Effective elastic properties of doubly periodic array of functionally graded inclusions by an iterative FE-BE coupling method

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Abstract. This paper is devoted to the study of the effective elastic properties of doubly periodic array of functionally graded inclusion problems by an iterative FE-BE coupling method. This method has some advantages compared with boundary element method (BEM), e.g. (1) inclusions can be isotropic, orthotropic, anisotropic or functionally graded materials; (2) various shaped inclusions can be easily solved; (3) the fundamental solutions for functionally graded inclusion materials are not needed. Some numerical examples are used to validate the applicability and reliability of the adopted scheme.

1. Introduction

Determining the elastic effective properties of a composite material has been the subject of numerous works [1]. Functionally graded materials (FGM) which were first reported [2] in the 1980s are a new generation of composite materials characterized by a continuously varying property. The concept of FGM is to take advantage of certain desirable features of the continuously and functionally varying volume composition of constituent particles and optimize the distribution of material properties such as strength, hardness, thermal resistance, etc, so that one can expect that functionally graded inclusion composites, which is formed when graded inclusions are embedded randomly in a host medium, will be applied in various fields such as mechanical, civil, medicine and so on.

A review about functionally graded materials and their successful implementation in a wide range of areas is available in [3]. The effect of the interphase zones around inclusions on effective elastic properties of the heterogeneous materials has been studied [4]. There are some works addressing the response of functionally graded composite systems with different geometries such as hollow cylinders, coatings on substrate and sandwich panels. In these researches, some benchmark results for the accuracy of numerical solutions have been obtained [5], which can be used to examine numerical solution accuracy from finite element method (FEM), BEM or other numerical methods.

Periodic or approximately periodic microstructures can be found in many natural or manmade materials. For this, micromechanics for solids with periodic microstructures has attracted the attention of many scholars. In principle, few exact solutions exist, except when the microstructure of the composite is known completely, for example, periodic arrays of spheres [6] or fiber-reinforced composites in the case of specific geometries [7]. Unfortunately, in most situations, this is not the case. Numerical methods, including finite element method [8] and boundary element method [9] etc., are naturally used to calculate the effective properties. Maxwell’s scheme [10] has been developed in several works of Mogilevskaya \textit{et al} [11] for the case of a material reinforced with parallel infinite...
fibers and Kushch et al [12] for a material with spherical particles. Also, Mogilevskaya and Crouch [13] combine Maxwell’s methodology with the BEM for evaluating the effective properties of composite, porous, and microcracked isotropic materials with periodic or random structure.

Dong [14] adopted an effective symmetric-iterative FE-BE coupling method to investigate elastic problems and obtained series of ideal results by comparing with analytical or exist results. In the present paper, this FE-BE coupling method is further developed and combined with unit cell models to evaluate the effective elastic properties of functionally graded inclusion problems with doubly periodic array.

Following [15], a rectangular cell containing single inclusion is cut from doubly periodic inclusion problems. The boundary condition on the rectangular cell is available for two loading systems, i.e. the tension and the in-plane shear. Then, the iterative FE-BE coupling method [16] is utilized to calculate those models. In calculation, a small stiffness matrix from BE domain is taken as a corresponding loading matrix in FE region. We first obtain displacements of the common interface nodes by iteratively calculating the coupled equations. After that, stress and displacement responses of the medium can be acquired by adopting the BE method. Numerical results are obtained for three examples. These results are compared with those from ABAQUS to demonstrate the validity of the adopted method.

2. Statement of the problem

![Figure 1](image)

**Figure 1.** (a) Doubly periodic inclusion problems; (b) rectangular cell containing single inclusion and its boundary condition for tension loading; (c) rectangular cell containing single inclusion and its boundary condition for shear force.

We refer to an infinite isotropic elastic matrix containing functionally graded inclusions with three
different cross sections (circular, elliptic or hollow cylinder). Inclusions are arranged in a regular lattice determined by two families of parallel lines, respectively parallel to the \( x_1 \) axis and \( x_2 \) axis, as sketched in figure 1(a). This geometrical structure subject to remote tension and in-plane shear forces can be interpreted as a two dimensional array of unit cells, developing periodically along the \( x_1 \) and \( x_2 \) directions. The geometry and the boundary conditions of the unit cell, i.e. the microstructure of the composite, can be displayed in figures 1(b) and 1(c). The cell sides measure \( 2l \) and \( 2h \) respectively, and the inclusion outer and inner radius are \( R_1 \) and \( R_2 \).

The materials of the matrix medium are assumed to be linearly elastic and isotropic, while for the functionally graded materials of inclusions, Young’s modulus depends only on the radial direction and Poisson’s ratio is assumed to be constant. To this end, a family of problems is introduced, indexed by parameter \( n \) scaling the microstructure. When solving these problems, we consider the following variation law cases.

- circular inclusions

\[
E(r) = E_o \left( \frac{r}{R} \right)^n, \quad 0 \leq r \leq R
\]

- elliptic inclusions

\[
E(x, y) = E_o \left( \frac{x^2}{c_1^2} + \frac{y^2}{c_2^2} \right)^n, \quad 0 \leq x \leq c_1, 0 \leq y \leq c_2
\]

- hollow cylinder inclusions

\[
E(r) = E_o \left( \frac{r}{R_2} \right)^n, \quad R_1 \leq r \leq R_2
\]

where \( n \) is the inhomogeneity parameter that can be assigned to be an arbitrary value between -5 and 5, and the homogenization limit is obtained by letting \( n \) go to zero. \( E_o \) is the elastic modulus of the matrix medium. In equation (1) \( R \) is the radius of circular inclusions. \( c_1 \) and \( c_2 \) in equation (2) are respectively the \( x \) semi-major axis and \( y \) semi-major axis of elliptic inclusions.

The main objective of this work is to obtain the effective elastic properties of doubly periodic functionally graded inclusion problems by using the FE-BE coupling method. A medium containing doubly periodic array of inclusions of various shapes can be considered as a homogeneous orthotropic medium. The detailed formula for calculating the effective elastic properties can be found in paper [19]. Here I won’t repeat.

### 3. Formulation of iterative FE-BE coupling algorithm

Consider a linear elastic solid of domain \( \Omega \) enclosed by a boundary \( \Gamma \). The solid can be regarded as two decomposed portions \( \Omega_f \) and \( \Omega_b \), which are modeled by FEM and BEM, respectively. For the FEM, the algebraic assembled element equations are given by

\[
\begin{bmatrix}
K_{ff} & K_{fc} \\
K_{cf} & K_{cc}
\end{bmatrix}
\begin{bmatrix}
U_f \\
U_c
\end{bmatrix} =
\begin{bmatrix}
F_f \\
F_c
\end{bmatrix}
\]

Where \( U_f \) and \( U_c \) denote the vectors of non-interface and interface displacements in the FE region, \( F_f \) and \( F_c \) are the associated loading vectors, respectively. \( K_{ff}, K_{fc}, K_{cf} \) and \( K_{cc} \) are respectively
sub-matrices of the global stiffness matrix of the FE region related to the vectors \( \mathbf{U}_f \) and \( \mathbf{U}_c \).

Neglecting body forces and after the process of boundary dicretization, the BEM partitioned equations are obtained

\[
\begin{bmatrix}
A_{bb} & A_{bc} \\
0 & A_{cc}
\end{bmatrix}
\begin{bmatrix}
\mathbf{X}_b \\
\mathbf{T}_b^c
\end{bmatrix}
= +
\begin{bmatrix}
\mathbf{f}_b \\
\mathbf{f}_b^c
\end{bmatrix}
+ \begin{bmatrix}
A_b \\
A_c
\end{bmatrix}
\begin{bmatrix}
\mathbf{U}_b^c \\
\mathbf{U}_b^c
\end{bmatrix}
\]

(5)

Where \( \mathbf{X}_b \) is the vector of non-interface in the BE region; \( \mathbf{U}_b^c \) and \( \mathbf{T}_b^c \) are the interface displacements and tractions, respectively; \( \mathbf{f}_b \) and \( \mathbf{f}_b^c \) are two vectors which contains the contribution of the known values.

It is understood that the governing equations in the FE domain involve nodal displacements and forces; in contrast, the primary unknowns in the BE domain are displacements and tractions. To facilitate the coupling, the relations between tractions and forces need to be established. After this, the displacement compatibility and force equilibrium along the common interface are adopted. Finally, the FE-BE coupling system of equations are obtained as follows

\[
\begin{bmatrix}
\mathbf{K}_{ff} & \mathbf{K}_{fc} \\
\mathbf{K}_{cf} & \mathbf{K}_{cc} + \mathbf{K}_f
\end{bmatrix}
\begin{bmatrix}
\mathbf{U}_f^{n+1} \\
\mathbf{U}_c^{n+1}
\end{bmatrix}
= +
\begin{bmatrix}
\mathbf{F}_f \\
\mathbf{F}_c
\end{bmatrix}
+ \begin{bmatrix}
0 \\
-\mathbf{K}_{cc} + \mathbf{K}_f
\end{bmatrix}
\begin{bmatrix}
\mathbf{U}_f^n \\
\mathbf{U}_c^n
\end{bmatrix}
\]

(6)

Where \( \mathbf{K}_i = (1/2)\left(\mathbf{K}_b^c + \mathbf{K}_b^c\right) \), \( \tilde{\mathbf{F}}_c = \mathbf{M}^{-1} \mathbf{f}_b^c \) in which \( \mathbf{M} \) is the converting matrix from tractions to nodal forces on the BE side of the FE-BE interface.

4. Numerical examples
In this section, an example is given to demonstrate the feasibility and effectiveness of the present scheme by comparing results with the ABAQUS. In order to ensure the numerical accuracy of the present method, some convergence tests have to be done.

![Figure 2](image_url)

**Figure 2.** The rectangular cell with one circular inclusion: (a) node distribution of the medium in BEM region; (b) element distribution of the inclusion in FEM region.

As shown in figure 2, this example is studied for doubly periodic circular inclusion whose radius is taken as \( R = 10 \). The elastic parameters of the medium are taken as \( E_0 = 1 \) and \( \nu_0 = 0.3 \), while for inclusions, they possess elastic modulus \( E_i \) that vary continuously as a function of position referring to equation (1), and the Poisson’s ration \( \nu_i \) is assuming to be the same as medium’s. The cell boundary lengths are respectively taken as \( 2l \) and \( 2h \) (see figures 1(b) and 1(c)).

In numerical implementation, each edge of the outer rectangular cell for various cases is divided into 32 quadratic boundary elements. The BE node distribution of this region is shown in figure 2(a).

The numbers of finite elements in the inclusion are chosen as 20, 56, 96, 192 and 256, respectively.
The corresponding numbers of the interface elements between the inclusion and the matrix are 8, 12, 24, 32 and 32, respectively. The convergence results for different values of effective properties are displayed in figures 3 and 4. With the increase of the number of finite elements, one can find that the convergence results have been obtained for 192 finite elements within the inclusion together with 32 interface elements. Therefore, we will choose this type of subdivision, the FE distribution of which is depicted in figure 2(b), to carry out further numerical analysis.

In ABAQUS model, an element type of CPS8R (one 8-node biquadratic plane stress quadrilateral, reduced integration) is used. The whole model is divided into 5261 elements, 609 of which are belonged to the inclusion region. The medium and the inclusion are tied together. By using the coupling method, we plot the effective elastic properties for \( R/c = 0.4 \) (\( c = \min(l, h) \)) and different values of \( n \) in figures 5-8. Here we keep the radius of inclusion be \( R = 10 \), so \( h \) and \( l \) are changed according to the relationship of \( R/c \) and \( h/l = 0.4 \).

![Figure 3](image1) ![Figure 4](image2)  
**Figure 3.** Normalized effective elastic modulus \( E_{\text{f}}/E_{\text{o}} \) with increasing finite element numbers \((n=-2, R/c=0.8)\).  
**Figure 4.** Effective Poisson’s ratio \( \nu_{\text{f}} \) with increasing finite element numbers \((n=4, R/c=0.8)\).
Figure 7. Effective Poisson’s ratio $v_{12}$ of doubly periodic circular inclusions for various $n$ and $c$; A: ABAQUS, P: the present method.

Figure 8. Effective Poisson’s ratio $v_{21}$ of doubly periodic circular inclusions for various $n$ and $c$; A: ABAQUS, P: the present method.

Figure 9. Mises stress contours of doubly periodic circular inclusions for $n = 4$ and $R / c = 0.4$.

It can be seen that the results from the present method are in excellent agreement with ABAQUS’s. Besides, we can observe for different values of $n$ that the harder the inclusion becomes, the larger the effective elastic moduli $E_1$, $E_2$ and $G_{12}$. The bigger the inclusion becomes, i.e. $R / c$ is increased, the larger the effective elastic moduli $E_1$ and $E_2$ and $G_{12}$. One can also notice that when $E_i / E_0 = 1$, i.e. the homogeneous medium, the normalized effective elastic and shear moduli reduce to 1, respectively, as expected. Effective Poisson’s ratios $v_{12}$ and $v_{21}$ are shown in figures (7) and (8), respectively. By
checking the results (see figures (5)-(8)), one can find that the relationship of \( E_1 v_{21} / (E_2 v_{12}) = 1 \) has been well approximated. Figure 9 depicts the plot contours of Mises stress with \( n = 4 \) and \( R / c = 0.4 \). We can observe the “obvious layer structures” which shows the outstanding merit of functionally graded materials, i.e. their properties can be tailored via the design of the material gradients. Its significance, for the development of engineering components, relies on the fact that properly tuning the fibre grading profile allows to modulate the material stress state, thus enhancing material performance and durability.

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