Controllability and maximum matchings of complex networks

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Previously, the controllability problem of a linear time-invariant dynamical system is mapped to the maximum matching (MM) problem on the bipartite representation of the underlying directed graph, and the sizes of MMs on bipartite random graphs, the basic quantity of the controllability problem, are calculated analytically with the cavity method at zero temperature limit. Here we present an alternative theory to estimate MM sizes based on the core percolation theory and the perfect matching of cores. Our theory is much simplified and easily interpreted, and can estimate MM sizes on general random graphs with or without symmetry between out- and in-degree distributions. Our result helps to illuminate the fundamental connection between the controllability problem and the underlying structure of complex systems.

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CONTENTS

I. Introduction 3
II. Model 3
III. Theory 4
IV. Results 6
V. Conclusion 9
VI. Acknowledgements 9

VII. Appendix A: Equations on model random graphs
   A. Erdös-Rényi random graphs 9
   B. Regular random graphs 9
   C. Asymptotical scale-free networks 10
   D. Random graphs with different forms of out- and in-degree distributions 11

VIII. Appendix B: Description of the real network dataset 11
      References 13
I. INTRODUCTION

Graph theory and network science [1–6] provide a consistent framework to understand the structure and the dynamics of complex connected systems. In the past few years, the problems of controllability and control on complex networks and their application on real-world networked systems are among the main focuses of complex systems community [7–25]. A rather comprehensive review on control issues in complex systems can be found in [20]. Here we focus on the first controllability problem that has been analyzed in the recent network community, the controllability of linear-time invariant systems with nodal dynamics [7, 8]. In the paper [9], the problem of finding the minimal driver node sets (MDNSs) to steer system dynamics is mapped to the maximum matching (MM) problem [27] (finding a maximal set of edges with no shared vertices) on the underlying directed graph. On the analytical side, the cavity method at the zero temperature limit is derived to estimate MM sizes [9, 28–30]. On the simulation side, the Hopcroft-Karp algorithm [31] is adopted to find the MM solutions. Yet the analytical framework presented in [9] suffers from two points. First, the final equation connecting structure (mainly the degree distribution) and MM sizes on random graphs is explicit, yet its derivation is quite complicated, burying a potentially simple and intrinsic explanation of the problem. Second, the degree asymmetry, or the disparity between out- and in-degree distributions, is ubiquitous in real-world networked systems. Even its implication in controllability and MM sizes has been touched in some literature [9, 12, 14], the degree asymmetry is still not well fit in the analytical framework.

In this paper, we try to resolve the above two points in a single framework. Our main contribution is the derivation of an alternative analytical framework to estimate MM sizes on general random graphs based on the intuition of the Karp-Sipser algorithm [32, 33] and the related core percolation theory [34–37]. Our theory is much simplified and easily interpreted. On random graphs with degree symmetry, our theory simply retrieves the result for MDNS sizes derived with the cavity method at the zero temperature limit in [9]. On random graphs without degree symmetry, our theory works naturally, thus provides an analytical perspective to the implication of degree asymmetry in the network controllability problem. In all, our framework helps to clarify a theoretical understanding of the roles of network topology and structure in the controllability of complex connected systems.

Here is the structure of the paper. In section II, we explain the controllability problem, the MM problem, the MDNS, and an algorithm to extract MM solutions. In section III, we present an analytical theory to estimate the MM sizes on undirected bipartite random graphs. In section IV, we show the results from simulation and theory on some model random graphs and real-world networks. In section V, we conclude the paper.

II. MODEL

We explain the controllability and the MM problems on underlying directed graphs. A directed graph $D = \{V, E\}$ consists of a vertex set $V$ of a size $N(=|V|)$ and an arc set $E$ of a size $M(=|E|)$ among vertices. An ordered pair $\{i, j\}$ of vertices $i$ and $j$ denotes an arc (a directed edge) starting from $i$ and arriving at $j$. The undirected bipartite representation $\Gamma$ of $D$ is derived as follows: each vertex $i \in V$ is split into an out-vertex $i^+$ and an in-vertex $i^-$ in $\Gamma$; each arc $\{i, j\} \in E$ leads to an undirected edge $(i^+, j^-)$ as an unordered pair of vertices in $\Gamma$. Thus $\Gamma = \{V^+ \cup V^-, E^+\}$, in which $V^+$, $V^-$, and $E^+$ are the out-vertex set, the in-vertex set, and the undirected edge set constructed as above, respectively. Once we find a MM of the undirected bipartite graph $\Gamma$, the unmatched vertices in the in-vertex set constitute a MDNS [9]. Specifically, $N_D = \max\{1, N - N_{MM}\}$ in which $N_{MM}$ is the size of a MM and $N_D$ is the size of the corresponding MDNS. On large graphs, we have $n_D = 1 - y$ in which $y(\equiv N_{MM}/N)$ is the fraction of MMs and $n_D(\equiv N_D/N)$ is the fraction of MDNSs. An illustration of the procedure is in (a) - (d) of figure 1.

On the simulation method to find MM solutions on undirected bipartite graphs, instead of the usually adopted Hopcroft-Karp algorithm [31], we adopt the Karp-Sipser algorithm [32, 33], which has a simple interpretation and finds MMs on a given undirected graph with high probability. A basic component of the Karp-Sipser algorithm is the greedy leaf removal (GLR) procedure, in which iteratively any vertex with only one nearest neighbor (a leaf) is removed along with its sole nearest neighbor (a root). The GLR procedure leads to the core percolation on graphs [35, 36], and its variants have implication in various combinatorial optimization and satisfiability problems [29, 30, 32, 33, 37, 41]. The Karp-Sipser algorithm is basically a randomized local algorithm on undirected graphs, and it goes as: (1) the GLR procedure is applied on a graph until there is a residual subgraph or a core; (2) an edge in the core is randomly chosen and matched, and all the edges adjacent to it are removed as satisfied local constraints; (3) the two steps are carried out sequentially and iteratively until there is no edge left. Each root, along with one of the leaves which leads to it, corresponds to a matched edge. Thus a MM can be reconstructed from the roots of the GLR procedures and those randomly matched edges in the shrinking core structure. An example of the Karp-Sipser algorithm is in (e) - (f) of figure 1.
III. THEORY

Here we derive an analytical framework to estimate the fraction of MMs on undirected bipartite random graphs. Before presenting our theory, we explain some graphical notions. On an undirected bipartite graph $\Gamma$ of a directed graph $D$, for any edge $(i,j)$ of vertices $i$ and $j$, $i$ is a nearest neighbor of $j$, and vice versa. The degree of $i^+$ ($i^-$) in the out-(in-)vertex set $k^{i+}$ ($k^{i-}$) is the size of its nearest neighbors in the in-(out-)vertex set $|\partial i^+|$ ($|\partial i^-|$). The degree distribution $P_+(k_+)$ ($P_-(k_-)$) of $\Gamma$ is the probability of randomly finding a vertex with a degree $k_+$ ($k_-$) in the out-(in-)vertex set. The excess degree distribution $Q_+(k_+)$ ($Q_-(k_-)$) of $\Gamma$ is the probability of arriving at a vertex with a degree $k_+$ ($k_-$) in the out-(in-)vertex set following a randomly chosen edge. We simply have the equivalence $Q_{\pm}(k_{\pm}) = k_{\pm} P_{\pm}(k_{\pm})/c$ as $c \equiv M/N$ is the arc density. When an edge $(i,j)$ is removed from the original graph $\Gamma$, we consider the residual subgraph as the cavity graph $\Gamma \setminus (i,j)$. When a vertex $i$ is removed along with all its adjacent edges from $\Gamma$, we consider the residual subgraph as the cavity graph $\Gamma \setminus i$.

Our analytical framework is based on the intuition of the Karp-Sipser algorithm and has two components: the core percolation theory and the perfect matching of cores. The core percolation theory is the analytical theory of the GLR procedure, which can estimate the sizes of vertices in cores and roots as the two faces of the GLR procedure on both undirected and directed uncorrelated random graphs. We follows the language of the cavity method to present the core percolation theory, in which the main results are basically marginal probabilities calculated from the stable solutions of pertinent cavity probabilities defined on random graphs. For the problem here on undirected
bipartite random graphs, a set of four cavity probabilities is introduced. Following a randomly chosen edge \((i,j)\)
arriving at the out-vertex, say \(i\), \(\alpha^+ (\beta^+)\) is defined as the probability that \(i\) becomes a leaf (a root) in the GLR procedure in the cavity graph \(\Gamma \setminus (i,j)\); following a randomly chosen edge \((i,j)\) arriving at the in-vertex, say \(j\), \(\alpha^- (\beta^-)\) is defined as the probability that \(j\) becomes a leaf (a root) in the GLR procedure in \(\Gamma \setminus (i,j)\). On undirected bipartite random graphs without degree-degree correlations, with the locally tree-like structure approximation \[28\], we have the self-consistent equations for \(\alpha^\pm\) and \(\beta^\pm\).

\[
\alpha^\pm = \sum_{k^\pm=1}^{\infty} Q^\pm(k^\pm)(\beta^\mp)^{k^\pm-1}, \\
\beta^\pm = 1 - \sum_{k^\pm=1}^{\infty} Q^\pm(k^\pm)(1 - \alpha^\mp)^{k^\pm-1}.
\]

With the stable fixed solutions of \((\alpha^\pm, \beta^\pm)\), we can calculate the fractions of out- and in-vertices in the core as \(n^\pm\), and the fraction of all roots in the out- and in-vertex sets as \(w\).

\[
n^\pm = \sum_{k^\pm=2}^{\infty} P^\pm(k^\pm) \sum_{s^\pm=2}^{k^\pm} \binom{k^\pm}{s^\pm} (\beta^\mp)^{k^\pm-s^\pm}(1 - \alpha^\mp - \beta^\mp)^s, \\
w = \left[1 - \sum_{k^\mp=0}^{\infty} P^+(k^+) (1 - \alpha^-)^{k^+}\right] + \left[1 - \sum_{k^\mp=0}^{\infty} P^-(k^-)(1 - \alpha^+)^{k^-}\right] - c\alpha^+\alpha^-.
\]

The equation for \(n^\pm\) can be further simplified as follows. We first remove the summation \(\sum_{k^\pm=2}^{\infty}\), then move the summation \(\sum_{k^\pm=2}^{\infty}\) to \(\sum_{k^\pm=0}^{\infty}\). With the equivalence \(Q^\pm(k^\pm) = k^\pm P^\pm(k^\pm)/c\) and equation (4), we have a concise form for \(n^\pm\).

\[
n^\pm = \sum_{k^\pm=0}^{\infty} P^\pm(k^\pm)\left[(1 - \alpha^\mp)^{k^\pm} - (\beta^\mp)^{k^\pm}\right] - c\alpha^+(1 - \alpha^- - \beta^-).
\]

Equations (1), (2), and (3) are first derived in [36], while equation (4) is our contribution here. Here we present a simple explanation of the four equations. We should mention that the following explanation applies on both out-vertices and in-vertices. On a cavity graph \(\Gamma \setminus i\) as \(i\) is a randomly chosen vertex in an uncorrelated undirected random \(\Gamma\), the locally tree-like structure approximation assumes that the states of \(i\)'s nearest neighbors or \(\partial i\) are independent of each other in the GLR procedure. We first consider the case from a cavity graph \(\Gamma \setminus i\) to another cavity graph \(\Gamma \setminus (i,j)\) after some edges are added in which \((i,j)\) is a randomly chosen edge in \(\Gamma\). If \(i\) is a leaf in \(\Gamma \setminus (i,j)\), its nearest neighbors other than \(j\) should be all roots in \(\Gamma \setminus i\). Thus we have equation (1). If \(i\) is a root in \(\Gamma \setminus (i,j)\), there should be at least one leaves in \(\partial i\setminus j\) in \(\Gamma \setminus i\). Thus we have equation (2). We then consider the case from a cavity graph \(\Gamma \setminus i\) to the original graph \(\Gamma\) after some edges are added in which \(i\) is a randomly chosen vertex in \(\Gamma\). If \(i\) is in the core in \(\Gamma\), then along all its nearest neighbors or simply \(\partial i\) in \(\Gamma \setminus i\), there should be no leaves and also at least two vertices in the core to forbid the GLR procedure. Thus we have equation (3). If \(i\) is a root in \(\Gamma\), there should be at least one leaves in \(\partial i\setminus j\) in \(\Gamma \setminus i\). Yet a recounting happens in which the contribution of an isolated edge (for example, the edge \(2^+, 3^-\)) in \((e)\) and \((f)\) of figure 4 to a matching is counted twice. To calculate the probability of the formation of isolated edges, we consider the case from a cavity graph \(\Gamma \setminus (i,j)\) to the original graph \(\Gamma\) after the edge \((i,j)\) is added, in which \((i,j)\) is a randomly chosen edge in \(\Gamma\). If \((i,j)\) is an isolated edge in \(\Gamma\), both \(i\) and \(j\) should be leaves in \(\Gamma \setminus (i,j)\). Thus we have equation (4).

Here we explain a little more on the configurations of core and roots on an undirected bipartite graph. The core from the GLR procedure is well defined [35], that is to say, the configuration of a core is independent of the pruning process. Thus it is reasonable to quantify the fractions of out- and in-vertices in a core as \(n^\pm\). Yet the configuration of the roots is dependent on the pruning process. For example, a specific pruning process, in which all leaves in the in-vertex set trigger GLR steps before the leaves in the out-vertex set, simply leads to a larger size of out-vertices as roots on an undirected bipartite graph with degree symmetry. Yet the size of all roots of a bipartite graph, on average, is independent of the pruning process of the GLR procedure [35]. This is why we calculate the fraction of roots on a whole graph as \(w^\pm\).
The second component of our analytical framework is the perfect matching of the core structure \[27, 32\]. On the core of an undirected bipartite graph of \(\Gamma\), in the case of \(n^+ > n^-\), the perfect matching states that the in-vertices in the core are all matched, leading to the MM size of the core structure simply as \(n^-\). Vice versa for the case of \(n^- > n^+\).

For a given undirected bipartite random graph \(\Gamma\), summing the fraction of matched edges reconstructed from the roots of the GLR procedure, which is simply \(w\), and the fraction of matched edges in the core structure, which is just \(\min\{n^+, n^-\}\), we have the fraction of MMs on \(\Gamma\). Equivalently, we have

\[
y = w + \min\{n^+, n^-\}.
\]

(6)

Taken together, equations (1), (2), (3), (4), and (6) constitute our analytical framework of MM fractions on undirected bipartite random graphs.

It is easy to see that, our theory assumes a general form of \(P_\pm(k_\pm)\) for a bipartite graph. In the case of degree symmetry, we have \(P_+(k) = P_-(k)\) for any \(k\). From equations (1) and (2), we have \(\alpha^+ = \alpha^-\) and \(\beta^+ = \beta^-\), further leading to \(n^+ = n^-\) from equation (6). Equation (6) can be equivalently formulated as

\[
y = w + \frac{1}{2}(n^+ + n^-).
\]

(7)

With equations (3) and (4) inserted into the above equation, we have

\[
y = 1 - \frac{1}{2} \left\{ \left[ \sum_{k_+=0}^{\infty} P_+(k_+)(1 - \alpha^-)^{k_+} + \sum_{k_+=0}^{\infty} P_+(k_+)(\beta^-)^{k_+} - 1 \right] 
+ \left[ \sum_{k_-=0}^{\infty} P_-(k_-)(1 - \alpha^+)^{k_-} + \sum_{k_-=0}^{\infty} P_-(k_-)(\beta^+)^{k_-} - 1 \right] 
+ c[\alpha^+(1 - \beta^-) + \alpha^- (1 - \beta^+)] \right\}.
\]

(8)

We can compare equation (8) with the estimation of \(n_D\) of equation (S37) in the paper [9]. We substitute the parameters \(w_1\) in equation (S26) as \(\alpha^+\), \(w_2\) in equation (S27) as \(\beta^+\), \(\hat{w}_1\) in equation (S29) as \(\alpha^-\), and \(\hat{w}_2\) in equation (S30) as \(\beta^-\). The MDNS fraction \(n_D\) of equation (S37) simply reduces to \(1 - y\) as \(y\) is shown in equation (8). Thus our theory retrieves the estimation of MDNS sizes on directed random graphs with degree symmetry based on the cavity method at zero temperature limit. An intuitive understanding of this correspondence is that our analytical approach as a combination of the core percolation theory and the perfect matching of cores is essentially a ‘cavity method’ calculation of MM sizes on random graphs directly at the the zero temperature, in which the coarse-grained parameters of cavity messages approximated at zero temperature limit from a finite-temperature statistical physical framework reduce to the cavity probabilities defined in our analytical framework. An implication of the correspondence is that the equation of \(n_D\) of equation (S37) in [9] only applies on directed random graphs with degree symmetry, while our theory also applies on directed graphs without degree symmetry. On graphs with degree asymmetry, adopting equation (7) rather than equation (6) leads to an overestimation of \(y\) by a difference of \(|n^+ - n^-|/2\).

IV. RESULTS

We test the simulation (the GLR procedure and the Karp-Sipser algorithm) and our analytical framework on some model directed random graphs. The details of graph construction and simplified equations of our theory are left in Appendix A. First, we consider the directed random graphs with symmetric out- and in-degree distributions. Examples are the directed Erdős-Rényi random graphs \[45, 46\], the directed regular random graphs, and the directed scale-free networks \[47\] generated with the static model \[48-50\] with the same out-degree exponent \(\gamma^+\) and in-degree exponent \(\gamma^-\). See the results of \(y\), \(n^\pm\), and \(w\) in (a) - (c) in figure 2. We then consider two cases of directed random graphs with degree asymmetry. The first case considers a same form for the out- and in-degree distributions: the directed scale-free networks generated with the static model with degree exponents \(\gamma^+ \neq \gamma^-\). The second case considers different forms for the out- and in-degree distributions: the directed random graphs with a Poissonian out-degree distribution and a power-law in-degree distribution generated from the static model. See the result in (d) - (f) in figure 2. In figure 2 we can see that, generally speaking, results coincide well between finite-size
FIG. 2. The relative sizes of MM $y$, the fractions of out- and in-vertices in cores $n^\pm$, and the fraction of roots $w$ on directed random graphs. (a) - (c) show results on the directed regular random (RR) graphs, directed Erdős-Rényi (ER) random graphs, and directed asymptotical scale-free networks generated with the static model (SM) with degree exponents $\gamma_+ = \gamma_- = 3.0$, which are listed from left to right, respectively. (d) - (f) show results on the directed random graphs with Poissonian out-degree distribution and power-law in-degree distribution with an exponent $\gamma_- = 3.0$ generated with the static model (ERSM) and the directed asymptotical scale-free networks generated with the static model (SM) with $\gamma_+ = 3.0$ and $\gamma_- = 2.7, 2.5, 2.4$, which are listed from left to right, respectively. (a) and (d) show results of $y$. (b) and (e) show results of $n^\pm$. (c) and (f) show results of $w$. Each sign is for the simulation result on a single graph instance with a vertex size $N = 10^5$. Each solid line is for the analytical result on infinitely large random graphs. Each dotted line in (e) and (f) is for a discontinuity in the analytical result.

simulation and infinite-size analytical theory, except in cases of power-law degree distributions with $\gamma_- < 3.0$ from the static model. This tendency of result discrepancy has a root in the intrinsic degree-degree correlations in the graph construction process of the static model [30].

From the result in [30], the core emergence ($n^\pm$) is continuous on random graphs with degree symmetry, and discontinuous on random graphs with degree asymmetry. In figure 2 we can see that $w$ follows a rise-and-fall pattern and undergoes a continuous decrease or a sudden drop with the same continuity of $n^\pm$ at the core percolation
FIG. 3. The differences of the MM sizes $\Delta y$, the core asymmetries $\Delta n$, and the root sizes $\Delta w$ between simulation and analytical theory on 24 real-world network instances. $\Delta y$, $\Delta n$, and $\Delta w$ are shown against the degree asymmetry $\Delta c$ in (a), (b), and (c), respectively.

transitions. Here we give an intuitive understanding of the pattern of $w$ and its behavior at the transition points. Before the formation of the giant connected component on a graph ($c < 1.0$), there are mainly trees with leaves in the graph. The GLR procedure is carried out basically based on the existing leaves in the graph, during which more added edges lead to more leaves and more GLR steps, thus an increasing $w$. With more edges added into the graph after the formation of the giant connected component, newly revealed leaves in the iterative GLR procedure play an increasingly important role, in which a larger arc density $c$ leads to more basic steps of the GLR procedure, thus an ever increasing $w$, until the formation of a core. A core is a subgraph which the GLR procedure cannot touch. A sudden emergence of core $n^\pm$ at the core percolation transition simply leads to a macroscopic fraction $((n^+ + n^-)/2)$ of vertices in the graph where the GLR procedure is excluded, thus a sudden drop of the root size $w$. With even more edges added into the graph beyond the core percolation transition, there is an increasing difficulty in triggering the GLR procedure and generating new leaves. Thus an increasing $n^\pm$ and a shrinking $w$ happen at the same time with a growing arc density $c$.

We further apply the simulation and our theory on real-world networks. A description of a dataset of 24 network instances is in Appendix B. We focus on the effect of the degree asymmetry on the matching sizes. For a directed network instance $D$ with degree distributions $P_\pm(k_\pm)$ for its undirected bipartite representation, we define a coefficient to measure its degree asymmetry as $\Delta c = \sum_{k=0}^{\infty} |P_+(k) - P_-(k)|/2$. It is easy to see that $\Delta c \in [0, 1]$, and a larger $\Delta c$ corresponds to a larger disparity between the out- and in-degree distributions. We also define the core asymmetry towards out-vertices in cores as $\delta n = n^+ - n^-$ as the difference between the sizes of the out- and in-vertex sets in a core after the GLR procedure. For any real-world network in the dataset, we count from simulation the fraction of roots $w_{\text{real}}$, and also calculate $y_{\text{theory}}$, $\delta n_{\text{theory}}$, and $w_{\text{theory}}$ respectively with the analytical theory with the empirical out- and in-degree distributions of the network instance as inputs. These analytical predictions based on empirical degree distributions can be approximately considered as averaged results of simulation on network instances with degree-constrained randomized wiring of arcs [9]. The difference $\Delta y = y_{\text{real}} - y_{\text{theory}}$, $\Delta n = \delta n_{\text{real}} - \delta n_{\text{theory}}$, and $\Delta w = w_{\text{real}} - w_{\text{theory}}$ can be considered as measures of the deviation of real-world networks from their randomized versions from the perspectives of the GLR procedure and MM sizes. Results on the real-world network dataset is in figure [8]. We can see that the 24 network instances show a wide range of degree asymmetry, and large differences of $\Delta y$, $\Delta n$, and $\Delta w$ show in those instances with moderate and small degree asymmetry. For the core asymmetry difference $\Delta n$, 16 instances show trivial values, 7 instances show clear positive values, and only 1 instance shows clear negative values. For the root size difference $\Delta w$, 8 instances show trivial values, only 1 instance shows clear positive values, and the other 15 instances show clear negative values. For the MM size difference $\Delta y$, 6 instances show trivial values, only 1 instance shows clear positive values, and the other 17 instances show clear negative values. Taken the above results in short, we have two major observations. First, the degree asymmetry has a relatively small influence on $\Delta n$ and a relatively large influence on both $\Delta w$ and $\Delta y$. Second, the network instances in our dataset show a rather clear tendency to have larger core asymmetries, smaller root sizes, and smaller MM sizes compared with their degree-constrained randomized versions.
V. CONCLUSION

Establishing the relation between the network structural properties and the size of actuators or driver nodes to guide a dynamical system to any final state is a fundamental problem in the network controllability problem, which is a recent example of the intricate interplay between structure and function of complex networked systems. In this paper, we derive a simple alternative framework to estimate the fraction of MMs on directed systems, thus the size of MDNSs, a basic quantity of the network controllability problem. Our simulation method is based on the Karp-Sipser algorithm on the undirected bipartite representations of the underlying directed networks. Our analytical theory adopts the core percolation theory and the perfect matching of cores, which works on random graphs with or without degree symmetry. We also find that real-world network instances show a clear tendency to larger core asymmetry and smaller root sizes and MM sizes compared with their degree-constrained randomizations, which is worthy for a further study. We hope that our theory contribute to clarify the fundamental role of the network structure in the controllability and control issues of complex connected systems.

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VII. APPENDIX A: EQUATIONS ON MODEL RANDOM GRAPHS

A. Erdos-Renyi random graphs

For directed Erdos-Renyi (ER) random graphs \[45, 46\], a graph instance \( D = \{V, E\} \) with a vertex set \( V \) of a size \( N(= |V|) \) and an arc set \( E \) of a size \( M(= |E|) \) can be constructed as: a null graph with \( N \) vertices and no edge is initialized; a number of \( M \) arcs are established by randomly choosing two distinct vertices, say vertices \( i \) and \( j \), and connecting by assigning a random direction with an equal probability, say \( \{i, j\} \) or \( \{j, i\} \).

On directed ER random graphs with arc density \( c(\equiv M/N) \), we have the degree distributions and the excess degree distributions of their undirected bipartite representations as

\[
P_{\pm}(k_{\pm}) = e^{-c}\frac{c^{k_{\pm}}}{k_{\pm}!},
\]

\[
Q_{\pm}(k_{\pm}) = e^{-c}\frac{c^{k_{\pm}-1}}{(k_{\pm} - 1)!}.
\]

We have the summation rules with above equations as

\[
\sum_{k_{\pm}=0}^{\infty} P_{\pm}(k_{\pm})x^{k_{\pm}} = e^{-c(1-x)},
\]

\[
\sum_{k_{\pm}=1}^{\infty} Q_{\pm}(k_{\pm})x^{k_{\pm} - 1} = e^{-c(1-x)}.
\]

Thus we have the simplified equations as follows.

\[
\alpha_{\pm} = e^{-c(1-\beta_{\mp})},
\]

\[
\beta_{\pm} = 1 - e^{-c\alpha},
\]

\[
n_{\pm} = 1 - \alpha_{\pm} - \beta_{\mp} - c\alpha_{\pm}(1 - \alpha_{\mp} - \beta_{\mp}),
\]

\[
w = \beta_{+} + \beta_{-} - c\alpha_{+}\alpha_{-}.
\]
B. Regular random graphs

A directed regular random (RR) graph instance $D = \{V,E\}$ with a vertex set $V$ of a size $N(=|V|)$ and an arc set $E$ of a size $M(=|E|)$ can be generated from its undirected counterpart: an undirected RR graph instance is constructed in which each vertex is connected to an integer $K(\equiv 2M/N)$ of randomly chosen distinct vertices; then each edge is assigned with a random direction with an equal probability.

For the directed RR graph instances with an arc density $c(\equiv K/2)$ as $K$ is the integer degree of the underlying undirected RR graphs, we have the degree distributions as

$$P_{\pm}(k_{\pm})=\binom{K}{k_{\pm}}/2^K,$$

$$Q_{\pm}(k_{\pm})=\binom{K-1}{k_{\pm}-1}/2^{K-1}.$$  (17) (18)

We have the summation as

$$\sum_{k_{\pm}=0}^{K} P_{\pm}(k_{\pm})x^{k_{\pm}} = \left(\frac{1+x}{2}\right)^K,$$  (19)

$$\sum_{k_{\pm}=1}^{K} Q_{\pm}(k_{\pm})x^{k_{\pm}-1} = \left(\frac{1+x}{2}\right)^{K-1}.$$  (20)

We thus have the simplified equations as

$$\alpha^+ = \left(\frac{1+\beta^+}{2}\right)K^{-1},$$  (21)

$$\beta^+ = 1 - \left(\frac{2-\alpha^+}{2}\right)K^{-1},$$  (22)

$$n^+ = \left(\frac{2-\alpha^+}{2}\right)K - \frac{1}{2}K\alpha^+K^{-1}(1-\alpha^+ - \beta^+),$$  (23)

$$w = 2 - \left(\frac{2-\alpha^+}{2}\right)K - \frac{1}{2}K\alpha^+\alpha^-.$$  (24)

C. Asymptotical scale-free networks

An uncorrelated directed scale-free (SF) network $D = \{V,E\}$ with $N(=|V|)$ vertices and $M(=|E|)$ arcs follows degree distributions $P_+(k_+) \propto k_+^{\gamma_+}$ and $P_-(k_-) \propto k_-^{\gamma_-}$ in which $\gamma_+$ and $\gamma_-$ are the out- and in-degree exponents, respectively. We adopt the static model [48-50] to generate asymptotical SF networks. We follow the procedure: a null graph with $N$ vertices and no edge is initialized in which each vertex has an index $i (=1,2,...N)$; each vertex is assigned with an out-weight $w_i^+ = \bar{w}_i^+ \equiv i^{-\xi_+}$ and an in-weight $w_i^- = \bar{w}_i^- \equiv i^{-\xi_-}$ as $\xi_+ \equiv 1/(\gamma_+ - 1)$; to further decouple the weights and the indices of vertices, the out-weights and in-weights are randomly shuffled between vertices respectively, then we have a new sequence of vertices with an out-weight $w_i^+$ and an in-weight $w_i^-$ for each vertex $i$; $M$ arcs are added into the null graph, while in each step a vertex $i$ is chosen from the out-vertex set with a probability proportional to its out-weight $w_i^+$, and a vertex $j$ is chose from the in-vertex set with a probability proportional to its in-weight $w_j^-$, then an arc $\{i,j\}$ is established if there is no $\{i,j\}$ nor $\{j,i\}$ before.

A directed SF network instance generated with the static model with an arc density $c$, an out-degree exponent $\gamma_+$, and an in-degree exponent $\gamma_-$ has the degree distributions $P_\pm(k_\pm)$ as

$$P_\pm(k_\pm) = \frac{1}{\xi_\pm} \frac{(c(1-\xi_\pm))^{k_\pm}}{k_\pm!} \int_1^\infty dt e^{-c(1-\xi_\pm)\frac{k_\pm}{t}} E_{-k_\pm+1+1/\xi_\pm}(c(1-\xi_\pm)).$$  (25)
The general exponential integral function $E_a(x) \equiv \int_1^\infty dt e^{-xt} t^{-a}$ can be calculated with the GNU Scientific Library (GSL) \cite{51}. For large $k_+$ and $k_-$, we have $P_+(k_+) \propto k_+^{\gamma_+}$ and $P_-(k_-) \propto k_-^{\gamma_-}$, respectively. The excess degree distributions are $Q_{\pm}(k_{\pm})$ has the form

$$Q_{\pm}(k_{\pm}) = \frac{1 - \xi_{\pm} (c(1 - \xi_{\pm}))(k_{\pm} - 1)!}{\xi_{\pm}} E_{-k_{\pm}+1+1/\xi_{\pm}}(c(1 - \xi_{\pm})).$$ \hspace{1cm} (26)

For the summation rules, we have

$$\sum_{k_{\pm}=1}^\infty P_{\pm}(k_{\pm})x^{k_{\pm}} = \frac{1}{\xi_{\pm}} E_{1+\xi_{\pm}}(c(1 - \xi_{\pm})(1-x)),$$ \hspace{1cm} (27)

$$\sum_{k_{\pm}=1}^\infty Q_{\pm}(k_{\pm})x^{k_{\pm}-1} = \frac{1 - \xi_{\pm}}{\xi_{\pm}} E_{1+\xi_{\pm}}(c(1 - \xi_{\pm})(1-x)).$$ \hspace{1cm} (28)

We thus have the simplified equations as

$$\alpha^+ = \frac{1 - \xi_+}{\xi_+} E_{1+\xi_+}(c(1 - \xi_+)(1 - \beta^+)),$$ \hspace{1cm} (29)

$$\beta^+ = 1 - \frac{1 - \xi_+}{\xi_+} E_{1+\xi_+}(c(1 - \xi_+)(1 - \beta^+)),$$ \hspace{1cm} (30)

$$n^+ = \frac{1}{\xi_+} E_{1+\xi_+}(c(1 - \xi_+)(1 - \beta^+)) - \frac{1 - \xi_+}{\xi_+} E_{1+\xi_+}(c(1 - \xi_+)(1 - \beta^-))$$

$$-\alpha^+(1 - \alpha^+ - \beta^+),$$ \hspace{1cm} (31)

$$w = 2 - \frac{1}{\xi_+} E_{1+\xi_+}(c(1 - \xi_+)(1 - \beta^+)) - \frac{1}{\xi_-} E_{1+\xi_-}(c(1 - \xi_-)(1 - \beta^+)) - \alpha^+ \alpha^-.$$ \hspace{1cm} (32)

D. Random graphs with different forms of out- and in-degree distributions

We can construct directed random graphs with a Poissonian out-degree distribution and a power-law in-degree distribution generated from the static model with an in-degree exponent $\gamma_-$. In the static model, we have $\xi_- \equiv 1/(\gamma_- - 1)$. With equations in the previous subsections and the arc density $c$, we have the following simplified equations.

$$\alpha^+ = e^{-c(1 - \beta^+)},$$ \hspace{1cm} (33)

$$\alpha^- = \frac{1 - \xi_-}{\xi_-} E_{1+\xi_-}(c(1 - \xi_-)(1 - \beta^+)),$$ \hspace{1cm} (34)

$$\beta^+ = 1 - e^{-\alpha^-},$$ \hspace{1cm} (35)

$$\beta^- = 1 - \frac{1 - \xi_-}{\xi_-} E_{1+\xi_-}(c(1 - \xi_-)(1 - \beta^+)),$$ \hspace{1cm} (36)

$$n^+ = 1 - \alpha^+ - \beta^+ - \alpha^-(1 - \alpha^- - \beta^-),$$ \hspace{1cm} (37)

$$n^- = \frac{1}{\xi_-} E_{1+\xi_-}(c(1 - \xi_-)(1 - \beta^+)) - \frac{1}{\xi_-} E_{1+\xi_-}(c(1 - \xi_-)(1 - \beta^+))$$

$$-\alpha^- (1 - \alpha^+ - \beta^+),$$ \hspace{1cm} (38)

$$w = \beta^+ + 1 - \frac{1}{\xi_-} E_{1+\xi_-}(c(1 - \xi_-)(1 - \beta^+)) - \alpha^+ \alpha^-.$$ \hspace{1cm} (39)

VIII. APPENDIX B: DESCRIPTION OF THE REAL NETWORK DATASET

We list some information about the real-world network dataset we use in the main text in table \ref{tab:1}. A major part of large network instances is from the collections in \cite{52}. In order to consider the skeleton of the interaction topology
among the constituents in the networked systems, we remove self-loops (self-interaction of a constituent) and merge multi-edges (multiple interactions with the same direction between two constituents) in the network instances of the dataset.

TABLE I. A list of 24 real-world directed network instances. For each network, we show its type and name, a brief description, and its size of vertices ($N$) and arcs ($M$).

| Type and Name                     | Description                                | N   | M    |
|----------------------------------|--------------------------------------------|-----|------|
| Regulatory networks              |                                            |     |      |
| *E. coli* [53]                   | Transcriptional regulatory network of *E. coli*. | 418 | 519  |
| *S. cerevisiae* [54]             | Transcriptional regulatory network of *S. cerevisiae*. | 688 | 1,079|
| PPI networks                     |                                            |     |      |
| PPI [55]                         | Protein-protein interaction network of human. | 6,339 | 34,814|
| Metabolic networks               |                                            |     |      |
| *C. elegans* [56]                | Metabolic network of *C. elegans*.         | 1,469 | 3,447|
| *S. cerevisiae* [56]             | Metabolic network of *S. cerevisiae*.      | 1,511 | 3,833|
| *E. coli* [56]                   | Metabolic network of *E. coli*.            | 2,275 | 5,763|
| Neuronal networks                |                                            |     |      |
| *C. elegans* [57]                | Neural network of *C. elegans*.            | 297  | 2,345|
| Food webs                        |                                            |     |      |
| St Marks [58]                    | Food web in St. Marks River Estuary.       | 54   | 353  |
| Everglades [59]                  | Food web in Everglades Graminoid Marshes.  | 69   | 911  |
| Florida Bay [60]                 | Food web in Florida Bay.                   | 128  | 2,106|
| Electronic circuits              |                                            |     |      |
| s208 [54]                        | Electronic sequential logic circuits.      | 122  | 189  |
| s420 [54]                        | Same as above.                             | 252  | 399  |
| s838 [54]                        | Same as above.                             | 512  | 819  |
| Ownership networks               |                                            |     |      |
| USCorp [61]                      | Ownership network of US corporations.      | 7,253 | 6,724|
| Citation networks                |                                            |     |      |
| cit-HepTh [62]                   | Citation network in HEP-TH category of ArXiv. | 27,769 | 352,768|
| cit-HepPh [62]                   | Citation network in HEP-PH category of ArXiv. | 34,546 | 421,534|
| Internet p2p networks            |                                            |     |      |
| p2p-Gnutella04 [63, 64]          | Gnutella peer-to-peer network from August 4, 2002. | 10,876 | 39,994|
| p2p-Gnutella31 [63, 64]          | Gnutella peer-to-peer network from August 31, 2002. | 62,586 | 147,892|
| Web graphs                       |                                            |     |      |
| Notre Dame [65]                  | Web graph of Notre Dame.                   | 325,729 | 1,469,679|
| Stanford [66]                    | Web graph of Stanford.edu.                 | 281,903 | 2,312,497|
| Google [66]                      | Web graph from Google.                     | 875,713 | 5,105,039|
| Social networks                  |                                            |     |      |
| WikiVote [67, 68]                | Who-vote-whom network of Wikipedia users.  | 7,115 | 103,689|
| Epinions [69]                    | Who-trust-whom network of Epinions.com users. | 75,879 | 508,837|
| Email-EuAll [63]                 | Email network from a EU research institution. | 265,009 | 418,956|
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