On the Hardy–Littlewood Majorant Problem

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Vienna, Preprint ESI 1296 (2003) March 31, 2003

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Ben Green\textsuperscript{1} and Imre Z. Ruzsa\textsuperscript{2}

Abstract
Let $\mathcal{M} = \{1, \ldots, N\}$, and let $\{a_n\}_{n \in \Lambda}$ be a sequence with $|a_n| \leq 1$ for all $n$. It is easy to see that
\[
\left\| \sum_{n \in \Lambda} a_n e(n\theta) \right\|_p \leq \left\| \sum_{n \in \Lambda} e(n\theta) \right\|_p
\]
for every even integer $p$. We give an example which shows that this statement can fail rather dramatically when $p$ is not an even integer. This answers in the negative a question known as the Hardy-Littlewood majorant conjecture, thereby ruling out a certain approach to the restriction and Kakeya families of conjectures.

1. Introduction
Let $\mathcal{M} = \{1, \ldots, N\}$, and let $\{a_n\}_{n \in \Lambda}$ be a sequence with $|a_n| \leq 1$ for all $n$. Hardy and Littlewood observed (as a simple consequence of Parseval’s identity) that we have
\[
\left\| \sum_{n \in \Lambda} a_n e(n\theta) \right\|_p \leq \left\| \sum_{n \in \Lambda} e(n\theta) \right\|_p
\]
for every even integer $p$. They asked what, if anything, can be said about other values of $p > 2$. Defining a constant $B_p(\Lambda)$ by
\[
\sup_{\{a_n\} : |a_n| \leq 1} \left\| \sum_{n \in \Lambda} a_n e(n\theta) \right\|_p = B_p(\Lambda) \left\| \sum_{n \in \Lambda} e(n\theta) \right\|_p
\]
their question may be interpreted as asking for the behaviour of $B_p(\Lambda)$. Thus for any $\Lambda$ one has $B_p(\Lambda) = 1$ for all even integers $p$. As Hardy and Littlewood \cite{4, 5} knew, it is possible for $B_3(\Lambda)$ to be larger than 1, so that a perfect analogue of (1) cannot hold. We will discuss an example later on.

Let us give a brief history of the problem since Hardy and Littlewood. It is natural to write
\[
B_p(N) = \sup_{\Lambda \subseteq \{1, \ldots, N\}} B_p(\Lambda)
\]
\textsuperscript{1}For some of the period during which this work was carried out the first author was resident in Budapest, and was supported by the \textit{Mathematics in Information Society} project carried out by Rényi Institute, in the framework of the European Community’s \textit{Confirming the International Role of Community Research} programme.
\textsuperscript{2}Both authors participated in the programme on combinatorial and number-theoretic methods in analysis at the Erwin Schrödinger Institute, Vienna. They would like to record here their appreciation of the hospitality accorded them by that institute.
and to ask for the behaviour of $B_p(N)$. Thus Hardy and Littlewood knew that $B_3(N) > 1$, and Boas [2] later showed that $B_p(N) > 1$ for any $p \not\in \{2, 4, 6, \ldots\}$. Disproving a conjecture of Hardy and Littlewood, Bachelis [1] (see also [8] p. 138) showed that in fact $B_p(N) \to \infty$ for any $p \not\in \{2, 4, 6, \ldots\}$. The idea behind this construction is to take a kind of product of Boas-type examples, an idea which Bachelis states was communicated to him by Katzelson. A more precise version of the same idea forms the heart of the present paper.

Recently, there has been renewed interest in the quantitative behaviour of $B_p(N)$ due to connections with two famous open questions in harmonic analysis, the restriction conjecture and the Kakeya conjecture. The Kakeya conjecture is a problem in geometric measure theory of much greater importance than one would at first sight think. In its weakest form, it asserts that a Besicovitch set, that is any compact subset of $\mathbb{R}^d$ containing a unit line segment in each direction, must have Minkowski dimension $d$. The restriction conjecture is, put simply, an assertion about the Fourier transforms of functions supported on the $(d-1)$-dimensional Euclidean sphere $S^{d-1} \subseteq \mathbb{R}^d$. One form of it is the following.

**Conjecture 1 (Local restriction conjecture)** There exists, for each $\epsilon > 0$, an absolute constant $\gamma = \gamma_{d, \epsilon}$ with the following property. Let $\sigma$ be the surface measure on $S^{d-1}$, and suppose that $f : S^{d-1} \to \mathbb{C}$ is a continuous function with $\|f\|_{\infty} \leq 1$. Then, for all $R$ sufficiently large,

$$\int_{|\xi| \leq R} |\hat{f}\sigma(\xi)|^{2d/(d-1)} \, d\xi \leq \gamma R^\epsilon.$$ 

This implies a rather more classical global restriction conjecture of the form

$$\int |\hat{f}\sigma(\xi)|^{2d/(d-1)+\delta} \, d\xi = O_{\delta,d}(1)$$

(3)

by the so-called $\epsilon$-removal technique of Tao [12]. Conjecture 1 is known to be true for $d = 2$ (see [13]) but not for any higher value of $d$. It would, by a classical method (essentially due to Fefferman – see [3], Chapter 5) imply the Kakeya conjecture.

The novice reader may be rather mystified by all of this, but by now there are several good introductions to this area of research, such as [10, 11]. See also [3].

By an argument which is sketched in [6], Conjecture 1 would follow if we could show that

$$B_p(\Gamma) \ll_{p, \epsilon} N^\epsilon,$$

(4)

where $\Gamma \subseteq \{1, \ldots, N\}$ is a set constructed by discretizing and then projecting the sphere $S^{d-1}$ in a suitable way. The resulting set $\Gamma$ is not particularly natural, but this would not concern us if we could prove that in fact (4) holds for any $\Lambda$ in place of $\Gamma$, that is that $B_p(N) \ll_{p, \epsilon} N^\epsilon$. The question of whether such a statement holds has become known as the Hardy-Littlewood majorant problem and, as we have stated, it would imply the restriction conjecture, and hence the Kakeya conjecture.
Problem 2 (Hardy-Littlewood majorant problem) Is it true that we have an estimate
\[ B_p(N) \ll_{p, \epsilon} N^\epsilon \]
for all \( \epsilon > 0 \) and \( p \geq 2 \)?

In [6] it is shown that \( B_p(N) \gg \exp(c \log N / \log \log N) \). In [7] a detailed investigation of the behaviour of \( B_p(\Lambda) \) for random subsets \( \Lambda \subseteq \{1, \ldots, N\} \) is conducted. In particular it is shown that for any fixed \( \lambda \in (0, 1) \) and for any \( p \in (2, 4) \) (the range of interest for applications) a random subset \( \Lambda \subseteq \{1, \ldots, N\} \) with cardinality \( N^\lambda \) will satisfy \( B_p(\Lambda) \ll_{p, \epsilon, \lambda} N^\epsilon \) almost surely.

The main theorem of this paper is that the answer to Problem 2 is no.

**Theorem 3** \( B_3(N) \gg N^\theta \) for some explicitly computable positive constant \( \theta \).

The proof is by construction of a set \( \Lambda \) with \( B_3(\Lambda) \) large, and is completely explicit. In the next section we describe the construction, and then we offer two alternative proofs of Theorem 3.

We have recently heard that Mockenhaupt and Schlag have independently obtained a proof of Theorem 3. In fact, they obtain the same result with 3 replaced by any \( p > 2 \) which is not an even integer. Our construction could be modified to obtain such a result, but we have not done so here. The two approaches are quite similar, which is perhaps rather unsurprising. Mockenhaupt and Schlag’s argument will appear in [7].

2. The construction. It seems to be quite well-known (see [8], p144) that if \( Q(\theta) = 1 + e(\theta) + e(3\theta) \) and \( q(\theta) = 1 + e(\theta) - e(3\theta) \) then
\[ \int_0^1 |q(\theta)|^3 d\theta > \int_0^1 |Q(\theta)|^3 d\theta. \]

In any case this statement is rather easy to check on a computer, and it confirms that the set \( \Lambda = \{0, 1, 3\} \) has \( B_3(\Lambda) > 1 \). It is rather natural to try taking a product of several copies of this set. Thus if \( D \) is a large positive integer, define the set
\[ \Lambda = \left\{ \sum_{i=0}^{k-1} \varepsilon_i D^i \mid \varepsilon_i \in \{0, 1, 3\} \text{ for all } i \right\}, \]
where \( k = \lfloor \log N / \log D \rfloor \). We will show that if \( D \) is large enough then these sets satisfy Theorem 3. In fact if \( a_n \) is defined to be \( (-1)^{W(n)} \), where \( W(n) \) is the number of 3s in the base \( D \) expansion of \( D \), then we will be able to prove that \( \| \sum a_n e(n\theta) \|_3 \) is much larger than \( \| \sum e(n\theta) \|_3 \). It is easy to see that
\[ \sum_{n \in \Lambda} e(n\theta) = \prod_{i=0}^{k-1} Q(D^i \theta) \] (5)
\[ \sum_{n \in \Lambda} a_n e(n\theta) = \prod_{i=0}^{k-1} q(D^i\theta). \quad (6) \]

We will write these two expressions as \( F(\theta) \) and \( f(\theta) \) respectively.

3. **Proof that the construction works.** To estimate the norms of \( F \) and \( f \) we prove the following.

**Lemma 4** Let \( p \) be any trigonometric polynomial and, for let \( D \) be a large positive integer. Define a trigonometric polynomial \( g \) by

\[ g(\theta) = \prod_{i=0}^{k-1} p(D^i\theta) \]

(viz. equation (5)). Let \( \alpha \) and \( \epsilon \) be positive numbers. Then there is a \( D_0 = D_0(p, \alpha, \epsilon) \) such that for all \( D > D_0 \) we have

\[ \|p\|_{\alpha} - \epsilon \leq \|g\|_{\alpha}^{1/k} \leq \|p\|_{\alpha} + \epsilon \quad (7) \]

for every \( k \).

**Remark.** Of course, we are interested in the case \( p = 1 + e(\vartheta) + e(3\vartheta) \) or \( 1 + e(\theta) - e(3\theta) \).

**Proof.** We proceed by finding functions \( r^+ \) and \( r^- \) satisfying

\[ 0 \leq r^-(\phi) \leq |p(\phi)|^\alpha \leq r^+(\phi) \quad (8) \]

and

\[ (\|p\|_{\alpha} - \epsilon)^{\alpha} \leq \|r^-\|_1 \leq \|r^+\|_1 \leq (\|p\|_{\alpha} + \epsilon)^{\alpha} \quad (9) \]

with the property that both \( r = r^+ \) and \( r = r^- \) satisfy

\[ \int_0^1 \prod_{i=0}^{k-1} r(D^i\vartheta)d\vartheta = \left( \int_0^1 r(\vartheta)d\vartheta \right)^k \quad (10) \]

for any positive integer \( k \). Once such functions have been given we can estimate \( \|g\|_{\alpha} \) by

\[ \|g\|_{\alpha}^\alpha = \int_0^1 \prod_{i=0}^{k-1} |p(D^i\vartheta)|^\alpha d\vartheta \]

\[ \leq \int_0^1 \prod_{i=0}^{k-1} r^+(D^i\vartheta)d\vartheta \]

\[ = \left( \int_0^1 r^+(\vartheta)d\vartheta \right)^k \]

\[ \leq (\|p\|_{\alpha} + \epsilon)^{\alpha k}, \]
which is the upper bound of (7). The lower bound follows similarly by using $r^-$. Properties (8) and (9) will follow from an assumption

$$\max(0, |p(\phi)|^\alpha - \delta) \leq r^-(\phi) \leq |p(\phi)|^\alpha \leq r^+(\phi) \leq |p(\phi)|^\alpha + \delta$$

(11)

with a suitably chosen positive $\delta$.

One class of functions $r$ that have property (10) is $A$, the functions that are constant on each interval $R_j = [j/D, (j + 1)/D)$. To see this, observe that

$$(\vartheta, D\vartheta, \ldots, D^{k-1}\vartheta) \in R_{j_0} \times \cdots \times R_{j_{k-1}}$$

(12)

precisely for those $\vartheta$ whose base $D$ expansion starts

$$\vartheta = \frac{j_0}{D} + \frac{j_1}{D^2} + \cdots + \frac{j_{k-1}}{D^k} + \cdots$$

Thus for fixed $j_0, \ldots, j_{k-1}$ the inclusion (12) holds for $\vartheta$ lying in some interval of length exactly $D^{-k}$, and this quickly implies (10). The continuity of $p$ ensures that we may find $r \in A$ which satisfy (11) as well, provided of course that $D$ is sufficiently large.

Another class of functions $r$ that can be used is $B$, the trigonometrical polynomials of degree less than $D/2$. It is easy to see that any $r \in B$ satisfies (10). If $p$ has no zero on the unit circle (which is the case for the polynomials which actually interest us) then (11) is an immediate consequence of Weierstrass’ approximation theorem. If $p$ does have a zero, some care is needed to find the lower function $r^-$. One can circumvent the obstacle by seeking $r^-$ in the form

$$r^-(\phi) = |p(\phi)|^m h(\phi),$$

where $m > \alpha$ is an even integer and $h$ is a trigonometric polynomial. If $h$ satisfies

$$\max\left(0, \frac{|p(\phi)|^\alpha - \delta}{|p(\phi)|^m}\right) \leq h(\phi) \leq \frac{|p(\phi)|^\alpha + \delta}{|p(\phi)|^m}$$

then $r^-$ will satisfy (11). The existence of such an $h$ does now follow from Weierstrass’ theorem.

Whether we use $A$ or $B$, the lemma has certainly been verified.

We remark that the use of $A$ is both simpler and works for every continuous function $p$, while the use of $B$ only works for functions that have only a finite number of roots, and if those roots have finite order. It has, however, some hidden advantages. If $p$ is a nice function (say, analytic), then approximation by polynomials of degree $D$ is typically much better than by functions of step $1/D$. Furthermore, if we use polynomials of higher degree, then, though
the simple equality (10) will no longer hold, it is still possible to calculate the integral on the left side by a linear recurrence. This was done in a different context by Pintz and the second author [9].

By the remarks in §2, the deduction of Theorem 3 is simply a matter of proving a bound of the form $\|f\|_3 \gg N^\eta \|F\|_3$. We apply the Lemma for the functions $p$ and $q$ defined in §2, $\alpha = 3$ and

$$\varepsilon = (\|q\|_3 - \|Q\|_3)/3,$$

say. Then we have $\|f\|_3/\|F\|_3 \geq c^k$ with

$$c = \frac{\|q\|_3 - \varepsilon}{\|Q\|_3 + \varepsilon} > 1.$$

Observing that $N < D^k$ we see that Theorem 3 holds with $\eta = \log c/\log D$. \hfill \Box

An explicit constant $\eta$ could be calculated, but it would be very small. To finish this section we show that $\eta \leq 1/18$ by modifying an argument sketched in [6].

**Proposition 5** Let $\Lambda \subseteq \{1, \ldots, N\}$, and let $(a_n)_{n \in \Lambda}$ be a sequence with $|a_n| \leq 1$ for all $n$. Then

$$\left\| \sum_{n \in \Lambda} a_n e(n\theta) \right\|_3 \ll N^{1/18} \left\| \sum_{n \in \Lambda} e(n\theta) \right\|_3.$$

**Proof.** Consider the map $T : l^\infty(\Lambda) \to C(\mathbb{T})$ defined by

$$(Ta)(\theta) = \sum_{n \in \Lambda} a_n e(n\theta)$$

for any $a = (a_n)_{n \in \Lambda}$. Write $F(\theta) = \sum_{n \in \Lambda} e(n\theta)$. Then it is easy to check, using Parseval’s identity, the bounds

$$\|Ta\|_2 \leq \|a\|_\infty \|F\|_2$$

and

$$\|Ta\|_4 \leq \|a\|_\infty \|F\|_4$$

for any sequence $a$. By the Riesz–Thorin interpolation theorem (one reference is [3], Ch. 7.) these imply a bound

$$\|Ta\|_3 \leq \|a\|_\infty \|F\|_2^{1/3} \|F\|_4^{2/3}. \quad (13)$$

Now observe that

$$\|F\|_2 \leq \|F\|_3 \quad (14)$$

and that

$$\|F\|_4 \leq \|F\|_\infty^{1/4} \|F\|_3^{3/4} \leq |\Lambda|^{1/4} \|F\|_3^{3/4}. \quad (15)$$
Equations (13), (14) and (15) immediately lead to
\[ \| Tu \|_3 \leq \| a \|_\infty | \Lambda |^{1/6} \| F \|_3^{5/6}. \]  \tag{16}

However, observing that \(| F(\theta) | \geq |\Lambda|/2\) for \(| \theta | \leq 1/10N\), we see that \( \| F \|_3 \gg |\Lambda|N^{-1/3} \). This provides an upper bound for \(|\Lambda|\) which, when substituted into (16), gives the bound
\[ \| Tu \|_3 \ll N^{1/18} \| a \|_\infty \| F \|_3. \]

This is what we wanted to prove.

\[ \square \]

We remark if one wished to replace the exponent 3 by some \( p \in (2, 4) \) then an identical argument would allow one to replace \( 1/18 \) by \( 2(1/p - 1/4)(1 - 2/p) \).

4. Further remarks on the majorant problem. Our result, Theorem 3, is rather negative since it does nothing more than rule out an approach to some well-known problems which was rather optimistic. The constants \( B_p(\Lambda) \) are really of interest for specific sets which arise “naturally”, for example from surfaces in Euclidean space.

Mockenhaupt [6] proves that if \( f : S^1 \to \mathbb{C} \) satisfies \( \| f \|_\infty \leq 1 \) then one has the inequality
\[ \int_{|\xi| \leq R} | \hat{f} \hat{d}\sigma(\xi)|^4 d\xi \leq \pi \int_{|\xi| \leq R} | \hat{d}\sigma(\xi)|^4 d\xi \]  \tag{17}
for \( R \) sufficiently large (recall that \( \sigma \) is the induced measure on \( S^1 \), considered as a subset of \( \mathbb{R}^2 \)). This may be regarded as an estimate of Hardy-Littlewood majorant type, with the rôle of the set \( \Lambda \) being played by \( S^1 \). We are not aware of any counterexamples to the corresponding statement in \( d \) dimensions. That is, we cannot be certain that there is no absolute constant \( C_d \) such that
\[ \int_{|\xi| \leq R} | \hat{f} \hat{d}\sigma(\xi)|^{2d/(d-1)} d\xi \leq C_d \int_{|\xi| \leq R} | \hat{d}\sigma(\xi)|^{2d/(d-1)} d\xi \]  \tag{18}
for all functions \( f : S^{d-1} \to \mathbb{C} \) with \( \| f \|_\infty \leq 1 \) and all \( R \geq R_0(d) \).

The relevance of the exponent \( 2d/(d-1) \) is that \( \hat{d}\sigma \) lies in \( L^q(\mathbb{R}^d) \) if and only if \( q > 2d/(d-1) \).

The estimate (18), if true, would be quite remarkable. It would immediately imply a strong quantitative form of the local restriction conjecture (Conjecture 1). It would also imply, by the classical method that if \( E \subseteq \mathbb{R}^d \) is a Besicovitch set then the \( \delta \)-neighbourhood \( N_\delta(E) \) has measure at least \( c_d (\log(1/\delta))^{1-d} \). This would be an extremely strong quantitative form of the Kakeya conjecture. Trying to disprove this might not be a bad way of showing that (18) is too optimistic, if indeed this is the case.
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