Brane world sum rules and the AdS soliton

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ABSTRACT

We consider “brane world sum rules” for compactifications involving an arbitrary number of spacetime dimensions. One of the most striking results derived from such consistency conditions is the necessity for negative tension branes to appear in five–dimensional scenarios. We show how this result is evaded for brane world models with more than five dimensions. As an example, we consider a novel realization of the Randall–Sundrum scenario in six dimensions involving only positive tension branes. A complete account of our results appeared in hep–th/0106140.

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I. INTRODUCTION

It is not a priori excluded that effective theories directly related to the brane world idea might reside on some corner of the moduli space of string/M–theory. Attempts to derive phenomenologically interesting warped compactifications of the latter were presented at this conference [1]. Our approach to the problem is of the bottom–up kind though. In recent years, considerable efforts have been invested in the study of localized gravity using five-dimensional general relativity (see, for example, Refs. [2,3]). However, Gibbons et al. [4] were able to show that negative tension branes are a necessary ingredient for a very wide class of five-dimensional warped compactifications. This is a rather disappointing conclusion as negative tension branes are inherently unstable objects. Here we generalize the work of Ref. [4] to find consistency conditions or “sum rules” that must be satisfied by brane world models with an arbitrary number of spacetime dimensions. This allows us to show that the necessity of introducing negative tension branes is only an artifact of five-dimensional spacetime physics. As an application, we present a six–dimensional realization of the Randall–Sundrum model based on the geometry of the AdS soliton. A complete account of this work appears in Ref. [5].

II. CONSISTENCY CONDITIONS FOR BRANE WORLDS IN ARBITRARY DIMENSIONS

The ansatz we use for the brane world consists of the $D$-dimensional warped product metric of a $(p + 1)$-dimensional spacetime with coordinates $x^\mu$ and a $(D - p - 1)$-dimensional compact internal space with coordinates $y^m$, 

$$ds^2 = G_{MN}(X)dx^M dx^N = g_{mn}(y)dy^m dy^n + W^2(y)g_{\mu\nu}(x)dx^\mu dx^\nu.$$  

(1)

The stress-energy tensor generating this configuration is assumed to have the following simple form, 

$$T_{MN} = -\frac{\Lambda G_{MN}}{8\pi G_D} - \sum_i T_q^{(i)} P[G_{MN}]^q_i \Delta^{(D-q-1)}(y-y_i).$$  

(2)

which contains a cosmological constant and a collection of branes of various dimensions. To simplify the presentation here, we have not included any matter fields in the bulk or on the branes — see, however, Ref. [3]. The $i^{th}$ brane is a $q$–brane ($p \leq q \leq D - 2$) with tension $T_q^{(i)}$ (units of energy/length$^q$) and transverse coordinates $y_i$. $P[G_{MN}]^q_i$ is the pull–back of the bulk metric to the worldvolume of the $q$–brane. In this ansatz, $\Delta^{(D-q-1)}(y-y_i)$ denotes that covariant combination of delta functions and (geo)metric factors necessary to position the brane. Note that implicitly we are assuming that all of the branes are extended in the $x^\mu$ directions, and if $q > p$ for a particular brane, it spans a $(q-p)$–cycle in the internal space.

Substituting Eqs. (1) and (2) into Einstein’s equations, we can derive a one–parameter ($\alpha$) family of consistency equations by integrating over the compact internal space [3],

$$\int W^{\alpha+1} \left( \alpha \bar{R} W^{-2} + (p-\alpha) \bar{R} - [\gamma + (D-p-1)\bar{\gamma}] \Lambda 
\right.

-8\pi G_D \sum_i (\gamma + (q-p)\bar{\gamma}) T_q^{(i)} \Delta^{(D-q-1)}(y-y_i) \right) = 0,$$  

(3)
where $\mathcal{R}$ and $\tilde{\mathcal{R}}$ denote the Ricci scalars of the “brane” and internal space metrics, $g_{\mu\nu}$ and $g_{mn}$, respectively. We have also introduced the constants:

$$\gamma = \frac{p + 1}{D - 2} \left[ (p - 2\alpha)(D - p - 1) + 2\alpha \right], \quad \tilde{\gamma} = \frac{p(2\alpha - p + 1)}{D - 2}.$$ 

Note that Eq. (3) is merely a convenient re-expression of some components of the Einstein equations as an infinite set of consistency equations relating the geometry of the brane world to its stress-energy content. To gain some insight into these consistency equations, we consider the phenomenologically interesting case $p = 3$. We also make the choice $\alpha = -1$ since it simplifies the equation considerably by removing the warp factor from most terms. With these choices, the constraint may be written as:

$$\oint \left( -\mathcal{R}W^{-2} + 4\tilde{\mathcal{R}} - \frac{8(D - 5)}{D - 2} \Lambda \right) = \frac{32\pi G_D}{D - 2} \sum_i (5D - 13 - 3q)L_i T_q^{(i)},$$

where $L_i$ is the area of the $(q-p)$–cycle in the internal space spanned by the $i^{th}$ brane. If $q = p$ (i.e., the brane is not extended in the internal space), then $L_i = 1$.

If $D = 5$, the consistency equation (4) is independent of $\Lambda$, and $\tilde{\mathcal{R}} = 0$ since the internal space is one-dimensional. Eq. (4) then reduces to the sum rule:

$$-\mathcal{R} \oint W^{-2} = 32\pi G_D \sum_i T_3^{(i)}.$$  

(5)

Our result here essentially reproduces that given in Ref. [4]. In particular, if the curvature on the branes is positive or vanishes, we have $\sum_i T_3^{(i)} \leq 0$ and so we must include a negative tension brane for a consistent model.

For $D = 6$, Eq. (4) becomes

$$\oint \left( -\mathcal{R}W^{-2} + 4\tilde{\mathcal{R}} - 2\Lambda \right) = 8\pi G_D \sum_i (17 - 3q)L_i T_q^{(i)}.$$  

(6)

Note that on the RHS, there are contributions coming from three– and four–branes, all with positive coefficients. On the LHS, however, we have additional contributions coming from the cosmological constant and the curvature of the two–dimensional internal space. Certainly, these contributions afford us much more leeway in constructing consistent brane world models (even when no matter fields are present). For instance, a positive $\tilde{\mathcal{R}}$ and/or negative $\Lambda$ can produce an overall positive contribution on the LHS which could then accommodate the appearance of only positive tensions on the RHS. Similar contributions from the cosmological constant and internal curvature appear in Eq. (4) for all higher dimensions $D \geq 6$. Hence the sum rules are obviously much less restrictive when we go beyond $D = 5$.

III. $D = 6$ BRANE WORLD BASED ON THE ADS SOLITON METRIC

We now consider a novel realization of the Randall-Sundrum scenario in six dimensions based on the geometry of the AdS soliton [6] — for related work, see Refs. [7,8]. The line element of interest is

$$ds^2 = \frac{r^2}{L^2} (\eta_{\mu\nu}dx^\mu dx^\nu + f(r)dr^2) + \frac{L^2}{r^2} \frac{dr^2}{f(r)},$$

(7)
where $f(r) = 1 - \omega^5/r^5$ and $\mu, \nu = 0, 1, 2, 3$. $L$ is related to the cosmological constant by $\Lambda = -10/L^2$. The warp factor is $W(r) = r/L$ and the brane metric is flat, i.e., $g_{\mu\nu} = \eta_{\mu\nu}$. The internal geometry closes off at $r = \omega$, and the coordinate $\tau$ is periodic with period

$$\Delta\tau = \frac{4\pi L^2}{5\omega} \left(1 - \frac{\delta}{2\pi}\right).$$  

The spacetime exhibits a conical singularity at $r = \omega$ (with a deficit angle $\delta$ in $\tau$) which can be thought of as being generated by a three-brane. To construct a brane world geometry with a compact internal space, we truncate the AdS soliton at some value $r = R$ and paste two identical copies back-to-back along the revealed hypersurface. The $r$ and $\tau$ directions then form an internal space with spherical topology. The resulting metric is continuous but not differentiable at the interface. The source of the discontinuity in the extrinsic curvature is interpreted as a bound state of positive tension branes at $r = R$. If we assume that the tensions add linearly, the system is composed of a four-brane with tension

$$T_4 = \frac{1}{\pi G_D L} \sqrt{1 - \left(\frac{\omega}{R}\right)^5} > 0,$$

and a three brane smeared over the $\tau$-direction with tension

$$T_3^{(3)} = \frac{1}{4\pi G_D} \left[\left(\frac{\omega}{R}\right)^4 \left(2\pi - \delta\right)\right] > 0.$$  

The spacetime also contains two three-branes at both “ends” of the internal space ($r = \omega$). Their tension is given by (where we assume $\delta > 0$):

$$T_3^{(1,2)} = \frac{\delta}{8\pi G_D} > 0.$$  

In the present case, the sum rule (14) reduces to

$$16\pi + V_2|\Lambda| = 32\pi G_D \sum_{i=1}^{3} T_3^{(i)} + 20\pi G_D L_\tau T_4,$$

where $V_2$ is the volume of the internal space, and $L_\tau = \Delta\tau f(R)^{1/2}R/L$. We have used the property that the integral of the internal curvature over the two-dimensional compact space yields a topological invariant, the Euler character

$$\chi = \frac{1}{4\pi} \oint \tilde{R} = 2,$$  

where the two is fixed by the spherical topology. Eq. (12) is satisfied for the brane tensions and the cosmological constant associated with the spacetime introduced in this section. This consistency condition makes explicit the fact that the curvature of the internal space (yielding the $16\pi$ in Eq. (12)) and the negative cosmological constant contribute counterterms to positive tension branes. We also showed that conditions corresponding to other values of the $\alpha$ parameter are satisfied.  

The AdS soliton provides an interesting realization of the Randall-Sundrum mechanism to generate a large hierarchy [4]. To see this we apply the coordinate transformation, $r(y) = \omega \cosh^{2/5}(5y/2L)$, which modifies the line element (7) such that
\[
\frac{L^2}{r^2} \frac{dr^2}{f(r)} = dy^2.
\]

(14)

Then the warp factor becomes

\[
W = \frac{r}{L} = \frac{\omega}{L} \cosh^{2/5} \left( \frac{5y}{2L} \right) \sim \frac{\omega}{L} \exp \left( \frac{y}{L} \right),
\]

(15)

where the final approximation on the RHS applies for \( y \gg L \). Hence there is a large gravitational redshift between the branes at \( y_{IR} = 0 \) \( (r = \omega) \) and those at the interface \( (y_{UV} = 2L/5 \arccosh(R/\omega)^{5/2}) \). Hence this model easily generates a large hierarchy between physics scales at the visible brane, taken to be one of the two three-branes generating a conical singularity at \( r = \omega \), and the six-dimensional Planck scale. The corresponding brane world model then provides a six-dimensional realization of the RS I scenario [4] while including only positive tension branes.

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REFERENCES

[1] See, for example, the talks of S. Kachru and E. Silverstein at this conference.
[2] L. Randall and R. Sundrum, Phys. Rev Lett. 83, 3370 (1999) [hep-ph/9905221].
[3] L. Randall and R. Sundrum, Phys. Rev Lett. 83, 4690 (1999) [hep-th/9906064].
[4] G. Gibbons, R. Kallosh and A. Linde, JHEP 0101, 022 (2000) [hep-th/0011223].
[5] F. Leblond, R.C. Myers and D.J. Winters, Consistency conditions for brane worlds in arbitrary dimensions, [hep-th/0106140].
[6] G.T. Horowitz and R.C. Myers, Phys. Rev. D 59, 026005 (1999) [hep-th/9808079].
[7] A.E. Nelson and Z. Chacko, Phys. Rev. D 62, 085006 (2000) [hep-th/9912186].
[8] J. Chen, M.A. Luty and E. Ponton, JHEP 0009, 012 (2000) [hep-th/0003067].