Schur-positive sets of permutations via products and grid classes

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Abstract

Characterizing sets of permutations whose associated quasisymmetric function is symmetric and Schur-positive is a long-standing problem in algebraic combinatorics. In this paper we present a general method to construct Schur-positive sets and multisets, based on geometric grid classes and the product operation. Our approach produces many new instances of Schur-positive sets, and provides a broad framework that explains the existence of known such sets that until now were sporadic cases.

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1 Introduction

1.1 Background

Given any subset $A$ of the symmetric group $S_n$, define the quasisymmetric function

$$Q(A) := \sum_{\pi \in A} F_{n, \text{Des}(\pi)},$$

where $\text{Des}(\pi) := \{i : \pi(i) > \pi(i + 1)\}$ is the descent set of $\pi$ and $F_{n, \text{Des}(\pi)}$ is Gessel’s fundamental quasisymmetric function (see Section 2.3 for more background and definitions). The quasisymmetric function $Q(A)$ was introduced in [15], where Gessel was concerned with the case when $A$ is the set of linear extensions of a labeled poset $P$. The following long-standing problem was first posed in [16].

**Problem 1.1.** For which subsets of permutations $A \subseteq S_n$ is $Q(A)$ symmetric?

A symmetric function is called *Schur-positive* if all coefficients in its expansion in the Schur basis are nonnegative. The problem of determining whether a given symmetric function is Schur-positive is a major problem in contemporary algebraic combinatorics [32].

By analogy, a set (or, more generally, a multiset) $A$ of permutations in $S_n$ is called *Schur-positive* if $Q(A)$ is symmetric and Schur-positive. Classical examples of Schur-positive sets of permutations include inverse descent classes and Knuth classes [15, Theorem 5.5] and permutations of fixed inversion number [3, Prop. 9.5].

An exotic example of a Schur-positive set, different from all the above ones, was recently found: the set of arc permutations, which may be characterized as those avoiding the patterns $\{\sigma \in S_4 : |\sigma(1) - \sigma(2)| = 2\}$. A bijective proof of its Schur-positivity is given in [11]. Inspired by this example,
Woo and Sagan raised the problem of finding other Schur-positive pattern-avoiding sets [23]. Our goal in this paper is to provide a conceptual approach that explains the existing results and produces new examples of Schur-positive pattern-avoiding sets of permutations. An important tool in our approach will be to consider products of geometric grid classes.

A geometric grid class consists of those permutations that can be drawn on a specified set of line segments of slope ±1, whose locations are determined by the positions of the corresponding entries in a matrix $M$ with entries in \{0, 1, −1\}. An example of a matrix and its corresponding line segments is given in Figure 1.1 Let $G_n(M)$ be the set of permutations in $S_n$ that can be obtained by placing $n$ dots on the segments in such a way that there are no two dots on the same vertical or horizontal line, labeling the dots with 1, 2, . . . , $n$ from bottom to top, and then reading them from left to right. Geometric grid classes may be characterized by a finite set of forbidden patterns [5]. It follows from [16, Theorem 5.5] together with elementary combinatorial arguments that all one-column grid classes, as well as layered and colayered permutation classes are Schur-positive; see Section 4 below.

\begin{align*}
M &= \begin{pmatrix}
0 & 1 \\
-1 & 0 \\
1 & -1
\end{pmatrix}
\end{align*}

Figure 1.1: A matrix $M$, its corresponding grid of segments, and a drawing of the permutation 62354781 $\in G_8(M)$.

1.2 Summary of main results

In general, the product of Schur-positive subsets of $S_n$ does not give a Schur-positive multiset or set. We are interested in finding families of subsets whose product is Schur-positive, either as a multiset or as a set.

After introducing some definitions in Section 2, a little background on fine sets in Section 3, and some simple examples of Schur-positive grid classes in Section 4, our first main result about products of Schur-positive sets appears in Section 5 as Theorem 5.12. For each $J \subseteq \{1, \ldots, n-1\}$, define the descent class $D_{n,J} := \{\pi \in S_n : \text{Des}(\pi) = J\}$, and its inverse $D_{n,J}^{-1} := \{\pi^{-1} : \pi \in D_{n,J}\}$. It was shown by Gessel [15] that $D_{n,J}^{-1}$ is Schur-positive. Theorem 5.12 includes the following statement, which appears also as Theorem 5.4.

**Theorem 1.2.** For every Schur-positive set $B \subseteq S_n$ and every $J \subseteq \{1, \ldots, n-1\}$, the multiset product $BD_{n,J}^{-1}$ is Schur-positive.

We provide a representation-theoretic proof that involves Solomon’s descent representations and Stanley’s shuffling theorem.

If instead of considering multiset products we are interested in the underlying sets being Schur-positive, we have a more restricted theorem. Let $c$ be the $n$-cycle $(1, 2, \ldots, n)$, and let $C_n = \langle c \rangle = \{c^k : 0 \leq k < n\}$, the cyclic subgroup generated by $c$, which is shown to be Schur-positive in Corollary 4.11. We can interpret $D_{n-1,J}^{-1}$ as a subset of $S_n$ by identifying $S_{n-1}$ as the set of
the permutations in $S_n$ that fix $n$. The following theorem about multiset products appears as Theorem 7.1.

**Theorem 1.3.** For every $J \subseteq \{1, \ldots, n-2\}$, the set product $D_{n-1,J}^{-1}C_n$ is Schur-positive.

The proof of this theorem combines a descent-set-preserving bijection together with sieve methods and representation-theoretic arguments.

Schur-positivity of various set and multiset products of grid classes follows from the above two theorems. In Section 6 we discuss applications of Theorem 1.2 to vertical rotations of grids, of which arc permutations are a special case, and thus we obtain a short proof of their Schur-positivity. Applications of Theorem 1.3 to horizontal rotations are discussed in Section 7, together with equidistribution phenomena.

Theorems 1.2 and 1.3 imply Schur-positivity of certain grid classes. One such class is $G(M_k)$, the grid obtained by vertical rotation of the grid whose matrix is a $k \times 1$ matrix of ones (see the drawing of the grid $G(M_3)$ in Figure 6.1). The following consequence is stated in Corollary 7.6.

**Corollary 1.4.** For every positive integers $k$ and $n$, $Q(G_n(M_k))$ is Schur-positive.

A closely related application involves cyclic descents, which were introduced by Cellini [9] and studied by Petersen, Dilks and Stembridge [19, 10] (see Definition 6.3). Even though inverse cyclic descent classes are not necessarily Schur-positive, it follows from Theorem 1.3 that subsets of permutations with fixed inverse cyclic descent number are Schur-positive (Corollary 7.7).

Other geometric operations on grid classes, such as reflections and stacking, are applied in Section 8 to get more examples of Schur-positive grid classes. Section 9 presents an explicit compact description of the set product of a one-column grid class with an arbitrary grid class. This result generalizes [4, Theorem 6]. The paper concludes with a list of some open problems, questions, conjectures, and ideas for further work in Section 10.

Table 1 summarizes our main results and conjectures about Schur-positive sets and multisets, together with their location in the paper.

## 2 Preliminaries

### 2.1 Descent classes and ribbons

Let $[n] := \{1, 2, \ldots, n\}$, and let $S_n$ denote the symmetric group on $[n]$. The descent set of a permutation $\pi \in S_n$ is defined by

$$\text{Des}(\pi) := \{i : \pi(i) > \pi(i + 1)\}.$$ 

For $J \subseteq [n-1]$, define the descent class

$$D_{n,J} := \{\pi \in S_n : \text{Des}(\pi) = J\}$$

and the corresponding inverse descent class $D_{n,J}^{-1} := \{\pi^{-1} : \pi \in D_{n,J}\}$.

For a skew shape $\lambda/\mu$, let $\text{SYT}(\lambda/\mu)$ be the set of standard Young tableaux of shape $\lambda/\mu$. We use the English notation in which row indices increase from top to bottom.

For $T \in \text{SYT}(\lambda/\mu)$, define its descent set by

$$\text{Des}(T) := \{i : i + 1 \text{ lies southwest of } i \text{ in } T\}.$$
A ribbon or zigzag shape is a connected skew shape that does not contain a $2 \times 2$ square. For example, every hook is a ribbon. There is a natural bijection between the set of all subsets of $[n-1]$ and the set of all ribbons of size $n$, where the size is defined to be the number of cells.

**Definition 2.1.** Given a subset $J \subseteq [n-1]$, let $Z_{n,J}$ be the ribbon with $n$ cells labeled $1, \ldots, n$ increasing in the northeast direction, where cell $i + 1$ is immediately above cell $i$ if $i \in J$, and immediately to the right of cell $i$ otherwise.

For example, if $n = 9$ and $J = \{1, 3, 5, 6\}$, then

$$Z_{9,J} = \begin{array}{cccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}$$

Consider the map from the set of standard Young tableaux (SYT for short) of all ribbons of size $n$ to permutations in $S_n$ defined by taking the reading word of the SYT, that is, listing its entries starting from the southwest corner and moving along the shape. The restriction of this map to the
set SYT($Z_{n,J}$) of tableaux of a fixed ribbon shape $Z_{n,J}$ is a bijection to permutations in $S_n$ with descent set $J$. This bijection has the property that the descent set of the SYT becomes the descent set of the inverse of the associated permutation. To see this, suppose that the entries of a SYT $T$ starting from the SW corner are $\pi = \pi(1)\pi(2)\ldots\pi(n)$. Then, $i$ is a descent of $\pi^{-1}$ if $i+1$ appears to the left of $i$ in this sequence, which is equivalent to saying that $i+1$ appears in a box lower than $i$ in $T$, that is, $i$ is a descent of $T$. We conclude that for every $J \subseteq [n-1]$, the distribution of the statistic Des is the same over $D_{n,J}^{-1}$ and over SYT($Z_{n,J}$).

### 2.2 Knuth classes and shuffles

The well-known Robinson–Schensted correspondence maps each permutation $\pi \in S_n$ to a pair $(P_\pi, Q_\pi)$ of standard Young tableaux of the same shape $\lambda \vdash n$. Recall that this correspondence is a Des-preserving bijection in the following sense.

**Lemma 2.2 (30, Lemma 7.23.1).** For every permutation $\pi \in S_n$,

$$\text{Des}(P_\pi) = \text{Des}(\pi^{-1}) \quad \text{and} \quad \text{Des}(Q_\pi) = \text{Des}(\pi).$$

The Knuth class corresponding to a standard Young tableau $T$ of size $n$ is the set of permutations

$$C_T = \{ \pi \in S_n : P_\pi = T \}.$$ 

If $T$ has shape $\lambda$ then we say that $C_T$ is a Knuth class of shape $\lambda$. By Lemma 2.2, inverse descent classes are disjoint unions of Knuth classes.

A Knuth relation on a permutation $\pi$ is a switch of two adjacent letters $ac$, where $bac$ or $acb$ are adjacent in $\pi$ and $a < b < c$ or $c < b < a$. Two permutations in $S_n$ are Knuth-equivalent if one can be obtained from the other by applying a sequence of Knuth relations.

**Theorem 2.3 (22, Theorem 3.4.3).** The Knuth-equivalence classes resulting from the above Knuth relations on $S_n$ are precisely the Knuth classes.

**Definition 2.4.** Let $U$ and $V$ be disjoint sets of letters and let $\sigma$ and $\tau$ be two permutations of $U$ and $V$ respectively. A shuffle of $\sigma$ and $\tau$, denoted by $\sigma \shuffle \tau$, is the set of all permutations of $U \cup V$ where the letters of $U$ appear in same order as in $\sigma$ and the letters of $V$ appear in same order as in $\tau$. For sets (or multisets) $A$ and $B$ of permutations on disjoint finite sets of letters, denote by $A \shuffle B$ the (multi)set of all shuffles of a permutation in $A$ with a permutation in $B$.

For example, if $A$ and $B$ are the multisets $A = \{\{12, 12\}\}$ and $B = \{\{3\}\}$, then $A \shuffle B = \{\{312, 312, 132, 132, 123, 123\}\}$; if $A$ and $B$ are the sets $A = \{12, 21\}$ and $B = \{43\}$, then $A \shuffle B = \{1243, 1423, 1432, 4123, 4132, 4312, 2143, 2413, 2431, 4213, 4231, 4321\}$. Shuffles will play an important role in Section 5.

For partitions $\mu \vdash k$ and $\nu \vdash n-k$, let $(\mu, \nu)$ be the skew Young diagram obtained by placing Young diagrams of shape $\mu$ and $\nu$ so that the upper-right vertex of the Young diagram of shape $\mu$ coincides with the lower-left vertex of the Young diagram of shape $\nu$. A Knuth class on the letters $k+1, \ldots, n$ is an equivalence class of the symmetric group on these letters resulting from the Knuth relations (equivalently, a set obtained from a Knuth class in $S_{n-k}$ by shifting the letters up by $k$). The following result belongs to mathematical folklore. Some generalizations of it can be found in [8, 21].
Theorem 2.5. Let $A$ be a Knuth class of shape $\mu$ on the letters $1, \ldots, k$ and $B$ a Knuth class of shape $\nu$ on the letters $k+1, \ldots, n$. The following hold.

1. $A \sqcup B$ is closed under Knuth relations.

2. The distribution of $\Des$ is the same over $A \sqcup B$ and over $\SYT((\mu, \nu))$.

Proof. For the first part, we will prove that for $\pi \in A \sqcup B$ containing adjacent letters $bac$ with $1 \leq a < b < c \leq n$, switching $a$ and $c$ gives another permutation $\overline{\pi} \in A \sqcup B$. The other cases are identical. First notice that there exist unique $\sigma \in A$ and $\tau \in B$ such that $\pi \in \sigma \sqcup \tau$. If $a < k < c$, then clearly $\overline{\pi} \in \sigma \sqcup \tau$. If $c \leq k$, then $bac$ is a sequence of adjacent letters in $\sigma$, so switching $a$ and $c$ in $\sigma$ we get a permutation $\overline{\sigma} \in A$ such that $\overline{\pi} \in \overline{\sigma} \sqcup \tau$. The case $k < a$ is similar, completing the proof of part 1.

For the second part, define a map $f : A \sqcup B \rightarrow \SYT((\mu, \nu))$ as follows. For a permutation $\pi \in A \sqcup B$, there exist unique $\sigma \in A$ and $\tau \in B$ such that $\pi \in \sigma \sqcup \tau$. Let $Q_\sigma$ and $Q_\tau$ be the recording tableaux that correspond to $\sigma$ and $\tau$ under Robinson–Schensted, which have shape $\mu$ and $\nu$, respectively.

Let $a_1 \cdots a_k$ be the list of positions of the letters of $\sigma$ in $\pi$, that is, the ordered set $\{\pi^{-1}(i) : 1 \leq i \leq k\}$. Similarly, let $b_1 \cdots b_{n-k}$ be the list of positions of the letters of $\tau$ in $\pi$. Place $Q_\sigma$ and $Q_\tau$ so that the upper-right vertex of $Q_\sigma$ and the lower-left vertex of $Q_\tau$ coincide, and replace each letter $i$ in the resulting skew SYT as follows: if $i \leq k$ replace $i$ by $a_i$, if $k < i$ replace $i$ by $b_{i-k}$. For example, if $\pi = 16783245$ and $k = 3$, we have $\sigma = 132$, $\tau = 67845$ and

$$f(\pi) = \begin{array}{cccc}
1 & 5 & 7 & 8 \\
2 & 3 & 4 & \\
6 & & & 
\end{array}.$$ 

One can verify that $f$ is a Des-preserving bijection.

The following result, due to Stanley, will be used in Section 5. For bijective proofs, see [17, 27].

Proposition 2.6 ([29, Ex. 3.161]). Given two permutations $\sigma$ and $\tau$ of disjoint sets of integers, the distribution of the descent set over all shuffles of $\sigma$ and $\tau$ depends only on $\Des(\sigma)$ and $\Des(\tau)$.

Remark 2.7. Theorem 2.5 and Proposition 2.6 may be generalized to shuffles of any number of Knuth classes and permutations, respectively.

2.3 Symmetric and quasisymmetric functions

Schur functions indexed by partitions of $n$ form a distinguished basis for $\Lambda^n$, the vector space of homogeneous symmetric functions of degree $n$; see, e.g., [30, Corollary 7.10.6]. A symmetric function is Schur-positive if all the coefficients in its expansion in the basis of Schur functions are nonnegative.

The irreducible characters of $S_n$ are also indexed by partitions of $n$. The Frobenius image of an $S_n$-character $\chi = \sum_{\lambda \vdash n} c_\lambda \lambda^\lambda$ is the symmetric function $f = \sum_{\lambda \vdash n} c_\lambda s_\lambda$, which we denote by $\ch(\chi)$.

It is clear from the combinatorial definition of Schur functions [30, Definition 7.10.1 and discussion in p. 339] that for every pair of partitions $\mu$ and $\nu$ we have

$$s_{(\mu, \nu)} = s_\mu s_\nu. \quad (1)$$
Since the characteristic map is an algebra isomorphism [30 Proposition 7.18.2], it follows that if the size of the skew shape \((\mu, \nu)\) is \(n\), then \(\chi^{(\mu, \nu)} = (\chi^\mu \otimes \chi^\nu)^\uparrow_{S^n}\). This identity is naturally generalized to skew shapes with \(t\) disconnected components as follows.

**Lemma 2.8.** Consider a skew shape \(\lambda/\mu\) of size \(n\), which consists of disconnected diagrams \(\nu^{(1)} \vdash k_1, \ldots, \nu^{(t)} \vdash k_t\). Then \(s_{\lambda/\mu} = s_{\nu^{(1)}} s_{\nu^{(2)}} \cdots s_{\nu^{(t)}}\) and

\[
\chi^{\lambda/\mu} = (\chi^{\nu^{(1)}} \otimes \chi^{\nu^{(2)}} \otimes \cdots \otimes \chi^{\nu^{(t)}})^\uparrow_{S_{k_1} \times S_{k_2} \times \cdots \times S_{k_t}}.
\]

Of special interest is the case of two connected components, the first consisting of one box and the second being an arbitrary partition \(\mu \vdash n-1\). Then \(\chi^{((1), \mu)}\) is equal to the induced character \(\chi^\mu \uparrow_{S^n}\), which amounts to the multiplicity-free sum of all irreducible characters indexed by partitions obtained by adding a box to the Young diagram of \(\mu\) [22 Theorem 2.8.3]. Thus, for every \(\mu \vdash n-1\),

\[
s_{\lambda} s_{\mu} = \sum_{\lambda/(\lambda/\mu) = (1)} s_{\lambda}.
\]

**Quasisymmetric functions** were introduced by Gessel, see [30 Section 7.19] for definitions and background. For each subset \(D \subseteq [n-1]\), define the quasisymmetric function

\[
F_{n,D} := \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} x_{i_1} x_{i_2} \cdots x_{i_n}.
\]

The set \(\{F_{n,D} : D \subseteq [n-1]\}\) is a basis of the vector space of homogeneous quasisymmetric functions of degree \(n\).

Let \(B\) be a (multi)set of combinatorial objects, equipped with a descent map \(\text{Des}\) which associates with each element \(b \in B\) a subset \(\text{Des}(b) \subseteq [n-1]\). Define the quasisymmetric function

\[
Q(B) := \sum_{b \in B} m(b, B) F_{n,\text{Des}(b)},
\]

where \(m(b, B)\) is the multiplicity of the element \(b\) in \(B\). Note that for two (multi)sets \(B\) and \(B'\), the equality \(Q(B) = Q(B')\) is equivalent to the fact that \(\text{Des}\) has the same distribution over \(B\) and over \(B'\).

The following key observation is due to Gessel.

**Proposition 2.9** ([30 Theorem 7.19.7]). For every skew shape \(\lambda/\mu\),

\[
Q(\text{SYT}(\lambda/\mu)) = s_{\lambda/\mu}.
\]

The next result about inverse descent classes is again due to Gessel. Here we include a proof because the same idea will be used in Section 4.1. For a symmetric function \(f = \sum_{\lambda \vdash n} c_{\lambda} s_{\lambda}\), let \(\langle f, s_{\mu} \rangle = c_{\lambda}\).

**Lemma 2.10** ([15 Theorem 7]). For \(J \subseteq [n-1]\) and \(\lambda \vdash n\),

\[
Q(D_{n,J}^{-1}) = Q(\text{SYT}(Z_{n,J})) = s_{Z_{n,J}}
\]

and

\[
\langle Q(D_{n,J}^{-1}), s_{\lambda} \rangle = |\{P \in \text{SYT}(\lambda) : \text{Des}(P) = J\}|.
\]
Proof. The expression as a skew Schur function follows from the Des-preserving bijection described at the end of Section 2.1 and Proposition 2.9, which imply that \( Q(D_{n,j}^{-1}) = Q(SYT(Z_{n,j})) = s_{Z_{n,j}} \).

The expansion as a sum of Schur functions can be obtained by noting, using Lemma 2.2, that the image of \( D_{n,j}^{-1} \) by RSK is the set of pairs of standard Young tableaux \( (P,Q) \) with shape \( P \vdash n \) and \( \text{Des}(P) = J \). By Lemma 2.2 and Proposition 2.9, it follows that

\[
Q(D_{n,j}^{-1}) = \sum_{\pi \in D_{n,j}^{-1}} \prod_{\lambda \vdash n} \sum_{\sum_{P \in \text{SYT}(\lambda)} \text{Des}(P) = J} \prod_{Q \in \text{SYT}(\lambda)} \prod_{\sum_{\text{Des}(Q)} = J} \prod_{\sum_{\lambda \vdash n} \prod_{\{P \in \text{SYT}(\lambda) : \text{Des}(P) = J\}} s_{\lambda}}. \]

In particular, inverse descent classes are symmetric and Schur-positive. The following variation of Problem 1.1 was proposed in [3].

**Problem 2.11.** For which subsets of permutations \( A \subseteq S_n \) is \( Q(A) \) Schur-positive?

### 2.4 Grid classes

A useful tool in our construction of Schur-positive sets is the concept of geometric grid classes, introduced and studied by Albert et al. [5]. To each matrix \( M \) with entries in \( \{0, 1, -1\} \), we associate the grid obtained by placing line segments of slope 1 and \(-1\) in the locations of the ones and negative ones in \( M \), respectively. See Figure 1.1 for an example. A geometric grid class consists of those permutations that can be drawn on such a grid.

**Definition 2.12.**

1. For any matrix \( M \) with entries in \( \{0, 1, -1\} \), let \( \mathcal{G}_n(M) \) be the set of permutations in \( S_n \) that can be obtained by placing \( n \) dots on the segments of the grid corresponding to \( M \) (in such a way that no two dots have the same \( x \)- or \( y \)-coordinate), labeling the dots with 1, 2, \ldots, \( n \) by increasing \( y \)-coordinate, and then reading them by increasing \( x \)-coordinate.

2. Let \( \mathcal{G}(M) = \bigcup_{n \geq 0} \mathcal{G}_n(M) \). We call \( \mathcal{G}(M) \) a geometric grid class, or simply a grid class for short. All grid classes that appear in this paper are geometric grid classes.

**Example.** Left-unimodal permutations, defined as those for which every prefix forms an interval in \( \mathbb{Z} \), are those in the grid class

\[
\mathcal{G} \left( \begin{array}{c} 1 \\ -1 \end{array} \right). \]

A drawing of a permutation on this grid is shown in Figure 2.1. Denote by \( \mathcal{L} \) the set of left-unimodal permutations.

Let \( \text{st} \) be the standardization operation that, given a sequence of distinct integers, replaces the smallest entry with a 1, the second smallest with a 2, and so on. A permutation \( \pi \) contains another permutation \( \sigma \) if there is a subsequence of \( \pi \) whose standardization is \( \sigma \). For example, 6425173 contains 2314 because \( \text{st}(4517) = 2314 \).

Since removing dots from the drawing of a permutation on a grid yields drawings of the permutations that it contains, it is clear that every geometric grid class is closed under pattern containment. Thus, it is characterized by its set of minimal forbidden patterns, which is always finite, as shown in [5].
Figure 2.1: A drawing of the permutation 4532617 on the grid for left-unimodal permutations.

3 Background on fine sets

3.1 Fine sets and Schur-positivity

Definition 3.1. 1. A set (or multiset) of combinatorial objects \( B \) equipped with a descent map \( \text{Des} : B \to 2^{[n-1]} \) is called set-fine (or multiset-fine) if the quasisymmetric function \( Q(B) \) in Equation (3) is symmetric and Schur-positive. We use the term fine to mean set-fine or multiset-fine when there is no confusion.

2. The (multi)set \( B \) is fine for a complex \( S_n \)-representation \( \rho \) if \( \text{ch}(\chi^\rho) = Q(B) \).

This paper considers fine sets and multisets of permutations and standard Young tableaux. Since the sum of Schur-positive symmetric functions is Schur-positive, we have the following.

Observation 3.2. Unions of disjoint fine sets, as well as unions of fine multisets, are fine.

We say that a sequence \((a_1, \ldots, a_n)\) of distinct positive integers is co-unimodal if there exists \(1 \leq m \leq n\) such that
\[ a_1 > a_2 > \ldots > a_m < a_{m+1} < \ldots < a_n. \]

Let \( \mu = (\mu_1, \ldots, \mu_t) \) be a composition of \( n \). A sequence of \( n \) positive integers is called \( \mu \)-modal if the first \( \mu_1 \) entries form a co-unimodal sequence, the next \( \mu_2 \) entries form a co-unimodal sequence, and so on. A permutation \( \pi \in S_n \) is called \( \mu \)-modal if the sequence \((\pi(1), \ldots, \pi(n))\) is \( \mu \)-modal. An element \( b \in B \) is \( \mu \)-modal if there exists a \( \mu \)-modal permutation \( \pi \in S_n \), such that \( \text{Des}(b) = \text{Des}(\pi) \). Denote by \( B_\mu \) the sub(multi)set of all \( \mu \)-modal elements in \( B \).

Example. The standard Young tableau
\[
T = \begin{array}{cccc}
1 & 3 & 5 & 8 \\
2 & 4 & 7 \\
6 
\end{array}
\]
is \((3,1,4)\)-modal, since \( \pi = 21346578 \in S_8 \) is \((3,1,4)\)-modal and \( \text{Des}(T) = \{1,3,5\} = \text{Des}(\pi) \).

Let \( \text{Comp}(n) \) denote the set of all compositions of \( n \). For \( \mu \in \text{Comp}(n) \), let \( S(\mu) := \{\mu_1, \mu_1 + \mu_2, \ldots, \mu_1 + \cdots + \mu_{t-1}\} \).

Theorem 3.3 (\cite[Theorem 1.5]{B}). For every set (or multiset) of combinatorial objects \( B \) equipped with a descent map \( \text{Des} : B \to 2^{[n-1]} \), the following are equivalent.
(i) \( \mathcal{B} \) is fine, that is, the quasisymmetric function \( Q(\mathcal{B}) \) is symmetric and Schur-positive.

(ii) The function \( \chi^\mathcal{B} : \text{Comp}(n) \rightarrow \mathbb{Z} \) defined by

\[
\chi^\mathcal{B}(\mu) := \sum_{b \in \mathcal{B}_\mu} m(b, \mathcal{B})(-1)^{|\text{Des}(b)\setminus S(\mu)|}
\]

is an \( S_n \)-character, i.e., it does not depend on the order of parts in \( \mu \) and is a linear combination, with nonnegative integer coefficients, of the irreducible characters of \( S_n \).

(iii) There exist a (multi)set partition \( \mathcal{B} = \mathcal{B}_1 \sqcup \ldots \sqcup \mathcal{B}_m \) and Des-preserving bijections from each \( \mathcal{B}_i \) to the set of all standard Young tableaux of some shape \( \lambda(i) \vdash n \), for each \( 1 \leq i \leq m \).

(iv) There exists a basis \( \{ C_b : b \in \mathcal{B} \} \) for an \( S_n \)-representation space \( V \) such that the linear action of every simple reflection \( s_i \) has the form

\[
s_i(C_b) = \begin{cases} 
-C_b, & \text{if } i \in \text{Des}(b); \\
C_b + \sum_{b' \in \mathcal{B} : i \in \text{Des}(b')} a_i(b, b') C_{b'}, & \text{otherwise},
\end{cases}
\]

for suitable coefficients \( a_i(b, b') \).

Remark 3.4. By Lemma 2.2, the distribution of the descent set over all standard Young tableaux of some shape \( \lambda \) is equal to its distribution over all permutations in a Knuth class of shape \( \lambda \). Thus, condition (iii) in Theorem 3.3 is equivalent to the existence of a (multi)set partition \( \mathcal{B} = \mathcal{B}_1 \sqcup \ldots \sqcup \mathcal{B}_m \) and Des-preserving bijections from each \( \mathcal{B}_i \) to the set of permutations in a Knuth class of shape \( \lambda(i) \vdash n \), for each \( 1 \leq i \leq m \).

Theorem 3.5 ([1, Theorem 3.2]). A set (or multiset) \( \mathcal{B} \) is fine for an \( S_n \)-representation \( \rho \), namely \( \text{ch}(\chi^\rho) = Q(\mathcal{B}) \), if and only if

\[
\chi^\rho(\mu) = \sum_{b \in \mathcal{B}_\mu} (-1)^{|\text{Des}(b)\setminus S(\mu)|}
\]

for every partition \( \mu \vdash n \).

Remark 3.6. Theorems 3.3 and 3.5 were proved in [3] and [1] for sets. It can be checked that the proofs hold for multisets as well.

The next proposition lists a few examples of set-fine subsets of \( S_n \).

Proposition 3.7 ([3 Prop. 9.5]). The following subsets of \( S_n \) are fine:

(i) all permutations of fixed Coxeter length (equivalently, fixed inversion number);

(ii) subsets closed under conjugation (e.g., conjugacy classes, the set of involutions, and the set of permutations with fixed cycle number);

(iii) subsets closed under Knuth relations (e.g., Knuth classes, inverse descent classes, and the set of permutations with fixed inverse descent number).
3.2 Arc permutations

An example of a fine set that does not fall in any of the cases in Proposition 3.7 is the set of arc permutations in \( S_n \). In the following definition, an interval in \( Z_n \) refers to the set obtained by reducing a interval in \( Z \) modulo \( n \).

**Definition 3.8.** A permutation \( \pi \in S_n \) is an *arc permutation* if, for every \( 1 \leq j \leq n \), the first \( j \) letters in \( \pi \) form an interval in \( Z_n \). Denote by \( A_n \) the set of arc permutations in \( S_n \).

For example, the permutation 12543 is an arc permutation in \( S_5 \), but 125436 is not an arc permutation in \( S_6 \), since \( \{1, 2, 5\} \) is an interval in \( Z_5 \) but not in \( Z_6 \).

Arc permutations were introduced in the study of flip graphs of polygon triangulations \([2]\). Some combinatorial properties of these permutations, including their description as a union of grid classes and their descent set distribution, are studied in \([11]\). In particular, it follows from \([11, Theorem 5]\) that \( A_n \) is a Schur-positive set. One of the goals of this paper is to explain this result by providing a general recipe for constructing Schur-positive subsets of \( S_n \).

4 Elementary examples of Schur-positive grid classes

We remark that, by definition, grid classes are always sets of permutations, that is, there are no multiplicities.

4.1 One-column grid classes

Grid classes whose matrix consists of one column are particularly interesting because they are unions of inverse descent classes, as we will see in Proposition 4.6 and thus fine sets by Lemma 2.10. Since we can ignore zero entries in the matrix without changing the grid class, we will assume that the entries of the matrix are in \( \{1, -1\} \). If \( v \in \{1, -1\}^k \), let \( G^v \) denote the one-column grid class where the entries of the matrix are given by \( v \) from bottom to top, and let \( G^v_n = G^v \cap S_n \). Sometimes we only write the signs of the entries of \( v \) for convenience. For example, \( G^{++} = L \) is the class of left-unimodal permutations, and \( G^{++} \) is the class of shuffles of two increasing sequences, with all the letters in one sequence being smaller than all the letters in the other. In general, we denote by \( G^{+k} \) the class of shuffles of \( k \) increasing sequences, each sequence consisting of a consecutive set of letters.

The following observation will be useful throughout this section.

**Observation 4.1.** For a given grid class \( H \), the set \( \{ \pi^{-1} : \pi \in H \} \) is a grid class, whose grid is obtained by reflecting the grid for \( H \) across the southwest-northeast diagonal.

We can easily characterize \( G^{+k} \) and \( G^{-k} \) as unions of inverse descent classes, allowing us to express \( Q(G^{+k}) \) and \( Q(G^{-k}) \) in terms of skew Schur functions, as follows.

**Observation 4.2.** For every \( \pi \in S_n \) and \( k \geq 1 \),

\[
\pi \in G^{+k} \iff \text{des}(\pi^{-1}) \leq k - 1
\]

and

\[
\pi \in G^{-k} \iff \text{des}(\pi^{-1}) \geq n - k.
\]
In the following proposition, the height and the width of a ribbon refer to the number of rows and columns, respectively.

**Proposition 4.3.** \( Q(G^+ k) \) (resp. \( Q(G^- k) \)) is the multiplicity-free sum of symmetric functions of ribbons of height (resp. width) at most \( k \). In terms of Schur functions,

\[
Q(G^+ k) = \sum_{\lambda \vdash n} |\{ P \in \text{SYT}((\lambda) : |\text{Des}(P)| \leq k-1\}| s_{\lambda},
Q(G^- k) = \sum_{\lambda \vdash n} |\{ P \in \text{SYT}((\lambda) : |\text{Des}(P)| \geq n-k\}| s_{\lambda}.
\]

**Proof.** By Observation 4.2

\[
G^+ k = \bigsqcup_{|J| < k} D_{n,J}^{-1}.
\]

Thus, by Lemma 2.10

\[
Q(G^+ k) = \sum_{|J| < k} Q(D_{n,J}^{-1}) = \sum_{|J| < k} Q(\text{SYT}(Z_{n,J})) = \sum_{|J| < k} s_{Z_{n,J}}
\]

which proves the first statement for \( Q(G^+ k) \). The statement for \( Q(G^- k) \) is proved similarly by taking the sum over sets \( J \) with \( |J| \geq n-k \). The expressions in terms of Schur functions also follow from Lemma 2.10.

We obtain a particularly simple expression when \( k = 2 \).

**Corollary 4.4.** We have

\[
Q(G^+ 2) = s_n + \sum_{a=1}^{\lfloor n/2 \rfloor} (n - 2a + 1)s_{n-a,a}.
\]

**Proof.** To apply Proposition 4.3, we find all SYT with at most one descent. The one-row tableau is the only one with no descents, and for each \( 1 \leq a \leq \lfloor n/2 \rfloor \), there are \( n - 2a + 1 \) tableaux \( P \) with \( \text{shape}(P) = (n-a,a) \) and one descent.

Recall the class of left-unimodal permutations from Section 2.4 and let \( L_n = \mathcal{L} \cap S_n \).

**Lemma 4.5.** We have

\[
Q(L_n) = \sum_{k=0}^{n-1} s_{n-k,1^k}.
\]

**Proof.** By Observation 4.1 \( L_n = G_n^- \) is a disjoint union of inverse descent classes. Specifically,

\[
L_n = \bigsqcup_{k=0}^{n-1} D_{n,k}^{-1}.
\]

We apply Lemma 2.10 noting that the only tableau \( P \) with \( \text{Des}(P) = [k] \) is the hook with entries 1, 2, \ldots, \( k, k+1 \) in its first column.

General one-column grid classes can be expressed as unions of inverse descent classes as follows.
Proposition 4.6. For every $v \in \{+, -\}^k$,
\[
G^v_n = S_n \setminus \bigcup_{u \in \{+, -\}^{n-1}} D^{-1}_{n,J_u} = \bigcup_{u \in \{+, -\}^{n-1}} D^{-1}_{n,J_u},
\]
where $v \leq u$ denotes that $v$ is a subsequence of $u$, and $J_u := \{i : u_i = +\}$. Consequently, by Lemma 2.10, $G^v_n$ is a fine set.

Example. We have
\[
G^{++}_5 = S_5 \setminus \left( D^{-1}_{5,\{2,3\}} \sqcup D^{-1}_{5,\{2,4\}} \sqcup D^{-1}_{5,\{3,4\}} \sqcup D^{-1}_{5,\{1,3,4\}} \sqcup D^{-1}_{5,\{2,3,4\}} \right).
\]

Proof of Proposition 4.6. It will be convenient to record the descent set of a permutation in $S_n$ as a word in $\{a, d\}^{n-1}$ which has $ds$ exactly at the positions of the descents. Let $W(v)$ be the set of words in $\{a, d\}^{n-1}$ that can be obtained from $v$ by replacing each maximal block of the form $+r$ with a word with at most $r - 1$ ds, and each maximal block of the form $-r$ with a word with at most $r - 1$ as. For example, $W(- - + + + -)$ is the set of words that can be obtained by concatenating a word with at most one $a$, a word with at most two $ds$, and a word with no $as$.

It follows easily from the definitions and Observation 4.2 that $\pi \in G^v_n$ if and only if the descent word of $\pi^{-1}$ belongs to $W(v)$. On the other hand, words in $W(v)$ are precisely those that avoid the subsequence obtained from $v$ by replacing each $+$ with a $d$ and each $-$ with an $a$. Indeed, this subsequence and any word containing it cannot be decomposed as a concatenation of subwords as described above. Conversely, any word that avoids this subsequence can be broken up into subwords as follows: if $v$ starts with $+r-s \cdots$ (the other case if analogous), let the first break up point occur right before the $r$th occurrence of $d$ in the word (or at the end of the word if there are not enough occurrences), the second break up point right before the $s$th occurrence of $a$ after that, and so on.

It follows that $\pi \notin G^v_n$ if and only the descent word of $\pi^{-1}$ contains this subsequence. This latter condition is equivalent to the fact that $\pi \in D^{-1}_{n,J_u}$ for some $u$ that contains $v$ as a subsequence. \qed

Applying Lemma 2.10 gives the following corollary.

Corollary 4.7. For every $v \in \{+, -\}^k$,
\[
Q(G^v_n) = \sum_{u \in \{+, -\}^{n-1}} s_{Z_{n,J_u}}.
\]

For example, $Q(G^{++})$ equals the multiplicity-free sum of ribbons $Z_{n,J}$ where $J$ is a prefix of $1, \ldots, n - 1$. These ribbons are precisely the hooks, agreeing with Lemma 4.5.

Recall from [20, Ch. 9.4] the Solomon descent subalgebra of $\mathbb{C}[S_n]$, which is spanned by the elements of the group algebra
\[
d_{n,J} := \sum_{\pi \in D_{n,J}} \pi. \tag{5}
\]
For $v \in \{+, -\}^k$ and $n > k$, let
\[
g_{n,v} := \sum_{\pi \in G^v_n} \pi^{-1} \in \mathbb{C}[S_n].
\]
Letting $k = n - 1$, we get the following immediate consequence of Proposition 4.6.
Corollary 4.8. The set
\[ \{ g_{n,v} : v \in \{+,-\}^{n-1} \} \]
forms a basis for the Solomon descent subalgebra in \( \mathbb{C}[S_n] \).

Proof. Proposition 4.6 for the special case of \( v \in \{+,-\}^{n-1} \), states that \( G_v^n = S_n \setminus D_{n,J_v}^{-1} \). Since the set \( \{ d_{n,J} : J \subseteq [n-1] \} \) forms a basis for the Solomon descent subalgebra, the result follows. \( \square \)

4.2 Colayered permutations

Another family of Schur-positive grid classes is the following.

Definition 4.9. The \( k \)-colayered grid class \( \mathcal{Y}^k \) is determined by the \( k \times k \) identity matrix:

\[
\mathcal{Y}^k = \mathcal{G} \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix}.
\]

Let \( \mathcal{Y} = \bigcup_{k \geq 1} \mathcal{Y}^k \) be the set of colayered permutations with an arbitrary number of layers. Let \( \mathcal{Y}^k_n = \mathcal{Y}^k \cap S_n \) and \( \mathcal{Y}_n = \mathcal{Y} \cap S_n \).

Figure 4.1 shows the 2- and 3-colayered grids.

![Figure 4.1: The grids for \( \mathcal{Y}^2 \) (left) and \( \mathcal{Y}^3 \) (right).](image)

Example. We have that \( \mathcal{Y}^1_3 = \{12345\} \) and \( \mathcal{Y}^2_5 = \{12345, 51234, 45123, 34512, 23451\} \).

Proposition 4.10. For every \( 1 \leq k \leq n \),
\[
Q(\mathcal{Y}^k_n \setminus \mathcal{Y}^{k-1}_n) = s_{n-k+1,1^{k-1}},
\]
with the convention that \( \mathcal{Y}^0_n = \emptyset \).

Proof. A permutation \( \pi \in \mathcal{Y}^k \) does not belong to \( \mathcal{Y}^{k-1} \) if and only if \( \text{des}(\pi) = k - 1 \). Permutations \( \pi \in \mathcal{Y}^k \) with \( \text{des}(\pi) = k - 1 \) are in bijection with subsets of \([n-1]\) of cardinality \( k - 1 \), via the map \( \pi \mapsto \text{Des}(\pi) \). We conclude that
\[
Q(\mathcal{Y}^k_n \setminus \mathcal{Y}^{k-1}_n) = \sum_{\pi \in \mathcal{Y}^k_n \setminus \mathcal{Y}^{k-1}_n} F_{n,\text{Des}(\pi)} = \sum_{J \subseteq [n-1]} \sum_{|J| = k-1} F_{n,J} = \sum_{T \in \text{SYT}((n-k+1,1^{k-1}))} F_{n,\text{Des}(T)} = s_{n-k+1,1^{k-1}}.
\]
\( \square \)
Corollary 4.11. For every $1 \leq k \leq n$, $\mathcal{Y}_n^k$ is a fine set and

$$Q(\mathcal{Y}_n^k) = s_n + s_{n-1,1} + s_{n-2,1,1} + \cdots + s_{n-k+1,1,k-1}.$$ 

In particular, $C_n$ is fine and $Q(C_n) = s_n + s_{n-1,1}$.

Remark 4.12. $Q(\mathcal{Y}_n)$ is the Frobenius image of the character of the exterior algebra $\wedge V$, where $V$ is the $n$-dimensional natural representation space of $S_n$. This is because the exterior algebra $\wedge V$ is isomorphic to the multiplicity-free sum of all hooks [13, Exer. 4.6].

5 Products of fine sets

Given two subsets $A, B \subseteq S_n$, define their product $AB$ to be the multiset of all permutations obtained as $\pi\sigma$ where $\pi \in A$ and $\sigma \in B$. To denote the underlying set, without multiplicities, we write $\{AB\}$. We call $AB$ the multiset product and $\{AB\}$ the set product of $A$ and $B$.

We are interested in pairs of fine subsets $A, B \subseteq S_n$ whose product is fine, either as a multiset or as a set.

5.1 Basic examples

In general, the product of fine subsets of $S_n$ does not give a fine multiset or set. For example, the subsets $A = \{2134, 3412, 1243\}$ and $B = \{2143, 3412\}$ in $S_4$ are fine, but $AB = \{AB\}$ is not fine (although $Q(AB)$ is symmetric).

For positive integers $n$ and $k$ let $B_{n,k}$ be the set of permutations in $S_n$ whose Coxeter length is at most $k$. Notice that $B_{n,k}$ is a union of sets each of which consists of all permutations of a fixed Coxeter length, and so it is set-fine by Proposition 3.7(ii) together with Observation 3.2. Noticing that $\{B_{n,k}B_{n,r}\} = B_{n,k+r}$, we have the following.

Observation 5.1. For every positive integers $k$ and $r$, the set product $\{B_{n,k}B_{n,r}\}$ is set-fine.

Note, however, that the multiset product of subsets of permutations of bounded Coxeter length is not necessarily fine. For example, letting $A = \{\pi \in S_4 : \ell(\pi) \leq 1\}$, the multiset $AA$ is not fine (in fact, $Q(A)$ is not even symmetric), and the same holds if we let $A = \{\pi \in S_5 : \ell(\pi) \leq 2\}$. The next lemma shows that conjugacy classes behave better with respect to products.

Lemma 5.2. Multiset and set products of conjugacy classes in $S_n$ are fine.

Proof. Set products of conjugacy classes are closed under conjugation, thus, by Proposition 3.7(ii), they are set-fine. Similarly, multiset products of conjugacy classes are multiset unions of conjugacy classes, thus by Proposition 3.7(ii) together with Observation 3.2 they are multiset-fine. \qed

Conjugacy classes span the center of the group algebra $\mathbb{C}[S_n]$. Inverse descent classes span the Solomon descent subalgebra, from where the next result follows.

Proposition 5.3. Multiset and set products of inverse descent classes in $S_n$ are fine.

Proof. For every triple $I, J, K$ of subsets of $[n-1]$, the multiplicity of $D_{n-K}^{-1}$ in the multiset product $D_{n-I}^{-1}D_{n-J}^{-1}$ equals the the multiplicity of $D_{n,K}$ in the multiset product $D_{n,I}D_{n,J}$, which is a nonnegative integer (see [15 Cor. 15]). Thus, both $D_{n-I}^{-1}D_{n-J}^{-1}$ and $\{D_{n,I}^{-1}D_{n,J}^{-1}\}$ are unions of inverse descent classes (possibly with multiplicities). The result now follows from Lemma 2.10. \qed
The main theorem in this section is a significant strengthening of Proposition 5.3 in the multiset case.

**Theorem 5.4.** For every fine set $B$ of permutations in $S_n$ and every $J \subseteq [n-1]$, 

$$BD_{n,J}^{-1}$$

is a fine multiset.

In the rest of this section we prove this result, which is part of Theorem 5.12 below. The proof relies on the fact that right multiplication of a permutation $\pi \in S_n$ by a union of inverse descent classes $\bigsqcup_{I \subseteq J} D_{n,I}^{-1}$ amounts to shuffling contiguous segments of appropriate lengths in the sequence $\pi(1), \ldots, \pi(n)$. On the other hand, shuffling fine sets on disjoint sets of letters may be interpreted as an algebraic operation on the corresponding representations, as described in Lemma 5.6 below. This will be used to show that if $B$ is a fine multiset of permutations in $S_n$, then the product of $B$ by the above union of inverse descent classes is multiset-fine. Finally we deduce that $BD_{n,J}$ is multiset-fine by using the inclusion-exclusion principle and applying representation-theoretic arguments to keep track of Schur-positivity.

### 5.2 Shuffles of fine sets

Recall from Subsection 2.2 the notation $A \sqcup B$ for the shuffle of two sets (or multisets) of permutations, and $(\mu, \nu)$ for the skew Young diagram consisting of two components of shapes $\mu$ and $\nu$.

**Remark 5.5.** When we refer to the group $S_k \times S_{n-k}$ in this section, we consider its natural embedding in $S_n$, where $S_k$ permutes the letters 1, $\ldots$, $k$ and $S_{n-k}$ permutes the letters $k+1, \ldots, n$. Similarly, $S_{\eta_1} \times \cdots \times S_{\eta_m}$ is embedded in $S_n$, where $S_{\eta_1}$ permutes the first $\eta_1$ letters, $S_{\eta_2}$ permutes the next $\eta_2$ letters, etc.

**Lemma 5.6.** 1. Fix a set partition $U \sqcup V = [n]$ with $|U| = k$. Let $A$ and $B$ be fine (multi)sets of the symmetric groups on $U$ and $V$, respectively. Then $A \sqcup B$ is a fine (multi)set of $S_n$, and 

$$Q(A \sqcup B) = Q(A)Q(B).$$

In other words, if $A$ is a fine (multi)set for the $S_k$-representation $\phi$ and $B$ is a fine (multi)set for the $S_{n-k}$-representation $\psi$, then $A \sqcup B$ is a fine (multi)set for the induced representation $(\phi \otimes \psi) \uparrow^{S_k \times S_{n-k}}$.  

2. More generally, let $\eta = (\eta_1, \ldots, \eta_m)$ be a composition of $n$ and let $\bigsqcup_{i=1}^m U_i = [n]$ be a set partition with $|U_i| = \eta_i$. If for all $1 \leq i \leq m$ the set $A_i$ is fine in the symmetric group on $U_i$ for the $S_{\eta_i}$-representation $\phi_i$, then $\bigsqcup_{i=1}^m A_i$ is a fine set for the induced representation $(\bigotimes_{i=1}^m \phi_i) \uparrow^{S_{\eta_1} \times \cdots \times S_{\eta_m}}$.

**Proof.** To prove part 1, it suffices to prove Equation (6). By Proposition 2.6 we may assume that $U = \{1, \ldots, k\}$ and $V = \{k+1, \ldots, n\}$.

By Theorem 3.3 together with Remark 3.4, $A$ is (multi)set-fine if and only if there exist a (multi)set partition $A = A_1 \sqcup \cdots \sqcup A_r$ and Des-preserving bijections from each $A_i$ to a Knuth class of a suitable shape $\mu^{(i)} \vdash n$, for $1 \leq i \leq r$. The same holds for $B$, with (multi)set partition
B = B_1 \sqcup \ldots \sqcup B_s$ and Des-preserving bijections from each $B_j$ to a Knuth class of a shape $\nu^{(j)} \vdash n$, for each $1 \leq j \leq s$. Then, by Proposition 2.9

$$Q(A) = \sum_i s_{\mu^{(i)}}, \quad Q(B) = \sum_j s_{\nu^{(j)}},$$

and so

$$Q(A)Q(B) = \sum_{i,j} s_{\mu^{(i)}}s_{\nu^{(j)}}.$$

But by Equation (11) and Proposition 2.9 we have

$$s_{\mu^{(i)}}s_{\nu^{(j)}} = Q(SYT((\mu^{(i)}, \nu^{(j)}))) = \sum_{T \in SYT((\mu^{(i)}, \nu^{(j)}))} F_{\nu, \Des(T)}. \quad (7)$$

Finally, by Theorem 2.5, the RHS of Equation (7) is equal to $Q(A_i \sqcup B_j)$, where $A_i$ and $B_j$ are the factors of the above-mentioned (multi)set partitions of $A$ and $B$. We conclude that

$$Q(A)Q(B) = \sum_{i,j} Q(A_i \sqcup B_j) = Q(A \sqcup B).$$

By Remark 2.7 the proof of part 1 may be generalized to shuffles of any number of fine sets on distinct sets of letters, proving part 2.

### 5.3 Fine sets of Young subgroups

For a composition $\eta = (\eta_1, \ldots, \eta_m)$ of $n$, denote by $S_\eta \subseteq S_n$ the corresponding Young subgroup $S_\eta := S_{\eta_1} \times \cdots \times S_{\eta_m}$. Next we consider the restriction of $S_n$-representations on fine sets to Young subgroups.

For compositions $\mu$ of $k$ and $\nu$ of $n-k$ denote by $\mu, \nu$ their concatenation, which is a composition of $n$. Note that for every $\pi \in S_n$, $\Des(\pi) \setminus S(\mu, \nu) \subseteq [k-1] \cup [k+1, n-1]$.

**Definition 5.7.** A (multi)set of permutations $B$ in $S_n$ is $S_k \times S_{n-k}$-fine for its complex representation $\rho$ if for each pair of compositions $\mu$ of $k$ and $\nu$ of $n-k$, the character value of $\rho$ at a conjugacy class of cycle type $\mu, \nu$ satisfies

$$\chi^\rho(\mu, \nu) = \sum_{\pi \in B_{\mu, \nu}} m(\pi, B)(-1)^{|\Des(\pi) \setminus S(\mu, \nu)|},$$

where $B_{\mu, \nu}$ is the set of $\mu, \nu$-modal permutations in $B$ and $m(\pi, B)$ is the multiplicity of $\pi \in B$.

**Remark 5.8.** Note that an $S_k \times S_{n-k}$-fine set may actually not be a subset of $S_k \times S_{n-k}$. For example, the singleton $\{3124\}$ which consists of one permutation in $S_4$ is fine for the $S_2 \times S_2$-irreducible representation $S(1^2) \otimes S(2)$.

**Proposition 5.9.** A (multi)set of permutations $B$ in $S_n$ is $S_k \times S_{n-k}$-fine if and only if there exist a (multi)set partition $B = B_1 \sqcup \ldots \sqcup B_m$ and, for each $1 \leq i \leq m$, a Des-preserving bijection from $B_i$ to the set of all pairs of standard Young tableaux $(T^1, T^2)$ of suitable shapes $\lambda^{(i)} \vdash k$ and $\mu^{(i)} \vdash n-k$, respectively, where $T^1$ consists of the letters $1, \ldots, k$, $T^2$ consists of the letters $k+1, \ldots, n$, and $\Des(T^1, T^2) := \Des(T^1) \sqcup \Des(T^2)$. 

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The proof is similar to the proof of [3, Theorem 1.5] and is omitted.

Lemma 5.10. If $B \subseteq S_n$ is a fine (multi)set for the $S_n$-representation $\rho$, then for every $1 \leq k \leq n$, $B$ is a fine (multi)set of $S_k \times S_{n-k}$ for the restricted representation $\rho : S_k \times S_{n-k}$.

Proof. Consider the natural embedding of $S_k \times S_{n-k}$ in $S_n$, where $S_k$ permutes the letters $1, \ldots, k$ and $S_{n-k}$ permutes the letters $k+1, \ldots, n$. By definition, for every $\sigma \in S_k$ of cycle type $\mu$ and every $\tau \in S_{n-k}$ of cycle type $\nu$,

$$\chi_{S_k \times S_{n-k}}(\sigma, \tau) = \sum_{\pi \in B_{\mu, \nu}} (-1)^{|\text{Des}(\pi)\setminus S_{\mu, \nu}|},$$

and so $B$ is a fine (multi)set for the restricted representation $\rho : S_k \times S_{n-k}$.

Remark 5.11. All statements in this subsection may be easily generalized from the Young subgroup $S_k \times S_{n-k}$ to any Young subgroup $S_\eta$ of $S_n$.

5.4 Right multiplication by an inverse descent class

Now we are ready to prove an extended version of Theorem 5.4. We use the notation $\{j_1, \ldots, j_t\} \prec$ to indicate that the elements of the set satisfy $j_1 < j_2 < \cdots < j_t$.

For $J = \{j_1, \ldots, j_t\} \prec \subseteq [n-1]$, let $S_J$ denote the Young subgroup $S_{j_1} \times S_{j_2-j_1} \times \cdots \times S_{n-j_t}$. Let

$$R_{n,J} := \{\pi \in S_n : \text{Des}(\pi) \subseteq J\} = \bigcup_{I \subseteq J} D_{n,I},$$

and let $R_{n,J}^{-1} := \{\pi^{-1} : \pi \in R_{n,J}\}$.

Theorem 5.12. Let $B \subseteq S_n$ be a fine set for the $S_n$-representation $\rho$. Then for every $J \subseteq [n-1]$, the following hold.

1. The multiset $BR_{n,J}^{-1}$ is a fine multiset of $S_n$ for $(\rho : S_J) \uparrow S_n$.

2. The multiset $BD_{n,J}^{-1}$ is a fine multiset of $S_n$ for $\rho \otimes S_{Z_{n,J}}$, where $Z_{n,J}$ is the ribbon as in Definition 2.1, and $S_{Z_{n,J}}$ is the corresponding Specht module.

The Kronecker product of two symmetric functions $f, h \in \Lambda^n$ is defined by

$$f \ast h := \sum_{\mu, \nu, \lambda \vdash n} \langle f, s_\mu \rangle \langle h, s_\nu \rangle g_{\mu, \nu, \lambda} s_\lambda,$$

where

$$g_{\mu, \nu, \lambda} := \langle \chi^\mu \otimes \chi^\nu, \chi^\lambda \rangle.$$

With this definition, part 2 in Theorem 5.12 is equivalent to the fact that

$$Q(BD_{n,J}^{-1}) = Q(B) \ast Q(D_{n,J}^{-1}).$$
Proof of Theorem 5.12. For the sake of clarity, let us first prove part 1 for a singleton \( J = \{ k \} \).

First, notice that \( R_{n(k)}^{−1} \) consists of all shuffles of \( 1, \ldots , k \) with \( k+1, \ldots , n \). Hence \( BR_{n(k)}^{−1} \) consists of all shuffles of \( (\pi(1), \ldots , \pi(k)) \) with \( (\pi(k+1), \ldots , \pi(n)) \) over all \( \pi \in B \).

Since \( B \) is a fine set, it follows from Lemma 5.10 that it is also a fine set of \( S_k \times S_{n-k} \) for the restricted representation \( \rho \downarrow_{S_k \times S_{n-k}} \). Proposition 5.9 and Remark 3.3 imply that there exist a multiset partition \( B = B_1 \sqcup \ldots \sqcup B_m \) and Des-preserving bijections, for each \( 1 \leq i \leq m \), from \( B_i \) to the cartesian product \( K^{(i)} \times L^{(i)} \) of a pair of Knuth classes of suitable shapes \( \lambda^{(i)} \vdash k \) and \( \mu^{(i)} \vdash n-k \), where \( K^{(i)} \) is on the letters \( 1, \ldots , k \) and \( L^{(i)} \) is on the letters \( k+1, \ldots , n \). Together with Proposition 2.6, this implies that the distribution of the descent set over \( BR_{n(k)}^{−1} \), that is over all shuffles of \( (\pi(1), \ldots , \pi(k)) \) with \( (\pi(k+1), \ldots , \pi(n)) \) for all \( \pi \in B \) is equal to its distribution over the union of all shuffles \( K^{(i)} \sqcup L^{(i)} \) for \( 1 \leq i \leq m \). Lemma 5.8 and Proposition 3.7(ii) complete the proof of the first part for \( J = \{ k \} \).

By Remarks 2.7 and 5.11 and part 2 of Lemma 5.6, this proof easily generalizes to any subset \( J = \{ j_1, \ldots , j_t \} \subseteq \{ n-1 \} \). Now \( BR_{n,J}^{-1} \) consists of all shuffles of \( (\pi(1), \ldots , \pi(j_1); \pi(j_1+1), \ldots , \pi(j_2); \ldots ; \pi(j_t+1), \ldots , \pi(n)) \) over all \( \pi \in B \), and pairs of Knuth classes and shapes are replaced by \( t+1 \)-tuples.

Next we prove part 2. By part 1, \( Q(BR_{n,J}^{-1}) \) is symmetric and Schur-positive. Since \( Q(BR_{n,J}^{-1}) = \sum_{I \subseteq J} Q(BD_{n,I}^{-1}) \), it follows by the inclusion-exclusion principle that

\[
Q(BD_{n,J}^{-1}) = \sum_{I \subseteq J} (-1)^{|J \setminus I|} Q(BR_{n,I}^{-1}).
\]  

We conclude that \( Q(BD_{n,J}^{-1}) \) is symmetric.

To prove that it is also Schur-positive, recall the following reciprocity result [24 Ch. 3.3 Ex. 5]: if \( H \) is a subgroup of a group \( G \), and \( \psi \) and \( \phi \) are representations of \( G \) and \( H \), respectively, then

\[
(\phi \otimes \psi \downarrow_H)^G \cong \phi \uparrow^G \otimes \psi.
\]

Hence, in our setting,

\[
(\rho \downarrow_{S_J}) \uparrow^{S_n} \cong (1_{S_J} \otimes \rho \downarrow_{S_J}) \uparrow^{S_n} \cong 1_{S_J} \uparrow^{S_n} \otimes \rho.
\]

It follows from Equation 9 that \( Q(BD_{n,J}^{-1}) \) is the Frobenius image of the representation

\[
\sum_{I \subseteq J} (-1)^{|J \setminus I|} (\rho \downarrow_{S_I}) \uparrow^{S_n} \cong \sum_{I \subseteq J} (-1)^{|J \setminus I|} 1_{S_I} \uparrow^{S_n} \otimes \rho \cong \rho \otimes \sum_{I \subseteq J} (-1)^{|J \setminus I|} 1_{S_I} \uparrow^{S_n}.
\]

Recalling that the Frobenius image of the induced representation \( 1_{S_I} \uparrow^{S_n} \) is the corresponding homogeneous symmetric function [30 Corollary 7.18.3], it follows from [15 p. 295] that

\[
ch \left( \sum_{I \subseteq J} (-1)^{|J \setminus I|} 1_{S_I} \uparrow^{S_n} \right) = s_{Z_{n,J}} = Q(D_{n,J}^{-1}).
\]

See also [25 Theorem 2] and [28 Theorem 1.2]. It follows that \( Q(BD_{n,J}^{-1}) \) is a symmetric function corresponding to a tensor product of two non-virtual representations and hence Schur-positive. 

\[\square\]
6 Vertical rotations

In this section we explore some applications of Theorem 5.12 with a geometric flavor. We use the term vertical rotation of $\pi \in S_n$ to mean a permutation obtained by replacing each entry $\pi(i)$ with $\pi(i) + k \mod n$ for some fixed $k$.

6.1 Vertical rotations of inverse descent classes

If we choose the fine set $B$ to be a colayered grid class $Y^k_n$, which is fine by Corollary 4.11, a consequence of Theorem 5.12 is that $Y^k_n D_{n,J}^{-1}$ is a fine multiset of $S_n$ for every $k \geq 1$ and every $J \subseteq [n-1]$.

Recall the notation $C_n = \langle c \rangle = \{ c^k : 0 \leq k < n \}$, where $c$ is the $n$-cycle $(1, 2, \ldots, n)$. Note that $C_n = Y^2_n = G_n(1 0 0 1)$. For $A \subseteq S_n$, the product $C_n A$ is the multiset of vertical rotations of the elements of $A$. The following is a consequence of Theorem 5.12.

Corollary 6.1. For every $J \subseteq [n-1]$, the multiset $C_n D_{n,J}^{-1}$ of vertical rotations of an inverse descent class is a fine multiset for $S_{Z_n,J} \downarrow S_{n-1} \uparrow S_n$.

Proof. By Corollary 4.11 together with Equation (2), $C_n$ is a fine set with $Q(C_n) = s_n + s_{n-1,1} = s_1 s_{n-1} = \text{ch}(1s_{n-1} \uparrow S_n)$. By Lemma 2.10 $Q(D_{n,J}^{-1}) = \text{ch}(S_{Z_n,J})$. Thus, by Theorem 5.12 and Equation (10), $C_n D_{n,J}^{-1}$ is a fine multiset for $1s_{n-1} \uparrow S_n \otimes S_{Z_n,J} \cong (1s_{n-1} \otimes S_{Z_n,J} \downarrow s_{n-1}) \uparrow S_n \cong S_{Z_n,J} \downarrow s_{n-1} \uparrow S_n$.

\[\blacksquare\]

6.2 Vertical rotations of grids

By Proposition 4.6 every one-column grid class $G^\gamma_v$ is a disjoint union of inverse descent classes. Thus, we get the following consequence of Theorem 5.12.

Corollary 6.2. For every one-column grid class $G^\gamma_v$ and every $k \geq 1$, $Y^k_n G^\gamma_v$ is a fine multiset of $S_n$.

Taking $k = 2$, Corollary 6.2 implies that the multiset of vertical rotations $C_n G^\gamma_v$ is fine. For an arbitrary $v$, we do not know if the underlying set is always fine, see Conjecture 10.1. However, we will show that this is the case sometimes.

Consider now the grid class $G^{+k}$, whose elements are shuffles of $k$ increasing sequences. By the above paragraph, $C_n G^{+k}_n$ is a fine multiset. The underlying set $\{ C_n G^{+k}_n \}$ is the grid class $G_n(M_k)$, where $M_k$ is the $2k \times 2$ matrix whose odd-numbered rows are $(1,0)$ and whose even-numbered rows are $(0,1)$. The grid $G(M_3)$ is drawn in Figure 6.1. We will show in Section 7 that $G_n(M_k)$ is a fine set. In the rest of this subsection we make some initial steps towards this goal.

Cyclic descents were introduced by Cellini [9] and further studied in [19, 10].
Definition 6.3. The cyclic descent set of a permutation $\pi \in S_n$ is

$$\text{cDes}(\pi) = \begin{cases} 
\text{Des}(\pi) & \text{if } \pi(n) < \pi(1), \\
\text{Des}(\pi) \cup \{n\} & \text{if } \pi(n) > \pi(1).
\end{cases}$$

The cyclic descent number is $\text{cdes}(\pi) := |\text{cDes}(\pi)|$.

The next lemma will be useful here and in Section 7. Recall the notation $c = (1, 2, \ldots, n) \in C_n$.

Lemma 6.4. For every $\sigma \in S_n$ and $j$, we have $\text{cdes}(\sigma) = \text{cdes}(\sigma c^j) = \text{cdes}(c^j \sigma)$.

Proof. It is clear by definition that horizontal rotations preserve $\text{cdes}$, and so $\text{cdes}(\sigma) = \text{cdes}(\sigma c^j)$. For vertical rotations, note that to obtain $c\sigma$ from a permutation $\pi \in S_n$ we add 1 from the values of all the entries, except that the entry $n$ becomes 1. If $i$ is the position of $n$ in $\pi$, so that $\pi(i) = n$ and $c\pi(i) = 1$, then the elements of $\text{cDes}(\pi)$ and $\text{cDes}(c\pi)$ other than $i-1,i$ are the same (where we are defining $i-1 = n$ if $i = 1$). On the other hand, $\text{cDes}(c\pi) \cap \{i-1,i\} = \{i\}$ and $\text{cDes}(c\pi) \cap \{i-1,i\} = \{i-1\}$. It follows that $\text{cdes}(\pi) = \text{cdes}(c\pi)$, and so, by iteration, $\text{cdes}(\sigma) = \text{cdes}(c^j \sigma)$. \qed

Recall from Observation 4.2 that $\mathcal{G}_n^{+k} = \{\pi \in S_n : \text{des}(\pi^{-1}) \leq k - 1\}$. One can characterize $\mathcal{G}_n(M_k)$ similarly as follows.

Lemma 6.5. For every $k \geq 1$,

$$\mathcal{G}_n(M_k) = \{\pi \in S_n : \text{cdes}(\pi^{-1}) \leq k\}.$$ 

In particular, by Lemma 6.4, $\mathcal{G}_n(M_k)$ is closed under vertical and horizontal rotations.

Proof. By construction of the grid for $\mathcal{G}(M_k)$, we have that $\pi \in \mathcal{G}_n(M_k)$ if and only if $\pi$ is a vertical rotation of some $\sigma \in \mathcal{G}_n^{+k}$, that is, $\pi = c^j \sigma$ for some $j$ and some $\sigma$ with $\text{des}(\sigma^{-1}) \leq k - 1$. By Lemma 6.4, $\text{cdes}(\pi^{-1}) = \text{cdes}(\sigma^{-1} c^{-j}) = \text{cdes}(\sigma^{-1}) \leq \text{des}(\sigma^{-1}) + 1 \leq k$.

For the converse, suppose that $\text{cdes}(\pi^{-1}) \leq k$. By rotating $\pi$ vertically until its rightmost entry is $n$, we can write $\pi = c^j \sigma$ for some $j$ and some $\sigma$ with $\sigma(n) = n$. Now, since $\text{cdes}(\sigma^{-1}) = \text{cdes}(\pi^{-1}) \leq k$ and $\text{des}(\sigma^{-1}) = \text{cdes}(\sigma^{-1}) - 1$, we have that $\text{des}(\sigma^{-1}) \leq k - 1$, and so $\sigma \in \mathcal{G}_n^{+k}$. \qed

It follows from Lemma 6.5 and Observation 4.2 or directly by looking at the drawings of the grids, that $\mathcal{G}(M_{k-1}) \subset \mathcal{G}^{+k} \subset \mathcal{G}(M_k)$. 22
**Proposition 6.6.** For every \( k \geq 1 \), the quasisymmetric function \( Q(\mathcal{G}_n(M_k)) \) is symmetric.

**Proof.** As mentioned after Corollary 6.2, \( C_n \mathcal{G}_n^{+k} \) is a fine multiset, and \( \mathcal{G}_n(M_k) = \{ C_n \mathcal{G}_n^{+k} \} \). Our proof is by induction on \( k \). For \( k = 1 \), \( \mathcal{G}_n(M_1) = C_n = \mathcal{Y}_n^2 \); thus, by Corollary 6.1

\[
Q(\mathcal{G}_n(M_1)) = Q(\mathcal{Y}_n^2) = s_n + s_{n-1,1}. 
\]  

(12)

For \( k \geq 2 \), we look at the multiplicities of the elements in \( C_n \mathcal{G}_n^{+k} \). Take \( \pi \in \mathcal{G}_n^{+k} \), and consider two cases:

- If \( \pi \in \mathcal{G}_n(M_{k-1}) \), then its orbit \( C_n \pi \) is contained in \( \mathcal{G}_n(M_{k-1}) \) by Lemma 6.5, hence in \( \mathcal{G}_n^{+k} \).

Thus, since each of the \( n \) permutations in \( C_n \pi \) generate the same orbit, the permutations in this orbit appear with multiplicity \( n \) in \( C_n \mathcal{G}_n^{+k} \).

- If \( \pi \in \mathcal{G}_n^{+k} \setminus \mathcal{G}_n(M_{k-1}) \) (equivalently, \( \text{des}(\pi^{-1}) = k - 1 \) and \( \text{cdes}(\pi^{-1}) = k \), by Lemma 6.5), then its orbit \( C_n \pi \) contains exactly \( k \) elements of \( \mathcal{G}_n^{+k} \). Indeed, these are elements of the form \( c^j \pi \), with \( 0 \leq j < n \), satisfying \( \text{des}(c^j \pi^{-1}) \leq k - 1 \), which, using Lemma 6.4 is equivalent to the condition \( \text{des}(\pi^{-1} c^{-j}) \leq \text{cdes}(\pi^{-1} c^{-j}) - 1 \). But this happens precisely when \( n \in \text{cDes}(\pi^{-1} c^{-j}) \), which holds for exactly \( k \) values of \( j \). Since each of the \( k \) elements of \( C_n \pi \cap \mathcal{G}_n^{+k} \) generate the same orbit \( C_n \pi \), the permutations in this orbit appear with multiplicity \( k \) in \( C_n \mathcal{G}_n^{+k} \).

It follows that

\[
Q(C_n \mathcal{G}_n^{+k}) = n Q(\mathcal{G}_n(M_{k-1})) + k Q(\mathcal{G}_n(M_k) \setminus \mathcal{G}_n(M_{k-1})),
\]

and so

\[
Q(\mathcal{G}_n(M_k)) = Q(C_n \mathcal{G}_n^{+k}) - (n - k) Q(\mathcal{G}_n(M_{k-1})).
\]

(13)

We conclude that \( Q(\mathcal{G}_n(M_k)) \) is symmetric.

For \( k = 2 \), the above argument can be used to show that \( Q(\mathcal{G}_n(M_2)) \) is Schur-positive. In Section 7 it will be proved that \( Q(\mathcal{G}_n(M_k)) \) is Schur-positive for all \( k \).

**Proposition 6.7.** \( \mathcal{G}_n(M_2) \) is a fine set, and

\[
Q(\mathcal{G}_n(M_2)) = s_n + (n - 1)s_{n-1,1} + \sum_{a=2}^{\frac{n}{3}-1} (2n - 4a + 2)s_{n-a,a} + \sum_{a=1}^{\frac{n}{3}} (n - 2a)s_{n-a-1,a,1} + \begin{cases} 2s_{\frac{n}{2},\frac{n}{2}} & \text{for even } n \geq 4, \\ 4s_{\frac{n+1}{2},\frac{n-1}{2}} & \text{for odd } n \geq 5, \\ 0 & \text{for } n \leq 3. \end{cases}
\]

(14)

**Proof.** Using Equations (12) and (13) for \( k = 2 \), we have

\[
Q(\mathcal{G}_n(M_2)) = \frac{1}{2} \left( Q(C_n \mathcal{G}_n^{+2}) - (n - 2)(s_n + s_{n-1,1}) \right).
\]

Since \( \mathcal{G}_n^{+2} \) is a union of descent classes, Corollaries 6.1 and 4.4 imply that

\[
Q(C_n \mathcal{G}_n^{+2}) = \text{ch} \left( \chi^{(n)} \downarrow s_{n-1} \uparrow s_n + \sum_{a=1}^{\frac{n}{2}} (n - 2a + 1) \chi^{(n-a,a)} \downarrow s_{n-1} \uparrow s_n \right).
\]

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Counting all the ways to remove and add a box in a two-row shape, we obtain

\[ Q(G_n^{\pm 2}) = ns_n + (3n - 4)s_{n-1,1} + \sum_{a=2}^{\frac{n-1}{2}} (4n - 8a + 4)s_{n-a,a} + \sum_{a=1}^{\frac{n-1}{2}} (2n - 4a)s_{n-a-1,a,1} \]

\[ + \begin{cases} 
4s_{\frac{n}{2},\frac{n}{2}} & \text{for even } n \geq 4, \\
8s_{\frac{n+1}{2},\frac{n-1}{2}} & \text{for odd } n \geq 5, \\
0 & \text{for } n \leq 3.
\end{cases} \]

The formula for \( Q(G_n(M_2)) \) follows now from Equation (14).

\[ \square \]

### 6.3 Arc permutations revisited

In general, for an arbitrary one-column grid class \( G_v \), the set \( \{C_n G_v\} \) of its vertical rotations may not be a grid class, but it is a union of grid classes. For example, taking the class \( L = G^{-+} \) of left-unimodal permutations, we obtain the following.

**Observation 6.8.** Arc permutations are obtained as vertical rotations of left-unimodal permutations, that is,

\[ A_n = \{C_n L_n\}, \]

and so they can be described as a union of grid classes, namely the two drawn in Figure 6.2.

\[ A_n = G_n \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & -1 \\ 0 & 1 \end{pmatrix} \cup G_n \begin{pmatrix} 0 & -1 \\ 0 & 1 \\ 1 & 0 \\ -1 & 0 \end{pmatrix}. \]

![Figure 6.2: Grids for arc permutations.](image)

The following result reformulates [11, Theorem 5]. Here we give a short proof, as an immediate consequence of our approach.

**Proposition 6.9.** \( A_n \) is a fine set, and

\[ Q(A_n) = s_n + s_{1^n} + \sum_{k=2}^{n-2} s_{n-k,2,1^{k-2}} + 2 \sum_{k=1}^{n-2} s_{n-k,1^k}. \]  \( (15) \)
Proof. By Corollary 6.1 together with Lemma 4.3, \( C_n \mathcal{L}_n \) is a fine multiset for the \( S_n \)-character
\[
\sum_{k=0}^{n-1} \chi^{(n-k,1^k)} \downarrow \mathcal{S}_{n-1} \uparrow \mathcal{S}_n = \left( 2 \sum_{k=0}^{n-2} \chi^{(n-k,1^k)} \right) \downarrow \mathcal{S}_n = 2(\chi(n) + \chi(1^n)) + 2 \sum_{k=2}^{n-2} \chi^{(n-k,2,1^{k-2})} + 4 \sum_{k=1}^{n-2} \chi^{(n-k,1^k)}.
\]
To show that the underlying \( \mathcal{A}_n = \{ C_n \mathcal{L}_n \} \) set is fine, we count multiplicities of permutations in the multiset. For every \( \pi \in \mathcal{L}_n \), exactly two elements of the orbit \( C_n\pi \) belong to \( \mathcal{L}_n \), namely \( \pi \) and either \( c\pi \) or \( c^{-1}\pi \), depending on whether \( \pi(n) = n \) or \( \pi(n) = 1 \), respectively. Thus, every element in \( C_n \mathcal{L}_n \) appears with multiplicity two, and so
\[
\mathcal{Q}(\mathcal{A}_n) = \mathcal{Q}(\{ C_n \mathcal{L}_n \}) = \frac{1}{2} \mathcal{Q}(C_n \mathcal{L}_n) = s_n + s_1 n + \sum_{k=2}^{n-2} s_{n-k,2,1^{k-2}} + 2 \sum_{k=1}^{n-2} s_{n-k,1^k}.
\]

Remark 6.10. The set
\[
\mathcal{G}_n = \begin{pmatrix}
1 & 0 \\
-1 & 0 \\
0 & -1 \\
0 & 1
\end{pmatrix},
\]
which is one of the two grid classes for arc permutations, is not a fine set (or even symmetric) already for \( n = 4 \).

7 Horizontal rotations

In this section we identify \( \mathcal{S}_{n-1} \) with the subset of \( \mathcal{S}_n \) consisting of those permutations \( \pi \) with \( \pi(n) = n \). In particular, subsets of \( \mathcal{S}_{n-1} \) such as \( D_{n-1,J} \) and \( R_{n-1,J} \) will be considered as subsets of \( \mathcal{S}_n \) as well.

While Section 6 discussed vertical rotations of inverse descent classes and grid classes as an application of Theorem 5.12, this section deals with horizontal rotations. Two important differences are that the rotated sets are now in \( \mathcal{S}_{n-1} \), and that the proofs do not rely on Theorem 5.12 but instead use bijective techniques. In Section 7.1 we prove the following result, and in Section 7.2 we use it to derive a significant strengthening of Proposition 6.6 by showing that \( \mathcal{G}_n(M_k) \) is fine.

Theorem 7.1. For every \( J \subseteq [n-2] \), \( D_{n-1,J}^{-1} C_n \) is a fine set for \( S^{Z_{n-1,J}} \uparrow \mathcal{S}_n \).

Remark 7.2. For every \( A \subseteq \mathcal{S}_{n-1} \), the multiset \( AC_n \) is in fact a set. For example, \( D_{n-1,J}^{-1} C_n = \{ D_{n-1,J}^{-1} C_n \} \). In particular, since \( \mathcal{S}_{n-1} C_n = \mathcal{S}_n \), every \( \pi \in \mathcal{S}_n \) can be written uniquely as \( \pi = \sigma c^j \) where \( \sigma \in \mathcal{S}_{n-1} \) and \( 0 \leq k < n \).

7.1 Horizontal rotations of inverse descent classes in \( \mathcal{S}_{n-1} \)

For \( J = \{ j_1, \ldots, j_t \} \subseteq [n-2] \), let \( L_{n,J} \) be the skew shape of size \( n \) consisting of disconnected horizontal strips of sizes \( j_1, j_2 - j_1, \ldots, n - 1 - j_t, 1 \) from left to right, each strip touching the next one at a single point. Let \( T \) be a SYT of shape \( L_{n,J} \) and let \( T_1 \) the entry in its upper-right box.

Example. The skew SYT
\[
T = \begin{array}{ccc}
1 & 3 & 4 \\
2 & & \\
\end{array}
\]
has shape \( L_{5,\{1\}} \) and \( T_1 = 4 \).
Recall the definition of $R_{n,j}$ from Equation \[5\].

Lemma 7.3. For every $J \subseteq [n-2]$, $I \subseteq [n-1]$, and $1 \leq k \leq n$,

$$\{|\{\pi \in R_{n-1,J}^{1}: \pi^{-1}(n) = k, \text{Des}(\pi) = I\}| = |\{T \in \text{SYT}(L_{n,J}) : T_1 = k, \text{Des}(T) = I\}|.$$  

Proof. We will construct a Des-preserving bijection from $R_{n-1,J}^{1}$ to $\text{SYT}(L_{n,J})$, under which the position of $n$ in the permutation becomes $T_1$ in the SYT. In this proof, addition and subtraction will be done modulo $n$, so that the entries of the tableaux are always between 1 and $n$.

For each $\sigma \in R_{n-1,J}^{1} \subseteq S_n$, let $T^\sigma$ be the SYT of shape $L_{n,J}$ whose entries from left to right are $\sigma^{-1}(1), \sigma^{-1}(2), \ldots, \sigma^{-1}(n) = n$. Let $T^\sigma + k$ be the tableau obtained from $T^\sigma$ by adding $k$ to each cell (and reducing modulo $n$ if necessary).

Given $\pi = \sigma c^{-k} \in R_{n-1,J}^{1}$, let $0 \leq k < n$, let $T^\pi \in \text{SYT}(L_{n,J})$ be obtained from $T^\sigma + k$ by rearranging the entries in each row in increasing order, so that it becomes a SYT.

To show that the map $\pi \mapsto T^\pi$ is a bijection, note that given $T \in \text{SYT}(L_{n,J})$ one can recover $\pi$ such that $T^\pi = T$ as follows. Letting $k = T_1$, subtract $k$ from each entry of $T$ and then rearrange the entries in each row in increasing order. Let $\sigma$ be the inverse of the permutation obtained by reading the resulting SYT from left to right, and let $\pi = \sigma c^{-k}$. Note that $\sigma \in R_{n-1,J}^{1}$.

It is immediate from the construction of $T^\pi$ that its rightmost entry equals the position of $n$ in $\pi$, namely, $T_1^\pi = \pi^{-1}(n)$.

It remains to prove that Des$(T^\pi) = \text{Des}(\pi)$. Note that $i \in \text{Des}(\pi)$ if and only if $i+1$ appears to the left of $i$ in the sequence $\pi^{-1}(1), \ldots, \pi^{-1}(n)$. If $\pi = \sigma c^{-k}$, then $\pi^{-1} = c^k \sigma^{-1}$, so $\pi^{-1}(1), \ldots, \pi^{-1}(n)$ is the list of entries of $T^\sigma + k$ from left to right. Since $i \in \text{Des}(T^\pi)$ if and only if $i+1$ appears to the left of $i$ in $T^\pi$, it only remains to show that $i+1$ appears to the left of $i$ in $T^\sigma + k$ if and only if the same is true in $T^\pi$. The only way for this to fail would be if $i+1$ and $i$ are in the same row of $T^\sigma + k$, with $i+1$ to the left of $i$, so that rearranging the row in increasing order changes their relative position. Since the rows of $T^\sigma$ are increasing, this is only possible if the corresponding entries in $T^\sigma$, which are $i+1-k$ and $i-k$ (mod $n$), equal 1 and $n$, respectively. But this contradicts the fact that $n$ is in a row by itself in $T^\sigma$. \[\Box\]

Example. Let $n = 5$, $J = \{1\} \subseteq [3]$ and $\sigma = 2314 \in R_{(1),4}^{1} \subseteq S_5$, so $\sigma^{-1} = 3124$. Then

$$\sigma C_5 = \{\sigma c^0, \sigma c^{-1}, \sigma c^{-2}, \sigma c^{-3}, \sigma c^{-4}\} = \{23145, 52314, 45231, 14523, 31452\}.$$  

The descent sets of these horizontal rotations are $\{2\}$, $\{1,3\}$, $\{2,4\}$, $\{3\}$, $\{1,4\}$, respectively. The SYT corresponding to these permutations are

$$T^{23145} = \begin{array}{ccc}3 & 1 & 2 \\ 4 & 5 & \end{array}, \quad T^{52314} = \begin{array}{ccc}2 & 3 & 5 \\ 1 & 4 & \end{array}, \quad T^{45231} = \begin{array}{ccc}1 & 3 & 4 \\ 2 & 5 & \end{array},$$

$$T^{14523} = \begin{array}{ccc}2 & 4 & 5 \\ 1 & 3 & \end{array}, \quad T^{31452} = \begin{array}{ccc}1 & 3 & 4 \\ 2 & 5 & \end{array},$$

whose descent sets are $\{2\}$, $\{1,3\}$, $\{2,4\}$, $\{3\}$, $\{1,4\}$, respectively.

For $J = \{j_1, \ldots, j_t\} \subseteq [n-2]$, recall the notation $S_J := S_{j_1} \times S_{j_2-j_1} \times \cdots \times S_{n-1-j_t} \subseteq S_{n-1}$.  

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Corollary 7.4. For every $J \subseteq [n-2]$, $R^{-1}_{n-1,J}C_n$ is a fine set for $1_{S_J} \uparrow^{S_n}$.

Proof. By Lemma 7.3, the distribution of Des over $R^{-1}_{n-1,J}C_n$ is the same as over SYT($L_n,J$). By Lemma 2.8 together with Proposition 2.9, the set SYT($L_n,J$) is fine for the induced character $1_{S_J} \uparrow^{S_n}$, and thus so is $R^{-1}_{n-1,J}C_n$. \hfill \blacksquare

Now we are ready to prove Theorem 7.1.

Proof of Theorem 7.1. Since $R^{-1}_{n,J} = \bigsqcup I \subseteq J D^{-1}_{n,I}C_n$, we have that

$$Q(R^{-1}_{n,J}C_n) = \sum_{I \subseteq J} Q(D^{-1}_{n,I}C_n).$$

By the inclusion-exclusion principle,

$$Q(D^{-1}_{n,I}C_n) = \sum_{I \subseteq J} (-1)^{|J \setminus I|} Q(R^{-1}_{n,I}C_n).$$

Hence, by Corollary 7.4, $Q(D^{-1}_{n,I}C_n)$ is symmetric, and in fact it is the Frobenius image of the alternating sum of representations

$$\sum_{I \subseteq J} (-1)^{|J \setminus I|} 1_{S_J} \uparrow^{S_n} = \sum_{I \subseteq J} (-1)^{|J \setminus I|} 1_{S_J} \uparrow^{S_n-1} \uparrow^{S_n} = \left( \sum_{I \subseteq J} (-1)^{|J \setminus I|} 1_{S_J} \uparrow^{S_n-1} \right) \uparrow^{S_n} = S^{Z_{n-1,J}} \uparrow^{S_n},$$

where the last equality uses Equation (11). This completes the proof. \hfill \blacksquare

7.2 Horizontal rotations of grids

In this subsection we prove that $G_n(M_k)$ is a fine set by interpreting the corresponding grid as a horizontal rotation of a grid in $S_{n-1}$.

Lemma 7.5. For every $k \geq 1$, $G_n(M_k) = G_{n-1}^{+k}C_n$.

Proof. Recall from Remark 7.2 that every $\pi \in S_n$ can be written uniquely as $\pi = \sigma c^j$ for some $\sigma \in S_{n-1} \subset S_n$ and $0 \leq j < n$. We will show that $\pi \in G_n(M_k)$ if and only if $\sigma \in G_{n-1}^{+k}$. By Lemma 6.5 and Observation 4.2, this is equivalent to showing that $cdes(\pi^{-1}) \leq k$ if and only if $\text{des}(\sigma^{-1}) \leq k - 1$. By Lemma 6.4 gives $\text{des}(\sigma^{-1}) = cdes(\sigma^{-1}) - 1 = cdes(\pi^{-1}) - 1$, and the result follows. \hfill \blacksquare

Now we can prove the following consequences of Theorem 7.1.

Corollary 7.6. For every $k \geq 1$, the set $G_n(M_k)$ is fine; namely, $Q(G_n(M_k))$ is symmetric and Schur-positive.

Proof. By Lemma 7.5 together with Observation 4.2 and Remark 7.2,

$$G_n(M_k) = \bigcup_{J \subseteq [n-2]} D_{n-1,J}^{-1}C_n.$$ 

Since $D_{n-1,J}^{-1}C_n$ is a fine set for every $J$ by Theorem 7.1, it follows that $G_n(M_k)$ is a fine set as well. \hfill \blacksquare

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Corollary 7.7. For every \( k \geq 1 \), the set
\[
\{ \pi \in S_n : \ cdes(\pi^{-1}) = k \}
\]
is fine for the multiplicity-free sum of induced representations \( S^n \uparrow S_n \) over all ribbons \( Z \) of height \( k \) and size \( n - 1 \).

**Proof.** By Lemmas 6.5 and 7.5, together with Remark 7.2 and Observation 4.2
\[
\{ \pi \in S_n : \ cdes(\pi^{-1}) = k \} = G_n(M_k) \setminus G_n(M_{k-1}) = G_{n-1}^+ C_n \setminus G_{n-1}^{k-1} C_n
\]
\[
= \{ \pi \in S_{n-1} : \ des(\pi^{-1}) = k - 1 \} C_n = \bigcup_{J \subseteq [n-2] \atop |J|=k-1} D_{n-1,j}^{-1} C_n.
\]
By Theorem 7.1, \( D_{n-1,j}^{-1} C_n \) is a fine set for \( S^{Z_{n-1,j}} \uparrow S_n \), completing the proof. \( \Box \)

### 7.3 Applications

This subsection discusses two more consequences of Theorem 7.1. The first one involves arc permutations and left-unimodal permutations in \( S_n \). It is easy to see that \( A_n = C_n L_{n-1} \). Indeed, as in Remark 7.2 but with \( C_n \) multiplying from the left instead of from the right, every \( \pi \in S_n \) can be written uniquely as \( \pi = c^k \sigma \) for \( 0 \leq k < n \) and some \( \sigma \in S_{n-1} \). In such an expression, the condition of every prefix of \( \pi \) being an interval in \( Z_n \) is equivalent to every prefix of \( \sigma \) being an interval in \( Z \). Thus, \( \pi \in A_n \) if and only if \( \sigma \in L_{n-1} \).

On the other hand, \( A_n \neq L_{n-1} C_n \) for \( n \geq 4 \). For example, \( 1342 \notin A_4 \) is the product of \( 213 \in L_3 \) with \( (1, 2, 3, 4) \in C_4 \). Note also that, by Observation 6.8, \( \bigcup_n A_n \) is a union of grid classes, but this is not the case for the set \( \bigcup_n L_{n-1} C_n \), since it is not closed under pattern containment. Nevertheless, the following equidistribution phenomenon holds.

**Corollary 7.8.** For every positive integer \( n \),
\[
Q(L_{n-1} C_n) = Q(A_n).
\]
In particular, \( L_{n-1} C_n \) is set-fine.

**Proof.** By Equation (4) together with Lemma 4.5 and Theorem 7.1, \( L_{n-1} C_n \) is fine and
\[
Q(L_{n-1} C_n) = c h \left( \sum_{k=0}^{n-2} \chi_{\left( n-1,k \right)^{\uparrow S_n}} \right) = s_n + s_1 n + \sum_{k=2}^{n-2} s_{n-k,2,k-2} + 2 \sum_{k=1}^{n-2} s_{n-k,1,k} = Q(A_n).
\]
The last equality follows from Proposition 6.9. \( \Box \)

**Corollary 7.9.** For every \( J \subseteq [n-2] \), if \( A \subset S_{n-1} \) is such that \( Q(A) = Q(D_{n-1,j}^{-1}) \), then \( AC_n \) is a fine set for \( S^{Z_{n-1,j}} \uparrow S_n \).

In particular, for every \( 2 \leq k \leq n - 1 \),
\[
Q((J_{n-1}^k \setminus J_{n-1}^{k-1}) C_n) = s_{n-k,1,k} + s_{n-k+1,1,k-1} + s_{n-k,2,k-2}.
\]

**Proof.** For every \( 0 \leq j < n \) and \( \pi \in S_{n-1} \), we have that \( i \in \text{Des}(\pi c^{-j}) \) if either \( i - j \) mod \( n \) belongs to \( \text{Des}(\pi) \) or \( i = j \). It follows that the distribution of \( \text{Des} \) on \( \pi C_n \) is determined by \( \text{Des}(\pi) \). Thus, if \( Q(A) = Q(D_{n-1,j}^{-1}) \), then \( Q(AC_n) = Q(D_{n-1,j}^{-1} C_n) \). Hence, by Theorem 7.1, \( AC_n \) is a fine set for \( S^{Z_{n-1,j}} \uparrow S_n \). The second statement follows now using Proposition 4.10. \( \Box \)

Some conjectured generalizations of Corollaries 7.8 and 7.9 will be discussed in Section 10.
8 Other geometric operations on grids

Aside from vertical and horizontal rotations, there are other operations on grids that preserve Schur-positivity in some circumstances. In this section we study some of them.

8.1 Vertical and horizontal reflections

Before we state our result about reflections of grids, let us introduce a few definitions and a lemma. For a set $D \subseteq [n-1]$, define $n - D := \{n - i : i \in D\}$. Let $w_0 := n(n - 1) \ldots 1 \in S_n$. For $\pi \in S_n$, we have $w_0\pi = (n + 1 - \pi(1))(n + 1 - \pi(2)) \ldots (n + 1 - \pi(n))$, $\pi w_0 = \pi(n) \ldots \pi(2)\pi(1)$, and $w_0\pi w_0 = (n + 1 - \pi(n)) \ldots (n + 1 - \pi(2))(n + 1 - \pi(1))$. In particular,

$$\text{Des}(w_0\pi w_0) = n - \text{Des}(\pi). \quad (16)$$

Lemma 8.1. Let $B$ be a fine set of $S_n$. Then the statistics $\text{Des}$ and $n - \text{Des}$ are equidistributed over $B$.

Proof. By Theorem 5.3 together with Remark 5.4 it suffices to prove that the statement holds when $B$ is a Knuth class. Observe that for every Knuth class $K \subseteq S_n$ of shape $\lambda$, the set $w_0Kw_0$ is also a Knuth class of shape $\lambda$. Finally, use Equation (16). □

Proposition 8.2. Let $G$ be a grid class, and let $G^\text{ver}$ and $G^\text{hor}$ be the grid classes obtained by reflecting the grid of $G$ vertically and horizontally, respectively. If $G_n$ is fine for the $S_n$-representation $\rho$, then $G_n^\text{ver}$ and $G_n^\text{hor}$ are fine for $S^{(1^n)} \otimes \rho$.

Proof. Note that $G_n^\text{hor} = G_n w_0$. Since $\{w_0\} = D_n^{-1} \cong S^{(1^n)} \otimes S_n$, Theorem 5.12 implies that $G_n w_0$ is a fine set for $S^{(1^n)} \otimes \rho$.

Similarly, $G_n^\text{ver} = w_0 G_n = w_0(G_n w_0)w_0$. By Equation (16), the distribution of Des over $w_0 G_n$ equals the distribution of $n - \text{Des}$ over $G_n w_0$, which in turn equals the distribution of Des over $G_n w_0$ by Lemma 8.1 applied to the fine set $G_n w_0$. This shows that the distribution of Des is the same over $G_n^\text{ver}$ and $G_n^\text{hor}$, completing the proof. □

Corollary 8.3. Let $G$ be a Schur-positive grid class, and let $G^\text{rot}$ be the grid class obtained by rotating the grid for $G$ by 180 degrees. Then

$$Q(G_n) = Q(G_n^\text{rot}). \quad (17)$$

Note that if $G = G(M)$ for some $k \times l$ matrix $M$, then $G^\text{rot} = G(M^*)$, where $M^*$ is defined by $m_{i,j}^* = m_{k+1-i,l+1-j}$. Interestingly, Equation (17) does not necessarily hold on general grid classes. For example, letting $G = G(-1 1)$, we have

$$Q(G_3) = F_3,0 + 2F_3,1 + F_3,2 \neq F_3,0 + 2F_3,2 + F_3,1 = Q(G_3^\text{rot}).$$

In fact, for any any non-palindromic one-row matrix $M$ one can easily verify that $Q(G_n(M)) \neq Q(G_n(M^*))$ if $n$ is large enough.

For $v = (v_1, \ldots, v_k) \in \{1, -1\}^k$ let $v^R := (v_k, \ldots, v_1)$. A special case of Corollary 8.3 is the fact that

$$Q(G_n^v) = Q(G_n^{v^R}).$$
8.2 Stacking grids

Another natural geometric operation on grids consists of stacking one grid on top of another. In the case of two multiple-column grids, there is an ambiguity in the choice of width of the columns. Thus, we will only consider the case where one of the stacked grids has a single column.

Definition 8.4 (The stacking operation). For a grid class \( \mathcal{H} \) and \( v \in \{+,-\}^k \), let \( G^v,\mathcal{H} \) be the grid class obtained by placing the grid for \( G^v \) below (atop) the one for \( \mathcal{H} \).

Figure 8.1 shows an example of the stacking operation.

![Figure 8.1: The grid \( G^{G(M_3),(+,−)} \).](image)

By Corollary 8.3 for any grid class \( G(M) \) and any \( v \in \{+,-\}^k \) such that \( G^v \) is fine, we have

\[ Q(G^v_n) = Q(G^R_n) \]

Question 8.5. Let \( \mathcal{H} \) be a fine grid and let \( v \in \{+,-\}^k \). Is the stacked grid \( G^v,\mathcal{H} \) necessarily fine?

Computer experiments hint at an affirmative answer. Note that \( G^v,\mathcal{H} \) is the underlying set of the multiset obtained by shuffling \( G^v \) and \( \mathcal{H} \). This multiset is fine by Lemma 5.6. In the special case that \( \mathcal{H} \) has a single column, the grid \( G^v,\mathcal{H} \) has a single column as well, and so it is a fine set by Corollary 4.7. In this section we consider some other instances when \( G^v,\mathcal{H} \) is fine.

Let

\[ J := G^{3^2,+} = G \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad K := G^{-3^2} = G \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}. \]

The corresponding grids are drawn in Figure 8.2. In terms of shuffles, \( J_n \) is the set of permutations in \( S_n \) resulting from shuffling a 2-colayered permutation with the identity permutation, and \( K_n \) is the set of permutations resulting from shuffling the reverse identity permutation with a 2-colayered permutation.

Proposition 8.6. We have \( |J_n| = |K_n| = (n-2)2^{n-1} + 2 \).

Proof. First, we prove that \( |J_n| = (n-2)2^{n-1} + 2 \). Considering the left half of the grid for \( J_n \), it is clear that \( G_n^{3^2} \subset J_n \). Note that \( |G_n^{3^2}| = 2^n - n \), since every \( \pi \in G_n^{3^2} \) is uniquely determined by the choice of dots \( \pi(i) \) placed on the upper and lower segments of the grid, except for the identity permutation, which is obtained in \( n+1 \) ways. It remains to count the permutations in \( J_n \) that are not in \( G_n^{3^2} \).
For $\pi \in \mathcal{J}_n \setminus \mathcal{G}_n^{+2}$, let $j$ be the index such that $\pi(j) = 1$. Then $\pi(1)\pi(2)\ldots\pi(j-1)$ is not increasing, because any drawing of $\pi$ on the grid for $\mathcal{J}$ where the entries to the left of the lowest dot are increasing would have to be a shuffle of two increasing sequences. It follows that any drawing of $\pi$ on the grid for $\mathcal{J}$ has $\pi(j) = 1$ as the leftmost dot on the lowest segment. The permutation $\text{st}(\pi(1)\pi(2)\ldots\pi(j-1))$, where $\text{st}$ is the standardization operation defined at the end of Section 2.4, can then be any permutation in $\mathcal{G}_{j-1}^{+2}$ other than the identity, and so there are $|\mathcal{G}_{j-1}^{+2}| - 1 = 2^{j-1} - j$ choices for it. To the right $\pi(j)$, any choice for which dots are drawn on the upper segment and which are drawn on the lower segment is valid and gives a different permutation $\pi$.

It follows that

$$|\mathcal{J}_n \setminus \mathcal{G}_n^{+2}| = \sum_{j=3}^{n} (2^{j-1} - j)2^{n-j} = (n-4)2^{n-1} + n + 2,$$

and so

$$|\mathcal{J}_n| = 2^n - n + (n-4)2^{n-1} + n + 2 = (n-2)2^{n-1} + 2.$$

Next we prove that $|\mathcal{K}_n| = (n-2)2^{n-1} + 2$ as well. Considering the left half of the grid for $\mathcal{K}_n$, we see that $\mathcal{L}_n \subset \mathcal{K}_n$. It is known that $|\mathcal{L}_n| = 2^{n-1}$. It remains to count the permutations in $\mathcal{K}_n$ that are not left-unimodal.

For $\pi \in \mathcal{K}_n \setminus \mathcal{L}_n$, let $j$ be the smallest index such that $\text{st}(\pi(1)\pi(2)\ldots\pi(j)) \notin \mathcal{L}_j$. Equivalently, $j$ is the smallest index with the property that $\pi(j)$ is neither the largest nor the smallest of all the preceding entries. Note that $j \geq 3$, and that in any drawing of $\pi$ on the grid, the dot for $\pi(j)$ must lie on the middle segment (the one whose left endpoint is in the center of the grid). We can assume that $\pi(j)$ is the leftmost dot on the middle segment, since the only exception would happen if $\pi(j-1)$ was on that segment as well, in which case it could be moved to the lower segment without changing the permutation. The choices for the remaining dots are as follows. To the left of $\pi(j)$, we can choose a subset of $[j-1]$ to be the positions of the dots drawn on the upper segment. The empty set and $[j-1]$ are not valid subsets, by the choice of $j$, but all other subsets will result in different permutations, so there are $2^{j-1} - 2$ choices for these dots. To the right of $\pi(j)$, any choice for which dots are drawn on the middle and lower segments is valid and gives a different permutation, so there are $2^{n-j}$ choices.

It follows that

$$|\mathcal{K}_n \setminus \mathcal{L}_n| = \sum_{j=3}^{n} (2^{j-1} - 2)2^{n-j} = (n-3)2^{n-1} + 2,$$

and so

$$|\mathcal{K}_n| = 2^{n-1} + (n-3)2^{n-1} + 2 = (n-2)2^{n-1} + 2.$$
Remark 8.7. We do not know if the equality $|\mathcal{J}_n| = |\mathcal{K}_n|$ can be explained as part of a more general setting to produce grid classes with the same cardinality but different quasisymmetric functions.

Proposition 8.8. $\mathcal{J}_n$ is a fine set, and

$$Q(\mathcal{J}_n) = s_n + \sum_{a=1}^{\lfloor \frac{n}{2} \rfloor} (n - 2a + 1)s_{n-a,a} + \sum_{a=1}^{\lfloor \frac{n-1}{2} \rfloor} (n - 2a)s_{n-a-1,a,1}.$$  

Proof. We construct a Des-preserving bijection $f$ from $\mathcal{J}_n$ to the multiset of SYT of the shapes that index the Schur functions in the formula.

For $\pi \in \mathcal{G}^+_{n} \setminus \mathcal{G}^+_{n-1}$, let $f(\pi)$ be the permutation obtained when applying RSK to $\pi$. As in Corollary 4.4, $f$ is a bijection between $\mathcal{G}^+_{n} \setminus \mathcal{G}^+_{n-1}$ and the SYT whose shapes are given by the multiset where each two-row partition $(n-a,a)$ has multiplicity $n-2a+1$ for each $1 \leq a \leq \lfloor \frac{n}{2} \rfloor$, and the shape $(n)$ has multiplicity 1.

For $\pi \in \mathcal{J}_n \setminus \mathcal{G}^+_{n} \setminus \mathcal{G}^+_{n-1}$, let $j$ be such that $\pi(j) = 1$. Then $\pi(1)\pi(2)\ldots\pi(j-1)$ is a shuffle (other than the trivial increasing one) of two increasing sequences, say $r+1, r+2, \ldots, s$ and $s+1, s+2, \ldots, t$, with $r < s < t$. Let $\pi'$ be the permutation obtained from $\pi$ by decreasing the entries $r+1, r+2, \ldots, s$ by 1, and decreasing the entries $2, 3, \ldots, r$ by $s-r$. Then $\pi'$ consists of a non-trivial shuffle of the sequences $2, 3, \ldots, s-r+1$ and $s+1, s+2, \ldots, t$, followed by the entry $\pi'(j) = 1$, followed by a shuffle of the sequence $s-r+2, s-r+3, \ldots, s$ and $t+1, t+2, \ldots, n$. Equivalently, $\pi'$ is an arbitrary element of $\mathcal{G}^+_{n-1} \setminus \{12\ldots(n-1)\}$ with its entries increased by 1 and the letter 1 inserted anywhere after the first descent. Note that $\text{Des}(\pi') = \text{Des}(\pi)$.

To describe $f(\pi)$, first apply RSK to the permutation $\pi'$ with the entry $\pi'(j) = 1$ removed, and let $Q'$ be the recording tableau with entries $[n] \setminus \{j\}$. Let $f(\pi)$ be the tableau obtained from $Q'$ by adding a third row consisting of a box with the entry $j$.

By construction, $\text{Des}(f(\pi)) = \text{Des}(Q') \cup \{j-1\} = \text{Des}(\pi') = \text{Des}(\pi)$. Recall from Corollary 4.4 that taking the recording tableau under RSK gives a bijection between $\mathcal{G}^+_{n-1} \setminus \{12\ldots(n-1)\}$ and the multiset of SYT where each tableau of shape $(n-a-1,a)$ appears $n-2a$ times for $1 \leq a \leq \lfloor \frac{n-1}{2} \rfloor$. It follows that the map $f$ is a bijection between $\mathcal{J}_n \setminus \mathcal{G}^+_{n} \setminus \mathcal{G}^+_{n-1}$ and the multiset of SYT with 3 rows and exactly one box on the third row, where every tableau of shape $(n-a-1,a,1)$ appears $n-2a$ times for each $1 \leq a \leq \lfloor \frac{n-1}{2} \rfloor$.

□

Proposition 8.9. $\mathcal{K}_n$ is a fine set, and

$$Q(\mathcal{K}_n) = s_n + s_{1,n} + 2 \sum_{k=1}^{n-2} s_{n-k,1^k} + 2 \sum_{k=1}^{n-3} s_{n-k-2,1^{k-1}}.$$  

Proof. Again, we construct a Des-preserving bijection $g$ from $\mathcal{K}_n$ to the appropriate multiset of SYT.

For $\pi \in \mathcal{L}_n$, let $g(\pi)$ be the recording tableau $Q$ obtained when applying RSK to $\pi$. As in Lemma 4.5, $g$ is a bijection between $\mathcal{L}_n$ and the set of SYT whose shape is a hook, and the contribution of these permutations to the quasisymmetric function is $\sum_{k=1}^{n-1} s_{n-k,1^k}$.

For $\pi \in \mathcal{K}_n \setminus \mathcal{L}_n$, let again $j$ be the smallest such that $\text{st}(\pi(1)\pi(2)\ldots\pi(j)) \notin \mathcal{L}_j$, and consider the drawing of $\pi$ on the grid where the dot for $\pi(j)$ is the leftmost dot on the right segment. As shown in the proof of Proposition 8.6, placing $\pi(j)$ there uniquely determines the segments on the grid in which each of the other dots lie. Consider three cases:
- If \( \pi(1)\pi(2)\ldots \pi(j-1) \) is increasing, which means that the dot for \( \pi(1) \) is the only one among these in the lower segment, let \( g(\pi) \) be the hook whose entries in the first column are the indices \( i \) such that \( \pi(i) \leq \pi(j) \). This construction is a bijection between permutations in this case and hooks with at least two rows and with the entry 2 in the first row.

- If \( \pi(1)\pi(2)\ldots \pi(j-1) \) is decreasing, which means that the dot for \( \pi(1) \) is the only one among these in the upper segment, let \( g(\pi) \) be the hook whose entries in the first row are the indices \( i \) such that \( \pi(i) \geq \pi(j) \). This construction is a bijection between permutations in this case and hooks with at least two columns and with the entry 2 in the first column.

- Finally, if \( \pi(1)\pi(2)\ldots \pi(j-1) \) is neither increasing nor decreasing, let \( g(\pi) \) be the SYT whose first \( j-1 \) entries are given by the recording tableau of \( \pi(1)\pi(2)\ldots \pi(j-1) \), the entry \( j \) is in the box in the second row and second column, and the remaining values \( i > j \) are appended to the first row if \( \pi(i) > \pi(j) \) and to the first column if \( \pi(i) < \pi(j) \). This construction gives a 2-to-1 map between permutations in this case and SYTs of shape consisting of a hook plus a box. Indeed, placing the dot \( \pi(1) \) in the top or the bottom segment of the grid results in the same tableau \( f(\pi) \) but in a different permutation \( \pi \).

The contribution of the permutations where \( \pi(1)\pi(2)\ldots \pi(j-1) \) is increasing or decreasing is \( \sum_{k=1}^{n-2} s_{n-k,1^k} \). The contribution of the remaining permutations is \( \sum_{k=1}^{n-3} s_{n-k-1,2,1^{k-1}} \). In each of the three cases it easy to check that \( \text{Des}(g(\pi)) = \text{Des}(\pi) \).

**Remark 8.10.** It follows from Proposition 8.8 together with Young’s rule [30 Prop. 7.18.7] that \( Q(J_n) \) is given by an alternating sum of permutation representations:

\[
Q(J_n) = \text{ch} \left( 2 \sum_{a=1}^{\lfloor \frac{n}{2} \rfloor - 1} \left( 1 \uparrow S_n \uparrow S_{n-a-1,a,1} - 1 \uparrow S_n \uparrow S_{n-a-1,a+1} \right) + 1 \uparrow S_n \uparrow_{\lfloor \frac{n}{2} \rfloor , \lfloor \frac{n}{2} \rfloor \mod 2} \right).
\]

Similarly, letting \( V_i \) be an \( i \)-dimensional vector space over \( \mathbb{C} \) and recalling that the exterior algebra \( \wedge(V_i) \) is isomorphic, as an \( S_i \)-module, to the multiplicity-free sum of all hooks of size \( i \) [13 Ex. 4.6], Proposition 8.9 implies that

\[
Q(K_n) = \text{ch} \left( 2 \left( \wedge(V_{n-1}) \uparrow S_n \uparrow S_{n-1} - \wedge(V_n) \right) + \chi(n) + \chi(1^n) \right).
\]

Computing the corresponding character degrees provides an alternative proof to Proposition 8.6.

### 9. Explicit multiplication of one-column grid classes

Multiset and set products of one-column grid classes are fine, as can bee seen using Propositions 4.6 and 5.3 and also the fact, which appears in the proof of the latter proposition, that set products of inverse descent classes are unions of inverse descent classes. Furthermore, it follows from Proposition 4.6 and Corollary 4.8 that the multiset product of one-column grid classes in \( S_n \) is spanned by one-column grid classes with \( v \in \{+,-\}^{n-1} \), with nonnegative constants, which are essentially the structure constants of the Solomon descent algebra. In this section we give an explicit compact description of the set product of one-column grid classes and, more generally, of the set product of a one-column grid class with an arbitrary grid class.
Given a grid class $\mathcal{G}(M)$, let $M^c$ be the matrix obtained by flipping $M$ upside down and changing the signs of its entries. Then $\mathcal{G}(M^c) = \{w_0\pi : \pi \in \mathcal{G}(M)\} = \mathcal{G}(M)^{\text{ver}}$, the grid class obtained from $\mathcal{G}(M)$ by vertical reflection.

**Proposition 9.1.** Let $v \in \{+, -\}^r$ and $A = \mathcal{G}(M)$ be a grid class. Then $\{\mathcal{G}^vA\}$ is the grid class whose matrix is obtained by stacking copies of $M$ and $M^c$ on top of each other, where the $i$th copy from the bottom is $M$ if $v_i = +$ and $M^c$ if $v_i = -$.

An example of a product of the form $\{\mathcal{G}^vA\}$ is given in Figure 9.1. The dots represent the permutations $\pi = 4532617$, $\sigma = 4176235$, and their product $\pi\sigma = 2471536$.

![Figure 9.1: The product of one-column grid classes.](image)

**Proof of Proposition 9.1.** Let $M^v$ be the matrix described in the statement, obtained by stacking copies of $M$ and $M^c$ on top of each other according to the vector $v$. The $k$th copy of the grid $\mathcal{G}(M)$ or $\mathcal{G}(M^c)$ in $\mathcal{G}(M^v)$ starting from the bottom will be called the $k$th block of $\mathcal{G}(M^v)$. Similarly, when referring to the $j$th segment of the grid $\mathcal{G}^v$, we consider that the segments are ordered from bottom to top. Our goal is to show that $\{\mathcal{G}^v\mathcal{G}(M)\} = \mathcal{G}(M^v)$.

To prove the inclusion $\{\mathcal{G}^v\mathcal{G}(M)\} \subseteq \mathcal{G}(M^v)$, let $\pi \in \mathcal{G}^v$ and $\sigma \in \mathcal{G}(M)$, and consider drawings of these permutations on their corresponding grids. Let us show how to draw the permutation $\pi\sigma$ on the grid $\mathcal{G}(M^v)$. For every $1 \leq k \leq r$, let $J_k$ be the set of indices $j$ such that the dot for $\pi(j)$ is on the $k$th segment of the grid $\mathcal{G}^v$. Consider the dots on the grid $\mathcal{G}(M)$ corresponding to entries $\sigma(i)$ such that $\sigma(i) \in J_k$, and copy these dots to the $k$th block of the grid $\mathcal{G}(M^v)$ in the same $x$-coordinates that they had in the grid $\mathcal{G}(M)$. If $v_k = -$, this block is a copy of the grid $\mathcal{G}(M^c)$, in which case the dots are flipped accordingly. Doing this for every $k$ results in a drawing of the permutation $\pi\sigma$ on the grid $\mathcal{G}(M^v)$. Indeed, let us show that if $\pi\sigma(i) > \pi\sigma(i')$ then the dot corresponding to $\pi\sigma(i)$ is placed higher than the dot for $\pi\sigma(i')$. If $\sigma(i)$ and $\sigma(i')$ belong to the same set $J_k$, then $\pi\sigma(i) > \pi\sigma(i')$ is equivalent to $\sigma(i) > \sigma(i')$ if $v_k = +$ (in which case our drawing preserves the relative height of the dots $\sigma(i)$ and $\sigma(i')$ in $\mathcal{G}(M)$), and to $\sigma(i) < \sigma(i')$ if $v_k = -$ (in which case our drawing reverses the relative height). If $\sigma(i)$ and $\sigma(i')$ belong to different sets $J_k$ and $J_{k'}$, respectively, then $\pi(\sigma(i)) > \pi(\sigma(i'))$ implies that $k > k'$, and our drawing places $\pi(\sigma(i))$ higher than $\pi(\sigma(i'))$.

Next we show that $\mathcal{G}(M^v) \subseteq \{\mathcal{G}^v\mathcal{G}(M)\}$. Given a drawing of a permutation $\tau$ on the grid $\mathcal{G}(M^v)$, let $\sigma$ be the permutation whose drawing is obtained by overlaying the $r$ blocks which are copies of $\mathcal{G}(M)$ or $\mathcal{G}(M^c)$, flipping those corresponding to $\mathcal{G}(M^c)$. Then we construct $\pi$ as follows. For every $i$, if the dot for $\tau(i)$ is in the $j$th block in the grid $\mathcal{G}(M^v)$, draw the dot for $\pi(j)$, where $j = \sigma(i)$, in the $k$th segment of the grid $\mathcal{G}^v$. One can check that the resulting permutations $\sigma$ and $\pi$ satisfy $\pi\sigma = \tau$.  

\[\square\]
One-column grid classes are, up to taking the inverse of its elements, equivalent to the so-called $W$-sets considered in \[6\] \[4\], which also appear in \[7\] under the name $\sigma$-classes. Compositions (i.e. products) of $W$-sets were studied in \[4\]. The special case of Proposition \[9.1\] where $A$ is a one-column grid class $G^w$, stated as Corollary \[7.2\] below, is equivalent to \[4\] Theorem 6. It gives an explicit description of the set product of one-column grid classes.

For a vector $w = (w_1, w_2, \ldots, w_n) \in \{1, -1\}^n$, we use the notation $w^1 = w$ and $w^{-1} = (-w_1, \ldots, -w_2, -w_1)$. Given $v = (v_1, \ldots, v_r) \in \{1, -1\}^r$ and $w \in \{1, -1\}^s$, define the product $v \star w := (w^{v_1}, w^{v_2}, \ldots, w^{v_r})$. For example, $(-1, 1) \star (1, -1, -1) = (1, 1, -1, 1, -1, -1)$.

**Corollary 9.2** (\[4\]). For every $v \in \{1, -1\}^r$ and $w \in \{1, -1\}^s$,

$$\{G^v, G^w\} = G^{v \star w}.$$  

Figure 9.1 shows how $G^{-+}G^{-++} = G^{+++}$ as sets.

**Remark 9.3.** Even if $A$ is a fine grid class, it is not necessarily true that $\{G^v A\}$ is a fine set. For example, if $A = \mathcal{Y}^2$ and $v = ++$, we have that

$$\{G^v A\} = \{\mathcal{L}\mathcal{Y}^2\} = G^J = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ -1 & 0 \end{pmatrix},$$

which is not a fine set (or even symmetric) already for $n = 6$.

On the other hand, the question of whether $G^v A$ is always a fine multiset remains open. An affirmative answer would follow from Conjecture 10.4.

## 10 Final remarks and open problems

We have shown in Corollary 7.6 that vertically rotated one-column grid classes are fine when all slopes have the same sign, that is, $\{C_n G_v^r\}$ is a fine set when $v = +^k$ or $v = -^k$ (the latter case follows by symmetry using Proposition 8.2). By Propositions 6.9 and 8.2, this phenomenon also holds when $v = -+$ or $v = +-$. Computer experiments suggest that the following more general statement is true.

**Conjecture 10.1.** For every one-column grid class $G^v$, the set $\{C_n G^v\}$ is a fine set.

The following conjecture suggests a far-reaching generalization of Corollary 7.8. Recall that one can interpret $D_{n-1,J}^{-1}$ as a subset of $S_n$ consisting of permutations that fix $n$.

**Conjecture 10.2.** For every $J \subseteq [n-2]$,

$$Q(C_n D_{n-1,J}^{-1}) = Q(C_n D_{n-1,J}^{-1}C_n).$$

Thus, by Theorem 7.1, $C_n D_{n-1,J}^{-1}$ is a fine set for $S_{n-1} \subseteq S_n$.

Note that both $C_n D_{n-1,J}^{-1}$ are $D_{n-1,J}^{-1}C_n$ are sets (that is, elements have multiplicity one). Conjecture 10.2 no longer holds when $C_n$ is replaced by a general fine set $B \subseteq S_n$.

The following generalization of Theorem 7.4 and Corollary 7.9 will be proved in a forthcoming paper \[12\].
Theorem 10.3. If $B \subseteq S_{n-1}$ is a fine set for the $S_{n-1}$-representation $\rho$, then $BC_n$ is a fine set for $\rho \uparrow S_n$.

It should be noted that an analogous statement for vertical rotation does not hold. For example, \{2143, 2413\} is a Knuth class in $S_4$, thus fine, but $C_5\{2143, 2413\}$ is not fine.

Regarding multiset products of fine sets and inverse descent classes, computer experiments support the following conjecture.

Conjecture 10.4. Let $B \subset S_n$ be fine. Then, for every $J \subseteq [n - 1]$,

$$Q(D_{n,J}^{-1}B) = Q(BD_{n,J}^{-1}).$$

In particular, by Theorem 5.4, the multiset $D_{n,J}^{-1}B$ is fine.

Note that when $B = C_n$, the conjecture involves horizontal and vertical rotations of an inverse descent class.

Given a pair of Knuth classes $A$ and $B$ in $S_n$ of shapes $\lambda$ and $\mu$, in many cases the multiset $AB$ is fine for the tensor product $\chi^\lambda \otimes \chi^\mu$. Equivalently, we have that $Q(AB) = Q(A) \ast Q(B)$, where $\ast$ denotes the Kronecker product. For example, consider the Knuth class $A = \{21435, 21453, 24135, 24153, 24513\} \subseteq S_5$, which satisfies $Q(A) = s_{3,2}$. Then $A^2 = \{A^2\}$ and

$$Q(A^2) = s_5 + s_{4,1} + s_{3,2} + s_{3,1,1} + s_{2,2,1} + s_{2,1,1,1} = \text{ch}(\chi^{3,2} \otimes \chi^{3,2}) = s_{3,2} \ast s_{3,2}.$$

The identity $Q(AB) = Q(A) \ast Q(B)$ holds for all Knuth classes $A$ and $B$ in $S_4$, but not in general. For $n \geq 5$, there are pairs of Knuth classes $A, B \in S_n$ for which $Q(AB)$ is not even symmetric.

Question 10.5. For which pairs of Knuth classes are their set and multiset products fine?

We conclude with a natural question regarding restriction of Schur-positive grids.

Question 10.6. Let $\mathcal{G}$ be a grid class. Does $Q(\mathcal{G}_n)$ being Schur-positive imply that $Q(\mathcal{G}_{n-1})$ is Schur-positive?

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