Hypergeometric Groups and Dynamics on K3 Surfaces

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Abstract

A hypergeometric group is a matrix group modeled on the monodromy group of a generalized hypergeometric differential equation. This article presents a fruitful interaction between the theory of hypergeometric groups and dynamics on K3 surfaces by showing that a certain class of hypergeometric groups and related lattices lead to a lot of K3 surface automorphisms of positive entropy, especially such automorphisms with Siegel disks.

1 Introduction

This article originates from a simple question: What happens if we put the following two topics together? One is the theory of hypergeometric groups due to Beukers and Heckman [4] and the other is dynamics on K3 surfaces due to McMullen [16, 18, 19]; see also Gross and McMullen [8]. In this article we present a fruitful interaction between them by showing that a certain class of hypergeometric groups and related lattices produce a lot of K3 surface automorphisms of positive entropy, especially such automorphisms with Siegel disks.

A hypergeometric group is a group modeled on the monodromy group of a generalized hypergeometric differential equation. It is a matrix group \( H = \langle A, B \rangle \subset \text{GL}(n, \mathbb{C}) \) generated by two invertible matrices \( A \) and \( B \) such that \( \text{rank}(A - B) = 1 \), which is equivalent to the condition \( \text{rank}(I - C) = 1 \) for the third matrix \( C := A^{-1}B \). In the context of an \( n \)-th order hypergeometric equation the matrices \( A, B, C \) are the local monodromy matrices around the regular singular points \( z = 0, \infty \), respectively. The rank condition for \( C \) is then a consequence of the property that the differential equation has \( n - 1 \) linearly independent holomorphic solutions around \( z = 1 \). Beukers and Heckman [4] established several fundamental properties of hypergeometric groups such as irreduciblity, invariant Hermitian form, signature, etc. and went on to classify finite hypergeometric groups. Along the way they also determined differential Galois groups of hypergeometric equations, that is, Zariski closures of hypergeometric groups.

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On the other hand, McMullen [16] synthesized examples of K3 surface automorphisms with Siegel disks. His constructions were based upon (i) K3 lattices and K3 structures in Salem number fields, (ii) Lefschetz and Atiyah-Bott fixed point formulas, (iii) Siegel-Sternberg theory on linearizations of nonlinear maps and small divisor problems, and (iv) Gel’fond-Baker method in transcendence theory and Diophantine approximation. In his work the characteristic polynomials of the constructed automorphisms were Salem polynomials of degree 22, so the topological entropies of them were logarithms of Salem numbers of degree 22 by the Gromov-Yomdin theorem [4, 25]. McMullen [18, 19] went on to construct K3 surface automorphisms with Salem numbers of lower degrees, especially ones with Lehmer’s number $\lambda_L$ in [14], whose logarithm was the minimum of the positive entropy spectrum for all automorphisms on compact complex surfaces [17]. He discussed the non-projective cases in [18] and the projective ones in [19], though these papers did not touch on Siegel disks. Here we should recall from [18, Theorem 7.2] that an automorphism on a projective K3 surface never admits a Siegel disk.

Our chief idea in this article is to use hypergeometric groups and associated lattices, in place of Salem number fields in item (i) above which was one of the main ingredients of [16]. To outline our idea we need to review the minimal basics about K3 surfaces and their automorphisms (see Barth et al. [8, Chap. VIII]). The middle cohomology group $L = H^2(X, \mathbb{Z})$ of a K3 surface $X$ equipped with the intersection form is an even unimodular lattice of rank 22 and signature $(3, 19)$. The geometry of $X$ then defines a triple, called the K3 structure on $L$, consisting of Hodge structure $L \otimes \mathbb{C} = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$, positive cone $C^+$ and Kähler cone $\mathcal{K}$. It is accompanied by the related concepts of Picard lattice (or Néron-Severi lattice), root system and Weyl group. Any automorphism $f : X \to X$ of the K3 surface $X$ induces a lattice automorphism $f^* : L \to L$ preserving the K3 structure. Conversely, thanks to the Torelli theorem and surjectivity of the period mapping, any automorphism of a K3 lattice preserving a given K3 structure is realized by a unique K3 surface automorphism up to isomorphisms.

We now turn our attention to an irreducible hypergeometric group $H = \langle A, B \rangle$ of rank $n$. Under a suitable condition there exists an $H$-invariant Hermitian form $h$ on $\mathbb{C}^n$. Suppose that $H$ is integral, that is, defined over $\mathbb{Z}$. Under a certain additional condition the matrix $C := A^{-1}B$ is a real reflection in a vector $r$. Consider the $\mathbb{Z}$-linear span $L := \langle r, Ar, \ldots, A^{n-1}r \rangle_{\mathbb{Z}}$, which is equal to $\langle r, Br, \ldots, B^{n-1}r \rangle_{\mathbb{Z}}$. With a suitable normalization of it, the Hermitian form $h$ is $\mathbb{Z}$-valued on $L$ and makes $L$ an even lattice, which is referred to as a hypergeometric lattice. It is unimodular if and only if the characteristic polynomials of $A$ and $B$ have resultant $\pm 1$. The generators $A$ and $B$ and hence the whole group $H$ act on $L$ as lattice automorphisms.

We focus on the specific rank $n = 22$. It is natural to ask when a hypergeometric lattice $L$ becomes a K3 lattice. If so, we wonder whether the matrix $F = A$ or $B$ not only acts on $L$ as a lattice automorphism but also acts as the producer of a K3 structure on $L$, making itself an automorphism of the K3 structure. If this is the case then we get a K3 surface automorphism $f : X \to X$ via the Torelli theorem and surjectivity of the period mapping. If the characteristic polynomial $\chi(z)$ of $F$ contains a Salem factor then $f$ has a positive entropy. Very often, however, this plan should be twisted because usually $F$ does not preserve any Weyl chamber (hence the Kähler cone) it produces. We must modify $F$ by a Weyl group element $w_F$ so that the resulting matrix $\tilde{F} := w_F \circ F$ satisfies the Kähler cone condition. So the automorphism $f$ is induced by $\tilde{F}$ rather than $F$. If the characteristic polynomial $\tilde{\chi}(z)$ of $\tilde{F}$ is nice then we can go on to discuss the existence of Siegel disks. When the root system in the Picard lattice is nonempty, the map $f$ may have some fixed curves in $X$. In such cases we shall use an analytic version of S. Saito’s fixed point formula [22] and the Toledo-Tong fixed point formula [21].
We sketch our main results of this article. The first part of the article is devoted to
enriching infrastructure in the theory of hypergeometric groups for its efficient applications to
the dynamics on K3 surfaces in the second part. In §3 a formula for the index of the invariant
Hermitian form $h$ is given in terms of the “clusters” of eigenvalues of $A$ and $B$ (Theorem 4.2),
namely, of roots of their characteristic polynomials $\varphi(z)$ and $\psi(z)$. When the group $H$ is
real, that is, defined over $\mathbb{R}$, the matrices $A$ and $B$ are asymmetric to the effect that $\varphi(z)$ is
anti-palindromic, $z^n\varphi(z^{-1}) = -\varphi(z)$, while $\psi(z)$ is palindromic, $z^n\psi(z^{-1}) = \varphi(z)$. If $n = 2N$
is even then there exist monic polynomials $\Phi(w)$ and $\Psi(w)$ of degrees $N - 1$ and $N$ such that
\[
\varphi(z) = (z^2 - 1)z^{N-1}\Phi(z + z^{-1}), \quad \psi(z) = z^N\Psi(z + z^{-1}).
\]
We refer to $\Phi(w)$ and $\Psi(w)$ as the trace polynomials of $\varphi(z)$ and $\psi(z)$. It is more convenient to
express the index of $h$ in terms of “trace clusters” of roots of $\Phi(w)$ and $\Psi(w)$. We remark that
clusters lie on the unit circle $S^1 \subset \mathbb{C}$, while trace clusters belong to the interval $[-2, 2] \subset \mathbb{C}$;
they are related by the Joukowsky transformation $z \mapsto w := z + z^{-1}$ with some care at $z = \pm 1$
and $w = \pm 2$. In §4 a formula for the index is given in terms of trace clusters (Theorem 4.2). A
formula for local indices (Proposition 4.5) is also included for later use in Hodge structures.

The second part of the article is an application of hypergeometric groups to dynamics on
K3 surfaces. The main results of this part contain the following seven items.

1. Classification of all real hypergeometric groups of rank 22 and index $\pm 16$ (Theorem 6.2).

2. Necessary and sufficient condition, in terms of trace clusters, for a hypergeometric lattice
$L$ of rank 22 to be a K3 lattice with a Hodge structure such that the matrix $A$ becomes
a Hodge isometry of elliptic, parabolic or hyperbolic type (Theorem 6.12).

3. Similar condition for the matrix $B$ (Theorem 6.13). Here, however, $B$ can be a Hodge
isometry of hyperbolic type only, reflecting the asymmetry of $A$ and $B$.

4. Recipe to twist $F = A$ or $B$ by a Weyl group element $w_F$ so that the resulting matrix
$\tilde{F} := w_F \circ F : L \to L$ preserves the Kähler cone and hence the K3 structure, inducing a
K3 surface automorphism $f : X \to X$ (Algorithm in §8.1).

5. Illustration of our method by two specific settings for $\Phi(w)$ and $\Psi(w)$ having Lehmer’s
trace polynomial and/or cyclotomic trace polynomials as their irreducible factors. Enu-
nmerations, under these settings, of the hypergeometric lattices $L$ that induce K3 surface
automorphisms $f$ of minimum entropy $h(f) = \log \lambda_j$ (Theorem 8.4 in which the matrix
$A$ acts as the Hodge isometry; Theorem 8.7 in which $B$ plays that role).

6. Another illustration by 289 examples of hypergeometric lattices such that $\tilde{B}$ (identical
with $B$ in this case) induces a K3 surface automorphism $f$ of entropy $h(f) = \log \lambda_j$,
$j = 1, \ldots, 10$, among which 256 have a Siegel disk, where $\lambda_j$ are ten Salem numbers of
degree 22 in McMullen [16, Table 4] reproduced in this article as Table 3 (Theorem 9.4).

7. A goodly number of hypergeometric lattices leading to K3 surface automorphisms of
minimum entropy with Siegel disks; ones with a Siegel disk and exceptional set of type
$E_6 \oplus E_6$ (Theorem 9.5); and ones with a three-cycle of Siegel disks and exceptional set of
type $E_8 \oplus A_2 \oplus A_2$ or $E_6$ (Theorem 9.7), where an exceptional set of type $\Gamma$ is the union
of $(-2)$-curves whose dual graph is a Dynkin diagram of type $\Gamma(-1)$, the negative of $\Gamma$.  

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A couple of remarks should be in order at this stage. The conditions in items (2) and (3) say that at least two eigenvalues of \( B \) must be algebraic units off the unit circle in order for a hypergeometric group to yield a K3 lattice. This differs from the situations in which all eigenvalues of \( A \) and \( B \) are roots of unity, such as in the classification of finite hypergeometric groups by Beukers and Heckman [4] and in their treatment of Lorentzian hypergeometric groups by Fuchs, Meiri and Sarnak [6]. In items (5) and (6) the use of Lehmer’s number \( \lambda_L \) and that of Salem numbers from [16, Table 4] are only for the sake of illustrations; our method works for various other settings with other Salem numbers. Related to item (7), Oguiso [20] gives an example of K3 surface automorphism with a Siegel disk and exceptional set of type \( E_8 \) that realizes the third smallest Salem number.

2 Hypergeometric Group

The theory of hypergeometric groups is developed by Beukers and Heckman [4]. A lucid explanation of this concept can also be found in Heckman’s lecture notes [9]. A hypergeometric group is a group \( H = \langle A, B \rangle \) generated by two invertible matrices \( A, B \in \text{GL}(n, \mathbb{C}) \) such that \( \text{rank}(A - B) = 1 \). Let \( a = \{a_1, \ldots, a_n\} \) and \( b = \{b_1, \ldots, b_n\} \) be the eigenvalues of \( A \) and \( B \) respectively (they are multi-sets). Then \( H \) acts on \( \mathbb{C}^n \) irreducibly if and only if

\[
a \cap b = \emptyset \tag{1}
\]

(see [9, Theorem 3.8]). Hereafter we always assume condition (1). Let \( a^\dagger := \overline{a}^{-1} \) for \( a \in \mathbb{C}^\times \). There then exists a non-degenerate \( H \)-invariant Hermitian form on \( \mathbb{C}^n \) if and only if

\[
a^\dagger = a, \quad b^\dagger = b, \tag{2}
\]

where \( a^\dagger := \{a_1^\dagger, \ldots, a_n^\dagger\} \) (see [4, Theorem 4.3] and [9, Theorem 3.13]; the “only if” part is not mentioned there but it is an easy exercise). Hereafter we also assume condition (2).

Let \( \chi(z; P) := \det(zI - P) \) be the characteristic polynomial of \( P \in \text{GL}(n, \mathbb{C}) \). It is interesting to describe conditions (1) and (2) in terms of \( \varphi(z) := \chi(z; A) \) and \( \psi(z) := \chi(z; B) \). Note that \( \varphi(0) = (-1)^n \det A \) and \( \psi(0) = (-1)^n \det B \) are non-zero. Condition (1) is equivalent to

\[
\text{Res}(\varphi, \psi) \neq 0, \tag{11}
\]

where \( \text{Res}(\varphi, \psi) \) denotes the resultant of \( \varphi(z) \) and \( \psi(z) \), while condition (2) is equivalent to

\[
z^n \bar{\varphi}(z^{-1}) = \bar{\varphi}(0) \cdot \varphi(z), \quad z^n \bar{\psi}(z^{-1}) = \bar{\psi}(0) \cdot \psi(z), \tag{22}
\]

where \( \bar{f}(z) := f(\overline{z}) \) for \( f(z) \in \mathbb{C}[z] \). Comparing the constant terms in (22) we have

\[
|\varphi(0)| = 1, \quad |\psi(0)| = 1, \tag{3}
\]

because \( \varphi(z) \) and \( \psi(z) \) are monic polynomials.

Given a monic polynomial \( f(z) = z^n + f_1 z^{n-1} + \cdots + f_n \in \mathbb{C}[z] \) with \( f_n \neq 0 \), put

\[
Z(f) := \begin{pmatrix} 0 & -f_n \\ 1 & 0 \\ \vdots & \vdots \\ 1 & -f_1 \end{pmatrix}.
\]
Note that $Z(f) \in \text{GL}(n, \mathbb{C})$ has characteristic polynomial $f(z)$. Levelt’s theorem [15] states that if $\varphi(z)$ and $\psi(z)$ are monic polynomials with nonzero constant term that satisfy condition \(1\) then $Z(\varphi)$ and $Z(\psi)$ generate an irreducible hypergeometric group

$$H(\varphi, \psi) := \langle Z(\varphi), Z(\psi) \rangle,$$

and conversely any irreducible hypergeometric group $H = \langle A, B \rangle$ is conjugate in $\text{GL}(n, \mathbb{C})$ to this one with $\varphi(z) := \chi(z; A)$ and $\psi(z) := \chi(z; B)$ (see [4, Theorem 3.5] and [9, Theorem 3.9]). In this sense $H$ is uniquely determined by the characteristic polynomials $\varphi$ and $\psi$, or equivalently by the eigenvalue sets $a$ and $b$, so it may be written as $H(a, b)$ or as $H(\varphi, \psi)$.

The structure of the invariant Hermitian form is discussed in [4, §4] and [9, §3.3] when $a$ and $b$ lie on the unit circle $S^1$. We can extend the discussion to the general case without this restriction. Since $C := A - 1 B$ is a complex reflection, that is, $\text{rank}(I - C) = 1$, its determinant $c := \det C = \psi(0) / \varphi(0) = b_1 \cdots b_n / a_1 \cdots a_n \in S^1$ is an eigenvalue of $C$, called the distinguished eigenvalue. In this article we assume that $c \neq 1$.

Let $r$ be an eigenvector of $C$ corresponding to the eigenvalue $c$. As in the proof of [9, Theorem 3.14] we have $(r, r) \in \mathbb{R}^\times$ and

$$Cv = v - \zeta(v, r)r, \quad v \in \mathbb{C}^n,$$

$$\frac{\psi(z)}{\varphi(z)} = 1 + \zeta((zI - A)^{-1}Ar, r). \quad \text{where} \quad \zeta := \frac{1 - c}{(r, r)}. \tag{7b}$$

The invariant Hermitian form is determined up to scalar multiplications in $\mathbb{R}^\times$. To eliminate this ambiguity we take the normalization

$$(r, r) = |1 - c| > 0 \quad \text{so that} \quad \zeta = \frac{1 - c}{|1 - c|} \in S^1. \tag{8}$$

Let $\{\xi_i\}_{i=1}^\infty$ be the sequence defined by the Taylor series expansion

$${\psi(z) \over \varphi(z)} = 1 + \zeta \sum_{i=1}^{\infty} \xi_i z^{-i} \quad \text{around} \quad z = \infty. \tag{9}$$

**Theorem 2.1** The invariant Hermitian pairing $g_{ij} := (A^{i-1}r, A^{j-1}r)$ is given by

$$g_{ij} = \begin{cases} \xi_{i-j} & (i \geq j \geq 1), \\ \xi_{j-i} & (1 \leq i < j), \end{cases} \tag{10}$$

with convention $\xi_0 := |1 - c|$. The determinant of $G := (g_{ij})_{i,j=1}^n$ has absolute value

$$|\det G| = |\text{Res}(\varphi, \psi)|, \tag{11}$$

which is non-zero by assumption \(1\). In particular $r, Ar, \ldots, A^{n-1}r$ form a basis of $\mathbb{C}^n$. 5
Proof. First we show formula (10). Taylor expansion of (7) around \( z = \infty \) reads

\[
\frac{\psi(z)}{\varphi(z)} = 1 + \zeta \left( (I - z^{-1}A)^{-1}z^{-1}Ar, r \right) = 1 + \zeta \sum_{i=1}^{\infty} (A^i r, r)z^{-i}.
\]

Comparing this with expansion (9) together with convention \( \xi_0 = |1 - c| \) yields \( (A^i r, r) = \xi_i \) for every \( i \in \mathbb{Z}_{\geq 0} \). Since the Hermitian form is \( A \)-invariant, we have

\[
g_{ij} = (A^{i-1}r, A^{j-1}r) = \begin{cases} 
(A^{i-j}r, r) = \xi_{i-j} & (i \geq j \geq 1), 

(A^{j-i}r, r) = \xi_{j-i} & (1 \leq i < j).
\end{cases}
\]

This together with normalization (8) leads to formula (10).

Next we show formula (11). Suppose for the time being that \( a_1, \ldots, a_n \) are mutually distinct. Let \( r = r_1 + \cdots + r_n \) be the decomposition of \( r \) into eigenvectors of \( A \), where \( r_i \) correspond to the eigenvalue \( a_i \). By condition \( a^\dagger = a \) in (2) there exists a permutation \( \sigma \in S_n \) such that \( \sigma^2 = 1 \) and \( a_{\sigma(i)} = a_i^\dagger \) for \( i = 1, \ldots, n \). Note that \( \sigma(i) = i \) if and only if \( a_i \in S^1 \). It follows from non-degeneracy and \( A \)-invariance of the Hermitian form that \( (r_i, r_{\sigma(i)}) \), \( i = 1, \ldots, n \), are non-zero while all the other Hermitian parings \( (r_i, r_j) \) vanish. Thus equation (11) leads to

\[
\frac{\psi(z)}{\varphi(z)} = 1 + \zeta \sum_{i=1}^{n} a_i (r_i, r_{\sigma(i)}) z^{-a_i}.
\]

Taking residue at \( z = a_i \) we have for \( i = 1, \ldots, n \),

\[
\lambda_i := (r_i, r_{\sigma(i)}) = \frac{\psi(a_i)}{\zeta a_i \varphi_i(a_i)} \quad \text{with} \quad \varphi_i(z) := \prod_{j \neq i} (z - a_j).
\]

After rearranging \( a_1, \ldots, a_n \) if necessary, we may assume that \( \sigma \) fixes \( 1, \ldots, l \) and exchanges \( l + 2i - 1 \) and \( l + 2i \) for \( i = 1, \ldots, m \), where \( n = l + 2m \). Then \( r_1, \ldots, r_n \) form a basis of \( \mathbb{C}^n \) with respect to which the Gram matrix of the invariant Hermitian form is given by

\[
A = (\lambda_1) \oplus \cdots \oplus (\lambda_l) \oplus A_1 \oplus \cdots \oplus A_m \quad \text{with} \quad A_i := \begin{pmatrix} 0 & \lambda_{l+2i-1} \\ \lambda_{l+2i} & 0 \end{pmatrix}.
\]

Note that \( \lambda_1, \ldots, \lambda_l \in \mathbb{R} \) and \( \lambda_{l+2i-1} = \lambda_{l+2i} \) for \( i = 1, \ldots, m \). From (12) and (13) we have

\[
|\det A| = |(-1)^m \lambda_1 \cdots \lambda_n| = \frac{|(\zeta^n a_1 \cdots a_n \varphi_1(a_1) \cdots \varphi_n(a_n))|}{\prod_{i<j} |a_i - a_j|},
\]

where \( |\zeta| = 1 = |a_1 \cdots a_n| \) is used. Moreover we have \( (r, Ar, \ldots, A^{n-1}r) = (r_1, \ldots, r_n)V \) with \( V := (a_i^{-1})_{i,j=1}^{n} \) being a Vandermonde matrix. This implies \( G = V A V^\dagger \) and hence

\[
|\det G| = |\det A||\det V|^2 = |\det A| \prod_{i<j} |a_i - a_j|^2 = |\text{Res}(\varphi, \psi)|,
\]

which proves formula (11) when \( a_1, \ldots, a_n \) are distinct. The formula in the general case follows by a continuity argument, which works as far as conditions (11) and (21) are fulfilled. \( \square \)

Remark 2.2 It is obvious that if \( H = \langle A, B \rangle \subset \text{GL}(n, \mathbb{C}) \) is a hypergeometric group then so is \( H^a := \langle -A, -B \rangle \). We refer to \( H^a \) as the antipode of \( H \). Note that \( \varphi^a(z) = (-1)^n \varphi(-z) \) and \( \psi^a(z) = (-1)^n \psi(-z) \), hence \( a^a = -a \) and \( b^a = -b \). It follows from \( C^a = C \) and formula (10) that \( H \) and \( H^a \) have the same invariant Hermitian form.
3 Index of the Hermitian Form

Let $a_{\text{on}}$ resp. $a_{\text{off}}$ be the component of $a$ whose elements lie on resp. off $S^1$. We define $b_{\text{on}}$ and $b_{\text{off}}$ in a similar manner for $b$. If both of $a_{\text{on}}$ and $b_{\text{on}}$ are nonempty then they dissect each other into an equal number of components $a_1, \ldots, a_t$ and $b_1, \ldots, b_t$ so that $a_1, b_1, a_2, b_2, \ldots, a_t, b_t$ are located consecutively on $S^1$ in the positive direction (anti-clockwise) as in Figure 1. Each $a_i$ is called a cluster of $a_{\text{on}}$ and is said to be simple, double, triple, etc. if $|a_i| = 1, 2, 3$, and so on, where $|x|$ denotes the cardinality counted with multiplicities of a multi-set $x$. We write

$$[a_{\text{on}}] = 1^{\nu_1} 2^{\nu_2} 3^{\nu_3} \cdots$$

if $a_{\text{on}}$ consists of $\nu_1$ simple clusters, $\nu_2$ double clusters, $\nu_3$ triple clusters, etc. Note that $|a_{\text{on}}| = \nu_1 + 2\nu_2 + 3\nu_3 + \cdots$, $|a_{\text{off}}|$ is even and $|a_{\text{on}}| + |a_{\text{off}}| = n$; the same is true for $b$. Taking a branch-cut $\ell$ separating $b_t$ and $a_1$ as in Figure 1 we define the argument of $z \in \mathbb{C}$ so that

$$\Theta \leq \arg z < \Theta + 2\pi,$$

where $\Theta \in [-\pi, \pi)$ is the angle of the ray $\ell$ to the positive real axis.

Let $\arg a_i = 2\pi \alpha_i$ and $\arg b_i = 2\pi \beta_i$ for $i = 1, \ldots, n$. Formula (5) allows us to write

$$c = e^{2\pi i \gamma} \in S^1 \quad \text{with} \quad \gamma := \sum_{i=1}^{n} \beta_i - \sum_{i=1}^{n} \alpha_i \in \mathbb{R},$$

where $i := \sqrt{-1}$, hence condition (5) is equivalent to $\gamma \in \mathbb{R} \setminus \mathbb{Z}$, that is, $\sin \pi \gamma \in \mathbb{R}^\times$.

**Remark 3.1** Taking another branch of arg has no effect on the contribution of $a_{\text{off}}$ and $b_{\text{off}}$ to the value of $\sin \pi \gamma$, because any pair $\lambda, \lambda^\dagger \in a_{\text{off}}$ has a common argument so the sum $\arg \lambda + \arg \lambda^\dagger$ alters only by an even multiple of $2\pi$; the same is true for $\lambda, \lambda^\dagger \in b_{\text{off}}$.
3.1 Clusters and Index

Let \((p, q)\) be the signature of the \(H(a, b)\)-invariant Hermitian form on \(\mathbb{C}^n\). Under the condition

\[a_{\text{off}} = b_{\text{off}} = \emptyset,\]

Beukers and Heckman [1] Theorem 4.5] gave a formula for the index \(p - q\) (up to sign). We can state a refined version of it in terms of the clusters of \(a_{\text{on}}\) and \(b_{\text{on}}\) without assuming [16].

**Theorem 3.2** If \(a_{\text{on}}\) and \(b_{\text{on}}\) are nonempty then the invariant Hermitian form has index

\[p - q = \varepsilon \sum_{k \in K} (-1)^{\tau_k} \quad \text{with} \quad K := \{ k = 1, \ldots, t : |a_k| \equiv 1 \text{ mod } 2 \},\]

where \(\varepsilon = \pm 1\) is the sign of \(\sin \pi \gamma \in \mathbb{R}^\times\) with \(\gamma\) given in [15] and \(\tau_k\) is defined by

\[\tau_1 := 0; \quad \tau_k := |a_1| + |b_1| + \cdots + |a_{k-1}| + |b_{k-1}|, \quad k = 2, \ldots, t.\]

If at least one of \(a_{\text{on}}\) and \(b_{\text{on}}\) is empty then the index \(p - q\) is zero.

**Proof.** We may assume that \(a_1, \ldots, a_n\) are mutually distinct, since the general case can be treated by a perturbation argument (see e.g. Kato [13, Chapter II, §1.4]). After rearranging the indices of \(a_i\) and \(b_j\) if necessary, we may further assume that

\[a_{\text{on}} = \{a_1, \ldots, a_l\}, \quad a_{\text{off}} = \{a_{l+1}, \ldots, a_n\}, \quad a_{l+2i-1} = a_{l+2i}^\dagger, \quad i = 1, \ldots, m,\]

\[b_{\text{on}} = \{b_1, \ldots, b_d\}, \quad b_{\text{off}} = \{b_{d+1}, \ldots, b_n\}, \quad b_{d+2i-1} = b_{d+2i}^\dagger, \quad i = 1, \ldots, e,\]

where \(n = l + 2m = d + 2e\). Suppose that both of \(a_{\text{on}}\) and \(b_{\text{on}}\) are nonempty, that is, \(l \geq 1\) and \(d \geq 1\). Since the Hermitian matrix \(A_i\) in [13] has null index, we have

\[p - q = l_+ - l_-, \quad l_\pm := \#\{ i = 1, \ldots, l : \pm \lambda_i > 0 \}.\]

For \(x, y \in \mathbb{R}^\times\) we write \(x \sim y\) if \(x\) and \(y\) have the same sign. We claim that

\[\lambda_i \sim \sigma_i := \varepsilon \cdot \frac{\prod_{j=1}^d \sin \pi (\beta_j - \alpha_i)}{\prod_{j=1}^l \sin \pi (\alpha_j - \alpha_i)} \in \mathbb{R}^\times, \quad i = 1, \ldots, l,\]

where \(\prod_j^i\) is the product avoiding \(j = i\). Indeed, equation (12) together with (8) yields

\[\lambda_i = \frac{|1 - c| \prod_{j=1}^d (a_i - b_j) \prod_{j=1}^l (a_i - b_{d+j}) (a_i - b_{d+2j})}{(1 - c) a_i \prod_{j=1}^l (a_i - a_j) \prod_{j=1}^m (a_i - a_{l+j}) (a_i - a_{l+2j})}\]

for \(i = 1, \ldots, l\). To evaluate the right-hand side we use the following identities:

\[n = l + 2m = d + 2e,\]
\[1 - c = -c^\frac{1}{2} (c^\frac{1}{2} - c^{-\frac{1}{2}}) = -2i \cdot c^\frac{1}{2} \cdot \sin \pi \gamma,\]
\[u - v = -u^\frac{1}{2} v^\frac{1}{2} (u^{-\frac{1}{2}} v^{-\frac{1}{2}} - u^\frac{1}{2} v^\frac{1}{2}) = -2i \cdot u^\frac{1}{2} v^\frac{1}{2} \cdot \sin \pi (\phi - \theta),\]
\[(u - w^\dagger)(u - w) = -uw \cdot |u - w^\dagger|^2 = -u(w^\dagger w)^\frac{1}{2} \cdot |w||u - w^\dagger|^2,\]
for $u = e^{2\pi i \theta} \in S^1$, $v = e^{2\pi i \phi} \in S^1$ and $w \in \mathbb{C}^\times$. Some calculations yield

$$\lambda_i = \mu_i \cdot \sigma_i, \quad \mu_i := 2^{d_i - l + 1} \cdot \prod_{j=1}^l |b_{d_i+2j}||a_i - b_{d_i+2j}^\dagger|^2 \prod_{j=1}^m |a_{l+i+2j}||a_i - a_{l+i+2j}^\dagger|^2 > 0$$

for $i = 1, \ldots, l$ and hence claim (19) is proved.

Relation (19) readily shows that the sign $\varepsilon_i = \pm 1$ of $\lambda_i$ is determined by

$$\varepsilon_i = \varepsilon \cdot (-1)^{\delta_i}, \quad \delta_i := \#\{j = 1, \ldots, n : \alpha_j < \alpha_i\} + \#\{j = 1, \ldots, n : \beta_j < \alpha_i\}, \quad (20)$$

where $\arg a_j = 2\pi \alpha_j$ and $\arg b_j = 2\pi \beta_j$. If $a_i$ is the $d_i$-th smallest element of $a_k$ in the argument then $\varepsilon = \varepsilon \cdot (-1)^{\tau_k + d_i - 1}$. Since $d_i$ ranges over 1, $\ldots, |a_k|$ as $a_i$ runs through $a_k$, one has

$$\sum_{a_i \in a_k} (-1)^{d_i - 1} = \begin{cases} 1 & (|a_k| \equiv 1 \text{ mod } 2), \\ 0 & (|a_k| \equiv 0 \text{ mod } 2). \end{cases}$$

Thus it follows from formula (18) that

$$p - q = l_+ - l_- = \sum_{i=1}^l \varepsilon_i = \varepsilon \sum_{k=1}^l \sum_{a_i \in a_k} (-1)^{d_i - 1} = \varepsilon \sum_{k \in K} (-1)^{\tau_k},$$

which establishes formula (17). When $a_{on}$ is empty, the index is zero as we have $l = 0$ in (13). When $b_{on}$ is empty, replace $a$ with $b$ and proceed in a similar manner.

\begin{remark}

The following remarks are helpful in applying Theorem 3.2

(1) Formula (17) is invariant under any cyclic permutation of the indices $k$ for $a_k$ and $b_k$.

(2) Note that $|p - q| \leq |K| \leq t \leq n$ and $|p - q| \equiv |K| \equiv n \text{ mod } 2$; moreover $|K| = t$ if and only if all $a_{on}$-clusters $a_1, \ldots, a_t$ have odd cardinalities.

(3) We may exchange the roles of $a$ and $b$ in Theorem 3.2.

\end{remark}

### 3.2 Lorentzian Hypergeometric Groups

It is interesting to consider when the invariant Hermitian form is definite or Lorentzian. Under condition (16) Beukers and Heckman [4, Corollary 4.7] obtained the interlacing criterion for the definiteness while Fuchs, Meiri and Sarnak [6, §2.2] derived the almost interlacing criterion for the Lorentzian. Even without assuming (16) a priori, Theorem 3.2 readily shows that the Hermitian form is definite if and only if $|a_{on}| = |b_{on}| = 1^n$ and $a_{off} = b_{off} = \emptyset$. For the Lorentzian cases we have the following classification, in which types 1 and 2 appear in [6].

\begin{theorem}
The Lorentzian cases are classified into five types in Table 1. In type 1 we mean by “doubles adjacent” that the double cluster in $a_{on}$ and the one in $b_{on}$ must be adjacent to each other. For the other types there are no constraints on the location of multiple clusters.

\end{theorem}
Table 1: Lorentzian cases.

Proof. It follows from (2) of Remark 3.3 that $|p - q| = n - 2 \leq |K| \leq t \leq n$ and $|K| \equiv n \mod 2$. So we have either $|K| = t = n$ or $|K| = n - 2$, but $t = n$ is ruled out as it would lead to the definite case. Thus $|K| = n - 2$ and $t = n - 2, n - 1$.

First we consider the case $t = n - 1$. If follows from $n \geq |a_{\text{on}}| \geq t = n - 1$; $n \equiv |a_{\text{on}}| \mod 2$ and $|K| = n - 2$ that $|a_{\text{on}}| = 1^{n-2}2^{1}$ and $|a_{\text{off}}| = 0$. One has also $|b_{\text{on}}| = 1^{n-2}2^{1}$ and $|b_{\text{off}}| = 0$ by (3) of Remark 3.3. Let $k, l \in \{1, \ldots, n - 1\}$ be the indices such that $|a_{k}| = 2$ and $|b_{l}| = 2$. By (1) of Remark 3.3 we may assume $k = n - 1$ and hence $K = \{1, \ldots, n - 2\}$. Index formula (17) now reads $| \sum_{k=1}^{n-2} a_{k} | = n - 2$, which implies $\tau_{k} = 0 \mod 2$ for $k = 2, \ldots, n - 2$ and hence $|a_{k}| + |b_{k}| = 1 + |b_{k}| \equiv 0 \mod 2$ for $k = 1, \ldots, n - 3$, which in turn forces $l = n - 2$ or $l = n - 1$, that is, $a_{k}$ and $b_{l}$ must be adjacent. This case falls into type 1 of Table 1.

Next we proceed to the case $t = n - 2$. Since $|K| = n - 2 = t$, all of $|a_{1}|, \ldots, |a_{n-2}|$ must be odd by (2) of Remark 3.3. It then follows from $n \geq |a_{\text{on}}| \geq n - 2$ and $n \equiv |a_{\text{on}}| \mod 2$ that $a$ must satisfy either (A1) $|a_{\text{on}}| = 1^{n-3}2^{3}$, $|a_{\text{off}}| = 0$; or (A2) $|a_{\text{on}}| = 1^{n-2}$, $|a_{\text{off}}| = 2$. By (3) of Remark 3.3 $b$ must also satisfy either (B1) $|b_{\text{on}}| = 1^{n-3}2^{3}$, $|b_{\text{off}}| = 0$; or (B2) $|b_{\text{on}}| = 1^{n-2}$, $|b_{\text{off}}| = 2$. Then the combinations (A1)-(B1), (A1)-(B3), (A3)-(B1), (A3)-(B3) lead to types 2, 3, 4, 5 in Table 1 respectively. The converse implication is easy to verify. □

3.3 Local Index

Let $E(\nu)$ be the generalized eigenspace of $A$ corresponding to an eigenvalue $\nu \in \alpha$. Note that $m(\nu) := \dim E(\nu)$ is the multiplicity of $\nu$ in the multi-set $\alpha$. Put $E(\mu, \mu^{\dagger}) := E(\mu) \oplus E(\mu^{\dagger})$ for $\mu \in \alpha_{\text{off}}$. Some linear algebra shows that the non-degeneracy and $A$-invariance of the Hermitian form lead to an orthogonal direct sum decomposition

$$\mathbb{C}^{n} = \bigoplus_{|\lambda| = 1} E(\lambda) \oplus \bigoplus_{|\mu| > 1} E(\mu, \mu^{\dagger}),$$

where $\lambda$ ranges over all distinct elements in $\alpha_{\text{on}}$ and $\mu$ ranges over all distinct elements in $\alpha_{\text{off}}$ such that $|\mu| > 1$. The Hermitian form is non-degenerate on $E(\lambda)$ and $E(\mu, \mu^{\dagger})$, though it is null on $E(\mu)$ and $E(\mu^{\dagger})$ individually. Note that $m(\mu) = m(\mu^{\dagger})$. It is interesting to find the index of the Hermitian form restricted to $E(\lambda)$ or $E(\mu, \mu^{\dagger})$. The same problem also makes sense with $B$ and $b$ in place of $A$ and $a$, where if $\nu \in b$ then $E(\nu)$ is understood to be the generalized $\nu$-eigenspace of $B$. Thanks to (i) or (ii) the common notation $m(\nu)$ is allowed for $\nu \in \alpha \cup b$, because $m(\nu)$ is the same as the multiplicity of $\nu$ in $\varphi(z) \cdot \psi(z)$.

For two distinct elements $\lambda, \lambda^{'} \in S^{1}$ we say that $\lambda^{'}$ is smaller than $\lambda$ if $\arg \lambda^{'} < \arg \lambda$ with respect to the argument defined in (14). For any $\lambda \in S^{1}$ let

$$r(\lambda) := \# \text{ of all elements in } \alpha_{\text{on}} \cup b_{\text{on}} \text{ that are smaller than } \lambda,$$

where $r(\lambda)$ is the index of the Hermitian form on $E(\lambda)$. The cases $t = n - 2$ and $t = n - 1$ are considered. If $t = n - 1$ then $r(\lambda) = 0$ for $\lambda \in S^{1}$; if $t = n - 2$ then $r(\lambda) = 0$ for $\lambda \in S^{1}$.
where \# denotes the cardinality counted with multiplicities.

**Proposition 3.5** For each \( \lambda \in a_{on} \cup b_{on} \) the Hermitian form restricted to \( E(\lambda) \) has index

\[
\text{idx}(\lambda) := \begin{cases} 
\varepsilon \cdot (-1)^{r(\lambda)} & \text{if } \lambda \in a_{on} \text{ and } m(\lambda) \text{ is odd}, \\
\varepsilon \cdot (-1)^{r(\lambda)+1} & \text{if } \lambda \in b_{on} \text{ and } m(\lambda) \text{ is odd}, \\
0 & \text{if } m(\lambda) \text{ is even},
\end{cases}
\]

(23)

where \( \varepsilon = \pm 1 \) is the sign of \( \sin \pi \gamma \) mentioned in Theorem 3.2. For each \( \mu \in a_{off} \cup b_{off} \) with \(|\mu| > 1\) the Hermitian form restricted to \( E(\mu, \mu^t) \) has null index, that is

\[
\text{idx}(\mu) = 0.
\]

(24)

**Proof.** First we show claims (23) and (24) for \( \lambda \in a_{on} \) and \( \mu \in a_{off} \). In the special case where \( a_1, \ldots, a_n \) are distinct and hence \( \lambda \) and \( \mu \) are simple, they are direct consequences of (20) and (13) respectively. In the general case they are then obtained by using perturbation theory of eigenprojections in Kato [13, Chapter II, §1.4], since \( E(\lambda) \) and \( E(\mu, \mu^t) \) are the total eigenspaces for the \( \lambda \)-group and \( \mu \)-group (in Kato’s terminology) respectively.

Next we can show the results for \( \lambda \in b_{on} \) and \( \mu \in b_{off} \) in a similar manner by exchanging the roles of \( a \) and \( b \). Notice that \( r(\lambda) =: r_a(\lambda) \) and \( \varepsilon =: \varepsilon_a \) in (23) are defined with respect to the branch cut \( \ell = \ell_a \) lying between \( b \) and \( a_1 \) as in Figure 1. Changing the roles of \( a \) and \( b \) we should replace \( \ell_a \) by a new branch cut \( \ell_b \) lying between \( a_1 \) and \( b_1 \) and consider the corresponding \( r_b(\lambda) \) and \( \varepsilon_b \). For \( \lambda \in b_{on} \) one has \( r_b(\lambda) = r_a(\lambda) - |a_1| \) and \( \varepsilon_b = -\varepsilon_a \cdot (-1)^{|a_1|} \), where Remark 3.1 is used to obtain the latter relation. Thus \( \varepsilon_b \cdot (-1)^{r_b(\lambda)} = \varepsilon_a \cdot (-1)^{r_a(\lambda)+1} \), which proves (23) for \( \lambda \in b_{on} \). The proof of (24) for \( \mu \in b_{off} \) is the same as that for \( \mu \in a_{off} \). \( \square \)

**Remark 3.6** Due to (24) the sum of \( \text{idx}(\lambda) \) over all distinct elements \( \lambda \) of \( a_{on} \) (or of \( b_{on} \)) is equal to the global index \( p - q \) of the Hermitian form on the whole space \( \mathbb{C}^n \).

4 Real Hypergeometric Groups

A hypergeometric group \( H = H(\varphi, \psi) = H(a, b) = \langle A, B \rangle \) is said to be real if

\[
\varphi(z) \in \mathbb{R}[z], \quad \psi(z) \in \mathbb{R}[z], \quad \text{or equivalently } \quad \bar{a} = a, \quad \bar{b} = b.
\]

Let \( L_\mathbb{R} \) be the \( \mathbb{R} \)-linear span of \( r, Ar, \ldots, A^{n-1}r \). These vectors form an \( \mathbb{R} \)-linear basis of \( L_\mathbb{R} \) as they are a \( \mathbb{C} \)-linear basis of \( \mathbb{C}^n \) by Theorem 2.3. Obviously \( A \) preserves \( L_\mathbb{R} \). Since \( \xi_i \in \mathbb{R} \) for every \( i \in \mathbb{Z}_{\geq 0} \), formula (10) shows that the Hermitian form is \( \mathbb{R} \)-valued on \( L_\mathbb{R} \). The matrix \( C = A^{-1}B \) acts on \( L_\mathbb{R} \) as a real action because formulas (7a) and (8) read

\[
Cv = v - (v, r)r, \quad v \in L_\mathbb{R} \quad \text{with} \quad (r, r) = 2.
\]

(25)

So \( L_\mathbb{R} \) is also preserved by \( B = AC \) and hence by the whole group \( H \), thus \( H \subset O(L_\mathbb{R}) \). In this section we always assume that \( H \) is a real hypergeometric group.

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4.1 Trace Polynomials

Conditions (3) implies \( \varphi(0) = \pm 1 \) and \( \psi(0) = \pm 1 \), while assumption (4) forces

\[
c = -1, \quad \zeta = 1, \quad (r, r) = 2,
\]

in (3), hence \( \varphi(0) \) and \( \psi(0) \) must have opposite signs. Thus after exchanging \( \varphi(z) \) and \( \psi(z) \) if necessary we may assume \( \varphi(0) = -1 \) and \( \psi(0) = 1 \) so that (21) becomes \( z^n \varphi(z^{-1}) = -\varphi(z) \) and \( z^n \psi(z^{-1}) = \psi(z) \), that is, \( \varphi(z) \) is anti-palindromic while \( \psi(z) \) is palindromic.

In general a palindromic polynomial \( f(z) \) of even degree \( 2d \) can be expressed as \( f(z) = z^d F(z + z^{-1}) \) for a unique polynomial \( F(w) \) of degree \( d \), a palindromic polynomial \( f(z) \) of odd degree factors as \( f(z) = (z + 1)g(z) \) with \( g(z) \) being palindromic of even degree, and an anti-palindromic polynomial \( f(z) \) factors as \( f(z) = (z-1)g(z) \) with \( g(z) \) being palindromic. Hence there exist unique monic real polynomials \( \Phi(w) \) and \( \Psi(w) \) such that

\[
\varphi(z) = (z^2 - 1)z^{N-1} \Phi(z + z^{-1}), \quad \psi(z) = z^N \Psi(z + z^{-1}), \quad \text{if } n = 2N; \tag{27a}
\]

\[
\varphi(z) = (z - 1)z^N \Phi(z + z^{-1}), \quad \psi(z) = (z + 1)z^N \Psi(z + z^{-1}), \quad \text{if } n = 2N + 1. \tag{27b}
\]

We refer to \( \Phi(w) \) and \( \Psi(w) \) as the trace polynomials of \( \varphi(z) \) and \( \psi(z) \). It is easily seen from (27) that the resultant of \( (\varphi, \psi) \) and that of \( (\Phi, \Psi) \) are related by

\[
\text{Res}(\varphi, \psi) = (-1)^N \cdot \Psi(2) \cdot \Psi(-2) \cdot \text{Res}(\Phi, \Psi)^2, \quad \text{if } n = 2N; \tag{28a}
\]

\[
\text{Res}(\varphi, \psi) = 2(-1)^N \cdot \Psi(2) \cdot \Phi(-2) \cdot \text{Res}(\Phi, \Psi)^2, \quad \text{if } n = 2N + 1. \tag{28b}
\]

The real hypergeometric group \( H = H(\varphi, \psi) \) can also be expressed as \( H = H(\Phi, \Psi) \).

It follows from (11) and (28) that \( \text{Res}(\Phi, \Psi) \neq 0 \) hence \( \Phi(w) \) and \( \Psi(w) \) have no root in common. By formulas (27) the roots \( \lambda \neq \pm 1 \) of \( \varphi(z) \) are in two-to-one correspondence with the roots \( \tau \neq \pm 2 \) of \( \Phi(w) \) via the relation \( \tau = \lambda + \lambda^{-1} \), since \( w - \tau = z^{-1}(z - \lambda)(z - \lambda^{-1}) \) with \( w = z + z^{-1} \). The same statement is true for \( \psi(z) \) and \( \Psi(w) \). Moreover we have

\[
m(\lambda) = \begin{cases} 
M(\tau) & \text{if } \lambda \neq \pm 1, \text{ i.e. } \tau \neq \pm 2, \\
2M(\tau) + 1 & \text{if } \lambda = \pm 1, \text{ i.e. } \tau = \pm 2,
\end{cases} \tag{29}
\]

under \( \tau := \lambda + \lambda^{-1} \) where \( M(\tau) \) is the multiplicity of \( w = \tau \) in \( \Phi(w) \cdot \Psi(w) = 0 \).

4.2 Trace Clusters and Index

In the real case the index formula (17) in Theorem 3.2 can be restated in terms of what we call trace clusters. In view of formulas (27), if \( n \) is even then \( \pm 1 \in a_{on} \) and hence \( a_{on} \) is nonempty (but \( b_{on} \) may be empty), while if \( n \) is odd then \( 1 \in a_{on}, -1 \in b_{on} \) and hence both of \( a_{on} \) and \( b_{on} \) are nonempty. In any case, as far as both of them are nonempty, the clusters \( a_1, b_1, \ldots, a_t, b_t \) can be indexed so that \( 1 \in a_1 \). With this convention it is easy to see that

if \( n \) is even then \( t = 2s \) is also even, \(-1 \in a_{s+1} \) and \( |a_1| \equiv |a_{s+1}| \equiv 1 \mod 2, \tag{30a} \)

if \( n \) is odd then \( t = 2s - 1 \) is also odd, \(-1 \in b_s \) and \( |a_1| \equiv |b_s| \equiv 1 \mod 2. \tag{30b} \)
In either case, with the convention \(a_{i+1} = a_1, b_{i+1} = b_1\), we have
\[
\bar{a}_i = a_{i+2-i}, \quad \bar{b}_i = b_{i+1-i}, \quad i = 1, \ldots, s. \tag{31}
\]
Note that \(a_2, \ldots, a_s\) are exactly those \(a_{on}\)-clusters which lie in the upper half-plane \(\text{Im} \, z > 0\).

Let \(A\) be the multi-set of all roots in \(\mathbb{C}\) of \(\Phi(w)\) and \(B\) be its \(\Psi(w)\)-counterpart. Let \(A_{on}\) resp. \(A_{off}\) be the component of \(A\) whose elements lie on resp. off \([-2, 2]\). Let \(B_{on}\) and \(B_{off}\) be defined in a similar manner for \(B\). Notice that both of \(a_{on}\) and \(b_{on}\) are nonempty if and only if either \(n\) is even and \(B_{on}\) is nonempty, or \(n\) is odd, \(\tag{32}\)
in which case \(A_{on}\) and \(B_{on}\) dissect each other into interlacing components called trace clusters,
\[
A_{s+1}, B_s, A_s, \ldots, B_1, A_1 \quad \text{if} \quad n \text{ is even}; \quad B_s, A_s, \ldots, B_1, A_1 \quad \text{if} \quad n \text{ is odd},
\]
where one or both of the end clusters may be empty but all the other clusters must be nonempty.
Put \(A_m := A_2 \cup \cdots \cup A_s\). Finally, let \(A_{>2}\) be the components of \(A_{off}\) whose elements are real numbers greater than 2 and \(B_{>2}\) be defined in a similar manner for \(B\).

**Lemma 4.1** Let \(\gamma \in \mathbb{R} \setminus \mathbb{Z}\) be the number defined in (15). Under assumption (32) we have
\[
\varepsilon = \sin \pi \gamma = (-1)^{|A_{on}|+|A_{>2}|+|B_{>2}|}. \tag{33}
\]

**Proof.** We consider how each component of \(a\) contributes to the sum \(2\pi \alpha := 2\pi \alpha_1 + \cdots + 2\pi \alpha_n\), where \(2\pi \alpha_i := \arg a_i\). The 1’s in \(a_{on}\) has no contribution. The \(-1\)’s in \(a_{on}\) has contribution \(\pi m_a^-\) if \(m_a^-\) is the multiplicity of \(-1\) in \(a\). For each non-real pair \(\lambda, \bar{\lambda} \in a_{on}\) with \(\text{Im} \, \lambda > 0\), the sum \(\arg \lambda + \arg \bar{\lambda}\) is 0 if \(\lambda \in a_1\) and \(2\pi\) if \(\lambda \not\in a_1\). So the total contribution of \(a_{on}\) is given by
\[
\pi m_a^- + 2\pi \cdot \frac{|a_{on}| - |a_1| - m_a^-}{2} = \pi (|a_{on}| - |a_1|).
\]
Let \(a_{<1}\) resp. \(a_{>1}\) be the component of \(a_{off}\) whose elements are real numbers \(< -1\) resp. \(> 1\). For each real pair \(\lambda, \lambda^t \in a_{off}\), the sum \(\arg \lambda + \arg \lambda^t\) is 0 if \(\lambda \in a_{>1}\) and \(2\pi\) if \(\lambda \in a_{<1}\). For each non-real quartet \(\lambda, \bar{\lambda}, \lambda^t, \bar{\lambda}^t \in a_{off}\) with \(|\lambda| > 1\) and \(\text{Im} \, \lambda > 0\) we have
\[
\arg \lambda + \arg \bar{\lambda} + \arg \lambda^t + \arg \bar{\lambda}^t = \begin{cases} 0 & (0 < \arg \lambda \leq |\Theta|), \\ 4\pi & (|\Theta| < \arg \lambda < \pi), \end{cases}
\]
where \(\Theta\) is the number appearing in (13), which belongs to the interval \((-\pi, 0)\) due to assumption (32). Therefore the contribution of \(a_{off}\) to the sum \(2\pi \alpha = 2\pi |a_{<1}| \mod 4\pi \mathbb{Z}\). In total we have \(2\pi \alpha = \pi (|a_{on}| - |a_1|) + 2\pi |a_{<1}| \mod 4\pi \mathbb{Z}\), which yields a modulo 2 congruence
\[
\alpha = \frac{|a_{on}| - |a_1|}{2} + |a_{<1}| \mod 2.
\]

In a similar manner we consider how each component of \(b\) contributes to the sum \(2\pi \beta := 2\pi \beta_1 + \cdots + 2\pi \beta_n\), where \(2\pi \beta_i := \arg b_i\). Taking \(1 \not\in b\) into account we find that
\[
\beta = \frac{|b_{on}|}{2} + |b_{<1}| \mod 2.
\]
Since $\gamma = \beta - \alpha$, we use relations $|a_{on}| + |a_{off}| = n$, $|a_{off}| \equiv 2|a_{<1}| + 2|a_{>1}| \mod 4$ and their $b$-counterparts together with $|a_1| = 2|A_1| + 1$ to obtain a modulo 2 congruence

$$
\gamma \equiv \frac{|b_{on}| - |a_{on}| + |a_1|}{2} + |b_{<1}| - |a_{<1}| \equiv \frac{|a_1|}{2} + |a_{>1}| - |b_{>1}|
$$

$$
= \frac{1}{2}(2|A_1| + 1) + |A_{>2}| - |B_{>2}| \equiv \frac{1}{2} + |A_1| + |A_{>2}| + |B_{>2}| \mod 2.
$$

This establishes formula (33). \hfill \Box

In the real case the index formula (17) in Theorem 3.2 can be restated as follows.

**Theorem 4.2** Let $H = H(\Phi, \Psi) = \langle A, B \rangle$ be a real hypergeometric group of rank $n$. If condition (32) is satisfied then the index of the $H$-invariant Hermitian form is given by

$$
p - q = \varepsilon(1 + \delta - 2S), \tag{34}
$$

where $\varepsilon = \pm 1$ is given in formula (33) while $\delta$ and $S$ are defined by

$$
\delta := \begin{cases} (-1)^{|A_{on}|+|B_{on}|+1} & \text{if } n \text{ is even}, \\ 0 & \text{if } n \text{ is odd}, \end{cases} \tag{35a}
$$

$$
S := \sum_{i \in I} (-1)^{\sigma_i} \quad \text{with } I := \{ i = 2, \ldots, s : |A_i| = 1 \mod 2 \}, \tag{35b}
$$

with $\sigma_i$ being defined by $\sigma_2 := |B_1|$ and

$$
\sigma_i := |B_1| + |A_2| + |B_2| + \cdots + |A_{i-1}| + |B_{i-1}|, \quad i = 3, \ldots, s.
$$

If $n$ is even and $B_{on}$ is empty then the index $p - q$ is zero.

**Proof.** If $n$ is even and $B_{on}$ is nonempty, then (30a) and (31) imply $|a_1| = 2|A_1| + 1$, $|a_{s+1}| = 2|A_{s+1}| + 1$, $|a_i| = |a_{s+2-i}| = |A_i|$ for $i = 2, \ldots, s$, and $|b_i| = |b_{s+1-i}| = |B_i|$ for $i = 1, \ldots, s$, hence $I = K \cap \{2, \ldots, s\}$ and $K = \{1, s+1\} \cup I \cup \{2s + 2 - i : i \in I\}$, where $K$ is defined in (17). For each $i \in I$ we have $\tau_i = |a_i| + \sigma_i \equiv 1 + \sigma_i \mod 2$ and

$$
\tau_{2s+2-i} = |a_1| + \cdots + |a_{2s+1-i}| + |b_1| + \cdots + |b_{2s+1-i}|
\equiv |a_1| + \cdots + |a_s| + |a_{s+1}| + |b_1| + \cdots + |b_{s+1}| \mod 2
= \tau_s + |a_s| + |a_{s+1}| \equiv \tau_s \mod 2.
$$

Moreover, $\tau_{s+1} = |a_1| + |A_{on}| + |B_{on}| \equiv 1 + |A_{on}| + |B_{on}| \mod 2$. Formula (17) then yields

$$
\varepsilon(p - q) = \sum_{i \in K} (-1)^{\tau_i} = \sum_{i \in I} (-1)^{\tau_i} + (-1)^{\tau_{s+1}} + \sum_{i \in I} (-1)^{\tau_{2s+2-i}}
= 1 + (-1)^{\tau_{s+1}} + 2 \sum_{i \in I} (-1)^{\tau_i} = 1 + \delta - 2 \sum_{i \in I} (-1)^{\sigma_i}.
$$

14
If $n$ is odd then (30b) and (31) imply $|a_1| = 2|A_1| + 1$, $|a_i| = |a_{2s+2-i}| = |A_i|$ for $i = 2, \ldots, s$, $|b_s| = 2|B_s| + 1$, and $|b_i| = |b_{2s-i}| = |B_i|$ for $i = 1, \ldots, s - 1$, hence $I = K \cap \{2, \ldots, s\}$ and $K = \{1\} \cup I \cup \{2s + 1 - i : i \in I\}$. For each $i \in I$ we have $\tau_i = |a_1| + \sigma_i \equiv 1 + \sigma_i \mod 2$ and

$$
\tau_{2s+1-i} = |a_1| + \cdots + |a_{2s-i}| + |b_1| + \cdots + |b_{2s-i}|
$$

$$
\equiv |a_1| + \cdots + |a_i| + |b_1| + \cdots + |b_{i-1}| + |b_s| \mod 2
$$

$$
= \tau_i + |a_i| + |b_s| \equiv \tau_i \mod 2.
$$

Thus formula (17) in Theorem 3.2 then yields

$$
\varepsilon(p - q) = \sum_{i \in K} (-1)^{\tau_i} = (-1)^{\tau_1} + \sum_{i \in I} (-1)^{\tau_i} + \sum_{i \in I} (-1)^{2s+1-i}
$$

$$
= 1 + 2 \sum_{i \in I} (-1)^{\tau_i} = 1 + \delta - 2 \sum_{i \in I} (-1)^{\sigma_i}.
$$

In either case we have obtained formula (34). If $n$ is even and $B_{on}$ is empty, then $b_{on}$ is empty and hence the index $p - q$ is zero by the last part of Theorem 3.2.

\begin{lemma}
There are the following numerical constraints

$$
S \equiv |I| \equiv |A_{in}| \mod 2, \quad |S| \leq |I| \leq s - 1 \leq \frac{|I| + |A_{in}|}{2} \leq |A_{in}|, \quad s \leq |B_{on}|.
$$

\end{lemma}

\begin{proof}
Congruence $S \equiv |I| \mod 2$ follows from the definition of $S$ in (35b) and $(-1)^{\sigma_i} \equiv 1 \mod 2$. Since $|A_i| \geq 1$ for $i = 2, \ldots, s$, the definition of $S$ and the inclusion $I \subset \{2, \ldots, s\}$ imply $|S| \leq |I| \leq s - 1 \leq |A_s| + \cdots + |A_i| = |A_{in}|$. As $|A_i|$ is odd for $i \in I$ and even for $i \in \{2, \ldots, s\} \setminus I$, we have $|I| \equiv |A_{in}| \mod 2$ and $|I| + 2(s - 1 - |I|) \leq |A_{in}|$, that is, $s - 1 \leq \frac{1}{2}(|I| + |A_{in}|)$. Moreover we have $s \leq |B_1| + \cdots + |B_s| = |B_{on}|$ because $|B_i| \geq 1$ for $i = 1, \ldots, s$. Putting all these together lead to the constraints (36).

\end{proof}

\begin{remark}
If $n$ is even then the reflection of the trace clusters in $A_{in} \cup B_{on}$,

$$
B_s, A_s, B_{s-1}, \ldots, B_2, A_2, B_1 \quad \text{reflection} \quad B_1, A_2, B_2, \ldots, B_{s-1}, A_s, B_s
$$

results in the change of signs $S \rightarrow \delta S$ and $p - q \rightarrow \delta(p - q)$, where $\delta$ is defined in (35a).

\end{remark}

\subsection{Local Index in the Real Case}

Proposition 3.3 gives formula (23) for the local index $\text{idx}(\lambda)$ at $\lambda \in a_{on} \cup b_{on}$. It can be restated in terms of $\tau := \lambda + \lambda^{-1} \in A_{on} \cup B_{on}$, where we are allowed to write $\text{idx}(\lambda) = \text{idx}(\lambda^{-1}) = \text{Idx}(\tau)$ if $\lambda \neq \pm 1$, i.e. $\tau \neq \pm 2$. To state the result let $\rho : [-2, 2] \rightarrow \mathbb{Z}_{\geq 0}$ be a function defined by

$$
\rho(\tau) := \# \text{ of all real roots of } \Phi(w) \cdot \Psi(w) \text{ that are } \begin{cases} 
> \tau & \text{if } \tau \in [-2, 2), \\
\geq 2 & \text{if } \tau = 2
\end{cases}
$$

where $\#$ denotes the cardinality counted with multiplicities.
Proposition 4.5 For any $\tau \in A_{on} \cup B_{on}$ with $\tau \neq \pm 2$ we have

$$\text{Idx}(\tau) := \begin{cases} 
(\frac{1}{2})^{|\tau|+1} & \text{if } \tau \in A_{on} \text{ and } M(\tau) \text{ is odd,} \\
(\frac{1}{2})^{|\tau|} & \text{if } \tau \in B_{on} \text{ and } M(\tau) \text{ is odd,} \\
0 & \text{if } M(\tau) \text{ is even,}
\end{cases}$$

(39)

where $M(\tau)$ is the multiplicity of $\tau$ in $A_{on} \cup B_{on}$. Moreover we have

$$\text{Idx}(1) = (-1)^{\rho(2)}, \quad \text{Idx}(-1) = (-1)^{\rho(-2)+n+1}.$$  

(40)

Proof. First we prove (39). Take an element $\lambda \in a_{on} \cup b_{on}$ such that $\lambda + \lambda^{-1} = \tau$. We may assume $\text{Im} \lambda > 0$ since $\text{Idx}(\lambda) = \text{Idx}(\lambda^{-1})$. If $\tau \in A_{on}$, that is, $\lambda \in a_{on}$ then definition (22) and relation $\varepsilon(\lambda) = 2|A_{on} \cup B_{on}| + |A_{1}| + 1$, which together with (35) and (38) yields $\varepsilon \cdot (-1)^{|\lambda|} = (\frac{1}{2})|A_{1}| + |A_{2}| + |A_{1}| + 1$. In a similar manner, if $\tau \in B_{on}$ then $\varepsilon \cdot (-1)^{|\lambda|} = (\frac{1}{2})^{|\lambda|}$. Thus (39) follows from formula (23).

Next we prove (40). By (22), (35) and (38) we have

$$r(1) = |A_{1}| - M(2), \quad r(-1) = |A_{on}| + |B_{on}| - M(-2) + |A_{1}| + 1, \\
r(2) = M(2) + |A_{2}| + |B_{2}|, \quad \rho(-2) = |A_{on}| + |B_{on}| - M(-2) + |A_{2}| + |B_{2}|.$$

Putting these equations into (23) and using (29) and (35) we establish (40), where we take into account that $1 \in a_{on}$ while $-1 \in b_{on}$ if $n$ is even.

Suppose that the rank $n$ is even; the odd case is omitted as it is not necessary in this article. Let $A_{i}^{s} := (A_{i})_{<2}$ and $A_{i+1}^{s} := (A_{i+1})_{<2}$. For a multi-set $X$ given in boldface its calligraphic style $\mathcal{X}$ denotes the ordinary set of all distinct elements in $X$ of odd multiplicity. We apply this rule to $X = A_{1}^{o}, A_{s+1}^{o}, A_{in}, B_{on}$ to define $\mathcal{X} = A_{1}^{o}, A_{s+1}^{o}, A_{in}, B_{on}$ respectively. For these sets we put $\text{Idx}(\mathcal{X}) := \sum_{\tau \in \mathcal{X}} \text{Idx}(\tau)$ and note that $0 \leq |X| - |\mathcal{X}| \equiv 0 \text{ mod } 2$. Moreover we denote by $\text{Par}(m)$ the parity of $m \in \mathbb{Z}$, that is, $\text{Par}(m) = 1$ when $m$ is even and $\text{Par}(m) = 1$ when $m$ is odd.

Theorem 4.6 In the situations of Theorem 4.2 with the rank $n$ being even we have

$$\text{Idx}(1) = \varepsilon \cdot (\frac{1}{2})^{|A_{1}|}, \quad \varepsilon = \text{Idx}(1) + 2\text{Idx}(A_{1}^{o}),$$

(41a)

$$\text{Idx}(1) = \varepsilon \cdot \text{Par}(A_{1}^{o}),$$

(41b)

$$\text{Idx}(A_{in}) = -\varepsilon S, \quad \text{Idx}(B_{on}) = (p - q)/2.$$  

(41c)

Proof. Formulas in (11a) are easy consequences of (10). The right-hand side of index formula (31) consists of three parts $\varepsilon, \delta$ and $-2\varepsilon S$. They are the contributions of $a_{1}$, $a_{s+1}$ and $a_{in} := a_{2} \cup a_{3} \cup \cdots \cup a_{s} \cup a_{in}$, and hence equal to the sums of local indices of all distinct elements in the three respective sets. Since $\lambda \mapsto \tau = \lambda + \lambda^{-1}$ induces two-to-one maps $(a_{1})_{\neq 1} \to A_{1}^{o}, (a_{s+1})_{\neq -1} \to A_{s+1}^{o}, a_{in} \to A_{in} \text{ and } \text{idx}(\lambda) = \text{Idx}(\lambda^{-1}) = \text{Idx}(\tau)$, we have

$$\varepsilon = \text{Idx}(1) + 2\text{Idx}(A_{1}^{o}), \quad \varepsilon \delta = \text{Idx}(-1) + 2\text{Idx}(A_{s+1}^{o}), \quad -2\varepsilon S = 2\text{Idx}(A_{in})$$

which in turn lead to the formulas in (41a) and the first formula in (41c). Finally, the total index $p - q$ is the sum of local indices over all distinct elements in $b_{on}$. In view of the two-to-one correspondence $b_{on} \to B_{on}$ we have the second formula in (41c).
5 Hypergeometric Lattices

A hypergeometric group \( H = H(\varphi, \psi) = H(\Phi, \Psi) = \langle A, B \rangle \) is said to be integral if
\[
\varphi(z) \in \mathbb{Z}[z], \quad \psi(z) \in \mathbb{Z}[z], \quad \text{or equivalently} \quad \Phi(w) \in \mathbb{Z}[w], \quad \Psi(w) \in \mathbb{Z}[w].
\]
In this case let \( L = L(\varphi, \psi) = L(\Phi, \Psi) \) be the \( \mathbb{Z} \)-linear span of \( r, Ar, \ldots, A^{n-1}r \). Then \( L \) is a free \( \mathbb{Z} \)-module with free basis \( r, Ar, \ldots, A^{n-1}r \). Then \( L \) is a free \( \mathbb{Z} \)-module with free basis \( r, Ar, \ldots, A^{n-1}r \), the Hermitian form is \( \mathbb{Z} \)-valued on \( L \) and \( H \subset O(L) \).

It follows from (25) that \( B_r = -A_r \) and \( B_k r = -A_k r + \mathbb{Z} \)-linear combination of \( A_r, \ldots, A^{k-1}r \) for every \( k \geq 2 \), hence \( r, B_r, \ldots, B_{n-1}r \) also form a \( \mathbb{Z} \)-linear basis of \( L \). Thus we have
\[
L = \langle r, Ar, \ldots, A^{n-1}r \rangle \mathbb{Z} = \langle r, B_r, \ldots, B_{n-1}r \rangle \mathbb{Z}.
\]

By the \( A \)-invariance of the Hermitian form and the normalization \( (r, r) = 2 \) we have \( (v, v) \in 2\mathbb{Z} \) for all \( v \in L \). Thus \( L \) equipped with the invariant form becomes an even lattice, called a hypergeometric lattice. We say that the group \( H \) is unimodular if so is the lattice \( L \).

### 5.1 Unimodularity

Formula (11) in Theorem 2.1 implies that the hypergeometric lattice \( L \) is unimodular if and only if \( \text{Res}(\varphi, \psi) = \pm 1 \). Then (28a) shows that when \( n \) is even this condition is equivalent to
\[
\Psi(2) = \pm 1, \quad \Psi(-2) = \pm 1; \quad \text{Res}(\Phi, \Psi) = \pm 1,
\]
while (28b) shows that when \( n \) is odd \( L \) cannot be unimodular. We say that \( \Psi \) is unramified if it satisfies the first and second conditions in (43). Note that there is no unramified polynomial of degree one. With irreducible decompositions \( \Phi(w) = \prod_i \Phi_i(w) \) and \( \Psi(w) = \prod_j \Psi_j(w) \) in \( \mathbb{Z}[w] \) the conditions (43) can be restated as
\[
\Psi_j(2) = \pm 1, \quad \Psi_j(-2) = \pm 1; \quad \text{Res}(\Phi_i, \Psi_j) = \pm 1 \quad \text{for all} \quad i, j.
\]

This observation provides us with a recipe to construct a unimodular hypergeometric lattice.

**Recipe 5.1** Given an integer \( N \in \mathbb{Z}_{\geq 2} \), find a finite set of monic irreducible polynomials \( \Phi_i(w), \Psi_j(w) \in \mathbb{Z}[w] \) that satisfies the unimodularity condition (43) and the degree condition
\[
\sum_i \deg \Phi_i = N - 1, \quad \sum_j \deg \Psi_j = N.
\]
Take the products \( \Phi(w) := \prod_i \Phi_i(w) \) and \( \Psi(w) := \prod_j \Psi_j(w) \) and consider the integral hypergeometric group \( H = H(\Phi, \Psi) \). Then the associated lattice \( L \) is an even unimodular lattice of rank \( n = 2N \).

We are especially interested in the settings where the irreducible factors \( \Phi_i(w) \) and \( \Psi_j(w) \) are cyclotomic trace polynomials or Salem trace polynomials.
Table 2: Cyclotomic trace polynomials $CT_k(z)$ of degree $\leq 10$.

| deg | $k$   | deg | $k$   |
|-----|-------|-----|-------|
| 1   | 1, 2, 3, 4, 6 | 6   | 13, 21, 26, 28, 36, 42 |
| 2   | 5, 8, 10, 12  | 8   | 17, 32, 34, 40, 48, 60 |
| 3   | 7, 9, 14, 18  | 9   | 19, 27, 38, 54          |
| 4   | 15, 16, 20, 24, 30 | 10  | 25, 33, 44, 50, 66      |
| 5   | 11, 22        |     |                   |

5.2 Cyclotomic Trace Polynomials

For $k \in \mathbb{Z}_{\geq 3}$ let $C_k(z) \in \mathbb{Z}[z]$ denote the $k$-th cyclotomic polynomial. It is a monic irreducible polynomial of degree $\phi(k)$, where $\phi(k)$ is Euler’s totient function. For $k = 1, 2$, in view of our purpose, it is convenient to employ an unconventional definition

$$C_1(z) := (z - 1)^2, \quad C_2(z) := (z + 1)^2, \quad \phi(1) = \phi(2) = 2.$$  

Note that $C_k(x) \geq 0$ for every $x \in \mathbb{R}$. We say that $C_k(z)$ is unramified if $C_k(\pm 1) = 1$.

For any $k \in \mathbb{Z}_{\geq 1}$ Euler’s totient $\phi(k)$ is an even integer and $C_k(z)$ is a monic palindromic polynomial of degree $\phi(k)$, so there exists a unique monic polynomial $CT_k(w) \in \mathbb{Z}[w]$ of degree $\phi(k)/2$, called the $k$-th cyclotomic trace polynomial, such that

$$C_k(z) = z^{\phi(k)/2} CT_k(z + z^{-1}).$$

Note that $CT_1(w) = w - 2$, $CT_2(w) = w + 2$ and $CT_k(w)$ is irreducible for $k \geq 1$. We say that $CT_k(w)$ is unramified if so is $C_k(z)$, in which case $CT_k(2) = 1$ and $CT_k(-2) = (-1)^{\phi(k)/2}$.

The following lemma is helpful in checking the unimodularity condition (43\′) for cyclotomic trace factors of $\Phi(w)$ and $\Psi(w)$.

**Lemma 5.2** Let $k$ and $m$ be positive integers such that $k > m$.

1. $\text{Res}(CT_k, CT_m) = \pm 1$ if and only if the ratio $k/m$ is not a prime power,
2. $CT_m(w)$ is unramified if and only if neither $m$ nor $m/2$ is a prime power,

where a prime power is an integer of the form $p^l$ with a prime $p$ and a positive integer $l$.

**Proof.** Apostol [1, Theorems 1, 3, 4] evaluates the resultants of cyclotomic polynomials:

$$\text{Res}(C_k, C_m) = \begin{cases} p^{\phi(m)} & \text{if } k/m \text{ is a power of a prime } p, \\ 1 & \text{otherwise,} \end{cases}$$

for $k > m$. In particular $C_k(z)$ is unramified if and only if neither $k$ nor $k/2$ is a prime power. Lemma 5.2 then readily follows from the relation $\text{Res}(C_k, C_m) = \text{Res}(CT_k, CT_m)^2$. \hfill $\Box$

**Lemma 5.3** There are exactly 41 cyclotomic trace polynomials $CT_k(w)$ of degree $\leq 10$, among which 15 are unramified. They are given in Table 2 with unramified ones being underlined.
The Salem trace polynomial \( R_i(w) \) of degree 11 from McMullen [16, Table 4].

| \( i \) | \( \lambda_i \) | Salem trace polynomial \( R_i(w) \) |
|---|---|---|
| 1 | 1.37289 | \( w(w-1)(w+1)^2(w^2-4)(w^5-6w^3+8w-2)-1 \) |
| 2 | 1.45099 | \( (w+1)(w^2-4)(w^8-8w^6-w^5+19w^4+3w^3-12w^2+1)-1 \) |
| 3 | 1.48115 | \( w(w-1)(w+1)^2(w^2-4)(w^5-6w^3-w^2+8w+1)-1 \) |
| 4 | 1.52612 | \( w^2(w+1)(w^2-4)(w^2+w-1)(w^4-w^3-5w^2+3w+5)-1 \) |
| 5 | 1.55377 | \( (w-1)(w+1)^2(w^2-4)(w^6-7w^4-w^3+12w^2+2w-1)-1 \) |
| 6 | 1.60709 | \( w(w+1)^2(w^2-4)(w^2+w-1)(w^4-2w^3-3w^2+6w-1)-1 \) |
| 7 | 1.6298 | \( (w+1)(w^2-4)(w^8-8w^6-w^5+19w^4+2w^3-14w^2+2)-1 \) |
| 8 | 1.6458 | \( w(w+1)^2(w^2-4)(w^6-w^5-7w^4+5w^3+13w^2-6w-2)-1 \) |
| 9 | 1.66566 | \( w(w+1)(w^2-4)(w^7-9w^5-2w^4+25w^3+9w^2-20w-8)-1 \) |
| 10 | 1.69496 | \( w(w+1)^2(w^2-3)(w^2-4)(w^4-w^3-4w^2+2w+1)-1 \) |

Table 3: Salem trace polynomials of degree 11 from McMullen [16, Table 4].

**Proof.** Indeed, the cases \( k = 1 \) and \( k = 2 \) are trivial. For \( k \geq 3 \), \( \phi(k) \) admits a lower bound

\[
\phi(k) > \phi_0(k) := \frac{k}{e\gamma \log \log k + \frac{3}{\log \log k}},
\]

where \( \gamma \) is Euler’s gamma constant (see [2, Theorem 8.8.7] and [21, Theorem 15]). Thus the condition \( 20 \geq \phi(k) \) implies \( 20 \geq \phi_0(k) \) and a careful analysis of the function \( \phi_0(x) \) in real variable \( x \geq 3 \) shows that the latter condition holds exactly for \( k = 3, \ldots, 93 \). A case-by-case check in this range gives all solutions \( k \) to the bound \( \deg \text{CT}_k \leq 10 \) as in Table 2.

**5.3 Salem Trace Polynomials**

A Salem number is an algebraic unit \( \lambda > 1 \) whose conjugates other than \( \lambda^{\pm 1} \) lie on the unit circle \( S^1 \) (see Salem [23]). The monic minimal polynomial of a Salem number is called a Salem polynomial. A Salem polynomial is a palindromic polynomial of even degree. A Salem trace is an algebraic integer \( \tau > 2 \) whose other conjugates lie in the interval \([-2, 2]\). The monic minimal polynomial of a Salem trace is called a Salem trace polynomial. Salem numbers \( \lambda \) and Salem traces \( \tau \) are in one-to-one correspondence via the relation \( \tau = \lambda + \lambda^{-1} \). A Salem polynomial \( S(z) \) and the associated Salem trace polynomial \( R(w) \) are related by

\[
S(z) = z^d R(z + z^{-1}) \quad \text{with} \quad d := \deg R(w).
\]

We say that \( S(z) \) and \( R(w) \) are unramified if \( |S(\pm 1)| = 1 \) and \( |R(\pm 2)| = 1 \) respectively.

**Example 5.4** McMullen [16, Table 4] gives a list of ten unramified Salem numbers \( \lambda_i \) of degree 22 and the associated Salem polynomials \( S_i(z) \) and Salem trace polynomials \( R_i(w) \). Approximate values of \( \lambda_i \) and exact formulas for \( R_i(w) \) are given in Table 3.
Example 5.5 Lehmer’s number $\lambda_L \approx 1.17628$ discovered in [14] is the smallest Salem number ever known (see e.g. Hironaka [10]). The associated Salem polynomial and Salem trace polynomial, that is, Lehmer’s polynomial and Lehmer’s trace polynomial are given by

$$L(z) = z^{10} + z^{9} - z^{7} - z^{6} - z^{5} - z^{3} + z + 1,$$

$$\text{LT}(w) = (w + 1)(w^{2} - 1)(w^{2} - 4) - 1,$$

respectively. Note that they are unramified. Lehmer’s trace $\tau_L$ is approximately 2.02642.

6 Hypergeometric K3 Lattices

An even unimodular lattice of rank 22 and signature (3,19), that is, index $-16$ is called a K3 lattice. It is well known that the second cohomology group $H^{2}(X,\mathbb{Z})$ of a K3 surface $X$ equipped with the intersection form is a K3 lattice. We wonder whether a K3 lattice can be realized as a hypergeometric lattice. To discuss this problem we make the following.

Definition 6.1 A hypergeometric K3 lattice is a unimodular hypergeometric lattice of rank 22 and index ±16, where the index is calculated with respect to the invariant Hermitian form normalized by $(r, r) = 2$ as in (26), which we call hypergeometric normalization. In the context of K3 lattice we should employ another normalization that makes the index always to be $-16$, which we call K3 normalization. When the index is $+16$ in the former normalization we can switch to the latter one by negating the invariant Hermitian form; this amounts to taking the reversed normalization $(r, r) = -2$.

6.1 Real of Rank 22 and Index ±16

Putting aside the integral structure and unimodularity condition for the moment we shall classify all real hypergeometric group of rank 22 and index ±16. We employ normalization (26) and use the notations in §4. We now have $\deg \Phi = |A| = 10$ and $\deg \Psi = |B| = 11$.

Theorem 6.2 A real hypergeometric group of rank $n = 22$ has index $p - q = \pm 16$ if and only if the configuration of $A_{\text{in}}$ and $B_{\text{on}}$ is just as in Table 4, where in case 5 we mean by “doubles

| case | $s$ | $|A_{\text{in}}|$ | $|B_{\text{on}}|$ | $|A_{\text{in}}|$ | $|B_{\text{on}}|$ | constraint | $\varepsilon(p - q)$ |
|------|-----|-----------------|-----------------|-----------------|-----------------|-------------|------------------|
| 1    | 8   | 7               | 8               | 1$^7$           | 1$^8$           |             | 16               |
| 2    | 8   | 9               | 8               | 1$^6$3$^1$      | 1$^8$           |             | 16               |
| 3    | 8   | 7               | 10              | 1$^7$           | 1$^7$3$^1$      |             | 16               |
| 4    | 8   | 9               | 10              | 1$^6$3$^1$      | 1$^7$3$^1$      |             | 16               |
| 5    | 9   | 9               | 10              | 1$^7$2$^1$      | 1$^8$2$^1$      | doubles adjacent | 16               |
| 6    | 9   | 8               | 10              | 1$^8$           | 1$^8$2$^1$      | $|B_{5\pm4}| = 2$ | ±16              |
| 7    | 9   | 10              | 10              | 1$^7$3$^1$      | 1$^8$2$^1$      | $|B_{5\pm4}| = 2$ | ±16              |
| 8    | 10  | 10              | 10              | 1$^8$2$^1$      | 1$^{10}$        | $|A_{6\pm4}| = 2$ | ±16              |

Table 4: Real of rank 22 and index ±16, where $\varepsilon = \pm$ is defined in (33).
adjacent” that the unique double cluster in $A_{in}$ and the one in $B_{on}$ must be adjacent to each other. Each of cases 6, 7, 8 divides into two subcases as indicated in the constraint column.

Proof. We use Theorem 4.2 and Lemma 4.3 with $n = 22$. Equation (34) yields $|1 + \delta - 2S| = |p - q| = 16$, which is the case if and only if

$$(\delta, S) = (1, -7), (1, 9), (-1, 7). \quad (45)$$

If $|B_{on}|$ is odd then $\delta = (-1)|A_{in}|$ by (35a) and hence $|A_{in}| \neq S$ mod 2 by (45), which contradicts the congruence $S \equiv |A_{in}|$ mod 2 in (36). Hence $|B_{on}|$ must be even and $\delta = -(1)|A_{in}|$. It follows from (36) that $8 \leq |S| + 1 \leq |B_{on}| \leq |B|$ = 11, so that we have either

$$|B_{on}| = 8, \quad s = 8; \quad \text{or} \quad |B_{on}| = 10, \quad s = 8, 9, 10. \quad (46)$$

A careful inspection shows that the only ten cases in Table 5 can meet the constraints (36), (45), (46) and $|A_{in}| \leq |A|$ = 10. Let us make a case-by-case treatment.

| case | $s$ | $|I|$ | $|A_{in}|$ | $B_{on}$ | $\delta$ | $S$ | $|A_{in}|$ | $|B_{on}|$ |
|------|-----|-----|----------|---------|--------|-----|----------|---------|
| 1    | 8   | 7   | 7        | 8       | 1      | -7  | 1        | 8       |
| 2    | 8   | 7   | 9        | 8       | 1      | -7  | 1^63^1  | 8       |
| 3    | 8   | 7   | 7        | 10      | 1      | -7  | 1^7    | 1^62^2 |
| 4    | 8   | 7   | 9        | 10      | 1      | -7  | 1^63^1  | 1^62^2 |
| 5    | 9   | 7   | 9        | 10      | 1      | -7  | 1^72^1 | 1^82^1 |
| 6    | 9   | 8   | 8        | 10      | -1    | 8   | 1        | 1^82^1 |
| 7    | 9   | 8   | 10       | 10      | -1    | 8   | 1^73^1  | 1^82^1 |
| 8    | 10  | 8   | 10       | 10      | -1    | 8   | 1^82^1  | 1^10    |
| 9    | 10  | 9   | 9        | 10      | 1     | -7  | 1^9    | 1^10    |
| 10   | 10  | 9   | 9        | 10      | 1     | 9   | 1^9    | 1^10    |

Table 5: Ten cases.

In cases 1–4 we have $I = \{2, \ldots, 8\}$. In cases 1 and 2, since $\sigma_i$ is odd for every $i = 2, \ldots, 8$, $S = -7$ is actually realized. The same is true in cases 3 and 4 with $|B_{on}| = 1^73^1$. In cases 3 and 4 with $|B_{on}| = 1^62^2$, let $|B_k| = |B_l|$ = 2 with $1 \leq k < l \leq 8$. Up to reflection (37) we may assume (i) $k = 1$ and $l = 8$; or (ii) $2 \leq k < l \leq 8$. In case (i) $\sigma_i$ is even for every $i = 2, \ldots, 8$, so that $S = 7$ contradicting $S = -7$. In case (ii) $\sigma_i$ is odd for $2 \leq i \leq k$ or $l + 1 \leq i \leq 8$ and even for $k + 1 \leq i \leq l$, so $S = -(k - 1) - (8 - l) + (l - k) = 2(l - k) - 7 \geq -5$, which again contradicts $S = -7$. Thus cases 3 and 4 with $|B_{on}| = 1^62^2$ cannot occur.

In case 5 let $|A_k| = |B_l| = 2$ with $2 \leq k < 9$ and $1 \leq l < 9$, in which case $I = \{2, \ldots, k, \ldots, 9\}$. Up to reflection (37) we may assume $l < k$. Then $\sigma_i$ is odd for $2 \leq i \leq l$ or $k + 1 \leq i \leq 9$ and even for $l + 1 \leq i \leq k - 1$, so that $S = -(l - 1) - (9 - k) + (k - l - 1) = 2(k - l) - 9 = -7$ implies $l = k - 1$. Taking reflection (37) we have also $l = k$. Thus $S = -7$ is realized if and only if $|A_k| = |B_{k-1}| = 2$ or $|A_k| = |B_{k-1}| = 2$ holds for some $2 \leq k < 9$, that is, the double cluster in $A_{in}$ and the one in $B_{on}$ must be adjacent to each other.

In cases 6 and 7 we have $I = \{2, \ldots, 9\}$. Let $|B_k| = 2$ with $1 \leq k \leq 9$. Up to reflection (37) we may assume $5 \leq k \leq 9$. Then $\sigma_i$ is odd for $2 \leq i \leq k$ and even for $k + 1 \leq i \leq 9$, so that

$$|\sigma_i| = 1^9.$$
\( S = -(k - 1) + (9 - k) = 2(5 - k) = \mp 8 \) implies \((k, S) = (9, -8)\). Taking reflection \(57\) we have also \((k, S) = (1, 8)\). In summary \( S = \mp 8 \) forces \(|B_{5\pm 4}| = 2\).

In case 8 let \(|A_i| = 2\) and \(I = \{2, \ldots, k, \ldots, 10\}\) with \(2 \leq k \leq 10\). Then \(\sigma_i\) is odd for \(2 \leq i \leq k - 1\) and even for \(k + 1 \leq i \leq 10\), so \(S = -(k - 2) + (10 - k) = 2(6 - k) = \mp 8\), which implies \(k = 6 \pm 4\), that is, \(|A_{6\pm 4}| = 2\).

In cases 9 and 10 we have \(|B_{on}| = 10\) and \(I = \{2, \ldots, 10\}\), so \(\sigma_i\) is odd for \(i = 2, \ldots, 10\), and hence \(S = -9\), i.e. neither \(S = -7\) nor \(S = 9\) occurs. Thus these cases cannot happen. \(\square\)

Some important information about \(A\) and \(B\) can be extracted from Table 4. Recall that an end cluster in \(A_{on}\) may be empty. If this is the case then it is called a null cluster.

**Lemma 6.3** The following are valid for \(A\) and the same is true with \(B\).

1. Any non-null \(A_{on}\)-cluster is simple except for at most one cluster of multiplicity 2 or 3.
2. We have \(|A_{off}| \leq 3\) and if \(|A_{off}| \geq 2\) then any non-null \(A_{on}\)-cluster is simple.
3. Any element of \(A\) is simple except for at most one element of multiplicity 2 or 3.

**Proof.** Table 4 tells us \(|A_{in}| \geq 7\) and so \(|A_1| + |A_{s+1}| + |A_{off}| = 10 - |A_{in}| \leq 3\), in particular \(|A_{off}| \leq 3\), which gives the first part of assertion (2). We have also \(|A_1| + |A_{s+1}| \leq 10 - |A_{in}|\). Using this inequality we consider how \(A_{in}\) can be extended to \(A_{on}\) by adding \(A_1\) and \(A_{s+1}\). A careful inspection of all cases in Table 4 leads to assertion (1). Suppose \(|A_{off}| \geq 2\) and hence \(|A_{in}| \leq 8\). Then we must be in one of the cases 1, 3, 6 in Table 4 where \(A_{in}\) consists of simple clusters only. Then \(A_{on}\) can also contain simple or null clusters only, since \(|A_1| + |A_{s+1}| \leq 1\). This proves the second part of assertion (2). By assertion (1) any element of \(A_{on}\) is simple except for at most one element which is of multiplicity 2 or 3. Moreover it follows from \(|A_{off}| \leq 3\) that \(A_{off}\) can contain at most one multiple element, which is of multiplicity 2 or 3. If there is one then the second part of assertion (2) implies that there is no multiple element in \(A_{on}\). This proves assertion (3). As for \(B\) assertions (1) and (2) can be seen directly from Table 4 and then assertion (3) is proved just in the same manner as in the case of \(A\). \(\square\)

We now take the integral structure and unimodularity condition into account.

**Lemma 6.4** If \(L(\Phi, \Psi)\) is a hypergeometric K3 lattice then any root of \(\Phi(w)\) is simple except for at most one integer root of multiplicity 2 or 3, whereas any root of \(\Psi(w)\) is simple.

**Proof.** It follows from (3) of Lemma 6.3 that \(\Phi(w)\) admits at most one multiple root. If \(\Phi(w)\) actually contains one, say \(\tau\), then any conjugate \(\tau'\) of it is also a multiple root, so uniqueness forces \(\tau' = \tau\), which means that \(\tau\) must be an integer. For the same reason any root of \(\Psi(w)\) is simple except for at most one integer root, but unramifiedness of \(\Psi(w)\) rules out this exception because there is no unramified polynomial of degree one. \(\square\)

**Remark 6.5** If \(\Phi(w)\) admits a multiple root \(\tau \in \mathbb{Z}\) then the last condition in (43) yields \(\Psi(\tau) = \pm 1\). This equation help us find the multiple root of \(\Phi(w)\) if it exists.
6.2 Hodge Isometry and Special Eigenvalue

Let $L$ be a K3 lattice and $L_{\mathbb{C}} := L \otimes \mathbb{C}$ be its complexification equipped with the induced Hermitian form. A Hodge structure on $L$ is an orthogonal decomposition

$$L_{\mathbb{C}} = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$$

(47)
of signatures $(1,0) \oplus (1,19) \oplus (1,0)$ such that $\overline{H^\mathbb{C}} = H^\mathbb{C}$. We remark that $H^{1,1}_{\mathbb{R}} := H^{1,1} \cap L_{\mathbb{R}}$ with $L_{\mathbb{R}} := L \otimes \mathbb{R}$ is a real Lorentzian space of signature $(1,19)$ and the set of time-like vectors $C := \{v \in H^{1,1}_{\mathbb{R}} : (v,v) > 0\}$ consists of two disjoint connected cones, one of which is referred to as the positive cone $C^+$ and the other as the negative cone $C^- = -C^+$.

A Hodge isometry is a lattice automorphism $F : L \to L$ preserving the Hodge structure $\tag{17}$. We then have either $F(C^+) = C^+$ or $F(C^+) = C^-$, according to which $F$ is said to be positive or negative. For any positive Hodge isometry $F$ there is a trichotomy:

(E) There exists a line in $C \cup \{0\}$ preserved by $F$. In this case the line is fixed pointwise by $F$ and all eigenvalues of $F$ lie on $S^1$.

(P) There exists a unique line in $C$ preserved by $F$ and this line is on the light-cone $\partial C$. In this case the line is fixed pointwise by $F$ and all eigenvalues of $F$ lie on $S^1$.

(H) There exists a real number $\lambda > 1$ such that $\lambda^{\pm 1}$ are the only eigenvalues of $F$ outside $S^1$.

In this case the eigenvalues $\lambda^{\pm 1}$ are simple and their eigen-lines are on the light-cone $\partial C$.

In the cases above $F$ is said to be of elliptic, parabolic or hyperbolic type respectively. Since $F$ preserves the lattice $L$ defined over $\mathbb{Z}$, any eigenvalue of $F$ is a root of unity in elliptic and parabolic cases, whereas it is a conjugate of a unique Salem number $\lambda > 1$ or a root of unity in hyperbolic case. In any case $F$ restricted to $H^{1,1}$ has a real eigenvalue $\lambda \geq 1$.

Remark 6.6 A Hodge isometry $F$ is necessarily positive and falls into the case of hyperbolic type, when $F|_{H^{1,1}}$ has a real eigenvalue $\lambda > 1$.

For any Hodge isometry $F : L \to L$ there exists a number $\xi \in S^1$ such that $F|_{H^{2,0}} = \xi I$ and $F|_{H^{0,2}} = \xi^{-1} I$, where $I$ is the identity map on the respective spaces. Note that if $\xi = \pm 1$ then $F|_{H^{2,0} \oplus H^{0,2}} = \pm I$. We refer to $\xi^{\pm 1}$ and $\delta := \xi + \xi^{-1} \in [-2, 2]$ as the special eigenvalues and the special trace of $F$ respectively. This observation leads us to the following.

Definition 6.7 Let $F : L \to L$ be a lattice automorphism of a K3 lattice $L$. An eigenvalue $\xi \in S^1$ of $F$ is said to be special if $\xi \neq \pm 1$ and there is a 1-dimensional subspace $\ell \subset L_{\mathbb{C}}$ such that $F|_{\ell} = \xi I$ and the induced Hermitian form is positive definite on $\ell$; or if $\xi = \pm 1$ and there is a 2-dimensional subspace $P \subset L_{\mathbb{C}}$ such that $F|_{P} = \pm I$, $\overline{P} = P$ and the induced Hermitian form is positive definite on $P$. We refer to $\tau := \xi + \xi^{-1} \in [-2, 2]$ as the special trace of $F$.

Remark 6.8 Since the K3 lattice $L$ has signature $(3,19)$, if $F$ admits a special trace $\tau$ then it is unique. For the same reason, if moreover $\tau \neq \pm 2$ then the pair $(\xi, \ell)$ in Definition 6.7 is uniquely determined by $F$ up to the exchange of $(\xi, \ell)$ and $(\xi^{-1}, \ell)$.

If $F$ admits a special eigenvalue $\xi \neq \pm 1$ with an associated line $\ell \subset L_{\mathbb{C}}$ then

$$L_{\mathbb{C}} = H^{2,0} \oplus H^{1,1} \oplus H^{0,2} := \ell \oplus (\ell \oplus \overline{\ell})^\perp \oplus \overline{\ell}$$

(48)
gives a Hodge structure with respect to which $F$ is a Hodge isometry. Thus a lattice automorphism with a special eigenvalue makes itself a Hodge isometry. The description of the case $\xi = \pm 1$ is omitted as it will not be used in this article.

Let $L = L(A, B)$ be a hypergeometric K3 lattice in K3 normalization (see Definition 6.1). It is natural to ask whether $A$ or $B$ admits a special eigenvalue or equivalently a special trace. First let us focus on the matrix $A$. Recall from §3.3 that $E(\lambda)$ stands for the generalized eigenspace of $A$ corresponding to an eigenvalue $\lambda \in \mathcal{a}_{\text{on}}$ and $m(\lambda) = \dim E(\lambda)$ stands for the multiplicity of $\lambda$. We denote by $V(\lambda)$ the $\lambda$-eigenspace of $A$ in the narrow sense.

**Lemma 6.9** For any $\lambda \in \mathcal{a}_{\text{on}}$ we have $\dim V(\lambda) = 1$ and if $m(\lambda) \geq 2$ then the invariant Hermitian form is null on $V(\lambda)$. In particular any special eigenvalue $\xi$ of $A$, if it exists, is simple and different from $\pm 1$, so the corresponding special trace $\tau$ is different from $\pm 2$.

**Proof.** The last part of Theorem 2.1 says that $r$ is a cyclic vector of the matrix $A$. Let $v$ be its projection down to $E(\lambda)$ with respect to the direct sum decomposition $\mathcal{H}$. Then $v$ is a cyclic vector of $A|_{E(\lambda)}$, so if we put $v_j := (A - \lambda I)^{-j}v$ for $j \in \mathbb{Z}_{\geq 1}$ then $v_1, \ldots, v_m$ form a basis of $E(\lambda)$ where $m := m(\lambda)$. Since $v_{m+1} = 0$, we have $Av_m = \lambda v_m$ and hence $V(\lambda) = \mathbb{C}v_m$ is 1-dimensional. If $m \geq 2$ then using $v_m = Av_{m-1} - \lambda v_{m-1}$ we have

$$\lambda(v_m, v_m) = (Av_m, Av_{m-1} - \lambda v_{m-1}) = (Av_m, Av_{m-1}) - (\lambda v_m, \lambda v_{m-1})$$

$$= (v_m, v_{m-1}) - |\lambda|^2(v_m, v_{m-1}) = 0,$$

by $A$-invariance of the Hermitian form and $\lambda \in S^1$, so the Hermitian form is null on $V(\lambda)$.

Let $\xi$ be a special eigenvalue of $A$. If $\xi = \pm 1$ the corresponding plane $P$ in Definition 6.7 must be contained in the line $V(\xi)$, but this is impossible. So we have $\xi \neq \pm 1$ and $\ell = V(\xi)$. Since the Hermitian form is positive-definite on $V(\xi)$, we must have $m(\xi) = 1$. It is clear that the corresponding special trace $\tau := \xi + \xi^{-1}$ is different from $\pm 2$. \hfill $\square$

**Remark 6.10** If $A$ admits a special trace then the recipe (48) allows us to construct a Hodge structure on $L$ with respect to which $A$ is a Hodge isometry. This structure is uniquely determined by $A$ up to the exchange of $H^{2,0}$ and $H^{0,2}$. Notice that $\pm A$ induce the same Hodge structure, so it makes sense to speak of $A$ being positive or negative. The hypergeometric group $H = (A, B)$ can be normalized so that $A$ is positive, otherwise by replacing $H$ with its antipode $H^s$ (see Remark 2.2). Lemma 6.9 and these remarks also apply to the matrix $B$.

### 6.3 Determination of Special Trace

To discuss when a special trace exists and to determine it explicitly, we begin with the following.

**Lemma 6.11** If $A$ has a special trace and is positive with respect to the Hodge structure (48), then $(M(2), \text{id}(1))$ is $(0, 1)$ in elliptic case; $(1, -1)$ in parabolic case; and $(0, -1)$ in hyperbolic case respectively, while $M(-2) = 0$ and $\text{id}(1) = -1$ in all three cases, where the index is taken in K3 normalization (see Definition 6.1); moreover every element of $A$ is simple.

**Proof.** Let $\xi^{\pm 1}$ be the special eigenvalues of $A$. We consider the associated Hodge structure (48) and use Lemma 6.9 repeatedly. Since $\xi \neq \pm 1$ we have $V(\pm 1) \subset E(\pm 1) \subset H^{1,1}$. Let $V_\mathbb{R}(\pm 1) := V(\pm 1) \cap L_\mathbb{R} \subset H^{1,1}_\mathbb{R}$. If the line $V_\mathbb{R}(\pm 1)$ lies in $\mathcal{C}$ then $A$ sends $\mathcal{C}^\pm$ to $\mathcal{C}^\mp$ as $A = -1$ on $V_\mathbb{R}(\pm 1)$. This contradicts the positivity of $A$, so $V_\mathbb{R}(\pm 1)$ must be in the space-like region. Thus
the Hermitian form \( h \) is negative-definite on \( V(-1) \) and we must have \( m(-1) = 2M(-2) + 1 = 1 \), i.e. \( M(-2) = 0 \) and \( \text{idex}(-1) = -1 \). In elliptic and parabolic cases \( \mathcal{V}_{\mathbb{R}}(1) \) is the unique line in \( \mathcal{C} \) preserved by \( A \). Accordingly, in elliptic case \( h \) is positive definite on \( V(1) \), so we have \( m(1) = 2M(2) + 1 = 1 \), i.e. \( M(2) = 0 \) and \( \text{idex}(1) = 1 \). Similarly, in parabolic case \( h \) is null on \( V(1) \) and has signature \( (u, v) = (M(2), M(2) + 1) \) or \( (M(2) + 1, M(2)) \) on \( E(1) \), which forces \( M(2) = 1 \) and \( (u, v) = (1, 2) \), i.e. \( \text{idex}(1) = -1 \), since \( E(1) \subset H^{1,1} \) and \( H^{1,1} \) has signature \( (1, 19) \). In hyperbolic case \( h \) must be negative-definite on \( E(1) \) because \( h \) has signature \( (1, 1) \) on \( E(\mu) \oplus E(\mu^{-1}) \subset H^{1,1} \) where \( \mu \) is the eigenvalue of \( A \) with \( \mu > 1 \). This makes \( M(2) = 0 \) and \( \text{idex}(1) = -1 \). If \( U \) be \( E(1) \) in elliptic or parabolic case and \( E(\mu) \oplus E(\mu^{-1}) \) in hyperbolic case then \( h \) is negative-definite on \( U^\perp \cap H^{1,1} \). This implies that any element of \( a_{\text{on}} \) other than \( \xi^\pm_1, \pm 1 \) must be simple. The last part of the lemma follows readily from this. \( \square \)

To state our theorems we give some remarks about notation and terminology. Recall that one or both of the end clusters of \( A_{\text{on}} \) may be empty, so for example \( [A_{\text{on}}] = 1^82^1; 0^11^73^1; 0^21^82^1 \) indicates that none, one, or both of them is null respectively. Let \( A_i \) and \( B_j \) be an adjacent pair of double clusters in \( A_{\text{on}} \cup B_{\text{on}} \). If \( A_i \cup B_j \) consists of distinct elements \( \lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 \) then \( \lambda_2 \) and \( \lambda_3 \) are referred to as the inner elements of the adjacent pair.

**Theorem 6.12** Let \( L(A, B) \) be a unimodular hypergeometric lattice of rank 22. It is a hypergeometric K3 lattice such that \( A \) admits a special trace and is positive with respect to the Hodge structure \( [45] \), if and only if all elements of \( A \) and \( B \) are simple, \( A \) does not contain \( -2 \), and according to the type of \( A \) the configuration of \( A \) and \( B \) is as follows:

(E) In elliptic case, all entries in Table 6 such that \( |A_1| \) does not contain \( 2 \).

(P) In parabolic case, those entries of Table 6 which have “Yes” in the “\(|A_1| \geq 1\)” column and such that \( |A_1| \) does contain \( 2 \).

(H) In hyperbolic case, all entries of Table 6 such that \( |A_1| \) does not contain \( 2 \).

The special trace is indicated in the last column of Tables 6 and 7 where we mean by “middle of TC” that it is the middle element of the triple cluster in \( A_{\text{on}} \), and by “inner of AP” that it is the inner element in \( A_{\text{on}} \) of the adjacent pair of double clusters in \( A_{\text{on}} \cup B_{\text{on}} \).

| No. | s | \([A_{\text{on}}]\) | \([B_{\text{on}}]\) | \(|A_1| \geq 1\) | \(|B_{\text{off}}|\) | constraints | special trace |
|-----|---|--------------------|--------------------|----------------|----------------|-------------|-------------|
| 1   | 8 | 0^11^73^1          | 1^8                | Yes            | 3              | middle of TC |
| 2   | 8 | 0^11^73^1          | 1^73^1             | Yes            | 1              | middle of TC |
| 3   | 9 | 0^21^73^1          | 1^8^2^1            | No             | 1              | middle of TC |
| 4   | 8 | 1^8^2^1            | 1^8                | Yes            | 3              | \(\text{min } A_9\) |
| 5   | 8 | 1^8^2^1            | 1^73^1             | Yes            | 1              | \(\text{min } A_9\) |
| 6   | 9 | 0^11^8^2^1         | 1^8^2^1            | Yes            | 1              | \(\text{inner of } A\) |
| 7   | 9 | 0^11^8^2^1         | 1^8^2^1            | No             | 1              | \(\text{min } A_{10}\) |
| 8   | 10| 0^21^8^2^1         | 1^10               | No             | 1              | \(\text{max } A_2\) |
| 9   | 10| 1^10               | 1^8^2^1            | Yes            | 1              | \(\text{element of } A_{10}\) |

Table 6: Positively normalized matrices \( A \) of elliptic or parabolic type.
Lemma 6.11 implies and all the other elements have local index $-\ell$.

These equations are combined with Lemma 6.11 to determine the parities of $A_1$ and $A_{s+1}$. Then $\text{Idx}(A_1^s)$ and $\text{Idx}(A_{s+1}^o)$ are evaluated by the formulas (41d), which now look like

\[
\text{Idx}(A_1^s) = -\text{Par}(|A_1^s|), \quad \text{Idx}(A_{s+1}^o) = -\text{Par}(|A_{s+1}^o|) \quad \text{in cases 1–5,}
\]

\[
\text{Idx}(A_1^s) = \mp \text{Par}(|A_1^s|), \quad \text{Idx}(A_{s+1}^o) = \pm \text{Par}(|A_{s+1}^o|) \quad \text{in cases 6–8.}
\]

The information so obtained and Lemma 6.11 confine the possibilities of $A_1$ and $A_{s+1}$. In particular, subcase $|B_9| = 2$ of case 7 and subcase $|A_{10}| = 10$ of case 8 are excluded in elliptic case, whereas cases 7 and 8 are altogether ruled out in parabolic and hyperbolic cases.

In cases 1–5 the first formula in (41d) reads $\text{Idx}(A_{in}) = -7$. In cases 1 and 3 where $|A_{in}| = 7$, this implies that all elements of $A_{in}$ have local index $-1$, so the special root $\tau$ must lie in $A_1$ or $A_{s+1}$. From the data we have already had it is easy to know which of them contains $\tau$. In cases 2, 4, 5 where $|A_{in}| = 9$, there is a unique element of $A_{in}$ with local index 1 (which is $\tau$) and all the other elements have local index $-1$. Cases 6–8 can be treated in a similar manner, where the first formula in (41d) reads $\text{Idx}(A_{in}) = -8$ with $|A_{in}|$ being either 8 or 10. In any case, once the configuration of clusters is fixed, formula (41d) tells us exactly where $\tau$ is located. In particular $\tau$ lies in the unique multiple cluster, if it exists, and $\tau$ is its middle element if it is triple. Exhausting all possibilities we have the assertions of the theorem. \hfill \Box

Next we turn our attention to the matrix $B$, whose treatment is simpler than that of $A$.

**Theorem 6.13** Let $L(A, B)$ be a unimodular hypergeometric lattice of rank 22. It is a hypergeometric K3 lattice such that $B$ admits a special trace and is positive with respect to the Hodge structure \([15]\), if and only if all elements of $B$ are simple, $|B_{\geq 2}| = 1$, and the configuration of $A_{in}$ and $B_{on}$ is precisely as in cases 3–8 of Table 4. The special trace is then given by

| No. | $s$ | $[A_{on}]$ | $[B_{on}]$ | $|B_{off}|$ | constraints | special trace |
|-----|-----|------------|------------|-----------|-------------|---------------|
| 1   | 8   | $0^21^31$ | $1^8$      | 3         | middle of TC |
| 2   | 8   | $0^21^31$ | $1^3$      | 1         | middle of TC |
| 3   | 8   | $0^11^21$ | $1^8$      | 3         | $|A_1| = 2$ | max $A_1$     |
| 4   | 8   | $0^11^21$ | $1^8$      | 3         | $|A_0| = 2$ | min $A_0$     |
| 5   | 8   | $0^11^21$ | $1^8$      | 3         | $|A_1| = 2$ | max $A_1$     |
| 6   | 8   | $0^11^21$ | $1^8$      | 3         | $|A_0| = 2$ | min $A_0$     |
| 7   | 9   | $0^21^2$  | $1^8$      | 1         | doubles adjacent | inner of AP |
| 8   | 9   | $0^11^9$  | $1^8$      | 1         | $|A_1| = 1$, $|B_1| = 2$ | element of $A_1$ |
| 9   | 9   | $0^11^9$  | $1^8$      | 1         | $|A_{10}| = 1$, $|B_9| = 2$ | element of $A_{10}$ |

Table 7: Positively normalized matrices $A$ of hyperbolic type.
\begin{itemize}
  \item the middle element of the unique triple cluster in \( B_{\text{on}} \) in cases 3 and 4,
  \item the inner element in \( B_{\text{on}} \) of the adjacent pair of double clusters in \( A_{\text{in}} \cup B_{\text{on}} \) in case 5,
  \item the \( \left\{ \min, \max \right\} \) element of the double cluster \( B_{5\pm 4} \) in cases 6 and 7,
  \item the unique element of the simple cluster \( B_{(11\pm 9)/2} \) in case 8.
\end{itemize}

In any case \( B \) is a Hodge isometry of hyperbolic type and any element of \( A \) is simple except for at most one integer element of multiplicity 2 or 3.

\begin{proof}
From Table \[ \text{4} \] we have \( |B_{\text{off}}| = 3 \) in cases 1–2, while \( |B_{\text{off}}| = 1 \) in cases 3–8. On the other hand, if \( A \) admits a special trace \( \tau \) then we have \( |B_{\text{off}}| = 0 \) in elliptic or parabolic case, while \( |B_{\text{off}}| = 1 \) in hyperbolic case. Thus cases 1–2 are ruled out and we are exclusively in hyperbolic case. Since \( B \) is positively normalized, we have \( |B_{\text{off}}| = |B_{>2}| = 1 \). Assertions on simplessons of elements of \( A \) and \( B \) follow from Lemma \[ \text{6.4} \]. With \( B_{\text{on}} = B_{\text{on}} \) the second formula in \[ \text{27} \] adapted in K3 normalization reads \( \text{Idx}(B_{\text{on}}) = -8 \). Since \( |B_{\text{on}}| = 10 \) in cases 3–8, this implies that there exists a unique element of \( B_{\text{on}} \) with local index 1 (which is \( \tau \)) and all the other elements have index \(-1\). The location of \( \tau \) can be determined by formula \[ \text{29} \]. \( \square \)
\end{proof}

7 K3 Structure

To discuss dynamics on K3 surfaces we need the concepts of Picard lattice (or Néron-Severi lattice) and Kähler cone in addition to Hodge structure and positive cone discussed in \[ \text{§6.2} \].

Given a K3 lattice \( L \) with a Hodge structure as in \[ \text{15} \], the Picard lattice and the root system are defined by \( \text{Pic} := H^{1,1} \cap L \) and \( \Delta := \{ u \in \text{Pic} : (u, u) = -2 \} \) respectively. Given a positive cone \( C^+ \subset C \), a subset \( \Delta^+ \subset \Delta \) is a set of positive roots if \( \Delta = \Delta^+ \cup (-\Delta^+) \) and

\[ \mathcal{K} := \{ v \in C^+ : (v, u) > 0, \forall u \in \Delta^+ \} \]

is nonempty, in which case \( \mathcal{K} \) is called the Kähler cone associated with \( \Delta^+ \). A Picard-Lefschetz reflection is a lattice automorphism \( \rho_u : L \to L \) defined by \( \rho_u (v) := v + (v, u)u \) for a positive root \( u \in \Delta^+ \). The group \( W := \langle \rho_u : u \in \Delta^+ \rangle \) generated by those reflections is called the Weyl group. It acts on \( C^+ \) properly discontinuously and the closure in \( C^+ \) of the Kähler cone \( \mathcal{K} \) is a fundamental domain of this action. Each fundamental domain is called a Weyl chamber.

\begin{definition}
A K3 structure on a K3 lattice \( L \) is a specification of (i) a Hodge structure, (ii) a positive cone \( C^+ \) and (iii) a set of positive roots \( \Delta^+ \). The Kähler cone \( \mathcal{K} \) is then associated with these data. Alternatively one can specify (i), (ii) and (iii)’ a Weyl chamber \( \mathcal{K} \subset C^+ \) named the Kähler cone, in place of (iii). In this case the set of positive roots \( \Delta^+ \) is associated afterwards. See McMullen \[ \text{[18], §6} \].
\end{definition}

Any K3 surface \( X \) induces a K3 structure on \( H^2(X, \mathbb{Z}) \); its Hodge structure is given by the Hodge-Kodaira decomposition of \( H^2(X, \mathbb{C}) \); its Kähler cone \( \mathcal{K}(X) \) is the set of all Kähler classes on \( X \); its positive cone \( C^+(X) \) is the connected component containing \( \mathcal{K}(X) \); the set of positive roots \( \Delta^+(X) \) is the set of all effective \((-2)\)-classes. Any automorphism \( f : X \to X \) induces a lattice automorphism \( f^* : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z}) \) that preserves the K3 structure. Conversely, the Torelli theorem and surjectivity of the period mapping tells us the following.
Theorem 7.2  Let $L$ be a fixed K3 lattice. Any K3 structure on $L$ is realized by a unique marked K3 surface $(X, \iota)$ up to isomorphism and any lattice automorphism $F : L \to L$ preserving the K3 structure is realized by a unique K3 surface automorphism $f : X \to X$ up to conjugacy:

$$
\begin{align*}
H^2(X, \mathbb{Z}) &\xrightarrow{\iota} L \\
\downarrow f^* &\quad \downarrow F \\
H^2(X, \mathbb{Z}) &\xrightarrow{\iota} L.
\end{align*}
$$

(49)

The map $f : X \to X$ is referred to as the K3 surface automorphism induced by $F$ (with the given K3 structure being understood). Thus constructing a K3 surface automorphism amounts to constructing an automorphism of a K3 lattice that preserves a K3 structure.

Let $L$ be a K3 lattice and $F : L \to L$ be a lattice automorphism admitting a special trace. We can then think of the Hodge structure [48], with respect to which we may assume that $F$ is positive. For any K3 structure having [48] as its Hodge structure, $F$ obviously preserves the Hodge structure and it also preserves the positive cone $C^+$ because $F$ is positively normalized. However, it is not always true that $F$ preserves the set of positive roots $\Delta^+$. We remark that preservation of $C^+$ and $\Delta^+$ is equivalent to that of the corresponding Kähler cone $K$. If this is the case then $F$ is said to satisfy the Kähler cone condition — positivity in McMullen’s terminology [18, 19], which we use in a different sense, that is, for preservation of the positive cone $C^+$. In any case there exists a unique element $w_F \in W$ that brings $F(K)$ back to $K$, since the Weyl group $W$ acts simply transitively on the set of Weyl chambers. The modified map

$$
\tilde{F} := w_F \circ F
$$

(50)

preserves the Kähler cone and hence the K3 structure, so we can apply Theorem 7.2 to $\tilde{F}$. We are interested in how to find the element $w_F \in W$, especially in the context of hypergeometric K3 lattice. Before discussing this problem in [48, 1] we need to determine the Picard lattice.

Let $L = L(\varphi, \psi) = L(A, B)$ be a hypergeometric K3 lattice such that $A$ admits a special trace $\tau = \xi + \xi^{-1}$ and is positively normalized. Let $\varphi_0(z)$ be the minimal polynomial of the special eigenvalues $\xi^{\pm 1}$. Then there exists a factorization of polynomials over $\mathbb{Z}$,

$$
\varphi(z) = \varphi_0(z) \cdot \varphi_1(z), \quad l := \deg \varphi_0, \quad m := \deg \varphi_1, \quad l + m = 22.
$$

(51)

By Lemma 6.11 and formula (27a) any root of $\varphi(z)$ is simple, except for the triple root $z = 1$ when $A$ is of parabolic type. Recall also from Lemma 6.9 that $\xi \neq \pm 1$. Thus $\varphi_0$ and $\varphi_1$ have no factors in common; $(z - 1)(z + 1) | \varphi_1(z)$ when $A$ is of elliptic or hyperbolic type; and $(z - 1)^3(z + 1) | \varphi_1(z)$ when $A$ is of parabolic type. In any case we have $m \geq 2$.

Lemma 7.3  If $\text{Pic} := H^{1,1} \cap L$ be the Picard lattice associated with the Hodge structure [48] then the vectors $s_1, \ldots, s_m$ form a $\mathbb{Z}$-basis of $\text{Pic}$, where

$$
s := \varphi_0(A)r, \quad s_j := A^{j-1}s, \quad j = 1, \ldots, m.
$$

(52)

Proof. Let $L_i(K) := \{ v \in L \otimes K : \varphi_i(A)v = 0 \}$ for a subfield $K \subset \mathbb{C}$ and $i = 0, 1$. We have an orthogonal decomposition $L_{\mathbb{C}} = L_0(\mathbb{C}) \oplus L_1(\mathbb{C})$ and an inclusion $L_1(\mathbb{C}) \subset H^{1,1}$. Let $u \in L_0(\mathbb{Q})$ be a $\xi$-eigenvector of $A$. Since $u \in H^{2,0}$ and $\text{Pic} \subset H^{1,1}$, we have $u \perp \text{Pic}$ and so $\sigma(u) \perp \text{Pic}$ for every $\sigma \in G := \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. This implies $L_0(\mathbb{C}) \perp \text{Pic}$ and hence $\text{Pic} \subset L_1(\mathbb{C})$, 28
because the vectors \( \{\sigma(u)\}_{\sigma \in G} \) span the space \( L_0(\mathbb{C}) \). Therefore \( \text{Pic} = \{ v \in L : \varphi_1(A)v = 0 \} \).

In view of (42) any element \( v \in L \) can be expressed as \( v = \varphi_2(A)r \) in a unique way in terms of a polynomial \( \varphi_2(z) \in \mathbb{Z}[z] \) with \( \deg \varphi_2 < 22 \). Then \( v \in \text{Pic} \) if and only if \( \varphi_1(A)\varphi_2(A)r = 0 \), which is the case precisely when \( \varphi(z) \) divides \( \varphi_1(z)\varphi_2(z) \), that is, \( \varphi_0(z) \) divides \( \varphi_2(z) \). Upon writing \( \varphi_2(z) = \varphi_0(z)\varphi_3(z) \), any element \( v \in \text{Pic} \) can be expressed as \( v = \varphi_3(A)s \) in a unique way in terms of a polynomial \( \varphi_3(z) \in \mathbb{Z}[z] \) with \( \deg \varphi_3 < m \).

Suppose that the matrix \( A \) is of hyperbolic type and in (51) the factor \( \varphi_0(z) \) is a Salem polynomial,

\[ (53) \]

that is, the special eigenvalues \( \xi^{\pm 1} \) are conjugate to a Salem number. There then exists an orthogonal decomposition \( H^{1,1}_R = V_R \oplus \text{Pic}_R \) of signatures \( (1,19-m) \oplus (0,m) \), where \( \text{Pic}_R := \text{Pic} \otimes \mathbb{R} \). Note that the Weyl group \( W \) acts on \( V_R \) trivially. The invariant Hermitian form \( (u,v) \) restricted to \( \text{Pic}_R \) is negative-definite. For the sake of convenience we turn it positive-definite by putting \( (u,v) := -(u,v) \). The essential part of the Kähler cone \( \mathcal{K} \) is given by

\[ (54) \]

Indeed, if we put \( C_0^+ := C^+ \cap V_R \) then the positive cone is given by

\[ C^+ = \{ v = v_0 - v_1 : v_0 \in C_0^+, v_1 \in \text{Pic}_R, (v_0,v_0) > (v_1,v_1) \}, \]

and the vector \( v = v_0 - v_1 \in C^+ \) belongs to \( \mathcal{K} \) if and only if \( v_1 \in \mathcal{K}_1 \).

Remark 7.4 We can focus on the matrix \( B \) instead of \( A \), in which case we suppose that \( B \) admits a special trace and is positively normalized. Factorization (51) is then replaced by

\[ (55) \]

and definition (52) should read \( s := \psi_0(B)r \) and \( s_j := B^{j-1}s \). In this setting Lemma 7.3 and the remarks after it are also valid for \( B \). Unlike the case of \( A \) the Picard number \( m \) may be zero. If \( m = 0 \) then \( \text{Pic} \) is trivial, \( \mathcal{K} \) is the entire positive cone \( C^+ \) and Kähler cone condition is automatically satisfied. In this case \( \psi(z) = \psi_0(z) \) is a Salem polynomial of degree 22.

8 Root System in Picard Lattice

Under hypothesis (53) we develop an algorithmic study of the root system in Picard lattice which makes it possible to calculate many things explicitly. We present an algorithm to find the Weyl group element \( w_A \in W \) by computers, when the map \( F \) is the matrix \( A \) in (50). The case where \( F \) is the matrix \( B \) can be dealt with in the same manner.

8.1 Algorithm

As the matrices \( A \) and \( B \) we take the standard ones \( A = Z(\varphi) \) and \( B = Z(\psi) \) as in (41).

1. Gram matrix. The Gram matrix \( \langle s_i, s_j \rangle \) of the basis \( s_1, \ldots, s_m \) for \( \text{Pic} \) (see Lemma 7.3) can be evaluated explicitly by using formula (10) in Theorem 2.1. Note that \( \langle s_i, s_j \rangle \) depends only on \( |i-j| \). We give an algorithm to find all roots of the root system \( \Delta \).
2. Finding all roots. We have an even, positive-definite quadratic form
\[ Q(t_1, \ldots, t_m) := \langle u, u \rangle = \sum_{i,j=1}^{m} \langle s_i, s_j \rangle t_i t_j \quad \text{in} \quad (t_1, \ldots, t_m) \in \mathbb{Z}^m, \]
by expressing each element \( u \in \text{Pic} \) as a \( \mathbb{Z} \)-linear combination \( u = t_1 s_1 + \cdots + t_m s_m \). Finding all roots in \( \Delta \) amounts to finding all integer solutions to the inequality \( Q(t_1, \ldots, t_m) \leq 2 \). Indeed, all but the trivial solution \( 0 = (0, \ldots, 0) \) lead to the solutions of the equation \( Q = 2 \), because \( Q \) cannot take value 1. As a positive-definite quadratic form, \( Q \) can be expressed as
\[ Q(t_1, \ldots, t_m) = \sum_{j=1}^{m} c_j \{ t_j - p_j(t_1, \ldots, t_{j-1}) \}^2, \quad c_j \in \mathbb{Q}_{>0}, \quad j = 1, \ldots, m, \tag{55} \]
where \( p_j = 0 \) and for \( j = 2, \ldots, m \), \( p_j(t_1, \ldots, t_{j-1}) \) is a linear form over \( \mathbb{Q} \) in \( t_1, \ldots, t_{j-1} \). Note that \( c_j \) and \( p_j \) can be calculated explicitly in terms of the coefficients of \( Q \).

Let \( Q_k(t_1, \ldots, t_{k-1}; t_k) \) be the partial sum of \( \sum_{j=1}^{k} \) summed over \( j = 1, \ldots, k \); it is a quadratic function of \( t_k \), once \( t_1, \ldots, t_{k-1} \) are given. We define a rooted forest in the following manner.

First, let all integer solutions \( t_1 \) of the inequality \( Q_1(t_1) \leq 2 \) be the roots (in graph theory) of the forest. Next, for each root \( t_1 \), let all integer solutions \( t_2 \) of \( Q_2(t_1; t_2) \leq 2 \) be the children of \( t_1 \). Inductively, given a parent \( t_k \) with its ancestors \( t_1, \ldots, t_{k-1} \), let all integer solutions \( t_{k+1} \) of \( Q_{k+1}(t_1, \ldots, t_k; t_{k+1}) \leq 2 \) be the children of \( t_k \). Consider all paths from roots to leaves, say, \( (t_1, \ldots, t_k) \). Some of them may continue to the \( m \)-th generation, that is, \( k = m \), while others may not. All paths \( (t_1, \ldots, t_m) \) that continue to the \( m \)-th generation yield all integer solutions to the inequality \( Q(t_1, \ldots, t_m) \leq 2 \) and hence all elements \( u \in \Delta \) along with the origin \( 0 \).

3. Positive roots, simple roots and the Kähler cone. Provide Pic with a lexicographic order in the following manner: for \( u = t_1 s_1 + \cdots + t_m s_m, \ u' = t'_1 s_1 + \cdots + t'_m s_m \in \text{Pic}, \)
\[ u \succ u' \overset{\text{def}}{=} \exists i \in \{1, \ldots, m\} \text{ such that } t_i > t'_i \text{ and } t_j = t'_j \text{ for all } j < i. \tag{56} \]
Then in the previous step all solutions \( u = t_1 s_1 + \cdots + t_m s_m > 0 \) give the set of positive roots \( \Delta^+ \). An element \( u \in \Delta^+ \) is a simple root if and only if \( u - u' \not\in \Delta^+ \) for any \( u' \in \Delta^+ \). This test can easily be carried out by computer and we obtain the set of simple roots, say \( \Delta_b \), that is, the basis of \( \Delta \) relative to \( \Delta^+ \). Looking at \( \Delta_b \) we can draw the Dynkin diagram of \( \Delta \), which indicates the irreducible decomposition of \( \Delta \) as well as the Dynkin-type of each irreducible component. The essential \( K_1 \) of the Kähler cone \( K \) is given by formula \[ 11 \]. It is the Weyl chamber \( C(\delta) \) containing the regular vector \( \delta := \frac{1}{2} \sum_{u \in \Delta^+} u \in \text{Pic}_\mathbb{R} \).

4. Bringing back. The matrix \( A \) sends \( K_1 = C(\delta) \) to the Weyl chamber \( C(d) \) containing the regular vector \( d := A\delta \in \text{Pic}_\mathbb{R} \). Let \( w_A \in W \) be an element maximizing \( \langle w(d), \delta \rangle \) for \( w \in W \). We claim that \( w_A \) brings \( C(d) \) back to \( C(\delta) \). Indeed, for any \( u \in \Delta_b \) one has \( \rho_u(\delta) = \delta - u \) by Humphreys \[ 11 \] §10.2, Lemma B] and hence the defining property of \( w_A \) and \( \rho_u \)-invariance of the inner product yield
\[ \langle w_A(d), \delta \rangle \geq \langle (\rho_u w_A)(d), \delta \rangle = \langle w_A(d), \rho_u(\delta) \rangle = \langle w_A(d), \delta - u \rangle = \langle w_A(d), \delta \rangle - \langle w_A(d), u \rangle, \]
that is, \( \langle w_A(d), u \rangle \geq 0 \). This implies more strictly that \( \langle w_A(d), u \rangle > 0 \) for any \( u \in \Delta_b \), since \( w_A(d) \) is a regular vector. Thus we have \( w_A(d) \in C(\delta) \) and so \( w_A(C(d)) = C(\delta) \). Note that the element \( w_A \in W \) is unique because \( W \) acts on the set of Weyl chambers simply transitively.
5. Product of Picard-Lefschetz reflections. To determine \( w_A \) explicitly, let PL be the set of all Picard-Lefschetz reflections together with the identity transformation 1. Start with \( d_0 := d \) and find an element \( \rho_1 \in \text{PL} \) maximizing \( \langle \rho(d_0), \delta \rangle \) for \( \rho \in \text{PL} \). If \( \rho_1 = 1 \), stop here; otherwise, put \( d_1 := \rho_1(d_0) \) and find \( \rho_2 \in \text{PL} \) maximizing \( \langle \rho(d_1), \delta \rangle \) for \( \rho \in \text{PL} \). Inductively, given \( \rho_k \in \text{PL} \) and \( d_{k-1} \in \text{Pic} \), if \( \rho_k = 1 \), stop at this stage; otherwise, put \( d_k := \rho_k(d_{k-1}) \) and find yet another \( \rho_{k+1} \in \text{PL} \) maximizing \( \langle \rho(d_k), \delta \rangle \) for \( \rho \in \text{PL} \). We claim that

\[
\text{if } \rho_{k+1} = 1 \text{ then } d_k \in C(\delta); \text{ otherwise, } \langle \rho_{k+1}(d_k), \delta \rangle > \langle d_k, \delta \rangle.
\] (57)

Indeed, if \( \rho_{k+1} = 1 \) then for any \( u \in \Delta_b \) we have from Humphreys [11 §10.2, Lemma B],

\[
\langle d_k, \delta \rangle \geq \langle \rho_u(d_k), \delta \rangle = \langle d_k, \rho_u(\delta) \rangle = \langle d_k, \delta - u \rangle = \langle d_k, \delta \rangle - \langle d_k, u \rangle,
\]

that is, \( \langle d_k, u \rangle \geq 0 \), which in turn implies \( \langle d_k, u \rangle > 0 \) for any \( u \in \Delta_b \), since \( d_k \) is a regular vector. Thus \( d_k \in C(\delta) \) and the first part of (57) is proved. Next, suppose that \( \rho_{k+1} \neq 1 \) is the Picard-Lefschetz reflection associated with a positive root \( u_{k+1} \in \Delta^+ \). Then

\[
\langle \rho_{k+1}(d_k), \delta \rangle = \langle d_k - (d_k, u_{k+1})u_{k+1}, \delta \rangle = \langle d_k, \delta \rangle - \langle d_k, u_{k+1} \rangle \langle u_{k+1}, \delta \rangle,
\]

where \( \langle d_k, u_{k+1} \rangle \langle u_{k+1}, \delta \rangle \) is nonzero, since \( d_k \) and \( \delta \) are regular vectors. On the other hand, by the defining property of \( \rho_{k+1} \) we have \( \langle \rho_{k+1}(d_k), \delta \rangle \geq \langle d_k, \delta \rangle \) and thus \( \langle \rho_{k+1}(d_k), \delta \rangle > \langle d_k, \delta \rangle \).

Since \( \{ \langle w(d), \delta \rangle : w \in W \} \) is a finite set, it follows from (57) that the step-by-step procedure mentioned above eventually terminates with \( \rho_{k+1} = 1 \) and leads to the desired representation \( w_A = \rho_k \rho_{k-1} \cdots \rho_1 \) as a product of Picard-Lefschetz reflections.

6. Modified matrix. Let \( w_A \) act on the whole lattice \( L \) by extending it as identity to the orthogonal complement of Pic. The modified matrix \( \tilde{A} := w_A A \) then satisfies the Kähler cone condition \( \tilde{A}(K) = K \) and hence preserves the K3 structure constructed from \( A \). Thanks to factorization (51) and assumption (53) the characteristic polynomial \( \tilde{\varphi}(z) \) of \( \tilde{A} \) factors as

\[
\tilde{\varphi}(z) = \varphi_0(z) \cdot \varphi_1(z), \quad l = \text{deg } \varphi_0, \quad m = \text{deg } \varphi_1,
\] (58)

where the Salem factor \( \varphi_0(z) \) is the same as that in (51) while \( \varphi_1(z) \) is the characteristic polynomial of \( \tilde{A}|_{\text{Pic}} \). Thus \( \tilde{A} \) has the same spectral radius as \( A \).

7. Action on the Dynkin diagram. Since \( \tilde{A} \) preserves \( \Delta^+ \), it also preserves \( \Delta_b \). Thus \( \tilde{A} \) induces an automorphism of the corresponding Dynkin diagram. Describe this action explicitly.

8.2 Minimum Entropy

McMullen [17, Theorem (A.1)] showed that any automorphism \( f : X \to X \) of a compact complex surface \( X \) with a positive entropy \( h(f) > 0 \) had a lower bound \( h(f) \geq \log \lambda_L \), where \( \lambda_L \) is Lehmer’s number in Example 5.5. In [18] he obtained some non-projective K3 examples that actually attain this lower bound and he went on to construct projective ones in [19].

Now our method enables us to synthesize a goodly number of non-projective K3 surface automorphisms with minimum entropy. Especially we are able to construct such maps with Siegel disks. To see this we apply the Algorithm in [8.1] to two settings mentioned below. Our method works in various other situations including the ones involving other Salem numbers.
| \( \Psi \) | No. | \( k \) | ST | root system | \( \tilde{\varphi}_1(z) \) | \( \text{Tr} \tilde{A} \) | S/H |
|---|---|---|---|---|---|---|---|
| \( R_1 \) | 7 | 4,20 | \( x_4 \) | \( E_6 \oplus E_6 \) | \((z - 1)^4(z + 1)^4(z^2 + 1)^2\) | -1 | H |
| \( R_2 \) | 7 | 4,6,7 | \( x_2 \) | \( D_{10} \) | \((z - 1)^6(z + 1)(z^2 + 1)\) | 7 | - |
| \( R_3 \) | 7 | 3,15 | \( x_4 \) | \( A_2 \) | \((z - 1)^2(z^2 + z + 1)^2C_{15}(z)\) | 1 | - |
| \( R_3 \) | 7 | 3,4,6,8 | \( x_3 \) | \( E_6 \oplus E_6 \) | \((z - 1)^4(z + 1)^4(z^2 + 1)^2\) | -1 | S |
| \( R_3 \) | 7 | 4,6,18 | \( x_2 \) | \( D_{10} \) | \((z - 1)^6(z + 1)(z^2 + 1)\) | 7 | - |
| \( R_4 \) | 7 | 3,4,9 | \( x_4 \) | \( D_{10} \) | \((z - 1)^6(z + 1)(z^2 + 1)\) | 7 | - |
| \( R_4 \) | 7 | 4,24 | \( x_2 \) | \( E_8 \oplus A_2 \oplus A_2 \) | \((z - 1)^6(z + 1)(z^2 + 1)\) | 7 | S |
| \( R_4 \) | 7 | 4,20 | \( x_1 \) | \( E_6 \oplus E_6 \) | \((z - 1)^4(z + 1)^4(z^2 + 1)^2\) | -1 | S |
| \( R_5 \) | 7 | 7,12 | \( x_3 \) | \( E_8 \oplus A_2 \oplus A_2 \) | \((z - 1)^6(z + 1)(z^2 + 1)\) | 7 | H |
| \( R_5 \) | 7 | 4,20 | \( x_1 \) | \( E_6 \oplus E_6 \) | \((z - 1)^6(z + 1)(z^2 + 1)\) | 7 | H |
| \( R_9 \) | 7 | 12,18 | \( x_3 \) | \( E_8 \oplus A_2 \oplus A_2 \) | \((z - 1)^6(z + 1)(z^2 + 1)\) | 7 | H |
| \( R_9 \) | 7 | 4,30 | \( x_1 \) | \( E_8 \oplus A_2 \oplus A_2 \) | \((z - 1)^6(z + 1)(z^2 + 1)\) | 7 | H |
| \( R_{10} \) | 7 | 4,16 | \( x_4 \) | \( E_6 \oplus E_6 \) | \((z - 1)^4(z + 1)^4(z^2 + 1)^2\) | -1 | H |
| \( R_{10} \) | 7 | 4,24 | \( x_2 \) | \( E_8 \oplus A_2 \oplus A_2 \) | \((z - 1)^6(z + 1)(z^2 + 1)\) | 7 | S |
| \( R_{10} \) | 7 | 3,15 | \( x_1 \) | \( A_2 \) | \((z - 1)^2(z^2 + z + 1)^2C_{15}(z)\) | 1 | - |

Table 8: Minimum entropy from matrix \( \tilde{A} \).

### 8.2.1 Use of Matrix \( A \)

Suppose that \( \Phi(w) = \text{LT}(w) \cdot \text{CT}_k(w) \), where \( \text{LT}(w) \) is Lehmer’s trace polynomial in (44b) and \( \text{CT}_k(w) \) is a product of cyclotomic trace polynomials of the form

\[
\text{CT}_k(w) := \prod_{k \in k} \text{CT}_k(w) \quad \text{with} \quad \sum_{k \in k} \deg \text{CT}_k(w) = 5, \tag{59}
\]

where \( k \) is a set of positive integers whose elements \( k \) come from Table 2 with \( \deg \leq 5 \), while \( \Psi(w) = R_i(w) \) is one of the Salem trace polynomials in Table 3. We remark that \( k \) is not a multi-set but an ordinary set because an assertion of Theorem 6.12 rules out the occurrence of multiple elements. To construct K3 surface automorphisms we utilize the matrix \( A \) and its modification \( \tilde{A} \). Let \( x_4 < x_3 < x_2 < x_1 \) denote the roots in \([-2, 2]\) of Lehmer’s trace polynomial \( \text{LT}(w) \) in (44b), whose numerical values are given by

\[
x_4 \approx -1.88660, \quad x_3 \approx -1.46887, \quad x_2 \approx -0.584663, \quad x_1 \approx 0.913731. \tag{60}
\]

**Theorem 8.1** In the setting mentioned above \( L(\Phi, \Psi) = L(A, B) \) is a hypergeometric K3 lattice such that the matrix \( A \) admits a special trace coming from the Lehmer factor, if and only if \( k \) and \( \Psi(w) = R_i(w) \) are as in Table 3, where the special trace is given in the “ST” column. In this case the matrix \( A \) is positive with respect to the Hodge structure 15. The modified matrix \( \tilde{A} := w_A \circ A \) has characteristic polynomial of the form \( \tilde{\varphi}(z) = L(z) \cdot \tilde{\varphi}_1(z) \), where \( L(z) \) is Lehmer’s polynomial in (44a) and \( \tilde{\varphi}_1(z) \) is given in Table 4. The Dynkin types of the root system \( \Delta \subset \text{Pic} \) and the values of \( \text{Tr} \tilde{A} \) are also included in the table.

**Proof.** First, pick out all pairs \((i, k)\) such that \( \Phi(w) = \text{LT}(w) \cdot \text{CT}_k(w) \) and \( \Psi(w) = R_i(w) \) satisfy unimodularity condition (43), where \( k \) must be subject to the degree constraint in (59).
Theorem 6.12. Since the special trace comes from the Lehmer factor, the matrix $A$ is positive by Remark 6.6. Secondly, determine all the correct solutions as well as their special traces by using case (H) of $\rho$ where $\rho$ are rather rare. The following two examples are of No. 1 and No. 9, respectively:

Remark 6.6. Thirdly, for each solution run the Algorithm in §8.1 to find the set of positive roots $\Delta^+$, its basis $\Delta_b$, its Dynkin type and the modified matrix $\hat{A}$. The characteristic polynomial of $\hat{A}$ is determined by the formula $[58]$.

We illustrate how the Algorithm in §8.1 works for an entry in Table 8.

Example 8.2 Consider the case where $\Phi(w) = LT(w) \cdot CT_k(w)$ with $k = \{3, 4, 6, 8\}$ and $\Psi(w) = R_3(w)$ in Table 8. Steps 1, 2, 3 of the Algorithm return us 72 elements $0 \prec u_1 \prec u_2 \prec \cdots \prec u_{72}$ in the lexicographic order $[60]$ for the set of positive roots $\Delta^+$ and then 12 elements for the basis $\Delta_b$, implying that the root system is of type $E_6 \oplus E_6$. If the basis $\Delta_b = \{e_1, \ldots, e_6, e'_1, \ldots, e'_6\}$ is specified as in Figure 2 then the simple roots are given by

$$e_1 = u_{23}, \; e_2 = u_8, \; e_3 = u_1, \; e_4 = u_3, \; e_5 = u_{16}, \; e_6 = u_{25},$$
$$e'_1 = u_7, \; e'_2 = u_{24}, \; e'_3 = u_2, \; e'_4 = u_5, \; e'_5 = u_9, \; e'_6 = u_{35}.$$

Steps 4, 5, 6 tell us that the Weyl group element $w_A \in W$ bringing $A(K)$ back to $K$ is

$$w_A = \rho_5 \circ \rho_{23} \circ \rho_{35} \circ \rho_{41} \circ \rho_{62} \circ \rho_{57} \circ \rho_{72},$$

where $\rho_j$ denotes the Picard-Lefschetz reflection with respect to the $j$-th positive root $u_j$. Step 7, that is, how the modified matrix $\hat{A}$ acts on $\Delta_b$ will be mentioned in §8.2.3.

Remark 8.3 The second column in Table 8 shows that all of its entries fall into the case of No. 7 of Table 7. However, other cases can occur in other settings, although empirically they are rather rare. The following two examples are of No. 1 and No. 9, respectively:

$$\Phi(w) = LT(w) \cdot CT_{18}(w) \cdot CT_3(w), \; \Psi(w) = \{(w + 1)(w^2 - 4) - 1\} \cdot CT_{60}(w),$$
$$\Phi(w) = LT(w) \cdot CT_{15}(w) \cdot CT_4(w), \; \Psi(w) = \{w(w^2 - 1)(w^2 - 3)(w^2 - 4) - 1\} \cdot CT_{24}(w).$$

8.2.2 Use of Matrix B

Suppose that $\Phi(w)$ is a product of cyclotomic trace polynomials of the form

$$\Phi(w) = CT_k(w) := \prod_{k \in k} CT_k(w) \quad \text{with} \quad \sum_{k \in k} \deg CT_k(w) = 10, \tag{61}$$

where $k$ is a multi-set of positive integers whose elements $k$ come from Table 2 while $\Psi(w)$ is a product of Lehmer’s trace polynomial and cyclotomic trace polynomials

$$\Psi(w) = LT(w) \cdot CT_l(w) \quad \text{such that} \quad \sum_{l \in l} \deg CT_l(w) = 6.$$
To construct K3 surface automorphisms we utilize the matrix $B$ and its modification $\tilde{B}$. By Theorem 6.13 any element of $l$ must be simple and the unramifiedness condition in (13) reduces the possibilities of $l$ considerably, confining $\Psi(w)$ into one of the following possibilities.

\[
\begin{align*}
L_1(w) &= LT(w) \cdot CT_{21}(w), \\
L_2(w) &= LT(w) \cdot CT_{28}(w), \\
L_3(w) &= LT(w) \cdot CT_{36}(w), \\
L_4(w) &= LT(w) \cdot CT_{12}(w), \\
L_5(w) &= LT(w) \cdot CT_{12}(w) \cdot CT_{15}(w), \\
L_6(w) &= LT(w) \cdot CT_{12}(w) \cdot CT_{20}(w), \\
L_7(w) &= LT(w) \cdot CT_{12}(w) \cdot CT_{24}(w), \\
L_8(w) &= LT(w) \cdot CT_{12}(w) \cdot CT_{30}(w).
\end{align*}
\]

| $\Psi$ | case | $k$ | ST | root system | $\tilde{\psi}_1(z)$ | $\text{Tr } \tilde{B}$ | S/H |
|--------|------|-----|----|-------------|-----------------|-----------------|-----|
| $L_3$  | 4    | 3, 6, 10, 21 | $x_4$ | $E_6 \oplus E_6$ | $(z - 1)^4(z + 1)^4(z^2 + 1)^2$ | -1 | H |
| $L_3$  | 4    | 1, 6, 8, 28 | $x_2$ | $E_6 \oplus E_6$ | $(z - 1)^4(z + 1)^4(z^2 + 1)^2$ | -1 | S |
| $L_3$  | 4    | 2, 6, 8, 28 | $x_2$ | $E_6 \oplus E_6$ | $(z - 1)^4(z + 1)^4(z^2 + 1)^2$ | -1 | S |
| $L_3$  | 4    | 8, 10, 42 | $x_1$ | $E_6 \oplus E_6$ | $(z - 1)^4(z + 1)^4(z^2 + 1)^2$ | -1 | S |
| $L_3$  | 5    | 1, 3, 5, 6, 11 | $x_2$ | $E_6 \oplus E_6$ | $(z - 1)^4(z + 1)^4(z^2 + 1)^2$ | -1 | S |
| $L_3$  | 5    | 1, 3, 7, 11 | $x_2$ | $E_6 \oplus E_6$ | $(z - 1)^4(z + 1)^4(z^2 + 1)^2$ | -1 | S |
| $L_3$  | 5    | 2, 3, 5, 6, 11 | $x_2$ | $E_6 \oplus E_6$ | $(z - 1)^4(z + 1)^4(z^2 + 1)^2$ | -1 | S |
| $L_3$  | 5    | 2, 3, 7, 11 | $x_2$ | $E_6 \oplus E_6$ | $(z - 1)^4(z + 1)^4(z^2 + 1)^2$ | -1 | S |
| $L_3$  | 5    | 50 | $x_2$ | $E_6 \oplus E_6$ | $(z - 1)^4(z + 1)^4(z^2 + 1)^2$ | -1 | S |
| $L_6$  | 3    | 1, 1, 8, 13 | $x_4$ | $E_8 \oplus A_2 \oplus A_2$ | $(z - 1)^9(z + 1)(z^2 + 1)$ | 7 | S |
| $L_6$  | 3    | 1, 2, 8, 13 | $x_4$ | $E_8 \oplus A_2 \oplus A_2$ | $(z - 1)^9(z + 1)(z^2 + 1)$ | 7 | S |
| $L_6$  | 3    | 2, 2, 8, 13 | $x_4$ | $E_8 \oplus A_2 \oplus A_2$ | $(z - 1)^9(z + 1)(z^2 + 1)$ | 7 | S |
| $L_6$  | 4    | 3, 3, 8, 13 | $x_4$ | $E_8$ | $(z - 1)^8C_{12}(z)$ | 7 | S |
| $L_6$  | 4    | 4, 4, 8, 13 | $x_4$ | $\emptyset$ | $C_{12}(z)C_{20}(z)$ | -1 | H |
| $L_6$  | 4    | 6, 6, 8, 13 | $x_4$ | $E_8$ | $(z - 1)^8C_{12}(z)$ | 7 | S |
| $L_7$  | 3    | 1, 27 | $x_4$ | $E_6 \oplus E_6$ | $(z - 1)^4(z + 1)^4(z^2 + 1)^2$ | -1 | H |
| $L_7$  | 3    | 2, 27 | $x_4$ | $E_6 \oplus E_6$ | $(z - 1)^4(z + 1)^4(z^2 + 1)^2$ | -1 | H |
| $L_7$  | 5    | 16, 42 | $x_2$ | $E_6 \oplus E_6$ | $(z - 1)^4(z + 1)^4(z^2 + 1)^2$ | -1 | S |
| $L_7$  | 5    | 50, 30 | $x_2$ | $E_6 \oplus E_6$ | $(z - 1)^4(z + 1)^4(z^2 + 1)^2$ | -1 | S |
| $L_8$  | 3    | 1, 1, 1, 7, 16 | $x_3$ | $E_8 \oplus A_2 \oplus A_2$ | $(z - 1)^9(z + 1)(z^2 + 1)$ | 7 | H |
| $L_8$  | 3    | 1, 1, 2, 7, 16 | $x_3$ | $E_8 \oplus A_2 \oplus A_2$ | $(z - 1)^9(z + 1)(z^2 + 1)$ | 7 | H |
| $L_8$  | 3    | 2, 2, 1, 7, 16 | $x_3$ | $E_8 \oplus A_2 \oplus A_2$ | $(z - 1)^9(z + 1)(z^2 + 1)$ | 7 | H |
| $L_8$  | 3    | 2, 2, 2, 7, 16 | $x_3$ | $E_8 \oplus A_2 \oplus A_2$ | $(z - 1)^9(z + 1)(z^2 + 1)$ | 7 | H |
| $L_8$  | 4    | 3, 3, 1, 7, 16 | $x_3$ | $E_8$ | $(z - 1)^8C_{12}(z)$ | 7 | H |
| $L_8$  | 4    | 3, 3, 2, 7, 16 | $x_3$ | $E_8$ | $(z - 1)^8C_{12}(z)$ | 7 | H |
| $L_8$  | 4    | 4, 4, 1, 7, 16 | $x_3$ | $E_8$ | $(z - 1)^8C_{12}(z)$ | 7 | H |
| $L_8$  | 4    | 4, 4, 2, 7, 16 | $x_3$ | $E_8$ | $(z - 1)^8C_{12}(z)$ | 7 | H |
| $L_8$  | 4    | 6, 6, 1, 7, 16 | $x_3$ | $\emptyset$ | $C_{12}(z)C_{20}(z)$ | -2 | $\emptyset$ |
| $L_8$  | 4    | 6, 6, 2, 7, 16 | $x_3$ | $\emptyset$ | $C_{12}(z)C_{20}(z)$ | -2 | $\emptyset$ |
| $L_8$  | 4    | 7, 9, 20 | $x_3$ | $E_8 \oplus A_2 \oplus A_2$ | $(z - 1)^9(z + 1)(z^2 + 1)$ | 7 | H |

Table 9: Minimum entropy from matrix $\tilde{B}$.  

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Theorem 8.4 In the above setting \( L(\Phi, \Psi) = L(A, B) \) is a hypergeometric K3 lattice such that the matrix \( B \) has a special trace coming from the Lehmer factor, if and only if \( k \) and \( \Psi(w) \) are as in Table 9, where the second column refers to the “cases” in Table 4 and the special trace is given in the “ST” column. In this case the matrix \( B \) is positive with respect to the Hodge structure \((48)\). The modified matrix \( \tilde{B} := w_B \circ B \) has characteristic polynomial of the form \( \tilde{\psi}(z) = L(z) \cdot \tilde{\psi}_1(z) \) with \( \tilde{\psi}_1(z) \) being given in Table 9. The Dynkin-types of the root system \( \Delta \subset \text{Pic} \) and the values of \( \text{Tr} \tilde{B} \) are also included in the table.

Proof. First, pick out all pairs \((k, i)\) such that \( \Phi(w) = CT_k(w) \) and \( \Psi(w) = L_i(w) \) satisfy unimodularity condition \((45)\), where \( k \) must be subject to the degree constraint in \((61)\). Secondly, determine all the correct solutions as well as their special traces by using Theorem 6.13. Since the special trace comes from the Lehmer factor, the matrix \( B \) is positive by Remark 6.6. Thirdly, for each solution run the \( B\)-version of Algorithm in §8.1 to find the set of positive roots \( \Delta^+ \), its basis \( \Delta_b \), its Dynkin type and the modified matrix \( \tilde{B} \). The characteristic polynomial of \( \tilde{B} \) is determined by the \( \tilde{B}\)-version of \((58)\), which follows from \((51)\) and \((53)\).

In Table 9 only four polynomials \( L_3(w), L_6(w), L_7(w), L_8(w) \) appear as \( \Psi(w) \).

8.2.3 Action on Dynkin Diagram

We denote by \( \tilde{F} \) the modified matrix \( \tilde{A} \) or \( \tilde{B} \). A glance at Tables 8 and 9 shows that for all entries of the same Dynkin type their modified matrices \( \tilde{F} \) have the same characteristic polynomial, in particular the same trace \( \text{Tr} \tilde{F} \). Carrying out step 7 of the Algorithm in §8.1 we also find that for the same Dynkin type those \( \tilde{F} \) act on \( \Delta_b \) in the same manner.

Observation 8.5 According to the Dynkin types we have the following observations, where we mean by \((c_1, c_2, \ldots, c_k)\) the cyclic permutation \( c_1 \to c_2 \to \cdots \to c_k \to c_1 \).

1. In case of type \( E_6 \oplus E_6 \) the matrix \( \tilde{F} \) acts on the simple roots in Figure 2 by \( (e_1, e'_1, e_6, e'_6)(e_3, e'_3, e_5, e'_5)(e_2, e'_2)(e_4, e'_4) \).

In particular \( \tilde{F} \) exchanges the two connected \( E_6\)-components.

2. In case of type \( E_8 \oplus A_2 \oplus A_2 \) the matrix \( \tilde{F} \) fixes \( e_1, \ldots, e_8 \) in the \( E_8\)-component while it acts on the simple roots in the \( A_2 \oplus A_2\)-component by \((c_1, d_1, c_2, d_2)\) in Figure 3.

3. In case of type \( D_{10} \) the matrix \( \tilde{F} \) fixes \( e_1, \ldots, e_8 \) and exchanges \( e_9 \) and \( e_{10} \) in Figure 4.

4. In cases of types \( A_2 \) and \( E_8 \) the matrix \( \tilde{F} \) fixes all simple roots.
9 Siegel Disks

Let \( \mathbb{D} \) and \( S^1 \) be the unit open disk and the unit circle in \( \mathbb{C} \) respectively. Given \( (\alpha_1, \alpha_2) \in T := S^1 \times S^1 \), the map \( g : \mathbb{D}^2 \to \mathbb{D}^2, (z_1, z_2) \mapsto (\alpha_1 z_1, \alpha_2 z_2) \) is said to be an irrational rotation if \( g : T \to T \) has dense orbits; this condition is equivalent to saying that \( \alpha_1 \) and \( \alpha_2 \) are multiplicatively independent, meaning that \( \alpha_1 m_1 \alpha_2 m_2 = 1 \), \( m_1, m_2 \in \mathbb{Z} \), implies \( m_1 = m_2 = 0 \).

Let \( f : X \to X \) be an automorphism of a complex surface \( X \), and \( U \) be an open neighborhood of a point \( p \in X \). We say that \( f \) has a Siegel disk \( U \) centered at \( p \) if \( f|U,p) \) is biholomorphically conjugate to an irrational rotation \( g|_{\mathbb{D}^2,0} \). Namely, a Siegel disk is an invariant subset modeled on an irrational rotation around the origin.

9.1 Existence of a Siegel Disk

McMullen [16] synthesized examples of K3 surface automorphisms with a Siegel disk from Salem numbers of degree 22. To this end he established a sufficient condition for the existence of a Siegel disk (see [16 Theorems 7.1 and 9.2]). We shall relax his criterion so that it may be applied to Salem numbers of degree \( \leq 22 \) in the presence of nontrivial root system.

Let \( L \) be a K3 lattice and \( F : L \to L \) be a positive automorphism of hyperbolic type preserving a K3 structure and having the special eigenvalue \( \delta \in S^1 \) such that \( F|_{H^{2,0}} = \delta I \).

Suppose that \( \delta \) comes from the Salem factor. (62)

Let \( f : X \to X \) be the K3 surface automorphism induced by \( F \). In [16] the number \( \delta \) is called the determinant of \( f \) because \( \delta \) is equal to the determinant of the holomorphic tangent map \( (df)_p : T_p X \to T_p X \) at any fixed point \( p \) of \( f \). Thus the eigenvalues of \( (df)_p \) can be expressed as \( \alpha_1 = \delta^{1/2} \alpha \) and \( \alpha_2 = \delta^{1/2} \alpha^{-1} \) for some \( \alpha \in \overline{\mathbb{Q}} \). Suppose that there exists a rational function \( q(w) \in \mathbb{Q}(w) \) such that \( (\alpha + \alpha^{-1})^2 = q(\tau) \) with special trace \( \tau := \delta + \delta^{-1} \in (-2, 2) \).

Proposition 9.1 Under condition (62), if \( \tau \) satisfies \( 0 \leq q(\tau) \leq 4 \) and admits a conjugate \( \tau' \in (-2, 2) \) such that \( q(\tau') > 4 \), then the fixed point \( p \in X \) is the center of a Siegel disk for \( f \).

On the other hand, if \( \tau \) satisfies \( q(\tau) > 4 \) then \( p \) is a hyperbolic fixed point of \( f \).

Proof. Since \( \tau \in (-2, 2) \), we have \( \delta \in S^1 \) and \( \delta^{1/2} \in S^1 \). It follows from \( 0 \leq q(\tau) \leq 4 \) that \( \alpha \in S^1 \) and hence \( \alpha_1, \alpha_2 \in S^1 \). To show \( \alpha_1, \alpha_2 \in \overline{\mathbb{Q}} \) are multiplicatively independent we mimic the proof of [16 Lemma 7.5]. Let \( \delta' \) and \( \alpha' \) be the conjugates of \( \delta \) and \( \alpha \) corresponding to \( \tau' \). We have \( \delta' \in S^1 \) and \( \alpha' \notin S^1 \) by \( \tau' \in (-2, 2) \) and \( \{\alpha' + (\alpha')^{-1}\}^2 = q(\tau') > 4 \). If \( \alpha_1^m \alpha_2^n = \delta^{(m+n)/2} \alpha^{m-n} = 1 \) with \( m, n \in \mathbb{Z} \) then \( \delta^2(m+n)/2(\alpha')^m = 1 \) and hence \( |\alpha'|^{m-n} = 1 \), which forces \( m = n \) and \( \delta^{m} = 1 \), but the latter implies \( m = 0 \) as \( \delta \) is not a root of unity. The Gel’fond-Baker method then shows that they are jointly Diophantine and the Siegel-Sternberg

Figure 4: Dynkin Diagram of type D_{10}.
theory tells us that there exists a Siegel disk centered at $p$ (see \[16\] Theorems 5.2 and 5.3). On the other hand, if $q(\tau) > 4$ then $\delta^{1/2} \in S_1$ and $\alpha \not \in S_1$, so we have $|\alpha_i| < 1 < |\alpha_j|$ for $(i, j) = (1, 2)$ or $(2, 1)$. Thus $p$ is a hyperbolic fixed point of $f$. 

When $f$ has no fixed curves in $X$, McMullen \[16\] uses Proposition 9.1 with the setting

$$
(a) \quad \text{Tr} [F : L \to L] = -1; \quad \text{and} \quad (b) \quad q(w) = \frac{(w + 1)^2}{w + 2},
$$

(63)

by showing that under condition (a) the Lefschetz fixed point formula implies the existence of a unique transverse fixed point $p \in X$ of $f$ and the Atiyah-Bott holomorphic Lefschetz formula determines $q(w)$ in the form (b). To guarantee the non-existence of fixed curves, he considers Salem numbers of degree 22 so that the Picard lattice and the root system in it are empty. His conclusion is that the unique fixed point $p$ is the center of a Siegel disk, provided

$$
\tau > \tau_0 := 1 - 2\sqrt{2} \approx -1.8284271 \text{ and } \tau \text{ has a conjugate } \tau' < \tau_0.
$$

(64)

In general, however, there may be curves preserved by $f$ and some of them may be fixed pointwise. In this respect all irreducible curves in $X$ are enumerated as follows.

**Lemma 9.2** Under condition (62) any irreducible curve $C \subset X$ is a $(-2)$-curve and hence the unique effective divisor representing the nodal class $[C] \in \text{Pic}(X)$. In particular, the irreducible curves in $X$ are in one-to-one correspondence with the elements of $\Delta_b \subset L$, namely, the simple roots in $\Delta^+$, and how those curves are transformed by $f$ is faithfully represented by the action of $F$ on $\Delta_b$, via the marking isomorphism $\iota$ in diagram (49).

**Proof.** Since the special trace $\tau$ of $F$ comes from the Salem factor, the Picard lattice $\text{Pic} \subset L$ is even and negative-definite, hence so is $\text{Pic} \subset H^2(X, \mathbb{Z})$. Since any irreducible curve $C$ represents a non-trivial effective class $[C] \in \text{Pic}(X)$, its self-intersection number $(C, C)$ is an even negative integer. But the irreducibility of $C$ forces $(C, C) \geq -2$ and so $(C, C) = -2$, which in turn implies that $C$ is a $(-2)$-curve. The remaining assertions of the lemma follow from Barth et al. \[8\] Chap. VIII, (3.7) Proposition].

To any simple root $u \in \Delta_b$ the corresponding $(-2)$-curve in $X$ is denoted by the same symbol $u$. If $F$ fixes a simple root $u$ then $f$ preserves the curve $u \cong \mathbb{P}^1$, inducing a Möbius transformation on it, so that $u$ is a fixed curve of $f$ precisely when the induced map is identity. In the possible occurrence of fixed curves we have to use fixed point formulas stronger than the classical Lefschetz and Atiyah-Bott formulas (see \[8, 13\]). Any version of Lefschetz-type formula involves the value of $\text{Tr} f^*|H^2(X)$ and the following remark is helpful in this respect.

**Remark 9.3** The trace of a monic polynomial $P(z) = z^d + c_1 z^{d-1} + \cdots + c_d$ is defined by $\text{Tr} P := -c_1$, the sum of its roots. A palindromic polynomial of even degree and its trace polynomial have the same trace. Let $\chi(z)$ be the characteristic polynomial of $F : L \to L$. It is a palindromic polynomial of degree 22. We can calculate $\text{Tr} f^*|H^2(X) = \text{Tr} F$ as the trace of $\chi(z)$ or equivalently as the trace of its trace polynomial.

### 9.2 Salem Numbers of Degree 22

Things are simpler with a Salem polynomial of degree 22, because in this case the Kähler cone condition is automatically satisfied. McMullen \[16\] Table 4] gave a list of ten unramified
Salem polynomials $S_i(z)$ of degree 22, $i = 1, \ldots, 10$, for each of which he was able to construct a K3 surface automorphism $f : X \to X$ with a Siegel disk such that the induced map $f^* : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ has $S_i(z)$ as its characteristic polynomial (see [16, Theorem 10.1]). Here the Salem trace polynomials $R_i(w)$ associated with $S_i(z)$ are given in Table 3. Applying our method of hypergeometric lattices to Salem trace polynomials of McMullen, we are able to construct a much greater number of K3 surface automorphisms with a Siegel disk.

**Setup and Tests.** Let $R(w)$ be an unramified Salem trace polynomial of degree 11. We are looking for all integral hypergeometric K3 lattices $L = L(\Phi, \Psi) = L(A, B)$ such that $\Phi(w)$ is a product of cyclotomic trace polynomials as in (61) and $\Psi(w) = R(w)$. To construct K3 surface automorphisms we utilize the matrix $B$. By the last part of Theorem 6.13 any element of $k$ is simple except for at most one multiple element of multiplicity 2 or 3, which must be one of 1, 2, 3, 4, 6. An inspection of Table 2 shows that $k$ has at most seven distinct elements and this number reduces to six if $k$ has a triple element.

1. Find all $k$’s satisfying the unimodularity condition (13), which now reads
   $$\text{Res}(\text{CT}_k, R) = \pm 1 \quad \text{for every} \quad k \in k.$$

2. Judge which of the $k$’s above are K3 lattices according to the criterion in Theorem 6.13.

3. Identify the special trace $\tau \in (-2, 2)$ of the matrix $B$ by using Theorem 6.13 again.

For each of the data in (61) passing two tests in steps (1) and (2), Remark 6.10 and the information from step (3) allow us to provide $L$ with the Hodge structure (48) preserved by $B$, with respect to which $B$ is necessarily positive by Remark 6.6 since $\Psi(w) = R(w)$ is a Salem trace polynomial. Specify a component of $C$ as the positive cone $C^+$ as well as the Kähler cone $K$. Theorem 7.2 then implies that there exist a K3 surface automorphism $f : X \to X$ and a marking $\iota : H^2(X, \mathbb{Z}) \to L$ such that the following diagram is commutative:

$$
\begin{array}{ccc}
H^2(X, \mathbb{Z}) & \xrightarrow{\iota} & L \\
\downarrow f^* & & \downarrow B \\
H^2(X, \mathbb{Z}) & \xrightarrow{\iota} & L.
\end{array}
$$

(65)

The above argument can be applied to the Salem trace polynomials in Table 3. For each $i = 1, \ldots, 10$, let $y_{10} < \cdots < y_2 < y_1$ denote the roots of $R_i(w)$ in the interval $(-2, 2)$. Numerical values of them are given in Table 10, where the roots $\tau_0 := 1 - 2\sqrt{2}$ are separated from the roots $< \tau_0$ by a line. The smallest root $y_{10}$ is always smaller than $\tau_0$.

**Theorem 9.4** Consider a hypergeometric lattice $L = L(\Phi, \Psi) = L(A, B)$ such that $\Phi(w)$ is a product of cyclotomic trace polynomials as in (61) and $\Psi(w) = R_i(w)$ is a Salem trace polynomial in Table 3. Then those $k$’s for which $L$ becomes a K3 lattice are listed in Tables 11–14, where the “case” column refers to which case occurs in Table 2 and the “ST” column tells which of the roots $y_i$ is the special trace $\tau$. For each entry of the tables there exists a K3 surface automorphism $f : X \to X$ such that the diagram (65) is commutative. The map $f$ has a unique fixed point $p \in X$, which is either the center of a Siegel disk or a hyperbolic fixed point; the former cases is indicated by “S” and the latter by “H” in the last column of the tables.
For each entry of type $E_6 \oplus E_6$ in Tables $8$ and $9$ the K3 surface automorphism $f : X \to X$ has no fixed point on $E$ and a unique fixed point $p \in X \setminus E$. If the special trace $\tau$...
| $\Psi$ | case | $k$ | ST | S/H |
|-------|------|-----|----|-----|
| $R_1$ | 3    | 1,1,1,3,4,6,16 | $y_8$ | S   |
|       | 3    | 1,1,2,3,4,6,16  | $y_8$ | S   |
|       | 3    | 2,2,1,3,4,6,16  | $y_8$ | S   |
|       | 4    | 3,3,1,3,4,6,16  | $y_8$ | S   |
|       | 4    | 3,3,2,3,4,6,16  | $y_8$ | S   |
|       | 4    | 4,4,1,3,4,6,16  | $y_8$ | S   |
|       | 4    | 4,4,2,3,4,6,16  | $y_8$ | S   |
|       | 4    | 6,6,1,3,4,6,16  | $y_8$ | S   |
|       | 4    | 6,6,2,3,4,6,16  | $y_8$ | S   |
|       | 5    | 1,3,16,30       | $y_2$ | S   |
|       | 5    | 2,3,16,30       | $y_2$ | S   |
|       | 5    | 1,3,5,7,9       | $y_2$ | S   |
|       | 5    | 2,3,5,7,9       | $y_2$ | S   |
|       | 6    | 1,2,3,4,5,6,7   | $y_{10}$ | H |
|       | 6    | 1,1,3,4,5,6,7   | $y_{10}$ | H |
|       | 6    | 2,2,3,4,5,6,7   | $y_{10}$ | H |
|       | 7    | 3,3,3,4,5,6,7   | $y_{10}$ | H |
|       | 7    | 4,4,3,4,5,6,7   | $y_{10}$ | H |
|       | 7    | 6,6,3,4,5,6,7   | $y_{10}$ | H |
|       | 8    | 3,6,16,30       | $y_1$  | S   |
|       | 8    | 4,5,7,20        | $y_1$  | S   |
|       | 8    | 3,5,6,7,9       | $y_1$  | S   |
| $R_2$ | 3    | 1,1,1,9,24      | $y_7$  | S   |
|       | 3    | 1,1,2,9,24      | $y_7$  | S   |
|       | 3    | 2,2,1,9,24      | $y_7$  | S   |
|       | 3    | 2,2,2,9,24      | $y_7$  | S   |
|       | 4    | 4,4,1,9,24      | $y_7$  | S   |
|       | 4    | 4,4,2,9,24      | $y_7$  | S   |
|       | 4    | 6,6,1,9,24      | $y_7$  | S   |
|       | 4    | 6,6,2,9,24      | $y_7$  | S   |
|       | 5    | 1,3,5,13        | $y_8$  | S   |
|       | 5    | 1,3,5,42        | $y_8$  | S   |
|       | 5    | 2,3,5,13        | $y_8$  | S   |
|       | 5    | 2,3,5,42        | $y_8$  | S   |
|       | 5    | 3,3,1,9,24      | $y_7$  | S   |
|       | 5    | 3,3,2,9,24      | $y_7$  | S   |
|       | 5    | 1,3,5,12,24     | $y_4$  | S   |
|       | 5    | 2,3,5,12,24     | $y_4$  | S   |
|       | 5    | 1,3,12,13       | $y_3$  | S   |
| $R_3$ | 3    | 1,1,1,4,36      | $y_8$  | S   |
|       | 3    | 1,1,2,4,36      | $y_8$  | S   |
|       | 3    | 2,2,1,4,36      | $y_8$  | S   |
|       | 3    | 2,2,2,4,36      | $y_8$  | S   |
|       | 3    | 1,1,4,1,3       | $y_7$  | S   |
|       | 3    | 1,1,2,4,13      | $y_7$  | S   |
|       | 3    | 2,2,1,4,13      | $y_7$  | S   |
|       | 3    | 2,2,2,4,13      | $y_7$  | S   |
|       | 4    | 3,3,1,4,36      | $y_8$  | S   |
|       | 4    | 3,3,2,4,36      | $y_8$  | S   |
|       | 4    | 4,4,1,4,36      | $y_8$  | S   |
|       | 4    | 4,4,2,4,13      | $y_7$  | S   |
|       | 4    | 4,4,2,4,13      | $y_7$  | S   |
|       | 4    | 6,6,1,4,13      | $y_7$  | S   |
|       | 4    | 6,6,2,4,13      | $y_7$  | S   |
|       | 4    | 1,3,8,42        | $y_3$  | S   |
|       | 4    | 2,3,8,42        | $y_3$  | S   |
|       | 5    | 1,3,7,11        | $y_9$  | S   |
|       | 5    | 2,3,7,11        | $y_9$  | S   |
|       | 5    | 1,3,4,7,30      | $y_9$  | S   |
|       | 5    | 2,3,4,7,30      | $y_9$  | S   |
|       | 5    | 3,3,1,4,13      | $y_7$  | S   |
|       | 5    | 3,3,2,4,13      | $y_7$  | S   |

Table 11: Siegel disks for Salem numbers of degree 22, Part 1.
Table 12: Siegel disks for Salem numbers of degree 22, Part 2.
| $\Psi$ | case | $k$ | ST | S/H |
|------|-----|-----|----|-----|
| $R_7$ | 3   | 2,2,7,24 | $y_7$ | S   |
| 4    | 4,4,1,7,16 | $y_7$ | S   |
| 4    | 4,4,1,7,24 | $y_7$ | S   |
| 4    | 4,4,2,7,16 | $y_7$ | S   |
| 4    | 4,4,2,7,24 | $y_7$ | S   |
| 4    | 6,6,1,7,16 | $y_7$ | S   |
| 4    | 6,6,1,7,24 | $y_7$ | S   |
| 4    | 6,6,2,7,16 | $y_7$ | S   |
| 4    | 6,6,2,7,24 | $y_7$ | S   |
| 5    | 3,3,1,7,16 | $y_7$ | S   |
| 5    | 3,3,1,7,24 | $y_7$ | S   |
| 5    | 3,3,2,7,16 | $y_7$ | S   |
| 5    | 3,3,2,7,24 | $y_7$ | S   |
| 5    | 1,3,7,11  | $y_7$ | S   |
| 5    | 2,3,7,11  | $y_7$ | S   |
| 5    | 1,3,5,7,9 | $y_5$ | S   |
| 5    | 2,3,5,7,9 | $y_5$ | S   |
| 5    | 1,3,7,11 | $y_5$ | S   |
| 5    | 2,3,7,11 | $y_5$ | S   |
| 5    | 1,3,5,7,9 | $y_5$ | S   |
| 5    | 2,3,5,7,9 | $y_5$ | S   |
| 5    | 1,3,16,30 | $y_2$ | S   |
| 5    | 1,3,24,30 | $y_2$ | S   |
| 5    | 2,3,16,30 | $y_2$ | S   |
| 5    | 2,3,24,30 | $y_2$ | S   |
| 5    | 1,3,16,30 | $y_2$ | S   |
| 5    | 1,3,24,30 | $y_2$ | S   |
| 5    | 2,3,16,30 | $y_2$ | S   |
| 5    | 2,3,24,30 | $y_2$ | S   |
| $R_8$ | 3   | 1,1,4,12,30 | $y_6$ | S   |
| 3    | 1,1,4,12,30 | $y_6$ | S   |
| 3    | 2,2,4,12,30 | $y_6$ | S   |
| 3    | 2,2,4,12,30 | $y_6$ | S   |
| 3    | 1,1,1,3,12,30 | $y_5$ | S   |
| 3    | 1,1,2,3,12,30 | $y_5$ | S   |
| 3    | 2,2,1,3,12,30 | $y_5$ | S   |
| 3    | 2,2,2,3,12,30 | $y_5$ | S   |
| 4    | 4,4,1,4,12,30 | $y_6$ | S   |
| 4    | 4,4,2,4,12,30 | $y_6$ | S   |
| 4    | 6,6,1,4,12,30 | $y_6$ | S   |

| $\Psi$ | case | $k$ | ST | S/H |
|------|-----|-----|----|-----|
| $R_8$ | 4   | 6,6,2,4,12,30 | $y_6$ | S   |
| 4    | 1,12,14,16 | $y_5$ | S   |
| 4    | 2,12,14,16 | $y_5$ | S   |
| 4    | 3,3,1,3,12,30 | $y_5$ | S   |
| 4    | 3,3,2,3,12,30 | $y_5$ | S   |
| 4    | 6,6,1,3,12,30 | $y_5$ | S   |
| 4    | 6,6,2,3,12,30 | $y_5$ | S   |
| 4    | 1,3,12,36 | $y_4$ | S   |
| 5    | 2,3,12,36 | $y_4$ | S   |
| 5    | 1,3,4,7,30 | $y_9$ | S   |
| 5    | 2,3,4,7,30 | $y_9$ | S   |
| 5    | 1,3,5,42 | $y_8$ | S   |
| 5    | 2,3,5,42 | $y_8$ | S   |
| 5    | 1,3,5,18 | $y_8$ | S   |
| 5    | 2,3,5,18 | $y_8$ | S   |
| 5    | 3,3,1,4,12,30 | $y_6$ | S   |
| 5    | 3,3,2,4,12,30 | $y_6$ | S   |
| 5    | 4,4,1,3,12,30 | $y_5$ | S   |
| 5    | 4,4,2,3,12,30 | $y_5$ | S   |
| 5    | 1,3,12,42 | $y_3$ | S   |
| 5    | 2,3,12,42 | $y_3$ | S   |
| 5    | 1,3,7,12,18 | $y_3$ | S   |
| 5    | 2,3,12,18 | $y_3$ | S   |
| 5    | 1,3,4,5,7,12 | $y_2$ | S   |
| 5    | 2,3,4,5,7,12 | $y_2$ | S   |
| 7    | 3,4,7,12,14 | $y_1$ | S   |
| $R_9$ | 3   | 1,2,3,4,28 | $y_9$ | H   |
| 3    | 1,1,3,4,28 | $y_9$ | H   |
| 3    | 2,2,3,4,28 | $y_9$ | H   |
| 3    | 1,1,1,4,8,24 | $y_8$ | S   |
| 3    | 1,1,2,4,8,24 | $y_8$ | S   |
| 3    | 2,2,1,4,8,24 | $y_8$ | S   |
| 3    | 2,2,2,4,8,24 | $y_8$ | S   |
| 3    | 1,1,1,7,24 | $y_7$ | S   |
| 3    | 1,1,2,7,24 | $y_7$ | S   |
| 3    | 1,1,1,4,12,24 | $y_7$ | S   |
| 3    | 1,1,2,4,12,24 | $y_7$ | S   |
| 3    | 2,2,1,7,24 | $y_7$ | S   |
| 3    | 2,2,2,7,24 | $y_7$ | S   |
| 3    | 2,2,1,4,12,24 | $y_7$ | S   |

Table 13: Siegel disks for Salem numbers of degree 22, Part 3.
is any of \( x_1, x_2, x_3 \) then \( p \) is the center of a Siegel disk, while if \( \tau = x_4 \) then \( p \) is a hyperbolic fixed point; this information is indicated in the “S/H” column of the tables.

**Proof.** It follows from item (1) of Observation [S10] that \( f \) exchanges the two \( \mathfrak{E}_6 \)-components of \( \mathcal{E} \), having no fixed point there. Thus \( f \) has no fixed curve on \( X \) and the framework of (63)-(64) can be applied to show the existence of a unique transverse fixed point \( p \in X \setminus \mathcal{E} \). In view of \( x_4 < \tau_0 < x_3 \) Proposition [9.1] leads to the theorem. \( \square \)

**Remark 9.6** There are three entries in Table 9 for which the root system and hence \( \mathcal{E} \) are empty. We have \( \text{Tr} \bar{B} = -1 \) for one of them and \( \text{Tr} \bar{B} = -2 \) for the other two. In the former case the same logic as in Theorems [9.1] and [9.2] shows that \( f \) has a unique fixed point \( p \in X \), which is hyperbolic by \( \tau = x_4 \), while in the latter cases \( f \) has no fixed point on \( X \).

To discuss the cases of types \( \mathfrak{E}_8 \oplus \mathfrak{A}_2 \oplus \mathfrak{A}_2 \) and \( \mathfrak{E}_8 \), we use S. Saito’s fixed point formula [22, (0.2)]. Originally it was stated for projective surfaces, but it works for compact Kähler surfaces.
as well (Dinh et al. [5, Theorem 4.3]). In the notation of [12, Theorem 1.2] the formula reads

$$L(f) := \sum_{i=0}^{4} (-1)^i \text{Tr} f^i |H^i(X) = \sum_{p \in X_0(f)} \nu_p(f) + \sum_{C \in X_1(f)} \chi_C \cdot \nu_C(f) + \sum_{C \in X_{\eta}(f)} \tau_C \cdot \nu_C(f),$$

(66)

where $X_0(f)$ is the set of fixed points of $f$ while $X_1(f)$ and $X_{\eta}(f)$ are the sets of irreducible fixed curves of types I and II respectively, $\chi_C$ is the Euler number of the normalization of $C$ and $\tau_C$ is the self-intersection number of $C$. We refer to [12, §3] for the definitions of the indices $\nu_p(f)$ and $\nu_C(f)$ as well as for a detailed account of the terminology used here.

We also use the Toledo-Tong fixed point formula [24, Theorem (4.10)] in the special case where $X$ is a compact complex surface, $f : X \to X$ is a holomorphic map, $E$ is a holomorphic line bundle on $X$ and $\phi : f^*E \to E$ is a holomorphic bundle map. Suppose that any isolated fixed point $p$ is transverse and any connected component of the 1-dimensional fixed point set is a smooth curve $C$ such that the induced differential map $d^Nf$ on the normal line bundle $N = N_C$ to $C$ does not have eigenvalue 1. The formula is then stated as

$$L(f, \phi) := \sum_{i=0}^{2} (-1)^i \text{Tr} f, \phi^i |H^i(X, \mathcal{O}(E)) = \sum_{p} \nu_p(f, \phi) + \sum_{C} \nu_C(f, \phi),$$

(67)

where the sums are taken over all isolated fixed points $p$ and all connected fixed curves $C$. If $d^Nf$ has eigenvalue $\lambda_C$ on $N_C$ while $\phi$ has eigenvalues $\mu_p$ and $\mu_C$ on $E_p$ and $E|_C$ respectively, then the indices $\nu_p(f, \phi)$ and $\nu_C(f, \phi)$ are given by

$$\nu_p(f, \phi) = \frac{\mu_p}{1 - \text{Tr}(df)_p + \text{det}(df)_p},$$

(68a)

$$\nu_C(f, \phi) = \int_C \text{td}(C) \cdot \{1 - \lambda_C \text{ch}(\tilde{N})\}^{-1} \cdot \mu_C \text{ch}(E),$$

(68b)

where $\text{td}(C)$ is the Todd class of $C$, $\text{ch}(E)$ is the Chern character of $E$, $\tilde{N}$ is the dual line bundle to $N$ and the integral sign stands for evaluation on the fundamental cycle of $C$.

A three-cycle of Siegel disks for $f$ is a sequence of open subsets $U$, $f(U)$, $f^2(U)$ in $X$ such that $U$ is a Siegel disk for $f^3$ centered at a point $p \in U$ which is a periodic point of period 3, that is, $p$, $f(p)$, $f^2(p)$ are mutually distinct and $f^3(p) = p$. We remark that $U$, $f(U)$, $f^2(U)$ are Siegel disks for $f^3$ centered at the points $p$, $f(p)$, $f^2(p)$, respectively.

**Theorem 9.7** For each entry of types $E_8 \oplus A_2 \oplus A_2$ and $E_8$ in Tables 8 and 9 the K3 surface automorphisms $f : X \to X$ has a unique periodic orbit $p$, $f(p)$, $f^2(p) \in X$ of period 3, and those three points lie in $X \setminus \mathcal{E}$. If the special trace $\tau$ is either $x_2$ or $x_4$ then they are the centers of a three-cycle of Siegel disks, while if $\tau$ is either $x_1$ or $x_3$ then they are hyperbolic periodic points; this information is indicated in the “S/H” column of the tables.

**Proof.** We give a proof for the case of type $E_8 \oplus A_2 \oplus A_2$. The proof for the $E_8$-case is just the same as the previous one upon forgetting the existence of the $A_2 \oplus A_2$-component of $\mathcal{E}$.

It follows from (2) of Observation 9.3 that $f$ has no fixed point on the $A_2 \oplus A_2$-component of $\mathcal{E}$. On the other hand $f$ preserves each $(−2)$-curve on the $E_8$-component, inducing a Möbius transformation on it. A Möbius transformation falls into one of the three categories according to the number $n$ of its fixed points; parabolic for $n = 1$, non-parabolic for $n = 2$, and the
identity for $n \geq 3$. In the parabolic case the derivative at the unique fixed point is 1. In the non-parabolic case, if the derivatives at one fixed points is $c$ then the derivative at the other fixed point is $c^{-1}$, where $c \neq 1$. Notice that $e_4$ is a fixed curve of $f$ since $e_4$ is fixed at the three intersections $e_4 \cap e_2$, $e_4 \cap e_3$, $e_4 \cap e_5$ (see Figure 5).

Consider the arm $a := e_5 \cup e_6 \cup e_7 \cup e_8$ emanating from $e_4$. Let $q_0, q_1, q_2, q_3$ be the points at $e_4 \cap e_5$, $e_5 \cap e_0$, $e_6 \cap e_7$, $e_7 \cap e_8$. Note that they are fixed points of $f$. We use the fact that $\det(df)_q = \delta$ at any fixed point $q$ of $f$. Since $(df)_{q_0} = 1$ along $e_4$ we have $(df)_{q_1} = \delta \neq 1$ along $e_5$. Thus $f$ induces a non-parabolic Möbius transformation on $e_5$ having derivative $\delta^{-1}$ at $q_1$. Then $f$ induces a non-parabolic Möbius transformation on $e_6$ having derivatives $\delta^2$ at $q_1$ and $\delta^{-2}$ at $q_2$. Repeating this argument shows that $f$ has four fixed points on $a$, three of which are $q_1, q_2, q_3$ and the final one $q_4$ is on $e_8$, and that $(df)_{q_j}$ has eigenvalues $\delta^{-j}$ and $\delta^{j+1}$ for $j = 1, 2, 3, 4$. Similar statements can be made for the shorter arms $b := e_1 \cup e_3$ and $e_2$. In total there are seven isolated fixed points indicated by $\bullet$ and one fixed curve $e_4$ on $\mathcal{E}$.

We apply Saito’s formula (66) to $f$. For each entry of type $E_8 \oplus A_2 \oplus A_2$ in Tables 8 and 9 we have $\text{Tr} F = 7$ where $F = A$ or $B$, and so $L(f) = 2 + \text{Tr} F = 9$. The seven isolated fixed point on $\mathcal{E}$ are transverse and hence of index 1. The fixed curve $e_4$ is of type I and has index 1, since $df$ has eigenvalue $\delta \neq 1$ in its normal direction. Any point on $e_4$ has index 0. The Euler number of $e_4 \cong \mathbb{P}^1$ is 2. There is no fixed curve of type II. Thus formula (66) reads

$$9 = 7 + \sum_{p \in X_0(f) \backslash \mathcal{E}} \nu_p(f) + 2, \quad \text{i.e.} \quad \sum_{p \in X_0(f) \backslash \mathcal{E}} \nu_p(f) = 0, \quad (69)$$

where 7 and 2 on the RHS are the contributions of the seven isolated fixed point on $\mathcal{E}$ and the fixed curve $e_4$. This implies that $f$ has no fixed point on $X \setminus \mathcal{E}$.

Next we apply the Toledo-Tong formula (67) to $f$ upon setting $E = K_X := \wedge^2 T^* X$ and $\phi = f^*$, where $K_X \cong \mathcal{O}_X$ as $X$ is a K3 surface. Notice that $\mu_p = \lambda_C = \mu_C = \delta$ with $C = e_4$ in (68). Let $\eta$ be a nowhere vanishing holomorphic 2-form on $X$. At each $x \in C$ the map $T_x X \to T^*_x C$, $v \mapsto \eta_x(v, \cdot)$ induces an isomorphism $N_x := T_x X / T_x C \cong T^*_x C$; hence $N \cong TC$. A little calculation yields $\nu_C(f, f^*) = \delta(\delta + 1) / (\delta - 1)^2$ with $C = e_4$ in (68). It is easy to see
that $L(f, f^*) = 1 + \delta$. Formula (67) then asserts that $z = \delta$ is a solution to the equation

$$1 + z = \sum_{j=1}^{4} \frac{z}{1 - (z^{-j} + z^{j+1})} + z$$

$$+ \sum_{j=1}^{2} \frac{z}{1 - (z^{-j} + z^{j+1})} + z$$

$$+ \frac{z}{1 - (z^{-1} + z^{2})} + \frac{z(z + 1)}{(z - 1)^2},$$

where the first three terms in the RHS are the contributions of the fixed points on the arms $a, b, e_2$ and the last term is that of the fixed curve $e_4$. A careful inspection shows that the difference $D(z)$ of the LHS from the RHS above admits a clean factorization

$$D(z) = \frac{L(z)}{(z + 1) \cdot C_1(z) \cdot C_3(z) \cdot C_5(z)} = \frac{z \cdot LT(w)}{(z + 1) \cdot CT_1(w) \cdot CT_3(w) \cdot CT_5(w)}, \quad (70)$$

where $w := z + z^{-1}$. So the Toledo-Tong formula for $f$ is nothing other than the tautological fact that $\delta$ is a root of Lehmer’s polynomial, but the formula (70) itself will be useful later.

We again apply Saito’s formula (66) this time to $f^3$. If $\tilde{F}$ has characteristic polynomial $\chi(z) = z^{22} - e_1 z^{21} + e_2 z^{20} - e_3 z^{19} + \cdots$ then $\text{Tr}(F^3) = p_3 = 3(e_3 - e_1 e_2) + e_3^3$ by the relation between power sums and elementary symmetric polynomials. We have $e_1 = 7, e_2 = 20, e_3 = 29$ and $p_3 = 10$ in the case of type $E_8 \oplus A_2 \oplus A_2$ (whereas $e_1 = 7, e_2 = 19, e_3 = 22$ and $p_3 = 10$ for type $E_8$). This implies $L(f) = 2 + \text{Tr}(F^3) = 12$, hence the equation (69) turns into

$$12 = 7 + \sum_{p \in X_0(f^3) \setminus \mathcal{E}} \nu_p(f^3) + 2, \quad \text{i.e.} \quad \sum_{p \in X_0(f^3) \setminus \mathcal{E}} \nu_p(f^3) = 3.$$

Thus $f^3$ has three fixed points on $X \setminus \mathcal{E}$ counted with multiplicity. Let $p$ be one of them. Since $f$ has no fixed point on $X \setminus \mathcal{E}$, we have $p \neq f(p)$ and hence $f(p) \neq f^2(p)$ and $f^2(p) \neq f^3(p) = p$. Accordingly, $p, f(p), f^2(p)$ must be distinct and $\nu_{f(p)}(f^3) = 1$ for $j = 0, 1, 2$.

The determinant for $f^3$ is $\delta^3$ and for $j = 0, 1, 2$ the tangent maps $(df^3)_{f(p)}$ have common eigenvalues of the form $\delta^{3/2} \alpha^j$. The Toledo-Tong formula (67) for $f^3$ is then expressed as

$$D(\delta^3) = \frac{3\delta^3}{1 - \delta^{3/2}(\alpha + \alpha^{-1}) + \delta^3},$$

in terms of the rational function $D(z)$ in (70). Solving this equation we have

$$q(\tau) := (\alpha + \alpha^{-1})^2 = \delta^{-3} \left(1 + \delta^3 - \frac{3\delta^3}{D(\delta^3)}\right)^2 = \frac{(1 + \delta^3)^2 M(\delta^3)^2}{\delta^3 L(\delta^3)^2}$$

$$= \frac{(\delta^3 + \delta^{-3} + 2) \text{MT}(\delta^3 + \delta^{-3})^2}{\text{LT}(\delta^3 + \delta^{-3})^2} = \frac{(\tau + 2)(\tau - 1)^2 \text{MT}(\tau^3 - 3\tau)^2}{\text{LT}(\tau^3 - 3\tau)^2},$$

where $M(z)$ and $\text{MT}(z)$ are Salem polynomial and its trace polynomial defined by

$$M(z) := z^{10} - 2z^9 - z^7 + 2z^6 - z^5 + 2z^4 - z^3 - 2z + 1,$$

$$\text{MT}(w) := (w + 1)(w - 2)(w^3 - w^2 - 4w + 1) - 1.$$

Here we have $0 < q(x_j) < 4$ for $j = 2, 4$ and $q(x_j) > 4$ for $j = 1, 3$. Thus Proposition 9.1 leads to the conclusion of the theorem. \[\square\]
Remark 9.8 In case of type D_10 it follows from (3) of Observation 8.5 that \( f \) has no fixed curve and exactly eight transverse fixed points on \( E \). The Lefschetz formula implies that \( f \) has a unique transverse fixed point \( p \in X \setminus E \). The Atiyah-Bott formula tells us that at \( p \) we have
\[
q(w) = \frac{(w^4 - 4w^2 + 2)^2}{(w + 2)(w - 2)(2w^3 + 3w^2 - 2w - 2)^2}.
\]
Unfortunately, however, Proposition 9.1 is not applicable to the current \( q(w) \), since \( 0 < q(x_j) < 4 \) for all \( j = 1, 2, 3, 4 \). Regrettably, moreover, we have \( q(\tau_L) > 4 \); otherwise, the situation might still have been manageable. We hope that \( p \) is the center of a Siegel disk.

Remark 9.9 In case of type A_2, let \( e_\pm \) be the two \((-2)\)-curves in \( E \) and \( p_0 \) be their intersection. Saito’s formula (66) and the Toledo-Tong formula (67) rule out the possibility that either \( e_+ \) or \( e_- \) is fixed by \( f \). Notice that \( p_0 \) is a fixed point of \( f \), which is either (i) transverse or (ii) of multiplicity 2. In case (i), thinking of the Möbius transformations on \( e_\pm \) induced by \( f \) and a use of the Lefschetz formula show that \( f \) has two transverse fixed points \( p_\pm \in e_\pm \setminus \{p_0\} \) and no fixed point on \( X \setminus E \). If the eigenvalues of \((df)_{p_0}\) are \( 1 / 2 \alpha^{\pm 1} \) then those of \((df)_{p_\varepsilon}\) are \( 1 / 2 (\delta^\varepsilon \alpha)^{\pm 1} \) for \( \varepsilon = \pm 1 \). The Atiyah-Bott formula yields \( q(w) = (w^2 - 3)^2 / (w + 2) \) at the point \( p_0 \). We have \( 0 < q(x_j) < 4 \) for \( j = 1, 3, 4 \) and \( q(x_2) > 4 \), hence Proposition 9.1 is applicable. The multiplicative independence of the eigenvalues of \((df)_{p_\varepsilon} \), \( \varepsilon = \pm 1 \), can be verified in the same manner as that of the eigenvalues of \((df)_{p_0} \) (see the proof of Proposition 9.1). We can now conclude that \( p_0 \) and \( p_\pm \) are the centers of Siegel disks, since we have \( \tau = x_1, x_4 \) for the entries of type A_2 in Table 8; there is no entry of type A_2 in Table 9. Unfortunately, however, as of this writing we do not know how to rule out the case (ii), although it is not likely to happen.

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