Towards a Degeneration Formula for the
Gromov-Witten Invariants of Symplectic Manifolds

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Abstract
In this paper, we outline a project started in [7] aimed at defining Gromov-Witten (GW) invariants relative to normal crossings symplectic divisors, and GW-type invariants for normal crossings symplectic varieties. Furthermore, we use the latter to propose a degeneration formula that relates the GW invariants of smooth fibers to the GW invariants of central fiber, in a semistable degeneration with a normal crossings central fiber. In the case of “basic” degenerations, the degeneration formula proposed in this article coincides with the Jun Li’s formula.

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1 Introduction
Gromov-Witten theory, which is the modern manifestation of the enumerative geometry in complex algebraic geometry, is about counting holomorphic curves of a specified type in a complex projective variety $X$, or more generally, in a symplectic manifold $(X, \omega)$ equipped with a compatible almost complex structure $J$. The resulting (rational) numbers are invariants of the deformation equivalence class of the target. For example, there are 2875 degree 1 genus 0 curves (complex lines) inside any smooth quintic threefold in $\mathbb{P}^4$. Since the resulting numbers (or virtual fundamental class) are invariant under the deformation of the target, it is natural to seek for an equivalent count of such curves in the limit, whenever a family of smooth targets degenerates to a singular one in a smooth family. Normal crossings (NC) varieties are the most basic and important class of such singular

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\[1\] i.e. the total space of the deformation space is smooth.
objects in complex algebraic (or Kähler) geometry. An NC divisor in a smooth complex variety $X$ is a sub-variety with local defining equations of the form

$$z_1 \cdots z_k = 0$$

in some holomorphic coordinate chart $(z_1, \ldots, z_n)$ on $X$. An NC variety of complex dimension $n$ is a variety that can be locally embedded as an NC divisor in $\mathbb{C}^{n+1}$. A simple normal crossings (SC) divisor/variety is an NC divisor/variety without self-intersections. In other words, an SC divisor $D$ is transverse union of smooth divisors

$$D = \bigcup_{i \in S} D_i \subset X,$$

and an SC variety $X_\emptyset$ is a global transverse union of smooth varieties $\{X_i\}_{i \in S}$ along SC divisors $X_{i;\emptyset} = \bigcup_{j \in S - i} X_{ij}$ in $X_i$, i.e.

$$X_\emptyset = \left( \bigsqcup_{i \in S} X_i \right) / \sim, \quad X_{ij} \approx X_{ji} \quad \forall \ i, j \in S, \ i \neq j,$$

with the singular locus

$$X_\emptyset = \bigcup_{i, j \in S, \ i \neq j} X_{ij} \subset X_\emptyset.$$

A 3-fold SC variety is shown in Figure 1. Then, a semistable degeneration is a smooth one-parameter family $\pi: \mathcal{Z} \to \Delta$, where $\Delta$ is a small disk around the origin in $\mathbb{C}$, the central fiber $\mathcal{Z}_0 \equiv \pi^{-1}(0)$ is an NC variety, and the fibers over $\Delta^* = \Delta - \{0\}$ are smooth.

In [11]-[14], we defined and studied the analogue of these spaces for the symplectic topology. An SC symplectic divisor in a symplectic manifold $(X, \omega)$ is a finite transverse union $D \equiv \bigcup_{i \in S} D_i$ of smooth symplectic divisors $\{D_i\}_{i \in S}$ such that for every $I \subset S$ the submanifold

$$D_I \equiv \bigcap_{i \in I} D_i \subset X$$

is symplectic and its symplectic and “intersection orientations” are the same; see [11] Definition 2.1]. An SC symplectic variety is a pair $(X_\emptyset, \omega_\emptyset \equiv (\omega_i)_{i \in S})$, where

$$X_\emptyset = \left( \bigsqcup_{i \in S} X_i \right) / \sim, \quad X_{ij} \approx X_{ji} \quad \forall \ i, j \in S, \ i \neq j,$$

\footnote{i.e. $\mathcal{Z}$ is smooth.}
for a finite collection \((X_i, \omega_i)_{i \in S}\) of symplectic manifolds, some SC symplectic divisor \(X_{i,0} = \bigcup_{j \in S - i} X_{ij}\) in \(X_i\) for each \(i \in S\), and symplectic identifications \(X_{ij} \cong X_{ji}\) for all \(i, j \in S\) distinct; see \([11],\) Definition 2.5]. Arbitrary NC symplectic divisors/varieties in \([13]\) are defined in terms of compatible local SC charts. Then, a **symplectic semistable degeneration**, or a "nearly regular symplectic fibration" as we call it in \([12],\) Definition 2.5], is a smooth symplectic manifold \((Z, \omega_Z)\) with a smooth surjective map \(\pi : Z \to \Delta\) such that \(Z_0\) is an NC symplectic divisor, \(\pi\) is a submersion outside of the singular locus \(Z^\text{sing}_0 \subset Z_0 \subset Z\), and for every \(\lambda \in \Delta^*\), the restriction \(\omega_\lambda\) of \(\omega_Z\) to \(Z_\lambda = \pi^{-1}(\lambda)\) is non-degenerate (In \([9]\), we will impose more restrictions on \(\pi\)).

From the analytic point of view, for a smooth manifold \(X, g, k \in \mathbb{N}, A \in H_2(X, \mathbb{Z})\), and an almost complex structure\(^3\) \(J\) on \(X\), a (nodal) \(k\)-marked genus \(g\) degree \(A\) \(J\)-holomorphic map into \(X\) is a tuple \((u, \Sigma, j, z^1, \ldots, z^k)\), where

- \((\Sigma, j)\) is a connected nodal Riemann surface of the arithmetic genus \(g\), with complex structure \(j\) and \(k\) distinct ordered marked points \(z^1, \ldots, z^k\) away from the nodes,

- \(u : (\Sigma, j) \to (X, J)\) is a continuous and component-wise smooth map satisfying the Cauchy-Riemann equation

\[
\bar{\partial} u = \frac{1}{2} (du + J du \circ j) = 0
\]

on each smooth component, and

- the map \(u\) represents the homology class \(A\).

Two such tuples

\((u, \Sigma, j, z^1, \ldots, z^k)\) and \((u', \Sigma', j', w^1, \ldots, w^k)\)

are equivalent if there exists a biholomorphic isomorphism \(h : (\Sigma, j) \to (\Sigma', j')\) such that \(h(z^i) = w^i\), for all \(i = 1, \ldots, k\), and \(u = u' \circ h\). Such a tuple is called stable if the group of self-automorphisms is finite. Let \(\mathcal{M}_{g,k}(X, A, J)\) (or simply \(\mathcal{M}_{g,k}(X, A)\) when \(J\) is fixed in the discussion) denote the space of equivalence classes of stable \(k\)-marked genus \(g\) degree \(A\) \(J\)-holomorphic maps into \(X\).

By a celebrated theorem of Gromov \([20],\) Theorem 1.5.B], and its subsequent refinements, for every smooth closed (i.e. compact and without boundary) symplectic manifold \((X, \omega)\) and an almost complex structure \(J\) compatible\(^4\) with \(\omega\), the moduli space \(\mathcal{M}_{g,k}(X, A, J)\) has a natural sequential convergence topology, called the **Gromov topology**, which is compact, Hausdorff, and furthermore metrizable. The symplectic structure only gives an energy bound which is needed for establishing the compactness. The specific choice of \(\omega\), up to deformation, is not important in GW theory. If \(\mathcal{M}_{g,k}(X, A)\) has a "nice" orbifold structure of the expected real dimension

\[
2(c_1^TX(A) + (n - 3)(1 - g) + k), \tag{3}
\]

Gromov-Witten (or GW) invariants are obtained by the integration of certain cohomology classes against its fundamental class. These numbers are independent of \(J\) and only depend on the deformation equivalence class of \(\omega\). These allow the formulation of symplectic analogues of enumerative questions from algebraic geometry, as well-defined invariants of symplectic manifolds. However, in general, such moduli spaces can be highly singular. This issue is known as the **transversality**

\(^3\)i.e. \(J\) is a real-linear endomorphism of \(TX\) lifting the identity map and satisfying \(J^2 = -\text{id}_{TX}\).

\(^4\)i.e. \(\omega(\cdot, J\cdot)\) is a metric.
problem. Fortunately, it has been shown (e.g. see [31, 32]) that \( \overline{M}_{g,k}(X, A) \) still carries a rational homology class, called virtual fundamental class (VFC); integration of cohomology classes against VFC gives rise to GW-invariants. We denote such an VFC by \([\overline{M}_{g,k}(X, A)]^{VFC}\). In [10], we sketch the construction of VFC via the method of Kuranishi structures in [19].

For a symplectic semistable degeneration \( \mathcal{Z} \) as above with compact fibers, let \( J_\mathcal{Z} \) be an almost complex structure on \( \mathcal{Z} \) which is compatible with both \( \omega_\mathcal{Z} \) and the fibration; i.e. \( \omega_\mathcal{Z}(\cdot, J_\mathcal{Z} \cdot) \) is a metric and every \( \mathcal{Z}_\lambda \) is \( J_\mathcal{Z} \)-holomorphic. One may furthermore require \( \pi \) to be \((i, J)\)-holomorphic. For every \( \lambda \in \Delta \), we denote the restriction of \( J_\mathcal{Z} \) to \( T \mathcal{Z}_\lambda \) by \( J_\lambda \). We say \( A \in H_2(\mathcal{Z}, \mathbb{Z}) \) is vertical if \( A[\mathcal{Z}_\lambda]=0 \); the latter is independent of the choice of \( \lambda \). If \( A \) is vertical, every \( J_\mathcal{Z} \)-holomorphic map in \( \overline{M}_{g,k}(\mathcal{Z}, A, J_\mathcal{Z}) \) has image in a fiber \( \mathcal{Z}_\lambda \), for some \( \lambda \in \Delta \). Therefore, we get a fibration

\[
\overline{M}_{g,k}(\mathcal{Z}, A, J_\mathcal{Z}) = \bigcup_{\lambda \in \Delta} \overline{M}_{g,k}(\mathcal{Z}_\lambda, A, J_\lambda) \to \Delta,
\]

where for \( \lambda = 0 \) by \( \overline{M}_{g,k}(\mathcal{Z}_0, A, J_0) \) we simply mean the space of \( J_\mathcal{Z} \)-holomorphic maps in \( \mathcal{Z} \) whose image lie inside \( \mathcal{Z}_0 \). Let \( \overline{M}_{g,k}(\mathcal{Z}_s, A, J_s) \) be the complement of \( \overline{M}_{g,k}(\mathcal{Z}_0, A, J_0) \) in \( \overline{M}_{g,k}(\mathcal{Z}, A, J_\mathcal{Z}) \).

For \( \lambda, \lambda' \in \Delta^* \) and any path \( \gamma \) in \( \Delta^* \) connecting \( \lambda \) and \( \lambda' \),

\[
\pi^{-1}(\gamma) \subset \overline{M}_{g,k}(\mathcal{Z}_s, A, J_s)
\]

gives a cobordism between

\[
[\overline{M}_{g,k}(\mathcal{Z}_\lambda, A)]^{VFC} \quad \text{and} \quad [\overline{M}_{g,k}(\mathcal{Z}_{\lambda'}, A)]^{VFC}
\]

in the sense of [10] Definition 3.5.1; thus the corresponding GW invariants are equal. Since the space \( \mathcal{Z}_0 \) is not smooth, the classical construction of GW invariants does not extend to \( \overline{M}_{g,k}(\mathcal{Z}_0, A) \). Moreover, the expected dimension of the (virtually) main components of \( \overline{M}_{g,k}(\mathcal{Z}_0, A) \) could be quite different from [39]. The question is:

\((\star \star)\) how to replace \( \overline{M}_{g,k}(\mathcal{Z}_0, A) \) with a refined compact space \( \overline{M}_{g,k}^{good}(\mathcal{Z}_0, A) \) (ideally, still a subset of \( \overline{M}_{g,k}(\mathcal{Z}_0, A) \)) such that

\[
\overline{M}_{g,k}^{good}(\mathcal{Z}, A) = \overline{M}_{g,k}(\mathcal{Z}_s, A) \cup \overline{M}_{g,k}^{good}(\mathcal{Z}_0, A)
\]

is still naturally compact, each “main component” of \( \overline{M}_{g,k}^{good}(\mathcal{Z}_0, A) \) has the same expected dimension [39] and is (virtually) smooth enough to admit a natural VFC, and there exists a similar cobordism relation between \( [\overline{M}_{g,k}^{good}(\mathcal{Z}_0, A)]^{VFC} \) and \( [\overline{M}_{g,k}(\mathcal{Z}_{\lambda}, A)]^{VFC} \), for any \( \lambda \in \Delta^* \)?

If \( \mathcal{Z}_0 = X_0 \) is a simple normal crossings, it decomposes into a transverse union of smooth varieties \( \{X_i\}_{i \in S} \) along SC divisors \( X_{i;\partial} = \bigcup_{j \in S_i} X_{ij} \subset X_i \) as in [11]. The inclusion \( Z_0 \subset Z \) is also an SC symplectic divisor. Every \( J_0 \)-holomorphic map in \( \mathcal{Z}_0 \) can be decomposed (not necessarily uniquely) into a union of \( J_i \)-holomorphic maps into \( X_i \) that are “compatible” at the nodes in a suitable sense. The question above thus brings up the following discussion.

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5There might be different homology classes in \( \mathcal{Z}_\lambda \) that are equal to \( A \), as homology classes in \( \mathcal{Z} \). In this case, \( \overline{M}_{g,k}(\mathcal{Z}, A, J_\mathcal{Z}) \) is the union over all the representatives of \( A \) in \( H_2(\mathcal{Z}_\lambda, \mathbb{Z}) \); see [17].
Assume \( D = \bigcup_{i \in S} D_i \) is an SC symplectic divisor in \((X, \omega)\), \( J \) is an almost complex structure “compatible” with \( \omega \) and \( D \), \( A \in H_2(X, \mathbb{Z}) \), \( g, k \in \mathbb{N} \), and
\[
\mathfrak{s} = (s_i = (s_{ij})_{j \in S})_{i=1}^k \in (\mathbb{N}^S)^k,
\]
such that
\[
A \cdot D_j = \sum_{i=1}^k s_{ij} \quad \forall \ j \in S. \tag{4}
\]
Define \( \mathcal{M}_{g,s}(X, D, A) \) (in the stable range) to be the space of equivalence classes of \( k \)-marked degree \( A \) genus \( g \) \( J \)-holomorphic maps into \( X \) (with smooth domain) which have contact order of \( s_{ij} \) with \( D_j \) at the marked point \( z_i \) for all \( 1 \leq i \leq k \) and \( j \in S \); in particular, by (4),
\[
u^{-1}(D) \subset \{z_1, \ldots, z_k\}.
\]
The expected real dimension of \( \mathcal{M}_{g,s}(X, D, A) \) is
\[
2 \left( c_1^{TX}(A) + (n - 3)(1 - g) + k - A \cdot D \right). \tag{5}
\]
Note that the marked points \( z_i \) with \( s_i = 0 \in \mathbb{N}^S \) correspond to the classical marked points with image away from the divisor. Similarly to (⋆), the question is:

(⋆) how to define a “good” compactification \( \overline{\mathcal{M}}_{g,s}^{\text{good}}(X, D, A) \) of \( \mathcal{M}_{g,s}(X, D, A) \) so that the definition of contact order \( \mathfrak{s} \) naturally extends to every element of \( \overline{\mathcal{M}}_{g,s}^{\text{good}}(X, D, A) \), \( \overline{\mathcal{M}}_{g,s}^{\text{good}}(X, D, A) \) is (virtually) smooth enough to admit a natural class of cobordant (for various choices of \( J \), etc.) Kuranishi structures of the real dimension (5), and the resulting GW invariants are invariants of the deformation equivalence class of \((X, D, \omega)\)?

Once we answer (⋆) and (⋆⋆), the natural question is:

(⋆⋆⋆) whether the resulting GW invariants of \( Z_0 \), and thus any \( Z_\lambda \), can be expressed as a function of the relative GW invariants of the pairs \((X_i, X_i; \partial)\)?

In this article, we outline a project aimed at answering (⋆) and (⋆⋆). We review the known results in Section 2. In order to address (⋆), in [7], we introduce a notion of log pseudoholomorphic map relative to simple normal crossings symplectic divisors. We review this notion in Section 3. In [7], for an appropriate class of almost complex structures, we show that the moduli space of “stable” log maps of any fixed type is compact and metrizable with respect to an enhancement of the Gromov topology. In the case of smooth symplectic divisors, our compactification is often smaller than the relative compactification and there is a projection map from the former onto the latter. The latter is constructed via expanded degenerations of the target, however, our construction does not need any modification of (or an extra structure on) the target. In Section 4, we will use the decomposition (2) and the space of relative log maps in each piece \((X_i, X_i; \partial)\) to define a notion of log pseudoholomorphic map for “smoothable” SC symplectic varieties. In [9], we will use the compactness result for the relative case to prove a similar compactness result for the moduli spaces of stable log maps into SC symplectic varieties. Unlike in the case of classical moduli spaces of stable maps, log moduli spaces are often virtually singular. We describe an explicit toric model for the space of gluing parameters along each stratum in terms of the defining combinatorial data of that stratum. In an upcoming paper, we will define a natural Fredholm operator which gives us
the deformation/obstruction spaces of each stratum and prove a gluing theorem for smoothing log maps in the normal direction to each stratum. With minor modifications to the theory of Kuranishi structures, the latter allows us to construct a virtual fundamental class for every such log moduli space. In Section 5, we propose an explicit degeneration formula that relates the GW invariants (or VFC) of the smooth fibers to the expected log GW invariants (resp. VFC) of the central fiber, in a “suitable” semistable degeneration \( \pi : Z \to \Delta \), with a simple normal crossings central fiber \( Z_0 \); see (37). The explicit coefficients in (37) come from calculating the degree of the projection map induced by \( \pi \) from the space of gluing parameters onto \( \mathbb{C} \). In Remark 5.4, we briefly compare our formula to that of Abramovich-Chen-Gross-Siebert in \([2, (1.1.1)]\). In Section 6, we re-study a non-trivial (i.e. not of the classical type) example about the degeneration of degree 3 rational curves in a pencil of cubic surfaces, studied in \([2, Section 6]\), to illustrate (37) and highlight the similarities/differences of our approach with the algebraic approach. Finally, by examples such as Example 6.2, a positive answer to question (\( \star \star \star \)) seems very unlikely in complex dimensions higher than 2.

2 The state of the art

First informal mention and application of such a degeneration formula date back to the mid-1990 papers of Tian \([40]\) and Caporaso-Harris \([4]\). In algebraic geometry, for a smooth divisor \( D \subset X \), Jun Li \([28]\) introduced a notion of stable relative map whose image lives in a natural simple normal crossings “expanded degeneration” associated to \((X, D)\). Similarly, for a smoothable basic SC variety \( Z_0 = X_1 \cup X_{12} X_2 \), he constructed a compactification \( \overline{M}_{g,k}(Z_0, A) \) whose main components are appropriate fiber products of the relative moduli spaces \( \overline{M}_{g,k}(X_1, X_{12}, A_1) \) and \( \overline{M}_{g,k}(X_2, X_{12}, A_2) \). For any semistable degeneration with a basic SC central fiber, in \([29]\), he proved a decomposition formula which expresses (a combination of) the GW invariants of the smooth fibers in term of products of relative GW invariants of \((X_1, X_{12})\) and \((X_2, X_{12})\). Therefore, the relative theory of \([28, 29]\) answers (*) for smooth divisors and (**)-(\( \star \star \star \)) for basic normal crossings degenerations. On the symplectic (analytical) side, similar moduli spaces are defined in \([23, 24, 30]\). After some fixes, definition of relative moduli spaces in \([23]\) for smooth symplectic divisors \( D \subset (X, \omega) \) includes the holomorphic case and is thus a generalization of the algebraic definition. However, proof of the compactness is not complete and the construction of relative GW invariants is limited to a narrow class of semi-positive pairs; see \([15, Definition 4.7]\) for the definition of semi-positivity which is not provided in \([23]\). Similarly, the definition of \( \overline{M}_{g,k}(X_1 \cup X_{12} X_2, A) \) in \([24]\) includes the holomorphic case, but the decomposition formula stated there is limited to the semi-positive range, and both its statement and proof are incorrect; see \([15]\) and \([26]\). Definition of the relative compactification in \([30]\) adapts the ideas of SFT for stretching of the target. This idea uses almost complex structures on \( X-D \) with translational and rotational symmetry in the cylindrical end as in SFT \([6]\). These almost complex structures are often not holomorphic. They are however more suitable for analytical purposes such as proving the essential gluing theorem. The statement of the degeneration formula in \([30]\) essentially coincides with the Jun Li’s formula but the paper contains no proof of the crucial statements. We believe that the analytical details of SFT can be adapted to make a complete proof.

More generally, an SC variety \( Z_0 \) is basic and the same results hold if the singular locus is smooth and locally separable.

Over possibly disconnected domains of Euler characteristic \( \chi_1 \) and \( \chi_2 \).
It has been long known that the method of expanded degenerations does not work (well) for arbitrary SC divisors/varieties. On the algebraic side, log Gromov-Witten theory of Gross-Siebert and Abramovich-Chen (initially proposed by Siebert about 15 years ago) answers (∗) and (⋆⋆) and works even for a larger class of “log smooth objects”. Every (complex) NC divisor \( D \subseteq X \) defines a natural “fine saturated log structure” on \( X \), and the log GW theory of [1, 21] constructs a good compactification with a perfect obstruction theory for every fine saturated log variety \( X \). Unlike in [28], the log compactification does not require any expanded degeneration of the target. Instead, it uses the extra log structure on \( X \) (and various log structures on the domains) to keep track of the curves sinking into the support of the log structure. The unpublished degeneration formula of [2, (1.1.1)] and the one that we propose in this article are a sum over the same set of combinatorial data, but a priori, with different coefficients; see Remark 5.4. Unlike the relative compactification, which has a geometric presentation, definition of algebraic log moduli spaces involves certain intricate algebraic structures that can not easily be translated into analytical/topological structures suitable for the symplectic category. The project outlined in this article is inspired by the log GW theory but it is completely different in definitions and details.

On the analytical side, in [36], Brett Parker uses his enriched almost Kähler category of “exploded manifolds”, defined in [35], to construct such a compactification relative to an almost Kähler NC divisor. His approach can be considered as a direct translation of log geometry into the almost Kähler category. With the “exploded semiring” \( C^R \) defined in [35, Section 2], an abstract exploded space \( B \) (cf. [35, Definition 3.1]) consists of the following data:

- A (possibly non-Hausdorff) topological space \( B \) whose topology is induced from a surjective map to a Hausdorff topological space, \( B \rightarrow \lceil B \rceil \);
- A sheaf \( \mathcal{E}^x(B) \) of Abelian groups on \( B \), called the sheaf of exploded functions on \( B \), such that
  
  \( (a) \) each element \( f \in \mathcal{E}^x(U) \) is a map \( f:U \rightarrow \mathbb{C}^*t^R \),
  
  \( (b) \) multiplication is given by the point-wise multiplication in \( \mathbb{C}^*t^R \),
  
  \( (c) \) \( \mathcal{E}^x(U) \) includes the constant functions if \( U \neq \emptyset \),
  
  \( (d) \) and restriction maps are given by restriction of functions.

The sheaf \( \mathcal{E}^x(B) \) plays the role of the sheaf of monoids in log geometry. A smooth exploded manifold is an abstract exploded space that can be covered with some specific type of local (toroidal) charts; see [35, Definition 3.13]. Similarly to the log geometry, every complex NC divisor \( D \subseteq X \) defines a natural exploded structure \( B \) on \( X = \lceil B \rceil \). Beyond the complex case, his construction still requires some “holomorphicity” near \( D \) and it is not clear whether there is a way to associate an exploded structure to every arbitrary NC symplectic divisor/variety. In addition to this rigidity issue, as the complicated definition above indicates, the moduli spaces arising from exploded structures are more complicated than desirable and have so far proved too unwieldy for practical applications. In [37, (1)-(2)], Brett Parker claims two “tropical” degeneration formulas, but we have not been able to understand these formulas and compare them to (37).

In [22], Eleny Ionel approaches (∗) by considering expanded degenerations similar to [23]. The definition of NC divisor in [22, Definition 1.3] is almost Kähler, i.e. the resulting invariants will not a priori be symplectic invariants; our topological notion of NC symplectic divisor in [11] together with the isomorphism [11 (1.7)] can be used to address this shortcoming. However, we believe that the proof of the main result [22, Theorem 8.1] is still deficient for the following reasons.

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By [22, Remark 2.3], this paper uses Ruan-Tian type \( (J, \nu) \)-holomorphic maps to avoid constructing VFC. For this method to work, one has to ensure that there are enough “appropriate” perturbations \( \nu \) to achieve transversality. This is typically done in the literature by either assuming a semi-positivity condition on the target as in [38, 39] or by uniformly stabilizing all the domains via some Donaldson divisor as in [5]. The proof of [22, Theorem 8.1] refers to [22, Remark 2.2] in this regard, which is about using Donalds on divisors for any arbitrary tuple \((X, D, g, A)\). However, even if \( D \) is smooth, by the counter examples in [16], the method of stabilizing the domains with a Donaldson divisor does not work in all cases; this issue has been acknowledged by Ionel-Parker in [25].

The main goal of [22] is to prove a compactness result and the author repeats the following improper argument in [24] about using symplectic sum fibrations. She uses various undefined deformation spaces \( X \rightarrow B \) (see top of [22, Page 53]) to reduce the proof of compactness to the classical Gromov compactness. As we explain in Page 75 of [15], even in the case of classical symplectic sum smoothing, using symplectic sum families associated to different expanded degenerations to prove compactness (i.e. [22, Definition 5.5]) does not make any sense. The general fibers of such smoothings are deformation equivalent to the original symplectic manifold \( X \), but not canonically. In addition, the so called “symplectic sum” family \( X_\lambda \) in [22, Section 4] (for that particular type of central fiber) does not exist anywhere in the literature.

Nevertheless, the main motivation behind the log GW theory of Gross-Siebert-Abramovich-Chen, exploded theory of Parker, and the current article is that the idea of expanded degenerations does not sound promising for proving a degeneration formula.

### 3 Log pseudoholomorphic maps relative to SC divisors

In this section, for an arbitrary SC symplectic divisor \( D \subset (X, \omega) \) and an appropriate choice of almost complex structure \( J \) on \( X \), we review the definition of log moduli spaces \( \overline{M}^{\log}_{g,s}(X, D, A) \) in [7]. We refer to [7] for more details and various examples.

For a smooth symplectic divisor \( D \subset (X, \omega) \), define \( \mathcal{J}(X, D, \omega) \) to be the space of almost complex structures \( J \) on \( X \) that are

1. compatible with \( \omega \), i.e. \( \omega(\cdot, J\cdot) \) is a metric,
2. compatible with \( D \), i.e. \( JTD=TD \), and
3. integrable to the first order in the normal direction to \( D \) in the sense that

\[ N_J(v_1, v_2) \in T_x D \quad \forall x \in D, \quad v_1, v_2 \in T_x X, \]

where

\[ N_J \in \Gamma(X, \Omega_X^2 \otimes TX), \quad N_J(u, v) \equiv [u, v] + J[u, Jv] + J[u, v] - [Ju, Jv] \quad \forall u, v \in TX, \]

is the Nijenhueis tensor of \( J \).

It can be shown using the Symplectic Neighborhood Theorem [33, Theorem 3.30] that this space is non-empty and contractible; see [23, Appendix]. Every almost complex structure satisfying (2) induces a complex structure \( i_{\mathcal{N}_X D} \) and a \( \bar{\partial} \)-operator \( \bar{\partial}_{\mathcal{N}_X D} \) on the normal bundle \( \mathcal{N}_X D \) of \( D \)

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\(^8\)One may consider “tame” almost complex structures.
$X$; see [11]. This $\bar{\partial}$-operator plays a crucial role in defining the relative and log moduli spaces. The Nijenhuis condition is needed for the proof of compactness/or equally to show that certain operators are complex linear.

For an SC symplectic divisor $D = \bigcup_{i \in S} D_i \subset (X, \omega)$, let

$$\mathcal{J}(X, D, \omega) = \bigcap_{i \in S} \mathcal{J}(X, D_i, \omega).$$

This space could be empty for some choices of $\omega$. Let $\text{Symp}^+(X, D)$ be the space of all symplectic structures $\omega$ on $X$ such that $D$ is an SC symplectic divisor in $(X, \omega)$ and

$$\mathcal{J}(X, D) = \{ (J, \omega) : \omega \in \text{Symp}^+(X, D), J \in \mathcal{J}(X, D, \omega) \}.$$

In the following, given a transverse union $D = \bigcup_{i \in S} D_i$ of smooth real codimension 2 submanifolds $\{D_i\}_{i \in S}$ in $X$, we say $D$ is an SC symplectic divisor in $X$ if $\text{Symp}^+(X, D) \neq \emptyset$. The particular choice of $\omega$ in its deformation equivalence class is not important in GW theory. In [11], we introduced a subclass $\text{AK}(X, D)$ of $\mathcal{J}(X, D)$ such that the projection map

$$\text{AK}(X, D) \longrightarrow \text{Symp}^+(X, D)$$

is a weak homotopy equivalence; see [11, Theorem 2.13]. While the definitions of this section work for any almost complex structure in $\mathcal{J}(X, D)$, the subspace $\text{AK}(X, D)$ is better for analytical purposes. Furthermore, because of the theorem [11, Theorem 2.13] stated above, by restricting to the subclass $\text{AK}(X, D)$, the eventual GW invariants would be invariants of the deformation equivalence class of $\omega$ in $\text{Symp}^+(X, D)$. I expect the same statements to be true for the entire $\mathcal{J}(X, D)$.

For an SC symplectic divisor $D$ in $X$ and $(J, \omega) \in \mathcal{J}(X, D)$, any $J$-holomorphic map $u: (\Sigma, j) \longrightarrow (X, J)$ with smooth domain has image in a minimal stratum $D_I = \bigcap_{i \in I} D_i$, for some $I \subset S$. Then, we say $u$ is a map of depth $I$. In this situation, for every $i \in I$, the pull-back $\bar{\partial}$-operator

$$u^*\bar{\partial}_{N_X D_i} = u^*\bar{\partial}_{N_{D_{I_i} D_i}}$$

defines a holomorphic structure on the pull-back complex line bundle $u^*N_X D_i$. Define

$$\Gamma_{\text{mero}}(\Sigma, u^*N_X D_i) \quad \forall \ i \in I$$

to be the space of non-trivial meromorphic sections of $u^*N_X D_i$ with respect to $u^*\bar{\partial}_{N_X D_i}$. Given a tuple

$$(u, \zeta = (\zeta_i)_{i \in I}, \Sigma ; j), \quad \zeta_i \in \Gamma_{\text{mero}}(\Sigma, u^*N_X D_i) \quad \forall \ i \in I,$$

for each $x \in \Sigma$, the order vector

$$\text{ord}_{u, \zeta}(x) = (\text{ord}_{u, \zeta}^j(x))_{j \in S} \in \mathbb{Z}^S$$

is defined by

$$\text{ord}_{u, \zeta}^j(x) = \text{ord}_{x}(u, D_j) \quad \forall \ j \in S - I \quad \text{and} \quad \text{ord}_{u, \zeta}^j(x) = \text{ord}_{x}(\zeta_j) \quad \forall \ j \in I. \quad \text{(7)}$$

The first item in (7) is the order of tangency of $u$ to $D_j$ at $x$; this is zero if $u(x) \notin D_j$ and is positive otherwise. The second item in (7) only depends on the $\mathbb{C}^*$-equivalence class $[\zeta_j]$ of the section $\zeta_j$.
i.e. changing $\zeta_j$ with a non-zero constant multiple of that does not change $\text{ord}_{\nu, \zeta}^j(x)$.

Before we discuss similar notions for nodal domains, we need to fix some notation. Associated to every $k$-marked nodal domain $(\Sigma, z^1, \ldots, z^k)$, we construct a decorated dual graph of the following sort. This dual graph plays a crucial role in defining and studying log maps.

Let $\Gamma = (V, E, L)$ be a graph with the set of vertices $V$, edges $E$, and legs $L$; the latter, also called flags or roots, are half edges that have a vertex at one end and an open at the other end. Let $E$ be the set of edges with an orientation. Given $e \in E$, let $\bar{e}$ denote the same edge with the opposite orientation. For each $e \in E$, let $v_1(e)$ and $v_2(e)$ in $V$ denote the starting and ending points of the arrow, respectively. For $v, v' \in V$, let $E_{v, v'}$ denote the subset of edges between the two vertices and $E_{v, v'}$ denote the subset of oriented edges from $v$ to $v'$. For every $v \in V$, let $E_v$ denote the subset of oriented edges starting from $v$.

A genus labeling of $\Gamma$ is a function $g: V \to \mathbb{N}$. An ordering of the legs of $\Gamma$ is a bijection $o: L \to \{1, \ldots, |L|\}$. If a graph $\Gamma$ with a genus labeling is connected, the arithmetic genus of $\Gamma$ is

$$g = g_{\Gamma} = \sum_{v \in V} g_v + \text{rank } H_1(\Gamma, \mathbb{Z}),$$

where $H_1(\Gamma, \mathbb{Z})$ is the first homology group of the underlying topological space of $\Gamma$. Figure 2-left illustrates a labeled graph with 2 flags.

Such decorated graphs $\Gamma$ characterize different topological types of nodal marked surfaces

$$(\Sigma, z = (z^1, \ldots, z^k))$$

in the following way. Each vertex $v \in V$ corresponds to a smooth component $\Sigma_v$ of $\Sigma$ with genus $g_v$. Each edge $e \in E$ corresponds to a node $q_e$ obtained by connecting $\Sigma_v$ and $\Sigma_{v'}$ at the points $q_{e_v} \in \Sigma_v$ and $q_{e_{v'}} \in \Sigma_{v'}$, where $e \in E_{v, v'}$ and $\bar{e}$ is an orientation on $e$ with $v_1(e) = v$. The last condition uniquely specifies $e$ unless $e$ is a loop connecting $v$ to itself. Finally, each leg $l \in L$ connected to the vertex $v(l)$ corresponds to a marked point $z^{v(l)} \in \Sigma_{v(l)}$ disjoint from the connecting nodes. If $\Sigma$ is connected, then $g_{\Gamma}$ is the arithmetic genus of $\Sigma$. Thus we have

$$(\Sigma, z) = \prod_{v \in V} (\Sigma_v, z^v_v, q_{e_v}) / \sim, \quad q_e \sim q_{\bar{e}} \quad \forall e \in E,$$

\footnote{We mean a smooth closed oriented surface.}
where \( \bar{z}_v = \bar{z} \cap \Sigma_v \) and \( q_v = \{q_e : e \in E_v\} \quad \forall \ v \in V. \)

We treat \( q_v \) as an un-ordered set of marked points on \( \Sigma_v \). In this situation, we say \( \Gamma \) is the (decorated) dual graph of \( (\Sigma, \bar{z}) \).

A complex structure \( j \) on \( \Sigma \) is a set of complex structures \( (j_v)_v \in V \) on its components. By a (complex) marked nodal curve, we mean a marked nodal real surface together with a complex structure \( (\Sigma, j, \bar{z}) \). Figure 3 illustrates a nodal curve with \( (g_1, g_2, g_3, g_4, g_5) = (0, 2, 0, 1, 0) \) corresponding to Figure 2-left.

Similarly, for nodal marked surfaces mapping into a topological space \( X \), we consider similar decorated graphs where the vertices carry an additional degree labeling \( \text{A} : V \rightarrow H_2(X, \mathbb{Z}) \), \( v \rightarrow A_v \), recording the homology classes of the corresponding maps. Figure 2-right illustrates a dual graph associated to a marked nodal map over the graph on the left.

Assume \( D = \bigcup_{i \in S} D_i \subset X \) is an SC symplectic divisor, \( (J, \omega) \in \mathcal{J}(X, D) \), and \( u = (u_v)_v \in V \) a nodal \( J \)-holomorphic map. Let \( \mathcal{P}(S) \) be the set of subsets of \( S \). In this situation, the dual graph of \( (u, \Sigma) \) carries additional labelings

\[
I : V, E \rightarrow \mathcal{P}(S), \quad v \rightarrow I_v \quad \forall v \in V, \quad e \rightarrow I_e \quad \forall e \in E,
\]

recording the depth of \( u_v \) and \( u(q_e) \), respectively.

**Definition 3.1.** Suppose \( D = \bigcup_{i \in S} D_i \subset (X, \omega) \) is an SC symplectic divisor and \( (J, \omega) \in \mathcal{J}(X, D) \).

A log \( J \)-holomorphic tuple \( (u, [\zeta], \Sigma, j, w) \) consists of a smooth (closed) connected curve \( (\Sigma, j) \), distinct points \( w = \{w^1, \ldots, w^\ell\} \) on \( \Sigma \), a \( (J, j) \)-holomorphic map \( u : (\Sigma, j) \rightarrow (X, J) \) of depth \( I \), and

\[
[\zeta] = ([\zeta_i])_{i \in I} \in \bigoplus_{i \in I} \left( \Gamma_{\text{mero}}(\Sigma, u^* \mathcal{N}_X D_i)/\mathbb{C}^* \right),
\]

such that

\[
\text{ord}_{u, \zeta}(x) = 0 \text{ if } x \notin w.
\]

For all \( i = 1 \ldots \ell \), if \( \text{ord}_{u, \zeta}(w^i) = s_i \in \mathbb{Z}^S \), we say \( (u, [\zeta], \Sigma, j, w) \) is of contact (order) type \( s = (s_1, \ldots, s_\ell) \in (\mathbb{Z}^S)^\ell \).
In particular, if \( u \) is of degree \( A \in H_2(X, \mathbb{Z}) \), Condition (10) implies that
\[
(A \cdot D_j)_{j \in S} = \sum_{i=1}^\ell s_i \in \mathbb{Z}^S.
\]

**Remark 3.2.** For every \( J \)-holomorphic map \( u : (\Sigma, j) \rightarrow (X, J) \) with smooth domain, \( \ell \) distinct points \( w^1, \ldots, w^\ell \) in \( \Sigma \), and \( s_1, \ldots, s_\ell \in \mathbb{Z} \), if \( \text{Im}(u) \subseteq D_i \), up to \( \mathbb{C}^* \)-action there exists at most one meromorphic section \( \zeta_i \in \Gamma_{\text{mero}}(\Sigma, u^*\mathcal{N}_X D_i) \) with zeros/poles of order \( s_i \) at \( w^i \) (and nowhere else).

**Definition 3.3.** Suppose \( D = \bigcup_{i \in S} D_i \subset (X, \omega) \) is an SC symplectic divisor, \( (J, \omega) \in J(X, D) \), and
\[
C \equiv (\Sigma, j, \tilde{z}) = \prod_{v \in V} C_v \equiv (\Sigma_v, j_v, \tilde{z}_v, q_v)/\sim, \quad q_e \sim q_{e'} \quad \forall \; e \in \mathcal{E},
\]
is a connected nodal \( k \)-marked curve with smooth components \( C_v \) and dual graph \( \Gamma = \Gamma(\mathcal{V}, \mathcal{E}, \mathbb{L}) \) as in (3). A pre-log \( J \)-holomorphic map of contact type \( s = (s_i)_{i=1}^k \in (\mathbb{Z}^S)^k \) from \( C \) to \( X \) is a collection
\[
f \equiv (f_v \equiv (u_v, [\zeta_v], C_v))_{v \in \mathcal{V}},
\]
such that
\begin{enumerate}
  \item for each \( v \in \mathcal{V} \), \( (u_v, [\zeta_v]) = ([\zeta_{v,i}])_{i \in I_v}, \Sigma_v, j_v, z_v \cup \{q_v\} \) is a log \( J \)-holomorphic tuple as in Definition 3.1;
  \item \( u_v(q_e) = u_{v'}(q_e) \in X \) for all \( v, v' \in \mathcal{V} \) and \( e \in \mathcal{E}_{v,v'} \);
  \item \( s_e \equiv \text{ord}_{u_v, \zeta_v}(q_e) = -\text{ord}_{u_{v'}, \zeta_{v'}}(q_e) \equiv -s_e \) for all \( v, v' \in \mathcal{V} \) and \( e \in \mathcal{E}_{v,v'} \);
  \item and \( \text{ord}_{u_v, \zeta_v}(z^i) = s_i \) for all \( v \in \mathcal{V} \) and \( z^i \in \tilde{z}_v \).
\end{enumerate}

**Remark 3.4.** For every \( v \in \mathcal{V} \) and \( e \in \mathcal{E}_{v,v'} \), let
\[
s_e = (s_{e,i})_{i \in S} = \left((\text{ord}_{u_v, \zeta_v}(q_e))_{i \in S-I_v}, (\text{ord}_{\zeta_{v,i}}(q_e))_{i \in I_v}\right) \in \mathbb{Z}^S.
\]
For \( e \in \mathcal{E}_{v,v'} \), if \( u_v \) and \( u_{v'} \) have image in \( D_{I_v} \) and \( D_{I_{v'}} \), respectively, by Condition (2) we have
\[
u(q_e) = u_v(q_e) = u_{v'}(q_e) \in D_{I_v} \cap D_{I_{v'}} = D_{I_v \cup I_{v'}};
\]
i.e. \( I_e \supset I_v \cup I_{v'} \) (see (3) for the notation). If \( i \in S \setminus I_v \cup I_{v'} \), by (7), we have
\[
s_{e,i} \geq 0.
\]
Therefore, by Condition (3) they are both zero, i.e.
\[
I_e = I_v \cup I_{v'} \quad \text{and} \quad s_e \in \mathbb{Z}^{I_e} \times \{0\}^{S-I_e} \subset \mathbb{Z}^S \quad \forall \; e \in \mathcal{E}_{v,v'}.
\]

The space of (equivalence classes) of pre-log maps of a fixed combinatorial type is too big; see [7: Example 3.6]. In Definition 3.5 below, we will take out a subspace that would give us a nice compactification with the correct expected dimension. In the future, we plan to completely address the question (?) in the introduction, using using this compactification. Before we state the definition, we need to introduce some combinatorial structures, that are explicitly defined in terms of the
decorated dual graph of a pre-log map.

Corresponding to a decorated dual graph \( \Gamma = \Gamma(V, E, L) \) as in Definition 3.3 and an arbitrary orientation \( O \equiv \{ \xi \}_{e \in E} \subset E \) on the edges, we define a homomorphism of \( \mathbb{Z} \)-modules

\[
\mathbb{D} = \mathbb{D}(\Gamma) \equiv \mathbb{Z}^E \oplus \bigoplus_{v \in V} \mathbb{Z}^{I_v} \xrightarrow{\varrho = \varrho_O} \mathbb{T} = \mathbb{T}(\Gamma) \equiv \bigoplus_{e \in E} \mathbb{Z}^{I_e}
\]

(13)
in the following way. For every \( e \in E \), let

\[
\varrho(1_e) = s_e \in \mathbb{Z}^{I_e},
\]

where \( 1_e \) is the generator of \( \mathbb{Z}^e \) in \( \mathbb{Z}^E \) and \( e \) is the chosen orientation on \( e \) in \( O \). In particular, \( \varrho(1_e) = 0 \) for any \( e \) with \( I_e = \emptyset \) (a classical node). Similarly, for every \( v \in V \) and \( i \in I_v \), let \( 1_{v,i} \) be the generator of the \( i \)-th factor in \( \mathbb{Z}^{I_v} \), and define

\[
\varrho(1_{v,i}) = \xi_{v,i} \in \bigoplus_{e \in E} \mathbb{Z}^{I_e}
\]

to be the vector which has \( 1_{e,i} \in \mathbb{Z}^{I_e} \subset \mathbb{Z}^S \) in the \( e \)-th factor, if \( v = v_1(e) \) and \( e \) is not a loop, it has \(-1_{e,i} \in \mathbb{Z}^{I_e} \) in the \( e \)-th factor, if \( v = v_2(e) \) and \( e \) is not a loop, and is zero elsewhere. This is well-defined by the first equality in (12). Let

\[
\Lambda = \Lambda(\Gamma) = \text{image}(\varrho), \quad K = K(\Gamma) = \text{Ker}(\varrho) \quad \text{and} \quad CK = CK(\Gamma) = \mathbb{T}/\Lambda = \text{coker}(\varrho).
\]

(14)

By Definition 3.3 (3), the \( \mathbb{Z} \)-modules \( \Lambda, K, \) and \( CK \) are independent of the choice of the orientation \( O \) on \( E \) and are invariants of the decorated graph \( \Gamma \). In particular,

\[
K = \left\{ \left( (\lambda_e)_{e \in E}, (s_v)_{v \in V} \right) \in \mathbb{Z}^E \oplus \bigoplus_{v \in V} \mathbb{Z}^{I_v} : s_v - s_{v'} = \lambda_e s_e, \quad \forall \ v, v' \in V, \ e \in E_{v,v'} \right\}.
\]

In this equation, via the first identity in (12) and the inclusion \( \mathbb{Z}^{I_v} \cong \mathbb{Z}^{I_e} \times \{0\}^{I_v-I_e} \subset \mathbb{Z}^{I_e} \) we think of \( s_v \) as a vector also in \( \mathbb{Z}^{I_e} \), for all \( e \in E_v \). For any field of characteristic zero \( F \), let

\[
\mathbb{D}_F = \mathbb{D} \otimes_\mathbb{Z} F, \quad \mathbb{T}_F = \mathbb{T} \otimes_\mathbb{Z} F, \quad \Lambda_F = \Lambda \otimes_\mathbb{Z} F, \quad K_F = K \otimes_\mathbb{Z} F, \quad \text{and} \quad CK_F = CK \otimes_\mathbb{Z} F,
\]

(15)

be the corresponding \( F \)-vector spaces, and \( \varrho_F \colon \mathbb{D}_F \longrightarrow \mathbb{T}_F \) be the corresponding \( F \)-linear map. Via the exponentiation map, let

\[
\exp(\Lambda_C) \subset \prod_{e \in E} (\mathbb{C}^*)^{I_e}
\]

be the subgroup corresponding to the sub-Lie algebra \( \Lambda_C \subset T_C \), and denote the quotient group by \( \mathcal{G} = \mathcal{G}(\Lambda) = \exp(CK_C) \). In [7, Lemma 3.7], to every (equivalence class of) pre-log map \( f \) as in Definition 3.3 map we associate a group element

\[
\text{ob}_\Gamma(f) \in \mathcal{G}(\Gamma).
\]

(16)

**Definition 3.5.** Suppose \( D = \bigcup_{e \in E} D_e \subset (X, \omega) \) is an SC symplectic divisor and \( (J, \omega) \in \mathcal{J}(X, D) \). A log \( J \)-holomorphic is a pre-log \( J \)-holomorphic map \( f \) with the decorated dual graph \( \Gamma \) such that
We only require such functions to exist for $\Gamma$ in order for $\Gamma$ in order for

$$s: \mathcal{V} \rightarrow \mathbb{R}^S, \ v \mapsto s_v, \ \text{and} \ \lambda: \mathcal{E} \rightarrow \mathbb{R}_+, \ e \mapsto \lambda_e,$$

such that

(a) $s_v \in \mathbb{R}^I \times \{0\}^{S-I}$ for all $v \in \mathcal{V},$

(b) $s_{v_2(\epsilon)} - s_{v_1(\epsilon)} = \lambda_e s_\epsilon$ for every $\epsilon \in \mathbb{E};$

(2) and $\text{ob}_\Gamma(f) = 1 \in \mathcal{G}(\Gamma).$

Remark 3.6. Note that the functions $s$ and $\lambda$ in (1) are not part of the defining data of a log map. We only require such functions to exist for $\Gamma$ in order for $\Gamma$ to define a log map. From the tropical perspective of \cite[Definition 2.5.3]{2}, the existence of such a function $s$ is equal to the existence of a suitable tropical map from the tropical curve associated to $(\Gamma, \lambda)$ into $\mathbb{R}^S_{\geq 0}.$

Two marked log maps are equivalent if they are related by a “reparametrization of the domain”. A marked log map is stable if it has a finite “automorphism group”. We denote the space of equivalence classes of stable $k$-marked degree $A$ genus $g$ log maps of contact type $s$ by $\overline{\mathcal{M}}_{g,s}^{\log}(X, D, A).$ Given $s \in (\mathbb{Z}^S)^k,$ it follows from Remark 3.2 that for every $k$-marked stable nodal map $f$ in $\overline{\mathcal{M}}_{g,k}(X, A)$ with dual graph $\Gamma$ and a choice of decorations $\{s_\epsilon\}_{\epsilon \in \mathbb{E}}$ satisfying $s_\epsilon = -s_\epsilon'$ for all $\epsilon \in \mathbb{E},$ there exists at most one element $f_{\log} \in \overline{\mathcal{M}}_{g,s}^{\log}(X, D, A)$ lifting $f$ with this decorated dual graph. Furthermore, $f_{\log}$ is stable if and only if $f$ is stable (and the automorphism groups are often the same). In \cite[7], we prove the following compactness result.

Theorem 3.7 (\cite[Theorem 1.1]{7}). Suppose $D \subseteq X$ is an SC symplectic divisor. If $(J, \omega) \in \mathcal{AK}(X, D)$ or if $(J, \omega)$ is Kähler, then for every $A \in H_2(X, \mathbb{Z}),$ $g, k \in \mathbb{N},$ and $s \in (\mathbb{Z}^S)^k,$ the Gromov sequential convergence topology on $\overline{\mathcal{M}}_{g,k}(X, A)$ lifts to a compact sequential convergence topology on $\overline{\mathcal{M}}_{g,s}^{\log}(X, D, A)$ such that the forgetful map

$$\nu: \overline{\mathcal{M}}_{g,s}^{\log}(X, D, A) \rightarrow \overline{\mathcal{M}}_{g,k}(X, A)$$

(17)

is a continuous local embedding. In particular, $\overline{\mathcal{M}}_{g,s}^{\log}(X, D, A)$ is metrizable. If $g = 0,$ then (17) is a global embedding.

Except for \cite[Proposition 4.11]{7}, proof of \cite[Theorem 1.1]{7} works for arbitrary $(J, \omega) \in \mathcal{J}(X, D).$ We are working to extend this compactness result to arbitrary $(J, \omega) \in \mathcal{J}(X, D).$

For $s \in (\mathbb{N}^S)^k,$ a simple calculation shows that the expected real dimension of $\overline{\mathcal{M}}_{g,s}^{\log}(X, D, A)$ is

$$2(c_1^{TX(-\log D)}(A) + (\dim_C X - 3)(1 - g) + k) = 2(c_1^{TX}(A) + (\dim_C X - 3)(1 - g) + k - A \cdot D),$$

where $TX(-\log D)$ is the log tangent bundle defined in \cite[14]{14}. Moreover, the only stratum with top virtual dimension is $\mathcal{M}_{g,s}^{\log}(X, D, A).$ The expected complex codimension of any other stratum $\mathcal{M}_{g,s}^{\log}(X, D, A)$ is equal to $\dim_{\mathbb{R}} \mathbb{K}_\mathbb{R}(\Gamma)$ and is positive because of Definition 3.5(1) see \cite[Lemma 3.19]{7}.

Examples 3.8 and 3.9 below illustrate the necessity of Conditions (1) and (2) in Definition 3.5 to get a compactification of the correct expected dimension.
Example 3.8 ([7], Example 3.6). Let \( X = \mathbb{P}^2 \) with projective coordinates \([x_1, x_2, x_3]\) and let \( D = D_1 \cup D_2 \) be a transverse union of two hyperplanes (lines). For

\[
g = 0, \quad s = ((3, 2), (0, 1)) \in (\mathbb{N}^{(1, 2)})^2, \quad \text{and} \quad A = [3] \in H_2(X, \mathbb{Z}) \cong \mathbb{Z},
\]

the (virtually) main stratum \( \mathcal{M}_{0, s}(X, D, [3]) \) is a manifold of complex dimension 4. If \( D_1 = (x_1 = 0) \) and \( D_2 = (x_2 = 0) \), every element in \( \mathcal{M}_{0, s}(X, D, [3]) \) is equivalent to a holomorphic map of the form

\[
[z, w] \mapsto [z^3, z^2 w, a_2 z^2 w + a_1 z w^2 + a_0 w^2]. \tag{18}
\]

Let \( \Gamma \) be the dual graph with three vertices \( v_1, v_2, v_3 \), and two edges \( e_1, e_2 \) connecting \( v_1 \) to \( v_3 \) and \( v_2 \) to \( v_3 \), respectively. Furthermore, choose the orientations \( e_1 \) and \( e_2 \) to end at \( v_3 \) and assume

\[
I_{v_1} = I_{v_2} = \emptyset, \quad I_{v_3} = \{1, 2\}, \quad s_{e_1} = (2, 1), \quad s_{e_2} = (1, 1), \quad A_{v_1} = [2], \quad A_{v_2} = [1];
\]

see Figure 4. Note that \( u_{v_3} \) is map of degree 0 from a sphere with three special points, two of which are the nodes connecting \( \Sigma_{v_3} \) to \( \Sigma_{v_1} \) and \( \Sigma_{v_2} \) and the other one is the first marked point \( z^1 \) with contact order \((3, 2)\). The second marked point with contact order \((0, 1)\) lies on \( \Sigma_{v_1} \). A simple calculation shows that the stratum of pre-log maps of type \( \Gamma \), denoted by \( \mathcal{M}_0^{\mathsf{plog}}(X, D, [3])_\Gamma \), is also a manifold of complex dimension 4 (something we do not want). Image of \( u_2 \) could be any line different from \( D_1 \) and \( D_2 \) passing through \( D_{12} \), and every such \( u_1 \) is equivalent to a holomorphic map of the form

\[
[z, w] \mapsto [z^2, z w, a_2 z^2 w + a_1 z w + a_0 w^2].
\]

This \( \Gamma \) does not satisfy Definition 3.5(1). Since \( I_{v_1} = I_{v_2} = \emptyset \), we should have \( s_{v_1} = s_{v_2} = (0, 0) \). Then Definition 3.5(1) requires \( s_{e_1} = (2, 1) \) and \( s_{e_2} = (1, 1) \) to be positive multiples of \( s_{v_3} \), which is impossible. A straightforward calculation shows that the line component \( u_{v_2} \) in any limit of \( (18) \) with a component \( u_{v_3} \) as in Figure 4 should lie in \( D_1 \). Then the function \( s: \mathbb{V} \rightarrow \mathbb{R}^2 \) given by \( s_{v_1} = (0, 0), \ s_{v_2} = (1, 0), \) and \( s_{v_3} = (2, 1) \) satisfies Definition 3.5(1). □

Example 3.9 ([7], Example 3.15)). Let

\[
X = \mathbb{P}^3, \quad D_1 \cup D_2 = \mathbb{P}^2 \cup \mathbb{P}^2, \quad A = [2d] \in H_2(X, \mathbb{Z}) \cong \mathbb{Z}, \quad g = (d - 1)^2;
\]

\[
s = ((1, 0), \ldots, (1, 0), (0, 1), \ldots, (0, 1)) \in (\mathbb{Z}^{(1, 2)})^{4d},
\]

and \( \Gamma \) be the decorated dual graph illustrated in Figure 5. This decorated graph satisfies all the necessary combinatorial conditions of a log map; in particular, the function \( s: \mathbb{V} \rightarrow \mathbb{R}^2 \) given by \( s_{v_1} = (1, 0) \) and \( s_{v_2} = (0, 1) \) satisfies Definition 3.5(1). Every element of \( \mathcal{M}_0^{\mathsf{plog}}(X, D, A)_{\Gamma} \) is supported on two generic degree \( d \) plane curves in \( D_1 \) and \( D_2 \) intersecting at \( d \) points along
For every log map \( f \) (a neighborhood of the origin in \( a \)) an (possibly non-irreducible and non-reduced) affine toric variety could be quite complicated. In the following, we describe this space and show that it is essentially out in direction of Figure 5: A decorated dual graph defining a stratum of log maps in \( \mathbb{P}^3 \) relative to two hyperplanes.

\[ D_{12} \]. The expected complex dimension of \( \mathcal{M}_{g,s}(X, D, A) \) and \( \mathcal{M}^{\text{log}}_{g,s}(X, D, A) \) are \( 8d \) and \( 9d-2 \), respectively. The latter is bigger than the former if \( d>2 \).

Orient each edge such that \( v_1(\xi_i) = v_1 \) for all \( i = 1, \ldots, d \). Then \( \Lambda \) is generated by the vectors \( s_{\xi_1}, \ldots, s_{\xi_d}, \xi_{v_1} = \xi_{v_1,1}, \xi_{v_2} = \xi_{v_2,2} \), such that the only relation is

\[
\xi_{v_1} + \xi_{v_2} + (s_{\xi_1} + \ldots + s_{\xi_d}) = 0.
\]

We conclude that the group \( G(\Gamma) \) is \((d-1)\)-dimensional. Therefore, the subset of log maps

\[
\mathcal{M}^{\text{log}}_{g,s}(X, D, A) \subset \mathcal{M}^{\text{log}}_{g,s}(X, D, A)
\]

is of expected complex dimension \((9d-2)-(d-1) = 8d-1 < 8d\).

Going back to the question \((\star)\) in the introduction, a main step in establishing \((\star)\) is to prove a gluing theorem for smoothing the nodes. For a classical nodal \( J \)-holomorphic map with \(|E|\) nodes, the space of gluing parameters is a neighborhood of the zero in \( \mathbb{C}^E \). For a log map \( f \) as in \((\Pi)\), the gluing procedure involves a simultaneous smoothing of the nodes, together with pushing \( u_v \) out in direction of \( \zeta_{v,i} \) for some \( v \in V \) and \( i \in I_v \). Thus, a priori, the space of gluing parameters could be quite complicated. In the following, we describe this space and show that it is essentially (a neighborhood of the origin in \( a \)) an (possibly non-irreducible and non-reduced) affine toric variety.

For every log map \( f \in \overline{\mathcal{M}}_{g,s}(X, D, A) \) with the decorated dual graph \( \Gamma \), fix a representative

\[
(u_v, (\zeta_{v,i})_{i \in I_v}, C_v = (\Sigma_v, [v], z_v))_{v \in V}
\]

and a set of local coordinates \( \{z_\xi\}_{\xi \in \mathbb{E}} \) around the nodes. For each \( v \in V \), \( \xi \in \mathbb{E}_v \), and \( i \in I_v \), \( \zeta_{v,i} \) has a local expansion of the form

\[
\eta_{\xi,i} z_\xi^{s_{\xi,i}} + \text{higher order terms},
\]

where \( 0 \neq \eta_{\xi,i} \in N_X D_{i,v}(q_v) \). Similarly, for \( i \in S-I_v \), the map \( u_v \) has a well-defined non-zero \( s_{\xi,i} \)-th derivative \( \eta_{\xi,i} \in N_X D_{i,v}(q_v) \) (with respect to the coordinate \( z_\xi \)) in the normal direction to \( D_i \) at the nodal point \( q_v \). Since \( \text{obr}(f) = 1 \), by the proof of \([7, \text{Lemma 3.7}]\), we can choose the representatives \( \zeta_{v,i} \) and \( z_\xi \) such that the leading coefficient vectors \( \eta_\xi = (\eta_{\xi,i})_{i \in S} \) satisfy

\[
\eta_\xi = \eta_\xi \quad \forall \, \xi \in \mathbb{E}.
\]

For every \( v \in V \) and \( i \in S-I_v \), let \( t_{v,i} = 1 \) in \((21)\). Then the space of gluing parameters for \( f \) is a
sufficiently small neighborhood of the origin in
\[
\mathcal{G}_\Gamma = \left\{ \left( (\epsilon_e)_{e \in E}, (t_{v,i})_{v \in V, i \in I_v} \right) \in \mathbb{C}^E \times \prod_{v \in V} \mathbb{C}^{I_v} : \begin{array}{l}
\epsilon_e^{-1} t_{v,i} = t_{v',i} \\
\forall v, v' \in \mathcal{V}, \epsilon_e \in \mathbb{E}, \left( t_{v,i} \right) v \in \mathcal{V}, i \in \mathcal{I}_v, \quad s_{e,i} \geq 0 \end{array} \right\} \subset \mathbb{C}^E \times \prod_{v \in V} \mathbb{C}^{I_v}. \tag{20}
\]

The complex numbers \(\epsilon_e\) are the gluing parameters for the nodes of \(\Sigma\) and \(t_{v,i}\) are the parameters for pushing \(u_v\) out in the direction of \(\zeta_{v,i}\). In \([8]\), given a set of representatives \(\{z_e\}_{e \in \mathbb{E}}, \{\zeta_{v,i}\}_{v \in \mathcal{V}, i \in \mathcal{I}_v}\) satisfying \((19)\) and a sufficiently small \((\epsilon, t) \equiv (\epsilon_e, t_{v,i})_{e \in \mathbb{E}, v \in \mathcal{V}, i \in \mathcal{I}_v} \in \mathcal{G}_\Gamma\), we will construct a pre-gluing almost log map \(\tilde{f}_{\epsilon, t}\) and\(^{10}\) show that there is an actual log pseudo-holomorphic map “close” to it.

Let
\[
\rho^\vee : T^\vee \rightarrow \mathbb{D}^\vee
\]
be dual of the linear map associated to \(\Gamma\) in \((13)\) (for any fixed choice of orientation \(O\) on \(E\)). For the kernel subspace \(K = \ker(\rho) \subset \mathbb{D}\) as in \((14)\), let
\[
K^\perp = \{ m \in \mathbb{D}^\vee : \langle m, \alpha \rangle = 0 \quad \forall \alpha \in K \} \subset \mathbb{D}^\vee.
\]
Then \(\text{Im}(\rho^\vee) \subset K^\perp\) is finite index sub-lattice. Let
\[
\sigma = \sigma(\Gamma) := K_\mathbb{R} \cap \left( \mathbb{R}^E_\geq \oplus \bigoplus_{v \in \mathcal{V}} \mathbb{R}^{I_v}_{\geq 0} \right). \tag{21}
\]
In \([7, \text{Lemma 3.15}]\), we show that \(\sigma\) is a maximal convex rational polyhedral cone in \(K_\mathbb{R}\). Every element of the interior of this cone is a tuple
\[
((\lambda_e)_{e \in \mathbb{E}}, (s_{v,i})_{v \in \mathcal{V}, i \in \mathcal{I}_v})
\]
satisfying Definition \(3.5(1)\). In other words, \(\sigma\) is the closure of the set of all tuples satisfying Definition \(3.5(1)\). Let \(Y_\sigma\) be the toric variety with toric fan \(\sigma \subset K_\mathbb{R}\). In \([7, \text{Proposition 3.24}]\), we prove that the space of gluing parameters \(\mathcal{G}_\Gamma\) in \((20)\) is a possibly non-irreducible and non-reduced toric sub-variety of \(\mathbb{C}^E \times \prod_{v \in \mathcal{V}} \mathbb{C}^{I_v}\) that is isomorphic to \(|K^\perp / \text{image}(\rho^\vee)|\) copies of (the reduced and irreducible affine toric variety) \(Y_\sigma\), counting with multiplicities\(^{11}\). Replacing \(\{z_e\}_{e \in \mathbb{E}}\) and \(\{\zeta_{v,i}\}_{i \in \mathcal{I}_v}\) with another choice satisfying \((19)\) corresponds to a torus action on \(\mathcal{G}_\Gamma\).

If \(D\) is smooth (i.e. \(|S| = 1\)) and \(s \in \mathbb{N}^k\), in \([7, \text{Proposition 3.20}]\), we prove that there exists a natural projection map
\[
\pi : \overline{M}_{g,s}^{rel}(X, D, A) \rightarrow \overline{M}_{g,s}^{log}(X, D, A).
\]
This is expected, since the notion of nodal log map involves more \(\mathbb{C}^*\)-quotients on the set of meromorphic sections. In the algebraic case, \([3, \text{Theorem 1.1}]\) shows that an algebraic analogue of this projection map induces an equivalence of the virtual fundamental classes. We expect the same to hold for invariants arising from our log compactification.

\(^{10}\) After taking care of the transversality issue.

\(^{11}\) We don't know of any example, arising from such dual graphs, such that the multiplicities are bigger than 1.
4 Log maps into smoothable SC varieties

In this section, for a compact smoothable SC symplectic variety \((X_\emptyset, \omega_\emptyset) = \bigcup_{i \in S}(X_i, \omega_i)\), an appropriate choice of almost complex structure \(J_\emptyset\) on \(X_\emptyset\), \(g, k \in \mathbb{N}\), and \(A \in H_2(X_\emptyset)\), we define the moduli space genus \(g\) \(k\)-marked degree \(A\) log \(J\)-holomorphic maps

\[
\overline{M}_{g,k}^{\log}(X_\emptyset, A)
\]  

(22)

which would address the first part of the question (**) in the introduction. By [12, Theorem 2.7], every compact smoothable SC variety sits as an SC symplectic divisor inside a smooth symplectic manifold \(Z\). Thus, we can essentially use the log moduli spaces of the previous section and restrict to the central fiber \(X_\emptyset\) to define the compactification (22); however, we will give an intrinsic definition without reference to any smoothing.

For an SC symplectic variety \((X_\emptyset, \omega_\emptyset) = \bigcup_{i \in S}(X_i, \omega_i)\) (cf. see [11, Section 2.1]), let \(\mathcal{J}(X_\emptyset, \omega_\emptyset)\) be the space of tuples \(J_\emptyset \equiv (J_i)_{i \in S}\) such that \(J_i \in \mathcal{J}(X_i, X_i; \omega_i)\), for each \(i \in S\), and

\[
J_i|_{T X_i} = J_j|_{T X_j} \quad \forall \ i, j \in S.
\]  

(23)

Let \(\text{Symp}^+(X_\emptyset)\) be the space of all symplectic structures \(\omega_\emptyset = (\omega_i)_{i \in S}\) with respect to which \(X_\emptyset\) is an SC symplectic variety and

\[
\mathcal{J}(X_\emptyset) = \{ (J_\emptyset, \omega_\emptyset) \colon \omega_\emptyset \in \text{Symp}^+(X_\emptyset), \ J_\emptyset \in \mathcal{J}(X_\emptyset, \omega_\emptyset) \}. 
\]

In [12], associated to every SC symplectic variety \(X_\emptyset\) we define a (deformation equivalence class of) complex line bundle over its singular locus \(O_{X_\emptyset}(X_\emptyset) \rightarrow X_\emptyset\) such that

\[
O_{X_\emptyset}(X_\emptyset)|_{X_i} = \bigotimes_{i \in I} N_{X_{I \setminus i}, X_I} \otimes \bigotimes_{i \in S \setminus I} O_{X_I}(X_{I \cup \{i\}}) \quad \forall \ I \subset S, \ |I| \geq 2,
\]

where \(O_{X_I}(X_{I \cup \{i\}})\) is the line bundle associated to the smooth divisor \(X_{I \cup \{i\}} \subset X_I\). By [12, Theorem 2.7], an SC symplectic variety is symplectically smoothable if and only if \(O_{X_\emptyset}(X_\emptyset)\) is trivializable. The last condition is the direct analogue of the d-semistability (necessary but not sufficient) condition of Friedman in the complex case; see [13, Definition (1.13)] .

In [9], in order to define the moduli space \(\overline{M}_{g,k}^{\log}(X_\emptyset, A)\), we will restrict to a subset \(\mathcal{J}_*(X_\emptyset) \subset \mathcal{J}(X_\emptyset)\) whose elements \(J_\emptyset\) have the following crucial property. For every \(i \in I \subset S\) with \(|I| \geq 2\), as in the previous section, (the \(i\)-th component \(J_i\) of) every \(J_\emptyset \in \mathcal{J}(X_\emptyset)\) induces a complex structure \(i_{I;i}\) and a \(\bar{\partial}\)-operator \(\bar{\partial}_{I;i}\) on the normal bundle

\[
N_{I;i} = N_{X_{I \setminus i}, X_I}
\]

of the smooth divisor \(X_I\) in \(X_{I \setminus i}\). Let \(J_I = J_i|_{T X_i}\) for any \(i \in I\); this is well-defined by (23). For any \(J_I\)-holomorphic map \(u \colon (\Sigma, i) \rightarrow X_I\) and \(i \in I\), \(u^* \bar{\partial}_{I;i}\) defines a holomorphic structure on \(u^*N_{I;i}\). For \(i \in S \setminus I\), let

\[
O_{\Sigma}(u^{-1}(X_{I \cup i}))
\]

be the holomorphic line bundle corresponding to the divisor

\[
\sum_{x \in u^{-1}(X_{I \cup i})} \ord_x(u, X_{I \cup i}) x.
\]
The latter admits a canonical (\(\mathbb{C}^*\)-equivalence class of) meromorphic section \(s_{I;i}\) with zeros/poles of the order \(\ord X_j(u, X_{I,j})\) at each \(x \in u^{-1}(X_{I,j})\). By assumption,

\[
u^*\mathcal{O}_{X_{\emptyset}}(X_{\emptyset}) \cong \bigotimes_{i \in I} \nu^*\mathcal{N}_{X_{i-1},X_i} \otimes \bigotimes_{i \in S-I} \mathcal{O}_{S}(u^{-1}(X_{I,j})) \tag{24}\]

is a degree 0 holomorphic line bundle; i.e. it defines an element of \(\text{Pic}^0(\Sigma, j)\). In [9], we will define \(\mathcal{J}_*(X_{\emptyset})\) in a way that for every \(J_{\emptyset} \in \mathcal{J}_*(X_{\emptyset})\), \((24)\) would be the trivial holomorphic line bundle.

For a set \(S\) and a group \(R\), let

\[
R^S_\bullet = \{ r = (r_i)_{i \in S} : \sum_{i \in S} r_i = 0 \} \subset R^S.
\]

For every \(i \in S\), the projection

\[
\pi_i : R^S_\bullet \rightarrow R^{S-i}, \quad (r_i)_{i \in S} \mapsto (r_i)_{i \in S-i} \tag{25}
\]

is an isomorphism. For every subset \(S' \subset S\), the inclusion homomorphism \(R^{S'} \hookrightarrow R^S\) restricts to an inclusion homomorphism \(R^{S'}_\bullet \hookrightarrow R^S_\bullet\).

Assume \(X_{\emptyset} = \bigcup_{i \in S} X_i\) is a smoothable SC symplectic variety, \((J_{\emptyset}, \omega_{\emptyset}) \in \mathcal{J}_*(X_{\emptyset})\), and

\[
\mathfrak{s} = (s_i = (s_{ij})_{j \in S})_{i \in [\ell]} \in (\mathbb{Z}^S_\bullet)^\ell.
\]

For every \(I \in S\) with \(|I| \geq 2\) and \(i \in I\), let

\[
(u : (\Sigma, j) \rightarrow (X_I, J_I), ([\zeta_{I,a}])_{a \in I-I-i}, w^1, \ldots, w^\ell)
\]

be a log tuple of the contact order type \(\pi_i(\mathfrak{s})\) for the target space \((X_i, X_{i,j}, J_i)\) in the sense of Definition 3.1. Since \((24)\) is holomorphically trivial, there exists a (unique) \(\mathbb{C}^*\)-equivalence class

\[
[\zeta_{I,i}] \in \Gamma_{\text{mero}}(\Sigma, u^*\mathcal{N}_{I,i})/\mathbb{C}^*
\]

such that \((u, ([\zeta_{I,a}])_{a \in I-I-i}, \Sigma, j, w^1, \ldots, w^\ell)\) is a log tuple of the contact order \(\pi_j(\mathfrak{s})\) with respect to the target space \((X_j, X_{j,j}, J_j)\), for any \(j \in S\). In fact, \([\zeta_{I,i}]\) is the unique \(\mathbb{C}^*\)-equivalence class such that

\[
\bigotimes_{a \in I} \zeta_{I,a} \otimes \bigotimes_{a \in S-I} s_{I,a}
\]

is a constant section. The relation

\[
(u, ([\zeta_{I,a}])_{a \in I-I-j}, \Sigma, j, w^1, \ldots, w^\ell) \sim (u, ([\zeta_{I,a}])_{a \in I-I-i}, \Sigma, j, w^1, \ldots, w^\ell) \quad \forall \ i, j \in S,
\]

defines an equivalence relation between depth \(I\) log \(J_j\)-holomorphic tuples of contact order \(\pi_j(\mathfrak{s})\) in \((X_j, X_{j,j})\), for all \(j \in I\). We denote the equivalence class by

\[
(u, ([\zeta_{I,a}])_{a \in I}, \Sigma, j, w^1, \ldots, w^\ell)_\bullet \tag{26}
\]

and call it a depth \(I\) log \(J_{\emptyset}\)-holomorphic tuple of contact order \(\mathfrak{s}\).
Definition 4.1. Suppose $X_0 = \bigcup_{i \in S} X_i$ is a smoothable SC symplectic variety, $(J_0, \omega_0) \in \mathcal{J}_*(X_0)$, and
\[ C \equiv (\Sigma, j, \bar{z}) = \prod_{v \in V} C_v \equiv (\Sigma_v, j_v, \bar{z}_v, q_v)/\sim, \quad q_v \sim q_{\bar{v}} \quad \forall \bar{v} \in \mathbb{E}, \]
is a nodal $k$-marked genus $g$ curve with smooth components $C_v$ and dual graph $\Gamma = \Gamma(V, E, L)$ as in (8). A pre-log $J_0$-holomorphic map from $C$ to $X$ is a collection
\[ f \equiv \left( (f_v \equiv (u_v, [\zeta_v], C_v))_{v \in V, |I_v| \geq 2}, (f_v \equiv (u_v, C_v))_{v \in V, |I_v| = 1} \right), \quad C_v = (\Sigma_v, j_v, \bar{z}_v) \quad \forall v \in V, \quad (27) \]
such that
1. for each $v \in V$ with $I_v = \{i\}$, $u_v : (\Sigma_v, j_v) \rightarrow (X_i, J_i)$ is a marked $J_i$-holomorphic map of depth \{i\} and $\bar{z}_v \subset \Sigma_v - u_v^{-1}(X_{i; 0})$,
2. for each $v \in V$ with $|I_v| \geq 2$, $(u_v, [\zeta_v] = ([\zeta_i])_{i \in I_v}, \Sigma_v, j_v, q_v)$ is a depth $I_v$ log $J_0$-holomorphic tuple of some contact order type
\[ (s_v \equiv \text{ord}_{u_v, [\zeta_v]}(q_v))_{\bar{v} \in \mathbb{E}_v} \subset (\mathbb{Z}^S)_{\bar{v}} \]
as in (26),
3. $u_v(q_v) = u_{v'}(q_v) \in X$ for all $\bar{v} \in \mathbb{E}_{v,v'}$;
4. $s_v = -s_{\bar{v}} \in \mathbb{Z}^S$ for all $v, v' \in V$ and $\bar{v} \in \mathbb{E}_{v,v'}$.

Note that Conditions (1)-(2) and Equation (10) imply that $\text{ord}_{u_v, [\zeta_v]}(z^i) = 0$ for all $v \in V$ and $z^i \in \bar{z}_v$. In other words, $\bar{z}$ is an ordered set of the classical-type marked points. The space of (equivalence classes) of pre-log maps of a fixed combinatorial type is too big. In Definition 4.2 below, we take out a subspace that would give us a nice compactification, suitable for addressing (★★). Similarly to the previous section, in order to define the notion of log map into an SC symplectic variety, we need to introduce some combinatorial structures associated the decorated dual graph of a pre-log map.

Corresponding to a decorated dual graph $\Gamma = \Gamma(V, E, L)$ as in Definition 4.1 and an arbitrary orientation $O \equiv \{\xi\}_{\bar{v} \in \mathbb{E}} \subset \mathbb{E}$ on the edges, the map $\varphi$ defined in (13) restricts to a homomorphism
\[ D_* \equiv \mathbb{Z}^F \oplus \bigoplus_{v \in V} \mathbb{Z}^{I_v} \xrightarrow{\varphi_*} T_* \equiv \bigoplus_{\bar{v} \in \mathbb{E}} \mathbb{Z}^{L_{\bar{v}}} \equiv \bigoplus_{\bar{v} \in \mathbb{E}} \mathbb{Z}^L_{\bar{v}}. \quad (28) \]

Let
\[ \Lambda_* = \text{image}(\varphi_*), \quad \mathbb{K}_* = \text{Ker}(\varphi_*) \quad \text{and} \quad \mathbb{C}K_* = T_*/\Lambda_* = \text{coker}(\varphi_*). \quad (29) \]
The $\mathbb{Z}$-modules $\Lambda_*$, $\mathbb{K}_*$, and $\mathbb{C}K_*$ are independent of the choice of the orientation $O$ on $E$ and are invariants of the decorated graph $\Gamma$. In particular,
\[ \mathbb{K}_* = \{ (s_v)_{v \in V} : s_v \in \mathbb{Z}^E \bigoplus \bigoplus_{v \in V} \mathbb{Z}^{I_v} : s_v - s_{v'} = \lambda_{v,v'} \forall v, v' \in V, \bar{v} \in \mathbb{E}_{v,v'} \}. \quad (30) \]

For any field of characteristic zero $F$, let
\[ D_{\bullet,F} = D_* \otimes F, \quad T_{\bullet,F} = T_* \otimes F, \quad \Lambda_{\bullet,F} = \Lambda_* \otimes F, \quad \mathbb{K}_{\bullet,F} = \mathbb{K}_* \otimes F, \quad \mathbb{C}K_{\bullet,F} = \mathbb{C}K_* \otimes F \quad (31) \]
be the corresponding \( F \)-vector spaces. Via the exponentiation map, let
\[
\exp(\Lambda_\bullet \cdot C) \subset \prod_{e \in \mathbb{E}} (\mathbb{C}^*)_{t_e}, \quad \text{with} \quad (\mathbb{C}^*)_t = \{(t_i)_{i \in I} \in (\mathbb{C}^*)^I : \prod_{i \in I} t_i = 1\},
\]
be the subgroup corresponding to the sub-Lie algebra \( \Lambda_\bullet \cdot C \subset T_\bullet \cdot C \), and denote the quotient group by \( G_\bullet = G_\bullet(\Lambda) = \exp(CK_\bullet \cdot C) \).

In [9], similarly to (16), to every pre-log \( f \) as in Definition 4.1 we will associate a group element \( \text{ob}_{\bullet, \Gamma}(f) \in G_\bullet(\Gamma) \).

**Definition 4.2.** Let \( X_\emptyset = \bigcup_{i \in S} X_i \) be a smoothable SC symplectic variety and \( (J_\emptyset, \omega_\emptyset) \in J_\bullet(X_\emptyset) \).

A log \( J \)-holomorphic map is a pre-log map \( f \) with the decorated dual graph \( \Gamma \) such that

1. there exist functions
   \[
s : V \longrightarrow \mathbb{R}^S, \quad v \longrightarrow s_v, \quad \text{and} \quad \lambda : E \longrightarrow \mathbb{R}^+_{\cdot}, \quad e \longrightarrow \lambda_e,
\]
such that
   a. \( s_v \in \mathbb{R}^I_+ \times \{0\}^{S-I} \) for all \( v \in V \),
   b. \( s_{v_2(e)} - s_{v_1(e)} = \lambda_e s_e \) for every \( e \in \mathbb{E} \);

2. and \( \text{ob}_{\bullet, \Gamma}(f) = 1 \in G_\bullet(\Gamma) \).

Since \( \Gamma \) is connected, Condition (b) and \( s_e \in \mathbb{Z}^S_e \), for all \( e \in \mathbb{E} \), imply that
\[
\sum_{i \in S} s_{v,i} = c \tag{32}
\]
is a fixed positive constant \( c \in \mathbb{R}^+_{\cdot} \) independent of \( v \in V \).

Two marked log maps are equivalent if they are related by a “reparametrization of the domain”. A marked log map is stable if it has a finite “automorphism group”. For \( g, k \in \mathbb{N} \) and \( A \in H_2(X_\emptyset, \mathbb{Z}) \), we denote the space of equivalence classes of stable \( k \)-marked degree \( A \) genus \( g \) log maps by \( \mathcal{M}_{g,k}^\log(X_\emptyset, A) \). Every element of \( \mathcal{M}_{g,k}^\log(X_\emptyset, A) \) belongs to a fiber product of relative moduli spaces \( \mathcal{M}_{g,s_i}^\log(X_i, X_i; \partial, A_i) \), with \( i \in S \). In [9], we will show that the sequential convergence topology of relative log moduli spaces in Theorem 3.7 give rise to a compact metrizable topology on \( \mathcal{M}_{g,k}^\log(X_\emptyset, A) \).

A similar calculation shows that the expected real dimension of (each virtually main component of) \( \mathcal{M}_{g,k}^\log(X_\emptyset, A) \) is equal to
\[
2(c_1^{T_\log X_\emptyset}(A) + (\dim_{\mathbb{C}} X - 3)(1 - g) + k),
\]
where \( T_\log X_\emptyset \) is the log tangent bundle defined in [14]. For any symplectic smoothing \( \pi : Z \longrightarrow \Delta \) with the central fiber \( X_\emptyset, c_1^{T_\log X_\emptyset}(A) \) coincides with \( c_1^{T_\mathbb{Z}_\lambda}(A) \), for all \( \lambda \in \Delta^* \).
5 The decomposition formula

In this section, we propose an explicit degeneration formula to address the last part of the question (**) in the introduction.

Assume $X_\emptyset = \bigcup_{i \in S} X_i$ is a smoothable SC symplectic variety and $(J_\emptyset, \omega_\emptyset) \in J_* (X_\emptyset)$. Unlike the classical case, any log moduli space $\overline{M}_{g,k}^{\log} (X_\emptyset, A)$ typically has many (virtually) main strata.

**Definition 5.1.** We say an admissible decorated dual graph $\Gamma$ is a main graph, if

$$K_* (\Gamma) = 0,$$

where $K_*$ is the torsion-free $\mathbb{Z}$-module in (30).

For a main graph, up to scaling, there is a unique set of functions $(s: V \to \mathbb{R}^S, \lambda: E \to \mathbb{R}_+)$ satisfying Definition 4.2.(1). The condition (33) implies that the image of the $\mathbb{Z}$-linear map

$$T^\vee \xrightarrow{\psi^\vee} D^\vee$$

(34)
dual to (28) is of finite index. Let

$$m(\Gamma) = |D^\vee / \text{Im}(\psi^\vee)| \in \mathbb{Z}_+.$$

(35)

For each $g, k \in \mathbb{N}$ and $A \in H_2 (X_\emptyset, \mathbb{Z})$, the moduli space $\overline{M}_{g,k}^{\log} (X_\emptyset, A)$ decomposes into a union of (virtually) main components

$$\overline{M}_{g,k}^{\log} (X_\emptyset, A) = \bigcup_{\text{main } \Gamma} \overline{M}_{g,k}^{\log} (X_\emptyset, A)_\Gamma.$$

(36)

Then the degeneration formula (37) below suggests that the latter gives rise to a similar but weighted decomposition formula for VFCs in such a semistable degeneration.

Before we state the decomposition formula, we need to state some facts about the behavior of homology classes in a semistable degeneration. Let $\pi: Z \to \Delta$ be a semistable degeneration as in the introduction. For every $\lambda \in \Delta$, the embedding $Z_\lambda \to Z$ gives us a homomorphism

$$\iota_\lambda: H_2 (Z_\lambda, \mathbb{Z}) \to H_2 (Z, \mathbb{Z}).$$

For $\lambda \in \Delta^*$ and $A, A' \in H_2 (Z_\lambda, \mathbb{Z})$, we write $A \sim A'$ whenever $\iota_\lambda (A - A') = 0$. We denote the equivalence class of $A$ with respect to this relation by $[A]$. For example, every two classes $A, A' \in H_2 (Z_\lambda, \mathbb{Z})$ related by the monodromy map are equivalent; therefore, $H_2 (Z_\lambda, \mathbb{Z})/ \sim$ is invariant under the deformation. In the following, we denote the deformation equivalence class of $Z_\lambda$ by $X^\#_\lambda$ and $H_2 (Z_\lambda, \mathbb{Z})/ \sim$ by $H_2 (X^\#_\lambda, \mathbb{Z})$. The induced map

$$\iota^\#: H_2 (X^\#_\lambda, \mathbb{Z}) \to H_2 (Z, \mathbb{Z})$$

is injective and well-defined.

Once we construct an appropriate VFC for each (virtually) main component of the decomposition (36), the claim is that the following degeneration formula holds:
Assume $\pi: (\mathcal{Z}, \omega_{\mathcal{Z}}) \rightarrow \Delta$ is a symplectic semistable degeneration with the SC central fiber $(X_0, \omega_0)$ and $J_\mathcal{Z}$ is a “nice” $\omega_\mathcal{Z}$-compatible (or tame) almost complex structure on $\mathcal{Z}$ such that $J_0 = J_\mathcal{Z} |_{x_0} \in J_*(X_0, \omega_0)$. Then, for every $[A] \in H_2(X_\mathcal{Z}, \mathbb{Z})$ and $g, k \in \mathbb{N}$, we have

$$\left[\overline{\mathcal{M}}_{g,k}(X_\mathcal{Z}, [A])\right]_{\text{VFC}}^{\mathcal{M}} = \sum_{A_0: \pi_0(A_0) = \tau} \sum_{\text{main } \Gamma} \frac{m(\Gamma)}{|\text{Aut}(\Gamma)|} \left[\overline{\mathcal{M}}_{g,k}^\log(X_0, A_0)(\tau)\right]_{\text{VFC}},$$

(37)

where $|\text{Aut}(\Gamma)|$ is the order of the automorphism group of the decorated dual graph $\Gamma$.

The equality (37) should be thought of as an equality of Čech cohomology classes in $\overline{\mathcal{M}}_{g,k}^\log(\mathcal{Z}, [A])$ in the sense of [31] Remark 8.2.4. In the case of basic degenerations, i.e. $S = \{1, 2\}$, this formula coincides with the Jun Li’s formula. In other words, the only decorated graphs with $\mathbb{K}_*(\Gamma) = 0$ are bipartite graphs with one group of vertices indexed by $\{1\}$, the opposite group indexed by $\{2\}$, and $s_e = (-\alpha_{\mathcal{Z}}, \alpha_{\mathcal{Z}}) \neq 0 \in \mathbb{Z}^{(1, 2)}$ for all $e \in \mathcal{E}$. Moreover, $m(\Gamma)$ is the product of contact orders, i.e.

$$m(\Gamma) = \prod_{e \in \mathcal{E}(\Gamma)} |\alpha_e|.$$

(38)

Unlike the basic case, a main decorated dual graph for SC varieties with non-trivial 3-fold (and higher) strata may have components mapped into a stratum $X_I$ with $|I| \geq 2$; see Section 6.

A main step in establishing (37) is to prove a gluing theorem for smoothing the nodes of a log map $f$ as in Definition 4.2 to get $J$-holomorphic maps in $\mathcal{Z}$. Similarly to the relative case, the space of gluing parameters for a fixed log map $f$ with the decorated dual graph $\Gamma$ is a sufficiently small neighborhood of the origin in

$$\mathcal{G}_\Gamma = \left\{((\varepsilon_e)_{e \in \mathcal{E}}, (t_{v,i})_{v \in \mathcal{V}, i \in I_v}) \in \mathbb{C}^\mathcal{E} \times \prod_{v \in \mathcal{V}} \mathbb{C}^{I_v}: \prod_{i \in I_v} t_{v,i} = \prod_{i \in I_{v'}} t_{v',i} \quad \text{and} \quad \varepsilon_e s_{v,i} t_{v,i} = \varepsilon_{v',i} t_{v',i} \right\} \subset \mathbb{C}^\mathcal{E} \times \prod_{v \in \mathcal{V}} \mathbb{C}^{I_v}.$$  

(39)

In (39), if $i \in I_v - I_{v'}$, by $t_{v,i}$ we mean 1. The complex numbers $\varepsilon_e$ are the gluing parameters for the nodes of $\Sigma$ and $t_{v,i}$ are the parameters for pushing $u_v$ out in the direction of $\zeta_{e,i}$. The common value $\lambda = \prod_{i \in I_v} t_{v,i}$ describes the fiber $\mathcal{Z}_\lambda$ in which the smoothing lies; in other words, the projection map $\pi: \mathcal{G}_\Gamma \rightarrow \mathbb{C}$ induced by $\pi: \mathcal{Z} \rightarrow \mathbb{C}$ is the map

$$((\varepsilon_e)_{e \in \mathcal{E}}, (t_{v,i})_{v \in \mathcal{V}, i \in I_v}) \rightarrow \lambda = \prod_{i \in I_v} t_{v,i}. $$

(40)

For any non-empty set $\Omega$ and a ring $R$, the quotient space

$$\frac{R^\Omega}{R} := R^\Omega / \{(a, \ldots, a) \in R^\Omega: \ a \in R\}.$$

is the dual of $R^\Omega$. Let

$$\mathbb{D} \equiv \mathbb{Z}^\mathcal{E} \oplus \bigoplus_{v \in \mathcal{V}} \mathbb{Z}^{I_v} \xrightarrow{\mu = e \oplus e'} \widehat{\mathbb{T}} \equiv \bigoplus_{e \in \mathcal{E}} \mathbb{Z}^{I_e} \oplus \mathbb{Z}^{I_e}$$

(41)
be the $\mathbb{Z}$-linear map where $\varrho$ is the map in (13) (for any fixed choice of orientation $O$ on $E$) and $\varrho'$ is given by
\[
\left((\lambda_e)_{e \in E}, (\lambda_{v,i})_{v \in V, i \in I_v}\right) \mapsto \left[\sum_{i \in I_v} \lambda_{v,i}\right]_{v \in V} \in \frac{\mathbb{Z}^V}{\mathbb{Z}}.
\]
Similarly to (15), let
\[
\mathcal{D}_R \xrightarrow{\mu = \varrho \oplus \varrho_k} \mathcal{T}_R \equiv \bigoplus_{e \in E} \mathbb{R}^{I_e} \oplus \frac{\mathbb{R}^V}{\mathbb{R}}
\]
be the tensor product of (41) with $\mathbb{R}$. By Definition 4.1.(4),
\[
\text{Ker}(\mu) = \text{Ker}(\varrho) = K := \{(\lambda_e)_{e \in E}, (s_v)_{v \in V}\} \in \mathbb{Z}^E \oplus \bigoplus_{v \in V} \mathbb{Z}^{I_v} : s_{v'} - s_v = \lambda_e s_e \quad \forall v, v' \in V, \ e \in E, e'
\]
(42)
The same identity holds over $\mathbb{R}$. Therefore, with notation as in (21), we have
\[
\sigma = \sigma(\Gamma) = \text{Ker}(\mu) \cap (\mathbb{R}^E_{\geq 0} \oplus \bigoplus_{v \in V} \mathbb{R}_{\geq 0}^{I_v})
\]
Remark 5.2. Similarly to [7, Proposition 3.27], $\mathcal{G}_\Gamma$ is the zero set in $\mathbb{C}^E \times \prod_{v \in V} \mathbb{C}^{I_v}$ of the “binomial ideal” corresponding to $\text{Im}(\mu^\vee)$.

Let $Y_{\sigma(\Gamma)}$ be the toric variety with toric fan $\sigma(\Gamma)$. Then, similarly to the paragraph after (21), the space of gluing parameters $\mathcal{G}_\Gamma$ in (39) is a possibly non-irreducible and non-reduced affine toric sub-variety of $\mathbb{C}^E \times \prod_{v \in V} \mathbb{C}^{I_v}$ that is isomorphic to $|K^\perp / \text{image}(\mu^\vee)|$ copies of $Y_{\sigma(\Gamma)}$, counting with multiplicities.

The projection map
\[
\pi : K_{\cdot, \mathbb{R}} \rightarrow \mathbb{R}, \quad ((\lambda_e)_{e \in E}, (\lambda_{v,i})_{v \in V, i \in I_v}) \mapsto \sum_{i \in I_v} \lambda_{v,i},
\]
(43)
is well-defined and $\text{Ker}(\pi) = K_{\cdot, \mathbb{R}}$. The map (43) sends $\sigma$ onto $\mathbb{R}_{\geq 0}$; thus it defines a projection map
\[
\pi : Y_{\sigma} \rightarrow \mathbb{C}.
\]
(44)
This projection map is the restriction to $Y_{\sigma} \subset \mathcal{G}_\Gamma$ of the projection map (30). For a main dual graph $\Gamma$, $K_{\cdot}$ is trivial and $K \cong \mathbb{Z}$, therefore $Y_{\sigma}$ is isomorphic to $\mathbb{C}$ and $\mathcal{G}_\Gamma$ is isomorphic to $|K^\perp / \text{image}(\mu^\vee)|$ copies of $\mathbb{C}$, counting with multiplicities. Let $m_{\text{red}}(\Gamma)$ be the degree of (44). The following lemma explains the coefficient $m(\Gamma)$ in (37).

Lemma 5.3. For a main dual graph $\Gamma$, (40) is a map of order $m(\Gamma)$, i.e.
\[
m_{\text{red}}(\Gamma) \left| K^\perp / \text{image}(\mu^\vee) \right| = m(\Gamma).
\]
(45)
Proof. The following diagram commutes:

We don't know of any example, arising from such dual graphs, such that the multiplicities are bigger than 1.
where \( \nu_\mathbb{D} \) and \( \nu_\mathcal{T} \) are the obvious inclusion maps. In the dual commutative diagram,

\[
\begin{array}{ccc}
\mathbb{D}^\vee \cong \mathbb{Z}^E \oplus \bigoplus_{v \in \mathcal{V}} \mathbb{Z}^{I_v} & \xrightarrow{\mu^\vee} & \mathcal{T}^\vee \cong \bigoplus_{v \in \mathcal{E}} \mathbb{Z}^{I_v} \oplus \mathbb{Z}^\vee \\
\bigoplus_{v \in \mathcal{E}} \mathbb{Z}^{I_v} & \xrightarrow{\nu} & \bigoplus_{v \in \mathcal{V}} \mathbb{Z}^{I_v} \\
\mathbb{D} \cong \mathbb{Z}^E \oplus \bigoplus_{v \in \mathcal{V}} \mathbb{Z}^{I_v} & \xrightarrow{\psi^\vee} & \mathcal{T} \cong \bigoplus_{v \in \mathcal{E}} \mathbb{Z}^{I_v} \oplus \mathbb{Z}^\vee 
\end{array}
\]

\( \psi_\mathbb{D} \) and \( \psi_\mathcal{T} \) are the obvious surjective projection maps. By definition, \( \text{Im}(\mu^\vee) \) is a sub-lattice of finite index \( |\mathbb{K}^\bot/\text{image}(\mu^\vee)| \) in \( \mathbb{K}^\bot \), and \( \text{Im}(\varphi^\vee) \) is a sub-lattice of finite index \( m(\Gamma) \). Since \( \psi_\mathbb{D} \) is surjective, in order to prove \( \text{[35]} \), we have to show that \( \text{Im}(\psi_\mathbb{D}|_{\mathbb{K}^\bot}) \) is a sub-lattice of the index \( m_{\text{red}}(\Gamma) \).

Since \( \Gamma \) is a main dual graph, the kernel \( \mathbb{K} \) is one-dimensional and is generated by a prime vector \( \eta = ((\lambda_e)_{e \in \mathcal{E}}, (s_{v,i})_{v \in \mathcal{V}, i \in I_v}) \) with positive coefficients, such that

\[
c = \sum_{v \in \mathcal{V}} s_{v,i}
\]

is independent of the choice of \( v \in \mathcal{V} \). Then, be \( \text{[33]} \), \( m_{\text{red}}(\Gamma) = c \). By definition,

\[
\mathbb{K}^\bot = \left\{ \eta^\vee = ((\alpha_e)_{e \in \mathcal{E}}, (\beta_{v,i})_{v \in \mathcal{V}, i \in I_v}) \in \mathbb{Z}^E \oplus \bigoplus_{v \in \mathcal{V}} \mathbb{Z}^{I_v}; \langle \eta, \eta^\vee \rangle = \sum_{e \in \mathcal{E}} \alpha_e \lambda_e + \sum_{v \in \mathcal{V}} \sum_{i \in I_v} \beta_{v,i} s_{v,i} = 0 \right\}.
\]

For any

\[
\eta^\vee = ((\alpha_e)_{e \in \mathcal{E}}, (\beta_{v,i})_{v \in \mathcal{V}, i \in I_v}) \in \mathbb{Z}^E \oplus \bigoplus_{v \in \mathcal{V}} \mathbb{Z}^{I_v},
\]

pre-images of \( \eta^\vee \) in \( \mathbb{D}^\vee \) are elements of the form

\[
\eta^\vee_k := \eta^\vee_0 + \sum_{v \in \mathcal{V}} k_v (0, (0)_{v^\prime \in \mathcal{V}, v \neq v^\prime}, (1_{v,i})_{i \in I_v}), \quad k = (k_v)_{v \in \mathcal{V}} \in \mathbb{Z}^\vee,
\]

where \( \eta^\vee_0 = ((\alpha_e)_{e \in \mathcal{E}}, (\beta_{v,i})_{v \in \mathcal{V}, i \in I_v}) \) is one fixed pre-image. If

\[
\langle \eta, \eta^\vee_k \rangle = a
\]

then \( \langle \eta, \eta^\vee_k \rangle = a + c \sum_{v \in \mathcal{V}} k_v \), for all \( k = (k_v)_{v \in \mathcal{V}} \in \mathbb{Z}^\vee \). We conclude that \( \eta^\vee \in \text{Im}(\psi_\mathbb{D}|_{\mathbb{K}^\bot}) \) if and only if \( a \equiv 0 \mod c \). Therefore, \( \text{Im}(\psi_\mathbb{D}|_{\mathbb{K}^\bot}) \) is a sub-lattice of index \( m_{\text{red}}(\Gamma) = c \). \( \square \)

**Remark 5.4.** Except for the coefficients \( m(\Gamma) \), the degeneration formula \( \text{[37]} \) coincides with \( \text{[2]} \) (1.1.1) \( \text{(There, it is called a “decomposition” formula).} \) The multiplicity \( m_\mathcal{T} \) in \( \text{[2]} \) (1.1.1) is \( m_{\text{red}}(\Gamma) \) in our notation. It is not clear to us how the extra factor \( |\mathbb{K}^\bot/\text{image}(\mu^\vee)| \) emerges from their definitions. For example, with notation as in \( \text{[35]} \), in the case of basic decompositions, \( m_{\text{red}}(\Gamma) \) is equal to

\[
\text{l.c.m}(\{ |\alpha_e| \}_{e \in \mathcal{E}})
\]

which could be different from the product in \( \text{[38]} \). We last heard from H. Ruddat in June 16 of 2017 that he, B. Kim, and H. Lho are working on a paper \( \text{[27]} \) to prove that \( \text{[2]} \) (1.1.1) decomposes to the Jun Li’s formula in the case of basic degenerations.

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Figure 6: On left, the central fiber of $Z'$ with its 9 singular points. On right, the central fiber of $Z$ with its 9 exceptional curves.

6 Rational curves in a pencil of cubic surfaces

In this section, we re-study a non-trivial (i.e. not of the classical type) example of the degeneration of degree 3 rational curves in a pencil of cubic surfaces, studied in [2, Section 6]. As our calculations show, we can easily identify the space of such log maps without any need for considering blow-ups and other sophisticated tricks used in [2, Section 6].

Let $P$ be a homogenous cubic polynomial in $x_0, \ldots, x_3$ and

$$Z' = \{ (t, [x_0, x_1, x_2, x_3]) \in \mathbb{C} \times \mathbb{P}^3 : x_1 x_2 x_3 = tP(x_0, x_1, x_2, x_3) \} \subset \mathbb{C} \times \mathbb{P}^3.$$

Let $\pi' : Z' \to \mathbb{C}$ be the projection map to the first factor. For a generic $P$ and $t \neq 0$, $\pi'^{-1}(t)$ is a smooth cubic hypersurface (divisor) in $\mathbb{P}^3$. For $t = 0$, $\pi'^{-1}(0)$ is the SC variety $X_{\emptyset} \equiv \{(x_i = 0) \approx \mathbb{P}^2 \quad \forall \ i \in \{1, 2, 3\}, \quad X'_{ij} \equiv X'_i \cap X'_j \approx \mathbb{P}^1 \quad \forall \ i, j \in \{1, 2, 3\}, \ i \neq j.$

However, the total space $Z'$ of $\pi'$ is not smooth at the 9 points of

$$Z'_{\text{sing}} \equiv \{0\} \times (X_0' \cap (P = 0)) \subset X_0,$$

where $X_0' = X'_{12} \cup X'_{13} \cup X'_{23} \subset \mathbb{P}^3$.

A small resolution $Z$ of $Z'$ can be obtained by blowing up each singular point on $X'_{ij}$ in either $X_i'$ or $X_j'$. The map $\pi'$ then induces a projection $\pi : Z \to \Delta$ and defines a semistable degeneration. Every fiber of $\pi$ over $\mathbb{C}^*$ is a smooth cubic surface. The central fiber $\pi^{-1}(0)$ is the SC variety $X_0 \equiv X_1 \cup X_2 \cup X_3$ with 3 smooth components, each a blowup of $\mathbb{P}^2$ at some number of points. If each singular point on $X'_{ij}$ is blown up in $X'_i$ with $i < j$, then $Z$ is obtained from $Z'$ through two global blowups of $\mathbb{C} \times \mathbb{P}^3$ and is thus projective; see Figure 6.

For the count of degree 1 and degree 2 rational curves in $Z_t$, it can be shown that limiting curves are of the classical type (i.e. they do not pass through the triple intersection). For example, the broken curve $\alpha$ in Figure 6 is one of the 27 degree 1 rational curves in the limit. For each $t \neq 0$, the moduli space $M_{0,2}(Z_t, [3])$ of 2-marked genus 0 degree 3 maps in $Z_t$ is of the (expected) complex dimension 4. In degree 3, for generic $t$, there are 84 such curves passing through 2 generic points of $Z_t$ at the marked points. In the limiting SC variety $X_\emptyset$, assuming that the two point constraints
move to $X_1$ and $X_2$, 81 of these 84 maps can be identified among the maps that do not intersect $X_{123}$. There is, however, a new type of main graph $\Gamma$ contributing to the degeneration formula (37) that has no analogue in the Jun Li’s formula. We are going to describe this $\Gamma$, identify the space of log maps $\mathcal{M}^\log_{0,2}(X_0, [3])_\Gamma$, and calculate the coefficient $m(\Gamma)$.

Let $\Gamma$ be the graph with the set of vertices $V = \{v_1, v_2, v_3, v_0\}$ and the set of edges $E = \{e_1, e_2, e_3\}$ such that $e_i$ connects $v_0$ and $v_i$, for all $i = 1, 2, 3$. Choose the orientations $e_i$ to end at $v_0$, for all $i = 1, 2, 3$, and assume

\[
\begin{align*}
I_{v_0} &= S = \{1, 2, 3\}, \\
I_{v_i} &= \{i\}, \\
A_{v_i} &= [1] \in H_2(X_i, \mathbb{Z}), \quad \forall i = 1, 2, 3,
\end{align*}
\]

where, for each $i = 1, 2, 3$, $[1] \in H_2(X_i, \mathbb{Z})$ is the pre-image in $X_i$ of the class of a line in $X_i^\prime$ away from the blow-up points. The two legs corresponding to the two marked points are attached to $v_1$ and $v_2$; see Figure 7. A log map with this dual graph is made of a line $\ell_i = \text{Im}(u_{v_0})$ in $X_i^\prime = \mathbb{P}^2$ passing through the point $X_{123}$ (so the choice of resolution and the exceptional curves are irrelevant in the following discussion), for each $i \in S$, and a log tuple

\[
(u_{v_0}, \zeta_i, i \in S, (\Sigma_{v_0}, \mathbf{i}_0)) \cong \mathbb{P}^1, q_{v_0} = \{q_{\mathbf{e}_i}\}_{i \in S},
\]

where $u_{v_0}$ is the constant map onto $X_{123}$ and $\zeta_i$, for each $i \in S$, is a meromorphic section of the trivial bundle

\[
u_{v_0}^* N_{S; i} \cong \Sigma_{v_0} \times \mathbb{C}
\]

with a zero of order 2 at $q_{\mathbf{e}_i}$ and poles of order 1 at $\{q_{\mathbf{e}_j}\}_{j \in S - i}$. The dual graph $\Gamma$ satisfies Definition 4.2(1) The function $s: V \rightarrow \mathbb{R}^S$ given by

\[
\begin{align*}
s_{v_1} &= (3, 0, 0), \\
s_{v_2} &= (0, 3, 0), \\
s_{v_3} &= (0, 0, 3), \quad \text{and} \quad s_{v_0} = (1, 1, 1)
\end{align*}
\]

has the required properties. Up to rescaling, the function $s$ above is the only function satisfying Definition 4.2(1) therefore, $\Gamma$ is a main dual graph (i.e. $K_\bullet = 0$ or $K \cong \mathbb{Z}$). Since the domain of the injective map (28) is 5-dimensional and its target is 6-dimensional, we conclude that the obstruction group $\mathcal{G}_\bullet$ is isomorphic to $\mathbb{C}^\ast$.

The dual map

\[
\begin{align*}
\mathbb{T}_\bullet^\vee &= \bigoplus_{i \in S} \left(\frac{\mathbb{Z}I_{\mathbf{e}_i}}{\mathbb{Z}}\right)^3 \\
&\cong \left(\frac{\mathbb{Z}^3}{\mathbb{Z}}\right)^3 \\
&\xrightarrow{\mathbb{D}_\bullet^\vee} \left(\frac{\mathbb{Z}^3}{\mathbb{Z}}\right)^3 \\
&\cong \mathbb{Z}^3 \\
&\cong \left(\frac{\mathbb{Z}^3}{\mathbb{Z}}\right)^3
\end{align*}
\]
in (34) is given by
\[ g^\vee([\eta_1], [\eta_2], [\eta_3]) = \left( -2\eta_{11} + \eta_{12} + \eta_{13}, (\eta_{21} - 2\eta_{22} + \eta_{23}), (\eta_{31} + \eta_{32} - 2\eta_{33}), -(|[\eta_1] + [\eta_2] + [\eta_3]) \right), \]
where \( \eta_i = [\eta_{i1}, \eta_{i2}, \eta_{i3}] \in \mathbb{Z}^3 \), for any \( i \in \{1, 2, 3\} \). It is straightforward to check that
\[ \text{Im}(g^\vee) = \left\{ (a, b, c, [x, y, z]) \in \mathbb{Z}^3 + \mathbb{Z}^3 : a + b + c \equiv x + y + z \mod 3 \right\}. \]

Therefore, the quotient group \( \mathbb{D}^\vee / \text{Im}(g^\vee) \) is isomorphic to \( \mathbb{Z}_3 \) and is generated by the class of \((1, 0, 0, [0, 0, 0])\); i.e. \( m(\Gamma) = 3 \).

**Remark 6.1.** Going back to Remark 5.4, the extra factor \( |K^\perp/\text{image}(\mu^\vee)| \) is equal 1 in this example. Therefore, \( m_{\text{red}}(\Gamma) = m(\Gamma) \); that explains why our coefficient \( m(\Gamma) = 3 \) coincides with the one calculated in [2, Section 6].

In the pre-log space \( \mathcal{M}^\log_{0,2}(X_0, [3])_\Gamma \), the three lines \( \ell_1, \ell_2, \ell_3 \) are allowed to be any line passing through the point \( X_{123} \) with some slope in \( \mathbb{C}^* \). However, the condition \( \text{ob}_{\star, \Gamma}(f) \in G_{\star} \cong \mathbb{C}^* \) in Definition [4,2] puts a restriction on the set of lines \( \ell_1, \ell_2, \ell_3 \) that give rise to a log map.

For each \( i = S \), the line \( \ell_i \) is the completion of the image of a map of the form
\[ \mathbb{C} \to \mathbb{C}^3, \quad z \to (z_{ij})_{j=1,2,3} \subset \mathbb{C}^3, \quad z_{ii} = 0, \quad z_{ij} = a_{ij} z, \quad a_{ij} \in \mathbb{C}^*, \quad \forall j \in S - i. \]

It follows from definition of \( \text{ob}_{\star, \Gamma}(f) \) in [9] that \( \text{ob}_{\star, \Gamma}(f) = 1 \) if and only if for any set of 3 distinct points \( \text{pt}_1, \text{pt}_2, \text{pt}_3 \in \mathbb{P}^1 \) and local coordinates \( w_1, w_2, w_3 \) around them, respectively, there exists a set of meromorphic sections \( (\zeta_i)_{i \in S} \) of \( \mathbb{P}^1 \times \mathbb{C} \) (holomorphic away from \( \text{pt}_1, \text{pt}_2, \text{pt}_3 \)) such that the product \( \zeta_1 \zeta_2 \zeta_3 \) is a constant section and
\[ \zeta_i(w_j) = a_{ij} w^{-1}_j, \quad \forall i \in S, j \in S - i. \]

A straightforward calculation shows that this is possible if and only if
\[ \frac{a_{12} a_{31} a_{23}}{a_{13} a_{23} a_{21}} = -1, \]
\[ \text{i.e. the product of the slopes of } \ell_1, \ell_2, \ell_3 \text{ (in a certain order) is } -1. \]

In the degeneration formula (37), imposing two generic point constraints in \( X_1 \) and \( X_2 \) on the image of the two marked points fixes \( \ell_1 \) and \( \ell_2 \). Then the slope condition above fixes \( \ell_3 \). Therefore, since \( m(\Gamma) = 3 \) and \( \text{Aut}(\Gamma) = 1 \), the contribution of such a star-shaped log map to the GW count of degree 3 rational curves in a smooth cubic surface passing through two generic points is 3.

We finish with some comments on the question (⋆⋆⋆) in the introduction. After removing the trivial component \( u_v : \Sigma_v \to X_{123} \), the moduli space \( \mathcal{M}^\log_{0,2}(X_0, [3])_\Gamma \) decomposes into the relative spaces
\[ \mathcal{M}^\log_{0,((0,0),(1,1))}(X_1, X_1, [1]), \quad \mathcal{M}^\log_{0,((0,0),(1,1))}(X_2, X_2, [1]), \quad \text{and } \mathcal{M}^\log_{0,((1,1))}(X_3, X_3, [1]). \]
So one might still hope to be able to get a decomposition formula in a situation like this. In general, specially in higher dimensions, there seems to be no obvious way to get such a decomposition. The following modification of the previous example highlights the issue.
Example 6.2. Consider the family \( \mathcal{Y} = \mathbb{Z} \times \mathbb{P}^1 \longrightarrow \mathbb{C} \), where \( \mathbb{Z} \) is as above and \( \pi \) is the lift of the projection map \( \pi: \mathbb{Z} \longrightarrow \mathbb{C} \). Let \( Y_I = X_I \times \mathbb{P}^1 \), for all \( \emptyset \neq I \subset \{1,2,3\} \). Consider the same dual graph but with \( k=3 \) (i.e. with a third marked point on \( \Sigma_{v_3} \)),

\[
A_{v_0} = [0,1] \in H_2(Y_{123}, \mathbb{Z}) \cong H_2(\{\text{point}\} \times \mathbb{P}^1, \mathbb{Z}) \cong \{0\} \times \mathbb{Z},
\]

and

\[
A_{v_i} = [1,0] \in H_2(Y_i, \mathbb{Z}) \cong H_2(X_i \times \mathbb{P}^1, \mathbb{Z}) \cong H_2(X_i, \mathbb{Z}) \times \mathbb{Z} \quad \forall \ i \in \{1,2,3\};
\]

see Figure 8. The moduli space \( \overline{M}_{0,3}(Y_{\emptyset}, [3,1])_r \) is complex 8 dimensional with the same contributing factor \( m(\Gamma)=3 \) to (37). A smooth fiber \( Y_\lambda \) of \( \mathcal{Y} \) is the product of the smooth cubic surface \( \mathbb{Z}_\lambda \) and \( \mathbb{P}^1 \). Let

\[
GW_{0,3}^{Y_\lambda}(\text{pt, pt, } \alpha \times \text{pt})
\]

be the number of degree \([3,1]\) rational curves in \( Y_\lambda \) with two point constraints and

\[
\alpha \times \text{pt} \in H_2(Y_\lambda, \mathbb{Z}) = H_2(\mathbb{Z}_\lambda \times \mathbb{P}^1, \mathbb{Z}),
\]

where \( \alpha \) is the (homology class of the) smoothing of the limiting line shown in Figure 6-Right. Since \( m(\Gamma)=3 \) as before, and there is a unique \( \Gamma \)-type log map in \( Y_{\emptyset} \) with those constraints, we conclude that the contribution of \( \Gamma \)-type curves to (46) is again 3. In examples like this, where there is a non-constant map \( u_{v_0} \) in a stratum \( Y_I \) with \( |I| \neq 1 \), for any decomposition of \( \overline{M}_{g,k}(Y_{\emptyset}, A)_r \) into a fiber product of relative spaces, either \( u_{v_0} \) should be considered in one of the relative moduli spaces (which would result in relative spaces with \( s \notin \mathbb{N}^3 \)), or it should be removed while its non-trivial GW contribution affects the matching conditions of the remaining parts. Both options sound very complicated!

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