A Faster Approximation Algorithm for the Gibbs Partition Function

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Abstract

We consider the problem of estimating the partition function $Z(\beta) = \sum_x \exp(-\beta H(x))$ of a Gibbs distribution with a Hamilton $H(\cdot)$, or more precisely the logarithm of the ratio $q = \ln Z(0)/Z(\beta)$. It has been recently shown how to approximate $q$ with high probability assuming the existence of an oracle that produces samples from the Gibbs distribution for a given parameter value in $[0, \beta]$. The current best known approach due to Huber [9] uses $O(q \ln n \cdot [\ln q + \ln \ln n + \varepsilon^{-2}])$ oracle calls on average where $\varepsilon$ is the desired accuracy of approximation and $H(\cdot)$ is assumed to lie in $\{0\} \cup [1, n]$. We improve the complexity to $O(q \ln n \cdot \varepsilon^{-2})$ oracle calls. By a standard argument, the same complexity can be achieved if exact oracles are replaced with approximate sampling oracles that are within $O(\varepsilon^2 q \ln n)$ variation distance from exact oracles.

1 Introduction

It is known that for large classes of problems, e.g. self-reducible problems [14], there is an intimate connection between approximate counting and sampling: the ability to solve one problem allows solving the other one. This paper explores this connection for Gibbs distributions.

Let $\Omega$ be some finite set and $H(\cdot)$ be some real-valued function on $\Omega$ called a Hamiltonian. The Gibbs distribution for such a system is a family of distributions $\{\mu_\beta\}$ on $\Omega$ parameterized by $\beta$, where

$$\mu_\beta(x) = \frac{1}{Z(\beta)} \exp(-\beta H(x)) \quad \forall x \in \Omega$$

(1)

The normalizing constant $Z(\beta)$ is called the partition function:

$$Z(\beta) = \sum_{x \in \Omega} \exp(-\beta H(x))$$

(2)

Estimating this function for a given value of $\beta$ is a widely studied computational problem with applications in many areas. In particular, it is a key computational task in statistical physics. Evaluations of $Z(\cdot)$ yield estimates of important thermodynamical quantities, such as the free energy. Note, parameter $\beta$ corresponds to the inverse temperature. A classical example of a Gibbs distribution in physics is the Ising model.

Example 1. Given an undirected graph $(V, E)$, let $\Omega = \{-1, +1\}^V$ and $H(x) = \sum_{\{i, j\} \in E} [x_i \neq x_j]$ where $[\cdot]$ is 1 if its argument is true, and 0 otherwise. Distribution (1) for such a Hamiltonian is called the Ising model. It is ferromagnetic if $\beta > 0$, and antiferromagnetic if $\beta < 0$ (although in the latter case the function $H'(x) = -H(x)$ is usually treated as the Hamiltonian). Computing $Z(\beta)$ exactly is a $\#P$-complete problem, and is even hard to approximate in the antiferromagnetic case [13].

The problem of counting various combinatorial objects such as proper $k$-coloring and matchings in graphs can also be naturally phrased as estimating the partition function.

Example 2. Let $\Omega = \{1, \ldots, k\}^{|V|}$ be the set of all colorings in an undirected graph $G = (V, E)$. Define $H(x) = \sum_{\{i, j\} \in E} [x_i = x_j]$, then $Z(+\infty)$ gives the number of proper $k$-colorings.

Example 3. Let $\Omega$ be the set of matchings $M \subseteq E$ in an undirected graph $G = (V, E)$. Define $H(M) = |M|$, then $Z(0) = |\Omega|$. 
A related problem is that of sampling from the distribution $\mu_\beta$ for a given value of $\beta$. There is a vast literature on designing sampling algorithms from Gibbs distributions, see e.g. [17 19 8 6] or [1] for an overview. For the ferromagnetic Ising model there exists a polynomial-time approximate sampling algorithm [13] and also an exact sampling algorithm that appears to run efficiently at or above the critical temperature [18]. Approximate sampling of $k$-colorings in low-degree graphs is addressed in [12 21] for $\beta = +\infty$, though techniques are potentially extendable to other values of $\beta$, and for matchings polynomial-time approximate sampling is described in [16 Section 2.3.5].

It is known that the ability to sample can be used for designing a randomized approximation scheme for estimating the partition function. By definition, it is an algorithm that for a given $\varepsilon > 0$ produces an estimate $\hat{Q}$ of the desired quantity $Q$ such that $\hat{Q} \in \left[\frac{Q}{1+\varepsilon}, Q(1+\varepsilon)\right]$ with probability at least $3/4$. (The value $3/4$ is arbitrary: by repeating the algorithm multiple times and taking the median of the outputs the probability can be boosted to any other constant in $[0, 1)$.) This paper studies the following question: how many samples are needed to approximate $Z(\beta)$ with a given accuracy $\varepsilon$?

**Formal description** To state the complexity of different approaches, we need to introduce several quantities. First, we assume that $H(x) \in \{0\} \cup [1, n]$ for any $x \in \Omega$ where $n$ is a known number. Non-negativity of the Hamiltonian implies that $Z(\cdot)$ is a decreasing function. Our goal will be to estimate the ratio $Q = Z(\beta_{\min})/Z(\beta_{\max})$ for given values $\beta_{\min} < \beta_{\max}$. Note that computing $Z(\beta)$ for some specific value of $\beta$ is usually an easy task, so this will allow estimating $Z(\beta)$ for any other $\beta$. In particular, in Examples 1, 2 and 3 we have $Z(0) = 2^{|V|}$, $Z(0) = k^{|V|}$ and $Z(+\infty) = 1$ respectively.

Let us denote $q = \log Q$, and assume that there exists an oracle that can produce a sample $X \sim \mu_\beta$ for a given value $\beta \in [\beta_{\min}, \beta_{\max}]$. When stating asymptotic complexities, we will always assume that $q = \Omega(1)$, $n = 1 + \Omega(1)$ and $\varepsilon = O(1)$ to simplify the expressions. Bezáková et al. [2] showed that $Q$ can be estimated using $O(q^2(\ln n)^2)$ oracle calls in the worst case (for a fixed $\varepsilon$). This was improved to $O(q(\ln q + \ln n)^5\varepsilon^{-2})$ expected number of calls by Štefankovič et al. [22] and then to $O(q \ln n \cdot (\ln q + \ln n + \varepsilon^{-2}))$ by Huber [9].

The contribution of this paper is to improve the complexity further to $O(q \ln n \cdot \varepsilon^{-2})$ oracle calls (on average). This is achieved by a better analysis of the algorithm in [9]. The formal statement of our result is given in Section 3 as Theorems 5 and 7.

In many applications we only have an access to approximate sampling oracles. Using a standard coupling argument, in Section 3.1 we show that the same complexity can be achieved with approximate oracles assuming that they are within $O(\frac{\varepsilon^2}{q_{\min}})$ variation distance from exact oracles.

**Remark 1.** The assumption that $H(\cdot)$ lies in $\{0\} \cup [1, n]$ can be relaxed using a standard trick. Suppose, for example, that $H(x) \in \{h_{\min}, h_{\min}+1, \ldots, h_{\max}\}$ where $h_{\min}$ and $h_{\max}$ are known integers. Let $n = h_{\max} - h_{\min}$. We claim that the problem can be solved using $O(q(\ln n \cdot \varepsilon^{-2}))$ oracle calls (on average), where either (i) $q' = q - (\beta_{\max} - \beta_{\min}) \cdot h_{\min}$, or (ii) $q' = -q + (\beta_{\max} - \beta_{\min}) \cdot h_{\max}$.

Indeed, to achieve the first complexity, define new Hamiltonian $H'(x) = H(x) - h_{\min}$. The partition function for the new problem is $Z'(\beta) = e^{\beta h_{\min}} \cdot Z(\beta)$, and so $q'$ is as defined in (i). (We use “primes” to denote all quantities related to the new problem). We have $H'(x) \in \{0, 1, \ldots, n\}$, so the algorithm claimed above can be applied to give an estimate of $q'$ and thus of $q$. Note that distributions $\mu'_{\beta}$ and $\mu_{\beta}$ coincide, and so sampling from $\mu_{\beta}$ allows to sample from $\mu'_{\beta}$.

To achieve the second complexity, define $H'(x) = -H(x) + h_{\max}$ and also change the bounds: $\beta'_{\min} = -\beta_{\max}$ and $\beta'_{\max} = -\beta_{\min}$. There holds $Z'(\beta) = e^{-\beta h_{\max}} \cdot Z(-\beta)$, and $q'$ is as defined in (ii). We again have $H'(x) \in \{0, 1, \ldots, n\}$, and distributions $\mu'_{\beta}$ and $\mu_{-\beta}$ coincide. We can now use the same argument as before.

## 2 Background and preliminaries

We will assume for simplicity that $H(\cdot) \neq const$. Let us denote $z(\beta) = \ln Z(\beta)$. It can be easily checked that

$$z'(\beta) = \mathbb{E}_{X \sim \mu_{\beta}}[-H(X)]$$
Since $H(\cdot)$ is non-negative and non-constant, we have $z'(\beta) < 0$ for any $\beta$ and thus $z(\cdot)$ and $Z(\cdot)$ are strictly decreasing functions. It is also known [24, Proposition 3.1] that function $z(\cdot)$ is convex for any $H(\cdot)$, and in fact strictly convex if $H(\cdot) \neq \text{const}$.

Next, we discuss previous approaches for estimating $Z(\beta_{\min})/Z(\beta_{\max})$, closely following [9].

It is well-known that for given values $\beta_1, \beta_2$ an unbiased estimator $W$ of $Z(\beta_2)/Z(\beta_1)$ can be obtained as follows: first sample $X \sim \mu_{\beta_1}$ and then set $W = \exp((\beta_1 - \beta_2)H(X))$. Indeed,

$$
\mathbb{E}[W] = \sum_{x \in \Omega} \frac{\exp(-\beta_1 H(x))}{Z(\beta_1)} \cdot \frac{\exp((\beta_1 - \beta_2)H(x))}{Z(\beta_1)} = \frac{Z(\beta_2)}{Z(\beta_1)}
$$

Applying this estimator directly to $(\beta_1, \beta_2) = (\beta_{\min}, \beta_{\max})$ or to $(\beta_1, \beta_2) = (\beta_{\max}, \beta_{\min})$ is problematic since it usually has a huge relative variance. A standard approach to reduce the relative variance is via the multistage sampling method of Valleau and Card [20]. First, a sequence $\beta_{\min} = \beta_0 \leq \beta_1 \leq \ldots \leq \beta_\ell = \beta_{\max}$ is selected; it is called a cooling schedule. We then have

$$
\frac{Z(\beta_{\min})}{Z(\beta_{\max})} = \frac{Z(\beta_0)}{Z(\beta_1)} \cdot \frac{Z(\beta_1)}{Z(\beta_2)} \cdot \ldots \cdot \frac{Z(\beta_{\ell-1})}{Z(\beta_\ell)}
$$

Throughout the paper we refer to $[\beta_i, \beta_{i+1}]$ as “interval $i$”, where $i \in \{0, 1, \ldots, \ell - 1\}$. The ratio $Z(\beta_i)/Z(\beta_{i+1})$ for each such interval can be estimated independently as described above, and then multiplied to give the final estimate. Fishman calls an estimate of this form a product estimator [7]. Its mean and variance are given by the lemma below. In this lemma we use the following notation: if $X$ is a random variable then $S[X] \triangleq \frac{\mathbb{E}[X^2]}{(\mathbb{E}[X])^2} = \frac{\text{Var}(X)}{(\mathbb{E}[X])^2} + 1$ (the relative variance of $X$ plus 1).

**Lemma 1** ([5, page 136]). For $P = \prod_i P_i$ where the $P_i$ are independent,

$$
\mathbb{E}[P] = \prod_i \mathbb{E}[P_i], \quad S[P] = \prod_i S[P_i]
$$

Using a fixed cooling schedule, Bezáková et al. [2] obtained an approximation algorithm that needs $O(q^2(\ln n)^2)$ samples in the worst case (for a fixed $\varepsilon$). Štefankovič et al. [22] asymptotically improved this to $10^6q(\ln q + \ln n)^5\varepsilon^{-2}$ samples on average. They used an adaptive cooling schedule where the values $\beta_i$ depend on the outputs of sampling oracles. A further improvement to $O(q \ln n \cdot \ln q + \ln n + \varepsilon^{-2})$ was given by Huber [9]. One of the key ideas in [9] was to replace the product estimator with the paired product estimator, which is described next.

### 2.1 Paired product estimator

One run of this estimator can be described as follows:

- sample $X_i \sim \mu_{\beta_i}$ for each $i \in [0, \ell]$
- for each interval $i \in [0, \ell - 1]$ compute
  $$
  W_i = \exp\left(-\frac{\beta_{i+1} - \beta_i}{2} H(X_i)\right), \quad V_i = \exp\left(\frac{\beta_{i+1} - \beta_i}{2} H(X_{i+1})\right)
  $$
- compute $W = \prod_i W_i$ and $V = \prod_i V_i$.

An easy calculation (see [9]) shows that

$$
\mathbb{E}[W_i] = \frac{Z(\beta_{i+1})}{Z(\beta_i)}, \quad \mathbb{E}[V_i] = \frac{Z(\beta_{i+1})}{Z(\beta_i)}, \quad \mathbb{E}[V_i]/\mathbb{E}[W_i] = \frac{Z(\beta_i)}{Z(\beta_{i+1})}, \quad \mathbb{E}[V]/\mathbb{E}[W] = \frac{Z(\beta_{\min})}{Z(\beta_{\max})}
$$

where we denoted $\beta_{i+1} = \beta_i + \beta_{i+1}/2$. Also,

$$
S[W_i] = S[V_i] = \frac{Z(\beta_i)Z(\beta_{i+1})}{Z(\beta_{i+1})^2}, \quad S[W] = S[V] = \prod_i \frac{Z(\beta_i)Z(\beta_{i+1})}{Z(\beta_{i+1})^2}
$$

(3)
Although \( \mathbb{E}[V]/\mathbb{E}[W] = \frac{Z(\beta_{\text{min}})}{Z(\beta_{\text{max}})} = Q \), using \( V/W \) as the estimator of \( Q \) would be a poor choice since it is biased in general. Instead, \cite{9} uses the following procedure.

**Algorithm 1:** Paired product estimator. **Input:** schedule \((\beta_0, \ldots, \beta_\ell)\), integer \( r \geq 1 \).

1. compute \( r \) independent samples of \((W, V)\) as described above
2. take their sample averages \( \bar{W} \) and \( \bar{V} \) and output \( \hat{Q} = \bar{V}/\bar{W} \) as the estimator of \( Q \)

The argument from \cite{9} gives the following result.

**Lemma 2.** Suppose that

\[
S[W] = S[V] \leq 1 + \frac{1}{2} \gamma r \bar{\varepsilon}^2
\]

where \( \bar{\varepsilon} = 1 - (1 + \varepsilon)^{-1/2} = \frac{1}{2} \varepsilon + O(\varepsilon^2) \) and \( \gamma > 0 \). Then \( \mathbb{P}(\hat{Q}/Q \in (\frac{1}{1+\varepsilon}, 1+\varepsilon)) \geq 1 - \gamma \).

**Proof.** We have \( \mathbb{E}[\bar{W}] = \mathbb{E}[W] \) and \( \text{Var}(\bar{W}) = \frac{1}{r} \text{Var}(W) \), and so \( S[\bar{W}] = \frac{1}{r} (S[W] - 1) + 1 \). By Chebyshev’s inequality, \( \mathbb{P}(|\bar{W} - \mathbb{E}[\bar{W}]| \geq \bar{\varepsilon}) \leq (S[\bar{W}] - 1)/\bar{\varepsilon}^2 = \frac{1}{r} (S[W] - 1)/\bar{\varepsilon}^2 \leq \gamma/2 \). Similarly, \( \mathbb{P}(|\bar{V}/\mathbb{E}[\bar{V}] - 1| \geq \bar{\varepsilon}) \leq \gamma/2 \).

Denote \( S = \bar{W}/\mathbb{E}[W] \) and \( T = \bar{V}/\mathbb{E}[\bar{V}] \). The union bound gives \( \mathbb{P}(\max\{|S - 1|, |T - 1| \geq \bar{\varepsilon}\}) \leq \gamma \).

Observe that condition \( \max\{|S - 1|, |T - 1| \} < \bar{\varepsilon} \) implies \( \{S, T\} \subset (1 - \bar{\varepsilon}, 1 + \bar{\varepsilon}) \subset (1 - \frac{1}{1+\varepsilon}, 1 + \frac{1}{1+\varepsilon}) \) and thus \( \hat{Q} = \frac{T}{S} \in (\frac{1}{1+\varepsilon}, 1+\varepsilon) \). The claim follows.

Recall that \( S[W] = S[V] \) is a deterministic function of the schedule \((\beta_0, \ldots, \beta_\ell)\) (see eq. (3)). We say that the schedule is good (with respect to fixed constants \( r \) and \( \gamma \)) if the resulting quantity \( S[W] = S[V] \) satisfies (1). Huber presented in \cite{9} a randomized algorithm that produces a good schedule with probability at least 0.95 (with respect to \( r = O(\varepsilon^{-2}) \) and \( \gamma = 0.2 \)). By Lemma 2, the output \( \hat{Q} \) of the resulting algorithm lies in \((\frac{Q}{1+\varepsilon}, Q(1+\varepsilon))\) with probability at least 0.95·(1 − \( \gamma \)) > 0.75, as desired.

Huber’s algorithm for producing schedule \((\beta_0, \ldots, \beta_\ell)\) is reviewed in the next section. It makes \( O(q \ln n \cdot \ln q + \ln \ln n) \) calls to the sampling oracle (on average). Then in Section 3 we will describe how to reduce the number of oracle calls to \( O(q \ln n) \) while maintaining the desired guarantees.

### 2.2 TPA method

The algorithm of \cite{9} for producing a schedule is based on the TPA method of Huber and Schott \cite{10,11}. (The abbreviation stands for the “Tootsie Pop Algorithm”). Let us review the application of the method to the Gibbs distribution with a non-negative Hamiltonian \( H(\cdot) \).

Its key subroutine is procedure \( \text{TPAstep}(\beta) \) that for a given constant \( \beta \) produces a random variable in \([\beta, +\infty)\) as follows:

- **sample** \( X \sim \mu_{\beta} \), **draw** \( U \in [0, 1] \) **uniformly at random**, **return** \( \beta - \ln U/H(X) \) (or \( +\infty \) if \( H(X) = 0 \)).

The motivation for this sampling rule comes from the following fact (which we prove here for completeness).

**Lemma 3.** Consider random variable \( U = Z(\text{TPAstep}(\beta)) \). If \( H(\cdot) \) is strictly positive (implying that \( Z(+\infty) = 0 \)) then \( U \) has the uniform distribution on \([0, Z(\beta)]\). If \( H(x) = 0 \) for some \( x \in \Omega \) (implying that \( Z(+\infty) > 0 \)) then \( U \) has the same distribution as the following random variable \( U' \): sample \( U' \in [0, Z(\beta)] \) uniformly at random and set \( U' \leftarrow \max\{U', Z(+\infty)\} \).

**Proof.** It suffices to prove \( \mathbb{P}(\text{TPAstep}(\beta) \geq \alpha) = Z(\alpha)/Z(\beta) \) for any \( \alpha \in [\beta, +\infty) \). We have

\[
\text{TPAstep}(\beta) \geq \alpha = [\ln U/H(X) \leq \beta - \alpha] = [\ln U \leq (\beta - \alpha)H(X)] = [U \leq \exp((\beta - \alpha)H(X))]
\]
Therefore,
\[
\mathbb{P}(\text{TPAstep}(\beta) \geq \alpha) = \sum_{x \in \Omega} \mathbb{P}(\text{TPAstep}(\beta) \geq \alpha | X = x) \mathbb{P}(X = x)
\]
\[
= \sum_{x \in \Omega} \mathbb{P}(U \leq \exp((\beta - \alpha)H(x))) \cdot \frac{\exp(-\beta H(\beta))}{Z(\beta)}
\]
\[
= \sum_{x \in \Omega} \exp((\beta - \alpha)H(x)) \cdot \frac{\exp(-\beta H(\beta))}{Z(\beta)}
\]
\[
= \sum_{x \in \Omega} \frac{\exp(-\alpha H(x))}{Z(\beta)} = \frac{Z(\alpha)}{Z(\beta)}
\]

Roughly speaking, the TPA method counts how many steps are needed to get from $\beta_{\text{min}}$ to $\beta_{\text{max}}$.

**Algorithm 2:** One run of TPA. **Output:** a multiset $B$ of values in the interval $[\beta_{\text{min}}, \beta_{\text{max}}]$.

1 set $\beta_0 = \beta_{\text{min}}$, let $B$ be the empty multiset
2 for $i = 1$ to $+\infty$ do
3 sample $\beta_i = \text{TPAstep}(\beta_{i-1})$
4 if $\beta_i \in [\beta_{\text{min}}, \beta_{\text{max}}]$ then add $\beta_i$ to $B$, otherwise output $B$ and terminate

The output of Algorithm 2 will be denoted as TPA(1), and the union of $k$ independent runs of TPA(1) as TPA($k$). For a multiset $B$ we define multiset $z(B) \equiv \{z(\beta) | \beta \in B\}$ in a natural way. (Recall that $z(\cdot)$ is a continuous strictly decreasing function). It is known [10, 11] that $z(\text{TPA}(k))$ is a Poisson Point Process (PPP) on $[z(\beta_{\text{max}}), z(\beta_{\text{min}})]$ of rate $k$, starting from $z(\beta_{\text{min}})$ and going downwards. In other words, the random variable $z = z(\text{TPA}(k))$ is generated by the following process.

**Algorithm 3:** Equivalent process for generating $z(\text{TPA}(k))$.

1 set $z_0 = z(\beta_{\text{min}})$, let $z$ be the empty multiset
2 for $i = 1$ to $+\infty$ do
3 draw $\eta$ from the exponential distribution of rate $k$ (and with the mean $\frac{1}{k}$), set $z_i = z_{i-1} - \eta$
4 if $z_i \in [z(\beta_{\text{max}}), z(\beta_{\text{min}})]$ then add $z_i$ to $z$, otherwise output $z$ and terminate

One way to use the TPA method is to simply count the number of points in TPA($k$). Indeed, $|\text{TPA}(k)|$ is distributed according to the Poisson distribution with rate $k \cdot (z(\beta_{\text{min}}) - z(\beta_{\text{max}})) = k \cdot q$, so $\frac{1}{k} |\text{TPA}(k)|$ is an unbiased estimator of $q$. Unfortunately, obtaining a good estimate of $q$ with this approach requires a fairly large number of samples, namely $O(q^2)$ for a given accuracy and the probability of failure [10, 11]. A better application of TPA was proposed in [9], where the method was used for generating a schedule $(\beta_0, \ldots, \beta_\ell)$ as follows.

**Algorithm 4:** Generating a schedule $(\beta_0, \ldots, \beta_\ell)$. **Input:** integers $k, d \geq 1$.

1 sample $B \sim \text{TPA}(k)$
2 sort the values in $B$ and then keep every $d$th successive value
3 add values $\beta_{\text{min}}$ and $\beta_{\text{max}}$ and output the resulting sequence $(\beta_0, \ldots, \beta_\ell) = (\beta_{\text{min}}, \ldots, \beta_{\text{max}})$

Note that the resulting sequence $(z_1, \ldots, z_{\ell-1}) = (z(\beta_1), \ldots, z(\beta_{\ell-1}))$ can be described by a process in Algorithm 3 where $\eta$ is drawn as the sum of $d$ exponential distributions each of rate $k$; this is the gamma (Erlang) distribution with shape parameter $d$ and rate parameter $k$.

Huber showed in [9] that if $d = \Theta(\ln q + \ln \ln n)$ and $k = \Theta(d \ln n)$ (with appropriate constants) then Algorithm 4 produces a good schedule with high probability. Since $q$ is unknown in practice, [9] uses a two-stage procedure: first an estimate $\hat{q} = 2 \cdot \frac{\text{TPA}(5)}{5} + 1$ is computed, which is shown to be an upper bound on $q$ with probability at least 0.99. This estimate is then used for setting $d$ and $k$. 
In the next section we prove that the algorithm has desired guarantees for smaller parameter values, namely \( d = \Theta(1) \) and \( k = \Theta(\ln n) \). This allows to reduce the complexity of Algorithm 4 by a factor of \( \Theta(\ln q + \ln \ln n) \), and also eliminates the need for a two-stage procedure.

### 3 Our results

For technical reasons we will need to make the following assumption for line 2 of Algorithm 4 if \( \beta_1, \beta_2, \ldots \) is the sorted sequence of points in \( B \) then the index of the first point to be taken is sampled uniformly from \( \{1, \ldots, d\} \) (and after that the index is always incremented by \( d \)).

Denote \( m = \frac{k}{d} \) and \( z_i = z(\beta_i) \) for \( i \in [0, \ell] \). We treat \( m \) and \( d \) as being fixed, and \( k = md \) as their function. Also let \( \delta = \ln \mathbb{S}[W] = \ln \mathbb{S}[V] \). From (3) we get

\[
\delta = \sum_i \delta_i, \quad \delta_i = z(\beta_i) - 2z\left(\frac{\beta_i + \beta_{i+1}}{2}\right) + z(\beta_{i+1}).
\]  

Since \( z(\cdot) \) is convex, we have \( \delta_i \geq 0 \) for all \( i \).

**Case 1: \( H(x) \in [1, n] \) for all \( x \in \Omega \)** First, let us assume that \( H(\cdot) \) does not take value 0. In this case the proofs become somewhat simpler, and we will get slightly smaller constants.

Huber showed that for \( d = \Theta(\ln(q \ln n)) \) the schedule is well-balanced with probability \( \Theta(1) \), meaning that all intervals \( i \) satisfy \( z_i - z_{i+1} \leq \tau \cdot \frac{1}{m} \) for a constant \( \tau = \frac{1}{2} \). (Note that \( \mathbb{E}[z_i - z_{i+1}] \approx \frac{1}{m} \), ignoring boundary effects). It was then proved that a well-balanced schedule satisfies \( \delta \leq \frac{\tau}{2} \cdot \frac{\ln n}{m} \), leading to condition (4). We improve on this result as follows.

Choose a constant \( \tau > 0 \) (to be specified later), and say that interval \( i \) is large if \( z_i - z_{i+1} > \tau \cdot \frac{1}{m} \), and small otherwise. Let \( \delta^+ \) be the sum of \( \delta_i \) over large intervals and \( \delta^- \) be the sum of \( \delta_i \) over small intervals (so that \( \delta = \delta^+ + \delta^- \)). In Section 4.1 we prove the following fact. (Recall that \( \delta^+, \delta^- \) are deterministic functions of the schedule \( (\beta_0, \ldots, \beta_\ell) \)).

**Lemma 4.** There holds \( \delta^- \leq \frac{\tau}{2} \cdot \frac{\ln n}{m} \) and \( \mathbb{E}[\delta^+] \leq \frac{\Gamma(d+2, \tau d)}{2d \cdot d!} \cdot \frac{\ln n}{m} \) for the schedule \( (\beta_0, \ldots, \beta_\ell) \) produced by Algorithm 4 with parameters \( k = md \) and \( d \), where \( \Gamma(\cdot, \cdot) \) is the upper incomplete gamma function:

\[
\Gamma(a, b) = \int_b^{\infty} t^{a-1} e^{-t} dt \quad (\text{with } \Gamma(a, 0) = \Gamma(a) = (a-1)!) \]

Using Markov’s inequality, we can now conclude that for any \( \tau^+ > 0 \) we have

\[
\mathbb{P}(\delta^+ \geq \frac{\tau^+}{2} \cdot \frac{\ln n}{m}) \leq \frac{1}{\tau^+} \cdot \frac{\ln n}{m} \cdot \mathbb{E}[\delta^+] \leq \frac{\Gamma(d+2, \tau d)}{\tau^+ \cdot d!} \cdot \frac{\ln n}{m}.
\]

Thus, with probability at least \( 1 - \frac{\Gamma(d+2, \tau d)}{\tau^+ \cdot d!} \), Algorithm 4 produces a schedule satisfying \( \delta \leq \frac{\tau + \tau^+}{2} \cdot \frac{\ln n}{m} \).

Recall that we want Algorithm 4 to succeed with probability at least \( \rho = \frac{1}{2} \) to make the overall probability of success at least 0.75. (Here \( \gamma \) is the constant from Lemma 2). Let us define function \( \tau_\rho(d) \) as follows:

\[
\tau_\rho(d) = \min_{\tau \geq 0, \tau^+ > 0} \left\{ \tau + \tau^+ \mid \frac{\Gamma(d+2, \tau d)}{\tau^+ \cdot d!} \leq 1 - \rho \right\} = \min_{\tau \geq 0} \left[ \tau + \frac{\Gamma(d+2, \tau d)}{(1-\rho) \cdot d} \right].
\]

\(^1\)More precisely, this is what the argument of [9] would give assuming that \( H(\cdot) \) does not take value 0.
The table below shows some values of this function for $\gamma = 0.24$ and $\rho = \frac{0.75}{1-\gamma} = \frac{75}{60}$ (computed with the code of [3]).

| $d$   | 1   | 2   | 4   | 8   | 16  | 32  | 64  | 128 | 256 | 512 |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| upper bound on $\tau_r(d)$ | 9.903 | 6.052 | 4.000 | 2.860 | 2.197 | 1.794 | 1.539 | 1.372 | 1.260 | 1.184 |
| achieved with $\tau =$    | 8.645 | 5.384 | 3.634 | 2.653 | 2.075 | 1.720 | 1.492 | 1.342 | 1.241 | 1.170 |

We can now formulate our main result for case I.

**Theorem 5.** Let $\hat{Q}$ be the estimate given by Algorithm I (with parameter $r$) applied to the schedule produced by Algorithm A (with parameters $k = md$ and $d$). Suppose that

$$m \geq \frac{\tau_p(d) \cdot \ln n}{2 \ln \left(1 + \frac{2}{\gamma} \tau \hat{\varepsilon}^2\right)} \quad \text{for some } \gamma \in (0, 0.25) \text{ and } \rho = \frac{0.75}{1-\gamma}$$

(6)

where $\hat{\varepsilon} = 1 - (1 + \varepsilon)^{-1/2} = \frac{1}{2} \varepsilon + O(\varepsilon^2)$. Then $\hat{Q} \in (\frac{Q}{1+\varepsilon}, Q(1 + \varepsilon))$ with probability at least 0.75. The expected number of oracle calls that this algorithm makes is $mq(r + d) + 2r + 1$.

In particular, (6) will be satisfied for $d = 64$, $m \geq 3.6 \cdot \ln n$ and $r = \left[2\varepsilon^{-2}\right] = 8(1 + o(1))\varepsilon^{-2}$.

**Proof.** As we just showed,

$$\mathbb{P} \left( \delta \leq \frac{\tau_r(d)}{2} \cdot \frac{\ln n}{m} \right) \geq \rho$$

(7)

Condition $\delta \leq \frac{\tau_r(d)}{2} \cdot \frac{\ln n}{m}$ implies condition $\delta \leq \ln \left(1 + \frac{2}{\gamma} \tau \hat{\varepsilon}^2\right)$ (by (6)), which is in turn equivalent to $S[W] \leq 1 + \frac{2}{\gamma} \tau \hat{\varepsilon}^2$. Thus, from Lemma 2 we get

$$\mathbb{P} \left( \hat{Q} \in (\frac{Q}{1+\varepsilon}, Q(1 + \varepsilon)) \mid \delta \leq \frac{\tau_r(d)}{2} \cdot \frac{\ln n}{m} \right) \geq 1 - \gamma$$

(8)

Multiplying (7) and (8) gives the first claim.

A PPP of rate $k$ on an interval $[z(\beta_{\max}), z(\beta_{\min})]$ produces $k[z(\beta_{\min}) - z(\beta_{\max})] = mdq$ points on average. Thus, Algorithm 4 makes $mdq + 1$ oracle calls on average and produces a sequence $(\beta_0, \ldots, \beta_{\ell})$ with $\mathbb{E}[\ell] = mq + 1$. Algorithm I then makes $(\ell + 1) r$ oracle calls, i.e. $(mq + 2) r$ calls on average. This gives the second claim. \hfill $\square$

**Case II:** $H(x) \in \{0\} \cup [1, n]$ for all $x \in \Omega$

We now consider the general case. In Section 4.2 we prove the following fact.

**Lemma 6.** For any constant $\lambda \in (0, 1)$ there exists a decomposition $\delta = \delta^- + \delta^+$ with $\delta^-, \delta^+ \geq 0$ such that

$$\delta^- \leq \ln \frac{1}{1 - \lambda} + \frac{\tau}{2} \cdot \frac{2 + \ln \frac{n}{m}}{\lambda} \quad \text{and} \quad \mathbb{E}[\delta^+] \leq \frac{\Gamma(d + 2, \tau d)}{2d \cdot d!} \cdot \frac{2 + \ln \frac{n}{m}}{m}$$

As in the first case, we conclude from the Markov’s inequality that with probability at least $1 - \frac{\Gamma(d + 2, \tau d)}{2d \cdot d!}$ Algorithm I produces a schedule satisfying $\delta \leq \ln \frac{1}{1 - \lambda} + \frac{\tau + \tau^+}{2} \cdot \frac{2 + \ln \frac{n}{m}}{m}$. This leads to

**Theorem 7.** The conclusion of Theorem 5 holds if

$$m \geq \frac{\tau_p(d) \cdot (2 + \ln \frac{n}{m})}{2 \ln \left(1 + \frac{2}{\gamma} \tau \hat{\varepsilon}^2\right) \cdot (1 - \lambda)} \quad \text{for some } \gamma \in (0, 0.25), \rho = \frac{0.75}{1-\gamma} \text{ and } \lambda \in (0, 1)$$

(9)

For example, (9) will be satisfied for $d = 64$, $m \geq 3.6 \cdot (9 + \ln n)$ and $r = \left[2\varepsilon^{-2}\right] = 8(1 + o(1))\varepsilon^{-2}$ (where we used $\gamma = 0.24$ and $\lambda = e^{-7}$).
3.1 Approximate sampling oracles

So far we assumed that exact sampling oracles $\mu_\beta$ are used. For many applications, however, we only have approximate sampling oracles $\tilde{\mu}_\beta$ that are sufficiently close to $\mu_\beta$ in terms of the variation distance $||\cdot||_{TV}$ defined via

$$||\tilde{\mu}_\beta - \mu_\beta||_{TV} = \max_{A \subseteq \Omega} |\tilde{\mu}_\beta(A) - \mu_\beta(A)| = \frac{1}{2} \sum_{x \in \Omega} |\tilde{\mu}_\beta(x) - \mu_\beta(x)|.$$ 

The analysis can be extended to approximate oracles using a standard trick (see e.g. [22, Remark 5.9]).

Theorem 8. Let $\tilde{Q}$ be the output of the algorithm with parameters $d,m,r$ satisfying the conditions of Theorem 5 or 7 (depending on whether $H(\cdot) \in [1,n]$ or $H(\cdot) \in \{0\} \cup [n]$), where exact sampling oracles $\mu_\beta$ are replaced with approximate sampling oracles $\tilde{\mu}_\beta$ satisfying $||\tilde{\mu}_\beta - \mu_\beta||_{TV} \leq \frac{\delta}{mq(r+d)+3r+1}$.

Then $\tilde{Q} \in \left[\frac{Q}{1+\epsilon}, Q(1+\epsilon)\right]$ with probability at least $0.75 - \kappa$.

As mentioned in the introduction, probability $0.75 - \kappa$ can be boosted to any other probability in $(0.5,1)$ by repeating the algorithm a constant number of times and taking the median (assuming that $\kappa$ is a constant in $(0,0.25)$). Alternatively, one can tweak parameters in Theorems 5 and 7 to get the desired probability directly.

**Proof.** It is known that there exists a coupling between $\mu_\beta$ and $\tilde{\mu}_\beta$ such that they produce identical samples with probability at least $1 - ||\tilde{\mu}_\beta - \mu_\beta||_{TV} \geq 1 - \delta$, where we denoted $\delta = \frac{\delta}{mq(r+d)+3r+1}$. Let $\mathbb{A}$ and $\tilde{\mathbb{A}}$ be the algorithms that use respectively exact and approximate samples, where the $k$-th call to $\mu_\beta$ in $\mathbb{A}$ is coupled with the $k$-th call to $\tilde{\mu}_\beta$ in $\tilde{\mathbb{A}}$ when $\beta = \tilde{\beta}$. We say that the $k$-th call is **good** if the produced samples are identical. Note, $\mathbb{P}[k$-th call is good | all previous calls were good] $\geq 1 - \delta$, since the conditioning event implies $\beta = \tilde{\beta}$. Also, if all calls are good then $\mathbb{A}$ and $\tilde{\mathbb{A}}$ give identical results.

Let $N$ be the number of points inside $[z(\beta_{\text{max}}), z(\beta_{\text{min}})]$ produced by the call TP$(md)$ in Algorithm 3. Then $N$ follows the Poisson distribution of rate $\lambda = mdq$, i.e. $\mathbb{P}(N = n) = \frac{\lambda^n e^{-\lambda}}{n!}$. Algorithm 3 makes $N + 1$ oracle calls, and produces a sequence $(\beta_0, \ldots, \beta_{\ell})$ with $\ell \leq \frac{N}{d} + 2$. Thus, the total number of oracle calls is $N + 1 + (\ell + 1)r \leq Nc + 3r + 1$ where $c = 1 + \frac{\ell}{d}$. Denoting $\mu = \lambda(1-\delta)^c$, we can write

$$\mathbb{P}[\text{all calls are good}] \geq \sum_{n=0}^{\infty} \mathbb{P}(N = n) \cdot (1-\delta)^{nc+3r+1} = \sum_{n=0}^{\infty} \frac{\lambda^n e^{-\lambda}}{n!} \cdot (1-\delta)^{nc+3r+1}$$

$$= \sum_{n=0}^{\infty} \frac{\mu^n e^{-\mu}}{n!} \cdot e^{\mu-\lambda(1-\delta)^{3r+1}} = e^{\mu-\lambda(1-\delta)^{3r+1}} = e^{-\lambda(1-(1-\delta)^c)}(1-\delta)^{3r+1}$$

$$\geq e^{-\lambda(1-(1-c\delta))}(1-\delta)^{3r+1} \geq (1-\lambda c\delta)(1-\delta)^{3r+1} \geq 1 - \lambda c\delta - (3r+1)\delta \geq 1 - \kappa$$

where we used the facts that $(1 - x)^c \geq 1 - cx$ and $e^{-x} \geq 1 - x$ for $x \geq 0$ and $c \geq 1$. Using the union bound, we obtain the claim of the theorem. \qed

4 Proofs

4.1 Proof of Lemma 4

We will assume that the sequence $(\beta_0, \ldots, \beta_{\ell})$ is strictly increasing (this holds with probability 1). Accordingly, the sequence $(z_0, \ldots, z_{\ell})$ is strictly decreasing. The following has been shown in [22,9].

Lemma 9. For any $i \in [0, \ell - 1]$ there holds $\delta_i \leq z_i - z_{i+1}$ and also

$$\frac{-z'(\beta_i)}{-z'(\beta_{i+1})} \geq \exp(2\delta_i/(z_i - z_{i+1}))$$
Proof. Denote $\beta = (\beta_i + \beta_{i+1})/2$ and $\bar{z} = z(\beta)$, then $\delta_i = z_{i+1} - 2\bar{z} + z_i$. Since $z(\cdot)$ is a convex strictly decreasing function, we have

$$-z'(\beta_i) \geq \frac{z_i - \bar{z}}{\beta - \beta_i} \quad -z'(\beta_{i+1}) \leq \frac{\bar{z} - z_{i+1}}{\beta_{i+1} - \beta}$$

Since $\bar{\beta} - \beta_i = \beta_{i+1} - \bar{\beta}$, taking the ratio gives the second claim of the lemma:

$$\frac{-z'(\beta)}{-z'(\beta_{i+1})} = \frac{z_i - \bar{z}}{\bar{z} - z_{i+1}} = \frac{1}{2} \left( \frac{z_i - z_{i+1} + \delta_i}{z_i - z_{i+1} - \delta_i} \right) = 1 + \lambda \geq e^{2\lambda}$$

where we denoted $\lambda = \frac{\delta_i}{z_i - z_{i+1}} \geq 0$ and observed that $\lambda < 1$ since $1 - \lambda = 2 \frac{\bar{z} - z_{i+1}}{z_i - z_{i+1}} > 0$. The fact that $\lambda < 1$ also gives the first claim of the lemma.

Let us define $s(\beta) = \ln[-z'(\beta)]$ and $s_i = \ln[-z'(\beta_i)]$ for $i \in [0, \ell]$, then function $s(\cdot)$ and the sequence $(s_0, \ldots, s_\ell)$ are strictly decreasing. Since $z(\beta)$ and $s(\beta)$ are continuous strictly decreasing functions of $\beta$, we can uniquely express $z$ via $s$ and define a continuous strictly increasing function $z(s)$ on the interval $S \overset{\text{def}}{=} [s_\ell, s_0]$ (Fig. 1b). Note, with some abuse of notation we use $z(\cdot)$ for two different functions: one of argument $\beta$, and another one of argument $s$. The exact meaning should always be clear from the context.

The inequality in the last lemma for an interval $i \in [0, \ell - 1]$ can be rewritten as follows:

$$2\delta_i \leq (z_i - z_{i+1}) \cdot (s_i - s_{i+1})$$

Equivalently, we have $2\delta_i \leq \text{Area}(\Delta_i)$ where $\Delta_i \subseteq [s_i, s_0] \times [z_i, z_0]$ is the rectangle with the top right corner at $(s_i, z_i)$ and the bottom left corner at $(s_{i+1}, z_{i+1})$ (Fig. 1b). Let $\Delta^+$ be the union of rectangles $\Delta_i$ corresponding to large intervals $i$ (with $|z_i - z_{i+1}| > \tau \cdot \frac{1}{m}$), and $\Delta^-$ be the union of $\Delta_i$ corresponding to small intervals $i$. Then $2\delta^+ \leq \text{Area}(\Delta^+)$ and $2\delta^- \leq \text{Area}(\Delta^-)$.

By geometric considerations it should be clear that

$$\text{Area}(\Delta^-) \leq \max \{|z_i - z_{i+1}| : \text{i is small}\} \cdot |S| \leq \tau \cdot \frac{1}{m} \cdot |S|$$

Observe that $-z'(\beta) = \mathbb{E}_{X \sim \mu_\beta}[H(X)] \in [1, n]$ for any $\beta$, and therefore $S = [s_\ell, s_0] \subseteq [0, \ln n]$ and so $|S| \leq \ln n$. This establishes the first claim of Lemma 4. Next, we focus on proving the second claim.

For a point $s \in S$ let $\eta_s$ be the length of the interval $(z_{i+1}, z_i)$ into which $z(s)$ falls (or 0, if $z(s) \in \{z_\ell, \ldots, z_0\}$). Also let $\eta_s^+ = \psi[\eta_s]$ where $\psi[\cdot]$ is the following function: $\psi[a] = a$ if $a > \tau \cdot \frac{1}{m}$, and
\( \psi[a] = 0 \) otherwise. Thus, if \( z(s) \in (z_i+1, z_i) \) for some large interval \( i \) then \( \eta^+_s = z_i - z_{i+1} \) (Fig. 1(c)), otherwise \( \eta^+_s = 0 \). We have

\[
\text{Area}(\Delta^+) = \int_S \eta^+_s ds
\]

The linearity of expectation gives

\[
2\mathbb{E}[\delta^+] \leq \mathbb{E}(\text{Area}(\Delta^+)) = \int_S \mathbb{E}[\eta^+_s] ds \leq \max_{s \in S} \mathbb{E}[\eta^+_s] \cdot |S|
\]

(11)

Now let \( X_0, X_1, X_2, \ldots \) be a Poisson process on \([0, +\infty)\) and \( X_{−1}, X_{−2}, \ldots \) be a Poisson process on \((-\infty, 0]\) (both with rate \( k \)). Thus, \( X_i = \xi_0 + \ldots + \xi_i \) for \( i \geq 0 \) and \( X_i = −\xi_1 − \xi_2 − \ldots − \xi_i \) for \( i \leq −1 \), where \( \xi_j \) are i.i.d. variables from the exponential distribution of rate \( k \). By the superposition theorem for Poisson processes \([15, \text{page 16}]\), bidirectional sequence \( \mathbf{X} = \ldots, X_{−2}, X_{−1}, X_0, X_1, X_2, \ldots \) is a Poisson process on \((-\infty, +\infty)\) (again with rate \( k \)), and in particular it is translation-invariant.

Let \( \mathbf{Y} = \ldots, Y_{−2}, Y_{−1}, Y_0, Y_1, Y_2, \ldots \) be the following process: draw an integer \( c \in \{0, \ldots, d − 1\} \) uniformly at random and then set \( Y_i = X_{di+c} \) for each \( i \). It can be seen that \( \mathbf{Y} \) models the output \((\beta_0, \ldots, \beta\ell)\) of Algorithm H as follows: take the sequence \( z(\beta_{\min}) − Y_0, z(\beta_{\min}) − Y_1, z(\beta_{\min}) − Y_2, \ldots \), restrict to \([z(\beta_{\max}), z(\beta_{\min})]\) and append \( z(\beta_{\min}) \) and \( z(\beta_{\max}) \). Then the resulting sequence has the same distribution as \((z(\beta_0), \ldots, z(\beta\ell))\). We assume below that \((\beta_0, \ldots, \beta\ell)\) is generated by this procedure.

For a point \( \alpha \in \mathbb{R} \) let \( \theta_\alpha \) be the length of the interval \((Y_i, Y_{i+1})\) in which \( a \) falls (or 0, if no such interval exists). Note, the distribution of random variable \( \theta_\alpha \) does not depend on \( a \) (since process \( \mathbf{Y} \) is translation-invariant). We also denote \( \theta^+_a = \psi[\theta_\alpha] \), and let \( \theta \) and \( \theta^+ = \psi[\theta] \) be random variables with the same distributions as \( \theta_\alpha \) and \( \theta^+_a \), respectively (for any fixed \( a \)). Clearly, for each \( s \in [s_\ell, s_0] \) we have \( \eta_s \leq \theta_\alpha \) and \( \eta^+_s \leq \theta^+_a \) for a suitably chosen \( a \), namely, \( a = z(\beta_{\min}) − z(s) \). (Note, if \( z(s) \in (z_{\ell−1}, z_i) \) then \( \eta_s = \theta_\alpha \) and \( \eta^+_s = \theta^+_a \), but at the boundaries the inequalities may be strict). We thus have

\[
\mathbb{E}[\eta^+_s] \leq \mathbb{E}[\theta^+]
\]

(12)

**Lemma 10.** Variable \( \theta \) has the gamma (Erlang) distribution with shape parameter \( d + 1 \) and rate \( k \), whose probability density is \( f(t) = k^{d+1}t^d e^{-kt}/d! \) for \( t \geq 0 \).

**Proof.** We prove this fact for variable \( \theta_\alpha \) with \( a = 0 \). We know that \( Y_{−1} = X_{c−d} \leq 0 \) and \( Y_0 = X_c \geq 0 \), so \( \theta_0 = Y_0 − Y_{−1} \) (with probability 1). By construction, \( X_c − X_{c−d} = \xi_c − \xi_{c−d} + \ldots + \xi_c \), i.e. \( \theta_0 \) is a sum of \( d+1 \) i.i.d. exponential random variables each of rate \( k \). This implies the claim.

**Remark 2.** It may seem counterintuitive that all intervals \( \xi_i = Y_i − Y_{i−1} \) are distributed as a sum of \( d \) exponential random variables with the exception of \( i = 0 \), in which case it is a sum of \( d + 1 \) variables (even though \( \mathbf{Y} \) is translation-invariant). This can be viewed as an instance of the “inspection paradox”, discussed e.g. at [3]. Below we describe an alternative approach, which may help to understand this phenomenon.

Let \( \xi \) be a sum of \( d \) exponential random variables each of rate \( k \) and \( g(\cdot) \) be the probability density of \( \xi \). Then the following (non-rigorous) argument shows that the probability density of \( \theta \) is \( t g(t)/\mathbb{E}[\xi] \) (after which a simple calculation would prove the claim).

Let \( L \) be some large number. Since the distribution of \( \theta_\alpha \) does not depend on \( a \), we can define \( \theta \) as the output of the following process: sample \( \mathbf{Y} \), sample \( a \in [0, L] \) uniformly at random, and then set \( \theta = \theta_\alpha \) (i.e. the length of the interval in \( \mathbf{Y} \) into which \( a \) falls). Let us compute the probability that \( \theta \in [t, t + dt] \). Process \( \mathbf{Y} \) will have \( L/\mathbb{E}[\xi] \) intervals in \([0, L]\) on average, and out of those \((L/\mathbb{E}[\xi]) \cdot (g(t)dt) \) intervals will have length in the range \([t, t + dt]\). The combined length of such intervals is \((L/\mathbb{E}[\xi]) \cdot (g(t)dt) \cdot t \). Thus, point \( a \) will fall into one of those intervals with probability \((L/\mathbb{E}[\xi]) \cdot (g(t)dt) \cdot t/L = (tg(t)/\mathbb{E}[\xi])dt \). Therefore, the density of \( \theta \) is \( t g(t)dt/\mathbb{E}[\xi] \).
Recall that \( \theta^+ = \theta \) if \( \theta > \tau/m \), and \( \theta^+ = 0 \) otherwise. Lemma 6 now gives

\[
\mathbb{E}[\theta^+] = \int_{\tau/m}^{+\infty} tf(t) dt = \int_{\tau/m}^{+\infty} \frac{k^{d+1}t^{d+1}e^{-kt}}{d!} dt = \int_{\tau d/k}^{+\infty} \frac{(kt)^{d+1}e^{-(kt)}}{k \cdot d!} d(kt) = \frac{1}{k \cdot d!} \int_{\tau d}^{+\infty} u^{d+1} e^{-u} du = \frac{\Gamma(d+2, \tau d)}{k \cdot d!} = \frac{\Gamma(d+2, \tau d)}{md \cdot d!}
\]

Combining this with (11) and (12) and observing again that \(|S| \leq \ln n\) finally gives the second claim of Lemma 4.

4.2 Proof of Lemma 6

We will use the same notation as in the previous section. Since \( H(\cdot) \) can now take value 0, we have 
\[-z'(\beta) = \mathbb{E}_{X \sim \mu_\beta}[H(X)] \in [0, n] \] and so \([s_s, s_o] \subseteq [-\infty, \ln n]\) (instead of \([s_s, s_o] \subseteq [0, \ln n]\), as in the previous section). We will deal with small values of \( s(\beta) \) exactly as in [9].

Recall that \( z'(\beta) \) is a strictly increasing function of \( \beta \). Let \( \hat{\beta} \) be the unique value with \( z'(\hat{\beta}) = -\lambda \). (If it does not exist, then we take \( \hat{\beta} \in \{-\infty, +\infty\} \) using the natural rule). Denote \( \hat{z} = z(\hat{\beta}) \) and \( \hat{s} = \ln[-z'(\hat{\beta})] \). Now introduce the following terminology for an interval \( i \in [0, \ell - 1] \):

- interval \( i \) is steep if \( \beta_{i+1} \leq \hat{\beta} \), or equivalently \( s_{i+1} \geq \hat{s} \);
- interval \( i \) is flat if \( \beta_i \geq \hat{\beta} \), or equivalently \( s_i \leq \hat{s} \);
- interval \( i \) is crossing if \( \hat{\beta} \in (\beta_i, \beta_{i+1}) \), or equivalently \( \hat{s} \in (s_{i+1}, s_i) \).

If steep intervals exist then \( \hat{\beta} \geq \beta_{\min} \) and \( z'(\hat{\beta}) \geq -\lambda \). (The inequality may be strict if \( \hat{\beta} = +\infty \)).

We thus have \([s_{i+1}, s_i] \subseteq [\hat{s}, s_o] \subseteq [\ln \lambda, \ln n]\) for all steep intervals \( i \). The argument from the previous section gives that

\[
\sum_{i: \text{ \( i \) is steep and small}} \delta_i \leq \frac{\tau}{2} \cdot \frac{\ln n}{m}
\]

(We just need to assume that \( \beta_{\max} \) was replaced with \( \min\{\beta_{\max}, \hat{\beta}\} \), then we would have \( S = [s_s, s_o] \subseteq [\ln \lambda, \ln n] \) and \( |S| \leq \ln \frac{n}{e} \) instead of \( |S| \leq \ln n \), the rest is the same as in the previous section).

Let us now consider flat intervals. The argument from [12] gives the following fact.

**Lemma 11.** The sum of \( \delta_i \) over flat intervals \( i \) is at most \( \ln \frac{1}{1-X} \).

**Proof.** Assume that flat intervals exist, then \( \hat{\beta} \leq \beta_{\max} \) and \( z'(\hat{\beta}) \geq -\lambda \). (The inequality may be strict if \( \hat{\beta} = -\infty \)). Denote \( \Omega_0 = \{x \in \Omega \mid H(x) = 0\} \) and \( \Omega_+ = \{x \in \Omega \mid H(x) \geq 1\} \), then \( \Omega = \Omega_0 \cup \Omega_+ \) and

\[
\mathbb{E}_{X \sim \mu_\beta}[H(X)] = \frac{\sum_{x \in \Omega_+} H(x) e^{-\hat{\beta}H(x)}}{Z(\hat{\beta})} \geq \frac{\sum_{x \in \Omega_+} e^{-\hat{\beta}H(x)}}{Z(\hat{\beta})} = 1 - \frac{\sum_{x \in \Omega_0} e^{-\hat{\beta}H(x)}}{Z(\hat{\beta})} \geq 1 - \frac{Z(\beta_{\max})}{Z(\hat{\beta})}
\]

On the other hand, \( \mathbb{E}_{X \sim \mu_\beta}[H(X)] = -z'(\hat{\beta}) \leq \lambda \) and so \( \frac{Z(\beta_{\max})}{Z(\hat{\beta})} \geq 1 - \lambda \) and \( z(\hat{\beta}) - z(\beta_{\max}) \leq \ln \frac{1}{1-X} \).

For all flat intervals \( i \) we have \([z_{i+1}, z_i] \subseteq [z(\beta_{\max}), z(\hat{\beta})]\) and also \( \delta_i \leq z_i - z_{i+1} \). This gives the claim of the lemma.

\[\Box\]
It remains to consider crossing intervals. Let us define values $\delta^-_c$ and $\delta^+_c$ as follows. If there are no crossing intervals then $\delta^-_c = \delta^+_c = 0$. Otherwise let $i$ be the unique crossing interval; if $i$ is small then set $\delta^-_c, \delta^+_c = (\delta_i, 0)$, and if $i$ is large then set $\delta^-_c, \delta^+_c = (0, \delta_i)$. In all cases we have $\delta^-_c \leq \tau_m$ (since $\delta_i \leq z_i - z_{i+1}$). Also, $E[\delta^+_c] \leq E[\psi(z_i - z_{i+1})] \leq E[\theta^+] \leq \Gamma(d+2, r_d \cdot \frac{1}{md!})$ where function $\psi(\cdot)$ and random variable $\theta^+$ were defined in the previous section.

We can finally prove Lemma 6. Define $\delta^-$ as $\delta^-_c$ plus the sum of $\delta_i$ over small steep intervals $i$ and flat intervals $i$. Define $\delta^+$ as $\delta^+_c$ plus the sum of $\delta_i$ over large steep intervals $i$. By collecting inequalities above we obtain the desired claim.

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