Position Dependent Mass Schrödinger Equation and Isospectral Potentials : 
Intertwining Operator approach

Bikashkali Midya\textsuperscript{\dag}, B. Roy\textsuperscript{\dag} and R. Roychoudhury\textsuperscript{\dag}

Physics & Applied Mathematics Unit
Indian Statistical Institute
Kolkata 700108
India

Here we have studied first and second-order intertwining approach to generate isospectral partner potentials of position-dependent (effective) mass Schrödinger equation. The second-order intertwiner is constructed directly by taking it as second order linear differential operator with position dependent coefficients and the system of equations arising from the intertwining relationship is solved for the coefficients by taking an ansatz. A complete scheme for obtaining general solution is obtained which is valid for any arbitrary potential and mass function. The proposed technique allows us to generate isospectral potentials with the following spectral modifications: (i) to add new bound state(s), (ii) to remove bound state(s) and (iii) to leave the spectrum unaffected. To explain our findings with the help of an illustration, we have used point canonical transformation (PCT) to obtain the general solution of the position dependent mass Schrodinger equation corresponding to a potential and mass function. It is shown that our results are consistent with the formulation of type A $\mathcal{N}$-fold supersymmetry \cite{14,18} for the particular case $\mathcal{N} = 1$ and $\mathcal{N} = 2$ respectively.

PACS numbers:

I. INTRODUCTION

There is a growing interest nowadays to design systems whose Hamiltonians have given spectral characteristics. In this context, the idea of designing potentials with prescribed energy spectra is worth investigating. Some progress in this area has been done by restricting the construction of potentials isospectral to a given initial one except a few energy values through the usage of Darboux transformation \cite{1}, factorization method \cite{2}, supersymmetric quantum mechanics (SUSYQM) \cite{3,4} and other related techniques. The underlying idea of most of these procedures has been summarized in an algebraic scheme known as intertwining approach. In general, the objective of the intertwining is to construct the so-called intertwining operator $\mathcal{L}$ which performs an intertwining between an initial solvable Hamiltonian $\mathcal{H}$ and a new solvable one $\tilde{\mathcal{H}}$ with slightly modified spectrum such that

$$\mathcal{L} \mathcal{H} = \tilde{\mathcal{H}} \mathcal{L}, \quad \tilde{\psi}(x) = \mathcal{L}\psi(x)$$

(1)

The ingredients to implement the intertwining are seed solutions of the initial stationary Schrödinger equation associated to factorization energies less than or equal to the ground state energy of $\mathcal{H}$. If $\mathcal{L}$ is a first order differential operator, the standard SUSYQM, with supercharges built of first order Darboux transformation operators, and the factorization method are recovered. On the other hand, if higher order differential operators are involved in the construction of $\mathcal{L}$, it gives rise to higher order SUSYQM \cite{24}. It is possible to generate

\textsuperscript{*}Electronic address: bikash.midya@gmail.com
\textsuperscript{\dag}Electronic address: barnana@isical.ac.in
\textsuperscript{\ddag}Electronic address: raj@isical.ac.in
families of isospectral Hamiltonians by either of the two ways:

(i) by iteration of first order Darboux transformations. Every chain of $\mathcal{N}$ first order Darboux transformation creates a chain of exactly solvable Hamiltonians $H_0, H_1, H_2, ..., H_N$. Hence the intertwining operator $\mathcal{L}^{(N)}$ between the initial Hamiltonian $H_0$ and the final Hamiltonian $H_N$ can always be presented as a product of $N$ first order Darboux transformation operators between every two juxtaposed Hamiltonians $H_0, H_1, ..., H_N$,

$$\mathcal{L}^{(N)} = \mathcal{L}_N \mathcal{L}_{N-1} ... \mathcal{L}_2 \mathcal{L}_1, \quad \mathcal{H}_p \mathcal{L}_p = \mathcal{L}_p \mathcal{H}_{p-1}, \quad p = 1, 2, ..., \mathcal{N}$$

(ii) by looking for the $\mathcal{N}$-th order intertwining operator directly, expressing the intertwiner as a sum of $\mathcal{N} + 1$ terms $g_i(x) \frac{d}{dx}$, $i = 0, 1, \cdots, \mathcal{N}$, and solving the system of equations resulting from the intertwining relationship for the $g_i(x)$’s.

At this point it is appropriate to mention that the quantum mechanical systems with position dependent (effective) mass \(^[6]\) have attracted a lot of interest due to their relevance in describing the physics of many microstructures of current interest such as semiconductor heterostructures \(^[7]\), quantum dots \(^[8]\), helium clusters and metal crystals \(^[9]\) etc. Recently, the intertwining operator method has been applied to Schrödinger equation with position dependent (effective) mass to construct first-order and chains(iterations) of first-order Darboux transformations and the connection between the first-order Darboux transformation and effective mass supersymmetry (factorization) was shown \(^[10]\). Subsequently Darboux transformations of arbitrary order for position dependent mass Schrödinger equation was derived and factorization of the $n$-th order transformation into first order transformations and existence of a reality condition for the transformed potential was shown \(^[11]\). In the standard supersymmetric (SUSY) approach for effective mass Hamiltonians \(^[12]\), the ladder operators are taken as first order differential operators similar to constant mass case, but now they depend on both superpotential and mass function. As a result one obtains two partner potentials with the same effective mass sharing identical spectra up to the zero mode of the supercharge. Second order supersymmetric approach (2-SUSY) was used in ref \(^[13]\) for describing dynamics of a quantum particle with a position dependent mass. A compact expression for 2-SUSY isospectral pairs was derived in terms of second order superpotential and the mass function. A detailed analysis has been given about zero mode equations of second order supercharges and possible reduction of 2-SUSY scheme to first order SUSY. Recently, a generalization of standard SUSY known as higher derivative SUSY or $\mathcal{N}$-fold SUSY \(^[14]\) was given for position dependent mass Hamiltonians. This method keeps the basic superalgebra intact but differs from the standard first order SUSY in that the supercharges are represented as $\mathcal{N}$-th order ($\mathcal{N} > 1$) differential operators.

In this article an attempt is made to generate isospectral potentials of position dependent mass Schrödinger equation (PDMSE) by applying first and second order intertwining technique. Specifically, the second order intertwiner is constructed by taking it as

$$\mathcal{L} = \frac{1}{M(x)} \frac{d^2}{dx^2} + \eta(x) \frac{d}{dx} + \gamma(x)$$

where $M(x)$ is the mass function and $\eta(x), \gamma(x)$ are to be determined. Substituting this in the intertwining relationship \(^[11]\), we have been able to solve the apparently intricate system of equations for $\eta(x)$ and $\gamma(x)$ by assuming an ansatz. As mentioned earlier, the closed form formulas for $\mathcal{N}$-th order intertwining operators and a pair of isospectral Hamiltonians with position dependent mass was already reported by Tanaka \(^[14]\), for an arbitrary value of $\mathcal{N}$ without recourse to any ansatz.

The motivation for constructing second order intertwining operator directly comes from the observation that although an $\mathcal{N}$-th order intertwining operator can be expressed formally as a product of $\mathcal{N}$ first order intertwining operators, it does not necessarily mean that a system constructed by an $\mathcal{N}$-th order intertwining operator is equivalent to one constructed by $\mathcal{N}$ successive applications of each 1st order operator. In fact, it was shown in ref \(^[15]\), by comparing the two approaches in the constant mass scenario, that the former is more general than the latter. The advantage of the direct method used here over the iterative method is that one can generate second-order isospectral partner potentials directly from the initial potential i.e. one need not
go through the first-order intertwining technique. We shall see that as in the case of constant mass scenario, it is possible to generate isospectral potentials with some spectral modification: (i) to add new bound state(s) (ii) to remove bound state(s) and (iii) to leave the spectrum unaffected. In this context a natural question is: What is the utility of finding the isospectral potentials in the position dependent mass background? To answer this let us note that in different areas of possible applications of low dimensional structures as already mentioned, there is need to have energy spectrum which is predetermined. For example, in the quantum well profile optimization, isospectral potentials (by deleting or creating bound states at a particular energy of the original potential) are generated through supersymmetric quantum mechanics. This is necessary because a particular effect (such as intersubband optical transitions in a quantum well) may be grossly enhanced by achieving the resonance conditions e.g. appropriate spacings between the most relevant states and also by tailoring the wave functions so that the (combinations of) matrix elements relevant for this particular effect are maximized \cite{10}. This is particularly important for studying higher order nonlinear processes.

The organization of the paper is as follows: in section II and III we have explained the first and second-order intertwining techniques respectively with possible spectral modifications, with the help of a suitable example given in the Appendix. Also, the connection of our approach to type A $N$-fold SUSY is shown in these sections. Section IV is kept for discussions and comments.

II. FIRST ORDER INTERTWINING

We consider the following two one-dimensional effective mass Schrödinger Hamiltonians (Bendaniel-Duke form) \cite{17} with the same spectrum but with different potential

\begin{equation}
\mathcal{H}\psi = E\psi, \quad \mathcal{H} = -\left[\frac{1}{M(x)} \frac{d}{dx}\right] \frac{d}{dx} + V(x)
\end{equation}

and

\begin{equation}
\bar{\mathcal{H}}\bar{\psi} = E\bar{\psi}, \quad \bar{\mathcal{H}} = -\left[\frac{1}{M(x)} \frac{d}{dx}\right] \frac{d}{dx} + \bar{V}(x)
\end{equation}

We connect the Hamiltonians \cite{2} and \cite{3} by means of the intertwining technique. To this end, we look for an operator $\mathcal{L}$ that satisfies the relation (1). Without loss of generality let us consider the first order intertwining operator \cite{10}, as

\begin{equation}
\mathcal{L} = \frac{1}{\sqrt{M(x)}} \frac{d}{dx} + A(x)
\end{equation}

Now using the intertwining relation (1) and equating the coefficients of like order of derivatives we obtain

\begin{equation}
\bar{V} = V + \frac{2A'}{\sqrt{M}} - \frac{3M'^2}{4M^3} + \frac{M''}{2M^2}
\end{equation}

and

\begin{equation}
A(\bar{V} - V) = \frac{A'M'}{M^2} + \frac{V'}{\sqrt{M}} + \frac{A''}{M}
\end{equation}

where ‘prime’ denotes differentiation with respect to $x$.

Now using (4), equation (6) reduces to

\begin{equation}
\frac{A''}{\sqrt{M}} - \frac{A'M'}{M^{3/2}} - \frac{AM''}{2M^{3/2}} + \frac{3AM'^2}{4M^2} - 2AA' + V' = 0
\end{equation}
Integrating equation (7) we get

\[
\frac{A'}{\sqrt{M}} - \frac{AM'}{2M^2} - A^2 + V = \mu
\]

where \(\mu\) is a constant of integration. Now we substitute

\[
A(x) = -\frac{K}{\sqrt{M}}
\]

where \(K=K(x)\) being an auxiliary function, in equation (8) we obtain the following Riccati equation

\[
-\frac{K'}{M} + \frac{KM'}{M^2} - \frac{K^2}{M} + V = \mu
\]

The equation (10) can be linearized by the substitution \(K(x) = \frac{U'(x)}{U(x)}\). Substituting this value of \(K\) in equations (9) and (10) we get

\[
A(x) = -\frac{U'}{\sqrt{MU}}
\]

and

\[
-\frac{1}{M}U'' - \left(\frac{1}{M}\right)' U' + VU = \mu U
\]

respectively. The equation (12) is similar to equation (2) with \(E = \mu\), \(\mu\) is sometimes called factorization energy and \(U(x)\) is called seed solution. It should be noted here that \(U(x)\) need not be normalizable solution of (12). However, for \(A(x)\) to be well defined (without singularity), \(U\) must not have any zeroes on the real line. For this, we shall restrict \(\mu \leq E_0\) throughout this article, \(E_0\) being the ground state energy eigenvalues of the equation (2). Once we can determine the solution \(U\) of (12) then we shall able to construct the intertwiner \(L\), the isospectral partner \(\bar{V}\) and its bound state eigenvalues \(\bar{\psi}(x)\) with the help of following relations

\[
\bar{V} = V - \frac{2U''}{MU} + \frac{2U'^2}{MU^2} + \frac{MU'}{2MU} + \frac{M''}{2M^2} - \frac{3M'^2}{4M^3}
\]

\[
\mathcal{L} = \frac{1}{\sqrt{M}} \left(\frac{d}{dx} - \frac{U'}{U}\right)
\]

and

\[
\bar{\psi}(x) \propto \mathcal{L}\psi = \frac{1}{\sqrt{M}} \left[\frac{d}{dx} - (\ln U)'\right] \psi
\]

The intertwiner \(\mathcal{L}\) cannot be used to generate wave function of \(\bar{H}\) at the factorization energy \(\mu\), because \(\mathcal{L}U = 0\). We are showing below with the help of supersymmetry how \(\bar{\psi}_\mu\) can be obtained from the relation \(\mathcal{L}^\dagger \psi_\mu = 0\), \(\mathcal{L}^\dagger\) being the adjoint of \(\mathcal{L}\) and is given by

\[
\mathcal{L}^\dagger = \frac{1}{\sqrt{M}} \left(-\frac{d}{dx} - \frac{U'}{U} + \frac{M'}{2M}\right)
\]
For this we calculate $L^\dagger L$ and $LL^\dagger$ given by
\begin{equation}
L^\dagger L = -\frac{1}{M} \frac{d^2}{dx^2} + \frac{M'}{M^2} \frac{d}{dx} + \left( \frac{U''}{MU} - \frac{M'U'}{M^2U} \right) \tag{17}
\end{equation}
and
\begin{equation}
LL^\dagger = -\frac{1}{M} \frac{d^2}{dx^2} + \frac{M'}{M^2} \frac{d}{dx} + \left( \frac{2U'^2}{MU^2} - \frac{U''}{MU} + \frac{M''}{2M^2} - \frac{3M'^2}{4M^3} \right) \tag{18}
\end{equation}
Now from the equation (12) we have
\begin{equation}
V = \frac{U''}{MU} - \frac{M'U'}{M^2U} + \mu \tag{19}
\end{equation}
Substituting this value of $V$ in (13) we obtain
\begin{equation}
\bar{V} = -\frac{U''}{MU} + \frac{2U'^2}{MU^2} + \frac{M''}{2M^2} - \frac{3M'^2}{4M^3} + \mu \tag{20}
\end{equation}
Now using (19) and (20) in (17) and (18) we get
\begin{equation}
L^\dagger L = -\frac{1}{M} \frac{d^2}{dx^2} + \frac{M'}{M^2} \frac{d}{dx} + V - \mu = \bar{H} - \mu \tag{21}
\end{equation}
and
\begin{equation}
LL^\dagger = -\frac{1}{M} \frac{d^2}{dx^2} + \frac{M'}{M^2} \frac{d}{dx} + \bar{V} - \mu = \check{\bar{H}} - \mu \tag{22}
\end{equation}
respectively. It is clear from the equation (22) that the wave function of $\bar{H}$ at the factorization energy $\mu$ can be obtained by $L^\dagger \bar{\psi}_\mu = 0$ i.e.,
\begin{equation}
\bar{\psi}_\mu \propto \exp \left[ \int \left( -\frac{U'}{U} + \frac{M'}{2M} \right) dx \right] = \frac{\sqrt{M}}{U} \tag{23}
\end{equation}
It is to be noted that if $U$ corresponds to the bound state of $H$, the wave function $\bar{\psi}_\mu(x)$ defined in (23) is not normalized so that $\mu$ does not belong to the bound state spectrum of $\bar{H}$. If $U$ corresponds to the ground state wavefunction of $H$ then then the potential $\bar{V}$ has no new singularity, except the singularity due to $V$, provided $M$ is not singular and $M \neq 0$. However, if we consider $U$ to an arbitrary state other than ground state of $H$ then $\bar{V}$ might contain extra singularities, which are not present in $V$. If $U$ is nodeless and unbounded at the both end points then $\bar{\psi}_\mu(x)$ defined in (23) is normalizable, so that $\mu$ can be included in the bound state spectrum of $\bar{H}$ to generate $\bar{V}$. In this case maximal set of bound state wavefunctions of $\bar{H}$ are given by $\{ \bar{\psi}_\mu, L\psi \}$.

A. First-order intertwining and type A 1-fold SUSY

To show that the results obtained in the previous section are consistent with the results of type A $\mathcal{N}$-fold SUSY, we are going to mention the brief results of type A $\mathcal{N}$-fold SUSY formalism (for details see [14] and references there). Type A $\mathcal{N}$-fold SUSY is characterized by the type A monomial space
\begin{equation}
\hat{\nu}_\mathcal{N} = \{ 1, z, \ldots z^{\mathcal{N}-1} \} \tag{24}
\end{equation}
preserved by $\tilde{\mathcal{H}}_N$:

$$\tilde{\mathcal{H}}_N = -A(z) \frac{d^2}{dz^2} - B(z) \frac{d}{dz} - C(z)$$  \hspace{1cm} (25)$$

where

$$A(z) = a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0, \quad N \geq 3$$

$$B(z) = Q(z) - \frac{N-2}{2} A'(z)$$

$$C(z) = \frac{(N-1)(N-2)}{12} A''(z) - \frac{(N-1)}{2} Q'(z) + R$$

$$Q(z) = b_2 z^2 + b_1 z + b_0, \quad N \geq 2$$  \hspace{1cm} (26)$$

$R, a_i, b_i$ are being constants. Applying the algorithm for constructing type A $N$-fold SUSY in PDM system [14], one can construct the most general form of type A $N$-fold SUSY PDM quantum systems ($\mathcal{H}, \tilde{\mathcal{H}}, \mathcal{L}$) or equivalently ($\mathcal{H}_N^+, \mathcal{H}_N^-, \mathcal{P}_N$):

$$\mathcal{H}_N^+ = -\frac{1}{M} \frac{d^2}{dx^2} + \frac{M'}{M^2} \frac{d}{dx} + V^+(x)$$  \hspace{1cm} (27)$$

$$\mathcal{P}_N = M(x)^{-\frac{N}{2}} \prod_{k=0}^{N-1} \left[ \frac{d}{dx} + W(x) - \frac{4}{M(x)} \frac{\dot{M}(x)}{2M(x)} \right]$$  \hspace{1cm} (28)$$

where

$$V_+ = V^- + 2N \left( \frac{W'(x)}{M(x)} - \frac{M'(x)W(x)}{2M(x)^2} \right)$$  \hspace{1cm} (29)$$

$$W(x) = \frac{dW_-(x)}{dx} - \frac{z''(x)}{z'(x)} + \frac{M'(x)}{2M(x)}$$  \hspace{1cm} (30)$$

$$\frac{dW_-(x)}{dx} = \frac{z''(x)}{2z'(x)} - \frac{M(x)B(z)}{2z'(x)} - \frac{M'(x)}{2M(x)}$$  \hspace{1cm} (31)$$

$$z'(x)^2 = M(x)A(z)$$  \hspace{1cm} (32)$$

and the product of operators are ordered as

$$\prod_{k=0}^{N-1} F_k = F_{N-1} F_{N-2} ... F_0$$

The solution space of the type A Hamiltonians $\mathcal{H}_N^+$ are given by

$$\nu_N^+ = e^{-W_+(x)(1, z, ..., z^N)}|_{x=z(x)}$$, where $W_N^+ = -W_+ + (N-1) \ln|z'(x)| - \frac{N}{2} \ln|M(x)|$$  \hspace{1cm} (33)$$

It is easily seen that ($\tilde{V} - V$) obtained in (13) coincides with ($V^+ - V^-$) in equation (29) (with $N = 1$) if one takes (comparing $\mathcal{L}$ with $P_1$)

$$\frac{d}{dx} \ln U(x) = -W(x) + \frac{M'(x)}{4M(x)}$$  \hspace{1cm} (34)$$
B. Example of first-order intertwining

It may be emphasized that the results mentioned in section II are most general and valid for any potential \( V(x) \). However, to illustrate the above procedure with the help of an example we shall need non-normalizable solutions of (12) corresponding to a particular mass function \( M(x) \). In Appendix A we have used point canonical transformation approach (PCT) to solve the equation (12). Here we are going to construct the isospectral partners of the following potential obtained in Appendix A (we have considered \( p = \lambda = 1 \) for simplicity)

\[
V(x) = \frac{[(a + b - c)^2 - 1]}{4}e^x + \frac{c(c - 2)}{4}e^{-x}
\]  

(35)
corresponding to the mass function

\[
M(x) = \frac{1}{4} \text{sech}^2 \left( \frac{1}{2}x \right)
\]  

(36)
The bound state solutions and eigenstates of the equation (2) are given by (see Appendix A)

\[
\psi_n(x) = \left( \frac{(2n + \sigma + \delta + 1)n!}{\Gamma(n + \sigma + 1)\Gamma(n + \delta + 1)} \right)^{1/2} \frac{e^{(\sigma+\delta)x}}{(1 + e^x)^{\sigma+\delta}} P_n^{(\sigma,\delta)} \left( \frac{1 - e^x}{1 + e^x} \right)
\]  

(37)
and

\[
E_n = n^2 + n(\sigma + \delta + 1) + \frac{(\sigma + 1)(\delta + 1)}{2}, \quad n = 0, 1, 2, ...
\]  

(38)
respectively, where \( b = 1 - a + \sigma + \delta, \ c = 1 + \sigma \) with \( c > \frac{1}{2} \) and \( a + b - c + \frac{1}{2} > 0 \). The seed solution \( U(x) \) and factorization energy \( \mu \) are given by

\[
U(x) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - b)\Gamma(c - a)} \frac{e^{\frac{a+\delta-1}{2}x}}{2F_1 \left( a, b, c, e^x \right)} \frac{e^{(1-\delta)x}}{1 + e^x} P_{n}^{(\sigma,\delta)} \left( \frac{1 - e^x}{1 + e^x} \right)
\]  

(39)
\[
\mu = -ab + \frac{(a + b + 1)c}{2} - \frac{c^2}{2}
\]  

(40)
respectively. The asymptotic behavior of the solution \( U(x) \) given in (39), at both end points \( \pm \infty \) are given by

\[
U(x) \sim (A_1 \alpha + B_1 \beta)e^{-\frac{a+b-\sigma+1}{2}x} + (A_2 \alpha + B_2 \beta)e^{-\frac{a-b+1}{2}x} \quad \text{as} \quad x \to -\infty
\]  

(41)
where

\[
A_1 = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - b)\Gamma(c - a)}, \quad B_1 = \frac{\Gamma(2-c)\Gamma(c - a - b)}{\Gamma(1-a)\Gamma(1 - b)}
\]
\[
A_2 = \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)}, \quad B_2 = \frac{\Gamma(2-c)\Gamma(a + b - c)}{\Gamma(a - c + 1)\Gamma(b - c + 1)}
\]

and

\[
U(x) \sim \alpha e^{\frac{a+\delta-1}{2}x} + \beta e^{(1-\delta)x} \quad \text{as} \quad x \to \infty
\]  

(42)
From these asymptotic behaviors it is clear that \( U(x) \) will unbounded at \( x \to \infty \) if \(|a + b - c| > 1\) and it is unbounded at \( x \to -\infty \) if \( c < 0 \) or \( c > 2\). Therefore \( U(x) \) will nodeless at the finite part of the \( x \) axis if \( A_1\alpha + B_1\beta, A_2\alpha + B_2\beta, \alpha \) and \( \beta \) are all positive and \(|a + b - c| > 1, c < 0 \) or \( c > 2\).

Now we are going to generate isospectral potentials of the potential \((35)\) with various possible spectral modifications.

**Deletion of the initial ground state**: In this case the factorization energy \( \mu \) is equal to the ground state energy \( E_0 \) giving \( a \) and/or \( b = 0 \) and \( U(x) \) becomes the ground state wavefunction \( \psi_0(x) \) which is obtained from \((37)\) as

\[
U(x) = \psi_0(x) \propto e^{\frac{\pi x^2}{4}}
\]

The isospectral partner of \( V(x) \) given in equation \((35)\), is obtained using equation \((13), (35), (36)\) and \((43)\) and is given by

\[
\bar{V}(x) = \frac{c^2 - 1}{4} e^{-x} + \frac{(a + b - c)(2 + a + b - c)}{4} e^x + \frac{a + b}{2}
\]

The above potential \((44)\) can also be obtained from the initial potential \((35)\) by making the changes \( a \to a + 1, b \to b + 1, c \to c + 1\), this property is known as shape invariance \([4]\). Since \( \psi_0(x) \) is bounded solution \( \bar{\psi}_\mu(x) = \frac{x}{\sqrt{\psi_0}} \) is unbounded at \( x \to \pm \infty \), so we have deleted the ground state energy of \( H \) to obtain \( \bar{V}(x) \). Therefore the eigenvalues of \( \bar{H} \) are given by

\[
\bar{E}_n = E_{n+1} = (n + 1)^2 + (n + 1)(a + b) + \frac{c(a + b - c + 1)}{2}, \quad n = 0, 1, 2...
\]

Corresponding bound state wavefunctions of \( \bar{V}(x) \) are obtained using equation \((15)\) as

\[
\bar{\psi}_n(x) \propto \frac{e^{(1 + \frac{c}{2})x}}{(1 + e^x)(\frac{a + b + c + 1}{2}) \sech(\frac{x}{2})} P_n^{(c, a + b - c + 1)}(-\tan \frac{x}{2}), \quad n = 0, 1, 2...
\]

We have plotted the potentials \( V(x) \) given in \((35)\) and \( \bar{V}(x) \) given in \((44)\) for \( a = 5, b = 0, c = 3, \alpha = 1, \beta = 0 \) in figure 1.

![Plot of the potential V(x) (solid line) given in (35) and its first order isospectral partners \( \bar{V}(x) \) (dashed line) given in (44) by deleting the ground state \( E_0 = 4.5 \), we have considered here \( a = 5, b = 0, c = 3, \alpha = 1, \beta = 0 \).](image)
Strictly isospectral potentials: The strictly (strict in the sense that the spectrum of the initial potential and its isospectral potential are exactly the same) isospectral potentials can be generated with the help of those seed solutions which vanish at one of the ends of the x-domain. Now for \( \beta = 0 \) and \( \alpha > 0 \), it is seen from (41) that \( U \) is unbounded at \( x \to \infty \) if \( |a + b - c| > 1 \). But the solution (37) become unbounded for \( a + b - c < -1 \). So we must take \( a + b - c > 1 \). On the other hand from (42) it is observed that \( U(x) \to 0 \) at \( x \to -\infty \) if \( c < 2 \) or \( c > 0 \) but \( \psi_n(x) \) are not normalizable for the values of \( c < 2 \) so we must take \( c > 0 \). So \( U(x) \) vanishes at \( x \to -\infty \) and unbounded at \( x \to \infty \) if \( a + b - c > 1 \) and \( c > 0 \). In this case the spectrum of the isospectral potential as well as original potential are identical i.e. \( E_n = \bar{E}_n \), \( n = 0, 1, 2 \ldots \). Considering the seed solution as

\[
U(x) = \frac{e^{\beta x}}{(1 + e^{\beta x})^{a + b + c}}, \quad a + b - c > 1, c > 0
\]

we have calculated the explicit form of the partner potential using (13) as

\[
\bar{V}(x) = \frac{1}{8} \left[ 2(c(c - 2)e^{-x} + 2((a + b - c)^2 - 1)e^x \right.
\]

\[
- \frac{sech^2(\frac{x}{2})}{2F_1(a, b, c, e^{-x})}\{-4a^22^b(1 + c) \left( 2F_1 \left( 1 + a, 1 + b, 1 + c, \frac{e^x}{1 + e^x} \right) \right)^2 + 4abc 2F_1 \left( a, b, c, \frac{e^x}{1 + e^x} \right) \}
\]

\[
\left( (a + 1)(b + 1) \left( 2F_1 \left( 2 + a, 2 + b, 2 + c, \frac{e^x}{1 + e^x} \right) \right)^2 - (1 + c)sinh \left( 2F_1 \left( a + 1, b + 1, c + 1, \frac{e^x}{1 + e^x} \right) \right) \right)^2 \]

\[
- 8c^2(1 + c)cosh^2x \left( 2F_1 \left( a, b, c, \frac{e^x}{1 + e^x} \right) \right)^2 ((a + b)cosh(\frac{x}{2}) + (1 + a + b - 2c)sinh(\frac{x}{2})) \}\]

(47)

In particular for \( a = 3, b = 5, c = 4, \alpha = 1, \beta = 0 \) and using (55), (13) we have obtained

\[
V(x) = \frac{1}{4}(23coshx + 7sinhx)
\]

\[
\bar{V}(x) = \frac{15e^{-x}}{4} + 2e^x + \frac{4c^2(1 + c)^2sech^4x}{(1 + e^x)^2}
\]

(48)

respectively, which are plotted in figure 2. In this case eigenfunctions and eigenvalues of the above partner potential \( \bar{V}(x) \) are given by

\[
\bar{\psi}_n(x) \propto \frac{e^{3x}((2 + e^x)(n + 7)P_n^{(4, 4)}(-\tanh(\frac{x}{2})) + (1 + e^x)(5 + 3e^x)P_n^{(3, 3)}(-\tanh(\frac{x}{2}))}{(1 + e^x)^6(2 + e^x)sech^2(\frac{x}{2})}
\]

(49)

respectively.

Creation of a new ground state: In this case we shall consider \( \mu < E_0 \). The new state can be created below the ground state of the initial potential with the help of those seed solutions which satisfies the following two conditions: (i) it should be nodeless throughout the x-domain and (ii) it should be unbounded at both the end points of the domain of definition of the given potential \( V(x) \). From the asymptotic behaviors of the seed solution \( U \), given in equations (41) and (42) we have, for \( |a + b - c| > 1 \) together with either \( c < 0 \) or \( c > 2 \), the above two conditions are satisfied. But to get \( \psi_n(x) \) as physically acceptable, we shall take \( c > 2 \) and \( a + b - c > 1 \). In this case the spectrum of the partner potential is \( \{\mu, \bar{E}_n, n = 0, 1, 2 \ldots \} \), \( E_n \) being the energy eigenvalues of the original potential \( V(x) \) given in (55). Corresponding bound state wavefunctions are \( \{\psi_{\mu}(x), \bar{\psi}_n(x), n = 0, 1, 2, \ldots \} \), where \( \psi_{\mu} \) and \( \bar{\psi}_n \) are given by (29) and (15) respectively. For \( a + b - c > 1 \), \( c > 2 \) and the seed solution \( U \) given in (39), the general expression of the isospectral potential becomes too involved so instead of giving the explicit expression of the partner potential we have plotted in figure 3 the original potential \( V(x) \) given in (55) and its partner potential \( \bar{V}(x) \) (which is obtained using (13)).
considering the particular values $a = 2.8, b = 20, c = 4.4$ and $\alpha = \beta = 1$. In this case the energy eigenvalues of $\tilde{V}(x)$ are given by

$$\tilde{E}_n = \{-13.32, E_n, n = 0, 1, 2...\} = \{-13.32, n^2 + 22.8n + 42.68, n = 0, 1, 2...\}$$

(50)

Corresponding eigenfunctions can be obtained using the formulae (23) and (15).

FIG. 2: Plot of the potential $V(x)$ (solid line) and its first order isospectral partner (dashed line) given in (48).

FIG. 3: Plot of the potential $V(x)$ (solid line) given in (35) and its first order isospectral partner $\tilde{V}$ (dashed line) by inserting the state $\mu = -13.32$. We have considered here $a = 2.8, b = 20, c = 4.4, \alpha = \beta = 1$. 
III. SECOND ORDER INTERTWINNING

Now we assume the existence of a second order intertwining operator

\[ \mathcal{L} = \frac{1}{M} \frac{d^2}{dx^2} + \eta(x) \frac{d}{dx} + \gamma(x) \]  

(51)

where \( \eta(x), \gamma(x) \) are to be determined. Substitution of this intertwiner in equation \( \mathbb{I} \) and comparison of the coefficients of like order derivatives leads to a set of following equations

\[ \tilde{V} = V + 2\eta' + \frac{M'}{M} \eta - \frac{3M'^2}{M^3} + \frac{2M''}{M^2} \]  

(52)

\[ (\tilde{V} - V) \eta = \frac{2V'}{M} + \frac{2\gamma'}{M} + \frac{\eta''}{M} - \frac{M''}{M^2} + 
\frac{2M'\eta}{M^2} - \frac{2M'\eta}{M^2} - \frac{2\gamma'}{M} + \frac{M''}{M^2} \]  

(53)

\[ (\tilde{V} - V) \gamma = \frac{V''}{M} + \frac{V''}{M} + \frac{\gamma''}{M} - \frac{M''}{M^2} \]  

(54)

Now using (52) the equations and (54) reads

\[ \frac{2\eta'}{M} + \frac{M'\eta^2}{M^2} - \frac{3\eta M'^2}{M^3} + \frac{2M''}{M^2} + \frac{2M''}{M^2} - \frac{2\gamma'}{M} + \frac{M''}{M^2} \]  

(55)

\[ + \frac{2M'\eta}{M^2} - \frac{\eta''}{M} - \frac{M''}{M^2} - \frac{2M'\eta}{M^2} - \frac{M''}{M^2} - \frac{M''}{M^2} - \frac{M''}{M^2} = 0 \]

and

\[ \gamma \left( \frac{2\eta'}{M} + \frac{M'\eta^2}{M^2} + \frac{2M''}{M^2} \right) + \frac{M'\gamma'}{M^2} - \frac{\gamma''}{M} - \frac{V''}{M} = 0 \]  

(56)

respectively. Equation \( \mathbb{I} \) can be integrated to obtain

\[ \gamma = \frac{M\eta^2}{2} + \frac{M'\eta}{2} - \frac{\eta'}{2} - V + \frac{M^2}{2M} - \frac{M''}{2M} + C_1 \]  

(57)

where \( C_1 \) is an arbitrary constant. Using (57) in (55) we obtain

\[ \frac{\eta''}{2M} + M' \eta M' + \frac{\eta M''}{M} - \frac{3\eta M'^2}{M^3} - \frac{2M''}{M^2} + \frac{M''}{M^2} + \frac{M''}{M^2} - \frac{2\gamma'}{M} + \frac{M''}{M^2} \]  

(58)

\[ + \frac{2M'\eta}{M^2} - \frac{\eta''}{M} - \frac{M''}{M^2} - \frac{2M'\eta}{M^2} + 2C_1 \eta' + \frac{2C_1\eta M'}{3M^2} - \frac{M^2}{M^2} - \frac{M^2}{M^2} - \frac{M^2}{M^2} - \frac{M''}{M^3} - \frac{M''}{M^3} - \frac{M''}{M^3} = 0 \]

Multiplying by \( \left( M + \frac{M'}{M} \right) \), above equation \( \mathbb{I} \) can be integrated to obtain

\[ \frac{\eta''}{2} - \frac{\eta^2}{4} + \frac{M'^2}{4} - \frac{M'\eta M^2}{M} - M^2 V + C_1 M^2 + \frac{C_1 M^2}{M^3} + \frac{C_1 M'^2}{M^3} + \frac{M'' V}{M^2} - \frac{M'' V}{M^2} - \frac{M'' M''}{M^3} + \frac{M'' M''}{M^3} - \frac{M'' M''}{M^3} \]  

(59)

\[ + \frac{M'^4}{4M^4} + \frac{M'^4}{2M^4} + C_2 = 0 \]
where \( C_2 \) is the constant of integration. For a given potential \( V(x) \), the new potential \( \bar{V}(x) \) and \( \gamma(x) \) can be obtained from (52) and (57) if the solution \( \eta(x) \) of (59) is known. To obtain \( \eta(x) \) we take the Ansatz

\[
\eta' = M\eta^2 + 2\left( \eta + \frac{M'}{M^2} \right) \tau + \frac{M'}{M} \eta + \frac{2M'^2}{M^3} - \frac{M''}{M^2} + \xi
\]

(60)

where \( \xi \) is a constant to be determined and \( \tau \) is a function of \( x \). Using above ansatz in equation (59) we obtain the following equation

\[
M \left( \frac{\tau'}{M} + \frac{\tau^2}{M} - \frac{M'\tau}{M^2} - V + C_1 - \frac{\xi}{2} \right) \eta^2 + \frac{2M'}{M} \left( \frac{\tau'}{M} + \frac{\tau^2}{M} - \frac{M'\tau}{M^2} - V + C_1 - \frac{\xi}{2} \right) \eta \\
+ \frac{M'^2}{M^3} \left( \frac{\tau'}{M} + \frac{\tau^2}{M} - \frac{M'\tau}{M^2} - V + C_1 - \frac{\xi}{2} \right) + \left( C_2 - \frac{\xi^2}{4} \right) = 0
\]

(61)

Since equation (61) is valid for arbitrary \( \eta \), the coefficients of each power of \( \eta \) must vanish, which give \( \xi^2 = 4C_2 \) and

\[
\frac{\tau'}{M} + \frac{\tau^2}{M} - \frac{M'\tau}{M^2} - V + C_1 - \frac{\xi}{2} = 0
\]

(62)

Now defining \( \mu = C_1 - \frac{\xi}{2} \), the above equation can be written as

\[
\frac{\tau'}{M} + \frac{\tau^2}{M} - \frac{M'\tau}{M^2} = V - \mu \,, \quad \mu = C_1 - \frac{\xi}{2} = C_1 \mp \sqrt{C_2}
\]

(63)

The equation (63) is a Riccati equation which can be linearized by defining \( \tau = \frac{U'}{U} \). Making this change in equation (63) we obtain

\[
- \frac{1}{M} u'' - \left( \frac{1}{M} \right)' U' + VU = \mu U
\]

(64)

Depending on whether \( C_2 \) is zero or not, \( \xi \) vanishes or takes two different values \( \pm \sqrt{C_2} \). If \( C_2 = 0 \), we need to solve one equation of the form (63) and then the equation (60) for \( \eta(x) \). If \( C_2 \neq 0 \), there will be two different equations of type (63) for two factorization energies \( \mu_{1,2} = C_1 \mp \sqrt{C_2} \). Once we solve them, it is possible to construct algebraically a common solution \( \eta(x) \) of the corresponding pair of equations (60). There is an obvious difference between the real case with \( C_2 > 0 \) and the complex case \( C_2 < 0 \); thus there follows a natural scheme of classification for the solutions \( \eta(x) \) based on the sign of \( C_2 \). In our present article we shall not discuss the case \( C_2 = 0 \).

(i) **Real Case** \( (C_2 > 0) \)
Here we have \( \mu_{1,2} \in \mathbb{R}, \mu_1 \neq \mu_2 \). Let the corresponding solutions of the Riccati equation (63) be denoted by \( \tau_{1,2}(x) \). Now the associated pair of equations (60) become

\[
\eta' = M\eta^2 + 2\left( \eta + \frac{M'}{M^2} \right) \tau_1 + \frac{M'}{M} \eta + \frac{2M'^2}{M^3} - \frac{M''}{M^2} + \mu_2 - \mu_1
\]

(65)

and

\[
\eta' = M\eta^2 + 2\left( \eta + \frac{M'}{M^2} \right) \tau_2 + \frac{M'}{M} \eta + \frac{2M'^2}{M^3} - \frac{M''}{M^2} + \mu_1 - \mu_2
\]

(66)
respectively. Subtracting (65) from (66) and using (64) we obtain \( \eta(x) \) as

\[
\eta(x) = \frac{\mu_1 - \mu_2}{\tau_1 - \tau_2} - \frac{M'}{M^2} = - \frac{W'(U_1, U_2)}{MW(U_1, U_2)}
\] (67)

where \( U_1, U_2 \) are the seed solutions of the equation (64) corresponding to the factorization energy \( \mu_1 \) and \( \mu_2 \) respectively and \( W(U_1, U_2) = U_1U_2 - \mathcal{U}_1U_2 \), is the Wronskian of \( U_1 \) and \( U_2 \).

Now it is clear from (52) and (67) that mass function \( M(x) \) is nonsingular and does not vanish at the finite part of the \( x \)-domain, so that the new potential \( \bar{V}(x) \) has no extra singularities (i.e. the number of singularities in \( V \) and \( \bar{V} \) remains the same) if \( W(U_1, U_2) \) is nodeless there. The spectrum of \( \mathcal{H} \) depends on whether or not its two eigenfunctions \( \tilde{\psi}_{\mu_1}, \tilde{\psi}_{\mu_2} \) which belongs as well to the kernel of \( \mathcal{L}^\dagger \) can be normalized [21], namely

\[
\mathcal{L}^\dagger \tilde{\psi}_{\mu_j} = 0 \quad \text{and} \quad \mathcal{H} \tilde{\psi}_{\mu_j} = \mu_j \tilde{\psi}_{\mu_j}, \quad j = 1, 2
\] (68)

where \( \mathcal{L}^\dagger \) is the adjoint of \( \mathcal{L} \) and is given by [20]

\[
\mathcal{L}^\dagger = \frac{1}{M} \frac{d^2}{dx^2} - \left( \eta + \frac{2M'}{M^2} \right) \frac{d}{dx} + \left( \frac{2M'^2}{M^3} - \frac{M''}{M^2} - \eta' + \gamma \right)
\]

For \( j = 1 \) the explicit expression of the two equation mentioned in (68) are

\[
\frac{1}{M} \frac{d^2 \psi_{\mu_1}}{dx^2} - \left( \eta + \frac{2M'}{M^2} \right) \frac{d\psi_{\mu_1}}{dx} + \left( \frac{2M'^2}{M^3} - \frac{M''}{M^2} - \eta' + \gamma \right) \psi_{\mu_1} = 0
\] (69)

and

\[
- \frac{1}{M} \psi''_{\mu_1} - \left( \frac{1}{M} \right)^' \psi'_{\mu_1} + (\bar{V} - \mu_1) \psi_{\mu_1} = 0
\] (70)

respectively. Adding (69) from (70) we obtain

\[
- \left( \frac{M'}{M^2} + \eta \right) \frac{d\psi_{\mu_1}}{dx} + \left( \bar{V} - \mu_1 + \frac{2M'^2}{M^3} - \eta' + \gamma - \frac{M''}{M^2} \right) \psi_{\mu_1} = 0
\] (71)

Substituting the values of \( \bar{V} \) and \( \gamma \) from (52) and (57) with \( 2C_1 = \mu_1 + \mu_2 \), in the above equation (71), we get

\[
\frac{d}{dx} (log \psi_{\mu_1}) = \frac{\eta' + 3\eta \frac{M'}{M^2} + \frac{M''}{M^2} + M\eta^2 + 2(C_1 - \mu_1)}{2(\eta + \frac{M'}{M^2})}
\] (72)

Now using our ansatz (60) in (72) and then integrating we obtain

\[
\psi_{\mu_1} \propto \frac{M (\eta + \frac{M'}{M^2})}{U_1} \propto \frac{MU_2}{W(U_1, U_2)}
\] (73)

Above procedure can be applied to obtain \( \tilde{\psi}_{\mu_2} \) as

\[
\tilde{\psi}_{\mu_2} \propto \frac{\eta M + \frac{M'}{M^2}}{U_2} \propto \frac{MU_1}{W(U_1, U_2)}
\] (74)

If both \( \tilde{\psi}_{\mu_1,2} \) are normalizable then we get the maximal set of eigenfunctions of \( \mathcal{H} \) as \( \{ \tilde{\psi}_{\mu_1}, \tilde{\psi}_{\mu_2}, \tilde{\psi}_n \propto \mathcal{L} \psi_n \} \).

Among the several spectral modifications which can be achieved through the real second order SUSYQM for
PDMSE, some cases are worth to be mentioned.

**Deletion of first two energy levels**: For \( \mu_1 = E_0 \) and \( \mu_2 = E_1 \) the two solutions of equation (64) are the normalizable solutions of equation (63) i.e., \( U_1 = \psi_0(x) \) and \( U_2 = \psi_1(x) \) respectively. It turns out that the Wronskian is nodeless but two solutions \( \bar{\psi}_{\mu_1} \) and \( \bar{\psi}_{\mu_2} \) are non-normalizable. Thus \( Sp(\mathcal{H}) = Sp(\mathcal{H}) \setminus \{E_0, E_1\} = \{E_2, E_3, E_4, \ldots\} \), i.e., the two levels \( E_0 \) and \( E_1 \) are deleted to generate \( \bar{V} \).

**Isospectral transformations**: If we take \( \mu_1 < \mu_2 < E_0 \) and choose \( U_1 \) and \( U_2 \) such way that either \( U_{1,2}(x_l) = 0 \) or \( U_{1,2}(x_r) = 0 \), \( x_l \) and \( x_r \) being the end points of the domain of definition of \( V(x) \), then the Wronskian \( W(U_1, U_2) \) vanishes at \( x_l \) or \( x_r \). Hence \( \bar{\psi}_{\mu_1} \) and \( \bar{\psi}_{\mu_2} \) become non-normalizable so that \( Sp(\mathcal{H}) = Sp(\mathcal{H}) \). Thus the normalizable solutions of equation (64) are the normalizable solutions of equation (2) i.e.,

\[
(64) \quad \text{are the normalizable solutions of equation (2) i.e.,}
\]

if we take \( \mu_1 < \mu_2 < E_0 \) and choose \( U_1 \) and \( U_2 \) such way that either \( U_{1,2}(x_l) = 0 \) or \( U_{1,2}(x_r) = 0 \), \( x_l \) and \( x_r \) being the end points of the domain of definition of \( V(x) \), then the Wronskian \( W(U_1, U_2) \) vanishes at \( x_l \) or \( x_r \). Hence \( \bar{\psi}_{\mu_1} \) and \( \bar{\psi}_{\mu_2} \) become non-normalizable so that \( Sp(\mathcal{H}) = Sp(\mathcal{H}) \). Thus the normalizable solutions of equation (64) are the normalizable solutions of equation (2) i.e.,

\[
(64) \quad \text{are the normalizable solutions of equation (2) i.e.,}
\]

if we take \( \mu_1 < \mu_2 < E_0 \) and choose \( U_1 \) and \( U_2 \) such way that either \( U_{1,2}(x_l) = 0 \) or \( U_{1,2}(x_r) = 0 \), \( x_l \) and \( x_r \) being the end points of the domain of definition of \( V(x) \), then the Wronskian \( W(U_1, U_2) \) vanishes at \( x_l \) or \( x_r \). Hence \( \bar{\psi}_{\mu_1} \) and \( \bar{\psi}_{\mu_2} \) become non-normalizable so that \( Sp(\mathcal{H}) = Sp(\mathcal{H}) \).

**Creation of two new levels below the ground state**: For \( \mu_2 < \mu_1 < E_0 \) and choosing \( U_1 \) and \( U_2 \) in such way that \( U_2 \) has exactly one node and \( U_1 \) is nodeless then the Wronskian \( W(U_1, U_2) \) becomes nodeless, also two wavefunctions \( \bar{\psi}_{\mu_1} \) and \( \bar{\psi}_{\mu_2} \) are normalizable. Therefore the spectrum of \( \mathcal{H} \) becomes \( Sp(\mathcal{H}) = Sp(\mathcal{H}) \bigcup \{\mu_1, \mu_2\} \) i.e. two new levels have been inserted to the spectrum of \( V(x) \) to obtain \( \bar{V}(x) \).

(ii) **Complex case** (\( C_2 < 0 \))

For \( C_2 < 0 \) the two factorization energies \( \mu_1 \) and \( \mu_2 \) become complex. In order to construct real \( \bar{V} \) we shall choose \( \mu_1 \) and \( \mu_2 \) as complex conjugate to each other i.e., \( \mu_1 = \mu \in \mathbb{C} \) and \( \mu_2 = \bar{\mu} \). For the same reason we shall take \( \tau_1(x) = \tau(x) \) and \( \tau_2(x) = \bar{\tau}(x) \). Hence the real solution \( \eta(x) \) of (63) generated from the complex \( \tau(x) \) of (63) becomes

\[
\eta(x) = \frac{\mu - \bar{\mu}}{\tau - \bar{\tau}} - \frac{M'}{M^2} \frac{\text{Im}(\mu)}{\text{Im}(\tau)} - \frac{M'}{M^2} = -\frac{W'(U, \bar{U})}{MW(U, \bar{U})}
\]

(75)

Defining \( w(x) = \frac{W(U, \bar{U})}{M(\mu - \bar{\mu})} \), \( \eta(x) \) becomes

\[
\eta(x) = -\frac{w'}{Mw} - \frac{M'}{M^2}
\]

(76)

For the factorization energies \( \mu \) and \( \bar{\mu} \) the equation (63) becomes

\[
-\frac{1}{M} U'' - \left(\frac{1}{M}\right)' U' + VU = \mu U \quad \text{and} \quad -\frac{1}{M} \bar{U}'' - \left(\frac{1}{M}\right)' \bar{U}' + \bar{V}\bar{U} = \bar{\mu} \bar{U}
\]

Multiplying first equation by \( \bar{U} \) and second equation by \( U \) and then subtracting we obtain

\[
\frac{W'(U, \bar{U})}{M(\mu - \bar{\mu})} - \frac{M'W(U, \bar{U})}{M^2(\mu - \bar{\mu})} = |U|^2
\]

(77)

Using above relation (77) we have

\[
w'(x) = \frac{W(U, \bar{U})}{M(\mu - \bar{\mu})} - \frac{M'W(U, \bar{U})}{M^2(\mu - \bar{\mu})} = |U|^2
\]

(78)

which implies that \( w(x) \) is a non-decreasing function. So it is sufficient to choose

\[
\lim_{x \to x_l} U = 0 \quad \text{or} \quad \lim_{x \to x_r} U = 0
\]

(79)

for the Wronskian \( W \) to be nodeless. It is to be noted here that in this case we can only construct potentials which are strictly isospectral with the initial potential.
A. Second-order intertwining and type A 2-fold SUSY

The second-order intertwiner $L$ in equation (51) coincides with $P_2$ given in equation (28) if one takes
\[ \eta(x) = \frac{2W(x)}{M(x)} - \frac{M'(x)}{M(x)^2} \]  
(80)

It is now easy to verify that for this $\eta(x)$, $(V^+ - V^-)$ given in (29) (with $N = 2$) agree with $(\bar{V} - V)$ given in (52).

Now it is to be shown that the Hamiltonian $H$ given in equation (2) admits two eigenfunctions $U_{1,2}(x)$ corresponding to two factorization energies $\mu_{1,2}$ respectively i.e.,
\[ H U_i(x) = \mu_i U_i(x), \quad i = 1, 2 \]  
(81)

will belong to type A 2-fold SUSY in PDM background (in constant mass scenario this was already proved in ref.[18]). For this we define
\[ z(x) = \frac{U_2(x)}{U_1(x)}, \quad W_2^-(z) \equiv W(z) = -\ln U_1(x) \]  
(82)

For this $W(z)$, it is evident that the gauged Hamiltonian $\hat{H}_2^-$ defined by
\[ \hat{H}_2^- = e^W H e^{-W} \]  
(83)

must be diagonal in the basis $\nu_2 = \langle 1, z \rangle$ because of the assumption (81) and the choice (82). From equation (81), its immediate consequence (for $N = 2$) is
\[ B(z) = \frac{z''(x)}{M(x)} - \frac{z'(x)M'(x)}{M(x)^2} - \frac{2z'(x)^2}{M(x)} \frac{dW(z)}{dz} \]  
(84)

Using equations (82) and (81) in the above equation (81) it can be shown that
\[ B(z) = (\mu_1 - \mu_2) z(x) \]  
(85)

For this value of $B(z)$ it is also easy to verify that the expression of $\eta(x)$ in equation (80) and (67) are same. Now it is evident that the gauged Hamiltonian $\hat{H}_2^-$ preserves the vector space $\nu_2 = \langle 1, z \rangle$. Hence it is possible to get type A 2-fold SUSY system $(H, \hat{H}, L)$ following the prescription given in ref. [14], with the choice of $z(x)$, $W(z)$ and $\hat{H}_2^-$ given by (82) and (25) respectively.

B. Example of second-order intertwining for real factorization energies

It may be emphasized that the results mentioned in section III are most general and valid for any potential $V(x)$. However to illustrate the above procedure with the help of an example we shall need non-normalizable solutions of (64) (which is similar to equation (12) but with two factorization energies) corresponding to a particular mass function $M(x)$. To illustrate the second order intertwining with an example we have considered the potential (35) as an initial potential. Corresponding seed solution for the factorization energy $\mu = \mu_1$ which is obtained in Appendix A, is
\[ \mu_1 = -ab + \frac{(a + b + 1)c}{2} - \frac{c^2}{2} \]  
(86)
\[ \mathcal{U}_1(x) = \alpha \frac{e^{\frac{\beta}{2}x}}{(1 + e^x)^{\frac{3}{2}}} 2F_1 \left( a, b, c, \frac{e^x}{1 + e^x} \right) + \beta \frac{e^{\frac{1 - \beta}{2}x}}{(1 + e^x)^{\frac{3}{2}}} 2F_1 \left( a - c + 1, b - c + 1, 2 - c, \frac{e^x}{1 + e^x} \right) \] 

We notice that the potential \[35\] and corresponding Hamiltonian are invariant under the transformation \( a \rightarrow a + \nu \) and \( b \rightarrow b - \nu \), \( \nu \in \mathbb{R} - \{0\} \). But the solution \[39\] of the corresponding Schrödinger equation changes to

\[ \mathcal{U}_2(x) = \alpha \frac{e^{\frac{\beta}{2}x}}{(1 + e^x)^{\frac{3}{2}}} 2F_1 \left( a + \nu, b - \nu, c, \frac{e^x}{1 + e^x} \right) + \beta \frac{e^{\frac{1 - \beta}{2}x}}{(1 + e^x)^{\frac{3}{2}}} 2F_1 \left( a + \nu - c + 1, b - \nu - c + 1, 2 - c, \frac{e^x}{1 + e^x} \right) \] 

and the corresponding factorization energy is given by

\[ \mu_2 = -ab + \frac{(a + b + 1)c}{2} - \frac{c^2}{2} + \nu(a - b) + \nu^2 \] 

Thus the general solutions of the equation \[63\] for the two factorization energies \( \mu_1 \) and \( \mu_2 \), are given by \[87\] and \[88\] respectively. The asymptotic behaviors of the seed solution \( \mathcal{U}_2 \) remains same as \( \mathcal{U}_1 \), which are given in \[11\] and \[12\].

**Deletion of first two energy levels:** Let us take \( \mu_1 = E_0 \) and \( \mu_2 = E_1 \), \( \mathcal{U}_1 = \psi_0(x) \) and \( \mathcal{U}_2 = \psi_1(x) \) which are given in \[37\]. The Wronskian \( W(\mathcal{U}_1, \mathcal{U}_2) \) is given by

\[ W(\mathcal{U}_1, \mathcal{U}_2) \propto \frac{e^{(c+1)x}}{(1 + e^x)^{\alpha+b+3}} \] 

which is nodeless and bounded in \((-\infty, \infty)\) as \( c > -\frac{1}{2} \) and \( a + b - c + \frac{1}{2} > 0 \) (these conditions are mentioned at the end of the Appendix A). The second-order SUSY partner of \( V(x) \) is obtained using equation \[52\] and is given by

\[ \tilde{V}(x) = \frac{1}{4} \left[ (c^2 - 2c(a + b + 2) + (a + b + 1)(a + b + 3)) e^x + c(c + 2)e^{-x} + 4(a + b + 1) \right] \] 

Clearly the eigenfunctions \( \tilde{\psi}_{\mu_1} \propto \frac{\mathcal{M}_1}{W(\mathcal{U}_1, \mathcal{U}_2)} \) and \( \tilde{\psi}_{\mu_2} \propto \frac{\mathcal{M}_2}{W(\mathcal{U}_1, \mathcal{U}_2)} \) of \( \tilde{\mathcal{H}} \) associated to \( \mu_1 = E_0 \) and \( \mu_2 = E_1 \) are not normalizable since

\[ \lim_{x \rightarrow -\infty, \infty} \tilde{\psi}_{\mu_1,2}(x) = \infty \] 

Thus \( \text{Sp}(\tilde{\mathcal{H}}) = \text{Sp}(\mathcal{H}) - \{E_0, E_1\} = \{E_2, E_3, \ldots\} \).

In particular taking \( a = 5, b = 0, c = 3 \) we have plotted the potential \( V(x) \) and its second-order SUSY partner \( \tilde{V}(x) \) given in \[65\] and \[91\] respectively, in figure 4.

**Strictly isospectral potentials:** The strictly isospectral partner potentials can be constructed by creating two new energy levels in the limit when each seed vanishes at one of the ends of the \( x \)-domain. Now from the asymptotic behaviors of the seed solutions, we note that both the seed solutions vanish at \( x \rightarrow -\infty \) for \( \beta = 0, \alpha > 0 \) if \( a + b - c > 1 \) and \( c > 0 \). Considering \( \beta = 0, \alpha = 1 \) in \[87\] and \[88\] we take two seed solution as

\[ \mathcal{U}_1(x) = \frac{e^{\frac{\beta}{2}x}}{(1 + e^x)^{\frac{3}{2}}} 2F_1 \left( a, b, c, \frac{e^x}{1 + e^x} \right) \] 

and

\[ \mathcal{U}_2(x) = \frac{e^{\frac{\beta}{2}x}}{(1 + e^x)^{\frac{3}{2}}} 2F_1 \left( a + \nu, b - \nu, c, \frac{e^x}{1 + e^x} \right) \]
FIG. 4: Plot of the original potential (solid line) for $a = 5, b = 0, c = 3$ and its first-order SUSY partner (dashed line) by deleting the ground state $E_0 = 4.5$ and second-order SUSY partner (dotted line) by deleting two successive states $E_0 = 4.5, E_1 = 10.5$.

Since $\hat{U}_{1,2}(x) \to 0$ at $x \to -\infty$, from the expressions (73) and (74) we can conclude that

$$\lim_{x \to -\infty} \bar{\psi}_{\mu_{1,2}}(x) = \infty$$

which implies that $\mu_{1,2}$ does not belong to $Sp(\mathcal{H})$ i.e. $\bar{V}(x)$ is strictly isospectral to $V(x)$. Here the general expression of the partner potential is too involved so instead of giving the explicit expression we have considered particular values $a = 3, b = 5, c = 4, \nu = 1, \alpha = 1, \beta = 0$. Corresponding expression of the partner potential and its energy spectrum are

$$\bar{V}(x) = 1 + \frac{3}{4}(9\cosh x - 7 \sinh x), \quad E_n = \bar{E}_n = n^2 + 8n + 10, \quad n = 0, 1, 2...$$

respectively. In figure 5, we have plotted the initial potential, its first and second-order strictly isospectral partner potentials for the parameter values $a = 3, b = 5, c = 4, \alpha = 1, \beta = 0, \nu = 1$. 

FIG. 5: Plot of the original potential (solid line) and its first-order (dashed line) and second-order (dotted line) SUSY partner by making the isospectral transformation for $a = 3, b = 5, c = 4, \alpha = 1, \beta = 0, \nu = 1$. 
Creation of two new levels below the ground state: Two energy levels can be created taking $\mu_2 < \mu_1 < E_0$ and using those seed solutions $U_1$ and $U_2$ for which the Wronskian become nodeless. In this case the expressions of the Wronskian contains several Hypergeometric function, so it is very difficult to mention the range of $a, b, c$ and $\nu$ for which it is nodeless. In particular for $a = 2.8, b = 20, c = 4.4, \alpha = 1, \beta = 1$ we have the Wronskian is found to be nodeless. For the same values of $a, b, c$ we have plotted the potential and its second order partner in figure 6. The second-order isospectral partner is obtained using equation (52).

![Diagram](Image)

FIG. 6: Plot of the original potential (solid line) for $a = 2.8, b = 20, c = 4.4$ and its first-order SUSY partner (dashed line) by creating a new level $\mu_1 = -13.32$ and second-order SUSY partner (dotted line) by creating two new levels $\mu_1 = -13.32, \mu_2 = -85.32$. 
C. Example of second order intertwining for complex factorization energies

As mentioned earlier, in this case we can only construct the strictly isospectral partner potentials. The complex factorization energy \( \mu_1 \) and \( \mu_2 \) given by equation (86) and (89), can be made conjugate to each other in several ways. One of the way is by making following restrictions on \( a, b, c, \nu \):

\[
\begin{align*}
    c &\in \mathbb{R}, \\
    \text{Im}(a) &= -\text{Im}(b), \\
    \nu &= \text{Re}(b) - \text{Re}(a).
\end{align*}
\]

But in order to keep the initial potential real we have to made two more restrictions e.g. \( \text{Re}(a) + \text{Re}(b) - c > 1 \) and \( c > 2 \). In particular taking \( a = 6.1 - 5i, b = 8 + 5i, c = 4.1, \nu = 1.9 \) we have two factorization energy \( \mu_1(= \mu) = -51.25 + 9.5i \) and \( \mu_2(= \bar{\mu}) = -51.25 - 9.5i \). For these values of \( a, b, c, \nu \) and \( \alpha = 1, \beta = 0 \) the seed solution \( U \) becomes

\[
U(x) = \frac{e^{2.05x}}{(1+e^x)^{7.55}} 2F1 \left( \begin{array}{c} 6.1 - 5i, 8 + 5i, 4.1 \\ \frac{e^x}{1+e^x} \end{array} \right) \tag{94}
\]

Clearly \( U(-\infty) = 0 \) and \( |U| \to \infty \) as \( x \to \infty \) so this seed \( U \) and its conjugate \( \bar{U} \) can be used to obtain the second-order SUSY partner potential \( \bar{V}(x) \) with the help of equations (52) and (75). In figure 7 we have plotted the initial potential \( V(x) \) given in (35) and its isospectral partner \( \bar{V}(x) \) for the parameter values mentioned earlier.

![Figure 7](image_url)

FIG. 7: Plot of the original potential (solid line) for \( a = 6.1 - 5i, b = 8 + 5i, c = 4.1, \nu = 1.9, \mu_1 = -51.25 + 9.5i, \mu_2 = -51.25 - 9.5i \) and its second-order isospectral partner (dashed line).

IV. SUMMARY AND OUTLOOK

In this article we have discussed the possibilities for designing quantum spectra of position dependent mass Hamiltonians offered by the intertwining technique. For doing this, we start with the non-normalizable solution of position dependent mass Schrödinger equation with the initial potential (obtained by using the point canonical transformation approach). To generate spectral modifications by first order intertwining, we have used solutions to the position dependent mass Schrödinger equation corresponding to factorization energy (not belonging to the physical spectrum of the initial problem) less than or equal the ground state energy in order to avoid singularity in the isospectral partner potential provided the mass function is not singular and is not equal to zero in the real line. Thus it is possible to generate isospectral partner potentials (a) with the ground state of the original potential deleted (b) with a new state created below the ground state of the original potential (c) with the spectrum of the original potential unaffected. In ref [10], the first order intertwining technique was illustrated by considering the free particle case.
In the case of second order intertwining, instead of using the iterative method used in [10], the second order intertwiner is constructed directly by taking it as second order linear differential operator with unknown coefficients which are functions of $x$. The main advantage of this construction is that one can generate second-order isospectral partner potentials directly from the initial potential without generating first-order partner potentials. The apparently intricate system of equations arising from the intertwining relationship is solved for the coefficients by taking an ansatz. In this case the spectral modifications are done by taking appropriately chosen factorization energies which may be real or complex. For real unequal factorization energies, it is possible to generate potentials (a) with deletion of first two energy levels (b) with two new levels embedded below the ground state of the original potential (c) with identical spectrum as of the original potential. For complex factorization energies, it is shown how to obtain strictly isospectral potentials. It must be mentioned here that in all the above cases the conditions for having spectral modifications remain the same as in the case of constant mass scenario [21] provided the mass function $M$ is nonsingular and is not equal to zero in the finite part of the real line.

In this article, the equivalence of our formalism to type A $N$-fold ($N = 1, 2$) SUSY in PDM background is shown. Also, it is shown that an arbitrary one body quantum PDM Hamiltonian which admits two eigenfunctions in closed form belongs to type A 2-fold SUSY as was previously done in constant mass scenario [18].

Some of the interesting issues to be investigated in future are
(i) to obtain spectral changes that appear above the ground state energy of the initial potential. Specifically, how to create/delete a pair of levels between any two neighboring initial ones, how to move an arbitrary level or delete an arbitrary level. Specially interesting will be the possibility of embedding a single level at any arbitrary position.
(ii) to obtain spectral modifications when the two factorization energies are equal.

Appendix A: Construction of exactly solvable effective potential via PCT

In order to find the general (unbounded) solution of the equation (12) we shall use PCT method in PDM background [22] to solve this equations. Let us find the solution of equation (12) of the form

$$U(x) = f(x) F(a, b, c, g)$$

where $f(x), g(x)$ are two function of $x$ to be determined and $F(a, b, c, g)$ is the Hypergeometric function which satisfies second order differential equation of the type

$$\frac{d^2 F}{dg^2} + Q(g) \frac{dF}{dg} + R(g)F = 0,$$  

with

$$Q(g) = \frac{c - (a + b + 1)g}{g(1 - g)}$$

and

$$R(g) = -\frac{ab}{g(1 - g)}$$

Comparing equation (A2) and (A3) we get

$$Q(g(x)) = \frac{g''}{g} + \frac{2f'}{fg} - \frac{M'}{Mg}$$

and

$$R(g(x)) = \frac{f''}{fg} + (\mu - V)\frac{M}{Mfg} - \frac{M'f'}{Mfg^2}$$

After simplification of the above equation (A4) we obtain

$$f(x) \propto \sqrt{\frac{M}{g}} \ exp\left(\frac{1}{2} \int^{g(x)} Q(t)dt\right)$$
and

\[ \mu - V = \frac{g'''}{2Mg'} - \frac{3}{4M} \left( \frac{g''}{g'} \right)^2 + \frac{g'^2}{M} \left( R - \frac{1}{2} \frac{dQ}{dg} \frac{Q^2}{4} \right) - \frac{M''}{2M^2} + \frac{3M'^2}{4M^3} \]  

(A6)

respectively. Now in PCT approach there are many options for choosing \( M(x) \) \([22]\), for example \( M(x) = \lambda g^2(x) \), \( M = \lambda g(x) \), \( M = \frac{\lambda}{g'(x)} \), \( \lambda \) being a constant. Here we choose \( M(x) = \lambda g'(x) \). For this choice of the mass function and using the values of \( Q(g), R(g) \) given in equation (A2), equation (A6) reduces to

\[
\mu - V = \frac{g'}{\lambda} \left[ -\frac{ab}{g(1 - g)} - \frac{(c - (a + b + 1)g)^2}{4g^2(1 - g)^2} + \frac{a + b + 1}{2g(1 - g)} + \frac{c - (a + b + 1)g}{2g^2(1 - g)} - \frac{c - (a + b + 1)g}{g(1 - g)^2} \right]
\]

(A7)

Now in order to generate a constant term on the right hand side of the above equation which will correspond to \( \mu \) on the left-hand side, we set \( \frac{g}{\lambda g(1 - g')} = p \), where \( p \) is a positive constant. This gives

\[
g(x) = \frac{e^{p\lambda x}}{1 + e^{p\lambda x}} \quad \text{and} \quad M(x) = \frac{p\lambda^2}{4} \operatorname{sech}^2 \left( \frac{p\lambda}{2} x \right), \quad -\infty < x < \infty
\]

(A8)

For these values of \( g(x) \) and \( M(x) \) we obtain from equation (A7) new potential \( V(x) \) and factorization energy \( \mu \) as

\[
V(x) = \frac{[(a + b - c)^2 - 1]p}{4} e^{p\lambda x} + \frac{cp(c - 2)}{4} e^{-p\lambda x}, \quad -\infty < x < \infty
\]

(A9)

and

\[
\mu = -abp + \frac{(a + b + 1)cp}{2} - \frac{c^2p}{2}
\]

(A10)

Also from (A5) we get

\[
f(x) = \frac{e^{p\lambda x}}{(1 + e^{p\lambda x})^{\frac{a+b+1}{2}}}
\]

Hence the solution of the equation (12) at the factorization energy \( \mu \) is given by

\[
U(x) = \frac{e^{p\lambda x}}{(1 + e^{p\lambda x})^{\frac{a+b+1}{2}}} \ {}_2F_1 \left( a, b, c, \frac{e^{p\lambda x}}{1 + e^{p\lambda x}} \right)
\]

(A11)

Another linearly independent solution of (12) at the same factorization energy can be written as (19)

\[
U(x) = \frac{e^{p\lambda (1 - \frac{x}{2})} x}{(1 + e^{p\lambda x})^{\frac{a+b+3}{2}}} \ {}_2F_1 \left( a - c + 1, b - c + 1, 2 - c, \frac{e^{p\lambda x}}{1 + e^{p\lambda x}} \right)
\]

The linear combination of above two solutions can be taken as the most general non-normalizable solution of the equation (12) at the factorization energy \( \mu \), and is

\[
U(x) = a \frac{e^{p\lambda x}}{(1 + e^{p\lambda x})^{\frac{a+b+1}{2}}} \ {}_2F_1 \left( a, b, c, \frac{e^{p\lambda x}}{1 + e^{p\lambda x}} \right) + \beta \frac{e^{p\lambda (1 - \frac{x}{2})} x}{(1 + e^{p\lambda x})^{\frac{a+b+3}{2}}} \ {}_2F_1 \left( a - c + 1, b - c + 1, 2 - c, \frac{e^{p\lambda x}}{1 + e^{p\lambda x}} \right)
\]

(A12)
where \( \alpha \) and \( \beta \) are two arbitrary constants. Consequently the bound state solutions of the equation (2) for the potential (A9), are obtained from equation (A12) by putting \( \alpha = 1 \), \( \beta = 0 \) and \( a = -n \), \( b = 1 - a + \sigma + \delta \), \( c = 1 + \sigma \) (see 15.4.6 of ref. [19])

\[
\psi_n(x) = \left( \frac{\mu(2n + \sigma + \delta + 1)}{\Gamma(n + \sigma + 1)\Gamma(n + \delta + 1)} \right)^{1/2} \frac{e^{p\lambda x}}{(1 + e^{p\lambda x})^{\frac{n+\sigma+\delta}{2}}} P_n^{(\sigma,\delta)} \left( \frac{1 - e^{p\lambda x}}{1 + e^{p\lambda x}} \right) \tag{A13}
\]

and the energy eigenvalues are given by

\[
E_n = n^2p + np(\sigma + 1) + \frac{(\sigma + 1)p(\delta + 1)}{2}, \quad n = 0, 1, 2, \ldots \tag{A14}
\]

It should be mentioned here that for \( \psi_n(x) \) to be a physically acceptable solution it should satisfy the following two conditions:

(i) It should be square integrable over domain of definition \( D \) of \( M(x) \) and \( \psi(x) \) i.e.,

\[
\int_D |\psi_n(x)|^2 dx < \infty
\]

(ii) The Hermiticity of the Hamiltonian (2) in the Hilbert space spanned by the eigenfunctions of the potential \( V(x) \) is ensured by the following extra condition [23]

\[
\frac{|\psi_n(x)|^2}{\sqrt{M(x)}} \to 0
\]

at the end points of the interval where \( V(x) \) and \( \psi_n(x) \) are defined. This condition imposes an additional restriction whenever the mass function \( M(x) \) vanishes at any one or both the end points of \( D \). In order to satisfy this two conditions we have to impose a restriction \( \sigma > -\frac{1}{2} \) and \( \delta > -\frac{1}{2} \) or equivalently \( c > \frac{1}{2} \) and \( a + b - c + \frac{1}{2} > 0 \).

Acknowledgments

One of us (RR) is grateful to the Council of Scientific and Industrial Research (CSIR) New Delhi, for a grant (project No. 21/0659/06/EMR-II).

[1] M. G. Darboux, C. R. Acad. Sci. Paris 94 1456 (1882)
[2] E. Schrödinger, Proc. Roy. Irish. Acad. A 46 9 (1940); L. Infeld and T. E. Hull, Rev. Mod. Phys. 23, 21 (1951); B. Mielnik, J. Math. Phys. 25, 3387 (1984)
[3] E. Witten, Nucl. Phys. B 188, 543 (1981); C.V. Sukumar, J. Phys. A18, L57 (1985); ibid 2917; C.V. Sukumar, J. Phys. A 19, 2297 (1986)
[4] P. Cooper, A Khare and U. Sukhatme, Phys. Rep. 251, 267 (1995)
[5] B.F. Samsonov, Phys. Lett. A 263, 273 (1999); Mod. Phys. Lett A 19, 1563 (1996)
[6] O. Von Roos, Phys. Rev. B 27, 7547 (1983)
[7] G. Bastard, Wave Mechanics Applied to Semiconductor Heterostructure, (Les Editions de Physique, Les Ulis, France, 1988); R. A. Morrow and K.R. Brownstein, Phys. Rev. B30 678 (1984)
[8] L. Serra and E. Lipparani, Europhys. Lett, 40 667 (1997)
[9] F. Aris de Saavedra, J. Boronat, A. Polls and A. Fabrocini, Phys. Rev. B 50, 4248 (1994); M. Baranco et. al. Phys. Rev. B 56, 8997 (1997); A. Puente, L. Serra and M. Casas, Z. Phys. D 31, 283 (1994)
[10] A. A. Suzko and A. S. Halberg, Phys. Lett. A 372, 5865 (2008)
[11] A. S. Halberg, E. Pozdeeva and A. Suzko, J. Phys. A: Math. Gen. 42, 115211 (2009)
[12] A. R. Plastino, et. al., Phys. Rev. A 60, 4318 (1999)
[13] A. Ganguly and L.M. Nieto, J. Phys. A: Math. Theor. 40, 7265 (2007)
[14] T. Tanaka, J. Phys. A: Math. Gen. 39, 219 (2006)
[15] H. Aoyama, M. Sato and T. Tanaka, Phys. Letts. B 503, 423 (2001)
[16] P. Bois et.al., Superlatt. Microstruct. 8 (1990)369
[17] D.J. BenDaniel and C.B. Duke, Phys. Rev. B152, 683(1966)
[18] A. Gonzalez-Lopez and T. Tanaka, J. Phys. A 39, 3715 (2006)
[19] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, (Dover publivcations, New York, 1965)
[20] If $T$ is a linear $n$th order differential operator $Tu = \sum_{k=0}^{n} a_k D^k u$, then its adjoint is given by $T^* u = \sum_{k=0}^{n} (-1)^k D^k(a_k u)$.
[21] D.J Fernandez and N. F. Garcia, AIP Conf Proc 744, 236 (2005)
[22] B. Bagchi, P. Gorain, C. Quesne and R. Roychoudhury, Eur. Phys. Lett. 72, 155 (2005); B. Midya and B. Roy, Phys. Lett. A 373, 4117 (2009)
[23] B. Bagchi, A. Banergy, C. Quesne and V. M. Tkachuk, J. Phys. A: Math. Gen. 38, 2929 (2005)
[24] A. A. Andrianov and F. Canata, J. Phys. A 37, 10297 (2004), and references therein.