Lie algebroids associated to Poisson actions

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1 Introduction

This work is motivated by a result of Drinfeld in [Dr2]. Recall [Dr1] that a Poisson Lie group is a Lie group $G$ together with a Poisson structure such that the group multiplication map

$$G \times G \rightarrow G$$

is a Poisson map. Given a Poisson Lie group $G$ and a Poisson manifold $P$, an action

$$\sigma : G \times P \rightarrow P$$

of $G$ on $P$ is called a Poisson action if the action map $\sigma$ is a Poisson map. When the action is transitive, we say that $P$ is a Poisson homogeneous $G$-space. Poisson $G$-spaces are the semi-classical analogs of quantum spaces with quantum group actions. Special cases of Poisson homogeneous $G$-spaces can be found in [Da-So] [Lu1] [Za].

Let $P$ be a Poisson homogeneous $G$-space. In [Dr2], Drinfeld shows that corresponding to each $p \in P$, there is a maximal isotropic Lie subalgebra $t_p$ of the Lie algebra $\mathfrak{g}$, the double Lie algebra of the tangent Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ of $G$. Moreover, for $g \in G$, the two Lie algebras $t_p$ and $t_g p$ are related by $t_g p = Ad_g t_p$ via the Adjoint action of $G$ on $\mathfrak{g}$. In particular, they are isomorphic as Lie algebras. The Lie algebra $t_p$ determines the Poisson structure on $P$, and it can be used to classify Poisson homogeneous $G$-spaces [Dr2].

The purpose of this note is to find an invariant setting for these Lie algebra $t_p$’s. We construct, for every Poisson manifold $P$ with a Poisson $G$-action (not necessarily $G$-homogeneous), a Lie algebroid structure on the direct sum vector bundle $(P \times \mathfrak{g}) \oplus T^*P$.
over \(P\). This Lie algebroid will be denoted by \(A = (P \times g) \triangleright \triangleright T^*P\) to indicate the fact that it is built out of the two Lie algebroids \(P \times g\) and \(T^*P\) and a pair of representations of them on each other. Moreover, the Lie algebroid \(A\) is naturally equipped with an action of \(G\), making it into a Harish-Chandra Lie algebroid of \(G\) \(\mathfrak{g}-\mathfrak{B-B}\). It follows immediately that the kernel \(t_p\) of the anchor map of \(A\) at each \(p \in P\) has a natural Lie algebra structure, and \(t_{gp} = g \cdot t_p\), where \(g \cdot t_p\) comes from the action of \(G\) on \(A\).

When the anchor map of \(A\) has full rank everywhere (it is said to be transitive in such a case), the subbundle of \(A\) defined by the kernel of the anchor map is a Lie algebra bundle in the sense that local trivializations exist \([\text{Mcz1}]\). This is the case when \(P\) is \(G\)-homogeneous or when \(P\) is symplectic. In both cases, each Lie algebra \(t_p\) can be naturally embedded in the Lie algebra \(\mathfrak{a}\), the double of the tangent Lie bialgebra of \(G\), as a maximal isotropic Lie subalgebra.

When \(P\) is \(G\)-homogeneous, these maximal isotropic Lie subalgebras of \(\mathfrak{a}\) are precisely those described in \([\text{Dr2}]\).

When \(P\) is symplectic, we show that the Lie algebra \(t_p\)'s define an isomorphic family of Lie bialgebras over \(\mathfrak{g}^*\).

As further applications, we describe the symplectic leaves of a Poisson homogeneous \(G\)-space \(P\) in terms of the Lie algebra \(t_p\). We show that the \(G\)-invariant Poisson cohomology of \(P\) can be realized as relative Lie algebra cohomology of \(t_p\) relative to the stabilizer subgroup of \(G\) at \(p\). As an example, we calculate the \(K\)-invariant Poisson cohomology of the Bruhat Poisson structure \([\text{Lu-We}]\) on the generalized flag manifold \(K/T\), where \(K\) is a compact semisimple Lie group and \(T \subset K\) is a maximal torus of \(K\), and we show that there is exactly one cohomology class for each element in the Weyl group of \(K\). We also describe the Lie groupoid corresponding to the Lie algebroid \(A = (P \times g) \triangleright \triangleright T^*P\) and show that it is an example of a double Lie groupoid in the sense of Mackenzie \([\text{Mcz2}]\).

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2 Lie algebroids

In this section, we collect some relevant facts about Lie algebroids. More systematic treatments of them can be found in [Mcz1] [H-M] and [X-M].

A Lie algebroid over a smooth manifold $P$ is a vector bundle $A$ over $P$ together with a Lie algebra structure on the space $\Gamma(A)$ of smooth sections of $A$ and a bundle map $\rho : A \to TP$ (called the anchor map of the Lie algebroid) such that

1) $\rho$ defines a Lie algebra homomorphism from $\Gamma(A)$ to the space $\chi(P)$ of vector fields with the commutator Lie algebra structure, and

2) for $f \in C^\infty(P), \omega_1, \omega_2 \in \Gamma(A)$ the following derivation law holds:

\[
\{\omega_1, f\omega_2\} = f\{\omega_1, \omega_2\} + (\rho(\omega_1)f)\omega_2.
\]

Immediate examples of Lie algebroids are 1) the tangent bundle $TP$ as a Lie algebroid over $P$ with the identity map as the anchor map, and 2) a Lie algebra $g$ as a Lie algebroid over a one point space.

Transformation Lie algebroids. Let $\sigma : G \times P \to P$ be an action of a Lie group $G$ on a manifold $P$. For each $x \in g$, the Lie algebra of $G$, denote by $\sigma_x$ the vector field on $P$ defined by

\[
\sigma_x(p) = \frac{d}{dt}|_{t=0}\sigma(\exp tx, p), \quad p \in P.
\]

Then there is a natural Lie algebroid structure on the trivial vector bundle $P \times g$ with $-\sigma$ as the anchor map. Here we regard $\sigma : x \mapsto \sigma_x$ as a bundle map from $P \times g$ to $TP$. The Lie bracket on the space $\Gamma(P \times g) \cong C^\infty(P, g)$ of smooth sections of $P \times g$ is given by

\[
\{\bar{x}, \bar{y}\} = [\bar{x}, \bar{y}]_g - \sigma_x \cdot \bar{y} + \sigma_y \cdot \bar{x},
\]

(1)

where the first term on the right hand side denotes the pointwise Lie bracket in $g$, and the second term denotes the derivative of the $g$-valued function $\bar{y}$ in the direction of the vector field $\sigma_x$.

Cotangent bundle Lie algebroids. Let $(P, \pi)$ be a Poisson manifold with Poisson bivector field $\pi$. We use $\pi^\#$ to denote the bundle map

\[
\pi^\#(p) : T^*_pP \to T_pP : \alpha_p \mapsto -\alpha_p \cdot \pi(p),
\]

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or
\[ (\beta_p, \pi^\#(\alpha_p)) = \pi(p)(\beta_p, \alpha_p), \quad \alpha_p, \beta_p \in T^*_p P. \]
(Note the sign convention here). Then, with \(-\pi^\#\) as the anchor map, the cotangent bundle \(T^*P\) becomes a Lie algebroid over \(P\), where the Lie bracket on the space \(\Omega^1(P)\) of 1-forms on \(P\) is given by
\[
\{\alpha, \beta\} = d\pi(\alpha, \beta) - \pi^\# \alpha \bigwedge d\beta + \pi^\# \beta \bigwedge d\alpha \quad (2)
\]
\[
= -d\pi(\alpha, \beta) - L_{\pi^\# \alpha} \beta + L_{\pi^\# \beta} \alpha, \quad (3)
\]
where \(L_{\pi^\# \alpha} \beta\) denotes the Lie derivative of the 1-form \(\beta\) in the direction of the vector field \(\pi^\# \alpha\).

**Lie algebroid morphisms.** The definition of Lie algebroid morphisms between Lie algebroids over different bases is rather involved \([H-M]\) \([X-M]\), the reason being that a bundle map does not necessarily induce a map between sections. We will only need the definition for the following special case. Let \(A\) be a Lie algebroid over \(P\) with anchor map \(\rho\). Let \(\mathfrak{g}\) be a Lie algebra, considered as a Lie algebroid over a one point space. A smooth map
\[
\phi : A \to \mathfrak{g}
\]
which is linear on each fiber is said to be a Lie algebroid morphism if for any sections \(\omega_1\) and \(\omega_2\) of \(A\),
\[
\phi(\omega_1, \omega_2) = [\phi(\omega_1), \phi(\omega_2)]_\mathfrak{g} + \rho(\omega_1) \cdot \phi(\omega_2) - \rho(\omega_2) \cdot \phi(\omega_1),
\]
where the first term on the right hand side denotes the pointwise Lie bracket in \(\mathfrak{g}\), and the second term denotes the Lie derivative of the \(\mathfrak{g}\)-valued function \(\phi(\omega_2)\) in the direction of the vector field \(\rho(\omega_1)\).

**Example 2.1** In the case of a transformation Lie algebroid \(A = P \times \mathfrak{g}\), the projection map \(P \times \mathfrak{g} \to \mathfrak{g}\) is a Lie algebroid morphism.

**Representations of Lie algebroids.** Let \(A\) be a Lie algebroid over \(P\) with anchor map \(\rho\). Let \(E\) be a vector bundle over \(P\). A representation of \(A\) on \(E\) is a \(k\)-linear map
\[
\Gamma(A) \otimes \Gamma(E) \to \Gamma(E) : a \otimes s \mapsto D_\alpha s,
\]
where $\Gamma(A)$ and $\Gamma(E)$ denote respectively the spaces of smooth sections of $A$ and $E$, such that for any $a, b \in \Gamma(A), s \in \Gamma(E)$ and $f \in C^\infty(P)$,

1. $D_{fa}(s) = f D_{a}s$;
2. $D_{a}(fs) = f D_{a}s + (\rho(a)f)s$;
3. $D_{a}(D_{b}s) - D_{b}(D_{a}s) = D_{\{a,b\}}s$.

If $(\cdot, \cdot)$ is a smooth section of $S^2(E^*)$, the second symmetric power of the dual bundle of $E$, we say that the representation of $A$ on $E$ is $(\cdot, \cdot)$-preserving if for any $a \in \Gamma(A), s_1, s_2 \in \Gamma(E),$

$$\rho(a)(s_1, s_2) = \langle D_{a}s_1, s_2 \rangle + \langle s_1, D_{a}s_2 \rangle.$$

**Example 2.2** A representation of the tangent bundle $TP$ over $P$ is a vector bundle $E$ over $P$ together with a flat linear connection.

**Example 2.3** Any transformation Lie algebroid $P \times \mathfrak{g}$ has a natural representation on the tangent bundle $TP$ of $P$ via

$$D_{\bar{x}}V = -[\sigma_{\bar{x}}, V] - \sigma_{V \cdot \bar{x}},$$

(4)

where $V$ is a vector field on $P$, $\bar{x}$ is a section of $P \times \mathfrak{g}$, considered as a $\mathfrak{g}$-valued function of $P$, and $V \cdot \bar{x}$ denotes the Lie derivative of $\bar{x}$ in the direction of $V$. Correspondingly, there a representation of $P \times \mathfrak{g}$ on the cotangent bundle $T^*P$ satisfying

$$-\sigma_{\bar{x}}(\alpha, V) = \langle D_{\bar{x}}\alpha, V \rangle + \langle \alpha, D_{\bar{x}}V \rangle$$

(5)

for any 1-form $\alpha$ on $P$. Or, equivalently,

$$\langle D_{\bar{x}}\alpha, V \rangle = -\sigma_{\bar{x}}(\alpha, V) + \langle \alpha, [\sigma_{\bar{x}}, V] + \sigma_{V \cdot \bar{x}} \rangle.$$  

(6)

Notice that when $\bar{x}$ is a constant section of $P \times \mathfrak{g}$ corresponding to $x \in \mathfrak{g}$, the 1-form $D_{\bar{x}}\alpha$ is given by the Lie derivative

$$D_{\bar{x}}\alpha = -L_{\sigma_x}\alpha.$$
Example 2.4 Let $A$ be a Lie algebroid over $P$ with anchor map $\rho$. Suppose that $\phi : A \to g$ is a Lie algebroid morphism from $A$ to a Lie algebra $g$. Then for any vector space $U$ with a $g$-action, there is a representation of $A$ on the trivial vector bundle $P \times U$ given by

$$D_a \bar{u} = \rho(a) \cdot \bar{u} + \phi(a)(\bar{u}),$$

(7)

where $a$ is a section of $A$, $\bar{u}$ is a section of $P \times U$, and $\phi(a)(\bar{u})$ denotes the action of $\phi(a) \in C^\infty(P, g)$ on $\bar{u} \in C^\infty(P, U)$ taken pointwise over $P$. In particular, there is a representation of $A$ on the trivial vector bundle $P \times g^*$ given by

$$D_a \bar{\xi} = \rho(a) \cdot \bar{\xi} + ad^*_\phi\xi,$$

where $\bar{\xi} \in C^\infty(P, g^*)$ is a section of $P \times g^*$, and the co-adjoint representation of $g$ on $g^*$ is defined by

$$(ad^*_\xi, y) = -(\xi, [x, y]), \quad x, y \in g, \ \xi \in g^*.$$

**Lie algebroid cohomology** Let $A$ be a Lie algebroid over $P$ with anchor map $\rho$. Let $E$ be a representation of $A$. Define

$$d : \Gamma(\wedge^{k-1} A^* \otimes E) \longrightarrow \Gamma(\wedge^k A^* \otimes E), \quad k = 1, 2, 3, ...$$

by

$$df(a_1, ..., a_k) = \sum_{i} (-1)^{i+1} D_{a_i} f(a_1, ..., \hat{a_i}, ..., a_k) + \sum_{i<j} (-1)^{i+j} f(\{a_i, a_j\}, ..., \hat{a_i}, ..., \hat{a_j}, ..., a_k).$$

(8)

Then $d$ is well-defined and that $d^2 = 0$. The cohomology of $(C^\bullet(A, E), d)$ is called the Lie algebroid cohomology of $A$ with coefficients in $E$, and it is denoted by $H^\bullet(A, E)$. Elements of $H^0(A, E)$ are called $A$-parallel sections of $E$. When $E$ is the trivial 1-dimensional vector bundle with the representation of $A$ given by

$$D_a f = \rho(a) f, \quad f \in C^\infty(P),$$

the cohomology of $A$ with coefficients in $E$ is called the cohomology of $A$ with trivial coefficients.

In the case when $A = TP$ is the tangent bundle of $P$, the Lie algebroid cohomology of $A$ with trivial coefficients is nothing but the de Rham cohomology of $P$. 
In the case when \( A = T^* P \) is the cotangent bundle Lie algebroid of a Poisson manifold \((P, \pi)\), the Lie algebroid cohomology of \( A \) with trivial coefficients is called the Poisson cohomology of \((P, \pi)\). The cochain complex is the space \( \Gamma(\wedge \cdot TP) \) of multi-vector fields on \( P \), and the coboundary operator \( d \) can be expressed as \( d = [\pi, \cdot] \), where \([, ,]\) stands for the Schouten bracket \([\text{Ku}]\) on the multi-vector fields on \( P \) \([\text{Li}]\).

In the case when \( A = P \times \mathfrak{g} \) is a transformation Lie algebroid, the Lie algebra cohomology of \( A \) with trivial coefficients is simply the Lie algebra cohomology of \( \mathfrak{g} \) with coefficients in \( C^\infty(P) \).

**Transitive Lie algebroids.** A Lie algebroid \( A \) over a manifold \( P \) with anchor map \( \rho : A \rightarrow TP \) is said to be transitive if \( \rho \) has full rank at every point of \( P \). In this case, the subbundle \( L \) of \( A \) defined by the kernel of \( \rho \) is a Lie algebra bundle in the sense that each fiber has a Lie algebra structure and local trivializations exist \([\text{Mcz}]\). Any section \( \gamma : TP \rightarrow A \) of \( \rho \) defines a linear connection \( \nabla \) of \( L \) by

\[
\nabla^\gamma_V(l) = \{\gamma(V), \ l\}, \quad V \in \chi(P), \ l \in \Gamma(L),
\]

and it satisfies

\[
\nabla^\gamma_V[l_1, l_2] = [\nabla^\gamma_Vl_1, l_2] + [l_1, \nabla^\gamma_Vl_2], \quad \forall l_1, l_2 \in \Gamma(L). \quad (9)
\]

3 Poisson Lie groups and Poisson actions

In this section, we collect some facts about Poisson Lie groups that will be used in this paper. Details can be found in \([\text{Dr}1\] \([\text{STS}\] \([\text{KS}\] and \([\text{Lu1}\].

Assume that \((G, \pi_G)\) is a Poisson Lie group with Poisson bi-vector field \( \pi_G \). For each \( g \in G \), the element \( r_{g^{-1}} \pi_G(g) \) is in \( \wedge^2 \mathfrak{g} \), where \( r_{g^{-1}} \) denotes the right translation on \( \wedge^2 TG \) by \( g^{-1} \). The derivative of the map

\[
G \rightarrow \wedge^2 \mathfrak{g} : \quad g \mapsto r_{g^{-1}} \pi_G(g)
\]

at the identity element \( e \) of \( G \) is denoted by \( \delta \), so we have

\[
\delta : \quad \mathfrak{g} \rightarrow \wedge^2 \mathfrak{g} : \quad \delta(x) = \frac{d}{dt} |_{t=0} r_{\exp(-tx)} \pi_G(\exp tx).
\]

The dual map of \( \delta \) defines a Lie bracket on \( \mathfrak{g}^* \), i.e.,

\[
([\xi, \eta], x) = (\delta(x), \xi \wedge \eta), \quad x \in \mathfrak{g}, \ \xi, \eta \in \mathfrak{g}^*.
\]

\( (\text{Ku}) \)
The pair \((g, \delta)\) (or the pair \((g, g^*)\)) is called the tangent Lie bialgebra of the Poisson Lie group \(G\). For \(x \in g\) and \(\xi \in g^*\), denote by \(ad^*_x \xi \in g^*\) and \(ad^*_\xi x \in g\) respectively the elements given by

\[
\langle ad^*_x \xi, y \rangle = -\langle \xi, [x, y] \rangle \quad \forall y \in g \\
\langle ad^*_\xi x, \eta \rangle = -\langle x, [\xi, \eta] \rangle \quad \forall \eta \in g^*.
\]

The double Lie algebra \(\mathfrak{d}\) is the vector space \(g \oplus g^*\) with the Lie bracket

\[
[x_1 + \xi_1, x_2 + \xi_2] = [x_1, x_2] + ad^*_{\xi_1} x_2 - ad^*_{\xi_2} x_1 + [\xi_1, \xi_2] + ad^*_x \xi_1 - ad^*_x \xi_2. \quad (12)
\]

It is the unique Lie bracket on the vector space \(g \oplus g^*\) characterized by the properties that 1) when restricted to \(g\) and \(g^*\), it coincides with the given Lie brackets on \(g\) and \(g^*\), and 2) the scalar product \(\langle \cdot, \cdot \rangle\) on \(g \oplus g^*\) defined by

\[
\langle x_1 + \xi_1, x_2 + \xi_2 \rangle_{\mathfrak{d}} = (x_1, \xi_2) + (\xi_1, x_2), \quad x_1, x_2 \in g, \xi_1, \xi_2 \in g^* \quad (13)
\]

is ad-invariant with respect to this Lie bracket. We use \(\mathfrak{d} = g \bowtie g^*\) to denote this Lie algebra, indicating the fact that the Lie bracket is built out of the Lie brackets on \(g\) and on \(g^*\) and their co-adjoint actions on each other.

Let \(G^*\) be the simply-connected Lie group with Lie algebra \(g^*\). Then it is also a Poisson Lie group with tangent Lie bialgebra \((g^*, g)\). It is called the dual Poisson Lie group of \(G\).

Assume that \(D\) is a connected Lie group with Lie algebra \(\mathfrak{d}\). Assume also that both \(G\) and \(G^*\) can be embedded in \(D\) as subgroups corresponding to the Lie algebra inclusions \(g \subset \mathfrak{d}\) and \(g^* \subset \mathfrak{d}\). Assume furthermore that every element of \(D\) can be uniquely written as a product \(gu\) for some \(g \in G\) and \(u \in G^*\). Then we can identify \(G\) with the quotient space \(D/G^*\). Consequently, there is a left action of \(D\) on \(G\) by left translations. We will denote it by

\[
D \times G \longrightarrow G : \quad (d, g) \longmapsto ^d g. \quad (14)
\]

Infinitesimally, this defines a Lie algebra anti-homomorphism from \(\mathfrak{d}\) to the Lie algebra \(\chi(G)\) of vector fields on \(G\):

\[
\mathfrak{d} \longrightarrow \chi(G) : \quad x + \xi \longmapsto ^{\rho_{x+\xi}}(g) = r_g x - r_g \left(\xi \land (r_{g^{-1}} \pi_G(g))\right), \quad g \in G. \quad (15)
\]

This action of \(\mathfrak{d}\) on \(G\) will be used in Example 5.2. The vector fields \(-\rho_\xi, \xi \in g^*\), are called the right dressing vector fields on \(G\). When the global decomposition \(D = GG^*\) does not hold, we will assume that these vector fields integrate to a left action of \(D\) on \(G\).
Similarly, by identifying $G$ with the quotient space $G^* \backslash D$, we get a right action of $D$ on $G$ which we denote by

$$G \times D \longrightarrow G : (g, d) \longmapsto gd. \quad (16)$$

The corresponding infinitesimal right action of $D$ on $G$ is given by

$$\mathfrak{d} \longrightarrow \chi(G) : x + \xi \longmapsto \lambda x + \xi(g) = \lambda g x + r_g(Ad_{g^{-1}}^\ast \xi \rfloor (r_{g^{-1}} \pi_G(g))). \quad (17)$$

This action will be used in Theorem 7.2. The vector fields $-\lambda \xi$, for $\xi \in \mathfrak{g}^*$, are called the left dressing vector fields on $G$.[Lu1].

On the other hand, the group $G$ acts on $\mathfrak{d}$ by Lie algebra automorphisms via

$$Ad_g(x + \xi) = Ad_g x + (Ad_g^*_{g^{-1}} \xi) \rfloor (r_{g^{-1}} \pi_G(g)) + Ad_g^* \xi. \quad (18)$$

The corresponding action of $\mathfrak{g}$ on $\mathfrak{d}$ is simply the adjoint action of $\mathfrak{g}$ on itself restricted to $\mathfrak{g}$. This action will be used in Proposition 4.4. Similarly, the group $G^*$ acts on $\mathfrak{d}$ by Lie algebra automorphisms via

$$Ad_u(x + \xi) = Ad_{u^{-1}} x + Ad_u \xi + (Ad_{u^{-1}}^* x) \rfloor (r_{u^{-1}} \pi_{G^*}(u)). \quad (19)$$

This action will be used in Examples 4.7 and 4.8.

We now turn to Poisson actions by $G$. For any action $\sigma : G \times P \rightarrow P$ of $G$ on a manifold $P$, we can define

$$\phi : T^* P \longrightarrow \mathfrak{g}^* : (\phi(\alpha_p), x) = (\alpha_p, \sigma_x(p)), \quad x \in \mathfrak{g}, \alpha_p \in T^*_p P, \ p \in P. \quad (20)$$

Each 1-form $\alpha$ on $P$ then defines a $\mathfrak{g}^*$-valued function $\phi(\alpha)$ on $P$:

$$\phi(\alpha)(p) = \phi(\alpha_p).$$

Assume now that $(P, \pi)$ is a Poisson manifold and $G$ is a Poisson Lie group. Recall that an action $\sigma : G \times P \rightarrow P$ is said to be Poisson if for any $g \in G, p \in P$

$$\pi(gp) = g_* \pi(p) + p_* \pi_G(g),$$

where $p_*$ is the differential of the map from $G$ to $P$ given by $g \mapsto gp$. When $G$ is connected, this is equivalent to the following infinitesimal criterion [Lu-We]:

$$L_{\sigma_x} \pi = \sigma_{\delta(x)} \quad (21)$$
for all $x \in \mathfrak{g}$, where $\delta(x) \in \mathfrak{g} \wedge \mathfrak{g}$ is given by (10), and $\sigma \delta(x) = \sum_i \sigma x_i \wedge \sigma y_i$ if $\delta(x) = \sum_i x_i \wedge y_i$.

This is then equivalent to

$$\phi(\{\alpha, \beta\}) = [\phi(\alpha), \phi(\beta)]_{\mathfrak{g}^*} - \pi^#(\alpha) \cdot \phi(\beta) + \pi^#(\beta) \cdot \phi(\alpha)$$

(22)

for any 1-forms $\alpha$ and $\beta$ on $P$. In other words, we have the following Proposition:

**Proposition 3.1** Assume that $G$ is connected. The action $\sigma$ is a Poisson action if and only if the map

$$\phi : T^*P \rightarrow \mathfrak{g}^*$$

defined by (22) is a Lie algebroid morphism, where $T^*P$ has the cotangent bundle Lie algebroid structure defined by the Poisson structure $\pi$ on $P$, and $\mathfrak{g}^*$ has the Lie algebra structure given by (11), considered as a Lie algebroid over a one point space.

**4 The Lie algebroid $A = (P \times \mathfrak{g}) \bowtie T^*P$**

In this section, we describe, for every Poisson manifold $P$ with a Poisson action $\sigma : G \times P \rightarrow P$ of a Poisson Lie group $G$, a Lie algebroid structure over $P$ on the direct sum vector bundle $(P \times \mathfrak{g}) \oplus T^*P$.

First, the action of $G$ on $P$ defines a representation of the transformation Lie algebroid $P \times \mathfrak{g}$ on $T^*P$ given by (6) as in Example 2.3.

Secondly, using the Lie groupoid homomorphism

$$\phi : T^*P \rightarrow \mathfrak{g}^*$$

in Proposition 3.4, we get (see Example 2.4) a representation of the cotangent bundle Lie algebroid $T^*P$ on the trivial vector bundle $P \times \mathfrak{g}$ induced from the co-adjoint representation of $\mathfrak{g}^*$ on $\mathfrak{g}$. It is given by

$$D_\alpha \bar{x} = -\pi^#(\alpha) \cdot \bar{x} + \text{ad}_{\phi(\alpha)}^* \bar{x}$$

(23)

where $(\text{ad}_{\phi(\alpha)}^* \bar{x})(p) = \text{ad}_{\phi(\alpha)(p)}^* \bar{x}(p)$ for $p \in P$.

**Theorem 4.1** Let $\sigma : G \times P \rightarrow P$ be a Poisson action of the Poisson Lie group $G$ on the Poisson manifold $P$. Let

$$A \overset{\text{def}}{=} (P \times \mathfrak{g}) \oplus T^*P$$

(24)
be the direct sum vector bundle over \( P \). Then there is a Lie algebroid structure on \( A \) such that both the transformation Lie algebroid \((P \times \mathfrak{g}, -\sigma)\) and the cotangent bundle Lie algebroid \((T^*P, -\pi^\#)\) are Lie subalgebroids of \( A \). The anchor map of \( A \) is \(-\sigma - \pi^\#\). The Lie bracket between a section \( \bar{x} \) of \( P \times \mathfrak{g} \) and a section \( \alpha \) of \( T^*P \) is given by

\[
\{\bar{x}, \alpha\} = -D_{\alpha} \bar{x} + D_{\bar{x}} \alpha,
\]

where \( D_{\alpha} \bar{x} \) is given by (23) and \( D_{\bar{x}} \alpha \) by (6);

**Definition 4.2** We use \( A = (P \times \mathfrak{g}) \triangleleft \triangleright T^*P \) to denote this Lie algebroid, indicating the fact that the Lie bracket on the sections of \( A \) uses both the representation of \( P \times \mathfrak{g} \) on \( T^*P \) and the representation of \( T^*P \) on \( P \times \mathfrak{g} \).

Examples will be given in Section 5. The proof of Theorem 4.1 will be given in Section 6. We now discuss some properties of the Lie algebroid \( A \).

By considering each element of \( \mathfrak{g} \) as a constant section of \( A \), we get a Lie algebra homomorphism

\[
i : \mathfrak{g} \longrightarrow \Gamma(A) : i(x)_p = x, \quad \forall p \in P.
\]

It induces an action of \( \mathfrak{g} \) on \( \Gamma(A) \) by

\[
g \ni x : (\bar{y}, \beta) \mapsto \{i(x), (\bar{y}, \beta)\} = \left([x, \bar{y}]_{\mathfrak{g}} - \sigma_x \cdot \bar{y} - ad_{\phi(\beta)}^* x, \ - L_{\sigma_x} \beta\right),
\]

where \( \bar{y} \in C^\infty(P, \mathfrak{g}) \) and \( \beta \) is a 1-form on \( P \).

**Proposition 4.3** The following defines a left action of \( G \) on \( A \):

\[
g \cdot (x, \alpha_p) = (Ad_gx + Ad_g^{-1}\phi(\alpha_p), \bigcup_{p \in P}(r_g^{-1}\pi_c(g)), \ (g^{-1})^* \alpha_p), \quad g \in G, \ p \in P,
\]

where \( x \in \mathfrak{g}, \alpha_p \in T^*_pP \). It has the following properties:

1. The action of \( \mathfrak{g} \) on \( \Gamma(A) \) induced by this \( G \)-action on \( A \) coincides with the action given by (23).
2. The action of \( G \) commutes with the anchor map \(-\sigma - \pi^\# \) of \( A \), and \( G \) acts on \( \Gamma(A) \) as automorphisms with respect to the Lie bracket \( \{\ , \ \} \).
3. For any \( x \in \mathfrak{g} \) and \( g \in G \), \( i(Ad_gx) = g \cdot i(x) \).
   In other words, \( A \) is a Harish-Chandra \( G \)-Lie algebroid \([B-B]\).
4. The action of \( G \) preserves the scalar product on \( A \) given by

\[
\langle x + \alpha_p, \ y + \beta_p \rangle_p = (\beta_p, \sigma_x(p)) + (\alpha_p, \sigma_y(p)) = \phi(\beta)(x) + \phi(\alpha)(y).
\]
Proof. The proof is straightforward. We just point out that to prove (28) defines an action, one only needs the fact that the Poisson structure $\pi_G$ on $G$ makes $G$ into a Poisson Lie group, i.e.,

$$\pi_G(gh) = l_g \pi_G(h) + r_h \pi_G(g), \quad \forall g, h \in G.$$ 

In the proof of (1), one needs the fact that $\pi^#$ vanishes at the identity point of $G$. (2) is a reformulation of the fact that $\sigma$ is a Poisson action. (3) is obvious. (4) is straightforward.

Q.E.D.

Proposition 4.4 1) the map

$$\Phi : A = (P \times g) \bowtie T^* P \rightarrow \mathfrak{d} : \quad (x, \alpha_p) \mapsto x + \phi(\alpha_p)$$ (30)

is a Lie algebroid morphism from $A$ to the Lie algebra $\mathfrak{d} = g \bowtie g^*$, considered as a Lie algebroid over a one point space. With respect to the following action of $G$ on $\mathfrak{d}$ (see (13) in Section 3:

$$Ad_g(x + \xi) = Ad_g x + Ad_{g^{-1}}^* \xi \cdot (r_{g^{-1}} \pi_G(g)) + Ad_{g^{-1}}^* \xi,$$

where $g \in G, x \in g$ and $\xi \in g^*$, the map $\Phi$ is in fact a morphism of Harish-Chandra $G$-Lie algebroids;

2) The map $\Phi$ induces the following representation of $A$ on the trivial vector bundle $P \times \mathfrak{d}$ via the adjoint representation of $\mathfrak{d}$ on itself:

$$D^A_{x+\alpha}(\bar{y} + \bar{\xi}) = (-\sigma_x - \pi^# \alpha) \cdot (\bar{y} + \bar{\xi}) + [\bar{x} + \phi(\alpha), \bar{y} + \bar{\xi}]$$ (31)

where $\bar{y} + \bar{\xi}$ is a section of $P \times \mathfrak{d}$ with $\bar{y} \in C^\infty(P, g)$ and $\bar{\xi} \in C^\infty(P, g^*)$. This representation of $A$ on $P \times \mathfrak{d}$ preserves the symmetric product $\langle \cdot, \cdot \rangle_{\mathfrak{d}}$ on $P \times \mathfrak{d}$ given by (13) on each fiber.

Proof. 1) is easily reduced to the fact that for any $\bar{x} \in C^\infty(P, g)$ and any 1-form $\alpha$ on $P$, 

$$\phi(D_{\bar{x}}\alpha) = -\sigma_{\bar{x}} \cdot \phi(\alpha) + ad^*_x \phi(\alpha).$$ (32)

This identity can be proved directly from the definition of $D_{\bar{x}}\alpha$. If we equip the trivial vector bundle $P \times g^*$ the representation of the transformation Lie algebroid $P \times g$ given by

$$D_{\bar{x}}\bar{\xi} = -\sigma_{\bar{x}} \cdot \bar{\xi} + ad^*_x \bar{\xi}.$$
Then (32) is saying that the bundle map
\[ T^*P \longrightarrow P \times \mathfrak{g}^* : \alpha_p \longmapsto (p, \phi(\alpha_p)) \]
is a \((P \times \mathfrak{g})\)-morphism. Proof of 2) is straightforward.

Q.E.D.

In general, if \(A\) is any Lie algebroid over \(P\) with anchor map \(\rho\), the kernel \(\rho\) in each fiber \(A_p\) has an induced Lie algebra structure, namely, for any \(a_p, b_p \in \ker(\rho|_{A_p})\), extend them arbitrarily to sections \(a\) and \(b\) of \(A\) and define
\[
[a_p, b_p] = \{a, b\}(p).
\]
This is independent of the extensions.

For a transformation Lie algebroid \(P \times \mathfrak{g}\), the kernels of the anchor map \(-\sigma\) are the stabilizer Lie subalgebras
\[
\mathfrak{g}_p = \{x \in \mathfrak{g} : \sigma_x(p) = 0\},
\]
For the cotangent bundle Lie algebroid \(T^*P\) of a Poisson manifold \((P, \pi)\), we get the Lie algebra
\[
t_p = \{\alpha_p \in T^*_p P : \pi^#(\alpha_p) = 0\}
\]
at each point \(p \in P\). They are called the transversal Lie algebras to the symplectic leaves of the Poisson structure on \(P\).

For the Lie algebroid \(A = (P \times \mathfrak{g}) \bowtie T^*P\) constructed in Theorem 4.1, the kernel of the anchor map \(-\sigma - \pi^#\) in the fiber \(A_p\) over \(p \in P\) is the space
\[
l_p = \{(x, \alpha_p) : x \in \mathfrak{g}, \alpha_p \in T^*_p P : \sigma_x(p) + \pi^#(\alpha_p) = 0\}.
\]
It is an isotropic subspace of \(A_p\) with respect to \(\langle , \rangle_p\) given by (29).

**Proposition 4.5** (1) For each \(p \in P\), the map
\[
\Phi : l_p \longrightarrow \mathfrak{d} = \mathfrak{g} \bowtie \mathfrak{g}^* : (x, \alpha_p) \longmapsto x + \phi(\alpha_p)
\]
is a Lie algebra homomorphism from \(l_p\) to the double Lie algebra \(\mathfrak{d}\) of \((\mathfrak{g}, \mathfrak{g}^*)\).

(2) For any \(g \in G\) and \(p \in P\), we have \(g \cdot l_p = l_{gp}\), where \(g \cdot l_p\) denotes the image of \(l_p\) in \(A_{gp}\) under the action of \(g\) on \(A\) given by (28).
The proof of this Proposition is again straightforward.

**Proposition 4.6** When \( P \) is symplectic or when \( P \) is \( G \)-homogeneous, the Lie algebroid \( A = (P \times g) \triangleright T^*P \) is transitive. Consequently, the subbundle \( L \) of \( A \) defined by the kernel of the anchor map \(-\sigma - \pi^\#\) is a Lie algebra bundle of rank \( = \dim g \).

In general, the dimensions of the Lie algebra \( l_p \)'s may vary. Consider again the stabilizer Lie subalgebra \( g_p \) and the transversal Lie algebra \( t_p \) at each point \( p \in P \). Clearly,

\[
g_p \subset l_p, \quad t_p \subset l_p
\]
as Lie subalgebras, and

\[
g_p \cap t_p = 0.
\]

Set

\[
O_p = \{\sigma_x(p) : x \in g\} \subset T_pP, \quad S_p = \{\pi^\#(\alpha_p) : \alpha_p \in T^*_pP\} \subset T_pP.
\]

Then

\[
\dim t_p = \dim g + \dim P - \dim(O_p + S_p) = (\dim g - \dim O_p) + (\dim P - \dim S_p) + \dim(O_p \cap S_p) = \dim g_p + \dim t_p + \dim(O_p \cap S_p).
\]

**Example 4.7** When \( P \) is symplectic, i.e., when \( \pi^\# : T^*_pP \to T_pP \) is one-to-one for each \( p \in P \), the map \( \Phi : t_p \to \frak{d} \) is injective, so \( \Phi(t_p) \) is a maximal isotopic Lie subalgebra of \( \frak{d} \).

Let \( \omega \) be the symplectic 2-form on \( P \) related to \( \pi \) by

\[
\omega(\pi^\#\alpha, \pi^\#\beta) = \pi(\alpha, \beta).
\]

Then

\[
\Phi(t_p) = \{x - x \cdot \Phi(\omega_p) : x \in g\} \subset \frak{d},
\]

where \( \Phi(\omega_p) = (\Phi \wedge \Phi)(\omega_p) \subset g^* \wedge g^* \). Notice that \( \Phi(t_p) \) is transversal to \( g^* \) in \( \frak{d} \). Thus \( (\frak{d}, g^*, \Phi(t_p)) \) form a Manin triple [Dr1], and \((g^*, \Phi(t_p))\) becomes a Lie bialgebra.
Let \((G^*, \pi_{G^*})\) be the simply-connected Poisson Lie group dual to \(G\). The tangent Lie bialgebra of \(G^*\) is \((g^*, g)\). Define
\[
\pi_{G^*, p} = \pi_{G^*} + (\Phi(\omega_p))^r - (\Phi(\omega_p))^l,
\]
where \((\Phi(\omega_p))^r\) (resp. \((\Phi(\omega_p))^l\)) is the right invariant bi-vector field on \(G^*\) with value \(\Phi(\omega_p) \in g^* \wedge g^*\) at the identity element \(e \in G^*\). Then \((G^*, \pi_{G^*, p})\) is also a Poisson Lie group, and its tangent Lie bialgebra is \((g^*, \Phi(l_p))\) \cite{Lu1} \cite{Da-Sc}.

The bi-vector field \(\pi_{G^*} + (\Phi(\omega_p))^r\) is also Poisson. It is an example of an affine Poisson structure on \(G^*\) \cite{Lu1} \cite{Da-Sc}. When \(P\) is connected and simply-connected, it is shown in \cite{Lu1} that for any \(p \in P\) there exists a unique map \(J_p : P \to G^*\) such that

1. \(J_p(p) = e;\)
2. \(\pi^#(J_p^*x^l) = \sigma_x\) for every \(x \in g\), where \(x^l\) is the left invariant 1-form on \(G^*\) such that \(x^l(e) = x;\)
3. \(J_p : (P, \pi) \to (G^*, \pi_{G^*} + (\Phi(\omega_p))^r)\) is a Poisson map.

The map \(J_p\) is called a moment map for the Poisson action \(\sigma\) of \(G\) on \(P\) \cite{Lu2}.

If \(p_1\) is any other point in \(P\), the maps \(J_p\) and \(J_{p_1}\) are related by
\[
J_p(q) = J_p(p_1) J_{p_1}(q), \quad q \in P,
\]
and
\[
\Phi(\omega_p) = -r_{u^{-1}} \pi_{G^*}(u) + Ad_u \Phi(\omega_{p_1}).
\]
The Manin triples \((\mathfrak{d}, g^*, \Phi(l_p))\) and \((\mathfrak{d}, g^*, \Phi(l_{p_1}))\) are related by
\[
(\mathfrak{d}, g^*, \Phi(l_p)) = Ad_{J_p(p_1)} (\mathfrak{d}, g^*, \Phi(l_{p_1}))
\]
where \(J_p(p_1) \in G^*\) acts on \(\mathfrak{d}\) by \((19)\).

**Example 4.8** The left dressing vector fields on \(G^*\) are defined by
\[
\sigma_x(u) = \pi_{G^*}^#(x^l)
\]
where \(x \in g\). The map
\[
g \to \chi(G^*) : x \mapsto \sigma_x
\]
defines a Lie algebra anti-homomorphism from \(g\) to the Lie algebra \(\chi(G^*)\) of vector fields on \(G^*\). Assume that these vector fields integrate to an action of \(G\) on \(G^*\). It is called the
left dressing action of \( G \) on \( G^* \). It is a Poisson action. It in fact has the identity map \( G^* \to G^* \) as a moment map \(^2\). The orbits of the dressing action are exactly the symplectic leaves of \( G^* \). Let \( P \) be such an orbit. Then the action \( G \times P \to P \) of \( G \) on \( P \) is a Poisson action.

For \( u \in P \subset G^* \), the Lie algebra \( \mathfrak{t}_u \) can be identified with the Lie subalgebra \( \text{Ad}_{u^{-1}} \mathfrak{g} \) of \( \mathfrak{d} \), where, again, \( \text{Ad}_{u^{-1}} : \mathfrak{d} \to \mathfrak{d} \) denotes the action of \( u^{-1} \in G^* \) on \( \mathfrak{d} \) given by \(^3\).

**Example 4.9** When \( P \) is a homogeneous \( G \)-space, we have \( \dim \mathfrak{t}_p = \dim \mathfrak{g} \) for each \( p \in P \), and the map \( \Phi : \mathfrak{t}_p \to \mathfrak{d} \) is injective. Therefore, each \( \Phi(\mathfrak{t}_p) \) is a maximal isotropic Lie subalgebra of \( \mathfrak{d} \). They are used in \(^4\) to classify Poisson homogeneous \( G \)-spaces. We will treat them in more details in Section 7.

**Example 4.10** When the Poisson bi-vector field \( \pi \) on \( P \) vanishes at a certain point \( p \in P \), we have

\[
\mathfrak{t}_p = \{ (x, \alpha_p) : x \in \mathfrak{g}, \alpha_p \in T_p^*P, \sigma_x(p) = 0 \} \cong \mathfrak{g}_p \oplus T^*_p P.
\]

To describe the Lie algebra structure on \( \mathfrak{t}_p \), we first note that the action of \( G_p \) on \( P \), where \( G_p \) is the stabilizer subgroup of \( G \) at \( p \), linearizes to an action of \( G_p \) on \( T_p P \) and thus on \( T^*_p P \). Denote the corresponding action of \( \mathfrak{g}_p \) on \( T^*_p P \) by

\[
\mathfrak{g}_p \times T^*_p P \to T^*_p P : (x, \alpha_p) \mapsto x \cdot \alpha_p.
\]

On the other hand, since the action is Poisson, we know from \(^2\) that

\[
\sigma_{\delta(x)}(p) = 0, \quad \forall x \in \mathfrak{g}_p.
\]

Thus

\[
(x, [\phi(\alpha_p), \phi(\beta_p)]_{\mathfrak{g}^*}) = 0, \quad \forall x \in \mathfrak{g}_p, \alpha_p, \beta_p \in T^*_p P.
\]

In other words,

\[
ad^*_\phi(\alpha_p) \mathfrak{g}_p \subset \mathfrak{g}_p, \quad \forall \alpha_p \in T^*_p P.
\]

The Lie algebra structure on \( \mathfrak{t}_p \) is now given by

\[
[(x, \alpha_p), (y, \beta_p)] = ([x, y] + ad^*_\phi(\alpha_p)y - ad^*_\phi(\beta_p)x, \ [\alpha_p, \beta_p] + x \cdot \beta_p - y \cdot \alpha_p).
\]
Note that the Lie algebra $l_p = g_p + T^*_pP$ is an example of a double Lie algebra \([Lu-We]\) \([KS]\) (called a matched pair of Lie algebras in \([Mj]\)) in the sense that it contains $g_p$ and $T^*_pP$ as Lie subalgebras and it is isomorphic to $g_p \oplus T^*_pP$ as vector spaces. We can think of $l_p$ as being built out of the Lie algebras $g_p$ and $T^*_pP$ together with the action of $g_p$ on $T^*_pP$ given by \((40)\) and the action of $T^*_pP$ on $g_p$ by

$$T^*_pP \times g_p \longrightarrow g_p : (\alpha_p, x) \longmapsto ad^*_\alpha x.$$

**Example 4.11** Combining the situations in Example 4.9 and Example 4.10, we consider the case when $P$ is a Poisson homogeneous $G$ space and the Poisson bivector field $\pi$ on $P$ vanishes at some point $p \in P$. In this case, we can identify $P$ with the quotient space $G/G_p$, where $G_p$ is the stabilizer subgroup of $G$ at $p$. The Poisson structure $\pi$ is the unique one on $G/G_p$ such that the projection from $G$ to $G/G_p$ is a Poisson map. From Example 4.10 we see that $g^\perp_p = \{ \xi \in g^* : (\xi, x) = 0 \ \forall x \in g_p \} \subset g^*$ is a Lie subalgebra of $g^*$. The Lie subalgebra $g_p$ of $g$ is said to be coisotropic relative to the Lie bialgebra $(g, g^*)$ \([Lu-We]\). The Lie algebra $l_p$ is now isomorphic to the Lie subalgebra $g_p + g_p^\perp$ of $\mathfrak{d} = g \bowtie g^*$.

The fact that the Lie algebra $l_p = g_p + g_p^\perp$ is a double Lie algebra will be used in Section 7 to calculate the relative Lie algebra cohomology of $l_p$ relative to the Lie subalgebra $g_p$, which will be shown to be isomorphic to the $G$-invariant Poisson cohomology of the Poisson homogeneous space $G/G_p$. See Theorem 7.8.

A special case of this situation is when $g_p^\perp$ is a Lie ideal of $g^*$. In such a case, $G_p$ is a Poisson Lie subgroup of $G$. The Lie bracket on $g_p + g_p^\perp$ is a little simpler:

$$[x + \xi, y + \eta] = [x, y] + [\xi, \eta] + ad^*_\eta \xi - ad^*_\xi \eta$$

(42)

for $x, y \in g_p$ and $\xi, \eta \in g_p^\perp$. In particular, $g_p^\perp$ is a Lie ideal of $l_p \cong g_p + g_p^\perp$.

This example will be carried out further in Example 7.6 in Section 7.

5 Examples

In this section, we give examples of the Lie algebroid $A = (P \times g) \bowtie T^*P$ constructed in Section 4.
Example 5.1 Any Lie group $G$ can be regarded as a Poisson Lie group with the zero Poisson structure. In this case, an action of $G$ on a Poisson manifold is Poisson if and only if each $g \in G$ acts on $P$ as a Poisson diffeomorphism. In particular, each $x \in \mathfrak{g}$ acts on the space of 1-forms on $P$ as a derivation. The corresponding Lie algebroid structure on the vector bundle $(P \times \mathfrak{g}) \oplus T^*P$ described in Theorem 4.1 becomes a semi-direct product Lie algebroid structure \[H-M\].

Example 5.2 Let $G$ be a Poisson Lie group. Consider the left action of $G$ on itself by left translations. This is a Poisson action by definition. By trivializing the cotangent bundle $T^*G$ to $G \times \mathfrak{g}^*$ via,

$$T^*G \rightarrow G \times \mathfrak{g}^* : \xi_g \mapsto \phi(\xi_g) = r_g^* \xi_g, \quad g \in G, \xi_g \in T^*_gG,$$

we can trivialize the vector bundle $(P \times \mathfrak{g}) \oplus T^*G$ to $G \times (\mathfrak{g} \oplus \mathfrak{g}^*) = G \times \mathfrak{d}$ by the map

$$\Phi : (x, \xi_g) \mapsto x + \phi(\xi_g) = x + r_g^* \xi_g, \quad g \in G, \xi_g \in T^*_gG.$$

In this case, the Lie algebroid structure on the bundle $A$ described in Theorem 4.1 is the transformation Lie algebroid structure on $G \times \mathfrak{d}$ given by the left infinitesimal action of $\mathfrak{d}$ on $G$ given by \[13\].

The maximal isotropic Lie subalgebra $\mathfrak{t}_g$ of $\mathfrak{d}$ in this case is simply the Lie subalgebra $Ad_g \mathfrak{g}^*$, the image of $\mathfrak{g}^*$ under the action of $g$ on $\mathfrak{d}$ given by \[18\].

Example 5.3 Assume that the action $\sigma : G \times P \rightarrow P$ is transitive, so $P$ is a Poisson homogeneous $G$-space. In this case, the map

$$T^*P \rightarrow P \times \mathfrak{g}^* : \alpha_p \mapsto (p, \phi(\alpha_p))$$

embeds $T^*P$ into $P \times \mathfrak{g}^*$ as a subbundle whose fiber over the point $p \in P$ is

$$\mathfrak{g}^*_p = \{ \xi \in \mathfrak{g}^* : (\xi, x) = 0, \forall x \in \mathfrak{g}_p \}.$$

Thus via the map

$$\Phi : A = (P \times \mathfrak{g}) \bowtie T^*P \rightarrow \mathfrak{d} = \mathfrak{g} \bowtie \mathfrak{g}^* : (x, \alpha_p) \mapsto x + \phi(\alpha_p)$$

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we can also embed \( A \) into \( P \times (\mathfrak{g} \oplus \mathfrak{g}^*) \) as a subbundle whose fiber over \( p \in P \) is \( \mathfrak{g} \oplus \mathfrak{g}_p^* \). Under such an embedding, a section of \( A \) has the form \( a = \bar{x} + \bar{\xi} \) where \( \bar{x} \in C^\infty(P, \mathfrak{g}) \) and \( \bar{\xi} \in C^\infty(P, \mathfrak{g}^*) \) is such that \( \bar{\xi}(p) \in \mathfrak{g}_p^* \) for all \( p \in P \). Denote by \( \pi^\#(\bar{\xi}) \) the vector field \( \pi^\#(\alpha) \) if \( \phi(\alpha) = \bar{\xi} \). The anchor of \( A \) is then the vector field \(-\sigma_{\bar{x}} - \pi^\#(\bar{\xi})\). Since \( \Phi \) is a Lie algebroid morphism from \( A \) to the Lie algebra \( \mathfrak{d} = \mathfrak{g} \triangleright \triangleleft \mathfrak{g}^* \) by Proposition 4.4, the Lie bracket on the sections of \( A \) now looks like that for a transformation Lie algebroid:

\[
\{ \bar{x} + \bar{\xi}, \ y + \bar{\eta} \} = [\bar{x} + \bar{\xi}, \ y + \bar{\eta}]_g - (\sigma_{\bar{x}} + \pi^\#(\bar{\xi})) \cdot (\bar{y} + \bar{\eta}) + (\sigma_{\bar{y}} + \pi^\#(\bar{\eta})) \cdot (\bar{x} + \bar{\xi}).
\]

Example 5.4 Consider now the example of the left dressing action of \( G \) on its dual Poisson Lie group \( (G^*, \pi_{G^*}) \) (see Example 4.8). Identify \( T^* G^* \) with the trivial vector bundle \( G^* \times \mathfrak{g} \) by left translations on \( G \). Then the cotangent bundle Lie algebroid \( T^* G^* \) becomes the same as the transformation Lie algebroid \( G^* \times \mathfrak{g} \) defined by the left dressing action of \( \mathfrak{g} \) on \( G^* \). Thus the Lie algebroid \( A \), whose underlying vector bundle is now identified with the trivial vector bundle \( G^* \times (\mathfrak{g} \oplus \mathfrak{g}) \), is built out of two copies of the transformation Lie algebroid \( G^* \times \mathfrak{g} \) and a pair of representations \( D \) and \( D' \) of it on itself. These two representations are respectively given by

\[
D_{\bar{x}} \bar{y} = -\sigma_{\bar{x}} \cdot \bar{y} + [\bar{x}, \bar{y}]_g + \text{ad}^*_{\tau(\bar{y})}(\bar{x})
\]

\[
D'_{\bar{y}} \bar{x} = -\sigma_{\bar{y}} \cdot \bar{x} + \text{ad}^*_{\tau(\bar{y})}(\bar{x}),
\]

where

\[
\tau : C^\infty(G^*, \mathfrak{g}) \rightarrow C^\infty(G^* \mathfrak{g}^*) : \tau(\bar{y})(u) = \bar{y}(u) \cdot (l_u^{-1} \pi_{G^*}(u)), \quad u \in G^*.
\]

The resulting Lie algeroid structure on \( G^* \times (\mathfrak{g} \oplus \mathfrak{g}) \) has the property that the bundle map

\[
G^* \times (\mathfrak{g} \oplus \mathfrak{g}) \rightarrow G^* \times \mathfrak{g} : (x_u, y_u) \mapsto x_u + y_u
\]

is a Lie algebroid homomorphism.

6 Proof of Theorem

We give the proof of Theorem 4.1 in this section. Following the definition of a Lie algebroid, we need to prove
1) for any \( f \in C^\infty(P) \), \( \bar{x} \in C^\infty(P, g) \) and any 1-form \( \alpha \) on \( P \),
\[
\{ f \bar{x}, \alpha \} = f \{ \bar{x}, \alpha \} + (\pi^\#(\alpha)f)\bar{x} \\
\{ \bar{x}, f\alpha \} = f \{ \bar{x}, \alpha \} - (\sigma_{\bar{x}} f)\alpha;
\]

2) the bracket on the sections of \( A \) described in Theorem 1.1 satisfies the Jacobi identity;

3) the map \(-\sigma - \pi^\#\) is a Lie algebra homomorphism from the Lie algebra of sections of \( A \) to the Lie algebra of vector fields on \( P \).

1) follows from the fact that \( D_\alpha \bar{x} \) and \( D_{\bar{x}} \alpha \) define representations of Lie algebroids. This fact can also be used to reduce 2) to Lemma 6.1 and 3) to Lemma 6.2 as follows.

**Lemma 6.1** For any \( \bar{x}, \bar{y} \in C^\infty(P, g) \) and any 1-forms \( \alpha \) and \( \beta \) on \( P \), we have
\[
D_\alpha \{ \bar{x}, \bar{y} \} = \{ D_\alpha \bar{x}, \bar{y} \} + \{ \bar{x}, D_\alpha \bar{y} \} + D_{D_{\bar{y}}\alpha} \bar{x} - D_{D_{\bar{x}}\alpha} \bar{y} \quad (47)
\]
\[
D_{\bar{x}} \{ \alpha, \beta \} = \{ D_{\bar{x}} \alpha, \beta \} + \{ \alpha, D_{\bar{x}} \beta \} + D_{D_{\bar{x}}\beta} \alpha - D_{D_{\bar{x}}\alpha} \beta. \quad (48)
\]

**Lemma 6.2** For any \( \bar{x} \in C^\infty(P, g) \) and any 1-form \( \alpha \) on \( P \), we have
\[
\sigma_{D_{\bar{x}}\bar{x}} - \pi^\#(D_{\bar{x}}\alpha) = [\sigma_{\bar{x}}, \pi^\#(\alpha)]. \quad (49)
\]

We prove Lemma 6.2 first, since it is easier to prove and it will be used in the proof of Lemma 6.1.

**Proof of Lemma 6.2** Fix the 1-form \( \alpha \) on \( P \). For any \( \bar{x} \in C^\infty(P, g) \), set
\[
V_\alpha(\bar{x}) = \sigma_{D_{\bar{x}}\bar{x}} - \pi^\#(D_{\bar{x}}\alpha) - [\sigma_{\bar{x}}, \pi^\#(\alpha)].
\]
It follows from the axioms for Lie algebroid representations that
\[
V_\alpha(f\bar{x}) = fV_\alpha(\bar{x})
\]
for any \( f \in C^\infty(P) \). Thus it suffices to show that \( V_\alpha(x) = 0 \) for any constant function \( x \in C^\infty(P, g) \) corresponding to \( x \in g \). In other words, we need to show that
\[
\sigma_{ad^\psi(x)x} + \pi^\#(L_{\sigma_x} \alpha) - [\sigma_x, \pi^\#(\alpha)] = 0 \quad (50)
\]
for any \( x \in g \). By pairing the left hand side with an arbitrary 1-form \( \beta \) on \( P \), we see that this is equivalent to
\[
(L_{\sigma_x} \pi)(\alpha, \beta) = (x, [\phi(\alpha), \phi(\beta)]_{\psi^*}),
\]
which is just the infinitesimal condition (21) for the action \( \sigma \) to be Poisson.
Proof of Lemma 6.1. For $\bar{x}, \bar{y}, \alpha$ and $\beta$ as given, set

$$V_\alpha(\bar{x}, \bar{y}) = D_\alpha\{\bar{x}, \bar{y}\} - \{D_\alpha\bar{x}, \bar{y}\} - \{\bar{x}, D_\alpha\bar{y}\} - D_D\bar{y}\alpha\bar{x} + D_D\bar{x}\alpha\bar{y}$$

$$W_{\alpha,\beta}(\bar{x}) = D_{\bar{x}}\{\alpha, \beta\} - \{D_{\bar{x}}\alpha, \beta\} - \{\alpha, D_{\bar{x}}\beta\} - D_{D_{\bar{x}}}\alpha + D_{D_{\bar{x}}}\beta.$$

Using Identity (50) and the axioms for Lie algebroid representations, we see that

$$V_\alpha(f\bar{x}, \bar{y}) = V_\alpha(\bar{x}, f\bar{y}) = fV_\alpha(\bar{x}, \bar{y})$$

for any $f \in C^\infty(P)$. Thus it suffices to show that $V_\alpha(x, y) = 0$ for any constant functions $x, y \in C^\infty(P, g)$ corresponding to $x, y \in g$. But this can be deduced from the following fact about the Lie bialgebra $(g, g^*)$:

$$ad^*_\xi[x, y] = [ad^*_\xi x, y] + [x, ad^*_\xi y] + ad^*_\xi [x, y] - ad^*_\xi x$$

for any $x, y \in g$ and $\xi \in g^*$. The proof of (51) can be obtained from the explicit formula (12) for the Lie bracket on the double $\mathfrak{d}$ of $(g, g^*)$. It can also be found in [Am].

For $W_{\alpha,\beta}$, it again follows from the axioms for Lie algebroid representations that

$$W_{\alpha,\beta}(f\bar{x}) = fW_{\alpha,\beta}(\bar{x})$$

for any $f \in C^\infty(P)$. Thus it suffices to show that $W_{\alpha,\beta}(x) = 0$ for constant functions $x \in C^\infty(P, g)$ corresponding to $x \in g$. In other words, we need to show that

$$L_{\sigma_x}\{\alpha, \beta\} - \{L_{\sigma_x}\alpha, \beta\} - \{\alpha, L_{\sigma_x}\beta\} + D_{\tilde{x}_\beta}\alpha - D_{\tilde{x}_\alpha}\beta = 0,$$

where

$$\tilde{x}_\beta = ad^*_{\phi(\beta)}x, \quad \tilde{x}_\alpha = ad^*_{\phi(\alpha)}x.$$

Using Formula (3) for the Lie bracket on 1-forms on $P$ and Identity (50), we can reduce $W_{\alpha,\beta}(x) = 0$ to

$$(L_{\sigma_x}\pi)(\alpha, \beta) = (x, [\phi(\alpha), \phi(\beta)]_g)$$

which is nothing but the infinitesimal criterion (21) for $\sigma$ being Poisson.

Q.E.D.
7 Poisson homogeneous spaces

In this section, we assume that $G$ is a connected Poisson Lie group and that $P$ is a homogeneous $G$-space with a Poisson structure $\pi$ such that the action of $G$ on $P$ is Poisson. We will treat two aspects of such a Poisson manifold: its symplectic leaves and its $G$-invariant Poisson cohomology. We first look at its symplectic leaves.

For each $p \in P$, embed $l_p$ into $\mathfrak{d}$ via

$$\Phi : l_p \longrightarrow \mathfrak{d} : (x, \alpha_p) \mapsto x + \phi(\alpha).$$

Then we get a Lie algebra homomorphism from $l_p$ to $\chi(G)$:

$$l_p \longrightarrow \chi(G) : (x, \alpha_p) \mapsto \lambda_{x+\phi(\alpha_p)}(g) = r_g \left( Ad_g x + (Ad_g^\ast \phi(\alpha_p) \right) \right).$$

Recall (see (17) in Section 3) that the vector fields $\lambda_{x+\xi}$, for $x \in \mathfrak{g}$, $\xi \in \mathfrak{g}^*$, define a right action of $\mathfrak{d}$ on $G$. Set

$$\sigma_p : G \longrightarrow P : g \mapsto gp. \tag{52}$$

Lemma 7.1 For any $p \in P$, $g \in G$, the image of the subspace

$$\{\lambda_{x+\phi(\alpha_p)}(g) : (x, \alpha_p) \in l_p\} \subset T_g G$$

of $T_g G$ under the differential of the map $\sigma_p$ is exactly the subspace $im(\pi^\# | T_{gp}^* P)$ of $T_{gp} P$.

Proof. We know from Proposition 4.3 and Proposition 4.7 that

$$\sigma_p(\lambda_{x+\phi(\alpha_p)}(g)) = - \pi^\# ((g^{-1})^\ast \alpha_p) \in im(\pi^\# | T_{gp}^* P).$$

The fact that we get all elements in $im(\pi^\# | T_{gp}^* P)$ this way is because the action of $G$ on $P$ is transitive.

Q.E.D.

We now have the following description of the symplectic leaves in $P$.

Theorem 7.2 Fix any $p \in P$. Assume that $L_p$ is a connected Lie group and that the vector fields

$$\lambda_{x+\phi(\alpha_p)}, \quad (x, \alpha_p) \in l_p$$
integrate to a right action of $L_p$ on $G$. Then the images of the $L_p$ orbits in $G$ under the map

$$\sigma_p : G \rightarrow P : g \mapsto gp$$

give all the symplectic leaves in $P$.

**Proof.** By definition, the symplectic leaves of $P$ are the integral submanifolds of the distribution $\{\text{im}(\pi^#|_{T_p^*P})\}_{p \in P}$ on $P$. By Lemma 7.1, we know that this distribution is the image under $\sigma_p$ of the distribution on $G$ defined by the vector fields $\lambda_{x+\phi(\alpha_p)}$, $(x, \alpha_p) \in l_p$, on $G$. The integral submanifolds of the latter distribution are exactly all the $L_p$-orbits on $G$. Thus these orbits project to $P$ under $\sigma_p$ to give all the symplectic leaves in $P$.

Q.E.D.

**Remark 7.3** Note that when the double group $D$ of $G$ and $G^*$ have a global decomposition $D = GG^*$, the right action of $L_p$ on $G$ described in Theorem 7.2 is simply the restriction to $L_p$ of the right action of $D$ on $G = G^* \backslash D$ (see (16) in Section 3).

We now turn to the $G$-invariant Poisson cohomology of $P$. Recall that the Poisson cohomology of $P$ is by definition the Lie algebroid cohomology of the cotangent bundle Lie algebroid $T^*P$ with trivial coefficients. The corresponding coboundary operator on $\wedge^\bullet TP$ is also given by

$$V \mapsto [\pi, V]$$

where $[\ , \ ]$ denotes the Schouten bracket $[Ku]$. If $V$ is $G$-invariant, then so is $[\pi, V]$. This is because

$$L_{\sigma_x}[\pi, V] = [L_{\sigma_x}, V] = [\sigma_{\delta(x)}; V] = 0$$

for any $x \in g$ (see (21)). Hence, since $G$ is connected, the space $\Gamma(\wedge TP)^G$ of $G$-invariant multi-vector fields on $P$ is closed under the coboundary operator $[\pi, \bullet]$.

**Definition 7.4** The cohomology of $(\Gamma(\wedge TP)^G, [\pi, \bullet])$ is called the $G$-invariant Poisson cohomology of $P$. We denote it by $H_{\pi,G}(P)$.

Assume now that $P$ is a Poisson homogeneous $G$-space. We have seen in Example 4.9 that the kernel of the anchor map $-\sigma - \pi^#$ at each $p \in P$ is a Lie algebra $t_p$ of dimension
Let $G_p$ be the stabilizer Lie subgroup of $G$ at $p$, and let $\mathfrak{g}_p$ be its Lie algebra. The group $G_p$ acts on $\mathfrak{t}_p$ via the action of $G$ on $A$ given by (28), and the corresponding action of $\mathfrak{g}_p$ is the adjoint action of $\mathfrak{g}_p$ on $\mathfrak{t}_p$ via the Lie algebra embedding $\mathfrak{g}_p \subset \mathfrak{t}_p$. Denote by $H^\bullet(\mathfrak{t}_p, G_p)$ the relative Lie algebra cohomology of $\mathfrak{t}_p$ relative to $G_p$.

**Theorem 7.5**

\[ H^\bullet_{\pi, G}(P) \cong H^\bullet(\mathfrak{t}_p, G_p) \]

for every $p \in P$.

**Proof.** Denote by $L$ the kernel of $-\sigma - \pi$, so $L$ is a subbundle of $A$ whose fiber at $p \in P$ is the Lie algebra $\mathfrak{t}_p$. By taking fiber-wise Lie bracket of sections of $L$, we can think of $L$ as a Lie algebroid itself with the zero anchor map. As a such, it is a “totally intransitive” Lie algebroid [Mcz1]. The inclusion of $L$ into $A$ makes it into a Lie subalgebroid of $A$. We know from Theorem 4.1 and Proposition 4.5 that $L$ is a $G$-vector bundle and that the action of $G$ on $L$ preserves the Lie brackets on the fibers of $L$.

Consider now the coboundary operator $d_L$ for the Lie algebroid cohomology of $L$ with trivial coefficients (see (8)). Since $L$ has the zero anchor map, the operator $d_L$ is $C^\infty(P)$-linear, i.e., $d_L$ is given by the field $d_{\mathfrak{t}_p}, p \in P$, of fiber-wise operators, where $d_{\mathfrak{t}_p}$ is the coboundary operator for the cohomology of the Lie algebra $\mathfrak{t}_p$.

For each $p \in P$, set

\[ \mathfrak{g}_p^0 = \{ f \in \mathfrak{t}_p^* : f(x) = 0, \forall x \in \mathfrak{g}_p \} \subset \mathfrak{t}_p^*. \tag{53} \]

Then the subspace $(\wedge^\bullet \mathfrak{g}_p^0)^G_p$ of $G_p$-invariant vectors in $\wedge^\bullet \mathfrak{g}_p^0$ is invariant under $d_{\mathfrak{t}_p}$. The relative Lie algebra cohomology of $\mathfrak{t}_p$ relative to $G_p$ is by definition the cohomology of the cochain complex $((\wedge^\bullet \mathfrak{g}_p^0)^G_p, d_{\mathfrak{t}_p})$.

Denote by $L_0^0_p$ the subbundle of $L^*$ whose fiber at $p$ is $\mathfrak{g}_p^0$. It is a $G$-invariant subbundle of $L^*$. By dimension counting, the map

\[ TP \rightarrow L^* : v_p \mapsto f_{v_p} : (x, \alpha_p) \mapsto (v_p, \alpha_p) \]

gives a bundle isomorphism from $TP$ to $L_0^0$. It is also a $G$-bundle morphism. Therefore we get a vector space isomorphism

\[ \Gamma(\wedge^k TP)^G \rightarrow \Gamma(\wedge^k L_0^0)^G \subset \Gamma(\wedge^k L^*), \tag{54} \]

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where both sides denote $G$-invariant sections in the corresponding vector bundles. Denote this map by $\theta$. It remains to show that

$$d_L \theta(V) = \theta([\pi, V])$$

for any $V \in \Gamma(\wedge^k TP)^G$, for then $\theta$ would give a cochain complex isomorphism from $\Gamma(\wedge^k TP)^G$ to $((\wedge^k \mathfrak{g})^G, d)$ for each $p \in P$, and thus inducing an isomorphism between their cohomology spaces.

Let $V$ be a $G$-invariant $k$-vector field on $P$. Then

$$D\bar{x}V = 0, \quad \forall \bar{x} \in C^\infty(P, \mathfrak{g}).$$

Consequently, for any 1-forms $\alpha_i, i = 1, \ldots, k$, on $P$, we have

$$-\sigma_{\bar{x}}(V, \alpha_1 \wedge \ldots \wedge \alpha_k) = (V, D\bar{x}(\alpha_1 \wedge \ldots \wedge \alpha_k)).$$

It follows then that for any $l_i = (\bar{x}_i, \alpha_i) \in \Gamma(L), i = 1, \ldots, k$

$$(d_L \theta(V))(l_1 \wedge \ldots \wedge l_k) = \sum_{i<j} (-1)^{i+j} (\theta(V), \{l_i, l_j\} \wedge l_1 \wedge \ldots \hat{l}_i \ldots \hat{l}_j \ldots \wedge l_k)$$

$$= \sum_{i<j} (-1)^{i+j} (V, \{\alpha_i, \alpha_j\} \wedge \alpha_1 \wedge \ldots \hat{\alpha}_i \ldots \hat{\alpha}_j \ldots \wedge \alpha_k)$$

$$+ \sum_{i<j} (-1)^{i+j} (V, (D_{\bar{x}_i} \alpha_j - D_{\bar{x}_j} \alpha_i) \wedge \alpha_1 \wedge \ldots \hat{\alpha}_i \ldots \hat{\alpha}_j \ldots \wedge \alpha_k)$$

$$= \sum_{i<j} (-1)^{i+j} (V, \{\alpha_i, \alpha_j\} \wedge \alpha_1 \wedge \ldots \hat{\alpha}_i \ldots \hat{\alpha}_j \ldots \wedge \alpha_k)$$

$$+ \sum_{j} (-1)^{j+1} \sigma_{\bar{x}_j}(V, \alpha_1 \wedge \ldots \hat{\alpha}_j \ldots \wedge \alpha_k))$$

$$= \sum_{i<j} (-1)^{i+j} (V, \{\alpha_i, \alpha_j\} \wedge \alpha_1 \wedge \ldots \hat{\alpha}_i \ldots \hat{\alpha}_j \ldots \wedge \alpha_k)$$

$$+ \sum_{j} (-1)^{j+1} \sigma_{\bar{x}_j}(V, \alpha_1 \wedge \ldots \hat{\alpha}_j \ldots \wedge \alpha_k))$$

$$= \sum_{i<j} (-1)^{i+j} (V, \{\alpha_i, \alpha_j\} \wedge \alpha_1 \wedge \ldots \hat{\alpha}_i \ldots \hat{\alpha}_j \ldots \wedge \alpha_k)$$

$$+ \sum_{j} (-1)^{j+1}(-\pi^\#(\alpha_j))(V, \alpha_1 \wedge \ldots \hat{\alpha}_j \ldots \wedge \alpha_k))$$

$$= [\pi, V](\alpha_1 \wedge \ldots \wedge \alpha_k)$$

$$= \theta([\pi, V])(l_1 \wedge \ldots \wedge l_k).$$
Therefore
\[ d_L \theta(V) = \theta([\pi, V]). \]

This finishes the proof of the theorem.

\[ \text{Q.E.D.} \]

**Example 7.6** We continue with Example 4.11. Let \( G \) be a connected Poisson Lie group, and let \( H \subset G \) (denoted by \( G_p \) in Example 4.11) be a connected closed subgroup of \( G \) with Lie algebra \( \mathfrak{h} \) such that
\[ \mathfrak{h}^\perp = \{ \xi \in \mathfrak{g}^* : (\xi, x) = 0, \ \forall x \in \mathfrak{h} \} \subset \mathfrak{g}^* \]
is a Lie subalgebra of \( \mathfrak{g}^* \). In this case, there is a unique Poisson structure on the quotient space \( G/H \) such that the projection from \( G \) to \( G/H \) is a Poisson map, and the left action of \( G \) on \( G/H \) by left translations is a Poisson action. We have seen in Example 4.11 that the Lie algebra \( \mathfrak{l}_p \) corresponding to the point \( p = eH \) in \( G/H \) is the Lie subalgebra \( \mathfrak{h} + \mathfrak{h}^\perp \) of \( \mathfrak{d} = \mathfrak{g} \bowtie \mathfrak{g}^* \). By Theorem 7.3, we know that the \( G \)-invariant Poisson cohomology of \( G/H \) is isomorphic to the relative Lie algebra cohomology of \( \mathfrak{h} + \mathfrak{h}^\perp \) relative to the Lie subalgebra \( \mathfrak{h} \). We also observed in Example 4.11 that the Lie algebra \( \mathfrak{h} + \mathfrak{h}^\perp \) is an example of a double Lie algebra. We first prove the following fact on relative cohomology in the case of a double Lie algebra.

**Lemma 7.7** Let \( \mathfrak{l} \) be a Lie algebra and let \( \mathfrak{h} \) and \( \mathfrak{n} \) be Lie subalgebras of \( \mathfrak{l} \) such that
\[ \mathfrak{l} = \mathfrak{h} \oplus \mathfrak{n} \]
as vector spaces. For \( x \in \mathfrak{h} \) and \( \xi \in \mathfrak{n} \), write
\[ [x, \xi] = -\xi \cdot x + x \cdot \xi \]
where \( \xi \cdot x \in \mathfrak{h} \) and \( x \cdot \xi \in \mathfrak{n} \). The maps
\begin{align*}
\mathfrak{h} \times \mathfrak{n} &\to \mathfrak{n} : (x, \xi) \mapsto x \cdot \xi \\
\mathfrak{n} \times \mathfrak{h} &\to \mathfrak{h} : (\xi, x) \mapsto \xi \cdot x
\end{align*}
define a pair of actions of $\mathfrak{h}$ and $\mathfrak{n}$ on each other. Denote by $(\wedge \mathfrak{n}^*)^\mathfrak{h}$ the subspace of $\wedge \mathfrak{n}^*$ of $\mathfrak{h}$-invariant vectors with respect to the action of $\mathfrak{h}$ on $\mathfrak{n}^*$ contragradient to the action of $\mathfrak{h}$ on $\mathfrak{n}$. Let

$$d_n : \wedge^k \mathfrak{n}^* \longrightarrow \wedge^{k+1} \mathfrak{n}^*$$

be the Chevalley-Eilenberg coboundary operator for the Lie algebra $\mathfrak{n}$. Then the subspace $(\wedge \mathfrak{n}^*)^\mathfrak{h}$ of $\wedge \mathfrak{n}^*$ is invariant under $d_n$ (even though the action of $\mathfrak{h}$ on $\mathfrak{n}$ is not by Lie algebra derivations). The cohomology $H^\bullet((\wedge \mathfrak{n}^*)^\mathfrak{h}, d_n)$ of the cochain subcomplex $((\wedge \mathfrak{n}^*)^\mathfrak{h}, d_n)$ is isomorphic to the relative Lie algebra cohomology of $\mathfrak{l}$ relative to the Lie subalgebra $\mathfrak{h}$.

**Proof.** By embedding $\mathfrak{n}^*$ into $\mathfrak{l}^*$ via $$\mathfrak{n}^* \ni f \mapsto \tilde{f}(x + y) = f(y), \quad x \in \mathfrak{h}, \ y \in \mathfrak{n}$$

we can identify $\mathfrak{n}^*$ with $\mathfrak{h}^\perp \subset \mathfrak{l}^*$ and $(\wedge \mathfrak{n}^*)^\mathfrak{h}$ with $(\wedge \mathfrak{h}^\perp)^\mathfrak{h} \subset \wedge \mathfrak{l}^*$. Denote again by $f$ an arbitrary element in $(\wedge \mathfrak{n}^*)^\mathfrak{h}$ and by $\tilde{f}$ its image in $(\wedge \mathfrak{h}^\perp)^\mathfrak{h} \subset \wedge \mathfrak{l}^*$, i.e., $$\tilde{f}(x_1 + y_1, \ldots, x_k + y_k) = f(y_1, \ldots, y_k).$$

We need to show that $$d_1 \tilde{f} = d_n f,$$

where $d_1 : \wedge^k \mathfrak{l}^* \rightarrow \wedge^{k+1} \mathfrak{l}^*$ is the Chevalley-Eilenberg coboundary operator for $\mathfrak{l}$. Let $l_i = x_i + y_i \in \mathfrak{h} + \mathfrak{n}$ with $x_i \in \mathfrak{h}$ and $y_i \in \mathfrak{n}$, $i = 1, \ldots, k + 1$. Then

$$(d_1 \tilde{f})(l_1, \ldots, l_{k+1}) = \sum_{i < j} (-1)^{i+j} \tilde{f}([l_i, l_j], \ldots, \hat{l_i}, \ldots, \hat{l_j}, \ldots, l_{k+1})$$

$$= \sum_{i < j} (-1)^{i+j} f([y_i, y_j] + x_i \cdot y_j - x_j \cdot y_i, \ldots, \hat{y}_i, \ldots, \hat{y}_j, \ldots, y_{k+1})$$

$$= d_n f(l_1, \ldots, l_{k+1}) + \sum_j (-1)^j f(x_j \cdot (y_1 \wedge \cdots \wedge \hat{y}_j \wedge \cdots \wedge y_{k+1}))$$

$$= d_n f(l_1, \ldots, l_{k+1}).$$

In the last step we use that fact that $f$ is $\mathfrak{h}$-invariant.

Q.E.D.

The following Theorem now follows from Lemma 7.7.
Theorem 7.8 Let $G$ be a connected Poisson Lie group, and let $H \subset G$ be a connected closed subgroup of $G$ with Lie algebra $\mathfrak{h}$ such that

$$\mathfrak{h}^\perp = \{\xi \in \mathfrak{g}^* : (\xi, x) = 0, \ \forall x \in \mathfrak{h}\} \subset \mathfrak{g}^*$$

is a Lie subalgebra of $\mathfrak{g}^*$. Denote by $(\wedge (\mathfrak{h}^\perp)^*)^\mathfrak{h}$ the subspace in $\wedge (\mathfrak{h}^\perp)^* \mathfrak{h}$-invariant vectors with respect to the adjoint action of $\mathfrak{h}$ on $(\mathfrak{h}^\perp)^* \cong \mathfrak{g}/\mathfrak{h}$. Let

$$d_{\mathfrak{h}^\perp} : \wedge^k (\mathfrak{h}^\perp)^* \longrightarrow \wedge^{k+1}(\mathfrak{h}^\perp)^*$$

be the Chevalley-Eilenberg coboundary operator for the Lie algebra $\mathfrak{h}^\perp$. Then $(\wedge (\mathfrak{h}^\perp)^*)^\mathfrak{h}$ is invariant under $d_{\mathfrak{h}^\perp}$, and the the cohomology of $((\wedge(\mathfrak{h}^\perp)^*)^\mathfrak{h}, d_{\mathfrak{h}^\perp})$ is isomorphic to the $G$-invariant Poisson cohomology of the Poisson homogeneous space $G/H$.

We now apply Theorem 7.8 to the Bruhat Poisson structure on a generalized flag manifold [Lu-We].

Let $G$ be a connected finite dimensional semisimple complex Lie group and let $G = KAN$ be an Iwasawa decomposition of $G$ as a real semisimple Lie group. Then there are natural Poisson structures on $K$ and on $AN$ making them into Poisson Lie groups. Moreover, as Poisson Lie groups, they are dual to each other, and the group $G$ is the double of $K$ and $AN$. On the Lie algebra level, we have the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$. The imaginary part of the (complex-valued) Killing form of $\mathfrak{g}$ gives a nondegenerate pairing between $\mathfrak{k}$ and $\mathfrak{a} + \mathfrak{n}$. In other words, the pair $(\mathfrak{k}, \mathfrak{a} + \mathfrak{n})$ form a Lie bialgebra and $\mathfrak{g}$ is its double Lie algebra.

Let $T$ be the maximal torus in $K$, and let $\mathfrak{t}$ be its Lie algebra. The subspace

$$\mathfrak{t}^\perp = \{\xi \in \mathfrak{t}^* : (\xi, x) = 0, \ \forall x \in \mathfrak{t}\}$$

of $\mathfrak{t}^* \cong \mathfrak{a} + \mathfrak{n}$ is clearly $\mathfrak{n}$ which is an ideal in $\mathfrak{a} + \mathfrak{n}$. Thus there is a unique Poisson structure on $K/T$ such that the projection from $K$ to $K/T$ is a Poisson map. Moreover, the left action of $K$ on $K/T$ is a Poisson action, so $K/T$ is a Poisson homogeneous $K$-space.

Symplectic leaves in $K/T$ for this Poisson structure are known to be exactly the Bruhat cells in $K/T$ (thus the name Bruhat Poisson structure). This can be seen directly from Theorem 7.2, for in this case, the Lie group $L_p$, for $p = eT \in K/T$, is $TN$, and the right action of it on $K$ described in Theorem 7.2 is just the restriction to $TN$ of the right action of $G$ on $K \cong (AN)\backslash G$ by right translations. The projection of these orbits from $K$ to $K/T$ are exactly the Bruhat cells in $K/T$. 28
Consider now the $K$-invariant Poisson cohomology space of the Bruhat Poisson structure with complex coefficients. By Theorem 7.8, we get the following

**Theorem 7.9** The (complex-valued) $K$-invariant Poisson cohomology $H_{\pi,K}(K/T)$ of the Bruhat Poisson structure on $K/T$ is isomorphic to the space $\text{End}_{\mathfrak{h}}(H(n))$, where $\mathfrak{h}$ is the complexification of $\mathfrak{t}$, and $n$ here is considered as a complex Lie algebra. Thus the dimension of $H_{\pi,K}^{k}(K/T)$ is 0 if $k$ is odd and is equal to the number of Weyl group element of length $k/2$ if $k$ is even. In particular, it is isomorphic to the (complex-valued) de Rham cohomology of $K/T$.

**Proof.** From Theorem 7.8 we know that

$$H_{\pi,K}(K/T) \cong H((\wedge(n + n_-)^*)^{\mathfrak{h}}, d_n) \cong \text{End}_{\mathfrak{h}}(H(n)).$$

By Kostant’s theorem [Kt1], there is exactly one cohomology class for the Lie algebra $n$ corresponding to each element in the Weyl group $W$ of $K$. Choose root vectors $\{E_\alpha, E_{-\alpha} : \alpha > 0\}$. For each $w \in W$, set

$$\Phi_w := \{\beta > 0 : w^{-1}\beta > 0\} = \{\beta_1, \cdots, \beta_k\}.$$

Then the (complex) $K$-invariant $2k$-vector field

$$E_{-\beta_1} \wedge \cdots \wedge E_{-\beta_k} \wedge E_{\beta_1} \wedge \cdots \wedge E_{\beta_k}$$

on $K/T$ is a representative of a cohomology class of the $K$-invariant Poisson cohomology of the Bruhat Poisson structure on $K/T$. The fact that the space $\text{End}_{\mathfrak{h}}(H(n))$ is isomorphic to the de Rham cohomology of $K/T$ is another theorem of Kostant’s [Kt2].

### 8 The Lie groupoid of $A$

We first recall the definition of a Lie groupoid. Details can be found in [Mcz1].

A groupoid over a set $P$ is a set $M$, together with

1. surjections $s, t : M \to P$ (called the source and target maps respectively);
2. $\mu : M \ast M \to M$ (called the multiplication), where

   $$M \ast M = \{(m_2, m_1) \in M \times M : s(m_2) = t(m_1)\};$$
each pair \((m_2, m_1)\) in \(M \star M\) is said to be “composable”;

(3) an injection \(\varepsilon : P \to M\) (identities);

(4) \(\iota : M \to M\) (inversion).

These maps must satisfy

(1) associative law: \(\mu(\mu(m_3, m_2), m_1) = \mu(m_3, \mu(m_2, m_1))\) (if one side is defined, so is the other);

(2) identities: for each \(m \in M\), \((\varepsilon(t(m)), m) \in M \star M\), \((m, \varepsilon(s(m))) \in M \star M\), and \(\mu(\varepsilon(t(m)), m) = \mu(m, \varepsilon(s(m))) = m\);

(3) inverses: for each \(m \in M\), \((m, \iota(m)) \in M \star M\), \((\iota(m), m) \in M \star M\), and \(\mu(m, \iota(m)) = \varepsilon(t(m))\) and \(\mu(\iota(m), m) = \varepsilon(s(m))\).

A Lie groupoid (or differential groupoid) \(M\) over a manifold \(P\) is a groupoid with a differential structure such that (1) \(s\) and \(t\) are differentiable submersions (this implies that \(M \star M\) is a submanifold of \(M \times M\)), and (2) \(\mu, \varepsilon\) and \(\iota\) are differentiable maps.

Given a Lie groupoid \(M\) over \(P\), the normal bundle of \(P\) in \(M\) has a Lie algebroid structure over \(P\) whose sections can be identified with left invariant vector fields on \(M\). It is called the Lie algebroid of \(M\).

A left action of a Lie groupoid \((M, P, s, t)\) on a smooth submersion \(f : Q \to P\) is a map

\[
M \star_f Q = \{(m, q) : s(m) = f(q)\} \to Q : (m, q) \mapsto mq
\]

such that

(1) \(f(mq) = t(m)\)
(2) \(m_2(m_1q) = (m_2m_1)q\)
(3) \(\varepsilon(f(q))q = q\)

for all \(m, m_1, m_2 \in M\) and \(q \in Q\) which are suitably compatible.

Given such an action, one can construct the transformation groupoid on \(M \star_f Q\) with base \(Q\) by defining

\[
s'(m, q) = q, \quad t'(m, q) = mq
\]

\[(m_2, m_1q)(m_1, q) = (m_2m_1, q).\]

The map

\[
M \star_f Q \to M : (m, q) \mapsto m
\]

is a morphism of Lie groupoid over \(f : Q \to P\).
A right action of a groupoid and the corresponding transformation groupoid are similarly defined.

**Example 8.1** A Lie group is a Lie groupoid over a one point space. A Lie group action $G \times P \to P$ is an example of a Lie groupoid action. The Lie algebroid of the corresponding transformation groupoid is the transformation Lie algebroid discussed in Section 2.

**Example 8.2** Let $P$ be a Poisson manifold. We have seen in Section 2 that the cotangent bundle $T^*P$ of $P$ has a Lie algebroid structure. If there is a Lie groupoid $(N, s_N, t_N)$ over $P$ whose Lie algebroid is $T^*P$ and whose $s_N$-fibers are simply-connected, we say that $P$ is integrable as a Poisson manifold. It turns out that there is always a symplectic structure on $N$ making it into a symplectic groupoid [C-D-W]. We call $N$ the symplectic groupoid of $P$.

Assume now that $G$ is a Poisson Lie group, $P$ is an integrable Poisson manifold with symplectic groupoid $(N, P, s_N, t_N)$, and $\sigma : G \times P \to P$ is a Poisson action. Denote by $(M, s_M, t_M)$ the transformation Lie groupoid over $P$ defined by $\sigma$. The purpose of this section is to describe a Lie groupoid whose Lie algebroid is the Lie algebroid $A = (P \times g) \bowtie T^*P$ given in Theorem 4.1. Not surprisingly, this groupoid will be built out of $M$ and $N$ and a pair of actions of $M$ and $N$ on each other.

Recall that the map

$$\phi : T^*P \longrightarrow g^*: (\phi(\alpha_p), x) = (\alpha_p, \sigma_x(p))$$

is a Lie algebroid homomorphism. Assume that it can be integrated to a Lie groupoid homomorphism

$$\varphi : N \longrightarrow G^*,$$

where $G^*$ is the dual Poisson Lie group of $G$ (see [Xu]). The map $\varphi$ is also a Poisson map. As such, it induces a left action of $G$ on $N$ [Lu2] which we will denote by $(g, n) \mapsto gn$. The map $\varphi$ and the action of $G$ on $N$ have the following properties: for any $g \in G$, $n, n_1, n_2 \in N$,

$$t_N(gn) = gt_N(n)$$

$$s_N(gn) = g\varphi(n)s_N(n)$$
\[ \varepsilon_N(gn) = g\varepsilon_N(n) \]
\[ g(n_2n_1) = (gn_2)(g\varphi(n_2)n_1) \]
\[ \varphi(\varepsilon_N(p)) = 1 \in G^* \]
\[ \varphi(n_2n_1) = \varphi(n_2)\varphi(n_1) \]
\[ \varphi(gn) = g\varphi(n). \]

Here \( g_u \) and \( g^u \), for \( g \in G \) and \( u \in G^* \), denote respectively the left action of \( G \) on \( G^* \) and the right action of \( G^* \) on \( G \) defined by the decomposition

\[ gu = g_u g^u \in D \]

in the group \( D \) with Lie algebra \( \mathfrak{d} \) (see Section 3).

We can now describe the pair of actions of the groupoids \( M \) and \( N \) on each other.

The left action of \( M \) on \( t_N : N \to P \) is given by

\[ M * t_N : (m, t_N(n)), n) \mapsto gn. \quad (57) \]

The right action of \( N \) on \( M \) is given by

\[ M * s_M N = M * t_N N : (m, t_N(n)), n) \mapsto (g\varphi(n), s_N(n)). \quad (58) \]

For notational simplicity, we denote the manifold \( M * t_N N = M * s_M N \) by \( M * N \) and denote the two actions respectively by

\[ M * N \to N : (m, n) \mapsto^m n \]
\[ M * N \to M : (m, n) \mapsto^m n. \]

This pair of actions are compatible in the following sense:

1. \( s_N(mn) = t_M(nm), \quad \forall (m, n) \in M * N \)
2. \( ^m(\varepsilon_N(s_M(m))) = \varepsilon_N(t_M(m)), \quad \forall m \in M \)
3. \( (\varepsilon_M(t_N(n)))^n = \varepsilon_M(s_N(n)), \quad \forall n \in N \)
4. \( ^m(n_2n_1) = (m_2)(m_2^n_1), \quad \forall (m, n_2) \in M * N, (n_2, n_1) \in N * N \)
5. \( ^m(m_2m_1) = (m_2^m_1)(m_1^n), \quad \forall (m_2, m_1) \in M * M. \)
According to [Mcz2], two groupoids $M$ and $N$ over the same base $P$ with a pair of actions of them on each other satisfying conditions (1)-(5) as above are called a pair of groupoids with an interaction or a “matched pair” of groupoids. Given such data, one can construct a third groupoid on the space $N \ast M = \{(n, m) : s_N(n) = t_M(m)\}$ over $P$ by defining
\[
s(n, m) = s_M(m), \quad t(n, m) = t_N(n)
\]
\[
(n_1, m_1)(n_2, m_2) = (n_1 (m_1 m_2), (m_1 m_2) m_2).
\]

The natural maps
\[
M \to N \ast M : \quad m \mapsto (\varepsilon_N(t_M(m)), m)
\]
\[
N \to N \ast M : \quad n \mapsto (n, \varepsilon_M(s_N(n))
\]
embed $M$ and $N$ into $N \ast M$ as subgroupoids.

**Theorem 8.3** Let $\sigma : G \times P \to P$ be a Poisson action of a Poisson Lie group $G$ on an integrable Poisson manifold $P$. Let $N$ be the symplectic groupoid of $P$. Let $N \ast M$ be the Lie groupoid over $P$ as described above. Then the Lie algebroid of $N \ast M$ is the Lie algebroid $A = (P \times g) \bowtie T^*P$ as given in Theorem [4.7].

We will skip the proof of this theorem. The main part of the proof consists of showing that the linearizations of the two actions of $M$ and $N$ on each other at their identity sections are exactly the Lie algebroid representations of $P \times g$ and $T^*P$ on each other which were used in the construction of the Lie algebroid $A = (P \times g) \bowtie T^*P$.

**Remark 8.4** For $p \in P$, denote by $(N \ast M)_p$ the intersection of the $s$-fiber and the $t$-fiber in $N \ast M$ over $p$. It can be identified with the space
\[
(N \ast M)_p = \{(n, g) : n \in N, g \in G : t_N(n) = p, s_N(n) = gp\}.
\]

We know from general groupoid theory that it has a group structure:
\[
(n_2, g_2)(n_1, g_1) = (n_2(g_2 n_1), g_2^\sigma(n_1) g_1), \quad (n_2, g_2), (n_1, g_1) \in (N \ast M)_p.
\]

The Lie algeba of $(N \ast M)_p$ is the Lie algebra $t_p$ introduced in Section 4.
Remark 8.5 We remark that the space $N \ast M$ is an example of what is called a vacant double groupoid in \cite{Mcz2}. On the space $N \ast M$, there is also a groupoid structure over $M$ and a groupoid structure over $N$, namely the transformation groupoids of the actions of $M$ and $N$ on each other. These two groupoid structures are compatible, making $N \ast M$ into a double groupoid. The groupoid structure on $P$ we just described is called the diagonal groupoid of the double groupoid $M \ast N$. See \cite{Mcz2} for more details. For a detailed treatment of “matched pairs” of Lie algebroids, see \cite{MF}.

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