Incoherent bound states in an infinite XXZ chain at

\[ \Delta = -1/2 \]

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Abstract

For an infinite XXZ chain with \( \Delta = -1/2 \) we have obtained a family of translationary invariant three-magnon states which do not satisfy the string conjecture. All of them have the same energy.

1 Introduction

We shall study an infinite XXZ spin chain [1] related to the Hamiltonian

\[ H = \sum_{n=-\infty}^{\infty} H_{n,n+1}, \]

where

\[ H_{n,n+1} = S_{n}^{x}S_{n+1}^{x} + S_{n}^{y}S_{n+1}^{y} + \Delta \left( S_{n}^{z}S_{n+1}^{z} - \frac{1}{4} \right). \]

The corresponding Hilbert space is an infinite tensor product of \( \mathbb{C}^2 \) spaces associated with the lattice sites. In every such space we shall use the following basis

\[ S_{n}^{z}|\pm\rangle_n = \pm \frac{1}{2} |\pm\rangle_n. \]
Here and in (2) \( S_n \) denotes a triple of \( S = 1/2 \) spin operators associated with \( n \)-th site. \( \Delta \) is a real parameter. The following transformation

\[
H \rightarrow -UHU^{-1},
\]

where

\[
U = \prod_n \sigma^z_{2n},
\]

\((\sigma^j_n = 2S^j_n, \text{for } j = x, y, z \text{ are the Pauli matrices})\) is equivalent to the substitution \( \Delta \rightarrow -\Delta \). The result of our paper corresponds to the special case \( \Delta = -1/2 \).

Traditionally the model (1) is treated on a finite chain related to the Hilbert space \((\mathbb{C}^2)^{\otimes N}\) (\( N \) is the number of sites). Usually there supposed periodic boundary conditions \([2]-[4]\)

\[
H^{(\text{period})} = \sum_{n=1}^{N} H_{n,n+1}, \quad N + 1 \equiv 1,
\]

(see however \([3]\) where the chain with open boundaries was studied).

Since both the Hamiltonians (1) and (6) commute with \( S^z \) the \( z \) component of the total spin

\[
S = \sum_n S_n
\]

their spectrums split on subsectors corresponding to different values of \( S^z \). Bethe Ansatze is used as an effective method for treating the Hamiltonian (1) \([1]\) or (6) \([2]-[4]\) separately in all subsectors. Within this approach first of all is considered the highest \( S^z \) state

\[
|\Omega\rangle = \prod_n |+\rangle_n,
\]

which is an eigenvector of both (1) and (6). The next sector is generated by quasiparticles (magnons). Since both the Hamiltonians (1) and (6) commute with lattice translations one can readily obtain an explicit form of the one-magnon state with quasi momentum \( k \)

\[
|1, k\rangle = \sum_n e^{ikn} \left( \prod_{m=n_{\text{min}}}^{n-1} \otimes |+_m\rangle \right) \otimes |-_n\rangle \otimes \left( \prod_{m=n+1}^{n_{\text{max}}} \otimes |+_m\rangle \right),
\]

where for the infinite chain \( n_{\text{min}} = -\infty, n_{\text{max}} = \infty \) while for the finite one \( n_{\text{min}} = 1, n_{\text{max}} = N \). The corresponding dispersion

\[
E_{\text{magn}}(k) = \cos k - \Delta,
\]
readily follows from the local action formulas

\[
H_{n,n+1} |\mp\rangle_n |\pm\rangle_{n+1} \cdots = -\frac{\Delta}{2} |\mp\rangle_n |\pm\rangle_{n+1} \cdots + \frac{1}{2} |\pm\rangle_n |\mp\rangle_{n+1} \cdots,
\]

\[
H_{n,n+1} |\pm\rangle_n |\pm\rangle_{n+1} \cdots = 0,
\]

which are consequences of (2).

The exponent \(e^{ikn}\) in (9) is a one-magnon wave function. Within Bethe Ansatz wave functions of all eigenstates are represented as sums of Bethe exponents. For example for a general two-magnon state

\[
|2, k_1, k_2\rangle = \sum_{m<n} \psi(m, n; k_1, k_2) |\mp\rangle_m |\pm\rangle_n \cdots,
\]

(where \(\cdots\) denote a product of suitable \(|+\rangle_l\), \(l \neq m, n\) similar to the products in (9)) the wave function should be a superposition of two Bethe exponents

\[
\psi(m, n; k_1, k_2) = C_{12}(k_1, k_2)e^{i(k_1m + k_2n)} + C_{21}(k_1, k_2)e^{i(k_2m + k_1n)}.
\]

The parameters \(k_1\) and \(k_2\) have sense of magnon quasi momentums. They must be either real or complex. The former case correspond to a scattering state while the later to a bound one. Correspondingly a \(n\)-magnon wave function is a linear combination of \(n!\) Bethe exponents depending on \(n\) different parameters \(k_1, \ldots, k_n\). The corresponding total quasi momentum \(k = \sum_{j=1}^{n} k_j\) must be real. The dispersion is

\[
E(k_1, \ldots, k_n) = \sum_{j=1}^{n} E_{magn}(k_j).
\]

Both for (1) and (6) the parameters \(k_j\) can not be arbitrary. However the corresponding restrictions on them are different. In the infinite case one should postulate that the wave function is bounded. This requirement results in some inequalities on imaginary parts of \(k_j\) (see for example Eq. (18) below). In other respects the parameters \(k_j\) may be arbitrary. From the opposite side in the finite case the parameters \(k_j\) are solutions of a transcendental system of Bethe equations. That is why the finite problem is much more complicated than the infinite one where the Bethe equations are not essential at all.

Of course physically relevant results usually belongs to infinite chains. However within some approaches they may be obtained only after suitable finite chain calculations before passing to the \(N \to \infty\) limit [2], [4]. Of course the latter must be defined correctly. First of all one has to control disappearance of all ”bad” exponents resulting unbounded wave
functions. But there is another very important statement relevant to $N \to \infty$ behavior of quasi momentums. Namely this is the so called string conjecture which asserts that at $N \to \infty$ all quasi momentums considered as solutions of the Bethe equations group into special complexes ”strings”. Within each of them all $k_j$ have similar real parts while their imaginary parts form equidistant lattices symmetric with respect to the real axis. For example for a two-magnon bound state related to the wave function (13) there must be $k_1 = u - iv$, $k_2 = u + iv$ ($v > 0$). Since in (13) $m < n$ the second term is ”bad” (unbounded). So there should be $C_{21}(k_1, k_2) = 0$. This condition produce a relation between $u$ and $v$.

All magnons within the same complex have a similar spatial dependence of phase. That is why a complex may be considered as a coherent bound state of the corresponding magnons.

Usually it is assumed that for the XXZ model the string conjecture is right. Within this assumption thermodynamics of the infinite XXZ chain was studied in [6] for $|\Delta| \geq 1$ and in [7] for $|\Delta| < 1$. However in the present paper we show that the string conjecture fails in the special point $\Delta = -1/2$. Namely we shall present a family of three-magnon infinite-chain incoherent bound states with total zero quasi momentum.

The $\Delta = -1/2$ XXZ chain is now intensively studied in various aspects (see the recent articles [8]-[10] and references therein). We believe that our result shed an additional light on this model.

## 2 Three-magnon incoherent bound states

First of all let us utilize the translation invariance and represent a three magnon state with total quasimomentum $k$ in the following general form

$$|3, k\rangle = \sum_{m<n<p} e^{ik(m+n+p)/3} a(k, n - m, p - n) \ldots |-\rangle_m \ldots |-\rangle_n \ldots |-\rangle_p \ldots$$ (15)

Reduced (to the center mass frame) wave function $a(k, m, n)$ has a physical sense only at $m, n > 0$ but may be continued to $m = 0, n > 0$ and $m > 0, n = 0$ according to Bethe conditions

$$2\Delta a(k, 1, n) = e^{ik/3} a(k, 0, n) + e^{-ik/3} a(k, 0, n + 1),$$
$$2\Delta a(k, m, 1) = e^{-ik/3} a(k, m, 0) + e^{ik/3} a(k, m + 1, 0).$$ (16)
Under (16) the Schrödinger equation in the whole region $m, n > 0$ takes the following form

$$
-3\Delta a(k, m, n) + \frac{1}{2} \left[ e^{-ik/3} a(k, m + 1, n) + e^{i k/3} a(k, m - 1, n) \\
+ e^{-ik/3} a(k, m - 1, n + 1) + e^{ik/3} a(k, m + 1, n - 1) + e^{-ik/3} a(k, m, n - 1) \\
+ e^{ik/3} a(k, m, n + 1) \right] = E a(k, m, n).
$$

(17)

The following trial bounded wave function

$$
a(m, n) = e^{(iu_1 - v_1)m + (iu_2 - v_2)n},
$$

(18)

satisfy (17) for

$$
E(k, u_1, u_2, v_1, v_2) = \cosh v_1 \cos (k/3 - u_1) + \cosh v_2 \cos (k/3 + u_2) \\
+ \cos (v_1 - v_2) \cos (k/3 + u_1 - u_2) - 3\Delta.
$$

(19)

Normalization condition

$$
\sum |a(m, n)|^2 < \infty,
$$

(20)

results in

$$
v_{1,2} > 0.
$$

(21)

Form the other side the system (16) gives

$$
x_1 = e^{i(k/3 + u_1) - v_1} + e^{-ik/3} - 2\Delta_1 e^{iu_2 - v_2} = 0, \\
x_2 = e^{ik/3} + e^{i(u_2 - k/3) - v_2} - 2\Delta_1 e^{iu_1 - v_1} = 0.
$$

(22)

Treating $x_1 - \bar{x}_2$ one may readily obtain

$$
2\Delta F = -\bar{F} e^{ik/3},
$$

(23)

where $F = e^{iu_2 - v_2} - e^{-iu_1 - v_1}$.

At $4\Delta^2 \neq 1$ Eq. (23) gives $F = 0$ or equivalently $u_1 = -u_2$ and $v_1 = v_2$. In this case the string conjecture is satisfied. However in two special points $\Delta = \pm 1/2$ connected by the symmetry (4) there should be additional solutions. Taking $\Delta = -1/2$ and treating $x_1 - e^{ik/3}x_2$ one gets

$$
k = 0.
$$

(24)

Now the system (22) results in

$$
e^{-v_1} \cos u_1 + e^{-v_2} \cos u_2 = -1, \quad e^{-v_1} \sin u_1 = -e^{-v_2} \sin u_2,
$$

(25)
or
\[ e^{v_1} = \frac{\sin(u_1 - u_2)}{\sin u_2}, \quad e^{v_2} = \frac{\sin(u_2 - u_1)}{\sin u_1}. \] (26)

According to (26) \( \sin u_1 \) and \( \sin u_2 \) have opposite signs. Without loss of generality one may put
\[ 0 < u_1 < \pi, \quad -\pi < u_2 < 0. \] (27)

Under this assumption both \( \cos u_{1,2}/2 > 0 \) and the system (21) is reducible to
\[ \sin \left( \frac{u_1 - u_2}{2} \right) < 0, \quad \sin \left( \frac{u_1}{2} - u_2 \right) < 0, \] (28)
or equivalently
\[ \pi < u_1 - \frac{u_2}{2} < 2\pi, \quad \pi < \frac{u_1}{2} - u_2 < 2\pi. \] (29)

It may be readily proved from (19) and (26) that all these states have zero energy. According to (15) and (18) they describe magnon triples with corresponding quasi-momentums
\[ k_1 = -u_1 - iv_1, \quad k_2 = u_1 - u_2 + i(v_1 - v_2), \quad k_3 = u_2 + iv_2. \] (30)

The string conjecture obviously is failed.

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