Existence, uniqueness and regularity of piezoelectric partial differential equations

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ABSTRACT
Piezoelectric devices belong to the most prominent examples of smart materials. They find their applications in sensors and actuators, e.g. in the context of ultrasonic applications, tomography, cavitation-based cleaning. Nowadays, the design of new piezoelectric devices is generally accompanied by a computer-aided design, i.e. some models are used to predict the mechanical–electrical coupling of the new products. The coupling is described by a set of second-order coupled partial differential equations. For the mechanical part, this system comprises the equation of motion for the mechanical displacement in three dimensions and for the electric part, an electrostatic potential equation is employed. Coupling terms and an additional Rayleigh damping approach ensure the validness of the model. In this work, we analyze the existence, uniqueness and regularity of the solutions to these equations and give a result concerning the long-term behavior. The assumptions mainly on the material parameters involved are quite natural and allow meaningful physical interpretation.

1. Introduction
In modern industry, it has become increasingly popular to generate a digital twin of a device of interest as it is much cheaper and easier to experiment with a virtual object than to conduct experiments on – and possibly damage or destroy – the real existing counterpart. Thus, there is distinct interest in how to accurately predict the behavior of a device at hand. This is especially true for piezoelectric devices some of which only have a limited life-cycle and oftentimes contain led which makes real physical experiments with those devices cumbersome.

The fundamental property of piezoelectricity is the transformation of mechanical energy into electrical energy and vice-versa. Hence, piezoelectric devices are usually applied as actuators or transducers. In the past century, piezoelectric devices have become an essential role in our modern everyday life.

Typical applications as actuators range from piezo-igniters over ultrasonic toothbrushes to diesel fuel injectors as well as many others, e.g. as part of intelligent sensory equipment.

In the past, many authors have examined and modeled the behavior of piezoelectric devices. However, the examination is usually restricted to the specific use-case at hand. The work presented here specifically deals with a model for piezoelectric ceramics where damping in time is taken into account and temperature-related effects can be neglected due to only small loads. This guarantees a more
realistic and accurate description and prediction than other available models for the specific use-case of the authors. This model is currently used by two of the authors in a project funded by the German Research Foundation (DFG) as a means to identify the material parameters of real piezoelectric devices, see e.g. [1–3].

As mentioned previously many authors have examined partial differential equations governing piezoelectricity: Some authors consider different physical effects which the authors chose to neglect for their ongoing work. These effects may result from applying large loads onto the piezoelectric device, e.g. local change in temperature [4], Hysteresis effects [5, 6], incorporation of magnetism (leading to Maxwell’s equations) [7, 8], consideration of space-dependant material parameters [9] and also if the device is examined on a macro- or micro-mechanical scale, see also [10] for an overview from a mechanical point of view. However, for many of the resulting partial differential equations only specific results may be available which may not be applicable to a different specific use-case.

More closely related to the authors’ area of application the following publications cover analysis in time and frequency domain. For frequency-dependent formulations results can be found in [9, 11, 12] as well as [13] (with frequency \( \omega = 0 \)) and extended by the full Maxwell equations in [7]. In [12] the authors state well-posedness results with real-and complex-valued material parameters. Furthermore, in this article damping is also considered for a frequency-dependent formulation. However, this does not provide any information about the long-term behavior in time of the solutions of the governing equations. In time domain there are numerous publications dealing existence, uniqueness and regularity of solutions, see e.g. [8, 9, 14–18]. However, in these publications neither damping in time is considered nor is the geometry and the nonsmooth excitation which the authors require for further practical work addressed.

Hence, the aim of this contribution is to add results on existence, uniqueness and regularity of solutions of the governing piezoelectric partial differential equations in time and space for a formulation that includes a damping model, further provide information on expected regularity for given practical excitation and give results on the expected long-term behavior. Some of these results have been addressed by the authors of this contribution in the PhD theses [11, 19]. The considerations presented here are consistent with theoretical results gained for models without additional damping, e.g. [9, 15, 17]. Furthermore, the findings are also consistent with numerical simulations.

The structure of this paper is as follows: In the second section, we state the general setting of the problem and provide notation. In the third section, we state and prove a theorem that guarantees unique weak solutions of the governing piezoelectric equations, we provide a theorem for higher regularity of these weak solutions and give a result for the expected long-term behavior of solutions. The section ends with a numerical example that demonstrates that the theoretical results are consistent with numerical simulations of the governing equations. Lastly, in the fourth section we give a conclusion.

2. Setting

Before we can begin to solve any partial differential equation we must first establish an exact setup – the geometry \( \Omega \), the boundary \( \partial \Omega \), the boundary conditions and initial values of the partial differential equations in question. We consider the case of a mechanically unclamped piezoceramic which is excited by prescribing a voltage on a part of the boundary. Let \( \Omega \subseteq \mathbb{R}^3 \) be an open domain describing the piezoelectric ceramic and let \( \partial \Omega =: \Gamma \) be the nonempty boundary of \( \Omega \). The boundary is divided into nonempty, disjunct, covering subsets of \( \Gamma \) (see also Figure 1) which are assumed to have a positive 2D measure. Let \( \Gamma_c \) be the section of the boundary which is electrically excited, \( \Gamma_g \) the section of the boundary which is grounded, \( \Gamma_r = \Gamma \setminus (\Gamma_c \cup \Gamma_g) \) the remaining boundary section.

For the readers convenience, the usual definitions of common function spaces which will be required later on are stated in the Appendix A. Only the newly defined function spaces for the
Figure 1. Domain and boundaries of a disk-shaped piezoceramic (front quarter cut out).

Considered differential equation system are described now:

\[ H_{0,1}^{1}(\Omega) := \{ \sigma_{1} + \sigma_{2} : \sigma_{1} \in H_{0}^{1}(\Omega) \text{ and } \sigma_{2} \in H^{1}(\Omega) \} \]

\[ H_{B}^{1}(\Omega) := \{ \sigma : \Omega \to \mathbb{R}^{3} : \|\sigma\|_{H_{B}^{1}(\Omega)}^{2} := \|\sigma\|_{L^{2}(\Omega)}^{2} + \|B\sigma\|_{L^{2}(\Omega)}^{2} < \infty \}, \]

where

\[ B := \begin{pmatrix}
\frac{\partial}{\partial x} & 0 & 0 \\
0 & \frac{\partial}{\partial y} & 0 \\
0 & 0 & \frac{\partial}{\partial z} \\
\frac{\partial}{\partial z} & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \\
\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0
\end{pmatrix} \]

with \( x, y, z \) referring to Cartesian coordinates. In this paper we denote derivatives with respect to time by the dot symbol e.g. \( \dot{\sigma} \) and derivatives with respect to space by the nabla or \( B \) symbol, e.g. \( \nabla \sigma \) or \( B\sigma \). Here \( B \) denotes the symmetric gradient in Voigt notation. It should be noted that the last three entries of the matrix vector product \( Bu \) still contains the factor 2, but for simplicity, no attention is paid here. The factor can be included in the definition of the linear strain vector \( S \), where \( S = Bu \). All derivatives in the above are understood in the distributional sense. In addition, the dual space of a Hilbert space \( X \) is denoted by \( X' \). In particular, \( H^{-1}(\Omega) \) denotes the dual space of \( H_{0}^{1}(\Omega) \). Note that in order to simplify the notation superscripts indicating the dimension of \( u \) or \( Bu \), which are 3 and 6, respectively, are omitted. This is reasonable as the vectorial scalar product inside \( \int_{\Omega} \sigma^{T} \sigma \, d\Omega \) always returns a scalar no matter what dimensions \( \sigma \) has.

Let \( \vec{n} := (n_{x}, n_{y}, n_{z}) \) be the normal vector and

\[ \mathbf{N} := \begin{pmatrix}
n_{x} & 0 & 0 \\
0 & n_{y} & 0 \\
0 & 0 & n_{z} \\
n_{z} & 0 & n_{x} \\
n_{y} & n_{x} & 0
\end{pmatrix}. \]
Definition 2.1: The material parameters \( c^E, \epsilon^S \) and \( \epsilon \) ([\( c^E = N \cdot m^{-2} \), \( \epsilon^S = F \cdot m^{-1} \), \( \epsilon = C \cdot m^{-2} \)]) are given by

\[
\begin{pmatrix}
    c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\
    c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\
    c_{13} & c_{11} & c_{33} & 0 & 0 & 0 \\
    0 & 0 & 0 & c_{44} & 0 & 0 \\
    0 & 0 & 0 & 0 & c_{44} & 0 \\
    0 & 0 & 0 & 0 & 0 & \frac{1}{2}(c_{11} - c_{12})
\end{pmatrix} \in \mathbb{R}^{6 \times 6}
\]

\[
\begin{pmatrix}
    \epsilon_{11} & 0 & 0 \\
    0 & \epsilon_{11} & 0 \\
    0 & 0 & \epsilon_{33}
\end{pmatrix} \in \mathbb{R}^{3 \times 3}
\]

\[
\begin{pmatrix}
    0 & 0 & 0 & 0 & \epsilon_{15} & 0 \\
    0 & 0 & \epsilon_{15} & 0 & 0 & 0 \\
    \epsilon_{13} & \epsilon_{13} & \epsilon_{33} & 0 & 0 & 0
\end{pmatrix} \in \mathbb{R}^{6 \times 3}.
\]

The material parameters are said to fulfill nonnegativity conditions if \( c^E \) and \( \epsilon^S \) are positive definite matrices.

The sparsity pattern and nonnegativity conditions are correct for many piezoelectric ceramics and correspond to the assumption of a transversely isotropic solid [10, 20].

The three-dimensional transient linear piezoelectric equations with Rayleigh damping parameters \( \alpha, \beta > 0 \) (chosen sufficiently large enough so that the system is parabolic) and density \( \rho > 0 \) describing the mechanical displacement \( u \in \mathbb{R}^3 \) and the electrical potential \( \phi \in \mathbb{R} \) with given boundary conditions are stated as:

\[
\rho \ddot{u}(t) + \alpha \rho \dot{u}(t) - B^T (c^E \dot{u}(t) + \beta c^E B \dot{u}(t) + \epsilon^S \nabla \phi(t)) = 0 \text{ in } \Omega \times [0, T],
\]

\[
- \nabla \cdot \left( \epsilon B \dot{u}(t) - \epsilon^S \nabla \phi(t) \right) = 0 \text{ in } \Omega \times [0, T],
\]

\[
\phi(t) = 0 \text{ on } \Gamma_g \times [0, T],
\]

\[
\phi(t) = \phi^e(t) \text{ on } \Gamma_e \times [0, T],
\]

\[
\nabla \cdot \left( \epsilon B \dot{u}(t) - \epsilon^S \nabla \phi(t) \right) = 0 \text{ on } \Gamma_r \times [0, T],
\]

\[
\mathcal{N}^T (c^E \dot{u}(t) + \beta c^E B \dot{u}(t) + \epsilon^S \nabla \phi(t)) = 0 \text{ on } \partial \Omega \times [0, T],
\]

\[
\begin{align*}
    u(0) &= u_0, \\
    \dot{u}(0) &= \dot{u}_1.
\end{align*}
\]

The weak form of the equations above can easily be obtained [11] by testing with appropriate functions \( \nu \in \mathbb{R}^3 \) (for the first line) and \( w \in \mathbb{R} \) (for the second line), integration by parts and using boundary conditions:

\[
\int_{\Omega} (B^T \sigma)^T \nu \, d\Omega = -\int_{\Omega} \sigma^T B \nu \, d\Omega + \int_{\partial \Omega} \left( \mathcal{N}^T \sigma \right)^T \nu \, d\Omega.
\]

First, we use a Dirichlet lift ansatz to homogenize the Dirichlet boundary condition for \( \phi(t) \): Let \( t \in [0, T] \) and let \( \chi \in H^1(\Omega) \) where \( \chi|_{\Gamma_g} = 0 \) and \( \chi|_{\Gamma_e} = 1 \). Such a \( \chi \) exists if we assume that \( \Omega \) is at least a Lipschitz domain. Let \( \phi(t) \) consist of two parts \( \phi(t) = \phi_0(t) + \phi_{\phi^e}(t) \) where \( \phi_0(t) \in H^2_0(\Omega) \) and \( \phi_{\phi^e}(t) \in H^1(\Omega) \). We then rewrite \( \phi_{\phi^e}(t) = \phi^e(t) \chi \). Therefore we set \( \phi_0(t) := \phi(t) - \phi^e(t) \chi \).
As $\phi^e(t)$ is a given value $\phi^e(t)\chi$ can be taken out of the left-hand side of the weak form and added to the right-hand side. The weak form of the piezoelectric system for all $t \in [0, T]$ a.e. and for all test functions $(v, w) \in H^1_0(\Omega) \times H^0_1(\Omega)$ is given by

\[
\int_\Omega \rho \ddot{u}^T v \, d\Omega + \alpha \int_\Omega \rho T \dot{v} \, d\Omega + \int_\Omega (e^E B \dot{u})^T B v \, d\Omega + \beta \int_\Omega (e^E \dot{B} \dot{u})^T B v \, d\Omega
\]
\[+ \int_\Omega (e^E \nabla \phi_0)^T B v \, d\Omega + \int_\Omega (e^E B u)^T \nabla w \, d\Omega - \int_\Omega (e^S \nabla \phi_0)^T \nabla w \, d\Omega
\]
\[= \phi^e \int_\Omega - (e^T \nabla \chi)^T B v + (e^S \nabla \chi)^T \nabla w \, d\Omega. \tag{1}
\]

### 3. Existence, uniqueness and regularity of solutions

Before we attempt to show existence, uniqueness and regularity of solutions some additional tools are required:

**Lemma 3.1 (Young inequality):** Let $1 < p, q < \infty, \frac{1}{p} + \frac{1}{q} = 1$. Then for $a, b > 0$ the following inequality holds:

\[ab \leq \frac{a^p}{p} + \frac{b^q}{q}.
\]

**Proof:** See [21, Appendix B.2].

**Lemma 3.2 (Hölder inequality):** Let $1 \leq p, q \leq \infty, \frac{1}{p} + \frac{1}{q} = 1$. Then for $u \in L^p(\Omega), v \in L^q(\Omega)$ the following inequality holds:

\[
\int_\Omega |uv| \, dx \leq \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}.
\]

**Proof:** See [21, Appendix B.2].

**Remark 3.3:** The latter two inequalities are especially true for $p = q = 2$. The latter inequality is then known as Cauchy–Schwarz (C.S.) inequality.

**Lemma 3.4 (Gronwall inequality, integral form):**

a) Let $\eta : [0, T] \rightarrow \mathbb{R}^{\geq 0}$ be a nonnegative, summable function on $[0, T]$, which satisfies for almost every $t$ the differential inequality

\[\eta(t) \leq C_1 \int_0^t \eta(s) \, ds + C_2 \]

for constants $C_1, C_2 \geq 0$. Then

\[\eta(t) \leq C_2 \left(1 + C_1 t e^{C_1 t}\right)
\]

for a.e. $0 \leq t \leq T$.

b) In particular, if

\[\eta(t) \leq C_1 \int_0^t \eta(s) \, ds
\]

for a.e. $t \in [0, T]$, then

\[\eta(t) = 0 \text{ a.e.}
\]

**Proof:** See [21, Appendix B.2].
Remark 3.5 (Sufficiently smooth boundary): We say the boundary $\partial \Omega$ is sufficiently smooth if it permits application of the trace theorem (cf. [21]).

Thus, a $C^1$-boundary is sufficient. However, it is possible to utilize a variation of the trace theorem under less strict requirements (cf. [22]). We note that the boundary for our specific application (see Figure 1) satisfies the special Lipschitz condition stated in Definition 5 of [22] and thus it can also be considered sufficiently smooth.

The proof follows the usual guideline as seen for many partial differential equations (e.g. [21]): We get existence and uniqueness of a weak solution by the usual procedure:

1. Discretization via Galerkin approximation of infinite-dimensional function spaces,
2. energy estimates via Gronwall inequality in discretized space which provide finiteness of the discretized solution,
3. weak limit of discretized solution provides weak existence of a solution in infinite-dimensional function space,
4. uniqueness of the solution is shown by applying estimates to the difference $w := w_1 - w_2$ of two solutions $w_1$ and $w_2$. Thus, the only solution to the homogeneous case is the trivial solution.

Theorem 3.6: Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with sufficiently smooth boundary as specified in Remark 3.5. Let the real valued material parameters $c^E, c$ and $\varepsilon^S$ be defined as in Definition 2.1 and let $c^E$ and $\varepsilon^S$ be symmetric and positive definite. The Rayleigh coefficients $\alpha$ and $\beta$ are assumed to be nonnegative. Let $T > 0$ and $\rho > 0$.

Then there exists a $C > 0$ such that for any $u_0 \in H^1_B(\Omega), u_1 \in L^2(\Omega)$ and $\phi^\varepsilon \in H^1(0, T; H^{1/2}(\Gamma))$ there exists a unique solution

$$\begin{equation}
(u, \phi) \in L^\infty(0, T; H^1_B(\Omega)) \times L^\infty(0, T; H^1_{0, \Gamma}(\Omega))
\end{equation}$$

with

$$\begin{equation}
\hat{u} \in L^\infty(0, T; L^2(\Omega)) \text{ and } \check{u} \in L^2(0, T; (H^1_B(\Omega))')
\end{equation}$$

to Equation (1) satisfying the initial conditions

$$u(0) = u_0, \quad \hat{u}(0) = u_1 \text{ on } \Omega$$

and the following estimate holds:

$$\|u\|_{L^\infty(0, T; H^1_B(\Omega))} + \|\hat{u}\|_{L^\infty(0, T; L^2(\Omega))} + \|\check{u}\|_{L^2(0, T; (H^1_B(\Omega))')} + \|\phi\|_{L^\infty(0, T; H^1_{0, \Gamma}(\Omega))} \leq C \left(\|u_0\|_{H^1_B(\Omega)} + \|u_1\|_{L^2(\Omega)} + \|\phi^\varepsilon\|_{H^1(0, T; H^{1/2}(\Gamma))}\right).$$

Proof: Note that many concepts of this proof are taken from [21, chapter 7] and information regarding involved spaces can be found in [23].

In the following constants denoted by the letter $C$ or $\tilde{C}$ are used. Unless explicitly specified otherwise we note that all these constants are positive $C_i > 0, i \geq 1$.

Weak solutions are functions $u, \hat{u}, \check{u}$ and $\phi_0$ as in Equations (2) and (3) where $\phi_0 = \phi + \phi_\varepsilon$ such that for almost all $t \in [0, T]$ for all $(v, w) \in H^1_B(\Omega) \times H^1_0(\Omega)$ the following equation holds:

$$\langle \rho \hat{u}(t), v \rangle + \alpha \langle \hat{u}(t), v \rangle + \langle c^E B u(t), B v \rangle + \beta \langle c^E \check{u}(t), B v \rangle + \langle e^\varepsilon \nabla \phi_0(t), B v \rangle + \langle e B \hat{u}(t), \nabla w \rangle - \langle e^\varepsilon \nabla \phi_0(t), \nabla w \rangle = \langle f(t), v \rangle + \langle g(t), w \rangle$$

(5)
with
\[ \{f(t), \nu\} := -\phi^e(t) \int_{\Omega} (e^T \nabla \chi)^T B \nu \, d\Omega, \]
and
\[ \{g(t), w\} := \phi^e(t) \int_{\Omega} (e^S \nabla \chi)^T \nabla w \, d\Omega. \]

Note that by the Riesz representation theorem there exists a unique representation for the latter functionals as an inner product, i.e. \( \{f, \cdot\}\) and \( \{g, \cdot\}\). As is common in the field of partial differential equation for convenience we will also use the same symbols \( f \) and \( g \) to refer to the Riesz-representative as well as the functionals \( \{f, \cdot\}\) and \( \{g, \cdot\}\). Furthermore, we remember that \( \chi \in H^1(\Omega) \) and that \( e^S, e \) are constant. The integrals of the right-hand side \( \int_{\Omega} (e^T \nabla \chi)^T B \nu \, d\Omega \), \( \int_{\Omega} (e^S \nabla \chi)^T \nabla w \, d\Omega \) are finite, their values \( c_1(\Omega), c_2(\Omega) < \infty \) depend, e.g. only on domain \( \Omega \) but not on time \( t \). Thus, by integrating this constant value over time we can estimate the Bochner-space norm of \( f \) by
\[ \|f\|_{H^1(0,T; (H^1_0(\Omega))^\prime)} \leq c_1(\Omega) \|\phi^e\|_{H^1(0,T)}, \]
and analogously we get
\[ \|g\|_{H^1(0,T; H^{-1}(\Omega))} \leq c_2(\Omega) \|\phi^e\|_{H^1(0,T)}. \]

**Phase 1: Galerkin approximation**

The weak form is tested with test functions \( \nu_j \in H^1_{L^2}(\Omega) \) and \( w_j \in H^1_0(\Omega) \), \( j \in \mathbb{N} \), with
\[ u(t) \approx u_m(t) = \sum_{j=1}^{m} u^j_{m}(t) \nu_j, \]
and
\[ \phi_0(t) \approx \phi_m(t) = \sum_{j=1}^{m} \phi^j_{m}(t) w_j, \]
where \( \approx \) is to be understood in the sense of an orthogonal projection in the appropriate spaces. The finite-dimensional spaces spanned by the test functions are defined as
\[ V_m := \text{span}\{\nu_1, \ldots, \nu_m\} \quad \text{and} \quad W_m := \text{span}\{w_1, \ldots, w_m\}. \]

We can assume that the dimension of the test function spaces \( \dim(V_m) = \dim(W_m) = m \) are the same, for \( V_m \) in each vectorial component. So the test functions can be selected to be linearly independent. Furthermore, the functions can be chosen such that
\[ \bigcup_{m=1}^{\infty} V_m = H^1_{L^2}(\Omega) \quad \text{and} \quad \bigcup_{m=1}^{\infty} W_m = H^1_0(\Omega). \]

Then via standard theory for ordinary differential equations (see e.g. [21] or [9]) for all \( m \in \mathbb{N} \) and for all \((\nu_m, w_m) \in V_m \times W_m\) there exists a unique solution
\[ (u_m, \phi_m) \in C^2([0, T]; V_m) \times C([0, T]; W_m) \]
to the discretized version of Equation (5) that fulfills the initial conditions \( u_m(0) = (u_0)_m, \dot{u}_m(0) = (u_1)_m. \)

**Phase 2: Energy estimates**

The aim of this phase is to use Gronwall inequality to show an energy estimate from which the finiteness of the finite-dimensional solutions \((u_m(t), \phi_m(t)) \in L^\infty(0, T; H^1_{L^2}(\Omega)) \times L^\infty(0, T; H^1_0(\Omega)), \)
\( \dot{u}_m \in L^\infty(0, T; L^2(\Omega)) \) and \( \ddot{u}_m \) in \( L^2(0, T; (H^1_{L^2}(\Omega))^\prime) \) can be deduced:
Let 
\[
\eta(t) := \left( \| \dot{u}_m(t) \|_{L^2(\Omega)}^2 + \| u_m(t) \|_{H^1_0(\Omega)}^2 + \| \phi_m(t) \|_{H^1_0(\Omega)}^2 \right).
\]

In order to use Gronwall inequality, we must show that there are constants \( p, q \geq 0 \) such that \( \eta(t) \leq p \int_0^t \eta(s) \, ds + q \) holds. If this condition is true, then it can be shown that
\[
\| \dot{u}_m(t) \|_{L^2(\Omega)}^2 + \| u_m(t) \|_{H^1_0(\Omega)}^2 + \| \phi_m(t) \|_{H^1_0(\Omega)}^2 \\
\leq (1 + pte^{pt}) \left( \| \dot{u}_m(0) \|_{L^2(\Omega)}^2 + \| u_m(0) \|_{H^1_0(\Omega)}^2 + \| \phi_m(0) \|_{H^1_0(\Omega)}^2 \\
+ \| f \|_{L^2(0,T;H^1_0(\Omega))}^2 \right)
\]
holds almost everywhere in \([0, T]\). Thus, this must also be true for the essential supremum over \( 0 \leq t \leq T \) and we will get finiteness in the \( L^\infty(0, T; X) \) norm for the appropriate sub-spaces \( X \). In order to show the requirement we consider the following:

First, the discretized version of the weak form Equation (5) is supposed to hold for all test functions \((v_m, w_m)\). Thus, it should also hold for \((\tilde{u}_m(t), 0)\):
\[
\langle \rho \dot{u}_m(t), \dot{u}_m(t) \rangle + \alpha \langle \rho \dot{u}_m(t), \ddot{u}_m(t) \rangle + \{ c^E B u_m(t), B \ddot{u}_m(t) \} \\
+ \beta \{ c^E B \dot{u}_m(t), E \dot{u}_m(t) \} + \{ e^T \nabla \phi_m(t), E u_m(t) \} = \{ f(t), \dot{u}_m(t) \}.
\]

By transposing the inner product and direct computation, it is easy to see that one can swap the placement of constant scalars or matrices such as \( \rho, c^S, c^E \) etc. (which are symmetric) in this bilinear form, e.g. the following holds:
\[
\{ c^E B u_m(t), B \dot{u}_m(t) \} = \{ (c^E)^T B \dot{u}_m(t), B u_m(t) \} = \{ c^E B \dot{u}_m(t), B u_m(t) \}.
\]
Thus, by bilinearity of the inner product
\[
2 \{ c^E B \dot{u}_m(t), B u_m(t) \} = \frac{d}{dt} \{ c^E B u_m(t), B u_m(t) \}.
\]
Hence, the above equation simplifies to
\[
\frac{1}{2} \frac{d}{dt} \left( \langle \rho \dot{u}_m(t), \ddot{u}_m(t) \rangle + \{ c^E B u_m(t), B \dot{u}_m(t) \} \right) + \alpha \langle \rho \dot{u}_m(t), \dot{u}_m(t) \rangle \\
+ \beta \{ c^E B \dot{u}_m(t), E \dot{u}_m(t) \} + \{ e^T \nabla \phi_m(t), E u_m(t) \} = \{ f(t), \dot{u}_m(t) \}.
\]
Now we differentiate the weak form Equation (5) with respect to \( t \) and test it with \((0, \phi_m(t))\), taking into account that the test functions \( v_m, w_m \) do not depend on time \( t \), therefore the time derivatives \( \dot{v}_m, \dot{w}_m \equiv 0\):
\[
\langle cB \dot{u}_m(t), \nabla \phi_m(t) \rangle - \frac{1}{2} \frac{d}{dt} \{ c^S \nabla \phi_m(t), \nabla \phi_m(t) \} = \{ \dot{g}(t), \phi_m(t) \}.
\]
A subtraction of Equations (7) and (8) gives
\[
\frac{1}{2} \frac{d}{dt} \left( \langle \rho \ddot{u}_m(t), \ddot{u}_m(t) \rangle + \{ c^E B u_m(t), B \dot{u}_m(t) \} + \{ c^S \nabla \phi_m(t), \nabla \phi_m(t) \} \right) \\
+ \alpha \langle \rho \dot{u}_m(t), \dot{u}_m(t) \rangle + \beta \{ c^E B \dot{u}_m(t), E \dot{u}_m(t) \}
\]
\[
= \{ f(t), \dot{u}_m(t) \} - \{ \dot{g}(t), \phi_m(t) \}.
\]
The last equation Equation (9) is integrated with respect to \( t \).

\[
\mathcal{F}_t(t) := \langle \rho \dot{u}_m(t), \dot{u}_m(t) \rangle + \langle c^E B u_m(t), B u_m(t) \rangle + \left( \varepsilon S \nabla \phi_m(t), \nabla \phi_m(t) \right) \\
+ 2 \alpha \int_0^t \langle \rho \dot{u}_m(s), \dot{u}_m(s) \rangle \, ds + 2 \beta \int_0^t \langle c^E B \dot{u}_m(s), \dot{B} u_m(s) \rangle \, ds \\
= \langle \rho \dot{u}_m(0), \dot{u}_m(0) \rangle + \langle c^E B u_m(0), B u_m(0) \rangle + \left( \varepsilon S \nabla \phi_m(0), \nabla \phi_m(0) \right) \\
+ 2 \int_0^t \left( f(t), \phi_m(s) \right) \, ds =: \mathcal{F}_r(t).
\] (10)

Hence, in short we can write

\[
\mathcal{F}_t(t) = \mathcal{F}_r(t).
\]

Now the aim is to use this equation to show that the requirements for Gronwall inequality are met.

We start by showing that the left-hand side \( \mathcal{F}_r(t) \) of Equation (10) has a lower bound. With \( \lambda_{1,\text{mech}} \) the smallest eigenvalue of \( c^E \) (which is strictly positive) one estimates

\[
\int_{\Omega} (B u_m(t))^T c^E B u_m(t) \, d\Omega \geq \lambda_{1,\text{mech}} \int_{\Omega} (B u_m(t))^T B u_m(t) \, d\Omega \\
= \lambda_{1,\text{mech}} \| B u_m(t) \|^2_{L^2(\Omega)} \\
= \lambda_{1,\text{mech}} \left( \| u_m(t) \|^2_{H^1_0(\Omega)} - \| u_m(t) \|^2_{L^2(\Omega)} \right). \tag{11}
\]

With \( \lambda_{1,\text{elec}} \) the smallest eigenvalue of \( \varepsilon S \) (which is strictly positive) one estimates

\[
\int_{\Omega} (\nabla \phi_m(t))^T \varepsilon S \nabla \phi_m(t) \, d\Omega \geq \lambda_{1,\text{elec}} \int_{\Omega} (\nabla \phi_m(t))^T \nabla \phi_m(t) \, d\Omega \\
= \lambda_{1,\text{elec}} \| \nabla \phi_m(t) \|^2_{L^2(\Omega)}.
\]

From the Poincaré inequality (see e.g. [24]), we obtain \( c_1, c_2 \in \mathbb{R} \) such that

\[
\int_{\Omega} (\nabla \phi_m(t))^T \varepsilon S \nabla \phi_m(t) \, d\Omega \geq \lambda_{1,\text{elec}} \| \nabla \phi_m(t) \|^2_{L^2(\Omega)} \\
= (1 + c_2) c_1 \| \nabla \phi_m(t) \|^2_{L^2(\Omega)} = c_1 \left( c_2 \| \nabla \phi_m(t) \|^2_{L^2(\Omega)} + \| \nabla \phi_m(t) \|^2_{L^2(\Omega)} \right) \\
\geq C_{\text{elec}} \| \phi_m(t) \|^2_{H^1_0(\Omega)} \tag{12}
\]

By nonnegativity of \( \rho, \alpha, \beta \) and the two inequalities Equations (11) and (12) one can now estimate

\[
C_1 \left( \| \dot{u}_m(t) \|^2_{L^2(\Omega)} + \| u_m(t) \|^2_{H^1_0(\Omega)} + \| \phi_m(t) \|^2_{H^1_0(\Omega)} - \| u_m(t) \|^2_{L^2(\Omega)} \right) \leq \mathcal{F}_r(t)
\]

with a positive constant \( C_1 > 0 \). Furthermore, by the inequalities Equation (11) and Equation (12) and Cauchy–Schwarz and Young inequalities the right-hand side \( \mathcal{F}_r(t) \) can be bounded from above with \( \tilde{c}, \tilde{c} > 0 \):

\[
\mathcal{F}_r(t) = \langle \rho \dot{u}_m(0), \dot{u}_m(0) \rangle + \langle c^E B u_m(0), B u_m(0) \rangle + \left( \varepsilon S \nabla \phi_m(0), \nabla \phi_m(0) \right) \\
\leq \tilde{c} \| u_m(0) \|^2_{H^1_0(\Omega)} \leq \tilde{c} \| \phi_m(0) \|^2_{H^1_0(\Omega)}
\]
\[ + 2 \int_0^t \langle f(s), \dot{u}_m(s) \rangle \, ds - 2 \int_0^t \| \dot{g}(s), \phi_m(s) \| \, ds \]

\[ \leq \widehat{C}_2 \left( \| \dot{u}_m(0) \|_{L^2(\Omega)}^2 + \| u_m(0) \|_{H^1(\Omega)}^2 + \| \phi_m(0) \|_{H^1(\Omega)}^2 \right) 
+ 2 \int_0^t \| f(s), \dot{u}_m(s) \| \, ds + 2 \int_0^t \| g(s), \phi_m(s) \| \, ds \]

\[ \leq \widehat{C}_2 \left( \| \dot{u}_m(0) \|_{L^2(\Omega)}^2 + \| u_m(0) \|_{H^1(\Omega)}^2 + \| \phi_m(0) \|_{H^1(\Omega)}^2 \right) 
+ \int_0^t \| \dot{u}_m(s) \|_{L^2(\Omega)}^2 + \| u_m(s) \|_{H^1(\Omega)}^2 + \| \phi_m(s) \|_{H^1(\Omega)}^2 \, ds \]

\[ + 2 \| f \|^2_{L^2(0,T;(H^1(\Omega))')} + 2 \| g \|^2_{H^1(0,T;H^{-1}(\Omega))} \]

Hence, we get

\[ F_r(t) \leq C_2 \left( \| \dot{u}_m(0) \|_{L^2(\Omega)}^2 + \| u_m(0) \|_{H^1(\Omega)}^2 + \| \phi_m(0) \|_{H^1(\Omega)}^2 \right) 
+ \int_0^t \| \dot{u}_m(s) \|_{L^2(\Omega)}^2 + \| u_m(s) \|_{H^1(\Omega)}^2 + \| \phi_m(s) \|_{H^1(\Omega)}^2 \, ds \]

\[ + 2 \| f \|^2_{L^2(0,T;(H^1(\Omega))')} + \| g \|^2_{H^1(0,T;H^{-1}(\Omega))} \tag{13} \]

with a positive constant \( C_2 > 0 \). As \( F_l(t) = F_r(t) \) it is now clear that

\[ C_1 \left( \| \dot{u}_m(t) \|_{L^2(\Omega)}^2 + \| u_m(t) \|_{H^1(\Omega)}^2 + \| \phi_m(t) \|_{H^1(\Omega)}^2 - c_{\text{mech}} \| u_m(t) \|_{L^2(\Omega)}^2 \right) \]

\[ \leq C_2 \left( \| \dot{u}_m(0) \|_{L^2(\Omega)}^2 + \| u_m(0) \|_{H^1(\Omega)}^2 + \| \phi_m(0) \|_{H^1(\Omega)}^2 \right) 
+ \int_0^t \| \dot{u}_m(s) \|_{L^2(\Omega)}^2 + \| u_m(s) \|_{H^1(\Omega)}^2 + \| \phi_m(s) \|_{H^1(\Omega)}^2 \, ds \]

\[ + 2 \| f \|^2_{L^2(0,T;(H^1(\Omega))')} + \| g \|^2_{H^1(0,T;H^{-1}(\Omega))} \tag{14} \]

Utilizing the inequality \( \| u(t) \|_{L^2(\Omega)}^2 \leq 2 \| u(0) \|_{L^2(\Omega)}^2 + 2T \int_0^t \| \dot{u}(s) \|_{L^2(\Omega)}^2 \, ds \) (see [25, p. 425] or [15]) for \( T \) large enough, we can remove \( c_{\text{mech}} \| u_m(t) \|_{L^2(\Omega)}^2 \) from the left-hand side of the inequality to obtain:

\[ C_1 \left( \| \dot{u}_m(t) \|_{L^2(\Omega)}^2 + \| u_m(t) \|_{H^1(\Omega)}^2 + \| \phi_m(t) \|_{H^1(\Omega)}^2 \right) \]

\[ \leq C_3 \left( \| \dot{u}_m(0) \|_{L^2(\Omega)}^2 + \| u_m(0) \|_{H^1(\Omega)}^2 + \| \phi_m(0) \|_{H^1(\Omega)}^2 \right) 
+ C_4 \int_0^t \| \dot{u}_m(s) \|_{L^2(\Omega)}^2 + \| u_m(s) \|_{H^1(\Omega)}^2 + \| \phi_m(s) \|_{H^1(\Omega)}^2 \, ds \]

\[ + 2 \| f \|^2_{L^2(0,T;(H^1(\Omega))')} + \| g \|^2_{H^1(0,T;H^{-1}(\Omega))} \tag{15} \]

where \( C_4 > 0 \) now also depends on the fixed value \( T \).
Let
\[ \eta(t) := \|\ddot{u}_m(t)\|^2_{L^2(\Omega)} + \|u_m(t)\|^2_{H^1_0(\Omega)} + \|\phi_m(t)\|^2_{H^1_0(\Omega)} \]
and let
\[ \tilde{C}_2 := \frac{1}{C_1} \left( \frac{C_3 \eta(0)}{C_1} + \|f\|^2_{L^2(0,T;H^1_0(\Omega))'} + \|g\|^2_{H^1(0,T;H^{-1}(\Omega))} \right) \geq 0. \]
Then the above inequality simplifies to
\[ \eta(t) \leq \frac{C_4}{C_1} \int_0^t \eta(s) \, ds + \tilde{C}_2. \]
Hence, all requirements for Gronwall inequality have been shown to hold and it can now be safely applied and the result simplified to:
\[
\begin{align*}
\|\ddot{u}_m(t)\|^2_{L^2(\Omega)} + \|u_m(t)\|^2_{H^1_0(\Omega)} + \|\phi_m(t)\|^2_{H^1_0(\Omega)} & \leq \left( \frac{C_3}{C_1} + \frac{C_4 \tilde{C}_3}{C_1} t e^{\tilde{C}_1 t} \right) \left( \|\ddot{u}_m(0)\|^2_{L^2(\Omega)} + \|u_m(0)\|^2_{H^1_0(\Omega)} + \|\phi_m(0)\|^2_{H^1_0(\Omega)} \right) \\
& \quad + \|f\|^2_{L^2(0,T;H^1_0(\Omega))'} + \|g\|^2_{H^1(0,T;H^{-1}(\Omega))} \tag{16}
\end{align*}
\]
holds almost everywhere in \([0, T]\).

We will return to this inequality shortly after considering the bilinear form
\[ A : H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbb{R}, \quad A(\phi_m(t), w) := \left( \epsilon \nabla \phi_m(t), \nabla w \right) \tag{17} \]
and the continuous linear functional on \(H^1_0(\Omega)\) for a fixed \(u_m(t)\)
\[ b(w) := \langle \epsilon \nabla u_m(t), \nabla w \rangle - \langle g(t), w \rangle \]
which together form the weak form Equation (5) tested by \((0, w)\). This bilinear form \(A\) is coercive (inequality Equation (12)) and continuous:
\[
\begin{align*}
\left| \left( \epsilon \nabla \phi_m(t), \nabla w \right) \right| & \leq \lambda_{\text{max}} \|\nabla \phi_m(t)\|_{L^2(\Omega)} \cdot \|\nabla w\|_{L^2(\Omega)} \\
& \leq \lambda_{\text{max}} \|\nabla \phi_m(t)\|_{L^2(\Omega)} \cdot \|w\|_{H^1_0(\Omega)} \\
& \leq \lambda_{\text{max}} \|\phi_m(t)\|_{H^1_0(\Omega)} \cdot \|w\|_{H^1_0(\Omega)}
\end{align*}
\]
Using Lax–Milgram lemma and the Young inequality we get the estimate for \(A(\phi_m(t), w) = b(w)\) \(\forall w \in H^1_0(\Omega)\):
\[
\begin{align*}
\|\phi_m(t)\|^2_{H^1_0(\Omega)} & \leq \tilde{M} \|b\|^2_{H^{-1}(\Omega)} \\
& = \tilde{M} \sup_{\|w\|_{H^1_0(\Omega)} \leq 1} \|b(w)\|^2_{H_0^1(\Omega)} \\
& = \tilde{M} \sup_{\|w\|_{H^1_0(\Omega)} \leq 1} \left| \langle \epsilon \nabla u_m(t), \nabla w \rangle - \langle g(t), w \rangle \right|^2 \\
& \leq \tilde{M} \sup_{\|w\|_{H^1_0(\Omega)} \leq 1} \left( \|\epsilon \nabla u_m(t), \nabla w \| + \|g(t), w\| \right)^2
\end{align*}
\]
\[ \leq M \sup_{\|w\|_{H^1_0(\Omega)} \leq 1} \left( 2 \frac{|\langle eB \mathbf{u}_m(t), \nabla w \rangle|^2}{C_S \leq \|eB \mathbf{u}_m(t)\|_{L^2(\Omega)}^2 \|w\|_{H^1_0(\Omega)}^2} + 2 \|g(t), w\|^2 \right) \]

\[ \leq 2M \left( \|eB \mathbf{u}_m(t)\|_{L^2(\Omega)}^2 + \|g(t)\|_{H^{-1}(\Omega)}^2 \right). \]  

(18)

Furthermore, for \( t = 0 \) we get

\[ \|\phi_m(0)\|_{H^1_0(\Omega)}^2 \leq 2M \left( \|eB \mathbf{u}_m(0)\|_{L^2(\Omega)}^2 + \|g(0)\|_{H^{-1}(\Omega)}^2 \right). \]  

(19)

Hence, we obtain

\[ \|\phi_m(0)\|_{H^1_0(\Omega)}^2 \leq C_5 \left( \|(\mathbf{u}_0)_m\|_{H^1_0(\Omega)}^2 + \|\phi^e(0)\|_{H^{1/2}(\Omega)}^2 \right). \]

Finally, from Gronwall inequality we can thus deduce

\[ \|\dot{\mathbf{u}}_m\|_{L^\infty(0,T;L^2(\Omega))} + \|\mathbf{u}_m\|_{L^\infty(0,T;H^1_0(\Omega))} + \|\phi_m\|_{L^\infty(0,T;H^1_0(\Omega))} \]

\[ \leq C_6 \left( \|(\mathbf{u}_1)_m\|_{L^2(\Omega)} + \|(\mathbf{u}_0)_m\|_{H^1_0(\Omega)}^2 + \|\phi^e\|_{L^\infty(0,T;H^{1/2}(\Omega))} \right). \]  

(20)

Now knowing that all these values are finite we can deduce from Equation (10) with \( \beta > 0 \) that also

\[ \beta \|B \dot{\mathbf{u}}_m\|_{L^2(0,T;L^2(\Omega))} < \infty. \]  

(21)

It now remains to show that \( \|\dot{\mathbf{u}}_m\|_{L^2(0,T;H^1_0(\Omega))'} \) is finite.

Fix any \( \tilde{v} \in H^1_0(\Omega) \) with \( \|\tilde{v}\|_{H^1_0(\Omega)} \leq 1 \) and \( \tilde{v} := \tilde{v}^1 + \tilde{v}^2 \) with \( \tilde{v}^1 \in \text{span}\{v_i\}_{i=1}^m \) and \( \langle \tilde{v}^2, v_i \rangle = 0 \) for all \( 1 \leq i \leq m \). Since \( \{v_i\}_{i=0}^m \) can be assumed orthogonal in \( H^1_0(\Omega) \),

\[ \|\tilde{v}^1\|_{H^1_0(\Omega)} \leq \|\tilde{v}\|_{H^1_0(\Omega)} \leq 1. \]

Now with \( \mathbf{u}_m = \sum_{i=0}^m u^i_m(t) v_i \) the following holds almost everywhere in \( [0, T] \):

\[ \langle \dot{\mathbf{u}}_m(t), \tilde{v} \rangle_{H^1_0(\Omega)} = \langle \dot{\mathbf{u}}_m(t), \tilde{v} \rangle = \langle \dot{\mathbf{u}}_m(t), \tilde{v}^1 \rangle \]

\[ = \langle f(t), \tilde{v} \rangle - \langle e^T B \mathbf{u}_m(t), B \tilde{v} \rangle - \langle e^T \nabla \phi_m(t), B \tilde{v} \rangle \]

\[ - \alpha \langle \rho \dot{\mathbf{u}}_m(t), \tilde{v}^1 \rangle - \beta \langle e^T B \dot{\mathbf{u}}_m(t), B \tilde{v} \rangle, \]

where the subscript \( H^1_0(\Omega) \) denotes the duality pairing between \( (H^1_0(\Omega))' \) and \( H^1_0(\Omega) \).

Using the Cauchy–Schwarz inequality, we can deduce

\[ \left| \langle \dot{\mathbf{u}}_m(t), \tilde{v} \rangle_{H^1_0(\Omega)} \right| \]

\[ \leq C_7 \left( \|f(t)\|_{(H^1_0(\Omega))'} + \|\mathbf{u}_m(t)\|_{H^1_0(\Omega)} + \|\dot{\mathbf{u}}_m(t)\|_{L^2(\Omega)} + \|B \dot{\mathbf{u}}_m(t)\|_{L^2(\Omega)} \right) \]

\[ + \left| \langle e^T \nabla \phi_m(t), B \tilde{v} \rangle \right|. \]  

(22)
Using Lax–Milgram lemma again on the form Equation (17) for an arbitrary \( t \in [0, T] \), we can further deduce with analogous arguments as in Equation (18) that the following holds
\[
\left| \left( e^T \nabla \phi_m(t), \mathcal{B} \tilde{v}^1 \right)_{H_B^1(\Omega)} \right| \leq M \left( \| \mathcal{B} u_m(t) \|_{L^2(\Omega)}^2 + \| \phi^e(t) \|_{H^{1/2}(\Gamma_e)}^2 \right).
\]

Thus by repetitive application of the Young inequality we get for the norm
\[
\| \tilde{u}_m(t) \|_{H_B^1(\Omega)}^2 = \sup_{\| v \|_{H_B^1(\Omega)} \leq 1} |(\tilde{u}_m(t), v)_{H_B^1(\Omega)}^2 \leq C_8 \left( \| f(t) \|_{H_B^1(\Omega)}^2 + \| u_m(t) \|_{H_B^1(\Omega)}^2 + \| \tilde{u}_m(t) \|_{L^2(\Omega)}^2 + \| \mathcal{B} \tilde{u}_m(t) \|_{L^2(\Omega)}^2 \right.
\]
\[
\left. + \| \phi^e(t) \|_{H^{1/2}(\Gamma_e)}^2 \right).
\]

Now we have finiteness for all components, hence we can finally integrate inequality Equation (22) over \([0, T]\).

We rearrange the terms and apply the estimates Equations (20) and (21).
\[
\int_0^T \| \tilde{u}_m(s) \|_{H_B^1(\Omega)}^2 \, ds \leq C_9 \int_0^T \left( \| f(s) \|_{H_B^1(\Omega)}^2 + \| u_m(s) \|_{H_B^1(\Omega)}^2 + \| \tilde{u}_m(s) \|_{L^2(\Omega)}^2 + \| \mathcal{B} \tilde{u}_m(s) \|_{L^2(\Omega)}^2 \right.
\]
\[
\left. + \| \phi^e(s) \|_{H^{1/2}(\Gamma_e)}^2 \right) \, ds \leq C_{10} \left( \| (u_0)_{m} \|_{H_B^1(\Omega)}^2 + \| (u_1)_{m} \|_{L^2(\Omega)}^2 + \| \phi^e \|_{H^1(0,T;H^{1/2}(\Gamma_e))}^2 \right) \tag{23}
\]

Thus, it is now clear that
\[
(u_m, \phi_m) \in L^\infty(0, T; H_B^1(\Omega)) \times L^\infty(0, T; H_0^1(\Omega)),
\]
\[
\dot{u}_m \in L^\infty(0, T; L^2(\Omega)),
\]
\[
\ddot{u}_m \in L^2(0, T; (H_B^1(\Omega))^\prime). \tag{24}
\]

**Phase 3: Weak limit**

Following e.g. [21, 24] from the energy estimates Equations (20) and (23) we get the boundedness of the sequences
\[
(u_m, \phi_m)_{m=1}^{\infty} \text{ in } L^\infty(0, T; H_B^1(\Omega)) \times L^\infty(0, T; H_0^1(\Omega)),
\]
\[
(\dot{u}_m)_{m=1}^{\infty} \text{ in } L^\infty(0, T; L^2(\Omega)) \text{ and}
\]
\[
(\ddot{u}_m)_{m=1}^{\infty} \text{ in } L^2(0, T; (H_B^1(\Omega))^\prime). \]

Thus there exist subsequences
\[
(u_{m_1}, \phi_{m_1})_{1=1}^{\infty} \subseteq (u_m, \phi_m)_{m=1}^{\infty}, \quad (\dot{u}_{m_1})_{1=1}^{\infty} \subseteq (\dot{u}_m)_{m=1}^{\infty} \text{ and } (\ddot{u}_{m_1})_{1=1}^{\infty} \subseteq (\ddot{u}_m)_{m=1}^{\infty},
\]
with \((u, \phi) \in L^\infty(0, T; H_B^1(\Omega)) \times L^\infty(0, T; H_0^1(\Omega)),\) \(\dot{u} \in L^\infty(0, T; L^2(\Omega))\) and \(\ddot{u} \in L^2(0, T; (H_B^1(\Omega))^\prime)\) such that
\[
u_{m_1} \rightharpoonup u \text{ weakly } - \ast \text{ in } L^\infty(0, T; H_B^1(\Omega)),
\]
\[
\phi_{m_1} \rightharpoonup \phi_0 \text{ weakly } - \ast \text{ in } L^\infty(0, T; H_0^1(\Omega)),
\]
\[ \mathbf{u}_{m_l} \rightharpoonup \mathbf{u} \text{ weakly} - \ast \text{ in } L^\infty(0, T; L^2(\Omega)), \]
\[ \mathbf{u}_{m_l} \rightharpoonup \mathbf{u} \text{ weakly in } L^2(0, T; (H^1_0(\Omega))^\prime). \]

We now proceed to show that the weak limit is a solution of the weak form. Following [21, p. 384] we fix a \( N \in \mathbb{N} \) and choose functions \( v \in C^1(0, T; H^1_0(\Omega)) \) and \( w \in C^1(0, T; H^1_0(\Omega)) \) having the form

\[ v(t) := \sum_{k=1}^{N} u_m^k(t) v_k, \quad w(t) := \sum_{k=1}^{N} \phi_m^k(t) w_k. \]

We choose \( m \geq N \), multiply the discretized versions for each pair \((v_k, w_k)\) of the weak form Equation (5) with \((u_m^k(t), \phi_m^k(t))\), sum over \( k = 1, \ldots, N \), integrate with respect to \( t \). This yields

\[ \int_0^T \left( \int_\Omega \rho \mathbf{u}_m(s)^T v \, d\Omega + \alpha \int_\Omega \rho \mathbf{u}_m(s)^T v \, d\Omega + \int_\Omega (c^E \mathbf{b}_m(s))^T B v \, d\Omega \right. \]
\[ + \, \beta \int_\Omega (c^E \mathbf{b}_m(s))^T B v \, d\Omega + \int_\Omega (e^T \nabla \phi_m(s))^T B v \, d\Omega \]
\[ + \int_\Omega (e \mathbf{b}_m(s))^T \nabla w \, d\Omega - \int_\Omega \left( e \nabla \phi_m(s) \right)^T \nabla w \, d\Omega \right) \, ds 
= \int_0^T \{f(s), v\} \, ds + \int_0^T \{g(s), w\} \, ds. \]

(27)

Fixing \( m = m_l \) and using Equation (25) we obtain in the limit \( m \to \infty \) along the subsequence \( m_l \)

\[ \int_0^T \left( \int_\Omega \rho \mathbf{u}(s)^T v \, d\Omega + \alpha \int_\Omega \rho \mathbf{u}(s)^T v \, d\Omega + \int_\Omega (c^E \mathbf{b}_m(s))^T B v \, d\Omega \right. \]
\[ + \, \beta \int_\Omega (c^E \mathbf{b}_m(s))^T B v \, d\Omega + \int_\Omega (e^T \nabla \phi_0(s))^T B v \, d\Omega \]
\[ + \int_\Omega (e \mathbf{b}_m(s))^T \nabla w \, d\Omega - \int_\Omega \left( e \nabla \phi_0(s) \right)^T \nabla w \, d\Omega \right) \, ds 
= \int_0^T \{f(s), v\} \, ds + \int_0^T \{g(s), w\} \, ds. \]

(28)

Noting that all functions of form Equation (26) are dense in the according spaces this equality holds for all functions \( v \in L^2(0, T; H^1_0(\Omega)), w \in L^2(0, T; H^1_0(\Omega)) \). In particular, it follows that also

\[ \int_\Omega \rho \mathbf{u}^T v \, d\Omega + \alpha \int_\Omega \rho \mathbf{u}^T v \, d\Omega + \int_\Omega (c^E \mathbf{b})^T B v \, d\Omega + \beta \int_\Omega (c^E \mathbf{b})^T B v \, d\Omega \]
\[ + \int_\Omega (e^T \nabla \phi_0) \, B v \, d\Omega + \int_\Omega (e \mathbf{b})^T \nabla w \, d\Omega - \int_\Omega \left( e \nabla \phi_0 \right)^T \nabla w \, d\Omega \]
\[ = \{f, v\} + \{g, w\} \]

(29)

almost everywhere \( t \in [0, T] \) for all \( v \in H^1_0(\Omega) \) and \( w \in H^1_0(\Omega) \).

Following [21, section 7.2.2 Theorem 3] we confirm that the initial conditions are also met. Choose any function \((\nu, 0)\) with \( \nu \in C^2([0, T]; H^1_0(\Omega)) \) and \( \nu(T) = \dot{\nu}(T) = 0 \). By integrating by parts twice with respect to \( t \) of Equation (27) we get

\[ \int_0^T \left( \int_\Omega \rho \mathbf{u}(t)^T \dot{v} \, d\Omega - \alpha \int_\Omega \rho \mathbf{u}(t)^T \dot{v} \, d\Omega + \int_\Omega (c^E \mathbf{b}_m(t))^T B v \, d\Omega \right. \]
\[ - \beta \int_\Omega (c^E \mathbf{b}_m(t))^T B \dot{v} \, d\Omega + \int_\Omega \left( e^T \nabla \phi_m(t) \right)^T B v \, d\Omega \right) \, dt 

Noticing that by property Equation (21) $\| \|_{\Omega}$ instead. Passing to limits, we substitute

$$= \int_0^T \{ f(t), v \} \, dt - \langle \rho u_m(0), \dot{v}(0) \rangle + \langle \rho \dot{u}_m(0), v(0) \rangle + \alpha \langle \rho u_m(0), v(0) \rangle + \beta \{ e^E Bu_m(0), Bv(0) \}$$

and analogously using Equation (28) we get

$$= \int_0^T \left( \int_\Omega \rho u(t)^T \dot{v} \, d\Omega - \alpha \int_\Omega \rho u(t)^T \dot{v} \, d\Omega + \int_\Omega \left( e^E Bu(t) \right)^T B v \, d\Omega - \beta \int_\Omega \left( e^E Bu(t) \right)^T B v \, d\Omega + \int_\Omega \left( e^T \nabla \phi_0(t) \right)^T B v \, d\Omega \right) \, dt$$

$$= \int_0^T \{ f(t), v \} \, dt - \langle \rho u(0), \dot{v}(0) \rangle + \langle \rho \dot{u}(0), v(0) \rangle + \alpha \langle \rho u(0), v(0) \rangle + \beta \{ e^E Bu(0), Bv(0) \}.$$  

(30)

For Equation (30) we set $m = m_1$ and recall Equation (25) to deduce

$$= \int_0^T \left( \int_\Omega \rho u(t)^T \dot{v} \, d\Omega - \alpha \int_\Omega \rho u(t)^T \dot{v} \, d\Omega + \int_\Omega \left( e^E Bu(t) \right)^T B v \, d\Omega - \beta \int_\Omega \left( e^E Bu(t) \right)^T B v \, d\Omega + \int_\Omega \left( e^T \nabla \phi_0(t) \right)^T B v \, d\Omega \right) \, dt$$

$$= \int_0^T \{ f(t), v \} \, dt - \langle \rho u_0, \dot{v}(0) \rangle + \langle \rho u_1, v(0) \rangle + \alpha \langle \rho u_0, v(0) \rangle + \beta \{ e^E Bu_0, Bv(0) \}.$$  

(31)

By equating coefficients of Equations (31) and (32) (set either $v(0)$ or $\dot{v}(0)$ to zero) we conclude $u(0) = u_0$ and $\dot{u}(0) = u_1$.

**Phase 4: Uniqueness**

Following e.g. [21, p. 385], it suffices to show that the only weak solution with

$$f \equiv 0, \ g \equiv 0, \ \phi^e \equiv 0, \ u_0 = u_1 \equiv 0$$

is

$$u \equiv 0, \ \phi \equiv 0.$$

Notice that by property Equation (21) $\| B \dot{u}_m \|_{L^2(0,T;L^2(\Omega))}$ is finite. Hence, the remark in [21, remark below Thm 4, section 7.2.2 c), p. 385] does not apply to our case and we can continue in the fashion of [21, Theorem 4, section 7.1.2c), p. 358] instead. Passing to limits, we substitute $v = u$ and $w = \phi_0$ in the original weak form. This is not prohibited as by property Equation (21) all components exist also in the limit. Hence, we can deduce that the following non-discretized inequality holds

$$C_4 \left( \| \dot{u}(t) \|_{L^2(\Omega)}^2 + \| u(t) \|_{H^1_0(\Omega)}^2 + \| \phi_0(t) \|_{H^1_0(\Omega)}^2 \right)$$

$$\leq C_3 \left( \| \dot{u}(0) \|_{L^2(\Omega)}^2 + \| u(0) \|_{H^1_0(\Omega)}^2 + \| \phi_0(0) \|_{H^1_0(\Omega)}^2 \right)$$

$$+ C_4 \int_0^T \left( \| \dot{u}(s) \|_{L^2(\Omega)}^2 + \| u(s) \|_{H^1_0(\Omega)}^2 + \| \phi_0(s) \|_{H^1_0(\Omega)}^2 \right) \, ds$$

$$+ \| f \|_{L^2(0,T;H^{-1}(\Omega))}^2 + \| g \|_{H^1(0,T;H^{-1}(\Omega))}^2.$$
In the case \( t = 0 \) we get from Equation (19) that \( \| \phi(0) \|_{H^1_0(\Omega)}^2 = 0 \). Hence, we now note that
\[
\bar{C}_2 = \frac{1}{C_1} \left( C_3 \eta(0) + \| f \|_{L^2(0,T;H^1_0(\Omega))}^2 + \| g \|_{H^1(0,T;H^{-1}(\Omega))}^2 \right) = 0.
\]

Finally, we can apply the second part of Gronwall inequality to conclude that
\[
\eta(t) = \| \dot{u}(t) \|_{L^2(\Omega)}^2 + \| u(t) \|_{H^1_0(\Omega)}^2 + \| \phi(t) \|_{H^1_0(\Omega)}^2 = 0 \quad \text{a.e. } t \in [0, T].
\]

Thus, the only solution can be the trivial solution.

Through the theorem we know what requirements we need to get existence of a solution of the weak form. Now prerequisites can be derived to achieve higher regularities of the solutions. The following theorem is inspired by Theorem 5, chapter 7.2 in [21]. The proof uses ideas from [9] adapted for additional Rayleigh damping.

**Theorem 3.7:** Let all requirements of Theorem 3.6 hold. If additionally \( u_0 \in H^2(\Omega), \ u_1 \in H^1(\Omega), \beta u_1 \in H^2(\Omega), \phi^e \in H^2(0,T;H^{1/2}(\Gamma_e)), \) then
\[
\begin{align*}
\dot{u} & \in L^{\infty}(0,T;H^1_0(\Omega)), \quad \ddot{u} \in L^{\infty}(0,T;H^1_0(\Omega)), \quad \dddot{u} \in L^{\infty}(0,T;L^2(\Omega)), \\
\phi & \in L^{\infty}(0,T;H^{1,1}(\Omega)), \quad \dot{\phi} \in L^{\infty}(0,T;H^1_0(\Omega)).
\end{align*}
\] (33)

**Proof:** We differentiate the weak form Equation (5) once with respect to time \( t \) and test the result first with \( (\dddot{u}_m(t),0) \) to obtain
\[
\frac{1}{2} \frac{d}{dt} \left( \langle \dot{\rho} \dddot{u}_m(t), \dddot{u}_m(t) \rangle + \langle e^S \dot{B} \dddot{u}_m(t), \dot{B} \dddot{u}_m(t) \rangle \right) + \alpha \langle \dot{\rho} \dddot{u}_m(t), \dddot{u}_m(t) \rangle + \beta \langle e^S \dot{B} \dddot{u}_m(t), \dot{B} \dddot{u}_m(t) \rangle + \langle e \nabla \dot{\phi}_m(t), \nabla \dddot{u}_m(t) \rangle = \langle f(t), \dddot{u}_m(t) \rangle.
\]

Then we differentiate the weak form Equation (5) twice with respect to time \( t \) and test the result first with \( (0,\dot{\phi}_m(t)) \) to obtain
\[
\langle e \dot{B} \dddot{u}_m(t), \nabla \dddot{u}_m(t) \rangle - \frac{1}{2} \frac{d}{dt} \left( e \nabla \dot{\phi}_m(t), \nabla \dddot{u}_m(t) \rangle = \langle \dddot{f}(t), \dddot{u}_m(t) \rangle.
\]

Analogously to the proof of Theorem 3.6, we subtract these two results and integrate with respect to \( t \) to obtain in analogy to Equation (10)
\[
\begin{align*}
&\left( \langle \dot{\rho} \dddot{u}_m(t), \dddot{u}_m(t) \rangle + \langle e^S \dot{B} \dddot{u}_m(t), \dot{B} \dddot{u}_m(t) \rangle + \langle e \nabla \dot{\phi}_m(t), \nabla \dddot{u}_m(t) \rangle \right) \\
&+ 2\alpha \int_0^t \langle \dot{\rho} \ddot{u}_m(s), \ddot{u}_m(s) \rangle \ ds + 2\beta \int_0^t \langle e^S \dot{B} \dddot{u}_m(s), \dot{B} \dddot{u}_m(s) \rangle \ ds \\
&= \left( \langle \dot{\rho} \dddot{u}_m(0), \dddot{u}_m(0) \rangle + \langle e^S \dot{B} \dddot{u}_m(0), \dot{B} \dddot{u}_m(0) \rangle + \langle e \nabla \dot{\phi}_m(0), \nabla \dddot{u}_m(0) \rangle \right) \\
&+ 2 \int_0^t \langle \dddot{f}(s), \dddot{u}_m(s) \rangle \ ds - 2 \int_0^t \langle \dddot{g}(s), \dddot{u}_m(s) \rangle \ ds
\end{align*}
\] (34)

or, again, abbreviated as \( F_I = F_r \). Analogously to inequality Equation (15) we then can obtain
\[
C_1 \left( \| \dddot{u}_m(t) \|_{L^2(\Omega)}^2 + \| \dddot{u}_m(t) \|_{H^1_0(\Omega)}^2 + \| \dddot{\phi}_m(t) \|_{H^1_0(\Omega)}^2 \right) - c_{\text{mech}} \| \dddot{u}_m(t) \|_{L^2(\Omega)}^2
\]
\[ \leq C_3 \left( \| \ddot{u}_m(0) \|_{L^2(\Omega)}^2 + \| \ddot{u}_m(0) \|_{H^1_0(\Omega)}^2 + \| \dot{\phi}_m(0) \|_{H^1_0(\Omega)}^2 \right) + C_4 \int_0^t \left( \| \ddot{u}_m(s) \|_{L^2(\Omega)}^2 + \| \ddot{u}_m(s) \|_{H^1_0(\Omega)}^2 + \| \dot{\phi}_m(s) \|_{H^1_0(\Omega)}^2 \right) \, ds + \| f \|_{H^1(0,T;H^1_0(\Omega))'}^2 + \| g \|_{H^2(0,T;H^{-1}(\Omega))} \]

for some \( C_1, C_3, C_4 > 0 \). Note that by deriving the weak form which we then test by \((0, \phi_m(t))\) we additionally obtain a bilinear form similar to Equation (17) and can analogously deduce with Lax–Milgram lemma that

\[ \| \dot{\phi}_m(0) \|_{H^1_0(\Omega)}^2 \leq 2M \left( \| eB \dot{u}_m(0) \|_{L^2(\Omega)}^2 + \| \dot{g}(0) \|_{H^{-1}(\Omega)}^2 \right). \]

This is only possible because of the added requirement of increased regularity of \( \dot{u}(0) \) and \( \dot{g} \). Furthermore, by the additional requirements on \( u_m(0) \in H^2(\Omega) \) we also obtain (estimating the \( H^2 \) norm by the norm of the Laplacian, see e.g. [26])

\[ \| \phi_m(0) \|_{H^1(\Omega)} \leq C \left( \| u_m(0) \|_{H^2(\Omega)} + \| g(0) \|_{H^{-1}(\Omega)} \right). \]

In order to utilize Gronwall lemma, we are left to show finiteness of \( \| \ddot{u}_m(0) \|_{L^2(\Omega)}^2 \). Notice that by the increased regularity of \( \ddot{u}_m(0) \in H^2(\Omega) \) and \( \phi_m(0) \in H^2(\Omega) \) the weak solution is also a strong solution, i.e. not quite a classical solution but solves the classical equations in \( t = 0 \) almost everywhere, see e.g. [27, section 2.3 and 3.5]. Thus, by evaluating the strong system in \( t = 0 \) and using the initial data and previously deduced inequalities we obtain

\[ \| \rho \ddot{u}_m(0) \|_{L^2(\Omega)} = \| \alpha \rho \ddot{u}_m(0) - B^T \left( cE B u_m(0) + \beta cE B u_m(0) + \gamma E \nabla \phi_m(0) \right) + f(0) \|_{L^2(\Omega)} \leq \| \alpha \rho \ddot{u}_m(0) \|_{L^2(\Omega)} + \| B^T cE B u_m(0) \|_{L^2(\Omega)} + \| B^T \gamma E \nabla \phi_m(0) \|_{L^2(\Omega)} + \| f(0) \|_{L^2(\Omega)}. \]

Note that this \( f \) is given by the Dirichlet lift ansatz for the strong system. Therefore we choose \( \chi \in H^2(\Omega) \) where \( \chi |_{\Gamma_{e}} = 0 \) and \( \chi |_{\Gamma_{s}} = 1 \). With this requirement the right-hand sight of the above inequality is bounded independently of \( m \).

Since all components are finite, analogously to inequality Equation (16) with

\[ \eta(t) := \| \ddot{u}_m(t) \|_{L^2(\Omega)}^2 + \| \ddot{u}_m(t) \|_{H^1_0(\Omega)}^2 + \| \dot{\phi}_m(t) \|_{H^1_0(\Omega)}^2 \]

we can apply Gronwall lemma to obtain that

\[ \| \ddot{u}_m(t) \|_{L^2(\Omega)}^2 + \| \ddot{u}_m(t) \|_{H^1_0(\Omega)}^2 + \| \dot{\phi}_m(t) \|_{H^1_0(\Omega)}^2 \leq \left( \frac{C_3}{C_1} + \frac{C_4}{C_1^2} e^{C_4 t} \right) \left( \| \ddot{u}(0) \|_{L^2(\Omega)}^2 + \| \ddot{u}(0) \|_{H^1_0(\Omega)}^2 + \| \dot{\phi}(0) \|_{H^1_0(\Omega)}^2 \right) + \| f(0) \|_{H^1(0,T;H^1_0(\Omega))'}^2 + \| g(0) \|_{H^2(0,T;H^{-1(\Omega))}}^2 \]

holds almost everywhere in \([0, T]\).
Using results from Theorem 3.6 it is now clear that
\[
\begin{align*}
\mathbf{u} &\in L^\infty(0,T;H^1_B(\Omega)), \quad \mathbf{\dot{u}} \in L^\infty(0,T;H^1_B(\Omega)), \\
\phi &\in L^\infty(0,T;H^1_0,\Gamma(\Omega)), \quad \dot{\phi} \in L^\infty(0,T;H^1_0,\Gamma(\Omega)).
\end{align*}
\] (36)

**Remark 3.8:** One may think that in order to achieve \( \mathbf{\dot{u}} \in L^\infty(0,T;L^2(\Omega)) \) it is only required that \( \mathbf{u}_1(0) = \mathbf{u} \in H^1(\Omega) \) instead of \( \beta \mathbf{u}_1 \in H^2(\Omega) \) (or more precisely \( \|B^T \beta \mathbf{e}^2 \mathbf{u}_1\|_{L^2(\Omega)} < \infty \)). However, this is not the case.

The condition \( \beta \mathbf{u}_1 \in H^2(\Omega) \) is required to show that \( \|\mathbf{\dot{u}}_m(0)\|_{L^2(\Omega)} \) is finite, such that Gronwall inequality can be applied.

**Remark 3.9:** In e.g. [21, p. 390, Equation (59)] a \( H^2 \) regularity for \( \mathbf{u} \) is achieved by selecting the test functions for \( \mathbf{u} \) to be the complete eigenfunction sequence of \( -\Delta \mathbf{u} \) which, indirectly, allows an estimation of \( \|\mathbf{u}\|_{H^2(\Omega)} \). A similar argument should also be possible for \( B^T B \) (or more precisely the operator that works on the solution vector \( (\mathbf{u},\phi)^T \) and contains \( B^T B \)). This would directly increase the regularity of \( \phi \) so that not only \( \mathbf{u} \in L^\infty(0,T;H^2(\Omega)) \) but also \( \phi \in L^\infty(0,T;H^2(\Omega) \cap H^1_0,\Gamma(\Omega)) \).

However, the authors did not follow that argumentation. Note that the here occurring differential operators are slightly different from the Laplacian. Hence, this leads to rather unpleasant changes due to the now very technical arguments and spaces. In that case, it would be possible to reduce the regularity requirements, however this would also change the resulting spaces and increase the cost of technical proof steps.

With the estimations and equations in Theorem 3.6 and the corresponding proof, a long-time behavior of the energy function \( \eta \) and in particular of each component can be derived.

**Corollary 3.10:** Let all requirements of Theorem 3.6 hold and let \( \alpha, \beta > 0 \) strictly. If additionally there exists a \( t_0 \in \mathbb{R} \), \( t_0 \geq 0 \) such that \( \phi^\epsilon(t) = 0 \) for \( t \geq t_0 \), then
\[
\|\mathbf{\dot{u}}_m(t)\|_{L^2(\Omega)} \to 0, \quad \|B\mathbf{u}_m(t)\|_{L^2(\Omega)} \to 0, \quad \|\phi_m(t)\|_{H^1_0(\Omega)} \to 0
\]
for \( t \to \infty \).

Furthermore, the energy of the system
\[
\eta(t) = \|\mathbf{\dot{u}}_m(t)\|^2_{L^2(\Omega)} + \|\mathbf{u}_m(t)\|^2_{H^1_B(\Omega)} + \|\phi_m(t)\|^2_{H^1_0(\Omega)}
\]
converges to a constant \( \eta(t) \to c \in \mathbb{R}^+ \) for \( t \to \infty \).

**Proof:** The right-hand side \( \mathcal{F}_r(t) \) of the energy balance Equation (10) is constant for \( t \geq t_0 \) as no new energy is given into the system starting from time \( t_0 \), i.e. \( \mathcal{F}_r(t) = c_1 \in \mathbb{R}^{\geq 0} \) for \( t \geq t_0 \). Let
\[
\gamma(t) := 2\alpha \int_0^t \langle \rho \mathbf{\dot{u}}_m(s), \mathbf{\dot{u}}_m(s) \rangle \ ds + 2\beta \int_0^t \langle cB\mathbf{\dot{u}}_m(s), B\mathbf{\dot{u}}_m(s) \rangle \ ds,
\]
and let
\[
\tilde{\eta}(t) := \langle \rho \mathbf{\dot{u}}_m(t), \mathbf{\dot{u}}_m(t) \rangle + \langle cB\mathbf{u}_m(t), B\mathbf{u}_m(t) \rangle + \langle \epsilon S \nabla \phi_m(t), \nabla \phi_m(t) \rangle.
\] (37)

Then Equation (10) implies that \( \tilde{\eta}(t) + \gamma(t) = c_1 \) for \( t \geq t_0 \). As \( \gamma(t) \) is monotonically increasing, it follows that \( \tilde{\eta}(t) \) is monotonically decreasing. Both \( \tilde{\eta}(t), \gamma(t) \) are bounded below and above by zero and \( c_1 \), respectively. Hence, \( \gamma(t) \) and \( \tilde{\eta}(t) \) must converge. Based on these results we get \( 0 \leq \tilde{\eta}(t) \to \)
We have to show that one of the other two summands converges and determine the limit values.

From Equations (40) and (41) we get
\[
\begin{align*}
\langle \rho \dot{u}_m(s), u_m(s) \rangle &\to 0, \\
\langle c^E \dot{B}u_m(s), B u_m(s) \rangle &\to 0.
\end{align*}
\]

Through these results and by utilizing positive definiteness of \( c^E \) and estimations similar to Equation (11) we also get the following convergence result
\[
\dot{u}_m(s) \to 0 \quad B \dot{u}_m(s) \to 0.
\] (38)

For the next steps it is already known that
\[
\tilde{\eta}(t) := \langle \rho \dot{u}_m(t), u_m(t) \rangle + \langle c^E \dot{B}u_m(t), B u_m(t) \rangle + \langle \epsilon^S \nabla \phi_m(t), \nabla \phi_m(t) \rangle \to c_2 \geq 0.
\]

We have to show that one of the other two summands converges and determine the limit values.

We get the convergence of \( \langle c^E \dot{B}u_m(t), B u_m(t) \rangle \) by taking advantage of Equation (38) and using the characteristic of the time derivative of \( \dot{B}u_m(t) \). As all other summands converge this implies that also \( \langle \epsilon^S \nabla \phi_m(t), \nabla \phi_m(t) \rangle \) must converge. In order to specify the limit values, we test the weak form Equation (5) first with \( (u_m(t), 0) \) and get
\[
\begin{align*}
\langle \rho \dot{u}_m(t), u_m(t) \rangle + \alpha \langle \rho \dot{u}_m(t), u_m(t) \rangle + \langle c^E \dot{B}u_m(t), B u_m(t) \rangle \\
+ \beta \langle c^E \dot{B}u_m(t), B u_m(t) \rangle + \langle \epsilon^T \nabla \phi_m(t), B u_m(t) \rangle &= \langle f(t), u_m(t) \rangle \\
\to &0
\end{align*}
\] (39)

to obtain
\[
\lim_{t \to \infty} \langle \epsilon^T \nabla \phi_m(t), B u_m(t) \rangle = \lim_{t \to \infty} -\langle c^E \dot{B}u_m(t), B u_m(t) \rangle \leq 0
\] (40)
and then a second time with \( (0, \phi_m(t)) \)
\[
\langle e \dot{B}u_m(t), \nabla \phi_m(t) \rangle - \left\langle \epsilon^S \nabla \phi_m(t), \nabla \phi_m(t) \right\rangle = \langle g(t), \phi_m(t) \rangle
\]
yielding
\[
\lim_{t \to \infty} \langle e \dot{B}u_m(t), \nabla \phi_m(t) \rangle = \lim_{t \to \infty} \left\langle \epsilon^S \nabla \phi_m(t), \nabla \phi_m(t) \right\rangle \geq 0.
\] (41)

From Equations (40) and (41) we get
\[
\lim_{t \to \infty} \langle c^E \dot{B}u_m(t), B u_m(t) \rangle = \lim_{t \to \infty} \left\langle \epsilon^S \nabla \phi_m(t), \nabla \phi_m(t) \right\rangle = 0.
\]

Note that from Equation (18) we are aware that \( \| e \dot{B}u_m(t) \|_{L^2(\Omega)} \to 0 \) and using the requirement for \( \phi^E(t) \) implies that also \( \| \phi_m(t) \|_{H^1_0(\Omega)} \to 0 \). It is clear that \( \tilde{\eta}(t) \to 0 \) and with the characteristics of the material parameters
\[
\rho > 0, \quad c^E \text{ and } \epsilon^S \text{ symmetric, positive definite,}
\]
we conclude
\[
\| \dot{u}_m(t) \|_{L^2(\Omega)} \to 0, \quad \| \dot{B}u_m(t) \|_{L^2(\Omega)} \to 0, \quad \| \dot{f}_m(t) \|_{H^1_0(\Omega)} \to 0
\]
for \( t \to \infty \). Finally, we know that the derivatives in time and space of \( u_m(t) \) converge to zero, so \( \| u_m(t) \|_{L^2(\Omega)} \to \tilde{c} \in \mathbb{R} \).
Then we can conclude that $\eta(t) \to c \in \mathbb{R}^+$ for $t \to \infty$. 

**Example 3.11:** We use a numerical simulation of the piezoceramic displayed in Figure 1 with homogeneous boundary conditions for the mechanical displacement $u_0, u_1 = 0$ and electrically excite the piezoceramic on $\Gamma_e$ via $\phi^e(t)$ using a nonsmooth *spike function*, see Figure 2. This amounts to a common setup for the purposes of simulating the electrical impedance curve of a piezoceramic and is used in practice for material parameter estimation, see [10]. In Figure 3, the simulation result for the energy $\tilde{\eta}(t)$ is shown. As predicted by Corollary 3.10, we observe that the energy $\tilde{\eta}(t)$ decreases (almost) monotonically in time due to damping. The small oscillating numerical errors are due to the HHT time integration method (see [10]) used for the simulation.

**Remark 3.12:** By using similar techniques as in the second part of the proof of Theorem 3.6, the last Corollary can be extended to non-discretized solutions of the partial differential equations.

### 4. Conclusion

Piezoelectric materials are widely diversified in their applications. Since trial-and-error designs of new specimens are rather expensive, computer-aided design techniques are used instead.
However, in order to confidently use computer simulations, the underlying partial differential equation must be analyzed before. In the setting treated here, damping terms are additionally introduced in order to model the correct treatment of any type of energy dissipation. This is in particular essential for soft ceramics.

In this paper, we prove existence, uniqueness and regularity of weak solutions of the governing partial differential equations and present results on the long-term behavior of solutions.

It can be stated, that under physically motivated conditions on the material properties, boundary and initial values the existence, uniqueness and regularity of the given equations can be proven. The proof employs techniques generally used in the analysis of uncoupled one-field potential and elasto-dynamics and scalar potential problems. Their combination in a direct coupled system is one particular contribution of this work which may find its applications in related other coupled problem, e.g. wave propagation in poroelastic materials.

The obtained theoretical findings are further consistent with numerical results gained from a computational simulation of the model as well as theoretical results of different authors for similar cases. With this, the basis is formed for ongoing computer-aided design optimization of piezoelectric transducers.

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Appendix A. Definitions

Let $d, k \in \mathbb{N}$ be integers and let $\alpha$ be a multi-index. Then we define the functional spaces

$$C^k(\Omega) := \left\{ \sigma : \Omega \to \mathbb{R}^d : \sigma \text{ is } k \text{ - times continuously differentiable} \right\},$$

$$L^2(\Omega) := \left\{ \sigma : \Omega \to \mathbb{R}^d : \|\sigma\|_{L^2(\Omega)}^2 := \int_\Omega \sigma^T \sigma \, d\Omega < \infty \right\},$$

$$H^1(\Omega) := \left\{ \sigma : \Omega \to \mathbb{R} : \|\sigma\|_{H^1(\Omega)}^2 := \|\sigma\|_{L^2(\Omega)}^2 + \|\nabla \sigma\|_{L^2(\Omega)}^2 < \infty \right\},$$

$$H^1_0(\Omega) := \left\{ \sigma \in H^1(\Omega) : \sigma |_{\Gamma_0} = 0 \text{ with } \|\sigma\|_{H^1_0(\Omega)} := \|\sigma\|_{H^1(\Omega)} < \infty \right\},$$

$$H^{-1}(\Omega) := \left\{ f \text{ continuous linear functional on } H^1_0(\Omega) : \sup_{\|\sigma\|_{H^1_0(\Omega)} \leq 1} |\langle f, \sigma \rangle| < \infty \right\}.$$

Let $\sigma : [0, T] \to X$ be Bochner-measurable. Then

$$L^2(0, T; X) := \left\{ \sigma : [0, T] \to X : \int_{[0, T]} \|\sigma(t)\|_X^2 \, dt < \infty \right\},$$

$$L^\infty(0, T; X) := \left\{ \sigma : [0, T] \to X : \text{ess sup}_{0 \leq t \leq T} \|\sigma(t)\|_X < \infty \right\},$$

$$H^1(0, T; X) := \left\{ \sigma : [0, T] \to X : \int_{[0, T]} \|\sigma(t)\|_X^2 + \|\dot{\sigma}(t)\|_X^2 \, dt < \infty \right\},$$

$$H^2(\Omega) := \left\{ \sigma : \Omega \to \mathbb{R}^3 : \|\sigma\|_{H^2(\Omega)} := \left( \sum_{|\alpha| \leq 2} \|D^{(\alpha)}(\sigma)\|_{L^2(\Omega)}^2 \right)^{1/2} < \infty \right\}.$$