High-Energy Approximation of One-Loop Feynman Integrals

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Abstract:
We provide high-energy approximations for all one-loop scalar 3- and 4-point functions and the corresponding tensor integrals that appear in scattering processes with four external on-shell particles. Our expressions are valid if all kinematical invariants are much larger than the internal and external masses. They contain all leading-order terms of the integrals.
1 Introduction

The experiments at LEP and the SLC have provided a large number of high-precision experimental data. In order to compare these data with theoretical predictions the inclusion of one-loop radiative corrections has been inevitable, and even the leading two-loop corrections had to be taken into account [1]. Although forthcoming experiments as e.g. at LEP2 or a Next Linear Collider (NLC) will not reach such a high experimental accuracy, nevertheless the inclusion of one-loop radiative corrections will in general be required in the corresponding theoretical predictions. In fact, as the radiative corrections typically involve logarithms of the energy, their relevance even grows with increasing energy.

The radiative corrections to $\text{e}^+\text{e}^- \to f\bar{f}$ ($f \neq t$) are relatively simple even off the Z resonance, because only external fermions are involved, and their masses can be neglected at sufficiently high energies. Once the masses of the external particles are relevant, as e.g. for top-pair production, or external gauge bosons are present, the number of Feynman diagrams is considerably increased, and the evaluation as well as the results for the one-loop radiative corrections are much more complicated. In addition, owing to the presence of different energy scales, numerical instabilities eventually arise in the TeV range or beyond when the standard calculational procedures are used. These instabilities are caused by numerical cancellations originating from the recursive reduction of the tensor integrals and—in reactions involving longitudinal gauge bosons—additionally from unitarity cancellations. Evidently, it is desirable to circumvent these problems.

As the energy scale for future experiments is large compared to the electroweak scale, a natural approach consists in neglecting all masses as compared to the centre-of-mass energy whenever possible. Such an approximation has been worked out for the process $\text{e}^+\text{e}^- \to \text{W}^+\text{W}^-$ [2]. This calculation has shown that the one-loop corrections indeed simplify a lot in a high-energy approximation, and that such an approximation can already be useful for energies around 500 GeV.

With purely algebraic manipulations, which can easily be performed by computer-algebra packages such as e.g. FeynCalc [3], all one-loop Feynman amplitudes can be expressed in terms of scalar and tensor integrals. The tensor integrals can be algebraically reduced to scalar integrals [4]. Moreover, all scalar integrals with more than four internal propagators can be related to scalar 4-point functions [5]. Consequently, all one-loop Feynman amplitudes can be reduced to scalar 1-, 2-, 3-, and 4-point functions. For these functions complete analytical results exist [6].

Thus, the expansions of the scalar integrals for high energies are an essential ingredient of a high-energy approximation. The example of W-pair production has shown that the evaluation of these approximations requires a substantial amount of work. In order to facilitate future calculations in the high-energy limit, it is therefore desirable to calculate and tabulate the high-energy expansion of these integrals.

In the literature, methods have been described to evaluate Feynman integrals in the limit of large masses and/or momenta [7]. However, these methods in general do not apply when the external particles are on their mass shell [8]. In this case (IR or mass) singularities can arise if all masses are neglected. For UV-singular Feynman integrals, including all 2-point integrals, no IR or mass singularities show up such that the corre-
sponding high-energy approximations are simply obtained by putting all masses to zero in the exact results.

One can, of course, use the known exact results for the 3- and 4-point functions to derive the approximations. However, the necessity to keep the masses finite in order to extract the correct singularities renders the expansion of the exact results a tedious exercise.

On the other hand, the Mellin-transform technique allows to construct approximations in a relatively simple and direct way as long as only one of the kinematical invariants gets large, as it is the case for 3-point functions. Moreover, it turns out that this approach can also be used for the scalar 4-point-function despite of the fact that two invariants become large in the corresponding high-energy limit. However, the Mellin-transform technique fails for the tensor 4-point integrals. Therefore, we used the tensor-integral reduction algorithm to construct the corresponding high-energy approximations. In this way we have calculated high-energy approximations of the basic one-loop integrals necessary for scattering processes of two on-shell particles into two on-shell particles, i.e. for the most important processes at high energy colliders.

The paper is organized as follows: In Sect. 2 we describe our evaluation of the high-energy approximation for the scalar and tensor 3-point functions and list the corresponding results. In Sect. 3 the same is done for the 4-point functions. In the appendix we summarize our conventions for the Feynman integrals.

2 High-energy approximation of one-loop 3-point functions

The scalar and tensor 3-point functions are defined in App. A. We first describe the evaluation of their high-energy expansions and then list the results.

2.1 Calculation

In order to calculate the high-energy approximation of the scalar 3-point function, we start from the Feynman-parameter representation

$$ C_0(p_1, p_2, m_0, m_1, m_2) = - \int_0^\infty dx_0 dx_1 dx_2 \frac{\delta(1 - x_0 - x_1 - x_2)}{g(x_0, x_1, x_2) - (p_1 - p_2)^2 x_1 x_2} $$

with

$$ g(x_0, x_1, x_2) = m_0^2 x_0 + m_1^2 x_1 + m_2^2 x_2 - p_1^2 x_0 x_1 - p_2^2 x_0 x_2 - i\epsilon. $$

We search for an asymptotic expansion in the limit

$$ r = |(p_1 - p_2)^2| \gg m_0^2, m_1^2, m_2^2, p_1^2, p_2^2, $$

including all terms of order $1/(p_1 - p_2)^2$. Looking at the specific results derived for the high-energy approximation of the W-pair production [2], we notice that the scalar 3-point function has the asymptotic form:

$$ C_0(p_1, p_2, m_0, m_1, m_2) = \frac{1}{r} \left( \frac{c_2}{2} \ln^2(r) + c_1 \ln(r) + c_0 \right) + O\left( \frac{1}{r^2} \right), \quad r \to \infty. $$
This asymptotic expansion cannot be extracted by performing a simple Taylor expansion or by using the general methods described in Ref. [1]. It is convenient to use the Mellin-transform technique, because each pole of the Mellin transform is uniquely related to one of the terms in the asymptotic expansion (4).

If the function \( f(r) \) fulfills certain integrability conditions (see Ref. [2]), the Mellin transform
\[
\mathcal{M}[f, \xi] = \int_0^\infty r^{\xi-1} f(r) dr
\]
converges absolutely and is holomorphic in a vertical strip \( \alpha < \text{Re}(\xi) < \beta \) of the complex plane. The corresponding analytical continuation contains poles in the complex half-planes \( \text{Re}(\xi) \geq \beta \) and \( \text{Re}(\xi) \leq \alpha \). The inversion of the Mellin transform
\[
f(r) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} r^{-\xi} \mathcal{M}[f, \xi] d\xi \quad \text{with} \quad \alpha < c < \beta
\]
can be performed by closing the integration contour over the right-hand complex half-plane at infinity. According to residue theorem \( f(r) \) is obtained from the poles of the Mellin transform in the right-hand half-plane \( \text{Re}(\xi) \geq \beta \).

The inverse Mellin transform of a single pole of order \((n+1)\) is given by
\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} d\xi r^{-\xi} \frac{n!}{(\xi_0 - \xi)^{n+1}} = \frac{1}{r^{\xi_0}} \ln^n(r),
\]
i.e. a term that appears in the asymptotic expansion (5). From this it is obvious that we get an asymptotic expansion for \( r \to \infty \) by closing the integration contour in the right-hand complex half-plane. Moreover, the leading terms originate from the poles of the Mellin transform \( \mathcal{M}[f, \xi] \) on the right edge of the convergence domain, i.e. with \( \text{Re}(\xi_0) = \beta \). Thus, we can write down the poles of the Mellin transform that determine the asymptotic expansion (5) of the scalar 3-point function
\[
\mathcal{M}[C_0, \xi] = \frac{c_2}{(1 - \xi)^2} + \frac{c_1}{(1 - \xi)^{2}} + \frac{c_0}{1 - \xi} + \mathcal{O}((1 - \xi)^0)
\]
with the same constants \( c_i \) as in (5) [see (7)].

In order to evaluate (multiple) Mellin transforms of one-loop integrals we use the generalized Feynman-parameter representation [10]
\[
\int_0^\infty dr_0 \cdots dr_n \frac{\prod_{i=0}^n \delta(\xi - \sum_{i=0}^n A_i r_i)^{\xi_i-1}}{(\sum_{i=0}^n A_i r_i)^{\xi_i}} = \frac{\prod_{i=0}^n \Gamma(\xi_i)}{\Gamma(\xi) \prod_{i=0}^n A_i^{\xi_i}},
\]
where \( \xi = \sum_{i=0}^n \xi_i \), \( \text{Re}(\xi_i) > 0 \), \( \alpha_i \geq 0 \), and the \( A_i \) are not on the negative real axis. Choosing \( \alpha_i = 0 \) for \( i > 0 \) and \( \alpha_0 = 1 \) and performing the integration over \( r_0 \) we get the following formula for a multiple Mellin transform:
\[
\int_0^\infty dr_1 \cdots dr_n \frac{r_1^{\xi_1-1} \cdots r_n^{\xi_n-1}}{(A_0 + A_1 r_1 + \cdots + A_n r_n)^{\xi}} = \frac{\prod_{i=0}^n \Gamma(\xi_i)}{\Gamma(\xi) \prod_{i=0}^n A_i^{\xi_i}},
\]
with \( \xi_0 = \xi - \sum_{i=1}^n \xi_i \). This formula can directly be used to evaluate the Mellin transform of (5) upon identifying:
\[
A_0 = g(x_0, x_1, x_2), \quad A_1 = (\sigma - i\epsilon) x_1 x_2 \quad \text{and} \quad r_1 = r = |(p_1 - p_2)^2| \]
with \(\sigma = (p_1 - p_2)^2/r = \pm 1\) and \(n = 1\). In order to ensure the validity of \(\text{(10)}\), an infinitesimal imaginary part of the \(\text{i}\)-prescription is given to both \(A_0\) and \(A_1\). After integration over \(x_0\) we find the Mellin transform of the 3-point function, which is holomorphic in the vertical strip \(0 < \text{Re}(\xi) < 1\),

\[
\mathcal{M}[C_0, \xi] = -\frac{\Gamma(\xi)\Gamma(1 - \xi)}{(-\sigma - \text{i}\epsilon)^\xi} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{1}{h(x_1, x_2)^{1-\xi}x_1^\xi x_2^\xi} \quad (12)
\]

with \(h(x_1, x_2) = g(1-x_1-x_2, x_1, x_2)\). According to the above discussion, the asymptotic expansion is determined from the poles at \(\xi = 1\). As \(\Gamma(1 - \xi)\) includes a single pole, we need the expansion of the integral about \(\xi = 1\) including the order \(O((1 - \xi)^0)\).

First we assume \(h(0, 0) = m^2_0 \neq 0\). Naive expansion of \(h(x_1, x_2)^{-(1-\xi)}\) about \(x_1 = 0\) and \(x_2 = 0\) would yield two infinite series of pole terms contributing to the high-energy approximation namely the Taylor expansions of \(h(x_1, 0)^{-(1-\xi)}\) and \(h(0, x_2)^{-(1-\xi)}\). Therefore, we use

\[
\frac{1}{h(x_1, x_2)^{1-\xi}} = \frac{1}{h(x_1, 0)^{1-\xi}} + \frac{1}{h(0, x_2)^{1-\xi}} - \frac{1}{h(0, 0)^{1-\xi}} + O(x_1 x_2) O(1 - \xi), \quad (13)
\]

such that the terms that are not explicitly written out involve the product of \(x_1\) and \(x_2\), and get a compact expansion of the Mellin transform:

\[
\mathcal{M}[C_0, \xi] = -\frac{\Gamma(\xi)\Gamma(1 - \xi)}{(-\sigma - \text{i}\epsilon)^\xi} \left\{ \frac{1}{h(0, 0)^{1-\xi}} - \frac{\pi^2}{6} \right. \\
- \int_0^1 dx_1 \frac{1}{x} \ln \left( \frac{h(x, 0)}{h(0, 0)} \right) - \int_0^1 dx_2 \frac{1}{x} \ln \left( \frac{h(0, x)}{h(0, 0)} \right) \left. \right\} + O(1 - \xi). \quad (14)
\]

From the inversion formula \(\text{(3)}\) or by comparing \(\text{(4)}\) with \(\text{(8)}\) we obtain the high-energy approximation of the 3-point function:

\[
C_0(p_1, p_2, m_0, m_1, m_2) = \frac{1}{(p_1 - p_2)^2} \left\{ \frac{1}{2} \ln^2 \left( \frac{-(p_1 - p_2)^2 - \text{i}\epsilon}{h(0, 0)} \right) - \int_0^1 dx_1 \frac{1}{x} \ln \left( \frac{h(x, 0)}{h(0, 0)} \right) - \int_0^1 dx_2 \frac{1}{x} \ln \left( \frac{h(0, x)}{h(0, 0)} \right) \right\} + O\left( \frac{1}{(p_1 - p_2)^4} \right). \quad (15)
\]

For \(h(0, 0) = m^2_0 = 0\) the expansion \(\text{(13)}\) breaks down. Therefore, we extend the integration domain of the Mellin transform \(\text{(12)}\) to a square \((0 \leq x_1 \leq 1, i = 1, 2)\) and divide this square into two sectors:

\[
x_1 > x_2 \quad \text{and} \quad x_2 > x_1. \quad (16)
\]

It can easily be seen that the extension has no poles at \(\xi = 1\), apart from the one contained in the prefactor \(\Gamma(1 - \xi)\), and the integral yields

\[
\int_0^1 dx_1 \int_0^1 dx_2 \frac{1}{h(x_1, x_2)^{1-\xi}x_1^\xi x_2^\xi} = \frac{\pi^2}{6} + O(1 - \xi). \quad (17)
\]

\[\text{1}\]This sector decomposition is similar to the one used in the proof of the convergence theorem for Feynman amplitudes \[\text{1}\].
By substituting \( x_2 \to x_1 \tilde{x}_2 \) in the first sector and \( x_1 \to x_2 \tilde{x}_1 \) in the second sector we get the following result for the Mellin transform:

\[
\mathcal{M}[C_0, \xi] = -\frac{\Gamma(\xi)\Gamma(1-\xi)}{(-\sigma-i\epsilon)^\xi} \left\{ \int_0^1 \frac{dx_1}{x_1} \int_0^1 \frac{d\tilde{x}_2}{\tilde{x}_2} \frac{1}{(h(x_1, x_1 \tilde{x}_2)/x_1)^{1-\xi} x_1 \tilde{x}_2^\xi} \right. \\
+ \left. \int_0^1 \frac{d\tilde{x}_1}{\tilde{x}_1} \int_0^1 \frac{dx_2}{x_2} \frac{1}{(h(x_2 \tilde{x}_1, x_2)/x_2)^{1-\xi} \tilde{x}_1 x_2^\xi} - \frac{\pi^2}{6} + \mathcal{O}(1-\xi) \right\}
\]

(18)

with

\[
h(x_1, x_1 \tilde{x}_2)/x_1 = m_1^2 - p_1^2 + (m_2^2 - p_2^2) \tilde{x}_2 + (p_1^2 + p_2^2 \tilde{x}_2)(1 + \tilde{x}_2)x_1 - i\epsilon,
\]

\[
h(x_2 \tilde{x}_1, x_2)/x_2 = m_2^2 - p_2^2 + (m_1^2 - p_1^2) \tilde{x}_1 + (p_1^2 \tilde{x}_1 + p_2^2)(\tilde{x}_1 + 1)x_2 - i\epsilon.
\]

(19)

For \( m_1^2 \neq p_1^2 \) and \( m_2^2 \neq p_2^2 \) the two integrals in (18) can be evaluated similarly to (12) in the case \( m_0^2 \neq 0 \).

If \( m_1^2 = p_1^2 \neq 0 \) we perform a further sector decomposition in the first integral of (18). The case \( m_2^2 = p_2^2 \neq 0 \) is covered by exchanging the indices 1 and 2. The remaining cases are either IR-singular (\( m_0 = 0, m_1^2 = p_1^2, m_2^2 = p_2^2 \)) or mass-singular (\( m_0^2 = m_1^2 = p_1^2 = 0 \) or \( m_0^2 = m_2^2 = p_2^2 = 0 \)). Thus, we can construct high-energy approximations for all scalar 3-point functions that are neither IR- nor mass-singular. The singular integrals are also covered by our results if the singularities are regularized with masses. Our general results were checked by comparing with the specific expressions in Ref. [3].

The high-energy approximation of tensor 3-point integrals can be evaluated in the same way:

Consider first the coefficient functions \( C_{1\ldots 1} \). They have the same Feynman-parameter representation as the scalar 3-point function (1), but with an additional factor \( x_1^3 \) in the numerator [see (A20)] which prevents the Mellin transform from being singular at \( x_1 = 0 \). Consequently, the integrand can be expanded with respect to \( x_2 \), and terms of order \( \mathcal{O}(x_2) \) are not contributing to the high-energy behaviour. As a consequence, the results are independent of \( m_2^2 \) and \( p_2^2 \).

The functions \( C_{1\ldots 12\ldots 2} \) have no singularities at \( x_1 = 0 \) and \( x_2 = 0 \) owing to the numerator factor \( x_1^3 x_2^2 \). The function \( \Gamma(1-\xi) \) contributes the single pole \( 1/(1-\xi) \). Since we can set \( \xi = 1 \) in the integral, the result is independent of the internal and external masses.

The remaining coefficient functions of the 2- and 3-point functions are UV-divergent. They have no IR or mass singularities, because the integrand of the Feynman-parameter representation has a logarithmic structure. It can be easily checked with the help of the Mellin-transform technique that the high-energy approximation is obtained by neglecting the internal and external masses.

In contrast to the asymptotic expansion of the scalar 3-point function, those of the coefficient functions of the tensor 2- and 3-point integrals include no dilogarithms and need not be split into several different cases. All results for these tensor integrals were derived using the Mellin-transform technique and checked by tensor-integral reduction.

### 2.2 Results

The following results hold for 3-point functions in the limit

\[
|(p_1 - p_2)| \gg m_0^2, m_1^2, m_2^2, p_1^2, p_2^2,
\]

(20)
and for 2-point functions in the limit
\[ |p^2| \gg m_0^2, m_1^2. \]  

### 2.2.1 Scalar 3-point function

For the scalar 3-point function we have to distinguish several different cases:

- **\( m_0^2 \neq 0 \):**
  \[
  C_0(p_1, p_2, m_0, m_1, m_2) \sim \frac{1}{(p_1 - p_2)^2} \left\{ \frac{1}{2} \ln^2 \left( \frac{-(p_1 - p_2)^2 - i\epsilon}{m_0^2 - i\epsilon} \right) + I_C(p_1^2, m_0, m_1) + I_C(p_2^2, m_0, m_2) \right\},
  \tag{22}
  \]

  where
  \[
  I_C(p_1^2, m_0, m_1) = - \int_0^1 dx \frac{1}{x} \ln \left( 1 + \frac{m_1^2 - m_0^2 - p_1^2}{m_0^2 - i\epsilon} x + \frac{p_1^2}{m_0^2 - i\epsilon} x^2 \right) = \sum_{\pm} \text{Li}_2 \left( \frac{2p_1^2}{m_0^2 - m_1^2 + p_1^2 \pm \kappa(p_1^2, m_0^2 - i\epsilon, m_1^2 - i\epsilon)} \right)
  \tag{23}
  \]

  with \( \kappa(a, b, c) = \sqrt{a^2 + b^2 + c^2 - 2ab - 2ac - 2bc} \);

- **\( m_0^2 = 0, m_1^2 \neq p_1^2, m_2^2 \neq p_2^2 \):**
  \[
  C_0(p_1, p_2, m_0, m_1, m_2) \sim \frac{1}{(p_1 - p_2)^2} \left\{ \ln \left( \frac{-(p_1 - p_2)^2 - i\epsilon}{m_1^2 - p_1^2 - i\epsilon} \right) \ln \left( \frac{-(p_1 - p_2)^2 - i\epsilon}{m_2^2 - p_2^2 - i\epsilon} \right) + \text{Li}_2 \left( -\frac{p_1^2}{m_1^2 - p_1^2 - i\epsilon} \right) + \text{Li}_2 \left( -\frac{p_2^2}{m_2^2 - p_2^2 - i\epsilon} \right) \right\}.
  \tag{24}
  \]

- **\( m_0^2 = 0, m_1^2 = p_1^2 \neq 0, m_2^2 \neq p_2^2 \):**
  \[
  C_0(p_1, p_2, m_0, m_1, m_2) \sim \frac{1}{(p_1 - p_2)^2} \left\{ \ln \left( \frac{-(p_1 - p_2)^2 - i\epsilon}{m_2^2 - p_2^2 - i\epsilon} \right) \ln \left( \frac{-(p_1 - p_2)^2 - i\epsilon}{p_1^2 - i\epsilon} \right) + \frac{1}{2} \ln^2 \left( \frac{-(p_1 - p_2)^2 - i\epsilon}{m_2^2 - p_2^2 - i\epsilon} \right) + \text{Li}_2 \left( -\frac{p_2^2}{m_2^2 - p_2^2 - i\epsilon} \right) + \frac{\pi^2}{6} \right\}.
  \tag{25}
  \]

The remaining cases are found by exchanging the indices 1 and 2 or are IR- or mass-singular. Specific examples of these general results can be found in App. A of the second paper of Ref. \[2\].

### 2.2.2 Tensor 2- and 3-point functions

The coefficient functions of the tensor 2- and 3-point integrals, defined in App. A.1, read in the high-energy limit \( (k, j > 0) \):

\[ B_{\ldots}(p, m_0, m_1) \sim B_{\ldots}(p, 0, 0), \tag{26} \]
The most important functions are explicitly given by:

\[ C_{00\ldots0} (p_1^2, (p_1 - p_2)^2, p_2^2, m_1, m_0, m_2) \sim C_{00\ldots0}(0, (p_1 - p_2)^2, 0, 0, 0), \quad (27) \]
\[ C_{1j\ldots2}(p_1, p_2, m_0, m_1, m_2) \sim (-1)^{k+j}(k-1)!(j-1)! \frac{m_1}{(p_1 - p_2)^2 (k+j)!}, \quad (28) \]
\[ C_{k\ldots1}(p_1, p_2, m_0, m_1, m_2) \sim \frac{1}{(p_1 - p_2)^2} \{ B_{k\ldots1}(p_1 - p_2, 0, 0) - B_{k\ldots1}(p_1, m_0, m_1) \}. \quad (29) \]

The most important functions are explicitly given by:

\[ B_0(p, 0, 0) = \frac{\gamma_E}{\pi} \ln \frac{\mu^2}{m_0^2} + 2, \quad (30) \]
\[ B_1(p, 0, 0) = -\frac{1}{2} B_0(p, 0, 0), \quad (31) \]
\[ B_{11}(p, 0, 0) = \frac{1}{3} B_0(p, 0, 0) + \frac{1}{18}, \quad (32) \]
\[ B_{00}(p, 0, 0) = -\frac{p^2}{12} B_0(p, 0, 0) - \frac{p^2}{18}, \quad (33) \]

and

\[ C_{12}(p_1, p_2, m_0, m_1, m_2) \sim \frac{1}{2(p_1 - p_2)^2}, \quad (34) \]
\[ C_{112}(p_1, p_2, m_0, m_1, m_2) \sim -\frac{1}{6(p_1 - p_2)^2}, \quad (35) \]
\[ C_{00}(p_1, p_2, m_0, m_1, m_2) \sim \frac{1}{4} B_0(p_1 - p_2, 0, 0) + \frac{1}{4}, \quad (36) \]
\[ C_{001}(p_1, p_2, m_0, m_1, m_2) \sim \frac{1}{12} B_0(p_1 - p_2, 0, 0) - \frac{1}{18}. \quad (37) \]

3 High-energy approximations of one-loop 4-point functions

The definitions of the scalar and tensor 4-point functions are given in App. [A.1]. The high-energy expansion of the scalar 4-point function can also be evaluated using the Mellin-transform technique. Therefore, we sketch only the most important aspects and, in particular, the differences to the case of the 3-point function discussed above. For the tensor 4-point functions the Mellin-transform technique breaks down and we have to rely on tensor-integral reduction.

3.1 Calculation

High-energy approximation of the scalar 4-point function means that both \(|(k_1 + k_2)^2|\) and \(|(k_1 + k_4)^2|\) are large compared with the internal and external masses:

\[ |(k_1 + k_2)^2|, |(k_1 + k_4)^2| \gg m_0^2, m_1^2, m_2^2, m_3^2, k_1^2, k_2^2, k_3^2, k_4^2. \quad (38) \]

Therefore, we need a double Mellin transform in order to evaluate the expansion. The scalar 4-point function has the Feynman-parameter representation

\[ D_0(k_1, k_1 + k_2, k_1 + k_2 + k_3, m_0, m_1, m_2, m_3) = \]
\[ \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \delta(1 - x_0 - x_1 - x_2 - x_3) dx_0 dx_1 dx_2 dx_3 \frac{g(x_0, x_1, x_2, x_3) - (\sigma_1 + i \epsilon) x_0 x_2 - (\sigma_2 + i \epsilon) x_2 x_3}{g(x_0, x_1, x_2, x_3) - (\sigma_1 + i \epsilon) x_0 x_2 - (\sigma_2 + i \epsilon) x_2 x_3} \quad (39) \]
Mellin transform into four sectors and expand each sector separately:

\[ g(x_0, x_1, x_2, x_3) = \sum_{n=0}^{3} m_n^2 x_0 x_1 - k_1^2 x_0 x_1 - k_2^2 x_1 x_2 - k_3^2 x_2 x_3 - k_4^2 x_3 x_0 - \iota, \quad (40) \]

With

\[ r_1 = |(k_1 + k_2)^2|, \quad r_2 = |(k_1 + k_4)^2|, \quad \sigma_1 = (k_1 + k_2)^2/r_1, \quad \sigma_2 = (k_1 + k_4)^2/r_2, \]

where \( k_4 = -k_1 - k_2 - k_3 \). As above, we have included an infinitesimal imaginary part in the terms involving the large parameters \((k_1 + k_2)^2\) and \((k_1 + k_4)^2\). Thus, we can use the generalized Feynman-parameter representation \((10)\) in order to perform the double Mellin transform resulting in:

\[
\mathcal{M}[D_0, \xi_1, \xi_2] = \frac{\Gamma(\xi_1)\Gamma(\xi_2)\Gamma(2 - \xi_1 - \xi_2)}{(-\sigma_1 - \iota)^{\xi_1}(-\sigma_2 - \iota)^{\xi_2}}
\times \int_{0}^{\infty} dx_0 dx_1 dx_2 dx_3 \frac{\delta(1 - x_0 - x_1 - x_2 - x_3)}{g(x_0, x_1, x_2, x_3)^{\xi_1} x_1^{\xi_2} x_2^{\xi_3} x_3^{\xi_4}. \quad (41)\]

From the explicit results of Ref. \([2]\) we expect the following form for the high-energy expansion of the scalar 4-point function:

\[ D_0 \sim \frac{1}{r_1 r_2} \sum_{i,j=0}^{2} c_{ij} \ln^i(r_1) \ln^j(r_2) + O\left(\frac{1}{r_1 r_2}\right) + O\left(\frac{1}{r_1 r_2}\right), \quad r_1, r_2 \to \infty \quad (42)\]

with \( c_{22} = c_{12} = c_{21} = 0 \). Accordingly, the Mellin transform should have the following meromorphic structure at \( \xi_1 \approx 1 \) and \( \xi_2 \approx 1 \):

\[
\mathcal{M}[D_0, \xi_1, \xi_2] = \sum_{i,j=0}^{2} \frac{c_{ij}}{(1 - \xi_1)^{i+1}(1 - \xi_2)^{j+1}} + O((1 - \xi_1)^0) + O((1 - \xi_2)^0). \quad (43)\]

Note that terms that involve only poles in one of the variables \( \xi_i \) are irrelevant for the asymptotic expansion, as they do not contribute to the inverse Mellin transform.

In contrast to the scalar 3-point function, we divide the integration domain of the Mellin transform into four sectors and expand each sector separately:

\[ x_0 > x_1, x_2, x_3, \quad x_1 > x_2, x_3, x_0, \quad x_2 > x_3, x_0, x_1, \quad x_3 > x_0, x_1, x_2. \quad (44)\]

The contribution of each sector can be associated to one side of the box and must be decomposed into several cases like for the scalar 3-point function. The third Gamma function in \((10)\) involves a pole \(1/(2 - \xi_1 - \xi_2)\). This pole does not fit into the structure \((13)\). It is essential for our approach that this pole is cancelled in the total result for the Mellin transform, although it is present in each single sector.

Because all sectors yield integrals of the same form, we consider only the sector \( x_0 > x_1, x_2, x_3 \). We substitute \( x_i \to x_0 x_i, \ i = 1, 2, 3, \) in \((11)\) and obtain for the integral:

\[
\mathcal{S}_0(\xi_1, \xi_2) = \int_{0}^{1} dx_1 dx_2 dx_3 \frac{1}{h(x_1, x_2, x_3)^{2 - \xi_1 - \xi_2} x_1^{\xi_1} x_2^{\xi_2} x_3^{\xi_3}} \quad (45)\]

with

\[ h(x_1, x_2, x_3) = m_0^2 + (m_1^2 + m_0^2 - k_1^2)x_1 + (m_2^2 + m_0^2)x_2 + (m_3^2 + m_0^2 - k_3^2)x_3
+ (m_1^2 x_1 + m_2^2 x_2 + m_3^2 x_3)(x_1 + x_2 + x_3) - k_2^2 x_1 x_2 - k_3^2 x_2 x_3 - \iota. \quad (46)\]
As for the scalar 3-point function, we consider first the case $m_0^2 \neq 0$ and use

$$
\frac{1}{h(x_1, x_2, x_3)^{2-\xi_1-\xi_2}} = \frac{1}{h(x_1, 0, 0)^{2-\xi_1-\xi_2}} + \frac{1}{h(0, 0, x_3)^{2-\xi_1-\xi_2}} - \frac{1}{h(0, 0, 0)^{2-\xi_1-\xi_2}} + (2 - \xi_1 - \xi_2)(O(x_2) + O(x_1 x_3)).
$$

(47)

The terms that have not been written down explicitly involve only poles in one of the variables $\xi_i$ and are hence not relevant for the asymptotic expansion. Thus, we find

$$
S_0(\xi_1, \xi_2) \sim \frac{(2 - \xi_1 - \xi_2)}{(1 - \xi_1)(1 - \xi_2)} \left\{ \frac{1}{h(0, 0, 0)^{1-\xi_2}(1-\xi_2)^2} - \int_0^1 dx \frac{1}{x} \ln \left( \frac{h(x, 0, 0)}{h(0, 0, 0)} \right) - \int_0^1 dx \frac{1}{x} \ln \left( \frac{h(0, 0, x)}{h(0, 0, 0)} \right) \right\}
$$

$$
\quad - \frac{1}{(1-\xi_2)^3} + (2 - \xi_1 - \xi_2)(O((1-\xi_1)^0) + O((1-\xi_2)^0)).
$$

(48)

The factor $(2 - \xi_1 - \xi_2)$ in the first and third terms cancels the pole contained in $\Gamma(2 - \xi_1 - \xi_2)$ in (47). If we combine all four sectors this pole is also cancelled in the contributions of the second term:

$$
\sum_{n=0}^3 S_n(\xi_1, \xi_2) = -\frac{2}{(1-\xi_1)^3} - \frac{2}{(1-\xi_2)^3} + O(2 - \xi_1 - \xi_2)
$$

$$
\quad = -\frac{2(2 - \xi_1 - \xi_2)}{(1 - \xi_1)(1 - \xi_2)} \left\{ \frac{1}{(1-\xi_1)^2} + \frac{1}{(1-\xi_2)^2} - \frac{1}{(1-\xi_1)(1-\xi_2)} \right\}
$$

$$
\quad + O(2 - \xi_1 - \xi_2).
$$

(49)

This holds as well in the other cases with $m_0^2 = 0$ considered below.

It is then straightforward to construct the expansion of the Mellin transform and evaluate the high-energy approximation by comparing (12) and (13). Because the calculation is lengthy but analogous to the one for the scalar 3-point function, we only make some remarks. The integrals involving the logarithms in (48) lead to contributions of the form

$$
I_D(k_1^2, m_0, m_1) = -\int_0^1 dx \frac{1}{x} \ln \left( 1 + \frac{m_1^2 + m_0^2 - k_1^2}{m_0^2 - i\epsilon} x + \frac{m_1^2}{m_0^2 - i\epsilon} x^2 \right).
$$

(50)

Each of these integrals can be associated with one corner of the box and for each corner two integrals appear in the sum of all sectors. Using the relation

$$
I_D(k_1^2, m_0, m_1) + I_D(k_1^2, m_1, m_0) = I_C(k_1^2, m_0, m_1) + I_C(k_1^2, m_1, m_0) - \frac{\pi^2}{3}
$$

(51)

with $m_0^2, m_1^2 \neq 0$, all these integrals can be expressed by those appearing in the asymptotic expansion of the 3-point functions. Finally the asymptotic expansion of the scalar 4-point function can be written in terms of the asymptotic expansions of the four corresponding scalar 3-point functions and a universal term.

Next we consider the case $m_0^2 = 0$. We decompose the integration domain of the integral $S_0$ into three sectors

$$
\begin{align*}
x_1 > x_2, x_3, & \quad x_2 > x_3, x_1, \quad x_3 > x_1, x_2,
\end{align*}
$$

(52)
and arrive at
\[
S_0 = \int_0^1 dx_1 dx_2 dx_3 \frac{1}{h_1(x_1, x_2, x_3)^{2-\xi_1-\xi_2-\xi_3}} h_1(x_1, x_2, x_3)^{2-\xi_1-\xi_2-\xi_3} \]
\[
+ \int_0^1 dx_1 dx_2 dx_3 \frac{1}{h_2(x_1, x_2, x_3)^{2-\xi_1-\xi_2-\xi_3}} h_2(x_1, x_2, x_3)^{2-\xi_1-\xi_2-\xi_3} \]
\[
+ \int_0^1 dx_1 dx_2 dx_3 \frac{1}{h_3(x_1, x_2, x_3)^{2-\xi_1-\xi_2-\xi_3}} h_3(x_1, x_2, x_3)^{2-\xi_1-\xi_2-\xi_3},
\]
with the polynomials
\[
h_1(x_1, x_2, x_3) = h(x_1, x_2, x_3)/x_1,
\]
\[
h_2(x_1, x_2, x_3) = h(x_2, x_1, x_3)/x_2,
\]
\[
h_3(x_1, x_2, x_3) = h(x_3, x_2, x_3)/x_3.
\]

If \(m_1^2 \neq k_1^2\) and \(m_3^2 \neq k_2^2\), the first and third integral are analogous to (43). The second integral can be evaluated by performing an expansion of the integrand with respect to \((2-\xi_1-\xi_2)\),

\[
\int_0^1 dx_1 dx_2 dx_3 \frac{1}{h_2(x_1, x_2, x_3)^{2-\xi_1-\xi_2-\xi_3}} h_2(x_1, x_2, x_3)^{2-\xi_1-\xi_2-\xi_3} = \frac{1}{(1-\xi_2)^3} + (2-\xi_1-\xi_2)O((1-\xi_1)^0),
\]

which is true no matter whether \(h_2(0, 0, 0) = 0\) or not. Notice that (49) holds also after performing sector decomposition, because the pole \(1/(1-\xi_2)^3\) of the second integral in (53) [see (53)] cancels one of the poles in the first and third integral [see (48)].

If \(m_1^2 = k_1^2 \neq 0\) or \(m_3^2 = k_2^2 \neq 0\) we obviously have to apply a further sector decomposition to the first or third integral, respectively. In this way, all 4-point functions which contain neither IR nor mass singularities can be approximated for high energies. Again the singular ones are covered by our approach once they are regularized with masses. It turns out, that in all cases the scalar 4-point function can be expressed in a universal way in terms of the four corresponding 3-point functions.

In contrast to the case of the scalar 4-point function, the pole \(1/(2-\xi_1-\xi_2)\) does not cancel in the Mellin transform of the coefficient functions of the tensor 4-point integrals. Because the poles of the variable \(\xi_1\) and \(\xi_2\) are coupled in the Mellin transform, the resulting asymptotic expansion depends on the order of the inversion of the Mellin transforms with respect to \(\xi_1\) and \(\xi_2\). Integrating first over \(\xi_1\) and then over \(\xi_2\) results in an approximation for \(r_1 \gg r_2 \gg m_i^2, k_i^2\), integrating first over \(\xi_2\) and then over \(\xi_1\) yields an approximation for \(r_2 \gg r_1 \gg m_i^2, k_i^2\). While these approximations coincide for the scalar 4-point function, where the poles in \(\xi_1\) and \(\xi_2\) factorize, they differ for the tensor 4-point functions. Therefore, we have evaluated the approximations for the tensor 4-point integrals by reducing them to scalar integrals with FeynCalc [3] and determining the high-energy limit by using the approximations for the scalar integrals with the help of Mathematica [12]. As can be seen from the results in Sect. 3.2.2, the pole \(1/(2-\xi_1-\xi_2)\) in the Mellin transform corresponds to a factor \(1/(\sigma_1 r_1 + \sigma_2 r_2)\) \((1/u)\) in the notation of Sect. 3.2.2 in the high-energy approximation of the integrals.
3.2 Results

The results are valid for

\[ |s|, |t|, |u| \gg k_1^2, k_2^2, k_3^2, k_4^2, m_0^2, m_1^2, m_2^2, m_3^2 \]  

with \( s = (k_1 + k_4)^2 \), \( t = (k_1 + k_2)^2 \) and \( u = (k_1 + k_3)^2 \). All 3-point functions in this section are implicitly understood in the high-energy limit.

3.2.1 Scalar 4-point function

The high-energy approximation of the scalar 4-point function can be reduced to the approximations of the four corresponding 3-point functions as follows:

\[
D_0(k_1, k_1 + k_2, k_1 + k_2 + k_3, m_0, m_1, m_2, m_3) \sim -\frac{1}{st} \left[ \ln^2 \left( \frac{-t - i\epsilon}{-s - i\epsilon} \right) + \pi^2 \right] + \frac{1}{s} C_0(k_1, -k_2, m_1, m_0, m_2) + \frac{1}{t} C_0(k_2, -k_3, m_2, m_1, m_3) \\
+ \frac{1}{s} C_0(k_3, -k_4, m_3, m_2, m_0) + \frac{1}{t} C_0(k_4, -k_1, m_0, m_3, m_1). 
\]  

(57)

Note that this result holds for all non-singular cases, in particular for arbitrary masses. The necessary split-up into different cases has to be done only at the level of 3-point functions. Specific examples of this general result can be found in App. A of the second paper of Ref. [2].

3.2.2 Tensor 4-point functions

The high-energy approximations of the coefficient functions of the tensor 4-point integrals, defined in App. A.1, can be written as:

\[
D_1 \sim -\frac{1}{2su}(L^2 + \pi^2) - \frac{1}{s} C_0, 
\]

(58)

\[
D_{11} \sim -\frac{t}{2su^2}(L^2 + \pi^2) - \frac{1}{su} L + \frac{1}{s} (C_0 + C_1 + C_2), 
\]

(59)

\[
D_{12} \sim -\frac{1}{2u^2}(L^2 + \pi^2) + \frac{1}{su} L - \frac{1}{s} C_2, 
\]

(60)

\[
D_{13} \sim \frac{1}{2u^2}(L^2 + \pi^2) - \frac{1}{su} L, 
\]

(61)

\[
D_{111} \sim -\frac{t^2}{2su^3}(L^2 + \pi^2) - \frac{3t + s}{2su^2} L + \frac{t + 2s}{2tsu} \\
- \frac{1}{s} (C_0 + C_1 + C_2 - C_{11} - C_{22}), 
\]

(62)

\[
D_{112} \sim -\frac{t}{2u^3}(L^2 + \pi^2) + \frac{t - s}{2su^2} L - \frac{1}{2tu} - \frac{1}{s} C_{22}, 
\]

(63)

\[
D_{113} \sim \frac{t}{2u^3}(L^2 + \pi^2) - \frac{t - s}{2su^2} u - \frac{1}{2su}, 
\]

(64)

\[
D_{123} \sim \frac{s - t}{4u^3}(L^2 + \pi^2) - \frac{1}{u^2} L + \frac{1}{2su}, 
\]

(65)
The arguments of the 3- and 4-point functions are given by:

\[ D_{111} \sim -\frac{t^3}{2su^4}(L^2 + \pi^2) - \frac{11t^2 + 7ts + 2s^2}{6su^3}L + \frac{2t^2 + 7ts + 4s^2}{2tu^2} \]
\[ + \frac{1}{s}(C_0 + C_1 + C_2 - C_{11} - C_{22} + C_{111} + C_{222}), \tag{66} \]

\[ D_{112} \sim -\frac{t^2}{2u^4}(L^2 + \pi^2) + \frac{2t^2 - 5ts - s^2}{6su^3}L - \frac{2t + s}{2tu^2} - \frac{1}{s}C_{222}, \tag{67} \]

\[ D_{113} \sim \frac{t^2}{2u^4}(L^2 + \pi^2) - \frac{2t^2 - 5ts - s^2}{6su^3}L - \frac{t}{2su^2}, \tag{68} \]

\[ D_{112} \sim -\frac{ts}{2u^4}(L^2 + \pi^2) + \frac{t^2 + 5ts - 2s^2}{6su^3}L - \frac{t^2 - ts + s^2}{6tu^2} \]
\[ + \frac{1}{s}(C_{22} + C_{222}), \tag{69} \]

\[ D_{113} \sim \frac{2ts - t^2}{6u^4}(L^2 + \pi^2) - \frac{5t - s}{6u^3}L + \frac{t - 2s}{6su^2}, \tag{70} \]

\[ D_{112} \sim \frac{2t^2 - ts}{6u^4}(L^2 + \pi^2) - \frac{t^2 - 5ts}{6su^3}L - \frac{2t - s}{6su^2}, \tag{71} \]

\[ D_{122} \sim \frac{s^2 - 2ts}{6u^4}(L^2 + \pi^2) + \frac{t - 5s}{6u^3}L + \frac{t + 4s}{6su^2}, \tag{72} \]

\[ D_{00} \sim -\frac{1}{4u}(L^2 + \pi^2), \tag{73} \]

\[ D_{001} \sim -\frac{t}{8u^2}(L^2 + \pi^2) - \frac{1}{4u}L, \tag{74} \]

\[ D_{001} \sim -\frac{2t^2}{12u^3}(L^2 + \pi^2) - \frac{3t + s}{12u^2}L - \frac{1}{12u}, \tag{75} \]

\[ D_{002} \sim -\frac{ts}{12u^3}(L^2 + \pi^2) + \frac{t - s}{12u^2}L + \frac{1}{12u}, \tag{76} \]

\[ D_{003} \sim \frac{ts - t^2}{24u^3}(L^2 + \pi^2) - \frac{t}{6u^2}L - \frac{1}{12u} \tag{77} \]

\[ D_{000} \sim -\frac{ts}{48u^2}(L^2 + \pi^2) - \frac{s}{24u}L + \frac{5}{72} + \frac{1}{24}B_0(t, 0, 0), \tag{78} \]

where

\[ L = \ln \left( \frac{-t - i\epsilon}{-s - i\epsilon} \right). \tag{79} \]

The arguments of the 3- and 4-point functions are given by:

\[ D_{...} = D_{...}(k_1, k_1 + k_2, k_1 + k_2 + k_3, m_0, m_1, m_2, m_3), \]
\[ C_{...} = C_{...}(k_1, -k_2, m_1, m_0, m_2). \tag{80} \]

For the 3-point integrals the approximations given in Sect. 2.2 have to be inserted. Note that only one kind of 3-point functions appears. Moreover, the coefficient functions with indices 00 or 1 and 3 are independent of all internal and external masses and do not involve dilogarithms.

4 Conclusion

We have calculated high-energy approximations of the basic one-loop integrals necessary for scattering processes of two on-shell particles into two on-shell particles (2 → 2
processes). All kinematical variables have been assumed to be large compared with the internal and external masses, and all terms with the leading power of the energy have been kept.

We used the Mellin-transform technique and overcame the difficulties in expanding the Mellin transform by sector decomposition. As a consequence, the results for the scalar integrals fall into several different cases. It turned out that the high-energy approximation of the scalar 4-point functions is just given by the four corresponding 3-point functions and an universal term. The tensor 2- and 3-point integrals were expanded with the help of the Mellin-transform technique and checked by using tensor-integral reduction. The approximations for these integrals are, in contrast to the one for the scalar 3-point function, very short and include no dilogarithms. The tensor 4-point integrals were first reduced to scalar functions and then approximated resulting in compact expressions in terms of 3-point functions and logarithms.

We have listed high-energy approximations for the complete set of scalar one-loop integrals that appears in $2 \rightarrow 2$ processes. In addition, we have provided explicit results for the corresponding tensor integrals. In the case of the 4-point function we have restricted ourselves to tensor integrals with at most four Lorentz indices. This set is sufficient for calculations in the ’t Hooft–Feynman gauge. The tensor integrals with more indices that appear in more general gauges can be easily reduced to those given here using tensor-integral reduction.

With the help of our results and tensor-integral reduction any one-loop matrix element for $2 \rightarrow 2$ scattering processes can be approximated for energies much larger than the internal and external masses. Moreover, our approximations are applicable to the decay of a very heavy particle into two light ones when all internal masses are also light. Since our results contain only the leading terms they allow to extract the leading contributions to matrix elements in the high-energy limit, i.e. the non-vanishing terms for $2 \rightarrow 2$ matrix elements.

A Appendix

A.1 Definition of one-loop integrals and tensor-coefficient functions

We use dimensional regularization denoting the space–time dimension by $D$. The one-loop functions are defined by:

$$A_0(m_0) = \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^D q \frac{1}{D_0}$$

$$B_{\{0,\mu,\nu\}}(p_1, m_0, m_1) = \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^D q \frac{\{1, q_\mu, q_\nu\}}{D_0D_1}$$

$$C_{\{0,\mu,\nu,\mu\nu\}}(p_1, p_2, m_0, m_1, m_2) = \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^D q \frac{\{1, q_\mu, q_\nuq_\nu, q_\muq_\nuq_\nu\}}{D_0D_1D_2}$$

$$D_{\{0,\mu,\nu,\mu\nu,\mu\nu\nu\}}(p_1, p_2, p_3, m_0, m_1, m_2, m_3) = \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^D q \frac{\{1, q_\mu, q_\nuq_\nu, q_\muq_\nuq_\nu, q_\muq_\nuq_\nuq_\nu\}}{D_0D_1D_2D_3}$$
with \( D_0 = q^2 - m_0^2 + i \epsilon \), \( D_i = (q + p_i)^2 - m_i^2 + i \epsilon \), \( i \geq 1 \). The tensor integrals can be decomposed into Lorentz tensors constructed of (linearly independent) external momenta \( p_{i\mu} \) and the metric tensor \( g_{\mu\nu} \) and tensor-coefficient functions as follows:

\[
B_{\mu} = B_1 p_{1\mu},
\]

\[
B_{\mu\nu} = B_{11} p_{1\mu} p_{1\nu} + B_{00} g_{\mu\nu},
\]

\[
C_{\mu} = \sum_{i=1}^{2} C_i p_{i\mu},
\]

\[
C_{\mu\nu} = \sum_{i,j=1}^{2} C_{ij} p_{i\mu} p_{j\nu} + C_{00} g_{\mu\nu},
\]

\[
C_{\mu\nu\rho} = \sum_{i,j,k=1}^{2} C_{ijk} p_{i\mu} p_{j\nu} p_{k\rho} + \sum_{i=1}^{2} C_{0i} \{ g_{\mu\nu} p_{i\rho} + g_{\nu\rho} p_{i\mu} + g_{\mu\rho} p_{i\nu} \},
\]

\[
D_{\mu} = \sum_{i=1}^{3} D_i p_{i\mu},
\]

\[
D_{\mu\nu} = \sum_{i,j=1}^{3} C_{ij} p_{i\mu} p_{j\nu} + D_{00} g_{\mu\nu},
\]

\[
D_{\mu\nu\rho} = \sum_{i,j,k=1}^{3} D_{ijk} p_{i\mu} p_{j\nu} p_{k\rho} + \sum_{i=1}^{3} D_{0i0} \{ g_{\mu\nu} p_{i\rho} + g_{\nu\rho} p_{i\mu} + g_{\mu\rho} p_{i\nu} \},
\]

\[
D_{\mu\nu\rho\sigma} = \sum_{i,j,k,l=1}^{3} D_{ijkl} p_{i\mu} p_{j\nu} p_{k\rho} p_{l\sigma} + D_{0000} \{ g_{\mu\nu} g_{\rho\sigma} + g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\rho} \}
\]

\[
+ \sum_{i,j=1}^{3} D_{00ij} \{ g_{\mu\nu} p_{i\rho} p_{j\sigma} + g_{\mu\rho} p_{i\nu} p_{j\sigma} + g_{\mu\sigma} p_{i\nu} p_{j\rho} + g_{\nu\sigma} p_{i\mu} p_{j\rho} \}.
\]

Since the tensor-coefficient functions are scalars and depend only on kinematical invariants, internal and external masses, we can write their arguments as:

\[
C_{\cdots}(p_1, p_2, m_0, m_1, m_2) = C_{\cdots}(p_1^2, (p_2 - p_1)^2, p_2^2, m_1, m_0, m_2),
\]

\[
D_{\cdots}(k_1, k_1 + k_2, k_1 + k_2 + k_3, m_0, m_1, m_2, m_3) =
\]

\[
= D_{\cdots}(k_1^2, k_2^2, k_3^2, (k_1 + k_2)^2, (k_1 + k_4)^2, m_0, m_1, m_2, m_3),
\]

where for convenience we introduce the momenta \( k_i \) of the external particles (cf. Fig. 1).

### A.2 Feynman-parameter representation of one-loop functions

The UV divergences of the following integrals are contained in the quantity

\[
\Delta = \frac{2}{4-D} - \gamma_E + \ln(4\pi),
\]

where \( \gamma_E \) is Euler’s constant.
A.2.1 1- and 2-point functions

The tensor-coefficient functions of the 1- and 2-point integrals can be represented as $(i \geq 0)$:

\[ A_0(m_0) = m_0^2 \left\{ \Delta - \ln \left( \frac{m_0^2 - i \epsilon}{\mu^2} \right) + 1 \right\}, \]  
(A17)

\[ B_{1\ldots i}(p_1, m_0, m_1) = \left( -1 \right)^i \left\{ \frac{1}{i + 1} \Delta - \int_0^\infty dx_0 dx_1 x_1^i \delta(1 - x_0 - x_1) \ln \left( \frac{M_B^2}{\mu^2} \right) \right\}, \]  
(A18)

\[ B_{001\ldots i}(p_1, m_0, m_1) = \left( -1 \right)^i \frac{1}{2} \left\{ \frac{m_0^2}{(i + 1)(i + 2)} + \frac{m_1^2}{(i + 2)} - \left( \frac{p_1^2}{(i + 2)(i + 3)} \right)(\Delta + 1) - \int_0^\infty dx_0 dx_1 x_1^i \delta(1 - x_0 - x_1)M_B^2 \ln \left( \frac{M_B^2}{\mu^2} \right) \right\} \]  
(A19)

with $M_B^2 = m_0^2 x_0 + m_1^2 x_1 - p_1^2 x_0 x_1 - i \epsilon$. Note that the scalar 2-point function $B_0$ is given by (A18) with $i = 0$.

A.2.2 3-point functions

The tensor-coefficient functions of the 3-point integrals $C_{\ldots}(p_1, p_2, m_0, m_1, m_2)$ allow for the following Feynman-parameter representations $(i, j \geq 0)$:

\[ C_{1\ldots 1 \ldots 1 \ldots j}(p_1, p_2, m_0, m_1, m_2) = -\left( -1 \right)^{i+j} \int_0^\infty dx_0 dx_1 dx_2 \frac{x_1^i x_2^j \delta(1 - x_0 - x_1 - x_2)}{M_C^2}, \]  
(A20)

\[ C_{001\ldots 1 \ldots 2\ldots j}(p_1, p_2, m_0, m_1, m_2) = \left( -1 \right)^{i+j} \frac{1}{2} \left\{ \frac{ijl}{(i + j + 2)!} \Delta - \int_0^\infty dx_0 dx_1 dx_2 x_1^i x_2^j \delta(1 - x_0 - x_1 - x_2) \ln \left( \frac{M_C^2}{\mu^2} \right) \right\} \]  
(A21)
with \( M_C^2 = m_0^2 x_0 + m_1^2 x_1 + m_2^2 x_2 - p_1^2 x_0 x_1 - p_2^2 x_0 x_2 - (p_1 - p_2)^2 x_1 x_2 - i \epsilon \). The scalar 3-point function is given by (A20) with \( i = j = 0 \).

A.2.3 4-point functions

The Feynman-parameter representations for the tensor-coefficient functions of the 4-point integrals \( D_{...}(k_1, k_1 + k_2, k_1 + k_2 + k_3, m_0, m_1, m_2, m_3) \) read \( (i, j, k \geq 0) \):

\[
D_{1\ldots12\ldots23\ldots3} = (-1)^{i+j+k} \int_0^\infty dx_0 dx_1 dx_2 dx_3 \frac{x_1^i x_2^j x_3^k \delta (1 - \sum_{n=0}^3 x_n)}{M_D^4},
\]

\[
D_{001\ldots12\ldots23\ldots3} = \frac{1}{2} (-1)^{i+j+k} \int_0^\infty dx_0 dx_1 dx_2 dx_3 \frac{x_1^i x_2^j x_3^k \delta (1 - \sum_{n=0}^3 x_n)}{M_D^4},
\]

\[
D_{0000\ldots12\ldots23\ldots3} = \frac{1}{4} (-1)^{i+j+k} \left\{ \frac{i! j! k!}{(i+j+k+3)!} \Delta 
- \int_0^\infty dx_0 dx_1 dx_2 dx_3 x_1^i x_2^j x_3^k \delta (1 - \sum_{n=0}^3 x_n) \ln \left( \frac{M_D^2}{\mu^2} \right) \right\}
\]

with \( M_D^2 = \sum_{n=0}^3 m_n^2 x_n - k_1^2 x_0 x_1 - k_2^2 x_1 x_2 - k_3^2 x_2 x_3 - k_4^2 x_3 x_0 - (k_1 + k_2)^2 x_0 x_2 - (k_1 + k_4)^2 x_1 x_3 - i \epsilon \)
and \( k_4 = -k_1 - k_2 - k_3 \). The scalar 4-point function is given by (A22) with \( i = j = k = 0 \).

Acknowledgement

We thank S. Dittmaier for useful discussions and a careful reading of the manuscript.

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