THE BETHE EQUATION AT \( q = 0 \), THE MÖBIUS INVERSION FORMULA, AND WEIGHT MULTIPlicITIES: III. THE \( X_N^{(r)} \) CASE

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Abstract. It is shown that the numbers of off-diagonal solutions to the \( U_q(X_N^{(r)}) \) Bethe equation at \( q = 0 \) coincide with the coefficients in the recently introduced canonical power series solution of the \( Q \)-system. Conjecturally the canonical solutions are characters of the KR (Kirillov-Reshetikhin) modules. This implies that the numbers of off-diagonal solutions agree with the weight multiplicities, which is interpreted as a formal completeness of the \( U_q(X_N^{(r)}) \) Bethe ansatz at \( q = 0 \).

1. Introduction

Enumerating the solutions to the Bethe equation began with the invention of the Bethe ansatz [Be], where Bethe himself obtained a counting formula for \( sl_2 \)-invariant Heisenberg chain. His calculation is based on the string hypothesis and has been generalized to higher spins [K1], \( sl_n \) [K2] and a general classical simple Lie algebra \( X_n \) [KR]. These works concern the rational Bethe equation [OW], or in other words, \( U_q(X_n^{(1)}) \) Bethe equation at \( q = 1 \).

On the other hand, a systematic count at \( q = 0 \) started rather recently [KN1, KN2]. The two approaches are contrastive in many respects. To explain them, recall the general setting where integrable Hamiltonians associated with \( U_q(X_n^{(1)}) \) act on a finite dimensional module called the quantum space. At \( q = 1 \), the Hamiltonians are invariant and the Bethe vectors are singular with respect to the classical subalgebra \( X_n \), while for \( q \neq 1 \), such aspects are no longer valid in general. Consequently, by completeness at \( q = 1 \) (resp. \( q = 0 \)) we mean that the number of solutions to the Bethe equation coincides with the multiplicity of irreducible \( X_n \) modules (resp. weight multiplicities) in the quantum space.

In this paper we study the Bethe equation associated with the quantum affine algebra \( U_q(X_N^{(r)}) \) [RW] at \( q = 0 \). By extending the analyses of the nontwisted case [KN1, KN2], an explicit formula \( R(\nu, N) \) is derived for the number of off-diagonal solutions of the string center equation. Moreover we relate the result to the \( Q \)-system for \( U_q(X_N^{(r)}) \) introduced in [KR, K3, HKOTT]. It is a (yet conjectural in general) family of character identities for the KR modules (Definition 2.4). Our main finding is that \( R(\nu, N) \) is identified with the coefficients in the canonical solution of the \( Q \)-system obtained in [KNT]. Under the Kirillov-Reshetikhin conjecture [KR] (cf. Conjecture 3.4), it leads to a character formula for tensor products of KR modules, which may be viewed as a formal completeness at \( q = 0 \).

The outline of the paper is as follows. In Section 2 we study the \( U_q(X_N^{(r)}) \) Bethe equation at \( q = 0 \). For a generic string solution, the string centers satisfy the key equation (2.18), which we call the string center equation (SCE). There is a one-to-one correspondence between the generic string solutions to the Bethe equation and the generic solutions
to the SCE (Theorem 2.13). We then enumerate the off-diagonal solutions of the SCE, and obtain the formula $R(\nu, N)$ in Theorem 2.13. In Section 3 we recall the Q-system for $U_q(X_N^{(r)})$. It corresponds to a special case (called KR-type) of a more general system considered in [KNT]. There, power series solutions are studied, and the notion of the definition one has $d(\epsilon \rightarrow n)$ and enumerate the nodes of the Dynkin diagram of $R$ by $(r > 1)$. The diagrams (and the enumeration of the nodes for $\mathfrak{g}$, $\mathfrak{h}$) are realized as a ratio of two power series [KNT], which matches the enumeration at $q = 1$ for the nontwisted cases.

In this paper we omit most of the proofs and calculations, which are parallel with those in [KN1, KN2, KNT].

2. Bethe equation at $q = 0$

2.1. Preliminary. Let $\mathfrak{g} = X_N$ be a finite-dimensional complex simple Lie algebra of rank $N$. We fix a Dynkin diagram automorphism $\sigma$ of $\mathfrak{g}$ of order $r = 1, 2, 3$. The affine Lie algebras of type $X_N^{(r)} = A_{n+1}^{(1)}(n \geq 1)$, $B_n^{(1)}(n \geq 3)$, $C_n^{(1)}(n \geq 2)$, $D_n^{(1)}(n \geq 4)$, $E_6^{(1)}(n = 6, 7, 8)$, $F_4^{(1)}$, $G_2^{(1)}$, $A_{2n}^{(2)}(n \geq 1)$, $A_{2n-1}^{(2)}(n \geq 2)$, $D_{n+1}^{(2)}(n \geq 2)$, $E_6^{(2)}$ and $D_4^{(3)}$ are realized as the canonical central extension of the loop algebras based on the pair $(\mathfrak{g}, \sigma)$. Let $\mathfrak{g}_0$ be the finite-dimensional $\sigma$-invariant subalgebra of $\mathfrak{g}$; namely,

| $\mathfrak{g}$ | $X_n$ | $A_{2n}$ | $A_{2n-1}$ | $E_6$ | $D_4$ |
| $r$ | 1 | 2 | 2 | 2 | 3 |
| $\mathfrak{g}_0$ | $X_n$ | $B_n$ | $C_n$ | $B_n$ | $F_4$ | $G_2$ |

Let $A' = (A'_{ij})$ ($i, j \in I$) and $A = (A_{ij})$ ($i, j \in I_\sigma$) be the Cartan matrices of $\mathfrak{g}$ and $\mathfrak{g}_0$, respectively, where $I_\sigma$ is the set of $\sigma$-orbits of $I$. We define the numbers $d'_i$, $d_i$, $\epsilon'_i$, $\epsilon_i$ ($i \in I$) as follows: $d'_i$ ($i \in I$) are coprime positive integers such that $(d'_i A'_{ij})$ is symmetric; $d_i$ ($i \in I_\sigma$) are coprime positive integers such that $(d_i A_{ij})$ is symmetric, and we set $d_i = d_{\pi(i)}$ ($i \in I$), where $\pi : I \to I_\sigma$ is the canonical projection. $\epsilon'_i = r$ if $\sigma(i) = i$, and 1 otherwise; $\epsilon_i = 2$ if $A'_{\sigma(i)} \neq 0$, and 1 otherwise. Let $\kappa_0 = 2$ if $X_N^{(r)} = A_{2n}^{(2)}$, and 1 otherwise. By the definition one has $d'_i = d_i$ and $\epsilon'_i = 1$ if $r = 1$; $d'_i = 1$ if $r > 1$; $\epsilon_i = 1$ if $X_N^{(r)} \neq A_{2n}^{(2)}$.

In this paper we let $\{1, 2, \ldots, N\}$ and $\{1, 2, \ldots, n\}$ label the sets $I$ and $I_\sigma$, respectively, and enumerate the nodes of the Dynkin diagram of $X_N^{(r)}$ by $I_\sigma \cup \{0\}$ as specified in Table 1. The diagrams (and the enumeration of the nodes for $r > 1$) coincide with TABLE Aff1-3 in [Kac], except the $A_{2n}^{(2)}$ case. We fix an injection $\iota : I_\sigma \to I$ such that $\pi \circ \iota = \text{id}_{I_\sigma}$ and $A_{ab} < 0 \iff A'_{\iota(a) \iota(b)} < 0$ for any $a, b \in I_\sigma$. To be specific, assume that the labeling of the nodes for the Dynkin diagram of $\mathfrak{g}$ are given by dropping the 0-th ones from $X_N^{(1)}$ case in Table 1. Then we simply set $\iota(a) = a$ and regard $\iota$ as the embedding of the subset $\{1, \ldots, n\} \to \{1, \ldots, N\}$. The symbols $d'_a$, $\epsilon'_a$ and $A'_{ab}$ for $a, b \in I_\sigma = \{1, \ldots, n\}$ should
Table 1. Dynkin diagrams for $X_{r}^{(r)}$. The enumeration of the nodes with $I_{\sigma} \cup \{0\} = \{0, 1, \ldots, n\}$ is specified under or the right side of the nodes. In addition, the numbers $d_{a}$ ($a \in I_{\sigma}$) are attached above the nodes if and only if $d_{a} \neq 1$.

$A_{1}^{(1)}$: 0 1

$A_{n}^{(1)}$: (n ≥ 2)

$B_{n}^{(1)}$: (n ≥ 3)

$C_{n}^{(1)}$: (n ≥ 2)

$D_{n}^{(1)}$: (n ≥ 4)

$E_{6}^{(1)}$: 1 2 3 5 6

$E_{7}^{(1)}$: 0 1 2 3 4 5 6

$E_{8}^{(1)}$: 0 1 2 3 4 5 6 7

$F_{4}^{(1)}$: 0 1 2 3 4

$G_{2}^{(1)}$: 0 1 2

$A_{2}^{(2)}$: 0

$A_{2n-1}^{(2)}$: (n ≥ 3)

$A_{2n}^{(2)}$: (n ≥ 2)

$D_{n+1}^{(2)}$: (n ≥ 2)

$E_{6}^{(2)}$: 0 1 2 3 4

$D_{4}^{(3)}$: 0 1 2

be interpreted accordingly. One can check

$$\kappa_{0} e'_{a} d'_{a} = \epsilon_{a} d_{a},$$

$$\sum_{s=1}^{r} A'_{a_{\sigma_{s}}(b)} = \frac{e'_{a}}{\epsilon_{a}} A_{ab}.$$  

We use the notation:

$$H = \{(a, m) \mid a \in I_{\sigma}, m \in \mathbb{Z}_{\geq 1}\}.$$

Let $U_{q}(X_{r}^{(r)})$ be the quantum affine algebra. The irreducible finite-dimensional $U_{q}(X_{r}^{(r)})$-modules are parameterized by $N$-tuples of polynomials $(P_{i}(u))_{i \in I}$ (Drinfeld polynomials) with unit constant terms $[CP1, CP2]$. They satisfy the relation $P_{\sigma(i)}(u) = P_{i}(\omega^{e'_{i}}u)$,
where \( \omega = \exp(2\pi \sqrt{-1}/r) \). Thus it is enough to specify \((P_b(u))_{b \in I_\sigma}\). Following [KNT] we introduce

**Definition 2.1.** For each \((a, m) \in H\) and \( \zeta \in \mathbb{C}^x \), let \( W_m^{(a)}(\zeta) \) be the finite-dimensional irreducible \( U_q(X_{N}^{(r)}) \)-module whose Drinfeld polynomials \( P_b(u) \) \((b = 1, \ldots, n)\) are specified as follows: \( P_a(u) = 1 \) for \( b \neq a \), and

\[
P_a(u) = \prod_{k=1}^{m}(1 - \zeta q^{\epsilon a d_a(m+2-2k)} u).
\]

We call \( W_m^{(a)}(\zeta) \) a KR (Kirillov-Reshetikhin) module.

2.2. The \( U_q(X_{N}^{(r)}) \) Bethe equation. Let

\[
\mathcal{N} = \{ N = (N_m^{(a)})(a, m) \in H \mid N_m^{(a)} \in \mathbb{Z}_{\geq 0}, \sum_{(a, m) \in H} N_m^{(a)} < \infty \}.
\]

Given \( \nu = (\nu_m^{(a)}) \in \mathcal{N} \), we define a tensor product module:

\[
W^\nu = \bigotimes_{(a, m) \in H} (W_m^{(a)}(\zeta_m^{(a)}))^{\otimes \nu_m^{(a)}},
\]

where \( \zeta_m^{(a)} \in \mathbb{C}^x \). In the context of solvable lattice models [B], one can regard \( W^\nu \) as the quantum space on which the commuting family of transfer matrices act. Reshetikhin and Wiegmann [RW] wrote down the \( U_q(X_{N}^{(r)}) \) Bethe equation and conjectured its relevance to the spectrum of those transfer matrices. In our formulation, it is the simultaneous equation on the complex variables \( x_i^{(a)} \) \((i \in \{1, 2, \ldots, M_a\}, a \in I_\sigma)\) having the form:

\[
\prod_{a=m}^{r} \prod_{k=1}^{\infty} \left( \frac{\omega^s(x_i^{(a)})^{-\frac{1}{2}} q^{m a d_a(x_i^{(a)} - \epsilon a a^{(a)})} - 1}{\omega^s(x_i^{(a)})^{-\frac{1}{2}} q^{m a d_a(x_i^{(a)} - \epsilon a a^{(a)})} - \omega^{s a^{(a)}} x_j^{(a)} x_j^{(a)} q^{a a^{(a)}} A_{a a^{(a)}} - (x_j^{(a)})^{\frac{1}{2}} q^{a a^{(a)}} A_{a a^{(a)}}} \right) = \prod_{a=m}^{r} \prod_{k=1}^{\infty} \left( \frac{\omega^s(x_i^{(a)})^{-\frac{1}{2}} q^{m a d_a(x_i^{(a)} - \epsilon a a^{(a)})} - 1}{\omega^s(x_i^{(a)})^{-\frac{1}{2}} q^{m a d_a(x_i^{(a)} - \epsilon a a^{(a)})} - \omega^{s a^{(a)}} x_j^{(a)} x_j^{(a)} q^{a a^{(a)}} A_{a a^{(a)}}} \right).
\]

For the nontwisted case \( r = 1 \), this reduces to eq.(2.3) in [KNT]. The both sides are actually rational functions of \( (x_i^{(a)}) \). In the sequel we consider a polynomial version of (2.3) specified as follows:

\[
F_{i+}^{(a)} G_{i-}^{(a)} = F_{i-}^{(a)} G_{i+}^{(a)},
\]

\[
F_{i+}^{(a)} = \prod_{k=1}^{\infty} (x_i^{(a)} q^{k a a^{(a)} d_a} - 1)\nu_k^{(a)},
\]

\[
F_{i-}^{(a)} = \prod_{k=1}^{\infty} (x_i^{(a)} - q^{k a a^{(a)} d_a})\nu_k^{(a)},
\]

\[
G_{i+}^{(a)} = \prod_{b=1}^{\tilde{n}} M_b \prod_{j=1}^{M_b} ((x_i^{(a)} q^{j b a^{(a)} d_a} - (x_j^{(a)})^{\frac{1}{2}} q^{j b a^{(a)} d_a} A_{a b})),
\]

\[
G_{i-}^{(a)} = \prod_{b=1}^{\tilde{n}} M_b \prod_{j=1}^{M_b} ((x_i^{(a)} - (x_j^{(a)}))^{\frac{1}{2}} q^{j b a^{(a)} d_a} A_{a b})),
\]
where \( \epsilon'_{ab} = \max(\epsilon'_a, \epsilon'_b) \), and \( \tilde{n} = n \) except for \( \tilde{n} = n + 1 \) for \( A^{(2)}_{2n} \). When \( X^{(r)}_N = A^{(2)}_{2n} \), we have set \( x^{(n+1)}_j = -x^{(n)}_j \) and \( M_{n+1} = M_n \).

**Remark 2.2.** Let \( P_m^{(a)}(u) \) denote the \( a \)-th Drinfeld polynomial of the KR module \( W_m^{(a)}(1) \). Then we have

\[
\frac{F^{(a)}_{i-}}{F^{(a)}_{i+}} = \prod_{(a,m) \in H} \left( q^{\epsilon_a d_a} \frac{P_m^{(a)}(q^{-2\epsilon_a d_a} x_i^{(a)})}{P_m^{(a)}(x_i^{(a)})} \right)^{x_m^{(a)}}.
\]

In view of this, we expect without proof that the solutions of \( (2.3) \) determine the spectrum of transfer matrices acting on \( (2.2) \) with the choice \( z_m^{(a)} = 1 \).

We consider a class of solutions \( (x_i^{(a)}) \) of \( (2.4) \) such that \( x_1^{(a)} = x_i^{(a)}(q) \) is meromorphic function of \( q \) around \( q = 0 \). For a meromorphic function \( f(q) \) around \( q = 0 \), let \( \text{ord}(f) \) be the order of the leading power of the Laurent expansion of \( f(q) \) around \( q = 0 \), i.e.,

\[
f(q) = q^{\text{ord}(f)}(f^0 + f^1 q + \cdots), \quad f^0 \neq 0,
\]

and let \( \tilde{f}(q) := f^0 + f^1 q + \cdots \) be the normalized series. When \( f(q) \) is identically zero, we set \( \text{ord}(f) = \infty \). For each \( N = (N_m^{(a)}) \in \mathcal{N} \), we set

\[
H' = H'(N) := \{ (a, m) \in H \mid N_m^{(a)} > 0 \}
\]

where \( H \) is defined in \( (2.1) \). We have \( |H'| < \infty \).

**Definition 2.3.** Let \( (M_a)_{a=1}^n \) be the one in the Bethe equation \( (2.4) \), and let \( N = (N_m^{(a)}) \in \mathcal{N} \) satisfy \( \sum_{m=1}^{\infty} m N_m^{(a)} = M_a \). A meromorphic solution \( (x_i^{(a)}) \) of \( (2.4) \) around \( q = 0 \) is called a **string solution of pattern** \( N \) if

1. \( \text{ord}(F_i^{(a)} G_{i-}^{(a)}) < \infty \) for any \( (a, i) \).
2. \( (x_i^{(a)}) \) can be arranged as \( (x_{\max}^{(a)}) \) with

\[
(a, m) \in H', \quad a = 1, \ldots, N_m^{(a)}, \quad i = 1, \ldots, m
\]

such that

\[
(a) \quad d_{\max}^{(a)} := \text{ord}(x_{\max}^{(a)}) = (m + 1 - 2i) \kappa_0 \epsilon'_a d'_a,
\]

\[
(b) \quad z_m^{(a)} := x_{\max}^{(a)0} = x_{\max}^{(a)0} = \cdots = x_{\max}^{(a)m0} (\neq 0), \text{ where } x_{\max}^{(a)0} \text{ is the coefficient of the leading power of } x_{\max}^{(a)}.
\]

For each \( (a, m, \alpha) \), \( (x_{\max}^{(a)})_{i=1}^m \) is called an **\( m \)-string of color** \( a \), and \( z_{\max}^{(a)} \) is called the **string center** of the **\( m \)-string** \( (x_{\max}^{(a)})_{i=1}^m \). Thus, \( N_m^{(a)} \) is the number of the \( m \)-strings of color \( a \).

For a string solution \( x_{\max}^{(a)}(q) = q^{d_{\max}^{(a)} x_{\max}^{(a)}(q)} \) of pattern \( N \), the Bethe equation \( (2.4) \) reads

\[
F_{\max}^{(a)} G_{\max}^{(a)} = F_{\max}^{(a)} G_{\max}^{(a)},
\]

where \( \epsilon'_{ab} = \max(\epsilon'_a, \epsilon'_b) \), and \( \tilde{n} = n \) except for \( \tilde{n} = n + 1 \) for \( A^{(2)}_{2n} \). When \( X^{(r)}_N = A^{(2)}_{2n} \), we have set \( x^{(n+1)}_j = -x^{(n)}_j \) and \( M_{n+1} = M_n \).
Definition 2.4. A string solution \((x_{mai})^{(a)}\) to (2.6) is called generic if

\[
\text{ord}(F_{mai}^{(a)}) = \xi_{mai}^{(a)},
\]

\[
\text{ord}(G_{mai}^{(a)}) = \eta_{mai}^{(a)} + \xi_{mai}^{(a)} = \eta_{mai-}^{(a)} + \xi_{mai+}^{(a)},
\]

where \(\xi_{mai}^{(a)} := \text{ord}(\bar{x}_{mai}^{(a)} - \bar{x}_{mai-1}^{(a)})\) for \(2 \leq i \leq m\), and \(\xi_{ma1}^{(a)} = \xi_{ma,m+1}^{(a)} = 0\).
Lemma 2.5. We have
\[
\hat{\gamma}_m^{(a)}(\nu) = \sum_{k=1}^{\infty} \min(m, k)\nu_k^{(a)}.
\]

Proposition 2.6. For a generic string solution, one can determine the order \( \zeta_{\text{mai}} \) from (2.6), (2.13) and Lemma 2.3. Requiring that the resulting \( \zeta_{\text{mai}}^{(a)} \) should be positive and finite (cf. Definition 2.3), one has

\[
\Delta_j^{(a)} = \begin{cases} 
-\Delta_{j-1}^{(a)} & \text{if } d_a = 2 \\
-\Delta_{j-1}^{(a)} + N_{3j-1}^{(a')} + N_{3j+1}^{(a')} & \text{if } d_a = 3.
\end{cases}
\]

For a generic string solution, (2.11) becomes an equation for the string centers \( (z_{\text{mai}}^{(a)}) \). We call it the string center equation (SCE).

Proposition 2.7. Let \( (x_{\text{mai}}^{(a)}) \) be a generic string solution of pattern \( N \). Then its string centers \( (z_{\text{mai}}^{(a)}) \) satisfy the following equations ((\( a, m \)) \( \in H', 1 \leq \alpha \leq N_m^{(a)} \)):

\[
\prod_{(b,k) \in H'} \prod_{\beta=1}^{N_{k}^{(b)}} (z_{k,\beta}^{(b)})^{A_{ama, bk\beta}} = (-1)^{\hat{\rho}_m^{(a)} + N_m^{(a)} + 1},
\]

\[
A_{ama, bk\beta} = \delta_{ab} \delta_{mk} \delta_{\alpha\beta} (P_m^{(a)} + N_m^{(a)}) + \frac{A_{ba}}{\epsilon_b d_a^{(b)}} \min(d_a^{(a)}, d_b^{(b)}) - \delta_{ab} \delta_{mk}.
\]
Note that all the quantities in (2.13), (2.16) and (2.19) are integers. As in [KN2], Proposition 2.7 is derived by explicitly evaluating the ratio (2.11) by

**Lemma 2.8.** For \( a \in \{1, 2, \ldots, n\} \) and \( b \in \{1, 2, \ldots, \bar{n}\} \), we have

\[
\prod_{i=1}^{m} F_{ma}^{(a)0} = \begin{cases} \frac{f_{ama}}{f_{ama}} & \epsilon = + \\ \frac{f_{ama}}{f_{ama}} & \epsilon = - \end{cases},
\]

\[
\prod_{i=1}^{k} \prod_{i=1}^{m} \left( (z_{ma}^{(a)} - \bar{z}_{ab}^{(a)} q^{-1}_{aa} d_{ma}^{(a)} + \frac{1}{2} (1 + \epsilon) \kappa_{ab} d_{ab}^{(a)} A'_{ab} - (z_{k\beta}^{(b)} - \bar{z}_{ab}^{(b)} q^{-1}_{bb} d_{k\beta}^{(b)} + \frac{1}{2} (1 - \epsilon) \kappa_{ab} d_{ab}^{(b)} A'_{ab}) \right) \right)^{\epsilon = 1} \prod_{i=1}^{m} \prod_{i=1}^{k} \left( (z_{ma}^{(a)} - \bar{z}_{ab}^{(a)} q^{-1}_{aa} d_{ma}^{(a)} + \frac{1}{2} (1 + \epsilon) \kappa_{ab} d_{ab}^{(a)} A'_{ab} - (z_{k\beta}^{(b)} - \bar{z}_{ab}^{(b)} q^{-1}_{bb} d_{k\beta}^{(b)} + \frac{1}{2} (1 - \epsilon) \kappa_{ab} d_{ab}^{(b)} A'_{ab}) \right) \right)^{\epsilon = -1},
\]

for some \( f_{ama} \) and \( g_{ama}^{(b)b} \), where we have set \( \bar{z}_{ma}^{(n+1)} := -z_{ma}^{(n)} \).

The quantities \( f_{ama} \) and \( g_{ama}^{(b)b} \) depend on the string centers \((z_{ma}^{(a)}), \) whose explicit formulae are available in [KN2] for nontwisted case. However we do not need them here. A string solution is generic if and only if \( f_{ama} \neq 0 \) and \( g_{ama}^{(b)b} \neq 0 \) for any \( a \in \{1, \ldots, n\}, b \in \{1, \ldots, \bar{n}\}, m, k \in \mathbb{Z}_{\geq 1}, 1 \leq \alpha \leq N_{m}^{(a)}, 1 \leq \beta \leq N_{k}^{(b)}. \) These conditions are equivalent to

\[(2.20)\]

\[z_{ma}^{(a)} \neq 1 \text{ if there is } k \geq 1 \text{ such that } \nu_{k}^{(a)} > 0 \text{ and } k \in \langle m \rangle,
\]

\[\bar{z}_{ma}^{(a)} - \bar{z}_{ab}^{(b)} q^{-1}_{aa} d_{ma}^{(a)} + \frac{1}{2} (1 + \epsilon) \kappa_{ab} d_{ab}^{(a)} A'_{ab} \text{ and } d_{ab}^{(b)} A'_{ab} \text{ are } \langle m \rangle \text{, except for the exceptional case } (a, m, \alpha) = (b, k, \beta), \]

where \( \langle m \rangle = \{m - 1, m - 3, \ldots, -m + 1\}. \) Apart from the exceptional case \( (a, m, \alpha) = (b, k, \beta), \) the condition (2.20) says that the two terms in each factor in (2.7) - (2.10) possess different leading terms whenever their orders coincide.

**Definition 2.9.** A solution to the SCE (2.18) is called generic if it satisfies (2.20).

Let \( A \) be the matrix with the entry \( A_{ama,bk\beta} \) in (2.19). The main theorem in this subsection is

**Theorem 2.10.** Suppose that \( N \in \mathcal{N} \) satisfies the conditions (2.13) and \( \det A \neq 0. \) Then, there is a one-to-one correspondence between generic string solutions of pattern \( N \) to the Bethe equation (2.6) and generic solutions to the SCE (2.18) of pattern \( N. \)

**Remark 2.11.** Given the Bethe equation (2.3), the choice of \( F_{i+}^{(a)} \) and \( G_{i-}^{(a)} \) in (2.4) is not the unique one. For example one may restrict the product in \( G_{i+}^{(a)} \) to those satisfying \( A'_{ab} \neq 0. \) Such an ambiguity influences Definition 2.3 (i), (2.7) - (2.10), (2.12), (2.20), hence Definition 2.3. However, the ratio in (2.11) is left unchanged, and all the statements in Lemma 2.7, Propositions 2.6, 2.7 and Theorem 2.10 remain valid.

2.3. Counting of off-diagonal solutions to SCE. For \( k \in \mathbb{C} \) and \( j \in \mathbb{Z}, \) we define the binomial coefficient by

\[
\binom{k}{j} = \frac{\Gamma(k+1)}{\Gamma(k-j+1)\Gamma(j+1)},
\]
For each \( \nu, N \in \mathcal{N} \), we define the number \( R(\nu, N) \) by

\[
R(\nu, N) = \det_{(a,m),(b,k) \in H'} F_{am,bk} \prod_{(a,m) \in H'} \frac{1}{N_m^{(a)}} \left( \frac{P_m^{(a)} + N_m^{(a)} - 1}{N_m^{(a)} - 1} \right),
\]

\[
F_{am,bk} = \sum_{\beta=1}^{N_k^{(b)}} A_{ama,bk} = \delta_{ab} \delta_{mk} P_m^{(a)} + \frac{A_{ba}}{\epsilon_d d_a} \min(d'_a m, d'_b k) N_k^{(b)},
\]

for \( N \neq 0 \). Here \( H' = H'(N) \) and \( P_m^{(a)} = P_m^{(a)}(\nu, N) \) are given by (2.18) and (2.19). For \( N = 0 \), we set \( R(\nu, 0) = 1 \) irrespective of \( \nu \). It is easy to see that \( R(\nu, N) \) is an integer.

**Definition 2.12.** A solution \( (z_m^{(a)}) \) to the SCE is called off-diagonal (diagonal) if \( z_m^{(a)} = z_m^{(a)} \) only for \( \alpha = \beta \) (otherwise).

Our main result in this subsection is

**Theorem 2.13.** Suppose \( P_m^{(a)}(\nu, N) \geq 0 \) for any \( (a,m) \in H' \). Then the number of off-diagonal solutions to the SCE (2.18) of pattern \( N \) divided by \( \prod_{(a,m) \in H'} N_m^{(a)} \) is equal to \( R(\nu, N) \).

The proof is due to the inclusion-exclusion principle and an explicit evaluation of the Möbius inversion formula similar to [KN1, KN2].

### 3. \( R(\nu, N) \) and \( Q \)-system

So much for the Bethe equation, we now turn to the \( Q \)-system. For \( a, b \in I_\sigma \) and \( m, k \in \mathbb{Z} \), set

\[
G_{am,bk} = \begin{cases}
-\frac{1}{t_\nu} A_{ba} \delta_{m,k} & r > 1 \\
-A_{ba}(\delta_{m,2k-1} + 2\delta_{m,2k} + \delta_{m,2k+1}) & d_b / d_a = 2 \\
-A_{ba}(\delta_{m,3k-2} + 2\delta_{m,3k-1} + 3\delta_{m,3k} + 2\delta_{m,3k+1} + \delta_{m,3k+2}) & d_b / d_a = 3 \\
-A_{ab} \delta_{d_a m, d_b k} & \text{otherwise}.
\end{cases}
\]

Let \( \alpha_a \) and \( \Lambda_a \) \((a \in I_\sigma)\) be the simple roots and the fundamental weights of \( \mathfrak{g}_0 \). We set

\[
x_a = e^{\epsilon_a \Lambda_a}, \quad y_a = e^{-\alpha_a},
\]

which are related as

\[
y_a = \prod_{b=1}^n x_b^{-A_{ba}/\epsilon_b}.
\]

**Definition 3.1.** The system of equations \( (Q_0^{(a)}(y) = 1) \)

\[
(Q_m^{(a)}(y))^2 = Q_{m+1}^{(a)}(y) Q_{m-1}^{(a)}(y) + y_a^m (Q_m^{(a)}(y))^2 \prod_{(b,k) \in H} (Q_m^{(b)}(y))^{G_{am,bk}}
\]

for a family \( (Q_m^{(a)}(y))_{(a,m) \in H} \) of power series of \( y = (y_a)_{a=1}^n \) with unit constant terms is called the \( Q \)-system.
The factor $y^n$ in the RHS is absorbed away if (3.2) is written in terms of the combination $x^n Q_m^\nu(y)$. The resulting form of the $Q$-system has originally appeared in [KR] ($A^{(1)}_n, B^{(1)}_n, C^{(1)}_n, D^{(1)}_n$), [KS] ($E^{(1)}_{6,7,8}, F^{(4)}_4, G^{(1)}_2$) and [HKOTT] (twisted case). Definition 3.1 corresponds to an infinite $Q$-system in the terminology of [KNT]. Its solution is not unique in general. Following [KNT] we introduce

**Definition 3.2.** A solution of (3.2) is **canonical** if the limit $\lim_{m \to \infty} Q_m^\nu(y)$ exists in the ring $\mathbb{C}[[y]]$ of formal power series of $y = (y_a)_{a=1}^n$ with the standard topology.

**Theorem 3.3.** ([KNT]) There exists a unique canonical solution $(Q_m^\nu(y))_{(a,m) \in H}$ of the $Q$-system (3.3). Moreover, for any $\nu \in \mathbb{N}$, it admits the formula:

$$\prod_{(a,m) \in H} (Q_m^\nu(y))^{\nu_m(a)} = R_\nu(y),$$

where the power series $R_\nu(y)$ is defined by

$$R_\nu(y) = \sum_{\nu \in \mathbb{N}} R(\nu, N) \prod_{a=1}^n y_a^{\sum_{m=0}^\infty m N_m^a}$$

in terms of the integer $R(\nu, N)$ in (2.21).

In the proof of the theorem [KNT], the expression $R(\nu, N)$ emerges from a general argument on the $Q$-system, which is independent of the Bethe equation. Our main finding in this paper is that it coincides with the number of off-diagonal solutions to the SCE obtained in Theorem 2.13.

Let us state the consequence of this fact in the light of the Kirillov-Reshetikhin conjecture. Let $\text{ch}_m(x)$ denote the Laurent polynomial of $x = (x_a)_{a=1}^n$ representing the $g_0$-character of the KR module $W_m^\nu(\zeta)$. Then, $Q_m^\nu(y)$ is the inverse map of (3.3), is a polynomial of $y = (y_a)_{a=1}^n$ with the unit constant term. We call $Q_m^\nu(y)$ the **normalized** $g_0$-character of $W_m^\nu(\zeta)$. The normalized character of the $g_0$-module $W^\nu$ in (2.2) is given by

$$Q^\nu(y) = \prod_{(a,m) \in H} (Q_m^\nu(y))^{\nu_m(a)}.$$  

The Kirillov-Reshetikhin conjecture [KR] is formulated in [KNT] as

**Conjecture 3.4.** $Q_m^\nu(y) = Q_m^\nu(y)$ for any $(a, m) \in H$.

Combining Theorem 3.3 and Conjecture 3.4, we relate the weight multiplicity in the tensor product of KR modules to the number of off-diagonal solutions to the SCE:

**Corollary 3.5** (Formal completeness of the Bethe ansatz at $q = 0$). Under Conjecture 3.4 one has

$$Q\nu(y) = R_\nu(y).$$

Conjecture 3.4 implies that $(\prod_{(a,m) \in H} x_{a\alpha}^{\nu_{\alpha}(m)}) R_\nu(y(x))$ is a Laurent polynomial invariant under the Weyl group of $g_0$. In fact canonical solutions have also been obtained as linear combinations of characters of irreducible finite dimensional $g_0$-modules for $X_N^{(r)} = A^{(1)}_n, B^{(1)}_n, C^{(1)}_n, D^{(1)}_n$ [KR, HKOTY] and for $X_N^{(r)} = A^{(2)}_{2n-1}, A^{(2)}_{2n}, D^{(2)}_{n+1}, D^{(3)}_4$ [HKOTT]. For the current status of Conjecture 3.4, see section 5.7 of [KNT].
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