Effective Field Theory with Differential Operator Technique for Dynamic Phase Transition in Ferromagnetic Ising Model

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Abstract. The non-equilibrium phase transition in a ferromagnetic Ising model is investigated by use of a new type of effective field theory (EFT) which correctly accounts for all the single-site kinematic relations by differential operator technique. In the presence of a time dependent oscillating external field, with decrease of the temperature the system undergoes a dynamic phase transition, which is characterized by the period averaged magnetization \(Q\), from a dynamically disordered state \(Q = 0\) to the dynamically ordered state \(Q \neq 0\). The results of the dynamic phase transition point \(T_c\) determined from the behavior of the dynamic magnetization and the Liapunov exponent provided by EFT are improved than that of the standard mean field theory (MFT), especially for the one dimensional lattice where the standard MFT gives incorrect result of \(T_c = 0\) even in the case of zero external field.

1. Introduction
The nonequilibrium phase transition characterized by the period time averaged magnetization is called as the dynamic phase transition (DPT) [1]. The DPT was first observed in a study of mean-field calculation of the time dependent magnetization for an Ising ferromagnetic system in an oscillating field [2]. Since then, many investigations have been performed by using various methods including mean-field type and Monte Carlo calculation [3, 4, 5], for various systems such as Blume-Capel model [6] and Ising ferrimagnets [7] as well as standard Ising ferromagnetic systems. The DPT has been also experimentally observed in Co films on a Cu (001) ultrathin magnetic films [8] and [Co/Pt]\(^3\) multilayers [9]. From these studies it is now well established that there appears a genuine continuous phase transition between the symmetric and asymmetric dynamic phase for some region in the parameter space spanned by temperature and magnetic field amplitude and frequency.

In this paper, we have investigated the nonequilibrium DPT by use of a new type effective field theory (EFT) [10] for the kinetic Ising ferromagnet in an oscillating field. The EFT correctly accounts for all the single-site kinematic relations by differential operator technique and is expected to improve the results of the standard simplest mean-field theory (MFT).
2. Theory

We consider a simple $s = \frac{1}{2}$ ferromagnetic Ising system under an oscillating magnetic field and the Hamiltonian is given by

$$\mathcal{H} = - \sum_{(ij)} J_{ij} s_i s_j - h(t) \sum_i s_i. \quad (1)$$

In equation (1), $J_{ij}$ is the ferromagnetic interaction between the nearest neighboring site $i$ and $j$, and $h(t) = h_0 \cos \omega t$ where $h_0$ and $\omega$ are the amplitude and the frequency of the oscillating field, respectively. The evolution equation for the expectation value $\langle s_i \rangle$ is directly obtained by the master equation, $\frac{d}{dt} \langle s_i \rangle = -2 \langle s_i w(s_i, t) \rangle$, where $w(s_i, t)$ is a transition probability per unit time that $i$-th spin changes its spin state from $s_i$ to $-s_i$, and it is expressed as

$$w(s_i, t) = \frac{1}{2} \{1 - 2s_i \tanh \beta E_i\},$$

where $E_i = \sum_j J_{ij} s_j + h(t)$, $\beta = (k_B T)^{-1}$ and $\tau$ represents the relaxation time. Therefore, the master equation for the system becomes

$$\tau \frac{d}{dt} \langle s_i \rangle = -\langle s_i \rangle + \frac{1}{2} \left\{ \tanh \frac{\beta E_i}{2} \right\}. \quad (2)$$

In the simple MFT, the second term in the right hand side of equation (2) is approximated as $\frac{1}{2} \tanh \frac{\beta E_i}{2}$. Hence, in addition to the negligence of the fluctuation and correlation effects, the single site kinematic relation $(2s_i)^2 = 1$ is not considered at all.

On the other hand, in the EFT described below, we can correctly account for all the single-site kinematic relations by using differential operator technique. Here we use the mathematical relation for arbitrary function $\varphi(x)$, $\exp(\lambda D) \varphi(x) = \varphi(x + \lambda)$ with $D = d/dx$ and the exact relation $\exp(a s_j) = \cosh(a/2) + 2s_j \sinh(a/2)$ with $s_j = \pm \frac{1}{2}$, and get the following polynomial expression for the second term in the right hand side of equation (2),

$$\left\{ \tanh \frac{\beta E_i}{2} \right\} = \left\{ \prod_{j=1}^z \left[ \cosh \left( \frac{D_{ij}}{4} \right) + 2s_j \sinh \left( \frac{D_{ij}}{4} \right) \right] \right\} \tanh \left( x + \frac{\beta h(t)}{2} \right) \bigg|_{x=0}, \quad (3)$$

where $z$ is the coordination number and $t_{ij} = \beta J_{ij}$. The right hand side of equation (3) contains unmanageable thermal average of multiple spin correlations. In order to make the problem manageable, let us use the following decoupling approximation (Zernike approximation),

$$\langle s_j s_k \cdots s_l \rangle \sim \langle s_j \rangle \langle s_k \rangle \cdots \langle s_l \rangle \quad (4)$$

which means that nearest neighbors of a site $i$ are assumed to be completely independent of each other. Introducing this approximation (4) and putting $J_{ij} = J$ and $\langle s_i \rangle = m$ for all sites, finally, we get the following relation,

$$\tau \frac{d}{dt} m = -m + \frac{1}{2} \left\{ \tanh \frac{\beta E_i}{2} \right\},$$

$$= -m + \frac{1}{2} \sum_{n=0}^z C_n \left( \frac{1}{2} + m \right)^{z-n} \left( \frac{1}{2} - m \right)^n \times \tanh \left[ \frac{1}{4} (z - 2n) \beta J + \frac{\beta h(t)}{2} \right]. \quad (5)$$

Hereafter we use the dimensionless parameters which are defined by $\xi = \omega t$, $T = (\beta J)^{-1}$, $h = h_0/J$, and $\Omega = \omega \tau$. In the next section, by use of equation (5), we will discuss the nonequilibrium DPT for the present system.
3. Results and Discussion

The stationary solutions of equation (5) will be a periodic function of $\xi$ with period $2\pi$, that is $m(\xi + 2\pi) = m(\xi)$, and those solutions are also distinguished by whether they satisfy $m(\xi) = -m(\xi + \pi)$ or not. Here we introduce the dynamical order parameter $Q$ which is defined by the average of the total magnetization in a period, $Q = \frac{1}{2\pi} \int_0^{2\pi} m(\xi) d\xi$. Then, we can classify the solutions in two types: a symmetric one where $m(\xi)$ completely follows the field and oscillates around zero leading $Q = 0$, and an asymmetric one where $m(\xi)$ does not follow the field and oscillates around a finite value leading $Q \neq 0$. For several fixed values of $h = h_0/J$ and $\Omega = \omega \tau$, the equation (5) is solved by using fourth order Runge-Kutta method and the dynamical order parameter $Q$ is obtained as a function of $T = (\beta J)^{-1}$. From the behavior of $Q$ vs $T$ we can locate the DPT point separating symmetric dynamically disorder ($Q = 0$) and asymmetric dynamically ordered ($Q \neq 0$) phase. In the calculation, we also checked and verified the results and stability of the solutions by estimating the well known Liapunov exponent adapted for the present periodic system.

![Figure 1](image1.png)

**Figure 1.** The phase diagram projected in $(T, h)$ plane for the 3-d Ising ferromagnetic system ($\Omega/2\pi = 1.0$). The P, F and P+F denote the region of the paramagnetic, ferromagnetic and the coexistence phase, respectively. TCP is denoted by cross mark (×). The phase boundaries obtained by EFT(MFT) are show by solid (dashed) lines.

![Figure 2](image2.png)

**Figure 2.** A part of the phase diagram projected in $(\Omega, h)$ plane for the 3-d Ising ferromagnetic system ($T = 0.2$). The P, F and P+F denote the region of the paramagnetic, ferromagnetic and the coexistence phase, respectively. The phase boundaries obtained by EFT(MFT) are show by solid (dashed) lines.

The phase diagram in $(T, h)$ plane by the present EFT is shown in figure 1 for the 3-d Ising ferromagnetic system. For comparison, the result of MFT are also shown by dashed line. We found that, as expected, present EFT gives DPT in the region of sufficiently small $h$ and small $T$. As well as MFT, in addition to the F and P phase there appears the coexistence phase F+P. The DPT between the F+P phase and F (P) phase is first order, while the DPT between F and P phase is second order. Therefore, both of EFT and MFT give a tricritical point in their respective diagrams.

Now it is believed and confirmed by various Monte Carlo simulation that for the present system the P and F phase do not coexist and DPT tricritical point does not appear. Then, it is considered that the coexistence of P and F phase is an artifact of the effective field type approximation of EFT and MFT. While the EFT cannot avoid the appearance of F+P coexistence phase, we can see that the coexistence region is strongly reduced as an effect of
taking into account the single site kinematic relation by EFT. In figure 2, we present a part of phase diagram in (Ω, h) plane, we can also confirm that the coexistence region in the phase diagram is definitely reduced than that of MFT.

In the 1-d equilibrium system of h = 0, the effect of taking into account the kinematic relation is very evident and EFT gives correct critical temperature $T_c = 0$ while MFT gives incorrect result of $T_c \neq 0$. In order to see whether it is true or not in the nonequilibrium system, we investigated the DPT in the 1-d system. For the 1-d present system, from the phase diagram obtained by EFT, we found that for the case of $h \neq 0$ the DPT does appear but only in the extremely narrow region. This fact suggests that more precise treatment than EFT may conclude the disappearance of the DPT for all parameter regions. Recently, M. Khorrami and A. Aghamohammadi have investigated the DPT of 1-d Ising system in which oscillating magnetic field is applied only to the boundary spins [11]. From the detailed analysis with changing system size, they concluded that the system shows a DPT in a finite system but it disappears in the thermodynamic limit. While their system is somewhat different with ours, their result seems to support our suggestion on the disappearance of DPT for 1-d Ising model.

4. Concluding Remarks

By use of new effective field theory (EFT), we have analyzed the dynamical phase transition (DPT) for d-dimensional ferromagnetic Ising system in an oscillating magnetic field h(t). The DPT temperature and the (T, h) and (Ω, h) phase diagrams are well improved than those of the standard molecular field calculation, especially for the one dimensional lattice.

To our knowledge this is the first EFT treatment of the kinetic Ising model, and we confirmed that EFT is well applicable to such nonequilibrium system as well as equilibrium system.

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