THE FOKAS METHOD TO THE SASA-SATSUMA EQUATION ON THE HALF-LINE

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Abstract. We present a Riemann-Hilbert problem formalism for the initial-boundary value problem for the Sasa-Satsuma(SS) equation: on the half-line. And we also analysis the global relation in this paper.

1. Introduction

Several of the most important PDEs in mathematics and physics are integrable. Integrable PDEs can be analyzed by means of the Inverse Scattering Transform (IST) formalism. Until the 1990s the IST methodology was pursued almost entirely for pure initial value problems. However, in many laboratory and field situations, the wave motion is initiated by what corresponds to the imposition of boundary conditions rather than initial conditions. This naturally leads to the formulation of an initial-boundary value (IBV) problem instead of a pure initial value problem.

In 1997, Fokas announced a new unified approach for the analysis of IBV problems for linear and nonlinear integrable PDEs [1, 2](see also [3]). The Fokas method provides a generalization of the IST formalism from initial value to IBV problems, and over the last fifteen years, this method has been used to analyze boundary value problems for several of the most important integrable equations with $2 \times 2$ Lax pairs, such as the Kortewegde Vries, the nonlinear Schrödinger, the sine-Gordon, and the stationary axisymmetric Einstein equations, see

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e.g. [4, 9]. Just like the IST on the line, the unified method yields an expression for the solution of an IBV problem in terms of the solution of a Riemann-Hilbert problem. In particular, the asymptotic behavior of the solution can be analyzed in an effective way by using this Riemann-Hilbert problem and by employing the nonlinear version of the steepest descent method introduced by Deift and Zhou [15].

It is well known that the nonlinear Schrödinger (NLS) equation

\[ iq_T + \frac{1}{2} q_{XX} + |q|^2 q = 0 \]  \hspace{1cm} (1.1)

describes slowly varying wave envelopes in dispersive media and arises in various physical systems such as water waves, plasma physics, solid-state physics and nonlinear optics. One of the most successful among them is the description of optical solitons in fibers. But, by the advancement of expermenal accuracy, several phenomena which can not be explained by equation (1.1) have been observed. In order to understand such phenomena, Kodama and Hasegawa proposed a higher-order nonlinear Schrödinger equation

\[ iq_T + \frac{1}{2} q_{XX} + |q|^2 q + i \varepsilon \{ \beta_1 q_{xxx} + \beta_2 |q|^2 q_X + \beta_3 q(|q|^2)_X \} = 0. \]  \hspace{1cm} (1.2)

In general, equation (1.2) may not be completely integrable. However, if some restrictions are imposed on the real parameters \( \beta_1, \beta_2 \) and \( \beta_3 \), then we can apply the IST to solve its initial value problems. Until now, the following four cases besides the NLS equation itself are known to be solvable:

- the derivative NLS equation-type I(\( \beta_1 : \beta_2 : \beta_3 = 0:1:1 \)),
- the derivative NLS equation-type II(\( \beta_1 : \beta_2 : \beta_3 = 0:1:0 \)),
- the Hirota equation(\( \beta_1 : \beta_2 : \beta_3 = 1:6:0 \)),
- the Sasa-Satsuma equation(\( \beta_1 : \beta_2 : \beta_3 = 1:6:3 \)).

\[ iq_T + \frac{1}{2} q_{XX} + |q|^2 q + i \varepsilon (q_{XXX} + 6|q|^2 q_X + 3q(|q|^2)_X) = 0 \]  \hspace{1cm} (1.3)

Recently, Lenells develop a methodology for analyzing IBV problems for integrable evolution equations with Lax pairs involving \( 3 \times 3 \) matrices [12]. He also used this method to analyze the Degasperis-Procesi
equation in [13]. In this paper we analyze the initial-boundary value problem of the Sasa-Satsuma equation on the half-line by using this method. The IST formalism for the initial value problem of the Sasa-Satsuma equation has been obtained in [10].

According to [10] we introduce variable transformations,

\[ u(x, t) = q(X) \exp \left\{ -i \frac{T}{6\epsilon} (X - \frac{T}{18\epsilon}) \right\}, \]  
\[ t = T, \]  
\[ x = X - \frac{T}{12\epsilon}. \]

Then equation (1.2) is reduced to a complex modified KdV-type equation

\[ u_t + \epsilon \{ u_{xxx} + 6|u|^2u_x + 3u(|u|^2)_x \} = 0. \] (1.5)

**Organization of the paper.** In section 2 we perform the spectral analysis of the associated Lax pair. And we formulate the main Riemann-Hilbert problem in section 3. We also analysis the global relation in section 4.

2. **Spectral analysis**

The Lax pair of equation (1.5) is [10],

\[ \Psi_x = UP, \quad \Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix}, \]  
\[ \Psi_t = VP. \] (2.1a)

where

\[ U = -ik\Lambda + V_1. \] (2.2)

and

\[ V = -4i\epsilon k^3\Lambda + V_2. \] (2.3)
here
\[
\Lambda = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix},
V_1 = \begin{pmatrix}
0 & 0 & u \\
0 & 0 & \bar{u} \\
-\bar{u} & -u & 0
\end{pmatrix},
V_2 = k^2 V_2^{(2)} + k V_2^{(1)} + V_2^{(0)}.
\]

(2.4)

where
\[
V_2^{(2)} = 4 \varepsilon \begin{pmatrix}
0 & 0 & u \\
0 & 0 & \bar{u} \\
-\bar{u} & -u & 0
\end{pmatrix},
V_2^{(1)} = 2i \varepsilon \begin{pmatrix}
|u|^2 & u^2 & u_x \\
\bar{u}^2 & |\bar{u}|^2 & \bar{u}_x \\
\bar{u}_x & u_x & -2|u|^2
\end{pmatrix},
V_2^{(0)} = -4|u|^2 \varepsilon \begin{pmatrix}
0 & 0 & u \\
0 & 0 & \bar{u} \\
-\bar{u} & -u & 0
\end{pmatrix} - \varepsilon \begin{pmatrix}
0 & 0 & u_{xx} \\
0 & 0 & \bar{u}_{xx} \\
-\bar{u}_{xx} & -u_{xx} & 0
\end{pmatrix} + \varepsilon (\bar{u}u_x - u\bar{u}_x) \begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

(2.5)

In the following, we let \( \varepsilon = 1 \) for the convenient of the analysis.

2.1. The closed one-form. Suppose that \( u(x, t) \) is sufficiently smooth function of \((x, t)\) in the half-line domain \( \Omega = \{0 < x < \infty, 0 < t < T\} \)
which decay as \( x \to \infty \). Introducing a new eigenfunction \( \mu(x, t, k) \) by
\[
\Psi = \mu e^{-i\Lambda k x - 4i k^3 t}
\]

(2.6)

then we find the Lax pair equations
\[
\begin{cases}
\mu_x + [ik\Lambda, \mu] = V_1 \mu, \\
\mu_t + [4ik^3\Lambda, \mu] = V_2 \mu.
\end{cases}
\]

(2.7)

the equations in (A.2) can be written in differential form as
\[
d(e^{i(kx+4ik^3t)}\hat{\Lambda} \mu) = W,
\]

(2.8)

where \( W(x, t, k) \) is the closed one-form defined by
\[
W = e^{i(kx+4ik^3t)}(V_1 dx + V_2 dt)\mu.
\]

(2.9)
2.2. The $\mu_j$’s. We define three eigenfunctions $\{\mu_j\}^3_1$ of (A.2) by the Volterra integral equations

$$
\mu_j(x, t, k) = I + \int_{\gamma_j} e^{(-ikx-4ik^3t)\Lambda} W_j(x', t', k). \quad j = 1, 2, 3. \quad (2.10)
$$

where $W_j$ is given by (2.9) with $\mu$ replaced with $\mu_j$, and the contours $\{\gamma_j\}^3_1$ are showed in Figure 1. The first, second and third column of the matrix equation (2.10) involves the exponentials

$$
[\mu_j]_1: e^{2ik(x-x')+8ik^3(t-t')},
[\mu_j]_2: e^{2ik(x-x')+8ik^3(t-t')},
[\mu_j]_3: e^{-2ik(x-x')-8ik^3(t-t')}, e^{-2ik(x-x')-8ik^3(t-t')}.
$$

And we have the following inequalities on the contours:

$$
\gamma_1: x - x' \geq 0, t - t' \leq 0,
\gamma_2: x - x' \geq 0, t - t' \geq 0,
\gamma_3: x - x' \leq 0.
$$

And we have the following inequalities on the contours:

So, these inequalities imply that the functions $\{\mu_j\}^3_1$ are bounded and analytic for $k \in \mathbb{C}$ such that $k$ belongs to

$$
\mu_1: (D_2, D_2, D_3),
\mu_2: (D_1, D_1, D_4),
\mu_3: (D_3 \cup D_4, D_3 \cup D_4, D_1 \cup D_2).
$$

where $\{D_n\}^4_1$ denote four open, pairwisely disjoint subsets of the Riemann $k$–sphere showed in Figure 2. And the sets $\{D_n\}^4_1$ has the fol-
Figure 2. The sets $D_n$, $n = 1, \ldots, 4$, which decompose the complex $k$–plane.

Following properties:

\[ D_1 = \{ k \in \mathbb{C} | \text{Re}l_1 = \text{Re}l_2 > \text{Re}l_3, \text{Re}z_1 = \text{Re}z_2 > \text{Re}z_3 \}, \]
\[ D_2 = \{ k \in \mathbb{C} | \text{Re}l_1 = \text{Re}l_2 > \text{Re}l_3, \text{Re}z_1 = \text{Re}z_2 < \text{Re}z_3 \}, \]
\[ D_1 = \{ k \in \mathbb{C} | \text{Re}l_1 = \text{Re}l_2 < \text{Re}l_3, \text{Re}z_1 = \text{Re}z_2 > \text{Re}z_3 \}, \]
\[ D_1 = \{ k \in \mathbb{C} | \text{Re}l_1 = \text{Re}l_2 < \text{Re}l_3, \text{Re}z_1 = \text{Re}z_2 < \text{Re}z_3 \}, \]

where $l_i(k)$ and $z_i(k)$ are the diagonal entries of matrices $-ik\Lambda$ and $-4ik^3\Lambda$, respectively.

In fact, for $x = 0$, $\mu_1(0, t, k)$ has enlarged domain of boundedness: $(D_2 \cup D_4, D_2 \cup D_4, D_1 \cup D_3)$, and $\mu_2(0, t, k)$ has enlarged domain of boundedness: $(D_1 \cup D_3, D_1 \cup D_3, D_2 \cup D_4)$.

2.3. The $M_n$’s. For each $n = 1, \ldots, 4$, define a solution $M_n(x, t, k)$ of (A.2) by the following system of integral equations:

\[
(M_n)_{ij}(x, t, k) = \delta_{ij} + \int_{D_n} (e^{(-ikx-4ik^3t)\hat{\Lambda}}W_n(x', t', k))_{ij}, \quad k \in D_n, \quad i, j = 1, 2, 3.
\]

(2.14)
where $W_n$ is given by (2.9) with $\mu$ replaced with $M_n$, and the contours $\gamma_{n}^{ij}$, $n = 1, \ldots, 4$, $i, j = 1, 2, 3$ are defined by

\[
\gamma_{n}^{ij} = \begin{cases} 
\gamma_1 & \text{if } \text{Re}(i(k)) < \text{Re}(j(k)) \text{ and } \text{Re}(z_i(k)) \geq \text{Re}(z_j(k)), \\
\gamma_2 & \text{if } \text{Re}(i(k)) < \text{Re}(j(k)) \text{ and } \text{Re}(z_i(k)) < \text{Re}(z_j(k)), \\
\gamma_3 & \text{if } \text{Re}(i(k)) \geq \text{Re}(j(k)) \text{.}
\end{cases}
\]

for $k \in \bar{D}_n$.

(2.15)

The following proposition ascertains that the $M_n$’s defined in this way have the properties required for the formulation of a Riemann-Hilbert problem.

Proposition 2.1. For each $n = 1, \ldots, 4$, the function $M_n(x, t, k)$ is well-defined by equation (2.14) for $k \in \bar{D}_n$ and $(x, t) \in \Omega$. For any fixed point $(x, t)$, $M_n$ is bounded and analytic as a function of $k \in D_n$ away from a possible discrete set of singularities $\{k_j\}$ at which the Fredholm determinant vanishes. Moreover, $M_n$ admits a bounded and continuous extension to $\bar{D}_n$ and

\[
M_n(x, t, k) = \mathbb{I} + O\left(\frac{1}{k}\right), \quad k \to \infty, \quad k \in D_n.
\]  

(2.16)

Proof. The boundedness and analyticity properties are established in appendix B in [12]. And substituting the expansion

\[
M = M_0 + \frac{M^{(1)}}{k} + \frac{M^{(2)}}{k^2} + \cdots, \quad k \to \infty.
\]

into the Lax pair (A.2) and comparing the terms of the same order of $k$ yield the equation (2.16). \qed

2.4. The jump matrices. We define spectral functions $S_n(k)$, $n = 1, \ldots, 4$, and

\[
S_n(k) = M_n(0, 0, k), \quad k \in D_n, \quad n = 1, \ldots, 4.
\]

(2.17)

Let $M$ denote the sectionally analytic function on the Riemann $k$–sphere which equals $M_n$ for $k \in D_n$. Then $M$ satisfies the jump conditions

\[
M_n = M_m J_{m,n}, \quad k \in \bar{D}_n \cap \bar{D}_m, \quad n, m = 1, \ldots, 4, \quad n \neq m,
\]

(2.18)
where the jump matrices $J_{m,n}(x,t,k)$ are defined by

$$J_{m,n} = e^{(-ikx - 4ik^3t)\hat{\Lambda}}(S_m^{-1}S_n).$$

(2.19)

According to the definition of the $\gamma^n$, we find that

$$\gamma^1 = \begin{pmatrix} \gamma_3 & \gamma_3 & \gamma_3 \\ \gamma_3 & \gamma_3 & \gamma_3 \\ \gamma_2 & \gamma_2 & \gamma_3 \end{pmatrix},$$

$$\gamma^2 = \begin{pmatrix} \gamma_3 & \gamma_3 & \gamma_3 \\ \gamma_3 & \gamma_3 & \gamma_3 \\ \gamma_1 & \gamma_1 & \gamma_3 \end{pmatrix},$$

$$\gamma^3 = \begin{pmatrix} \gamma_3 & \gamma_3 & \gamma_1 \\ \gamma_3 & \gamma_3 & \gamma_1 \\ \gamma_3 & \gamma_3 & \gamma_3 \end{pmatrix},$$

$$\gamma^4 = \begin{pmatrix} \gamma_3 & \gamma_3 & \gamma_2 \\ \gamma_3 & \gamma_3 & \gamma_2 \\ \gamma_3 & \gamma_3 & \gamma_3 \end{pmatrix}.$$

(2.20)

2.5. The adjugated eigenfunctions. We will also need the analyticity and boundedness properties of the minors of the matrices $\{\mu_j(x,t,k)\}_1^3$. We recall that the adjugate matrix $X^A$ of a $3 \times 3$ matrix $X$ is defined by

$$X^A = \begin{pmatrix} m_{11}(X) & -m_{12}(X) & m_{13}(X) \\ -m_{21}(X) & m_{22}(X) & -m_{23}(X) \\ m_{31}(X) & -m_{32}(X) & m_{33}(X) \end{pmatrix},$$

where $m_{ij}(X)$ denote the $(ij)$th minor of $X$.

It follows from (A.2) that the adjugated eigenfunction $\mu^A$ satisfies the Lax pair

$$\begin{cases} 
\mu_x^A - [ik\hat{\Lambda}, \mu^A] = -V_1^T \mu^A, \\
\mu_t^A - [4ik^3\hat{\Lambda}, \mu^A] = -V_2^T \mu^A.
\end{cases}$$

(2.21)

where $V^T$ denote the transform of a matrix $V$. Thus, the eigenfunctions $\{\mu_j^A\}_1^3$ are solutions of the integral equations

$$\mu_j^A(x,t,k) = \mathbb{I} - \int_{\gamma_j} e^{ik(x-x') + 4ik^3(t-t')\hat{\Lambda}}(V_1^Tdx + V_2^T)\mu^A, \quad j = 1,2,3.$$

(2.22)

Then we can get the following analyticity and boundedness properties:

$$\begin{align*}
\mu_1^A &: (D_3, D_3, D_2), \\
\mu_2^A &: (D_4, D_4, D_1), \\
\mu_3^A &: (D_1 \cup D_2, D_1 \cup D_2, D_3 \cup D_4).
\end{align*}$$

(2.23)
In fact, for \( x = 0 \), \( \mu_1^A(0, t, k) \) has enlarged domain of boundedness: \((D_1 \cup D_3, D_1 \cup D_3, D_2 \cup D_4)\), and \( \mu_2^A(0, t, k) \) has enlarged domain of boundedness: \((D_2 \cup D_4, D_2 \cup D_4, D_1 \cup D_3)\).

2.6. The \( J_{m,n} \)'s computation. Let us define the \( 3 \times 3 \)-matrix value spectral functions \( s(k) \) and \( S(k) \) by

\[
\mu_3(x, t, k) = \mu_2(x, t, k)e^{(-ikx-4ik^3)t}A_s(k), \quad (2.24a)
\]

\[
\mu_1(x, t, k) = \mu_2(x, t, k)e^{(-ikx-4ik^3)t}A S(k), \quad (2.24b)
\]

Thus,

\[
s(k) = \mu_3(0, 0, k), \quad S(k) = \mu_1(0, 0, k). \quad (2.25)
\]

And we deduce from the properties of \( \mu_j \) and \( \mu_j^A \) that \( s(k) \) and \( S(k) \) have the following boundedness properties:

\[
s(k) : (D_3 \cup D_4, D_3 \cup D_4, D_1 \cup D_2),
\]

\[
S(k) : (D_2 \cup D_4, D_2 \cup D_4, D_1 \cup D_3),
\]

\[
s^A(k) : (D_1 \cup D_2, D_1 \cup D_2, D_3 \cup D_4),
\]

\[
S^A(k) : (D_1 \cup D_3, D_1 \cup D_3, D_2 \cup D_4).
\]

Moreover,

\[
M_n(x, t, k) = \mu_2(x, t, k)e^{(-ikx-4ik^3)t}A s_n(k), \quad k \in D_n. \quad (2.26)
\]

**Proposition 2.2.** The \( S_n \) can be expressed in terms of the entries of \( s(k) \) and \( S(k) \) as follows:

\[
S_1 = \begin{pmatrix}
m_{22}(s) & m_{21}(s) & s_{13} \\
m_{12}(s) & m_{11}(s) & s_{23} \\
s_{33} & 0 & s_{33}
\end{pmatrix},
\]

\[
S_2 = \begin{pmatrix}
m_{22}(s)m_{33}(s) - m_{32}(s)m_{23}(s) & m_{21}(s)m_{33}(s) - m_{31}(s)m_{23}(s) & s_{13} \\
m_{12}(s)m_{33}(s) - m_{32}(s)m_{13}(s) & m_{11}(s)m_{33}(s) - m_{31}(s)m_{13}(s) & s_{23} \\
m_{12}(s)m_{33}(s) - m_{32}(s)m_{13}(s) & m_{11}(s)m_{33}(s) - m_{31}(s)m_{13}(s) & s_{33}
\end{pmatrix}. \quad (2.27a)
\]
\[
S_3 = \begin{pmatrix}
  s_{11} & s_{12} & \frac{S_3}{(S^3 s^4)_3x}
s_{21} & s_{22} & \frac{S_3}{(S^3 s^4)_3x}
s_{31} & s_{32} & \frac{S_3}{(S^3 s^4)_3x}
\end{pmatrix}, \quad S_4 = \begin{pmatrix}
  s_{11} & s_{12} & 0 \\
  s_{21} & s_{22} & 0 \\
  s_{31} & s_{32} & \frac{1}{\max(s)}
\end{pmatrix}.
\]

(2.27b)

Proof. Let \(\gamma_3^{X_0}\) denote the contour \((X_0, 0) \to (x, t)\) in the \((x, t)\)-plane, here \(X_0 > 0\) is a constant. We introduce \(\mu_3(x, t, k; X_0)\) as the solution of (2.10) with \(j = 3\) and with the contour \(\gamma_3\) replaced by \(\gamma_3^{X_0}\). Similarly, we define \(M_n(x, t, k; X_0)\) as the solution of (2.14) with \(\gamma_3\) replaced by \(\gamma_3^{X_0}\). We will first derive expression for \(S_n(k; X_0) = M_n(0, 0, k; X_0)\) in terms of \(S(k)\) and \(s(k; X_0) = \mu_3(0, 0, k; X_0)\). Then (2.27) will follow by taking the limit \(X_0 \to \infty\).

First, we have the following relations:

\[
\begin{align*}
M_n(x, t, k; X_0) &= \mu_1(x, t, k)e^{(i k x - 4 i k^3 t)\Lambda} R_n(k; X_0), \\
M_n(x, t, k; X_0) &= \mu_2(x, t, k)e^{(i k x - 4 i k^3 t)\Lambda} S_n(k; X_0), \\
M_n(x, t, k; X_0) &= \mu_3(x, t, k)e^{(i k x - 4 i k^3 t)\Lambda} T_n(k; X_0).
\end{align*}
\]

(2.28)

Then we get \(R_n(k; X_0)\) and \(T_n(k; X_0)\) are defined as follows:

\[
R_n(k; X_0) = e^{4 i k^3 T \Lambda} M_n(0, T, k; X_0),
\]

(2.29a)

\[
T_n(k; X_0) = e^{i k x \Lambda} M_n(X_0, 0, k; X_0).
\]

(2.29b)

The relations (2.28) imply that

\[
s(k; X_0) = S_n(k; X_0) T_n^{-1}(k; X_0), \quad S(k) = S_n(k; X_0) R_n^{-1}(k; X_0).
\]

(2.30)

These equations constitute a matrix factorization problem which, given \(\{s, S\}\) can be solved for the \(\{R_n, S_n, T_n\}\). Indeed, the integral equations (2.14) together with the definitions of \(\{R_n, S_n, T_n\}\) imply that

\[
\begin{align*}
(R_n(k; X_0))_{ij} = 0 & \quad if \quad \gamma_{ij}^{n} = \gamma_1, \\
(S_n(k; X_0))_{ij} = 0 & \quad if \quad \gamma_{ij}^{n} = \gamma_2, \\
(T_n(k; X_0))_{ij} = 0 & \quad if \quad \gamma_{ij}^{n} = \gamma_3.
\end{align*}
\]

(2.31)

It follows that (2.30) are 18 scalar equations for 18 unknowns. By computing the explicit solution of this algebraic system, we find that
The spectral functions \( S(k) \) and \( s(k) \) are not independent but satisfy an important relation. Indeed, it follows from (2.24) that

\[
\mu_1(x, t, k) e^{(-ikx - 4ik^3t)\hat{\Lambda}} S^{-1}(k) s(k) = \mu_3(x, t, k), \quad k \in (D_3 \cup D_4, D_3 \cup D_4, D_1 \cup D_2). 
\]

(2.32)

Since \( \mu_1(0, T, k) = \mathbb{I} \), evaluation at \((0, T)\) yields the following global relation:

\[
S^{-1}(k) s(k) = e^{4ik^3T\hat{\Lambda}} c(T, k), \quad k \in (D_3 \cup D_4, D_3 \cup D_4, D_1 \cup D_2). 
\]

(2.33)

where \( c(T, k) = \mu_3(0, T, k) \).

2.8. The residue conditions. Since \( \mu_2 \) is an entire function, it follows from (2.26) that \( M \) can only have singularities at the points where the \( S_n \)'s have singularities. We infer from the explicit formulas (2.27) that the possible singularities of \( M \) are as follows:

- \([M]_1\) could have poles in \( D_1 \cup D_2 \) at the zeros of \( s_{33}(k) \);
- \([M]_1\) could have poles in \( D_2 \) at the zeros of \((s^T S^A)_{33}(k)\);
- \([M]_2\) could have poles in \( D_1 \cup D_2 \) at the zeros of \( s_{33}(k) \);
- \([M]_2\) could have poles in \( D_2 \) at the zeros of \((s^T S^A)_{33}(k)\);
- \([M]_3\) could have poles in \( D_3 \) at the zeros of \((S^T s^A)_{33}(k)\);
- \([M]_3\) could have poles in \( D_3 \cup D_4 \) at the zeros of \( m_{33}(s)(k) \);

We denote the above possible zeros by \( \{k_j\}_1^N \) and assume they satisfy the following assumption.

**Assumption 2.3.** We assume that

- \( s_{33}(k) \) has \( n_0 \) possible simple zeros in \( D_1 \) denoted by \( \{k_j\}_{1}^{n_0} \);
- \( s_{33}(k) \) has \( n_1 - n_0 \) possible simple zeros in \( D_2 \) denoted by \( \{k_j\}_{n_0 + 1}^{n_1} \);
- \((s^T S^A)_{33}(k)\) has \( n_2 - n_1 \) possible simple zeros in \( D_2 \) denoted by \( \{k_j\}_{n_1 + 1}^{n_2} \).
(\textit{S}^T \textit{s}^A)_{33}(k) \text{ has } n_3 - n_2 \text{ possible simple zeros in } D_3 \text{ denoted by } \{k_j\}_{n_2+1}^{n_3}; \\
m_{33}(s)(k) \text{ has } n_4 - n_3 \text{ possible simple zeros in } D_3 \text{ denoted by } \{k_j\}_{n_3+1}^{n_4}; \\
m_{33}(s)(k) \text{ has } n_5 - n_4 \text{ possible simple zeros in } D_3 \text{ denoted by } \{k_j\}_{n_4+1}^{n_5}; \\
m_{33}(s)(k) \text{ has } N - n_5 \text{ possible simple zeros in } D_4 \text{ denoted by } \{k_j\}_{n_5+1}^{N}; \\
and that none of these zeros coincide. Moreover, we assume that none of these functions have zeros on the boundaries of the } D_n \text{'s.}

We determine the residue conditions at these zeros in the following:

**Proposition 2.4.** Let \( \{M_n\}_1^4 \) be the eigenfunctions defined by (2.14) and assume that the set \( \{k_j\}_1^N \) of singularities are as the above assumption. Then the following residue conditions hold:

\[
Res_{k=k_j}[M_1] = \frac{m_{12}(s)(k_j)}{s_{33}(k_j)s_{23}(k_j)} e^{\theta_{12}(k_j)} [M(k_j)]_3, \quad 1 \leq j \leq n_0, k_j \in D_1
\]

\[
Res_{k=k_j}[M_2] = \frac{m_{12}(s)(k_j)}{s_{33}(k_j)s_{13}(k_j)} e^{\theta_{12}(k_j)} [M(k_j)]_3, \quad 1 \leq j \leq n_0, k_j \in D_1
\]

\[
Res_{k=k_j}[M_1] = \frac{m_{21}(s)(k_j)m_{33}(s)(k_j) - m_{32}(s)(k_j)m_{13}(s)(k_j)}{(s^T A^T)_{33}(k_j)s_{23}(k_j)} e^{\theta_{13}(k_j)} [M(k_j)]_3
\]

\[
Res_{k=k_j}[M_2] = \frac{m_{21}(s)(k_j)m_{33}(s)(k_j) - m_{31}(s)(k_j)m_{23}(s)(k_j)}{(s^T A^T)_{33}(k_j)s_{13}(k_j)} e^{\theta_{23}(k_j)} [M(k_j)]_3
\]

\[
Res_{k=k_j}[M_3] = \frac{S_{13}(k_j)s_{32}(k_j) - S_{33}(k_j)s_{12}(k_j)}{(s^T A^T)_{33}(k_j)m_{23}(s)(k_j)} e^{\theta_{13}(k_j)} [M(k_j)]_1
\]

\[
Res_{k=k_j}[M_3] = \frac{S_{33}(k_j)s_{13}(k_j) - S_{13}(k_j)s_{31}(k_j)}{(s^T A^T)_{33}(k_j)m_{23}(s)(k_j)} e^{\theta_{23}(k_j)} [M(k_j)]_2
\]

\[
Res_{k=k_j}[M_3] = \frac{s_{12}(k_j)}{m_{33}(s)(k_j)m_{23}(s)(k_j)} e^{\theta_{13}(k_j)} [M(k_j)]_1 - \frac{s_{11}(k_j)}{m_{33}(s)(k_j)m_{23}(s)(k_j)} e^{\theta_{23}(k_j)} [M(k_j)]_2
\]

\[
n_4 + 1 \leq j \leq N, k_j \in D_4.
\]
where \( \dot{f} = \frac{df}{dt} \), and \( \theta_{ij} \) is defined by

\[
\theta_{ij}(x, t, k) = (l_i - l_j)x + (z_i - z_j)t, \quad i, j = 1, 2, 3.
\] (2.35)

that implies that

\[
\theta_{ij} = 0, \quad i, j = 1, 2; \quad \theta_{13} = \theta_{23} = -\theta_{32} = -\theta_{31} = -2ikx - 8ik^3t.
\]

**Proof.** We will prove (2.34a), (2.34c), (2.34e), (2.34f), the other conditions follow by similar arguments. Equation (2.26) implies the relation

\[
M_1 = \mu_2 e^{(-ikx-4ik^3t)A} S_1, \quad (2.36a)
\]

\[
M_2 = \mu_2 e^{(-ikx-4ik^3t)A} S_2, \quad (2.36b)
\]

\[
M_3 = \mu_2 e^{(-ikx-4ik^3t)A} S_3, \quad (2.36c)
\]

\[
M_4 = \mu_2 e^{(-ikx-4ik^3t)A} S_4, \quad (2.36d)
\]

In view of the expressions for \( S_1 \) and \( S_2 \) given in (2.27), the three columns of (2.36a) read:

\[
[M_1]_1 = [\mu_2]_1 \frac{m_{22}(s)}{s_{33}} + [\mu_2]_2 e^{\theta_{12}} \frac{m_{12}(s)}{s_{33}}, \quad (2.37a)
\]

\[
[M_1]_2 = [\mu_2]_1 e^{\theta_{13}} \frac{m_1(s)}{s_{33}} + [\mu_2]_2 e^{\theta_{23}} \frac{m_{11}(s)}{s_{33}}, \quad (2.37b)
\]

\[
[M_1]_3 = [\mu_2]_1 e^{\theta_{13}} s_{13} + [\mu_2]_2 e^{\theta_{23}} s_{23} + [\mu_2]_3 s_{33}. \quad (2.37c)
\]

while the three columns of (2.36b) read:

\[
[M_2]_1 = [\mu_2]_1 \frac{m_{22}(s)m_{33}(s) - m_{32}(s)m_{23}(s)}{(s^t S^3)_{33}^{A_{33}}} e^{\theta_{12}} + [\mu_2]_2 \frac{m_{12}(s)m_{33}(s) - m_{32}(s)m_{13}(s)}{(s^t S^3)_{33}^{A_{33}}} e^{\theta_{21}} + [\mu_2]_3 \frac{m_{12}(s)m_{33}(S) - m_{32}(s)m_{13}(S)}{(s^t S^3)_{33}^{A_{33}}} e^{\theta_{31}}, \quad (2.38a)
\]

\[
[M_2]_2 = [\mu_2]_1 \frac{m_{22}(s)m_{33}(S) - m_{32}(s)m_{23}(S)}{(s^t S^3)_{33}^{A_{33}}} e^{\theta_{12}} + [\mu_2]_2 \frac{m_{12}(s)m_{33}(S) - m_{32}(s)m_{13}(S)}{(s^t S^3)_{33}^{A_{33}}} e^{\theta_{21}} + [\mu_2]_3 \frac{m_{12}(s)m_{33}(S) - m_{32}(s)m_{13}(S)}{(s^t S^3)_{33}^{A_{33}}} e^{\theta_{31}} \quad (2.38b)
\]

\[
[M_2]_3 = [\mu_2]_1 s_{13} e^{\theta_{13}} + [\mu_2]_2 s_{23} e^{\theta_{23}} + [\mu_2]_3 s_{33}. \quad (2.38c)
\]

and the three columns of (2.36c) read:

\[
[M_3]_1 = [\mu_2]_1 s_{11} + [\mu_2]_2 s_{21} e^{\theta_{21}} + [\mu_2]_3 s_{31} e^{\theta_{31}}, \quad (2.39a)
\]

\[
[M_3]_2 = [\mu_2]_1 s_{12} + [\mu_2]_2 s_{22} e^{\theta_{22}} + [\mu_2]_3 s_{32} e^{\theta_{32}}, \quad (2.39b)
\]

\[
[M_3]_3 = [\mu_2]_1 s_{13} + [\mu_2]_2 s_{23} e^{\theta_{23}} + [\mu_2]_3 s_{33} e^{\theta_{33}}. \quad (2.39c)
\]
\[ [M_3]_2 = [\mu_2]_1 s_{12} e^{\theta_{12}} + [\mu_2]_2 s_{22} + [\mu_2]_3 s_{32} e^{\theta_{32}}, \quad (2.39b) \]

\[ [M_3]_3 = [\mu_2]_1 \frac{S_{13}}{(S^T S^A)_{33}} e^{\theta_{13}} + [\mu_2]_2 \frac{S_{23}}{(S^T S^A)_{33}} e^{\theta_{23}} + [\mu_2]_3 \frac{S_{33}}{(S^T S^A)_{33}} e^{\theta_{33}}. \quad (2.39c) \]

the three columns of (2.36d) read:

\[ [M_4]_1 = [\mu_2]_1 s_{11} + [\mu_2]_2 s_{21} e^{\theta_{21}} + [\mu_2]_3 s_{31} e^{\theta_{31}}, \quad (2.40a) \]

\[ [M_4]_2 = [\mu_2]_1 s_{12} e^{\theta_{12}} + [\mu_2]_2 s_{22} + [\mu_2]_3 s_{32} e^{\theta_{32}}, \quad (2.40b) \]

\[ [M_4]_3 = [\mu_2]_3 \frac{1}{m_{33}(s)}. \quad (2.40c) \]

We first suppose that \( k_j \in D_1 \) is a simple zero of \( s_{33}(k) \). Solving (2.37c) for \([\mu_2]_2\) and substituting the result in to (2.37a), we find

\[ [M_1]_1 = \frac{m_{12}(s)}{s_{33}s_{23}} e^{\theta_{31}} [M_1]_3 + \frac{m_{32}(s)}{s_{23}} [\mu_2]_2 - \frac{m_{12}(s)}{s_{23}} e^{\theta_{31}} [\mu_2]_3. \]

Taking the residue of this equation at \( k_j \), we find the condition (2.34a) in the case when \( k_j \in D_1 \). Similarly, solving (2.38a) for \([\mu_2]_2\) and substituting the result in to (2.38a), we find

\[ [M_2]_1 = \frac{m_{12}(s)m_{33}(S) - m_{32}(s)m_{13}(S)}{(s^T S^A)_{33}s_{23}} e^{\theta_{31}} [M_1]_3 - \frac{m_{32}(s)}{s_{23}} [\mu_2]_1 - \frac{m_{12}(s)}{s_{23}} e^{\theta_{31}} [\mu_2]_3. \]

Taking the residue of this equation at \( k_j \), we find the condition (2.34a) in the case when \( k_j \in D_2 \).

In order to prove (2.34c), we solve (2.39a) and (2.39b) for \([\mu_2]_1\) and \([\mu_2]_3\), then substituting the result into (2.39c), we find

\[ [M_3]_3 = \frac{S_{13}s_{32} - S_{33}s_{12}}{(S^T S^A)_{33}m_{23}(s)} e^{\theta_{31}} [M_3]_1 + \frac{S_{33}s_{31} - S_{13}s_{31}}{(S^T S^A)_{33}(k_j)m_{23}(s)} e^{\theta_{31}} [M_3]_2 + \frac{1}{m_{23}(s)} [\mu_2]_3. \]

Taking the residue of this equation at \( k_j \), we find the condition (2.34c) in the case when \( k_j \in D_3 \). Similarly, solving (2.40a) and (2.40b) for \([\mu_2]_1\) and \([\mu_2]_3\), then substituting the result into (2.40c), we find

\[ [M_4]_3 = \frac{s_{12}}{m_{33}(s)m_{23}(s)} e^{\theta_{13}} [M_4]_1 - \frac{s_{11}}{m_{33}(s)m_{23}(s)} e^{\theta_{31}} [M_4]_2 - \frac{1}{m_{23}(s)} e^{\theta_{31}} [\mu_2]_2. \]

Taking the residue of this equation at \( k_j \), we find the condition (2.34b) in the case when \( k_j \in D_4 \).
3. The Riemann-Hilbert problem

The sectionally analytic function $M(x, t, k)$ defined in section 2 satisfies a Riemann-Hilbert problem which can be formulated in terms of the initial and boundary values of $u(x, t)$. By solving this Riemann-Hilbert problem, the solution of (1.5) (then (1.3)) can be recovered for all values of $x, t$.

**Theorem 3.1.** Suppose that $u(x, t)$ is a solution of (1.5) in the half-line domain $\Omega$ with sufficient smoothness and decays as $x \to \infty$. Then $u(x, t)$ can be reconstructed from the initial value $\{u_0(x)\}$ and boundary values $\{g_0(t), g_1(t), g_2(t)\}$ defined as follows,

$$u_0(x) = u(x, 0), \quad g_0(t) = u(0, t), \quad g_1(t) = u_x(0, t), \quad g_2(t) = u_{xx}(0, t).$$

(3.1)

Use the initial and boundary data to define the jump matrices $J_{m,n}(x, t, k)$ as well as the spectral $s(k)$ and $S(k)$ by equation (2.24). Assume that the possible zeros $\{k_j\}_{j=1}^N$ of the functions $s_{33}(k), (s^T S^A)_{33}(k), (S^T s^A)_{33}(k)$ and $m_{33}(s)(k)$ are as in assumption 2.3.

Then the solution $\{u(x, t)\}$ is given by

$$u(x, t) = 2i \lim_{k \to \infty} (kM(x, t, k))_{13}.$$

(3.2)

where $M(x, t, k)$ satisfies the following $3 \times 3$ matrix Riemann-Hilbert problem:

- $M$ is sectionally meromorphic on the Riemann $k-$sphere with jumps across the contours $\bar{D}_n \cap \bar{D}_m, n, m = 1, \cdots, 4$, see Figure 2.
- Across the contours $\bar{D}_n \cap \bar{D}_m$, $M$ satisfies the jump condition

$$M_n(x, t, k) = M_m(x, t, k)J_{m,n}(x, t, k), \quad k \in D_n \cap D_m, n, m = 1, 2, 3, 4.$$  

(3.3)

- $M(x, t, k) = I + O(\frac{1}{k}), \quad k \to \infty.$
- The residue condition of $M$ is showed in Proposition 2.4.
Proof. It only remains to prove (3.2) and this equation follows from the large $k$ asymptotics of the eigenfunctions, see the appendix A. □

4. Non-linearizable Boundary Conditions

A major difficulty of initial-boundary value problems is that some of the boundary values are unknown for a well-posed problem. All boundary values are needed for the definition of $S(k)$, and hence for the formulation of the Riemann-Hilbert problem. Our main result expresses the spectral function $S(k)$ in terms of the prescribed boundary data and the initial data via the solution of a system of nonlinear integral equations.

4.1. Asymptotics. An analysis of (A.2) shows that the eigenfunctions $\{\mu_j\}^3_{j=1}$ have the following asymptotics as $k \to \infty$ (see the appendix A):

\[
\mu_j(x, t, k) = \Pi + \frac{1}{k} \left( \begin{array}{c}
\frac{i}{2} \int_{(x_j, t_j)} (x, t) \Delta \mu_{11}^{(2)} \\
\frac{i}{2} \int_{(x_j, t_j)} (x, t) \Delta \mu_{12}^{(2)} \\
\frac{i}{2} \int_{(x_j, t_j)} (x, t) \Delta \mu_{13}^{(2)} \\
\frac{i}{2} \int_{(x_j, t_j)} (x, t) \Delta \mu_{21}^{(3)} \\
\frac{i}{2} \int_{(x_j, t_j)} (x, t) \Delta \mu_{22}^{(3)} \\
\frac{i}{2} \int_{(x_j, t_j)} (x, t) \Delta \mu_{23}^{(3)} \\
\frac{i}{2} \int_{(x_j, t_j)} (x, t) \Delta \mu_{31}^{(3)} \\
\frac{i}{2} \int_{(x_j, t_j)} (x, t) \Delta \mu_{32}^{(3)} \\
\frac{i}{2} \int_{(x_j, t_j)} (x, t) \Delta \mu_{33}^{(3)} \\
\end{array} \right) + O\left(\frac{1}{k^4}\right)
\]

(4.1a)

where

\[
\Delta = -|u|^2 dx + (u \bar{u}_{xx} + u_{xx} \bar{u} - u_x \bar{u}_x + 6|u|^4) dt
\]
\[
\Delta_1 = -u^2 dx + (uu_{xx} + u_{xx} u - (u_x)^2 + 6|u|^2 u^2) dt
\]
\[
\Delta_2 = -\bar{u}^2 dx + (\bar{u}\bar{u}_{xx} + \bar{u}_{xx} \bar{u} - (\bar{u}_x)^2 + 6|u|^2 \bar{u}^2) dt
\]

(4.2a)
\[ \begin{align*}
\mu_{13}^{(2)} &= -\frac{1}{2} u \int_{(x,t)} \Delta + \frac{1}{4} u_x \\
\mu_{23}^{(2)} &= -\frac{1}{2} \bar{u} \int_{(x,t)} \Delta + \frac{1}{4} \bar{u}_x \\
\mu_{31}^{(2)} &= \frac{1}{4} (\bar{u} \int_{(x,t)} \Delta + u \int_{(x,t)} \Delta_2) - \frac{1}{4} \bar{u}_x \\
\mu_{32}^{(2)} &= \frac{1}{4} (\bar{u} \int_{(x,t)} \Delta_1 + u \int_{(x,t)} \Delta) - \frac{1}{4} u_x.
\end{align*} \] (4.2b)

\[ \eta = d\left[ \frac{1}{2} \left( \int_{(x,t)} \Delta \right)^2 \right] \]

\[ \begin{align*}
\nu_1 &= \Delta_1 \int_{(x,t)} \Delta_2 + (u \bar{u}_x) dx + (u \bar{u}_t - 2|u|^2(u \bar{u}_x - u_x \bar{u})) dt \\
\eta_1 &= \int_{(x,t)} \Delta \int_{(x,t)} \Delta_1 + u^2 \\
\eta_2 &= \int_{(x,t)} \Delta \int_{(x,t)} \Delta_2 + \bar{u}^2 \\
\nu_2 &= \Delta_2 \int_{(x,t)} \Delta_1 + (\bar{u} u_x) dx + (\bar{u} u_t - 2|u|^2(\bar{u} u_x - \bar{u}_x u)) dt, \\
\eta_3 &= d\left[ -\frac{1}{2} \left( \int_{(x,t)} \Delta \right)^2 \right] - \frac{1}{2}|u|^2. \tag{4.2c}
\end{align*} \]

and in the following we just use \( \mu_{13}^{(3)}, \mu_{23}^{(3)}, \mu_{31}^{(3)} \) and \( \mu_{32}^{(3)} \), so we only compute these functions

\[ \begin{align*}
\mu_{13}^{(3)} &= \frac{1}{2} u \mu_{33}^{(2)} + \frac{1}{2} u_x \mu_{33}^{(1)} + \frac{i}{4} |u|^2 u + \frac{i}{8} u_{xx} \\
\mu_{23}^{(3)} &= \frac{1}{2} \bar{u} \mu_{33}^{(2)} + \frac{1}{2} \bar{u}_x \mu_{33}^{(1)} + \frac{i}{4} |\bar{u}|^2 \bar{u} + \frac{i}{8} \bar{u}_{xx} \\
\mu_{31}^{(3)} &= \frac{1}{2} (\bar{u} \mu_{21}^{(2)} + u \mu_{21}^{(2)}) - \frac{1}{4} (\bar{u}_x \mu_{11}^{(1)} + u_x \mu_{12}^{(1)}) + \frac{i}{4} |u|^2 \bar{u} + \frac{i}{8} \bar{u}_{xx} \\
\mu_{32}^{(3)} &= \frac{1}{2} (\bar{u} \mu_{22}^{(2)} + u \mu_{22}^{(2)}) - \frac{1}{4} (\bar{u}_x \mu_{11}^{(1)} + u_x \mu_{12}^{(1)}) + \frac{i}{4} |u|^2 u + \frac{i}{8} u_{xx}.
\end{align*} \] (4.2d)

From the global relation (2.33) and replacing \( T \) by \( t \), we find

\[ \mu_2(0, t, k) e^{-i k^3 t \Lambda} s(k) = c(t, k), \quad k \in (D_3 \cup D_4, D_3 \cup D_4, D_1 \cup D_2). \] (4.3)

We define functions \( \{ \Phi_{13}(t, k), \Phi_{23}(t, k), \Phi_{33}(t, k) \} \) and \( \{ c_j(t, k) \}_1^3 \) by:

\[ \mu_2(0, t, k) = \left( \begin{array}{ccc}
\Phi_{11}(t, k) & \Phi_{12}(t, k) & \Phi_{13}(t, k) \\
\Phi_{21}(t, k) & \Phi_{22}(t, k) & \Phi_{23}(t, k) \\
\Phi_{31}(t, k) & \Phi_{32}(t, k) & \Phi_{33}(t, k)
\end{array} \right), \quad \frac{[c(t, k)]_3}{s_{33}(k)} = \left( \begin{array}{c}
c_1(t, k) \\
c_2(t, k) \\
c_3(t, k)
\end{array} \right). \] (4.4)

we can write the (13) and (23) entries of the global relation as

\[ \Phi_{11}(t, k) e^{-i k^3 t} \frac{s_{13}}{s_{33}} + \Phi_{12}(t, k) e^{-i k^3 t} \frac{s_{23}}{s_{33}} + \Phi_{13}(t, k) = c_1(t, k), \quad k \in D_1 \cup D_2, \] (4.5a)
$$\Phi_{21}(t, k)e^{-\frac{8i\xi t}{3}s_{33}^{13}} + \Phi_{22}(t, k)e^{-\frac{8i\xi t}{3}s_{33}^{23}} + \Phi_{23}(t, k) = c_2(t, k), \quad k \in D_1 \cup D_2,$$

(4.5b)

The functions \(\{c_j(t, k)\}_j^3\) are analytic and bounded in \(D_1 \cup D_2\) away from the possible zeros of \(s_{33}(k)\) and of order \(O(\frac{1}{k})\) as \(k \to \infty\).

From the asymptotic of \(\mu_j(x, t, k)\) in (4.11a) we have

\[
\begin{pmatrix}
    s_{13}(k) \\
    s_{23}(k) \\
    s_{33}(k)
\end{pmatrix} = \begin{pmatrix}
    0 \\
    0 \\
    1
\end{pmatrix} + \frac{1}{2ik} \begin{pmatrix}
    u(0, 0) \\
    \tilde{u}(0, 0) \\
    2 \int_{(0,0)}^{(\infty,0)} \Delta
\end{pmatrix} + O\left(\frac{1}{k^2}\right). \quad (4.6)
\]

and

\[
\Phi_{j3}(t, k) = \frac{\Phi_{j3}^{(1)}(t)}{k} + \frac{\Phi_{j3}^{(2)}(t)}{k^2} + \frac{\Phi_{j3}^{(3)}(t)}{k^3} + O\left(\frac{1}{k^4}\right), \quad (4.7a)
\]

\[
\Phi_{33}(t, k) = 1 + \frac{\Phi_{33}^{(1)}(t)}{k} + \frac{\Phi_{33}^{(2)}(t)}{k^2} + O\left(\frac{1}{k^3}\right), \quad k \to \infty, k \in D_1 \cup D_2. \quad (4.7b)
\]

where

\[
\Phi_{j3}^{(1)}(t) = \frac{1}{2T}g_0(t)^T, \quad \Phi_{j3}^{(2)}(t) = \frac{1}{2}g_1(t)^T - \frac{1}{2}g_0^T \int_{(0,0)}^{(x,t)} \Delta
\]

\[
\Phi_{j3}^{(3)}(t) = \frac{1}{2T}g_0^T \Phi_{33}^{(2)} + \frac{1}{4}g_1^T \Phi_{33}^{(1)} + \frac{i}{4}|u|^2 g_0^T + \frac{i}{8}g_2^T,
\]

\[
\Phi_{33}^{(1)}(t) = -i \int_{(0,0)}^{(x,t)} \Delta, \quad \Phi_{33}^{(2)}(t) = \int_{(x,t)}^{(x,t)} \eta_3.
\]

Here the definition of \(\Phi_{j3}(t, k)\) can be found in the appendix \(A\).

In particular, we find the following expressions for the boundary values:

\[
g_0^T = 2i\Phi_{j3}^{(1)}(t), \quad (4.8a)
\]

\[
g_1^T = 2i g_0^T \Phi_{33}^{(1)}(t) + 4\Phi_{j3}^{(2)}(t), \quad (4.8b)
\]

\[
g_2^T = -2|g_0|^2 g_0^T + 2ig_1^T \Phi_{33}^{(1)}(t) + 4g_0^T \Phi_{33}^{(2)}(t) - 8i\Phi_{j3}^{(3)}(t). \quad (4.8c)
\]

We will also need the asymptotic of \(c_j(t, k)\),

**Lemma 4.1.** The global relation (4.3) implies that the large \(k\) behavior of \(c_j(t, k)\) satisfies

\[
c_j(t, k) = \frac{\Phi_{j3}^{(1)}(t)}{k} + \frac{\Phi_{j3}^{(2)}(t)}{k} + \frac{\Phi_{j3}^{(3)}(t)}{k} + O\left(\frac{1}{k^4}\right), \quad k \to \infty, k \in D_1. \quad (4.9)
\]

**Proof.** See the appendix \(B\) \(\square\)
4.2. The Dirichlet and Neumann problems. We can now derive effective characterizations of spectral function $S(k)$ for the Dirichlet ($g_0$ prescribed), the first Neumann ($g_1$ prescribed), and the second Neumann ($g_2$ prescribed) problems.

Define $\alpha$ by $\alpha = e^{\frac{2\pi i}{3}}$ and let $\{\Pi_j(t,k), \hat{\Pi}_j(t,k), \tilde{\Pi}_j(t,k)\}_1^3$ denote the following combinations formed from $\{\Phi_j(t,k)\}_1^3$:

$$
\Pi_j(t,k) = \Phi_j(t,k) + \alpha \Phi_j(t,\alpha k) + \alpha^2 \Phi_j(t,\alpha^2 k), \quad j = 1, 2, 3,
\hat{\Pi}_j(t,k) = \Phi_j(t,k) + \alpha^2 \Phi_j(t,\alpha k) + \alpha \Phi_j(t,\alpha^2 k), \quad j = 1, 2, 3,
\tilde{\Pi}_j(t,k) = \Phi_j(t,k) + \Phi_j(t,\alpha k) + \Phi_j(t,\alpha^2 k), \quad j = 1, 2, 3.
$$

(4.10)

And let $R(k) = \Phi_{11,33} + \Phi_{12,33}$.

Let $D_1 = D'_1 \cup D''_1$ where $D'_1 = D_1 \cap \{\text{Re } k > 0\}$ and $D''_1 = D_1 \cap \{\text{Re } k < 0\}$. Similarly, let $D_4 = D'_4 \cup D''_4$ where $D'_4 = D_4 \cap \{\text{Re } k > 0\}$ and $D''_4 = D_4 \cap \{\text{Re } k < 0\}$.

**Theorem 4.2.** Let $T < \infty$. Let $u_0(x), u \geq 0$, be a function of Schwartz class.

For the Dirichlet problem it is assumed that the function $g_0(t), 0 \leq t < T$, has sufficient smoothness and is compatible with $u_0(x)$ at $x = t = 0$.

For the first Neumann problem it is assumed that the function $g_1(t), 0 \leq t < T$, has sufficient smoothness and is compatible with $u_0(x)$ at $x = t = 0$.

Similarly, for the second Neumann problem it is assumed that the function $g_2(t), 0 \leq t < T$, has sufficient smoothness and is compatible with $u_0(x)$ at $x = t = 0$.

Suppose that $s_{33}(k)$ has a finite number of simple zeros in $D_1$.

Then the spectral function $S(k)$ is given by

$$
S(k) = \begin{pmatrix}
A(k) & B(k) & e^{8ik^3T}C(k) \\
D(k) & E(k) & e^{8ik^3T}F(k) \\
e^{-8ik^3T}G(k) & e^{-8ik^3T}H(k) & I(k)
\end{pmatrix}
$$

(4.11)
where
\[ A(k) = \Phi_{22}(k)\Phi_{43}(k) - \Phi_{23}(k)\Phi_{32}(k) \]
\[ B(k) = \Phi_{13}(k)\Phi_{22}(k) - \Phi_{12}(k)\Phi_{33}(k) \]
\[ C(k) = \Phi_{12}(k)\Phi_{23}(k) - \Phi_{13}(k)\Phi_{22}(k) \]
\[ D(k) = \Phi_{23}(k)\Phi_{31}(k) - \Phi_{21}(k)\Phi_{33}(k) \]
\[ E(k) = \Phi_{11}(k)\Phi_{33}(k) - \Phi_{13}(k)\Phi_{31}(k) \]
\[ F(k) = \Phi_{21}(k)\Phi_{13}(k) - \Phi_{11}(k)\Phi_{23}(k) \]
\[ G(k) = \Phi_{21}(k)\Phi_{32}(k) - \Phi_{22}(k)\Phi_{31}(k) \]
\[ H(k) = \Phi_{12}(k)\Phi_{31}(k) - \Phi_{11}(k)\Phi_{32}(k) \]
\[ I(k) = \Phi_{11}(k)\Phi_{22}(k) - \Phi_{12}(k)\Phi_{21}(k) \]

and the complex-value functions \( \{\Phi_{13}(t, k)\}_{t=1}^{3} \) satisfy the following system of integral equations:
\[ \Phi_{13}(t, k) = \int_{0}^{t} e^{-8ik^{3}(t-t')} \left[ (2ik|g_{0}|^{2} + (g_{0}g_{1} - g_{1}g_{0}))\Phi_{13} + g_{0}^{2}\Phi_{23} + (4k^{2}g_{0} + 2ikg_{1} - 4|g_{0}|^{2}g_{0} - g_{2})\Phi_{33} \right] (t', k)dt' \]  
\[ \Phi_{23}(t, k) = \int_{0}^{t} e^{-8ik^{3}(t-t')} \left[ (2ik|g_{0}|^{2} - (g_{0}g_{1} - g_{1}g_{0}))\Phi_{13} + g_{0}^{2}\Phi_{23} + (4k^{2}g_{0} + 2ikg_{1} - 4|g_{0}|^{2}g_{0} - g_{2})\Phi_{33} \right] (t', k)dt' \]  
\[ \Phi_{33}(t, k) = 1 + \int_{0}^{t} \left[ (-4k^{2}g_{0} + 2ikg_{1} + 4|g_{0}|^{2} + g_{2})\Phi_{13} + (-4k^{2}g_{0} + 2ikg_{1} + 4|g_{0}|^{2} + g_{2})\Phi_{23} + -4ik|g_{0}|^{2}\Phi_{33} \right] (t', k)dt' \]

and \( \{\Phi_{11}(t, k)\}_{t=1}^{3}, \{\Phi_{12}(t, k)\}_{t=1}^{3} \) satisfy the following system of integral equations:
\[ \Phi_{11}(t, k) = 1 + \int_{0}^{t} \left[ (2ik|g_{0}|^{2} + (g_{0}g_{1} - g_{1}g_{0}))\Phi_{11} + g_{0}^{2}\Phi_{21} + (4k^{2}g_{0} + 2ikg_{1} - 4|g_{0}|^{2}g_{0} - g_{2})\Phi_{31} \right] (t', k)dt' \]  
\[ \Phi_{21}(t, k) = \int_{0}^{t} \left[ (2ik|g_{0}|^{2} - (g_{0}g_{1} - g_{1}g_{0}))\Phi_{11} + g_{0}^{2}\Phi_{21} + (4k^{2}g_{0} + 2ikg_{1} - 4|g_{0}|^{2}g_{0} - g_{2})\Phi_{31} \right] (t', k)dt' \]  
\[ \Phi_{33}(t, k) = \int_{0}^{t} e^{8ik^{3}(t-t')} \left[ (-4k^{2}g_{0} + 2ikg_{1} + 4|g_{0}|^{2} + g_{2})\Phi_{11} + (-4k^{2}g_{0} + 2ikg_{1} + 4|g_{0}|^{2} + g_{2})\Phi_{21} + -4ik|g_{0}|^{2}\Phi_{31} \right] (t', k)dt' \]  
\[ \Phi_{12}(t, k) = \int_{0}^{t} \left[ (2ik|g_{0}|^{2} + (g_{0}g_{1} - g_{1}g_{0}))\Phi_{12} + g_{0}^{2}\Phi_{22} + (4k^{2}g_{0} + 2ikg_{1} - 4|g_{0}|^{2}g_{0} - g_{2})\Phi_{32} \right] (t', k)dt' \]  
\[ (4.13a) \]
\[ (4.13b) \]
\[ (4.13c) \]
\[ (4.14a) \]
\( \Phi_{22}(t, k) = 1 + \int_0^t \left[ (2ik|g_0|^2 - (g_0\bar{g}_1 - g_1\bar{g}_0))\Phi_{12} + g_0^2\Phi_{22} + (4k^2\bar{g}_0 + 2ik\bar{g}_1 - 4|g_0|^2\bar{g}_0 - \bar{g}_2)\Phi_{32} \right] (t', k) dt' \)  
(4.14b)

\( \Phi_{32}(t, k) = \int_0^t e^{8ik(t-t')} \left[ (-4k^2\bar{g}_0 + 2ik\bar{g_1} + 4|g_0|^2 + \bar{g}_2)\Phi_{12} + (-4k^2\bar{g}_0 + 2ik\bar{g}_1 + 4|g_0|^2 + g_2)\Phi_{22} - 4ik|g_0|^2\Phi_{32} \right] (t', k) dt' \)  
(4.14c)

(i) For the Dirichlet problem, the unknown Neumann boundary values \( g_1(t) \) and \( g_2(t) \) are given by

\[
g_1(t) = \frac{2g_0(t)}{\pi} \int_{\partial D_3} \Pi_3(t, k) dk + \frac{2}{\pi i} \int_{\partial D_3} \left[ k\Pi_1(t, k) - \frac{3g_0(t)}{2t} \right] dk
\]

and

\[
g_2(t) = g_0(t)^3 - \frac{4}{\pi} \int_{\partial D_3} \left[ k^2\Pi_1(t, k) - \frac{3k\Pi_1(t)}{2t} \right] dk
\]

+ \frac{4}{\pi} \int_{\partial D_3} k^2 e^{-8ikt} \left[ (1 - \alpha) R(\alpha k) + (1 - \alpha^2) R(\alpha^2 k) \right] dk
\]

\[
-8i \left\{ (1 - \alpha) \sum_{k_j \in D'_1} + (1 - \alpha^2) \sum_{k_j \in D''_1} \right\} k_j^2 e^{-8ikt} Res_{k_j} R(k)
\]

(4.15a)

(ii) For the first Neumann problem, the unknown boundary values \( g_0(t) \) and \( g_2(t) \) are given by

\[
g_0(t) = \frac{1}{\pi} \int_{\partial D_3} \tilde{\Pi}_1(t, k) dk - \frac{1}{\pi} \int_{\partial D_3} e^{-8ikt} \left[ (\alpha - \alpha\bar{a}^2) R(\alpha k) + (\alpha^2 - \alpha R(\alpha^2 k) \right] dk
\]

+ 2i \left\{ (1 - \alpha) \sum_{k_j \in D'_1} + (1 - \alpha^2) \sum_{k_j \in D''_1} \right\} e^{-8ikt} Res_{k_j} R(k),
\]  
(4.16a)

and

\[
g_2(t) = g_0^3(t) - \frac{4}{\pi} \int_{\partial D_3} \left( k^2\tilde{\Pi}_1(t, k) - \frac{3k\tilde{\Pi}_1(t)}{2t} \right) \tilde{\Pi}_1(t, l) dl \right) dk
\]

+ \frac{4}{\pi} \int_{\partial D_3} k^2 e^{-8ikt} \left[ (1 - \alpha^2) R(\alpha k) + (1 - \alpha R(\alpha^2 k) \right] dk
\]

\[
-8i \left\{ (1 - \alpha) \sum_{k_j \in D'_1} + (1 - \alpha^2) \sum_{k_j \in D''_1} \right\} k_j^2 e^{-8ikt} Res_{k_j} R(k)
\]

+ \frac{4g_0(t)}{\pi} \int_{\partial D_3} \tilde{\Pi}_3(t, k) dk + \frac{2g_0(t)}{\pi} \int_{\partial D_3} \Pi_3(t, k) dk.
\]  
(4.16b)
(iii) For the second Neumann problem, the unknown boundary values \(g_0(t)\) and \(g_1(t)\) are given by

\[
g_0(t) = \frac{1}{\pi} \int_{\partial D_3} \hat{\Pi}_1(t, k) dk - \frac{1}{\pi} \int_{\partial D_4} e^{-8i\mu k^2 t} \left[ (\alpha - \alpha^2) R(\alpha k) + (\alpha^2 - \alpha) R(\alpha^2 k) \right] dk
\]

\[
+ 2i \left\{ (1 - \alpha) \sum_{k_j \in D'_1} (1 - \alpha^2) \sum_{k_j \in D''_1} \right\} e^{-8i\mu k^2 t} \text{Res}_{k_j} R(k),
\]

and

\[
g_1(t) = \frac{2g_0(t)}{\pi} \int_{\partial D_3} \Pi_3(t, k) dk + \frac{2}{\pi} \int_{\partial D_3} k \hat{\Pi}_1(t, k) dk
\]

\[
- \frac{2}{\pi} \int_{\partial D_4} k e^{-8i\mu k^2 t} \left[ (\alpha^2 - 1) R(\alpha k) + (\alpha - 1) R(\alpha^2 k) \right] dk
\]

\[
+ 4 \left\{ (1 - \alpha) \sum_{k_j \in D'_1} (1 - \alpha^2) \sum_{k_j \in D''_1} \right\} k_j e^{-8i\mu k^2 t} \text{Res}_{k_j} R(k).
\]

Proof. The representations (4.11) follow from the relation \(S(k) = e^{8i\mu k^2 T} \mu_2^A(0, T, k)^T\). And the system (4.12) is the direct result of the Volteral integral equations of \(\mu_2(0, t, k)\).

(i) In order to derive (4.15a) we note that equation (4.8b) expresses \(g_1\) in terms of \(\Phi_{33}^{(1)}(1)\) and \(\Phi_{13}^{(2)}\). Furthermore, equation (4.17) and Cauchy theorem imply

\[
- \frac{2\pi i}{3} \Phi_{33}^{(1)}(t) = 2 \int_{\partial D_2} [\Phi_{33}(t, k) - 1] dk = \int_{\partial D_4} [\Phi_{33}(t, k) - 1] dk
\]

and

\[
- \frac{2\pi i}{3} \Phi_{13}^{(2)}(t) = 2 \int_{\partial D_2} \left[ k \Phi_{13}(t) - \frac{g_0(t)}{2i} \right] dk = \int_{\partial D_4} \left[ k \Phi_{13}(t) - \frac{g_0(t)}{2i} \right] dk.
\]

Thus,

\[
i\pi \Phi_{33}^{(1)}(t) = - \left( \int_{\partial D_2} + \int_{\partial D_4} \right) [\Phi_{33}(t, k) - 1] dk = \left( \int_{\partial D_2} + \int_{\partial D_3} \right) [\Phi_{33}(t, k) - 1] dk
\]

\[
= \int_{\partial D_3} [\Phi_{33}(t, k) - 1] dk + \alpha \int_{\partial D_3} [\Phi_{33}(t, k) - 1] dk + \alpha^2 \int_{\partial D_3} [\Phi_{33}(t, k) - 1] dk
\]

\[
= \int_{\partial D_3} \Pi_3(t, k) dk.
\]

(4.18)

Similarly,

\[
i\pi \Phi_{13}^{(2)}(t) = \left( \int_{\partial D_3} + \int_{\partial D_4} \right) \left[ k \Phi_{13}(t) - \frac{g_0(t)}{2i} \right] dk
\]

\[
= \left( \int_{\partial D_3} + \alpha^2 \int_{\partial D'_1} + \alpha \int_{\partial D''_1} \right) \left[ k \Phi_{13}(t) - \frac{g_0(t)}{2i} \right] dk + I(t)
\]

\[
= \int_{\partial D_3} \left[ k \Pi_1(t, k) - \frac{3g_0(t)}{2i} \right] dk + I(t).
\]

(4.19)
where \( I(t) \) is defined by

\[
I(t) = \left( (1 - \alpha^2) \int_{\partial D'_1} + (1 - \alpha) \int_{\partial D''_1} \right) \left[ k\Phi_{13}(t) - \frac{g_0(t)}{2i} \right] dk
\]

The last step involves using the global relation to compute \( I(t) \)

\[
I(t) = \left( (1 - \alpha^2) \int_{\partial D'_1} + (1 - \alpha) \int_{\partial D''_1} \right) \left[ k\Phi_{13}(t) - \frac{g_0(t)}{2i} \right] dk
\]

(4.20)

Using the asymptotic (4.9) and Cauchy theorem to compute the first term on the right-hand side of equation (4.20) and using the transformation \( k \rightarrow \alpha k \) and \( k \rightarrow \alpha^2 k \) in the second term on the right-hand side of (4.20), we find

\[
I(t) = -i\pi \Phi_{13}^{(2)}(t) - \int_{\partial D'_3} \left[ k\Phi_{13}(t, k) - \frac{g_0(t)}{2i} \right] dk
\]

(4.21)

Equations (4.19) and (4.21) imply

\[
\Phi_{13}^{(2)}(t) = \frac{1}{2\pi i} \int_{\partial D_3} k^2\Pi_{13}(t, k) - \frac{3g_0(t)}{2i} dk
\]

\[
-\frac{1}{2\pi i} \int_{\partial D_3} k^2e^{-8ik^3t} \left[ (1 - \alpha^2) R(\alpha^2 k) \right] dk
\]

(4.22a)

This equation together with (4.8b) and (4.18) yields (4.15a).

In order to derive (4.15b), we note that (4.8c) expresses \( g_2 \) in terms of \( \Phi_{13}^{(3)} \), \( \Phi_{33}^{(2)} \) and \( \Phi_{33}^{(1)} \). Equation (4.15b) follows from the expression (4.18) for \( \Phi_{33}^{(1)} \) and the following formulas:

\[
\Phi_{33}^{(2)}(t) = \frac{1}{\pi i} \int_{\partial D_3} k\Pi_{33} dk,
\]

(4.22a)

\[
\Phi_{13}^{(3)}(t) = \frac{1}{2\pi i} \int_{\partial D_3} k^2\Pi_{13}(t, k) - \frac{3g_0(t)}{2i} dk
\]

\[
-\frac{1}{2\pi i} \int_{\partial D_3} k^2e^{-8ik^3t} \left[ (1 - \alpha^2) R(\alpha^2 k) \right] dk
\]

(4.22b)
(ii) In order to derive the representations (4.16) relevant for the first Neumann problem, we use (4.8) together with (4.18), (4.22a) and the following formulas:

\[
\Phi_{13}^{(1)}(t) = \frac{1}{2\pi i} \int_{\partial D_3} \tilde{\Pi}_1(t, k) dk
\]

\[
-\frac{1}{2\pi i} \int_{\partial D_3} e^{-8ik^3t} \left[ (\alpha - \alpha^2)R(\alpha k) + (\alpha^2 - \alpha)R(\alpha^2 k) \right] dk
\]

\[
\left\{ (1 - \alpha) \sum_{k_j \in D_1'} + (1 - \alpha^2) \sum_{k_j \in D_1''} \right\} e^{-8ik^3t} \text{Res}_{k_j} R(k).
\]

(4.23a)

\[
\Phi_{13}^{(2)}(t) = \frac{1}{\pi i} \int_{\partial D_3} k \tilde{\Pi}_1 dk,
\]

(4.23b)

\[
\Phi_{13}^{(3)}(t) = \frac{1}{2\pi i} \int_{\partial D_3} \left[ k^2 \tilde{\Pi}_1(t, k) - 3\Phi_{13}^{(2)} \right] dk
\]

\[
-\frac{1}{2\pi i} \int_{\partial D_3} k^2 e^{-8ik^3t} \left[ (1 - \alpha)R(\alpha k) + (1 - \alpha^2)R(\alpha^2 k) \right] dk
\]

\[
\left\{ (1 - \alpha) \sum_{k_j \in D_1'} + (1 - \alpha) \sum_{k_j \in D_1''} \right\} k^2 e^{-8ik^3t} \text{Res}_{k_j} R(k).
\]

(4.23c)

(iii) In order to derive the representations (4.17) relevant for the second Neumann problem, we use (4.8) together with (4.18) and the following formulas:

\[
\Phi_{13}^{(1)}(t) = \frac{1}{2\pi i} \int_{\partial D_3} \tilde{\Pi}_1(t, k) dk
\]

\[
-\frac{1}{2\pi i} \int_{\partial D_3} e^{-8ik^3t} \left[ (\alpha^2 - 1)R(\alpha k) + (\alpha - 1)R(\alpha^2 k) \right] dk
\]

\[
\left\{ (1 - \alpha) \sum_{k_j \in D_1'} + (1 - \alpha^2) \sum_{k_j \in D_1''} \right\} e^{-8ik^3t} \text{Res}_{k_j} R(k).
\]

(4.24a)

\[
\Phi_{13}^{(2)}(t) = \frac{1}{2\pi i} \int_{\partial D_3} k \tilde{\Pi}_1(t, k) dk
\]

\[
-\frac{1}{2\pi i} \int_{\partial D_3} k e^{-8ik^3t} \left[ (\alpha^2 - 1)R(\alpha k) + (\alpha - 1)R(\alpha^2 k) \right] dk
\]

\[
\left\{ (1 - \alpha) \sum_{k_j \in D_1'} + (1 - \alpha^2) \sum_{k_j \in D_1''} \right\} e^{-8ik^3t} \text{Res}_{k_j} R(k).
\]

(4.24b)

\[\square\]

4.3. Effective characterizations. Substituting into the system (4.12) the expressions

\[
\Phi_{ij} = \Phi_{ij}^{(0)} + \varepsilon \Phi_{ij}^{(1)} + \varepsilon^2 \Phi_{ij}^{(2)} + \cdots, \quad i, j = 1, 2, 3.
\]

(4.25a)

\[
g_0 = \varepsilon g_{01} + \varepsilon^2 g_{02} + \cdots,
\]

(4.25b)

\[
g_1 = \varepsilon g_{11} + \varepsilon^2 g_{12} + \cdots,
\]

(4.25c)

\[
g_2 = \varepsilon g_{21} + \varepsilon^2 g_{22} + \cdots,
\]

(4.25d)
where $\varepsilon > 0$ is a small parameter, we find that the terms of $O(1)$ give $\Phi^{(0)}_{13} = \Phi^{(0)}_{23} = 0$ and $\Phi^{(0)}_{33} = 1$. Moreover, the terms of $O(\varepsilon)$ give $\Phi^{(1)}_{33} = 0$ and

$$O(\varepsilon): \quad \Phi^{(1)}_{13}(t, k) = \int_0^t e^{-8i k^3 (t-t')} (4k^2 g_{01} + 2ik g_{11} - g_{21})(t', k) dt',$$

(4.26)

From the above equation (4.26) we can get

$$\Pi^{(1)}_1(t, k) = 12k^2 \int_0^t e^{-8i k^3 (t-t')} g_{01}(t') dt',$$

(4.27a)

$$\hat{\Pi}^{(1)}_1(t, k) = 6ik \int_0^t e^{-8i k^3 (t-t')} g_{11}(t') dt',$$

(4.27b)

$$\tilde{\Pi}^{(1)}_1(t, k) = -3 \int_0^t e^{-8i k^3 (t-t')} g_{11}(t') dt',$$

(4.27c)

The Dirichlet problem can now be solved perturbatively as follows: assuming for simplicity that $s_{33}(k)$ has no zeros and expanding (4.15a) and (4.15b), we find

$$g_{11} = \frac{2}{\pi i} \int_{\partial D_3} \left[ k \Pi^{(1)}_1(t, k) - \frac{3g_{01}(t)}{2i} \right] dk$$

$$- \frac{2}{\pi i} \int_{\partial D_3} ke^{-8i k^3 t} [(\alpha^2 - \alpha)s_{131}(\alpha k) + (\alpha - \alpha^2)s_{131}(\alpha^2 k)] dk,$$

(4.28a)

$$g_{21} = -\frac{4}{\pi} \int_{\partial D_3} k^2 \Pi^{(1)}_1(t, k) - \frac{3k g_{01}(t)}{2i} \right] dk$$

$$+ \frac{4}{\pi} \int_{\partial D_3} k^2 e^{-8i k^3 t} [(1 - \alpha)s_{131}(\alpha k) + (1 - \alpha^2)s_{131}(\alpha^2 k)] dk,$$

(4.28b)

Using equation (4.27a) to determine $\Pi^{(1)}_1$, we can determine $g_{11}, g_{21}$ from (4.28), then $\Phi^{(1)}_{13}$ can be found from (4.26). And these arguments can be extended to higher orders and also can be extended to the systems (4.13a) and (4.14a), thus yields a constructive scheme for computing $S(k)$ to all orders.

Similarly, these arguments also can be used to the first Neumann problem and the second Neumann problem. That is to say, in all cases, the system can be solved perturbatively to all orders.
Appendix A. The asymptotic behavior of the functions
\{μ_j(x, t, k)\}_1^3

We denote some symbols as follows:

\[ \Lambda = \begin{pmatrix} I_{2 \times 2} & 0 \\ 0 & -1 \end{pmatrix}, \] (A.1a)

\[ V_1 = \begin{pmatrix} 0 & U^T \\ -\bar{U} & 0 \end{pmatrix}, \]
\[ V_2^{(2)} = 4 \begin{pmatrix} 0 & U^T \\ -\bar{U} & 0 \end{pmatrix}, \]
\[ V_2^{(1)} = 2i \begin{pmatrix} U^T \bar{U} & U_x^T \\ \bar{U}_x & -2|u|^2 \end{pmatrix}, \]
\[ V_2^{(0)} = -4|u|^2 \begin{pmatrix} 0 & U^T \\ -\bar{U} & 0 \end{pmatrix} - \begin{pmatrix} 0 & U_{xx}^T \\ -\bar{U}_{xx} & 0 \end{pmatrix} + (u\bar{u}_x - u_x\bar{u}) \begin{pmatrix} \sigma_3 & 0 \\ 0 & 0 \end{pmatrix}. \] (A.1b)

where \( I_{2 \times 2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) and \( U = (u, \bar{u}) \).

From the Lax pair of \( \mu \)
\[
\begin{cases}
\mu_x + [i\Lambda, \mu] = V_1\mu, \\
\mu_t + [4ik^3\Lambda, \mu] = V_2\mu.
\end{cases}
\] (A.2)

Suppose that
\[
\mu(x, t, k) = D_0 + \frac{D_1}{k} + \frac{D_2}{k^2} + \frac{D_3}{k^3} + \cdots.
\] (A.3)

We substitute the equation (A.3) into the Lax pair (A.2), and compare the order of \( k \), we find that:

\[ O(k) : \quad [i\Lambda, D_0] = 0, \]
\[ O(1) : \quad D_{0x} + [i\Lambda, D_1] = V_1D_0, \]
\[ O(k^{-1}) : \quad D_{1x} + [i\Lambda, D_2] = V_1D_1, \]
\[ O(k^{-2}) : \quad D_{2x} + [i\Lambda, D_3] = V_1D_2, \] (A.4a)

\( O(k^3) \): \[ 4i\Lambda, D_0 \] = 0,
\( O(k^2) \): \[ 4i\Lambda, D_1 \] = \( V_2^{(2)} \)\( D_0 \),
\( O(k^1) \): \[ 4i\Lambda, D_2 \] = \( V_2^{(2)} \)\( D_1 + V_2^{(1)} \)\( D_0 \),
\( O(1) \): \[ D_0 t + 4i\Lambda, D_3 \] = \( V_2^{(2)} \)\( D_2 + V_2^{(1)} \)\( D_1 + V_2^{(0)} \)\( D_0 \),
\( O(k^{-1}) \): \[ D_1 t + 4i\Lambda, D_4 \] = \( V_2^{(2)} \)\( D_3 + V_2^{(1)} \)\( D_2 + V_2^{(0)} \)\( D_1 \),
\( O(k^{-2}) \): \[ D_2 t + 4i\Lambda, D_5 \] = \( V_2^{(2)} \)\( D_4 + V_2^{(1)} \)\( D_3 + V_2^{(0)} \)\( D_2 \),

And we denote the \( D_l \) by \( D_l = \begin{pmatrix} D_{2\times 2}^{(l)} & D_{3j}^{(l)} \\ D_{3j}^{(l)} & D_{33}^{(l)} \end{pmatrix} \), \( j = 1, 2 \).

Then, from \( O(k^3) \), we have
\( D_{3j}^{(0)} = 0, \quad D_{3j}^{(0)} = 0 \). \hspace{1cm} (A.5)

\( O(k^2) \), we get
\[ 4i \begin{pmatrix} 0 & 2D_{3j}^{(1)} \\ -2D_{3j}^{(1)} & 0 \end{pmatrix} = 4 \begin{pmatrix} 0 & U^T D_{33}^{(0)} \\ -\bar{U} D_{2\times 2}^{(0)} & 0 \end{pmatrix}, \hspace{1cm} (A.6a) \]

this implies that
\[ \begin{cases} D_{3j}^{(1)} = -\frac{i}{2} U^T D_{33}^{(0)} \\ D_{3j}^{(1)} = -\frac{i}{2} \bar{U} D_{2\times 2}^{(0)}. \end{cases} \hspace{1cm} (A.6b) \]

\( O(k) \), we find
\[ 4i \begin{pmatrix} 0 & 2D_{3j}^{(2)} \\ -2D_{3j}^{(2)} & 0 \end{pmatrix} = 4 \begin{pmatrix} U^T D_{3j}^{(1)} & U^T D_{33}^{(1)} \\ -\bar{U} D_{2\times 2}^{(1)} & -\bar{U} D_{3j}^{(1)} \end{pmatrix} + 2i \begin{pmatrix} U^T \bar{U} D_{2\times 2}^{(0)} & U^T x D_{33}^{(0)} \\ -\bar{U} x D_{2\times 2}^{(0)} & -2|u|^2 D_{33}^{(0)} \end{pmatrix}, \hspace{1cm} (A.7a) \]

this implies that
\[ \begin{cases} D_{3j}^{(2)} = -\frac{i}{2} U^T D_{33}^{(1)} + \frac{i}{4} U^T x D_{33}^{(0)} \\ D_{3j}^{(2)} = -\frac{i}{2} \bar{U} D_{2\times 2}^{(1)} - \frac{i}{4} \bar{U} x D_{2\times 2}^{(0)}. \end{cases} \hspace{1cm} (A.7b) \]
\[ O(1), \text{ we have} \]
\[
\begin{pmatrix}
D_{2\times 2t}^{(0)} & 0 \\
0 & D_{33t}^{(0)}
\end{pmatrix} + 4i
\begin{pmatrix}
0 & 2D_{j3}^{(3)} \\
-2D_{3j}^{(3)} & 0
\end{pmatrix} = \\
4
\begin{pmatrix}
U^T D_{3j}^{(2)} & U^T D_{33}^{(2)} \\
-\bar{U} D_{2\times 2}^{(2)} & -\bar{U} D_{j3}^{(2)}
\end{pmatrix} + 2i
\begin{pmatrix}
U^T \bar{U} D_{2\times 2}^{(1)} + U^T D_{3j}^{(1)} & U^T \bar{U} D_{j3}^{(1)} + U^T D_{33}^{(1)} \\
-\bar{U} x D_{2\times 2}^{(1)} - 2|u|^2 D_{3j}^{(1)} & \bar{U} x D_{j3}^{(1)} - 2|u|^2 D_{33}^{(1)}
\end{pmatrix} \\
-4|u|^2
\begin{pmatrix}
0 & U^T D_{33}^{(0)} \\
-\bar{U} D_{2\times 2}^{(0)} & 0
\end{pmatrix} - \\
+(u\bar{u}_x - u_x \bar{u})
\begin{pmatrix}
\sigma_3 D_{2\times 2}^{(0)} & 0 \\
0 & 0
\end{pmatrix}.
\]
\[ (A.8a) \]

this implies that
\[
D_{2\times 2t}^{(0)} = 0, \quad D_{33t}^{(0)} = 0 \\
\begin{cases}
D_{3j}^{(3)} = -\frac{i}{2} U^T D_{33}^{(2)} + \frac{i}{4} U^T D_{3j}^{(1)} + \frac{i}{4}|u|^2 U^T D_{33}^{(0)} + \frac{i}{8} U^T D_{33}^{(0)} \\
D_{3j}^{(3)} = -\frac{i}{2} \bar{U} D_{2\times 2}^{(2)} - \frac{i}{4} \bar{U} D_{2\times 2}^{(1)} + \frac{i}{4}|u|^2 \bar{U} D_{2\times 2}^{(0)} + \frac{i}{8} \bar{U} D_{2\times 2}^{(0)}.
\end{cases}
\]
\[ (A.8b) \]

\[ O(k^{-1}), \text{ we get} \]
\[
\begin{pmatrix}
D_{2\times 2t}^{(1)} & D_{j3t}^{(1)} \\
D_{3jt}^{(1)} & D_{33t}^{(1)}
\end{pmatrix} + 4i
\begin{pmatrix}
0 & 2D_{j3}^{(4)} \\
-2D_{3j}^{(4)} & 0
\end{pmatrix} = \\
4
\begin{pmatrix}
U^T D_{3j}^{(3)} & U^T D_{33}^{(3)} \\
-\bar{U} D_{2\times 2}^{(3)} & -\bar{U} D_{j3}^{(3)}
\end{pmatrix} + 2i
\begin{pmatrix}
U^T \bar{U} D_{2\times 2}^{(2)} + U^T D_{3j}^{(2)} & U^T \bar{U} D_{j3}^{(2)} + U^T D_{33}^{(2)} \\
-\bar{U} x D_{2\times 2}^{(2)} - 2|u|^2 D_{3j}^{(2)} & \bar{U} x D_{j3}^{(2)} - 2|u|^2 D_{33}^{(2)}
\end{pmatrix} \\
-4|u|^2
\begin{pmatrix}
0 & U^T D_{33}^{(1)} \\
-\bar{U} D_{2\times 2}^{(1)} & -\bar{U} D_{j3}^{(1)}
\end{pmatrix} - \\
+(u\bar{u}_x - u_x \bar{u})
\begin{pmatrix}
\sigma_3 D_{2\times 2}^{(1)} & \sigma_3 D_{j3}^{(1)} \\
0 & 0
\end{pmatrix}.
\]
\[ (A.9a) \]

this implies that
\[
\begin{cases}
D_{2\times 2t}^{(1)} = \frac{i}{2}\{U^T \bar{U} x + U x \bar{U} - U^T \bar{U} x + 6|u|^2 U^T \bar{U}\} D_{2\times 2}^{(0)} \\
D_{33t}^{(1)} = -i\{u\bar{u}_x + u_x \bar{u} - u_x \bar{u}_x + 6|u|^4\} D_{33}^{(0)}.
\end{cases}
\]
\[
\begin{cases}
D_{3j}^{(4)} = \frac{1}{16} U^T D_{33}^{(3)} - \frac{1}{2} U^T D_{3j}^{(3)} + \frac{i}{8} U^T D_{33}^{(1)} + \frac{i}{2}|u|^2 U^T D_{3j}^{(1)} + \frac{i}{8} U^T D_{33}^{(3)} + \frac{i}{8}|u|^2 U^T D_{33}^{(3)} \\
D_{3j}^{(3)} = \frac{1}{16} \bar{U} D_{2\times 2}^{(3)} - \frac{1}{2} \bar{U} D_{2\times 2}^{(3)} + \frac{1}{4} \bar{U} D_{2\times 2}^{(2)} + \frac{i}{4}|u|^2 \bar{U} D_{2\times 2}^{(2)} + \frac{i}{8} \bar{U} D_{2\times 2}^{(1)} + \frac{i}{8}|u|^2 \bar{U} D_{2\times 2}^{(1)}.
\end{cases}
\]
\[ (A.9b) \]
Then from the integral contours $\gamma_j$, we get
\[
\begin{pmatrix}
D^{(2)}_{2 \times 2t} & D^{(2)}_{3j3t} \\
D^{(2)}_{3j} & D^{(2)}_{33t}
\end{pmatrix} + 4i \begin{pmatrix}
0 & 2D^{(5)}_{j3} \\
-2D^{(5)}_{j3} & 0
\end{pmatrix} = \\
4 \begin{pmatrix}
U^T D^{(4)}_{3j} & U^T D^{(4)}_{33} \\
-\bar{U} D^{(4)}_{2 \times 2} & -\bar{U} D^{(4)}_{j3}
\end{pmatrix} + 2i \begin{pmatrix}
U^T \bar{U} D^{(3)}_{2 \times 2} + U^T D^{(3)}_{j3} & U^T \bar{U} D^{(3)}_{j3} + U^T D^{(3)}_{33} \\
-\bar{U} D^{(3)}_{2 \times 2} - 2|u|^2 D^{(3)}_{j3} - \bar{U} D^{(3)}_{j3} - 2|u|^2 D^{(3)}_{33}
\end{pmatrix} \\
-4|u|^2 \begin{pmatrix}
U^T D^{(2)}_{3j} & U^T D^{(2)}_{33} \\
-\bar{U} D^{(2)}_{2 \times 2} & -\bar{U} D^{(2)}_{j3}
\end{pmatrix} + \begin{pmatrix}
0 & \sigma_3 D^{(2)}_{2 \times 2} \\
\sigma_3 D^{(2)}_{j3} & 0
\end{pmatrix}.
\]
\hspace{1cm} (A.10a)

this implies that
\[
D^{(2)}_{2 \times 2t} = \frac{i}{2} \{U^T \bar{U} x + U_{xx} \bar{U} - U^T \bar{U} x + 6|u|^2 U^T \bar{U} \} D^{(1)}_{2 \times 2} \\
+ \left\{ -\frac{1}{4} U^T \bar{U} t + \frac{1}{2}|u|^2 (\bar{u} u_x - u_{xx} - u_x \bar{u}) \sigma_3 + \frac{1}{4} (u_{xx} u_x - u_{xx} \bar{u}) \sigma_3 \right\} D^{(2)}_{33} \\
D^{(2)}_{33t} = -i \{u \bar{u}_{xx} + u_{xx} \bar{u} - u_x \bar{u}_x + 6|u|^4 \} D^{(1)}_{33} - \frac{1}{4} (|u|^2) t D^{(0)}_{33}.
\]
\hspace{1cm} (A.10b)

Also, from the $x$–part of the Lax pair, we have the following equations
\[
D^{(0)}_{2 \times 2x} = 0, \quad D^{(0)}_{33x} = 0. \hspace{1cm} (A.11a)
\]
\[
\begin{pmatrix}
D^{(1)}_{2 \times 2x} \\
D^{(1)}_{33x}
\end{pmatrix} = \frac{i}{2} U^T \bar{U} D^{(0)}_{2 \times 2} \\
D^{(1)}_{33} = i |u|^2 D^{(0)}_{33}. \hspace{1cm} (A.11b)
\]
\[
\begin{pmatrix}
D^{(2)}_{2 \times 2x} \\
D^{(2)}_{33x}
\end{pmatrix} = -\frac{i}{2} U^T \bar{U} D^{(1)}_{2 \times 2} - \frac{1}{4} U^T \bar{U} x D^{(0)}_{2 \times 2} \\
D^{(2)}_{33} = i |u|^2 D^{(1)}_{33} - \frac{1}{4} (|u|^2) x D^{(0)}_{33}. \hspace{1cm} (A.11c)
\]

Then from the integral contours $\gamma_j$, we can get
\[
D^{(0)}_{2 \times 2x} = \mathbb{1}_{2 \times 2}, \quad D^{(0)}_{33x} = 1. \hspace{1cm} (A.12)
\]

\section*{Appendix B. The asymptotic behavior of $c_j(t, k)$}

Let
\[
\mu_2(0, t, k) = \begin{pmatrix}
\Phi_{2 \times 2} & \Phi_{j3} \\
\Phi_{3j} & \Phi_{33}
\end{pmatrix}.
\]

The global relation shows that
\[
\Phi_{2 \times 2} s^{(3)}_{j3} e^{-8ik^2 t} + \Phi_{j3} = c_j. \hspace{1cm} (B.1)
\]
And from equation

$$\mu_t + [4ik^3\Lambda, \mu] = V_2\mu.$$ 

we get

$$\begin{pmatrix}
\Phi_{2\times2} & \Phi_{j3} \\
\Phi_{3j} & \Phi_{33}
\end{pmatrix}_t + 4ik^3 \begin{pmatrix}
0 & 2\Phi_{j3} \\
-2\Phi_{3j} & 0
\end{pmatrix} = 4k^2 \begin{pmatrix}
U^T\Phi_{3j} & U^T\Phi_{33} \\
-\bar{U}\Phi_{2\times2} & -\bar{U}\Phi_{j3}
\end{pmatrix}$$

$$+ 2ik \begin{pmatrix}
\bar{U}^T\Phi_{2\times2} + U^T_x\Phi_{3j} & U^T\bar{U}\Phi_{j3} + U^T_x\Phi_{33} \\
\bar{U}_x\Phi_{2\times2} - 2|u|^2\Phi_{3j} & \bar{U}_x\Phi_{j3} - 2|u|^2\Phi_{33}
\end{pmatrix} - 4|u|^2 \begin{pmatrix}
U^T\Phi_{3j} & U^T\Phi_{33} \\
-\bar{U}\Phi_{2\times2} & -\bar{U}\Phi_{j3}
\end{pmatrix}$$

$$- \begin{pmatrix}
U^T_{xx}\Phi_{3j} & U^T_{xx}\Phi_{33} \\
-\bar{U}_{xx}\Phi_{2\times2} & -\bar{U}_{xx}\Phi_{j3}
\end{pmatrix} + (u\bar{u}_x - u_x\bar{u})(\sigma_3\Phi_{2\times2} + \sigma_3\Phi_{j3})$$

$$\begin{pmatrix}
\Phi_{3j3} \\
\Phi_{333}
\end{pmatrix} = \begin{pmatrix}
\Phi_{j3} \\
\Phi_{33}
\end{pmatrix} = \begin{pmatrix}
(\alpha_0(t) + \frac{\alpha_1(t)}{k} + \frac{\alpha_2(t)}{k^2} + \cdots) + (\beta_0(t) + \frac{\beta_1(t)}{k} + \frac{\beta_2(t)}{k^2} + \cdots) e^{-ik^3t}
\end{pmatrix}$$

(B.4)

where the coefficients $\alpha_l(t)$ and $\beta_l(t)$, $l \geq 0$, are independent of $k$.

To determine these coefficients, we substitute the above equation into equation (B.3) and use the initial conditions

$$\alpha_0(0) + \beta_0(0) = (0_{1\times2}, 1)^T, \quad \alpha_1(0) + \beta_1(0) = (0_{1\times2}, 0)^T.$$ 

Then we get

$$\begin{pmatrix}
\Phi_{j3} \\
\Phi_{33}
\end{pmatrix} = \begin{pmatrix}
0_{1\times2} \\
1
\end{pmatrix} + \frac{1}{k} \begin{pmatrix}
\Phi_{j3}^{(1)} \\
\Phi_{33}^{(1)}
\end{pmatrix} + \frac{1}{k^2} \begin{pmatrix}
\Phi_{j3}^{(2)} \\
\Phi_{33}^{(2)}
\end{pmatrix} + \cdots$$

$$+ \left[ -\frac{1}{k} \begin{pmatrix}
\Phi_{j3}^{(1)}(0) \\
0
\end{pmatrix} + \cdots \right] e^{-ik^3t}$$

(B.5)
From the first column of the equation (B.2) we get
\[
\begin{align*}
\Phi_{2\times 2t} &= 4k^2 U^T \Phi_{3j} + 2ik (U^T \bar{U} \Phi_{2\times 2} + U^T \bar{x} \Phi_{3j}) \\
-4|u|^2 U^T \Phi_{3j} - U^T \Phi_{3j} + (u \bar{u} - u_x \bar{u}) \sigma_3 \Phi_{2\times 2} \\
\Phi_{3jt} - 8ik^3 \Phi_{3j} &= -4k^2 \bar{U} \Phi_{2\times 2} + 2ik (\bar{U}_x \Phi_{2\times 2} - 2|u|^2 \Phi_{3j}) + 4|u|^2 \bar{U} \Phi_{2\times 2} + \bar{U}_x \Phi_{2\times 2}.
\end{align*}
\] (B.6)

Suppose
\[
\begin{pmatrix}
\Phi_{2\times 2}

\Phi_{3j}
\end{pmatrix}
= (\xi_0(t) + \frac{\xi_1(t)}{k} + \frac{\xi_2(t)}{k^2} + \cdots) + (\nu_0(t) + \frac{\nu_1(t)}{k} + \frac{\nu_2(t)}{k^2} + \cdots) e^{8ik^3 t}
\] (B.7)

where the coefficients $\xi_l(t)$ and $\nu_l(t)$, $l \geq 0$, are independent of $k$.

To determine these coefficients, we substitute the above equation into equation (B.6) and use the initial conditions
\[
\xi_0(0) + \nu_0(0) = (I_{2\times 2}, 0_{2\times 1})^T,
\]

Then we get
\[
\begin{pmatrix}
\Phi_{2\times 2}

\Phi_{3j}
\end{pmatrix}
= \begin{pmatrix}
I_{2\times 2}

0_{2\times 1}
\end{pmatrix} + \frac{1}{k} \begin{pmatrix}
\Phi_{2\times 2}^{(1)}

\Phi_{3j}^{(1)}
\end{pmatrix} + \cdots
\]
\[
+ \left[ \frac{1}{k^2} \begin{pmatrix}
\nu_0^{(2)}

0
\end{pmatrix} + \cdots \right] e^{8ik^3 t}
\] (B.8)

So, from the equation (B.1) and the asymptotic of $s_{j3}(k)$ and $s_{33}(k)$, we get the asymptotic behavior of $c_j(t, k)$ as $k \to \infty$,
\[
c_j(t, k) = \frac{\Phi_{j3}^{(1)}}{k} + \frac{\Phi_{j3}^{(2)}}{k^2} + \frac{\Phi_{j3}^{(3)}}{k^3} + \cdots.
\] (B.9)

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