Scattering and bound state solutions of the asymmetric Hulthén potential

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Abstract

The one-dimensional time-independent Schrödinger equation is solved for the asymmetric Hulthén potential. The reflection and transmission coefficients and bound state solutions are obtained in terms of the hypergeometric functions. It is observed that the unitary condition is satisfied in the non-relativistic region.

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1. Introduction

The solutions of the wave equations for the scattering and/or bound states have been of great interest in quantum mechanical systems [1–22]. To obtain full information, one has to investigate the bound as well as scattering state problem. In [3], Wei et al have obtained the analytical scattering state solutions of the ℓ-wave Schrödinger equation for the Eckart potential. The ℓ-wave continuum states of the Schrödinger equation for the modified Morse potential have been studied by Wei and Chen [7]. They have obtained the normalized analytical radial wave functions and derived a corresponding calculation of phase shifts. Chen et al [4] have found the exact solutions of scattering states for the s-wave Schrödinger equation with the Manning–Rosen potential by using a standard method. The scattering solutions of the Klein–Gordon equation for the Woods–Saxon potential in one dimension (1D) have been obtained in terms of hypergeometric functions by Rojas and Villalba [10], and the authors have derived the condition for the existence of transmission resonances. In an arbitrary dimension, Chen et al [16] have presented the properties of the scattering state solutions of the Klein–Gordon equation for a Coulomb-like scalar plus vector potentials. In [19], low-momentum scattering in the Dirac equation has been studied. Villalba and Greiner [17] investigated the transmission resonances and supercritical states by solving the two-component Dirac equation for the cusp potential. In this manner, we intend to search for the transmission and reflection coefficients and eigenvalues of the Schrödinger equation for the asymmetric Hulthén potential (ASHp).

The ‘usual’ Hulthén potential [23] is one of the significant exponential potentials that behave like a Coulomb potential for small values of a spatial coordinate. The Hulthén potential has applications in many areas of physics, for example atomic physics [24, 25], nuclear and high-energy physics [26], solid-state physics [27] and chemical physics [28]. In addition, the Hulthén potential and its various forms are used in the relativistic and non-relativistic regions [8, 9, 12–14]. In [8], the approximate analytical scattering state solutions of the Schrödinger equation with the generalized Hulthén potential for any ℓ-state have been obtained. Saad [9] has studied the bound states of a spin-0 particle in D dimensions and found the normalization constant in terms of the incomplete beta function. The scattering solutions of the Klein–Gordon equation for the general Hulthén potential have been obtained and the transmission resonances investigated in [13]. Guo et al [21] found the transmission resonances for a Dirac particle in the presence of the Hulthén potential in 1D. On the other hand, the solutions of the bound and scattering states of the wave equations for the asymmetric potentials were examined recently [18, 22, 29]. In [18], Jiang et al have investigated the low-momentum scattering of a Dirac particle in the presence of the cusp potential. In (1 + 1) dimensions, transmission resonances in the Duffin–Kemmer–Petiau (DKP) equation for an asymmetric cusp potential have been obtained in [29]. Recently, Sogut and Havare examined the scattering and bound state solutions of the DKP equation in the presence of the ASHp [22].

In the present paper, we study the scattering and bound state solutions of the 1D Schrödinger equation for the
We summarize our results in section 4. We give some numerical values of the energy eigenvalues for the transcendental equation that can be solved numerically. We hold \( q_1 \) and \( q_2 \) as parameters and \( a \) and \( b \) as positive parameters. \( \theta(x) \) is the Heaviside step function, and for the parameters \( q \) and \( \tilde{q} \) hold \( q < 1 \) and \( \tilde{q} < 1 \). Figure 1 shows the dependence of the ASHP barrier on these parameters.

The organization of this paper is as follows. In section 2, we search for the reflection and transmission coefficients in terms of hypergeometric functions for the ASHP barrier by using the form of the wave functions for \( x \to \pm \infty \). In section 3, we obtain a condition for extracting energy eigenvalues for the ASHP well. This condition is a transcendental equation that can be solved numerically. We give some numerical values of the energy eigenvalues for the bound states for selected values of the potential parameters. We summarize our results in section 4.

2. Reflection and transmission coefficients

The 1D time-independent Schrödinger equation for a particle with mass \( m \) moving in a potential \( V(x) \) reads

\[
\left\{ \frac{d^2}{dx^2} + 2m(E - V(x)) \right\} \Psi(x) = 0. \tag{2}
\]

We now look for the solution of the ASHP barrier for the region \( x < 0 \). Inserting equation (1) into (2) yields

\[
\left\{ \frac{d^2}{dx^2} + 2m \left[ E - \frac{V_0}{1 - q e^{a x} - \tilde{q}} \right] \right\} \Psi_L(x) = 0. \tag{3}
\]

Using a new variable \( y = q e^{a x} \) in equation (4), one obtains the following equation:

\[
y(1 - y)\psi''_L(y) + (1 - y)\psi'_L(y) + \frac{1}{y(1 - y)} \times \left\{ \beta_1 - \beta_2 y + \beta_3 y^2 \right\} \psi_L(y) = 0, \tag{4}
\]

where

\[
\beta_1 = \frac{2mE}{a^2}, \quad \beta_2 = \frac{4mE}{a^2} + \frac{2mV_0}{qa^2}, \quad \beta_3 = \frac{2mE}{a^2} + \frac{2mV_0}{qa^2}. \tag{5}
\]

Taking the trial wave function

\[
\psi_L(y) = y^\nu (1 - y)^\nu f(y) \tag{6}
\]

and inserting it into equation (4), we have

\[
y(1 - y)f''(y) + [1 + 2\mu - (2\mu + 2v + 1)y]f'(y) - (\mu + v + \nu) (\mu + v - \nu)f(y) = 0, \tag{7}
\]

which has the form of the hypergeometric-type equation [30]

\[
x(1 - s)x'' + [\zeta_1 - (\zeta_1 + \zeta_2 + 1)s]x' - \zeta_1 \zeta_2 x = 0 \tag{8}
\]

whose solution is given as 

\[
\psi_L(x) = A_12F_1(\mu + v - \gamma, \mu + v + \gamma; 1 + 2\mu; y) + A_2 y^{-\mu} 2F_1(-\mu + v - \gamma, -\mu + v + \gamma; 1 - 2\mu; y), \tag{9}
\]

and the whole solution for the region \( x < 0 \):

\[
\psi_L(x) = A_1 y^\nu (1 - y)^\nu 2F_1(\mu + v - \gamma, \mu + v + \gamma; 1 + 2\mu; y) + A_2 y^{-\mu} (1 - y)^\nu 2F_1(-\mu + v - \gamma, -\mu + v + \gamma; 1 - 2\mu; y), \tag{10}
\]

where

\[
\mu = \frac{i}{a}, \quad k = \sqrt{2mE}, \quad v = 1, \quad \gamma = \frac{i}{a} \sqrt{2m \left( E - \frac{V_0}{q} \right)}. \tag{11}
\]

We have to obtain the asymptotic form of the above wave function, since we seek the reflection and transmission coefficients. As \( x \to -\infty \), \( y \to 0 \) and \( (1 - y)^\nu \to 1 \), we obtain from equation (10)

\[
\psi_L(x \to -\infty) \sim A_1 y^\mu e^{i\mu x} + A_2 y^{-\mu} e^{-i\mu x} \sim A_1 q^{i/2 \alpha} e^{i k x} + A_2 q^{-i/2 \alpha} e^{-i k x}, \tag{12}
\]

where we have used \( 2F_1(\zeta_1, \zeta_2, \zeta_3; 0) = 1 \). To obtain the solution of the ASHP barrier for the region \( x > 0 \), we insert equation (1) into (2) and obtain

\[
\left\{ \frac{d^2}{dx^2} + 2m \left[ E - \frac{V_0}{e^{b x} - \tilde{q}} \right] \right\} \Psi_R(x) = 0. \tag{13}
\]
Defining the new variable \( z = \tilde{q} e^{-b x} \) yields
\[
(1 - z) \psi_R''(z) + (1 - z) \psi_R'(z) + \frac{1}{z(1 - z)} \times (\tilde{\beta} - \tilde{\beta} z + \tilde{\beta} z^2) \psi_R(z) = 0,
\]
where
\[
\tilde{\beta}_1 = \frac{2mE}{b^2}, \quad \tilde{\beta}_2 = \frac{4mE + 2mV_0}{q b^2}, \quad \tilde{\beta}_3 = \frac{2mE + 2mV_0}{q b^2}.
\]
By using a trial wave function \( \psi_R(z) = z^{\mu_i} (1 - z)^{\nu} h(z) \) in equation (14), we obtain the whole solution of the ASHP for the region \( x > 0 \):
\[
\psi_R(z) = A_3 z^{\mu_i} (1 - z)^{\nu} 2F_1(\mu_1 + v_1 - \gamma_1, \mu_1 + v_1 + \gamma_1; 1 + 2\mu_1 + z) + A_2 z^{\mu_i} (1 - z)^{\nu - 1} 2F_1(-\mu_1 + v_1 - \gamma_1, -\mu_1 + v_1 + \gamma_1; 1 - 2\mu_1; z),
\]
where
\[
\mu_1 = \frac{k}{b}, \quad k = \sqrt{2mE}, \quad v_1 = 1, \quad \gamma_1 = \frac{i}{b} \sqrt{2m \left( E + \frac{V_0}{q} \right)}.
\]
In order to define a plane wave traveling from left to right, we have to set \( A_3 = 0 \) in equation (16), so
\[
\psi_R(z) = A_2 z^{\mu_i} (1 - z)^{\nu} 2F_1(-\mu_1 + v_1 - \gamma_1, -\mu_1 + v_1 + \gamma_1; 1 - 2\mu_1; z).
\]
Now we give the form of the wave function at \( x \to +\infty \) for the region \( x > 0 \). As \( x \to +\infty, z \to 0 \) and \( (1 - z)^\nu \to 1 \), we have from equation (18)
\[
\psi_R(x \to +\infty) \sim A_4 (\tilde{q})^{-\mu_i} e^{b x} \sim A_4 (\tilde{q})^{-\mu_i} e^{b x}.
\]
As a result, we can summarize the wave function for the limit \( x \to \pm \infty \) from equations (12) and (19) as
\[
\psi(x) = \begin{cases} A_1 q^{\mu_i} e^{b x} + A_2 \tilde{q}^{\mu_i} e^{-b x}, & x \to -\infty, \\ A_4 (\tilde{q})^{-\mu_i} e^{b x}, & x \to +\infty. \end{cases}
\]
The wave function in equation (10) can be written as \( \psi_L = \psi_{inc} + \psi_{ref} \) in the limit \( x \to -\infty \), where \( \psi_{inc} \) is the incident and \( \psi_{ref} \) is the reflected wave. Similarly, as \( x \to +\infty \) the wave function in equation (18) is \( \psi_R = \psi_{trans} \), where \( \psi_{trans} \) is the transmitted wave. These definitions give us the reflection and transmission coefficients as
\[
R = \left| \frac{\psi_{ref}}{\psi_{inc}} \right|^2 = \left| \frac{A_2}{A_1} \right|^2,
\]
\[
T = \left| \frac{\psi_{trans}}{\psi_{inc}} \right|^2 = \left| \frac{A_4}{A_1} \right|^2.
\]
In order to give explicit expressions for the coefficients used in the above equations, we need to use the continuity conditions on the wave function given as \( \psi_R(x = 0) = \psi_L(x = 0) \) and \( \psi_R'(x = 0) = \psi_L'(x = 0) \), where the prime denotes the derivative with respect to \( x \). The matching of the wave functions at \( x = 0 \) gives
\[
A_1 C_1 F_1 + A_2 C_2 F_2 = A_4 C_3 F_3,
\]
and the matching of derivatives of the wave functions reads
\[
a q A_1 C_1 (D_1 F_1 + D_2 F_2) + a q A_2 C_2 (D_2 F_2 + D_3 F_3) = b q A_4 C_3 (D_3 F_3 - D_6 F_6),
\]
where we have used the property of the hypergeometric functions as \( \frac{d}{dx} F_1(\zeta_1; \zeta_2; \zeta_3; s) = \frac{\zeta_1}{\zeta_3} F_1(\zeta_1 + 1, \zeta_2 + 1; \zeta_1 + 1; s) \).
Combining the last two equations, we obtain the following for the coefficients written in equation (21):
\[
\begin{align*}
A_2 &= C_1 \left[ b q F_1(D_1 F_3 - D_6 F_6) - a q F_1(D_1 F_1 + D_2 F_2) \right], \\
A_4 &= a q C_1 [F_1(D_2 F_1 + D_3 F_3) - F_2(D_1 F_1 + D_4 F_4)] \\
A_1 &= C_2 \left[ a q F_3(D_2 F_2 + D_5 F_5) - b q F_2(D_3 F_3 - D_6 F_6) \right], \\
A_3 &= C_3 \left[ a q F_3(D_2 F_2 + D_5 F_5) - b q F_2(D_3 F_3 - D_6 F_6) \right],
\end{align*}
\]
where the following abbreviations have been used in the above equations:
\[
C_1 = q^{\mu_i} (1 - q)^\nu, \quad C_2 = q^{\mu_i} (1 - q)^\nu, \quad C_3 = (\tilde{q})^{-\mu_i} (1 - \tilde{q})^{\nu_i}.
\]
and the complete solution of equation (29) is given as
\[
\psi_L(y) = A_5 y \psi_1 (1 - y)^{\frac{1}{2}} 2 F_1 (\mu_2 + v_2 - y_2, \mu_2 + v_2 + y_2; 1 + 2 \mu_2; y) \\
+ A_6 y^{-\mu_2} 2 F_1 (-\mu_2 + v_2 - y_2, -\mu_2 + v_2 + y_2; 1 - 2 \mu_2; y). \quad (32)
\]

Next, we search for the solutions of the following form of equation (13) at \( x > 0 \):
\[
\frac{d^2}{dx^2} + 2m \left[ E + \frac{V_0}{e^{\alpha x} - q} \right] \psi(x) = 0. \quad (33)
\]
By using the variable \( z = \hat{q} e^{-bx} \) and putting \( \psi(z) = z^\nu \psi(x) \) in equation (33), we obtain
\[
\omega(z) = A_7 z F_1 (\mu_3 + v_3 - y_3, \mu_3 + v_3 + y_3; 1 + 2 \mu_3; z) \\
+ A_8 z^{-2\mu_3} 2 F_1 (-\mu_3 + v_3 - y_3, -\mu_3 + v_3 + y_3; 1 - 2 \mu_3; z), \quad (34)
\]
with the parameters
\[
\mu_3 = \frac{i}{b} \sqrt{2mE} = \mu_1, \quad v_3 = v_1 = 1, \\
\gamma_3 = \frac{1}{b} \sqrt{2m \left( E + \frac{V_0}{q} \right)}. \quad (35)
\]
Finally, we obtain the complete bound state solution of the Schrödinger equation for \( x > 0 \):
\[
\psi_R(z) = A_7 z^{\nu_3} (1 - z)^{\nu_3} 2 F_1 (\mu_3 + v_3 - y_3, \mu_3 + v_3 + y_3; 1 + 2 \mu_3; z) \\
+ A_8 z^{-2\mu_3} 2 F_1 (-\mu_3 + v_3 - y_3, -\mu_3 + v_3 + y_3; 1 - 2 \mu_3; z). \quad (36)
\]
The bound state wave functions in equations (32) and (36) should satisfy the boundary condition being zero at infinity which gives \( A_6 = A_8 = 0 \) and then we obtain
\[
\psi_L(y) \sim A_5 y^{\psi_1} (1 - y)^{\frac{1}{2}} 2 F_1 (\mu_2 + v_2 - y_2, \mu_2 + v_2 + y_2; 1 + 2 \mu_2; y), \quad (37)
\]
\[
\psi_R(z) \sim A_7 z^{\nu_3} (1 - z)^{\nu_3} 2 F_1 (\mu_3 + v_3 - y_3, \mu_3 + v_3 + y_3; 1 + 2 \mu_3; z). \quad (38)
\]
Matching the last two expressions in \( x = 0 \) requiring continuity of the wave function and of its first derivative gives
\[
A_5 F_1 (\mu_2, v_2, y_2, q) - A_7 F_2 (\mu_3, v_3, y_3, e^{bx}, q) = 0, \quad (39a)
\]
\[
A_5 \left( \frac{\mu_2}{q} - \frac{v_2}{1 - q} \right) F_1 (\mu_2, v_2, y_2, q) + F_3 (\mu_2, v_2, y_2, q) \\
- A_7 \left( \frac{\mu_3}{q} - \frac{v_3}{1 - q} \right) F_2 (\mu_3, v_3, y_3, q) \\
+ F_4 (\mu_3, v_3, y_3, q) = 0. \quad (39b)
\]

3. Energy eigenvalues

In this section, we deal with the bound state solutions of the ASHP well, which means that \( V_0 \rightarrow -V_0 \). Equation (3) for \( x < 0 \) becomes
\[
\left[ \frac{d^2}{dx^2} + 2m \left( E + \frac{V_0}{e^{\alpha x} - q} \right) \right] \psi(x) = 0. \quad (29)
\]
Using the transformation \( y = \hat{q} e^{ax} \) and taking the trial wave function \( \psi(y) = y^{\psi_1} (1 - y)^{\psi_2} \psi(y) \), the solution of equation (29) becomes
\[
g(y) = A_5 2 F_1 (\mu_2 + v_2 - y_2, \mu_2 + v_2 + y_2; 1 + 2 \mu_2; y) \\
+ A_6 y^{-\mu_2} 2 F_1 (-\mu_2 + v_2 - y_2, -\mu_2 + v_2 + y_2; 1 - 2 \mu_2; y), \quad (30)
\]
with the parameters
\[
\mu_2 = \frac{i}{a} \sqrt{2mE} = \mu, \quad v_2 = v = 1, \quad \gamma_2 = \frac{i}{a} \sqrt{2m \left( E - \frac{V_0}{q} \right)}. \quad (31)
\]

**Figure 2.** Variation of the transmission (T) and reflection (R) coefficients with \( E \) for \( a = 0.4, b = 0.5, q = 0.6, \hat{q} = 0.7, m = 1 \) and \( V_0 = 2 \).
Figure 3. Variation of the transmission coefficient with different potential parameters for $m = 1$, $V_0 = 2$. Left panel: $a = b = 0.5$; $a = 0.8, b = 0.3$; $a = 0.3, b = 0.8$; $q = \bar{q} = 0.7$. Right panel: $q = \bar{q} = 0.5$; $q = 0.6, \bar{q} = 0.4$; $q = 0.4, \bar{q} = 0.6$; $a = b = 0.5$.

Figure 4. Variation of the transmission coefficient with energy $E$ and potential parameter $V_0$ for $a = b = q = \bar{q} = 0.5, m = 1$.

where

$$F_1(\mu_2, v_2, \gamma_2, q) = q^{\mu_2} (1 - q)^{v_2} F_1(\mu_2 + v_2 - |\gamma_2|),$$

$$F_2(\mu_3, v_3, \gamma_3, \bar{q}) = (\bar{q})^{\mu_3} (1 - \bar{q})^{v_3} F_1(\mu_3 + v_3 - |\gamma_3|),$$

$$F_3(\mu_2, v_2, \gamma_2, q) = q^{\mu_2} (1 - q)^{v_2} \frac{(\mu_2 + v_2 - |\gamma_2|)((\mu_2 + v_2 + |\gamma_2|))}{1 + 2\mu_2} \times F_1(\mu_2 + v_2 - |\gamma_2| + 1, \mu_2 + v_2 + |\gamma_2| + 1; 2 + 2\mu_2; q),$$

$$F_4(\mu_3, v_3, \gamma_3, \bar{q}) = (\bar{q})^{\mu_3} (1 - \bar{q})^{v_3} \frac{(\mu_3 + v_3 - |\gamma_3|)((\mu_3 + v_3 + |\gamma_3|))}{1 + 2\mu_3} \times F_1(\mu_3 + v_3 - |\gamma_3| + 1, \mu_3 + v_3 + |\gamma_3| + 1; 2 + 2\mu_3; \bar{q}).$$

Equation (39) has a non-trivial solution only if its determinant is zero. Using this equation, one can determine the energy eigenvalues of the ASHp numerically. Here, we give our numerical results for the energy eigenvalues as a list for some values of the parameters, for example $m = 1$, $a = 0.5$, $b = 0.75$, $V_0 = 5$, $q = 0.1$ and $\bar{q} = 0.5$, taking into account that $-|V_0| < E < 0$: $E_1 = -2.453010$, $E_2 = -2.251290$, $E_3 = -0.924802$, $E_4 = -0.491271$, $E_5 = -0.001356$ (in atomic units).

4. Results and conclusions

We solved the one-dimensional Schrödinger equation for the asymmetric Hulthén potential. We found the transmission and reflection coefficients for the ASHp barrier in terms of hypergeometric functions and gave the plots showing the dependence of these coefficients on the potential parameters $a$, $b$, $q$, $\bar{q}$ and $V_0$ and on the energy $E$. We found that the unitarity condition is exactly satisfied in all cases. We also computed the energy eigenvalues for the bound states extracting an eigenvalue equation which can be solved numerically. We calculated five different energy eigenvalues by taking into account that $-|V_0| < E < 0$.

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