EXPLICIT IDENTITIES ON ZETA VALUES OVER IMAGINARY QUADRATIC FIELD

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Abstract. In this article, we study special values of the Dedekind zeta function over an imaginary quadratic field. The values of the Dedekind zeta function at any even integer over any totally real number field is quite well known in literature. In fact, in one of the famous article, Zagier obtained an explicit formula for Dedekind zeta function at point 2 and conjectured an identity at any even values over any number field. We here exhibit the identities for both even and odd values of the Dedekind zeta function over an imaginary quadratic field which are analogous to Ramanujan’s identities for even and odd zeta values over \( \mathbb{Q} \). Moreover, any complex zeta values over imaginary quadratic field may also be evaluated from our identities.

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1. Introduction

We begin with the famous quote by Zagier \cite{29} about zeta function that “Zeta functions of various sorts are all-pervasive objects in modern number theory, and an ever-recurring theme is the role played by their special values at integer arguments, which are linked in mysterious ways to the underlying geometry and often seem to dictate the most important properties of the objects to which the zeta functions are associated.” In the literature, the special values of the Riemann zeta function are well studied. The zeta values at even integers were established by Euler in 1735 which precisely states that for all \( m \in \mathbb{N} \), we have

\[
\zeta(2m) = (-1)^{m+1} \frac{(2\pi)^{2m} B_{2m}}{2(2m)!} \tag{1.1}
\]
where $B_{2m}$ denotes the $2m$-th Bernoulli numbers. More surprisingly, the value of Riemann zeta function at odd integer is still mysterious, even the question whether the zeta values at odd integers are rational or irrational, is solved only for the value $\zeta(3)$ by Apery [2]. Zudilin [30] has shown that at least one of the four members $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational. A celebrated identity due to Ramanujan for odd zeta values as [23] pp. 319-320, formula (28), states that for any $\alpha, \beta > 0$ with $\alpha \beta = \pi^2$, we have

$$
\alpha^{-m} \left\{ \frac{1}{2} \zeta(2m + 1) + \sum_{n=1}^{\infty} \frac{n^{-2m-1}}{e^{2\pi n \alpha} - 1} \right\} = (-\beta)^{-m} \left\{ \frac{1}{2} \zeta(2m + 1) + \sum_{n=1}^{\infty} \frac{n^{-2m-1}}{e^{2\pi n \beta} - 1} \right\}
- 2^{-2m} \sum_{k=0}^{\infty} \frac{(-1)^k B_{2k} B_{2m+2-2k}}{(2k)! (2m+2-2k)!} \alpha^{m+1-k} \beta^k.
$$

(1.2)

The zeta values over a number field have also been studied extensively. The result (1.1) of Euler was generalized further for any totally real number field by Klingen [14] and Siegel [24], who precisely showed that for any totally real number field $K$ of degree $n$ with discriminant $D,$

$$
\zeta_K(2m) = \frac{q_m \pi^{2mn}}{\sqrt{D}} (m \in \mathbb{N}),
$$

where $q_m$ is some fixed non-zero rational number. In particular, for a real quadratic field $K$ one can obtain a more precise evaluation [cf. 5] such as

$$
\zeta_K(2m) = \frac{\tau(\chi_D)(2\pi)^{4m} B_{2m} B_{2m, \chi_D}}{4 ((2m)!)^2 D^{2m}}
$$

where $\tau(\chi_D)$ is the Gauss sum associated to $\chi_D$ and $B_{2m, \chi_D}$ is the $2m$-th generalized Bernoulli number associated to $\chi_D$. Zagier [28] obtained an explicit formula for $\zeta_K(2)$ over any number field $K,$ which precisely states that for any number field $K$ with discriminant $D$ and signature $(r_1, r_2),$ the following finite sum

$$
\zeta_K(2) = \frac{\pi^{2r_1+2r_2}}{\sqrt{D_K}} \sum_{\nu} c_{\nu} A(x_{\nu,1}) \cdots A(x_{\nu,r_2})
$$

(1.3)

holds, where $A(x)$ is the real-valued function given by the following integral

$$
A(x) := \int_0^x \frac{1}{1 + t^2} \log \frac{4}{1 + t^2} \, dt,
$$

c_{\nu}$ are rational, and $x_{\nu,j}$ are real algebraic numbers.

Recently, Dixit et al. [11] studied the series $\sum_{n=1}^{\infty} \sigma_a(n) e^{-ny}$ associated to the divisor function $\sigma_a(n)$ and obtained an explicit transformation of this series for any complex number $a$. As a special case, the result provides the transformation formulas for Eisenstein series, Eichler integrals, Dedekind eta function and Ramanujan’s formula (1.2). On the other hand, for $a$ even new transformation formulas have been obtained in [11] Theorem 2.11, Corollary 2.13.

In this article, we investigate zeta values over an imaginary quadratic field through a series which is analogous to $\sum_{n=1}^{\infty} \sigma_a(n) e^{-ny}.$ Throughout the paper, we let our imaginary quadratic field be $K$ with discriminant $D$ (absolute value $D_K$), class number $h$ and number of roots of unity to be $\omega.$ Let $\mathcal{O}_K$ be its ring of integers and $v_K(m)$ denote the number of non-zero ideal divisors in $\mathcal{O}_K$ with norm $m.$ Let $\mathfrak{N}$ be the norm map of $K$ over $\mathbb{Q}$ and $\mathfrak{N}_{K/Q}(I)$ denotes the absolute norm of any non-zero ideal ideal $I \subseteq \mathcal{O}_K.$ We denote the Dedekind zeta function over any imaginary quadratic field $K$ by $\zeta_K(s)$ and the $L$-function associated to the quadratic character $\chi_D = (2)$ by $L(s, \chi_D)$ where $(\cdot)$ denotes the Jacobi symbol. We define the general divisor function over $\mathbb{K}$ by

$$
\sigma_{K,a}(n) := \sum_{I \subseteq \mathcal{O}_K} (\mathfrak{N}_{K/Q}(I))^{a} \sum_{d|n} v_K(d) d^a
$$

(1.4)

where $a$ is any complex number.
For an imaginary quadratic field \( K \), it immediately follows from \([1,3]\) that \( \zeta_K(2) \) can be expressed by the finite sum
\[
\zeta_K(2) = \frac{\pi^2}{\sqrt{D_K}} \sum_{\nu} c_\nu A(x_\nu).
\]
We obtain an alternate expression for \( \zeta_K(2) \) over an imaginary quadratic field in the following theorem.

**Theorem 1.3.** Let \( \text{Re}(y) > 0 \). Then, we have
\[
\zeta_K(2) = \frac{y}{2} \left\{ L'(1, \chi_D) + L(1, \chi_D) \left( 2\gamma - \log \left( \frac{2\pi y}{12\pi} \right) \right) + \sum_{n=1}^{\infty} \sigma_{K,-1}(n) e^{-ny} \right\} + \sum_{n=1}^{\infty} \frac{\sigma_{K,1}(n)}{n} \text{kei} \left( 4\pi \sqrt{\frac{2n\pi}{yD_K}} \right),
\]
where the function \( \text{kei}(x) \) is the Kelvin function defined in \([2] \).

In the same article \([28]\), Zagier conjectured an explicit identity to evaluate the Dedekind zeta function at any even integer over any number field.

**Conjecture 1.2 (Zagier).** For each \( m \in \mathbb{N} \), let \( A_m(x) \) be the real valued function
\[
A_m(x) = \frac{2^{2m-1}}{(2m-1)!} \int_{0}^{\infty} \frac{t^{2m-1}}{x \sin^2(t) + x^{-1} \cosh^2(t)} \, dt.
\]
Then the value of \( \zeta_K(2m) \) for an arbitrary number field \( K \) with signature \((r_1, r_2)\) and discriminant \( D \) may equal \( \pi^{2m(r_1+r_2)}/\sqrt{|D|} \) times a rational linear combination of products of \( r_2 \) values of \( A_m(x) \) at algebraic arguments.

We provide an explicit expression for \( \zeta_K(s) \) at even and odd arguments in the following theorem which may be considered as an analogue of Ramanujan’s formula \([1,2]\) over an imaginary quadratic field.

**Theorem 1.3.** For any natural number \( m \) and any complex number \( \alpha, \beta \) with \( \text{Re}(\alpha), \text{Re}(\beta) > 0 \) and \( \alpha \beta = \frac{D_K}{16\pi^2} \), we have
\[
\alpha^{-m} \left\{ \frac{1}{2} \zeta_K(2m+1) + \sum_{n=1}^{\infty} \sigma_{K,-2m-1}(n) e^{-A\pi n \alpha} - \frac{1}{A\pi \alpha} \zeta_K(2m+2) \right\}
= (-\beta)^{-m} \left\{ \frac{\pi h}{w \sqrt{D_K}} \zeta(2m+1) - \frac{4}{\sqrt{D_K}} \sum_{n=1}^{\infty} \frac{\sigma_{K,2m+1}(n)}{n^{2m+1}} \text{kei} \left( A \sqrt{n \beta} \right) - \frac{2h}{w A \beta} \zeta(2m+2) \right\}
+ \frac{\pi^3}{D_K^{m-2} \pi^4} \sum_{k=1}^{m} (-1)^{m-k} \zeta(2m+2-2k) \zeta_K(2k) \alpha^{m+3-k} \beta^{m-1+k}
\]
and
\[
\alpha^{-(m-\frac{1}{2})} \left\{ \frac{1}{2} \zeta_K(2m) + \sum_{n=1}^{\infty} \sigma_{K,-2m}(n) e^{-A\pi n \alpha} - \frac{1}{A\pi \alpha} \zeta_K(2m+1) \right\} = (-1)^{m+1} \beta^{-(m-\frac{1}{2})}
\times \left\{ \frac{1}{\pi} \zeta(2m) \left( \gamma + \log \left( \frac{A\beta}{2} \right) L(1, \chi_D) + L'(1, \chi_D) \right) - \frac{4\pi^2}{\sqrt{D_K}} \sum_{n=1}^{\infty} \frac{\sigma_{K,2m}(n)}{n^{2m}} \text{ker} \left( A \sqrt{n \beta} \right) \right\}
- \frac{2h}{w \sqrt{D_K}} \zeta'(2m) \right\} + \pi^{2m-3} \sum_{k=1}^{m-1} (-1)^{m-1-k} (2\pi)^{2m-2k} \zeta(2m-2k) \zeta_K(2k+1) \alpha^{1-k-m} \beta^{k-2m+\frac{3}{2}}
\]
where \( A = \frac{\pi h}{w \sqrt{D_K}} \) and the functions \( \text{ker}(x), \text{kei}(x) \) are the Kelvin functions which are defined in \([2] \).
Remark. We are not claiming here that the above theorem solves the conjecture over an imaginary quadratic field but certainly it provides an alternate expression for \( \zeta_K(2m) \) over any imaginary quadratic field. An analogue to Lerch’s result \([17]\) over an imaginary quadratic field can be obtained from (1.6) by substituting \( m \) by \( 2m + 1 \).

The following corollary provides a representation for \( \zeta_K(3) \) in terms of \( \zeta_K(2) \). The latter is well-known due to Zagier’s identity \((1.5)\).

**Corollary 1.4.** We have

\[
\zeta_K(3) = 2\pi \left\{ \sum_{n=1}^{\infty} \sigma_{K,-2}(n) e^{-2\pi n} + \frac{4}{\sqrt{D_K}} \sum_{n=1}^{\infty} \frac{\sigma_{K,2}(n)}{n^2} \ker \left( 4\pi \sqrt{\frac{n}{D_K}} \right) \right\} - \frac{\pi^3}{3} (\gamma + L'(1, \chi_D)) + \frac{4\pi h}{w \sqrt{D_K}} \zeta'(2) + \pi \zeta_K(2).
\]

The above corollary follows immediately by letting \( m = 1 \) and \( \alpha = \beta = \frac{D_K}{4\pi} \) in (1.7). It is natural to ask whether it is possible to find an explicit identity for Dedekind zeta function over any imaginary quadratic field \( K \) at complex arguments. The next theorem answers the question.

**Theorem 1.5.** For \( \Re(y) > 0 \) and \( \Re(a) > -1 \), the identity

\[
\sum_{n=1}^{\infty} \sigma_{K,a}(n) e^{-ny} + \frac{1}{2} \zeta_K(-a) - \frac{\zeta_K(1-a)}{y} - \frac{L(1, \chi_D) \Gamma(a+1) \zeta(a+1)}{y^{a+1}} = 4\pi^2 - 2a D_K^\frac{a-1}{2} \sum_{n=1}^{\infty} \sigma_{K,-a}(n) \left\{ \frac{2^{-2a}}{\Gamma^2(1-a)} {\left( \frac{1}{2} - \frac{a}{2}, \frac{1}{2}, \frac{1-a}{2}, \frac{1-a}{2} \right)} \right\} - \frac{4\pi^6 n^2}{y^2 D_K^2} \right)
\]

holds, where the functions \( \text{ber}(x) \), \( \text{bei}(x) \) are the Kelvin functions and \( \psi F_q \) denotes the hypergeometric function which are defined in \([2]\).

**Remark.** An analogous version of the above theorem over \( \mathbb{Q} \) was obtained in \([11]\).

We next abbreviate \( \sigma_{K,0} \) by \( \sigma_K \) and obtain the following important corollary from the above theorem by substituting \( a = 0 \).

**Corollary 1.6.** Let \( \gamma \) be Euler’s constant. For \( \Re(y) > 0 \) and \( \Re(a) > -1 \), we have

\[
\sum_{n=1}^{\infty} \sigma_K(n) e^{-ny} - \frac{h}{2w} \frac{L'(1, \chi_D) + L(1, \chi_D)(\gamma - \log(y))}{y} = \frac{8\pi}{y \sqrt{D_K}} \sum_{n=1}^{\infty} \sigma_K(n) \ker \left( 4\pi \sqrt{\frac{2\eta n}{y D_K}} \right) .
\]

Theorem 1.5 can be extended in the half-plane \( \Re(a) > -2m - 3 \), where \( m \) is any non-negative integer through analytic continuation.

**Theorem 1.7.** If \( \Re(y) > 0 \) and \( \Re(a) > -2m - 3 \) with \( m \in \mathbb{N} \cup \{0\} \), then the following identity holds:

\[
\sum_{n=1}^{\infty} \sigma_{K,a}(n) e^{-ny} + \frac{1}{2} \zeta_K(-a) - \frac{\zeta_K(1-a)}{y} - \frac{2\pi h}{w \sqrt{D_K}} \frac{\Gamma(a+1) \zeta(a+1)}{y^{a+1}} = 4\pi^2 - 2a D_K^\frac{a-1}{2} \sum_{n=1}^{\infty} \sigma_{K,-a}(n) \left\{ \frac{2^{-2a} \left( -\frac{64\pi^6 n^2}{y^2 D_K^2} \right)^{-m}}{\Gamma^2(1-a-2m)} \right\} - \frac{4\pi^6 n^2}{y^2 D_K^2} \right)
\]

\[
- 2^{4m}(a+2m)^2(a+2m+1)^2 \left( \frac{64\pi^6 n^2}{y^2 D_K^2} \right)^{-1} \right\} - \frac{4\pi^6 n^2}{y^2 D_K^2} \right) \}
\]

\[
\sum_{n=1}^{\infty} \sigma_{K,-a}(n) \left\{ \frac{2^{-2a} \left( -\frac{64\pi^6 n^2}{y^2 D_K^2} \right)^{-m}}{\Gamma^2(1-a-2m)} \right\} - \frac{4\pi^6 n^2}{y^2 D_K^2} \right) \}
\]
\[-\sin \left(\frac{\pi a}{2}\right) \text{bei} \left(4\pi \sqrt{\frac{2n\pi}{yD}}\right) + \frac{yD_{K}^{\frac{a+3}{2}}}{(2\pi)^{2a+1} \sin(\pi a)} \sum_{k=0}^{m} \frac{(-1)^k \zeta(2k+2) \zeta(2k+a+2)}{\Gamma^2(-a-1-2k)} \left(\frac{8\pi^3}{yD_{K}}\right)^{-2k}.\]

(1.9)

The series \(\sum_{n=1}^{\infty} \sigma_{K,a}(n)e^{-ny}\) over any imaginary quadratic field \(K\) appearing in the above theorem, can be considered as an analogue of a series \(\sum_{n=1}^{\infty} \sigma_{a}(n)e^{-ny}\) in \(Q\) which plays a significant role in the theory of modular forms. For instance, for \(a = 2m - 1\) with \(m \in N\) and \(y = -2\pi iz\) with \(z\) lying in the upper half plane, the series in \(Q\) essentially represents the Eisenstein series of weight \(2m\) over the full modular group, and for \(a = -2m - 1\) with \(m \in N\) and \(y = -2\pi iz\) the same series in \(Q\) represents the Eichler integral corresponding to the weight \(2m + 2\) Eisenstein series [7, Section 5]. Moreover, the series \(\sum_{n=1}^{\infty} \sigma_{-1}(n)e^{2\pi iz}\) appears in the transformation formula of the logarithm of Dedekind eta function [6, Equation (3.10)].

In the following theorem, we investigate the transformation for the above series over an imaginary quadratic field \(K\) for \(a\) being any natural number.

**Theorem 1.8.** For any natural number \(m\) and any complex number \(\alpha, \beta\) with \(\text{Re}(\alpha), \text{Re}(\beta) > 0\) and \(\alpha\beta = \frac{D_{K}}{16\pi^2}\), the transformations

\[
\alpha^{m} \sum_{n=1}^{\infty} \sigma_{K,2m-1}(n) e^{-A\pi n\alpha} = -(-\beta)^{m} \left\{ \frac{4}{\sqrt{D_{K}}} \sum_{n=1}^{\infty} \sigma_{K,1-2m}(n) \frac{\text{kei}(A\sqrt{\pi n\beta})}{n^{1-2m}} \right\} (1.10)
\]

and

\[
\alpha^{m+\frac{1}{2}} \sum_{n=1}^{\infty} \sigma_{K,2m}(n) e^{-A\pi n\alpha} = \beta^{m+\frac{1}{2}} \left\{ \frac{4(-1)^{m}}{\sqrt{D_{K}}} \sum_{n=1}^{\infty} \sigma_{K,-2m}(n) \frac{\text{ker}(A\sqrt{\pi n\beta})}{n^{-2m}} \right\} (1.11)
\]

hold true, where \(A = \frac{8\pi}{D_{K}}\).

**Remark.** One can conclude by a quick observation in the above theorem that (1.10) provides transformation formula analogous to that for Eisenstein series over an imaginary quadratic field and (1.11) provides an explicit formula for \(\zeta(2m+1)\).

### 2. Preliminaries

Throughout the paper, we require some basic tools of analytic number theory and complex analysis.

#### 2.1. Schwartz function

A function is said to be a Schwartz function if all of its derivatives exist and decay faster than any polynomial. We denote the space of Schwartz functions on \(R\) by \(S(R)\). For \(f \in S(R)\), we let the Mellin transform of \(f\) be \(M(f)\) i.e,

\[
M(f)(s) = \int_{0}^{\infty} f(x)x^{s-1}dx.\]

(2.1)

The following lemma provides the analytic behaviour of the Mellin transform of any Schwartz function.

**Lemma 2.1.** The function \(F(s)\) is absolutely convergent for \(\text{Re}(s) > 0\). It can be analytically continued to the whole complex plane except for simple poles at every non-positive integers. It also satisfies the functional equation:

\[
M(f')(s + 1) = -s M(f)(s),
\]

Proof. The functional equation follows immediately from (2.1) by applying integration by parts on the integral. Moreover, the functional equation yields

\[
M(f^{m})(s + m) = (-1)^{m+1}s(s+1)\cdots(s+m-1)M(f)(s),
\]

(2.2)

which implies that \(M(f)(s)\) has an analytic continuation to the whole complex plane except for the possible simple poles at \(s = 0, 1, \cdots\).
**Example.** One of the most popular example of Schwartz function is \( e^{-x} \). The Mellin transform of \( e^{-x} \) is known as Gamma function which can be defined for \( \text{Re}(s) > 0 \) via the convergent improper integral as

\[
\Gamma(s) = \int_0^\infty e^{-x}x^{s-1}dx. \tag{2.3}
\]

The analytic properties and functional equation of the \( \Gamma \)-function are given in the following proposition which follows immediately from the previous Lemma.

**Proposition 2.2.** [3, Appendix A] The integral in (2.3) is absolutely convergent for \( \text{Re}(s) > 0 \). It can be analytically continued to the whole complex plane except for simple poles at every non-positive integers. It also satisfies the functional equation:

\[
\Gamma(s + 1) = s\Gamma(s).
\]

The \( \Gamma \)-function satisfies many important properties. Here we mention two of them.

(i) Euler’s reflection formula:

\[
\Gamma(s)\Gamma(1 - s) = \frac{\pi}{\sin \pi s} \quad \text{where } s \notin \mathbb{Z}.
\]

(ii) Legendre’s duplication formula:

\[
\Gamma(s)\Gamma \left( s + \frac{1}{2} \right) = 2^{1-2s}\sqrt{\pi}\Gamma(2s). \tag{2.5}
\]

Proofs of these properties can be found in [3, Appendix A].

2.2. **Dedekind zeta function.** The *Dedekind zeta function* attached to an imaginary quadratic field \( \mathbb{K} \) can be defined as

\[
\zeta_{\mathbb{K}}(s) = \sum_{a \in \mathcal{O}_\mathbb{K}} \frac{1}{N(a)^s} = \prod_{p \in \mathcal{O}_\mathbb{K}} \left( 1 - \frac{1}{N(p)^s} \right)^{-1},
\]

for all \( s \in \mathbb{C} \) with \( \text{Re}(s) > 1 \), where \( a \) and \( p \) run over the non-zero integral ideals and prime ideals of \( \mathcal{O}_\mathbb{K} \) respectively. For \( v_{\mathbb{K}}(m) \) denoting the number of non-zero integral ideals in \( \mathcal{O}_\mathbb{K} \) with norm \( m \), \( \zeta_{\mathbb{K}} \) can also be expressed as

\[
\zeta_{\mathbb{K}}(s) = \sum_{m=1}^\infty \frac{v_{\mathbb{K}}(m)}{m^s}.
\]

The following proposition provides the analytic behaviour and the functional equation satisfied by the Dedekind zeta function.

**Proposition 2.3.** [10] pp. 254-255] The function \( \zeta_{\mathbb{K}}(s) \) is absolutely convergent for \( \text{Re}(s) > 1 \). It can be analytically continued to the whole complex plane except for a simple pole at \( s = 1 \) with residue \( L(1, \chi_D) \). It also satisfies the functional equation

\[
\zeta_{\mathbb{K}}(s) = (2\pi)^{2s-1}D_{\mathbb{K}}^{\frac{1}{2}-s}\Gamma(\frac{1}{2}-s)\Gamma(s)\zeta_{\mathbb{K}}(1-s). \tag{2.6}
\]

The famous Dirichlet class number formula for the Dedekind zeta function over an imaginary quadratic field is given in the following proposition.

**Proposition 2.4.** The quadratic \( L \)-function \( L(s, \chi_D) \) of \( \mathbb{K} \) satisfies

\[
L(1, \chi_D) = \frac{2\pi h}{w\sqrt{D_{\mathbb{K}}}}.
\]
2.3. **Special functions.** The mathematical functions which are non-elementary and are useful due to their applications in mathematical analysis, functional analysis, geometry, physics, and other fields are known as special functions. These mainly appear as solutions of differential equations or integrals of elementary functions.

One of the most important families of special functions are the Bessel functions. The Bessel functions of the first kind and the second kind of order $\nu$ are defined by \[ J_\nu(z) := \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m+\nu}}{m! \Gamma(m+1+\nu)} \quad (z, \nu \in \mathbb{C}), \]

\[ Y_\nu(z) := \frac{J_\nu(z) \cos(\pi \nu) - J_{-\nu}(z)}{\sin \pi \nu} \quad (z \in \mathbb{C}, \nu \notin \mathbb{Z}), \]

along with $Y_\nu(z) = \lim_{\nu \to \nu} Y_{\nu}(z)$ for $\nu \in \mathbb{Z}$. The modified Bessel functions of the first and second kinds are defined by \[ I_\nu(z) := \begin{cases} e^{-\frac{1}{2} \pi i \nu} J_\nu(e^{\frac{1}{2} \pi i} z), & \text{if } -\pi < \arg(z) \leq \frac{\pi}{2}, \\ e^{\frac{1}{2} \pi i \nu} J_\nu(e^{-\frac{1}{2} \pi i} z), & \text{if } \frac{\pi}{2} < \arg(z) \leq \pi, \end{cases} \]

\[ K_\nu(z) := \pi \frac{I_{-\nu}(z) - I_\nu(z)}{2 \sin \pi \nu} \tag{2.7} \]

respectively. When $\nu \in \mathbb{Z}$, $K_\nu(z)$ is interpreted as a limit of the right-hand side of (2.7). The real and imaginary parts of Bessel functions are known as Kelvin functions \[20, p. 267\]. More precisely, for any $x \geq 0$ and $\nu \in \mathbb{R}$, the Kelvin functions are defined as

\[ \text{ber}_\nu(x) + i \text{bei}_\nu(x) = J_\nu(x e^{\pm \pi i/4}) \]

and

\[ \text{ker}_\nu(x) + i \text{kei}_\nu(x) = e^{-\nu \pi i/2} K_\nu(x e^{\pm \pi i/4}) \]

where $J_\nu$ (resp. $K_\nu$) denotes the Bessel function of first kind (resp. modified Bessel function of second kind) of order $\nu$.

The generalized hypergeometric function is defined by the following power series:

\[ _pF_q \left( a_1, a_2, \ldots, a_p \mid b_1, b_2, \ldots, b_q \mid z \right) := \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n \cdots (a_p)_n z^n}{(b_1)_n(b_2)_n \cdots (b_q)_n n!} \]

where $(a)_n$ denotes the Pochhammer symbol defined by $(a)_n := a(a+1) \cdots (a+n-1) = \Gamma(a+n)/\Gamma(a)$. It is well-known \[1\, p. 62, \text{Theorem 2.1.1}\] that the above series converges absolutely for all $z$ if $p \leq q$ and for $|z| < 1$ if $p = q + 1$, and it diverges for all $z \neq 0$ if $p > q + 1$ and the series does not terminate.

The following proposition states an important result due to Slater \[19, p. 56-59\] which precisely evaluates inverse Mellin transforms of certain functions in terms of generalized hypergeometric functions. We give its statement below to make the paper self-contained. To begin with we need some notations. Let

\[ \Gamma \left[ a_1, a_2, \ldots, a_A \mid b_1, b_2, \ldots, b_B \right] := \Gamma([a); (b)] = \frac{\Gamma(a_1) \Gamma(a_2) \cdots \Gamma(a_A)}{\Gamma(b_1) \Gamma(b_2) \cdots \Gamma(b_B)}, \]

\[ (a)_n + s := a_1 + s, a_2 + s, \ldots, a_A + s, \]

\[ (b)_n - b_k := b_1 - b_k, \ldots, b_{n-k} - b_k, b_{k+1} - b_k, \ldots, b_B - b_k, \]

\[ \Sigma_A(z) := \sum_{j=1}^{A} z^{a_j} \Gamma \left[ (a') - a_j, (b) + a_j \mid (c) - a_j, (d) + a_j \right] B^{C} F_{A+D-1} \left( \left( b + a_j, 1 + a_j - c \mid 1 + a_j - (a') + (d) + a_j \right) (-1)^{C-A} z \right), \]

\[ \Sigma_B(1/z) := \sum_{k=1}^{B} z^{-b_k} \Gamma \left[ (b') - b_k, (a) + b_k \mid (d) - b_k, (c) + b_k \right] A^{D} F_{B+C-1} \left( \left( a + b_k, 1 + b_k - d \mid 1 + b_k - (b') + (c) + b_k \right) (-1)^{D-B} z \right), \]
Proposition 2.5 (Slater’s Theorem). Let
\[ H(s) = \Gamma \left[ \frac{(a) + s, (b) - s}{(c) + s, (d) - s} \right], \tag{2.8} \]
where the vectors \((a), (b), (c), \text{ and } (d)\) have, respectively, \(A, B, C, \text{ and } D\) components \(a_j, b_k, c_l, \text{ and } d_m\). Then if the following two groups of conditions hold:
\[ -\text{Re}(a_j) < \text{Re}(s) < \text{Re}(b_k) \quad (j = 1, 2, \ldots, A, \quad k = 1, 2, \ldots, B), \tag{2.9} \]
\[ \begin{align*}
  &A + B > C + D, \\
  &A + B = C + D, \quad \text{Re}(s(A + D - B - C)) < -\text{Re}(\eta) \\
  &A = C, \quad B = D, \quad \text{Re}(\eta) < 0,
\end{align*} \tag{2.10} \]
where
\[ \eta := \sum_{j=1}^{A} a_j + \sum_{k=1}^{B} b_k - \sum_{l=1}^{C} c_l - \sum_{m=1}^{D} d_m, \]
then for these \(s\) we have
\[ H(s) = \begin{cases} 
  \int_{0}^{\infty} x^{s-1} \Sigma_A(x) \, dx, & \text{if } A + D > B + C, \\
  \int_{0}^{\infty} x^{s-1} \Sigma_A(x) \, dx + \int_{1}^{\infty} x^{s-1} \Sigma_B(1/x) \, dx, & \text{if } A + D = B + C, \\
  \int_{0}^{\infty} x^{s-1} \Sigma_B(1/x) \, dx, & \text{if } A + D < B + C,
\end{cases} \]
\(\Sigma_A(1) = \Sigma_B(1)\) if \(A + D = B + C, \text{ Re}(\eta) + C - A + 1 < 0, A \geq C.\)

Corollary 2.6. [19] p. 58] Under the conditions (2.9) and (2.10), the inverse Mellin transform of the function in (2.8) is a function \(H(x)\) of hypergeometric type given by
\[ H(x) = \begin{cases} 
  \Sigma_A(x) \text{ for } x > 0, & \text{if } A + D > B + C, \\
  \Sigma_A(x) \text{ for } 0 < x < 1, \quad \text{or} \quad \Sigma_B(1/x) \text{ for } x > 1, & \text{if } A + D = B + C, \\
  \Sigma_B(1/x) \text{ for } x > 0, & \text{if } A + D < B + C,
\end{cases} \]
\(H(1) = \Sigma_A(1) = \Sigma_B(1)\) if \(A + D = B + C, \text{ Re}(\eta) + C - A + 1 < 0, A \geq C.\)

3. Generalization of a Voronoi-type identity over an imaginary quadratic field

In this section, we setup our main ingredients to prove the identities provided in §1. Dirichlet introduced the problem of counting the number of lattice points inside or on the hyperbola. In other words, he studied the asymptotic behaviour of the summatory function of the divisor function. Let \(d(n)\) denotes the divisor function i.e, \(d(n) = \sum_{d|n} 1\). He obtained an asymptotic formula with the main term \(x \log x + (2\gamma - 1)x + \frac{1}{4}\) and an error term of order \(\sqrt{x}\). The problem of estimating the error term is known as the Dirichlet hyperbola problem or the Dirichlet divisor problem. The bound on the error term has been further improved by many mathematicians. At this writing, the best estimate \(O(x^{131/416+\epsilon})\), for each \(\epsilon > 0\), as \(x \to \infty\), is due to M. N. Huxley [12].

Voronoï [25] introduced a new phase into the Dirichlet divisor problem. He was able to express the error term as an infinite series containing the Bessel functions. More precisely, letting \(Y_\nu\) (resp. \(K_\nu\)) denote the Bessel function of the second kind (resp. modified Bessel function of second kind) of order \(\nu\) and \(\gamma\) denote the Euler constant, a celebrated identity of Voronoï is given by
\[ \sum_{n \leq x} d(n) = x \log x + (2\gamma - 1)x + \frac{1}{4} - \sum_{k=1}^{\infty} \frac{d(k)}{k} \left( Y_1(4\pi \sqrt{xk}) + \frac{2}{\pi} K_1(4\pi \sqrt{xk}) \right) \sqrt{xk}, \tag{3.1} \]
where $\sum'$ means that the term corresponding to $n = x$ is halved. In the same article [25], Voronoi also obtained a more general form of (3.1), namely

$$
\sum_{\alpha < n < \beta} d(n) f(n) = \int_{\alpha}^{\beta} (2\gamma + \log t) f(t) \, dt + 2\pi \sum_{n=1}^{\infty} d(n) \int_{\alpha}^{\beta} f(t) \left( \frac{2}{\pi} K_0(4\pi \sqrt{nt}) - Y_0(4\pi \sqrt{nt}) \right) \, dt,
$$

where $f(t)$ is a function of bounded variation in $(\alpha, \beta)$ and $0 < \alpha < \beta$. A shorter proof of the above identity for $0 < \alpha < \beta$ with $\alpha, \beta \notin \mathbb{Z}$ was offered by Koshliakov in [15] where he assumed $f$ to be any analytic function lying inside a closed contour strictly containing the interval $[\alpha, \beta]$. The identity (3.1) can be generalized by generalizing the divisor function in different directions (cf. [4], [26]).

The identity (3.2) was generalized in [8, Section 6, 7] for the general divisor function $\sigma_a(n)$ which can be defined as $\sigma_a(n) := \sum_d n^d$ where $a$ is any complex number. The function $\sigma_{K,a}(n)$ defined in (1.4) is basically the function which is analogous to $\sigma_a(n)$ over an imaginary quadratic field. The following theorem states an analogous identity of (3.2) associated to the divisor function $\sigma_{K,a}(n)$. To the best of our knowledge, the result is new. Before stating our result, we define the function

$$
H_{K,\nu}(x) := \frac{\sqrt{\pi}}{\sin(2\pi \nu)} \left\{ \frac{2^{1-4\nu}}{\Gamma^2(1-2\nu)} \left( \frac{x}{4} \right)^{-\nu} \phantom{0} _0F_5 \left( \begin{array}{c} -1, -\frac{1}{2} \nu \end{array} \right| \frac{x^2}{16} \right) - \frac{2^{1+2\nu} \cos(\pi \nu)}{\Gamma(1+2\nu)} \left( \frac{x}{4} \right)^{\nu} \phantom{0} _0F_5 \left( \begin{array}{c} 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1 \end{array} \right| \frac{x^2}{16} \right) - \frac{2^{4+2\nu} \sin(\pi \nu)}{\Gamma(2+2\nu)} \left( \frac{x}{4} \right)^{1+\nu} \phantom{0} _0F_5 \left( \begin{array}{c} \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, 1 \end{array} \right| \frac{x^2}{16} \right) \right\}.
$$

**Theorem 3.1.** Let $a$ be any complex number with $-1 < \text{Re}(a) < 1$. Then for any Schwarz function $f$, the identity

$$
\sum_{n=1}^{\infty} \sigma_{K,a}(n) f(n) = \int_{0}^{\infty} \left( \zeta_{K}(1-a) + \frac{2\pi h \zeta(1+a)}{w\sqrt{D_K}} \right) f(t) \, dt - \frac{1}{2} \zeta_{K}(-a) f(0^+) + 2\pi \frac{3-a}{2} D_K^{-1/2} \sum_{n=1}^{\infty} \sigma_{K,-a}(n)n^{a/2} \int_{0}^{\infty} e^{a/2 H_{K,a/2}} \left( \frac{4\pi^3 nt}{D_K} \right) f(t) \, dt.
$$

holds, provided the Mellin transform of $f$ decays faster than any polynomial in any bounded vertical strip.

**Proof.** For $\text{Re}(s) > 1$ and $\text{Re}(s - a) > 1$, the Dirichlet series associated to the divisor function $\sigma_{K,a}(n)$ is given by

$$
\sum_{n=1}^{\infty} \frac{\sigma_{K,a}(n)}{n^s} = \zeta(s) \zeta_K(s - a).
$$

(3.3)

For $f \in \mathcal{S}(\mathbb{R})$, its inverse Mellin transform on $F$ yields

$$
I_{K,a} = \sum_{n=1}^{\infty} \sigma_{K,a}(n) f(n) = \sum_{n=1}^{\infty} \sigma_{K,a}(n) \frac{1}{2\pi i} \int_{(c)} F(s)n^{-s} \, ds = \frac{1}{2\pi i} \int_{(c)} F(s) \zeta(s) \zeta_K(s - a) \, ds
$$

(3.4)

where $c > \max(1, 1 + \text{Re}(a))$. We next consider the contour $C$ given by the rectangle with vertices \{ $c - iT, c + iT, \lambda + iT, \lambda - iT$ \} in the anticlockwise direction as $T \to \infty$ where $-1 < \lambda < 0$. It follows from Lemma 2.1 the analytic behaviour of $\zeta(s)$ and Proposition 2.3 that the integrand is analytic inside the contour except for the possible simple poles at $s = 0, 1$ and $1 + a$. Employing the Cauchy residue theorem, we have

$$
\frac{1}{2\pi i} \int_{C} F(s) \zeta(s) \zeta_K(s - a) \, ds = \mathcal{R}_0 + \mathcal{R}_1 + \mathcal{R}_{1+a}
$$

(3.5)
where $\mathcal{R}_{z_0}$ denotes the residue of the integrand at $z_0$. We next evaluate the values of $\mathcal{R}_0$, $\mathcal{R}_1$ and $\mathcal{R}_{1+a}$ using Lemma 2.1, Proposition 2.3 and 2.4 respectively, which are given by

$$
\mathcal{R}_0 = \lim_{s \to -1} s F(s) \zeta(s) \zeta_K(s - a) = \frac{1}{2} M(f')(1) \zeta_K(-a) = \frac{\zeta_K(-a)}{2} \int_{0}^{\infty} f'(t) \, dt = -\frac{\zeta_K(-a) f(0^+)}{2},
$$

$$
\mathcal{R}_1 = \lim_{s \to 1} (s - 1) F(s) \zeta(s) \zeta_K(s - a) = F(1) \zeta_K(1 - a) = \zeta_K(1 - a) \int_{0}^{\infty} f(t) \, dt
$$

and

$$
\mathcal{R}_{1+a} = \lim_{s \to 1+a} (s - 1 - a) F(s) \zeta(s) \zeta_K(s - a) = (1 + a) \zeta(1 + a) \frac{2\pi h}{w \sqrt{D_K}} = \frac{2\pi h \zeta(1 + a)}{w \sqrt{D_K}} \int_{0}^{\infty} f(t) t^a \, dt
$$

Inserting the values of $\mathcal{R}_0$, $\mathcal{R}_1$ and $\mathcal{R}_{1+a}$ in (3.5), the equations (3.4) and (3.5) together imply

$$
I_{K,a} = \int_{0}^{\infty} \left( \zeta_K(1 - a) + \frac{2\pi h \zeta(1 + a)}{w \sqrt{D_K}} \right) f(t) \, dt - \frac{1}{2} \zeta_K(0) f(0^+) + \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{V}
$$

where $\mathcal{H}_1 := \lim_{T \to \infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} F(s) \zeta(s) \zeta_K(s - a) \, ds$ and $\mathcal{H}_2 := \lim_{T \to \infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} F(s) \zeta(s) \zeta_K(s - a) \, ds$ are the horizontal integrals and $\mathcal{V} := \frac{1}{2\pi i} \int_{(\lambda)} F(s) \zeta(s) \zeta_K(s - a) \, ds$ is the vertical integral.

It follows from a standard argument of the Phragmen-Lindelöf principle [cf. 13, Chapter 5] and the functional equation of both zeta functions that for $s = \sigma + it$ with $\lambda < \sigma < c$ and for some $\theta \in \mathbb{R}$,

$$
|\zeta(\sigma + it)\zeta_K(\sigma + it)| \ll t^{\theta(1-\sigma)}, \text{ as } t \to \infty.
$$

On the other hand according to our hypothesis, $F(s)$ decays faster than any polynomial in $t$ in the above vertical strip. Thus, the horizontal integrals $\mathcal{H}_1$ and $\mathcal{H}_2$ vanish.

We next concentrate on the vertical integral $\mathcal{V}$. The functional equation of the Riemann zeta function

$$
\zeta(s) = 2^s \pi^{s-1} \Gamma(1-s) \sin \left( \frac{\pi s}{2} \right) \zeta(1-s)
$$

and that of the Dedekind zeta function in (2.6) together imply that

$$
\zeta(s)\zeta_K(s - a) = D_K^{a-\frac{1}{2}} \frac{1}{2\pi i} \int_{(\lambda)} F(s) \Gamma(1-s) \Gamma(1-s+a) \sin \left( \frac{\pi s}{2} \right) \zeta(1-s) \zeta_K(1-s+a) \left( \frac{8\pi^3}{D_K} \right)^s \, ds
$$

Substituting (3.7) into $\mathcal{V}$ and changing the variable $s$ by $1 - s$ in the next step, the vertical integral becomes

$$
\mathcal{V} = 2D_K^{a-\frac{1}{2}} \frac{1}{2\pi i} \int_{(\lambda)} (1-s) \Gamma(s+a) \frac{(8\pi^3)}{D_K} ds.
$$

We now replace $s$ by $s - a$ and assume $\lambda^* := 1 - \lambda + \text{Re}(a)$ in the above integral to obtain

$$
\mathcal{V} = 2\frac{(2\pi)^{a+1}}{\sqrt{D_K}} \frac{1}{2\pi i} \int_{(\lambda^*)} F(1-s) \Gamma(s-a) \frac{(8\pi^3)}{D_K} ds.
$$

For $\text{Re}(s) > 1$ and $\text{Re}(s-a) > 1$, it follows that

$$
\zeta(s-a) \zeta_K(s) = \sum_{n=1}^{\infty} \frac{\sigma_{K,-a}(n)}{n^{s-a}},
$$

therefore the integral (3.8) can be written as

$$
\mathcal{V} = 2\frac{(2\pi)^{a+1}}{\sqrt{D_K}} \sum_{n=1}^{\infty} \frac{\sigma_{K,-a}(n)}{n^{s-a}} I_{K,a}(n).
$$
where,

\[ I_{K,a}(n) := \frac{1}{2\pi i} \int_{(\lambda^*)} F(1 + a - s)N_{K,a}(s) \left( \frac{8\pi^3 n}{D_K} \right)^{-s} ds, \]  

(3.10)

and

\[ N_{K,a}(s) := \frac{\Gamma(s-a)\Gamma(s)}{\Gamma(1-s)} \cos \left( \frac{\pi}{2} (s-a) \right). \]

We apply (2.4) and (2.5) together on the above factor \( N_{K,a}(s) \) to obtain

\[ N_{K,a}(s) = 2^{3s-a-2}\pi^\frac{a}{2} \frac{\Gamma \left( \frac{\pi}{2} - \frac{s}{2} \right) \Gamma \left( \frac{\pi}{2} \right) \Gamma \left( \frac{\pi}{2} + \frac{1}{2} \right)}{\Gamma \left( \frac{\pi}{2} - \frac{s}{2} \right) \Gamma(1-\frac{s}{2})\Gamma \left( \frac{\pi}{2} + \frac{s}{2} \right)} \]  

(3.11)

On the other hand, using (2.2) into the integral (3.10), we evaluate

\[ I_{K,a}(n) = -\frac{1}{2\pi i} \int_{(\lambda^*)} \int_0^\infty \frac{N_{K,a}(s)f(t)t^{1+a-s}}{1+a-s} \left( \frac{8\pi^3 n}{D_K} \right)^{-s} dt \ ds \]

\[ = -\frac{1}{n^{1+a}} \int_0^\infty f(t) \left( \frac{1}{2\pi i} \int_{\lambda^*} \frac{N_{K,a}(s)(nt)^{1+a-s}}{1+a-s} \left( \frac{8\pi^3 n}{D_K} \right)^{-s} ds \right) dt \]

\[ = -\frac{1}{n^{1+a}} \int_0^\infty f(t)J_{K,a}(nt) dt, \]  

(3.12)

where

\[ J_{K,a}(x) := \frac{1}{2\pi i} \int_{(\lambda^*)} \frac{N_{K,a}(s)x^{1+a-s}}{1+a-s} \left( \frac{8\pi^3 n}{D_K} \right)^{-s} ds. \]

We perform integration by part in (3.12) considering \( J_{K,a}(nt) \) as first function and \( f(t) \) as second to obtain

\[ I_{K,a}(n) = \frac{1}{n^{a+1}} \int_0^\infty f(t) \frac{d}{dt} (J_{K,a}(nt)) \ dt. \]  

(3.13)

Differentiating \( J_{K,a}(nt) \) with respect to \( t \), we get

\[ \frac{d}{dt} (J_{K,a}(nt)) = \frac{n^{a+1}a}{2\pi i} \int_{(\lambda^*)} N_{K,a}(s) \left( \frac{8\pi^3 nt}{D_K} \right)^{-s} ds. \]  

(3.14)

We next insert the factor \( N_{K,a}(s) \) from (3.11) and replace \( s \) by \( \frac{a}{2} - 2s \) into (3.14) to deduce that

\[ \frac{d}{dt} (J_{K,a}(nt)) = \frac{n^{a+1}a}{2\pi i} \int_{(\lambda^*)} \frac{\Gamma \left( \frac{a}{2} - s \right) \Gamma \left( \frac{a}{2} - s \right) \Gamma \left( \frac{1}{2} + \frac{a}{2} - s \right)}{\Gamma \left( \frac{1}{2} - \frac{s}{2} + 1 \right) \Gamma(1-\frac{s}{2})\Gamma \left( \frac{1}{2} + \frac{a}{2} + s \right) \left( \frac{\pi^6 n^2 t^2}{D_K^2} \right)^s} ds. \]

Invoking Proposition 2.5 and applying (2.4) and (2.5) both in the next step, we write the above integral as

\[ \frac{1}{2\pi i} \int_{(-\frac{a}{2}+\frac{1}{2})} \frac{\Gamma \left( \frac{a}{2} - s \right) \Gamma \left( \frac{a}{2} - s \right) \Gamma \left( \frac{1}{2} + \frac{a}{2} - s \right)}{\Gamma \left( \frac{1}{2} - \frac{s}{2} + 1 \right) \Gamma(1-\frac{s}{2})\Gamma \left( \frac{1}{2} + \frac{a}{2} + s \right) \left( \frac{\pi^6 n^2 t^2}{D_K^2} \right)^s} ds \]

\[ = \frac{\Gamma \left( \frac{a}{2} \right) \Gamma \left( \frac{1}{2} + \frac{a}{2} \right)}{\Gamma \left( \frac{1}{2} - \frac{a}{2} \right) \Gamma \left( \frac{1}{2} + \frac{a}{2} \right)} \frac{\Gamma(1) \left( 8\pi^3 nt \right)^{-\frac{a}{2}}}{D_K^2} \left. 0F_5 \left( \begin{array}{c} \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, 1 \end{array} \right| \left. \frac{\pi^6 n^2 t^2}{D_K^2} \right) \right| \]

\[ + \frac{\Gamma \left( \frac{a}{2} \right) \Gamma \left( \frac{1}{2} + \frac{a}{2} \right)}{\Gamma \left( \frac{1}{2} - \frac{a}{2} \right) \Gamma \left( \frac{1}{2} + \frac{a}{2} \right)} \left( \frac{8\pi^3 nt}{D_K^2} \right)^{\frac{a}{2}} \left. 0F_5 \left( \begin{array}{c} \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, 1 \end{array} \right| \left. \frac{\pi^6 n^2 t^2}{D_K^2} \right) \right| \]

\[ + \frac{\Gamma \left( \frac{a}{2} \right) \Gamma \left( \frac{1}{2} - \frac{a}{2} \right) \Gamma \left( -\frac{a}{2} \right)}{\Gamma \left( \frac{1}{2} + \frac{a}{2} \right) \Gamma \left( -\frac{a}{2} \right)} \left( \frac{8\pi^3 nt}{D_K^2} \right)^{\frac{a}{2}} \left. 0F_5 \left( \begin{array}{c} \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, 1 \end{array} \right| \left. \frac{\pi^6 n^2 t^2}{D_K^2} \right) \right|. \]
Thus the derivative of $J_{\kappa,a}(nt)$ reduces to
\begin{equation}
\frac{d}{dt} (J_{\kappa,a}(nt)) = \frac{n^{a/2} \Gamma(\frac{a}{2})}{2^{a+1} \pi^{a/2}} H_{\kappa,a/2} \left( \frac{4\pi^3 nt}{\Gamma(\kappa)} \right).
\end{equation}

Employing (3.16) into (3.13) and inserting the resulting expression into (3.9), we evaluate the vertical integral as
\begin{equation}
\mathcal{V} = 2\pi^{3-a} D_{\kappa} \frac{a}{2} \sum_{n=1}^{\infty} \sigma_{\kappa,-a}(n) \frac{n^{a/2}}{2} \int_{0}^{\infty} t^{a/2} H_{\kappa,a/2} \left( \frac{4\pi^3 nt}{\Gamma(\kappa)} \right) f(t) \, dt
\end{equation}

Finally, the above evaluation (3.17) and equation (3.6) together concludes our theorem.

4. Identities for the Dedekind zeta function over an imaginary quadratic field

In this section, we mainly investigate the transformation formulas for the series $\sum_{n=1}^{\infty} \sigma_{\kappa,a}(n)e^{-ny}$, where $a$ and $y$ are any complex numbers with $\text{Re}(y) > 0$. The following lemma provides the growth of the function which is mainly involved inside the series of right hand side of Theorem 1.5. It plays a significant role in proving Theorem 3.3 and Theorem 3.7.

Lemma 4.1. For any complex number $a$ and any non-negative integer $m$, we have
\begin{equation}
\frac{2^{-2a}}{\Gamma^2(1-a)} F_1 \left( \begin{array}{c} 1 \ rac{1}{2}, 1 - \frac{a}{2}, 1 - \frac{a}{2} \\ 1 - \frac{a}{2}, 1 - \frac{a}{2}, 1 - \frac{a}{2} \end{array} \right) - z^{a/2} \left( \cos \left( \frac{\pi a}{2} \right) \text{ber}(4z^{1/4}) - \sin \left( \frac{\pi a}{2} \right) \text{bei}(4z^{1/4}) \right)
\end{equation}

Proof. We first apply
\begin{equation}
\text{ber}(4z^{1/4}) = a F_3 \left( \begin{array}{c} 1/2, 1, 1 \\ 1, 1, 1 \end{array} \right) z \quad \text{and} \quad \text{bei}(4z^{1/4}) = 4\sqrt{z} a F_3 \left( \begin{array}{c} 1/2, 1, 1 \\ 1/2, 1, 1 \end{array} \right) z
\end{equation}
[cf. 22 Formula (13), (17), p. 516] together to write the left-hand side of (4.1) as
\begin{equation}
\frac{2^{-2a}}{\Gamma^2(1-a)} F_1 \left( \begin{array}{c} 1 \ rac{1}{2}, 1 - \frac{a}{2}, 1 - \frac{a}{2} \\ 1 - \frac{a}{2}, 1 - \frac{a}{2}, 1 - \frac{a}{2} \end{array} \right) - z^{a/2} \left( \cos \left( \frac{\pi a}{2} \right) \text{ber}(4z^{1/4}) - \sin \left( \frac{\pi a}{2} \right) \text{bei}(4z^{1/4}) \right)
\end{equation}

Invoking Proposition 2.5, the above equation reduces to
\begin{equation}
\frac{2^{-2a}}{\Gamma^2(1-a)} F_1 \left( \begin{array}{c} 1 \ rac{1}{2}, 1 - \frac{a}{2}, 1 - \frac{a}{2} \\ 1 - \frac{a}{2}, 1 - \frac{a}{2}, 1 - \frac{a}{2} \end{array} \right) - z^{a/2} \left( \cos \left( \frac{\pi a}{2} \right) \text{ber}(4z^{1/4}) - \sin \left( \frac{\pi a}{2} \right) \text{bei}(4z^{1/4}) \right)
\end{equation}

\begin{equation}
= \frac{\sin(\pi a) z^{\frac{3}{4}}}{2\pi} \left\{ \frac{\Gamma \left( \frac{3}{2} \right) \Gamma \left( 1 + a \right)}{\Gamma \left( \frac{1}{2} - \frac{a}{4} \right) \Gamma \left( \frac{3}{2} - \frac{a}{4} \right)} z^{-a/4} F_3 \left( \begin{array}{c} 1 - a \ rac{1}{2}, 1 - a \\ 1 - a, 1 - a, 1 - a \end{array} \right) \right\}.
\end{equation}

where $-1 - \text{Re}(a)/4 < \eta < \min \left\{ \pm \frac{\text{Re}(a)}{4}, 1 + \frac{\text{Re}(a)}{4} \right\}$. The definition of Meijer G-function [18], p. 143] readily implies that the above integral can be expressed as
\begin{equation}
\frac{1}{2\pi i} \int_{(\eta)} \frac{\Gamma(1 + a/4 + s) \Gamma(-a/4 - s) \Gamma(a/4 - s) \Gamma(1/2 + a/4 - s)}{\Gamma(1/2 - a/4 + s) \Gamma(1 - a/4 + s)} z^{s} \, ds = G_{3,1}^{1,3} \left( \begin{array}{c} \frac{a}{4} \\ -\frac{a}{4}, 1 + \frac{a}{4}, 1 + \frac{a}{4} \end{array} \right) z \right).
\end{equation}
We next find the asymptotics of Meijer G-function. For $1 \leq h \leq p < q$, $1 \leq g \leq q$ and $|\arg(z)| \leq \rho \pi - \delta$ with $\rho > 0$ and $\delta \geq 0$, it follows from [18] Theorem 2, p. 179] that for $|z| \to \infty$, we have

\[ G_{p,q}^{q,h} \left( \frac{a_1, \ldots, a_q}{b_1, \ldots, b_q} \middle| z \right) \sim \sum_{j=1}^{h} \exp(-i\pi(v+1)a_j) \Delta_{q}^{h,j}(v) E_{p,q}(z \exp(i\pi(v+1)|a_j|) \right), \tag{4.5} \]

where $\nu = q - g - h$,

\[ E_{p,q}(z||a_j|) := \frac{z^{a_j-1}}{\Gamma(1 + a_j)} \sum_{k=0}^{m} \frac{\Gamma(1 + b_k - a_j)}{k! \prod_{\ell=1}^{p} (1 + a_\ell - a_j)} \left( \frac{1}{z} \right)^k + O \left( \frac{1}{z^{m+2-a_j}} \right) \]

and $\Delta_{q}^{h,j}(v) := (-1)^{v+1} \left( \prod_{\ell=1}^{h} (a_\ell - a_j) \Gamma(1 + a_\ell - a_j) \right) / \left( \prod_{\ell=g+1}^{q} \Gamma(a_j - b_\ell) \Gamma(1 + b_\ell - a_j) \right)$.  

Letting $g = 3, h = p = 1$ and $q = 5$ in (4.5), we have

\[ G_{1,5}^{3,1} \left( \frac{-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}}{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}} \middle| z \right) = \frac{\Gamma(1 + 3 a - \frac{3}{2}) z^{-\frac{a}{2} - 1}}{\Gamma(-a) \Gamma(\frac{3 + a}{2}) - \frac{1}{2}} \sum_{k=0}^{m} (-1)^k \left( 1 + \frac{a}{2} \right)_k^2 \left( \frac{3 + a}{2} \right)_k^2 z^{-k} + O \left( \frac{1}{z^{m+\frac{3}{2}+2}} \right) \tag{4.6} \]

for $n \to \infty$. Finally (4.3), (4.4) and (4.6) together with the application of (2.3) and (2.5) on the gamma factors inside the integral, we conclude our Lemma. □

4.1. Proof of Theorem 1.5 We first prove the result for $0 < \text{Re}(a) < 1$ and $y > 0$, later we extend it to $\text{Re}(a) > -1$ and $\text{Re}(y) > 0$ respectively by analytic continuation. We consider the particular Schwartz function $f(n) = e^{-ny}$ with $y > 0$ in Theorem 4.1 which yields the following identity

\[ \sum_{n=1}^{\infty} \sigma_{a,n}(n)e^{-ny} = \frac{\zeta_{a}(1-a)}{y} + \frac{2\pi h \Gamma(a+1) \zeta(a+1)}{y^{a+1} w D_{a}} - \frac{1}{2} \zeta_{a}(-a) + 2\pi e^{\frac{a}{2}} D_{a}^{\frac{a-1}{2}} \sum_{n=1}^{\infty} \sigma_{a,-n}(n) I_{a,n}(a) \tag{4.7} \]

where $I_{a,n}(a) = \int_{0}^{\infty} t^{a/2} H_{a,n/2} \left( \frac{4\pi^{2}nt}{D_{a}} \right) e^{-ty} \, dt$. We now concentrate on simplifying the integral $I_{a,n}(a)$. Considering two functions $h_1(t) = t^{a/2}e^{-ty}$ and $h_2(t) = H_{a,n/2} \left( \frac{4\pi^{2}nt}{D_{a}} \right)$, the integral can be expressed in the form

\[ I_{a,n}(a) := \int_{0}^{\infty} h_1(t)h_2(t) \, dt. \]

The Mellin transform associated to $h_1(t)$ and $h_2(t)$ is denoted by $H_1(s)$ and $H_2(s)$ respectively, which we need to evaluate next. We first obtain the Mellin transform of $h_1(t)$ as

\[ H_1(s) := \int_{0}^{\infty} h_1(t)t^{s-1} \, dt = \int_{0}^{\infty} e^{\frac{-a/2}{y}t}t^{s-1} \, dt = \frac{\Gamma(a/2 + s)}{ya^{a/2+s}} \]

where the integral is valid for $\text{Re}(s) > -\frac{\text{Re}(a)}{2}$. It follows from (3.15) that

\[ h_2(t) := \frac{1}{2\pi i} \int_{(\lambda-a)} \frac{\Gamma\left(-\frac{a}{2} - s\right) \Gamma\left(\frac{a}{2} + s\right)}{\Gamma\left(\frac{1}{2} - \frac{a}{2} + s\right) \Gamma\left(\frac{1}{2} + \frac{a}{2} + s\right)} \left( \frac{\pi n^{2}t^{2}}{D_{a}} \right)^{-s} \, ds \]

\[ = \frac{1}{4\pi i} \int_{(\lambda-a)} \frac{\Gamma\left(-\frac{a}{2} + \frac{a}{2}\right) \Gamma\left(\frac{a}{2} + \frac{a}{2}\right) \Gamma\left(\frac{a}{2} + \frac{a}{2}\right) \Gamma\left(\frac{a}{2} + \frac{a}{2}\right)}{\Gamma\left(\frac{1}{2} - \frac{a}{2} - \frac{a}{2}\right) \Gamma\left(\frac{1}{2} - \frac{a}{2} - \frac{a}{2}\right) \Gamma\left(\frac{1}{2} + \frac{a}{2} - \frac{a}{2}\right) \Gamma\left(\frac{1}{2} + \frac{a}{2} - \frac{a}{2}\right)} \left( \frac{\pi n^{2}t^{2}}{D_{a}} \right)^{-s} \, ds. \]
Thus the Mellin transform of \( h_2(t) \) can be evaluated as

\[
H_2(s) := \int_0^\infty h_2(t) t^{s-1} dt = \frac{\Gamma\left(\frac{1}{4} + \frac{s}{2}\right) \Gamma\left(\frac{1}{2} + \frac{s}{2}\right) \Gamma\left(\frac{3}{4} + \frac{s}{2}\right)}{2 \Gamma\left(\frac{3}{4} - \frac{s}{2}\right) \Gamma\left(1 - \frac{s}{2}\right) \Gamma\left(\frac{1}{2} - \frac{s}{2}\right) \Gamma\left(\frac{1}{4} - \frac{s}{2}\right)} \left(\frac{\pi^3 n^2}{D_\mathcal{K}}\right)^{-s}
\]

where the integral is valid for \( 1 < \text{Re}(s) < 2 \). On the other hand, the region of convergence for \( H_1(1-s) \) is \( \text{Re}(s) < 1 + \frac{\text{Re}(a)}{2} \). Thus applying Parseval’s formula [cf. [21] p. 83] for any real \( \mu \) satisfying \( 1 < \mu < 1 + \frac{\text{Re}(a)}{2} \), we obtain

\[
I_{\mathcal{K},a}(n) = \frac{1}{2 \pi y^{a/2+1}} \int_{(-\frac{1}{2})}^{(\frac{1}{2})} \frac{\Gamma\left(\frac{1}{4} + 1 - s\right) \Gamma\left(-\frac{1}{4} + \frac{s}{2}\right) \Gamma\left(\frac{1}{2} + \frac{s}{2}\right) \Gamma\left(\frac{3}{4} + \frac{s}{2}\right)}{\Gamma\left(\frac{1}{2} - \frac{s}{2}\right) \Gamma(1 - \frac{s}{2}) \Gamma\left(\frac{1}{4} - \frac{s}{2}\right)} \left(\frac{\pi^3 n y}{D_\mathcal{K}}\right)^{-s} ds. \tag{4.8}
\]

We apply (2.5) on the first gamma factor in the numerator and replace \( s \) by \(-2s\) in (4.8) to deduce the above integral as

\[
I_{\mathcal{K},a}(n) = \frac{2^{\alpha/2}}{\sqrt{\pi} y^{a/2+1}} \frac{1}{2 \pi i} \int_{(-\frac{1}{2})}^{(\frac{1}{2})} \frac{\Gamma\left(1 + \frac{\alpha}{4} + s\right) \Gamma\left(-\frac{\alpha}{4} - \frac{s}{2}\right) \Gamma\left(\frac{1}{4} - \frac{s}{2}\right) \Gamma\left(\frac{1}{4} + s\right)}{\Gamma\left(\frac{1}{2} - \frac{s}{2}\right) \Gamma(1 - \frac{s}{2}) \Gamma\left(\frac{1}{4} + \frac{s}{2}\right)} \left(\frac{4 \pi^6 n^2 y^2}{D_\mathcal{K}}\right)^{s} ds.
\]

Thus (4.3) readily implies that

\[
I_{\mathcal{K},a}(n) = \frac{2 \pi^{-\frac{\alpha a}{2}} D_\mathcal{K}^{a/2}}{n^{a/2} y \sin(\pi a)} \left[ \frac{2^{-2a}}{\Gamma^2(1-a)} I_1 F_4\left(1 - \frac{\alpha}{4}, 1 - \frac{\alpha}{4}, \frac{1}{2}, \frac{1}{2}; -\frac{4 \pi^6 n^2 y^2}{D_\mathcal{K}} \right) \right.
\]

\[
- \frac{4 \pi^6 n^2 y^2}{y^2 D_\mathcal{K}} \right]^{a/2} \left\{ \cos\left(\frac{\pi a}{2}\right) \text{ber}\left(4 \pi \sqrt{\frac{2 n \pi y}{y D_\mathcal{K}}}\right) - \sin\left(\frac{\pi a}{2}\right) \text{bei}\left(4 \pi \sqrt{\frac{2 n \pi y}{y D_\mathcal{K}}}\right) \right\}.
\]

Invoking the above evaluation of \( I_{\mathcal{K},a}(n) \) into (1.1), we conclude our result for \( 0 < \text{Re}(a) < 1 \) and \( y > 0 \). It remains to show next that the result is also valid for \( \text{Re}(a) > -1 \) and \( \text{Re}(y) > 0 \). The result in [9] Corollary 7.19, p. 430] implies that \( \sigma_{\mathcal{K},-a}(n) \leq \sum_{d \mid n} \sigma_0(d) d^{-a} \), where \( \sigma_0(d) \) is the divisor function \( d(n) \). Using the elementary bound of divisor function we can bound \( \sigma_{\mathcal{K},-a}(n) \) as

\[
\sigma_{\mathcal{K},-a}(n) \ll \begin{cases} n^\epsilon & \text{for } \text{Re}(a) > 0 \\ n^{\epsilon - \text{Re}(a)} & \text{for } \text{Re}(a) < 0 \end{cases} \tag{4.9}
\]

where \( \epsilon \) is arbitrarily small positive quantity. We next employ the bounds from Lemma [4.1] and (4.9) together to conclude that the series on the right hand side of (1.8) converges uniformly as long as \( \text{Re}(a) > -1 \). Since the summand of the series is analytic for \( \text{Re}(a) > -1 \), by Weierstrass’ theorem on analytic functions, we see that it represents an analytic function of \( a \) when \( \text{Re}(a) > -1 \).

On the other hand, the left-hand side of (1.8) is also analytic for \( \text{Re}(a) > -1 \), hence by the principle of analytic continuation (1.8) holds for \( \text{Re}(a) > -1 \) and \( y > 0 \). The both sides of (1.8) are seen to be analytic as a function of \( y \), in \( \text{Re}(y) > 0 \). Therefore the principle of analytic continuation concludes (1.8) for \( \text{Re}(a) > -1 \) and \( \text{Re}(y) > 0 \).

In the following lemma we prove an identity, which is crucial in proving Theorem [1.7]

**Lemma 4.2.** For any \( a, z \in \mathbb{C} \), we have

\[
1 F_4\left(1 - \frac{a}{2}, 1 - \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1; -z\right) = \Gamma^2(1-a) \sum_{k=0}^{m-1} \frac{(-1)^k (16z)^{-k-1}}{\Gamma^2(-1-a-2k)} \frac{1}{2^{4m} \Gamma^2(1-a-2m)} \times 1 F_4\left(1 - \frac{a}{2} - m, 1 - \frac{1}{2} - m, \frac{1-a}{2} - m, \frac{1-a}{2} - m; 1; -z\right). \tag{4.10}
\]

**Proof.** We use the following reduction formula repeatedly for \( 1 F_4 \), which is given by

\[
1 F_4\left(\frac{a+1}{b_1+1, b_2+1, b_3+1, b_4+1}; x\right) = - \frac{b_1 b_2 b_3 b_4}{x} \left[ 1 F_4\left(\frac{a}{b_1, b_2, b_3, b_4}; x\right) - 1 F_4\left(\frac{a+1}{b_1, b_2, b_3, b_4}; x\right) \right]. \tag{4.10}
\]
The above formula with \( a = 0 \), \( b_1 = b_2 = -\frac{a}{2} \), \( b_3 = b_4 = -\frac{1+a}{2} \) and \( x = -z \) provides

\[
1 F_4 \left( 1 - \frac{a}{2}, 1 - \frac{1}{2}, \frac{1-a}{2}, \frac{1-a}{2} \bigg| -z \right) = \frac{(-\frac{a}{2})^2 \left( -\frac{1+a}{2} \right)^2}{z} \left[ 1 - 1 F_4 \left( -\frac{a}{2}, -\frac{a}{2}, 1 + \frac{a}{2}, -\frac{1+a}{2} \bigg| -z \right) \right]
\]

\[
= \Gamma^2 \left( 1 - \frac{a}{2} \right) \Gamma^2 \left( -\frac{a}{2} \right) \Gamma^2 \left( \frac{1-a}{2} \right) \left[ (z)^{-1} \Gamma^2 \left( -\frac{a}{2} \right) \Gamma^2 \left( -\frac{1+a}{2} \right) - (z)^{-1} \Gamma^2 \left( -\frac{a}{2} - 1 \right) \Gamma^2 \left( -\frac{1+a}{2} \right) \right].
\]

Applying (4.10) on the right-hand side of the above equation, we get

\[
1 F_4 \left( 1 - \frac{a}{2}, 1 - \frac{1}{2}, \frac{1-a}{2}, \frac{1-a}{2} \bigg| -z \right) = \Gamma^2 \left( 1 - \frac{a}{2} \right) \Gamma^2 \left( -\frac{a}{2} \right) \Gamma^2 \left( \frac{1-a}{2} \right) \left[ (z)^{-1} \Gamma^2 \left( -\frac{a}{2} \right) \Gamma^2 \left( -\frac{1+a}{2} \right) - (z)^{-2} \Gamma^2 \left( -\frac{a}{2} - 1 \right) \Gamma^2 \left( -\frac{1+a}{2} \right) \right] + \frac{(z)^{-2}}{\Gamma^2 \left( -\frac{a}{2} - 1 \right) \Gamma^2 \left( -\frac{1+a}{2} \right)} 1 F_4 \left( -\frac{a}{2} - 1, -\frac{a}{2} - 1, -\frac{3+a}{2}, -\frac{3+a}{2} \bigg| -z \right).
\]

We repeat this process \( m \)-times and obtain

\[
1 F_4 \left( 1 - \frac{a}{2}, 1 - \frac{1}{2}, \frac{1-a}{2}, \frac{1-a}{2} \bigg| -z \right) = \Gamma^2 \left( 1 - \frac{a}{2} \right) \Gamma^2 \left( -\frac{a}{2} \right) \Gamma^2 \left( \frac{1-a}{2} \right) \left[ \sum_{j=1}^{m} (-1)^{j-1} \frac{(z)^{-j}}{\Gamma^2 \left( 1 - \frac{a}{2} - j \right) \Gamma^2 \left( \frac{1-a}{2} - j \right)} \right] + \frac{(z)^{m}}{\Gamma^2 \left( 1 - \frac{a}{2} - m \right) \Gamma^2 \left( \frac{1-a}{2} - m \right)} 1 F_4 \left( 1 - \frac{a}{2}, 1 - \frac{1}{2}, \frac{1-a}{2}, \frac{1-a}{2} \bigg| -z \right).
\]

Substituting \( j = k - 1 \) in the finite sum and applying (2.5), we conclude our lemma. \( \square \)

4.2. Proof of Theorem 1.7. For \( \text{Re}(a) > -1 \), we rewrite the identity (1.3) as

\[
\sum_{n=1}^{\infty} \sigma_{\mathcal{K},a}(n) e^{-ny} + \frac{1}{2} \zeta_{\mathcal{K}}(-a) - \frac{\zeta_{\mathcal{K}}(1-a)}{y} - \frac{2\pi h}{y \sqrt{D_{\mathcal{K}}}} \frac{\Gamma(a+1) \zeta(a+1)}{y^{a+1}} = 4\pi^2 2a D_{\mathcal{K}}^{-1} \sum_{n=1}^{\infty} \sigma_{\mathcal{K},-a}(n) \left( \frac{2^{-2a}}{\Gamma^2(1-a)} \right) 1 F_4 \left( 1 - \frac{a}{2}, 1 - \frac{1}{2}, \frac{1-a}{2}, \frac{1-a}{2} \bigg| -\frac{4\pi^2 a^2}{y^2 D_{\mathcal{K}}} \right) - \left( 4\pi^2 n^2 \right)^{\frac{1}{2}} \left( \cos \left( \frac{\pi a}{2} \right) \right) \ber(4\pi \sqrt{\frac{2n\pi}{y D_{\mathcal{K}}}} \cos \left( \frac{\pi a}{2} \right)) \ber(4\pi \sqrt{\frac{2n\pi}{y D_{\mathcal{K}}}})
\]

\[- \frac{1}{2\pi^{2a}} \sum_{k=0}^{m} \left( -1 \right)^k \frac{\left( 64\pi^6 n^2 \right)^{k-1}}{\Gamma^2(-1-a-2k)} + \left( \frac{2\pi}{\sin(\pi a)} \right) \sum_{n=1}^{\infty} \sigma_{\mathcal{K},-a}(n) \sum_{k=0}^{m} \left( -1 \right)^k \frac{\left( 64\pi^6 n^2 \right)^{k-1}}{\Gamma^2(-1-a-2k)} \zeta(2k+2) \zeta_{\mathcal{K}}(2k+a+2).
\]

The last term of the above expression can be simplified using (3.3) as

\[
\sum_{k=0}^{m} \left( -1 \right)^k \frac{\left( 64\pi^6 n^2 \right)^{k-1}}{\Gamma^2(-1-a-2k)} \sigma_{\mathcal{K},-a}(n) n^{2k+2} = \sum_{k=0}^{m} \left( -1 \right)^k \frac{\left( 64\pi^6 n^2 \right)^{k-1}}{\Gamma^2(-1-a-2k)} \zeta(2k+2) \zeta_{\mathcal{K}}(2k+a+2).
\]

Therefore for \( \text{Re}(a) > -1 \), (4.11) can be written as

\[
\sum_{n=1}^{\infty} \sigma_{\mathcal{K},a}(n) e^{-ny} + \frac{1}{2} \zeta_{\mathcal{K}}(-a) - \frac{\zeta_{\mathcal{K}}(1-a)}{y} - \frac{2\pi h}{y \sqrt{D_{\mathcal{K}}}} \frac{\Gamma(a+1) \zeta(a+1)}{y^{a+1}} = 4\pi^2 2a D_{\mathcal{K}}^{-1} \sum_{n=1}^{\infty} \sigma_{\mathcal{K},-a}(n) \left( \frac{2^{-2a}}{\Gamma^2(1-a)} \right) 1 F_4 \left( 1 - \frac{a}{2}, 1 - \frac{1}{2}, \frac{1-a}{2}, \frac{1-a}{2} \bigg| -\frac{4\pi^2 a^2}{y^2 D_{\mathcal{K}}} \right).
\]
\[
- \left( \frac{4\pi^6 n^2}{y^2 D_K^2} \right)^{\frac{1}{2}} \left( \cos \left( \frac{\pi a}{2} \right) \operatorname{ber} \left( 4\pi \sqrt{\frac{2n\pi}{y^2 D_K^2}} \right) - \sin \left( \frac{\pi a}{2} \right) \operatorname{bei} \left( 4\pi \sqrt{\frac{2n\pi}{y^2 D_K^2}} \right) \right) \\
- \frac{1}{2^{2a}} \sum_{k=0}^{m} \left( -1 \right)^k \left( \frac{64 \pi^6 \rho n^2}{y^2 D_K^2} \right)^{-k-1} \frac{y D_K^{2a+\frac{3}{2}}}{(2\pi)^{2a+4} \sin(\pi a)} \sum_{k=0}^{m} \frac{(-1)^k \zeta(2k+2) \zeta(2k+2a+2)}{\Gamma^2(-a-1-2k)} \left( \frac{8\pi^3}{y^2 D_K^2} \right)^{-2k} 
\]

Invoking Lemma 4.1 and (4.9), we have
\[
\sigma_{K,-a}(n) \left\{ \frac{2^{-2a}}{\Gamma^2(1-a)} \right\} F_4 \left( 1 - \frac{a}{2}, 1 - \frac{a}{2}, 1 - \frac{a}{2}, 1 - \frac{a}{2} \left| \frac{4\pi^6 n^2}{y^2 D_K^2} \right) \right) - \left( \frac{4\pi^6 n^2}{y^2 D_K^2} \right)^{\frac{1}{2}} \left( \cos \left( \frac{\pi a}{2} \right) \operatorname{ber} \left( 4\pi \sqrt{\frac{2n\pi}{y^2 D_K^2}} \right) \right) \\
- \frac{1}{2^{2a}} \sum_{k=0}^{m} \left( -1 \right)^k \left( \frac{64 \pi^6 \rho n^2}{y^2 D_K^2} \right)^{-k-1} \frac{y D_K^{2a+\frac{3}{2}}}{(2\pi)^{2a+4} \sin(\pi a)} \sum_{k=0}^{m} \left( -1 \right)^k \zeta(2k+2) \zeta(2k+2a+2) \frac{8\pi^3}{y^2 D_K^2} \right)^{-2k} 
\]

\[
\ll \begin{cases} 
\left| n^{-2m-4+\epsilon} \right| & \text{for } \Re(a) \geq 0 \\
\left| n^{-2m-4-\Re(a)+\epsilon} \right| & \text{for } \Re(a) < 0.
\end{cases}
\]

This shows that the infinite series on the right-hand side of (4.12) is uniformly convergent for \( \Re(a) \geq -2m - 3 + \epsilon \) where \( \epsilon > 0 \). Since the summand of the above series is analytic for \( \Re(a) > -2m - 3 \), it follows from Weierstrass' theorem that this series represents an analytic function of \( a \) for \( \Re(a) > -2m - 3 \).

The left-hand side of (4.12) as well as the finite sum on its right-hand side are also analytic for \( \Re(a) > -2m - 3 \), hence by the principle of analytic continuation, (4.12) holds for \( \Re(a) > -2m - 3 \) with \( m \geq 0 \). Finally we apply Lemma 4.3 in (4.12) to conclude our theorem.

The following lemma is crucial in proving the next results and seems new in the literature.

**Lemma 4.3.** Let \( \ell \in \mathbb{Z} \). Then for \( z > 0 \), we have
\[
\frac{d}{da} F_4 \left( 1 - \frac{a}{2} + \ell, 1 - \frac{a}{2} + \ell, 1 - a, 1 - a \left| -z^4 \right) \right\} \bigg|_{a=2\ell-1} = \frac{1}{2z^2} \left( \gamma - 1 + \log(2z) \right) \operatorname{bei}(4z) + \frac{\pi}{4} \operatorname{ber}(4z) + \frac{\pi}{2} \operatorname{bei}(4z) + 2 \operatorname{ker}(4z). 
\]

**Proof.** The series definition of \( F_4 \) and (2.5) yields
\[
F_4 \left( 1 - \frac{a}{2} + \ell, 1 - \frac{a}{2} + \ell, 1 - a, 1 - a \left| -z^4 \right) \right) = \sum_{k=0}^{\infty} \frac{1}{\left( 1 - a + 2\ell + 2k \right)^2 \Gamma^2(1 - a + 2\ell + 2k)(2z)^{4k}}. 
\]

Differentiating both sides of the above equation with respect to \( a \), we obtain
\[
\frac{d}{da} F_4 \left( 1 - \frac{a}{2} + \ell, 1 - \frac{a}{2} + \ell, 1 - a, 1 - a \left| -z^4 \right) \right) = -2 \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma^2(1 - a + 2\ell)(1 - a + 2\ell)}{\Gamma^2(1 - a + 2\ell + 2k)} (2z)^{4k} 
\]
Thus the above expression at $a = 2\ell - 1$ becomes

$$
\frac{d}{da} F_1 \left( 1 - \frac{a}{2} + \ell, 1 - \frac{a}{2} + \ell, \frac{1-a}{2} + \ell, \frac{1-a}{2} + \ell \right) \left| -z^4 \right| = 2(\gamma - 1) \sum_{k=0}^{\infty} \frac{(-1)^k (2z)^{4k}}{\Gamma^2(2k+2)} + 2 \sum_{k=0}^{\infty} \frac{(-1)^k \psi(2k+2)(2z)^{4k}}{\Gamma^2(2k+2)}.
$$

(4.15)

Applying (4.2), the first infinite series on the right hand side of the above equation reduces to

$$
\sum_{k=0}^{\infty} \frac{(-1)^k (2z)^{4k}}{\Gamma^2(2k+2)} = F_3 \left( \frac{3}{2}, \frac{3}{2}, 1 \left| -z^4 \right| \right) = \frac{1}{4z^2} \ber(4z),
$$

(4.16)

It follows from [10, p. 439, Formula 78] that for $t > 0$, we have

$$
\sum_{k=0}^{\infty} \frac{(-1)^k \psi(2k+2)(2z)^{4k}}{\Gamma^2(2k+2)} = \frac{4}{4z^2} \left\{ \pi \ber(2t) + 4 \log(t) \ber(2t) + 4 \kei(2t) \right\}.
$$

(4.17)

Thus (4.17) with $t = 2z$ and (4.16) together implies (4.14). We can also obtain (4.14) in a similar way by substituting $a$ by $2\ell$ in (4.15) and applying the following relation

$$
\sum_{k=0}^{\infty} \frac{\psi(2k+1)(-1)^k t^{4k}}{((2k)!)^2} = \log(t) \ber(2t) - \frac{\pi}{4} \ber(2t) + \ker(2t)
$$

for $t > 0$, which follows from [10, p. 439, Formula 77].

4.3. **Proof of Theorem 1.1.** The main idea here is to take limit on both sides of the identity in Theorem 1.7 as $a \to -1$. Therefore, it is sufficient to consider $m = 0$ in Theorem 1.7. Applying the Laurent series expansion at $s = 1$ of the functions $\zeta(s)$, $\zeta(s)$ and the expansion of $\Gamma(s)$ at $s = 0$, the following limit reduces as

$$
\lim_{a \to -1} \left\{ \frac{1}{2} \zeta(-a) - \frac{2\pi h}{\sqrt{D_K}} \frac{\Gamma(a+1)\zeta(a+1)}{y^{a+1}} \right\} = \frac{1}{2} \left\{ L'(1, \chi_D) + L(1, \chi_D) \left( 2\gamma - \log \left( \frac{2\pi}{y} \right) \right) \right\}.
$$

(4.18)

We next evaluate the following limit

$$
L_{-1} := \lim_{a \to -1} \frac{1}{\sin(\pi a)} \left\{ 2^{-2a} \Gamma^2(1-a) \left( \frac{1}{2} - \frac{a}{2} \right) \left( 1 - \frac{a}{2}, \frac{1-a}{2} \right) - \frac{4\pi^6 n^2}{y^2 D_K^2} - a^2(a+1)^2 \left( \frac{64\pi^6 n^2}{y^2 D_K^2} \right) \right\}
$$

$$
- \left( \frac{4\pi^6 n^2}{y^2 D_K^2} \right)^{\frac{3}{2}} \left\{ \cos \left( \frac{\pi a}{2} \right) \ber \left( \frac{2n\pi}{y D_K} \right) - \sin \left( \frac{\pi a}{2} \right) \be \left( \frac{2n\pi}{y D_K} \right) \right\}.
$$

(4.19)

For $a = -1$, it follows from (4.12) that

$$
2^{-2a} \Gamma^2(1-a) \left\{ 1F_4 \left( 1 - \frac{a}{2}, 1 - \frac{a}{2}, \frac{1-a}{2}, \frac{1-a}{2} \right) - \frac{4\pi^6 n^2}{y^2 D_K^2} - a^2(a+1)^2 \left( \frac{64\pi^6 n^2}{y^2 D_K^2} \right) \right\}
$$

$$
= \left( \frac{4\pi^6 n^2}{y^2 D_K^2} \right)^{\frac{3}{2}} \left\{ \cos \left( \frac{\pi a}{2} \right) \ber \left( \frac{2n\pi}{y D_K} \right) - \sin \left( \frac{\pi a}{2} \right) \be \left( \frac{2n\pi}{y D_K} \right) \right\}.
$$

Thus we have $0/0$ form in the limit (4.19). Applying L'Hopital’s rule and (4.13) with $\ell = 0$, the limit evaluates as

$$
L_{-1} = -\frac{y D_K}{n\pi} \ker \left( 4\pi \sqrt{\frac{2n\pi}{y D_K}} \right).
$$

(4.20)
Finally, taking limit as \( a \to -1 \) on the both sides of (1.8) and applying (1.18), (1.20) and (1.21) together we conclude our theorem.

4.4. Proof of Corollary 1.6. In this case, we need to take limit on both sides of the identity in Theorem 1.5 as \( a \to 0 \). It follows by change of variable that

\[
\lim_{a \to 0} \left\{ \frac{\zeta_K(1-a)}{y} + \frac{L(1,\chi_D)(a+1)n}{\Gamma(1-a)} \right\} = \frac{1}{y} \lim_{s \to 1} \left\{ \frac{\zeta_K(s)}{y} + \frac{L(1,\chi_D)(2-s)n}{\Gamma(1-s)} \right\}.
\]

Inserting the Laurent series expansion at \( s = 1 \) of the functions \( \zeta_K(s) \), \( \zeta(s) \) and \( \Gamma(s) \), the above limit reduces to

\[
\lim_{a \to 0} \left\{ \frac{\zeta_K(1-a)}{y} + \frac{L(1,\chi_D)(a+1)n}{\Gamma(1-a)} \right\} = \frac{L'(1,\chi_D) + L(1,\chi_D)(\gamma - \log(y))}{y}.
\]

We also have singularities in the first and second term of the summand on the right-hand side of (1.8). Therefore, we need to evaluate the following limit

\[
L_0 := \lim_{a \to 0} a \left\{ \frac{2^{-2a}\Gamma(a+1)}{\Gamma(1-a)} \left( \frac{4\pi^6 n^2}{y^2 D_K^2} \right)^{-\frac{a}{2}} \right\} 1F_4 \left( 1 - \frac{a}{2}, 1 - \frac{a}{2}, 1 - \frac{a}{2}, 1; 1 \mid -\frac{4\pi^6 n^2}{y^2 D_K^2} \right)
\]

where we have applied (2.4) on the gamma factors. It can be noted that for \( a = 0 \), we have

\[
\frac{2^{-2a}\Gamma(a+1)}{\Gamma(1-a)} \left( \frac{4\pi^6 n^2}{y^2 D_K^2} \right)^{-\frac{a}{2}} 1F_4 \left( 1 - \frac{a}{2}, 1 - \frac{a}{2}, 1 - \frac{a}{2}, 1; 1 \mid -\frac{4\pi^6 n^2}{y^2 D_K^2} \right)
\]

\[
= \Gamma \left( 1 - \frac{a}{2} \right) \Gamma \left( 1 + \frac{a}{2} \right) \left( \frac{4\pi^6 n^2}{y^2 D_K^2} \right)^{-\frac{a}{2}} \text{ber} \left( 4\pi \sqrt{\frac{2n\pi}{yD_K}} \right).
\]

thus we have \( 0/0 \) form in (4.23). Applying L’Hopital’s rule, we evaluate the limit as

\[
L_0 = \left( -2\gamma - \log \left( \frac{8\pi^3 n}{yD_K} \right) \right) \text{ber} \left( 4\pi \sqrt{\frac{2n\pi}{yD_K}} \right) + \frac{d}{da} \left( 1F_4 \left( 1 - \frac{a}{2}, 1 - \frac{a}{2}, 1 - \frac{a}{2}, 1; 1 \mid -\frac{4\pi^6 n^2}{y^2 D_K^2} \right) \right) \bigg|_{a=0}.
\]

We next employ (4.14) with \( \ell = 0 \) in the above equation to get

\[
L_0 = 2\text{ker} \left( 4\pi \sqrt{\frac{2n\pi}{yD_K}} \right) - \frac{\pi}{2} \text{bei} \left( 4\pi \sqrt{\frac{2n\pi}{yD_K}} \right).
\]

Finally, taking limit as \( a \to 0 \) on the both sides of (1.8) and applying \( \zeta_K(0) = -h/w \), (4.22) and (4.24) together conclude our corollary.

5. Analogues of transformation formulas for Eisenstein series

In this section, we mainly study the infinite series associated to \( \sigma_{K,a}(n) \), which is analogous to Eisenstein series.
5.1. Proof of Theorem 1.8. We first prove the identity (1.10) and for that we need to take the limit for \(a \to 2m - 1\) on the both sides of (1.8). Substituting \(a\) by \(a - 2m\) in Lemma 1.2 we apply it on the right-hand-side of (1.8) to obtain

\[
2^{1 + \frac{a}{2}} \pi^{1 - \frac{a}{2}} \frac{D_{K}^{2}}{y^{1 + \frac{a}{2}}} \sum_{n=1}^{\infty} \frac{\sigma_{\infty,-a}(n)}{n^{-\frac{a}{2}}} \left[ \frac{\Gamma \left( \frac{a}{2} \right) \Gamma \left( \frac{1 + a}{2} \right)}{\Gamma \left( 1 - \frac{a}{2} \right) \Gamma \left( \frac{1 - a}{2} \right)} \left( \frac{4\pi^{6} n^{2} y^{2}}{y^{2} D_{K}^{2}} \right)^{-\frac{a}{2}} - \frac{\Gamma \left( \frac{a}{2} \right) \Gamma \left( \frac{1 + a}{2} \right) \Gamma \left( 1 - \frac{a}{2} \right) \Gamma \left( \frac{1 - a}{2} \right)}{\Gamma \left( 1 - \frac{a}{2} \right) \Gamma \left( \frac{1 - a}{2} \right) \Gamma \left( 1 + \frac{a}{2} \right) \Gamma \left( \frac{1 + a}{2} \right)} \right] \times \left( \frac{4\pi^{6} n^{2} y^{2}}{y^{2} D_{K}^{2}} \right)^{-\frac{a}{2}} \sum_{k=0}^{m-2} \frac{(-1)^{k}}{\Gamma \left( 2 - \frac{a}{2} + k \right)^{2}} \left( \frac{4\pi^{6} n^{2} y^{2}}{y^{2} D_{K}^{2}} \right)^{k+1} \frac{1}{\Gamma \left( 1 - \frac{a}{2} + m \right)^{2} \Gamma \left( \frac{1 - a}{2} + m \right)^{2}} \right) \times \left( \frac{4\pi^{6} n^{2} y^{2}}{y^{2} D_{K}^{2}} \right)^{m} \left[ \left( \frac{\pi a}{2} \right) \text{ber} \left( 4\pi \sqrt{\frac{2\pi n}{y D_{K}}} \right) - \left( \frac{\pi a}{2} \right) \text{bei} \left( 4\pi \sqrt{\frac{2\pi n}{y D_{K}}} \right) \right].
\]

(5.1)

Firstly, we evaluate the limit

\[
L_{2m-1} := \lim_{a \to 2m-1} \frac{1}{(a-2m+1)} \left( (a-2m+1) \Gamma \left( \frac{a}{2} \right) \Gamma \left( \frac{1 + a}{2} \right) \Gamma \left( 1 - \frac{a}{2} \right) \Gamma \left( \frac{1 - a}{2} \right) \right) \left( \frac{4\pi^{6} n^{2} y^{2}}{y^{2} D_{K}^{2}} \right)^{m-\frac{a}{2}} \times \frac{(-1)^{m}}{\Gamma \left( 1 - \frac{a}{2} + m \right)^{2} \Gamma \left( \frac{1 - a}{2} + m \right)^{2}} \left[ 1 \right] F_{1} \left( 1 - \frac{a}{2} + m, 1 - \frac{a}{2} + m, 1 - \frac{a}{2} + m \left| - \frac{4\pi^{6} n^{2} y^{2}}{y^{2} D_{K}^{2}} \right. \right) \right.
\]

and

\[
L_{2m-1} = \lim_{a \to 2m-1} \frac{1}{(a-2m+1)} \left( 2 \Gamma \left( \frac{a}{2} \right) \Gamma \left( \frac{1 - a}{2} \right) \Gamma \left( \frac{1 + a}{2} \right) \Gamma \left( 1 - \frac{a}{2} \right) \right) \left( \frac{4\pi^{6} n^{2} y^{2}}{y^{2} D_{K}^{2}} \right)^{m-\frac{a}{2}} \times \left[ 1 \right] F_{1} \left( 1 - \frac{a}{2} + m, 1 - \frac{a}{2} + m, 1 - \frac{a}{2} + m \left| - \frac{4\pi^{6} n^{2} y^{2}}{y^{2} D_{K}^{2}} \right. \right)
\]

(5.2)

Using (2.4), the following gamma factors can be reduced as

\[
(a - 2m + 1) \Gamma \left( \frac{1 - a}{2} \right) \Gamma \left( \frac{1 + a}{2} \right) = 2(-1)^{m} \Gamma \left( \frac{a - 2m + 1}{2} + 1 \right) \Gamma \left( \frac{1 - a + 2m}{2} \right)
\]

(5.3)

and

\[
(a - 2m + 1) \Gamma \left( \frac{3 + a}{2} \right) = 2(-1)^{m+1} \Gamma \left( \frac{a - 2m + 1}{2} + 1 \right) \Gamma \left( \frac{1 - a + 2m}{2} \right).
\]

(5.4)

We then plug back (5.3) and (5.4) into (5.2) to obtain

\[
L_{2m-1} = \lim_{a \to 2m-1} \frac{1}{(a-2m+1)} \left( 2 \Gamma \left( \frac{a}{2} \right) \Gamma \left( \frac{1 - a}{2} \right) \Gamma \left( \frac{1 + a}{2} \right) \Gamma \left( 1 - \frac{a}{2} \right) \right) \left( \frac{4\pi^{6} n^{2} y^{2}}{y^{2} D_{K}^{2}} \right)^{m-\frac{a}{2}} \times \left[ 1 \right] F_{1} \left( 1 - \frac{a}{2} + m, 1 - \frac{a}{2} + m, 1 - \frac{a}{2} + m \left| - \frac{4\pi^{6} n^{2} y^{2}}{y^{2} D_{K}^{2}} \right. \right) \right.
\]

and

\[
+ 2(-1)^{m} \Gamma \left( \frac{a - 2m + 1}{2} + 1 \right) \Gamma \left( \frac{1 - a + 2m}{2} \right) \left( \frac{4\pi^{6} n^{2} y^{2}}{y^{2} D_{K}^{2}} \right)^{m-\frac{a}{2}} \times \left[ 1 \right] F_{1} \left( 1 - \frac{a}{2} + m, 1 - \frac{a}{2} + m, 1 - \frac{a}{2} + m \left| - \frac{4\pi^{6} n^{2} y^{2}}{y^{2} D_{K}^{2}} \right. \right)
\]

(5.5)

Applying (1.2) it is easy to see that for \(a = 2m - 1\), we have 0/0 form in (5.5), hence we use L’Hopital’s rule to evaluate the limit as

\[
L_{2m-1} = (-1)^{m} \left( \frac{2\pi^{3} n}{y D_{K}} \right)^{m-\frac{1}{2}} \left( 4(\gamma - 1) + \log \left( \frac{64\pi^{6} n^{2}}{y^{2} D_{K}^{2}} \right) \right) \times \left[ 1 \right] F_{1} \left( 1 - \frac{a}{2} + m, 1 - \frac{a}{2} + m, 1 - \frac{a}{2} + m \left| - \frac{4\pi^{6} n^{2} y^{2}}{y^{2} D_{K}^{2}} \right. \right) \right.
\]

and

\[
\times \frac{d}{da} \left[ 1 \right] F_{1} \left( 1 - \frac{a}{2} + m, 1 - \frac{a}{2} + m, 1 - \frac{a}{2} + m \left| - \frac{4\pi^{6} n^{2} y^{2}}{y^{2} D_{K}^{2}} \right. \right) \right.
\]

(5.6)
Substituting (4.13) into (5.6), we obtain

\[ L_{2m-1} = (-1)^{m+2} \pi^{3m} \left( \frac{n}{yD_k} \right)^{m-\frac{1}{2}} \left\{ \pi \text{ber} \left( 4\pi \sqrt{\frac{2n\pi}{yD_k}} \right) + 4\text{kei} \left( 4\pi \sqrt{\frac{2n\pi}{yD_k}} \right) \right\}. \]  

(5.7)

It is straightforward to see that

\[
\lim_{a \to 2m-1} \left\{ \frac{\Gamma \left( \frac{a}{2} \right) \Gamma \left( \frac{1+a}{2} \right) \left( 4\pi^2 n^2 \right)^{-\frac{a}{2}}}{\Gamma \left( \frac{1+a}{2} \right) \Gamma \left( \frac{1-a}{2} \right) \Gamma \left( \frac{1-a}{2} \right) \Gamma \left( \frac{1-a}{2} \right) \left( 4\pi^2 n^2 \right)^{-\frac{a}{2}}} \times \sum_{k=0}^{m-2} \frac{(-1)^k}{\Gamma \left( \frac{2-a}{2} + k \right)^2 \Gamma \left( \frac{3-a}{2} + k \right)^2} \left( 4\pi^2 n^2 \right)^{k+1} \right\} = 0. 

(5.8)

Thus taking limit as \( a \to 2m - 1 \) on the both sides of (1.8) and using the fact that \( \zeta_k(s) \) has zeros on the negative integers, (5.1), (5.7) and (5.8) together yield

\[
\sum_{n=1}^{\infty} \sigma_{2m-1}(n) e^{-ny} = \frac{L(1, \chi_D) \Gamma(2m) \zeta(2m)}{y^{2m}} + \frac{4(-1)^{m+1}}{\pi} \sum_{n=1}^{\infty} \sigma_{1-2m}(n) \text{kei} \left( 4\pi \sqrt{\frac{2n\pi}{yD_k}} \right). 

(5.9)

Finally, we substitute \( y \) by \( \frac{8\pi^2}{10\pi} \) with \( \alpha \beta = \frac{D_k^2}{10\pi} \) in (5.9) to conclude (1.10).

We next show the second part of our theorem. The idea of the proof goes along the similar direction as in the previous part by taking limit as \( a \to 2m \) on the both sides of (1.8). It follows from (5.1) that the following limit needs to be evaluated:

\[
L_{2m} := \lim_{a \to 2m} \frac{1}{(a-2m)} \left\{ (a-2m) \Gamma \left( \frac{1+a}{2} \right) \Gamma \left( \frac{1-a}{2} \right) \Gamma \left( \frac{1-a}{2} \right) \left( 4\pi^2 n^2 \right)^{m-\frac{1}{2}} \times \sum_{k=0}^{m-2} \frac{(-1)^k}{\Gamma \left( \frac{2-a}{2} + k \right)^2 \Gamma \left( \frac{3-a}{2} + k \right)^2} \left( 4\pi^2 n^2 \right)^{k+1} \right\}. 

(5.10)

\[
L_{2m} = \lim_{a \to 2m} \frac{1}{(a-2m)} \left\{ 2\Gamma \left( \frac{1+a}{2} - m \right) \Gamma \left( \frac{1-a}{2} + m \right) \right\} \left( 4\pi^2 n^2 \right)^{m-\frac{1}{2}} \times F_4 \left( 1 - \frac{a}{2} + m, 1 - \frac{a}{2} + m, 1 - \frac{a}{2} + m, 1 - \frac{a}{2} + m \right) \left( 4\pi^2 n^2 \right)^{-\frac{1}{2}} \text{ber} \left( 4\pi \sqrt{\frac{2n\pi}{yD_k}} \right). 

(5.11)

Equation (4.2) implies that the above equation reduces to 0/0 form for \( a = 2m \). Thus we can use L’Hopital’s rule to evaluate the limit as

\[
L_{2m} = (-1)^{m+1} 2^m \pi^{3m} \left( \frac{n}{yD_k} \right)^{m} \left\{ \text{ber} \left( 4\pi \sqrt{\frac{2n\pi}{yD_k}} \right) \left( 4\gamma + \log \left( \frac{64\pi^2 n^2}{y^2 D_k^2} \right) \right) \right\}.

\]

Equation (5.11) implies that the above equation reduces to 0/0 form for \( a = 2m \). Thus we can use L’Hopital’s rule to evaluate the limit as

\[
L_{2m} = (-1)^{m+1} 2^m \pi^{3m} \left( \frac{n}{yD_k} \right)^{m} \left\{ \text{ber} \left( 4\pi \sqrt{\frac{2n\pi}{yD_k}} \right) \left( 4\gamma + \log \left( \frac{64\pi^2 n^2}{y^2 D_k^2} \right) \right) \right\}.

\]

Equation (5.11) implies that the above equation reduces to 0/0 form for \( a = 2m \). Thus we can use L’Hopital’s rule to evaluate the limit as

\[
L_{2m} = (-1)^{m+1} 2^m \pi^{3m} \left( \frac{n}{yD_k} \right)^{m} \left\{ \text{ber} \left( 4\pi \sqrt{\frac{2n\pi}{yD_k}} \right) \left( 4\gamma + \log \left( \frac{64\pi^2 n^2}{y^2 D_k^2} \right) \right) \right\}.

\]
Proof of Theorem 1.3.  

6.1. The proof follows by taking derivative on the both sides of the functional equation (2.6).

Proof. Thus taking limit as \( n \to \infty \)

\[
\lim_{n \to \infty} \left\{ \frac{\Gamma (\frac{\alpha}{2}) \Gamma (\frac{1+\beta}{2})}{\Gamma (1 - \frac{\alpha}{2}) \Gamma (\frac{1-\beta}{2})} \left( \frac{4\pi^6 n^2}{y^2 D_{K}} \right)^{-\frac{\alpha}{4} - m} - \Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{1+\beta}{2} \right) \Gamma \left( \frac{1}{2} - \frac{\alpha}{2} - m \right) \right\} = 0.
\]

(5.11)

Thus taking limit as \( a \to 2m \) on the both sides of (1.8) and using the fact that \( \zeta_K(s) \) has zeros on the negative integers, (5.1), (5.10) and (5.11) together yield

\[
\sum_{n=1}^{\infty} \sigma_{K,2m}(n)e^{-ny} = \frac{L(1, \chi_D)(2m+1)\zeta(2m+1)}{y^{2m+1}} + \frac{4(-1)^m}{\sqrt{D_K}} \sum_{n=1}^{\infty} \sigma_{K,-2m}(n) \text{Ker} \left( 4\pi \sqrt{\frac{2n\pi}{yD_K}} \right).
\]

(5.12)

Finally, we substitute \( y \) by \( \frac{8\pi a^2}{D_K} \) with \( \alpha = \frac{D_K}{16\pi^2} \) in (5.12) to conclude (1.11).

6. Transformation formulas analogous to Ramanujan’s identity for \( \zeta(2m+1) \)

In this section, we exhibit an identity over imaginary quadratic field which is analogue to Ramanujan’s identity (1.2).

Lemma 6.1. For any natural number \( n > 1 \), we have

\[
\zeta_K'(1 - n) = (-1)^{n-1} F_K^{\frac{n-1}{2}} (2\pi)^{1-2n} ((n-1)!)^2 \zeta_K(n).
\]

Proof. The proof follows by taking derivative on the both sides of the functional equation (2.6). \( \square \)

6.1. Proof of Theorem 1.3. For the first part, the idea here is to take limit \( a \to -2m - 1 \) on the both sides of (1.9). We first evaluate the following limit:

\[
L_{-2m-1} := \lim_{a \to -2m-1} \frac{1}{a + 2m + 1} \left\{ (-1)^m \frac{2\pi^2}{\sin(\pi a)} \frac{\Gamma(a + 2m + 1)}{\Gamma(1 - \frac{\alpha}{2} - m)} \frac{4\pi^6 n^2}{y^2 D_{K}}^{-\frac{\alpha}{4} - m} \right\}
\]

\[
\times \left| F_4 \left( 1 - \frac{\alpha}{2} - m, 1 - \frac{\alpha}{2} - m, 1 - \frac{\alpha}{2} - m, 1 - \frac{\alpha}{2} - m ; 1 - \frac{\alpha}{2} - m \right) \right| - \frac{4\pi^6 n^2}{y^2 D_{K}}^{\frac{\alpha}{2}} \text{bei} \left( 4\pi \sqrt{\frac{2n\pi}{yD_K}} \right).
\]

Invoking (2.4) on the gamma factors, the above equation yields

\[
L_{-2m-1} = \lim_{a \to -2m-1} \frac{1}{a + 2m + 1} \left\{ (-1)^{m+1} \frac{2\pi^2}{\Gamma(1 - \frac{\alpha}{2}) \Gamma(1 - \frac{\alpha}{2} - m)} \frac{\Gamma(a + 2m + 2)\Gamma(-a - 2m)}{\Gamma(1 - \frac{\alpha}{2} - m)} \frac{4\pi^6 n^2}{y^2 D_{K}}^{-\frac{\alpha}{4} - m} \right\}
\]

\[
\times \left| F_4 \left( 1 - \frac{\alpha}{2} - m, 1 - \frac{\alpha}{2} - m, 1 - \frac{\alpha}{2} - m, 1 - \frac{\alpha}{2} - m ; 1 - \frac{\alpha}{2} - m \right) \right| - \frac{4\pi^6 n^2}{y^2 D_{K}}^{\frac{\alpha}{2}} \text{bei} \left( 4\pi \sqrt{\frac{2n\pi}{yD_K}} \right).
\]

(21)
We now apply (4.2) on the above equation to show that the above limit is of the form 0/0 for \( a = -2m - 1 \). Thus we can use L'Hopital's rule and apply \( \ell = -m \) to evaluate the limit as

\[
L_{-2m-1} = (-1)^{m+1} \left( \frac{2\pi^3 n}{yD_K} \right)^{-\frac{1}{2} - m} \left\{ \pi \text{ber} \left( 4\pi \sqrt{\frac{2n\pi}{yD_K}} \right) + 4\text{kei} \left( 4\pi \sqrt{\frac{2n\pi}{yD_K}} \right) \right\}. \tag{6.1}
\]

It can also be observed that

\[
\text{We next apply (4.2) on the above equation to show that the above limit is of the form 0/0 for a = -2m - 1, for 0 \leq k \leq m - 1 due to the zeros of \( \sin(\pi a) \) and \( \zeta_K(a + 2k + 2) \) but for \( k = m \), the term reduces to } \infty/\infty \text{ form. Thus we evaluate these two limits separately using L' Hopital rule as}
\]

\[
\lim_{a \to -2m-1} \left\{ \frac{2^{-a-3}}{\sin(\pi a)} \left( \frac{2\pi^3 n}{yD_K} \right)^{-\frac{1}{2} - m} \left( \frac{-1}{\Gamma^2(-a - 2m)} \left( \frac{4\pi^6 n^2}{yD_K^2} \right)^{-m} \right) \right\} = 0. \tag{6.2}
\]

We next investigate the last term on the right-hand side of (1.9) which is of the form 0/0 as \( a \to -2m - 1 \), substitute \( \sum_{k=0}^{m-1} \left( -1 \right)^k \zeta(2k + 2) \zeta(a + 2k + 2) \) as the previous part by taking limit as \( k \to -m \) due to the zeros of \( \sin(\pi a) \) and \( \zeta_K(a + 2k + 2) \). We next show the second part of our theorem. The idea of the proof goes along the similar direction as the previous part by taking limit as \( a \to -2m \) on both sides of (1.9), the evaluations (6.1), (6.2), (6.3) and (6.4) together yield

\[
\sum_{n=1}^{\infty} \sigma_{K,-2m-1}(n)e^{-ny} = -\frac{1}{2} \zeta_K(2m + 1) + \frac{1}{y} \zeta_K(2m + 2) + \frac{(-1)^m 2^{-2m} \zeta(2m + 1)}{wD_K \pi^{2m-1} y^{2m-2}} \left( \frac{8\pi^3}{yD_K} \right)^{-2m} + \frac{(-1)^{m+1} 2^{-2m} \pi^{-2m} \zeta(2m + 1)}{wD_K \pi^{2m-1} y^{2m-2}} \sum_{n=1}^{\infty} \frac{\sigma_{K,2m+1}(n)}{n^{2m+1}} \text{kei} \left( 4\pi \sqrt{\frac{2n\pi}{yD_K}} \right) \left( \frac{8\pi^3}{yD_K} \right)^{-2k}.
\tag{6.5}
\]

Finally, we replace \( k \) by \( m - k \) and apply Lemma 6.1 in the last term of the above equation then substitute \( y \) by \( \frac{8\pi^3}{yD_K} \) with \( \alpha \beta = \frac{D_K^2}{wD_K^2} \) in (6.5) to arrive at (1.6).

We next show the second part of our theorem. The idea of the proof goes along the similar direction as the previous part by taking limit as \( a \to -2m \) on both sides of (1.9). In this case, we need to determine the following limit:

\[
L_{-2m} := \lim_{a \to -2m} \left( \frac{1}{a + 2m} \right) \left( \frac{2(-1)^m \pi^2(a + 2m)}{\sin(\pi a) \Gamma^2 \left( 1 - m \right) \Gamma^2 \left( \frac{1-a}{2} - m \right)} \right) \left( \frac{4\pi^6 n^2}{yD_K^2} \right)^{-\frac{1}{2} - m} \times _1F_4 \left( \frac{1}{2} - a - m, 1 - \frac{a}{2} - m, \frac{1-a}{2} - m, \frac{1-a}{2} - m; \frac{4\pi^6 n^2}{yD_K^2} \right).
\]

\[ + (a + 2m) \Gamma \left( -\frac{a}{2} \right) \Gamma \left( 1 + \frac{a}{2} \right) \left( \frac{4\pi n^2}{y^2 D_k^2} \right)^{\frac{a}{2}} \mathrm{ber} \left( 4\pi \sqrt{\frac{2n\pi}{y D_k}} \right) \].

Applying (2.4) on the gamma factors of the above equation, we obtain

\[ L_{-2m} = \lim_{a \to -2m} \frac{1}{(a + 2m)} \left\{ \frac{2\pi(-1)^m \Gamma(a + 2m + 1)\Gamma(1 - a - 2m)}{\Gamma^2(1 - a - 2m)\Gamma^2(1 - a - 2m)} \right\} \left( \frac{4\pi n^2}{y^2 D_k^2} \right)^{-\frac{a}{2} - m} 
\times F_4 \left( 1 - \frac{a}{2} - m, 1 - \frac{a}{2} - m, \frac{1-a}{2} - m, \frac{1-a}{2} - m \right) \left( \frac{4\pi n^2}{y^2 D_k^2} \right)^{\frac{a}{2}} \mathrm{ber} \left( 4\pi \sqrt{\frac{2n\pi}{y D_k}} \right) \]

\[ - 2(-1)^m \Gamma \left( 1 + \frac{a + 2m}{2} \right) \Gamma \left( 1 - a + 2m \right) \left( \frac{4\pi n^2}{y^2 D_k^2} \right)^{\frac{a}{2}} \mathrm{ber} \left( 4\pi \sqrt{\frac{2n\pi}{y D_k}} \right) \].

It is clear by (4.2) that the above limit reduces to 0/0 form. Thus, L'Hopital’s rule is applicable to evaluate the limit. Applying it and using (4.4) with \( \ell = -m \) after simplification on the above limit, we have

\[ L_{-2m} = (-1)^m 2^{-m} \pi^{-3m} \left( \frac{y D_k}{n} \right)^m \left\{ 4 \ker \left( 4\pi \sqrt{\frac{2n\pi}{y D_k}} \right) - \pi \ber \left( 4\pi \sqrt{\frac{2n\pi}{y D_k}} \right) \right\}. \quad (6.6) \]

It is easy to see that

\[ \lim_{a \to -2m} \left\{ \frac{2^{-2a-3\pi}}{\sin(\pi a)} \left( \frac{2\pi n}{y D_k} \right)^{-\frac{a}{2} - 2} \frac{(-1)^m}{\Gamma^2(-a - 1 - 2m)} \left( \frac{4\pi n^2}{y^2 D_k^2} \right)^{-m} \right\} = 0. \quad (6.7) \]

Next, we evaluate the finite sum on the right-hand side of (1.9) as \( a \to -2m \), which is 0/0 form for \( 0 \leq k \leq m - 2 \) due to the zeros of \( \sin(\pi a) \) and \( \zeta_k(a + 2k + 2) \). The \( m \)-th term of the finite sum goes to zero as \( s \to -2m \) because of the double pole of \( \Gamma^2(-1 - a - 2k) \) in the denominator. Next we show that the addition of \( (m - 1) \)-th term of the finite sum and the fourth term on the left-hand side of (1.9) provides 0/0 form and for that we use the functional equation of \( \zeta(s) \) in the asymmetric form to obtain

\[ \frac{(-1)^{m-1} y D_k^{a + \frac{3}{2}} \zeta(2m) \zeta(2m + a)}{(2\pi)^{2a+4} \sin(\pi a) \Gamma^2(1 - a - 2m)} \left( \frac{64\pi^6}{y^2 D_k^2} \right)^{1-m} \frac{2\pi h \Gamma(a + 1) \zeta(a + 1)}{w \sqrt{D_k} y^{a+1}} \]

\[ = \frac{1}{2(2\pi)^{2a+4} \sin(\pi a) \Gamma^2(1 - a - 2m)} \left( \frac{64\pi^6}{y^2 D_k^2} \right)^{1-m} - \frac{2a+1}{w \sqrt{D_k} y^{a+1}} \frac{\pi a^2 + 2h \zeta(-a)}{w \sqrt{D_k} y^{a+1}}. \]

The fact \( \zeta_k(0) = \frac{-h}{w} \) exhibits that the term inside the bracket on the right-hand side of the above expression is 0 for \( a = -2m \). Thus we have 0/0 form on the above limit where we can apply L'Hopital’s rule to evaluate the limit as

\[ \lim_{a \to -2m} \left\{ \frac{(-1)^{m-1} y D_k^{a + \frac{3}{2}} \zeta(2m) \zeta(2m + a)}{(2\pi)^{2a+4} \sin(\pi a) \Gamma^2(1 - a - 2m)} \left( \frac{64\pi^6}{y^2 D_k^2} \right)^{1-m} \frac{2\pi h \Gamma(a + 1) \zeta(a + 1)}{w \sqrt{D_k} y^{a+1}} \right\} \]

\[ = \frac{2(-1)^m}{\pi} \left\{ \frac{(-1)^{m-1} y D_k^3 \zeta(2m)}{2(2\pi)^4} \left( \frac{64\pi^6}{y^2 D_k^2} \right)^{1-m} \left[ (-1)^m \zeta_k(0) \left( \frac{D_k}{4\pi^2} \right)^{-2m} \left( \log \left( \frac{D_k}{4\pi^2} \right) - 2\gamma \right) \right. \right. \]

\[ + \left. \left. (-1)^m \zeta_k(0) \left( \frac{D_k}{4\pi^2} \right)^{-2m} \right] + \frac{2\pi^2 h}{yw \sqrt{D_k}} \left( \frac{2\pi}{y} \right)^{-2m} \left( \zeta'(2m) - \zeta(2m) \log \left( \frac{2\pi}{y} \right) \right) \right\} \].
The facts $\zeta_{K}(0) = -\sqrt{\frac{\pi}{2\pi}}L(1, \chi D)$, $\zeta'_{K}(0) = \sqrt{\frac{\pi}{2\pi}}L'(1, \chi D) - \sqrt{\frac{\pi}{2\pi}}(\gamma - 2 \log \left(\frac{\sqrt{D_K}}{2\pi}\right))L(1, \chi D)$ and Proposition 2.3 reduce the above equation as

$$
\lim_{a \to -2m} \left\{ \frac{(-1)^{m-1}yD_{K}^{a+\frac{3}{2}}\zeta(2m)\zeta_K(2m+a)}{(2\pi)^{2a+4}\sin(\pi a)\Gamma^{2}(1-a-2m)} \left( \frac{64\pi^{6}}{y^{2}D_{K}^{2}} \right)^{1-m} - \frac{2\pi h\Gamma(a+1)\zeta(a+1)}{w\sqrt{D_{K}y^{a+1}}} \right\} 
$$

$$
= \frac{(-1)^{m}}{\pi} \left( \frac{y}{2\pi} \right)^{2m-1} \left\{ \left( \zeta'(2m) - \gamma \zeta(2m) - \zeta(2m) \log \left(\frac{2\pi}{y}\right) \right) L(1, \chi D) - L'(1, \chi D) \zeta(2m) \right\}. \quad (6.8)
$$

We also have

$$
\lim_{a \to -2m} \left\{ \frac{A_{K}^{a+\frac{3}{2}}y}{(2\pi)^{2a+4}\sin(\pi a)\Gamma^{2}(-1-a-2k)} \left( \frac{64\pi^{6}}{y^{2}D_{K}^{2}} \right)^{-k} \right\} 
$$

$$
= \frac{yD_{K}^{\frac{3}{2}-2m}}{\pi(2\pi)^{4-\delta}m} \sum_{k=0}^{m-2} \left\{ (-1)^{k} \frac{\zeta(2k+2)\zeta_K(2k-2m+2)}{\Gamma^{2}(2m-2k-1)} \frac{8\pi^{3}}{y^{2}D_{K}^{2}} \right\}. \quad (6.9)
$$

Invoking (6.6), (6.7), (6.8) and (6.9) and taking limit as $a \to -2m$ in Theorem 1.9, we arrive at

$$
\sum_{n=1}^{\infty} \sigma_{K,-2m}(n) e^{-ny} = \frac{1}{2} \zeta_{K}(2m) + \frac{1}{2} \zeta_{K}(2m+1) + \frac{4(-1)^{m}}{\pi D_{K}} \left( \frac{2\pi}{y} \right)^{1-2m} \sum_{n=1}^{\infty} \frac{\sigma_{K,2m}(n)}{n^{2m}} \ker \left( \frac{4\pi}{y^{2}D_{K}} \right) 
$$

$$
+ \frac{(-1)^{m}}{\pi} \left( \frac{y}{2\pi} \right)^{2m-1} \left\{ \left( \zeta'(2m) - \gamma \zeta(2m) - \zeta(2m) \log \left(\frac{2\pi}{y}\right) \right) L(1, \chi D) - L'(1, \chi D) \zeta(2m) \right\} 
$$

$$
+ \frac{y}{2} \left( \frac{2\pi}{\sqrt{D_{K}}} \right)^{4m-3} \sum_{k=0}^{m-2} \frac{(-1)^{k} \zeta(2k+2)\zeta_K(2k-2m+2)}{\Gamma^{2}(2m-2k-1)} \left( \frac{64\pi^{6}}{y^{2}D_{K}^{2}} \right)^{-k}. \quad (6.10)
$$

Finally, we replace $k$ by $m-1-k$ and apply Lemma 6.1 in the last term of the above equation then substitute $y$ by $\frac{8\pi^{2}a}{D_{K}}$ with $\alpha \beta = \frac{D_{K}^{2}}{16\pi^{2}}$ in (6.10) to conclude (1.7). This completes the proof of Theorem 1.3.

### 7. Concluding Remarks

Zagier [28] asked whether there is a formula for $\zeta_{K}(4), \zeta_{K}(6)$ etc. attached an arbitrary imaginary quadratic field $K$ similar to (1.3). He also remarked that the answer to this question is possible if one can prove the result using methods available in analytic number theory. Here, we have obtained the relation between two zeta values for any complex arguments in terms of infinite series. Moreover, Theorem 1.3 provides an explicit relation between even and odd zeta values over an imaginary quadratic field. Thus the expression of $\zeta_{K}(2)$ (cf. Theorem 1.1) together with Theorem 1.3 expresses any zeta value at positive integers over an imaginary quadratic field in terms of Lambert series:

$$
\sum_{n=1}^{\infty} \sigma_{K,a}(n) e^{-ny}, \quad \sum_{n=1}^{\infty} \sigma_{K,a}(n) \ker(\sqrt{ny}) \quad \text{and} \quad \sum_{n=1}^{\infty} \sigma_{K,a}(n) \kei(\sqrt{ny}).
$$

These series demand independent study since their behaviour may lead to some important information about the arithmetic nature of Dedekind zeta function over an imaginary quadratic field.

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