Codimension Two Compactifications and the Cosmological Constant Problem

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We consider solutions of six dimensional Einstein equations with two compact dimensions. It is shown that one can introduce 3-branes in this background in such a way that the effective four dimensional cosmological constant is completely independent of the brane tensions. These tensions are completely arbitrary, without requiring any fine tuning. We must, however, fine tune bulk parameters in order to obtain a sufficiently small value for the observable cosmological constant. We comment in the effective four dimensional description of this effect at energies below the compactification scale.

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1. Introduction. Extra dimensional models have proven to be very fruitful in providing new ways of attacking old problems. Despite this, some of them, like the cosmological constant problem (CCP), still lack a compelling solution. In this paper we report in what we believe could be some progress in obtaining an eventual solution to this formidable puzzle.

It has been known for some time that if one introduces a 3-brane in a codimension 2 bulk around which the background solution is rotationally symmetric, the only effect of a non-zero brane tension will be to induce a conical singularity (or a deficit angle) in the transverse space, and the 4 dimensional effective cosmological constant will be independent of this tension. This scenario realizes the idea of E: the curvature associated with the energy carried by the brane is measurable by "bulk observers" only, being spent in curving the extra dimensional manifold, and not the brane worldvolume. However, this scenario has never been realized satisfactorily in the literature. In [1] a fine tuning relation between different brane tensions had to be imposed. In [2] the hope was for solutions that localized gravity in a 3-brane in a rotationally symmetric 6D bulk, but the solutions found always involved naked spacetime singularities or did not compactify the extra space at all (see, however, [3] for a proposal along these lines). Other scenarios considered in the literature for localizing gravity in a 3-brane in a 6D bulk that are free of singularities [4] are not insensitive to the deficit angle, so some fine tuning between brane and bulk parameters has to be imposed.

In this letter we consider solutions of 6 dimensional gravity with two compact dimensions, along the lines of the spontaneous compactification models of [7, 8]. One of them is periodic and the background is rotationally symmetric around it, so we are able to realize the aforementioned scenario, recovering 4D gravity at low energies and in a setup free of singularities (except for the conical ones). As has been previously discussed [2], this is not a complete solution to the CCP, since we still need to fine tune bulk parameters to obtain a small enough value for the 4D cosmological constant. However, it is progress because one could hope that the required fine tunings, not involving brane parameters, are consequence of some unbroken symmetry in the bulk (e.g. supersymmetry). The outline of the paper is as follows: in the next section we present the solution, explain the induced conical singularity and comment on possible bulk topologies. In section 3 we comment on the effective low energy 4D description, and we will try to get some insight in the effective 4D mechanism that cancels the vacuum energy coming from "brane fields". Finally we write the conclusions.

2. Bulk Solution. The metric ansatz we take is of the factorizable form

\[ ds^2 = \gamma(x)_{\mu\nu} dx^\mu dx^\nu + \kappa(z)_{ij} dz^i dz^j \] (1)

where latin indices run over the two extra dimensions and greek indices over the conventional ones. We will consider an energy-momentum tensor with a vacuum expectation value given by

\[ T_{MN} = - \left( \gamma_{\mu\nu} \Lambda_\gamma \kappa_{ij} \Lambda_\kappa \right) . \] (2)

Inhomogeneous forms for this tensor have been considered previously in the literature [2, 3, 7, 8, 9], and while we will not discuss here the origin of this energy-momentum tensor, we will simply note that the inhomogeneity can be due to different contributions to the casimir energy in the different directions [8], or to a vacuum expectation value of some p-form field [2, 7]. From Einstein equations one can check that the metric admits a solution where the space is the direct product of a four dimensional manifold (with metric $\gamma_{\mu\nu}$) times a two dimensional one (with metric $\kappa_{ij}$), both of constant curvature, being this curvatures

\[ R(\gamma_{\mu\nu}) = 2\Lambda_\gamma , \quad R(\kappa_{ij}) = 2\Lambda_\kappa - \Lambda_\kappa . \] (3)

In this expressions and in what follows powers of the higher dimensional Planck mass, $M_{Pl}^{(6D)}$, should be understood where needed. Of direct phenomenological interest will be the case in which the four dimensional

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the manifold is de Sitter space, $dS_4$, or Minkowski space, $E^{1,3}$, and the two dimensional one is compact. Since Einstein equations do not fix the topology of the space we have to make some assumptions about it. For this compactification manifold we will consider two different possibilities: a sphere, $S^2$, or the disk, $D_2 \equiv S^2/Z_2$. This second manifold is an orbifold obtained from the sphere by the identification of points in the northern and southern hemispheres that are symmetric under reflections through the equatorial plane. The equator of the sphere, being invariant under this reflection, is an orbifold fixed curve that forms the boundary of the manifold. For these spaces the metric will take the form

$$ds^2 = dt^2 - e^{2Ct}d\vec{x}^2 - R_0^2(d\theta^2 + \beta^2 \sin^2 \theta \, d\varphi^2) \quad (4)$$

with $\varphi$ taking values in $[0, 2\pi)$ and $\theta$ ranging from $0$ to $\pi$ in the case of a sphere or from $0$ to $\pi/2$ for the case of the disk. The constants $C$ and $R_0$ can be determined as

$$C^2 = -\frac{2}{3}\Lambda_\kappa, \quad R_0^{-2} = \frac{1}{2}\Lambda_\kappa - \Lambda_\gamma. \quad (5)$$

It is clear that if we want the parameter $C$ to be small enough to agree with observations we have to fine tune $\Lambda_\kappa$ to be very small. From the previous equations it is apparent that in principle we could choose $\Lambda_\kappa = \Lambda_\gamma$, so the required energy-momentum tensor would be derived from a conventional 6D cosmological constant. But in this case, if we fix $\Lambda_\kappa$ to obtain a value of $C$ compatible with observations we would get a size for the compactification manifold that is too large. In this scenario we have to assume that $|\Lambda_\gamma| \gg |\Lambda_\kappa|$ in order to stabilize the size of the extra dimensions to a small enough value. It is interesting to note that for some values of $\Lambda_\gamma$ one could find a solution for the gauge hierarchy problem in the form of (1). This problem would be rephrased here in finding a theory that gives naturally values of $\Lambda_\gamma$ in the appropriate range.

Note also that we included an arbitrary constant $\beta$ in the $(\varphi, \varphi)$ component of the metric. For any value of this constant different from one there will be a conical singularity at $\theta = 0$ (and in $\theta = \pi$ for the case of the sphere). The metric for the two dimensional compact manifold will take the usual form with $\beta = 1$ if we redefine $\varphi$ so it ranges from $0$ to $2\pi \beta$, so we can think of this manifold as being a disk (or sphere) with a "wedge" cut out. This deficit angle gives a $\delta$-like contribution to the curvature tensor localized at the points with $\sin(\theta) = 0$. This $\delta$-like curvature singularity will be cancelled if we introduce a 3-brane at this positions with tension $T_0 = 2\pi(1 - \beta)$, i.e., if we add a piece to the Lagrangian given by

$$\Delta \mathcal{L} = \frac{T_0}{2\pi \sqrt{\kappa}} \delta(\sin(\theta)) \quad (6)$$

where $\sqrt{\kappa}$ is the volume element in the compactification manifold (see [12] for a discussion of delta functions in curved manifolds). We can see that the contribution to the energy-momentum tensor of a piece in the Lagrangian given by eq. (6) will be cancelled by the contribution to the Einstein tensor induced by the deficit angle in a number of different ways. One can compute directly the curvature tensor for the metric (1) and check that a term proportional to $(1 - \beta)\delta(\sin(\theta))$ is present in a way analogous to (2). For the sake of completeness, and to show that this singularity can be consistently smoothed out, here we will consider a regularization of a 3-brane placed in $\theta = 0$ and we will see that in the infinitely thin limit, the only effect of this brane will be to induce this conical singularity. We will not have to assume any particular regularization procedure, since in the thin limit the result is independent of it. We proceed as follows: we add a finite term $(\Delta T_{MN}(\theta, \epsilon))$ to the energy-momentum tensor such that for $\theta > \epsilon \Rightarrow \Delta T_{MN} = 0$ and

$$\lim_{\epsilon \to 0} \Delta T_{MN}(\theta, \epsilon) = -\left( \gamma_{\mu\nu} \frac{T_0}{2\pi \sqrt{\kappa}} \delta(\theta) \right)_{0}, \quad (7)$$

that is, a regularization of the brane with width $< \epsilon$. For $\theta > \epsilon$ the solution is given by eqs. (10), while for $\theta < \epsilon$ it depends on the regularization used. Considering an ansatz for the metric like

$$ds^2 = dt^2 - e^{2Ct}d\vec{x}^2 - R_0^2(d\theta^2 + \rho(\theta)^2 \, d\varphi^2) \quad (8)$$

the $(0, 0)$ component of Einstein equations is

$$\rho''(\theta) - \frac{R_0^2 (3C^2 + 4\Lambda)}{4} \rho(\theta) = -R_0^2 \rho(\theta) \Delta T_{00}. \quad (9)$$

It is always possible to consider regularizations of the brane such that one finds solutions with the ansatz (4), and it is a straightforward exercise to construct a particular regularization and solve for the function $\rho(\theta)$. We integrate now the previous equation from $0$ to $\epsilon$, and take the limit $\epsilon \to 0$. For doing this we only need to take into account the following facts: $\rho(0) = 0$ ($\theta = 0$ corresponds to a single point in the transverse space), $\rho'(0) = 1$ (the geometry is regular at the origin as it should in any regularization of the $\delta$-like brane), for $\theta > \epsilon$ the solution is given by (10) and in the infinitely thin limit the extra contribution to the energy-momentum tensor is given by eq. (4). Then one obtains

$$1 - \beta = \frac{T_0}{2\pi}. \quad (10)$$

For the rest of the $(\mu, \nu)$ equations we obtain the same result, independently of the regularization used, while the $(\theta, \theta)$ and $(\varphi, \varphi)$ components of the equations give a zero contribution to the matching condition in the thin limit, since $\rho''(\theta)$ do not appear in them, and this is the only term that contributes.
For the case in which the compactification manifold is a sphere, the same treatment applies to the singularity at $\theta = \pi$, so we have to consider configurations with two "twin branes" at the north and south poles, with equal tensions. One could argue that the fine tuning of the two brane tensions against each other could be regarded as natural in such symmetric configurations, but one can also consider the compactification in the disk, so no second 3-brane is needed. In this case one has to look at the matching conditions for Einstein equations at the orbifold fixed points, those with $\theta = \pi/2$. If there was a jump in the first derivatives of the metric components with respect to the $\theta$ coordinate at these points, we would need to assume the presence of a 4-brane with non-zero energy-momentum tensor in $\theta = \pi/2$. This is a subtle issue, since this could reintroduce a fine tuning between bulk and brane parameters if the energy carried by this 4-brane depends in the deficit angle $\theta$. However, it is straightforward to see from (4) that the first derivatives of the metric components are zero at the orbifold fixed points, and the matching conditions are satisfied trivially, without having to consider the presence of a 4-brane carrying any energy at this position.

3. 4D Effective Theory and the Cosmological Constant. From the six dimensional point of view, it is clear that the inflation rate in the brane worldvolume only depends on $\Lambda_x$, and not on the brane tension, since the parameter $C$ in the solution is fixed by eq. (6), and it is completely independent of $T_0$. The effect of the three brane at $\theta = 0$ with nonzero tension can only be to induce a conical singularity, since it gives a contribution to the 6D Einstein equations proportional to $\delta(\theta)$, just like a deficit angle in the transverse space. However, from the 4D point of view, it is not transparent how this mechanism works. Consider the low energy theory at energies below the compactification scale. We will have a massless 4D graviton. The equation of motion for this mode can be extracted from the metric

$$ds^2 = \tilde{g}_{\mu\nu}(x)dx^\mu dx^\nu - R_0^2(\delta \theta^2 + \beta^2 \sin^2 \theta \, d\varphi^2)$$ \hspace{1cm} (11)

for which the 6D curvature scalar can be expressed as

$$R^{(6D)} = R^{(4D)}(\tilde{g}) - \frac{2}{R_0^2} - (1 - \beta) \frac{\delta(\theta)}{\sqrt{\kappa}}.$$ \hspace{1cm} (12)

It is then straightforward to extract the equation for the 4D graviton

$$R_{\mu\nu} - \frac{1}{2} \tilde{g}_{\mu\nu} R = -\tilde{g}_{\mu\nu} \left( \frac{1}{2} \Lambda_x + A_2^{-1}(T_0 - 2\pi(1 - \beta)) \right),$$ \hspace{1cm} (13)

where $A_2$ is the area of the compactification manifold. We know from eq. (13) that the deficit angle $(1 - \beta)$ exactly cancels the brane tension. This, however, was extracted from the matching conditions in the full 6D theory. Looking at eqs. (11, 12), that are intrinsically six dimensional, it is clear why this happens, since these terms are the only ones with a $\delta(\theta)$. But from the point of view of the 4D low energy graviton it is a mystery why there is a contribution to the vacuum energy, coming from the curvature of some internal manifold, that exactly cancels the contribution coming from fields localized in the brane, and not the contribution coming from those living in the bulk. It is obvious from the equation above that the 4D graviton makes no distinction between this different contributions. We will not attempt here to describe completely the cancellation mechanism for the "brane part" of the vacuum energy in the low energy 4D theory, but we will give an argument why this must happen in order to obtain a sensible theory. If the conical singularity contribution to the effective 4D vacuum energy did not cancel the contribution coming from the brane, we know that the 6D Einstein equations would not be satisfied, and this must have a reflection in the effective 4D theory.

Remember that the bulk solution is rotationally symmetric around the 3-brane, and this fact is crucial for the mechanism to work. It allows us to choose the deficit angle in such a way that it cancels the contribution to the 6D Einstein equations coming from the 3-brane, without reintroducing any fine tuning between bulk and brane parameters. So let’s consider now the metric (11) with the replacement

$$\varphi \rightarrow \varphi + \alpha(x)$$ \hspace{1cm} (15)

$$A_\mu(x) \rightarrow A_\mu(x) - \partial_\mu \alpha(x)$$ \hspace{1cm} (16)

If the energy-momentum tensor is also invariant under these transformations, after compactification we retain this $U(1)$ gauge symmetry for the field $A_\mu(x)$, and we can expect a massless graviphoton in the spectrum. In this case the graviphoton will also be present in the low energy theory and we can extract the equation of motion for this field from the $(\mu, \varphi)$ component of the 6D Einstein equations. At first order in the fluctuations $A_\mu(x)$ it is

$$\nabla_\mu F^{\mu\nu}(x) = (R(\tilde{g}) - 2\Lambda_x) A^\nu(x),$$ \hspace{1cm} (17)

where derivatives are covariant with respect to $\tilde{g}_{\mu\nu}$. The previous equation shows that the graviphoton can distinguish between contributions to the vacuum energy coming from the brane and the bulk, since it only couples to the latter ones. If it is to be massless, the right hand side of eq. (17) must vanish. This condition is nothing but eq. (13), that was obtained in the full 6D theory, obtained
from a 4D point of view. We see that the requirement of the graviphoton to be massless forces the deficit angle in eq. (13) to cancel exactly the brane tension. As we said, if this was not the case, full 6D Einstein equations would not be satisfied, and we can expect some pathologic behaviour in the 4D effective theory, for instance, from (17) we see that the graviphoton could acquire a negative mass squared.

4. Conclusions. We have presented solutions of six dimensional gravity with two compact dimensions. Allowing for an inhomogeneous form for the energy-momentum tensor, eq. (2), we found that the 4D inflation rate and the size of the compact dimensions are independent parameters. Some components of this energy-momentum tensor have to be very small to obtain a Hubble expansion that does not contradict observations. We did not assume any particular origin for the required energy-momentum tensor, but there are examples in the literature of theories were a inhomogeneous form for this tensor is generated.

In this background it is possible to consider the presence of δ-like brane sources in such a way that the bulk solution is rotationally symmetric around them. This fact allows the existence of an arbitrary deficit angle in the transverse space, that can be chosen to cancel exactly the contribution of this sources to the 6D Einstein equations, without producing any other effect. In this way the solution is able to adjust freely this parameter to cancel the contribution to the 4D vacuum energy coming from the brane tension. This is significant progress in the longstanding CCP, since one can think of Standard Model fields being localized on the brane and not contributing to the effective vacuum energy. Still, one has to assume that some of the components of the six dimensional energy-momentum tensor are extremely small but the hope is that a model can be constructed where this is the consequence of some (almost) unbroken (super)symmetry in the bulk. It is remarkable also that the gauge hierarchy problem can be related in this context to the parameter $\Lambda_\gamma$ that appears in the bulk energy-momentum tensor.

While in the full 6D theory it is clear how this mechanism works, it is not easy to imagine how can this be seen from a 4D perspective, since the four dimensional graviton makes no distinction between contributions to the effective vacuum energy coming from the brane or from the bulk. However, we have seen that there are fields in the 4D effective theory (the graviphoton) that do make a distinction between this different contributions to the vacuum energy. While we have not shown explicitly the cancellation mechanism from a 4D perspective we have seen that some pathologic behaviour can occur in the low energy effective theory if the brane tension was to contribute to the vacuum energy.

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Note Added. While this paper was being prepared for publication the preprint [13] appeared, where similar solutions to the ones presented here have been considered.

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