A homology theory for Smale spaces: a summary

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Abstract

We consider Smale spaces, a particular class of hyperbolic topological dynamical systems, which include the basic sets for Smale’s Axiom A systems. We present a homology theory for such systems which is based on Krieger’s dimension group in the special case of shifts of finite type. This theory provides a Lefschetz formula relating trace data with the number of periodic points of the system.

1 Introduction

Smale introduced the notion of an Axiom A diffeomorphism of a compact manifold [13]. For such a system, a basic set is an invariant subset of the non-wandering set which is irreducible in a certain sense. One of Smale’s key observations was that such a set need not be a submanifold. Typically, it is some type of fractal object. The aim of this article is to introduce a new type of homology theory for such spaces. This takes as its starting point the notion of the dimension group of a shift of finite type introduced by Krieger [7] and the fundamental result of Bowen [2] that every basic set is a factor of a shift of finite type.

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In an effort to give a purely topological (i.e. without reference to any smooth structure) description of the dynamics on a basic set, Ruelle introduced the notion of a Smale space \([12]\): a Smale space is a compact metric space, \((X, d)\), and a homeomorphism, \(\varphi\), of \(X\), which possesses canonical coordinates of contracting and expanding directions. The precise definition involves the existence of a map \([,]\) giving canonical coordinates. Here, we review only the features necessary for the statements of our results.

There is a constant \(\epsilon_X > 0\) and, for each \(x\) in \(X\) and \(0 < \epsilon \leq \epsilon_X\), there are sets \(X^s(x, \epsilon)\) and \(X^u(x, \epsilon)\), called the local stable and unstable sets, respectively, whose product is homeomorphic to a neighbourhood of \(x\). As \(\epsilon\) varies, these form a neighbourhood base at \(x\). Moreover, there is a constant \(0 < \lambda < 1\) such that

\[
    d(\varphi(y), \varphi(z)) \leq \lambda d(y, z), \quad y, z \in X^s(x, \epsilon_X) \\
    d(\varphi^{-1}(y), \varphi^{-1}(z)) \leq \lambda d(y, z), \quad y, z \in X^u(x, \epsilon_X)
\]

The bracket \([x, y]\) is the unique point in the intersection of \(X^s(x, \epsilon_X)\) and \(X^u(y, \epsilon_X)\). We say that \((X, \varphi)\) is non-wandering if every point of \(X\) is non-wandering for \(\varphi\) \([6]\).

Stable and unstable equivalence relations are defined by

\[
    R^s = \{(x, y) \mid \lim_{n \to +\infty} d(\varphi^n(x), \varphi^n(y)) = 0\} \\
    R^u = \{(x, y) \mid \lim_{n \to +\infty} d(\varphi^{-n}(x), \varphi^{-n}(y)) = 0\}.
\]

We let \(X^s(x)\) and \(X^u(x)\) denote the stable and unstable equivalence classes of \(x\) in \(X\).

The main examples of such systems are shifts of finite type (of which we will say more in a moment), hyperbolic toral automorphisms, solenoids, substitution tiling spaces (under some hypotheses) and, most importantly, the basic sets for Smale’s Axiom A systems \([13, 3]\).

Let \((Y, \psi)\) and \((X, \varphi)\) be Smale spaces. A factor map from \((Y, \psi)\) to \((X, \varphi)\) is a function \(\pi : Y \to X\) which is continuous, surjective and satisfies \(\pi \circ \psi = \varphi \circ \pi\). It is clear that, for any \(y\) in \(Y\), \(\pi(X^s(y)) \subset X^s(\pi(y))\) and \(\pi(X^u(y)) \subset X^u(\pi(y))\). David Fried \([5]\) defined \(\pi\) to be \(s\)-resolving (or \(u\)-resolving) if, for every \(y\) in \(Y\), the restriction of \(\pi\) to \(X^s(y)\) (or to \(X^u(y)\), respectively) is injective. We say that \(\pi\) is \(s\)-bijective (or \(u\)-bijective) if, for every \(y\) in \(Y\), \(\pi\) is a bijection from \(X^s(y)\) to \(X^s(\pi(y))\) (or from \(X^u(y)\) to \(X^u(\pi(y))\), respectively). This actually implies that \(\pi\) is a local homeomorphism from
the local stable sets (or unstable sets, respectively) in $Y$ to those in $X$. In the case that $(X, \varphi)$ is non-wandering, $s$-resolving ($u$-resolving) and $s$-bijective ($u$-bijective, respectively) are equivalent.

2 Shifts of Finite type

Shifts of finite type are described in detail in [8]. We consider a finite directed graph $G$. This consists of a finite vertex set $G^0$, a finite edge set $G^1$ and maps $i, t$ (for initial and terminal) from $G^1$ to $G^0$. The associated shift space

$$\Sigma_G = \{e = (e^k)_{k \in \mathbb{Z}} | e^k \in G^1, t(e^k) = i(e^{k+1}), k \in \mathbb{Z}\}$$

consists of all bi-infinite paths in $G$, specified as an edge list. The map $\sigma$ is the left shift on $\Sigma_G$ defined by $\sigma(e)^k = e^{k+1}$, for all $e$ in $\Sigma_G$ and $k$ in $\mathbb{Z}$. By a shift of finite type, we mean any system topologically conjugate to $(\Sigma_G, \sigma)$, for some graph $G$. This is not the usual definition, but is equivalent to it (see Theorem 2.3.2 of [8]). We observe that such systems are Smale spaces by noting that the bracket operation is defined as follows. For $e, f$ in $\Sigma_G$, $[e, f]$ is defined if $t(e^0) = t(f^0)$ and then it is the sequence, $(\ldots, f^{-1}, f^0, e^1, e^2, \ldots)$. For any $l \geq 1$ and $e_0$ in $\Sigma_G$, the local stable and unstable sets are given by

$$\Sigma^s_G(e_0, 2^{-l}) = \{e \in \Sigma | e^k = e_0^k, k \geq -l\},$$
$$\Sigma^u_G(e_0, 2^{-l}) = \{e \in \Sigma | e^k = e_0^k, k \leq l\}.$$ 

**Theorem 2.1.** Shifts of finite type are exactly the zero-dimensional (i.e. totally disconnected) Smale spaces.

The fundamental rôle of shifts of finite type is demonstrated by the following universal property, due to Bowen [2].

**Theorem 2.2 (Bowen).** Let $(X, \varphi)$ be a non-wandering Smale space. Then there exists a non-wandering shift of finite type $(\Sigma, \sigma)$ and a finite-to-one factor map $\pi : (\Sigma, \sigma) \rightarrow (X, \varphi)$.

3 Krieger’s dimension group invariant

Krieger [7] defined the past and future dimension groups of a shift of finite, $(\Sigma, \sigma)$, as follows. Consider $\mathcal{D}^s(\Sigma, \sigma)$ to be the collection of compact open
subsets of $\Sigma^*(e, e)$, as $e$ varies over $\Sigma$ and $0 < \epsilon \leq \epsilon_X$. We let $\sim$ denote the smallest equivalence relation on $D^*(\Sigma, \sigma)$ such that

1. If $E, F$ are in $D^*(\Sigma, \sigma)$ with $[E, F] = F, [F, E] = E$ (meaning both are defined), then $E \sim F$.

2. If $E, F, \varphi(E)$ and $\varphi(F)$ are all in $D^*(\Sigma, \sigma)$, then $E \sim F$ if and only if $\varphi(E) \sim \varphi(F)$.

We generate a free abelian group on the equivalence classes of elements, $E$, of $D^s(\Sigma, \sigma)$ (denoted $[E]$) subject to the additional relation that $[E \cup F] = [E] + [F]$, if $E \cup F$ is in $D^s(\Sigma, \sigma)$ and $E$ and $F$ are disjoint. The result is denoted by $D^s(\Sigma, \sigma)$. There is an analogous definition of $D^u(\Sigma, \sigma)$.

**Theorem 3.1** (Krieger [7]). Let $G$ be a finite directed graph and $(\Sigma_G, \sigma)$ be the associated shift of finite type. Then $D^s(\Sigma_G, \sigma)$ (or $D^u(\Sigma_G, \sigma)$, respectively) is isomorphic to the inductive limit of the sequence

$$\mathbb{Z}G^0 \xrightarrow{\gamma^s} \mathbb{Z}G^0 \cdots$$

where $\mathbb{Z}G^0$ denotes the free abelian group on the vertex set $G^0$ and the map $\gamma(s) = \sum_{t(e) = v} i(e)$, for any $v$ in $G^0$ (or respectively, replacing $\gamma^s$ by $\gamma^u$, whose definition is the same, interchanging the rôles of $i$ and $t$).

The first crucial result for the development of our theory is the following functorial property of $D^s$ and $D^u$, which can be found in [4].

**Theorem 3.2.** Let $(\Sigma, \sigma)$ and $(\Sigma', \sigma)$ be shifts of finite type and let $\pi : (\Sigma, \sigma) \to (\Sigma', \sigma)$ be a factor map.

1. If $\pi$ is $s$-bijective, then there are natural homomorphisms

$$\pi^s : D^s(\Sigma, \sigma) \to D^s(\Sigma', \sigma),$$
$$\pi^{u*} : D^u(\Sigma', \sigma) \to D^u(\Sigma, \sigma).$$

2. If $\pi$ is $u$-bijective, then there are natural homomorphisms

$$\pi^u : D^u(\Sigma, \sigma) \to D^u(\Sigma', \sigma),$$
$$\pi^{s*} : D^s(\Sigma', \sigma) \to D^s(\Sigma, \sigma).$$
The idea is simple enough: in the covariant case, the induced map sends the class of a set \( E \) in \( D^s(\Sigma, \sigma) \) to the class of \( \pi(E) \), while in the contravariant case, the map sends the class of \( E' \) in \( D^s(\Sigma', \sigma) \) to the class of \( \pi^{-1}(E') \). The latter is not correct since \( \pi^{-1}(E') \) may not even be contained in a single stable equivalence class, but it suffices that it may be written as a finite union of elements of \( D^s(\Sigma, \sigma) \). These ideas work correctly under the stated hypotheses.

4 \( s/u \)-bijective pairs

The key ingredient in our construction is the following notion.

**Definition 4.1.** Let \((X, \varphi)\) be a Smale space. An \( s/u \)-bijective pair, \( \pi \), for \((X, \varphi)\) consists of Smale spaces \((Y, \phi)\) and \((Z, \zeta)\) and factor maps

\[
\pi_s : (Y, \psi) \to (X, \varphi), \quad \pi_u : (Z, \zeta) \to (X, \varphi)
\]

such that

1. \( Y^u(y, \epsilon) \) is totally disconnected, for all \( y \) in \( Y \) and \( 0 < \epsilon \leq \epsilon_Y \),
2. \( \pi_s \) is \( s \)-bijective,
3. \( Z^s(z, \epsilon) \) is totally disconnected, for all \( z \) in \( Z \) and \( 0 < \epsilon \leq \epsilon_Z \),
4. \( \pi_u \) \( u \)-bijective.

To summarize the idea in an informal way, the space \( Y \) is an extension of \( X \), where the local unstable sets are totally disconnected, while the local stable sets are homeomorphic to those in \( X \). The existence of such \( s/u \)-bijective pairs, at least for non-wandering \((X, \varphi)\), can be deduced from the results of [11]. It can be viewed as a coordinate-wise version of Bowen’s theorem.

**Theorem 4.2.** If \((X, \varphi)\) is non-wandering, then there exists an \( s/u \)-bijective pair for \((X, \varphi)\).

**Definition 4.3.** Let \( \pi = (Y, \psi, \pi_u, Z, \zeta, \pi_s) \) be an \( s/u \)-bijective pair for \((X, \varphi)\). For each \( L, M \geq 0 \), we define

\[
\Sigma_{L,M}(\pi) = \{(y_0, \ldots, y_L, z_0, \ldots, z_M) \mid y_l \in Y, z_m \in Z, \pi_s(y_l) = \pi_u(z_m), 0 \leq l \leq L, 0 \leq m \leq M\}
\]
For simplicity, we also denote $\Sigma_{0,0}(\pi)$ by $\Sigma(\pi)$. We define $\sigma_{L,M} : \Sigma_{L,M}(\pi) \to \Sigma_{L,M}(\pi)$ by

$$\sigma_{L,M}(y_0, \ldots, y_L, z_0, \ldots, z_M) = (\psi(y_0), \ldots, \psi(y_L), \zeta(z_0), \ldots, \zeta(z_M)).$$

For $L \geq 1$ and $0 \leq l \leq L$, we let $\delta_l : \Sigma_{L,M}(\pi) \to \Sigma_{L-1,M}(\pi)$ be the map which deletes entry $y_l$. Similarly, the map $\delta_m : \Sigma_{L,M}(\pi) \to \Sigma_{L,M-1}(\pi)$ deletes entry $z_m$, for $M \geq 1, 0 \leq m \leq M$.

The important properties of these systems and maps is summarized as follows.

**Theorem 4.4.** For every $L, M \geq 0$, $(\Sigma_{L,M}(\pi), \sigma_{L,M})$ is a shift of finite type. For $L \geq 1$, $0 \leq l \leq L$, the map $\delta_l$ is an $s$-bijective factor map. For $M \geq 1$, $0 \leq m \leq M$, the map $\delta_m$ is a $u$-bijective factor map.

## 5 Homology

There are actually two homology theories here. One, based on the dimension group $D^s$ will be denoted by $H^s_\pi$ and the other, based on $D^u$, will be denoted by $H^u_\pi$. We will concentrate on the former for the remainder of this note.

**Definition 5.1.** Let $(X, \varphi)$ be a Smale space and suppose that $\pi = (Y, \psi, \pi_s, Z, \zeta, \pi_u)$ is an $s/u$-bijective pair for $(X, \varphi)$. We define

$$C^s_N(\pi) = \oplus_{N = 0}^{L-M} D^s(\Sigma_{L,M}(\pi), \sigma_{L,M})$$

for every $N$ in $\mathbb{Z}$ and a boundary map $\partial^s_N(\pi) : C^s_N(\pi) \to C^s_{N-1}(\pi)$ by

$$\partial^s_N(\pi)|D^s(\Sigma_{L,M}, \sigma_{L,M}) = \sum_{l=0}^{L} (-1)^l \delta_{l}^s + \sum_{m=0}^{M+1} (-1)^{m+L} \delta_{m}^s,$$

where, in the special case $L = 0$, we set $\delta_{0}^s = 0$.

We define $H^s_\pi(\pi)$ to be the homology of the complex $(C^s_\pi(\pi), \partial^s_\pi(\pi))$.

We want to establish some basic properties of our theory. The first crucial result is the following. It is stated in a slightly informal manner, but it conveys the main idea.

**Theorem 5.2.** $H^s_\pi(\pi)$ is independent of the $s/u$-bijective pair $\pi = (Y, \pi_s, Z, \pi_u)$ and depends only on $(X, \varphi)$. 

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In light of this, we denote $H^*_s(\pi)$ by $H^*_N(X, \varphi)$ instead. It is defined provided that there exists an $s/u$-bijective pair for $(X, \varphi)$, which is true for all non-wandering Smale spaces.

**Theorem 5.3.** The homology theory $H^*_s$ is functorial in the following sense: if $\pi : (Y, \psi) \rightarrow (X, \varphi)$ is an $s$-bijective factor map, then there are induced group homomorphisms

$$\pi^* : H^*_N(Y, \psi) \rightarrow H^*_N(X, \varphi),$$

for all $N$ in $\mathbb{Z}$. If the map $\pi$ is a $u$-bijective factor, then there are induced group homomorphisms

$$\pi^{**} : H^*_N(X, \varphi) \rightarrow H^*_N(Y, \psi),$$

for all $N$ in $\mathbb{Z}$.

To describe the next property, we assume that we have chosen a graph $G$ which presents the shift of finite type $(\Sigma(\pi), \sigma)$ associated with our $s/u$-bijective pair. More specifically, for each $(y, z)$ in $\Sigma(\pi)$ and each integer $k$, $e^k(y, z)$ is an edge in $G$ and the map sending $(y, z)$ in $\Sigma(\pi)$ to $(e^k(y, z))_{k \in \mathbb{Z}}$ is a conjugacy between $(\Sigma(\pi), \sigma)$ and $(\Sigma_G, \sigma)$. We make the additional assumption that the presentation is regular in the following sense. For any $(y, z), (y', z')$ in $\Sigma(\pi)$ such that $t(e^0(y, z)) = t(e^0(y', z'))$, it follows that $[\pi_s(y), \pi_s(y')]$ is defined in $X$, $[y, y']$ is defined in $Y$ and $[z, z']$ is defined in $Z$ and we have

$$\pi_s[y, y'] = \pi_u[z, z'] = [\pi_s(y), \pi_s(y')].$$

This may always be achieved by passing to a higher block presentation of $G$ [8]. As a consequence, it is fairly easy to see that, for all $L, M \geq 0$, there exists a graph $G_{L,M}$ consisting of $L + 1$ by $M + 1$ arrays of vertices and edges from $G$ such that the map sending $(y_0, \ldots, y_L, z_0, \ldots, z_M)$ in $\Sigma_{L,M}(\pi)$ to the sequence $(e^k(y_l, z_m))_{k \in \mathbb{Z}}$ in a conjugacy with $\Sigma_{G_{L,M}}$.

We note that the groups $S_{L+1}$ and $S_{M+1}$ act on both $\Sigma_{L,M}(\pi)$ and $G_{L,M}$. We let $\mathbb{Z}[G_{L,M}]$ be the quotient $\mathbb{Z}G_{L,M}^0$ by the relations that $v = 0$ if two distinct rows of $v$ are equal and $\alpha \cdot v = sgn(\alpha)v$, for any $v$ in $G_{L,M}^0$ and $\alpha$ in $S_{L+1}$. The map $\gamma^*_s \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \·
⊕_{L-M=N} D^s[GL,M], and we denote the result by $\partial^s[\pi]$. There analogous complexes $(C^s[\pi], \partial^s[\pi])$ which treats the actions of the various $S_{M+1}$’s in an analogous way. In this case, it is a subcomplex of our original. Finally, there is a quotient of this, denoted $(C^s[\pi], \partial^s[\pi])$ which includes both $S_{L+1}$ and $S_{M+1}$ actions.

**Theorem 5.4.** The natural maps between the complexes $(C^s(\pi), \partial^s(\pi)), (C^s[\pi], \partial^s[\pi]), (C^s(\pi), \partial^s[\pi])$ and $(C^s[\pi], \partial^s[\pi])$ all induce isomorphisms at the level of homology.

This result is very useful for computational purposes since the groups $\mathbb{Z}[G_{L,M}^0]$ have many fewer generators $\mathbb{Z}[G_{L,M}^0]$. In addition, every s-bijective or u-bijective map is finite-to-one, meaning that there is a uniform upper bound on the cardinality of a pre-image. Hence, there exists some $N_0$ such that, if either $L$ or $M$ exceed $N_0$, any element of $\Sigma_{L,M}(\pi)$ will contain a repeated entry. In consequence, the group $\mathbb{Z}[G_{L,M}^0]$, for all such $L, M$, will be trivial.

**Corollary 5.5.** For any Smale space $(X, \varphi)$ which has an s/u-bijective pair, the groups $H^s_N(X, \varphi)$ are finite rank and are non-zero for only finite many values of $N$.

One of the most important aspects of Krieger’s theory is that, for any shift of finite type $(\Sigma, \sigma)$, the group $D^s(\Sigma, \sigma)$ has a natural order structure. This carries over to our theory as follows.

**Theorem 5.6.** If $(X, \varphi)$ is a non-wandering Smale space, then the group $H^s_N(X, \varphi)$ has a natural order structure, which is preserved under the functorial properties of Theorem 5.3.

Finally, we have the following analogue of the Lefschetz formula. Given $(X, \varphi)$, we can regard $\varphi$ as a factor map from this system to itself. It is both s-bijective and u-bijective and so, by Theorem 5.3, induces an automorphism of our invariant, denoted $\varphi^*$. The following result, already known in the case of shifts of finite type, uses ideas of Manning [9].

**Theorem 5.7.** For any non-wandering Smale space $(X, \varphi)$ and $p \geq 1$, we have

$$\sum_{N \in \mathbb{Z}} (-1)^{NT} r[(\varphi^*)^p] : H^s_N(X, \varphi) \otimes \mathbb{Q} \to H^s_N(X, \varphi) \otimes \mathbb{Q}$$

$$= \#\{x \in X \mid \varphi^p(x) = x\}.$$
6 Examples

We present four examples where the computations above may be carried out quite explicitly. The full details of the last three are in preparation [1]. All of the examples are computed using the double complex $C^*[\pi]$; it is that one which is described in each example.

Example 6.1. Suppose $(\Sigma, \sigma)$ is a shift of finite type. In this case, an s/u-bijective pair is just $(Y, \psi) = (Z, \zeta) = (\Sigma, \sigma)$. Only the 0,0-term in the double complex is non-zero and it is just $D^*(\Sigma, \sigma)$. Hence, $H^*_N(\Sigma, \sigma)$ is just $D^*(\Sigma, \sigma)$, for $N = 0$, and zero otherwise.

Example 6.2. For $m \geq 2$, let $(X, \varphi)$ be the $m^\infty$-solenoid. More specifically, we let

$$X = \{(z_0, z_1, \ldots) \mid z_n \in T, z_n = z_{n+1}^m, n \geq 0\},$$

with the map

$$\varphi(z_0, z_1, \ldots) = (z_0^m, z_1^m, z_2^m, \ldots),$$

for $(z_0, z_1, \ldots)$ in $X$. In this case, there is an s-bijective factor map onto $(X, \varphi)$ from the full $m$-shift (i.e. $G$ is the graph with one vertex and $m$ edges, although it is necessary to pass to a higher block presentation for the map to be regular). The s/u-bijective pair here is $(Y, \psi) = (\Sigma_G, \sigma)$ and $(Z, \zeta) = (X, \varphi)$. The only non-zero groups in the double complex occur for $L = 0 = M$ and $L = 1, M = 0$ and these are $\mathbb{Z}[m^{-1}]$ and $\mathbb{Z}$, respectively. The boundary maps are all zero (only one needs to be computed) and $H^*_N(X, \varphi)$ is isomorphic to $\mathbb{Z}[m^{-1}]$, for $N = 0$, $\mathbb{Z}$, for $N = 1$ and zero for all other $N$.

Example 6.3. Let $X$ be the 2-torus, $\mathbb{T}^2$, and $\varphi$ be the hyperbolic automorphism determined by the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

In this case, a Markov partition can be chosen so that the natural quotient factors in two ways, as an s-bijective followed by u-bijective and vice versa. (In fact, it is the Markov partition with three rectangles which appears in many dynamics texts.) The only non-zero terms in the double complex are in positions $(L, M) = (0, 0), (1, 0), (0, 1), (1, 1)$. The calculation yields $H^*_N(X, \varphi)$ is $\mathbb{Z}$ for $N = 1$ and $N = -1$ and is $\mathbb{Z}^2$ for $N = 0$. Notice that the homology coincides with that of the torus, except with a dimension shift.
Example 6.4. There is an example, \((X, \varphi)\), roughly based on the Sierpinski gasket. We do not give any details except to mention that it is not a shift of finite type, but its homology is the same as the full 3-shift.

7 Concluding remarks

Remark 7.1. It is certainly a natural question to ask whether this theory can be computed from other (already existing) machinery. A more specific question would be to relate our homology to, say, the Čech cohomology of the classifying space of the topological equivalence relation \(\mathcal{R}^s\). (For a discussion of the topology, see [10].) There are examples, such as the first three above, where they are different, but only up to a dimension shift (depending on the space under consideration).

Remark 7.2. An important motivation in the construction of this theory was to compute the \(K\)-theory of certain \(C^*\)-algebras associated with the Smale space \((X, \varphi)\). See [10] for a discussion of these \(C^*\)-algebras. At present, there seems to be a spectral sequence which relates the two; this work is still in progress.

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