Bremsstrahlung by static charges outside a static black hole?

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Abstract

We show that in complete analogy with the usual bremsstrahlung process, when studied from the coaccelerated observer’s point of view, a charge moving along the integral curves of the static Killing field in the exterior of a static black hole gives rise to the emission of zero-energy photons, induced by the thermal bath of Hawking Radiation.
The relation between acceleration, radiation, and the equivalence principle has for a long time been the source of much confusion and discussion. In particular, in the case of flat spacetime, the natural question that arises is related to the compatibility of the following two facts: First, an accelerated charge is known to radiate from the point of view of Minkowski observers, and second, according to the equivalence principle, the same charge is seen by comoving observers as a static charge in a "gravitational field". This question has been successfully answered, first in the classical context by Rohrlich [1] and further elaborated by Boulware [2], who showed that the presence of an horizon for the collection of comoving observers describing the spacetime as static serves to explain the apparent contradictions, as due to the fact that the radiation zone (as described by the Minkowski observers) lies completely beyond the comoving observers horizon and is thus unobservable for them. In the quantum mechanical context, the solution to the apparent paradox (which now is cast in terms of photon emission rates) has been solved by the authors [3], by recalling that, as seen by the comoving observers the static charge (which has in fact constant proper acceleration) is immersed now in a Fulling-Davies-Unruh thermal bath [4–7] that describes, according to such observers, the Minkowski vacuum state of the Maxwell field, and that the interaction of the static charge with this bath of particles results in the stimulated emission of zero energy Rindler photons that completely account for the ordinary QED bremsstrahlung.

The purpose of this paper is to note that, in complete analogy to the result obtained in the case of the static charge in Rindler spacetime, which interacts with the thermal bath representing the Minkowski vacuum, the analysis of a static charge in a static black hole spacetime, which interacts with a thermal bath representing the Hartle-Hawking vacuum, yields a finite and nonzero response rate that is completely analogous to the result known to correspond to bremsstrahlung.

The procedure used to obtain the rate of emission and absorption into and from the thermal bath that was successfully employed in [3] is the following: First, we note that
the introduction of a regulator is required in order to make sense of the expression of the form $0 \times \infty$ that arises from the simultaneous consideration of the rate of emission into the Rindler vacuum of "zero energy photons", and the emission-stimulating effect of "the number of zero energy photons present in the thermal bath". The regulating procedure consisted in introducing a fictitious oscillation in the value of the charge, with frequency $E$, and then to take the limit $E \to 0$ after the corresponding rate was evaluated. In this analysis, we will proceed in precisely the same fashion, with the added complication that, due to the fact that we do not know the explicit form of the mode functions in the Schwarzschild metric, we need to introduce a simulated potential that mimics the main features of the true effective potential, and that is simple enough that the modes can be found explicitly.

We will concentrate for simplicity in the case of a scalar field $\Phi$ that interacts with a source $j$ and is described by the action:

$$S_0 = \int_M d^4x \sqrt{-g} \left( \nabla^\mu \Phi \nabla_\mu \Phi + j \Phi \right)$$

with the background spacetime corresponding to a Schwarzschild black hole with mass $M$ and horizon at $r_H = 2M$. Thus we write the metric:

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 - \frac{2M}{r}\right)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

We proceed to quantize the free scalar field in a canonical way using for this the notion of positive energy provided by the Killing field $\partial/\partial t$. We will thus consider the field only in the exterior of the black hole. The scalar field satisfies the equation of motion in the corresponding spacetime metric:

$$\Box \Phi_0(x) = 0.$$ 

The solution of (3) can be written in terms of the positive frequency modes

$$u_{\omega lm} = \frac{\psi_{\omega l}(r)}{r} Y_{lm}(\theta, \phi)e^{-i\omega t}.$$
Here $\psi_\omega(l)$ is the solution of the ordinary equation
\[
\frac{d}{dr} \left[ f(r) \frac{d\psi_\omega(r)}{dr} \right] - s(r) \psi_\omega(r) - l(l+1) \frac{\psi_\omega(r)}{r^2} = 0,
\]
where $f(r) = (1 - 2M/r)$ and $s(r) = -f^{-1}(r)\omega^2 + 2M/r^3$.

For each pair of values $\omega, l$ there are actually two independent solutions of (5). We choose to use as mode I the mode that is purely incoming from the past horizon $H^-$, and as mode II the mode that is purely incoming from past null infinity $J^-$. These two modes are then automatically orthogonal to each other (with respect to the natural Klein Gordon inner product).

Thus, it is possible to expand the scalar field in terms of positive and negative energy modes as
\[
\Phi_{\omega lm}(t, r, \theta, \phi) = \sum_{l,m,\alpha} \int_{0}^{+\infty} d\omega (a_{\omega lm\alpha} u_{\omega lm\alpha} + H.c.),
\]
where $a_{\omega lm\alpha}$ and $a_{\omega lm\alpha}^\dagger$ are annihilation and creation operators of particles with quantum numbers $\omega, l, m$ and $\alpha = I, II$. They satisfy the usual commutation relations
\[
[ a_{\omega lm\alpha}, a_{\omega' l'm'\alpha'}^\dagger ] = \delta(\omega - \omega')\delta_{ll'}\delta_{mm'}\delta_{\alpha\alpha'}.
\]

In order for (7) to follow from the canonical commutation relations for the field and its conjugate momentum, it is necessary that the normal modes (4) are Klein Gordon orthonormalized [9]. This normalization condition corresponds to:
\[
(\omega + \omega') \int_{r_H}^{+\infty} dr f^{-1}(r) \psi_\omega(r) \psi_{\omega'}^*(r) = \delta(\omega - \omega').
\]

Making use of the mode equation (5), we rewrite the condition (8) in the form
\[
\lim_{L \to \infty} \left\{ \left[ -\psi_\omega \frac{d\psi_{\omega'}^*}{dr} + \psi_{\omega'}^* \frac{d\psi_\omega}{dr} \right] \frac{f(r)}{(\omega' - \omega)} \right\}_{r_H}^L = \delta(\omega - \omega').
\]

Now, we consider the interaction of the field with a charge $q$, that is at rest with respect to static observers, i.e, it is following an orbit of the timelike Killing field $\partial_t$. This is described, in the above coordinates, by the density
\[ j(x) = q\delta(r - r_0)\delta(\theta - \theta_0)\delta(\phi - \phi_0)/\sqrt{h}, \]  
\[ (10) \]

where \( h = -\text{det}(h_{\mu\nu}) \), and \( h_{\mu\nu} \) is the spatial metric induced over the equal time hypersurface \( \Sigma_t \). With this definition, we obtain

\[ \int_{\Sigma_t} j = q, \]
\[ (11) \]

for any \( \Sigma_t \), where the natural volume element over \( \Sigma_t \) is understood.

Our aim is to evaluate the particle emission and absorption rates to and from the thermal bath in which the charge is immersed. Since the charge is static, it is clear that the spontaneous emission rate vanishes. However, this does not imply that the induced emission and absorption rates must also vanish. This is because these rates will depend on the number of particles present in the bath, and which interact effectively with the source. In our case, the relevant modes are the zero-energy modes, because the static current \((10)\) cannot interact with any of the other modes. Since the number of zero-energy modes per unit volume in a thermal bath diverges, the induced emission and absorption rates are indefinite. Here, as we mentioned before, we will use the same “regularization procedure” used in \([3]\) and \([8]\), because it has already led to physically and mathematically sound results. The procedure consists in replacing the static current \((10)\) by an oscillating one

\[ j(x) = q\sqrt{2}\cos(tE)\delta(r - r_0)\delta(\theta - \theta_0)\delta(\phi - \phi_0)/\sqrt{h}, \]
\[ (12) \]

and taking the limit \( E \to 0 \) at the end of the calculations. The \( \sqrt{2} \) factor appears because of the fact that, at tree level, the emission and absorption response rates are a function of \( q^2 \), and, by the requirement that the time average of the square of this current is equal to \( q^2 \), the charge interacts with the scalar field through the interaction corresponding to the second term in \((1)\).

Now, let us calculate the emission amplitude of a Boulware mode \( |\omega l m \alpha \rangle \), when our source is in the Boulware vacuum \( |0\rangle \); i.e., the quantum state annihilated by \( a_{\omega l m \alpha} \). At tree level, we have
\[ A_{\omega lm\alpha}^{em} = \langle \omega lm\alpha | S_I | 0 \rangle. \]  

(13)

Therefore, we obtain

\[ A_{\omega lm\alpha}^{em} = q \sqrt{2 \pi^2 f(r_0)/r_0^2} \psi_{\omega l\alpha}^*(r_0) Y_{lm}^*(\theta_0, \phi_0) \delta(\omega - \omega_0), \]  

(14)

where we have defined \( \omega_0 \equiv E \). We recall that \((r_0, \theta_0, \phi_0)\) are the spatial coordinates of the charge’s position. We note that a static charge can only interact with zero-energy modes, since in the limit \( E \to 0 \), the amplitude is proportional to \( \delta(\omega) \).

The thermal bath is characterized by a temperature \( \beta^{-1} = \mathcal{K}/2\pi \), where \( \mathcal{K} \), the surface gravity, is \( 1/4M \) (for a Schwarzschild black hole). Thus, the emission rate per total proper time \( T^{\text{tot}} \) of particles with fixed angular momentum is, in this case,

\[ \frac{P_{em}^{lm} T^{\text{tot}}}{T^{\text{tot}}} = \frac{1}{T^{\text{tot}}} \int_0^{+\infty} d\omega |A_{\omega lm\alpha}^{em}|^2 \left[ 1 + \frac{1}{e^{\omega/\beta} - 1} \right]. \]  

(15)

The first term inside the brackets corresponds to the spontaneous emission contribution, while the second one corresponds to the induced emission contribution. Notice that, for small \( \omega \), the induced emission dominates over the spontaneous emission. Substituting (14) in (13), we obtain

\[ \frac{P_{em}^{lm} T^{\text{tot}}}{T^{\text{tot}}} = \sum_\alpha q^2 \pi f^{1/2}(r_0)(1/r_0^2) |\psi_{\omega l\alpha}(r_0)|^2 |Y_{lm}(\theta_0, \phi_0)|^2 \left[ 1 + \frac{1}{e^{\omega_0/\beta} - 1} \right], \]  

(16)

where we have used \( T^{\text{tot}} = 2\pi f^{1/2}(r_0) \delta(0) \). We are interested, however, in the limit \( E \to 0 \), \( (\omega_0 \to 0) \) so (16), becomes

\[ \frac{P_{em}^{lm} T^{\text{tot}}}{T^{\text{tot}}} = \sum_\alpha q^2 \pi f^{1/2}(r_0) |Y_{lm}(\theta_0, \phi_0)|^2 \lim_{\omega_0 \to 0} \left[ \begin{array}{c} |\psi_{\omega l\alpha}(r_0)|^2 \\ \omega_0 \end{array} \right]. \]  

(17)

Analogously, the absorption rate per total proper time of particles with fixed angular momentum is

\[ \frac{P_{abs}^{lm} T^{\text{tot}}}{T^{\text{tot}}} = \frac{1}{T^{\text{tot}}} \int_0^{+\infty} d\omega |A_{\omega lm\alpha}^{abs}|^2 \left[ 1 + \frac{1}{e^{\omega/\beta} - 1} \right]. \]  

(18)

Unitarity implies that \( A_{\omega lm\alpha}^{abs} = A_{\omega lm\alpha}^{em} \), hence, the absorption rate of zero-energy particles by the static scalar charge is given simply by
\[ \frac{P_{\text{abs}}^{\text{lm}}}{T_{\text{tot}}} = \sum_{\alpha} q^2 \pi f^{1/2}(r_0) |Y_{lm}(\theta_0, \phi_0)|^2 \lim_{\omega_0 \to 0} \frac{|\psi_{\omega_0 l \alpha}(r_0)|^2}{\omega_0}. \] (19)

We conclude that a static charge outside a black hole emits and absorbs zero-energy modes with identical rates given by (17), which we could calculate explicitly, if we knew the exact form of \( \psi_{\omega l}^{\alpha} \).

In order to get more explicit information, in particular, to ascertain whether these rates are finite, divergent or zero, we proceed to replace the radial equations for the modes by an analogous equation with the effective potential replaced by a simulated potential with similar features (See [11] for other uses of this method).

It will be convenient to use the dimensionless Wheeler tortoise coordinate \( x \equiv y + \ln(y - 1) \), where \( y \equiv r/2M \). The desired calculation for the exact Schwarzschild modes consists then in finding the solutions \( \phi(x)_{\tilde{\omega} l}^{\alpha} \) of the equation:

\[ \frac{d^2}{dx^2} \phi(x) + [\tilde{\omega}^2 - V_{\text{eff}}(x)] \phi(x) = 0, \] (20)

where \( \tilde{\omega} = 2M \omega \) and where the effective potential is:

\[ V_{\text{eff}}(x) = (1 - 1/y)(1/y^3 + l(l + 1)/y^2), \] (21)

and normalize them so

\[ \lim_{L \to \infty} [\phi^*(x)_{\tilde{\omega} l} \frac{d}{dx} \phi(x)_{\tilde{\omega} l} - \phi(x)_{\tilde{\omega} l} \frac{d}{dx} \phi^*(x)_{\tilde{\omega} l}]^L_L = 2M (\tilde{\omega}' - \tilde{\omega}) \delta(\tilde{\omega}' - \tilde{\omega}). \] (22)

Then to recover the modes, we just put

\[ \psi_{\omega l}^{\alpha}(r) = \phi(x(r))_{\tilde{\omega} l}^{\alpha}, \] (23)

where \( x(r) = r/2M + \ln(r/2M - 1) \) and \( \omega = \tilde{\omega}/2M \). The approximate calculation for Schwarzschild modes consists in solving Eq. (20), but with the effective potential replaced by a simulated potential which is simpler, and yet preserves the essential features of the true effective potential. We will take the following form for the simulated potential:

\[ V(x) = \frac{l(l + 1)}{x^2}, \] (24)
for $x > 1$ and $V(x) = 0$ otherwise.

We can now write explicitly the two solutions corresponding to $\alpha = I, II$ for each $\omega$ and $l$.

1) The mode incoming from the past horizon $H^-$

$$\phi^I_{\omega l}(x) = a^I_{\omega l}x h^{(1)}_{\omega}(\bar{\omega}x)$$

for $x > 1$

$$\phi^I_{\omega l}(x) = a^I_{\omega l}(\beta^I_{\omega l} e^{i\bar{\omega}x} + \gamma^I_{\omega l} e^{-i\bar{\omega}x})$$

for $x < 1$.

2) The mode incoming from past null infinity $J^-$:

$$\phi^H_{\omega l}(x) = a^H_{\omega l}(\beta^H_{\omega l} x h^{(2)}_{\omega}(\bar{\omega}x) + \gamma^H_{\omega l} x h^{(1)}_{\omega}(\bar{\omega}x))$$

for $x > 1$ and

$$\phi^H_{\omega l}(x) = a^H_{\omega l} e^{-i\bar{\omega}x}$$

for $x < 1$.

Here $h^{(1)}_{\omega}(x) = j_{\omega}(x) + i\eta_{\omega}(x)$ and $h^{(2)}_{\omega}(x) = j_{\omega}(x) - i\eta_{\omega}(x)$ are spherical Bessel functions (See [12] for properties and asymptotia).

From the continuity of the mode-functions and their derivatives at $x = 1$ we find:

$$\beta^I_{\omega l} = (1/2)e^{-i\bar{\omega}}[(1 - i/\bar{\omega})h^{(1)}_{\omega}(\bar{\omega}) - i(h^{(1)}_{\omega})'(\bar{\omega})]$$

$$\gamma^I_{\omega l} = (1/2)e^{i\bar{\omega}}[(1 + i/\bar{\omega})h^{(1)}_{\omega}(\bar{\omega}) + i(h^{(1)}_{\omega})'(\bar{\omega})].$$

Here the prime indicates the derivative of the function with respect to its argument.

We note that:

$$|\beta^I_{\omega l}|^2 - |\gamma^I_{\omega l}|^2 = (\bar{\omega})^{-2}.$$
For the mode II we find

\[ \beta_{\tilde{\omega}l}^H = (\tilde{\omega}^2/2i)e^{-i\tilde{\omega}}[(h_l^{(1)})'(\tilde{\omega}) + (i + 1/\tilde{\omega})h_l^{(1)}(\tilde{\omega})] \tag{32} \]

\[ \gamma_{\tilde{\omega}l}^H = (-\tilde{\omega}^2/2i)e^{-i\tilde{\omega}}[(h_l^{(2)})'(\tilde{\omega}) + (i + 1/\tilde{\omega})h_l^{(2)}(\tilde{\omega})], \tag{33} \]

and note that

\[ |\beta_{\tilde{\omega}l}^H|^2 - |\gamma_{\tilde{\omega}l}^H|^2 = (\tilde{\omega})^2. \tag{34} \]

Thus normalizing the modes according to (22) we find:

\[ |a_{\tilde{\omega}l}^I|^2 = \frac{M}{2\pi\tilde{\omega}}|\beta_{\tilde{\omega}l}^I|^{-2}, \tag{35} \]

and

\[ |a_{\tilde{\omega}l}^H|^2 = \frac{M\tilde{\omega}}{2\pi}|\beta_{\tilde{\omega}l}^H|^{-2}. \tag{36} \]

For \( l \neq 0 \) we have for small \( \tilde{\omega} \)

\[ |\beta_{\tilde{\omega}l}^I|^2 \approx (1/4)l^2d_l^2\tilde{\omega}^{-(2l+4)} \tag{37} \]

\[ |\beta_{\tilde{\omega}l}^H|^2 \approx (1/4)l^2d_l^2\tilde{\omega}^{-(2l)}. \tag{38} \]

Now to compute the response rate per unit proper time we need the the wave function as for all values of \( x \) in the small \( \tilde{\omega} \) limit.

First analyze mode I

For \( x < 1 \) we have:

\[ |\phi_{\tilde{\omega}l}(x)|^2 = \frac{M}{2\pi\tilde{\omega}}[2 - \tilde{\omega}^{-2}|\beta_{\tilde{\omega}l}^I|^2 + 2|\beta_{\tilde{\omega}l}^I|^{-2}\Re(\beta_{\tilde{\omega}l}^I(\gamma_{\tilde{\omega}l}^I)^*e^{2i\tilde{\omega}x})], \tag{39} \]

and after some calculation we find

\[ |\phi_{\tilde{\omega}l}(x)|^2 = \frac{M\tilde{\omega}}{\pi}2[(1 - x) + l^{-1}]^2 + O(\tilde{\omega}^3). \tag{40} \]
For $x > 1$:

$$|\phi^I_{\omega l}(x)|^2 = \frac{2M\tilde{\omega}}{l^2\pi} x^{-2l} + \mathcal{O}(\tilde{\omega}^3). \quad (41)$$

Note that $|\phi^I_{\omega l}(x)|^2$ is proportional to $\tilde{\omega}$, therefore, as can be seen from Eq. (17), this mode yields a finite and non-zero rate.

Next we analyze mode II;

For $x > 1$ we have:

$$|\phi^{II}_{\omega l}(x)|^2 \approx \frac{M\tilde{\omega}}{2\pi} x^2 |\beta^{II}_{\omega l}|^2 \times$$

$$[|\beta^{II}_{\omega l} - \gamma^{II}_{\omega l}|^2 \eta(\tilde{\omega}x)^2 + |\beta^{II}_{\omega l} + \gamma^{II}_{\omega l}|^2 j_l(\tilde{\omega}x)^2 + 4j_l(\tilde{\omega}x)\eta(\tilde{\omega}x)\Im(\beta^{II}_{\omega l}(\gamma^{II}_{\omega l})^*)], \quad (43)$$

and after some calculation

$$|\phi^{II}_{\omega l}(x)|^2 \approx \frac{2M\tilde{\omega}}{\pi} \tilde{\omega}^{2l} x^2 (c_l^2/l^2)[(l + 1)x^{-(l+1)} + lx^l]. \quad (44)$$

For $x < 1$:

$$|\phi^{II}_{\omega l}(x)|^2 \approx \frac{M\tilde{\omega}}{2\pi} 4\tilde{\omega}^{2l}/(d_l^2l^2). \quad (45)$$

Thus $|\phi^{II}_{\omega l}(x)|^2$ is proportional to $\tilde{\omega}^{2l+1}$ and according to Eq. (17) this mode yields zero emission and absorption rates.

Finally, we write explicitly the response rate of the charge to zero-energy modes in first order perturbation theory, and with the approximation corresponding to the substitution of the true effective potential of Eq. (21) by the simulated potential of Eq. (24).

Case I: The charge is at $x > 1$ (or $r/2M > 1.56$)

$$P^{em}_{l_0l_1}(x > 1)/T^{tot} = P^{abs}_{l_0l_1}(x > 1)/T^{tot} = \frac{q^2M}{2\pi} \frac{|Y_{lm}(\theta_0, \phi_0)|^2}{l^2} \times$$

$$\frac{(1 - 2M/r_0)^{1/2}}{r_0^2} \left[r_0/2M + ln(r_0/2M - 1)\right]^{-2l} \quad (46)$$

where $(r_0, \theta_0, \phi_0)$ are the coordinates of the charge’s position.
Case II: The charge is at \( x \leq 1 \) (or \( r/2M \leq 1.56 \))

\[
P_{lm}^{em}(x \leq 1)/T_{tot}^{em} = P_{lm}^{abs}(x \leq 1)/T_{tot}^{abs} = \frac{q^2 M}{2\pi} |Y_{lm}(\theta_0, \phi_0)|^2 \times
\]

\[
\frac{(1 - 2M/r_0)^{1/2}}{r_0^2} \left[ t^{-1} + 1 - r_0/2M - ln(r_0/2M - 1) \right]^2,
\]

(47)

Notice that (i) these rates are finite and nonzero, (ii) if \( r_0 \rightarrow +\infty \) the responses vanish (iii) if \( r_0 \rightarrow 2M \) the responses also vanish because \( \lim_{r \rightarrow 0} \sqrt{r} \ln^2 \epsilon = 0 \).

Also, it is interesting to note that since the only mode that generates a response is the mode I (coming from the past horizon \( H^- \)) the response is going to be the same in the Hartle Hawking vacuum or the Unruh vacuum.

The interpretation of these results is, however, not as straightforward as in the case of the charge undergoing constant proper acceleration in Minkowski spacetime, because in that case we have both the Killing field associated with the comoving observers, and a second Killing field associated with the global inertial system, so the comparison of the corresponding results confirmed the interpretation of the response rate as ordinary bremsstrahlung. However, the lesson learned in that exercise strongly supports the interpretation of the response rate in this case as bremsstrahlung by static charges in a static black hole spacetime. It will be interesting to investigate if this result can be considered as a \( 0^{th} \) order approximation to the back reaction effect in the case of a black hole resulting from gravitational collapse, i.e. considering these response rates as the interaction of the Hawking radiation with the infalling matter considered as quasistatic.

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