NORMAL LATTICE OF CERTAIN METABELIAN $p$-GROUPS $G$ WITH $G/G' \simeq (p,p)$

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Abstract. Let $p$ be an odd prime. The lattice of all normal subgroups and the terms of the lower and upper central series are determined for all metabelian $p$-groups with generator rank $d = 2$ having abelianization of type $(p,p)$ and minimal defect of commutativity $k = 0$. It is shown that many of these groups are realized as Galois groups of second Hilbert $p$-class fields of an extensive set of quadratic fields which are characterized by principalization types of $p$-classes.

1. Introduction

Let $p \geq 3$ be an odd prime number, and $G = \langle x, y \rangle$ be a two-generated metabelian $p$-group having an elementary bicyclic derived quotient $G/G'$ of type $(p,p)$.

Assume further that $G$ is of order $|G| = p^n$ with $n \geq 2$, and of nilpotency class $cl(G) = m - 1$ with $m \geq 2$. Then $G$ is of coclass $cc(G) = n - m + 1 = e - 1$ with $e \geq 2$. Denote by

$$G = \gamma_1(G) > \gamma_2(G) = G' > \ldots > \gamma_{m-1}(G) > \gamma_m(G) = 1$$

the (descending) lower central series of $G$, where $\gamma_j(G) = [\gamma_{j-1}(G), G]$ for $j \geq 2$, and by

$$1 = \zeta_0(G) < \zeta_1(G) < \ldots < G' = \zeta_m-2(G) < \zeta_m-1(G) = G$$

the (ascending) upper central series of $G$, where $\zeta_j(G)/\zeta_{j-1}(G) = \text{Centre}(G/\zeta_{j-1}(G))$ for $j \geq 1$.

Let $s_2 = t_2 = [y, x]$ denote the main commutator of $G$, such that $\gamma_2(G) = \langle s_2, \gamma_3(G) \rangle$. By means of the two series $s_j = [s_{j-1}, x]$ for $j \geq 3$ and $t_{\ell} = [t_{\ell-1}, y]$ for $\ell \geq 3$ of higher commutators and the subgroups $\Sigma_j = \langle s_j, \ldots, s_m \rangle$ with $j \geq 3$ and $T_\ell = \langle t_\ell, \ldots, t_{\ell+1} \rangle$ with $\ell \geq 3$, we obtain the following fundamental distinction of cases.

1. The uniserial case of a CF group (cyclic factors) of coclass $cc(G) = 1$ (maximal class), where $t \in \Sigma_3$; $\gamma_3(G) = \langle s_3, \gamma_4(G) \rangle$, $e = 2$, and $m = n$. There are two subcases:
   (1.1) $t = 1 \in \gamma_m(G)$, where $G$ contains an abelian maximal subgroup and $k = 0$,
   (1.2) $\neq t \in \gamma_{m-k}(G)$, $1 \leq k \leq m - 4$, where all maximal subgroups are non-abelian.

2. The biserial case of a non-CF or BCF group (bicyclic or cyclic factors) of coclass $cc(G) \geq 2$, where $t \notin \Sigma_3$; $\gamma_3(G) = \langle s_3, t_3, \gamma_4(G) \rangle$, $e \geq 3$, and $m < n$. Again there exist two subcases, characterized by the defect of commutativity $k$ of $G$:
   (2.1) $t_{\ell+1} = 1 \in \gamma_m(G)$, where $\Sigma_3 \cap T_3 = 1$ and $k = 0$,
   (2.2) $\neq t_{\ell+1} \in \gamma_{m-k}(G)$, for some $k \geq 1$, where $\Sigma_3 \cap T_3 \leq \gamma_{m-k}(G)$.

In this article, we are interested in two-generator metabelian $p$-groups $G = \langle x, y \rangle$ of coclass $cc(G) \geq 2$ having the convenient property $\Sigma_3 \cap T_3 = 1$, resp. $k = 0$, where the product $\Sigma_3 \times T_3$ is direct and coincides with the major part of the normal lattice of $G$, as shown in Figure 1.

Definition 1.1. A pair $(U, V)$ of normal subgroups of a $p$-group $G$, such that $V < U \leq G$ and $(U : V) = p^2$, is called a diamond if the quotient $U/V$ is abelian of type $(p,p)$.

If $(U, V)$ is a diamond and $U = \langle u_1, u_2, V \rangle$, then the $p+1$ intermediate subgroups of $G$ between $U$ and $V$ are given by $\langle u_2, V \rangle$ and $\langle u_1 u_2^{i-2}, V \rangle$ with $2 \leq i \leq p+1$.

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2. The normal lattice

In this section, let $G = \langle x, y \rangle$ be a metabelian $p$-group with two generators $x, y$, having abelianization $G/G'$ of type $(p, p)$ and satisfying the independence condition $\Sigma_3 \cap T_3 = 1$, that is, $G$ is a metabelian $p$-group with defect of commutativity $k = 0$ [14] § 3.1.1, p. 412, and § 3.3.2, p. 429]. We assume that $G$ is of coclass $cc(G) \geq 2$, since the normal lattice of $p$-groups of maximal class has been determined by Blackburn [5].

**Theorem 2.1.** The complete normal lattice of $G$ contains the heading diamond $(G, G')$ and the rectangle $\{(P_{j, t}, P_{j+1, t+1})\}_{3 \leq j \leq m-1, 3 \leq t \leq e}$ of trailing diamonds, where $P_{j, t} = \Sigma_j \times T_t$ for $3 \leq j \leq m$ and $3 \leq t \leq e + 1$. The structure of the normal lattice is visualized in Figure 7.

Note that $P_{j, t} = \langle s_j, \ldots, s_{m-1} \rangle \times \langle t_1, \ldots, t_e \rangle = \langle s_j, t_e, P_{j+1, t+1} \rangle$ for $3 \leq j \leq m - 1, 3 \leq t \leq e$.

**Conjecture 2.1.** The complete normal lattice of $G$ consists exactly of the normal subgroups given in Theorem 2.1.

**Corollary 2.1.** The total number of normal subgroups of $G$ is given by

$me - (m + 2e) + 6 + [me - (2m + 3e) + 7] \cdot (p - 1),$

in particular, for $p = 3$ it is given by

$3me - (5m + 8e) + 20.$

**Corollary 2.2.** Blackburn’s two-step centralizers of $G$ [5] are given by

$$\chi_j(G) = \begin{cases} G' & \text{for } 1 \leq j \leq e - 1, \\ \langle y, G' \rangle & \text{for } e \leq j \leq m - 2, \\ G & \text{for } j \geq m - 1, \end{cases}$$

in particular, none of the maximal subgroups of $G$ occurs as a two-step centralizer, when $e = m - 1$.

1. The factors of the lower central series of $G$ are given by

$$\gamma_j(G)/\gamma_{j+1}(G) \simeq \begin{cases} (p, p) & \text{for } j = 1 \text{ and } 3 \leq j \leq e, \\ (p) & \text{for } j = 2 \text{ and } e + 1 \leq j \leq m - 1. \end{cases}$$

2. The terms of the lower central series of $G$ are given by

$$\gamma_j(G) = \begin{cases} \langle x, y, G' \rangle & \text{for } j = 1, \\ \langle s_2, \gamma_3(G) \rangle & \text{for } j = 2, \\ P_{j, j} & \text{for } 3 \leq j \leq e, \\ \Sigma_j & \text{for } e + 1 \leq j \leq m - 1. \end{cases}$$

3. The factors of the upper central series of $G$ are given by

$$\zeta_j(G)/\zeta_{j-1}(G) \simeq \begin{cases} (p, p) & \text{for } 1 \leq j \leq e - 2 \text{ and } j = m - 1, \\ (p) & \text{for } e - 1 \leq j \leq m - 2. \end{cases}$$

4. The terms of the upper central series of $G$ are given by

$$\zeta_j(G) = \begin{cases} P_{m-j, e+1-j} & \text{for } 1 \leq j \leq e - 2, \\ P_{m-j, 3} & \text{for } e - 1 \leq j \leq m - 3, \\ \langle s_2, \zeta_{m-3}(G) \rangle & \text{for } m - 2, \\ \langle x, y, \zeta_{m-2}(G) \rangle & \text{for } j = m - 1. \end{cases}$$
Proof. We prove the invariance of all claimed normal subgroups under inner automorphisms of $G = \langle x, y \rangle$.

It is well known that the subgroups in the heading diamond are normal, since they contain the commutator subgroup $G' = \gamma_2(G)$.

We start the proof with the tops of trailing diamonds. For $g \in P_{j,\ell}$ and $s \in G'$ we have $s^{-1}gs = s^{-1}g$, since $P_{j,\ell} < G'$, for $j \geq 3$, $\ell \geq 3$, and $G$ was assumed to be metabelian. Now, $P_{j,\ell}$ is the direct product of $\Sigma_j$ and $T_\ell$, since we suppose that $\Sigma_j \cap T_3 = 1$. So it suffices to show invariance of $\Sigma_j$ and $T_\ell$ under conjugation with the generators $x$ and $y$ of $G$. We have $x^{-1}s_jx = s_j[s_j, x] = s_j s_{j+1} \in \Sigma_j$ and $y^{-1}s_jy = s_j[s_j, y] = s_j \in \Sigma_j$ for $j \geq 3$. And similarly we have $x^{-1}t_\ell x = t_\ell[t_\ell, x] = t_\ell \in T_\ell$ and $y^{-1}t_\ell y = t_\ell[t_\ell, y] = t_\ell t_{\ell+1} \in T_\ell$ for $\ell \geq 3$.

Next we prove invariance of intermediate groups between top and bottom of trailing diamonds. They are of the shape $\langle t_\ell, P_{j+1,\ell+1} \rangle$ or $\langle s_j t_\ell, P_{j+1,\ell+1} \rangle$ with $0 \leq i \leq p - 1$. For $t_\ell$, invariance has been shown above. So we investigate $s_j t_\ell$. We have $x^{-1}s_j t_\ell x = x^{-1}s_j x(x^{-1}t_\ell x) = s_j s_{j+1} t_\ell$, where $s_j s_{j+1} \in P_{j+1,\ell+1}$, and $y^{-1}s_j t_\ell y = y^{-1}s_j y(y^{-1}t_\ell y) = s_j t_\ell t_{\ell+1}$, where $t_\ell t_{\ell+1} \in P_{j+1,\ell+1}$. (Here we probably are tacitly using power conditions like $s_j^p \in \Sigma_{j+1}$ for $j \geq 3$ and $t_\ell^p \in T_{\ell+1}$ for $\ell \geq 3$.)

Thus we have proved the invariance of all claimed normal subgroups under inner automorphisms.

The number of all (heading and trailing) diamonds of the normal lattice is $1 + (m - 1 - 2)(e - 2) = 1 + (m - 3) \cdot (e - 2) = 1 + me - 2m - 3e + 6 = me - (2m + 3e) + 7$.

There are $p - 1$ inner vertices of valence 2 in each diamond, which gives us a total of $(me - [2m + 3e] + 7) \cdot (p - 1)$ inner vertices.

The remaining (outer) vertices form the heading square and the trailing rectangle with

$$4 + (m - 1 + 1 - 2) \cdot (e + 1 - 2) = 4 + (m - 2) \cdot (e - 1) = 4 + me - m - 2e + 2 = me - (m + 2e) + 6$$

vertices.

Outer and inner vertices together form a lattice of $me - (m + 2e) + 6 + (me - [2m + 3e] + 7) \cdot (p - 1)$ normal subgroups.

For $p = 3$, this formula yields $me - m - 2e + 6 + 2me - 4m - 6e + 14 = 3me - (5m + 8e) + 20$.

For each $j \geq 2$, Blackburn’s two-step centralizer $\chi_j(G)$ is defined as the biggest intermediate group between $G$ and $G' = \gamma_2(G)$ such that $[\gamma_j(G), \chi_j(G)] \leq \gamma_{j+2}(G)$. Since $[\gamma_j(G), \gamma_2(G)] \leq \gamma_{j+2}(G)$, for any $j \geq 2$, $\chi_j(G)$ certainly contains $\gamma_2(G)$. Since $[s_j, x] = s_{j+1} \notin [\gamma_2(G)$ for $2 \leq j \leq m - 2$, $[t_\ell, y] = t_{\ell+1} \notin [\gamma_2(G)$ for $2 \leq \ell \leq e - 1$, and $e \leq m - 1$, neither $x$ nor $y$ can be an element of $\chi_j(G)$ for $2 \leq j \leq e - 1$. However, since $[t_\ell, y] = t_{\ell+1} = 1 \in [\gamma_2(G)$ and $[s_j, y] = 1 \in [\gamma_2(G)$, we have $\chi_j(G) = [y, \gamma_2(G)]$ for $j \leq 2$ and $e \leq m - 1$, both provided that $e \leq m - 2$.

Finally, since $[s_{m-1}, x] = s_m \in [\gamma_m(G) = \gamma_{m+1}(G) = 1$, the two-step centralizers $\chi_j(G)$ with $j \geq 2$ coincide with the entire group $G$.

The members of the lower central series can be constructed recursively by $\gamma_j(G) = [\gamma_{j-1}(G), G]$.

There is a unique ramification generating the series $\Sigma_3$ and $T_3$ for $j = 3$, since $\gamma_3(G) = [\gamma_2(G), G] = [s_2, \gamma_3(G)], [s_2, y], [s_2, \gamma_4(G)] = \{s_3, T_3, \gamma_4(G)\}$. Otherwise the series $\Sigma_3$ and $T_3$ do not mix and we have $[\gamma_j(G)] = [\gamma_{j-1}(G), G] = [s_{j-1}, t_{j-1}, \gamma_j(G), G]$.

$$\gamma_j(G) = [s_j, t_j, y, \gamma_{j+1}(G)]$$

for $j \geq 4$. For $j = e + 1$ the bicyclic factors stop, since $t_{e+1} = [t_e, y] = 1$, and $\gamma_{e+1}$ is simply given by $\Sigma_{e+1}$.

The members of the upper central series can be constructed recursively by $\zeta_j(G) = [\gamma_{j-1}(G), G] = Centre(G/\gamma_{j-1}(G))$. All groups $G$ with the assigned properties have a bicyclic centre $\zeta_j(G) = [s_{m-1}, t_{e+1}]$, since $[s_{m-1}, x] = [t_e, y] = 1$.

Generally, the equations $[s_{j-1}, x] = s_{m-j-1}, [s_{m-j}, y] = 1, [t_{e+1-j}, x] = 1, [t_{e+1-j}, y] = t_{e+1-j},$ whose right sides are elements of $\zeta_j(G)$, show that $s_{m-j}$ and $t_{e+1-j}$ commute with all elements of $G$ modulo $\gamma_{j-1}(G)$. Therefore, we have $\zeta_j(G) = P_{m-j, e+1-j}$.

However, for $j = e - 1$ the bicyclic factors stop, since $[t_{e+1-j}, x] = [t_{e+1-j}, x] = [s_2, x] = s_3$, which is not contained in $\zeta_{-2}(G)$, except for $e = m - 1$. Consequently, $\zeta_j(G) = P_{m-j, 3}$ for $j \geq e - 1$, since it cannot contain $t_2 = s_2$.

□
Figure 1. Full normal lattice, including lower and upper central series, of a $p$-group $G$ with $G/G' \simeq (p, p)$, cl$(G) = m - 1$, cc$(G) = e - 1$, dl$(G) = 2$, $k(G) = 0$. 
NORMAL LATTICE OF CERTAIN METABELIAN $p$-GROUPS $G$ WITH $G/G' \simeq (p,p)$

3. Applications in Algebraic Number Theory

Let $K = \mathbb{Q}(\sqrt{D})$ be a quadratic number field with discriminant $D$ and denote by $G = \text{Gal}(F_p^2(K)|K)$ the Galois group of the second Hilbert $p$-class field $F_p^2(K)$ of $K$, that is, the maximal metabelian unramified $p$-extension of $K$. We recall that coclass and class of $G$ are given by the equations $\text{cc}(G) = r = e - 1$ and $\text{cl}(G) = m - 1$ in terms of the invariants $e$ and $m$. Due to our extensive computations for the papers [12, 14], we are able to underpin the present theory of normal lattices by numerical data concerning the 2020 complex and the 2576 real quadratic fields with 3-class group of type $(3,3)$ and discriminant in the range $-10^6 < D < 10^7$.

Figure 2 shows several examples of normal lattices of 3-groups $G$ with bicyclic and cyclic factors of the central series. They are located on coclass trees of coclass graphs $G(3,r)$ [15, p. 189 ff].

Here, the length of the rectangle of trailing diamonds is bigger than the width, $m - 1 > e$, the upper central series is different from the lower central series, and the last lower central $\gamma_m(G)$ is cyclic, whence the parent $\pi(G) = G/\gamma_m(G)$ is of the same coclass. Such groups were called core groups in [14]. Concerning the principalization type $\zeta(K)$ of $K$ which coincides with the transfer kernel type (TKT) $\tau(G)$ of $G$, see [13, 14]. Different TKTs can give rise to equal normal lattices.

**Figure 2.** 3-groups $G = \text{Gal}(F_p^2(K)|K)$ with bicyclic and cyclic factors.

| order $3^n$ | $e = 5$, $m = 8$ | $e = 5$, $m = 7$ | $e = 4$, $m = 6$ |
|-------------|-----------------|-----------------|-----------------|
| 177 147     |                 |                 |                 |
| 59 049      |                 |                 |                 |
| 19 683      |                 |                 |                 |
| 6 561       |                 |                 |                 |
| 2 187       |                 |                 |                 |
| 729         |                 |                 |                 |
| 243         |                 |                 |                 |
| 81          |                 |                 |                 |
| 27          |                 |                 |                 |
| 9           |                 |                 |                 |
| 3           |                 |                 |                 |

**Example 3.1.** 3-groups $G$ of coclass $3 \leq \text{cc}(G) \leq 4$.

- Coclass $\text{cc}(G) = 4$, class $\text{cl}(G) = 7$:
  a total of 14 complex quadratic fields, e. g.,
  $D = -159 208$ with principalization type F.13,
  $D = -249 371$ with principalization type F.12,
  $D = -469 787$ with principalization type F.11,
  $D = -469 816$ with principalization type F.7,
  and a single real quadratic field of discriminant
  $D = 8 127 208$ with principalization type F.13,
  branch groups of depth 1, visualized by Figure 2 $e = 5$, $m = 8$.

- Coclass $\text{cc}(G) = 4$, class $\text{cl}(G) = 6$:
  a single real quadratic field of discriminant
  $D = 8 491 713$ with principalization type d*.25,
  mainline group, visualized by Figure 2 $e = 5$, $m = 7$.

- Coclass $\text{cc}(G) = 3$, class $\text{cl}(G) = 5$:
  two real quadratic fields of discriminant
  $D = 1 535 117$ with principalization type d.23,
  $D = 2 328 721$ with principalization type d.19,
  branch groups of depth 1, visualized by Figure 2 $e = 4$, $m = 6$. 
In Figure 3 we display numerous examples of normal lattices of $p$-groups $G$ with *bicyclic factors* of the central series, except the bottle neck $\gamma_2(G)/\gamma_3(G)$. They are located as vertices on the sporadic part $G_0(p, r)$ of coclass graphs $G(p, r)$, outside of coclass trees, [14, Fig. 3.5, p. 439].

Here, the rectangle of trailing diamonds degenerates to a square with $e = m - 1$, the upper central series is the reverse lower central series, and thus the last lower central $\gamma_{m-1}(G)$ is bicyclic, whence the (generalized) parent $\tilde{\pi}(G) = G/\gamma_{m-1}(G)$ is of lower coclass. Such groups were called *interface groups* in [14].

**Example 3.2.** $p$-groups $G = \text{Gal}(F_p^2(K)|K)$ with bicyclic factors only.

- $p = 3$, coclass $\text{cc}(G) = 6$, class $\text{cl}(G) = 7$: a single complex quadratic field of discriminant $D = -423,640$ with principalization type F.12, sporadic group, visualized by Figure 3, $e = 7$, $m = 8$.
- $p = 3$, coclass $\text{cc}(G) = 4$, class $\text{cl}(G) = 5$: a total of 78 complex quadratic fields, e. g., $D = -27,156$ with principalization type F.11, $D = -31,908$ with principalization type F.12, $D = -67,480$ with principalization type F.13, $D = -124,363$ with principalization type F.7, and a single real quadratic field of discriminant $D = 8,321,505$ with principalization type F.13, sporadic groups, visualized by Figure 3, $e = 5$, $m = 6$.
- $p = 3$, coclass $\text{cc}(G) = 2$, class $\text{cl}(G) = 3$: a total of 936 complex quadratic fields, e. g., $D = -4,027$ with principalization type D.10, $D = -12,131$ with principalization type D.5, and a total of 140 real quadratic fields, e. g., $D = 422,573$ with principalization type D.10, $D = 631,769$ with principalization type D.5, sporadic groups, visualized by Figure 3, $e = 3$, $m = 4$.
- $p = 5$, coclass $\text{cc}(G) = 2$, class $\text{cl}(G) = 3$: see [14, Tbl. 3.13, p. 450].
- $p = 7$, coclass $\text{cc}(G) = 2$, class $\text{cl}(G) = 3$: see [14, Tbl. 3.14, p. 450].
Figure 4 shows many examples of normal lattices of “small” $p$-groups $G$ with bicyclic and cyclic factors of the central series. They are located on coclass trees of coclass graphs $G(p, r)$ Fig. 3.6–3.7, pp. 442–443.

**Figure 4.** Small $p$-groups $G = \text{Gal}(F_p^2(K)|K)$ with bicyclic and cyclic factors.

**Example 3.3.** Small $p$-groups $G$ with $p \in \{3, 5, 7\}$.

- $p = 3$, coclass $cc(G) = 2$, class $cl(G) = 7$: a total of 28 complex quadratic fields, e. g.,
  - $D = -262744$ with principalization type E.14,
  - $D = -268040$ with principalization type E.6,
  - $D = -297079$ with principalization type E.9,
  - $D = -370740$ with principalization type E.8,
  branch groups of depth 1, visualized by Figure 4 $e = 3$, $m = 8$.

- $p = 3$, coclass $cc(G) = 2$, class $cl(G) = 6$: two real quadratic fields, e. g.,
  - $D = 1001957$ with principalization type c.21,
  mainline groups, visualized by Figure 4 $e = 3$, $m = 7$.

- $p = 3$, coclass $cc(G) = 2$, class $cl(G) = 5$: a total of 383 complex quadratic fields, e. g.,
  - $D = -9748$ with principalization type E.9,
  - $D = -15544$ with principalization type E.6,
  - $D = -16627$ with principalization type E.14,
  - $D = -34867$ with principalization type E.8,
  and a total of 21 real quadratic fields, e. g.,
  - $D = 342664$ with principalization type E.9,
  - $D = 3918837$ with principalization type E.14,
  - $D = 5264069$ with principalization type E.6,
  - $D = 6098360$ with principalization type E.8,
  branch groups of depth 1, visualized by Figure 4 $e = 3$, $m = 5$.

- $p = 3$, coclass $cc(G) = 2$, class $cl(G) = 4$: a total of 54 real quadratic fields, e. g.,
  - $D = 534824$ with principalization type c.18,
  - $D = 540365$ with principalization type c.21,
  mainline groups, visualized by Figure 4 $e = 3$, $m = 5$.

- $p = 5$, coclass $cc(G) = 2$, class $cl(G) = 5$: see [14] Tbl. 3.13, p. 450.

- $p = 7$, coclass $cc(G) = 2$, class $cl(G) = 5$: see [14] Tbl. 3.14, p. 450.
4. Final Remarks

• Among the 2 020 complex quadratic fields with 3-class group of type \((3, 3)\) and discriminant in the range \(-10^6 < D < 0\), the dominating part of 1 440, that is 71.29\%, has a second 3-class group with minimal defect of commutativity \(k = 0\). The remaining 28.71\% have \(k = 1\) and TKTs G.16, G.19 and H.4.

• Among the 2 576 real quadratic fields with 3-class group of type \((3, 3)\) and discriminant in the range \(0 < D < 10^7\), a modest part of 273, i.e. 10.6\%, has a second 3-class group of coclass at least 2. A dominating part of 222 among these 273 second 3-class groups, that is 81.3\%, has minimal defect of commutativity \(k = 0\), whereas 18.7\% have \(k = 1\) and TKTs b.10, G.16, G.19 and H.4.

• It should be pointed out that the power-commutator presentations which we used for proving Theorem 2.1 and its Corollaries are rudimentary, since in fact they consist of commutator relations only. Thus they define an isoclinism family of \(p\)-groups of fixed order, rather than a single isomorphism class of \(p\)-groups.

On the other hand, experience shows that the transfer kernel type (TKT) of a \(p\)-group mainly depends on the power relations. This explains why different TKTs frequently give rise to equal normal lattices.

References

[1] J. A. Ascione, G. Havas, and C. R. Leedham-Green, A computer aided classification of certain groups of prime power order, Bull. Austral. Math. Soc. 17 (1977), 257–274, Corrigendum Supplement p. 320.
[2] J. A. Ascione, On 3-groups of second maximal class (Ph.D. Thesis, Australian National University, Canberra, 1979).
[3] J. A. Ascione, On 3-groups of second maximal class, Bull. Austral. Math. Soc. 21 (1980), 473–474.
[4] H. U. Besche, B. Eick, and E. A. O’Brien, The SmallGroups Library — a Library of Groups of Small Order, 2005, an accepted and refereed GAP 4 package, available also in MAGMA.
[5] N. Blackburn, On a special class of \(p\)-groups, Acta Math. 100 (1958), 45–92.
[6] W. Bosma, J. Cannon, and C. Playoust, The Magma algebra system. I. The user language, J. Symbolic Comput. 24 (1997), 235–265.
[7] W. Bosma, J. J. Cannon, C. Fieker, and A. Steels (eds.), Handbook of Magma functions (Edition 2.19, Sydney, 2013).
[8] B. Eick, C. R. Leedham-Green, M. F. Newman, and E. A. O’Brien, On the classification of groups of prime-power order by coclass: The 3-groups of coclass 2, to appear in Int. J. Algebra and Computation, 2013.
[9] The GAP Group, GAP — Groups, Algorithms, and Programming — a System for Computational Discrete Algebra, Version 4.4.12, Aachen, Braunschweig, Fort Collins, St. Andrews, 2008, (\texttt{http://www.gap-system.org}).
[10] G. Gamble, W. Nickel, and E. A. O’Brien, ANU \(p\)-Quotient — \(p\)-Quotient and \(p\)-Group Generation Algorithms, 2006, an accepted GAP 4 package, available also in MAGMA.
[11] The MAGMA Group, MAGMA Computational Algebra System, Version 2.19-9, Sydney, 2013, (\texttt{http://magma.maths.usyd.edu.au}).
[12] D. C. Mayer, The second \(p\)-class group of a number field, Int. J. Number Theory 8 (2012), no. 2, 471–505, DOI 10.1142/S179304211250025X.
[13] D. C. Mayer, Transfers of metabelian \(p\)-groups, Monatsh. Math. 166 (2012), no. 3–4, 467–495, DOI 10.1007/s00605-010-0277-x.
[14] D. C. Mayer, The distribution of second \(p\)-class groups on coclass graphs, J. Théor. Nombres Bordeaux 25 (2013) no. 2, 401–456 (27th Journées Arithmétiques, Faculty of Mathematics and Informatics, Vilnius University, Vilnius, Lithuania, 2011).
[15] B. Nebelung, Klassifikation metabelscher \(3\)-Gruppen mit Faktorkommutatorgruppe vom Typ \((3, 3)\) und Anwendung auf das Kapitulationsproblem (Inauguraldissertation, Universität zu Köln, 1989).
[16] M. F. Newman, Groups of prime-power order, Groups — Canberra 1989, Lecture Notes in Mathematics, vol. 1456, Springer, 1990, pp. 49–62.
[17] M. F. Newman and E. A. O’Brien, Classifying 2-groups by coclass, Trans. Amer. Math. Soc. 351 (1999), 131–169.
[18] E. A. O’Brien, The \(p\)-group generation algorithm, J. Symbolic Comput. 9 (1990), 677–698.