FIRST-ORDER HYPERBOLIC PSEUDODIFFERENTIAL EQUATIONS WITH GENERALIZED SYMBOLS

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Abstract. We consider the Cauchy problem for a hyperbolic pseudodifferential operator whose symbol is generalized, resembling a representative of a Colombeau generalized function. Such equations arise, for example, after a reduction-decoupling of second-order model systems of differential equations in seismology. We prove existence of a unique generalized solution under log-type growth conditions on the symbol, thereby extending known results for the case of differential operators [17, 19].

Keywords: Colombeau algebra, generalized solution, hyperbolic pseudodifferential Cauchy problem
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1. Introduction

This paper establishes existence and uniqueness of a generalized solution to the scalar hyperbolic pseudodifferential Cauchy problem

\[ \frac{\partial}{\partial t} u + A(t, x, D_x) u = f \quad \text{when } t \in (0, T), \]

\[ u(0) = g. \]

The data \( f \) and \( g \) are Colombeau generalized functions and \( A \) is a generalized pseudodifferential operator of order 1. Its symbol is represented by a family of smooth regularizations, which may (but need not) be convergent to a distributional symbol. Problem (1-2) represents an extension of the scalar case of the partial differential equations considered by Lafon and Oberguggenberger in [17, 19].

One may think of problem (1-2) as resulting from a system of second-order (partial differential) equations by reduction to first-order followed by a decoupling into scalar equations (cf. [22, Section IX.1]). This is a standard technique in applications, for example, in mathematical geophysics, where one decouples the modes of seismic propagation and subjects these to further refined analysis (cf. [21]). As they stand, these reduction-decoupling methods are rigorously applicable in the case of models with smooth coefficients or symbols, but cease to be well-defined under the realistic assumptions of only measurable (bounded) coefficients, which are to represent the elastic or acoustic properties of the earth’s subsurface. Moreover, the initial value and the right-hand side are distributions corresponding to the original seismic source and force terms, which are, by nature, strongly singular, e.g., delta-like. If the original model coefficients are replaced by regularizations, then we may carry out all transformations within algebras of generalized functions from the outset and arrive at (1-2) in a well-defined way. The purpose of the current paper is to investigate the general feasibility of rigorously solving the resulting decoupled,
so-called one-way wave equation, by generalized functions. Future work will be
devoted to the regularity analysis of the solutions and their asymptotic relations
with distributions.

A word on conventions and notations concerning the Fourier transform: if $u$ is a
temperate distribution on $\mathbb{R}^n$ we denote its Fourier transform by $\hat{u}$ or $\mathcal{F}u$; occasion-
ally, when several variables and parameters are involved, we write expressions of the form
$\mathcal{F}_{x_\rightarrow \xi}(u(y, x))$ to indicate that the transform acts on the partial function
(or distribution) $u(y, \cdot)$ and yields a function (or distribution) in $(y, \xi)$; the inte-
gral formulas for the transforms follow the convention $\mathcal{F}u(-x)/(2\pi)^n = \mathcal{F}^{-1}u(x) =
\int \exp(ix\xi) u(\xi) d\xi$, where $d\xi = d\xi/(2\pi)^n$.

Subsections 1.1-3 serve to review Colombeau theory, fix our notations for general-
ized symbols, and also recall the corresponding result on the Cauchy problem for
hyperbolic differential operators with generalized coefficients. Section 2 establishes
precise energy estimates, which are at the heart of the existence and uniqueness
properties below. As general references and for discussions of the overall properties
of Colombeau algebras we refer to the literature (e.g. [4, 9, 20]).

We consider the space-time domain $X_T := \mathbb{R}^n \times (0, T)$. The basic objects defining
our generalized functions are regularizing families $(u_\varepsilon)_{\varepsilon \in (0, 1]}$ of smooth functions
$u_\varepsilon \in H^\infty(X_T)$ for $0 < \varepsilon \leq 1$, where $H^\infty$ denotes the intersection over all Sobolev
spaces. To simplify the notation, we shall write $(u_\varepsilon)_\varepsilon$ in place of $(u_\varepsilon)_{\varepsilon \in (0, 1]}$
throughout. We single out the following subalgebras:

- **Moderate families**, denoted by $\mathcal{E}_{M,L^2}(X_T)$, are defined by the property:
  \[
  \forall \alpha \in \mathbb{N}_0^n \exists p \geq 0 : \|\partial^\alpha u_\varepsilon\|_{L^2} = O(\varepsilon^{-p}) \text{ as } \varepsilon \to 0.
  \]

- **Null families**, denoted by $\mathcal{N}_{L^2}(X_T)$, are the families in $\mathcal{E}_{M,L^2}(X_T)$ having the fol-
lowing additional property:
  \[
  \forall q \geq 0 : \|u_\varepsilon\|_{L^2} = O(\varepsilon^q) \text{ as } \varepsilon \to 0.
  \]

Hence moderate families satisfy $L^2$-estimates with at most polynomial divergence
as $\varepsilon \to 0$, together with all derivatives, while null families vanish faster than any
power of $\varepsilon$ in the $L^2$-norms. For the latter, one can show that, equivalently, all
derivatives satisfy estimates of the same kind (cf. [4]). The null families form a
differential ideal in the collection of moderate families. The *Colombeau algebra*
$\mathcal{G}_{L^2}(X_T)$ is the factor algebra
\[
\mathcal{G}_{L^2}(X_T) = \mathcal{E}_{M,L^2}(X_T)/\mathcal{N}_{L^2}(X_T).
\]

(The notation in [1] is $\mathcal{G}_{2,2}$, and correspondingly for moderate and negligible nets,
where the variability of $L^2$-norms in the definitions was essential.) The algebra
$\mathcal{G}_{L^2}(\mathbb{R}^n)$ is defined in exactly the same way and its elements can be considered as elements of $\mathcal{G}_{L^2}(X_T)$. On the other hand, as explained in [1] Remark 2.2(i) and
Definition 2.8, the restriction of a generalized function from $\mathcal{G}_{L^2}(X_T)$ to $t = 0$
is well-defined: for any representative $(u_\varepsilon)_\varepsilon$ in $\mathcal{E}_{M,L^2}(X_T)$ we have $u_\varepsilon \in C^\infty(\mathbb{R}^n \times
Distributions in $H^{-\infty}(\mathbb{R}^n) = \bigcup_{s \in \mathbb{R}} H^s(\mathbb{R}^n)$ are embedded in $\mathcal{G}_{L^2}(\mathbb{R}^n)$ by convolution: $u(w) = [(w * (\rho_\varepsilon))_\varepsilon]$, where

$$\rho_\varepsilon(x) = \varepsilon^{-n} \rho(x/\varepsilon)$$

is obtained by scaling the fixed mollifier $\rho$, i.e., a test function $\rho \in \mathcal{S}(\mathbb{R}^n)$ of integral one with all moments (of order 1 and higher) vanishing. This embedding renders $H^\infty(\mathbb{R}^n)$ a faithful subalgebra (cf. [11 Theorem 2.7(ii)]). In fact, given $f \in H^\infty(\mathbb{R}^n)$, one can define the corresponding element of $\mathcal{G}_{L^2}(\mathbb{R}^n)$ by $[(f)_\varepsilon]$ (with representative independent of $\varepsilon$). In the same way we may consider $H^\infty(X_T)$ a faithful subalgebra of $\mathcal{G}_{L^2}(\mathcal{D}')$.

Some Colombeau generalized functions behave macroscopically like a distribution. We say that $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}_{L^2}$ is associated with the distribution $w \in \mathcal{D}'$, denoted by $u \approx w$, if $u_\varepsilon \to w$ in $\mathcal{D}'$ as $\varepsilon \to 0$.

Intrinsic regularity theory for Colombeau generalized functions has been a subject of active research. Its foundation is [29, Section 25] with the definition of the subalgebra $\mathcal{G}_\infty$ of $\mathcal{G}$, which plays the same role for $\mathcal{G}$ as $C^\infty$ does within $\mathcal{D}'$. The basic idea is to couple the generalized regularity notion to uniform $\varepsilon$-growth in all derivatives and it leads to the important compatibility relation

$$\mathcal{G}_\infty \cap \mathcal{D}' = C^\infty.$$

Similarly, we define here the subalgebra $\mathcal{G}_{L^2}^\infty$ of regular elements of $\mathcal{G}_{L^2}$ by the following condition: $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}_{L^2}$ if and only if

$$\exists p \geq 0 \forall \alpha \in \mathbb{N}^n_0 : \|\partial^\alpha u_\varepsilon\|_{L^2} = O(\varepsilon^{-p}) \text{ as } \varepsilon \to 0.$$

Observe that $p$ can be chosen uniformly over all $\alpha$. In particular, if $u_\varepsilon = v * \rho_\varepsilon$ with $v \in H^\infty$, then $p = 0$ is possible when we let fall all derivatives on the factor $v$.

Concerning sources for recent and related research in Colombeau theory, with a diversity of directions, including such topics as pseudodifferential operators with generalized symbols, regularity theory, and microlocal analysis of nonlinear singularity propagation we refer to [7, 11, 12, 13, 14, 18].

1.2. Generalized pseudodifferential operators. For comprehensive theories of approaches to pseudodifferential operators with Colombeau generalized functions as symbols we may refer to the recent literature on the subject [7, 11, 12, 13, 14, 18]. However, the purpose of the present paper is to present a short and self-contained discussion of the solution to the hyperbolic pseudodifferential Cauchy problem. Therefore we do not need to call on the full theory of generalized symbol classes, mapping properties, and symbol calculus, as it has been extended systematically and with strong results in [7, 11, 12, 13, 14, 18]. Nevertheless, this background will be substantial in further development, refinements, and applications of the current work, in particular, concerning regularity theory and microlocal analysis.

We will use families of smooth symbols satisfying uniform estimates with respect to the $x$ (and $t$) variable as described in [11, 12]. To fix notation, let us review the definition. A complex valued function $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ belongs to the symbol class $S^m$ of order $m \in \mathbb{R}$ if for all $(\alpha, \beta) \in \mathbb{N}^n_0$

$$c^m_{\alpha, \beta}(a) := \sup_{(x, \xi) \in \mathbb{R}^{n+p}} (1 + |\xi|)^{-m+|\alpha|} |\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| < \infty.$$
$S^m$ is a Fréchet space when equipped with the semi-norms
\begin{equation}
q^{m}_{k,l}(a) := \max_{|\alpha| \leq k, |\beta| \leq l} \epsilon^{\alpha \beta} c_{\alpha \beta}^{m}(a),
\end{equation}
a notation we will make use of freely in several estimates in the sequel. \((\text{Observe that compared to the semi-norms and notation used in [16] we have } |a|^{(m)} = \max \{q^{m}_{k,r}(a) : k + r \leq l\}.\) In fact, we will use symbols which depend smoothly on time, considered as a parameter. More precisely, we consider the space of symbols $a(t, x, \xi)$ where $a \in C^\infty([0, T], S^m)$ (i.e., each $t$-derivative on $(0, T)$ is continuous into $S^m$ up to the boundary $t = 0$ and $t = T$) with the semi-norms
\begin{equation}
Q^{m}_{j,k,l}(a) := \max_{0 \leq t \leq T} \sup_{j \leq j} q^{m}_{j,k,l}(\partial_x^t a(t, .., )).
\end{equation}

By a \textit{generalized symbol} we mean a family $(a_\varepsilon)_{\varepsilon \in [0, 1)}$ of smooth symbols in $S^m$ (the same $m$ for all $\varepsilon$) which satisfy moderate semi-norm estimates, i.e., for all $k$ and $l$ in $\mathbb{N}_0$ there is $N \in \mathbb{N}_0$ such that
\begin{equation}
q^{m}_{k,l}(a_\varepsilon) = O(\varepsilon^{-N}) \quad (\varepsilon \to 0).
\end{equation}
Generalized symbols with parameter $t \in [0, T]$ are given by families $(t, x, \xi) \mapsto a_\varepsilon(t, x, \xi)$ \((\varepsilon \in (0, 1)]\) such that $a_\varepsilon \in C^\infty([0, T], S^m)$ with moderate semi-norm estimates: for all $j, k, l$ in $\mathbb{N}_0$ there is $N \in \mathbb{N}_0$ such that
\begin{equation}
Q^{m}_{j,k,l}(a_\varepsilon) = O(\varepsilon^{-N}) \quad (\varepsilon \to 0).
\end{equation}
Obviously, no major changes would be required to incorporate more general types of symbols, especially the Hörmander’s classes $S^m_{\rho, \delta}$ would mainly require changes in notation (at least when $0 \leq \rho < \delta < 1$).

Let $(a_\varepsilon)_\varepsilon$ be a generalized symbol with parameter $t \in [0, T]$. We define the corresponding linear operator
\[ A : S_{L^2}(X_T) \to S_{L^2}(X_T) \]
in the following way. On the representative level, $A$ acts as the diagonal operator
\[ (u_\varepsilon)_\varepsilon \mapsto (a_\varepsilon(t, x, D_x)u_\varepsilon)_\varepsilon \quad \forall (u_\varepsilon)_\varepsilon \in E_{M, L^2}(X_T). \]
Here, $a_\varepsilon(t, x, D_x)$ acts as an operator in the $x$ variable with parameter $t$. The moderateness of $(a_\varepsilon(t, x, D_x)u_\varepsilon)_\varepsilon$ follows from \(8\) and the fact that operator norms of $\partial_x^\beta \circ a_\varepsilon(x, t, D_x)$ on Sobolev spaces are bounded (linearly) by finitely many semi-norms of the symbol (cf. [16] Ch. 3, Theorem 2.7)). In the same way, it follows that null families are mapped into null families, so that $A$ is well-defined on equivalence classes. We call $A$ the \textit{generalized pseudodifferential operator} with generalized symbol $(a_\varepsilon)_\varepsilon$.

1.3. \textbf{Review of hyperbolic partial differential equations with generalized coefficients.} We briefly review the situation for symmetric hyperbolic systems of partial differential equations in Colombeau algebras. The heart of this theory was developed in [17] [19], from where we recall the main result on the Cauchy problem.

The theory is placed in $\mathcal{G}$ instead of $\mathcal{G}_{L^2}$, i.e., the data $f, g$ as well as all coefficients satisfy asymptotic local $L^\infty$-estimates of the kind described in the introduction. In view of our intended generalization of the scalar case to pseudodifferential operators, let us simply focus on this situation in the Cauchy problem [12]. We have $f \in \mathcal{G}(\mathbb{R}^{n+1})$ and $g \in \mathcal{G}(\mathbb{R}^n)$ and the spatial operator $A$ is a differential operator of the form
\[ A = \sum_{j=1}^n a_j(x, t)\partial_{x_j} + b(x, t) \]
where the coefficients $a_j, b$ are in $\mathcal{S}(\mathbb{R}^{n+1})$, $a_j$ real. Note that a generalized symbol for $A$ is given by

\begin{equation}
(10) \quad i \sum_{j=1}^{n} a_{j,\varepsilon}(x,t)\xi_j + b_{\varepsilon}(x,t),
\end{equation}

where $a_{j,\varepsilon}, b_{\varepsilon}$ are any representatives of $a_j, b$; $a_{j,\varepsilon}$ may taken to be real-valued.

Sufficient conditions for existence and uniqueness of a solution $u \in \mathcal{S}(\mathbb{R}^{n+1})$ to (1-2) are as follows:

(i) $a_j, b$ are equal to a (classical) constant for large $|x|$ (any kind of uniform boundedness in $x$ and $\varepsilon$ for large $|x|$ would do; it ensures uniqueness and enables one to use partition of unity arguments in the proof)

(ii) $b$ as well as $D_k a_j$ are of log-type, i.e., the asymptotic norm estimates (of order 0) have bounds $O(\log(1/\varepsilon))$ (this ensures existence by guaranteeing moderateness from energy estimates).

Counter examples show that none of the two conditions can be dropped without losing existence or uniqueness in general.

**Remark 1.1.** It turns out that the non-uniqueness effect as constructed in [19] Example 1.4 disappears in $\mathcal{S}_{L^2}$. (In mentioned example, the constructed initial value $v(x,0) = [(\chi(x+1/\varepsilon))]$, $\chi \in \mathcal{D}(\mathbb{R})$ with $\chi(0) = 1$, is negligible in $\mathcal{S}$ but gives $\|v\|_{L^2} = \|\chi\|_{L^2} > 0$, hence is nonzero on $\mathcal{S}_{L^2}$.) As a matter of fact, the $L^2$-energy estimates, to be discussed in the following section, directly yield uniqueness; this holds even with coefficients that allow for logarithmic growth as $\varepsilon \to 0$ throughout the entire domain.

The non-locality of pseudodifferential operators seems to prohibit an adaption of the proof technique of [17], where one is able to pass from $L^2$-energy estimates to local $L^\infty$-estimates. On the other hand, when working in $\mathcal{S}_{L^2}$, there is also the structural advantage of having good mapping properties of pseudodifferential operators with uniform symbol estimates on Sobolev spaces.

### 2. Preparatory energy estimates

Our proof of unique solvability of the Cauchy problem will be based on energy estimates, with precise growth estimates of all appearing constants depending on the regularization parameter $\varepsilon$ as $\varepsilon \to 0$. This in turn is solely encoded into the generalized symbol in form of the semi-norm estimates of the regularizing (resp. defining) family of symbols. Therefore, and also to make the structure more transparent, we will first state the preparatory estimates for smooth symbols in terms of explicit dependencies on symbol semi-norms and insert the $\varepsilon$-asymptotics only later on.

In order to maintain close resemblance in notation with the cases of differential operators or decoupled systems, we shall write the symbol of $A$ in the form $i a(t, x, \xi)$ with $a \in C^\infty([0, T], S^1)$; in other words, we review energy estimates for the operator

\begin{equation}
(11) \quad P := \partial_t + i a(t, x, D_x)
\end{equation}

under the hyperbolicity assumption

\begin{equation}
(12) \quad a = a_1 + a_0 \quad \text{with } a_1 \text{ real-valued, } a_0 \text{ of order 0},
\end{equation}
or equivalently, that
\[ a(t, x, D_x) - a(t, x, D_x)^* \text{ is of order 0.} \]

Besides stating the general case in the following proposition we also give details on
two special instances. These are of interest in applications and allow for certain
improvements concerning the regularity assumptions in terms of symbol derivatives,
which are required in the constants of the basic energy estimate.

**Proposition 2.1.** Assume that \( P \) is the operator given in (11) and such that
(12) holds. Let \( u \in C([0, T], H^1(\mathbb{R}^n)) \cap C^1([0, T], L^2(\mathbb{R}^n)) \) and define \( f := Pu \in C([0, T], L^2(\mathbb{R}^n)) \). Then we have the energy estimate

\[
\begin{align*}
\|u(t)\|_{L^2}^2 &\leq \|u(0)\|_{L^2}^2 + \int_0^t \|f(\tau)\|_{L^2}^2 d\tau + \\
&+ C \left( 1 + Q_{0, k', l_n}(a_0) + Q_{0, k_n, l_n}(a_1) \right) \int_0^t \|u(\tau)\|_{L^2}^2 d\tau,
\end{align*}
\]

where the constant \( C > 0 \) as well as \( k', l_n, k_n, l_n \) are independent of \( u \) and can be
chosen according to certain assumptions on the symbol \( a \) as follows:

(a) **General case:** We have \( k'_n = l'_n = \lfloor n/2 \rfloor + 1, k_n = 3\lfloor n/2 \rfloor + 1, l_n = 2(n + 2) \) and \( C \) depends only on the dimension \( n \).

(b) **Constant for large \(|x|\):** If there is \( r_0 \geq 0 \) such that

\[
a(t, x, \xi) = h(t, \xi) \quad \text{whenever } |x| \geq r_0,
\]

where \( h \) is a symbol of order 1 (with parameter \( t \) and no \( x \) variable), then
\( C \) depends only on \( n, r_0 \) and the semi-norm orders are at most \( k'_n = 0, l'_n = n + 1, k_n = 1, l_n = n + 2 \).

(c) **Real symbol:** If in addition (to any of the assumptions above) the symbol \( a \) is real-valued, so that \( a_0 \) is real as well in (12), then the term
\( Q_{0, k', l_n}(a_0) \) can be dropped in (13).

**Proof.** Using the standard decomposition of the operator \( a_1(t, x, D_x) \) into self- and
skew-adjoint part, \( a_1 = (a_1 + a_1^*)/2 + (a_1 - a_1^*)/2 \), we obtain

\[
\begin{align*}
\frac{d}{dt}\|u(t)\|_{L^2}^2 &= 2 \text{Re}\langle \partial_t u(t)|u(t) \rangle \\
&= 2 \text{Re}\langle f(t)|u(t) \rangle + 2 \text{Im}\langle a_1(t, x, D_x)u(t)|u(t) \rangle + 2 \text{Im}\langle a_0(t, x, D_x)u(t)|u(t) \rangle \\
&\leq \|f(t)\|_{L^2}^2 + \|u(t)\|_{L^2}^2 + \|a_1(t, x, D_x) - a_1(t, x, D_x)^*\| \|u(t)\|_{L^2}^2 \\
&\quad + 2\|a_0(x, D)\| \|u(t)\|_{L^2}^2.
\end{align*}
\]

The operator norms (taken with respect to \( L^2 \)) are finite by (12) and (12)' (cf. [10] Theorem 18.1.11 or [16] Ch.2, Theorem 4.1]) and we will derive explicit estimates
for these.

For the proof of case (c), observe that we have \( \text{Im}\langle a_0(t, x, D_x)u(t)|u(t) \rangle = 0 \) if \( a_0 \)
is real, so that the last term on the right-hand side of (15) can be dropped from all
further considerations.

**Case (a):** We use a representation of the symbol of \( b_1(t, x, D_x) := a_1(t, x, D_x) - a_1(t, x, D_x)^* \) with integral remainder terms as it is developed in [10] Ch.2.1-3] or
[5] Ch.1.5-6]. According to this (or as sketched in the Appendix below), the zero
order symbol $b_1(t, x, \xi)$ is given by
\begin{equation}
(16) \quad b_1(t, x, \xi) := a_1(t, x, \xi) - a_1(t, x, \xi) - i \sum_{j=1}^{n} \int_{0}^{1} r_{j, \theta}(t, x, \xi) d\theta
\end{equation}
\begin{equation*}
= -i \sum_{j=1}^{n} \int_{0}^{1} r_{j, \theta}(t, x, \xi) d\theta,
\end{equation*}
since $a_1(t, x, \xi)$ is real-valued, where
\begin{equation}
(17) \quad r_{j, \theta}(t, x, \xi) = \int e^{-is\eta} \frac{\partial_j}{\partial x_j} a_1(t, x + y, \xi + \theta \eta) dy d\eta
\end{equation}
in the sense of oscillatory integrals.

As a close inspection of the proof of [16] Ch.2, Lemma 2.4 shows (which we detail in the Appendix), we have the following estimate for all $d \in \mathbb{N}_0$, $(\alpha, \beta) \in \mathbb{N}_0^{2n}$
\begin{equation}
(18) \quad |\partial_t^d \partial_{\xi}^d \partial_x^2 r_{j, \theta}(t, x, \xi)| \leq C_{d, \alpha, \beta} (1 + |\xi|)^{-|\alpha|} Q^1_{d, n+2+|\alpha|, n+2+|\alpha|+|\beta|}(a_1),
\end{equation}
which is uniform with respect to $\theta \in [0, 1]$. Combined with formula (16) for $b_1$ this yields
\begin{equation*}
Q^0_{d, k, l}(b_1 + 2a_0) \leq 2 Q^0_{d, k, l}(a_0) + C_{k, l} Q^1_{d, n+2+k, n+2+k+l}(a_1).
\end{equation*}
By the theorem of Calderón-Vaillancourt (or one of its variants, cf. [2, Ch.I, Théorème 3], [13, Ch.3, Corollary 1.3]), we have the general $L^2$-operator norm estimate
\begin{equation*}
\|b(t, x, D_x)\| \leq C_n Q^0_{0, [n/2]+1, [n/2]+1}(b)
\end{equation*}
whenever $b \in C^\infty([0, T], S^0)$. Therefore we conclude that
\begin{equation*}
\|b_1(t, x, D_x)\| + 2\|a_0(x, D)\| \leq C_n \left( Q^0_{0, [n/2]+1, [n/2]+1}(a_0) + Q^1_{0, k, l}(a_1) \right)
\end{equation*}
for any $k \geq n + |n/2| + 3$ and $l \geq n + 2|n/2| + 4$. This completes the proof of the general case.

Case (b): Let $\chi \in \mathcal{D}(\mathbb{R}^n)$ such that $\chi(x) = 1$ for $|x| \leq r_0$ and $0 \leq \chi \leq 1$. Then the term $a(t, x, \xi) - a(t, x, \bar{\xi})$ occurring in (16) can be written in the form
\begin{equation*}
\chi(x)(a_0(t, x, \xi) - a_0(t, x, \bar{\xi}) + (1 - \chi(x))(h(t, \xi) - \bar{h}(t, \xi))
\end{equation*}
\begin{equation*}
:= b_0(t, x, \xi) + (1 - \chi(x))h_0(t, \xi).
\end{equation*}
The second part in this decomposition is the operator symbol of a convolution with bounded symbol (since $h_0 := h - \bar{h}$ is of order 0), composed with multiplication by $1 - \chi$ from the left. Hence the $L^2$ operator norm corresponding to this second summand has the following upper bound
\begin{equation*}
\|(1 - \chi(x))h_0(t, D_x)\| \leq \|1 - \chi\|_{L^\infty} \|h_0\|_{L^\infty} \leq 2 \|a_0\|_{L^\infty}.
\end{equation*}
Note that $b_0$ is a symbol of order zero with support contained in $|x| \leq r_0$. We will estimate the operator norm of $B_0 := b_0(t, x, D_x)$ (on $L^2(\mathbb{R}^n)$), uniformly with respect to $t \in [0, T]$ via the Schwartz kernel $\widetilde{K}_0$ of the “Fourier transformed” operator $\tilde{B}_0 := \mathcal{F} \circ B_0 \circ \mathcal{F}^{-1}$ and using the fact that
\begin{equation*}
\|B_0\| = (2\pi)^{n/2} \|\tilde{B}_0\|.
\end{equation*}
As a distribution in $C^\infty([0, T], \mathcal{S}'(\mathbb{R}^{2n}))$, the kernel is computed from the symbol by the formula
\begin{equation}
(19) \quad \widetilde{K}_0(t, \xi, \eta) = (2\pi)^{-n} \mathcal{F}(b_0(t, \xi, \eta))(\xi - \eta).
\end{equation}
Since $x \mapsto b_0(t, \xi, \eta)$ has compact support it follows that $\widetilde{K}_0$ is smooth on $[0, T] \times \mathbb{R}^{2n}$; in fact, we will see that it is an integrable kernel and hence we may apply a classical
We return to the scalar pseudodifferential Cauchy problem

\[
\partial_t u + Au = f \quad \text{in } X_T,
\]

\[
u(0) = g,
\]

where \(X_T = \mathbb{R}^n \times (0, T)\) and with data \(f \in \mathcal{S}_{L^2}(X_T)\) and \(g \in \mathcal{S}_{L^2}(\mathbb{R}^n)\). \(A\) is a generalized pseudodifferential operator of order 1. More precisely, we assume that \(A: \mathcal{S}_{L^2}(X_T) \to \mathcal{S}_{L^2}(X_T)\) is given by \((u_\varepsilon)_\varepsilon \mapsto (i a_\varepsilon(t, x, D_x) u_\varepsilon)_\varepsilon\),

Consider formula (17) and introduce the short-hand notation \(b_j := \partial_\xi, \partial_x \pi\). Then \(z \mapsto b_j(t, z, \zeta)\) has compact support in \(|z| \leq r_0\) and we may write

\[
r_{j, \theta}(t, x, \xi) = \mathcal{F}^{-1}_{\eta \to z} \left( \mathcal{F}(b_j(t, \cdot, \xi + \theta \eta))(\eta) \right).
\]

Now let \(R_{j, \theta} := r_{j, \theta}(t, x, D_x)\) and define, exactly as above, the corresponding operator \(\tilde{R}_{j, \theta}\) with intertwining Fourier transforms; denote by \(\tilde{K}_{j, \theta}\) its Schwartz kernel. The above representation for the symbol \(r_{j, \theta}\) in terms of \(b_j\) and direct computation yields the formula

\[
\tilde{K}_{j, \theta}(t, \xi, \eta) = (2\pi)^{-n} \mathcal{F}(b_j(t, \cdot, \eta + \theta(\xi - \eta)))(\xi - \eta).
\]

Equations (19) and (20) have the following structure in common: we have a symbol \(d \in C^\infty([0, T], S^0)\) which vanishes when \(|x| \geq r_0\) and a smooth kernel \(\tilde{K}\) defined by

\[
\tilde{K}(t, \xi, \eta) := (2\pi)^{-n} \mathcal{F}(d(t, \cdot, f(\xi, \eta)))(\xi - \eta),
\]

where \(f : \mathbb{R}^{2n} \to \mathbb{R}^n\) is a linear map. In order to apply Schur’s lemma we estimate the partial \(L^1\)-norms of the kernel and obtain

\[
(2\pi)^n \int_{\mathbb{R}^n} |\tilde{K}(t, \xi, \eta)| \, d\xi = \int_{\mathbb{R}^n} |\mathcal{F}(d(t, \cdot, f(\xi, \eta)))(\xi - \eta)| \, d\xi
\]

\[
= \int_{\mathbb{R}^n} |\mathcal{F}(d(t, \cdot, f(\xi + \eta, \eta)))(\xi)| \, d\xi \leq \int_{\mathbb{R}^n} \sup_{\xi} |\mathcal{F}(d(t, \cdot, \zeta))|(\xi) \, d\xi,
\]

and similarly

\[
(2\pi)^n \int_{\mathbb{R}^n} |\tilde{K}(t, \xi, \eta)| \, d\eta \leq \int_{\mathbb{R}^n} \sup_{\xi} |\mathcal{F}(d(t, \cdot, \zeta))(\eta)| \, d\eta.
\]

Assertion: There exists a constant \(c(n, r_0)\), depending only on \(n\) and \(r_0\), such that

\[
\int_{\mathbb{R}^n} \sup_{\xi} |\mathcal{F}(d(t, \cdot, \zeta))(\mu)| \, d\mu \leq c(n, r_0) \, Q^0_{0, n+1, 0}(d).
\]

Noting that \(\int |\partial^2_x d(t, x, \xi)| \, dx \leq c_n r_0^n \|\partial^2_x d(t, \cdot, \xi)\|_{L^\infty}\), the proof is exactly as in [10, Theorem 18.1.11].

In summary, applying (21) and the general integral kernel estimates above to the kernels given by (19) and (20) (note that \(b_j\) involves first-order derivatives of \(a\) in \(x\) and \(\xi\) already) we have proved the claims of case (b) in the proposition. \(\square\)

3. Colombeau solutions

We return to the scalar pseudodifferential Cauchy problem

\[
\partial_t u + Au = f \quad \text{in } X_T,
\]

\[
u(0) = g,
\]

where \(X_T = \mathbb{R}^n \times (0, T)\) and with data \(f \in \mathcal{S}_{L^2}(X_T)\) and \(g \in \mathcal{S}_{L^2}(\mathbb{R}^n)\). \(A\) is a generalized pseudodifferential operator of order 1. More precisely, we assume that \(A: \mathcal{S}_{L^2}(X_T) \to \mathcal{S}_{L^2}(X_T)\) is given by \((u_\varepsilon)_\varepsilon \mapsto (i a_\varepsilon(t, x, D_x) u_\varepsilon)_\varepsilon\),
where \((a_{\varepsilon}(t,x,\xi))\) is a generalized symbol of order 1 with parameter \(t \in [0,T]\). In addition, we impose the hyperbolicity assumption

\[
\forall \varepsilon \in (0,1]: \quad a_{\varepsilon} = a_{1,\varepsilon} + a_{0,\varepsilon} \quad \text{a}_{1,\varepsilon} \text{ real-valued, } a_{0,\varepsilon} \text{ of order 0.}
\]

The semi-norms in the basic energy estimate [13] now depend on \(\varepsilon \in (0,1]\), and, upon applying Gronwall’s inequality, will appear as exponents in the \(L^2\)-norm estimates of a prospective generalized solution; this suggests to assume logarithmic bounds on the symbols. We say that a generalized symbol \(b_{\varepsilon}\) of order \(m\) (with parameter \(t \in [0,T]\)) is of log-type up to order \((k,l)\) if

\[
Q_{0,k,l}^m(b_{\varepsilon}) = O(\log(1/\varepsilon)) \quad (\varepsilon \to 0).
\]

**Theorem 3.1.** Let \(A\) be a generalized first-order pseudodifferential operator, defined by the generalized symbol \((ia_{\varepsilon})_{\varepsilon \in (0,1]}\) with parameter \(t \in [0,T]\), and satisfying the hyperbolicity assumption [24]. Assume that \((a_{1,\varepsilon})\) is of log-type up to order \((k_n,l_n+1)\) and that \((a_{0,\varepsilon})\) is of log-type up to order \((k'_n,l'_n)\).

Then for any given \(f \in \mathcal{S}_{L^2}(X_T), g \in \mathcal{S}_{L^2}(\mathbb{R}^n)\) the Cauchy problem [22-23] has a unique solution \(u \in \mathcal{S}_{L^2}(X_T)\) if \(k_n = 3([n/2]+1), l_n = 2(n+2), k'_n = l'_n = [n/2]+1\).

Furthermore, we have variants of the log-type requirements in the following two cases:

(i) If there is \(r_0 \geq 0\) and an \(x\)-independent generalized symbol \((h_{\varepsilon}(t,\xi))\) such that

\[
a_{\varepsilon}(t,x,\xi) = h_{\varepsilon}(t,\xi) \quad \text{when } |x| \geq r_0,
\]

then we may put \(k_n = 1, l_n = n+2, k'_n = 0, l'_n = n+1\).

(ii) If \(a_{\varepsilon}\) is real-valued for every \(\varepsilon \in (0,1]\) then no log-type assumption on \(a_{0,\varepsilon}\) is required.

**Proof.** Let \((g_{\varepsilon}) \in g, (f_{\varepsilon}) \in f\) be representatives. At fixed, but arbitrary, \(\varepsilon \in (0,1]\) we consider the smooth Cauchy problem

\[
\begin{align*}
\partial_t u_{\varepsilon} + i a_{\varepsilon}(t,x,D)u_{\varepsilon} &= f_{\varepsilon} \quad \text{in } X_T, \\
u_{\varepsilon}(0) &= g_{\varepsilon}.
\end{align*}
\]

It has a unique solution \(u_{\varepsilon} \in \mathcal{C}^\infty([0,T], H^\infty(\mathbb{R}^n))\), thus constituting a solution candidate \((u_{\varepsilon})\) (cf. [16] Ch.7, Theorem 3.2 or [15] Ch.6, Theorem 2.1 with additional \(t\)-regularity following directly from the equation). We have to show that \((u_{\varepsilon}) \in \mathcal{E}_{M,L^2}(X_T)\).

Denote by \(C_{\varepsilon} := C(1 + Q_{0,k'_n,l'_n}^1(a_{0,\varepsilon}) + Q_{0,k_n,l_n}^1(a_{\varepsilon}))\) the constant occurring in the energy estimate [13] applied to \(u_{\varepsilon}\). Gronwall’s lemma implies

\[
\|u_{\varepsilon}(t)\|_{L^2}^2 \leq \left(\|g_{\varepsilon}\|_{L^2}^2 + \int_0^T \|f_{\varepsilon}(\tau)\|_{L^2}^2 d\tau\right) \exp(C_{\varepsilon}T).
\]

By hypothesis we have \(C_{\varepsilon} = O(\log(1/\varepsilon))\) as \(\varepsilon \to 0\). Thus we obtain uniqueness immediately from [24] – once moderateness is established – because null family estimates for \(f_{\varepsilon}, g_{\varepsilon}\) then imply such for \(u_{\varepsilon}\) as well.

For the proof of existence, we first observe that the basic estimate for \(\|u_{\varepsilon}\|_{L^2(X_T)} \leq T \sup_{t \in [0,T]} \|u_{\varepsilon}(t)\|_{L^2} = O(\varepsilon^{-N})\) follows at once from [24] by the moderateness of the data. It remains to prove moderateness estimates for the higher order derivatives of \(u_{\varepsilon}\).
\( x \)-derivatives: Let \( 0 \neq \alpha \in \mathbb{N}_0^n \) and apply \( \partial_\alpha^a t \) to equation \((24)\). It follows by induction and simple commutator relations of \( a_e(t, x, D_x) \) with \( \partial_x \) that this produces an equation of the following structure. Denote by \( e_j = (\delta_{i,j})_{k=1}^{n} \) the \( j \)th standard basis vector in \( \mathbb{R}^n \) then

\[
(30) \quad \partial_t \partial_\alpha^a u_e + i a_e(t, x, D_x) \partial_\alpha^a u_e + i \sum_{1 \leq j \leq n \atop a_j \neq 0} (\partial_{x_j} a_{1,e})(t, x, D_x) \partial_\alpha^a e_j u_e = F_{e,\alpha},
\]

where \( F_{e,\alpha} \) equals the sum of \( \partial_\alpha^a f_e \) plus, if \( |\alpha| \geq 2 \), a linear combination of terms of the form

\[
(31) \quad (\partial_\alpha^a a_e)(t, x, D_x) \partial_\alpha^a u_e \quad \text{with } \beta \leq \alpha \text{ and } 2 \leq |\beta|,
\]

and

\[
(32) \quad (\partial_{x_j} a_{0,e})(t, x, D_x) \partial_\alpha^a e_j u_e \quad \text{where } \alpha_j \neq 0.
\]

Assume that moderateness of \( \| \partial_\alpha^a u_e \|_{L^2} \) has been established already when \( |\gamma| < |\alpha| \). Since \( \partial_\alpha^a a_e \) is of order 1 we have

\[
\| (\partial_\alpha^a a_e)(t, x, D_x) \partial_\alpha^a e_j u_e(t) \|_{L^2} \leq C_1 Q_{0,m,m'}(\partial_\alpha^a a_e) \| \partial_\alpha^a e_j u_e(t) \|_{H^1} \\
\leq C_1' Q_{0,m,m'}(\partial_\alpha^a a_e) \max_{|\gamma| < |\alpha|} \| \partial_\gamma^a u_e(t) \|_{L^2},
\]

where \( C_1, C_1' \), and \( m \) depend only on the dimension \( n \) (16 Ch.3, Theorem 2.7)). Similarly, since \( \partial_{x_j} a_{0,e} \) is of order 0 we also have

\[
\| (\partial_{x_j} a_{0,e})(t, x, D_x) \partial_\alpha^a e_j u_e(t) \|_{L^2} \leq C_2 Q_{0,m',m'}(\partial_{x_j} a_{0,e}) \| \partial_\alpha^a e_j u_e(t) \|_{L^2} \\
\leq C_2' Q_{0,m',m'}(\partial_{x_j} a_{0,e}) \max_{|\gamma| < |\alpha|} \| \partial_\gamma^a u_e(t) \|_{L^2},
\]

where \( C_2, C_2' \), depend only on the dimension and \( m' = \lfloor n/2 \rfloor + 1 \). Hence, by the induction hypothesis, we have \( \| F_{e,\alpha}(t) \|_{L^2} = O(\varepsilon^{-N}) \) as \( \varepsilon \to 0 \) uniformly in \( t \in [0, T] \) for some \( N \).

We return to equation \((30)\), consider it as an equation for \( v_e := \partial_\alpha^a u_e \), and supply the initial value \( v_e(0) = \partial_\alpha^a u_e(0) = \partial_\alpha^a g_e \). Applying the basic technique from the beginning of the proof of the energy estimate \((13)\) to equation \((30)\) we obtain

\[
\frac{d}{dt} \| v_e(t) \|_{L^2}^2 = 2 \text{Re}(\partial_t v_e(t))v_e(t) \\
\leq \| F_{e,\alpha}(t) \|_{L^2}^2 + \| v_e(t) \|_{L^2}^2 + \| a_e(t, x, D_x) - a_e(t, x, D_x)^* \| \| v_e(t) \|_{L^2}^2 \\
+ \sum_{1 \leq j \leq n \atop a_j \neq 0} \| (\partial_{x_j} a_{1,e})(t, x, D_x) - (\partial_{x_j} a_{1,e})(t, x, D_x)^* \| \| \partial_\alpha^a e_j u_e(t) \|_{L^2}^2 + \| v_e(t) \|_{L^2}^2.
\]

If we define \( G(\partial_{x_j} a_{1,e})(t) := \sum_{j=1}^{n} \| (\partial_{x_j} a_{1,e})(t, x, D_x) - (\partial_{x_j} a_{1,e})(t, x, D_x)^* \| \) then we get

\[
\frac{d}{dt} \| v_e(t) \|_{L^2}^2 \leq \| F_{e,\alpha}(t) \|_{L^2}^2 + G(\partial_{x_j} a_{1,e})(t) \max_{|\gamma| < |\alpha|} \| \partial_\gamma^a u_e(t) \|_{L^2}^2 \\
+ \left( 1 + \| a_e(t, x, D_x) - a_e(t, x, D_x)^* \| + G(\partial_{x_j} a_{1,e})(t) \right) \| v_e(t) \|_{L^2}^2.
\]
The term \( H_{\varepsilon, \alpha}(t) := \| F_{\varepsilon, \alpha}(t) \|_{L^2}^2 + G(\partial_x a_{1, \varepsilon})(t) \cdot \max_{|\gamma| < |\alpha|} \| \partial_x^\gamma u_{\varepsilon}(t) \|_{L^2}^2 \) is of moderate growth and, by the proof of Proposition 3.1, we have

\[
1 + \| a_{\varepsilon}(t, x, D_x) - a_{\varepsilon}(t, x, D_x) \|_{L^2}^2 + G(\partial_x a_{1, \varepsilon})(t)
\]

\[
\leq C \left( 1 + Q^0_{0, k_n, l_n}(a_{\varepsilon}) + Q^1_{0, k_n, l_n}(a_{\varepsilon}) + \sum_{j=1}^n Q^j_{0, k_n, l_n}(\partial_x a_{1, \varepsilon}) \right) := \overline{C}_\varepsilon,
\]

which is a log-type constant by the hypotheses of the theorem. Note that the specifications of \( k_n, l_n, k'_n, l'_n \) for the general case match those of Proposition 3.1 case (a), whereas the hypotheses in (i), (ii) match cases (b), (c) there. Thus, we prove all assertions of the theorem simultaneously when the notation is understood in this way. Finally, integration with respect to \( t \) and Gronwall’s lemma yield the estimate

\[
\| \partial_x^a u_{\varepsilon}(t) \|_{L^2}^2 \leq \left( \| \partial_x^a g \|_{L^2}^2 + \int_0^T H_{\varepsilon, \alpha}(\tau) d\tau \right) \exp(C_{\varepsilon}T) = O(\varepsilon^{-M})
\]

for some \( M \) and \( \varepsilon \) sufficiently small. Hence \( \| \partial_x^a u_{\varepsilon} \|_{L^2(X,T)} \) satisfies a similar estimate.

In particular, we have the same bounds on the spatial Sobolev norms \( \| u_{\varepsilon}(t) \|_{H^k} \) for \( k \) arbitrary and uniformly in \( t \in [0, T] \).

\( t \)- and mixed derivatives: Equations (24) and (30) directly imply estimates of the form \( \| \partial_\tau \partial_x^a u_{\varepsilon}(t) \|_{L^2} = O(\varepsilon^{-N}) \) for any \( \alpha \in \mathbb{N}_0^d \) (and uniformly in \( t \)). To proceed to higher order \( t \)-derivatives, we simply differentiate equations (24), resp. (30), with respect to \( t \). The Sobolev mapping properties of the operators \( (\partial_\tau^a \partial_x^a) a_{\varepsilon}(t, x, D_x) \) and moderateness assumptions on the symbols then yield the desired estimates for \( \| \partial_\tau^a \partial_x^a u_{\varepsilon}(t) \|_{L^2} \) successively for \( l = 0, 1, 2, \ldots \) and \( \alpha \) arbitrary. \( \square \)

**Remark 3.2.** (i) The key assumptions in Theorem 3.1 are the log-type estimates on the symbol. We know already from the differential operator case that they cannot be dropped completely. However, these are sufficient conditions and merely reflect the various operator norm bounds available for zero order symbols (as used in proving the energy estimates). Thus they cannot be expected to be sharp. In fact, the value of the theorem lies in a general feasibility proof and any special structure inherent in a concrete symbol under consideration in applications may allow for improvement.

(ii) In order to meet the log-type conditions of the above theorem in a specific symbol regularization one may call on a re-scaled mollification as described in [10]. To illustrate this procedure, let us assume that the non-smooth symbol of order \( m \) is given as the measurable bounded function \( a(x, \xi) \) such that for almost all \( x \) the partial function \( \xi \mapsto a(x, \xi) \) is smooth and satisfies for all \( \alpha \in \mathbb{N}_0^d \) an estimate

\[
\| \partial_\xi^a a(., \xi) \|_{L^\infty} \leq C_\alpha (1 + |\xi|)^{m-|\alpha|}.
\]

Let \( \rho \) be a mollifier and let \( 0 < \omega \leq (\log(1/\varepsilon))^{1/k} \) for some \( k \in \mathbb{N} \), \( \omega \to \infty \) as \( \varepsilon \to 0 \). Let \( \rho^\varepsilon(y) := \omega^\varepsilon \rho(\omega \varepsilon y) \) and define the regularized symbol by \( a_{\varepsilon}(x, \xi) := (\rho^\varepsilon * a_{\varepsilon}(., \xi))(x) \) (convolution with respect to the \( x \)-variable only). Then it is easy to check that \( a_{\varepsilon} \in S^m \) and of log-type up to order \( (\infty, k) \).

As in [17], essentially by inspection of the above existence proof, we establish compatibility with distributional or smooth solutions, that is macroscopic regularity in a certain sense, when the symbol is smooth.

**Corollary 3.3.** In Theorem 3.1 assume that \( A \) is given by a smooth symbol \( a \in C^\infty([0, T], S^1) \), i.e., \( a_{\varepsilon} = a \) for all \( \varepsilon \in (0, 1] \).
(i) If \( f \in H^\infty(X_T), g \in H^\infty(\mathbb{R}^n) \), then the generalized solution \( u \in S_{L^2}(X_T) \) is equal to the classical smooth solution.

(ii) Let \( f \in C([0,T], H^s(\mathbb{R}^n)) \) and \( g \in H^s(\mathbb{R}^n) \) for some \( s \in \mathbb{R} \), and \( v \)
be the unique distributional solution to \((\text{P}2)\) in \( C([0,T], H^s) \). Define generalized data for problem \((\text{P}2)\) by \( f := [(f_\varepsilon)_\varepsilon] \in S_{L^2}(X_T) \) (resp. \( g := [(g_\varepsilon)_\varepsilon] \in S_{L^2}(\mathbb{R}^n) \)), where \( f_\varepsilon \in H^\infty(X_T) \) (resp. \( g_\varepsilon \in H^\infty(\mathbb{R}^n) \)) are moderate regularizations such that \( f_\varepsilon \to f \) in \( C([0,T], H^s(\mathbb{R}^n)) \) (resp. \( g_\varepsilon \to g \) in \( H^s(\mathbb{R}^n) \)) as \( \varepsilon \to 0 \). If \( u \) is the corresponding generalized solution in \( S_{L^2}(X_T) \) then it is associated with the distributional solution \( v \).

\[ \epsilon \]

**Proof.** Part (i): Since we may choose the constant nets \((f_\varepsilon)_\varepsilon\), \((g_\varepsilon)_\varepsilon\) as representatives of the classes of \( f \) and \( g \) in \( S_{L^2} \), and \( a_\varepsilon = a \) by assumption, we obtain the classical smooth solution to equation \((\text{P}2)\) as a representative of the unique Colombeau solution.

Part (ii): The unique solution \( v \in C([0,T], H^s) \) to \((\text{P}1)\) depends continuously on the data \( f \) and \( g \) by the closed graph theorem. Hence the solution representative \( u_\varepsilon \), defined as the solution to \((\text{P}4)\), converges to \( v \) in \( C([0,T], H^s) \) as \( \varepsilon \to 0 \). □

Finally, we prove that the intrinsic regularity property for the generalized solution holds if the data are in \( \tilde{S}_{L^2}^\infty \) and the generalized symbol is only mildly generalized, namely satisfies additional slow scale conditions. This notion was introduced and investigated in some detail in \cite{13} and found to be crucial for regularity theory of partial differential equations. Recall that a net \((r_\varepsilon)_\varepsilon\) of complex numbers is said to be of **slow scale** if it satisfies

\[ \forall \varepsilon \geq 0: |r_\varepsilon|^p = O(1/\varepsilon) \quad (\varepsilon \to 0). \]

In the proposition below, we call a net \((s_\varepsilon)_\varepsilon\) of complex numbers a **slow-scale log-type net** if there is a slow scale net \((r_\varepsilon)_\varepsilon\) of real numbers, \( r_\varepsilon \geq 1 \), such that

\[ |s_\varepsilon| = O(\log(r_\varepsilon)) \quad (\varepsilon \to 0). \]

**Proposition 3.4.** In Theorem 3.1 assume all log-type conditions to be replaced by slow-scale log-type estimates and, in addition, that \((a_\varepsilon)_\varepsilon\) is of slow scale in each derivative. By the latter, we mean that for all \( j, k, l \), we can find a slow scale net \((r_\varepsilon)_\varepsilon\) positive real numbers such that

\[ Q^1_{j,k,l}(a_\varepsilon) = O(r_\varepsilon) \quad (\varepsilon \to 0). \]

Then \( f \in \tilde{S}_{L^2}^\infty(X_T) \) and \( g \in \tilde{S}_{L^2}^\infty(\mathbb{R}^n) \) implies \( u \in \tilde{S}_{L^2}^\infty(X_T) \). In particular, this is always true when the symbol of \( A \) is smooth (as in the Corollary above).

**Proof.** Thanks to the explicit assumptions this is straightforward by an inspection of the proof of Theorem 3.1. To be more precise, assume that we have a uniform \( \varepsilon \)-growth, say \( \varepsilon^{-M} \), for the derivatives of \( f \) and \( g \); i.e., for all \( k, \alpha \), we have

\[ \|\partial_t^k \partial_x^\alpha f_\varepsilon\|_{L^2} = O(\varepsilon^{-M}) \quad \text{as well as} \quad \|\partial_x^\alpha g_\varepsilon\|_{L^2} = O(\varepsilon^{-M}). \]

Note that all constants involving \( a_\varepsilon \) in the energy and Sobolev estimates throughout the proof yield only slow scale factors. (Observe again, that in the exponential factors in all energy estimates we only need a fixed finite order of derivatives, corresponding to \( k_n \), \( l_n \) etc.) Thus, the same induction argument shows that we obtain for all \( k, \alpha \) a certain slow scale net \((r_\varepsilon)_\varepsilon\) of positive real numbers, such that

\[ \|\partial_t^k \partial_x^\alpha u_\varepsilon\|_{L^2} = O(r_\varepsilon^{-M}) = O(\varepsilon^{-M-1}), \]

which proves the assertion. □
Remark 3.5. (i) The somewhat extensive slow-scale log-type conditions in the above proposition are by far not necessary for regularity, but are suited to make the energy estimates, with their exponential constants, directly applicable. We expect that these can be relaxed at least to plain slow scale conditions by appealing to pseudodifferential parametrix techniques (cf. [7, 8]).

(ii) A slow-scale property of $(a_x)_\varepsilon$ (in all derivatives) is implied, for example, by the log-type assumptions on $(a_x)_\varepsilon$ if, in addition, only a $S^{\infty}$-type regularity of $(a_x)_\varepsilon$ is assumed. This follows from [13, Proposition 1.6] and the fact that $\log(1/\varepsilon)$ is a slow scale net.

Appendix: Remainder term estimates

We briefly outline a proof of [10] and verify the precise form of the estimate [13]; it is an adaption of the reasoning in [10, Ch. 2, Sections 2-3]; thereby, we also recall the precise meaning of the oscillatory integral (17). We may suppress the dependence of all symbols on the parameter $t$, since it will be clear that all steps in the process respect continuity (or smoothness) with respect to it and yield uniform bounds in all estimates when $t$ varies in $[0, T]$.

Let $a(x, \xi)$ be a (smooth) symbol of order 1. The starting point is the following formula for the adjoint of $a(x, D)$, e.g. valid for $u \in \mathcal{S}(\mathbb{R}^n)$ as iterated integral,

$$a(x, D)^* u(x) = \int \int e^{ix(y-x)} a(y, \eta) u(y) dyd\eta.$$

Writing $u(y) = \int e^{iy\xi} \hat{u}(\xi) d\xi$ we obtain, now in the sense of oscillatory integrals,

$$a(x, D)^* u(x) = \int e^{ix\xi} \hat{u}(\xi) \int \int e^{iy(y-x)} a(y, \xi + \eta) dyd\eta d\xi =: \int e^{ix\xi} \hat{u}(\xi) a^*(x, \xi) d\xi,$$

by which we define the symbol $a^*$. Using Taylor expansion

$$a(y, \xi + \eta) = a(y, \xi) + \int_0^1 \eta \cdot \partial_\xi a(y, \xi + \theta \eta) \theta$$

and (the oscillatory integral interpretation of) the Fourier identity $\mathcal{F}^{-1}(\mathcal{F}(a(x, \xi)))(x) = a(x, \xi)$ leads to

$$a^*(x, \xi) = \frac{a(x, \xi)}{a(x, \xi)} + \int_0^1 \int \eta \cdot e^{iy(y-x)} \partial_\xi a(y, \xi + \theta \eta) dyd\eta d\theta.$$

Noting that $\eta e^{i(x-y)n} = D_\eta (e^{i(x-y)n})$ and integrating by parts yields equations [10] and [17]. We use the notation $\partial_\xi \partial_\eta = \sum_{j=1}^n \partial_{x_j} \partial_{\xi_j}$, $r_\theta = \sum_j r_j, \theta$, and recall that [17] can be defined as the classical integral

$$r_\theta(x, \xi) = \int \int e^{-iy\eta}(1 + |y|^2)^{-\lambda} (1 - \Delta_\eta) a(x + y, \xi + \theta \eta) dyd\eta,$$

where $\lambda > n/2$, so that $s_\theta(x, \xi; y, \eta) := (1 + |y|^2)^{-\lambda} (1 - \Delta_\eta) a(x + y, \xi + \theta \eta)$ is integrable. We have the estimate

$$|s^\lambda_{x, x} s_\theta(x, \xi; y, \eta)| \leq c_{n, \alpha, \beta} q^{2n+1+|\alpha|, 1+|\beta|}(a) (1 + |y|)^{-2\lambda} (1 + |\xi + \theta \eta|^2)^{-|\alpha|/2},$$

where $q$ is a suitable power of $\mathcal{X}$-norms, $\mathcal{X} = \left\{ a(y, \eta) \mid \lambda \eta \in \mathbb{R}^n \right\}$.
where $c_{n,\alpha,\beta}$ is uniform in $\theta \in [0, 1]$. In order to prove (18) we have to estimate
\[
\nabla_x^n \nabla_\xi^\beta \rho_\theta(x, \xi) = \int \int e^{-iy\eta} \nabla_x^n \nabla_\xi^\beta s_\theta(x, \xi; y, \eta) \, dy \, d\eta \nabla \theta \nabla \eta \nabla \theta \nabla \eta
= \int \int \ldots \theta \nabla \eta \nabla \theta \nabla \eta
= I_1 + I_2.
\]
For an upper bound of $I_1$ we use (33) and the implication $|\eta| \leq |\xi|/2 \Rightarrow |\xi + \theta\eta| \geq |\xi|/2$ (when $\theta \in [0, 1]$) to find
\[
|I_1| \leq c_{n,\alpha,\beta,\lambda} \rho_\alpha^1 2^{2\lambda+1+|\alpha|,1+|\beta|}(a) (1 + |\xi|)^{-|\alpha|},
\]
uniformly in $\theta$. To estimate $I_2$, we first use that $e^{-iy\eta} = |\eta|^{-2l}(-\Delta_y)^l(e^{-iy\eta})$ and integrate by parts to obtain
\[
|I_2| \leq \int \int \int_{\eta \geq |\xi|/2} |\eta|^{-2l}(-\Delta_y)^l \nabla_x^n \nabla_\xi^\beta s_\theta(x, \xi; y, \eta) \, dy \, d\eta.
\]
We apply (33) with $\beta$ replaced by $\beta+2l\epsilon_j$ ($j = 1, \ldots, n$) to the integrand and arrive at
\[
|I_2| \leq c_{n,\alpha,\beta,\lambda} \rho_\alpha^1 2^{2\lambda+1+|\alpha|,2l+1+|\beta|}(a) \int_{\eta \geq |\xi|/2} |\eta|^{-2l}(1 + |\xi + \theta\eta|^2)^{-|\alpha|/2} \, d\eta.
\]
By Peetre’s inequality $(1 + |\xi + \theta\eta|^2)^{-|\alpha|/2} \leq 2^{\alpha/2} (1 + |\xi|^2)^{-|\alpha|/2} (1 + |\theta\eta|^2)^{\alpha/2}$, so that
\[
|I_2| \leq c_{n,\alpha,\beta,\lambda} \rho_\alpha^1 2^{2\lambda+1+|\alpha|,2l+1+|\beta|}(a) (1 + |\xi|)^{-|\alpha|} \int_{\eta \geq |\xi|/2} |\eta|^{-2l}(1 + |\theta\eta|^2)^{\alpha/2} \, d\eta,
\]
where the remaining integral is finite if $2l > n + |\alpha|$. Summing up, and combining the conditions $2\lambda > n$, $2l > n + |\alpha|$, we have shown (18).

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