Perturbative Analysis of Possible Failures in the Traditional Adiabatic Conditions

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Abstract

Recently, Marzlin and Sanders (2004) demonstrated an inconsistency when the adiabatic approximation was applied to specific, "inverse" time-evolving systems. Following that, Tong et al. (2005) showed that the widely used traditional adiabatic conditions are insufficient to guarantee the validity of the adiabatic approximation for this class of systems. In this article we explore the origin of these observations by a perturbative approach and find that in first order approximation certain nonzero terms appear in the solution which gives rise to the breakdown of the adiabatic approximation (despite the fact that the traditional adiabatic conditions are satisfied). We argue that in this case the Hamiltonian of Marzlin and Sanders’ inverse time evolving system cannot be written in terms of $t/T$, where $T$ denotes the total evolution time. It is further demonstrated that the new qualitative adiabatic condition of Ye et al. (2005) performs well in some cases when the traditional conditions fail to describe properly non-adiabatic evolution.

1 Introduction

The quantum adiabatic theorem is one of the oldest fundamental results in quantum physics [1]. It concerns development of systems where the nondegenerate Hamiltonian evolves slowly in time. In the limit when the change
of Hamiltonian $H(t)$ is made infinitely slow the system, which started from one of the eigenstates of $H(0)$, passes through the corresponding instantaneous eigenstate of $H(t)$ [2, 3, 4, 5]. The adiabatic theorem underlies the adiabatic approximation scheme, which states that if the Hamiltonian $H(t)$ evolves slowly enough by satisfying the adiabatic condition in time interval $t \in [0, T]$, then the evolving state of the system will remain close to its instantaneous eigenstate up to a multiplicative phase factor in the interval $[0, T]$. The adiabatic approximation has potential applications in several areas of physics such as the Landau-Zener transition in molecular physics [6], Gell-Mann-Low theorem in quantum field theory [7] or in the lore of Berry phase [8]. Recently, with the emergence of the new field of quantum information theory, new interest has arisen in the application of the quantum adiabatic theorem [9]. An alternative scheme appeared beside the usual quantum algorithms, based on an adiabatic time evolution of the state of the Hamiltonian where the final Hamiltonian encodes the solution for a given problem [10]. In another application, the Berry phase has been proposed to perform quantum information processing tasks [11]. Both of these concepts exploit the circumstance, that the adiabatically evolving ground state is very robust against decoherence and small perturbations [12]. It is thus important to explore the limits of the adiabatic conditions in order to understand better when the evolution of a quantum system can be considered adiabatic.

In a recent Letter Marzlin and Sanders [13] demonstrated on a dual pair of systems (called a-system and b-system), that if the evolution operator of the b-system is the Hermitian conjugate of an adiabatically evolving a-system, then the application of the adiabatic approximation in the b-system can lead to contradiction. Tong et al. [14] explained this inconsistency by pointing out that the widely used traditional adiabatic conditions are not sufficient to guarantee adiabaticity.

In the present paper our aim is to find the common root of the problems by analyzing the time evolution of the dual systems in a perturbative manner. In a first order approximation we obtain certain non-vanishing terms in the regime where the traditional adiabatic conditions hold. We point out that these nonzero terms are responsible for the violation of the adiabatic approximation. We further argue that in this case the Hamiltonian of the b-system cannot be written in terms of $t/T$, where $T$ denotes the total evolution time. In addition, an explicit example is provided which on one hand illustrates our arguments, and on the other hand allows a new adiabatic condition of Ye et al. [16] to be tested on.
2 Analysis of dual pair of systems

2.1 Expansion of state

Consider a closed $N$-dimensional quantum system in a state $|\psi(t)\rangle$, which evolves through the time-dependent Schrödinger equation

$$i\frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle,$$

where $H(t)$ denotes the time-dependent non-degenerate Hamiltonian of the system and we set $\hbar = 1$. Let us introduce the normalized time $s$ by the variable-transformation $t = sT$, $0 \leq s \leq 1$, and rewrite equation (1) as

$$i \frac{\partial}{\partial s}|\psi(s,T)\rangle = TH(s,T)|\psi(s,T)\rangle,$$

where $T$ denotes the total evolution time. The instantaneous eigenstates $|E_n(s,T)\rangle$ of the Hamiltonian $H(s,T)$ satisfy

$$H(s,T)|E_n(s,T)\rangle = E_n(s,T)|E_n(s,T)\rangle, \quad n = 1,\ldots,N,$$

and the elements of the non-adiabatic coupling matrix are defined by

$$\tau_{nk}(s,T) = \langle E_k(s,T)|\frac{\partial}{\partial s}E_n(s,T)\rangle.$$

We now expand the state $|\psi(s,T)\rangle$ in the basis of the instantaneous eigenstates of $H(s,T)$,

$$\psi(s,T) = \sum_{n=1}^{N} \phi_n(s,T)e^{-iT\int_{0}^{s} E_n(s',T)ds'}|E_n(s,T)\rangle.$$

(We shall frequently neglect in the text, but not in the formulae, the variables $(s,T)$ from the expressions following from equations (2,3,4), and the possible $(s,T)$-dependence is understood without denoting it.)

2.2 Parallel transport

Let us now introduce the following local phase change for the $n$’th instantaneous eigenstate

$$|\tilde{E}_n(s,T)\rangle = e^{i\Theta_n(s,T)}|E_n(s,T)\rangle, \quad n = 1,\ldots,N,$$
where \( \Theta_n \) are real, \((s,T)\)-dependent parameters. Plugging (6) into the definition (4) the transformation formula for the non-adiabatic coupling terms [17] reads

\[
\tilde{\tau}_{nk}(s, T) = e^{i(\Theta_n(s,T) - \Theta_k(s,T))} \tau_{nk}(s, T) + i \frac{\partial \Theta_n(s,T)}{\partial s} \delta_{nk} . \tag{7}
\]

This simple relation says that the diagonal element \( \tilde{\tau}_{nn} \) is boosted with respect to \( \tau_{nn} \) by \( i \frac{\partial \Theta_n(s,T)}{\partial s} \), while the non-diagonal elements of \( \tilde{\tau} \) take up a phase with respect to those of \( \tau \). If we choose the phase \( \Theta_n(s,T) = i \int_0^s \tau_{nn}(s', T) ds' \), then under relation (7) \( \tilde{\tau}_{nn} \) becomes zero, and by definition (4) the \( n \)th eigenstate satisfies the parallel transport law \( \langle E_n(s, T) | \partial/\partial s | E_n(s, T) \rangle = 0 \). Let us denote in this gauge the matrix elements of \( \tau \) by \( \tau^\parallel_{nk} \), then we get

\[
\tau^\parallel_{nk}(s, T) = e^{\int_0^s \tau_{kk}(s', T) - \tau_{nn}(s', T) ds'} \tau_{nk}(s, T) , \tag{8}
\]

for \( n \neq k \). Substituting \( |\psi(s, T)\rangle \) expressed in the rotating frame (5) into (2), and performing some algebra we obtain for the complex amplitudes \( \phi_n \) the following differential equation:

\[
\frac{\partial \phi_k(s)}{\partial s} = - \sum_{n \neq k} \phi_n(s, T) \tau^\parallel_{nk}(s, T) e^{-iT \int_0^s g_{nk}(s', T) ds'} , \tag{9}
\]

where \( g_{nk}(s, T) \equiv E_n(s, T) - E_k(s, T) \). Now let us define the matrix elements \( A_{nk}(s, T) \) by

\[
A_{nk}(s, T) = \frac{\tau^\parallel_{nk}(s, T)}{g_{nk}(s, T)} . \tag{10}
\]

The widely used, traditional condition for the adiabatic approximation, if the system starts its evolution in the \( n \)'th instantaneous eigenstate of the Hamiltonian \( H(t) \), is then encoded in the statement [3, 13, 14, 18, 19]

\[
\left| \frac{\langle E_k(t) | \dot{E}_n(t) \rangle}{E_k(t) - E_n(t)} \right| \ll 1 , \quad k \neq n , \quad t \in [0, T] , \tag{11}
\]

which under the variable transformation \( t = sT \) is equivalent to the condition [15]

\[
|A_{nk}(s, T)| \ll T , \quad k \neq n , \quad s \in [0, 1] . \tag{12}
\]
2.3 $a$ and $b$ systems

Let us now turn our attention to a pair of $N$-dimensional quantum systems \[13, 14, 20\], where the $a$-system defined by Hamiltonian $H^a(t)$ and its dual $b$-system with Hamiltonian $H^b(t)$ are related through the following formula:

\[ H^b(t) = -U^a(t)H^a(t)U^a(t) , \]

where we assumed that the spectrum of $H^a(t)$ is entirely discrete and non-degenerate. Formula (13) implies $U^b(t) = U^a(\mathbb{T})$ between the evolution operators of the $a$-system and the $b$-system, and links the eigenvalues and the eigenvectors of the dual systems through the equations

\[ E^b_n(s,T) = -E^a_n(s,T) \quad (\text{and equivalently} \quad g^b_{nk}(s,T) = -g^a_{nk}(s,T)) \quad \text{and} \quad |E^b_n(s,T)\rangle = U^a(\mathbb{T})|E^a_n(s,T)\rangle. \]

According to Duki et al. \[20\] after some manipulation of these formulae one may arrive at the following correspondence between the matrix elements of $\tau^b$ for the dual pair of systems:

\[ \tau^b_{nk}(s,T) = \tau^a_{nk}(s,T) e^{-iT \int_0^s g^a_{nk}(s',T)ds'}. \]

(14)

It is noted that since according to the definition (13) the $a$-system and the $b$-system are interchangeable, the equations numbered $(13-20)$ have dual pairs by exchanging the labels $(a)$ and $(b)$. Now let us write out equation (9) specifically for the $b$-system,

\[ \frac{\partial \phi^b_k(s)}{\partial s} = -\sum_{n \neq k} \phi^b_n(s,T)\tau^b_{nk}(s,T) e^{-iT \int_0^s g^a_{nk}(s',T)ds'}. \]

(15)

By use of (14) this results in a simpler expression for the evolution equation of the $b$-system \[20\],

\[ \frac{\partial \phi^b_k(s,T)}{\partial s} = -\sum_{n \neq k} \phi^b_n(s,T)\tau^a_{nk}(s,T). \]

(16)

2.4 Perturbational solution

In the following we consider the above pair of equations (15,16) and solve each of them perturbatively. Let us consider that the initial state of the $b$-system is $|E^b_n(0,T)\rangle$ with $\phi^b_k(0,T) = \delta_{nk}$. As zeroth order solution, the amplitudes $\phi^b_k$ are assumed not to be evolving, i.e. $\phi^b_k(s,T) \approx \phi^b_k(0,T)$, which constitutes the adiabatic approximation, and in turn by insertion
into the rhs of (15) and (16) yields for \( \phi_k^b \) the parallel set of first order solutions shown below. Performing an integration by parts as well, the first order solution for (15) is given by

\[
\phi_k^b(s, T) = \phi_k^b(0, T) + \frac{i}{T} \sum_{n \neq k} \left( P_{nk}^b(s, T) + Q_{nk}^b(s, T) \right),
\]

(17)

where the terms \( P_{nk}^b \) and \( Q_{nk}^b \) are specified by

\[
P_{nk}^b(s, T) = A_{nk}^b(0, T) - A_{nk}^b(s, T) e^{-iT \int_0^s g_{nk}^b(s', T) ds'},
\]

(18)

and

\[
Q_{nk}^b(s, T) = \int_0^s e^{-iT \int_0^{s'} g_{nk}^b(s'', T) ds''} \frac{\partial A_{nk}^b(s', T)}{\partial s'} ds',
\]

(19)

for \( n \neq k \). On the other hand, the first order solution of (16) is

\[
\phi_k^b(s, T) = \phi_k^b(0, T) + \sum_{n \neq k} \int_0^s \tau_{nk}^a(s', T) ds'.
\]

(20)

(The first term on the rhs of (17) and of (20) corresponds to the zeroth order solution.) We note that equation (17) would be formally identical to Eq. (21) in ref. [15] if Sarandy et al. fixed the gauge to obey the parallel-transport law \( \tau_{mn} = 0 \). However, the main difference lies in the fact that Sarandy et al. do not allow \( H^b(s, T) \) to depend on the total evolution time \( T \). By contrast, in our treatment the Hamiltonian of the b-system and all the other quantities derived from it accommodate a possible \( T \)-dependence. (It is noted that from the notion of \( T \)-dependence we exclude the trivial situation when \( T \) appears as \( T/T' \), where \( T' \) is different from but the same order as \( T \), since here the presence of \( T \) is immaterial.)

Now let us consider the case [15, 19] when the Hamiltonian of the a-system \( H^a \) is a (continuous) function of \( s \) alone. Then both \( \tau_{nk}^a \) and \( E_n^a \) depend only on \( s \), which in turn entails that \( \partial A_{nk}^a(s)/\partial s \), appearing in the dual equation of (19), is a function of \( s \) alone, as well. However, the phase of the exponential in this equation does depend on \( T \). Then it follows from the Riemann-Lebesgue lemma [21] that \( Q_{nk}^a(s) \) goes to zero in the limit \( T \to \infty \). This is the argument of Sarandy et al. [15], which allows us to keep only the first \( P_{nk}^a(s) \) terms under the summation in the dual equation of (17) and then uses the adiabatic condition (12) for the a-system, which in turn guarantees that the adiabatic approximation is accurate.
Next let us focus our attention on the Hamiltonian of the dual b-system $H^b$, while we still assume, as above, that $H^a$ is $T$-independent. One can see through the relation (14) that $\tau^a_{nk}$ depends on $T$ on a rapid oscillatory manner and through the relation (10) that this applies also to $\partial A^b_{nk}(s)/\partial s$. Namely, we obtain after some algebra

$$\frac{\partial A^b_{nk}(s,T)}{\partial s} = e^{-iT\int^s_0 g^a_{nk}(s')ds'} \left( iT\tau^a_{nk}(s) - \frac{(\partial \tau^a_{nk}(s)/g^a_{nk}(s))}{\partial s} \right), \quad (21)$$

where the notation $g^a_{nk}(s)$ and $\tau^a_{nk}(s)$ refer to their $T$-independence. By substitution of this equation into definition (19), in the limit $T \to \infty$ we obtain

$$Q^b_{nk}(s,T) \simeq iT \int^s_0 \tau^a_{nk}(s')ds'.$$  \quad (22)

Thus, provided that $|\tau^a_{nk}|$ in the above equation is not the constant zero function, $Q^b_{nk}(s,T)$ may go to infinity in the large $T$ limit. The reason that for $T \to \infty$, $Q^b_{nk}$ behaves extremely differently than its counterpart $Q^a_{nk}$ (which rather tends to zero), can be understood by the fact that (contrary to $\partial A^a_{nk}/\partial s$) $\partial A^b_{nk}/\partial s$ is not a function of $s$ alone, and moreover it depends on $T$ in a fast oscillatory manner (see equation (21)). Hence, in the limit of $T \to \infty$, $\partial A^b_{nk}/\partial s$ does not converge for any $s \in [0, 1]$ and in turn the Riemann-Lebesgue lemma is not applicable, unlike in the $a$-system.

### 2.5 T-dependence of the b-Hamiltonian

It is also possible to prove that when $H^a$ depends on $s$ only (as we assumed above), then $H^b$ depends also on $T$, and consequently $H^b$ can not be written in the only terms of $s = t/T$. The proof proceeds by reductio ad absurdum. Suppose that $H^b$ is $T$-independent. Hence the eigenvalues $E^b_n$ (or equivalently $g^b_{nk}$) and the eigenstates $|E^b_n\rangle$ of $H^b$ are also independent on $T$. According to definition (4) the $T$-independence also applies to $\tau^b_{nk}$, and by virtue of relation (8) it applies to $\tau^{a}_{nk}$ as well. However, due to definition (10) $A^b_{nk}$ and then $\partial A^b_{nk}/\partial s$ should be also $T$-independent, which is the contradiction since $\partial A^b_{nk}/\partial s$ under (21) depends on $T$.

### 2.6 Extended case

So far, we have examined the b-system in the large $T$ limit, with the constraint that the Hamiltonian of its dual a-system is independent on $T$. Now
we turn to a more general situation, by imposing the following pair of conditions on the $a$-system: 1. it satisfies the adiabatic condition (12), 2. $|\tau_{nk}^{a}|$ is strictly greater than zero $\forall s \in [0,1]$. As we can see, now the possible $T$-dependence of the $a$-system is not ruled out. In the following we prove that if these conditions are met, then $P_{nk}^b$ is much smaller than $Q_{nk}^b$ and then $P_{nk}^b$ can be neglected in the first order approximate solution with respect to $Q_{nk}^b$. Let us remark that this situation is opposite to the one, discussed before in the $T$-independent $a$-system, when $Q_{nk}^a$ were negligible in relation to $P_{nk}^a$. Consequently, let us analyze the fulfillment of

$$\frac{|Q_{nk}^b(s,T)|}{|P_{nk}^b(s,T)|} \gg 1.$$  

Subtraction of (17) from (20) and further algebra lead just to the quantity

$$\frac{|Q_{nk}^b(s,T)|}{|P_{nk}^b(s,T)|} = 1 + \frac{iT \int_0^s \tau_{nk}^{a}(s',T)ds'}{|P_{nk}^a(s,T)|}, n \neq k.$$  

The term $P_{nk}^b$ in (18) can be further written with the aid of (14) and with the use of $g_{nk}^a(s,T) = -g_{nk}^a(s,T)$, to obtain

$$P_{nk}^b(s,T) = -(A_{nk}^0(0,T) + A_{nk}^a(s,T)).$$  

Inserting this result back to equation (24) entails the formula

$$\frac{|Q_{nk}^b(s,T)|}{|P_{nk}^b(s,T)|} = 1 - \frac{T}{(A_{nk}^0(0,T) + A_{nk}^a(s,T))} i \int_0^s \tau_{nk}^{a}(s',T)ds'.$$  

Due to the pair of conditions prescribed for the $a$-system above, the usual adiabatic conditions (12) are satisfied, implying $|T/(A_{nk}^0(0,T) + A_{nk}^a(s,T))| \gg 1$ in formula (26) and $|\tau_{nk}^{a}(s)|$ is strictly greater then zero $\forall s \in [0,1]$, hence the rhs of (26) is large enough so that inequality (23) holds true and in turn $P_{nk}^b$ can be neglected in (17). In this case subtracting again (17) from (20) yields

$$Q_{nk}^b(s,T) \simeq iT \int_0^s \tau_{nk}^{a}(s',T)ds',$$  

which formula can be considered as a $T$-dependent extension of (22). Thus now we have two expressions for $Q_{nk}$: an exact formula (19) expressed by the quantities of the $b$-system, and an approximate formula (27) defined in terms of the non-adiabatic couplings of the $a$-system, where the approximation
is valid provided the a-system satisfies the adiabatic conditions (12) and besides $|\tau_{nk}(s)|$ is not an extreme small quantity $\forall s \in [0,1]$. It is easy to construct an a-system which meets the above requirements. However, according to the approximate formula (27), and the pair of constraints which guarantee the accuracy of (27), $Q_{nk}/T$ is non-vanishing. This implies that in the first order solution (17) the amplitudes $\phi_k^n$ are not constant in time, and in turn the evolution of the dual b-system cannot be considered adiabatic.

Next let us examine, whether the traditional adiabatic conditions, despite of the invalidity of the adiabatic approximation, are fulfilled in this b-system. According to definition (10), relation (14) and $g_{nk}^b = -g_{nk}^a$, we obtain

$$|A_{nk}^b(s,T)| = |A_{nk}^a(s,T)|.$$  \hspace{1cm} (28)

Thus the b-system satisfies the traditional adiabatic condition (12) only when the a-system satisfies it [14].

We can summarize that although in the b-system (constructed by relation (13) and by imposing the double conditions above) the standard adiabatic conditions are satisfied, due to the nonzero values $Q_{nk}^b/T$ the state of the b-system is not evolving adiabatically. Therefore it represents an example that the standard, traditional adiabatic conditions may not guarantee adiabaticity (Tong et al. [14]). On the other hand, we argued that the Hamiltonian of this b-system can not be written in terms of $t/T$ quantities, unlike the case when the standard adiabatic conditions are sufficient in guaranteeing adiabaticity [15, 19].

### 3 Illustration of the Theory

In the following the general analysis of the previous section is supported by a simple example to demonstrate that the non-vanishing $Q_{nk}^b/T$ terms has a key role, in that the standard adiabatic conditions are not sufficient to guarantee the validity of the adiabatic theorem. To this end imagine that the a-system is a spin-half particle at rest at the origin in the presence of a constant magnitude magnetic field rotating in a plane at constant angular velocity $\omega$. This type of system was also considered by refs. [22, 23]. The corresponding effective Hamiltonian in the rest frame is

$$H^a(t) = -\frac{\omega_0}{2} \begin{pmatrix} 0 & e^{-i\omega t} \\ e^{i\omega t} & 0 \end{pmatrix},$$  \hspace{1cm} (29)
where $\omega_0$ is defined by the magnetic moment of the spin and the strength of the magnetic field. The corresponding eigenvalues are $E_1^a(s, T) = -E_2^a(s, T) = \omega_0/2$. Identifying $T = 2\pi/\omega$ and switching to normalized time $s$ by the transformation $t = sT$, we obtain for the normalized eigenspinors

$$|E_1^a(s, T) = \frac{\sqrt{2}}{2} \left( e^{-i\pi s} \right), |E_2^a(s, T) = \frac{\sqrt{2}}{2} \left( e^{i\pi s} \right), \quad (30)$$

so that neither the Hamiltonian $H^a$ nor the corresponding eigenvectors are $T$-dependent. The local phases of the eigenstates has been chosen so that they obey the parallel transport law ($\tau^a_{nn}(s, T) = 0$ for $n = 1, 2$). In this case by direct substitution of (30) into definition (4) $\tau^a_{21} = \tau^a_{12} = -i\pi$.

Calculating the Hamiltonian $H^b$ of the dual system from (13) to the first order in the small quantity $\omega/\omega_0$ one finds

$$H^b(s, T) - \frac{\omega}{2} = \frac{\omega_0}{2} \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) - \frac{\omega}{2} \left( \begin{array}{cc} \cos \omega_0 T s & i \sin \omega_0 T s \\ -i \sin \omega_0 T s & -\cos \omega_0 T s \end{array} \right) \quad (31)$$

Evidently $H^b$ has an irremovable $T$-dependence. Let us now treat the $b$-system perturbatively, by the general method given above. We consider that the dual $b$-system is initially in its instantaneous $|E_1^b(0)\rangle$ eigenstate, and calculate the relevant $Q^b_{21}(s, T)$ term both exactly and approximately by the mean of equations (19) and (27), respectively. The approximate solution according to the previous section is good if the adiabatic condition (12) holds for the $a$-system and that $|\tau^a_{21}|$ is nonzero. In our two-level model the absolute value of $\tau^a_{21}$ is $\pi$, whereas condition (12) implies $\omega \ll \omega_0$. If $T$ is big enough so that $\omega \ll \omega_0$ fulfills, then the exact solution is approximated sufficiently well. However, in the concrete calculations both the exact and approximate solutions yield $Q^b_{21}(s, T)/T = -\pi s$, which surprising result is due to the fact that the above pair of conditions are in fact not necessary for the approximation would be faithful (i.e. they are merely sufficient conditions). Thus the formula $Q^b_{21}/T = -\pi s$ tells us that regardless of the value of $T$, $Q^b_{21}/T$ may become relevant provided $s$ is greater than then zero.

Let us now calculate how faithful the adiabatic approximation is in this case. The adiabatic approximation is acceptable in the interval $t \in [0, T]$ if the overlap $|\langle \psi(t)|E_1(t)\rangle|^2 = |\langle \psi(s, T)|E_1(s, T)\rangle|^2 \simeq 1$ for $s \in [0, 1]$ provided the state of the system is prepared initially in the eigenstate $|E_1(0)\rangle$. However,

$$|\langle \psi(s, T)|E_1(s, T)\rangle|^2 = 1 - |\langle \psi(s, T)|E_2(s, T)\rangle|^2 = 1 - |\phi_2(s, T)|^2, \quad (32)$$

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where the first equality is valid for two-level systems, and the expansion formula (5) was used to obtain the second equality. For our concrete example, according to (20) in first order approximation the amplitude $\phi_2^b$ reads

$$\phi_2^b(s, T) = -\int_0^s \tau_{21}^a(s', T) ds' = i\pi s,$$  \hspace{1cm} (33)

and by returning to the original time variable $t = \frac{2\pi s}{\omega}$ we can write this further as

$$|\langle \psi^b(t) | E_1^b(t) \rangle|^2 = 1 - \left( \frac{\omega t}{2} \right)^2.$$  \hspace{1cm} (34)

In our simple example $\tau_{21}^a$ does not depend on $s$ and equation (16) can be easily solved (without applying perturbation theory) to obtain the exact solution $\phi_2^b(s, T) = \sin(\pi s) = \sin(\frac{\omega t}{2})$, which yields the exact fidelity

$$|\psi^b(t) | E_1^b(t) \rangle|^2 = 1 - \sin^2 \frac{\omega t}{2},$$  \hspace{1cm} (35)

for our particular example. If we compare (35) with (34), one can see that the approximate formula (34) reproduces the first term of the Taylor series expansion of the sine function in the exact formula (35). Performing higher order perturbation by means of an iterative solution of equation (15), the resulting series will approach more and more accurately to the exact expression for the overlap. This implies that in order to solve (15) by perturbation method one has to take into account at each iterative step non-negligible off-diagonal terms (where in the first order the corresponding term is $Q_{nk}^b$ for $n \neq k$). These terms when we go beyond the first order approximation accumulate and according to the exact formula (35) for the overlap (alternatively see Tong et al. [14]) for sufficiently large $t$ results in a great deviation from the ideal overlap 1, pertaining to the pure adiabatic evolution.

In summary, this simple example shows that correct to first order it is indeed $Q_{nk}^b, (n = 1, k = 2)$ term in (17) that is responsible for the breakdown of the adiabatic approximation, despite the fulfillment of the traditional adiabatic conditions.
4 Test of the novel adiabatic condition

Finally, we test a new quantitative adiabatic condition, proposed by Ye et al. [16]. The new quantity, introduced in ref. [16], looks as

\[
\tilde{A}_{nk}(s,T) = \frac{\tau_{nk}(s,T)}{g_{nk}(s,T) - \frac{1}{T} \frac{d}{ds} \arg \tau_{nk}(s,T)} ,
\]

and the new quantitative adiabatic condition is encoded in the requirement [16]

\[
|\tilde{A}_{nk}(s,T)| \ll T , \quad k \neq n , \quad s \in [0,1] ,
\]

in order the adiabatic approximation be faithful in the time range \( t \in [0, T] \).

By substitution of the relations (14) and \( g_{nk} = -g_{nk}^b \) between the dual systems, the following simpler condition is given for the b-system,

\[
\frac{|\tilde{A}_{nk}(s,T)|}{T} = \left| \frac{\tau_{nk}^b(s,T)}{T g_{nk}^b(s,T) - \frac{d}{ds} \arg \tau_{nk}^b(s,T)} \right| = \left| \frac{\tau_{nk}^a(s,T)}{\frac{d}{ds} \arg \tau_{nk}^a(s,T)} \right| \ll 1 .
\]

Next the above condition is applied for our special two-level b-system, discussed in the previous section. Since \( \tau_{21}^b(s,T) = -i \pi \), its argument remains constant in time, and therefore quantity \( \frac{|\tilde{A}_{nk}(s,T)|}{T} \) is infinite for any \( s \in [0,1] \) and in turn (38) is not fulfilled. This implies that the novel adiabatic condition of Ye et al. [16] indicates correctly that the adiabatic approximation is violated in the b-system, whereas the traditional adiabatic condition fails to detect it.

5 Conclusion

In conclusion we investigated through a first order perturbative solution the state evolution of a dual pair of systems, which construction was considered first by Marzlin and Sanders [13]. The a-system is assumed to satisfy the usual adiabatic criteria, and the b-system performs a reversed time evolution with respect to its dual pair. In this case we argue that in the b-system due to the fast oscillations of the off-diagonal non-adiabatic coupling terms \( (\tau_{nk}^b) \), the terms \( Q_{nk}^b \) defined by (19) may become large and dominate in the first order solution of (15). This results in the breakdown of the adiabatic approximation in the b-system, although the standard adiabatic conditions
are satisfied in it. Thus, we could find via perturbative treatment the underlying reason why the traditional adiabatic conditions are insufficient in the b-system. It is further argued that in this case the Hamiltonian of the b-system cannot be written in the form of $H(t/T)$. The above findings are illustrated on a simple two-level model as well, whereon a new adiabatic condition was also tested and showed that in this particular case it performs well.

References

[1] M. Born and V. Fock, Z. Phys. 51 165 (1928)
[2] T. Kato, J. Phys. Soc. Jpn. 5 435 (1950)
[3] A. Messiah, Quantum Mechanics Vol. 2, Chapter XVII section 11
[4] J.E. Avron and A. Elgart, Commun. Math. Phys. 203 445 (1999)
[5] A. Ambainis and O. Regev, arXiv: quant-ph/0411152
[6] L.D. Landau, Zeitschrift 2 46 (1932), C. Zener, Proc. R. Soc. A 137 696 (1932)
[7] M. Gell-Mann and F. Low, Phys. Rev. 84 350 (1951)
[8] M.V. Berry, Proc. Roy. Soc. (London) A 392 45 (1984)
[9] M.A. Nielsen and I.L. Chuang, Quantum Computation and Quantum Information, (Cambridge University Press, 2000)
[10] E. Farhi, J. Goldstone, S. Gutmann, and M. Sipser, arXiv: quant-ph/0001106
[11] A. Jones, V. Vedral, A. Ekert, and G. Castagnoli, Nature 403 869 (2000)
[12] G. De Chiara and G.M. Palma, Phys. Rev. Lett. 91 090404 (2003)
[13] Karl-Peter Marzlin and Barry C. Sanders, Phys. Rev. Lett. 93 160408 (2004)
[14] D.M. Tong, K. Singh, L.C. Kwek, and C.H. Oh, Phys. Rev. Lett. 95 110407 (2005)
[15] M.S. Sarandy, L.-A. Wu, and D.A. Lidar, Quant. Inform. Proc. 3 331 (2004)

[16] M-Y. Ye, X-F Zhou, Y-S. Zhang, and G-C. Guo, arXiv: quant-ph/05091083 v2

[17] R. Baer, J. Chem. Phys. 117 7406 (2002)

[18] Y. Aharonov and J. Anandan, Phys. Rev. Lett. 58 1593 (1987)

[19] R. MacKenzie, E. Marcotte and H. Paquette, arXiv: quant-ph/0510024

[20] S. Duki, H. Mathur, and O. Narayan, arXiv: quant-ph/05100131

[21] The Riemann-Lebesgue lemma states that if $f : [a, b] \to \mathbb{C}$ is integrable on $[a, b]$, then $\int_a^b f(x)e^{inx}dx \to 0$ as $n \to \pm \infty$

[22] Z. Wu and H. Yang, arXiv: quant-ph/0410118 v2

[23] A.K. Pati and A.K. Rajagopal, arXiv: quant-ph/0405129 v2

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