Limits of residually irreducible $p$-adic Galois representations

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Abstract: In this note we produce examples of converging sequences of Galois representations, and study some of their properties.

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1 Introduction

Consider a continuous representation

$$\rho : G_L \to GL_m(K)$$

of the absolute Galois group $G_L$ of a number field $L$, with $K$ a finite extension of $\mathbb{Q}_p$, with $O$ its ring of integers, $|\cdot|$ its norm, and $k$ its residue field. Then $\rho$ has an integral model taking values in $GL_m(O)$, and the semisimplification of its reduction modulo the maximal ideal $m$ of $O$, denoted by $\overline{\rho}$, is independent of the choice of integral model. We assume that $\overline{\rho}$ is absolutely irreducible and in fact will assume that all the $p$-adic representations considered in this paper are residually absolutely irreducible.

Definition 1 An infinite sequence of (residually absolutely irreducible) continuous representations $\rho_i : G_L \to GL_m(K)$ tends to $\rho : G_L \to GL_m(K)$, if $|\text{tr}(\rho_i(g)) - \text{tr}(\rho(g))| \to 0$ uniformly for all $g \in G_L$. We also say that the $\rho_i$'s converge to $\rho$, or $\rho$ is their limit point.

By Theorem 1 of [Ca], which we can apply because of our blanket assumption of residual absolute irreducibility, this is equivalent to saying that given any integer $n$, for all $i >> 0$, the reduction mod $m^n$, $\rho_{i,n}$, of (an integral model of) $\rho_i$ is isomorphic to the reduction mod $m^n$, $\rho_n$, of (an integral
model of) $\rho$. Note that we are not assuming that the $\rho_i$’s (or $\rho$) are \textit{finitely ramified}, though we do know by the main theorem of [KhRa] that the density of primes which ramify in a given $\rho_i$ is 0.

In this note we study the limiting behavior of the lifts produced in [R1] and completely characterise the limit points of these lifts (see Theorem 1 below). This suggests another approach to certain special cases of the modularity lifting theorems of Wiles, Taylor-Wiles et al. In the process we construct many sequences of converging $p$-adic Galois representations (of fixed determinant and fixed ramification behaviour at $p$). This raises many questions that can be posed far more easily than answered.

Consider $\bar{\rho} : G_{\mathbb{Q}} \to \text{GL}_2(k)$ that satisfies the conditions of [R1], namely:

- $\bar{\rho}$ and $\text{Ad}^0(\bar{\rho})$ are absolutely irreducible Galois representations, and the finite field $k$ of characteristic $p$ is the minimal field of definition of $\bar{\rho}$.

- The (prime to $p$) Artin conductor $N(\bar{\rho})$ of $\bar{\rho}$ is minimal amongst its twists. Denote by $S$ set of primes given by the union of the places where $\bar{\rho}$ is ramified and $\{p, \infty\}$.

- If $\bar{\rho}$ is even then for the decomposition group $G_p$ above $p$ we assume that $\bar{\rho}|_{G_p}$ is not twist equivalent to $\left(\begin{array}{cc} \chi & 0 \\ 0 & 1 \end{array}\right)$ or twist equivalent to the indecomposable representation $\left(\begin{array}{cc} \chi^{p-2} & \ast \\ 0 & 1 \end{array}\right)$ where $\chi$ is the mod $p$ cyclotomic character.

- If $\bar{\rho}$ is odd we assume $\bar{\rho}|_{G_p}$ is not twist equivalent to the trivial representation or the indecomposable unramified representation given by $\left(\begin{array}{cc} 1 & \ast \\ 0 & 1 \end{array}\right)$.

- $p \geq 7$ and the order of the projective image of $\bar{\rho}$ is a multiple of $p$.

Let $Q = \{q_1, \cdots, q_n\}$ be a finite set of primes such that $q_i \neq \pm 1 \mod p$, unramified in $\bar{\rho}$, and the ratio of the eigenvalues of $\bar{\rho}(\text{Frob}_{q_i})$ equal to $q_i^{\pm 1}$. We will call the primes as in $Q$ above \textit{Ramakrishna primes for $\bar{\rho}$} or \textit{R-primes} for short (suppressing the $\bar{\rho}$ which is fixed). We consider the deformation ring $R_{S\cup Q}^{Q_{new}}$ of [KR] (see Definition 1 of loc. cit.). To orient the reader we recall the definition of $R_{S\cup Q}^{Q_{new}}$. For this we need:
Definition 2 If $q$ is a prime, $G_{Q_q}$ the absolute Galois group of $Q_q$ and $R$ a complete Noetherian local ring with residue field $k$, a continuous representation $\rho : G_{Q_q} \to GL_2(R)$ is said to be special if up to conjugacy it is of the form \[
abla \varepsilon \chi' \ *
\]
for $\varepsilon$ the $p$-adic cyclotomic character, and $\chi' : G_{Q_q} \to R^\times$ a continuous character. A continuous representation $\tilde{\rho} : G_{Q} \to GL_2(R)$, is said to be special at a prime $q$ if $\tilde{\rho}|_{D_q}$, with $D_q$ a decomposition group at $q$, is special.

Then $R_{Q\cup Q}^{Q_{new}}$ is the universal ring that parametrises deformations of $\overline{\rho}$ that are minimally ramified at $S$ and such that at primes $q \in Q$ these deformations are special. The ring $R_{Q\cup Q}^{Q_{new}}$ is a complete Noetherian local $W(k)$-algebra, with $W(k)$ the Witt vectors of $k$. The deformation rings considered here are for the deformation problem with a certain fixed (arithmetic) determinant character, and all the deformations of $\overline{\rho}$ we consider will have this fixed determinant character.

Definition 3 A finite set of $R$-primes $Q$ is said to be auxiliary if $R_{Q\cup Q}^{Q_{new}} \simeq W(k)$.

In [R1] auxiliary sets $Q$ of the above type were proven to exist. The representation corresponding to $R_{Q\cup Q}^{Q_{new}} \simeq W(k)$ is denoted by $\overline{\rho}_{Q\cup Q}^{Q_{new}}$. We will call these lifts Ramakrishna lifts of $\overline{\rho}$ or $R$-lifts for short (suppressing the $\rho$ which is fixed).

Theorem 1 A continuous representation $\rho : G_{Q} \to GL_2(W(k))$ that is a deformation of $\overline{\rho}$, is a limit point of distinct $R$-lifts, if and only if $\rho$ is unramified outside $S$ and the set of all $R$-primes, and minimally ramified at primes of $S$.

Remark: Thus we have a complete description of the “$p$-adic closure” of $R$-lifts. Note that in particular each $R$-lift is a limit point of other $R$-lifts. Note also that any deformation $G_{Q} \to GL_2(K)$ of $\overline{\rho}$ that is a limit point of $R$-lifts has a model that takes values in $GL_2(W(k))$. The above theorem can be viewed in a sense as producing an “infinite fern” structure (cf., [M]) in the set of all $R$-lifts of a given $\overline{\rho}$ as above. From the proof of Theorem 1 above, we in fact can deduce that each $R$-lift gives rise to infinitely many “splines” passing through it, where a “spline” consists of a sequence of $R$-lifts converging to it, and each element in a spline gives rise to its own infinitely many splines.
Missing from the picture are the limit points of $R$-lifts which themselves are not $R$-lifts and which the theorem above characterises completely.

Here is the plan of the paper. In Section 2 we prove Theorem 1 which is a simple consequence of the methods of [R1] and [T1]. In Section 3 we prove a result about converging sequences of representations arising from newforms, point out a possible approach to the lifting theorems of Wiles et al that is suggested by the work here. In Section 4 we raise questions about rationality and motivic properties of converging sequences of $p$-adic Galois representations.

2 Converging sequences of Galois representations

We now prove Theorem 1. It follows from the methods of [R1] and [T1]. For the proof we need the following lemma which follows from the methods of [R1] (see also Lemma 1.2 of [T1]) and Lemma 8 of [KR].

Lemma 1 Let $\rho_n : G_{\mathbb{Q}} \rightarrow GL_2(W(k)/(p^n))$ be a lift of $\overline{p}$ that is unramified outside $S$ and the set of all $R$-primes, minimally ramified at primes of $S$, and special at all the primes outside $S$ at which it is ramified. Let $Q'_n$ be any finite set primes that includes the primes of ramification of $\rho_n$, such that $Q'_n \setminus S$ contains only $R$-primes and such that $\rho_n|_{D_q}$ is special for $q \in Q'_n \setminus S$. Then there exists a finite set of primes $Q_n$ that contains $Q'_n$, such that $\rho_n|_{D_q}$ is special for $q \in Q_n \setminus S$, $Q_n \setminus S$ contains only $R$-primes and $Q_n \setminus S$ is auxiliary.

Proof: We use [R1] and Lemma 8 of [KR] to construct an auxiliary set of primes $T_n$ such that $\rho_n|_{D_q}$ is special for $q \in T_n$. Then as $Q'_n \setminus S$ contains only $R$-primes, it follows (using notation of [R1]) from Proposition 1.6 of [W] that the kernel and cokernel of the map

\[ H^1(G_{S \cup T_n \cup Q_n}, \text{Ad}^0(\overline{p})) \rightarrow \bigoplus_{v \in S \cup T_n \cup Q_n} H^1(G_v, \text{Ad}^0(\overline{p}))/N_v \]

have the same cardinality. Then using Proposition 10 of [R1], or Lemma 1.2 of [T1], and Lemma 8 of [KR], we can augment the set $S \cup T_n \cup Q'_n$ to get a set $Q_n$ as in the statement of the lemma.

We are now ready to prove Theorem 1. If $\rho : G_{\mathbb{Q}} \rightarrow GL_2(W(k))$ is a limit point of $R$-lifts, then it is clear that $\rho$ is unramified outside $S$ and the set of
all $R$-primes, and minimally ramified at primes of $S$. We prove the converse. So let $\rho$ satisfy the conditions of Theorem 4 and recall that we denote by $\rho_n$ the reduction modulo $p^n$ of $\rho$. It is easily checked that if $q$ is a $R$-prime any deformation of $\rho|_{D_q}$ to a ramified $p$-adic representation is special: this follows from the structure of tame inertia and the fact that $q^2 \neq 1 \pmod{p}$. Further from the method of proof of Proposition 1 of [KhRa], we easily deduce that the set of primes $q$ for which $\rho|_{D_q}$ is special is of density 0. Thus using Cebotarev and the assumptions on $\overline{\rho}$ in the introduction, we choose a finite set of primes $Q'_n$ such that

- $Q'_n \setminus S$ consists of $R$-primes and $\rho_n|_{D_q}$ is special for $q \in Q'_n \setminus S$,
- $Q'_n$ contains all the ramified primes of $\rho_n$,
- for some prime $q \in Q'_n \setminus S$, $\rho|_{D_q}$ is not special.

Using Lemma 4 we complete $Q'_n$ to a set $Q_n$ such that $Q_n \setminus S$ is auxiliary and $\rho_n|_{D_q}$ is special for $q \in Q_n \setminus S$. Then we claim $\rho_{Q_n \setminus S \cup Q_n} \equiv \rho \pmod{p^n}$. The claim is true as there is a unique representation $G_Q \to GL_2(W(k)/(p^n))$ (with the determinant that we have fixed) that is unramified outside $S \cup Q_n$, minimal at $S$ and special at primes of $Q_n \setminus S$ (as $R_{S \cup Q_n}^{Q_n \setminus S \cup new} \simeq W(k)$). By construction the sets $Q_n$ contain at least one prime at which $\rho$ is not special. Thus we see that we can pick a subsequence of mutually distinct representations $\rho_i$ from the $\rho_{Q_n \setminus S \cup new}$’s such that $\rho_i \to \rho$.

**Remark:** It is of vital importance that $\rho$ is $GL_2(W(k))$-valued as otherwise we would not be able to invoke the disjointness results that are used in the proof of Lemma 4 (Lemma 8 of [KR]).

**Remark:** Theorem 4 can be applied in practise to give many examples of converging sequences of $p$-adic representations: for a non-CM elliptic curve $E_\mathbb{Q}$ for most primes $p$ the mod $p$ representation satisfies the conditions given in the introduction, and the corresponding $p$-adic representation is minimally ramified and $GL_2(\mathbb{Z}_p)$ valued.

We end this section with a result that refines the main result of [KhRa].

**Proposition 1** If $\rho_i : G_L \to GL_m(K)$ is a sequence of (residually absolutely irreducible) continuous representations that converges to $\rho$, then the set of primes where any of the $\rho_i$’s is ramified (i.e., $\bigcup \text{Ram}(\rho_i)$ where $\text{Ram}(\rho_i)$ is the set of primes at which $\rho_i$ is ramified) is of density zero.
**Proof:** Denote by $\rho_{i,n}$ (resp., $\rho_n$) the reduction mod $m^n$ of an integral model of $\rho_i$ (resp., $\rho$). The proof consists in applying Theorem 1 of [KhRa] twice: more precisely first its statement, and then its proof. By an application of its statement we conclude that the density of $\bigcup_{i=1}^n$Ram$(\rho_i)$ is 0 for any $n$. Now applying the proof of Theorem 1 of [KhRa], we define $c_{\rho,n}$ to be the upper density of the set $S_{\rho,n}$ of primes $q$ of $L$ that

- lie above primes which split in $L/\mathbb{Q}$
- are unramified in $\rho_1$ and $\neq p$,
- $\rho_n|_{D_q}$ is unramified, but there exists a “lift” of $\rho_n|_{D_q}$, with $D_q$ the decomposition group at $q$, to a representation $\tilde{\rho}_q$ of $D_q$ to $\text{GL}_m(K)$ that is ramified at $q$: by a lift we mean some conjugate of $\tilde{\rho}_q$ reduces mod $m^n$ to $\rho_n|_{D_q}$.

We have from [KhRa] (see Proposition 1 of loc. cit. which was stated in greater generality than needed there with the present application in mind):

**Lemma 2** Given any $\varepsilon > 0$, there is an integer $N_\varepsilon$ such that $c_{\rho,n} < \varepsilon$ for $n > N_\varepsilon$.

To prove Proposition [1] it is enough to show that given any $\varepsilon > 0$, the upper density of the set $\bigcup$Ram$(\rho_i)$ is $< \varepsilon$. As $\bigcup_{i=1}^n$Ram$(\rho_i)$ has density 0 for (the finite) $n$ that is the supremum of the $i$’s such that $\rho_{i,N_\varepsilon}$ is not isomorphic to $\rho_{N_\varepsilon}$, and $\rho_{N_\varepsilon}$ is finitely ramified, it follows from the lemma above that the upper density of $\bigcup$Ram$(\rho_i)$ is $< \varepsilon$. Hence Proposition [1].

**Remark:** One can ask for more refined information about the asymptotics of ramified primes in (limits of) residually absolutely irreducible $p$-adic Galois representations. For instance in Theorem 1 of [KhRa] one can ask (clued by Theorem 10 of [S1]) if the order of growth of ramified primes can be proved to be bounded by $O(x^{1-N/p+\epsilon})$, where $N$ is the $p$-adic analytic dimension of im$(\rho)$, for any $\epsilon > 0$. Such quantitative refinements were asked for by Serre in an e-mail message to the author and seem hard and will require a new idea (that goes beyond [KhRa]) and a strong use of effective versions of the Cebotarev density theorem.
3 Finite and infinite ramification

Let $L$ be a number field and $K$ a finite extension of $\mathbb{Q}_p$ as before.

**Definition 4** We say that a residually absolutely irreducible continuous representation $\rho : G_L \to GL_n(K)$ is motivic if $\rho$ arises as a subquotient of the $i^{th}$ étale cohomology $H^i(X \times_L L, K)$ of a smooth projective variety $X$ defined over a number field $L$.

A motivic representation is finitely ramified. In [R] examples of residually irreducible representations $\rho : G_{\mathbb{Q}} \to GL_2(K)$ were constructed that were infinitely ramified (see also the last section of [KR]). Infinitely ramified $p$-adic representations cannot be motivic. But they can arise as limits of $p$-adic representations that are motivic. Fix an embedding $\mathbb{Q} \to \overline{\mathbb{Q}}_p$. Then as in [R] (and the last section of [KR]), there is a sequence of eigenforms $f_i \in S_2(\Gamma_0(N_i))$, for a sequence of squarefree integers $N_i$ such that $N_i \to \infty$ and $(p, N_i) = 1$, new of level $N_i$ such that the corresponding $p$-adic representations $\rho_{f_i} : G_{\mathbb{Q}} \to GL_2(\mathbb{Z}_p)$ have a $p$-adic limit $\rho$, with $\rho$ infinitely ramified. Such a $\rho$ is non-motivic, but is the limit of motivic $p$-adic representations. Such limits of eigenforms (in the works of Serre and Katz for instance, cf., [Ka]) have been considered when varying weights or varying the $p$-power level, while fixing the prime-to-$p$ part of the level.

**Proposition 2** Let $f_i \in S_2(\Gamma_0(N_i))$ be a sequence of eigenforms with coefficients in a finite extension $K$ of $\mathbb{Q}_p$ with $(N_i, p) = 1$ and $p \geq 3$, that tend in the $p$-adic $q$-expansion topology to an element $f \in K[[q]]$, such the corresponding residual representation $\overline{\rho}$ satisfies the conditions in the introduction. The element $f$, that gives rise naturally to a Galois representation $\rho_f : G_{\mathbb{Q}} \to GL_2(K)$, is the $q$-expansion of a classical eigenform (of weight 2) if and only if $\rho_f$ is finitely ramified.

**Proof:** The only if part is clear. The if part follows from the methods of Wiles (see Chapter 3 of [W] and also [TW]) and their refinements: note that $\rho_f$ is finite, flat at $p$.

**Remark:** Applying Theorem 1 when $\overline{\rho}$ is odd and finite flat at $p$, in which case the $R$-lifts are modular by Theorem 1 of [K], we can construct systematically many examples of sequences of eigenforms $f_i \to f$ ($f \in K[[q]]$), with the levels of $f_i$ unbounded and such that $\rho_f$ is finitely ramified ($f$ in fact then
a classical eigenform as above). On the other hand as recalled above in [R] (see also last section of [KR]) we have examples of situations as above with \( \rho_f \) infinitely ramified.

It will be of interest to see if Proposition 2 could be proved in a more self-contained manner. The proof above does not use seriously the fact that one does know that \( f \) arises as a limit of the classical forms \( f_i \). If such a proof could be devised, in conjunction with Theorem 1 above and Theorem 1 of [K] (which is due to Ravi Ramakrishna) it would give in special cases a simpler approach using \( R \)-primes to the modularity lifting theorems of Wiles et al (see also [K]) that directly works with the \( p \)-adic Galois representation that needs to be proved modular, and if it could be implemented would avoid (in special cases albeit) the sophisticated deformation theoretic approach of [W].

We elaborate on this: Assume that \( \overline{\rho} \) is modular. In Theorem 1 we have characterised the limit points of \( R \)-lifts. By Theorem 1 of [K] which proves that the representation corresponding to \( R_{Q_{new}} \approx W(k) \) is modular as a consequence of the isomorphism \( R_{Q_{new}} \approx T_{Q_{new}} \) (using notation of [K]), we know that \( R \)-lifts are modular. Hence limits of \( R \)-lifts do arise as limits of \( p \)-adic representations arising from classical newforms. It only (!) remains to prove that a limit of a converging sequence of \( p \)-adic representations arising from newforms (say of weight 2 and level prime to \( p \) to avoid delicate considerations at \( p \)) that is finitely ramified itself arises from a newform (i.e., prove Proposition 2 without appealing directly to [W]). Note that for a semistable elliptic curve \( E \), for all large enough primes \( p \) (bigger than 3 for the methods here to directly work unfortunately!), \( T_p(E) \) is a limit point of \( R \)-lifts.

Note: In recent work we have indeed been able to give a self-contained approach to a result like Proposition 2 above under some technical restrictions: see [K1].

4 Questions

Proposition 2 suggests that a representation that arises as a limit of motivic representations (of “bounded weights”: see Definition 3 below) is finitely ramified if and only if it is motivic. We first recall one of the main conjectures in [FM] in a form that is most pertinent for the considerations here.
Conjecture 1 (Fontaine-Mazur) Consider a continuous residually absolutely irreducible representation $\rho: G_L \to GL_m(K)$ that is potentially semistable at places above $p$. Then the following are equivalent:

1. $\rho$ is motivic
2. $\rho$ is finitely ramified.

From our earlier considerations it is natural to ask the following weaker question.

Question 1 Consider a continuous residually absolutely irreducible representation $\rho: G_L \to GL_m(K)$ that is potentially semistable at places above $p$ that arises as the limit of motivic representations $\rho_i$. Then if $\rho$ is finitely ramified, is $\rho$ motivic?

It seems unlikely that the infinitely ramified representations produced in [R] are algebraic (see definition below). This motivates the following considerations.

Definition 5 A continuous (residually absolutely irreducible) representation $\rho: G_L \to GL_m(K)$ is said to be algebraic if there is a number field $F$ such that the characteristic polynomial of $\rho(Frob_q)$ has coefficients in the ring of integers of $F$ for all primes $q$ which are unramified in $\rho$. The minimal such field is the field of definition of $\rho$.

As by the main theorem of [KhRa], the set of primes at which $\rho$ ramifies is of density 0, the definition above is a sensible one.

Definition 6 A continuous (residually absolutely irreducible) algebraic representation $\rho: G_L \to GL_m(K)$ is said to be of weight $\leq t$ ($t \in \mathbb{Z}$) if for primes $q$ that are unramified in $\rho$, any root $\alpha$ of the characteristic polynomial of $\rho(Frob_q)$ satisfies $|\iota(\alpha)| \leq |k_q|^{\frac{1}{2t-1}}$ for any embedding $\iota : \overline{\mathbb{Q}} \to \mathbb{C}$, with $k_q$ the residue field at $q$.

Question 2 If $\rho_i: G_L \to GL_m(K)$ is an infinite sequence of (residually absolutely irreducible) distinct algebraic representations, all of weight $\leq t$ for some fixed integer $t$, converging to $\rho: G_L \to GL_m(K)$, and $K_i$ the field of definition of $\rho_i$, does $[K_i : \mathbb{Q}] \to \infty$ as $i \to \infty$?
Remark:

- It is observed in [R] (this is a remark of Fred Diamond) that in the situation of Question 2 only finitely many of the \( \rho_i \)'s can arise from elliptic curves: this is a consequence of the Mordell conjecture which gives that suitable twists of the classical modular curves \( X(p^n) \) for \( n >> 0 \) have finitely many \( L \)-valued points for a given number field \( L \).

- If Question 2 has a negative answer, using Proposition 1, we deduce that for a set of primes \( \{ r \} \) of density one, the characteristic polynomials of \( \rho_i(\text{Frob}_r) \) are eventually constant. Hence we deduce that the characteristic polynomials of \( \rho(\text{Frob}_r) \) are defined and integral over a fixed number field \( F \), i.e., \( \rho \) is algebraic (in the case when \( \rho \) is infinitely ramified this is linked to the questions below).

**Question 3** Let \( \rho : G_L \to GL_m(K) \) be a continuous, residually absolutely irreducible representation that is potentially semistable at places above \( p \). Then are the following equivalent:

1. \( \rho \) is motivic
2. \( \rho \) is finitely ramified
3. \( \rho \) is algebraic?

In the question above, the equivalence of 1 and 2 is the Fontaine-Mazur conjecture recalled above: the possible equivalence of 3 to 1 and 2 is the main thrust of the question. One might even ask the stronger question: If \( \rho : G_L \to GL_m(K) \), a continuous, residually absolutely irreducible representation, is algebraic, then is \( \rho \) forced to be both finitely ramified, and potentially semistable at places above \( p \)? All the questions of this section have a positive answer when \( m = 1 \).

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