Article

Global well-posedness and analyticity for generalized porous medium equation in critical Fourier-Besov-Morrey spaces

Mohamed Toumlilin

1 FST FES, Laboratory AAFA, Department of Mathematics, University Sidi Mohamed Ben Abdellah, Fes, Morocco.
* Correspondence: mohamed.toumlilin@usmba.ac.ma

Received: 23 July 2019; Accepted: 29 September 2019; Published: 19 October 2019.

Abstract: In this paper, we study the generalized porous medium equations with Laplacian and abstract pressure term. By using the Fourier localization argument and the Littlewood-Paley theory, we get global well-posedness results of this equation for small initial data $u_0$ belonging to the critical Fourier-Besov-Morrey spaces. In addition, we also give the Gevrey class regularity of the solution.

Keywords: Porous medium equation, well-posedness, analyticity, Fourier-Besov-Morrey space.

MSC: 35K55, 74G25, 76S05.

1. Introduction

We investigate the generalized porous medium equation in the whole space $\mathbb{R}^3$,

$$ \begin{cases} 
 u_t + \mu \Lambda^s u = \nabla \cdot (u \nabla P u); & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\
 u(0, x) = u_0 & x \in \mathbb{R}^3, 
\end{cases} 
$$

(1)

where $u = u(t, x)$ is a real-valued function, which denotes a density or concentration. The dissipative coefficient $\mu > 0$ corresponds to the viscous case, while $\mu = 0$ corresponds to the inviscid case. The fractional Laplacian operator $\Lambda^s$ is defined by Fourier transform as $\hat{\Lambda^s u} = |\xi|^s \hat{u}$, and $P$ is an abstract operator.

The equation (1) was introduced in the first by Zhou et al. [1]. In fact, Equation (1) is obtained by adding the fractional dissipative term $\mu \Lambda^s u$ to the continuity equation (PME) $u_t + \nabla \cdot (uV) = 0$ given by Caffarelli and Vázquez [2], where the velocity $V$ derives from a potential, $V = -\nabla p$ and the velocity potential or pressure $p$ is related to $u$ by an abstract operator $p = Pu$ [3].

For $\mu = 0$ and $Pu = (-\Delta)^{-s} u = \Lambda^{-2s} u$, $0 < s < 1$; X. Zhou et al. [4] were interested in finding the strong solutions of the equation (1) which becomes the fractional porous medium equation in the Besov spaces $B^s_{p,\infty}$ and they obtained the local solution for any initial data in $B^s_{1,\infty}$. Moreover, in the critical case $s = 1$, the Equation (1) leads to a mean field equation [4,5]. Let’s take this opportunity to briefly quote some works on the well-posedness and regularity on those equations such as [4,6] and the references therein.

On the other hand, another similar model occurs in the aggregation equation, and plays a fundamental role in applied sciences such as physics, biology, chemistry, population dynamics. It describes a collective motion and aggregation phenomena in biology and in mechanics of continuous media [7,8]. In the aggregation equation, the abstract form pressure term $Pu$ can also be represented by convolution with a kernel $K$ as $Pu = K * u$. The typical kernels are the Newton potential $|x|^s$ [9,10], and the exponent potential $-e^{-|x|}$ [11,12]. For more results on this equation, we refer to [13,14] and the references therein.

Recently, Zhou et al. [1] obtained the local well-posedness in Besov spaces for large initial data, and proved that the solution becomes global if the initial data is small, also, they studied a blowup criterion for the solution.
In addition, we can represent the Equation (1) with the same initial data by
\[
\begin{align*}
u_1 + \mu \Lambda^2 u + \nu \cdot \nabla u &= -u(\nabla \cdot v); \\
v &= -\nabla Pu.
\end{align*}
\]
(2)

As a consequence, this equation must be compared to the geostrophic model. So, the convective velocity is not absolutely divergence-free for the generalized porous medium equation. Additionally, if we assume that \( v \) is divergence-free vector function (\( \nabla \cdot v = 0 \)), the form (2) can contain the quasi-geostrophic (Q-G) equation [15,16].

Inspired by the works [1,17]; the aim of this paper is to prove the well-posedness results of Equation (1) and to give the Gevrey class regularity of the solution in homogeneous Fourier Besov-Morrey spaces under the condition that the abstract operator \( P \) is commutative with the operator \( e^{-\sqrt{\Delta} t} \) and
\[
\| \varphi_j \nabla Pu \|_{\mathcal{M}^p_j} \leq C 2^{jq} \| \varphi_j \mu \|_{\mathcal{M}^p_j}.
\]
(3)

Clearly, for the fractional porous medium equation, i.e. \( Pu = \Lambda^{-2s} u \), we get \( \sigma = 1 - 2s \). If \( Pu = K \ast u \) in the aggregation equation, Wu and Zhang [18] proved a similar result under the condition \( \nabla K \in W^{1,1} \), \( a \in (0,1) \). Corresponding to their case we give a same result for \( \sigma = 0 \) when \( \nabla K \in L^1 \), and also a similar result for \( \sigma = 1 \) when \( K \in L^1 \).

Throughout this paper, we use \( \mathcal{F} \mathcal{N}^p_{\mu,\lambda,\sigma} \) to denote the homogenous Fourier Besov-Morrey spaces, \( C \) will denote constants which can be different at different places, \( U \lesssim V \) means that there exists a constant \( C > 0 \) such that \( U \leq CV \), and \( p' \) is the conjugate of \( p \) satisfying \( \frac{1}{p} + \frac{1}{p'} = 1 \) for \( 1 \leq p \leq \infty \).

2. Preliminaries and main results

We start with a dyadic decomposition of \( \mathbb{R}^n \). Suppose \( \chi \in C_0^\infty(\mathbb{R}^n) \), \( \varphi \in C_0^\infty(\mathbb{R}^n \setminus \{0\}) \) satisfying
\[
\begin{align*}
supp \chi &\subset \left\{ \xi \in \mathbb{R}^n : |\xi| \leq \frac{4}{3} \right\}, \\
supp \varphi &\subset \left\{ \xi \in \mathbb{R}^n : |\xi| \leq \frac{8}{3} \right\}, \\
\chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j} \xi) &= 1, \quad \xi \in \mathbb{R}^n, \\
\sum_{j \in \mathbb{Z}} \varphi(2^{-j} \xi) &= 1, \quad \xi \in \mathbb{R}^n \setminus \{0\},
\end{align*}
\]
and denote \( \varphi_j(\xi) = \varphi(2^{-j} \xi) \) and \( \mathcal{P} \) the set of all polynomials.

First, we recall the definition of Morrey spaces which are a complement of \( L^p \) spaces.

**Definition 1** ([19]). For \( 1 \leq p < \infty \), \( 0 \leq \lambda < n \), the Morrey spaces \( \mathcal{M}^p_\lambda = \mathcal{M}^p_\lambda(\mathbb{R}^n) \) is defined as the set of functions \( f \in L^p_{\text{loc}}(\mathbb{R}^n) \) such that
\[
\| f \|_{\mathcal{M}^p_\lambda} = \sup_{x_0 \in \mathbb{R}^n} \sup_{r > 0} r^{-\frac{\lambda}{p}} \| f \|_{L^p(B(x_0,r))} < \infty,
\]
(4)
where \( B(x_0, r) \) denotes the ball in \( \mathbb{R}^n \) with center \( x_0 \) and radius \( r \).

It is easy to see that the injection \( \mathcal{M}^\lambda_{p_1} \hookrightarrow \mathcal{M}^\mu_{p_2} \) provided \( \frac{n-\mu}{p_1} \geq \frac{n-\lambda}{p_2} \) and \( p_2 \leq p_1 \), and \( \mathcal{M}^0_p = L^p \).

If \( 1 \leq p_1, p_2, p_3 < \infty \) and \( 0 \leq \lambda_1, \lambda_2, \lambda_3 < n \) with \( \frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2} \) and \( \frac{\lambda_1}{p_1} + \frac{\lambda_2}{p_2} = \frac{\lambda_3}{p_3} \), then we have the Hölder type inequality
\[
\| fg \|_{\mathcal{M}^\lambda_{p_3}} \leq \| f \|_{\mathcal{M}^{\lambda_1}_{p_1}} \| g \|_{\mathcal{M}^{\lambda_2}_{p_2}}.
\]

Also, for \( 1 \leq p < \infty \) and \( 0 \leq \lambda < n \),
\[
\| \varphi * g \|_{\mathcal{M}^\lambda_p} \leq \| \varphi \|_{L^1} \| g \|_{\mathcal{M}^\lambda_p},
\]
(5)
for all $q \in L^1$ and $g \in M^q_p$.

**Definition 2.** (homogeneous Fourier-Besov-Morrey spaces) Let $s \in \mathbb{R}$, $0 \leq \lambda < n$, $1 \leq p < +\infty$ and $1 \leq q \leq +\infty$. The space $\mathcal{F}N^\sigma_{p,\lambda,q}(\mathbb{R}^n)$ denotes the set of all $u \in S' (\mathbb{R}^n) / \mathcal{P}$ such that

$$\|u\|_{\mathcal{F}N^\sigma_{p,\lambda,q}(\mathbb{R}^n)} = \left\{ \sum_{j \in \mathbb{Z}} 2^{jqs} \|\varphi_j \hat{u}\|^{q}_p \right\}^{1/q} < +\infty,$$

(6)

with suitable modification made when $q = \infty$.

Note that the space $\mathcal{F}N^\sigma_{p,\lambda,0}(\mathbb{R}^n)$ equipped with the norm (6) is a Banach space. Since $M^q_p = L^p$, we have $\mathcal{F}N^0_{p,0,q} = \mathcal{F}B^q_{p,q}$. The space $\mathcal{F}N^{-1}_{1,0,1}$ is the Fourier-Herz space $\mathcal{B}^0_q$ and $\mathcal{F}N^{-1}_{1,0,1} = \chi^{-1}$ where $\mathcal{B}^0_q$ is the Fourier-Herz space and $\chi^{-1}$ is the Lei-Lin space $[20]$.

Now, we recall the definition of the mixed space-time spaces.

**Definition 3.** Let $s \in \mathbb{R}$, $1 \leq p < \infty$, $1 \leq q, \rho \leq \infty$, $0 \leq \lambda < n$, and $I = [0, T)$, $T \in (0, \infty]$. The space-time norm is defined on $u(t,x)$ by

$$\|u(t,x)\|_{L^p(I;\mathcal{F}N^\sigma_{p,\lambda,q})} = \left\{ \sum_{j \in \mathbb{Z}} 2^{jqs} \|\varphi_j \hat{u}\|^{q}_p \right\}^{1/q},$$

and denote by $L^p(I;\mathcal{F}N^\sigma_{p,\lambda,q})$ the set of distributions in $S' (\mathbb{R} \times \mathbb{R}^n) / \mathcal{P}$ with finite $\|\cdot\|_{L^p(I;\mathcal{F}N^\sigma_{p,\lambda,q})}$ norm.

According to Minkowski inequality, we have

$$L^p(I;\mathcal{F}N^\sigma_{p,\lambda,q}) \hookrightarrow L^p(I;\mathcal{F}N^\sigma_{p,\lambda,q}), \quad \text{if } \rho \leq q,$$

$$L^p(I;\mathcal{F}N^\sigma_{p,\lambda,q}) \hookrightarrow L^p(I;\mathcal{F}N^\sigma_{p,\lambda,q}), \quad \text{if } \rho \geq q,$$

where $\|u(t,x)\|_{L^p(I;\mathcal{F}N^\sigma_{p,\lambda,q})} := \left( \int_I \|u(\tau,\cdot)\|_{\mathcal{F}N^\sigma_{p,\lambda,q}}^p d\tau \right)^{1/p}$.

Our first main result is the following theorem.

**Theorem 4.** Assume that the abstract operator $P$ satisfies the condition (3). If $0 \leq \lambda < 3$, $1 \leq q \leq \infty$, $1 \leq p < \infty$ and

$$\max \{1 + \sigma, 0\} < a \leq 2 + \frac{2}{p} + \frac{\lambda}{p} + \frac{\sigma}{r}$$

then there exists a constant $C_0$ such that for any $u_0 \in \mathcal{F}N^{-a+\frac{2}{p}+\frac{\lambda}{p}+\frac{\sigma}{r}}_{p,\lambda,q}$ satisfies $\|u_0\|_{\mathcal{F}N^{-a+\frac{2}{p}+\frac{\lambda}{p}+\frac{\sigma}{r}}_{p,\lambda,q}} \leq C_0\mu$, the equation (1) admits a unique global solution $u$,

$$\|u\|_{L^\infty([0,\infty);\mathcal{F}N^{-a+\frac{2}{p}+\frac{\lambda}{p}+\frac{\sigma}{r}}_{p,\lambda,q})} + \mu \|u\|_{L^1([0,\infty);\mathcal{F}N^{-a+\frac{2}{p}+\frac{\lambda}{p}+\frac{\sigma}{r}}_{p,\lambda,q})} \leq 2C\|u_0\|_{\mathcal{F}N^{-a+\frac{2}{p}+\frac{\lambda}{p}+\frac{\sigma}{r}}_{p,\lambda,q}}$$

where $C$ is a positive constant.

Now, we give some remarks about this result.

**Remark 1.** The result stated in Theorem 4 is based on the works [3]. In particular, this result remains true if we replace the Fourier-Besov-Morrey space $\mathcal{F}N^\sigma_{p,\lambda,q}$ by other functional spaces such as Fourier-Herz space $\mathcal{B}^0_q$, Fourier-Besov space $\mathcal{F}B^q_{p,q}$ and Lei-Lin space $\chi^{-1}$.

The analyticity of the solution is also an important subject developed by several researchers, particularly with regard to the Navier-Stokes equations, see [17] and its references. In this paper, we will prove the Gevrey class regularity for (1) in the Fourier-Besov-Morrey space. Inspired by this, we have obtained the following specific results.
Theorem 5. Let \(0 \leq \lambda < 3, 1 \leq q \leq \infty, 1 \leq p < \infty\) and \(\max\{1 + \sigma, 0\} < \alpha < \min\{2, 2 + \frac{3}{p} + \frac{1}{q} + \sigma\}\). There exists a constant \(C_0\) such that, if \(u_0 \in \mathcal{F}\mathcal{N}^{1-\alpha+p_1, 1+p_1-\sigma}_{p, \lambda, q}\) satisfies \(\|u_0\|_{\mathcal{F}\mathcal{N}^{1-\alpha+p_1, 1+p_1-\sigma}_{p, \lambda, q}} < C_0\), then the Cauchy problem (1) admits a unique analytic solution \(u\), in the sense that

\[
\|e^{\mu \sqrt{\gamma}|D|^\frac{2}{p} u}\|_{L^\infty((0,\infty);\mathcal{F}\mathcal{N}^{1-\alpha+p_1, 1+p_1-\sigma}_{p, \lambda, q})} + \mu \|e^{\mu \sqrt{\gamma}|D|^\frac{2}{p} u}\|_{L^1((0,\infty);\mathcal{F}\mathcal{N}^{1-\alpha+p_1, 1+p_1-\sigma}_{p, \lambda, q})} \leq 2C\|u_0\|_{\mathcal{F}\mathcal{N}^{1-\alpha+p_1, 1+p_1-\sigma}_{p, \lambda, q}}.
\]

We finish this section with a Bernstein type lemma in Fourier variables in Morrey spaces.

Lemma 6 ([21]). Let \(1 \leq q \leq p < \infty, 0 \leq \lambda_1, \lambda_2 < n, \frac{n-\lambda_1}{p} \leq \frac{n-\lambda_2}{q}\), and let \(\gamma\) be a multiindex. If supp\((\hat{f})\) \(\subset\{|\xi| \leq N\}\) then there is a constant \(C > 0\) independent of \(f\) and \(j\) such that

\[
\|((i\xi)^\gamma \hat{f})\|_{M^{q,p}_\gamma} \leq C 2^{\|\gamma\| + \|j\| \left(\frac{n-\lambda_2}{p} - \frac{n-\lambda_1}{q}\right)} \|f\|_{M^{q,p}_\gamma}.
\]

3. The well-posedness

First, we consider the linear nonhomogeneous dissipative equation

\[
\begin{align*}
\left\{ \begin{array}{l}
u_t + \mu \Lambda^\alpha u &= f(t,x) \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3 \\
u(0,x) &= u_0(x) \quad x \in \mathbb{R}^3,
\end{array} \right.
\end{align*}
\]

for which we recall the following result.

Lemma 7 ([22]). Let \(I = [0,T), 0 < T \leq \infty, s \in \mathbb{R}, 0 \leq \lambda < 3, 1 \leq p < \infty\), and \(1 \leq q, \rho \leq \infty\). Assume that \(u_0 \in \mathcal{F}\mathcal{N}^s_{p, \lambda, q}\) and \(f \in L^p(I; \mathcal{F}\mathcal{N}^{s-\alpha+\frac{3}{p}}_{p, \lambda, q})\). Then the Cauchy problem (8) has a unique solution \(u(t,x)\) such that for all \(\rho_1 \in [\rho, +\infty)\)

\[
\begin{align*}
\mu^{\frac{1}{\rho}} \|u\|_{L^\rho(I; \mathcal{F}\mathcal{N}^{s-\alpha+\frac{3}{p}}_{p, \lambda, q})} &\leq \left(\frac{4}{3}\right)^{\alpha} \left(\|u_0\|_{\mathcal{F}\mathcal{N}^s_{p, \lambda, q}} + \mu^{\frac{1}{\rho}-1} \|f\|_{L^\rho(I; \mathcal{F}\mathcal{N}^{s-\alpha+\frac{3}{p}}_{p, \lambda, q})}\right) \\
\|u\|_{L^\infty(I; \mathcal{F}\mathcal{N}^{s-\alpha+\frac{3}{p}}_{p, \lambda, q})} + \mu \|u\|_{L^1(I; \mathcal{F}\mathcal{N}^{s-\alpha+\frac{3}{p}}_{p, \lambda, q})} &\leq (1 + \left(\frac{4}{3}\right)^{\alpha}) \left(\|u_0\|_{\mathcal{F}\mathcal{N}^s_{p, \lambda, q}} + \|f\|_{L^1(I; \mathcal{F}\mathcal{N}^{s-\alpha+\frac{3}{p}}_{p, \lambda, q})}\right).
\end{align*}
\]

If in addition \(q\) is finite, then \(u\) belongs to \(C(I; \mathcal{F}\mathcal{N}^s_{p, \lambda, q})\).

Proposition 8. Let \(1 \leq p < \infty, 1 \leq \rho, q \leq \infty, 1 + \sigma < \alpha < \frac{2+\frac{3}{p}+\frac{1}{q}+\sigma}{2-\frac{3}{p}}\), \(0 \leq \lambda < 3\), \(I = [0,T), T \in (0,\infty)\), and set

\[
X = L^\infty(I; \mathcal{F}\mathcal{N}^{1-\alpha+p_1, 1+p_1-\sigma}_{p, \lambda, q}) \cap L^p(I; \mathcal{F}\mathcal{N}^{1-\alpha+p_1, 1+p_1-\sigma}_{p, \lambda, q}),
\]

with the norm

\[
\|u\|_X = \|u\|_{L^\infty(I; \mathcal{F}\mathcal{N}^{1-\alpha+p_1, 1+p_1-\sigma}_{p, \lambda, q})} + \mu \|u\|_{L^p(I; \mathcal{F}\mathcal{N}^{1-\alpha+p_1, 1+p_1-\sigma}_{p, \lambda, q})}.
\]

There exists a constant \(C = C(p, q) > 0\) depending on \(p, q\) such that

\[
\|u_0 P_{\theta} v\|_{L^p(I; \mathcal{F}\mathcal{N}^{1-\alpha+p_1, 1+p_1-\sigma}_{p, \lambda, q})} \leq C \mu^{-1} \|u\|_X \|v\|_X.
\]
Proof. Let us introduce some notations about the standard localization operators. We set
\[ u_j = \Delta_j u = (\mathcal{F}^{-1} \varphi_j) * u, \quad \bar{S}_j u = \sum_{k \leq j-1} \Delta_k u, \quad \bar{\Delta}_j u = \sum_{|k-j| \leq 1} \Delta_k u, \quad \forall j \in \mathbb{Z}. \]

Using the decomposition of Bony’s paraproducts for the fixed \( j \), we have
\[
\Delta_j(u\partial_1 P\nu) = \sum_{|k-j| \leq 4} \Delta_j(\bar{S}_{k-1} u \Delta_k (\partial_1 P\nu)) + \sum_{|k-j| \leq 4} \Delta_j(\bar{S}_{k-1} (\partial_1 P\nu) \Delta_k u) + \sum_{k \geq j-3} \Delta_j(\Delta_k u \bar{\Delta}_k (\partial_1 P\nu)) = I_j + II_j + III_j.
\]

To prove this proposition, we can write
\[
\|u\partial_1 P\nu\|_{L^p(I,F^N_{\rho,\lambda,q})} \lesssim \left\{ \sum_{j \in \mathbb{Z}} 2^{(j-k-1)\alpha + \frac{3p}{q'} + \frac{p}{q'} + \sigma)q} \|\tilde{T}_j\|_{L^q(I,M^p_\rho)} \right\}^{1/q} + \left\{ \sum_{j \in \mathbb{Z}} 2^{(j-k-1)\alpha + \frac{3p}{q'} + \frac{p}{q'} + \sigma)q} \|\tilde{I}_j\|_{L^q(I,M^p_\rho)} \right\}^{1/q} + \left\{ \sum_{j \in \mathbb{Z}} 2^{(j-k-1)\alpha + \frac{3p}{q'} + \frac{p}{q'} + \sigma)q} \|\tilde{H}_j\|_{L^q(I,M^p_\rho)} \right\}^{1/q}. \tag{10}
\]

We treat the above three terms differently. First, using Young’s inequality (5) in Morrey spaces, and Lemma 6 with \(|\gamma| = 0\), we get
\[
\|\tilde{T}_j\|_{L^q(I,M^p_\rho)} \leq \sum_{|k-j| \leq 4} \|\tilde{S}_{k-1} u \Delta_k (\partial_1 P\nu)\|_{L^q(I,M^p_\rho)} \sum_{l \leq k-2} \|\varphi_l\tilde{u}\|_{L^\infty(I,L^1)} \leq \sum_{|k-j| \leq 4} \|\varphi_k \mathcal{F}(\partial_1 P\nu)\|_{L^q(I,M^p_\rho)} \sum_{l \leq k-2} 2^{l(\frac{3}{p'} + \frac{p}{q'})} \|\tilde{u}\|_{L^\infty(I,M^p_\rho)} \lesssim \sum_{|k-j| \leq 4} 2^{k(\alpha-1)} \|\tilde{\varphi}_k\|_{L^q(I,M^p_\rho)} \|\tilde{u}\|_{L^\infty(I,F^N_{\rho,\lambda,q})} \lesssim \sum_{|k-j| \leq 4} 2^{k(\alpha-1)} \|\tilde{\varphi}_k\|_{L^q(I,M^p_\rho)} \|\tilde{u}\|_{L^\infty(I,F^N_{\rho,\lambda,q})}. \tag{11}
\]

Multiplying by \( 2^{j(-2(\alpha-1)+\frac{3}{p'} + \frac{p}{q'} + \frac{p}{q'} + \sigma)} \), and taking \( L^q \)-norm of both sides in the above estimate, we obtain
\[
\left\{ \sum_{j \in \mathbb{Z}} 2^{j(-2(\alpha-1)+\frac{3}{p'} + \frac{p}{q'} + \frac{p}{q'} + \sigma)q} \|\tilde{T}_j\|_{L^q(I,M^p_\rho)} \right\}^{1/q} \leq \left\{ \sum_{j \in \mathbb{Z}} \left( \sum_{|k-j| \leq 4} 2^{k(\alpha-1) - \frac{3}{p'} + \frac{p}{q'} + \sigma)} \|\tilde{\varphi}_k\|_{L^q(I,M^p_\rho)} \|\tilde{u}\|_{L^\infty(I,F^N_{\rho,\lambda,q})} \right)^q \right\}^{1/q} \lesssim \|u\|_{L^\infty(I,F^N_{\rho,\lambda,q})} \|\tilde{\nu}\|_{L^p(I,F^N_{\rho,\lambda,q})}. \tag{12}
\]

Likewise, we prove that
\[
\left\{ \sum_{j \in \mathbb{Z}} 2^{j(-2(\alpha-1)+\frac{3}{p'} + \frac{p}{q'} + \frac{p}{q'} + \sigma)q} \|\tilde{I}_j\|_{L^q(I,M^p_\rho)} \right\}^{1/q} \lesssim \|v\|_{L^\infty(I,F^N_{\rho,\lambda,q})} \|\tilde{u}\|_{L^p(I,F^N_{\rho,\lambda,q})}. \tag{13}
\]
To evaluate $III_j$, we apply the Young inequality (5) in Morrey spaces and Lemma 6 with $|\gamma| = 0$, we obtain
\[
2^{j(2(a-1) + \frac{p}{p} + \frac{1}{p} + \sigma)} \|III_j\|_{L^p(L^m_0)} \leq 2^{j(2(a-1) + \frac{p}{p} + \frac{1}{p} + \sigma)} \sum_{k \geq j-3} \sum_{|k| \leq 1} \|\mathcal{F}(\Delta_k u \Delta_l (\partial_j P\nu))\|_{L^p(L^m_0)} \\
\leq 2^{j(2(a-1) + \frac{p}{p} + \frac{1}{p} + \sigma)} \sum_{k \geq j-3} \sum_{|k| \leq 1} \|\hat{u}_k\|_{L^p(L^m_0)} \|\mathcal{F}(\partial_j P\nu)\|_{L^{2\gamma}(L^1)} \\
\leq 2^{j(2(a-1) + \frac{p}{p} + \frac{1}{p} + \sigma)} \sum_{k \geq j-3} \sum_{|k| \leq 1} 2^{l(\frac{1}{p'} + \frac{1}{p})} \|\hat{u}_k\|_{L^p(L^m_0)} 2^{|l|} \|\mathcal{F}(\partial_j P\nu)\|_{L^{2\gamma}(L^1)} \\
\leq \sum_{k \geq j-3} \sum_{l=1}^1 2^{j(a-1) + \frac{p}{p} + \frac{1}{p} + \sigma)} \|\hat{u}_k\|_{L^p(L^m_0)} (2^{j(a-1) + \frac{p}{p} + \frac{1}{p} + \sigma)} \|\mathcal{F}(\partial_j P\nu)\|_{L^{2\gamma}(L^1)} .
\]

Taking the $l^q$-norm on both sides in the above estimate and using Hölder’s inequalities for series with $-2(a-1) + \frac{p}{p} + \frac{1}{p} + \sigma > 0$, we get
\[
\left( \sum_{j \in \mathbb{Z}} 2^{j(a-1) + \frac{p}{p} + \frac{1}{p} + \sigma)q} \|\hat{u}_j\|_{L^p(L^m_0)} \right)^{\frac{1}{q}} \\
\leq \left( \sum_{j \in \mathbb{Z}} \left( \sum_{m \geq j-1} \sum_{l=1}^1 2^{j(a-1) + \frac{p}{p} + \frac{1}{p} + \sigma) m} \sum_{2^{j(a-1) + \frac{p}{p} + \frac{1}{p} + \sigma)} 2^{j(a-1) + \frac{p}{p} + \frac{1}{p} + \sigma)} \|\hat{u}_j\|_{L^p(L^m_0)} \|\mathcal{F}(\partial_j P\nu)\|_{L^{2\gamma}(L^1)} \right) \frac{q}{p} .
\]

Since $l^q \hookrightarrow l^\infty$, we obtain
\[
\left( \sum_{j \in \mathbb{Z}} 2^{j(a-1) + \frac{p}{p} + \frac{1}{p} + \sigma)q} \|\hat{u}_j\|_{L^p(L^m_0)} \right)^{\frac{1}{q}} \leq \|\mathcal{F}\|_{L^{p'}(L^{p'}(L^{p'}(L^{p'})))} \|\mathcal{F}\|_{L^{p'}(L^{p'}(L^{p'}(L^{p'})))} \|\mathcal{F}\|_{L^{p'}(L^{p'}(L^{p'}(L^{p'})))} .
\]

Estimates (10), (11), (12) and (13) yield (9). \qed

Lemma 9. Let $X$ be a Banach space with norm $\|\cdot\|_X$ and $B : X \times X \to X$ be a bounded bilinear operator satisfying
\[
\|B(u, v)\|_X \leq \eta \|u\|_X \|v\|_X
\]
for all $u, v \in X$ and a constant $\eta > 0$. Then, if $0 < \epsilon < \frac{1}{2\eta}$ and if $y \in X$ such that $\|y\|_X \leq \epsilon$, the equation $x := y + B(x, x)$ has a solution $\overline{x}$ in $X$ such that $\|\overline{x}\|_X \leq 2\epsilon$. This solution is the only one in the ball $B(0, 2\epsilon)$. Moreover, the solution depends continuously on $y$ in the sense: if $\|y\|_X \leq \epsilon$, $x' = y' + B(x', x')$, and $\|x'\|_X \leq 2\epsilon$, then
\[
\|\overline{x} - x'\|_X \leq \frac{1}{1 - 4\eta} \|y - y'\|_X .
\]

Proof of theorem 4

Proof. To ensure the existence of global solutions with small initial data, we will use Lemma 9.
In the following, we consider the Banach space

\[ X = L^\infty \left( [0, +\infty); \mathcal{F} \mathcal{N}^{1-a+\frac{3}{p}+\frac{1}{p}+\sigma}_{\rho, \lambda, d} \right) \cap L^1 \left( [0, +\infty); \mathcal{F} \mathcal{N}^{1-x+\frac{3}{p}+\frac{1}{2}+\sigma}_{\rho, \lambda, d} \right). \]

First, we start with the integral equation

\[
u = e^{-\mu \Lambda^s} u_0 + \int_0^t e^{-\mu (t-\tau) \Lambda^s} \nabla \cdot (u(\tau) \nabla P u(\tau)) \, d\tau = e^{-\mu \Lambda^s} u_0 + B(u, u). (14)\]

We notice that \( B(u, v) \) can be thought as the solution to the heat Equation (8) with \( u_0 = 0 \) and force \( f = \nabla \cdot (u(\tau) \nabla P v(\tau)) \). According to Lemma 7 with \( s = 1 - a + \frac{3}{p} + \frac{1}{3} + \sigma \) and Proposition 8 with \( \rho = 1 \), we obtain

\[
\|B(u, v)\|_X \leq \left( 1 + \left( \frac{4}{3} \right)^a \right) \|\nabla \cdot (u \nabla v)\| \mathcal{L}^1 \left( [0, +\infty); \mathcal{F} \mathcal{N}^{1-\frac{3}{4}+\frac{1}{3}+\sigma}_{\rho, \lambda, d} \right)
\leq \left( 1 + \left( \frac{4}{3} \right)^a \right) C \mu^{-1} \|u\|_X \|v\|_X.
\]

By Lemma 9, we know that if \( \|e^{-\mu \Lambda^s} u_0\|_X < R \) with \( R = \frac{\mu}{4(1+\frac{3}{4})C} \), then the equation (14) has a unique solution in \( B(0, 2R):= \{ x \in X : \|x\|_X \leq 2R \} \). To prove \( \|e^{-\mu \Lambda^s} u_0\|_X < R \), notice that \( e^{-\mu \Lambda^s} u_0 \) is the solution to the dissipative equation with \( u_0 = u_0 \) and \( f = 0 \). So, Lemma 7 yields

\[
\|e^{-\mu \Lambda^s} u_0\|_X \leq \left( 1 + \left( \frac{4}{3} \right)^a \right) \|u_0\| \mathcal{F} \mathcal{N}^{1-a+\frac{3}{p}+\frac{1}{2}+\sigma}_{\rho, \lambda, d}. (15)
\]

If \( \|u_0\| \mathcal{F} \mathcal{N}^{1-a+\frac{3}{p}+\frac{1}{2}+\sigma}_{\rho, \lambda, d} \leq C_0 \mu \) with \( C_0 = \frac{1}{4(1+\frac{3}{4})C} \), then (14) has a unique global solution \( u \in X \) satisfying

\[
\|u\|_X \leq 2 \left( 1 + \left( \frac{4}{3} \right)^a \right) \|u_0\| \mathcal{F} \mathcal{N}^{1-a+\frac{3}{p}+\frac{1}{2}+\sigma}_{\rho, \lambda, d}.
\]

□

**Proof of theorem 5**

**Proof.** To prove Theorem 5, we note \( a(t, x) := e^{\mu \sqrt{\tau} |D|^{\frac{3}{2}} - \frac{1}{2} \Lambda^s} u(t, x) \). Using the integral Equation (14), we obtain

\[
a(t, x) = e^{\mu \sqrt{\tau} |D|^{\frac{3}{2}} - \frac{1}{2} \Lambda^s} e^{-\mu t \Lambda^s} u_0 + \int_0^t e^{\mu \sqrt{\tau} |D|^{\frac{3}{2}} - \frac{1}{2} \Lambda^s} e^{-\mu (t-\tau) \Lambda^s} e^{\mu \sqrt{\tau} |D|^{\frac{3}{2}} - \frac{1}{2} \Lambda^s} \nabla \cdot (u \nabla (P u)) \, d\tau \nonumber
\]

\[
:= Lu_0 + B(u, u). 
\]

In order to obtain the Gevrey class regularity of the solution, we use Lemma 9. Firstly, we start by estimating the term \( L u_0 = e^{-\mu t |D|^{\frac{1}{2}} - \frac{1}{2} \Lambda^s} e^{-\mu \frac{1}{4} t \Lambda^s} u_0 \).

Using the Fourier transform, multiplying by \( \varphi_j \) and taking the \( M^{1}_{p} \)-norm we obtain

\[
\|\varphi_j \tilde{u}_0\|_{M^{1}_{p}} \leq C e^{-\frac{1}{4} t \frac{p}{2} (3/4)^{\sigma}} \|\varphi_j \tilde{u}_0\|_{M^{1}_{p}}.
\]

□
Multiplying by $2^{(1-a+\frac{2}{p} + \frac{1}{q}+\sigma)}$ and taking $l^{\infty}$-norm we get

$$
\|Lu_0\|_{L^{\infty}} \leq C \|u_0\|_{\mathcal{F}_{p,\lambda,q}^{1-a+\frac{2}{p} + \frac{1}{q}+\sigma}}.
$$

Similarly

$$
2^{(1-a+\frac{2}{p} + \frac{1}{q}+\sigma)} \|\varphi_jLu_0\|_{L^1([0,\infty);M^j_\lambda)} \leq \left( \int_0^{\infty} e^{-\frac{1}{2}\mu t^{2\mu}(3/4)\alpha^2} dt \right) 2^{(1-a+\frac{2}{p} + \frac{1}{q}+\sigma)} \|\varphi_ju_0\|_{M^j_\lambda}.
$$

We conclude by taking $l^{\infty}$-norm that

$$
\mu \|Lu_0\|_{L^1([0,\infty);\mathcal{F}_{p,\lambda,q}^{1-a+\frac{2}{p} + \frac{1}{q}+\sigma})} \leq C \|u_0\|_{\mathcal{F}_{p,\lambda,q}^{1-a+\frac{2}{p} + \frac{1}{q}+\sigma}}.
$$

Finally,

$$
\|Lu_0\|_X \leq C \|u_0\|_{\mathcal{F}_{p,\lambda,q}^{1-a+\frac{2}{p} + \frac{1}{q}+\sigma}}.
$$

On the other hand, we notice that $\tilde{B}(u, v)$ as $\tilde{B} \left( e^{-\frac{1}{2}\mu t^{2\mu}(3/4)\alpha^2} e^{-\frac{1}{2}\mu t^{2\mu}(3/4)\sigma b} \right)$ with $b := e^{\frac{1}{2}\mu t^{2\mu}(3/4)\sigma b}$. Since $e^{\frac{1}{2}\mu t^{2\mu}(3/4)\alpha^2} \frac{d}{dt} \left( e^{-\frac{1}{2}\mu t^{2\mu}(3/4)\sigma b} \right)$ is uniformly bounded on $t \in (0, \infty)$ and $\tau \in [0, t]$, it sufficient to consider the estimate of $\|e^{\frac{1}{2}\mu t^{2\mu}(3/4)\sigma b} \partial_i(P\nu)\|_{L^1(\mathcal{F}_{p,\lambda,q}^{2-a+\frac{2}{p} + \frac{1}{q}+\sigma})}$ for which we prove the flowing lemma.

**Lemma 10.** Let $1 \leq p < \infty$, $1 \leq q \leq \infty$, $0 \leq \lambda < 3$, $1 + \sigma < \alpha < \min\{2, 2 + \frac{2}{p} + \frac{1}{q}+\sigma\}$, $I = [0, T)$, $T \in (0, \infty)$, and set

$$
X = L^\infty \left( I; \mathcal{F}_{p,\lambda,q}^{2-a+\frac{2}{p} + \frac{1}{q}+\sigma} \right) \cap L^1 \left( I; \mathcal{F}_{p,\lambda,q}^{1-a+\frac{2}{p} + \frac{1}{q}+\sigma} \right).
$$

There exists a constant $C = C(p, q) > 0$ depending on $p, q$ such that

$$
\|e^{\frac{1}{2}\mu t^{2\mu}(3/4)\sigma b} \partial_i(P\nu)\|_{L^1(\mathcal{F}_{p,\lambda,q}^{2-a+\frac{2}{p} + \frac{1}{q}+\sigma})} \leq C \mu^{-1} \|a\|_X \|b\|_X.
$$

**Proof.** Based on the same procedure in the proof of Proposition 8, we evaluate the estimate of $\|e^{\frac{1}{2}\mu t^{2\mu}(3/4)\sigma b} \partial_i(P\nu)\|_{L^1(\mathcal{F}_{p,\lambda,q}^{2-a+\frac{2}{p} + \frac{1}{q}+\sigma})}$, in fact, we have for fixed $j$.

$$
\Delta \hat{R}^{\frac{1}{2}} \hat{R}^{\frac{1}{2}} (u \partial_i(P\nu)) = \sum_{|k-j| \leq 4} \hat{R}^{\frac{1}{2}} \hat{R}^{\frac{1}{2}} \left( S_{k-1} u \Delta_k \partial_i(P\nu) \right)
+ \sum_{|k-j| \leq 4} \hat{R}^{\frac{1}{2}} \hat{R}^{\frac{1}{2}} \left( S_{k-1} \partial_i(P\nu) \Delta_k u \right)
+ \sum_{k \geq j-3} \hat{R}^{\frac{1}{2}} \hat{R}^{\frac{1}{2}} \left( \Delta_k u \Delta_0 \partial_i(P\nu) \right)
:= S_{1,j} + S_{2,j} + S_{3,j}.
$$
Since \( e^{\mu \sqrt{|\xi|^2 - |\xi - \eta|^2 - |\eta|^2}^2} \) is uniformly bounded on \( \tau \) when \( \alpha \in [0, 2] \), we obtain
\[
\| S_{2,j} \|_{M^p_{\alpha}} = \| \sum_{|k-j| \leq 4} \phi_j e^{\mu \sqrt{|\xi|^2 - |\xi - \eta|^2 - |\eta|^2}^2} \mathcal{F} (S_{k-1} u \partial_t (Pv)) \|_{M^p_{\alpha}} \\
= \| \sum_{|k-j| \leq 4} \phi_j e^{\mu \sqrt{|\xi|^2 - |\xi - \eta|^2 - |\eta|^2}^2} \left[ \left( \sum_{l \leq k-2} e^{-\mu \sqrt{|\xi|^2 - |\xi - \eta|^2 - |\eta|^2}^2} \right) \left( \sum_{l \leq k-2} \hat{a}_l \right) (\xi - \eta) \mathcal{F} (\Delta_k \partial_t (Pb)) \right] \|_{M^p_{\alpha}} \\
\leq \| \sum_{|k-j| \leq 4} \mathcal{F} (S_{k-1} a \partial_t \partial_t (Pb)) \|_{M^p_{\alpha}}.
\]

The same calculus as in Proposition 8 gives
\[
\left\{ \sum_{j \in \mathbb{Z}} 2^{(2-\alpha + \frac{3}{p} + \frac{1}{p'} + \sigma)q} \| S_{3,j} \|_{L^1(1,M^p_{\alpha})} \right\}^{1/q} \lesssim \| a \|_{L^\infty \left( I,FN^{1-a + \frac{3}{p} + \frac{1}{p'} + \sigma}_{p, \lambda, q} \right)} \| b \|_{L^1 \left( I,FN^{1-a + \frac{3}{p} + \frac{1}{p'} + \sigma}_{p, \lambda, q} \right)}.
\]

Similarly, we show that
\[
\left\{ \sum_{j \in \mathbb{Z}} 2^{(2-\alpha + \frac{3}{p} + \frac{1}{p'} + \sigma)q} \| S_{3,j} \|_{L^1(1,M^p_{\alpha})} \right\}^{1/q} \lesssim \| b \|_{L^\infty \left( I,FN^{1-a + \frac{3}{p} + \frac{1}{p'} + \sigma}_{p, \lambda, q} \right)} \| a \|_{L^1 \left( I,FN^{1-a + \frac{3}{p} + \frac{1}{p'} + \sigma}_{p, \lambda, q} \right)}.
\]

Similarly,
\[
\| S_{3,j} \|_{M^p_{\alpha}} \leq \sum_{k \geq j-3} \sum_{|k-l| \leq 1} \| \mathcal{F} (\Delta_k \partial_l (Pb)) \|_{M^p_{\alpha}}.
\]

Using again the same procedure described in the proof of Proposition 8 we obtain
\[
\left\{ \sum_{j \in \mathbb{Z}} 2^{(2-\alpha + \frac{3}{p} + \frac{1}{p'} + \sigma)q} \| S_{3,j} \|_{L^1(1,M^p_{\alpha})} \right\}^{1/q} \lesssim \| a \|_{L^\infty \left( I,FN^{1-a + \frac{3}{p} + \frac{1}{p'} + \sigma}_{p, \lambda, q} \right)} \| b \|_{L^1 \left( I,FN^{1-a + \frac{3}{p} + \frac{1}{p'} + \sigma}_{p, \lambda, q} \right)}.
\]

Finally,
\[
\left\| e^{\mu \sqrt{|\xi|^2 - |\xi - \eta|^2 - |\eta|^2}^2} \right\|_{L^1 \left( I,FN^{1-a + \frac{3}{p} + \frac{1}{p'} + \sigma}_{p, \lambda, q} \right)} \leq C \mu^{-1} \| a \|_{X} \| b \|_{X}.
\]

To finish the proof of Theorem 5, it is easy to obtain the requested result by repeating the same step in the proof of Theorem 4 and Proposition 8. □

Conflicts of Interest: “The author declare no conflict of interest.”

References

[1] Zhou, X., Xiao, W., & Zheng, T. (2015). Well-posedness and blowup criterion of generalized porous medium equation in Besov spaces. Electronic Journal of Differential Equations, 2015(261), 1-14.
[2] Caffarelli, L. A., & Vázquez, J. L. (2011). Nonlinear porous medium flow with fractional potential pressure. Archive for Rational Mechanics and Analysis, 202 (2011), 537-565.
[3] Xiao, W., & Zhou, X. (2016). On the generalized porous medium equation in Fourier-Besov spaces. arXiv preprint arXiv:1612.03304.
[4] Zhou, X., Xiao, W., & Chen, J. (2014). Fractional porous medium and mean field equations in Besov spaces. Electron Journal fo Differential Equations, 2014(199), 1-14.
[5] Lin, F., & Zhang, P. (2002). On the hydrodynamic limit of Ginzburg-Landau wave vortices. Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences, 55(7), 831-856.
[6] Biler, P., Imbert, C., & Karch, G. (2011). Barenblatt profiles for a nonlocal porous medium equation. *Comptes Rendus Mathematique*, 349(11-12), 641-645.

[7] Blanchet, A., Carrillo, J. A., & Masmoudi, N. (2008). Infinite time aggregation for the critical Patlak-Keller-Segel model in $\mathbb{R}^2$. *Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences*, 61(10), 1449-1481.

[8] Topaz, C. M., Bertozzi, A. L., & Lewis, M. A. (2006). A nonlocal continuum model for biological aggregation. *Bulletin of mathematical biology*, 68(7), 1601-1623.

[9] Huang, Y., & Bertozzi, A. L. (2010). Self-similar blowup solutions to an aggregation equation in $\mathbb{R}^n$. *SIAM Journal on Applied Mathematics*, 70(7), 2582-2603.

[10] Li, D., & Zhang, X. (2010). Global wellposedness and blowup of solutions to a nonlocal evolution problem with singular kernels. *Communications on Pure & Applied Analysis*, 9(6), 1591-1606.

[11] Bertozzi, A. L., & Laurent, T. (2007). Finite-time Blow-up of Solutions of an Aggregation Equation in $\mathbb{R}^n$. *Communications in mathematical physics*, 274(3), 717-735.

[12] Li, D., & Rodrigo, J. L. (2010). Wellposedness and regularity of solutions of an aggregation equation. *Revista Matemática Iberoamericana*, 26(1), 261-294.

[13] Karch, G., & Suzuki, K. (2010). Blow-up versus global existence of solutions to aggregation equations. *Applied Mathematics (Warsaw)*, 38 (2011), 243-258.

[14] Laurent, T. (2007). Local and global existence for an aggregation equation. *Communications in Partial Differential Equations*, 32(12), 1941-1964.

[15] Chen, Q., & Zhang, Z. (2007). Global well-posedness of the 2D critical dissipative quasi-geostrophic equation in the Triebel-Lizorkin spaces. *Nonlinear Analysis: Theory, Methods & Applications*, 67(6), 1715-1725.

[16] Wang, H., & Zhang, Z. (2011). A frequency localized maximum principle applied to the 2D quasi-geostrophic equation. *Communications in Mathematical Physics*, 301(1), 105-129.

[17] Wang, W., & Wu, G. (2018). Global mild solution of the generalized Navier-Stokes equations with the Coriolis force. *Applied Mathematics Letters*, 76, 181-186.

[18] Wu, G., & Zhang, Q. (2013). Global well-posedness of the aggregation equation with supercritical dissipation in Besov spaces. *ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik*, 93(12), 882-894.

[19] Kato, T. (1992). Strong solutions of the Navier-Stokes equation in Morrey spaces. *Boletim da Sociedade Brasileira de Matemática-Bulletin/Brazilian Mathematical Society*, 22(2), 127-155.

[20] Cannone, M., & Wu, G. (2012). Global well-posedness for Navier-Stokes equations in critical Fourier-Herz spaces. *Nonlinear Analysis: Theory, Methods & Applications*, 75(9), 3754-3760.

[21] Ferreira, L. C., & Lima, L. S. (2014). Self-similar solutions for active scalar equations in Fourier-Besov-Morrey spaces. *Monatshefte für Mathematik*, 175(4), 491-509.

[22] El Baraka, A., & Toumlilin, M. (2017). Global Well-Posedness for Fractional Navier-Stokes Equations in critical Fourier-Besov-Morrey Spaces. *Moroccan Journal of Pure and Applied Analysis*, 3(1), 1-13.