Geometrodynamics of Information on Curved Statistical Manifolds and its Applications to Chaos

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A novel information-geometrodynamical approach to chaotic dynamics (IGAC) on curved statistical manifolds based on Entropic Dynamics (ED) is presented and a new definition of information geometrodynamical entropy (IGE) as a measure of chaoticity is proposed. The general classical formalism is illustrated in a relatively simple example. It is shown that the hyperbolicity of a non-maximally symmetric $6N$-dimensional statistical manifold $M_s$ underlying an ED Gaussian model describing an arbitrary system of $3N$ degrees of freedom leads to linear information-geometric entropy growth and to exponential divergence of the Jacobi vector field intensity, quantum and classical features of chaos respectively. An information-geometric analogue of the Zurek-Paz quantum chaos criterion in the classical reversible limit is proposed. This analogy is illustrated applying the IGAC to a set of $n$-uncoupled three-dimensional anisotropic inverted harmonic oscillators characterized by a Ohmic distributed frequency spectrum.

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I. INTRODUCTION

The lack of a unified characterization of chaos in classical and quantum dynamics is well-known. In the Riemannian [1] and Finslerian [2] (a Finsler metric is obtained from a Riemannian metric by relaxing the requirement that the metric be quadratic on each tangent space) geometrodynamical approach to chaos in classical Hamiltonian systems, an active field of research concerns the possibility of finding a rigorous relation among the sectional curvature, the Lyapunov exponents, and the Kolmogorov-Sinai dynamical entropy (i.e. the sum of positive Lyapunov exponents) [3]. The largest Lyapunov exponent characterizes the degree of chaoticity of a dynamical system and, if positive, it measures the mean instability rate of nearby trajectories averaged along a sufficiently long reference trajectory. Moreover, it is known that classical chaotic systems are distinguished by their exponential sensitivity to initial conditions and that the absence of this property in quantum systems has lead to a number of different criteria being proposed for quantum chaos. Exponential decay of fidelity, hypersensitivity to perturbation, and the Zurek-Paz quantum chaos criterion of linear von Neumann’s entropy growth [4] are some examples [5]. These criteria accurately predict chaos in the classical limit, but it is not clear that they behave the same far from the classical realm.

The present work makes use of the so-called Entropic Dynamics (ED) [6]. ED is a theoretical framework that arises from the combination of inductive inference (Maximum relative Entropy Methods, [7]) and Information Geometry (Riemannian geometry applied to probability theory) (IG) [8]. As such, ED is constructed on statistical manifolds. It is developed to investigate the possibility that laws of physics - either classical or quantum - might reflect laws of inference rather than laws of nature.

This article is a follow up of a series of the authors works [9, 10, 11]. In this paper, the ED theoretical framework is used to explore the possibility of constructing a unified characterization of classical and quantum chaos. We investigate a system with $3N$ degrees of freedom (microstates), each one described by two pieces of relevant information, its mean expected value and its variance (Gaussian statistical macrostates). This leads to consider an ED model on a non-maximally symmetric $6N$-dimensional statistical manifold $M_s$. It is shown that $M_s$ possesses a constant negative Ricci curvature that is proportional to the number of degrees of freedom of the system, $R_{M_s} = -3N$. It is shown that the system explores statistical volume elements on $M_s$ at an exponential rate. We define a dynamical information-geometric entropy $S_{M_s}$ of the system and we show it increases linearly in time (statistical evolution parameter) and is moreover, proportional to the number of degrees of freedom of the system. The geodesics on $M_s$ are hyperbolic trajectories. Using the Jacobi-Levi-Civita (JLC) equation for geodesic spread, it is shown that the Jacobi vector field intensity $J_{M_s}$ diverges exponentially and is proportional to the number of degrees of freedom of the system.
Thus, $R_{M_r}$, $S_{M_r}$ and $J_{M_r}$ are proportional to the number of Gaussian-distributed microstates of the system. This proportionality leads to conclude there is a substantial link among these information-geometric indicators of chaoticity.

Finally, an information-geometric analog of the Zurek-Paz quantum chaos criterion is suggested. We illustrate this point by use of an $n$-set of inverted harmonic oscillators (IHO). In the ED formalism, the IHO system is described by a curved $n$-dimensional statistical manifold that is conformally related to an Euclidean one.

## II. SPECIFICATION OF THE GAUSSIAN ED-MODEL

Maximum relative Entropy (ME) methods are used to construct an ED model that follows from an assumption about what information is relevant to predict the evolution of the system. Given a known initial macrostate (probability distribution) and that the system evolves to a final known macrostate, the possible trajectories of the system are examined. A notion of distance between two probability distributions is provided by IG. As shown in [12, 13] this distance is quantified by the Fisher-Rao information metric tensor.

We consider an ED model whose microstates span a $3N$-dimensional space labelled by the variables $\{\vec{x}\} = \{\vec{x}^{(1)}, \vec{x}^{(2)}, \ldots, \vec{x}^{(N)}\}$ with $\vec{x}^{(\alpha)} = (x_1^{(\alpha)}, x_2^{(\alpha)}, x_3^{(\alpha)})$, $\alpha = 1, \ldots, N$ and $x_\alpha^{(\alpha)} \in \mathbb{R}$ with $\alpha = 1, 2, 3$. We assume the only testable information pertaining to the quantities $x_\alpha^{(\alpha)}$ consists of the expectation values $\langle x_\alpha^{(\alpha)} \rangle$ and variance $\Delta x_\alpha^{(\alpha)} = \sqrt{\langle (x_\alpha^{(\alpha)} - \langle x_\alpha^{(\alpha)} \rangle)^2 \rangle}$. The set of these expectation values define the $6N$-dimensional space of macrostates of the system. A measure of distinguishability among the states of the ED model is obtained by assigning a probability distribution $P(\vec{x} | \vec{\theta})$ to each macrostate $\vec{\theta}$ where $\{\vec{\theta}\} = \{(1)^{\theta_a^{(\alpha)}}, (2)^{\theta_a^{(\alpha)}}\}$ with $\alpha = 1, 2, \ldots, N$ and $\alpha = 1, 2, 3$. The process of assigning a probability distribution to each state endows $\mathcal{M}_S$ with a metric structure. Specifically, the Fisher-Rao information metric defined in [12] is a measure of distinguishability among macrostates. It assigns an IG to the space of states.

### A. The Gaussian statistical manifold $\mathcal{M}_S$

We consider an arbitrary system evolving over a $3N$-dimensional space. The variables $\{\vec{x}\} = \{\vec{x}^{(1)}, \vec{x}^{(2)}, \ldots, \vec{x}^{(N)}\}$ label the $3N$-dimensional space of microstates of the system. All information relevant to the dynamical evolution of the system is assumed to be contained in the probability distributions. For this reason, no other information is required. Each macrostate may be viewed as a point of a $6N$-dimensional statistical manifold with coordinates given by the numerical values of the expectations $(1)^{\theta_a^{(\alpha)}} = \langle x_\alpha^{(\alpha)} \rangle$ and $(2)^{\theta_a^{(\alpha)}} = \Delta x_\alpha^{(\alpha)} = \sqrt{\langle (x_\alpha^{(\alpha)} - \langle x_\alpha^{(\alpha)} \rangle)^2 \rangle}$. The available information is contained in the following $6N$ information constraint equations,

$$\langle x_\alpha^{(\alpha)} \rangle = \int_{-\infty}^{+\infty} dx_a^{(\alpha)} x_a^{(\alpha)} P_a^{(\alpha)}(x_a^{(\alpha)}) \big| (1)^{\theta_a^{(\alpha)}}, (2)^{\theta_a^{(\alpha)}} \big),$$

$$\Delta x_\alpha^{(\alpha)} = \left[ \int_{-\infty}^{+\infty} dx_a^{(\alpha)} (x_a^{(\alpha)} - \langle x_a^{(\alpha)} \rangle)^2 P_a^{(\alpha)}(x_a^{(\alpha)}) \big| (1)^{\theta_a^{(\alpha)}}, (2)^{\theta_a^{(\alpha)}} \big) \right]^{\frac{1}{2}},$$

(1)

where $(1)^{\theta_a^{(\alpha)}} = \langle x_a^{(\alpha)} \rangle$ and $(2)^{\theta_a^{(\alpha)}} = \Delta x_a^{(\alpha)}$ with $\alpha = 1, 2, \ldots, N$ and $\alpha = 1, 2, 3$. The probability distributions $P_a^{(\alpha)}$ are constrained by the conditions of normalization,

$$\int_{-\infty}^{+\infty} dx_a^{(\alpha)} P_a^{(\alpha)}(x_a^{(\alpha)}) \big| (1)^{\theta_a^{(\alpha)}}, (2)^{\theta_a^{(\alpha)}} \big) = 1.$$  

(2)

The Gaussian distribution is identified by information theory as the maximum entropy distribution if only the expectation value and the variance are known. ME methods allows to associate a probability distribution $P(\vec{x} | \vec{\theta})$ to each
point in the space of states $\Theta$. The distribution that best reflects the information contained in the prior distribution $m(\vec{X})$ updated by the information $\left(\left(\vec{x}^{(a)}_a, \Delta x^{(a)}_a\right)\right)$ is obtained by maximizing the relative entropy

$$S(\Theta) = -\int d^nX P(\vec{X} | \Theta) \log \left( \frac{P(\vec{X} | \Theta)}{m(\vec{X})} \right). \quad (3)$$

As a working hypothesis, the prior $m(\vec{X})$ is set to be uniform since we assume the lack of prior available information about the system (postulate of equal a priori probabilities). Upon maximizing (3), given the constraints (1) and (2), we obtain

$$P(\vec{X} | \Theta) = \prod_{\alpha=1}^{N} P^{(a)}_\alpha \left( \vec{x}^{(a)}_a | \mu^{(a)}_a, \sigma^{(a)}_a \right) \quad (4)$$

where

$$P^{(a)}_\alpha \left( \vec{x}^{(a)}_a | \mu^{(a)}_a, \sigma^{(a)}_a \right) = \left( 2\pi \sigma^{(a)}_a \right)^{-\frac{N}{2}} \exp \left[ -\frac{\left( \vec{x}^{(a)}_a - \mu^{(a)}_a \right)^2}{2 \sigma^{(a)}_a^2} \right] \quad (5)$$

and (1) $\theta^{(a)} = \mu^{(a)}_a$, (2) $\theta^{(a)} = \sigma^{(a)}_a$. For the rest of the paper, unless stated otherwise, the statistical manifold $M$ will be defined by the following expression,

$$M = \left\{ P \left( \vec{X} | \Theta \right) \text{ in } [1] : \vec{X} \in \mathbb{R}^{3N}, \Theta \in D_\Theta = \left( -\infty, +\infty \right)_\mu \times (0, +\infty)^{3N} \right\}. \quad (6)$$

The probability distribution (4) encodes the available information concerning the system. Note we assumed uncoupled constraints among microvariables $x^{(a)}_a$. In other words, we assumed that information about correlations between the microvariables need not to be tracked. This assumption leads to the simplified product rule (4). However, coupled constraints would lead to a generalized product rule in (1) and to a metric tensor (12) with non-trivial off-diagonal elements (covariance terms). For instance, the total probability distribution $P(x, y | \mu_x, \sigma_x, \mu_y, \sigma_y)$ of two dependent Gaussian distributed microvariables $x$ and $y$ reads

$$P(x, y | \mu_x, \sigma_x, \mu_y, \sigma_y) = \frac{1}{2\pi \sigma_x \sigma_y \sqrt{1-r^2}} \times$$

$$\times \exp \left\{ -\frac{1}{2(1-r^2)} \left[ \frac{(x - \mu_x)^2}{\sigma_x^2} - 2r \frac{(x - \mu_x)(y - \mu_y)}{\sigma_x \sigma_y} + \frac{(y - \mu_y)^2}{\sigma_y^2} \right] \right\}, \quad (7)$$

where $r \in (-1, +1)$ is the correlation coefficient given by

$$r = \frac{\langle (x - \langle x \rangle)(y - \langle y \rangle) \rangle}{\sqrt{\langle (x - \langle x \rangle)^2 \rangle} \sqrt{\langle (y - \langle y \rangle)^2 \rangle}} = \frac{\langle xy \rangle - \langle x \rangle \langle y \rangle}{\sigma_x \sigma_y}. \quad (8)$$

The metric induced by (7) is obtained by use of (12), the result being

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sigma^2}{r(r^2-1)} & \frac{\sigma^2}{2(r^2-1)} & \frac{\sigma^2}{2(r^2-1)} \\ 0 & \frac{\sigma^2}{2(r^2-1)} & \sigma^2 & 0 \\ 0 & \frac{\sigma^2}{2(r^2-1)} & 0 & \frac{\sigma^2}{2(r^2-1)} \end{bmatrix}, \quad (9)$$

where $i, j = 1, 2, 3, 4$. The Ricci curvature scalar associated with manifold characterized by (6) is given by

$$R = g^{ij} R_{ij} = -\frac{8 (r^2 - 2) + 2r^2 (3r^2 - 2)}{8 (r^2 - 1)}. \quad (10)$$
It is clear that in the limit $r \to 0$, the off-diagonal elements of $g_{ij}$ vanish and the Scalar $R$ reduces to the result obtained in [10], namely $R = -2 < 0$. Correlation terms may be fictitious. They may arise for instance from coordinate transformations. On the other hand, correlations may arise from external fields in which the system is immersed. In such situations, correlations among $x^{(a)}_a$ effectively describe interaction between the microvariables and the external fields. Such generalizations would require more delicate analysis. Before proceeding, a comment is in order. Most probability distributions are generated in this manner however. Some distributions are generated by combining the results of simple cases (multinomial from a binomial) while others are found as a result of a change of variables (Cauchy distribution). For instance, the Weibull and Wigner-Dyson distributions can be obtained from an exponential distribution as a result of a power law transformation [15].

1. Metric structure of $\mathcal{M}_S$

We cannot determine the evolution of microstates of the system since the available information is insufficient. Not only is the information available insufficient but we also do not know the equation of motion. In fact there is no standard "equation of motion". Instead we can ask: how close are the two total distributions with parameters $(\mu^{(a)}, \sigma^{(a)})$ and $(\mu^{(a)} + d\mu^{(a)}, \sigma^{(a)} + d\sigma^{(a)})$? Once the states of the system have been defined, the next step concerns the problem of quantifying the notion of change from the state $\Theta$ to the state $\Theta + d\Theta$. A convenient measure of change is distance. The measure we seek is given by the dimensionless distance $ds$ between $P(\vec{X} \mid \Theta)$ and $P(\vec{X} \mid \Theta + d\Theta)$,

$$ds^2 = g_{\mu\nu}d\Theta^\mu d\Theta^\nu$$

where

$$g_{\mu\nu} = \int d\vec{X} P(\vec{X} \mid \Theta) \frac{\partial \log P(\vec{X} \mid \Theta)}{\partial \Theta^\mu} \frac{\partial \log P(\vec{X} \mid \Theta)}{\partial \Theta^\nu}$$

is the Fisher-Rao information metric. Substituting (11) into (12), the metric $g_{\mu\nu}$ on $\mathcal{M}_S$ becomes a $6N \times 6N$ matrix $M$ made up of $3N$ blocks $M_{2 \times 2}$ with dimension $2 \times 2$ given by,

$$M_{2 \times 2} = \begin{pmatrix} (\sigma^{(a)})^{-2} & 0 \\ 0 & 2 \times (\sigma^{(a)})^{-2} \end{pmatrix}$$

with $\alpha = 1, 2, \ldots, N$ and $a = 1, 2, 3$. From (12), the "length" element (11) reads,

$$ds^2 = \sum_{\alpha=1}^{N} \sum_{a=1}^{3} \left[ \frac{1}{(\sigma^{(a)})^2} d\mu^{(a)}_a^2 + \frac{2}{(\sigma^{(a)})^2} d\sigma^{(a)}_a^2 \right].$$

We bring attention to the fact that the metric structure of $\mathcal{M}_S$ is an emergent (not fundamental) structure. It arises only after assigning a probability distribution $P(\vec{X} \mid \Theta)$ to each state $\Theta$.

2. Curvature of $\mathcal{M}_S$

Given the Fisher-Rao information metric, we use standard differential geometry methods applied to the space of probability distributions to characterize the geometric properties of $\mathcal{M}_S$. Recall that the Ricci scalar curvature $R$ is given by,

$$R = g^{\mu\nu} R_{\mu\nu},$$

where $g^{\mu\nu}g_{\nu\rho} = \delta^\mu_\rho$ so that $g^{\mu\nu} = (g_{\mu\nu})^{-1}$. The Ricci tensor $R_{\mu\nu}$ is given by,

$$R_{\mu\nu} = \partial_\gamma \Gamma^\gamma_\mu_\nu - \partial_\nu \Gamma^\gamma_\mu_\gamma + \Gamma^\gamma_\mu_\rho \Gamma^\rho_\nu_\gamma - \Gamma^\gamma_\nu_\rho \Gamma^\rho_\mu_\gamma.$$
The Christoffel symbols $\Gamma^\rho_{\mu\nu}$, appearing in the Ricci tensor are defined in the standard manner as,

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left( \partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right).$$

(17)

Using (13) and the definitions given above, we can show that the Ricci scalar curvature becomes

$$R_{\mathcal{M}_s} = R^\sigma_\alpha = \sum_{\rho \neq \sigma} K (\epsilon_\rho, \epsilon_\sigma) = -3N < 0.$$ 

(18)

The scalar curvature is the sum of all sectional curvatures of planes spanned by pairs of orthonormal basis elements $\{\epsilon_\rho = \partial_{\theta_\rho(p)}\}$. The tangent space $T_p \mathcal{M}_s$ with $p \in \mathcal{M}_s$,

$$K (a, b) = \frac{R_{\mu\nu\rho\sigma} a^\mu b^\nu a^\rho b^\sigma}{(g_{\alpha\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}) a^\mu b^\nu a^\rho b^\sigma}, \quad a = \sum_\rho \langle a, h^\rho \rangle \epsilon_\rho,$$

(19)

where $\langle a, h^\rho \rangle = \delta^\rho_\rho$. Notice that the sectional curvatures completely determine the curvature tensor. From (18) we conclude that $\mathcal{M}_s$ is a $6N$-dimensional statistical manifold of constant negative Ricci scalar curvature. A detailed analysis on the calculation of Christoffel connection coefficients using the ED formalism for a four-dimensional manifold of Gaussians can be found in [10].

3. Anisotropy and Compactness

It can be shown that $\mathcal{M}_s$ is not a pseudosphere (maximally symmetric manifold). The first way this can be understood is from the fact that the Weyl Projective curvature tensor [16] (or the anisotropy tensor) $W_{\mu\nu\rho\sigma}$ defined by

$$W_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{R_{\mathcal{M}_s}}{n(n-1)} (g_{\alpha\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}),$$

(20)

with $n = 6N$ in the present case, is non-vanishing. In (20), the quantity $R_{\mu\nu\rho\sigma}$ is the Riemann curvature tensor defined in the usual manner by

$$R^\alpha_{\beta\rho\sigma} = \partial_\sigma \Gamma^\alpha_{\beta\rho} - \partial_\rho \Gamma^\alpha_{\beta\sigma} + \Gamma^\alpha_{\lambda\beta} \Gamma^\lambda_{\rho\sigma} - \Gamma^\alpha_{\lambda\rho} \Gamma^\lambda_{\beta\sigma}.$$ 

(21)

Considerations regarding the negativity of the Ricci curvature as a strong criterion of dynamical instability and the necessity of compactness of $\mathcal{M}_s$ in "true" chaotic dynamical systems is under investigation [14].

The issue of symmetry of $\mathcal{M}_s$ can alternatively be understood from consideration of the sectional curvature. In view of (19), the negativity of the Ricci scalar implies the existence of expanding directions in the configuration space manifold $\mathcal{M}_s$. Indeed, from (18) one may conclude that negative principal curvatures (extrema of sectional curvatures) dominate over positive ones. Thus, the negativity of the Ricci scalar is only a sufficient (not necessary) condition for local instability of geodesic flow. For this reason, the negativity of the scalar provides a strong criterion of local instability. Scenarios may arise where negative sectional curvatures are present, but the positive ones could prevail in the sum so that the Ricci scalar is non-negative despite the instability in the flow in those directions. Consequently, the signs of the sectional curvatures are of primary significance for the proper characterization of chaos.

Yet another useful way to understand the anisotropy of the $\mathcal{M}_s$ is the following. It is known that in $n$ dimensions, there are at most $\frac{n(n+1)}{2}$ independent Killing vectors (directions of symmetry of the manifold). Since $\mathcal{M}_s$ is not a pseudosphere, the information metric tensor does not admit the maximum number of Killing vectors $K_\nu$ defined as

$$\mathcal{L}_K g_{\mu\nu} = D_\mu K_\nu + D_\nu K_\mu = 0,$$

(22)

where $D_\mu$, given by

$$D_\mu K_\nu = \partial_\mu K_\nu - \Gamma^\rho_{\nu\mu} K_\rho,$$

(23)

is the covariant derivative operator with respect to the connection $\Gamma$ defined in (17). The Lie derivative $\mathcal{L}_K g_{\mu\nu}$ of the tensor field $g_{\mu\nu}$ along a given direction $K$ measures the intrinsic variation of the field along that direction (that is, the metric tensor is Lie transported along the Killing vector) [15]. Locally, a maximally symmetric space of Euclidean signature is either a plane, a sphere, or a hyperboloid, depending on the sign of $R$. In our case, none of these scenarios
occur. As will be seen in what follows, this fact has a significant impact on the integration of the geodesic deviation equation on $M_s$. At this juncture, we emphasize it is known that the anisotropy of the manifold underlying system dynamics plays a crucial role in the mechanism of instability. In particular, fluctuating sectional curvatures require also that the manifold be anisotropic. However, the connection between curvature variations along geodesics and anisotropy is far from clear and is currently under investigation.

Krylov was the first to emphasize [18] the use of $R < 0$ as an instability criterion in the context of an $N$-body system (a gas) interacting via Van der Waals forces, with the ultimate hope to understand the relaxation process in a gas. However, Krylov neglected the problem of compactness of the configuration space manifold which is important for making inferences about exponential mixing of geodesic flows [19]. Why is compactness so significant in the characterization of chaos? True chaos should be identified by the occurrence of two crucial features: 1) strong dependence on initial conditions and exponential divergence of the Jacobi vector field intensity, i.e., stretching of dynamical trajectories; 2) compactness of the configuration space manifold, i.e., folding of dynamical trajectories. Compactness [2, 20] is required in order to discard trivial exponential growths due to the unboundedness of the "volume" available to the dynamical system. In other words, the folding is necessary to have a dynamics actually able to mix the trajectories, making practically impossible, after a finite interval of time, to discriminate between trajectories which were very nearby each other at the initial time. When the space is not compact, even in presence of strong dependence on initial conditions, it could be possible in some instances (though not always), to distinguish among different trajectories originating within a small distance and then evolved subject to exponential instability.

The statistical manifold defined in (9) is compact. This can be seen as follows. It is known from IG that there is a one-to-one relation between elements of the statistical manifold and the parameter space. More precisely, the statistical manifold $M_s$ is homeomorphic to the parameter space $D_\Theta$. This implies the existence of a continuous, bijective map $h_{M_s, D_\Theta}$,

$$h_{M_s, D_\Theta} : M_s \ni P(X|\Theta) \rightarrow \Theta \in D_\Theta$$  

where $h_{M_s, D_\Theta}^{-1}(\Theta) = P(X|\Theta)$. The inverse image $h_{M_s, D_\Theta}^{-1}$ is the so-called homeomorphism map. In addition, since homeomorphisms preserve compactness, it is sufficient to restrict ourselves to a compact subspace of the parameter space $D_\Theta$ in order to ensure that $M_s$ is itself compact.

III. CANONICAL FORMALISM FOR THE GAUSSIAN ED-MODEL

The geometrization of a Hamiltonian system by transforming it to a geodesic flow is a well-known technique of classical mechanics associated with the name of Jacobi [21]. Transformation to geodesic motion is obtained in two steps: 1) conformal transformation of the metric; 2) rescaling of the time parameter [22]. The reformulation of dynamics in terms of a geodesic problem allows the application of a wide range of well-known geometrical techniques in the investigation of the solution space and properties of equations of motions. The power of the Jacobi reformulation is that all of the dynamical information is collected into a single geometric object - the manifold on which geodesic flow is induced - in which all the available manifest symmetries are retained. For instance, integrability of the system is connected with the existence of Killing vectors and tensors on this manifold [23, 24].

In this section we study the trajectories of the system on $M_s$. We emphasize ED can be derived from a standard principle of least action (of Maupertuis-Euler-Lagrange-Jacobi type) [6, 27]. The main differences are that the dynamics being considered here, namely ED, is defined on a space of probability distributions $M_s$, not on an ordinary linear space $V$ and the standard coordinates $q_\mu$ of the system are replaced by statistical macrovariables $\Theta^\mu$. The geodesic equations for the macrovariables of the Gaussian ED model are given by,

$$\frac{d^2\Theta^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho} \frac{d\Theta^\nu}{d\tau} \frac{d\Theta^\rho}{d\tau} = 0$$  

with $\mu = 1, 2, ..., 6N$. Observe the geodesic equations are nonlinear second order coupled ordinary differential equations. They describe a reversible dynamics whose solution is the trajectory between an initial and a final macrostate. The trajectory can be equally well traversed in both directions.

A. Geodesics on $M_s$

We determine the explicit form of [22] for the pairs of statistical coordinates $(\mu^\alpha_\alpha, \sigma^\alpha_\alpha)$. Substituting the expression of the Christoffel connection coefficients into [25], the geodesic equations for the macrovariables $\mu^\alpha_\alpha$ and $\sigma^\alpha_\alpha$
associated to the microstate \( x^{(\alpha)}_a \) become,

\[
\frac{d^2 \mu_a^{(\alpha)}}{d\tau^2} - \frac{2}{\sigma_a^{(\alpha)}} \frac{d\mu_a^{(\alpha)}}{d\tau} \frac{d\sigma_a^{(\alpha)}}{d\tau} = 0, \quad \frac{d^2 \sigma_a^{(\alpha)}}{d\tau^2} - \frac{1}{\sigma_a^{(\alpha)}} \left( \frac{d\sigma_a^{(\alpha)}}{d\tau} \right)^2 + \frac{1}{2\sigma_a^{(\alpha)}} \left( \frac{d\mu_a^{(\alpha)}}{d\tau} \right)^2 = 0, \tag{26}
\]

with \( \alpha = 1, 2, \ldots, N \) and \( a = 1, 2, 3 \). This is a set of coupled ordinary differential equations, whose solutions are

\[
\mu_a^{(\alpha)}(\tau) = \frac{(B_a^{(\alpha)})^2}{2\beta_a^{(\alpha)}} \cosh \left( \frac{2\beta_a^{(\alpha)} \tau}{s_a^{(\alpha)}} \right) - \sinh \left( \frac{2\beta_a^{(\alpha)} \tau}{s_a^{(\alpha)}} \right),
\]

\[
\sigma_a^{(\alpha)}(\tau) = B_a^{(\alpha)} \frac{\cosh \left( \frac{2\beta_a^{(\alpha)} \tau}{s_a^{(\alpha)}} \right) - \sinh \left( \frac{2\beta_a^{(\alpha)} \tau}{s_a^{(\alpha)}} \right)}{\cosh \left( \frac{2\beta_a^{(\alpha)} \tau}{s_a^{(\alpha)}} \right) - \sinh \left( \frac{2\beta_a^{(\alpha)} \tau}{s_a^{(\alpha)}} \right) + \frac{\beta_a^{(\alpha)}}{s_a^{(\alpha)}}}. \tag{27}
\]

The quantities \( B_a^{(\alpha)}, C_a^{(\alpha)}, \beta_a^{(\alpha)} \) are real integration constants that can be evaluated upon specification of boundary conditions. We are interested in the stability of the trajectories on \( \mathcal{M}_s \). It is known \cite{25} that the Riemannian curvature of a manifold is intimately related to the behavior of geodesics on it. If the Riemannian curvature of a manifold is negative, geodesics (initially parallel) rapidly diverge from one another. For the sake of simplicity, we assume very special initial conditions: \( B_a^{(\alpha)} \equiv \Lambda, \beta_a^{(\alpha)} \equiv \lambda \in \mathbb{R}^+ \), \( C_a^{(\alpha)} = 0 \), \( \forall \alpha = 1, 2, \ldots, N \) and \( a = 1, 2, 3 \). However, the conclusions drawn can be generalized to more arbitrary initial conditions. We observe that since every maximal geodesic is well-defined for all temporal parameters \( \tau, \mathcal{M}_s \) constitute a geodesically complete manifold \cite{26}. It is therefore a natural setting within which one may consider global questions and search for a weak criterion of chaos \cite{2}.

### IV. EXPONENTIAL DIVERGENCE OF THE JACOBI VECTOR FIELD INTENSITY

The actual interest of the Riemannian formulation of the dynamics stems from the possibility of studying the instability of natural motions through the instability of geodesics of a suitable manifold, a circumstance that has several advantages. First of all a powerful mathematical tool exists to investigate the stability or instability of a geodesic flow: the Jacobi-Levi-Civita equation for geodesic spread \cite{24}. The JLC-equation describes covariantly how nearby geodesics locally scatter. It is a familiar object both in Riemannian geometry and theoretical physics (it is of fundamental interest in experimental General Relativity). Moreover the JLC-equation relates the stability or instability of a geodesic flow with curvature properties of the ambient manifold, thus opening a wide and largely unexplored field of investigation of the connections among geometry, topology and geodesic instability, hence chaos.

Consider the behavior of the one-parameter family of neighboring geodesics \( \mathcal{F}_{G_{\mathcal{M}_s}}(\lambda) \equiv \{ \Theta^{\mu}_{\mathcal{M}_s}(\tau; \lambda) \}_{\lambda \in \mathbb{R}^+}^{\mu = 1 \ldots 6N} \) where

\[
\mu_a^{(\alpha)}(\tau; \lambda) = \frac{\Lambda^2}{2\lambda} \frac{1}{\cosh (2\lambda \tau) - \sinh (2\lambda \tau) + \frac{\Delta^2}{8\lambda^2}},
\]

\[
\sigma_a^{(\alpha)}(\tau; \lambda) = \Lambda \frac{\cosh (\lambda \tau) - \sinh (\lambda \tau)}{\cosh (2\lambda \tau) - \sinh (2\lambda \tau) + \frac{\lambda^2}{8\lambda^2}}, \tag{28}
\]

with \( \alpha = 1, 2, \ldots, N \) and \( a = 1, 2, 3 \). The relative geodesic spread on a (non-maximally symmetric) curved manifold as \( \mathcal{M}_s \) is characterized by the Jacobi-Levi-Civita equation, the natural tool to tackle dynamical chaos \cite{17, 27},

\[
\frac{D^2 \delta \Theta^\mu}{D\tau^2} + R^\nu_{\epsilon\rho\sigma} \frac{\partial \Theta^\nu}{\partial \tau} \delta \Theta^\rho \frac{\partial \Theta^\sigma}{\partial \tau} = 0 \tag{29}
\]

where the Jacobi vector field \( J^\mu \) is defined as,

\[
J^\mu \equiv \delta \Theta^\mu \equiv \delta_\lambda \Theta^\mu \left. \left( \frac{\partial \Theta^\mu (\tau; \lambda)}{\partial \lambda} \right) \right|_{\tau = \text{const}} \delta \lambda. \tag{30}
\]

Notice that the JLC-equation appears intractable already at rather small \( N \). For isotropic manifolds, the JLC-equation can be reduced to the simple form,

\[
\frac{D^2 J^\mu}{D\tau^2} + K J^\mu = 0, \quad \mu = 1, \ldots, 6N \tag{31}
\]
where $K$ is the constant value assumed throughout the manifold by the sectional curvature. The sectional curvature of manifold $\mathcal{M}_s$ is the $6N$-dimensional generalization of the Gaussian curvature of two-dimensional surfaces of $\mathbb{R}^3$. If $K < 0$, unstable solutions of equation (31) assumes the form

$$J (\tau) = \frac{1}{\sqrt{-K}} \omega (0) \sinh \left( \sqrt{-K} \tau \right)$$

once the initial conditions are assigned as $J (0) = 0$, $\frac{d J (0)}{d \tau} = \omega (0)$ and $K < 0$. Equation (29) forms a system of $6N$ coupled ordinary differential equations linear in the components of the deviation vector field (30) but nonlinear in derivatives of the metric (12). It describes the linearized geodesic flow: the linearization ignores the relative velocity of the geodesics. When the geodesics are neighboring but their relative velocity is arbitrary, the corresponding geodesic deviation equation is the so-called generalized Jacobi equation [28, 29].

The nonlinearity is due to the existence of velocity-dependent terms in the system. Neighboring geodesics accelerate relative to each other with a rate directly proportional to the space of probability distributions $\mathcal{P} (\vec{X} | \vec{\Theta})$ labeled by $6N$ statistical parameters $\vec{\Theta}$. These parameters are the coordinates for the point $P$, and in these coordinates a volume element $d V_{\mathcal{M}_s}$ reads,

$$d V_{\mathcal{M}_s} = \sqrt{g} d^6 \vec{\Theta} = \prod_{a=1}^N \prod_{a=1}^3 \sqrt{\frac{2}{\sigma_a (0)}} d \mu_a (0) d \sigma_a (0)$$

The volume of an extended region $\Delta V_{\mathcal{M}_s} (\tau; \lambda)$ of $\mathcal{M}_s$ is defined by,

$$\Delta V_{\mathcal{M}_s} (\tau; \lambda) = \prod_{a=1}^N \prod_{a=1}^3 \int_{\mu_a (0)}^{\mu_a (\tau)} \int_{\sigma_a (0)}^{\sigma_a (\tau)} \sqrt{2} d \mu_a (0) d \sigma_a (0)$$

where $\mu_a (\tau)$ and $\sigma_a (\tau)$ are given in (28). The quantity that encodes relevant information about the stability of neighboring volume elements is the average volume $\bar{V}_{\mathcal{M}_s} (\tau; \lambda)$,

$$\bar{V}_{\mathcal{M}_s} (\tau; \lambda) \equiv \langle \Delta V_{\mathcal{M}_s} (\tau; \lambda) \rangle |_{\tau} = \frac{1}{\tau} \int_0^\tau \Delta V_{\mathcal{M}_s} (\tau'; \lambda) d \tau' \tau \approx e^{3N \lambda \tau}$$

This asymptotic regime of diffusive evolution in (37) describes the exponential increase of average volume elements on $\mathcal{M}_s$. The exponential instability characteristic of chaos forces the system to rapidly explore large areas (volumes) of the

where $\Theta$ is the space of probability distributions $\mathcal{P} (\vec{X} | \vec{\Theta})$.
statistical manifold. It is interesting to note that this asymptotic behavior appears also in the conventional description of quantum chaos where the entropy increases linearly at a rate determined by the Lyapunov exponents \(\lambda\). The linear increase of entropy as a quantum chaos criterion was introduced by Zurek and Paz \([4]\). In our information-geometric approach a relevant quantity that can be useful to study the degree of instability characterizing the ED model is the information-geometric entropy defined as,

\[
S_{M_s} \overset{\text{def}}{=} \lim_{\tau \to \infty} \log \hat{V}_{M_s}(\tau; \lambda).
\]

Substituting (37) in (38), we obtain

\[
S_{M_s} = \lim_{\tau \to \infty} \log \left\{ \frac{1}{\tau} \int_0^\tau \prod_{\alpha=1}^N \prod_{a=1}^3 \mu_{\alpha}^{(a)}(\tau') \int \sigma_{\alpha}^{(a)}(0) \frac{\sqrt{2}}{\sigma_{\alpha}(0)} \frac{\int \mu_{\alpha}^{(a)}(\tau') \int \sigma_{\alpha}^{(a)}(0)}{\sqrt{2}} d\mu_{\alpha}^{(a)} d\sigma_{\alpha}^{(a)} \right\} \approx \int_0^\tau \approx 3N\lambda \tau.
\]

The entropy \(S_{M_s}\) in (39) is the asymptotic limit of the natural logarithm of the statistical weight \(\langle \Delta V_{M_s} \rangle\) defined on \(M_s\). Its linear growth in time is reminiscent of the aforementioned quantum chaos criterion. Indeed, equation (39) may be considered the information-geometric analog of the Zurek-Paz quantum chaos criterion.

In conclusion, we have shown,

\[
R_{M_s} = -3N, \quad S_{M_s} \approx 3N\lambda \tau, \quad J_{M_S} \approx 3Ne^{\lambda \tau}.
\]

The Ricci scalar curvature \(R_{M_s}\), the information-geometric entropy \(S_{M_s}\), and the Jacobi vector field intensity \(J_{M_S}\) are proportional to the number of Gaussian-distributed microstates of the system. This proportionality leads to the conclusion that there exists a substantial link among these information-geometric measures of chaoticity, namely

\[
R_{M_s} \sim S_{M_s} \sim J_{M_S}.
\]

Equation (41), together with the information-geometric analog of the Zurek-Paz quantum chaos criterion, equation (39), represent the fundamental results of this work. We believe our theoretical modelling scheme may be used to describe actual systems where transitions from quantum to classical chaos scenario occur, but this requires additional analysis. In the following section, we briefly consider some similarities among the von Neumann, Kolmogorov-Sinai and Information-Geometrodynamical entropies.

**VI. ON THE VON NEUMANN, KOLMOGOROV-SINAI AND INFORMATION GEOMETRODYNAMICAL ENTROPIES**

In conventional approaches to chaos, the notion of entropy is introduced, in both classical and quantum physics, as the missing information about the systems fine-grained state \([3][31]\). For a classical system, suppose that the phase space is partitioned into very fine-grained cells of uniform volume \(\Delta v\), labelled by an index \(j\). If one does not know which cell the system occupies, one assigns probabilities \(p_j\) to the various cells; equivalently, in the limit of infinitesimal cells, one can use a phase-space density \(\rho(X_j) = \frac{p_j}{\Delta v}\). Then, in a classical chaotic evolution, the asymptotic expression of the information needed to characterize a particular coarse-grained trajectory out to time \(\tau\) is given by the Shannon information entropy (measured in bits),

\[
S^{(\text{chaotic})}_{\text{classical}} = -\int dX \rho(X) \log_2 (\rho(X) \Delta v) = -\sum_j p_j \log_2 p_j \sim K_\tau.
\]

where \(\rho(X)\) is the phase-space density and \(p_j = \frac{\rho(X_j)}{\Delta v}\) is the probability for the corresponding coarse-grained trajectory. \(S^{(\text{chaotic})}_{\text{classical}}\) is the missing information about which fine-grained cell the system occupies. The quantity \(K\) represents the linear rate of information increase and it is called the Kolmogorov-Sinai entropy (or metric entropy) \((K\) is the sum of positive Lyapunov exponents, \(K = \sum_j \lambda_j\)). \(K\) quantifies the degree of classical chaos. It is worthwhile emphasizing that the quantity that grows asymptotically as \(K\tau\) is really the average of the information on the left side of equation (42). This distinction can be ignored however, if we assume that the chaotic system has roughly constant Lyapunov exponents over the accessible region of phase space. In quantum mechanics the fine-grained alternatives are normalized state vectors in Hilbert space. From a set of probabilities for various state vectors, one can construct a density operator

\[
\hat{\rho} = \sum_j \lambda_j |\psi_j\rangle \langle \psi_j|, \quad \hat{\rho} |\psi_j\rangle = \lambda_j |\psi_j\rangle.
\]
The normalization of the density operator, \( tr (\hat{\rho}) = 1 \), implies that the eigenvalues make up a normalized probability distribution. The von Neumann entropy of the density operator \( \hat{\rho} \) (measured in bits) \( [32] \),

\[
S_{\text{quantum}}^{(\text{chaotic})} = -tr (\hat{\rho} \log_2 \hat{\rho}) = -\sum_j \lambda_j \log_2 \lambda_j
\]  

(44)
can be thought of as the missing information about which eigenvector the system is in. Entropy quantifies the degree of unpredictability about the system's fine-grained state.

Recall that decoherence is the loss of phase coherence between the set of preferred quantum states in the Hilbert space of the system due to the interaction with the environment. Moreover, decoherence induces transitions from quantum to classical systems. Therefore, classicality is an emergent property of an open quantum system. Motivated by such considerations, Zurek and Paz investigated implications of the process of decoherence for quantum chaos.

They considered a chaotic system, a single unstable harmonic oscillator characterized by a potential \( V (x) = -\frac{\lambda x^2}{2} \) (\( \lambda \) is the Lyapunov exponent), coupled to an external environment. In the \textit{reversible classical limit} \( [32] \), the von Neumann entropy of such a system increases linearly at a rate determined by the Lyapunov exponent,

\[
S_{\text{quantum}}^{(\text{chaotic})} (\text{Zurek-Paz}) \sim \frac{3}{2} \lambda \tau.
\]  

(45)

Notice that the consideration of \( 3N \) uncoupled identical unstable harmonic oscillators characterized by potentials \( V_i (x) = -\frac{\lambda_i x^2}{2} \) (\( \lambda_i = \lambda_j \); \( i, j = 1, 2, \ldots, 3N \)) would simply lead to

\[
S_{\text{quantum}}^{(\text{chaotic})} (\text{Zurek-Paz}) \sim 3N \lambda \tau.
\]  

(46)

The resemblance of equations \( [30] \) and \( [46] \) is remarkable. In what follows, we apply our information geometrical method to an \( n \)-set (for \( n = 2 \)) of uncoupled inverted harmonic oscillators, each with different frequency, and show we obtain asymptotic linear IGE growth. The case for arbitrary \( n \)-set in three dimensions is presented in the Appendix.

\section{VII. THE INFORMATION GEOMETRY OF A 2-SET OF INVERTED HARMONIC OSCILLATORS (IHO)}

In this section, our objective is to characterize chaotic properties of a 2-set of one-dimensional inverted harmonic oscillators, each with different frequency \( \omega_1 \neq \omega_2 \) using the formalism presented in this paper. We will study the asymptotic behavior of the geometrodynamical entropy and the functional dependence of the Ricci scalar curvature of the 2-dimensional manifold \( \mathcal{M}^{(2)}_{\text{IHO}} \) underlying the ED model of the IHOs on the frequencies \( \omega_i \), \( i = 1, 2 \). Recent investigation explore the possibility of using well established principles of inference to derive Newtonian dynamics from relevant prior information codified into an appropriate statistical manifold \( [34] \). In that work the basic assumption is that there is an irreducible uncertainty in the location of particles so that the state of a particle is defined by a probability distribution. The corresponding configuration space is a statistical manifold the geometry of which is defined by the information metric. The trajectory follows from a principle of inference, the method of Maximum Entropy. There is no need for additional "physical" postulates such as an action principle or equation of motion, nor for the concept of mass, momentum and of phase space, not even the notion of time. The resulting "entropic" dynamics reproduces Newton’s mechanics for any number of particles interacting among themselves and with external fields. Both the mass of the particles and their interactions are explained as a consequence of the underlying statistical manifold.

In what follows, we introduce the basics of the general formalism for an \( n \)-set of IHOs. This approach is similar (mathematically but not conceptually) to the geometrization of Newtonian dynamics used in the Riemannian geometrodynamical to chaos \( [1, 35] \).

\subsection{A. Informational geometrization of Newtonian dynamics}

Newtonian dynamics can be recast in the language of Riemannian geometry applied to probability theory, namely, Information Geometry. In our case, the system under investigation has \( n \) degrees of freedom and a point on the \( n \)-dimensional configuration space manifold \( \mathcal{M}^{(n)}_{\text{IHO}} \) is parametrized by the \( n \) Lagrangian coordinates \( (\theta_1, \ldots, \theta_n) \). Moreover, the system under investigation is described by the Lagrangian \( \mathcal{L} \),

\[
\mathcal{L} = T \left( \dot{\theta}_1, \ldots, \dot{\theta}_n \right) - \Phi (\theta_1, \ldots, \theta_n) = \frac{1}{2} \sum_{i,j} \delta_{ij} \theta_i \theta_j + \frac{1}{2} \sum_{j=1}^n \omega_j^2 \theta_j^2
\]  

(47)
so that the Hamiltonian function $H = T + \Phi \equiv E$ is a constant of motion. For the sake of simplicity, let us set $E = 1$. According to the principle of stationary action - in the form of Maupertuis - among all the possible isoenergetic paths $\gamma(t)$ with fixed end points, the paths that make vanish the first variation of the action functional

$$\mathcal{I} = \int_{\gamma(t)} \frac{\partial L}{\partial \dot{\theta}_i} \dot{\theta}_i \, dt$$

are natural motions. As the kinetic energy $T$ is a homogeneous function of degree two, we have $2T = \dot{\theta}_i \frac{\partial L}{\partial \theta_i}$, and Maupertuis’ principle reads

$$\delta \mathcal{I} = \delta \int_{\gamma(t)} 2T \, dt = 0.$$  

(49)

The manifold $\mathcal{M}^{(n)}_{1HO}$ is naturally given a proper Riemannian structure. In fact, let us consider the matrix

$$g_{ij} (\theta_1, ..., \theta_n) = [1 - \Phi(\theta_1, ..., \theta_n)] \delta_{ij}$$

(50)

so that Maupertuis’ principle becomes

$$\delta \int_{\gamma(t)} T \, dt = \delta \int_{\gamma(t)} (T^2)^{\frac{1}{2}} \, dt \delta = \delta \int_{\gamma(t)} \left\{ [1 - \Phi(\theta_1, ..., \theta_n)] \delta_{ij} \dot{\theta}_i \dot{\theta}_j \right\}^{\frac{1}{2}}$$

$$= \delta \int_{\gamma(t)} (g_{ij} \dot{\theta}_i \dot{\theta}_j)^{\frac{1}{2}} \, dt = \delta \int_{\gamma(s)} ds = 0, \; ds^2 = g_{ij} d\theta^i d\theta^j.$$  

(51)

thus natural motions are geodesics of $\mathcal{M}^{(n)}_{1HO}$, provided we define $ds$ as its arclength. The metric tensor $g_J(\cdot, \cdot)$ of $\mathcal{M}^{(n)}_{1HO}$ is then defined by

$$g = g_{ij} d\theta^i \otimes d\theta^j$$

(52)

where $(d\theta^1, ..., d\theta^n)$ is a natural base of $T^*_\theta \mathcal{M}^{(n)}_{1HO}$ - the cotangent space at the point $\theta$ - in the local chart $(\theta^1, ..., \theta^n)$. This is known as the Jacobi metric (or kinetic energy metric). Denoting by $\nabla$ the canonical Levi-Civita connection, the geodesic equation

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0$$

(53)

becomes, in the local chart $(\theta^1, ..., \theta^n)$,

$$\frac{d^2 \theta^i}{ds^2} + \Gamma^i_{jk} \frac{d\theta^j}{ds} \frac{d\theta^k}{ds} = 0$$

(54)

where the Christoffel coefficients are the components of $\nabla$ defined by

$$\Gamma^i_{jk} = \langle d\theta^i, \nabla_j e_k \rangle = \frac{1}{2} g^{im} (\partial_j g_{km} + \partial_k g_{mj} - \partial_m g_{jk}),$$

(55)

with $\partial_i = \frac{\partial}{\partial \theta_i}$. Since $g_{ij} (\theta_1, ..., \theta_n) = [1 - \Phi(\theta_1, ..., \theta_n)] \delta_{ij}$, from the geodesic equation we obtain

$$\frac{d^2 \theta^i}{ds^2} + \frac{1}{2 (1 - \Phi)} \left[ 2 \frac{\partial (1 - \Phi) \frac{d \theta^j}{ds}}{\partial \theta_j} \frac{d \theta^i}{ds} - g^{ij} \frac{\partial (1 - \Phi)}{\partial \theta_j} g_{km} \frac{d \theta^k}{ds} \frac{d \theta^m}{ds} \right] = 0,$$

(56)

whereupon using $ds^2 = (1 - \Phi) \, dt^2$, we verify that (56) reduces to

$$\frac{d^2 \theta^i}{d\tau^2} + \frac{\partial \Phi (\theta_1, ..., \theta_n)}{\partial \theta_i} = 0, \; i = 1, ..., n.$$  

(57)

Equation (57) are Newton’s equations. It is worthwhile emphasizing that the transformation to geodesic motion on a curved statistical manifold is obtained in two key steps: the \textit{conformal transformation of the metric}, $\delta_{ij} \rightarrow g_{ij} = (1 - \Phi) \, \delta_{ij}$ and, the \textit{rescaling of the temporal evolution parameter}, $d\tau^2 \rightarrow ds^2 = 2 (1 - \Phi) \, d\tau^2$. 
B. The 2-set of inverted anisotropic one-dimensional harmonic oscillators

As a simple physical example, we examine the IG associated with a 2-set configuration of IHOs. In this case, the metric tensor \( g_{ij} \) appearing in (50) takes the form

\[
g_{ij} (\theta_1, \theta_2) = [1 - \Phi (\theta_1, \theta_2)] \cdot \delta_{ij} (\theta_1, \theta_2) \text{ with } i, j = 1, 2.
\] (58)

where the function \( \Phi (\theta_1, \theta_2) \) is given by,

\[
\Phi (\theta_1, \theta_2) = \sum_{j=1}^{2} \Phi_j (\theta_j), \quad \Phi_j (\theta_j) = -\frac{1}{2} \omega_j^2 \theta_j^2.
\] (59)

Hence the metric tensor \( g_{ij} \) on \( M_{IHO}^{(2)} \) becomes,

\[
g_{ij} = \begin{pmatrix}
1 + \frac{1}{2} (\omega_1^2 \theta_1^2 + \omega_2^2 \theta_2^2) & 0 \\
0 & 1 + \frac{1}{2} (\omega_1^2 \theta_1^2 + \omega_2^2 \theta_2^2)
\end{pmatrix}.
\] (60)

Using the standard definition of the Ricci scalar (15), we obtain

\[
R_{M_{IHO}^{(2)}} (\omega_1, \omega_2) = \frac{4 (\theta_1^2 \omega_1^4 + \theta_2^2 \omega_2^4) - 4 (\theta_1^4 + \theta_2^4) \omega_1^2 \omega_2^2 - 8 (\omega_1^4 + \omega_2^4)}{(\theta_1^2 \omega_1^2 + \theta_2^2 \omega_2^2 + 2)^3}.
\] (61)

In the limit of a flat frequency spectrum, \( \omega_1 = \omega_2 = \omega \), the scalar curvature (61) is constantly negative,

\[
R_{M_{IHO}^{(2)}} (\omega) = \frac{-16 \omega^2}{[2 + (\theta_1^2 + \theta_2^2) \omega^2]} < 0, \forall \omega \geq 0.
\] (62)

However, in presence of distinct frequency values, \( \omega_1 \neq \omega_2 \), it is possible to properly choose the \( \omega \)'s so that \( R_{M_{IHO}^{(2)}} (\omega_1, \omega_2) \) becomes either negative or positive. In addition, we notice that the manifold underlying the IHO model is anisotropic since its associated Weyl projective curvature tensor components are non-vanishing. For the special case, \( \omega_1 = \omega_2 \), we obtain

\[
W_{1212} (\omega) = \frac{8 \omega^4 (\theta_1^2 + \theta_2^2) + 2 \omega^6 (\theta_1^4 + \theta_2^4) + 4 \omega^6 \theta_1^2 \theta_2^2}{(\theta_1^2 \omega^2 + \theta_2^2 \omega^2 + 2)^3}.
\] (63)

Clearly, the frequency parameter \( \omega \) drives the degree of anisotropy of the statistical manifold \( M_{IHO}^{(2)} \) and, as expected, in the limit of vanishing \( \omega \), we recover the flat \((R = 0),\) isotropic \((W = 0)\) Euclidean manifold characterized by metric \( \delta_{ij} \). This result is a concrete example of the fact that conformal transformations change the degree of anisotropy of the ambient statistical manifold underlying the Newtonian dynamics. Our only remaining task is to compute the information geometrodynamical entropy \( S_{M_{IHO}^{(2)}} (\tau; \omega_1, \omega_2) \), defined as

\[
S_{M_{IHO}^{(2)}} (\tau; \omega_1, \omega_2) \overset{\text{def}}{=} \lim_{\tau \to \infty} \log \left[ \langle \Delta V_{M_{IHO}^{(2)}} (\tau; \omega_1, \omega_2) \rangle \right].
\] (64)

The quantity \( \langle \Delta V_{M_{IHO}^{(2)}} (\tau; \omega_1, \omega_2) \rangle \) appearing in (64) is the average volume element, defined by

\[
\langle \Delta V_{M_{IHO}^{(2)}} (\tau; \omega_1, \omega_2) \rangle = \frac{1}{\tau} \int_{0}^{\tau} \Delta V_{M_{IHO}^{(2)}} (\tau; \omega_1, \omega_2) \, d\tau',
\] (65)

with the statistical volume element \( \Delta V_{M_{IHO}^{(2)}} \) given by

\[
\Delta V_{M_{IHO}^{(2)}} (\tau; \omega_1, \omega_2) = \int_{\{\bar{\theta}'\}} \left[ 1 + \frac{1}{2} (\omega_1^2 \theta_1^2 + \omega_2^2 \theta_2^2) \right] d\theta_1' d\theta_2'
\] (66)

\[
\approx \frac{1}{6} \omega_1^2 \omega_2^2 (\bar{\theta}_1^2 \omega_1^2 + \bar{\theta}_2^2 \omega_2^2).
\]
Recall that the two Newtonian equations of motion for each inverted harmonic oscillator are given by,

\[ \frac{d^2 \theta_j}{dt^2} - \omega_j^2 \theta_j = 0, \forall j = 1, 2. \]  
(67)

Hence, the asymptotic behavior of such macrovariables on manifold \( \mathcal{M}_{IHO}^{(2)} \) is given by,

\[ \theta_j (\tau) \xrightarrow{\tau \to \infty} \Xi_j e^{\omega_j \tau}, \Xi_j \in \mathbb{R}, \forall j = 1, 2. \]  
(68)

Substituting \( \theta_1 (\tau') = \Xi_1 e^{\omega_1 \tau'} \) and \( \theta_2 (\tau') = \Xi_2 e^{\omega_2 \tau'} \) into (66), we obtain

\[ \Delta V_{\mathcal{M}_{IHO}^{(2)}} (\tau; \omega_1, \omega_2) \xrightarrow{\tau \to \infty} \frac{\Xi_1 \Xi_2}{6} e^{(\omega_1 + \omega_2)\tau} \left( \Xi_1^2 e^{2\omega_1 \tau} \omega_1^2 + \Xi_2^2 e^{2\omega_2 \tau} \omega_2^2 \right). \]  
(69)

By direct computation, we find the average of (69) is given by,

\[ \left\langle \Delta V_{\mathcal{M}_{IHO}^{(2)}} (\tau; \omega_1, \omega_2) \right\rangle \xrightarrow{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \left[ \frac{\Xi_1 \Xi_2}{6} e^{(\omega_1 + \omega_2)\tau'} \left( \Xi_1^2 e^{2\omega_1 \tau'} \omega_1^2 + \Xi_2^2 e^{2\omega_2 \tau'} \omega_2^2 \right) \right] d\tau'. \]  
(70)

Assuming as a working hypothesis that \( \Xi_1 = \Xi_2 = \Xi \), we obtain

\[ \frac{1}{\tau} \int_0^\tau \left[ \frac{\Xi_1 \Xi_2}{6} e^{(\omega_1 + \omega_2)\tau'} \left( \Xi_1^2 e^{2\omega_1 \tau'} \omega_1^2 + \Xi_2^2 e^{2\omega_2 \tau'} \omega_2^2 \right) \right] d\tau' = \begin{cases} \frac{1}{18} \Xi^6 e^{\exp(3\omega_1 \tau)}, & \text{if } \omega_1 = \omega_2, \\ \frac{1}{18} \Xi^6 \omega_1 e^{\exp(3\omega_1 \tau)}, & \text{if } \omega_1 \gg \omega_2, \\ \frac{1}{18} \Xi^6 \omega_2 e^{\exp(3\omega_2 \tau)}, & \text{if } \omega_2 \gg \omega_1. \end{cases} \]  
(71)

Finally, substituting (71) in (64), we obtain

\[ S_{\mathcal{M}_{IHO}^{(2)}} (\tau; \omega_1, \omega_2) \xrightarrow{\tau \to \infty} \begin{cases} 2\omega_1 \tau, & \text{if } \omega_1 = \omega_2, \\ \omega_1 \tau, & \text{if } \omega_1 \gg \omega_2, \\ \omega_2 \tau, & \text{if } \omega_2 \gg \omega_1. \end{cases} \]  
(72)

It is clear that the information-geometrodynamical entropy \( S_{\mathcal{M}_{IHO}^{(2)}} (\tau; \omega_1, \omega_2) \) exhibits classical linear behavior in the asymptotic limit, with proportionality coefficient \( \Omega = \omega_1 + \omega_2 \).

\[ S_{\mathcal{M}_{IHO}^{(2)}} (\tau; \omega_1, \omega_2) \xrightarrow{\tau \to \infty} \Omega \tau. \]  
(73)

Equation (73) expresses the asymptotic linear growth of our information geometrodynamical entropy for the IHO system considered. This result (for \( n = 2 \)) extends the result of Zurek-Paz [10]. This result, together with the authors previous works [10, 11] lend substantial support for the IG approach advocated in the present article.

**VIII. FINAL REMARKS**

A Gaussian ED statistical model has been constructed on a 6\( N \)-dimensional statistical manifold \( \mathcal{M}_s \). The macrocoordinates on the manifold are represented by the expectation values of microvariables associated with Gaussian distributions. The geometric structure of \( \mathcal{M}_s \) was studied in detail. It was shown that \( \mathcal{M}_s \) is a curved manifold of constant negative Ricci curvature \(-3N\). The geodesics of the ED model are hyperbolic curves on \( \mathcal{M}_s \). A study of the stability of geodesics on \( \mathcal{M}_s \) was presented. The notion of statistical volume elements was introduced to investigate the asymptotic behavior of a one-parameter family of neighboring volumes \( \mathcal{F}_{V_{\mathcal{M}_s}} (\lambda) = \{ V_{\mathcal{M}_s} (\tau; \lambda) \}_{\lambda \in \mathbb{R}^+} \). An information-geometric analog of the Zurek-Paz chaos criterion was suggested. It was shown that the behavior of geodesics is characterized by exponential instability that leads to chaotic scenarios on the curved statistical manifold. These conclusions are supported by a study based on the geodesic deviation equations and on the asymptotic behavior of the Jacobi vector field intensity \( J_{\mathcal{M}_s} \) on \( \mathcal{M}_s \). A Lyapunov exponent analog similar to that appearing in the Riemannian geometric approach to chaos was suggested as an indicator of chaoticity. On the basis of our analysis
a relationship among an entropy-like quantity, chaoticity and curvature is proposed, suggesting to interpret the statistical curvature as a measure of the entropic dynamical chaoticity.

The results obtained in this work are significant, in our opinion, since a rigorous relation among curvature, Lyapunov exponents and Kolmogorov-Sinai entropy is still under investigation\cite{3}. In addition, there does not exist a well defined unifying characterization of chaos in classical and quantum physics\cite{2} due to fundamental differences between the two theories. In addition, the role of curvature in statistical inference is even less understood. The meaning of statistical curvature for a one-parameter model in inference theory was introduced in\cite{36}. Curvature served as an important tool in the asymptotic theory of statistical estimation. Therefore the implications of this work is twofold. Firstly, it helps understanding possible future use of the statistical curvature in modelling real processes by relating it to conventionally accepted quantities such as entropy and chaos. On the other hand, it serves to cast what is already known in physics regarding curvature in a new light as a consequence of its proposed link with inference.

As a simple physical example, we considered the information-geometry \(M_{IHO}^{(2)}\) associated with a 2-set configuration of inverted harmonic oscillators. It was determined that in the limit of a flat frequency spectrum (\(\omega_1 = \omega_2 = \omega\)), the scalar curvature \(R_{M_{IHO}^{(2)}}(\omega_1, \omega_2)\) is constantly negative. In the case of distinct frequencies, i.e., \(\omega_1 \neq \omega_2\), it is possible - for appropriate choices of \(\omega_1\) and \(\omega_2\) - to obtain either negative or positive values of \(R_{M_{IHO}^{(2)}}(\omega_1, \omega_2)\). Moreover, it was shown that \(M_{IHO}^{(2)}\) is an anisotropic manifold since the Weyl projective curvature tensor has a non-vanishing component \(W_{1212}\). It was found that the information geometrodynamical entropy of the IHO system exhibits asymptotic linear growth. This IHO example is generalized to arbitrary values of \(n\) in the Appendix.

The descriptions of a classical chaotic system of arbitrary interacting degrees of freedom, deviations from Gaussianity and chaoticity arising from fluctuations of positively curved statistical manifolds are being investigated\cite{14}. The work here presented is shown to be useful to investigate chaotic quantum spectra arising, for instance, from the Poisson and Wigner-Dyson quantum level spacing distributions\cite{13,37,38}. We remark that based on the results obtained from the chosen ED models, it is not unreasonable to think that should the correct variables describing the true degrees of freedom of a physical system be identified, perhaps deeper insights into the foundations of models of physics and reasoning (and their relationship to each other) may be uncovered.

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**X. APPENDIX**

**A. The \(n\)-set of inverted anisotropic three-dimensional harmonic oscillators**

We now generalize the results obtained in this article for the \(n\)-set of IHOs. The information metric on the \(3n\)-dimensional statistical manifold \(M_{IHO}^{(3n)}\) is given by

\[
g_{ij}(\theta_1,\ldots,\theta_{3n}) = [1 - \Phi(\theta_1,\ldots,\theta_{3n})] \cdot \delta_{ij}(\theta_1,\ldots,\theta_{3n}),
\]

where

\[
\Phi(\theta_1,\ldots,\theta_{3n}) = \sum_{j=1}^{3n} \Phi_j(\theta_j), \quad \Phi_j(\theta_j) = -\frac{1}{2} \omega_j^2 \theta_j^2.
\]

The information geometrodynamical entropy \(S_{M_{IHO}^{(3n)}}(\tau; \omega_1,\ldots,\omega_{3n})\) is defined as

\[
S_{M_{IHO}^{(3n)}}(\tau; \omega_1,\ldots,\omega_{3n}) \defeq \lim_{\tau \to \infty} \log \left[ \left\langle \Delta V_{M_{IHO}^{(3n)}}(\tau; \omega_1,\ldots,\omega_{3n}) \right\rangle \right],
\]

where the average volume element \(\Delta V_{M_{IHO}^{(3n)}}(\tau)\) is given by

\[
\left\langle \Delta V_{M_{IHO}^{(3n)}}(\tau; \omega_1,\ldots,\omega_{3n}) \right\rangle = \frac{1}{\tau} \int_0^\tau \Delta V_{M_{IHO}^{(3n)}}(\tau'; \omega_1,\ldots,\omega_{3n}) d\tau',
\]
and the statistical volume element $\Delta V_{M_{IHO}}^{(3n)}$ is defined as

$$
\Delta V_{M_{IHO}}^{(3n)} (\tau; \omega_1, \ldots, \omega_n) = \int_{\{\vec{\theta}'\}} d^{3n}\vec{\theta}' \left( 1 + \frac{1}{2} \sum_{j=1}^{3n} \omega_j^2 \theta_j^2 \right)^{\frac{3n}{2}}. \tag{78}
$$

Substituting (77) and (78) in (76) we obtain the general expression for $S_{M_{IHO}}^{(3n)} (\tau; \omega_1, \ldots, \omega_{3n})$,

$$
S_{M_{IHO}}^{(3n)} (\tau; \omega_1, \ldots, \omega_{3n}) \equiv \lim_{\tau \to -\infty} \log \left\{ \frac{1}{\tau} \int_0^\tau \left[ \int_{\{\vec{\theta}'\}} d^{3n}\vec{\theta}' \left( 1 + \frac{1}{2} \sum_{j=1}^{3n} \omega_j^2 \theta_j^2 \right)^{\frac{3n}{2}} \right] d\tau' \right\}. \tag{79}
$$

To evaluate (79) we observe $\Delta V_{M_{IHO}}^{(3n)}$ can be written as

$$
\Delta V_{M_{IHO}}^{(3n)} (\tau; \omega_1, \ldots, \omega_{3n}) = \int_{\{\vec{\theta}'\}} d^{3n}\vec{\theta}' \left( 1 + \frac{1}{2} \sum_{j=1}^{3n} \omega_j^2 \theta_j^2 \right)^{\frac{3n}{2}},
$$

$$
= \int d\theta'_1 \int d\theta'_2 \ldots \int d\theta'_{3n-1} \left[ \int \left( 1 + \frac{1}{2} \sum_{j=1}^{3n} \omega_j^2 \theta_j^2 \right)^{\frac{3n}{2}} d\theta'_{3n} \right],
$$

$$
\approx \frac{1}{3n} \frac{1}{2^{3n}} \left( \prod_{i=1}^{3n} \theta_i' \right)^{\frac{3n}{2}} \left[ \sum_{j=1}^{3n} \omega_j^2 \theta_j^2 \right]. \tag{80}
$$

Since the $n$-Newtonian equations of motions for each IHO are given by

$$
\frac{d^2 \theta_j}{d\tau^2} - \omega_j^2 \theta_j = 0, \forall j = 1, \ldots, 3n, \tag{81}
$$

the asymptotic behavior of such macrovariables on manifold $M_{IHO}^{(3n)}$ is given by

$$
\theta_j (\tau) \approx \Xi_j e^{\omega_j \tau}, \Xi_j \in \mathbb{R}, \forall j = 1, \ldots, 3n. \tag{82}
$$

We therefore obtain

$$
\Delta V_{M_{IHO}}^{(3n)} (\tau; \omega_1, \ldots, \omega_{3n}) \approx \frac{1}{3n} \frac{1}{2^{3n}} \left( \prod_{i=1}^{3n} \Xi_i \right) \cdot \exp \left( \sum_{i=1}^{3n} \omega_i \tau \right) \left[ \sum_{j=1}^{3n} \Xi_j^{2} e^{2\omega_j \tau} \Xi_j^2 \right]^{\frac{3n}{2}}. \tag{83}
$$

Upon averaging (83) we find

$$
\langle \Delta V_{M_{IHO}}^{(3n)} (\tau; \omega_1, \ldots, \omega_{3n}) \rangle_{\tau} \approx \frac{1}{\tau} \int_0^\tau \left\{ \frac{1}{3n} \frac{1}{2^{3n}} \left( \prod_{i=1}^{3n} \Xi_i \right) \cdot \exp (\Omega \tau') \left[ \sum_{j=1}^{3n} \Xi_j^{2} e^{2\omega_j \tau'} \Xi_j^2 \right] \right\} d\tau'. \tag{84}
$$

where $\Omega = \sum_{i=1}^{3n} \omega_i$. As a working hypothesis, we assume $\Xi_i = \Xi_j = \Xi \forall i, j = 1, \ldots, 3n$. Furthermore, assume that $n \to \infty$ so that the spectrum of frequencies becomes continuous and, as an additional working hypothesis, assume this spectrum is linearly distributed,

$$
\rho (\omega) = \omega \text{ with } \int_0^{\Omega_{\text{cut-off}}} \rho (\omega) d\omega = 1, \Omega_{\text{cut-off}} = \xi \Omega, \xi \in \mathbb{R}. \tag{85}
$$
Therefore, we obtain
\[
\Delta V_{\mathcal{A}_{IHO}}^{(n)}(\tau; \omega_1, ..., \omega_{3n}) \approx \frac{1}{3n} \frac{1}{2^{2n} \pi^{6n}} \left( \frac{\xi^2 \Omega^2}{2} \right)^{\frac{3}{4}} \exp \left( \frac{-\mu \xi \Omega \tau}{\tau} \right).
\]
(86)

Finally, substituting (86) into (76), we obtain the remarkable result
\[
S_{\mathcal{A}_{IHO}}^{(n)}(\tau; \omega_1, ..., \omega_{3n}) \approx \Omega \tau, \quad \Omega = \sum_{i=1}^{3n} \omega_i.
\]
(87)

Equation (87) displays the asymptotic, linear information geometrodynamical entropy growth of the generalized $n$-set of inverted harmonic oscillators and extends the result of Zurek-Paz to an arbitrary set of anisotropic inverted harmonic oscillators [4].

XI. REFERENCES

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