. \( d \)) be the smallest (resp. largest) integer such that \( \nu^c\rho \cong b(\Delta) \) (resp. \( \nu^d\rho \cong b(\Delta) \) for some \( \Delta \in m \). For each \( e = c, \ldots, d \), let \( k_e \) be the number of segments in \( m \) with \( b(\Delta) \cong \nu^e\rho \). Then \( D^d \circ \ldots \circ D^{k_{e+1}} \circ D^{k_e}(\pi) \cong \pi^{-} \).

The above proposition can be computed directly by using e.g. [Mi09, Théorème 7.5] (also see [MW86] or [LM16]). We omit the details and indeed it is based on the following lemma:

**Lemma 3.8.** Let \( m \in \text{Mult}_\rho \). Let \( \pi = \langle m \rangle \). Let \( B(\pi,c) \) be the number of segments \( \Delta \) in \( m \) such that \( b(\Delta) \cong \nu^c\rho \). Suppose, for some \( e \in \mathbb{Z} \) such that \( B(\pi,e-1) = 0 \).

Then \( \varepsilon_e(\pi) = B(\pi,e) \).

**Proof.** It is a consequence of [Mi09, Théorème 7.5]. \( \square \)

3.5. Left-right Bernstein-Zelevinsky derivatives. Recall that Bernstein-Zelevinsky derivative is defined in Section 1.1. We also define a left version (c.f. [CS21, Ch21]). The \( i \)-th Bernstein-Zelevinsky derivative is defined as:

\[ (i)\pi = \theta_{n-i}(\theta_n(\pi)^{(i)}). \]

Note that one can use the transpose \( R_i^t \) of \( R_i \) to define the left derivative as in [L11]. One may then apply a conjugation on an antidiagonal element to obtain the following formulation:

\[ (i)\pi = \delta_{R_i}^{-1/2} \cdot \frac{\pi}{[x.v - \psi(x)v : x \in R_i, v \in \pi]}, \]

where \( R_i = aR_i^t a^{-1} \). Here \( a \) is the matrix with 1 in the antidiagonal entries and 0 elsewhere.

We shall only prove the results for the 'right' version, and the 'left' version can be formulated and proved similarly.

3.6. Properties of derivatives. From the multiplicity-one theorem [AGRS10] (see [Ch21, Proposition 2.5], [CS21, Lemma 2.3]) and a self-dual property (see [CS21, Lemma 2.4]), we deduce that:

**Lemma 3.9.** [Ch21, Proposition 2.5] Let \( \pi \in \text{Irr}(G_n) \). Then \( \text{soc}(\pi^{(i)}) \) is multiplicity-free. The same holds for \( \text{soc}^{(i)}\pi \), \( \text{cosoc}(\pi^{(i)}) \) and \( \text{cosoc}^{(i)}\pi \).

**Lemma 3.10.** [CS21, Lemma 2.4] Let \( \pi \in \text{Irr}(G_n) \). Then, for any \( i \) with \( \pi^{(i)} \neq 0 \), \( \text{soc}(\pi^{(i)}) \cong \text{cosoc}(\pi^{(i)}) \).
Let $m \in \text{Mult}_\rho$. For any $i$,
\[
m^{(i)} = \left\{ \left\{ \Delta^{(i_1)}, \ldots, \Delta^{(i_r)} \right\} : i_k = 0 \text{ or } n(\rho), \quad i_1 + \ldots + i_r = i \right\}.
\]
and
\[
(i)m = \left\{ \left\{ (i_1)\Delta, \ldots, (i_r)\Delta \right\} : i_k = 0 \text{ or } n(\rho), \quad i_1 + \ldots + i_r = i \right\}.
\]

Lemma 3.11. Let $\pi \in \text{Irr}(G_n)$. For any socle $\tau$ of $\pi^{(i)}$ (resp. $(i)\pi$), $\tau \cong \langle n \rangle$ for some $n \in m^{(i)}$ (resp. $n \in (i)m$).

Lemma 3.11 can be proved by embedding $\pi$ to $\zeta(m)$ and then applying geometric lemma. See, for example, [Ch21] Lemma 7.3.

3.7. Notations for derivatives. For a segment $\Delta = [a, b]_\rho$, write
\[
\Delta^- = [a + 1, b]_\rho, \quad \Delta^- = [a, b-1]_\rho.
\]
For a multisegment $m = \{\Delta_1, \ldots, \Delta_k\}$ in $\text{Mult}$, write
\[
-m = \{-\Delta_1, \ldots, -\Delta_k\}.
\]

3.8. Submodule of derivatives from Jacquet functor. The author would like to thank G. Savin for a discussion on the following proposition.

Proposition 3.12. Let $\pi$ be an admissible representation of $G_n$ of finite length. Let $\tau$ be an irreducible submodule of $\pi^{(i)}$. Then there exists cuspidal representations $\rho_k$ of $G_{n_k}$ ($k = 1, \ldots, r$) such that $\rho_i \neq \rho_j$ for any $i < j$ and
\[
\tau \boxtimes \rho_r \boxtimes \rho_{r-1} \boxtimes \ldots \boxtimes \rho_1 \hookrightarrow \pi_{N'},
\]
where $N'$ is the unipotent radical of the standard parabolic subgroup associated to $(n - n_1 - \ldots - n_r, n_1, \ldots, n_r)$.

Proof. Again we only consider $\pi$ to be in $\text{Irr}_\rho$. The module $\tau$ determines a set of cuspidal representations $\rho_1, \ldots, \rho_r$ such that
\[
\text{csupp}(\tau) + \rho_1 + \ldots + \rho_r = \text{csupp}(\pi).
\]
We shall relabel $\rho_1, \ldots, \rho_r$ such that $\rho_i \neq \rho_j$ for $i < j$.

Using the Hecke algebra realization [CS19] of Bernstein-Zelevinsky derivatives (which is described as the composition of restriction functor and taking a sign projector), there is a submodule of $\pi_{N'}$ of the form
\[
\tau \boxtimes \omega,
\]
where $\omega$ contains a generic representation. Hence we have a subspace
\[
\tau \boxtimes \omega \hookrightarrow \pi_{N_i}
\]
as $G_{n-i} \times G_i$-module. Since $\omega$ contains a generic representation, it is standard to obtain a non-zero map
\[
\rho_1 \times \ldots \times \rho_r \rightarrow \omega.
\]
Hence, $\omega_{N''}$ has as submodule $\rho_1 \boxtimes \ldots \boxtimes \rho_r$, where $N''$ is the unipotent radical of the parabolic subgroup associated to the partition $(n_1, \ldots, n_r)$. Thus, by Jacquet functors in stages,
\[
\tau \boxtimes \rho_r \boxtimes \ldots \boxtimes \rho_1 \hookrightarrow \pi_{N''}.
as desired.

We also prove a kind of converse of the above statement.

**Definition 3.13.** A sequence of segments $\Delta_1, \ldots, \Delta_k$ is said to be in an ascending order if for any $i < j$, $\Delta_j \not< \Delta_i$. (This is an opposite ordering which usually defines a standard representation.)

**Proposition 3.14.** Let $\pi \in \text{Irr}$. Let $\Delta_1, \ldots, \Delta_k$ be an ascending sequence of segments. Let $n_1, \ldots, n_k$ be the absolute lengths of $\Delta_1, \ldots, \Delta_k$. Let $N$ be the unipotent radical associated to the partition $(n - n_1 - \ldots - n_k, n_1, \ldots, n_k)$ and let $N' = N_{n_1+\ldots+n_k}$. Let $n' = n_1 + \ldots + n_k$. Then,

1. For any $\tau \in \text{Irr}(G_{n-n_1-\ldots-n_k})$,
   \[
   \dim \text{Hom}_G(\tau \boxtimes \text{St}(\Delta_k) \boxtimes \ldots \boxtimes \text{St}(\Delta_1), \pi_N) \leq 1,
   \]
   where $G = G_{n-n'} \times G_{n_1} \times \ldots \times G_{n_k}$.
2. For any $\tau \in \text{Irr}(G_{n-n'})$,
   \[
   \dim \text{Hom}_{G'}(\tau \boxtimes (\text{St}(\Delta_1) \times \ldots \times \text{St}(\Delta_k)), \pi_{N'}) \leq 1,
   \]
   where $G' = G_{n-n'} \times G_{n'}$.
3. If the dimensions above are non-zero, then $\tau$ is a submodule of $\pi^{(n')}$.

**Proof.** Note that (1) and (2) are equivalent by Frobenius reciprocity. We consider (2). Suppose

\[
\dim \text{Hom}_{G'}(\tau \boxtimes (\text{St}(\Delta_1) \times \ldots \times \text{St}(\Delta_k)), \pi_{N'}) \geq 2.
\]

Then there exists quotients $\kappa_1, \kappa_2$ of $\text{St}(\Delta_k) \times \ldots \times \text{St}(\Delta_1)$ such that

\[
\tau \boxtimes \kappa_1 \oplus \tau \boxtimes \kappa_2 \hookrightarrow \pi_{N'}.
\]

Note that $\kappa_1$ and $\kappa_2$ have a generic representation as the unique quotient by [JS83] (also see [Ch21 Proposition 2.3]) and no other composition factor of $\kappa_1$ and $\kappa_2$ is generic. Hence, now taking the (exact) $(U_{n'}, \psi_{n'})$-twisted Jacquet functor, we obtain an embedding

\[
\tau \oplus \tau \hookrightarrow \pi^{(n')}.
\]

This contradicts Lemma 3.9. Hence, we have (3).

3.9. Simple consequence.

**Lemma 3.15.** Let $m \in \text{Mult}_\rho$ and let $\pi = \langle m \rangle \in \text{Irr}_\rho$. Let $\Delta_1, \ldots, \Delta_r$ in Seg$_\rho$ be an ascending sequence of segments such that $D_{\Delta_r} \circ \ldots \circ D_{\Delta_1}(\pi) \neq 0$. For an integer $c$, let $B(\pi, c)$ be the number of segments in $m$ such that $b(\Delta) \cong \nu^c \rho$; and let $x_c$ be the total number of segments $\Delta$ in $\{\Delta_1, \ldots, \Delta_r\}$ such that $\nu^c \rho \in \Delta$. Then, for any $c$,

\[
x_c \leq B(\pi, c).
\]

**Proof.** Let $\tau = D_{\Delta_r} \circ \ldots \circ D_{\Delta_1}(\pi)$. By Proposition 3.14 $\tau \in \pi^{(i)}$, where $i = \sum_{k=1}^r l_a(\Delta_k)$. Now the lemma follows from Lemma 3.11 and a cuspidal support condition.

□
4. On commutativity of derivatives

4.1. \( \epsilon_{\Delta} \)-invariant. Let \( \Delta \) be a segment of absolute length \( m \). Let \( \pi \in \text{Irr}(G_n) \) with \( n \geq m \). Let \( \tau \) be the maximal semisimple representation of \( G_{n-m} \) such that

\[
\tau \boxtimes \text{St}(\Delta) \hookrightarrow \pi_{N_m},
\]

or equivalently \( \tau \) is the maximal semisimple representation of \( G_{n-m} \) such that

\[
\pi_{N_m} \rightarrow \tau \boxtimes \text{St}(\Delta).
\]

It is known that \( \tau \) is either irreducible or zero (see Lemma 3.3). We shall denote the irreducible submodule by \( D_{\Delta}(\pi) \).

Let \( \epsilon_{\Delta}(\pi) \) be the maximal integer \( k \) such that

\[
D_{\Delta}^k(\pi) := \underbrace{D_{\Delta} \circ \ldots \circ D_{\Delta}}_{k \text{ times}}(\pi) \neq 0.
\]

When \( \Delta = [a]_\rho \), \( \epsilon_{\Delta} \) coincides with \( \epsilon_a \) defined in Section 3.2. Indeed, it is known that \( \epsilon_{\Delta}(\pi) \) coincides with the maximal integer \( k' \) such that \( \pi \) is a submodule of

\[
\tau' \times \underbrace{\text{St}(\Delta) \times \ldots \times \text{St}(\Delta)}_{k' \text{ times}},
\]

where \( \tau' \) is some irreducible representation of \( G_{n-k'l_\rho(\Delta)} \). To see this, we need the fact that \( \tau' \times \text{St}(\Delta) \times \ldots \times \text{St}(\Delta) \) has a unique simple submodule (Lemma 3.3), where \( \text{St}(\Delta) \) appears for arbitrary times. The uniqueness implies that \( \pi \) is a submodule of

\[
\omega_r \times \underbrace{\text{St}(\Delta) \times \ldots \times \text{St}(\Delta)}_{r \text{ times}}
\]

\[
\times \underbrace{\text{St}(\Delta) \times \ldots \times \text{St}(\Delta)}_{k'-r \text{ times}},
\]

where \( \omega_r \) is the unique submodule of \( \tau' \times \text{St}(\Delta) \times \ldots \times \text{St}(\Delta) \). Then, inductively, we obtain that \( \omega_r \cong D_{\Delta}^r(\pi) \) and so \( k' = k \).
4.2. Maximal multisegments at a point. Recall that a multisegment at a point is those multisegment in Mult\(_{\rho,c}\) for some integer \(c\).

We define an ordering \(\leq_c\) on the collection of multisegments at a point \(c\). For two multisegments \(m_1\) and \(m_2\) at a point \(c\), we write \(m_1 \leq_c m_2\) if for

\[
m_1 = \{[c, b_1]_\rho, \ldots, [c, b_r]_\rho\}, \quad m_2 = \{[c, b'_1]_\rho, \ldots, [c, b'_s]_\rho\}
\]

satisfies:

\[
b_1 \leq \ldots \leq b_r, \quad b'_1 \leq \ldots \leq b'_s
\]

and, \(r \leq s\) and,

\[
b_1 \leq b'_1, \quad b_2 \leq b'_2, \ldots, \quad b_r < b'_r.
\]

For a multisegment \(m = \{\Delta_1, \ldots, \Delta_r\}\) at a point \(\nu^c\rho\), we define: for any \(\pi \in \text{Irr}(G_n)\),

\[
D_m(\pi) := D_{\Delta_r} \circ \ldots \circ D_{\Delta_1}(\pi).
\]

When \(m\) is an empty set, \(D_m\) is just the identity functor. Since \(\tau \times \text{St}(m)\) has unique submodule for any irreducible \(\tau\), the same argument as in Section 4.1 shows that \(D_m\) is independent of the choice of ordering of segments.

We also adapt the convention that:

\[
D_0(\pi) = \pi.
\]

**Lemma 4.2.** Let \(\pi \in \text{Irr}\). Fix \(c \in \mathbb{Z}\). There exists at most one (nonempty) maximal multisegment \(m\) in Mult\(_{\rho,c}\) such that \(D_m(\pi) \neq 0\).

**Proof.** Let \(m_1\) and \(m_2\) be maximal multisegments at a point \(\nu^c\rho\). Let

\[
m_1 = \{[c, b_1]_\rho, \ldots, [c, b_r]_\rho\}, \quad m_2 = \{[c, b'_1]_\rho, \ldots, [c, b'_s]_\rho\}
\]

with \(b_1 \geq \ldots \geq b_r\) and \(b'_1 \geq \ldots \geq b'_s\). Write \(\Delta_i = [c, b_i]_\rho\) and \(\Delta'_i = [c, b'_i]\).

If \(b_1 = b'_1\), then one can proceed inductively since \(m_1 - [c, b_1]_\rho\) and \(m_2 - [c, b_2]_\rho\) are still maximal for \(D_{[c, b_1]}(\pi)\). If \(b_1 > b'_1\), then one considers, by Lemma 4.1

\[
\pi \mapsto D_{m_2}(\pi) \times \text{St}(m_2).
\]

Now one applies the functor \(N_{(b_1-c-1)}(\rho)\) and uses geometric lemma and Section 2.7 to see that the only possible layers of the form \(\omega \times \text{St}([c, b_1])\) as a submodule gives the following possible embedding:

\[
D_{[c, b_1]}(\pi) \hookrightarrow \omega' \times \text{St}(m_2 - \{\Delta\}),
\]

for some \(\omega_2 \in \text{Irr}\), where \(\Delta\) is one of the segments in \(m_2\). However, then \(m' := m_2 - \{\Delta\} + \{[c, b_1]_\rho\}\) also satisfies \(D_{m'}(\pi) \neq 0\). This gives a contradiction to the maximality of \(m_2\). The case for \(b'_1 > b_1\) is similar. 

**Definition 4.3.** Let \(\pi \in \text{Irr}_\rho\). Let \(c \in \mathbb{Z}\). If \(D_c(\pi) \neq 0\), we shall write \(m(\pi, c)\) to be the unique maximal multisegment at the point \(\nu^c\rho\).

**Proposition 4.4.** Let \(\pi \in \text{Irr}_\rho\) and let \(n \in \text{Mult}_\rho\). Let \(c \in \mathbb{Z}\). Suppose \(n \leq_c m(\pi, c)\).

Then

\[
D_n(\pi) \neq 0.
\]
Proof. We have the embedding:

\[ \pi \hookrightarrow D_{m(\pi, c)}(\pi) \times \text{St}(m(\pi, c)). \]

For \( n \leq m(\pi, c) \), we have that:

\[ \tau \boxtimes \text{St}(n) \hookrightarrow \text{St}(m(\pi, c))_{N_i}, \]

for some \( \tau \in \text{Irr} \). Here \( i = l_a(n) \). One may prove the last statement by discussions in Section 4.2 (or see [Ch21, Corollary 2.6]) and we omit the details. Thus we have a non-zero map:

\[ D_{m(\pi, c)}(\pi) \boxtimes \tau \boxtimes \text{St}(n) \hookrightarrow \pi_{N_i}, \]

where \( N = N_{n(\pi) - l_a(m(\pi, c)), l_a(m(\pi, c)) - i, i} \). By Frobenius reciprocity, we have

\[ \tau' \boxtimes \text{St}(n) \hookrightarrow \pi_{N_i} \]

for some \( \tau \in \text{Irr} \), as desired. \( \square \)

4.3. Right maximal multisection at a point.

Definition 4.5. A multisection \( n \in \text{Mult}_\rho \) is said to be at the right point \( \nu^c \rho \) if for any segment \( \Delta \) in \( n \), \( \Delta = [a, c]_{\rho} \) for some \( a \leq c \).

Similar to the argument in Lemma 4.2, one can deduce that there is a unique maximal multisection \( n \) at the right point \( \nu^c \rho \) such that \( D_n(\pi) \neq 0 \). We shall denote such multisection by \( m^R(\pi, c) \).

For convenience, we define the following notion. Let \( c \in \mathbb{Z} \). An irreducible representation \( \pi \in \text{Irr}_\rho \) is said to have maximal point \( \nu^c \rho \) if for any integer \( d > c \), \( \nu^d \rho \notin \text{csupp}(\pi) \).

Proposition 4.6. Let \( \pi \in \text{Irr}_\rho(G_n) \). Let \( c \in \mathbb{Z} \). Suppose \( m^R(\pi, c) = 0 \). For any \( i < n, \pi_i \) does not have a simple composition factor of the form \( \tau \boxtimes \omega \) for some \( \omega \in \text{Irr}_\rho(G_i) \) having maximal point \( \nu^c \rho \) (defined above) and for some \( \tau \in \text{Irr}_\rho(G_{n-i}) \).

Proof. Suppose not. Then \( \pi_i \) also have a simple composition factor of the form \( \tau \boxtimes \omega \) for some \( \omega \in \text{Irr}_\rho(G_i) \) having maximal point \( \nu^c \rho \) and for some \( \tau \in \text{Irr}_\rho(G_{n-i}) \). Since \( \omega \) have maximal point \( \nu^c \rho \), there exists \( j > 0 \) such that \( \omega_{N_j} \) have an irreducible quotient isomorphic to \( \tau' \boxtimes \text{St}(n) \) for some multisection \( n \) at the right point \( \nu^c \rho \) and \( \tau' \in \text{Irr}(G_{i-j}) \).

Hence, by taking Jacquet functors in stage, we have a surjection:

\[ \pi_{N'} \rightarrow \tau \boxtimes \tau' \boxtimes \text{St}(n), \]

where \( N' \) is the unipotent radical of the parabolic subgroup \( P_{n-i, j-i, j} \). Now applying Frobenius reciprocity, we have that

\[ \pi_{N_j} \rightarrow (\tau \times \tau') \boxtimes \text{St}(n). \]

Thus \( D_n(\pi) \neq 0 \), giving a contradiction. \( \square \)
4.4. Commutation of derivatives.

Lemma 4.7. Let $\pi \in \text{Irr}$. Let $\Delta_1$ and $\Delta_2$ be unlinked segments. Then $D_{\Delta_1} \circ D_{\Delta_2}(\pi) \cong D_{\Delta_2} \circ D_{\Delta_1}(\pi)$.

Proof. Let $\omega_1 = D_{\Delta_2} \circ D_{\Delta_1}(\pi)$ and $\omega_2 = D_{\Delta_1} \circ D_{\Delta_2}(\pi)$. Then, by Proposition 4.4 (twice),

$$\pi \hookrightarrow \omega_1 \times \text{St}(\Delta_2) \times \text{St}(\Delta_1)$$

and similarly,

$$\pi \hookrightarrow \omega_2 \times \text{St}(\Delta_1) \times \text{St}(\Delta_2) \cong \omega_2 \times \text{St}(\Delta_2) \times \text{St}(\Delta_1).$$

But then uniqueness implies $\text{soc}(\omega_1 \times \text{St}(\Delta_1)) \cong \text{soc}(\omega_2 \times \text{St}(\Delta_1)) \cong D_{\Delta_2}(\pi)$. Then Lemma 3.3 implies $\omega_1 \cong \omega_2$.

Lemma 4.8. Let $\pi \in \text{Irr}_\rho$. Let $\Delta_1 = [a_1, b_1]_\rho$ and $\Delta_2 = [a_2, b_2]_\rho$ be linked with $a_2 > a_1$. Suppose $D_{\Delta_1}(\pi) \neq 0$ and $D_{\Delta_2}(\pi) \neq 0$. Then

$$D_{\Delta_1} \circ D_{\Delta_2}(\pi) \neq 0, \quad D_{\Delta_2} \circ D_{\Delta_1}(\pi) \neq 0.$$

Proof. Let $\omega_1 = D_{\Delta_1}(\pi)$ and let $\omega_2 = D_{\Delta_2}(\pi)$. Let $l = n(\rho)$ and let $k = b_1 - a_1 + 1$. Then

$$\pi \hookrightarrow \omega_1 \times \text{St}(\Delta_1), \quad \pi \hookrightarrow \omega_2 \times \text{St}(\Delta_2).$$

Checking the second one is easier by geometric lemma and cuspidal support. Some similar computation appears for the first one and so we omit the details for the first one. Now when one applies a geometric lemma on

$$(\omega_2 \times \text{St}(\Delta_2))_{N_{kl}},$$

to reduce the possibility contributing to a factor of the form $\tau \boxtimes \text{St}(\Delta_1)$ to those of the form

$$(\ast) \quad \tau' \times \text{St}([b_1 + 1, b_2]_\rho) \boxtimes \text{St}([a_1, a_2 - 1]_\rho) \times \text{St}([c + 1, b_1]_\rho) \times \text{St}([a_2, c]_\rho),$$

for some $c$, or simply

$$(\ast\ast) \quad \tau'' \times \text{St}([a_2, b_2]_\rho) \boxtimes \text{St}([a_1, b_1]_\rho).$$

Here one of the irreducible factor in (\ast) has to take the form $\text{St}([a_1, a_2 - 1]_\rho) \times \text{St}([c + 1, b_1]_\rho)$ by picking the unique generic representation with given cuspidal support, and another irreducible factor $\text{St}([a_2, c]_\rho)$ comes from the Jacquet module of $\text{St}(\Delta_2)_{N_{(c-a_2+1)l}}$ by Section 2.7.

Here $\tau'$ is an irreducible representation such that $\tau' \boxtimes \text{St}([a_1, a_2 - 1]_\rho)$ is a composition factor of $(\omega_2)_{N_{(a_2-a_1+1)l}}$; and $\tau''$ is an irreducible representation such that $\tau'' \boxtimes \text{St}([a_1, b_1]_\rho)$. However, for Case (\ast), by Frobenius reciprocity, the $\text{St}([a_1, b_1]_\rho)$ does not appear in the submodule of $\text{St}([a_1, a_2 - 1]_\rho) \times \text{St}([a_2, b_1]_\rho)$. Thus only (\ast\ast) can contribute to a submodule of the form $\kappa \boxtimes \text{St}(\Delta_1)$ in $\pi_{N_{(b_1-a_1+1)l}}$ (which we know such submodule exists by $D_{\Delta_1}(\pi) \neq 0$). Thus the socle of $(\omega_2)_{N_{(b_1-a_1+1)l}}$ has a factor of the form $\tau'' \boxtimes \text{St}(\Delta_1)$ (see Lemma 5.1 below for more details). This shows $D_{\Delta_2} \circ D_{\Delta_1}(\pi) \neq 0$ and finishes the proof of the claim. \qed
We shall need the following in Section \textbf{8.2}. We also remark that dropping the condition that $D_{\Delta_1}(\pi) \neq 0$ or $D_{\Delta_2}(\pi) \neq 0$ below will make the statement fail in general (e.g. consider the derivatives on a Speh representation).

**Lemma 4.9.** Let $\pi \in \text{Irr}_\rho$. Let $\Delta_1 = [a_1, b_1]_\rho$ and $\Delta_2 = [a_2, b_2]_\rho$ be linked with $a_2 > a_1$. Suppose $D_{\Delta_1}(\pi) \neq 0$ and $D_{\Delta_2}(\pi) \neq 0$. Let $\Delta = \Delta_1 \cup \Delta_2$. If $D_{\Delta}(\pi) = 0$, then

$$D_{\Delta_1} \circ D_{\Delta_2}(\pi) \cong D_{\Delta_2} \circ D_{\Delta_1}(\pi).$$

**Proof.** By Lemma \textbf{4.8} the two terms $D_{\Delta_1} \circ D_{\Delta_2}(\pi)$ and $D_{\Delta_2} \circ D_{\Delta_1}(\pi)$ are non-zero.

Now let $\tau_1 = D_{\Delta_2} \circ D_{\Delta_1}(\pi)$ and let $\tau_2 = D_{\Delta_1} \circ D_{\Delta_2}(\pi)$. Then we have the embeddings:

$$\pi \hookrightarrow \tau_1 \times \text{St}(\Delta_2) \times \text{St}(\Delta_1), \quad \pi \hookrightarrow \tau_2 \times \text{St}(\Delta_1) \times \text{St}(\Delta_2).$$

By applying the Frobenius reciprocity, we have non-zero maps:

$$\pi_N \rightarrow \tau_1 \boxtimes (\text{St}(\Delta_2) \times \text{St}(\Delta_1)), \quad \pi_N \rightarrow \tau_2 \boxtimes (\text{St}(\Delta_1) \times \text{St}(\Delta_2))$$

Since $\text{St}(\Delta_2) \times \text{St}(\Delta_1)$ as well as $\text{St}(\Delta_1) \times \text{St}(\Delta_2)$ have only two composition factors, $\sigma_1 := \text{St}(\{\Delta_1, \Delta_2\})$ and $\sigma_2 := \text{St}(\{\Delta_1 \cap \Delta_2, \Delta_1 \cup \Delta_2\})$, the socle of $\pi_N$ has either $\tau_1 \boxtimes \sigma_1$ or $\tau_1 \boxtimes \sigma_2$, and the socle of $\pi_N$ also has either $\tau_2 \boxtimes \sigma_1$ or $\tau_2 \boxtimes \sigma_2$.

We now show that $\tau_1 \boxtimes \sigma_2$ and $\tau_2 \boxtimes \sigma_2$ cannot be simple quotients of $\pi_N$. Otherwise, by Frobenius reciprocity, we obtain a non-zero map:

$$\pi_{N,(b_2-a_1+1)d} \rightarrow \tau_k \times \text{St}(\Delta_1 \cap \Delta_2) \boxtimes \text{St}(\Delta_1 \cup \Delta_2).$$

for $k = 1$ or 2. This contradicts $D_{\Delta}(\pi) = 0$.

Hence, we have $\pi_N$ has quotient $\tau_1 \boxtimes \sigma_1$ and $\tau_2 \boxtimes \sigma_1$. Now $\tau_1 \cong \tau_2$ follows from the uniqueness in Lemma \textbf{3.3}.

\[ \square \]

4.5. **Commutations in another form.** As mentioned before, Lemma \textbf{4.9} requires the assumption that $D_{\Delta_1}(\pi) \neq 0$ and $D_{\Delta_2}(\pi) \neq 0$. It is not convenient for some application purpose. We now prove another version of commutativity, and one may compare with the proof of Lemma \textbf{4.9}.

**Lemma 4.10.** Let $\pi \in \text{Irr}_\rho$. Let $c \in \mathbb{Z}$. Let $\tau = D_{m(\pi,c)}(\pi)$. Let $d > c$ be an integer. Let $n \in \text{Mult}_{\rho,d}$. If $D_n(\tau) \neq 0$, then $D_n(\pi) \neq 0$.

**Proof.** Let $\tau' = D_n(\tau)$. By Lemma \textbf{4.14} we have embeddings:

$$\pi \hookrightarrow \tau \times \text{St}(m(\pi, c)), \quad \tau \hookrightarrow \tau' \times \text{St}(n).$$

Hence,

$$\pi \hookrightarrow \tau' \times \text{St}(n) \times \text{St}(m(\pi, c)).$$

The lemma will follow from the following claim.

**Claim:** $\pi \hookrightarrow \tau' \times \text{St}(m(\pi, c)) \times \text{St}(n)$.

**Proof of Claim:** We shall write the segments in $m(\pi, c)$ as:

$$\Delta_k \leq_c \ldots \leq_c \Delta_1,$$
and write the segments in \( n \) as:

\[ \overline{\Delta}_1 \leq_c \ldots \leq_c \overline{\Delta}_j. \]

(Note that the order is opposite to \( m(\pi, c) \).)

We shall inductively show that:

\[ \pi \hookrightarrow \tau^* \times A_{ij} \times \text{St}(\overline{\Delta}_j) \times B_{ij}, \]

where

\[ A_{ij} = (\text{St}(\overline{\Delta}_l) \times \ldots \times \text{St}(\overline{\Delta}_{j+1})) \times (\text{St}(\Delta_k) \times \ldots \times \text{St}(\Delta_l)) \]

and

\[ B_{ij} = \text{St}(\Delta_{i-1}) \times \ldots \times \text{St}(\Delta_1) \times \text{St}(\overline{\Delta}_{j-1}) \times \ldots \times \text{St}(\overline{\Delta}_1). \]

The basic case has been given before the claim. Suppose the case is proved for \( i = i^* \) and \( j = j^* \). To prove the case that \( i = i^*-1 \) and \( j = j^* \) (if \( i^* = 1 \), then we proceed \( i = k+1 \) and \( j = j^* + 1 \) and the argument is similar), we consider two cases:

1. \( \overline{\Delta}_{i^*} \subset \Delta_{i^*-1} \). Then it follows from \((i^*, j^*)\) case and the fact that \( \text{St}(\Delta_{j^*}) \times \text{St}(\Delta_{i^*-1}) \cong \text{St}(\Delta_{j^*}) \times \text{St}(\Delta_{i^*-1}) \).
2. \( \overline{\Delta}_{i^*} \not\subset \Delta_{i^*-1} \). Then there are two composition factors for \( \text{St}(\Delta_{j^*}) \times \text{St}(\Delta_{i^*-1}) \), which are

\[ R = \text{St}(\Delta_{j^*} \cup \Delta_{i^*-1}) \times \text{St}(\Delta_{j^*} \cap \Delta_{i^*-1}), \]

and a non-generic factor denoted by \( S \).

Now, by induction hypothesis,

\[ \pi \hookrightarrow \tau^* \times A_{i^*, j^*} \times \text{St}(\overline{\Delta}_{j^*}) \times \text{St}(\Delta_{i^*-1}) \times \times B_{i^*-1, j^*} \]

and so the above discussion implies that

\[ (\bigcirc) \quad \pi \hookrightarrow \tau^* \times A_{i^*, j^*} \times R \times B_{i^*-1, j^*}, \]

or

\[ (\bigstar) \quad \pi \hookrightarrow \tau^* \times A_{i^*, j^*} \times S \times B_{i^*-1, j^*}. \]

We first prove the former case is impossible. Suppose the former case happens. We write \( \overline{\Delta} = \overline{\Delta}_{j^*} \cup \Delta_{i^*-1} \). We choose all the segments \( \Delta_1, \ldots, \Delta_p \) in \( m(c) \) such that those \( \overline{\Delta} \subset \Delta_x \ (x = 1, \ldots, p) \). Then, by the ordering above, we also have

\[ \overline{\Delta}_{j^*-1} \subset \ldots \subset \overline{\Delta}_1 \subset \overline{\Delta} \subset \Delta_p \subset \ldots \subset \Delta_1. \]

We also further have that \( \overline{\Delta} \) is unlinked to \( \Delta_y \) for any \( y \). Thus, using \([23]\) several times, we have that:

\[ A_{i^*, j^*} \times R \times B_{i^*-1, j^*} \cong A_{i^*, j^*} \times \widehat{R} \times \text{St}(\overline{\Delta}) \times \text{St}(\Delta_1) \times \ldots \times \text{St}(\Delta_p), \]

where \( \widehat{R} \) is \( R \times B_{i^*-1, j^*} \) with the terms \( \text{St}(\overline{\Delta}), \text{St}(\Delta_1), \ldots, \text{St}(\Delta_p) \) dropped. This implies that, by Frobenius reciprocity,

\[ \{ \Delta_1, \Delta_2, \ldots, \Delta_p \} \leq_c m(\pi, c), \]

but this gives a contradiction to the unique maximality in Lemma \([12]\).

Thus we must lie in the \((\bigstar)\) case. Now combining

\[ S \hookrightarrow \text{St}(\Delta_{i^*-1}) \times \text{St}(\Delta_{j^*}) \]
with (*), we obtain the case that $i = i^* - 1$ and $j = j^*$, as desired. \qed

5. Highest derivative multisegment

In this section, we construct the highest derivative of an irreducible representation by a sequence of St-derivatives. One may compare with the construction using $\rho$-derivatives in Proposition 3.7.

5.1. Submodules from a layer in geometric lemma.

Lemma 5.1. Let $p, q, r$ be integers. Let $\omega_1 \in \text{Irr}(G_{p+q})$ and let $\omega_2 \in \text{Irr}(G_r)$. Let $\pi \in \text{Irr}(G_{p+r})$ and let $\tau \in \text{Irr}(G_q)$. Suppose

$\omega_1 \boxtimes \omega_2 \hookrightarrow \text{Ind}_{P_{p,q,r}}^{G_{p+q} \times G_r}(\pi \boxtimes \tau)^\phi$.

Here $(\pi_N, \boxtimes \tau)^\phi$ is a $G_p \times G_q \times G_r$-representation with the action defined by:

$\text{diag}(g_1, g_2, g_3).v = \text{diag}(g_1, g_3, g_2).v$,

where the latter action is the original action of $G_p \times G_r \times G_q$. The notion $\phi$ is to represent the twisted action:

$\phi(\text{diag}(g_1, g_2, g_3)) = \text{diag}(g_1, g_3, g_2)$.

Then there exists $\omega \in \text{Irr}(G_p)$ such that

$\omega \boxtimes \omega_2 \hookrightarrow \pi_N$.

Proof. One applies Frobenius reciprocity on the injection:

$\omega_1 \boxtimes \omega_2 \hookrightarrow \text{Ind}_{P_{p,q,r}}^{G_{p+q} \times G_r}(\pi_N \boxtimes \tau)^\phi$

to obtain a non-zero map

$(\omega_1)_N \boxtimes \omega_2 \hookrightarrow (\pi_N \boxtimes \tau)^\phi$.

Then the non-zero map must contain $(\kappa \boxtimes \tau)^\phi$ for some submodule $\kappa$ of $\pi_N$. Hence, it is of the form $\lambda \boxtimes \lambda' \boxtimes \omega_2$ for $\lambda \in \text{Irr}(G_p)$ and $\lambda' \in \text{Irr}(G_q)$. Hence

$\lambda \boxtimes \omega_2 \cong \kappa \hookrightarrow \pi_N$,

as desired. \qed

5.2. A computation of maximal multisegment at a point.

Lemma 5.2. Let $m \in \text{Mult}_\rho$ and let $\pi = \langle m \rangle$. Let $c \in \mathbb{Z}$ with $\epsilon_c(\pi) \neq 0$. Then, for any $d > c$, the maximal multisegment at $\nu^d \rho$ of $D_{m(\pi,c)}(\pi)$ is equal to $m(\pi,d)$.

Proof. We simply abbreviate $m_c$ as $m(\pi,c)$ and $m_d$ as $m(\pi,d)$. Let $n$ be the maximal multisegment at $\nu^d \rho$ for $D_{m_c}(\pi)$. We have that:

$\pi \hookrightarrow D_{m_c}(\pi) \times \text{St}(m_c)$.

This implies that

$D_{m_d}(\pi) \boxtimes \text{St}(m_d) \hookrightarrow (D_{m_c}(\pi) \times \text{St}(m_c))_{N_{n_2}}$,

where $n_2 = l_a(m_d)$. 

By the geometric lemma and comparing cuspidal support, $D_{m_d}(\pi) \boxtimes \text{St}(m_d)$ can only come from the layer

$$\text{Ind}_{G}^{G_n}(D_{m_c}(\pi)_{N_{n_2}} \boxtimes \text{St}(m_c))^\phi,$$

where $P = P_{n-n_1-n_2,n_1,n_2}$ for $n_1 = l_0(m_c)$. Other notions are analogous to the ones in Lemma 5.1. Thus, this implies that $D_{m_d} \circ D_{m_c}(\pi) \neq 0$. Hence, $n \geq m_d$.

Now the opposite inequality follows from Lemmas 4.10 and 4.2, and hence we are done.

\[\square\]

5.3. **Highest derivative by St-derivatives.** Recall that in Proposition 3.14, we have shown that an ascending sequence of segments can be used to construct simple quotient of derivatives. On the other hand, the highest derivative is known to be irreducible. Thus, the strategy of the following result is to construct an ascending sequence of segments, with sum of absolute length of segments equal to the level of the representation.

**Theorem 5.3.** Let $m \in \text{Mult}_\rho$. Let $\pi = \langle m \rangle$ in $\text{Irr}_\rho$. We choose the smallest integer $c$ such that $\nu^c \rho \cong b(\Delta)$ for some $\Delta$ in $m$ and choose the largest integer $d$ such that $\nu^d \rho \cong b(\Delta)$ for some $\Delta$ in $m$. Then

$$D_{m_d}(\pi,d) \circ \ldots \circ D_{m_c}(\pi) \cong \pi^{-}.$$

**Proof.** For simplicity, let $m_e = m(\pi,e)$.

**Step 1:** Claim: The following two conditions:

- $D_{m_d} \circ \ldots \circ D_{m_c}(\pi) \neq 0$; and
- $m_d + \ldots + m_c$ is a multisegment whose sum of absolute lengths of all segments is equal to the level of $\pi$.

imply the theorem.

**Proof of the claim:** Let

$$\tau = D_{m_d} \circ \ldots \circ D_{m_c}(\pi).$$

The first bullet and Lemma 4.11 gives that

$$\pi \hookrightarrow \tau \times \text{St}(m_d) \times \ldots \times \text{St}(m_c),$$

and so, by Frobenius reciprocity again,

$$\pi_{N_e} \hookrightarrow \tau \boxtimes \text{St}(m_d) \times \ldots \times \text{St}(m_c),$$

where $k$ is the sum of the absolute lengths of all multisegments $m_d, \ldots, m_c$. By Proposition 3.14, $\tau$ is a submodule of $\pi^{(k)}$. By the second bullet, $\pi^{(k)}$ is the highest derivative and so it is irreducible. Thus $\tau \cong \pi^{(k)}$.

Note that the first bullet follows from repeatedly using Lemma 5.2. It remains to prove the second bullet in the claim.

**Step 2:** We now prove the second bullet. Let $x_e$ be the total number of segments
in $m_c + \ldots + m_{e-1}$ has a segment containing $\nu^e \rho$. Let $B(\pi, e)$ be the number of segments in $m$ such that $b(\Delta) \cong \nu^e \rho$. We shall show by induction on $e$ that

\[(*) \quad x_e + \text{number of segments in } m(e, \pi) = B(\pi, e).\]

The second bullet will then follow from $(*)$.

When $e = c$, one can compute quite directly by Lemma 3.8 (also see [MW86, Ja07, Mi09]). Now again let

$$\tau := D_{m_{e-1}} \circ \ldots \circ D_{m_c}(\pi).$$

As argued in Step 1, we have that $\tau \rightarrow \pi(j)$, where $j$ is the sum of the lengths of all segments in $m_c + \ldots + m_{e-1}$. We also have, by the induction hypothesis (and a cuspidal support calculation using Lemma 3.11), that $B(\pi, e - 1)$ (i.e. all) segments in $m$ with $b(\Delta) \cong \nu^{e-1} \rho$ are truncated to produce the segments in $m(\tau) \in m^{(j)}$ (see Definition). In other words, $B(\tau, e - 1) = 0$. By Lemma 3.8 (and Lemma 4.2), this implies that $m_k$ contains at least $B(\tau, e)$ number of segments $B(\tau, e)$ times

since $\{\nu^e \rho, \ldots, \nu^c \rho\} \leq_e m_c$, but indeed contains exactly $B(\tau, e)$-number by Lemma 3.15.

By Lemma 3.11 and a cuspidal support consideration again, the multisegment associated to $D_{m_e}(\tau)$ does not have any segment $\Delta$ satisfying $b(\Delta) \cong \nu^e \rho$. Again 3.11 and a cuspidal support consideration on

$$D_{m_e} \circ \ldots \circ D_{m_c}(\pi) = D_{m_e}(\tau),$$

we have:

$$x_e + \text{number of segments in } m_e = B(\pi, e).$$

Now one uses Lemma 5.2 to obtain $(*)$. \[\Box\]

6. Constructing some derivatives

In this section, we prove two constructions of derivatives. While we do not use directly, the constructions give a more conceptual explanation on some steps in the proof of Theorem 8.2.

6.1. Enlarging to derivative non-extendable one.

**Definition 6.1.** Let $\pi \in \text{Irr}_\rho$. Let $c$ be an integer such that $\nu^c \rho \geq \nu^b \rho$ for any $\nu^b \rho \in \text{csupp}(\pi) - \text{csupp}(\tau)$. (Here $\text{csupp}(\pi) - \text{csupp}(\tau)$ is those cuspidal representations 'removed' from the derivatives.) Let $\tau$ be a simple submodule of $\pi^{(i)}$ for some $i$. We call $\tau$ to be $c$-derivative-extendable or simply $c$-extendable if there exists a segment $\Delta = [a, c]_\rho$ (for $a \leq c$) such that $D_\Delta(\pi) \neq 0$. Otherwise, we call $\tau$ to be not $c$-extendable. Note that the notion depends on $\pi$ and $i$, but it should be clear from the content.

The above terminology is suggested by the following lemma:

**Proposition 6.2.** Let $\tau$ be a simple submodule of $\pi^{(i)}$. Let $c$ be an integer such that $\nu^c \rho \in \text{csupp}(\pi) - \text{csupp}(\tau)$. Suppose $\tau$ is $c$-extendable. Then there exists segments $\Delta_1, \ldots, \Delta_r$ such that the followings are satisfied:
(1) $b(\Delta_k) \cong \nu^k \rho$ for each $k$;
(2) $D_{\Delta_r} \circ \cdots \circ D_{\Delta_1}(\tau)$ is a submodule of $\pi^{(i+j)}$, where $j$ is the sum of the absolute lengths of those $\Delta_k$;
(3) $D_{\Delta_r} \circ \cdots \circ D_{\Delta_1}(\tau)$ is not c-extendable;
(4) $\{\Delta_1, \ldots, \Delta_r\} \not\leq_R m^R(\pi, c)$.

Proof. By discussions in Section 4, one obtains segments $\Delta_1, \ldots, \Delta_r$ such that $\Delta_k \cong \nu^k \rho$ and $D_{\Delta_r} \circ \cdots \circ D_{\Delta_1}(\tau)$ is not derivative-extendable (and non-zero).

Let $j$ be the sum of absolute lengths of all $\Delta_1, \ldots, \Delta_r$. To prove (2) and (3), we consider, by Proposition 3.12,
$$\tau \boxtimes \rho_s \boxtimes \cdots \boxtimes \rho_1 \hookrightarrow \pi N',$$
where $N'$ is a unipotent radical and $\rho_k$ are cuspidal representations as in Proposition 3.12. Let $\tau' = D_{\Delta_r} \circ \cdots \circ D_{\Delta_1}(\pi)$. Hence, we have:
$$\tau' \boxtimes St(\Delta_r) \boxtimes \cdots \boxtimes St(\Delta_1) \boxtimes \rho_s \boxtimes \cdots \boxtimes \rho_1 \hookrightarrow \pi N'',$$
where $N''$ is another unipotent radical. By Frobenius reciprocity or Proposition 3.14, we have a non-zero map:
$$\tau' \boxtimes \rho_1 \times \cdots \times \rho_s \times \text{St}(\Delta_1) \times \cdots \times \text{St}(\Delta_r) \rightarrow \pi N_{i+j}.$$
By Proposition 3.14, $\tau'$ is a submodule of $\pi^{(i+j)}$.

Then there is a composition factor $\omega$ in $\text{St}(\Delta_1) \times \cdots \times \text{St}(\Delta_r)$ such that $\tau' \boxtimes \omega$ embeds to $\pi N_{i+j}$. Suppose that
$$\{\Delta_1, \ldots, \Delta_r\} \not\leq_R m^R(\pi, c)$$
and we shall arrive a contradiction. Let $n$ be the multisegment such that $\omega \cong \text{St}(n)$. Then $n$ is obtained by intersection-union processes of segments $[\rho_1], \ldots, [\rho_s], \Delta_1, \ldots, \Delta_r$, and set $n_c = n(c) = m^R(\omega, c)$. By definition of the intersection-union process, we also have:
$$\{\Delta_1, \ldots, \Delta_r\} \not\leq_R m^R(\pi, c).$$

Now we consider the map
$$\tau' \boxtimes \lambda(n) \rightarrow \tau \boxtimes \omega \hookrightarrow \pi N_{i+j}$$
and so we have a non-zero map:
$$\tau' \boxtimes (\lambda(n_c) \times \lambda(n-n_c)) \rightarrow \pi N_{i+j}.$$
By applying Frobenius reciprocity and using $\text{St}(n_c) = \lambda(n_c)$,
$$\tau'' \boxtimes \text{St}(n_c) \hookrightarrow \pi N_{j'},$$
where $j'$ is the sum of absolute lengths of all segments in $n_c$. However, this contradicts (*) (see Definition 4.5). \(\square\)
6.2. Shrinking from derivative non-extendable case.

**Proposition 6.3.** Let \( \pi \in \text{Irr}_{\rho} \). Let \( \tau \) in \( \text{soc}(\pi(i)) \). Let \( c \in \mathbb{Z} \) be the largest integer such that \( \nu^c \rho \in \text{csupp}(\pi) - \text{csupp}(\tau) \). Suppose \( \tau \) is a non-c-extendable submodule of \( \pi(i) \). Then there exists a submodule \( \tau' \) of \( \pi(i-j) \) such that \( D_{mR(\pi,c)}(\tau') \cong \tau \), where \( j = l_a(mR(\pi,c)) \).

**Proof.** Let \( n = mR(\pi,c) \). Let \( \omega = D_n(\pi) \). Then

\[ \pi \hookrightarrow \omega \times \text{St}(n). \]

We first claim that:

**Claim 1:** \( \text{Hom}(\tau,(\omega \times \text{St}(n))(i)) \cong \text{Hom}(\tau,\omega(i-j)) = \mathbb{C} \), where \( j \) is the level of \( \text{St}(n) \).

**Proof of Claim 1:** By geometric lemma, \((\omega \times \text{St}(n))(i)\) admits a filtration of the form

\[ \omega^{(i-l)} \times \text{St}(n) \]

for some \( l \geq 0 \). Since \( mR(\tau,c) = \emptyset \), by Proposition 4.6 and comparing cuspidal support, for any \( k \),

\[ \text{Ext}^k(\tau,\omega^{(i-l)} \times \text{St}(n)) = 0 \]

unless \( l \) is the level \( j \) of \( \text{St}(n) \). Thus, a long exact sequence argument gives that:

\[ \text{Hom}(\tau,\omega^{(i-j)}) \cong \text{Hom}_{G_a}(\tau,\omega \times \text{St}(n)). \]

Hence, it suffices to determine the left hand side. But, by Lemma 4.9, it remains to show that \( \text{Hom}(\tau,\omega^{(i-j)}) \neq 0 \). This indeed follows from the embeddings:

\[ \tau \hookrightarrow \pi(i) \hookrightarrow (\omega \times \text{St}(n))(i), \]

where the first embedding follows from the definition of \( \tau \) and the second embedding follows from applying the \( i \)-th derivative on \( \pi \hookrightarrow \omega \times \text{St}(n) \).

We now claim another one:

**Claim 2:** There exists a submodule \( \bar{\tau} \) of \((\omega \times \text{St}(n))(i-j)\) such that \( D_n(\bar{\tau}) \cong \tau \).

**Proof of Claim 2:** We have the following

\[ \tau \times \text{St}(n) \hookrightarrow \omega^{(i-j)} \times \text{St}(n) \hookrightarrow (\omega \times \text{St}(n))^{(i-j)}, \]

where the first map induced from applying the (exact) parabolic induction on \( \tau \hookrightarrow \omega^{(i-j)} \) and the second map comes from the embedding of the bottom layer in the geometric lemma. Now let \( \bar{\tau} \) be the unique submodule of \( \tau \times \text{St}(n) \), and by Lemma 4.1 \( D_n(\bar{\tau}) \cong \tau \), as desired.

We now return to the proof. Let \( Q \) be the of \( \omega \times \text{St}(n) \) lies in the image of the map:

\[ 0 \to \pi \to (\omega \times \text{St}(n)) \xrightarrow{q} Q \to 0 \]

Let \( A = \omega \times \text{St}(n) \).
Claim 3: For any \( f : \tilde{\tau} \boxtimes (\rho_1 \times \ldots \times \rho_r) \to A_{N'} \), \( f \) factors through a map \( f' : \tilde{\tau} \boxtimes (\rho_1 \times \ldots \times \rho_r) \to \pi \). Here \( \rho_1, \ldots, \rho_r \) are cuspidal representations satisfying that for \( i < j \), \( \rho_i \nless \rho_j \) (also see Proposition 3.12).

Suppose Claim 3 in the meanwhile. Then, by Proposition 3.12 and Claim 2, we obtain a map as in Claim 3. Now by Proposition 3.14, \( \tilde{\tau} \) is a submodule \( \pi^{(i-j)} \). Now Claim 2 guarantees that \( \tilde{\tau} \) satisfies the property that \( D_n(\tilde{\tau}) = \tau \).

Proof of Claim 3: It remains to prove Claim 3. Suppose not. Then the composition
\[
\tilde{\tau} \boxtimes (\rho_1 \times \ldots \times \rho_r) \xrightarrow{f} A_{N'} \xrightarrow{q'} Q_{N'}
\]
is non-zero. By adjointness of functors, the composition
\[
\tilde{\tau} \boxtimes \rho_r \boxtimes \ldots \boxtimes \rho_1 \xrightarrow{f} A_{\tilde{\tau}} \xrightarrow{q} Q_{\tilde{\tau}}
\]
is still non-zero.

Note that \( \tilde{\tau} \boxtimes \rho_r \boxtimes \ldots \boxtimes \rho_1 \) is irreducible and so both \( \tilde{f} \) and \( \tilde{q} \circ \tilde{f} \) are non-zero. Thus, using the exactness of Jacquet functors and the embedding
\[
\tau \boxtimes \text{St}(n) \hookrightarrow \tilde{\tau}_{N_j},
\]
the composition
\[
\tau \boxtimes \text{St}(n) \boxtimes \rho_r \boxtimes \ldots \boxtimes \rho_1 \hookrightarrow A_{N''} \to Q_{N''}
\]
is still non-zero. By adjointness, we still have non-zero composition
\[
\tau \boxtimes (\rho_1 \times \ldots \times \rho_r \times \text{St}(n)) \to A_{N_i} \to Q_{N_i}.
\]
Taking the twisted Jacquet functor on \( U_i \)-part, we have a map
\[
\tau \hookrightarrow A^{(i)} \to Q^{(i)},
\]
whose composition is again non-zero.

On the other hand, we have maps
\[
\tau \hookrightarrow \pi^{(i)} \hookrightarrow A^{(i)} \to Q^{(i)}
\]
such that the composition is zero by the exactness. Here the first embedding comes from that \( \tau \) is a submodule of \( \pi^{(i)} \).

Hence, we obtain \( \dim \text{Hom}_{G_{n-1}}(\tau, A^{(i)}) > 1 \), giving a contradiction to Claim 1.

\[\square\]

7. Operations of \( \text{St} \)-derivative on \( \mathfrak{hd} \)

7.1. Highest derivative multisegment \( \mathfrak{hd} \). For \( \pi \in \text{Irr}_\rho \), recall that for \( c \in \mathbb{Z}, m(\pi, c) \) is the maximal multisegment \( m \) at \( \nu^c \rho \) such that \( D_m(\pi) \neq 0 \) (Definition 4.3).

Define \( \mathfrak{hd}(\pi) \) to be the multisegment
\[
\mathfrak{hd}(\pi) = \sum_{c \in \mathbb{Z}} m(\pi, c),
\]
which is called to be the highest derivative multisegment for \( \pi \). Note that there are finitely many \( c \) such that \( m(\pi, c) \neq \emptyset \).
By definitions, we have that 
\[ h_\mathcal{D}(\pi)[a] = m(\pi, a). \]
However, \( h_\mathcal{D}(\pi)(b) \) is not necessarily equal to \( m^R(\pi, b) \).

7.2. A combinatorial removal process.

**Definition 7.1.** Given a multisegment \( h \in \text{Mult}_\rho \), a segment \( \Delta = [a, b]_\rho \) is said to be admissible to \( h \) if there exists a segment \( \Delta' \) in \( h \) of the form \( [a, c]_\rho \) for some \( c \geq b \).

**Remark 7.2.** Suppose \( h = h_\mathcal{D}(\pi) \) for some \( \pi \in \text{Irr} \). Then \( \Delta \) is admissible to \( h \) if and only if \( D_\Delta(\pi) \neq 0 \).

**Definition 7.3.** Let \( h \in \text{Mult}_\rho \). Let \( \Delta = [a, b]_\rho \) be a segment admissible to \( h \). The removal process on \( h \) by \( \Delta \) is to obtain a new multisegment \( r(\Delta, h) \) given by the following steps:

1. Choose a segment \( \Delta_1 \) in \( h \) which has shortest relative length among all segments of the form \( [a, b']_\rho \) for some \( b' \geq b \). (In particular, \( \nu^b_\rho \Delta \in \text{Mult}_\rho \).)
2. (Minimality condition and nesting condition) Inductively choose segments \( \Delta_2, \ldots, \Delta_r \) such that \( \Delta_i \) is the minimal segment (with respect to the ordering \( \leq^L \) in Section 2.3) among all segments \( \Delta \) in \( m \) satisfying \( \Delta \subset \Delta_{i-1} \) and \( \nu^b_\rho \Delta \notin \Delta' \) and \( b(\Delta) \neq b(\Delta_{i-1}) \). (Here we pick those segments until no further one can be found.)
3. Write \( \Delta_i = [a_i, b_i]_\rho \). Obtain new segments \( \Delta'_0, \Delta'_1, \ldots, \Delta'_r \) defined as:
   - for \( 0 \leq i \leq r - 1 \), \( \Delta'_i = [a_{i+1}, b_i]_\rho \) (as convention, if \( a_{i+1} = b_i + 1 \), \( [a_{i+1}, b_i]_\rho = \emptyset \));
   - \( \Delta'_r = [b + 1, b_r]_\rho \).
4. The new multisegment \( r(\Delta, h) \) is defined as:
   \[ r(\Delta, h) = h - \sum_{i=0}^r \Delta_i + \sum_{i=0}^r \Delta'_i. \]

**Remark 7.4.**
(a) \( \Delta_0 \) in Step (1) above is guaranteed to exist by the assumption that \( \Delta \) is admissible to \( h \).
(b) In (2), the condition that \( b(\Delta_i) \equiv b(\Delta_{i-1}) \) indeed can be dropped, and the resulting multisegment defined in such way is the same as the way defined in Definition 7.3. Imposing such condition is more convenient for the proofs.

**Definition 7.5.** In the above notation, we shall call that \( \Delta_1, \ldots, \Delta_r \) is a removal sequence for \( (\Delta, h) \). The nesting condition refers to the condition that \( \Delta_{i+1} \subset \Delta_i \) for any \( i \). The minimal condition refers to the minimal choice of \( \Delta_i \) in Step (2).

**Example 7.6.** Let \( h = \{[0, 4]_\rho, [2, 5]_\rho, [2, 3]_\rho, [2]_\rho \} \). Then
1. \( r([0, 2]_\rho, h) = \{[2, 4]_\rho, [2, 5]_\rho, [2, 3]_\rho \}; \)
2. \( r([0, 3]_\rho, h) = \{[2, 4]_\rho, [2, 5]_\rho, [2]_\rho \}; \)
3. \( r([0, 5]_\rho, h) \) is not defined since \([0, 5]_\rho \) is not admissible.

**Example 7.7.**
1. Let \( h = \{[0, 7], [1, 4], [1, 6]\} \). Let \( \Delta = [0, 5] \) and let \( \Delta' = [1, 4] \). The removal sequence for \( (\Delta, h) \) is \([0, 7], [1, 6]\). The removal sequence for \( (\Delta', h) \) is \([1, 6]\).
(2) Let \( \mathbf{h} = \{[0, 7], [1, 5], [1, 6]\} \). Let \( \Delta = [0, 5] \) and let \( \Delta' = [1, 4] \). The removal sequence for \((\Delta, \mathbf{h})\) is \([0, 7], [1, 5]\). The removal sequence for \((\Delta', \mathbf{h})\) is \([1, 5]\).

(3) Let \( \mathbf{h} = \{[0, 7], [1, 5], [1, 8]\} \). Let \( \Delta = [0, 5] \) and let \( \Delta' = [1, 4] \). The removal sequence for \((\Delta, \mathbf{h})\) is \([0, 7], [1, 5]\), and the removal sequence for \((\Delta', \mathbf{h})\) is \([1, 5]\).

### 7.3. Properties of removal process.

A simple but useful computation is the following:

**Lemma 7.8.** (Removal of a cuspidal point at one time) Let \( \mathbf{h} \in \text{Mult}_\rho \). Let \( \Delta \) be an admissible segment for \( \mathbf{h} \). Let \( \Delta_1 \) be the first segment in the removal sequence for \((\Delta, \mathbf{h})\). Let \( \mathbf{h}^* = \mathbf{h} - \{\Delta_1\} + \{-\Delta_1\} \). Then

\[
\tau(\Delta, \mathbf{h}) = \tau(-\Delta, \mathbf{h}^*).
\]

(As convention, \( \tau(\emptyset, \mathbf{h}^*) = \mathbf{h}^* \).)

**Proof.** Suppose \( \mathbf{h} \) has no segment \( \Delta \) satisfying \( a(\Delta) \cong \nu a(\Delta) \). Then the first segment in the removal sequence for \((-\Delta, \mathbf{h}^*)\) is simply \(-\Delta_1\). Since the only difference between \( \mathbf{h} \) and \( \mathbf{h}^* \) is on one segment, one checks that the remaining sequences are picked in the same way by the minimality and nesting condition. This gives \( \tau(-\Delta, \mathbf{h}^*) = \tau(\Delta, \mathbf{h}) \).

Suppose \( \mathbf{h} \) has some segments \( \Delta \) satisfying \( a(\Delta) \cong \nu a(\Delta) \). Let \( \Delta^* \) be the shortest such segment. We further have two cases:

1. Suppose \( \Delta^* \subset \Delta_1 \).
2. Suppose \( \Delta^* \not\subset \Delta_1 \) or \( \Delta^* = -\Delta_1 \).

In Case (1), the first one in the removal sequence for \((-\Delta, \mathbf{h}^*)\) is \( \Delta^* \), coinciding with the second one in the removal sequence. By the minimality and nesting conditions, the subsequent segments in the removal sequence for \((-\Delta, \mathbf{h}^*)\) are the same as those (starting from the third one) in the removal sequence for \((\Delta, \mathbf{h})\).

In Case (2), the first one in the removal sequence for \((-\Delta, \mathbf{h}^*)\) is \(-\Delta_1\), and, by the nesting condition, the subsequent segments in the removal sequence for \((-\Delta, \mathbf{h}^*)\) are those (starting from the second one) in the removal sequence for \((\Delta, \mathbf{h})\).

In any such case, we will then obtain \( \tau(\Delta, \mathbf{h}) = \tau(-\Delta, \mathbf{h}^*) \).

We prove some further properties in Lemmas 7.9 to 7.11.

**Lemma 7.9.** (No effect on previous segments) Let \( \mathbf{h} \in \text{Mult}_\rho \). Let \( \Delta = [a, b]_\rho \) be an admissible segment for \( \mathbf{h} \). Then for any \( a' < a \), \( \mathbf{h}[a'] = \tau(\Delta, \mathbf{h})[a'] \).

**Proof.** This follows directly from Definition 7.3 since those segments do not involve in the removal process.

**Lemma 7.10.** (Removing a whole segment in \( \mathbf{h} \)) Let \( \mathbf{h} \in \text{Mult}_\rho \). Let \( \Delta \in \mathbf{h} \). Then

\[
\tau(\Delta, \mathbf{h}) = \mathbf{h} - \{\Delta\}.
\]

**Proof.** Write \( \Delta = [a, b]_\rho \). Note that the first segment in the removal sequence for \( \tau(\Delta, \mathbf{h}) \) is \( \Delta \). The nesting property guarantees that there is no other segments.
Lemma 7.11. (Commutativity for unlinked segments) Let \( h \in \text{Mult}_\rho \). Let \( \Delta, \Delta' \in \text{Seg}_\rho \) such that \( \Delta' \subseteq \Delta \). Suppose \( r(\Delta, h) \neq 0 \) and \( r(\Delta', r(\Delta, h)) \neq 0 \). Then

\[
r(\Delta', r(\Delta, h)) = r(\Delta, r(\Delta', h)).
\]

Proof. We shall prove by an induction on the sum of all segments in \( h \). For induction purpose, we also allow \( \Delta, \Delta' \) to be an empty set and in such case, it is trivial. We now assume both are not empty sets. Let \( \Delta_1 \) be the first segment in the removal sequence for \( (\Delta, h) \).

Case 1: \( a(\Delta) \not\equiv a(\Delta'), \nu^{-1}a(\Delta') \). Hence \( a(\Delta') > a(\Delta) \). Now we consider

\[
h^* = h - \{\Delta_1\} + \{-\Delta_1\},
\]

and so, by Lemma 7.8, \( r(\Delta, h) = r(-\Delta, h^*) \). Thus,

\[
(\ast) \quad r(\Delta', r(\Delta, h)) = r(\Delta', r(-\Delta, h^*)).
\]

Now using Lemma 7.9, the first segment in the removal sequence for \( (\Delta, r(\Delta', h^*)) \) is still \( \Delta_1 \). Hence, we have that:

\[
(7.4) \quad r(\Delta, r(\Delta', h)) = r(-\Delta, r(\Delta', h^*)),
\]

where we also use \( r(\Delta', h) - \{\Delta_1\} + \{-\Delta_1\} = r(\Delta', h^*) \) by Lemma 7.9.

Now,

\[
r(\Delta', r(-\Delta, h^*)) = r(-\Delta, r(\Delta', h^*)) = r(-\Delta, r(\Delta', h^*)),
\]

where the middle equality follows from the inductive case. Hence, we are done.

Case 2: \( a(\Delta) \equiv a(\Delta') \). Let \( \overline{\Delta}_1 \) be the first segment in the removal sequence for \( (\Delta', h) \).

Case (a): \( \overline{\Delta}_1 \not\subseteq \Delta_1 \). Then \( \Delta_1 \subseteq \overline{\Delta}_1 \) and \( \overline{\Delta}_1 \not\subseteq \Delta \). Let

\[
h^* = h - \{\Delta_1\} + \{-\Delta_1\}, \quad h^{**} = h - \{\overline{\Delta}_1, \Delta_1\} + \{-\Delta_1, -\overline{\Delta}_1\}.
\]

One first checks, by the minimality condition,

\[
r(-\Delta', h^{**}) = r(-\Delta', h^*) - \{\overline{\Delta}_1\} + \{-\overline{\Delta}_1\}.
\]

Then, by Lemma 7.8 twice,

\[
r(-\Delta, r(-\Delta', h^{**})) = r(\Delta, r(-\Delta', h^*)) = r(\Delta, r(\Delta', h)).
\]

Using the minimality for \( \overline{\Delta}_1 \) and \( \overline{\Delta}_1 \not\subseteq \Delta \), we similarly have:

\[
r(-\Delta, h^{**}) = r(-\Delta, h^*) - \{\overline{\Delta}_1\} + \{-\overline{\Delta}_1\},
\]

where \( h^* = h - \{\Delta_1\} + \{-\Delta_1\} \). Now similar argument as above gives that

\[
r(-\Delta', r(-\Delta, h^{**})) = r(\Delta', r(\Delta, h)).
\]

Now, by induction on the cardinality of \( h^{**} \),

\[
r(-\Delta', r(-\Delta, h^{**})) = r(-\Delta, r(-\Delta', h^{**})).
\]

Combining the equations, we obtain this case.
Case (b): $\Delta_1 = \tilde{\Delta}_1$ and $\Delta_1$ appears with multiplicity at least 2 in $\mathfrak{h}$. The argument is similar to Case (a).

Case (c): $\Delta_1 = \tilde{\Delta}_1$ and appears with multiplicity 1 in $\mathfrak{h}$. Let $\Delta_1(\geq \tilde{\Delta}_1)$ be the first segment in the removal sequence for $(\Delta, r(\Delta', \mathfrak{h}))$. Note that, by using minimality, the first segment in the removal segment for $(\Delta', r(\Delta, \mathfrak{h}))$ is also $\Delta_1$.

Again, define
$$\mathfrak{h}^* = \mathfrak{h} - \tilde{\Delta}_1, \quad \mathfrak{h}^{**} = \mathfrak{h} - \tilde{\Delta}_1 - \Delta_1.$$

Now again using minimality condition, we have that
$$r(-\Delta', \mathfrak{h}^{**}) = r(-\Delta', \mathfrak{h}^*) - \{\tilde{\Delta}_1\} + \{-\tilde{\Delta}_1\}.$$ 

Now similar argument as above,
$$r(-\Delta, r(-\Delta', \mathfrak{h}^{**})) = r(\Delta, r(\Delta', \mathfrak{h})).$$

With similar argument, we also have that
$$r(-\Delta', r(-\Delta, \mathfrak{h}^{**})) = r(\Delta', r(\Delta, \mathfrak{h})).$$

The induction hypothesis gives that $r(-\Delta', r(-\Delta, \mathfrak{h}^{**})) = r(-\Delta, r(-\Delta', \mathfrak{h}^{**}))$. Combining the equations, we are done.

**Case 3:** $a(\Delta) \equiv \nu^{-1}a(\Delta')$. In view of the argument (and notations) in Case 1, it suffices to show that
$$r(-\Delta, r(\Delta', \mathfrak{h}) - \{-\tilde{\Delta}_1\} + \{-\tilde{\Delta}_1\}) = r(-\Delta, r(\Delta', \mathfrak{h}^*)).$$

We shall denote $l^* = r(\Delta', \mathfrak{h}) - \{-\tilde{\Delta}_1\} + \{-\tilde{\Delta}_1\}$.

We need to consider the following cases:

1. There is a segment $\hat{\Delta}$ such that $a(\hat{\Delta}) \equiv \nu a(\Delta)$ and $\Delta' \subset \hat{\Delta} \subset \tilde{\Delta}_1$. In such case, one has the equality \([4.2]\) as in Case (1).

2. There is no such segment in the above case. Let $\Delta_1$ be the first segment in the removal sequence for $(\Delta', \mathfrak{h})$. Then
   - the first segment in the removal sequence for $(\Delta', \mathfrak{h}^*)$ (resp. $(\Delta, r(\Delta', \mathfrak{h}^*))$) is $\tilde{\Delta}_1$ (resp. $\tilde{\Delta}_1$). We have proved in Step 2 that
     $$r(-\Delta, r(\Delta', \mathfrak{h}^*)) = r(-\Delta, r(-\Delta', \mathfrak{h}^*)), $$

     where
     $$\mathfrak{h}^{**} = \mathfrak{h}^* + \{-\tilde{\Delta}_1, \Delta_1\} + \{-\tilde{\Delta}_1, \Delta_1\}.$$

   - the first segment in the removal sequence for $(\Delta', l^*)$ is $\tilde{\Delta}_1$. By Lemma \([7,8]\) twice, this gives that
     $$r(-\Delta, l^*) = r(-\Delta, l^{**}).$$

     where
     $$l^{**} = r(\Delta', \mathfrak{h}) - \{-\tilde{\Delta}_1\} + \{-\tilde{\Delta}_1\} = r(-\Delta', \mathfrak{h}^*) - \{-\tilde{\Delta}_1\} + \{-\tilde{\Delta}_1\}$$
     and $\mathfrak{h}^* = \mathfrak{h} - \{-\Delta_1\} + \{-\tilde{\Delta}_1\}$.

Now one proceeds inductively to show $r(-\Delta, r(-\Delta', \mathfrak{h}^{**})) = r(-\Delta, l^{**})$. 

7.4. Derivative resultant multisegment.

**Definition 7.12.** (1) Let \( h \in \text{Mult}_\rho \). A multisegment \( n \) is said to be admissible for \( h \), if we label the segments \( \Delta_1, \ldots, \Delta_k \) in \( n \) in an ascending order,

\[
\tau(\Delta_k, \tau(\Delta_{k-1}, \ldots \tau(\Delta_1, h) \neq 0.
\]

By Lemma 7.11 it is independent of a choice of an ascending order.

(2) We also write \( r(n, h) = r(\{\Delta_1, \ldots, \Delta_k\}, h) \) for \( r(\Delta_k, r(\Delta_{k-1}, \ldots r(\Delta_1, h) \ldots) \).

For any multisegment \( n \) to \( h \), we say call \( r(n, h) \) to be a derivative resultant multisegment.

7.5. Shrinking derivative resultant multisegment.

**Lemma 7.13.** Let \( h \in \text{Mult}_\rho \). Let \( c \in \mathbb{Z} \) such that, for any \( i \geq 1 \), \( \nu^{c+i} \rho \) is not in any segment of \( n \). Let \( s = r(n, h) \).

Recall that \( h(c) \) (resp. \( s(c) \)) is the submultisegment of \( h \) (resp. \( s \)) containing all the segments \( \Delta \) in \( h \) (resp. \( s \)) satisfying \( b(\Delta) \geq \nu^c \rho \). Furthermore, we assume that \( s(c) = h(c) \) for any \( e \geq c+1 \). Then \( s - s(c) + h(c) \) is also a derivative resultant multisegment for \( h \).

**Proof.** We shall prove by an induction on \( l_a(h) \). We write the segments in \( n \) in the following ascending order:

\[
(*) \quad \Delta_1 \preceq \ldots \preceq \Delta_p.
\]

Let \( \tilde{\Delta} \) be the first segment in the removal sequence for \((\Delta_1, h)\).

We first consider the case that \( \tilde{\Delta} \notin h(c) \). Note that

\[
\tau(n - \Delta_1, \tau(\Delta_1, h)) = \tau(n, h).
\]

Let \( h' = \tau(\Delta_1, h) \). The assumptions in the lemma and \( \tilde{\Delta} \notin h(c) \) imply that \( h(c) = h'(c) \). Now induction hypothesis with (*) gives that

\[
s - s(c) + h(c) = s - s(c) + h'(c)
\]

is still a derivative resultant multisegment i.e.

\[
\tau(\tilde{n}, h') = s - s(c) + h(c).
\]

It remains to observe from (*) that we still have

\[
\tau(\tilde{n} + \Delta_1, h) = \tau(\tilde{n}, h').
\]

We now consider the case that \( \tilde{\Delta} \in h(c) \). Let

\[
h^* = h - \{\tilde{\Delta}\} + \{-\tilde{\Delta}\},
\]

\[
n^* = n - \{\Delta_1\} + \{-\Delta_1\}.
\]
By Lemma 7.8
\[ \tau(\Delta_1, h) = \tau(\neg \Delta_1, h^*) \]
and so, by Lemma 7.11
\[ (\star) \quad \tau(n, h) = \tau(n^*, h^*). \]

Now, by induction hypothesis, we have that
\[ s - s(c) + h^*(c) \]
is a derivative resultant multisegment, say there exists a multisegment \( \tilde{\eta} \) such that
\[ \tau(\tilde{\eta}, h^*) = s - s(c) + h^*(c). \]

We now claim:
Claim: \( \tau(\tilde{\eta}, h^*))[a] = h^*[a] \), where \( a \) satisfies \( a(\Delta) \cong \nu^\rho \).

Proof of claim: Since \( \Delta_1 \) is minimal in \( \preceq \), we have that (see the proof of Lemma 9.3 for more details)
\[ s = \tau(n, h) = \tau(n - n[a] + \neg(n[a]), h) = \{\tilde{\Delta}, \ldots, \tilde{\Delta} \} + \{\neg\tilde{\Delta}, \ldots, \neg\tilde{\Delta} \}, \]
where both \( \tilde{\Delta} \) and \( \neg\tilde{\Delta} \) appear \( \lfloor n[a] \rfloor \)-times. Thus, by Lemma 7.9
\[ s[a] = h[a] - \{\tilde{\Delta}, \ldots, \tilde{\Delta} \} = h^*[a] - \{\Delta, \ldots, \Delta \}. \]
Hence, \( s[a] - s[a](c) = h[a] - h[a](c) = h^*[a] - h^*[a](c) \) and so \( (s - s(c) + h(c))[a] = h^*[a] \). This proves the claim.

Now the claim with our minimal choice of \( \Delta_1 \), we must have that
\[ (\star\star) \quad \tilde{\eta}[d] = \emptyset, \text{ for any integer } d \leq a. \]

Then one observes that \( \tau(\tilde{\eta}, h) = \tau(\tilde{\eta}, h^*) + \{\tilde{\Delta} \} - \{\neg\tilde{\Delta} \} \) by using Lemma 7.9
(\star\star) and the fact that any segment in the removal sequence for \( (\tilde{\eta}, h^*) \) does not involve \( \neg\tilde{\Delta} \). Hence
\[ \tau(n^*, h) = s - s(c) + h^*(c) + \{\tilde{\Delta} \} - \{\neg\tilde{\Delta} \} = s - s(c) + h(c) \]
is still a derivative resultant multisegment.

Example 7.14. Let \( h = \{[1, 5], [2, 4], [4, 5] \} \).

1. Let \( \Delta = [1, 3] \). Then \( \tau(\Delta, h) = \{[2, 5], [4, 5] \} \). Then \( \{[1, 5], [4, 4, 5] \} \) is also a derivative resultant multisegment.

2. Let \( n = \{[1, 3], [2] \} \). Then \( \tau(n, h) = \{[3, 5], [4, 5] \} \). Then \( \{[1, 5], [4, 4, 5] \} \) is a derivative resultant multisegment.

One may compare the following lemma with Proposition 6.3 while it is not an exact analogue.

Lemma 7.15. We use notations in the previous lemma. Let \( n' \in \text{Mult}_{\nu} \) such that
\[ \tau(n', h) = s - s(c) + h(c). \]
Then
\[ \tau(h(c) + n', h) = s - s(c). \]
Note that
\[ r(h(c) + n', h) = r(h(c), r(n', h)) = r(h(c), s - s(c) + h(c)). \]
Then the lemma follows from Lemma 7.10.

7.6. Effect of St-derivatives. We shall now compute the effect of St-derivatives on the invariant \( \epsilon_\Delta \). We need a preparation lemma first, which allows one does the induction. For a segment \( \Delta = [a, b \rho] \), define \( + \Delta = [a - 1, b \rho] \).

**Lemma 7.16.** Let \( m \in \text{Mult}_\rho \) and let \( \pi = \langle m \rangle \). Let \( c \) be the an integer such that \( b(\Delta) \equiv \nu^c \rho \) for some \( \Delta \in m \) and let \( k = \epsilon_c(\pi) \). Let \( \widetilde{\pi} = D^k_c(\pi) \). For \( \Delta = [a, b \rho] \) with \( b > 0 \),

1. if \( a > c + 1 \), then \( \epsilon_\Delta(\widetilde{\pi}) = \epsilon_\Delta(\pi) \);
2. if \( a = c + 1 \), then \( \epsilon_\Delta(\widetilde{\pi}) = \epsilon_\Delta(\pi) + \epsilon_\Delta(\pi) \).

**Proof.** We consider (1). Let \( l = \epsilon_\Delta(\widetilde{\pi}) \). Then we have
\[ (\nu^c \rho)^k \times \pi' \to \pi \Rightarrow \pi' \times (\text{St}(\Delta))^l \]
By geometric lemma and comparing cuspidal support, we have a non-zero composition factor on \( \pi'\Delta_i \) of the form \( \tau \otimes (\text{St}(\Delta))^l \), where \( i \) is equal to \( l \) times the absolute length of \( \Delta \) and \( \tau \in \text{Irr} \), and hence \( \pi'\Delta_i \) also has a quotient of such form \( \tau \otimes (\text{St}(\Delta))^l \). Thus \( l \leq \epsilon_\Delta(\pi) \).

Let \( l' = \epsilon_\Delta(\pi) \). We have an embedding:
\[ \pi \hookrightarrow \pi' \times (\nu^c \rho)^k \hookrightarrow \pi'' \times \text{St}(\Delta)^{\times l'} \times (\nu^c \rho)^k \equiv \pi'' \times (\nu^c \rho)^k \times \text{St}(\Delta)^{\times l'}, \]
where the last isomorphic follows from [Ze80, Theorem 4.2]. Hence, by Frobenius reciprocity, \( l = \epsilon_\Delta(\pi) \geq l' \). Thus \( l = l' \) and this proves (1).

We now consider (2). We shall write \( m_c = m(\pi, c) \) and \( m_{c+1} = m(\pi, c+1) \) (see Definition 4.3). Note that by definition of maximal multisegment and \( \epsilon_c \), \( m_c \) has exactly \( k \)-segments. By Lemma 5.2, we have an embedding:
\[ \pi \hookrightarrow \omega \times \text{St}(m_{c+1}) \times \text{St}(m_c) \]
and so we also have:
\[ \pi \hookrightarrow \omega \times \text{St}(m_{c+1} + (m_c)) \times (\nu^c \rho)^k. \]
(see Section 3.7 for the notation \( -(m_c) \)). Thus, by Corollary 3.6
\[ \widetilde{\pi} \hookrightarrow \omega \times \text{St}(m_{c+1} + (m_c)). \]
Then
\[ (\ast) \quad \epsilon_\Delta(\widetilde{\pi}) \geq \epsilon_\Delta(\pi) + \epsilon_\Delta(\pi). \]
To prove the opposite inequality, suppose it fails for some \( \Delta = [c + 1, b \rho] \). Let \( l = \epsilon_\Delta(\widetilde{\pi}) \). Then we can write
\[ \pi \hookrightarrow D^k_\Delta(\pi) \times (\text{St}(\Delta))^l \times (\nu^c \rho)^k, \]
which implies that
\[ \pi \hookrightarrow D^k_\Delta(\pi) \times \omega, \]
for some composition factor \( \omega \) in \( St(\Delta)^{\times l} \times (\nu^c \rho)^{\times k} \). Since the only possible segments appearing in \( m(\omega) \) include \([c, b]_\rho\), \([c + 1, b]_\rho\) or \([c]_\rho\), we must have:

\[
\pi \mapsto D^l_\Delta(\pi) \times (\nu^c \rho)^{\times r} \times St([c, b]_\rho)^{\times s} \times St([c + 1, b]_\rho)^{\times t},
\]

where \( s + t = \epsilon_\Delta(\pi) = l \). Hence, by our assumption, we have either:

\[ s > \epsilon_\Delta(\pi) \quad \text{or} \quad t > \epsilon_+\Delta(\pi). \]

However, one applies Frobenius reciprocity and obtains a contradiction to the definition of \( \epsilon_\Delta(\pi) \) or \( \epsilon_+\Delta(\pi) \), as desired.

\[ \square \]

The following is a key property that allows one to deduce that the derivative resultant multisegments 'matching' the effect of St-derivatives by an induction.

**Lemma 7.17.** Let \( m \in \text{Mult}_\rho \) and let \( \pi = \langle m \rangle \). Let \( c \) be an integer such that \( \epsilon_c(\pi) \neq 0 \). Let \( \Delta = [c, b]_\rho \) for some \( b \). Let \( \tilde{\pi} = D^k_\Delta(\pi) \), where \( k = \epsilon_c(\pi) \). Then

\[ \pi \mapsto \tilde{\pi} \times (\nu^c \rho)^k. \]

Then

\[ D_\Delta(\pi) \mapsto D_{-\Delta}(\tilde{\pi}) \times (\nu^c \rho)^{k-1}. \]

**Proof.** A simple application of geometric lemma gives that:

\[ (* \quad D_\Delta(\pi) \mapsto \text{Hom}_{G_\epsilon}(\text{St}(-\Delta), \tilde{\pi}_{N_\epsilon}) \times (\nu^c \rho)^{k-1}, \]

where \( r = l_a(-\Delta) \).

By using Lemma 7.17 we have that \( \epsilon_c(D_\Delta(\pi)) = k - 1 \). Hence, Corollary 3.6 and (* \ give

\[ D^k_{c-1} \circ D_\Delta(\pi) \mapsto \text{Hom}_{G_\epsilon}(\text{St}(-\Delta), \tilde{\pi}_{N_\epsilon}) \]

and hence \( D^k_{c-1} \circ D_\Delta(\pi) \cong D_{-\Delta}(\tilde{\pi}) \), which gives the lemma. \[ \square \]

**Theorem 7.18.** Let \( m \in \text{Mult}_\rho \) and let \( \Delta = [a, b]_\rho \) be a segment. Let \( \pi = \langle m \rangle \) and let \( h = h_\partial(\pi) \). Then

1. Suppose \( \Delta \) is not admissible to \( h \). Then \( D_\Delta(\pi) = 0 \).
2. Suppose \( \Delta \) is admissible to \( h \). Let \( \pi' = D_\Delta(\pi) \). Then
   - For any \( c \geq a \), the multisegment of \( \pi' \) at the point \( \nu^c \rho \) is equal to \( \nu(\Delta, \pi)[c] \).
   - (See Section 2.2 for the notations.)
   - For any \( c < a \), \( m(\pi', c) \) satisfies \( h[c] \leq_c m(\pi', c) \).

**Proof.** (1) follows from definitions. The second bullet of (2) can be proved by a similar manner as the first case in the proof of Lemma 4.8 and we omit the details.

We shall prove the first bullet of (2) by an induction on \( n(\pi) \). When \( n(\pi) = 0, 1 \), it is trivial. Let \( \Delta = [a, b]_\rho \) be an admissible segment for \( \pi \). Let \( \tilde{\pi} = D^k_\Delta(\pi) \), where \( k = \epsilon_a(\pi) \). Now we have that:

\[ (* \quad \pi \mapsto \tilde{\pi} \times (\nu^a \rho)^{\times k} \]
By Lemma 7.17
\[ D_{\Delta}(\pi) \hookrightarrow D_{-\Delta}(\tilde{\pi}) \times (\nu^{a}\rho)^{(k-1)}. \]

Now we consider the two cases:

(i) Suppose \( c \geq a + 2 \). In such case, \( D_{\Delta} \circ D^{k-1}_{\tilde{\Delta}}(\pi) \cong D^{k-1}_{\tilde{\Delta}} \circ D_{\Delta}(\pi) \). Let \( n \) be a multisegment at \( \nu^{c} \rho \). Then
\[
D_{n} \circ D_{\Delta} \circ D^{k-1}_{\tilde{\Delta}}(\pi) \neq 0 \iff D^{k-1}_{\tilde{\Delta}} \circ D_{n} \circ D_{\Delta}(\pi) \neq 0 \iff D_{n} \circ D_{\Delta}(\pi) \neq 0,
\]
where the first 'if and only if' condition follows by applying Lemma 4.7 twice, and the second 'if and only if' condition follows by an analogous statement for Lemma 4.8 in the linked case. Now one deduces the maximal multisegment at \( \nu^{c} \rho \) (\( c \geq a \)) for \( D_{\Delta}(\pi) \) by the inductive case and Lemma 7.16.

(ii) Suppose \( c = a + 1 \). Let
\[ s = m(\pi, c) - \Delta_{0}, \]
where \( \Delta_{0} \) is the shortest segment in \( m(\pi, c) \) that \( \Delta \subset \Delta_{0} \).

By Lemma 7.16
\[ n := m(\tilde{\pi}, c + 1) = -(m(\pi, c)) + m(\pi, c + 1). \]

By Proposition 4.4
\[ D_{s} \circ D_{\tilde{\Delta}}(\pi) \neq 0 \]
and now by repeated uses of Lemma 7.17
\[ D_{s} \circ D_{\Delta}(\pi) \cong D_{-s} \circ D_{-\Delta}(\tilde{\pi}). \]
Hence, by the inductive case,
\[ m(D_{s} \circ D_{\Delta}(\pi), a + 1) = m(D_{-s} \circ D_{-\Delta}(\tilde{\pi}), a + 1) \]
is precisely
\[ r(\bar{s} + \bar{\Delta}, \bar{\pi})[a + 1] = r(s + \Delta, \pi)[a + 1]. \]
The last equality follows from the rules of removal process (see Lemma 9.3 below for the detail) and Lemma 7.16.

Indeed we also have
\[ r(s + \Delta, \pi)[a + 1] = r(s, r(\Delta, \pi))[a + 1] = r(\Delta, \pi)[a + 1], \]
where the second equality follows from that applying \( D_{\Delta'} \) for each \( \Delta' \in s \) will simply remove the segment \( \Delta' \) in \( r(\Delta, \pi) \) by Lemma 7.10.

Since \( s \) is the maximal multisegment at \( \nu^{c} \rho \) of \( D_{\Delta}(\pi) \), Lemma 5.2 implies that
\[ m(D_{\Delta}(\pi), a + 1) = m(D_{s} \circ D_{\tilde{\Delta}}(\pi), a + 1). \]
Now, combining all the equations, we have that:
\[ m(D_{\Delta}(\pi), a + 1) = r(\Delta, \pi)[a + 1]. \]

(iii) Suppose \( c = a \). Then \( m(\pi', a) \) follows from Proposition 4.4 and Lemma 4.2.

\[ \square \]
8. Isomorphic quotients of some simple Bernstein-Zelevinsky derivatives

8.1. Complementary sequence of St-derivatives. For \( \pi \in \text{Irr} \) and a multisegment \( n \), we shall write:

\[
\mathfrak{t}(n, \pi) := \mathfrak{t}(n, h\partial(\pi)),
\]

where the latter term is defined in Definition 7.3. We prove a main property of the derivative resultant multisegment.

**Theorem 8.1.** Let \( \pi \in \text{Mult}_\rho \). Let \( \Delta_1, \ldots, \Delta_k \) be an ascending admissible sequence of segments for \( \pi \). Let

\[
\omega = D_{\Delta_k} \circ \ldots \circ D_{\Delta_1}(\pi).
\]

Write \( \mathfrak{t}(\{\Delta_1, \ldots, \Delta_k\}, \pi) = \{\tilde{\Delta}_1, \ldots, \tilde{\Delta}_l\} \) and we label in the way such that \( \tilde{\Delta}_1, \ldots, \tilde{\Delta}_l \) forms an ascending sequence of segments. Then

\[
D_{\tilde{\Delta}_i} \circ \ldots D_{\tilde{\Delta}_1}(\pi) \cong \pi^-. \]

**Proof.** We shall prove by a backward induction on the sum of the lengths of all those \( \Delta_1, \ldots, \Delta_k \). Since the sequence is admissible, the sum must be not greater than the level of \( \pi \). If the sum is equal to the level of \( \pi \), then \( \omega \cong \pi^- \) by the irreducibility of the highest derivative of \( \pi \).

Let \( c^* \) be the largest integer such that \( \nu^{c^*} \rho \leq a(\Delta) \) for any \( \Delta \in \mathfrak{t}(\{\Delta_1, \ldots, \Delta_k\}, \pi) \). In other words, \( \nu^{c^*} \rho \) is isomorphic to a minimal element in

\[
\{a(\Delta') : \Delta' \in \mathfrak{t}(\{\Delta_1, \ldots, \Delta_k\}, \pi)\}.
\]

Let \( r \leq k \) such that \( \Delta_1, \ldots, \Delta_r \) be all the segments such that \( \nu^{c^*} \rho \geq a(\Delta_i) \) for \( i = 1, \ldots, r \). We rearrange the segments \( \Delta_{\delta(r+1)}, \ldots, \Delta_{\delta(k)} \) so that

\[
b(\Delta_{\delta(k)}) \geq \ldots \geq b(\Delta_{\delta(r+1)}),
\]

where \( \delta \) is a permutation on \( \{r+1, \ldots, k\} \).

Let \( \tau = D_{\Delta_r} \circ \ldots \circ D_{\Delta_1}(\pi) \). Note that the sequence \( \Delta_{\delta(r+1)}, \ldots, \Delta_{\delta(k)} \) can be obtained from \( \Delta_{r+1}, \ldots, \Delta_k \) by repeatedly switching two adjacent unlinked segments. Hence, by Lemma 4.7,

\[
(\ast) \quad D_{\Delta_{\delta(k)}} \circ \ldots \circ D_{\Delta_{\delta(r+1)}}(\tau) \cong D_{\Delta_k} \circ \ldots \circ D_{\Delta_{r+1}}(\tau) \cong \omega.
\]

Now we let \( \tilde{\Delta} \) be the longest segment \( \Delta \) in \( \mathfrak{t}(\{\Delta_1, \ldots, \Delta_k\}, \pi) \) satisfying \( a(\Delta) \cong \nu^{c^*} \rho \). We claim that: for \( i \geq 2 \),

\[
D_{\tilde{\Delta}} \circ D_{\Delta_{\delta(i)}} \circ \ldots \circ D_{\Delta_{\delta(1)}}(\tau) \cong D_{\Delta_{\delta(i)}} \circ D_{\tilde{\Delta}} \circ D_{\Delta_{\delta(i-1)}} \circ \ldots \circ D_{\Delta_{\delta(1)}}(\tau),
\]

and for \( i = 1 \),

\[
D_{\tilde{\Delta}} \circ D_{\Delta_{\delta(1)}}(\tau) \cong D_{\Delta_{\delta(1)}} \circ D_{\tilde{\Delta}}(\tau).
\]

Suppose the claim for the meanwhile. Then we obtain a new ascending sequence of segment,

\[
\Delta_1, \ldots, \Delta_r, \tilde{\Delta}, \Delta_{r+1}, \ldots, \Delta_k,
\]

which is admissible since the composition of their corresponding derivatives is non-zero by (\ast) and the claim. Now one applies induction hypothesis to obtain an ascending sequence of segments. Any segment \( \tilde{\Delta} \) in such sequence satisfies \( a(\tilde{\Delta}) \cong \nu^{c^*} \rho \)
for some \( e \) with \( e \geq e^* \). Hence, adjoining \( \tilde{\Delta} \) still gives an ascending sequence of segments and so we are done.

It remains to prove the claim. Indeed, it will follow from Lemma 4.7 or Lemma 4.9 if we could check those conditions in the lemma. Use the notations in the claim. If \( \Delta \) and \( \Delta_{\delta(i)} \) are unlinked, then we use Lemma 4.7 and we are done. Now suppose \( \Delta \) and \( \Delta_{\delta(i)} \) are linked. Note that, by Lemma 7.9,

\[
\bullet \quad \tau(\{\Delta_1, \ldots, \Delta_k\}, \pi)[e^*] = \tau(\{\Delta_1, \ldots, \Delta_r, \Delta_{\delta(r+1)}, \ldots, \Delta_{\delta(i)}\}, \pi)[e^*].
\]

This implies that, by Theorem 7.18,

\[
(+\quad D_{\Delta}(\kappa) \neq 0,
\]

where

\[
\kappa = D_{\Delta_{\delta(i-1)}} \circ \ldots \circ D_{\Delta_{\delta(1)}} \circ D_{\Delta_r} \circ \ldots \circ D_{\Delta_1}(\pi).
\]

Now let \( \Delta' = \Delta_{\delta(i)} \cup \Delta \) and we have to check that \( D_{\Delta'}(\kappa) = 0 \). Note that \( b(\Delta') > b(\Delta) \). Thus, we have:

1. by the maximality of our choice on \( \Delta \) in \( m(\pi, \Delta_1, \ldots, \Delta_k) \) and the above bullet,
   \[
   D_{\Delta'}(\tau) = 0
   \]
2. our arrangement on \( \Delta_{\delta(p)} \) gives that \( \Delta' \) and \( \Delta_{\delta(x)} \) are unlinked for \( x = 1, \ldots, i-1 \).

Hence,

\[
(++) \quad D_{\Delta'}(\kappa) = D_{\Delta'} \circ D_{\Delta_{\delta(i-1)}} \circ \ldots \circ D_{\Delta_{\delta(1)}}(\tau) = D_{\Delta_{\delta(i-1)}} \circ \ldots \circ D_{\Delta_{\delta(1)}} \circ D_{\Delta'}(\tau) = 0,
\]

where the second equality follows from (3) above with Lemma 4.7 and the last equality follows from (1) above.

Since \( D_{\Delta_k} \circ \ldots \circ D_{\Delta_1}(\pi) \neq 0 \), we also have

\[
(+ + +) \quad D_{\delta(i)}(\kappa) \neq 0.
\]

Hence, the conditions \((+), (++), (+++)\) guarantee conditions in Lemma 4.9 and this completes the proof of the claim. \( \square \)

### 8.2. Isomorphic quotients under St-derivatives

We now prove a main result using the highest derivative segment and the removal process to determine when two sequences of St-derivatives give rise to isomorphic simple quotient of a Bernstein-Zelevinsky derivative. The strategy for 'if' direction is that we use Theorem 8.1 to construct isomorphic modules by taking same sequence of St-derivatives. The strategy for 'only if' direction is to find some St-derivatives that kill one, but not another one. However, in order to do so, we need to do it on some other derivatives via the constructions.

**Theorem 8.2.** Let \( \Delta_1, \ldots, \Delta_k \) and \( \Delta'_1, \ldots, \Delta'_l \) be two ascending admissible sequences of segments. Then

\[
D_{\Delta_k} \circ \ldots \circ D_{\Delta_1}(\pi) \cong D_{\Delta'_l} \circ \ldots \circ D_{\Delta'_1}(\pi)
\]
if and only if
\[ \tau(\{\Delta_1, \ldots, \Delta_k\}, \pi) = \tau(\{\Delta'_1, \ldots, \Delta'_l\}, \pi). \]

**Proof.** Let \( \omega_1 = D_{\Delta_k} \circ \cdots \circ D_{\Delta_1}(\pi), \quad \omega_2 = D_{\Delta'_l} \circ \cdots \circ D_{\Delta'_1}(\pi). \)

For if direction, we write \( \tilde{\Delta}_1, \ldots, \tilde{\Delta}_r \) to be all the segments in \( \tau(\{\Delta_1, \ldots, \Delta_k\}, \pi) \) as an ascending sequence. It follows from Lemma 8.1 that
\[ D_{\tilde{\Delta}_r} \circ \cdots \circ D_{\tilde{\Delta}_1}(\omega_1) \cong D_{\tilde{\Delta}_r} \circ \cdots \circ D_{\tilde{\Delta}_1}(\omega_2) \cong \pi^- . \]

Hence, both \( \omega_1 \) and \( \omega_2 \) are isomorphic to
\[ I_{\tilde{\Delta}_1} \circ \cdots \circ I_{\tilde{\Delta}_r}(\pi^-) , \]
where \( I_{\tilde{\Delta}}(\pi) \) (for \( \pi \in \text{Irr} \)) denotes the unique irreducible submodule of \( \pi \times \text{St}(\tilde{\Delta}) \).

In particular, they are isomorphic.

We now consider the only if direction. We denote by
\[ \tau_1 = \tau(\{\Delta_1, \ldots, \Delta_k\}, \pi), \quad \tau_2 = \tau(\{\Delta'_1, \ldots, \Delta'_l\}, \pi). \]

For \( p = 1, 2 \), let \( \tau_p(c) \) be the sub-multiset of \( \tau_p \) containing all the segments \( \Delta \) satisfying \( b(\Delta) \cong \nu^p \circ \rho \). Suppose \( c^* \) is the smallest integer such that
\[ \tau_1(c^*) \neq \tau_2(c^*) . \]

Let \( \tau^p(c) \) be the largest integer such that \( \nu^p \circ \rho \in \text{csupp}(\pi) \). Let \( c_i = c_0 - i \), and let \( z \) be the integer such that \( c_z = c^* \). Set
\[ \omega_{1,0} = D_{\Delta_k} \circ \cdots \circ D_{\Delta_1}(\pi), \quad \omega_{2,0} = D_{\Delta'_l} \circ \cdots \circ D_{\Delta'_1}(\pi). \]

For \( p = 1, 2 \), we inductively, for \( c_i = c_0, \ldots, c^* \), define representations (c.f. Proposition 6.2):
\[ \kappa_{1,i} = D_{\pi_{i-1}}(\omega_{1,i-1}), \quad \kappa_{2,i} = D_{\pi_{i-1}}(\omega_{2,i-1}) , \]
where \( \pi_{i-1} = \tau_1(c_{i-1}) = \tau_2(c_{i-1}) \) (possibly empty and the equality follows from our choice of \( c^* \)), and
\[ \omega_{1,i}, \quad \omega_{2,i} . \]

are the representations with the derivative resultant multiset obtained in Lemma 7.13 (we are in the special case that \( \varepsilon(c) = 0 \) in the notation of Lemma 7.13) i.e. by Lemma 7.15 and the (proved) if direction of this theorem, \( \omega_{1,i} \) and \( \omega_{2,i} \) are those satisfying that
\[ \kappa_{1,i} \cong D_{\pi_{i-1}}(\omega_{1,i}), \quad \kappa_{2,i} \cong D_{\pi_{i-1}}(\omega_{2,i}) , \]
where \( \pi_{i-1} = \pi(c_{i-1}) \).

We have either
\[ D_{\tau_1(c^*)}(\kappa_{1,c^*}) = 0, \quad \text{or} \quad D_{\tau_2(c^*)}(\kappa_{2,c^*}) = 0 . \]

This implies that \( \kappa_{1,c^*} \neq \kappa_{2,c^*} \), and so we obtain that, for \( p = 1, 2 \),
\[ I_{q_{z-1}} \circ D_{q_{z-1}} \circ \cdots \circ I_{q_0} \circ D_{q_0}(\omega_{p,0}) \cong \kappa_{p,c^*} . \]

Here \( I_{q_{z-1}}(\kappa_{p,c^*}) \) is the unique simple submodule of \( \kappa_{p,c^*} \times \text{St}(q_{z-1}) \). Hence, we must have \( \omega_{1,0} \neq \omega_{2,0} . \)

\[ \square \]
9. Intersection-union process and derivative resultant multisegments

Recall that $S(\pi, \tau)$ is defined in Section 1.2. In this section, we give an application of Theorem 8.2 to deduce some combinatorial structure on $S(\pi, \tau)$.

9.1. Removal segments. Let $h \in \text{Mult}_\rho$. Let $n \in \text{Mult}_\rho$. Let $a$ be the smallest integer such that $n[a] \neq 0$. For the segments $\overline{\Delta}_1, \ldots, \overline{\Delta}_k$ in $n[a]$, we label as:

$$\overline{\Delta}_1 \subset \ldots \subset \overline{\Delta}_k.$$  

Let $\Delta'_i$ be the first segment in the removal sequence for $r(\Delta_i, r(\{\Delta_{i-1}, \ldots, \overline{\Delta}_1\}, h))$.

If the last term exists i.e. $\Delta'_i$ exists, then we define $s(n, h) = \{\Delta'_1, \ldots, \Delta'_k\}$. Define $s(n, h) = \emptyset$ otherwise.

Remark 9.1. Indeed, the ordering in (9.5) is not important by Lemma 7.11 (and its proof).

Following from definitions,

**Lemma 9.2.** With the notations as above,

$$s(n, h) = s(n[a], h) = s(n[a], h[a]).$$

We define

$$\tau(n, h) = h - s(n, h) + -(s(n, h)),$$

and

$$\text{cp}(n, h) = n - n[a] + -(n[a])$$

**Lemma 9.3.** Let $n, h, a$ be as above. Then

$$r(n, h) = r(\text{cp}(n, h), \tau(n, h)).$$

**Proof.** Write $n[a]$ as in (9.5). We write

$$s(n, h) = \{\Delta'_1, \ldots, \Delta'_k\},$$

with $\Delta'_i$ being the first segment in the removal sequence of $r(\overline{\Delta}_i, r(\{\overline{\Delta}_{i-1}, \ldots, \overline{\Delta}_1\}, h))$.

Define

$$h'_1 = h - \{\overline{\Delta}_1\} + \{-\overline{\Delta}_1\}.$$ 

We observe that:

$$r(n[a], h) = r(\{\overline{\Delta}_2, \ldots, \overline{\Delta}_k\}, r(\overline{\Delta}_1, h))$$

$$= r(\{\overline{\Delta}_2, \ldots, \overline{\Delta}_k\}, r(-\overline{\Delta}_1, h'_1))$$

$$= r(-\overline{\Delta}_1, r(\{\overline{\Delta}_2, \ldots, \overline{\Delta}_k\}, h'_1))$$

$$= r(-\overline{\Delta}_1, r(\{-\overline{\Delta}_2, \ldots, -\overline{\Delta}_k\}, \tau(n, h)))$$

$$= r(-n[a], \tau(n, h)),$$

where the second equation follows from Lemma 7.8, the first, third and last equations follow from Lemma 7.11 and the forth equation follows from the induction
hypothesis (where the first case is Lemma \ref{lem:cond}). (It is straightforward to check from definitions that $\text{tr}(n,h) = \text{tr}(\{\Delta_2, \ldots, \Delta_k\}, h^r_1).$

The lemma then follow by applying $\tau(n-n[a])$ on the first and last terms. □

**Definition 9.4.** We modify $<_c$ to define an ordering $<_c$ on $\text{Mult}_{\rho,c}$ which includes the empty set: For $m_1, m_2$ be in $\text{Mult}_{\rho,c}$ $\cup \{\emptyset\}$. Write the segments in $m_i$ $(i = 1, 2)$ as

$$\Delta_{i,k} = [c, b_{i,k}]_\rho$$

with $b_{i,1} \geq b_{i,2} \geq \ldots$. Define $m_1 <_c m_2$ if either (1) $m_2 = \emptyset$, or (2) $m_2 \neq \emptyset$, and $m_1 <_c m_2$.

**Definition 9.5.** Let $h \in \text{Mult}_\rho$. Let $n \in \text{Mult}_\rho$. Set $n_0 = n$ and $h_0 = h$. We recursively define:

$$h_i = \text{tr}(h_{i-1}, n_{i-1}), \quad n_i = \text{cp}(h_{i-1}, n_{i-1}).$$

We shall call

$$s(n_0, h_0), s(n_1, h_1), \ldots$$

to be a fine chain of removal segments (or simply fine chains) for $(n,h)$.

**Lemma 9.6.** Let $h \in \text{Mult}_\rho$. Let $m_1 \in \text{Mult}_{\rho,c}$. Let $m_2 \in \text{Mult}_{\rho,c}$ be obtained from $m_1$ by replacing one segment in $m_1$ with a longer segment of the form $[c, b]_\rho$ for some $b$. Then either

$$s(m_1, h) = s(m_1, h), \quad s(m_2, h) <_c s(m_2, h).$$

**Proof.** Suppose $s(m_1, h) \neq \emptyset$. If $s(m_2, h) = \emptyset$, then it is clear. Suppose $s(m_2, h) \neq \emptyset$. We write segments in $m_1 \in \text{Mult}_{\rho,c}$ as: $\Delta_1, \ldots, \Delta_k$. Relabeling if necessary, we assume that $\Delta_k$ is the segment replaced by another one, say $\Delta^*$ to obtain $m_2$, and $\Delta_k \subset \Delta^*$.

For $i = 1, \ldots, k$, set

$$r_i = r(\Delta_i, r(\Delta_{i-1}, \ldots, r(\Delta_1, h) \ldots)).$$

By Remark \ref{rem:cond}

1. $s(m_1, h)$ is obtained by the first segments in the sequences of removal segments for $(\Delta_i, r_{i-1})$ for $i = 1, \ldots, k$,

2. $s(m_2, h)$ is obtained by the first segments in the sequences of removal segments for $(\Delta_i, r_{i-1})$ for $i = 1, \ldots, k-1$ and one more from the first segment in the removal sequence for $(\Delta^*, r_{k-1})$.

Thus, $s(m_1, h)$ and $s(m_2, h)$ are only differed by only one segments. Since $\Delta_k \subset \Delta^*$, one readily check that the condition in the lemma has to hold.

We now consider that $s(m_1, h) = \emptyset$. But this is equivalent to

$$h[c] <_c m_1.$$

Since $m_1 <_c m_2$, we also have $h[c] <_c m_2$, which implies that $s(m_2, h) = \emptyset$ as desired. □
Definition 9.7. Let \( \mathfrak{h} \in \text{Mult}_\rho \) and let \( \mathbf{n}, \mathbf{n'} \in \text{Mult}_\rho \). Write the fine chains for \((\mathbf{n}, \mathfrak{h})\) and \((\mathbf{n'}, \mathfrak{h})\) respectively as \(\{s(\mathbf{n}_0, \mathfrak{h}_0), s(\mathbf{n}_1, \mathfrak{h}_1), \ldots\}\) and \(\{s(\mathbf{n}'_0, \mathfrak{h}'_0), s(\mathbf{n}'_1, \mathfrak{h}'_1), \ldots\}\).

We say that two fine chains coincide if

1. \(\tau(\mathbf{n}, \mathfrak{h}), \tau(\mathbf{n}', \mathfrak{h}) \neq 0\); and
2. for all \(i, s(\mathbf{n}_i, \mathfrak{h}_i) = s(\mathbf{n}'_i, \mathfrak{h}'_i)\). (In particular, this implies that there exists an integer \(c\) such that \(s(\mathbf{n}_i, \mathfrak{h}_i), s(\mathbf{n}'_i, \mathfrak{h}'_i) \in \text{Mult}_{\rho, c}\)).

Lemma 9.8. Let \( \mathfrak{h} \in \text{Mult}_\rho \). Let \( \mathbf{n}, \mathbf{n'} \in \text{Mult}_\rho \). Then \(\tau(\mathbf{n}, \mathfrak{h}) = \tau(\mathbf{n'}, \mathfrak{h}) \neq 0\) if and only if the fine chains of removal segments for \((\mathbf{n}, \mathfrak{h})\) and \((\mathbf{n}, \mathfrak{h})\) coincide.

Proof. Let \(\mathbf{n}_0 = \mathbf{n}\) and let \(\mathbf{n}'_0 = \mathbf{n'}\) and let \(\mathfrak{h}_0 = \mathfrak{h}_0' = \mathfrak{h}\). As in the above definition, we write the fine chain of removal segments as

\[\{s(\mathbf{n}_0, \mathfrak{h}_0), s(\mathbf{n}_1, \mathfrak{h}_1), \ldots\}, \quad \{s(\mathbf{n}'_0, \mathfrak{h}'_0), s(\mathbf{n}'_1, \mathfrak{h}'_1), \ldots\}\]

for \((\mathbf{n}, \mathfrak{h})\) and \((\mathbf{n'}, \mathfrak{h})\) respectively.

We first prove the only if direction. Assume two fine chains do not coincide. If \(s(\mathbf{n}_0, \mathfrak{h}_0) \neq s(\mathbf{n}'_0, \mathfrak{h}_0)\), then we count the number of segments of \(\tau(n[a^*], \mathfrak{h})\) and \(\tau(n'[a^*], \mathfrak{h})\) of the form \([a^*, c]_\rho\), where \(a^*\) is the smallest integer such that \(n[a^*] \neq 0\). To see that \(\tau(n[a^*], \mathfrak{h}) \neq \tau(n'[a^*], \mathfrak{h})\), we apply Lemma 7.9. Then we have

\[\tau(\mathbf{c}(\mathbf{n}, \mathfrak{h}), \mathfrak{h}_1) = \tau(\mathbf{c}(\mathbf{n'}, \mathfrak{h}), \mathfrak{h}_1)\]

Now \(\tau(\mathbf{n}, \mathfrak{h}) = \tau(\mathbf{n'}, \mathfrak{h})\) by Lemma 9.3.

Lemma 9.9. Let \( \mathfrak{h} \in \text{Mult}_\rho \). Let \( \mathbf{n} \in \text{Mult}_\rho \) and let \( \mathbf{n'} \) be a multisegment obtained by an elementary intersection-union operation from \( \mathbf{n} \). We write the fine chain as in Definition 9.7 for \((\mathbf{n}, \mathfrak{h})\) and \((\mathbf{n'}, \mathfrak{h})\) respectively:

\[\{s(\mathbf{n}_0, \mathfrak{h}_0), s(\mathbf{n}_1, \mathfrak{h}_1), \ldots\}, \quad \{s(\mathbf{n}'_0, \mathfrak{h}'_0), s(\mathbf{n}'_1, \mathfrak{h}'_1), \ldots\}\].

Then either one of the following holds:

1. the two fine chain of removal segments for \((\mathbf{n}, \mathfrak{h})\) and \((\mathbf{n'}, \mathfrak{h})\) coincide; or
2. there exists some \(i\) such that for any \(j < i\), \(s(\mathbf{n}_j, \mathfrak{h}_j) = s(\mathbf{n}_j, \mathfrak{h}_j'; i')\) and \(s(\mathbf{n}_j, \mathfrak{h}_j) <_{c'} s(\mathbf{n}'_i, \mathfrak{h}_i)\), where \(c'\) is the smallest integer such that \(n_j[c'] \neq 0\).

Proof. Let \(\Delta_1\) and \(\Delta_2\) be two linked segments involved in the elementary intersection-union operation on \(\mathbf{n}\) to obtain \(\mathbf{n'}\). Relabelling if necessary, we write:

\[\Delta_1 = [a_1, b_1]_\rho, \quad \Delta_2 = [a_2, b_2]_\rho,\]

with \(a_1 < a_2\) and \(b_1 \leq b_2\).

We shall prove by induction on the sum of lengths of the segments in \(\mathfrak{h}\). When the sum is \(0\) or \(1\), it is easy. Suppose \(a^*\) is the smallest integer such that \(n[a^*] \neq 0\). If \(a_1 \neq a^*\), then \(n[a^*] = n'[a^*]\). We obtain that \(s(\mathbf{n}, \mathfrak{h}) = s(\mathbf{n'}, \mathfrak{h})\). Let \(\mathfrak{h}_1 := \tau(\mathbf{n}, \mathfrak{h}) = \tau(\mathbf{n'}, \mathfrak{h}) =: \mathfrak{h}_1'\). Let \(\mathbf{n}_1 = \mathbf{c}(\mathbf{n}, \mathfrak{h})\) and let \(\mathbf{n}'_1 = \mathbf{c}(\mathbf{n'}, \mathfrak{h})\). Note that,
by definition of \( cp \), \( n'_1 \) can be obtained by the intersection-union process on those two segments in \( n_1 \). Thus now one proceeds by an induction to obtain the required comparison.

Now we consider that \( a_1 = a^* \). Note that \( n'[a^*] \) is obtained from \( n[a^*] \) by replacing \( \Delta \) with a longer segment. Then by Lemmas 9.6 and 9.8 either

\[
s(n, h) = s(n', h) \quad \text{or} \quad s(n, h) < s(n', h).
\]

In the latter case, we are done. In the former case, we again then proceeds inductively.

\[ \square \]

**Corollary 9.10.** Let \( n, n' \in \text{Mult}_\rho \). Suppose \( n' \leq_Z n \). Let \( n'' \in \text{Mult}_\rho \) be another multisegment such that

\[
n' \leq_Z n'' \leq_Z n.
\]

Then, if \( r(n, h) = r(n', h) \), then \( r(n, h) = r(n'', h) \).

**Proof.** From definition of \( \leq_Z \), there exists a sequence of multisegments in \( \text{Mult}_\rho \)

\[
m_0 = n, m_1, \ldots, m_k = n'
\]

such that each \( m_i \) is obtained from \( m_{i-1} \) by an elementary intersection-union operation. Suppose the fine chain sequence of removal segments for \( (m_{j-1}, h) \) does not coincide with that for \( (m_j, h) \). Then we must fall in the second possibility of Lemma 9.9. But then using Lemma 9.9 again and the transitivity of \( <_c \), we inductively also have that the fine chain for \( (m_1, h) \) does not coincide with that for \( (m_k, h) \). By Lemma 9.8 we have that \( r(n', h) \neq r(n, h) \), giving a contradiction.

Hence, the fine chain for \( (m_{j-1}, h) \) coincides with that for \( (m_j, h) \) for all \( i \). Hence, by Lemma 9.8 we have that

\[
r(n, h) = r(m_1, h) = \ldots = r(n', h).
\]

In particular, one of \( m_j \) can be chosen to be \( n'' \) and we are done.

\[ \square \]

**Corollary 9.11.** Let \( \pi \in \text{Irr}_\rho \) and let \( \tau \) be a simple quotient of \( \pi^{(i)} \). Recall that \( S(\pi, \tau) \) is defined in Section 8.2. Suppose \( n, n' \in S(\pi, \tau) \) and \( n' \leq_Z n \). For any \( n'' \in \text{Mult} \) such that \( n' \leq_Z n'' \leq_Z n \), we have \( n'' \in S(\pi, \tau) \).

**Proof.** This follows from Theorem 8.2 and Corollary 9.10. \[ \square \]

**9.2. Proof of Theorem 1.1.** In view of Theorem 5.3, it remains to prove that \( h\delta(\pi) \) is minimal. Corollary 9.11 reduces to show that if \( n \) is a multisegment obtained by an elementary intersection-union process, then \( D_n(\pi) = 0 \).

Let \( \Delta_1 = [a_1, b_1], \Delta_2 = [a_2, b_2] \) be two linked segments in \( h\delta(\pi) \). Relabelling if necessary, we assume that \( a_1 < a_2 \). Define

\[
n = h\delta(\pi) - \{\Delta_1, \Delta_2\} + \{\Delta_1 \cup \Delta_2, \Delta_1 \cap \Delta_2\}.
\]

Then, \( n[e] = h\delta(\pi)[e] \) for any \( e < a_1 \) and \( n[a_1] \not< a_1 \) \( h\delta(\pi)[a_1] \). Hence, by Lemma 5.2

\[
D_{n[a_1]} \cdots D_{n[e]}(\pi) = D_{n[a_1]} \circ D_{h\delta(\pi)[a_1-1]} \circ \cdots \circ D_{h\delta(\pi)[e]}(\pi) = 0,
\]

and so \( n \not\in S(\pi, \pi^-) \). Here \( c \) is the smallest integer such that \( h\delta(\pi)[c] \neq 0 \).
10. Examples of highest derivative multisegments

10.1. Generic representations. An irreducible representation $\pi$ of $G_n$ is said to be generic if $\pi^{(n)} \neq 0$. According to [Ze80], for $m \in \text{Mult}_\rho$, $\langle m \rangle$ is generic if and only if all the segments are singletons. Equivalently, $\langle m \rangle \cong \langle n \rangle$ for a multisegment whose all segments are unlinked. One can compute $n$, for example, by Meeglin-Waldspurger algorithm. In this case, $\mathfrak{h}\mathfrak{d}(\pi) = n$ (e.g. use [23]).

10.2. Arthur type representations. We write

$$\Delta_\rho(d) = \left[\nu^{-(d-1)/2}, \nu^{(d-1)/2}\right].$$

Let

$$u_\rho(d, m) = \langle \left\{ \nu^{(m-1)/2} \Delta_\rho(d), \ldots, \nu^{-(m-1)/2} \Delta_\rho(d) \right\} \rangle.$$

Let $Y(u_\rho(d, m)) = \nu^{(d-m)/2}$. The representations $u_\rho(d, m)$ are so-called Speh representations. For each Speh representation $u_\rho(d, m)$, it associates with a segment

$$\Delta(u_\rho(d, m)) := [(d - m)/2, (d + m - 2)/2].$$

It follows from [LM16] (also see [CS19]) that

$$\mathfrak{h}\mathfrak{d}(u_\rho(d, m)) = \{\Delta(u_\rho(d, m))\}.$$

**Proposition 10.1.** Let $\pi$ be a Arthur type representation in $\text{Irr}_\rho$, i.e.

$$\pi = \pi_1 \times \ldots \times \pi_r,$$

where each $\pi_a$ is a Speh representation. Then

$$\mathfrak{h}\mathfrak{d}(\pi) = \Delta(\pi_1) + \ldots + \Delta(\pi_r).$$

**Proof.** To compute the lower bound of the multisegment at a point $\nu^c\rho$ (see Proposition 4.4), we rearrange the Speh representations such that

$$\pi_1 \times \ldots \times \pi_k \times \pi_{k+1} \times \ldots \times \pi_r,$$

satisfying that $Y(\pi_1) \cong \ldots \cong Y(\pi_k) \cong \nu^c\rho$ and, for $i = k + 1, \ldots, r$, $Y(\pi_i) \not\cong \nu^c\rho$.

One a composition factor $(\pi_i^- \times \ldots \times \pi_k^-) \boxtimes (\text{St}(\Delta(\pi_1)) \times \ldots \times \text{St}(\Delta(\pi_k)))$ in

$$(\pi_1 \times \ldots \times \pi_k)^N_i.$$

Now, using geometric lemma,

$$((\pi_1 \times \ldots \times \pi_k)^N_i \times (\pi_{k+1} \times \ldots \times \pi_r))^\phi \leftrightarrow (\pi_1 \times \ldots \times \pi_r)^N_i,$$

where $\phi$ is a twist so that the resulting representation is a $G_{n-i} \times G_i$-representation. Here $n = n(\pi)$. Combining with the previous paragraph, we have that

$$\tau \boxtimes (\text{St}(\Delta(\pi_1)) \times \ldots \times \text{St}(\Delta(\pi_k))) \leftrightarrow \pi^N_i.$$

Hence, the maximal multisegment of $\pi$ at the point $\nu^{(d-m)/2}\rho$ is

$$\geq_{(d-m)/2} \Delta(\pi_1) + \ldots + \Delta(\pi_k).$$

We obtain the lower bound for each multisegment at each point. We conclude that the lower bound is also the upper bound by the level of $\pi$ and using Theorem 5.3. □
10.3. **Ladder representations.** As we saw above, the highest derivative multi-segment for a Speh representation is simply a segment. The highest derivative multi-segment for a ladder representation is sum of segments which are mutually unlinked and mutually disjoint.

10.4. □-irreducible representations. An irreducible representation $\pi$ of $G_n$ is said to be □-irreducible if $\pi \times \pi$ is still irreducible [LM18].

**Proposition 10.2.** Let $\pi \in \text{Irr}$. Suppose $\pi$ is □-irreducible. Then

$$\mathfrak{h}(\pi \times \pi) = \mathfrak{h}(\pi) + \mathfrak{h}(\pi).$$

**Proof.** One can, for example, deduce from Proposition 4.6. We omit the details. □

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