Vector coherent state theory of the generic representations of \( \mathfrak{so}(5) \) in an \( \mathfrak{so}(3) \) basis

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For applications of group theory in quantum mechanics, one generally needs explicit matrix representations of the spectrum generating algebras that arise in bases that reduce the symmetry group of some Hamiltonian of interest. Here we use vector coherent state techniques to develop an algorithm for constructing the matrices for arbitrary finite-dimensional irreps of the SO(5) Lie algebra in an SO(3) basis. The SO(3) subgroup of SO(5) is defined by Regarding SO(5) as linear transformations of the five-dimensional space of an SO(3) irrep of angular momentum two. A need for such irreps arises in the nuclear collective model of quadrupole vibrations and rotations. The algorithm has been implemented in MAPLE, and some tables of results are presented.

I. INTRODUCTION

A vector coherent state (VCS) representation is a representation of a group (or Lie algebra) on a space of vector-valued functions. It is a representation induced from a multi-dimensional representation of a subgroup. Such representations have been used widely in the construction of explicit representations of Lie algebras and Lie groups in the construction of shift tensors, and for the computation of Clebsch-Gordan coefficients for reducing tensor product representations.

The VCS construction for a representation of a group \( G \) involves two subgroups which play quite different roles. A so-called “intrinsic” subgroup (sometimes called the “core” subgroup) acts in a known way on a subspace of the representation of interest. A second “orbiter” subgroup acts upon this subspace to generate the larger representation of the group \( G \). A prototypical example of the construction is that for the dynamical group of the rigid rotor given by the semi-direct product \( G = \mathbb{R}^3 \rtimes \text{SO}(3) \) of an intrinsic \( \mathbb{R}^3 \) subgroup, which describes the quadrupole moments (hence the shape) of an object, and an orbiter group \( \text{SO}(3) \), corresponding to physical rotations of the object, which describes its possible orientations. The quantum mechanics of such a rotor are then described by the unitary irreducible representations (irreps) of \( G \).

The key requirement is that the Lie algebras of the intrinsic and orbiter groups, together with those elements of the complexified Lie algebra of \( G \) which leave the intrinsic space invariant, span the complex extension of the Lie algebra. Finding such groups is often easier in the complex extension of \( G \). In many cases there are then mathematically naturally choices of intrinsic and orbiter subgroups for which the VCS construction of an induced representation is straightforward. Unfortunately a mathematically natural choice often produces a representation in a basis which is not adapted to the symmetries of a physical problem. A goal of this paper is to show how to construct representations of \( \mathfrak{so}(5) \) in a basis which reduces a physically relevant \( \mathfrak{so}(3) \) \( \subset \mathfrak{so}(5) \) subalgebra.

The group SO(5) and its Lie algebra \( \mathfrak{so}(5) \) arise in many physical contexts. For example, they are needed for the classification of states in the Bohr-Mottelson model and Interacting Boson model of nuclear collective states. They arise in a charge-independent pairing theory and in the use of isospin for the classification of nuclear shell model basis states. They have also been used for the study of algebraic many-body equations of motion methods and high-temperature superconductivity. Depending on the context, \( \mathfrak{so}(5) \) irreps may be required in an \( \mathfrak{su}(2) \) (e.g., isospin) or an \( \mathfrak{so}(3) \) (angular momentum) basis. The isospin \( \mathfrak{su}(2) \) algebra is embedded in \( \mathfrak{so}(5) \) as a subalgebra \( \mathfrak{su}(2) \subset \mathfrak{so}(2) \times \mathfrak{su}(2) \cong \mathfrak{so}(4) \subset \mathfrak{so}(5) \). Thus, the required \( \mathfrak{so}(5) \) \( \subset \mathfrak{su}(2) \) irreps are given in a basis that reduces the Gel’fand chain \( \mathfrak{so}(5) \supset \mathfrak{so}(4) \supset \mathfrak{so}(3) \supset \mathfrak{so}(2) \). Such irreps were constructed years ago and reconstructed more simply by VCS methods in Refs. \(^c\) \(^d\) \(^e\). Note, however, that the \( \mathfrak{so}(3) \subset \mathfrak{so}(4) \) subalgebra of the SO(3) group in the Gel’fand chain is not the same as the above-mentioned angular momentum algebra which generates a “geometrical” \( \mathfrak{so}(3) \subset \mathfrak{so}(5) \) subgroup of rotations of an associated three-dimensional space; as a result the construction of \( \mathfrak{so}(5) \) irreps in an angular-momentum basis is more challenging.

The so-called one-rowed representations that occur in the decomposition of the Hilbert space of the 5-dimensional harmonic oscillator can be inferred in an \( \mathfrak{so}(3) \) basis from the results of ChaóN, Moshinsky and others or from the \( \mathfrak{so}(5) \) hyperspherical harmonics and Clebsch-Gordan coefficients given in Ref. \(^f\). An explicit VCS (vector coherent state) construction of irreps with highest weights of the type \((v,0)\) and \((0,f)\) (this notation is explained below) was...
given by Rowe and Hecht [22].

In this paper we give a systematic construction of the generic \((v,f)\) irreps in an \(\mathfrak{so}(3)\) basis. In addition to its obvious relevance to the representation theory of \(\mathfrak{so}(5)\), the construction is a prototype for a relatively sophisticated application of VCS theory.

II. VECTOR COHERENT STATE REPRESENTATIONS

Vector coherent state (VCS) theory is a generalisation of standard (scalar) coherent state theory [2,3]. It was introduced [1] for the purpose of providing an explicit systematic construction of the irreducible unitary representations of the compact and non-compact symplectic Lie algebras. Simplicity and efficiency were achieved in the construction by making use of the already well-known representation theory of the unitary subalgebras. Important aspects of the theory were also introduced independently by Deenen and Quesne in their partial coherent state representations [24]. Subsequently, VCS has been used to construct representations of a large number of Lie algebras, groups, and superalgebras (cf. Ref. [6] for a review). Early applications of VCS theory gave realisations of the so-called holomorphic representations (reviewed by Hecht [25]). A more general class of VCS representations was later used in the construction of \(\mathfrak{su}(3)\) irreps in an \(\mathfrak{so}(3)\) basis [26]. It has also been shown that VCS theory is compatible with the theory of induced representations [6] and the theory of geometric quantisation [27]. A more general perspective on the theory was given in Ref. [28].

The construction of the finite-dimensional irreps of \(\mathfrak{so}(5)\) in an \(\mathfrak{so}(3)\) basis has much in common with the construction of irreps of \(\mathfrak{su}(3)\) in an \(\mathfrak{so}(3)\) basis. The \(\mathfrak{su}(3)\) Lie algebra is spanned by the components of \(\mathfrak{so}(3)\) tensors of angular momentum \(L = 1\) and \(L = 2\), while the \(\mathfrak{so}(5)\) Lie algebra is spanned by \(\mathfrak{so}(3)\) tensors of angular momentum \(L = 1\) and \(L = 3\). However, whereas for \(\mathfrak{su}(3)\) it was possible to use scalar coherent state wave functions, a special case of VCS functions, it proves to be essential to use vector-valued wave functions for \(\mathfrak{so}(5)\).

The VCS theory of \(\mathfrak{su}(3)\) relies on the fact that the carrier space for an \(\mathfrak{su}(3)\) irrep is spanned by the set of states generated by \(\text{SO}(3)\) rotations of a highest weight state. The carrier space for a generic irrep of \(\mathfrak{so}(5)\) is generated by \(\text{SO}(3)\) rotations of a set of highest grade states.

A. Highest grade states for an \(\mathfrak{so}(5)\) irrep

The \(\mathfrak{so}(5)\) Lie algebra is semisimple, of rank 2, and has the root diagram shown in figure 1. It is conventional to separate the roots of a semisimple Lie algebra into positive and negative roots and to regard the corresponding root vectors as raising and lowering operators, respectively. Every irrep is then characterised by a highest weight state.

Let \(\alpha_1\) be the root corresponding to the vector \(T_+\) and \(\alpha_2\) be that corresponding to \(F_+\). Then in the standard labelling scheme, the highest weight \(\Lambda\) for an irrep is given by two integers \(\lambda_1\) and \(\lambda_2\) such that \(\Lambda = (\lambda_1 + \frac{1}{2}\lambda_2)\alpha_1 + (\lambda_1 + \lambda_2)\alpha_2\). In keeping with the nuclear structure notation, we use the label \(v = \lambda_1\) (the “seniority”), and find it convenient to introduce the half-integer \(f = \frac{1}{2}\lambda_2\) since it also labels a \(\mathfrak{u}(2)\) irrep, as is shown below. Thus, we label an \(\mathfrak{so}(5)\) irrep \((v,f)\); the highest weight is then given by \(\Lambda = (v + f)(\alpha_1 + \alpha_2) + f\alpha_2\).

For present purposes, we separate the root vectors into grade raising, grade conserving, and grade lowering operators as shown in figure 2. The horizontal grade-conserving root vectors \(\{F_\pm, F_0, X_0\}\) then define what we shall refer to as an intrinsic or core \(\mathfrak{u}(2)\) subalgebra. This grading of the \(\mathfrak{so}(5)\) Lie algebra generates a grading of any irrep. Each

![FIG. 1: Root diagram for the \(\mathfrak{so}(5)\) Lie algebra.](image)
irrep has a set of highest grade states \( \{|(vf)m\}\} \) that are annihilated by the grade-raising operators \( \hat{T}_+, \hat{X}_+, \) and \( \hat{S}_+ \), and carry an irrep of the above-mentioned intrinsic \( \mathfrak{u}(2) \) algebra; the highest grade states satisfy the equations

\[
\begin{align*}
\hat{S}_+|(vf)m\rangle &= \hat{X}_+|(vf)m\rangle = \hat{T}_+|(vf)m\rangle = 0, \\
\hat{X}_0|(vf)m\rangle &= (v + f)|(vf)m\rangle, \\
\hat{F}_0|(vf)m\rangle &= \sqrt{(f \pm m)(f \pm m + 1)}|(vf)(m \pm 1)\rangle.
\end{align*}
\]

The weights for the highest grade states of a generic irrep of \( \mathfrak{so}(5) \) are as illustrated in figure 2. A set of wave functions \( \{\xi^{(vf)}_m; m = -f, \ldots, f\} \) for these highest grade states are regarded as intrinsic wave functions in the VCS construction — it is in the Hilbert space of these intrinsic functions that the VCS wave functions take their vector values.

**B. Holomorphic VCS wave representations**

Let \( |\psi\rangle \) be a state in the carrier space of an \( \mathfrak{so}(5) \) irrep \( (vf) \). Then a holomorphic VCS wave function is defined for this state by

\[
\Psi(z) = \sum_m \xi^{(vf)}_m \langle(vf)m|e^{\hat{z}}|\psi\rangle,
\]

where

\[
\hat{z} = z_1 \hat{S}_+ + z_2 \hat{X}_+ + z_3 \hat{T}_+
\]

and \( z = (z_1, z_2, z_3) \) is a set of complex numbers. The corresponding VCS representation \( \Gamma \) of the \( \mathfrak{so}(5) \) Lie algebra is defined by

\[
[\Gamma(X)\Psi](z) = \sum_m \xi^{(vf)}_m \langle(vf)m|e^{\hat{z}}X|\psi\rangle, \quad X \in \mathfrak{so}(5).
\]

Such holomorphic representations are natural generalisations of the familiar Bargmann-Segal representations [29] of the Heisenberg-Weyl algebras. They were the first to be considered in the formulation of VCS theory [1]. However, in practical applications they are not always the most useful. In particular, they do not reduce the \( \mathfrak{so}(3) \subset \mathfrak{so}(5) \) angular momentum subalgebra.

**C. VCS wave functions in an SO(3)-coupled basis**

The group \( \text{SO}(3) \) can be embedded as a subgroup in \( \text{SO}(5) \) in many ways. We consider the \( \text{SO}(3) \) subgroup defined up to conjugation by regarding \( \text{SO}(5) \) as a group of orthogonal transformations of the five-dimensional carrier space
for an \( L = 2 \) irrep of SO(3). This embedding is motivated by the rotational properties of the five quadrupole degrees of freedom in the nuclear collective model. The construction of an SO(3)-coupled basis for a VCS irrep of \( \mathfrak{so}(5) \) then parallels a similar construction of an SO(3)-coupled basis for a VCS irrep of \( \mathfrak{su}(3) \). \( \mathfrak{so}(5) \) construction makes use of the following theorem, which constrains the choice of the SO(3) subgroup.

**Theorem 1:** Provided no \( \mathfrak{so}(3) \) angular momentum operator lies within the \( \mathfrak{u}(2) \) intrinsic subalgebra, the set of states \( \{ \hat{R}(\Omega)|\psi\rangle; m = -f, \ldots, f; \Omega \in \text{SO}(3) \} \) obtained by all SO(3) rotations of an orthonormal basis for the highest grade subspace spans the Hilbert space for the \( \mathfrak{so}(5) \) irrep \((vf)\).

**Proof:** The set of states generated by repeated application of the lowering operators \( \{ \hat{S}_-, \hat{X}_-, \hat{T}_- \} \) to the highest grade states spans the Hilbert space of the irrep. Now, if \( \{ \hat{L}_i; i = 1, 2, 3 \} \) is a hermitian basis for the \( \mathfrak{so}(3) \subset \mathfrak{so}(5) \) subalgebra, then each \( \hat{L}_i \) can be expanded \( \hat{L}_i = \hat{L}_i^- + \hat{L}_i^0 + \hat{L}_i^+ \), where \( \hat{L}_i^- \) is a grade lowering operator, \( \hat{L}_i^0 \) is of grade zero, and \( \hat{L}_i^+ \) is a grade raising operator. By hermiticity, if \( \hat{L}_i \) has a non-zero component \( \hat{L}_i^+ \), it must also have a non-zero \( \hat{L}_i^- \) component. Thus, if no \( \hat{L}_i \) lies in the zero grade \( \mathfrak{u}(2) \) subalgebra, then each \( \hat{L}_i \) must have a nonzero \( \hat{L}_i^- \) component. By linear independence, it must be that the span of \( \{ \hat{L}_i^- \} \) equals the span of \( \{ \hat{S}_-, \hat{X}_-, \hat{T}_- \} \). QED

This theorem means that an arbitrary state \(|\psi\rangle\) in an irrep \((vf)\) is defined by the set of overlaps \(\langle (vf)m|\hat{R}(\Omega)|\psi\rangle; m = -f, \ldots, f; \Omega \in \text{SO}(3) \)\), provided that the SO(3) subgroup is chosen as required by the theorem, which we assume from now on. It also means that, if \(\{ (vf)\tau \mathcal{R} \} \) is an SO(3)-coupled basis for an \( \mathfrak{so}(5) \) irrep and \(\{ (vf)m \}\), with wave functions \(\{ \xi_{m}^{(vf)} \}\), is an orthonormal \( \mathfrak{u}(2) \) basis of highest grade states for this irrep, then the basis states \(\{ (vf)\tau \mathcal{R} \} \) have VCS wave functions given as vector-valued functions over SO(3) by

\[
\Phi_{\tau \mathcal{R}LM}(\Omega) = \sum_{m} \xi_{m}^{(vf)} \langle (vf)m|\hat{R}(\Omega)-(vf)\tau \mathcal{R} \rangle .
\]

These wave functions are very much like rotor-model wave functions \([11]\). Indeed, with basis states chosen to have good SO(3) transformation properties, they can be expanded

\[
\Phi_{\tau \mathcal{R}LM}(\Omega) = \sum_{mK} \xi_{m}^{(vf)} \langle (vf)m|(vf)\tau \mathcal{R} \rangle \mathcal{D}^{L}_{K M}(\Omega) ,
\]

where \( \mathcal{D}^{L}(\Omega) \) is a Wigner rotation matrix. It follows that a basis state \((vf)\tau \mathcal{R} \) is characterised by the set of expansion coefficients

\[
b_{mK}^{(vf)} (\tau \mathcal{R}) = \langle (vf)m|(vf)\tau \mathcal{R} \rangle .
\]

The following gives a systematic procedure for determining these coefficients and for deriving the transformations of these coefficients by elements of the \( \mathfrak{so}(5) \) Lie algebra as defined by the VCS representation

\[
[\Gamma(X)\Phi_{\tau \mathcal{R}LM}(\Omega)] = \sum_{m} \xi_{m}^{(vf)} \langle (vf)m|\hat{R}(\Omega)X|(vf)\tau \mathcal{R} \rangle , \quad X \in \mathfrak{so}(5) .
\]

**III. REPRESENTATION SPACES FOR \( \mathfrak{so}(5) \)**

**A. A subspace of harmonic oscillator states**

Irreps of \( \mathfrak{so}(5) \) can be built up from its two fundamental irreps with highest weights \((10)\) and \((0, \frac{1}{2})\). The former is the fundamental five-dimensional irrep, the latter is the fundamental four-dimensional irrep. Both weight diagrams are shown in figure 4. These fundamental irreps are carried by the spaces generated by the raising operators \(\{ \eta_{\nu}^{\dagger}; \nu = 0, \pm 1, \pm 2 \}\) of a five-dimensional harmonic oscillator with symmetry group U(5) and by the raising operators \(\{ \zeta_{m}^{\dagger}; m = \pm \frac{1}{2}, \pm \frac{3}{2} \}\) of a four-dimensional harmonic oscillator with symmetry group U(4), respectively. The operators \(\{ \eta_{\nu}^{\dagger} \}\) and \(\{ \zeta_{m}^{\dagger} \}\), together with the corresponding lowering operators, satisfy the usual boson commutation relations

\[
[[\eta^{\nu}, \eta^{\mu}] = \delta_{\mu\nu} I , \quad [[\eta^{\mu}, \eta^{\nu}] = [\eta^{\mu}, \eta^{\nu}] = 0 , \quad (11)
\]

\[
[[\zeta^{m}, \zeta^{n}] = \delta_{mn} I , \quad [[\zeta^{m}, \zeta^{n}] = [\zeta^{m}, \zeta^{n}] = 0 , \quad (12)
\]

where we use the notation \(\eta^{\nu} = (\eta_{\nu}^{\dagger})^{\dagger}\) and \(\zeta^{m} = (\zeta_{m}^{\dagger})^{\dagger}\).
The invariants of $U(5)$ and $U(4)$ are given by their respective number operators

\[ \hat{n}_\eta = \eta^\dagger \cdot \eta = \sum_\nu \eta^\dagger_\nu \eta^\nu, \quad \hat{n}_\zeta = \zeta^\dagger \cdot \zeta = \sum_m \zeta^\dagger_m \zeta^m. \] (13)

For the natural $SO(3)$ embedding (defined to within conjugation), we can regard these number operators as coupled products, e.g.,

\[ \hat{n}_\zeta = 2 \sum_m (\frac{3}{2}, m, \frac{3}{2}, m) |0, 0\rangle \zeta^\dagger_m \zeta^m = \sum_m (-1)^{\frac{1}{2}+m} \zeta^\dagger_m \zeta^m, \] (14)

where

\[ \zeta^m = (-1)^{\frac{1}{2}+m} \zeta^m. \]

Similarly, we can define

\[ \eta_{-\nu} = (-1)^{\nu} \eta^\nu. \] (15)

The fundamental five-dimensional irrep of the group $SO(5)$ can be realised as the group of special orthogonal transformations of the creation operators $\{ \eta^\dagger_\nu ; \nu = 0, \pm 1, \pm 2 \}$ that leave invariant the quantity $\eta^\dagger \cdot \eta^\nu = \sum_\nu (-1)^\nu \eta^\dagger_\nu \eta^\nu$. This realisation exhibits $SO(5)$ as a subgroup of $U(5)$. The four-dimensional fundamental irrep is a spinor irrep of $SO(5)$. It can be realised as the group of special orthogonal transformations of the (boson) creation operators $\{ \zeta^\dagger_m ; m = \pm \frac{1}{2}, \pm \frac{3}{2} \}$ that leave invariant the quantity $\zeta^\dagger \cdot \zeta^m = \sum_m (-1)^{\frac{1}{2}+m} \zeta^\dagger_m \zeta^m$. This realisation exhibits $USp(4)$, the two-fold cover of $SO(5)$, as a subgroup of $U(4)$. Because every irrep of a group is contained in a tensor product of copies of its fundamental irreps, it follows that every irrep of $so(5)$ can be realised on a subspace of the tensor product $\mathbb{H} = \mathbb{H}^{(5)} \otimes \mathbb{H}^{(4)}$, where $\mathbb{H}^{(n)}$ is the Hilbert space of an $n$-dimensional harmonic oscillator. Highest grade states for an $so(5)$ irrep in $\mathbb{H}$ are given by

\[ |(vf)m\rangle = (\frac{\zeta^1_{f/2} \zeta^1_{f+m/2} \eta_{f+m}^{(v)\nu}}{\sqrt{(f+m)!(f-m)!v!}} |0\rangle, \quad m = -f, \ldots, f. \] (16)

B. A model space of VCS wave functions

A model space for a compact Lie group $G$ is a representation of $G$ which is a direct sum of irreps comprising precisely one copy from each equivalence class of irreps $[30]$. As emphasised by Biedenharn and Flath $[31]$, a model space provides a valuable framework for a realisation of the tensor algebra of the group. We now show that a Hilbert space of VCS wave functions for the states of $\mathbb{H}$ provides a model space with very useful properties.

It follows from Eq. (7) that, provided the conditions of Theorem 1 are satisfied, any state $|\psi\rangle$ in $\mathbb{H}$ has VCS wave function $\Psi$ given by

\[ \Psi(\Omega) = \sum_{vf,m} \xi_{m}^{(vf)} \langle(vf)m|\hat{R}(\Omega)|\psi\rangle. \] (17)

Now observe that, if the wave functions $\{ \xi_{m}^{(vf)} \}$ of the highest grade states are expressed in Bargmann form as the holomorphic functions

\[ \xi_{m}^{(vf)}(x,y) = \frac{x_{1}^{f+m} x_{2}^{-m} y^{v}}{\sqrt{(f+m)!(f-m)!v!}}, \] (18)
then, using Eq. (16), the \(u(2)\)-intertwining operator \(\sum_{v f m} \xi_m^{(v f)} \langle (v f) m \rangle\) can be expressed

\[
\sum_{v f m} \xi_m^{(v f)} \langle (v f) m \rangle = \langle 0 | \hat{\mathcal{X}} |angle,
\]

(19)

where

\[
\hat{\mathcal{X}} = x_1 \zeta^{3/2} + x_2 \zeta^{1/2} + y \eta^2.
\]

(20)

Thus, any state \(|\psi\rangle\) in \(\mathbb{H}\) has VCS wave function defined by

\[
\Psi(\Omega) = \langle 0 | \hat{\mathcal{X}} R(\Omega) | \psi \rangle.
\]

(21)

Let \(\mathcal{F}^{(v f)}\) denote the Hilbert space of VCS wave functions for an \(\mathfrak{so}(5)\) irrep of highest weight \(\langle v f \rangle\), relative to the inner product inherited from that of \(\mathbb{H}\). The space of all VCS wave functions for states in \(\mathbb{H}\) is then the direct sum

\[
\mathcal{F} = \bigoplus_{\langle v f \rangle} \mathcal{F}^{(v f)},
\]

(22)

of all such Hilbert spaces. Hence, by construction, it is a model space for \(\mathfrak{so}(5)\). The following theorem, which generalises a theorem of Rowe and Hecht [22], shows that \(\mathcal{F}\) is also a ring and that it can be generated from the VCS wave functions of the fundamental irreps \((1,0)\) and \((0,1)\).

**Theorem 2:** If \(\Phi = \Phi_1 \circ \Phi_2\) is the model product of VCS wave functions defined by

\[
\Phi(x, y, \Omega) = \Phi_1(x, y, \Omega) \Phi_2(x, y, \Omega),
\]

(23)

then, if \(\Phi_1\) is in \(\mathcal{F}^{(\nu_1 \nu_1)}\) and \(\Phi_2\) is in \(\mathcal{F}^{(\nu_2 \nu_2)}\), the product \(\Phi\) is in \(\mathcal{F}^{(\nu_1 + \nu_1, \nu_1 + \nu_1)}\). Moreover, if \(\mathcal{F}^{(\nu_1 \nu_1)} \circ \mathcal{F}^{(\nu_2 \nu_2)}\) denotes the linear span of model products of the functions of \(\mathcal{F}^{(\nu_1 \nu_1)}\) with the functions of \(\mathcal{F}^{(\nu_2 \nu_2)}\), then

\[
\mathcal{F}^{(\nu_1 \nu_1)} \circ \mathcal{F}^{(\nu_2 \nu_2)} = \mathcal{F}^{(\nu_1 + \nu_1, \nu_1 + \nu_1)}.
\]

(24)

**Proof:** Let \(\tilde{Z}^{(\nu_1 \nu_1)}_1\) be a homogeneous polynomial of degree \(\nu_1\) in the \(\{\eta^1\}\) raising operators and degree \(2 \nu_1\) in the \(\{\zeta_m\}\) raising operators which creates a state with VCS wave function

\[
\Phi^{(\nu_1 \nu_1)}(\Omega) = \langle 0 | \hat{\mathcal{X}} R(\Omega) \tilde{Z}^{(\nu_1 \nu_1)}_1 | 0 \rangle.
\]

(25)

Let

\[
\tilde{Z}^{(\nu_1 \nu_1)}_2(\Omega) = R(\Omega) \tilde{Z}^{(\nu_1 \nu_1)}_1 R(\Omega) \Omega^{-1},
\]

(26)

denote the corresponding rotated operator. Then

\[
\Phi^{(\nu_1 \nu_1)}(\Omega) = \langle 0 | \hat{\mathcal{X}} \tilde{Z}^{(\nu_1 \nu_1)}_1(\Omega) e^{-\hat{\mathcal{X}}} | 0 \rangle
\]

\[= \langle 0 | \left( \tilde{Z}^{(\nu_1 \nu_1)}_1(\Omega) + [\hat{\mathcal{X}}, \tilde{Z}^{(\nu_1 \nu_1)}_1(\Omega)] + \frac{1}{2!} [\hat{\mathcal{X}}, [\hat{\mathcal{X}}, \tilde{Z}^{(\nu_1 \nu_1)}_1(\Omega)]] + \cdots \right) | 0 \rangle.
\]

(27)

The successive terms in the sequence of multiple commutators inside the brackets are homogenous polynomials of decreasing degree in the raising operators. This sequence terminates with a polynomial of degree zero (a number) at the \((\nu_1 + 2 \nu_1)\)th term. Moreover, this last term is the only term in the sequence with non-vanishing vacuum expectation value. It follows that

\[
\Phi^{(\nu_1 \nu_1)}(\Omega) = \frac{1}{(\nu_1 + 2 \nu_1)!} \left[ \hat{\mathcal{X}}, [\hat{\mathcal{X}}, [\cdots, \tilde{Z}^{(\nu_1 \nu_1)}_1(\Omega)] \cdots] \right].
\]

(28)

If \(\tilde{Z}^{(\nu_2 \nu_2)}_2\) is similarly defined for the wave function \(\Phi_2^{(\nu_2 \nu_2)}\), then the wave function \(\Phi = \Phi_1^{(\nu_1 \nu_1)} \circ \Phi_2^{(\nu_2 \nu_2)}\) has values given by

\[
\Phi(\Omega) = \langle 0 | e^{\hat{\mathcal{X}}} \tilde{Z}^{(\nu_1 \nu_1)}_1(\Omega) e^{-\hat{\mathcal{X}}} \tilde{Z}^{(\nu_2 \nu_2)}_2(\Omega) e^{-\hat{\mathcal{X}}} | 0 \rangle = \langle 0 | e^{\hat{\mathcal{X}}} R(\Omega) \tilde{Z}^{(\nu_1 \nu_1)}_1 \tilde{Z}^{(\nu_2 \nu_2)}_2 | 0 \rangle.
\]

(29)
The state \( \hat{Z}_1^{(v_1,f_1)} \hat{Z}_2^{(v_2,f_2)} |0\) does not necessarily belong to an irreducible \( \mathfrak{so}(5) \) subspace. However, it is of degree \((v_1 + v_2) \) and \(2(f_1 + f_2)\) in the \( \{ \eta_1^\dagger \} \) and \( \{ \zeta_1 \} \) operators, respectively, and, therefore, only the component of the state \( \hat{Z}_1^{(v_1,f_1)} \hat{Z}_2^{(v_2,f_2)} |0\) that does lie within the irreducible \( \text{SO}(5) \) subspace of highest weight \((v_1 + v_2, f_1 + f_2)\) will have non-zero overlap with \( \langle 0 | e^\hat{R}(\Omega) \rangle \). Thus, \( \Phi \) is a VCS wave function for a state belonging to an \( \text{SO}(5) \) irrep of highest weight \((v_1 + v_2, f_1 + f_2)\) as claimed by the theorem. The second part of the theorem follows from the observation that, if \( \{ \hat{Z}_a^{(v_f)} \} \) is a basis for the linear space \( P^{(v_f)} \) of homogeneous polynomials of degree \( v \) in \( \{ \eta_\nu \} \) and degree \( 2f \) in \( \{ \zeta_m \} \), then the space \( P^{(v_1+v_2,f_1+f_2)} \) is spanned by the products \( \{ \hat{Z}_a^{(v_1,f_1)} \hat{Z}_b^{(v_2,f_2)} \} \). QED

The theorem shows that \( \mathcal{F} \) can be constructed from the two fundamental irreps, and that this process will generate all the irreps of \( \mathfrak{so}(5) \). The importance of a model space is that, since it is multiplicity free, calculations performed in \( \mathcal{F} \) are not complicated by any need to keep track of equivalent copies of the same \( \mathfrak{so}(5) \) irreps.

IV. BASES FOR THE \( \mathfrak{so}(5) \) LIE ALGEBRA

The VCS representations of \( \mathfrak{so}(5) \) in an \( \text{SO}(3) \)-coupled basis make use of two bases for the \( \mathfrak{so}(5) \) Lie algebra: a Cartan basis of root vectors and a basis of components of \( \mathfrak{so}(3) \) tensors.

A. A Cartan basis

Starting from two copies of the \((10)\) irrep, one spanned by the creation operators \( \{ \eta_\nu \} \) and one spanned by the annihilation operators \( \{ \eta_\nu^\dagger \} \), one can construct a realisation of the the \( \mathfrak{so}(5) \) Lie algebra, which carries a \((01)\) irrep, by taking an antisymmetric tensor product of the two. This gives the \( \mathfrak{so}(5) \) root vectors of figure 2 as a subset of \( \mathfrak{u}(5) \) operators

\[
\hat{S}_+ = \eta_1^\dagger \eta_2 - \eta_2^\dagger \eta_1 = \eta_1^\dagger \eta_2^2 - \eta_2^\dagger \eta_1^2, \\
\hat{T}_+ = \eta_{-1}^\dagger \eta_2 - \eta_2^\dagger \eta_{-1} = \eta_{-1}^\dagger \eta_2^2 - \eta_2^\dagger \eta_{-1}^2, \\
\hat{X}_+ = \sqrt{2} (\eta_2^\dagger \eta_0 - \eta_0^\dagger \eta_2) = \sqrt{2} (\eta_2^\dagger \eta_0^0 - \eta_0^\dagger \eta_2^0), \\
\hat{F}_+ = \sqrt{2} (\eta_1^\dagger \eta_0 - \eta_0^\dagger \eta_1) = \sqrt{2} (\eta_1^\dagger \eta_0^0 + \eta_0^\dagger \eta_1^0), \\
\hat{S}_- = (\hat{S}_+)^\dagger, \quad \hat{T}_- = (\hat{T}_+)^\dagger, \quad \hat{X}_- = (\hat{X}_+)^\dagger, \quad \hat{F}_- = (\hat{F}_+)^\dagger.
\]

A basis for the Cartan subalgebra is given by

\[
\hat{X}_0 = \frac{1}{2} [ \hat{X}_+, \hat{X}_-] = \eta_2^\dagger \eta_2 - \eta_2^\dagger \eta_2, \\
\hat{F}_0 = \frac{1}{2} [ \hat{F}_+, \hat{F}_-] = \eta_1^\dagger \eta_1 - \eta_1^\dagger \eta_1.
\]

A realisation of the \( \mathfrak{so}(5) \) algebra as a subalgebra of \( \mathfrak{u}(4) \) is similarly obtained from the symmetric tensor product of two copies of the \((0, \frac{1}{2})\) irrep:

\[
\hat{S}_+ = -\zeta_3^\dagger \zeta_3^{-3/2}, \quad \hat{T}_+ = \zeta_1^\dagger \zeta_1^{-1/2}, \\
\hat{X}_+ = \zeta_1^\dagger \zeta_3^{-3/2} - \zeta_3^\dagger \zeta_1^{-1/2}, \\
\hat{F}_+ = \zeta_3^\dagger \zeta_1^1/2 + \zeta_1^\dagger \zeta_3^{-1/2}, \\
\hat{X}_0 = \frac{1}{2} (\zeta_3^\dagger \zeta_3^{3/2} + \zeta_1^\dagger \zeta_1^{1/2} - \zeta_1^\dagger \zeta_3^{1/2} - \zeta_3^\dagger \zeta_1^{1/2} - \zeta_1^\dagger \zeta_3^{-1/2} - \zeta_3^\dagger \zeta_1^{-1/2} - \zeta_3^\dagger \zeta_3^{-1/2}) \\
\hat{F}_0 = \frac{1}{2} (\zeta_3^\dagger \zeta_3^{3/2} - \zeta_1^\dagger \zeta_1^{1/2} + \zeta_1^\dagger \zeta_3^{1/2} - \zeta_3^\dagger \zeta_1^{1/2} - \zeta_3^\dagger \zeta_3^{1/2} - \zeta_1^\dagger \zeta_3^{-1/2} - \zeta_3^\dagger \zeta_3^{-1/2}).
\]

B. An \( \text{SO}(3) \) tensor basis

Let \( \{ \hat{L}_k \} \) denote a set of angular momentum operators for the \( \mathfrak{so}(3) \) subalgebra of \( \mathfrak{so}(5) \) and let \( \{ d_M^1 \}; M = 0, \pm 1, \pm 2 \) and \( \{ p_M^1 \}; M = \pm \frac{1}{2}, \pm \frac{3}{2} \) denote linear combinations of the \( \{ \eta_\nu \} \) and \( \{ \zeta_m \} \) operators, respectively, which satisfy the
commutation relations

\[ [\hat{L}_0, d^l_M] = M d^l_M, \quad [\hat{L}_\pm, d^l_M] = \sqrt{(2 \mp M)(3 \pm M)} d^l_{M \pm 1}, \tag{43} \]

\[ [\hat{L}_0, p^l_M] = M p^l_M, \quad [\hat{L}_\pm, p^l_M] = \sqrt{(\frac{3}{2} \mp M)(\frac{5}{2} \pm M)} p^l_{M \mp 1}. \tag{44} \]

A basis for a realisation of the \( \mathfrak{so}(5) \) Lie algebra on the combined (tensor product) Hilbert spaces of the four- and five-dimensional harmonic oscillators is then provided by the (\( L_r \) of \( \mathfrak{so}(3) \subset \mathfrak{so}(5) \)) spanned by the \( \{ \hat{L}_k \} \) angular momentum operators in terms of the \( \{ \eta^l_M \} \) and \( \{ \zeta^l_M \} \) operators then defines the embedding of \( \text{SO}(3) \subset \text{SO}(5) \) as the subgroup with Lie algebra \( \mathfrak{so}(3) \) spanned by the \( \{ \hat{L}_k \} \) angular momentum operators. From now on, all references to \( \text{SO}(3) \) or \( \mathfrak{so}(3) \) will mean this subgroup or its Lie algebra.

C. Relationships between the two bases

Many choices of relationship are possible. However, the simple relationship defined by \( d^l_M = \eta^l_M \) and \( p^l_M = \zeta^l_M \) is unsatisfactory because, for this choice, it is found that \( \hat{L}_0 = 2 \hat{X}_0 + \hat{F}_0 \); this means that \( \hat{L}_0 \) lies in the intrinsic \( \mathfrak{u}(2) \) subalgebra and the conditions for Theorem 1 are violated. A satisfactory relationship is given by setting

\[ d^l_M = e^{\frac{\tau}{2}(\hat{S}_+ - \hat{S}_-)} \eta^l_M e^{-\frac{\tau}{2}(\hat{S}_+ - \hat{S}_-)}, \tag{51} \]

\[ p^l_M = e^{\frac{\tau}{2}(\hat{S}_+ - \hat{S}_-)} \zeta^l_M e^{-\frac{\tau}{2}(\hat{S}_+ - \hat{S}_-)}. \tag{52} \]

This relationship gives

\[ \eta^l_2 = \sqrt{\frac{1}{2}} (d^l_2 + d^l_{-1}), \quad \eta^l_{-2} = \sqrt{\frac{1}{2}} (d^l_{-2} - d^l_1), \tag{53} \]

\[ \eta^l_0 = d^l_0, \tag{54} \]

\[ \eta^l_1 = \sqrt{\frac{1}{2}} (d^l_1 + d^l_{-2}), \quad \eta^l_{-1} = \sqrt{\frac{1}{2}} (d^l_{-1} - d^l_2), \tag{55} \]

and

\[ \zeta^l_2 = \sqrt{\frac{1}{2}} (p^l_2 + p^l_{-2}), \quad \zeta^l_{-2} = \sqrt{\frac{1}{2}} (p^l_{-2} + p^l_2), \tag{56} \]

\[ \zeta^l_0 = p^l_0, \quad \zeta^l_1 = p^l_1. \tag{57} \]

The relationship between the two bases for \( \mathfrak{so}(5) \) is then given by

\[ \hat{L}_0 = \frac{1}{2}(\hat{X}_0 - \hat{F}_0) - \frac{i}{2}(\hat{S}_+ + \hat{S}_-), \tag{58} \]

\[ \hat{L}_\pm = 2 \hat{T}_\pm + \sqrt{\frac{3}{2}} (\hat{F}_\pm + \hat{X}_\mp), \tag{59} \]

\[ \hat{O}_0 = \frac{i}{2}(\hat{X}_0 - \hat{F}_0) + \frac{i}{2}(\hat{S}_+ - \hat{S}_-), \tag{60} \]

\[ \hat{O}_{\pm 1} = \mp \sqrt{3} \hat{T}_\pm \pm \sqrt{\frac{3}{2}} (\hat{F}_\pm + \hat{X}_\mp), \tag{61} \]

\[ \hat{O}_{\pm 2} = \frac{3}{2} (\hat{X}_\pm - \hat{F}_\mp), \tag{62} \]

\[ \hat{O}_{\pm 3} = \frac{\sqrt{3}}{2} (\mp \hat{X}_0 \mp \hat{F}_0 - \hat{S}_+ + \hat{S}_-), \tag{63} \]
where $\hat{L}_\pm = \mp \sqrt{2} \hat{L}_{\pm 1}$.

V. CONSTRUCTION OF VCS BASIS WAVE FUNCTIONS

Orthonormal basis states $\{|(v f)\tau LM\rangle\}$ for the (10) and $(0\frac{1}{2})$ irreps, for which the multiplicity index $\tau$ is redundant, are given by

$$| (10) 2M \rangle = d^1_M |0\rangle, \quad M = 0, \pm 1, \pm 2, \quad (64)$$
$$| (0 \frac{1}{2}) 2M \rangle = p^1_M |0\rangle, \quad M = \pm \frac{1}{2}, \pm \frac{3}{2}. \quad (65)$$

Thus, from the definitions (18)-(21), the corresponding VCS wave functions are given by

$$\Psi^{(10)}_{2M}(\Omega) = \xi^{(10)}_0 (0) \eta^2 \hat{R}(\Omega) d^1_M |0\rangle = \sum_K \xi^{(10)}_0 (0) \eta^2 d^1_K |0\rangle D^2_{KM}(\Omega) \equiv \frac{1}{\sqrt{2}} \xi^{(10)}_0 \left[ D^2_{2M}(\Omega) + D^2_{1, M}(\Omega) \right], \quad (66)$$

and

$$\Psi^{(0 \frac{1}{2})}_{\frac{1}{2}M}(\Omega) = \xi^{(0 \frac{1}{2})}_0 (0) \zeta \hat{R}(\Omega) p^1_M |0\rangle + \xi^{(0 \frac{1}{2})}_0 (0) \zeta \hat{R}(\Omega) p^1_M |0\rangle = \frac{1}{\sqrt{2}} \xi^{(0 \frac{1}{2})}_0 \left[ D^2_{\frac{1}{2}M}(\Omega) - D^2_{-\frac{1}{2}, M}(\Omega) \right] + \zeta D^2_{-\frac{1}{2}} \frac{1}{2} M(\Omega). \quad (67)$$

From Eq. (66), it follows that

$$\xi^{(v_1 f_1)}_{m_1} \otimes \xi^{(v_2 f_2)}_{m_2} = \sqrt{\frac{(f + m)! (f - m)! v!}{(f_1 + m_1)! (f_1 - m_1)! (f_2 + m_2)! (f_2 - m_2)! v_1 v_2}} \xi^{(v f)}_{m}, \quad (68)$$

where

$$v = v_1 + v_2, \quad f = f_1 + f_2, \quad m = m_1 + m_2. \quad (69)$$

From the properties of the Wigner rotation matrices, it also follows that

$$D^{L_1}_{K_1 M_1} \otimes D^{L_2}_{K_2 M_2} = \sum_{L = |L_1 - L_2|}^{L_1 + L_2} (L_2 M_2, L_1 M_1 |LM\rangle / (L_2 K_2, L_1 K_1 |LK\rangle) D^L_{KM}, \quad (70)$$

where

$$K = K_1 + K_2, \quad M = M_1 + M_2. \quad (71)$$

Thus, Theorem 2 and these expressions lead naturally to an algorithm for constructing (non-orthonormal) basis wave functions for an arbitrary $\mathfrak{so}(5)$ irrep.

It is convenient to start by constructing a non-orthonormal basis of VCS wave functions of the form

$$\Phi^{(v f)}_{\tau LM} = \sum_{m_k} \xi^{(v f)}_{m_k} b^{(v f)}_{m K}(\tau L) D^L_{KM}, \quad (72)$$

with $b^{(v f)}_{m K}(\tau L)$ coefficients conveniently chosen to be real and normalised such that

$$\sum_{m_k} b^{(v f)}_{m K}(\tau L) b^{(v f)}_{m K}(\sigma L) = \delta_{\tau \sigma}. \quad (73)$$

The functions $\Phi$ so normalised will be related to the orthonormal $\mathfrak{so}(5)$ functions $\Psi$ in section VII.

In order to carry out this basis construction efficiently, it is useful to know the values of the angular momentum and the multiplicity of their occurrence in any given irrep. In other words, we need to know the $\mathfrak{so}(5) \downarrow \mathfrak{so}(3)$ branching
rules which give the $\mathfrak{so}(3)$ irreps contained in any given $\mathfrak{so}(5)$ irrep. The $\mathfrak{so}(5) \downarrow \mathfrak{so}(3)$ branching rules for irreps of the type $(v0)$ were conveniently summarised by Williams and Pursey [32] in the form

\[
L = 2K, \, 2K - 2, \, 2K - 3, \ldots, \, K
\]

\[
K = v, \, v - 3, \, v - 6, \ldots, \, K_{\text{min}},
\]

(74)

where $K_{\text{min}} = 0, \, 1, \, \text{or} \, 2$. The branching rules for the irreps of type $(0f)$ are given [33] by

\[
L = 3K, \, 3K - 2, \, 3K - 3, \ldots, \, K
\]

\[
K = f, \, f - 2, \, f - 4, \ldots, \, K_{\text{min}},
\]

(75)

where $K_{\text{min}} = 0 \text{ or } 1$. The branching rules for a generic irrep can be inferred by use of character theory [34] or by a simple ‘peeling-off’ program [35] which uses knowledge of the number of eigenvalues of the $\hat{L}_0$ operator. The $\mathfrak{so}(3)$ content of some low-dimensional $\mathfrak{so}(5)$ irreps is given in Table I.

TABLE I: The $\mathfrak{so}(3)$ content of some low-dimensional $\mathfrak{so}(5)$ irreps; note that the spinor irreps are labelled in the table by $2L$ for convenience.

| $(v, f)$ | $L$   | $(v, f)$ | $2L$   |
|---------|------|---------|-------|
| (1, 0)  | 2    | (0, 1/2)| 3     |
| (0, 1)  | 1, 3 | (1, 1/2)| 1, 5, 7|
| (2, 0)  | 2, 4 | (0, 3/2)| 3, 5, 9|
| (1, 1)  | 1, 2, 3, 4, 5 | (1, 3/2)| 1, 3, 5, 7, 9, 11, 13 |
| (0, 2)  | 0, 2, 3, 4, 6 | (2, 1/2)| 3, 5, 7, 9, 11 |
| (3, 0)  | 0, 3, 4, 6   | (0, 5/2)| 3, 5, 7, 9, 11, 15 |
| (2, 1)  | 1, 2, 3, 4, 5, 6, 7 | (1, 5/2)| 1, 3, 5, 7, 9, 11, 11, 13, 13, 15, 17, 19 |
| (1, 2)  | 1, 2, 3, 4, 5, 6, 7, 8 | (2, 3/2)| 1, 3, 5, 7, 9, 9, 11, 11, 13, 13, 15, 17 |
| (0, 3)  | 1, 3, 4, 5, 6, 7, 9 | (3, 1/2)| 3, 5, 7, 9, 11, 13, 15 |
| (4, 0)  | 2, 4, 5, 6, 8  | (0, 7/2)| 3, 5, 7, 9, 11, 13, 15, 17, 21 |

A basis for any irrep of highest weight $(vf)$ can now be built up by taking coupled products of the above basic wave functions. For example, for the $(20)$ irrep,

\[
\Phi_{20}^{(20)} \propto \left[ \psi_2^{(10)} \otimes \psi_2^{(10)} \right]_{LM}, \quad L = 2, 4,
\]

(76)

gives

\[
\Phi_{LM}^{(20)} \propto \epsilon_0^{(20)} \sum_{K_1K_2} b_0^{(10)}(K_1) b_0^{(10)}(K_2) (2K_1, 2K_2 | L, K_1 + K_2) D_{K_1+K_2,M}^{L}
\]

\[
\propto \epsilon_0^{(20)} \left[ (22, 22 | L4) D_{LM}^{4M} + 2(22, 2, -1 | L1) D_{LM}^{1M} + (2, -1, 2, -1 | L, -2) D_{LM}^{-2M} \right].
\]

(77)

Thus, we obtain the results shown in Table I. Similarly, we have for the $(01)$ irrep

\[
\Phi_{LM}^{(01)} \propto \xi_1^{(01)} \sum_{K_1K_2} b_0^{(01)}(K_1) b_0^{(01)}(K_2) (\frac{1}{2} K_1, \frac{1}{2} K_2 | L, K_1 + K_2) D_{L}^{K_1+K_2,M}
\]

\[
+ \sqrt{2} \xi_0^{(01)} \sum_{K_1K_2} b_0^{(01)}(K_1) b_0^{(01)}(K_2) (\frac{1}{2} K_1, \frac{3}{2} K_2 | L, K_1 + K_2) D_{L}^{K_1+K_2,M}
\]

\[
+ \xi_{-1}^{(01)} \sum_{K_1K_2} b_0^{(01)}(K_1) b_0^{(01)}(K_2) (\frac{3}{2} K_1, \frac{1}{2} K_2 | L, K_1 + K_2) D_{L}^{K_1+K_2,M},
\]

(78)

which leads to a set of coefficients given in Table III.

In general we can form basis states for an irrep $(vf)$ from the coupled products

\[
\left[ \Phi_{\tau L_2}^{(v)} \otimes \Phi_{\tau L_1}^{(f)} \right]_{LM} = \sum_{m} \xi_0^{(v)} \otimes \xi_m^{(f)} \sum_{K_1K_2} b_0^{(v)}(\tau_2 L_2) b_0^{(f)}(m_{K_1} (\tau_1 L_1) (L_1 K_1, L_2 K_2 | L, K_1 + K + 2) D_{L}^{K_1+K_2,M}.
\]

(79)

Some examples are given in Table IV.

It is important to recognise that the basis $\{ \Phi_{\tau L}^{(v)} \}$ is not orthonormal relative to the appropriate inner product for VCS wave functions. An orthonormal basis is one relative to which the VCS representations are unitary; i.e., for which the elements of the $\mathfrak{so}(5)$ Lie algebra are represented by Hermitian operators.
is given by the standard action of the form

\[ \langle \tau L, \nu | \Phi \rangle = \sum_{mK} \xi_{mK}^{(vf)} (\tau L) D_{K M}^L (\Omega) = \sum_{mK} \xi_{mK}^{(vf)} (\tau L) D_{K M}^L (\Omega) \]

is given by the standard action

\[ \begin{align*}
[\Gamma(L_0) \Phi^{(vf)}(\tau L)](\Omega) &= M \Phi^{(vf)}(\tau L, \nu)(\Omega), \\
[\Gamma(L_{\pm}) \Phi^{(vf)}(\tau L)](\Omega) &= \sqrt{(L \mp M) L \pm M + 1} \Phi^{(vf)}(\tau L, M \pm 1)(\Omega).
\end{align*} \]

The action of the octupole operators is given by

\[ \Gamma(O_v) \Phi^{(vf)}(\tau L, \nu)(\Omega) = \sum_{mK} \xi_{mK}^{(vf)} \langle \nu | (vf) m | (vf) \tau L K \rangle D_{\mu K M}^L (\Omega) \]

or, in coupled form,

\[ \left[ \Gamma(O) \otimes \Phi^{(vf)} \right]_{L'M}^{L} = \sum_{mK\mu} \xi_{mK}^{(vf)} \langle \nu | (vf) m | (vf) \tau L K \rangle \langle \Omega, \nu | (vf) \tau L K \rangle D_{\mu K M}^{L'} (\Omega). \]

Thus, it remains to determine the matrix elements \{ \langle (vf) m | (vf) \tau L K \rangle \} to define an so(5) irrep \( (vf) \). From the definition of the states \{ \langle (vf) m \rangle \} as highest grade states, cf. Eqns. (14), we have the identities

\[ \langle (vf) m | (vf) \tau L K \rangle = \langle (vf) m | (vf) \tau L K \rangle = \langle (vf) m | (vf) \tau L K \rangle = 0, \]

and from the standard action of the angular momentum operators on the states \{ \langle (vf) m | (vf) \tau L K \rangle \}, we have

\[ \langle (vf) m | (vf) \tau L K \rangle = K \langle (vf) m | (vf) \tau L K \rangle, \]

\[ \langle (vf) m | (vf) \tau L K \rangle = \sqrt{(L \mp K)(L \pm K + 1)} \langle (vf) m | (vf) \tau L, K \pm 1 \rangle. \]
TABLE III: The $b_{mK}^{(0f)}(\tau L)$ coefficients for some irreps of type $(0f)$. Note that for an $\mathfrak{so}(5)$ irrep $(vf)$ the coefficients among different $\mathfrak{so}(3)$ irreps can all be indexed by the same values of $K - m$, hence this is used to label columns.

$$\begin{array}{ccc}
(vf) = (0_1^1) & & \\
 & K - m & \\
 L \ m & 1 & -2 \\
\frac{1}{2} \ + \frac{1}{2} & 1/2 & -1/2 \\
\frac{1}{2} \ - \frac{1}{2} & 1/\sqrt{2} & \\
\end{array}$$

$$\begin{array}{ccc}
(vf) = (01) & & \\
 & K - m & \\
 L \ m & 2 & -1 & -4 \\
1 \ + 1 & 0 & 3/\sqrt{23} & 0 \\
0 \ 0 & 0 & \sqrt{6}/\sqrt{23} \\
-1 \ 2\sqrt{2}/\sqrt{23} & 1/\sqrt{23} \ 1/\sqrt{23} & \\
3 \ + 1 & 1/\sqrt{5}/\sqrt{37} & -1/\sqrt{5}/\sqrt{37} & \sqrt{5}/\sqrt{37} \\
0 \ 10/\sqrt{37} & -2/\sqrt{37} & \\
-1 \ 2\sqrt{3}/\sqrt{37} & \\
\end{array}$$

$$\begin{array}{ccc}
(vf) = (0_3^3) & & \\
 & K - m & \\
 L \ m & 3 & 0 & -3 & -6 \\
\frac{1}{2} \ + \frac{1}{2} & 0 & \sqrt{27}/152 & \sqrt{27}/152 & 0 \\
+ \frac{1}{2} & 0 & 3/\sqrt{2}/152 & 0 \\
- \frac{1}{2} & 0 & 4\sqrt{3}/152 & \\
-1 \ 4\sqrt{2}/152 & -4/\sqrt{2}/152 & \\
\frac{1}{2} \ + \frac{1}{2} & 0 & -\sqrt{27}/232 & -\sqrt{27}/232 & 0 \\
+ \frac{1}{2} & 0 & 4\sqrt{3}/232 & -30/232 \\
- \frac{1}{2} & 0 & 2\sqrt{5}/232 & 2\sqrt{2}/232 \\
-1 \ 6\sqrt{2}/232 & 6\sqrt{2}/232 & \\
\frac{1}{2} \ + \frac{1}{2} & 2\sqrt{7}/328 & -3/\sqrt{328} & \sqrt{3}/\sqrt{328} & -2\sqrt{7}/328 \\
+ \frac{1}{2} & 2\sqrt{14}/328 & -4/\sqrt{1/328} & \sqrt{14}/328 & \\
- \frac{1}{2} \ 2\sqrt{21}/328 & -6/\sqrt{328} & \\
-1 \ 6\sqrt{2}/328 & \\
\end{array}$$

Eqs. (58-59) give the relationships

$$\hat{S_+} = \frac{1}{3}(\hat{X_0} - \hat{F_0}) - \hat{S}_- - \frac{2}{3} \hat{L_0},$$  \hspace{1cm} (90)$$

$$\hat{X_+} = \sqrt{2/3} \hat{L_-} - 2\sqrt{2/3} \hat{T_-} - \hat{F_-},$$  \hspace{1cm} (91)$$

$$\hat{T_+} = \frac{1}{2} \hat{L_+} - \frac{1}{2} \sqrt{\frac{3}{2}} (\hat{F_+} + \hat{X_-})$$  \hspace{1cm} (92)$$

which makes it possible to rewrite Eqs. (60-63) in the form

$$\hat{O}_0 = \frac{5}{3}(\hat{X}_0 - \hat{F}_0) - \frac{1}{3} \hat{L}_0,$$  \hspace{1cm} (93)$$

$$\hat{O}_1 = -\sqrt{\frac{3}{2}} \hat{L}_+ + \frac{5}{2\sqrt{2}} (\hat{F}_+ + \hat{X}_-),$$  \hspace{1cm} (94)$$

$$\hat{O}_{-1} = \frac{1}{\sqrt{3}} \hat{L}_- + \frac{5}{\sqrt{3}} \hat{T}_-, $$  \hspace{1cm} (95)$$

$$\hat{O}_2 = \sqrt{\frac{5}{6}} \hat{L}_- - \sqrt{\frac{10}{3}} \hat{T}_- - \sqrt{5} \hat{F}_-, $$  \hspace{1cm} (96)$$
TABLE IV: The $b_{mK}^{(v_f)}(\tau L)$ coefficients for some generic irreps.

\[
\begin{array}{c|cc}
(v_f) &= (1^{\frac{1}{2}}) & \\
 L & m & K - m \\
\frac{1}{2} & +\frac{1}{2} & 0 \\
-\frac{1}{2} & 0 & \sqrt{3}/5 \\
\frac{1}{2} & +\frac{1}{2} & -4/\sqrt{47} \\
-\frac{1}{2} & 0 & \sqrt{15}/47 \\
\frac{1}{2} & +\frac{1}{2} & \sqrt{7}/22 \\
-\frac{1}{2} & 0 & \sqrt{6}/55 \\
\end{array}
\]

\[
\begin{array}{c|cc}
(v_f) &= (11) & \\
 L & m & K - m \\
1 & +1 & 0 \\
-1 & 0 & 3/\sqrt{29} \\
2 & +1 & 0 \\
-1 & 0 & -2\sqrt{3}/29 \\
3 & +1 & \sqrt{15}/109 \\
-1 & 2\sqrt{10}/109 & 2\sqrt{6}/109 \\
4 & +1 & 0 \\
-1 & -4\sqrt{7}/829 & 11\sqrt{2}/829 \\
5 & +1 & \sqrt{70}/313 \\
-1 & 2\sqrt{14}/313 & 2\sqrt{10}/313 \\
\end{array}
\]

\[
O_{-2} = \frac{\sqrt{5}}{2} \hat{X}_- - \frac{\sqrt{5}}{2} \hat{F}_+ ,
\]

\[
O_3 = -\frac{2\sqrt{5}}{3} \hat{X}_0 + \frac{\sqrt{5}}{3} \hat{F}_0 + \sqrt{\frac{5}{3}} \hat{S}_+ + \frac{\sqrt{5}}{3} \hat{L}_0 ,
\]

\[
O_{-3} = \frac{\sqrt{5}}{3} \hat{X}_0 + \frac{2\sqrt{5}}{3} \hat{F}_0 + \frac{\sqrt{5}}{3} \hat{S}_+ + \frac{\sqrt{5}}{3} \hat{L}_0 .
\]

We now have all of the $\mathfrak{so}(5)$ operators in terms of the basis operators \{\(S_-, T_-, X_-, X_0, F_0, F_{\pm}, L_0, L_{\pm}\)\} with known algebraic actions on either the highest grade states or on the $\mathfrak{so}(3)$-coupled states. Note that Theorem 1 ensures that this set is linearly independent and spans $\mathfrak{so}(5)$. This is a general feature of a useful VCS construction.

We obtain

\[
\langle (v_f)m|\hat{O}_0|(v_f)\tau L\rangle = \frac{1}{3}(5v + 5f - 5m - K) b_{mK}^{(v_f)}(\tau L),
\]

\[
\langle (v_f)m|\hat{O}_1|(v_f)\tau L\rangle = \frac{2}{3} \sqrt{\frac{1}{2}(f + m)(f - m + 1)} b_{m-1,K}^{(v_f)}(\tau L)
\]

\[
-\frac{1}{2} \sqrt{3(L - K)(L + K + 1)} b_{m,K+1}^{(v_f)}(\tau L),
\]

\[
\langle (v_f)m|\hat{O}_{-1}|(v_f)\tau L\rangle = -\sqrt{\frac{1}{2}(L + K)(L - K + 1)} b_{m,K-1}^{(v_f)}(\tau L),
\]

\[
\langle (v_f)m|\hat{O}_2|(v_f)\tau L\rangle = \frac{1}{2} \sqrt{(L + K)(L - K + 1)} b_{m+1,K}^{(v_f)}(\tau L)
\]

\[
-\frac{1}{2} \sqrt{5(f - m)(f + m + 1)} b_{m-1,K}^{(v_f)}(\tau L),
\]

\[
\langle (v_f)m|\hat{O}_{-2}|(v_f)\tau L\rangle = -\frac{1}{2} \sqrt{5(f + m)(f - m + 1)} b_{m+1,K}^{(v_f)}(\tau L),
\]

(100, 101, 102, 103, 104)
\[
\langle (vf)m|\hat{O}_3|(vf)\tau LK \rangle = -\frac{\sqrt{3}}{3}(2v+2f+m-K)b^{(vf)}_{mK}(\tau L),
\]
\[
\langle (vf)m|\hat{O}_{-3}|(vf)\tau LK \rangle = \frac{\sqrt{3}}{3}(v+f+2m+K)\nu^{(vf)}_{mK}(\tau L).
\]

It follows that Eq. \ref{eq:ma} has the explicit expansion given by
\[
\left[ \Gamma(O) \otimes \Phi^{(vf)}_{\tau L}\right]_{L'M} = \sum_{mKLm'K'} b^{(vf)}_{mK}(\tau L) M^{(vf)}_{mKL,m'K'L'} \xi^{(vf)}_{m'} D^{L'}_{K'M},
\]
where \( M^{(vf)} \) is a matrix with non-zero entries
\[
\begin{align*}
M^{(vf)}_{mKL,m+1K+1L'} &= \frac{\sqrt{3}}{6}(5v+5f-5m-K)(LK,3|0L'K) \\
&- \frac{\sqrt{3}}{6}(L+K)(L-K+1) (L K-1,3|1L'K), \\
M^{(vf)}_{mKL,m-1K+2L'} &= -\frac{\sqrt{6}}{6}(f-m)(f+m+1) (LK,3|1L' K+1), \\
M^{(vf)}_{mKL,mK+3L'} &= \frac{\sqrt{3}}{3}(2v+2f+m-K) (LK,3|3L' K+3), \\
M^{(vf)}_{mKL,m+1K-2L'} &= -\frac{\sqrt{3}}{6}(L-K)(L+K+1) (LK,3|2L' K-2), \\
M^{(vf)}_{mKL,mK-3L'} &= \frac{\sqrt{3}}{3}(v+f+2m+K) (LK,3|3L' K-3).
\end{align*}
\]

Now, if the basis wave functions \( \{\Phi_{\tau LM}^{(vf)}\} \) are chosen, as defined in Sect. \ref{sect:vii} with \( b^{(vf)}_{mK}(\tau L) \) coefficients that are real and satisfy Eq. \ref{eq:es}, it follows that Eq. \ref{eq:ta} can be expressed
\[
\left[ \Gamma(O) \otimes \Phi^{(vf)}_{\tau L}\right]_{L'M} = \sum_{\sigma} \Phi^{(vf)}_{\sigma L'M} O^{(vf)}_{\sigma L',\tau L},
\]
where
\[
\begin{align*}
O^{(vf)}_{\sigma L',\tau L} &= \sum_{mKLm'K'} b^{(vf)}_{mK}(\sigma L') M^{(vf)}_{mKL,m'K'L'} b^{(vf)}_{m'K'}(\tau L).
\end{align*}
\]

Thus, together with Eqs. \ref{eq:ma} \ref{eq:es}, these equations give the explicit transformations of the \( \{\Phi_{\tau LM}^{(vf)}\} \) basis wave functions for any \( so(5) \) irrep.

Unfortunately, these matrices do not satisfy the Hermiticity conditions required of a unitary representation. This is because the \( \{\Phi_{\tau LM}^{(vf)}\} \) basis is not orthonormal relative to the appropriate VCS inner product.

\section{The Matrices of Unitary Irreps}

We now suppose that \( \{(vf)\alpha LM\} \) is an orthonormal basis for an irrep \( (vf) \) and that the corresponding VCS wave functions have expansions
\[
\Psi^{(vf)}_{\alpha LM} = \sum_{mK} \xi^{(vf)}_{m}(vf)m|(vf)\alpha LK \rangle D^{L}_{K} = \sum_{mK} \xi^{(vf)}_{m} a^{(vf)}_{mK}(\alpha L) D^{L}_{K}.
\]

Note that for an orthonormal basis we have called the expansion coefficients of the VCS wave functions \( a^{(vf)}_{mK} \) in order to distinguish them from the \( b^{(vf)}_{mK} \) coefficients of the non-orthonormal \( \Phi_{\sigma LM}^{(vf)} \) basis. The Wigner-Eckart theorem for matrix elements in an orthonormal basis
\[
\langle (vf)\beta L'M'|\hat{O}_\nu|(vf)\alpha LM \rangle = \frac{1}{\sqrt{2L'+1}} (LM,3\nu|L'M') \langle (vf)\beta L'|\hat{O}_\nu|(vf)\alpha L \rangle
\]
then implies that
\[
[\text{I}(O) \otimes \Psi_{\alpha L}^{(v_f)}]_{L'M'} = \frac{1}{\sqrt{2L' + 1}} \sum_{\beta} \Psi_{\beta L'}^{(v_f)} \langle (v_f)\beta L' \| \hat{O} \| (v_f)\alpha L \rangle,
\]
It follows from Eq. (119) that if the orthonormal basis wave functions are expanded
\[
\Psi_{\alpha L M}^{(v_f)} = \sum_{\sigma} \Phi_{\sigma L M}^{(v_f)} K_{\sigma \alpha}^{(v_f)} (L)
\]
then the desired reduced matrix elements, relative to the orthonormal basis, are given by
\[
\langle (v_f)\beta L' \| \hat{O} \| (v_f)\alpha L \rangle = \sqrt{2L' + 1} \sum_{\sigma \tau} \hat{K}_{\beta\tau}^{(v_f)} O_{\tau,\sigma L}^{(v_f)} K_{\sigma \alpha}^{(v_f)},
\]
where \( \hat{K}^{(v_f)} \) is the inverse of the matrix \( K^{(v_f)} \), i.e., it is defined such that
\[
\sum_{\sigma} \hat{K}_{\beta\sigma}^{(v_f)} K_{\sigma \alpha}^{(v_f)} = \delta_{\alpha\beta}.
\]
The \( K^{(v_f)}(L) \) matrices are chosen in VCS theory such that the reduced matrix elements satisfy the Hermiticity condition
\[
\langle (v_f)\alpha L' \| \hat{O} \| (v_f)\beta L \rangle^* = (-1)^{L-L'} \langle (v_f)\beta L \| \hat{O} \| (v_f)\alpha L' \rangle
\]
required of a unitary representation. Such a transformation is found in two steps.

First observe that the submatrices \( O(L) \) with elements
\[
O_{\sigma\tau}(L) = O_{\alpha L,\tau L}^{(v_f)}
\]
are real and symmetric. Thus, we first make an orthogonal transformation
\[
\hat{\Phi}_{\alpha L M}^{(v_f)} = \sum_{\sigma} \Phi_{\sigma L M}^{(v_f)} K_{\sigma \alpha}^{(v_f)} (L),
\]
which diagonalises the \( O(L) \) matrices. The transformed matrix
\[
\hat{\mathcal{O}}_{\beta\tau L',\alpha L}^{(v_f)} = \sum_{\sigma \tau} K_{\tau\sigma}^{(v_f)} (L') O_{\tau,\sigma L}^{(v_f)} K_{\sigma \alpha}^{(v_f)} (L),
\]
then satisfies the equality
\[
\hat{\mathcal{O}}_{\beta\alpha L,\alpha L}^{(v_f)} = \delta_{\alpha\beta} \mathcal{O}_{\beta L,\beta L}^{(v_f)}.
\]
Before proceeding, it is important to note that states of different \( L \) and \( M \) are automatically orthogonal by virtue of their transformation properties under \( SO(3) \). Moreover, if the reduced matrices for \( L = L' \) are diagonal, this subset of matrices automatically satisfies the Hermiticity condition.

Thus, in general, it only remains to apply suitable scale factors to the basis vectors \( \{ \Phi_{\alpha L M}^{(v_f)} \} \) to obtain an orthonormal basis. The required scale factors \( \{ k_{\alpha L}^{(v_f)} \} \) must be such that
\[
(k_{\alpha L}^{(v_f)})^{-1} \hat{\mathcal{O}}_{\alpha L,\beta L}^{(v_f)} k_{\beta L'}^{(v_f)} = (-1)^{L-L'} (k_{\beta L'}^{(v_f)})^{-1} \hat{\mathcal{O}}_{\alpha L',\beta L}^{(v_f)} k_{\alpha L}^{(v_f)}
\]
The desired \( K^{(v_f)} \) matrices are then given by
\[
K_{\tau\alpha}^{(v_f)} (L) = k_{\alpha L}^{(v_f)} k_{\tau\alpha}^{(v_f)} (L)
\]
with
\[
\frac{|k_{\alpha L}^{(v_f)}|^2}{|k_{\beta L'}^{(v_f)}|^2} = (-1)^{L-L'} \frac{\hat{\mathcal{O}}_{\alpha L,\beta L'}^{(v_f)}}{\hat{\mathcal{O}}_{\beta\alpha L,\beta L'}^{(v_f)}}.
\]
Special consideration must be given to the relatively few irreps for which there is a multiplicity of states of \( L = 0, \frac{1}{2} \) or 1. This is because the \( O(L) \) matrices are identically zero for these \( L \) values. However, to ensure the orthogonality and correct normalisation of, for example, the \( L = 0 \) states it is sufficient to determine linear combinations of these states which satisfy the equations
\[
\langle (v_f)\alpha 0 \| \hat{O} \| (v_f)\beta L \rangle = (-1)^L \langle (v_f)\beta L \| \hat{O} \| (v_f)\alpha 0 \rangle,
\]
where \( \{ (v_f)\beta L M \} \) is a small subset of states that have already been orthonormalised.
VIII. SAMPLE RESULTS

In this section we tabulate the a-coefficients \( a^{(vf)}_{mK}(\tau L) \) of the unitary basis wave functions and the unitary SO(3)-reduced so(5) matrix elements of the \( \hat{O} \) operator for the simplest generic so(5) irreps as well as the table for the first irrep with a multiplicity, namely \( (1\frac{1}{2}) \), (we index the multiple so(3) irreps as \( L_{\tau} \)). Because of the SO(3) reduction, we need only provide values between states of different \( L \), since the Wigner-Ekart theorem provides the rest. Note that the matrix elements of the \( \hat{L} \) operator are easily computed using Eqs. (81) and (82), and so they are not included in the tables. Since the diagonalisation of Eq. (119) must be done numerically, the values are given in floating point form.

| TABLE V: Expansion coefficients and reduced matrix elements for so(5) irrep \((1, \frac{1}{2})\). |
|---|
| \( m \) | \( K - m \) | \( L \) |
| \( \frac{1}{2} \) | 0 | 0.474341 | 0 |
| \( \frac{1}{2} \) | 0 | -0.387298 | 0 |
| \( \frac{1}{2} \) | 0 | 0.517549 | 0 |
| \( \frac{1}{2} \) | -0.534522 | 0.422577 | 0 |
| \( \frac{1}{2} \) | 0.499999 | 0.084515 | -0.377964 |
| \( \frac{1}{2} \) | 0.462910 | 0.414039 | 0 |

| TABLE VI: Expansion coefficients and reduced matrix elements for so(5) irrep \((2, \frac{1}{2})\). |
|---|
| \( m \) | \( K - m \) | \( L \) |
| \( \frac{1}{2} \) | 0 | 0.439155 | 0 |
| \( \frac{1}{2} \) | 0 | -0.358568 | 0 |
| \( \frac{1}{2} \) | 0 | 0.462910 | 0 |
| \( \frac{1}{2} \) | 0 | 0.306186 | 0 |
| \( \frac{1}{2} \) | 0.353553 | 0.082572 | 0 |
| \( \frac{1}{2} \) | 0.261116 | 0.190692 | 0 |

| TABLE VII: Expansion coefficients and reduced matrix elements for so(5) irrep \((1, 1)\). |
|---|
| \( m \) | \( K - m \) | \( L \) |
| \( \frac{1}{2} \) | 0 | 0.380319 | 0 |
| \( \frac{1}{2} \) | 0 | -0.358568 | 0 |
| \( \frac{1}{2} \) | 0 | 0.462910 | 0 |
| \( \frac{1}{2} \) | 0 | 0.306186 | 0 |
| \( \frac{1}{2} \) | 0.353553 | 0.082572 | 0 |
| \( \frac{1}{2} \) | 0.261116 | 0.190692 | 0 |
TABLE VIII: Expansion coefficients and reduced matrix elements for \( \mathfrak{so}(5) \) irrep (1, \( \frac{2}{2} \)).

| \( m \) | 5 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \( K - m \) | 0 | 0 | 0.396130 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0.560213 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0.411252 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0.662853 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0.441902 | -0.441902 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | -0.360811 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( L \) | 0 | 0 | -8.964214 | -7.598895 | -5.949760 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

IX. CONCLUDING REMARKS

As mentioned in the introduction, the group SO(5) and its Lie algebra \( \mathfrak{so}(5) \) arise in many physical contexts and, depending on the situation, their irreps are needed in different bases. In particular, they are needed in bases which reduce one of the following: (i) the U(2) subgroup whose Lie algebra is the grade-conserving intrinsic subalgebra shown in Fig. 2; (ii) the SO(4) subgroup generated by the \( S \) and \( T \) root vectors shown in Fig. 2 and (iii) the geometric SO(3) subgroup considered in this paper. VCS theory has now been used in systematic ways to construct SO(5) irreps in all of these bases.

VCS theory was originally formulated to construct the holomorphic representations of the compact and non-compact discrete-series representations of the symplectic algebras. A first application constructed the irreps of the non-compact \( \mathfrak{sp}(6, \mathbb{R}) \) algebra (also called \( \mathfrak{sp}(3, \mathbb{R}) \)) in a U(3) coupled basis. A next application, by Hecht and Elliott, derived similar results for the compact \( \mathfrak{sp}(4) \) algebra (isomorphic to \( \mathfrak{so}(5) \)) in a U(2)-coupled basis.

The common feature of the holomorphic VCS representations, which makes them particularly simple, is their use of Abelian orbiter groups with Lie algebras spanned by commuting sets of grade-lowering (or grade-raising) operators to complement the grade conserving intrinsic subgroups which generate highest- (or lowest-) grade intrinsic vectors for a VCS representation. A problem is that Abelian orbiter groups cannot be found in many important situations. For example, in constructing irreps of \( \mathfrak{so}(5) \) in an \( \mathfrak{so}(4) \)-coupled basis, it is natural to use SO(4) as the intrinsic subgroup. However, the complementary \( \mathfrak{so}(5) \) lowering operators (i.e., the operators \( F_- \) and \( X_- \) of Fig. 2) do not then commute. Fortunately, it was possible to extend the VCS construction to accommodate this situation. It was also discovered that VCS theory could be extended to give the much needed irreps of, for example, \( \mathfrak{su}(3) \) in an \( \mathfrak{so}(3) \) basis by using SO(3) itself as an orbiter group to complement an intrinsic U(2) group. This method was subsequently used to construct the so-called one-rowed irreps of \( \mathfrak{so}(5) \) (the irreps of type \( (v, 0) \) in the notation of this paper) for which only scalar valued VCS wave functions are needed. The techniques have been further developed in this paper to give the generic \( \mathfrak{so}(5) \) irreps in what is the most sophisticated explicit VCS construction of irreps to date.

Some novel features which considerably simplify the construction of VCS representations were introduced in Ref. and further developed here. In particular, we have found that the Hilbert space \( \mathcal{H} \) of all SO(5) VCS wave functions is a model space for SO(5), in the sense that it contains precisely one copy of every irrep. A particularly valuable
feature of this model space is that it is a ring in as much as the product of any two VCS wave functions is another VCS wave function in \( \mathcal{H} \). This observation greatly facilitates the construction of an \( \text{SO}(5) \supset \text{SO}(3) \)-coupled basis for \( \mathcal{H} \); one has simply to construct multiple \( \text{SO}(3) \)-coupled products of the generating functions given by the VCS wave functions for the fundamental \((1,0)\) and \((0,\frac{1}{2})\) irreps. As a result, it is very easy to construct basis wave functions for any \( \mathfrak{so}(5) \) irrep. It is also shown that, once a basis has been determined, the matrices of the \( \mathfrak{so}(5) \) Lie algebra in this basis can be determined algebraically, cf. Eqs. \( \text{117} \) and \( \text{118} \). These results are remarkable in view of the fact that there is generally a multiplicity in the \( \text{SO}(5) \supset \text{SO}(3) \) reduction of the representation space. It must be remembered, however, that the basis in which these results are obtained is not orthonormal.

An important component of VCS theory is its incorporation of algorithms, that go under the name of \( K \)-matrix theory \[30\], whose purpose is to determine the inner product of a Hilbert space, construct an orthonormal basis, and transform a VCS irrep into a unitary representation whenever it is equivalent to a unitary irrep (or more generally to an isometric irrep when it is equivalent to an isometric irrep.) For the irreps considered in this paper, a particularly simple version of the \( K \)-matrix transformation is given \[30\] by first finding an orthogonal basis in which off-diagonal matrix elements of the \( \text{SO}(5) \) octupole operator are zero between states of the same \( \text{SO}(3) \) angular momentum. This is simply achieved by diagonalisation of sub-blocks of the octupole matrices. The renormalisation factors needed to give an orthonormal basis can then be read off as shown in Section \[\text{VII} \]. Unfortunately, this simple method does not work for states of angular momentum \( L = 0, \frac{1}{2}, \) or 1, when there are multiple states of these \( L \) values. One must then resort to less simple \( K \)-matrix methods.

To summarise, the construction of the \( \text{SO}(3) \)-reduced matrices of the octupole operators in a generic \( \mathfrak{so}(5) \) irrep is achieved in three simple steps: (i) construction of basis wave functions with good \( \text{SO}(3) \) angular momentum quantum numbers; (ii) calculation of the reduced matrix elements of the \( \mathfrak{so}(5) \) octupole operators in this basis; and (iii) transforming these matrices to those of a unitary representation in an orthonormal basis.

Step (i) is simply achieved for a \((v0)\) irrep by first computing the basis wave functions for the \((v0)\) and \((0f)\) irreps and then taking their \( \text{SO}(3) \)-coupled products, as indicated in Section \[\text{VI} \]. Step (ii) is achieved by use of the analytical expressions given in Section \[\text{VII} \] in terms of \( \text{SU}(2) \) Clebsch-Gordan coefficients and the coefficients of the wave functions of step (i). Finally step (iii) is achieved by the unitarisation process described above and in Section \[\text{VII} \]. The routine has been coded in MAPLE which is amenable to both algebraic and numeric computations. Steps (i) and (ii) are carried out algebraically; thus the results in Tables \[\text{VI}-\text{VIII} \] are given in exact arithmetic. Step (iii), in which diagonalisations are used to orthogonalise states of a common angular momentum, is done numerically; thus the results given in Section \[\text{VIII} \] are floating point numbers. The current code has not been designed to handle large-dimensional irreps. However, should large irreps be of interest, the routine could be coded entirely in a numerical language such as FORTRAN, MATLAB, or C++.

The representations of \( \mathfrak{so}(5) \) in an \( \text{SO}(3) \)-coupled basis are of interest for several reasons. The so-called one-row irreps of type \((v0)\) feature in the Bohr-Mottelson and IBM-1 collective models and, consequently, have received much attention \[20\]-\[22\]. The generic irreps, constructed in this paper, show up in these collective models whenever the neutron and proton degrees of freedom are considered independently. They could also prove useful in the classification of shell-model states of fermions in an angular momentum \( l = 2 \) orbital. Generic \( \mathfrak{so}(5) \) irreps also show up in supersymmetric boson-fermion models \[33\]-\[39\]. For example, the irreps of the orthosymplectic \( \mathfrak{osp}(5/4) \) superalgebra of Iachello’s model contain irreps of \( \mathfrak{so}(5) \) subalgebra contained in the chain

where \( \mathfrak{so}(5) \) here signifies the subalgebra of \( \mathfrak{so}(5) \oplus \mathfrak{sp}(4) \) obtained by adding the corresponding infinitesimal generators of the isomorphic \( \mathfrak{so}(5) \) and \( \mathfrak{sp}(4) \) algebras. The construction of VCS representations of orthosymplectic superalgebras was considered by LeBlanc and Rowe \[40\] in a natural extension of the holomorphic representations to include representations over Grassmann as well as complex variables. Thus, it would be useful for the development of supersymmetric models of coupled boson-fermion systems to extend the construction given in this paper for \( \mathfrak{so}(5) \cong \mathfrak{usp}(4) \) irreps in an \( \text{SO}(3) \) basis to the irreps of the orthosymplectic algebras.

It is interesting to note that, in the present construction of \( \mathfrak{so}(5) \) irreps, the \( \mathfrak{so}(5) \) algebra has been realised as a combination of \( L = 2 \) and \( L = \frac{3}{2} \) boson operators. The use of bosons of half-odd integer angular momentum is admittedly unusual in physics. But, in spite of the spin-statistics theorem, there is no algebraic reason forbidding their use in this way. The fact is that bilinear products of either boson or fermion operators of any given angular momentum obey precisely the same commutation relations. However, the number of \( \mathfrak{so}(5) \) irreps that can be built up with fermions is severely limited by the Pauli principle whereas with bosons there is no such limitation.

A major motivation for the present study was that the methods developed for \( \mathfrak{so}(5) \) would serve as prototype examples of what can be done in more general situations. Constructing the irreps of a Lie algebra (or a Lie group) in a basis which reduces a non-canonical subgroup chain is generally much more challenging than in a basis which reduces a canonical subgroup chain. Thus, it is an order of magnitude simpler to construct \( \mathfrak{so}(5) \) irreps in a canonical \( \text{SO}(5) \supset \text{SO}(4) \supset \text{SO}(3) \) basis than in the geometrical \( \text{SO}(3) \) basis considered in this paper. For example, analytical expressions
are readily derived for the irreps of $\mathfrak{su}(3)$ in the canonical $\text{SU}(3) \supset \text{SU}(2)$ basis. But, prior to the development of opposite VCS techniques \cite{24}, special ad hoc methods were needed to obtain these irreps in $\text{SU}(3) \supset \text{SO}(3)$ bases. A possible application of the methods developed in this paper might be to construct irreps of the exceptional Lie algebra $\mathfrak{g}_2$ in an $\text{SU}(3) \supset \text{SO}(3)$-coupled basis. This would appear to be possible by choosing intrinsic states which carry an irrep of a $\mathfrak{u}(2)$ subalgebra, corresponding to a pair of short $\mathfrak{g}_2$ roots, and employing $\text{SU}(3)$ as the orbiter group. Construction of such irreps of $\mathfrak{g}_2$ is of physical interest as a step towards the construction of $\mathfrak{so}(7)$ irreps in a $G_2 \supset \text{SO}(3)$-coupled basis. The need for such irreps would surface if it were considered desirable to formulate a collective model of octupole dynamics analogous to Bohr’s quadrupole model or to classify shell-model states of fermions in an angular momentum $l = 3$ orbital.

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