Sparse PCA from Inaccurate and Incomplete Measurements

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Abstract

We consider the problem of recovering an unknown effectively sparse low-rank matrix from a small set of incomplete and inaccurate linear measurements of the form \( y = A(X) + \eta \), where \( \eta \) is an ineliminable noise. This problem encompasses and fuses two important classes of machine learning and signal processing challenges: sparse principal component analysis and compressed sensing. More specifically, we aim at recovering low-rank matrices with effectively \((s_1, s_2)\)-sparse non-orthogonal rank-1 decompositions. We describe an optimization problem for matrix recovery under the considered model and propose a novel algorithm, called Alternating Tikhonov regularization and Lasso (A-T-LAS\(_{s_1},s_2\)), to solve it. The algorithm is based on a multi-penalty regularization, which is able to leverage both structures (low-rankness and sparsity) simultaneously. The algorithm is a fast first order method, and straightforward to implement. We prove global convergence for any linear measurement model to stationary points and local convergence to global minimizers of the multi-penalty objective functional. Global minimizers balance effective sparsity of their rank-1 decompositions and the fidelity to data, up to noise level. By adapting the concept of restricted isometry property from compressed sensing to our novel model class, we prove error bounds between global minimizers and ground truth, up to noise level, from a number of subgaussian measurements scaling as \( R(s_1 + s_2) \), up to log-factors in the dimension, and relative-to-diameter distortion. Simulation results demonstrate both the accuracy and efficacy of the algorithm, as well as its superiority to the state-of-the-art algorithms in strong noise regimes and for matrices, whose singular vectors do not possess exact (joint-) sparse support.

1 Introduction

Due to data deluge, the existing amounts of data increasingly exceed the processing capacities, leading much of the recent research to focus on compressive data acquisition and storage. In this case, data recovery typically requires finding a solution for an underdetermined system of linear equations, which becomes tractable only when the data possess some special structure. This general paradigm encompasses two important problem classes, which have received significant scientific attention recently: compressed sensing and (sparse) principal component analysis. The work of this paper stands at the intersection of these problems, incorporating the challenges from both of them and extending their framework. Specifically, we are interested in recovering an effectively sparse matrix of low rank \( X \) from an incomplete and inaccurate set of linear measurements \( y = A(X) + \eta \). This problem is relevant in several areas such as blind deconvolution, machine learning, and data mining, as we illustrate below with a couple of motivating examples.

Sparse Principal Component Analysis from inaccurate and incomplete linear measurements.
Principal Component Analysis (PCA) [19] is a classical tool for processing large amounts of data and performing data analysis such as dimensionality reduction and factor extraction. Its scope of application ranges from engineering and technology to social sciences, and biology.

PCA and, more generally, matrix completion [8] has been widely used for recommendation systems as popularised by the so-called Netflix prize problem [5]. We illustrate PCA by considering a simple example of such recommendation system for a grocery store, which has \( n_1 \) regular customers and \( n_2 \) products. Let

\(^1\)With “ineliminable” we mean that the focus of the paper is not on recovery from exact \( y = A(X) \) measurements, rather on the stable recovery under severe measurement noise.
\( X \in \mathbb{R}^{n_1 \times n_2} \) be a matrix whose components \( X_{i,j} \) encode the probability of customer \( i \) buying product \( j \). It is reasonable to assume that there are only \( R \ll \min\{n_1, n_2\} \) underlying basic factors like age, income, family size, etc. which govern the customer’s purchase behavior. For each basic factor \( r \in [R] := \{1, \ldots, R\} \) one defines two vectors: a vector \( u^r \in \mathbb{R}^{n_1} \) of components \( u^r_i \) encoding for each user \( i \in [n_1] \) how much they are affected by the factor \( r \), and a vector \( v^r \in \mathbb{R}^{n_2} \) encoding the probability of buying product \( j \) if having factor \( r \). Then, one can decompose

\[
X \approx UV^T = \sum_{r=1}^{R} u^r (v^r)^T, \tag{1}
\]
as the product of two matrices \( U \in \mathbb{R}^{n_1 \times R} \) and \( V \in \mathbb{R}^{n_2 \times R} \) with columns \( u^r \) and \( v^r \). Even if the product \( UV^T \) is only approximately \( X \), the decomposition into orthogonal principal components \( U \) and loadings \( V \) is appealing for more interpretability and having less data to store (\( O(\max\{n_1, n_2\} R) \) instead of \( O(n_1 n_2) \)). However, if we want to understand which factors mostly affect customer’s behaviour, PCA might not be the best option, since principal components are usually a linear combination of all original variables. To further improve interpretability and reduce the number of explicitly used variables, sparse PCA \cite{9}, which promotes sparsity of the loadings \( v^r \) in \( (1) \), has been proposed. Sparse PCA trades orthogonality of the principal components for sparse solutions. In the aforementioned example of the grocery store, it is quite reasonable to assume sparsity of the probability distributions \( v^r \), as certain factors normally are more correlated with the probability of purchase of few specific items.

For some applications one may not have access to the complete matrix \( X \) but only to a partial indirect information, i.e., one has only \( m \ll n_1 n_2 \) scalars encoding information about \( X \). In the example of the grocery store this may model the situation where customers do not possess all a fidelity card, which allows to identify them individually, and the grocery store still wishes to learn the matrix \( X \) from aggregated revenues. Each day \( d \in [D] \) the store caches in a certain amount of money \( y^d_i \) corresponding to purchases of a random subset \( T_d \subset [n_1] \) of its customers \( (\ell \in \mathbb{N} \) is a fixed index, whose role will soon become clear). If \( P^\ell \in \mathbb{R}^{n_2} \) is a vector encoding the prices \( P^\ell_j \) of each product \( j \) and \( P_{i,d} \subset [n_2] \) is the random set of products purchased by customer \( i \) on a day \( d \), we can express the takings as

\[
y^d_i = \sum_{i \in T_d} \sum_{j \in P_{i,d}} P^\ell_j.
\]

If we assume that each customer \( i \) visits the grocery store with probability \( q_i \), we can compute the expected takings as

\[
\mathbb{E}_{T_d, P_{i,d}} \left[ \sum_{i \in T_d} \sum_{j \in P_{i,d}} P^\ell_j \right] = \sum_{i=1}^{n_1} q_i \sum_{j=1}^{n_2} X_{i,j} P^\ell_j.
\]

Choosing \( D \) sufficiently large, the law of large numbers guarantees that

\[
\lim_{D \to \infty} \frac{1}{D} \sum_{d=1}^{D} y^d_i = \mathbb{E}_{T_d, P_{i,d}} \left[ \sum_{i \in T_d} \sum_{j \in P_{i,d}} P^\ell_j \right],
\]
in probability and almost surely. Moreover, by Central Limit Theorem, we may model the average takings of \( D \) days as

\[
\frac{1}{D} \sum_{d=1}^{D} y^d_i = \sum_{i=1}^{n_1} q_i \sum_{j=1}^{n_2} X_{i,j} P^\ell_j + \eta^D, \tag{2}
\]

for a suitable Gaussian noise \( \eta^D \). By defining \( y^\ell = \frac{1}{D} \sum_{d=1}^{D} y^d_i \), we can rewrite the above equation as

\[
y^\ell = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} (q_i P^\ell_j) X_{i,j} + \eta_{D,l} = \langle A\ell, X \rangle_F + \eta^D
\]
where the matrix $A_{\ell} \in \mathbb{R}^{n_1 \times n_2}$ has entries $(q_{\ell} P_{ji})_{i,j}$, and $(\cdot, \cdot)^{\mathcal{F}}$ is the Frobenius scalar product.

Tracking the daily sales over a time period of $m \cdot D$ days and perturbing the prices in each subperiod $\ell \in [m]$ randomly would result in $m$ inaccurate linear measurements, where each single measurement is a random average over the entries of $X$ with an ineliminable additive noise $\eta_{D}$. The whole measurement process can be written as

$$y = A(X) + \eta$$

where $A : \mathbb{R}^{n_1 \times n_2} \rightarrow \mathbb{R}^m$ is a linear operator defined by the matrices $A_1, \ldots, A_m$ and $\eta = (\eta_{D}^{1}, \ldots, \eta_{D}^{m})^T \in \mathbb{R}^m$ models the noise. As we demonstrate in this paper, it is possible by means of our resource efficient algorithm to recover a low-rank-$R$ matrix $X$ with effectively $(s_1, s_2)$-sparse non-orthogonal rank-1 decomposition, from a number $m \approx R(s_1 + s_2)$ of random noisy measurements. This would offer a plausible solution to the grocery store problem. (We should stress that in general our algorithm will converge to a data fitting solution of low rank and sparse components for arbitrary linear measurement operators $A$.)

Blind deconvolution in signal processing In blind deconvolution [16] one is interested in recovering two unknown vectors $w$ and $s$ solely from their convolutional product

$$y = w * s + \eta = \left( \sum_{i=1}^{m} w_i s_{(k-i) \mod m} \right)_{k=1}^{m} + \eta,$$

where $\eta$ is again measurement noise. In imaging applications, $s$ represents the picture and $w$ – an unknown blurring kernel [32]. In signal transmission, $s$ is a coded message and $w$ models the properties of the transmission channel [13]. Independently of the concrete application, problem (3) is highly under-determined and contains ambiguities.

In [1] the authors used that by bilinearity of the convolution, (3) can be represented as a linear map acting on the tensor product $ws^T$, a technique commonly known as lifting. They assumed in addition that the channel properties $w$ and the message $s$ are drawn from lower dimensional subspaces and are of the form $w = Bh$ and $s = Cx$ with $h \in \mathbb{C}^{n_1}$ and $x \in \mathbb{C}^{n_2}$ being coefficient vectors encoding channel and message ($B$ and $C$ are suitable transformation matrices). Accordingly, they re-write (3) as

$$y = A(X) + \eta,$$

where the rank-1 matrix $X = h x^T \in \mathbb{C}^{n_1 \times n_2}$ has to be recovered from $m$ linear measurements, a quite popular model in compressed sensing literature [30]. Under suitable assumptions on $A$ the recovery of $X$ is solved by (convex) nuclear norm minimization.

In blind demixing [24, 25, 20] or MIMO channel identification [11] a receiver gets the overlay of $R$ different convolutions which translates the above mentioned formulation into the recovery of rank-$R$ matrices from linear measurements of the type

$$y = A \left( \sum_{r=1}^{R} h_r x_r^T \right) + \eta.$$

As already mentioned in [20], one can typically impose extra structure like sparsity on the channel impulse responses $h$ to further reduce the number of measurements $m$. In this case one wants to benefit from exploiting two different structures at the same time, low-rankness and sparsity.

The recovery from linear measurements of low-rank matrices without sparsity constraints has been well studied as an extension of classical compressed sensing theory [7, 31]. When the unknown matrix is assumed to have both low-rankness and sparsity, Oymak et. al. in [27] showed that a mere convex combination of regularizers for different sparsity structures does not allow in general outperform the recovery guarantees of the “best” one of them alone. Consequently, in order to improve recovery further, one has to go beyond

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2The random fluctuation of prizes is applied by groceries also for rotating promotions on products. Periodic price reductions, or sales, constitute a widely observed phenomenon in retailing. Sales occur on a regular basis, which suggests that they are not entirely due to random variations such as shocks to inventory holdings or demand.
linear combinations of already known convex regularizers. In \cite{4} the authors overcome the aforementioned limitations of purely convex approaches by assuming a nested structure of the measurement operator $A$ and applying basic solvers for low-rank resp. row-sparse recovery in two consecutive steps which is an elegant approach but clearly restricts possible choices for $A$. Lee et. al. \cite{23} propose and analyze the so-called Sparse Power Factorization (SPF), a modified version of Power Factorization (see \cite{18}) for recovery of low-rank matrices by representing them as product of two matrices $X = UV^T$ and then applying alternating minimization over the (de)composing matrix $U, V$. SPF introduces Hard Thresholding Pursuit to each of the alternating steps to enforce additional sparsity of the columns of $U$ and/or $V$. Lee et. al. were able to show that using suitable initializations and assuming the noise level to be small enough, SPF approximates low-rank and row- and/or column-sparse matrices $X$ from a nearly optimal number of measurements: If $X$ is rank-$R$, has $s_1$-sparse columns and $s_2$-sparse rows $m \geq R(s_1 + s_2) \log(\max\{en_1/s_1, en_2/s_2\})$ measurements suffice for robust recovery, which is up to the log-factor at the information theoretical bound. Note that all columns (resp. rows) need to share a common support in this setting, and this seems to be an empirically necessary requirement, see Section 5.3. On the one hand, empirically, it has also been shown in \cite{23} that SPF outperforms methods based on convex relaxation. On the other side, SPF is heavily based on the assumption that the operator $A$ possesses a suitable restricted isometry property and cannot be applied to arbitrary inverse problems of type \cite{2}, as it may even fail to converge otherwise. The reason is that SPF is based on hard-thresholding \cite{6}, which is not a Lipschitz continuous (nonexpansive) map.

**Contribution** Some recent work provide theoretical and numerical evidences of superior performance of multi-penalty regularization, see \cite{26} \cite{15} \cite{11} and references therein, for correct modeling and separation of the superposition of the signals $u + v$. Additionally, sparse PCA introduced a while ago \cite{35} \cite{9} uses such kind of multi-penalty regularization in a multiplicative setting $uw^T$ to successfully compute sparse nonorthogonal 1-rank decompositions of matrices. Motivated by these results, we propose to recover and decompose $X$ by a variational approach based on alternating minimizing the following multi-penalty functional $J_{\alpha,\beta}^R: \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \ldots \times \mathbb{R}^{n_2} \to \mathbb{R}$ defined, for $\alpha, \beta > 0$, by

$$J_{\alpha,\beta}^R(u^1, \ldots, u^R, v^1, \ldots, v^R) := \left\| y - A \left( \sum_{r=1}^{R} u^r (v^r)^T \right) \right\|_2^2 + \alpha \sum_{r=1}^{R} \|u^r\|_2^2 + \beta \sum_{r=1}^{R} \|v^r\|_1, \quad (4)$$

where $\alpha, \beta$ are regularization parameters. We denote the global minimizer of (4) by

$$(u^1_{\alpha,\beta}, \ldots, u^R_{\alpha,\beta}, v^1_{\alpha,\beta}, \ldots, v^R_{\alpha,\beta}).$$

As the reader can easily realize, it is perhaps the most natural generalization of sparse PCA \cite{35} \cite{9} in case of incomplete and inaccurate measurements.

The main contribution of this paper is the theoretical and numerical analysis of the iterative alternating algorithm, based on simple iterative soft-thresholding, for the minimisation of (4), which we dub (A-T-LAS_{2,1}) for Alternating Tikhonov regularization and Lasso or simply ATLAS throughout the rest of the paper. At first glance, ATLAS may seem to be quite similar to Sparse Power Factorization (SPF), as they both rely on alternating minimization. However, ATLAS exhibits important positive differences: by using convex relaxation ($\ell_1$-norm minimization) at each iteration, instead of solving a non-convex problem ($\ell_0$-minimization) as SPF, and by virtue of the Lipschitz continuity of soft-thresholding, we obtain convergence and approximation guarantees, Section 5.1 below, for any inverse problem, where neither restricted isometry property of the measurement operator $A$ nor conditions on the support distribution of $X$ are assumed. In fact, as we already mentioned above, SPF may even fail to converge in general. Moreover, while SPF can be considered as an alternating minimization over matrices, ATLAS alternates on $R$ pairs of vectors, very much as sparse PCA does \cite{35} \cite{9}. As sparse PCA, ATLAS trades as well effective sparsity of the components $(u^1_{\alpha,\beta}, \ldots, u^R_{\alpha,\beta}, v^1_{\alpha,\beta}, \ldots, v^R_{\alpha,\beta})$ with their orthogonality, which is instead an extremely restrictive requirement of SPF. In fact, SPF assumes the exact sparsity of orthogonal singular vectors. As a consequence of the freedom of giving up the orthogonality, the componentwise procedure of ATLAS enables us also to drop the assumption of a common support for all columns (resp. rows) as required in SPF. In the case of measurements fulfilling additionally a suitable restricted isometry property, we obtain recovery guarantees also for high level of noise, while the theoretical analysis of SPF ensures recovery for low level of noise or for vanishing noise.
only, which is not a realistic assumption in practice (see for instance the problem of the grocery store above). Before presenting in details the main results of this paper, let us compare right away the empirical performance of ATLAS and SPF on a few explicated cases of low-rank sparse matrix recovery. As mentioned above, SPF can be considered as state-of-the-art benchmark for such problems, as it has been shown to outperform most of the popular recovery algorithms based on convex relaxation. We compare the performance of the algorithms in terms of empirical recovery probability and mean approximation error. The comparison is displayed in Fig. 1 and Fig. 2. As can be seen from the results, ATLAS shows a higher level of robustness with respect to noise, in contrast to SPF that has a better performance in noise-free cases. Additionally, ATLAS outperforms SPF as soon as singular vectors of $X$ do not share common support, which is a quite restrictive assumption for SPF to be successful. More details on implementation and experimental settings are described in Section 5.

In summary, we show that differently from a pure compressed sensing framework where vanishing noise and exact measurements are particularly emphasized, ATLAS is able to deal with highly inaccurate measurements, which are immanent in many real-life problems. Moreover, we show that ATLAS converges to meaningful solutions also for $A$ not fulfilling a restricted isometry property. Finally, similarly to the sparse PCA, for our recovery results, we do not require that $X$ possesses a sparse singular value decomposition, but it is sufficient that it has an arbitrary effectively sparse nonorthogonal 1-rank decomposition.

**Outline** The organization of the paper is as follows. Section 2 presents the setting of this paper, clarifies notation, and introduces the algorithm for the minimisation of (4). In Section 3 we give an overview of the main results. The corresponding proofs can be found in Section 4. In Section 5 we present the actual implementation of (10) and provide numerical experiments, confirming our theoretical results and showing extensive comparisons to SPF. We conclude in Section 6 with a discussion on open problems and future work. Finally, the Appendix contains proofs of some technical results from Sections 3 and 4 which are mainly generalization and extension of other existing works.

## 2 Problem Formulation and Notation

We recall that the **Singular Value Decomposition (SVD)** of a matrix $Z \in \mathbb{R}^{n_1 \times n_2}$ is given by

$$Z = U \Sigma V^T = \sum_{r=1}^{\text{rank}(Z)} \sigma_r u^r (v^r)^T,$$

(5)
Figure 2: Phase transition diagrams comparing SPF and ATLAS with and without noise on the measurements (see Section 5.3). Empirical recovery probability is depicted by color from zero (blue) to one (yellow).

where $\Sigma$ is a diagonal matrix containing the singular values $\sigma_1 \geq \ldots \geq \sigma_{\text{rank}(Z)} > 0$ while $U \in \mathbb{R}^{n_1 \times \text{rank}(Z)}$ and $V \in \mathbb{R}^{n_2 \times \text{rank}(Z)}$ have orthonormal columns which are called left and right singular vectors. Hence, each singular value $\sigma_r$ has one left singular vector $u_r$ and one right singular vector $v_r$.

In the following, we assume $\hat{X}$ of rank $R > 0$ and to possess a decomposition of the form

$$\hat{X} = \sum_{r=1}^{R} \hat{u}_r (\hat{v}_r)^T,$$

where $\hat{v}_r$ are effectively $s$-sparse, meaning that $||\hat{v}_r||_1 \leq \sqrt{s}||\hat{v}_r||_2$ for $r = 1, \ldots, R$. Note that we require $\hat{v}_r$ neither to be exactly sparse nor to share a common support. Moreover, we do not assume them to be mutually orthogonal. We call the vectors $\hat{u}_r$ (resp. $\hat{v}_r$) the left (resp. right) component vectors of $\hat{X}$. From the context it will be clear to which decomposition they are referred. In fact, (6) does not need to be the SVD of $\hat{X}$, although this case is also covered by our analysis. We focus on decompositions (6) with effectively sparse right component vectors. Conceptually straightforward, but perhaps tedious modifications of the arguments lead to similar results in the left-sided and both-sided sparse case (see Section 4.2).

Furthermore, we are given some linear measurement operator $\mathcal{A} : \mathbb{R}^{n_1 \times n_2} \to \mathbb{R}^m$ and the vector of measure-
ments $y$, which is obtained from $\hat{X}$ by

$$y = A(\hat{X}) + \eta = \frac{1}{\sqrt{m}} \begin{pmatrix} \langle A_1, \hat{X} \rangle_F \\ \vdots \\ \langle A_m, \hat{X} \rangle_F \end{pmatrix} + \eta. \tag{7}$$

The operator $A$ is completely characterized by the $m$ matrices $A_i \in \mathbb{R}^{n_1 \times n_2}$ and individual measurements correspond to Frobenius products $\langle A_i, \hat{X} \rangle_F = \text{trace}(A_i \hat{X}^T)$. Noise comes into play by the additive vector $\eta \in \mathbb{R}^m$ of which only the $\ell_2$-norm is known.

**Notation** We fix all notation used in this paper in order to ease readability. For a matrix $Z \in \mathbb{R}^{n_1 \times n_2}$, we denote its transpose by $Z^T$. A variety of norms are used throughout this paper: $\|Z\|_p$ is the Schatten-$p$ quasi-norm ($\ell_p$-quasi-norm of the vector of singular values $(\sigma_1, ..., \sigma_R)^T$); $\|Z\|_F$ is the Frobenius norm (the $\ell_2$-norm of the vector of singular values); $\|\cdot\|_{2 \to 2}$ is the operator norm of $Z$ (the top singular value).

Note that for $0 < p < 1$, Schatten-$p$ norm is only a quasi-norm. For $p = 2$, the Schatten norm is equal to the Frobenius norm, whereas the $\infty$-Schatten norm corresponds to the operator norm. We use the shorthand notation $[R] = \{1, ..., R\}$ to write index sets. The relation $a \gtrsim b$ is used to express $a \geq Cb$ for some positive constant $C$, and $a \simeq b$ stands for $a \gtrsim b$ and $b \gtrsim a$.

If $\hat{X}$ of rank $R > 0$ possesses an SVD as in (6) for $\|\hat{v}^r\|_2 = \sigma_r$, $r \in [R]$, then for any $0 < p < \infty$

$$\|X\|_F^p = \sum_{r=1}^R (\|\hat{u}^r\|_2 \|\hat{v}^r\|_2)^p. \tag{8}$$

If the decomposition (6) does not coincide with the SVD of $\hat{X}$, then $\hat{u}^1, ..., \hat{u}^R$ are anyhow linearly independent and

$$\|X\|_p^2 = \sum_{j=1}^{n_2} \sum_{i=1}^{n_1} \sum_{r=1}^R \|\hat{u}^r_i \hat{v}^r_j\|^2 = \sum_{j=1}^{n_2} \sum_{r=1}^R \|\hat{u}^r\|_2 \|\hat{v}^r\|_2 \|\hat{u}^r\|_2 \|\hat{v}^r\|_2 \simeq \sum_{j=1}^{n_2} \sum_{r=1}^R \|\hat{u}^r\|_2 \|\hat{v}^r\|_2^2 = \sum_{r=1}^R (\|\hat{u}^r\|_2 \|\hat{v}^r\|_2)^2.$$

From this equivalence and the equivalence of $\ell_p$-quasi-norms and Schatten-$p$-quasi-norms for $0 < p \leq 2$, one further obtains as a relaxation of (8)

$$c_G^{-1} R^{p/2 - 1} \sum_{r=1}^R (\|\hat{u}^r\|_2 \|\hat{v}^r\|_2)^p \leq \|X\|_p^p \leq c_F R^{1-p/2} \sum_{r=1}^R (\|\hat{u}^r\|_2 \|\hat{v}^r\|_2)^p, \tag{9}$$

for positive constants $c_G, c_F > 0$, which depend on the largest and smallest eigenvalues of the Gramian of the vectors $\hat{u}^1/\|\hat{u}^1\|_2, ..., \hat{u}^R/\|\hat{u}^R\|_2$. Below we shall use (9) mostly for $p = 2/3$.

**Recovery algorithm** Following promising results on multi-penalty functionals for unmixing problems [26] and generalizing sparse PCA [35, 9], we propose to approximate $\hat{X}$ by global minimizers of the functional $J^R_{\alpha, \beta}$ defined in (4), which combines one quadratic least-squared error term on the measurements with several convex regularizers applied to vectors (not matrices). Note that $J^R_{\alpha, \beta}$ does apply to matrices implicitly by viewing each $2R$-uple $(u^1, ..., u^R, v^1, ..., v^R)$ as the matrix $X = \sum_{r=1}^R u^r (v^r)^T$, and we denote $X_{\alpha, \beta} = \sum_{r=1}^R u^r_{\alpha, \beta} (v^r_{\alpha, \beta})^T$ the one corresponding to global minimizer $(u^1_{\alpha, \beta}, ..., u^R_{\alpha, \beta}, v^1_{\alpha, \beta}, ..., v^R_{\alpha, \beta})$. In such a way, instead of combining convex regularizers for sparse and low-rank matrices, we enforce low-rankness by restricting the domain of $J^R_{\alpha, \beta}$ properly (the decomposition can only consist of $R$ vector pairs) and promote sparsity by $\ell_1$-norm regularization directly on vectors of the decomposition, while fit to the measurements $y$ is ensured by the first term. Despite the convex multi-penalty regularization term $\alpha \sum_{r=1}^R ||v^r||_1 + \beta \sum_{r=1}^R ||v^r||_1$, the functional (4) is highly non-convex, hence, it provides hope for better performances than lifting and convex relaxation. At the same time, one notices that $J^R_{\alpha, \beta}$ becomes convex when all but one $u^r$ and/or $v^r$ are
fulfilling the noisy measurements

Proposition 3.1
The first result bounds measurement misfit by

\[
\begin{align*}
\|y - A(\sum_{r=2}^{R} u_{k}^r v_{k+1}^T)\|_2^2 &+ \alpha\|u\|_2^2 + \frac{1}{2\lambda_k}\|u - u_{k}^1\|_2^2, \\
\|y - A(\sum_{r=2}^{R} u_{k+1}^r v_{k}^T)\|_2^2 &+ \beta\|v\|_1 + \frac{1}{2\sigma_k}\|v - v_{k}^1\|_2^2.
\end{align*}
\]

(A-TLAS)\textsubscript{2,1}

In each iteration above, the terms \(\|u - u_{k}^r\|_2^2\) and \(\|v - v_{k}^r\|_2^2\) are added to provide theoretical convergence guarantees for the sequence \((u_{k}^1, \ldots, u_{k}^R)\) with suitable choice of the \(2R\) positive sequences of parameters \((\lambda_k)_{k \in \mathbb{N}}, \ldots, \mu_k, \ldots, (\mu_k^R)_{k \in \mathbb{N}} > 0\). In practice, ATLAS converges without those terms. As most of the non-convex minimization algorithms, empirical performances of ATLAS likely depends on a proper initialization \((u_{k}^1, \ldots, u_{k}^R, v_{k}^1, \ldots, v_{k}^R)\). Initialization by the leading right singular vectors of \(A^*(y)\), where \(A^*\) denotes the adjoint of \(A\), ensures empirically stable recovery in the experiments (Section \ref{sec:experiments}). However, we do not provide any theoretical guarantees for this observation. We prove local convergence of ATLAS, see Theorem \ref{thm:local_convergence} below.

One of the virtues of the algorithm is the explicit formulas for computation of the successive iterations, resulting in low computational complexity.

3 Main Results

In this section we state the main results of the paper. First, we show how minimizers of \(J_{\alpha, \beta}^R\) yield under minimal assumptions solutions to the inverse problem \(\hat{X}\). Second, to explain the performance observed in Figure \ref{fig:performance}, we introduce a versatile matrix model class and come up with a suitable restricted isometry property (RIP), which captures both low-rankness and sparsity. Third, we provide bounds on a sufficient number of measurements for subgaussian operators to fulfill the RIP with high probability. Finally, local convergence of ATLAS to global minimizers of \(J_{\alpha, \beta}^R\) is proved.

3.1 Properties of Minimizers of \(J_{\alpha, \beta}^R\)

Let us begin with some basic properties that minimizers of \(J_{\alpha, \beta}^R\) have under very general assumptions. For a given minimizer \((u_{\alpha, \beta}^1, \ldots, u_{\alpha, \beta}^R, v_{\alpha, \beta}^1, \ldots, v_{\alpha, \beta}^R)\) of \(J_{\alpha, \beta}^R\) we denote

\[
X_{\alpha, \beta} = U_{\alpha, \beta} \Sigma_{\alpha, \beta} V_{\alpha, \beta}^T = \sum_{r=1}^{R} (\sigma_{\alpha, \beta})_r \frac{u_{\alpha, \beta}^r}{\|u_{\alpha, \beta}^r\|_2} \left(\frac{v_{\alpha, \beta}^r}{\|v_{\alpha, \beta}^r\|_2}\right)^T
\]

where \((\sigma_{\alpha, \beta})_r = \|u_{\alpha, \beta}^r\|_2\|v_{\alpha, \beta}^r\|_2\), for all \(r \in [R]\), and \(\Sigma_{\alpha, \beta}\) is the diagonal matrix defined by the vector \(\sigma_{\alpha, \beta}\). The first result bounds measurement misfit by \(X_{\alpha, \beta}\).

Proposition 3.1 (Measurement misfit), Assume \((u_{\alpha, \beta}^1, \ldots, u_{\alpha, \beta}^R)\) is a global minimizer of \(J_{\alpha, \beta}^R\) and \(\hat{X}\) is fulfilling the noisy measurements \(y = A(\hat{X}) + \eta\). Then,

\[
\|y - A(X_{\alpha, \beta})\|_2^2 \leq \|\eta\|_2^2 + C_{2,1} \sqrt{\alpha \beta^2} \sum_{r=1}^{R} (\|\hat{u}^r\|_2\|\hat{v}^r\|_1)^{\frac{3}{2}},
\]

where \(C_{2,1}\) is the constant from Lemma \ref{lem:existence} below.

By similar proof techniques, one can also control the norms of the components \(u_{\alpha, \beta}^r\) and \(v_{\alpha, \beta}^r\), for \(r \in [R]\).
Lemma 3.2 (Boundedness). Assume \((u^1_{\alpha,\beta}, \ldots, v^R_{\alpha,\beta})\) is a global minimizer of \(J^R_{\alpha,\beta}\) and \(\hat{X}\) is fulfilling the noisy measurements \(y = A(\hat{X}) + \eta\). If \(\|y - A(X_{\alpha,\beta})\|_2 \geq \|\eta\|_2\), we have

\[
\sum_{r=1}^{R} \|u^r_{\alpha,\beta}\|_2^2 \leq C_{2.1} \sqrt{\frac{\beta}{\alpha}} \sum_{r=1}^{R} (\|\hat{u}^r\|_2 \|\hat{v}^r\|_1)_1^2
\]

and

\[
\sum_{r=1}^{R} \|v^r_{\alpha,\beta}\|_1 \leq C_{2.1} \sqrt{\frac{\alpha}{\beta}} \sum_{r=1}^{R} (\|\hat{u}^r\|_2 \|\hat{v}^r\|_1)_1^2
\]

where \(C_{2.1}\) is the constant from Lemma 4.1.

The two estimates in (13) point out an interesting property of \(J^R_{\alpha,\beta}\). If one chooses the parameters \(\alpha\) and \(\beta\) of different magnitude, either the left or the right components of a minimizer \((u^1_{\alpha,\beta}, \ldots, v^R_{\alpha,\beta})\) can be forced to become smaller in norm, while the grip on the others is lost. If \(\alpha\) and \(\beta\) are chosen to be equal the norm bounds are balanced and one obtains

\[
\sum_{r=1}^{R} (\|u^r_{\alpha,\beta}\|_2 + \|v^r_{\alpha,\beta}\|_1) \leq C_{2.1} \sum_{r=1}^{R} (\|\hat{u}^r\|_2 \|\hat{v}^r\|_1)_1^2
\]

The assumption \(\|y - A(X_{\alpha,\beta})\|_2 \geq \|\eta\|_2\) is not restrictive. As soon as \(\|y - A(X_{\alpha,\beta})\|_2 = \|\eta\|_2\) one does not have to decrease \(\alpha\) and \(\beta\) any further. There is no hope of obtaining an accuracy below noise level and any smaller parameter choice will lead to overfitting phenomena.

The \(\ell_1\)-regularization in \(J^R_{\alpha,\beta}\) provides means to bound \(\|v^r_{\alpha,\beta}\|_1\), for \(r \in [R]\). By this, one can control effective sparsity of the minimizer’s right components.

Lemma 3.3 (Sparsity control). Assume \(A: \mathbb{R}^{n_1 \times n_2} \rightarrow \mathbb{R}^m\) is a linear operator and \(y \in \mathbb{R}^m\). Let \((u^1_{\alpha,\beta}, \ldots, v^R_{\alpha,\beta})\) be a minimizer of \(J^R_{\alpha,\beta}\). For all \(r \in [R]\) we have that if \(\|v^r_{\alpha,\beta}\|_2 \geq \|y\|_2^2 / \gamma\) for some \(\gamma > 0\), then

\[
\frac{\|v^r_{\alpha,\beta}\|_1}{\|v^r_{\alpha,\beta}\|_2} < \frac{\gamma}{\beta}
\]

The above Lemma states that those vector components \(v^r_{\alpha,\beta}\), which lie not too close to zero, are effectively sparse. Numerical experiments suggest that if \(\hat{X}\) has \(s\)-sparse right components \(\hat{v}^r\), ATLAS yields solutions with exactly sparse right components \(v^r_{\alpha,\beta}\). The theoretical necessity of considering effective sparsity also when \(\hat{X}\) has \(s\)-sparse right components is caused by the difficulty of obtaining better bounds on the support size of the vectors \(v^r_{\alpha,\beta}\).

To conclude, by considering the results of \([6]\) and \([7]\), Proposition 3.1 and Lemma 3.3 we can claim that \(X_{\alpha,\beta}\) is, even without any more specific requirements on \(A\), a reasonable approximation of \(\hat{X}\), in the sense that it is of rank \(R\), fulfills the measurements up to noise level, and has effectively sparse right components. However, the parameters \(\alpha\) and \(\beta\) have to be chosen with care, neither too small nor too large. Moreover, Lemma 3.2 shows that \(\alpha\) and \(\beta\) have to be chosen of similar magnitude. Otherwise either left or right components of \(X_{\alpha,\beta}\) cannot be controlled.

3.2 Recovery Properties of Minimizers of \(J^R_{\alpha,\beta}\) with RIP

To explain the performance of ATLAS illustrated in Figure 2 we introduce two sets of matrices, which are sums of few rank-one matrices with sparse singular vectors. At this point we need to stress that, as an innovation with respect to usual practice, we are not requiring the orthogonality of the components. We also define corresponding additive RIPS, which are useful for proving the approximation result and can be seen as a generalization of the rank-R and \((s_1, s_2)\)-sparse RIP of Lee et. al. in \([29]\).
Matrix models. The first matrix set is, for $\Gamma \geq 1$,
\[
S^{R,\Gamma}_{s_1,s_2} = \{ Z \in \mathbb{R}^{n_1 \times n_2} : \exists u^1, \ldots, u^R \in \mathbb{R}^{n_1}, v^1, \ldots, v^R \in \mathbb{R}^{n_2}, \text{ and } \sigma = (\sigma_1, \ldots, \sigma_R)^T \in \mathbb{R}^R, \text{ s.t.} \]
\[
\begin{align*}
Z &= \sum_{r=1}^R \sigma_r u^r(v^r)^T, \\
\text{where } |\text{supp}(u^r)| &\leq s_1, |\text{supp}(v^r)| \leq s_2, \|u^r\|_2 = \|v^r\|_2 = 1, \text{ for all } r \in [R], \text{ and } \|\sigma\|_2 \leq \Gamma.
\end{align*}
\] (14)

It contains all matrices $Z$ which can be decomposed into three matrices $U\Sigma V^T$ such that $U \in \mathbb{R}^{n_1 \times R}$ and $V \in \mathbb{R}^{n_2 \times R}$ have $s_1$-sparse (resp. $s_2$-sparse) unit norm columns and $\Sigma \in \mathbb{R}^{R \times R}$ is the diagonal matrix defined by $\sigma$. The set is restricted to decompositions with $\|\Sigma\|_F \leq \Gamma$, and $\Gamma \geq 1$ is a natural condition as explained in Remark 4.1 below.

The important difference w.r.t. [23] is that the columns do not need to share a common support. Moreover, we do not require $U$ and $V$ to be orthogonal matrices. Nevertheless, all matrices $X$ with rank less or equal $R$, $s_1$-sparse (resp. $s_2$-sparse) left and right singular vectors, and $\|X\|_F \leq \Gamma$ are in $S^{R,\Gamma}_{s_1,s_2}$. In this case $\|\Sigma\|_F = \|X\|_F$. We call such an admissible decomposition $U\Sigma V^T$ in (14) a Sparse Decomposition (SD) of $Z$.

Note that the SD is not unique and that the SVD of $Z$ is not necessarily a SD of $Z$.

The second set is a further generalization of $S^{R,\Gamma}_{s_1,s_2}$. We drop the sparsity assumption and replace it by effective sparsity, a useful concept introduced by Plan and Vershynin in [28].

Definition 3.4 (Effectively Sparse Vectors). Let
\[
K_{n,s} = \{ z \in \mathbb{R}^n : \|z\|_2 \leq 1 \text{ and } \|z\|_1 \leq \sqrt{s} \}.
\]
The set of effectively $s$-sparse vectors of dimension $n$ is defined by $\{ z \in \mathbb{R}^n : \|z\|_1/\|z\|_2 \leq 1 \text{ in } K_{n,s} \}.$

Remark 3.5. Note that any $s$-sparse vector is also effectively $s$-sparse. Effectively sparse vectors are well approximated by sparse vectors as made precise in [28, Lemma 3.2].

Based on the just introduced relaxation of sparsity we can define, for $\Gamma \geq 1$, a new set of matrices by
\[
K^{R,\Gamma}_{s_1,s_2} = \{ Z \in \mathbb{R}^{n_1 \times n_2} : \exists u^1, \ldots, u^R \in K_{n_1,s_1}, v^1, \ldots, v^R \in K_{n_2,s_2}, \text{ and } \sigma = (\sigma_1, \ldots, \sigma_R)^T \in \mathbb{R}^R, \text{ s.t.} \]
\[
\begin{align*}
Z &= \sum_{r=1}^R \sigma_r u^r(v^r)^T, \\
\|u^r\|_2 &= \|v^r\|_2 = 1, \text{ for all } r \in [R], \text{ and } \|\sigma\|_2 \leq \Gamma
\end{align*}
\] (15)

which is a relaxed version of $S^{R,\Gamma}_{s_1,s_2}$ as $S^{R,\Gamma}_{s_1,s_2} \subset K^{R,\Gamma}_{s_1,s_2}$. One of the most important features of the class $K^{R,\Gamma}_{s_1,s_2}$ is that it is to a certain extent closed under summation: in fact if $Z \in K^{R,\Gamma}_{s_1,s_2}$ and $\hat{Z} \in K^{R,\Gamma}_{\hat{s}_1,\hat{s}_2}$ then
\[
Z + \hat{Z} \in K^{2R,\Gamma \sqrt{\Gamma^2 + \Gamma^2}}_{\max\{s_1,\hat{s}_1\}, \max\{s_2,\hat{s}_2\}}.\]

(16)

Let us stress very much that this additivity property is a technical change of perspective, as it would not be possible if we were insisting, as it is done usually in the literature, e.g., [23], on singular value decompositions, whose components need to be orthogonal. We call such an admissible decomposition $Z = U\Sigma V^T$ in (15) an effectively Sparse Decomposition of $Z$ and use the same shorthand notation, i.e., SD. The context makes clear which decomposition is meant. Any $X$ decomposed as in (6) belongs to $K^{R,\Gamma}_{n_1,s_2}$ if $\sum_{r=1}^R \|\hat{u}^r\|_2\|\hat{v}^r\|_2 \leq \Gamma^2$. Having the sets $S^{R,\Gamma}_{s_1,s_2}$ and $K^{R,\Gamma}_{s_1,s_2}$ at hand we now define corresponding RIPS.

Definition 3.6 (Additive Rank-$R$ and (effectively) $(s_1,s_2)$-sparse RIP$_\Gamma$). A linear operator $A : \mathbb{R}^{n_1 \times n_2} \to \mathbb{R}^m$ satisfies the additive rank-$R$ and $(s_1,s_2)$-sparse RIP$_\Gamma$ with isometry constant $\delta > 0$ if
\[
\|\|A(Z)\|_2^2 - \|Z\|_F^2\| \leq \delta,
\] (17)

for all $Z \in S^{R,\Gamma}_{s_1,s_2}$. If (17) holds for all $Z \in K^{R,\Gamma}_{s_1,s_2}$, we say $A$ has the additive rank-$R$ and effectively $(s_1,s_2)$-sparse RIP$_\Gamma$. Note that the rank-$R$ and effectively $(s_1,s_2)$-sparse RIP$_\Gamma$ implies the rank-$R$ and $(s_1,s_2)$-sparse RIP$_\Gamma$ as $S^{R,\Gamma}_{s_1,s_2} \subset K^{R,\Gamma}_{s_1,s_2}$.
Recovery results We are ready now to state the main recovery result: If one assumes RIP, any appropriate global minimizer of \( J_{\alpha,\beta}^R \) provides an approximation to \( \hat{X} \), with an error bound depending on the magnitude of \( \alpha \) and \( \beta \), the sparsity \( s \), the RIP constant \( \delta \), and the magnitude of \( \hat{X} \) measured in an appropriate Schatten quasi-norm. The approximation is worsened in an additive way by noise level.

**Theorem 3.7 (Approximation of \( \hat{X} \)).** Fix the positive constants \( \alpha, \beta > 0 \), \( \Gamma \geq 1 \), and the effective sparsity indicator level \( 1 \leq s \leq n_2 \). Let \( A \) have the additive rank-2R effectively \( (n_1, \max\{s, (\gamma/\beta)^2\}) \)-sparse RIP with RIP-constant \( 0 < \delta < 1 \), for a fixed choice of \( \gamma > 0 \) and \( c \geq 1 \).
If \( \hat{X} \in K_{n_1,3}^R \) of rank \( R \) and \( y = A(X) + \eta \in \mathbb{R}^m \), then
\[
\|\hat{X} - X_{\alpha,\beta}\|_F \leq \sqrt{s^2 R^2 C_{2,1} C_\delta} \sqrt{\alpha \beta^2} \|\hat{X}\|_F + \left( \|\hat{X}\|_F + \|\eta\|_2 + \sqrt{\delta} \right)^2 / \gamma \tag{18}
\]
for any global minimizer \((v^R_{\alpha,\beta}, \ldots, v^R_{\alpha,\beta})\) of \( J_{\alpha,\beta}^R \) that fulfills \( \|v^R_{\alpha,\beta}\|_2 \geq \|X\|_F \) or \( \|\eta\|_2 + \sqrt{\delta} \) for all \( r \in |R| \) and \( \|\sigma_{\alpha,\beta}\|_F \leq c \Gamma \) in \([11]\). In this case, in particular, \( X_{\alpha,\beta} \in K_{n_1,3}^{R,\gamma^2} \) with the SD in \([11]\).

There are some aspects of this result we would like to discuss before we proceed:
(a) If we could take the limits \( \alpha \to 0 \) and \( \beta \to 0 \), the error in \([18]\) would vanish up to noise-level and RIP-constant. However, this limit cannot be performed as there are important restrictions dictated by the need of fulfilling simultaneously the RIP and the assumptions on \( X_{\alpha,\beta} \). If \( \beta \) is getting small the conditions for having RIP degenerate, i.e., reconstruction for a fixed number of measurements only works up to a minimal \( \beta \). Letting \( \alpha \) to zero while keeping \( \beta \) fixed leads to minimizers which violate the lower bound \( \|v^R_{\alpha,\beta}\|_2 \geq \|\hat{X}\|_F + \|\eta\|_2 + \sqrt{\delta} \) or the upper bound \( \|\sigma_{\alpha,\beta}\|_F \leq c \Gamma \). To see this, note that by Lemma 3.2 small \( \alpha \) leads to strict bounds on \( \|v^R_{\alpha,\beta}\|_2 \) and weak bounds on \( \|\sigma_{\alpha,\beta}\|_F \).
(b) Let us mention that in case \( \hat{X} \in K_{n_1,3}^{R,\gamma^2} \) and the SD of \( \hat{X} \) coincides with its SVD, then in view of the identity \([8]\) the factor \( c \sqrt{n R^2} \) in the error estimates \([18]\) and \([19]\) can be substituted by 1, hence there would be no dependence on the rank \( R \).
(c) In order to clarify how \( (\gamma/\beta)^2 \) and \( s \) are related in the RIP in Theorem 3.7 (and Corollary 3.8 below), let us assume for simplicity that the SD of \( \hat{X} \) coincides with its SVD and \( \alpha = \beta \). Consequently, to get an error bound independent of \( s \) in \([18]\), \( \alpha \) and \( \beta \) have to be chosen of order \( O(s^{-\frac{1}{2}}) \), i.e., \( (\gamma/\beta)^2 \) is of order \( O(s^2) \) which means that an \((n_1, \gamma s)-\)sparse RIP \((c+1)\Gamma \) is sufficient for recovery.
(d) The result only applies to minimizers whose scaling matrix \( \Sigma_{\alpha,\beta} \) is bounded in Frobenius norm and whose right components \( v^R_{\alpha,\beta} \) are not too close to zero. The first requirement is necessary as the RIP is restricted to SDs with scaling matrices within a ball around zero. The second one is needed to show some level of effective sparsity of the minimizers \( X_{\alpha,\beta} \) (see also the discussion in Section 3.1). While effective sparsity of (right) component vectors of \( X_{\alpha,\beta} \) is naturally wished and expected if \( \hat{X} \in K_{n_1,3}^{R,\gamma^2} \), we were not able in all cases to show exact sparsity of (right) component vectors of \( X_{\alpha,\beta} \) if \( \hat{X} \in S_{n_1,3}^{R,\Gamma} \), but again only their effective sparsity. Hence, we are bound to using as an artifact of the proof the stronger effectively \((s_1, s_2)-\)sparse RIP \( \Gamma \) for theoretical analysis also in this case. In numerical experiments, however, for \( \hat{X} \in S_{n_1,3}^{R,\Gamma} \) the obtained minimizers \( X_{\alpha,\beta} \) are empirically exactly sparse (not just effectively sparse) and, hence, the weaker rank-2R \((s_1, s_2)-\)sparse RIP \( \Gamma \) might suffice in practice. The latter can already be guaranteed for a smaller number of measurements.
(e) As argued in Section 4.2 the above theorem can be straightforwardly extended to sparsity on left component vectors. In this case \( J_{\alpha,\beta}^R \) has to be adapted by considering \( \ell_1 \)-norm penalties on the \( u \)-components.
(f) It is important to require \( \text{rank}(\hat{X}) = R \) as otherwise the equivalence of Schatten-norm and normed SD cannot be guaranteed as \([9]\). If the SD of \( \hat{X} \) coincides with its SVD though, the rank condition may be dropped.

By choosing \( \alpha \) and \( \beta \) in relation to the noise-to-signal ratio \( \|\eta\|_2^2 / \|\hat{X}\|_F^2 \) we obtain the following version of Theorem 3.7 which has the form of a typical compressed sensing recovery bound. Assuming the RIP, the approximation error is linear in noise level while the slope of the linear function depends on sparsity level and possibly the rank. However, peculiarly, for fixed number of measurements the RIP fails for exceedingly small noise. Hence, the result is valid only for sufficiently small signal-to-noise ratio. As we will show in
Section 5 with numerical experiments, this apparently counterintuitive result is factual and not an artifact of the proof technique. A possible intuitive explanation is that \( J_{R_{\alpha,\beta}}^R \) becomes a mere least-squares without sparsifying effect for \( \alpha \) and \( \beta \) close to zero, which is caused by vanishing noise.

**Corollary 3.8.** Let \( \hat{X} \in K_{n_1,s}^{R_{\alpha,\beta}} \) with \( \text{rank}(\hat{X}) = R \) fulfill the noisy measurements \( y = A(\hat{X}) + \eta \) and let \( \alpha = \beta = \frac{\|\eta\|^2}{\|\hat{X}\|^2} < 1 \). Assume \( A \) has for some \( \gamma > 0 \) and \( c \geq 1 \) the additive rank-2R effectively \( (n_1, \max\{s, \gamma^2(\|\hat{X}\|^2/\|\eta\|^2)^2\} ) \)-sparse RIP-constant \( 0 < \delta < 1 \). Then, for \( X_{\alpha,\beta} \) with \( \|\Sigma_{\alpha,\beta}\|_F \leq c\Gamma \) and \( \|v_{\alpha,\beta}\|_2 \geq (\|\hat{X}\|_F + \|\eta\|_2 + \sqrt{\delta}^2)/\gamma \), \( r \in [R] \), we have

\[
\|\hat{X} - X_{\alpha,\beta}\|_F \leq \left( 2\sqrt{c_0 R^{2/3}s^{1/3}} + 2 \right) \|\eta\|_2 + \sqrt{\delta}. \tag{19}
\]

**Remark 3.9.** One could object that the simple zero solution \( \hat{X} = 0 \) is already a competitor in case of large noise \( \|\eta\|_2 \geq \Xi(m)\|\hat{X}\|_F \), i.e.,

\[
\|\hat{X} - \hat{X}\|_F \leq \Xi(m)^{-1}\|\eta\|_2. \tag{20}
\]

However, for a larger number \( m \) of measurements we can consider lower level of noise, i.e., \( \Xi(m) \to 0 \) and the bound \( \|\hat{X} - \hat{X}\|_F \leq \Xi(m)^{-1}\|\eta\|_2 \) would explode, while \( \|\hat{X} - \hat{X}\|_F \) would remain effective. Moreover, our numerical experiments show empirically that also in case of larger noise level, computing \( X_{\alpha,\beta} \) gives a solution, which outperforms not only trivial competitors as \( \hat{X} \), but also state-of-the-art methods such as SPF.

### 3.3 RIP Results for Subgaussian Operators

As already mentioned above, a linear operator \( A \) of the form \( \hat{A} \) which is drawn from a subgaussian distribution fulfills the above introduced RIPs with high probability. This is stated in the following Lemma. We first recall the definition of subgaussian random variables (for further details see [33]).

**Definition 3.10 (Subgaussian Random Variable).** A random variable \( \xi \in \mathbb{R} \) is called \( K \)-subgaussian if the tail bound \( \Pr(|\xi| > t) \leq C \exp(-ct^2/K^2) \) holds where \( c, C > 0 \) are absolute constants. The smallest possible number for \( K > 0 \) is called subgaussian norm of \( \xi \) and denoted by \( \|\xi\|_{\psi_2} \).

**Remark 3.11.** The class of subgaussian random variables covers important special cases as Gaussian, Bernoulli, and more generally all bounded random variables (see [23]).

**Lemma 3.12 (RIP for Subgaussian Operators).** Let \( \Gamma \geq 1 \) and let \( A: \mathbb{R}^{n_1 \times n_2} \to \mathbb{R}^m \) be the linear measurement operator of form \( \hat{A} \). Assume, all \( A_i \), for \( 1 \leq i \leq m \), have i.i.d. \( K \)-subgaussian entries \( a_{i,j,k} \) with mean 0 and variance 1. If

\[
m \gtrsim \left( \frac{\delta}{\Gamma^2R} \right)^{-2} R(s_1 + s_2) \log(\max\{n_1, n_2\}) \tag{21}
\]

then \( A \) has the additive rank-\( R \) and \( (s_1, s_2) \)-sparse RIP\(_\Gamma \) with isometry constant \( \delta \in (0, \Gamma^2R) \) with probability at least \( 1 - 2\exp(-C(\delta/\Gamma^2R)m) \) where \( C > 0 \) is a constant depending on \( K \). If

\[
m \gtrsim \left( \frac{\delta}{\Gamma^2R} \right)^{-2} R(s_1 + s_2) \log^3(\max\{n_1, n_2\}) \tag{22}
\]

then \( A \) has the additive rank-\( R \) and effectively \( (s_1, s_2) \)-sparse RIP\(_\Gamma \) with isometry constant \( \delta \in (0, \Gamma^2R) \) with probability at least \( 1 - 2\exp(-C'(\delta/\Gamma^2R)m) \) where \( C' > 0 \) is a constant depending on \( K \).

**Remark 3.13.** Lemma 3.12 states, for \( \delta = \Delta(\Gamma^2R), \Delta \in (0, 1) \), that, up to log-factors, \( m \approx \mathcal{O} \left( \Delta^{-2}R(s_1 + s_2) \right) \) subgaussian measurements are sufficient to have \( \delta \)-stable embeddings of \( S_{n_1,s_2}^{R,\Gamma} \) and \( K_{n_1,s_2}^{R,\Gamma} \) (cf. [29, Def. 1.1 Thm. 1.5]). Note that \( \Gamma^2R \) is the squared Frobenius diameter of \( S_{n_1,s_2}^{R,\Gamma} \) and \( K_{n_1,s_2}^{R,\Gamma} \). As we restrict ourselves below to \( s \)-effective sparse right component vectors of \( \hat{X} \), we only use the rank-\( R \) and (effectively) \( (n_1, s) \)-sparse RIP\(_\Gamma \). For the presented results to have some meaning, a typical dimensional setting is \( R \ll s \approx n_1 \ll n_2 \). In fact, if \( n_1 \) were close to \( n_2 \) in magnitude, the sparsity \( s \) of the right component vectors would not be useful
to reduce the order of the measurements \( m \approx O(R(n_1 + s)) \approx O(R(n_1)) \approx O(R(n_2)) \). Moreover, if \( R \) were close to \( n_1 \), the matrix would not be low-rank as \( n_1 \) would be the maximal possible rank.

In [23] the authors give information theoretical lower bounds on the necessary number of measurements for reconstructing low-rank matrices with sparse singular vectors sharing a common support, namely \( m \gtrsim R(s_1 + s_2) \). As we do not require orthogonality of SDs in \( S^{R,\Gamma}_{s_1,s_2} \) resp. \( K^{R,\Gamma}_{s_1,s_2} \) (excluding a scaling invariant RIP which is independent of the set diameter, see Remark 3.14), the bounds in [21] and [22] are up to log-factors at the information theoretic limit for the class of matrices in [23]. We are not aware of any information theoretical lower bounds for the more general class of matrices considered in the present paper.

**Remark 3.14.** The additive RIP in (17) differs from the commonly used multiplicative RIPS of the form

\[
(1 - \delta) \|Z\|_F^2 \leq \|A(Z)\|_2^2 \leq (1 + \delta) \|Z\|_F^2
\]

as it is not scaling invariant and \( A(Z) = A(Z') \) does not imply \( Z = Z' \) but only \( \|Z - Z'\|_2^2 \leq \delta \). In fact it is not possible to derive a classical scaling invariant RIP like (23) on \( K^{R,\Gamma}_{s_1,s_2} \) under similar conditions as [22].

The main problem is non-orthogonality of the SD. A simple example illustrates this point: Assume \( R = 2 \) and \( m \approx 2(n_1 + s) \log^3(\max\{n_1, n_2\}) \) and the linear operator \( A \) fulfills (23) for all \( Z \in K^{2,1}_{n_1,s} \). Choose some \( u \in \mathbb{R}^{n_1}, v_1 \in \mathbb{R}^{n_2} \) of unit norm and \( |v_1| \leq \sqrt{s}/2 \). Define \( v_2 := -v_1 + \varepsilon w \) for any \( w \in \mathbb{R}^{n_2} \) and choose \( \varepsilon > 0 \) sufficiently small to ensure \( \|v_2\|_1 \leq \sqrt{s} \) and \( \|v_2\|_2 \approx 1 \). Then \( Z := (1/2)uw^T + (1/2)uv_2^T \in K^{2,1}_{n_1,s} \) and (23) holds. But this implies by definition of \( Z \) and scaling invariance of (23) that

\[
(1 - \delta) \|uw^T\|_F^2 \leq \|A(uw^T)\|_2^2 \leq (1 + \delta) \|uw^T\|_F^2
\]

which means the RIP directly extends to all rank-1 matrices (not only those with sparse right component). If \( n_1, s \ll n_2 \), this is a clear contradiction to information theoretical lower bounds, as corresponding RIPS would require at least \( m \approx \max\{n_1, n_2\} \) (see [2] Section 2.1).

### 3.4 Convergence of ATLAS

So far an important question has not been posed. The above results only apply to global minimizers of \( J^{R}_{\alpha,\beta} \) which is a highly non-convex functional. One wonders if the alternating minimization defined in [10] is able to provide minimizers. We give a partial answer to this issue. By adapting results of Attouch et al. in [2] we show convergence of ATLAS and that there is a neighborhood \( U(u^0_{\alpha,\beta},...,v^R_{\alpha,\beta}) \) of a global minimizer \((u^1_{\alpha,\beta},...,v^R_{\alpha,\beta})\) such that the sequence \((u^1,...,v^r)\) defined by [10] converges to \((u^1_{\alpha,\beta},...,v^R_{\alpha,\beta})\) of \( J^{R}_{\alpha,\beta} \) if the initialization lies within \( U(u^0_{\alpha,\beta},...,v^R_{\alpha,\beta}) \). However, we do not give proof for any initialization to fulfill the requirement. This is an open issue for future research, but recent promising results [12] may shed light on how to attack this problem also for ATLAS. The techniques in [2] also might be adjusted for an analysis of rate of convergence of ATLAS, but this would go beyond the scope of this work and is a topic for future investigation. We begin by a generalization of the basic conditions of [2]. Let \( L \) be a functional of the following form:

\[
(H) \quad \begin{cases}
L(u^1,...,u^R,\ldots,v^1,...,v^R) = \sum_{r=1}^{R} f_r(u^r) + Q(u^1,...,u^R) + \sum_{r=1}^{R} g_r(v^r), \\
f_r : \mathbb{R}^{n_1} \to \mathbb{R} \cup \{\infty\}, \ g_r : \mathbb{R}^{n_2} \to \mathbb{R} \cup \{\infty\} \text{ are proper lower semicontinuous, for } 1 \leq r \leq R, \\
Q : \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_2} \to \mathbb{R} \text{ is a } C^1 \text{ function,} \\
\nabla Q \text{ is Lipschitz continuous on bounded subsets of } \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_2}.
\end{cases}
\]

For given \((u^0_1,...,u^0_R) \in (\mathbb{R}^{n_1})^R \times (\mathbb{R}^{n_2})^R \) and fixed sequences \((\lambda^1_k)_{k \in \mathbb{N}},\ldots,(\lambda^R_k)_{k \in \mathbb{N}},(\mu^1_k)_{k \in \mathbb{N}},\ldots,(\mu^R_k)_{k \in \mathbb{N}}\) assume that

\[
(H1) \quad \begin{cases}
\inf L > -\infty, \\
L(\cdot, u^0_1,...,u^0_R) \text{ is proper,} \\
\text{for some positive } r_- < r_+ \text{ the sequences } \lambda^1_k,\ldots,\mu^R_k \text{ belong to } (r_-,r_+).
\end{cases}
\]
The adapted main result of [2] now guarantees convergence of the so-called Proximal Alternating Minimization

\[
\begin{cases}
    u_{k+1}^j = \underset{u \in \mathbb{R}^n_+}{\arg \min} L(u, u_{k}^2, \ldots, u_{k}^R, v_{k}^1, \ldots, v_{k}^R) + \frac{1}{2\mu_k} ||u - u_{k}^j||_2^2, \\
v_{k+1}^j = \underset{v \in \mathbb{R}^n_+}{\arg \min} L(u_{k+1}^1, u_{k}^2, \ldots, u_{k}^R, v, v_{k}^2, \ldots, v_{k}^R) + \frac{1}{2\mu_k} ||v - v_{k}^j||_2^2, \\
    \vdots \\
    u_{k+1}^R = \underset{u \in \mathbb{R}^n_+}{\arg \min} L(u_{k+1}^1, \ldots, u_{k+1}^{R-1}, u, v_{k+1}^1, \ldots, v_{k+1}^R) + \frac{1}{2\mu_k} ||u - u_{k}^R||_2^2, \\
v_{k+1}^R = \underset{v \in \mathbb{R}^n_+}{\arg \min} L(u_{k+1}^1, \ldots, u_{k+1}^{R-1}, v_{k+1}^1, \ldots, v_{k+1}^R) + \frac{1}{2\mu_k} ||v - v_{k}^R||_2^2,
\end{cases}
\]

(24)
to a stationary point of $L$ (resp. convergence to a global minimizer $(u_1^*, \ldots, v_R^*)$ of $L$ if the initialization $(u_0^1, \ldots, v_0^R)$ of (PAM) lies sufficiently close to $(u_1^*, \ldots, v_R^*)$) if $L$ fulfills (H), (H1) and the so called Kurdyka-
Lojasiewicz Property, which requires $L$ to behave well around stationary points.

Definition 3.15 (Kurdyka-Lojasiewicz Property). A proper lower semicontinuous function $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is said to have the KL-property at $\bar{x} \in \text{dom} \partial f[\Sigma]$ if there exist $\eta \in (0, \infty)$, a neighborhood $U$ of $\bar{x}$ and a continuous concave function $\varphi : [0, \infty) \to \mathbb{R}_+$ such that
- $\varphi(0) = 0$,
- $\varphi(t) > 0$, for all $t \in (0, \eta)$,
- and, for all $x \in U \cap \{x \in \mathbb{R}^n : f(x) < f(\bar{x}) < f(x) + \eta\}$, the KL-inequality holds:
\[
\varphi(f(x) - f(\bar{x})) \text{dist}(0, \partial f(x)) \geq 1.
\]

Theorem 3.16 (Local Convergence to Global Minimizers). Assume that $L$ satisfies (H), (H1). If $L$ has the Kurdyka-Lojasiewicz property at its global minimizer $(u_1^*, \ldots, v_R^*)$, then there exist $\varepsilon, \eta > 0$, such that the initial conditions
\[
\|(u_0^1, \ldots, v_0^R) - (u_1^*, \ldots, v_R^*)\|_2 < \varepsilon, \quad \min L < L(u_0, v_0) < \min L + \eta,
\]

imply that the iterations $(u_1^1, \ldots, v_R^R)$ generated by (PAM) converge to $(u_1^*, \ldots, v_R^*) \in \arg \min L$. If $L$ has the Kurdyka-Lojasiewicz at each point of its domain, then either $\|(u_1^1, \ldots, v_R^R)\|_2 \rightarrow \infty$ or $(u_k^1, \ldots, v_k^R)$ converges to a stationary point of $L$.

By applying Theorem 3.16 to $L = J_{\alpha, \beta}^R$ and ATLAS we obtain convergence to stationary points and local convergence to global minimizers as the sequence $(u_k^1, \ldots, v_k^R)$ is bounded by coercivity of $J_{\alpha, \beta}^R$. One can check that conditions (H), (H1) are fulfilled by $J_{\alpha, \beta}^R$ and ATLAS for a suitable choice of the sequences $(\lambda_k^1)_{k \in \mathbb{N}}, \ldots, (\lambda_k^R)_{k \in \mathbb{N}}, (\mu_k^1)_{k \in \mathbb{N}}, \ldots, (\mu_k^R)_{k \in \mathbb{N}}$. It remains to validate the KL-property. As mentioned in [2] Section 4.3], all semialgebraic functions satisfy the KL-property at each point with $\varphi(t) = ct^{1-\theta}$ for some $\theta \in [0, 1) \cap \mathbb{Q}$ and $c > 0$. Hence, by showing that $J_{\alpha, \beta}^R$ is semialgebraic, we get the KL-property for free. But we pay the price of having no better knowledge on the parameters $\varepsilon$ and $\eta$ in Theorem 3.16 which characterize the convergence radius. Therefore, let us conclude by showing that $J_{\alpha, \beta}^R$ is semialgebraic, i.e., graph$(J_{\alpha, \beta}^R) \subset \mathbb{R}^{Rn_1+Rn_2} \times \mathbb{R}$ is a semialgebraic set.

A set in $\mathbb{R}^d$ is called semialgebraic if it can be written as a finite union of sets of the form
\[
\{x \in \mathbb{R}^d : p_i(x) = 0, q_i(x) > 0, i = 1, \ldots, p\},
\]
where $p_i, q_i$ are real polynomials. First, the absolute value of one component of a vector $\hat{h}(x) := |x_i|$ is a semialgebraic function as
\[
\text{graph}(\hat{h}) = \{(x, r) \in \mathbb{R}^d \times \mathbb{R} : x_i + r = 0, x_i < 0 \} \cup \{(x, r) \in \mathbb{R}^d \times \mathbb{R} : x_i = 0, r = 0 \} \cup \{(x, r) \in \mathbb{R}^d \times \mathbb{R} : x_i - r = 0, -x_i < 0 \}.
\]

Here $\partial f$ denotes the subdifferential of $f$ and dom $\partial f$ the domain on which $\partial f$ takes finite values.
Second, it is clear that polynomials $p$ are semialgebraic as \( \text{graph}(p) = \{(x,r) \in \mathbb{R}^d \times \mathbb{R} : p(x) - r = 0\} \) and, third, composition, finite sums and finite products of semialgebraic functions are semialgebraic. The semialgebraicity of \( J_{\alpha,\beta}^R \) follows as
\[
J_{\alpha,\beta}^R(u^1, \ldots, v^R) = \sum_{l=1}^m |y_l| - \sum_{r=1}^R (A_l, u^r v^T_T) f_r^2 + \alpha \sum_{r=1}^R \sum_{l=1}^{n_{1r}} |u^r_l|^2 + \beta \sum_{r=1}^R \sum_{l=1}^{n_{2r}} |v^r_l|^2
\]
is just a finite composition of semialgebraic basic units.

4 Proofs

This section provides proofs for the main results from Section 3. Some merely technical parts are moved to the Appendix to ease the reading. We begin by showing the general properties of global minimizers \((u^1_{\alpha,\beta}, \ldots, v^R_{\alpha,\beta})\) of \( J_{\alpha,\beta}^R \) and proving Theorem 3.7. Then, we present the proof of Lemma 3.12. The proof of Theorem 3.16 can be found in the Appendix, as it is based on straightforward modifications of the arguments in [2].

4.1 Bounds on Minimizers

Recall the SD related representation \( \hat{X} = \sum_{r=1}^R \hat{u}r(\hat{v}r)^T \) in (6) where \( \hat{\sigma}_r = \|\hat{u}r\|_2\|\hat{v}r\|_2 \) and the notation \( X_{\alpha,\beta} = \sum_{r=1}^R u^r_{\alpha,\beta}(v^r_{\alpha,\beta})^T \). For proving Proposition 3.1 and Lemma 3.2 we need following technical lemma.

Lemma 4.1. Let \( \alpha, \beta, a, b, p, q > 0 \). Then
\[
f : \mathbb{R}^+ \to \mathbb{R}, \quad f(\lambda) := \lambda^p a + \frac{1}{\lambda^q} b,
\]
attains its minimum at \( \hat{\lambda} = \left(\frac{a \beta b}{p \alpha a}\right)^{\frac{1}{p+q}} \) and has the minimal value
\[
\min f = f(\hat{\lambda}) = C_{p,q}(\alpha a)^{\frac{q}{p+q}} (\beta b)^{\frac{p}{p+q}},
\]
where \( C_{p,q} = \left(\frac{a}{p}\right)^{\frac{q}{p+q}} + \left(\frac{b}{q}\right)^{\frac{p}{p+q}} \).

Proof of Lemma 4.1: The result is obtained by differentiation of \( f \) and by searching for its derivative’s zeros.

Proof of Proposition 3.1: By applying Lemma 4.1 \( R \) times using \( p = 2, q = 1, a = \|\hat{u}r\|^2_2, b = \|\hat{v}r\|_1 \) we get \( \hat{\lambda}_1, \ldots, \hat{\lambda}_R \), such that
\[
J_{\alpha,\beta}(\hat{\lambda}_1 \hat{u}^1, \ldots, \hat{\lambda}_R \hat{u}^R, \frac{1}{\hat{\lambda}_1} \hat{v}^1, \ldots, \frac{1}{\hat{\lambda}_R} \hat{v}^R) = \|y - A(\hat{X})\|^2_2 + \sum_{r=1}^R C_{2,1} \sqrt[4]{\alpha \beta^2} \sqrt[3]{\|\hat{u}r\|_2^2\|\hat{v}r\|_1^3}
\]
\[
= \|\eta\|_2^2 + C_{2,1} \sqrt[4]{\alpha \beta^2} \sum_{r=1}^R \left(\|\hat{u}r\|_2\|\hat{v}r\|_1\right)^{\frac{3}{2}}.
\]
Note that, although not explicitly labeled, each \( \hat{\lambda}_r \) depends on the choice of \( \alpha \) and \( \beta \) as well as on \( a, b, p, \) and \( q \). The minimality of \((u^1_{\alpha,\beta}, \ldots, v^R_{\alpha,\beta})\) implies
\[
\|y - A(X_{\alpha,\beta})\|^2_2 \leq J_{\alpha,\beta}(u^1_{\alpha,\beta}, \ldots, v^R_{\alpha,\beta}) \leq J_{\alpha,\beta}(\hat{\lambda}_1 \hat{u}^1, \ldots, \hat{\lambda}_R \hat{u}^R, \frac{1}{\hat{\lambda}_1} \hat{v}^1, \ldots, \frac{1}{\hat{\lambda}_R} \hat{v}^R)
\]
\[
= \|\eta\|_2^2 + C_{2,1} \sqrt[4]{\alpha \beta^2} \sum_{r=1}^R \left(\|\hat{u}r\|_2\|\hat{v}r\|_1\right)^{\frac{3}{2}}
\]
which is the claim.
The proof of Lemma 3.2 works in a similar way.

**Proof of Lemma 3.2:** From (25) in the proof of Proposition 3.1 we obtain
\[ \|y - A(X_{\alpha,\beta})\|_2^2 + \sum_{r=1}^R (\alpha\|u_{\alpha,\beta}^r\|_2^2 + \beta\|v_{\alpha,\beta}^r\|_1) = J_{\alpha,\beta}^R(u_{\alpha,\beta}^1, ..., v_{\alpha,\beta}^R) \]
\[ \leq J_{\alpha,\beta}^R(\tilde{\lambda}_1 \hat{u}^1, ..., \tilde{\lambda}_R \hat{u}^R, 1 \frac{1}{\lambda_1}, ..., 1 \frac{1}{\lambda_R}) \]
\[ = \|\eta\|_2^2 + C_{2,1} \sqrt{\alpha\beta^2} \sum_{r=1}^R (\|\hat{u}^r\|_2\|\hat{v}^r\|_1)\frac{2}{3} \]

The first part of the claim follows by subtracting \(\|y - A(X_{\alpha,\beta})\|_2^2\) on both sides, leaving out half of the terms on the left-hand side, and dividing by \(\alpha\) (resp. \(\beta\)). To show the second part, note that by minimality of \((u_{\alpha,\beta}^1, ..., v_{\alpha,\beta}^R)\) and Lemma 4.1
\[ \sum_{r=1}^R (\alpha\|u_{\alpha,\beta}^r\|_2^2 + \beta\|v_{\alpha,\beta}^r\|_1) = C_{2,1} \sqrt{\alpha\beta^2} \sum_{r=1}^R (\|u_{\alpha,\beta}^r\|_2\|v_{\alpha,\beta}^r\|_1)\frac{2}{3} \]
and hence
\[ \|y - A(X_{\alpha,\beta})\|_2^2 + C_{2,1} \sqrt{\alpha\beta^2} \sum_{r=1}^R (\|u_{\alpha,\beta}^r\|_2\|v_{\alpha,\beta}^r\|_1)\frac{2}{3} = J_{\alpha,\beta}^R(u_{\alpha,\beta}^1, ..., v_{\alpha,\beta}^R) \]
\[ \leq J_{\alpha,\beta}^R(\tilde{\lambda}_1 \hat{u}^1, ..., \tilde{\lambda}_R \hat{u}^R, 1 \frac{1}{\lambda_1}, ..., 1 \frac{1}{\lambda_R}) \]
\[ = \|\eta\|_2^2 + C_{2,1} \sqrt{\alpha\beta^2} \sum_{r=1}^R (\|\hat{u}^r\|_2\|\hat{v}^r\|_1)\frac{2}{3} \].

Subtracting \(\|y - A(X_{\alpha,\beta})\|_2^2\) on both sides and dividing by \(C_{2,1} \sqrt{\alpha\beta^2}\) concludes the proof.

To show the effective sparsity as in Lemma 3.3 we combine the fact that \(X_{\alpha,\beta}\) is a minimizer with the assumed lower bound on \(v_{\alpha,\beta}^r\).

**Proof of Lemma 3.3:** By comparing \(J_{\alpha,\beta}^R(u_{\alpha,\beta}^1, ..., v_{\alpha,\beta}^R)\) to \(J_{\alpha,\beta}^R(0, ..., 0)\), we get
\[ \sum_{r=1}^R (\alpha\|u_{\alpha,\beta}^r\|_2^2 + \beta\|v_{\alpha,\beta}^r\|_1) \leq J_{\alpha,\beta}^R(u_{\alpha,\beta}^1, ..., v_{\alpha,\beta}^R) \leq J_{\alpha,\beta}^R(0, ..., 0) = \|y\|_2^2 \]
This implies \(\|v_{\alpha,\beta}^r\|_1 < \|y\|_2^2/\beta\). As by assumption \(\|v_{\alpha,\beta}^r\|_2 \geq \|y\|_2^2/\gamma\), we conclude
\[ \frac{\|v_{\alpha,\beta}^r\|_1}{\|v_{\alpha,\beta}^r\|_2} < \frac{\|y\|_2^2}{\beta} \frac{\gamma}{\|y\|_2^2} = \frac{\gamma}{\beta}. \]

### 4.2 Proof of Theorem 3.7
We have now all necessary tools at hand to prove our main approximation result. Most of the technical work has been already presented in Proposition 3.1 and Lemma 3.3. By combining the RIP with the above bounds on norms and sparsity of minimizers, we can estimate the worst-case distance between \(X\) and \(X_{\alpha,\beta}\) depending on the size of \(\alpha\) and \(\beta\), the sparsity \(s\), the RIP constant \(\delta\), and the size of \(X\) measured in a Schatten quasi-norm.

As the reader may notice, all technical results of Section 4.1 can be adapted to effective sparsity of the left components \((u_{\alpha,\beta}^1, ..., u_{\alpha,\beta}^R)\) as well. This can be done by replacing \(\ell_2\)-norms by corresponding \(\ell_1\)-norms in \(J_{\alpha,\beta}^R\). The proof Lemma 3.3 which guarantees effective sparsity of the right components, is independent of the minimization of the left components. Therefore, Lemma 3.3 applies also to the left components if \(\ell_2\)-norms are replaced by \(\ell_1\)-norms in \(J_{\alpha,\beta}^R\). Theorem 3.7 then can be adapted to this setting in a straightforward way.
**Proof of Theorem 3.7:** As \( \|y\|_2 \leq \|A(\hat{X})\|_2 + \|\eta\|_2 \leq (\|X\|_F + \sqrt{\delta}) + \|\eta\|_2 \), Lemma 3.3 applies and yields that \( X_{\alpha,\beta} \) is in \( K_{R_n,\ell^1(\gamma/\beta)^2} \). Combined with \( \hat{X} \in K_{R_n,\ell^1(\gamma/\beta)^2} \), we know from \([16]\) that the difference \( \hat{X} - X_{\alpha,\beta} \in K_{R_n,\max(s,\gamma/\beta)^2} \). Hence, we apply the rank-\( 2R \) and effectively \( (n_1, \max\{s, \gamma/\beta\}) \)-sparse RIP\(_{(\epsilon+1)R} \) of \( \mathcal{A} \) to obtain (note that \( |a^2 - b^2| \leq \delta \) implies \( |a - b| \leq \sqrt{\delta} \), for \( a, b > 0 \))

\[
\|\hat{X} - X_{\alpha,\beta}\|_F \leq \|A(\hat{X}) - A(X_{\alpha,\beta})\|_2 + \sqrt{\delta} \leq (\|y - A(X_{\alpha,\beta})\|_2 + \|\eta\|_2) + \sqrt{\delta}
\]

\[
\leq \sqrt{s^2 R^2 C_{2,1} c_D \frac{1}{\gamma} \alpha \beta^2 \|\hat{X}\|_2 + \|\eta\|_2 + \|\eta\|_2} + \sqrt{\delta}
\]

\[
\leq \sqrt{s^2 R^2 C_{2,1} c_D \frac{1}{\gamma} \alpha \beta^2 \|\hat{X}\|_2 + 2\|\eta\|_2} + \sqrt{\delta}.
\]

In the third inequality we used Proposition 3.1 in combination with \( \|\hat{\sigma}\|_1 \leq \sqrt{s}\|\hat{\sigma}\|_2 \) and

\[
\sum_{r=1}^{R} (\|\hat{\sigma}^r\|_2 \sqrt{\|\hat{\sigma}^r\|_2}) \leq s^2 \sum_{r=1}^{R} (\|\hat{\sigma}^r\|_2 \sqrt{\|\hat{\sigma}^r\|_2}) \leq C_D s^2 \frac{1}{\gamma} \hat{X} \|_2
\]

where we used again \([\theta]\) for \( p = 2/3 \).

\[\blacksquare\]

### 4.3 Proof of Lemma 3.12

For proving Lemma 3.12, we need bounds on the covering numbers of \( S_{s_1,s_2}^{R,0} \) and \( K_{s_1,s_2}^{R,0} \). The covering number \( N(M, \| \cdot \|, \varepsilon) \) of a set \( M \) is the minimal number of \( \| \cdot \| \)-balls of radius \( \varepsilon \) that are needed to cover the set \( M \) completely. The cardinality of any \( \varepsilon \)-net \( M \) of \( M \), i.e., for all \( z \in M \) there is \( \hat{z} \in M \) with \( \|z - \hat{z}\| < \varepsilon \), yields an upper bound for \( N(M, \| \cdot \|, \varepsilon) \). The bound for \( N(S_{s_1,s_2}^{R,0}, \| \cdot \|, \varepsilon) \) below is an adaption of Lemma 3.1 in \([17]\) and its proof can be found in the Appendix.

**Lemma 4.2 (Covering Number for Low-Rank Matrices with Sparse Rank-R Decomposition).** Let \( S_{s_1,s_2}^{R,0} \) be the set defined in \([14]\). Then, for all \( 0 < \varepsilon < 1 \), one has

\[
\log(N(S_{s_1,s_2}^{R,0}, \| \cdot \|_F, \varepsilon)) \leq R(s_1 + s_2 + 1) \log \left( \frac{18R \varepsilon}{\varepsilon} \right) + Rs_1 \log \left( \frac{c_{s_1}}{s_1} \right) + Rs_2 \log \left( \frac{c_{s_2}}{s_2} \right).
\]

(26)

To derive a similar bound on \( N(K_{s_1,s_2}^{R,0}, \| \cdot \|_F, \varepsilon) \) we need information on the covering number of the set of effectively \( s \)-sparse vectors \( K_{n,s} \subset \mathbb{R}^n \). Plan and Vershynin derived several interesting properties of \( K_{n,s} \) in \([28]\). Among those \([28]\) Lemma 3.4] gives the following bound for \( N(K_{n,s}, \| \cdot \|_2, \varepsilon) \).

**Lemma 4.3.** For \( 0 < \varepsilon < 1 \) the covering number of \( K_{n,s} \) is bounded by

\[
\log(N(K_{n,s}, \| \cdot \|_2, \varepsilon)) \leq \begin{cases} n \log \left( \frac{\varepsilon}{2s} \right) & 0 < \varepsilon < 2\sqrt{\frac{n}{n}}, \\ \log \left( \frac{9n}{s} \right) & 2\sqrt{\frac{n}{n}} \leq \varepsilon < 1. \end{cases}
\]

**Lemma 4.4 (Covering Number for Matrices with Effectively Sparse Decomposition).** Let \( K_{s_1,s_2}^{R,0} \) be the set defined in \([15]\). Assume \( w.l.o.g. \) that \( s_1/n_1 \leq s_2/n_2 \). Then, for all \( 0 < \varepsilon < 1 \), one has

\[
\log(N(K_{s_1,s_2}^{R,0}, \| \cdot \|_F, \varepsilon)) \leq \begin{cases} \frac{R(n_1 + n_2 + 1)}{\varepsilon} \log \left( \frac{36R \varepsilon}{\varepsilon} \right) & 0 < \varepsilon < 12\varepsilon \sqrt{\frac{Rs_2}{n_2}}, \\ 144R^2 s_1 \log \left( \frac{9n_1}{61\varepsilon R s_1} \right) + R(n_2 + 1) \log \left( \frac{36R \varepsilon}{\varepsilon} \right) & 12\varepsilon \sqrt{\frac{Rs_2}{n_2}} \leq \varepsilon < 12\varepsilon \sqrt{\frac{Rs_2}{n_2}}, \\ 144R^2 s_1 (s_1 + s_2) \log \left( \frac{9n_1}{61\varepsilon R s_1} \right) + R \log \left( \frac{18R \varepsilon}{\varepsilon} \right) & 12\varepsilon \sqrt{\frac{Rs_2}{n_2}} \leq \varepsilon < 1. \end{cases}
\]

(27)

**Proof:** Let \( \hat{K}_{n,s} \) be a minimal \( \varepsilon/(6\sqrt{R}) \)-net for \( K_{n,s} \) in Euclidean norm. Let \( D_T \) be the set of \( R \times R \) diagonal matrices with Frobenius-norm less or equal \( \Gamma \). It is well known that \( N(D_T, \| \cdot \|_F, \varepsilon) \leq (3\Gamma/\varepsilon)^R \).

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Denote by $\hat{D}_T$ a minimal $(\varepsilon/(6R))$-net of $D_T$ and define the sets

$$K = \{ Z \in \mathbb{R}^{n_1 \times n_2} : U \Sigma V^T \text{ with } u^r \in K_{n_1, s_1} \text{, } v^r \in K_{n_2, s_2} \text{ for all } r \in [R] \text{, and } ||\Sigma||_F \leq \Gamma \}$$

$$\tilde{K} = \{ \tilde{Z} \in \mathbb{R}^{n_1 \times n_2} : \tilde{Z} = \tilde{U} \tilde{\Sigma} \tilde{V}^T \text{ with } \tilde{u}^r \in \tilde{K}_{n_1, s_1} \text{, } \tilde{v}^r \in \tilde{K}_{n_2, s_2} \text{ for all } r \in [R] \text{, and } \tilde{\Sigma} \in \hat{D}_T \}.$$ 

We first show that $\tilde{K}$ is an $(\varepsilon/2)$-net of $K$. Let $Z = U \Sigma V^T \in K$ be given. There exists $\tilde{Z} = \tilde{U} \tilde{\Sigma} \tilde{V}^T \in \tilde{K}$ with $\|u^r - \tilde{u}^r\|_2 \leq \varepsilon/(6\sqrt{R})$, $\|v^r - \tilde{v}^r\|_2 \leq \varepsilon/(6\sqrt{R})$, and $\|\Sigma - \tilde{\Sigma}\|_F \leq \varepsilon/(6R)$, for all $r \in [R]$, and $\|\Sigma - \tilde{\Sigma}\|_F \leq \varepsilon/(6R)$. Therefore, $\|U - \tilde{U}\|_2^2 = \sum_{r=1}^{R} \|u^r - \tilde{u}^r\|_2^2 \leq (\varepsilon/(6\Gamma))^2$ and $\|V - \tilde{V}\|_2^2 \leq (\varepsilon/(6\Gamma))^2$. Moreover, $\|U\|_F^2 = \sum_{r=1}^{R} \|u^r\|_2^2 \leq R$ (the same holds for $V, \tilde{U}, \tilde{V}$) and $\|\Sigma\|_F \leq \|\Sigma\|_F$ (the same holds for $\Sigma V^T, \tilde{U} \tilde{\Sigma} \tilde{V}^T$).

We now obtain by triangle inequality and the fact that $\|AB\|_F \leq \|A\|_F \|B\|_F$

$$\|Z - \tilde{Z}\|_F \leq \|(U - \tilde{U}) \Sigma V^T\|_F + \|\tilde{U} (\Sigma - \tilde{\Sigma}) V^T\|_F + \|\tilde{U} \tilde{\Sigma} (V - \tilde{V})\|_F$$

$$\leq \frac{\varepsilon \Gamma}{6} + \frac{\varepsilon}{6R} \sqrt{\Gamma} + \frac{\varepsilon}{6} \leq \frac{\varepsilon}{2}$$

Since $K_{s_1, s_2} \subset K$ one has $N(K_{s_1, s_2}, \|\cdot\|_F, \varepsilon) \leq N(K, \|\cdot\|_F, \varepsilon/2)$. Hence,

$$N(K_{s_1, s_2}, \|\cdot\|_F, \varepsilon) \leq |\tilde{K}_{n_1, s_1}|^R |\tilde{D}_T| |\tilde{K}_{n_2, s_2}|^R$$

which yields the claim by applying Lemma 4.3.

Lemma 3.12 can be proven by applying the following mighty bound on suprema of chaos processes [21] Theorems 1.4 & 3.1 in combination with the bounds on the covering numbers $N(S, \|\cdot\|_F, \varepsilon)$ and $N(K_{s_1, s_2}, \|\cdot\|_F, \varepsilon)$ of $S$ and $K$ of Lemma 4.2 and Lemma 4.4. We recall below the relevant result in the form presented in [20]. The appearing $\gamma_2$-functional is defined in [21] and can be bounded by

$$\gamma_2(\mathcal{H}, \|\cdot\|_{2 \to 2}) \lesssim \int_0^{d_{2 \to 2}(\mathcal{H})} \sqrt{\log N(\mathcal{H}, \|\cdot\|_{2 \to 2}, \varepsilon)} d\varepsilon,$$

in the case of a set of matrices $\mathcal{H}$ equipped with the operator norm. Here and below $d_{\square}(\mathcal{H}) = \sup_{H \in \mathcal{H}} \|H\|_{\square}$, where $\square$ is a generic norm.

**Theorem 4.5.** Let $\mathcal{H}$ be a symmetric set of matrices, i.e., $\mathcal{H} = -\mathcal{H}$, and let $\xi$ be a random vector whose entries $\xi_i$ are independent $\mathcal{K}$-subgaussian random variables with mean 0 and variance 1. Set

$$E = \gamma_2(\mathcal{H}, \|\cdot\|_{2 \to 2}) (\gamma_2(\mathcal{H}, \|\cdot\|_{2 \to 2}) + d_{\mathcal{H}}(\mathcal{H}))$$

$$V = d_{2 \to 2}(\mathcal{H}) (\gamma_2(\mathcal{H}, \|\cdot\|_{2 \to 2}) + d_{\mathcal{H}}(\mathcal{H}))$$

$$U = d_{2 \to 2}(\mathcal{H})$$

Then, for $t > 0$,

$$\Pr \left[ \sup_{H \in \mathcal{H}} \|H\xi\|_2^2 - E \left[ \|H\xi\|_2^2 \right] \geq c_1 E + t \right] \leq 2 \exp \left( -c_2 \min \left( \frac{t^2}{V^2}, \frac{t}{U} \right) \right).$$

The constants $c_1$ and $c_2$ are universal and only depend on $\mathcal{K}$.

We refer the reader to [21] and [20] for further details.

**Proof of Lemma 3.12:** The proof consists of three main parts. We start in (I) by fitting our setting into the one of Theorem 4.5 in (IIa) resp. (IIb) the $\gamma_2$-functional gets bounded for $S^{R, T}_{s_1, s_2}$ and $K^{R, T}_{s_1, s_2}$, and in (III) we conclude by applying Theorem 4.5.

(I) We first switch the roles of our random measurement operator $A$ applied to the fixed matrices $Z$ to have fixed operators $H_Z^T$ applied to a random vector $\xi$. Denote by $\text{vec}(Z) \in \mathbb{R}^{n_1 n_2}$ the vectorization of $Z$. Observe, for all $Z \in \mathbb{R}^{n_1 \times n_2}$,

$$A(Z) = \frac{1}{\sqrt{m}} \begin{pmatrix} \langle \text{vec}(A_1), \text{vec}(Z) \rangle \\ \vdots \\ \langle \text{vec}(A_m), \text{vec}(Z) \rangle \end{pmatrix} = \frac{1}{\sqrt{m}} \begin{pmatrix} \text{vec}(Z)^T 0 & \cdots & 0 \\ 0 & \ddots & \cdots \\ \cdots & \cdots & \text{vec}(Z)^T \end{pmatrix} \begin{pmatrix} \text{vec}(A_1) \\ \vdots \\ \text{vec}(A_m) \end{pmatrix} = H_Z \cdot \xi.$$
where $H_Z \in \mathbb{R}^{m \times mn_1n_2}$ is a matrix depending on $Z$ and $\xi \in \mathbb{R}^{mn_1n_2}$ has i.i.d. $\mathcal{K}$-subgaussian entries $\xi_i$ of mean 0 and variance 1. We define $\mathcal{H}_S = \{H_Z : Z \in S^{R \Gamma}_{s_1, s_2}\}$. Note that the mapping $Z \mapsto H_Z$ is an isometric linear bijection. In particular, we have $\|H_Z\|_F = \|Z\|_F$ and $\|H_Z\|_{2 \rightarrow 2} = \|Z\|_F/\sqrt{m}$. For $Z \in S^{R \Gamma}_{s_1, s_2}$ it holds that $\|Z\|_F \leq \|U\|_F \|\Sigma V^T\|_F \leq \Gamma \sqrt{R}$. Hence, $d_F(\mathcal{H}_S) \leq \Gamma \sqrt{R}$ and $d_{2 \rightarrow 2}(\mathcal{H}_S) \leq \Gamma \sqrt{R}/\sqrt{m}$.

(IIa) Since $\|H_Z\|_{2 \rightarrow 2} = \|Z\|_F/\sqrt{m}$ and $Z \mapsto H_Z$ is a linear bijection, it follows that $N(\mathcal{H}_S, \| \cdot \|_{2 \rightarrow 2}, \varepsilon) = N(S, \| \cdot \|_F, \varepsilon/m)$. We can estimate by (28) and Lemma 7.1

$$
\gamma_2(\mathcal{H}_S, \| \cdot \|_{2 \rightarrow 2}) \lesssim \int_0^{\Gamma \sqrt{R}/\sqrt{m}} \sqrt{\log N(\mathcal{H}_S, \| \cdot \|_{2 \rightarrow 2}, \varepsilon)} d\varepsilon = \int_0^{\Gamma \sqrt{R}/\sqrt{m}} \sqrt{\log N(S^{R \Gamma}_{s_1, s_2}, \| \cdot \|_F, \varepsilon/m)} d\varepsilon
$$

$$
\leq \frac{C_2 \Gamma^2 R^2 (s_1 + s_2) \log (\max\{n_1, n_2\})}{m} =: \mathcal{L}_S
$$

for some constant $C_S > 0$.

(IIb) In the same manner we obtain a bound on $\gamma_2(\mathcal{H}_K, \| \cdot \|_{2 \rightarrow 2})$ where $\mathcal{H}_K = \{H_Z : Z \in K^{R \Gamma}_{s_1, s_2}\}$. Recall that $\|H_Z\|_F = \|Z\|_F, \|H_Z\|_{2 \rightarrow 2} = \|Z\|_F/\sqrt{m}$ and $Z \mapsto H_Z$ is a linear bijection. This implies $N(\mathcal{H}_K, \| \cdot \|_{2 \rightarrow 2}, \varepsilon) = N(K^{R \Gamma}_{s_1, s_2}, \| \cdot \|_F, \varepsilon/m)$. Note that $d_F(\mathcal{H}_K) \leq \Gamma \sqrt{R}$ and $d_{2 \rightarrow 2}(\mathcal{H}_K) \leq \Gamma \sqrt{R}/\sqrt{m}$. We obtain by (28) and Lemma 7.1

$$
\gamma_2(\mathcal{H}_K, \| \cdot \|_{2 \rightarrow 2}) \lesssim \int_0^{\Gamma \sqrt{R}/\sqrt{m}} \sqrt{\log N(\mathcal{H}_K, \| \cdot \|_{2 \rightarrow 2}, \varepsilon)} d\varepsilon = \int_0^{\Gamma \sqrt{R}/\sqrt{m}} \sqrt{\log N(K^{R \Gamma}_{s_1, s_2}, \| \cdot \|_F, \varepsilon/m)} d\varepsilon
$$

$$
\leq \frac{C_K \Gamma^2 R^2 (s_1 + s_2) \log (\max\{n_1, n_2\})}{m} =: \mathcal{L}_K
$$

for some constant $C_K > 0$.

(III) The final part of the proof is now equal for both sets $S^{R \Gamma}_{s_1, s_2}$ and $K^{R \Gamma}_{s_1, s_2}$. We write $\mathcal{L}$ for $\mathcal{L}_S$ resp. $\mathcal{L}_K$ and assume $m > C_S \Delta^{-2} R (s_1 + s_2) \log (\max\{n_1, n_2\})$ resp. $m > C_K \Delta^{-2} R (s_1 + s_2) \log (\max\{n_1, n_2\})$, for some $0 < \Delta < 1$. Then, $\mathcal{L} \leq \Gamma \sqrt{R}$ and

$$
\mathcal{L}^2 + \Gamma \sqrt{R} \mathcal{L} \leq \Gamma^2 R (\Delta^2 + \Delta) \leq 2 \Gamma^2 R \Delta.
$$

(29)

We obtain the following bounds on the quantities (cf. Theorem 4.5):

$$
E \leq \mathcal{L}^2 + \Gamma \sqrt{R} \mathcal{L}, \quad V \leq \frac{\Gamma \sqrt{R} \mathcal{L} + \Gamma^2 R}{\sqrt{m}}, \quad U \leq \frac{\Gamma^2 R}{m}.
$$

(30)

Observing now that $E[\|H_Z \xi\|_2^2] = \|H_Z\|_F^2 = \|Z\|_F^2$ and recalling $\Gamma \geq 1$ we finally get, for $\delta \geq 3c_1 \Gamma^2 R \Delta$ (which implies by (29) that $\delta \geq c_1 E + c_1 \Gamma^2 R \Delta$),

$$
\Pr \left[ \sup_{Z \in S} \| A(Z) \|_2^2 - \| Z \|_F^2 \right] \geq \delta \right] \leq \Pr \left[ \sup_{H_Z \in \mathcal{H}} \| H_Z \xi \|_2^2 - E[\|H_Z \xi\|_2^2] \right] \geq c_1 E + c_1 \Gamma^2 R \Delta
$$

$$
\leq \exp \left( -c_2 \min \left\{ \frac{m}{\Gamma^2 R (\mathcal{L} + \Gamma \sqrt{R})^2}, \frac{m c_1^2 \Gamma^2 R \Delta}{\Gamma^2 R} \right\} \right)
$$

$$
\leq \exp \left( -C \Delta^2 m \right)
$$

where $C > 0$ is a positive constant which depends on $\mathcal{K}$. In the last step we used that $\mathcal{L} + \Gamma \sqrt{R} \in [\Gamma \sqrt{R}, 2 \Gamma \sqrt{R}]$ (because $0 < \mathcal{L} < \Gamma \sqrt{R}$). 

\]
Remark 4.6. The condition $1 \leq \Gamma \leq \Gamma^2$ is used in a crucial way in part (III) of the proof above. Additionally, without condition $\Gamma \geq 1$, any $\hat{X} \in S_{n_1,n_2}^{R,\Gamma}$ could be decomposed also as follows

$$
\hat{X} = \sum_{r=1}^{R} \sigma_r u^r(v^r)^T = \sum_{r=1}^{R} \sum_{j=1}^{K} \frac{\sigma_r}{K} u^r(v^r)^T, \quad \left( \sum_{r=1}^{R} \sum_{j=1}^{K} \frac{\sigma^2_r}{K^2} \right)^{1/2} \leq \frac{\Gamma}{\sqrt{K}}
$$

for any $K \in \mathbb{N}$, implying $\hat{X} \in S_{n_1,n_2}^{R,K,\Gamma/\sqrt{K}}$ as well, which would result in a larger number of necessary measurements $m \gtrsim (\frac{\delta}{\Gamma^2})^{-2} RK (s_1 + s_2) \log^3 (\max\{n_1, n_2\})$. Hence, $\Gamma \geq 1$ emerges as a natural condition, in order to have a correct proof of part (III) and to avoid ambiguities on the necessary measurements.

5 Implementation and Numerical Experiments

After having obtained some theoretical insight on the proposed optimization problem, we provide an implementation of\(^4\) and discuss its predicted behavior in numerical experiments. Therefore, we begin by presenting the implementation that has been used in all experiments\(^5\). As in practice ATLAS converges even without the auxiliary terms $\frac{1}{\delta X} \|u - u^r_k\|_2^2$ and $\frac{1}{\delta X} \|v - v^r_k\|_2^2$ introduced in (10), for sake of simplicity we drop those terms. By the alternating form of (10), one has to solve a certain number of Tikhonov regularization resp. $\ell_1$-LASSO problems. Note that for the Tikhonov regularization

$$
u = \text{arg min}_{z \in \mathbb{R}^n} \|y - Az\|_2^2 + \alpha \|z\|_2^2,$$

with $A \in \mathbb{R}^{m \times n}$, $y \in \mathbb{R}^m$, and $\alpha > 0$, the solution is explicitly given by $u = (\alpha \text{Id} + A^T A)^{-1} A^T y$. Solutions to $\ell_1$-LASSO

$$
v = \text{arg min}_{z \in \mathbb{R}^n} \|y - Az\|_2^2 + \beta \|z\|_1,$$

for some $A \in \mathbb{R}^{m \times n}$, $y \in \mathbb{R}^m$ and $\beta > 0$ can be well approximated by the so-called Iterative Soft-Thresholding Algorithm (ISTA) which is based on the soft-thresholding operator $S_\beta$

$$
S_\beta(z) = \begin{pmatrix} S_\beta(z_1) \\ \vdots \\ S_\beta(z_n) \end{pmatrix}, \quad \text{where} \quad S_\beta(z_i) = \begin{cases} 
\frac{z_i - \beta}{2} & z_i > \frac{\beta}{2} \\
0 & |z_i| \leq \frac{\beta}{2} \\
\frac{z_i + \beta}{2} & z_i < -\frac{\beta}{2} \end{cases}.
$$

Hence, a suitable implementation of (10) is given by Algorithm 1 whereas Algorithm 2 describes ISTA for the reader’s convenience. Necessary modifications in case of sparse left component vectors of $\hat{X}$ are rather straightforward.

Let us turn toward numerical simulations. First, we check if the main theoretical results stated in Theorem 3.7 and Corollary 3.8 describe the qualitative and quantitative behavior of the approximation error well. Then, we compare ATLAS to the already mentioned Sparse Power Factorization (SPF). We used the leading singular vectors of $A^*(y)$ to initialize both algorithms, which is likely not an optimal choice and certainly may cause loss of performance for both algorithms, but it is nevertheless sufficient to illustrate certain comparisons numerically.

5.1 Validation of Corollary 3.8

Figure 3 shows the average approximation error of 100 randomly drawn $\hat{X} \in \mathbb{R}^{16 \times 100}$, $\|\hat{X}\|_F = 10$, with $\text{rank}(\hat{X}) = 1$ (resp. $\text{rank}(\hat{X}) = 5$) and 10-sparse right singular vector(s) from $m = 90$ (resp. $m = 400$) noisy measurements $y = A \hat{X} + \eta$. The parameters have been chosen exemplarily for purpose of illustration. The operator $A$ is drawn once at random. The error bound from Corollary 3.8 is plotted as dashed red

\(^4\)The corresponding Matlab code is provided at https://www-m15.ma.tum.de/Allgemeines/SoftwareSite
red, whereas the average approximation errors are in blue. Though not tight the theoretical bound seems to describe the linear dependence of the approximation error on noise level appropriately. In addition, Figure 3(b) shows a breakdown of approximation for noise to signal ratios below \( \approx 0.25 \). This occurrence is not surprising as the assumptions of Corollary 3.8 include a lower-bound on the noise-to-signal ratio for a fixed number of measurements. Below a certain value the RIP requirements will be too strong for \( \mathcal{A} \) to fulfill it, the RIP breaks down, and the recovery guarantees fail.

5.2 Validation of Theorem 3.7

In the second experiment, we study the influence of parameters \( \alpha \) and \( \beta \) on the reconstruction accuracy. In particular, we vary the parameters \( \alpha \) and \( \beta \) when reconstructing one randomly drawn \( \hat{X} \in \mathbb{R}^{16 \times 100} \), \( \| \hat{X} \|_F = 10 \), with rank(\( \hat{X} \)) = 1 and 10-sparse right singular vector from 90 measurements without noise. Again parameter choice is exemplary. We compare the three settings: (a) \( \alpha = \beta \), (b) \( \alpha = 0.01\beta \)  and (c) \( \alpha = 100\beta \) in Figure 4. One can observe a decrease of approximation error for \( \alpha, \beta \to 0 \) up to a certain threshold, under which the approximation seemingly fails. While this threshold lies at \( \beta \approx 0.15 \) in (a) and (b) it is hardly recognizable in (c). At the same time (a) and (b) show a much smaller approximation error. These observations suggest that the choice of \( \alpha \) strongly influences the approximation quality of ATLAS.

This is consistent with Theorem 3.7 as a smaller \( \alpha \) leads to a smaller theoretical approximation error bound. Even though (a) and (b) show a linear decrease in approximation error which is in contrast to the square-root behavior of the theoretical bound, (c) suggests that the error, indeed, behaves similar to the theoretical bound.

Figure 4 shows that the sparsity level remains stable for sufficiently large \( \beta \) and breaks down precisely at the same threshold as the approximation error, coinciding with the violation of the RIP conditions.

For a better understanding of ATLAS we made a third experiment reconstructing one randomly drawn \( \hat{X} \in \mathbb{R}^{16 \times 100} \) with rank(\( \hat{X} \)) = 1 and 10-sparse right singular vector for different values of \( \| \hat{X} \|_F \) from 90 measurements. The noise level was set to 0 and the parameters to \( \alpha = \beta = 0.5 \). The outcome is depicted in Figure 5. One can see the relative approximation error decreasing with the magnitude of \( \hat{X} \) as expected from the bound of Theorem 3.7. This seemingly confirms the theoretical dependence of reconstruction error on \( \| \hat{X} \|_2 \).
5.3 ATLAS vs SPF

After confirming the theoretical results numerically, we now turn to the comparison of ATLAS with its state-of-the-art counterpart SPF (see [23]). To our knowledge, SPF is the only algorithm available so far in matrix sensing, which exploits low-rankness and sparsity constraints together and comes with near-optimal recovery guarantees (not relying on a special structure of $A$ as in [4]). As [23] contains exhaustive numerical comparisons of SPF and low-rank (resp. sparse) recovery strategies based on convex relaxation, SPF suffices for numerical benchmark tests. From the structure of the algorithms and their respective theoretical analysis one would expect SPF to yield more accurate reconstruction in the noiseless-to-low-noise setting, while ATLAS should prove to be more reliable if noise becomes large. This theoretical expectation is confirmed by the following experiments.

In Figure 2, we compare for $s/n_2 \in [0, 1]$ and $m/(n_1 n_2)$ the number of successful recoveries of 30 randomly drawn $\hat{X} \in \mathbb{R}^{4 \times 128}$, $\|\hat{X}\|_F = 10$, with rank($\hat{X}$) = 1 and $s$-sparse right singular vectors from $m$ measurements. The dimensions of $\hat{X}$ were chosen accordingly to similar experiments in [23]. We set the noise level to 0 (resp. 0.3) and counted the recovery successful if $\|\hat{X} - X_{app}\|_F / \|\hat{X}\|_F \leq 0.2$ (resp. 0.4). In order to compare the noisy and noiseless cases, we fix $\alpha = \beta = 0.5$ for both, which is a reasonable choice for high noise level, but perhaps sub-optimal if the noise level is low. Selected quantiles are directly compared in Figure 6 for convenience.

As expected, SPF outperforms ATLAS if there is no noise. In case of strong noise on the measurements, the situation changes. In particular, we observe the improved performance of ATLAS, whereas the SPF performance remarkably deteriorates.

To further quantify this effect, we perform the experiments reflected in Figure 1. For varying number of measurements we compared average approximation error and recovery probability of SPF and ATLAS for 30 randomly chosen $\hat{X} \in \mathbb{R}^{16 \times 100}$, $\|\hat{X}\|_F = 10$, with rank($\hat{X}$) = 5 and 10-sparse right singular vectors which either share a common support or may have various support sets. The parameters are chosen as $\alpha = \beta = 0.5$. One can clearly see that SPF outperforms ATLAS even in the noisy case for common support sets of the singular vectors. This is not surprising as ATLAS makes no use of the additional information provided by shared support sets. If the singular vectors, however, do not share a common support set, ATLAS shows its strength in the noisy setting. SPF which needs pre-information on the row-/column-sparsity $\tilde{s}$ of $\hat{X}$ has to be initialized with $\tilde{s} = Rs$ as in the general case all support sets may differ.

Figure 3: Approximation quality depending on noise level (see Section 5.1). The $x$-axis shows noise to signal ratio $\|\eta\|_2 / \|\hat{X}\|_F$ while the $y$-axis presents approximation error relative to $\|\hat{X}\|_F$. One can see the comparison of approximation results (solid blue) and theoretical bound (dashed red).
(a) $\alpha = \beta$

(b) $\alpha = 0.01\beta$

(c) $\alpha = 100\beta$

Figure 4: Approximation quality and sparsity depending on parameter size (see Section 5.2). The approximation error (solid blue) and the theoretical bound (dashed red) are measured relative to $\|\hat{X}\|_F$ while sparsity of the right singular vector (dotted yellow) is relative to $n_2$.

Figure 5: Approximation error depending on the magnitude of $\hat{X}$ in Frobenius norm (see Section 5.2). Approximation error (solid blue) and theoretical bound (dashed red) are relative to $\|X\|_F$. 

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5.4 Initialization

We close the section by a simple test on the influence of the initialization. The plots in Figure 7 compared for $s/n_2 \in [0, 0.5]$ and $m/(n_1 n_2) \in [0, 1]$ the number of successful recoveries of 20 randomly drawn $X \in \mathbb{R}^{8 \times 128}$, $\|X\|_F = 10$, with rank($\hat{X}$) $\in \{1, 3\}$ and s-sparse right singular vectors from $m$ measurements. The noise level was set to $0.3\|\hat{X}\|_F$ and recovery was counted successful if $\|\hat{X} - X_{\text{appr}}\|_F/\|\hat{X}\|_F \leq 0.2$ (resp. 0.4). We compare the initializations by the leading singular vectors of $A^*(y)$ and by the leading singular vectors of $X + Z$ where $Z$ was drawn at random, and scaled to $\|Z\|_F = 100$ (strong perturbation) resp. $\|Z\|_F = 0.2$ (mild perturbation).

For rank($\hat{X}$) $= 1$ we note remarkably that the convergence radius of ATLAS is seemingly very large (yet not global), as the phase transition diagrams in Figure 7 do not show significant variations from choosing as initialization the leading singular vectors of $A^*(y)$ and those of small random perturbation. Instead for rank($\hat{X}$) $= 3$, initialization plays a more important role in performance and the initialization by leading singular vectors of $A^*(y)$ does not yield optimal performance.

6 Discussion and Open Questions

Motivated by challenging examples from recommendation systems and blind demixing in signal processing, in this paper we propose a multi-penalty approach to perform effectively sparse PCA from inaccurate and incomplete linear measurements. The considered problem stands at the intersection of the compressed sensing and sparse PCA framework, though significantly extending their settings. Our analysis results in general bounds on the performance of the proposed algorithm, ATLAS, and in a necessary number of subgaussian measurements to approximate effectively sparse and low-rank matrices. These theoretical results are confirmed in numerical experiments. ATLAS is especially of use and effective in the most realistic setting of ineliminable noise and, hence, it complements the state-of-the-art algorithm SPF of Lee et. al. in [23], which works well for low level of noise or exact measurements only. Moreover, ATLAS tackles the recovery of a significantly larger class of matrices than SPF, i.e., matrices with non-orthogonal rank-1 decompositions and effectively sparse components, which are of interest when performing sparse PCA. Nevertheless, the analysis of ATLAS is remarkably simple and it is easily prone to several extensions/generalizations. We mention a few of them as follows.

First, replacing the $\ell_2$- and $\ell_1$-norms by $\ell_p$- and $\ell_q$-(quasi)-norms, for $p \geq 2$ and $0 < q < 2$, yields to the
Figure 7: Comparison of different initializations for ATLAS for (7) with noise $\eta \neq 0$ on the measurements (see Section 5.4), namely, initialization with a strongly perturbed approximation $X_0 \approx \hat{X}$ (left), initialization by the leading singular vectors of $A^*(y)$ (middle), and initialization with a mildly perturbed approximation $X_0 \approx \hat{X}$ (right). Empirical recovery probability is depicted by color from zero (blue) to one (yellow).
and, in turn, the algorithm

\[ J_{\alpha,\beta}^{R,p,q}(u^1, \ldots, u^R, v^1, \ldots, v^R) := \left\| y - A \left( \sum_{r=1}^{R} u^r (v^r)^T \right) \right\|_2^2 + \alpha \sum_{r=1}^{R} \| u^r \|^p_p + \beta \sum_{r=1}^{R} \| v^r \|^q_q, \quad (31) \]

and, in turn, the algorithm

\[
\begin{align*}
    u^1_{k+1} & = \arg \min_u \left\{ \left\| y - A \left( \sum_{r=2}^{R} u^r_{k+1} v^r_{k+1}^T \right) \right\|_2^2 + \alpha \| u \|^p_p + \frac{1}{2\lambda_k} \| u - u^1_k \|_2^2, \\
    u^r_{k+1} & = \arg \min_u \left\{ \left\| y - A \left( \sum_{r=2}^{R} u^r_{k+1} v^r_{k+1}^T \right) \right\|_2^2 + \beta \| v \|^q_q + \frac{1}{2\lambda_k} \| v - v^r_{k+1} \|_2^2, \\
    \vdots & \\
    u^R_{k+1} & = \arg \min_u \left\{ \left\| y - A \left( \sum_{r=1}^{R-1} u^r_{k+1} v^r_{k+1}^T \right) \right\|_2^2 + \alpha \| u \|^p_p + \frac{1}{2\lambda_k} \| u - u^R_k \|_2^2, \\
    u^1_{k+1} & = \arg \min_v \left\{ \left\| y - A \left( \sum_{r=1}^{R} u^r_{k+1} v^r_{k+1}^T \right) \right\|_2^2 + \beta \| v \|^q_q + \frac{1}{2\lambda_k} \| v - u^1_{k+1} \|_2^2, \\
    u^r_{k+1} & = \arg \min_v \left\{ \left\| y - A \left( \sum_{r=1}^{R-1} u^r_{k+1} v^r_{k+1}^T \right) \right\|_2^2 + \beta \| v \|^q_q + \frac{1}{2\lambda_k} \| v - u^r_{k+1} \|_2^2, \right\},
\end{align*}
\]

As for \( q < 1 \) even the single component minimizations become non-convex, this setting needs special care. One would need non-standard iterative thresholding methods, which have been developed and studied, e.g., in [20]. As \( q \)-quasi-norms, for \( q < 1 \), have proved particularly effective in enforcing sparsity, this additional technical difficulties are worth to overcome.

Second, in recommendation systems, one usually imposes additionally non negativity constraints on the obtained matrices. We could easily implement them in ATLAS by asymmetric \( \ell_1 \)-regularization. Define for \( z \in \mathbb{R}^n \) and \( \theta > 0 \)

\[ \| z \|^+_{1,\theta} := \sum_{i=1}^{n} \max(0, z_i), \quad | x |^+_{\theta} := \begin{cases} x & x \geq 0 \\ \theta | x | & \text{else.} \end{cases} \]

For \( \theta \) becoming large, the regularization by \( \| \cdot \|_{1,\theta}^+ \) promotes sparsity and non-negativity. Replacing the \( \ell_1 \)-norm in ATLAS by \( \| \cdot \|_{1,\theta}^+ \) would result in the remarkably simple modification of ISTA (Algorithm 2), where the soft-thresholding operator \( S_\beta \) is substituted in line 2 with

\[ S_{\beta,\theta}(z_i) = \begin{cases} z_i - \frac{\theta}{\beta} & z_i > \frac{\beta}{\theta} \\ 0 & -\frac{\theta}{\beta} \leq z_i \leq \frac{\beta}{\theta} \\ z_i + \frac{\theta}{\beta} & z_i < -\frac{\theta}{\beta} \end{cases}. \]

Note that in the limit case \( \theta \to \infty \) the operator \( S_{\beta,\theta} \) is a shifted ReLU function. Choosing \( \theta \) sufficiently large or considering the limit \( \theta \to \infty \) would lead to non-negative sparse PCA [34] from incomplete and inaccurate measurements with further applications in economics [17], biology [3], and computer vision [22].

Third, as a byproduct of our generalization of sparse PCA, we introduce, in our view, the right class of matrices \( K_{\alpha,\beta}^{R,T} \) and corresponding RIP, Definition 3.6, which might allow to generalize SPF to matrices having non-orthogonal, effectively sparse decompositions. In fact, the assumption of SPF of model matrices with sparse SVD is quite restrictive. For \( q \to 0 \), the ATLAS algorithm can be seen as a generalization of SPF and be realized by iterative hard-thresholding.

The current results demand a careful choice of parameters at noise level. This drawback of multi-penalty regularization is well-known and could be attacked by implementing LASSO-path. LASSO-path has been recently extended to the multi-penalty setting in case of superposition of the signals [14], where the authors provided an efficient procedure for the construction of regions containing structurally similar solutions. In addition, \( J_{\alpha,\beta}^R \) depends by construction heavenly on pre-knowledge of the rank \( R \). One might ask how to get good estimates for \( R \) in case the rank is unknown.

As mentioned above, initialization is crucial for good performances of the algorithm. It is currently unclear how a good initialization can be obtained to guarantee convergence of the whole procedure to global minimizers. This question is closely connected to the fundamental problem in non-convex optimization how to initialize gradient-descent methods. In fact, alternating minimization is somewhat related to gradient-descent. While in gradient-descent one determines an optimal descent direction and then approximates the
optimal step size, alternating minimization strongly restricts the directions in space in order to calculate optimal step sizes. Lee et. al. proposed an initialization, which worked in their setting if one assumes a strong decay of the singular values. Possibly one could prove this initialization to be sufficiently good in our setting as well, also in the light of recently improved analysis [12].

7 Appendix

7.1 Proofs of Section 4

Proofs of two technical results used in [4] are provided here. The first result estimates possible coverings of \(S_{s_1,s_2}^{R_1,R_2}\) defined in (14) as stated in Lemma 4.2, while the second result contains two integral estimates used in the proof of Lemma 3.12.

**Proof of Lemma 4.2:** Recall, each \(Z \in S_{s_1,s_2}^{R_1,R_2}\) can be represented as \(Z = U\Sigma V^T\) with \(U = (u^1,\ldots,u^R)\), \(V = (v^1,\ldots,v^R)\) where all unit norm columns \(u^r \in \mathbb{R}^{n_1}\) are \(s_1\)-sparse, all unit norm columns \(v^r \in \mathbb{R}^{n_2}\) are \(s_2\)-sparse, and \(\|\Sigma\|_F \leq \Gamma\). Let us first consider the larger set \(S = \{Z = U\Sigma V^T : U \in Q_{n_1,s_1}^R, \Sigma \in D, V \in Q_{n_2,s_2}^R\}\) where \(D\) is the set of \(R \times R\) diagonal matrices with Frobenius norm less or equal \(\Gamma\) and \(Q_{n,s}^R = \{W \in \mathbb{R}^{n \times R} : \|W\|_F \leq \sqrt{R}\} and all columns \(w^r\) are \(s\)-sparse\}. Then, we know that \(S_{s_1,s_2}^{R_1,R_2} \subset S\). We construct an \((\varepsilon/2)\)-net \(\hat{S}\) of \(S\) by covering the sets of permissible \(U, \Sigma, V\) and conclude the proof by applying the well-known relation \(N(K,\|\cdot\|,\varepsilon) \leq N(K',\|\cdot\|,\varepsilon/2)\) which holds whenever \(K \subset K'\).

First note that if \(B\) is a unit ball in \(D\) dimensions (with respect to some norm \(\|\cdot\|_B\)) there exists an \(\varepsilon\)-net \(\hat{B}\) (i.e., for all \(b \in B\) there is some \(\hat{b} \in \hat{B}\) with \(\|b - \hat{b}\|_B \leq \varepsilon\)) with \(\hat{B} \subset B\) and \(|\hat{B}| \leq (3/\varepsilon)^D\). See for example [7, Begin Section 3]. Moreover, note that \(N(K,\|\cdot\|,\varepsilon) = N(cK,\|\cdot\|,\varepsilon/2c)\) for any set \(K\) and \(c > 0\). Hence, for any scaled unit ball \(cB\) there exists an \(\varepsilon\)-net \(\hat{B} \subset cB\) and \(|\hat{B}| \leq (3c/\varepsilon)^D\).

Let \(\hat{D}_R\) be an \((\varepsilon/(6R))\)-net of \(D\) which is of size \(|\hat{D}_R| \leq (18\Gamma R/\varepsilon)^R\). For \(W \in \mathbb{R}^{n \times R}\) denote by \(\text{supp}(W) = \{\text{supp}(w^1),\ldots,\text{supp}(w^R)\}\) and by \(\text{supp}(W) \subset \text{supp}(W')\) that \(\text{supp}(w^r) \subset \text{supp}(w'^r)\), for all \(r \in [R]\). Define the set of all possible supports of maximal size

\[T_{n,s}^R = \{\text{supp}(W) : W \in \mathbb{R}^{n \times R} and all columns w^r have exactly s non-zero entries\}.

For any fixed \(\theta \in T_{n,s}^R\) the set \(\{W \in Q_{n,s}^R : \text{supp}(W) \subset \theta\}\) is an \(\mathbb{R}^s\times\mathbb{R}\) Frobenius ball of radius \(\sqrt{R}\) embedded into \(\mathbb{R}^{n \times R}\) and \(Q_{n,s}^R = \bigcup_{\theta \in T_{n,s}^R} \{W \in Q_{n,s}^R : \text{supp}(W) \subset \theta\}\). Hence, there is an \((\varepsilon/(6\sqrt{R}))\)-net \(\hat{Q}_{n,s}^R\) of \(Q_{n,s}^R\) with

\[|\hat{Q}_{n,s}^R| \leq |T_{n,s}^R| \left(\frac{18\Gamma R}{\varepsilon}\right)^R s \leq \left(\frac{n}{s}\right)^R \left(\frac{18\Gamma R}{\varepsilon}\right)^R s \leq \left(\frac{e\eta}{s}\right)^R s \left(\frac{18\Gamma R}{\varepsilon}\right)^R s.

We define now \(\hat{S} = \{\hat{Z} = \hat{U}\hat{\Sigma}\hat{V}^T : \hat{U} \in \hat{Q}_{n_1,s_1}^R, \hat{\Sigma} \in \hat{D}_R, and \hat{V} \in \hat{Q}_{n_2,s_2}^R\}\). It is clear that

\[|\hat{S}| \leq |\hat{Q}_{n_1,s_1}^R| \cdot |\hat{D}_R| \cdot |\hat{Q}_{n_2,s_2}^R| \leq \left(\frac{18\Gamma R}{\varepsilon}\right)^{R(s_1+s_2+1)} \left(\frac{e\eta_1}{s_1}\right)^{R_1} \left(\frac{e\eta_2}{s_2}\right)^{R_2}.

Let us conclude by showing \(\hat{S}\) is indeed an \((\varepsilon/2)\)-net for \(S\). Given any \(Z = U\Sigma V^T \in S\), there exists \(\hat{Z} = \hat{U}\hat{\Sigma}\hat{V}^T \in \hat{S}\) with \(\|U - \hat{U}\|_F \leq \varepsilon/(6\sqrt{R})\), \(\|\Sigma - \hat{\Sigma}\|_F \leq \varepsilon/(6\sqrt{R})\), and \(\|V - \hat{V}\|_F \leq \varepsilon/(6\sqrt{R})\). We can estimate

\[\|Z - \hat{Z}\|_F \leq \|(U - \hat{U})\Sigma V^T\|_F + \|\hat{U}(\Sigma - \hat{\Sigma})V^T\|_F + \|\hat{U}\hat{\Sigma}(V - \hat{V})^T\|_F\]
\[\leq \frac{\varepsilon}{6\sqrt{R}} \Gamma \sqrt{R} + \frac{\varepsilon}{6\sqrt{R}} \sqrt{R} \sqrt{R} + \frac{\varepsilon}{6\sqrt{R}} \Gamma \leq \frac{\varepsilon}{2}.

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Lemma 7.1. If $\Gamma \geq 1$, we have for the sets $S_{s_1, s_2}^{R, \Gamma}$ and $K_{s_1, s_2}^{R, \Gamma}$ defined in (14) and (15) that

$$
\int_0^\frac{\Gamma\sqrt{m}}{m} \sqrt{\log N \left( S_{s_1, s_2}^{R, \Gamma}, \| \cdot \|_F, \sqrt{m\varepsilon} \right)} \, d\varepsilon \leq \sqrt{\int_0^\frac{\Gamma\sqrt{m}}{m} 1 \, d\varepsilon \int_0^\frac{\Gamma\sqrt{m}}{m} \log N \left( S_{s_1, s_2}^{R, \Gamma}, \| \cdot \|_F, \sqrt{m\varepsilon} \right) \, d\varepsilon}
$$

$$
\leq \frac{\Gamma^2 R^2(s_1 + s_2 + 1) \left( 1 + \log \left( 18\sqrt{R} \right) \right) + \Gamma^2 R^2 s_1 \log \left( \frac{m_1}{s_1} \right) + \Gamma^2 R^2 s_2 \log \left( \frac{m_2}{s_2} \right)}{m}
$$

$$
\leq \frac{C_S \Gamma^2 R^2 (s_1 + s_2) \log (\max \{n_1, n_2\})}{m}
$$

where $C_S, C_K > 0$ are constants.

**Proof:** For the first estimate apply Lemma 4.2 to obtain

$$
\int_0^\frac{\Gamma\sqrt{m}}{m} \sqrt{\log N \left( S_{s_1, s_2}^{R, \Gamma}, \| \cdot \|_F, \sqrt{m\varepsilon} \right)} \, d\varepsilon \leq \sqrt{\int_0^\frac{\Gamma\sqrt{m}}{m} 1 \, d\varepsilon \int_0^\frac{\Gamma\sqrt{m}}{m} \log N \left( S_{s_1, s_2}^{R, \Gamma}, \| \cdot \|_F, \sqrt{m\varepsilon} \right) \, d\varepsilon}
$$

$$
\leq \frac{\Gamma^2 R^2(s_1 + s_2 + 1) \left( 1 + \log \left( 18\sqrt{R} \right) \right) + \Gamma^2 R^2 s_1 \log \left( \frac{m_1}{s_1} \right) + \Gamma^2 R^2 s_2 \log \left( \frac{m_2}{s_2} \right)}{m}
$$

$$
\leq \frac{C_S \Gamma^2 R^2 (s_1 + s_2) \log (\max \{n_1, n_2\})}{m}
$$

where we used Cauchy-Schwarz inequality in the first step and the fact that $\sqrt{R} \leq \max \{n_1, n_2\}$ in the last inequality. $C_S > 0$ is an appropriate constant.

To obtain the second estimate let us first assume $s_1/n_1 \leq s_2/n_2$. We apply Lemma 4.4 and find

$$
\int_0^\frac{\Gamma\sqrt{m}}{m} \sqrt{\log N \left( K_{s_1, s_2}^{R, \Gamma}, \| \cdot \|_F, \sqrt{m\varepsilon} \right)} \, d\varepsilon
$$

$$
\leq \int_0^{\frac{\Gamma\sqrt{m}}{m_1}} \sqrt{R(n_1 + n_2 + 1) \log \left( \frac{36\Gamma R}{\sqrt{m\varepsilon}} \right)} \, d\varepsilon + \int_{\frac{\Gamma\sqrt{m}}{m_1}}^{\frac{\Gamma\sqrt{m}}{m_2}} \sqrt{\frac{144\Gamma^2 R^2 s_1}{m_2}} \log \left( \frac{9\sqrt{m\varepsilon n_1}}{6\Gamma \sqrt{R}s_1} \right) \, d\varepsilon
$$

$$
+ \int_{\frac{\Gamma\sqrt{m}}{m_2}}^{\frac{\Gamma\sqrt{m}}{m_1}} \sqrt{\frac{R(n_2 + 1)}{\log \left( \frac{36\Gamma R}{\sqrt{m\varepsilon}} \right)}} \, d\varepsilon + \int_{\frac{\Gamma\sqrt{m}}{m_2}}^{\frac{\Gamma\sqrt{m}}{m_1}} \sqrt{\frac{144\Gamma^2 R^2 s_2}{m_2}} \log \left( \frac{9\sqrt{m\varepsilon n_2}}{6\Gamma \sqrt{R}s_1} \right) \, d\varepsilon
$$

$$
= I_1 + I_2 + I_3 + I_4 + I_5.
$$

We now estimate the five integrals. We use the short notation $a_i = 12\Gamma \sqrt{\frac{R s_i}{m_{n_i}}}$ for $i = 1, 2$ and $b = \frac{\Gamma\sqrt{m}}{m}\sqrt{m\varepsilon}$. The first integral can be bounded by

$$
I_1 \leq \left( \int_0^{a_1} 1 \, d\varepsilon \int_0^{a_1} R(n_1 + n_2 + 1) \log \left( \frac{36\Gamma R}{\sqrt{m\varepsilon}} \right) \, d\varepsilon \right)^\frac{1}{2}
$$

$$
\leq \left( a_1 R(n_1 + n_2 + 1) \left[ \varepsilon \left( 1 + \log \left( \frac{36\Gamma R}{\sqrt{m\varepsilon}} \right) \right) \right]_{\varepsilon=0}^{a_1} \right)^\frac{1}{2}
$$

$$
= \left( \frac{144\Gamma^2 R^2 s_1(n_1 + n_2 + 1)}{m n_1} \left( 1 + \log \left( 3 \sqrt{\frac{R n_1}{s_1}} \right) \right) \right)^\frac{1}{2} \leq \left\{ \begin{array}{ll}
\frac{432\Gamma^2 R^2 s_1}{m} \left( 1 + \log \left( 3 \sqrt{\frac{R n_1}{s_1}} \right) \right)^\frac{1}{2} & n_1 \geq n_2 \\
\frac{432\Gamma^2 R^2 s_2}{m} \left( 1 + \log \left( 3 \sqrt{\frac{R n_2}{s_2}} \right) \right)^\frac{1}{2} & \text{else}
\end{array} \right.
$$
where we used in the last step the assumption \( s_1/n_1 \leq s_2/n_2 \). As can be seen later, the case distinction is irrelevant in the final estimate. Let us now turn to the second integral.

\[
I_2 = \sqrt{\frac{144 \Gamma^2 R^2 s_1}{m}} \int_{a_1}^{a_2} \frac{1}{\varepsilon} \sqrt{\log \left( \frac{9 \sqrt{\varepsilon} n_1}{6 \Gamma \sqrt{R}} \right)} \, d\varepsilon = \sqrt{\frac{144 \Gamma^2 R^2 s_1}{m}} \left[ \frac{2}{3} \log^2 \left( \frac{9 \sqrt{\varepsilon} n_1}{6 \Gamma \sqrt{R}} \right) \right]_{\varepsilon=a_1}^{a_2} = \left( \frac{64 \Gamma^2 R^2 s_1}{m} \right)^\frac{1}{2} \left( \log^2 \left( \frac{18 \sqrt{s_2 n_1}}{\sqrt{n_2 s_1}} \right) - \log^2 \left( \frac{18 \sqrt{s_1}}{\sqrt{s_1}} \right) \right) \leq \left( \frac{64 \Gamma^2 R^2 s_1}{m} \log^3 (18 n_1) \right)^\frac{1}{2}
\]

The third integral is similar to the first. Again the case distinction does not play a major role in the end.

\[
I_3 \leq \left[ (a_2 - a_1) R (n_2 + 1) \left[ \varepsilon \left( 1 + \log \left( \frac{36 \Gamma R}{\sqrt{m \varepsilon}} \right) \right) \right]_{\varepsilon=a_1}^{a_2} \right]^\frac{1}{2} = \left( a_2 - a_1 \right) R (n_2 + 1) \left[ a_2 \left( 1 + \log \left( \frac{3 \sqrt{R n_2}}{s_2} \right) \right) - a_1 \left( 1 + \log \left( \frac{R n_1}{s_1} \right) \right) \right]^\frac{1}{2} \leq \left( a_2 - a_1 \right)^2 R (n_2 + 1) \left( 1 + \log \left( \frac{R n_1}{s_1} \right) \right)^\frac{1}{2} \leq \left( a_2^2 + a_1^2 \right) R (n_2 + 1) \left( 1 + \log \left( \frac{R n_1}{s_1} \right) \right)^\frac{1}{2}
\]

\[
I_3 = \left( \frac{144 \Gamma^2 R^2}{m} \left( \frac{s_2 (n_2 + 1)}{n_2} + \frac{s_1 (n_2 + 1)}{n_1} \right) \left( 1 + \log \left( \frac{R n_1}{s_1} \right) \right) \right)^\frac{1}{2} \leq \left\{ \begin{array}{ll} \left( \frac{432 \Gamma^2 R^2 (s_1 + s_2)}{m} \left( 1 + \log \left( \frac{R n_1}{s_1} \right) \right) \right)^\frac{1}{2} & n_1 \geq n_2, \\
\left( \frac{432 \Gamma^2 R^2 s_2}{m} \right)^\frac{1}{2} & \text{else.}
\end{array} \right.
\]

In the third and the last line we again used \( s_1/n_1 \leq s_2/n_2 \). The fourth integral is similar to the second.

\[
I_4 = \sqrt{\frac{144 \Gamma^2 R^2 (s_1 + s_2)}{m}} \int_{a_2}^{b} \frac{1}{\varepsilon} \sqrt{\log \left( \frac{9 \sqrt{\varepsilon} n_1}{6 \Gamma \sqrt{R}} \right)} \, d\varepsilon = \sqrt{\frac{144 \Gamma^2 R^2 (s_1 + s_2)}{m}} \left[ \frac{2}{3} \log^2 \left( \frac{9 \sqrt{\varepsilon} n_1}{6 \Gamma \sqrt{R}} \right) \right]_{\varepsilon=a_2}^{b} = \left( \frac{64 \Gamma^2 R^2 (s_1 + s_2)}{m} \right)^\frac{1}{2} \left( \log^2 \left( \frac{3 n_1}{2 s_1} \right) - \log^2 \left( \frac{18 \sqrt{s_2 n_1}}{\sqrt{n_2 s_1}} \right) \right) \leq \left( \frac{64 \Gamma^2 R^2 (s_1 + s_2)}{m} \log^3 (18 n_1) \right)^\frac{1}{2}
\]

The last integral is similar to the third.

\[
I_5 \leq \left( (b - a_2) R \left[ \varepsilon \left( 1 + \log \left( \frac{18 \sqrt{R}}{m \varepsilon} \right) \right) \right]_{\varepsilon=a_2}^{b} \right)^\frac{1}{2} \leq \left( (b - a_2)^2 R \left( 1 + \log \left( 18 \sqrt{\frac{R n_2}{s_2}} \right) \right) \right)^\frac{1}{2} \leq \left( b^2 + a_2^2 \right) R \left( 1 + \log \left( 18 \sqrt{\frac{R n_2}{s_2}} \right) \right)^\frac{1}{2} \leq \left( \frac{145 \Gamma^2 R^2}{m} \left( 1 + \log \left( 18 \sqrt{\frac{R n_2}{s_2}} \right) \right) \right)^\frac{1}{2}
\]

Let us now put all estimates together. If \( s_1/n_1 \geq s_2/n_2 \), the involved entities would just switch their roles. Hence, we obtain

\[
\int_{0}^{\sqrt{\varepsilon m}} \sqrt{\log N(K, \|f\|_F, \sqrt{m \varepsilon})} \, d\varepsilon \leq \sqrt{\frac{C_K \Gamma^2 R^2 (s_1 + s_2) \log^4 (\max \{n_1, n_2\})}{m}}
\]

for some constant \( C_K > 0 \).
7.2 Proof of Theorem 3.16

In this subsection we show the convergence of ATLAS to global minimizers as presented in Theorem 3.16. To do that, we make use of the results from [2]. In particular, we first present two technical lemmas (Lemma 7.2 & Lemma 7.3), which are essentially generalizations of work [2]. These lemmas would be useful to prove the central theorem (here Theorem 7.4) of Attouch et. al. in our general setting. Finally, the theorem on local convergence, Theorem 3.16 can essentially be derived from Theorem 7.4 and combines the two statements [2 Theorem 9] and [2 Theorem 10]. We refer the interested reader to [2] for further details. We also provide a reference to the original work in brackets.

Lemma 7.2 ([2 Lemma 5]). Under assumptions (H) and (H1) the sequences \( u_k^1, \ldots, v_k^R \) are well-posed in the sense that all minimizations in (24) have unique and finite solutions. Moreover,

(i) \[
L(u_k^1, \ldots, v_k^R) + \sum_{r=1}^{R} \frac{1}{2\lambda_{k-1}} \|u_k^r - u_{k-1}^r\|^2 + \sum_{r=1}^{R} \frac{1}{2\mu_{k-1}} \|v_k^r - v_{k-1}^r\|^2 \leq L(u_{k-1}^1, \ldots, v_{k-1}^R),
\]

for all \( k \geq 1 \), hence \( L(u_k^1, \ldots, v_k^R) \) is non-increasing.

(ii) \[
\sum_{k=1}^{\infty} (\|u_k^1 - u_{k-1}^1\|^2 + \cdots + \|v_k^R - v_{k-1}^R\|^2) < \infty,
\]

hence \( \lim_{k \to \infty} (\|u_k^1 - u_{k-1}^1\|^2 + \cdots + \|v_k^R - v_{k-1}^R\|^2) = 0. \)

(iii) For \( k \geq 1 \), define

\[
(u_k^1, \ldots, v_k^R) := \begin{pmatrix}
\nabla u_1 Q(u_k^1, \ldots, v_k^R) - \nabla u_1 Q(u_k^1, u_k^2, \ldots, v_k^R) \\
\vdots \\
0
\end{pmatrix} - \begin{pmatrix}
\frac{1}{\lambda_{k-1}} (u_k^1 - u_{k-1}^1) \\
\vdots \\
\frac{1}{\mu_{k-1}} (v_k^R - v_{k-1}^R)
\end{pmatrix}.
\]

Then \( (u_k^1, \ldots, v_k^R) \in \partial L(u_k^1, \ldots, v_k^R) \) and for all bounded subsequences \( (u_k^1, \ldots, v_k^R) \) we have \( (u_k^1, \ldots, v_k^R) \to 0 \), hence \( \text{dist}(0, \partial L(u_k^1, \ldots, v_k^R)) \to 0 \), for \( k' \to \infty \).

Proof: From \( \inf L > -\infty \) and (H) it follows that the functions to be minimized in (24) are bounded below, coercive and lower semicontinuous and, therefore, the sequence \( (u_k^1, \ldots, v_k^R) \) is well-posed.

(i) Using the minimizing properties of \( u_k^1, \ldots, v_k^R \) from (24), we obtain

\[
L(u_k^1, \ldots, v_k^R) \geq L(u_k^1, u_k^2, \ldots, v_k^R) + \frac{1}{2\lambda_{k-1}} \|u_k^1 - u_{k-1}^1\|^2 \\
\geq \left( L(u_k^1, u_k^2, \ldots, v_k^R) + \frac{1}{2\mu_{k-1}} \|v_k^R - v_{k-1}^R\|^2 \right) + \frac{1}{2\lambda_{k-1}} \|u_k^1 - u_{k-1}^1\|^2 \\
\vdots \\
\geq L(u_k^1, \ldots, v_k^R) + \sum_{r=1}^{R} \frac{1}{2\lambda_{k-1}} \|u_k^r - u_{k-1}^r\|^2 + \sum_{r=1}^{R} \frac{1}{2\mu_{k-1}} \|v_k^r - v_{k-1}^r\|^2.
\]

(ii) From (i) and (H1) one has, for every \( K \in \mathbb{N} \),

\[
\frac{1}{2R} \sum_{k=1}^{K} (\|u_k^1 - u_{k-1}^1\|^2 + \cdots + \|v_k^R - v_{k-1}^R\|^2) \leq \sum_{k=1}^{K} (L(u_k^1, \ldots, v_k^R) - L(u_{k-1}^1, \ldots, v_{k-1}^R)) \\
= L(u_0^1, \ldots, v_0^R) - L(u_K^1, \ldots, v_K^R) \\
< L(u_0^1, \ldots, v_0^R) - \inf L < \infty.
\]

By letting \( K \to \infty \) we get the claim.
Lemma 7.3 (2 Proposition 6). Assume (H) and (H1) hold. Let \((u_k^1, \ldots, v_k^R)\) be a sequence defined by \((24)\) and \(\omega(u_k^1, \ldots, v_k^R)\) be a (possibly empty) set of limit points. Then,

(i) if \((u_k^1, \ldots, v_k^R)\) is bounded, then \(\omega(u_k^1, \ldots, v_k^R)\) is nonempty, compact and connected and \(\text{dist}((u_k^1, \ldots, v_k^R), \omega(u_k^1, \ldots, v_k^R)) \to 0\) as \(k \to \infty\),

(ii) \(\omega(u_0^1, \ldots, v_0^R) \subset \text{crit}L\), where \(\text{crit}L\) denotes a set of critical points of \(L\),

(iii) \(L\) is finite and constant on \(\omega(u_0^1, \ldots, v_0^R)\), equal to \(\inf_{k \in \mathbb{N}} L(u_k^1, \ldots, v_k^R) = \lim_{k \to \infty} L(u_k^1, \ldots, v_k^R)\).

Proof: (i) If \((u_k^1, \ldots, v_k^R)\) is bounded, there exists a convergent subsequence, which implies \(\omega(u_k^1, \ldots, v_k^R)\) is nonempty. It also follows \(\omega(u_0^1, \ldots, v_0^R)\) is bounded.

Let now \((\hat{u}^1, \ldots, \hat{v}^R) \notin \omega(u_0^1, \ldots, v_0^R)\) be given. There must exist some \(\varepsilon > 0\) with \((u_k^1, \ldots, v_k^R) \notin B((\hat{u}^1, \ldots, \hat{v}^R), \varepsilon)\), for all \(k \in \mathbb{N}\). But then \(\omega(u_0^1, \ldots, v_0^R) \cap B((\hat{u}^1, \ldots, \hat{v}^R), \varepsilon) = 0\). This proves \(\omega(u_0^1, \ldots, v_0^R)\) is closed and, hence, compact.

Let us assume \(\omega(u_0^1, \ldots, v_0^R)\) is not connected and let \(\omega_c(u_0^1, \ldots, v_0^R) \subset \omega(u_0^1, \ldots, v_0^R)\) be a connected component. Then, \(\omega(u_0^1, \ldots, v_0^R) \setminus \omega_c(u_0^1, \ldots, v_0^R) \neq \emptyset\) and there exists some \(\varepsilon > 0\) such that

\[
\omega_c(u_0^1, \ldots, v_0^R) \cap \omega(u_0^1, \ldots, v_0^R) \setminus \omega_c(u_0^1, \ldots, v_0^R) = \emptyset,
\]

where \(\omega_c(u_0^1, \ldots, v_0^R)\) is an \(\varepsilon\)-neighborhood of \(\omega_c(u_0^1, \ldots, v_0^R)\). We know from Lemma 7.2 (ii) that

\[
\lim_{k \to \infty} (||u_k^1 - u_{k-1}^1||_2 + \cdots + ||v_k^R - v_{k-1}^R||_2) = 0.
\]

Combined with \(\omega_c(u_0^1, \ldots, v_0^R)\) and \(\omega(u_0^1, \ldots, v_0^R) \setminus \omega_c(u_0^1, \ldots, v_0^R)\) being sets of limit points of \((u_k^1, \ldots, v_k^R)\), it implies the existence of a subsequence \((u_k^{\lambda}, \ldots, v_k^{\lambda}) \subset \omega_c(u_0^1, \ldots, v_0^R) \setminus \omega_c(u_0^1, \ldots, v_0^R)\). As this subsequence is bounded, it must have a limit point and \(\omega(u_0^1, \ldots, v_0^R) \cap \omega_c(u_0^1, \ldots, v_0^R) \setminus \omega_c(u_0^1, \ldots, v_0^R) \neq \emptyset\).

Contradiction.

The last part of (i) can be proven in a similar way. If \(\text{dist}((u_k^1, \ldots, u_k^R), \omega(u_0^1, \ldots, v_0^R)) \to 0\), there must exist a subsequence that keeps distance to \(\omega(u_0^1, \ldots, v_0^R)\). But this subsequence again must have a limit point which obviously lies in \(\omega(u_0^1, \ldots, v_0^R)\).
(ii) We have, for all $k \geq 1$, $\xi^r \in \mathbb{R}^n_+, \eta^r \in \mathbb{R}^n_-$

$$L(u_k^1, u_{k-1}^2, \ldots, v_k^R) + \frac{1}{2\lambda^r_{k-1}} \|u_k^1 - u_{k-1}^1\|_2^2 \leq L(\xi^1, u_{k-1}^2, \ldots, v_{k-1}^R) + \frac{1}{2\lambda^r_{k-1}} \|\xi^1 - u_{k-1}^1\|_2^2$$

$$\vdots$$

$$L(u_1^1, \ldots, v_k^R) + \frac{1}{2\mu^r_{k-1}} \|v_k^R - v_{k-1}^R\|_2^2 \leq L(u_1^1, \ldots, v_{k-1}^R, \eta^R) + \frac{1}{2\mu^r_{k-1}} \|\eta^R - v_{k-1}^R\|_2^2$$

Using the bounds on $\lambda^r_k$ and $\mu^r_k$ and the special form of $L$ one gets

$$f_1(u_k^1) + Q(u_k^1, u_{k-1}^2, \ldots, v_{k-1}^R) + \frac{1}{2r^+} \|u_k^1 - u_{k-1}^1\|_2^2 \leq f_1(\xi^1) + Q(\xi^1, u_{k-1}^2, \ldots, v_{k-1}^R) + \frac{1}{2r^+} \|\xi^1 - u_{k-1}^1\|_2^2$$

$$\vdots$$

$$g_R(v_k^R) + Q(u_k^1, \ldots, v_k^R) + \frac{1}{2r^+} \|v_k^R - v_{k-1}^R\|_2^2 \leq g_R(\eta^R) + Q(u_1^1, \ldots, v_{k-1}^R, \eta^R) + \frac{1}{2r^+} \|\eta^R - v_{k-1}^R\|_2^2$$

Let $(\bar{u}^1, \ldots, \bar{v}^R) \in \omega(u_0^1, \ldots, v_0^R)$. There exists a subsequence $(u_k^1, \ldots, v_k^R)$ of $(u_1^1, \ldots, v_1^R)$ with $(u_k^1, \ldots, v_k^R) \to (\bar{u}^1, \ldots, \bar{v}^R)$. Together with Lemma 7.2(ii) this gives

$$\liminf_{k' \to \infty} f_r(u^r_{k'}) + Q(\bar{u}^1, \ldots, \bar{v}^R) \leq f_r(\xi^r) + Q(\bar{u}^1, \ldots, \xi^r, \ldots, \bar{v}^R) + \frac{1}{2r^+} \|\xi^r - \bar{u}^r\|_2^2,$$

for all $1 \leq r \leq R$. We can now set $\xi^r = \bar{u}^r$ to obtain

$$\liminf_{k' \to \infty} f_r(u^r_{k'}) \leq f_r(\bar{u}^r).$$

This and $f_r$ being lower semicontinuous yields

$$\lim_{k' \to \infty} f_r(u^r_{k'}) = f_r(\bar{u}^r).$$

Repeating this for $g_r$, $1 \leq r \leq R$, and recalling the continuity of $Q$ we obtain $L(u_k^1, \ldots, v_k^R) \to L(\bar{u}^1, \ldots, \bar{v}^R)$. Combined with Lemma 7.2(iii) and the closedness properties of $\partial L$(see Remark 1(b) in [2]) proves $0 \in \partial L(\bar{u}^1, \ldots, \bar{v}^R)$.

(iii) As we just seen, for any point $(\bar{u}^1, \ldots, \bar{v}^R) \in \omega(u_0^1, \ldots, v_0^R)$, there exists a subsequence $(u_k^1, \ldots, v_k^R)$ of $(u_1^1, \ldots, v_1^R)$ with $L(u_k^1, \ldots, v_k^R) \to L(\bar{u}^1, \ldots, \bar{v}^R)$. Then $L(\bar{u}^1, \ldots, \bar{v}^R) = \inf L(u^1_k, \ldots, v^R_k)$ as $L(u^1_k, \ldots, v^R_k)$ is non-increasing. This holds for every limit point. Hence, $L$ is finite and constant on the set of limit points.

As in [2] we use the notation

$$z_k := (u^1_k, \ldots, v^R_k), \quad l_k := L(z_k),$$

$$\bar{z} := (\bar{u}^1, \ldots, \bar{v}^R), \quad \bar{l} := L(\bar{z}).$$

The next theorem essentially says that a sequence $z_k$ that starts in the neighborhood of a point $\bar{z}$ as described in (34) and that does not improve $L(\bar{z})$ as given in (33) converges to a critical point near $\bar{z}$.

**Theorem 7.4 (2 Theorem 8).** Let $L$ satisfy $(H), (H1)$ and have the KL-property at some $\bar{z}$. Denote by $U, \eta$ and $\varphi : [0, \eta] \to \mathbb{R}$ the objects connected to the KL-property of $L$ at $\bar{z}$. Let $\rho > 0$ be chosen such that $B(\bar{z}, \rho) \subset U$. Let $z_k$ be generated by (29) with $z_0$ as initial point. Let us assume that

$$\bar{l} < l_k < \bar{l} + \eta,$$

(33)
for all \( k \geq 0 \), and

\[
M \varphi(l_0 - \tilde{l}) + 2\sqrt{2r_+} \sqrt{l_0 - \tilde{l}} + \|z_0 - \tilde{z}\|_2 < \rho
\]  

(34)

where \( M = 2r_+(C\sqrt{2R} + \frac{1}{r_-}) \) and \( C \) is a Lipschitz-constant for \( \nabla Q \) on \( B(\tilde{z}, \sqrt{2R}\rho) \). Then, the sequence \( z_k \) converges to a critical point of \( L \) and the following holds, for all \( k \geq 0 \):

(i) \( z_k \in B(\tilde{z}, \rho) \)

(ii) \( \sum_{i=k+1}^{\infty} \|z_{i+1} - z_i\|_2 \leq M \varphi(l_k - \tilde{l}) + \sqrt{2r_+} \sqrt{l_k - \tilde{l}}. \)

**Proof:** We may without loss of generality assume \( \varphi(l) = 0 \) (just replace \( L \) by \( L - L(\tilde{z}) \)). With Lemma 7.2(i) we have

\[
l_i - l_{i+1} \geq \frac{1}{2r_+} \|z_{i+1} - z_i\|_2^2,
\]

for all \( i \geq 0 \). Moreover, \( \varphi'(l_i) \) makes sense in view of (33) and \( \varphi'(l_i) > 0 \). Hence,

\[
\varphi'(l_i)(l_i - l_{i+1}) \geq \frac{\varphi'(l_i)}{2r_+} \|z_{i+1} - z_i\|_2^2.
\]

Owing to \( \varphi \) being concave, we obtain

\[
\varphi(l_i) - \varphi(l_{i+1}) \geq \frac{\varphi'(l_i)}{2r_+} \|z_{i+1} - z_i\|_2^2,
\]

(36)

for all \( i \geq 0 \). Let us first check (i) for \( k = 0 \) and \( k = 1 \). We know from (34) that \( z_0 \) lies in \( B(\tilde{z}, \rho) \). Furthermore, (35) yields

\[
\frac{1}{2r_+} \|z_1 - z_0\|_2^2 \leq l_0 - l_1 \leq l_0
\]

which gives

\[
\|z_1 - \tilde{z}\|_2^2 \leq \|z_1 - z_0\|_2^2 + \|z_0 - \tilde{z}\|_2^2 \leq \sqrt{2r_+} \sqrt{l_0} + \|z_0 - \tilde{z}\|_2 < \rho.
\]

Let us now prove by induction that \( z_k \in B(\tilde{z}, \rho) \), for all \( k \geq 0 \). We assume this holds true up to some \( k \geq 0 \). Hence, for \( 0 \leq i \leq k \), using \( z_i \in B(\tilde{z}, \rho) \) and \( 0 < l_i < \eta \) we can write the KL-inequality

\[
\varphi'(l_i) \text{dist}(0, \partial L(z_i)) \geq 1.
\]

Lemma 7.2(iii) says

\[
z_i^* := \left( \begin{matrix} \nabla u_1 Q(u_1, v_1) - \nabla u_i Q(u_i, v_i) \\ \vdots \\ 0 \\ \frac{1}{\mu_i} (u_i - u_{i-1}) \\ \frac{1}{\mu_i} (v_i - v_{i-1}) \end{matrix} \right) - \left( \begin{matrix} (u_i^1 - u_{i-1}^1) \\ \vdots \\ (v_i^R - v_{i-1}^R) \end{matrix} \right),
\]

is an element of \( \partial L(z_i) \). So, we have

\[
\varphi'(l_i) \|z_i^*\|_2 \geq 1,
\]

(37)

for all \( 1 \leq i \leq k \). Let us now examine \( \|z_i^*\|_2 \), for \( 1 \leq i \leq k \). On the one hand,

\[
\left\| \left( \begin{matrix} \frac{1}{\lambda_i} (u_i - u_{i-1}) \\ \vdots \\ \frac{1}{\mu_i^R} (v_i^R - v_{i-1}^R) \end{matrix} \right) \right\|_2 \leq \frac{1}{r_-} \|z_i - z_{i-1}\|_2.
\]
On the other hand, for arbitrary \( s_i \in \{ i - 1, i \} \), \( t \in \{ 1, \ldots, 2R \} \),
\[
\left\| (u_{s_i}^1, \ldots, v_{s_2R}^R) - (\bar{u}^1, \ldots, \bar{v}^R) \right\|_2^2 = \left\| u_{s_1}^1 - \bar{u}^1 \right\|_2^2 + \cdots + \left\| v_{s_2R}^R - \bar{v}^R \right\|_2^2 \\
\leq \left\| z_{s_1} - \bar{z} \right\|_2^2 + \cdots + \left\| z_{s_2R} - \bar{z} \right\|_2^2 \leq 2R\rho^2.
\]
Hence, \((u_{s_i}^1, \ldots, v_{s_2R}^R)\) and \( z_i \) lie in \( B(\bar{z}, \sqrt{2R}\rho) \). We can use Lipschitz-continuity of \( \nabla Q \) to obtain
\[
\| \nabla Q(u_{s_i}^1, \ldots, v_{s_2R}^R) - \nabla Q(u_1^1, \ldots, v_1^R) \|_2 \leq C\| z_i - z_{i-1} \|_2,
\]
for any \( \theta \in \{ u^1, \ldots, u^R \} \), which implies
\[
\| z_i^* \|_2 \leq \left( (C\sqrt{2R} + \frac{1}{r_-}) \| z_i - z_{i-1} \|_2, \right.
\]
for all \( 1 \leq i \leq k \). Now (37) yields
\[
\varphi'(l_i) \geq \frac{1}{C\sqrt{2R} + \frac{1}{r_-}} \| z_i - z_{i-1} \|_2^{-1}, \quad 1 \leq i \leq k,
\]
and combined with (36)
\[
\varphi(l_i) - \varphi(l_{i+1}) \geq \frac{1}{M} \| z_{i+1} - z_i \|_2^2, \quad 1 \leq i \leq k.
\]
This is equivalent to
\[
\| z_i - z_{i-1} \|_2 \geq M \left( \varphi(l_i) - \varphi(l_{i+1}) \right) \geq \| z_{i+1} - z_i \|_2, \quad 1 \leq i \leq k.
\]
and, using \( ab \leq (a^2 + b^2)/2 \), gives
\[
\| z_i - z_{i-1} \|_2 + M(\varphi(l_i) - \varphi(l_{i+1})) \geq 2\| z_{i+1} - z_i \|_2, \quad 1 \leq i \leq k.
\]
(38)

Summation over \( i \) leads to
\[
\| z_1 - z_0 \|_2 + M(\varphi(l_1) - \varphi(l_{k+1})) \geq \sum_{i=1}^{k} \| z_{i+1} - z_i \|_2 + \| z_{k+1} - z_k \|_2.
\]
Therefore, by using the monotonicity properties of \( \varphi \) and \( l_k \)
\[
\| z_1 - z_0 \|_2 + M\varphi(l_0) \geq \sum_{i=1}^{k} \| z_{i+1} - z_i \|_2.
\]
Finally,
\[
\| z_{k+1} - \bar{z} \|_2 \leq \sum_{i=1}^{k} \| z_{i+1} - z_i \|_2 + \| z_1 - \bar{z} \|_2 \leq M\varphi(l_0) + 2\sqrt{2r_+\sqrt{l_0}} + \| z_0 - \bar{z} \|_2 < \rho
\]
which closes the induction and proves (i). Moreover, (38) holds for all \( i \geq 1 \). We can sum from \( k \) to \( K \) and get
\[
\| z_k - z_{k-1} \|_2 + M(\varphi(l_k) - \varphi(l_{K+1})) \geq \sum_{i=k}^{K} \| z_{i+1} - z_i \|_2 + \| z_{K+1} - z_k \|_2.
\]

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For $K \to \infty$, this becomes
\[
\sum_{i=k}^{\infty} \|z_{i+1} - z_i\|_2 \leq \|z_k - z_{k-1}\|_2 + M\varphi(l_k).
\]

We conclude with proving (ii)
\[
\sum_{i=k}^{\infty} \|z_{i+1} - z_i\|_2 \leq M\varphi(l_k) + \sqrt{2r_+\sqrt{l_{k-1}}} \leq M\varphi(l_{k-1}) + \sqrt{2r_+\sqrt{l_{k-1}}}.
\]

This implies $z_k$ is convergent and, therefore, its limit is a critical point. This was guaranteed by Lemma 7.3.

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