Refining the Elliptic Genus*

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We show how special forms of an $N = 2$ Landau-Ginzburg potential directly imply the presence of an $N = 2$ super-$W$ algebra. If the Landau-Ginzburg model has a super-$W$ algebra, we show how the elliptic genus can be refined so as to give much more complete information about the structure of the model. We study the super-$W_3$ model in some detail, and present some results and conjectures about more general models.

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1. Introduction

Some important new results were obtained in [1,2] about the relationship between $N = 2$ superconformal models and $N = 2$ Landau-Ginzburg models [3–8]. In [2] it was shown that one could compute the elliptic genus of the model by taking a free field limit in which the coefficient of the potential vanishes. From the elliptic genus one can then extract information about Ramond characters of the model and its orbifolds [9–12]. A second result of [2] was to show that one could obtain representatives of the superconformal energy-momentum tensor in the Landau-Ginzburg model. That is, even though Landau-Ginzburg action is not superconformal, one is able to identify operators that, by virtue of the Landau-Ginzburg equation of motion, generate the $N = 2$ superconformal algebra upon left-moving states up to right-moving states that are cohomologically trivial. The idea being that these operators would generate the exact $N = 2$ superconformal algebra at the infra-red fixed point of the model.

There are many natural extensions of this work, see for example, [9–12]. In this letter we wish to address the following issues. First, the $N = 2$ superconformal coset models based upon $CP_n$ have very particular Landau-Ginzburg potentials, and they also possess super-$W$ algebras. We will use the techniques of [2] to establish a more direct relationship between the special form of the potential and the presence of such an algebra. Indeed, the classical limit of this question has already been extensively analysed in [13]. Our approach is a little different, and we will establish results for the quantum theory.

Secondly, given that there is such an extended algebra, one can, in principle, refine the elliptic genus so as to give information about the quantum numbers of the $W$-charges in the Ramond sector. We will indeed show how to modify the simple formulas for the elliptic genus given in [3–11] so as to extract detailed information about the $W$-structure of the Ramond ground states.

2. Landau-Ginzburg formulation $N=2$ super-$W$ algebras

Consider an $N = 2$ supersymmetric Landau-Ginzburg model with action

$$ S = \int d^2 x \, d^4 \theta \, \sum_j \bar{\Phi}_j \Phi_j - \int d^2 x \, d^2 \theta \, W(\Phi_j) - \int d^2 x \, d^2 \bar{\theta} \, W(\bar{\Phi}_j), \quad (2.1) $$
where $\Phi_j, j = 1, \ldots, n$ are $N = 2$ chiral superfields. We will adopt the notation of [1,2,14] and in particular, super-derivatives are defined by:

\[
D_+ = \frac{\partial}{\partial \theta^+} - i \bar{\theta}^+ (\partial_0 + \partial_1) \quad D_- = \frac{\partial}{\partial \theta^-} - i \bar{\theta}^- (\partial_0 - \partial_1) ;
\]
\[
\bar{D}_+ = -\frac{\partial}{\partial \bar{\theta}^+} + i \theta^+ (\partial_0 + \partial_1) \quad \bar{D}_- = -\frac{\partial}{\partial \bar{\theta}^-} + i \theta^- (\partial_0 - \partial_1) .
\]

These super-derivatives satisfy the relations: $\{D_+, \bar{D}_+\} = 2i(\partial_0 + \partial_1)$, and $\{D_-, \bar{D}_-\} = 2i(\partial_0 - \partial_1)$. Imposing chirality on the fields $\Phi_j$ means requiring that $\bar{D}_+ \Phi_j = \bar{D}_- \Phi_j = 0$, which implies that these superfields have an expansion:

\[
\Phi_j(y, \theta) = \phi_j(y) + 2 \theta^\alpha \psi_{j,\alpha}(y) + \theta^\alpha \theta^\beta F_{j}(y) ,
\]

where $\alpha = \pm$ and $y^m = x^m + i \theta^\alpha \sigma^m_{\alpha\bar{\beta}} \bar{\theta}^\beta$. Note that we have normalized $\psi_{j,\alpha}$ differently from [2].

Given the kinetic term in (2.1), the short distance expansion of $\Phi_j$ with $\bar{\Phi}_j$ is given by:

\[
\Phi_j(x_1, \theta_1, \bar{\theta}_1)\bar{\Phi}_j(x_2, \theta_2, \bar{\theta}_2) \sim -ln(\tilde{x}_m \bar{x}_m) ,
\]

where

\[
\tilde{x}_m = (x_1 - x_2)^m + i \theta_1 \sigma^m \bar{\theta}_1 + i \theta_2 \sigma^m \bar{\theta}_2 - 2i \theta_1 \sigma^m \theta_2 .
\]

One should note that in terms of the component fields, the foregoing conventions lead to: $\phi_j(x) \phi_j(0) \sim -ln(x^m x_m)$, and the rather non-standard form: $\psi_{j,-}(x) \psi_{j,-}(0) \sim -i(\frac{1}{x^m - x_m})$.

The equations of motion derived from (2.1) have a very simple form:

\[
\bar{D}_+ \bar{D}_- \Phi_j = \frac{1}{2} \frac{\partial W}{\partial \Phi_i} \]
\[
D_+ D_- \Phi_j = \frac{1}{2} \frac{\partial W}{\partial \Phi_i} .
\]

Throughout this letter we will assume that the Landau-Ginzburg potential is quasi-homogeneous with indices $\omega_j$. That is,

\[
W(\Phi_j) = \lambda^{-1} W(\lambda^{\omega_j} \Phi_j) .
\]

The energy-momentum tensor, $T$, the supersymmetry generators, $G^\pm$, and the $U(1)$ current, $J(z)$, of a superconformal algebra can be incorporated into the various components of an $N = 2$ superfield $\mathcal{J}$. In Landau-Ginzburg model one can explicitly construct
a representative of the superfield $\mathcal{J}$, whose components generate the the $N = 2$ superconformal algebra on the left-movers (up to trivial cohomology on the right-movers) \cite{2}. The superfield, $\mathcal{J}$, is simply:

$$\mathcal{J} = \sum_j \left[ \frac{1}{2}(1 - \omega_j)D_\pm \Phi_j \bar{D}_\pm \bar{\Phi}_j - i \omega_j \Phi_j (\partial_0 - \partial_1) \bar{\Phi}_j \right], \quad (2.8)$$

and it has been constructed so as to satisfy

$$\bar{D}_+ \mathcal{J} = 0. \quad (2.9)$$

This equation basically requires that $\mathcal{J}$ be holomorphic (up to the cohomology of $\bar{D}_+$).

In particular, one has

$$\mathcal{J}(x_1, \theta_1, \bar{\theta}_1) \mathcal{J}(x_2, \theta_2, \bar{\theta}_2) = -\frac{c}{3\bar{x}_{12}^2} + 2 \left( \frac{\theta_{12} \bar{\theta}_{12}}{\bar{x}_{12}} + i \frac{\theta_{12}}{2 \bar{x}_{12}} D_- + i \frac{\bar{\theta}_{12}}{2 \bar{x}_{12}} \bar{D}_- + \frac{\theta_{12} \bar{\theta}_{12}}{\bar{x}_{12}} (\partial_0 - \partial_1) \right) \mathcal{J}(x_2, \theta_2), \quad (2.10)$$

where $\bar{x}_{12} = z_1 - z_2 + i(\bar{\theta}_1 \theta_2 - \theta_1 \bar{\theta}_2)$, $\theta_{12} = \theta_1 - \theta_2$ and $c$ is the charge of the $N = 2$ supersymmetric model. The normalization of $\mathcal{J}$ has been determined by fixing the leading singularity in (2.10).

Note that in order to establish (2.9) one needs to use quasi-homogeniety of $W$ along with the equations of motion (2.6).

For $n = 1$ and $W(\Phi) = \frac{1}{k+2} \Phi^{k+2}$, this Landau-Ginzburg model describes the $N = 2$ superconformal minimal models with central charge $c = 3k/(k + 2)$ \cite{2,3}. The currents in the superfield $\mathcal{J}$ constitute a complete chiral algebra for the theory, that is, the Hilbert space is finitely reducible as representation of the algebra. When $n \geq 2$, this is no longer true. However, for special Landau-Ginzburg potentials we know that the chiral algebra can be extended to an $N = 2$ super-$W$ algebra. In particular the $N = 2$ superconformal coset models \cite{15}:

$$\frac{SU_k(n + 1) \times SO_1(2n)}{SU_{k+1}(n) \times U(1)} \quad (2.11)$$

have an $N = 2$ super-$W$ algebra, and this generally believed to be a complete chiral algebra. Moreover, these models have a Landau-Ginzburg formulation. An easy way to compute the Landau-Ginzburg potential is as follows \cite{3,16}. The potential, $W$, is given by

$$W = \frac{1}{k + n + 1} \xi_p^{k+n+1} \quad (2.12)$$
where the $\Phi_j$ are defined by

$$
\Phi_j = \sum_{1 \leq p_1 < p_2 < \ldots < p_j \leq n} \xi_{p_1} \xi_{p_2} \ldots \xi_{p_j} .
$$

Since $W$ is a symmetric function of the $\xi_p$, one can write $W$ as a function of $\Phi_j$ and then $W(\Phi_j)$ is the requisite Landau-Ginzburg potential.

We note that $W$ is uniquely characterized (up to scaling of the $\Phi_j$) by its quasi-homegeneity and the differential equation

$$
\frac{\partial^2 W}{\partial \xi_p \partial \xi_q} = 0 \quad p \neq q .
$$

This implies obvious second order differential equations in terms of the fields $\Phi_j$. In particular, for $n = 2$, the Landau-Ginzburg potential is uniquely characterized by the scaling indices $\omega_1 = \frac{1}{k+3}$ and $\omega_2 = \frac{2}{k+3}$ and the differential equation,

$$
\frac{\partial^2 W}{(\partial \Phi_1)^2} + \Phi_1 \frac{\partial^2 W}{\partial \Phi_1 \partial \Phi_2} + \Phi_2 \frac{\partial^2 W}{(\partial \Phi_2)^2} + \frac{\partial W}{\partial \Phi_2} = 0 .
$$

For simplicity we will restrict our discussion to super-$W_3$ generators, but the generalization to higher spin elements of the chiral algebra should be straightforward though algebraically awful. One can determine the super-$W_3$ current in much the same way as one determines the current $J$. One can make an Ansatz as follows: The lowest component, $S$, of the superfield $\mathcal{W}_3$ has dimension two and therefore the realization of $\mathcal{W}_3$ in terms of the chiral superfields must consist of terms with four super-derivatives. As was the case for the current $J$, the fields $\bar{\Phi}_i, i = 1, 2$, never appear without a super-derivative. Futhermore, the number of the chiral fields, $\Phi_i$, is equal to the number of anti-chiral fields, $\bar{\Phi}_i$. These constraints leave one with about twenty possible terms. The constraint (2.9) was solved in the classical limit in [13] for a number of $W_n$ generators, (and not just $W_3$). Here we will show how to greatly simplify the Ansatz, and arrive at the full quantum result for $W_3$. From this we will be able to conjecture the result for general $W_n$.

Recall that the models (2.11) factorize into a tensor products according to

$$
\mathcal{M}_1 \times \mathcal{M}_2 \times \mathcal{M}_3 = \frac{SU_k(n+1)}{SU_k(n) \times U(1)} \times \frac{SU_k(n) \times SU_1(n)}{SU_{k+1}(n)} \times U(1) .
$$

Note that we have not yet restricted the number of superfields, we have simply focussed upon the simplest non-trivial extension of the chiral algebra.
Let $T_1$ and $T_2$ denote the energy-momentum tensors of $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. The corresponding central charges are $c_1 = \frac{n(k-1)(1+2k+n)}{(k+n)(k+n+1)}$ and $c_2 = (n-1)(1-\frac{n(n+1)}{(k+n)(k+n+1)})$. The lowest component, $S$, of the $\mathcal{N}=2$ super-$\mathcal{W}_3$ generator can be written, up to normalization, as

$$S = c_2 T_1 - c_1 T_2 .$$

(2.17)

The field $S$ manifestly has vanishing operator product with $J$, as is required by the super-$\mathcal{W}_3$ algebra. The relative coefficient of $T_1$ and $T_2$ in (2.17) is determined by requiring that $S$ be a good conformal field, i.e. with no anomalies). Our aim will be to construct a superfield $\mathcal{T}_2$, with lowest component $T_2$. Once we have $T_2$, we can reconstruct $S$ by writing,

$$S = c_2 T - \frac{3c_2}{2c} J^2 - (c_1 + c_2) T_2 ,$$

(2.18)

where $T$ is the energy-momentum tensor of the complete model. This then implies that the $\mathcal{W}_3$ generator is given by

$$\mathcal{W}_3 = -\frac{ic_2}{4}(D_--\bar{D}_- - \bar{D}_- D_-)\mathcal{J} + \frac{3c_2}{2c} J^2 - (c_1 + c_2) \mathcal{T}_2 .$$

(2.19)

We now specialize to $n = 2$, for which $\mathcal{M}_2$ is a standard minimal model. In making an Ansatz for $\mathcal{T}_2$ it is natural to assume that the $\mathcal{N}=2$ superfields provide the standard realization of the minimal model in terms of a single free boson. We, therefore, make an Ansatz for the superfield, $\hat{J}$, corresponding to this free boson. Apart from the fact it works, we have another reason for making this Ansatz and we will comment about this later. The most general form for $\hat{J}$, consistent with rules outlined above, is:

$$\hat{J} = aD_- \Phi_1 D_- \bar{\Phi}_1 + bD_- \Phi_2 \bar{D}_- \bar{\Phi}_2 + c\Phi_1 \partial \bar{\Phi}_1 + d\Phi_2 \partial \bar{\Phi}_2 ,$$

(2.20)

where $\partial = \partial_0 - \partial_1$. The coefficients $a, b, c$ and $d$ in (2.20) are determined by requiring that $\hat{J}$ have proper operator product expansion with $T$, and imposing the operator equation of motion

$$\bar{D}_+ \mathcal{T}_2 = 0 .$$

(2.21)

We find

$$\hat{J} = \frac{i}{2} \sqrt{1-\omega} D_- \Phi_1 D_- \bar{\Phi}_1 - \frac{i}{2} \frac{1}{\sqrt{1-\omega}} D_- \Phi_2 \bar{D}_- \bar{\Phi}_2 + \frac{\omega}{\sqrt{1-\omega}} \Phi_1 \partial \bar{\Phi}_1 ,$$

(2.22)
where \( \omega = \frac{1}{\pi+\sigma} \). For future reference we note that lowest component of the \( U(1) \) current is

\[
\hat{j} = \frac{1}{\sqrt{1-\omega}}(\omega \phi_1 \partial \bar{\phi}_1 - i(1-\omega)\bar{\psi}_1 \psi_1 \bar{\psi}_2. \bar{\psi}_2.
\]

(2.23)

It is important to note that we could not impose the operator equation of motion on \( \hat{j} \) itself because \( \hat{j}(z) \) does not commute with the screening charges and is not an operator in the minimal model.

There are several important aspects to the foregoing computation. First, the equation (2.21) is only satisfied by the virtue of quasi-homogeneity (2.7), the Landau-Ginzburg equations (2.6) and the constraint (2.15) upon the Landau-Ginzburg potential. In particular, if we had not used the constraint (2.15), then there would be no general solution. The second point is that the verification that \( T_2 \) satisfies (2.21) is complicated by the operator ordering after one uses the Landau-Ginzburg equation of motion, and some subtleties of screening operators in the minimal model. In this computation, it is elementary to fix \( a, b, c \) and \( d \), but to verify that \( T_2 \) satisfies the operator equation (2.21) to all orders in the Wick contractions is much more complicated. Indeed we ultimately confirmed our results by finding the translation table between the Landau-Ginzburg fields of [2] and the explicit formula for \( W_3 \) given in terms of Drinfeld-Sokolov reduction in [17,18]. These issues will be discussed fully in [19].

### 3. The elliptic genus

The elliptic genus of the model (2.1) is defined by [20,23]:

\[
E(q, \gamma) = Tr_{\mathcal{H}} \left( (-1)^F q^{H_L} \bar{q}^{H_R} \exp(i \gamma J_0) \right).
\]

(3.1)

In this expression \( \mathcal{H} \) is the complete Hilbert space of the model in the Ramond sector, \( H_L \) and \( H_R \) are the hamiltonians of the left-movers and right-movers, \( F \) is the total fermion number, and \( J_0 \) is the left-moving \( U(1) \) charge. Contrary to the conventions of [2], we will identify \( H_L \) with the Virasoro generator \( L_0 \), and \( H_R \) with \( \bar{L}_0 \). The standard index argument can be used to show that in the right-moving sector, only the ground-states contribute to the trace. As a result, the elliptic genus is a function of \( q \) alone (and not a function of \( \bar{q} \)), and consists of a sum of the (left-moving) Ramond ground-state characters.
The $N = 2$, $U(1)$ current is given obtained from the lowest component of (2.8). Specifically, one has $J = -i\mathcal{J}|_{\theta = \bar{\theta} = 0}$, and so:

$$J(z) = \sum_{j=1}^{n} \left[ i(1 - \omega_j) \bar{\psi}_{j,-}(z) \psi_{j,-}(z) - \omega_j \phi_j(z) \partial \bar{\phi}_j(z) \right].$$ \hspace{1cm} (3.2)

The action of the charge $J_0$ on the (left-moving) superfield components is:

$$\phi_j \to \exp(i\omega_j \gamma) \phi_j$$

$$\psi_{j,-} \to \exp(i(\omega_j - 1) \gamma) \psi_{j,-},$$ \hspace{1cm} (3.3)

where $\gamma$ is the parameter.

One of the key observations in \[2\] was that the elliptic genus could be computed in a Landau-Ginzburg model by taking the limit in which the coefficient of the Landau-Ginzburg potential goes to zero, and the Landau-Ginzburg fields become free. One therefore obtains a simple free-field expression for $E(q, \gamma)$:

$$E(q, \gamma) = e^{-i\gamma \frac{c}{24}} \prod_{j=1}^{n} \prod_{p=1}^{\infty} \frac{(1 - q^{p-1} e^{i\gamma (1 - \omega_j)})(1 - q^{p} e^{-i\gamma (1 - \omega_j)})}{(1 - q^{p-1} e^{i\gamma \omega_j})(1 - q^{p} e^{-i\gamma \omega_j})}.$$ \hspace{1cm} (3.4)

In this expression $c$ is the central charge of the model, and is given by (3.5):

$$c = 3 \sum_{j=1}^{n} (1 - 2\omega_j).$$ \hspace{1cm} (3.5)

It was verified in \[9-11\] that for $\omega_j = j/(k + n + 1)$, this is indeed the sum of the Ramond ground state characters of the $N = 2$ superconformal coset models (2.11).

The elliptic genus (3.1) has already been refined in the sense that it gives the $N = 2$, $U(1)$ charges of the states. For the minimal models ($n = 1$), knowledge of the $U(1)$ charges is sufficient to isolate individual Ramond characters from the elliptic genus \[2\]. This is no longer true when one has more superfields ($n > 1$). However, if such a model possesses an $N = 2$ super-$W$ algebra one should be able to once again resolve the elliptic genus into individual characters. To accomplish this, one seeks left-moving generators of the extended chiral algebra that commute with each other and with $J_0$ and $H_L$. One can then, in principle, insert exponentials of these additional charges into the elliptic genus
and completely refine it with respect to the extended algebra\(^1\). The obvious problem now is to find the generalization of (3.4). We will illustrate the procedure by restricting our attention to two superfields and the \( N = 2 \) super-\( W_3 \) algebra.

Let \( S_0 \) be the zero-mode of the left-moving spin-2 field that is the lowest component of the \( W_3 \)-supermultiplet. This charge commutes with \( J_0 \) and \( H_L = L_0 \), and so one can define an obvious refinement by inserting \( p^{S_0} \) into (3.1). It is also relatively easy to see that the quantum numbers of \( S_0 \) and \( J_0 \) are sufficient to resolve the Ramond ground-states in the \( N = 2 \) super-\( W_3 \) model \([24]\).

There does not appear to be a direct way to find a simple expression for this refinement of the elliptic genus. Instead we will construct a simpler character function with equivalent information. This approach will lead us to a simple generalization of the elliptic genus, and to a method that will easily generalize to higher super-\( W \) algebras. The first step is to use the fact that the \( N = 2 \) superconformal model factorizes as in (2.16). We then refine the elliptic genus using the operator \( L_0^{(2)} \), which is the zero-mode of the energy-momentum tensor, \( T_2 \), of \( M_2 \) in (2.16). That is, we define

\[
E(q,p,\gamma) = Tr_\mathcal{H} \left( (-1)^F q^{H_L} \bar{q}^{H_R} \ p^{L_0^{(2)}} \ exp(i\gamma J_0) \right) .
\]

(3.6)

As we saw in the previous section, the energy momentum tensor, \( T_2 \), appears in the superconformal model in terms of its standard realization in terms of a single boson, with associated \( U(1) \) current \( \hat{j}_0 \) defined by (2.23). Introduce the function:

\[
F(q,\nu,\gamma) = Tr_\mathcal{H} \left( (-1)^F q^{H_L} \bar{q}^{H_R} \ exp(i\nu \hat{j}_0) \ exp(i\gamma J_0) \right) ,
\]

(3.7)

and define its symmetrized form by:

\[
F_s(q,\nu,\gamma) = F(q,\nu,\gamma) + F(q,\nu,-\gamma) .
\]

(3.8)

It is this function (and its generalizations) that is easily computed in the free field limit of the Landau-Ginzburg model. The problem is that, unlike \( T_2(z) \), the \( U(1) \) current \( \hat{j}(z) \) is emphatically not in the chiral algebra of the \( N = 2 \) superconformal model. The current

\footnote{It is, of course, critical that one only insert into the elliptic genus operators that commute with the right-moving supercharge, otherwise the elliptic genus would no longer be an index, and would depend upon \( \bar{q} \). This is why we have restricted to the left moving chiral algebra here, but we note that there are discrete exponentials of right-moving charges that also commute with the right-moving supercharges.}
\( \hat{j}(z) \) does not satisfy any operator equations analogous to (2.9), or equivalently it does not commute with the requisite screening currents. Therefore \( F(q, \nu, \gamma) \) is not going to be any kind of character on the Hilbert space of the \( N = 2 \) superconformal model. However, the function \( F_s \) is a kind of character on the \( N = 2 \) superconformal model, and it contains exactly the same information as \( E(q, p, \gamma) \). Indeed, one has

\[
E(q, p, \gamma) = \sqrt{\frac{2}{\pi}} \frac{\eta(q)}{\eta(pq)} \int_{-\infty}^{\infty} e^{-\nu^2 \lambda} F_s(q, \gamma, \nu - \frac{a}{\lambda}) d\nu .
\]

The gaussian integral has the effect of replacing \( \exp(i\nu \hat{j}_0) \) by \( p^{\frac{1}{2}(\hat{j}_0+a)^2-\frac{1}{4}a^2} \), where \( p = e^{-1/\lambda} \). Thus for the proper choice of \( a \), each state in the trace is weighted by the power of \( p \) appropriate to the energy of the associated \( \hat{j}_0 \)-momentum state. The \( \eta \)-function prefactors in (3.9) take care of the oscillator contribution to the minimal model. One can invert the integral transform (3.9) by essentially performing the inverse Laplace transform. The symmetrization of \( F \) is necessary because \( E(q, p, \gamma) \) is an even function of the \( \hat{j}_0 \) eigenvalues, and so the inversion of (3.9) must yield an even function of \( \nu \).

To understand more generally what is transpiring here, we recall a basic theorem about Lie algebras [25]: Two weights of a Lie algebra are equal up to Weyl rotations if and only if all the Casimir invariants take the same values on the two weights. Essentially, if we have a bosonic realization of a \( W \)-algebra then the values of the \( W \)-charges on the bosonic momentum states are precisely the values of the Casimirs of the underlying Lie Algebra [26]. Thus knowing the \( W \)-charges of the momentum states is equivalent to knowing the Weyl symmetrized character of the bosonic Hilbert space.

Therefore, because the function \( F_s \) is precisely equivalent to a refined form of the elliptic genus, one can also compute it for the \( N = 2 \) superconformal model by taking the free field limit in which the coefficient of the Landau-Ginzburg potential vanishes.

Let \( y = \exp[\nu \hat{j}_0] \) and \( z = \exp[i\omega \gamma] \), then, in the free field limit one has:

\[
F(q, y, z) = y^{-1} z^k \prod_{p=1}^{\infty} \left\{ \frac{(1 - q^{p-1}y^{-(k+2)}z^{(k+2)}) (1 - q^p y^{(k+2)}z^{-(k+2)})}{(1 - q^{p-1}y^{-1}z) (1 - q^p y z^{-1})} \frac{(1 - q^{p-1}y^{k+3}z^{(k+1)}) (1 - q^p y^{-(k+3)}z^{-(k+1)})}{(1 - q^{p-1}z^2) (1 - q^p z^{-2})} \right\} .
\]

One can then extract the refined elliptic genus from:

\[
F_s(q, y, z) = F(q, y, z) + F(q, y^{-1}, z) = F(q, y, z) + F(q, y, z^{-1}) .
\]
Alternatively, and perhaps more usefully, one can use (3.11) to generate the characters of the first factor, $M_1$, in (2.16). That is, if one expands $F_s(q,y,z)$ and collects the coefficient of $y^a z^b$, then this will be $(\eta(q))^2$ times the character of a representation, $R$, of the model $SU_k(3)/SU_2(2) \times U(1)$. This representation, $R$, is the one that is paired in the $N = 2$ superconformal model with a state that has $N = 2$, $U(1)$ charge $b \frac{k + 3}{k + 3}$ and minimal model momentum $\frac{a}{\sqrt{2(k+2)(k+3)}}$. We have used Mathematica to verify this explicitly for $k = 1, 2$ and powers up to $q^4 y^{10} z^{10}$.

Thus one can extract the characters of the component models directly and easily from the elliptic genus.

One can also play various other games. For example, one can extract the complete $N = 2$ super-$W_3$ character above a single Ramond ground state. Recall that the Ramond ground states of (2.11) are in one-to-one correspondence with the $SU_k(3)$ highest weight labels $[5,27]$. One can obtain a Ramond ground state from a highest weight state, $\Lambda$, of $SU_k(3)$ by tensoring it with the $SO(4)$ spinor ground state of maximum charge, and then factoring out the $SU_{k+1}(2) \times U(1)$ highest weight state that is simply the projection onto $SU(2) \times U(1)$ of the sum of the $SU(3)$ and $SO(4)$ weights. Thus, if $\Lambda$ has Dynkin labels $(n_1, n_2)$, then the $SU(2)$ Dynkin label is $n_1$, and the $U(1)$ charge is $n_1 + n_2 + 3$ (the shift of +3 comes from the $SO(4)$ spinor contribution). Therefore, since the elliptic genus consists of purely Ramond ground state characters, we may isolate a particular such ground state by fixing the representation of $SU_{k+1}(2) \times U(1)$. Observe that in the decomposition (2.16), the factor $SU_{k+1}(2)$ appears in the denominator of the minimal model. Moreover, in the standard bosonic formulation, the minimal model momentum is:

$$ p = \sqrt{\frac{k+2}{2(k+3)}} m_2 - \sqrt{\frac{k+3}{2(k+2)}} m_1 + \sqrt{2(k+2)(k+3)} m , \quad (3.12) $$

with $m \in \mathbb{Z}$, $m_1 = 1, 2, \ldots, (k+1)$ and $m_2 = 1, 2, \ldots, (k+2)$. Such a bosonic momentum state contributes to the minimal model character of the $\Phi_{m_1, m_2}$ representation. To fix the label of $SU_{k+1}(2)$ we need to fix $m_2$, but to obtain the complete character, we must at the same time sum over all allowed values of $m_1$. Take $\nu = 2\pi i \sqrt{\frac{k+2}{k+3}} j$ and observe that with this choice one has $e^{i \nu j} = e^{i\sqrt{2}\nu} = e^{-2\pi i j m_2/(k+3)}$. Summing over $j$ will then project the elliptic genus onto a specific $SU_{k+1}(2)$ state. Therefore, the function

$$ E_\ell(q, \gamma) = \frac{1}{k + 3} \sum_{j=0}^{k+2} e^{2\pi i j (\ell+1)/(k+3)} F_s \left( q, \gamma, \nu = 2\pi i j \sqrt{\frac{k+2}{k+3}} \right) \quad (3.13) $$

provides the projection onto the Ramond ground states with $SU(2)$ Dynkin label equal to $\ell$. The $U(1)$ quantum number can also be fixed, exactly as was done in [2], by performing a similar finite sum over values of the parameter $\gamma$. 

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4. Generalizations

It is relatively simple to conjecture about the generalization of results to models of the form \( (2.11) \) for \( n \geq 3 \). The first step is to seek out the free bosonic realization of \( \mathcal{M}_2 \) within the \( N = 2 \) superfields. For \( \ell = 1, \ldots, n - 1 \), define:

\[
\hat{j}_\ell(z) = \frac{1}{\sqrt{1 - \omega}} \left[ \omega \phi_\ell \bar{\phi}_\ell - i(1 - \omega) \bar{\psi}_\ell, -(z) \psi_\ell, (z) + i\bar{\psi}_{\ell+1}, -(z) \psi_{\ell+1}, (z) \right],
\]

where \( \omega = 1/(k + n + 1) \). Observe that these currents are orthogonal to \( J(z) \), and satisfy

\[
\hat{j}_\ell(z) \hat{j}_m(w) = \frac{A_{\ell m}}{(z - w)^2} + \ldots,
\]

where \( A_{\ell m} \) is the Cartan matrix of \( SU(n) \). We may therefore set \( \hat{j}_\ell(z) = \tilde{\alpha}_\ell \cdot \partial \vec{X}(z) \), where the \( \tilde{\alpha}_\ell \) are the roots of \( SU(n) \) and \( \vec{X}(z) \) is a vector of \( n \) free bosons. We have not yet proved, but have a compelling body of evidence that these bosons generate, inside the \( N = 2 \) Hilbert space, the standard free bosonic realization of the \( W_n \) minimal model, \( \mathcal{M}_2 \) in \( (2.16) \). We also have confirmation of this from the corresponding refinements of the elliptic genus.

Define the character

\[
F(q, \nu, \gamma) = Tr_{\mathcal{H}} \left( (-1)^F q^{H_L} \bar{q}^{H_R} \exp \left( i \sum_\ell \nu_\ell \hat{j}_\ell, 0 \right) \exp(i\gamma J_0) \right),
\]

and symmetrize it with respect to the Weyl group of \( SU(n) \):

\[
F_s(q, \nu, \gamma) = \sum_{w \in W(SU(n))} F(q, w(\nu), \gamma).
\]

As above, we claim that this can be computed in the Landau-Ginzburg model in the limit where the coefficient of the potential vanishes. Therefore, we have

\[
F(q, \nu, \gamma) = \prod_{j=1}^n \frac{\vartheta_1(a_j|\tau)}{\vartheta_1(b_j|\tau)},
\]

where:

\[
a_j = (1 - \omega)\nu_j - \nu_{j-1} + (1 - j\omega)\gamma
\]

\[
b_j = -\omega\nu_j - j\omega\gamma.
\]

---

\(^3\) Remember that we have the somewhat unusual conventions: \( \phi_j(x) \bar{\phi}_j(0) \sim -\ln(x^m x_m) \), \( \psi_{j,-}(x) \bar{\psi}_{j,-}(0) \sim -i\frac{1}{(x^m x_m)} \).
and \( \nu_0 \equiv \nu_n \equiv 0 \).

Using *Mathematica*, we have checked the expansion of \( F_s \) explicitly for \( n = 3, 4; k = 1, 2 \) and find that it does indeed produce the proper generalization of the results of the previous section. It is also amusing to note that for \( n \geq 3 \) it is far from obvious that \( F_s \) is non-singular as \( \gamma \to 0 \). For \( n = 3 \) one can write the six terms in \( F_s \) over a common denominator, and the numerator becomes sums of products of four theta functions. The numerator vanishes in the limit \( \gamma \to 0 \) by virtue of the vanishing of a particular sum of three terms each consisting of a product of four theta functions. It is this same identity that is of particular importance in establishing that elliptic Boltzmann weights satisfy the Yang-Baxter equations.

5. Conclusion

In this letter we have refined the elliptic genus for \( \mathcal{N} = 2 \) superconformal models by including new charges arising from the super-\( W \) algebra of the Landau-Ginzburg models. We have argued that we could find these new generators only when the superpotential has a very specific form and satisfies additional second order differential equations such as (2.13).

The refinement of the elliptic genus enabled us to isolate characters of various component parts of the \( \mathcal{N} = 2 \) superconformal Hilbert space, and also isolate individual characters in the Ramond sector.

In writing this letter we have suppressed many of the technical details. A very useful guide in our computations has been the translation table between the Landau-Ginzburg fields of \([42]\) (which is essentially the same as \( \beta, \gamma \)-system of \([28,10]\)) and the fields that naturally appear in Drinfeld-Sokolov reduction. Questions about the operator equations of motion in the Landau-Ginzburg formulation can then be converted into fairly standard questions about commutations with screening charges. It was also detailed knowledge of the relationship between the Landau-Ginzburg fields and the superfields of Drinfeld-Sokolov reduction that led us to make the Ansatz for \( \hat{J} \) instead of working with a much more complicated Ansatz for \( \mathcal{W}_3 \).

We have also suppressed a rather interesting technical point about the embedding of the bosonic formulation of minimal models into the \( \mathcal{N} = 2 \) superfield Hilbert space. The minimal model screening charges are slightly non-standard, and this directly linked with the fact the decomposition (2.16) is not just a simple tensor product, but there is a “locking
together” of representations of $\mathcal{M}_1$ and $\mathcal{M}_2$ so as to make a non-trivial modular-invariant. All of the foregoing issues will be discussed in detail [19].

Our purpose in this letter has thus been to distill the essential ideas and some key results of our work and defer the technical details, and some of the subtleties, to a future publication [19].

**Note added:**

While working on this manuscript, we were advised that W. Lerche and A. Sevrin had also derived results related to ours about the connection between super-$W$ algebras and the form of the superpotential.
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