POINTED HOPF ALGEBRA (CO)ACTIONS ON RATIONAL FUNCTIONS

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Abstract. This article studies the construction of Hopf algebras $H$ acting on a given algebra $K$ in terms of algebra morphisms $\sigma: K \to M_n(K)$. The approach is particularly suited for controlling whether these actions restrict to a given subalgebra $B$ of $K$, whether $H$ is pointed, and whether these actions are compatible with a given $*$-structure on $K$. The theory is applied to the field $K = k(t)$ of rational functions containing the coordinate ring $B = k[t^2, t^3]$ of the cusp. An explicit example is described in detail and shown to define a new quantum homogeneous space structure on the cusp.

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1. Introduction

The question which Hopf algebras can (co)act on a commutative algebra goes back at least to Cohen’s article [Coh94]. For semisimple Hopf algebras $H$ over algebraically closed fields $k$ of characteristic 0, it was answered completely by Etingof and Walton [EW14], see also [EGMW17]: if $H$ acts inner faithfully on a commutative integral domain $K$ (i.e. the action does not arise from the action of a proper quotient Hopf algebra of $H$), then $H$ is a group algebra. In contrast to this, there are typically many inner faithful actions of pointed Hopf algebras on a commutative algebra $K$, even when $K$ admits only few automorphisms and derivations. For example, the coordinate rings $B$ of singular plane curves are conjecturally all quantum homogeneous spaces in the sense of [MS99], that is, can be embedded as right coideal subalgebras into Hopf algebras $A$ which are faithfully flat as modules over these subalgebras [KT17, KM20, BT21], and in many examples studied so far, these embeddings yield inner faithful actions of pointed Hopf algebras $H$ that are dually paired with $A$.

The starting point of the present article is to demand that like classical symmetries (given by actions of group algebras or universal enveloping algebras), the action of $H$ extends from the coordinate ring $B$ to the field $K$ of rational functions on the curve. As the classification of Hopf algebra (co)actions on fields is an interesting topic in its own right (see e.g. [EW15, EW16, GP87, TT19, Tsa19, CRV16] and the references therein), we felt it worthwhile to begin a systematic study of such actions on function fields that restrict to coordinate rings.

The approach we take was maybe first applied by Manin in his construction of quantum $SL(2)$ as a Hopf algebra (co)acting on the quantum plane [Man87]. More recently, it was used mostly in the $C^*$-algebraic quantum group community in the construction and study of compact quantum automorphism groups such as the quantum permutation groups or the liberations of compact Lie groups [BB10, BG10, Web17, BS09, BBC08, BSS15, BBDC12].

In this approach, a bialgebra action on a $k$-algebra $K$ is given by – or more precisely constructed from – an algebra morphism $K \to M_n(K)$; the bialgebra is a Hopf algebra if this morphism, viewed as an element of $M_n(\text{End}_k(K))$ is strongly invertible in the sense of Definition 2.1.1 below. The general approach is well-known, but some aspects of our presentation are to our knowledge novel, such as the connection to the theory of general linear groups over noncommutative rings (see Section 2.1), the treatment of $*$-structures in the pointed rather than
the semisimple setting (see Section 3.7), and the application to the field $K = k(t)$ of rational functions (see Section 2.4).

Our main focus is the construction of pointed Hopf algebra (co)actions on $K = k(t)$ which restrict to the coordinate ring $B = k[t^2, t^3]$ of the cusp. Working over a field $k$ whose characteristic is not 2 or 3, we construct Hopf algebras $A_\sigma$ and $H_\sigma$ which are generated as algebras by $a, b, c$ respectively.

These generators satisfy the defining relations

\[
\begin{align*}
ab + ba &= ac + ca = bc + cb = 0, \\
3b^2 &= a^6, \\
c^2 &= 1, \\
KD + DK &= KY - YK = 0, \\
Y^2 D - 2YDY + DY^2 &= 0, \\
K^2 &= 1, \\
D^2 &= 0.
\end{align*}
\]

In terms of these generators, the coproduct, counit, and antipode of $A_\sigma$ and $H_\sigma$ are given by

\[
\begin{align*}
\Delta(a) &= 1 \otimes a + a \otimes c, \\
\Delta(b) &= 1 \otimes b + a^2 \otimes a - a \otimes a^2 c + b \otimes c \\
\epsilon(a) &= \epsilon(b) = 0, \\
\epsilon(c) &= 1, \\
S(a) &= -ac, \\
S(b) &= -bc, \\
S(c) &= c, \\
\Delta(K) &= K \otimes K, \\
\Delta(D) &= 1 \otimes D + D \otimes K, \\
\Delta(Y) &= 1 \otimes Y - 6D \otimes DK + Y \otimes 1, \\
\epsilon(K) &= 1, \\
\epsilon(D) &= \epsilon(Y) = 0, \\
S(K) &= K, \\
S(D) &= -DK, \\
S(Y) &= -Y.
\end{align*}
\]

Our main results are summarised in the following theorem:

**Theorem.** We have:

1. The coalgebras $A_\sigma$ and $H_\sigma$ are pointed.
2. There is a dense Hopf algebra embedding $A_\sigma \to H_\sigma^\circ$.
3. For any point $(\lambda^2, \lambda^3), \lambda \in k$, of the cusp, there is an embedding $\iota: B = k[t^2, t^3] \to A_\sigma$ of its coordinate ring as a right coideal subalgebra such that $t^2 \mapsto \lambda^2 1 + 12a^2, \ t^3 \mapsto 6\lambda^2 a + 36a^3 + 36b + \lambda^3 c$ and $A_\sigma$ is faithfully flat over $B$.
4. The $H_\sigma$-action on $B$ dual to the resulting $A_\sigma$-coaction on $B$ is inner faithful.
5. This $H_\sigma$-action and this $A_\sigma$-coaction both extend to the field $K = k(t)$ of rational functions.
(6) If \( k = \mathbb{C} \), then \( A_\sigma \) and \( H_\sigma \) become Hopf \(*\)-algebras with
\[
K^* = K, \quad D^* = -D, \quad Y^* = -Y + 6iD,
\]
and the images of \( A_\sigma \) and of \( B \) in \( H_\sigma \) are \(*\)-subalgebras if \( \bar{\lambda} = \lambda \); the resulting \(*\)-structure on \( \mathbb{C}[t^2, t^3] \) is given by \( t^* = t \).

The paper is divided into two main sections: our presentation is arranged in such a way that large parts of the development of the general theory and of the computations carried out for the cusp only use linear algebra and the theory of polynomials and rational functions in one variable. No Hopf algebra theory or algebraic geometry is required for this material, which is gathered in Section 2. Section 2.6 contains the first steps towards a classification of pointed Hopf algebra actions on \( k[t^2, t^3] \) that extend to \( k(t) \), but we mostly focus on the specific example described in the theorem above. Section 3 contains the proof of this theorem.

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2. Quantum automorphisms of the cusp

This section contains those parts of the paper that can be formulated in terms of elementary algebra. The interpretation in terms of Hopf algebras is contained in Section 3.

2.1. Strongly invertible matrices. Recall that if \( \sigma \in \text{GL}_n(P) \) is an invertible \( n \times n \)-matrix with entries in a (unital associative) ring \( P \), then the transpose \( \sigma^T \) is in general not invertible:

**Example 2.1.1.** If \( a, d \in P \) satisfy \( da = 1 \neq ad \), then we have
\[
\begin{pmatrix} a & 1 \\ 0 & d \end{pmatrix} \in \text{GL}_2(P), \quad \begin{pmatrix} a & 0 \\ 1 & d \end{pmatrix} \notin \text{GL}_2(P).
\]

More precisely, \( \sigma^T \in \text{GL}_n(P) \) if and only if \( \sigma \in \text{GL}_n(P^{\text{op}}) \), where \( P^{\text{op}} \) denotes the opposite ring of \( P \) (the additive abelian group \( P \) equipped with the opposite multiplication \( a \cdot_{\text{op}} b := ba \)). Indeed, this follows from the straightforwardly verified fact that for \( \sigma, \tau \in \text{M}_n(P) \), we have
\[
(\sigma \tau)^T = \tau^T \cdot_{\text{op}} \sigma^T,
\]
where on the right hand side, \( \cdot_{\text{op}} \) is the multiplication in \( \text{M}_n(P^{\text{op}}) \). This yields the contragredient isomorphism of general linear groups
\[
\text{GL}_n(P) \to \text{GL}_n(P^{\text{op}}), \quad \sigma \mapsto \tilde{\sigma} := (\sigma^{-1})^T,
\]
see e.g. [HO89, Chapter 3] for a discussion of this map.

The theory of Hopf algebras motivates to study those matrices to which one can apply this operation arbitrarily often without leaving $\text{GL}_n(P)$; we are not aware of a standard name for such matrices, so we introduce a working terminology:

**Definition 2.1.1.** We call $\sigma \in M_n(P)$ **strongly invertible** if there exists a sequence $\{\sigma_d\}_{d \in \mathbb{Z}}$ in $\text{GL}_n(P)$ with $\sigma_0 = \sigma$ and $\sigma_{d+1} = \hat{\sigma}_d$.

Note this means that also all $\sigma_d^T$ are invertible with $(\sigma_d^T)^{-1} = \sigma_{d-1}$.

Note also that the strongly invertible matrices do not form a group:

**Example 2.1.2.** Let $k$ be a field and $P := k\langle x, y \rangle$ be the free $k$-algebra generated by $x, y$. Then

$$\sigma := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad \tau := \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}$$

are strongly invertible matrices, but $(\sigma \tau)^T$ is not invertible.

We furthermore remark that the set of strongly invertible matrices is in general a proper subset of $\text{GL}_n(P) \cap \text{GL}_n(P^{op})$:

**Example 2.1.3.** Assume that $P$ is any ring that contains elements $x, y$ and that $x$ is invertible. Consider

$$\sigma := \begin{pmatrix} x^{-1} + y^2 & y \\ y & 1 \end{pmatrix} \in \text{GL}_2(P), \quad \sigma^{-1} = \begin{pmatrix} x & -xy \\ -yx & 1 + yxy \end{pmatrix}.$$

Since $\sigma = \sigma^T$, we have $\sigma \in \text{GL}_2(P) \cap \text{GL}_2(P^{op})$. However,

$$\hat{\sigma} = (\sigma^{-1})^T = \begin{pmatrix} x & -yx \\ -xy & 1 + yxy \end{pmatrix} \in M_2(P)$$

is in general not invertible: indeed, assume that $\hat{\sigma} \in \text{GL}_2(P)$ with

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} := \hat{\sigma}^{-1}.$$

Then the condition $\hat{\sigma} \hat{\sigma}^{-1} = 1$ implies

$$b = x^{-1}yx, \quad 1 = (1 + y(xy - yx))d.$$

Similarly, $\hat{\sigma}^{-1} \hat{\sigma} = 1$ yields

$$c = dxyx^{-1}, \quad 1 = d(1 + yxy - xyx^{-1}yx).$$

Thus $d \in P$ is invertible with inverse

$$d^{-1} = 1 + yxy - xyx^{-1}yx = 1 + y(xy - yx).$$

However, this implies $(y - xyx^{-1})y = 0$ which does not hold in general.

In this paper, we will focus on upper triangular matrices, and for these, the condition of strong invertibility is easily controlled:
Proposition 2.1.1. If $\sigma \in \text{M}_n(P)$ is upper triangular, $\sigma_{ij} = 0$ for $i > j$, then the following statements are equivalent:

1. $\sigma$ is strongly invertible.
2. $\sigma$ is invertible.
3. $\sigma_{ii} \in P$ is invertible for $i = 1, \ldots, n$.

In this case, $\sigma^{-1}$ is upper triangular and all $(\sigma^{-1})_{ij}$ are contained in the subring of $P$ generated by the $\sigma_{ij}$ and the $\sigma_{ii}^{-1}$.

Proof. “$\Rightarrow$”: Suppose $\sigma$ is strongly invertible. Then $\sigma$ and $\sigma^T$ are invertible. As $\sigma_{ni} = 0$ for $i < n$, we then have

$$\sigma \sigma^{-1} = 1 = \sigma^{-1} \sigma.$$ 

Hence, $\sigma_{nn}$ is invertible in $P$ with inverse $(\sigma^{-1})_{nn} = ((\sigma^{-1})_{nn})_{nn}$. 

$\sigma \sigma^{-1}$ shows

$$\sigma_{nn}(\sigma^{-1})_{nj} = 0 \ \forall \ j = 1, \ldots, n - 1.$$ 

But $\sigma_{nn}$ is invertible, thus

$$(\sigma^{-1})_{nj} = 0 \ \forall \ j = 1, \ldots, n - 1.$$ 

Analogously, one shows that $((\sigma^T)^{-1})_{jn}$ vanishes for $j = 1, \ldots, n - 1$. So $\sigma$, $\sigma^{-1}$, and $(\sigma^T)^{-1}$ can be written in block matrix form as

$$\sigma = \begin{pmatrix} \alpha & \mu \\ 0^T & \sigma_{nn} \end{pmatrix}, \quad \sigma^{-1} = \begin{pmatrix} \beta & \gamma \\ 0^T & \sigma_{nn}^{-1} \end{pmatrix}, \quad (\sigma^T)^{-1} = \begin{pmatrix} \delta & 0 \\ \nu^T & \sigma_{nn}^{-1} \end{pmatrix},$$

where $\alpha, \beta, \delta \in \text{M}_{n-1}(P)$, $\mu, \gamma, \nu$ are column vectors in $P^{n-1}$, and 0 is the zero vector in $P^{n-1}$.

From $\sigma \sigma^{-1} = 1 = \sigma^{-1} \sigma$ we obtain that $\alpha$ is invertible with inverse $\beta$. Analogously, $\alpha^T$ is invertible with inverse $\delta$. We also obtain

$$\gamma = -\alpha^{-1} \mu \sigma_{nn}^{-1},$$

so the entries $\gamma_j$ are elements of the subring of $P$ generated by the entries of $\sigma$ and of $\alpha^{-1}$. Continuing inductively one obtains the claim.

“$\Leftarrow$”: Suppose the diagonal entries $\sigma_{ii}$ of $\sigma = \sigma_0$ are invertible. We show that the equation $\sigma \tau = 1$ can be solved inductively in the ring of upper triangular matrices with entries in $P$. Indeed, this equation means that

$$\sum_{m=1}^{n} \sigma_{im} \tau_{mn} = \delta_{ln}.$$
These equations can be solved by induction on \( n - l \). For \( n - l = 0 \) we obtain

\[
\tau_{ll} = \sigma_{ll}^{-1}.
\]

For \( n - l = i \), we obtain

\[
\tau_{l+i} = -\sigma_{ll}^{-1} \left( \sum_{m=l+1}^{n} \sigma_{lm} \tau_{mn} \right).
\]

Furthermore, solving the equation \( \tau \sigma = 1 \) inductively as above, we conclude that \( \tau = \sigma^{-1} \). Analogously, one can show by solving the equations \( \sigma^T \rho = 1 \) and \( \rho \sigma^T = 1 \) inductively in the ring of lower triangular matrices that \( \sigma^T \) is invertible with inverse \( \rho \).

\[\square\]

2.2. A quantum Galois group. Let now \( k \) be a commutative ring and \( B \) be a \( k \)-algebra. Let \( P = \text{End}_k(B) \) be the ring of \( k \)-linear maps \( B \to B \). A matrix \( \sigma \in M_n(\text{End}_k(B)) \) can alternatively be viewed as a map \( B \to M_n(B) \), and we can demand this to be a ring morphism:

**Definition 2.2.1.** A quantum automorphism of \( B \) over \( k \) is a strongly invertible matrix \( \sigma \in M_n(\text{End}_k(B)) \) satisfying

1. \( \sigma_{ij}(1) = \delta_{ij}, \quad \sigma_{ij}(ab) = \sum_{i=1}^{n} \sigma_{ii}(a)\sigma_{ij}(b) \quad \forall a, b \in B. \)

Given a quantum automorphism, we denote by

\[ U_\sigma \subseteq \text{End}_k(B) \]

the \( k \)-algebra generated by the entries \( \sigma_{d,ij} \) of the \( \sigma_d \in M_n(\text{End}_k(B)) \), \( \sigma_0 = \sigma \), \( \sigma_{d+1} = \hat{\sigma}_d \).

For \( n = 1 \), this just means that \( \sigma \) is a \( k \)-algebra automorphism of \( B \), and that \( U_\sigma \) is the group algebra of the subgroup of the Galois group \( \text{Gal}(B/k) \) of \( B \) over \( k \) that is generated by \( \sigma \). In this sense, the set of quantum automorphisms is a generalisation of \( \text{Gal}(B/k) \).

If \( B \) is noncommutative, \( \sigma^{-1} \) is in general not a ring morphism, but note that the set of quantum automorphisms is closed under \( \sigma \mapsto \hat{\sigma} \):

**Proposition 2.2.1.** If \( \sigma \) is a quantum automorphism, then so is \( \hat{\sigma} \).

**Proof.** The key point is to show that \( \hat{\sigma} \) is multiplicative. To see this, first apply \( \sigma_{pi}^{-1} \) to (3) and sum over \( i \). This yields

\[
\delta_{p,j}ab = \sum_{i,l=1}^{n} \sigma_{pi}^{-1}(\sigma_{il}(a)\sigma_{lj}(b)).
\]
Inserting into this equation \( a = \sigma^{-1}_{qr}(c), b = \sigma^{-1}_{jq}(d) \) for elements \( c, d \in B \) and summing over \( j \) and \( q \) yields the claim:

\[
\sum_{q=1}^{n} \hat{\sigma}_{rq}(c)\hat{\sigma}_{qp}(d) = \sum_{q=1}^{n} \sigma^{-1}_{qr}(c)\sigma^{-1}_{pq}(d) = \sum_{q,j=1}^{n} \delta_{pq} \sigma^{-1}_{qr}(c)\sigma^{-1}_{jq}(d)
\]

\[
= \sum_{i,j,l,q=1}^{n} \sigma^{-1}_{pi}(\sigma^{-1}_{qr}(c))\sigma_{ij}(\sigma^{-1}_{jq}(d))
\]

\[
= \sum_{i,l,q=1}^{n} \sigma^{-1}_{pi}(\sigma^{-1}_{il}(c))\delta_{lq}d = \sum_{i,l=1}^{n} \sigma^{-1}_{pi}(\sigma^{-1}_{il}(c))\delta_{lq}d
\]

\[
= \sum_{i=1}^{n} \sigma^{-1}_{pi}(\delta_{ir}cd) = \sigma^{-1}_{pr}(cd) = \hat{\sigma}_{rp}(cd). \quad \square
\]

However, even when \( B \) is commutative, the (matrix) product of quantum automorphisms is in general not a quantum automorphism, so quantum automorphisms do not form groups. As we will explain in Section 3, they instead generate quantum groups (Hopf algebras).

2.3. A quantum subgroup. Even for basic examples of ring extensions, the set of all quantum automorphisms is usually wild. There are various subsets that one can focus on, and we will in particular be interested in the following two attributes:

Definition 2.3.1. A quantum automorphism \( \sigma \) is

1. upper triangular if \( \sigma_{ij} = 0 \) for \( i > j \), and
2. locally finite if for all \( a \in B \), the set \( \{X(a) \mid X \in U_\sigma\} \subseteq B \) is contained in a finitely generated \( k \)-module.

Both conditions will be motivated and explained further in Section 3. For now, we only point out that the upper triangularity makes it particularly easy to find such quantum automorphisms:

Corollary 2.3.1. A \( k \)-algebra morphism \( \sigma : B \to M_n(B) \) with \( \sigma_{ij} = 0 \) for \( i > j \) is a quantum automorphism if and only if its diagonal entries \( \sigma_{ii} \) are invertible for \( i = 1, \ldots, n \). In this case, \( U_\sigma \) is generated as a \( k \)-algebra by the \( \sigma_{ij} \) together with the \( \sigma^{-1}_{ii} \).

Proof. This follows immediately from Proposition 2.1.1. \( \square \)

A typical situation in which local finiteness holds is the following:

Proposition 2.3.1. Suppose \( B \) is a \( k \)-algebra, and \( \{F_d\}_{d \in \mathbb{Z}} \) is an exhaustive filtration of the algebra \( B \) by finitely generated \( k \)-modules, \( F_d \), and that \( \sigma \in \text{End}_k(B) \) is a quantum automorphism with \( \sigma_{ij}(F_d) \subseteq F_d \). Then \( \sigma \) is locally finite.
2.4. Upper triangular quantum automorphisms of $k(t)$. We will be interested in quantum automorphisms of coordinate rings of singular plane curves whose field of fractional functions is the field $k(t)$, and we first classify the upper triangular quantum automorphisms of the latter.

**Proposition 2.4.1.** For any field $k$, the assignment $\sigma \mapsto \sigma(t) \in M_n(k(t))$ defines a bijection between upper triangular quantum automorphisms of $k(t)$ over $k$ and upper triangular matrices in $M_n(k(t))$ whose diagonal entries are of the form

$$\sigma(t)_{ii} = \frac{\alpha_i t + \beta_i}{\gamma_i t + \delta_i}$$

for some $\alpha_i, \beta_i, \gamma_i, \delta_i \in k, \alpha_i \delta_i - \beta_i \gamma_i \neq 0$.

**Proof.** For any $k$-algebra $M$, $\sigma \mapsto T := \sigma(t)$ defines a bijection between the set of $k$-algebra morphisms $\sigma : k[t] \to M$ and $M$. Such an algebra morphism extends in at most one way to an algebra morphism $\sigma : k(t) \to M$ given by $\frac{p}{q} \mapsto p(T)q(T)^{-1}$, and it does extend if and only if for any $q \in k[t] / \{0\}$ the element $q(T) \in M$ is invertible in $M$.

Specialising these general considerations to the case $M = M_n(k(t))$, we have furthermore by elementary linear algebra over fields:

1. $p(T)$ is upper triangular for all $p \in k[t]$ if and only if $T$ is so.
2. $q(T)$ is invertible if and only if $\det(q(T)) \neq 0$.
3. If $q(T)$ is invertible and upper triangular, so is $q(T)^{-1}$.
4. If $T$ is upper triangular, then $\det(q(T)) = q(T_{11}) \cdots q(T_{nn})$.

We conclude that $k$-algebra morphisms $k(t) \to M_n(k(t))$ that are upper triangular correspond bijectively to upper triangular matrices $T \in M_n(k(t))$ with $q(T_{ii}) \neq 0$ for all $i = 1, \ldots, n$ and all $q \in k[t] / \{0\}$.

Corollary 2.3.1 shows that such an algebra morphism is a quantum automorphism if and only if its diagonal entries $\sigma_{ii}$ are in the Galois group of $k(t)$ over $k$, which is well-known to be the group of Möbius transformations [Cox12, Theorem 7.5.7]. So if we are given an upper triangular matrix $T$ that defines an upper triangular quantum automorphism of $k(t)$, the $T_{ii}$ are necessarily of the form as stated. Conversely, if all $T_{ii}$ are of this form, then we also have $q(T_{ii}) \neq 0$ for all $q \in k[t] / \{0\}$, as the inverse of the unique $k$-algebra automorphism $\sigma_{ii} : k(t) \to k(t)$ that maps $t$ to $T_{ii}$ maps $q(T_{ii})$ to $q$; thus $T$ defines an algebra morphism $k(t) \to M_n(k(t))$ which by Corollary 2.3.1 is a quantum automorphism.
2.5. **Restriction to** $k[t^2, t^3]$. Assume $k$ is a field, $\sigma$ is a quantum automorphism of $k(t)$, and abbreviate as above $T := \sigma(t) \in M_n(k(t))$. Then $\sigma$ restricts to an intermediate ring $k \subseteq B \subseteq k(t)$ if and only if for all $b = \frac{p}{q} \in B$, we have

\[
\sigma(b) = b(T) = p(T)q(T)^{-1} \in M_n(B) \subseteq M_n(k(t)).
\]

The main example we are interested in is $B = k[t^2, t^3] = \text{span}_k \{t^i \mid i \neq 1\} \subseteq k[t]$.

It is evidently sufficient to test (4) only for a set of generators of the $k$-algebra $B$, so in this case for $b = t^2$ and $b = t^3$. In other words, we have:

**Corollary 2.5.1.** A quantum automorphism $\sigma$ of $k(t)$ over $k$ restricts to $k[t^2, t^3]$ if and only if $T^2, T^3 \in M_n(k[t^2, t^3])$, where $T = \sigma(t)$.

When classifying upper triangular quantum automorphisms of $k[t^2, t^3]$ that extend to $k(t)$, it is sufficient to consider matrices whose entries are Laurent polynomials:

**Proposition 2.5.1.** If an upper triangular matrix $T \in M_n(k(t))$ satisfies $T^2, T^3 \in M_n(k[t^2, t^3])$, then $T \in M_n(k[t, t^{-1}])$ whose entries contain no terms of degree less than $-3n + 4$.

**Proof.** We prove $T_{ij} \in k[t, t^{-1}]$ by induction on $j - i$.

- **$j - i = 0$:** We have shown that the diagonal entries are Möbius transformations, i.e., are of the form $T_{ii} = \frac{\alpha_i t + \beta_i}{\gamma_i t + \delta_i}$. The condition $T^2 \in M_n(k[t^2, t^3])$ and $\alpha_i \delta_i - \beta_i \gamma_i \neq 0$ force $\beta_i = \gamma_i = 0$, thus without loss of generality we have $T_{ii} = \alpha_i t$.

- **$j - i = n$:** Assume that $T_{i,i+r}$ is for all $r < n$ a Laurent polynomial and contains no term of degree less than $-3r + 1$.

By assumption, the elements

\[
(T^2)_{i,n+i} = (\alpha_i + \alpha_{n+i}) T_{i,n+i} + \sum_{r=1}^{n-1} T_{i,i+r} T_{i+r,n+i}
\]

\[
(T^3)_{i,n+i} = (\alpha_i^2 + \alpha_i \alpha_{n+i} + \alpha_{n+i}^2) t^2 T_{i,n+i} + \\
+ \sum_{r=1}^{n-1} T_{i,i+r} (T^2)_{i+r,n+i} + \alpha_i t T_{i,i+r} T_{i+r,n+i}
\]

must be in $k[t^2, t^3]$. As either $\alpha_i + \alpha_{n+i}$ or $\alpha_i^2 + \alpha_i \alpha_{n+i} + \alpha_{n+i}^2$ are non-zero, it follows from the induction hypothesis that $T_{i,n+i}$ is a Laurent polynomial which contains no term of degree less than $-3n + 1$. \hfill $\Box$

**Remark 2.5.1.** If $k \subset B \subset k(t)$ is any intermediate ring, then $B$ is a subring of a field, hence an integral domain, and its fraction field
embeds naturally into \( k(t) \); so by Lüroth’s theorem, the fraction field is isomorphic to \( k(t) \). If \( B \) is the coordinate ring of an algebraic set \( V \), this means that \( V \) is an irreducible curve which is birationally equivalent to the affine line. In particular, when \( k \) is algebraically closed, then this is the case if and only if \( B \) is finitely generated as a \( k \)-algebra (by Hilbert’s Nullstellensatz).

For \( B = k[t^2, t^3] \), the curve \( V \) is the cusp

\[
V = \{(\alpha, \beta) \in k^2 \mid \alpha^3 = \beta^2\} = \{(\lambda^2, \lambda^3) \mid \lambda \in k\} \subseteq k^2,
\]

so in geometric terms, we are talking about quantum automorphisms of the cusp that extend from its coordinate ring to its field of rational functions.

### 2.6. Classification for \( n = 2, 3 \)

Recall that for \( l \in \mathbb{N} \) and \( \beta \in k \), one defines the quantum numbers

\[
[\l]_\beta := 1 + \beta + \cdots + \beta^{l-1} = \frac{1 - \beta^l}{1 - \beta},
\]

where the last equality of course only applies when \( \beta \neq 1 \).

**Lemma 2.6.1.** If \( z = \sum_{i \in \mathbb{Z}} z_i t^i \), then the matrix

\[
T = \begin{pmatrix} \alpha t & z \\ 0 & \alpha \beta t \end{pmatrix}
\]

corresponds to a quantum automorphism of \( k[t^2, t^3] \) if and only if

1. \( [2]_\beta = 0 \iff \beta = -1 \) and \( z_{-1} = z_{-3} = z_{-4} = \ldots = 0 \), or
2. \( [3]_\beta = 0 \iff \beta = e^{\pm 2\pi i/3} \) and \( z_0 = z_{-2} = z_{-3} = \ldots = 0 \) or
3. \( [2]_\beta, [3]_\beta \neq 0 \) and \( z_0 = z_{-1} = z_{-2} = \ldots = 0 \).

**Proof.** We have

\[
T^2 = \alpha \begin{pmatrix} \alpha t^2 & [2]_\beta t z \\ 0 & \alpha \beta^2 t^2 \end{pmatrix}, \quad T^3 = \alpha^2 \begin{pmatrix} \alpha t^3 & [3]_\beta t^2 z \\ 0 & \alpha \beta^3 t^3 \end{pmatrix}
\]

so \( T \) corresponds to a quantum automorphism if and only if

\([2]_\beta t z, [3]_\beta t^2 z \in k[t^2, t^3]\)

which leads to the conditions as stated. \( \square \)

**Lemma 2.6.2.** If \( z = \sum_{i \in \mathbb{Z}} z_i t^i \), \( y = \sum_{j \in \mathbb{Z}} y_j t^j \), and \( x = \sum_{i \in \mathbb{Z}} x_i t^i \), then the matrix

\[
T = \begin{pmatrix} \alpha t & x \\ 0 & \alpha \beta t & y \\ 0 & 0 & \alpha \beta \gamma t \end{pmatrix}
\]

corresponds to a quantum automorphism of \( k[t^2, t^3] \) if and only if

1. \( \beta = -1 \) and \( x_{-1} = x_{-3} = x_{-4} = \ldots = 0 \), or
(b) $\beta = e^{\pm 2\pi i/3}$ and $x_0 = x_{-2} = x_{-3} = \ldots = 0$, or
(c) $\beta \neq -1, e^{\pm 2\pi i/3}$ and $x_0 = x_{-1} = x_{-2} = \ldots = 0$.

and

\begin{align*}
\text{(a) } & \gamma = -1 \text{ and } y_{-1} = y_{-3} = y_{-4} = \ldots = 0, \text{ or} \\
& \gamma = e^{\pm 2\pi i/3} \text{ and } y_0 = y_{-2} = y_{-3} = \ldots = 0, \text{ or} \\
& \gamma \neq -1, e^{\pm 2\pi i/3} \text{ and } y_0 = y_{-1} = y_{-2} = \ldots = 0,
\end{align*}

(3) and

\begin{align*}
\text{(a) } & \beta \neq -1 - \gamma^{-1} \text{ and there are } a, b \in k[t^2, t^3] \text{ with} \\
& z = (1 + \beta + \beta \gamma)t^{-1}a - t^{-2}b, \quad xy = \alpha(-[3]_{\beta \gamma}a + [2]_{\beta \gamma}t^{-1}b), \\
& \text{or} \\
\text{(b) } & \beta = -1 - \gamma^{-1} \text{ and} \\
& c := xy - \alpha \gamma tz \in k[t^2, t^3], \quad [3]_{\gamma}tc \in k[t^2, t^3].
\end{align*}

Proof. Applying Lemma 2.6.1 to the two matrices obtained by deleting from $T$ the first respectively third row and column leads to the conditions (1) and (2). Condition (3) arises from considering the $(1, 3)$-entry of $T^2, T^3$:

\[
\begin{pmatrix}
\alpha[2]_{\beta \gamma} \\
\alpha^2[3]_{\beta \gamma}t \alpha(1 + \beta[2]_{\gamma})t
\end{pmatrix}
\begin{pmatrix}
tz \\
yx
\end{pmatrix}
=:
\begin{pmatrix}
a \\
b
\end{pmatrix}
\in k[t^2, t^3]^2.
\]

The determinant of the coefficient matrix is

\[\alpha^2 \beta (1 + \gamma + \beta \gamma).\]

So we can invert the matrix over $k[t, t^{-1}]$ if

\[\beta \neq -1 - \gamma^{-1}.\]

In this regular case, we obtain

\[
\frac{1}{\alpha^2 \beta (1 + \gamma + \beta \gamma)}
\begin{pmatrix}
1 \\
-t^{-2}
\end{pmatrix}
\begin{pmatrix}
(1 + \beta[2]_{\gamma})t^{-1} \\
-\alpha[3]_{\beta \gamma}
\end{pmatrix}
\begin{pmatrix}
a \\
b
\end{pmatrix}
=:
\begin{pmatrix}
z \\
yx
\end{pmatrix}
\]

By rescaling $a, b$ this yields elements $a, b \in k[t^2, t^3]$ with

\[z = (1 + \beta + \beta \gamma)t^{-1}a - t^{-2}b\]

and

\[xy = \alpha(-[3]_{\beta \gamma}a + [2]_{\beta \gamma}t^{-1}b).\]

In the singular case $1 + \gamma + \beta \gamma = 0$, the equation reduces to

\[c := xy - \alpha \gamma tz \in k[t^2, t^3], \quad [3]_{\gamma}tc \in k[t^2, t^3].\]

The converse is verified by straightforward computation. \qed

2.7. An explicit example. From now on, we assume that $2, 3 \in k$ are invertible and that $k$ contains a square root $i$ of $-1$. We will study in detail the following example of a quantum automorphism of $k(t)$:
\[ \sigma(t) = T = \begin{pmatrix} t & t - i & -\frac{1}{2}t^{-1} - \frac{1}{2}t \\ 0 & -t & t + i \\ 0 & 0 & t \end{pmatrix}. \]

This does restrict to \( B = k[t^2, t^3] \); indeed, we have

\[ \sigma(x) = T^2 = \begin{pmatrix} x & 0 & \frac{1}{3} \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix}, \quad \sigma(y) = T^3 = \begin{pmatrix} y & y - ix & -\frac{1}{2}y \\ 0 & -y & y + ix \\ 0 & 0 & y \end{pmatrix}, \]

where \( x := t^2, \ y := t^3 \).

By definition, the resulting algebra \( U_\sigma \) has four generators that act as follows on the elements \( x, y \):

\[ K := \sigma_{22} : x \mapsto x, \ y \mapsto -y, \]
\[ E := \sigma_{12} : x \mapsto 0, \ y \mapsto y - ix, \quad F := \sigma_{23} : x \mapsto 0, \ y \mapsto y + ix, \]
\[ Z := \sigma_{13} : x \mapsto \frac{1}{3}, \ y \mapsto -\frac{1}{2}y. \]

Since \( \sigma \) is upper triangular, the operator \( K \) is an algebra automorphism, \( K \in \text{Gal}(k(t)/k) \), so for all \( f, g \in k(t) \), we have

\[ K(fg) = K(f)K(g), \]

the operators \( E, F \) are twisted derivations satisfying

\[ E(fg) = fE(g) + E(f)K(g), \quad F(fg) = K(f)F(g) + F(f)g, \]

and \( Z \) is a twisted differential operator of order 2,

\[ Z(fg) = fZ(g) + E(f)F(g) + Z(f)g. \]

This and the action (5) completely determines \( K, E, F, Z \) as \( k \)-linear maps:

**Lemma 2.7.1.** For all \( n \in \mathbb{N} \), we have

\[ K(t^n) = (-1)^n t^n, \]
\[ E(t^n) = \begin{cases} t^n - it^{n-1} & n \text{ is odd}, \\ 0 & n \text{ is even}, \end{cases} \quad F(t^n) = \begin{cases} t^n + it^{n-1} & n \text{ is odd}, \\ 0 & n \text{ is even}, \end{cases} \]
\[ Z(t^n) = \begin{cases} \frac{n-2}{6}t^{n-2} - \frac{1}{2}t^n & n \text{ is odd}, \\ \frac{n}{6}t^{n-2} & n \text{ is even}. \end{cases} \]

**Proof.** This is verified by a straightforward computation using induction on \( n \). \( \Box \)

In particular, one observes:
Corollary 2.7.1. The restriction of $\sigma$ to $B = k[t^2, t^3]$ is locally finite.

Proof. The algebra $k[t^2, t^3]$ inherits a grading from $k[t]$, so
\[ \text{deg}(x) = 2, \quad \text{deg}(y) = 3, \]
and if we denote by
\[ (7) \quad F_d := \text{span}_k \{ 1, t^2, t^3, \ldots, t^d \} \]
the resulting filtration of $k[t^2, t^3]$, then by the above formulas all assumptions of Proposition 2.3.1 are met. \qed

3. The quantum groups $H_\sigma$ and $A_\sigma$

This section contains the interpretation and motivation for the above computations: we explain how quantum automorphisms as described above give rise to (co)actions of Hopf algebras on $B$. The main goal is to prove that the explicit example discussed in Section 2.7 turns $k[t^2, t^3]$ into a quantum homogeneous space. From now on, we freely use standard terminology from the theory of bialgebras and Hopf algebras, see e.g. [Rad12, Mon93, KS97, Swe69] for the necessary definitions. For simplicity, “Hopf algebra” means for us “Hopf algebra with bijective antipode”.

3.1. The Hopf algebra $H_\sigma$. Recall that a quantum automorphism $\sigma \in M_n(\text{End}_k(B))$ of a $k$-algebra $B$ gives rise to a Hopf algebra $H_\sigma$ that acts inner faithfully on $B$, see [Man87, BB10, EW14]. This is constructed as follows:

1. Consider the free $k$-algebra $k\langle s_{d,ij} \rangle$ with generators $s_{d,ij}$, $i, j = 1, \ldots, n$, $d \in \mathbb{Z}$. This carries a unique bialgebra structure whose coproduct and counit are determined by
\[ (8) \quad \Delta(s_{d,ij}) = \sum_{r=1}^{n} s_{d,ir} \otimes s_{d,rj}, \quad \varepsilon(s_{d,ij}) = \delta_{ij}. \]

2. Define an action of this free algebra on $B$ in which the generators act by the entries of the quantum automorphisms $\sigma_d$:
\[ \triangleright: k\langle s_{d,ij} \rangle \otimes B \to B, \quad s_{d,ij} \triangleright a := \sigma_{d,ij}(a). \]
In view of (8), this turns $B$ into a $k\langle s_{d,ij} \rangle$-module algebra, that is, for any $X \in k\langle s_{d,ij} \rangle$ and $a, b \in B$, we have
\[ X \triangleright (ab) = (X_1 \triangleright a)(X_2 \triangleright b). \]

3. If $I \subseteq k\langle s_{d,ij} \rangle$ is the ideal generated by all elements of the form
\[ \sum_{r=1}^{n} s_{d,ir} s_{d+1,jr} - \delta_{ij}, \quad \sum_{r=1}^{n} s_{d+1,ri} s_{d,rj} - \delta_{ij} \]
for some $d, i, j$, then $k\langle s_{d,ij} \rangle/I$ becomes a Hopf algebra with (invertible) antipode induced by

$$S(s_{d,ij}) := s_{d+1,ji},$$

and the action $\triangleright$ of $k\langle s_{d,ij} \rangle$ on $B$ descends by construction to this quotient. That is, if by abuse of notation we also denote by $\sigma$ the action viewed as a morphism

$$\sigma : k\langle s_{d,ij} \rangle \to \text{End}_k(B), \quad X \mapsto X \triangleright -,$$

then $I \subseteq \ker \sigma$ so that $\sigma$ descends to an algebra morphism $k\langle s_{d,ij} \rangle/I \to \text{End}_k(B)$ that we still denote by $\sigma$.

(4) The Hopf algebra $k\langle s_{d,ij} \rangle/I$ depends only on the size $n$ of the matrix $\sigma \in M_n(\text{End}_k(B))$ and not on the choice of $\sigma$ or $B$. However, we finally define:

**Definition 3.1.1.** We denote by $H_{\sigma}$ the Hopf image of

$$\sigma : k\langle s_{d,ij} \rangle/I \to \text{End}_k(B).$$

That is, $H_{\sigma}$ is the universal quotient Hopf algebra that acts on $B$. In other words, $H_{\sigma}$ is the quotient of $k\langle s_{d,ij} \rangle/I$ by the sum $J_{\sigma}$ of all Hopf ideals contained in $\ker \sigma$, see [BB10, Theorem 2.1] for further information.

**Remark 3.1.1.** More abstractly, an algebra morphism $\sigma : B \to M_n(B)$ is the same as a measuring of $B$ by the coalgebra $C := M_n(k)^*$, the dual of the algebra $M_n(k)$. Steps (1)-(3) construct the free Hopf algebra with invertible antipode on this coalgebra, see e.g. [Chi11] and the references therein. The assumption of strong invertibility implies that the measuring of $B$ by $C$ extends to the free Hopf algebra, and taking the Hopf image yields the universal quotient Hopf algebra that has $B$ as a module algebra.

In the upper triangular case, the results obtained in the previous section show that $H_{\sigma}$ is a finitely generated pointed Hopf algebra:

**Proposition 3.1.1.** If $\sigma$ is upper triangular, then we have:

1. $H_{\sigma}$ is generated as an algebra by the classes $[s_{0,ij}]$ with $i \leq j$, together with $[s_{1,ii}] = [s_{0,ii}]^{-1}$.
2. $H_{\sigma}$ is pointed.

**Proof.** The first claim is shown in the same way as Proposition 2.1.1. The second claim uses a standard argument: define a Hopf algebra filtration $\{C_f\}$ of $H_{\sigma}$ by assigning to $[s_{d,ij}]$ the filtration degree $j - i$,

$$C_f = \text{span}_k \{ [s_{d_1,i_1,j_1}] \cdots [s_{d_l,i_l,j_l}] \mid \sum q_j - q_i \leq f \}. $$
This is an algebra filtration by definition and a coalgebra filtration as \([s_{d,ij}] = 0\) if \(i > j\). As the \([s_{d,ij}]\) generate \(H_\sigma\) as an algebra, it is exhaustive. If \(S \subseteq H_\sigma\) is a simple coalgebra, then \(\dim_k S < \infty\), so there exists a minimal \(f \geq 0\) with \(S \subseteq C_f, S \nsubseteq C_{f-1}\), and if \(f > 0\), it is immediately verified that \(S \cap C_{f-1}\) is a proper non-zero subcoalgebra of \(S\), contradicting the fact that \(S\) is simple. Finally, if \(S \subseteq C_0\) then \(S\) is spanned by group-likes and the span of any group-like is a subcoalgebra. So as \(S\) is simple, it is one-dimensional. \(\square\)

3.2. Application to the cusp. For the quantum automorphism described in Section 2.7, we abbreviate

\[K := [s_{0,22}], \quad E := [s_{0,12}], \quad F := [s_{0,23}], \quad Z := [s_{0,13}] \in H_\sigma.\]

By Proposition 3.1.1, \(H_\sigma\) is generated as an algebra by these elements whose images in \(U_\sigma\) are the operators \(K,E,F,Z\) from Lemma 2.7.1. Furthermore, the fact that \([s_{0,ij}] = 0\) for \(i > j\) implies that \(K\) is group-like, that is, its coproduct is given by

\[\Delta(K) = K \otimes K,\]

that \(E, F\) are \((1,K)\)- respectively \((K,1)\)-twisted primitive,

\[\Delta(E) = 1 \otimes E + E \otimes K, \quad \Delta(F) = K \otimes F + F \otimes 1,\]

and \(Z\) is of degree 2 with respect to the coradical filtration of \(H_\sigma\),

\[\Delta(Z) = 1 \otimes Z + E \otimes F + Z \otimes 1.\]

We will now obtain a presentation of \(H_\sigma\) as an algebra.

First, we observe that the definition of \(H_\sigma\) implies:

\[\text{Lemma 3.2.1.} \quad \text{We have } K^2 = 1 \text{ and } F = -KE.\]

\[\text{Proof.} \quad K^2 - 1 \text{ and } KE + F \text{ are in the kernel of the representation } \sigma: H_\sigma \to U_\sigma. \text{ The coproducts of these elements are }\]

\[\Delta(K^2 - 1) = K^2 \otimes K^2 - 1 \otimes 1 = K^2 \otimes (K^2 - 1) + (K^2 - 1) \otimes 1,\]

\[\Delta(KE + F) = K \otimes (KE + F) + (KE + F) \otimes 1,\]

It follows that each one generates a Hopf ideal in \(H_\sigma\) which is in the kernel of \(\sigma\), so by definition of \(H_\sigma\), these elements vanish. \(\square\)

Thus \(H_\sigma\) is generated as an algebra by \(K, E, Z\).

Second, we decompose \(E\) and \(Z\) into eigenvectors of the map given by conjugation by \(K\); that is, we define

\[E_\pm := \frac{1}{2}(E \pm KEK), \quad Z_\pm := \frac{1}{2}(Z \pm KZK).\]

By the definition of these elements, they (anti)commute with \(K\):
Lemma 3.2.2. We have $KE_\pm = \pm E_\pm K$ and $KZ_\pm = \pm Z_\pm K$.

The coproduct of these elements and their action on $B$ is given by
\[
\Delta(E_\pm) = 1 \otimes E_\pm + E_\pm \otimes K, \\
\Delta(Z_+) = 1 \otimes Z_+ - E_+ \otimes E_+ K - E_- \otimes E_- K + Z_+ \otimes 1, \\
\Delta(Z_-) = 1 \otimes Z_- - E_+ \otimes E_- K - E_- \otimes E_+ K + Z_- \otimes 1,
\]

\[
\sigma(E_+)(t^n) = \begin{cases} t^n & n \text{ is odd}, \\ 0 & n \text{ is even}, \end{cases} \quad \sigma(E_-)(t^n) = \begin{cases} -it^{n-1} & n \text{ is odd}, \\ 0 & n \text{ is even}, \end{cases}
\]

\[
\sigma(Z_+)(t^n) = \begin{cases} \frac{n-3}{6}t^{n-2} - \frac{1}{2}t^n & n \text{ is odd}, \\ \frac{n}{6}t^{n-1} & n \text{ is even}, \end{cases} \quad \sigma(Z_-)(t^n) = \begin{cases} 0 & n \text{ is odd}, \\ 0 & n \text{ is even}, \end{cases}
\]

From this, we obtain in a similar manner as in Lemma 3.2.1:

Lemma 3.2.3. We have $Z_- = -E_+ E_-$, $E^2 = 0$, $E_+ = -\frac{1}{2}(K - 1)$.

Proof. It follows from the above and the relation $K^2 = 1$ that the elements $Z_+ + E_+ E_-$ and $E^2$ are primitive while $E_+ + \frac{1}{2}(K - 1)$ is $(1, K)$-twisted primitive, and they are straightforwardly verified to be in $\text{ker } \sigma$, hence as in Lemma 3.2.1 it follows that they vanish in $H_\sigma$. □

So $H_\sigma$ is generated as an algebra by $K, E_-$ and $Z_+$.

Finally, we abbreviate
\[
Y := 6Z_+ - \frac{3}{2}(K - 1), \quad D := iE_-, \quad C := YD - DY.
\]

Their coproduct is given by
\[
\Delta(Y) = 1 \otimes Y - 6D \otimes DK + Y \otimes 1, \quad \Delta(C) = 1 \otimes C + C \otimes K
\]
and they act on $B$ by the operators
\[
(Y(t^n)) := \begin{cases} (n-3)t^{n-2} & n \text{ odd}, \\ nt^{n-2} & n \text{ even}, \end{cases} \quad (D(t^n)) := \begin{cases} t^{n-1} & n \text{ odd}, \\ 0 & n \text{ even}, \end{cases}
\]

as well as
\[
(C(t^n)) := \begin{cases} 2t^{n-3} & n \text{ odd}, \\ 0 & n \text{ even}. \end{cases}
\]

Their commutation relations (as elements in $H_\sigma$) are as follows:

Lemma 3.2.4. We have $YK = KY$, $KC = -CK$, $DC = -CD$, and $YC = CY$, $C^2 = 0$.

Proof. The relations $YK = KY, KC = -CK, DC = -CD$ follow from the definition of $Y, C, D$ and the commutation relations already
obtained. The final two relations follow as in Lemma 3.2.1; $YC - CY$ is $(1, K)$-twisted primitive while $C^2$ is primitive. □

**Remark 3.2.1.** Note that we can express the operators $Y$ and $D$ in terms of $K$, the differential operator $\frac{d}{dt}$, and the multiplication operators by $t^{-m}$ as

$$D = -\frac{1}{2}t^{-1}(K - 1), \quad Y = t^{-1}\frac{d}{dt} + \frac{3}{2}t^{-2}(K - 1).$$

The two summands in $Y$ can be considered separately as operators on $k(t^{-1})$ that both restrict to $k[t^{-1}]$, but only the sum restricts to $k[t^2, t^3]$. As operators on $k(t)$, $Y_0 := t^{-1}\frac{d}{dt}$ is a derivation and $Y_1 := \frac{3}{2}t^{-2}(K - 1)$ is a twisted derivation,

$$Y_0(fg) = fY_0(g) + Y_0(f)g, \quad Y_1(fg) = fY_1(g) + Y_1(f)K(g),$$

so it is a rather non-trivial fact that their sum is a twisted differential operator of order 2 on $k[t^2, t^3]$. The other generator $D$ is a twisted derivation,

$$D(fg) = fD(g) + D(f)K(g).$$

Our aim is to prove that the relations we have found are complete. In order to do so, we define the auxiliary Hopf algebra $\tilde{H}_\sigma := \langle \tilde{K}, \tilde{D}, \tilde{Y} \rangle / I$,

$$I := \langle \tilde{K}^2 - 1, \tilde{K}\tilde{D} + \tilde{D}\tilde{K}, \tilde{K}Y - Y\tilde{K}, \tilde{Y}^2\tilde{D} - 2\tilde{Y}\tilde{D}\tilde{Y} + \tilde{D}\tilde{Y}^2, \tilde{D}^2 \rangle$$

as the algebra generated by $\tilde{K}, \tilde{D}, \tilde{Y}$ satisfying the relations established in the lemmata in this subsection, equipped with the coproduct given on generators by the same formulas as in $H_\sigma$. Bergman’s diamond lemma [Ber77] immediately yields:

**Lemma 3.2.5.** If $\tilde{C} := \tilde{Y}\tilde{D} - \tilde{D}\tilde{Y}$, then the set

$$\{\tilde{C}^a\tilde{D}^b\tilde{K}^c\tilde{Y}^d \mid a, b, c \in \{0, 1\}, d \in \mathbb{N}\}$$

is a $k$-vector space basis of $\tilde{H}_\sigma$.

We now describe the algebra morphism $\sigma: \tilde{H}_\sigma \to U_\sigma$; by showing that its kernel contains no Hopf ideal, we will then prove that $\tilde{H}_\sigma = H_\sigma$.

By direct computation, one establishes that the generators $K, C, D, Y$ of $U_\sigma$ satisfy the following relations in addition to those satisfied by $\tilde{K}, \tilde{C}, \tilde{D}$ and $\tilde{Y}$:

**Lemma 3.2.6.** We have $CD = 0$, $KC = C$, $KD = D$.

A moment’s thought tells that this a complete presentation of $U_\sigma$:

**Proposition 3.2.1.** The above relations define a presentation of $U_\sigma$. 
Proof. The claim is that if we define an abstract algebra \( k\langle K, C, D, Y \rangle / R \), where \( R \) is the ideal generated by the relations in Lemma 3.2.6 together with those that follow from the ones between \( \tilde{K}, \tilde{C}, \tilde{D}, \tilde{Y} \) in \( I \), then the resulting algebra morphism \( k\langle K, C, D, Y \rangle / R \to U_\sigma \) is an isomorphism. To do so, observe that using the \( k \)-vector space basis of \( \tilde{H}_\sigma \) and the relations stated in the current proposition, we obtain a \( k \)-vector space basis of \( k\langle K, C, D, Y \rangle / R \) of the form

\[
\{ Y^a, CY^b, DY^c, KY^d \mid a, b, c, d \in \mathbb{N} \}.
\]

It is now straightforward to show that these operators are mapped to linearly independent elements of \( \text{End}_k(B) \). \( \square \)

Remark 3.2.2. Thus \( U_\sigma \) is the Ore extension of the 4-dimensional subalgebra spanned by \( \{ K, C, D, Y \} \) by the derivation \( C \mapsto 0, D \mapsto C, K \mapsto 0 \) (which is given by the commutator with \( Y \)).

Remark 3.2.3. Note also that \( U_\sigma \) carries a natural grading in which \( \deg K = 0, \deg C = 3, \deg D = 1, \deg Y = 2 \), and that \( B \) becomes a graded \( U_\sigma \)-module, \( (U_\sigma)_i B_j \subseteq B_{j-i} \).

In order to proceed with the proof that \( \tilde{H}_\sigma \cong H_\sigma \), note that by Lemma 3.2.5 the subalgebra of \( \tilde{H}_\sigma \) generated by \( \tilde{Y} \) is as an abstract algebra the polynomial algebra \( k[\tilde{Y}] \) and that \( \{ C^a D^b K^c \mid a, b, c \in \{0, 1\} \} \) is a basis of \( \tilde{H}_\sigma \) as a right \( k[\tilde{Y}] \)-module, so as such, \( \tilde{H}_\sigma \) has rank 8. Similarly, \( U_\sigma \) becomes a right \( k[\tilde{Y}] \)-module where \( \tilde{Y} \) acts via right multiplication by \( Y \), and by the above proposition, \( \{ K, C, D, K \} \) is a basis of this \( k[\tilde{Y}] \)-module, so this has rank 4. The map \( \sigma: \tilde{H}_\sigma \to U_\sigma \) is right \( k[\tilde{Y}] \)-linear, and we have:

**Lemma 3.2.7.** As a right \( k[\tilde{Y}] \)-module, \( \ker \sigma \) is free with basis given by

\[
\{ \tilde{C}\tilde{K} + \tilde{C}, \tilde{D}\tilde{K} + \tilde{D}, \tilde{C}\tilde{D}, \tilde{C}\tilde{D}\tilde{K} \}.
\]

As a last ingredient, we list the one-dimensional representations of \( \tilde{H}_\sigma \) and their left and right hit actions on \( \tilde{H}_\sigma \):

**Lemma 3.2.8.** For any \( s \in \{-1, 1\}, \lambda \in k \), there is an algebra morphism

\( \chi_{s, \lambda}: \tilde{H}_\sigma \to k, \quad \tilde{K} \mapsto s, \quad \tilde{D} \mapsto 0, \quad \tilde{Y} \mapsto \lambda \)

and any algebra morphism \( \tilde{H}_\sigma \to k \) is of this form.

**Proof.** Immediate. \( \square \)

These define algebra automorphisms \( L_{s, \lambda}, R_{s, \lambda}: \tilde{H}_\sigma \to \tilde{H}_\sigma \) given by

\[
R_{s, \lambda}(h) := \chi_{s, \lambda}(h(1))h(2), \quad L_{s, \lambda} := h(1)\chi_{s, \lambda}(h(2)).
\]
On the generators of $\tilde{H}_\sigma$, these automorphisms are given by
\[ L_{s,\lambda}(\tilde{K}) = R_{s,\lambda}(\tilde{K}) = s\tilde{K}, \]
\[ L_{s,\lambda}(\tilde{D}) = s\tilde{D}, \quad R_{s,\lambda}(\tilde{D}) = \tilde{D}, \]
\[ L_{s,\lambda}(\tilde{Y}) = R_{s,\lambda}(\tilde{Y}) = \tilde{Y} + \lambda. \]

All one-dimensional representations of $\tilde{H}_\sigma$ descend to $U_\sigma$:

**Lemma 3.2.9.** We have $\ker \sigma \subseteq \bigcap_{s,\lambda} \ker \chi_{s,\lambda}$.

**Proof.** This follows immediately from Lemma 3.2.7. \hfill \Box

We are now ready to prove:

**Proposition 3.2.2.** The quotient $\tilde{H}_\sigma \to H_\sigma$ is an isomorphism.

**Proof.** Assume that $J \subseteq \ker \sigma$ is a nontrivial Hopf ideal. Then for all $h \in J$, we have $\Delta(h) \in J \otimes \tilde{H}_\sigma + \tilde{H}_\sigma \otimes J$; by the last lemma, applying $\chi_{s,\lambda}$ to the left or to the right tensor component yields an element in $J$. That is, we have
\[ L_{s,\lambda}(J) = R_{s,\lambda}(J) = J. \]

The maps $R_{s,\lambda}, L_{s,\lambda}$ act on the basis elements from Lemma 3.2.5 by
\[ L_{s,\lambda}(\tilde{C}^a \tilde{D}^b \tilde{K}^c \tilde{Y}^d) = s^{a+b+c} \tilde{C}^a \tilde{D}^b \tilde{K}^c (\tilde{Y} + \lambda)^d, \]
\[ R_{s,\lambda}(\tilde{C}^a \tilde{D}^b \tilde{K}^c \tilde{Y}^d) = s^c \tilde{C}^a \tilde{D}^b \tilde{K}^c (\tilde{Y} + \lambda)^d. \]

Thus if
\[ X = \sum_{abcd} t_{abcd} \tilde{C}^a \tilde{D}^b \tilde{K}^c \tilde{Y}^d \in J, \quad t_{abcd} \in k \]
and $d_{\text{max}}(X)$ is the largest $d$ such that $t_{abcd} \neq 0$ for some $a, b, c$, then unless $d_{\text{max}} = 0$,
\[ X' := X - R_{1,1}(X) \in J \]
is a non-zero element with $d_{\text{max}}(X') = d_{\text{max}}(X) - 1$. So if $J \neq 0$, it necessarily contains a non-zero element of the form
\[ X = \sum_{abc} t_{abc} \tilde{C}^a \tilde{D}^b \tilde{K}^c. \]

Using now $R_{-1,0}$ instead of $R_{1,1}$, the analogous argument shows that $J$ contains a non-zero element of the form
\[ X = \sum_{ab} t_{ab} \tilde{C}^a \tilde{D}^b. \]

Considering finally
\[ X \pm L_{-1,0}(X) \]
we find that
\[ t_{00} + t_{11} \tilde{C} \tilde{D} \in J, \quad t_{01} \tilde{C} + t_{10} \tilde{D} \in J. \]
Since $C, D \in U_{\sigma}$ are linearly independent and $J \subseteq \ker \sigma$, the second element vanishes. Using that the coproduct of $\tilde{C} \tilde{D}$ is
\[
\Delta(\tilde{C} \tilde{D}) = 1 \otimes \tilde{C} \tilde{D} + \tilde{C} \otimes \tilde{K} \tilde{D} + \tilde{D} \otimes \tilde{C} \tilde{K} + \tilde{C} \tilde{D} \otimes 1
\]
and that $\tilde{C}, \tilde{D}$ are linearly independent modulo $\ker \sigma$, one also concludes that the first element vanishes, a contradiction. \qed

Remark 3.2.4. Thus $H_{\sigma}$ does not act faithfully on $B$ – for example, we now know that $CD \in H_{\sigma}$ is a non-zero element while $CD = \sigma(CD) = 0$. However, $H_{\sigma}$ acts by definition inner faithfully on $B$, that is, the action does descend to algebra, but not to Hopf algebra quotients of $H_{\sigma}$.

Remark 3.2.5. Thus $H_{\sigma}$ is the Ore extension of the finite-dimensional subalgebra generated by $K, C, D$ by the derivation given by
\[
K \mapsto 0, \quad D \mapsto C, \quad C \mapsto 0.
\]
In particular, $H_{\sigma}$ has Gelfand-Kirillov dimension 1, but note that it is not semiprime (the right ideal generated by $C$ is a nonzero ideal that squares to zero) so is not part of the recent classification of these Hopf algebras (see e.g. [BZ10, Liu09] and the references therein).

Remark 3.2.6. Note that the subalgebra generated by $K$ and $D$ (and similarly the subalgebra generated by $K$ and $C$) is isomorphic as Hopf algebra to Sweedler’s 4-dimensional Hopf algebra. Recall that for any $c \in k$,
\[
R_c := \frac{1}{2}(1 \otimes 1 + 1 \otimes K + K \otimes 1 - K \otimes K)
+ \frac{c}{2}(D \otimes D - D \otimes KD + KD \otimes D + KD \otimes KD)
\]
is a universal R-matrix (a quasitriangular structure) of this Hopf subalgebra (cf. [Rad12, Exercise 12.2.11]). This does however not define a quasitriangular structure on $H_{\sigma}$ as $R_c \Delta(Y)R_c^{-1} \neq \Delta^{\text{op}}(Y)$, but at least for $c = 0$, the corresponding braiding on $B \otimes B$ is a morphism of $H_{\sigma}$-modules. This braiding is simply the standard nontrivial symmetric braiding on the category of graded vector spaces,
\[
t^i \otimes t^j \mapsto (-1)^{ij}t^j \otimes t^i.
\]

3.3. The Hopf algebra $A_{\sigma}$. We now pass to a dual picture: assume that $\sigma$ is a locally finite quantum automorphism of a $k$-algebra $B$. Then the $H_{\sigma}$-action on $B$ arises by dualisation from an $H_{\sigma}^\circ$-coaction, where $H_{\sigma}^\circ$ denotes the Hopf dual of $H_{\sigma}$, that is, the universal Hopf algebra contained in the vector space dual $H_{\sigma}^\ast$. In other words, $B$ is a right $H_{\sigma}^\circ$-comodule algebra with a coaction that we denote by
\[
\rho: B \to B \otimes H_{\sigma}^\circ, \quad b \mapsto b_{(0)} \otimes b_{(1)}.
\]
Definition 3.3.1. We denote by $A_{\sigma} \subseteq H_{\sigma}^\circ$ the Hopf subalgebra generated by the matrix coefficients $\{f(b_{(0)})b_{(1)} \mid b \in B, f \in B^*\}$ of $\rho$.

When $H_{\sigma}$ is infinite-dimensional, $A_{\sigma}$ could be a proper Hopf subalgebra of $H_{\sigma}^\circ$, but note that it is always dense:

Proposition 3.3.1. The restriction of the pairing of $H_{\sigma}^\circ \otimes H_{\sigma} \longrightarrow k$ to $A_{\sigma} \otimes H_{\sigma}$ is non-degenerate.

Proof. The degeneration space
$$\{X \in H_{\sigma} \mid a(X) = 0 \forall a \in A_{\sigma}\}$$
is a Hopf ideal that acts trivially on $B$, hence vanishes by the definition of $H_{\sigma}$. □

Note that if $M \subseteq B$ is any finite-dimensional $H_{\sigma}$-submodule that generates $B$ as an algebra, then we have
$$B \cong TM/R$$
as an $H_{\sigma}$-module algebra, where $TM$ is the tensor algebra of $M$ (over $k$) and $R$ is the 2-sided ideal of relations that hold among the elements of $M$ in the algebra $B$. Then we have:

Lemma 3.3.1. $A_{\sigma}$ is generated as a Hopf algebra by the matrix coefficients of $M$.

Proof. The matrix coefficients of $M^{\otimes n}$ are sums of products of $n$ matrix coefficients of $M$, and the space of matrix coefficients of a quotient co-module $M^{\otimes n}/(R \cap M^{\otimes n})$ is a subspace of the space of matrix coefficients of $M^{\otimes n}$. □

Alternatively, this can be formulated in coordinates: if $e_1, \ldots, e_{\dim_k M}$ is a vector space basis of $M$, then $A_{\sigma}$ is generated as a Hopf algebra by the functionals $a_{ij} \in H_{\sigma}^\circ$, $i, j = 1, \ldots, \dim_k M$, for which
$$X \triangleright e_i = \sigma(X)(e_i) = \sum_{j=1}^{\dim_k M} a_{ji}(X)e_j, \quad X \in H_{\sigma},$$
so the coaction $\rho$ is given by
$$M \rightarrow M \otimes A_{\sigma}, \quad e_i \mapsto \sum_{j=1}^{\dim_k M} e_j \otimes a_{ji}.$$

Remark 3.3.1. In general, the matrix coefficients of $M$ do not generate $A_{\sigma}$ as an algebra – the subalgebra that they generate is a subbialgebra of $A_{\sigma}$ as the span of the matrix coefficients is a subcoalgebra, but this subbialgebra is not closed under the antipode in general. However, if
the matrix coefficients can be chosen to be upper triangular (that is, there is a vector space basis of $M$ such that $a_{ij} = 0$ for $i > j$), then the same arguments that were used in Proposition 3.1.1 show that $A_\sigma$ is generated by the $a_{ij}$ together with the $a_{ii}^{-1}$, and that $A_\sigma$ is a pointed Hopf algebra.

3.4. Embedding $B$ into $A_\sigma$. Now assume that $\chi: B \to k$ is an algebra map, that is, a one-dimensional representation of $B$. This induces a map (see [EW14, Section 3] for a more detailed discussion of this map)

$$\iota := (\chi \otimes \text{id}_{A_\sigma}) \circ \rho: B \to A_\sigma, \quad b \mapsto \chi(b(0))b(1).$$

By definition, this is a morphism of algebras and of right $A_\sigma$-comodules and hence it maps $B$ to a right coideal subalgebra of $A_\sigma$.

**Proposition 3.4.1.** $\iota$ is injective if and only if for all $b \in B$, $b \neq 0$, there exists $X \in H_\sigma$ with $\chi(X \triangleright b) \neq 0$.

**Proof.** The map $\iota$ is not injective if there exists $b \in B$, $b \neq 0$, with

$$\iota(b) = \chi(b(0))b(1) = 0.$$

This is an element in $A_\sigma \subseteq H_\sigma^\circ$, so it is zero if and only if it pairs trivially with all elements $X \in H_\sigma$. Thus $\iota$ is not injective if and only if there exists $b \in B$, $b \neq 0$, such that

$$X(\iota(b)) = \chi(b(0))b(1)(X) = \chi(X \triangleright b) = 0$$

for all $X \in H_\sigma$. \hfill \square

If this condition is satisfied, then $\iota$ embeds $B$ as a right coideal subalgebra into $A_\sigma$. In particular, when $B = k[V]$ is the coordinate ring of an algebraic set $V$, then $\chi$ corresponds to a point $p \in V$. The above proposition states that $B$ can be embedded as a right coideal subalgebra into $A_\sigma$ provided that there exists a point $p \in V$ such that for any non-zero regular function $b: V \to k$ there exists some $X \in H_\sigma$ such that the function $X \triangleright b$ does not vanish at $p$.

3.5. The cusp again. To compute a full presentation of $A_\sigma$ in a given example is tedious, but relatively straightforward. Like elsewhere, we illustrate the theory with our main example:

**Proposition 3.5.1.** If $B = k[t^2, t^3]$ and $M$ is the $H_\sigma$-module $F_3$ from (7) with basis $e_1 = 1, e_2 = t^2, e_3 = t^3$, then we have:

1. There is a surjective algebra morphism

$$\pi: k\langle \gamma, \varphi, \psi \rangle \to A_\sigma$$
given by $\pi(\gamma) = a_{13}$, $\pi(\varphi) = a_{23}$, $\pi(\psi) = a_{33}$ whose kernel is the ideal generated by
\[
\psi^2 - 1, \quad \gamma\psi + \psi\gamma, \quad \varphi\psi + \psi\varphi,
27\gamma^2 - \varphi^6, \quad 3(\gamma\varphi + \varphi\gamma) - \varphi^4.
\]
(11)

(2) In this presentation, the coalgebra structure of $A_\sigma$ is given by
\[
\Delta(\psi) = \psi \otimes \psi, \quad \Delta(\varphi) = 1 \otimes \varphi + \varphi \otimes \psi,
\Delta(\gamma) = 1 \otimes \gamma + \frac{1}{3}\varphi^2 \otimes \varphi + \gamma \otimes \psi,
\Delta(\delta) = 1 \otimes \delta + \delta \otimes 1,
\epsilon(\psi) = 1, \quad \epsilon(\gamma) = \epsilon(\varphi) = 0,
\]
and its antipode is given by
\[
S(\psi) = \psi, \quad S(\varphi) = -\varphi\psi, \quad S(\gamma) = (\frac{1}{3}\varphi^3 - \gamma)\psi.
\] (13)

For the proof that will be split into several lemmata, we introduce a redundant generator $\delta$ and first observe:

**Lemma 3.5.1.** Let $J \triangleleft k\langle \gamma, \varphi, \psi, \delta \rangle$ be the ideal generated by the elements (11) together with $\delta - \frac{1}{3}\varphi^2$. Then we have:

(1) $k\langle \gamma, \varphi, \psi, \delta \rangle$ carries a unique bialgebra structure such that
\[
\Delta(\psi) = \psi \otimes \psi, \quad \Delta(\varphi) = 1 \otimes \varphi + \varphi \otimes \psi,
\Delta(\gamma) = 1 \otimes \gamma + \delta \otimes \varphi + \gamma \otimes \psi, \quad \Delta(\delta) = 1 \otimes \delta + \delta \otimes 1,
\epsilon(\psi) = 1, \quad \epsilon(\gamma) = \epsilon(\varphi) = \epsilon(\delta) = 0.
\] (14)

(2) The ideal $J$ is a coideal, so this bialgebra structure descends to
\[
\tilde{A}_\sigma := k\langle \gamma, \varphi, \psi, \delta \rangle/J.
\]
(3) The bialgebra $\tilde{A}_\sigma$ is a Hopf algebra whose antipode is given on the generators by (13) and $S(\delta) = -\delta$.

**Proof.** All claims are verified by straightforward computations that we leave to the reader. \qed

Next, we note:

**Lemma 3.5.2.** There is a surjective bialgebra morphism
\[
\pi: k\langle \gamma, \varphi, \psi, \delta \rangle \to A_\sigma
\]
satisfying $\pi(\gamma) = a_{13}$, $\pi(\varphi) = a_{23}$, $\pi(\psi) = a_{33}$, and $\pi(\delta) = a_{12}$.

**Proof.** Recall first that the values of the functionals $a_{ji}$ on the generators $K, D, Y$ of $H_\sigma$ are by (9) and (6) given by the following matrices
As the generators of $H_\sigma$ act by upper triangular matrices, all elements in $H_\sigma$ act by upper triangular matrices, hence as elements of $H_\star^{\sigma}$, the $a_{ij}$ with $i>j$ vanish (this is just a restatement of the fact that $F_i$ is a filtration of $B$ by $H_\sigma$-submodules).

Finally, $a_{11} = a_{22} = \varepsilon_{H_\sigma} = 1_{A_\sigma}$ is the counit of the coalgebra $H_\sigma$ hence the unit of the algebra $A_\sigma$. Using this it is immediately verified that the algebra morphism $\pi$ defined on the generators $\gamma, \varphi, \psi, \delta$ as in the lemma is a bialgebra morphism, and Lemma 3.3.1 implies that $\pi$ is surjective. □

Next, we show that $\pi$ descends to a Hopf algebra surjection $\tilde{A}_\sigma \to A_\sigma$:

**Lemma 3.5.3.** $J \subseteq \ker \pi$.

**Proof.** The bialgebra map $\pi$ induces a pairing of bialgebras

$$\langle -, \rangle : H_\sigma \otimes k(\gamma, \varphi, \psi, \delta) \to k, \quad \langle X, a \rangle := \pi(a)(X).$$

To prove the lemma, one has to show that this pairing descends to a pairing between $H_\sigma$ and $\tilde{A}_\sigma$. In order to do so, it is sufficient to show that for each of the six relators $\xi$ that generate $J$ and for all $i, j, k \in \{0, 1\}, l \in \mathbb{N}$, we have

$$\langle C^i D^j K^k Y^l, \xi \rangle = 0.$$  

This is verified by straightforward computation. For example, using that $\langle -, \rangle$ is a pairing of bialgebras, one obtains

$$\langle C^i D^j K^k Y^l, \varphi^2 \rangle = \langle (C^i D^j K^k Y^l)(1), \varphi \rangle \langle (C^i D^j K^k Y^l)(2), \varphi \rangle$$

$$= \langle C^i D^j_k K^i_k Y^l(1), \varphi \rangle \langle C^i D^j_k K^i_k Y^l(2), \varphi \rangle$$

$$= \langle (C^i(1), \varphi(1)) \langle D^j_{(1)}, \varphi(2) \rangle \langle K^k(3), \varphi(4) \rangle \langle Y^l(1), \varphi(4) \rangle \rangle$$

$$\langle (C^i(2), \varphi(1)) \langle D^j_{(2)}, \varphi(2) \rangle \langle K^k(3), \varphi(4) \rangle \langle Y^l(2), \varphi(4) \rangle \rangle\rangle.$$  

Inserting the explicit coproducts and at last the values (15) of the pairings of the generators, one obtains that the above is equal to

$$\langle C^i D^j K^k Y^l, 3\delta \rangle = 3\delta_{i0} \delta_{j0} \delta_{l1}.$$
so that \( \langle - , \delta - \frac{1}{3} \phi^2 \rangle \) vanishes as a \( k \)-linear functional on \( H_\sigma \). The other five relators are treated in the same way.

The proof that \( \pi : \tilde{A}_\sigma \to A_\sigma \) is also injective relies on:

**Lemma 3.5.4.** The set \( \{ \gamma^a \phi^b \psi^c \mid a, c \in \{0, 1\}, b \in \mathbb{N} \} \) is a \( k \)-vector space basis of \( \tilde{A}_\sigma \).

**Proof.** Like Lemma 3.2.5, this is a standard application of Bergman’s diamond lemma. \( \square \)

Thus to prove the injectivity of \( \pi \), one has to show that the elements \( \pi(\gamma^a \phi^b \psi^c) \in A_\sigma \) are linearly independent over \( k \). This is maybe shown most easily by explicitly computing the values of the functionals:

**Lemma 3.5.5.** The dual pairing \( \langle - , - \rangle : H_\sigma \otimes \tilde{A}_\sigma \to k \) satisfies

\[
\langle C^i D^j K^k Y^l, \gamma^a \phi^b \psi^c \rangle = \delta_{j+2l,0} \delta_{ia} (-1)^{(a+c)+ib+k(a+b+c)} a^6 l!
\]

**Proof.** A direct computation and a nested induction on \( b \) and \( l \) shows

\[
\Delta(\gamma^a \phi^b \psi^c) = (1 \otimes \gamma + \frac{1}{3} \phi^2 \otimes \phi + \gamma \otimes \psi)^a \\
\cdot \sum_{l=0}^{b} (1 - \lfloor (b+1)l \rfloor) \left( \begin{array}{c} b \\lfloor l/2 \rfloor \\ l \end{array} \right) \phi^l \psi^c \otimes \phi^{b-l} \psi^{[b]_{-1+c}},
\]

where as before \( \lfloor n \rfloor_q = 1 + q + \cdots + q^{n-1} \), which for \( q = -1 \) is 0 if \( n \) is even and 1 if \( n \) is odd, and

\[
\lfloor n/2 \rfloor := \begin{cases} (n-1)/2 & \text{if } n \text{ odd}, \\ n/2 & \text{if } n \text{ even}. \end{cases}
\]

Using this and the formulas for the coproduct of and the relations between the generators of \( H_\sigma \) respectively \( \tilde{A}_\sigma \) as well as their pairing (15), one computes

\[
\langle C^i D^j Y^l K^k, \gamma^a \phi^b \psi^c \rangle = \langle C^i D^j Y^l \otimes K^k, \Delta(\gamma^a \phi^b \psi^c) \rangle \\
= \langle C^i D^j Y^l, \gamma^a \phi^b \psi^c \rangle \langle K^k, \psi^a \phi^{[b]_{-1+c}} \rangle \\
= \langle C^i \otimes D^j Y^l, \Delta(\gamma^a \phi^b \psi^c) \rangle (-1)^{k(a+c+[b]_{-1})}.
\]

If \( i = 0 \), the above is equal to

\[
\ldots = \langle D^j Y^l, \gamma^a \phi^b \psi^c \rangle (-1)^{k(a+c+[b]_{-1})} \\
= \langle \Delta(D^j Y^l), \gamma^a \phi^{b+c} \psi^c \rangle (-1)^{k(a+b+c)} \\
= \delta_{ia} \langle D^j Y^l, \phi^b \psi^c \rangle (-1)^{k(a+b+c)}.
\]
The last equality follows since the pairing of the first tensor component of $\Delta(D^jY^l)$ with $\gamma$ vanishes.

If $i = 1$, we have instead

$$
\ldots = \langle C \otimes D^jY^l, \Delta(\gamma^a \varphi^b \psi^c) \rangle (-1)^{k(a+c+[b^2]-1)}
= \langle D^jY^l, \varphi^b \psi^c \rangle 2\delta_{ia} (-1)^{ic} (-1)^{ac} (-1)^{k(a+c+[b^2]-1)}
= \langle D^jY^l, \varphi^b \psi^c \rangle 2\delta_{ia} (-1)^{ic} + ab + k(a+b+c).
$$

We have deliberately written $i$ instead of 0 respectively 1 in the above two cases as we now can merge them again:

$$
\langle C \otimes D^jY^l K^k, \gamma^a \varphi^b \psi^c \rangle = \langle D^jY^l, \varphi^b \psi^c \rangle 2\delta_{ia} (-1)^{ic} + ab + k(a+b+c)
= \langle \Delta(D^jY^l), \varphi^c \otimes \psi^a \rangle 2\delta_{ia} (-1)^{ic} + ab + k(a+b+c)
= \langle D^j \otimes Y^l, \Delta(\varphi^b) \rangle 2\delta_{ia} (-1)^{j(a+c)+ic} + ab + k(a+b+c)
= 6!l!(1 - [j + 2l + 1])_1(1 - 1)^{j} 2\delta_{j+2l} \delta_{ia}
\cdot (-1)^{j(a+c)+ic} + ab + k(a+b+c)
= \delta_{j+2l} \delta_{ia} (-1)^{j(a+c)+ic} + ab + k(a+b+c) 6!l!2^a.
$$

Therefore, if we define

$$
E_{uvw} := \frac{(-1)^{2[v/2](u+v)-v-u-w-w}}{6[v/2][u/2]!2u+1} C^a D^{v-2[v/2]} Y^{[v/2]} (1 + (-1)^{u+v+w} K^u),
$$

then these elements also form a basis of $H_\sigma$, and we have

$$
\langle E_{uvw}, \gamma^a \varphi^b \psi^c \rangle = \delta_{ua} \delta_{vb} \delta_{wc}.
$$

This shows that the elements $\pi(\gamma^a \varphi^b \psi^c) \in A_\sigma$ are linearly independent which finishes the proof of Proposition 3.5.1.

**Remark 3.5.1.** In our example, $\sigma$ is a $3 \times 3$-matrix with entries in $\text{End}_k(B)$ and $(a_{ij})$ is a $3 \times 3$-matrix with entries in $H_\sigma^0$, but be aware it is a pure coincidence that these sizes match.

Finally, we prove that the map $\iota: B \to A_\sigma$ is injective for every point $p$ on the cusp: the algebra morphisms $\chi: B = k[t^2, t^3] \to k$ are in bijection with the points $p = (\lambda^2, \lambda^3)$ on the cusp, $\lambda \in k$, and the algebra morphism $\iota$ is given on generators by

$$
t^2 \mapsto \lambda^2 + \frac{1}{3} \varphi^2, \quad t^3 \mapsto \gamma + \lambda^2 \varphi + \lambda^3 \psi.
$$

Now we observe:

**Lemma 3.5.6.** The map $\iota: B \to A_\sigma$ is injective for all algebra maps $\chi: B \to k$. 

Proof. The element \( X := Y + D \in H_\sigma \) acts by

\[
X(t^n) := \begin{cases} 
(n - 3)t^{n-2} + t^{n-1}, & n \text{ is odd}, \\
nt^{n-2} & n \text{ is even}.
\end{cases}
\]

In particular, if \( b \in F_d \setminus F_{d-1} \) is a polynomial of degree \( d > 0 \), then applying \( X^{d/2} \) when \( d \) is even and \( X^{(d+1)/2} \) when \( d \) is odd yields a non-zero scalar, that is, a constant regular function on the cusp. \( \square \)

Remark 3.5.2. To clean up the presentation of \( A_\sigma \), we finally introduce a new set of generators:

\[
a := \frac{1}{6} \varphi, \quad b := \frac{1}{6^2} \gamma - \frac{1}{6^3} \varphi^3, \quad c := \psi.
\]

The new relations between the new generators are

\[
ab + ba = ac + ca = bc + cb = 0, \quad 3b^2 = a^6, \quad c^2 = 1.
\]

Their coproduct, counit, and antipode are

\[
\Delta(a) = 1 \otimes a + a \otimes c, \quad \Delta(c) = c \otimes c
\]

\[
\Delta(b) = 1 \otimes b + a^2 \otimes a - a \otimes a^2 c + b \otimes c
\]

\[
\varepsilon(a) = \varepsilon(b) = 0, \quad \varepsilon(c) = 1,
\]

\[
S(a) = -ac, \quad S(b) = -bc, \quad S(c) = c.
\]

It is now evident that \( A_\sigma \) is a graded Hopf algebra with \( a, b, c \) of degree 1, 3, 0, respectively. The monomials \( a^l b^m c^n \), \( l \geq 0 \), \( m, n \in \{0, 1\} \) are a vector space basis, so \( A_\sigma \) has GK-dimension 1. One also observes that \( Z(A_\sigma) \) is the polynomial ring \( k[a^2] \). In terms of the new generators, the embedding of the cusp is given by

\[
t^2 \mapsto \lambda^2 1 + 12a^2, \quad t^3 \mapsto 6\lambda^2 a + 36a^3 + 36b + \lambda^3 c.
\]

3.6. Faithful flatness. To complete the proof that the embedding \( \iota : B \to A_\sigma \) turns \( B \) into a quantum homogeneous space in the sense of \([MS99]\), we remark that a result of Masuoka implies that \( A_\sigma \) is a faithfully flat \( B \)-module.

As a preliminary result, we need to compute the coradical of \( A_\sigma \):

**Proposition 3.6.1.** The Hopf algebra \( A_\sigma \) is pointed. Furthermore, for any point \( p \) in the cusp, \( A_\sigma \) is faithfully flat over \( \iota(B) \).

**Proof.** We just observed in Remark 3.5.2 that \( A_\sigma \) is a graded Hopf algebra with a vector space basis given by the monomials \( a^l b^m c^n \) which are of degree \( l + 3m \). It follows that the degree 0 part of \( A_\sigma \) is the group algebra \( \operatorname{span}_k \{1, c\} \cong k\mathbb{Z}_2 \).
By [Mas91, Theorem (1)], we only need to prove that the intersection of \(\iota(B)\) with the coradical span \(k\{1,c\}\) is invariant under the antipode \(S\) of \(A_\sigma\). However, the restriction of the antipode \(S\) to the coradical is the identity map (1 and \(c\) are their own inverses), so there is nothing to prove. \(\square\)

3.7. \(*\)-structures and involutions. As we have seen in Lemma 2.6.2, even the classification of the \(3 \times 3\) upper triangular quantum automorphisms of \(B = k[t^2, t^3]\) is rather involved. The explicit example studied since Section 2.7 was obtained by demanding in addition that the quantum automorphism is compatible with a chosen \(*\)-structure on \(B\). This might be of interest in its own right, for example as it is the starting point for the transition from the algebraic theory of Hopf algebras acting on rings to the analytic theory of locally compact quantum groups acting on \(C^*\)-algebras.

**Definition 3.7.1.** Assume that \(k\) is a field with a chosen field automorphism \(k \to k, \lambda \mapsto \bar{\lambda}\) that is involutive, i.e. which satisfies \(\bar{\bar{\lambda}} = \lambda\).

(1) For each \(k\)-vector space \(V\), we denote by \(\overline{V}\) the conjugate vector space which is the same abelian group but whose scalar multiplication is twisted by \(\bar{\cdot}\) to \(\lambda \overline{v} := \bar{\lambda}v, \lambda \in k, v \in V\).

(2) A \(*\)-structure on a \(k\)-algebra \(B\) is an involutive \(k\)-algebra isomorphism \(\ast: B \to \overline{B}^{op}\).

(3) An involution on a \(k\)-algebra \(P\) is an involutive \(k\)-algebra isomorphism \(\theta: P \to \overline{P}\).

So explicitly, a \(*\)-structure is a map \(B \to B, b \mapsto b^*\) satisfying for all \(a, b \in B, \lambda \in k\)
\[(\lambda a + b)^* = \bar{\lambda}a^* + b^*, \quad (ab)^* = b^*a^*, \quad a^{**} = a,
\]
while an involution is a map \(\theta: P \to P\) such that for all \(\lambda \in k, f, g \in P\)
\[\theta(\lambda f + g) = \bar{\lambda}\theta(f) + \theta(g), \quad \theta(fg) = \theta(f)\theta(g), \quad \theta(\theta(f)) = f.
\]

Typically, these notions are considered for \(k = \mathbb{C}\) with involution given by complex conjugation, see e.g. [KS97, Section 1.2.7].

The following is immediate:

**Lemma 3.7.1.** A \(*\)-structure on an algebra \(B\) induces an involution on \(P = \operatorname{End}_k(B)\) given by \(\theta(f) := \ast \circ f \circ \ast\).

On Hopf algebras, one demands the following compatibility between \(*\)-structures and involutions with the coalgebra structure:

**Definition 3.7.2.** Let \(H\) be a Hopf algebra.
(1) A Hopf $\ast$-structure on $H$ is a $\ast$-structure on the underlying algebra satisfying for all $h \in H$
\[(h^\ast)(1) \otimes (h^\ast)(2) = (h_{(1)})^\ast \otimes (h_{(2)})^\ast, \quad \varepsilon(h^\ast) = \overline{\varepsilon(h)}.\]

(2) A Cartan involution on $H$ is an involution $\theta$ on the underlying algebra such that for all $h \in H$, we have
\[\theta(h)(1) \otimes \theta(h)(2) = \theta(h_{(1)}) \otimes \theta(h_{(2)}), \quad \varepsilon(\theta(h)) = \overline{\varepsilon(h)}.\]

These structures correspond bijectively to each other:

**Lemma 3.7.2.** A map $\ast : H \to H$ is a Hopf $\ast$-structure if and only if $\ast \circ S$ is a Cartan involution.

**Proof.** This is mostly obvious; for a proof of $\ast \circ S \circ \ast \circ S = \text{id}_H$ that carries over verbatim to arbitrary ground fields, see e.g. [KS97, Proposition 1.2.7.10]. $\square$

**Remark 3.7.1.** More abstractly, a Cartan involution corresponds to a lift of the endofunctor $V \mapsto \overline{V}$ from the category of finite-dimensional $k$-vector spaces to that of finite-dimensional $H$-modules: since $H$ is a Hopf algebra, this category is a rigid monoidal category, that is, one can form tensor products and left and right duals of $H$-modules, and these structures are compatible with the forgetful functor to $k$-vector spaces. A Hopf $\ast$-structure on $H$ allows one to additionally define for any $H$-module $V$ the conjugate $\overline{V}$ to be the conjugate vector space with $H$-module structure given by $hv := \theta(h)v = S(h)^\ast v$. Note that as $\theta$ is a coalgebra morphism $H \to \overline{H}^{\text{cop}}$, the compatibility with the monoidal structure is $\overline{V} \otimes \overline{W} = \overline{W} \otimes \overline{V}$.

If $H$ is a Hopf $\ast$-algebra, then an $H$-module algebra $B$ which is also a $\ast$-algebra is called a module $\ast$-algebra if $(hb)^\ast = \theta(h)(b^\ast)$ holds for all $h \in H, b \in B$, that is, if the resulting map $H \to \text{End}_k(B)$ is a morphism of algebras with involution. The question we want to address in this subsection is whether for a given quantum automorphism $\sigma$ of a $\ast$-algebra $B$ the Hopf algebra $H_\sigma$ becomes naturally a Hopf $\ast$-algebra in such a way that $B$ is a module $\ast$-algebra over $H_\sigma$.

In order to do so, we first extend the $\ast$-structure $\ast$ respectively the associated involution $\theta$ to $M_n(B)$ respectively $M_n(\text{End}_k(B))$. This depends on the choice of an involutive permutation
\[\{1, \ldots, n\} \to \{1, \ldots, n\}, \quad i \mapsto \overline{i}, \quad \overline{i} = i\]
that will be used afterwards to be able to restrict the resulting $\ast$-structure on $M_n(B)$ to upper triangular matrices.

**Proposition 3.7.1.** Let $B$ be a $\ast$-algebra and assume $s \in S_n, s^2 = 1$. We abbreviate $\overline{i} := s(i)$.
(1) Setting $(\sigma^\dagger)_{ij} := \sigma_{ji}^*$ defines a $*$-structure $\dagger$ on $M_n(B)$.
(2) Setting $\vartheta(\sigma)_{ij} := \theta(\sigma_{ji})$ defines an involution on $M_n(\text{End}_k(B))$.

**Proof.** Clearly, $\dagger$ is involutive:

$$(\sigma^{\dagger\dagger})_{ij} = (\sigma^\dagger)^*_{ji} = \sigma_{ij}^{**} = \sigma_{ij},$$

where the last equality follows as $*$ is a $*$-structure on $B$. It is also a ring morphism $M_n(B) \to M_n(B)^{op}$: let $\sigma, \tau \in M_n(B)$ then

$$((\sigma \tau)^\dagger)_{ij} = ((\sigma \tau)_{ji})^* = \left( \sum_{r=1}^{n} \sigma_{jr} \tau_{ri} \right)^* = \sum_{r=1}^{n} \tau_{ri}^* \sigma_{jr}^* = \sum_{r=1}^{n} (\tau^\dagger)_{ir} (\sigma^\dagger)_{rij} = (\sigma^\dagger)(\tau^\dagger)_{ij}. $$

That $\dagger$ is a $k$-linear map $M_n(B) \to \overline{M_n(B)}$ follows from the fact that $*: B \to \overline{B}$ is linear. The second claim is shown analogously. \hfill $\Box$

The definition of this $*$-structure and of this involution is made in order to have the following:

**Lemma 3.7.3.** A $k$-linear map $\sigma: B \to M_n(B)$ satisfies

$$\sigma(b^*) = \sigma(b)^\dagger$$

if and only if $\vartheta(\sigma) = \sigma^T$ when $\sigma$ is viewed as an element in $M_n(\text{End}_k(B))$.

**Proof.** This holds as $(\sigma^\dagger)_{ij}(b) = \sigma_{ji}^*(b) = \theta(\sigma_{ji})(b^*) = \theta(\sigma)_{ji}(b^*)$. \hfill $\Box$

Now we apply the above to the study of quantum automorphisms:

**Proposition 3.7.2.** Let $\sigma: B \to M_n(B)$ be a quantum automorphism.

(1) $\vartheta(\sigma)^T = \vartheta(\sigma^T)$ is a quantum automorphism.
(2) If $\vartheta(\sigma)^T = \sigma$, then $H_\sigma$ is a Hopf $*$-algebra with $*$-structure given by $s_{d,ij} \mapsto s_{1-d,ij}$, and $B$ is a module $*$-algebra over $H_\sigma$. 

Proof. (1): Since $\sigma : B \to M_n(B)$ is an algebra morphism, we have

$$(\vartheta(\sigma)^T)_{ij}(ab) = \theta(\sigma_{ji})(ab) = \sigma_{ji}((ab)^*)^* = \sigma_{ji}(b^*a^*)^*$$

$$= \left(\sum_r \sigma_{jr}(b^*)\sigma_{ri}(a^*)\right)^*$$

$$= \sum_r (\sigma_{ri}(a^*))^*(\sigma_{jr}(b^*))^*$$

$$= \sum_r \theta(\sigma_{ri})(a)\theta(\sigma_{jr})(b)$$

$$= \sum_r (\vartheta(\sigma)^T)_{ir}(a)(\vartheta(\sigma)^T)_{rj}(b)$$

$$= \sum_r (\vartheta(\sigma)^T)_{ir}(a)(\vartheta(\sigma)^T)_{rj}(b).$$

In order to show that $\vartheta(\sigma)^T$ is strongly invertible, note first that as $\vartheta$ is an involution on $M_n(\text{End}_k(B))$, it is in particular multiplicative,

$$\vartheta(\sigma \tau) = \vartheta(\sigma)\vartheta(\tau),$$

so if $\sigma \in M_n(\text{End}_k(B))$ is invertible, then so is $\vartheta(\sigma)$ with inverse given by $\vartheta(\sigma)^{-1} = \vartheta(\sigma^{-1})$. Furthermore, it follows directly from its definition that $\vartheta$ commutes with taking transposes, so

$$\vartheta(\hat{\sigma}) = \vartheta((\sigma^{-1})^T) = \vartheta(\sigma^{-1})^T = (\vartheta(\sigma)^{-1})^T = \vartheta(\sigma).$$

It follows that if $\{\sigma_d\}$ is a sequence of invertible matrices with $\hat{\sigma}_d = \sigma_{d+1}$ then $\{\vartheta(\sigma_d)\}$ is a sequence of invertible matrices with $\vartheta(\hat{\sigma}_d) = \vartheta(\sigma_{d+1}).$

(2): Since $k\langle s_{d,ij} \rangle$ is a free algebra, there is a unique algebra morphism

$$\theta : k\langle s_{d,ij} \rangle \to k\langle s_{d,ij} \rangle, \quad s_{d,ij} \mapsto s_{-d,ji}.$$

That $k\langle s_{d,ij} \rangle \to k\langle s_{d,ij} \rangle^\text{cop}$ is a coalgebra morphism is verified by straightforward computation.

Our aim is to show that $\theta$ descends to a Cartan involution on $H_\sigma$. For this we first prove by induction on $d$ that

$$(16) \quad \theta(\sigma_{d,ij}) = \vartheta(\sigma_d)_{ij} = \sigma_{-d,ij}^T = \sigma_{-d,ji}.$$

Indeed, for $d = 0$ this holds by the assumption that $\vartheta(\sigma)^T = \sigma$. In the induction step we compute

$$\vartheta(\sigma_d) = \sigma_{d,ij}^T \Rightarrow \vartheta(\sigma_d^{-1}) = (\sigma_{d,ij}^T)^{-1} = \sigma_{-d,ji},$$

where the last equality follows from the comment after Definition 2.1.1. Hence

$$\vartheta(\sigma_{d+1}) = \vartheta(\hat{\sigma}_d) = \vartheta(\hat{\sigma}_d)_{ij} = \vartheta(\sigma_d^{-1})^T = \sigma_{-d-1,ij}^T.$$
which proves (16) for $d+1$. This means that by the definition of the involution $\theta$ on $k\langle s_{d,ij} \rangle$, the map $k\langle s_{d,ij} \rangle \to \text{End}_k(B)$, $s_{d,ij} \mapsto \sigma_{d,ij}$ is a morphism of algebras with involution. In particular, $\theta$ descends to $k\langle s_{d,ij} \rangle/I$ and to $H_\sigma$, and defines a Cartan involution and hence by Lemma 3.7.2 a Hopf $*$-algebra structure on $H_\sigma$. It also follows that $B$ is a module $*$-algebra over this Hopf $*$-algebra. □

Example 3.7.1. Throughout, we fixed an involutive permutation $s \in S_n$. The most obvious choice is the identity, $\bar{i} = i$. In this case, $\sigma^\dagger$ is the usual adjoint of a matrix $\sigma \in M_n(B)$ (transpose and apply $*$ to the entries). However, if $\sigma$ is upper triangular, the condition $\vartheta(\sigma)^T = \sigma$ cannot hold with respect to this involution, as we then have $\vartheta(\sigma)_{ij} = \theta(\sigma_{ij})$, so that $\vartheta(\sigma)^T$ is a lower triangular matrix. Hence for upper triangular quantum automorphisms we focus on the permutation $\bar{i} := n + 1 - i$.

In particular, consider $k = \mathbb{C}$ with involution given by complex conjugation and $B = \mathbb{C}[t^2, t^3]$ with $*$-algebra structure given by

$$*: B = \mathbb{C}[t^2, t^3] \longrightarrow B, \quad (\lambda t^n)^* := \bar{\lambda} t^n.$$ 

Geometrically, this describes the real points of the singular curve $V \subseteq \mathbb{C}^2$ in the sense that the points of the curve $V_\mathbb{R} = V \cap \mathbb{R}^2$ correspond to the one-dimensional $*$-representations of $B$, and these correspond further to the maximal ideals in $\mathbb{C}[t^2, t^3]$ which are invariant under $*$. The quantum automorphism $\sigma : B \to M_3(B)$ that we study since Section 2.7 satisfies $\vartheta(\sigma)^T = \sigma$ provided that we work with the involution $\bar{j} := 4 - j$ on $\{1, 2, 3\}$ as in Example 3.7.1. The Hopf $*$-structure on $H_\sigma$ is then given by

$$K^* = K, \quad D^* = -D, \quad Y^* = -Y + 6iD.$$ 

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