On two-dimensional magnetic reconnection with nonuniform resistivity

Leonid M Malyshkin¹ and Russell M Kulsrud²

¹ Department of Astronomy and Astrophysics, University of Chicago, 5640 S Ellis Avenue, Chicago, IL 60637, USA
² Princeton Plasma Physics Laboratory, Princeton, NJ 08543, USA
E-mail: leonmal@uchicago.edu

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Abstract

In this paper, two theoretical approaches for the calculation of the rate of quasi-stationary, two-dimensional magnetic reconnection with nonuniform anomalous resistivity are considered in the framework of incompressible magnetohydrodynamics (MHD). In the first, ‘global’ equations approach, the MHD equations are approximately solved for a whole reconnection layer, including the upstream and downstream regions and the layer center. In the second, ‘local’ equations approach, the equations are solved across the reconnection layer, including only the upstream region and the layer center. Both approaches give the same approximate answer for the reconnection rate. Our theoretical model is in agreement with the results of recent simulations of reconnection with spatially nonuniform resistivity.

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(Some figures in this article are in colour only in the electronic version.)

1. Introduction

Magnetic reconnection is one of the most important processes of plasma physics, and is believed to play a central role in many phenomena observed in laboratory and cosmic plasmas. Reconnection is generally a local process; it takes place in thin reconnection layers that are regions of space where the electric current is strong and dissipation processes are important. At the same time, magnetic reconnection frequently controls the overall plasma dynamics and large-scale equilibrium of magnetized physical systems. As a result, magnetic reconnection is directly related to other important physical and astrophysical processes, such as in dynamos and turbulence in magnetized plasmas, transport of charged particles and thermal conduction in plasmas, angular momentum transport in rotating systems (e.g. in accretion discs) and acceleration of cosmic ray particles.

There has been a long-standing debate about the correct theoretical model of magnetic reconnection. Most previous theoretical and numerical works focused on reconnection processes in two dimensions, in which all physical scalars and vectors are independent of the third coordinate (z). There exist two original and well-known models of magnetic reconnection with constant resistivity: firstly, the Sweet–Parker reconnection model [1, 2], which predicts a slow magnetic reconnection rate in hot low-density plasmas; secondly, the Petschek model [3], in which a fast reconnection rate is achieved by introducing switch-off magnetohydrodynamic (MHD) shocks attached to the ends of a relatively short reconnection layer in the downstream regions. Many numerical simulations have been carried out to discriminate between these two models. More recent high-resolution simulations generally favor the Sweet–Parker model of slow reconnection in the case of constant resistivity, and do not confirm the Petschek theoretical picture for the geometry of the reconnection layer with shocks [4–6]. However, reconnection becomes much faster and Petschek-like if resistivity is not constant and is enhanced locally in the reconnection layer (numerical studies of this case were pioneered by Ugai and Tsuda [7, 8], by Hayashi and Sato [9, 10] and by Scholer [11]).

These results have been called into question in a recent paper by Baty et al [15], which claims that Petschek reconnection can occur when the resistivity is not absolutely constant but varies by an arbitrarily small amount. This paper has motivated us to show, on the basis of an earlier paper...
by us [14], that the Petschek shocks are produced not by the variation of the resistivity but by the rate of variation, which for their case is very large and unlikely to happen naturally.

In 2001, Kulsrud [12] provided some theoretical insight into the magnetic reconnection process, which qualitatively explained the results of recent simulations. He considered quasi-stationary, two-dimensional magnetic reconnection in the classical Sweet–Parker–Petschek reconnection layer with zero guide field ($B_z = 0$), zero plasma viscosity and an anomalous resistivity that is a piecewise linear function of the electric current. Kulsrud was the first to suggest that one has to calculate the half-length of the reconnection layer $L'$ from the MHD equations and the jump conditions on the Petschek shocks, instead of treating $L'$ as a free parameter (as Petschek erroneously did when he chose $L'$ to be equal to its minimal possible value under the condition of no significant disruption to the plasma flow). As a result, in the case of constant resistivity, Kulsrud correctly estimated the layer half-length $L'$ to be approximately equal to the global magnetic field scale, $L' \approx L$. In this case, the Petschek reconnection rate reduces to the slow Sweet–Parker reconnection rate [12, 13], in agreement with numerical simulations. The second result obtained by Kulsrud is that in the case when resistivity is non-constant but anomalous and enhanced (e.g. by plasma instabilities), the reconnection rate can become considerably faster than the Sweet–Parker rate.

In our recent paper (Malyskin et al. [14]), to which we will hereafter refer to as the MLK2005 paper, we put Kulsrud’s derivations on a rigorous analytical basis and extended his model by using a new theoretical approach to calculate the reconnection rate. This approach is based on ‘local’ analytical derivations across a thin reconnection layer, and is applicable to the case when resistivity is anomalous, nonuniform and an arbitrary function of the electric current and of the spatial coordinates. We included the case of non-zero guide field $B_z \neq 0$ and non-zero plasma viscosity in our model. We found an approximate formula for the reconnection rate, which confirmed Kulsrud’s theoretical results.

This paper has two goals: firstly, to calculate the reconnection rate, and secondly, to compare it with the recent simulations of Baty et al. [15].

In the next section, we simultaneously follow two theoretical approaches to the calculation of the reconnection rate. In the first approach, which we call the ‘global’ equations approach, we derive and solve approximate MHD equations for a whole reconnection layer, including the upstream and downstream regions and the layer center. These theoretical derivations are similar to those done before, except for an important difference. Namely, we take into consideration an additional important equation, the spatial homogeneity of the $z$-component of electric field $E_z$ along the reconnection layer. This equation, together with the jump condition on Petschek shocks, allows us to find the reconnection layer length $L'$, which must be determined consistently [12]. The second theoretical approach, which we call the ‘local’ equations approach, basically coincides with the calculations done in the MLK2005 paper. In this approach, we derive and solve approximate MHD equations across a thin reconnection layer, including only the upstream region and the layer center.

Both approaches give the same approximate formula for the reconnection rate, which is valid in the general case of an arbitrary nonuniform anomalous resistivity; see equation (10). Using these two approaches simultaneously allows us to better understand the physics of a reconnection process.

In section 3, we pursue the second goal of this paper. We consider the case treated by Baty et al. [15], when resistivity is a prescribed function of the two spatial coordinates $x$ and $y$ (remember that we consider two-dimensional reconnection, so that no physical quantities depend on $z$). We argue that our theoretical model is in agreement with the results of these recent simulations of reconnection by Baty et al [15]. This agreement contradicts their theoretical conclusion that Petschek shocks exist with constant resistivity, and we explain why. Finally, in section 4, we discuss our results.

2. Reconnection with nonuniform anomalous resistivity

In this section, we consider magnetic reconnection with nonuniform anomalous resistivity in the classical two-dimensional Sweet–Parker–Petschek reconnection layer, shown in figure 1. The layer is in the $x$–$y$ plane with the $x$- and $y$-axis being perpendicular to and along the layer, respectively. The length of the layer is $2L'$, which is approximately equal to or smaller than the global magnetic field scale $L$ that will be introduced below. The thickness of the layer, $2\delta_0$, is much smaller than its length, i.e. $2\delta_0 \ll 2L'$. The classical Sweet–Parker–Petschek reconnection layer is assumed to have a point symmetry with respect to its geometric center point O in figure 1 and reflection symmetries with respect to the axis $x$ and $y$. Thus, for example, the $x$- and $y$-component of the plasma velocities $\mathbf{V}$ and of the magnetic field $\mathbf{B}$ have the following simple symmetries: $V_z(\pm x, \mp y) = \mp V_z(x, y)$, $V_y(\pm x, \mp y) = \mp V_y(x, y)$, $B_z(\pm x, \mp y) = \mp B_z(x, y)$, and $B_y(\pm x, \mp y) = \pm B_y(x, y)$. There could be a pair of Petschek shocks attached to each of the two reconnection layer ends in the downstream regions. Because of the MHD jump conditions on the Petschek shocks, there must be a nonzero perpendicular magnetic field $B'_z$ in the downstream region at point $O'$ in figure 1 [12, 13]. If the plasma viscosity is small, then the plasma outflow velocity $V_{\text{out}}$ in the downstream region at point $O'$ is approximately equal to the Alfvén velocity $V_A$ calculated in the upstream region at point $M$ (refer to figure 1). The plasma inflow velocity $V_R$ in the upstream region at point $M$, outside the reconnection layer,
is much smaller than the outflow velocity, $V_{A} \ll V_{A}$. Finally, the magnetic field $B_m$ at point $M_\ast$, outside the layer, is mostly in the direction of the layer (i.e. in the $y$-direction).

Now let us list four assumptions that we make about the reconnection process. Firstly, we assume that the characteristic Lundquist number is large, which (by our definition) means that resistivity is negligible outside the reconnection layer. Secondly, we assume that the plasma flow is incompressible. Thirdly, for simplicity, we neglect plasma viscosity. (The case of nonzero viscosity is treated in the MLK2005 paper [14].) Fourthly, we assume that the reconnection process is quasi-stationary. This is true if the reconnection is slow, $V_{A}/V_{A} \ll 1$, and that there are no plasma instabilities in the reconnection layer. For an incompressible viscousless plasma, the assumption of slow reconnection is equivalent to the assumption that the reconnection layer is thin, $\delta_{y}/L' \approx V_{A}/V_{A} \ll 1$. The above assumptions are standard in the Sweet–Parker and Petschek reconnection models. Note that we make no assumptions about the values of the guide field $B_{y}$. In other words, our derivations apply to what is called the ‘two-and-a-half dimensional’ reconnection.

For brevity, we use physical units in which the speed of light and four times $\pi$ are replaced by unity, $c = 1$ and $4\pi = 1$. In these units, the MHD equations we need for finding the reconnection rate are as follows. Faraday’s law $\mathbf{V} \times \mathbf{B} = -\partial \mathbf{B}/\partial t$ for the $x$- and $y$-components of magnetic field in two dimensions is

$$\partial E_{x}/\partial y = -\partial B_{y}/\partial t \approx 0, \quad \partial E_{y}/\partial x = -\partial B_{x}/\partial t \approx 0,$$

(1)

where $\partial \mathbf{B}/\partial t \approx 0$ because of the quasi-stationarity of reconnection. From (1), we see that the electric field $z$-component is constant in space, i.e. $E_{z} = E_{z}(t)$ is a function of time only. Next, neglecting the displacement current in the framework of MHD, Ampere’s law for the $z$-component of the current is

$$j_{z} = (\mathbf{V} \times \mathbf{B})_{z} = \partial B_{y}/\partial x - \partial B_{x}/\partial y.$$

(2)

Ohm’s law for the spatially uniform $z$-component of the electric field is

$$E_{z}(t) = -V_{x}B_{x} + V_{y}B_{y} + \eta j_{z} = \text{const in space},$$

(3)

where resistivity $\eta = \eta(j_{z}, x, y)$ is an arbitrary function of the electric current $z$-component and of the two-dimensional coordinates $x$. Next, the equation for plasma acceleration along the layer in the $y$-direction is

$$\rho(V \cdot \nabla)V_{y} = -(\partial V_{y}/\partial y)\left[ P + B_{x}^{2}/2 + B_{y}^{2}/2 \right] + (\mathbf{B} \nabla)B_{y},$$

(4)

where $\rho$ is the constant plasma density and $P$ is the sum of the plasma pressure and the guide field pressure $B_{z}^{2}/2$. In addition, we have

$$\partial V_{x} + \partial V_{y} = 0, \quad \partial B_{x} + \partial B_{y} = 0,$$

(5)

because the field and the velocity are divergence free.

Now, from MHD equations (2)–(5), we obtain ‘global’ and ‘local’ equations for magnetic reconnection that are given in table 1. Here quantities with subscript o are taken at the reconnection layer central point O, while quantities with the prime sign are taken at point O’ in the downflow region (refer to figure 1). The column ‘Global equations’ includes equations that are written only at points O, M and O’; these equations represent the ‘global’ equations approach to the calculation of the reconnection rate. The column ‘Local equations’ includes equations that are written only at points O and M and represent the ‘local’ equations approach. Note that equations on lines 1 and 5 enter both columns in the same form. The first line of the table includes the Ampere’s law equation (2), with the $\partial \mathbf{B}/\partial t$ term neglected because it is small, and the $\partial B_{y}$ term estimated at point O. The second line of the table includes the plasma incompressibility condition (5) in its ‘global’ and ‘local’ forms. The ‘global’ form is the mass conservation equation, while in the ‘local’ form the $\partial V_{y}$ term is estimated at point O. The third line contains equations for plasma acceleration given by equation (4). The ‘global’ equation $V_{out} \approx V_{A}$ reflects the well-known result that in the absence of viscosity the plasma outflow velocity in the downstream region is approximately equal to the Alfvén velocity calculated in the upstream region [1–3]. The ‘local’ equation for plasma acceleration results from differentiation of equation (4) with respect to $y$ and taking the symmetries of the reconnection layer into account. The $j_{z}(\partial B_{y})_{o}$ term in this equation is the magnetic tensor force, while the pressure term is equal to

$$\left(\partial^{2}P\right)_{o} = \left(\partial^{2}(B_{y}^{2}/2)_{m}\right) = B_{m}(\partial^{2}B_{y})_{m} < 0.$$
pressure drop of the parallel magnetic field outside the layer. This result follows from the force balance condition for the plasma across the reconnection layer (in analogy with the Sweet–Parker derivations), and a rigorous proof of it can be found in appendix A of the MLK2005 paper [14]. The last equality in equation (6) comes from the layer reflection symmetry with respect to the x-axis.

Next, the fourth line in table 1 includes the standard jump condition on the switch-off MHD Petschek shocks attached to the ends of the reconnection layer [12, 13]. This condition is a ‘global’ equation, and the plasma incoming velocity \( V_R \) at point \( M' \) is estimated to be approximately equal to the plasma incoming velocity \( V_R \) at point \( M \) (refer to figure 1). There is no corresponding ‘local’ equation because we do not consider the downstream region and Petschek shocks in the ‘local’ equations approach! Equations in the fifth and sixth lines of table 1 directly result from Ohm’s law, equation (3). Namely, we use the spatial homogeneity of \( E_\z \), which is a consequence of the quasi-stationarity of reconnection. To obtain the single equation in the fifth line and the ‘global’ equation in the sixth line, we equate the Ohm’s law expression for \( E_\z \) at points \( O \), \( M \) and \( O' \). To obtain the ‘local’ equation in the sixth line, we take the second-order partial derivative \( \partial_y^2 \) of equation (3) at point \( O \). Finally, the seventh line in table 1 lists all unknown physical quantities to be estimated in the ‘global’ and ‘local’ equations approaches. Note that quantities \( j_o, \delta_o, V_o, (\partial_x V_o)_o \) and \( (\partial_y B_y)_o \) are ‘local’ (i.e. defined at the layer central point \( O \) and at point \( M \) in the upstream region), while quantities \( L', V'_{o,\text{out}} \) and \( B'_o \) are ‘global’ (i.e. defined at point \( O' \) in the downstream region).

There are a few additional equations that we need. Firstly, we use the second-order Taylor expansion of \( \eta_j \) along the \( y \)-axis to estimate the \( \eta' j'_o \) term in the sixth line in table 1:

\[
\eta' j'_o \approx \eta_o j_o + \left( L^2 / 2 \right) j_o (\partial_y^2 \eta)_o + \left( \eta_o + j_o (\partial_y \eta / \partial y)_o \right) (\partial_y j'_o).
\]  

(7)

Note that the first-order Taylor expansion terms are zero because of the symmetry. Secondly, we define the field global scale \( L \) and the resistivity scale \( l_q \) as \(^5\)

\[
L^2 \equiv -2B_m / (\partial_y^2 B_y)_m, \quad l_q^2 \equiv -2\eta_o / (\partial_y^2 \eta)_o.
\]  

(8)

Thirdly, the \( y \)-scale of the current \( j_o \), to a factor of order unity, turns out to be almost the same as the \( y \)-scale of the outside magnetic field:

\[
j_o^{-1} (\partial_y^2 j_o)_m \approx B_m^{-1} (\partial_y^2 B_y)_m = -2/L^2.
\]  

(9)

This result can be understood by taking the second-order partial derivative \( \partial_y^2 \) of the Ampère’s law equation \( j_o \approx B_m / \delta_o \) given in the first line of table 1, while keeping \( \delta_o \) constant because the partial derivative in \( y \) is to be taken at a constant value of \( x = \delta_o \). A detailed proof of equation (9) is given in appendix B of the MLK2005 paper [14].

Now we have all the equations necessary for finding the reconnection rate and all other unknown physical parameters.

\(^5\) Here we consider the natural case when \( (\partial y / \partial y)_o \geq 0 \) and \( (\partial y^2 / \partial y^2)_o \leq 0 \), because plasma conductivity decreases as current increases and we are interested in anomalous reconnection that is faster than the Sweet–Parker reconnection.

In the column ‘Global equations’ of table 1, we have six equations and six unknowns, and in the column ‘Local equations’ we have five equations and five unknowns. Using the ‘local’ equations and equations (6), (8) and (9), we obtain the following approximate algebraic equation for the \( z \)-current \( j_o \) at the reconnection layer central point \( O \):

\[
3 + \frac{j_o}{\eta_o} \left( \frac{\partial \eta}{\partial y}_o \right) + \left( l_q^2 / \eta_o \right) \approx \frac{B_m^2 \delta_o^2 L^2}{V_R^2 B_x^2},
\]  

(10)

where resistivity \( \eta_o \equiv \eta(j_o, x = 0, y = 0) \) is a function of \( j_o \). The ‘global’ equations give a similar result with 3 replaced by 1 in equation (10), which is a less accurate result due to additional approximations made in equation (7) and in the formula \( V_R' \approx V_R \) in the fourth line of table 1.

Given the resistivity function \( \eta = \eta(j_z, x, y) \), as well as the magnetic field \( B_m \) and its global scale \( L \equiv -2B_m / (\partial_y^2 B_y)_m \), both calculated at point \( M \) in the upstream region, we can solve the algebraic equation (10) for the current \( j_o \). Once \( j_o \) is calculated, we can easily find the reconnection rate, which is the rate of destruction of magnetic flux at point \( O \) and is equal to the electric field \( z \)-component \( E_\z \). We can also find all the other physical parameters by using the equations in table 1:

\[
\delta_o \approx B_m / j_o,
\]  

(11)

\[
V_R \approx \eta_o j_o / B_m \ll V_A,
\]  

(12)

\[
(\partial_y V_y)_o \approx \eta_o j_o^2 / B_m^2 \approx V_R / \delta_o,
\]  

(13)

\[
(\partial_y B_x)_o \approx j_o \left( V_R^2 / V_A^2 - 2B_m^2 / j_o^2 L^2 \right),
\]  

(14)

\[
B'_o \approx V_\text{out} \sqrt{\rho_0} \approx B_m (V_R / V_A),
\]  

(15)

\[
V'_{\text{out}} \approx V_A,
\]  

(16)

\[
L' \approx \delta_o (V_A / V_R) \approx V_A / (\partial_x V_x)_o \approx B'_o / (\partial_y B_x)_o.
\]  

(17)

The last approximate equality in equation (17) is valid because the second term on the right-hand side of equation (14) can be neglected. This is because \( 3B_m^2 / j_o^2 L^2 \ll V_R^2 / V_A^2 \), which follows from equations (12) and (10).

Equations (10)–(17) are very general results for quasi-stationary magnetic reconnection with no assumptions about the functional form of resistivity and the guide field value. Now let us look at the three terms on the left-hand side of equation (10). When resistivity is constant, the only term left is the first term. The second term is clearly related to the dependence of anomalous resistivity on the current. The third term becomes important when resistivity is ad hoc localized in space. As a result, in the end of this section, we consider three special cases of magnetic reconnection, in which equations (10)–(17) reduce to simpler formulae. These three cases correspond to domination of the first, second and third terms, respectively, on the left-hand side of equation (10), and they are as follows.
The first case is the quasi-uniform resistivity case:

if $1 \gg \max\{(j_o/\eta_o)(\partial \eta/\partial j_o), L^2/\eta_o^2\}$,

then $j_o \approx (B_m/L)^{3/2}$, where $S \equiv V_A L/\eta_o$. (18)

$V_R/V_A \approx S^{-1/2}$, $\delta_o \approx LS^{-1/2}$, $L' \approx L$.

Here we introduce the Lundquist number $S_o \equiv V_A L/\eta_o$ and assume for our estimates that $3^{1/4} \approx 1$. Equations (18) are the familiar Sweet–Parker results [1, 2]. Thus, if resistivity is quasi-uniform or uniform, then reconnection is Sweet–Parker.

The second case is the Petschek–Kulsrud reconnection:

if $(j_o/\eta_o)(\partial \eta/\partial j_o) \gg \max[1, L^2/\eta_o^2]$, 

then $V_R/V_A \approx \delta_o/L' \approx [(B_m/V_A L^2)(\partial \eta/\partial j_o)/(\partial \eta/\partial j_o)]^{1/3}$, (19)

$L' \approx L \left[(j_o/\eta_o)(\partial \eta/\partial j_o)\right]^{-1/2} \ll L$.

This is the case of fast reconnection with Petschek geometry and shocks. Equations (19) were first derived by Kulsrud [12]. That is why we call this case the Petschek–Kulsrud reconnection.

The third case is the case of reconnection with spatially localized resistivity:

if $L^2/\eta_o^2 \gg \max[1, (j_o/\eta_o)(\partial \eta/\partial j_o)]$,

then $j_o \approx (B_m/\eta_o)^{1/2}$, where $S_o \equiv V_A L/\eta_o$, (20)

$V_R/V_A \approx S_o^{1/2}$, $\delta_o \approx \eta_o S_o^{-1/2}$, $L' \approx \eta_o \ll L$.

Here we introduce the effective Lundquist number $S_o \equiv V_A L/\eta_o$ that is based on the resistivity scale $\eta_o$ given by equation (8). Equations (20) are the same as the Sweet–Parker equations (18) with the global field scale $L$ replaced by the resistivity scale $\eta_o$. When resistivity is strongly localized, $\eta_o \ll L$, the reconnection becomes fast-yielding Petschek geometry and shocks.

We postpone detailed analysis and discussion of our theoretical results to the last section of the paper.

3. Reconnection with spatially localized resistivity

In this section, we compare our theoretical results to the results of recent simulations of reconnection with spatially nonuniform resistivity by Baty et al [15]. These authors took resistivity as

$$\eta(x, y) = \eta_1 + (\eta_0 - \eta_1) \exp\left[-(x/l_y)^2 - (y/l_y)^2\right]$$

(21)

and considered different values of the parameters $\eta_0$, $\eta_1$, $l_x$ and $l_y$. They found that the reconnection rate does not depend on the value of $l_y$. This is in agreement with our equation (10), which includes only $\partial^2 \eta(x, y)$ derivative via resistivity scale $\eta_o$ given by equation (8). As for the dependence on the other parameters, in all their simulation runs Baty et al found the Petschek geometrical configuration with shocks and reconnection rate faster than the Sweet–Parker rate. They paid special attention to the case when $\eta_0 - \eta_1$ in equation (21) is small and the resistivity is weakly nonuniform. They called this case a ‘quasi-uniform resistivity’ case, and observed the Petschek solution in this case as well. At the same time, we theoretically derived the Sweet–Parker solution for the quasi-uniform resistivity case, given by equations (18). Thus, there is disagreement between our theoretical results and the claims by Baty et al which needs to be addressed.

The reason for this disagreement is that the resistivity used by Baty et al was not actually quasi-uniform in all their simulation runs. In fact, when resistivity depends only on coordinates, the condition of a quasi-uniform resistivity is $l_y^2/\eta_o^2 \ll 1$, refer to equations (18). In other words, the resistivity localization scale $l_y$ must be much larger than the field global scale $L$. In their most uniform resistivity simulation run, Baty et al used $\eta_o = 10^{-4}$, $\eta_1 = 9.3 \times 10^{-5}$, $l_y = 0.1$ and $L \approx 1$. According to equation (8), in this case $l_y = l_y \sqrt{\eta_o/(\eta_0 - \eta_1)} = 0.378$, which is considerably smaller than $L \approx 1$. Thus, while the resistivity function $\eta(x, y) = 9.3 \times 10^{-5} + 7 \times 10^{-6} \exp[-(y/0.1)^2 - (x/l_y)^2]$ would indeed appear close to uniform when its graph is plotted, its second derivative $\partial^2 \eta$ is large due to small value of $l_y = 0.1$. As a result, this resistivity should be viewed as far from uniform and as rather well localized, $L^2/\eta_o^2 \approx 7$. In their other simulation runs, Baty et al used even stronger localized resistivity with larger values of $\eta_0 - \eta_1$ in equation (21). In fact, they were not able to run any simulations with strictly uniform resistivity $\eta = \text{const}$ because of an instability of the reconnection layer. The reason for this instability possibly lies in the boundary conditions that Baty et al used, which might conflict with our equation (10). The latter must be satisfied for quasi-stationary reconnection.

We would like to quantitatively compare our theoretical results to the results obtained by Baty et al in their most uniform resistivity simulation run. They had $l_y = 0.378$, $L \approx 1$, $\eta_o = 10^{-4}$, $V_A \approx 1$, $B_m \approx 0.9$ and they found that $j_o \approx 120$ for the current at the reconnection layer central point O [15, 16]. Our approximate equations (20), derived for the localized resistivity case, give $j_o \approx (B_m/\eta_o)(V_A l_y/\eta_o)^{1/2} = 146$, which is close to $j_o \approx 120$ observed in the simulation. The small disagreement could be due to plasma compressibility, or due to the finite Lundquist number used in the simulation and due to the approximate nature of our theoretical model.

We conclude that our theoretical model for magnetic reconnection is in agreement with the simulations by Baty et al [15]. Our model is in reasonable agreement with several other previous numerical simulations of reconnection with spatially nonuniform resistivity [7, 8, 11, 17].

4. Discussion

Equation (10), that determines the reconnection current $j_o$ and the quasi-stationary reconnection rate $\eta_o j_o$, is derived by using the ‘local’ equations theoretical approach (see the last column of table 1). This approach involves only ‘local’ quantities and equations that are defined on the interval OM across the reconnection layer; refer to figure 1. Thus, the quasi-stationary reconnection rate in a thin two-dimensional layer is determined locally, in the layer central point O and in the upstream region outside the layer at point M. In other words, the rate is fully determined by a particular
functional form of anomalous resistivity $\eta(j_z, x, y)$ and by the local configuration of the magnetic field in the upstream region. The latter determines field $B_m$ and its scale $L \equiv [-2B_m/(\partial^2 \eta \partial x^2)]^{1/2}$ at point M. As a result, the global properties of the reconnection layer (e.g. its length $L'$, the plasma outflow velocity $V_{out}$ and the presence or absence of Petschek shocks) do not directly matter for the quasi-stationary reconnection rate. This is a very important result because the reconnection layer global geometry can be very complicated (e.g. in turbulent plasmas). Of course, there exists an indirect dependence of the reconnection rate on the field global properties because the local field configuration in the upstream region is determined by the field global configuration. The above statements are also true for all other ‘local’ parameters—for the layer thickness $\delta_0$ and for the reconnection velocity $V_R$, refer to equations (11) and (12).

The reconnection rate can also be estimated by using the ‘global’ equations theoretical approach, in which the whole reconnection layer is considered, including the downstream region at point O (refer to figure 1). The equations used in this approach are presented in the second column of table 1. We would like to point out that two of these equation play a key role in correct determination of the reconnection layer length $L'$ and geometry. The first key equation is the jump condition on the Petschek shocks, $B'_y/\sqrt{\rho} \approx V_R \approx V_R$, whose importance was first pointed out by Kul murad [12]. The second key equation is the constancy of the electric field $z$-component along the reconnection layer, $\eta_o j_0 = \eta' j'_z + V_{out} B'_y$, which has been overlooked in the previous theoretical models (e.g. in the Petschek model).

For the case of a strong dependence of resistivity on the current, i.e. when $(j_0/\eta_o)(\partial \eta/\partial j_z)_o \gg \max(1, L'/L)^2$, our results, given by equations (19), coincide with the results obtained by Kul murad [12]. Thus, both the ‘local’ and ‘global’ theoretical approaches confirm Kul murad’s results and ideas, contrary to the doubts raised in the paper by Baty et al [15].

We found that in the case of uniform or quasi-uniform resistivity the magnetic reconnection rate is the slow Sweet–Parker rate and not the fast Petschek rate; see equations (18). This theoretical result follows from rigorous analytical derivations and agrees with numerical simulations. At the same time it contradicts the original Petschek theoretical model. Let us consider both the ‘global’ and ‘local’ analytical approaches in the case of a strictly uniform resistivity $\eta = \text{const} = \eta_o$, and let us explain why the Petschek reconnection layer geometry is not realized in this case. We take the ‘global’ equations approach first. In the case of constant resistivity, equations (7) and (9) result in the equation, $(\eta_o j_0 - \eta' j'_z)/\eta_o \approx (L'/L)^2$, for the fractional drop of the $j_z$ term along the reconnection layer. On the other hand, the constancy of the electric field $z$-component along the reconnection layer implies that this fractional drop is $(\eta_o j_0 - \eta' j'_z)/\eta_0 \approx V_{out} B'_y/\eta_0 \approx 1$, where we use the ‘global’ equations on lines 3–6 in the second column of table 1. These two equations agree only if $L' \approx L$, which means that the geometry of the reconnection layer is Sweet–Parker and not Petschek. Next, we take the ‘local’ equations approach. In this ‘local’ approach, we prefer not to consider the reconnection layer geometry and any ‘global’ parameters, such as the layer length $L'$, for the calculation of the reconnection rate. Instead we argue as follows. Referring to the ‘local’ equations in the last column of table 1, equations on lines 1, 2 and 5 result in an estimate of the plasma outflow velocity derivative $(\partial \eta \partial j_z)_o \approx \eta_o j'_z/B_m^2$, see equation (13). The plasma acceleration equation on line 3 results in the upper estimate for the $B_y$ field derivative, $(\partial \eta \partial j_z)_o \approx \rho (\partial \eta \partial V_y)/j_o$. As a result, we can find the upper estimate for the reconnection current $j_0$ from the ‘local’ equation on line 6, which is the condition of constancy of the electric field $z$-component along the reconnection layer. If resistivity is uniform, this estimate turns out to be the Sweet–Parker value, $j_0 \approx (B_m/L)(V_A/L/\eta_o)^{-1/2}$, as given by equation (18). In addition, if one estimates the reconnection layer length as $L' \approx B'_y/(\partial \eta \partial j_z)_o \approx V_A/(\partial \eta \partial V_y)_o$, one would again recover the Sweet–Parker result $L' \approx L$ [14]. We conclude that in the case of constant resistivity both the ‘global’ and ‘local’ equations approaches consistently lead to the Sweet–Parker reconnection rate and the Sweet–Parker geometry of the reconnection layer ($L' \approx L$).

Let us finally point out that whether reconnection is unforced (free) or forced does not matter for our results and conclusions. Indeed, on the one hand, in the case of unforced reconnection one solves equations (10) and (12) for the current $j_o$ and for the reconnection velocity $V_R$. The solution will depend on the magnetic field $B_m$ in the upstream region at point M. On the other hand, in the case of forced magnetic reconnection the reconnection velocity $V_R$ is prescribed and fixed. In this case the field $B_m$ in the upstream region should be treated as an unknown parameter, and equations (10) and (12) are to be solved together in order to find the correct quasi-stationary values of $j_o$ and $B_m$. In other words, in the forced reconnection case, an initially weak outside magnetic field $B_m$ gets piled up to higher values until the resulting current $j_0$ in the reconnection layer becomes large enough to be able to match the prescribed velocity $V_R$ of magnetic flux and energy supply in the upstream region.

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References

[1] Sweet P A 1958 Electromagnetic Phenomena in Ionized Gases ed B Lehner (New York: Cambridge University Press) p 123
[2] Parker E N 1963 Astrophys. J. Suppl. Ser. 8 177
[3] Petschek H E 1964 AAS–NASA Symp. on Solar Flares NASA SP50 (Washington, DC: National Aeronautics and Space Administration) p 425
[4] Biskamp D 1986 Phys. Fluids 29 1520
[5] Breslau J A and Jardin S C 2003 Phys. Plasmas 10 1291
[6] Uzdensky D A and Kulsrud R M 2000 Phys. Plasmas 7 4018
[7] Ugai M and Tsuda T 1977 J. Plasma Phys. 17 337
[8] Tsuda T and Ugai M 1977 J. Plasma Phys. 18 451
[9] Hayashi T and Sato T 1978 J. Geophys. Res. 83 217
[10] Sato T and Hayashi T 1979 Phys. Fluids 22 1189
[11] Scholer M 1989 J. Geophys. Res. 94 8805
[12] Kulsrud R M 2001 Earth, Planets Space 53 417
    Kulsrud R M 2000 arXiv:astro-ph/0007075
[13] Kulsrud R M 2005 Plasma Physics for Astrophysics (Princeton, NJ: Princeton University Press)
[14] Malyshkin L M, Linde T and Kulsrud R M 2005 Phys. Plasmas 12 102902
[15] Baty H, Priest E R and Forbes T G 2006 Phys. Plasmas 13 022312
[16] Baty H 2006 private communication
[17] Biskamp D and Schwarz E 2001 Phys. Plasmas 8 4729