HOPF ALGEBRAS AND POLYNOMIAL IDENTITIES

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Abstract. This is a survey of results obtained jointly with E. Aljadeff and published in [2]. We explain how to set up a theory of polynomial identities for comodule algebras over a Hopf algebra, and concentrate on the universal comodule algebra constructed from the identities satisfied by a given comodule algebra. All concepts are illustrated with various examples.

Key Words: Polynomial identity, Hopf algebra, comodule, localization

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Introduction

As has been stressed many times (see, e.g., [19]), Hopf Galois extensions can be viewed as non-commutative analogues of principal fiber bundles (also known as G-torsors), where the role of the structural group is played by a Hopf algebra. Such extensions abound in the world of quantum groups and of non-commutative geometry. The problem of constructing systematically all Hopf Galois extensions of a given algebra for a given Hopf algebra and of classifying them up to isomorphism has been addressed in a number of papers, such as [4, 7, 9, 12, 13, 14, 15, 18] to quote but a few.

A new approach to the classification problem of Hopf Galois extensions was recently advanced by Eli Aljadeff and the present author in [2]; this approach uses classical techniques from non-commutative algebra such as polynomial identities (such techniques had previously been used in [1] for group-graded algebras). In [2] we developed a theory of identities for any comodule algebra over a given Hopf algebra $H$, hence for any Hopf Galois extension. As a result, out of the identities for an $H$-comodule algebra $A$, we obtained a universal $H$-comodule algebra $U_H(A)$. It turns out that if $A$ is a cleft $H$-Galois object (i.e., a comodule algebra obtained from $H$ by twisting its product with the help of a two-cocycle) with trivial center, then a suitable central localization of $U_H(A)$ is an $H$-Galois extension of its center. We thus obtain a “non-commutative principal fiber bundle” whose base space is the spectrum of some localization of the center of $U_H(A)$.

This survey is organized as follows. After a preliminary section on comodule algebras, we define the concept of an $H$-identity for such algebras in §2. We illustrate this concept with a few examples and we attach a universal $H$-comodule algebra $U_H(A)$ to each $H$-comodule algebra $A$.

In §3, turning to the special case where $A = \alpha H$ is a twisted comodule algebra, we exhibit a universal comodule algebra map that allows us to detect the $H$-identities for $A$. 

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In §4 we construct a commutative domain \( B_H^{\alpha} \) and we state that under some natural extra condition, \( B_H^{\alpha} \) is the center of a suitable central localization of \( \mathcal{U}_H(A) \); moreover after localization, \( \mathcal{U}_H(A) \) becomes a free module over its center.

Lastly in §5 we illustrate all previous constructions with the help of the four-dimensional Sweedler algebra, thus giving complete answers in this simple, but non-trivial example. We end the paper with an open question on Taft algebras.

The material of the present text is mainly taken from [2], for which it provides an easy access. The reader is advised to complement it with [10, 11].

1. Hopf algebras and coactions

1.1. Standing assumption. We fix a field \( k \) over which all our constructions are defined. In particular, all linear maps are supposed to be \( k \)-linear and unadorned tensor products mean tensor products over \( k \). Throughout the survey we assume that the ground field \( k \) is infinite.

By algebra we always mean an associative unital \( k \)-algebra. We suppose the reader familiar with the language of Hopf algebra, as expounded for instance in [20]. As is customary, we denote the coproduct of a Hopf algebra by \( \Delta \), its counit by \( \varepsilon \), and its antipode by \( S \). We also make use of a Heyneman-Sweedler-type notation for the image

\[ \Delta(x) = x_1 \otimes x_2 \]

of an element \( x \) of a Hopf algebra \( H \) under the coproduct, and we write

\[ \Delta^{(2)}(x) = x_1 \otimes x_2 \otimes x_3 \]

for the iterated coproduct \( \Delta^{(2)} = (\Delta \otimes \text{id}_H) \circ \Delta = (\text{id}_H \otimes \Delta) \circ \Delta \), and so on.

1.2. Comodule algebras. Let \( H \) be a Hopf algebra. Recall that an \( H \)-comodule algebra is an algebra \( A \) equipped with a right \( H \)-comodule structure whose (coassociative, counital) coaction

\[ \delta : A \to A \otimes H \]

is an algebra map. The subalgebra \( A^H \) of coinvariants of an \( H \)-comodule algebra \( A \) is defined by

\[ A^H = \{ a \in A \mid \delta(a) = a \otimes 1 \} . \]

Given two \( H \)-comodule algebras \( A \) and \( A' \) with respective coactions \( \delta \) and \( \delta' \), an algebra map \( f : A \to A' \) is an \( H \)-comodule algebra map if

\[ \delta' \circ f = (f \otimes \text{id}_H) \circ \delta . \]

We denote by \( \text{Alg}^H \) the category whose objects are \( H \)-comodule algebras and arrows are \( H \)-comodule algebra maps.

Let us give a few examples of comodule algebras.

Example 1.1. If \( H = k \), then an \( H \)-comodule algebra is nothing but an ordinary (associative, unital) algebra.
Example 1.2. The algebra \( H = k[G] \) of a group \( G \) is a Hopf algebra with coproduct, counit, and antipode given for all \( g \in G \) by
\[
\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad S(g) = g^{-1}.
\]
It is well-known (see [5, Lemma 4.8]) that an \( H \)-comodule algebra \( A \) is the same as a \( G \)-graded algebra
\[
A = \bigoplus_{g \in G} A_g, \quad A_g A_h \subset A_{gh}.
\]

The coaction \( \delta : A \to A \otimes H \) is given by \( \delta(a) = a \otimes g \) for all \( a \in A_g \) and \( g \in G \).
We have \( A^H = A_e \), where \( e \) is the neutral element of \( G \).

Example 1.3. Let \( G \) be a finite group and \( H = k^G \) be the algebra of \( k \)-valued functions on a finite group \( G \). This algebra can be equipped with a Hopf algebra structure that is dual to the Hopf algebra \( k[G] \) above. An \( H \)-comodule algebra \( A \) is the same as a \( G \)-algebra, i.e., an algebra equipped with a left action of \( G \) on \( A \) by group automorphisms.

If we denote the action of \( g \in G \) on \( a \in A \) by \( a \cdot g \), then the coaction \( \delta : A \to A \otimes H \) is given by
\[
\delta(a) = \sum_{g \in G} a \cdot g \otimes e_g,
\]
where \( \{e_g\}_{g \in G} \) is the basis of \( H \) consisting of the functions \( e_g \) defined by \( e_g(h) = 1 \) if \( h = g \), and 0 otherwise.

The subalgebra of coinvariants of \( A \) coincides with the subalgebra of \( G \)-invariant elements: \( A^H = A^G \).

Example 1.4. Any Hopf algebra \( H \) is an \( H \)-comodule algebra whose coaction coincides with the coproduct of \( H \):
\[
\delta = \Delta : H \to H \otimes H.
\]

In this case the coinvariants of \( H \) are exactly the scalar multiples of the unit of \( H \); in other words, \( H^H = k^1 \).

2. Identities

2.1. Polynomial identities. Let \( A \) be an algebra. A polynomial identity for an algebra \( A \) is a polynomial \( P(X, Y, Z, \ldots) \) in a finite number of non-commutative variables \( X, Y, Z, \ldots \) such that
\[
P(x, y, z, \ldots) = 0
\]
for all \( x, y, z, \ldots \in A \).

Examples 2.1. (a) The polynomial \( XY - YX \) is a polynomial identity for any commutative algebra.

(b) If \( A = M_2(k) \) is the algebra of \( 2 \times 2 \)-matrices with entries in \( k \), then
\[
(\text{XYZ} - X \cdot \text{Z} - \text{Z} \cdot \text{XYZ})^2
\]
is a polynomial identity for \( A \). (Use the Cayley-Hamilton theorem to check this.)
The concept of a polynomial identity first emerged in the 1920’s in an article [6] on the foundation of projective geometry by Max Dehn, the topologist. The above polynomial identity for the algebra of $2 \times 2$-matrices appeared in 1937 in [22]. Today there is an abundant literature on polynomial identities; see for instance [8, 17].

For algebras graded by a group $G$ there exists the concept of a graded polynomial identity (see [1, 3]). In this case we need to take a family of non-commutative variables $X_g, Y_g, Z_g, \ldots$ for each element $g \in G$. Given a $G$-graded algebra $A = \bigoplus_{g \in G} A_g$, a graded polynomial identity is a polynomial $P$ in these indexed variables such that $P$ vanishes upon any substitution of each variable $X_g$ appearing in $P$ by an element of the $g$-component $A_g$.

In general, we should keep in mind that in order to define polynomial identities for a class of algebras, we need to single out

(i) a suitable algebra of non-commutative polynomials and

(ii) a suitable notion of specialization for these polynomials.

The algebras of interest to us in this survey are comodule algebras over a Hopf algebra $H$. The non-commutative variables we wish to use will be indexed by the elements of some linear basis of $H$. Since in general a Hopf algebra does not have a natural basis, we find it preferable to use a more canonical construction, namely the tensor algebra over $H$, and to resort to a given basis only when we need to perform computations.

2.2. Definition and examples of $H$-identities. Let $H$ be a Hopf algebra. We pick a copy $X_H$ of the underlying vector space of $H$ and we denote the identity map from $H$ to $X_H$ by $x \mapsto X_x$ for all $x \in H$.

Consider the tensor algebra $T(X_H)$ of the vector space $X_H$ over the ground field $k$: $$T(X_H) = \bigoplus_{r \geq 0} T^r(X_H),$$ where $T^r(X_H) = X_H^\otimes r$ is the tensor product of $r$ copies of $X_H$ over $k$, with the convention $T^0(X_H) = k$. If $\{x_i\}_{i \in I}$ is some linear basis of $H$, then $T(X_H)$ is isomorphic to the algebra of non-commutative polynomials in the indeterminates $X_{x_i}$ ($i \in I$).

Beware that the product $X_x X_y$ of symbols in the tensor algebra is different from the symbol $X_{xy}$ attached to the product of $x$ and $y$ in $H$; the former is of degree 2 while the latter is of degree 1.

The algebra $T(X_H)$ is an $H$-comodule algebra equipped with the coaction $$\delta : T(X_H) \to T(X_H) \otimes H ; \; X_x \mapsto X_{x_1} \otimes x_2.$$ Note that $T(X_H)$ is graded with all generators $X_x$ in degree 1. The coaction preserves the grading, where $T(X_H) \otimes H$ is graded by $$(T(X_H) \otimes H)_r = T^r(X_H) \otimes H$$ for all $r \geq 0$.

We now give the main definition of this section.

Definition 2.2. Let $A$ be an $H$-comodule algebra. An element $P \in T(X_H)$ is an $H$-identity for $A$ if $\mu(P) = 0$ for all $H$-comodule algebra maps $$\mu : T(X_H) \to A.$$
To convey the feeling of what an $H$-identity is, let us give some simple examples.

**Example 2.3.** Let $H = k$ be the one-dimension Hopf algebra as in Example 1.1. An $H$-comodule algebra $A$ is then the same as an algebra. In this case, $T(X_H)$ coincides with the polynomial algebra $k[X_1]$ and an $H$-comodule algebra map is nothing but an algebra map. Therefore, an element $P(X_1) \in T(X_H) = k[X_1]$ is an $H$-identity for $A$ if and only if all $P(a) = 0$ for all $a \in A$. Since $k$ is assumed to be infinite, it follows that there are no non-zero $H$-identities for $A$.

**Example 2.4.** Let $H = k[G]$ be a group Hopf algebra as in Example 1.2. We know that an $H$-comodule algebra is a $G$-graded algebra $A = \bigoplus_{g \in G} A_g$. Since $\{g\}_{g \in G}$ is a basis of $H$, the tensor algebra $T(X_H)$ is the algebra of non-commutative polynomials in the indeterminates $X_g$ ($g \in G$).

It is easy to check that an algebra map $\mu : T(X_H) \to A$ is an $H$-comodule algebra map if and only if $\mu(X_g) \in A_g$ for all $g \in G$. This remark allows us to produce the following examples of $H$-identities.

(a) Suppose that $A$ is *trivially graded*, i.e., $A_g = 0$ for all $g \neq e$. Then any non-commutative polynomial in the indeterminates $X_g$ with $g \neq e$ is killed by any $H$-comodule algebra map $\mu : T(X_H) \to A$. Therefore, such a polynomial is an $H$-identity for $A$.

(b) Suppose that the trivial component $A_e$ is *central* in $A$. We claim that

$$X_g X_{g^{-1}} X_h - X_h X_g X_{g^{-1}}$$

is an $H$-identity for $A$ for all $g, h \in G$. Indeed, for any $H$-comodule algebra map $\mu : T(X_H) \to A$, we have

$$\mu(X_g) \in A_g \quad \text{and} \quad \mu(X_{g^{-1}}) \in A_{g^{-1}};$$

therefore, $\mu(X_g X_{g^{-1}}) = \mu(X_g) \mu(X_{g^{-1}})$ belongs to $A_e$, hence commutes with $\mu(X_h)$. One shows in a similar fashion that if $g$ is an element of $G$ of finite order $N$, then for all $h \in G$,

$$X_g^N X_h - X_h X_g^N$$

is an $H$-identity for $A$.

**Example 2.5.** Let $H$ be an arbitrary Hopf algebra, and let $A$ be an $H$-comodule algebra such that the subalgebra $A^H$ of coinvariants is central in $A$ (the twisted comodule algebras of §3.11 satisfy the latter condition). For $x, y \in H$ consider the following elements of $T(X_H)$:

$$P_x = X_{x_1} X_{S(x_2)} \quad \text{and} \quad Q_{x, y} = X_{x_1} X_{y_1} X_{S(x_2 y_2)}.$$

Then for all $x, y, z \in H$,

$$P_x X_z - X_z P_x \quad \text{and} \quad Q_{x, y} X_z - X_z Q_{x, y}$$

are $H$-identities for $A$. Indeed, $P_x$ and $Q_{x, y}$ are coinvariant elements of $T(X_H)$; see [2] Lemma 2.1. It follows that for any $H$-comodule algebra map $\mu : T(X_H) \to A$, the elements $\mu(P_x)$ and $\mu(Q_{x, y})$ are coinvariant, hence central, in $A$.

More sophisticated examples of $H$-identities will be given in §5.
2.3. The ideal of $H$-identities. Let $H$ be a Hopf algebra and $A$ an $H$-comodule algebra. Denote the set of all $H$-identities for $A$ by $I_H(A)$. By definition,

$$I_H(A) = \bigcap_{\mu \in \text{Alg}_H^H(T(X_H), A)} \ker \mu.$$ 

A proof of the following assertions can be found in [2, Prop. 2.2].

**Proposition 2.6.** The set $I_H(A)$ has the following properties:

(a) it is a graded ideal of $T(X_H)$, i.e.,

$$I_H(A) T(X_H) \subset I_H(A) \supset T(X_H) I_H(A).$$

and

$$I_H(A) = \bigoplus_{r \geq 0} \left( I_H(A) \cap T^r(X_H) \right);$$

(b) it is a right $H$-coideal of $T(X_H)$, i.e.,

$$\delta(I_H(A)) \subset I_H(A) \otimes H.$$ 

Note that for any $H$-comodule algebra map $\mu : T(X_H) \to A$, we have $\mu(1) = 1$; therefore, the degree 0 component of $I_H(A)$ is always trivial:

$$I_H(A) \cap T^0(X_H) = 0.$$ 

If, in addition, there exists an injective $H$-comodule map $H \to A$, then the degree 1 component of $I_H(A)$ is also trivial:

$$I_H(A) \cap T^1(X_H) = 0.$$

**Remark 2.7.** Right from the beginning we required the ground field $k$ to be infinite. This assumption is used for instance to establish that $I_H(A)$ is a graded ideal of $T(X_H)$. Let us give a proof of this fact in order to show how the assumption is used. Indeed, expand $P \in I_H(A)$ as

$$P = \sum_{r \geq 0} P_r$$

with $P_r \in T^r(X_H)$ for all $r \geq 0$. To prove that $I_H(A)$ is a graded ideal, it suffices to check that each $P_r$ is in $I_H(A)$. Given a scalar $\lambda \in k$, consider the algebra endomorphism $\lambda_x$ of $T(X_H)$ defined by $\lambda(x) = \lambda x$ for all $x \in H$; clearly, $\lambda_x$ is an $H$-comodule map. If $\mu : T(X_H) \to A$ is an $H$-comodule algebra map, then so is $\mu \circ \lambda_x$. Since $P \in I_H(A)$, we have

$$\sum_{r \geq 0} \lambda^r \mu(P_r) = (\mu \circ \lambda_x)(P) = 0.$$

The $A$-valued polynomial $\sum_{r \geq 0} \lambda^r \mu(P_r)$ takes zero values for all $\lambda \in k$. By the assumption on $k$, this implies that its coefficients are all zero, i.e., $\mu(P_r) = 0$ for all $r \geq 0$. Since this holds for all $\mu \in \text{Alg}_H^H(T(X_H), A)$, we obtain $P_r \in I_H(A)$ for all $r \geq 0$.

If the ground field is finite, then Definition 2.2 still makes sense, but the ideal $I_H(A)$ may no longer be graded. Indeed, let $k$ be the finite field $F_p$ and $H = k$. Then for $q = p^N$, the finite field $F_q$ is an $H$-comodule algebra. In view of Example 3.3, the polynomial $X_1^q - X_1$ is an $H$-identity for $F_q$, but clearly the homogeneous summands in this polynomial, namely $X_1^q$ and $X_1$, are not $H$-identities.
2.4. The universal $H$-comodule algebra. Let $A$ be an $H$-comodule algebra and $I_H(A)$ the ideal of $H$-identities for $A$ defined above. Since $I_H(A)$ is a graded ideal of $\mathcal{T}(X_H)$, we may consider the quotient algebra
$$
\mathcal{U}_H(A) = \mathcal{T}(X_H)/I_H(A).
$$
The grading on $\mathcal{T}(X_H)$ induces a grading on $\mathcal{U}_H(A)$. As $I_H(A)$ is a right $H$-coideal of $\mathcal{T}(X_H)$, the quotient algebra $\mathcal{U}_H(A)$ carries an $H$-comodule algebra structure inherited from $\mathcal{T}(X_H)$.

By definition of $\mathcal{U}_H(A)$, all $H$-identities for $A$ vanish in $\mathcal{U}_H(A)$. For this reason we call $\mathcal{U}_H(A)$ the universal $H$-comodule algebra attached to $A$.

The algebra $\mathcal{U}_H(A)$ has two interesting subalgebras:

(i) The subalgebra $\mathcal{U}_H(A)^H$ of coinvariants of $\mathcal{U}_H(A)$.

(ii) The center $\mathcal{Z}_H(A)$ of $\mathcal{U}_H(A)$.

We now raise the following question. Suppose that the comodule algebra $A$ is free as a module over the subalgebra of coinvariants $A^H$ (or over its center); is $\mathcal{U}_H(A)$, or rather some suitable central localization of it, then free as a module over some localization of $\mathcal{U}_H(A)^H$ (or of $\mathcal{Z}_H(A)$)? An answer to this question will be given below (see Theorem 4.5) for a special class of comodule algebras, which we introduce in the next section.

3. Detecting $H$-identities

Fix a Hopf algebra $H$. We now define a special class of $H$-comodule algebras for which we can detect all $H$-identities.

3.1. Twisted comodule algebras. Recall that a two-cocycle $\alpha$ on $H$ is a bilinear form $\alpha : H \times H \to k$ such that
$$
\alpha(x_1, y_1) \alpha(x_2 y_2, z) = \alpha(y_1, z_1) \alpha(x, y_2 z_2)
$$
for all $x, y, z \in H$. We assume that $\alpha$ is convolution-invertible and write $\alpha^{-1}$ for its inverse. For simplicity, we also assume that $\alpha$ is normalized, i.e.,
$$
\alpha(x, 1) = \alpha(1, x) = \varepsilon(x)
$$
for all $x \in H$.

Any Hopf algebra possesses at least one normalized convolution-invertible two-cocycle, namely the trivial two-cocycle $\alpha_0$, which is defined by
$$
\alpha_0(x, y) = \varepsilon(x) \varepsilon(y)
$$
for all $x, y \in H$.

Let $u_H$ be a copy of the underlying vector space of $H$. Denote the identity map from $H$ to $u_H$ by $x \mapsto u_x$ ($x \in H$). We define the twisted algebra $^\alpha H$ as the vector space $u_H$ equipped with the associative product given by
$$
u_x u_y = \alpha(x_1, y_1) u_{x_2 y_2}
$$
for all $x, y \in H$. This product is associative because of the above cocycle condition; the two-cocycle $\alpha$ being normalized, $u_1$ is the unit of $^\alpha H$.

The algebra $^\alpha H$ is an $H$-comodule algebra with coaction $\delta : ^\alpha H \to ^\alpha H \otimes H$ given for all $x \in H$ by
$$
\delta(u_x) = u_{x_1} \otimes x_2.
$$
It is easy to check that the subalgebra of coinvariants of $^\alpha H$ coincides with $k u_1$, which lies in the center of $^\alpha H$. 


Note that if $\alpha = \alpha_0$ is the trivial two-cocycle, then $^aH = H$ is the $H$-comodule algebra of Example [1.3].

The twisted comodule algebras of the form $^aH$ coincide with the so-called cleft $H$-Galois objects; see [16, Prop. 7.2.3]. It is therefore an important class of comodule algebras. We next show how we can detect $H$-identities for such comodule algebras.

3.2. The universal comodule algebra map. We pick a third copy $t_H$ of the underlying vector space of $H$ and denote the identity map from $H$ to $t_H$ by $x \mapsto t_x \ (x \in H)$. Let $S(t_H)$ be the symmetric algebra over the vector space $t_H$. If $\{x_i\}_{i \in I}$ is a linear basis of $H$, then $S(t_H)$ is isomorphic to the (commutative) algebra of polynomials in the indeterminates $t_{x_i} \ (i \in I)$.

We consider the algebra $S(t_H) \otimes {}^aH$. As a $k$-algebra, it is generated by the symbols $t_z u_x \ (x, z \in H)$ (we drop the tensor product sign $\otimes$ between the $t$-symbols and the $u$-symbols).

The algebra $S(t_H) \otimes {}^aH$ is an $H$-comodule algebra whose $S(t_H)$-linear coaction extends the coaction of $^aH$:

$$\delta(t_z u_x) = t_z u_{x_1} \otimes x_2.$$  

Define an algebra map $\mu_\alpha : T(X_H) \to S(t_H) \otimes {}^aH$ by

$$\mu_\alpha(x) = t_{x_1} u_{x_2}$$

for all $x \in H$. The map $\mu_\alpha$ possesses the following properties (see [2, Sect. 4]):

**Proposition 3.1.** (a) The map $\mu_\alpha : T(X_H) \to S(t_H) \otimes {}^aH$ is an $H$-comodule algebra map.

(b) For every $H$-comodule algebra map $\mu : T(X_H) \to {}^aH$, there is a unique algebra map $\chi : S(t_H) \to k$ such that

$$\mu = (\chi \otimes \mathrm{id}) \circ \mu_\alpha.$$  

In other words, any $H$-comodule algebra map $\mu : T(X_H) \to {}^aH$ can be obtained from $\mu_\alpha$ by specialization. For this reason we call $\mu_\alpha$ the universal comodule algebra map for $^aH$.

**Theorem 3.2.** An element $P \in T(X_H)$ is an $H$-identity for $^aH$ if and only if $\mu_\alpha(P) = 0$; equivalently,

$$I_{H}(^aH) = \ker(\mu_\alpha).$$

This result is a consequence of Proposition 3.1. It allows us to detect the $H$-identities for any twisted comodule algebra: it suffices to check them in the easily controllable algebra $S(t_H) \otimes {}^aH$. In §3 we shall show how to apply this result in an interesting example.

Let us derive some consequences of Theorem 3.2. To simplify notation, we denote the ideal of $H$-identities $I_H(^aH)$ by $I_H^a$, the universal $H$-comodule algebra $U_H(^aH)$ by $U_H^a$, and the center $Z_H(^aH)$ of $U_H^a$ by $Z_H^a$.

**Corollary 3.3.** (a) The map $\mu_\alpha : T(X_H) \to S(t_H) \otimes {}^aH$ induces an injection of comodule algebras

$$\mu_\alpha : U_H^a \hookrightarrow S(t_H) \otimes {}^aH.$$
(b) An element of \( \mathcal{U}_H^\alpha \) belongs to the subalgebra \((\mathcal{U}_H^\alpha)_H\) of coinvariants if and only if its image under \( \overline{\pi}_\alpha \) sits in the subalgebra \( S(t_H) \otimes u_1 \).

We also proved that an element of \( \mathcal{U}_H^\alpha \) belongs to the center \( Z_H^\alpha \) if and only if its image under \( \overline{\pi}_\alpha \) sits in the subalgebra \( S(t_H) \otimes Z(\mathcal{Z}^\alpha H) \), where \( Z(\mathcal{Z}^\alpha H) \) is the center of \( \alpha H \) (see [2, Prop. 8.2]). In particular, since \( u_1 \) is central in \( \alpha H \), it follows that all coinvariant elements of \( \mathcal{U}_H^\alpha \) belong to the center \( Z_H^\alpha \).

We mention another consequence: it asserts that there always exist non-zero \( H \)-identities for any non-trivial finite-dimensional twisted comodule algebra.

**Corollary 3.4.** If \( 2 \leq \dim_k H < \infty \), then \( I_H^\alpha \neq \{0\} \).

**Proof.** Suppose that \( I_H^\alpha = \{0\} \). Then in view of \( \mathcal{U}_H^\alpha = T(X_H)/I_H^\alpha \) and of Corollary 3.3, we would have an injective linear map

\[
T^r(X_H) \hookrightarrow S^r(X_H) \otimes \alpha H
\]

for all \( r \geq 0 \). (Here \( S^r(X_H) \) is the subspace of elements of degree \( r \) in \( S(t_H) \).) Taking dimensions and setting \( \dim_k H = n \), we would obtain the inequality

\[
n^r \leq n \left( \binom{r + n - 1}{n - 1} \right),
\]

which is impossible for large \( r \). \( \square \)

4. LOCALIZING THE UNIVERSAL COMODULE ALGEBRA

We now wish to address the question raised in §2.4 in the case \( A \) is a twisted comodule algebra of the form \( \alpha H \), where \( H \) is a Hopf algebra and \( \alpha \) is a normalized convolution-invertible two-cocycle on \( H \).

4.1. **The generic base algebra.** Recall the symmetric algebra \( S(t_H) \) introduced in §3.2. By [2, Lemma A.1] there is a unique linear map \( x \mapsto t^{-1}_x \) from \( H \) to the field of fractions \( \text{Frac} S(t_H) \) of \( S(t_H) \) such that for all \( x \in H \),

\[
\sum_{(x)} t_{x(1)} t_{x(2)}^{-1} = \sum_{(x)} t_{x(1)}^{-1} t_{x(2)} = \varepsilon(x) 1.
\]

(The algebra of fractions generated by the elements \( t_x \) and \( t_x^{-1} \) \((x \in H)\) is Takeuchi’s free commutative Hopf algebra on the coalgebra underlying \( H \); see [21].)

**Examples 4.1.** (a) If \( g \) is a group-like element, i.e., \( \Delta(g) = g \otimes g \) and \( \varepsilon(g) = 1 \), then

\[
t_g^{-1} = \frac{1}{t_g},
\]

(b) If \( x \) is a skew-primitive element, i.e., \( \Delta(x) = g \otimes x + x \otimes h \) for some group-like elements \( g, h \), then

\[
t_x^{-1} = \frac{t_x}{t_g h}.~
\]
For \( x, y \in H \), define the following elements of the fraction field \( \text{Frac} S(t_H) \):

\[
\sigma(x, y) = \sum_{(x), (y)} t_{x(1)} t_{y(1)} \alpha(x(2), y(2)) t_{x(3)}^{-1} y_{(3)}^1
\]

and

\[
\sigma^{-1}(x, y) = \sum_{(x), (y)} t_{x(1)} y_{(1)} \alpha^{-1}(x(2), y(2)) t_{x(3)}^{-1} t_{y(3)}^{-1}
\]

where \( \alpha^{-1} \) is the inverse of \( \alpha \).

The map \( (x, y) \in H \times H \mapsto \sigma(x, y) \in \text{Frac} S(t_H) \) is a two-cocycle with values in the fraction field \( \text{Frac} S(t_H) \).

**Definition 4.2.** The generic base algebra is the subalgebra \( B_\alpha H \) of \( \text{Frac} S(t_H) \) generated by the elements \( \sigma(x, y) \) and \( \sigma^{-1}(x, y) \), where \( x \) and \( y \) run over \( H \).

Since \( B_\alpha H \) is a subalgebra of the field \( \text{Frac} S(t_H) \), it is a domain and the Krull dimension of \( B_\alpha H \) cannot exceed the Krull dimension of \( S(t_H) \), which is \( \dim_k H \). Actually, it is proved in [11, Cor. 3.7] that if the Hopf algebra \( H \) is finite-dimensional, then the Krull dimension of \( B_\alpha H \) is exactly equal to \( \dim_k H \). More properties of the generic base algebra are given in [11].

**Example 4.3.** If \( H = k[G] \) is the Hopf algebra of a group \( G \) and \( \alpha = \alpha_0 \) is the trivial two-cocycle, then the generic base algebra \( B_\alpha H \) is the algebra generated by the Laurent polynomials

\[
\left( \frac{t_g t_h}{t_{gh}} \right)^{\pm 1}
\]

where \( g, h \) run over \( G \). A complete computation for the (in)finite cyclic groups \( G = \mathbb{Z} \) and \( G = \mathbb{Z}/N \) was given in [10, Sect. 3.3].

### 4.2. Non-degenerate cocycles

We now restrict to the case when \( \alpha \) is a non-degenerate two-cocycle, i.e., when the center of the twisted algebra \( {}^\alpha H \) is one-dimensional. In this case, the center of \( {}^\alpha H \) coincides with the subalgebra of coinvariants.

Recall the injective algebra map \( \overline{\pi}_\alpha : U_\alpha^H \to S(t_H) \otimes {}^\alpha H \) of Corollary [3] By this corollary and the subsequent comment, it follows that in the non-degenerate case the center \( Z_\alpha^H \) of \( U_\alpha^H \) coincides with the subalgebra \( (U_\alpha^H)^H \) of coinvariants, and we have

\[
Z_\alpha^H = (U_\alpha^H)^H = \overline{\pi}_\alpha^{-1}(S(t_H) \otimes u_1) .
\]

The following result connects \( Z_\alpha^H \) to the generic base algebra \( B_\alpha^H \) introduced in §[4.1] (see [2 Prop. 9.1]).

**Proposition 4.4.** If \( \alpha \) is a non-degenerate two-cocycle on \( H \), then \( \overline{\pi}_\alpha \) maps \( Z_\alpha^H \) into \( B_\alpha^H \otimes u_1 \).

This result allows us to view the center \( Z_\alpha^H \) of \( U_\alpha^H \) as a subalgebra of the generic base algebra \( B_\alpha^H \). It follows from the discussion in §[4.1] that \( Z_\alpha^H \) is a domain whose Krull dimension is at most \( \dim_k H \).
We may now consider the $B_αH$-algebra

$$B_αH \otimes Z_αH \otimes U_αH.$$ 

It is an $H'$-comodule algebra, where $H' = B_αH \otimes H$.

The following answers the question raised in § 2.4.

**Theorem 4.5.** If $H$ is a Hopf algebra and $α$ is a non-degenerate two-cocycle on $H$ such that $B_αH$ is a localization of $Z_αH$, then $B_αH \otimes Z_αH \otimes U_αH$ is a cleft $H$-Galois extension of $B_αH$. In particular, there is a comodule isomorphism

$$B_αH \otimes Z_αH \otimes U_αH \cong B_αH \otimes H.$$ 

It follows that under the hypotheses of the theorem, a suitable central localization of the universal comodule algebra $U_αH$ is free of rank $\dim_k H$ as a module over its center.

5. An example: the Sweedler algebra

We assume in this section that the characteristic of $k$ is different from 2.

5.1. Presentation and twisted comodule algebras. The Sweedler algebra $H_4$ is the algebra generated by two elements $x, y$ subject to the relations

$$x^2 = 1, \quad xy + yx = 0, \quad y^2 = 0.$$ 

It is four-dimensional. As a basis of $H_4$, we take the set $\{1, x, y, z\}$, where $z = xy$.

The algebra $H_4$ carries the structure of a non-commutative, non-cocommutative Hopf algebra with coproduct, counit, and antipode given by

$$\begin{align*}
\Delta(1) &= 1 \otimes 1, & \Delta(x) &= x \otimes x, \\
\Delta(y) &= 1 \otimes y + y \otimes x, & \Delta(z) &= x \otimes z + z \otimes 1, \\
\varepsilon(1) &= \varepsilon(x) = 1, & \varepsilon(y) &= \varepsilon(z) = 0, \\
S(1) &= 1, & S(x) &= x, \\
S(y) &= z, & S(z) &= -y.
\end{align*}$$

The tensor algebra $T(H_4)$ is the free non-commutative algebra on the four symbols

$$E = X_1, \quad X = X_x, \quad Y = X_y, \quad Z = X_z,$$

whereas $S(t_{H_4})$ is the polynomial algebra on the symbols $t_1, t_x, t_y, t_z$.

Masuoka [13] (see also [7]) showed that any twisted $H_4$-comodule algebra as in § 3.1 has, up to isomorphism, the following presentation:

$$^αH_4 = k \left\langle u_x, u_y \mid u_x^2 = au_1, \quad u_xu_y + u_yu_x = bu_1, \quad u_y^2 = cu_1 \right\rangle$$

for some scalars $a, b, c$ with $a \neq 0$. To indicate the dependence on the parameters $a, b, c$, we denote $^αH_4$ by $A_{a,b,c}$.

The center of $A_{a,b,c}$ consists of the scalar multiples of the unit $u_1$ for all values of $a, b, c$. In other words, all two-cocycles on $H_4$ are non-degenerate.

The coaction $δ : A_{a,b,c} → A_{a,b,c} \otimes H_4$ is determined by

$$δ(u_x) = u_x \otimes x \quad \text{and} \quad δ(u_y) = u_1 \otimes y + u_y \otimes x.$$ 

As observed in § 3.1, the coinvariants of $A_{a,b,c}$ consists of the scalar multiples of the unit $u_1$. Therefore, coinvariants and central elements of $A_{a,b,c}$ coincide.
5.2. Identities. In this situation, the universal comodule algebra map
\[ \mu_\alpha : T(X_H) \to S(t_H) \otimes A_{a,b,c} \]
is given by
\[
\begin{align*}
\mu_\alpha(E) &= t_1 u_1, & \mu_\alpha(X) &= t_x u_x, \\
\mu_\alpha(Y) &= t_1 u_y + t_y u_x, & \mu_\alpha(Z) &= t_x u_z + t_z u_1.
\end{align*}
\]

Let us set
\[
R = X^2, \quad S = Y^2, \quad T = XY + YX, \quad U = X(XZ + ZX).
\]

**Lemma 5.1.** In the algebra \( S(t_H) \otimes A_{a,b,c} \) we have the following equalities:
\[
\begin{align*}
\mu_\alpha(R) &= a t_2^2 u_1, \\
\mu_\alpha(S) &= (a t_y^2 + b t_y t_x + c t_1^2) u_1, \\
\mu_\alpha(T) &= t_x (2 a t_y + b t_1) u_1, \\
\mu_\alpha(U) &= a t_x^2 (2 t_z + b t_x) u_1.
\end{align*}
\]

**Proof.** This follows from a straightforward computation. Let us compute \( \mu_\alpha(S) \) as an example. We have
\[
\begin{align*}
\mu_\alpha(S) &= \mu_\alpha(Y)^2 = (t_1 u_y + t_y u_x)^2 \\
&= t_y^2 u_x^2 + t_1 t_y (u_x u_y + u_y u_x) + t_1^2 u_y^2 \\
&= (a t_y^2 + b t_y t_x + c t_1^2) u_1
\end{align*}
\]
in view of the definition of \( A_{a,b,c} \). \( \square \)

We now exhibit two non-trivial \( H_4 \)-identities.

**Proposition 5.2.** The elements
\[ T^2 - 4RS - \frac{b^2 - 4ac}{a} E^2 R \quad \text{and} \quad ERZ - RXY - \frac{EU - RT}{2} \]
are \( H_4 \)-identities for \( A_{a,b,c} \).

**Proof.** It suffices to check that these two elements are killed by \( \mu_\alpha \), which is easily done using Lemma 5.1. \( \square \)

Since \( E, R, S, T, U \) are sent under \( \mu_\alpha \) to \( S(t_H) \otimes u_1 \), their images in \( U_\alpha^H \) belong to the center \( Z_\alpha^H \). We assert that after inverting the elements \( E \) and \( R \), all relations in \( Z_\alpha^H \) are consequences of the leftmost relation in Proposition 5.2. More precisely, we have the following (see [2, Thm. 10.3]).

**Theorem 5.3.** There is an isomorphism of algebras
\[
Z_\alpha^H[E^{-1}, R^{-1}] \cong k[E, E^{-1}, R, R^{-1}, S, T, U]/(D_{a,b,c}),
\]
where
\[
D_{a,b,c} = T^2 - 4RS - \frac{b^2 - 4ac}{a} E^2 R.
\]

To prove this theorem, we first check that the generic base algebra \( B_\alpha^H \) (whose generators we know) is generated by \( E, E^{-1}, R, R^{-1}, S, T, U \); this implies that \( B_\alpha^H \) is the localization
\[ B_\alpha^H = Z_\alpha^H[E^{-1}, R^{-1}] \]
of $Z^n_H$. In a second step, we establish that all relations between the above-listed generators of $B^n_H$ follow from the sole relation $D_{a,b,c} = 0$.

Let us now turn to the universal comodule algebra $U^n_H$. By Proposition 5.2, we have the following relation in $U^n_H$, where we keep the same notation for the elements of $T(X^n_H)$ and their images in $U^n_H$:

\[(ER)Z = (R)XY + \left( \frac{EU - RT}{2} \right) \text{ in } U^n_H.\]

The elements in parentheses being central, it follows from the previous relation that if we again invert the central elements $E$ and $R$, then $Z$ is a linear combination of 1 and $XY$ with coefficients in $B^n_H = Z^n_H[E - 1, R - 1]$. Noting that $YX = -XY + T \in -XY + Z^n_H \subset -XY + B^n_H$, we easily deduce that after inverting $E$ and $R$ any element of $U^n_H$ is a linear combination of 1, $X$, $Y$, $XY$ over $B^n_H$.

In [2] the following more precise result was established (see loc. cit., Thm. 10.7). It answers positively the question of §2.4.

**Theorem 5.4.** The localized algebra $U^n_H[E^{-1}, R^{-1}]$ is free of rank 4 over its center $B^n_H = Z^n_H[E^{-1}, R^{-1}]$, and there is an isomorphism of algebras

\[U^n_H[E^{-1}, R^{-1}] \cong B^n_H \langle \xi, \eta \rangle / \left( \xi^2 - R, \xi \eta + \eta \xi - T, \eta^2 - S \right).\]

Note that the algebra $B^n_H$ coincides with the subalgebra of coinvariants of $U^n_H[E^{-1}, R^{-1}]$.

5.3. **An open problem.** To complete this survey, we state a problem who will hopefully attract the attention of some researchers.

Fix an integer $n \geq 2$ and suppose that the ground field $k$ contains a primitive $n$-th root $q$ of 1. Consider the Taft algebra $H^{a2}$, which is the algebra generated by two elements $x$, $y$ subject to the relations $x^n = 1$, $yx = qxy$, $y^n = 0$.

This is a Hopf algebra of dimension $n^2$ with coproduct determined by

$\Delta(x) = x \otimes x$ and $\Delta(y) = 1 \otimes y + y \otimes x$.

The twisted comodule algebras $H^{a2}$ have been classified in [7, 13]. (All two-cocycles of $H^{a2}$ are non-degenerate.)

Give a presentation by generators and relations of the generic base algebra $B^n_{H^{a2}}$ and show that $B^n_{H^{a2}}$ is a localization of $Z^n_{H^{a2}}$. (By [11] Rem. 2.4 (c) it is enough to consider the case where $\alpha$ is the trivial cocycle.)

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