Light Quark Masses from Exclusive Tau Decays: An Experimental Proposal

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Abstract

A method of indirect measurement of the light quark running mass $\hat{m} = (m_d + m_u) / 2$ is elaborated in detail. It is based on measuring 1%-level azimuthal angular asymmetries in the decay $\tau \to \nu_\tau + 3\pi$. The latter are then used in QCD sum rules to obtain experimental lower bounds for $\hat{m}$. For a sample of $2.5 \times 10^5 \tau \to \nu_\tau + 3\pi$ decays free of background, the resulting statistical error in the bound for $\hat{m}$ is estimated to be 1 MeV, i.e., comparable to the systematic error due to the use of QCD sum rules.

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1. INTRODUCTION

The QCD Lagrangian contains seven parameters (apart from the vacuum angle \( \theta \)) which are not determined by theoretical considerations: The gauge coupling constant \( g \) and the quark masses \( m_a \), one for each flavor \((a = u, d, s, c, b, t)\). After renormalization, these constants become scale dependent:

\[
g \rightarrow \alpha_s(\mu^2) = \frac{12\pi}{33 - 2N_f} (\ln \frac{\mu^2}{\Lambda^2})^{-1} \{1 + \ldots\}
\]

\[
m_a \rightarrow m_a(\mu^2) = \bar{m}_a \left(\frac{1}{2} \ln \frac{\mu^2}{\Lambda^2}\right)^{-\frac{1}{33 - 2N_f}} \{1 + \ldots\}.
\]

(The dots stand for higher loop corrections.) Experimental study of these parameters is necessarily indirect, since quarks and gluons do not exist as free particles. The measurement of the running coupling constant \( \alpha_s(\mu^2) \) in different processes and at different scales provides a nontrivial test of QCD at short distances. Heavy quark \((c, b, t)\) masses can – in principle – be investigated in the framework of heavy flavor quarkonia spectroscopy. Finally, as argued below, hadronic \( \tau \) decays seem to provide a unique source of experimental information on light (current) quark running masses \( m_u, m_d, m_s \). The purpose of the present communication is to indicate how this information can be extracted in practice from experiment.

Light quark masses (LQM) have a reputation of being ”not well-measurable” and, indeed, their experimental determination has so far not even been attempted. On the other hand, there exists a huge number of theoretical estimates of their values, some of which claim an amazing accuracy. These estimates will be commented on shortly. At any rate, LQM are certainly tiny compared to the scale \( \Lambda_H \sim 1 \) GeV at which massive bound states of QCD are formed. This fact represents the major difficulty in the experimental study of LQM, since their contribution to hadron masses (except pions) is small and hard to estimate theoretically. On the other hand, LQM measure the absolute strength of chiral symmetry breaking. This follows from the fact that in QCD the divergences of observable axial and vector weak-transition currents are given by the equations

\[
\partial^\mu (\bar{d} \gamma_\mu \gamma_5 u) = (m_d + m_u) \bar{d} i \gamma_5 u
\]

\[
\partial^\mu (\bar{s} \gamma_\mu \gamma_5 u) = (m_s + m_u) \bar{s} i \gamma_5 u
\]

\[
\partial^\mu (\bar{d} \gamma_\mu u) = (m_d - m_u) i \bar{d} u
\]

\[
\partial^\mu (\bar{s} \gamma_\mu u) = (m_s - m_u) i \bar{s} u
\]

which are valid at all scales. The experimental determination of LQM is based on these equations, combined with the short distance properties of QCD.

Before we develop the details of our argument that experimental investigation of LQM is feasible, let us briefly summarize why it is important:

(i) It is important to know all parameters of the standard model.
(ii) The values of LQM are closely related to the value of the quark-antiquark condensate $\langle \bar{\psi} \psi \rangle$, which is an important quantitative characteristic of the nonperturbative chiral order in the QCD vacuum.

(iii) It is important to have a direct experimental check of the theoretical prediction for the ratios of LQM that is based on standard chiral perturbation theory. The existence of a possible disagreement between this prediction and some low-energy data has recently been pointed out.

(iv) The claim that “$m_u$ is not equal to zero” should be framed in terms of a bound referred to a confidence level with a firm experimental basis. This issue is important for the understanding of the strong CP violation problem.

2. HOW BIG IS THE DIVERGENCE OF THE AXIAL CURRENT?

We shall mainly concentrate on the average of the $u$ and $d$ quark masses,

$$ \hat{m} = (m_u + m_d)/2, \quad (3) $$

which, according to Eq. (2a), controls the strength of the divergence of the axial current. (The remaining cases of $m_s - m_u$ and $m_d - m_u$ will be briefly mentioned in Section 5.)

The object of our concern will be the spectral function

$$ \rho(Q^2) = \frac{1}{2\pi} \sum_n (2\pi)^4 \delta^{(4)}(Q - P_n) |\langle n| \partial^{\mu}(\bar{d}\gamma_{\mu}\gamma_5 u)|0\rangle|^2, \quad (4) $$

where the sum extends over all states with the quantum numbers of the pion and with squared invariant mass $Q^2$:

$$ n = \pi^-, \pi^-\pi^+, \pi^-\pi^0\pi^0, \pi^-\pi^+\pi^-\pi^0, \ldots \quad (5) $$

The spectral function $\rho(Q^2)$ measures the amount of explicit chiral symmetry breaking at squared momentum transfer $Q^2$. For large $Q^2$, it is given by QCD perturbation theory:

$$ \rho(Q^2) \to \frac{3}{2\pi^2} [\hat{m}(Q^2)]^2 Q^2 \left\{ 1 + \frac{17}{3} \frac{\alpha_s(Q^2)}{\pi} + \ldots \right\}. \quad (6) $$

Hence, $\hat{m}(Q^2)$ is directly measurable to the extent that $\rho(Q^2)$ is measurable for sufficiently large $Q^2$. The spectral function is comprised of individual exclusive components

$$ \rho(Q^2) = 2F_\pi^2 M_\pi^4 \delta(Q^2 - M_\pi^2) + \rho_{3\pi}(Q^2) + \rho_{K\bar{K}\pi}(Q^2) + \rho_{5\pi}(Q^2) + \ldots. \quad (7) $$

The one-pion contribution is the only one which is known; it is given by the pion mass, $M_\pi = 139$ MeV, and by the pion decay constant, $F_\pi = 93.1$ MeV. The remaining components, $\rho_{3\pi}, \rho_{K\bar{K}\pi}$, etc. are terra incognita of particle physics; they hide the experimental information on the quark mass $\hat{m}$ that we are looking for.
The only way to measure the unknown components of the spectral function Eq. (4) seems to be via exclusive hadronic $\tau$ decays, such as $\tau \rightarrow \nu_\tau + 3\pi, \nu_\tau + 5\pi, \nu_\tau + \pi K\bar{K}$, etc. (See Fig. 1.) In order to extract the desired information from these decays, one faces two distinct problems:

(i) The final hadronic state (say $3\pi$) excited from the vacuum by the virtual $W$ consists of $J = 1$ and $J = 0$ waves. The $J = 1$ wave, which is large and resonant (cf. the $a_1$ resonance), is a subject of current experimental study [12]; however, for our purposes, it is of little interest. $\rho_{3\pi}(Q^2)$ is given by the square of the $J = 0$ part of the corresponding amplitude, which is $O(\hat{m}^2)$, i.e. too small to be directly measured. A model-independent solution of this problem is presented in the next section. $\rho_{3\pi}$ can be reconstructed from the interference of $J=0$ and $J=1$ waves, which is $O(\hat{m})$, and can be measured even for $\hat{m}$ as small as 7 MeV, provided one has large enough statistics.

(ii) The second difficulty is related to the use of the asymptotic formula Eq. (6) in a not quite asymptotic region of $Q^2$. Not only is $Q^2$ limited by the $\tau$ mass $m_\tau$, but the differential decay rate is strongly suppressed near $Q^2 = m_\tau^2$. This difficulty may be resolved by using QCD sum rules [13] for the two-point correlator of the axial current divergence. In general, these sum rules can be put into the form

$$\hat{m}^2(s_0) = H^{-1}(w, s_0) \int_0^\infty dQ^2 \, w(Q^2, s_0) \rho(Q^2).$$

(8)

Here, $w$ denotes a positive weight function that selects contributions with $Q^2 < s_0$. In practice, one uses $w = exp(-Q^2/s_0)$ or $w = Q^{2\eta}(s_0 - Q^2)$. $H(w, s_0)$ is then defined by the large $Q^2$ behavior of the two-point correlator. It consists of a part given by QCD perturbation theory and of a nonperturbative part parametrized in terms of vacuum condensates. The theoretical uncertainty in $H(w, s_0)$ decreases with increasing $s_0$.

Sum rules of the type shown in Eq. (8) have been extensively used in the past to estimate quark masses. These studies show a reasonable stability with respect to different choices of the weight function $w$, and they are rather insensitive to the uncertainty in the nonperturbative part of $H(w, s_0)$. The main problem in these estimates is not the sum-rule technique itself but rather the complete absence of experimental information on the magnitude of $\rho(Q^2)$ beyond the one-pion contribution. Retaining just the pion contribution to the integral of Eq. (8), and using the positivity of individual components Eq. (7) of $\rho(Q^2)$, one finds [3] a lower bound

$$\hat{m}(1 \text{ GeV}^2) \geq (4 - 5) \text{ MeV}.$$  

(9)

We propose to exploit the sum rules Eq. (8) once more, and to improve the lower bound Eq. (9) by using the measured component $\rho_{3\pi}(Q^2)$ as input. $\tau$ decays make it possible to work at higher $s_0$ ($s_0 \leq m_\tau^2$), thus reducing the systematic error due to the nonperturbative part of $H(w, s_0)$. 

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3. RECONSTRUCTION OF $\rho_{3\pi}$ FROM $\tau$-DECAY EXPERIMENTS

There are two $3\pi$ contributions to $\rho(Q^2)$: $\pi^-\pi^-\pi^+$ and $\pi^0\pi^0\pi^-$. They are not related by any symmetry, and consequently they should be measured separately and added afterwards:

$$\rho_{3\pi}(Q^2) = \rho_{\pi\pi\pi}(Q^2) + \rho_{\pi\pi\pi}(Q^2).$$

The kinematic analysis of the corresponding $\tau^- \rightarrow \nu_\tau 3\pi$ decays is identical in both cases; the difference resides merely in possible experimental difficulty in detecting neutral pions with sufficient efficiency. Even if the latter could not be attained, experimental information on $\rho_{\pi\pi\pi}$ would still be valuable. This is because isospin symmetry implies the inequality

$$\rho_{3\pi}(Q^2) \geq \frac{5}{4} \rho_{\pi\pi\pi}(Q^2),$$

which could lead to a considerable improvement of the lower bound Eq. (9) even if no experimental information on $\rho_{\pi\pi\pi}$ were available.

3a. Azimuthal asymmetries

The kinematics of exclusive decays of $\tau$ into three hadrons has been analyzed in full generality in Ref. [14] (hereafter referred to as KM). Here, we will use the same notation, except for a few simplifications which are appropriate for the special case of the decays

$$\tau^- \rightarrow \pi^a(q_1)\pi^b(q_2)\pi^\nu(q_3)\nu_\tau,$$

where $a = -(0), \ b = +(-)$. These decays proceed via the axial weak current

$$\langle \pi^-(q_1)\pi^-(q_2)\pi^+(q_3)|d\gamma_\mu\gamma_5u|0 \rangle \equiv A_\mu(q_1, q_2, q_3)$$

and are described by three independent form factors. (The anomalous vector current contribution vanishes because of G-parity conservation.) Following KM, it is convenient to work in the hadronic center of mass frame, in which the three pions move in the $x-y$ plane. The $x$ axis is chosen parallel to $\vec{q}_3$ and the hadronic plane is oriented so that the $z$ axis points in the direction of $\vec{q}_1 \times \vec{q}_2$, where $|\vec{q}_1| > |\vec{q}_2|$. In this frame, the $z$-component of Eq. (13) vanishes and the three independent components $A_x, A_y$ and $A_t$ are functions of the scalar hadronic variables

$$Q^2 = (q_1 + q_2 + q_3)^2, \ s_1 = (Q - q_1)^2, \ s_2 = (Q - q_2)^2.$$

The three form factors $A_x, A_y$ and $A_t$ are related to the form factors $F_1, F_2$ and $F_4$ of KM by

$$A_x = x_1 F_1 + x_2 F_2, \ A_y = x_3 (F_1 - F_2), \ A_t = \sqrt{Q^2} F_4,$$
where the $x_i$ are kinematic functions defined in Eq. (33) of KM. The space components $A_x$ and $A_y$ describe the $J = 1$ part of the matrix element. They are large and insensitive to the quark mass. The time component $A_t$ is proportional to the matrix element of the divergence of the axial current, and consequently to the running quark mass $\hat{m}$.

We do not assume a polarized $\tau$ beam, and the knowledge of the direction of flight of the $\tau$ in the lab system (i.e., of the $\tau$ rest frame) is not needed. One should just measure the momenta of the three pions and reconstruct their center of mass frame as defined above. This determines $Q^2$, the Dalitz plot variables $s_1$ and $s_2$, and the angles $\beta$ (polar) and $\gamma$ (azimuthal) which define the direction of flight $\vec{n}_L$ of the laboratory with respect to the oriented center of mass hadronic plane (see Fig. 2). The remaining variable is then $\theta$ (the $\tau$ decay angle) given in terms of the hadronic energy $E_h$ and the energy $E_\tau$ of the $\tau$, both in the laboratory frame.

The hadronic structure functions of interest are

\[
W_{ab}(Q^2, \Delta_i) \equiv \int_{\Delta_i} ds_1 ds_2 \text{Re}(A^*_a A_b), \quad a, b = x, y, t,
\]

(16)

where $\Delta_i$ denotes a bin of the Dalitz plot. $W_{xx}, W_{yy}$ and $W_{xy}$ are independent of the quark mass, while the interference terms $W_{xt}$ and $W_{yt}$ are $O(\hat{m})$, and $W_{tt}$ is $O(\hat{m}^2)$. The latter determines the contribution to the spectral function Eq. (4),

\[
\rho_{--}(Q^2) = \frac{1}{512\pi^4} \sum_i W_{tt}(Q^2, \Delta_i),
\]

(17)

where the sum extends over all bins of the Dalitz plot. (By definition, the sum in (17) is independent of the manner in which the phase space is divided into bins.) The most straightforward measurement of the relevant structure functions $W_{ab}(Q^2, \Delta_i)$ involves the differential decay rate integrated over the polar angle $\beta$ and over the $\tau$-decay angle $\theta$:

\[
\Gamma(Q^2, \Delta_i, \gamma) = \frac{1}{(2\pi)^5} \frac{C_F^2}{128 m_\tau} |V_{ud}|^2 \left( \frac{m_\tau^2 - Q^2}{Q^2} \right)^2 \frac{m_\tau^2 + 2Q^2}{3m_\tau^2} W(Q^2, \Delta_i) f(\gamma) \frac{d\gamma}{2\pi} dQ^2,
\]

(18)

where

\[
W(Q^2, \Delta_i) \equiv W_{xx}(Q^2, \Delta_i) + W_{yy}(Q^2, \Delta_i) + \frac{3m_\tau^2}{m_\tau^2 + 2Q^2} W_{tt}(Q^2, \Delta_i)
\]

(19)

describes the differential decay rate integrated over all angles. The normalized distribution in the azimuthal angle $\gamma$ is of the form

\[
f(\gamma) = 1 + \lambda_2 (A \cos 2\gamma + B \sin 2\gamma) + \lambda_1 (C_{LR} \cos \gamma + C_{UD} \sin \gamma).
\]

(20)

The coefficients $A$ and $B$ are “large,” i.e., they are $O(1)$ in the chiral limit:

\[
A = \frac{m_\tau^2 - Q^2}{m_\tau^2 + Q^2} \frac{W_{xx} - W_{yy}}{W}, \quad B = \frac{m_\tau^2 - Q^2}{m_\tau^2 + Q^2} \frac{2W_{xy}}{W}.
\]

(21)
The small chiral symmetry breaking shows up through the left-right and up-down asymmetry coefficients

\[
C_{LR} = \frac{\pi}{2} \frac{3m^{2}_{\tau}}{m^{2}_{\tau} + 2Q^{2}} \frac{W_{xt}}{W}, \quad C_{UD} = -\frac{\pi}{2} \frac{3m^{2}_{\tau}}{m^{2}_{\tau} + 2Q^{2}} \frac{W_{yt}}{W},
\]

which are proportional to the quark mass \( \hat{m} \). Measurement of these azimuthal asymmetries represents the hard core of our method of determining \( \hat{m} \). The coefficients \( \lambda_{1} \) and \( \lambda_{2} \) in Eq. (20) are kinematic functions of \( Q^{2} \) and of the velocity of the \( \tau \) in the laboratory frame, \( \beta_{\tau} = \sqrt{1 - m^{2}_{\tau}/E_{0}^{2}} \), where \( E_{0} \) is the energy of the beam. The \( \lambda_{n} \) result from integration over the \( \tau \) decay angle \( \theta \),

\[
\lambda_{n}(Q^{2}, \beta_{\tau}) = \int_{-1}^{1} \frac{d\cos \theta}{2} P_{n}(\cos \psi),
\]

where \( P_{n} \) are Legendre polynomials and \( \cos \psi \) is a function of \( \cos \theta, Q^{2} \) and \( \beta_{\tau} \) as defined in KM. The shape of these functions depends strongly upon the energy of the machine, \( E_{0} \). For \( \tau \) leptons produced at rest \( \cos \psi = 1 \), so one has

\[
\lambda_{n}(Q^{2}, 0) \equiv 1,
\]

whereas in the ultrarelativistic limit \( \beta_{\tau} = 1 \) these functions become (see Fig. 3.)

\[
\lambda_{1}(Q^{2}, 1) = \frac{m^{4}_{\tau} - Q^{4} + 2m^{2}_{\tau}Q^{2}\ln Q^{2}/m^{2}_{\tau}}{(m^{2}_{\tau} - Q^{2})^{2}}
\]

\[
\lambda_{2}(Q^{2}, 1) = -2 + 3 \frac{m^{2}_{\tau} + Q^{2}}{m^{2}_{\tau} - Q^{2}} \lambda_{1}(Q^{2}, 1).
\]

For low-energy machines such as tau-charm factories, the integration over the decay angle \( \theta \) represents a simplification which does not lower the sensitivity to the azimuthal asymmetries (21) and (22). On the other hand, for high-energy machines such as at LEP (\( \beta_{\tau} \approx 0.999 \)), CESR (\( \beta_{\tau} \approx 0.93 \)) or B-factories, Eq. (24b) indicates an important loss in sensitivity. In this case, one should make use of the knowledge of the decay angle distribution rather than simply integrating over it.

3b. The trick

One may take advantage of the smallness of the quark mass and neglect \( W_{tt} \) compared to \( W_{xx} + W_{yy} \) in Eq. (19). Within this approximation, the measurement described above yields the structure functions \( W_{xx}, W_{yy}, W_{xy} \) and \( W_{xt}, W_{yt} \) for a given \( Q^{2} \) and for a given bin \( \Delta_{i} \) of the Dalitz plot. On the other hand, similar information on the interference terms \( Im A^{*}_{x}A \) and \( Im A^{*}_{y}A \) would require both a known nonzero \( \tau \) polarization and the reconstruction of the \( \tau \) rest frame. This information is, fortunately, not needed. We will prove that, provided the binning of the Dalitz plot is sufficiently fine, the \( O(\hat{m}^{2}) \) quantity \( W_{tt} \) can be reconstructed from the experimentally accessible \( O(\hat{m}) \) interference terms.
\[
\phi(Q^2, \Delta_i) = \begin{pmatrix} W_{xt} \\ W_{yt} \end{pmatrix}
\] (25a)

and from the spin-1 structure functions
\[
K(Q^2, \Delta_i) = \begin{pmatrix} W_{xx} & W_{xy} \\ W_{yx} & W_{yy} \end{pmatrix} = K^T.
\] (25b)

Given two complex numbers \(x\) and \(z\), it is obviously impossible to reconstruct \(|z|^2\) from \(|x|^2\) and \(Re(x^*z)\) alone. However, for three complex numbers \(x_1, x_2\) and \(z\) one has the identity
\[
|z|^2 = \sum_{i,j=1}^{2} Re(x_i^*z)(k^{-1})_{ij}Re(z^*x_j),
\] (26)

where \(k^{-1}\) denotes the inverse of the matrix \(k_{ij} = Re(x_i^*x_j)\). This identity applies to the decay \(\tau \rightarrow 3\pi\nu\tau\), which is characterized by three (complex) form factors. For each bin \(\Delta_i\), let us define the quantity
\[
\mathbf{W}_{tt}(Q^2, \Delta) = \phi^T(Q^2, \Delta)K^{-1}(Q^2, \Delta_i)\phi(Q^2, \Delta),
\] (27)

given entirely in terms of observables (25a) and (25b). If a bin \(\Delta\) shrinks to a point \(P\) of the Dalitz plot, one has
\[
W_{ab}(Q^2, \Delta) = Re(A_a^*(P)A_b(P)) \epsilon(\Delta)
\] (28)

where
\[
\epsilon(\Delta) = \int_{\Delta} ds_1ds_2 \to 0.
\] (29)

Consequently, in the small bin limit, the identity (26) implies
\[
W_{tt}(Q^2, \Delta) = \mathbf{W}_{tt}(Q^2, \Delta) + O(\epsilon^2).
\] (30)

It is now straightforward to reexpress this result in terms of the spectral function Eq. (17). For a given division of the Dalitz plot into \(N\) bins, define
\[
\rho^{(N)}_{-+}(Q^2) \equiv \frac{1}{512\pi^4} \sum_{i=1}^{N} \mathbf{W}_{tt}(Q^2, \Delta_i),
\] (31)

where \(\mathbf{W}_{tt}\) is defined by Eq. (27). Consider now the limit \(N \rightarrow \infty\) such that \(\epsilon(\Delta) \sim O(N^{-2})\). The "true" spectral function is then obtained as
\[
\rho_{-+}(Q^2) = \lim_{N \rightarrow \infty} \rho^{(N)}_{-+}(Q^2).
\] (32)
In practice, however, one deals with a finite number of bins of a finite size, and it is important to analyze what can be concluded in this case. Independently of the size of $\Delta$, Eq. (16) defines a scalar product which satisfies the Schwarz inequality

$$ W_{xt}(Q^2, \Delta) \geq \frac{|W_{xt}(Q^2, \Delta)|^2}{W_{xx}(Q^2, \Delta)}. \quad (33) $$

A similar lower bound holds under the substitutions $W_{xt} \rightarrow W'_{xt}$ and $W_{xx} \rightarrow W'_{xx}$, where $W'_{xt}$ and $W'_{xx}$ are obtained by the replacement

$$ A_x \rightarrow A'_x = g_1 A_x + g_2 A_y, \quad (34) $$

$g_1, g_2$ being two arbitrary real constants. In this way, one generates a whole class of lower bounds, and one can then ask which one is the best. Remarkably, the answer leads to the reappearance of the expression Eq. (27); one easily finds

$$ \max_{(g_1, g_2)} \frac{(W'_{xt})^2}{W'_{xx}} = W_{xt}(Q^2, \Delta_i). \quad (35) $$

Consequently, for any finite size bin $\Delta_i$, one has the lower bound

$$ W_{xt}(Q^2, \Delta_i) \geq W_{xt}(Q^2, \Delta_i), \quad (36) $$

which, according to Eq. (30), is saturated in the limit of small bins. For any splitting of the Dalitz plot into $N$ bins, one can sum Eq. (36) over all bins and divide by $512 \pi^4$; by Eq. (31), the left hand side is simply $\rho_{--}(Q^2)$, which is, of course, independent of binning, while the right hand side is precisely $\rho^{(N)}_{--}(Q^2)$. One thus obtains the lower bound

$$ \rho_{--}(Q^2) \geq \rho^{(N)}_{--}(Q^2), \quad (37) $$

which is saturated with increasing number of bins.

It is worth noting the independence of our method on the geometry chosen in the hadronic plane. This is a reflection of the fact that the expression Eq. (27) is invariant under

$$ \phi \rightarrow g\phi, \quad K \rightarrow gKg^T, \quad (38) $$

where $g$ is a two-by-two real matrix, representing a general linear (nonsingular) real transformation of a two-dimensional space spanned by $A_x$ and $A_y$.

3c. Back to the sum rules

Suppose that the above program has been completed for both $\tau^- \rightarrow \pi^-\pi^-\pi^+\nu_\tau$ and $\tau^- \rightarrow \pi^0\pi^0\pi^-\nu_\tau$ decays. That is, the azimuthal asymmetries have been measured in $N$ bins of the Dalitz plot and the corresponding functions $\rho_{--}$ and $\rho_{00--}$ have been
constructed according to Eq. (31) for \( N \leq N_{\text{max}} \) allowed by the available statistics. One can then return to the QCD sum rules and define the quantity

\[
\hat{m}_N^2(\mu, s_0) \equiv \left( \frac{\ln s_0/\Lambda^2}{\ln \mu^2/\Lambda^2} \right)^{24/29} H^{-1}(w, s_0) \int_0^{m_0^2} dQ_2 w(Q^2, s_0) \{ 2F_{\pi}^2 M_{\pi}^2 \delta(Q^2 - M_{\pi}^2) \\
+ \rho_{-+}^{(N)}(Q^2) + \rho_{00}^{(N)}(Q^2) \}.
\]

One expects that \( \hat{m}_N(\mu, s_0) \) will depend rather weakly on the choice of the weight function \( w \), and on \( s_0 \) in a typical range

\[
2 \text{ GeV}^2 \leq s_0 \leq m_r^2 \simeq 3.18 \text{ GeV}^2,
\]

especially for large \( N \). For \( s_0 \) in the range (40), the running quark mass \( \hat{m}(\mu) \) obeys

\[
\hat{m}(\mu) \geq \hat{m}_N(\mu, s_0),
\]

and this lower bound should saturate rather rapidly with increasing \( N \). The saturation as well as the variation of the right hand side of Eq. (39) with \( s_0 \) can be controlled experimentally. The latter variation can be considered as a source of systematic error arising from imperfections in the method. Another source of error, which is more difficult to estimate, is related to both perturbative (higher orders \([15]\)) and nonperturbative (condensates, instantons \([16]\)) uncertainties in the high-energy factor \( H(w, s_0) \).

4. WHAT IS EXPECTED

Light quark masses are the only entries in the Particle Data Group compilations \([17]\) that are not based on a measurement but on theoretical estimates. We first briefly recall why a direct estimate of \( \hat{m} \) is problematic without experimental information on \( \rho_{3\pi}(Q^2) \). Then we proceed to a model-dependent numerical study of the various steps of the method described in the preceding section. In particular, the resulting statistical error of a measurement of \( \hat{m}(1 \text{ GeV}) \) will be estimated, using the maximum likelihood method.

For definiteness, the finite energy version of QCD sum rules will be used here, closely following Ref. \([3]\); Eq. (31) then takes the form

\[
\hat{m}^2(s_0) = \frac{4\pi^2}{3s_0^2} \left[ 1 + R_2(s_0) + 2C_4(\alpha_4)/s_0^2 \right]^{-1} \int_0^{s_0} dQ^2 \rho(Q^2),
\]

and similarly, the definition Eq. (39) becomes

\[
\hat{m}_N^2(\mu, s_0) \equiv \left( \frac{\ln s_0/\Lambda^2}{\ln \mu^2/\Lambda^2} \right)^{24/29} \frac{4\pi^2}{3s_0^2} \left[ 1 + R_2(s_0) + 2C_4(\alpha_4)/s_0^2 \right]^{-1} \times
\left[ 2F_{\pi}^2 M_{\pi}^4 + \int_0^{s_0} dQ^2 \left\{ \rho_{-+}^{(N)}(Q^2) + \rho_{00}^{(N)}(Q^2) \right\} \right],
\]

where the two-loop expression for \( R_2(s_0) \) as well as a discussion of the value of the dimension-4 condensate \( C_4(\alpha_4) \) can be found in Ref. \([3]\). We will use the value for the QCD coupling constant \( \alpha_s \) as determined in Ref. \([3]\).
4a. Chiral Perturbation theory

Unlike $\hat{m}$, the order of magnitude of the difference $m_s - \hat{m}$ is fairly easy to estimate: From the size of $SU_V(3)$ symmetry breaking, one may infer

$$(m_s - \hat{m})(1 \text{ GeV}^2) = 100 - 300 \text{ MeV},$$

in agreement with sum-rule results. The estimate Eq. (44) can be combined with the presumed value of the quark mass ratio $r = m_s / \hat{m} = 26$ to conclude that $\hat{m}(1 \text{ GeV}^2) = 4 - 12 \text{ MeV}$, in agreement with the lower bound (9). Actually, even this rather conservative and crude estimate of $\hat{m}$ is doubtful. The value $r = 26$ is based on the standard chiral perturbation theory which assumes that the quark antiquark condensate $-\langle \bar{\psi} \psi \rangle$ is large compared to $F_\pi^2 m_s$. The latter hypothesis has no clear experimental or theoretical support, and in fact it need not be correct for the actual value of $m_s$. The generalized chiral perturbation theory which admits a lower value of $-\langle \bar{\psi} \psi \rangle$ does not fix the quark mass ratio $r = m_s / \hat{m}$; any value $6.3 \leq r \leq 25.9$ is consistent with the mass spectrum of pseudoscalar mesons, the lower bound for $r$ arising from the condition of vacuum stability. Hence, accepting the estimate Eq. (44) constrains $\hat{m}$ to the range

$$4 \text{ MeV} \leq \hat{m}(1 \text{ GeV}^2) \leq 50 \text{ MeV}.$$ (45)

However, it is clear that finding $\hat{m}$ significantly higher than – say – 10 MeV would imply a considerably lower value of $-\langle \bar{\psi} \psi \rangle$ and of $r = m_s / \hat{m}$ than the standard chiral perturbation theory could support. (Independent, though indirect, measurements of the quark mass ratio $r$ are possible in low-energy $\pi - \pi$ scattering, $K_{\mu4}$ decays and from observed corrections to the Goldberger-Treiman relations.) This illustrates once more why even an experimental lower bound on $\hat{m}$ would be of considerable interest.

It is instructive to illustrate the variation of $\hat{m}$ and $\langle \bar{\psi} \psi \rangle$ within a simple model, in which the $3\pi$ contribution to the spectral function is described by a narrow $J^P = 0^-$ resonance – the $\pi'$:

$$\rho_{3\pi}(Q^2) = 2F_{\pi'}^2 M_{\pi'}^4 \delta(Q^2 - M_{\pi'}^2).$$

(Such a resonance does indeed exist for $M_{\pi'} \approx 1.3 \text{ GeV}$, but it is not narrow.) The constant $F_{\pi'}$ – the $\pi'$ analogue of $F_\pi = 93.1 \text{ MeV}$ – describes the coupling of the $\pi'$ to the axial current. It is proportional to $\hat{m}$ ($F_{\pi'} = 0$ in the chiral limit) and it is expected to be small compared to $F_\pi$. Its value is, however, unknown and there is no reliable model available to pin it down. (In particular, quark model estimates (e.g., Ref. [18]) of $F_{\pi'}$ are trustworthy only to the extent that the model would correctly describe the small breaking of chiral symmetry.) Considering $F_{\pi'}$ as a free parameter, one may use the narrow $\pi'$ model Eq. (46) within the FESR Eq. (42) in order to investigate the sensitivity of $\hat{m}$ to $F_{\pi'}$. Setting $M_{\pi'} = 1.3 \text{ GeV}$ and fixing $s_0$ from the higher moment sum rules (as explained in Ref. [3]), one finds that as $F_{\pi'}$ increases from 1.5 MeV to 15 MeV, $\hat{m}(1 \text{ GeV})$ slowly rises from 7 MeV to 45 MeV. At the same time, the value of the $\bar{q}q$ condensate $-F_{\pi'}^{-2}\langle \bar{\psi} \psi \rangle$ rapidly decreases from $\sim 1.3 \text{ GeV}$ to $\sim 40 \text{ MeV}$. A priori, there is no reason
to exclude a value for $F_\pi$ as large as 10 or 15 MeV. However, it is clear that, in this case, the $\pi'$ (or $3\pi$) contribution to the quark mass would largely dominate over the single pion contribution. Moreover, the slow rise of $\hat{m}$ would not compensate for the drop of $-\langle \bar{\psi}\psi \rangle$, so that $-2\hat{m}\langle \bar{\psi}\psi \rangle$ would be considerably lower than $F_\pi^2 M_\pi^2$. Such a violation of the Gell-Mann, Oakes, Renner formula is allowed and expected within the generalized chiral perturbation theory, [10] and its experimental confirmation would be an argument in favor of the latter.

It is conceivable that the $\pi'$ width cannot be neglected, and that a more realistic model of $\rho_3\pi\left(Q^2\right)$ is provided by a Breit-Wigner formula. However, even if the corresponding width were known, the absolute normalization of $\rho_3\pi\left(Q^2\right)$ remains unspecified. Dominguez and de Rafael [6] have proposed to normalize $\rho_3\pi\left(Q^2\right)$ by its low $Q^2$ behavior

$$\rho_{3\pi}(Q^2) \rightarrow \frac{1}{768\pi^4} \frac{M_\pi^4}{F_\pi^2} Q^2,$$

(47)

as given by chiral perturbation theory. In this way, they obtain $\hat{m}(1 \text{ GeV}) = 7.8 \pm 1.0$ MeV, where the quoted error merely arises from the input data ($M_\pi, \Gamma_\pi$) and from the uncertainty brought in by the sum rules. It does not involve the possible error in the absolute normalization of $\rho_{3\pi}(Q^2)$ based on Eq. (47).

Apart from doubts about using the $Q^2 \sim 0$ behavior of $\rho_{3\pi}$ to fix its value at $Q^2 \sim M_{\pi'}^2 \sim 1.69 \text{ GeV}^2$, the main uncertainty in the above method of normalizing $\rho_{3\pi}$ resides in the formula Eq. (47) itself. This formula is obtained using the standard chiral perturbation theory in an experimentally unexplored domain of low-energy $\pi-\pi$ interaction, where results strongly depend on the value of the quark antiquark condensate $\langle \bar{\psi}\psi \rangle$. The generalized chiral perturbation theory, [10] which parametrizes this dependence in a model independent way, leads to a modification of Eq. (47):

$$\rho_{3\pi}(Q^2) \rightarrow \frac{1}{768\pi^4} \frac{M_\pi^4}{F_\pi^2} \frac{5\alpha_{\pi\pi}^2 + 1}{6} Q^2.$$

(48)

Here, $\alpha_{\pi\pi}$ is a parameter introduced in [10] which is a function of the quark mass ratio $r = m_s/\hat{m}$, varying from $\alpha_{\pi\pi} = 1 \ (r = 25.9)$ to $\alpha_{\pi\pi} = 4 \ (r = 6.3)$, and which can be measured in low-energy $\pi-\pi$ scattering. [10] Hence, $\rho_{3\pi}(Q^2)$ cannot be normalized using its low-$Q^2$ behavior, since the latter is only known up to a factor of 1–13.5. $\rho_{3\pi}(Q^2)$ has to be determined experimentally.

4b. A model-dependent theoretical experiment

In order to show how the method above might work in practice, we have generated data using a model for the form factors of Eq. (15) and performed the analysis including an estimate of statistical errors. We do not expect any model to give better than an order of magnitude estimate of the observable quantities that we seek. The form factors $F_1, F_2$ are chosen to be exactly those of KM (cf. [18,19]). For the $J = 0$ form factor $F_4$ we follow KM in assuming the dominance of the $\pi'$ resonance that decays into $\rho\pi$; however, we use the minimal $\pi'\pi\rho$ coupling (which they do not), so that
\[ F_4(s_1, s_2, Q^2) = -35i \xi BW_{\pi'}(Q^2)[(s_2-s_3)B_\rho(s_1) + (s_1-s_3)B_\rho(s_2)] \] (49)

where \( s_3 = Q^2 + 3M_{\pi}^2 - s_1 - s_2 \) and the \( \pi' \) Breit-Wigner is given by

\[ BW_{\pi'}(Q^2) = \frac{M_{\pi'}^2}{M_{\pi'}^2 - Q^2 - i\sqrt{Q^2}\Gamma_{\pi'}(Q^2)} \]

\[ \Gamma_{\pi'}(Q^2) = \Gamma_{\pi'} \frac{M_{\pi'}^2}{Q^2}[p(Q^2)/p(M_{\pi'}^2)]^3 \]

\[ p(Q^2) = \sqrt{(Q^2 - (M_\rho + M_{\pi'}^2))(Q^2 - (M_\rho - M_{\pi'})^2)/(4Q^2)}. \] (50)

The parameters \( M_{\pi'} \) and \( \Gamma_{\pi'} \) as well as the spin-1 function, \( B_\rho \), are taken directly from KM without alteration. For simplicity, we take the pion mass to be zero. We have introduced a dimensionless parameter \( \xi \) which sets the scale of the \( J = 0 \) form factor (the numerical factor of 35 is for convenience, with units GeV\(^{-3}\) understood here). The parameter \( \xi \) plays the same role here as the constant \( F_{\pi'} \) did in the narrow-resonance model described in the preceding subsection; that is, it is \( \xi \) which determines the contribution of the \( J = 0 \) spectral function to the quark mass, and so it is an unknown quantity. We will see below that our normalization is chosen so that, within the framework of the model we have adopted, \( \xi \) is of order unity for values of \( \hat{m} \) in the range (45). Note that the model treats the \( \pi^-\pi^-\pi^+ \) and \( \pi^0\pi^0\pi^- \) modes similarly, so we may simply combine the effects of the two modes by absorbing all normalization into the single parameter \( \xi \).

The first step in our experiment is to generate the data, i.e., compute the differential decay rate \( \Gamma(Q^2, \Delta t, \gamma) \) of Eq. (18). We do this for three cases, \( \xi = 0.5, 1 \) and 2. Using the form factors \( F_1, F_2, F_4 \) described above, we compute from Eq. (14) the functions \( A_x, A_y, A_t \), these are then used to determine from Eq. (16) the functions \( W_{ab} \), which then give \( \Gamma \). For our three choices of \( \xi \), we show in Fig. 4 the asymmetry coefficients \( A, B, C_{LR}, C_{UD} \) of Eqs. (21),(22), where the \( W_{ab} \) are integrated over the entire Dalitz plot. Note that the azimuthal asymmetry coefficients \( C_{LR}, C_{UD} \) are on the order of a few percent or less, much smaller than \( A \) (\( B \) is accidentally small here, which is due to the choice of axes in the hadronic plane). This is expected, since \( C_{LR}, C_{UD} \) are \( O(\hat{m}) \), while \( A, B \) are \( O(1) \) in the chiral limit. Similarly, we may compute the functions \( W_{xx}, W_{yy}, W_{xy}, W_{xt}, W_{yt} \) as functions of \( Q^2, s_1, s_2 \). In Fig. 5 we show (for \( \xi = 1 \)) the functions \( W_{ab} \) integrated over the entire Dalitz plot; clearly \( W_{tt} \) is negligible.

Next, we study the lower bound of Eq. (43) for four choices of binning of the Dalitz plot: divide the square \( 0 \leq s_1, s_2 \leq m_{\pi}^2 \) into 1, 4, 16, or 64 equal-sized squares; this gives a total of \( N = 1, 3, 10 \) and 36 bins respectively for these four cases, since the region of the Dalitz plot is simply a triangular half of this square region. For a given choice of binning, we compute for each bin \( \Delta t \) the lower bound \( \widetilde{W}_{tt}(Q^2, \Delta t) \) from Eq. (27). After summing over bins, we show (for \( \xi = 1 \)) in Fig. 6 how \( \sum_{i=1}^{N} \widetilde{W}_{tt}(Q^2, \Delta t) \) approaches \( W_{tt}(Q^2) \) as the number of bins \( N \) grows. Then, we obtain \( \rho^{(N)}_{++} (Q^2) \) from Eq. (31), and thus determine \( \hat{m}_N(\mu, s_0) \) from Eq. (43). The latter is shown (for \( \xi = 1 \)) in Fig. 7b as a function of \( s_0 \) for various \( N \). (We take \( \mu = 1 \) GeV.) Note that the bound is essentially independent of \( s_0 \) for \( 2 \) GeV\(^2 \leq s_0 \leq m_{\pi}^2 \approx 3.18 \) GeV\(^2 \). The figure clearly indicates that the choice of \( \xi = 1 \) corresponds to a lower bound on \( \hat{m}(\mu) \) of about 14 MeV. Similar results are presented in
Fig. 7a for $\xi = 0.5$ (a lower bound of 7 MeV) and in Fig. 7c for $\xi = 2$ (a lower bound of 28 MeV). We see that the bound is roughly proportional to $\xi$ in this range of $\xi$. For smaller $\xi$ the pion contribution to the spectral function eventually dominates, and the bound on $\hat{m}(\mu)$ reduces to that of Ref. [5], independent of $\xi$.

4c. Maximum likelihood estimate of error

Apart from any systematic experimental error, there will be some error arising from finite statistics; this can be estimated by the maximum likelihood method. For $N_{\text{evt}}$ events, the likelihood function is

$$L(\hat{m}_N(\mu, s_0)) = \prod_{i=1}^{N_{\text{evt}}} \Gamma(X_i; \hat{m}_N(\mu, s_0)), \tag{51}$$

where $X_i$ denotes the measured values of the phase space variables $Q^2, s_1, s_2, \gamma$ for the event $i$, and $\Gamma$ is normalized to unity. For fixed $\mu = 1$ GeV and fixed $s_0$, there is a one-to-one correspondence between the lower bound $\hat{m}_N(\mu, s_0)$ and $\xi$; therefore, the measurement of $\xi$ is equivalent to the measurement of $\hat{m}_N(\mu, s_0)$. The best estimate for $\xi$ is the value which maximizes the likelihood $L$, or equivalently $\ln L$; the estimated standard deviation for the measurement of $\xi$ is simply

$$\sigma_{\xi} = \left[ N_{\text{evt}} \int \frac{1}{\Gamma} \left( \frac{\partial \Gamma}{\partial \xi} \right)^2 \right]^{-\frac{1}{2}}. \tag{52}$$

Hence, the standard deviation for the measurement of the lower bound $\hat{m} \equiv \hat{m}_N(\mu, s_0)$ is

$$\sigma_{\hat{m}} = \sigma_{\xi}(d\hat{m}/d\xi); \tag{53a}$$

then, using Eq. (43) to compute $(d\hat{m}/d\xi)$ [the $\xi$ dependence of $\hat{m}$ is contained in $\rho_{++}^{(N)}(Q^2)$ and $\rho_{00}^{(N)}(Q^2)$],

$$\sigma_{\hat{m}} = \sigma_{\xi}(\hat{m}/\xi)[1 - \hat{m}_0^2/\hat{m}^2], \tag{53b}$$

where $\hat{m}_0$ is obtained from $\hat{m}$ by putting $\xi = 0$, i.e., including only the contribution of the pion to the integral of Eq. (43). For our model calculation, the standard deviation $\sigma_{\hat{m}}$ varies as $1/\sqrt{N_{\text{evt}}}$ and is weakly dependent on $s_0$ and $\xi$ ($\sigma_{\xi}$ and $\hat{m}/\xi$ are roughly independent of $\xi$). For example, from 250000 $\tau \to 3\pi \nu_\tau$ events with $\beta_\tau \approx 0$ at the $\tau$-charm factory, one finds $\sigma_{\hat{m}} \approx 1$ MeV for $2 \text{ GeV}^2 \leq s_0 \leq m_\tau^2$. The standard deviation $\sigma_{\hat{m}}$ (in MeV) is displayed in the table below as a function of the number of bins $N$ and of $\xi$:

| N   | N=1 | N=3 | N=10 | N=36 | N=∞ |
|-----|-----|-----|------|------|-----|
| $\xi=0.5$ | 0.5 | 0.6 | 0.8 | 0.9 | 0.9 |
| $\xi=1.0$ | 0.6 | 0.7 | 0.9 | 0.9 | 1.0 |
| $\xi=2.0$ | 0.6 | 0.7 | 0.9 | 1.0 | 1.0 |
It is important to realize that $\sigma_{m}$ increases rapidly with $\beta\tau$, i.e., with beam energy, because of the factor $\lambda_1$ (see Eq. (20) and Fig. 3). Thus, the design of an experiment to measure the quark mass must be optimized with respect to the competing effects of increasing $\tau$ production and decreasing sensitivity as the beam energy is raised from threshold.

In the preceding analysis, no use was made of any information coming from measurement of the decay angle $\theta$. In the computation of $\sigma_{m}$, before integration of $\theta$, $\partial\Gamma/\partial\xi$ is proportional to $\cos\psi$. If one makes no use of the measurement of $\theta$, then one first integrates $\Gamma$ over $\theta$, so $\frac{(\partial\Gamma/\partial\xi)^2}{2}$ is proportional to $\lambda_1^2 = (\int \cos^2\psi)$. As pointed out in Sec. 3a, for $\beta\tau = 0$ there is no loss in sensitivity from this procedure; however, this is no longer true for $\beta\tau \neq 0$, and for $\beta\tau \approx 1$ (characteristic of LEP and CESR) we find that $\sigma_{m}$ is increased by about a factor of 4. So, for example, with $10^5 \tau \rightarrow 3\pi\nu_\tau$ events with $\beta \approx 1$ at CESR, $\sigma_{m} \approx 6$ MeV.

On the other hand, if one makes use of the measurement of $\theta$, then one integrates $(\partial\Gamma/\partial\xi)^2/\Gamma$ over $\theta$ in order to estimate the sensitivity. One can see that this will lead to a much less dramatic degradation of sensitivity: ignoring the relatively weak $\theta$ dependence of the $1/\Gamma$ factor, the $\theta$ integral is proportional to $\int \cos^2\psi = (2\lambda_2 + 1)/3$. A detailed study of this question is beyond the scope of the present work.

5. OTHER COMBINATIONS OF LIGHT QUARK MASSES

In principle, exclusive $\tau$ decays allow the measurement of the divergences of all four currents $\bar{d}\gamma_\mu\gamma_5u$, $\bar{s}\gamma_\mu\gamma_5u$, $\bar{d}\gamma_\mu u$, and $\bar{s}\gamma_\mu u$ which appear in Eqs. (2a-d). This suggests that the method described in detail in the previous section for the case $m_d + m_u$ could be extended to other combinations of LQM: $m_s + m_u$, $m_s - m_u$ and $m_d - m_u$. We shall briefly comment on each of these cases.

5a. $m_s + m_u$

Equation (2b) seems at first sight to allow a straightforward extension of the measurement of $m_d + m_u$ to $m_s + m_u$. The corresponding $J = 0$ spectral function defined as in Eq. (4) receives contributions from single $K$ state and the continuum starts with $K\pi\pi$. The corresponding component $\rho_{K\pi\pi}$ of the spectral function could, in principle, be measured in $\tau \rightarrow \nu_\tau + K\pi\pi$ decays. The Cabibbo suppression of the latter may be partially compensated by a considerably larger ratio of signal to background ($J = 0$ to $J = 1$), which is due to $m_s \gg m_s$. Unfortunately, this decay receives an anomaly contribution $V_z$ from the vector current in addition to the three usual axial-vector form factors $A_x$, $A_y$ and $A_t$. The presence of $V_z$ essentially complicates the reconstruction of $W_{tt}$ and of $\rho_{K\pi\pi}$ from observable quantities. Actually, this reconstruction turns out to be possible only using a polarized $\tau$ beam. [14] (This example emphasizes once more the lucky circumstances which make the measurement of $\rho_{3\pi}$ and of $m_d + m_u$ possible.)

5b. $m_s - m_u$

The difference $m_s - m_u$ controls the divergence of the vector current $\bar{s}\gamma_\mu u$ – cf. Eq. (2d). There is no single-particle contribution to the corresponding $J=0$ spectral func-
tion and the continuum starts with the $K\pi$ state. The component $\rho_{K\pi}(Q^2)$ can be measured in exclusive $\tau$ decays

$$\tau^- \to K^- \pi^0 + \nu_{\tau}, \quad \tau^- \to \bar{K}^0 \pi^- + \nu_{\tau}$$

(54)
described by two form factors representing analytic continuations of the well-known $K\ell_3$ form factors $f_{\pm}(Q^2)$. The $J = 0$ and $J = 1$ combinations of $f_{\pm}$ can be separated by measuring the polar angle between $\vec{n}_L$ and the direction of the $K$ (or $\pi$) momentum in the hadronic center of mass frame. Notice that $\rho_{K\pi}(Q^2)$ – which is not expected to be as small as $\rho_{3\pi}(Q^2)$, since $m_s \gg \hat{m}$ – should now be measured directly. (Since there are only two form factors, it is impossible to reconstruct $\rho_{K\pi}$ from the $s-p$ interference. The latter is, however, interesting in itself; it provides model-independent information on $K-\pi$ phase shifts. \[21\]) Finally, a lower bound for $m_s - m_u$ is provided by the sum rules \[\text{7}\] analogous to (8).

5c. $m_d - m_u$

The leading contribution to the spectral function including the divergence (2c) of the vector current $\bar{d} \gamma_\mu u$ comes from the $\eta\pi$ state. \[22\] The kinematics of the decay

$$\tau^- \to \pi^- \eta + \nu_{\tau}$$

(55)
is completely analogous to the decays (54). However, in this case, the $J=0$ form factor containing the information about $m_d - m_u$ has no particular reason to be suppressed with respect to the $J=1$ form factor. \[22\] The latter also vanishes as $m_d - m_u \to 0$, and is given by isospin breaking effects such as $\eta - \pi^0$ mixing. \[8\] The decay (55) is obviously rare, but its observation and study would provide new information both about $m_d - m_u$ and about the ratio $(m_d - m_u)/(m_s - \hat{m})$.

6. CONCLUSION

In the present paper, we have argued that it may be possible to obtain an experimental determination of the running quark mass $\hat{m} = (m_d + m_u)/2$ based on the measurement of angular asymmetries at the 1% level in the decay $\tau \to \nu_{\tau} + 3\pi$. The hard core of our argument is the trick, described in Section 3b, which allows to reconstruct the dominant $3\pi$ component $\rho_{3\pi}(Q^2)$ to the order $O(\hat{m}^2)$ spectral function $\rho(Q^2)$, associated with the divergence of the axial current, from the measurement of the order $O(\hat{m})$ angular asymmetries.

Combining $\rho_{3\pi}(Q^2)$ as extracted from experiment with QCD sum rules leads to a lower bound for the running quark mass $\hat{m}(\mu)$ as shown by the inequality of Eq.(41). The appearance of only a lower bound arises from the following: We have restricted our attention to the exclusive contributions to $\rho(Q^2)$ coming from one and three pion intermediate states only. Other contributions, such as $K\bar{K}\pi$ or $5\pi$ intermediate states, could also be studied experimentally, but they are expected to be small due to the lack of phase space. Also, in reconstructing $\rho_{3\pi}(Q^2)$ we worked with a finite number of bins in
the fixed \( Q^2 \) Dalitz plot, just as one does in practice. Again, the question of convergence with increasing number of bins can be addressed experimentally.

In order both to study the different steps of the above method and to get an order of magnitude estimate of the quantities we are looking for, we have generated a set of data from a theoretical model described in Section 4. We wish to stress that the model on which these estimates are based should not be given any further significance beyond its illustrative purpose. We have found that a sample of about 250,000 background free \( \tau \to \nu_\tau + 3\pi \) events is needed in order to reduce the statistical error on the lower bound for \( \hat{m} \) down to 1 Mev. This error becomes then comparable to the theoretical uncertainties associated with the sum rules analysis.

As it stands, our analysis is mainly suited for the low energy machines, such as a Tau/Charm Factory: In this case it is possible to integrate over the \( \tau \) decay angle \( \theta \) without loss of sensitivity for the measured angular asymmetries. If, on the other hand, one would like to use a high energy source, such as LEP, CESR, B-factories, the method described above would have to be adjusted and include the measurement of the \( \tau \) decay angle \( \theta \), in order to compensate for the loss of sensitivity.

In the present study, we have focused on the mass combination \( m_u + m_d \), which is rather special, due to the particular role of G-parity. It also remains the most interesting combination from the point of view of its theoretical and phenomenological implications. In contrast, \( m_s + m_u \) cannot be given a lower bound via the same analysis, because the vector current contributes an additional form factor to the corresponding \( \tau \to \nu_\tau + K\pi\pi \) decays. On the other hand, the quark mass differences \( m_s - m_u \) and \( m_d - m_u \) can be investigated in the quasi-two-body decays \( \tau \to \nu_\tau + K\pi \) and in the rare decays \( \tau \to \nu_\tau + \eta\pi \), respectively.

We conclude that a high statistics sample of tagged, exclusive \( \tau \) decays (\( \tau \to \nu_\tau + \pi^-\pi^-\pi^+, \pi^0\pi^0\pi^-, K\bar{K}\pi^- \), ...) would qualify as an authentic light quark mass spectrometer.

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FIGURES

FIG. 1. Exclusive hadronic $\tau$ decays.

FIG. 2. The oriented center of mass hadronic plane.
FIG. 3. (a) The function $\lambda_1(Q^2, \beta)$ as a function of $Q^2/m^2$ for selected values of $\beta$.

FIG. 3. (b) The function $\lambda_2(Q^2, \beta)$ as a function of $Q^2$ for $\beta = 1$. 
FIG. 4. (a) The asymmetry coefficients $A, B$ as functions of $Q^2$ (in GeV).

FIG. 4. (b) The asymmetry coefficient $C_{LR}$ as a function of $Q^2$ (in GeV), for $\xi = 0.5, 1 \text{ and } 2$. 
FIG. 4. (c) The asymmetry coefficient $C_{UD}$ as a function of $Q^2$ (in GeV), for $\xi = 0.5, 1$ and 2.

FIG. 5. (a) $W_{xx}, W_{xy}, W_{yy}$ integrated over the entire Dalitz plot, for $\xi = 1$, as a function of $Q^2$ (in GeV).
FIG. 5. (b) $W_{xt}, W_{yt}, W_{tt}$ integrated over the entire Dalitz plot, for $\xi = 1$, as a function of $Q^2$ (in GeV).

FIG. 6. How $\sum_{i=1}^{N} W_{tt}(Q^2, \Delta_i)$ approaches $W_{tt}(Q^2)$ as the number of bins $N$ grows.
FIG. 7. (a) Lower bounds $\hat{m}_N(\mu, s_0)$ (in MeV) as a function of $s_0$ (in GeV$^2$) and for selected values of $N$, for fixed $\mu = 1$ GeV and $\xi = 0.5$.

FIG. 7. (b) Lower bounds $\hat{m}_N(\mu, s_0)$ (in MeV) as a function of $s_0$ (in GeV$^2$) and for selected values of $N$, for fixed $\mu = 1$ GeV and $\xi = 1.0$. 
FIG. 7. (c) Lower bounds \( \hat{m}_N(\mu, s_0) \) (in MeV) as a function of \( s_0 \) (in GeV\(^2\)) and for selected values of \( N \), for fixed \( \mu = 1 \) GeV and \( \xi = 2.0 \).
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