INFLUENCE OF FEEDBACK CONTROLS ON THE GLOBAL STABILITY OF A STOCHASTIC PREDATOR-PREY MODEL WITH HOLLING TYPE II RESPONSE AND INFINITE DELAYS

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ABSTRACT. In this work a stochastic Holling-II type predator-prey model with infinite delays and feedback controls is investigated. By constructing a Lyapunov function, together with stochastic analysis approach, we obtain that the stochastic controlled predator-prey model admits a unique global positive solution. We then utilize graphical method and stability theorem of stochastic differential equations to investigate the globally asymptotical stability of a unique positive equilibrium for the stochastic controlled predator-prey system. If the stochastic predator-prey system is globally stable, then we show that using suitable feedback controls can alter the position of the unique positive equilibrium and retain the stable property. If the predator-prey system is destabilized by large intensities of white noises, then by choosing the appropriate values of feedback control variables, we can make the system reach a new stable state. Some examples are presented to verify our main results.

1. Introduction. The traditional deterministic predator-prey system with Holling type II response can be described as

\[
\begin{align*}
\dot{x}(t) &= x(t) \left( b_1 - a_1 x(t) - \frac{\beta_1 y(t)}{1 + k x(t)} \right), \\
\dot{y}(t) &= y(t) \left( \frac{\beta_2 x(t)}{1 + k x(t)} - b_2 - a_2 y(t) \right),
\end{align*}
\]

where \(x(t)\) and \(y(t)\) respectively represent the densities of predator and prey at time \(t\); \(b_1, a_i, \beta_i, k\), \(i = 1, 2\) are positive constants. For biological meaning of each coefficient we refer the reader to [7, 17, 18, 34]. Owning to its theoretical and practical significance, the dynamic behaviors of system 1 have received great attention and been studied extensively, see [7, 6, 17, 18, 21, 24, 33, 34] for example. Particularly, from [18, 34] we know that system 1 has a unique positive equilibrium \((\bar{x}, \bar{y})\) provided its coefficients satisfy

\[
\frac{b_1}{a_1} > \frac{b_2}{\beta_2 - b_2 k} > 0.
\]

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In addition, by constructing Lyapunov functions or see [7, 33], we derive that if the condition
\[ a_1(1 + k\bar{x}) > \beta_1\bar{y} \]
holds, then the unique coexisting point \((\bar{x}, \bar{y})\) is globally stable. However, in the real world, time delays frequently occur in almost every situation and research has shown that time delays have a great destabilizing influence on species populations [3, 10, 14, 11, 20, 32, 35, 38, 41]. Thus, it is essential to take the cumulative effect of the past of a biological species during a time period \(\tau\) (including the case \(\tau = \infty\)) into account, which means a more accurate model is 
\[
\dot{x}(t) = rx(t)(1 - \frac{1}{\bar{x}} \int_{-\infty}^{t} x(t + s) d\mu(s)), \quad \text{where} \quad \mu(s) \text{ is a probability measure on } (-\infty, 0].
\]
For more details in this direction, please see [14, 38].

In [15], the authors argued that in a situation where the equilibrium is not the desirable one (or affordable), thus one may wish to change the position of positive equilibrium but to retain its stability. This is of significance in the control procedure to keep the balance of ecology. One of the techniques used to achieve the aim is to alter the system structurally by introducing a feedback control variable [1, 22], which can be implemented by means of a biological control or some harvesting procedure. During the last decade, the single-species or multispecies systems with feedback controls have been extensively studied in many literatures [4, 5, 9, 25, 37, 39, 40, 42]. Particularly, in [5, 9, 25], authors showed that feedback controls have no influence on the permanence or global stability of corresponding systems, while in [40], authors argued that by choosing suitable values of feedback control variables, stable species of Holling-II type cooperative system could become extinct, or still keep the property of globally stable. However, so far as we know a very few work has been done to investigate the influence of feedback controls on the stability property of predator-prey model with Holling type II response and infinite delay, which takes the following forms:

\[
\begin{align*}
\dot{x}(t) &= x(t) \left( b_1 - a_{11} x(t) - a_{12} \int_{-\infty}^{0} x(t + \theta) d\mu_1(\theta) - \frac{\beta_2 y(t)}{1 + k x(t)} - c_1 u_1(t) \right), \\
\dot{y}(t) &= y(t) \left( \frac{\beta_2 x(t)}{1 + k x(t)} - b_2 - a_{21} y(t) - a_{22} \int_{-\infty}^{0} y(t + \theta) d\mu_2(\theta) - c_2 u_2(t) \right), \\
u_1(t) &= -f_1 u_1(t) + g_1 x(t), \\
u_2(t) &= -f_2 u_2(t) + g_2 y(t),
\end{align*}
\]
where \(u_1(t)\) and \(u_2(t)\) are feedback control variables; \(\mu_1(\theta)\) and \(\mu_2(\theta)\) denote probability measures on \((-\infty, 0] ; b_1, a_{ij}, \beta_1, k, c_i, f_i, g_i, i,j = 1,2\) are positive constants. For more detailed meanings of coefficients of the system, one can refer to [5, 14, 15, 38] and the reference cited therein.

On the other hand, population dynamics are inevitably affected by environmental noise, which is an important component in an ecosystem. The early research by Levin [23] considered an autonomous two species Lotka-Volterra predator-prey dispersal system and showed the dispersion could destabilize the system. Later, lots of authors introduce stochastic perturbation into deterministic models to reveal the effect of environmental variability on biological models [2, 8, 12, 13, 19, 27, 26, 28, 30, 31, 36, 43].

In practice, the intrinsic growth rate \(b_1\) or mortality rate \(b_2\) of the species at time \(t\) is estimated by an average value plus an error term. In general, by the
well-known central limit theorem, the error term follows a normal distribution and is sometimes dependent on how much the current population sizes differ from the equilibrium state [12, 13, 26, 30, 43]. In other words, we can replace the rate $b_i$ by an average value plus a random fluctuation term

$$b_1 + \sigma_1(x - x^*)\dot{B}_1(t), \quad b_2 + \sigma_2(y - y^*)\dot{B}_2(t),$$

where $(x^*, y^*)$ is an equilibrium state of system 2, which is assumed to have a unique positive equilibrium $(x^*, y^*, u^*_1, u^*_2)$; $B_i(t)$ $(i = 1, 2)$ are mutually independent one-dimensional standard Brownian motions defined on the complete probability space $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ with its filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions, and $\sigma_i > 0$ $(i = 1, 2)$ are the intensities of the white noises in [29].

Motivated by the above discussions, how do feedback controls affect the global stability of Holling-II type predator-prey system with random perturbations and infinite delays? To answer this question, in this work, we consider a stochastic predator-prey system with infinite delays and feedback controls, that is,

$$dx(t) = x(t)\left(b_1 - a_{11}x(t) - a_{12}\int_{-\infty}^{0} x(t + \theta)d\mu_1(\theta) - \frac{\beta_1 y(t)}{1 + b x(t)} - c_1 u_1(t)\right)dt + \sigma_1 x(t)(x(t) - x^*)dB_1(t),$$

$$dy(t) = y(t)\left(\frac{\beta_2 x(t)}{1 + b x(t)} - b_2 - a_{21} y(t) - a_{22}\int_{-\infty}^{0} y(t + \theta)d\mu_2(\theta) - c_2 u_2(t)\right)dt + \sigma_2 y(t)(y(t) - y^*)dB_2(t),$$

$$du_1(t) = (-f_1 u_1(t) + g_1 x(t))dt,$$

$$du_2(t) = (-f_2 u_2(t) + g_2 y(t))dt,$$

The initial conditions of model 3 are:

$$(x(\theta), y(\theta)) = (\varphi_1, \varphi_2) \in BC((\infty, 0]; \mathbb{R}^2_+), \quad u_i(0) > 0, \quad i = 1, 2,$$

where $BC((\infty, 0]; \mathbb{R}^2_+)$ represents the family of bounded and continuous functions from $(\infty, 0]$ to $\mathbb{R}^2_+$ with the norm $\|\varphi\| = \sup_{\theta \leq 0} |\varphi(\theta)|$.

We attempt to organize this paper by several parts: we will prove that model 3 admits a unique global positive solution in the next section. Then sufficient conditions guarantee the global stability property of model 3 will be established in Section 3. As a consequence, we derive the conclusions and demonstrate several examples and their numerical simulations in order to support the main results of this paper.

2. Existence and uniqueness of the global positive solution.

Lemma 2.1. For any given initial condition $4$, model 3 has a unique local positive solution $(x(t), y(t), u_1(t), u_2(t))$ on $(-\infty, \tau_\varepsilon)$ almost surely (a.s.), where $\tau_\varepsilon$ denotes the explosion time.

Because the proof of this lemma is similar to that in [27], we omit it here.

Theorem 2.2. For any given initial data $4$, there exists a unique solution $(x(t), y(t), u_1(t), u_2(t))$ to model 3 on $\mathbb{R}$ and the solution will remain in $\mathbb{R}^4_+$ with probability one.
Proof. The following proof is motivated by the work of Liu and Wang [26]. In order to show that the solution is global, it is sufficient to show \( \tau_\epsilon = \infty \) a.s.. Let \( n_0 \geq 1 \) be so large that \( \nu_i(\theta)(i = 1, 2), \theta \in (-\infty, 0], u_i(0)(i = 1, 2) \) lying within the interval \( \left[ \frac{1}{n_0}, n_0 \right] \). For each integer \( n > n_0 \), we define the stopping time

\[
\tau_n = \inf \left\{ t \in (-\infty, \tau_\epsilon) : \min \{ x(t), y(t), u_1(t), u_2(t) \} < \frac{1}{n} \text{ or } \max \{ x(t), y(t), u_1(t), u_2(t) \} > n \right\}.
\]

Obviously, \( \tau_n \) is an increasing function as \( n \to \infty \). We denote \( \tau_\infty = \lim_{n \to \infty} \tau_n \), according to the definition of the stopping time and the fact that \( \tau_\epsilon \) is the explosion time, we derive that \( \tau_\infty \leq \tau_\epsilon \). In order to prove the assertion \( \tau_\epsilon = \infty \) holds almost surely, we need to check that \( \tau_\infty = \infty \) will be valid almost surely. Otherwise, there is a constant \( \varepsilon \in (0, 1) \) such that \( P\{\tau_\infty < \infty\} > \varepsilon \). Then, there exist an integer \( n_1 \geq n_0 \) and a constant \( T > 0 \) satisfying

\[
P\{\tau_n \leq T\} \geq \varepsilon, \text{ for all } n \geq n_1.
\]

We define a \( C^2 \)-function \( V : \mathbb{R}^4_+ \to \mathbb{R}_+ \) as follows:

\[
V(x(t), y(t), u_1(t), u_2(t)) = \sqrt{x(t)} - 1 - \frac{1}{2} \ln x(t)
+ \sqrt{y(t)} - 1 - \frac{1}{2} \ln y(t) + \frac{c_1}{2f_1} u_1(t) + \frac{c_2}{2f_2} u_2(t),
\]

The generalized Itô’s formula then gives that

\[
dV(x, y, u_1, u_2) = \frac{\sqrt{x} - 1}{2} \left( b_1 - a_{11} x - a_{12} \int_{-\infty}^0 x(t + \theta) d\mu_1(\theta) - \frac{\beta_1 y}{1 + kx} - c_1 u_1 \right) dt
+ \frac{c_1}{2f_1} (-f_1 u_1 + g_1 x) dt + \frac{\sigma_1^2 (2 - \sqrt{x})}{8} (x - x^*)^2 dt
+ \frac{\sigma_1 (\sqrt{x} - 1)}{2} (x - x^*) dB_1(t) + \frac{\sqrt{y} - 1}{2} \left( \frac{\beta_2 y}{1 + kx} - b_2 - a_{21} y \right)
- a_{22} \int_{-\infty}^0 y(t + \theta) d\mu_2(\theta) - c_2 u_2 \right) dt + \frac{c_2}{2f_2} (-f_2 u_2 + g_2 y) dt
+ \frac{\sigma_2^2 (2 - \sqrt{y})}{8} (y - y^*)^2 dt + \frac{\sigma_2 (\sqrt{y} - 1)}{2} (y - y^*) dB_2(t)
\leq \frac{1}{2} \left( b_1 x^\frac{1}{2} - a_{11} x^\frac{3}{2} + a_{11} x + a_{12} \int_{-\infty}^0 x(t + \theta) d\mu_1(\theta) + \beta_1 y + \frac{c_1 g_1}{2f_1} x
- c_1 u_1 x^\frac{1}{2} \right) dt + \frac{\sigma_1^2}{8} \left( - x^\frac{3}{2} + 2x^2 + 2x^* x^\frac{3}{2} + 2(x^*)^2 \right) dt
+ \frac{1}{2} \left( \beta_2 y^\frac{1}{2} - b_2 y^\frac{3}{2} - a_{21} y^\frac{3}{2} + b_2 + a_{21} y + a_{22} \int_{-\infty}^0 y(t + \theta) d\mu_2(\theta)
+ \frac{c_2 g_2}{2f_2} y - c_2 u_2 y^\frac{1}{2} \right) dt + \frac{\sigma_2^2}{8} \left( - y^\frac{3}{2} + 2y^2 + 2y^* y^\frac{3}{2} + 2(y^*)^2 \right) dt
\]
3. Global stability.

Lemma 3.1. If the assumption

\[
\frac{b_1}{a_{11} + a_{12} + \frac{c_{11}y_1}{A}} > \frac{b_2}{\beta_2 - b_2k} > 0
\]  

(7)

holds, then system 3 has a unique positive equilibrium \( E^*(x^*, y^*, u_1^*, u_2^*) \).

Proof. We determine the positive equilibrium of system 3 through solving the following equations

\[
\frac{d}{dt} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} a_{11}x + a_{12}y + \frac{c_{11}y}{A} - b_1x \\ a_{21}x + a_{22}y + \frac{c_{22}y}{A} - b_2y \end{pmatrix},
\]

where \( K_1, K_2, K_3 \) are positive constants. Integrating both sides of 6 from 0 to \( \tau_k \land T \) and then taking the expectation yields that

\[
EV(x(\tau_n \land T), y(\tau_n \land T), u_1(\tau_n \land T), u_2(\tau_n \land T)) 
\leq V(x(0), y(0), u_1(0), u_2(0)) + (K_1 + K_2 + K_3)T.
\]

We set \( \Omega_n = \{ \tau_n \leq T \} \) for \( n \geq n_1 \), then 5 turns into \( P(\Omega_n) \geq \varepsilon \). Note that for \( \omega \in \Omega_n \), each component of \( (x(\tau_n, \omega), y(\tau_n, \omega), u_1(\tau_n, \omega), u_2(\tau_n, \omega)) \) equals either \( n \) or \( \frac{1}{n} \), and hence

\[
V(x(\tau_n, \omega), y(\tau_n, \omega), u_1(\tau_n, \omega), u_2(\tau_n, \omega)) \geq \min \left\{ n - 1 - \ln n, \frac{1}{n} - 1 + \ln n \right\}.
\]

Therefore,

\[
V(x(0), y(0), u_1(0), u_2(0)) + (K_1 + K_2 + K_3)T 
\geq \mathbb{E}\left(1_{\Omega_n}V(x(\tau_n, \omega), y(\tau_n, \omega), u_1(\tau_n, \omega), u_2(\tau_n, \omega)) \right) 
\geq \varepsilon \min \left\{ n - 1 - \ln n, \frac{1}{n} - 1 + \ln n \right\},
\]

the contradiction is derived when letting \( n \to \infty \). \( \square \)
\[
b_1 - a_{11}x - a_{12}x - \frac{\beta_1 y}{1+kx} - c_1u_1 = 0, \\
\frac{\beta_2 x}{1+kx} - b_2 - a_{21}y - a_{22}y - c_2u_2 = 0, \\
- f_1u_1 + g_1x = 0, \\
- f_2u_2 + g_2y = 0,
\]
which is equivalent to
\[
l_1 : y = \frac{1}{\beta_1} (1+kx)(b_1 - \hat{a}x), \\
l_2 : y = \frac{1}{b} \left( \frac{\beta_2 x}{1+kx} - b_2 \right), 
\]
where \( \hat{a} = a_{11} + a_{12} + \frac{c_1g_1f_1}{b_1} \), \( \hat{b} = a_{21} + a_{22} + \frac{c_2g_2f_2}{b_2} \).

It is easy to see that the curve \( l_1 \) of 9 is a parabola whose axis is vertical and whose vertex (maximum of \( y \)) is at \( x = \hat{a}, y = \beta_1 \left( \hat{a} - b \right) \). This parabola intersects the \( y \)-axis only at \( y_p = \frac{b_1}{\beta_1} \) and the \( x \)-axis at \( x_{p1} = -\frac{1}{k}, x_{p2} = \frac{b_1}{\beta_1} \). What is more, the curve \( l_2 \) of 9 is a hyperbola whose (mutually orthogonal) asymptotes are \( x_h = -\frac{1}{k}, y_h = \frac{1}{k} \left( \frac{\beta_2}{b} - b_2 \right) \). Thus, the left branch of this hyperbola does not belong to the first quadrant and its right branch may cross this quadrant only if \( y_h > 0 \), that is, \( \beta_2 - b_2k > 0 \), and the intersection of the right branch with the axes are \( x_q = \frac{b_2}{\beta_2 - b_2k}, y_q = -\frac{b_2}{b} \). Therefore, if the assumption (7) holds, the parabola \( l_1 \) and the hyperbola \( l_2 \) have a unique intersection \((x^*, y^*)\) in positive quadrant. We present the parabola \( l_1 \) and the hyperbola \( l_2 \) of 9 in Figure 1. It follows that from the equation of 8, system 3 admits a unique positive equilibrium \( E^*(x^*, y^*, u_1^*, u_2^*) \).

![Figure 1. The parabola \( l_1 \) (green line) and the hyperbola \( l_2 \) (blue line) of 9 provided that the condition 7 holds.](image)

Note that: (i) \( x_{p1} \) and \( y_p \) are independent of control parameters, whereas \( x_{p2} \) and \( x_q \) decrease as \( \frac{c_1g_1f_1}{b_1} \) (and, thus, \( \hat{a} \)) increases, and (ii) \( x_h \) and \( x_q \) are independent of control parameters, whereas \( y_q \) increases while \( y_h \) decreases as \( \frac{c_2g_2f_2}{b_2} \) (and, thus, \( \hat{b} \)) increases.
Therefore, the symmetric axis of parabola \( l_1 \) which is \( x_v \) and the positive intersection of \( l_1 \) with \( x \)-axis which is \( x_p \) both move left as control parameter \( \frac{c_1g_1}{f_1} \) increases, and the right branch of hyperbola \( l_2 \) corresponding to \( \frac{c_2g_2}{f_2} = 0 \) is above all the hyperbola corresponding to positive values of \( \frac{c_2g_2}{f_2} \) in the first quadrant.

A consequence of the previous analysis is the following.

**Proposition 1.** The region \( R_0 \) of the first quadrant where equilibria \((x^*, y^*)\) will occur under feedback controls (that is, \( c_ig_i > 0, \ i = 1, 2 \)) is formed by the points below both the parabola \( l_1 \) and the hyperbola \( l_2 \) of 9 with \( c_ig_i = 0, \ i = 1, 2 \).

We show the region \( R_0 \) in shaded area in Figure 2. Here the blue dash lines represent \( l_1 \) and \( l_2 \) with \( c_ig_i = 0 \) and the red solid lines represent \( l_1 \) and \( l_2 \) with \( c_ig_i > 0(i = 1, 2) \).

![Figure 2](image)

**Figure 2.** The region \( R_0 \) where positive equilibria \((x^*, y^*)\) will occur under feedback controls.

To any chosen point \((x^*, y^*) \in R_0\), we can determine the control parameters of system 3 since there corresponds a unique pair of control parameters \((\frac{c_1g_1}{f_1}, \frac{c_2g_2}{f_2})\) according to

\[
\begin{align*}
c_1g_1 &= \frac{1}{x^*} \left( b_1 - \frac{\beta_1y^*}{1+kx^*} \right) - a_{11} - a_{12}, \\
c_2g_2 &= \frac{1}{y^*} \left( \frac{\beta_2x^*}{1+kx^*} - b_2 \right) - a_{21} - a_{22},
\end{align*}
\]

(10)

where the parameters appearing at the right-hand sides only depend on the controlled process.

Next, we will derive some sufficient conditions to determine whether a chosen equilibrium in the region \( R_0 \) is globally stable.

**Theorem 3.2.** If the parameters of system 3 satisfy the following inequalities

\[
\begin{align*}
a_{11} - a_{12} - \frac{k\beta_1y^*}{1+kx^*} &> \frac{1}{2} \sigma_1^2 x^*, \\
a_{21} - a_{22} &> \frac{1}{2} \sigma_2^2 y^*.
\end{align*}
\]

(11)
Then the unique positive equilibrium \( E^*(x^*, y^*, u_1^*, u_2^*) \) of model 3 is globally stochastically asymptotically stable, that is, for any given initial value \( \mathbf{x} \), the solution of system 3 has the property that

\[
\lim_{t \to \infty} x(t) = x^*, \quad \lim_{t \to \infty} y(t) = y^*, \quad \lim_{t \to \infty} u_i(t) = u_i^*, \quad i = 1, 2
\]

almost surely (a.s.).

Proof. From the theory of stability of stochastic differential equations, we only need to find a Lyapunov function \( V(z) \) satisfying \( LV(z) \leq 0 \) and the identity holds if and only if \( z = z^* \) (see [29]), where \( z = z(t) \) is the solution of the \( n \)-dimensional stochastic differential equation

\[
dz(t) = f(z(t), t)dt + g(z(t), t)dB(t).
\]

(12)

\( z^* \) is the equilibrium position of 12, and

\[
LV(z) = V_z(z)f(t, z) + 0.5\text{trace}[g^T(t, z)V_{zz}(z)g(t, z)].
\]

Now define a Lyapunov function as follows

\[
V_1(x, y, u_1, u_2) = d_1 \left( x - x^* - x^* \ln \frac{x}{x^*} \right) + d_2 \left( y - y^* - y^* \ln \frac{y}{y^*} \right) + \sum_{i=1}^{2} e_i (u_i - u_i^*)^2,
\]

(13)

where \( d_1 = 1, \ d_2 = \frac{(1+\kappa x^*)^2}{\beta_2}, \ e_i = \frac{c_1 d_1}{2g_i}, \ i = 1, 2. \)

Obviously, this function is nonnegative on \( \mathbb{R}_+ \). Making use of Itô’s formula yields

\[
LV_1(x, y, u_1, u_2)
\]

\[
= \frac{d_1}{1+kx} \left( b_1 - a_{11} x - a_{12} \int_{-\infty}^{t} x(t+\theta) d\mu_1(\theta) - \frac{\beta_1 y}{1+kx} - c_1 u_1 \right)
\]

\[
+ 0.5d_1 \sigma_x^2 \left( x - x^* \right)^2 + d_2 \left( y - y^* \right) \left( \frac{\beta_2 y}{1+kx} - b_2 - a_{21} y - c_2 u_1 \right)
\]

\[
- a_{22} \int_{-\infty}^{t} y(t+\theta) d\mu_2(\theta) + 0.5d_2 \sigma_y^2 \left( y - y^* \right)^2
\]

\[
+ 2e_1 (u_1 - u_1^*) (-f_1 u_1 + g_1 x) + 2e_2 (u_2 - u_2^*) (-f_2 u_2 + g_2 y)
\]

\[
= d_1 \left( x - x^* \right) \left( -a_{11} (x - x^*) - c_1 (u_1 - u_1^*) - \frac{\beta_1 y}{1+kx} + \frac{\beta_1 y^*}{1+kx^*} \right)
\]

\[
+ a_{12} \int_{-\infty}^{t} (x - x(t+\theta)) d\mu_1(\theta) + 0.5d_1 \sigma_x^2 (x - x^*)^2
\]

\[
+ d_2 \left( y - y^* \right) \left( \frac{\beta_2 y}{1+kx} - a_{21} (y - y^*) - c_2 (u_2 - u_2^*) \right)
\]

\[
+ a_{22} \int_{-\infty}^{t} (y - y(t+\theta)) d\mu_2(\theta) + 0.5d_2 \sigma_y^2 (y - y^*)^2
\]

\[
+ \frac{2g_1 e_1}{u_1^*} (u_1 - u_1^*) \left( -x^* (u_1 - u_1^*) + u_1^* (x - x^*) \right)
\]

\[
+ \frac{2g_2 e_2}{u_2^*} (u_2 - u_2^*) \left( -y^* (u_2 - u_2^*) + u_2^* (y - y^*) \right)
\]

\[
= \left( -d_1 a_{11} + 0.5d_1 \sigma_x^2 (x^*) + \frac{d_1 \beta_1 ky^*}{(1+kx^*)(1+kx)} \right) (x - x^*)^2
\]
all trajectories in $\mathbb{R}^4$. Proposition 1 and Theorem 3.2 demonstrate that for the autonomous Holling-II type predator-prey system, feedback controls not only alter the position of positive equilibrium, but also have influence on the stability of system 2 and 3, that is, for both deterministic system and stochastic system, we can only locate the unique positive equilibrium in a certain area in the region $R_0$ by feedback controls.

\begin{align*}
+ \left( -a_{21}d_2 + 0.5d_2\sigma_2^2 y^* \right)(y - y^*)^2 - \frac{2g_1e_1x^*}{u_1^*}(u_1 - u_1^*)^2 \\
- \frac{2g_2e_2y^*}{u_2^*}(u_2 - u_2^*)^2 - a_{12}d_1 \int_{-\infty}^{0} (x - x^*)(x(t + \theta) - x^*)d\mu_1(\theta) \\
- a_{22}d_2 \int_{-\infty}^{0} (y - y^*)(y(t + \theta) - y^*)d\mu_2(\theta).
\end{align*}

Noting that $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$, $\theta > 0$, then it follows from 14 that

\[ LV_1(x, y, u_1, u_2) \leq d_1 \left( -a_{11} + 0.5\sigma_1^2 x^* + 0.5a_{12} \frac{\beta_1ky^*}{(1+kx^*)(1+kx)} \right)(x - x^*)^2 \\
+ 0.5a_{12}d_1 \int_{-\infty}^{0} (x(t + \theta) - x^*)^2d\mu_1(\theta) - \frac{2g_1e_1x^*}{u_1^*}(u_1 - u_1^*)^2 \\
- \frac{2g_2e_2y^*}{u_2^*}(u_2 - u_2^*)^2 + d_2 \left( -a_{21} + 0.5\sigma_2^2 y^* + 0.5a_{22} \right)(y - y^*)^2 \\
+ 0.5a_{22}d_2 \int_{-\infty}^{0} (y(t + \theta) - y^*)^2d\mu_2(\theta).
\]

Let

\[ V_2(t) = 0.5d_1a_{12} \int_{-\infty}^{0} \int_{t+\theta}^{t} (x(s) - x^*)^2dsd\mu_1(\theta) \\
+ 0.5d_2a_{22} \int_{-\infty}^{0} \int_{t+\theta}^{t} (y(s) - y^*)^2dsd\mu_2(\theta).
\]

Then

\[ LV_2(t) = 0.5d_1a_{12}(x(t) - x^*)^2 - 0.5a_{12}d_1 \int_{-\infty}^{0} (x(t + \theta) - x^*)^2d\mu_1(\theta) \\
+ 0.5d_2a_{22}(y(t) - y^*)^2 - 0.5a_{22}d_2 \int_{-\infty}^{0} (y(t + \theta) - y^*)^2d\mu_2(\theta).
\]

Then define

\[ V(x, y, u_1, u_2, t) = V_1(x, y, u_1, u_2) + V_2(t).
\]

It follows from 15 and 17 that

\begin{align*}
LV(x, y, u_1, u_2, t) &\leq -d_1 \left( a_{11} - a_{12} - 0.5\sigma_1^2 x^* - \frac{\beta_1ky^*}{(1+kx^*)(1+kx)} \right)(x - x^*)^2 \\
&\quad - d_2 \left( a_{21} - a_{22} - 0.5\sigma_2^2 y^* \right)(y - y^*)^2 \\
&\quad - \frac{2g_1e_1x^*}{u_1^*}(u_1 - u_1^*)^2 - \frac{2g_2e_2y^*}{u_2^*}(u_2 - u_2^*)^2.
\end{align*}

Clearly, if 11 holds then the above inequality implies $LV(x, y, u_1, u_2, t) < 0$ along all trajectories in $\mathbb{R}_+^4$ except $(x^*, y^*, u_1^*, u_2^*)$. \qed

**Remark 1.** Proposition 1 and Theorem 3.2 demonstrate that for the autonomous Holling-II type predator-prey system, feedback controls not only alter the position of positive equilibrium, but also have influence on the stability of system 2 and 3, that is, for both deterministic system and stochastic system, we can only locate the unique positive equilibrium in a certain area in the region $R_0$ by feedback controls.
to ensure the stable property. Moreover, for stochastic system, this certain area is relevant to the insensities of white noises.

Similar to the analysis of Theorem 3.3, we have the following corollary.

**Corollary 1.** If the parameters of system 3 without feedback control satisfy the inequalities (11), then the unique positive equilibrium \( E^* (x^*, y^*) \) of model 3 is globally stochastically asymptotically stable, that is, for any given initial value \( 4 \), the solution of system 3 has the property that
\[
\lim_{t \to \infty} x(t) = x^*, \quad \lim_{t \to \infty} y(t) = y^*
\]
almost surely (a.s.).

**Remark 2.** Corollary 1 implies that if the positive equilibrium of the deterministic system is globally stable, then the stochastic model will preserve this nice property provided the noises are sufficiently small. In this case, according to Theorem 3.2 and relationships (10), we can alter this globally stable positive equilibrium to a desirable (or affordable) position in some areas of \( R_0 \) and system 20 exists a unique positive equilibrium \( E_0 \).

**Remark 3.** We can also derive from Corollary 1 and Theorem 3.2 that if the stochastic system is destabilized by large intensities of white noises, feedback control can make the controlled system to be globally stable at a new positive equilibrium in the region \( R_0 \), which means the system can return to a stable community in which all species could coexist.

4. **Examples and simulations.** Several examples and the corresponding numerical simulations will be presented to support the main results by means of Milstein Method [16].

Consider the following equations
\[
\dot{x}(t) = x(t) \left( 1.5 - 0.6x(t) - 0.1 \int_{-\infty}^{0} e^{\theta} x(t + \theta) d\theta - \frac{0.6y(t)}{1 + 0.5x(t)} \right), \\
\dot{y}(t) = y(t) \left( \frac{1.5x(t)}{1 + 0.5x(t)} - 0.5 - 0.5y(t) - 0.1 \int_{-\infty}^{0} e^{\theta} y(t + \theta) d\theta \right). \tag{20}
\]

It is easy to verify that \( \frac{b_1}{a_{11} + a_{12}} = 2.143 > \frac{b_2}{\beta_3 - \beta_2 k} = 0.4 > 0 \), thus condition (7) holds and system (20) exists a unique positive equilibrium \( (x_{e0}, y_{e0}) = (1.501, 1.311) \). Then we derive \( a_{11} - a_{12} - \frac{k \beta_3 y_{e0}}{1 + k x_{e0}} = 0.2754 > 0 \), \( a_{21} - a_{22} = 0.4 > 0 \). In view of Corollary 1, this equilibrium position is globally asymptotically stable. Figure 3 shows the dynamic behavior of system (20). We also present the region \( R_0 \) in which equilibria will occur under feedback control in Figure 4.

**Example 1.** We assume that system (20) is affected by small noises and set \( \sigma_1 = \sigma_2 = 0.1 \). Obviously, conditions (11) still hold, since \( a_{11} - a_{12} - \frac{k \beta_3 y_{e0}}{1 + k x_{e0}} = 0.2754 > \frac{1}{2} \sigma_1^2 x_{e0} = 0.0075 \), \( a_{21} - a_{22} = 0.4 > \frac{1}{2} \sigma_2^2 y_{e0} = 0.0066 \). Therefore, the stochastic model will keep the stable property at unique positive equilibrium \( (x_{e0}, y_{e0}) = (1.501, 1.311) \), which can be observed in Figure 5.

Now, we use feedback control to locate the positive equilibrium to a lower position in the region \( R_0 \), which is \( (x_{e1}, y_{e1}) = (1.4, 1) \). It is not difficult to verify that conditions (11) of Theorem 3.2 are still being satisfied under \( (x_{e1}, y_{e1}) \). And by
relationships 10, we can determine the corresponding control parameters, which are \( \frac{c_1 g_1}{f_1} = 0.1193, \frac{c_2 g_2}{f_2} = 0.1353 \). If we set \( c_1 = 0.15, c_2 = 0.17, f_1 = 1, f_2 = 1.2, g_1 = 0.8, g_2 = 0.96 \), then the following controlled stochastic system can be obtained:

\[
\begin{align*}
\frac{dx(t)}{dt} &= x(t) \left( 1.5 - 0.6x(t) - 0.1 \int_{-\infty}^{0} e^{\theta} x(t + \theta) d\theta - \frac{0.6y(t)}{1 + 0.5x(t)} - 0.15u_1(t) \right) dt \\
&\quad + 0.1x(t)(x(t) - 1.4)dB_1(t), \\
\frac{dy(t)}{dt} &= y(t) \left( \frac{1.5x(t)}{1 + 0.5x(t)} - 0.5 - 0.5y(t) - 0.1 \int_{-\infty}^{0} e^{\theta} y(t + \theta) d\theta - 0.17u_2(t) \right) dt \\
&\quad + 0.1y(t)(y(t) - 1)dB_2(t),
\end{align*}
\]

Figure 3. Dynamic behavior of the solution \((x(t), y(t))^T\) of system 20 with the initial condition \((\varphi_1(\theta), \varphi_2(\theta)) = (1.2e^{\theta}, 0.8e^{\theta}), \theta \in (-\infty, 0]\) .

Figure 4. The region \( R_0 \) where positive equilibria of system 20 will occur under feedback controls.
Figure 5. Dynamic behavior of the solution \((x(t), y(t))^T\) of system 20 with small perturbations \(\sigma_1 = \sigma_2 = 0.1\) and the initial condition \((\varphi_1(\theta), \varphi_2(\theta)) = (1.2e^\theta, 0.8e^\theta), \theta \in (-\infty, 0]\).

\[
du_1(t) = (-u_1(t) + 0.8x(t))dt, \\
du_2(t) = (-1.2u_2(t) + 0.96y(t))dt.
\]

This system is globally asymptotically stable at unique positive equilibrium \((x_e, y_e, u_{1e}, u_{2e}) = (1.4, 1.12, 0.8)\). Figure 6 shows the dynamic behavior of system 21.

Example 2. If the intensities of white noises become larger such as \(\sigma_1 = \sigma_2 = 1\), then conditions 11 are not being satisfied since \(a_{11} - a_{12} - k_1y_{e_0} = 0.2754 < \frac{1}{2}\sigma_1^2x_{e_0} = 0.75, a_{21} - a_{22} = 0.4 < \frac{1}{4}\sigma_2^2y_{e_0} = 0.65\). Therefore, by Corollary 1 we derive that the stochastic system without feedback control is destabilized at positive equilibrium \((x_e, y_e)\) and it is illustrated in Figure 7.

We choose a new positive equilibrium \((x_{e_2}, y_{e_2}) = (0.8, 0.4)\), which belongs to the region \(R_0\) and satisfies conditions 11 of Theorem 3.2. According to relationships 10, suitable control parameters can be obtained \(\frac{c_{12}}{f_{11}} = 0.9607, \frac{c_{22}}{f_{22}} = 0.2929\). We
set $c_1 = 1.2, c_2 = 0.366, f_i = 1, g_i = 0.8, i = 1, 2$ and the corresponding controlled stochastic system is as following:

$$dx(t) = x(t) \left(1.5 - 0.6x(t) - 0.1 \int_{-\infty}^{0} e^{\theta} x(t + \theta)d\theta - \frac{0.6y(t)}{1 + 0.5x(t)} - 1.2u_1(t) \right)dt + x(t)(x(t) - 0.8)dB_1(t),$$

$$dy(t) = y(t) \left(\frac{1.5x(t)}{1 + 0.5x(t)} - 0.5 - 0.5y(t) - 0.1 \int_{-\infty}^{0} e^{\theta} y(t + \theta)d\theta - 0.366u_2(t) \right)dt + y(t)(y(t) - 0.4)dB_2(t),$$

$$du_1(t) = (- u_1(t) + 0.8x(t))dt,$$

$$du_2(t) = (- u_2(t) + 0.8y(t))dt,$$

which is globally asymptotically stable at unique positive equilibrium $(x_{e_2}, y_{e_2}, u_{1e_2}, u_{2e_2}) = (0.8, 0.4, 0.64, 0.32)$ and the dynamic behavior of system 22 is demonstrated in Figure 8.

$$\text{Figure 7. Dynamic behavior of the solution } (x(t), y(t))^T \text{ of system 20 with big perturbations } \sigma_1 = \sigma_2 = 1 \text{ and the initial condition } (\varphi_1(\theta), \varphi_2(\theta)) = (1.2e^\theta, 0.8e^\theta), \theta \in (-\infty, 0].$$

5. Conclusion. In this paper, we consider a stochastic Holling-II type predator-prey system with infinite delays and feedback controls. We show that model 3 admits a unique and global solution for any given initial value 4 by using of Lyapunov function method. We also use graphical description to obtain the convex region where positive equilibria of predator-prey models 3 can be assigned through feedback controls and the values of the control parameters corresponding to a given equilibrium state are determined. By constructing a suitable Lyapunov functional, we derive sufficient conditions to guarantee the global stability of the equilibrium of system 3 and show the influence of feedback controls on the Holling-II type predator-prey system with random perturbations.

There are still many interesting questions that deserve further investigation. For example, for the predator-prey system incorporating with colored noises, what is
the influence of feedback controls on the global stability of the solutions and how does it happen? This is a further problem to be studied in the future.

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