JÓNSSON POSETS

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To the memory of Bjarni Jónsson

Abstract. According to Kearnes and Oman (2013), an ordered set $P$ is Jónsson if it is infinite and the cardinality of every proper initial segment of $P$ is strictly less than the cardinality of $P$. We examine the structure of Jónsson posets.

1. Introduction

An ordered set $P$ is Jónsson if it is infinite and the cardinality of every proper initial segment of $P$ is strictly less than the cardinality of $P$. This notion is due to Kearnes and Oman (2013). Jónsson posets, notably the uncountable one, appear in the study of algebras with the Jónsson property (algebras for which proper subalgebras have a cardinality strictly smaller than the algebra). The study of these algebras are the motivation of the paper by Kearnes and Oman [15] where the reader will find detailed information. Countable Jónsson posets occur naturally at the interface of the theory of relations and symbolic dynamics as Theorem 5 and Theorem 4 below illustrate. They were considered in the thesis of the second author [26], without bearing this name, and characterized in [27] under the name of minimal posets.

This characterization involves first the notion of well-quasi-order (w.q.o. for short) – a poset $P$ is w.q.o. if it is well founded, that is every non-empty subset $A$ of $P$ has at least a minimal element, and $P$ has no infinite antichain –, next, the decomposition of a well founded poset into levels (for each ordinal $\alpha$, the $\alpha$-th level is defined by setting $P_\alpha := \text{Min}(P \setminus \bigcup_{\beta < \alpha} P_\beta)$ so that $P_0$ is the set Min($P$) of minimal elements of $P$; each element $x \in P_\alpha$ has height $\alpha$, denoted by $h_P(x)$; the height of $P$, denoted by $h(P)$, is the least ordinal $\alpha$ such that $P_\alpha = \emptyset$) and, finally, the notion of ideal of a poset (every non-empty initial segment which is up-directed).

The following result reproduced from [27] gives a full description of countable Jónsson posets.

Proposition 1. Let $P$ be an infinite poset. Then, the following properties are equivalent:

(i) $P$ is w.q.o. and all ideals distinct from $P$ are principal;
(ii) $P$ has no infinite antichain and all ideals distinct from $P$ are finite;

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Every proper initial segment of $P$ is finite;
(iv) Every linear extension of $P$ has order type $\omega$ (the order type of chain $\mathbb{N}$ of non negative integers);
(v) $P$ is level-finite, of height $\omega$, and for each $n < \omega$ there is $m < \omega$ such that each element of height at most $n$ is below every element of height at least $m$;
(vi) $P$ embeds none of the following posets: an infinite antichain; a chain of order type $\omega^*$ (the dual of the chain of non-negative integer); a chain of order type $\omega + 1$; the direct sum $\omega \oplus 1$ of a chain of order type $\omega$ and a one element chain.

The equivalence between item (iii), (iv) and (v) was given in [26]. One proves

$$(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (vi) \Rightarrow (i)$$

using straightforward arguments.

Posets as above are said minimal or having minimal type.

A new characterization involving semiorders (posets which do not embed $2 \oplus 2$, the direct sum of two 2-element chains, nor $3 \oplus 1$, the direct sum of a 3-element chain and a 1-element chain) is given in Subsection 3.4. In order to present this characterization, we say that an order $L$, considered as a set of ordered pairs, is between two orders $A$ and $B$ on the same set if $A \subseteq L \subseteq B$. We prove:

**Proposition 2.** A poset has minimal type iff the order is between a semiorder with no maximal element and a linear order of order type $\omega$.

The requirement that there is no maximal element is essential (otherwise an antichain will satisfies the stated conditions).

An easy way of obtaining posets with minimal type is given by the following corollary of Proposition 2.

**Corollary 3.** Let $n$ be a non-negative integer and $P$ be a poset. The order on $P$ is the intersection of $n$ linear orders of order type $\omega$ if and only if $P$ is the intersection of $n$ linear orders and $P$ has minimal type.

We illustrate first the role of minimal posets in the theory of relations. A binary relational structure is a pair $R := (V, (\varrho_i)_{i \in I})$ made of a set $V$ and a family of binary relations $\varrho_i$ on $V$. A subset $A$ of $V$ is an interval of $R$ if for every $x, x' \in A$, $y \in V \setminus A$, $i \in I$, $x \varrho_i y$ if and only if $x' \varrho_i y$ and $y \varrho_i x'$ if and only if $y \varrho_i x'$. The structure $R$ is indecomposable if its only intervals are the empty set, the singletons and the whole set $V$. Fix a set $I$; relational structures of the form $R := (V, (\varrho_i)_{i \in I})$ will have type $I$. If $R'$ is an other structure of type $I$, then $R$ is embeddable into $R'$ and we set $R \leq R'$ if $R$ is isomorphic to an induced substructure of $R'$. The age of $R$ is the set $\text{Age}(R)$ of isomorphic types of finite structures which are embeddable to $R$, these finite structures being considered up to isomorphism. Let
Ind(Ω(I)) be the set of finite indecomposable structures of type I. A subset \( D \) of Ind(Ω(I)) is hereditary if \( R \in \text{Ind}(Ω(I)), R \leq R' \in D \) imply \( R \in D \).

The following result was obtained by D. Oudrar in her thesis [24] in collaboration with the second author.

**Theorem 4.** An infinite hereditary subset \( D \) of Ind(Ω(I)) contains a hereditary subset \( D' \) having minimal type iff it contains only finitely many members of size 1 or 2. If \( D' \) has minimal type then \( \downarrow D' \) (the set of isomorphic types of \( S \) which are embeddable into some \( S' \in D \)) is the age of an infinite indecomposable structure; this age is well-quasi-ordered by embeddability.

For example, if \( R \) is the infinite path on the set of non-negative integers, the age of \( R \) consists of direct sums of finite paths. Those finite paths are the indecomposable members of Age\((R)\). They form a set having minimal types. Uncountably many sets of binary relations having minimal type are given in Chapter 5 of [24]. Recent results of [7, 16, 20] suggest that a complete characterization is attainable.

Let us turn to symbolic dynamic. Let \( S : A^ω \to A^ω \) be the shift operator on the set \( A^ω \) of infinite sequences \( s := (s_n)_{n<ω} \) of elements of a finite set \( A \) (that is \( S(s) := (s_{n+1})_{n<ω} \)). A subset \( F \) of \( A^ω \) is shift-invariant if \( S(F) \subseteq F \) where \( S(F) := \{ S(x) : x \in F \} \). It is minimal if it is non-empty, compact, shift-invariant and if no proper subset has the same properties (cf [3]). As it is well-known, every compact (non-empty) invariant subset contains a minimal one. An infinite word \( u \) is called uniformly recurrent if the adherence of the set of its translates via the shift operator is minimal (hence, all \( u \) belonging to a minimal set are uniformly recurrent. For example, infinite Sturmian words are uniformly recurrent and the set \( S_α \) of infinite Sturmian words with slope \( α \) is minimal (see Chapter 6 of [12]).

To a compact invariant subset \( F \) we may associate the set \( A(F) \) of finite sequences \( s := (s_0, \ldots, s_{n−1}) \) such that \( s \) is an initial segment of some member of \( F \). Looking as these sequences as words, we may order \( A(F) \) by the factor ordering: a sequence \( s \) being a factor of a sequence \( t \) if \( t \) can be obtained from \( s \) by deleting an initial segment and a final segment of \( t \).

We have then:

**Theorem 5.** \( A(F) \) has minimal type if and only if \( F \) is minimal.

This result about words can be viewed as a special instance of Theorem 4. Indeed, to an infinite sequence \( s \in A^ω \) we may associate the relational structure \( R_s := (\mathbb{N}, c, (u_i)_{i\in A}) \), where \( c \) is the binary relation on \( \mathbb{N} \) defined by \( n cm \) if \( m = n + 1 \) and \( u_i \) is the unary relation such that \( u_i n \) iff \( s(n) = i \). Let Ind(Ω) be the collection of finite indecomposable relational structures with the same type as \( R \). Then, \( s \) is uniformly recurrent iff Age\((R_s) \cap \text{Ind}(Ω) \) has minimal type.

For more about the combinatorial aspects of symbolic dynamic, see [3, 12, 18].

Posets of minimal type are related to a notion of Jaco graph introduced by Johann Kok in 2014 and studied by him and his collaborators [17]. A directed
graph $G$ is a Jaco graph if its vertex set $V(G)$ is the set of positive integers and there is a nondecreasing sequence $(a_n)_{n \in \mathbb{N}^*}$ of positive integers such that a pair $(n, m)$ forms an arc of $G$ iff $n < m \leq a_n + n$. Given a Jaco graph $G$, let $G^d$ be the directed complement of $G$, that is the graph made of directed pairs $(n, m)$ such that $n < m$ but $(n, m)$ is not an arc of $G$. All pairs $(n, m)$ such that $a_n + n < m$ belong to this graph, hence they define a strict order (i.e., a irreflexive and transitive relation) on $\mathbb{N}^*$. This ordered set has minimal type. In fact, an order is minimal iff it can be labelled in such a way that it extends the directed complement of a Jaco graph.

Concerning the structure of Jónsson posets, we note that countable Jónsson chains are isomorphic to the chain $\omega$. Jónsson chains which are well founded are isomorphic to initial ordinals, alias cardinals. But there are uncountable Jónsson chains which are not well founded. For an example, the chain $\omega^* \cdot \omega_1$, lexicographical sum along the chain $\omega_1$ of copies of $\omega^*$ (the dual of $\omega$), is Jónsson but not well founded. Next, uncountable Jónsson posets may contain infinite antichains as for an example $\Delta_{\aleph_0} \cdot \omega_1$, the sum along the chain $\omega_1$ of copies of the countable antichain $\Delta_{\aleph_0}$.

Still, uncountable Jónsson posets, w.q.o. or not, retain several properties of minimal posets. In this paper, we give several characterizations of Jónsson posets (e.g. Theorem 14). We give a description of those whose cardinality is regular (see Theorem 24): we observe that a poset $P$ of regular cardinality $\kappa > \aleph_0$ is Jónsson if and only if it decomposes into a lexicographic sum $\sum_{\alpha \in C} P_{\alpha}$ where $C$ is a chain of type $\kappa$ and every $P_{\alpha}$ is a non empty poset of cardinality $\kappa_{\alpha}$ strictly less than $\kappa$. Introducing pure posets, we extend this characterization to posets of singular cofinality and give an extension of Proposition 2 (Theorem 30).

The case of singular cardinality is more subtle, especially when the cofinality is countable. A description of Jónsson w.q.o. posets of singular cardinality seems to be difficult in regard of the following example.

Let $\kappa$ be a singular cardinal with cofinality $\nu$; let $(\kappa_\alpha)_{\alpha < \nu}$ be a sequence of cardinals cofinal in $\kappa$; let $[\nu]^2 := \{(\alpha, \beta) : \alpha < \beta < \nu\}$ be ordered componentwise and for each $u := (\alpha, \beta) \in [\nu]^2$, let $P_u$ be any w.q.o. of cardinality $\kappa_\alpha$. Then $P := \sum_{u \in [\nu]^2} P_u$ is w.q.o. and Jónsson (but not pure, cf. definition 21).

Our motivation for looking at w.q.o. Jónsson posets is a beautiful conjecture of Abraham, Bonnet and Kubis 2 relating the notion of w.q.o. and the stronger notion of better-quasi-order (b.q.o.) invented by Nash-Williams 22, 23. Up to now, our attempt has been unsuccessful.

Part of this work is based on an unpublished study of spectra of posets 4.

2. Terminology, notation

Our terminology is based on 13 and 14. We denote cardinal numbers by greek letters, like $\kappa$, $\lambda$, $\mu$ and by $|X|$ the cardinality of set $X$. We identify a binary relation $\rho$ on a set $X$ with a set of ordered pairs and we set $xy$ if
We say that $\varrho$ is a quasi-order (or a preorder) if it is reflexive and transitive, in which case we say that the set $X$ is quasi-ordered. Let $\leq$ be a quasi-order on $X$; we say that $x$ and $y$ are comparable if $x \leq y$ or $y \leq x$ and we set $x \sim y$ (despite the fact that this relation is not an equivalence relation); otherwise we say that $x$ and $y$ are incomparable and we set $x \not\sim y$. The relation $\equiv$ defined by $x \equiv y$ if $x \leq y$ and $y \leq x$ is an equivalence relation, whereas the relation $\prec$ defined by $x \prec y$ if $x \leq y$ and $y \not\leq x$ is a strict order. The relation $\prec$ is transitive and irreflexive; in fact, every transitive and irreflexive relation is a strict order. The relation $\leq$ is total if for every two elements $a, b \in X$ either $a \leq b$ or $b \leq a$ holds. The quasi-order $\leq$ is a partial order and the pair $P := (V, \leq)$ is partially ordered (poset for short) if $\leq$ is antisymmetric. A set of pairwise incomparable elements of a poset is called an antichain. A chain is a totally ordered set. Let $P$ be a poset. A strengthening (also called an extension) of $P$ is any order $\leq'$ on $X$ containing the original order. A linear extension of $P$ is any linear order containing this order. When we mention elements or subsets of a poset, we usually mean elements or subsets of its vertex set. Sometimes, we use the same terminology for orders and posets. We feel free to say Jónsson order as well as Jónsson poset. A subset $A$ of a poset $P$ is a final (resp. initial) segment of $P$ if whenever $a \in A$ and $x \geq a$ (resp. $x \leq a$), then $x \in A$. For $A \subseteq P$ we set $\uparrow A := \{x \in P : \exists a \in A (x \geq a)\}$, this is the final segment generated by $A$; we denotes by $\downarrow A$ the corresponding initial segment generated by $A$; $\uparrow a$ and $\downarrow a$ abbreviate $\uparrow \{a\}$ and $\downarrow \{a\}$. We denote by $\mathcal{I}(P)$, resp. $\mathcal{F}(P)$, the set of initial, resp. final, segments of $P$. A subset $A \subseteq P$ of a poset $P$ is cofinal in $P$ if every $x \in P$ is majorized by some $y \in A$, and the cofinality of $P$, denoted by $\text{cf}(P)$, is the least cardinal $\nu$ such that $P$ contains a cofinal subset of cardinality $\nu$. If $\kappa$ is a cardinal, the cofinality of $\kappa$, denoted by $\text{cf}(\kappa)$, is the cofinality of $\kappa$ viewed as an initial ordinal. A cardinal is regular if it is equal to its cofinality, otherwise it is singular. A basic property of linearly ordered sets, observed by Hausdorff (see [14]), is that they contain well ordered cofinal subsets; it follows that either they have a maximum element or their cofinality is an infinite regular cardinal. For an arbitrary poset $P$, the corresponding fact is that it contains a well founded cofinal subset, but in this case, the cofinality, $\text{cf}(P)$, might be a singular cardinal.

3. Characterizations and descriptions of Jónsson posets

3.1. Strengthening of Jónsson posets. We start with the following observation:

**Lemma 6.** A poset $P$ is Jónsson iff $P$ is infinite and for every $x \in P$, $|P \setminus \uparrow x| < |P|$.

The proof is immediate: every proper initial segment $A$ is contained into a proper initial segment of the form $P \setminus \uparrow x$ for some $x \in P \setminus A$.

As a special consequence of this lemma, note that a Jónsson poset cannot have a maximal element (if $a$ is a maximal element of $P$ then $P \setminus \{a\}$ is an initial segment of $P$).
The following proposition gives a description of Jónsson linear order; the straightforward proof is omitted.

**Proposition 7.** A well founded linear order \( L \) is Jónsson iff and only if its order type is an initial ordinal. Furthermore, a linear order \( L \) of cardinality \( \kappa \) is Jónsson iff \( L \) is a lexicographic sum \( \sum_{\alpha<\mu} C_\alpha \) of chains \( C_\alpha \) of cardinality less than \( \kappa \) indexed by a regular ordinal \( \mu \).

The relationship between Jónsson posets and Jónsson linear orders is at the bottom of properties of Jónsson posets:

**Proposition 8.** Let \( P := (X, \leq) \) be a poset. Then the following properties are equivalent:

(i) \( P \) is Jónsson;

(ii) Every strengthening of \( P \) is Jónsson;

(iii) Every linear extension of \( P \) is Jónsson.

*Proof.* (i) \( \Rightarrow \) (ii). Let \( \leq' \) be a strengthening of the order \( \leq \) of \( P \). Let \( A \) be a proper initial segment of \( P' := (X, \leq) \). Then \( A \) is an initial segment of \( P \) (indeed, if \( x \in A \) and \( y \leq x \), then, since \( \leq' \) contains \( \leq \), \( y \leq' x \) hence \( y \in A \)). Hence \( |A| < |X| \). Thus \( P' \) is Jónsson.

(ii) \( \Rightarrow \) (iii). Obvious.

(iii) \( \Rightarrow \) (i). Let \( A \) be an initial segment of \( P \). Let \( B \) be the complement. Let \( P'' \) be the sum \( A + B \) that is the poset in which the order extends the order \( \leq \) on \( P \) and every element of \( A \) is smaller than every element of \( B \). A linear extension \( \leq' \) of this order is a linear extension of \( \leq \), furthermore \( A \) is an initial segment of \( P' := (X, \leq) \). Since \( P' \) is Jónsson, \( |A| < |X| \). Thus \( P \) is Jónsson. \( \square \)

Since any countable Jónsson linear order has order type \( \omega \), this proposition yields the equivalence between (ii) and (iv) of Proposition 7.

**Lemma 9.** On a poset of cardinality \( \kappa \) the order is Jónsson whenever it is the intersection of less than \( \text{cf}(\kappa) \) Jónsson orders.

*Proof.* Let \( P := (X, \leq) \). And let \( (\leq_i)_{i<\nu} \), with \( \nu < \text{cf}(\kappa) \), be a family of \( \nu \) orders such that \( \leq = \bigcap_{i<\nu} \leq_i \). Let \( x \in X \). For \( i < \nu \) set \( \uparrow_i x := \{ y \in X : x \leq_i y \} \). Then \( \uparrow x = \bigcap_{i<\nu} \uparrow_i x \). Hence, \( X \setminus \uparrow x = \bigcup_{i<\nu} X \setminus \uparrow_i x \). from which follows:

\[
|X \setminus \uparrow x| \leq \sum_{i<\nu} |X \setminus \uparrow_i x|.
\]

Since each \( \leq_i \) is Jónsson, \( |X \setminus \uparrow_i x| < |X| \). Since \( \nu < \text{cf}(\kappa) \) and \( \kappa = |X| \), \( |X \setminus \uparrow x| < |X| \). Hence \( \leq \) is Jónsson. \( \square \)

**Corollary 10.** Let \( P \) be a poset of cardinality \( \kappa \) and let \( \mu := \text{cf}(\kappa) \). The order on \( P \) is the intersection of strictly less that \( \mu \) linear Jónsson orders if and only if \( P \) is the intersection of strictly less than \( \mu \) linear orders and \( P \) is Jónsson.

Since any countable Jónsson linear order has order type \( \omega \), this corollary yields Corollary 8.
We recall that a poset $P$ is w.q.o. iff all its linear extensions are well ordered. Hence, it follows from Proposition 8 that a w.q.o. is Jónsson iff it is infinite and all its linear extensions have the same order type and this order type is an initial ordinal. If $P$ is w.q.o. there is a largest ordinal type of the linear extensions of $P$, a famous and non trivial result due to de Jongh and Parikh. This order type, denoted by $o(P)$, is the ordinal length of $P$. With this notion, a w.q.o. poset $P$ is Jónsson iff $P$ is infinite and $o(P)$ is the cardinality of $P$, viewed as an initial ordinal. For an illustration of Lemma 9, we mention that every poset whose order is the intersection of finitely many linear orders of ordinal length $\kappa$, where $\kappa$ is an initial ordinal, is a w.q.o. with ordinal length $\kappa$ (see [9]).

**Proposition 11.** Every w.q.o. of infinite cardinality contains an initial segment with the same cardinality which is Jónsson.

**Proof.** Let $P$ be a w.q.o. The set of initial segments of $P$, once ordered by inclusion, is well founded (Higman 1952, see [13]). Among the initial segments of $P$ with the same cardinality, take a minimal one. This is a Jónsson poset. □

### 3.2. Kearnes-Oman result and cofinality.

Kearnes and Oman [15] proved the following result.

**Theorem 12.** If $P$ is a Jónsson poset of cardinality $\kappa$ then, for every cardinal $\lambda < \kappa$, $P$ contains some principal initial segment of cardinality at least $\lambda$.

This is a significant result in the sense that several apparent strengthenings follow easily from it. We present in Theorem 14 a slight improvement.

There are posets with singular cofinality (e.g. an antichain with singular cardinality). This cannot be the case with a Jónsson poset. As said in Corollary 15, Jónsson posets have a regular cofinality.

The first reason of the relevance of the cofinality notion is this:

**Lemma 13.** For an infinite poset $P$ the following properties are equivalent:

(i) $P$ is Jónsson;

(ii) Every subset of $P$ of cardinality $|P|$ is cofinal in $P$;

(iii) There is some cofinal subset $C$ of $P$ such that $|P \uparrow x| < |P|$ for every $x \in C$.

**Proof.** (i) ⇒ (ii). Let $P$ be a Jónsson poset and $A$ be a subset of $P$ with cardinality $|P|$. The cardinality of the initial segment $\downarrow A$ is $|P|$. Since $P$ is Jónsson, $\downarrow A = P$, hence $A$ is cofinal. (ii) ⇒ (iii). Let $C$ be any subset of $P$. Let $x \in C$. Since $P \uparrow x$ cannot be cofinal in $P$, (ii) ensures that $|P \uparrow x| < |P|$. (iii) ⇒ (i). Let $A$ be a proper initial segment. Let $x \in P \setminus A$. Since $x$ is dominated by some $y \in C$ we have $|A| \leq |P \uparrow x| \leq |P \uparrow y| < |P|$, thus $|A| < |P|$. Hence $P$ is Jónsson.

In fact, in every Jónsson poset, some cofinal subset is a chain. This is our first result:
Theorem 14. A poset $P$ of infinite cardinality $\kappa$ is Jónsson iff there is a cofinal chain $C$ with cofinality $\text{cf}(\kappa)$ such that $|P \setminus \uparrow x| < |P|$ for every $x \in C$.

A straightforward proof based on König’s Lemma (Theorem 36) is given for level-finite posets in Subsection 4.2. As an immediate corollary of Theorem 14 we have:

Corollary 15. The cofinality of a Jónsson poset is the cofinality of its cardinality; in particular, this is a regular cardinal.

The proof of Theorem 14 relies on the following two lemma, the first one being well known:

Lemma 16. Every poset $P$ of cofinality $\nu$ in which every subset of cardinality strictly less than $\nu$ has an upper bound has a cofinal well ordered chain of order type $\nu$.

For the second lemma, we introduce the set $\mathcal{NC}(P)$ of non cofinal subsets of a poset $P$. We observe that for a given cardinal $\nu$, every subset of $P$ of cardinality strictly less than $\nu$ has an upper bound in $P$ if and only if $\mathcal{NC}(P)$ is a $\langle < \nu \rangle$-ideal of subsets of $P$, that is $\mathcal{NC}(P)$ is closed under inclusion and unions of less than $\nu$ members.

Lemma 17. Let $P$ be a Jónsson poset of cardinality $\kappa$ and let $\nu := \text{cf}(\kappa)$. Then:

1. every subset $A$ of cardinality strictly less than $\nu$ has an upper bound;
2. $\text{cf}(P) = \nu$.

Proof. (1) We prove that $\bigcap_{x \in A} \uparrow x$ is non-empty. We have:

$$X := P \setminus \bigcap_{x \in A} \uparrow x = \bigcup_{x \in A} (P \setminus \uparrow x).$$

Hence $|X| \leq \sum_{x \in A} |P \setminus \uparrow x|$. Since $P$ is Jónsson, each member of the sum is strictly less than $\kappa$; since $|A| < \nu = \text{cf}(\kappa)$, the sum is strictly less than $\kappa$. Hence $X \neq P$.

(2-1) $\nu \leq \text{cf}(P)$. Suppose by contradiction that $\text{cf}(P) < \nu$. Let $(x_\alpha)_{\alpha < \text{cf}(P)}$ be an enumeration of a cofinal subset of $P$ with size $\text{cf}(P)$. Then $P = \bigcup_{\alpha < \text{cf}(P)} \downarrow x_\alpha$. Since $P$ is Jónsson, $|x_\alpha| < \kappa$ for every $\alpha < \text{cf}(P)$. Since $\text{cf}(P) < \text{cf}(\kappa)$ we have $\kappa = |P| \leq \sum_{\alpha < \text{cf}(P)} |x_\alpha| < \kappa$. A contradiction.

(2-2) $\text{cf}(P) \leq \nu$. For each cardinal $\mu < \kappa$, let $P_\mu := \{ x \in P : |P \setminus \uparrow x| \leq \mu \}$.

Claim 3.1. There is a set $Z_\mu$ of cardinality a most $\nu$ such that $P_\mu \subseteq \downarrow Z_\mu$.

Proof of Claim 3.1 Let $X \subseteq P_\mu$. Then $|\bigcup_{x \in X} P \setminus \uparrow x| \leq \mu |X|$. If $|X| < \kappa$, $|\bigcup_{x \in X} P \setminus \uparrow x| < \kappa$ hence $\bigcup_{x \in X} P \setminus \uparrow x \neq P$. Any $z \in P \setminus \bigcup_{x \in X} (P \setminus \uparrow x) = \bigcap_{x \in X} \uparrow x$ dominates $X$. If $|P_\mu| < \kappa$, set $X := P_\mu$ and set $Z_\mu := z$. If $|P_\mu| = \kappa$, enumerate it by a sequence $(x_\nu)_{\nu < \kappa}$. Let $(\mu_\alpha)_{\alpha < \nu}$ be a cofinal sequence in $\kappa$. For each $\alpha < \nu$, select $z_\alpha$ which dominates $X_\alpha := \{ x_\nu : \nu < \mu_\alpha \}$ and set $Z_\mu := \{ z_\alpha : \alpha < \nu \}$. With that, the proof of Claim 3.1 is complete. □
We conclude the proof of (2–2) as follows. Since $P$ is Jónsson, $P = \bigcup_{\alpha < \nu} P_{\mu_{\alpha}}$. Hence, with Claim 19, $P \subseteq \bigcup_{\alpha < \nu} \downarrow Z_{\mu_{\alpha}} = \downarrow Z$ where $Z := \bigcup_{\alpha < \nu} Z_{\mu_{\alpha}}$. The set $Z$ is cofinal and has cardinality at most $\nu$, hence $\text{cf}(P) \leq \nu$. □

**Remarks 18.** The proof of (2–2) is reminiscent of the proof that if a poset $P$ of cardinality $\kappa$ is well founded, $| \downarrow x | < \kappa$ for every $x \in P$ and $\text{cf}(P) = \kappa$ is a singular cardinal then there is some $x \in P$ such that $\text{cf}(P \uparrow x) = \kappa$ (see [21]). A fact from which it readily follows that if the cofinality of a poset is a singular cardinal, the poset contains an infinite antichain, a result due to the second author and reproduced for example in [13].

**Lemma 19.** Let $P$ be a poset of cardinality $\kappa$.

1. If $P$ has a cofinal chain then for every $\lambda < \kappa$ it contains a principal initial segment of cardinality at least $\lambda$.
2. If $P$ is non-empty with no largest element, then $P$ contains a subset $Q$ of the same cardinality $\kappa$ as $P$ which is a lexicographical sum $\sum_{\alpha < \text{cf}(\kappa)} P_{\alpha}$ of $\text{cf}(\kappa)$ sets $P_{\alpha}$ of cardinality strictly less than $\kappa$ provided that for every $\lambda < \kappa$ every proper final segment of $P$ contains a principal initial segment of cardinality at least $\lambda$ and every subset of $P$ of cardinality strictly less than $\text{cf}(\kappa)$ is majorized. The converse holds if $Q$ is cofinal in $P$.
3. If $P$ is Jónsson then the conditions in Item (2) are satisfied.

**Proof.** (1). Let $C$ be a cofinal chain. With no loss of generality we may suppose that $C$ is well ordered and that its order type is an initial ordinal $\nu$, with $\nu$ regular. If $\nu = \kappa$ the conclusion holds for $C$ hence for $P$. Suppose $\nu < \kappa$. We have $P = \bigcup_{x \in C} \downarrow x$, hence $|P| \leq \sum_{x \in C} | \downarrow x |$. If the conclusion does not hold, there is some $\lambda < \kappa$ with $| \downarrow x | < \lambda$ for every $x \in C$, hence $\kappa \leq \lambda \cdot \nu < \kappa$, which is impossible.

(2). Suppose that $P$ contains a lexicographical sum $Q$ as described in the sentence. Let $x \in P$. Then $\uparrow x$ contains the sum $\sum_{\alpha_0 < \alpha < \text{cf}(\kappa)} P_{\alpha}$ where $\alpha_0$ is such that $\uparrow x \cap P_{\alpha_0} \neq \emptyset$. Hence, $| \uparrow x | \geq \kappa$. If $Q$ is cofinal and $A$ is any subset of cardinality strictly less than $\text{cf}(\kappa)$, then $A$ is majorized. On the other hand, suppose that the two conditions are satisfied. Let $\nu := \text{cf}(\kappa)$. If $\nu = \kappa$, then according to Lemma 16, $P$ contains a cofinal chain with order type $\kappa$ and $P$ contains a poset $Q$ as described. Suppose that $\nu < \kappa$. Let $(\kappa_{\alpha})_{\alpha < \nu}$ be an increasing sequence of cardinal numbers whose supremum is $\kappa$. We define a sequence $(x_{\alpha})_{\alpha < \nu}$ of elements of $P$ such that $|Q_{\alpha}| \geq \kappa_{\alpha}$ where $Q_{\alpha} := \downarrow x_{\alpha} \cap (\bigcup_{\beta < \alpha} \uparrow x_{\beta})$.

(3). If $P$ is Jónsson then every non-empty final segment of $P$ is Jónsson too. Hence, according to Theorem 12 the first condition of item (1) holds. Now, according to item (1) of Lemma 17, every subset of size strictly less than $\text{cf}(\kappa)$ is majorized. From this, $P$ contains a poset $Q$ as described. Note that since $Q$ has the same cardinality as $P$ and $P$ is Jónsson, $Q$ is cofinal in $P$. □

**Remark 20.** With the help of Item (1) of Lemma 14, Theorem 12 follows immediately from Theorem 14. On the other hand, if $P$ is Jónsson then, according to Item 3 of Lemma 19 above, it contains a poset $Q$ as described in Item (2).
According to Lemma [13] this poset $Q$ is cofinal in $P$, hence $P$ has a cofinal chain thus Theorem [14] holds.

An improvement of Item (3) is given in Theorem [24].

3.3. Purity. In order to describe (some) Jónsson posets we start with the following notion.

**Definition 21.** A poset $P$ is pure if every proper initial segment $I$ of $P$ is strictly bounded above (that is some $x \in P \setminus I$ dominates $I$).

This condition amounts to the fact that every non cofinal subset of $P$ is strictly bounded above (indeed, if a subset $A$ of $P$ is not cofinal, then $\nabla A \not\subseteq P$ hence from purity, $\nabla A$, and thus $A$, is strictly bounded above. The converse is immediate).

An equivalent condition is this:

- *For every $x \in P$ there is some $y \geq x$ such that $P \setminus x \subseteq \nabla y$.*

Every poset with a largest element is pure. Every chain is pure. *Every pure poset has a cofinal chain.* This last fact is consequence of the following lemma.

**Lemma 22.** Let $P$ be a poset with infinite cofinality $\nu$. Then $P$ is pure iff it contains an increasing cofinal sequence $(x_\alpha)_{\alpha < \nu}$ such that

\begin{equation}
P \setminus \uparrow x_\alpha \subseteq \nabla x_{\alpha+1}
\end{equation}

for all ordinal $\alpha$ such that $\alpha < \nu$.

*Proof.* Suppose that $P$ contains such a sequence. Let $x$ any element of $P$. Then $x \leq x_\alpha$ for some $\alpha < \nu$. Let $y := x_{\alpha+1}$. We have $P \setminus \uparrow x \subseteq P \setminus \uparrow x_\alpha \subseteq \nabla x_{\alpha+1} \subseteq \nabla y$. This proves that $P$ is pure. Conversely, suppose that $P$ is pure. As any poset, $P$ contains a cofinal sequence $(y_\alpha)_{\alpha < \nu}$ which is non decreasing in the sense that $y_{\beta'} \not\leq y_{\alpha}$ for $\alpha < \beta$. From the purity of $P$, we may extract a subsequence satisfying Inequality [1]. Indeed, define $\varphi : \nu \to \nu$ as follows. Set $\varphi(0) := 0$. Suppose $\varphi$ be defined for all $\beta'$ with $\beta' < \beta$. The set $\{y_{\beta'} : \beta' < \beta\}$ cannot be cofinal in $P$, hence there is some $y_\gamma \in P \setminus \nabla \{y_{\beta'} : \beta' < \beta\}$. If $\beta$ is a limit ordinal distinct of $0$, set $\varphi(\beta) = \gamma$. If not, $\beta = \beta' + 1$ and there is some $y \geq y_{\beta'}$ such that $P \setminus \uparrow y_{\beta'} \subseteq \nabla y$. If there is no element strictly above $y$ then $y$ is the largest element of $P$, a case we have excluded. Hence $y < y_\delta$ for some $\delta$ and in fact $\delta > \beta'$. We set $\varphi(\beta) = \delta$. Setting $x_\alpha := y_{\varphi(\alpha)}$, Inequality [1] is then satisfied. Let us check that the sequence is increasing. If not, let $\alpha < \beta$ with $x_\alpha \not\leq x_\beta$. We have $x_\beta \in P \setminus \uparrow x_\alpha \subseteq \nabla x_{\alpha+1}$ hence $x_\beta \leq x_{\alpha+1}$ contradicting the fact that the sequence $(y_\alpha)_{\alpha < \nu}$ is non decreasing. \hfill \box

- *(a)* If a poset $P$ is a strengthening of a pure poset $Q$ then $P$ is pure. The same conclusion holds if $P$ is a cofinal subset of a pure poset.

Indeed, let $A$ be a proper initial segment of $P$. Then $A$ is an initial segment of $Q$ hence it is proper. Since $Q$ is pure, $A$ is majorized by some element $x$. Since $P$ is a strengthening of $Q$, $x$ majorizes $A$ in $P$. Thus $P$ is pure. Now if $P$ is a cofinal subset of $Q$ and $A$ a proper initial segment of $P$ then $\nabla A \not\subseteq Q$ hence
Claim 3.2. If \( \alpha \) is a strengthening of a lexicographical sum which is Jónsson (indeed, it contains a cofinal chain).

(b) If a poset with no largest element is pure it contains a cofinal subset which is Jónsson.

(c) If \( P \) is pure then \( P \) is Jónsson iff \( P \) is infinite and \( |\downarrow x| < |P| \) for every \( x \in P \).

Indeed, if \( P \) is Jónsson then trivially, \( |\downarrow x| < |P| \) for every \( x \in P \). Conversely, let \( A \) be a proper initial segment of \( P \). Since \( P \) is pure, \( A \) is bounded above, that is \( A \subseteq \downarrow x \) for some \( x \in P \). According to the second condition, \( |\downarrow x| < |P| \) hence \( |A| < |P| \). This proves that \( P \) is Jónsson.

Purity and the condition above on principal initial segments imply Jónsson. The converse holds if the cardinality is regular. In particular, every minimal poset is pure. This is a consequence of Theorem 14 or (3) of Lemma 19.

Theorem 23. If the cardinality of \( P \) is an infinite regular cardinal \( \kappa \), then \( P \) is Jónsson if and only if \( P \) is pure and \( |\downarrow x| < \kappa \) for every \( x \in P \).

Proof. Suppose that \( P \) is Jónsson. Trivially, \( |\downarrow x| < \kappa \) for every \( x \in P \). Now, we show that \( P \) is pure. Let \( Q \) be a proper initial segment of \( P \) and let \( \lambda := |Q| \). According to Lemma 17, \( P \) contains a well ordered cofinal chain \( C \) of order type \( \kappa \). Since \( C \) is cofinal in \( P \), every \( x \in Q \) is majorized by some \( y_x \in C \). Let \( Q' := \{ y_x : x \in Q \} \). We have \( |Q'| \leq |Q| = \lambda \). Since \( P \) is Jónsson, \( \lambda < \kappa \). Since \( \kappa \) is regular, \( Q' \) is not cofinal in \( C \), thus it is majorized and hence \( Q \) is majorized. Thus \( P \) is pure.

If \( P \) is not pure, it could happen that by deleting some initial segment the remaining set is pure. But this is not general.

(d) Let \( P \) be a poset and \( Q \) be a proper initial segment of \( P \). Then \( P \) is Jónsson iff \( P \setminus Q \) is Jónsson, \( |Q| < |P| \) and \( Q \subseteq \downarrow (P \setminus Q) \).

Theorem 24. Let \( P \) be a poset with infinite cofinality \( \nu \). Then \( P \) is pure iff \( P \) is a strengthening of a lexicographical sum \( \sum_{a \in K} P_a \) where \( K \) is a chain of order type \( \nu \) if \( \nu \) is uncountable or a minimal poset if \( \nu \) is countable and every \( P_a \) is a non empty poset. If \( P \) is pure, then \( P \) is Jónsson iff each member \( P_a \) of the sum above has cardinality \( \kappa_a \) strictly less than \( |P| \).

Proof. A lexicographical sum as above is pure; thus from (a), every strengthening is pure. Moreover, if each member of the sum has cardinality less that \( |P| \), the sum is Jónsson hence, by Proposition 8, every extension is Jónsson.

For the converse, let \( N := (\nu, \leq_2) \) where \( \leq_2 \) is the order defined on \( \nu \) by \( \alpha \leq_2 \beta \) if \( \alpha = \beta \) or \( \alpha + 2 \leq \beta \). Then \( N \) is pure and Jónsson.

Claim 3.2. If \( P \) is pure then \( P \) is a strengthening of a lexicographical sum \( \sum_{a \in N} P_a \) where each \( P_a \) is a non empty poset.
Proof of Claim \[3.2\] Let \((x_\alpha)_{\alpha<\kappa}\) be the sequence given by Lemma 22. For every \(x \in P\), let \(h(x)\) be the least ordinal \(\alpha\) such that \(x \leq x_\alpha\), and for \(\alpha < \nu\), let \(P_\alpha := \{x \in P : h(x) = \alpha\}\). The order on \(P\) extends the order on the sum \(\sum_{\alpha \in \mathbb{N}} P_\alpha\) provided that for every \(x \in P_\alpha\), \(y \in P_\beta\), \(\alpha + 2 \leq \alpha\) implies \(x \leq y\) in \(P\). This is trivial: we have \(y \geq x_\alpha\), (otherwise \(y \not\leq x_\alpha\) and since \(P \uparrow x_\alpha \subseteq x_{\alpha+1}\), we have \(y \leq x_{\alpha+1}\) giving \(h(y) \leq \alpha + 1\) while \(h(y) = \beta > \alpha + 1\). Since \(x \leq x_\alpha\) we get \(x \leq y\) by transitivity, as claimed.

If \(\nu = \omega\) then \(N\) is minimal and the sentence in the theorem holds. If \(\nu\) is uncountable, let \((\mu_\lambda)_{\lambda<\nu}\) be an increasing cofinal sequence in \(\nu\) of limit ordinals; set \(N_\lambda := \{\alpha : \mu_\lambda \leq \alpha < \mu_{\lambda+1}\}\). Then \(N\) is the lexicographical sum of its restrictions to the \(N_\lambda\)'s indexed by the chain \(\nu\). Setting \(Q_\lambda := \bigcup_{\alpha \in N_\lambda} P_\alpha\), we get that \(P\) is the lexicographical sum of the \(Q_\lambda\)'s.

Note that if \(P\) is a pure Jónsson poset of cardinality \(\kappa\) with \(cf(\kappa) > \aleph_0\), one can easily show that the incomparability graph of \(P\) decomposes into at least \(cf(\kappa)\) connected components, each of cardinality strictly less than \(\kappa\). This yields another proof of Theorem 23 in this case.

With Theorem 23 and Theorem 24, we have:

**Corollary 25.** If \(P\) is a Jónsson poset and \(\kappa := |P|\) is a successor cardinal then \(P\) is the union of strictly less than \(\kappa\) chains.

**Remark 26.** It is not true that an uncountable pure and Jónsson poset \(P\) with countable cofinality is the lexicographical sum \(\sum_{n \in M} P_n\) where \(M\) is a minimal poset, each \(P_n\) is a non empty poset of cardinality \(\kappa_n\) strictly less than \(\kappa := |P|\) and the supremum of \(\kappa_n\) is \(\kappa\). The reason is that a strengthening of such a poset is pure and Jónsson but not necessarily a lexicographical sum. We give an example below.

Let \(Q\) be the lexicographical sum \(\sum_{n \in M} Q_n\) where \(M\) is the poset on the set \(\mathbb{N}\) of non-negative integers, with \(n \leq m\) if either \(n = m\) or \(n + 2 \leq m\), each \(Q_n\) is an antichain of cardinality the \(n\)-th number \(2^n\) (where \(2^n := \aleph_0\), \(2^{n+1} := 2^{2^n}\)). Trivially, \(Q\) is pure and Jónsson. We define a strengthening \(P\) of \(Q\) by adding just well chosen comparabilities between pairs of consecutive levels \(Q_n\) and \(Q_{n+1}\), for \(n \in \mathbb{N}\). Hence \(P\) will be pure and Jónsson. In order to do so, we suppose that \(Q_{n+1} = \emptyset(Q_n)\) and we add all pairs \((x, y)\) such that \(x \in Q_n\), \(y \in Q_{n+1}\) and \(x \in y\). We claim that the resulting poset \(P\) does not decompose into a non trivial sum. Indeed, otherwise some factor of the sum would be a proper autonomous subset of \(P\), that is, a subset \(F\) distinct from the empty set, any singleton and the whole set, and such that for every \(x, x' \in F\) and \(y \not\in F\), \(x \leq y\) iff \(x' \leq y\) and also \(y \leq x\) iff \(y \leq x'\). This is impossible. For each non-negative integer, the ordering induces a bipartite graph on \(Q_n \cup Q_{n+1}\) which is connected and such that distinct vertices have distinct neighborhoods. Since this graph has more than three vertices, it has no proper autonomous subset (e.g. see Proposition 1 of [28]). Hence, for each \(n\), \(F \cap (Q_n \cup Q_{n+1})\) is either empty, a
3.4. Semiorders and Jónsson posets. The poset $N := (\nu, \leq_2)$ which appears in the proof of Theorem [24] is an example of semiorder. We examine below the role of these orders in the present study.

Posets which do not embed the direct sum $2 \oplus 2$ of two 2-element chains are called interval orders, whereas posets which do not embed $2 \oplus 2$ nor $3 \oplus 1$, the direct sum of a 3-element chain and a 1-element chain, are called semiorders. Semiorders were introduced and applied in mathematical psychology by Luce [19]. For a wealth of information about interval orders and semiorders the reader is referred to [10] and [25].

The name interval order comes from the fact that the order of a poset $P$ is an interval order iff $P$ is isomorphic to a subset $J$ of the set $\text{Int}(C)$ of nonempty intervals of a totally ordered set $C$, ordered as follows: if $I, J \in \text{Int}(C)$, then

\[(2) \quad I < J \text{ if } x < y \text{ for every } x \in I \text{ and every } y \in J.\]

This result is due to Fishburn [11]. See also Wiener [32].

The Scott and Suppes Theorem [31] states that the order of a finite poset $P$ is a semiorder if and only if $P$ is isomorphic to a collection of intervals of a totally ordered set $C$, ordered as above. Extension of this result to infinite semiorders have been considered (see [6]).

Interval orders and semiorders can be characterized in terms of the quasi-orders $\leq_{\text{pred}}$ and $\leq_{\text{succ}}$ associated with a given order. They are defined as follows. Let $P$ be a poset. Set $x \leq_{\text{pred}} y$ if $z < x$ implies $z < y$ for all $z \in P$ and set $x \leq_{\text{succ}} y$ if $y < z$ implies $x < z$ for all $z \in P$. The relations $\leq_{\text{pred}}$ and $\leq_{\text{succ}}$ are quasi-orders. The strict orders associated to $\leq_{\text{pred}}$ and $\leq_{\text{succ}}$ extend the strict order associated to $\leq$, that is:

\[(3) \quad x < y \Rightarrow x <_{\text{pred}} y \text{ and } x <_{\text{succ}} y\]

for all $x, y \in P$.

We recall the following result (see Theorems 2 and 7 of [30]).

**Lemma 27.** Let $P$ be a poset. Then $P$ is an interval order if and only if the quasi-order $\leq_{\text{pred}}$ is a total quasi-order; equivalently $\leq_{\text{succ}}$ is a total quasi-order. Furthermore, $P$ is a semiorder if and only if the quasi-order intersection of $\leq_{\text{pred}}$ and $\leq_{\text{succ}}$ is total.

Note that the intersection of $\leq_{\text{pred}}$ and $\leq_{\text{succ}}$ can be total and these quasi-orders can be distinct. For an example, look at the direct sum of a 2-element chain and a 1-element chain. For an other example, if $\leq$ is the order $\leq_2$ on the ordinal $\nu$, then $\leq_<$ coincide with the natural order on $\nu$, $\leq_{\text{pred}}$ coincide with the natural order on all pairs distinct from the pair $(0, 1)$ but do not distinguish between 0 and 1. Orders $\leq$ such that $\leq_{\text{pred}}$ and $\leq_{\text{succ}}$ are total orders and equal are studied in [29], under the name of threshold orders.

We present an other characterization which is relevant to our purpose.
Theorem 28. A poset $P := (X, \preceq)$ is an interval order, resp. a semiorder, iff there is an order-preserving map $h$ from $P$ into a chain $K$ and a map, resp. an order-reversing map, $\Psi : K \to \mathcal{F}(K)$ such that:

1. $k \notin \Psi(k)$ for every $k \in K$;
2. $x < y$ in $P$ iff $h(y) \in \Psi(h(x))$ for every $x, y \in P$.

Proof. Suppose that $P$ is an interval order. By Lemma 27 we may select a total order $\preceq$ included into $\preceq_{\text{pred}}$. Set $K := (X, \preceq_{\text{pred}})$, $h$ be the identity and for each $x \in X$, set $\Psi(x) := \{y \in X : x \prec y\}$. Then $\Psi(x)$ is a final segment of $(X, \preceq)$; indeed, first $\Psi(x)$ is a final segment of $(X, \preceq_{\text{pred}})$ (if $y \in \Psi(x)$ and $z$ are such that $y \preceq_{\text{pred}} z$, then since $x < y$ we have $x < z$ hence $z \in \Psi(x)$, proving our assertion). Next, since $\preceq$ is included into $\preceq_{\text{pred}}$, it follows that $\Psi(x)$ is a final segment of $K := (X, \preceq)$. Conditions (1) and (2) hold trivially. If $P$ is a semiorder then, according to Lemma 27 the quasi-order intersection of $\preceq_{\text{pred}}$ and $\preceq_{\text{succ}}$ is total. Hence, we may select a total order $\preceq$ included into this intersection. In this case, the map $\Psi : P \to K := (X, \preceq)$ is order decreasing (indeed, let $x < y$ and let $z \in \Psi(y)$; by definition, $y < z$. Since $x < y$, $x < z$ hence $z \in \Psi(x)$ proving $\Psi(y) \subseteq \Psi(x)$).

In order to prove that the converse holds, suppose that there is an order-preserving map $h$ from $P$ into a chain $K$ and a map $\Psi : K \to \mathcal{F}(K)$ such that:

1. $k \notin \Psi(k)$ for every $k \in K$ and
2. $x < y$ in $P$ whenever $h(y) \in \Psi(h(x))$ for every $x, y \in P$.

Set $x \prec \Psi y$ if $h(y) \in \Psi(h(x))$. Hence Condition (2') ensures that $x \prec \Psi y$ implies $x < y$.

With the claims below, we prove that the relation $\prec \Psi$ is a strict order and the corresponding order $\preceq \Psi$ an interval order. Since Condition (2) expresses that $\prec \Psi$ and $<$ coincides, this proves that the converse of the lemma holds.

Claim 3.3. The relation $\prec \Psi$ is a strict-order on $X$.

Proof of Claim 3.3. First, this relation is irreflexive: from (1'), $h(x) \notin \Psi(h(x))$, that is $x \notin \Psi x$. Next, it is transitive. Indeed, suppose $x \prec \Psi y$ and $y \prec \Psi z$. We have $h(y) \in \Psi(h(x))$ and $h(z) \in \Psi(h(y))$. Since $K$ is a chain, $\Psi(h(x))$ and $\Psi(h(y))$ are comparable w.r.t. set inclusion. If $\Psi(h(x)) \subseteq \Psi(h(y))$, then since $h(y) \in \Psi(h(x))$ we have $h(y) \in \Psi(h(y))$, a fact which is excluded by (1'). Hence $\Psi(h(y)) \subset \Psi(h(x))$. Since $h(z) \in \Psi(h(y))$, we have $h(z) \in \Psi(h(x))$ that is $x \prec \Psi z$, proving the transitivity.

Let $\preceq \Psi$ be the order associated to $\prec \Psi$, that is $x \preceq \Psi y$ if $x = y$ or $x \prec \Psi y$; let $\preceq_{\Psi_{\text{pred}}}$ and $\preceq_{\Psi_{\text{succ}}}$ be the preorder "pred" and "succ" associated with $\preceq \Psi$. Set $\Phi(x) := \{y \in X : h(y) \in \Psi(h(x))\}$.

Claim 3.4. If $h(x) \leq h(y)$ then $x \preceq_{\Psi_{\text{pred}}} y$.

Proof of Claim 3.4. Let $z \in X$ such that $z \prec \Psi x$. We need to prove that $z \prec \Psi y$ that is $h(y) \in \psi(h(z))$. Since $z \prec \Psi x$, we have $h(x) \in \psi(h(z))$. Since $\psi(h(z)) \in \mathcal{F}(K)$ and $h(x) \leq h(y)$ we have $h(y) \in \psi(h(z))$, as required.
Claim 3.5. \( \Psi(h(y)) \cap h(P) \subseteq \Psi(h(x)) \) iff \( x \leq_{\Psi^{\text{succ}}} y \).

Proof of Claim 3.5. Let \( z \in X \) such that \( y <_{\Psi} z \). This means \( h(z) \in \Psi(h(y)) \). Since \( \Psi(h(y)) \cap h(P) \subseteq \Psi(h(x)) \), it follows \( h(z) \in \Psi(h(x)) \) that is \( x <_{\Psi(h(x))} z \). Hence \( x \leq_{\Psi^{\text{succ}}} y \) as claimed. Conversely, suppose \( x \leq_{\Psi^{\text{succ}}} y \). Let \( z \in X \) such that \( h(z) \in \Psi(h(y)) \cap h(P) \). We have \( y <_{\Psi} z \). Since \( x \leq_{\Psi^{\text{succ}}} y \) it follows \( x <_{\Psi} z \), that is \( z \in \Psi(h(x)) \), proving that \( \Psi(h(y)) \cap h(P) \subseteq \Psi(h(x)) \). \( \square \)

Claim 3.6. The order \( \leq_{\Psi} \) is an interval order. This is a semiorder provided that \( h \) is order-reversing.

Proof of Claim 3.6. Since \( K \) is a chain, the images via \( h \) of any two elements \( x \) and \( y \) of \( P \) are comparable in \( K \). According to Claim 3.4, \( x \) and \( y \) are comparable via the quasi order \( \leq_{\Psi^{\text{pred}}} \), hence from Lemma 27, \( \leq_{\Psi} \) is an interval order.

With this claim the proof is complete.

Lemma 29. Let \( Q \) be semiorder.

1. If \( Q \) has no maximal element then it is pure.
2. \( Q \) is Jónsson iff \( Q \) has no maximal element and the order on \( Q \) has a strengthening into a Jónsson linear order.

Proof. (1). If \( Q \) is empty, it is pure. Suppose that \( Q \) is non-empty. Our aim is to prove that for every \( x \in Q \) there is some \( y \in Q \) which majorizes \( Q \setminus \uparrow x \). Let \( x \in Q \). Since \( Q \) has no maximal element, \( x \) is not maximal hence there is some \( x' \in Q \) with \( x < x' \); again \( x' \) is not maximal, hence there is some \( y \in Q \) with \( x' < y \). This element \( y \) will do. Indeed, let \( z \in Q \setminus \uparrow x \). If \( z \notin y \) then \( z \) is incomparable to \( x, x' \) and \( y \), hence \( Q \) contains a 3\@1, contradicting the fact that \( Q \) is a semiorder.

(2) A Jónsson poset has no maximal element (cf. Lemma 6) and every strengthening is Jónsson (Proposition 5). For the converse, our aim is to prove that \( |Q \setminus \uparrow x| < |Q| \) for every \( x \in Q \). Let \( x \in Q \). Since \( Q \) is a semiorder with no maximal element then, according to (1), it is pure, hence there is some element \( y \in Q \) such that \( Q \setminus \uparrow x \subseteq y \). If \( L \) is any strengthening of the order on \( Q \) we have \( \downarrow y \subseteq_L y \). If \( L \) is a Jónsson order, we have \( | \downarrow_L y | < | Q | \). Hence, \( |Q \setminus \uparrow x| < |Q| \). Proving that \( Q \) is Jónsson. \( \square \)

Theorem 30. A poset \( P \) is pure and Jónsson iff the order on \( P \) is a strengthening of a semiorder with no maximal element and has a strengthening into a Jónsson linear order.

Proof. Suppose that the order on \( P \) is a strengthening of a semiorder \( Q \) with no maximal element and has a strengthening into a Jónsson linear order. According to Lemma 29, \( Q \) is pure and Jónsson. According to \( \bullet(a) \) and Proposition 8, \( P \) is pure and Jónsson. Conversely, suppose that the order on \( P \) is pure and Jónsson. Apply Claim 3.2 of Theorem 24 \( P \) is a strengthening of a lexicographical sum \( \Sigma_{\alpha \in N} P_{\alpha} \) where \( N := (\nu, \leq_2) \). We may suppose that the \( P_{\alpha} \)'s are antichains; since
$N$ is a semiorder with no largest element, this lexicographical sum is a semiorder with no maximal element. Since $N$ has a linear extension of order type $\nu$, $P$ has a strengthening into a Jónsson order. □

3.5. Conclusion. Since every countable Jónsson poset is pure and every countable Jónsson chain has order type $\omega$, it follows from Theorem 30 above that a countable poset $P$ is Jónsson iff the order of $P$ is between a semiorder with no maximal element and a linear order of type $\omega$. This is Proposition 2.

If the cardinality $\kappa$ of $P$ is uncountable, there are two cases to consider: $\kappa$ is regular, $\kappa$ is singular.

If $\kappa$ is a regular cardinal, then by Theorem 30, Proposition 2 extends verbatim: $P$ is Jónsson iff the order of $P$ is between a semiorder with no maximal element and a Jónsson linear order.

But, in this case, $\kappa$ being uncountable, we have a much more precise result: $P$ is a lexicographical sum indexed by the ordinal $\kappa$ of posets of cardinality less than $\kappa$ (cf. Theorem 24).

If $\kappa$ is singular, then

- either $\text{cf}(\kappa)$ is uncountable and this conclusion still holds provided that $P$ is pure,
- or $\text{cf}(\kappa) = \aleph_0$. In this case, $P$ is a pure Jónsson poset iff the order of $P$ is a strengthening of a lexicographic sum $\sum_{a \in K} P_a$ where $K$ is a minimal poset and $|P_a| < \kappa$ for every $a \in K$ (Theorem 24).

**Problem 31.** Find a characterization of Jónsson posets of singular cardinality $\kappa$.

4. Uniformity

Behind the properties of a pure and Jónsson poset $P$ (as in (v) of Proposition 1 or in the proof of Claim 3.2 of Theorem 24) are the properties of a function $h$ from $P$ into the ordinal numbers. This suggests the following development.

4.1. Uniformity and purity.

**Definitions 32.** Let $P$ be a poset and $h$ be an order-preserving map from $P$ onto a chain $K$. We say that $P$ is:

- $h$-weakly uniform if there is a map $\varphi : K \to K$ such that:

$$\forall x, y \in P (\varphi(h(x)) < h(y) \implies x < y).$$

- $h$-uniform if there is a map $\varphi : K \to K$ such that:

$$\forall x, y \in P (\alpha \in K, h(x) \leq \alpha \text{ and } h(y) > \varphi(\alpha) \implies x < y).$$

- $h$-minimal if the image of every non cofinal subset of $P$ is non cofinal in $K$, that is

$$\downarrow \{h(x) : x \in A\} \neq K$$

for every proper initial segment $A$ of $P$. 

These notions, with a slight variation, were originally defined in [4] for well founded posets with the height function \( h_P \) as an order preserving function.

For an example, if \( P \) is \( h \)-uniform with \( \varphi \) being the identity then it is the lexicographical sum of posets indexed by the chain \( K \). The existence of a map \( \varphi \) satisfying Condition (\( \psi \)) is a specialization of part of the conditions in Theorem 28. Indeed, to \( \varphi \) associate the map \( \psi : K \to \mathcal{F}(K) \), defined by setting \( \psi(k) := ]\varphi(k) \) for all \( k \in K \). And Condition (\( \psi \)) amounts to \( x < y \) whenever \( h(y) \in \psi(h(x)) \). A map \( \varphi \) witnessing that \( P \) is \( h \)-weakly uniform or \( h \)-uniform is extensive, that is \( \varphi(\alpha) \geq \alpha \) for every \( \alpha \in K \) (if, otherwise \( \varphi(\alpha) < \alpha \) for some \( \alpha \) then let \( x \) and \( y \) such that \( h(x) = \alpha \) and \( h(y) = \varphi(\alpha) \); we have \( \varphi(h(x)) = h(y) \). Since \( h \) is order preserving, \( x \neq y \), so neither Condition (\( \psi \)) nor Condition (\( \psi \)) may hold). If \( P \) is \( h \)-uniform and \( K \) well ordered, we may suppose that some map \( \varphi \) witnessing it is order preserving: indeed, for each \( \alpha \in K \), set \( P_\alpha := h^{-1}(\alpha) := \{ x \in P : h(x) = \alpha \} \), \( P_{\geq \alpha} := \bigcup_{\gamma \geq \alpha} P_{\gamma} \), \( P_{> \alpha} := \bigcup_{\gamma > \alpha} P_{\gamma} \), and let \( \beta \) be the least member of \( K \) such that \( P_{\geq \beta} \) is dominated by every element of \( P_{> \beta} \). This process defines an order preserving map for which \( P \) is \( h \)-uniform. Clearly, if \( P \) is \( h \)-uniform then it is \( h \)-weakly uniform. Conversely, if \( P \) is \( h \)-weakly uniform and some \( \varphi \) witnessing it is order preserving, then \( P \) is \( h \)-uniform. Also, if \( \varphi \) witnesses that \( P \) is \( h \)-weakly uniform, and \( K \) is well ordered, set \( \varphi^*(\alpha) := \sup_{\beta < \alpha} \varphi(\nu) \); if \( \varphi^*(\alpha) \in K \) for every \( \alpha \in K \) then \( P \) is \( h \)-uniform. In particular, if the order type of \( K \) is a regular initial ordinal then \( P \) is \( h \)-uniform.

**Theorem 33.** A poset \( P \) with no largest element is pure if and only if it is \( h \)-uniform for some order-preserving map \( h \) from \( P \) into some infinite limit ordinal \( \nu \). If \( P \) is well founded, the height function will do.

**Proof.** Suppose that \( P \) is \( h \)-uniform. We prove that it is pure. Let \( A \) be a non cofinal subset of \( P \). Pick some \( x \in P \setminus A \). Let \( \alpha := h(x) \). Since \( P \) is \( h \)-uniform, \( \downarrow A \) is disjoint from \( P_{\delta} := \bigcup_{\beta \in \delta} P_{\beta} \) where \( \delta := \varphi(\alpha) \), \( \varphi \) witnesses that \( P \) is \( h \)-uniform, \( P_{\beta} := h^{-1}(\beta) \). Hence, \( \downarrow A \subseteq P_{\geq \delta} := \bigcup_{\gamma \leq \delta} P_{\gamma} \). Since \( P \) is \( h \)-uniform, any \( y \in P_{\geq \delta} \) dominates \( \subseteq P_{> \delta} \), hence dominates \( A \). Hence \( P \) is pure. For the converse, let \( \nu := \text{cf}(P) \). Since \( P \) is pure with no largest element, \( \nu \) is an infinite limit ordinal. Let \( h : P \to \nu \) be defined as in the proof of Claim 28. Let \( x, y \in P \) with \( h(x) \leq \alpha \) and \( h(y) > \alpha + 1 \). Then, we have \( x < y \). Hence, \( P \) is \( h \)-uniform with \( \varphi : \nu \to \nu \) defined by \( \varphi(\alpha) := \alpha + 1 \).

Suppose that \( P \) is well founded and is pure. To prove that \( P \) is \( h_P \)-uniform, it suffices to prove that for every \( \alpha < h_P(P) \) there is some \( \beta < h_P(P) \) such that every element of \( P_{\beta} \) dominates \( P_{\geq \alpha} := \bigcup_{\gamma \geq \alpha} P_{\gamma} \) (indeed, for every \( \beta' > \beta \), every element of \( P_{\beta'} \) dominates some element of \( P_{\beta} \), hence dominates \( P_{> \alpha} \)). Supposing that this does not holds, then there is some \( \alpha \) such that for every \( \beta > \alpha \) there is some \( x_{\beta} \in P_{\beta} \) which does not dominates \( P_{\geq \alpha} \), hence there is some \( \alpha_\beta \leq \alpha \) and some \( y_{\beta} \in P_{\alpha_\beta} \) which is not majorized by \( x_{\beta} \). The set \( P_{\geq \alpha} \) is not cofinal in \( P \) hence, since \( P \) is pure, it is majorized. Let \( x \) be a such an element. The set \( \{ x : \alpha < \beta < h(P) \} \) is not majorized in \( P \), hence again, since \( P \) is pure, it is
Lemma 34. Let $P$ be a poset, $h : P \to K$ be an order preserving map. Suppose that $h$ is onto and $K$ has no largest element. Then $P$ is $h$-weakly uniform iff $P$ is $h$-minimal and, for every $\alpha \in K$, $P_\alpha := h^{-1}(\alpha)$ is majorized. If, furthermore, $P$ is wqo and $h$ is the height function $h_P$, then these conditions amounts to the fact that every chain $C$ in $P$ with order type $h(C)$ is cofinal in $P$.

Proof. Suppose that $P$ is $h$-weakly uniform. Let $A$ be a proper initial segment of $P$. Pick $x \in P \setminus A$. Let $\alpha := h(x)$. If $h(A)$ is cofinal in $K = h(P)$ there is some $y \in A$ with $h(y) \geq \varphi(\alpha)$. This element $y$ dominates all members of $P_\alpha$, in particular it dominates $x$, which is impossible since $x \notin A$. Hence $P$ is $h$-minimal. Trivially, for every $\alpha \in K$, $P_\alpha$ is majorized by every element $y$ such that $\varphi(h(y)) \geq \alpha$.

For the converse, note that in order to prove that $P$ is $h$-weakly uniform, it suffices to prove that for every $\alpha \in K$ there is some $\beta \in K$ such that every element of $P_\alpha$ dominates $P_\beta$ and choose $\beta$ for $\varphi(\alpha)$. Supposing that this does not holds, there is some $\alpha$ such that for every $\beta > \alpha$ there is some $x_\beta \in P_\beta$ which does not dominates $P_\alpha$, hence there some $y_\beta \in P_\alpha$ which is not majorized by $x_\beta$. According to our hypothesis, $P_\alpha$ is majorized by some $x \in P$; since $K$ has no largest element and $P$ is $h$-minimal, the set $\{ x_\beta : \alpha < \beta \in K \}$ is cofinal in $P$; hence $x$ is majorized by some $x_\beta$. This $x_\beta$ majorizes $y_\alpha$, which is impossible.

Suppose that $P$ is w.q.o. Suppose that $P$ is $h_P$-minimal. Let $C \subseteq P$ be a chain with order type $\kappa := h(P)$. Then $h_P(C) = \kappa$ hence $h(C)$ is cofinal in $h(P)$. Since $P$ is $h_P$-minimal, $C$ is cofinal in $P$. Conversely, let $A \subseteq P$ such that $h_P(A)$ is cofinal in $h(P)$. According to König’s Lemma (cf. Theorem 36), $\downarrow A$ contains a chain $C$ with order type $h(P)$. According to $(iv)$, $C$ is cofinal in $P$. Since $C \subseteq \downarrow A$, $A$ is cofinal in $P$. Finally, these conditions imply that every level $P_\alpha$ is majorized. Indeed, an element $x$ of $P$ majorizes $P_\alpha$ iff $x \notin A_\alpha := \bigcup_{y \in P_\alpha} P \setminus \uparrow y$. Since $P_\alpha$ is finite, $A_\alpha$ is not cofinal in $P$, hence $P \setminus A_\alpha \neq \emptyset$. \]

Let $\kappa$ be an ordinal and $\varphi : \kappa \to \kappa$ be an order-preserving and extensive map. Denote by $\varphi^{(n)}$ the $n$-th iterate of $\varphi$. Set $A_0 := \{ \varphi^{(n)}(0) : n < \omega \}$ and set $A_\alpha := \{ \varphi^{(n)}(\delta_\alpha) : n < \omega \}$ where $\delta_\alpha$ is the least element of $\kappa \setminus \bigcup_{\beta < \alpha} A_\beta$. Let $h(\varphi)$ be the least $\alpha$ such that $A_\alpha = \emptyset$ (equivalently $\kappa = \bigcup_{\beta < \alpha} A_\beta$). Clearly each $A_\alpha$ is an interval of $\kappa$ preserved under $\varphi$.

Lemma 35. Let $P$ be a well founded poset and $\kappa := h(P)$. Then $P$ is $h_P$-uniform iff $P$ is a lexicographical sum $\sum_{\alpha < \kappa} Q_\alpha$ where each $Q_\alpha$ is $h_{Q_\alpha}$-uniform.

Proof. Let $\varphi : \kappa \to \kappa$ be an order-preserving and extensive map $\varphi$ witnessing that $P$ is $h_P$-uniform. For each $\alpha < h(\varphi)$, let $Q_\alpha$ be the restriction of $P$ to $h_P^{-1}(A_\alpha)$. Then $P$ is the lexicographic sum $\sum_{\alpha < h(\varphi)} Q_\alpha$ and each $Q_\alpha$ is $h_{Q_\alpha}$-uniform. The fact that every element of $Q_\alpha$ is dominated by every element of $Q_{\alpha'}$ for $\alpha < \alpha'$
follows from the uniformity of $\varphi$. Hence $P$ is the lexicographical sum of the $Q_\alpha$’s. We have $h_P(x) = \alpha + h_{Q_\alpha}(x)$ for every $x \in Q_\alpha$. Hence, $\varphi_\alpha$ defined by setting $\varphi_\alpha(\nu) := \varphi(\alpha + \nu)$ witnesses that $Q_\alpha$ is $h_{Q_\alpha}$-uniform.

This result yields an other proof of Theorem 24 in the case of well founded posets.

4.2. Spectrum and uniformity. The spectrum of a poset $P$ is the set $\text{Spec}(P)$ of order types of its linear extensions (this notion was introduced in [4]). If $P$ is finite, $|P| = n$, then all linear extensions of $P$ are isomorphic to the $n$-element chain, hence $|\text{Spec}(P)| = 1$. The cardinality of the spectrum of an infinite antichain of size $\kappa$ is the number of isomorphic types of chains of cardinality $\kappa$. This number is $2^\kappa$; consequently, $|\text{Spec}(P)| \leq 2^{|P|}$ for every infinite poset $P$. As shown in [4], the equality holds if $P$ contains an antichain of cardinality $|P|$. According to de Jongh-Parikh’s theorem, if $P$ is w.q.o., the cardinality of its spectrum is at most $|P|$. If in addition $P$ is Jónsson then, according to Proposition [11] its spectrum reduces to a single element. Hence, among w.q.o. posets, those whose spectrum reduces to a single element generalize Jónsson posets. As we will see below, some properties of Jónsson posets, as Theorem 24, extend to w.q.o.’s.

A well founded poset $P$ is level-finite if $P_\alpha$ is finite for every ordinal $\alpha$. Let us recall the following version of König’s Lemma.

**Theorem 36.** Every well founded and level-finite poset $P$ contains a chain which intersects each level. Consequently, the supremum of the lengths of chains contained in $P$ is attained and is equal to $h(P)$.

From this result, it follows immediately that a Jónsson poset $P$ of cardinality $\kappa$ which is level-finite has a cofinal chain of order type $\text{cf}(\kappa)$ (Theorem 14). Indeed, let $C$ be a chain going through all the levels. Since $P$ is infinite, $C$ has cardinality $\kappa$. Since $P$ is Jónsson, $C$ is cofinal in $P$ and, as a chain, has order type $\kappa$.

Let $\alpha$ be an ordinal; as it is well-known there is unique pair of ordinals $\beta, r$ such that $\alpha = \omega \cdot \beta + r$ and $r < \omega$. The ordinal $\omega \cdot \beta$, denoted by $\ell(\alpha)$, is the limit part of $\alpha$, the ordinal $r$, denoted $\alpha \mod \omega$, is the remainder.

Let $P$ be a well founded poset. Let $\alpha := h(P)$. Set $\ell(P) := \{x \in P : h_P(x) < \ell(\alpha)\}$ and $\text{res}(P) := \{x \in P : h(x) \geq \ell(\alpha)\}$.

**Lemma 37.** If $P$ is well founded and level finite then $\ell(h(P)) + |\text{res}(P)|$ is the least order type among the linear extensions of $P$.

**Proof.** The lexicographical sum $\sum_{\alpha < h(P)} P_\alpha$, where $P_\alpha$ is a linear order on $P_\alpha$, is a linear extension of $P$. Its type is $\ell(h(P)) + |\text{res}(P)|$. If $\overline{P}$ is a linear extension of $P$, let $a$ be the least element of $\text{res}(P)$ in that linear extension. This element must be minimal in $\text{res}(P)$, hence $h_P(a) = \ell(h(P))$. Due to Theorem 36, the
initial segment $\downarrow a$ of $P$ contains a chain with order type $\ell(h(P))$. Hence $\overline{P}$ contains a chain with order type $\ell(h(P)) + \lvert \text{res}(P) \rvert$. □

- A well founded poset $P$ with $\lvert \text{Spec}(P) \rvert < 2^{\aleph_0}$ is w.q.o. Indeed, if $P$ has an infinite antichain it has a countable one, say $A$. Let $A^- := \{ x \in P : x < a \text{ for some } a \in A \}$ and $A^+ = P \setminus (A \cup A^+)$. Since $P$ is well founded, $A^-$ and $A^+$ are well founded too, hence they have a well-ordered linear extension, say $\overline{A^-}$ and $\overline{A^+}$. Each linear order $\overline{A}$ on $A$ yields the linear extension $\overline{A^-} + \overline{A^+}$. To get $2^{\aleph_0}$ distinct linear extensions, select linear orders on $A$ of the form $Q_1 + B + Q_2$ where $Q_1$ and $Q_2$ are isomorphic to the chain $Q$ of rational numbers and $B$ is a countable scattered chain (i.e. does not contain a copy of the chain $Q$). From the fact that there are $2^{\aleph_0}$ such $B$ which are pairwise non isomorphic the conclusion follows.

Let $P$ be a w.q.o. and let $m(P)$ be the least order type among the linear extensions of $P$. By definition $m(P) \leq o(P)$; the equality hold iff $\lvert \text{Spec}(P) \rvert = 1$. As shown below, the description of wqo’s $P$ which have a unique order type of linear extension reduces to those for which the height is a limit ordinal.

**Lemma 38.** Let $P$ be a wqo. If $h(P)$ is not a limit ordinal then $\lvert \text{Spec}(P) \rvert = 1$ iff $P$ is the lexicographical sum $\ell(P) + \text{res}(P)$ and $\text{Spec}(\ell(P)) = 1$.

**Proof.** If $P$ is the lexicographical sum $\ell(P) + \text{res}(P)$ then every linear extension $\overline{P}$ of $P$ is the sum of a linear extension of $\ell(P)$ and a linear extension of $\text{res}(P)$. Since $\text{res}(P)$ is finite, its linear extensions are isomorphic to a $m$-element chain where $m = \lvert \text{res}(P) \rvert$, hence $\text{Spec}(P) = \text{Spec}(\ell(P))$. Suppose that $\text{Spec}(P) = 1$. Then, according to Lemma 37, the linear extensions of $P$ have order type $\ell(h(P)) + \lvert \text{res}(P) \rvert$. Let $a$ be a minimal element of $\text{res}(P)$. We claim that $a$ dominates $\ell(P)$. Otherwise, $\ell(P) \setminus \downarrow a$ is non-empty, hence $(\downarrow a) \cap \ell(P) + \ell(P) \setminus a + \text{res}(P)$ is an extension of $P$ whose every linear extension has type larger that $\ell(h(P)) + \lvert \text{res}(P) \rvert$. A contradiction. Since every minimal element of $\text{res}(P)$ dominates $\ell(P)$, $P$ is the lexicographical sum $\ell(P) + \text{res}(P)$. □

**Proposition 39.** Let $P$ be a w.q.o. such that $\nu := h(P)$ is a limit ordinal. If $\lvert \text{Spec}(P) \rvert = 1$ then $P$ is $h_P$-weakly uniform.

**Proof.** If $P$ is not $h_P$-weakly uniform then by Lemma 34 $P$ is not $h_P$-minimal, hence there is some $A$ not cofinal in $P$ with $h_P(A)$ cofinal in $\nu$. Let $\overline{A} := \downarrow A$. Then $\overline{A} \not\subseteq P$. Since $h_P \upharpoonright \overline{A} = h_P(\overline{A})$ we have $h_P(\overline{A}) = \nu$. Hence $o(P) > o(\overline{A}) \geq h_P(\overline{A}) = h(P) = m(P)$. This yields $\lvert \text{Spec}(P) \rvert \neq 1$. □

The converse of this property does not hold. Let $\alpha$ be an ordinal, denote by $C_\alpha$ any chain of order type $\alpha$. Let $D_{\alpha, \beta} := (C_\alpha \oplus C_\alpha) + C_\beta$ be the lexicographic sum of the direct sum of two copies of $C_\alpha$ with a copy of $C_\beta$. Then $h(D_{\alpha, \beta}) = \alpha + \beta$ and $D_{\alpha, \beta}$ is $h_{D_{\alpha, \beta}}$-uniform. Also $o(D_{\alpha, \beta}) = (\alpha \oplus \alpha) + \beta$ (the symbol $\oplus$ denotes the Heissenberg sum). If $\alpha = \beta$ then this quantity is strictly greater than $m(P)$.
hence $|\text{Spec}(P)| > 1$. If $\beta$ is indecomposable and $\alpha < \beta$ then $o(P) = \beta$ and $|\text{Spec}(P)| = 1$.

Here is an example of well founded poset $Q$ which is $h_Q$-weakly uniform but not $h_Q$-uniform. Let $Q$ be disjoint union of a chain $C_\delta$ of order type $\delta := \omega_\beta$, and a chain $C_\gamma$ of order type $\gamma := \delta^\omega := \sum_{\alpha < \delta} \delta^\alpha$. Select in $C_\gamma$ a strictly increasing and cofinal sequence $(c_\alpha)_{\alpha < \delta}$ with $c_0 \geq \omega$. And for $x, y \in Q$, set $x < y$ if either $x, y$ are ordered according to one of the two chains, or $x := \alpha \in C_\delta$ and $y \in C_\gamma$ with $y \geq c_\alpha$. Then $h(Q) := \gamma = o(Q)$ and $Q$ is $h_Q$-weakly uniform but not $h_Q$-uniform.

5. A conjecture on w.q.o.'s

C.St.J.A. Nash-Williams introduced the notion of better-quasi-ordering (b.q.o.), a strengthening of the notion of w.q.o. (cf. [22], [23]). The operational definition is not intuitive; since we are not going to prove results about b.q.o.s, an intuitive definition is enough. Let $P$ be a poset and $P^{< \omega_1}$, the set of maps $f : \alpha \to P$, where $\alpha$ is any countable ordinal. If $f$ and $g$ are two such maps, we set $f \leq g$ if there is a one-to-one preserving map $h$ from the domain $\alpha$ into the domain $\beta$ of $g$ such that $f(\gamma) \leq g(h(\gamma))$ for all $\gamma < \alpha$. This relation is a quasi-order; the poset $P$ is a better quasi-order if $P^{< \omega_1}$ is w.q.o. B.q.o.s are w.q.o.s. As w.q.o.s, finite sets and well-ordered sets are b.q.o. (do not try to prove it with this definition), finite unions, finite products, subsets and images of b.q.o. are b.q.o. But, contrarily to w.q.o., if $P$ is b.q.o. then $I(P)$ is bqo.

In [2] was made the following:

**Conjecture 40.** Every w.q.o. is a countable union of b.q.o.

In order to attack this conjecture, the second author asked more: is every w.q.o. a countable union of posets, each one being a strengthening of some finite dimensional poset?

To prove the validity of these conjectures it is natural to use induction. Induction can be on the ordinal type or, perhaps better, on the ordinal rank of antichains of w.q.o.’s: if $P$ is a poset, order the set $\mathcal{A}(P)$ of antichains by reverse of the inclusion, if $P$ has no infinite antichain, $\mathcal{A}(P)$ is well founded, so the empty antichain $\emptyset$ has a height in this poset, we denote it by $\text{rank}(\mathcal{A}(P))$. It was shown by Abraham [1] that $\text{rank}(\mathcal{A}(P)) < \omega_1^2$ iff $P$ is a countable union of chains. So for this value of the rank, the conjecture holds. In particular it holds for w.q.o. posets for which $o(P) < \omega_1^2$ simply because $\text{rank}(\mathcal{A}(P)) \leq o(P)$. In fact, if $o(P) < \omega_1^2$, Abraham’s result is immediate, indeed $P$ decomposes into a countable union of posets $P_\alpha$, each of ordinal length a most $\omega_1$. Hence $P$ decomposes into a countable union of Jónsson posets. According to Corollary 25 each one is a countable union of chains hence $P$ is a countable union of chains. The first instance of poset for which the conjecture poses problem is a w.q.o. which is an uncountable union of chains of type $\omega_1$ and not less.
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