SOME PROPERTIES OF OPTIMAL FUNCTIONS
FOR SPHERE PACKING IN DIMENSIONS 8 AND 24

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Abstract. We study some sequences of functions of one real variable and
conjecture that they converge uniformly to functions with certain positivity
and growth properties. Our conjectures imply a conjecture of Cohn and Elkies,
which in turn implies the complete solution to the sphere packing problem
in dimensions 8 and 24. We give numerical evidence for these conjectures as
well as some arithmetic properties of the hypothetical limiting functions. The
conjectures are of greatest interest in dimension 24, in light of Viazovska’s
recent solution to the Cohn-Elkies conjecture (and consequently the sphere
packing problem) in dimension 8.

1. Introduction

One of the fundamental problems in geometry is to determine the densest sphere
packing in Euclidean space. In other words, how large a fraction of $\mathbb{R}^n$ can be
covered by equal-sized, non-overlapping balls? The answer is known so far only for
$n \leq 3$ (see [FT] and [H]), and very recently for $n = 8$ as well [V]. A remarkable
feature of this problem is that each dimension has its own idiosyncrasies. Even
setting aside the issue of proofs, the best packings known do not seem to follow any
simple pattern.

Perhaps the most striking packings are those formed by centering spheres at
the points of the $E_8$ root lattice and the Leech lattice. Both have been known
for some years now to be the densest lattice packings in their dimensions. The
$E_8$ case was proved by Blichfeldt in his 1935 paper [B], and the Leech lattice case
was proved by Cohn and Kumar in [CK3] (see also [CK1]). The latter work was
based on an analytic approach introduced in [CE] by Cohn and Elkies, who in fact
studied the general sphere packing problem (including non-lattice packings, which
may improve on the density of lattice packings in some dimensions). Cohn and
Elkies proved that $E_8$ and the Leech lattice are optimal among all sphere packings
if there exist functions from $\mathbb{R}$ to $\mathbb{R}$ satisfying certain sign and regularity conditions;
they furthermore conjectured that such functions do indeed exist. In this paper we
introduce explicit sequences of functions which we conjecture converge to functions
satisfying the Cohn-Elkies conditions. (We of course note that the $n = 8$ case of the
Cohn-Elkies conjecture was solved in [V].)

Our functions depend on a parameter $n$, the dimension of the sphere packing
problem. One advantage of our approach is that our conjectures appear to hold for
a broader range of values of $n$, not only for $n = 8$ and $n = 24$. Although they have
no sphere packing implications except in those two cases, existence might be easier
to prove because they no longer depend on delicate facts about these particular dimensions.

A second advantage is that our approach does not rely on numerical optimization. By contrast, the Leech lattice optimality proof makes use of a carefully optimized polynomial of degree 803 with 3000-digit coefficients. The computer-assisted proof in [CK3] reads this polynomial from a file and verifies that it has the desired properties to complete the proof, but there is no conceptual description of the polynomial or simple method to construct it from scratch. (It was found by combining numerous ad hoc techniques to locate a starting point from which Newton’s method would converge.) Using our approach, one could replace this complicated polynomial with a polynomial that has a much simpler description. That would not remove the need for computer verification of its properties, but it is a step towards simplifying the proof.

Our lack of need for optimization also enables us to carry out much larger computations than in previous papers. For example, we arrive at density bounds that are sharp to over fifty decimal places in \( \mathbb{R}^8 \) and \( \mathbb{R}^{24} \), compared with the fourteen and twenty-nine decimal places from [CK3]. Strictly speaking our new bounds are not theorems, because we have not bothered to verify them using exact arithmetic, but our floating point calculations leave no reasonable doubt. We are confident that the approach from Appendix A in [CK3] could be used to provide a proof (should a rigorous bound be needed for some purpose).

The results of these large calculations display intricate and surprising structure. Most interestingly, in Section 5 we find that the second Taylor coefficients appear to be rational. If the pattern governing the higher coefficients could be identified, it would yield a direct construction by power series of functions satisfying the Cohn-Elkies conjecture.

In the next section we review background from [CE]. Our functions are introduced in Section 3. In Section 4, we provide experimental evidence that our sequences of functions are converging rapidly (despite the failure of a related, naive construction), and we study this numerical data in detail. In Section 5 we examine the Taylor coefficients and values of the Mellin transform of the optimal functions, both of which exhibit some unexplained rationality properties. In Section 6 we study the closely related problem of potential energy minimization. Finally we conclude in Section 7 by describing some related but simpler sequences of functions, which serve as a testing ground for our main conjectures.

2. Background

Define the Fourier transform of a function \( f : \mathbb{R}^n \to \mathbb{R} \) by

\[
\hat{f}(t) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i \langle x, t \rangle} \, dx.
\]  

We call a continuous function \( f \) admissible if both \( |f(x)| \) and \( |\hat{f}(x)| \) are bounded by a constant times \( (1 + |x|)^{n-\delta} \) for some \( \delta > 0 \). This bound ensures, for example, that the integral defining \( \hat{f} \) converges. It also guarantees that both sides of the Poisson summation formula

\[
\sum_{x \in \Lambda} f(x) = \frac{1}{|\Lambda|} \sum_{t \in \Lambda^*} \hat{f}(t)
\]
converge absolutely and are equal. Here \( \Lambda \) denotes a lattice in \( \mathbb{R}^n \), \( \Lambda^* = \{ t \in \mathbb{R}^n : \langle t, x \rangle \in \mathbb{Z} \text{ for all } x \in \Lambda \} \) its dual, and \( |\Lambda| = \text{vol}(\mathbb{R}^n/\Lambda) \) its covolume.

Our primary connection between sphere packing and Fourier analysis is the following theorem of Cohn and Elkies (Theorem 3.1 in [CE]; see also [C1]):

**Theorem 2.1.** Suppose there exists an admissible function \( f : \mathbb{R}^n \to \mathbb{R} \) and a constant \( r \) such that

1. \( f(0) = \hat{f}(0) \neq 0 \),
2. \( f(x) \leq 0 \) for \( |x| \geq r \), and
3. \( \hat{f}(t) \geq 0 \) for all \( t \).

Then every sphere packing in \( \mathbb{R}^n \) has density at most

\[
\frac{\pi^{n/2}}{(n/2)!} \left( \frac{r}{2} \right)^n.
\]

As usual \((n/2)!\) is to be interpreted as \( \Gamma(n/2 + 1) \) when \( n \) is odd. The density of a sphere packing refers to the fraction of space covered by the packing.

We will briefly explain how to prove Theorem 2.1 using Poisson summation, because the conditions for a sharp bound will be important later in the paper.

**Proof.** First, we give the proof for lattice packings, after which we will sketch the general proof.

Suppose \( \Lambda \subset \mathbb{R}^n \) is a lattice. We can assume without loss of generality that the minimal nonzero vector length in \( \Lambda \) is \( r \), because sphere packing density is invariant under scaling. That amounts to using balls of radius \( r/2 \) in the sphere packing.

By Poisson summation,

\[
\sum_{x \in \Lambda} f(x) = \frac{1}{|\Lambda|} \sum_{t \in \Lambda^*} \hat{f}(t).
\]

Applying the inequalities on \( f \) and \( \hat{f} \) yields

\[
f(0) \geq \sum_{x \in \Lambda} f(x) = \frac{1}{|\Lambda|} \sum_{t \in \Lambda^*} \hat{f}(t) \geq \frac{\hat{f}(0)}{|\Lambda|}.
\]

Thus,

\[
|\Lambda| \geq 1.
\]

In other words, there is at most one lattice point per unit volume in \( \mathbb{R}^n \). It follows that the density is at most the volume of a sphere of radius \( r/2 \), i.e.,

\[
\frac{\pi^{n/2}}{(n/2)!} \left( \frac{r}{2} \right)^n
\]

(because the density equals the volume of a sphere times the number of spheres per unit volume in space).

For the general case, one can assume without loss of generality that the sphere packing is periodic, i.e., a union of translates of a lattice packing. Suppose it is the disjoint union of \( \Lambda + v_1, \ldots, \Lambda + v_n \). Then applying the identity

\[
\sum_{j,k=1}^N \sum_{x \in \Lambda} f(x + v_j - v_k) = \frac{1}{|\Lambda|} \sum_{t \in \Lambda^*} \hat{f}(t) \left| \sum_{j=1}^N e^{2\pi i(v_j, t)} \right|^2,
\]
which follows from Poisson summation and some manipulation, completes the proof as above.

\[ \square \]

One can weaken the hypothesis of admissibility in this theorem, at the cost of complicating the proof (see Proposition 9.3 in [CK2], which is set in the more general context of potential energy minimization, or the proof in [CZ2], which does not even use Poisson summation). However, the applications in this paper will use only admissible functions.

Unfortunately, Theorem 2.1 does not address the issue of how to find functions \( f \) that lead to good sphere packing bounds (i.e., that minimize \( r \)). Doing so amounts to grappling with an infinite-dimensional optimization problem, which has a simple solution when \( n = 1 \) but is unsolved and appears difficult for \( n > 1 \). Cohn and Elkies performed a computer search to locate explicit functions that improve on the previously known density upper bounds for \( 4 \leq n \leq 36 \). (For \( 4 \leq n \leq 7 \) and \( n = 9 \), a refinement of this approach from [LOV] yields slightly better bounds.) These functions are probably nearly optimal in terms of minimizing the values \( r \) achieved by functions satisfying the hypotheses of Theorem 2.1. However, in most cases these bounds are still far above the densities of the best packings known.

The most remarkable application of Theorem 2.1 occurs when the dimension \( n \) is 8 or 24. In those dimensions, Cohn and Elkies found functions that come tantalizingly close to solving the sphere packing problem completely. Using more sophisticated search techniques, Cohn and Kumar [CK3] later achieved a bound within a factor of \( 1 + 1.65 \times 10^{-30} \) of the conjectured optimum for \( n = 24 \) and a factor of \( 1 + 10^{-14} \) for \( n = 8 \). Typically it is harder to get more accurate bounds for larger values of \( n \); the reason the bound for \( n = 24 \) is so much better is that Cohn and Kumar required that level of accuracy for their application and thus devoted much more computer time to optimizing this case.

One may ask whether the functions produced by these computer searches asymptotically produce a sharp sphere packing bound in these dimensions. That appears to be true, and Cohn and Elkies conjectured an even stronger statement, namely that the sphere packing problem in dimensions 2, 8, and 24 can be solved exactly by the use of a single function \( f \) in Theorem 2.1:

**Conjecture 2.2** (Conjecture 7.3 in [CE]; now a theorem when \( n = 8 \) [V]). When \( n \in \{2, 8, 24\} \), there exists a function \( f \) satisfying the hypotheses of Theorem 2.1 with

\[
\begin{align*}
  r &= \begin{cases} 
    (4/3)^{1/4} & \text{if } n = 2, \\
    \sqrt{2} & \text{if } n = 8, \text{ and} \\
    2 & \text{if } n = 24.
  \end{cases}
\end{align*}
\]

The sphere packing problem is of course trivial for \( \mathbb{R}^1 \), where

\[
f(x) = \frac{1}{1 - x^2} \left( \frac{\sin \pi x}{\pi x} \right)^2
\]

gives an optimal function for use in Theorem 2.1. At first glance it may seem quite unlikely that Theorem 2.1 leads to a sharp sphere packing bound in any other dimension \( n > 1 \). For example, positivity arguments such as its proof (which involve dropping a number of terms to get an inequality) nearly always lose information; in analytic number theory it is essentially a given that they will not produce sharp results.
Despite this, there is ample numerical evidence that Conjecture 2.2 is true in the special dimensions \( n = 2, 8 \) (where it was proved in [V]), and 24. Similarly sharp solutions have been found for related problems in \( \mathbb{R}^2, \mathbb{R}^8, \) and \( \mathbb{R}^{24} \) such as the kissing problem (see [Lev, OS]), and there are many analogies with error-correcting codes (see, for example, [CZ1]).

The main purpose of this paper is to introduce explicit sequences which we conjecture converge to functions satisfying Conjecture 2.2. We will focus on \( n = 8 \) and 24, not only because these cases are more interesting, but also because they appear to be more similar to each other than either is to the \( n = 2 \) case.

3. Explicit functions

The conditions on \( f \) in Theorem 2.1 are radially symmetric, so any function satisfying them can be rotationally symmetrized. Thus, without loss of generality we will assume that \( f \) is a radial function, and we will sometimes write \( f(r) \) for the common value \( f(x) \) with \( |x| = r \). A convenient family of functions to consider are products of polynomials with Gaussians. If we write

\[
(3.1) \quad f(x) = p(|x|^2) e^{-\pi|x|^2}
\]

with \( p \) a polynomial, then a calculation shows

\[
\hat{f}(t) = (T p)(|t|^2) e^{-\pi|t|^2}
\]

for some polynomial \( T p \) depending on \( p \). In other words, \( T \) is the linear map given by

\[
(3.2) \quad (T p)(|t|^2) = e^{\pi|t|^2} \int_{\mathbb{R}^n} p(|x|^2) e^{-\pi|x|^2} e^{-2\pi i \langle x, t \rangle} \, dx,
\]

which one can check maps polynomials to polynomials.

The functions \( f \) used in [CE] are of the form (3.1). They are created by requiring that \( f(0) = \hat{f}(0) = 1 \) and also that \( f \) and \( \hat{f} \) must have forced single and double roots at certain locations. Together these can be interpreted as a set of linear conditions satisfied by the coefficients of the polynomial \( p \), which can be solved when the degree of \( p \) is appropriately large compared to the number of forced roots. Cohn and Elkies used a computer search to choose locations for these forced roots in order to optimize the sphere packing bound obtained from Theorem 2.1.

This procedure works well in practice, but it is difficult to analyze. It is not at all obvious that these successively optimized functions \( f \) (coming from polynomials of higher and higher degree) even converge to a locally optimal choice of \( f \), let alone the global optimum. The numerical evidence is compelling, but a proof is completely lacking.

In this paper, we examine a simpler variant of this approach. Instead of carefully optimizing the forced root locations, we specify them \textit{a priori}. Specifying the roots is worse in practice, but not much worse: for example, using 200 roots we will come within a factor of \( 1 + 1.23 \times 10^{-27} \) of the Leech lattice’s density, compared with \( 1 + 1.65 \times 10^{-30} \) in [CK3] using 200 carefully optimized roots. Because our functions are explicit and do not involve a computer search, they can be computed more quickly and may be easier to analyze.
Table 1. Vector lengths in optimal lattices normalized with $|\Lambda| = 1$.

| dimension | lattice    | vector lengths                                      |
|-----------|------------|-----------------------------------------------------|
| 1         | $\mathbb{Z}$ | $\{k : k \geq 0\}$                                 |
| 2         | hexagonal  | $(\frac{4}{3})^{1/4} \sqrt{k^2 + k\ell + \ell^2} : (k, \ell) \in \mathbb{Z}^2$ |
| 8         | $E_8$      | $\{\sqrt{2k} : k \geq 0\}$                        |
| 24        | Leech      | $\{\sqrt{2k} : k \geq 0, k \neq 1\}$              |

In order to describe where and why we force roots of $f$ and $\hat{f}$, it is helpful to recall the proof of Theorem 2.1. There we proved the inequality

$$f(0) \geq \sum_{x \in \Lambda} f(x) = \frac{1}{|\Lambda|} \sum_{t \in \Lambda^*} \hat{f}(t) \geq \frac{\hat{f}(0)}{|\Lambda|}$$

using the conditions that $f(x) \leq 0$ for $|x| \geq r$ and $\hat{f}(t) \geq 0$ for all $t$. If the lattice $\Lambda$ is actually the densest sphere packing in $\mathbb{R}^n$, and if this method proves a sharp bound, then both inequalities must actually be equalities. For that to happen, one must first have $|\Lambda| = 1$. (Recall that in the proof, we scaled $\Lambda$ so its minimal vector length is $r$.) For this scaling of $\Lambda$, the terms $f(x)$ and $\hat{f}(t)$ must vanish whenever $x \in \Lambda \neq 0$ and $t \in \Lambda^* \neq 0$. In other words,

1. $f$ must vanish at all nonzero vector lengths, and
2. $\hat{f}$ must vanish at all nonzero dual vector lengths.

In order to preserve the sign constraints (2) and (3) from Theorem 2.1, the order of vanishing at every vector length must be even, with the exception of $f(x)$ at $|x| = r$, where a sign change should in fact occur.

Note that even if one did not assume that $f$ is radial, it would still vanish on concentric spheres through the lattice points, not simply at the individual lattice points. This is because the above argument applies not only to $\Lambda$, but to any rotation of it; consequently, $f$ must vanish at each rotated lattice point.

Table 1 lists the lengths of nonzero vectors in the optimal lattices in dimensions 1, 2, 8, and 24 (scaled so that $|\Lambda| = 1$, which is the usual scaling except in $\mathbb{R}^2$); these lattices are undoubtedly the densest sphere packings in their respective dimensions, but of course this has not been proved in 8 or 24 dimensions. For each of these lattices, the dual vector lengths are the same as the vector lengths: in each case except dimension 2, $\Lambda^* = \Lambda$, and in dimension 2, $\Lambda^*$ is a rotation of $\Lambda$.

One naive approach to constructing optimal functions would be to force roots at exactly these locations. Specifically, let $r_1 < r_2 < \ldots$ be the nonzero vector lengths in the last column of Table 1. (In other words, $r_1 = \sqrt{2}$ if $n = 8$ and $r_1 = 2$ if $n = 24$, etc.) For any integer $k \geq 1$ we define the function $f_k(x)$ to be of the form (3.1), with $p(x) = q_k(x)$ a polynomial of degree $4k - 1$, subject to the following 4k constraints:

$$f_k(0) = 1,$$
$$f_k(x) \text{ vanishes to order 1 at } |x| = r_1,$$
$$f_k(x) \text{ vanishes to order } 2 \text{ at } |x| = r_2, \ldots, r_k, \text{ and}$$
$$\hat{f}_k(x) \text{ vanishes to order } 2 \text{ at } |x| = r_1, \ldots, r_k.$$
Such a function is designed to satisfy the requirements of (3.3) and thereby be used in Theorem 2.1. However condition (1) of the theorem has not been addressed; i.e., we have not forced $\hat{f}_k(0) = 1$ as well. This condition in fact holds automatically for the limit $f$ of the functions $f_k$, provided it exists; the reason is that $f$ and $\hat{f}$ vanish at all non-zero lattice points, and Poisson summation over the lattice $\Lambda$ implies $\hat{f}(0) = f(0) = 1$. If one wishes to use the functions $f_k$ themselves to prove sphere packing bounds, then one must rescale them to force condition (1) to hold. This rescaling changes the bound to

$$\frac{\pi^{n/2}}{(n/2)!} \left( \frac{r_1}{2} \right)^n \frac{f_k(0)}{\hat{f}_k(0)}$$

(i.e., it introduces a factor of $f_k(0)/\hat{f}_k(0)$).

Unfortunately, this sequence of functions fails, at first subtly and then dramatically: the functions do not converge as $k \to \infty$, and for sufficiently large $k$ they do not even prove packing bounds at all (because they develop unwanted sign changes). See Section 4 for a discussion of the numerical evidence.

Instead of using the exact vector lengths in the definition of $f_k$, we modify them as follows. Let $\ell_m$ denote the actual $m$-th vector length. Given $k$, we define modified root locations $r_1, \ldots, r_k$ (depending on $k$) as follows:

$$r_m = \begin{cases} 
\ell_m & \text{if } m < \lfloor 2k/3 \rfloor, \\
\sqrt{\ell_m^2 + \frac{1}{4} \ell_k^2 \left( \frac{m-2k/3}{k-2k/3} \right)^2} & \text{if } \lfloor 2k/3 \rfloor \leq m \leq k.
\end{cases}$$

In other words, the first two-thirds of the root locations are left unchanged, while the squares of the others are perturbed by a quadratically growing amount culminating in making the final one 25% larger. The numbers $2/3$ and $1/4$ in (3.5) are somewhat arbitrary, but these choices appear to work well in practice. The rescaling (3.5) was motivated by the empirical location of the roots of the optimized functions of particular degrees mentioned earlier, as well as the similar spacing of large roots of orthogonal polynomials (see [D]).

We can now use these modified root locations to define functions $f_k$. Unlike the naive definition using $\ell_m$, the improved definition using $r_m$ appears to work well. In Section 4 we will examine numerical evidence and make conjectures, but before that we must resolve one theoretical issue: it is not obvious that the functions $f_k$ even exist, because the linear equations defining them may have no solution. In fact, if the forced root locations $r_1, r_2, \ldots$ were chosen differently, then this difficulty could occur. For example, for $n = 1$, $k = 2$, $r_1 = 1$, and $r_2 = 1.3403207576 \ldots$ (chosen to satisfy a certain polynomial equation with coefficients in $\mathbb{Q}[\pi]$), the constraints (3.4) defining $f_2$ have no solution. Fortunately, existence and uniqueness do hold in our cases:

**Lemma 3.1.** For any algebraic numbers $0 < r_1 < \cdots < r_k$, there exists a unique polynomial $q_k$ of degree $4k - 1$ such that the constraints (3.4) hold for $f_k(x) = q_k(|x|^2)e^{-\pi|x|^2}$.

For the proof of this lemma, we will need to diagonalize the transform $T$ defined in (3.2). Define $p_j(x) = L_j^{n/2-1}(2\pi x)$, where $L_j^\alpha$ is the Laguerre polynomial of degree $j$ and index $\alpha = n/2 - 1$. Recall that the polynomials $L_j^\alpha$ are orthogonal...
polynomials with respect to the measure $x^{-\alpha} e^{-x} \, dx$ on $[0, \infty)$, which can be written as

$$L_\alpha^j(x) = \frac{x^{-\alpha} e^x \, dj}{j!} (x^{\alpha+j} e^{-x}).$$

The product $p_j(|x|^2) e^{-\pi |x|^2}$ is a radial eigenfunction of the Fourier transform (2.1) with eigenvalue $(-1)^j$. In other words,

$$T p_j = (-1)^j p_j.$$

Writing an arbitrary polynomial as a linear combination of the polynomials $p_j$ makes it easy to apply $T$.

Proof. Write the polynomial $q_k$ as a linear combination

$$q_k = \sum_{j=0}^{4k-1} c_j p_j.$$

The constraints (3.4) amount to the following linear equations in $c_0, \ldots, c_{4k-1}$:

$$
\sum_{j=0}^{4k-1} c_j p_j(0) = 1
$$

$$
\sum_{j=0}^{4k-1} c_j p_j(2\pi r_m^2) = 0 \quad \text{for } 1 \leq m \leq k
$$

$$
\sum_{j=0}^{4k-1} c_j p_j'(2\pi r_m^2) = 0 \quad \text{for } 2 \leq m \leq k
$$

$$(3.6)
\sum_{j=0}^{4k-1} (-1)^j c_j p_j(2\pi r_m^2) = 0 \quad \text{for } 1 \leq m \leq k
$$

$$
\sum_{j=0}^{4k-1} (-1)^j c_j p_j'(2\pi r_m^2) = 0 \quad \text{for } 1 \leq m \leq k.
$$

To prove the lemma, we need only show that the determinant of the $4k \times 4k$ matrix of coefficients is nonzero. View the coefficients as polynomials in $\pi$ (recall that $p_j(x) = L_j^{n/2-1}(2\pi x)$, where $L_j^{n/2-1}$ has coefficients in $\mathbb{Q}$, and that the forced root locations $r_1, \ldots, r_k$ are algebraic). We will use the transcendence of $\pi$ to prove that the determinant is nonzero, by identifying its leading coefficient as a polynomial in $\pi$ and showing that it does not vanish.

Each column of the matrix corresponds to $p_j$ for some $j$, with entries of the form $p_j(0)$, $p_j(2\pi r_m^2)$, $p_j'(2\pi r_m^2)$, $(-1)^j p_j(2\pi r_m^2)$, and $(-1)^j p_j'(2\pi r_m^2)$ for suitable values of $m$. If we write $p_j$ as a linear combination of monomials, then we can expand the determinant as a corresponding linear combination, with the highest power of $\pi$ coming from the monomial $x^j$ of highest degree. Thus, if we can show that the determinant is nonzero after replacing $p_j(x)$ with $x^j$ for all $j$, then it must have been nonzero to start with.

This replacement dramatically simplifies the equations, because we can reinterpret them as describing a more tractable interpolation problem. The new equations ask for the coefficients of a polynomial of degree $4k - 1$ with the following constraints. Its value at 0 is specified, its value at $2\pi r_1^2$ is specified, its values and first derivatives at
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2πr^2, ..., 2πr^2_k are specified, and its values and first derivatives at −2πr^2, ..., −2πr^2_k are specified. For the negative cases, note that replacing p_j(x) with x^j transforms (−1)^j p_j(x) into (−x)^j. This interpolation problem is a special case of Hermite interpolation, and the determinant of the coefficient matrix is therefore nonzero. (See Subsection 2.1 of [CK2] for a review of Hermite interpolation.)

It follows that the coefficient matrix of the original equations also has a nonzero determinant, so there exists a unique solution. □

4. Numerical evidence

In this section we will examine the numerical evidence for convergence. Our calculations are based on floating-point arithmetic, with no rigorous bounds on the rounding error, but we believe all reported digits are correct. (We believe that these calculations could be made rigorous if necessary, for example by using interval arithmetic or the techniques from Appendix A in [CK3].) When using k forced root locations, we carried out all computations to 8k + 75 digits of precision using PARI/GP. Experimentation suggests that 8k + 75 digits is far more precision than is actually needed, but it is easier to pick an unnecessarily high bound than to calibrate how little precision we could get away with.

First, consider the naive approach discussed in the previous section, in which one takes the forced root locations r_1, ..., r_k to be the first k nonzero vector lengths in the optimal lattice. Though at first this approach gives good bounds, it subtly reverses course and eventually fails completely for large n (see Table 2). In the \( \mathbb{R}^8 \) case, the bound improves as k grows until k = 40, at which point it is slightly better than the bound proved in [CE] (and much better than the previous record bound of \( \approx 1.012 \)). However, after k = 40 the bound steadily gets worse. By k = 130, the bound would be less than 1, which is impossible and indicates that the function must have developed an unwanted sign change by that point. In the \( \mathbb{R}^{24} \) case, the problems are even more dramatic.

This failure demonstrates the difficulty of making predictions based on limited numerical data. If one looked at only the data for k ≤ 40 and n = 8, one might reasonably conjecture that the bound was converging to 1 (although a sophisticated analysis would indicate that the convergence was happening uncomfortably slowly as k neared 40).

This effect is reminiscent of Runge’s phenomenon from interpolation theory (see [Ep]). Although the problem is not literally overconstrained, forcing too many roots at the limiting locations constrains the function so much that it develops undesired oscillations to compensate. Pushing the larger roots towards infinity seemingly relaxes the constraints, dampens the oscillations, and allows convergence.

We have been unable to analyze the asymptotic behavior of the functions \( f_k \) defined using the roots (3.5), but they lead to excellent bounds (see Table 3) and appear to converge rapidly. In what follows, we refer to \( f_k \) and \( \hat{f}_k \), as well as their hypothetical limits as \( k \to \infty \), as analytic functions of a radial variable.

**Conjecture 4.1.** As \( k \to \infty \), \( f_k \) converges to a function \( f \) and \( \hat{f}_k \) converges to \( \hat{f} \), on some neighborhood of the real line in \( \mathbb{C} \). The convergence is uniform on compact subsets of this neighborhood.

The evidence for uniform convergence is of course not as strong as that for convergence, but it implies that \( f \) and \( \hat{f} \) are analytic and thus rapidly decreasing
Table 2. Supposed upper bounds for packing density using the exact vector lengths as forced root locations, without checking for unwanted sign changes. Bounds are expressed as a multiple of the density of the optimal lattice.

| $k$ | naive packing bound in $\mathbb{R}^8$ | naive packing bound in $\mathbb{R}^{24}$ |
|-----|---------------------------------|---------------------------------|
| 10  | 1.0001507518\ldots           | 1.3706005433\ldots           |
| 20  | 1.000052091\ldots           | 1.1082380574\ldots           |
| 30  | 1.000013138\ldots           | 1.1109658270\ldots           |
| 40  | 1.000009656\ldots           | 1.2417952436\ldots           |
| 50  | 1.0000014330\ldots          | 2.1249579472\ldots           |
| 60  | 1.0000035296\ldots          | -3.7219923464\ldots          |
| 70  | 1.000012844\ldots           |                                 |
| 80  | 1.0000634933\ldots          |                                 |
| 90  | 1.0004126231\ldots          |                                 |
| 100 | 1.0031219206\ldots          |                                 |
| 110 | 1.0256918168\ldots          |                                 |
| 120 | 1.557203878\ldots           |                                 |
| 130 | 0.9163797290\ldots          |                                 |

Table 3. Upper bounds for packing density using the modified vector lengths as forced root locations. Bounds are expressed as a multiple of the density of the optimal lattice. Note the contrast with Table 2.

| $k$ | packing bound in $\mathbb{R}^8$ | packing bound in $\mathbb{R}^{24}$ |
|-----|---------------------------------|---------------------------------|
| 25  | $1 + 2.013636284513588\ldots \times 10^{-10}$ | $1 + 1.276838479911905\ldots \times 10^{-6}$ |
| 50  | $1 + 5.35689304673532\ldots \times 10^{-16}$ | $1 + 4.112485306793651\ldots \times 10^{-11}$ |
| 75  | $1 + 2.8432795834257\ldots \times 10^{-20}$ | $1 + 1.034793038360603\ldots \times 10^{-14}$ |
| 100 | $1 + 6.131875484794015\ldots \times 10^{-24}$ | $1 + 6.03683281483083\ldots \times 10^{-18}$ |
| 200 | $1 + 7.957229644125821\ldots \times 10^{-35}$ | $1 + 1.224810072437178\ldots \times 10^{-27}$ |
| 300 | $1 + 8.043925729944741\ldots \times 10^{-43}$ | $1 + 6.139675825632854\ldots \times 10^{-35}$ |
| 400 | $1 + 1.554622153413999\ldots \times 10^{-49}$ | $1 + 3.603562346839\ldots \times 10^{-41}$ |
| 500 | $1 + 4.477920519243749\ldots \times 10^{-55}$ | $1 + 2.5113482848921\ldots \times 10^{-46}$ |
| 600 | $1 + 6.319153710652842\ldots \times 10^{-60}$ | $1 + 7.27698908362016\ldots \times 10^{-51}$ |

(because their Fourier transforms are analytic and hence smooth). It follows that they are admissible.

As evidence for Conjecture 4.1, we offer Figure 1, which demonstrates steady convergence as $k$ increases from 30 to 100, at a selection of sample points with real parts up to 5 and imaginary parts up to 0.2. In fact, convergence seems to hold even for somewhat larger imaginary parts; for example, Table 4 shows the values at $i/2$. However, convergence does not occur when the imaginary part is 1 or more; for example, for $n = 8$ we have

$$f_{500}(i) = 1786219116279967.87\ldots$$
Figure 1. Number $N$ of digits to which the values of $f_k$ and $\hat{f}_k$ agree with those of $f_{k-5}$ and $\hat{f}_{k-5}$ at all of the points $x/10 + (y/10)i$, for integers $0 \leq x \leq 50$ and $0 \leq y \leq 2$. Data points for $n = 8$ are gray and those for $n = 24$ are black.

Table 4. Values of $f_k(i/2)$ and $\hat{f}_k(i/2)$.

| $n$ | $k$ | $f_k(i/2)$ | $\hat{f}_k(i/2)$ |
|-----|-----|------------|------------------|
| 8   | 100 | 0.939432541969057457603843... | 0.526774741363446491086599... |
| 8   | 200 | 0.939432541959478373173290... | 0.526774741373025575517211... |
| 8   | 300 | 0.939432541959477686529550... | 0.526774741373026262160950... |
| 8   | 400 | 0.939432541959477685343726... | 0.526774741373026263346775... |
| 8   | 500 | 0.939432541959477685338723... | 0.52677474137302633351777...  |
| 8   | 600 | 0.939432541959477685338602... | 0.52677474137302633351898...  |
| 24  | 100 | 0.909504018094605062955468... | 0.543934528596990605074180... |
| 24  | 200 | 0.909504017149389039571803... | 0.543934529542206632888889... |
| 24  | 300 | 0.909504017149302144551677... | 0.543934529542293527909015... |
| 24  | 400 | 0.909504017149301977449339... | 0.543934529542293696501135... |
| 24  | 500 | 0.909504017149301976704937... | 0.543934529542293695575575... |
| 24  | 600 | 0.909504017149301976686050... | 0.543934529542293695774641... |

while

$$f_{600}(i) = 474994401497433517.69...$$

Conjecture 4.2. The limiting functions $f$ and $\hat{f}$ from Conjecture 4.1 have no real roots other than the forced roots.

When $n = 24$, $\hat{f}_2$ has another real root, but we have found no other case in which $f_k$ or $\hat{f}_k$ has any non-forced real roots. If Conjecture 4.1 holds and there is a neighborhood of the real line into which the complex roots never intrude, then that is enough to imply Conjecture 4.2. However, it is unclear whether this stronger hypothesis is true. As one can see from the data in Table 5, the complex roots are growing steadily closer to the real axis, and they might reach it around $k = 1400$. Even if they eventually reach the axis, we conjecture that any unwanted sign changes will occur far from the origin and will disappear in the limit as $k \to \infty$. It is plausible that one could remove them entirely by modifying (3.5).
Table 5. The minimal distance between the complex roots of \( f_k \) or \( \hat{f}_k \) and the real axis.

|   |   | minimal imaginary part for \( f_k \) | minimal imaginary part for \( \hat{f}_k \) |
|---|---|-------------------------------------|-------------------------------------|
| 8 | 100 | 0.62170638623230323... | 0.6217063862269778... |
| 8 | 200 | 0.5288857517088769... | 0.5288857517088764... |
| 8 | 300 | 0.4616044778908506... | 0.4616044778908506... |
| 8 | 400 | 0.4160654620013710... | 0.4160654620013710... |
| 8 | 500 | 0.367852942190859... | 0.367852942190859... |
| 8 | 600 | 0.3248411054701392... | 0.3248411054701392... |
| 24 | 100 | 0.623613221273594... | 0.6236132212943291... |
| 24 | 200 | 0.5282754706164285... | 0.5282754706164232... |
| 24 | 300 | 0.4605618680853859... | 0.4605618680853859... |
| 24 | 400 | 0.4148144356278994... | 0.4148144356278994... |
| 24 | 500 | 0.3664964681747482... | 0.3664964681747482... |
| 24 | 600 | 0.3234610116479302... | 0.3234610116479302... |

The complex root locations have several mysterious properties. See Figure 2 for plots with \( k = 600 \) and \( n = 8 \), and Figure 3 for plots with \( n = 24 \) (which are very similar to the \( n = 8 \) case). The roots lie on several clear curves, and they are most likely accumulating on the boundary of the domain of holomorphy. Note that their nearest approach to the real axis is quite far from the origin, as we asserted above.

One surprising observation is that \( f_k \) and \( \hat{f}_k \) have nearly the same roots away from the origin. In the third part of these two figures, we show the roots of one of \( f_k \) or \( \hat{f}_k \) that do not agree to six decimal places with any root of the other. Only the roots relatively near the origin appear in these plots. See also Table 5, in which the \( f_k \) and \( \hat{f}_k \) columns become nearly identical as \( k \) grows.

For comparison, Table 6 shows the nearest roots to the origin. In each case, \( \hat{f}_k \) has a purely imaginary root that is probably converging as \( k \to \infty \) (the numbers show clear convergence when \( n = 8 \) and possible convergence when \( n = 24 \)). The other roots are roughly paired up for \( f_k \) and \( \hat{f}_k \), but these pairs are not nearly as close to each other as those further from the origin. We see no reason to think any of the non-real roots are converging except for the purely imaginary roots.

It is clear from this data that the roots have considerable structure, which we are unable to explain conceptually. More data could help, but calculations for large \( k \) are very time consuming. We have computed \( f_k \) and \( \hat{f}_k \) for \( k = 700, 800, \) and \( 900 \), but we have not located their roots. If they have no unexpected sign changes, then with \( k = 900 \) we get sphere packing bounds within a factor of \( 1 + 5.33 \times 10^{-72} \) of the density of \( E_8 \) or \( 1 + 3.04 \times 10^{-62} \) of that of the Leech lattice. We expect that these bounds are true and could be proved given enough computing power, but the evidence is not as conclusive as it is in the cases for which we have located the roots.

It follows from Conjecture 4.2 that \( f \) and \( \hat{f} \) have no unexpected sign changes. Thus, Conjectures 4.1 and 4.2 for \( n = 8 \) or 24 would solve the sphere packing problem in \( \mathbb{R}^n \).

It is interesting to note that the parameter \( n \) in these conjectures can be varied, while leaving the forced root locations fixed. Of course there is no connection with
Figure 2. The non-real roots of $f_{600}$ (above) and $\hat{f}_{600}$ (middle) in the right half-plane, for $n = 8$. The lower graph shows the points in either of the previous two graphs for which no point in the other agrees to within $10^{-6}$ in real and imaginary parts.

Table 6. The leftmost roots of $f_k$ and $\hat{f}_k$ in the right half-plane. Each of the four parts of the table corresponds to the values of $n$ and $k$ specified in square brackets.

| roots of $f_k$ | roots of $\hat{f}_k$ |
|---------------|---------------------|
| $[n = 8, k = 500]$ | $\pm 0.6817374606\ldots i$ |
| 0.01864424055\ldots $\pm$ 0.7968630734\ldots $i$ | 0.01969453528\ldots $\pm$ 0.7922341309\ldots $i$ |
| 0.05589098197\ldots $\pm$ 0.7966856319\ldots $i$ | 0.05879879542\ldots $\pm$ 0.792798039\ldots $i$ |
| $[n = 8, k = 600]$ | $\pm 0.6817374605\ldots i$ |
| 0.01690822801\ldots $\pm$ 0.7854640149\ldots $i$ | 0.01788437319\ldots $\pm$ 0.7806599882\ldots $i$ |
| 0.05069999170\ldots $\pm$ 0.7853472831\ldots $i$ | 0.05339644678\ldots $\pm$ 0.7812202661\ldots $i$ |
| $[n = 24, k = 500]$ | $\pm 0.7236064057\ldots i$ |
| 0.01960883579\ldots $\pm$ 0.7855031742\ldots $i$ | 0.0222383238\ldots $\pm$ 0.7756885639\ldots $i$ |
| 0.0587932631\ldots $\pm$ 0.7854671622\ldots $i$ | 0.06451464105\ldots $\pm$ 0.7782266879\ldots $i$ |
| $[n = 24, k = 600]$ | $\pm 0.7235866774\ldots i$ |
| 0.01772546664\ldots $\pm$ 0.774627816\ldots $i$ | 0.02044183069\ldots $\pm$ 0.7644882404\ldots $i$ |
| 0.05314786265\ldots $\pm$ 0.7746157584\ldots $i$ | 0.05868760933\ldots $\pm$ 0.7671872439\ldots $i$ |
sphere packing for general $n$ (it does not even have to be an integer). If a limiting $f$ exists, it also does not follow in general that $f(0) = \hat{f}(0)$, since that requires Poisson summation over an appropriate lattice. However, the analogues of Conjectures 4.1 and 4.2 do seem to hold in all small dimensions (although we have not investigated them as carefully as the $n = 8$ and $n = 24$ cases). In particular, we conjecture that if $f_k$ is defined with forced roots based on the $E_8$ vector lengths, then these conjectures hold for $0 < n < 10$ (for $n = 10$ there in fact appear to be extraneous real roots). This flexibility is encouraging, because it suggests that a proof need not depend on specific facts about $\mathbb{R}^8$, but rather could hold for much more general reasons. Similarly, for the Leech lattice vector lengths the conjectures seem to hold for $0 < n < 26$. More generally, many of the phenomena we study in this paper are not restricted to $n = 8$ and $n = 24$. For example, we make the following conjecture:

**Conjecture 4.3.** For $0 < n < 10$, forcing roots at the $E_8$ vector lengths yields a limiting function $f$ satisfying

\[
\frac{f(0)}{\hat{f}(0)} = -\frac{n^4 - 56n^3 + 1184n^2 - 11200n + 40320}{16(n - 10)(n - 14)(n - 18)}.
\]

For $0 < n < 26$, using the Leech lattice vector lengths yields instead

\[
\frac{f(0)}{\hat{f}(0)} = -\frac{p_{24}(n)}{32(n - 26)(n - 34)(n - 38)(n - 42)(n^3 - 116n^2 + 4480n - 57024)}.
\]
where
\[ p_{24}(n) = n^8 - 284n^7 + 35312n^6 - 2510720n^5 + 111652352n^4 - 3180064256n^3 + 56651266048n^2 - 577142292480n + 257449947952. \]

This conjecture is evidence that the limiting functions have even more intricate structure than is apparent just from the \( n = 8 \) and \( n = 24 \) cases.

Note that for reasons of computational efficiency, one should never solve the equations (3.6) directly. Instead, it is more convenient to solve two systems, each of half the size. To form them, we write the polynomial \( q_k \) from the definition \( q_k(|x|^2)e^{-\pi|x|^2} \) of \( f_k(x) \) as the sum \( q_k^0 + q_k^1 \), where
\[ q_k^\varepsilon = \frac{q_k + (-1)^\varepsilon Tq_k}{2} \]
for \( \varepsilon \in \{0,1\} \). Then
\[ Tq_k^\varepsilon = (-1)^\varepsilon q_k. \]

(In other words, we have diagonalized the Fourier transform.)
We can express \( q_k^0 \) as a linear combination of the rescaled Laguerre polynomials \( p_j \) with \( j \) even, and \( q_k^1 \) as a linear combination with \( j \) odd. The constraints on \( f_k \) and \( \hat{f}_k \) amount to the following individual constraints on \( q_k^\varepsilon \):
\[ q_k^\varepsilon \text{ vanishes to order 1 at } r_1^2 \text{ and order 2 at } r_2^2, r_3^2, \ldots, r_k^2. \]
The only missing constraint is that \( \hat{f}_k \) must have a double root at \( r_1 \) ((4.1) forces only a single root). The issue is that given only the constraints above, \( q_k^0 \) and \( q_k^1 \) are only determined up to scaling, and may be scaled independently; to produce the double root the scalings must be compatible.
The following determinant gives a formula for \( q_k^\varepsilon(x) \), up to scaling (it follows using the approach of Lemma 3.1 that this determinant is not identically zero):
\[
\begin{vmatrix}
    p_e(x) & p_{2+\varepsilon}(x) & p_{4+\varepsilon}(x) & \cdots & p_{4k-2+\varepsilon}(x) \\
    p_e(r_1^2) & p_{2+\varepsilon}(r_1^2) & p_{4+\varepsilon}(r_1^2) & \cdots & p_{4k-2+\varepsilon}(r_1^2) \\
    p_e(r_2^2) & p_{2+\varepsilon}(r_2^2) & p_{4+\varepsilon}(r_2^2) & \cdots & p_{4k-2+\varepsilon}(r_2^2) \\
    p_e(r_3^2) & p_{2+\varepsilon}(r_3^2) & p_{4+\varepsilon}(r_3^2) & \cdots & p_{4k-2+\varepsilon}(r_3^2) \\
    \vdots & & & & \\
    p_e(r_k^2) & p_{2+\varepsilon}(r_k^2) & p_{4+\varepsilon}(r_k^2) & \cdots & p_{4k-2+\varepsilon}(r_k^2) \\
    p_e'(r_1^2) & p_{2+\varepsilon}'(r_1^2) & p_{4+\varepsilon}'(r_1^2) & \cdots & p_{4k-2+\varepsilon}'(r_1^2) \\
    p_e'(r_2^2) & p_{2+\varepsilon}'(r_2^2) & p_{4+\varepsilon}'(r_2^2) & \cdots & p_{4k-2+\varepsilon}'(r_2^2) \\
    p_e'(r_3^2) & p_{2+\varepsilon}'(r_3^2) & p_{4+\varepsilon}'(r_3^2) & \cdots & p_{4k-2+\varepsilon}'(r_3^2) \\
    \vdots & & & & \\
    p_e'(r_k^2) & p_{2+\varepsilon}'(r_k^2) & p_{4+\varepsilon}'(r_k^2) & \cdots & p_{4k-2+\varepsilon}'(r_k^2) 
\end{vmatrix}
\]
It is tempting to take the limit as \( k \to \infty \) and hope to write down an infinite determinant for the limiting function. However, we see no way to make sense of this idea.

Computing \( q_k^0 \) and \( q_k^1 \) independently is substantially faster than computing \( q_k \) (approximately four times faster using a cubic-time algorithm). So far, it has not led to any theoretical advances, but in Section 7 we will see a closely related example in which it is theoretically important to separate the Fourier eigenfunctions.
5. Rationality

Although we are unable to identify the proposed limiting functions \( f \) for dimensions 8 and 24, we can say two things about their special values. In fact, the analysis we provide applies to the functions in the statement of Conjecture 2.2, and in particular the function explicitly exhibited in \([V]\) for the \( n = 8 \) case. The first is a property we can derive, while the second has been observed only numerically and so far lacks an explanation.

The first observation is that we can predict the value of \( f'(r_1) \), where \( r_1 \) is the first forced root. Here we view \( f \) as a function of a single radial variable, so \( f' \) is the radial derivative. By condition (3.4), knowing \( f'(r_1) \) means we know the values of both \( f \) and \( f' \) at every vector length in the respective lattices (\( E_8 \) and Leech) for dimensions 8 and 24.

**Lemma 5.1.** Let \( n \in \{2, 8, 24\} \), and let \( f \) be a hypothetical optimal function for use in Theorem 2.1, as in Conjecture 2.2. Then

\[
\left(5.1\right) \quad f'(r_1) = -\frac{n}{Nr_1} \widehat{f}(0),
\]

where

\[
r_1 = \text{the minimal vector length} = \begin{cases} 
\left(\frac{4}{3}\right)^{1/4} & \text{for } n = 2, \\
\sqrt{2} & \text{for } n = 8, \text{ and} \\
2 & \text{for } n = 24
\end{cases}
\]

and

\[
N = \text{the number of minimal vectors} = \begin{cases} 
6 & \text{for } n = 2, \\
240 & \text{for } n = 8, \text{ and} \\
196560 & \text{for } n = 24.
\end{cases}
\]

Note that without loss of generality, we assume that \( f \) is radial.

**Proof.** Define rescaled functions for \( r > 0 \) by

\[
f_r(x) = f(rx) \quad \text{and} \quad \widehat{f}_r(t) = r^{-n} \widehat{f}\left(\frac{t}{r}\right).
\]

Let

\[
F(x) = \frac{d}{dr}\bigg|_{r=1} f_r(x) = |x| f'(x),
\]

so that

\[
\widehat{F}(t) = -n \widehat{f}(t) - i \widehat{f}'(t).
\]

Now apply Poisson summation to \( F \) over optimal lattice \( \Lambda \). Removing terms where \( F \) or \( \widehat{F} \) is forced to vanish, this identity states

\[
\sum_{x \in \Lambda, |x|=r_1} |x| f'(x) = -n \widehat{f}(0),
\]

which is (5.1). \( \square \)

The second—and perhaps more interesting—feature we have noticed is that the Taylor series for \( f \) and \( \widehat{f} \), normalized so that \( f(0) = \widehat{f}(0) = 1 \), have rational quadratic coefficients. Table 7 shows numerical evidence for this. It displays the second and fourth Taylor coefficients for \( f \) and \( \widehat{f} \) in dimensions 8 and 24. (We cannot be certain that all the reported digits are correct for the limiting functions, but they
Table 7. Approximate Taylor series coefficients of \( f \) and \( \hat{f} \) about 0.

| \( n \) | function | order | coefficient | conjecture |
|---|---|---|---|---|
| 8 | \( f \) | 2 | \(-2.70000000000000000000000000000000\ldots\) | \(-27/10\) |
| 8 | \( \hat{f} \) | 2 | \(-1.50000000000000000000000000000000\ldots\) | \(-3/2\) |
| 24 | \( f \) | 2 | \(-2.6276556776556776556776556776\ldots\) | \(-14347/5460\) |
| 24 | \( \hat{f} \) | 2 | \(-1.3141025641025641025641025641\ldots\) | \(-205/156\) |
| 8 | \( f \) | 4 | \(4.216750124096829821099896562\ldots\) | ? |
| 8 | \( \hat{f} \) | 4 | \(-1.239796907029598002622059689\ldots\) | ? |
| 24 | \( f \) | 4 | \(3.8619903167183007758184168473\ldots\) | ? |
| 24 | \( \hat{f} \) | 4 | \(-0.73767277890153223037995397\ldots\) | ? |

agree for \( k = 300 \) and \( k = 600 \). One can see from the decimal expansions that the quadratic coefficients are rational, but the quartic coefficients remain mysterious.

**Conjecture 5.2.** For \( n = 8 \), the limiting functions \( f \) and \( \hat{f} \) have quadratic Taylor coefficients \(-27/10\) and \(-3/2\), respectively (when normalized so that \( f(0) = \hat{f}(0) = 1 \)). For \( n = 24 \), the corresponding coefficients are \(-14347/5460\) and \(-205/156\).

The same is true when \( n = 8 \) for the functions studied in [V]. We do not know whether the higher Taylor coefficients are rational or even given by simple expressions at all. Needless to say, it would be interesting to have explicit formulas for the general coefficients, because this would give a direct construction of \( f \) and \( \hat{f} \) by power series and analytic continuation.

To put this conjecture in a slightly broader context, consider the Mellin transform

\[
M_f(s) = \int_0^\infty f(x)x^{s-1} \, dx.
\]

When \( f \) is smooth and rapidly decreasing (as Conjecture 4.1 implies), the integral converges to a holomorphic function for \( \Re s > 0 \). It is a standard fact that \( M_f(s) \) can be meromorphically continued to \( \mathbb{C} \), with at most simple poles at \( s \in \mathbb{Z}_{\leq 0} \); furthermore, for integers \( j \geq 0 \) its residue at \( s = -j \) is the \( j \)-th Taylor coefficient of \( f \). To see why, note that if \( f(x) \) has the Taylor series expansion \( \sum_{j \geq 0} c_j x^j \) about \( x = 0 \), then

\[
M_f(s) = \int_0^1 \left( f(x) - \sum_{j=0}^{\ell} c_j x^j \right) x^{s-1} \, dx + \sum_{j=0}^{\ell} \frac{c_j}{s+j} + \int_1^\infty f(x)x^{s-1} \, dx,
\]

where both integrals converge as long as \( \Re s > -(\ell + 1) \). Since our function \( f \) is radial, its Taylor coefficients \( c_j \) vanish if \( j \) is odd.

A short calculation (see [LL, Theorem 5.9]) shows that if \( \hat{f} \) is the \( n \)-dimensional Fourier transform of \( f \) (interpreted as a radial function), then

\[
M_{\hat{f}}(s) = \frac{\pi^{n/2-s} \Gamma(s/2)}{\Gamma((n-s)/2)} M_f(n-s),
\]
valid as an identity of meromorphic functions on $\mathbb{C}$. In particular, computing the residue of $M_f(s)$ at $s = -j$ shows that the $j$-th Taylor coefficient of $f$ equals

$$(-1)^{j/2} \frac{2\pi^{j+n/2}}{\Gamma(j/2 + 1)\Gamma((n+j)/2)} M_f(n+j),$$

and vice versa with $f$ and $\hat{f}$ switched.

Thus, in $n$ dimensions Conjecture 5.2 amounts to specifying $M_f(n+2)$ and $M_{\hat{f}}(n+2)$. The values $M_f(n)$ and $M_{\hat{f}}(n)$ are easy consequences of $f(0) = \hat{f}(0) = 1$. We have identified one other value, namely the midpoint 4 of the $s \leftrightarrow n-s$ symmetry when $n = 8$:

**Conjecture 5.3.** For $n = 8$, the limiting functions satisfy

$$M_f(4) = M_{\hat{f}}(4) = \frac{1}{15}$$

when normalized with $f(0) = \hat{f}(0) = 1$.

The equality $M_f(4) = M_{\hat{f}}(4)$ follows from (5.2), but not the value $1/15$. It is natural to expect a corresponding conjecture for $n = 24$, but we have been unable to identify the numerical value $M_f(12) = M_{\hat{f}}(12) = 0.177860964729650276645646126241\ldots$ in that case.

6. Energy minimization

One natural generalization of sphere packing is potential energy minimization. Given a radial potential function $\varphi: \mathbb{R}^n \to \mathbb{R}$ and a set $\mathcal{P}$ of point particles, the energy $E_\varphi(\mathcal{P}, x)$ of a particle $x \in \mathcal{P}$ is defined to be

$$\sum_{y \in \mathcal{P}, y \neq x} \varphi(x-y),$$

and the energy $E_\varphi(\mathcal{P})$ is defined as the average of $E_\varphi(\mathcal{P}, x)$ over all $x \in \mathcal{P}$. (Of course some hypotheses are needed for this to make sense, but it is well defined when $\mathcal{P}$ is a periodic discrete set and $\varphi$ is rapidly decreasing.) The question of how to choose $\mathcal{P}$ so as to minimize energy with a fixed density, arises naturally in physics; see [C2] for a survey.

Cohn and Kumar [CK2] defined a configuration $\mathcal{P}$ to be universally optimal if it minimizes energy whenever $\varphi(x)$ is completely monotonic as a function of $|x|^2$ and decreases sufficiently quickly. For example, $\varphi$ could be a sufficiently steep inverse power law. As explained in Section 9 of [CK2], it suffices to check optimality for the Gaussians $\varphi(x) = e^{-c|x|^2}$ with $c > 0$, i.e., the Gaussian core model [S] from mathematical physics. Cohn and Kumar conjectured that the hexagonal lattice, $E_8$, and the Leech lattice are universally optimal. (See [CKS] for information about ground states in other dimensions.)

Proposition 9.3 of [CK2] offers an approach to proving this conjecture by linear programming bounds, which Cohn and Kumar conjectured were sharp in these special dimensions (much like the case of sphere packing). Given an admissible auxiliary
function $h: \mathbb{R}^n \to \mathbb{R}$ satisfying $h \leq \varphi$ and $\hat{h} \geq 0$ everywhere, this proposition says that every configuration $\mathcal{P}$ of density 1 satisfies

$$E_\varphi(\mathcal{P}) \geq \hat{h}(0) - h(0).$$

We can construct $h$ by imitating the sphere packing construction: let $h(x)$ be a radial polynomial times $e^{-\pi|x|^2}$, with the polynomial chosen with minimal degree so that $\varphi - h$ and $\hat{h}$ have double roots at the modified root locations from Section 3. We conjecture that as the number of roots tends to infinity, these functions converge and the limiting functions prove a sharp bound for energy.

The closest analogue of Conjectures 5.2 and 5.3 we have found is the following.

**Conjecture 6.1.** For the potential function $\varphi(x) = e^{-c|x|^2}$ in $\mathbb{R}^n$ with $n = 8$ or 24, the limiting auxiliary function $h$ satisfies

$$\hat{h}(0) = \frac{2c}{n} E_\psi(\Lambda_n),$$

where $\Lambda_n$ is $E_8$ or the Leech lattice when $n = 8$ or 24, respectively, and $\psi(x) = |x|^2 \varphi(x)$.

Besides numerical evidence, one reason to believe this conjecture is that it is compatible with duality symmetry. If the auxiliary function $h$ proves an energy bound for an integrable potential function $\varphi$, then $g := \hat{\varphi} - \hat{h}$ does so for $\hat{\varphi}$. Specifically, $g \leq \hat{\varphi}$ since $\hat{h} \geq 0$, and $\hat{g} \geq 0$ since $h \leq \varphi$. This duality transformation preserves optimality: if $h$ proves that a lattice $\Lambda$ of covolume 1 minimizes $E_\varphi$, then $g$ proves that the dual lattice $\Lambda^*$ minimizes $E_{\hat{\varphi}}$. To see why, note that $\varphi(0) + \hat{h}(0) - h(0) = \hat{\varphi}(0) + \hat{g}(0) - g(0)$, from which it follows by Poisson summation that

$$\hat{h}(0) - h(0) = E_\varphi(\Lambda)$$

if and only if

$$\hat{g}(0) - g(0) = E_{\hat{\varphi}}(\Lambda^*).$$

This duality is compatible with Conjecture 6.1, in the sense that $h$ satisfies the conjecture if and only if $g$ does; the compatibility is not obvious, but it follows from a short calculation using Poisson summation.

**7. Forcing single roots**

In this section we discuss a related problem: constructing functions with forced single roots (instead of the forced double roots used earlier in the paper). Such functions do not have direct applications to sphere packing, but they can be explicitly written down in some cases and thus serve as a testing ground for ideas concerning our main conjectures. Furthermore, their properties are quite a bit more interesting than one would guess from their definition.

The structure in this problem is best seen by forcing single roots for Fourier eigenfunctions. The use of eigenfunctions was merely a computational convenience in Section 4, but in this section it will play an essential role in our conjectures.

For $\varepsilon \in \{0, 1\}$, let

$$g^\varepsilon_{n,k} = r^\varepsilon_{n,k}(|x|^2)e^{-\pi|x|^2},$$

where $r^\varepsilon_{n,k}$ is a polynomial of degree at most $2k + \varepsilon$ that is not identically zero, vanishes at $2, 4, \ldots, 2k$, and is a linear combination of the polynomials $p^{\varepsilon}_{2j+\varepsilon}$ for $0 \leq j \leq k$. The last condition means that $\hat{g}^\varepsilon_{n,k} = (-1)^\varepsilon g^\varepsilon_{n,k}$. The same arguments
as in Lemma 3.1 shows that these functions exist and are unique, up to scaling. We see no canonical way to scale them, so we will not choose a preferred scaling.

The choice of $2, 4, \ldots, 2k$ as forced root locations is inspired by the norms of the vectors in the $E_8$ lattice. One could also study the analogous functions for the Leech lattice, but we have focused on the simplest case. Note that we use the exact vector norms, with no need to modify them along the lines of (3.5).

**Conjecture 7.1.** As $k \to \infty$ with $n$ and $\varepsilon$ fixed, $g_{n,k}^\varepsilon$ converges (when suitably normalized) to a Fourier eigenfunction $g_n^\varepsilon$ (not identically zero) that vanishes at all radii of the form $\sqrt{2j}$. If we view $g_{n,k}^\varepsilon$ as an entire function of $|x|$, then the convergence is uniform on all compact subsets of $\mathbb{C}$.

Uniform convergence implies that $g_n^\varepsilon(x)$ is an entire function of $|x|$.

These limiting functions are mysterious in general, but when $n$ is a multiple of 4 we can conjecture explicit formulas for half of them. The remaining functions appear to be much more subtle, as we will see shortly.

**Conjecture 7.2.** If the scaling is chosen appropriately, then

$$g_4^0(x) = \frac{\sin \pi |x|^2/2}{|x|^2/2} e^{-\pi \sqrt{3} |x|^2/2}.$$  

If $n > 4$ is a multiple of 4 and $\varepsilon \not\equiv n/4 \pmod{2}$, then (again up to scaling)

$$g_n^\varepsilon(x) = \left(\frac{\sin \pi |x|^2/2}{|x|^2/2}\right) e^{-\pi \sqrt{3} |x|^2/2}$$  

if $n \equiv 0 \pmod{3}$,

$$g_n^\varepsilon(x) = \left(\frac{\sin \pi |x|^2/2}{|x|^2/2}\right) \left(|x|^2 - \frac{(n+2)\sqrt{3}}{6\pi}\right)^2 e^{-\pi \sqrt{3} |x|^2/2}$$  

if $n \equiv 1 \pmod{3}$, and

$$g_n^\varepsilon(x) = \left(\frac{\sin \pi |x|^2/2}{|x|^2/2}\right) \left(|x|^2 - n/(2\pi \sqrt{3})\right) e^{-\pi \sqrt{3} |x|^2/2}$$  

if $n \equiv 2 \pmod{3}$.

We have no explanation for the exceptional behavior in four dimensions.

**Proposition 7.3.** The functions listed in Conjecture 7.2 are all eigenfunctions of the Fourier transform, with the appropriate eigenvalues.

**Sketch of proof.** This can be verified by straightforward calculation. Because $x \mapsto e^{-\pi |x|^2}$ is its own Fourier transform, it follows that when $\text{Re}(\alpha) > 0$, the Fourier transform of $x \mapsto e^{-\pi |x|^2/\alpha}$ is $x \mapsto e^{-\pi |x|^2/\alpha} / \alpha^{n/2}$. Differentiating with respect to $\alpha$ allows one to compute the Fourier transform of $x \mapsto |x|^{2k} e^{-\pi |x|^2/\alpha}$ for $k \in \{1, 2, \ldots\}$. Finally, we write

$$\left(\frac{\sin \pi |x|^2/2}{|x|^2/2}\right) e^{-\pi \sqrt{3} |x|^2/2} = \frac{e^{-\pi |x|^2}}{2i}.$$  

The result when $n \equiv 0 \pmod{3}$ follows easily from the fact that $\sqrt{3}/2 + i/2$ is a 12-th root of unity, and the results when $n \not\equiv 0 \pmod{3}$ follow from similar but slightly more elaborate calculations. The trickiest case is when $n = 4$, because it involves dividing by $|x|^2$. That can be handled by integrating with respect to $\alpha$ instead of differentiating.  

□
Note that the limiting functions in Conjecture 7.2 are entire and have imaginary roots at $\sqrt{-2j}$ for each $j > 0$. It is not clear where these imaginary roots come from. There are also a finite number of extraneous real roots, which appear to be needed to create Fourier eigenfunctions. If one plots the roots of these functions (as in Figure 4), one sees the expected roots on the real axis, the surprising purely imaginary roots, and a V-shaped collection of non-real roots spreading out from the imaginary axis. It seems that as $k \to \infty$, that final collection tends to infinity and contributes no roots in the limit. It does serve, however, to reduce the exponent in the Gaussian from the original $-\pi$ to $-\pi\sqrt{3}/2$.

Note also that the root of $g_{18}^1$ at the origin must occur by Poisson summation over $E_{8m}$ (so it is not extraneous in the same sense, even though it was not deliberately forced).

Whenever the dimension is a multiple of 4, Conjecture 7.2 predicts one of the two eigenfunctions Conjecture 7.1 asserts exists. The other eigenfunction is more mysterious. It does not have imaginary roots at $\sqrt{-2j}$. Instead, it almost has roots at $\sqrt{-(2j-1)}$, but not quite. For example, $g_8^0$ appears to vanish at the square roots of the numbers from Table 8. The precise perturbations away from the odd integers depend on the dimension. We do not know an explanation for this interesting behavior. The most natural possibility is that these functions are given by a dominant term, which has roots at exactly $\sqrt{-(2j-1)}$, plus some lower order terms. However, we have been unable to conjecture a formula of this sort.

More general convergence theorems are likely true. For example, in four dimensions, if we force an extra factor of $|x|^2 - c$ in the +1 eigenfunction, then the resulting functions seem to converge to

$$g_4^0(x) \left( |x|^2 - c \right) \left( \pi^2 |x|^4 + (c\pi^2 - 2\pi \sqrt{3}) |x|^2 + c(c\pi^2 - 2\pi \sqrt{3}) \right).$$

(Note that now the extraneous roots need not be real.) It seems plausible that in each case covered by Conjecture 7.2, forcing some additional finite set of roots simply creates an additional factor corresponding to a finite set of extraneous roots. That does not seem to be true for the mysterious eigenfunctions not predicted by Conjecture 7.2 (i.e., forcing another root does not seem to multiply them by a polynomial factor, but instead changes the imaginary root perturbations as well).
Table 8. Squares of the imaginary roots of \( g_0^2 \).

\[
\begin{array}{l}
-0.980217784819734913 \\
-2.999513816437548808 \\
-4.999987267218782800 \\
-6.999996513484933208 \\
-8.999999904718346910 \\
-10.999999997413895690 \\
-12.999999999993018222 \\
-14.999999999981049306 \\
-16.999999999999454080 \\
-18.999999999999985904 \\
-20.999999999999999608 \\
-22.999999999999999999 \\
\end{array}
\]

In dimensions that are not multiples of four, we do not know any closed form expressions for the limiting eigenfunctions. It appears that they behave somewhat like the mysterious eigenfunctions in the multiple-of-four case (i.e., those not predicted by Conjecture 7.2). For example, \( g_0^2 \) appears to have imaginary roots at some perturbation of the square roots of \(-2.5, -4.5, -6.5, \ldots\), and \( g_1^2 \) at some perturbation of the square roots of \(-1.5, -3.5, -5.5, \ldots\). We have focused on the multiples of four because they seem simpler (and perhaps more relevant to sphere packing).

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