Using stochastic order to compare different Euclidean Random Assignment Problems

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Abstract

This paper provides a theorem to compare the minimum total cost of two different Euclidean Random Assignment Problems with the same number of points, using the stochastic order of the costs of one of the pairs in these two problems. The subsequent sections provide two applications of the theorem, including studies of the problem on the one-dimensional $k$-star graph and comparisons between some problems in higher dimensions. More possible applications and limitations of the theorem are also discussed.

1 Introduction

Many random combinatorial optimization problems [21], like the traveling salesman problem [24] and random K-satisfiability (K-SAT) [32] have long been questions of great interest in a wide range of fields. The assignment problem [26] is also a classic type of them and has received increased attention across a number of disciplines in recent years, like in education [20], transportation [36], and healthcare [19]. Among the several extensions of the assignment problem, a prominent one is the Euclidean Random Assignment Problem (ERAP) [17], which goes as follows. Consider an $n$-sample $\mathcal{B} = (B_1, \ldots, B_n)$ of i.i.d. random variables of law $\mu_B$ over the defined domain $\Omega_b \subset \mathbb{R}^d$, and another $n$-sample $\mathcal{R} = (R_1, \ldots, R_n)$ of i.i.d. r.v.s of law $\mu_R$ over $\Omega_r \subset \mathbb{R}^d$. We are interested in the statistical properties of the random variable

\[
H^{\Omega_{b},\Omega_{r}}_{\text{opt}} := \min_{\pi \in S_n} \sum_{i=1}^{n} D^p(B_i, R_{\pi(i)}) ,
\]

where $S_n$ is the symmetric group over $n$ elements, and $D(B_1, R_1)$ is the Euclidean distance between points $B_1$ and $R_1$. In some cases, we use the notation $H^{\Omega_{b},\Omega_{r}}_{\text{opt}}$ to stress the dependence on $n$. However, without causing any misunderstanding, we will also abbreviate it to $H_{\text{opt}}$.

Interest in ERAPS has arisen originally in statistical physics, under the name of “Euclidean Bipartite Matching Problem”, which were introduced by M´ezard and Parisi as toy models of “spin-glass” [30, 31]. Moreover, the ERAP is the discrete counterpart of the well-known Monge-Kantorovich problem via the identification

\[
H^{\Omega_{b},\Omega_{r}}_{\text{opt}} d = nW_p^{\mathcal{B}}(\rho_{\mathcal{B}}, \rho_{\mathcal{R}}) ,
\]

where $\rho_{\mathcal{B}}, \rho_{\mathcal{R}}$ are the empirical measures of $\mathcal{B}$ and $\mathcal{R}$ and $W_p^{\mathcal{B}}(\mu, \nu)$ is the $p$-th power of the Wasserstein distance $W_p$ among probability measures $\mu, \nu$ (for the relevant definitions, see e.g. [22]). It turns out that establishing the law of large numbers or the (generalized) central limit theorem for the random variable $H^{\Omega_{b},\Omega_{r}}_{\text{opt}}$ are already highly non-trivial questions and one can focus on the order of the expected value $E[H^{\Omega_{b},\Omega_{r}}_{\text{opt}}]$ depending on the choice of disorder, $\Omega_{b},\Omega_{r}$, and $p$. However, despite such an elementary formulation, already the study of $H^{\Omega_{b},\Omega_{r}}_{\text{opt}}$ (or equivalently, $nW_p^{\mathcal{B}}(\rho_{\mathcal{B}}, \rho_{\mathcal{R}})$) turned out to be rather challenging in general, and attracted interest of several communities. Results include convergence in general dimension with uniform disorder as $n$ grows [4, 23], asymptotics for $E[H^{\Omega_{b},\Omega_{r}}_{\text{opt}}]$ for $\Omega$ a one-dimensional interval, the real line or the unit circle [8–10, 14, 29], also for a possibly non-uniform disorder [6, 7, 11, 18].

Of special interest is also the 2d case, which has a long history since a pioneering work of Ajtai-Komlós-Tusnády [1]. Here, it is now known that, if $\Omega$ is a 2d compact manifold and $p \geq 1$,

\[
E[H^{\Omega_{b},\Omega_{r}}_{\text{opt}}] = c_p n^{1-rac{2}{p}} (\log n)^{\frac{2}{p}} (1 + o(1)) , \quad n \to \infty .
\]

On the basis of a celebrated linearisation ansatz of the Monge-Ampère equation to the Poisson equation proposed by Caracciolo-Lucibello-Parisi-Sicuro (CLPS) [13] (see also [15]), which was later proven under appropriate assumptions on the domain by Ambrosio-Stra-Trevisan [3] via PDE methods, it is known that $c_2 = \frac{1}{2\pi}$. More recently, there has
been an increasing amount of literature on PDE methods for studying this 2d problem [2, 23, 27]. Further consequences of the CLPS ansatz have been investigated in the case Ω a 2d manifold [5]. See [17] for further details.

In this paper, we prove a theorem to compare $E[\mathcal{H}^{\Omega_1, \Omega_2}]$ and $E[\mathcal{H}^{\Omega_2, \Omega_1}]$ by using stochastic order of $c(B_1^{(\Omega_1)}, R_1^{(\Omega_2)})$ and $c(B_1^{(\Omega_2)}, R_1^{(\Omega_1)})$, i.e., the cost functions of a single couple $(B_1, R_1)$ without the assignment constraint. We then focus on the comparison of $E[\mathcal{H}^{\Omega_1, \Omega_2}]$ vs $E[\mathcal{H}^{\Omega_2, \Omega_1}]$, with $\Omega \neq \Omega'$. More precisely, we discuss a special one-dimensional structure in Section 3 and Theorem 2 in Section 4 for comparing a particular class of problems $(|\Omega| = |\Omega'|$, in dimension $d > 1$).

A motivation of our study comes from recent numerical experiments regarding one dimensional $k$-star graph (see Section 3.2 for definition and Figure 3.1($\beta$) for a visualisation).

2 Basic definition and main result

Stochastic order is useful for research in many branches of probability theory, such as stochastic processes [34, Section 9], discrete probability distributions [25], and decision theory [33]. In this paper, we only use the basic definition of the stochastic order of random variables. Further properties and applications about stochastic dominances can be found e.g. in [35].

Definition 1 (Stochastic order of random variables). Two random variables $X$ and $Y$ are in stochastic order, written $X \preceq_{st} Y$, if

$$\mathbb{P}(X \geq t) \leq \mathbb{P}(Y \geq t), \quad \forall t \in \mathbb{R}.$$ 

Equivalently, $X \preceq_{st} Y \iff F_X(t) \geq F_Y(t), \quad \forall t \in \mathbb{R}$, where $F_X(t)$ and $F_Y(t)$ are the cumulative distribution functions.

The next theorem is the main result of this paper, and its proof is reported in Section 5.1.

Theorem 1. Let $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ be four domains. $B_i^{(\Omega_i)}$ are $n$ i.i.d. r.v.s over $\Omega_i$. Analogously, $R_i^{(\Omega_i)}$, $B_i^{(\Omega_i)}$ and $R_i^{(\Omega_i)}$ are respectively $n$ i.i.d. r.v.s over $\Omega_2, \Omega_3,$ and $\Omega_4$. Consider the cost function defined by

$$c(\Omega_i, \Omega_j) = c(x,y) \rightarrow c(x,y),$$

and let $F_c(x,y)$ be its cumulative distribution function. Assume that, if we take arbitrary points $B_i^{(\Omega_1)}, R_i^{(\Omega_2)}, B_i^{(\Omega_3)}$ and $R_i^{(\Omega_4)}$, one in each $n$-sample $B_i^{(\Omega_1)}, R_i^{(\Omega_2)}, B_i^{(\Omega_3)}$ and $R_i^{(\Omega_4)}$, then the following inequality holds

$$c(B_i^{(\Omega_1)}, R_i^{(\Omega_2)}) \leq_{st} c(B_i^{(\Omega_3)}, R_i^{(\Omega_4)}).$$

Then

$$E[\mathcal{H}^{\Omega_1, \Omega_2, \Omega_3, \Omega_4}] \leq E[\mathcal{H}^{\Omega_2, \Omega_1, \Omega_3, \Omega_4}], \forall n \geq 1. \quad (2)$$

Remark 1. (i) The domains $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ in the hypothesis are generalised, that means, for instance, the boundaries could be less than regular or the domains could be not bounded, as eq.(2) is a very strong assumption. This theorem holds even if the cost function $c(x,y)$ can be equal to $+\infty$.

(ii) If neither $c(B_i^{(\Omega_1)}, R_i^{(\Omega_2)})$ nor $c(B_i^{(\Omega_3)}, R_i^{(\Omega_4)})$ stochastically dominates the other, then the direction of the inequality in eq.(3) generally depends on $n$. We illustrate in section 3.2 a one-dimensional example.

(iii) Even if $\Omega_1 = \Omega_3$ and $\Omega_2 = \Omega_4$, it is possible that $c(B_i^{(\Omega_1)}, R_i^{(\Omega_2)})$ is not equal to $c(B_i^{(\Omega_3)}, R_i^{(\Omega_4)})$ by definition of the cost function $c(x,y)$. For instance, one can set $c(B_i^{(\Omega_1)}, R_i^{(\Omega_2)}) = D^2(B_i^{(\Omega_3)}, R_i^{(\Omega_4)})$ and $c(B_i^{(\Omega_3)}, R_i^{(\Omega_4)}) = D^4(B_i^{(\Omega_3)}, R_i^{(\Omega_4)})$.

(iv) Theorem 1 holds for any choice of disorder (as long as $B_i(R_j) \perp B_j(R_l), i \neq j, i, j = 1, \ldots, n$).

(v) As in the case of Remark 3 (iii) (as an example), if we have equality

$$F_c(B_i^{(\Omega_1)}, R_i^{(\Omega_2)})(t) = F_c(B_i^{(\Omega_3)}, R_i^{(\Omega_4)})(t), \quad \forall t \geq 0,$$

then we get equality among all moments

$$\mathbb{E}\left[\left(\mathcal{H}^{\Omega_1, \Omega_2, \Omega_3, \Omega_4}^n\right)^m\right] = \mathbb{E}\left[\left(\mathcal{H}^{\Omega_2, \Omega_1, \Omega_3, \Omega_4}^n\right)^m\right], \quad \forall n \geq 1, m \in \mathbb{N}. \quad (3)$$

3 Applications of Theorem 1 in dimension $d = 1$

We show in section 3.2 how Theorem 1 can be used to study some structures (see Figure 3.1 as examples) in one dimension. Before the applications, our discussion starts with possibly the simplest case, that is, both domains $\Omega$ are line segments. In this section, we assume that $B (R)$ are distributed with uniform disorder unless otherwise stated.
\section{\Omega_b \text{ and } \Omega_r \text{ are line segments: Results valid in general}}

In dimension $d = 1$, one can get a lot more information due to the particularly simple combinatorial properties of the optimal permutation $\pi_{\text{opt}}$. For example, it is well-known that for an ERAP over an interval, if $p \geq 1$, (strict) convexity of the cost function and optimality imply $\pi_{\text{opt}}(i) = i, \forall i = 1, \ldots, n$, if the points are sorted in natural order (see e.g. \cite{10,17}). More precisely, in the case $\Omega_r = \Omega_b = \Omega = [0, 1]$, the reformulation of ERAP in terms of generalized Selberg integrals \cite{12} allows to write, for $p \geq 1$,

$$E[\mathcal{H}_{\text{opt}}^{\Omega_b, \Omega_r}] = \frac{\Gamma \left( 1 + \frac{2}{p} \right)}{(p + 1)} \frac{\Gamma(n + 1)}{(n + 1 + \frac{2}{p})}, \quad \forall n \in \mathbb{N}. \quad (4)$$

We can thus address (here in the case $p = 2$ but with minor modifications for generic $p \geq 1, p$ even) the evaluation of $E[\mathcal{H}_{\text{opt}}^{\Omega_b, \Omega_r}]$, which we resume in the following.

\textbf{Proposition 1.} Let $\Omega_b = [0, l_b]$ and $\Omega_r = [0, l_r]$, for $l_b, l_r > 0$, and let $\Omega = [0, 1]$. Then, at $p = 2$,

$$E[\mathcal{H}_{\text{opt}}^{\Omega_b, \Omega_r}] = (n + 1)(l_b^2 + l_r^2) - (2n + 1)l_bl_r \quad E[\mathcal{H}_{\text{opt}}^{\Omega_b, \Omega_r}], \quad \forall n \in \mathbb{N}. \quad (5)$$

\textbf{Proof of Proposition 1.} Let $\rho_b, \rho_r$ be the probability density functions for red and blue points respectively, $R_{\rho_b}, R_{\rho_r}$ the cumulative distribution functions, and $R_{\rho_b}^{-1}, R_{\rho_r}^{-1}$ their inverse functions (usually called “quantile functions” in Statistics\footnote{This proof follows the assumptions and ideas of \cite[Lemma 2.6.1]{17}’s proof.}). More precisely,

$$\rho_b(x) = \frac{1}{l_b} 1_{(0,l_b)}(x) \Rightarrow R_{\rho_b}(x) = \frac{x}{l_b} 1_{(0,l_b)}(x) \Rightarrow R_{\rho_b}^{-1}(u) = l_b u \cdot 1_{(0,1)}(u).$$

$$\rho_r(y) = \frac{1}{l_r} 1_{(0,l_r)}(y) \Rightarrow R_{\rho_r}(y) = \frac{y}{l_r} 1_{(0,l_r)}(y) \Rightarrow R_{\rho_r}^{-1}(v) = l_r v \cdot 1_{(0,1)}(v).$$

After re-labeling the $n$ points in order of distance from an endpoint, the probability of the $k$-th point being in interval $[u, u + du]$, denoted by $P_{n,k}(u)$, is

$$P_{n,k}(u)du = \frac{n!}{(k-1)!(n-k)!} u^{k-1}(1-u)^{n-k}du.$$ 

Then, for $p \geq 2$ even, we can write

$$E[\mathcal{H}_{\text{opt}}^{\Omega_b, \Omega_r}] = \sum_{k=1}^{n} \int_{0}^{1} \int_{0}^{1} P_{n,k}(u)P_{n,k}(v) |R_{\rho_b}^{-1}(u) - R_{\rho_r}^{-1}(v)|^p du dv$$

$$= \sum_{k=1}^{n} \sum_{q=0}^{p} \binom{p}{q} \int_{0}^{1} \int_{0}^{1} \frac{n!u^{k-1}(1-u)^{n-k}k!v^{k-1}(1-v)^{n-k}}{(k-1)!(n-k)!}\left(R_{\rho_b}^{-1}(u)\right)^q \left(-R_{\rho_r}^{-1}(v)\right)^{p-q} du dv \quad (6)$$

At $p = 2$, eq.(6) simplifies to

$$E[\mathcal{H}_{\text{opt}}^{\Omega_b, \Omega_r}] = \frac{n!}{3} + \frac{n!}{3} - \frac{n(2n + 1)}{3(n + 1)} l_bl_r,$$

which is a homogeneous quadratic polynomial of $(l_b, l_r)$, invariant under $l_b \leftrightarrow l_r$. Upon recalling the standard result at $p = 2$, namely $E[\mathcal{H}_{\text{opt}}^{\Omega_b, \Omega_r}] = \frac{1}{4} \frac{n^2}{p+1}$, eq.(5) follows by simple algebra. \hfill $\square$

\subsection{On the $k$-star graph}

As an application of Theorem 1, we study now the upper and lower bounds of $E[\mathcal{H}_{\text{opt}}]$ for the ERAP on the $k$-star graph presented in \cite{28}, that is, $B$ and $R$ are uniformly distributed over a tree with $k$ edges of length $\frac{1}{k}$ that share a common vertex $O'$, $k \in \mathbb{N}^*$. In this structure, distances are defined by cases, i.e.,

$$D(B_1, R_1) = \begin{cases} |(D(B_1, O') - D(R_1, O')|, & \text{if } B_1 \text{ and } R_1 \text{ are on the same edge}, \\ D(B_1, O') + D(R_1, O'), & \text{otherwise}. \end{cases}$$
**Proposition 2.** Let $\Omega^*$ be 3-star graph and let the cost function be $D^2(B^{(\Omega^*)}, R^{(\Omega^*)})$. Then we have

$$\frac{4n}{27(n + 1)} \leq E[H_{\text{opt}}^{\Omega^*, \Omega^*, n}] \leq \frac{n}{3(n + 1)}, \forall n \geq 1.$$  

**Proof.** We use Theorem 1 to prove this. We bound the ERAP on $\Omega^*$ above and below using two ERAPs with the same cost function but defined, respectively, on the line segments:

$$\Omega := \Omega_b = \Omega_r = [0, 1], \quad \Omega' := \Omega_b = \Omega_r = \left[0, \frac{2}{3}\right].$$

![Figure 3.1](image1.png)

**Figure 3.1:** (a): $\Omega_b = \Omega_r = [0, 1]$, (b): $\Omega_b, \Omega_r$ are 3-star graphs, (c): $\Omega_b = \Omega_r = \left[0, \frac{2}{3}\right]$.

Now we use the results of Section 3.1 for domains of the form $\Omega_b = \Omega_r = [0, l]$ in the case of the 3-star graph. There are 3 choices of 2 arms among 3 available (see Figure 3.2 (a)-(c)). Once this choice is made, the blue $B^{(\Omega)}$ and the red $R^{(\Omega)}$ will fall on some interval of length $\frac{2}{3}$. Notice that case (a), (b) and (c) are not independent: each one carries a probability $\frac{2}{3}$ that that two points are distributed on exactly the bold edges (either on the same edge or not). And with reference to Figure 3.2 (d)-(f), it is clear that each case has a probability of occurring with $\frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$. That is, an edge is specified, and both points are distributed on this edge.

![Figure 3.2: 3-star graphs.](image2.png)

Therefore, with reference to Figure 3.1, we can write the cumulative distribution functions as

$$\begin{align*}
\text{(a)}: \quad &P\left(D\left(B^{(\Omega)}, R^{(\Omega)}\right) \leq t\right) = \begin{cases}
0, &\forall t < 0 \\
n^2 + 2t, &\forall t \in [0, 1] \\
1, &\forall t \geq 1
\end{cases} \\
\text{(b)}: \quad &P\left(D\left(B^{(\Omega')}, R^{(\Omega')}\right) \leq t\right) = 3 \times \frac{4}{9} P\left(D\left(B^{(\Omega')}, R^{(\Omega')}\right) \leq t\right) - 3 \times \frac{1}{9} P\left(D\left(B^{(\Omega)}, R^{(\Omega)}\right) \leq t\right) \\
&= \frac{4}{3} P\left(D\left(B^{(\Omega)}, R^{(\Omega)}\right) \leq \frac{3}{2} t\right) - \frac{1}{3} P\left(D\left(B^{(\Omega)}, R^{(\Omega)}\right) \leq 3t\right), \quad \text{where } \Omega := \left[0, \frac{1}{3}\right].
\end{align*}$$

and

$$\begin{align*}
\text{(c)}: \quad &P\left(D\left(B^{(\Omega')}, R^{(\Omega')}\right) \leq t\right) = P\left(D\left(B^{(\Omega)}, R^{(\Omega)}\right) \leq \frac{3}{2} t\right).
\end{align*}$$

Therefore,

$$D\left(B^{(\Omega')}, R^{(\Omega')}\right) \leq_{\text{st}} D\left(B^{(\Omega')}, R^{(\Omega')}\right) \leq_{\text{st}} D\left(B^{(\Omega)}, R^{(\Omega)}\right),$$

this means

$$c\left(B^{(\Omega')}, R^{(\Omega')}\right) \leq_{\text{st}} c\left(B^{(\Omega')}, R^{(\Omega')}\right) \leq_{\text{st}} c\left(B^{(\Omega)}, R^{(\Omega)}\right).$$

By applying Theorem 1 and Proposition 1,

$$\frac{4n}{27(n + 1)} = E[H_{\text{opt}}^{\Omega', \Omega', n}] \leq E[H_{\text{opt}}^{\Omega', \Omega', n}] = E[H_{\text{opt}}^{\Omega, \Omega, n}] = \frac{n}{3(n + 1)}, \forall n \geq 1.$$

**Remark 2.** An analogous argument (with minor technical modifications) will give upper and lower bounds for the ERAP on the k-star graph for general exponent $p \geq 1$. The k-star graph also provides an easy situation for illustrating Remark 1-(ii).
Example 1. By reusing the previous notations and by setting \( \Omega'' = [0, \frac{2}{3}] \), we have

\[
P \left( D \left( B^{(\Omega''}, R^{(\Omega'')} \right) \leq t \right) = P \left( D \left( B^{(\Omega)}, R^{(\Omega)} \right) \leq \frac{3}{24} t \right).
\]

One instance can show that neither \( c \left( B^{(\Omega''}, R^{(\Omega'')} \right) \leq \frac{2}{3} \) holds:

\[
P \left( D \left( B^{(\Omega''}, R^{(\Omega'')} \right) \leq \frac{2}{3} \right) = 1 \leq P \left( D \left( B^{(\Omega)}, R^{(\Omega)} \right) \leq \frac{2}{3} \right),
\]

while,

\[
P \left( D \left( B^{(\Omega''}, R^{(\Omega'')} \right) \leq \frac{1}{4} \right) = \frac{5.4}{10.24} \leq \frac{5.12}{10.24} = \frac{1}{2} = P \left( D \left( B^{(\Omega)}, R^{(\Omega)} \right) \leq \frac{1}{4} \right).
\]

Using Proposition 1 and Proof of Proposition 2, we get for \( n = 1 \),

\[
E[H_{\text{opt}}^{\Omega', \Omega''}] = \frac{5.76}{54}, \quad E[H_{\text{opt}}^{\Omega', \Omega''}] = 3 \times \frac{4}{9} E(H_{\text{opt}}^{[0,2/3],[0,2/3]}) - 3 \times \frac{1}{9} E(H_{\text{opt}}^{[0,1/3],[0,1/3]}) = \frac{5}{54},
\]

and

\[
\lim_{n \to +\infty} E[H_{\text{opt}}^{\Omega', \Omega''}] = \frac{5.76}{27}.
\]

The numerical results in [28] have suggested that

\[
\lim_{n \to +\infty} E[H_{\text{opt}}^{\Omega', \Omega''}] = \frac{7}{27},
\]

Therefore,

\[
\lim_{n \to +\infty} E[H_{\text{opt}}^{\Omega', \Omega''}] < \lim_{n \to +\infty} E[H_{\text{opt}}^{\Omega', \Omega''}], \quad \text{and for } n = 1, \quad E[H_{\text{opt}}^{\Omega', \Omega''}] > E[H_{\text{opt}}^{\Omega', \Omega''}].
\]

4 Applications of Theorem 1 in general dimension \( d \geq 2 \)

Theorem 2. Let \( \Omega \) be a domain with dimension \( d \), and consider one random blue point and one random red point \( B_1 = (B_{1,1}, \ldots, B_{1,d})^T, \ R_1 = (R_{1,1}, \ldots, R_{1,d})^T \in \Omega \). We suppose that \( \forall 1 \leq i \leq d, \ \forall t \geq 0, \ P(D(B_{1,i}, R_{1,i}) \leq t) = P(D(B_{1,i}, R_{1,i}) \leq t) \).

Then the 3 following statements hold:

1. For \( (\lambda_1, \lambda_2, \ldots, \lambda_d) \in \mathbb{R}^d \backslash \{0\} \), consider a diagonal matrix \( \Lambda \in \mathcal{M}_d(\mathbb{R}) \) with determinant \( \pm 1 \),

\[
\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_d),
\]

which acts as a linear map on \( B_1 \) and \( R_1 \). Set \( B'_1 = \Lambda B_1 \in \Omega' \) and \( R'_1 = \Lambda R_1 \in \Omega' \) for some domain \( \Omega' \). Then \( \forall p \in \mathbb{R}^+ \), we prove in Section 5.2 that \( D(B_1, R_1) \leq D(B'_1, R'_1) \) (and hence by strict monotonicity \( D^p(B_1, R_1) \leq D^p(B'_1, R'_1) ) \), \( \forall p \in \mathbb{R}^+ \). Therefore, by Theorem 1,

\[
E[H_{\text{opt}}^{\Omega, \Omega'}] = E[H_{\text{opt}}^{\Omega', \Omega'}], \quad \forall n \geq 1.
\]

2. Let \( \Lambda' \) and \( \Lambda'' \) be two diagonal \( d \times d \)-matrices with determinant \( \pm 1 \),

\[
\Lambda' = \text{diag}(\lambda'_1, \lambda'_2, \ldots, \lambda'_d), \quad \Lambda'' = \text{diag}(\lambda''_1, \lambda''_2, \ldots, \lambda''_d),
\]

which transform the points in \( \Omega \) into the domain \( \Omega' \) and \( \Omega'' \), as follows,

\[
R'_1 = \Lambda' R_1 \in \Omega', \quad B'_1 = \Lambda' B_1 \in \Omega', \quad R''_1 = \Lambda'' R_1 \in \Omega'', \quad B''_1 = \Lambda'' B_1 \in \Omega''.
\]

If \( \text{Tr}(\text{abs}(\Lambda'')) \leq \text{Tr}(\text{abs}(\Lambda')) \), where \( \text{abs}(M) := (|M_{ij}|)_{1 \leq i, j \leq d} \), then \( \forall p \in \mathbb{R}^+ \), \( D^p(B'_1, R'_1) \leq D^p(B''_1, R''_1) \) and by Theorem 1,

\[
E[H_{\text{opt}}^{\Omega', \Omega''}] \leq E[H_{\text{opt}}^{\Omega', \Omega'}], \quad \forall n \geq 1.
\]

3. Let \( M \in \text{GL}_d(\mathbb{R}) \). We assume that \( M \) transforms the points in \( \Omega \) into the domain \( \bar{\Omega} \), thus

\[
M R_1 \in \bar{\Omega}, \quad MB_1 \in \bar{\Omega}.
\]

Then \( \forall p > 0, \ D^p(B_1, R_1) \leq D^p(MB_1, MR_1) \) and by Theorem 1,

\[
E[H_{\text{opt}}^{\Omega, \bar{\Omega} \bar{\Omega}}] \leq E[H_{\text{opt}}^{\Omega', \Omega''}], \quad \forall n \geq 1.
\]
We now attempt to make Theorem 2 more visible through an example, and to add some comments on this result. The example is comparing the $\mathbb{E}[\mathcal{H}_{\text{opt}}]$ in the ellipses with different eccentricities and equal areas. Unless otherwise specified, we assume in the rest of Section 4 that $\mathcal{B}(\mathcal{R})$ are uniformly distributed over $\Omega_0$ ($\Omega_r$), and that the cost function is $D^2(x, y)$.

Consider the one-parameter family of ellipses $\mathcal{E}_\lambda = \{(x, y) \in \mathbb{R}^2, \text{s.t. } (\lambda x)^2 + (\lambda y)^2 \leq \frac{1}{\pi^2}, \lambda \in (0, 1]\}$. Notice that $|\mathcal{E}_\lambda| = 1$, $\forall \lambda$, and that $\mathcal{E}_1 = \mathcal{B}\left((0, 0), \frac{1}{\sqrt{\pi}}\right)$ is the unit-area 2-ball. The eccentricity of $\mathcal{E}_\lambda$ is $\sqrt{1-\lambda^2}$. Let $\mathcal{H}_{\text{opt}}^\lambda$ be the ground state energy of the ERAP at $p = 2$ on $\mathcal{E}_\lambda$, that is

$$\mathcal{H}_{\text{opt}}^\lambda := \min_{\pi \in \mathcal{S}_n} \sum_{i=1}^{n} D^2(B_i, R_{\pi(i)}), \text{ with } B_i, R_j \text{ i.i.d. } \sim \mathcal{U}(\mathcal{E}_\lambda) \text{ rvs}, 1 \leq i, j \leq n.$$  

**Remark 3.**

(i) In $\mathcal{E}_1$, we get

$$\forall 1 \leq i \leq d, \forall t \geq 0, \mathbb{P}(D(B_{1,i}, R_{1,i}) \leq t) = \mathbb{P}(D(B_{1,1}, R_{1,1}) \leq t).$$

Then, by Theorem 2-1,

$$\mathbb{E}[\mathcal{H}_{\text{opt}}^\lambda] \geq \mathbb{E}[\mathcal{H}_{\text{opt}}^1], \forall n \geq 1, \forall \lambda \in (0, 1]. \quad (9)$$

(ii) Theorem 2-2 tells us that, the closer the eccentricity of the boundary is to 0 (when the boundary is a circle), in other words, the closer $\lambda$ is to 1, the lower $\mathbb{E}[\mathcal{H}_{\text{opt}}^\lambda]$.

(iii) In the proof of Theorem 2-3, if $M = I I^T$, $I$ is an identity matrix and $U \in \text{SO}_d(\mathbb{R})$, then $\mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega, \pi}] = \mathbb{E}[\mathcal{H}_{\text{opt}}^\lambda], \forall n \geq 1$. It means $\mathbb{E}[\mathcal{H}_{\text{opt}}]$ is the identical in the two congruent regions.

(iv) Further, we observe that for the second part of Theorem 2, if the absolute value of determinant is $\alpha \in \mathbb{R}_+$ instead of 1, the stochastic order holds, so does eq.(8). Therefore, we can compare any two ellipses that are equal in area, but the area is not necessarily equal to 1.

Let us now consider the case when $\Omega$ and $\Omega'$ are triangles, $\mathcal{R}(\Omega)$, $\mathcal{B}(\Omega)$ are uniformly distributed on $\Omega$, and $\mathcal{R}(\Omega')$, $\mathcal{B}(\Omega')$ are uniformly distributed on $\Omega'$. We define the Cartesian coordinate system so that two vertices of the triangle are located at the same position on the positive half-axis of the horizontal and vertical axes, respectively. This means, their coordinates are $(\alpha, 0)$ and $(0, \alpha)$, with $\alpha \in \mathbb{R}_+$. When the third vertex of the triangle is on the line $y = x$, we get

$$\forall t \geq 0, \mathbb{P}(D(B_{i,x}, R_{i,x}) \leq t) = \mathbb{P}(D(B_{i,y}, R_{i,y}) \leq t). \quad (10)$$

In this case, $\Omega$ is an isosceles triangle (see Figure 4.1 for examples). We do the transformation $\Lambda = \text{diag}(\lambda, 1/\lambda)$, $\lambda \in \mathbb{R}_+$ on $\mathcal{R}(\Omega)$, $\mathcal{B}(\Omega)$ to get points $\mathcal{R}(\Omega')$, $\mathcal{B}(\Omega')$ (examples of $\Omega'$ with $\lambda = \frac{1}{4}, \frac{3}{4}$ are shown in Figure 4.1). Therefore, by using Theorem 2,

$$\mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega, \pi}] \leq \mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega', \pi'}], \forall n \geq 1.$$  

We then discuss this according to the shape of the triangle:

1. If $\Omega$ is a right triangle, so is $\Omega'$. $\Omega'$ is an isosceles triangle only if $\lambda = 1$;
2. If $\Omega$ is an obtuse triangle, so is $\Omega'$. $\Omega'$ is an isosceles triangle only if $\lambda = 1$;
3. If $\Omega$ is an acute triangle, $\Omega'$ might be acute, right or obtuse. And except for the case when $\Omega$ is an equilateral triangle, there is only one $\lambda > 0$ such that $\Omega'$ is an isosceles triangle. The isosceles triangle $\Omega'$, which might be acute, right or obtuse (see Figure 4.2 for examples), and $\Omega$ are not congruent, since the base of $\Omega$ gets longer and becomes one leg of $\Omega'$.  

![Figure 4.1: Examples of $\Omega$ (with red boundary) as right obtuse, or acute triangle. And examples of $\Omega'$ (with pink boundary)](https://example.com/figure4.1.png)
In summary, every triangle $\Omega_1$ (including the isosceles triangle) can be obtained by picking a suitable Cartesian coordinate system then transforming from an isosceles triangle $\Omega_2$ with $\lambda > 1$, except for the equilateral triangle. Theorem 2-1 tells us that this transformation leads to a reduction in energy. That is, for every triangle $\Omega_1$ which is not an equilateral triangle, we can find an acute isosceles triangle $\Omega_2$ such that
\[
E[\mathcal{H}^n_{\text{opt}}_{\Omega_2}] \leq E[\mathcal{H}^n_{\text{opt}}_{\Omega_1}], \forall n \geq 1.
\]

We can therefore conclude the following.

![Diagram](image)

**Figure 4.2:** Examples of $\Omega$ (with red boundary) as acute triangles and $\Omega'$ as acute, right or obtuse triangles (with pink boundary)

**Proposition 3.** If $\Omega$ is an equilateral triangle and $\Omega'$ is a triangle with the same area, we always have
\[
E[\mathcal{H}^n_{\text{opt}}_{\Omega}] \leq E[\mathcal{H}^n_{\text{opt}}_{\Omega'}], \forall n \geq 1.
\]

To verify Proposition 3, we only need to compare $E[\mathcal{H}_{\text{opt}}]$ over isosceles triangles for $n = 1$ (see Lemma 1).

**Lemma 1.** Let the family of isosceles triangles be
\[
\Lambda \Omega = \Omega^{(\lambda)} = \left\{(x, y) \in \mathbb{R}^2 \text{ s.t. } \frac{1}{\lambda^2}(x - \lambda) \leq y \leq \frac{1}{\lambda^2}(\lambda - x), \lambda \in [0, \lambda], \lambda > 0\right\}.
\]

Then, we have, for $n = 1$,
\[
E[\mathcal{H}^n_{\text{opt}}_{\Omega}] \geq E[\mathcal{H}^{3/4}_{\text{opt}}]. \tag{11}
\]

**Proof.** Remark that the uniform distribution over $\Omega^{(1)}$ has the following probability density function
\[
\rho_{X,Y}(x, y) = 1_{\Omega}(x, y)
\]

and hence the following marginals
\[
\rho_X(x) = 2(1 - x)1_{[0,1]}(x), \quad \rho_Y(y) = (1 + y)1_{[-1,0]}(y) + (1 - y)1_{[0,1]}(y). \tag{13}
\]

For general $\lambda$, the marginals read
\[
\rho_X^{(\lambda)}(x) = \frac{2}{\lambda^2}(\lambda - x)1_{[0,\lambda]}(x), \quad \rho_Y^{(\lambda)}(y) = (1 + \lambda y)1_{[-1,0]}(y) + (1 - \lambda y)1_{[0,1]}(y), \tag{14}
\]

which recover eq.(13) at $\lambda = 1$. Now, for $s \in \mathbb{N}$, the moments write
\[
E[X^s] = 2 \frac{\lambda^s}{(s + 2)(s + 1)}, \quad E[Y^s] = \begin{cases} \frac{(\lambda^s)^*}{2s(s + 1)} & \text{s even} \\ 0 & \text{s odd} \end{cases}. \tag{15}
\]

As $B_x, R_x \sim X$; $B_y, R_y \sim Y$ and $B \perp R$,
\[
E[\mathcal{H}^n_{\text{opt}}_{\Omega}] = \mathbb{E}[(B_x - R_x)^2 + (B_y - R_y)^2] = \mathbb{E}[(B_x - R_x)^2] + \mathbb{E}[(B_y - R_y)^2]
\]
\[
= 2\mathbb{E}[X^2] - 2\mathbb{E}[X]^2 + 2\mathbb{E}[Y^2] - 2\mathbb{E}[Y]^2 = 2 \left(\mathbb{E}[X^2] - \mathbb{E}[X]^2 + \mathbb{E}[Y^2] \right)
\]
\[
= 2 \left(\frac{\lambda^2}{6} - \frac{(\lambda/3)^2}{6} + \frac{1}{6}\right) = \frac{\lambda^2}{9} + \frac{1}{3\lambda^2}. \tag{16}
\]

Then, when $\lambda = 3^{1/4}$, eq.(16) reaches the minimum. Thus, eq.(12) is proven, and $\Omega^{(3^{1/4})}$ is an equilateral triangle. \hfill \Box

**Remark 4.** Similarly, we can extend Proposition 3 to the $n$-sided convex polygons, i.e., if $\Omega$ is an $n$-sided regular convex polygon, and $\Omega'$ is an $n$-sided convex polygon with the same area, then the eq.(11) holds.
5 Proofs

5.1 Proof of Theorem 1

Let \{X_i, 1 \leq i \leq m\} be \textit{m} r.v.s, and \{Y_i, 1 \leq i \leq m\} be \textit{m} r.v.s such that

\[ F_{X_i}(t) \geq F_{Y_i}(t), \forall t \geq 0, \forall 1 \leq i \leq m. \]  

We will discuss some findings about \{X_i\} and \{Y_i\} in Steps 1 and 2. Then applying these findings, by substitution in Steps 3 and 4, we will get the stochastic order of \( \min_{\pi \in S_n} \left\{ \sum_{i=1}^{n} c(B_i^{(\pi_1)}, R_i^{(\pi_2)}) \right\} \) and \( \min_{\pi \in S_n} \left\{ \sum_{i=1}^{n} c(B_i^{(\pi_1)}, R_{\pi(i)}^{(\pi_2)}) \right\} \). In Step 5 we will finish the proof.

Step 1. We now prove that if \{X_i\} are i.i.d. r.v.s, \{Y_i\} as well, then \( \forall 1 \leq k \leq m, \)

\[ F_{X_k} \left( \sum_{j=1}^{k} X_j \right)(t) \geq F_{Y_k} \left( \sum_{j=1}^{k} Y_j \right)(t), \forall t \geq 0. \]  

In fact, \( \forall t \geq 0, \)

\[ F_{X_i}(t) \geq F_{Y_i}(t) \Rightarrow \mathbb{P}(X_i \leq t) \geq \mathbb{P}(Y_i \leq t). \]  

By independence, we have

\[ F_{X_k} \left( \sum_{j=1}^{k} X_j \right)(t) = \mathbb{P} \left( \sum_{j=1}^{k} X_j \leq t \right) = \prod_{j=1}^{k} \mathbb{P} \left( X_j \leq t - \sum_{i=1}^{j-1} X_i \right), \forall 1 \leq k \leq m. \]  

An analogous expression holds for the cdf of \( \sum_{j=1}^{k} Y_j \)'s. Therefore, we apply eq.(19) to the right-hand side of eq.(20) to get eq.(18).

Step 2. Now we prove the following inequality by contradiction,

\[ F_{\min_{1 \leq j \leq m} X_j}(t) \geq F_{\min_{1 \leq j \leq m} Y_j}(t), \forall t \geq 0. \]  

If there exist \( t \geq 0 \) and \( 1 \leq i_1, i_2 \leq m \) such that

\[ F_{\min_{1 \leq j \leq m} X_j}(t) = F_{X_{i_1}}(t), F_{\min_{1 \leq j \leq m} Y_j}(t) = F_{Y_{i_2}}(t), \text{ and } F_{X_{i_1}}(t) < F_{Y_{i_2}}(t), \]

then

\[ F_{X_{i_1}}(t) < F_{Y_{i_2}}(t) \leq F_{X_{i_1}}(t), \]

which reaches a contradiction.

Step 3. Due to the i.i.d. assumption,

\[ F_{c (B_i^{(\Omega_1)}, R_{\pi(i)}^{(\Omega_2)})}(t) = F_{c (B_i^{(\Omega_1)}, R_{\pi'(i)}^{(\Omega_2)})}(t), \forall 1 \leq i, j \leq n, \forall t \geq 0. \]  

It means, \( \forall \leq i, j \leq n, c \left( B_i^{(\Omega_1)}, R_j^{(\Omega_2)} \right) \) follow the identical distributions. We remark that, for every \( \pi, \pi' \in S_n, \)

\[ \left\{ c \left( B_i^{(\Omega_1)}, R_{\pi(i)}^{(\Omega_2)} \right), \forall 1 \leq i \leq n \right\} \text{ and } \left\{ c \left( B_i^{(\Omega_1)}, R_{\pi'(i)}^{(\Omega_2)} \right), \forall 1 \leq i \leq n \right\} \]

are sequences of \( n \) i.i.d. r.v.s. By the assumption (eq.(2)), \( \forall 1 \leq i, j \leq n, \)

\[ F_{c (B_i^{(\Omega_2)}, R_{\pi(i)}^{(\Omega_2)})}(t) \geq F_{c (B_i^{(\Omega_1)}, R_{\pi'(i)}^{(\Omega_2)})}(t), \forall t \geq 0, \]

and we bring the following substitutions into eq.(17) and eq.(18) of Step 1

\[ m \rightarrow n, \ k \rightarrow n, \ X_j \rightarrow c (B_i^{(\Omega_2)}, R_{\pi(i)}^{(\Omega_2)}), \ Y_j \rightarrow c (B_i^{(\Omega_1)}, R_{\pi(i)}^{(\Omega_2)}), \]

then,

\[ F_{\sum_{i=1}^{n} c (B_i^{(\Omega_2)}, R_{\pi(i)}^{(\Omega_2)})}(t) \geq F_{\sum_{i=1}^{n} c (B_i^{(\Omega_1)}, R_{\pi(i)}^{(\Omega_2)})}(t), \forall t \geq 0. \]  

Step 4. We remark that

\[ \left\{ \sum_{i=1}^{n} c (B_i^{(\Omega_1)}, R_{\pi(i)}^{(\Omega_2)}), \forall \pi \in S_n \right\} \]
are \( n! \) identically distributed r.v.s because \( \forall 0 \leq i, j \leq n, \ c\left(B_{i}^{(\Omega_{1})}, R_{j}^{(\Omega_{2})}\right) \) follow the identical distributions. By the same reason,

\[
\left\{ \sum_{i=1}^{n} c(B_{i}^{(\Omega_{1})}, R_{\pi(i)})^{\pi}, \pi \in S_{n} \right\}
\]

are \( n! \) identically distributed r.v.s. Using eq. (22), and we bring the following substitutions into eq. (17) and eq. (21) of Step 2

\[
\begin{align*}
m \to n!, \ \{X_{j}\} \to \left\{ \sum_{i=1}^{n} c(B_{i}^{(\Omega_{1})}, R_{\pi(i)})^{\pi}, \pi \in S_{n} \right\}, \quad \{Y_{j}\} \to \left\{ \sum_{i=1}^{n} c(B_{i}^{(\Omega_{1})}, R_{\pi(i)})^{\pi}, \pi \in S_{n} \right\}.
\end{align*}
\]

then,

\[
F_{\min}\left\{ \sum_{i=1}^{n} c(B_{i}^{(\Omega_{1})}, R_{\pi(i)})^{\pi}, \pi \in S_{n} \right\}(t) \geq F_{\min}\left\{ \sum_{i=1}^{n} c(B_{i}^{(\Omega_{1})}, R_{\pi(i)})^{\pi}, \pi \in S_{n} \right\}(t), \ \forall t \geq 0.
\]

(23)

Step 5. Since \( \min_{\pi \in S_{n}} \sum_{i=1}^{n} c(x, y) \) is non-negative, \( \mathbb{E} \left[ \min_{\pi \in S_{n}} \sum_{i=1}^{n} c(x, y) \right] = \int_{0}^{+\infty} \left( 1 - \mathbb{F}_{\min_{\pi \in S_{n}} \sum_{i=1}^{n} c(x, y)}(t) \right) dt. \) Thus,

\[
\begin{align*}
&\mathbb{E} \left[ \min_{\pi \in S_{n}} \sum_{i=1}^{n} c(B_{i}^{(\Omega_{1})}, R_{\pi(i)})^{\pi} \right] - \mathbb{E} \left[ \min_{\pi \in S_{n}} \sum_{i=1}^{n} c(B_{i}^{(\Omega_{2})}, R_{\pi(i)})^{\pi} \right] \\
&= \int_{0}^{+\infty} \left( 1 - \mathbb{F}_{\min_{\pi \in S_{n}} \sum_{i=1}^{n} c(B_{i}^{(\Omega_{1})}, R_{\pi(i)})^{\pi}}(t) \right) dt - \int_{0}^{+\infty} \left( 1 - \mathbb{F}_{\min_{\pi \in S_{n}} \sum_{i=1}^{n} c(B_{i}^{(\Omega_{1})}, R_{\pi(i)})^{\pi}}(t) \right) dt \\
&= \int_{0}^{+\infty} \left( F_{\min_{\pi \in S_{n}} \sum_{i=1}^{n} c(B_{i}^{(\Omega_{1})}, R_{\pi(i)})^{\pi}}(t) - F_{\min_{\pi \in S_{n}} \sum_{i=1}^{n} c(B_{i}^{(\Omega_{1})}, R_{\pi(i)})^{\pi}}(t) \right) dt \geq 0.
\end{align*}
\]

We conclude that

\[
\mathbb{E} \left[ \min_{\pi \in S_{n}} \sum_{i=1}^{n} c(B_{i}^{(\Omega_{1})}, R_{\pi(i)})^{\pi} \right] \geq \mathbb{E} \left[ \min_{\pi \in S_{n}} \sum_{i=1}^{n} c(B_{i}^{(\Omega_{1})}, R_{\pi(i)})^{\pi} \right], \ \forall n \geq 1.
\]

(24)

We notice that if

\[
\int_{0}^{+\infty} \left( 1 - \mathbb{F}_{\min_{\pi \in S_{n}} \sum_{i=1}^{n} c(B_{i}^{(\Omega_{1})}, R_{\pi(i)})^{\pi}}(t) \right) dt = +\infty, \ \text{or}, \ \int_{0}^{+\infty} \left( 1 - \mathbb{F}_{\min_{\pi \in S_{n}} \sum_{i=1}^{n} c(B_{i}^{(\Omega_{1})}, R_{\pi(i)})^{\pi}}(t) \right) dt = +\infty,
\]

eq. (24) holds naturally.

### 5.2 Proof of Theorem 2

1. By assumption, \( \forall 1 \leq i \leq d, \ \forall t \geq 0, \ \mathbb{P}(D_{p}(B_{i, d}, R_{1, d}) = t) = \mathbb{P}(D_{p}(B_{1, d}, R_{1, d}) = t). \) Thus,

\[
\begin{align*}
\mathbb{P}(\{\lambda_{1}^{p}|B_{1, 1} - R_{1, 1}|^{p} + |\lambda_{2}^{p}|B_{1, 2} - R_{1, 2}|^{p} + \cdots + |\lambda_{d}^{p}|B_{1, d} - R_{1, d}|^{p} \leq t\}) \\
= \mathbb{P}(\{\lambda_{1}^{p}|B_{1, 2} - R_{1, 2}|^{p} + |\lambda_{2}^{p}|B_{1, 3} - R_{1, 3}|^{p} + \cdots + |\lambda_{d}^{p}|B_{1, 1} - R_{1, 1}|^{p} \leq t\}) \\
= \mathbb{P}(\{\lambda_{1}^{p}|B_{1, 1} - R_{1, 1}|^{p} + |\lambda_{2}^{p}|B_{1, 2} - R_{1, 2}|^{p} + \cdots + |\lambda_{d}^{p}|B_{1, d} - R_{1, d}|^{p} \leq t\}).
\end{align*}
\]

We remark that, if for every \( t \leq 0, \ \mathbb{P}(X \leq t) = \mathbb{P}(Y \leq t) \) then \( \mathbb{P}(X \leq t) = \mathbb{P}(X + Y \leq 2t). \) Therefore,

\[
\begin{align*}
\mathbb{P}(\{\lambda_{1}^{p}|B_{1, 1} - R_{1, 1}|^{p} + \cdots + |\lambda_{d}^{p}|B_{1, d} - R_{1, d}|^{p} \leq t\}) \\
= \mathbb{P}\left(\sum_{i=1}^{d} |\lambda_{i}^{p}|B_{1, 1} - R_{1, 1}|^{p} + \cdots + \sum_{i=1}^{d} |\lambda_{i}^{p}|B_{1, d} - R_{1, d}|^{p} \leq d \times t\right) \\
= \mathbb{P}\left(\sum_{i=1}^{d} |\lambda_{i}^{p}|/d \times |B_{1, 1} - R_{1, 1}|^{p} + \cdots + \sum_{i=1}^{d} |\lambda_{i}^{p}|/d \times |B_{1, d} - R_{1, d}|^{p} \leq t\right).
\end{align*}
\]

Meanwhile,

\[
\begin{align*}
\mathbb{P}(\{B_{1, 1} - R_{1, 1}|^{p} + B_{1, 2} - R_{1, 2}|^{p} + \cdots + B_{1, d} - R_{1, d}|^{p} \leq t\}) \\
= \mathbb{P}\left(\sum_{i=1}^{d} |\lambda_{i}^{p}|/d \times |B_{1, 1} - R_{1, 1}|^{p} + \cdots + \sum_{i=1}^{d} |\lambda_{i}^{p}|/d \times |B_{1, d} - R_{1, d}|^{p} \leq \left(\sum_{i=1}^{d} |\lambda_{i}^{p}|\right) /d \times t\right).
\end{align*}
\]
As \( \prod_{i=1}^{d} \lambda_i \geq 1 \), by the inequality of arithmetic and geometric means, we get
\[
\left( \sum_{i=1}^{d} |\lambda_i|^p \right)^{1/d} \geq \left( \prod_{i=1}^{d} |\lambda_i|^p \right)^{1/d} = \prod_{i=1}^{d} |\lambda_i|^{p/d} = 1.
\]

In conclusion,
\[
F_{D^p}(B_1', R_1')(t) = \mathbb{P}(D^p(B_1', R_1') \leq t) = \mathbb{P}(\text{Tr}((\text{abs}(\Lambda B_1 - \Lambda R_1))^p) \leq t)
\]
\[
= \mathbb{P}(\|\lambda_1^p|B_{1,1} - R_{1,1}|p + |\lambda_2^p|B_{1,2} - R_{1,2}|p + \cdots + |\lambda_d^p|B_{1,d} - R_{1,d}|p \leq t)
\]
\[
\leq \mathbb{P}(|B_{1,1} - R_{1,1}|p + |B_{1,2} - R_{1,2}|p + \cdots + |B_{1,d} - R_{1,d}|p \leq t)
\]
\[
= \mathbb{P}(\text{Tr}((\text{abs}(\Lambda B_1 - R_1))^p) \leq t) = \mathbb{P}(D^p(B_1, R_1) \leq t) = F_{D^p}(B_1, R_1)(t), \quad \forall t \geq 0,
\]
i.e.,
\[
D^p(B_1, R_1) \leq_M D^p(B_1', R_1'), \quad \forall p \in \mathbb{R}_+,
\]
and hence, by Theorem 1,
\[
\mathbb{E}[\mathcal{H}_{\text{opt}}^{\lambda^p}] \leq \mathbb{E}[\mathcal{H}_{\text{opt}}^{\lambda^p'}], \quad \forall n \geq 1.
\]

2. We use monotonicity of trace functions (see e.g. [16, Section 2.2]) w.r.t. the function \( f(t) = t^p \) (which is monotone increasing \( \forall p > 0 \)) for any Hermitian matrix \( A \), the function \( A \rightarrow \text{Tr}[A^p] \) is monotone increasing on \( \mathbb{R}_+ \). In our case,
\[
\text{Tr}((\text{abs}(\Lambda')) \leq_M \text{Tr}((\text{abs}(\Lambda')) \implies \text{Tr}((\text{abs}(\Lambda'^p)) \leq_M \text{Tr}((\text{abs}(\Lambda')) \implies \sum_{i=1}^{d} |\lambda_i'^p| \leq \sum_{i=1}^{d} |\lambda_i^p|.
\]

Then, as above, by the assumption, \( \forall 1 \leq i \leq d, \forall t \geq 0, \mathbb{P}(D^p(B_1, R_1, i) = t) = \mathbb{P}(D^p(B_1, R_1, 1) = t) \), we have
\[
F_{D^p}(B_1', R_1')(t) = \mathbb{P}(D^p(B_1', R_1') \leq t) = \mathbb{P}(\text{Tr}((\text{abs}(\Lambda'' - \Lambda R_1))^p) \leq t)
\]
\[
= \mathbb{P}((\sum_{i=1}^{d} |\lambda_i''|^p) |B_{1,1} - R_{1,1}|^p + |\lambda_2''|^p|B_{1,2} - R_{1,2}|^p + \cdots + |\lambda_d''|^p|B_{1,d} - R_{1,d}|^p \leq t)
\]
\[
= \mathbb{P}\left( \frac{d}{d} \sum_{i=1}^{d} |\lambda_i''|^p |B_{1,1} - R_{1,1}|^p + \cdots + \left( \sum_{i=1}^{d} |\lambda_i''|^p \right) \frac{d}{d} |B_{1,d} - R_{1,d}|^p \leq t \right)
\]
\[
= \mathbb{P}\left( \frac{d}{d} |B_{1,1} - R_{1,1}|^p + \cdots + |B_{1,d} - R_{1,d}|^p \leq d \left( \sum_{i=1}^{d} |\lambda_i''|^p \right) \right), \quad \text{as } \sum_{i=1}^{d} |\lambda_i''|^p \leq \sum_{i=1}^{d} |\lambda_i|^p
\]
\[
= \mathbb{P}\left( \frac{d}{d} |\lambda_1''|^p |B_{1,1} - R_{1,1}|^p + \cdots + \left( \sum_{i=1}^{d} |\lambda_i''|^p \right) \frac{d}{d} |B_{1,d} - R_{1,d}|^p \leq t \right)
\]
\[
= \mathbb{P}(\text{Tr}((\text{abs}(\Lambda'' B_1 - R_1))^p) \leq t) = \mathbb{P}(D^p(B_1', R_1') \leq t) = F_{D^p}(B_1', R_1')(t), \quad \forall t \geq 0.
\]

Therefore, \( D^p(B_1', R_1') \leq_M D^p(B_1, R_1) \) and using again Theorem 1,
\[
\mathbb{E}[\mathcal{H}_{\text{opt}}^{\lambda''}] \leq \mathbb{E}[\mathcal{H}_{\text{opt}}^{\lambda'}], \quad \forall n \geq 1.
\]

3. Since \( M \) is real symmetric, it is diagonalizable by rotation matrices: \( M = U \Lambda U^T \), where
   
   (a) \( U \) is a diagonal \( d \times d \) matrix, and \( \det(A) \leq 1 \), since \( \det(M) \leq 1 \);
   
   (b) \( U \in \text{SO}_d(\mathbb{R}) \).

Clearly
\[
D(\Lambda U B_1, \Lambda U R_1) = D(\Lambda U^T B_1, \Lambda U^T R_1), \quad D(U^T B_1, U^T R_1) = D(B_1, R_1),
\]
so that, \( \forall p \), we have
\[
F_{D^p(U^T \Lambda U^T B_1, U^T \Lambda U^T R_1)}(t) = F_{D^p(M U^T B_1, M U^T R_1)}(t), \quad F_{D^p(U^T B_1, U^T R_1)}(t) = F_{D^p(B_1, R_1)}(t), \quad \forall t \geq 0.
\]

According to Theorem 2-1, \( \forall p > 0 \),
\[
F_{D^p(U^T \Lambda U^T B_1, U^T \Lambda U^T R_1)}(t) \leq F_{D^p(U^T B_1, U^T R_1)}(t), \quad \forall t \geq 0.
\]
In summary, \( \forall p > 0, \)
\[
F_{D^p(MB_1, MR_1)}(t) = F_{D^p(U_{AB_1}U_{AR_1})}(t) \leq F_{D^p(B_1, R_1)}(t), \forall t \geq 0,
\]
i.e. \( D^p(B_1, R_1) \leq_{st} D^p(MB_1, MR_1) \). Therefore, after using Theorem 1,
\[
E[H_{\Omega, \Omega}] \leq E[H_{\Omega, \Omega}^*], \forall n \geq 1.
\]

6 Some research perspectives

A number of limitations needs to be noted regarding Theorem 1. First, stochastic order does not appear to be a necessary condition for getting eq. (3). We believe that further study of this condition may help elucidate comparison of ERAPs in a general setting. Second, it would be interesting to study simpler sufficient conditions, as calculating the cumulative distribution function of \( c(B_1, R_1) \) might be challenging (analytically), particularly for the higher dimensional problems.

In spite of its limitations, Theorem 1 offers valuable insights into studying some difficult cases. In terms of future work, we are also considering several applications. First of all, it should be possible to find upper and lower bounds for \( E[H_{\Omega, \Omega}] \) on the \( k \)-star graphs that are closer to the exact solution using a similar method (with some refinement) to Proposition 2. Furthermore, setting \( \Omega \neq \Omega^* \), the comparison of \( E[H_{\Omega, \Omega}^*] \) and \( E[H_{\Omega, \Omega^*}] \) could be made using Theorem 1. Finally, Theorem 1 may be a useful tool for studying ERAPs with glued domains in higher dimensions by performing projections.

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References

[1] Miklós Ajtai, János Komlós, and Gábor Tusnády. On optimal matchings. *Combinatorica*, 4(4):259–264, 1984.
[2] Luigi Ambrosio, Michael Goldman, and Dario Trevisan. On the quadratic random matching problem in two-dimensional domains. *Electronic Journal of Probability*, 27:1–35, 2022.
[3] Luigi Ambrosio, Federico Stra, and Dario Trevisan. A PDE approach to a 2-dimensional matching problem. *Probability Theory and Related Fields*, 173(1):433–477, 2019.
[4] Franck Barthe and Charles Bordenave. Combinatorial optimization over two random point sets. In *Séminaire de Probabilités XLV*, pages 483–535. Springer, 2013.
[5] Dario Benedetto, Emanuele Caglioti, Sergio Caracciolo, Matteo D’Achille, Gabriele Sicuro, and Andrea Sportiello. Random assignment problems on 2d manifolds. *Journal of Statistical Physics*, 183, 2021.
[6] Philippe Berthet and Jean-Claude Fort. Exact rate of convergence of the expected \( W_2 \) distance between the empirical and true Gaussian distribution. *Electronic Journal of Probability*, 25(none):1–16, jan 2020.
[7] Sergey Bobkov and Michel Ledoux. One-dimensional empirical measures, order statistics, and kantorovich transport distances. *Memoirs of the American Mathematical Society*, 2019.
[8] Elena Boniolo, Sergio Caracciolo, and Andrea Sportiello. Correlation function for the Grid-Poisson Euclidean matching on a line and on a circle. *Journal of Statistical Mechanics: Theory and Experiment*, 2014(11):P11023, nov 2014.
[9] Sergio Caracciolo, Matteo D’Achille, Vittorio Erba, and Andrea Sportiello. The dyck bound in the concave 1-dimensional random assignment model. *Journal of Physics A: Mathematical and Theoretical*, 53(6):064001, 2020.
[10] Sergio Caracciolo, Matteo D’Achille, and Gabriele Sicuro. Random Euclidean matching problems in one dimension. *Physical Review E*, 96(4):042102, 2017.
[11] Sergio Caracciolo, Matteo D’Achille, and Gabriele Sicuro. Anomalous Scaling of the Optimal Cost in the One-Dimensional Random Assignment Problem. *Journal of Statistical Physics*, 174(4):846–864, feb 2019.
12 Sergio Caracciolo, Andrea Di Gioacchino, Enrico M Malatesta, and Luca G Molinari. Selberg integrals in 1D random Euclidean optimization problems. *Journal of Statistical Mechanics: Theory and Experiment*, 2019:063401, 2019.

13 Sergio Caracciolo, Carlo Lucibello, Giorgio Parisi, and Gabriele Sicuro. Scaling hypothesis for the euclidean bipartite matching problem. *Physical Review E*, 90(1):012118, 2014.

14 Sergio Caracciolo and Gabriele Sicuro. One-dimensional Euclidean matching problem: exact solutions, correlation functions, and universality. *Physical Review E*, 90(4):042112, 2014.

15 Sergio Caracciolo and Gabriele Sicuro. Quadratic Stochastic Euclidean Bipartite Matching Problem. *Physical Review Letters*, 115(23):230601, dec 2015.

16 Eric Carlen. Trace inequalities and quantum entropy: an introductory course. *Entropy and the quantum*, 529:73–140, 2010.

17 Matteo D’Achille. *Statistical Properties of the Euclidean Random Assignment Problem*. PhD thesis, Université Paris-Saclay, 2020.

18 Matteo D’Achille and Andrea Sportiello. Anomalous scaling of the optimal cost in the one-dimensional euclidean random assignment problem: some rigorous results. *In preparation*, 2020–.

19 A Errarhout, S Kharraja, and C Corbier. Two-stage stochastic assignment problem in the home health care. *IFAC-PapersOnLine*, 49(12):1152–1157, 2016.

20 Syakinah Faoudzi, Syariza Abdul-Rahman, and Rosshairy Abd Rahman. An assignment problem and its application in education domain: A review and potential path. *Advances in Operations Research*, 2018, 2018.

21 Leslie R Foulds. *Combinatorial optimization for undergraduates*. New York: Springer-Verlag, 1984.

22 Nicolas Fournier and Arnaud Guillin. On the rate of convergence in wasserstein distance of the empirical measure. *Probability Theory and Related Fields*, 162:707–738, 2015.

23 Michael Goldman and Dario Trevisan. Convergence of asymptotic costs for random Euclidean matching problems. *Probability and Mathematical Physics*, 2(2):341–362, may 2021.

24 Patrick Jaillet. A priori solution of a traveling salesman problem in which a random subset of the customers are visited. *Operations research*, 36(6):929–936, 1988.

25 Achim Klenke and Lutz Mattner. Stochastic ordering of classical discrete distributions. *Advances in Applied Probability*, 42(2):392–410, 2010.

26 Harold W Kuhn. The hungarian method for the assignment problem. *Naval research logistics quarterly*, 2(1-2):83–97, 1955.

27 Michel Ledoux and Jie-Xiang Zhu. On optimal matching of Gaussian samples III. *Probability and Mathematical Statistics*, 41(2):237–265, 2021.

28 Yuqi Liu. Universality in the one-dimensional Euclidean random assignment problem. Master 1 thesis, Université Paris-Est Créteil Val-de-Marne. Soon available on https://dumas.ccsd.cnrs.fr/.

29 Robert J McCann. Exact solutions to the transportation problem on the line. *Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences*, 455(1984):1341–1380, 1999.

30 Marc Mézard and Giorgio Parisi. The Euclidean matching problem. *Journal de Physique*, 49(12):2019–2025, 1988.

31 Marc Mézard, Giorgio Parisi, and Miguel Virasoro. *Spin glass theory and beyond: An Introduction to the Replica Method and Its Applications*. World Scientific Publishing Company, 1987.

32 Marc Mézard, Giorgio Parisi, and Riccardo Zecchina. Analytic and algorithmic solution of random satisfiability problems. *Science*, 297(5582):812–815, 2002.

33 Stylianos Perrakis. *Stochastic dominance option pricing: An alternative approach to option market research*. Palgrave Macmillan, 2019.

34 Sheldon M Ross. *Stochastic processes*. New York: John Wiley & Sons, 2nd edition, 1996.

35 Moshe Shaked and J George Shanthikumar. *Stochastic orders*. Springer, 2007.

36 Xi Xiang, Changchun Liu, and Lixin Miao. Reactive strategy for discrete berth allocation and quay crane assignment problems under uncertainty. *Computers & Industrial Engineering*, 126:196–216, 2018.