Emergence of Cascading Risk and Role of Spatial Locations of Collisions in Time-Delayed Platoon of Vehicles

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Abstract—We develop a framework to assess the risk of cascading collisions in a platoon of vehicles in the presence of exogenous noise and communication time-delay. The notion of Value-at-Risk (VaR) is adopted to quantify the risk of collision between vehicles in a pair conditioned on the knowledge of multiple previously occurred failures in the platoon. We show that the risk of cascading collisions depends on the Laplacian spectrum of the underlying communication graph, time-delay, and noise statistics. Furthermore, we exploit the structure of several standard graphs to show how the risk profile depends on the magnitude and spatial location of the prior collisions (failures). Our theoretical findings, supported by simulation examples, can be used to design safe platoons that minimize the risk of cascading collisions.

I. INTRODUCTION

Uncertainties, originating from the fundamental laws of Physics, prevail in every application in control systems. Even when a robust control system is designed to minimize its fragility, there is always a chance that the system will be driven into an undesired and even dangerous state in response to large exogenous shocks. The inherent stochasticity prevents one from precisely predicting the future state of the system. In such cases, one can still assess the chance or the estimated cost of the system experiencing failure via systemic risk analysis. The risk quantification plays a crucial role in almost all control systems as the evaluation of how “risky” a closed-loop system is will help both researchers and users to gain adequate knowledge about the safety and reliability of the system operation.

The risk analysis in real-world applications can be traced from when [9] built the algorithmic risk measure for surgical robots to when [22] showed how the involvement of risk-mitigation in robot operations could increase safety. A systematic and analytical approach, which adopts the widely used financial tool value-at-risk (VaR) measure and the conditional-value-at-risk (CVaR) measure [15], has recently been utilized to analyze the risk of certain events in the networked control system. Our previous works [20, 16, 19, 17, 18, 10] have shown that the VaR measure is an effective tool to evaluate and analyze the safety in networked control systems, e.g., a platoon of vehicles, synchronous power networks, and rendezvous problems for a team of robots.

In this work, we consider the problem of autonomous vehicle platooning as a motivational application. This type of application is omnipresent in multi-agent systems [7, 2] or, more specifically, the car platooning [21]. An effective communication network is essential for closing a control loop while sensing and communication time-delays are inherent in all real-world applications. This phenomenon has also been observed and considered in recent works about vehicle platooning [1, 25]. Hence, in this work, we consider platoon models that incorporate the effect of unified time-delay and noise in the dynamics of vehicles.

One fundamental design principle in a multi-vehicle platoon is to develop a communication topology that improves the platoon’s reliability and safety. In [17], the authors quantify the failure risk of one undesired event, e.g., inter-vehicle collision, and formulate several fundamental limits and trade-offs on the best achievable levels of risk. In a more recent work [12], we evaluate the collision risk of a pair of vehicles conditioned on the event of exactly one prior collision in the platoon, where it is shown that the risk of cascading collisions is not only dependent on the network parameters (e.g., Laplacian eigen-spectrum), time-delay and noise statistics, but it is also affected by the relation between the existing collision and the pair of interest. In this work, which is along with our previous works [20, 16, 19, 17, 18, 12] to analyze the risk of collisions in a platoon of vehicles, we propose a method that allows us to calculate cascading risk of collision between vehicles in a pair, which is conditioned on knowledge of an arbitrary number of prior collisions. We aim to explore how design paradigms will change while considering all possible collision scenarios.

Our contributions are twofold. First, we derive explicit formulas for the cascading risk of collision subject to a series of known malfunctioned vehicles (failures). Second, we focus on exploring how the topology of a communication graph, the scale, and the spatial distribution of the prior failures will affect the risk of cascading collision. We obtain closed-form risk representation for some particular graph topology by considering various scenarios for the spatial distribution of prior collisions. We should highlight that the behavior of cascading risk is significantly more complicated than the risk of a single event, and that may prevent one from drawing general conclusions about risk.

II. PRELIMINARIES

The $n$-dimensional Euclidean space is denoted by $\mathbb{R}^n$ and $\mathbb{R}^n_{++}$ represents the positive orthant of $\mathbb{R}^n$. We denote the vector of all ones by $1_n = [1, \ldots, 1]^T$. The set of standard
Euclidean basis for $\mathbb{R}^n$ is represented by $\{e_1, \ldots, e_n\}$ and $e_i := e_{i+1} - e_i$ for all $i = 1, \ldots, n - 1$.

Algebraic Graph Theory: A weighted graph is defined by $G = (\mathcal{V}, \mathcal{E}, \omega)$, where $\mathcal{V}$ is the set of nodes, $\mathcal{E}$ is the set of edges (feedback links), and $\omega : \mathcal{V} \times \mathcal{V} \to \mathbb{R}_{+}$ is the weight function that assigns a non-negative number (feedback gain) to every link. Two nodes are directly connected if and only if $(i, j) \in \mathcal{E}$.

**Assumption 1.** Every graph in this paper is connected. In addition, for every $i, j \in \mathcal{V}$, the following properties hold:

- $\omega(i, j) > 0$ if and only if $(i, j) \in \mathcal{E}$.
- $\omega(i, j) = \omega(j, i)$, i.e., links are undirected.
- $\omega(i, i) = 0$, i.e., links are simple.

The Laplacian matrix of $G$ is a $n \times n$ matrix $L = [l_{ij}]$ with elements

$$l_{ij} := \begin{cases} -k_{i,j} & \text{if } i \neq j \\ k_{i,1} + \ldots + k_{i,n} & \text{if } i = j \end{cases},$$

where $k_{i,j} := \omega(i, j)$. Laplacian matrix of a graph is symmetric and positive semi-definite [24]. Assumption 1 implies the smallest Laplacian eigenvalue is zero with algebraic multiplicity one. The spectrum of $L$ can be ordered as

$$0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n.$$

The eigenvector of $L$ corresponding to $\lambda_k$ is denoted by $q_k$. By letting $Q = [q_1 | \ldots | q_n]$, it follows that $L = QAQ^T$ with $A = \text{diag}([0, \lambda_2, \ldots, \lambda_n])$. We normalize the Laplacian eigenvectors such that $Q$ becomes an orthogonal matrix, i.e., $Q^TQ = QQ^T = I_n$ with $q_1 = \frac{1}{\sqrt{n}}1_n$.

**Probability Theory:** Let $L^2(\mathbb{R}^q)$ be the set of all $\mathbb{R}^q$-valued random vectors $z = [z^{(1)}, \ldots, z^{(q)}]^T$ of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with finite second moments. A normal random variable $y \in \mathbb{R}^q$ with mean $\mu \in \mathbb{R}^q$ and $q \times q$ covariance matrix $\Sigma$ is represented by $y \sim N(\mu, \Sigma)$. The error function erf : $\mathbb{R} \to (-1, 1)$ is $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$, which is invertible on its range as $\text{erf}^{-1}(x)$. We employ standard notation $d\xi_t$ for the formulation of stochastic differential equations.

## III. PROBLEM STATEMENT

Suppose that a finite number of vehicles $\mathcal{V} = \{1, \ldots, n\}$ form a platoon along the horizontal axis. Vehicles are labeled in descending order, where the $n$’th vehicle is assumed to be the leading vehicle in the platoon. The $i$’th vehicle’s state is determined by $[x_t^{(i)}, v_t^{(i)}]^T$, where $x_t^{(i)}$ is the position and $v_t^{(i)}$ is the velocity of vehicle $i \in \mathcal{V}$ at time $t$.

The dynamics of the $i$’th vehicle evolve according to the stochastic differential equation

$$\begin{align*}
\frac{dx_t^{(i)}}{dt} &= v_t^{(i)} \\
\frac{dv_t^{(i)}}{dt} &= u_t^{(i)} dt + g \, d\xi_t^{(i)}
\end{align*}$$

where $u_t^{(i)} \in \mathbb{R}$ is the control input at time $t$. The term $g \, d\xi_t^{(i)}$ represents the white noise generator that models the uncertainty diffused in the system. It is assumed that noise acts on every vehicle additively and independently from the other vehicles’ noises. The noise magnitude is represented by the diffusion $g \neq 0$, which is assumed to be identical for all $i \in \mathcal{V}$. The control objectives for the platoon are to guarantee the following global behaviors: (i) the relative distance between every two consecutive vehicles converges to a prescribed distance; and (ii) the platoon of vehicles attain the same constant velocity in the steady state.

To incorporate deficiencies in communication network, we assume all vehicles experience an identical, constant, and known time-delay $\tau \in \mathbb{R}_{++}$. It is known from [26] that the following feedback control law can achieve the platooning objectives

$$u_t^{(i)} = \sum_{j=1}^{n} k_{i,j} (v_{t-\tau}^{(j)} - v_{t-\tau}^{(i)}) + \beta \sum_{j=1}^{n} k_{i,j} (x_{t-\tau}^{(j)} - x_{t-\tau}^{(i)} - (j - i)d).$$

The parameter $\beta \in \mathbb{R}_{++}$ regulates the effect of the relative positions and the velocities. By defining the vector of positions, velocities, and noise inputs as $x_t = [x_t^{(1)}, \ldots, x_t^{(n)}]^T$, $v_t = [v_t^{(1)}, \ldots, v_t^{(n)}]^T$ and $\xi_t = [\xi_t^{(1)}, \ldots, \xi_t^{(n)}]^T$, respectively, we denote the target distance vector as $r = [d, 2d, \ldots, nd]^T$. Using the above control input, we represent the closed-loop dynamics as an initial value problem

$$\begin{align*}
\frac{dx_t}{dt} &= v_t dt \\
\frac{dv_t}{dt} &= -L v_{t-\tau} dt - \beta L (x_{t-\tau} - r) dt + g d\xi_t,
\end{align*}$$

for all $t \geq 0$ and given deterministic initial function of $\phi_t^{\tau}$ and $\phi^{\tau}_t \in \mathbb{R}^n$ for $t \in [-\tau, 0]$. It is well known that (3) generates a well-posed stochastic process $\{(x_t, v_t)\}_{t \geq -\tau}$, [13].

The problem is to quantify the collision risk of one particular pair of vehicles conditioned on the knowledge that a subset of pairs has already collided. We refer to this quantity as cascading collisions (failures) risk, and our goal is to characterize its value as a function of the underlying communication graph topology, time-delay, and statistics of noise. To this end, we will develop a general framework to assess cascading risk of collisions using the steady-state statistics of the closed-loop system (1) and (3).

## IV. PRELIMINARY RESULTS

To evaluate the risk of cascading collisions in a platoon, we briefly review some necessary concepts and results [17].
Fig. 2: This figure depicts the concept of steady-state distance $d_j$, conditional distance $d_j|d_{Im} = d_c$ and the collision set $C_δ$.

A. Platooning State and Stability Conditions

Keeping a safe, constant distance from each other while traversing with a constant velocity is commonly referred to as the target (or consensus) state in a platoon \[8, 5\]. We say that the group of vehicles with unperturbed dynamics, i.e., (3) with $g = 0$, forms a platoon if

$$\lim_{t \to \infty} |v^{(i)}_t - v^{(j)}_t| = 0 \quad \text{and} \quad \lim_{t \to \infty} |x^{(i)}_t - x^{(j)}_t| - (i - j)d = 0$$

for all $i, j \in \mathcal{V}$ and all initial functions. It is known \[26, 17\] that the deterministic time-delayed network of vehicles will converge and form a platoon if and only if $(\lambda, \tau, \beta \tau) \in S$ for all $i = 2, \ldots, n$ where

$$S = \left\{ (s_1, s_2) \in \mathbb{R}^2 \mid s_1 \in \left(0, \frac{\pi}{2}\right), \ s_2 \in \left(0, \frac{a}{\tan(a)}\right) \right\}$$

for $a \in \left(0, \frac{\pi}{2}\right)$ the solution of $a \sin(a) = s_1$.

Throughout the paper, we assume that the underlying deterministic network of vehicles forms a platoon.

B. Steady-State Inter-vehicle Distance

In the presence of stochastic exogenous noise, inter-vehicle failures are mainly due to large deviations in inter-vehicle distances from a desired safe distance [1]. Let us denote the steady-state value of the relative distance between vehicles $i$ and $i + 1$ by $d_i := \lim_{t \to +\infty} (d_i^{(i+1)} - \tilde{x}_i^{(i)})$ whenever it exists. These distances can be stacked as $\mathbf{d} = [d_1, \ldots, d_{n-1}]^T$.

**Lemma 1.** [12] Suppose that the unperturbed network of vehicles forms a platoon. Then, random variable $\mathbf{d}$ exists almost surely, and has a multivariate normal distribution in $\mathbb{R}^{n-1}$ that is given by

$$\mathbf{d} \sim \mathcal{N}(\mathbf{d} \mathbf{1}_n, \Sigma),$$

where the elements of the covariance matrix $\Sigma = [\sigma_{ij}]$ are

$$\sigma_{ij} = g^2 \frac{\tau^3}{2 \pi} \sum_{k=2}^{n} (\tilde{e}_k^T \mathbf{q}_k)(\tilde{e}_j^T \mathbf{q}_k)f(\lambda_k \tau, \beta \tau),$$

for all $i, j = 1, \ldots, n$ and

$$f(s_1, s_2) = \int_{\mathbb{R}} \frac{dr}{(s_1 s_2 - r^2 \cos(r))^2 + r^2(s_1 - r \sin(r))^2}.$$  

For the simplicity of notations, we use $\sigma^2$ instead of $\sigma_{ii}$.

C. VaR of a Single Collision

To quantify the uncertainty level encapsulated in the relative distances between vehicles, we employ distribution-based quantities of risk such as VaR \[6, 14, 20, 17\]. VaR indicates the chance of a random variable landing inside an undesirable set of values, i.e., a near-collision situation. The set of undesirable values is referred to as a systemic set, which is denoted as $C \subset \mathbb{R}$. In probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the set of systemic events of random variable $y : \Omega \to \mathbb{R}$ is defined as $\{\omega \in \Omega \mid y(\omega) \in C\}$. We define a collections of super-sets $\{C_\delta \mid \delta \in [0, \infty]\}$ of $C$ that satisfy the following conditions for any sequence $(\bar{\delta}_n)_{n=1}^\infty$ with property $\lim_{n \to \infty} \bar{\delta}_n = \infty$

- $C_\delta \subset C_{\delta_2}$ when $\delta_1 > \delta_2$.
- $\lim_{n \to \infty} C_{\bar{\delta}_n} = \bigcap_{n=1}^\infty C_{\bar{\delta}_n} = C$.

In practice, one can tailor the super-sets to cover a suitable neighborhood of $C$ to characterize alarm zones as a random variable approaches $C$. For a given $\delta > 0$, the chance of $\{y \in C_\delta\}$ indicates how close $y$ can get to $C$ in probability. For a given design parameter $\varepsilon \in (0, 1)$, the VaR measure $R_{\varepsilon} : \mathcal{F} \to \mathbb{R}_+$ is defined by

$$R_{\varepsilon} := \inf \left\{ \delta > 0 \mid \mathbb{P}\{y \in C_\delta\} < \varepsilon \right\},$$

where a smaller $\varepsilon$ indicates a higher confidence level on random variable $y$ to stay away from $C_\delta$. Let us elaborate and interpret the typical values of $R_{\varepsilon}$. The case $R_{\varepsilon} = 0$ signifies that the probability of observing $y$ dangerously close to $C$ is less than $\varepsilon$. We have $R_{\varepsilon} > 0$ if $y \in C_{\delta}$ for some $\delta > 0$ (in fact, $\delta > R_{\varepsilon}$) with probability greater than $\varepsilon$. The extreme case $R_{\varepsilon} = \infty$ indicates that the event that $y$ is to be found in $C$ is assigned with a probability greater than $\varepsilon$. In addition, several interesting properties (see for instance [3, 4, 17]), the risk measure is non-increasing with $\varepsilon$. We refer to Fig. 2 for an illustration.

V. RISK OF CASCADING COLLISIONS

In [12], we consider the collision risk of a particular pair conditioned on the knowledge of only one prior collision. However, in platoons, the chances of encountering multiple collisions are negligible, and it is desirable to design platoons that are robust w.r.t cascade of multiple collisions. Thus, as an organic outgrowth of [12, 17], we develop tools to assess cascading risk conditioned on an arbitrary number of prior collisions in this section.

Let us consider the collision risk between the vehicles in the $j$’th pair and assume that pairs with ordered indices $\mathcal{I}_m = \{i_1, \ldots, i_m\}$ for some $m < n$ have already experienced
collisions, where \( j \notin I_m \) and \( i_1 < i_2 < \ldots < i_m \). To evaluate the effect of the prior collisions on the \( j \)’th pair, we form the \((m+1) \times (m+1)\) block covariance matrix
\[
\hat{\Sigma} = \begin{pmatrix}
\hat{\Sigma}_{11} & \hat{\Sigma}_{12} \\
\hat{\Sigma}_{21} & \hat{\Sigma}_{22}
\end{pmatrix}
\] (5)
using (4), where \( \hat{\Sigma}_{11} = \sigma_j \), \( \hat{\Sigma}_{12} = \hat{\Sigma}_{21} = \{\sigma_{j1}, \ldots, \sigma_{jm}\} \), and \( \hat{\Sigma}_{22} = \{\sigma_{ij} | i, j \in I_m \} \in \mathbb{R}^{m \times m} \). Suppose that the states of the collided pairs are represented by vector \( d_c = [d_{c1}, \ldots, d_{cm}]^T \). For instance, when the vehicles in the \( i \)’th pair are detached [17], one may set \( d_{ci} \) to \( 2d \), and when they are about to collide, one may set \( d_{ci} \) to \( 0.1d \) or even 0. We calculate the conditional probability distribution of \( \tilde{d}_j \) given \( \tilde{d}_{I_m} = d_c \) in a multivariate normal distribution, where \( \tilde{d}_{I_m} = [d_{i1}, \ldots, d_{im}]^T \).

**Lemma 2.** Suppose that the assumptions of Lemma 1 hold. Then, the conditional distribution of \( \tilde{d}_j \) given \( \tilde{d}_{I_m} = d_c \) is a normal distribution \( \mathcal{N}(\tilde{\mu}, \tilde{\sigma}) \) with
\[
\tilde{\mu} = \bar{d} + \tilde{\Sigma}_{12} \tilde{\Sigma}_{22}^{-1}(d_c - d_{1m})
\] (6)
and
\[
\tilde{\sigma}^2 = \tilde{\Sigma}_{11} - \tilde{\Sigma}_{12} \tilde{\Sigma}_{22}^{-1} \tilde{\Sigma}_{21},
\] (7)
where the sub-blocks \( \tilde{\Sigma}_{11}, \ldots, \tilde{\Sigma}_{22} \) are defined in (5).

The proof of this lemma is eliminated due to the page limit; the main lines of proof rely on calculating the conditional distribution of a multivariate normal random variable [23].

We characterize the cascading risk of the \( j \)’th pair by
\[
\left\{ \bar{d}_j \in C_\delta \mid \tilde{d}_{I_m} = d_c \right\}
\]
with \( C_\delta := (-\infty, \frac{\bar{d}}{\delta + \varepsilon}) \) for \( \delta \in [0, \infty] \) and design parameter \( \varepsilon \geq 1 \). The systemic set of collision is \( C := (-\infty, 0) \). A visualization of this setup is depicted in Fig. 2. The value-at-risk measure for the cascading collision event is defined as
\[
\mathcal{R}^{I_m,j}_{c,C} := \inf \left\{ \delta > 0 \mid \mathbb{P}\left\{ \bar{d}_j \in C_\delta \mid \tilde{d}_{I_m} = d_c \right\} < \varepsilon \right\}
\] (8)
with \( \varepsilon \in (0, 1) \) and \( j \notin I_m \). The next theorem presents a closed-form representation of the cascading collision risk\(^1\).

**Theorem 1.** Suppose that the vehicles form a platoon and it is known that the \( i_1, \ldots, i_m \)’th pairs of vehicles have already collided. The risk of cascading collision of the \( j \)’th pair is
\[
\mathcal{R}^{I_m,j}_{c,C} = \begin{cases} 
0, & \text{if } \frac{\bar{d} - \varepsilon \tilde{\mu}}{\sqrt{2\varepsilon \tilde{\sigma}}} \leq \ell_c \\
\frac{\bar{d} - \varepsilon \tilde{\mu}}{\sqrt{2\varepsilon \tilde{\sigma}}}, & \text{if } \ell_c \in \left( \frac{-\tilde{\mu}}{\sqrt{2\varepsilon \tilde{\sigma}}}, \frac{d - \varepsilon \tilde{\mu}}{\sqrt{2\varepsilon \tilde{\sigma}}} \right), \\
\infty, & \text{if } \frac{-\tilde{\mu}}{\sqrt{2\varepsilon \tilde{\sigma}}} > \ell_c
\end{cases}
\] (9)
where \( \tilde{\mu} \) and \( \tilde{\sigma} \) are given in Lemma 2, \( \ell_c = \text{erf}^{-1}(2\varepsilon - 1) \).

**Proof.** Due to the page limit, the proof is presented in the extended version of this paper [11]. \( \square \)

\(^1\)The risk of other types of failures, e.g., detachments [17], can be derived using similar lines of analysis.

Based on the selected confidence level \((1 - \varepsilon)\), the cascading risk falls into three categories. In the first scenario, if the confidence level \((1 - \varepsilon)\) is too low, we have \( \mathbb{P}\{ \bar{d}_j \in C_\delta \mid \tilde{d}_{I_m} = d_c \} < \mathbb{P}\{ \bar{d}_j \in C_0 \mid \tilde{d}_{I_m} = d_c \} \leq \varepsilon \) for every \( \delta > 0 \). Hence, there is no risk of having a further collision with the confidence level \((1 - \varepsilon)\); see the yellow area in Fig. 3. The second case indicates that if the confidence level \((1 - \varepsilon)\) is too high, the risk of having the cascading collision will be infinitely large and no \( \delta \) can bound the value of \( \mathbb{P}\{ \bar{d}_j \in C_\delta \mid \tilde{d}_{I_m} = d_c \} \) with \( \varepsilon \); see the blue area in Fig. 3. The third case complements the above two extreme cases, which is when the risk assumes a finite value with some intermediate confidence level.

**VI. CASE STUDIES**

We consider a platooning problem with \( n = 50 \) vehicles. Systematic set parameters are set to \( c = 2, d = 3 \) and \( \varepsilon = 0.1 \), and the existing failures are considered as collisions, i.e., \( d_c = 0 \), for all case studies. To measure the risk of cascading collisions among the entire platoon, let us introduce the risk profile as follows
\[
\mathcal{R}^{I_m}_{c,C} = [\mathcal{R}^{I_m,1}_{c,C}, \ldots, \mathcal{R}^{I_m,n-1}_{c,C}]^T,
\]
in which \( \mathcal{R}^{I_m,j}_{c,C} = 0 \) if \( j \notin I_m \).

**A. Risk of Cascading Collisions**

The risk profile of cascading collision \( \mathcal{R}^{I_m}_{c,C} \) for all pairs of vehicles in the platoon is evaluated with the closed-form representation derived in Theorem 1. In Fig. 4, we assess the risk profile among various communication graphs discussed below.

**Complete Graph:** We assume the all-to-all communication is available, and set \( g = 10, \tau = 0.03, \) and \( \beta = 0.005 \).
2. It is shown that the vehicle pair that is adjacent to the failure group obtains a zero risk value, and the remaining pairs obtain the same value as the naive risk $R_{C,j}^{C}$ in [17].

Path Graph [24]: We assume vehicles can only communicate with their front and rear neighbors in the platoon, which can be interpreted as a car platoon on the highway. We set $g = 0.1, \tau = 0.03$, and $\beta = 2$. The results indicate that the impact from the failures in a path graph will first exasperate and then dilute as the distance to the failure increases.

p-Cycle Graph [24]: We assume vehicles communicate to their $p$ immediate graph neighbors, and set $g = 0.1, \tau = 0.01$, and $\beta = 2$. The behavior of the risk profile resembles the one in the path graph when using a 1-cycle graph for communication. However, when $p = 5$, the risk of the adjacent pairs is reduced to 0, which indicates it is relatively safe to stay close (in the communication graph) to the existing collisions.

**B. Impacts from the Characteristics of Collisions**

The existing collisions affect the cascading risk in a vehicle platoon in a compound manner since the failure’s dimension has been lifted from 1 to $m$. Hence, one should expect the risk of cascading collisions to be affected by the scale of existing failures, the relative position of failures, and the communication graph topology.

1) **Scale of Failures**: It is instinctive to notice that the number of collisions affects $R_{C,j}^{C,m}$ since the statistics of the conditional distribution $d_j$ is highly dependent on the value of $m$. To reveal this relation, we assume there exists $1, \ldots, 20$ failures at the beginning of the platoon and evaluate the risk profile $R_{C,j}^{C,m}$ over different communication graph topology, which is depicted in Fig. 5. A detailed discussion can be found in the extended version of this paper [11].

2) **Sparsity of Failures**: When the scale of failures is fixed, the position distribution of existing failures also affects the risk of new collisions. For exposition of the next result, let us consider an indicator vector $\chi = [\chi_k]$, where $k = i_1, i_1 + 1, \ldots, i_m$ and $i_1, \ldots, i_m \in I_m$ are the labels of the existing failures. The value of $\chi_k$ is given by the following indicator function:

$$\chi_k = 1_{I_m}(k) = \begin{cases} 1 & \text{if } k \in I_m \\ 0 & \text{if } k \notin I_m \end{cases}.$$ 

Then, the number of zeros, i.e., the sparsity of $\chi$, represents how sparse the positions of failures are. To reveal how it will affect the cascading risk in a vehicle platoon, we assume $m = 5$ in both a $n = 10$ platoon and a $n = 50$ platoon and evaluate the average risk value among all possible combinations of $\chi^3$ with the same sparsity level, as depicted in Fig. 6.

We highlight that in most of the tested graph structures, the average risk decreases with the increase of the sparsity of $\chi$ in most cases, indicating that the platoon is more vulnerable to clustered failures. Hence, in applications like highway cars or robot platoons, one wants to avoid congregate failures and keep them distributed if failures are inevitable.

**C. Adding New Edges to the Communication Graph**

When communication capability is limited, as in most real-world problems, one may have a chance to alter the existing communication graph by adding very few new edges. It is crucial to select smartly such that they will help to reduce the risk of cascading collisions in the system. In this problem, we assume the communication graph is given, and the platoon is allowed to add two more edges$^4$ to the same destination in the original graph. The cascading risk is evaluated on

$^2$Different $g, \tau$, and $\beta$ are selected for different communication graphs specifically to the risk is maintained within $(0, +\infty)$.

$^3$For the sparsity level $0 \leq k \leq n - m$, the total possible number of combinations is $\binom{n}{k}$.

$^4$Adding two edges is more meaningful than one since the distance $d_j$ involves two vehicles.
the platoon’s first pair for both path graph and 5-cycle graph, as shown in Fig. 7. In a path graph, the result suggests that the risk of cascading collision can be reduced if new communication is allowed with the vehicles before the collisions, and the risk deteriorates if the communication link is established with the vehicles located on another side of the collisions in the platoon.

VII. CONCLUSION

To open up the opportunity of probing into the risk of cascading collisions in a multi-vehicle platoon, we develop the generalized value-at-risk measure among multi-systemic events. Using a platoon of vehicles as a motivational application, the risk profile of cascading collisions (e.g., multi-vehicle accidents) is quantified using the steady-state statistics obtained from the system dynamics. Extensive simulations show the impact of multi-vehicle accidents with different scales and spatial distributions and comprehensively evaluate the platoon’s vulnerability to cascading collisions. Some possible variations of the communication graph are also presented by adding new edges to particular graph topology in favor of risk minimization.

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