Debiased Machine Learning of Set-Identified Linear Models

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Abstract

This paper provides estimation and inference methods for an identified set’s boundary (i.e., support function) where the selection among a very large number of covariates is based on modern regularized tools. I characterize the boundary using a semiparametric moment equation. Combining Neyman-orthogonality and sample splitting ideas, I construct a root-$N$ consistent, uniformly asymptotically Gaussian estimator of the boundary and propose a multiplier bootstrap procedure to conduct inference. I apply this result to the Partially Linear Model, the Partially Linear IV Model and the Average Partial Derivative with an interval-valued outcome.

1 Introduction and Motivation.

Interval-valued outcomes are ubiquitous in economic research. Examples of such outcomes include bidders’ valuation in English auctions (Haile and Tamer (2003)), income and wages (Trostel et al. (2002), Gafarov (2019)), house prices (Gamper-Rabindran and Timmins (2013), Beresteau and Sasaki (2020)), and county-level employment rates (Autor et al. (2013)). An outcome is interval-valued if the actual outcome $Y$ is missing, but there exist an observable lower bound $Y_L$ and an upper bound $Y_U$ so that

$$Y_L \leq Y \leq Y_U \ a.s. \quad (1.1)$$

When an outcome is interval-valued, the parameter of interest is a set, where each point corresponds to a possible random variable $Y$ in the band (1.1).

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The main contribution of this paper is to provide an estimator of the identified set’s boundary, where the selection among high-dimensional controls is based on modern machine learning/regularized methods. The paper focuses on identified sets whose boundary can be represented by a moment equation as in Beresteanu and Molinari (2008), Bontemps et al. (2012). In this paper, the equation depends on an identified functional nuisance parameter, for example, a conditional mean function. A naive approach would be to plug-in a machine learning estimate of the nuisance parameter into the moment equation and solve for the boundary. However, modern regularized methods (machine learning techniques) have bias converging slower than the parametric rate, which cannot be made small by classic techniques (e.g., undersmoothing). As a result, plugging such estimates into the moment equation produces a biased, suboptimal estimate of the boundary itself.

To overcome the transmission of the bias into the second stage, I adjust the moment equation to make it insensitive or, formally, Neyman-orthogonal, to the biased estimation of the nuisance parameter. While orthogonality has been extensively studied in the point-identified case, set identification presents several challenges. The first one is the non-smoothness of $x \rightarrow \min(x, 0)$ function at $x = 0$, often occurred in censored LAD (e.g., Powell (1984)). The second one is to establish uniformly valid inference over the boundary in addition to the pointwise one.

When the identified set is multi-dimensional, its boundary consists of continuum points. As a result, economists are interested in uniform inference in addition to the pointwise inference. Establishing uniform inference is not trivial. To control the speed at which an empirical sample average concentrates around the population mean, I invoke maximal inequalities of Chernozhukov et al. (2014) instead of Markov inequality that is typically sufficient in the point-identified case. I propose multiplier bootstrap algorithm to conduct inference. By virtue of orthogonality, only the moment function (the second stage), not the nuisance parameter estimate (the first stage), needs to be resampled in simulation. As a result, this multiplier bootstrap is faster to compute than the weighted bootstrap of Chandrasekhar et al. (2012), which is based on a non-orthogonal moment and involves resampling of both stages. I demonstrate the proposed approach in a simulation exercise and give a brief empirical illustration.

**Literature review. Set identification** This paper bridges the gap between three literatures: set-identified models, debiased/orthogonal machine learning, and non-smooth models. Set identification is a vast area of research, encompassing a wide variety of approaches: linear and quadratic programming, random set theory, support function, and moment inequalities (Manski (1990), Manski and Pepper (2000), Manski and Tamer (2002), Haile and Tamer (2003), Chernozhukov et al. (2007), Beresteanu and Molinari (2008), Molinari (2008), Ciliberto and Tamer (2009), Lee (2009), Stove (2009), Andrews and Shi (2013), Beresteanu et al. (2011), Chandrasekhar et al. (2012), Beresteanu et al. (2012), Chernozhukov et al. (2010), Bontemps et al. (2012), Chernozhukov et al. (2013),
Orthogonality. Next, this paper contributes to a large body of work on debiased inference for parameters following regularization or model selection (Neyman 1959, Neyman 1979, Hardle and Stoker 1989, Newey and Stoker 1993, Newey 1994, Robins and Rotnitzky 1994, Robinson 1988, Zhang and Zhang 2014, Javanmard and Montanari 2014, Chernozhukov et al. 2018, Chernozhukov et al. 2016, Belloni et al. 2017, Sasaki and Ura 2020, Sasaki et al. 2020, Chiang et al. 2019, Chiang et al. 2019, Ning et al. 2020, Chernozhukov et al. 2018a, Chernozhukov et al. 2018b, Semenova and Chernozhukov 2021, Nekipelov et al. 2022, Singh and Suri 2020, Colangelo and Lee 2020, Fan et al. 2019, Zimmert and Lechner 2021). A basic idea is to make the moment condition insensitive, or, formally, Neyman-orthogonal, to the biased estimation of the nuisance parameter. For a semiparametric GMM setting, the work by Ackerberg et al. (2014) derives an orthogonal moment condition whose nuisance functions are identified by conditional moment restriction, such as conditional mean and conditional quantiles. Combining Neyman-orthogonality and sample splitting, Chernozhukov et al. (2016) and Chernozhukov et al. (2018) derive a root-$N$ consistent and asymptotically normal estimator for a single target parameter. This has idea has been extended for many functional parameters in $Z$-estimation framework, in the context of distribution regression (Belloni et al. 2018) and quantile regression (Sasaki et al. 2020). Next, the paper is related to literature on the non-smooth estimating equations (Powell 1984, Powell 1986, Powell et al. 1989, Kaplan and Suri 2017, Franguridi et al. 2021). Finally, the paper contributed to a small, but growing literature on machine learning for bounds and partially identified models (Kallus and Zhou 2019, Jeong and Namkoong 2021, Semenova 2020, Bonvini and Kennedy 2021).

Structure of the paper. The paper is organized as follows. Section 2 demonstrates main points for the partially linear model of Robinson (1988). Section 3 states theoretical results. Section 4 applies the results to models with an interval-valued outcome. Section 5 presents finite-sample evidence. Section 6 contains an empirical illustration. Section 8 contains the proofs of main results.
2 Set-Up.

2.1 General Framework

I focus on parameters that are linear in an unobserved scalar outcome $Y$. The identified set takes the form

$$B = \{ \beta = \Sigma^{-1} \mathbb{E} V(\eta_0) Y | Y_L \leq Y \leq Y_U \}, \quad (2.1)$$

where the random vector $V(\eta_0) \in \mathbb{R}^d$ depends on a nuisance function $\eta_0$. Examples of the nuisance functions

$$\eta_0 = \eta_0(X)$$

include the propensity score, the conditional density, and the regression function, among others. The matrix $\Sigma \in \mathbb{R}^{d \times d}$ is identified by a moment equation

$$\Sigma = \mathbb{E} A(W, \eta_0) \quad (2.2)$$

and is assumed invertible. The key innovation of this framework is to allow the vector $V(\eta)$, the matrix function $A(W, \eta)$, and the bounds $Y_L, Y_U$ to depend on a functional nuisance parameter $\eta$, covering the models in Bontemps et al. (2012), Chandrasekhar et al. (2012), Kaido (2017), and many others as special cases.

2.2 Examples

Example 2.1 (Partially Linear Model). Consider the partially linear model of Robinson (1988)

$$Y = D' \beta_0 + f_0(X) + U, \quad \mathbb{E}[U | D, X] = 0, \quad (2.3)$$

where $D \in \mathbb{R}^d$ is a treatment (policy) variable, $\beta_0$ is a causal (structural) parameter,

$$X = (1, X_1, X_2, \ldots, X_p_X)$$

is a vector of covariates whose dimension may be large relative to the sample size (e.g., $p_X \gg N$), and $f_0(\cdot)$ is an integrable function. The parameter $\beta_0$ can be represented as the minimizer of the least squares criterion function

$$\beta_0 = \arg \min_{b \in \mathbb{R}^d} \mathbb{E} (Y - D'b - f(X))^2. \quad (2.4)$$

As in Lemma A.8, $\beta_0$ coincides with the minimizer of a shorter criterion function

$$\beta_0 = \arg \min_{b \in \mathbb{R}^d} \mathbb{E} (Y - (D - \eta_0(X))'b)^2. \quad (2.5)$$

Thus, the identified set $B$ is a special case of model (2.1)-(2.2) with $V(\eta)$ and $A(W, \eta)$ defined as follows. The treatment regression function is

$$\eta_0(X) = \mathbb{E}[D | X], \quad (2.6)$$
the treatment residual is

\[ V(\eta) = D - \eta(X), \quad (2.7) \]

the matrix function is

\[ A(W, \eta) = (D - \eta(X))(D - \eta(X))'. \quad (2.8) \]

**Example 2.2 (Partially Linear IV Model).** Consider the following partially linear IV model

\[
\begin{align*}
Y &= D'\beta_0 + f_0(X) + U, \quad \mathbb{E}[U | Z, X] = 0 \quad (2.9) \\
D &= m_0(X) + E, \quad \mathbb{E}[E | X] = 0, \quad (2.10) \\
Z &= \eta_0(X) + V, \quad \mathbb{E}[V | X] = 0, \quad (2.11)
\end{align*}
\]

where \( Z \in \mathbb{R}^d \) is the instrument for \( D \). The parameter \( \beta_0 \) can be characterized by a moment equation

\[
\mathbb{E}(Z - \eta_0(X))(Y - (D - m_0(X))'\beta_0) = 0,
\]

which gives a closed-form expression for \( \beta_0 \)

\[
\beta_0 = (\mathbb{E}(Z - \eta_0(X))(D - m_0(X))'\beta_0)^{-1}\mathbb{E}(Z - \eta_0(X))Y.
\]

The model is a special case of (2.1)-(2.2) with

\[
\begin{align*}
V(\eta) &= Z - \eta(X) \quad (2.12) \\
A(W, \eta, m) &= (Z - \eta(X))(D - m(X))', \quad (2.13)
\end{align*}
\]

where \( V = V(\eta_0) \) in (2.11). If \( Z = D \), the model (2.9)-(2.11) coincides with (2.3)-(2.6).

**Example 2.3 (Average Partial Derivative).** An important parameter in economics is the average partial derivative. This parameter shows the average effect of a small change in a variable of interest \( D \) on the outcome \( Y \) conditional on the covariates \( X \). To describe this change, define the conditional expectation function of an outcome \( Y \) given the variable \( D \) and exogenous variable \( X \) as

\[
\mu(D, X) := \mathbb{E}[Y | D, X]
\]

and its partial derivative with respect to \( D \) as \( \nabla_D \mu(D, X) := \nabla_D \mu(d, X)|_{d=D} \). Then, the average partial derivative is defined as

\[
\beta = \mathbb{E}\nabla_D \mu(D, X). \quad (2.14)
\]

For example, when \( Y \) is the logarithm of consumption, \( D \) is the logarithm of price, and \( X \) is the vector of other demand attributes, the average partial derivative stands for the average price elasticity.
Assume that the variable $D$ has bounded support $D \subset \mathcal{R}^d$. Furthermore, the conditional density $f(D \mid X)$ has positive density on this support a.s. in $X$. Hardle and Stoker (1989) have shown that the average partial derivative can be represented as $\beta = \mathbb{E} V Y$, where $V = -\nabla_D \log f(D \mid X) = -\frac{\nabla D f(D \mid X)}{f(D \mid X)}$ is the negative partial derivative of the logarithm of the density $f(D \mid X)$. When $Y$ is interval-valued, the identified set $B$ for $\beta$ is a special case of (2.1)-(2.2) with $A(W, \eta) = \Sigma = I_d$, the nuisance function $\eta_0(D, X) = \frac{\nabla D f(D \mid X)}{f(D \mid X)}$ and the vector $V(\eta) = -\eta$. Kaido (2017) studies a special case of this problem without covariates.

### 2.3 Single treatment

In this section, I derive an orthogonal moment equation for the upper bound $\beta_U$ on the causal parameter $\beta_0$ in Example 2.1. Because the bias due to sign mistake in $Y_{\text{best}}(\eta) \neq Y_{\text{best}}(\eta_0)$ proves to be second-order, I derive the orthogonal moment treating $Y_{\text{best}}(\eta)$ as observed. Next, I formally control the bias due to the sign mistake. The following two subsections elaborate on these points.

**Moment equation for $\beta_U$.** Consider Example 2.1 with a single treatment (i.e., $d = 1$). The identified set $B$ becomes a closed interval $[\beta_L, \beta_U]$. The upper bound $\beta_U$ is

$$\beta_U = \max_{(Y : Y_L \leq Y \leq Y_U)} \left\{ \frac{\mathbb{E}(D - \eta_0(X)) \cdot Y}{\mathbb{E}(D - \eta_0(X))^2} \right\}. \quad (2.15)$$

To maximize the numerator of (2.15), take $Y = Y_U$ for positive values of $D - \eta_0(X)$ and $Y = Y_L$ otherwise. Define the best-case outcome

$$Y_{\text{best}}(\eta) = \begin{cases} Y_L, & D - \eta_0(X) \leq 0, \\ Y_U, & D - \eta_0(X) > 0. \end{cases} \quad (2.16)$$

Plugging (2.16) into (2.15) gives the moment function for $\beta_U$

$$m(W; \beta_U, \eta) := (Y_{\text{best}}(\eta) - (D - \eta(X)) \beta_U)(D - \eta(X)). \quad (2.17)$$

**Establishing orthogonality.** In what follows, $\eta$ corresponds to an instance of the nuisance parameter whose true value is $\eta_0$. Consider an infeasible moment function

$$m_0(W; \beta_U, \eta) := (Y_{\text{best}}(\eta_0) - (D - \eta(X)) \beta_U)(D - \eta(X)), \quad (2.18)$$

where the best-case outcome $Y_{\text{best}}(\eta)$ is treated as if it was observed, and $\eta$ indicates the instance of the nuisance parameter treated as unknown. The moment equation (2.18) is not orthogonal to the perturbations of $\tilde{\eta}(X) - \eta_0(X)$

$$\partial_r [m_0(W; \beta_U, \eta(\eta - \eta_0) + \eta_0)]_{r=0} = -\mathbb{E}[Y_{\text{best}}(\eta_0)(\eta(X) - \eta_0(X))] \neq 0.$$
Therefore, the bias of the estimation error \(\hat{\gamma}(X) - \gamma_0(X)\) translates into the moment (2.18). To overcome the transmission of this bias, Robinson (1988) proposes an orthogonal moment equation

\[
g_0(W, \beta_U, \{\eta, \gamma_U\}) = (Y_{\text{best}}(\eta_0) - \gamma_U(X) - (D - \eta(X))\beta_U)(D - \eta(X)),
\]

where the true value of \(\gamma_U(x)\) is

\[
\gamma_{U,0}(X) = \mathbb{E}[Y_{\text{best}}(\eta_0) | X].
\]

The orthogonality condition is

\[
\hat{\gamma}_r \mathbb{E}[g_0(W, \beta_U, \r(\eta - \eta_0) + \eta_0, \gamma_{U,0})]_{r=0} = -\mathbb{E}[(Y_{\text{best}}(\eta_0) - \mathbb{E}[Y_{\text{best}}(\eta_0) | X])(\eta(X) - \eta_0(X))] = 0.
\]

Thus, the bias of the estimation error, \(\hat{\gamma}(X) - \eta_0(X)\), does not translate into the moment (2.18). The orthogonality condition with respect to \(\gamma_{U,0}\) can be verified in a similar way.

**Bias due to sign mistake.** I now discuss whether the mistake in the best-case outcome \(Y_{\text{best}}(\eta) \neq Y_{\text{best}}(\eta_0)\), which has been so far ignored, has any effect on the support function estimate. Consider the difference between the feasible moment \(g(W, \beta_U, \{\eta, \gamma_U\})\) and its infeasible analog

\[
g(W, \beta_U, \{\eta, \gamma_U\}) - g_0(W, \beta_U, \{\eta, \gamma_U\}) = (D - \eta(X))(Y_{\text{best}}(\eta) - Y_{\text{best}}(\eta_0))
\]

\[
= (D - \eta_0(X))(Y_{\text{best}}(\eta) - Y_{\text{best}}(\eta_0))
\]

\[
+ (\eta_0(X) - \eta(X))(Y_{\text{best}}(\eta) - Y_{\text{best}}(\eta_0)).
\]

Define the first-order bias \(B_1(\eta, \eta_0)\) as

\[
B_1(\eta, \eta_0) := \mathbb{E}[(D - \eta_0(X))(Y_{\text{best}}(\eta) - Y_{\text{best}}(\eta_0))]
\]

and the second-order one

\[
B_2(\eta, \eta_0) := \mathbb{E}[(\eta_0(X) - \eta(X))(Y_{\text{best}}(\eta) - Y_{\text{best}}(\eta_0))].
\]

Below, I describe the conditions under which \(B_1(\eta, \eta_0)\) and \(B_2(\eta, \eta_0)\) are negligible, (i.e., \(o(N^{-1/2})\)). If they hold, the feasible moment equation \(g(W, \beta_U, \{\eta, \gamma_U\})\) is insensitive to the biased estimation of \(\eta\). As a result, the support function estimator based on \(g(W, \beta_U, \{\eta, \gamma_U\})\) is asymptotically unbiased under plausible conditions.

Consider the best-case outcome \(Y_{\text{best}}(\eta)\)

\[
Y_{\text{best}}(\eta) = Y_L + (Y_U - Y_L)1\{D - \eta(X) \geq 0\}.
\]

The sign mistake \(Y_{\text{best}}(\eta) \neq Y_{\text{best}}(\eta_0)\) occurs on the events

\[
\mathcal{E}_- := \left\{ D - \eta_0(X) < 0 < D - \eta(X) \right\}, \quad (2.20)
\]

\[
\mathcal{E}_+ := \left\{ D - \eta(X) < 0 < D - \eta_0(X) \right\}. \quad (2.21)
\]
On these events, the residual cannot exceed estimation error in absolute value

\[ \left\{ \mathcal{E}_- \cup \mathcal{E}_+ \right\} \Rightarrow \left\{ 0 < |D - \eta_0(X)| < |\eta(X) - \hat{\eta}_0(X)| \right\}. \quad (2.22) \]

If the width \( Y_U - Y_L \) is bounded by \( M_{UL} \), the estimation error of \( Y_{\text{best}}(\eta) \) is bounded as

\[ |Y_{\text{best}}(\eta) - Y_{\text{best}}(\eta_0)| = (Y_U - Y_L)I\{\mathcal{E}_- \cup \mathcal{E}_+\} \leq M_{UL}1\{0 < |D - \eta_0(X)| < |\eta(X) - \hat{\eta}_0(X)|\}. \quad (2.23) \]

Suppose the conditional density \( h_{V(\eta_0)}(t, X) \) of \( V(\eta_0) \) is bounded by \( M \) \( a.s. \). Invoking (2.22) gives

\[ \mathbb{E}|\eta(X) - \eta_0(X)|1\{\mathcal{E}_+ \cup \mathcal{E}_-\} \leq \mathbb{E}_X \int_{-|\eta(X) - \eta_0(X)|}^{+|\eta(X) - \eta_0(X)|} th_{V(\eta_0)}(t)dt \leq 2M\mathbb{E}_X (\eta(X) - \eta_0(X))^2. \quad (2.24) \]

As a result, the bias terms \( B_1(\eta, \eta_0) \) and \( B_2(\eta, \eta_0) \) shrink at the quadratic rate. Combining (2.24) and (2.19) gives a feasible moment function

\[ g(W, [\eta, \gamma_U]) = (Y_{\text{best}}(\eta) - \gamma_U(X) - (D - \eta(X))\beta_U)(D - \eta(X)). \quad (2.25) \]

### 2.4 Multi-dimensional case

In this section, I derive an orthogonal moment for the support function, starting from a non-orthogonal one due to Bontemps et al. (2012) and Beresteanu and Molinari (2008).

**Moment Equation for Support Function.** As shown in Beresteanu and Molinari (2008), the identified set \( \mathcal{B} \) in (2.1) is a compact and convex set. Thus, it can be described by its projections onto a unit sphere

\[ S^{d-1} := \{ q \in \mathbb{R}^d, \ |q| = 1 \}. \quad (2.26) \]

For any direction \( q \in S^{d-1} \), define the support function as the upper bound on \( q'\beta_0 \)

\[ \sigma(q) := \sup_{b \in \mathcal{B}} q'b. \quad (2.27) \]

As proposed in Beresteanu and Molinari (2008) and Bontemps et al. (2012), define the projected weighting vector

\[ z(p, \eta) = p'V(\eta), \quad (2.28) \]

the best-case outcome \( Y(p, \eta) \)

\[ Y(p, \eta) = Y_L + (Y_U - Y_L)I\{z(p, \eta) > 0\}, \quad (2.29) \]
and the projection parameter \( p(q) \)
\[
p(q) = \Sigma^{-1} q.
\] (2.30)
Then, the moment equation for \( \sigma(q) \) is
\[
\sigma(q) = \mathbb{E}[z(q, \eta)Y(p, \eta)]_{p=p(q)}.
\] (2.31)

**Orthogonal Moment for Support Function.** The moment equation (2.31) is sensitive to the biased estimation of the nuisance parameter \( \eta \). To avoid the transmission of this bias into the second stage, I construct another moment function \( g(W, p, \xi) \), where \( \xi(p) \) is the nuisance parameter, in the following steps.

1. Starting from an infeasible, smooth moment
\[
m_0(W, p, \eta) := z(p, \eta)Y(p, \eta_0)
\] (2.32)
derive an infeasible orthogonal moment \( g_0(W, p, \xi(p)) \) obeying (2.19) for each \( p \).

2. Invoke Lemma 4.3 to bound the bias
\[
\sup_{p \in \mathcal{P}} \mathbb{E}|z(p, \eta) (Y(p, \eta) - Y(p, \eta_0))| = O(\mathbb{E}|\eta(X) - \eta_0(X)|^2),
\] (2.33)
where \( p \) belongs to a compact bounded set \( \mathcal{P} \) defined below. When \( \Sigma = I_d, \mathcal{P} = \mathcal{S}_d^{d-1} \).

3. Combine (2.33) and (2.32) to obtain the feasible orthogonal moment
\[
g(W, p, \xi(p)) = g_0(W, p, \xi(p)) + z(p, \eta)(Y(p, \eta) - Y(p, \eta_0)).
\] (2.34)

**Example (Example 2.2 cont.).** Consider Example 2.2. The projected weighting vector (2.28) is
\[
z(p, \eta) = p'(Z - \eta(X)).
\]
The infeasible orthogonal moment \( g_0(W, p, \xi(p)) \) is
\[
g_0(W, p, \xi(p)) = z(p, \eta)(Y(p, \eta_0) - \gamma(p, X)),
\] (2.35)
where
\[
\gamma_0(p, x) = \mathbb{E}[Y(p, \eta_0) \mid X]
= \mathbb{E}[Y_L \mid X = x] + \mathbb{E}[(Y_U - Y_L)1\{p'V(\eta_0) > 0\} \mid X = x]
\] (2.36)
The nuisance function \( \xi(p) = (\eta(\cdot), \gamma(p, \cdot)) \). Invoking (2.34) gives a feasible orthogonal moment
\[
g(W, p, \xi(p)) = z(p, \eta)(Y(p, \eta) - \gamma(p, X)).
\] (2.37)
Corollary 4.1 establishes the asymptotic theory for the support function estimator based on (2.37).
Example (Example 2.3 cont.). Consider Example 2.3. The projected weighting vector is
\[ z(q, \eta) = -q' \hat{q}_D \log f(D \mid X). \]

The infeasible orthogonal moment \( g_0(W, p, \xi(p)) \) is
\[ g_0(W, q, \xi(q)) = z(q, \eta)Y(q, \eta_0) + q' \hat{q}_D \log f(D \mid X) \mu(q, D, X) + q' \nabla_D \mu(q, D, X), \tag{2.38} \]
where
\[ \mu_0(q, D, X) = \mathbb{E}[Y_L \mid D, X] + \mathbb{E}[(Y_U - Y_L) \mid D, X]1\{-q' \hat{q}_D \log f(D \mid X) > 0\} = \gamma_{L,0}(D, X) + \gamma_{U,0}(D, X)1\{-q' \hat{q}_D \log f(D \mid X) > 0\}. \]

Invoking (2.34) gives a feasible moment equation
\[ g(W, p, \xi(p)) = g_0(W, q, \xi(q)) + z(q, \eta)(Y(q, \eta) - Y(q, \eta_0)) = z(q, \eta)Y(q, \eta) + q' \hat{q}_D \log f(D \mid X) \mu(q, D, X) + q' \nabla_D \mu(D, X). \tag{2.39} \]

In absence of the conditioning covariates, (2.39) coincides with the efficient score in Kaido (2017). Corollary 5.1 establishes the asymptotic theory for the support function estimator based on (2.38).

2.5 Overview of Main Results

The Support Function Estimator \( \hat{\sigma}(q) \) has two stages. In the first stage, I construct an estimate \( \hat{\xi} \) of the nuisance parameter \( \xi_0 \) using some regularized estimator. In the second stage, I compute the estimated values \( (\hat{\xi}_k)_{k=1}^K \) and the support function estimate. I use different samples in the first and the second stage in the form of cross-fitting. The number \( K \) of cross-fit partitions is assumed to be fixed/finite relative to \( N \).

Definition 2.1 (Cross-Fitting).

1. For a random sample of size \( N \), denote a \( K \)-fold random partition of the sample indices \( \{N\} = \{1, 2, \ldots, N\} \) by \( (J_k)_{k=1}^K \), where \( K \) is the number of partitions and the sample size of each fold is \( n = N/K \). For each \( k \in [K] = \{1, 2, \ldots, K\} \) define \( J_k^c = \{1, 2, \ldots, N\} \setminus J_k \).

2. For each \( k \in [K] \), construct estimates \( \hat{\xi}_k = \hat{\xi}(W_{i \in J_k}) \) and \( \hat{\eta}_k = \hat{\eta}(W_{i \in J_k}) \) of the nuisance parameters \( \xi_0 \) and \( \eta_0 \) using only the data \( \{W_{j} : j \in J_k^c\} \). For any observation \( i \in J_k \), define \( \hat{\xi}_i = \hat{\xi}_k(W_i) \) and \( \hat{\eta}_i = \hat{\eta}_k(W_i) \).
Definition 2.2 (Support Function Estimator). Let \( \hat{\xi} \) and \( \hat{\eta} \) be the estimates of \( \xi_0 \) and \( \eta_0 \). Define

\[
\hat{\Sigma} = \frac{1}{N} \sum_{i=1}^{N} A(W_i, \hat{\eta}_i), \quad \hat{p}(q) = (\hat{\Sigma}^{-1})' \quad q
\]

(2.40)

\[
\hat{\sigma}(q) = \frac{1}{N} \sum_{i=1}^{N} g(W_i, \hat{p}(q), \hat{\xi}(\hat{p}(q))).
\]

(2.41)

Definition 2.3 (Multiplier Bootstrap). Let \( (e_i)_{i=1}^{N} : e_i \) are i.i.d. truncated exponential random variables \( \text{Tr Exp}(1) \) on \([0, M]\) independent of the data. Define the bootstrap analog of \( \hat{\sigma}(q) \) as

\[
\hat{\Sigma} = \frac{1}{N} \sum_{i=1}^{N} e_i A(W_i, \hat{\eta}_i), \quad \hat{p}(q) = (\hat{\Sigma}^{-1})' \quad q
\]

(2.42)

\[
\hat{\sigma}(q) = \frac{1}{N} \sum_{i=1}^{N} e_i g(W_i, \hat{p}(q), \hat{\xi}(\hat{p}(q))).
\]

(2.43)

Under mild conditions on \( \xi \), the Support Function Estimator delivers a high-quality estimate \( \hat{\sigma}(q) \) of the support function \( \sigma(q) \) with the following properties

1. With probability (w.p.) \( \to 1 \), the estimator converges uniformly over the unit sphere \( S^{d-1} \)

\[
\sup_{q \in S^{d-1}} |\hat{\sigma}(q) - \sigma(q)| = O_P(1/\sqrt{N}) = o_P(1).
\]

(2.44)

2. The estimator \( \hat{\sigma}(q) \) is asymptotically Gaussian

\[
S_N(q) := \sqrt{N}(\hat{\sigma}(q) - \sigma(q)) = \mathcal{G}_N(q) + o_P(1) \quad \text{uniformly in } S^{d-1},
\]

(2.45)

where the empirical process \( \mathcal{G}_N(q) \) is approximated by a Gaussian process \( \mathcal{G}(q) \), which is a tight \( P \)-Brownian bridge in \( \ell^\infty(S^{d-1}) \).

Define the bootstrap statistic

\[
\hat{S}_N(q) := \sqrt{N}(\hat{\sigma}(q) - \hat{\sigma}(q))
\]

Pointwise asymptotics. The sharp identified set for \( q' \beta_0 \) is \([\sigma(-q), \sigma(q)]\). Its \((1 - \tau)\)-pointwise confidence region (CR) is

\[
[q, \bar{q}] := [\sigma(-q) + N^{-1/2} \hat{C}_{\tau/2}(q), \quad \sigma(q) + N^{-1/2} \hat{C}_{1-\tau/2}(q)],
\]

where the critical values \( \hat{C}_{\tau/2}(q) \) and \( \hat{C}_{1-\tau/2}(q) \) are the \( \tau/2 \) and \( 1 - \tau/2 \) quantiles of the bootstrapped statistic \( |\hat{S}_N(q)| \). Plugging

\[
q = e_k = (0, 0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^d, \quad k = 1, 2, \ldots, d
\]

gives the CR for the projection of the identified set \( B \).
Uniform asymptotics. The $(1-\tau)$-uniform confidence region (CR) for $[-\sigma(q), \sigma(q)]$ is

$$[\ell_t(q), u_t(q)]:=[-\hat{\sigma}(q) + N^{-1/2} \hat{C}_{\tau/2}, \hat{\sigma}(q) + N^{-1/2} \hat{C}_{1-\tau/2}],$$

(2.46)

where $\hat{C}_{\tau/2}$ and $\hat{C}_{1-\tau/2}$ are the quantiles of the bootstrapped statistic $\sup_{q \in S^i} |\tilde{S}_N(q)|$. Likewise, for any function $f(\cdot)$ and a critical value $\hat{c}_N = c_N + o_P(1)$ and $c_N = O_P(1)$,

$$P(f(S_N) \leq \hat{c}_N) - P^c(f(\tilde{S}_N) \leq \hat{c}_N) \to_P 0,$$

where $P^c(\cdot)$ is the probability conditional on the data.

3 Theoretical Results.

Notation. I use the empirical process notation. For a generic function $f$ and a generic sample $(W_i)_{i=1}^N$, denote the empirical sample average by

$$E_N f(W_i) := \frac{1}{N} \sum_{i=1}^N f(W_i)$$

and the scaled, demeaned sample average by

$$G_N f(W_i) := \frac{1}{\sqrt{N}} \sum_{i=1}^N [f(W_i) - \int f(w)dP(w)].$$

For two sequences of random variables $\{a_N, b_N, N \geq 1\} : a_N \lessdot_P b_N$ means $a_N = O_P(b_N)$. For two sequences of numbers $\{a_N, b_N, N \geq 1\}, a_N \lessdot b_N$ means $a_N = O(b_N)$. Let $a \wedge b = \min \{a, b\}, a \vee b = \max \{a, b\}$. The $\ell_2$ norm of a vector is denoted by $\| \cdot \|_2$, the $\ell_1$ norm is denoted by $\| \cdot \|_1$, the $\ell_\infty$ norm is denoted by $\| \cdot \|_\infty$, and $\ell_0$ norm is denoted by $\| \cdot \|_0$. For a matrix $Q$, let $\|Q\|$ be the maximal eigenvalue of $Q$ and $\|Q\|_f$ be the Frobenius norm of $Q$. For a random vector $W$, let $\|W\|_{P,2} := (\int |W|^2 dP)^{1/2}$. The random sample $(W_i)_{i=1}^N$ is a sequence of independent copies of a random element $W$ taking values in a measurable space $(W, \mathcal{A}_W)$ according to a probability law $P$. The $\|f\|_{P,2}$ is the empirical $\ell_2$-norm, denoted as $\|f\|_{P,2} := (N^{-1} \sum_{i=1}^N f^2(W_i))^{1/2}$. Define the projection set

$$\mathcal{P} = \left\{ p \in \mathbb{R}^d : \quad 1/2 \min \text{eig}(\Sigma^{-1}) \leq \|p\| \leq 2 \max \text{eig}(\Sigma^{-1}) \right\}$$

(3.1)

and let $C_P := 2 \max \text{eig}(\Sigma^{-1})$. Let $\mathcal{F}_c$ be the space of continuous functions obeying two conditions: (1) $f(Z)$ has a continuous distribution when $Z$ is a tight Gaussian process with non-degenerate covariance function and (b) $f(\xi_N + c) - f(\xi_N) = o(1)$ for any $c = o(1)$ and any $\|\xi_N\| = O_P(1)$. Note that $[p_X] := \{1, 2, \ldots, p_X\}$. 
3.1 Assumptions

Assumption 3.1 is a standard identification condition. It ensures that the eigenvalues of $\Sigma$ are bounded from above and below.

**Assumption 3.1 (Identification).** There exist constants $\lambda_{\text{min}} > 0$ and $\lambda_{\text{max}} < \infty$ that bound the eigenvalues of $\Sigma$ in (2.2) from above and below $0 < \lambda_{\text{min}} \leq \min \text{eig}(\Sigma) \leq \max \text{eig}(\Sigma) \leq \lambda_{\text{max}}$.

Assumption 3.2 ensures that the support function is differentiable on the unit sphere. It requires the distribution of the weighting vector $V_{p, \eta_0}$ to be sufficiently smooth. For example, if the vector $V_{p, \eta_0}$ has a symmetric (i.e., spherical) distribution around zero, the normalized vector $\frac{q}{\|q\|^2} V_{p, \eta_0}$ is uniformly distributed on the surface of the unit sphere $S^{d-1}$, and Assumption 3.2 holds. Assumption 3.2 is a common regularity condition in set-identified models (e.g., Condition C.1 in Chandrasekhar et al. (2012)) and censored median regression (e.g., Assumption R.2 in Powell (1984)).

**Assumption 3.2 (Smooth boundary).** Let $d \geq 2$. There exists a finite constant $C_V$ such that

$$\sup_{q \in S^{d-1}} P\left( \frac{|q' \Sigma^{-1/2} V_{p, \eta_0}|}{\|\Sigma^{-1/2} V_{p, \eta_0}\|} \leq \delta \right) \leq C_V \delta. \tag{3.2}$$

**Example 3.1 (Gaussian Noise).** Consider Example 2.1 with $V_{p, \eta_0} = Z - \eta_0(X) \sim N(0, \Sigma)$ independent of $X$. Then, $\Sigma^{-1/2} V_{p, \eta_0} \sim N(0, I_d)$ is standard Gaussian vector and $\frac{q}{\|q\|^2} \Sigma^{-1/2} V_{p, \eta_0}/\|\Sigma^{-1/2} V_{p, \eta_0}\|$ is uniformly distributed over the unit sphere $S^{d-1}$. For any $q \in S^{d-1}$, $|q' \Sigma^{-1/2} V_{p, \eta_0}|/\|\Sigma^{-1/2} V_{p, \eta_0}\|$ is uniformly distributed on $[0, 1]$ (Pitman and Ross (2012)). As a result, (3.2) holds with $C_V = 1$ conditional on $X$ uniformly in $X$.

Assumption 3.2 is a sufficient condition for the smoothness of the boundary. If it holds, the moment equation (2.37) is differentiable in $p$. The gradient

$$G(p) := EV_{\eta_0}(p, \eta_0)$$

is a uniformly continuous function of $p$ (see Lemma A.5 in Online Appendix). As a result, there exists a uniform Gaussian approximation for the support function estimator.

**Remark 3.1.** Consider Example 2.1 If $D$ and $X$ consist of discrete variables only, the distribution of $V_{p, \eta_0}$ cannot be continuous, and Assumption 3.2 fails. Discrete distributions imply flat surfaces on the identified set, which may not be compatible with uniform Gaussian approximation. In this case, Chandrasekhar et al. (2012) suggests adding a small amount of continuously distributed noise and work with a slightly expanded set with smooth boundary, while Gafarov (2019) provides an alternative approach.
The moment functions $A(W, \eta)$ in (2.2) and $g(W, p, \xi(p))$ depend on the nuisance parameters $\eta_0$ and $\xi_0$, respectively. Definition 3.4 introduces a sequence of nuisance realization sets $\Xi_N \subseteq \Xi$ and $T_N \subseteq T$ that contain $\xi_0$ and $\eta_0$, as well as their estimators $\hat{\xi}$ and $\hat{\eta}$, with probability $1 - \phi_N$. As the sample size increases, the sets $\Xi_N$ and $T_N$ shrink. The shrinkage speed is measured by the rates below.

**Assumption 3.3 (Regularity Conditions).** (1) There exist absolute constants $c' > 2$ and $B_A < \infty$ such that the following matrix norms are bounded

$$
\sup_{\eta \in T_N} \mathbb{E} \| A(W, \eta) \|^{c'} \leq B_A, \quad \mathbb{E} \| A(W, \eta_0) \|^2 \leq B_A
$$

(2) There exists a sequence $v_N = o(N^{-1/4})$ such that

$$
\mathbb{E} | N_i A(W_i, \eta_0) - \Sigma \mathbb{E} A(W, \eta_0) | = O_P(v_N) = o_P(1).
$$

Assumption 3.4 requires the rates of Definition 3.1 to decay sufficiently fast. The matrix moment function $A(W, \eta)$ is assumed to be already orthogonal with respect to $\eta$. This is the case for covariance matrices in Examples 2.1 and 2.2.

**Assumption 3.4 (Convergence Rates).** (1) For $c_3 > 2$ in Assumption 3.3 suppose the numerator moment rates $\mu_N, r''_N, r'_N$ obey the following bounds: $\mu_N = o(N^{-1/2})$ and

$$(r''_N + r'_N) \log^{1/2} (1/(r''_N + r'_N)) + N^{-1/2+1/c_3} \log N = o(1).$$

(2) The matrix rates $A_N$ and $\delta_N$ obey the following bounds: $A_N = o(N^{-1/2})$ and $\delta_N = o(1)$. 

\[ \text{Page 14} \]
Assumption 3.5 bounds the complexity of the function class
\[ \mathcal{G}_\xi = \{ g(W, p, \xi(p)), p \in \mathcal{P} \}. \]

**Assumption 3.5 (Complexity Conditions).** (1) There exists a measurable envelope function \( G_\xi = G_\xi(W) \) that almost surely bounds all elements in the class
\[ \sup_{p \in \mathcal{P}} |g(W, p, \xi(p))| \leq G_\xi(W) \quad \text{a.s.} \]

There exists \( c > 2 \) such that \( \|G_\xi\|_{P,c} := \left( \mathbb{E}_{W \sim \mathcal{P}} (G_\xi(W))^c \right)^{1/c} < \infty. \) (2) There exist constants \( a, \nu \) that do not depend on \( N \) such that the uniform covering entropy of the function class \( \mathcal{G}_\xi \) is bounded
\[ \log \sup_Q N(\epsilon \|G_\xi\|_{Q,2}, \mathcal{G}_\xi, \| \cdot \|_{Q,2}) \leq \nu \log(a/\epsilon), \quad \text{for all } 0 < \epsilon \leq 1. \] (3.9)

### 3.2 Results

Define the influence function \( h_g(W, q) \) as
\[ h_g(W, q) = g(W, p(q), \xi_0(p(q))) - \sigma(q) \] (3.10)

and the influence function for the matrix estimation
\[ h_A(W, q) = -G(p(q))/\Sigma^{-1}(A(W, \eta_0) - \Sigma)\Sigma^{-1}q, \] (3.11)

where \( G(p) \) is the gradient defined in (3.3). Finally, define
\[ h(W, q) = h_g(W, q) + h_A(W, q). \] (3.12)

**Theorem 3.1 (Limit Theory for the Support Function Process).** Suppose Assumptions 3.1-3.5 hold. Then, the support function process \( S_N(q) = \sqrt{N}(\hat{\sigma}(q) - \sigma(q)) \) is asymptotically linear uniformly on \( S^{d-1} \)
\[ S_N(q) = G_N[h(W, q)] + o_p(1) \text{ uniformly on } S^{d-1}, \]
where \( h(W, q) \) is as in (3.12). Furthermore, the process \( S_N(q) \) admits the following approximation
\[ S_N(q) \Rightarrow_d \mathbb{G}[h(q)] + o_p(1) \text{ in } \ell^2(S^{d-1}), \]
where the process \( \mathbb{G}[h(q)] \) is a tight \( P \)-Brownian bridge in \( \ell^2(S^{d-1}) \) with a non-degenerate covariance function
\[ \Omega(q_1, q_2) = \mathbb{E}[h(W, q_1)h(W, q_2)] - \mathbb{E}[h(W, q_1)]\mathbb{E}[h(W, q_2)], \quad q_1, q_2 \in S^{d-1}. \]

Theorem 3.1 is my first main result. It says that the Support Function Estimator is asymptotically equivalent to a tight Gaussian process with a non-degenerate covariance function. Previous work (e.g., Chandrasekhar et al. (2012), Kaido (2017)) has derived similar Gaussian approximations for the estimators based on classic nonparametric methods. Combing Neyman-orthogonality and sample splitting, Theorem 3.1 allows to accommodate both classic nonparametric and modern regularized/machine learning estimators. Corollary 3.1 states that the inference properties of support function estimator.

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**Corollary 3.1** (Limit Inference on Support Function Process). Suppose Assumptions 3.1–3.5 hold. For any \( f \in \mathcal{F}_c \), \( \hat{c}_N = c_N + o_P(1) \) and \( c_N = O_P(1) \),

\[
P(f(S_N) \leq \hat{c}_N) - P(f(\mathbb{G}[h(q)]) \leq \hat{c}_N) \to 0.
\]

If \( c_N(1 - \tau) \) is the \((1 - \tau)\)-quantile of \( f(\mathbb{G}[h(q)]) \) and \( \hat{c}_N(1 - \tau) = c_N(1 - \tau) + o_P(1) \) is any consistent estimate of this quantile, then

\[
P(f(S_N) \leq \hat{c}_N(1 - \tau)) \to 1 - \tau.
\]

**Theorem 3.2** (Limit Theory for the Bootstrap Support Function Process). Under conditions of Theorem 3.1, the bootstrap support function process \( \hat{S}_N(q) = \sqrt{N}(\hat{\sigma}(q) - \hat{\sigma}(q)) \) is asymptotically linear uniformly on \( S^{d-1} \)

\[
\hat{S}_N(q) = \mathbb{G}_N[(e - 1)h(W,q)] + o_P(1) \text{ uniformly on } S^{d-1}.
\]

Furthermore, the bootstrap support function process admits an approximation conditional on the data:

\[
\mathbb{E}[h(q)] + o_P(1) \text{ in } \ell^\infty(S^{d-1}), \text{ in probability } P,
\]

where \( \mathbb{G}[h(q)] \) is a tight \( P \)-Brownian bridge in \( \ell^\infty(S^{d-1}) \) with the same distribution as the process \( \mathbb{G}[h(q)] \) defined in Theorem 3.1 and independent of \( \mathbb{G}[h(q)] \).

**Theorem 3.2** is my second main result. It establishes the validity of multiplier bootstrap for uniform inference on the support function. In contrast to the weighted bootstrap of Chandrasekhar et al. (2012), the multiplier bootstrap does not require re-estimating the first-stage nuisance parameter in each bootstrap repetition. Instead, the nuisance parameter is estimated on an auxiliary sample once and plugged into the bootstrap sampling procedure.

**Corollary 3.2** (Limit Inference on Bootstrap Support Function Process). For any \( \hat{c}_N = c_N + o_P(1) \) and \( c_N = O_P(1) \),

\[
P(f(S_N) \leq \hat{c}_N) - P^e(f(\hat{S}_N) \leq \hat{c}_N) \to_P 0.
\]

If \( \hat{c}_N(1 - \tau) \) is the \((1 - \tau)\)-quantile of \( f(\hat{S}_N) \) under \( P^e \), then

\[
P(f(\hat{S}_N) \leq \hat{c}_N(1 - \tau)) \to_P 1 - \tau.
\]

### 4 Partially Linear IV Model

#### 4.1 The First Stage

In this section, I give examples of the first-stage estimators as well as the nuisance realization sets. I focus on the partially linear IV model of Example 2.2. This discussion automatically covers Example 2.1 that is a special case of Example 2.2 with \( D = Z \).
Definition 4.1 introduces sequences of nuisance realization sets \( \{T_N, N \geq 1\} \) and \( \{M_N, N \geq 1\} \) that contain the true value of \( \eta_0 \) and \( m_0 \) and their estimates \( \hat{\eta} \) and \( \hat{m} \) with probability \( 1 - o(1) \). As the sample size \( N \) increases, the sets \( T_N \) and \( M_N \) shrink. The shrinkage speed is measured by mean square rates \( \eta_N \) and \( m_N \), respectively.

**Definition 4.1 (Mean Square Rates).** Define the mean square rate for the expectation functions \( \eta_0 \) and \( \gamma_0 \).

\[
\sup_{\eta \in T_N} \left( \mathbb{E} \| \eta(X) - \eta_0(X) \|^2 \right)^{1/2} = \eta_N
\]

\[
\sup_{m \in M_N} \left( \mathbb{E} \| m(X) - m_0(X) \|^2 \right)^{1/2} = m_N
\]

Definition 4.1 introduces mean square convergence rates for the expectation functions \( \eta_0 \) and \( \gamma_0 \). The bounds on \( \eta_N \) are established for a wide variety of regularized estimators, including partition estimators (Cattaneo and Farrell (2013), Cattaneo et al. (2020)), \( \ell_2 \)-boosting (Luo and Spindler (2016)), deep neural networks (Schmidt-Hieber (2017), Farrell et al. (2021)), linear and nonlinear sieve estimators (Chen (2007), penalized sieve estimators Chen (2011)), random forest in small (Wager and Walther (2015)) and high (Syrgkanis and Zampetakis (2020)) dimensions with sparsity structure. For the sake of brevity, the examples of nuisance realization sets \( M_N \) and \( T_N \) are omitted in this text, but they can be found in Semenova and Chernozhukov (2021), Appendix B or Semenova et al. (2017), Section 5.

**First-Stage Fitted Values: Basic Case.** In this paragraph, I consider the case when the nuisance parameter \( \xi_0 \) does not depend on \( p \).

**Assumption 4.1 (Independent and Symmetric Residual).** The following conditions hold. (1) The interval width \( Y_U - Y_L \) is independent of \( V(\eta_0) \) conditional on \( X \):

\[
(Y_U - Y_L) \perp V(\eta_0) \mid X.
\]

(2) The vector \( V(\eta_0) \) is independent of \( X \), for any \( p \in \mathcal{P} \). (3) \( V(\eta_0) \) has a spherical distribution, which implies \( \mathbb{P}(p'V(\eta_0) > 0) = 1/2 \) for any \( p \in \mathcal{P} \).

Suppose Assumption 4.1 holds. Define

\[
\gamma_{L,0}(X) := \mathbb{E}[Y_L \mid X], \quad \gamma_{UL,0}(X) = \mathbb{E}[(Y_U - Y_L) \mid X].
\]

Then, the Riesz representer function takes the form

\[
\gamma_0(p, X) = \gamma_{L,0}(X) + 1/2 \gamma_{UL,0}(X)
\]

and the first-stage fitted values take the form

\[
\hat{\gamma}(X_i) := \hat{\gamma}_{L}(X_i) + 1/2 \hat{\gamma}_{UL}(X_i), \quad i = 1, 2, \ldots, N,
\]

where \( \hat{\gamma}_{L}(X_i), \hat{\gamma}_{UL}(X_i) \) are the cross-fit first-stage estimates of Definition 2.1.
First-Stage Fitted Values: High-Dimensional Sparse Case. In this paragraph, I sketch a possible estimator of the Riesz representer function without relying on Assumption 4.1. Define
\[
\rho_0(p, X) = \mathbb{E}[\gamma(p, \eta_0) \mid X] = \mathbb{E}[(Y_U - Y_L)1\{p'V(\eta_0) > 0\} \mid X].
\]
Assume that the expectation function is approximated as
\[
\rho_0(p, X) = \Lambda(Z(X)'\nu_0(p)) + R_p(\eta_0, X), \tag{4.4}
\]
where \(Z : \mathcal{X} \to \mathbb{R}^{p_X}\) is a set of \(p_X\) measurable basis functions of the covariates \(X\), \(\nu_0(p)\) is a \(p_X\)-dimensional vector, and \(R_p(\eta_0, X)\) is an approximation error, and \(\Lambda : \mathbb{R} \to \mathbb{R}\) is the linear link function (Example 4.2) and logistic link function (Example 4.3).

Example 4.2 (Linear Lasso Estimator). Let
\[
\gamma_i(p, \hat{\eta}) := (Y_{U,i} - Y_{L,i})1\{p'V(\hat{\eta}) > 0\}, \quad i = 1, 2, \ldots, N
\]
be an estimate of \(\gamma_i(p, \eta_0)\). Given the penalty level
\[
\lambda_Z = \frac{1.1}{N^{1/2}} \Phi^{-1} \left(1 - \frac{\bar{\gamma}}{2p_XN^d}\right), \quad \bar{\gamma} = .1/\log N
\]
and the diagonal matrix of penalty loadings \(\tilde{\Psi}_p\), define
\[
\hat{\nu}(p) \in \arg \min_{\nu \in \mathbb{R}^{p_X}} N^{-1} \sum_{i=1}^N (\gamma_i(p, \hat{\eta}) - Z(X_i)'\nu)^2 + \lambda_Z\|\tilde{\Psi}_p\nu\|_1
\]
and the fitted value as
\[
\hat{\rho}(p, X_i) := Z(X_i)'\hat{\nu}(p).
\]
Given the cross-fit estimate \(\hat{\gamma}_L(\cdot)\) of the \(\gamma_{L,0}(\cdot)\), define
\[
\hat{\gamma}(p, X_i) = \hat{\gamma}_L(X_i) + Z(X_i)'\hat{\nu}(p). \tag{4.5}
\]
An important special case occurs when the brackets have constant width
\[
Y_U - Y_L = \Delta \quad \text{a.s.,}
\]
and the nuisance parameter reduces to the conditional probability
\[
\mathbb{E}[M(p, \eta_0) \mid X] := P(p'V(\eta_0) > 0 \mid X).
\]
Suppose this probability can be approximated as
\[
P(p'V(\eta_0) > 0 \mid X) = \Lambda(Z(X)'\nu_0(p)) + R_p(\eta_0, X), \tag{4.6}
\]
where \(\Lambda(t) = \exp t/(\exp t + 1)\) is the logistic link function.
Example 4.3 (Logistic Lasso Estimator). Let
\[ L(y, t) := -1(1\{y = 1\} \log \Lambda(t) + 1\{y = 0\} \log(1 - \Lambda(t))) \]
be the logistic loss function and let \( M_i(p, \tilde{y}) = 1\{p' V_i(\tilde{y}) > 0\} \) be the estimated outcome. Given the penalty level \( \lambda_Z \) and the diagonal matrix of penalty loadings \( \hat{\Psi}_p \), define
\[
\hat{\nu}(p) \in \arg \min_{\nu \in \mathbb{R}^p} N^{-1} \sum_{i=1}^{N} L(M_i(p, \tilde{y}), Z(X_i)' \nu) + \lambda_Z \| \hat{\Psi}_p \nu \|_1
\]
The final estimator of fitted values is
\[
\hat{\gamma}(p, X_i) = \hat{\gamma}_L(X_i) + \hat{\bar{\nu}}(p, X_i) = \hat{\gamma}_L(X_i) + \Delta \Lambda(Z(X_i)' \hat{\nu}(p)). \tag{4.7}
\]
Lemma 4.1 provides the first-stage convergence rates for linear Lasso estimator. In contrast to regular linear Lasso model, the outcome \( Y(\eta_0) \) depends on the nuisance parameter \( \eta_0 \) and therefore has to be estimated. I show that the linear Lasso estimator remains valid as long as the first-stage mean square rate decays sufficiently fast.

Lemma 4.1 (Validity of Linear Lasso with Estimated Outcome). The following conditions hold for \( N \) large enough and a sequence \( \kappa_N = o(1) \) and \( \Lambda(t) = t \). (i) The model (4.4) is approximately sparse with \( s_\nu = s_\nu(N) \)
\[
\sup_{p \in \mathcal{P}} \| \nu_0(p) \|_0 \leq \kappa_N
\]
and \( \log(p_X \vee N) \leq \kappa_N N^{1/3} \). (ii) Heteroscedasticity. There exists a constant \( c_\xi > 0 \) so that \( 0 < c_\xi \leq \mathbb{E}[\xi^2(\eta_0) \mid X] \) a.s. (iii) Lipschitz property of \( \nu_0(p) \). For some finite constant \( C_L, \sup_{p_1, p_2 \in \mathcal{P}} \| \nu_0(p_1) - \nu_0(p_2) \|_1 \leq C_L \| p_1 - p_2 \| \). (iv) The approximation error decays fast:
\[
\sup_{p \in \mathcal{P}} (\mathbb{E} R_0^2(p, X))^{1/2} = o(\kappa_N), \quad \kappa_N := \sqrt{\kappa_N \log(p_X \vee N) / N}.
\]
(vi) First Stage. (a) The first-stage mean square rate is fast enough \( \eta_N = o(\kappa_N) \) and \( \eta_N \). Then, under additional conditions on \( Z(X) \) in Assumption B.4, the estimate \( \hat{\nu}(p) \) of Example 4.3 is uniformly sparse, that is \( \sup_{p \in \mathcal{P}} \| \hat{\eta}(p) \|_0 \leq C_X \kappa_N \), and the following performance bounds hold:
\[
\sup_{p \in \mathcal{P}} \| Z(X)'(\hat{\nu}(p) - \eta_0(p)) \|_{p, 2} \leq C_X \sqrt{\kappa_N \log p_X / N} \tag{4.8}
\]
\[
\sup_{p \in \mathcal{P}} \| \hat{\nu}(p) - \eta_0(p) \|_1 \leq C_X \sqrt{\kappa_N^2 \log p_X / N}. \tag{4.9}
\]

Lemma 4.2 (Validity of Logistic Lasso with Estimated Outcome). Suppose the conditions of Lemma 4.1 hold for (4.4) with logistic link function \( \Lambda(t) \). Then, under Assumption B.5 the estimate \( \hat{\nu}(p) \) of Example 4.3 is uniformly sparse, that is \( \sup_{p \in \mathcal{P}} \| \hat{\eta}(p) \|_0 \leq C \kappa_N \), and the bounds (4.8)–(4.9) hold.
4.2 The Second Stage

In this section, I describe the Support Function Estimator for the Partially Linear IV Model. The estimator for Example 2.1 is obtained by replacing Steps 1 and 2 and 4 of Algorithm 1 by their analogs in Algorithm 2. As an input, Algorithm 2 takes a direction \( q \in S^{d-1} \) and the first-stage fitted values \( (\widehat{\eta}(X_i), \widehat{m}(X_i), \widehat{\gamma}(p, X_i))_{i=1}^{N} \).

**Algorithm 1** Support Function Estimator for Partially Linear IV Model.

1: The treatment residual

\[
\widehat{V}_i := Z_i - \widehat{\eta}(X_i), \quad \widehat{E}_i := D_i - \widehat{m}(X_i) \quad i = 1, 2, \ldots, N.
\]

2: The sample covariance matrix:

\[
\widehat{\Sigma} := \frac{1}{N} \sum_{i=1}^{N} \widehat{V}_i \widehat{E}_i'.
\]

3: The best-case outcome as a function of \( p \)

\[
Y_i(p, \widehat{\eta}) := Y_{L,i} + (Y_{U,i}-Y_{L,i})1\{p\widehat{V}_i > 0\}, \quad i = 1, 2, \ldots, N.
\]

4: The IV coefficient of the second-stage residual \( Y_i(\hat{p}(q), \widehat{\eta}) - \widehat{\gamma}(\hat{p}(q), X_i) \) on the instrument residual \( \widehat{V}_i \)

\[
\hat{\beta}_q = \widehat{\Sigma}^{-1} \frac{1}{N} \sum_{i=1}^{N} \widehat{V}_i [Y_i(\hat{p}(q), \widehat{\eta}) - \widehat{\gamma}(\hat{p}(q), X_i)], \quad \hat{p}(q) = \widehat{\Sigma}^{-1} q \quad (4.10)
\]

5: Report: the projection of \( \hat{\beta}_q \) onto the direction \( q \):

\[
\widetilde{\sigma}(q) = q' \hat{\beta}_q.
\]
**Algorithm 2** Support Function Estimator for Partially Linear Model.

Input: a direction \( q \in S^{d-1} \), estimated values \((\hat{\eta}(X_i), \hat{\gamma}(p, X_i))\)\(_{i=1}^N\). Estimate the following quantities:

1', 2' The treatment residual and the sample covariance matrix

\[
\hat{V}_i := D_i - \hat{\eta}(X_i), \quad i = 1, 2, \ldots, N, \quad \hat{\Sigma} := \frac{1}{N} \sum_{i=1}^N \hat{V}_i \hat{V}_i^\top.
\]

4' The OLS coefficient of the second-stage residual \( Y_i(\hat{p}(q), \hat{\eta}) - \hat{\gamma}(\hat{p}(q), X_i) \) on the treatment residual \( \hat{V}_i \) as in (4.10).

### 4.3 Results

In this section, I verify Assumptions 3.4–3.5 for Example 2.2. Then, I establish asymptotic Gaussian approximation for the Support Function Estimator.

**Assumption 4.2** (Bounded Width \( Y_U - Y_L \)). The width \( Y_U - Y_L \) is bounded by a finite constant \( M_{UL} \) a.s., that is

\[
Y_U - Y_L \leq M_{UL} \text{ a.s.}
\]

**Assumption 4.3** (Regularity Conditions). (1) For every \( p \in \mathcal{P} \), the residual vector \( V(\eta_0) \) has a conditional density \( h(p \mid V(\eta_0)|X=x) \) that is bounded uniformly over \( X \) by \( M_h \). (2) The residual vector norms \( \|V(\eta_0)\| = \|Z - \eta_0(X)\| \) and \( \|E(\eta_0)\| = \|D - \eta_0(X)\| \) are subGaussian random variables conditional on \( X \). (3) The random variable \( Y_L \) has finite conditional second moment \( \sup_{x \in X} \mathbb{E}[Y_L^2 \mid X = x] \leq C_L \) and \( \|Y_L\|_{P,4} \leq C_L \) for some finite \( C_L < \infty \). Finally, for some \( p' > 2 \), \( \|\gamma(X)\|_{P,p'} \) and \( \|V(\eta_0)\|_{P,p'} \) are finite.

Lemma 4.3 shows that the moment equation (2.31) incurs only a second-order bias due to the sign mistake of \( V(\eta_0) \) as long as \( V(\eta_0) \) is continuously distributed.

**Lemma 4.3** (First-Order Bias). Under Assumptions 4.2 and 4.3 (1), the first-order bias shrinks at the quadratic speed

\[
\sup_{p \in \mathcal{P}} |\mathbb{E}[z(p, \eta_0)(Y(p, \eta) - Y(p, \eta_0))]| \leq 2C^2_p M_{UL} M_h \mathbb{E} \|\eta(X) - \eta_0(X)\|^2.
\]

Likewise, the second-order bias shrinks at the quadratic speed

\[
\sup_{p \in \mathcal{P}} |\mathbb{E}[(z(p, \eta) - z(p, \eta_0))(Y(p, \eta) - Y(p, \eta_0))]| \leq 2C^2_p M_{UL} M_h \mathbb{E} \|\eta(X) - \eta_0(X)\|^2.
\]

**Lemma 4.4** (Verification of Assumption 3.3). Suppose Assumption 4.3 (2) holds. For any sequence \( \ell_N \to \infty \), Assumption 3.3 is satisfied with \( v_N = \sqrt{\ell_N / N} \), in particular, one can take \( \ell_N = \log N \) to ensure that \( v_N = o(N^{-1/2} \log N) \).
Lemma 4.5 verifies the Assumption [3.4] for Example 2.2. Define the mean square rates for the expectation functions
\[
\sup_{\gamma \in \Gamma_{L,N}} \left( \mathbb{E}(\gamma_L(D, X) - \gamma_{L,0}(D, X))^2 \right)^{1/2} =: \gamma_{L,N}
\]
\[
\sup_{\gamma \in \Gamma_{UL,N}} \left( \mathbb{E}(\gamma_{UL}(D, X) - \gamma_{UL,0}(D, X))^2 \right)^{1/2} =: \gamma_{UL,N}
\]

**Lemma 4.5 (Verification of Assumptions [3.4–3.3])**. Let \( \eta_0(X) \) be as in (2.11), \( V(\eta) \) be as in (2.12), the matrix function \( A(W, \eta, m) \) be as in (2.13) and the orthogonal moment function be as in (2.37). Suppose Assumptions 4.2 and 4.3 hold. Furthermore, suppose (1) the elements of \( \Gamma_{L,N} \) are bounded by finite constant \( M \), in the sup-norm: \( \sup_{\gamma \in \Gamma_{L,N}} |\gamma_L(x)| \leq M \), and \( \sup_{\gamma \in \Gamma_{UL,N}} |\gamma_{UL}(x)| \leq M_{UL} \gamma \) (2) the elements of \( \mathcal{M}_N \) and \( \mathcal{T}_N \) are bounded by finite constant \( M_\eta \). Furthermore, suppose (1) the elements of \( \Gamma_{L,N} \) are bounded by finite constant \( M \), in the sup-norm: \( \sup_{\gamma \in \Gamma_{L,N}} |\gamma_L(x)| \leq M \), and \( \sup_{\gamma \in \Gamma_{UL,N}} |\gamma_{UL}(x)| \leq M_{UL} \gamma \) (2) the elements of \( \mathcal{M}_N \) and \( \mathcal{T}_N \) are bounded by finite constant \( M_\eta \). Then, Assumption 3.3(1) holds. Furthermore, the bias rates in Definition 3.1 can be bounded as follows with \( \gamma_0^2 := 2(\gamma_{L,N}^2 + \gamma_{UL,N}^2) \)
\[
\mu_N \leq 4C_2^2 M_{UL}\Sigma \eta_N^2 + C_P \eta_N \cdot \gamma_N
\]
\[
\gamma_N \leq \eta_N \cdot m_N.
\]
In addition, for \( \tau_N := N^{-1/2} \log N \), the sequences \( r_N^\delta, r_N^\gamma, \delta_N \) obey
\[
r_N^\delta = O(\eta_N + \gamma_N), \quad r_N^\gamma = O(N^{-1/2} \log N), \quad \delta_N = O(\eta_N + m_N).
\]
Thus, if \( \eta_N = O(N^{-1/4}) \) and \( (m_N + \gamma_N) \cdot \eta_N = o(N^{-1/2}) \) and \( m_N = o(1) \) and \( \gamma_N = o(1) \), Assumption [3.4] holds.

Combining the statements in Lemmas 4.4 and 4.5, I obtain the following corollary.

**Corollary 4.1 (Asymptotic Theory for Partially Linear IV Model with Interval-Valued Outcome)**. Suppose Assumptions [3.1–3.2] and [4.7] and the conditions of Lemma 4.5 hold with the fast enough first-stage rates \( \eta_N, m_N \) and \( \gamma_N := 2(\gamma_{L,N} + \gamma_{UL,N}) \) such that Assumption [3.4] holds. Then, Theorems 3.1 and 3.2 and Corollaries 3.1 and 3.2 hold for the Support Function Estimator of the Algorithm 1 with the first-stage fitted values \( \gamma_0 \), and with the influence function \( h(W, q) \) equal to \( (3.12) \).

**Corollary 4.2 (Asymptotic Theory for Partially Linear IV Model with Interval-Valued Outcome)**. Suppose Assumptions [3.1–3.2] and the conditions of Lemma 4.5 hold with the fast enough first-stage rates \( \eta_N, m_N \) and \( \gamma_N := 2(\gamma_{L,N} + \gamma_{UL,N}) \) such that Assumption [3.4] holds. Then, Theorems 3.1 and 3.2 and Corollaries 3.1 and 3.2 hold for the Support Function Estimator of the Algorithm 1 with the influence function \( h(W, q) \) equal to \( (3.12) \), where the first-stage fitted values are as in (4.3) (assuming Lemma 4.1 holds with \( \sigma_N = o(N^{-1/4}) \)) or as in (4.7) (assuming Lemma 4.2 holds with \( \sigma_N = o(N^{-1/4}) \)).
Remark 4.2. Consider Example 2.1. Note that the short least squares regression (2.5) uses only $d$ out of (infinitely) many restrictions implied by the exogeneity restriction (2.1). Therefore, the identified set

$$B_1 := \left\{ b \in \mathbb{R}^d : \exists f_b \in \mathcal{L} \text{ and } Y \in [Y_L, Y_U] : \mathbb{E}[Y - Db - f_b(X) \mid D, X] = 0 \right\}$$

is a convex subset of $B$, but may not coincide with $B$.

5 Average Partial Derivative

In this section, I present the identification, estimation and inference results for Example 2.3.

Assumption 5.1 states the sufficient conditions for compactness and convexity of the identified set $B$.

Assumption 5.1 (Regularity Conditions for Average Partial Derivative). The following conditions hold. (1) There exists a compact and convex set $D$ with nonempty interior containing the support of $D$, such that $f_0(d \mid X) = 0$ on the boundary of $D$ a.s. in $X$. (2) The random variables $D$ and $f_0(D \mid X)$ and $\nabla f_0(D \mid X)$ have the density conditional on $X$. (3) The random vector $\eta_0(D, X) = \nabla f_0(D \mid X)/f_0(D \mid X)$ is $L_{P,2}$-integrable, that is, there exists a finite constant $C_{APD} < \infty$ such that

$$\|V(\eta_0)\|_{P,2} := \|\nabla f_0(D \mid X)/f_0(D \mid X)\|_{P,2} \leq C_{APD}.$$  

(4) The functions $\gamma_{L,0}(D, X)$ and $\gamma_{UL,0}(D, X)$ are bounded by some finite constant $C_\gamma < \infty$ on the support of $D$ and $X$. (5) Furthermore, $\gamma_{L,0}(D, X)$ and $\gamma_{UL,0}(D, X)$ are continuously differentiable w.r.t to $D$ with bounded derivatives a.s. in $D$ and $X$. (5) The random variables $|V(\eta_0)|$ and $\|V(\eta_0)(Y_L - \gamma_{L,0}(D, X))\|$ are $L_{P,2}$-integrable and $L_{P,c'}$-integrable for $c' > 2$.

Lemma 5.1. Suppose Assumption 5.1 (1)-(4) hold. Then, (a) the identified set $B$ for $\beta_0$ is compact and convex and (b) the support function of $B$ is given by (2.31). In addition, if Assumption 5.2 holds, which implies that $B$ is strictly convex.

Lemma 5.1 extends the Theorem 2.1 of Kaido (2017) to allow the density $D$ to be conditioned on $X$.

Assumption 5.2 (Margin Condition). (1) There exists an absolute constant $C_f < \infty$ so that, in some neighborhood $(0, \delta)$ of zero,

$$\sup_{q \in S^{d-1}} P \left( |q' \nabla_D f_0(D \mid X)/f_0(D \mid X)| \leq \delta \right) \leq C_f \delta, \quad \delta \in (0, \bar{t}).$$

The margin condition is commonly used in classification analysis (Mammen and Tsybakov (1999), Tsybakov (2004)) and empirical welfare maximization (Kitagawa and Tetenov (2018), Mbakop and Tabord-Meehan (2021)) and bounds Semenova (2020). This paper is the first one to introduce it for the study of average partial derivative with an interval-valued variable.
**Definition 5.1 (Worst-Case Rates).** Let $f_0(D \mid X)$ and $\nabla D f_0(D \mid X)$ be the conditional density and its derivative. Let $\{F_N, \ N \geq 1\}$ and $\{F_N^1, \ N \geq 1\}$ be sequences of realization sets of the estimates of $f_0(D \mid X)$ and $\nabla D f_0(D \mid X)$, respectively. Assume that these sequences shrink at the following worst-case rates $f_N^\infty$ and $f_1^\infty$:

\[
\sup_{f \in F_N} \sup_{d, x} |f(d \mid x) - f_0(d \mid x)| \leq f_N^\infty
\]

\[
\sup_{f \in F_N^1} \sup_{d, x} \|\nabla_d f(d \mid x) - \nabla_d f_0(d \mid x)\| \leq f_1^\infty
\]

Define

\[
\eta_N^\infty := f_1^\infty + f_N^\infty.
\]

Furthermore, assume that the elements of $F_N$ are bounded by some $\bar{B}_f < \infty$:

\[
\sup_{f \in F_N} \sup_{d, x} |f^{-1}(d \mid x)|, |f(d \mid x)| \leq \bar{B}_f.
\]

and

\[
\sup_{f \in F_N^1} \sup_{d, x} \|\nabla_d f(d \mid x)\| \leq \bar{B}_f.
\]

5.1 The Algorithm

**Algorithm 3 Support Function Estimator for Average Partial Derivative.**

1. The best-case outcome as a function of $q$

\[
Y_i(q, \tilde{\eta}) := Y_{L,i} + (Y_{U,i} - Y_{L,i})1\{-q'\tilde{\eta}(D_i, X_i) > 0\}, \quad i = 1, 2, \ldots, N.
\]

2. The expectation function of $Y(q, \eta)$ and its derivative

\[
\hat{\mu}(q, D_i, X_i) = \tilde{\gamma}_L(D_i, X_i) + \tilde{\gamma}_U(L(D_i, X_i)1\{-q'\tilde{\eta}(D_i, X_i) > 0\}
\]

\[
\nabla_D \hat{\mu}(q, D_i, X_i) = \nabla_D \tilde{\gamma}_L(D_i, X_i) + \nabla_D \tilde{\gamma}_U(L(D_i, X_i)1\{-q'\tilde{\eta}(D_i, X_i) > 0\}
\]

3. The sample moment

\[
\hat{\beta}_q = \frac{1}{N} \sum_{i=1}^{N} -\tilde{\eta}(D_i, X_i)[Y_i(q, \tilde{\eta}) - \hat{\mu}(q, D_i, X_i)] + \nabla_D \hat{\mu}(q, D_i, X_i) \quad (5.1)
\]

4. Report: the projection of $\hat{\beta}_q$ onto the direction $q$:

\[
\hat{\sigma}(q) = q'\hat{\beta}_q.
\]
5.2 Results

Lemma 5.2 shows that the moment equation (2.31) incurs only a second-order bias due to the sign mistake of \( V_{p \eta_0} \) as long as \( V_{p \eta_0} \) is continuously distributed. In contrast to the setup of Lemma 4.3, the estimation error is not orthogonal to the space of \( V_{p \eta_0} \). As a result, the first-order bias is bounded by invoking the margin assumption and the \( \ell_8 \)-rate.

Lemma 5.2 (First-Order Bias). Suppose Assumptions 4.2 and 5.2 holds. Then, the first-order bias shrinks at quadratic speed, that is, for some constant \( \bar{C} \) large enough,

\[
\sup_{\eta \in \mathcal{T}_N} \left| \mathbb{E}[z(q, \eta_0)(Y(q, \eta) - Y(q, \eta_0))] \right| = O((\eta_N^8)^2).
\]

Likewise, the second-order bias shrinks at quadratic speed

\[
\sup_{\eta \in \mathcal{T}_N} \left| \mathbb{E}[(z(q, \eta) - z(q, \eta_0))(Y(q, \eta) - Y(q, \eta_0))] \right| = O((\eta_N^8)^2).
\]

Definition 5.2 (Mean Square Rates). Define the mean square rates for the expectation functions

\[
\sup_{\gamma_L \in \Gamma_L} \left( \mathbb{E}(\gamma_L(D, X) - \gamma_{L,0}(D, X))^2 \right)^{1/2} =: \gamma_{L,N}
\]

\[
\sup_{\gamma_{UL} \in \Gamma_{UL}} \left( \mathbb{E}(\gamma_{UL}(D, X) - \gamma_{UL,0}(D, X))^2 \right)^{1/2} =: \gamma_{UL,N}
\]

and for their derivatives

\[
\sup_{\gamma_L \in \Gamma_{UL}} \left( \mathbb{E}\|\nabla_D \gamma_L(X) - \nabla_D \gamma_{L,0}(X)\|^2 \right)^{1/2} =: \gamma_{L,N}^1
\]

\[
\sup_{\gamma_{UL} \in \Gamma_{UL}} \left( \mathbb{E}\|\nabla_D \gamma_{UL}(X) - \nabla_D \gamma_{UL,0}(X)\|^2 \right)^{1/2} =: \gamma_{UL,N}^1
\]

and let

\[
\gamma_N := \gamma_{L,N} + \gamma_{UL,N} + \gamma_{L,N}^1 + \gamma_{UL,N}^1.
\]

Corollary 5.1 (Asymptotic Theory for Average Partial Derivative with an Interval-Valued Outcome). Suppose Assumptions 5.1 and 4.2 and 5.2 hold. In addition, suppose \( |\gamma_L(D, X)| + \|\nabla_D \gamma_L(D, X)\| + |\gamma_{UL}(D, X)| + \|\nabla_D \gamma_{UL}(D, X)\| < M, \) a.s.. Then, the sequences \( \mu_N \) and \( r_N' \) can be bounded as follows

\[
\mu_N = O((\eta_N^8)^{3/2} + \eta_N^{8} \cdot \gamma_N).
\]

and \( r_N' = O(\gamma_N + (\eta_N^8)^{1/2}) \). In particular, if \( \eta_N^8 = o(N^{-1/3}) \) and \( \eta_N^8 \cdot \gamma_N = o(N^{-1/2}) \) and \( \gamma_N = o(1) \), Assumption 5.4 holds. Assumption 5.3 holds automatically since \( A(W, \eta_0) = \Sigma = I_4 \) is a known matrix. Finally, Assumption 3.5 holds. Then, Theorems 5.1 and 5.2 hold for the Support Function Estimator of Algorithm 3 and the influence function equal to

\[
h(W, q) = g(W, q, \xi(q)) - \mathbb{E}[g(W, q, \xi(q))],
\]

where \( g(W, q, \xi(q)) \) is given in (2.38).
6 Simulation Study

In this section, I compare the performance of the classic (series-based non-orthogonal) and the proposed (lasso-based orthogonal) approaches in a moderate-dimensional sparse design. The first approach is to plug a least squares series first-stage estimator into the non-orthogonal moment equation (2.31). The second one is to plug an \( \ell_1 \)-regularized least squares series into the orthogonal moment equation (2.37). Regularization helps to leverage the sparsity assumption and to reduce risk.

Consider the partially linear model of Example 2.1 with \( d = 2 \) treatments. The outcome equation is generated as

\[
Y = D_1 \beta_1 + D_2 \beta_2 + X \cdot (c_\theta \theta_0) + U, \tag{6.1}
\]

where \( D = (D_1, D_2) \) is the treatment vector, \( X \in \mathbb{R}^{p_X} \) is the \( p_X \)-covariate vector with \( p_X = 50 \), and \( U \sim N(0, \sigma_U^2) \) is a normal shock independent of \( D \) and \( X \) with \( \sigma_U = 1 \). For each treatment \( m = 1, 2 \), its reduced form is

\[
D_m = X \cdot (c_{D_m} \alpha_m) + V_m, \quad m = 1, 2, \tag{6.2}
\]

where the coefficients

\[
\theta_0 = \alpha_1 = \alpha_2 = (1, 1/2^2, \ldots, 1/j^2, \ldots, 1/p_X^2).
\]

The parameters in (6.1) and (6.2) are chosen as

\[
c_D = (c_{D_1}, c_{D_2}) = (2, 1), \quad c_\theta = 1, \quad \beta_0 = (1, 1).
\]

The covariates \( X \) are generated from \( N(0, \Omega) \), where \( \Omega \) is a Toeplitz matrix with correlation coefficient \( \rho = 0.5 \). That is, for every \( (i, j) \in \{1, 2, \ldots, p_X\}^2 \), \( \Omega_{ij} = \rho^{|i-j|} \). The vector \( V \) is independent of \( (X, U) \) and is drawn from the bivariate normal distribution

\[
V = (V_1, V_2) \sim N(0, \Sigma), \quad \Sigma = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}. \tag{6.3}
\]

The outcome \( Y \) is not included into the data. Instead, the support of \( Y \) is partitioned into the bins \( \cup_{s=1}^S \{b_s, b_{s+1}\} \) of width \( \Delta \):

\[
b_{s+1} = b_s + \Delta, \quad s = 1, 2, \ldots, S - 1,
\]

where \( b_0 = -\infty \) and \( b_{S+1} = \infty \). The observed bounds \( Y_L \) and \( Y_U \) are taken to be

\[
[Y_L, Y_U] := \sum_{s=1}^S \{b_s, b_{s+1}\} \cdot 1\{Y \in [b_s, b_{s+1}]\}.
\]

Thus, the observed data vector \( W = (X, D, Y_L, Y_U) \) but does not contain \( Y \).
I now derive the true (population) support function. Plugging \( Y_U - Y_L = \Delta \) into (2.31) gives
\[
\sigma(q) = q'\Sigma^{-1}EY(q, \eta_0) = q'\Sigma^{-1}EY_L + \Delta E \max(q'\Sigma^{-1}Y, 0).
\]
Invoking (6.3) gives
\[
\max(q'\Sigma^{-1}Y, 0) \sim N(0, q'\Sigma^{-1}\Sigma\Sigma^{-1}q) = \sqrt{q'\Sigma^{-1}q}N(0, 1),
\]
which implies
\[
E \max(q'V, 0) = \sqrt{q'\Sigma^{-1}q}/2\pi.
\]
Thus, the support function is
\[
\sigma(q) = q'\kappa_0 + \Delta \sqrt{q'\Sigma^{-1}q}/2\pi, \quad \kappa_0 := \Sigma^{-1}EY_L.
\]
(6.4)

The classic and the proposed approaches are implemented via Algorithm 1 with different sets of the first-stage fitted values. For each treatment \( m \in \{1, 2\} \), the first-stage regression parameter \( \alpha \) is estimated as
\[
\hat{\alpha}_m := \arg\min_{\beta \in \mathbb{R}^p} \frac{1}{N} \sum_{i=1}^N (D_{im} - X'\beta)^2 + \lambda_D \|\beta\|_1,
\]
where \( \lambda_D = 0 \) in the series-based case and \( \lambda_D > 0 \) in the lasso-based case. For the lasso estimator, the penalty parameter \( \lambda_D \) is chosen according to Algorithm 1 in Belloni et al. (2017) (i.e., the default value of rlasso package). In both cases, the treatment fitted values are
\[
\hat{\eta}(X) = X'\hat{\alpha}_1, \hat{\alpha}_2.
\]
In the orthogonal case (the proposed approach), the best-case outcome fitted values are
\[
\hat{\gamma}_U(X) = X'\hat{\gamma}_L + \frac{1}{2}\Delta,
\]
where \( \hat{\gamma}_L \) is estimated by Lasso regression of \( Y_L \) on \( X \) similarly to (6.5). Since the classic case does not require partialling out, the fitted values \( \hat{\gamma}_U(X) \) are set to zero.

The estimator’s performance is summarized in terms of its risk and coverage. The total risk of the estimator is defined as
\[
R_H := \sup \{\tilde{\sigma}(q) - \sigma(q)\}.
\]
(6.6)

It coincides with the Hausdorff distance \( d(\hat{B}, B) \) between the estimated \( \hat{B} \) and the true \( B \) sets. Furthermore, the outer and the inner risks are defined as
\[
R_O := \sup_{q \in \mathbb{S}^{d-1}} \max(\tilde{\sigma}(q) - \sigma(q), 0), \quad R_I := \sup_{q \in \mathbb{S}^{d-1}} \max(\sigma(q) - \tilde{\sigma}(q), 0).
\]
(6.7)

The rejection frequency is the share of rejected simulation draws
\[
\frac{1}{N_S} \sum_{s=1}^{N_S} 1\{R_H > c_{1-\alpha}\},
\]
(6.8)
where $c^*_1$ is the $(1 - \alpha)$-quantile of $R^b_H$ of the bootstrap process

$$R^b_H := \sup_{q \in S^{d-1}} |\hat{\sigma}^b(q) - \hat{\sigma}(q)|. \quad (6.9)$$

Table I compares the classic and the proposed estimators in terms of risk and coverage. Across the board, the proposed estimator has a smaller total risk than the classic one, by a factor of ranging from 2.0 for $\Delta = 1$ to 3.0 for $\Delta = 3$. The classic estimator has higher risk due to the excessive noise of series estimators $\hat{\alpha}_1$ and $\hat{\alpha}_2$ in a regime with $p_X = 50$ covariates.

I investigate the importance of the first-stage regularization and second-stage orthogonality, applying one at a time. Table C.1 compares the non-orthogonal (Columns (1)-(4)) and the orthogonal (Columns (5)-(8)) estimators based on the true treatment first stage. Across the board, the orthogonal estimator has smaller total risk than the non-orthogonal one, by a factor of 2.5 on average. The variance reduction occurs due to partialing out the relevant controls from the outcome $Y^{best}(\eta_h)$. In contrast, the risks of series-based non-orthogonal (Table I Column (3)) and orthogonal (Table C.1 Column (7)) estimators are close to each other. In particular, the high risk of the series-based estimator cannot be improved by orthogonalization.

Next, I compare the ortho (Table I Columns 5–8) and non-ortho (Table C.2 Columns 1–4) estimators based on the lasso-based first stage. Across the board, the risk of the ortho version is substantially smaller than the non-ortho one, by a factor ranging from 2 to 5.5. The non-ortho version has a higher risk, because the estimates of non-zero coefficients in $\alpha_m$ are shrunk to zero. While the shrinkage bias could be reduced by invoking the post-lasso instead of the lasso estimator, post-single-selection inference may not be robust to moderate deviations from zero.

### 7 Empirical illustration

This section demonstrates the proposed approach by estimating the gender wage gap with a bracketed wage variable. First, I show that a frequent empirical practice – midpoint regression – gives biased results. Instead of this approach, I propose reporting identified set (i.e., the lower and the upper bound) for the parameter of interest and demonstrate how to estimate the set.

The sample for the analysis comes from the U.S. March Supplement of the Current Population Survey (CPS) in 2015, as studied in Mulligan and Rubinstein (2008) and Chandrasekhar et al. (2012). The selected sample consists of white non-hispanic individuals, aged 25 to 64 years, working more than 35 hours per week during at least 50 weeks of the year. The resulting sample comprises 32,523 workers, including 18,137 men and 14,386 women. The object of interest is the gender wage gap – the average difference in log wages between men and women after controlling for the observed characteristics. The data set
is augmented by the lower bound $Y_L$ and the upper bound $Y_U$ defined as

$$[Y_L, Y_U] := \sum_{s=1}^{S} [b_s, b_{s+1}) \cdot 1\{Y \in [b_s, b_{s+1})\},$$

where $b_0 = 1$ and $b_{s+1} - b_s = \Delta \quad s = 1, 2, \ldots, S$.

I assume that the log wage variable follows the partially linear regression of Example 2.1

$$Y = D\beta_0 + X'\gamma_0 + U, \quad E[U \mid X, D] = 0.$$  

Here, the outcome variable $Y$ is the logarithm of the hourly wage rate, the treatment/policy variable $D$ is an indicator for female gender, and the vector of $p_X = 260$ controls includes demographic indicators, region, and experience indicators, as well as their interactions. The coefficients $\beta_0$ and $\gamma_0$ are the target and the nuisance parameters, respectively. The parameter $\beta_0$ is identified as a minimizer of the least squares loss function

$$\beta_0 := \arg\min_{b \in \mathbb{R}} E(Y - (D - \eta_0(X))b)^2.$$  

When $Y$ is unobserved, a frequent approach is to replace $Y$ by the bracket midpoint

$$Y_M := (Y_L + Y_U)/2.$$  

The “mid-point” regression parameter is taken to be

$$\beta_M := \arg\min_{b \in \mathbb{R}} E(Y_M - (D - \eta_0(X))b)^2,$$

which can differ from $\beta_0$. I report the estimate and the 95% CI of $\beta_0$ and $\beta_M$. The first-stage treatment and the outcome expectation functions are estimated via logistic and linear Lasso regression of Belloni et al. (2017) implemented in the hdm R package, respectively. In addition, I also consider the random forest estimator as implemented in the ranger R package. The final estimator is taken to be the Double Machine Learning estimator of Chernozhukov et al. (2018) with $K = 2$-fold cross-fitting. The Lasso-based first-stage fitted values (Columns (1)-(2)) and the random-forest-based (Columns (3)-(4))

Instead of reporting $\beta_M$ – which is a biased measure of $\beta_0$ – I propose reporting the lower and the upper bound on $\beta_0$. The moment equation for $\beta_U$ is given in (2.15) and its estimate is defined in the Algorithm 2 with the fitted values described below. The symmetry-based specification (SYM) imposes Assumption 4.1 and the first-stage fitted values are taken to be (4.1). The sparsity-based specification (SPRS) imposes the model (4.4), and the first-stage are taken to be as in Example 4.3 (4.7). The treatment expectation function $\eta_0(\cdot)$ is estimated the same as in the point-identified specifications.

Table 2 summarizes the findings for the bracket width $\Delta \in \{1, 2, 3\}$. The “ground-truth” gender wage gap ranges between 18% (Columns (3)-(4)) and
20% (Columns (1)-(2)). I call this estimate “ground-truth” because this is the estimate to be reported if the wage \( Y \) was observed. When the bracket width \( \Delta = 1 \) is small, the midpoint estimate \( \beta_M \) is close to the estimate of \( \beta_0 \). When \( \Delta = 2 \), the midpoint estimate \( \beta_M \) ranges between 25\% (RF) and 28\% (Lasso). Furthermore, the 95\% CI for \( \beta_M \) does not contain the “ground-truth” estimate of \( \beta_0 \) for either RF or Lasso first-stage method. When \( \Delta = 3 \), the magnitude of the bias remains substantial, which speaks against midpoint regression for large values of bracket width.

To estimate \([\beta_L, \beta_U]\), I consider four specifications: SYM-Lasso, SYM-RF, SPRS-Lasso and SPRS-RF. Across the board, the bounds contain \( \beta_0 \). Furthermore, the specifications yield close results despite being based on very different starting assumptions. That said, the bounds \([\beta_L, \beta_U]\) come out wide. As discussed in Remark 4.2, the bounds \([\beta_L, \beta_U]\) may not be sharp for \( \beta_0 \), since they are utilize only \( d \) (out of infinitely many) moment restriction implied by (2.1). The derivation of sharp bounds for \( \beta_0 \) is left for the future work.
Table 1: Finite-sample performance of the classic (series-based) and the proposed (Lasso-based) methods

|        | Series-based |          |          |          | Lasso-based |          |          |          |
|--------|--------------|----------|----------|----------|-------------|----------|----------|----------|
|        | Total | Outer | Inner | Rej.freq | Total | Outer | Inner | Rej.freq |
| $N$    |       |       |        |          |       |       |        |          |
| 250    | 0.24  | 0.18  | 0.23   | 0.03     | 0.11  | 0.11  | 0.09   |          |
| 500    | 0.19  | 0.09  | 0.19   | 0.25     | 0.08  | 0.08  | 0.08   |          |
| 700    | 0.19  | 0.07  | 0.19   | 0.43     | 0.07  | 0.07  | 0.04   |          |
| 1,000  | 0.18  | 0.06  | 0.18   | 0.99     | 0.06  | 0.06  | 0.10   |          |

Panel A: Bracket width $\Delta = 1$

|        | Series-based |          |          |          | Lasso-based |          |          |          |
|        | Total | Outer | Inner | Rej.freq | Total | Outer | Inner | Rej.freq |
| 250    | 0.35  | 0.24  | 0.34   | 0.05     | 0.14  | 0.13  | 0.09   |          |
| 500    | 0.33  | 0.13  | 0.33   | 0.85     | 0.12  | 0.09  | 0.09   |          |
| 700    | 0.32  | 0.09  | 0.32   | 0.99     | 0.10  | 0.08  | 0.03   |          |
| 1,000  | 0.32  | 0.07  | 0.32   | 1.00     | 0.09  | 0.07  | 0.10   |          |

Panel B: Bracket width $\Delta = 2$

|        | Series-based |          |          |          | Lasso-based |          |          |          |
|        | Total | Outer | Inner | Rej.freq | Total | Outer | Inner | Rej.freq |
| 250    | 0.48  | 0.32  | 0.47   | 0.10     | 0.21  | 0.17  | 0.16   | 0.07     |
| 500    | 0.46  | 0.16  | 0.46   | 0.98     | 0.15  | 0.12  | 0.12   | 0.07     |
| 700    | 0.46  | 0.12  | 0.46   | 1.00     | 0.13  | 0.11  | 0.10   | 0.04     |
| 1,000  | 0.46  | 0.09  | 0.46   | 1.00     | 0.11  | 0.09  | 0.09   | 0.06     |

Panel C: Bracket width $\Delta = 3$

Notes. Results are based on 10,000 simulation runs. Panels A, B and C correspond to the bin width $\Delta = 1, 2, 3$. Table shows the total risk (6.6), the outer and inner risks (6.7), and the rejection frequency (6.8) for the nominal size $\alpha = 0.05$. The supremum over $S^1$ is approximated by the maximum over the grid consisting of 50 evenly spaced points on unit circumference $S^1$. Columns (1–4) and (5–8) correspond to the classic and the proposed approach. The number of bootstrap repetitions $B = 2,000$. The true support function $\sigma(q)$ is in (6.4). For the description of estimators, see text.
Table 2: Bounds on gender wage gap with bracketed log wage

|                  | Lasso Estimated Set | 95% CI       | RF Estimated Set | 95% CI       |
|------------------|---------------------|--------------|-----------------|--------------|
| **Panel A: Bracket width \( \Delta = 1 \)** |                     |              |                 |              |
| True \( \beta_0 \) | -0.200 (-0.213, -0.186) | -0.180 (-0.194, -0.167) |               |              |
| \( \beta_M \)     | -0.199 (-0.215, -0.184) | -0.181 (-0.196, -0.165) |               |              |
| \([\beta_L, \beta_U] \text{ (SYM)}\) | [-1.195, 0.796] (-1.211, 0.813) | [-1.133, 0.771] (-1.150, 0.788) |       |              |
| \([\beta_L, \beta_U] \text{ (SPRS)}\) | [-1.193, 0.794] (-1.209, 0.810) | [-1.115, 0.754] (-1.131, 0.770) |       |              |

| **Panel B: Bracket width \( \Delta = 2 \)** |                     |              |                 |              |
| True \( \beta_0 \) | -0.200 (-0.213, -0.186) | -0.180 (-0.194, -0.167) |               |              |
| \( \beta_M \)     | -0.279 (-0.302, -0.256) | -0.251 (-0.273, -0.228) |               |              |
| \([\beta_L, \beta_U] \text{ (SYM)}\) | [-2.270, 1.712] (-2.295, 1.737) | [-2.155, 1.654] (-2.180, 1.679) |       |              |
| \([\beta_L, \beta_U] \text{ (SPRS)}\) | [-2.266, 1.708] (-2.290, 1.731) | [-2.120, 1.618] (-2.144, 1.643) |       |              |

| **Panel C: Bracket width \( \Delta = 3 \)** |                     |              |                 |              |
| True \( \beta_0 \) | -0.200 (-0.213, -0.186) | -0.180 (-0.194, -0.167) |               |              |
| \( \beta_M \)     | -0.158 (-0.179, -0.137) | -0.131 (-0.152, -0.110) |               |              |
| \([\beta_L, \beta_U] \text{ (SYM)}\) | [-3.145, 2.828] (-3.172, 2.854) | [-2.988, 2.725] (-3.015, 2.752) |       |              |
| \([\beta_L, \beta_U] \text{ (SPRS)}\) | [-3.139, 2.822] (-3.162, 2.844) | [-2.935, 2.672] (-2.959, 2.697) |       |              |

Notes. Estimated parameter (square brackets) and the 95% confidence bands (parentheses) for the parameter. The true parameter \( \beta_0 \) is based on the observed outcome \( Y \). The midpoint parameter \( \beta_M \) is based on the midpoint outcome \( Y_M \). The upper bound \( \beta_U \) is as defined in (2.15), and \( \beta_L \) is its analog. The first-stage treatment expectation function is estimated by linear Lasso (Columns (1)-(2)) and random forest (Columns (3)-(4)). The symmetry-based specification (SYM, Row 3) is based on Assumption 4.1, and the first-stage fitted values are given in (4.3). The sparsity-based specification (SPRS, Row 4) is based on Example 4.3, and the first-stage fitted values are given in (4.7). For more details, see text.
8 Proofs

Empirical process notation. Let \( \hat{\eta}_k \) and \( \hat{\xi}_k \), \( k = 1, 2, \ldots, K \) be as in Definition 2.1. Define an event

\[
\mathcal{E}_N := \{ \hat{\eta}_k, (\hat{\xi}_k(p))_{p \in P} \in \Xi \forall k = 1, 2, \ldots, K \}.
\]

By union bound, this event holds with probability approaching one

\[
P(\mathcal{E}_N) \geq 1 - K \epsilon_N = 1 - o(1).
\]

For a given partition \( k \) in \( \{1, 2, \ldots, K\} \), define the partition-specific averages

\[
\mathbb{E}_{n,k} f(W_i) := \frac{1}{n} \sum_{i \in J_k} f(W_i),
\]

\[
\mathcal{G}_{n,k} f(W_i) := \frac{1}{\sqrt{n}} \sum_{i \in J_k} [f(W_i) - \int f(w) dP(w)].
\]

Define the function \( \psi_0(p) \)

\[
\psi_0(p) = \psi(p, \xi_0) = \mathbb{E}[g(W, p, \xi_0(p))] \tag{8.1}
\]

and observe that plugging \( p_0(q) \) into \( \psi_0(p) \) gives the support function \( \sigma(q) \) at \( q \):

\[
\psi_0(p_0(q)) = \psi_0(\Sigma^{-1}q) = \sigma(q).
\]

For \( i \in J_k \) and \( k = 1, 2, \ldots, K \), define the partition-specific conditional expectation

\[
\psi(p, \hat{\xi}_k) := \mathbb{E}[g(W_i, p, \hat{\xi}_k(p)) \mid (W_i)_{i \in J_k}], \quad i \in J_k \tag{8.2}
\]

and its weighted sample analog

\[
\hat{\psi}_k(p, \hat{\xi}_k) := \mathbb{E}_{n,k} \psi(W_i, p, \hat{\xi}_k(p)). \tag{8.3}
\]

Finally, define the matrix error terms

\[
\hat{\Sigma}_k(\hat{\eta}) := \mathbb{E}_{n,k} A(W_i, \hat{\eta}_k), \quad \hat{\Sigma}(\hat{\eta}) := \frac{1}{K} \sum_{k=1}^{K} \hat{\Sigma}_k(\hat{\eta}_k)
\]

and the weighted matrix error

\[
\hat{\Sigma}^v_k(\hat{\eta}_k) := \mathbb{E}_{n,k} v_i A(W_i, \hat{\eta}_k), \quad \hat{\Sigma}^v(\hat{\eta}) := \frac{1}{K} \sum_{k=1}^{K} \hat{\Sigma}^v_k(\hat{\eta}_k).
\]
Empirical process remainder terms. Define the remainder term
\[ R_{1,k}(p) = \sqrt{N}(\hat{\psi}_k(p, \hat{\xi}_k) - \hat{\psi}_k(p_0, \xi_0)) - (\psi(p, \hat{\xi}_k) - \psi_0(p_0)), \quad k = 1, 2, \ldots, K, \]
the bias term
\[ R_{2,k}(p) = \sqrt{N}(\psi(p, \hat{\xi}_k) - \psi_0(p)), \quad k = 1, 2, \ldots, K, \]
the second-order remainder term
\[ R(p, p_0) = \sqrt{N}(\psi(p_0) - \psi(p_0) - G(p_0)'(p - p_0)) \]
and the bootstrap term
\[ R^{\ast}_{1,k}(p) = \sqrt{N}(\hat{\psi}_k^*(p, \hat{\xi}_k) - \hat{\psi}_k^*(p_0, \xi_0)) - (\psi(p, \hat{\xi}_k) - \psi_0(p_0)), \quad k = 1, 2, \ldots, K. \]
Thus, the support function process \( S_N(q) \) of Theorem 3.1 can be decomposed as
\[
S_N(q) := \sqrt{N}(\tilde{\sigma}(q) - \sigma(q))
= \sqrt{N}\left(\frac{1}{K} \sum_{k=1}^{K} \hat{\psi}_k(\hat{\rho}(q), \hat{\xi}_k) - \sigma(q)\right)
= \sqrt{N}\left(\frac{1}{K} \sum_{k=1}^{K} \hat{\psi}_k(p_0(q), \xi_0) - \sigma(q) + G(p_0)'(\hat{\rho}(q) - p_0(q))\right)
+ \frac{1}{K} \sum_{k=1}^{K} [R_{1,k}(\hat{\rho}(q)) + R_{2,k}(\hat{\rho}(q))] + R(\hat{\rho}(q), p_0(q)).
\]

Misclassification events. For \( p, p_0 \in \mathcal{P} \), define the events \( \mathcal{E}_+(p), \mathcal{E}_-(p), \mathcal{E}^+(p, p_0), \mathcal{E}^-(p, p_0) \)
\[
\mathcal{E}_+(p) = \left\{ p'V(\eta) < 0 < p'V(\eta_0) \right\}, \quad (8.4)
\]
\[
\mathcal{E}_-(p) = \left\{ p'V(\eta_0) < 0 < p'V(\eta) \right\} \quad (8.5)
\]
and
\[
\mathcal{E}^-(p, p_0) = \left\{ p_0'V(\eta_0) < 0 < p_0'V(\eta_0) \right\}, \quad (8.6)
\]
\[
\mathcal{E}^+(p, p_0) = \left\{ p_0'V(\eta_0) < 0 < p_0'V(\eta_0) \right\}. \quad (8.7)
\]

8.1 Proofs of Main Results.

Proof of Lemma 4.3. Define
\[
z(p, \eta) := p'V(\eta)
B_1(W, \eta, p) := p'V(\eta_0)(Y(p, \eta) - Y(p, \eta_0))
B_2(W, \eta, p) := p'(V(\eta) - V(\eta_0))(Y(p, \eta) - Y(p, \eta_0)).
\]
Observe that
\[ z(p, \eta)(Y(p, \eta) - Y(p, \eta_0)) = B_1(W, \eta, p) + B_2(W, \eta, p). \]
The mistake in \( Y(p, \eta) \) can only occur if \( p'V(\eta_0) \) is small enough
\[
\begin{align*}
\left\{ Y(p, \eta) \neq Y(p, \eta_0) \right\} &\iff \left\{ \mathcal{E}_+(p) \text{ or } \mathcal{E}_-(p) \right\} \\
&\Rightarrow \left\{ 0 < |p'V(\eta_0)| < |p'(V(\eta) - V(\eta_0))| \right\} \\
&\Rightarrow \left\{ 0 < |p'V(\eta_0)| < C_p \|V(\eta) - V(\eta_0)\| \right\} =: \mathcal{E}_-(p).
\end{align*}
\] (8.8)

Recall that
\[
Y(p, \eta) - Y(p, \eta_0) = (Y_U - Y_L)1\{\mathcal{E}_+(p) \cup \mathcal{E}_-(p)\}
\] (8.9)

Invoking \( |\mathbb{E}X| \leq \mathbb{E}|X| \) and (B.45) gives
\[
|\mathbb{E}B_1(W, \eta, p)| = |\mathbb{E}p'V(\eta_0)(Y(p, \eta) - Y(p, \eta_0))| \\
\leq \mathbb{E}|p'V(\eta_0)||Y_U - Y_L|1\{\mathcal{E}_+(p) \cup \mathcal{E}_-(p)\} \\
\leq \mathbb{E}|p'V(\eta_0)||Y_U - Y_L|1\{\mathcal{E}_-(p)\}
\] (8.10)

Invoking definition of \( \mathcal{E}_-(p) \) in (8.8) and \( Y_U - Y_L \leq M_{UL} \) a.s. gives
\[
\begin{align*}
\mathbb{E}|p'V(\eta_0)||Y_U - Y_L|1\{\mathcal{E}_-(p)\} &\leq C_p M_{UL}\mathbb{E}\|V(\eta) - V(\eta_0)\|1\{\mathcal{E}_-(p)\}.
\end{align*}
\] (8.11)

The second-order bias term is bounded as
\[
|\mathbb{E}B_2(W, \eta, p)| \leq C_p M_{UL}\mathbb{E}\|V(\eta) - V(\eta_0)\|1\{\mathcal{E}_+(p) \cup \mathcal{E}_-(p)\}.
\] (8.12)

Invoking Assumption 4.3 gives
\[
\begin{align*}
\mathbb{E}|\eta_0(X) - \eta(X)|1\{\mathcal{E}_+(p) \cup \mathcal{E}_-(p)\} &\leq 2C_p M_{h}\mathbb{E}\|\eta(X) - \eta_0(X)\|^2.
\end{align*}
\] (8.13)

Combining the bounds gives
\[
|\mathbb{E}[B_1(W, \eta, p) + B_2(W, \eta, p)]| \leq |\mathbb{E}B_1(W, \eta, p)| + |\mathbb{E}B_2(W, \eta, p)| \\
\leq 4C_p^2 M_{UL} M_{h}\mathbb{E}\|\eta(X) - \eta_0(X)\|^2.
\] (8.14)
Proof of Theorem 3.1. Step 1. This step is required only if \( \Sigma \) is unknown, such as in Example 2.2. As shown in Lemma A.10 for \( v = 1 \) (regular case) and \( v = e \) (bootstrap case),

\[
(\hat{\Sigma}^v(\hat{\eta}))^{-1} - \Sigma^{-1} = -\Sigma^{-1}(\hat{\Sigma}^v(\eta_0) - \Sigma)\Sigma^{-1} + M^v,
\]

(8.15)

where the remainder matrix \( M^v \) obeys \( \|M^v\| = o_P(N^{-1/2}) \) for both cases. Take \( v = 1 \). Post-multiplying the LHS above by \( q \) gives

\[
\sqrt{N}(\hat{p}(q) - p_0(q)) = -\Sigma^{-1}(\hat{\Sigma}(\eta_0) - \Sigma)\Sigma^{-1}q + o_P(1)
\]

Likewise, taking \( v = e \) gives

\[
\sqrt{N}(\hat{p}(q) - p_0(q)) = ((\hat{\Sigma}^e(\hat{\eta}))^{-1} - \Sigma^{-1})q
\]

\[-\Sigma^{-1}(\hat{\Sigma}^e(\eta_0) - \Sigma)\Sigma^{-1}q + o_P(1).
\]

For some \( N \) large enough, \( \hat{p}(q) \in \mathcal{P} \) \( \forall q \in \mathcal{S}^d \) and \( \hat{p}(q) \in \mathcal{P} \) \( \forall q \in \mathcal{S}^d \) with probability \( 1 - o(1) \).

Step 2. Let \( k = 1, 2, \ldots, K \) denote the partition index. We bound \( R_{1,k}(\hat{p}(q)) \) and \( R_{1,k}^e(\hat{p}(q)) \). Define the function class

\[
\mathcal{F}_{2k}^v := \{ v \cdot (g(\cdot, p, \hat{\xi}_k(p)) - g(\cdot, p_0, \xi_0(p_0))), \quad p, p_0 \in \mathcal{P}, \quad \|p - p_0\| \leq \tau_N \},
\]

where \( v = 1 \) (regular case) and \( v = e \) (bootstrap case). The class \( \mathcal{F}_{2k} \) is obtained as

\[
\mathcal{F}_{2k} \subset G_{\xi_k} - G_{\xi_0},
\]

where \( G_{\xi_k} \) and \( G_{\xi_0} \) are defined in Assumption 3.5. On the event \( \mathcal{E}_N \),

\[
\sup_{p \in \mathcal{P}}|g(W_i, p, \hat{\xi}_k(p)) - g(W_i, p_0, \xi_0(p_0))| \leq \sup_{p \in \mathcal{P}}|g(W_i, p, \hat{\xi}_k(p))| + \sup_{p \in \mathcal{P}}|g(W_i, p, \xi_0(p))| \leq G_{\hat{\xi}_k} + G_{\xi_0},
\]

and \( G_{\hat{\xi}_k, \xi_0} := G_{\hat{\xi}_k} + G_{\xi_0} \) is a measurable envelope for the class \( \mathcal{F}_{2k} \). Note that \( \|G_{\hat{\xi}_k, \xi_0}\|_{p,c} \leq \|G_{\xi_0}\|_{p,c} + \|G_{\hat{\xi}_k}\|_{p,c} \leq 2C_1 \). The uniform covering entropy of the function class \( G_{\hat{\xi}_k} - G_{\xi_0} \) is bounded as

\[
\log \sup_{Q} N(\epsilon \|G_{\hat{\xi}_k} + G_{\xi_0}\|_{Q,2}, G_{\hat{\xi}_k} - G_{\xi_0}, \|\cdot\|_{Q,2}) \leq \log \sup_{Q} N(\epsilon/2 \|G_{\xi_0}\|_{Q,2}, G_{\hat{\xi}_k} - G_{\xi_0}, \|\cdot\|_{Q,2}) + \log \sup_{Q} N(\epsilon/2 \|G_{\hat{\xi}_k}\|_{Q,2}, G_{\hat{\xi}_k} - G_{\xi_0}, \|\cdot\|_{Q,2}) \leq 2v \log(2\alpha/\epsilon)
\]

by the proof of Theorem 3 in Andrews (1994) and Assumption 3.5. The class \( \mathcal{F}_{2k} \) is obtained by multiplication of \( \mathcal{F}_{2k} \) by an integrable random variable independent of the data, and therefore retains \( P \)-Donsker and uniform covering.
properties of the class \(F_2\). In particular, \(G_{\xi_0}^\varepsilon := |v|(G_{\xi_k} + G_{\xi_0})\) is a valid envelope for \(F_{2k}^\varepsilon\). Next, for \(v = 1\) and \(v = e_e\),

\[
\sup_{p \in \mathcal{P}} \mathbb{E}v^2(g(W, p, \xi(p)) - g(W, p_0, \xi_0(p_0)))^2 \\
\leq 2 \left( \sup_{p \in \mathcal{P}} \mathbb{E}v^2(g(W, p, \xi(p)) - g(W, p, \xi_0(p)))^2 \\
+ \sup_{p_0, p \in \mathcal{P}, |p-p_0| \leq r_N} \mathbb{E}v^2(g(W, p, \xi_0(p)) - g(W, p_0, \xi_0(p_0)))^2 \right) \\
\leq 2((r''_N)^2 + (r''_N)^2).
\]

Invoking Lemma\(^{A.2}\) conditional on \((W_i)_{i \in I_k}\) and taking \(\sigma = 2(r''_N + r''_N')\) gives

\[
\sup_{f \in F_k^\varepsilon} |G_{n,k}[f]| = \sup_{p_0, p \in \mathcal{P}} |\hat{\psi}_k(p, \hat{\xi}_k) - \hat{\psi}_k(p_0, \xi_0) - (\overline{\psi}(p, \hat{\xi}_k) - \overline{\psi}(p_0, \xi_0))| \\
\leq_p (r''_N + r''_N') \log^{1/2}(1/(r''_N + r''_N')) + N^{-1/2 + 1/e} \log N \\
= o_P(1),
\]

where the last equality follows from Assumption\(^{3.5}\). By Lemma\(^{A.1}\) \(\sup_{f \in F_k^\varepsilon} |G_{n,k}[f]| = o_P(1)\) holds unconditionally. as shown in Step 1, wp 1 \(- o(1), \hat{\psi}(q) \in \mathcal{P} \forall q, and

\[
\sup_{f \in F_k^\varepsilon} |G_{n,k}[f]| = o_P(1).
\]

**Step 3. Bound on** \(R_{2,k}(\hat{\psi}(q))\). On the event \(E_N\), for any \(k = 1, 2, \ldots, K\)

\[
\sup_{q \in S^{d-1}} |R_{2,k}(\hat{\psi}(q))| \leq \sup_{p \in \mathcal{P}} \sqrt{N} |\overline{\psi}(p, \hat{\xi}_k) - \psi_0(p)| \\
\leq \sup_{p \in \mathcal{P}} \sup_{\xi \in \Xi} \sqrt{N} |\overline{\psi}(p, \xi) - \psi(p)| \leq \sqrt{N} \mu_N.
\]

Combining the bounds over a finite set \(k = 1, 2, \ldots, K\) gives

\[
\frac{1}{K} \sum_{k=1}^{K} \left[ R_{1,k}(\hat{\psi}(q)) + R_{2,k}(\hat{\psi}(q)) \right] = o_P(1).
\]

**Step 4. Bound on** \(R(\hat{\psi}(q), p_0)\). By Assumption\(^{3.2}\), Lemma\(^{A.5}\) and Step 1, on the event \(E_N\),

\[
\sup_{q \in S^{d-1}} |R(\hat{\psi}(q), p_0(q))| = o(\sqrt{N} |(\hat{\Sigma}(\hat{\eta}))^{-1} - \Sigma^{-1}|) = o_P(1).
\]

**Step 5. Conclusion.** For the influence function \(h(W, q)\) in \((3.12)\), the function class

\[
\mathcal{H} = \left\{ h(\cdot, q), \quad q \in S^{d-1} \right\} \subseteq G_{\xi_0} + \mathcal{H}_A
\]
is included into the sum $G_\xi + H_A$. The function classes $G_\xi$ and $H_A$ are Donsker classes with square integrable envelopes (by Assumption 3.5 and Lemma A.6 respectively). The statement of the lemma follows from the Skorohod-Dudley-Wichura construction, as in Skorohod (1956), Dudley (1968) and Wichura (1970).

The bootstrap support function process can be decomposed as

$$\tilde{S}_N(q) = \sqrt{N}((\hat{\sigma}(q) - \sigma(q)) - (\hat{\sigma}(q) - \sigma(q))).$$

The first summand can be decomposed as

$$\hat{\sigma}(q) - \sigma(q) = \frac{1}{K} \sum_{k=1}^{K} \tilde{\psi}_k^e(\tilde{p}(q), \tilde{\xi}_k) - \sigma(q)$$

$$= \frac{1}{K} \sum_{k=1}^{K} \tilde{\psi}_k^e(\tilde{p}(q), \tilde{\xi}_k)/(1 + o_P(1)) - \sigma(q)$$

$$= \frac{1}{K} \sum_{k=1}^{K} \tilde{\psi}_k^e(\tilde{p}(q), \tilde{\xi}_k) - \sigma(q) + o_P(1).$$

The weighted moment can be decomposed as

$$\sqrt{N} \frac{1}{K} \sum_{k=1}^{K} \tilde{\psi}_k^e(\tilde{p}(q), \tilde{\xi}_k) = \sqrt{N} \frac{1}{K} \sum_{k=1}^{K} \tilde{\psi}_k^e(p_0(q), \xi_0) + \sqrt{N} G(p_0(q))' (\tilde{p}(q) - p_0(q))$$

$$+ \frac{1}{K} \sum_{k=1}^{K} \left[ R_{1,k}^e(\tilde{p}(q)) + R_{2,k}(\tilde{p}(q)) \right] + R(\tilde{p}(q), p_0(q)).$$

**Proof of Theorem 3.2** By Comment B.1 in Chandrasekhar et al. (2012), if the bootstrap random element converges in probability $P$ unconditionally (i.e., $Z_N \xrightarrow{P} \xi$), then $Z_N \xrightarrow{P} \xi$ in $L^1(P)$ sense and hence in probability $P$, where $P^e$ denotes the probability measure conditional on the data.

**Step 1.** As shown in Step 1 of the proof of Theorem 3.1, $\tilde{p}(q) \in \mathcal{P}$ for all $q$. The bound on $R_{1,k}^e(\tilde{p}(q))$ is established in the proof of Theorem 3.1. **Step 2.** By construction,

$$\sqrt{N}(\psi(p, \hat{\xi}_k) - \psi(p, \xi_0)) = R_{2,k}(p) = E[v \cdot (g(W, p, \hat{\xi}_k) - g(W, p, \xi_0)) \mid (W_i)_{i \in J_k}].$$

Thus, the bounds on $R_{2,k}(p)$ and $R(p, p_0)$ are established in Steps 3 and 4 of Theorem 3.1.

**Step 2.** As shown in Step 5 of Theorem 3.1, the function class $H \subseteq G_\xi + H_A$ is a Donsker class with square-integrable envelopes. Then by the Donsker theorem for exchangeable bootstraps, weak convergence holds conditional on the data,

$$G_N((e - 1)h(W, q))/\bar{e} \Rightarrow G[h(q)] \text{ under } P^e \text{ in probability } P;$$
where $G[h(q)]$ is a P-Brownian bridge independent of $G[h(q)]$ with the same distribution as $G[h(q)]$. 

\[ \square \]
Abstract

This appendix contains supplementary statements for the paper “Debiasing Linear Models” by Vira Semenova.
Appendix A contains useful technical statements. Appendix B contains additional proofs. Appendix C contains supplementary tables.

Appendix A: Technical Lemmas

Lemma A.1 (Conditional Convergence Implies Unconditional). Let \( \{X_m\}_{m \geq 1} \) and \( \{Y_m\}_{m \geq 1} \) be sequences of random vectors. (i) If for \( \epsilon_m \to 0 \), \( P(\|X_m\| > \epsilon_m \mid Y_m) \to P \) 0, then \( P(\|X_m\| > \epsilon_m) \to P \) 0. In particular, this occurs if \( E[\|X_m\|^q / \epsilon_m^q \mid Y_m] \to P \) 0 for some \( q \geq 1 \), by Markov inequality. (ii) Let \( \{A_m\}_{m \geq 1} \) be a sequence of positive constants. If \( \|X_m\| = O_P(A_m) \) conditional on \( Y_m \), namely, that for any \( \ell_m \to \infty \), \( P(\|X_m\| > \ell_m A_m \mid Y_m) \to P \) 0, then \( X_m = O_P(A_m) \) unconditionally, namely, that for any \( \ell_m \to \infty \), \( P(\|X_m\| > \ell_m A_m) \to P \) 0.

Lemma A.1 is a restatement of Lemma 6.1 in Chernozhukov et al. (2018).

Lemma A.2 (Maximal Inequality). Let \( F \) be a set of suitably measurable functions \( f : W \to \mathbb{R} \). Suppose that \( F : W \to \mathbb{R} \), \( F \geq \sup_{f \in F} |f| \) is a measurable envelope for \( F \) with \( \|F\|_{P,c} < \infty \) for some \( c \geq 2 \). Let \( M = \max_{i \in N} F(W_i) \) and \( \sigma^2 \) be any positive constant such that \( \sup_{f \in F} \|f\|^2_{P,2} \leq \sigma^2 \leq \|F\|^2_{P,2} \). Suppose that there exist constants \( a \geq \epsilon \) and \( v' \geq 1 \) such that

\[
\log \sup_{Q} N(\epsilon \|F\|_{Q,2}, F, \cdot ; \|Q,2\|) \leq v' \log(a/\epsilon), \quad 0 < \epsilon \geq 1.
\]

Then,

\[
E_P[\|G_N\|_F] \leq K \left( \sqrt{v' \sigma^2 \log \left( \frac{a\|F\|_{P,2}}{\sigma^2} \right)} + \frac{v' \|M\|_{P,2}}{\sqrt{N}} \log \left( \frac{a\|F\|_{P,2}}{\sigma^2} \right) \right), \tag{A.1}
\]

where \( K \) is an absolute constant. Furthermore, with probability \( > 1 - c_2(\log N)^{-1} \),

\[
\|G_N\|_F \leq K(c, c_2) \left( \sqrt{v' \sigma^2 \log \left( \frac{a\|F\|_{P,2}}{\sigma^2} \right)} + \frac{v' \|M\|_{P,c}}{\sqrt{N}} \log \left( \frac{a\|F\|_{P,2}}{\sigma^2} \right) \right), \tag{A.2}
\]

where \( \|M\|_{P,c} \leq N^{1/c} \|F\|_{P,c} \) and \( K(c, c_2) > 0 \) is a constant depending only on \( c \) and \( c_2 \).

Lemma A.2 is a restatement of Lemma 6.2 in Chernozhukov et al. (2018) (cf. Chernozhukov et al. (2014)).

Define the subGaussian Orlicz norm

\[
\|X\|_{\psi_2} = \inf \{ t > 0 : E \exp(X^2/t^2) \leq 2 \}
\]
and the subExponential Orlicz norm
\[ \|X\|_{\psi_1} = \inf\{t > 0 : \mathbb{E} \exp(|X|/t) \leq 2\}. \]

A random variable \( X \) is \( K \)-subGaussian (\( X \) is \( K \)-subExponential) if there exist a finite \( K < \infty \) such that
\[ \|X\|_{\psi_1} \leq K, \]
where \( t = 2 \) corresponds to subGaussian case and \( t = 1 \) to subExponential case. For a \( d \)-vector \( X \), if \( \|X\| \) is \( K \)-subGaussian, since
\[ |\alpha'X| \leq \|X\|, \quad \alpha \in S^{d-1} \]
a\( X \) is \( K \)-subGaussian.

We rely on the following properties of Orlicz norms. The product \( XY \) of two \( K \)-subGaussian random variables \( X \) and \( Y \) obeys
\[ \|XY\|_{\psi_1} \leq \|X\|_{\psi_2}\|Y\|_{\psi_2} \leq K^2. \quad (A.3) \]
Furthermore, if \( X \) is \( K^2 \)-subExponential random variable, its centered analog \( X - \mathbb{E}X \) obeys
\[ \|X - \mathbb{E}X\|_{\psi_1} \leq C\|X\|_{\psi_1}. \quad (A.4) \]
(see the page 92 in Vershynin (2018), the proof of Theorem 4.6.1).

**Lemma A.3** (Tail Bound on SubGaussian Covariance Matrix). Let \( (E_i, V_i)_{i=1}^N \) be an i.i.d sequence of \( K \)-subGaussian vectors. Define
\[
\Sigma = EV'E' \quad \hat{\Sigma} = N^{-1} \sum_{i=1}^N V_i E_i'
\]
Then, with probability at least \( 1 - 2e^{-u^2} \),
\[
\|\hat{\Sigma} - \Sigma\| \leq CK^2 \left( \sqrt{\frac{d + u}{N}} + \sqrt{\frac{d + u}{N}} \right) = CK^2 \delta.
\]
As a result, for any \( \ell_N \to \infty \),
\[
\|\hat{\Sigma} - \Sigma\| \leq \ell_N \sqrt{\frac{d + \ell_N}{N}}. \quad (A.6)
\]

**Proof of Lemma A.3**

**Step 1.** As shown in Corollary 4.2.13 of Vershynin (2018), there exist an \(1/4\)-net \( \mathcal{N} \) on the sphere \( S^{d-1} \) with cardinality \( |\mathcal{N}| \leq 9^d \). Thus,
\[
\|A\| = \max_{\alpha \in S^{d-1}} |\alpha' A\alpha| \leq 2 \max_{\alpha \in \mathcal{N}} |\alpha' A\alpha|. \quad (A.7)
\]
**Step 2.** For any \( \alpha \in \mathcal{N} \), the random variables \( E(\alpha) = \alpha' E \) and \( V(\alpha) = \alpha' V \) are \( K \)-subGaussian random variables. Invoking (A.3) and (A.4) gives
\[
\|E(\alpha)V(\alpha) - \mathbb{E}E(\alpha)V(\alpha)\|_{\psi_1} \leq C\|E(\alpha)V(\alpha)\|_{\psi_1} \leq CK^2.
\]
Therefore, \( \{E_i(\alpha)V_i(\alpha) - \mathbb{E}E(\alpha)V(\alpha)\}_{i=1}^N \) is an i.i.d, mean zero, and subExponential random variables with its \( \psi_1 \)-norm bounded by \( CK^2 \). Take

\[
\epsilon := CK^2 \max(\delta, \delta^2)
\]

and invoke Bernstein’s inequality with \( \delta < 1 \), noting that

\[
\min(\max(\delta, \delta^2)^2, \max(\delta, \delta^2)) = \delta^2.
\]

Bernstein’s inequality gives

\[
P\left(|N^{-1} \sum_{i=1}^N E_i(\alpha)V_i(\alpha) - \mathbb{E}E(\alpha)V(\alpha)| \geq \epsilon/2\right) \leq 2 \exp^{-c_1 \min(\delta^2/C^2 K^4, \epsilon/CK^2)N}
\]

for \( C \) large enough. **Step 3.** For any fixed \( \alpha \in \mathcal{S}^{d-1} \),

\[
\alpha'(\hat{\Sigma} - \Sigma)\alpha = N^{-1} \sum_{i=1}^N [V_i(\alpha)E_i(\alpha) - \mathbb{E}V(\alpha)E(\alpha)].
\]

Invoking (A.7) and the union bound over the net \( \mathcal{N} \) gives

\[
P\left(\|\hat{\Sigma} - \Sigma\| \geq \epsilon\right) \leq P(\max_{\alpha \in \mathcal{N}} |\alpha'(\hat{\Sigma} - \Sigma)\alpha| \geq \epsilon/2)
\]

\[
\leq 9^d 2 \exp^{-c_1 \delta^2(N+u^2)} \leq 2 \exp^{-u^2}.
\]

**Lemma A.4 (Weighted Covariance Matrix).** Suppose \( \mathbb{E}\|A(W, \eta_0)\|_F^2 \) is finite. Let \( \{e_i\}_{i=1}^N \) be i.i.d \( \text{Exp}(1) \) draws independent from the data \( \{W_i\}_{i=1}^N \). Then, the following bound holds

\[
\frac{1}{N} \sum_{i=1}^N A(W_i, \eta_0)(e_i - 1) = O_P(N^{-1/2}). \tag{A.8}
\]

**Proof of Lemma A.4**

**Step 1.** Note that

\[
\mathbb{E}\|\frac{1}{N} \sum_{i=1}^N A(W_i, \eta_0)(e_i - 1)\|_F^2
\]

\[
= \sum_{k=1}^d \sum_{j=1}^d \mathbb{E}\left(\frac{1}{N} \sum_{i=1}^N A_{kj}(W_i, \eta_0)(e_i - 1)^2\right)
\]

\[
= N^{-1} \sum_{k=1}^d \sum_{j=1}^d \mathbb{E}A_{kj}^2(W, \eta_0) = N^{-1} \mathbb{E}\|A(W, \eta_0)\|_F^2.
\]

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Involving Markov inequality gives
\[
\frac{1}{N} \sum_{i=1}^{N} A(W_i, \eta_0)(e_i - 1) \leq \frac{1}{N} \sum_{i=1}^{N} A(W_i, \eta_0)(e_i - 1) \|_F = O_p(N^{-1/2}),
\]
which implies (A.24).

Lemma A.5 is similar to Lemma 3 in Chandrasekhar et al. (2012). For completeness, we provide it below with proof. Recall that

\[
\psi_0(p) = \mathbb{E}z(p, \eta_0)Y(p, \eta_0) = \mathbb{E}p'V(\eta_0)Y(p, \eta_0)
\]

and

\[
G(p_0) = \mathbb{E}V(\eta_0)Y(p, \eta_0).
\]

Let \( \mathcal{E}_-(p, p_0) \) and \( \mathcal{E}_+(p, p_0) \) be

\[
\mathcal{E}_-(p, p_0) = \{ p_0'V(\eta_0) < 0 < p'V(\eta_0) \}
\]

\[
\mathcal{E}_+(p, p_0) = \{ p'V(\eta_0) < 0 < p_0'V(\eta_0) \}.
\]

**Lemma A.5 (Uniform Gradient).** Let \( \psi_0 : \mathcal{P} \to \mathbb{R} \) be as defined in (8.1). If Assumption 3.2 holds, the function \( \psi_0(p) \) has a gradient that is uniformly continuous in \( p \). For any \( p, p_0 \in \mathcal{P} \),

\[
\psi_0(p) - \psi_0(p_0) = G(p_0)'(p - p_0) + R(p, p_0),
\]

where \( R(p, p_0) = o(||p - p_0||) \) uniformly over \( p, p_0 \in \mathcal{P} \).

**Proof of Lemma A.5** Note that

\[
\mathcal{E}_+(p, p_0) \cup \mathcal{E}_-(p, p_0) \Rightarrow \left\{ 0 < |p_0'V(\eta_0)| < |(p - p_0)'V(\eta_0)| \right\}
\]

\[
\leq \left\{ 0 < \frac{|p_0'V(\eta_0)|}{\Sigma^{-1/2}V(\eta_0)} < \|\Sigma^{1/2}\|_F \|p - p_0\| \right\}
\]

\[
\leq \left\{ 0 < \frac{|p_0'\Sigma^{-1/2}V(\eta_0)|}{\Sigma^{-1/2}p_0\|\Sigma^{-1/2}V(\eta_0))} < \|\Sigma^{1/2}\|/\|\Sigma^{1/2}p_0\| \|p - p_0\| \right\}
\]

\[
=: \mathcal{E}_+(p, p_0).
\]

Note that

\[
\psi_0(p) - \psi_0(p_0) = \mathbb{E}[p'V(\eta_0)Y(p, \eta_0) - p_0'V(\eta_0)Y(p_0, \eta_0)]
\]

\[
= \mathbb{E}[p'V(\eta_0)(Y(p, \eta_0) - Y(p_0, \eta_0))] + G(p_0)'(p - p_0)
\]

\[
=: R(p, p_0) + G(p_0)'(p - p_0).
\]

We bound the remainder term \( R(p, p_0) \) uniformly over \( p, p_0 \in \mathcal{P} \). On the event \( \mathcal{E}_+(p, p_0) \),

\[
\sup_{p, p_0 \in \mathcal{P}} |R(p, p_0)| \leq \mathbb{E}[p_0'V(\eta_0)(Y_U - Y_L)1\{\mathcal{E}_+(p, p_0)\}]
\]

\[
\leq \mathbb{E}[(p - p_0)'V(\eta_0)(Y_U - Y_L)1\{\mathcal{E}_+(p, p_0)\}].
\]
Invoking Cauchy inequality gives
\[
\sup_{p,p_0 \in \mathcal{P}} |R(p,p_0)| \leq \|p - p_0\| E |V(\eta_0)| (Y_U - Y_L) 1_{\{E_+ - (p,p_0)\}} \\
\leq \|p - p_0\| M_{UL} |V(\eta_0)| P^{1/2} (E_+ - (p,p_0))
\]
and Assumption 3.2 implies
\[
\sup_{p,p_0 \in \mathcal{P}} |R(p,p_0)| \leq \|p - p_0\| M_{UL} |V(\eta_0)| P^{1/2} (E_+ - (p,p_0))
\]
for \(C_V\) depending on \(\Sigma\) and \(C_V\) large enough.

Let \(\mathcal{P}\) be as in (3.1). Define the function classes
\[
\mathcal{L}_\eta := \left\{ p' V(\eta), \ p \in \mathcal{P} \right\}
\]
\[
\mathcal{I}_\eta := \left\{ 1_{\{p' V(\eta) > 0\}}, \ p \in \mathcal{P} \right\}
\]
\[
\mathcal{L}_\eta \cdot \mathcal{I}_\eta := \left\{ p' V(\eta) \cdot 1_{\{p' V(\eta) > 0\}}, \ p \in \mathcal{P} \right\}
\]
\[
\mathcal{R}_\eta := \left\{ (p, p' V(\eta)), \ p \in \mathcal{P} \right\}
\]
and
\[
\mathcal{Y}_\eta := (Y_U - Y_L) \cdot \mathcal{L}_\eta, \mathcal{I}_\eta = \left\{ Y(p, \eta), \ p \in \mathcal{P} \right\}
\]
\[
\mathcal{G}_{1,\eta} := \left\{ \mathbb{E}[Y(p, \eta) \mid X], \ p \in \mathcal{P} \right\}
\]
\[
\mathcal{F}_{1,2} = \left\{ W \rightarrow p' V(\eta_0) \gamma_0(p, X), \ p \in \mathcal{P} \right\} \subseteq \mathcal{L}_{\eta_0} : (\gamma_L,0(X) + \mathcal{G}_{1,\eta_0})
\]
\[
\mathcal{H}_\eta = \left\{ q' A(W, \eta) q, \ q \in S^{d-1} \right\}
\]
\[
\mathcal{H}_A = \left\{ G(\Sigma^{-1} q) \Sigma^{-1} (A(W, \eta_0) - \Sigma) \Sigma^{-1} q, \ q \in S^{d-1} \right\}
\]

**Lemma A.6 (Entropy Bounds).** Suppose Assumptions 3.1, 3.2, 3.3(2), 4.2–4.3 hold. Let \(Q\) be any probability measure whose support concentrates on a finite set. For every function class \(\mathcal{F} \in \{ \mathcal{L}_\eta, \mathcal{I}_\eta, \mathcal{R}_\eta, \mathcal{Y}_\eta + Y_L, \mathcal{G}_{1,\eta}, \mathcal{G}_{1,\eta} + \mathbb{E}[Y_L \mid X], \mathcal{F}_{1,2}, \mathcal{H}_\eta, \mathcal{H}_A \}\), there exists an integrable envelope \(F\) so that the following bound holds
\[
\log \sup_{Q} N(\epsilon, F, \mathcal{F}, \|\|_{Q,2}) \leq 1 + d \log(1/\epsilon), \quad \text{for all } 0 < \epsilon \leq 1. \quad (A.12)
\]

**Proof of Lemma A.6**  
**Step 1.** As shown in Andrews (1994), the statement holds for the linear class \(\mathcal{L}_\eta\), the class of indicators \(\mathcal{I}_\eta\), and \(\mathcal{L}_\eta \cdot \mathcal{I}_\eta\). Multiplying each element of \(\mathcal{L}_\eta \cdot \mathcal{I}_\eta\) by \(Y_U - Y_L\) gives \(\mathcal{Y}_\eta\). Therefore, (A.12) holds for the entropy
of $\mathcal{Y}_q$ and $\mathcal{Y}_q + Y_L$, by Lemma L.1 in Belloni et al. (2018) (cf. Andrews (1994)). $R_q$ is a cartesian product of the linear class and the index set $\mathcal{P}$.

**Step 2.** Each element of $\mathcal{G}_{1,q}$ is a conditional on $X$ expectation of an element in $\mathcal{Y}_q$. By Lemma L.2 from Belloni et al. (2018) (cf. van der Vaart (2000)), the covering entropy of class $\mathcal{G}_{1,q}$ is bounded. The class $\mathcal{F}_{1,2}$ is included into the product of $\mathcal{L}_{\eta_0}$ and $\mathcal{G}_{1,\eta_0} + \gamma_{L,0}(X)$ obeying (A.12). Define the envelope

$$F_{1,2} := C_F\|V(\eta_0)\|\gamma_{L,0}(X) + M_{UL}$$

and note that $\|F_{1,2}\|_p$ is finite for any $e \geq 2$ since $\|V(\eta_0)\|$ is subGaussian conditional on $X$. Therefore, (A.12) holds for $\mathcal{F}_{1,2}$ with $F_{1,2}$.

**Step 3.** For any two elements of $\mathcal{H}_q$,

$$|q_1 A(W, \eta q_1 - q_2 A(W, \eta) q_2| \leq 2\|A(W, \eta)\| q_1 - q_2|,$$

which implies $F^A(W) := 2\|A(W, \eta)\|$ is a valid envelope function for (A.12) to hold.

**Step 4.** Recall that $G(p) = EV(\eta_0)(Y_L + \mathcal{Y}(p, \eta_0))$. Therefore, $\|G(p)\| \leq E[V(\eta_0)](Y_L + M_{UL}) = C_G$, which is finite by Assumption 4.3. Then, the function class

$$\mathcal{H}_A \subseteq \{\mu(A(W, \eta_0) - \Sigma)\Sigma^{-1} q, \quad q \in \mathcal{S}^d, \mu \in \mathcal{R}^d, \quad \|\mu\| \leq C_G\}$$

has a bounded VC index. Each element of $\mathcal{H}_A$ is bounded by $2C_G\|A(W, \eta_0) - \Sigma\|\Sigma^{-1}\|$.  

**Lemma A.7 (Entropy Bounds, cont.).** Let $Z(X)$ be a vector of basis functions such that $\|Z(X)\|_\infty \leq K_N$ a.s. for some deterministic sequence $\{K_N \geq 1\}$. Suppose Assumptions 4.2 and 4.3 hold. Define

$$\zeta(p, \eta) = \mathcal{Y}(p, \eta) - E[\mathcal{Y}(p, \eta) | X]. \quad (A.13)$$

For $j \in [p_X] := \{1, 2, \ldots, p_X\}$, define

$$\mathcal{F}_{1,j} := \{W \rightarrow Z_j(X)\zeta(\eta) - \zeta(\eta), \quad p \in \mathcal{P}\}$$

$$\mathcal{F}_{2,j} := \{W \rightarrow Z_j^2(X)(\zeta(p, \eta) - \zeta(\eta)) \eta, \quad p \in \mathcal{P}\}$$

$$\mathcal{F}_{3,j} := \{W \rightarrow Z_j^2(X)(\mathcal{Y}(p, \eta) - \mathcal{Y}(p, \eta)) \eta, \quad p \in \mathcal{P}\}$$

$$\mathcal{F}_{4,j} := \{W \rightarrow Z_j^2(X)(\zeta(p_1, \eta_0) - \zeta(p_2, \eta_0))^2, \quad p_1, p_2 \in \mathcal{P}, \quad p_1 - p_2 \| \leq N^{-1}\}$$

and

$$\mathcal{M} = \left\{W \rightarrow Z(X)\zeta, \quad \zeta \in \mathcal{R}^{p_X}, \|\zeta\|_0 \leq C s_N, \|\zeta\| \leq C\right\}$$

$$\mathcal{F}_{1,1} = \left\{W \rightarrow p'V(\eta_0)\Lambda(Z(X)\zeta), \quad p \in \mathcal{P}, \zeta \in \mathcal{R}^{p_X}\right\}.$$
For $j \in [p_X]$, for every function class $\mathcal{F} \in \{ \mathcal{F}_{1j}, \mathcal{F}_{2j}, \mathcal{F}_{3j}, \mathcal{F}_{4j} \}$, there exists an integrable envelope $F$ so that with $a_N = p_X + N$,

$$\log \sup_Q N(\epsilon; F \|_Q, \mathcal{F}, \|_q; Q, 2) \leq 1 + \log(a_N/\epsilon), \quad \text{for all } 0 < \epsilon \leq 1. \quad (A.14)$$

Furthermore, for every function class $\mathcal{F} \in \{ \mathcal{M}, \mathcal{F}_{1,1} \}$, there exists an integrable envelope $F$ so that with $a_N = p_X + N$,

$$\log \sup_Q N(\epsilon; F \|_Q, \mathcal{F}, \|_q; Q, 2) \leq 1 + s_N \log(a_N/\epsilon), \quad \text{for all } 0 < \epsilon \leq 1. \quad (A.15)$$

**Proof of Lemma** **A.7**  
**Step 1.** Note that $0 \leq \mathcal{Y}(p, \eta) \leq M_{UL} \ a.s., \text{ which implies } \zeta^2(p, \eta) \leq M_{UL}^2 \ a.s. \text{ and } (\zeta(p, \eta) - \zeta(p, \eta_0))^2 \leq 2M_{UL}^2 \ a.s.$ Each class in $\mathcal{F}_j \in \{ \mathcal{F}_{1j}, \mathcal{F}_{2j}, \mathcal{F}_{3j}, \mathcal{F}_{4j} \}$ is derived by multiplying an element in $\mathcal{Y}_n - \mathcal{G}_n$ or $(\mathcal{Y}_n - \mathcal{G}_n)^2$ by an integrable random variable $Z_j(X)$ for $j \in [p_X]$. Therefore, for each $j \in [p_X]$ and with $F(W) := 4|Z_j(X)|M_{UL}$ for $F_{1j}$ and $F_j(W) := 2Z_j^2(X)M_{UL}$ for $\mathcal{F}_j \in \{ \mathcal{F}_{2j}, \mathcal{F}_{3j}, \mathcal{F}_{4j} \}$.

$$\log \sup_Q N(\epsilon; F_j \|_Q, \mathcal{F}_j, \|_q; Q, 2) \leq 1 + \log(e/\epsilon), \quad \text{for all } 0 < \epsilon \leq 1. \quad (A.15)$$

Hence, by Lemma L.1(2) in **Belloni et al. (2018)**, the uniform entropy numbers of

$$\mathcal{F} \in \{ \cup_{j \in [p_X]} \mathcal{F}_{1j}, \cup_{j \in [p_X]} \mathcal{F}_{2j}, \cup_{j \in [p_X]} \mathcal{F}_{3j}, \cup_{j \in [p_X]} \mathcal{F}_{4j} \}$$

obeys **(A.14)**

**Step 2.** The class $\mathcal{M}$ is a union over $(\binom{p_X}{C_{\eta_0}})$ VC-subgraph classes with indices $O(s_N)$. Therefore, **(A.15)** holds for $\mathcal{M}$. By Lemma L.3 in **Belloni et al. (2018)**, the rule holds for $\Lambda(\mathcal{M})$.

**Step 3.** The class $\mathcal{F}_{1,1} \subseteq \mathcal{L}_{\eta_0} \cdot \Lambda(\mathcal{M})$, where the class $\mathcal{L}_{\eta_0}$ obeys **(A.12)** and $\mathcal{M}$ obeys **(A.15)**. Define the envelope function

$$F_{1,1} := C_P \|V(\eta_0)\|_p K_N \quad \Lambda(t) = t$$

and note that for any $c \geq 2$, $\|F_{1,1}\|_{p,c} \leq C_P K_N \|V(\eta_0)\|_{p,c} < \infty$ since $|V(\eta_0)|$ is subGaussian by Assumption 4.3. Therefore, $\mathcal{F}_{1,1}$ obeys **(A.15)** with $F_{1,1}$.

**Lemma A.8** (Equivalence of Long and Short Definitions of the Partially Linear Parameter). The minimizers of **(2.4)** and **(2.5)** coincide.

**Proof.** Fix a random variable $Y$ in a random interval $[Y_L, Y_U]$. For each $b$ in **(2.4)**, we solve for $f(X) = f_b(X)$ as a function of $b$. The solution $f_b(X)$ is a conditional expectation function:

$$f_b(X) = \mathbb{E}[Y - D'b \mid X] = \mathbb{E}[Y \mid X] - \eta_0(X)'b.$$

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Substituting \( f_k(X) \) into (2.4) gives:
\[
\beta = \arg\min_{b \in \mathbb{R}^d} \mathbb{E}(Y - \mathbb{E}[Y | X] - (D - \eta_0(X))'b)^2. \quad \text{ (A.16)}
\]
Expanding \((m + n)^2 = m^2 + 2mn + n^2\) with \(m = Y - (D - \eta_0(X))'b\) and \(n = \mathbb{E}[Y | X]\) gives:
\[
\beta = \arg\min_{b \in \mathbb{R}^d} \mathbb{E}(Y - (D - \eta_0(X))'b)^2 \\
- 2\mathbb{E}(Y - (D - \eta_0(X))'b)\mathbb{E}[Y | X] \\
+ \mathbb{E}(\mathbb{E}[Y | X])^2 \\
= \arg\min_{b \in \mathbb{R}^d} \mathbb{E}(Y - (D - \eta_0(X))'b)^2 + \mathbb{E}(\mathbb{E}[Y | X])^2 \\
= \arg\min_{b \in \mathbb{R}^d} \mathbb{E}(Y - (D - \eta_0(X))'b)^2.
\]
Since \(\mathbb{E}[(D - \eta_0(X))'b]\mathbb{E}[Y | X] = 0\), \(ii\) follows. Since the third term does not depend on \(b\), \(iii\) follows.

Recall that
\[
\hat{\Sigma}_k(\hat{\eta}_k) := \mathbb{E}_{n,k} A(W_i, \hat{\eta}_k), \quad \hat{\Sigma}(\hat{\eta}) := \frac{1}{K} \sum_{k=1}^K \hat{\Sigma}_k(\hat{\eta}_k)
\]
and the weighted matrix error
\[
\hat{\Sigma}_k^v(\hat{\eta}_k) := \mathbb{E}_{n,k} v_i A(W_i, \hat{\eta}_k), \quad \hat{\Sigma}^v(\hat{\eta}) := \frac{1}{K} \sum_{k=1}^K \hat{\Sigma}_k^v(\hat{\eta}_k).
\]

**Lemma A.9** (Matrix Concentration). Let \( \delta_N \) and \( A_N \) be as in Definition [37]. For \( v = 1 \) (regular case) and \( v = e \) (bootstrap case), for every partition \( k = 1, 2, \ldots, K \),
\[
\| \hat{\Sigma}_k^v(\hat{\eta}) - \hat{\Sigma}_k^v(\eta_0) \| = O_P \left( N^{-1/2} \delta_N + N^{-1+1/\nu} + A_N \right) = o_P(1). \quad \text{ (A.17)}
\]
Combining the bounds gives
\[
\| \hat{\Sigma}_k^v(\hat{\eta}) - \hat{\Sigma}^v(\eta_0) \| = O_P \left( N^{-1/2} \delta_N + N^{-1+1/\nu} + A_N \right).
\]
**Proof of Lemma A.9**

**Step 1.** Note that
\[
\hat{\Sigma}_k^v(\hat{\eta}) - \hat{\Sigma}_k^v(\eta_0) = \frac{1}{K} \sum_{k=1}^K \mathbb{E}_{n,k} v_i [A(W_i, \hat{\eta}_k) - A(W_i, \eta_0)].
\]
For each \( k = 1, 2, \ldots, K \),
\[
\mathbb{E}_{n,k} [A(W_i, \hat{\eta}_k) - A(W_i, \eta_0)] v_i \\
= n_k^{-1/2} \mathbb{E}_{n,k} [A(W_i, \hat{\eta}_k) - A(W_i, \eta_0)] v_i \\
+ \Sigma(\hat{\eta}_k) - \Sigma(\eta_0) = I_1^v(\hat{\eta}_k) + I_2^v(\hat{\eta}_k).
\]

Under Assumption 3.4 on the event $E_N$,

$$
\|I^*_v(\hat{\eta}_k)\| = \|\Sigma(\hat{\eta}_k) - \Sigma(\eta_0)\| \leq \sup_{\eta \in \mathcal{T}_N} \|\Sigma(\eta) - \Sigma(\eta_0)\| \leq A_N.
$$

We establish the bound on $\|I^*_v(\hat{\eta}_k)\|$ with $v = 1$ in Step 2 and $v = e$ in Step 3, respectively.

**Step 2.** Consider the function class

$$
\mathcal{F}_A := \{ \alpha'(A(W, \eta) - A(W, \eta_0))\alpha, \quad \alpha \in \mathcal{S}_d\}
$$

and note that

$$
\sup_{\alpha \in \mathcal{S}_d} |\alpha'(A(W, \eta) - A(W, \eta_0))\alpha| \leq \|A(W, \eta) - A(W, \eta_0)\| =: F^A(W).
$$

The matrix operator norm $\|A(W, \eta)\|$ is integrable in $L_{p,c}$ for $c = 2$ and some $c' > 2$. Thus, for any $\eta \in \mathcal{T}_N$,

$$
\|F^A(W)\|_{p,c} \leq 2\|A(W, \eta)\|_{p,c} \leq 2\bar{B}_A.
$$

Note that $F_A$ is a parametric class obeying (A.12). Assumption 3.4 implies that, for any $\eta \in \mathcal{T}_N$,

$$
\sup_{\eta \in \mathcal{T}_N} (\mathbb{E}|A(W, \eta) - A(W, \eta_0)|^2)^{1/2} \leq \delta_N.
$$

**Step 3.** Consider the function class

$$
\mathcal{F}^e_A := \{ e(\alpha'(A(W, \eta) - A(W, \eta_0))\alpha, \quad \alpha \in \mathcal{S}_d\}
$$

and note that

$$
\sup_{\alpha \in \mathcal{S}_d} |e \cdot \alpha'(A(W, \eta) - A(W, \eta_0))\alpha| \leq |e|F^A(W),
$$

where $|e|F^A(W)$ is integrable in $L_{p,c}$ for $c = 2$ and some $c' > 2$. The function class $\mathcal{F}^e_A$ is obtained by multiplying $\mathcal{F}_A$ by an exponential random variable independent of the data, and, therefore, retains the $P$-Donsker and covering properties $\mathcal{F}_A$. Finally, for any $\eta \in \mathcal{T}_N$,

$$
\sup_{\eta \in \mathcal{T}_N} \mathbb{E}e^2\|A(W, \eta) - A(W, \eta_0)\|^2 = \sup_{\eta \in \mathcal{T}_N} 2\mathbb{E}\|A(W, \eta) - A(W, \eta_0)\|^2 \leq 2\delta_N^2.
$$

Invoking (A.2) in Lemma A.2 with

$$
\sigma^2 = \delta_N^2, \quad v = d, \quad a = e
$$

conditional on the data $(W_i)_{i \in \mathcal{I}_n^+}$ gives

$$
\sup_{f \in \mathcal{F}_A} \mathbb{E}|\mathcal{G}_{n,k}f(W_i)v_i| \lesssim P \delta_N + N^{-1/2 + 1/c'} = o(1).
$$

Since $K$ is finite,

$$
\|\tilde{\Sigma}^u(\hat{\eta}) - \tilde{\Sigma}^u(\eta_0)\| = O_P \left(K(N^{-1/2}\delta_N + N^{-1+1/c'} + A_N)\right) = o_P(1).
$$

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Lemma A.10 (Matrix Inverse Linearization). Under Assumption 3.2 for \( v = 1 \) (regular case) and \( v = e \) (bootstrap case)

\[
(\hat{\Sigma}^v(\hat{\eta}))^{-1} - \Sigma^{-1} = -\Sigma^{-1}(\hat{\Sigma}^v(\eta_0) - \Sigma)\Sigma^{-1} + M^v, \tag{A.18}
\]

where \( \|M^v\| = o_P(N^{-1/2}) \).

Proof of Lemma A.10 We invoke the identity

\[
(A + B)^{-1} - A^{-1} = -A^{-1}B(A + B)^{-1}
\]

with

\[
A = \Sigma, \quad B = \hat{\Sigma}^v(\hat{\eta}) - \Sigma.
\]

Furthermore, we decompose

\[
B = \hat{\Sigma}^v(\hat{\eta}) - \Sigma = (\hat{\Sigma}^v(\eta_0) - \Sigma) + (\hat{\Sigma}^v(\hat{\eta}) - \hat{\Sigma}^v(\eta_0)) = B_1^v + B_2^v.
\]

Plugging \( A \) and \( B = B_1^v + B_2^v \) gives

\[
((\hat{\Sigma}^v(\hat{\eta}))^{-1} - \Sigma^{-1}) = -A^{-1}B_1(A + B)^{-1} - A^{-1}B_2(A + B)^{-1}
\]

\[
= -\Sigma^{-1}(\hat{\Sigma}^v(\eta_0) - \Sigma)((\hat{\Sigma}^v(\hat{\eta}))^{-1} - \Sigma^{-1}(\hat{\Sigma}^v(\hat{\eta}) - \hat{\Sigma}^v(\eta_0))((\hat{\Sigma}^v(\hat{\eta}))^{-1}
\]

\[
= -\Sigma^{-1}(\hat{\Sigma}^v(\eta_0) - \Sigma)\Sigma^{-1}
\]

\[
- \Sigma^{-1}(\hat{\Sigma}^v(\hat{\eta}) - \hat{\Sigma}^v(\eta_0))/((\hat{\Sigma}^v(\hat{\eta}))^{-1}
\]

\[
- \Sigma^{-1}(\hat{\Sigma}^v(\eta_0) - \Sigma)((\hat{\Sigma}^v(\hat{\eta}))^{-1} - \Sigma^{-1})
\]

\[
= M_1^v + M_2^v + M_3^v. \tag{A.21}
\]

By Assumption 3.1 \( \|\Sigma^{-1}\| \leq \lambda_\text{min}^{-1} < \infty \) is bounded. For \( v = 1 \) (regular case), Assumption 3.3 gives

\[
\|\hat{\Sigma}(\eta_0) - \Sigma\| = O_P(v_N) = o_P(1).
\]

For \( v = e \) (bootstrap case), Lemma A.4 and Assumption 3.3 give

\[
\|B_2^v\| = \|\hat{\Sigma}^v(\eta_0) - \Sigma\| \leq \|\hat{\Sigma}^v(\eta_0) - \hat{\Sigma}(\eta_0)\| + \|\hat{\Sigma}(\eta_0) - \Sigma\| = O_P(v_N + N^{-1/2}). \tag{A.22}
\]

In both cases, for \( v = 1 \) and \( v = e \),

\[
\|M_1^v\| \leq \|\Sigma^{-1}\||\hat{\Sigma}^v(\eta_0) - \Sigma||\Sigma^{-1}\| = O_P(v_N + N^{-1/2}). \tag{A.23}
\]

As shown in Lemma A.9 for \( v = 1 \) and \( v = e \),

\[
\|\hat{\Sigma}^v(\hat{\eta}) - \hat{\Sigma}^v(\eta_0)\| = O_P(N^{-1/2} \delta_N + N^{-1/2+1/e} + N^{-1/2}A_N) = o_P(1). \tag{A.24}
\]

Combining (A.22) and (A.24) gives

\[
\|\hat{\Sigma}^v(\hat{\eta}) - \Sigma\| \leq \|\hat{\Sigma}^v(\hat{\eta}) - \hat{\Sigma}^v(\eta_0)\| + \|\hat{\Sigma}^v(\eta_0) - \Sigma\| = O_P(v_N + N^{-1/2}),
\]

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and \(\|\hat{\Sigma}^v(\hat{\eta})\| \geq \lambda_{\min}/2\|p\). On the same event, \(\|\hat{\Sigma}^v(\hat{\eta})^{-1}\|\) is bounded away from \(2/\lambda_{\min}\). Invoking Assumption 3.4 gives
\[
\|M_2^v\| \leq \|\Sigma^{-1}\|\|\hat{\Sigma}^v(\eta_0) - \Sigma\|\|\hat{\Sigma}^v(\hat{\eta})^{-1}\| - \Sigma^{-1}\|\].
\]
As shown above, \(\|\hat{\Sigma}^v(\hat{\eta})^{-1} - \Sigma^{-1}\| \leq \|\hat{\Sigma}^v(\hat{\eta})^{-1}\| + \|\Sigma^{-1}\| = O_P(1)\). Therefore,
\[
\|M_2^v\| \leq \|\Sigma^{-1}\|\|\hat{\Sigma}^v(\eta_0) - \Sigma\|O_P(1) = O_P(v_N + N^{-1/2}).
\]
(A.25)

Collecting the bounds gives
\[
\|\hat{\Sigma}^v(\hat{\eta})^{-1} - \Sigma^{-1}\| \leq \sum_{k=1}^{3} \|M_k^v\| = O_P(v_N + N^{-1/2}).
\]

The following bound holds
\[
\|M_3^v\| \leq \|\Sigma^{-1}\|\|\hat{\Sigma}^v(\eta_0) - \Sigma\|\|\hat{\Sigma}^v(\hat{\eta})^{-1} - \Sigma^{-1}\| = O_P((v_N + N^{-1/2})^2) = o_P(N^{-1/2}).
\]
(A.26)

As a result, \(\|M^v\| \leq \|M_2^v\| + \|M_3^v\| = o_P(N^{-1/2})\), and (A.18) holds.

**Proof of Corollary 3.2** Corollary 3.2 follows from Theorem 3.2 and Steps 4 and 5 of the proof of Theorems 3 and 4 of Chandrasekhar et al. (2012). Assumptions 3.4–3.3 are

**Proof of Corollary 3.3** Corollary 3.3 follows from Theorem 3.3 and Steps 4 and 5 of the proof of Theorems 3 and 4 of Chandrasekhar et al. (2012).

### B Proofs of Section 4

#### B.1 Proof of Lemmas 4.1 and 4.2

Assumption B.3 is a restatement of Assumption 6.1 in Belloni et al. (2017). In this subsection, Assumption B.3 is verified from the primitive conditions on the data generating process. Recall that (4.3) implies
\[
\mathcal{Y}(p, \eta_0) = \Lambda(Z(X)\eta_0(p)) + R_0(p, X) + \zeta(p, \eta_0).
\]

Decompose
\[
\mathcal{Y}(p, \eta) = \mathbb{E}[\mathcal{Y}(p, \eta) - \mathcal{Y}(p, \eta_0) \mid X] + \mathbb{E}[\mathcal{Y}(p, \eta_0) \mid X] + \zeta(p, \eta)
\]
\[
= \Lambda(Z(X)\eta_0(p)) + R_0(p, X) + R(p, \eta, X) + \zeta(p, \eta).
\]
Define the following terms

\[ A_{11N}(\eta) := \sup_{P \in \mathcal{P}, j \in [p_X]} |[E_N - E] Z_j^2(X) Y^2(p, \eta)| \]
\[ A_{12N}(\eta) := \sup_{P \in \mathcal{P}, j \in [p_X]} |[E_N - E] Z_j^2(X) \zeta^2(p, \eta)| \]
\[ B_N(\eta) := \sup_{p_1, p_2 \in \mathcal{P}, d_P(p_1, p_2) \leq 1/N} |[E_N - E] Z_j^2(X)(\zeta(p_1, \eta) - \zeta(p_2, \eta))^2| \]
\[ G_N(\eta) := \sup_{p_1, p_2 \in \mathcal{P}, d_P(p_1, p_2) \leq 1/N} |EZ_j^2(X)(\zeta(p_1, \eta) - \zeta(p_2, \eta))| \]
\[ C_N(\eta) := \sup_{p_1, p_2 \in \mathcal{P}, d_P(p_1, p_2) \leq 1/N} |E_N Z_j(X)(\zeta(p_1, \eta) - \zeta(p_2, \eta))| \]
\[ D_N(\eta) := \sup_{P \in \mathcal{P}, j \in [p_X]} |[E_N - E] Z_j(X)(\zeta(p, \eta) - \zeta(p, \eta_0))| \]
\[ E_N(\eta) := \sup_{P \in \mathcal{P}, j \in [p_X]} |[E_N - E] Z_j^2(X)(\zeta(p, \eta) - \zeta(p, \eta_0))^2| \]
\[ F_N(\eta) := \sup_{P \in \mathcal{P}, j \in [p_X]} |[E_N - E] Z_j^2(X)(\zeta(p, \eta) - \zeta(p, \eta_0))^2| \]
\[ L_N(\eta) := \sup_{P \in \mathcal{P}, j \in [p_X]} |EZ_j^2(X)(\zeta(p, \eta) - \zeta(p, \eta_0))| \]

In the statement of the following assumption, \(\Delta_N = o(1)\) and \(\zeta_N = o(1)\) are fixed sequences approaching zero from above at a speed at least polynomial in \(N\) (for example, \(\zeta_N \geq N^{-c}\) for some \(c > 0\), and \(c_\zeta, C_\zeta, c_\varepsilon, C_\varepsilon, C_\sigma\) are positive finite constants.

**Assumption B.3** (Assumption 6.1, [Belloni et al. 2017]). The following conditions hold for \(N \geq N_0\) and \(a_N := p_X\). (i) The model \((4.4)\) is approximately sparse with \(s_\nu = s_\nu(N)\)

\[
\sup_{P \in \mathcal{P}} |\nu_0(p)|_0 \leq s_N
\]
\[
\sup_{P \in \mathcal{P}} (E(R(p, \eta, X) + R_0(p, X))^2)^{1/2} \leq \sigma_N
\]

(ii) The set \(\mathcal{P}\) has uniform covering entropy obeying \(\log N(\mathcal{P}, d_P) \leq d \log(1/\epsilon) + 0\) with \(d_P(p_1, p_2) = \|p_1 - p_2\|\) and the collection \((\zeta(p, X), R(p, \eta, X), R_0(p, X))\) are suitably measurable transformations of \(W\) and \(p\). (iii) Uniformly over \(p \in \mathcal{P}\) and \(\eta \in T_N\), (a) the moments of the model are boundedly heteroscedastic, namely \(0 < c_\zeta \leq E[\zeta^2(p, \eta) | X] \leq C_\zeta < \infty\) a.s. for some positive finite constants that do not depend on \(\eta\) and (b) \(\max_{j \in [p_X]} E[|Z_j(X)\zeta(p, \eta)|^3 + |Z_j(X)Y(p, \eta)|^3] \leq C_\zeta\). (iv) For some sequence \(K_N\) such that \(K_N^{-1/3} \Delta_N \leq \zeta_N \Delta_N \leq C_N\) and \(\log(a_N) \leq C_N N^{1/3}\), the dictionary functions, approximation errors, and empirical errors obey the following regularity conditions: (a) \(c_2 \leq E Z_j^2(X) \leq C_2\) for \(j \in [p_X]\) and \(|Z(X)|_\infty \leq K_N\) a.s. and \(\max_{j \in [p_X]} E(Z_j^4(X) + Z_j^6(X)) \leq C_2\). (b) With probability \(1 - \Delta_N\), the following
statements hold

\[
\sup_{p \in \mathcal{P}} \mathbb{E}_N(R(p, \eta, X) + R_0(p, X))^2 \leq C \sigma_N^2 \tag{B.2}
\]

\[
A_{11N}(\eta) + A_{12N}(\eta) \leq \zeta_N \tag{B.3}
\]

\[
\log(a_N)(B_N(\eta) + G_N(\eta))^{1/2} \leq \zeta_N \tag{B.4}
\]

\[
C_N(\eta) = o_P(\zeta_N N^{-1/2}) \tag{B.5}
\]

(v) With probability \(1 - \Delta_N\), the empirical minimum and maximum sparse eigenvalues of the basis covariance matrix are bounded from zero and above.

Define the following rates

\[
\delta_{1N} = \left( \frac{\log(N p_{X} K_N)}{N} \right)^{1/2} + \frac{K_N^2 \log(N p_{X} K_N)}{N^{2/3}} \tag{B.6}
\]

\[
\delta_{2N} := \left( \frac{\log(N p_{X} K_N)}{N^{3/2}} \right)^{1/2} + \frac{K_N \log(N p_{X} K_N)}{N^{5/6}} \tag{B.7}
\]

\[
\lambda_{1N} = K_N^{2}(\sqrt{\eta_N \log a_N} + N^{-1/2} \log a_N). \tag{B.8}
\]

Lemma B.1 (Lemma J.5 in Belloni et al. (2018)). Let \(\zeta(p, \eta_0)\) be as in (A.13). Suppose that for all \(p \in \mathcal{P}\), we have that \(\mathcal{Y}(p, \eta_0) = H(Y, p)\), where \(Y\) is a random variable and \(\{H(\cdot, p), p \in \mathcal{P}\}\) is a VC-subgraph class of functions bounded by a constant \(M_{UL}\) with index \(C_Y\) for some constant \(C_Y \geq 1\). Moreover, suppose \(Z(X)\) and \(\zeta(p, \eta_0)\) obey Assumption B.3(iii) and (iv)(a). If Assumption 3.2 holds,

\[
G_N(\eta_0) \leq N^{-1/2}. \tag{B.9}
\]

and, with probability at least \(1 - (\log N)^{-1}\),

\[
A_{11N}(\eta_0) + A_{12N}(\eta_0) \leq \delta_{1N} \tag{B.10}
\]

\[
C_N(\eta_0) \leq \delta_{2N}. \tag{B.11}
\]

Lemma B.1 is not an exact restatement of Lemma J.5 in Belloni et al. (2018), but follows from its proof.

Proof of Lemma B.1. I verify the conditions of Lemma J.5 in Belloni et al. (2018) with

\[
q_Z = 6, K_N := K_N, \nu = 1, C := c_C c_Z, \tilde{C} := (1 + C_C)C_Z.
\]

I show that \(\sup_{p_1, p_2 \in \mathcal{P}} \mathbb{E}(\mathcal{Y}(p_1, \eta_0) - \mathcal{Y}(p_2, \eta_0))^4 \leq \tilde{C}_P \|p_1 - p_2\|\) with \(\tilde{C}_P\) large enough. Invoking

\[
|\mathcal{Y}(p_1, \eta_0) - \mathcal{Y}(p_2, \eta_0)| \leq M_{UL} \lambda_1 E_+(p_1, p_2) \cup E_-(p_1, p_2)
\]

implies

\[
\mathbb{E}|\mathcal{Y}(p_1, \eta_0) - \mathcal{Y}(p_2, \eta_0)|^4 \leq M_{UL}^k P(E_+(p_1, p_2) \cup E_-(p_1, p_2)) \leq \tilde{C}_P \|p_1 - p_2\|.
\]

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Then, the bound (B.10) is established similarly to (J.10), (B.11) similarly to (J.11) and (B.9) is established in the first line (unnumbered display) on page 94 of the Belloni et al. (2018).

**Lemma B.2** (Effect of estimation error of \( \eta \)). (1) The following inequalities hold for \( A_N(\eta), B_N(\eta), C_N(\eta) \):

\[
A_{11N}(\eta) \leq 2A_{12N}(\eta_0) + 2F_N(\eta) \quad (B.12)
\]
\[
A_{12N}(\eta) \leq 2A_{12N}(\eta_0) + 2E_N(\eta) \quad (B.13)
\]
\[
B_N(\eta) \leq 3(E_N(\eta) + B_N(\eta_0) + E_N(\eta)) \quad (B.14)
\]
\[
G_N(\eta) \leq 3(L_N(\eta) + G_N(\eta_0) + L_N(\eta)) \quad (B.15)
\]
\[
C_N(\eta) \leq C_N(\eta_0) + 2D_N(\eta) \quad (B.16)
\]

(2) The following properties hold for \( k = 1 \) and \( k = 2 \)

\[
\mathbb{E}[(\zeta(p, \eta) - \zeta(p, \eta_0))^{2k} | X] \leq C_{k, \zeta} \| \eta(X) - \eta_0(X) \| \quad (B.17)
\]
\[
\mathbb{E}[(\mathcal{Y}(p, \eta) - \mathcal{Y}(p, \eta_0))^{2k} | X] \leq C_{k, \mathcal{Y}} \| \eta(X) - \eta_0(X) \| \quad (B.18)
\]

where \( C_{k, \zeta} \) and \( C_{k, \mathcal{Y}} \) are some finite constants. (3) The following statements hold

\[
A_{11N}(\eta) + A_{12N}(\eta) = O_P(N^{-1/2}(\delta_1N + \lambda_{1N})), \quad (B.19)
\]
\[
B_N(\eta) = O_P(N^{-1/2}(\delta_1N + \lambda_{1N})), \quad (B.20)
\]
\[
C_N(\eta) = O_P(\delta_2N + N^{-1/2}(\lambda_{1N})), \quad (B.21)
\]
\[
G_N(\eta) = O(\eta_N) \quad (B.22)
\]

**Proof of Lemma B.2** Steps 1-2 establish \( (B.12), (B.16) \). Step 3 verifies (2). Steps 4–6 verifies (3).

**Step 1.** Decomposing

\[
(\zeta(p_1, \eta) - \zeta(p_2, \eta)) = ((\zeta(p_1, \eta) - \zeta(p_1, \eta_0))
\]
\[
+ (\zeta(p_1, \eta_0) - \zeta(p_2, \eta_0))
\]
\[
+ (\zeta(p_2, \eta) - \zeta(p_2, \eta_0)).
\]

For any \( f_j(\cdot) \) and \( g_j(\cdot), j \in [p_X] \),

\[
\sup_{p \in \mathcal{P}, j \in [p_X]} f_j(p) + g_j(p) \leq \sup_{p \in \mathcal{P}, j \in [p_X]} f_j(p) + \sup_{p \in \mathcal{P}, j \in [p_X]} g_j(p).
\]

Multiplying \( (\zeta(p_1, \eta) - \zeta(p_2, \eta)) \) by \( Z_j(X) \) and taking \( \sup_{p \in \mathcal{P}, j \in [p_X]} \) gives

\[
C_N(\eta) \leq \sup_{p_1, p_2 \in \mathcal{P}, j \in [p_X]} \left| \mathbb{E} N_j(X)(\zeta(p_1, \eta) - \zeta(p_1, \eta_0)) \right|
\]
\[
+ C_N(\eta_0) + \sup_{p_1, p_2 \in \mathcal{P}, j \in [p_X]} \left| \mathbb{E} N_j(X)(\zeta(p_2, \eta) - \zeta(p_2, \eta_0)) \right|
\]
\[
\leq 2D_N(\eta) + C_N(\eta_0),
\]

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which gives (B.16).

**Step 2.** Taking squares and multiplying by $Z_j^2(X)$ with $Z_j^2(X) \geq 0$ gives
\[
Z_j^2(X)(\zeta(p_1, \eta) - \zeta(p_2, \eta))^2 \leq 3Z_j^2(X)(\zeta(p_1, \eta) - \zeta(p_1, \eta_0))^2 \\
+ (\zeta(p_1, \eta_0) - \zeta(p_2, \eta_0))^2 + (\zeta(p_2, \eta) - \zeta(p_2, \eta_0))^2).
\]

Likewise,
\[
\zeta^2(p, \eta) = (\zeta(p, \eta) - \zeta(p, \eta_0) + \zeta(p, \eta_0))^2 \leq 2(\zeta(p, \eta) - \zeta(p, \eta_0))^2 + 2\zeta^2(p, \eta_0).
\]

A similar argument to Step 1 gives (B.12) and (B.13) and (B.14) and (B.15), which completes (1).

**Step 3. Verification of (2).** Invoking Assumptions A.3 and A.2 and (B.45) and (B.13) from Lemma 4.3 gives (B.18). (A.13) implies that $\zeta(p, \eta) - \zeta(p, \eta_0)$ is the demeaned version of $\mathcal{Y}(p, \eta) - \mathcal{Y}(p, \eta_0)$ conditional on $X$. Therefore, for any $p \in \mathcal{P}$ and for any $\eta \in \mathcal{T}_N$,
\[
\mathbb{E}[(\zeta(p, \eta) - \zeta(p, \eta_0))^2 \mid X] \leq \mathbb{E}[\mathcal{Y}(p, \eta) - \mathcal{Y}(p, \eta_0))^2 \mid X] \\
\leq 2C_P M_h M^2_U L \eta_N,
\]
which establishes (B.17). Next, for any $\eta \in \mathcal{T}_N$,
\[
\mathbb{E}[(\zeta(p, \eta) - \zeta(p, \eta_0))^4 \mid X] \leq 16\mathbb{E}[(\mathcal{Y}(p, \eta) - \mathcal{Y}(p, \eta_0))^2 \mid X] \\
\leq 32C_P M_h M^4_U L \eta_N.
\]

As a result, for any $\eta \in \mathcal{T}_N$,
\[
L_N(\eta) \leq 2C_2 C_P M_h M^2_U L \eta_N \tag{B.23}
\]

**Step 4. Verification of (B.19) and (B.21).** Let $(N^{c-\epsilon}) \leq \eta_N$ for some $c > 0$ as assumed. I verify the conditions of Lemma A.2 with the function class $\mathcal{F}_{1\bar{j}}, \mathcal{F}_{2\bar{j}}, \mathcal{F}_{3\bar{j}}$ in Lemma A.7, all of which obey (A.12). Take
\[
v' = 1, \quad a_N = p_N \vee N, \quad \sigma^2 = M^2_U L 2C_P M_h K^2_N \eta_N, \quad F(W) := 2K_N M^2_U L
\]
gives
\[
\log(a_N F(W) / \sigma') = O(\log(a_N)).
\]
Likewise for $E_N(\eta)$ and $F_N(\eta)$, take
\[
v' = 1, \quad a_N = p_N \vee N, \quad \sigma^2 = M^4_U L 2C_P M_h K^4_N \eta_N, \quad F(W) := 4K^2_N M^2_U L
\]
and $F(W) := 4K^2_N M^2_U L$ gives $\log(a_N F(W) / \sigma') = O(\log(a_N))$. Thus,
\[
D_N(\eta) + E_N(\eta) + F_N(\eta) = O_P(N^{-1/2} \lambda_{1N})
\]
where $\lambda_{1N}$ as in (B.8). Invoking (B.10) and (B.12)–(B.13) gives (B.19), and invoking (B.11) and (B.16) gives (B.21).
Step 5. Verification of (B.21) I verify the conditions of Lemma A.2 with $\mathcal{F}_{1j}$. Note that

$$\mathbb{E}Z_j^2(X)(\zeta(p_1, \eta_0) - \zeta(p_2, \eta_0))^4 \leq 16K_N^4 \mathbb{E}(\mathcal{Y}(p_1, \eta_0) - \mathcal{Y}(p_2, \eta_0))^4 \leq 32K_N^4 M_{UL}^4 C_\delta N^{-1}$$

Invoking Lemma A.2 with $\nu' = 1$, $a_N = p_X \vee N$, $\sigma'^2 = 32K_N^4 M_{UL}^4 C_\delta N^{-1}$ and $F(W) := K_N^2 M_{UL}^2$ gives $\log(aF(W)/\sigma') = O(\log(a_N))$

$$B_N(\eta_0) = O\left( N^{-1/2} \left( \sqrt{K_N^4 N^{-1} \log a_N} + N^{-1/2} K_N^2 \log a_N \right) \right) = O(N^{-1/2} \lambda_{1N})$$

Invoking (B.13) and (B.24) gives (B.20).

Step 6. Verification of (B.22) Invoking (B.9) and (B.23) and (B.15) gives (B.22).

Assumption B.4 (Additional assumptions for Lemma 4.1). Suppose the conditions of Lemma A.4 held. In addition, suppose the following conditions hold on the basis functions $Z(X)$. (1) There exists constant $c_Z \leq C_Z$ and deterministic sequence $\{K_N, N \geq 1\}$ where $c_Z \leq \mathbb{E}Z_j^2(X) \leq C_Z$ for all $j \in [p_X]$ and $\|Z(X)\|_\infty \leq K_N$ a.s. and $\max_{j \in [p_X]}(\mathbb{E}Z_j^p(X))^{1/2} \leq C_Z$. The sequences $\delta_{1N} + N^{1/2} \delta_{2N} + \lambda_{1N} + \log^{1/2} a_N((\eta_0)^{1/2} + N^{-1/4} (\sqrt{a_N} + \sqrt{\lambda_{1N}})) = o(1)$. (v) With probability $1 - \Delta_N$, the empirical minimum and maximum sparse eigenvalues of the basis covariance matrix are bounded from zero and above.

Proof of Lemma 4.1. Step 0. Define

$$R(p, \eta, X) := \mathbb{E}[\mathcal{Y}(p, \eta) - \mathcal{Y}(p, \eta_0) \mid X].$$

The function classes

$$\mathcal{R}_{1,1} := \{R(p, \eta, X), \quad p \in \mathcal{P}\}$$

is derived by taking expectation functions of $\{\mathcal{Y}(p, \eta) - \mathcal{Y}(p, \eta_0), \quad p \in \mathcal{P}\} \subseteq (Y_U - Y_L) \cdot (I_0 - I_{\eta_0})$, and, therefore, obeys (A.12). The function class

$$\mathcal{R}_{1,2} := \{R_0(p, X), \quad p \in \mathcal{P}\} \subseteq \mathcal{G}_{1,\eta_0} - \{Z(X)'\eta_0(p), \quad p \in \mathcal{P}\}$$

obeys (A.12) because $p \rightarrow \|\eta_0(p)\|_1$ is assumed Lipschitz in $p$. Thus, the function class

$$\{(R_0(p, X) + R(p, \eta, X))^2, \quad p \in \mathcal{P}\} \subseteq (\mathcal{R}_{1,1} + \mathcal{R}_{1,2}) \cdot (\mathcal{R}_{1,1} + \mathcal{R}_{1,2})$$

obeys (A.12) for any fixed $\eta \in \mathcal{T}_N$. 55
**Step 1.** (i)–(iii). Assumptions [B.3](i)–(ii) are directly assumed in Lemma 4.1. Note that (ii) automatically holds for a compact set \( P \subset \mathbb{R}^d \) with Euclidean distance. The bound (B.17) implies
\[
\mathbb{E}[(\zeta(p, \eta) - \zeta(p, \eta_0))^2 | X] = O(\eta \sqrt{N}).
\]
Therefore, the condition (iii) for \( \zeta(p, \eta_0) \) in Assumption [B.4](ii) implies the condition (iii) for \( \zeta(p, \eta) \) in Assumption [B.3](i) with \( c_1 \) and \( \bar{c}_1 \) large enough. Cauchy inequality and Assumption 4.2 gives
\[
\max_{j \in [J_X]} \mathbb{E}[|Z_j(X)\zeta(p, \eta)|^3 + |Z_j(X)\zeta(p, \eta)|^3] \leq 2 \max_{j \in [J_X]} \mathbb{E}[Z_j^4(X)]^{1/2} M_{UL}^3.
\]

**Step 2.** I verify (B.2). Note that for \( p \in P \),
\[
\sup_{p \in P} |R(p, \eta, X)| \leq M_{UL}\mathbb{P}(E_+(p) | X) \leq 2C_P M_{UL} M_{\eta} \|\eta(X) - \eta_0(X)\| \text{ a.s.}
\]
Therefore, for any \( k = 1 \) and \( k = 2 \),
\[
\sup_{p \in P} \mathbb{E}\|R^k(p, \eta, X)\| \leq (2C_P M_{UL} M_{\eta})^k \mathbb{E}\|\eta(X) - \eta_0(X)\|^k.
\]
\[
\sup_{p \in P} \mathbb{E} R_0^2(p, X) \leq \mathbb{E}(\rho_0(p, X) - Z(X)'\eta_0(p))^2 \leq C\mathbb{E}(\rho_0(p, X) - Z(X)'\eta_0(p))^2 = O(\bar{\sigma}_N^2),
\]
which ensures that the approximation error converges at rate
\[
\sup_{p \in P} \mathbb{E}(R_0(p, X) + R(p, \eta, X))^2 = O(\bar{\sigma}_N^2 + \eta_N^2) = O(\bar{\sigma}_N^2). \tag{B.25}
\]
Noting that
\[
\mathbb{E}(R_0(p, X) + R(p, \eta, X))^4 \leq 8\mathbb{E}(R_0^4(p, X) + R^4(p, \eta, X)) = O(\mathbb{E}\|\eta(X) - \eta_0(X)\|^4 + \bar{\sigma}_N^2) = O(\bar{\sigma}_N^2).
\]
Invoking Lemma [A.2] with
\[
\nu' = d, \quad \sigma' = C^2\bar{\sigma}_N^2, \quad F(W) := \sup_{p \in P} |R_0(p, X)| + M_{UL}
\]
gives
\[
\sup_{p \in P} |\mathbb{E}_N - \mathbb{E}[R_0(p, X) + R(p, \eta, X)]|^2 = O_P(N^{-1/2} \lambda_N), \tag{B.26}
\]
where
\[
\lambda_N = \sqrt{C^2\bar{\sigma}_N^2 \log(1/\bar{\sigma}_N)} + N^{-1/2} C_\sigma \log(1/\bar{\sigma}_N) \tag{B.27}
\]
obeys \( N^{-1/2} \lambda_N = O(\sigma_N^2) \). Indeed, \( \sqrt{\log(1/\sigma_N)/N} = O(\sqrt{\sigma_N \log a_N}/N) = O(\sigma_N) \). Likewise, \( N^{-1} C_\sigma \log(1/\sigma_N) = O(\sigma_N \log a_N/N) = O(\sigma_N^2) \). Combining (B.26) and (B.25) gives (B.2). Taking
\[
\zeta_N := \max(\delta_{1N} + N^{1/2} \delta_{2N} + \lambda_{1N} + \log^{1/2} a_N((\eta_N)/2)
+ N^{-1/4}(\sqrt{\delta_{1N}} + \sqrt{\lambda_{1N}}) - N^{-1/3} \log a_N)
\]
verifies the conditions (B.3) and (B.5) based on (B.19) and (B.21), Lemma B.2 To verify (B.4), note that \( B(\eta) \) and \( G(\eta) \) are non-negative, and
\[
\sqrt{B(\eta)} + G(\eta) \leq \sqrt{\log(1/\sigma_N)/N} + \sqrt{G(\eta)} = O_P((\eta_N)^{1/2} + N^{-1/4}(\sqrt{\delta_{1N}} + \lambda_{1N})) = O_P(\zeta_N).
\]
Therefore, Assumption B.3—which coincides with Assumption 6.1 in Belloni et al. (2017)—has been verified. Invoking Theorem 6.1 of Belloni et al. (2017) gives the result.

Assumption B.5 (Additional assumptions for Lemma 4.2). The following additional assumption hold. There exists a sequence \( \rho_N = o(1) \) such that \( \sup_{p \in P} |R_0(p, X)| \leq \rho_N = o(1) \) a.s.

Proof of Lemma 4.2 Define
\[
R(p, \eta, X) = \mathbb{E}[\mathcal{M}(p, \eta) - \mathcal{M}(p, \eta_0) | X].
\]
The function class
\[
\mathcal{R}_{1,1} := \{ R(p, \eta, X), \quad p \in P \}
\]
is derived by taking expectation functions of \( I_{\eta} \), and, therefore, obeys (A.12). The function class
\[
\mathcal{R}_{1,2} := \{ R_0(p, X), \quad p \in P \} \subseteq \mathcal{G}_{1, \rho_0} - \{ \Lambda(Z(X), \eta_0(p)), \quad p \in P \}
\]
obey (A.12) because \( \|\eta_0(p)\|_1 \) is assumed Lipschitz in \( p \) and \( \Lambda(\cdot) \) is Lipshitz (Lemma L.2 and L.3 in Belloni et al. (2018)). Invoking Lemma A.2 with
\[
\nu' = d, \quad \sigma' = C_\sigma \sigma_N^2, \quad F(W) := 4
\]
and invoking (B.26) gives
\[
\sup_{p \in P} |(\mathbb{E}[\mathcal{M}(p, \eta, X) - R(p, \eta, X)]^2) = o_P(N^{-1/2} \lambda_N) = o_P(\sigma_N^2)
\]
which verifies (iv) (a) of Assumption B.3. The condition (iv) (b) is verified in Lemma B.2 where
\[
\mathcal{Y}(p, \eta) := \mathcal{M}(p, \eta), \quad M_{UL} := 1.
\]
Finally,
\[
\sup_{x \in \mathcal{X}} |R(p, \eta, x)| \leq \rho_N \mathcal{Y}_N^0,
\]
which implies \( \sup_{x \in \mathcal{X}} |R(p, \eta, x) + R_0(p, x)| \leq \rho_N \mathcal{Y}_N^0 + \rho_N = o(1) \). The rest of the proof follows similarly to the proof of Lemma 4.1
B.2 Section 4.3 Proofs

Proof of Lemma 4.5. To cover the cases of Corollaries 4.1 and 4.2 I take

$$\gamma(p, X) := \gamma_L(X) + 1/2\gamma_U(X), \quad \gamma_N := 2(\gamma_L + \gamma_U + \gamma_N)$$

in Corollary 4.1 and

$$\gamma(p, X) := \gamma_L(X) + \rho_0(p, X), \quad \gamma_N := \gamma_L$$

in Corollary 4.2. Recall that

$$Y(p, \eta) = Y_L + (Y_U - Y_L)1\{p'V(\eta) > 0\} = Y_L + Y(p, \eta).$$

Define remainder terms

$$I_1(W, p, \eta) := p'V(\eta_0)(Y(p, \eta) - Y(p, \eta_0))$$ (B.28)
$$I_2(W, p, \eta) := p'V(\eta_0)(\log a_N - \gamma(p, X))$$ (B.29)
$$I_3(W, p, \eta) := p'(\eta_0(X) - \eta(X))(Y(p, \eta_0) - \gamma_0(p, X))$$ (B.30)
$$I_4(W, p, \eta) := p'(\eta_0(X) - \eta(X))(Y(p, \eta) - Y(p, \eta_0))$$ (B.31)
$$I_5(W, p, \eta) := p'(\eta_0(X) - \eta(X))(\log a_N - \gamma(p, X)).$$ (B.32)

The estimation error of the orthogonal moment can be decomposed as

$$g(W, p, \xi(p)) - g(W, p, \xi_0(p)) = \sum_{j=1}^5 I_j(W, p, \eta).$$ (B.33)

Likewise, define the remainder terms

$$R_1(W, p, p_0) := p_0'V(\eta_0)(Y(p, \eta_0) - Y(p_0, \eta_0))$$ (B.34)
$$R_2(W, p, p_0) := p_0'(\eta_0(X) - \eta_0)(\gamma_0(p, X) - \gamma_0(p_0, X))$$ (B.35)
$$R_3(W, p, p_0) := (p - p_0)'V(\eta_0)(Y(p_0, \eta_0) - \gamma_0(p_0, X))$$ (B.36)
$$R_4(W, p, p_0) := (p - p_0)'V(\eta_0)(Y(p, \eta_0) - Y(p_0, \eta_0))$$ (B.37)
$$R_5(W, p, p_0) := -(p - p_0)'V(\eta_0)(\gamma_0(p, X) - \gamma_0(p_0, X))$$ (B.38)

and note that

$$g(W, p, \xi_0(p)) - g(W, p_0, \xi_0(p_0)) = \sum_{j=1}^5 R_j(W, p, p_0).$$

Step 1. Bound on $\mu_N$. The terms $I_1(W, p, \eta)$ and $I_4(W, p, \eta)$ coincide with $B_1(W, p, \eta)$ and $B_2(W, p, \eta)$ in Lemma 4.3 that is

$$I_1(W, p, \eta) := B_1(W, p, \eta), \quad I_4(W, p, \eta) := B_2(W, p, \eta).$$

Invoking (8.14) from Lemma 4.3

$$\sup_{\eta \in \mathcal{T}_N} \|E[I_1(W, p, \eta) + I_4(W, p, \eta)]\| \leq \sup_{\eta \in \mathcal{T}_N} 4C_p^2 M U L M_h E \|\eta(X) - \eta_0(X)\|^2 \leq 4C_p^2 M U L M_h \eta_N^2.$$
The terms $I_2(W, p, \eta)$ and $I_3(W, p, \eta)$ are mean zero by Law of Iterated Expectations

$$
\mathbb{E}[I_2(W, p, \eta) + I_3(W, p, \eta)] = 0, \quad \forall p \in \mathbb{R}^d \quad \forall \eta \in \mathcal{T}_N.
$$

Finally, for any $\eta \in \mathcal{T}_N$, the term $I_5(W, p, \eta)$ is bounded as

$$
|\mathbb{E}I_5(W, p, \eta)| \leq Cp(\mathbb{E}\|\eta(X) - \eta_0(X)\|^2)^{1/2}(\mathbb{E}|\gamma(p, X) - \gamma_0(p, X)|^2)^{1/2}.
$$

Combining the bounds give a valid bound on $\mu_N$, that is

$$
\mu_N \leq 4C_p^2M_{UL}M_N\eta_N^2 + Cp\eta_N \cdot \gamma_N. \quad (B.39)
$$

**Step 2. Bound on $r''_N$.** Taking squares of $\mathbf{(B.33)}$ gives

$$
\mathbb{E}(g(W, p, \xi(p)) - g(W, p, \xi_0(p)))^2 = \mathbb{E}(\sum_{j=1}^{5} I_j(W, p, \eta))^2 \leq 5 \sum_{j=1}^{5} \mathbb{E}I_j^2(W, p, \eta).
$$

We establish the bound on $\mathbb{E}I_j^2(W, p, \eta)$ for each $j = 1, 2, \ldots, 5$. Invoking $\mathbf{(8.10)}$ in Lemma $\ref{lemma:bound_U}$ and $Y_U - Y_L \leq M_{UL} \text{ a.s.}$ gives the bound on the first term

$$
\mathbb{E}I_1^2(W, p, \eta) = \mathbb{E}(p'V(\eta_0))^{2}(Y(p, \eta) - Y(p, \eta_0))^2 \leq M_{UL}^2\mathbb{E}(p'V(\eta_0))^21\{\mathcal{E}_+- (p)\}.
$$

On the event $\mathcal{E}_+- (p)$, we have $|p'(\eta_0(X) - \eta(X))| \leq |p'(\eta_0(X) - \eta(X))|$. As a result,

$$
\mathbb{E}I_1^2(W, p, \eta) \leq M_{UL}^2C_p^2\mathbb{E}\|\eta_0(X) - \eta(X)\|^21\{\mathcal{E}_+- (p)\}
$$

$$
\leq M_{UL}^2C_p^2\eta_N^2 \quad \forall p \in \mathbb{R}^d \quad \forall \eta \in \mathcal{T}_N.
$$

Likewise,

$$
\mathbb{E}I_2^2(W, p, \eta) \leq \mathbb{E}(p'(\eta_0(X) - \eta(X)))^2(Y_U - Y_L)^21\{\mathcal{E}_+- (p)\}
$$

$$
\leq M_{UL}^2C_p^2\eta_N^2 \quad \forall p \in \mathbb{R}^d \quad \forall \eta \in \mathcal{T}_N.
$$

**Step 3. Bound on $r''_N$, cont.** Next, I bound $\mathbb{E}I_3^2(W, p, \eta)$ and $\mathbb{E}I_4^2(W, p, \eta)$. Let $U$ be a conditionally on $X$ subGaussian random variable. Then, $\sup_{x \in \mathcal{X}} \mathbb{E}[U^2 \mid X = x]$ is finite and bounded by some constant, and

$$
\mathbb{E}\Delta^2(X)U^2 \leq \sup_{x \in \mathcal{X}} \mathbb{E}[U^2 \mid X = x] \mathbb{E}\Delta^2(X). \quad (B.40)
$$

Invoking the bound above with

$$
\Delta(X) = \log a_N - \gamma(p, X), \quad U = \|V(\eta_0)\|, \quad \sup_{x} \mathbb{E}[U^2 \mid X = x] \leq \mathcal{C}_{V}
$$

gives

$$
\mathbb{E}I_3^2(W, p, \eta) \leq C_p^2\mathcal{C}_{V}\mathbb{E}(\log a_N - \gamma(p, X))^2 \leq \mathcal{C}_{V}C_p^2\gamma_N^2 \quad \forall p \in \mathcal{P} \forall \gamma \in \Gamma_N.
$$

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To bound $\mathbb{E}I_j^2(W, p, \eta)$, we take $U := Y(p, \eta_0) - \log a_N$ where $\log a_N = \mathbb{E}[Y(p, \eta_0) | X]$. Invoking
\[
\mathbb{E}[U^2 | X] = \text{Var}(Y(p, \eta_0) | X) \leq \mathbb{E}[Y^2(p, \eta_0) | X]
\]
and $(a + b)^2 \leq 2(a^2 + b^2)$ with $a = Y_L$ and $b = (Y_U - Y_L)1\{p'V(\eta_0) > 0\}$ gives
\[
\mathbb{E}[U^2 | X] \leq 2[\mathbb{E}[Y_L^2 | X] + M_2^2] \leq 2(C_L + M_2^2) =: \bar{C}_3 \text{ a.s.}
\]
Invoking (B.40) with $U := Y(p, \eta_0) - \log a_N$ and $\Delta(X) = p'(\eta_0(X) - \eta(X))$ gives
\[
\mathbb{E}I_j^2(W, p, \eta) \leq C_P^2\mathbb{E}||\eta_0(X) - \eta(X)||^2 \mathbb{E}[U^2 | X]
\]
\[
\leq \bar{C}_3 C_P^2\mathbb{E}||\eta_0(X) - \eta(X)||^2 \leq \bar{C}_3 C_P^2 \gamma_N^2 \quad \forall p \in \mathbb{R}^d \quad \forall \eta \in \mathcal{T}_N.
\]
Finally,
\[
\mathbb{E}I_j^2(W, p, \eta) \leq C_P^2\mathbb{E}||\eta_0(X) - \eta(X)||^2 (\log a_N - \gamma(p, X))^2
\]
\[
\leq C_P^2 A M_2^2 \mathbb{E}||\eta_0(X) - \eta(X)||^2 \leq 4M_2^2 C_P^2 \gamma_N^2.
\]
Collecting the bounds gives
\[
\sum_{j=1}^5 \mathbb{E}I_j^2(W, p, \eta) \leq C'(\gamma_N^2 + \gamma_N^2)
\]
for $C'$ large enough.

**Step 4. Bound on $r'_N$.** Let $(R_j(p, \eta))_{j=1}^5$ be as in [B.34]-[B.38] and $\mathcal{E}_-(p, p_0), \mathcal{E}_+(p, p_0)$ as in [B.6]-[B.7]. Observe that
\[
\mathbb{E}R_j^2(W, p, p_0) = \mathbb{E}(p_0'V(\eta_0))^2(Y_U - Y_L)^21\{\mathcal{E}_-(p, p_0)\}
\]
\[
\leq M_2^2 \mathbb{E}((p - p_0)'V(\eta_0))^21\{\mathcal{E}_-(p, p_0)\}
\]
\[
\leq M_2^2 \|p - p_0\|^2 \mathbb{E}|V(\eta_0)|^2
\]
(B.42)

Note that
\[
|\gamma(p, X) - \gamma(p_0, X)| = |\mathbb{E}[(Y_U - Y_L)1\{p'V(\eta_0) > 0\} | X]
\]
\[
- \mathbb{E}[(Y_U - Y_L)1\{p_0'V(\eta_0) > 0\} | X]|
\]
\[
\leq \mathbb{E}[(Y_U - Y_L)1\{p'V(\eta_0) > 0\} - 1\{p_0'V(\eta_0) > 0\} | X]
\]
\[
\leq \mathbb{E}[(Y_U - Y_L)1\{\mathcal{E}_-(p, p_0)\} | X]
\]
\[
\leq M_{UL} \mathbb{P}(\mathcal{E}_-(p, p_0) | X) \text{ a.s.}
\]
Taking the squares of each side gives
\[
|\gamma(p, X) - \gamma(p_0, X)|^2 \leq M_{UL}^2 \mathbb{P}^2(\mathcal{E}_-(p, p_0) | X) \leq M_{UL}^2 \mathbb{P}(\mathcal{E}_-(p, p_0) | X) \text{ a.s.}
\]
where the last bound follows from $t^2 \leq t$ if $t \in [0, 1]$. Invoking (5.40) with $U := \|V(\eta_0)\|$ and $\Delta(X) := \gamma_0(p, X) - \gamma_0(p_0, X)$ gives

$$
\begin{align*}
\mathbb{E}R_2^2(W, p, p_0) &= \mathbb{E}(p_0^2V(\eta_0)\gamma_0(p, X) - \gamma_0(p_0, X))^2 \\
& \leq M_{UL}^2 C_\mathcal{V}^2 \mathbb{E}_X[P(\mathcal{E}_-(p, p_0) \mid X)] \\
& = M_{UL}^2 C_\mathcal{V}^2 \mathbb{E}_X[P(\mathcal{E}_-(p, p_0))] \\
& \leq M_{UL}^2 \mathcal{C}_\mathcal{V}\|p - p_0\|.
\end{align*}
$$

Cauchy inequality gives a bound $\mathbb{E}R_3^2(W, p, p_0)$:

$$
\begin{align*}
\mathbb{E}R_3^2(W, p, p_0) &\leq C_p^2 \|p - p_0\|^{2\mathbb{E}}\|V(\eta_0)\|^2 (Y(p_0, \eta_0) - \gamma_0(p_0, X))^2 \\
& \leq C_p^2 \|p - p_0\|^2 \|V(\eta_0)\|_{p,4} \sup_{p_0 \in \mathcal{P}} \|Y(p_0, \eta_0) - \gamma_0(p_0, X)\|_{p,4},
\end{align*}
$$

where $\|V(\eta_0)\|_{p,4} < \infty$ follows from subGaussianity and $\|Y(p_0, \eta_0) - \gamma_0(p_0, X)\|_{p,4} \leq \|Y(p_0, \eta_0)\|_{p,4} + M_\gamma \leq \|Y_L\|_{p,4} + M_{UL} + M_\gamma$ is finite by assumption. Finally, for $j = 4$ and $j = 5$, the term $R_j(W, p, p_0)$ is a product of $(p - p_0)^jV(\eta_0)$ and a random variable bounded by $M_{UL}$ a.s. Therefore,

$$
\mathbb{E}R_j^2(W, p, p_0) \leq C_p^2 M_{UL}^2 \|p - p_0\|^2 \|V(\eta_0)\|^2, \quad j = 4, 5.
$$

Collecting the bounds shows that

$$
\begin{align*}
\sup_{p, p_0 \in \mathcal{P}, \|p - p_0\| \leq \tau_N} \sum_{j=1}^5 \mathbb{E}R_j^2(W, p, p_0) &\leq C\tau_N + \tau_N^2
\end{align*}
$$

for $C$ large enough.

**Step 5. Bound on $A_N$.** Define

$$
\begin{align*}
M_1(W, \eta, \eta_0, m, m_0) :& = (\eta_0(X) - \eta(X))(D - m_0(X))' \\
M_2(W, \eta, \eta_0, m, m_0) :& = (Z - \eta_0(X))(m_0(X) - m(X))' \\
M_3(W, \eta, \eta_0, m, m_0) :& = (\eta_0(X) - \eta(X))(m_0(X) - m(X))'
\end{align*}
$$

and note that

$$
A(W, \eta, m) - A(W, \eta_0, m_0) = (Z - \eta(X))(D - m(X))' - (Z - \eta_0(X))(D - m_0(X))' = \sum_{k=1}^3 M_k(W, \eta, \eta_0, m, m_0).
$$

The terms $M_1(W, \eta, \eta_0, m, m_0)$ and $M_2(W, \eta, \eta_0, m, m_0)$ are zero mean by LIE. The second-order term $M_3(W, \eta, \eta_0, m, m_0)$ is bounded as

$$
\begin{align*}
\|\mathbb{E}M_3(W, \eta, \eta_0, m, m_0)\| &\leq \mathbb{E}\|M_3(W, \eta, \eta_0, m, m_0)\| \\
& \leq \mathbb{E}\|\eta_0(X) - \eta(X)\|\|m_0(X) - m(X)\| \\
& \leq (\mathbb{E}(\eta_0(X) - \eta(X))^2)^{1/2}(\mathbb{E}\|m_0(X) - m(X)\|^2)^{1/2}.
\end{align*}
$$
Collecting the bounds ensures that the condition \((3.5)\) holds with \(A_N \leq \eta_N \cdot m_N\).

**Step 6. Bound on \(\delta_N\).**

\[
E\|M_1(W, \eta, \eta_0, m, m_0)\|^2_F \leq E\|M_1(W, \eta, \eta_0, m, m_0)\|^2_F = E\|\eta_0(X) - \eta(X)\|^2\|D - m_0(X)\|^2 \\
\leq \sup_x E[\|D - m_0(X)\|^2 \mid X = x]E\|\eta_0(X) - \eta(X)\|^2 \\
=: \tilde{C}_D E[\eta_0(X) - \eta(X)\|^2.
\]

A similar argument gives

\[
E\|M_2(W, \eta, \eta_0, m, m_0)\|^2_F \leq \tilde{C}_V E\|m_0(X) - m(X)\|^2.
\]

Finally,

\[
E\|M_3(W, \eta, \eta_0, m, m_0)\|^2_F \leq E\|m_0(X) - m(X)\|^2\|\eta_0(X) - \eta(X)\|^2 \leq 4M_2^2\eta_N^2.
\]

Collecting the bounds gives

\[
\|A(W, \eta, m) - A(W, \eta_0, m_0)\|_2 \leq \sum_{k=1}^{3} M_k(W, \eta, \eta_0, m, m_0)\|_2
\]

which implies

\[
E\|A(W, \eta, m) - A(W, \eta_0, m_0)\|_2^2 \leq 3 \sum_{k=1}^{3} E\|M_k(W, \eta, \eta_0, m, m_0)\|_2^2,
\]

which ensures that the condition \((3.5)\) holds with

\[
\delta_N^2 \leq 3(4M_2^2\eta_N^2 + \tilde{C}_D \eta_N^2 + \tilde{C}_V m_N^2).
\]

**Step 7. (Conclusion)** Steps 1-6 verify Assumption \((3.4)\). Take \(A(W, \eta_0, m_0) = (V - \eta_0(X))(D - m_0(X))'\). The bound on \(E\|A(W, \eta_0, m_0)\|^2_F\) is

\[
E\|A(W, \eta_0, m_0)\|^2_F = E\|Z - \eta_0(X)\|^2\|D - m_0(X)\|^2 \\
\leq \sqrt{E}[Z - \eta_0(X)]^2 \sqrt{E}\|D - m_0(X)\|^2 = \sqrt{E}\|V\|^2 \|E\| < \infty,
\]

since \(\|V\| = \|V(\eta_0)\|\) and \(\|E\| = \|E(m_0)\|\) are subGaussian. ■

**Proof of Corollary \((4.1)\)** Note that Assumptions \((3.4)\) and \((3.5)\) are verified in Lemmas \((4.5)\) and \((4.4)\) respectively. Below, I verify Assumption \((3.5)\). Let \(\mathcal{L}_\eta\) and \(\mathcal{Y}_\eta\) and \(\mathcal{G}_{1, \eta}\) be the function classes defined in Lemma \((A.6)\). The following classes

\[
\{p'V(\eta) \cdot Y(p, \eta), \ p \in \mathcal{P}\} \subseteq \mathcal{L}_\eta \cdot (Y_L + \mathcal{Y}_\eta)
\]

and

\[
\{p'V(\eta) \cdot (\gamma_L(X) + 1/2\gamma_{UL}(X)), \ p \in \mathcal{P}\} \subseteq \mathcal{L}_\eta \cdot (\gamma_L(X) + 1/2\gamma_{UL}(X))
\]
Lemma A.6. Take a whose $F$ obeys (A.12) with

Proof of Corollary 4.2. obeys (A.15) with $T$ realization set $\{W \to p'V(\eta)(Y(p, \eta) - \gamma_L(X) - 1/2\gamma_{UL}(X)), \ p \in \mathcal{P}\}$

The class $\mathcal{F}_{n,\gamma_L,\gamma_{UL}} := \{W \to p'V(\eta)(Y(p, \eta) - \gamma_L(X) - 1/2\gamma_{UL}(X)), \ p \in \mathcal{P}\}$

obeys (A.12) with the envelope function

$F_{n,\gamma_L,\gamma_{UL}} = C_P(\|V(\eta_0)\| + M_\eta)(Y_L - \gamma_{L_0}(X)) + C_P(\|V(\eta_0)\| + M_\eta)(M_\gamma + M_{UL} + M_{UL})$,

whose $\|F_{n,\gamma_L,\gamma_{UL}}\|_{p,c'}$ is integrable for $c' = 2$ and $c' > 2$ by Assumption 4.3.

Proof of Corollary 4.2. Step 1. Define the first-order term

$\tilde{S}_1^\eta(p) := \sqrt{N}E_Np'V_i(\eta_0)(\rho_0(p, X_i) - \hat{\rho}(p, X_i))v_i$

and the second-order term

$\tilde{S}_2^\eta(p) := \sqrt{N}E_Np'(V_i(\hat{\eta}) - V_i(\eta_0))(\rho_0(p, X_i) - \hat{\rho}(p, X_i))v_i$

and the cross-fit term

$\tilde{S}_3^\eta(p) := \sqrt{N}E_N(g(W_i, p, \{\hat{\eta}, \gamma_L, \rho_0(p, \cdot)\}) - g(W_i, p_0(q), \{\eta_0, \gamma_{L_0}, \rho_0(p_0(q))\}))v_i$.

Decompose the estimation error

$\sqrt{N}(E_Ng(W_i, \hat{\rho}(q), \hat{\xi}(\hat{\rho}(q))) - E_Ng(W_i, p_0(q), \xi_0(p_0(q))))v_i$

$= S_1^\eta(\hat{\rho}(q)) + S_2^\eta(\hat{\rho}(q)) + S_3^\eta(\hat{\rho}(q))$.

I show that $\sup_{p \in \mathcal{P}} |\tilde{S}_1^\eta(p)| = o_P(1)$ in Steps 2 and 3. Step 4 and Step 5 elaborates

on $\sup_{p \in \mathcal{P}} |\tilde{S}_2^\eta(p)| = o_P(1)$ and $\tilde{S}_3^\eta(\hat{\rho}(q))$.

Step 2. Let $\ell_\eta$ and $Y_\eta$ and $\mathcal{G}_{1,\eta}$ and $\mathcal{M}$ be the function classes defined in Lemma A.6. Take $a_N = p_X \vee N$ and $\overline{\sigma}_N = C\sqrt{s_N \log a_N/N}$. Define the nuisance realization set

$T_p := \{\rho_0(p, \cdot)\} \cup \{\|\eta\|_0 \leq C s_N, \|\eta - \eta_0(p)\| \leq \overline{\sigma}_N, \|\eta - \eta_0(p)\|_1 \leq \sqrt{s_N \overline{\sigma}_N}, \eta \in \mathbb{R}^{p_X}\}$.

The class

$\mathcal{F}_{2,1} := \{W \to p'V(\eta_0) \cdot \Lambda(Z(X')\eta), \ p \in \mathcal{P}, \eta \in T_p \setminus \{\rho_0(p, \cdot)\}\} \subseteq \ell_\eta \cdot \Lambda(\mathcal{M})$

obeys (A.13) with $F_{2,1} := C_P\|V(\eta_0)\|K_N$ (linear link) or $F_{2,1} := C_P\|V(\eta_0)\|$ (logistic link), and the class

$\mathcal{F}_{2,2} := \{W \to p'V(\eta_0) \cdot \rho_0(p, X), \ p \in \mathcal{P}\} \subseteq \ell_\eta \cdot \mathcal{G}_{1,\eta_0}$

obeys (A.12) with $F_{2,2} := C_P M_{UL}\|V(\eta_0)\|$. The function class

$\mathcal{F}_{2,3} := \{W \to p'V(\eta_0) \cdot (\Lambda(Z(X')\eta) - \rho_0(p, X)), \ p \in \mathcal{P}\} \subseteq \mathcal{F}_{2,1} - \mathcal{F}_{2,2}$

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obey (A.15) with $F_{2,3} := F_{2,1} + F_{2,2}$. For each $\mathcal{F} \in \{\mathcal{F}_{2,1}, \mathcal{F}_{2,2}, \mathcal{F}_{2,3}\}$, the class $\mathcal{F}^*$ is obtained by multiplying the elements of $\mathcal{F}$ by an integrable random variable, and therefore retains the entropy properties of $\mathcal{F}$. Finally, for $v = 1$ (regular case) and $v = e$ (bootstrap case)

$$\sup_{p \in \mathcal{P}} |\bar{S}_1^v(p)| = \|G_N f^v(W)\|_F.$$

**Step 3.** For both linear and logistic link, $\sup |\Lambda'(t)| \leq 1$. By intermediate value theorem,

$$|\Lambda(Z'(X)\eta(p)) - \Lambda(Z'(X)\eta_0(p))| \leq \sup_t |\Lambda'(t)||Z(X)\eta(p) - \eta_0(p)||$$

$$\leq |Z(X)\eta(p) - \eta_0(p)||.$$

Next, for any $p \in \mathcal{P}$,

$$\mathbb{E}(\Lambda(Z'(X)\eta(p)) - \rho_0(p, X))^2 \leq 2\mathbb{E}(Z'(X)\eta(p) - \eta_0(p))^2 + 2\mathbb{E}R_0^2(p, X)$$

$$\leq s_N \log a_N / N \equiv \sigma_N^2.$$

Notice that

$$\sup_{f \in \mathcal{F}} \mathbb{E}f^2(W) \leq C_P \sup_{x \in X} \mathbb{E}[\|V(\eta_0)\|^2 | X = x]\mathbb{E}(\Lambda(Z'(X)\eta(p)) - \rho_0(p, X))^2 \leq \sigma_N^2$$

**Invoking Lemma A.2** with $a_N = p_N \vee N$, $v' = s_N$, $\sigma' = C_0 \sigma_N$, $F_{2,3}(W) := F_{2,1} + F_{2,2}$ gives for $\Lambda(t) = t$

$$\sup_{p \in \mathcal{P}} |\bar{S}_1^v(p)| \leq P \sqrt{s_N \sigma_N^2 \log(a_N K_N / \sigma_N)} + s_N N^{-1/2 + 1/c'} \log(a_N K_N / \sigma_N) = o_P(1)$$

and for $\Lambda(t) = \exp t/(\exp t + 1)$

$$\sup_{p \in \mathcal{P}} |\bar{S}_1^v(p)| \leq P \sqrt{s_N \sigma_N^2 \log(a_N / \sigma_N)} + s_N N^{-1/2 + 1/c'} \log(a_N / \sigma_N) = o_P(1)$$

**Step 4.** I show that $\sup_{p \in \mathcal{P}} |\bar{S}_2^v(p)| = o_P(1)$. By the argument in Step 3,

$$\Lambda(Z'(X_i)\eta(p)) - \Lambda(Z'(X_i)\eta_0(p)))^2 \leq (Z'(X_i)(\hat{\eta}(p) - \eta_0(p)))^2$$

and summing over $i = 1, 2, \ldots, N$ gives

$$\sup_{p \in \mathcal{P}} \mathbb{E}_N (\Lambda(Z'(X_i)\eta(p)) - \Lambda(Z'(X_i)\eta_0(p)))^2 \leq \sup_{p \in \mathcal{P}} \|\hat{\eta}(p) - \eta_0(p)\|^2_{\mathcal{P}, 2}$$

and

$$\sup_{p \in \mathcal{P}} \mathbb{E}_N (\rho_0(p, X_i) - \hat{\rho}(p, X_i))^2 \leq \sup_{p \in \mathcal{P}} 2\|\hat{\eta}(p) - \eta_0(p)\|^2_{\mathcal{P}, 2} + N^{-1}2\mathbb{E}_N R_0^2(p, X_i).$$

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As assumed in (4.8), the RHS above is $O_P(\bar{\sigma}_N^2)$. By Assumption 4.3,

$$\sup_{p \in P} (E_N v_i^2 (p' (V_i (\tilde{\eta}) - V_i (\eta_0)))^{1/2} \leq C_P (E_N v_i^2 \| \tilde{\eta} (X_i) - \eta_0 (X_i) \|^2)^{1/2} = O_P(\eta_N + N^{-1/4}).$$

As a result,

$$\sup_{p \in P} \left\{ \left| \bar{S}_N^p \right| \right\} = \sup_{p \in P} \left\{ N^{1/2} \| E_N p' (V_i (\tilde{\eta}) - V_i (\eta_0)) \rho_0 (p, X_i) - \tilde{\rho}(p, X_i) v_i \|^2 \right\}$$

$$= \sup_{p \in P} N^{1/2} (E_N (\rho_0 (p, X_i) - \tilde{\rho}(p, X_i))^2)^{1/2} (E_N v_i^2 (p' (V_i (\tilde{\eta}) - V_i (\eta_0)))^2)^{1/2}$$

$$= o_P(N^{1/2} \bar{\sigma}_N \cdot (\eta_N + N^{-1/4})).$$

**Step 5.** I show that $\sup_{p \in P} |\bar{S}_N^p (\tilde{\rho}(q)) - G_N [h (W, q)]| = o_P(1)$. This follows from the Steps 1–5 of the proof of Theorems 3.1, where Assumptions 3.4 and 3.3 and 3.5 need to be verified. Assumptions 3.4 and 3.3 are verified in Lemmas 4.5 and 4.4, respectively. Assumption 3.5 is verified by the same argument as the proof of Corollary 4.1 with

$$\{ p' V(\eta) \cdot Y(p, \eta), \ p \in P \} \subseteq \mathcal{L}_\eta \cdot (Y_L + \mathcal{Y}_L)$$

and

$$\{ p' V(\eta) \cdot (\gamma_L(X) + \rho_0 (p, X)), \ p \in P \} \subseteq \mathcal{L}_\eta \cdot (\gamma_L(X) + \mathcal{G}_{1, \eta_0}),$$

where both classes obey (A.12). The envelope function

$$F(W) := (\| V(\eta_0) \| + M_\eta) (Y_L - \gamma_{L,0}(X) + M_\gamma + M_{UL})$$

is integrable by Assumption 4.3.

**B.3 Proof of Section 5**

**Proof of Lemma 5.1** The identified set can be written as

$$\mathcal{B} = \{ \beta : \beta = E V(\eta_0) \gamma(D, X) \}$$

where

$$P(\gamma_L(D, X) \leq \gamma(D, X) \leq \gamma_U(D, X)) = 1 \quad (B.44)$$

The convexity argument is similar to [Kaido (2017)]. For any $\beta_1$ and $\beta_2$, there exist $\gamma_1(D, X)$ and $\gamma_2(D, X)$ obeying (B.44) such that $\beta_j = E V(\eta_0) \gamma_j(D, X)$, \( j = 1, 2 \). Therefore, for any $\alpha \in [0, 1]$,

$$\alpha \beta_1 + (1 - \alpha) \beta_2 = E V(\eta_0) \gamma_\alpha(D, X),$$

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where $\gamma_\alpha(D, X) = \alpha \gamma_1(D, X) + (1 - \alpha) \gamma_2(D, X)$ also obeys (5.44). To show strict convexity, invoke Lemma A.5 with $\Sigma = I_d$ gives

$$\sigma(q) - \sigma(q_0) = G(q_0)'(q - q_0) + R(q, q_0), \quad R(q, q_0) = o(\|q - q_0\|).$$

By Corollary 1.7.3 in Schneider (1993), the set $B$ is strictly convex. Next, I show that the set $B$ is bounded. For each $j = 1, 2, \ldots, d$, take $l_j(D, X) := \nabla_j f_0(D \mid X)/f_0(D \mid X)$. Cauchy inequality implies

$$\|\beta_j\| \leq \mathbb{E} |\gamma(D, X)l_j(D, X)| \leq \tilde{C}_\gamma (\mathbb{E} l(D, X)^2)^{1/2} \leq C \gamma C_L.$$

Finally, the set $B$ contains its boundary by the argument of the proof of Proposition 2 in Bontemps et al. (2012) and Beresteanu and Molinari (2008) and the proof of Theorem 2.1 in Kaido (2017), page 23.

**Proof of Lemma 5.2.** For the Example 2.3 recall that

$$\mathcal{E}^+(q) = \{0 < |q' V(\eta)| \leq \|q'(V(\eta) - V(\eta_0))\|\}, \quad p := q, \quad P := S^{d-1}, \quad CP = 1.$$

Let $B_1(W, \eta, q)$ and $B_2(W, \eta, q)$ be the bias terms as defined in Lemma 5.3. Invoking (8.11) and (8.12) in Lemma 4.3 gives

$$Y(q, \eta) - Y(q, \eta_0) = (Y_U - Y_L)1\{\mathcal{E}^+(q) \cup \mathcal{E}^-(q)\}$$

and

$$\|\mathbb{E} B_j(W, \eta, q)\| \leq M U \mathbb{E} \|V(\eta) - V(\eta_0)\|1\{\mathcal{E}^-(q)\}, \quad j = 1, 2.$$

Assumption 3.2 and $\sup_{\eta \in T_N} \|V(\eta) - V(\eta_0)\| \leq \tilde{\eta}^\mathbb{Z}$ gives

$$\sup_{\eta \in T_N} \mathbb{E} \|V(\eta) - V(\eta_0)\|1\{\mathcal{E}^-(q)\} \leq \tilde{\eta}^\mathbb{Z} P(\mathcal{E}^-(q)) \leq \tilde{C}_f (\tilde{\eta}^\mathbb{Z})^2.$$

**Proof of Corollary 5.7.** Step 1 (Outline). Assumption 3.2 holds trivially with known $\Sigma = I_d$ and $v_N = 0$. Therefore, Assumption 3.4 can be invoked with

$$A_N = \delta_N = \tau_N = \nu_N = 0, \quad p = q \in S^{d-1}.$$

Recall that

$$\gamma_N = \gamma_{L,N} + \gamma_{UL,N} + \gamma_{UL,N}^1 + \gamma_{UL,L}^1.$$

Step 2 verifies Assumption 3.5. Steps 3 and 4 verify Assumption 3.4 the bounds on $\mu_N$ and $\nu_N''$, respectively. Define

$$\mu_0(q, D, X) := \gamma_{L,0}(D, X) + \gamma_{UL,0}(D, X)1\{-q' \eta_0(D, X) > 0\} \quad \nabla_D \mu_0(q, D, X) := \nabla_D \gamma_{L,0}(D, X) + \nabla_D \gamma_{UL,0}(D, X)1\{-q' \eta_0(D, X) > 0\}.$$
Define the first-stage error terms

\[ S_1(D, X) = \gamma_L(D, X) - \gamma_{L,0}(D, X) \]
\[ S_2(D, X) = \gamma_{UL}(D, X) - \gamma_{UL,0}(D, X) \]
\[ S_3(q, D, X) = 1\{q'V(q) > 0\} - 1\{q'V(q_0) > 0\} \]

and note that

\[ \mu(q, D, X) - \mu_0(q, D, X) = S_1(D, X) + S_2(D, X)1\{q'\eta(D, X) > 0\} + \gamma_{UL,0}(D, X)S_3(q, D, X). \]

\[ \nabla_D(\mu(q, D, X) - \mu_0(q, D, X)) = \nabla_DS_1(D, X) + \nabla_DS_2(D, X)1\{q'\eta(D, X) > 0\} + \nabla_D\gamma_{UL,0}(D, X)S_3(q, D, X). \]

**Step 2.** I verify Assumption 3.5. The function class

\[ F_\xi = \{g(W, q, \xi(q)), q \in \mathcal{S}_d\} \subseteq \mathcal{H}_\xi + \mathcal{Y}_\eta - \mathcal{M}_q, \]

where

\[ \mathcal{Y}_\eta = (Y_U - Y_L) \cdot \mathcal{L}_\eta \cdot \mathcal{I}_\eta + Y_L \]

and

\[ \mathcal{M}_q = \{q'\mathcal{V}(\eta)\mu(q, D, X), \; q \in \mathcal{S}_d\} \subseteq \mathcal{L}_\eta \cdot (\gamma_L(D, X) + \gamma_{UL}(D, X) \cdot \mathcal{I}_\eta) \]

and

\[ \mathcal{H}_\xi = \{q'\nabla_D\mu(q, D, X), \; q \in \mathcal{S}_d\} \subseteq \mathcal{L}\nabla_D\gamma_L + \mathcal{L}\nabla_{UL}\mathcal{I}_\eta, \]

where all the classes above are P-Donsker and obey (A.12), which implies Assumption 3.5 (2). The integrable envelope \( F_\xi \) for \( F_\xi \) can be taken as

\[ F_\Xi := \|\nabla_D\gamma_L(D, X)\| + \|\nabla_D\gamma_{UL}(D, X)\| + \|V(\eta)(Y_L - \gamma_L,0(D, X) + M_\gamma) + \|V(\eta)(M_{UL} + M_\gamma), \]

which is integrable by Assumption 5.1.

**Step 3. Bound on \( \mu_N \).** Decompose the first-stage estimation error

\[ g(W, q, \xi(q)) - g(W, q, \xi_0(q)) = q'(\tilde{\gamma}_D\mu(q, D, X) - \tilde{\gamma}_D\mu_0(q, D, X)) + q'\mathcal{V}(\eta_0)(Y(q, \eta) - Y(q, \eta_0)) - q'\mathcal{V}(\eta_0)(\mu(q, D, X) - \mu_0(q, D, X)) + (q'\mathcal{V}(\eta) - q'\mathcal{V}(\eta_0))(Y(q, \eta) - Y(q, \eta_0)) - (q'\mathcal{V}(\eta) - q'\mathcal{V}(\eta_0))(\mu(q, D, X) - \mu_0(q, D, X)) = \sum_{j=1}^{5} K_j(W, q, \xi), \]

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Integration by parts implies
\[ \mathbb{E}[K_1(W, q, \xi) + K_3(W, q, \xi)] = 0 \quad \forall q \in \mathcal{S}^{d-1}. \]

Note that the terms \( K_2(W, q, \xi) \) and \( K_4(W, q, \xi) \) coincide with \( B_1(W, q, \eta) \) and \( B_2(W, q, \eta) \), that is,
\begin{align*}
K_2(W, q, \xi) &= K_2(W, q, \eta) = B_1(W, q, \eta),
K_4(W, q, \xi) &= K_4(W, q, \eta) = B_2(W, q, \eta).
\end{align*}

Lemma 5.2 has shown that
\[ \sup_{q \in T_N} \mathbb{E}K_2(W, q, \xi) + K_4(W, q, \xi) \leq 2MUL C_f(\eta_N^2). \]

Cauchy inequality gives
\begin{align*}
&\mathbb{E}(\mu(q, D, X) - \mu_0(q, D, X))^2 \\
&\leq 3(\mathbb{E}S^2_1(D, X) + \mathbb{E}S^2_2(D, X) + \mathbb{E}\gamma^2_{UL, 0}(D, X)S^2_\delta(q, D, X)) \\
&\leq 3(\mathbb{E}S^2_1(D, X) + \mathbb{E}S^2_2(D, X) + M^2_{UL} P(\mathcal{E}_{-}\{q\})) \\
&= O(\gamma_N^2 + \eta_N^2).
\end{align*}

Cauchy inequality gives
\begin{align*}
\sup_{\xi \in \Xi_N} \mathbb{E}|K_3(W, q, \xi)| &\leq \sqrt{\mathbb{E}}\|V(\eta) - V(\eta_0)\|^2 \sqrt{(\mathbb{E}(\mu(q, D, X) - \mu_0(q, D, X))^2 \\
&= O((\eta_N^2)^{3/2} + \eta_N^2 \gamma_N).
\end{align*}

**Step 4. Bound on** \( r_n' \). For \( \eta \in T_N \), the following bounds hold for \( j = 2 \) and \( j = 4 \) follow from (B.45)
\begin{align*}
\sup_{q \in T_N} \mathbb{E}K^2_j(W, q, \eta) &\leq \mathbb{E}\|V(\eta) - V(\eta_0)\|^2 (Y_U - U_L)^2 \{\mathcal{E}_{-}\{q\}\} \leq M^2_{UL}(\eta_N^2)^2, \quad j = 2, 4.
\end{align*}

For \( \sup_{q \in T_N} \|V(\eta)\| \leq \bar{C} \) a.s. and \( \sup_{q \in T_N} \|V(\eta) - V(\eta_0)\| \leq 2\bar{C} \) a.s., note that
\begin{align*}
\sup_{\xi \in \Xi_N} \mathbb{E}K^2_j(W, q, \xi) &\leq 4\bar{C}^2 \sup_{\xi \in \Xi_N} \mathbb{E}(\mu(q, D, X) - \mu_0(q, D, X))^2 = O(\gamma_N^2 + \eta_N^2), \quad j = 3, 5.
\end{align*}

and
\begin{align*}
\sup_{\xi \in \Xi_N} \mathbb{E}K^2_j(W, q, \xi) &\leq 3(\mathbb{E}\|\nabla_D \gamma_{UL}(D, X) - \nabla_D \gamma_{UL, 0}(D, X)\|^2 \\
&+ \mathbb{E}\|\nabla_D \gamma_{UL}(D, X) - \nabla_D \gamma_{UL, 0}(D, X)\|^2 \\
&+ C^2_{UL} P(\mathcal{E}_{-}\{q\})) = O(\gamma_N^2 + \eta_N^2)
\end{align*}

Collecting the bounds gives \( r_n' = O(\gamma_N + (\eta_N^2)^{1/2}) \).
Table C.1: Finite-sample performance of the non-ortho and ortho approaches (true first-stage)

|       | Non-ortho |         | Ortho |         |         |         |         |
|-------|-----------|---------|-------|---------|---------|---------|---------|
|       | Total     | Outer   | Inner | Rej.freq | Total   | Outer   | Inner   | Rej.freq |
| N     |           |         |       |         |         |         |         |         |
| 250   | 0.46      | 0.45    | 0.44  | 0.07    | 0.14    | 0.13    | 0.12    | 0.06    |
| 500   | 0.34      | 0.33    | 0.33  | 0.05    | 0.10    | 0.09    | 0.08    | 0.06    |
| 700   | 0.28      | 0.27    | 0.27  | 0.03    | 0.08    | 0.07    | 0.07    | 0.04    |
| 1,000 | 0.24      | 0.23    | 0.23  | 0.03    | 0.07    | 0.06    | 0.06    | 0.08    |

Panel A: Bracket width $\Delta = 1$

|       | Non-ortho |         | Ortho |         |         |         |         |
|-------|-----------|---------|-------|---------|---------|---------|---------|
|       | Total     | Outer   | Inner | Rej.freq | Total   | Outer   | Inner   | Rej.freq |
| N     |           |         |       |         |         |         |         |         |
| 250   | 0.49      | 0.46    | 0.45  | 0.06    | 0.18    | 0.15    | 0.14    | 0.08    |
| 500   | 0.36      | 0.34    | 0.33  | 0.05    | 0.12    | 0.10    | 0.10    | 0.07    |
| 700   | 0.29      | 0.27    | 0.27  | 0.03    | 0.10    | 0.09    | 0.08    | 0.03    |
| 1,000 | 0.25      | 0.23    | 0.23  | 0.03    | 0.09    | 0.08    | 0.07    | 0.09    |

Panel B: Bracket width $\Delta = 2$

|       | Non-ortho |         | Ortho |         |         |         |         |
|-------|-----------|---------|-------|---------|---------|---------|---------|
|       | Total     | Outer   | Inner | Rej.freq | Total   | Outer   | Inner   | Rej.freq |
| N     |           |         |       |         |         |         |         |         |
| 250   | 0.52      | 0.48    | 0.46  | 0.06    | 0.22    | 0.18    | 0.16    | 0.04    |
| 500   | 0.38      | 0.34    | 0.33  | 0.06    | 0.15    | 0.13    | 0.12    | 0.05    |
| 700   | 0.31      | 0.28    | 0.28  | 0.03    | 0.13    | 0.11    | 0.10    | 0.03    |
| 1,000 | 0.26      | 0.24    | 0.23  | 0.03    | 0.11    | 0.09    | 0.09    | 0.06    |

Panel C: Bracket width $\Delta = 3$

Notes. Results are based on 10,000 simulation runs. Panels A, B and C correspond to the bin width $\Delta = 1, 2, 3$. Table shows the total risk [6.6], the outer and inner risks [6.7], and the rejection frequency [6.8] for the nominal size $\alpha = 0.05$. The supremum over the unit circumference is approximated by the maximum over the grid consisting of 50 evenly spaced points. Columns (1–4) and (5–8) correspond to the non-orthogonal and the orthogonal second-stage. The number of bootstrap repetitions $B = 2,000$. The true support function is in [6.4]. The estimated support function is based on the true treatment fitted values (i.e., true $\alpha_1, \alpha_2$), zero outcome fitted values (non-ortho) and Lasso outcome fitted values (ortho).
Table C.2: Finite-sample performance of estimators

|                  | Lasso-based (non-ortho) | Series-based (ortho) |
|------------------|-------------------------|----------------------|
|                  | Total       | Outer   | Inner   | Rej.freq | Total       | Outer   | Inner   | Rej.freq |
| **N**            |            |         |         |          |            |         |         |          |
| 250              | 0.65       | 0.64    | 0.63    | 0.02     | 0.21       | 0.16    | 0.20    | 0.17     |
| 500              | 0.48       | 0.47    | 0.47    | 0.00     | 0.19       | 0.09    | 0.19    | 0.43     |
| 700              | 0.42       | 0.41    | 0.41    | 0.00     | 0.18       | 0.08    | 0.18    | 0.62     |
| 1,000            | 0.36       | 0.35    | 0.35    | 0.00     | 0.18       | 0.06    | 0.18    | 1.00     |
| **Panel A: Bracket width ∆ = 1** |             |         |         |          |            |         |         |          |
| 250              | 0.67       | 0.65    | 0.64    | 0.03     | 0.34       | 0.24    | 0.33    | 0.24     |
| 500              | 0.49       | 0.48    | 0.47    | 0.00     | 0.33       | 0.13    | 0.33    | 0.87     |
| 700              | 0.43       | 0.42    | 0.41    | 0.00     | 0.32       | 0.11    | 0.32    | 0.99     |
| 1,000            | 0.37       | 0.36    | 0.35    | 0.00     | 0.32       | 0.08    | 0.32    | 1.00     |
| **Panel B: Bracket width ∆ = 2** |             |         |         |          |            |         |         |          |
| 250              | 0.69       | 0.67    | 0.64    | 0.04     | 0.48       | 0.33    | 0.47    | 0.18     |
| 500              | 0.51       | 0.49    | 0.48    | 0.00     | 0.47       | 0.18    | 0.47    | 0.95     |
| 700              | 0.44       | 0.42    | 0.42    | 0.00     | 0.47       | 0.14    | 0.47    | 1.00     |
| 1,000            | 0.38       | 0.36    | 0.36    | 0.00     | 0.47       | 0.11    | 0.47    | 1.00     |
| **Panel C: Bracket width ∆ = 3** |             |         |         |          |            |         |         |          |

Notes. Results are based on 10,000 simulation runs. Panels A, B and C correspond to the bin width ∆ = 1, 2, 3. Table shows the total risk (6.6), the outer and inner risks (6.7), and the rejection frequency (6.8) for the nominal size α = 0.05. The supremum over the unit circumference is approximated by the maximum over the grid consisting of 50 evenly spaced points. The number of bootstrap repetitions B = 2,000. The true support function is in (6.4).

References

Ackerberg, D., Chen, X., Hahn, J., and Liao, Z. (2014). Asymptotic efficiency of semiparametric two-step gmm. *The Review of Economic Studies*, 81(3):919–943.

Andrews, D. (1994). Asymptotics for semiparametric econometric models via stochastic equicontinuity. *Handbook of Econometrics*, (4):2247–2294.

Andrews, D. and Shi, X. (2013). Inference based on conditional moment inequalities. *Econometrica*, 81:609–666.

Andrews, D. and Shi, X. (2017). Inference based on many conditional moment inequalities. *Journal of Econometrics*, 196:275–287.
Andrews, I., Roth, J., and Pakes, A. (2020). Inference for linear conditional moment inequalities.

Autor, D., Dorn, D., and Gordon, H. (2013). The china syndrome: Local labor market effects of import competition in the united states. *American Economic Review*, 103(6):2121–2168.

Belloni, A., Chernozhukov, V., Chetverikov, D., and Wei, Y. (2018). Uniformly valid post-regularization confidence regions for many functional parameters in Z-estimation framework. *The Annals of Statistics*, 46(6B):3643 – 3675.

Belloni, A., Chernozhukov, V., Fernandez-Val, I., and Hansen, C. (2017). Program evaluation and causal inference with high-dimensional data. *Econometrica*, 85:233–298.

Belloni, A., Chernozhukov, V., and Hansen, C. (2010). Lasso methods for gaussian instrumental variables models. *arXiv preprint arXiv:1012.1297*.

Beresteanu, A., Molchanov, I., and Molinari, F. (2011). Sharp identification regions in models with convex moment predictions. *Econometrica*, 79:1785–1821.

Beresteanu, A., Molchanov, I., and Molinari, F. (2012). Partial identification using random set theory. *Journal of Econometrics*, 166(1):17–32.

Beresteanu, A. and Molinari, F. (2012). Partial identification using random set theory. *Journal of Econometrics*, 166(1):17–32.

Beresteanu, A. and Sasaki, Y. (2020). Quantile regression with interval data. *Econometrics Reviews*.

Bontemps, C., Magnac, T., and Maurin, E. (2012). Set identified linear models. *Econometrica*, 80:1129–1155.

Bonvini, M. and Kennedy, E. H. (2021). Sensitivity analysis via the proportion of unmeasured confounding. *Journal of the American Statistical Association*, page 1–11.

Bugni, F., Canay, I., and Shi, X. (2017). Inference for functions of partially identified parameters in moment inequality models. *Quantitative Economics*, 8:1–38.

Cattaneo, M., Farrell, M., and Feng, Y. (2020). Large sample properties of partitioning-based series estimators. *The Annals of Statistics*, 48(3):1718–1741.

Cattaneo, M. D. and Farrell, M. H. (2013). Optimal convergence rates, bahadur representation, and asymptotic normality of partitioning estimators. *Journal of Econometrics*, 174:127–143.

Chandrasekhar, A., Chernozhukov, V., Molinari, F., and Schrimpf, P. (2012). Inference for best linear approximations to set identified functions. *arXiv e-prints*, page arXiv:1212.5627.
Chen, X. (2007). Large sample sieve estimation of semi-nonparametric models. 6B.

Chen, X. (2011). Penalized sieve estimation and inference of semi-nonparametric dynamic models: A selective review.

Chen, X., Tamer, E., and Christensen, T. (2018). Mcmc confidence sets for identified sets. *Econometrica*, 86:1965–2018.

Chernozhukov, V., Chetverikov, D., Demirer, M., Duflo, E., Hansen, C., Newey, W., and Robins, J. (2018). Double/debiased machine learning for treatment and structural parameters. *Econometrics Journal*, 21:C1–C68.

Chernozhukov, V., Chetverikov, D., and Kato, K. (2014). Gaussian approximation of suprema of empirical processes. *Annals of Statistics*, 42(4):1564–1597.

Chernozhukov, V., Escanciano, J. C., Ichimura, H., Newey, W. K., and Robins, J. M. (2016). Locally Robust Semiparametric Estimation. *arXiv e-prints*, page arXiv:1608.00033.

Chernozhukov, V., Hong, H., and Tamer, E. (2007). Estimation and confidence regions for parameter sets in econometric models. *Econometrica*, 75:1243–1284.

Chernozhukov, V., Lee, S., and Rosen, A. (2013). Intersection bounds: Estimation and inference. *Econometrica*, 81:667–737.

Chernozhukov, V., Newey, W., and Singh, R. (2018a). Debiased machine learning of global and local parameters using regularized riesz representers. *arXiv e-prints*, page arXiv:1802.08667.

Chernozhukov, V., Newey, W. K., and Singh, R. (2018b). Automatic debiased machine learning of causal and structural effects. *arXiv e-prints*, page arXiv:1809.05224.

Chernozhukov, V., Rigobon, R., and Stoker, T. (2010). Set identification and sensitivity analysis with Tobin regressors. *Quantitative Economics*, 1(6B):255–277.

Chiang, H. D., Kango, K., Ma, Y., and Sasaki, Y. (2019). Multiway cluster robust double/debiased machine learning.

Chiang, H. D., Kato, K., Ma, Y., and Sasaki, Y. (2019). Multiway Cluster Robust Double/Debiased Machine Learning. *arXiv e-prints*, page arXiv:1909.03489.

Cilibero, F. and Tamer, E. (2009). Market structure and multiple equilibria in airline markets. *Econometrica*, 77:1791–1828.

Colangelo, K. and Lee, Y.-Y. (2020). Double Debiased Machine Learning Nonparametric Inference with Continuous Treatments. *arXiv e-prints*, page arXiv:2004.03036.
References

Dong, B., Hsieh, Y.-W., and Shum, M. (2021). Computing moment inequality models using constrained optimization. *The Econometrics Journal*, 24(3):399–416.

Dudley, R. M. (1968). Distances of probability measures and random variables. *The Annals of Mathematical Statistics*, 39:1563–1572.

Fan, Q., Hsu, Y.-C., Lieli, R. P., and Zhang, Y. (2019). Estimation of conditional average treatment effects with high-dimensional data. *arXiv e-prints*, page arXiv:1908.02399.

Fan, Y. and Park, S. S. (2010). Sharp bounds on the distribution of treatment effects and their statistical inference. *Econometric Theory*, 26(3):931–951.

Fan, Y. and Shi, X. (2021). Wald, qlr, and score tests when parameters are subject to linear inequality constraints.

Fan, Y., Shi, X., and Tao, J. (2020). Partial identification and inference in moment models with incomplete data.

Farrell, M., Liang, T., and Misra, S. (2021). Deep neural networks for estimation and inference. *Econometrica*, 89(1):181–213.

Franguridi, G., Gafarov, B., and Wuthrich, K. (2021). Conditional quantile estimators: A small sample theory.

Gafarov, B. (2019). Inference in high-dimensional set-identified affine models.

Gafarov, B., Meier, M., and Montiel Olea, J. L. (2018). Delta-method inference for a class of set-identified svars. *Journal of Econometrics*, 203:316–327.

Gamper-Rabindran, S. and Timmins, C. (2013). Does cleanup of hazardous waste sites raise housing values? evidence of spatially localized benefits. *Journal of Environmental Economics and Management*, 65(3):345–360.

Haile, P. A. and Tamer, E. (2003). Inference with an incomplete model of english auctions. *Journal of Political Economy*, 111(1):1–51.

Hardle, W. and Stoker, T. (1989). Investigating smooth multiple regression by the method of average derivatives. *Journal of American Statistical Association*, 84(408):986–995.

Hasminskii, R. Z. and Ibragimov, I. A. (1979). On the nonparametric estimation of functionals. In *Proceedings of the Second Prague Symposium on Asymptotic Statistics*.

Honore, B. and Hu, L. (2020). Selection without exclusion. *Econometrica*, 88(88):1007–1029.

Hsieh, Y.-W., Shi, X., and Shum, M. (2021). Inference on estimators defined by mathematical programming. *Journal of Econometrics*. 

73
Javanmard, A. and Montanari, A. (2014). Confidence intervals and hypothesis testing for high-dimensional regression. *Journal of Machine Learning Research*, 2(4):2869–2909.

Jeong, S. and Namkoong, H. (2020). Robust causal inference under covariate shift via worst-case subpopulation treatment effects. *arXiv e-prints*, page arXiv:2007.02411.

Kaido, H. (2016). A dual approach to inference for partially identified econometric models. *Journal of Econometrics*, 192:269–290.

Kaido, H. (2017). Asymptotically efficient estimation of weighted average derivatives with an interval censored variable. *Econometric Theory*, 33(5):1218–1241.

Kaido, H., Molinari, F., and Stoye, J. (2019). Confidence intervals for projections of partially identified parameters. *Econometrica*, 87(4):1397–1432.

Kaido, H., Molinari, F., and Stoye, J. (2021). Constraint qualifications in partial identification. *Econometric Theory*.

Kaido, H. and Santos, A. (2014). Asymptotically efficient estimation of models defined by convex moment inequalities. *Econometrica*, 82(1):387–413.

Kaido, H. and White, H. (2012). Estimating misspecified moment inequality models. *Recent Advances and Future Directions in Causality, Prediction, and Specification Analysis: Essays in Honour of Halbert L. White Jr*.

Kaido, H. and White, H. (2014). A two-stage procedure for partially identified models. *Journal of Econometrics*, 182:5–13.

Kallus, N., Mao, X., and Uehara, M. (2020). Localized debiased machine learning: Efficient inference on quantile treatment effects and beyond.

Kallus, N. and Zhou, A. (2019). Assessing disparate impacts of personalized interventions: Identifiability and bounds.

Kaplan, D. M. and Sun, Y. (2017). Smoothed estimating equations for instrumental variables quantile regression. *Econometric Theory*, 33(1):105–157.

Kasy, M. (2016). Partial identification, distributional preferences, and the welfare ranking of policies. *Review of Economics and Statistics*, 98:111–131.

Kitagawa, T. and Teteno, A. (2018). Who should be treated? empirical welfare maximization methods for treatment choice. *Econometrica*, 86:591–616.

Kline, B. and Tamer, E. (2016). Bayesian inference in a class of partially identified models. *Quantitative Economics*, 7:329–366.

Lee, D. (2009). Training, wages, and sample selection: Estimating sharp bounds on treatment effects. *Review of Economic Studies*, 76(3):1071–1102.
Li, Q., Molchanov, I., Molinari, F., and Peng, S. (2021). Local regression smoothers with set-valued outcome data. *International Journal of Approximate Reasoning*, 128:129–150.

Luo, Y. and Spindler, M. (2016). High-dimensional $L_2$-boosting: rate of convergence. *arXiv e-prints*, page arXiv:1602.08927.

Mammen, E. and Tsybakov, A. B. (1999). Smooth discrimination analysis. *The Annals of Statistics*, 27(6):1808 – 1829.

Manski, C. and Pepper, J. (2000). Monotone instrumental variables: With an application to the returns to schooling. *Econometrica*, 68(4):997–1010.

Manski, C. and Tamer, E. (2002). Inference on regressions with interval data on a regressor or outcome. *Econometrica*, 70(2):519–546.

Manski, C. F. (1990). Nonparametric bounds on treatment effects. *The American Economic Review*, 80(2):319–323.

Mbakop, E. and Tabord-Meehan, M. (2021). Model selection for treatment choice: Penalized welfare maximization. *Econometrica*, 89:825–848.

Molchanov, I. and Molinari, F. (2018). *Random Sets in Econometrics*. Cambridge University Press.

Molinari, F. (2008). Partial identification of probability distributions with misclassified data. *Journal of Econometrics*, 144:81–117.

Mulligan, C. B. and Rubinstein, Y. (2008). Selection, Investment, and Women’s Relative Wages Over Time*. *The Quarterly Journal of Economics*, 123(3):1061–1110.

Nekipelov, D., Semenova, V., and Syrgkanis, V. (2022). Regularized orthogonal estimation of nonlinear semiparametric models.

Newey, W. (1994). The asymptotic variance of semiparametric estimators. *Econometrica*, 62(6):245–271.

Newey, W. and Stoker, T. (1993). Efficiency of weighted average derivative estimators and index models. *Econometrica*, 61(5):1199–1223.

Neyman, J. (1959). Optimal asymptotic tests of composite statistical hypotheses. *Probability and Statistics*, 213(57):416–444.

Neyman, J. (1979). $c(\alpha)$ tests and their use. *Sankhya*, pages 1–21.

Ning, Y., Peng, S., and Tao, J. (2020). Doubly robust semiparametric difference-in-differences estimators with high-dimensional data.

Pakes, A., Porter, J., Ho, K., and Ishii, J. (2015). Moment inequalities and their application. *Econometrica*, 83:315–334.
Pitman, J. and Ross, N. (2012). Archimedes, Gauss, and Stein.

Powell, J. L. (1984). Least absolute deviations estimation for the censored regression model. *Journal of Econometrics*, 25:303–325.

Powell, J. L. (1986). Censored regression quantiles. *Journal of Econometrics*, 32:143–155.

Powell, J. L., Stock, J. H., and Stoker, T. M. (1989). Semiparametric estimation of index coefficients. *Econometrica*, 57(6):1403–1430.

Robins, J. and Rotnitzky, A. (1995). Semiparametric efficiency in multivariate regression models with missing data. *Journal of American Statistical Association*, 90(429):122–129.

Robinson, P. M. (1988). Root-n-consistent semiparametric regression. *Econometrica*, 56(4):931–954.

Sasaki, Y. and Ura, T. (2020). Estimation and inference for Policy Relevant Treatment Effects. *Journal of Econometrics*.

Sasaki, Y., Ura, T., and Zhang, Y. (2020). Unconditional quantile regression with high-dimensional data. *arXiv e-prints*, page arXiv:2007.13659.

Schick, A. (1986). On asymptotically efficient estimation in semiparametric models. *The Annals of Statistics*, 14(3):1139–1151.

Schmidt-Hieber, J. (2017). Nonparametric regression using deep neural networks with ReLU activation function. *arXiv e-prints*, page arXiv:1708.06633.

Semenova, V. (2020). Generalized lee bounds.

Semenova, V. and Chernozhukov, V. (2021). Debiased machine learning of conditional average treatment effect and other causal functions. *Econometrics Journal*, (24).

Semenova, V. and Chernozhukov, V. (2021). Debiased machine learning of conditional average treatment effects and other causal functions.

Semenova, V., Goldman, M., Chernozhukov, V., and Taddy, M. (2017). Estimation and inference about heterogeneous treatment effects in high-dimensional dynamic panels. *arXiv e-prints*, page arXiv:1712.09988.

Shi, X. and Shum, M. (2015). Simple two-stage inference for a class of partially identified models. *Econometric Theory*, 31:493–520.

Shi, X., Shum, M., and Song, W. (2018). Estimating semi-parametric panel multinomial choice models using cyclic monotonicity. *Econometrica*, 86:737–761.

Singh, R. and Sun, L. (2020). De-biased machine learning in instrumental variable models for treatment effects.
Skorohod, A. V. (1956). Limit theorems for stochastic processes. *Theo. Probab. Applications*, 1:261–290.

Stoye, J. (2009). More on confidence intervals for partially identified parameters. *Econometrica*, 77(4):1299–1315.

Syrgkanis, V., Tamer, E., and Ziani, J. (2018). Inference on auctions with weak assumptions on information.

Syrgkanis, V. and Zampetakis, M. (2020). Estimation and inference with trees and forests in high dimensions.

Tamer, E. (2010). Partial identification in econometrics. *Annual Reviews of Economics*, 2(1):167–195.

Torgovitsky, A. (2019). Partial identification by extending subdistributions. *Quantitative Economics*, 10:105–144.

Trostel, P., Walker, I., and Woolley, P. (2002). Estimates of the economic return to schooling for 28 countries. *Labour Economics*, 9(1):1–16.

Tsybakov, A. B. (2004). Optimal aggregation of classifiers in statistical learning. *The Annals of Statistics*, 32(1):135 – 166.

van der Vaart, A. (2000). *Asymptotic Statistics*. Cambridge University Press.

Vershynin, R. (2018). *High-Dimensional Probability: An Introduction with Applications in Data Science*. Cambridge University Press.

Wager, S. and Walther, G. (2015). Adaptive concentration of regression trees, with application to random forests. *arXiv e-prints*, page arXiv:1503.06388.

Wichura, M. (1970). On the construction of almost uniformly convergent random variables with given weakly convergent image laws. *The Annals of Mathematical Statistics*, 41:284–291.

Zhang, C.-H. and Zhang, S. (2014). Confidence intervals for low-dimensional parameters in high-dimensional linear models. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 76(1):217–242.

Zheng, W. and van der Laan, M. J. (2010). Asymptotic theory for cross-validated targeted maximum likelihood estimation. Technical report, UC Berkeley Division of Biostatistics.

Zimmert, M. and Lechner, M. (2019). Nonparametric estimation of causal heterogeneity under high-dimensional confounding. *arXiv e-prints*, page arXiv:1908.08779.