On the Convergence of Quantized Parallel Restarted SGD for Serverless Learning

Feijie Wu\textsuperscript{1}, Shiqi He\textsuperscript{1}, Yutong Yang\textsuperscript{1}, Haozhao Wang\textsuperscript{1,2}, Zhihao Qu\textsuperscript{1,3}, and Song Guo\textsuperscript{1}

\textsuperscript{1} The Hong Kong Polytechnic University, Hong Kong, China
{harli.wu, shiqi.he.he, yutong.yang}@connect.polyu.hk, song.guo@polyu.edu.hk
\textsuperscript{2} Huazhong University of Science and Technology, Wuhan, China
hz.wang@hust.edu.cn
\textsuperscript{3} Hohai University, Nanjing, China
quzhihao@hhu.edu.cn

Abstract. With the growing data volume and the increasing concerns of data privacy, Stochastic Gradient Decent (SGD) based distributed training of deep neural network has been widely recognized as a promising approach. Compared with server-based architecture, serverless architecture with All-Reduce (AR) and Gossip paradigms can alleviate network congestion. To further reduce the communication overhead, we develop Quantized-PR-SGD, a novel compression approach for serverless learning that integrates quantization and parallel restarted (PR) techniques to compress the exchanged information and to reduce synchronization frequency respectively. The underlying theoretical guarantee for the proposed compression scheme is challenging since the precision loss incurred by quantization and the gradient deviation incurred by PR interact with each other. Moreover, the accumulated errors that are not strictly controlled make the training not converging in Gossip paradigm. Therefore, we establish the bound of accumulative errors according to synchronization mode and network topology to analyze the convergence properties of Quantized-PR-SGD. For both AR and Gossip paradigms, theoretical results show that Quantized-PR-SGD are at the convergence rate of $O(1/\sqrt{NM})$ for non-convex objectives, where $N$ is the total number of iterations while $M$ is the number of nodes. This indicates that Quantized-PR-SGD admits the same order of convergence rate and achieves linear speedup with respect to the number of nodes. Empirical study on various machine learning models demonstrates that communication overhead has reduced by 90\%, and the convergence speed has boosted by up to 3.2\times under low bandwidth network compared with PR-SGD.

Keywords: Distributed Machine Learning · Non-convex Optimization · Quantization.

1 Introduction

In the era of big data, distributed machine learning is thriving with unprecedented prosperity and scalability since it can maximize the computational power
of a large number of workers [22]. Frequently, it attempts to minimize the cumulative expected loss over all workers, which can be formally written as follows:

$$\min_{x \in \mathbb{R}^d} F(x) = \frac{1}{M} \sum_{m=1}^{M} \mathbb{E}_{\xi_m \sim D_m} \left[ f_m(x, \xi_m) \right]$$ \quad (1)

where $M$ is the number of workers (or nodes) and $D_m$ is the predefined data distribution for Worker $m$. Problem (1) is commonly applicable for training deep learning models in a distributed manner.

The underlying network topology affects the performance of both computation and communication in distributed machine learning. Generally, the distributed cluster is categorized into two architectures – server-based architecture and serverless architecture. Parameter Server (PS) [18], as a paradigm of server-based architecture, is highly controlled by a central node, where all worker nodes independently send the local gradients to and receive the global parameters from PS. Such framework is easy to implement and maintain. However, since all workers have to communicate with the central node, the potential network congestion degrades the performance of PS. To avoid the demand of high bandwidth network raised by the central node, serverless architectures, including All-Reduce (AR) and Gossip, are attracting more and more attentions [2]. AR paradigm successfully releases the burden of the only central node to multiple transit nodes and reveals the same training efficiency as server-based architecture. One of its successful practices, ring AR [16], outperforms PS because it makes good use of overlapping computation and communication [2]. However, such instance requires specific underlying network topology or otherwise it losses its dominance [24]. By contrast, Gossip paradigm tends to be a generalized solution for arbitrary network topology. It also possesses higher scalability than PS and eliminates the risk of single point of failure.

Nevertheless, AR and Gossip still suffer from high communication cost and many researchers attempt to alleviate this problem. Current solutions include (1) diminishing the frequency of synchronization and (2) compressing the traffic data in each transmission. The first solution refers to PR [27], also known as model averaging. Instead of communicating parameters in each iteration, workers aggregate individual solutions in multiple iterations and synchronize averaged result once. Zhou and Cong [27] assert that multiple local updates can lead to a better convergence rate. Quantization [1] is one of the common strategies to the second solution, which lowers the gradient precision to reduce bandwidth consumption.

Apparently, PR-SGD cannot perform smoothly under low bandwidth, while assembling with quantization can break through this limitation. However, one may question whether the combination of these two strategies leads to an algorithm that can converge robustly under AR and Gossip paradigms. This concern raises from some evidence: the precision loss generated by quantization will be amplified by the PR process. Besides, considering that each worker synchronizes the model only with neighbours in Gossip paradigm, the compression errors ac-
cumulate and spread over the whole network, which seriously affect the convergence ability. In this paper, we propose and evaluate quantized parallel restarted SGD (Quantized-PR-SGD), which resolves all the concerns above by regulating system parameters based on rigorous theoretical analysis.

With the proposed algorithm, our contributions are listed as follows:

• We theoretically evaluate the convergence of Quantized-PR-SGD for non-convex objectives under AR paradigm. Its convergence rate is consistent with its original prototypes – PR-SGD [19,25] and QSGD[1,21], i.e. a linear speedup with respect to the number of workers. To our best knowledge, this is the first work that investigates the PR-SGD with quantization in AR paradigm.
• Since AR paradigm has restriction on network topology, we extend Quantized-PR-SGD to Gossip paradigm. We adapt Quantized-PR-SGD to still preserve the linear speedup and achieve the same order of convergence rate. This is another piece of pioneering work that evaluates PR-SGD with lossy-compression in Gossip paradigm.
• We further conduct an empirical study to compare it with its original prototypes. In terms of convergence rate, Quantized-PR-SGD achieves up to 2.3× and 3.2× convergence efficiency compared to QSGD and PR-SGD, respectively, under low bandwidth network. Besides, as for communication overhead, it is able to save more than 90% compared to PR-SGD.

The rest of the paper is organized as follows: In Section 2, related work is introduced to provide the overview of distributed optimization problem. Section 3 introduces the original prototypes and key notations used throughout the paper. We analyze the convergence rate and the communication cost of Quantized-PR-SGD under the of AR paradigm and the Gossip paradigm in Section 4 and Section 5 respectively. Furthermore, an empirical study is conducted to validate our theoretical analysis in Section 6. Section 7 concludes this paper.

2 Related Work

**Stochastic gradient decent (SGD)** The Stochastic Gradient Descent (SGD) is a popular optimizing algorithm to address large scale machine learning problems [13]. For general convex [13] and non-convex functions [7], it is proved that SGD guarantees a iteration complexity $O(1/\sqrt{N})$.

**Parallel stochastic gradient decent (PSGD)** The most classical method Parallel SGD [11], where each worker executes the mini-batch SGD locally and then gradients are aggregated to update the model for the next step computation. Such a method achieves a convergence rate of $O(1/\sqrt{NM})$.

**Aggregation Paradigm** PSGD requires to aggregate the gradients from workers. Parameter server (PS) [184] is a widely adopted server-based architecture, but its performance is largely determined by the bandwidth of the central node.
Alternatively, the workers are also able to preserve a consistent model using All-Reduced (AR) paradigm without the central node \[14\]. However, its long hand-shaking processes sometimes slow down the training in high latency network \[12\]. To tackle this, decentralized parallel stochastic gradient decent (D-PSGD) \[11,12\], which is also referred to as Gossip, is introduced and proved preserving the same computation complexity $O(1/\sqrt{NM})$ as PSGD for non-convex objectives.

**Compression schemes** Reducing the communication overhead by requiring the workers to pass selected or compressed gradients instead of raw gradients is a practical strategy to optimize communication cost as well, e.g., quantization \[1,23,26,20\] and sparsification \[5\]. The combination of quantization and D-PSGD has been studied and proposed as DCD-PSGD \[21\].

## 3 Preliminary

### 3.1 Parallel Restarted SGD (PR-SGD)

PR-SGD \[25\], also known as local SGD \[19\] or K-AVG SGD \[27\], is an algorithm that all nodes make a constant times of local updates before global synchronization. Generally, the flow of two successive global synchronizations works in every node is decribed as follows:

- **Step 1:** Initialized the start point $x_0$
- **Step 2:** Repeat the following steps for $K$ times
  - Generate the i.i.d. realizations of the random $\xi_k$
  - Compute the gradient for the next iteration: $x_{k+1} = x_k - \gamma \nabla f(x_k; \xi_k)$
- **Step 3:** Output the gradient $x_K$

This method performs well in AR paradigm and saves a great amount of communication overheads \[25\].

### 3.2 Quantized SGD (QSGD)

Quantization method compresses gradients that are exchanged through the network, while generally preserves the model convergence performance of optimization. We adopt $s$-level uniformly distributed quantized function, proposed by Alistarh et. al \[1\]. For vector $v \neq 0$, the function can be defined as follows:

$$Q_s(v) = [v'_1 \ v'_2 \ \ldots \ v'_d]$$

$$v'_i = \|v\|_2 \cdot \text{sgn}(v_i) \cdot \zeta(v_i, s)/s$$  \hspace{1cm} (2)

where $\text{sgn}(\cdot) \in \{-1, 1\}$ represents the sign bit of a real number and $\zeta(v_i, s)$ is defined as follows: Let $\ell$ be an integer such that $|v_i|/\|v\|_2 \in [\ell/s, (\ell + 1)/s]$ and

$$\zeta(v_i, s) = \begin{cases} 
\ell + 1, & \text{with probability of } P = \frac{|v_i|}{\|v\|_2} s - \ell \\
\ell, & \text{otherwise}
\end{cases} \hspace{1cm} (3)$$
How to compress the exchanged information is the key concern of quantization method. The following lemma is useful to find the second moment of averaging model among $M$ nodes.

**Lemma 1.** For any $m \in [1, M]$ and vector $w^{(m)} \in \mathbb{R}^d$ which is independent with others, we have

$$
\mathbb{E} \left[ \left\| \frac{1}{M} \sum_{m=1}^{M} Q_s(w^{(m)}) \right\|_2^2 \right] \leq \frac{d}{4s^2M^2} \sum_{m=1}^{M} \|w^{(m)}\|_2^2 + \left\| \frac{1}{M} \sum_{m=1}^{M} w^{(m)} \right\|_2^2
$$

(4)

This lemma reveals that it is not applicable to directly compress the whole model since the gradient itself may vary from worker to worker and there does not exist an upper bound for the second moment. It only considers exchanging the difference of the gradients in two successive global synchronizations.

### 3.3 Notation

We use the following notations throughout this paper:

- $\nabla F(\cdot)$ denotes the gradient of a function $F$
- $\partial F(X) := [\nabla F_1(x^{(1)}) \quad \nabla F_2(x^{(2)}) \quad \cdots \quad \nabla F_M(x^{(M)})]$
- $F^*$ denotes optimal solution to Problem (1).
- $\|\cdot\|_n$ denotes $\ell_n$ norm of a vector in $\mathbb{R}^d$. Specially, it represents the number of nonzeros when $n = 0$.
- $\|\cdot\|_F$ denotes the Frobenius norm in a matrix.
- $\langle \cdot, \cdot \rangle$ denotes inner product of two vectors in $\mathbb{R}^d$.
- $\mathbf{1}_n$ denotes an $n$-dimension column vector filled with 1s.
- $\lambda_i(\cdot)$ denotes the $i$-th largest eigenvalue of a matrix.

### 4 Quantized Parallel Restarted SGD

We first introduce a quantized parallel restart SGD (Quantized-PR-SGD) algorithm for AR paradigm. From the literal perspective, it assembles QSGD [1] and PR-SGD [27,27,25,19].

A main characteristic of AR is that all nodes possess the same gradient after a full update without a parameter server. Regardless of the details of AR paradigm, each node works independently in every two successive synchronizations:

- **(Pull):** pull the parameter $\tilde{x}$ from the last update as initial state $x_0$
- **(Compute):** compute the gradient using PR-SGD
- **(Push):** push the quantized variance $g$ by $g = x_0 - x_K$
- **(Aggregate):** aggregate the averaging stochastic gradients $g$ from all other nodes and summarize them into $\Delta$
- **(Update):** update the parameter $\tilde{x}$ by $\tilde{x} = \tilde{x} - \Delta$
Algorithm 1: Quantized-PR-SGD with AR paradigm (Worker $m$)

Input : Initial Point $\tilde{x}_{1}$, stepsize series $\{\gamma_{n}\}$, the interval value $K$, and the number of total iterations $N$

1. for $n \leftarrow 1$ to $N$
   2. $x_{n,0}^{(m)} \leftarrow \tilde{x}_{n}$;
   3. for $k \leftarrow 0$ to $K - 1$
      4. Generate a realization of the random variable $\xi^{(m)}_{k}$;
      5. $x_{n,k+1}^{(m)} \leftarrow x_{n,k}^{(m)} - \gamma_{n} \nabla f_{m}(x_{n,k}^{(m)}, \xi^{(m)}_{k})$;
   6. end
   7. $g_{n}^{(m)} \leftarrow \text{Quantize}(x_{n,0}^{(m)} - x_{n,K}^{(m)})$;
   8. Send $g_{n}^{(m)}$ to all other nodes;
   9. Receive $g_{n}^{(j)}, \forall j \in \{1, 2, ..., M\}$:
      $$\Delta \leftarrow \frac{1}{M} \sum_{j=1}^{M} g_{n}^{(j)}$$
   10. Update the gradient $\tilde{x}_{n+1} \leftarrow \tilde{x}_{n} - \Delta$
end

Whether it is possible to isolate the sum of local updates from the effect of quantization is a major challenge. As is mentioned in 3.2 Lemma 1, successfully resolves our concern. When the left hand side of Lemma 1 is

$$\left\| \frac{1}{M} \sum_{m=1}^{M} Q_{s} \left( \gamma_{n} \sum_{k=0}^{K-1} \nabla f_{m}(x_{n,k}^{(m)}, \xi^{(m)}_{k}) \right) \right\|_{2}^{2}$$

where the sum of $K$ multiple local updates is quantized, how to make the second term in the right hand side converge is another difficult task.

Before analyzing the convergence property of Algorithm 1 we make the following assumptions, which are commonly used for SGD-based distributed optimization 3:

Assumption 1 Problem (1) satisfies the following constraints:

1. Smoothness: All function $F_{m}(\cdot)$’s are continuous differentiable and their gradient functions are $L$-Lipschitz continuous with $L > 0$.
2. Bounded variance: For any worker $m$ and vector $x \in \mathbb{R}^{d}$, there exist scalars $\sigma \geq 0$ and $\kappa \geq 0$ such that:

$$\mathbb{E}_{\xi \sim \mathcal{D}_{m}} \| \nabla f_{m}(x, \xi) - \nabla F_{m}(x) \|_{2}^{2} \leq \sigma^{2}; \quad \frac{1}{M} \sum_{m=1}^{M} \| \nabla F(x) - \nabla F_{m}(x) \|_{2}^{2} \leq \kappa^{2}$$

Under Assumption 1 the following theorem establishes the upper bound of convergence rate for non-convex optimization with fixed stepsize.
Theorem 1. Consider Problem (1) for non-convex optimization. Under Assumption 1, suppose that Algorithm 1 is run with a constant stepsize $\bar{\gamma}$ satisfying

$$1 - 2L\bar{\gamma}K \geq 0 \quad \text{and} \quad L^2\bar{\gamma}^2K(K - 1) \leq 1 - \delta$$

and

$$\frac{L\bar{\gamma}K}{\delta} \left( L\bar{\gamma}(K - 1) + \frac{d}{2s^2} \right) \leq 1 - \varepsilon \quad \exists \varepsilon \in (0, 1)$$

then for all $N \geq 1$, we have

$$\frac{1}{N} \sum_{n=1}^{N} \mathbb{E}[||\nabla F(\tilde{x}_n)||_2^2] \leq K(K - 1)(\sigma^2 + 2\kappa^2) + \frac{D_1}{M\varepsilon} KL\bar{\gamma} + \frac{2[F(\tilde{x}_1) - F_*]}{\bar{\gamma}\varepsilon N}$$

where,

$$D_1 := 2\sigma^2 + \frac{(\sigma^2 + 2\kappa^2)d}{4s^2\delta}$$

To have a clear idea about the convergence result in Theorem 1, we select an appropriate stepsize which achieves linear speedup:

Corollary 1. Under Theorem 1, take

$$\bar{\gamma} := \sqrt{\frac{[F(\tilde{x}_1) - F_*]M}{D_1NLK^2}}$$

Then for any

$$N \geq \frac{L[F(\tilde{X}_1) - F_*]}{D_1} \cdot \max \left( 4M, \frac{(K - 1)M}{K(1 - \delta)}, \frac{4s^4(K - 1)^2M}{d^2k^2}, \frac{d^2\mu}{(1 - \varepsilon)^22s^2\delta} \right)$$

$$N \geq \frac{L[F(\tilde{X}_1) - F_*]}{D_1} \cdot \frac{(K - 1)^2(\sigma^2 + 2\kappa^2)^2M^3}{4\delta^2K^2(D_1)^2}$$

the output of Algorithm 1 achieves the following ergodic convergence rate:

$$\frac{1}{N} \sum_{n=1}^{N} \mathbb{E}[||\nabla F(\tilde{x}_n)||_2^2] \leq \frac{4}{\varepsilon} \sqrt{\frac{D_1L[F(\tilde{x}_1) - F_*]}{NM}}$$

where $D_1$ is the same as the one in Theorem 1.

Since $\varepsilon, D_1, L$ and $[F(\tilde{x}_1) - F_*]$ are constant, this corollary claims that the convergence rate achieves $O(1/\sqrt{NM})$ when total iterations $N$ is sufficiently large. With this result, we have the following observations:

**Linear Speedup** Since the only term of the convergent rate is $O(1/\sqrt{NM})$, which has the same result as PR-SGD [19,25], this indicates that Algorithm 1 achieves linear speedup with respect to the number of workers.
Algorithm 2: D-Quantized-PR-SGD (Worker $m$)

**Input**: Initial Point $\tilde{x}^{(j)}_1 = x_1, \forall j \in \{1, 2, \ldots, M\}$, step size series $\{\gamma_j\}$, weighted matrix $W$, the interval value $K$, and the number of total iterations $N$

1. for $n \leftarrow 1$ to $N$
   2. $x_{n,0}^{(m)} \leftarrow \tilde{x}_n^{(m)}$;
   3. for $k \leftarrow 0$ to $K - 1$
      4. Generate a realization of the random variable $\xi_k^{(m)}$;
      5. $x_{n,k+1}^{(m)} \leftarrow x_{n,k}^{(m)} - \gamma_n \nabla f_m(x_{n,k}^{(m)}, \xi_k^{(m)})$
   6. end
   7. Temporarily update the local model:
      \[ \tilde{x}_{n+\frac{1}{2}}^{(m)} \leftarrow \sum_{j=1}^M W_{mj} \tilde{x}_n^{(j)} - x_{n,0}^{(m)} + x_{n,K}^{(m)} \]
   8. $g_n^{(m)} \leftarrow \text{Quantize}(\tilde{x}_n^{(m)} - \tilde{x}_{n+\frac{1}{2}}^{(m)})$;
   9. Encode and send $g_n^{(m)}$ to its neighbours;
10. Decode $g_n^{(j)}$, $\forall j \in \{1, 2, \ldots, M\}$ and $W_{mj} \neq 0$ and update the gradient:
      \[ \tilde{x}_{n+1}^{(j)} \leftarrow \tilde{x}_n^{(j)} - g_n^{(j)} \]

5. Extension: Quantized-PR-SGD with Gossip Paradigm

Current AR paradigm compromises to the underlying network topology \[24\]. Thus, we consider a more general case for Gossip paradigm to implement Quantized-PR-SGD. This paradigm can be represented by an undirected graph with the value of: $(V, W)$. $V \in \{1, \ldots, M\}$ denotes the set of $M$ workers. $W \in \mathbb{R}^{M \times M}$ is a doubly stochastic matrix, which means (i) $W_{ij} \in [0, 1]$, (ii) $W = W^T$ and (iii) $\sum_j W_{ij} = 1$ for all $i$. In every update, each node exchanges the compressed difference of local model of two successive iterations to its neighbours.

Inspired by DCD-P SGD \[21\], we design Quantized-PR-SGD for Gossip paradigm (D-Quantized-PR-SGD), whose procedure in Worker $m$ is demonstrated as follows:

- **(Pull)**: pull the parameter $\tilde{x}_1^{(m)}$ from the last update as initial state $x_0$
- **(Compute)**: compute the gradient using PR-SGD
- **(Store)**: store the intermediate value by $\tilde{x}_{0.5}^{(m)} = \sum_{j=1}^M W_{mj} \tilde{x}_j^{(j)} - x_0 + x_K$
- **(Push)**: push the quantized variance $g_0^{(m)}$ by $g_0^{(m)} = x_0^{(m)} - \tilde{x}_{0.5}^{(m)}$
- **(Update)**: update the parameters $\tilde{x}^{(j)}$ by $\tilde{x}^{(j)} = \tilde{x}^{(j)} - g^{(j)}$ for all connected neighbours (i.e. $W_{mj} \neq 0$)

The full realization is presented in Algorithm \[2\]
Assumption 1 is not enough to analyze the convergence result in Algorithm 2 and therefore, we make an additional assumption, which is commonly adopted in [11,21].

**Assumption 2 (Spectral gap)** Given the symmetric doubly stochastic matrix $W \in \mathbb{R}^{M \times M}$, we assume $\rho := \max \{|\lambda_2(W)|, |\lambda_M(W)|\} < 1$.

It is challenging to intuitively find the recursive function of Algorithm 2. We first assemble multiple local updates into one single update and then use bounded noise to replace the quantization loss. Therefore, the recursive function can be represented as follows:

$$X_{n+1} = X_n W - \gamma_n \sum_{k=0}^{K-1} G(X_{n,k}, \xi_k) + C_n; \quad X_{n,t} = X_n - \gamma_n \sum_{j=0}^{\ell-1} G(X_{n,j}, \xi_j) \quad (9)$$

where

- $G(X_{n,k}, \xi_k) = [\nabla f(x_{n,k}^{(1)}, s_k^{(1)}) \ldots \nabla f(x_{n,k}^{(M)}, s_k^{(M)})]$ represents a stochastic matrix under a local update
- $\Delta X_n = X_n W - \gamma_n \sum_{k=0}^{K-1} G(X_{n,k}, \xi_k) - X_n$ represents the gradient difference
- $C_n = Q_s(\Delta X_n) - \Delta X_n$ represents the noise of compression with quantization

In Gossip paradigm, in order to get a tighter upper bound, we analyze the accumulated noise of (i.e. $\sum_{n=0}^{N} \|C_n\|^2_F$) over the whole training process. The upper bound indicates that when $K$ increases, the accumulated noise would result in a worse convergence rate.

Under Assumption 1 and Assumption 2, the convergence result for Algorithm 2 is derived as follows:

**Theorem 2.** The weighted matrix $W$ in Algorithm 2 is a symmetric double stochastic matrix satisfying $d\mu^2 - s^2(1 - \rho)^2 < 0$, and initial point is $x_1 = 0$. Under Assumption 1 and Assumption 2, by choosing the fixed stepsize $\gamma$ with which the following inequalities hold:

$$1 - 12\gamma^2 L^2 (K + 1)(K - 2) > 0 \quad \text{and} \quad 1 - 8L^2\gamma^2 K D'_1 D'_3 > 0 \quad \text{and} \quad D'_6 > 0$$

then for all $N \geq 1$, we have

$$\frac{1}{N} \left( (1 - D'_5) \sum_{n=1}^{N} \mathbb{E} \left\| \nabla F \left( \frac{X_n \cdot 1_M}{M} \right) \right\|_2^2 + \frac{D'_6}{K} \sum_{n=1}^{N} \mathbb{E} \left\| \frac{\partial F(X_n) \cdot 1_M}{M} \right\|_2^2 \right)$$

$$\leq \frac{2\gamma KLD'_4}{1 - 8L^2\gamma^2 K D'_1 D'_3} \left[ 1 + \frac{4L^2\gamma^2 (K - 1)}{M} + \frac{8(K - 1) L^2\gamma^2}{1 - 12\gamma^2 L^2 (K + 1)(K - 2)} \right] \sigma^2$$

$$+ \frac{8\gamma KLD'_4}{1 - 8L^2\gamma^2 K D'_1 D'_3} \left[ 1 + \frac{4(K - 1)(2K - 1)L^2\gamma^2}{1 - 12\gamma^2 L^2 (K + 1)(K - 2)} \right] \kappa^2 + \frac{2(F(x_1) - F_*)}{\gamma KN} \quad (10)$$
where

\[
D'_1 := \frac{2K + 24\bar{\gamma}^2L^2 - 1}{1 - 12\bar{\gamma}^2L^2(K + 1)(K - 2)}, \quad D'_2 := \frac{d(1 - \rho)^2 + 2d\mu^2}{2[\sigma^2(1 - \rho)^2 - d\mu^2]}
\]

\[
D'_3 := \frac{D'_2}{1 - \rho^2} + \frac{1}{(1 - \rho)^2}, \quad D'_4 := \frac{2\bar{\gamma}LK\gamma}{2} + \frac{\bar{\gamma}L(K - 1)}{2} + \frac{D'_2}{2M}
\]

\[
D'_5 := \frac{8KL\gamma\gamma}{(1 - 8L^2\bar{\gamma}^2K\gamma)}M, \quad D'_6 := 1 - 2\bar{\gamma}KL\left(\frac{4K(K - 1)D'_4}{1 - 8L^2\bar{\gamma}^2K\gamma}D'_1D'_3L^2\bar{\gamma}^2 + 1\right)
\]

To have an intuitive insight of Theorem 2, we choose a constant stepsize and obtain the following corollary:

**Corollary 2.** Under Theorem 2, choose the stepsize

\[
\bar{\gamma} := \left(\sigma\sqrt{N/M} + 3KL\sqrt{D'_2} + 16KLD'_3 + 6KL\right)^{-1}
\]

Then we have the following convergence rate:

\[
\frac{1}{N} \sum_{n=1}^{N} \mathbb{E} \left\| \nabla F \left( \frac{X_n \cdot 1_M}{M} \right) \right\|_2^2 \leq \frac{12KLD'_2(\sigma^2 + 4\kappa^2)}{\sigma\sqrt{NM}} + \frac{4\sigma(F(x_1) - F_*)}{K\sqrt{NM}} + \frac{4(F(x_1) - F_*)(3L\sqrt{D'_2} + 16LD'_3 + 6L)}{N}
\]

when \( N \) is sufficiently large, in particular,

\[
N \geq \frac{L^2M}{\sigma^2} \max \left( \frac{K^2M^2(4D'_3 + 1)^2}{(D'_2)^2}, 6(K - 1)(2K - 1) \right)
\]

Specially, if

\[
K \leq \sqrt{\frac{\sigma^2(F(x_1) - F_*)}{LD'_2(\sigma^2 + 4\kappa^2)}}
\]

The convergence rate can be represented by

\[
\frac{1}{N} \sum_{n=1}^{N} \mathbb{E} \left\| \nabla F \left( \frac{X_n \cdot 1_M}{M} \right) \right\|_2^2 \leq \frac{16(F(x_1) - F_*)}{K\sqrt{NM}} + \frac{4(F(x_1) - F_*)(3L\sqrt{D'_2} + 16LD'_3 + 6L)}{N}
\]

The result suggests that the convergent rate for D-Quantized-PR-SGD is \( O\left(\frac{1}{\sqrt{NM}} + \frac{1}{N}\right) \) when the total iterations \( N \) is large enough. The following discussions interpret the tightness of our result.
Linear Speedup  Apparently, the term $\frac{1}{\sqrt{NM}}$ dominates the term $\frac{1}{N}$ when $N$ is sufficiently large, leading to a convergent rate of $O(1/\sqrt{NM})$. This is consistent with PR-SGD [19,25], achieving linear speedup with respect to the number of workers.

Effect of $K$  We set up a specified constant upper bound in order to provide faster convergence, such that, the larger value of $K$ is, the better convergence effect Algorithm 2 has. The upper bound of $K$ is dependent on the variance of gradients among all workers (i.e. $\kappa$).

Consistence with DCD-PSGD  Setting $K = 1$ to match the same scenario of DCD-PSGD. In this case, D-Quantized-PR-SGD admits the convergence rate of $O\left(\frac{1}{\sqrt{NM}} + \frac{1}{N}\right)$. Apparently, the $\frac{1}{\sqrt{T^2}}$ term in DCD-PSGD [21] is released, indicating that D-Quantized-PR-SGD is sightly better.

6  Experiment

In this section we evaluate the performance of Quantized-PR-SGD by comparing with its original prototypes (i.e. PR-SGD and QSGD, notice that DCD-PSGD is the Gossip implementation of QSGD) and PSGD under AR paradigm and Gossip paradigm. We conduct extensive experiments in different network condition. The results show that the algorithm reaches the stable convergence rate with communication cost reduction rate at more than 90%.

6.1  Experimental Setup

We train both ResNet-18 [8] and VGG-11 [17] models on CIFAR-10 dataset [9], which is a image recognition task constituted by 50000 training images and 10000 testing images. We implement Quantized-PR-SGD under Gossip paradigm on a cluster equipped with GCC Linux Red Hat 4.8.5-16 and supporting OpenMPI:

- **AR paradigm**: We build the proposed algorithm on OpenMPI, where AR paradigm is predefined. A famous case using this method is Microsoft CNTK.
- **Gossip paradigm**: With OpenMPI, the implementation of Gossip paradigm follows the ring network topology, where each node only exchange gradients with its two certain neighbours.
- **Compression schemes**: Considering the compression function in 2, we encode the gradients with the following rules: Firstly, we put the 32-bit full precision of $\|v\|_2$ at the beginning of the segment. Then for each index $i \in \{1, 2, ..., d\}$, we use 1 bit for its sign and the rest representing Elias gamma code [6] of $\zeta(v_i, s)$.

We built up the environment on 8 Intel Xeon Gold 6130 GPU cores for each node, with Tesla P100 Nvidia GPU card for acceleration. In terms of communication compression, we use full precision (i.e. $s = 2^{32} - 1$) for PSGD and PR-SGD and lossy precision (i.e. $s = \sqrt{d}$) for QSGD and Quantized-PR-SGD, where $d$ is the number of parameters in a training model while $s$ is the quantization level.
Fig. 1: Experiments on different algorithms, loss w.r.t. epoch

Without special notation, local gradients update for 4 times (i.e. $k=4$) before each global synchronization in both PR-SGD and Quantized-PR-SGD. Furthermore, the $x$-axis Epoch in experimental diagrams refers to the number of global synchronizations instead of the number of total iterations.

6.2 Results on Epoch and Communication Cost

In this group of experiments, we present and discuss the convergence rate and the communication cost of these four algorithms under two different paradigms.

*Training quality* As is shown Fig. 1 for training machine learning model ResNet and VGG, Quantized-PR-SGD and PR-SGD convergence at a similar rate in terms of epoch, demonstrated as the overlapping lines (green & red) in the chart. Similarly, QSGD and PSGD (yellow & blue) has a similar convergence rate in all experiments. The result indicates our approach could remain the
same convergence performance as PR-SGD and PSGD. It also demonstrates the epoch efficiency improvement of Quantized-PR-SGD comparing to PSGD. For example, in Fig. 1(a) the convergence speed of Quantized-PR-SGD is over $3\times$ faster than PSGD. In the other three charts (Fig. 1(b-d)), the convergence speed-up of Quantized-PR-SGD diverse from $1.93\times$ to $4.1\times$.

| Model | Transfer Cost | Algorithm | No Compression | Quantization |
|-------|---------------|-----------|----------------|--------------|
| ResNet | 42.63MB       | PR-SGD    |                | 3.80MB       |
| VGG   | 37.20MB       | Quantized-PR-SGD |                | 3.18MB       |

Table 1: Average Communication Cost per Epoch

Fig. 2: Experiments on different algorithms, loss w.r.t. communication cost

*Communication Efficiency* Our Experiment indicates that Quantized-PR-SGD has a significant improvement of communication overhead comparing to other three algorithms. Table 1 shows that the quantization method could achieve a communication cost reduction rate of over 91% for a single transfer. Fig. 2 demonstrates the total communication cost required for each algorithm to reach convergence. For ResNet model in Gossip paradigm (Fig. 2(a)), Quantized-PR-SGD reduced communication cost by 91.7% comparing to PR-SGD, and reduced communication cost by 61% comparing to QSGD. A similar result is observed for VGG model in AR paradigm (Fig. 2(b)), Quantized-PR-SGD uses 92.3% less than PR-SGD to reach convergence (loss < 0.1), and 72.5% less comparing to QSGD. An interesting pattern is that the communication cost of Quantized-PR-SGD is less than QSGD under the same number of global synchronizations.
The phenomena indicates that our compression scheme requires less bits while approaching to the convergence.

The communication cost reduction for parallel restarted process emphasize on the convergence speed-up on epoch. Fig. 3 demonstrates the accuracy performance for ResNet and VGG in terms of different $K$-step.

**Test accuracy** In Fig. 3(a), when $K = 8$, the ResNet model in Gossip paradigm reached the accuracy 78% in Epoch 109, which is $2 \times$ faster than the case of $K = 4$ on Epoch 206 and $4 \times$ faster than the case of $K = 8$ on Epoch 418. All three tests achieve a same accuracy in the end of the experiment. In Fig. 3(b), when $K = 8$, the VGG model in AR paradigm reached the accuracy 80% in Epoch 525, which is $1.9 \times$ faster than the case of $K = 4$ on Epoch 984 and $3.8 \times$ faster than the case of $K = 8$ on Epoch 1997. Meanwhile, the final accuracy of $K = 8$ has a gap of 1.2% for $K = 4$, and a gap of 3.7% for $K = 2$.

### 6.3 Results on Convergence Time

To better evaluate the convergence time of Quantized-PR-SGD, we simulate network conditions under various bandwidths and compare the total time cost for convergence. In our experiment, the transfer speed of 100Mbps is the maximum speed that we could reach. In Figure 4, we test various algorithms on ResNet and VGG model under bandwidth ranging from 5Mbps to 100Mbps.

In ResNet model, Quantized-PR-SGD shows its robustness as bandwidth decrease, remaining the fastest convergence speed except the case with full bandwidth due to the quantization overhead. As a computation-emphasized model, the communication cost of ResNet is relatively low. As a result, the PR-SGD could still perform under 30 minutes when bandwidth is limited to 5Mbps.

As for VGG model, the communication cost is the main workload comparing to the computation time. As a result, the gap between PSGD and Quantized-PR-
SGD under full bandwidth is smaller than that in ResNet model. In addition, the convergence time for PR-SGD and PSGD increase sharply when the bandwidth is limited. In VGG model, the Quantized-PR-SGD remains almost the same convergence time for bandwidth from 5Mbps to 100Mpbs, and runs up to $6.7 \times$ faster than PSGD.

7 Conclusion

This paper investigates quantized parallel restarted SGD for AR and Gossip paradigm, which is the seamless combination of two famous models – QSGD and PR-SGD. This novel SGD algorithm is analyzed from theoretical and empirical aspects. We find that the algorithm can achieve linear speedup in both AR paradigm and Gossip paradigm. Furthermore, it significantly saves the total communication overhead and preserves the convergence rate comparing to its prototypes.

References

1. Alistarh, D., Grubic, D., Li, J., Tomioka, R., Vojnovic, M.: Qsgd: Communication-efficient sgd via gradient quantization and encoding. In: Advances in Neural Information Processing Systems. pp. 1709–1720 (2017)
2. Alqahtani, S., Demirbas, M.: Performance analysis and comparison of distributed machine learning systems. arXiv preprint arXiv:1909.02061 (2019)
3. Bottou, L., Curtis, F.E., Nocedal, J.: Optimization methods for large-scale machine learning (2016)
4. Dean, J., Corrado, G., Monga, R., Chen, K., Devin, M., Mao, M., Ranzato, M., Senior, A., Tucker, P., Yang, K., et al.: Large scale distributed deep networks. In: Advances in neural information processing systems. pp. 1223–1231 (2012)
5. Dryden, N., Moon, T., Jacobs, S.A., Van Essen, B.: Communication quantization for data-parallel training of deep neural networks. In: 2016 2nd Workshop on Machine Learning in HPC Environments (MLHPC). pp. 1–8. IEEE (2016)
6. Elias, P.: Universal codeword sets and representations of the integers. IEEE transactions on information theory 21(2), 194–203 (1975)
7. Ghadimi, S., Lan, G.: Stochastic first-and zeroth-order methods for nonconvex stochastic programming. SIAM Journal on Optimization 23(4), 2341–2368 (2013)
8. He, K., Zhang, X., Ren, S., Sun, J.: Deep residual learning for image recognition. In: Proceedings of the IEEE conference on computer vision and pattern recognition. pp. 770–778 (2016)
9. Krizhevsky, A., Hinton, G., et al.: Learning multiple layers of features from tiny images (2009)
10. Li, M., Andersen, D.G., Smola, A.J., Yu, K.: Communication efficient distributed machine learning with the parameter server. In: Advances in Neural Information Processing Systems. pp. 19–27 (2014)
11. Lian, X., Zhang, C., Zhang, H., Hsieh, C.J., Zhang, W., Liu, J.: Can decentralized algorithms outperform centralized algorithms? a case study for decentralized parallel stochastic gradient descent. In: Advances in Neural Information Processing Systems. pp. 5330–5340 (2017)
12. Lian, X., Zhang, W., Zhang, C., Liu, J.: Asynchronous decentralized parallel stochastic gradient descent. arXiv preprint arXiv:1710.06952 (2018)
13. Nemirovski, A., Juditsky, A., Lan, G., Shapiro, A.: Robust stochastic approximation approach to stochastic programming. SIAM Journal on optimization 19(4), 1574–1609 (2009)
14. Patarasuk, P., Yuan, X.: Bandwidth optimal all-reduce algorithms for clusters of workstations. Journal of Parallel and Distributed Computing 69(2), 117–124 (2009)
15. Robbins, H., Monro, S.: A stochastic approximation method. The annals of mathematical statistics pp. 400–407 (1951)
16. Sergeev, A., Del Balso, M.: Horovod: fast and easy distributed deep learning in tensorflow. arXiv preprint arXiv:1802.05799 (2018)
17. Simonyan, K., Zisserman, A.: Very deep convolutional networks for large-scale image recognition. arXiv preprint arXiv:1409.1556 (2014)
18. Smola, A., Narayanamurthy, S.: An architecture for parallel topic models. Proceedings of the VLDB Endowment 3(1-2), 703–710 (2010)
19. Stich, S.U.: Local sgd converges fast and communicates little. In: ICLR 2019 ICLR 2019 International Conference on Learning Representations. No. CONF (2019)
20. Suresh, A.T., Yu, F.X., Kumar, S., McMahan, H.B.: Distributed mean estimation with limited communication. In: Proceedings of the 34th International Conference on Machine Learning-Volume 70. pp. 3329–3337. JMLR. org (2017)
21. Tang, H., Gan, S., Zhang, C., Zhang, T., Liu, J.: Communication compression for decentralized training. In: Advances in Neural Information Processing Systems. pp. 7652–7662 (2018)
22. Wang, H., Qu, Z., Guo, S., Gao, X., Li, R., Ye, B.: Intermittent pulling with local compensation for communication-efficient federated learning. arXiv preprint arXiv:2001.08277 (2020)
23. Wen, W., Xu, C., Yan, F., Wu, C., Wang, Y., Chen, Y., Li, H.: Terngrad: Ternary gradients to reduce communication in distributed deep learning. In: Advances in neural information processing systems. pp. 1509–1519 (2017)
24. Yu, H., Jin, R., Yang, S.: On the linear speedup analysis of communication efficient momentum sgd for distributed non-convex optimization. In: International Conference on Machine Learning. pp. 7184–7193 (2019)
25. Yu, H., Yang, S., Zhu, S.: Parallel restarted sgd with faster convergence and less communication: Demystifying why model averaging works for deep learning. In: Proceedings of the AAAI Conference on Artificial Intelligence. vol. 33, pp. 5693–5700 (2019)

26. Zhang, H., Li, J., Kara, K., Alistarh, D., Liu, J., Zhang, C.: Zipml: Training linear models with end-to-end low precision, and a little bit of deep learning. In: Proceedings of the 34th International Conference on Machine Learning-Volume 70. pp. 4035–4043. JMLR. org (2017)

27. Zhang, J., De Sa, C., Mitliagkas, I., Ré, C.: Parallel sgd: When does averaging help? arXiv preprint arXiv:1606.07365 (2016)
A Supplementary Materials: Proofs

A.1 Proof of Lemma 1

The following two lemmas will show the unbiasness of the quantized method and its bound of the second moment.

**Lemma 2 (Unbiasness).** For any vector \( v \in \mathbb{R}^d \), we have

\[
E[Q_s(v)] = v
\]

*Proof.* For each index \( i \),

\[
E[\zeta(v_i, s)/s] = \frac{\ell}{s}(1 - P) + \frac{\ell + 1}{s}P = \frac{\ell}{s} + \frac{1}{s}P = \frac{\ell}{s}s \left( \frac{|v_i|}{||v||_2} - \ell \right) = \frac{|v_i|}{||v||_2}
\]

Thus,

\[
E(v'_i) = ||v||_2 \cdot sgn(v_i) \cdot E[\zeta(v_i, s)/s] = ||v||_2 \cdot sgn(v_i) \cdot \frac{|v_i|}{||v||_2} = v
\]

Obviously, the expected value of a quantized gradient is its original gradient.

**Lemma 3 (Second moment bound).** For any vector \( v \in \mathbb{R}^d \), we have

\[
E[|Q_s(v) - v|^2] \leq \frac{d}{4s^2}||v||_2^2
\]

*Proof.* In order to show the statement holds, we first find the expectation of the square of \( \zeta(v_i, s)/s \), i.e.

\[
E[(\zeta(v_i, s)/s)^2] = \frac{\ell^2}{s^2}(1 - P) + \frac{(\ell + 1)^2}{s^2}P = -\frac{1}{s^2}P^2 + \frac{1}{s^2}P + \frac{|v_i|^2}{||v||_2^2}
\]

\[
= -\frac{1}{s^2}(P - 1)^2 + \frac{1}{4s^2} + \frac{|v_i|^2}{||v||_2^2} \leq \frac{1}{4s^2} + \frac{|v_i|^2}{||v||_2^2}
\]

where (a) holds because the range of \( P \) is \([0, 1]\). Then,

\[
E[|Q_s(v)|_2^2] = \sum_{i=1}^{d} E[||v||_2^2 \cdot (\zeta(v_i, s)/s)^2] = ||v||_2^2 \cdot \sum_{i=1}^{d} E[(\zeta(v_i, s)/s)^2]
\]

\[
\leq ||v||_2^2 \cdot \sum_{i=1}^{d} \left( \frac{1}{4s^2} + \frac{|v_i|^2}{||v||_2^2} \right) = \left( \frac{d}{4s^2} + 1 \right)||v||_2^2
\]

Therefore, with Lemma 2, we get

\[
E[|Q_s(v) - v|^2] = E[|Q_s(v)|_2^2] - E[Q_s(v)]^2 = E[|Q_s(v)|_2^2] - ||E[Q_s(v)]||_2^2
\]

\[
\leq \left( \frac{d}{4s^2} + 1 \right)||v||_2^2 - ||v||_2^2 = \frac{d}{4s^2}||v||_2^2
\]
Main Proof of Lemma 1

Proof. Since \( w^{(i)} \) is independent with \( w^{(j)} \) for all \( i, j \in [1, M] \), in accordance with Lemma 2,

\[
\mathbb{E}\langle Q_s(w^{(i)}) - w^{(i)}, Q_s(w^{(j)}) - w^{(j)} \rangle = \mathbb{E}[Q_s(w^{(i)}) - w^{(i)}] \cdot \mathbb{E}[Q_s(w^{(j)}) - w^{(j)}] = 0
\]

Therefore, with Lemma 3,

\[
\mathbb{E}\left[\left\| \frac{1}{M} \sum_{m=1}^{M} Q_s(w^{(m)}) \right\|^2 \right] = \mathbb{E}\left[\left\| \frac{1}{M} \sum_{m=1}^{M} (Q_s(w^{(m)}) - w^{(m)}) \right\|^2 \right] + \left\| \frac{1}{M} \sum_{m=1}^{M} w^{(m)} \right\|^2 = 0
\]

Using Lemma 4.

Then, we have

\[
\mathbb{E}\left[\left\| \frac{1}{M} \sum_{m=1}^{M} Q_s(w^{(m)}) \right\|^2 \right] \leq \frac{d}{4s^2M^2} \sum_{m=1}^{M} \| w^{(m)} \|^2_2 + \left\| \frac{1}{M} \sum_{m=1}^{M} w^{(m)} \right\|^2_2
\]

A.2 Proof of Theorem 1 in AR Paradigm

To prove this theorem, the recursion function is therefore introduced:

\[
x^{(m)}(n, 0) = \bar{x}_n, \quad x^{(m)}(n, t) = \bar{x}_n - \gamma_n \sum_{j=0}^{t-1} \nabla f_m(x^{(m)}(n, j), \xi^{(m)}(n, j)), \quad (14)
\]

\[
\bar{x}_{n+1} = \bar{x}_n - \frac{1}{M} \sum_{m=1}^{M} Q_s \left( \gamma_n \sum_{k=0}^{K-1} \nabla f_m(x^{(m)}(n, k), \xi^{(m)}(n, k)) \right) \quad (15)
\]

Next, we prove some useful lemmas that contribute to the establishment of Theorem 4.

**Lemma 4.** Under Assumption 1, if stepsize \( \gamma_n \) satisfies the inequality

\[
1 - L^2 \gamma_n^2 K(K - 1) > 0
\]

Then, we have

\[
\frac{1}{M} \sum_{m=1}^{M} \sum_{k=0}^{K-1} \| \nabla f_m(x^{(m)}(n, k), \xi^{(m)}(n, k)) \|^2_2 \leq \frac{K \sigma^2}{1 - L^2 \gamma_n^2 K(K - 1)} + \frac{2K (\kappa^2 + \| \nabla F(\bar{x}_n) \|^2_2)}{1 - L^2 \gamma_n^2 K(K - 1)}
\]
Proof.

\[ ||\nabla f_m(x_{n;k}^{(m)}, \xi_k^{(m)})||_2^2 \leq \sigma^2 + ||\nabla F_m(x_{n;k}^{(m)})||_2^2 \]

\[ = \sigma^2 + ||(\nabla F_m(x_{n;k}^{(m)}) - \nabla F_m(\bar{x}_n)) + \nabla F_m(\bar{x}_n)||_2^2 \]

\[ \leq \sigma^2 + 2||\nabla F_m(x_{n;k}^{(m)}) - \nabla F_m(\bar{x}_n)||_2^2 + 2||\nabla F_m(\bar{x}_n)||_2^2 \]

\[ \leq \sigma^2 + 2L^2||\nabla F_m(\bar{x}_n)||_2^2 \]

\[ = \sigma^2 + 2L^2||x_{n;k}^{(m)} - \bar{x}_n||_2^2 + 2||\nabla F_m(\bar{x}_n)||_2^2 \]

\[ \leq \sigma^2 + 2L^2\gamma_n^2k\sum_{j=0}^{k-1}||\nabla f_m(x_{n;j}^{(m)}, \xi_j^{(m)})||_2^2 + 2||\nabla F_m(\bar{x}_n)||_2^2 \]

where (a) refers to \( ||a + b||_2^2 \leq 2||a||_2^2 + 2||b||_2^2 \), (b) is based on Assumption \[ \] and (c) is because of Cauchy-Schwarz inequality. Then, we sum all \( k \in \{0, 1, \ldots, K-1\} \) and have:

\[ \sum_{k=0}^{K-1} ||\nabla f_m(x_{n;k}^{(m)}, \xi_k^{(m)})||_2^2 \]

\[ \leq K\sigma^2 + 2L^2\gamma_n^2k\sum_{k=0}^{K-1} \sum_{j=0}^{k-1} ||\nabla f_m(x_{n;j}^{(m)}, \xi_j^{(m)})||_2^2 + 2K||\nabla F_m(\bar{x}_n)||_2^2 \]

\[ \leq K\sigma^2 + L^2\gamma_n^2K(K-1) \sum_{k=0}^{K-1} ||\nabla f_m(x_{n;k}^{(m)}, \xi_k^{(m)})||_2^2 + 2K||\nabla F_m(\bar{x}_n)||_2^2 \]

with the prerequisite \( 1 - L^2\gamma_n^2K(K-1) > 0 \),

\[ [1 - L^2\gamma_n^2K(K-1)] \sum_{k=0}^{K-1} ||\nabla f_m(x_{n;k}^{(m)}, \xi_k^{(m)})||_2^2 \leq K\sigma^2 + 2K||\nabla F_m(\bar{x}_n)||_2^2 \]

\[ \sum_{k=0}^{K-1} ||\nabla f_m(x_{n;k}^{(m)}, \xi_k^{(m)})||_2^2 \leq \frac{K\sigma^2}{1 - L^2\gamma_n^2K(K-1)} + \frac{2K||\nabla F_m(\bar{x}_n)||_2^2}{1 - L^2\gamma_n^2K(K-1)} \]

Under the variance bound of all individual workers in Assumption \[ \]

\[ \frac{1}{M} \sum_{m=1}^{M} \sum_{k=0}^{K-1} ||\nabla f_m(x_{n;k}^{(m)}, \xi_k^{(m)})||_2^2 \leq \frac{K\sigma^2}{1 - L^2\gamma_n^2K(K-1)} + \frac{2K\kappa^2 + ||\nabla F_m(\bar{x}_n)||_2^2}{1 - L^2\gamma_n^2K(K-1)} \]
Lemma 5. Under Assumption [1], the following equation holds:

\[
\mathbb{E} \left\| \frac{\gamma_n}{M} \sum_{m=1}^{M} \sum_{k=0}^{K-1} \nabla f_m(x_{n;k}^{(m)}, \xi_k^{(m)}) \right\|^2 \leq \frac{2\gamma_n^2 K^2}{M} \sigma^2 + 2\gamma_n^2 K \mathbb{E} \| \nabla F(\bar{x}_n) \|^2 + 2\gamma_n^2 K \sum_{k=1}^{K-1} \mathbb{E} \left\| \frac{1}{M} \sum_{m=1}^{M} \nabla F_m(x_{n;k}^{(m)}) \right\|^2
\]

Proof.

\[
\mathbb{E} \left\| \frac{\gamma_n}{M} \sum_{m=1}^{M} \sum_{k=0}^{K-1} \nabla f_m(x_{n;k}^{(m)}, \xi_k^{(m)}) \right\|^2 \leq \gamma_n^2 K \sum_{k=0}^{K-1} \mathbb{E} \left\| \frac{1}{M} \sum_{m=1}^{M} \nabla f_m(x_{n;k}^{(m)}, \xi_k^{(m)}) - \nabla F_m(x_{n;k}^{(m)}) \right\|^2
\]

\[
\leq 2\gamma_n^2 K \sum_{k=0}^{K-1} \mathbb{E} \left\| \frac{1}{M} \sum_{m=1}^{M} \nabla f_m(x_{n;k}^{(m)}, \xi_k^{(m)}) - \nabla F_m(x_{n;k}^{(m)}) \right\|^2 + 2\gamma_n^2 K \sum_{k=1}^{K-1} \mathbb{E} \left\| \frac{1}{M} \sum_{m=1}^{M} \nabla F_m(x_{n;k}^{(m)}) \right\|^2
\]

\[
= \frac{2\gamma_n^2 K}{M^2} \sum_{k=0}^{K-1} \sum_{m=1}^{M} \mathbb{E} \left\| \nabla f_m(x_{n;k}^{(m)}, \xi_k^{(m)}) - \nabla F_m(x_{n;k}^{(m)}) \right\|^2 + 2\gamma_n^2 K \sum_{k=0}^{K-1} \mathbb{E} \left\| \frac{1}{M} \sum_{m=1}^{M} \nabla F_m(x_{n;k}^{(m)}) \right\|^2
\]

where (a) follows \( \|a + b\|^2 \leq 2\|a\|^2 + 2\|b\|^2 \) and (b) is because of the variance bound of Assumption [7].

Lemma 6. Under Assumption [4], we have

\[
- \frac{1}{M} \sum_{m=1}^{M} \sum_{k=0}^{K-1} \mathbb{E} \left\langle \nabla F(\bar{x}_n), \nabla F_m(x_{n;k}^{(m)}) \right\rangle \leq - \frac{K+1}{2} \| \nabla F(\bar{x}_n) \|^2 - \frac{1}{2} \sum_{k=1}^{K-1} \mathbb{E} \left\| \frac{1}{M} \sum_{m=1}^{M} \nabla F_m(x_{n;k}^{(m)}) \right\|^2
\]
Lemma 7. Under Assumption 1, the following inequality holds for Algorithm 1

\[ \frac{L^2 \gamma^2}{4M} \sum_{k=0}^{K-1} \sum_{m=1}^{M} \left\| \nabla f_m \left( x_{n;k}^{(m)}, \xi_k^{(m)} \right) \right\|_2^2 \]

Proof.

\[ - \frac{1}{M} \sum_{m=1}^{M} \sum_{k=0}^{K-1} \mathbb{E} \left\langle \nabla f(x_n), \nabla f_m(x^{(m)}_{n;k}) \right\rangle = - \sum_{k=0}^{K-1} \mathbb{E} \left\langle \nabla f(x_n) \cdot \left( \frac{1}{M} \sum_{m=1}^{M} \nabla f_m(x^{(m)}_{n;k}) \right) \right\rangle \]

\[ = - \frac{1}{2} \sum_{k=0}^{K-1} (\| \nabla f(x_n) \|_2^2 + \| \frac{1}{M} \sum_{m=1}^{M} \nabla f_m(x^{(m)}_{n;k}) \|_2^2 \]

\[ - \| \nabla f(x_n) - \frac{1}{M} \sum_{m=1}^{M} \nabla f_m(x^{(m)}_{n;k}) \|_2^2 \]

\[ \leq - \frac{K}{2} \| \nabla f(x_n) \|_2^2 - \frac{1}{2} \sum_{k=0}^{K-1} \| \frac{1}{M} \sum_{m=1}^{M} \nabla f_m(x^{(m)}_{n;k}) \|_2^2 \]

\[ - \frac{1}{2} \sum_{k=0}^{K-1} \| \frac{1}{M} \sum_{m=1}^{M} \nabla f_m(x^{(m)}_{n;k}) \|_2^2 \]

\[ + \frac{L^2}{2M} \sum_{k=0}^{K-1} \sum_{m=1}^{M} \| \frac{1}{M} \sum_{m=1}^{M} \nabla f_m(x^{(m)}_{n;k}) \|_2^2 \]

\[ \leq - \frac{K}{2} \| \nabla f(x_n) \|_2^2 - \frac{1}{2} \sum_{k=1}^{K-1} \| \frac{1}{M} \sum_{m=1}^{M} \nabla f_m(x^{(m)}_{n;k}) \|_2^2 \]

\[ + \frac{L^2}{2M} \sum_{k=0}^{K-1} \sum_{m=1}^{M} \| \frac{1}{M} \sum_{m=1}^{M} \nabla f_m(x^{(m)}_{n;k}) \|_2^2 \]

\[ \leq - \frac{K}{2} \| \nabla f(x_n) \|_2^2 - \frac{1}{2} \sum_{k=1}^{K-1} \| \frac{1}{M} \sum_{m=1}^{M} \nabla f_m(x^{(m)}_{n;k}) \|_2^2 \]

\[ + \frac{L^2}{2M} \sum_{k=0}^{K-1} \sum_{m=1}^{M} \| \frac{1}{M} \sum_{m=1}^{M} \nabla f_m(x^{(m)}_{n;k}) \|_2^2 \]

\[ \leq - \frac{K}{2} \| \nabla f(x_n) \|_2^2 - \frac{1}{2} \sum_{k=1}^{K-1} \| \frac{1}{M} \sum_{m=1}^{M} \nabla f_m(x^{(m)}_{n;k}) \|_2^2 \]

\[ + \frac{L^2}{2M} \sum_{k=0}^{K-1} \sum_{m=1}^{M} \| \frac{1}{M} \sum_{m=1}^{M} \nabla f_m(x^{(m)}_{n;k}) \|_2^2 \]

where (a) follows \( \nabla f(w) = \frac{1}{M} \sum_{m=1}^{M} \nabla f_m(w) \), (b) is from the smoothness of Assumption 1, (c) is based on Cauchy-Schwarz inequality.

**Lemma 7.** Under Assumption 1, the following inequality holds for Algorithm 1.
\[ \mathbb{E}[F(\tilde{x}_{n+1}) - F(\tilde{x}_n)] \]
\[ \leq -\frac{\gamma_n}{2} [K + 1 - 2L\gamma_n K - \frac{L\gamma_n K^2}{1 - L^2\gamma_n^2 K(K - 1)}(L\gamma_n(K - 1) + \frac{d}{2s^2 M})] \mathbb{E}[\|\nabla F(\tilde{x}_n)\|^2_2] \]
\[ - \frac{\gamma_n}{2} [1 - 2L\gamma_n K] \sum_{k=1}^{K-1} \| \frac{1}{M} \sum_{m=1}^{M} \nabla f_m(x_{n;k}^{(m)}), \xi_k^{(m)} \|_2^2 + \frac{L\gamma_n^2 K^2 \sigma^2}{M} \]
\[ + \frac{L\gamma_n^2 K^2}{4} (L\gamma_n(K - 1) + \frac{d}{2s^2 M})(\frac{\sigma^2 + 2\kappa^2}{1 - L^2\gamma_n^2 K(K - 1)}) \]

**Proof.**
\[ \mathbb{E}[F(\tilde{x}_{n+1}) - F(\tilde{x}_n)] = \mathbb{E}[\nabla F(\tilde{x}_n), \tilde{x}_{n+1} - \tilde{x}_n] + \frac{1}{2} L \|\tilde{x}_{n+1} - \tilde{x}_n\|^2_2 \]
\[ = \mathbb{E}[\nabla F(\tilde{x}_n), \frac{1}{M} \sum_{m=1}^{M} Q_s \left( \gamma_n \sum_{k=0}^{K-1} \nabla f_m(x_{n;k}, \xi_k) \right)] \]
\[ + \frac{L}{2} \| \frac{1}{M} \sum_{m=1}^{M} Q_s \left( \gamma_n \sum_{k=0}^{K-1} \nabla f_m(x_{n;k}^{(m)}, \xi_k^{(m)}) \right) \|^2_2 \]
\[ \leq -\frac{(K + 1)\gamma_n}{2} \|\nabla F(\tilde{x}_n)\|^2_2 - \frac{\gamma_n}{2} \sum_{k=1}^{K-1} \| \frac{1}{M} \sum_{m=1}^{M} \nabla f_m(x_{n;k}) \|^2_2 \]
\[ + \frac{L^2 \gamma_n^3 K(K - 1)}{4M} \sum_{k=1}^{K-1} \sum_{m=1}^{M} \|\nabla f_m(X_{n;\tilde{x}}, \xi_j^{(m)})\|^2_2 + \frac{\gamma_n^2 dL K}{8s^2 M^2} \sum_{m=1}^{M} \mathbb{E}[\|\nabla f_m(x_{n;k}, \xi_k)\|^2_2] \]
\[ \leq \frac{\gamma_n^2 K^2 \sigma^2 L}{M} + L\gamma_n^2 K E[\|\nabla F(\tilde{x}_n)\|^2_2] + L\gamma_n^2 K \sum_{k=1}^{K-1} E[\| \frac{1}{M} \sum_{m=1}^{M} \nabla F(x_{n;k}) \|^2_2] \]
\[ = \left[ -\frac{(K + 1)\gamma_n}{2} - L\gamma_n^2 K \right] \|\nabla F(\tilde{x}_n)\|^2_2 - \left( \frac{\gamma_n}{2} - L\gamma_n^2 K \right) \sum_{k=1}^{K-1} \| \frac{1}{M} \sum_{m=1}^{M} \nabla f_m(x_{n;k}) \|^2_2 \]
\[ + \frac{L\gamma_n^2 K^2 \sigma^2}{M} + \left( \frac{L^2 \gamma_n^3 K(K - 1)}{4M} + \frac{dL\gamma_n^2 K}{8s^2 M^2} \right) \sum_{m=1}^{M} \sum_{k=0}^{K-1} \|\nabla f_m(x_{n;k}, \xi_k)\|^2_2 \]
\[ \leq \left[ -\frac{(K + 1)\gamma_n}{2} + L\gamma_n^2 K + \frac{L\gamma_n^2 K}{2} (L\gamma_n(K - 1) + \frac{d}{2s^2 M}) \right] \frac{K}{1 - L^2\gamma_n^2 K(K - 1)} \]
\[ \mathbb{E}[\|\nabla F(\tilde{x}_n)\|^2_2] \]
\[-\frac{\gamma_n}{2} \left( 1 - 2L\gamma_n K \right) \sum_{k=1}^{K-1} \left\| \frac{1}{M} \sum_{m=1}^{M} \nabla F_{m}(x_{n,k}^{(m)}) \right\|^2 + \frac{L\gamma_n^2 K^2 \sigma^2}{M} \]
\[+ \frac{L\gamma_n^2 K^2}{4} (L\gamma_n(K - 1) + \frac{d}{2s^2 M})(\frac{\sigma^2 + 2\kappa^2}{1 - L^2 \gamma_n^2 K(K - 1)}) \]

where (a) follows Lemma 5, Lemma 6 and Cauchy-Schwarz inequality, (b) follows Lemma 4.

**Main Proof of Theorem 1**

**Proof.** With \(12L\bar{\gamma}K > 0\) and \(L^2 \bar{\gamma}^2 K(K - 4) \leq 1 - \delta\),

\[
E[F(\tilde{x}_{n+1}) - F(\tilde{x}_n)] 
\leq \frac{\bar{\gamma}}{2} (K - L^2 \bar{\gamma}^2 K^2(K - 1) - \frac{K\bar{\gamma}^2 d}{2s^2 M\delta}) \|\nabla F(\tilde{x}_n)\|^2 
+ \frac{L\bar{\gamma}^2 K^2 \sigma^2}{M} + \frac{L\bar{\gamma}^2 K^2 (\sigma^2 + 2\kappa^2)}{4\delta} [L\bar{\gamma}(K - 1) + \frac{d}{2s^2 M}] \]

\[
F_* - F(\tilde{X}_1) \leq E[F(\tilde{X}_{N+1}) - F(\tilde{X}_1)] 
\leq \frac{\bar{\gamma}}{2} (K - L^2 \bar{\gamma}^2 K^2(K - 1) - \frac{K\bar{\gamma}^2 d}{2s^2 M\delta}) \sum_{n=1}^{N} \|\nabla F(\tilde{x}_n)\|^2 
+ \frac{L\bar{\gamma}^2 K^2 \sigma^2}{M} + \frac{L\bar{\gamma}^2 K^2 (\sigma^2 + 2\kappa^2)}{4\delta} [L\bar{\gamma}(K - 1) + \frac{d}{2s^2 M}] \]

Hence, with

\[
\frac{L\bar{\gamma} K}{\delta} (L\bar{\gamma}(K - 1) + \frac{d}{2s^2}) < 1 - \varepsilon \tag{16} \]

We have

\[
\frac{\bar{\gamma} K \varepsilon}{2} \sum_{n=1}^{N} \|\nabla F(\tilde{x}_n)\|^2 \leq \frac{\bar{\gamma} K}{2} (1 - \frac{L\bar{\gamma} K}{\delta} (L\bar{\gamma}(K - 1) - \frac{d}{2s^2 M})) 
\leq \left( \frac{L\bar{\gamma}^2 K^2 \sigma^2}{M} + \frac{L\bar{\gamma}^2 K^2 (\sigma^2 + 2\kappa^2)}{4\delta} \right) [L\bar{\gamma}(K - 1) + \frac{d}{2s^2 M}] \] \[N + F(\tilde{x}_1) - F_* \]

**Proof of Corollary 1** We choose the constant stepsize:

\[
\tilde{\gamma} = \sqrt{\frac{F(\tilde{x}_1) - F_*}{D_1 N L K^2}} \tag{17} \]

With the given \(N\), it can satisfy all the prerequisite in Theorem 1. Then:

\[
\frac{1}{N} \sum_{n=1}^{N} E[\|\nabla F(\tilde{x}_n)\|^2] \leq \frac{K(K - 1)(\sigma^2 + 2\kappa^2)}{2\varepsilon \delta} L^2 \tilde{\gamma}^2 + \frac{D_1}{M \varepsilon} KL \tilde{\gamma} + \frac{2[F(\tilde{x}_1) - F_*]}{\tilde{\gamma} K \varepsilon N} \]
\[
(\mathbf{K} - 1)\left(\sigma^2 + 2\kappa^2\right) \cdot L \cdot \frac{F(\tilde{x}_1)F_*}{D_1NK} + \frac{\sqrt{D_1}}{\sqrt{M\varepsilon}} \cdot \sqrt{\frac{F(\tilde{x}_1)F_*}{N}}
\]
\[
+ \frac{2\sqrt{\left[F(\tilde{x}_1)F_*\right]}}{\varepsilon\sqrt{N}\sqrt{\frac{M}{D_1L}}}
\]
\[
= \frac{(\mathbf{K} - 1)\left(\sigma^2 + 2\kappa^2\right) L [F(\tilde{x}_1)F_*] M}{2\varepsilon D_1NK} + \frac{3\sqrt{D_1L[F(\tilde{x}_1)F_*]}}{\varepsilon\sqrt{NM}}
\]

Since
\[
N \geq \frac{(\mathbf{K} - 1)^2(\sigma^2 + 2\kappa^2)^2LM^3(F(\tilde{x}_1)F_*)}{4\delta^2K^2D_1^2}
\]

Therefore,
\[
\frac{(\mathbf{K} - 1)(\sigma^2 + 2\kappa^2)LM^3(F(\tilde{x}_1)F_*)}{2\varepsilon D_1NK} \leq \frac{\sqrt{D_1L[F(\tilde{x}_1)F_*]}}{\varepsilon\sqrt{NM}}
\]

\[
\frac{1}{N} \sum_{n=1}^{N} E[||\nabla F(\tilde{x}_n)||^2] \leq \frac{4\sqrt{D_1L[F(\tilde{x}_1)F_*]}}{\varepsilon\sqrt{NM}}
\]

A.3 Proof of Theorem 2 in Gossip paradigm

Recursion function for Gossip paradigm:
\[
X_n = \begin{bmatrix}
\tilde{x}_n^{(1)} \\
\tilde{x}_n^{(2)} \\
\vdots \\
\tilde{x}_n^{(M)}
\end{bmatrix}
\]
\[
X_{n;k} = \begin{bmatrix}
x_{n;k}^{(1)} \\
x_{n;k}^{(2)} \\
\vdots \\
x_{n;k}^{(M)}
\end{bmatrix}
\]
\[
G(X_{n;k}, \xi_k) = \begin{bmatrix}
\nabla f_1(x_{n;k}, \xi_k^{(1)}) \\
\nabla f_2(x_{n;k}, \xi_k^{(2)}) \\
\vdots \\
\nabla f_M(x_{n;k}, \xi_k^{(M)})
\end{bmatrix}
\]
\[
\partial F(X_{n;k}) = \begin{bmatrix}
\nabla F_1(x_{n;k}) \\
\nabla F_2(x_{n;k}) \\
\vdots \\
\nabla F_M(x_{n;k})
\end{bmatrix}
\]

Therefore, the recursion function can be represented as follows:
\[
X_{n+1} = X_n W - \gamma_n \sum_{k=0}^{K-1} G(X_{n;k}, \xi_k) + C_n;
\]
\[
X_{n;t} = X_n - \gamma_n \sum_{j=0}^{t-1} G(X_{n;j}, \xi_j)
\]

where \(W = \) symmetric doubly stochastic matrix:
\[ \Delta X_n = X_n W - \gamma_n \sum_{k=0}^{K-1} G(X_{n,k}, \xi_k) - X_n \]
\[ C_n = Q_s(\Delta X_n) - \Delta X_n \]

**Lemma 8 (Lemma 5 in [21]).**
For any matrix \( X_t \in \mathbb{R}^{N \times n} \), decompose the confusion matrix \( W \) as \( W = \sum_{i=1}^{n} \lambda_i \mathbf{v}(i)(\mathbf{v}(i)^T) = P\Lambda P^T \), where \( P = (\mathbf{v}(1), \mathbf{v}(1), \ldots, \mathbf{v}(n)) \in \mathbb{R}^{N \times n} \), \( \mathbf{v}(i) \) is the normalized eigenvector of \( \lambda_i \) and \( \Lambda \) is a diagonal matrix with \( \lambda_i \) be its ith element. We have
\[
\sum_{m=1}^{M} \mathbb{E}\|X_n \cdot W^T \cdot e^{(m)} - X_n \cdot \frac{1}{M} \|_F^2 \leq \| \rho^2 \|_F \| X_n \|_F^2
\]

**Lemma 9 (Lemma 6 in [21]).**
Given two non-negative sequences \( \{a_t\}_{t=1}^{\infty} \) and \( \{b_t\}_{t=1}^{\infty} \) that satisfying
\[
a_\varepsilon = \sum_{\eta=1}^{\varepsilon} \rho^{\varepsilon-\eta} b_\eta
\]
with \( \rho \in [0, 1) \), we have
\[
\sum_{\varepsilon=1}^{k} a_\varepsilon \leq \frac{1}{1-\rho} \sum_{\varepsilon=1}^{k} b_\varepsilon
\]
\[
\sum_{\varepsilon=1}^{k} a_\varepsilon^2 \leq \frac{1}{(1-\rho)^2} \sum_{\varepsilon=1}^{k} b_\varepsilon^2
\]

**Lemma 10.** Given the fixed size \( \bar{\gamma} \), under Assumption 3 and Assumption 2 we have
\[
\sum_{n=1}^{N} \sum_{k=0}^{K-1} \sum_{m=1}^{M} \mathbb{E}\|X_n \cdot \frac{1}{M} - \tilde{x}_n^{(m)} \|_2 \leq \frac{2K}{1-\rho} \sum_{n=1}^{N} \mathbb{E}\|C_n\|_F^2
\]
\[
+ \frac{2K^2 \bar{\gamma}^2}{(1-\rho)^2} \sum_{n=1}^{N} \sum_{k=0}^{K-1} ||G(X_{n,k}, \xi_k)||_F^2
\]

**Proof.**
\[
X_n \cdot \frac{1}{M} = - \sum_{\varepsilon=1}^{n-1} [\bar{\gamma} \sum_{k=0}^{K-1} G(X_{\varepsilon;k}, \xi_k)] \cdot \frac{1}{M} + \sum_{\varepsilon=1}^{n-1} C_\varepsilon \cdot \frac{1}{M}
\]
\[
\tilde{X}_n^{(m)} = - \sum_{\varepsilon=1}^{n-1} \gamma \sum_{k=0}^{K-1} G(X_{\varepsilon:k}, \xi_k) W^{n-\varepsilon-1} e^{(m)} + \sum_{\varepsilon=1}^{n-1} C_\varepsilon W^{n-\varepsilon-1} e^{(m)}
\]

Then,

\[
\sum_{m=1}^{M} \mathbb{E} \left| \frac{X_n \cdot 1_M}{M} - \tilde{X}_n^{(m)} \right|^2 \\
= \sum_{m=1}^{M} \mathbb{E} \left| \left( \sum_{\varepsilon=1}^{n-1} C_\varepsilon \cdot \frac{1_M}{M} - \sum_{\varepsilon=1}^{n-1} C_\varepsilon W^{n-\varepsilon-1} e^{(m)} \right) \\
- \left( \sum_{s=1}^{\tilde{\gamma}} \sum_{t=0}^{K-1} G(X_{\varepsilon:t}, \xi_t) \cdot \frac{1_M}{M} - \sum_{s=1}^{n-1} \tilde{\gamma} \sum_{t=0}^{K-1} G(X_{\varepsilon:t}, \xi_t) W^{n-\varepsilon-1} e^{(m)} \right) \right|^2 \\
\leq (a) 2 \sum_{m=1}^{M} \mathbb{E} \left| \sum_{\varepsilon=1}^{n-1} C_\varepsilon \cdot \frac{1_M}{M} - \sum_{\varepsilon=1}^{n-1} C_\varepsilon W^{n-\varepsilon-1} e^{(m)} \right|^2 \\
+ 2 \sum_{m=1}^{M} \mathbb{E} \left| \tilde{\gamma} \sum_{t=0}^{K-1} \sum_{\varepsilon=1}^{n-1} (G(X_{\varepsilon:t}, \xi_t) - G(X_{\varepsilon:t}, \xi_t)) W^{n-\varepsilon-1} e^{(m)} \right|^2 \\
\leq (b) 2 \sum_{m=1}^{M} \sum_{\varepsilon=1}^{n-1} \mathbb{E} \left| C_\varepsilon \cdot \frac{1_M}{M} - C_\varepsilon W^{n-\varepsilon-1} e^{(m)} \right|^2 \\
+ 2 \sum_{m=1}^{M} \mathbb{E} \left| \tilde{\gamma} \sum_{t=0}^{K-1} \sum_{\varepsilon=1}^{n-1} (G(X_{\varepsilon:t}, \xi_t) - G(X_{\varepsilon:t}, \xi_t)) W^{n-\varepsilon-1} e^{(m)} \right|^2 \\
\leq (c) 2 \sum_{\varepsilon=1}^{n-1} \mathbb{E} \left| \rho^{n-\varepsilon-1} C_\varepsilon \right|^2 + 2 \mathbb{E} \left( \tilde{\gamma} \sum_{\varepsilon=1}^{n-1} \sum_{t=0}^{K-1} \rho^{n-\varepsilon-1} \left| G(X_{\varepsilon:t}, \xi_t) \right| \right)^2
\]

where (a) follows \(|a - b|^2 \leq 2|a|^2 + 2|b|^2\), (b) is according to the fact that the expected compression loss is 0 and all nodes work independently, (c) is based on Lemma. \[\text{With Lemma.} 10\] we have,

\[
\sum_{n=1}^{N} \sum_{\varepsilon=1}^{n-1} \mathbb{E} \left| \rho^{n-\varepsilon-1} C_\varepsilon \right|^2 \leq \frac{1}{1 - \rho^2} \sum_{n=1}^{N} \mathbb{E} \left| C_n \right|^2 \\
\sum_{m=1}^{M} \mathbb{E} \left( \tilde{\gamma} \sum_{\varepsilon=1}^{n-1} \sum_{t=0}^{K-1} \rho^{n-\varepsilon-1} \left| G(X_{\varepsilon:t}, \xi_t) \right| \right)^2 \leq \frac{1}{(1 - \rho)^2} \sum_{n=1}^{N} \sum_{t=0}^{K-1} \left| G(X_{n:t}, \xi_t) \right|^2 \\
\leq \frac{\tilde{\gamma}^2 K}{(1 - \rho)^2} \sum_{n=1}^{N} \sum_{t=0}^{K-1} \left| G(X_{n:t}, \xi_t) \right|^2 
\]

Therefore,
\[
\sum_{n=1}^{N} \sum_{k=0}^{K-1} \sum_{m=1}^{M} \mathbb{E}\left\| \frac{X_n \cdot 1_M}{M} - \bar{X}_n^{(m)} \right\|^2_2 \\
\leq 2 \sum_{n=1}^{N} \sum_{k=0}^{K-1} \sum_{\epsilon=1}^{n-1} \mathbb{E}\left\| \rho^{n-\epsilon} \sigma_{\epsilon} \right\|^2_2 + 2 \sum_{n=1}^{N} \sum_{k=0}^{K-1} \mathbb{E}\left\| \mu_{\epsilon} \sum_{m=1}^{M} \rho^{n-\epsilon-1} \|G(X_{\epsilon, t}, \xi_{\epsilon})\|_F^2 \right\|_F^2 \\
\leq \frac{2K}{1-\rho^2} \sum_{n=1}^{N} \mathbb{E}\|C_n\|^2_2 + \frac{4K^2 \bar{\gamma}^2}{(1-\rho^2)^2} \sum_{n=1}^{N} \sum_{k=0}^{K-1} \|G(X_{n,k}, \xi_k)\|_F^2 \\
\text{Lemma 11. Given the fixed stepsize } \bar{\gamma}, \text{ under Assumption 1 and Assumption 2, we have:} \\
\sum_{n=1}^{N} \sum_{k=0}^{K-1} \sum_{m=1}^{M} \mathbb{E}\left\| \frac{X_n \cdot 1_M}{M} x_{n,k}^{(m)} \right\|^2_2 \\
\leq \frac{4K}{1-\rho^2} \sum_{n=1}^{N} \mathbb{E}\|C_n\|^2_2 + \left[ \frac{4K^2 \bar{\gamma}^2}{(1-\rho^2)^2} + \bar{\gamma}^2 K (K-1) \right] \sum_{n=1}^{N} \sum_{k=0}^{K-1} \|G(X_{n,k}, \xi_k)\|_F^2 \\
\text{Proof.} \\
\sum_{m=1}^{M} \mathbb{E}\left\| \frac{X_n \cdot 1_M}{M} - x_{n,k}^{(m)} \right\|^2_2 \\
= \sum_{m=1}^{M} \mathbb{E}\left\| \frac{X_n \cdot 1_M}{M} - \bar{x}_n^{(m)} + \bar{\gamma} \sum_{j=0}^{k-1} \nabla f_m(x_{n,j}^{(m)}, \xi_j^{(m)}) \right\|^2_2 \\
\leq 2 \sum_{m=1}^{M} \mathbb{E}\left\| \frac{X_n \cdot 1_M}{M} - \bar{x}_n^{(m)} \right\|^2_2 + 2 \sum_{m=1}^{M} \mathbb{E}\left\| \bar{\gamma} \sum_{j=0}^{k-1} \nabla f_m(x_{n,j}^{(m)}, \xi_j^{(m)}) \right\|^2_2 \\
\leq 2 \sum_{m=1}^{M} \mathbb{E}\left\| \frac{X_n \cdot 1_M}{M} - \bar{x}_n^{(m)} \right\|^2_2 + 2 \bar{\gamma}^2 k \sum_{m=1}^{M} \mathbb{E}\left\| \nabla f_m(x_{n,j}^{(m)}, \xi_j^{(m)}) \right\|^2_2 \\
= 2 \sum_{m=1}^{M} \mathbb{E}\left\| \frac{X_n \cdot 1_M}{M} - \bar{x}_n^{(m)} \right\|^2_2 + 2 \bar{\gamma}^2 k \sum_{j=0}^{k-1} \mathbb{E}\left\| G(x_{n,j}^{(m)}, \xi_j^{(m)}) \right\|^2_2 \\
\text{where (a) follows } \|a + b\|^2_2 \leq 2\|a\|^2_2 + 2\|b\|^2_2, \text{ (b) is based on Cauchy-Schwarz inequality. Therefore,} \\
\sum_{n=1}^{N} \sum_{k=0}^{K-1} \sum_{m=1}^{M} \mathbb{E}\left\| \frac{X_n \cdot 1_M}{M} - x_{n,k}^{(m)} \right\|^2_2 \\
\leq 2 \sum_{n=1}^{N} \sum_{k=0}^{K-1} \sum_{m=1}^{M} \mathbb{E}\left\| \frac{X_n \cdot 1_M}{M} - \bar{x}_n^{(m)} \right\|^2_2 + 2 \bar{\gamma}^2 k \sum_{n=1}^{N} \sum_{k=0}^{K-1} \sum_{j=0}^{k-1} \mathbb{E}\left\| G(x_{n,j}^{(m)}, \xi_j^{(m)}) \right\|^2_2 \\
\]
Proof.

Lemma 12. Suppose $1 - (d\mu^2)/(s^2(1 - \rho)^2) > 0$. With the fixed stepsize, under Assumption 1 and Assumption 2, the ergodic second bound of the noise of compression is:

$$
= 2 \sum_{n=1}^{N} \sum_{k=0}^{K-1} \sum_{m=1}^{M} \mathbb{E}[\| \frac{X_n \cdot 1M}{M} - x_n^{(m)} \|_2^2] \\
+ 2\gamma^2 \sum_{n=1}^{N} \sum_{k=0}^{K-1} \frac{(K + k)(K - k - 1)}{2} \mathbb{E}[\| G(x_{n,j}^{(m)}, \xi_j^{(m)}) \|_F^2] \\
\leq \frac{2(2K}{1 - \rho^2} \sum_{n=1}^{N} \mathbb{E}[\| C_n \|_F^2] + \frac{2K^2\gamma^2}{(1 - \rho)^2} \sum_{n=1}^{N} \sum_{k=0}^{K-1} \mathbb{E}[\| G(x_{n,k}, \xi_k) \|_F^2] \\
+ \frac{2\gamma^2 K(K - 1)}{2} \sum_{n=1}^{N} \sum_{k=0}^{K-1} \mathbb{E}[\| G(x_{n,j}^{(m)}, \xi_j^{(m)}) \|_F^2]
$$

where (a) comes from Lemma 11.

Lemma 12. Suppose $1 - (d\mu^2)/(s^2(1 - \rho)^2) > 0$. With the fixed stepsize, under Assumption 1 and Assumption 2, the ergodic second bound of the noise of compression is:

$$
\sum_{n=1}^{N} \sum_{m=1}^{M} \mathbb{E}[\| C_n^{(m)} \|_2^2] = \sum_{n=1}^{N} \mathbb{E}[\| C_n \|_F^2] \\
\leq \gamma^2 K \frac{d(1 - \rho^2) + 2d\mu^2}{2(s^2(1 - \rho)^2 - d\mu^2)} \sum_{n=1}^{N} \sum_{k=0}^{K-1} \mathbb{E}[\| G(X_{n,k}, \xi_k) \|_F^2]
$$

Proof.

$$
\sum_{n=1}^{N} \sum_{m=1}^{M} \mathbb{E}[\| C_n^{(m)} \|_2^2] \leq \frac{d}{4s^2} \sum_{n=1}^{N} \sum_{m=1}^{M} \mathbb{E}[\| \Delta X_n^{(m)} \|_2^2] \\
= \frac{d}{4s^2} \sum_{n=1}^{N} \sum_{m=1}^{M} \mathbb{E}[\| X_n(W - I)e^{(m)} - \gamma_n \sum_{k=0}^{K-1} \nabla f_m(X_{n,k}^{(m)}, \xi_k^{(m)}) \|_2^2] \\
\leq \frac{d}{2s^2} \sum_{n=1}^{N} \sum_{m=1}^{M} \mathbb{E}[\| X_n(W - I)e^{(m)} \|_2^2] + \frac{d}{2s^2} \sum_{n=1}^{N} \sum_{m=1}^{M} \mathbb{E}[\| \gamma_n \sum_{k=0}^{K-1} \nabla f_m(X_{n,k}^{(m)}, \xi_k^{(m)}) \|_2^2]
$$

where (a) is based on Lemma 3 and (b) is from $\| a + b \|_2^2 \leq 2\| a \|_2^2$. Considering the first term,

$$
\sum_{m=1}^{M} \mathbb{E}[\| X_n(W - I)e^{(m)} \|_2^2] = \mathbb{E}[\| X_n(W - I) \|_F^2] \\
= \mathbb{E}[\| X_nP(A - I)P^T \|_F^2]
$$
\[= \mathbb{E}||X_n P||^2_F = \sum_{m=1}^M (\lambda_m - 1)^2 \mathbb{E}||\tilde{x}_n^{(m)}||^2_2 \]

\[\leq \mu^2 \sum_{m=2}^M \mathbb{E}||\tilde{x}_n^{(m)}||^2_2 \quad (\mu := \max_{i \in \{2,3,\ldots,M\}} |\lambda_i - 1|)\]

Then,

\[\sum_{n=1}^N \sum_{m=2}^M \mathbb{E}||c_n^{(m)}||^2_2 \leq \frac{d \mu^2}{2\delta^2} \sum_{n=1}^N \sum_{m=2}^M \mathbb{E}||\tilde{x}_n^{(m)}||^2_2 + \frac{d}{2\delta^2} \sum_{n=1}^N \sum_{m=1}^M \mathbb{E}||\gamma K-1 \sum_{k=0}^{K-1} \nabla f_m(x_{n,k}, \xi_k)||^2_2\]

With the recursion formula, we have the following recurrence one:

\[X_n = X_{n-1} W - \gamma_n \sum_{k=0}^{K-1} G(X_{n-1;k}, \xi_k) + C_n \]

\[= - \sum_{\epsilon=1}^{n-1} \bar{\gamma} \sum_{k=0}^{K-1} G(X_{\epsilon;k}, \xi_k) W^{n-\epsilon-1} + \sum_{\epsilon=1}^{n-1} C_\epsilon W^{n-\epsilon-1}\]

Therefore,

\[\sum_{m=2}^M \mathbb{E}||\tilde{x}_n^{(m)}||^2_2\]

\[= \sum_{m=2}^M \mathbb{E}|| - \sum_{\epsilon=1}^{n-1} \bar{\gamma} \sum_{k=0}^{K-1} G(X_{\epsilon;k}, \xi_k) W^{n-\epsilon-1} e^{(m)} + \sum_{\epsilon=1}^{n-1} C_\epsilon W^{n-\epsilon-1} e^{(m)} ||^2_2\]

\[\leq \sum_{m=2}^M 2\mathbb{E}|| \sum_{\epsilon=1}^{n-1} C_\epsilon W^{n-\epsilon-1} e^{(m)} ||^2_2 + 2 \sum_{m=2}^M \mathbb{E}|| \sum_{\epsilon=1}^{n-1} \bar{\gamma} \sum_{k=0}^{K-1} G(X_{\epsilon;k}, \xi_k) W^{n-\epsilon-1} e^{(m)} ||^2_2\]

Considering the following formula,

\[\sum_{n=1}^N \sum_{m=2}^M \mathbb{E}|| \sum_{\epsilon=1}^{n-1} C_\epsilon W^{n-\epsilon-1} e^{(m)} ||\]
\[
\begin{align*}
&= \sum_{n=1}^{N} \mathbb{E} \left[ \sum_{\varepsilon=1}^{n-1} C_\varepsilon \right] P \left[ \begin{array}{ccc}
0 & \lambda_2 - \varepsilon - 1 & \lambda_3 - \varepsilon - 1 \\
\lambda_2 & \ddots & \\
& & \lambda_n - \varepsilon - 1
\end{array} \right] F
\leq \sum_{n=1}^{N} \mathbb{E} \left[ \sum_{\varepsilon=1}^{n-1} \rho^{n-\varepsilon-1} C_\varepsilon \right] F^2 \\
&= \frac{1}{(1-\rho)^2} \sum_{n=1}^{N} \left| C_n \right|^2 F^2 \\
&= \frac{1}{(1-\rho)^2} \sum_{n=1}^{N} \sum_{m=1}^{M} \mathbb{E} \left[ \left| C_n^{(m)} \right|^2 \right] F^2 \\
&\leq \frac{1}{(1-\rho)^2} \sum_{n=1}^{N} \sum_{m=1}^{M} \mathbb{E} \left[ \left| \gamma \sum_{k=0}^{K-1} \nabla f_m(x_{n,k}, \xi_k^{(m)}) \right|^2 \right]
\end{align*}
\]

Therefore,
\[
\sum_{n=1}^{N} \sum_{m=2}^{M} \mathbb{E} \left[ \left| C_n^{(m)} \right|^2 \right] F^2 \leq \frac{d \mu^2}{s^2(1-\rho)^2} \sum_{n=1}^{N} \sum_{m=1}^{M} \mathbb{E} \left[ \left| C_n^{(m)} \right|^2 \right] F^2 \\
+ \frac{d \mu^2}{2s^2} \sum_{n=1}^{N} \sum_{m=1}^{M} \left\| \nabla f_m(x_{n,k}, \xi_k^{(m)}) \right\|^2 F^2
\]

With above result, we have:
\[
[1 - \frac{d \mu^2}{s^2(1-\rho)^2}] \sum_{n=1}^{N} \sum_{m=1}^{M} \mathbb{E} \left[ \left| C_n^{(m)} \right|^2 \right] F^2 \\
\leq \frac{d}{2s^2} \mathbb{E} \left[ \left| \gamma \sum_{k=0}^{K-1} \nabla f_m(X_{n,k}, \xi_k) \right|^2 \right] F^2 \\
\leq \gamma^2 K \left[ \frac{d}{2s^2} + \frac{d \mu^2}{s^2(1-\rho)^2} \right] \sum_{n=1}^{N} \sum_{m=1}^{M} \sum_{k=0}^{K-1} \left\| \nabla f_m(X_{n,k}, \xi_k) \right\|^2 F^2 \\
= \gamma^2 K \left[ \frac{d}{2s^2} + \frac{d \mu^2}{s^2(1-\rho)^2} \right] \sum_{n=1}^{N} \sum_{k=0}^{K-1} \left\| \nabla f_m(X_{n,k}, \xi_k) \right\|^2 F^2
\]
Thus, with $1 - \frac{d_f^2}{\sigma^2(1 - \rho)^2} > 0$, the theorem is proved.

**Lemma 13.** Suppose $1 - (d_f^2)/(\sigma^2(1 - \rho)^2) > 0$. Given the fixed stepsize $\gamma$, satisfying:

$$18L^2\gamma^2KD_1D_3 > 0 \quad \text{and} \quad 1 - 12\gamma^2L^2(K+1)(K-2) > 0$$

under Assumption 7 and Assumption 2, we have

$$\sum_{n=1}^{N} \sum_{k=0}^{K} ||G(X_{n,k}, \xi_k)||_2^2 \leq \frac{MNK(\sigma^2 + 4\kappa^2) + 4NL^2\sigma^2\gamma^2K(K-1)}{1 - 8L^2\gamma^2KD_1D_3}$$

$$+ \frac{8MNK(K-1)L^2\gamma^2[\sigma^2 + \kappa^2(2K-1)]}{(1 - 8L^2\gamma^2KD_1D_3)[1 - 12\gamma^2L^2(K+1)(K-2)]}$$

$$+ \frac{4MK(K-1)L^2\gamma^2}{1 - 8L^2\gamma^2KD_1D_3} \sum_{n=1}^{N} \sum_{k=0}^{K-1} ||\frac{\partial F(X_{n,k}) \cdot 1_M}{M}||_2^2$$

$$+ \frac{4K}{1 - 8L^2\gamma^2KD_1D_3} \sum_{n=1}^{N} ||\nabla F\left(\frac{X_n \cdot 1_M}{M}\right)||_2^2$$

where $D_1'$ and $D_3'$ are the same as the notation in Theorem 2.

**Proof.**

$$||G(x_{n,k}, \xi_k)||_2^2 = \sum_{m=1}^{M} ||\nabla f_m(x_{n,k}^{(m)}, \xi_k^{(m)})||_2^2$$

$$= \sum_{m=1}^{M} (||\nabla f_m(x_{n,k}^{(m)}, \xi_k^{(m)}) - \nabla F_m(x_{n,k}^{(m)})||_2^2 + ||\nabla F_m(x_{n,k}^{(m)})||_2^2)$$

$$= M\sigma^2 + \sum_{m=1}^{M} ||\nabla F_m(x_{n,k}^{(m)})||_2^2$$

$$= M\sigma^2 + \sum_{m=1}^{M} ||\nabla F_m(x_{n,k}^{(m)}) - \nabla F_m\left(\frac{X_{n,k} \cdot 1_M}{M}\right) + \nabla F_m\left(\frac{X_{n,k} \cdot 1_M}{M}\right)||_2^2$$

$$\leq M\sigma^2 + 4L^2 \sum_{m=1}^{M} ||x_{n,k}^{(m)} - \frac{X_{n,k} \cdot 1_M}{M}||_2^2 + 4\kappa^2M$$

$$+ 4Ml^2 ||x_{n,k}^{(m)} - \frac{X_{n,k} \cdot 1_M}{M}||_2^2 + 4||\nabla F\left(\frac{X_{n} \cdot 1_M}{M}\right)||_2^2$$

$$= M\sigma^2 + 4L^2 \sum_{m=1}^{M} ||x_{n,k}^{(m)} - \frac{X_{n,k} \cdot 1_M}{M}||_2^2 + 4\kappa^2M$$
On the Convergence of Quantized-PR-SGD for Serverless Learning

\[ + 4ML^2 \gamma^2 \| \frac{1}{M} \sum_{m=1}^{M} \sum_{t=0}^{k-1} \nabla f_m(x_{n,t}^{(m)}, \xi_t^{(m)}) \|^2 + 4\| \nabla F(\frac{X_n \cdot 1_M}{M}) \|^2 \]

\[ \leq M \sigma^2 + 4L^2 \sum_{m=1}^{M} \| x_{n,k}^{(m)} - \frac{X_{n,k} \cdot 1_M}{M} \|^2 + 4\kappa^2 M \]

\[ + 4ML^2 \gamma^2 \left( \frac{2k \sigma^2}{M} + 2k \sum_{t=0}^{k-1} \| \nabla F(x_{n,t}) \cdot \frac{1_M}{M} \|^2 \right) + 4\| \nabla F(\frac{X_n \cdot 1_M}{M}) \|^2 \]

\[ = M \sigma^2 + 4L^2 \sum_{m=1}^{M} \| x_{n,k}^{(m)} - \frac{X_{n,k} \cdot 1_M}{M} \|^2 + 4\kappa^2 M + 8kL^2 \sigma^2 \gamma^2 \]

\[ + 8MkL^2 \gamma \sum_{t=0}^{k-1} \| \nabla F(x_{n,t}) \cdot \frac{1_M}{M} \|^2 + 4\| \nabla F(\frac{X_n \cdot 1_M}{M}) \|^2 \]

where \((a)\) follows \(\| a + b + c + d \|^2 \leq 4\| a \|^2 + 4\| b \|^2 + 4\| c \|^2 + 4\| d \|^2\). Therefore,

\[
\sum_{n=1}^{N} \sum_{k=0}^{K-1} \| G(x_{n,k}, \xi_k) \|^2 \leq NKM \sigma^2 + 4L^2 \sum_{n=1}^{N} \sum_{k=0}^{K-1} \sum_{m=1}^{M} \| x_{n,k}^{(m)} - \frac{X_{n,k} \cdot 1_M}{M} \|^2 + 4\kappa^2 MNK \]

\[ + 8NL^2 \sigma^2 \sum_{k=0}^{K-1} k \gamma^2 + 8ML^2 \gamma^2 \sum_{n=1}^{N} \sum_{k=0}^{K-1} \sum_{m=1}^{M} k \| \nabla F(x_{n,t}) \cdot \frac{1_M}{M} \|^2 + 4K \sum_{n=1}^{N} \| \nabla F(\frac{X_n \cdot 1_M}{M}) \|^2 \]

(18)

Consider that,

\[
\sum_{m=1}^{M} \| x_{n,k}^{(m)} - \frac{X_{n,k} \cdot 1_M}{M} \|^2 \]

\[ = \sum_{m=1}^{M} \| \tilde{x}_n^{(m)} - \frac{X_n \cdot 1_M}{M} \|^2 + \gamma \sum_{j=0}^{k-1} \| \nabla f_m(x_{n,j}^{(m)}, \xi_j^{(m)}) \|^2 \]

\[ \leq (a) 2 \sum_{m=1}^{M} \| x_{n,k}^{(m)} - \frac{X_n \cdot 1_M}{M} \|^2 + 2 \sum_{m=1}^{M} \| \nabla f_m(x_{n,j}^{(m)}, \xi_j^{(m)}) \|^2 \]

\[ \leq (b) 2 \sum_{m=1}^{M} \| x_{n,k}^{(m)} - \frac{X_n \cdot 1_M}{M} \|^2 + 4\gamma^2 \sum_{m=1}^{M} \| \nabla f_m(x_{n,j}^{(m)}, \xi_j^{(m)}) - \frac{1}{M} \sum_{i=1}^{M} \nabla f_i(x_{n,j}^{(m)}, \xi_j^{(m)}) \|^2 \]

\[ - \frac{1}{M} \sum_{i=1}^{M} \| \nabla f_i(x_{n,j}^{(m)}, \xi_j^{(m)}) - \frac{1}{M} \sum_{i=1}^{M} \nabla f_i(x_{n,j}^{(m)}, \xi_j^{(m)}) \|^2 \]

\[ \leq (c) 2 \sum_{m=1}^{M} \| x_{n,k}^{(m)} - \frac{X_n \cdot 1_M}{M} \|^2 + 4k \sigma^2 \gamma^2 M + 4\gamma^2 (6kL^2 \sum_{m=1}^{M} \sum_{j=0}^{k-1} \| X_{n,j} \cdot 1_M - x_{n,j}^{(m)} \|^2 + 3k^2 \kappa^2 M) \]
where (a)/(b) follows \(|a + b|^2 \leq 2|a|^2 + 2|b|^2\), (c) follows the variance bound of Assumption 1.

when \(k = 0:\)
\[
\sum_{m=1}^{M} ||\tilde{x}^{(m)}_n - \frac{X_n \cdot 1_M}{M}||^2 \leq 2 \sum_{m=1}^{M} ||\tilde{x}^{(m)}_n - \frac{X_n \cdot 1_M}{M}||^2
\]

when \(k \leq 1:\)
\[
\sum_{k=1}^{K-1} \sum_{m=1}^{M} ||x^{(m)}_{n;k} - \frac{X_{n;k} \cdot 1_M}{M}||^2 \\
\leq 2 \sum_{k=1}^{K-1} \sum_{m=1}^{M} ||\tilde{x}^{(m)}_n - \frac{X_n \cdot 1_M}{M}||^2 + 4\gamma^2 \kappa^2 M \sum_{k=1}^{K-1} k \\
+ 24\gamma^2 L^2 \sum_{k=1}^{K-1} k \sum_{m=1}^{M} \sum_{j=0}^{K-1} ||X_{n;j} \cdot 1_M - x^{(m)}_{n;j}||^2 + 12\gamma^2 \kappa^2 M \sum_{k=1}^{K-1} k^2 \\
\leq (K - 1) \sum_{m=1}^{M} ||\tilde{x}^{(m)}_n - \frac{X_n \cdot 1_M}{M}||^2 + 2\gamma^2 \kappa^2 M K (K - 1) \\
+ 24\gamma^2 L^2 \frac{K(K - 1)}{2} \sum_{m=1}^{M} ||\tilde{x}^{(m)}_n - \frac{X_n \cdot 1_M}{M}||^2 \\
+ 12\gamma^2 L^2 (K + 1)(K - 2) \sum_{k=1}^{K-1} \sum_{m=1}^{M} ||x^{(m)}_{n;k} - \frac{X_{n;k} \cdot 1_M}{M}||^2 + 2\gamma^2 \kappa^2 M K (K - 1) (2K - 1) \\
= (2K - 2 + 12K(K - 1)\gamma^2 L^2) \sum_{m=1}^{M} ||\tilde{x}^{(m)}_n - \frac{X_n \cdot 1_M}{M}||^2 + 2\gamma^2 \kappa^2 M K (K - 1) \\
+ 2\gamma^2 \kappa^2 M (K - 1) K (2K - 1) + 12\gamma^2 L^2 (K + 1)(K - 2) \sum_{k=1}^{K-1} \sum_{m=1}^{M} ||x^{(m)}_{n;k} - \frac{X_{n;k} \cdot 1_M}{M}||^2
\]

Then, we have:
\[
[1 - 12\gamma^2 L^2 (K + 1)(K - 2)] \sum_{k=1}^{K-1} \sum_{m=1}^{M} ||x^{(m)}_{n;k} - \frac{X_{n;k} \cdot 1_M}{M}||^2 \\
\leq (2K - 2 + 12K(K - 1)\gamma^2 L^2) \sum_{m=1}^{M} ||\tilde{x}^{(m)}_n - \frac{X_n \cdot 1_M}{M}||^2 \\
+ 2\gamma^2 \kappa^2 M K (K - 1) + 2\gamma^2 \kappa^2 M (K - 1) K (2K - 1)
\]

Hence, note that \(1 - 12\gamma^2 L^2 (K + 1)(K - 2) > 0,\)
\[ \sum_{k=1}^{K-1} \sum_{m=1}^{M} \| x_{n,k}^{(m)} - \frac{X_{n,k} \cdot 1_M}{M} \|_2^2 \]
\[ \leq \frac{2K - 2 + 12K(K - 1)\gamma^2 L^2}{1 - 12\gamma^2 L^2(K + 1)(K - 2)} \sum_{m=1}^{M} \| x_{n}^{(m)} - \frac{X_n \cdot 1_M}{M} \|_2^2 \]
\[ + \frac{2\sigma^2 \gamma^2 M K(K - 1) + 2\gamma^2 \kappa^2 M(K - 1)K(2K - 1)}{1 - 12\gamma^2 L^2(K + 1)(K - 2)} \]
\[ \leq \frac{2K - 1 + 24\gamma^2 L^2}{1 - 12\gamma^2 L^2(K + 1)(K - 2)} \sum_{n=1}^{N} \sum_{m=1}^{M} \| \tilde{x}_n^{(m)} - \frac{X_n \cdot 1_M}{M} \|_2^2 \]
\[ + \frac{2\sigma^2 \gamma^2 M K(K - 1) + 2\gamma^2 \kappa^2 M(K - 1)K(2K - 1)}{1 - 12\gamma^2 L^2(K + 1)(K - 2)} \]

Thus,

\[ \sum_{n=1}^{N} \sum_{k=0}^{K-1} \sum_{m=1}^{M} \| x_{n,k}^{(m)} - \frac{X_{n,k} \cdot 1_M}{M} \|_2^2 \]
\[ \leq \frac{2K - 1 + 24\gamma^2 L^2}{1 - 12\gamma^2 L^2(K + 1)(K - 2)} \sum_{n=1}^{N} \sum_{m=1}^{M} \| \tilde{x}_n^{(m)} - \frac{X_n \cdot 1_M}{M} \|_2^2 \]
\[ + \frac{2\sigma^2 \gamma^2 M K(K - 1) + 2\gamma^2 \kappa^2 M(K - 1)K(2K - 1)}{1 - 12\gamma^2 L^2(K + 1)(K - 2)} N \]
\[ \leq \frac{2K - 1 + 24\gamma^2 L^2}{1 - 12\gamma^2 L^2(K + 1)(K - 2)} \left( \frac{2}{1 - \rho^2} \sum_{n=1}^{N} E \| C_n \|_F^2 \right) + \frac{2\sigma^2 \gamma^2}{(1 - \rho)^2} \sum_{n=1}^{N} \sum_{k=0}^{K-1} \| G(X_{n,k}, \xi_k) \|_F^2 \]
\[ + \frac{2\sigma^2 \gamma^2 M K(K - 1)K(2K - 1)}{1 - 12\gamma^2 L^2(K + 1)(K - 2)} \]
\[ \leq (a) \frac{2(2K + 24\gamma^2 L^2 - 1)\gamma^2 K}{(1 - \rho^2)(1 - 12\gamma^2 L^2(K + 1)(K - 2))} \left( \frac{d(1 - \rho)^2 + 2\mu^2}{2(s^2(1 - \rho)^2 - d\mu^2)} \right) \sum_{n=1}^{N} \sum_{k=0}^{K-1} \| G(X_{n,k}, \xi_k) \|_F^2 \]
\[ + \frac{2(2K + 24\gamma^2 L^2 - 1) \gamma^2 K}{(1 - \rho^2)(1 - 12\gamma^2 L^2(K + 1)(K - 2))} \sum_{n=1}^{N} \sum_{k=0}^{K-1} \| G(X_{n,k}, \xi_k) \|_F^2 \]
\[ + \frac{2\sigma^2 \gamma^2 M K(K - 1)K(2K - 1)}{1 - 12\gamma^2 L^2(K + 1)(K - 2)} N \]

where \((a)\) comes from Lemma. Let

\[ D_1 := \frac{(2K + 24\gamma^2 L^2 - 1)}{1 - 12\gamma^2 L^2(K + 1)(K - 2)} \]
\[ D_2 := \frac{d(1 - \rho)^2 + 2\mu^2}{2(s^2(1 - \rho)^2 - d\mu^2)} \]
Therefore,

\[
\sum_{n=1}^{N} \sum_{k=1}^{K-1} \sum_{m=1}^{M} ||X_{n;k}^{(m)} - \frac{X_{n;k} \cdot 1_M}{M}||_F^2 \\
\leq \frac{2D_1 D_2}{1 - \rho^2} + \frac{2D_1}{(1 - \rho)^2} \tilde{\gamma}^2 K \sum_{n=1}^{N} \sum_{k=0}^{K-1} ||G(X_{n;k}, \xi_k)||_F^2 \\
+ \frac{2\tilde{\gamma}^2 MK(K-1)(\sigma^2 + \kappa^2(2K-1))N}{1 - 12\tilde{\gamma}^2 L^2(K+1)(K-2)}
\]

Then, recall the Equation 18 we have:

\[
\sum_{n=1}^{N} \sum_{k=0}^{K-1} \sum_{m=0}^{M} ||X_{n;i}^{(m)} - \frac{X_{n;i} \cdot 1_M}{M}||_F^2 \\
\leq NKM\sigma^2 + 4L^2 \left( \frac{2D_1 D_2}{1 - \rho^2} + \frac{2D_1}{(1 - \rho)^2} \right) \tilde{\gamma}^2 K \sum_{n=1}^{N} \sum_{k=0}^{K-1} ||G(X_{n;k}, x_{ik})||_F^2 \\
+ \frac{2\tilde{\gamma}^2 MK(K-1)(\sigma^2 + \kappa^2(2K-1))N}{1 - 12\tilde{\gamma}^2 L^2(K+1)(K-2)} + 4\kappa^2 M NK + 4NL^2\sigma^2\tilde{\gamma}^2 K(K-1) \\
+ 8ML^2\tilde{\gamma}^2 \sum_{n=1}^{N} \sum_{k=0}^{K-1} \sum_{m=0}^{M} \sum_{t=0}^{t=0} k || \frac{\partial F(X_{n;i}) \cdot 1_M}{M} ||_2^2 + 4K \sum_{n=1}^{N} || \nabla F(X_{n} \cdot 1_M) ||_2^2
\]

\[
\leq NKM\sigma^2 + 8L^2 D_1 \left( \frac{2D_2}{1 - \rho^2} + \frac{2D_1}{(1 - \rho)^2} \right) \tilde{\gamma}^2 K \sum_{n=1}^{N} \sum_{k=0}^{K-1} ||G(X_{n;k}, x_{ik})||_F^2 \\
+ \frac{8L^2\tilde{\gamma}^2 MK(K-1)(\sigma^2 + \kappa^2(2K-1))N}{1 - 12\tilde{\gamma}^2 L^2(K+1)(K-2)} + 4\kappa^2 M NK + 4NL^2\sigma^2\tilde{\gamma}^2 K(K-1) \\
+ 4ML^2\tilde{\gamma}^2 (K-1)K \sum_{n=1}^{N} \sum_{k=0}^{K-1} \sum_{m=0}^{M} || \frac{\partial F(X_{n;k}) \cdot 1_M}{M} ||_2^2 + 4K \sum_{n=1}^{N} || \nabla F(X_{n} \cdot 1_M) ||_2^2
\]

Let \( D_3 = \frac{D_2}{1 - \rho^2} + \frac{1}{(1 - \rho)^2}, \)

\[
(18L^2\tilde{\gamma}^2 K D_1 D_3) \sum_{n=1}^{N} \sum_{k=0}^{K} ||G(X_{n;k}, x_{ik})||_F^2 \\
\leq NKM\sigma^2 + 4\kappa^2 M NK + 4NL^2\sigma^2\tilde{\gamma}^2 K(K-1) \\
+ \frac{8L^2\tilde{\gamma}^2 MK(K-1)(\sigma^2 + \kappa^2(2K-1))N}{1 - 12\tilde{\gamma}^2 L^2(K+1)(K-2)} + 4K \sum_{n=1}^{N} || \nabla F(X_{n} \cdot 1_M) ||_2^2
\]

Note that \( 18L^2\tilde{\gamma}^2 K D_1 D_3 > 0, \) the lemma is proved.
Lemma 14. Under Assumption 1 and Assumption 2, given the constant step-size $\bar{\gamma}$, we have:

$$
\mathbb{E}\left[\frac{1}{M} \sum_{m=1}^{M} \sum_{k=0}^{K-1} \nabla f_m(x_{n;k}^{(m)}, \xi_k^{(m)})\right]^2 \leq \frac{2K\sigma^2}{M} + 2K \sum_{k=0}^{K-1} \left\| \frac{\partial F(X_{n;k}) \cdot 1_M}{M} \right\|_2
$$

Proof.

$$
\mathbb{E}\left[\frac{1}{M} \sum_{m=1}^{M} \sum_{k=0}^{K-1} \nabla f_m(x_{n;k}^{(m)}, \xi_k^{(m)})\right]^2 = \mathbb{E}\left[\frac{1}{M} \sum_{m=1}^{M} \sum_{k=0}^{K-1} (\nabla f_m(x_{n;k}^{(m)}, \xi_k^{(m)}) - \nabla F_m(x_{n;k}^{(m)})) + \frac{1}{M} \sum_{m=1}^{M} \sum_{k=0}^{K-1} \nabla F_m(x_{n;k}^{(m)})\right]^2
$$

$$
\leq 2\mathbb{E}\left[\frac{1}{M} \sum_{m=1}^{M} \sum_{k=0}^{K-1} (\nabla f_m(x_{n;k}^{(m)}, \xi_k^{(m)}) - \nabla F_m(x_{n;k}^{(m)}))\right]^2 + 2\mathbb{E}\left[\frac{1}{M} \sum_{m=1}^{M} \sum_{k=0}^{K-1} \nabla F_m(x_{n;k}^{(m)})\right]^2
$$

$$
\leq 2 \frac{1}{M^2} \sum_{m=1}^{M} \sum_{k=0}^{K-1} \mathbb{E}\left[\nabla f_m(x_{n;k}^{(m)}, \xi_k^{(m)}) - \nabla F_m(x_{n;k}^{(m)})\right]^2 + 2K \sum_{k=0}^{K-1} \mathbb{E}\left[\left\| \frac{\partial F(X_{n;k}) \cdot 1_M}{M} \right\|_2\right]^2
$$

$$
= \frac{2K\sigma^2}{M} + 2K \sum_{k=0}^{K-1} \left\| \frac{\partial F(X_{n;k}) \cdot 1_M}{M} \right\|_2
$$

Main Proof of Theorem 2

$$
\mathbb{E}[F\left(\frac{X_{n+1} \cdot 1_M}{M}\right)]
$$

$$
\leq \mathbb{E}[F\left(\frac{X_n \cdot 1_M}{M}\right)] + \mathbb{E}\left[\nabla F\left(\frac{X_n \cdot 1_M}{M}\right), \frac{X_{n+1} \cdot 1_M - X_n \cdot 1_M}{M}\right] + \frac{L}{2} \left\| \frac{X_{n+1} \cdot 1_M - X_n \cdot 1_M}{M} \right\|_2
$$

$$
= \mathbb{E}[F\left(\frac{X_n \cdot 1_M}{M}\right)] + \mathbb{E}\left[\nabla F\left(\frac{X_n \cdot 1_M}{M}\right), -\frac{\bar{\gamma} \sum_{k=0}^{K-1} \nabla f_m(x_{n;k}^{(m)}, \xi_k^{(m)})}{M} + \frac{C_n \cdot 1_M}{M}\right]
$$

$$
+ \frac{L}{2} \left\| -\frac{\bar{\gamma} \sum_{k=0}^{K-1} \nabla f_m(x_{n;k}^{(m)}, \xi_k^{(m)})}{M} + \frac{C_n \cdot 1_M}{M}\right\|_2
$$

$$
= \mathbb{E}[F\left(\frac{X_n \cdot 1_M}{M}\right)] - \frac{\bar{\gamma} K}{M} \sum_{k=0}^{K-1} \mathbb{E}\left[\nabla F\left(\frac{X_n \cdot 1_M}{M}\right), \nabla F_m(x_{n;k}^{(m)})\right]
$$

$$
+ \frac{L}{2} \left\| -\frac{\bar{\gamma} \sum_{k=0}^{K-1} \nabla f_m(x_{n;k}^{(m)}, \xi_k^{(m)})}{M} + \frac{C_n \cdot 1_M}{M}\right\|_2
$$

$$
\leq \mathbb{E}[F\left(\frac{X_n \cdot 1_M}{M}\right)] - \frac{\bar{\gamma} K}{2M} \mathbb{E}\left[\left\| \nabla F\left(\frac{X_n \cdot 1_M}{M}\right)\right\|_2 - \frac{\bar{\gamma} K}{2} \sum_{k=0}^{K-1} \mathbb{E}\left[\left\| \frac{\partial F(X_{n;k} \cdot 1_M)}{M} \right\|_2\right]\right]
$$

$$
+ \frac{\bar{\gamma} K}{2M} \sum_{k=0}^{K-1} \mathbb{E}\left[\left\| \nabla F_m(x_{n;k}^{(m)})\right\|_2\right] + \frac{L\gamma^2}{2} \left\| \frac{1}{M} \sum_{m=1}^{M} \sum_{k=0}^{K-1} \nabla f_m(x_{n;k}^{(m)}, \xi_k^{(m)})\right\|_2
$$
\[
\begin{align*}
&+ \frac{L}{2} \left\| \frac{1}{M} \sum_{m=1}^{M} C_n^{(m)} \right\|_2^2 \\
\leq \ & \mathbb{E}[F(\frac{X_n \cdot 1_{M}}{M})] - \frac{\tilde{\gamma}K}{2} \mathbb{E}\|\nabla F(\frac{X_n \cdot 1_{M}}{M})\|_2^2 - \frac{\gamma}{2} \sum_{k=0}^{K-1} \mathbb{E}\|\frac{\partial F(X_{n,k} \cdot 1_{M})}{M}\|_2^2 \\
&+ \frac{\tilde{\gamma}L^2}{2M} \sum_{k=0}^{K-1} \sum_{m=1}^{M} \mathbb{E}\|X_n \cdot 1_{M} - x_{n,k}^{(m)}\|_2^2 + \frac{L\gamma^2}{2} \sum_{k=0}^{K-1} \sum_{m=1}^{M} \|\nabla f_m(x_{n,k}^{(m)} \cdot \zeta_k^{(m)})\|_2^2 \\
&+ \frac{L}{2M^2} \sum_{m=1}^{M} \|C_n^{(m)}\|_2^2
\end{align*}
\]

where (a) is based on the expected value of compression noise is 0, (b) follows that

\[
\left\| \frac{1}{M} \sum_{m=1}^{M} C_n^{(m)} \right\|_2^2 = \left\| \frac{1}{M} \sum_{m=1}^{M} (Q_s(\Delta X_n^{(m)}) - \Delta_n^{(m)}) \right\|_2^2
\]

\[
= \frac{1}{M^2} \sum_{m=1}^{M} \left\| (Q_s(\Delta X_n^{(m)}) - \Delta_n^{(m)}) \right\|_2^2 = \frac{1}{M^2} \sum_{m=1}^{M} \|C_n^{(m)}\|_2^2
\]

\[
F_s - F(X_1)
\]

\[
\leq \mathbb{E}[F(\frac{X_n+1 \cdot 1_{M}}{M})] - F(X_1)
\]

\[
\leq - \frac{\tilde{\gamma}K}{2} \sum_{n=1}^{N} \left\|\nabla F(\frac{X_n \cdot 1_{M}}{M})\right\|_2^2 - \frac{\gamma}{2} \sum_{n=1}^{K-1} \left\|\frac{\partial F(X_{n,k} \cdot 1_{M})}{M}\right\|_2^2 \\
+ \frac{\tilde{\gamma}L^2}{2M} \left[ 4K \mathbb{E}\|C_n\|_2^2 + (\frac{4K^2\gamma^2}{(1-P)^2} + \gamma^2 K - 1) \sum_{k=1}^{K-1} \left\|G(X_{n,k}, \xi_k)\right\|_F^2 \right] \\
+ \frac{\gamma^2 L^2}{2} \left[ 2K^2 N \cdot \mathbb{E}\|C_n\|_2^2 + 2K \mathbb{E}\|\frac{\partial F(X_{n,k} \cdot 1_{M})}{M}\|_2^2 \right] + \frac{L}{2M^2} \sum_{n=1}^{N} \|C_n^{(m)}\|_2^2
\]

\[
= - \frac{\tilde{\gamma}K}{2} \sum_{n=1}^{N} \left\|\nabla F(\frac{X_n \cdot 1_{M}}{M})\right\|_2^2 - \left( \frac{\tilde{\gamma} - \gamma^2 L K}{2} \right) \sum_{n=1}^{N} \sum_{k=0}^{K-1} \left\|\frac{\partial F(X_{n,k} \cdot 1_{M})}{M}\right\|_2^2
\]
\[+ \left( \frac{2\gamma LK}{M(1-P^2)} + \frac{L}{2M^2} \right) \sum_{n=1}^{N} \| C_n \|_F^2 \]
\[+ \frac{\bar{\gamma} L^2}{2M} \left[ \frac{4K^2\bar{\gamma}^2}{(1-\rho)^2} + \bar{\gamma}^2 K (K-1) \right] \sum_{n=1}^{N} \sum_{k=0}^{K-1} \| G(X_{n;k}; \xi_k) \|_F^2 \]
\[\leq \frac{\bar{\gamma} K}{2} \sum_{n=1}^{N} \| \nabla F(\frac{X_n \cdot 1_M}{M}) \|_2^2 - \left( \frac{\bar{\gamma}}{2} - \bar{\gamma}^2 LK \right) \sum_{n=1}^{N} \sum_{k=0}^{K-1} \| \frac{\partial F(X_{n;k} \cdot 1_M)}{M} \|_2^2 \]
\[+ \frac{\bar{\gamma} L^2}{2M} \left[ \frac{4K^2\bar{\gamma}^2}{(1-\rho)^2} + \bar{\gamma}^2 K (K-1) \right] \sum_{n=1}^{N} \sum_{k=0}^{K-1} \| G(X_{n;k}; \xi_k) \|_F^2 \]
\[\leq \frac{\bar{\gamma} K}{2} \sum_{n=1}^{N} \| \nabla F(\frac{X_n \cdot 1_M}{M}) \|_2^2 - \left( \frac{\bar{\gamma}}{2} - \bar{\gamma}^2 LK \right) \sum_{n=1}^{N} \sum_{k=0}^{K-1} \| \frac{\partial F(X_{n;k} \cdot 1_M)}{M} \|_2^2 \]
\[+ \bar{\gamma}^3 \frac{L^2 K}{2M} \left[ \frac{4K}{(1-\rho)^2} + K - 1 \right] + \bar{\gamma}^2 KL \left( \frac{2\bar{\gamma} LK}{M} \frac{1}{2(1-\rho^2)} + \frac{1}{2M} \right) D_2 \]
\[\left( MNK (\sigma^2 + \kappa^2) + 4NL^2 \sigma^2 \bar{\gamma}^2 K (K-1) \right) \frac{1}{1 - 8L^2 \bar{\gamma}^2 KD_1 D_3} \]
\[\frac{8MNK (K-1)L^2 \bar{\gamma}^2 [\sigma^2 + \kappa^2(2K-1)]}{(1 - 8L^2 \bar{\gamma}^2 KD_1 D_3)[1 - 12\bar{\gamma}^2 L^2 (K+1)(K-2)]} \]
\[\frac{4MK (k-1)L^2 \bar{\gamma}^2}{1 - 8L^2 \bar{\gamma}^2 KD_1 D_3} \sum_{n=1}^{N} \sum_{k=0}^{K-1} \| \frac{\partial F(X_{n;k} \cdot 1_M)}{M} \|_2^2 \]
\[+ \frac{4K}{1 - 8L^2 \bar{\gamma}^2 KD_1 D_3} \sum_{n=1}^{N} \| \nabla F(\frac{X_n \cdot 1_M}{M}) \|_2^2 \]
\[= -\frac{\bar{\gamma} K}{2} \left( 1 - \frac{8KL\bar{\gamma}}{(1 - 8L^2 \bar{\gamma}^2 KD_1 D_3)M} \right) \left[ \frac{\bar{\gamma} L}{2} \left( \frac{4K}{(1-\rho)^2} + K - 1 \right) \right] \]
\[+ \frac{2\bar{\gamma} KL}{1 - \rho^2} + \frac{1}{2M} \right) D_2 \right) \sum_{n=1}^{N} \| \nabla F(\frac{X_n \cdot 1_M}{M}) \|_2^2 \]
\[- \frac{\bar{\gamma}}{2} \left( 1 - \bar{\gamma} LK \right) - \frac{8KL^2 (K-1)L^2 \bar{\gamma}^3}{1 - 8L^2 \bar{\gamma}^2 KD_1 D_3} \frac{\bar{\gamma} L}{2} \left( \frac{4K}{(1-\rho)^2} + K - 1 \right) \]
\[+ \frac{2\bar{\gamma} KL}{1 - \rho^2} + \frac{1}{2M} \right) D_2 \right) \sum_{n=1}^{N} \sum_{k=0}^{K-1} \| \frac{\partial F(X_{n;k} \cdot 1_M)}{M} \|_2^2 \]
\[+ \bar{\gamma}^2 KL \left( \frac{4K}{(1-\rho)^2} + K - 1 \right) + \left( \frac{2\bar{\gamma} KL}{1 - \rho^2} + \frac{1}{2M} \right) D_2 \right) \]
\[(\sigma^2 + \kappa^2) + \frac{4L^2 \sigma^2 \bar{\gamma}^2 (K-1)}{M} + \frac{8KL^2 \sigma^2 \bar{\gamma}^2 [\sigma^2 + \kappa^2(2K-1)]}{(1 - 12\bar{\gamma}^2 L^2 (K+1)(K-2))} \frac{NK}{1 - 8L^2 \bar{\gamma}^2 KD_1 D_3} \]
where (a) follows Lemma 15 and Lemma 13, (b) follows Lemma 14 and (c) follows Lemma 12. Suppose $D_6 > 0$, therefore,

$$F^* - F(X_1) \leq -\bar{\gamma}K \left(1 - D_5\right) \sum_{n=1}^{N} E\|\nabla F(\frac{X_n \cdot 1_M}{M})\|_2^2 - \frac{\bar{\gamma}}{2} d_6 \sum_{n=1}^{N} E\|\partial F(\frac{X_n \cdot 1_M}{M})\|_2^2$$

$$+ \frac{\bar{\gamma}^2 K \tilde{L} D_4 N}{1 - 8L^2\bar{\gamma}^2 K D_1 D_3 }\left[1 + \frac{4L^2\bar{\gamma}^2 (K - 1)}{M} \right] + \frac{8(K - 1)L^2\bar{\gamma}^2}{1 - 12\bar{\gamma}^2 L^2(K + 1)(K - 2)} \sigma^2$$

$$+ \frac{4\bar{\gamma}^2 K^2 \tilde{L} D_4 N}{1 - 8L^2\bar{\gamma}^2 K D_1 D_3 }\left[1 + \frac{4(K - 1)(2K - 1)L^2\bar{\gamma}^2}{1 - 12\bar{\gamma}^2 L^3(K + 1)(K - 2)} \right] \bar{\gamma}^2$$

**Proof of Corollary 2** We selected the stepsize

$$\bar{\gamma} = (\sigma \sqrt{N/M} + 3KL^3 \sqrt{D_2} + 16KL D_3 + 6KL)^{-1}$$

Then, we have:

$$112\bar{\gamma}^2 L^2(K + 1)(K - 2) \geq \frac{2}{3}$$

$$D_1\bar{\gamma}L \leq \frac{2}{3}$$

$$D_3KL\bar{\gamma} \leq \frac{1}{16}$$

$$18L^2\bar{\gamma}^2 KD_1 D_3 \geq \frac{2}{3}$$

$$D_6 > 0, D_5 \leq \frac{1}{2}$$

Therefore,

$$\frac{1}{2N} \sum_{n=1}^{N} E\|\nabla F(\frac{X_n \cdot 1_M}{M})\|_2^2$$

$$\leq 3\bar{\gamma} K \tilde{L} D_4 \left[1 + \frac{4L^2\bar{\gamma}^2 (K - 1)}{M} \right] + 12(K - 1)L^2\bar{\gamma}^2 \sigma^2$$

$$+ 12\bar{\gamma} K \tilde{L} D_4 \kappa^2 (1 + 6(K - 1)(2K - 1)L^2\bar{\gamma}^2)$$

$$+ \frac{2(F(X_1) - F^*) (\sigma \sqrt{N/M} + 3KL^3 \sqrt{D_2} + 16KL D_3 + 6KL)}{K N}$$

$$= 3\bar{\gamma} K \tilde{L} D_4 \left[1 + \frac{4L^2\bar{\gamma}^2 (K - 1)}{M} \right] + 12(K - 1)L^2\bar{\gamma}^2 \sigma^2 + 4[1 + 6(K - 1)(2K - 1)L^2\bar{\gamma}^2] \kappa^2$$
\[ + \frac{2[F(X_1) - F_*]\sigma}{K\sqrt{NM}} + \frac{2(F(X_1) - F_*)(\sigma\sqrt{N/M} + 3L^3\sqrt{D_2} + 16LD_3 + 6L)}{N} \]

With the given range of \( N \), the following inequality holds:

1. 
\[
3\tilde{\gamma}KLD_4 = 3\tilde{\gamma}KL(2\tilde{\gamma}KLD_3 + \frac{\tilde{\gamma}L(K-1)}{2} + \frac{D_2}{2M}) \\
\leq 6\tilde{\gamma}^2K^2L^2D_3 + \frac{3\tilde{\gamma}^2L^2K^2}{2} + \frac{3\tilde{\gamma}KLD_2}{2M} \\
= \frac{3K^2L^2M}{\sigma^2N}(2D_3 + \frac{1}{2}) + \frac{3KLD_2}{2\sigma\sqrt{NM}} \leq \frac{3KLD_2}{\sigma\sqrt{NM}}
\]

2. 
\[
\frac{4L^2\tilde{\gamma}^2(K-1)}{M} + 12(K-1)L^2\tilde{\gamma}^2 = 4(K-1)L^2\tilde{\gamma}^2(\frac{1}{M} + 3) \\
\leq \frac{4(K-1)L^2}{\sigma^2N/M}(\frac{1}{M} + 3) = \frac{4(K-1)L^2}{\sigma^2N}(1 + 3M) \leq 1
\]

3. 
\[
6(K-1)(2K-1)L^2\tilde{\gamma}^2 \leq \frac{6(K-1)(2K-1)L^2}{\sigma^2N/M} \leq 1
\]

Therefore, the following inequality is derived:

\[
\frac{1}{2N} \sum_{n=1}^{N} E[||\nabla F(\frac{X_n \cdot 1_M}{M})||^2] \leq \frac{6KLD_2}{\sigma^2N}(\sigma^2 + 4K^2) + \frac{2[F(X_1) - F_*]\sigma}{K\sqrt{NM}} \\
+ \frac{2(F(X_1) - F_*)(3L^3\sqrt{D_2} + 16LD_3 + 6L)}{N}
\]

### A.4 Communication Cost of Quantized-PR-SGD

For the sake of reducing the number of bits in communication, we leverage Elias gamma coding [6] to compress the vector. In this part, we first introduce several lemmas and an alternative compression scheme is subsequently proposed.

**Lemma 15.** For any vector \( v \in \mathbb{R}^d \), the expected number of non-zero values in a vector should be:

\[
E[||Q_s(v)||_0] \leq \min(s^2 + s\sqrt{d}, d)
\]
Proof. Let \( u = v / \| v \|_2 \). Let \( I(u) \) be the set of index \( i \) where \( |u_i| < 1/s \). Since

\[
(d - |I(u)|)/s^2 \leq \sum_{i \not\in I(u)} u_i^2 \leq 1
\]

the inequality \( d - |I(u)| \leq s^2 \) holds. Then, with the definition, the probability that \( Q_s(v_i) \) is non-zero value is \( |u_i|s \) for all \( i \in I(u) \) and therefore, we have

\[
E[|\|Q_s(v)\|_0|] = d - |I(u)| + \sum_{i \in I(u)} s|u_i| \leq s^2 + s\|u\|_1 \leq s^2 + s\sqrt{d}
\]

Besides, it is easy to notice that \( v \) is a \( d \)-dimension vector so that:

\[
E[|\|Q_s(v)\|_0|] \leq d
\]

**Lemma 16.** Let \( v \in \mathbb{R}^d \) be a vector so that for all \( i \), \( v_i \) is a positive integer and moreover, \( \|v\|_p \leq \rho \). Then,

\[
\sum_{i=1}^{d} |\text{Elias}(v_i)| \leq \frac{2d}{\rho} \log \frac{\rho}{d} + d
\]

Proof.

\[
\sum_{i=1}^{d} |\text{Elias}(v_i)| = \sum_{i=1}^{d} (2\log(v_i) + 1) = 2\sum_{i=1}^{d} \log(v_i) + d \\
\leq \frac{2}{\rho} d \log \left( \frac{1}{d} \sum_{i=1}^{d} v_i^\rho \right) + d = \frac{2d}{\rho} \log \frac{\rho}{d} + d
\]

where (a) holds on account for Jensen’s inequality.

**Compression Schemes** For any integer \( k \), we use Elias gamma encoding \([6]\), denoted as Elias(\( k \)), to generate its code. The encoding process is simple: let \( \text{bin}(k) \) be the binary representation of \( k \) and \( \text{len} \) be the length of \( \text{bin}(k) \), the code Elias(\( k \)) would simply be \( \text{len} \) of zeros added before \( \text{bin}(k) \). Therefore, the encoding length —Elias(\( k \))— = \( O(2\log(k) - 1) = O(\log(k)) \). Such encoding scheme is used to encode positive integer whose upper-bound is unknown, since the actual length of its binary representation could be calculated by the number of 0s before the first 1 received.

For compressed vector \( Q_s(v) = [v'_1, v'_2, \ldots, v'_d] \), we have \( v'_i = \|v\|_2 \cdot \text{sgn}(v_i) \cdot \zeta(v_i, s)/s \). We use the following process to implement the encoding: firstly, we put the 32-bit full precision of \( \|v\| \) in the beginning of the transmission code. For \( i = 1 \ldots d \), we use 1 bit to represent \( \text{sgn}(v_i) \) and \( O(\log(\zeta(v_i, s))) \) bits for
Elias(ζ(v$_i$, s) + 1), which are concatenated after the end of previous code in order. The decoding scheme could be processed in the similar way: we first read the 32-bit precision of ||v||, then keep reading $sgn(v_i)$ and Elias(ζ(v$_i$, s) + 1) until the end of the coding.

**Theorem 3.** For any vector $v \in \mathbb{R}^d$, in compression scheme 2, the upper bound of the expected communication cost is

$$
\mathbb{E}[|Code(Q_s(v))|] \leq F + 2d + d \log \frac{s^2 + 2s\sqrt{d} + 1 + d/4}{d}
$$

**Proof.** Let $y = (y_1, y_2, ..., y_d)$. Then, we have:

$$
\mathbb{E}||y + 1||^2_2 \leq \mathbb{E}||y||^2_2 + 2\mathbb{E}[||y||_1] + 1 \leq \frac{d}{4} + s^2 + 2s\sqrt{d} + 1
$$

Therefore,

$$
\mathbb{E}[|Code(Q_s(v))|] = F + \sum_{i=1}^{d} (1 + |Elias(y_i + 1)|)
\leq F + 2d + d \log \frac{s^2 + 2s\sqrt{d} + 1 + d/4}{d}
$$