ON THE ALGEBRAIC FUNDAMENTAL GROUPS

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Abstract. Passing from arithmetic schemes to algebraic schemes, in a similar manner we will have the computation of the étale fundamental group of an algebraic scheme and then will define and discuss the qc fundamental group of an algebraic scheme in this paper. The qc fundamental group will also give a prior estimate of the étale fundamental group.

Contents

Introduction
1. Preliminaries
2. The Étale Fundamental Group
3. The qc Fundamental Group
References

INTRODUCTION

As an arithmetic scheme behaves frequently like the ring of algebraic integers for a number field, it has been seen that there exist many tricks arising from algebraic number theory which still work for the case of an arithmetic scheme; hence, we have obtained some properties of the profinite fundamental groups of arithmetic schemes (for example, see [2, 3, 4, 5, 6]).

Passing from arithmetic schemes to algebraic schemes, in this paper we will have the computation of the étale fundamental group of an algebraic scheme; then we will define and discuss the qc fundamental group of an algebraic scheme which will also give a prior estimate of the étale fundamental group. These results are related to the Section Conjecture of Grothendieck (see [10]).

Convention. By an algebraic variety we will understand an integral scheme $X$ over a field $k$ of finite type in the paper. In such a case, $X$ is also said to be an algebraic $k$-variety. Here, the field $k$ can have an arbitrary characteristic.

2000 Mathematics Subject Classification. Primary 14F35; Secondary 11G35.

Key words and phrases. arithmetic scheme, automorphism group, étale fundamental group, quasi-galois, unramified extension.
Acknowledgment. The author would like to express his sincere gratitude to Professor Li Banghe for his advice and instructions on algebraic geometry and topology.

1. Preliminaries

For convenience, let us fix notation and definitions in this subsection.

1.1. Notation. Fixed an integral domain $D$. In the paper, we let $Fr(D)$ denote the field of fractions on $D$.

If $D$ be a subring of a field $\Omega$, the field $Fr(D)$ will always assumed to be contained in $\Omega$.

Let $E$ be an extension of a field $F$ (not necessarily algebraic). $E$ is said to be Galois over $F$ if $F$ is the fixed subfield of the Galois group $Gal(E/F)$.

1.2. Affine Covering with Values. Fixed a scheme $X$. As usual, an affine covering of the scheme $X$ is a family

$$C_X = \{(U_\alpha, \phi_\alpha; A_\alpha)\}_{\alpha \in \Delta}$$

such that for each $\alpha \in \Delta$, $\phi_\alpha$ is an isomorphism from an open set $U_\alpha$ of $X$ onto the spectrum $Spec A_\alpha$ of a commutative ring $A_\alpha$. Each $(U_\alpha, \phi_\alpha; A_\alpha) \in C_X$ is called a local chart.

An affine covering $C_X$ of $X$ is said to be reduced if $U_\alpha \neq U_\beta$ holds for any $\alpha \neq \beta$ in $\Delta$.

Let $\mathsf{Comm}$ be the category of commutative rings with identity. Fixed a subcategory $\mathsf{Comm}_0$ of $\mathsf{Comm}$. An affine covering $\{(U_\alpha, \phi_\alpha; A_\alpha)\}_{\alpha \in \Delta}$ of $X$ is said to be with values in $\mathsf{Comm}_0$ if for each $\alpha \in \Delta$ there are $\mathcal{O}_X(U_\alpha) = A_\alpha$ and $U_\alpha = Spec(A_\alpha)$, where $A_\alpha$ is a ring contained in $\mathsf{Comm}_0$.

In particular, let $\Omega$ be a field and let $\mathsf{Comm}(\Omega)$ be the category consisting of the subrings of $\Omega$ and their isomorphisms. An affine covering $C_X$ of $X$ with values in $\mathsf{Comm}(\Omega)$ is said to be with values in the field $\Omega$.

Assume that $\mathcal{O}_X$ and $\mathcal{O}_X'$ are two structure sheaves on the underlying space of an integral scheme $X$. The two integral schemes $(X, \mathcal{O}_X)$ and $(X, \mathcal{O}_X')$ are said to be essentially equal provided that for any open set $U$ in $X$, we have

$$U \text{ is affine open in } (X, \mathcal{O}_X) \iff \text{ so is } U \text{ in } (X, \mathcal{O}_X')$$
and in such a case, \( D_1 = D_2 \) holds or there is \( Fr(D_1) = Fr(D_2) \) such that for any nonzero \( x \in Fr(D_1) \), either
\[
x \in D_1 \bigcap D_2
\]
or
\[
x \in D_1 \setminus D_2 \iff x^{-1} \in D_2 \setminus D_1
\]
holds, where \( D_1 = \mathcal{O}_X(U) \) and \( D_2 = \mathcal{O}_X'(U) \).

Two schemes \((X, \mathcal{O}_X)\) and \((Z, \mathcal{O}_Z)\) are said to be essentially equal if the underlying spaces of \( X \) and \( Z \) are equal and the schemes \((X, \mathcal{O}_X)\) and \((X, \mathcal{O}_Z)\) are essentially equal.

1.3. Quasi-Galois Closed. Fixed a field \( k \). Let \( X \) and \( Y \) be algebraic \( k \)-varieties and let \( f : X \rightarrow Y \) be a surjective morphism of finite type. Denote by \( Aut(X/Y) \) the group of automorphisms of \( X \) over \( Z \).

By a conjugate \( Z \) of \( X \) over \( Y \), we understand an algebraic \( k \)-variety \( Z \) that is isomorphic to \( X \) over \( Y \).

Definition 1.1. \( X \) is said to be quasi-galois closed over \( Y \) by \( f \) if there is an algebraically closed field \( \Omega \) and a reduced affine covering \( \mathcal{C}_X \) of \( X \) with values in \( \Omega \) such that for any conjugate \( Z \) of \( X \) over \( Y \) the two conditions are satisfied:
- \((X, \mathcal{O}_X)\) and \((Z, \mathcal{O}_Z)\) are essentially equal if \( Z \) has a reduced affine covering with values in \( \Omega \).
- \( \mathcal{C}_Z \subseteq \mathcal{C}_X \) holds if \( \mathcal{C}_Z \) is a reduced affine covering of \( Z \) with values in \( \Omega \).

Remark 1.2. We can prove the existence and the main property for an algebraic \( k \)-variety \( Z \) that is isomorphic to \( X \) over \( Y \).

2. The \( \acute{\text{E}} \text{tale Fundamental Group} \)

2.1. Definitions. Fixed an algebraic variety \( X \) over a field \( k \). Let \( L_1 \) and \( L_2 \) be two algebraic extensions over \( k(X) \), respectively.

Definition 2.1. \( L_2 \) is said to be a finite \( X \)-formally unramified Galois extension over \( L_1 \) if there are two algebraic \( k \)-varieties \( X_1 \) and \( X_2 \) and a surjective morphism \( f : X_2 \rightarrow X_1 \) such that
- \( \text{Sp}[X] = \text{Sp}[X_1] = \text{Sp}[X_2] \), i.e., \( X, X_1 \), and \( X_2 \) have a same \( sp \)-completion.
- \( k(X_1) = L_1, k(X_2) = L_2 \);
- \( X_2 \) is a finite \( \acute{\text{E}} \text{tale} \) Galois cover of \( X_1 \) by \( f \).

In such a case, \( X_2/X_1 \) are said to be a \( X \)-geometric model of the field extension \( L_2/L_1 \).
For \( L = k(X) \), set
- \( L^{al} \triangleq \) an algebraical closure of \( L \);
- \( L^{sep} \triangleq \) the separable closure of \( L \) contained in \( L^{al} \);
- \( L^{au} \triangleq \) the union of all the finite \( X \)-formally unramified subextensions over \( L \) contained in \( L^{al} \).

**Remark 2.2.** Let \( L \) be a finitely generated extension over a number field \( K \). Then \( L^{au} \) is a subfield of \( L^{al} \). In particular, it is seen that \( L^{au} \) is a Galois extension over \( L \).

Moreover, let \( \omega \in L^{sep} \) be unramified over \( L \). Then \( f(\omega) \in L^{sep} \) is also unramified over \( L \) for any element \( f \) of the absolute Galois group \( Gal(L^{al}/L) \).

Hence, by set inclusion, \( L^{au} \) is (equal to and then defined to be) the **maximal unramified subextensions** over \( L \) (contained in \( L^{al} \)).

2.2. **The Etale Fundamental Group.** By a trick similar to [4, 6], we have the following result.

**Theorem 2.3.** Fixed any algebraic \( k \)-variety \( X \). Then there exists an isomorphism
\[
\pi_{1}^{et}(X, s) \cong Gal\left(k(X)^{au}/k(X)\right)
\]
between groups for any geometric point \( s \) of \( X \) over the separable closure of the function field \( k(X) \).

3. **The qc Fundamental Group**

3.1. **Definitions.** Let \( X \) be an algebraic \( k \)-variety. Let \( \Omega \) be a separably closed field containing the function field \( k(X) \). Here, \( \Omega \) is not necessarily algebraic over \( k(X) \).

Define \( X_{qc}[\Omega] \) to be the set of algebraic \( k \)-varieties \( Z \) satisfying the following conditions:
- \( Z \) has a reduced affine covering with values in \( \Omega \);
- there is a surjective morphism \( f : Z \to X \) of finite type such that \( Z \) is quasi-galois closed over \( X \).

Set a partial order \( \leq \) in the set \( X_{qc}[\Omega] \) in such a manner:
Take any \( Z_{1}, Z_{2} \in X_{qc}[\Omega] \), we say
\[
Z_{1} \leq Z_{2}
\]
if there is a surjective morphism \( \varphi : Z_{2} \to Z_{1} \) of finite type such that \( Z_{2} \) is quasi-galois closed over \( Z_{1} \).

It is seen that \( X_{qc}[\Omega] \) is a directed set and
\[
\{Aut(Z/X) : Z \in X_{qc}[\Omega]\}
\]
is an inverse system of groups. Hence, we have the following definition.
Definition 3.1. Let $X$ be an algebraic $k$—variety. Take any separably closed field $\Omega$ containing $k(X)$. The inverse limit
\[ \pi_1^{qc}(X; \Omega) \triangleq \lim_{\leftarrow Z \in X_{qc}[\Omega]} Aut(Z/X) \]
of the inverse system $\{Aut(Z/X) : Z \in X_{qc}[\Omega]\}$ of groups is said to be the \textbf{qc fundamental group} of the scheme $X$ with coefficient in $\Omega$.

3.2. The qc Fundamental Group. By a trick similar to [5], we have the following results.

Theorem 3.2. Let $X$ be an algebraic $k$—variety. Take any separably closed field $\Omega$ containing $k(X)$. There are the following statements.

(i) There is a group isomorphism
\[ \pi_1^{qc}(X; \Omega) \cong Gal(\Omega/k(X)) \]

(ii) Take any geometric point $s$ of $X$ over $\Omega$. Then there is a group isomorphism
\[ \pi_1^{et}(X; s) \cong \pi_1^{qc}(X; \Omega)_{et} \]
where $\pi_1^{qc}(X; \Omega)_{et}$ is a subgroup of $\pi_1^{qc}(X; \Omega)$. Moreover, $\pi_1^{qc}(X; \Omega)_{et}$ is a normal subgroup of $\pi_1^{qc}(X; \Omega)$.

Remark 3.3. Let $X$ be an algebraic $k$—variety. Put
\[ \pi_1^{qc}(X) = \pi_1^{qc}(X; k(X)^{sep}) \]
Then there is a group isomorphism
\[ \pi_1^{qc}(X) \cong Gal(k(X)^{sep}/k(X)) \]

Definition 3.4. Let $X$ be an algebraic $k$—variety. The quotient group
\[ \pi_1^{br}(X) = \pi_1^{qc}(X; k(X)^{sep}) / \pi_1^{qc}(X; k(X)^{sep})_{et} \]
is said to be the \textbf{branched group} of the algebraic variety $X$.

The branched group $\pi_1^{br}(X)$ can reflect the topological properties of the scheme $X$, especially the properties of the associated complex space $X^{an}$ of $X$, for example, the branched covers of $X^{an}$.

Remark 3.5. Let $X$ be an algebraic $k$—variety. Then we have
\[ \pi_1^{br}(X) = \{0\} \]
if and only if $X$ has no finite branched cover.
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