New exact solutions for inflationary cosmological perturbations

Jérôme Martin\textsuperscript{a}\textsuperscript{1} & Dominik J. Schwarz\textsuperscript{b}\textsuperscript{2}

\textsuperscript{a} DARC, Observatoire de Paris-Meudon, UMR 8629 CNRS, 92195 Meudon Cedex, France.
\textsuperscript{b} Institut für Theoretische Physik, Technische Universität Wien, Wiedner Hauptstraße 8–10, 1040 Wien, Austria.

Abstract

From a general ansatz for the effective potential of cosmological perturbations we find new, exact solutions in single-scalar-field inflation: a three parameter family of exact inflationary solutions that encompasses all exact solutions that have been known previously (power-law inflation, Easther’s model, and a generalised version of Starobinsky’s solution). The main feature of this new family is that the spectral indices are scale dependent.

PACS numbers: 98.80.Cq, 98.70.Vc
Keywords: cosmology, inflation, cosmological perturbations

\textsuperscript{1}email: jmartin@iap.fr
\textsuperscript{2}email: dschwarz@hep.itp.tuwien.ac.at
1 Introduction

The observation of the first acoustic peak in the spectrum of anisotropies in the CMBR (Cosmic Microwave Background Radiation) \cite{1, 2} leaves inflation as the only mechanism that can provide seeds for the observed large-scale structure of the Universe. The theory of inflation predicts primordial density perturbations \cite{3} and primordial gravitational waves \cite{4} from quantum fluctuations of the vacuum during the inflationary epoch. So far, inflationary predictions have been based either on the slow-roll approximation \cite{5} or on power-law inflation \cite{6}. However, it is clear that inflation might occur in a much broader context and we should discriminate between generic predictions of inflation and predictions of a specific scenario.

For any model of inflation, the scalar power spectrum can be expressed as a Taylor series around some pivot scale $k_*$ \cite{7, 8}:

$$\ln(k^3 P) = \ln(k_*^3 P_*) + (n_* - 1) \ln\left(\frac{k}{k_*}\right) + \frac{1}{2} \frac{dn}{d\ln k} \ln^2\left(\frac{k}{k_*}\right) + \cdots. \quad (1)$$

A similar expression can be written for the power spectrum of tensor perturbations. The quadratic term represents the first deviation of the spectrum from a power-law shape and is often referred as the “running” of the spectral index $n$. No presently known exact solutions \cite{6, 9} gives rise to a “running” spectral index. In this letter, we present a class of exact solutions which do exhibit $k$-dependent spectral indices.

It is difficult to predict the coefficients of the series (1) in general. One has to rely either on approximations or on numerical mode-by-mode integration \cite{10, 11, 12}. The best currently available analytical method is the slow-roll approximation \cite{5, 13} where “running” only shows up at second order in the slow-roll parameters \cite{7, 8}. The precision of any approximation scheme should be controlled, this means that the errors should quantified. This can only be done by comparing the approximate power spectrum to an exact one. For inflationary models without “running” of the spectral indices, we have already analysed this question for the slow-roll approximation up to the first order by comparing to the exact solutions of power-law inflation \cite{14, 15}. Therefore, to go beyond the first order and to test the accuracy of methods that estimate the “running” of the spectral index, exact solutions with $k$-dependent spectral indices are needed. So far, only numerical solutions are available to perform an error analysis model by model for the second order of the slow-roll approximation and for models where the slow-roll conditions are not satisfied \cite{10, 11, 12}. The exact solutions presented here constitute a first step towards a more complete analysis.

The generic features of inflation might be studied from two sides: from the particle physics point of view the starting point is the inflaton potential $V(\varphi)$, together with initial conditions for $\varphi$ and $\dot{\varphi}$. From the cosmological side, one might start from the effective potential (or effective action) for the evolution of
cosmological perturbations, together with initial conditions for the Hubble rate $H$ and its time derivative. Here we choose the latter approach, which was also taken in Refs. [9, 16]. We find a family of exact inflationary solutions for the mode equations of cosmological perturbations that includes all previously known solutions as limiting cases and that allows us to gain new insight into the generic properties of inflation.

Let us now introduce some basic tools for the study of cosmological perturbations. During inflation, density (scalar) perturbations and gravitational waves (tensor perturbations) can be characterised by the quantities $\mu_{S,T}(\eta)$ [17], which obey the equations (we use the notation of [18, 14]):

$$\mu''_{S,T} + \left[k^2 - \frac{z''_{S,T}}{z_{S,T}}\right]\mu_{S,T} = 0,$$

(2)

where a prime denotes differentiation with respect to conformal time $\eta$ and $z_{S,T}$ are functions of the scale factor $a(\eta)$ and of its derivatives only:

$$z_{S,T}(\eta) \equiv a^{\sqrt{\gamma}}, \quad z_T(\eta) \equiv a,$$

(3)

with $\gamma(\eta) \equiv 1 - (a'^2/a^2)(a'/a)'$, which is nothing but the slow-roll parameter $\epsilon \equiv -\dot{H}/H^2$. Inflation occurs when $\gamma < 1$. Eq. (2) may be viewed as a “time-independent” Schrödinger equation with an effective potential given by

$$U_{S,T}(\eta) \equiv \frac{z''_{S,T}}{z_{S,T}}.$$

(4)

The assumption that the quantum fields are initially placed in the vacuum state, when the mode $k$ is subhorizon, fixes the initial conditions for the quantities $\mu_{S,T}$. They read:

$$\lim_{k/(aH) \to +\infty} \mu_{S,T}(\eta) = \mp 4\sqrt{\pi}l_P \frac{e^{-ik(\eta-\eta_i)}}{\sqrt{2k}},$$

(5)

where $\eta_i$ is an arbitrary initial time at the beginning of inflation. Then, an integration of Eq. (2) allows the determination of the power spectra $P(k)$. For density perturbations we choose to work with the power spectrum of the hypersurface independent quantity $\zeta$, which corresponds to the perturbation of the intrinsic curvature of a spatial, uniform density hypersurface. For gravitational waves we determine the power spectrum of the amplitude $h$. These power spectra are calculated according to:

$$k^3 P_\zeta \equiv \frac{k^3}{8\pi^2} \left|\frac{\mu_S}{z_S}\right|^2, \quad k^3 P_h \equiv \frac{2k^3}{\pi^2} \left|\frac{\mu_T}{z_T}\right|^2.$$

(6)

Measurements of the CMBR anisotropies and of the large-scale structure probe scales which were well beyond the Hubble radius at the end of inflation. Thus, we are interested in the modes which satisfy $k/(aH) \ll 1$ at the end of inflation.
It is easy to see from Eq. (2) that the spectra are time independent in this regime since $\mu_{S,T} \propto z_{S,T}$ once the subdominant mode can be neglected. Let us also note that the spectrum of the quantity $\zeta$ is related to the spectrum of the Bardeen potential today by $k^3 P_\zeta = (25/9) k^3 P_\Phi$. In agreement with Eq. (1) the spectral indices are defined by: $n_S - 1 \equiv d \ln(k^3 P_\zeta)/d \ln k$ and $n_T \equiv d \ln(k^3 P_h)/d \ln k$.

The behaviour of cosmological perturbations during inflation is completely fixed once the functions $z_{S,T}(\eta)$ are specified. The effective potential $U_{S,T}(\eta)$ follows from Eq. (4) and Eq. (2) can be solved. Integration of equation (3) provides the scale factor. Instead of assuming specific functions $z_{S,T}(\eta)$, one can also start from the effective potential $U_{S,T}(\eta)$ itself, such that Eq. (2) may be solved analytically. In this case the functions $z_{S,T}(\eta)$ are known explicitly since $z_{S,T}(\eta) = \mu_{S,T}(k = 0, \eta)$.

Let us now discuss the shape of $U_{S,T}(\eta)$. One expects that the effective potential possesses the following properties: $\lim_{|\eta| \to +\infty} U_{S,T}(\eta) \ll k^2$, for any wavenumber of interest, in order for the modes to be inside the horizon at the initial time $\eta_i$ (so that we can fix the normalisation from quantum-mechanical considerations); it seems also reasonable to assume that $\lim_{|\eta| \to 0} U_{S,T}(\eta) \gg k^2$ which guarantees that the quantum fluctuations are amplified (i.e. are frozen outside the horizon). These requirements do not single out a unique effective potential. However, a general simple ansatz satisfying these conditions is given by:

$$U_{S,T}(\eta) = \sum_{m=1}^M \frac{c_m}{|\eta|^m},$$

where the $c_m$’s should be determined for each specific model of inflation. It is necessary to have $c_{\text{min}}$ and $c_M$ positive. The coefficient $c_1$ defines a characteristic scale $k_C \approx c_1$. The corresponding term in the series (7) dominates the other ones when $|\eta| \to \infty$ and will therefore determine the power spectrum at very large scales.

Inflationary models for which the evolution of cosmological perturbations has been studied so far, power-law inflation and slow-roll inflation, are chosen such that $c_m = 0$ if $m \neq 2$. In this sense they are special models. In particular, they lead to $k$-independent spectral indices.

In this article we study the behaviour of cosmological perturbations for models characterised by an effective potential given in Eq. (7). A first step in the analysis of these models consists in studying the effective potential

$$U_{S,T}(\eta) = \frac{c_1}{|\eta|} + \frac{c_2}{|\eta|^2}.$$  

The final power spectra depend on the free parameters $c_1$ and $c_2$ plus one parameter contained in $z_{S,T}(\eta)$. We demonstrate that the general solution for this potential is given in terms of Whittaker functions and we show explicitly how all the previously known cases can be recovered from this more general solution.
The initial conditions together with Eq. (8) determine the region of the inflaton potential explored during the evolution of the field. We always take $\gamma_i < 1$, such that an inflationary phase is guaranteed. Although all the properties of the cosmological perturbations can be calculated analytically, it is not possible to determine the corresponding inflaton potential analytically. This issue is important since we would like to know whether the new class of solutions corresponds to generic potentials $V(\varphi)$ from the particle physics point of view. Therefore, we calculate the corresponding inflaton potentials numerically for the scalar case, with initial conditions inspired by chaotic inflation.

This letter is organised as follows. In section II, we briefly remind the reader about known analytical solutions. In section III, we present a new family of exact solutions and in section IV we study various limits of the parameters of the new solutions.

2 Known analytical solutions

In the simplest case all coefficients $c_m$ vanish and $U_{S,T} = 0$ (Easther’s solution [9]). The corresponding functions $z_{S,T}$ are given by $z_{S,T}(\eta) \equiv B\eta + A$, where $A$ and $B$ are free parameters. For the tensor sector, this gives just the radiation dominated Universe. For the scalar sector, the general solution of Eq. (2) can be written as $\mu_S(\eta) = C_1e^{ik\eta} + C_2e^{-ik\eta}$, where $C_1$ and $C_2$ are two arbitrary constants to be determined by the initial conditions (5). They read $C_1 = 0$ and $C_2 = -4\sqrt{\pi l_{pl}}e^{ik\eta}/\sqrt{2k}$. In the large scale limit we have $z_S \approx A$ and the spectrum of density perturbations reads:

$$k^3P_\zeta = \frac{l_{pl}^2}{\pi A^2}k^2.$$ (9)

The spectral index is given by $n_S = 3$, which is in obvious contradiction with observations. The analytic form of the corresponding scalar potential $V(\varphi)$ has been given in Ref. [9].

All other cases known so far assume a potential such that $U_{S,T} = \alpha(\alpha + 1)/\eta^2$, i.e. $c_m = 0$ if $m \neq 2$ and $c_2 = \alpha(\alpha + 1)$, where $\alpha$ is a free parameter [14]. The corresponding functions $z_{S,T}$ can be expressed as $z_{S,T}(\eta) = A/|\eta|^\alpha$, where $A$ is another free parameter. Thus, we have a two-parameter family characterised by $A$ and $\alpha$. For tensor perturbations all these models correspond to power-law inflation [9]. For scalar perturbations the subclass $\alpha = 1$ was studied by Starobinsky [16]. Another subclass is $A = l_0(\alpha - 1)/\alpha$, which corresponds to power-law inflation [4] with the scale factor given by $a(\eta) = l_0|\eta|^{-\alpha}$. The quantity $l_0$ has the dimension of a length. In this model inflation occurs if $\alpha \geq 1$. In the de Sitter case ($\alpha = 1$), $l_0$ is simply the constant Hubble radius during inflation. Only for power-law inflation scalar and tensor perturbations can be solved analytically at the same time. For scalar perturbations the general two-parameter family is
Figure 1: The generalised Starobinsky potentials for $\alpha = 1, 2, \alpha_{pl} \equiv -\beta - 1, 5$ (from top to bottom) and $A = 1$, where $\alpha_{pl} \approx 3.44$. The initial conditions are $\gamma_i = (\alpha_{pl} - 1)/\alpha_{pl} \approx 0.7$ and $H_i = \sqrt{8\pi G}/3$.  

not equivalent to power-law inflation. Of course, $a(\eta) \propto |\eta|^{-\alpha}$ is a particular solution of $z_S(\eta) = A/|\eta|^\alpha$, but it is not the general solution. The limit $A$ to zero and $\alpha$ close to one reproduces a slow-roll inflation model. Eq. (2) is solved in terms of Bessel functions, $\mu_S(\eta) = (k\eta)^{1/2}[C_1 J_{\alpha+\frac{1}{2}}(k\eta) + C_2 J_{-\alpha-\frac{1}{2}}(k\eta)]$, where and $C_1$ and $C_2$ are constants fixed from Eq. (5). The spectrum may be calculated exactly to read

$$k^3 P_\zeta(k) = \frac{l_{Pl}^2}{\pi^2 A^2} 2^{2\alpha} \Gamma^2\left(\alpha + \frac{1}{2}\right) k^{-2(\alpha-1)}.$$  

(10)

The corresponding spectral index is $n_S = 3 - 2\alpha$. In particular, we have $k^3 P_\zeta = l_{Pl}^2/(\pi A^2)$ and $n_S = 1$ for $\alpha = 1$. For gravitational waves the same ansatz gives a power-law model with spectral index $n_T = 2 - 2\alpha$.

The inflaton potential for the scalar sector is displayed in Fig. (1) for various values of $\alpha$. We use initial conditions motivated by the scenario of chaotic inflation, i.e. the potential and the kinetic energy densities of the inflaton field are of the order of the Planck energy density initially. The condition $\gamma_i < 1$ guarantees that inflation takes place. For power-law inflation, the potential is given by: $V(\varphi) = V_i \exp[4\sqrt{\pi}\gamma(\varphi - \varphi_i)/m_{Pl}]$.  

![](image.png)
3 A new analytical solution

We now turn to the main result of this article. The function $z_{S,T}$ corresponding to the effective potential \((8)\) can be written as:

$$z_{S,T}(\eta) = \frac{2A_c^{\xi-1/2}}{\Gamma(2\xi)} \sqrt{c_1|\eta|} K_{2\xi}(2\sqrt{c_1|\eta|}) , \quad (11)$$

where $A > 0$ and $\xi \equiv \sqrt{c_2 + 1/4} \geq 1/2$. $K_{2\xi}$ is a modified Bessel function of order $2\xi$. We now define $\tau$ as $\tau \equiv 2ik\eta$ and $\kappa$ as $\kappa \equiv -ic_1/(2k)$. $c_1$ defines a characteristic scale given by $k_C = c_1/2$. Then, the equation of motion \((2)\) can be expressed as:

$$\frac{d^2 \mu_{S,T}}{d\tau^2} + \left[ -\frac{1}{4} + \frac{\kappa}{\tau} + \left( \frac{1/4 - \xi^2}{\tau^2} \right) \right] \mu_{S,T} = 0. \quad (12)$$

The general solution to this equation is given in terms of Whittaker functions

$$\mu_{S,T}(\tau) = C_1 W_{\kappa,\xi}(\tau) + C_2 W_{-\kappa,\xi}(-\tau). \quad (13)$$

The constants $C_1$ and $C_2$ are fixed by the initial conditions. Using the asymptotic behaviour of the Whittaker functions in the small-scale limit, $\lim_{|\tau| \to \infty} W_{\kappa,\xi}(\tau) = e^{-\tau/2}$, Eq. (9.227) of Ref. \[19\], we find:

$$C_1 = \mp 4\sqrt{\pi l_p} \frac{e^{ikm + \pi c_1/(4k)}}{\sqrt{2k}}, \quad C_2 = 0 . \quad (14)$$

Expressing the Whittaker functions in terms of Kummer functions, Eqs. (9.220) of Ref. \[19\], we deduce that, if $-\xi + 1/2 < 0$, the growing mode in the large scale limit is given by: $\lim_{|\tau| \to 0} W_{\kappa,\xi}(\tau) = \Gamma(2\xi)/[\Gamma(1/2 + \xi - \kappa)](\tau)^{-\xi+1/2} e^{-\tau/2}$. Using that $z_{S,T}(\eta) \approx A|\eta|^{2-\xi}$ in this limit, it is straightforward to calculate the spectrum of density perturbations:

$$k^3 P_\zeta = \frac{l_p^2}{\pi A^2} \frac{\Gamma^2(2\xi)}{2^{2\xi-1}} \frac{k^{3-2\xi} e^{\pi k_C/k}}{[\Gamma(1/2 + \xi + ik_C/k)]^2} . \quad (15)$$

This spectrum corresponds to a new exact solution which contains all the previously known cases as particular cases, see the next section. Using Eq. (8.328.1) of Ref. \[19\], we see that for $k \ll k_C$ one has $[\Gamma(1/2 + \xi + ik_C/k)]^2 \approx 2\pi \exp(-\pi k_C/k)(k_C/k)^{2\xi}$. This implies that in the large-scale limit the spectrum is given by:

$$k^3 P_\zeta \approx \frac{l_p^2 k^3}{\pi^2 A^2} \frac{\Gamma^2(2\xi)}{(2k_C)^{2\xi}} e^{2\pi k_C/k} . \quad (16)$$

if $c_1 \neq 0$ and $c_0 = 0$. We see that it is not analytic in the region of small $k$. Of course, such an infra-red divergence of the spectrum is excluded from observation. However, the validity of our description fails for modes that leave the horizon at
the Planck epoch, i.e. at the beginning of inflation in the scenario of chaotic inflation. This sets a natural infra-red cut-off to the power spectrum \((15)\). In the small-scale limit \(k \gg k_C\) the spectrum tends to

\[
k^3 P_\zeta \approx \frac{l_{Pl}^2}{\pi^2 A^2} 2^{2\xi - 1} \Gamma^2(\xi) k^{3 - 2\xi}.
\] (17)

This is the spectrum of the two parameter family studied in the previous section, see Eq. \((10)\). The corresponding constant spectral index is given by \(n_S = 4 - 2\xi = 4 - 2\sqrt{c_2 + 1}/4\) and is entirely determined by the coefficient \(c_2\).

## 4 Limiting cases

### 4.1 \(c_1 \neq 0, c_2 = 0\): Coulomb solution

As already mentioned, the case usually treated in the literature is \(c_1 = 0, c_2 \neq 0\). On the other hand, the case \(c_1 \neq 0, c_2 = 0\) has never been studied before. Although it is of course just a particular case of the general solution given in the previous section, it is worth investigating its properties in some details. For this purpose, we restart from the beginning and consider the following function

\[
z_{S,T}(\eta) \equiv 2A \sqrt{|c_1| \eta} K_1 \left(2 \sqrt{|c_1| \eta} \right).
\] (18)

The link with Eq. \((11)\) is obvious since \(c_2 = 0\) corresponds to \(\xi = 1/2\). Let us now define \(\rho\) and \(\delta\) according to \(\rho \equiv i\tau/2\), \(\delta \equiv i\kappa\). Then the equation of motion \((2)\) for \(\mu_{S,T}(\eta)\) can be written as:

\[
\frac{d^2 \mu_{S,T}}{d\rho^2} + \left(1 - \frac{2\delta}{\rho}\right) \mu_{S,T} = 0.
\] (19)

This equation can be solved in terms of Coulomb functions \([20]\) with \(l = 0\), \(\mu_{S,T}(\rho) = C_1 F_0(\delta; \rho) + C_2 G_0(\delta; \rho)\). The coefficients \(C_1\) and \(C_2\) are fixed by the initial conditions, see Eq. \((3)\). We find that \(C_1 = \pm 4i \sqrt{\pi l_{Pl} e^{ikp}} / \sqrt{2k}\), \(C_2 = -iC_1\). In the large scale limit, the growing mode is given by the irregular Coulomb wave function \(G_0(\delta; \rho) \approx 1/C_0(\delta)\), where \(C_0(\delta) \equiv \sqrt{2\pi \delta / (e^{2\pi \delta} - 1)}\). Using the fact that \(K_1(x) \approx 1/x\) when \(x\) goes to zero, the spectrum is easily derived. For density perturbations, we find:

\[
k^3 P_\zeta = \frac{l_{Pl}^2 k^3}{2\pi^2 A^2 k_C^2} \left(e^{2\pi k_C/k} - 1\right).
\] (20)

This spectrum is of course nothing but Eq. \((15)\) for \(\xi = 1/2\). It is displayed in Fig. \(3\) for \(A = 1\) and \(c_1 = 0.5\). In the regime where \(k \ll k_C\) one recovers Eq. \((16)\) whereas if \(k \gg k_C\) this spectrum tends to the Easther’s solution \(k^3 P_\zeta = (l_{Pl}^2 k^2) / (\pi A^2)\) for which \(n_S = 3\) in accordance with Eq. \((17)\). The corresponding inflaton potentials for various values of \(c_1\) are displayed in Fig. \(3\). The initial conditions are chosen as in Fig. 1 and guarantee that inflation occurs.
Figure 2: The scalar power spectrum for the Coulomb case with $A = 1$ and $c_1 = 0.5$.

Figure 3: The potentials for the Coulomb case for $c_1 = 0.1, 0.5, 1, 1.5$ and $A = 1$ with the same initial conditions as before.
4.2 \( c_1 = 0, c_2 \neq 0 \): generalised power-law solution

Putting \( c_1 = 0 \) and \( \kappa = 0 \) in Eq. (15), one easily checks that the result is the generalised power-law spectrum, see Eq. (10), with \( \alpha = \xi - 1/2 \). The corresponding spectral index is \( n_S = 4 - 2\xi \).

4.3 \( c_1 \neq 0, c_2 = 2 \)

Let us finally investigate in more details the case \( c_2 = 2 \) which corresponds to \( \xi = 3/2 \). Using Eq. (8.332.1) of Ref. [19], we find that the exact spectrum for density perturbations is given by:

\[
k^3 P_\zeta^{(c_2=2)} = \frac{l^2_{Pl}}{2\pi^2 A^2 k_C} \frac{k}{1 + \left(\frac{k_D}{k_C}\right)^2} \left(e^{2\pi k_C/k} - 1\right).
\]

The spectrum is displayed in Fig. (4) for \( A = 1 \) and \( c_1 = 0.5 \).

In the limit \( k_C/k \ll 1 \), the previous spectrum reduces to \( k^3 P_\zeta^{(c_2=2)} = (l^2_{Pl})/(\pi A^2) \), i.e. the spectrum of Starobinsky’s solution \( (n_S = 1) \) as expected. The potential of the inflaton is displayed in Fig. (5) for various values of \( c_1 \) with the same initial conditions as in Figs. 1 and 3.

We briefly conclude in recalling that the main result of this article is the discovery of a new family of exact solutions for inflationary cosmological perturbations which encompasses all the previously known cases as limiting cases, see Table I. This new family could help to shed light on inflationary models which do not fulfill the slow-roll conditions. Moreover, it can be used to test...
methods of approximation that are valid for models possessing “running” of the spectral indices. For this purpose the knowledge of exact solutions allows us to avoid a model-by-model analysis, as is necessary for numerical methods. In this spirit, the precision of the slow-roll approximation was recently investigated using power-law inflation as an exact solution [14, 15].

**Acknowledgement**

D.J.S. would like to thank the Austrian Academy of Sciences for financial support. J.M. would like to thank the ITP, TU Wien (Vienna, Austria) for warm hospitality.

**References**

[1] P. de Bernardis *et al*, Nature (London) **404**, 955 (2000).

[2] S. Hanany *et al*, preprint astro-ph/0005123 (2000).

[3] V. Mukhanov and G. Chibisov, JETP Lett. **33**, 532 (1981); Sov. Phys. JETP **56**, 258 (1982); A. Guth and S. Y. Pi, Phys. Rev. Lett. **49**, 1110 (1982); S. W. Hawking, Phys. Lett. **115B**, 295 (1982); A. A. Starobinsky, Phys. Lett. **117B**, 175 (1982).

[4] A. A. Starobinsky, JETP Lett. **30**, 682 (1979).

[5] E. D. Stewart and D. H. Lyth, Phys. Lett. **302B**, 171 (1993).
Table 1: Spectra for different values of $c_1$ and $c_2$ and in different regimes.

| $c_1$ | $c_2$ | Regime                        | $k^3P_\zeta$       |
|-------|-------|-------------------------------|-------------------|
| $\neq 0$ | $\neq 0$ | Exact                        | Whittaker’s solution [15], $n_S = n_S(k)$ |
| $\neq 0$ | $\neq 0$ | $k \ll k_C$                  | Eq. [16], $n_S = n_S(k)$ |
| $\neq 0$ | $\neq 0$ | $k \gg k_C$                  | Generalised power-law [10], $n_S = 4 - 2\sqrt{c_2 + 1/2}$ |
| 0      | 0     | Exact                        | Easther’s solution [9], $n_S = 3$ |
| $\neq 0$ | 0     | Exact                        | Generalised power-law [10], $n_S = 4 - 2\sqrt{c_2 + 1/2}$ |
| 0      | 2     | Exact                        | Coulomb’s solution [20], $n_S = n_S(k)$ |
| $\neq 0$ | 2     | $k \gg k_C$                  | Easther’s solution [9], $n_S = 3$ |
| $\neq 0$ | 0     | Exact                        | Eq. [21], $n_S = n_S(k)$ |
| $\neq 0$ | 2     | $k \gg k_C$                  | Starobinsky’s solution [10], $n_S = 1$ |

[6] L. F. Abbott and M. B. Wise, Nucl. Phys. B244, 541 (1984).
[7] A. Kosowsky and M. S. Turner, Phys. Rev. D 52, 1739 (1995).
[8] E. J. Copeland, I. J. Grivell and A. R. Liddle, Mon. Not. R. Astron. Soc. 298, 1233 (1998).
[9] R. Easther, Class. Quant. Grav. 13, 1175 (1996).
[10] I. J. Grivell and A. R. Liddle, Phys. Rev. D 54, 7191 (1996); *ibid* 61, 081301 (2000).
[11] L. Wang, V. F. Mukhanov and P. J. Steinhardt, Phys. Lett. 414B, 18 (1997).
[12] E. J. Copeland, I. J. Grivell, E. W. Kolb and A. R. Liddle, Phys. Rev. D 58, 043002 (1998).
[13] J. E. Lidsey *et al.*, Rev. Mod. Phys. 69, 373 (1997).
[14] J. Martin and D. J. Schwarz, Phys. Rev. D 62, 103520 (2000).
[15] J. Martin, A. Riazuelo and D. J. Schwarz, Astrophys. J. Letters, to appear, preprint astro-ph/0006392 (2000).
[16] A. A. Starobinsky, talk given at DARC, Meudon, December 11, 1998.
[17] V. F. Mukhanov, H. A. Feldman and R. H. Brandenberger, Phys. Rep. 215, 203, (1992).
[18] J. Martin and D. J. Schwarz, Phys. Rev. D 57, 3302 (1998).
[19] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products*, Academic, New York, 1981.

[20] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, 1985.