Signal Velocity in Oscillator Arrays

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Abstract

We investigate decentralized systems of linear coupled oscillators on a circle. First we state and prove necessary and sufficient conditions for nearest neighbor systems to be asymptotically stable. We then investigate these asymptotically stable systems further. We establish that in this case the system behaves like a wave equation in which the high frequencies are damped (due to dispersion). Thus low frequency signals travel through the flock as $f(x - c_+ t)$ in the direction of increasing agent numbers and $f(x - c_- t)$ in the other. Here $c_+ > 0$ and $c_- < 0$ are called the signal velocities. The motivation for this work is in more realistic systems of finitely many coupled oscillators on the line, where the same is true away from the boundaries. We can thus study solutions of those systems simply by analyzing what happens near the boundaries.

1 Introduction

This note is part of a larger program to develop mathematical methods to quantitatively study performance of models for flocking. Our main interest in the current work is to develop methods that tell us how we can most effectively program driverless cars so that in situations of dense traffic they can coherently move at high speed. This is obviously an important problem, not only because it can lead to enormous cost savings to have smooth and dense traffic on our busier highways, but also because failures may cost lives.

Our approach is to assume that each car is programmed identically and that it can observe relative velocities and positions of nearby cars. In this note we take nearby to mean only the car in front and behind. However the methods we develop will be applicable to larger interactions (and these will be explored in future work). We will assume that the system is linearized. Various examples and analyses of nonlinear systems exist. But the emphasis here is on linear systems where we can allow for many parameters (to take the neighbors into account) and still perform a meaningful analysis.

There are two main aspects in our analysis. The first is the asymptotic stability. This can be analyzed via the eigenvalues of the matrix associated with the first order differential equation. Section 3 is devoted to establishing necessary and sufficient conditions for a class of systems to be asymptotically stable. Even though this is a fairly straightforward calculation, we have not found it in this generality in the literature. We use this as input for the next step of the analysis (see the remark before Proposition 4.1).

The second, more delicate aspect of the problem is related to the fact that we may have arbitrarily many cars following each other, hundreds or even thousands. In this situation, even if all our systems are known to be asymptotically stable, transients may still grow exponentially in the number of cars. The spectrum of the linear operator does not help us to recognize this problem (16). A dramatic example of this can be found in 18 where eigenvalues have real part bounded from above by a negative number and yet transients grow exponentially in $N$.

This kind of exponential growth underscores the need for different (non-spectral) methods to analyze these systems.

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The method we propose consists of establishing that for the parameter values of interest the system behaves like a wave equation in which the high frequencies are damped (due to dispersion). Thus low frequency signals travel through the flock as \( f(x - c_+t) \) in the direction of increasing agent numbers and \( f(x - c_-t) \) in the other. Here \( c_+ > 0 \) and \( c_- < 0 \) are called the signal velocities. This is what constitutes our main result (see Theorem 4.4). If we assume that this characterization also holds for finite systems on the line, and away from the boundaries, this allows us to study solutions in that case simply by analyzing what happens near the boundaries. This will be done in [4].

Ever since the inception ([6], [9]) of the subject, systems with periodic boundary conditions have been popular ([7], [8], and [1]) because they tend to be easier to study. Indeed they occur throughout physics. However the precise connection between these systems and more realistic systems with non-trivial boundary conditions has always been somewhat unclear. Our current work differs from earlier work in two crucial ways. The first is that we make precise what the impact of our analysis is for the (more realistic) systems on the line (see above). The second is that we consider all possible nearest neighbor interactions: we do not impose symmetries. This turns out to be of the utmost importance: when we apply these ideas in [4] it turns out that the systems with the best performance are asymmetric. Asymmetric systems (though not the same as ours) have also been considered by [2] and with similar results. However their methods are perturbative, and spectral based. Our methods are global and not spectral. In [10] and [11] asymmetric interactions are also studied, and it was shown that in certain cases they may lead to exponential growth (in \( N \)) in the perturbation. In the later of these, the model is qualitatively different because absolute velocity feedback is assumed (their method is also perturbative and not global). Signal velocities were employed in earlier calculations namely [19] and [12]. These calculations have in common that they were done for car-following models. We are interested in a more general framework, namely where automated pilots may pay attention also to their neighbor behind them or indeed other cars further afield.

Our model is strictly decentralized. There are two reasons to do that. First, in high speed, high/density traffic, small differences in measured absolute velocity may render that measurement useless, if not dangerous, for the feedback. Secondly, the desired velocity, even on the highway, may not be constant. It will depend on weather, time of day, condition of the road, and so on. For these reasons we limit ourselves to strictly decentralized models that only use information relative to the observers in the cars (see [17] and [18]). Many authors study models featuring a term proportional to velocity minus desired velocity (see e.g. [1], [2], [6], [7], [8], and [11]).

\section{Definitions}

In a decentralized flock the only information an agent receives are the position and velocity relative to it of some nearby agents, that is: the acceleration of the \( k \)th agent is an affine function of the differences \( (x_k - h_k) - (x_i - h_i) \) and \( \dot{x}_k - \dot{x}_i \), where \( i \) runs over the neighbors of \( k \). Here \( h_k \) is the (constant) desired distance from the leader, typically \( k \) times a fixed spacing \( \delta \). To simplify the equations, we now change coordinates to \( z_k \equiv x_k - h_k \). This gets rid of the constants \( h_k \). See [17] for more details. We leave it to the reader to check that the equations for a leaderless system of identical agents with periodic boundary conditions can be written as follows.

\begin{definition}[Leaderless, decentralized, identical agents, and periodic boundary conditions] The system \( S^*_N \) of differential equations is defined as follows. For \( 1 \leq k \leq N \)

\begin{align*}
\ddot{z}_k &= g_x \sum_{j=1}^{N} \rho_{x,j} \dot{z}_{k+j} + g_v \sum_{j=1}^{N} \rho_{v,j} \dot{z}_{k+j} = g_x \sum_{j=1}^{N} L_{x,k,j} \dot{z}_j + g_v \sum_{j=1}^{N} L_{v,k,j} \dot{z}_j
\end{align*}

where \( g_x \), \( g_v \), \( \rho_{x,j} \), and \( \rho_{v,j} \) are real numbers. We furthermore set:

\begin{align*}
\rho_{x,j} &= \rho_{x,j+N} \quad \text{and} \quad \rho_{v,j} = \rho_{v,j+N} \quad \text{and} \quad z_j = z_{j+N}
\end{align*}

and (decentralized):

\begin{align*}
\sum_{j=1}^{N} \rho_{x,j} &= \sum_{j=1}^{N} \rho_{v,j} = 0 \quad \text{for} \quad j \in \{-1, 0, 1\}.
\end{align*}

In the current work we limit ourselves to nearest neighbor systems, that is \( j \in \{-1, 0, 1\} \).

\begin{remark}
The Laplacians \( L_x \) and \( L_v \) of Definition 2.1 are circulant matrices. Circulant matrices have orthogonal eigenbases. In fact it is well known and easy to check that circulant matrices are diagonalized by the Fourier transform (see [10]). This is the reason periodic boundary conditions are so convenient.
\end{remark}
It will be useful to write the equations of $S^*_N$ as a first order system:

$$\frac{d}{dt} \binom{z}{\dot{z}} = M_N \binom{z}{\dot{z}} = \begin{pmatrix} 0 & I \\ g_x L_x & g_v L_v \end{pmatrix} \binom{z}{\dot{z}}$$

(2.2)

It is easy to see that this system has a 2-dimensional family of coherent solutions, namely:

$$\forall i \quad z_i(t) = v_0 t + x_0$$

where $v_0$ and $x_0$ are arbitrary elements of $\mathbb{R}$ and these correspond to the eigenspace of the eigenvalue 0 with algebraic multiplicity two. It is easy to see that all solutions converge to one of these coherent ones if and only if all other eigenvalues of $S^*_N$ have negative real part. With a slight abuse of notation we will call this case asymptotically stable (see [13] for precise definitions):

**Definition 2.2** The system in Equation 2.2 is called asymptotically stable if all eigenvalues except for one have (strictly) negative real parts. That one eigenvalue is 0, has algebraic multiplicity 2, and corresponds to coherent motion.

Define $\lambda_{x,m}$ and $\lambda_{v,m}$ as follows: denote $\theta \equiv \frac{2\pi}{N}$ and set

$$\lambda_{x,m} \equiv g_x \sum_{j=1}^{N} \rho_{x,j} e^{ijm\theta} \quad \text{and} \quad \lambda_{v,m} \equiv g_v \sum_{j=1}^{N} \rho_{v,j} e^{ijm\theta}$$

(2.3)

These are essentially the Fourier transform of the functions $\rho_x$ and $\rho_v$. Denote the vector $w_m$ by:

$$w_m \equiv \frac{1}{\sqrt{N}} \left(1, e^{i\theta}, e^{2i\theta}, \ldots, e^{(N-1)i\theta}\right)^T$$

We furthermore define the moments of $g_x \rho_x$ and $g_v \rho_v$:

$$I_{x,\ell} \equiv g_x \sum_{j=1}^{N} \rho_{x,j} j^\ell \quad \text{and} \quad I_{v,\ell} \equiv g_v \sum_{j=1}^{N} \rho_{v,j} j^\ell$$

and observe that the real and imaginary parts of $\lambda_{x,m}$ can be expanded as follows (with $\theta \equiv \frac{2\pi}{N}$):

$$\lambda_{x,m} = \lambda_{x,R} + i \lambda_{x,I} = im\theta I_{x,1} - \frac{m^2\theta^2}{2} I_{x,2} - i \frac{m^3\theta^3}{3!} I_{x,3} + \frac{m^4\theta^4}{4!} I_{x,4} + i \frac{m^5\theta^5}{5!} I_{x,5} \cdots$$

(2.4)

Analogous expressions for $\lambda_{v,m}$ can be given.

### 3 Asymptotic Stability

In this section we state and prove necessary and sufficient conditions for nearest neighbor systems to be asymptotically stable.

**Proposition 3.1** Let $L_x$ and $L_v$ be the Laplacians defined in Definition 2.1. The eigenvalues of $g_x L_x$ are $\lambda_{x,m}$ with associated eigenvector $w_m$ (where $m \in \{0, \cdots, N-1\}$). Similarly, $\lambda_{v,m}$ and $w_m$ form eigenpairs for $g_v L_v$.

**Remark:** Even though the Laplacians have a basis of normal eigenvectors, $M$ does not. What happens (and this is not hard to show) is that the $2N$ dimensional square matrix $M$ admits $N$ two-dimensional eigenspaces ($\mathbb{C}^2$) orthogonal to each other. Each of these planes may be spanned by two not necessarily orthogonal eigenvectors, or by an eigenvector and a (Jordan) generalized eigenvector. This is also clear from the following result.


Proposition 3.2 The eigenvalues \( \nu_{m,\pm} \) \((m \in \{0, \cdots N - 1\})\) of \( M \) are given by the solutions of
\[
\nu^2 - \lambda_{v,m} \nu - \lambda_{x,m} = 0 \quad \Rightarrow \quad \nu_{m,\pm} = \frac{\lambda_{v,m}}{2} \pm \sqrt{\frac{\lambda_{v,m}^2}{4} + \lambda_{x,m}}
\]
with associated eigenvectors given by \( \left( \frac{w_m}{\nu_{m,\pm} w_m} \right) \), and \( \lambda_{x,m}, \lambda_{v,m}, \) and \( w_m \) given in Proposition 3.1.

Proof: Let \( w_m \) be an eigenvector of the Laplacian (see Proposition 3.1). Solve for \( \nu \) in
\[
\begin{pmatrix}
  0 & I \\
  g_x L_x & g_v L_v
\end{pmatrix}
\begin{pmatrix}
  w_m \\
  \nu w_m
\end{pmatrix} = \nu
\begin{pmatrix}
  w_m \\
  \nu w_m
\end{pmatrix}
\]
(see Equation 2.2). This gives the quadratic equation of the Proposition. \( \blacksquare \)

Figure 3.1: (color online) A representative figure for the calculation of the eigenvalues for 500 agents. The values of the parameters are given in the figures. Green line: \( \lambda_{x,m} \), Blue ellipse: \( \lambda_{v,m} \).

Definition 3.3 Define \( \lambda_x : S^1 \to \mathbb{C} \) and \( \lambda_v : S^1 \to \mathbb{C} \) by
\[
\lambda_x(\phi) \equiv g_x \sum_{j=1}^{N} \rho_{x,j} e^{ij\phi} \quad \text{and} \quad \lambda_v(\phi) \equiv g_v \sum_{j=1}^{N} \rho_{v,j} e^{ij\phi}
\]
The curve \( \gamma \) is the solution set of
\[
\nu^2 - \lambda_v(\phi) \nu - \lambda_x(\phi) = 0
\]

Note that \( \gamma \) depends on all parameters except \( N \). As \( N \) grows and none of the other parameters is varied, the eigenvalues of the associated Laplacians tend to fill out curves denoted by \( \lambda_x \) and \( \lambda_v \). This is illustrated in Figure 3.1. As a consequence the same holds for the eigenvalues of \( M_N \) (see Figure 3.2).

Corollary 3.4 The set \( \text{spec} (M_N) \) of eigenvalues of \( M_N \) satisfies:
\[
\text{spec} (M_N) \subset \gamma \quad \text{and} \quad \lim_{N \to \infty} \text{spec} (M_N) = \gamma
\]
Similar results hold for the spectra of \( L_x \) and \( L_v \) and the curves \( \lambda_x \) and \( \lambda_v \).
Figure 3.2: (color online) A representative figure for the calculation of the eigenvalues for 500 agents. The values of the parameters are given in the figures. The eigenvalues of $M_N$ of Proposition 3.2.

**Proposition 3.5** For a flock as in Definition 2.1, a necessary condition for asymptotic stability is $I_{x,1} = 0$.

**Proof:** If $I_{x,1} \neq 0$ then for large enough $N$ (i.e. $\theta$ small enough), we get from Proposition 3.2

$$\nu_{m,\pm} \approx \pm \sqrt{\lambda_{x,m}} \approx \pm \sqrt{im\theta I_{x,1}}$$

The last expression has four branches, two of which have positive real part and therefore the system is not asymptotically stable.

In the remainder of this section we use a global method to determine a better condition for asymptotic stability for nearest neighbor systems.

**Proposition 3.6** The system $S_N^*$ of Definition 2.1 is asymptotically stable for all $N$ (all other parameters fixed) if

$$\forall \phi \neq 0 : \Re(\lambda_x(\phi)) < 0 \quad \text{and} \quad \Re(\lambda_v(\phi)) < 0$$

Instability will occur for large $N$ if either of the opposite inequalities holds for some $\phi \neq 0$.

**Proof:** By the Routh-Hurwitz criterion applied to the equation $\nu^2 - \lambda_{v,m} \nu - \lambda_{x,m} = 0$ with complex coefficients (see [5]), we see that all nonzero eigenvalues given in Proposition 3.2 all have negative real parts if and only if for all $m \in \{1, \cdots, N-1\}$ we have (in the notation of Equation 2.4):

$$\Re(\lambda_v) < 0 \quad \text{and} \quad 2\Re(\lambda_x) < |\lambda_x|^2 \quad \Re(\lambda_x)\Re(\lambda_v) + \Im(\lambda_x)\Im(\lambda_v) > 0 \quad \Re(\lambda_v)|\Re(\lambda_v)|^2 + \Re(\lambda_v)\Im(\lambda_v)\Im(\lambda_v) + |\Im(\lambda_x)|^2 < 0$$

If $\rho_{x,j}$ are symmetric then $\lambda_{x,1}$ is zero. In this case these requirements reduce to: for all $m \in \{1, \cdots, N-1\}$, $\lambda_{x,R} < 0$ and $\lambda_{v,R} < 0$. Since the values $\lambda_{x,m}$ and $\lambda_{v,m}$ are dense in the curves $\lambda_x(\phi)$ and $\lambda_v(\phi)$, we have that stability for all $N$ follows if $\forall \phi \neq 0$, $\Re(\lambda_x(\phi)) < 0$ and $\Re(\lambda_v(\phi)) < 0$. (The case that $\phi = 0$ is excluded because the zero eigenvalue is excluded from our definition of asymptotic stability (see Section 2).) A similar reasoning holds for the second statement.
We are now in a position to state and prove the main theorem of this section. Recall that we identify \( \rho_{x,-1} \) and \( \rho_{v,-1} \) with \( \rho_{x,N-1} \) and \( \rho_{v,N-1} \) in Definition 2.1.

**Theorem 3.7** Suppose \( S^*_N \) is as defined in Definition 2.1. Then

\[ S^*_N \text{ is asymptotically stable for all } N \iff (\rho_{x,-1} = \rho_{x,1} \text{ and } g_x \rho_{x,0} < 0 \text{ and } g_v \rho_{v,0} < 0) \]

**Proof:** By Proposition 3.5, asymptotic stability implies \( \rho_{x,-1} = \rho_{x,1} \). Using Equation 2.1 we have

\[ \Re(\lambda_x(\phi)) = g_x \rho_{x,0}(1 - \cos \phi) \text{ and } \Re(\lambda_v(\phi)) = g_v \rho_{v,0}(1 - \cos \phi) \]

Proposition 3.6 i) then gives the left-to-right implication. Part ii) of that proposition says that if \( I_{x,1} = 0 \) then

asymptotic stability \( \iff g_x \rho_{x,0} < 0 \text{ and } g_v \rho_{v,0} < 0 \)

Thus the only possibility for the statement to be false is if \( S^*_N \) has \( I_{x,1} \neq 0 \) and is still asymptotically stable. But that violates Proposition 3.5. \( \square \)

### 4 Signal Velocities

The main result of this section is the determination of the signal velocity in asymptotically stable systems of Theorem 3.7. The signal velocity is the velocity with which information carrying disturbances (such as a short pulse) in the flock propagate. Signal velocities are difficult to determine. The reason is that a pulse consists of a superposition of (infinite) plane waves, typically each with a different phase velocity. The phase velocity of a plane wave solution is the velocity of the phase (or of a crest) and which cannot carry information. The fact that plane waves may have different phase velocities, causes a pulse to spread out over time (dispersion). Thus the determination of arrival time of the signal becomes problematic. For details we refer to [3].

For nearest neighbor systems of Definition 2.1 we define:

\[ a \equiv \frac{I_{v,1}^2}{4} + \frac{I_{x,2}}{2} = \frac{(\rho_{v,0} + 2\rho_{v,1})^2 g_v^2}{4} + \frac{-g_x \rho_{x,0}}{2} \]

**Remark:** From now on we will restrict our attention to (stable) systems satisfying the conditions of Definition 2.1 and the conclusions of Theorem 3.7. Note that for these systems \( a > 0 \). In order to simplify notation we will also (without loss of generality, because of Theorem 3.7) re-scale \( g_x \) and \( g_v \) so that the values of \( \rho_{x,0} \) and \( \rho_{v,0} \) are 1 from now on.

**Remark:** From the definitions it is clear that \( \nu_{N-m,\pm} \) can be identified with \( \nu_{-m,\pm} \) and that \( \nu_{-m,\pm} \) is the complex conjugate of \( \nu_{m,\pm} \). It will be convenient in this section to relabel these eigenvalues so that \( m \) runs from \( -(N-1)/2 \) to \( (N-1)/2 \). For simplicity of notation, we will however write \( \sum_{m=-(N-1)/2}^{(N-1)/2} m^2 \) as \( \sum_{m=-N/2}^{N/2} \).
Proposition 4.1 Let $S_N^*$ as in Definition 2.1 and Theorem 3.7. Then $a > 0$ and the eigenvalues $\nu_{m,\varepsilon}$ of $M_N$ can be expanded as (with $\varepsilon = \pm 1$ and $\theta = \frac{2\pi}{N}$):

\[
\begin{align*}
im\theta & \left( \frac{I_{v,1}}{2} + \varepsilon e^{1/2} \right) + \\
m^2\theta^2 & \left( -\frac{I_{v,2}}{4} - \varepsilon \left( \frac{I_{v,1}I_{v,2}}{4} + \frac{I_{v,3}}{6} \right) \right) + \\
im^3\theta^3 & \left( -\frac{I_{v,3}}{12} - \varepsilon \left( \frac{I_{v,1}I_{v,3} + I_{v,2}^2}{24} + \frac{I_{v,5}}{120} \right) \right) + \\
m^4\theta^4 & \left( \frac{I_{v,4}}{48} + \varepsilon \left( \frac{I_{v,2}I_{v,3} + I_{v,2}I_{v,4} + I_{v,5}}{48} \right) \right) + \ldots
\end{align*}
\]

Proof: Expand $\nu_{m,\pm}$ given in Proposition 3.2 in powers of $\theta$ using

\[
a \neq 0 \Rightarrow \sqrt{z - a} = \pm i\sqrt{a} \left( 1 - \frac{z}{2a} - \frac{z^2}{8a^2} - \frac{z^3}{16a^3} \ldots \right)
\]

After a substantial but straightforward calculation the result is obtained.

Lemma 4.2 For systems satisfying Definition 2.1 and Theorem 3.7, phase velocities are given by

\[
c_{m+} = -\Im(\nu_{m,-})/m\theta > 0 \quad \text{and} \quad c_{m-} = -\Im(\nu_{m,+})/m\theta < 0
\]

for $1 \leq m \leq \frac{N}{2}$ and $\nu_{m,\pm}$ given by Proposition 3.2.

Proof: Suppose $\mu_j = \alpha_j + i\beta_j$ for $j \in 1, 2$, then

\[
(x - \mu_1)(x - \mu_2) = x^2 - (\alpha_1 + \alpha_2 + i(\beta_1 + \beta_2))x + \alpha_1\alpha_2 - \beta_1\beta_2 + i(\alpha_1\beta_2 + \alpha_2\beta_1)
\]

Suppose the following conditions are satisfied: $\alpha_1$ and $\alpha_2$ are negative, $\alpha_1\alpha_2 - \beta_1\beta_2 < 0$, and $\alpha_1\beta_2 + \alpha_2\beta_1 = 0$. Then $\beta_1$ and $\beta_2$ have opposite signs.

Now consider the eigenvalues of $M_N$. By Proposition 3.2, they are roots $\nu_{m,\pm}$ of $\nu^2 - \lambda_{v,m}\nu - \lambda_{x,m} = 0$. Furthermore Theorem 3.7 (see also the remark at beginning of this section) implies that for $m \neq 0$

\[
\Re(\nu_{m,\pm}) < 0 \quad \text{and} \quad \lambda_{x,m} = g_x(1 - \cos m\theta) < 0
\]

Thus the previous paragraph implies that for $m \neq 0$ the imaginary parts of $\nu_{m,\pm}$ have opposite signs. Redefine (if necessary, see Proposition 3.2) the subscripts “+$\varepsilon$” and “$-\varepsilon$” of these eigenvalues $\nu_{m,\pm}$ so that $\nu_{m,+}$ has positive imaginary part, and $\nu_{m,-}$ has negative imaginary part.

We now derive phase velocities where + denotes going from agent “0” towards agent “N”. The expression for the $j$-th entry of the time-evolution of the $m$-th eigenvector of the Laplacian (see Proposition 3.1) is:

\[
z_j = e^{\Re(\nu_{m,\pm})t} e^{i\Im(\nu_{m,\pm})t} e^{ijm\theta}
\]

We can rewrite this as:

\[
z_j = e^{\Re(\nu_{m,\pm})t} e^{im\theta(j-c_{m,\pm}t)}
\]
where we have set:

\[ c_{m+} = \frac{-\Im(\nu_{m,-})}{m\theta} \quad \text{and} \quad c_{m-} = \frac{-\Im(\nu_{m,+})}{m\theta} \]

From Proposition 4.1 we see that the eigenvalues close to the origin form four branches which intersect at the origin. Namely \( \varepsilon \) can be +1 or -1, and the counter \( m \) can be positive or negative. This is illustrated in Figure 3.2. So for given \( |m| \) we get two phase velocities: one in each direction.

Lemma 4.3 For systems satisfying Definition 2.1 and Theorem 3.7, the phase velocities \( c_{me} \) of Lemma 4.2 can be expanded as (\( \varepsilon \in \{-1, 1\} \)):

\[
c_{me} = \frac{-g_v(1 + 2\rho_v,1)}{2} + \varepsilon \sqrt{\frac{g_v^2(1 + 2\rho_v,1)^2}{4} - g_x + m^2\theta^2 \left( \frac{g_v(1 + 2\rho_v,1)}{12} - \varepsilon \frac{2g_v^2(1 + 2\rho_v,1)^2 - g_x + \frac{3}{2}g_v^2}{24[g_v^2(1 + 2\rho_v,1)^2 - 2g_x]^{1/2}} + \varepsilon \frac{g_v^2(1 + 2\rho_v,1)}{16[g_v^2(1 + 2\rho_v,1)^2 - 2g_x]^{3/2}} \right) + \cdots}
\]

The real parts of the associated eigenvalues can be expanded as:

\[
\Re(\nu_{m,\varepsilon}) = m^2\theta^2 \left( \frac{g_v}{4} + \varepsilon \frac{g_v^2(1 + 2\rho_v,1)}{4[g_v^2(1 + 2\rho_v,1)^2 - 2g_x]^{1/2}} \right) + \cdots
\]

Proof: This follows from combining the last two results together with the conditions for stability of Theorem 3.7

Theorem 4.4 Let \( S'_N \) as in Definition 2.1 and Theorem 3.7 and \( c_{\pm} \equiv c_{0\pm} \) given in Lemma 4.3 (\( m = 0 \)). Fix \( 0 < \alpha < 1/2 < \beta < 2/3 \). For large \( N \) we have the following. When \( t \) is of the order of \( N/c_{\pm} \) or \( N/c_{\mp} \) the solution is given by traveling waves:

\[
z_j(t) = f_+(j - c_+t) + f_-(j - c_-t) = \sum_{m} a_m e^{im\theta(j-c_+t)} + \sum_{m} b_m e^{im\theta(j-c_-t)}
\]

where wave numbers \( m \) with \( 0 \leq |m| < N^\beta \) are undamped, wave numbers with \( N^\alpha \leq |m| < N^\beta \) are partially damped, and greater wave numbers are absent.

Proof: The initial conditions on \( z_j(0) \) and \( \dot{z}_j(0) \) uniquely determine a decomposition into the eigenvectors given in Proposition 3.2. Thus (see the remarks at the beginning of this section)

\[
z_j(t) = \sum_{m=-N/2}^{N/2} a_m e^{im\theta_j} e^{\nu_{m,+}t} + \sum_{m=-N/2}^{N/2} b_m e^{im\theta_j} e^{\nu_{m,-}t}
\]

We will now show that the first sum represents a signal traveling to the left (and by the same token the second, a signal traveling to the right).

Choose initial conditions so that all \( b_m \) are zero. Then

\[
z_j(0) = \sum_{m=-N/2}^{N/2} a_m e^{im\theta_j} = \left( \sum_{|m| < N^\alpha} + \sum_{|m| \in [N^\alpha, N^\beta)} + \sum_{|m| \geq N^\beta} \right) a_m e^{im\theta}
\]

Consider the signal after time \( N/c_+ \):

\[
z_j(N/c_+) = \left( \sum_{|m| < N^\alpha} + \sum_{|m| \in [N^\alpha, N^\beta)} + \sum_{|m| \geq N^\beta} \right) a_m e^{im\theta_j} e^{\nu_{m,+}N/c_+}
\]

The term \( e^{\nu_{m,+}N/c_+} \) must be worked out in each of three sums of the above equation. Using Lemma 4.2 we obtain

\[
e^{\nu_{m,+}N/c_+} = e^{\Re(\nu_{m,-})N/c_+} e^{-im\theta N_{c_+}}
\]
This term must be estimated in each of the three sums. This is done using the conditions of $\alpha$ and $\beta$ and Lemma 4.3.

In the first sum ($|m| < N^\alpha$) we see that $\Re(\nu_{m,-}) N$ tends to 1 as $N$ tends to infinity. Furthermore

$$e^{-im\theta N c_{m+}/c_+} = e^{-im\theta N} e^{-im\theta N(c_{m+} - c_+)/c_+}$$

The first factor on the right side equals 1. The second has an exponent proportional to $im^3\theta^3 N$ which tends to 0 as $N$ tends to infinity. This shows that

$$\lim_{N \to \infty} \sum_{|m| < N^\alpha} a_m e^{im\theta \nu_{m,N}} = \sum_{|m| < N^\alpha} a_m e^{im\theta}$$

and establishes that the signal velocity for low-frequency signals is indeed $c_+$.

Similar considerations show that the second sum travels with the same signal velocity, but undergoes some damping. The third sum undergoes substantial damping and is shown to tend to zero as $N$ tends to infinity. Thus high frequencies die out and $c_+$ is the only signal velocity. A similar calculation can be done for $c_-$. 

![Figure 4.1](color online) A representative figure for the calculation of the phase velocities for 500 agents. The values of the parameters are given in the figures. Light blue: $-\Im(\nu_{m,+}) m\theta$, Blue: $-\Im(\nu_{m,-}) m\theta$, Orange: $\Re(\nu_{m,+})$, Red: $\Re(\nu_{m,-})$.

The maximum phase velocities occur at $m = 0$. These are the signal velocities $c_+ > 0$ and $c_- < 0$ of Theorem 4.4.

**Remark:** It is interesting to note that the signal velocity we determine is actually equal to the group velocity at $m = 0$. The group velocity is defined as $\frac{dc_m}{d(m\theta)}$. It is not necessarily true that group velocity in these kinds of systems equals signal velocity. In the system studied in [19] they are different. See [3] for more information.

**Remark:** A similar argument as the one in Theorem 4.4 easily shows that eigenfunctions with wave numbers $m$ greater than $N^{0.5+\delta}$ will die out before $t = N/c_+$. Thus for considerations on time-scales longer than that, these are irrelevant. It also (conveniently) turns out that very often the greatest phase velocities are associated with the lowest wave numbers. A typical case is seen in Figure 4.1. One can show that in those asymptotically stable cases where $\rho_{e,1}$ is close to $-1/2$, we have that $c_{m\pm}$ has a local maximum at $m = 0$. In fact Lemma 4.3 implies that for $\rho_{e,1} = -1/2$:

$$c_{m\pm} = \varepsilon \sqrt{-gx} + \varepsilon m^2 g^2 \left( \frac{gx - g_0^2}{24\sqrt{-2gx}} \right) + \cdots$$

which has a local maximum at $m = 0$. 

9
5 Conclusion

Though experiments with cars have been done on circular roads (see [14]), our interest in the system with periodic boundary conditions of \( S_N^* \) as defined in Definition 2.1 stems from the applicability to traffic systems with non-periodic boundary conditions. The only reason we study the former is that they enable us to analyze how disturbances propagate, and — under the assumption that this propagation does not depend on boundary conditions — apply that to the latter systems to find the transients. Some remarks on how that works are given in the Introduction and is the subject of [4]. A relative novelty here is that we consider all strictly decentralized systems, not just symmetric ones.

In Section 3 we give precise conditions on the parameters so that decentralized systems with periodic boundary condition are asymptotically stable. In its generality stated here this is new, though related observations have been made in [2] and [11]. The main importance here is that we use these conditions on the parameters to show that in these systems disturbances travel with constant a constant signal velocity, and — as our main result — we determine that velocity in Section 4. This explains why in these cases, approximations of these systems with large \( N \), by the wave equation are successful (see for example [2]). It can be shown however that for other parameter values diffusive behavior may occur (see [4]).

Finally we test our prediction of the signal velocity in a numerical experiment. The result can be seen in Figure 5.1. Here agent number \( 0 = N \) at time \( t = 0 \) is given a different initial velocity from the others. That signal propagates forward (in the direction 1,2,3,...) through the flock as well as backwards (in the direction \( N-1, N-2, N-3,... \)). In figure we color coded according to the speed of the agents, who are stationary until the signal reaches them. In black we mark when the signal is predicted to arrive. One can see the excellent agreement.

![Figure 5.1](image)

**Figure 5.1:** (color online) The orbits of 200 cars with specific choices for the parameters. (\( \Delta \) is the desired distance between cars.) At time 0 agent 0 receives a different initial condition. They are color coded according to the velocity of the agent. The black curves indicate the theoretical position of the wavefront calculated via the signal velocity. Note that these velocities depend on the direction, and that the signal velocity is measured in number of cars per time unit. Due to the different velocities of the cars, these curves are not straight lines.

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