Unconditionally Secure Multipartite Quantum Key Distribution

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We consider the problem of secure key distribution among $n$ trustful agents: the goal is to distribute an identical random bit-string among the $n$ agents over a noisy channel such that eavesdroppers learn little about it. We study the general situation where the only resources required are secure bipartite key distribution and authenticated classical communication. Accordingly, multipartite quantum key distribution can be proven unconditionally secure by reducing the problem to the bipartite case and invoking the proof of security of bipartite quantum key distribution.

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I. INTRODUCTION

With the growing use of the internet and other forms of electronic communication, the question of secure communication becomes one of considerable importance. Modern cryptographic techniques, based on the availability of ever increasing computational power, and the invention of public key cryptography, provide practical solutions for information security in various situations. But invariably these techniques are only computationally– and not unconditionally– secure, that is, they depend on the (unproven) hardness of certain mathematical problems. As a result, it cannot be guaranteed that future advances in computational power will not nullify their cryptographic protection. Nevertheless, there does exist a form of encryption with unconditional security: the use of one-time key pads. These are strings of random numbers added by the information sender to encode the message, to be subtracted by the receiver to decode. Provided that the key material is truly random and used only once, this system is unbreakable in the sense described by Shannon in the 1940s [1]. It is critically important that the pad is only used once, i.e., an encryption key can never be used twice. This restriction translates into the practical one of key distribution (KD). This need to securely distribute the key between the users makes it impractical in many applications. Where it is used in real life (e.g., in confidential communications between governments), the one-time key pads are actually delivered in person by some trusted third party [2], an arrangement prohibitively expensive for common usage and moreover not truly secure. Fortunately, recent advances in quantum information theory have shown that unconditionally secure key distribution is possible in principle.

That quantum information can, on account of quantum uncertainty and the no-cloning principle [3], be used to distribute cryptographic keys was realized two decades ago [4, 5]. More rigorous and comprehensive proofs of this task, generally called quantum key distribution (QKD), taking into consideration source, device and channel noise as well as an arbitrarily powerful eavesdropper, have been studied by various authors [6, 7, 8, 9, 10]. Recently, the issues of efficiency [11], security in the presence of an uncharacterized source [12] and high bit-error rate tolerance [13] of QKD have been considered. In particular, Lo and Chau [8] showed that, given fault-tolerant quantum computers, quantum key distribution over an arbitrarily long distance of a realistically noisy channel can be made unconditionally secure. This is a heartening development, since QKD is, among quantum information applications, relatively easy to implement, and some large scale implementations have already been achieved [14, 15]. The above mentioned works consider QKD between two parties (i.e., 2-QKD). It is of interest to consider its extension to more than two parties (i.e., n-QKD).

The problem of n-QKD is to determine how $n$ parties, who are able to communicate quantally, may share an identical and unconditionally secure, secret key among themselves in the presence of eavesdroppers. (A different generalization of 2-QKD gives multipartite quantum secret-sharing [16], which we do not consider here). In this work, we propose a protocol for this purpose and prove its unconditional security. We note that a simple extrapolation of 2-QKD to n-QKD would suggest that the agents should begin by sharing an $n$-partite entangled state. However, this proposition...
II. CLASSICAL REDUCTION OF $n$-KD TO 2-KD

As in 2-KD, the goal of $n$-KD is to show that $n$ trustful parties can securely share random, secret classical bits, even in the presence of noise and eavesdropping. It is assumed that the $n$ agents can share authenticated classical communication. It is convenient to treat the problem graph theoretically. The $n$ agents $A_i$ ($1 \leq i \leq n$) are considered as the vertices (or nodes) of an undirected graph. An instance of a secure bi-partite channel being shared between two parties is considered as an undirected edge between the two corresponding vertices. A graph so formed is called a security graph. It is obvious that if the security graph has a star topology (a hub vertex with all edges radiating from it to the other vertices), a simple $n$-KD protocol can be established. The agent at the hub vertex (say, called, Lucy) generates a random bit string and transmits it to every other agent along the edges to each of them. This will create a secure, identical random bit string with each agent.

In real life situations, because of practical and geographical constraints, the $n$ agents may not form a security graph with star topology. We describe a simple protocol that allows for more general secure bi-partite connectivity between the agents. In particular, from among the secure bipartite channels suppose a spanning tree (a graph connecting all vertices without forming a loop) can be constructed. This construction can be formalized in order to determine an optimal spanning tree. Some useful definitions are given below.

Weighted security graph: Given $n$ parties treated as nodes on a graph, we extend the definition of a security graph to the weighted security graph. A weight is associated with every edge and is defined to be some suitable measure of the cost of communicating by means of the channel corresponding to the edge.

Minimum spanning security tree: Consider the weighted security graph $G = (V, E)$. A spanning tree selected from $G$, given by $G_1 = (V, E_1), E_1 \subseteq E$ is called the minimum spanning security tree if it minimizes the total weight of the graph. It need not be unique and can be obtained using Kruskal’s or Prim’s algorithm. The minimum spanning security tree minimizes the resources needed in the protocol as well as the size of the sector eavesdroppers can potentially control.

Terminal agent: An agent that corresponds to a vertex of degree one (i.e., with exactly one edge linked to it). On the other hand, an agent that corresponds to a vertex of degree greater than one is called a non-terminal agent.

We now present a classical subroutine that allows $n - 1$ pair-wise shared random bits to be turned into a single random bit shared between the $n$ parties.

1. 2-KD: Along the $n - 1$ edges of a minimum spanning security tree, $n - 1$ random bits are securely shared by means of some secure 2-KD protocol.

2. Each non-terminal agent $A_i$ announces his uniformly randomized record: this is the list of edges emanating from the vertex along with the corresponding random bit values, to all of which a fixed random bit $x(i)$ is added.

3. This information is sufficient to allow every player, in conjunction with her/his own random bit record, to reconstruct the random bits of all parties. The protocol leader (say, Lucy) decides randomly on the terminal agent whose random bit will serve as the secret bit shared among the $n$ agents.

This subroutine consumes $n - 1$ pair-wise shared random numbers to give a one-bit secret key shared amongst the $n$-parties. To generate an $m$-bit string shared among the $n$ agents, the subroutine is repeated $m$ times.

Given that the initial bipartite sharing of randomness is secure, we will show that the above protocol subroutine allows some randomness to be shared between the $n$ agents without revealing anything to an eavesdropper. It involves each non-terminal agent announcing his uniformly randomized record. Suppose one such, $A_i$, has the random record 0,1,1 on the three edges linked to his vertex. He may announce 0,1,1 (for $x(i) = 0$) or 1,0,0 (for $x(i) = 1$). All
the three agents linked to him can determine which the correct string is by referring to their shared secret bit. It is a straightforward exercise to see that each of other agents linked to these three can determine the right bit string. Therefore, each agent can determine the random bits of all others. Eavesdroppers, on the other hand, lacking knowledge of any of the \( n - 1 \) shared random bits, can only work out the relative outcomes of all parties. The result is exactly two possible configurations for each secret bit, which are complements of each other. The eavesdropper “Eve” is thus maximally uncertain about which the correct configuration is. Hence, Lucy’s choice of a party to fix the secret bit reveals little to Eve. Insofar as the \( n \)-parties are able to communicate authenticated classical messages, the subroutine protocol is as secure as the underlying procedure for 2-KD.

It is obvious that the above protocol works for any spanning security tree. Clearly, a sufficient condition for turning shared bipartite randomness into randomness shared between \( n \) parties is that the weighted security graph should contain at least one spanning tree. On the other hand, if the security graph is disconnected, one easily checks that it is impossible to arrive at a definite random bit securely shared between both the disconnected pieces. Therefore, the existence of at least one spanning tree in the weighted security graph is both a necessary and sufficient condition for the required task.

The amount of securely shared randomness may be quantified by the length of shared random bit string multiplied by the number of sharing agents. In the above protocol, the \( n - 1 \) instances of pair-wise shared randomness is consumed to produce exactly one instance of a random bit shared between the \( n \) parties. We can then define the ‘random efficiency’ of the above protocol by \( \eta = \frac{(n \times 1)!}{(n - 1) \times 2} \), which tends to \((1/2)\) as \( n \to \infty \). Unconditional security of the above subroutine can in principle only be guaranteed in a protocol which includes in step 1 a quantum sub-routine that implements 2-QKD. In the following Section, we will present one such, based on the Shor-Preskill protocols [10], as an example.

III. QUANTUM PROTOCOL

As in 2-QKD, the goal of the proposed \( n \)-QKD protocol is to show that \( n \) trustful parties can securely distil random, shared, secret classical bits, whose security is to be proven inspite of source, device and channel noise and of Eve, an eavesdropper assumed to be as powerful as possible, and in particular, having control over all communication channels. From the result of the preceding Section, it follows that a quantum protocol is needed only in step 1 above. It will involve establishing 2-QKD along a minimum spanning tree in order to securely share pair-wise randomness along spanning tree’s edges and thence proceed to \( n \)-QKD. We assume as given the security of establishing pair-wise randomness along a spanning tree by means of a quantum communication network, based on a secure 2-QKD protocol [5, 6, 7, 8, 9, 10, 11, 12, 13]. In principle, these protocols guarantee security under various circumstances.

In an \( n \)-QKD scheme, the insecurity of even one of the players can undermine all. Hence additional classical processing like key reconciliation and privacy amplification [14] of the final key may be needed at the \( n \)-partite level. In the full \( n \)-QKD protocol that we present below, following Ref. [10] we exploit the connection of error correction codes [20] with key reconciliation and privacy amplification. These procedures have been extensively studied by classical cryptographers [14], and other possibilities exist.

In particular, we adopt a quantum protocol wherein pair-wise randomness is created by means of sharing EPR pairs (this follows the pattern set by the Ekert [5], Lo-Chau [8] and Modified Lo-Chau [10] protocols, but entanglement is not necessary, as seen in the original BB84 protocol). The basic graph theoretic definitions introduced above apply also for the quantum case, except that now the security channels correspond to shared EPR pairs. In place of a secure quantum channel, our classical subroutine described in the previous Section is capable of producing EPR pairs and is the weighted undirected graph \( \mathcal{G} \) as one whose every edge must contain a vertex drawn from the set \( A_i \) of all parties, and that of the minimum spanning security tree is the minimum spanning EPR tree. Let us enumerate the \( n \) parties as \( A_1, A_2, \ldots, A_n \). Suppose that only \( A_{i_1}, A_{i_2}, \ldots, A_{i_s} \) are capable of producing EPR pairs and \( S = \{ A_{i_1}, \ldots, A_{i_s} \} \) is the set of all such vertices, with \( S \neq \emptyset \). We construct a weighted undirected graph \( G = (V, E) \) as one whose every edge must contain a vertex drawn from the set \( S \), as follows: \( V = \{ A_i \mid i = 1, 2, \ldots, n \} \) and \( E = \{(A_i, A_j) \mid A_i, A_j \in S \text{ and } A_i \neq A_j \} \). And the weight of edge \((A_i, A_j)\) is defined to be \( w_{i,j} \propto k/m \), where \( k/m \) is the rate of the code. The classical subroutine described in the previous Section is some instance of EPR pair shared between two parties is considered as an undirected edge between the two corresponding vertices. A graph so formed is called an EPR graph [17]. The analog of the weighted security graph is the weighted EPR graph, and that of the minimum spanning security tree is the minimum spanning EPR tree.

Let \( C \) be a classical \( t \)-error correcting \([m, k]\)-code [20]. We now present a protocol that consumes \( n - 1 \) pair-wise securely shared sets of EPR pairs to create random bits shared between the \( n \) parties with asymptotic efficiency \( \eta = (1/2)k/m \), where \( k/m \) is the rate of the code. The classical subroutine described in the previous Section is some instance of EPR pair shared between two parties is considered as an undirected edge between the two corresponding vertices. A graph so formed is called an EPR graph [17].
adapted to include key reconciliation and privacy amplification at the $n$-partite level, that uses the group theoretic properties of $C$.

1. EPR protocol: Along the $n-1$ edges of the minimum spanning EPR tree, EPR pairs are shared (using eg., the Lo-Chau [8] or Modified Lo-Chau protocols [10]). Let the final, minimum number of EPR pairs distilled along any edge of the minimum spanning EPR tree be $2m$. A projective measurement in the computational basis is performed by all the parties on their respective qubits to obtain secure pair-wise shared randomness along the tree edges (making due adjustments according to whether the entangled spins are correlated or anti-correlated).

2. Classical subroutine of Section III. All non-terminal vertices announce their uniformly randomized outcome record. This information in principle allows every party, in conjunction with her/his outcome, to reconstruct the outcomes of all other parties, save for some errors of mismatch.

3. For each set of $n-1$ shared EPR pairs, protocol leader Lucy decides randomly on the terminal party whose outcome will serve as the secret bit.

4. Lucy decides randomly a set of $m$ bits to be used as check bits, and announces their positions.

5. All parties announce the value of their check bits. If too few of these values agree, they abort the protocol.

6. Lucy broadcasts $c_i \oplus v$, where $v$ is the string consisting of the remaining code (non-check) bits, and $c_i$ is a random codeword in $C$.

7. Each member $j$ from amongst the remaining $n-1$ parties subtracts $c_i \oplus v$ from his respective code bits, $v \oplus \epsilon_j$, and corrects the result, $c_i \oplus \epsilon_j$, to a codeword in $C$. Here $\epsilon_j$ is a possibly non-vanishing error-vector.

8. The parties use $i$ as the key.

A rigorous proof of the security of the $n$-QKD scheme requires: (a) the explicit construction of a procedure such that whenever Eve’s strategy has a non-negligible probability of passing the verification test by the $n$ parties, her information on the final key will be exponentially small. (b) the shared, secret randomness is robust against source, device and channel noise. By construction, our scheme combines a 2-QKD scheme to generate pair-wise shared randomness and a classical scheme to turn this into multipartite-shared randomness. The security of the latter (in its essential form) was proven in Section III. Therefore the security of the protocol with respect to (a) and (b) reduces to that of the 2-QKD in step 1. For various situations, 2-QKD can be secured, as proven in Refs [4, 5, 8, 9, 10]. For example, Lo and Chau [8] or Modified Lo-Chau protocols [10] have proved that EPR pairs can be prepared to be nearly perfect, even in the presence of Eve and channel noise. Their proofs essentially rely on the idea that sampling the coherence of the qubits allows one to place an upper bound on the effects due to noise and information leakage to Eve. Yet, subject to the availability of high quality quantum repeaters and fault-tolerant quantum computation, in principle 2-QKD can be made unconditionally secure [5].

In regard to the key reconciliation part: in step (3), each non-terminal vertex party announces his uniformly randomized outcome record. Here this consumes $m$ instances of $n-1$ pair-wise shared random bits into $k$ random bits shared between the $n$ agents while revealing little to Eve. The random efficiency is given by $\eta = (k \times n \times 1)/(m \times (n-1) \times 2)$, which tends to $(1/2)k/m$ as $n \to \infty$. The check bits, whose positions and values are announced in steps (4) and (5), are eventually discarded. Steps (7) and (8) involve purely local, classical operations [24]. If security of step (1) against Eve is guaranteed, the string $v$, and thereby the string $c_i \oplus v$ announced by Lucy in step (6), are completely random, as far as Eve can say. So, she (Eve) gains nothing therefrom. Hence her mutual information with any of the $n-1$ (sets of) random bits does not increase beyond what she has at the end of the EPR protocol [24].

Finally, step (5) permits with high probability to determine whether the key can be reconciled amongst the $n$ players. The check bits that the parties measure behave like a classical random sample of bits [10]. We can then use the measured error rates in a classical probability estimate. For any two parties, the probability of obtaining more than $(\delta + \epsilon)n$ errors on the code bits and fewer than $\delta n$ errors on the check bits is asymptotically less than $\exp[-0.25\epsilon^2 n/((\delta - \delta^2))]$. Noting that the errors on the $n$ check vectors are independent, it follows that probability that the check vectors are all scattered within a ball of radius $\delta n$ but one or more code vectors fall outside a scatter ball of radius $(\delta + \epsilon)n$ is exponentially small, and can be made arbitrarily small by choosing sufficiently small $\delta$. The decision criterion adopted in step (5) is calculated so that the Hamming weight of the error vectors $\epsilon_i$ estimated in the above fashion will be less than $t$ with high probability. Hence all parties correct their results to the same codeword $c_i$ in step (8) with high probability. This completes the proof of unconditionally security of $n$-QKD.
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