Aspects of Domain-Wall Fermion on the Lattice

Ting-Wai Chiu$^{1,2,3}$

$^1$ Physics Department, National Taiwan University, Taipei 10617, Taiwan
$^2$ Center for Quantum Science and Engineering, National Taiwan University, Taipei 10617, Taiwan
$^3$ Center for Theoretical Sciences, National Taiwan University, Taipei 10617, Taiwan

Two transparent layers are introduced at the boundaries of the fifth dimension, for the optimal domain-wall fermions. For the quark fields defined in terms of these two transparent layers, they obey the usual chiral projection rule in the continuum, independent of the gauge fields. Consequently, any observable constructed with the quark fields manifests the symmetries exactly as those of its counterpart in the continuum.

The basic idea of domain-wall fermion (DWF) [1, 2] is to use an infinite set of coupled Dirac fermion fields \( \psi_s(x), s \in (-\infty, \infty) \) with masses behaving like a step function \( m(s) = \theta(s) \) such that the Weyl fermion state can arise as zero modes bound to the mass defect at \( s = 0 \). However, if one uses a compact set of masses, then the boundary conditions of the mass (step) function must lead to the occurrence of both left-handed and right-handed chiral fermion fields, i.e., a vector-like theory. For lattice QCD with DWF [3], in practice, one can only use a finite number \( N_s \) of lattice Dirac fermion fields to set up the domain wall, thus the chiral symmetry of the quark fields (in the massless limit) is broken. Thereupon, a relevant question is whether one can construct a domain-wall fermion action such that the effective 4D lattice Dirac operator has the mathematically optimal chiral symmetry for finite \( N_s \).

In Ref. [4], the optimal domain-wall fermion (ODWF) is constructed such that the effective 4D lattice Dirac operator attains the mathematically optimal chiral symmetry for any finite \( N_s \), exponentially-locally-for sufficiently smooth gauge backgrounds [5, 6], and independent of the lattice spacing in the fifth dimension. The basic idea of ODWF is to construct a set of analytical weights, \( \{\omega_s, s = 1, \cdots, N_s\} \), one for each layer in the fifth dimension, such that the chiral symmetry breaking due to finite \( N_s \) can be reduced to the minimum. The 4-dimensional effective Dirac operator of massless ODWF is

\[
D = m_0[1 + \gamma_5 S_{\text{opt}}(H_w)],
\]

\[
S_{\text{opt}}(H_w) = \frac{1 - \prod_{s=1}^{N_s} T_s}{1 + \prod_{s=1}^{N_s} T_s}, \quad T_s = \frac{1 - \omega_s H_w}{1 + \omega_s H_w},
\]

which is exactly equal to the Zolotarev optimal rational approximation of the overlap Dirac operator [6, 7]. That is, \( S_{\text{opt}}(H_w) = H_w R_Z(H_w) \), where \( R_Z(H_w) \) is the optimal rational approximation of \( (H_w^2)^{-1/2} \) [8, 9].

However, in the original formulation [4], the valence quark propagator cannot be expressed in terms of the correlation function of the quark fields defined in terms of the boundary modes, unlike the conventional domain-wall fermion. In this paper, we solve this problem by introduced two transparent layers with \( \omega_s = 0 \), as boundary layers appending to the original action of ODWF such that the quark fields defined in terms of these two transparent layers obey the usual chiral projection rule in the continuum, independent of the gauge fields. Consequently, the valence quark propagator can be expressed in terms of the correlation function of the quark fields, and any observable constructed with the quark fields manifests the symmetries exactly as those of its counterpart in the continuum. The salient feature of a transparent layer (with \( \omega_s = 0 \), and \( T_s = 1 \)) is that its presence does not change the effective 4D Dirac operator.

With two additional transparent layers at \( s = 0 \) and \( s = N_s + 1 \), the action of ODWF can be written as

\[
\mathcal{A}_f = \sum_{s,s'=0}^{N_s+1} \sum_{x,x'} \langle \psi_{s',x'}(\omega_s D_w + 1)_{x,x'} \delta_{s,s'} \rangle + \langle \omega_s D_w - 1 \rangle_{x,x'} (P_- \delta_{s',s+1} + P_+ \delta_{s',s-1}) \psi_{s',s'}
\]

with boundary conditions

\[
P_+ \psi_{s,-1} = -m P_+ \psi_{s,N_s+1}, \quad m \equiv m_q/(2m_0),
\]

\[
P_- \psi_{s,N_s+2} = -m P_- \psi_{s,0},
\]

where \( P_\pm = (1 \pm \gamma_5)/2 \), \( m_q \) is the bare quark mass, \( D_w \) is the standard Wilson-Dirac operator plus a negative parameter \(-m_0 \) (\( 0 < m_0 < 2 \)), and the formula for the weights \( \{\omega_s, s = 1, \cdots, N_s\} \) is given in Ref. [4].

Now we define the quark fields in terms of the boundary modes

\[
q(x) = (2m_0)^{-1/2} (P_- \psi_{x,0} + P_+ \psi_{x,N_s+1}), \quad \bar{q}(x) = (2m_0)^{-1/2} (\bar{\psi}_{x,0} P_+ + \bar{\psi}_{x,N_s+1} P_-).
\]

In the following, we show that the valence quark propagator in a gauge background is equal to the correlation function of the quark fields, i.e.,

\[
\langle q(x) \bar{q}(y) \rangle = (D_c + m_q)_{x,y}^{-1},
\]

where

\[
D_c = 2m_0 \frac{1 + \gamma_5 S_{\text{opt}}}{1 - \gamma_5 S_{\text{opt}}},
\]

\[
S_{\text{opt}} = \frac{1 - \prod_{s=1}^{N_s} T_s}{1 + \prod_{s=1}^{N_s} T_s},
\]

\[
T_s = \frac{1 - \omega_s H_w}{1 + \omega_s H_w},
\]

\[\text{NTUTH-03-505A} \quad \text{arXiv:hep-lat/0303008v5 13 Sep 2012} \]
Obviously, the transparent layers (with \( \omega_s = 0 \)) do not change the effective 4D Dirac operator since \( T_s = 1 \).

The generating functional \( W \) for connected \( n \)-point Green’s function of the quark fields is defined as

\[
e^W[J,J] = Z[J,J] = \frac{\int e^{-A_s - A_f - A_{PV} + J_q + \bar{q}J}}{\int e^{-A_s - A_f - A_{PV}}}, \tag{7}
\]

where \( J \) and \( J \) are the Grassman sources of \( q \) and \( \bar{q} \) respectively, \( A_s = \int \equiv \int [dU] \left[ d\psi \right] \left[ d\bar{\psi} \right] \left[ d\phi \right] \left[ d\bar{\phi} \right] \), \( A_q \) is the gauge action, \( A_{PV} \) is the action of the Pauli-Villars fields \( \left\{ \phi_s, \bar{\phi}_s \right\} \) with \( m_q = 2m_0 + 1 \), i.e.,

\[
A_{PV} = \sum_{s,s' = 0}^{N_s+1} \sum_{x,x'} \bar{\phi}_{s,x'} \frac{d}{dx} \delta_{s,s'} + \left( \psi_{s,x'} \frac{d}{dx} \delta_{s,s'} \right) \phi_{s,x'}
\]

with boundary conditions

\[
P_+ \phi(x,-1) = -P_+ \phi(x, N_s + 1), \quad P_- \phi(x, N_s + 2) = -P_- \phi(x, 0).
\]

First we evaluate the fermionic integrals in (7). Using \( \gamma_5 \psi_x = \pm P_x \), \( P_+ + P_- = 1 \), and \( H_w = \gamma_5 D_w \), we can rewrite (7) as

\[
A_f = (m + 1) \bar{\psi} \gamma_5 P_+ \psi_{N_s+1} - \bar{\psi} \gamma_5 (P_+ \psi_0 + P_+ \psi_{N_s+1}) + \bar{\psi} \gamma_5 (P_- \psi_1 + P_+ \psi_0) + \bar{\psi} \gamma_5 (P_- \psi_1 + P_+ \psi_0)
\]

\[
+ \sum_{s=1}^{N_s} \left( \bar{\psi}_s \gamma_5 (\psi_0 H_w - 1)(P_+ \psi_0 + P_+ \psi_{s+1}) - \bar{\psi}_s \gamma_5 (P_- \psi_0 + P_+ \psi_{N_s}) + \bar{\psi}_s \gamma_5 (P_- \psi_0 + P_+ \psi_{N_s}) - (m + 1) \bar{\psi}_s \gamma_5 P_- \psi_0,
\]

where all indices are suppressed except the index in the 5-th dimension. Next we define

\[
\eta_0 = P_- \eta_0 + P_+ \psi_{N_s+1}, \quad \bar{\eta}_0 = -\bar{\psi}_0 \gamma_5,
\]

\[
\eta_s = P_- \psi_s + P_+ \psi_{s-1}, \quad \bar{\eta}_s = \bar{\psi}_s \gamma_5 (\psi_0 H_w - 1),
\]

\[
\eta_{N_s+1} = P_- \psi_{N_s+1} + P_+ \psi_{N_s}, \quad \bar{\eta}_{N_s+1} = -\bar{\psi}_{N_s+1} \gamma_5,
\]

where the index \( s \) in the second line runs from 1 to \( N_s \), and the inverse transform

\[
\psi_0 = P_- \eta_0 + P_+ \eta_1, \quad \bar{\psi}_0 = -\bar{\psi}_0 \gamma_5,
\]

\[
\psi_s = P_- \eta_s + P_+ \eta_{s+1}, \quad \bar{\psi}_s = \bar{\psi}_0 \gamma_5 (\psi_0 H_w - 1)^{-1} \gamma_5,
\]

\[
\psi_{N_s+1} = P_- \eta_{N_s+1} + P_+ \eta_0, \quad \bar{\psi}_{N_s+1} = -\bar{\psi}_{N_s+1} \gamma_5.
\]

Then the action (8) can be rewritten as

\[
A_f = \bar{\eta}_0 \left( P_- - m P_+ \right) \eta_0 - \bar{\eta}_0 \eta_1
\]

\[
+ \sum_{s=1}^{N_s} \left\{ \bar{\eta}_s \eta_s - \bar{\eta}_s T_s^{-1} \eta_{s+1} \right\} + \bar{\eta}_{N_s+1} \eta_{N_s+1} - \bar{\eta}_{N_s+1} \left( P_- - m P_+ \right) \eta_0,
\]

Thus the fermionic integral in the numerator of (7) can be written as

\[
\mathcal{J} \int [d\bar{\eta}][d\eta] e^{-A_f[\eta,\bar{\eta}]} + J_q + \bar{q}J
\]

\[
= \mathcal{J} \int [d\bar{\eta}][d\eta] e^{-A_f[\eta,\bar{\eta}]} + J_q + \bar{q}J - \bar{\eta}_0 P_+ J' + \bar{\eta}_{N_s+1} P_- J' \tag{10}
\]

where \( J' = (2m_0)^{-1/2} J \) and \( J' = (2m_0)^{-1/2} J \), and \( \mathcal{J} \) is the Jacobian of the transformation,

\[
\mathcal{J} = \prod_{s=1}^{N_s} \det(\omega_s H_w - 1).
\]

Now using the Grassman integral formula

\[
\int d\chi \gamma^5 e^{-\chi M + \bar{\chi} \bar{\chi}} = \int d\eta \gamma^5 e^{-\eta M \gamma^5 \eta} \det M,
\]

one can easily evaluate the Grassman integrals in (10), by integrating \( (\eta_s, \bar{\eta}_s) \) successively from \( s = N_s + 1 \) to \( s = 0 \). Explicitly, after integrating \( (\eta_{N_s+1}, \bar{\eta}_{N_s+1}) \), (10) becomes

\[
\mathcal{J} \prod_{s=0}^{N_s-1} \left[ \int [d\bar{\eta}_s][d\eta_s] \exp\left\{ -\bar{\eta}_0 (P_- - m P_+) \eta_0 + \bar{\eta}_0 \eta_1 + J_s \eta_s - \bar{\eta}_0 P_+ J' \right\} + \bar{\eta}_0 P_+ J' \right. \!
\]

\[
\left. - \sum_{s=1}^{N_s-2} (\bar{\eta}_s \eta_s - \bar{\eta}_s T_s^{-1} \eta_{s+1}) - \bar{\eta}_{N_s-1} \eta_{N_s-1} + \bar{\eta}_{N_s-1} T_{N_s-1}^{-1} \left[ (P_- - m P_+) \eta_0 + P_- J' \right] \right],
\]

Subsequent integrations over \( (\eta_{N_s-2}, \bar{\eta}_{N_s-2}) \) up to \( (\eta_1, \bar{\eta}_1) \) are similar to the above integration, and the result is

\[
\mathcal{J} \int [d\bar{\eta}_0][d\eta_0] \exp\left\{ -\bar{\eta}_0 (P_- - m P_+) \eta_0 + \bar{\eta}_0 P_+ J' \right\}
\]

\[
+ \bar{\eta}_0 \prod_{s=1}^{N_s} T_s^{-1} \left[ (P_- - m P_+) \eta_0 + P_- J' \right]
\]

Finally, integrating \( (\eta_0, \bar{\eta}_0) \) gives

\[
\mathcal{J} \det \left[ (P_- - m P_+) - T^{-1} (P_+ - m P_+) \right] e^{J_f \left[ (P_+ - T^{-1} P_+) - m \right] - T^{-1} P_-}
\]

where \( T^{-1} = \prod_{s=1}^{N_s} T_s^{-1} \). Using the simple identity

\[
(P_+ - T^{-1} P_-)^{-1} (P_- - T^{-1} P_+) = \frac{1 + \gamma_5 S_{opt}}{1 - \gamma_5 S_{opt}},
\]
the above result becomes
\[ K \det[(D_c + m_q)(2m_0)^{-1}] \exp\{\bar{J}(D_c + m_q)^{-1}J\}, \] (11)
where \(D_c\) is defined in (11), and
\[ K = \prod_{s=1}^{N_c} \det(\omega_s H_w - 1) \cdot \det \left[ (-P_+ + T^{-1}P_-) \right]. \]
Setting \(\bar{J} = J = 0\) in (11), we obtain the result for the fermionic integral in the denominator of (7).
\[ \int [d\bar{\psi}] [d\psi] e^{-A_f} = K \det[(D_c + m_q)(2m_0)^{-1}]. \] (12)

Next, we evaluate the integrals over the Pauli-Villars fields in (7). Using the Gaussian integration formula for the boson fields, and following the procedures similar to above for the fermion fields, we obtain
\[ \int [d\bar{\delta}] [d\delta] e^{-A_{PV}} = \pi^{N_c+2} \frac{1}{K} \det[1 + D_c/(2m_0)]^{-1}. \] (13)
Substituting (11), (13) into (7), we have
\[ e^{W[J,\bar{J}]} = \frac{\int [dU] e^{-\bar{A}_q} \det (D(m_q)) e^{J(D_c+m_q)^{-1}J}}{\int [dU] e^{-\bar{A}_q} \det (D(m_q))}, \] (14)
where
\[ D(m_q) = (D_c + m_q)[1 + D_c/(2m_0)]^{-1} = m_q + (m_0 - m_q/2)[1 + \gamma_5 S_{opt}(H_w)], \]
is the effective 4D operator of ODWF, in which \(S_{opt}(H_w)\) is exactly equal to the Zolotarev optimal rational approximation of the sign function in the overlap Dirac operator (7). That is, \(S_{opt}(H_w) = H_w R_Z(H_w)\), where \(R_Z(H_w)\) is the optimal rational approximation of \((H_w^2)^{-1/2}\). In the limit \(N_c \to \infty\), \(S_{opt}(H_w) = H_w R_Z(H_w)^{1/2}\), \(\gamma_5 S_{opt}(H_w) = V\) satisfying \(V^\dagger = \gamma_5 V\gamma_5 = V^{-1}\). Then \(D_c\) becomes \(D_c = 2m_0(1 + V)(1 - V)^{-1}\) which is chirally symmetric, and \(D(0) = D_c(1 + D_c)^{-1} = m_q(1 + V)\) is exactly equal to the overlap Dirac operator (7), satisfying the Ginsparg-Wilson relation
\[ D(0)\gamma_5 + \gamma_5 D(0) = \frac{1}{m_0} D(0)\gamma_5 D(0). \]
The quark propagator can be obtained by differentiating \(W[J,\bar{J}]\) with respect to \(J\) and \(\bar{J}\),
\[ \langle q(x)\bar{q}(y) \rangle = -\frac{\delta^2 W[J,\bar{J}]}{\delta J(x) \delta \bar{J}(y)} \Bigg|_{j=\bar{j}=0} \]
\[ = \frac{\int [dU] e^{-\bar{A}_q} \det (D(m_q)) [D_c + m_q]^{-1} \delta J_q(x,y) \delta \bar{J}_q(x,y)}{\int [dU] e^{-\bar{A}_q} \det (D(m_q))}, \] (16)
which reduces to (8) for a background gauge field.
In general, any observable involving quark fields can be obtained from \(Z[J,\bar{J}]\) by differentiation, and it possesses the symmetries exactly the same as its counterpart in the continuum. For example, the current-current correlator
\[ \langle \bar{d}(x)\gamma_4 P_- s(x)\bar{s}(0)\gamma_4 P_- d(0) \rangle = \frac{\delta}{\delta J_d(x)} \frac{\delta}{\delta J_s(0)} \frac{\delta}{\delta J_d(0)} \frac{\delta}{\delta J_s(0)} Z[J,\bar{J}] \]
\[ = \frac{\int [dU] e^{-A_{PV}} \prod_f \det (D(m_f)) O_K(x)}{\int [dU] e^{-A_{PV}} \prod_f \det (D(m_f))}, \]
where
\[ O_K(x) = -\text{tr}[(D_c + m_q)^{-1} \gamma_4 P_- (D_c + m_q)^{-1} \gamma_4 P_-]. \]

Obviously, the \(V - A\) structure of the left-handed quark currents is preserved exactly. This is one of the basic motivations to introduce the transparent layers with \(\omega_s = 0\). On the other hand, if one uses the Ginsparg-Wilson Dirac operator (satisfying \(D\gamma_5 + \gamma_5 D = D\gamma_5 D\)) to construct the quark action,
\[ A_F = \sum_{x,y} \bar{q}(x) D_{x,y} q(y) \]
\[ = \sum_{x,y} [\bar{q}(x) P_+ (D\hat{P}_-)_{x,y} q(y) + \bar{q}(x) P_- (D\hat{P}_+)_{x,y} q(y) \]
\[ = \sum_{x,y} \bar{q}(x) P_+ (D\hat{P}_-)_{x,y} q(y) + \bar{q}(x) P_- (D\hat{P}_+)_{x,y} q(y) \]
where \(P_\pm = (1 \pm \gamma_5)/2\) and \(\hat{P}_\pm = \frac{1}{2}[1 \pm \gamma_5 (1 - D)]\) are the chiral projectors for \(\bar{q}\) and \(q\) respectively. Then \(d(x)\gamma_\mu P_- s(x) \neq d(x)P_\mu s(x)\), and the left-handed quark current does not manifest the \(V - A\) structure. Consequently, these left-handed quark currents explicitly breaks the \(SU_L(2)\) gauge symmetry by \(O(a)\) effect \(10\), and the discrete CP symmetry in chiral gauge theories with Ginsparg-Wilson fermion is explicitly broken by \(O(a)\) effect \(11\).

Even though the transparent layers (with \(\omega_s = 0\)) are introduced as the boundary layers for defining the quark fields such that any observable involving the quark fields manifest the symmetries exactly as those of its counterpart in the continuum, in practice, one does not need to keep these transparent layers in the dynamical simulations of QCD, since their presence does not change the fermion determinant at all. Moreover, the valence quark propagator also can be obtained without using the transparent layers. In the original ODWF action \(4\), the quark fields are defined by the boundary modes at \(s = 1\) and \(s = N_c\) similar to \(2\). Then one can obtain the valence quark propagator by solving the following linear system
\[ D(m_q)|Y\rangle = D(2m_0) B^{-1} |\text{source vector}\rangle \] (17)
where \(B_{s,s',s'',s'''(s,s')^2}^{-1} = \delta_{s,s'}(P_-\delta_{s,s'} + P_+\delta_{s+1,s})\) with periodic boundary conditions in the fifth dimension. Then the solution of (17) gives the valence quark propagator
\[ (D_c + m_q)_{x,x'}^{-1} = (2m_0 - m_q)^{-1} [(BY)_{x,1;x',1} - \delta_{x,x'}]. \]

Nevertheless, the transparent layers with \(\omega_s = 0\) turn out to play a crucial role in the derivations of some analytical results which would be difficult to obtain otherwise. An example is the derivation of the axial Ward
identity for lattice QCD with ODWF [12]. Obviously, one can insert any numbers of transparent layers at any locations along the 5-th dimension. This opens new possibilities to tackle problems in domain-wall fermions.

This work is supported in part by the National Science Council (Grant No. NSC99-2112-M-002-012-MY3), and NTU-CQSE (Grant No. 10R80914-4).

[1] V. A. Rubakov and M. E. Shaposhnikov, Phys. Lett. B 125, 136 (1983).
[2] C. G. Callan and J. A. Harvey, Nucl. Phys. B 250, 427 (1985).
[3] D. B. Kaplan, Phys. Lett. B 288, 342 (1992); Nucl. Phys. B (Proc. Suppl.) 30, 597 (1993).
[4] T. W. Chiu, Phys. Rev. Lett. 90, 071601 (2003).
[5] T. W. Chiu, Phys. Lett. B 552, 97 (2003).
[6] H. Neuberger, Phys. Lett. B 417, 141 (1998); Phys. Lett. B 427, 353 (1998).
[7] R. Narayanan and H. Neuberger, Nucl. Phys. B 443, 305 (1995).
[8] N. I. Akhiezer, "Theory of approximation", Reprint of 1956 English translation, Dover, New York, 1992.
[9] T. W. Chiu, T. H. Hsieh, C. H. Huang and T. R. Huang, Phys. Rev. D 66, 114502 (2002).
[10] K. Fujikawa, M. Ishibashi and H. Suzuki, JHEP 0204, 046 (2002).
[11] P. Hasenfratz, Nucl. Phys. B (Proc. Suppl.) 106, 159 (2002).
[12] Y. C. Chen, T. W. Chiu [TWQCD Collaboration] arXiv:1205.6151 [hep-lat].