Price return auto-correlation and predictability in agent-based models of financial markets

Damien Challet$^{1, 2}$ and Tobias Galla$^{2, 3, 4}$

$^1$Nomura Centre for Quantitative Finance, Mathematical Institute, Oxford University, 24–29 St Giles', Oxford, United Kingdom
$^2$The Rudolf Peierls Centre for Theoretical Physics, Oxford University, 1–3 Keble Road, Oxford, OX1 3NP, United Kingdom
$^3$International Center for Theoretical Physics, Strada Costiera 11, 34014 Trieste, Italy
$^4$Istituto Nazionale per la Fisica della Materia (INFN), Trieste-SIS SA Unit, V. Beirut 2-4, 34014 Trieste, Italy

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We demonstrate that minority mechanisms arise in the dynamics of markets because of price impact; accordingly the relative importance of minority and delayed majority mechanisms depends on the frequency of trading. We then use mixed majority/minority games to illustrate that a vanishing price return auto-correlation function does not necessarily imply market efficiency. On the contrary, we stress the difference between correlations measured conditionally and unconditionally on external patterns.

Whether financial markets are predictable or not is a debate whose origin can be traced back to Bachelier’s hypothesis that prices follow a random walk. Later work, in particular by Fama and Samuelson, aimed at proving mathematically that markets are efficient, i.e. unpredictable, grounding their theories on perfect rationality. This is indeed the simplest view of a financial market, and a very convenient one. There are however good reasons to believe that markets are not perfectly efficient. The well-known January effect is a systematic deviation from efficiency and has been observed in real markets for many years, but is reportedly disappearing; other examples include the predictive power of moving averages. According to more recent theoretical studies, the cost of acquiring information or the presence of noise traders prohibits perfect efficiency, as perfect information can only be achieved at an infinite cost, or by accepting a considerable risk; this leads to the hypothesis of marginally efficient markets, where most of the easily detectable predictability is removed by traders.

Market efficiency is often illustrated by the fact that the price return auto-correlation function is essentially zero. Denoting the log-price at time $t$ by $p(t)$, the price return is defined as $r(t) = p(t+1) - p(t)$, and assuming that $r(t)$ is a stationary process, its auto-correlation reads

$$C(\tau) = \langle (r(t) + r(t+\tau)) \rangle - \langle r(t) \rangle^2 / \langle r(t)^2 \rangle. \quad (1)$$

Here $\langle \cdots \rangle$ stands for averages over time; note that we normalise by the average magnitude of returns. The units of time are arbitrary in this paper. If $C(\tau) \neq 0$, statistically significant predictions of future price changes are possible on time-scales $\tau$, that is, the knowledge of $r(t)$ allows to make probabilistic statements about $r(t+\tau)$. The analysis of real market data shows that $C(\tau)$ is typically for $\tau$ larger than a few minutes. However, the existing correlations on shorter time-scales are not exploitable in reality because of transaction costs.

This apparent efficiency however does not deter some practitioners from making statistically abnormal profits. In this paper we argue that the correlation function as defined in Eq. is a poor measure of predictability. It is indeed an unconditional measure of correlation and does not differentiate between technical analysis patterns, states of the market or of the economy. Finding and exploiting relevant information patterns generates arbitrage opportunities. As a consequence, correlation functions conditional on these patterns have to be used.

In the following we will use agent-based market models in which patterns play an essential role, to illustrate the differences between measures of correlation in price time series which are conditional or unconditional on these patterns.

**PATTERNS**

We now formalise the notions of conditional and unconditional averages as introduced above. Assume for example that there are $P$ patterns $\mu = 1, \cdots, P$ which are relevant (or believed to be relevant) for the behaviour of the market. The pattern or state of the market at time $t$ will be denoted by $\mu(t)$. Furthermore we will assume for simplicity that all patterns occur with equal probability $1/P$. Such patterns can then potentially be used in order to predict future price changes and in this sense money making consists essentially in identifying a suitable set of patterns and in exploiting the resulting predictability conditional on a given pattern, if it is present.

The average return conditional on $\mu$ will be denoted by $\langle r | \mu \rangle$ in the following, that is the time average $\langle r | \mu \rangle$ is only performed over instances $t$ for which $\mu(t) = \mu$. If at time $t$, $\mu(t) = \mu$, $r(t)$ is statistically predictable if $\langle |r| \mu \rangle > c$; in addition, if $\langle |r| \mu \rangle > c$ where $c$ denotes the transaction cost, the price return predictability given $\mu$...
can be exploited. We will neglect transaction costs in our discussion. Given a set of $P$ states, the predictability, $H$, can now be defined as follows:

$$H = \frac{1}{P} \sum_{\mu=1}^{P} \langle r | \mu \rangle^2. \quad (2)$$

Note that $H$ measures the average predictability of the next price return. This means that the time horizon over which this predictability can be exploited is fixed, and equal to the time interval over which returns are defined. In addition, $H$ fails to measure predictability associated with oscillatory behaviour. The latter is captured instead by the conditional price return auto-correlation function $K(\tau)$ defined as follows

$$K(\tau) = \left( \frac{1}{P} \sum_{\mu=1}^{P} \langle r(t) r(t+\tau) | \mu \rangle - H \right) / \langle r(t)^2 \rangle. \quad (3)$$

Here $\langle r(t) r(t+\tau) | \mu \rangle$ measures the correlation of price returns between two occurrences of a given pattern $\mu$, i.e., we impose $\mu(t) = \mu(t+\tau) = \mu$ when taking the above average. Although $H$ is a measure for conditional predictability, $K$ can provide additional predictability, as we will see agent based market models can display regimes in which $H$ vanishes, but where at the same time $K$ is non-zero. Secondly, $K$ is of interest when the signal-to-noise (or Sharpe) ratio $H / (\langle r^2 \rangle - H)$ is small, that is, when it is likely that for a fixed $\mu$, $A$ changes sign between two occurrences of that pattern.

The aim of the present paper is not to identify sets of predictability patterns in real market time series (see e.g., for a method of extracting patterns from real market data). We shall rather illustrate the difference between conditional and unconditional measures of predictability in agent-based models and will accordingly focus on models which are able to capture the relationship between predictability and price return auto-correlation. Agent-based models of financial market are appealing because they make it possible to study the relationship between the behaviour of the agents and the statistical properties of the resulting price time series. Early studies showed that such models are able to reproduce some typical market-like data, so-called stylized facts. Recently, modified Minority Games where the agents are allowed not to play if they do not believe to be able to make money, exhibit similar features. Such models are on the borderline of being exactly solvable by methods of statistical physics, while showing non-trivial realistic behaviour, and are hence of interest to both economists and statistical physicists. Volatility clustering obtained in agent based models has been related to strategy switching, and large price returns to market efficiency.

### PRICE EVOLUTION, MINORITY AND MAJORITY GAMES

Before turning to specific agent-based models we first define a price evolution mechanism and reward structure. Let us concentrate on one given trader who opens a position $a$ at time $t$, which can be either long ($a = 1$), or short ($a = -1$). After some waiting time, say at time $t' > t$, the same trader closes his position. His gain is therefore given by

$$a[p(t' + 1) - p(t + 1)], \quad (4)$$

as $p(t + 1)$ and $p(t' + 1)$ are the prices at which the transactions take place. We assume that the excess demand at times $t$ and $t'$ are given by $A(t)$ (which includes $a$), and $A(t')$ (which includes $-a$), respectively. It is reasonable to suppose that the excess demand $A(t)$ has a price impact of $f[A(t)]$, i.e. that $p(t + 1) = p(t) + f[A(t)]$. For convenience, we will consider a simple linear price impact function $f(x) = x/\lambda$, where $\lambda$ is the liquidity. We will set $\lambda = 1$ without loss of generality. Therefore, the gain of this trader in this round trip reads

$$a[p(t' + 1) - p(t + 1)] = -(-a)A(t') - aA(t) + a[p(t') - p(t)]. \quad (5)$$

The first two terms on the right hand side are typical of a minority game payoff, in which this trader is rewarded if his action $a$ is that of the minority at that time, i.e., the payoff $-A(t)$ is positive whenever $a$ and $A(t)$ have opposite signs. The last term represents the payoff that could be obtained if there were no price impact: its nature is that of a cumulative and delayed majority game. In other words, because of price impact, the market is a minority game when positions are opened or closed. When a position is held, the market is a retarded majority game, as it is favourable to hold a long position when the minority is large.

The Minority Game imposes to trade at times $t$ and $t'$, where the excess demand $A(t)$ is small, that is, whenever the price is increasing ($A > 0$) and vice versa. As a consequence, the relative importance of minority and majority mechanisms depends on the frequency of trading. The Minority Game imposes to trade at each time step, and hence focuses on price impact.

The first paper to apply the minority game framework to a two time-step payoff is where the payoff at time $t$ is $a[N_p(t) - N_p(t - 1)]$, $N_p(t)$ being the number of people with a long position at time $t$ (no short position is allowed) and is similar to our price $p(t)$; therefore, this payoff can be translated into $a(t)A(t + 1)$. The idea is to reward people for anticipating the crowd. Later work explicitly rewarded the agents with $a(t)A(t + 1)$, allowing also for short selling: games with this type of payoff are often referred to as ‘$\$-games’. Eq. therefore extends previous work in two crucial aspects: first by adding the holding of a position over several time steps and secondly by considering price impact. The former is reflected by payoffs.
\(a(t)A(t), a(t)A(t + 1), \ldots, a(t)A(t' - 1)\), i.e. a total gain of \(a(t) [A(t' - 1) + \ldots + A(t)] = a(t) [p(t') - p(t)]\) while holding the position, and price impact leads to the terms \(-a(t)A(t)\) and \(-a(t')A(t')\). It should be noted that equation \((\ref{eq:linear_price_impact})\) states that the real gain from a round-trip is composed of a sum of sequential payoffs; therefore it also contains a description of how profits and losses (P&L) and trading strategy gains are to be updated in real time.

Exact solutions for models with two-time step payoffs are currently unavailable, but sometimes an understanding of their qualitative properties is possible (e.g. \cite{21}). Although realistic speculative trading inevitably involves two or more time-steps, the perception of price dynamics by the agents can be captured by models involving only one-step payoffs: Ref. \cite{22} derives the \$-game (which does not include price impact) rigorously and argues that different players can have different types of one-step payoffs depending on their \textit{a priori} beliefs on the market: it concludes that fundamentalists, or value traders, believe that the market is a minority game, while trend-followers believe it to be a majority game. Thus, it is possible to consider a simple market with constant proportions of fundamentalists and trend-followers, respectively, that is, a mixed population of minority and majority players. An analysis along the lines of \cite{22} can be applied to the last term of Eq \((\ref{eq:linear_price_impact})\) of course, taking into account possible different beliefs of the agents on the future behaviour of the market. This procedure would not, however, lead to one-step payoffs as the two market impact terms would not disappear.

Since mixed minority/majority game models are exactly dynamically solvable \cite{23,24,25}, they will be the focus of this paper (see \cite{26} for a model of speculation based on Eq \((\ref{eq:linear_price_impact})\)).

**MINORITY/MAJORITY GAMES: DEFINITION**

A minority/majority game describes an ensemble of \(N\) agents, each of whom at every time-step has to take one of two possible actions, labelled by \(-1\) and \(+1\). We will denote the choice of agent \(i\) at time \(t\) by \(a_i(t)\). The excess demand is \(A(t) = \sum_{j=1}^{N} a_j(t)\). The payoff given to agent \(i\) is

\[
\phi_i a_i(t) A(t),
\]

where \(\phi_i = 1\) if agent \(i\) is a trend follower, thus prefers to be in the majority; conversely, if \(\phi_i = -1\), agent \(i\) is a fundamentalist, and plays a minority game: he is rewarded if his action \(a_i(t)\) has the opposite sign of the total bid \(A(t)\) \cite{22}. We assume that there is a constant fraction \(\theta\) of minority players, so that choosing \(\theta \in [0,1]\) allows to interpolate between pure majority populations \((\theta = 0)\) and pure minority games \((\theta = 1)\). We adopt the assumption of a linear price impact function as above so that price returns are taken to be proportional to the excess demand \(A\).

**Games without patterns**

As a warm-up we first compute the price return auto-correlation function of models without patterns as they are simple and analytically tractable. The adaptation abilities of the agents follow a simple rule: agent \(i\) computes the cumulative payoff, i.e the score of action \(+1\), denoted by \(\Delta_i(t)\); at time \(t\) his trading decision is probabilistic:

\[
P(a_i(t) = 1) = \frac{1 + \phi_i \tanh(\Gamma \Delta_i(t))}{2}.
\]

The scores themselves evolve according to

\[
\Delta_i(t+1) = \Delta_i(t) + A(t)/N.
\]

This defines a reinforcement learning known as the Logit model in economics \cite{27} and corresponds to Boltzmann weights in statistical physics \cite{43}. \(\Gamma\) is a learning rate. Note that if \(\phi_i = 1\) a large score \(\Delta_i\) will lead to a preference for action \(a_i(t) = 1\), while minority players \((\phi_i = -1)\) will predominantly play \(a_i(t) = -1\) if they find a positive \(\Delta_i\).

Minority games without patterns were introduced in \cite{28}, while pattern-less mixed minority/majority games are first found in \cite{22}. If all the agents have the same initial score valuation \(\Delta_i(0) = \Delta(0)\), all the \(\Delta_i\) are equal at all times, hence one can drop the index \(i\) in Eq. \((\ref{eq:score_evolution})\). \(A(t)\) is then the sum of \(N\) independent identically distributed variables. In particular \(A(t)/N\) converges to \((1 - 2\theta) \tanh(\Gamma \Delta)\) for large \(N\). This means that in this limit the dynamics of \(\Delta\) is well described by

\[
\Delta(t+1) = \Delta(t) + (1 - 2\theta) \tanh(\Gamma \Delta(t)).
\]

In the majority regime \((\theta < 1/2)\), one has \(1 - 2\theta > 0\) so that \(\Delta\) converges to \(\pm \infty\): all the majority players agree on a common decision and are happy, while the minority players agree on the opposite decision and are equally happy: this is a Nash equilibrium. \(A\) is constant; more precisely, \(\frac{A(t)}{N} = (1 - 2\theta)\), therefore \(\frac{(A(t)/N + 1)}{N} = (1 - 2\theta)^2\), which is confirmed by simulations, see Fig. \(\ref{fig:3}\).

The minority game regime \((\theta > 1/2)\) is slightly more complicated, as Eq. \((\ref{eq:score_evolution})\) has a unique fixed point which depends on \(\Gamma(2\theta - 1)\). If \(\Gamma(2\theta - 1) < 2\), the fixed point \(\Delta^* = 0\) is stable and unique, leading to \(\langle A \rangle = 0\) \cite{22}; this also implies that \(\langle A(t)A(t+1) \rangle = 0\). On the other hand, if \(\Gamma(2\theta - 1) > 2\), \(\Delta = 0\) becomes unstable, and a period two dynamics emerges \cite{22}. The oscillation amplitude of \(\Delta(t) = (-1)^t \Delta^*\) is determined by

\[
\Delta^* = \frac{(2\theta - 1)}{2} \tanh(\Gamma \Delta^*),
\]
which has of course two solutions of opposite signs. Neglecting fluctuations around $\Delta^*$, the resulting one-step correlation function is

$$\frac{(A(t)A(t+1))}{N^2} \simeq -[(2\theta - 1) \tanh(\Gamma \Delta^*)]^2. \quad (11)$$

In Fig. 1 we show $\langle A(t)A(t+1) \rangle/N^2$ computed from numerical simulations of the mixed minority/majority game without patterns, and compare the results with the simple approximation of Eq. (11). We find that both are in good agreement. As discussed in Refs. 23, 24, the sign of $\langle A(t)A(t+1) \rangle$ reveals the minority/majority nature of the market, that is, the fraction of value investors and trend followers: if the correlation function is positive majority players dominate and vice versa. As the outcome of the game in the majority regime is constant, its auto-correlation $C(1) = 0$. The case of the minority regime is more interesting: as $\langle A \rangle = 0$, the agents cannot predict the next outcome; in addition, if $\Gamma(2\theta - 1) \leq 2$, its correlation function $C(1) = 0$, hence, the game is really unpredictable: if $\Gamma(2\theta - 1) > 2$ and its correlation function $C(1) < 0$ measures predictability that cannot be exploited by the agents defined above, but could be by more sophisticated agents that would have strategies taking into account correlations.

### Games with patterns

When the agents believe that the next outcome of the game depends on some information such as economic forecast, the price history or the number of sun spots, their trading actions at time $t$ will depend on the pattern $\mu(t)$. Accordingly the future evolution of the price itself will be correlated with these pieces of information, hence the need for conditional averages.

In the original Minority Game [14], the agents are given pieces of information about the last price returns. The agents in this game are therefore technical analysts. More specifically, the public pattern $\mu$ at time $t$ consists in the signs of the past $M$ price returns; therefore in the original Minority Game, $\mu(t)$ can be seen as a binary string of length $M$ ($\mu(t) = 1, \cdots, 2^M$). Interestingly, Ref. 30 showed that the averaged price fluctuations are mostly unchanged if $\mu$ is chosen randomly from $\{1, \cdots, 2^M\}$. On the other hand crashes extending over several time-steps only occur if the patterns encode the real price history [31].

Here we will only consider random patterns. Since the agents act conditionally upon a given pattern $\mu(t)$ at time $t$ and are forced to trade at each time step, a strategy $a$ is a function which maps the pattern $\mu$ onto a trading action $a^\mu = +1$ or $-1$ for each pattern $\mu = 1, \cdots, P$. Strategies can hence be viewed as $P$-dimensional vectors with binary entries. We will assume that each agent holds two such strategies, which are assigned at random at the start of the game and are then kept fixed [44]. Following the now standard formalism, we denote the the strategy used by agent $i$ at time step $t$ by $s_i(t) = \pm 1$, so that upon presentation of pattern $\mu(t)$ at $t$ his action will read $a_i(t) = a_{s_i(t)}^\mu$. This can be written as $a_i(t) = \omega_i^\mu(t) + s_i(t)\xi_i(t)$ where $\omega_i^\mu(t) = \frac{1}{2}(a_{i,1}^\mu + a_{i,-1}^\mu)$ and $\xi_i(t) = \frac{1}{2}(a_{i,1}^\mu - a_{i,-1}^\mu)$. The excess demand at time $t$ can then be decomposed as follows:

$$A(t) = \Omega(t) + \sum_{i=1}^N \xi_i(t) s_i(t), \quad (12)$$

where $\Omega^\mu = \sum_{i=1}^N \omega_i^\mu$.

In order to decide which of his two strategies to play each agent assigns a virtual score to each of his strategies, based on the payoff he would have obtained had he always played this particular strategy. In the case where the agents have two strategies, only the score difference $q_i(t)$ matters, and evolves according to

$$q_i(t+1) = q_i(t) + 2\phi_i \xi_i(t) A(t)/P. \quad (13)$$

As in Eq. (9), $\phi_i = \pm 1$ describes the nature of the trader. At time $t$, agent $i$ then plays strategy $s_i(t) = \text{sgn} q_i(t)$.

This is the so-called online version of the game, where the agents adapt their scores after every round of the game. It was observed that the qualitative behaviour of the model does not change much if one allows the agents to switch strategies only after a large number of rounds have been played [32]. This effectively amounts to sampling all the patterns $\mu = 1, \cdots, P$ and leads to...
the so-called batch version of the game:

\[ q_i(t+1) = q_i(t) + 2 \frac{\phi_i}{P} \sum_{\mu=1}^{P} \left[ \xi_i^\mu \left( \Omega^\mu + \sum_{j=1}^{N} \xi_j^\mu s_j(t) \right) \right], \]

which is much less demanding to study analytically than the online game \([33, 34]\). Note that the time \( t \) in Eq. \([14]\) is not the same as that of Eq. \([13]\), as it has been rescaled by a factor \( P \) to take into account the fact that the agents are allowed to switch strategies once every \( P \) ‘online’ time-steps.

The statistical mechanics analysis of the minority game has revealed a phase transition between a predictable and an unpredictable phase. The relevant control parameter is the ratio \( \alpha = P/N \) between the number of different patterns \( P \) and the number of players \( N \). For \( \alpha < \alpha_c \approx 0.33 \) the market is unpredictable so that \( H \) as defined above vanishes. This is illustrated in the upper panels of Figs 2 and 5. For \( \alpha > \alpha_c \) one finds \( H > 0 \) so that the knowledge of \( \mu(t) \) is enough to predict the next price return \([35]\).

Online games

The setup of the online game makes it possible to compare the price return auto-correlation function \( C(\tau) \), with \( K(\tau) \), the conditional price return auto-correlation function as defined in Eq. \([16]\).

Both auto-correlation functions are negative for \( \tau = 1 \), as the minority game induces a mean-reverting process \([23, 36]\). The lower panel of Fig. 2 illustrates that the conditional correlation function \( K(1) \) is much larger than the unconditional one \( C(1) \) for all values of \( \alpha \). Note in particular that \( C(1) \) is close to zero in the predictable phase \( (H > 0) \) (and converges to 0 in the limit of infinite systems), whereas \( K(1) \approx -0.5 \), confirming that unconditional measures fail to detect existing arbitrage opportunities which can only be revealed by conditional approaches. Fixing \( P/N \) and plotting \( C(\tau) \) and \( K(\tau) \) as a function of the time-lag \( \tau \) confirms this statement (see Figs. 2 and 4); when \( \theta \) decreases, the mean reversion of the price decreases because the fraction of minority players becomes less and less important; accordingly the amplitude of \( K \) decreases as well, and converges to 0 when \( \theta = 1 \), since in that case the agents are all majority players, and only play one of their strategies, which does not produce any fluctuation of \( A \) around \( \langle A | \mu_i \rangle \). When \( H = 0 \), as \( \alpha \to 0 \), \( K(1) \to -1 \) for sufficiently large \( \theta \), meaning that for a given \( \mu \), \( A \) acquires an oscillatory behaviour. This type of predictability is not captured by \( H \), but by \( K(1) \), and, to a much lesser extent, by \( C(1) \).

Batch games

The setup of the batch game allows only the measurement of \( K(\tau) \), the conditional price return auto-correlation function \([33]\), because \( A \) is replaced by its weighted sum over the patterns in Eq. \([14]\). Fig. 6 shows that the behaviour of \( K(1) \) as a function of \( \alpha \) is qualitatively the same as in the online game (Fig. 2).
However, $K(\tau)$ itself is very different in batch games, as it oscillates as function of $\tau$ (Fig. 4). This is a signature of the mean-reverting process induced by the Minority Game players, and of the batch process, which updates the scores with respect to all the patterns at the same time.

The dynamics of the batch Minority Game was analyzed in Ref. [33], where it is proved that the $N$ time.

The correlation function $C(\tau)$ is indistinguishable from zero on this scale. Results are from simulations of the online game with $N = 300$ agents, run for $10^6$ time-steps and averaged over 10 random assignments of the strategies.

FIG. 4: Conditional price return autocorrelation function $K$ vs time-lag $\tau$ for the online game at $\alpha = 0.75$ and for different values of $\theta$. The unconditional correlation function $C(\tau)$ is indistinguishable from zero on this scale. Results are from simulations of the online game with $N = 300$ agents, run for $10^6$ time-steps and averaged over 10 random assignments of the strategies.

The dynamics of the batch Minority Game was analyzed in Ref. [33], where it is proved that the $N$ coupled equations (14) can be replaced, in the limit of $N \rightarrow \infty$, $P \rightarrow \infty$ at fixed $\alpha = P/N$, by a single equation describing the evolution of a representative agent. This result relies on the so-called generating functional analysis, a technique originally devised to study neural networks and other disordered systems [33]. This method consists in averaging the dynamics over the disorder (i.e., over all possible strategy assignments) and is exact. The resulting equation reads

$$q(t+1) = q(t) + \alpha \phi \sum_{t' \leq t} (1 + G(t))^{-1} sgn(q(t')) + \theta(t) + \sqrt{\alpha} \eta(t),$$

where $\phi = \pm 1$ again distinguishes between majority and minority traders. While the original batch game is Markovian, this equation contains two different types of memory: a retarded self-interaction term (the second term in the above equation) relating to earlier times $t' \leq t$ and coloured Gaussian noise $\eta(t)$ with temporal covariances

$$\langle \eta(t)\eta(t') \rangle = \left[ (1 + G)^{-1} (E + C)(1 + G^T)^{-1} \right]_{tt'}.$$  

The correlation function $C_{tt'} = \langle sgn(q(t))sgn(q(t')) \rangle$ and response function $G_{tt'} = \langle \frac{\partial \theta(t)}{\partial \theta(t')} \rangle$ to external perturbations $\theta(t)$ are to be computed self-consistently as averages over the noise. $1$ is the identity matrix, and $E$ has all entries equal to one, $E_{tt'} = 1$.

Because of the non-Markovian nature of the single agent process and the presence of coloured noise a full analytical solution of this self-consistent problem is in general impossible. Assuming independence from initial conditions $q(0)$, the predictability $H$ can be computed exactly, but the calculation of quantities like the volatility $\langle A^2 \rangle$ or one-step correlation functions requires the knowledge of both the persistent and transient parts of the correlation and response functions. Neglecting correlations leads to accurate approximations for the market volatility $\sigma$, but unsurprisingly analogous approximations are inadequate for the computation of price return correlation functions. One therefore has to resort to a Monte-Carlo integration of Eq. (14), and as shown in Fig. 5 we find nearly perfect agreement with simulations of the original batch process. The figure also illustrates that the behaviour of the conditional one-step correlation function is very similar in batch and online minority games. Fig. 5 reports the behaviour of the conditional auto-correlation as a function of the time lag $\tau$ from Monte-Carlo simulations. One observes an oscillatory behaviour for small $\tau$, the oscillation amplitude can be reasonably well fitted by exp $\left(-\tau^2/\kappa\right)$, with some constant $\kappa$. 

FIG. 5: Predictability (upper panel) and one-step autocorrelation function $K(1)$ (lower panel) vs $\alpha$ for the batch game at different values of $\theta$. Data displayed as open symbols are obtained from numerical simulations of the batch process with $N = 300$ players, run for 100 time-steps and averaged over 30 realizations of the strategy assignments. The solid lines are results from the Monte-Carlo integration of the self-consistent theory, displayed for comparison (60 time-steps of the integration are performed, and results are averages over $2 \cdot 10^5$ samples of the effective agent process). The vertical dashed line marks the transition between unpredictable and predictable states for the case of the pure minority game ($\theta = 1$).
CONCLUSIONS

We have used agent-based models to illustrate the differences between conditional and unconditional price return auto-correlation functions in financial markets. Unconditional indicators may be zero even though conditional measures reveal very large correlations. This emphasizes the inappropriateness of the use of purely unconditional auto-correlation functions for weak proofs of market efficiency. Therefore, a more correct statement about market efficiency would be that they are remarkably efficient on average (over patterns, over time), but not locally in time or given relevant market states.

As noted above, practitioners try to identify patterns in real markets and exploit resulting predictability in order to make profit. The non-parametric clustering method recently developed by Marsili for example automatically extracts a set of patterns from market data, which are at borderline of being predictive. It would be interesting to compute price return auto-correlation functions, conditional on patterns obtained by this method.

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[41] In practice of course one should also take into account the risk, i.e. fluctuations of $r$ conditional on $\mu$.
[42] In principle, a better measure of the predictability would be $\frac{1}{T} \sum_{\mu=1}^{\mu_T} |\langle r |\mu \rangle|$. However the definition used here is more convenient.
[43] The original minority game was defined with $\Gamma = \infty$. $\Gamma < \infty$ was first seen in [40].
[44] Minority games with larger pools of strategies per agent show qualitatively similar behaviour.