TWISTED TORUS KNOTS $T(p, q, 3, s)$ ARE TUNNEL NUMBER ONE

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Abstract. We show that twisted torus knots $T(p, q, 3, s)$ are tunnel number one. A short spanning arc connecting two adjacent twisted strands is an unknotting tunnel.

1. Introduction

A torus knot is a knot that can be embedded in a standard torus in $S^3$. For a fixed standard meridian and longitude system on a standard torus, let $T(p, q)$ denote a torus knot that runs $p$ times in longitudinal direction and $q$ times in meridional direction. By convention, we assume that $p > q > 0$ and $p$ and $q$ are relatively prime. (If $p < 0$ or $q < 0$, we consider it as a mirror image.)

Take $r$ ($1 < r < p$) adjacent parallel strands of $T(p, q)$ and replace them with $s$ times full twists. The resulting knot is called a twisted torus knot $T(p, q, r, s)$. The class of twisted torus knots is interesting in many aspects. For example, it gives strong candidates for non-minimal genus, weakly reducible and unstabilized Heegaard splittings obtained by boundary stabilization [1].

A knot $K$ in $S^3$ is said to be tunnel number one if there exists an arc $\gamma$ properly embedded in the exterior of $K$ such that $cl(S^3 - N(K \cup \gamma))$ is a genus two handlebody. It is well known that torus knots and $T(p, q, 2, s)$ are tunnel number one. It is also known that tunnel number of a twisted torus knot is one or two. As a next step to $T(p, q, 2, s)$, Morimoto asked about the tunnel number of $T(p, q, 3, s)$ ([2], Problem 5'). We answer the question.

Theorem 1.1. Twisted torus knots $T(p, q, 3, s)$ are tunnel number one.

As a corollary, every $T(p, q, 3, s)$ is a prime knot since tunnel number one knots are prime [3], [4].

Remark 1.2. There exist composite twisted torus knots [2].

2. Proof of Theorem 1.1

Consider a standard knot diagram of a twisted torus knot $T(p, q, 3, s)$ as in the Figure 1. (It is $T(17, 7, 3, -2)$.) There is $(p, q)$-braided part on the left, and $s$ times full-twisted part on the right. On the full-twisted part of the diagram, label the middle strand among the three full-twisted strands as 0. Label all the other strands with consecutive integers modulo $p$.

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Connect two strands labelled as 0 and \(-1\) with a short spanning arc \(\gamma\). It will be clear later that \(\gamma\) is an unknotting tunnel. Do slide and isotopy through the full-twisted part so that the twisted part becomes \(s\) times full twists on two strands. (The dotted arc in Figure 1. will disappear after the slide and isotopy.) Now we have two copies of \(\gamma\), \(\gamma_1\) and \(\gamma_2\), where \(\gamma_1\) is above the twisted part and \(\gamma_2\) is below the twisted part.

Let \(P_1\) and \(P_2\) be endpoints of \(\gamma_1\) and \(\gamma_2\) on the strand labelled 0, respectively. Suppose \(P_1\) moves along \(T(p, q, 3, s)\). Starting to the upper direction, suppose \(P_1\) arrives at the strand labelled 1 before arriving at the strands labelled 2 or \(-1\) in the bottom part of the twists. That implies \(\gamma_1\) is isotopic to an arc connecting two parallel strands of \(s\) times full twists, which is an unknotting tunnel for the type \(T(-, -, 2, s)\) and the proof is done. The same arguments holds for \(P_2\). So we only need to show that either \(P_1\) or \(P_2\) arrives at the strand labelled 1 before arriving at the strands labelled 2 or \(-1\).

Suppose it does not happen. Then one of the following properties holds.

(*) \(P_1\) arrives at the strand 2 first and \(P_2\) arrives at the strand \(-1\) first among \(\{1, 2, -1\}\), or

(**) \(P_1\) arrives at the strand \(-1\) first and \(P_2\) arrives at the strand 2 first among \(\{1, 2, -1\}\)
Case 1) Suppose (*) holds. When $P_1$ passes through the $(p, q)$-braided part from above to below, the labelled number on the strand it belongs increases by $q$ modulo $p$. For $P_2$, it decreases by $q$ modulo $p$. Hence by property (*), there exist some $k$ ($k < p$) and $j$ ($j < p$) satisfying the following equations.

\begin{align*}
(1) & \quad q \neq 1, -1, \quad 2q \neq 1, -1 \quad \ldots \quad kq = 2 \quad (\text{mod } p) \\
(2) & \quad -q \neq 1, 2, \quad -2q \neq 1, 2 \quad \ldots \quad -jq = -1 \quad (\text{mod } p)
\end{align*}

So $jq = 1$ (mod $p$) and we can see that $j > k$ from (1). Adding equations, we get $-(j - k)q = 1$ (mod $p$) and this contradicts equations (2).

Case 2) Suppose (***) holds. There exist some $k$ ($k < p$) and $j$ ($j < p$) satisfying the following equations.

\begin{align*}
(3) & \quad q \neq 1, 2, \quad 2q \neq 1, 2 \quad \ldots \quad kq = -1 \quad (\text{mod } p) \\
(4) & \quad -q \neq 1, -1, \quad -2q \neq 1, -1 \quad \ldots \quad -jq = 2 \quad (\text{mod } p)
\end{align*}

So $-kq = 1$ (mod $p$) and we can see that $k > j$ from (4). Adding equations, we get $(k - j)q = 1$ (mod $p$) and this contradicts equations (3).

The sign of $s$ does not effect much and the arguments are similar. This completes the proof.

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