THE VALUE RING OF GEOMETRIC MOTIVIC INTEGRATION, AND
THE IWAHORI HECKE ALGEBRA OF $SL_2$

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(with an appendix by Nir Avni)

1. Introduction

In [1], an integration theory for valued fields was developed with a Grothendieck group approach. Two types of categories were studied. The first was of semi-algebraic sets over a valued field, with all semi-algebraic morphisms. The Grothendieck ring of this category was shown to admit two natural homomorphisms, essentially into the Grothendieck ring of varieties over the residue field. These can be viewed as generalized Euler characteristics. The objects of the second category are semi-algebraic sets with volume forms; the morphisms are semi-algebraic bijections preserving the absolute value of the volume form. (Some finer variants were also studied.) The Grothendieck ring of bounded objects in this category can be viewed as a universal integration theory.

Even before the restriction to bounded sets, an isomorphism was shown between the semiring of semi-algebraic sets with measure preserving morphisms, and certain semirings formed out twisted varieties over the residue field, and rational polytopes over the value group. Though this description is very precise, the target remains complicated. With a view to representation-theoretic applications, we require a simpler description of the possible values of the integration, and in particular natural homomorphisms into fields. In the present paper we obtain such results after tensoring with $\mathbb{Q}$, in particular introducing additive inverses. Since this operation trivializes the full semiring, we restrict to bounded sets. We show that the resulting $\mathbb{Q}$-algebra is generated by its one-dimensional part. In the “geometric” case, i.e. working over an elementary submodel as a base, we determine the structure precisely. As a corollary we obtain useful canonical homomorphisms in the general case.

Let $F$ be a valued field of residue characteristic 0. Let $V$ be an $F$-variety. A semialgebraic subset of $V$ is a Boolean combination of subvarieties and of sets defined by valuation inequalities $\{x \in U : \text{val} f(x) \leq \text{val} g(x)\}$, where $U$ is a relatively closed $F$-subvariety of $V$, and $f, g$ are regular functions on $U$. (It is possible to think of the $F$-points defined by these equalities, but better to think of $K$-points where $K$ is an undetermined valued field extension of $F$.)

Let Vol$_F$ be the category of semi-algebraic sets with bounded semi-algebraic volume forms; see §3.21 for a precise definition. The Jacobian of any semi-algebraic map between such objects can then be defined, outside a lower dimensional variety; morphisms are semi-algebraic bijections whose Jacobian has valuation zero (outside a lower dimensional variety.) The Grothendieck ring $K(\text{Vol}_F)$ of this category can be viewed as a universal integration theory for semialgebraic sets and volume forms over $F$. This ring is graded by dimension, but one can form out of it a ring $K^{df}(\text{Vol}_F)$ of “pure numbers”, ratios of integrals of equal dimension (see §1.1). We state there a version of Theorem 3.24 in the case of a higher dimensional local field.

Let Var$_F$ be the category of algebraic varieties over the residue field of $F$. $K^{df}_Q(\text{Var}_F)$ is the dimension-free Grothendieck ring with rational coefficients this category. There exists a natural
homomorphism $L_{F}^{\text{var}} : K_{Q}^{df}(\text{Var} F) \to K_{Q}^{df}(\text{Vol} F)$, induced by taking the full pullback of a variety $V \subseteq \mathbb{A}^n(F)$ to the valuation ring, with the standard form $dx_1 \ldots dx_n$.

Assume $F$ has value group generated by $n$ elements $\gamma_1, \ldots, \gamma_n$. Extend $L_{F}^{\text{var}}$ to a homomorphism

$$L_{F} : K_{Q}^{df}(\text{Var} F)[t_1, \ldots, t_n, q_1, \ldots, q_n] \to K_{Q}^{df}(\text{Vol} F)$$

by mapping $q_i$ to the ratio of the annulus of valuative radius $\gamma_i$ to the unit annulus $U_0$; and $t_i$ to the logarithmic quantity $L_{F}(t_i) = [\{(x : 0 \leq \text{val}(x) < \gamma_i)\}, dx/x]/[(U_0, dx)]$.

Localizations by certain elements will be needed. They are explained in the text before the statement of Theorem 3.24. Here we will just denote them with a subscript $\text{loc}$. We denote by $L_{F}$ the homomorphism induced on localizations also.

**Theorem 1.1.** Assume $F$ has value group $\mathbb{Z}^n$. Let $F$ denote the residue field of $F$.

There exists a canonical homomorphism

$$I_{F} : K_{Q}^{df}(\text{Vol} F)_{\text{loc}} \to K_{Q}^{df}(\text{Var} F)[t_1, \ldots, t_n, q_1, \ldots, q_n]_{\text{loc}}$$

with $I_{F}J_{F} = 1d$.

The ring $K_{Q}^{df}(\text{Var} F)$ is a subring of the usual Grothendieck ring $K_{Q}(\text{Var}$ of varieties over $F$, localized at $[G_m]$; we have $K_{Q}^{df}(\text{Var} F) = \{a/[G_m]^k : k \in \mathbb{N}, a \in K_{Q}(\text{Var}), \dim(a) \leq k\}$. It includes an element $L = 1 + [G_m]$, corresponding on the left to the ratio of the volume of a closed and an open ball of the same radius. On the other hand, $q_i$ corresponds to the ratio of the ball val$(x) > \text{val}(t_i)$, to the unit ball val$(x) > 0$.

The quantities $L, q_1, \ldots, q_n$ are $\mathbb{Q}$-algebraically independent. This contrasts with the $p$-adic integration theories, and those of Denef, Denef-Loeser, Cluckers-Loeser, where one has $(n = 1$ and) $L^{-1} = q_1$. The reason for the additional degree of freedom is that we chose the “geometric” realization of the universal integral. It can already be seen via the the following functoriality in ramified extensions:

If $F \leq F'$ is a finite ramified field extension, whose value group is generated (for simplicity) by $\gamma_1/m_1, \ldots, \gamma_n/m_n$, then we have:

$$I_{F'} : K_{Q}^{df}(\text{Vol} F')_{\text{loc}} \to K_{Q}^{df}(\text{Var} F')[t'_1, \ldots, t'_n, q'_1, \ldots, q'_n]_{\text{loc}}$$

With $m_it_i = t_i$ and $(q_i')^{m_i} = q_i$. At the limit over all ramified extensions, or just a family whose value groups approach $\mathbb{Q}^n$, the homomorphisms $I_{F'}$ become an isomorphism. In fact the fundamental case here is really the case of divisible value group.

Viewed as an integral, $I_{F}$ satisfies Fubini and the usual change of variable formula, with respect to arbitrary semi-algebraic maps. It is also additive with respect to definable maps into the value group or residue field.

In the case of value group $\mathbb{Z}^n$ described above, the theorem should be compared to earlier integration theories of Fesenko and Parshin; see [3].

The above statements are all special cases of the results in [1], with improvement only in the description of the target ring. This depends on a closer study of the Grothendieck ring of bounded piecewise linear polytopes. We express in closed form the motivic volume of any bounded polytope over an ordered Abelian group, in terms of quantities $i(b)$ referring to the length of a one-dimensional segment $[0, b)$, and Boolean quantities $e(b)$ that can be viewed as referring to the existence or not of $b$ as a rational point. Note that $\frac{1}{m}i(x) \neq i(\frac{x}{m})$ in general. The formulas specialize (in their graded version) to standard integration formulas, and on the other hand formulas giving the number of integer points in bounded polytopes. But since they must also be valid in groups such as $\mathbb{Z}^n$, nothing can be assumed about the index of arithmetic
sequences. Nevertheless when sufficient care is taken with arithmetic issues, it turns out that
the formulas can be proved using integration by parts.

In [H], a parallel theory without volume forms, and without ignoring lower dimensional
sets, was also developed. On the one hand, a universal invariant was found, with values in a
Grothendieck ring formed out of $K(\text{Var}_F)$ and $K(\Gamma)$. (Theorem 1.1) On the other hand, two
homomorphisms were found, essentially into $K(\text{Var}_F)$; they were deduced from the universal
invariant and two “Euler characteristic” homomorphisms $K(\Gamma) \to \mathbb{Z}$, found earlier by [2] and
[3]. (Theorem 10.5) However, no universality property was shown for the latter. The two
Euler characteristics are known to be universal with respect to $GL_n(\mathbb{Q})$ transformations, but
it is $GL_n(\mathbb{Z})$ transformations that are relevant here; since it is these (along with translations
by values of rational points) that lift to the valued field. Theorem 3.13 fills this gap in the
rational coefficient case, by showing that even with respect to integral transformations alone,
$K^{df}(\Gamma) \cong \mathbb{Q}^2$.

In the appendix we define the Iwahori Hecke algebra of $SL_2$ over an algebraically closed
valued field. Iwahori Hecke algebras are usually defined for (quasi-)split algebraic groups over
non archimedian local fields as convolution algebras with respect to the Haar measure. Here,
instead, we use motivic integration. We give an analogue of the Bernstein presentation for the
algebra and find its center. In [5], a construction of the Iwahori Hecke algebra of $SL_2$ over a
two dimensional local field is given. We think this construction is unrelated to ours.

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2. The Grothendieck ring of bounded polytopes over an ordered Abelian group

2.1. The dimension-free part of a graded ring. While we are ultimately interested in
$\mathbb{Q}$-algebras, in the interest of simpler proofs we will also use semirings for the basic lemmas. Elements of the Grothendieck semiring are represented by definable sets, and equality corresponds
to definable bijections. For the corresponding ring representing an element $[X] - [Y]$ requires
two definable sets, and equality $[X] - [Y] = [X'] - [Y']$ invokes a third definable set $Z$ and
an isomorphism $X \cup Y' \cup Z \to X' \cup Y' \cup Z$. Thus a canonical isomorphism between semirings,
when available, is not only stronger but easier to prove than the isomorphism of rings it implies.

All semigroups in this paper will be commutative, with addition denoted by $+$, and a distinc-
ted element 0 (perhaps the term commutative monoid is more standard.)

Given a graded semiring $R = \oplus_{n \geq 0} R_n$, and an element $a_1 \in R_1$, $R[a_1^{-1}]$ is naturally $\mathbb{Z}$-
graded; let $R^{df}_1 = R[a_1^{-1}]_0$ be the zero'th homogeneous component. When $a_1$ is fixed we will
just write $R^{df}$. We think of the elements of $R^{df}$ as ratios or pure numbers, whereas the elements
of $R$ may have “units”.

As a semigroup, $R^{df}$ can also be described as the direct limit of the semigroups $R_d$ under
the maps $R_d \to R_{d+1}$ given by $x \mapsto a_1 x$. In some cases that will be encountered, e.g. when $R_d$
is the Grothendieck group of varieties of dimension $\leq d$, $R^{df}$ can be thought of as a stabilized
version of the Grothendieck group of varieties (of all dimensions at once.)

Define a semiring homomorphism $f : R \to R[a^{-1}]_0$ by $f(r) = \frac{r}{a_1}$ for $r \in R_n$. $R[a_1^{-1}]_0$ has
the universal property for semiring homomorphisms $g : R \to S$ such that $g(a_1) = 1$.

The Laurent polynomial semiring $R^{df}[t,t^{-1}]$ is isomorphic, as a $\mathbb{Z}$- graded semiring, to the localization $R[a_1^{-1}]$ (with $t \mapsto a_1$.)

If $f_1 : A \to B_1$ is a semiring homomorphism, $B_1 \otimes_A B_2$ is defined to be the universal semiring
$B$ with maps $g_1 : B_1 \to B$ such that $g_1 f_1 = g_2 f_2$. If $A, B, B_1$ are the ring canonically obtained
from $A, B, B_1$ by introducing additive inverses, one verifies immediately that the natural map
$B \to (B_1 \otimes_A B_2)$ is an isomorphism.
Lemma 2.1. Let $\phi : R_1 \otimes R_2 \to R_3$ be a surjective homomorphism of graded semirings with $\phi(e_1 \otimes 1) = \phi(1 \otimes e_2) = e_3$, with kernel $\sim$. Let $S_1 = R_1[e^{-1}_1]_0$.

If $\sim$ is generated by the single relation $1 \otimes e_2 \sim e_1 \otimes 1$, then $\phi$ induces an isomorphism $S_1 \otimes S_2 \to S_3$.

More generally, if $\sim$ is generated by $1 \otimes e_2 = e_1 \otimes 1$ and $1 \otimes f_2 = f_1 \otimes 1$, with $e_1, f_1 \in R_1[1]$, then $\phi$ induces a surjective homomorphism $S_1 \otimes S_2 \to S_3$, with kernel generated by $\frac{t_2}{t_1} \otimes 1 \sim 1 \otimes \frac{t_1}{t_2}$.

Proof. We pass to the localizations, and obtain a homomorphism of Laurent polynomial semirings

\[ S_1[t_1, t_2^{-1}] \otimes S_2[t_2, t_2^{-1}] \to S_3[t_3, t_3^{-1}] \]

restricting to a homomorphism $S_1 \otimes S_2 \to S_3$, with kernel generated by $1 \otimes t_2 \sim t_1 \otimes 1$ and (in the second case) an additional relation, that may be written $t_2 \frac{t_2}{t_1} \otimes 1 \sim 1 \otimes \frac{t_1}{t_2}$. But $S_1[t_1, t_1^{-1}] \otimes S_2[t_2, t_2^{-1}] / (1 \otimes t_2 = t_1 \otimes 1) = (S_1 \otimes S_2) [t, t^{-1}]$. We thus have a surjective homomorphism $(S_1 \otimes S_2)[t, t^{-1}] \to S_3[t, t^{-1}]$ with kernel generated by $(\frac{t_2}{t_1} \otimes 1) t \sim (1 \otimes \frac{t_1}{t_2}) t$, or equivalently by $\frac{t_2}{t_1} \otimes 1 \sim 1 \otimes \frac{t_1}{t_2}$. Restricting to $S_1 \otimes S_2$ we find a homomorphism into $S_3$; it is easy to see that it must be surjective, with kernel generated by the same relation. \qed

Lemma 2.2. Let $R$ be a graded ring, $a_1 \in R_1$, $R_{a_1} = R[a_1^{-1}] = R/I$. Let $b \in R_1$, $I = Rb$, $R = R/I$, $a_1 = a_1/I \in \mathbb{R}$. Let $I^{df} = R^{df}_{a_1}$. Then $R^{df} / I^{df} \cong \mathbb{R}^{df}$.

Proof. The homomorphism $R \to R/I$ extends to a homomorphism $h : R[a_1^{-1}] \to \mathbb{R}[a_1^{-1}]$ of Z-graded rings. $h$ is surjective on every homogeneous component. In particular $h$ restricts to a surjective ring homomorphism $h_0 : R[a_1^{-1}]_0 \to \mathbb{R}[a_1^{-1}]_0$. Any element of $R[a_1^{-1}]_0$ can be written as $\frac{r}{a_1}$ for some $r \in R_n$. If $h_0(\frac{r}{a_1}) = 0$ then $h(r)a_1^n = 0$ for some $m \geq 0$. So $h(r)a_1^n = 0$, i.e. $ra_1^n = ba_1^m$ for some $s$. Since $r, a_1, b$ are homogeneous of repsective degrees $n, 1, 1$, we can take $s$ to be homogeneous of degree $n + m - 1$. But then in $R[a_1^{-1}]_0$ we have $\frac{a_1}{a_1} \rightarrow \frac{a_1}{a_1} \in I^{df}$. This shows that $\ker(h_0) = I^{df}$, proving the lemma. \qed

We also have:

Lemma 2.3. Let $R, S$ be graded semirings, $e \in R_1, e' \in S_1$, and let $f : R \to S$ be an injective homomorphism, $f(e) = e'$. If for any $r' \in S$, for some $n, r'(e')^n \in f(R)$, then $f$ induces an isomorphism $R_{e'}^{df} \to S_{e'}^{df}$.

Proof. Clear. \qed

2.2. Two categories of bounded definable subsets of $\Gamma^n$. Throughout the text, $A$ denotes an ordered Abelian group, seen as a base subset of a model of the theory $DOAG$ of divisible ordered Abelian groups.

Definition 2.4. (1) An object of $\Gamma_A[n]$ is a subset of $\Gamma^n$ defined by linear equalities and inequalities with $Z$-coefficients and parameters in $A$. When $A$ is fixed, we write $\Gamma[n] = \Gamma_A[n]$.

Given $X, Y \in \text{Ob } \Gamma_A[n]$, $f \in \text{Mor}_\Gamma(X, Y)$ iff $f$ is a bijection, and there exists a partition $X = \cup_{i=1}^n X_i, M_i \in \text{GL}_n(\mathbb{Z}), a_i \in A^n$, such that for $x \in X_i$,

\[ f(x) = M_i x + a_i \]

(2) $\Gamma_A^{\text{bdf}}[\bullet]$ is the full subcategory of $\Gamma[\bullet]$ consisting of bounded sets, i.e. an element of $\text{Ob } \Gamma_A^{\text{bdf}}[\bullet]$ is a definable subset of $[-\gamma, \gamma]^n$ for some $\gamma \in \Gamma$.

(3) $\text{Ob } \text{vol}_\Gamma A[n] = \text{Ob } \Gamma A[n]$ Given $X, Y \in \text{Ob } \text{vol}_\Gamma A[n]$, $f \in \text{Mor}_\text{vol}_\Gamma A[n] \Gamma A[n] \Gamma A[n]$, and for any $x = (x_1, \ldots, x_n) \in X$, $y = (y_1, \ldots, y_n) = f(x)$ then $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$. 

(4) $\text{vol} \Gamma_{A}^{\text{bdd}}[n] = \text{Ob} \Gamma_{A}^{\text{bdd}}[n]$ is the full subcategory of $\text{vol} \Gamma_{A}[[\gamma, \infty)^{n}]$ for some $\gamma \in \Gamma$. (Such objects will be called semi-bounded.)

(5) $\text{vol} \Gamma_{A}[\ast]$ is the direct sum of the categories $\text{vol} \Gamma_{A}[[\gamma, \infty)^{n}]$ for $n \geq 0$; similarly for the other categories.

$K_{+}[\Gamma_{A}^{\text{bdd}}][n]$ denotes the Grothendieck semigroup of $\Gamma_{A}^{\text{bdd}}[n]$. By definition, it is the free semigroup generated by the objects of $\Gamma_{A}^{\text{bdd}}[n]$, subject to the relations: $[X_{1}] + [X_{2}] = [Z]$ when there exists a partition $Z = Z_{1} \cup Z_{2}$ of $Z$, $Z_{i} \in \Gamma_{A}^{\text{bdd}}[n]$, with $X_{i}, Z_{i}$ isomorphic in $\Gamma_{A}^{\text{bdd}}[n]$. The zero object is defined to be the class of $\emptyset$. It is easy to see (using boundedness) that for $n \geq 1$, $K_{+}[\Gamma_{A}^{\text{bdd}}][n]$ has finite direct sums (represented by disjoint unions). Hence any element of $K_{+}[\Gamma_{A}^{\text{bdd}}][n]$ is represented by an object of $\Gamma_{A}^{\text{bdd}}[n]$.

The semigroup $K_{+}[\Gamma_{A}^{\text{bdd}}][0]$ is $\mathbb{N}$; in this case only 0 and 1 are represented by an object of $\Gamma_{A}^{\text{bdd}}[0]$.

$K_{+}[\Gamma_{A}^{\text{bdd}}]$ is the graded semiring $\oplus_{n \in \mathbb{N}} K_{+}[\Gamma_{A}^{\text{bdd}}][n]$. Here $K_{+}[\Gamma_{A}^{\text{bdd}}][0] = \mathbb{N}$. $K_{+}[\Gamma_{A}^{\text{bdd}}]$ is the corresponding ring. Similar notation is used for the measured categories.

Observe that a disjoint union of $\text{vol} \Gamma_{A}[[\gamma, \infty)^{n}]$ isomorphisms is again a $\text{vol} \Gamma_{A}[[\gamma, \infty)^{n}$ isomorphism, provided that it is a $\Gamma[[\gamma, \infty)^{n}$ isomorphism.

Here we will be interested in dimension-free quantities, i.e. ratios of elements of $\Gamma[[\gamma, \infty)^{n}$ for each $n$, taking their direct limit over $n$. We will normalize $K_{+}[\Gamma_{A}^{\text{bdd}}]$ using the element $[0]_{1}$. Let

$$K_{0} = K_{+}[\Gamma_{A}^{\text{bdd}}][[[0]_{1}]^{-1}]_{0}$$

Let $K_{Q}^{\text{df}}(\Gamma_{A}^{\text{bdd}})$ be the corresponding ring, and

$$K_{Q}(\Gamma_{A}^{\text{bdd}}) = \mathbb{Q} \otimes K_{Q}^{\text{df}}(\Gamma_{A}^{\text{bdd}})$$

**Remark 2.5.** We defined the dimension free ring using a dehomogenizing element $a = [0]_{1}$, but could define a variant $K_{+}^{\text{df}(b)}$ using $b = [X]_{1}$, for any nonempty definable $X \subseteq \Gamma$. The choice $a = [0]_{1}$ has the following following universality property: $K_{0}^{\text{df}}(\Gamma_{A}^{\text{bdd}})$ embeds into a localization $K_{+}^{\text{df}}(\Gamma_{A}^{\text{bdd}})[(\mathbb{X})^{-1}]$ of $K_{0}^{\text{df}}(\Gamma_{A}^{\text{bdd}})$, $\frac{a}{a} \mapsto \frac{a}{b}$. This requires a lemma: if $[Y \times \{0\}^{m}] = [Y \times \mathbb{X}^{m}]$ then $[Y \times \mathbb{X}^{m}] = [Y \times \mathbb{X}^{m}]$.

For $a \in \mathbb{Q} \otimes \mathbb{A}$, let $e(a) = [a]_{1}/[0]_{1}$. We have $[a]_{1}[0]_{1} = [(a, 0)]_{2} = [(a, a)]_{2} = [a]_{1}^{2}$, using the $GL_{2}(\mathbb{Z})$ map $(x, y) \mapsto (x + y, y)$. Hence $e(a)$ is idempotent. For $a \in \mathbb{A}$, we have $e(a) = 1$. We also have an element $\iota(a) = [a, a]_{1}/[0]_{1}$, in $K_{Q}^{\text{df}}(\Gamma_{A}^{\text{bdd}})$. (Here $[0]_{1}$ is the closed-open interval, for $a > 0$; if $a < 0$ we let $\iota(a) = -\iota(-a)$, and $\iota(0) = 0$.) We will sometimes write $[a, b]$ to denote the class $\iota(b) - \iota(a)$. If $\phi(x_{1}, \ldots, x_{n})$ is a formula, we will sometimes write $[\phi]$ for the class of $\{(x_{1}, \ldots, x_{n}) : \phi(x_{1}, \ldots, x_{n})\}$.

We define the dimension-free Grothendieck ring as in the unmeasured case:

$$K_{Q}^{\text{df}}(\text{vol} \Gamma_{A}^{\text{bdd}}) = (K_{+}[\text{vol} \Gamma_{A}^{\text{bdd}}][[[0]_{1}]^{-1}]_{0}$$

Let $K_{Q}^{\text{df}}(\text{vol} \Gamma_{A}^{\text{bdd}})$ be the corresponding ring, and

$$K_{Q}^{\text{df}}(\text{vol} \Gamma_{A}^{\text{bdd}}) = \mathbb{Q} \otimes K_{Q}^{\text{df}}(\text{vol} \Gamma_{A}^{\text{bdd}})$$

It turns out that the measured ring can be constructed from the unmeasured one; we thus begin by studying the latter.

**2.3. Definable functions.** Recall the semigroup of functions $F_{n}(\Gamma, K_{+}[\Gamma_{A}^{\text{bdd}}])$. An element of this semiring is represented by a definable set $F \subseteq \Gamma \times \mathbb{X}^{m}$, such that $F(x) = \{y : (a, y) \in F\}$ is bounded for any $x$. $F$ represents a function in the following sense: given any ordered Abelian group extension $A(t)$ of $A$, generated over $A$ by a single element $t$, we obtain an element $[F(t)]$ of $K_{+}[\Gamma_{A}^{\text{bdd}}]_{A(t)}$. 
Similarly we define $F_n(\Gamma, K_{+0}(\Gamma_A^{\text{bdd}}))$. An element is again represented by a definable set $F \subseteq \Gamma \times \Gamma^m$, such that $F(x) = \{ y : (a, y) \in F \}$ is bounded for any $x$. Two such sets $F, F'$ represent the same function if for any $A'$ extending $A$ and $b \in A'$, $[F(b)]_{[0,1]^m} = [F'(b)]_{[0,1]^m}$ as elements of $K'^{\text{dif}}(\Gamma_{A'}^{\text{bdd}})$, i.e. if $[F(b)]_{m+m'} = [F'(b)]_{m+m'}$ for some $m'$. Note that $e(t)$ represents the function $1$ in this formalism, since $[b]_1 = [0]_1$ in $K'^{\text{dif}}(\Gamma_{A'}^{\text{bdd}})$, using the translation $x \mapsto x - b$. Hence $F(t), e(t)F(t)$ represent the same function. Since all $A'$ are at issue, we may take $A' = A(b)$. Addition is defined pointwise on representatives. There is more than one option for multiplication; at present we will use pointwise multiplication, yielding a semiring. The ring of functions $F_n(\Gamma, K'^{\text{dif}}(\Gamma_{A(h)}^{\text{bdd}})) = \mathbb{Z} \otimes F_n(\Gamma, K_{+0}(\Gamma_A^{\text{bdd}}))$ is the ring of formal differences; an element $[F_1] - [F_2]$ is represented by a pair $(F_1, F_2)$, with the obvious rules for equivalence, sum and product.

If $F(x)$ represents an element of $F_n(\Gamma, K_{+0}(\Gamma_A^{\text{bdd}}))$, and $h : \Gamma \to \Gamma$ is any definable function, consider $[F \circ h] \in F_n(\Gamma, K_{+0}(\Gamma_A^{\text{bdd}}))$. If $[F] = [F']$ then $[e(h(x))] [F \circ h] = [e(h(x))] [F' \circ h]$. In particular, if $h(x) = nx + a$, with $a \in A$ and $n \in \mathbb{N}$, then $[F] = [F']$ implies $[F \circ h] = [F' \circ h]$. But if $h$ has non-integral coefficients, this need not be the case.

2.4. Integral notation. Let $f$ be a function represented by $F$. If $a < b \in \Gamma$, write $\int_a^b f(x)dx$ for the class of $\{(t, y) : a \leq t < b, (t, y) \in F \}$. Note that $\int_a^b f(x)dx = \int_a^b f(x)e(dx)$. One can think of the element “$dx$” as denoting the idempotent $e(x)$.

If $a > b$, we let $\int_a^b dt = -\int_b^a dt$.

If $\alpha \in \mathbb{Q}$ and $c \in \mathbb{Q} \otimes A$, we have a term $e(\alpha t - c) \in F_n(\Gamma, K(\Gamma_A^{\text{bdd}}))$, mapping $b$ to the idempotent $e(\alpha b - c)$ of $K(\Gamma_A^{\text{bdd}})$.

We also use the notation of indefinite integrals\footnote{The useful notational element $dx$, along with the conventions of indefinite integration, led us to adopt integral rather than summation notation.}. We write:

$$\int f(x)dx = g(x)$$

to mean: for any $a, b$, $\int_a^b f(x)dx = g(b) - g(a)$.

Thus only $g|_a = g(x) - g(y)$ is defined, not $g(x)$. Nevertheless addition and composition on the right with a function make sense: $(g \circ h)|_y = g|_{h(y)}$, $(g + g')|_y = g|_y + (g')|_y$. Moreover, if $f, f'$ represent the same element of $F_n(\Gamma, K(\Gamma_A^{\text{bdd}}))$, then for any $h: \Gamma \to \Gamma$, $(\int f(x)dx) \circ h = (\int f'(x)dx) \circ h$. In particular, $\int_a^b f(t)dt$ induces a well-defined functional $F_n(\Gamma, K'^{\text{dif}}(\Gamma_A^{\text{bdd}})) \to F_n(\Gamma, K'^{\text{dif}}(\Gamma_A^{\text{bdd}}))$.

2.5. Dimension filtration. The ring $K'^{\text{dif}}(\Gamma_A^{\text{bdd}})$, unlike $K(\Gamma_A^{\text{bdd}})$, no longer keeps track of ambient dimension; but we still have a filtration based on intrinsic dimension:

$$F_n(K'^{\text{dif}}(\Gamma_A^{\text{bdd}})) = \{ \alpha[X]/[0,1]^{-m} - \beta[Y]/[0,1]^{-m} : \alpha, \beta \in \mathbb{Q}, X, Y \in \Gamma_A^{\text{bdd}}[m], \dim(X), \dim(Y) \leq n \}$$

Let $Gr_nK'^{\text{dif}}(\Gamma_A^{\text{bdd}}) = F_n(K'^{\text{dif}}(\Gamma_A^{\text{bdd}}))/F_{n-1}(K'^{\text{dif}}(\Gamma_A^{\text{bdd}}))$.

The graded version is not needed at the level of results; but it will simplify the proofs inasmuch as without it the integration by parts formulas become more complicated.

Lemma 2.6. Let $a < b$ be definable points. There exists a unique linear map

$$\text{gr} \int_a^b dt : F_n(\Gamma, Gr_{n-1}K'^{\text{dif}}(\Gamma_A^{\text{bdd}})) \to Gr_nK'^{\text{dif}}(\Gamma_A^{\text{bdd}}))$$
such that for any bounded, definable \( X \subseteq \Gamma \times \Gamma^n \), if \( \dim X_1 \leq n - 1 \), and \( f(t) \) is the class of \( X_1 \) in \( \text{Gr}_n \Gamma \mathcal{K}_Q \mathcal{K}_Q(\Gamma^{\text{bd}}_{A(t)}) \), then

\[
\text{gr} \int_{t=a}^{b} f(t) dt = [X]
\]

where \([X]\) is the class of \( X \) in \( \text{Gr}_n \Gamma \mathcal{K}_Q \mathcal{K}_Q(\Gamma^{\text{bd}}_{A(t)}) \).

2.6. Integration by parts. Let \( C \) be one of the categories: \( \Gamma_A, \Gamma_A^{\text{bd}}, \text{vol}_A^{\text{bd}} \). Let \( K \) be the Grothendieck ring of \( C \).

The category and the ring \( K \) are then \( \mathbb{N} \)-graded, with a canonical homogeneous element \([0]_1\) of grade 1, and we can form the dimension free ring \( K^{df} \). We also have canonical maps \( K[n] \rightarrow K[n+1] \), multiplication by \([0]_1\). Integrals over \( \Gamma \) of objects in \( \Gamma[n] \) do not in general exist in \( \Gamma[n] \), but if the objects come from \( \Gamma[n-1] \) they do: thus integration over \( \Gamma \) (or a definable interval in \( \Gamma \)) gives an operator \( \Gamma[n-1] \rightarrow \Gamma[n] \). The integral notation extends formally to \( K^{df} \).

For \( 1 \leq i \leq n \), let \( f_i \in \text{Fn}(\Gamma, K^{df}) \), \( F_i(x) = \int_0^{l_i(x)} f_i(t) dt \), where \( l_i \) is a monotone increasing definable function \( \Gamma \rightarrow \Gamma \). Also let \( F_i(x) = F_i(x) + f_i(l_i(x)) \).

**Lemma 2.7.** Let \( b \in \mathbb{Q} \otimes A \). We have equality of classes in \( K \):

\[
\prod_i F_i(b) = \sum_{i=1}^{n} \int_0^{l_i(b)} f_i(t) \prod_{j<i} F_j(l_i^{-1}(t)) \prod_{j>i} F_j(l_i^{-1}(t)) dt
\]

**Proof.** It suffices to prove the same statement for \( f_i \in \text{Fn}(\Gamma, K_+) \), since it is linear in each \( f_i \) and hence formally extends to \( K \), and thence to \( K^{df} \) by division.

For \( t = (t_1, \ldots, t_n) \in \Gamma^n \), let \( i(t) \) be an index \( i \in \{1, \ldots, n\} \) with \( l_i^{-1}(t_i) \) having the maximal value. In case there are several such indices, let \( i(t) \) be the smallest possible one. \( \prod_i F_i \) is the class of

\[
\sum\{f(t_1) \cdots f(t_n) : t \in X\}
\]

where \( X = \{(t_1, \ldots, t_n) : 0 \leq t_i < l_i(b)\} \). Let \( X_i = \{(t_1, \ldots, t_n) : i(t) = i\} \). Then \( X \) is the disjoint union of the \( X_i \), and

\[
X_i = \{t : 0 \leq t_i < l_i(b), l_j^{-1}(t_j) < l_i^{-1}(t_i)(j < i), l_j^{-1}(t_j) \leq l_i^{-1}(t_i)(j \leq i)\}
\]

The formula follows.

Now assume in addition that \( g \in \text{Fn}(\Gamma, K^{\text{bd}}_A) \), \( G(x) = \int_0^x g(x) \).

**Corollary 2.8.**

\[
\int_0^b g(t) \prod_i F_i(t) dt = G(b) \prod_i F_i(b) - \sum_{j=1}^{n} \int_0^{l_j(b)} G(l_j^{-1}(t)) f_j \prod_{1 \leq k < j} F_k(l_j^{-1}) \prod_{j < k \leq n} F_k(l_j^{-1}) dt
\]

\[
\int_0^b g(t) \prod_i F_i(t) dt = G(b) \prod_i F_i(b) - \sum_{j=1}^{n} \int_0^{l_j(b)} G(l_j^{-1}(t)) f_j \prod_{1 \leq k < j} F_k(l_j^{-1}) \prod_{j < k \leq n} F_k(l_j^{-1}) dt
\]

**Proof.** Obtained by subtraction from Lemma 2.7 in the case of \( n + 1 \) functions, with \( G = F_0 \) and \( l_0(x) = x \) for the first equation, \( G = F_{n+1}, l_{n+1} = x \) for the second.

We will often look at highest homogenous terms. The degree will be clear from the context, so we will write \( = \) for equality in the graded ring. In the graded ring there is no distinction between \( F_i, F_j \) and the formula simplifies to:
(1) \[ \int_0^b g \cdot \prod_i F_i(t) dt = \frac{G(b)}{g_r} \cdot \prod_i F_i(0) - \sum_{j=1}^n \int_0^{l_j(b)} G(l_j^{-1}(t)) f_j \prod_{1 \leq k \neq j} F_k(l_j^{-1}(t)) dt \]

The variable limits of integration are needed because of the expression below for \( \iota(\alpha x + c) \); it cannot be written as an integral with limits 0, \( x \) of a function.

**Claim 2.9.** Let \( \alpha = \frac{q}{p} \in \mathbb{Q} \) be a reduced fraction. Then

\[ [0, \alpha x + c] = \int_0^{\frac{q x + pc}{p}} e(\frac{r}{p}) dx \]

Now in (1) we take, for \( i \geq 1 \):

\[ f_i(x) = e(\frac{r}{p_i})(1 \leq p_i \in \mathbb{N}) \]

\[ l_i(x) = qi x + pi c_i \quad (c_i \in \mathbb{Q} \otimes A, 1 \leq q_i \in \mathbb{N}). \]

\[ \alpha_i = qi/p_i \]

By Lemma 2.9 we have \( F_i(x) = \int_0^{l_i(x)} f_i(x) = \iota(\alpha_i x + c_i) \). Hence (1) gives:

\[ \int_0^b g(t) \prod_{j=1}^n \iota(\alpha_j t + c_j) dt = \frac{G(b)}{g_r} \cdot \prod_{j=1}^n \iota(\alpha_j b + c_j) - \sum_{j=1}^n H_j \]

where \( H_j = \int_0^{l_j(b)} G(l_j^{-1}(t)) e(t/p_j) \prod_{1 \leq k \neq j} F_k(l_j^{-1}(t)) dt \). Now the change of variable \( s = t/p_j \) gives:

\[ H_j = \int_0^{\alpha_j b + c_j} G(\alpha_j^{-1}(s - c_j)) \prod_{1 \leq k \neq j} \iota(\alpha_k(\alpha_j^{-1}(s - c_j)) + c_k) ds \]

From this we retain:

**Lemma 2.10.**

\[ \int_0^b g(t) \prod_{j=1}^n \iota(\alpha_j t + c_j) dt = \frac{G(b)}{g_r} \prod_{j=1}^n \iota(\alpha_j b + c_j) - \sum_{j=1}^n \int_0^{b_j} G(\frac{s - c_j}{\alpha_j}) \prod_{1 \leq k \neq j} \iota(\frac{\alpha_k}{\alpha_j} s - c_{jk}) ds \]

where \( b_j = \alpha_j b + c_j, c_{jk} = c_k - \alpha_j^{-1} \alpha_k c_j \). Note that \( c_{jk} = c_k \) if \( c_j = 0 \).

**Corollary 2.11.** \( \int_0^b \iota(t)^n dt = \frac{\iota(b)^{n+1}}{n+1} \)

**Proof.** Let \( g(t) = 1, \alpha_j = 1, c_j = 0 \). Then \( G(b) = \iota(b) \), and Lemma 2.10 gives:

\[ \int_0^b \iota(t)^n dt = \frac{\iota(b) \iota(t)^n dt}{g_r} = \frac{n \iota(s) \iota(s)^{n-1} ds}{g_r} = \frac{\iota(b)^{n+1} - n \iota(s)^n}{g_r} \]

Changing sides, we obtain \((n + 1) \int_0^b \iota(t)^n dt = \iota(b)^{n+1} \), whence the corollary. \( \square \)

We will need a more precise version later. In any \( \mathbb{Q} \)-algebra, one can define \( c_n(x) := \binom{x}{n} = \frac{x(x-1) \ldots (x-n+1)}{n!} \). Note:

(2) \[ c_{n-1}(t)(t - (n - 1)) = nc_t(t) \]

Let \( C_n(x) = c_n(\iota(x)) \). Thus \( C_0(x) = 1, C_1(x) = \iota(x) \).

**Lemma 2.12.** For \( b \in \mathbb{Q} \otimes A \), \( \int_0^b C_n(t) dt = C_{n+1}(b) \).
Proposition 9.2, this in turn holds iff for all subgroups $T$ (3) iff for any $c$ ∈ $T$ (4) $e$ induction hypothesis and (2) for $t$ (1 + $C$) = 0 this is clear; we proceed by induction. By Lemma 2.8 with $g(x) = 1$, $G(x) = x + 1$, $l_0(x) = l_1(x) = x$, $f_1 = C_{n-1}$, $F_1 = C_n$, we have:

$$\int_0^b C_n(t)dt = \int_0^b 1 \cdot C_n(t)dt = \int_0^b (1 + t)C_{n-1}(t)dt$$

Now $(1 + t)C_{n-1}(t) = (t - (n - 1))C_{n-1}(t) + nC_{n-1}(t) = nC_n(t) + nC_{n-1}(t)$. Thus using the induction hypothesis and (2) for $n + 1$,

$$(n + 1) \int_0^b C_n(t)dt = \int_0^b (n + 1)C_n(b) - nC_n(b) = (n + 1)C_{n+1}(b)$$

2.7. Zero-dimensional functions. Consider elements of $Fn(\Gamma, K^A(\Gamma_{A_{\text{bdd}}}))$ of the form $e(\alpha x + \beta a)$, with $\alpha, \beta \in \mathbb{Q}$, $a \in A$. By definition, two such terms $e_1(c), e_2(c)$ are equal iff for all $M \models DOAG_A$ and $c \in M$, the idempotents $e_1(c), e_2(c)$ are equal elements of $K^A(\Gamma_{A_{\text{bdd}}})$. According to Proposition 9.2, this in turn holds iff for all subgroups $T$ of $\mathbb{Q} \otimes A(c)$ containing $A(c), e_1(c) \in T$ iff $e_2(c) \in T$; in other words, iff $A(c, e_1(c)) = A(c, e_2(c))$. More generally,

$$\prod_{i=1}^n e(\alpha_i x + \beta_i a_i) = \prod_{i=1}^{n'} e(\alpha_i' x + \beta_i' a_i) \in Fn(\Gamma, K^A(\Gamma_{A_{\text{bdd}}}))$$

iff for any $c \in M \models DOAG_A$,

$$A(c, \alpha c + \beta a_1, \ldots, \alpha c + \beta a_l) = A(c, \alpha' c + \beta' a_1', \ldots, \alpha' c + \beta' a_{l'})$$

As an application, note the equalities, for $m, m'$ relatively prime integers, $k \in \mathbb{Z}, b \in A$:

$$e\left(\frac{kb}{m}\right) e\left(\frac{kb}{mm'}\right) = e\left(\frac{kb}{mm'}\right)$$

$$e\left(\frac{kb}{m}\right) = e\left(\frac{kb}{m'}\right)$$

The term “piecewise” will refer to partitions of $\Gamma$ into definable points and open intervals, including all of $\Gamma$ or half-infinite intervals. By a constant term we mean a piecewise constant function, whose values on each piece are of the form $e\left(\frac{kb}{m}\right)$ with $m \in \mathbb{N}, b \in A$. By a standard divisibility term we mean a term $e\left(\frac{kb}{mm'}\right)e(b)$, with $m \in \mathbb{N}, b \in \mathbb{Q} \otimes A$. The integer $m$ is referred to as the denominator.

Lemma 2.13. Any term $e(\alpha x + \beta b) \in Fn(\Gamma, K^A(\Gamma_{A_{\text{bdd}}}))$ is equivalent to a product of a a constant term with a standard divisibility term. The denominator of the latter is equal to the denominator of $\alpha$ as a reduced fraction.

Proof. The term can be written as $e(mx + nb)/p$, with $b \in A, m, n, p \in \mathbb{Z}, p \neq 0$. Write $m = m_1m_2, p = m_1m_3$, with $m_2, m_3$ relatively prime. As in (1), we have:

$$e\left(\frac{mx + nb}{m_1m_3}\right) = e\left(\frac{nb}{m_1}\right)e\left(\frac{m_1x + nb}{m_3}\right)$$

Now since $m_2, m_3$ are relatively prime, there exists $m' \geq 1$ with $m_2m' = 1 \mod m_3$. In particular, $m', m_3$ are relatively prime. As in (1),

$$e\left(\frac{nb}{m_1}\right)e\left(\frac{m_1x + nb}{m_3}\right) = e\left(\frac{m_1x + nb}{m_3}\right) = e(m'nb/m_3)e\left(\frac{x + m'nb}{m_3}\right)$$
This is the product of the constant term $e(m'nb/m'm_1)$ with the standard term $e(m'nb/m_1)e\left(\frac{x+m'nb/m_1}{m_3}\right)$. Moreover, $\alpha = m/(m_1m_3) = m_2/m_3$ has denominator $m_3$. □

**Lemma 2.14.** Any finite product of terms $e(\alpha x + \beta y) \in \text{Fn}(\Gamma, K^{df}(\Gamma^{\text{bdd}}_A))$ equals a product of one standard divisibility term and a number of constant terms.

**Proof.** Using Lemma 2.13 and (4) (with $k = 1$), it suffices to consider products of terms $e\left(\frac{x+b}{m}\right)e(b)$ with $m$ a prime power, $b \in \mathbb{Q} \cap A$.

If $m|m'$, we have, using Criterion (6):

\[
e(b)e(b')e\left(\frac{x+b}{m}\right)e\left(\frac{x+b'}{m'}\right) = e(b)e(b')e\left(\frac{x+b'}{m'}\right)e\left(\frac{b-b'}{m}\right)
\]

Thus for each prime $p$, it suffices to consider one term $e\left(\frac{x+b}{p}\right)e(b)$, i.e. the highest occurring power can be used to reduce the others to constant terms. So we need only consider products of terms $e\left(\frac{x+b}{m_i}\right)e(b_i)$ with the $m_i$ relatively prime.

Now if $m_1, \ldots, m_k$ are relatively prime, find integers $l_i$ with $l_j = \delta_{ij}$ (mod $m_i$) (Where $\delta_{ij}$ is the Kronecker delta.) Given $b_1, \ldots, b_k \in A$, let $b^* = \sum l_i b_i$; then

\[
\prod_{i=1}^{k} e(b_i)e\left(\frac{x+b_i}{m_i}\right) = \prod_{i=1}^{k} e(b_i)e\left(\frac{x+b^*}{\prod_{i=1}^{k} m_i}\right)
\]

This finishes the proof. □

**Corollary 2.15.** Any element of $\text{Fn}(\Gamma, F_0K^{df}(\Gamma^{\text{bdd}}_A))$ is equivalent to a $\mathbb{Q}$-linear combination of products of the form of Lemma 2.14.

**Proof.** $F_0K^{df}(\Gamma^{\text{bdd}}_A)$ is generated by the classes of definable points $p = (p_1, \ldots, p_n)$. Each $p_i$ has the form $c_i/m_i$ with $c_i \in A$, and the class $[\{p\}] = e(p_1) \cdot \ldots \cdot e(p_n)$. Thus any $f \in \text{Fn}(\Gamma, F_0K^{df}(\Gamma^{\text{bdd}}_A))$ is piecewise of the form of Lemma 2.14 i.e. there exists a partition $I_1 \cup \ldots \cup I_k$ of $\Gamma$ such that $f|I_j = e_j$, with $e_j$ a $\mathbb{Q}$-linear combination of a finite product of terms $e(\alpha x + \beta y)$. Now the characteristic functions of the $I_k$ are also constant terms, and using them it is clear that $f$ itself is of the stated form. □

Zero-dimensional terms inside integrals can now be eliminated as follows.

**Lemma 2.16.** $e(b) \int e\left(\frac{x+b}{m}\right)h(x)dx = e(b)\left(\int h(mx - b)dx\right) \circ \left(\frac{x+b}{m}\right)$

**Proof.** It suffices to consider standard divisibility terms $e\left(\frac{x+b}{m}\right)$, with $b \in A, m \in \mathbb{N}$. The substitution $y = (x+b)/m$ leads to:

\[
e(b)\int_{x=a}^{u} e\left(\frac{x+b}{m}\right)h(x)dx = e(b)\int_{y=a+b/m}^{u+b/m} h(my - b)dy
\]

Note that the analogous formula with rational $m$ would not be valid; in effect we used the fact that $e(x)e(b)e\left(\frac{x+b}{m}\right) = e(b)e\left(\frac{x+b}{m}\right))$.

We note in passing a more direct approach to the computation of the length of a segment on lines through the origin; but this method, that ignores the arithmetic of the inhomogeneous part, does not work for other segments.
Lemma 2.17. Let $p, q$ be relatively prime integers. Then there exists $M \in GL_2(\mathbb{Z})$ with $M \cdot \begin{pmatrix} p/q \\ 1 \end{pmatrix} = \begin{pmatrix} 1/q \\ 1 \end{pmatrix}$.

Proof. $GL_2(\mathbb{Z})$ acts transitively on primitive integer vectors, since they may be completed to a lattice basis. Hence some $M \in GL_2(\mathbb{Z})$ takes $(p/q)^t$ to $(1, q)^t$. Thus $M$ takes a planar line of slope $p/q$ to one of slope $1/q$. For lines through the origin, the length is now just the length of a projection. □

2.8. One-dimensional functions.

Lemma 2.18. $F_n(\Gamma, F_0K^d_{\mathbb{Q}}(\Gamma^{bdd}))$ is generated as a $F_n(\Gamma, F_0K^d_{\mathbb{Q}}(\Gamma^{bdd}))$-module by the terms $\iota(\alpha x + b)$, $\alpha \in \mathbb{Q}$, $c \in \mathbb{Q} \otimes A$.

Proof. A bounded, definable, one-dimensional subset of $\Gamma^n$ is a finite union of points and bounded segments on lines in $\Gamma^n$, i.e. additive translates of 1-dimensional definable subspaces $(\alpha_1, \ldots, \alpha_n)\Gamma$, with $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Q}^n$.

We can take $\alpha$ to be a primitive element of $\mathbb{Z}^n$. All such elements are $GL_n(\mathbb{Z})$-conjugate, so in fact we can take $\alpha = (1, 0, \ldots, 0)$. In this case the translate has the form $\Gamma \times \{p\}$, with $p = (p_2, \ldots, p_n)$ a definable point of $\Gamma^{n-1}$. So the segment has the form $(a, b) \times \{p\}$, with $a, b \in \mathbb{Q} \otimes A$. Hence the class of the segment is $[(a, b) \times \{p\}] = \iota(b) - \iota(a) - e(a)e(p_2) \cdots e(p_n)$. So $F_nK^d_{\mathbb{Q}}(\Gamma^{bdd})$ is generated as an $F_0K^d_{\mathbb{Q}}(\Gamma^{bdd})$-module by the elements $\iota(b)$, $b \in \mathbb{Q} \otimes A$. The lemma follows. □

For later use, if $\alpha = p/d$ with $p, d \in \mathbb{N}$, and $b \in A$, we will say that $\iota(\alpha x + b)$ admits internal denominator $d$. A product of terms, each admitting internal denominator $d$, will also be said to admit this denominator. Note that in general $\iota((1/d)a) \neq (1/d)\iota(a)$ (even modulo $F_0$.)

2.9. Integration of higher dimensional functions. Recall the dimension filtration $(F_n)_A = F_n(K^d_{\mathbb{Q}}(\Gamma_A^{bdd}))$. Let $F_n(K^d_{\mathbb{Q}}(\Gamma_A^{bdd}))$ be the $\mathbb{Q}$-subspace of $F_n(K^d_{\mathbb{Q}}(\Gamma_A^{bdd}))$ generated by products of elements of $F_0(K^d_{\mathbb{Q}}(\Gamma_A^{bdd}))$ with $\leq n$ elements of $F_1(K^d_{\mathbb{Q}}(\Gamma_A^{bdd}))$. We seek to show (cf. Proposition [3]) that $F_n = F_n'$, i.e. $F_n(K^d_{\mathbb{Q}}(\Gamma_A^{bdd})) = F_n'(K^d_{\mathbb{Q}}(\Gamma_A^{bdd}))$ for all $A$ and $n$.

Let $F_n = F_n(\Gamma, F_n'(K^d_{\mathbb{Q}}(\Gamma_A^{bdd})))$. We will also use an arithmetic refinement: let $F_{n,d}$ be the $F_0$-submodule of $F_n$ generated by $F_{n-1}$ along with $n$-fold products of basic one-dimensional terms with internal denominator dividing $d$, i.e. terms $\iota(t x + b)$, $p \in \mathbb{N}$, $b \in \mathbb{Q} \otimes A$.

Lemma 2.19. Let $d, d', p_i \in \mathbb{N}$, $c_i \in \mathbb{Q} \otimes A$, $\alpha_i = p_i/d$, $\gamma = d/d'$, $f(t) = \prod_{i=1}^{n} \iota(\alpha_i t + c_i)$

Then $\int_0^{\gamma x + c} f(t) dt \in F_{n+1,d'}$

Proof. We use induction on $d$. Since $\iota((\alpha_1 + 1)t + c_i) = \iota(t) + \iota(\alpha_1 t + c_i)$ as functions of $t$ in $F_n(\Gamma, K^d_{\mathbb{Q}}(\Gamma_A^{bdd}))$, and using additivity of the integral, we may assume $p_i \leq d$. Similarly, $\iota(t + c_i) = \iota([t, t + c_i] + [0, t])/[0, t] = \iota(\alpha_i) + \iota(t)$; so we may assume that if $\alpha_1 = 1$ then $c_1 = 0$.

In case $d = 1$, we have $p_i = \alpha_i = 1$, so $c_1 = 0$ and $\iota(\alpha_1 t + c_i) = \iota(t)$. By Lemma 2.11, $\int_0^{\gamma x + c} \iota(t)^n = \frac{1}{g^r} \iota(c - c)^{n+1}$. Clearly this expression lies in $F_{n+1,d'}$.

In general, let $J_1 = \{j \leq n : \alpha_j = 1\}$, $J_2 = J \setminus J_1$. For $j \in J_1$ we have $c_j = 0$. 

\[ \int_0^{\gamma x + c} f(t) dt \]
Using Lemma \ref{lem:2.10} with \( g = 1 \), we have:

\[
\int_0^{\gamma x+c} \prod_{j=1}^n \iota(\alpha_j t + c_j)dt = \iota(\gamma x + c) \cdot \prod_{i} \iota(\alpha_i(\gamma x + c) + c_i) - \sum_{j=1}^n h_j(\alpha_j(\gamma x + c) + c_j)
\]

where

\[
h_j(y) = \int_0^y \iota\left(\frac{s-c_j}{\alpha_j}\right) \prod_{1 \leq k \neq j} \iota\left(\frac{\alpha_k s - c_{jk}}{\alpha_j}\right) ds = \int_0^y \iota\left(\frac{d}{p_j}(s - c_j)\right) \prod_{1 \leq k \neq j} \iota\left(\frac{p_k s - c_{jk}}{p_j}\right) ds
\]

Now if \( \alpha_j = 1 \) and \( c_j = 0 \), then \( c_{jk} = c_k \). Thus (using also \( p_j = d \)) each of the terms \( h_j(\alpha_j(\gamma x + c) + c_j) \) is identical with \( \int_0^{(\gamma x+c)} \prod_{j=1}^n \iota(\alpha_j t + c_j)dt \). Moving these terms to the left we have, with \( \nu = |J_1| + 1 \), \( c_j' = c_j/\alpha_j \):

\[
\nu \int_0^{\gamma x+c} \prod_{j=1}^n \iota(\alpha_j t + c_j)dt = \iota(\gamma x + c) \cdot \prod_{i} \iota(\alpha_i(\gamma x + c) + c_i) - \sum_{j \in J_2} h_j(\alpha_j(\gamma x + c) + c_j)
\]

For \( j \in J_2 \) we have \( p_j < d \), so the induction hypothesis applies. Since \( \alpha_j, \gamma = \frac{m}{\nu} \), we have \( h_j(\alpha_j(\gamma x + \alpha_j c)) \in \mathcal{F}_{n+1,d'} \). The remaining terms \( \iota(\gamma x+c), \iota(\alpha_j x + \alpha_j c + c_i) \) clearly have internal denominator \( d' \). This concludes the proof of the lemma. \( \square \)

**Lemma 2.20.** Assume \( F_n = F_n' \) for all ordered Abelian groups \( A \). Let \( f \in \mathcal{F}_n \). Then \( \int_0^b f(t)dt \in \mathcal{F}_{n+1} \).

**Proof.** It follows from the hypothesis, applied to the structure generated by an element \( b \), that \( f(b) \in F_n' \); it follows by compactness that \( f \) itself is a product of 0- and 1-dimensional generators. By Lemma \ref{lem:2.14} any product of 0-dimensional generators equals a product of one standard divisibility term \( e\left(\frac{x+b'}{m}\right)e(b') \) and constant terms. The constants commute with integration and may be ignored. So we may assume \( f = e\left(\frac{x+b'}{m}\right)e(b')g_1 \cdot \cdots \cdot g_n \) with \( g_i \) a basic one-dimensional term. Now with the change of variable \( s = \frac{b+b'}{m} \) we have \( e\left(\frac{b+b'}{m}\right)dt = e(s)ds = ds \), i.e.

\[
\int_0^b f(t)dt = \int_0^{b+b'} g_1(ms-b') \cdot \cdots \cdot g_n(ms-b')ds
\]

Since \( g_i(ms-b) \) is again a basic one-dimensional term, we may assume:

\[
f(t) = \prod_{i=1}^n \iota(\alpha_i t + c_i)
\]

in order to show: \( \int_0^x f(t) \in \mathcal{F}_{n+1} \). This follows from Lemma \ref{lem:2.19}. \( \square \)

**Proposition 2.21.** \( K^d_{\mathbb{Q}}(\Gamma_{A(n)}) \) is generated as a \( \mathbb{Q} \)-algebra by the elements \( e(a), \iota(a), a \in \mathbb{Q} \otimes A \).

**Proof.** We have seen that \( F_0', F_1' \) are contained in the algebra generated by these terms. Hence it suffices to show that \( F_n = F_n' \) for each \( n \). For \( n = 0, 1 \) this is true by definition; we proceed by induction. Assume \( F_n = F_n' \), and let \( X \subseteq \Gamma^n \) be definable and bounded, of dimension \( \leq n + 1 \). After a finite definable partition we may assume the first projection has fibers of dimension \( \leq n \). By induction, for any \( t, [X_t] \in F_n(K^d_{\mathbb{Q}}(\Gamma_{A(n)})) \). It follows that there exists a definable partition \( \Gamma = \cup_j I_j \) and \( f_j \in Fn(\Gamma, F_n) \) such that for \( t \in I_j, [X_t] = f_j(t) \). We may take \( I_j \) to
be an interval \((a_j, b_j)\) \((j \in J_0)\) or a singleton \(\{c_j\}\) \((j \in J_1)\), or \(f_j = 0\). Then \(X\) is the disjoint union of the pullbacks of the \(I_j\); so we may assume

\[
[X] = \sum_{j \in J_0} \int_{a_j}^{b_j} f_j + \sum_{j \in A} f_j(c_j)
\]

By Lemma 2.20,

\[
\text{Lemma 2.22.}
\]

\[
\text{Proof. We may assume}
\]

\[
\text{by Definition 2.4. Let } K_{+} \text{ be the subsemiring generated by the elements } e(a), \iota(a). \text{ Let } K_{Q}^d \text{ be the semiring obtained from } K_{+} \text{ by adding additive inverses to the elements of } K_{+}(\Gamma) \text{, and } K_{Q}^d(\Gamma_{A}) \text{ is also an isomorphism.}
\]

\[
\text{Now have natural homomorphisms}
\]

\[
K_{Q}^d(\Gamma_{A}) \to K_{Q}^d(\Gamma_{A})_{\nu} \to K_{Q}^d(\Gamma_{A})
\]

\[
\text{Lemma 2.22. The natural homomorphism } K_{Q}^d(\Gamma_{A})_{\nu} \to K_{Q}^d(\Gamma_{A}) \text{ is an isomorphism}
\]

\[
\text{Explicitly, for any } a \in K_{+}(\Gamma) \text{ there exists } m \in \mathbb{N} \text{ and } b, c \in K_{+}(\Gamma)' \text{ such that } ma + b = c \text{ in } K_{+}(\Gamma).
\]

\[
\text{Proof. All our integral equalities are valid in } K_{Q}^d(\Gamma_{A})_{\nu}. \text{ Hence the proof of Proposition 2.21 shows that } K_{Q}^d(\Gamma_{A})_{\nu} \to K_{Q}^d(\Gamma_{A}) \text{ is surjective. Since the same elements are inverted in these semirings, the homomorphism is also injective, hence bijective, and } K_{Q}^d(\Gamma_{A})_{\nu} \text{ is in fact a ring. Since } K_{Q}^d(\Gamma_{A}) \text{ is obtained from } K_{Q}^d(\Gamma_{A})_{\nu} \text{ by additively inverting elements, the homomorphism } K_{Q}^d(\Gamma_{A})_{\nu} \to K_{Q}^d(\Gamma_{A}) \text{ is also an isomorphism.}
\]

\[
\text{2.10. Subrings and quotients of } K_{Q}^d(\Gamma_{A}). \text{ Let } A \text{ be an ordered Abelian group, and let } T_{A} \text{ denote the symmetric algebra } \mathbb{Q} \oplus (\mathbb{Q} \otimes A) \oplus \text{Sym}^2(\mathbb{Q} \otimes A) \oplus \ldots. \text{ If } A = \mathbb{Z}^n, \text{ this is a polynomial ring in } n \text{ variables.}
\]

\[
\text{We have a homomorphism } \phi_{A} : T_{A} \to K_{Q}^d(\Gamma_{A}), a \mapsto \iota(a). \text{ The image contains the classes of points of } A \text{ (all equivalent to 1) and segments with endpoints in } A.
\]

\[
\text{Lemma 2.23. The natural homomorphism } \phi_{A} : T_{A} \to K_{Q}^d(\Gamma_{A}) \text{ is injective.}
\]

\[
\text{If } A \text{ is divisible, } \phi_{A} \text{ is an isomorphism.}
\]

\[
\text{Proof. We may assume } A \text{ is finitely generated. First consider the case } A \subseteq \mathbb{Q}. \text{ So } A \cong \mathbb{Z}, \text{ and we may take } A = \mathbb{Z}. \text{ The symmetric algebra } T_{A} \text{ can be identified with the polynomial ring } \mathbb{Q}[T]. \text{ Given a nonzero polynomial } f \in \mathbb{Q}[T], \text{ we must show that } f(\iota(1)) \neq 0 \in K_{Q}^d(\Gamma_{A}). \text{ Now for any } m, \text{ we have a homomorphism}
\]

\[
\text{count}_{m} : K_{+}(\Gamma_{A}) \to \mathbb{Q} : [X] \mapsto \#(X \cap ((1/m)\mathbb{Z})^n)
\]

\[
\text{counting points of a bounded definable set } X \subset \Gamma^n \text{ with coordinates in } (1/m)\mathbb{Z}. \text{ This is clearly } GL_{n}(\mathbb{Z}) \text{-invariant, and induces a ring homomorphism } count_{m} : K_{Q}^d(\Gamma_{A}) \to \mathbb{Q}. \text{ Composing with } f \mapsto f(\iota(1)) \text{ we have a homomorphism } c_{m} : \mathbb{Q}[T] \to \mathbb{Q}. \text{ Now } c_{m}(T) = \#(\iota([0,1) \cap (1/m)\mathbb{Z}) = m. \text{ So } c_{m}(f) = f(m). \text{ Since } \mathbb{Z} \text{ is Zariski dense in the affine line, } c_{m}(f) \neq 0 \text{ for some } m. \text{ It follows that } f(\iota(1)) \neq 0.
\]
For the general case we will use a statement of Van den Dries, Ealy, and Marikova. The proof is included in Proposition 9.10, with $\mathbb{R}$ in place of $\mathbb{Q}$, but this does not matter.

**Claim** Let $Q \in \mathbb{Q}[u_1, \ldots, u_n]$, $B \subset \Gamma^n$ a $DOAG$-definable set, and $Q$ vanishes on $B(\mathbb{Q})$, then $Q$ vanishes on $B$.

An element of $T_A$ can be written as $G(a)$, with $G \in \mathbb{Q}[X_1, \ldots, X_n]$ and $a = (a_1, \ldots, a_n) \in A$. Suppose $\phi_A(G(a)) = 0$. This is due to a finite number of $GL_k(\mathbb{Z})$-isomorphisms and $A$-translations between finite unions of products of the intervals $[0, a_i)$ and points, and possibly some auxiliary intervals and points with endpoints $a'_1, \ldots, a'_{n'}$, that cancel out. Hence there exists $DOAG$-definable set $B \subseteq \Gamma^{n+n'}$ such that $(a, a') \in B$, and for any ordered Abelian group $A'$, $\phi_A'(G(c)) = 0$ whenever $(c, c') \in B(A')$. Now suppose in addition that $G(a) \neq 0$. Then by the Claim, there exist $(c, c') \in \Gamma(\mathbb{Q})^{n+n'}$ with $G(c) = 0$. But $\phi_Q(G(c)) = 0$. This contradicts the case $A = \mathbb{Q}$ proved above.

If $A$ is divisible, the homomorphism $\phi_A$ is surjective. This follows from Proposition 2.21 all $\epsilon(a) = 1$, while $\iota(a) = \phi_A(a)$.

Denote $T_A = \phi_A(T_A)$. This is always a split subalgebra of $K^{df}_Q(\Gamma_A^{bdd})$, equal to it if $A$ is divisible. To clarify the full structure, we ask:

**Question 2.24.** For $n = 2$ we have $2\iota(a/2) = ([0, a/2] \cup (a/2, a]) = \iota(a) + 1 - \epsilon(a/2)$; so $\iota(a/2) = (1/2)(\iota(a) + 1 - \epsilon(a/2))$. Is this the first term of a sequence of polynomial relations?

The proof of Lemma 2.23 may give the impression that specializations of finitely generated subgroups of $\Gamma$ into $\mathbb{Z}$, followed by the maps $count_m$, resolve points on $K^{df}_Q(\Gamma_A^{bdd})$ and thus give decisive information. This is not the case, as the example below shows.

**Example 2.25.**

$$\int \int (e(\frac{s}{2}) - 1)(e(\frac{t}{2}) - 1)(e(\frac{s-t}{2}) - 1)dsdt$$

evaluates to 0 under any $count_m$, for any choice of $s, t \in \mathbb{Z}$, but is not identically 0.

Let $L_A$ be the field of fractions of $T_A$, where $A = \mathbb{Q} \otimes A$.

**Corollary 2.26.** There exists a natural homomorphism $\psi_A : K^{df}_Q(\Gamma_A^{bdd}) \to L_A$, injective on the image of $T_A$. The kernel is generated by the relations $\epsilon(a) = 1, \nu(\frac{a}{2}) = \iota(a)$.

**Proof.** If $A$ is divisible, the homomorphism $\phi_A$ of Proposition 2.23 has an inverse $\psi : K^{df}_Q(\Gamma_A^{bdd}) \simeq T_A$. It suffices to view $\psi$ as a homomorphism into the field of fractions $L_A$ of $T_A$.

In general, let $A = \mathbb{Q} \otimes A$. We have a natural surjective homomorphism $\nu : K^{df}_Q(\Gamma_A^{bdd}) \to K^{df}_Q(\Gamma_{\mathbb{A}}^{bdd}), [X] \mapsto [X]$. Composing with $\psi_{\mathbb{A}}$ we obtain a homomorphism $\psi_A : K^{df}_Q(\Gamma_A^{bdd}) \to L_A$ where $L_A = L_{\mathbb{A}}$. Since $\nu \phi_A = \phi_{\mathbb{A}}|T_A$, $\psi_A \phi_A = \psi_{\mathbb{A}} \phi_{\mathbb{A}}|T_A = Id_{T_A}$. This proves the injectivity on $T_A$.

The relations $\epsilon(a/n) = 1, \nu(\frac{a}{n}) = \iota(a)$ ($a \in A$) are already in the kernel of $\nu$; both are seen using the translation $x \mapsto x + a/n$. These relations suffice (using Proposition 2.21) to reduce any element of $K^{df}_Q(\Gamma_A^{bdd})$ to an element of the image of $T_A$. By the injectivity on $T_A$ no further relations intervene.

## 3. The measured Grothendieck ring

We turn to the dimension-free Grothendieck ring of the category $\text{vol}\Gamma_A[*]$ of Definition 2.4 (3-5). When possible we omit $A$ from the notation.

We begin by representing this Grothendieck ring as a ring of functions under convolution.
Recall the semigroup of definable functions $\Gamma \to K_+(\Gamma[n])$ of \cite{23} Define a convolution product
\[ F_n(\Gamma, K_+(\Gamma[n - 1])) \times F_n(\Gamma, K_+(\Gamma[m - 1])) \to F_n(\Gamma, K_+(\Gamma[n + m - 1])) \]

as follows: if $f$ is represented by a definable $F \subseteq \Gamma \times \Gamma^m$, in the sense that $f(\gamma) = [F(\gamma)]$, and $g$ by a definable $G \subseteq \Gamma \times \Gamma^n$, let
\[ f \ast g(\gamma) = \{(\alpha, b, c) : \alpha \in \Gamma, b \in F(\alpha), c \in G(\gamma - \alpha)\} \]

To distinguish this semiring from the semiring $F_n(\Gamma, K_+(\Gamma))$ with pointwise multiplication, we denote it $F_{n_0}(\Gamma, K_+(\Gamma))$. $F_{n_0}(\Gamma, K_+(\Gamma))$ is a vol\$\Gamma$-isomorphism, then clearly $h$ restricts to bijections $h_i : X_i \to Y_i$ which are in fact vol\$\Gamma\{t[n - 1]\}$-isomorphisms. Hence $\alpha(X)$ depends only on $[X]$, and a homomorphism $\alpha : K_+\text{vol}\Gamma[n] \to F_n(\Gamma, K_+(\Gamma[n - 1]))$ is induced.

Conversely given $F \subseteq \Gamma \times \Gamma^{n-1}$ representing an element of $F_n(\Gamma, K_+(\Gamma[n - 1]))$, let $\beta(F) = [F]$, the class in vol\$\Gamma[n]$ of the graph of $F$. If $F, F'$ represent the same element of $F_n(\Gamma, K_+(\Gamma[n - 1]))$, then for any $t, F_t, F'_t \in \Gamma A(t)[n - 1]$ - isomorphic. The isomorphism is given by a definable bijection $g_t : F_t \to F'_t$. By a standard compactness argument (cf. \cite{1} Lemma 2.3) we can take $g_t$ definable uniformly in $t$, and define $g(t, x_1, \ldots, x_{n-1}) = (t, g_t(x_1, \ldots, x_{n-1}))$; then $g : F \to F'$ is a definable bijection. Moreover for any $t$, there is a finite set of matrices $M_1(t), \ldots, M_k(t) \in GL_{n-1}(\mathbb{Z})$ and elements $c_i(t) \in A(t)$ such that for any $x \in \Gamma^{n-1}$, for some $i \leq k(t), g_t(x) = M_i(t)x + c_i(t)$. By compactness, $M_1(t), \ldots, M_k(t)$ can be chosen from a finite set $M_1, \ldots, M_k$ of matrices. So for any $t \in \Gamma$ and $x \in \Gamma^{n-1}$, for some $i \leq k$, $g_t(x) - M_i(t)x \in A(t)$. Now $A(t)$ is the group generated by $t$ over $A$, so any element of $A(t)$ has the form $a + mt$ for some $m \in \mathbb{Z}$. By compactness, there exist finite subset $A_0$ of $A$ and $Z_0$ of $\mathbb{Z}$ such that for any $t \in \Gamma$ and $x \in \Gamma^{n-1}$, for some $i \leq k$, some $a \in A_0$ and $m \in Z_0$, $g_t(x) = M_i(t)x + a + mt$.

Partition $X$ into finitely many pieces, such that $M_i, m, a$ are constant on each piece; then on each piece $g$ is given by $(t, x) \mapsto (t, Mx + a + mt)$ for some $a \in A^{n-1}$ and $m \in \mathbb{Z}^{n-1}$. This is clearly an affine $GL_n(\mathbb{Z})$-transformation. Thus $g$ is a vol\$\Gamma[n]-isomorphism. So $[F] = [F']$ in vol\$\Gamma[n]$, his allows us to define $\beta : F_n(\Gamma, K_+(\Gamma[n - 1])) \to K_+\text{vol}\Gamma[n]$. It is clear that $\alpha, \beta$ are inverse homomorphisms. So $\alpha$ is an isomorphism and shows (1).
Restricting $\alpha$ to bounded sets yields an isomorphism yields (2).

$$K_+ \text{vol} \Gamma^n \rightarrow \{ f \in F_n(\Gamma, K_+([n - 1]^\text{bddd})) : (\exists \gamma)(\forall \gamma < \gamma_0)(f(\gamma) = 0)\}$$

The direct sum of these isomorphisms over all $n \geq 1$ (2); in grade 0 we have $N$ on both sides. The verification that product goes to convolution product is straightforward. 

Let $q_0 \in F_n^\text{bdd}(\Gamma, K_+([0]^\text{bddd}))$ be the function with support at $\{0\}$ and value 1. Note that $f \in F_n^\text{bdd}(\Gamma, K_+([n]^\text{bddd}))$, $f * q_0$ is the element of $F_n^\text{bdd}(\Gamma, K_+([n]^\text{bddd}))$ satisfying $(f * q_0)(t) = f(t) \times [\{0\}]$. 

We can also define a convolution product on the semigroup $F_n^\text{bdd}(\Gamma, K_+^d(\Gamma^\text{bddd}))$. An element of this semigroup is represented by a pair $(f, n)$, where $f \in F_n^\text{bdd}(\Gamma, K_+([n]^\text{bddd}))$, and $(f, n)$ is identified with $(f * q_0^n, n + m)$. The pair $(f, n)$ is intended to represent the function $t \mapsto f(t)[0]_1^{-n}$. 

We get $(f, n) * (g, m) = (f * g, n + m + 1)$. This makes $F_n(\Gamma, K_+^d(\Gamma^\text{bddd}))$ into a semiring $F_n(\Gamma, K_+^d(\Gamma^\text{bddd}))$.

**Lemma 3.3.** $K_+^d(\text{vol} \Gamma^\text{bddd})$ is canonically isomorphic to $F_n^\text{bdd}(\Gamma, K_+^d(\Gamma^\text{bddd}))$

**Proof.** By Lemma 3.2 (3),

$$K_+^d(\text{vol} \Gamma^\text{bddd})[\star] \cong (\oplus_n F_n^\text{bdd}(\Gamma, K_+([n - 1]^\text{bddd})))[q_0^{-1}]_0$$

Let $(f, n)$ represent an element of $F_n^\text{bdd}(\Gamma, K_+^d(\Gamma^\text{bddd}))$. Let

$$[(f, n)] \mapsto f q_0^{-(n+1)}$$

This defines an injective semiring homomorphism

$$F_n^\text{bdd}(\Gamma, K_+^d(\Gamma^\text{bddd})) \rightarrow (\oplus_n F_n^\text{bdd}(\Gamma, K_+([n - 1]^\text{bddd})))[q_0^{-1}]_0$$

which is clearly also surjective. 

Given a definable function $h : \Gamma \rightarrow F_n(\Gamma, K_+(\Gamma))$, and a definable $Y \subseteq \Gamma$, we define $\int_Y h \in F_n(\Gamma, K_+(\Gamma))$ pointwise, i.e. $(\int_Y h)(\gamma) = \int_{t \in Y} ev_\gamma(h)(dt)$ where $ev_\gamma(h)(t) = h(t)(\gamma)$. This carries over to the groups and rings considered below.

Let $\mathcal{R}^\Gamma = Q \otimes F_n^\text{bdd}(\Gamma, K_+^d(\Gamma^\text{bddd})) = F_n^\text{bdd}(\Gamma, K_+^d(\Gamma^\text{bddd}))$ be $Q$-algebra of functions represented by elements whose support is bounded below. Then $\mathcal{R}^\Gamma$ also has a natural convolution structure, and forms a ring. We begin by developing some identities in $\mathcal{R}^\Gamma$. We denote convolution of functions $f, g$ by $fg$; we will not consider the pointwise product except when one of the functions is supported on \{0\}, in which case the two products are equal.

Let $\mathcal{R}_0^\Gamma$ be the subring of $\mathcal{R}^\Gamma$ consisting of elements with support \{0\} (and 0.) The map $a \mapsto aq(0)$ gives an homomorphism of rings $K_+^d(\Gamma^\text{bddd}) \rightarrow \mathcal{R}_0^\Gamma$. In fact, since equality of functions in $F_n(\Gamma, K_+^d(\Gamma^\text{bddd}))$ is defined pointwise, and implies equality of the value at 0, it is easy to see that this is an isomorphism.

(11)

$$K_+^d(\Gamma^\text{bddd}) \cong \mathcal{R}_0^\Gamma$$

Let $q(\gamma)$ denote the element supported on \{0\}, with $q(\gamma) = 1$.

Then $e(\gamma)q(\gamma + \gamma') = q(\gamma)q(\gamma')$. We have

(12) $f = \int_{t \in \Gamma} f(t)q(t)dt$
The elements of $K^d_f(\Gamma^{\text{bldd}})$ can be identified with constant functions with support $\{0\}$.

For $m \geq 1$, and $b \in \mathbb{Q} \otimes A$, let

$$
\theta_{m,b} = \int_{t \geq b} q(mt)dt \quad \text{and} \quad \theta_m = \theta_{m,0}, \quad \theta = \theta_1.
$$

Let $Q_m(b) = \int_0^b q(mt)dt$. So $Q_m(b) = \theta_m - \theta_{m,b}$.

The filtration on $K^d_f(\Gamma^{\text{bldd}})$ induces a filtration $F_\ast \mathcal{R}^\Gamma$ on $\mathcal{R}^\Gamma$. $F_0 \mathcal{R}^\Gamma$ consists of “purely exponential” sums; it has as a $\mathbb{Q}$-basis the elements $\theta_m$, $q(b)$, $Q_m(b)$. Let $F_n \mathcal{R}^\Gamma_0 = F_n \mathcal{R}^\Gamma \cap \mathcal{R}^\Gamma_0$.

Let $F_n \mathcal{R}^\Gamma$ be the $\mathbb{Q}$-space generated by products $q(b')a_1 \cdots a_n$, where $a_i \in F_1 \mathcal{R}^\Gamma_0$ or $a_i = \theta_{m,b}$ for some $m$ and some $b \in \mathbb{Q} \otimes A$.

As above we will write some of the identities in graded form.

Note that $e(b)\theta_{m,b} = e(b) \int_b^\infty q(mt)dt = e(b) \int_{mb}^\infty e(\frac{t}{m})q(s)ds$. Since $e(b)e(\frac{t+mb}{m}) = e(b)e(\frac{t}{m})$, we have: $e(b)\theta_{m,b} = e(b) \int_0^b \int_0^\infty e(\frac{s+mb}{m})q(s+mb)ds = e(b)q(mb) \int_0^\infty e(\frac{t}{m})q(s)ds = e(b)q(mb)\theta_m$.

Hence

\begin{equation}
(13)
e(b)Q_m(b) = e(b) \int_0^b q(mt)dt = e(b)(1 - q(mb))\theta_m
\end{equation}

Note that while $\int_0^\infty q(t)f(t)$ is defined, $\int_0^\infty f(t)$ is not. Thus integration by parts does not directly apply. To compute unbounded integrals (when $A(0) = 0$) we will use:

**Lemma 3.4.** Let $f(x) = \prod_{i=1}^n \nu(\alphaix + c_i)$. Let $m \in \mathbb{N}$ be such that $ma_i \in \mathbb{N}$, and let $a \in ma_i, a \neq 0$. Then $f(t - a) = f(t) - f_1(t)$ for some $f_1 \in F_{n-1}K^d_f(\Gamma^{\text{bldd}})$; and we have:

$$
(1 - q(a)) \int_0^\infty f(t)q(t)dt = \int_0^\infty f_1(t)q(t)dt - \int_0^a f_1(t)q(t)dt + \int_0^a f(t)q(t)dt
$$

**Proof.** Let $a_i = p_i/m$, $a = mb$. We have $\nu(\alphaix + c_i) = \nu(\alphaix + c_i - p_i b) = \nu(\alphaix - c_i - p_i (\nu)(b))$.

From this the existence of $f_1$ is clear. We compute:

$$
q(a) \int_0^\infty f(t)q(t)dt = \int_0^\infty f(t)q(t + a)dt = \int_a^\infty f(s - a)q(s)ds =
$$

$$
\int_a^\infty f_1(s)q(s)ds - \int_a^\infty f_1(s)q(s)ds = \int_0^\infty f(t)q(t)dt - \int_0^a f(t)q(t)dt + \int_0^a f_1(t)q(t)dt + \int_0^a f_1(t)q(t)dt
$$

and the lemma follows. \hfill \square

Assuming $A(0)$, fix an element $0 < a_0 \in A$. Define $\mathcal{R}^\Gamma_f = \mathcal{R}^\Gamma[(1 - q(ma))^{-1} : m \in \mathbb{N}]$.

Since the elements inverted are from $F_0 \mathcal{R}^\Gamma_f$, the filtration carries through to $\mathcal{R}^\Gamma_f$. Let $\mathcal{R}^{\text{bldd}}_f$ be the subring of $\mathcal{R}^\Gamma_f$ consisting of elements with two-sided bounded support:

$$
\mathcal{R}^{\text{bldd}}_f = \{ \int_b^bf(t)q(t)dt : f \in \mathcal{R}^\Gamma_f, \ b \in \mathbb{Q} \otimes A \}
$$

and $\mathcal{R}^{\text{bldd}}_{b,t}$ be the localization of $\mathcal{R}^{\text{bldd}}_f$ obtained by inverting the elements $(1 - q(ma_0))$, $m \in \mathbb{N}$.

**Corollary 3.5.** Assume $A(0)$. Then the inclusion $\mathcal{R}^{\text{bldd}}_f \to \mathcal{R}^\Gamma_f$ induces an isomorphism $\mathcal{R}^{\text{bldd}}_{b,t} \to \mathcal{R}^\Gamma_f$.

**Proof.** The surjectivity is clear from Lemma 3.4 and induction. Moreover, inspection of the proof shows that additive inverses are used only for elements in the image of $\mathcal{R}^{\text{bldd}}_f$. Thus if $A$ is the subsemiring of $\mathcal{R}^{\text{bldd}}_f$, generated by $\text{Fu}^{\text{bldd}}_\ast(\Gamma, K^d_f(\Gamma^{\text{bldd}}))$ and by the image of $\mathcal{R}^{\text{bldd}}_{b,t}$, then the
Lemma 3.7. \( R^\Gamma_{b,t} \rightarrow A \) is surjective; and in this case injectivity is evident. Hence \( R^\Gamma_{b,t} \cong A = R^\Gamma_{b,t}. \) \[\square\]

Now an analog of Lemma 2.10. We use integration by parts in \( K(\text{vol} \Gamma A) \). Products refer to the Grothendieck ring of these categories, or equivalently to convolution from the point of view of Lemma 5.2.

Lemma 3.6. Let \( \alpha = \alpha_0, \alpha_1, \ldots, \alpha_n \in \mathbb{Q}^{>0}, c = c_0, c_1, \ldots, c_n \in \mathbb{Q} \otimes A, b_j = \alpha_j b + c_j, c_{jk} = c_k - \alpha_j^{-1} \alpha_k c_j. \) Then

\[
\int_0^b Q_m(at + c) \prod_{j=1}^n \iota(\alpha_j t + c_j) dt = \iota(b) Q_m(ab + c) \prod_{i=1}^n \iota(\alpha_i b + c_i) + \]

\[
- \int_0^{b_j} q(ms) \iota(\frac{s - c_0}{\alpha_0}) \prod_{j=1}^n \iota(\frac{\alpha_j s - c_0 k}{\alpha_j}) ds + \]

\[
\sum_{j=1}^n \int_0^{b_j} \iota(\frac{s - c_j}{\alpha_j}) Q_m(\frac{\alpha_j s - c_j}{\alpha_j}) \prod_{1 \leq k \neq j} \iota(\frac{\alpha_k s - c_k}{\alpha_k}) ds \]

Proof. Let \( F_0(t) = Q(at + c), l_0(t) = at + c, f_0(t) = q(mt). \) We have by definition \( F_0(t) = \int_0^{t+c} q(ms) ds. \) We apply (1) (for indices \( 0, \ldots, n \)) with \( g = 1, G = t, f_0, F_1 \) as above, and for \( i \geq 1, \) writing \( \alpha_i = q_i/p_i, f_i(x) = e(\frac{x}{p_i}) l_i(x) = q_i x + p_i c_i, F_i(x) = \iota(\alpha_i x + c_i) \) as in the proof of Lemma 2.11. Thus:

\[
\int_0^b Q_m(at + c) \prod_{j=1}^n \iota(\alpha_j t + c_j) dt = \iota(b) Q_m(ab + c) \prod_{i=1}^n \iota(\alpha_i b + c_i) - H_0 - \sum_{j=1}^n H_j \]

where

\[
H_0 = \int_0^{\alpha b + c} \iota(l_0^{-1}(t)) q(mt) \prod_{k=1}^n F_k(l_0^{-1}(t)) dt \]

while for \( j \geq 1, \) \( H_j = \int_0^{l_j(b)} \iota(l_j^{-1}(t)) e(t/p_i) Q_m(\alpha_j^{-1}(t) + c) \prod_{1 \leq k \neq j} F_k(l_j^{-1}(t)) dt. \) Now \( Q_m(\alpha_j^{-1}(p_j s) + c) = Q_m(\alpha \alpha_j^{-1}(s - c), \) so the change of variable \( s = t/p_j \) gives:

\[
H_j = \int_0^{b_j} \iota(\alpha_j^{-1}(s - c_j)) Q_m(\alpha_j^{-1}(s - c) + c) \prod_{1 \leq k \neq j} \iota(\alpha_k(\alpha_j^{-1}(s - c_j)) + c_k) ds \]

\[\square\]

Lemma 3.7. \( b, c, c \in \mathbb{Q} \otimes A, \alpha, \alpha_i \in \mathbb{Q}^{>0}, b \in \mathbb{Q} \otimes A. \) Then

1. \( \int_0^b q(mt) \prod_{i=1}^n \iota(\alpha_i t + c_i) dt \in F_n^\Gamma \)
2. \( \int_0^b Q_m(at + c) \prod_{i=1}^n \iota(\alpha_i t + c_i) dt \in F_n^\Gamma \)

Proof. Let \( d, p, p_i \in \mathbb{N}, \alpha_i = p_i/d, \alpha = p/d. \) We use induction on \( n \) and on \( d. \)

If \( M_i \) is the largest integer \( \leq \alpha_i, \) we have: \( \iota(\alpha_i t + c_i) = M_i \iota(t) + \iota((\alpha_i - M_i) t + c_i). \) Using this relation, we immediately reduce to the case \( p_i \leq d. \)

(2.1) We begin with (2) in the case: \( \alpha = 1. \)

We have

\[
Q_m(t + c) = \int_0^{t+c} q(ms)ds = Q_m(t) + \int_t^{t+c} q(ms)ds
\]

Now \( e(t) e(mt) = e(t), \) and \( e(mt) q(m(t + s)) = q(mt) q(ms). \) Thus
The first summand on the right is evidently in $F_{\mathcal{E}}$. Recall (13):

\[ e(x)Q_m(x) = e(x)(1 - q(mx))\theta_m. \]

So

\[ Q_m(t + c)e(t) = Q_m(t)e(t) + e(t)q(mt)Q_m(c) = e(t)(1 - q(mt))\theta_m + e(t)q(mt)Q_m(c) \]

Thus the integral (2) equals:

\[ \theta_m \int_0^b (1 - q(mt)) \prod_{k=1}^{n-1} \epsilon(\alpha_k t - c_k) dt - Q_m(-c) \int_0^b q(mt) \prod_{k=1}^{n-1} \epsilon(\alpha_k s - c_k) ds \]

Both summands lie in $F'_{\mathcal{E}} \mathcal{R}^\Gamma$ by induction on $n$, and using Proposition 2.11. This finishes (2) in the case $\alpha = 1$.

(1) Let $b_j = \alpha_j b + c_j$, $f(t) = \prod_{i=1}^n \epsilon(\alpha_it + c_i)$. By Lemma 2.10 with $g(t) = q(mt)$,

\[ \int_0^b f(t)q(mt)e(t)dt = Q_m(b) \cdot \prod_{i=1}^n \epsilon(\alpha_i b + c_i) - \sum_{j=1}^n \int_0^{b_j} Q_m\left(\frac{s - c_j}{\alpha_j}\right) \prod_{1 \leq k \neq j} \epsilon(\frac{\alpha_k s - c_k}{\alpha_j}) ds \]

The first summand on the right is evidently in $F'_{\mathcal{E}} \mathcal{R}^\Gamma$. If $\alpha_j = 1$, so the second, by the case (2.1). If $\alpha_j < 1$, then $\alpha_k / \alpha_j = p_k / p_j$ have denominators $< d$, so induction on $d$ applies and (2) can be quoted. Hence $\int_0^b f(t)q(mt)e(t)dt \in F'_{\mathcal{E}} \mathcal{R}^\Gamma$.

(2) in the general case. We use Lemma 3.6 (for $n - 1$). The first summand on the right is clearly in $F'_{\mathcal{E}} \mathcal{R}^\Gamma$. By (1), so is the second. The remaining $n - 1$ summands are

\[ E_j = \int_0^{b_j} Q_m\left(\frac{\alpha}{\alpha_j} s - c_j\right) \prod_{1 \leq k \neq j} \epsilon(\frac{\alpha_k s - c_k}{\alpha_j}) ds \]

If $\alpha_j \neq 1$ then again the denominators are $< d$, and by induction $E_j \in F'_{\mathcal{E}} \mathcal{R}^\Gamma$. If $\alpha_j = 1$ then $E_j$ has the form (2), and so can be moved to the left as in Lemma 2.19.

\[ \square \]

Proposition 3.8. (1) $\mathcal{R}^{\text{bdd}}[\theta_m : m = 1, 2, \ldots]$ is generated as a $\mathcal{R}_0^\Gamma = K_0(\Gamma_{A}^{\text{bdd}})$-algebra by the elements $q(b), \theta_m$ and $Q_m(b), m \in \mathbb{N}, b \in \mathcal{Q} \otimes A$.

(2) If $A \neq (0)$, $\mathcal{R}_1^\Gamma$ is generated over $\mathcal{R}_0^\Gamma = K_0(\Gamma_{A}^{\text{bdd}})$ by the elements $q(b), (1 - q(ma_0))^{-1}$ and $\theta_{m,b}$.

Proof. (1) By Corollary 3.5, an element of $\mathcal{R}^{\text{bdd}}$ can be written as a difference of elements $\int_0^b f(t)q(t)dt$, with $f \in F_{\mathcal{E}}(\Gamma, K_0(\Gamma_{A}^{\text{bdd}}))$. By the proof of Proposition 2.21 we can take $f$ to be a product of zero-dimensional terms and basic one-dimensional terms, restricted to a point or an interval. Multiplying by an appropriate $q(c)$, we may assume the point is 0, or the interval is of the form $[0, c]$. If $b \leq c$ then the interval may be ignored; if $c < b$ we replace the integral by $\int_0^b f(t)q(t)dt$, so that $f$ is defined on $[0, b]$. Moreover the 0-dimensional terms can be collected together to form one basic term $e(\frac{t+b}{m})e(b)$. But

\[ \int e(\frac{t+b}{m})e(b)f(t)q(t)dt = \int e(\frac{s}{m})f(s)q(s)ds = \int f(t)q(mt)e(t)dt \]

So it suffices to show that for $\alpha_i \in \mathcal{Q}, c_i \in \mathcal{Q} \otimes A$,

\[ \int_0^b \prod_{i=1}^n \epsilon(\alpha_i t + c_i)q(mt)e(t)dt \]
lies in the $\mathbb{Q}$-algebra generated by the elements $q(b)$ and $\theta_{m,b}$, $m \in \mathbb{N}$, $b \in \mathbb{Q} \otimes A$. This follows from Lemma 3.7.

(2) Follows from Lemma 3.4. □

The next lemma suggests a way to look at unbounded functions; it will not be used further on. Let $\mathcal{R}^{\Gamma,\infty} = \mathbb{Q} \otimes \text{Fun}(\Gamma, K_{\infty}^d(\Gamma^{\text{bdd}})) \cong \text{Fun}(\Gamma, K_{\infty}^d(\Gamma^{\text{bdd}}))$. $\mathcal{R}^{\Gamma,\infty}$ is a $K_{\infty}^d(\Gamma^{\text{bdd}})$-module, under pointwise multiplication, and more generally an $\mathcal{R}^{\Gamma}$-module, under convolution. Thus we can define $\mathcal{R}^{\Gamma,\infty}_i := \mathcal{R}^{\Gamma,\infty}[(1 - q(ma_0))^{-1} : m \in \mathbb{N}]$. Note that $\mathcal{R}^{\Gamma}$ is no a priori a ring. However, it can be made into one using:

**Lemma 3.9.** Let $0 \neq a \in A$. The natural inclusion $\mathcal{R}^{\Gamma}_i \to \mathcal{R}^{\Gamma,\infty}_i$ is an $\mathcal{R}^{\Gamma}$-module isomorphism.

**Proof.** Using Proposition 3.8 together with the automorphism $\gamma \mapsto -\gamma$, the elements with negative support are generated by the $q(\gamma)$ together with the elements $\theta_{m,b} := \sum_{\gamma < b} e(\gamma)q(\gamma)$. But $\theta_{m,b} + \theta_{m,b} = \sum_{\gamma} e(\gamma)q(\gamma) = 0$. Since $q(\gamma) \sum_{\gamma} e(\gamma)q(\gamma) = \sum_{\gamma} e(\gamma)q(\gamma + ma) = \sum_{\gamma} e(\gamma)q(\gamma) = 0$, we have $(1 - q(ma))/\sum_{\gamma} e(\gamma)q(\gamma)) = 0$, so in the localized ring we have $\theta_{m,b} + \theta_{m,b} = 0$. Thus $\theta_{m,b} = -\theta_{m,b}$ lies in the image of the localization of $\mathcal{R}^{\Gamma}$. □

### 3.1. The elements $\theta_m$.

We will see later (Lemma 3.14) that $\theta$ is transcendental over the elements $q(a)$ and $i(a)$. But the various $\theta_m$ are rational over $1$:

**Lemma 3.10.** The identity $\theta_{n+1}(\theta + \theta - 1) = \theta_n\theta$ is valid in $\mathcal{R}^{\Gamma}$. Hence $\theta_n$ is invertible in $\mathcal{R}^{\Gamma}[\theta^{-1}]$, and we have

$$ (1 - \theta_n^{-1}) = (1 - \theta)^n $$

**Proof.** We have:

\begin{align*}
(14) & \quad \theta_n \theta_{n+1} = \theta_n + \int_{t=0}^{\infty} q(t) \int_{s=0}^{t} e(s) e\left(\frac{s-t}{n+1}\right) ds dt \\
(15) & \quad \theta_n \theta = \theta_n + \int_{t=0}^{\infty} q(t) \int_{s=0}^{t} e(s) ds = \theta_n + \int_{t=0}^{\infty} q(t) \xi\left(\frac{t}{n}\right) dt \\
(16) & \quad \theta_{n+1} \theta = \theta_{n+1} + \int_{t=0}^{\infty} q(t) \xi\left(\frac{t}{n+1}\right) dt
\end{align*}

Now by (5),

$$ e(s) e(t) e\left(\frac{s-t}{n+1}\right) \equiv e(s) e(t) e\left(\frac{s+nt}{n(n+1)}\right) $$

With the change of variables $s' = s + nt$ we obtain $e(s) e(t) = e(s') e(t)$, and

$$ \int_{0}^{t} e(t) e\left(\frac{s}{n+1}\right) ds = e(t) \int_{nt}^{t(n+1)} e\left(\frac{s}{n(n+1)}\right) ds $$

With a further change of variable $s'' = \frac{s}{n(n+1)}$,

$$ \int_{0}^{t} e(t) e\left(\frac{s}{n+1}\right) ds = e(t) \int_{nt}^{n} e\left(\frac{t}{n+1}\right) ds = e(t) \int_{nt}^{n} e\left(\frac{t}{n+1}\right) ds $$

so by (14),

$$ \theta_{n+1} \theta_n = \theta_n + \int_{t=0}^{\infty} q(t) \left[\xi\left(\frac{t}{n}\right) - \xi\left(\frac{t}{n+1}\right)\right] dt $$
By \cite{[15], [14]},
\[
\theta_{n+1} \theta_n - \theta_n \theta + \theta_{n+1} \theta = \theta_{n+1}
\]
This identity is equivalent to the one in the statement of the lemma. From this we see that 
\[\theta_{n+1}^{-1} \in \mathbb{Q}[\theta, \theta_n, \theta^{-1}, \theta_{n+1}^{-1}]\] and
\[1 - \theta_{n+1}^{-1} = (1 - \theta_n^{-1})(1 - \theta_{n+1}^{-1})\]
The lemma follows by induction. \hfill \Box

**Remark 3.11.** Lemma 3.10 will be used to show, after an appropriate localization, that elements of finite support generate the entire value ring of \( \Gamma \). This will go over to \( RV \) and to \( VF \). It is a generalization of the rationality of Poincaré series and similar rationality results for generating series, in the integration theory of Denef, Denef-Loësser, and Cluckers-Loeser (polynomials are power series with bounded support.)

**Remark 3.12.** Another aspect of Lemma 3.10 is that a certain localization of the elements of bounded support is forced geometrically. The element \( \theta_m \) can be written, according to the lemma, as \( \frac{\theta_m - \theta - 1 - \theta}{\theta - 1} \), or again as \( \frac{\theta_m}{\theta - 1} \).

Now (say over an algebraically closed field) the value ring of \( VF \) can be obtained as a tensor product of value rings of \( \text{RES} \) and of \( \Gamma \), modulo two linear homogeneous relations of degree one: the equality of the point 0 ∈ \( \Gamma \) with the class of the variety \([G_m]\) over the residue field, and the equality of the point \(1_k\) of the residue field with the class \([\Gamma^0]_L\). Using these relations, we find in \( K(\text{volVF}^{\text{bdd}})[\epsilon][[G_m]]^{-1} \) an element whose volume is: \( \frac{[A_1]^m}{([A_1]^m - [1_k]^m]} \), or \( \frac{L_m}{L - 1} \) in common notation. Subtracting 1 we find an inverse of \( L - 1 \). It follows that for any cyclotomic polynomial \( c_m(z) = (z - 1)^{-1}(z^m - 1) \) with \( m > 1 \), \( h_m([A_1]) \) is invertible in \( KQ(\text{volVF}^{\text{bdd}})[\epsilon] \); though it is not invertible in \( KQ(Var_k) \).

3.2. Unbounded sets. We briefly pause to describe the dimension-free Grothendieck ring of \( \Gamma \). The resulting homomorphisms on \( KVF \) were already described in \cite{[1]}; the present results confirms their uniqueness. Compare \cite{[6], [4]}.

We denote \( e(a) = \{[a]_1]/\{(0)\}_1, \iota(a) = [0, a)_1/[0]_1, \iota(\infty) = [0, \infty)_1/[0]_1 \).

**Theorem 3.13.** \( K_Q^d(\Gamma_A) \) is generated as a \( Q \)-algebra by the elements \( e(a), \iota(a) \quad (a \in Q \otimes A) \) and \( \iota(\infty) \).

For \( a \in A \), we have \( \iota(a) = 0 \). Also \( \iota(\infty)^2 = -\iota(\infty) \).

If \( A \) is divisible, then \( K_Q^d(\Gamma_A) \cong Q^2 \).

**Proof.** The proof of Lemma \ref{2.4} remains valid for \( K(\Gamma) \) with \( b = \infty \), letting \( F_1(\infty) = F_1(\infty) = \int_0^\infty f_i(t)dt \); and the subsequent lemmas through Proposition \ref{2.4} go through verbatim. This shows that \( K_Q^d(\Gamma_A) \) is generated by the elements \( e(a), \iota(a) \) and \( \iota(\infty) \).

The translation \( x \mapsto x + a \) shows that \( [0, \infty] = [a, \infty] \). Hence \( [0, a] = 0 \) in \( K(\Gamma) \), so \( \iota(a) = 0 \). See \cite{[1]} Proposition 9.4 for the relation \( \iota(\infty) = 0 \).

Thus if \( A = Q \otimes A \), \( K_Q^d(\Gamma_A) \) is generated by the element \( \iota(\infty) \). The relation \( \iota(\infty)^2 = \iota(\infty) \) shows that the \( Q \)-algebra is a quotient of \( Q^2 \); the two Euler characteristics in \cite{[1]} show that it is in fact \( Q^2 \). \hfill \Box

3.3. Subrings and quotients of \( K_Q^d(\text{vol}^{\text{bdd}}) \). Recall Lemma \ref{3.3}. \( K_Q^d(\text{vol}^{\text{bdd}}) \) := \( Q \otimes K_Q^d(\text{vol}^{\text{bdd}}) = F_n^{\text{bdd}}(\Gamma, K_Q^d(\Gamma^{\text{bdd}})) \) := \( \mathcal{R}_\Gamma \).

Let \( L_A \) be the field of Corollary \ref{2.20}. Let \( A = Q \otimes A \), and let \( L_A[q^A] \) be the formal Puiseux polynomial ring over \( L_A \) (i.e. the group ring of \( (A, +) \) over \( L_A \)). Let \( L_A(q^A) \) be the field of
fractions. Also form the polynomial ring \( L_A(q^A)[\theta] \), and rational function field \( L_A(q^A)(\theta) \) (with \( \theta \) viewed as an indeterminate.)

**Lemma 3.14.** \( \theta \) is transcendental over \( L_A(q^A) \).

*Proof.* \( \theta^n = \int_0^\infty j(t)q(t)dt \) with \( j \) of degree \( n \). Convolving by an element of \( L_A(q^A)(\theta) \) still leaves an expression of the same form, with \( j(t) \in F_n \setminus F_{n-1} \). The lemma follows from the linear independence of polynomials of distinct degrees over the functions with finite support. \( \square \)

**Proposition 3.15.** Assume \( A \neq 0 \). There is a natural homomorphism

\[
\psi_A^*: \mathcal{R}_A \to K_{\text{bdd}}^{\text{diff}}(\text{vol}\Gamma) 
\]

as well as a homomorphism \( \psi_A : T_A[q^A][\theta] \to \mathcal{R}_A \), with \( \psi_A^* \psi_A = \text{Id} \).

If \( A \) is divisible, \( \psi_A^* \) induces an isomorphism

\[
\mathcal{R}_A[\theta^{-1}] \to T_A[q^A][\theta^{-1}], (1 - q(ma_0))^{-1}, (1 - (1 - \theta^{-1})^m)^{-1}]_{m=1,2,...,n} 
\]

*Proof.* Composing the map \( \phi_A : T_A \to K_{\text{bdd}}^{\text{diff}}(\Gamma_A) \) with the homomorphism \( K_{\text{bdd}}^{\text{diff}}(\Gamma_A) \to \mathcal{R}_A \) of (11), we obtain a map \( \psi_A : T_A \to \mathcal{R}_A \). We have \( \mathcal{R}_A = K_{\text{bdd}}^{\text{diff}}(\Gamma, K_{\text{bdd}}^{\text{diff}}(\Gamma)) \). Extend \( \psi_A \) to a homomorphism \( \psi_A : T_A[q^A] \to \mathcal{R}_A \) with \( q^a \mapsto q(a) \). It is clear by support considerations, and using Lemma (2.23) that \( \psi_A \) is injective on \( T_A[q^A] \). Extend \( \psi_A \) further to the polynomial ring \( T_A[q^A][\theta] \) mapping \( \theta \to \theta \). By Lemma 3.14, \( \psi_A \) remains injective.

Next using Lemma (3.10) extend \( \psi_A \) to

\[
\psi_A' : T_A[q^A][\theta^{-1}], (1 - (1 - \theta^{-1})^m)^{-1}]_{m=1,2,...,n} \to \mathcal{R}_A[\theta^{-1}] 
\]

It is still injective, by Lemma (3.14). By Lemma (3.10) the image of \( \psi' \) contains \( \theta_n \) for each \( n \). By (13), for any \( a \in A \), since \( e(a) = 1 \in \mathcal{R}_A \), \( Q_m(a) = (1 - q(ma))\theta_m \) is also in the image of \( \psi' \). Hence so is \( \theta_{m,a} \).

Assume now that \( A \) is divisible. By Proposition (3.8) \( \mathcal{R}_{\text{bdd}}[\theta_m : m = 1,2,...,n] \) is contained in the image of \( \psi' \). Moreover if we let

\[
\psi'' : T_A[q^A][\theta^{-1}], (1 - q(ma_0))^{-1}, (1 - (1 - \theta^{-1})^m)^{-1}]_{m=1,2,...,n} \to \mathcal{R}_A[\theta^{-1}] 
\]

be the induced homomorphism, then \( \psi'' \) is surjective. It follows that \( \psi'' \) is an isomorphism. Let \( \psi^* \) be the inverse; restricting back to \( \mathcal{R}_A \) we obtain the lemma in the divisible case.

In general, define \( \psi_A^* \) to be the composition of the natural homomorphism

\[
\mathbb{Q} \otimes K_{\text{bdd}}^{\text{diff}}(\text{vol}\Gamma) \to \mathbb{Q} \otimes K_{\text{bdd}}^{\text{diff}}(\text{vol}\Gamma_A) 
\]

with \( \psi_A^* \).

\( \square \)

### 3.4. The Grothendieck ring of \( RV \)

As a step towards the valued field, consider the theory of extensions

\[
1 \to k^* \to RV \to_{\text{val}_k} \Gamma \to 0 
\]

of an ordered divisible Abelian group \( \Gamma \) (written additively) by the multiplicative group of an algebraically closed field. This is a complete theory; in a saturated model \( M \), the sequence is split, though of course the set of points in a given substructure need not be. See [1] for details.

We work over a base structure \( A_{RV} \), which as above is left out of the notation. Let \( A \) be the image of \( A_{RV} \) in \( \Gamma \). Let \( A_{\text{RES}} = A \cap \text{RES} \) where \( \text{RES} = \cup_{\gamma \in Q \otimes A_{\text{val}_k}}^{-1}(\gamma) \).
The following specializes Definitions 3.66 and 5.21 of [1]. Define \( \Sigma : \Gamma^n \rightarrow \Gamma \) by \( \Sigma((x_1, \ldots, x_n)) = \sum_{i=1}^n x_i \).

**Definition 3.16.**
1) \( RV[n] \) is the category of pairs \((U, f)\), with \( U \) a definable subset of \( RV^m \) for some \( m \), and \( f = (f_1, \ldots, f_n) : U \rightarrow RV^n \) a finite-to-one map. A morphism \( U \rightarrow V \) is a definable bijection \( U \rightarrow V \).

2) \( Ob \, volRV[n] = Ob 
\)

3) \( volRV^{bdd}[m] \) is the full subcategory of \( volRV[m] \) consisting of objects whose \( \Gamma \)-image is contained in \( [\gamma, \infty]^m \), for some definable \( \gamma \in \Gamma \). These will again be referred to as semi-bounded.

4) \( RES[n] \) (respectively \( volRES[n] \)) is the full subcategory of \( RV[n] \) (respectively \( volRV[n] \)) whose objects \( U \) are contained in \( RES[m] \) for some \( m \). Equivalently, such that \( val_{rv}(U) \) is finite.

The map \( val_{rv} : RV \rightarrow \Gamma \) induces maps \( RV^n \rightarrow \Gamma^n \). If \( X, Y \) are \( \Gamma[n] \)-isomorphic definable subsets of \( \Gamma^n \), then \( val_{rv}^{-1}X, val_{rv}^{-1}Y \) are definably isomorphic: both \( GL_n(\mathbb{Z}) \) transformations and \( A \)-translations obviously lift. The definition of the category \( \Gamma[n] \) was indeed engineered for this. Hence the pullback \( X \mapsto \Gamma^{-1}X \) induces a map

\[
K_+ \Gamma[n] \rightarrow K_+ RV[n], \ [X] \mapsto [val_{rv}^{-1}X]
\]

Semi-boundedness is preserved by the pullback; and also, again by definition, a \( vol\Gamma[n] \)-isomorphism lifts to a \( volRV[n] \) isomorphism. Thus we also have

\[
K_+ vol\Gamma^{bdd}[n] \rightarrow K_+ volRV^{bdd}[n], \ [X] \mapsto [val_{rv}^{-1}X]
\]

On the other hand the inclusion induces an obvious map

\[
K_+ RES[n] \rightarrow K_+ RV[n]
\]

and

\[
K_+ volRES[n] \rightarrow K_+ volRV[n]
\]

We obtain homomorphisms

\[
K_+(RES[*]) \otimes K_+(\Gamma[*]) \rightarrow K_+(RV[*])
\]

\[
K_+(volRES[*]) \otimes K_+(vol\Gamma^{bdd}[*]) \rightarrow K_+(volRV^{bdd}[*])
\]

These are shown in [7] to be surjective. If \( \gamma \in \Gamma[1] \) is a definable point, then \( [val_{rv}^{-1}(\gamma)] \in K_+ RES[1] \) has the same image under \((19)\) as \( \gamma \) has under \((17)\); and similarly in the measured case. Thus in both cases the kernel contains the elements \( 1 \otimes [val_{rv}^{-1}(\gamma)] - [\gamma] \otimes 1 \), \( \gamma \in \Gamma \) definable. By Corollary 10.3 and Proposition 10.10 of [1], these elements generate the kernel in both cases, \((19)\) and \((17)\).

\[2\]The definitions in [1] are more general in several respects. In particular several kinds of resolution on volume forms are considered; here we consider the type denoted \( vol \) in [1]. Since no other volumes are considered, the subscript becomes unnecessary. Similar results are possible for the other variants.
3.5. **Bounded definable subsets of RV.** We begin with a description of the Grothendieck ring of two-sided bounded definable subsets of RV in the divisible case, using Lemma 2.23. This does not immediately translate to a statement for VF, since the notion of boundedness is not preserved under arbitrary definable maps. The results of this subsection will not be used further on.

(23) \[ K_+(\text{RES}[s])([G_m(k)]_1^{-1}) \otimes K_+(\Gamma[s])([0]_1^{-1}) \to K(\text{RV}[s])([G_m(k)]_1^{-1}) \]
whose kernel is again generated by the elements \(1 \otimes [\text{val}_{\text{rv}}^{-1}(\gamma)]_1 - [\gamma]_1 \otimes 1, \gamma \in \Gamma \) definable, as one can see by multiplying an element of the kernel by a high enough power of \([G_m(k)]_1^{-1}\).

Hence we have a surjective homomorphism

(24) \[ K_+(\text{RES}[s])([G_m(k)]_1^{-1})_0 \otimes K_+(\Gamma[s])([0]_1^{-1})_0 \to K_+(\text{RV}[s])([G_m(k)]_1^{-1})_0 \]
whose kernel is generated by the relations \([\text{val}_{\text{rv}}^{-1}(\gamma)]_1 \otimes [\gamma]_1 \otimes 1, \gamma \in \Gamma \) definable.

Note that \(K(\text{RES}[n])_0\) is naturally isomorphic to the direct limit of the \(K(\text{RES}[n])_1\), where \(K(\text{RES}[n])_1\) is mapped to \(K(\text{RES}[n + 1])\) by the map \([X] \mapsto [X \times G_m(k)]_1\).

**Definition 3.17.** \(K^d(\text{RES}) := K(\text{RES}[s])([G_m(k)]_1^{-1})_0\) will be called the stabilized Grothendieck ring of \(\text{RES}\). Similarly \(K^d(\text{VAR}_F) = K(k[s])([G_m(k)]_1^{-1})_0\) and \(K^d(\text{RV}) = K(\text{RV}[s])([G_m(k)]_1^{-1})_0\), and similarly for the semirings.

**Proposition 3.18.** \((K^d(\text{RES}_{A_{\text{bv}}}) \otimes K^d(\Gamma^b_{A})) / I \cong K^d(\text{RV}_{A_{\text{bv}}})\) where \(I\) is the ideal generated by \(\{\frac{[\text{val}_{\text{rv}}^{-1}(\gamma)]_1}{[G_m(k)]_1} - e(\gamma) : \gamma \in \mathbb{Q} \otimes A\}\)

**Proof.** The homomorphism (23) is compatible with restriction to semi-bounded sets: \(K_+(\text{RES}[s]) \otimes K_+(\Gamma^b_{A}) \to K_+(\text{RV}^b_{A})\) is surjective and has kernel generated by the elements \(1 \otimes [\gamma] - [\text{val}_{\text{rv}}^{-1}(\gamma)]_1 \otimes 1\). Equations (23), (24) for semi-bounded sets follow in the same way. The proposition follows upon taking additive inverses.

Let \(T_A\) denote the symmetric algebra \(\mathbb{Q} \oplus (\mathbb{Q} \otimes A) \oplus \text{Sym}^2(\mathbb{Q} \otimes A) \oplus \ldots\)

**Corollary 3.19.** Assume \(A\) is divisible, and let \(F = A_{\text{RV}} \cap k\). Then \(K^d(\text{RV}^b_{A_{\text{bv}}}) \cong K^d(\text{VAR}_F) \otimes T_A\)

**Proof.** Assume \(A\) is divisible. In this case every definable set \(X \subseteq \text{RES}^m\) is definably isomorphic to a definable subset of a Cartesian power of \(k\), where \(k\) is the residue field. So \(K(\text{RES}[n])\) reduces to \(K(k)\), the Grothendieck ring of \(F\)-varieties. Moreover for any definable \(\gamma \in G, \text{val}_{\text{rv}}^{-1}(\gamma)\) is definable isomorphic \(G_m(k)\. Hence in this case the relations in Proposition 3.18 are redundant, and the tensor product is valid over \(\mathbb{Q}\). By Proposition 2.23 \(K^d(\Gamma^b_{A}) \cong T_A\).

The corollary follows.

3.6. **The measured Grothendieck ring of RV.** The connection between varieties with forms over the valued field, and the category \(\text{vol}\Gamma[n]\), is mediated by \(\text{volRV}[n]\). We now study the dimension-free Grothendieck ring of this category, incorporating in particular both \(\Gamma\) and the residue field.

Let \(F = A_{\text{RV}} \cap k\) be the base residue field, and \(\text{VAR}_F[n]\) the category of \(F\)-varieties of dimension \(\leq n\). (22) can be used to describe \(K^d(\text{volRV}^b_{A})\). We do this now in the case: \(A\) is divisible. Recall that the rings \(K^d(\text{VAR}_F), K^d(\text{vol}\Gamma^b_{A}), K^d(\text{volRV}^b_{A})\) are defined with respect to dehomogenizing elements \([G_m]_1, [0]_1\) and \([G_m]_1 \otimes 1 = 1 \otimes [0]_1\) respectively.
Proposition 3.20. Assume $A$ is divisible. Then

$$K^d_f(\text{volR}^{\text{bdd}}) \simeq K^d_f(\text{Var}_F) \otimes K^d_f(\text{volR}^{\text{bdd}})$$

Proof. Let $K(\text{Var}_F[*]) = \oplus_{n \geq 0} K(\text{Var}_F[n])$. In this case the natural map

$$K_+(\text{Var}_F[*]) \otimes K_+(\text{volR}^{\text{fin}}[*]) \to K_+(	ext{volRES}[*])$$

is a surjective homomorphism, with kernel generated by the single relation

$$R : [G_m]_1 \otimes 1 = 1 \otimes [0]_1$$

simplifies to:

$$K(\text{volR}^{\text{bdd}}[*]) \simeq K(\text{Var}_F[*]) \otimes K(\text{volR}^{\text{bdd}}[*])/R$$

The proposition follows using Lemma 2.1. \qed

3.7. The Grothendieck ring of bounded volume forms over valued fields. Let $T$ be a $V$-minimal theory; to simplify notation we will assume $T$ is effective. See [1] for the definitions of these notions. The principal example are the theory $ACVF_F$ of algebraically closed valued fields, over a base valued field $F$ with residue field $F'$ of characteristic 0. The reader may take $T$ to be $ACVF_F$; in this case “definable” is the same as “$F$-semi-algebraic”, and the category $\text{Vol}_F$ described below is $\text{Vol}_F$ of the introduction. Other examples are analytic expansions of L. Lipshitz and Z. Robinson.

If $V$ is a smooth $n$-dimensional variety, let $\Omega V = \bigwedge^n TV$, considered as a variety rather than a vector bundle. The notion of a bounded subset of $V$ and in the same way as in [7], §6.1. If $X \subseteq V$ is bounded, we consider definable sections $\omega : X \to \Omega V$ over $X$; we say $\omega$ is bounded if the graph of $\omega$ in $\Omega V$ is bounded.

Definition 3.21. $\text{Vol}_T[n]$ is the category whose objects are pairs $(X, \omega)$, with $X$ either empty or a definable bounded Zariski dense subset of a smooth $F$-variety $V$ of dimension $n$, and $\omega : X \to \Omega V$ a definable bounded section. A morphism $(X, \omega) \to (X', \omega')$ is a definable bijection $g$ between subsets of $X, X'$ whose complement has dimension $< \dim(V)$, such that (away from a set of dimension $< \dim(V)$) $\omega = cg^*\omega'$ for some definable function $c$ on $X$ with $\text{val}(c) = 0$.

For $b \in \Gamma$, let $U_b = \{x : \text{val}(x) = b\}$. In particular $U_0 = \{x : \text{val}(x) = 0\} = O \setminus M$. $M = \{x : \text{val}(x) > 0\}$.

$\text{Vol}_T$ is an $\mathbb{N}$-graded category, and yields a graded Grothendieck semiring $K_+(\text{Vol}_T)$. We take $e_1 = [(U_0, dx)]$, and form the dimension free semiring $K^d_f(\text{Vol}_T) = K^d_f(\text{Vol}_T)$. Let $K^{d}_Q(\text{Vol}_T) = Q \otimes K^d_f(\text{Vol}_T)$.

To facilitate the comparison to Definition 3.10 we need to compare $\text{Vol}_T$ to a more elementary version.

Definition 3.22. 1) $\text{VF}[n]$ is the category of pairs $(X, f)$, with $X$ a definable subset of $\text{VF}^m$ for some $m$, and $f = (f_1, \ldots, f_n) : X \to \text{VF}^n$ a finite-to-one map. A morphism $X \to Y$ is a definable bijection $X \to Y$.

2) $\text{Ob volVF}[n] = \text{Ob VF}[n]$. A morphism $(X, f) \to (Y, g)$ is a definable bijection $h : X \to Y$ such that $h^*g^*dx = f^*dx$ away from a variety of dimension $< n$, where $dx = dx_1 \wedge \ldots \wedge dx_n$ is the standard volume form on $\text{VF}^n$.

3) $\text{volVF}^{\text{bdd}}[n]$ is the full subcategory of $\text{volVF}[n]$ consisting of objects $(X, f)$ with $f(X)$ bounded.
volVF is dimension-graded, with distinguished element \([U_0], Id\), and we form \(K^d_f volVF\) using the dehomogenizing element \([U_0] \); similarly \(K^d_f volVF^{bdd}\) and \(K^d_f volVF^{bdd} = K(volVF^{bdd})_{\mid [U_0]}\).

**Lemma 3.23.** \(K^d_f volVF \cong K^d_f volT\) canonically; the isomorphism takes \(K^d_f volVF^{bdd}\) to \(K^d_f volT^{bdd}\), and induces an isomorphism \(K^d_f volVF \cong K^d_f volF\).

**Proof.** Let \((X, f) \in Ob volVF[n]\). Let \(V\) be the Zariski closure of \(X\), and \(\omega = f^* dx\); this is defined away from a subvariety of \(V\) of dimension \(< n\). \((X, f) \rightarrow (X, \omega)\) is a functor \(VF[n] \rightarrow VolT[n]\), inducing an injective graded semiring homomorphism \(K_+ volVF \rightarrow K_+ volT\).

An element of \(K_+ VolT[n]\) has the form \([(X, \omega)]\) with \(X\) a definable subset of a smooth affine variety \(V \subseteq VF^n\), admitting a finite-to-one projection \(f : V \rightarrow \mathbb{A}^n\), and \(\omega(v) = c(v) f^* dx\) for some definable \(c : V \rightarrow VF\). Let \(Y = \{(x, t) \in V \times \mathbb{A}^1 : val(t) = val(c(x))\}\), \(g(x, t) = (f(x), t)\). Then \((X, \omega) \times ([U_0], dx) \cong_{volVF} (Y, g^* (dx \land dt))\) and hence lies in the image of \(volVF\). Hence by Lemma 2.24 \(K^d_f volVF \cong K^d_f VolT\) canonically, and so \(K^d_f volVF \cong K^d_f volF\). \(\square\)

We write \(\theta_V^F = 1 + \frac{|M|}{|\bar{a}|}\), and for a definable \(b \in \Gamma\) we write \(q_{VF}(b) = \frac{|U_0|}{|\bar{a}|}\). These correspond under the canonical isomorphisms below to the classes \(\theta\) and \(q(b)\) of \(K^n_f (volF^{bdd})\), and when no confusion can be caused we will omit the subscript. We assume \(\Gamma\) has at least one definable element \(a_0 > 0\), and write \(q^{-m}\) for \(q_{VF}(ma_0)\). Note that \(q^{-m} = (q_{VF}^{-1})^m\).

Write \(\hat{q}^{-1}\) for \(1 - \theta_{VF}^{-1} \in K^d_f (volT)(\theta_{VF}^{-1})\). So \(1 - \hat{q}^{-1} = \theta_{VF}^{-1}\).

When no confusion can arise, we also write \(\hat{q}^{-m}\) for \(q(ma_0)\) and \(\hat{q}^{-1}\) for \(1 - \theta^{-1}\).

Recall \(T_A\) denotes the symmetric algebra \(\mathbb{Q} \otimes (\mathbb{Q} \otimes A) \otimes Sym^2(\mathbb{Q} \otimes A) \oplus \ldots\).

**Theorem 3.24.** Let \(T\) be an effective \(V\)-minimal theory. Let \(F\) be the field of \(VF\)-definable points of \(T\), \(A = val(F)\), \(A = \mathbb{Q} \otimes A\), and let \(0 < a_0 \in A\). Then there exists a canonical homomorphism

\[K^d_f (VolT) \rightarrow K^d_f (VarF^n)[\hat{q}^{-1}, (1 - \hat{q}^{-m})^{-1}]_{m=1,2,\ldots} \otimes T_A[\mathbb{Q}^A]_A[(1 - q(ma_0))^{-1}]_{m=1,2,\ldots} \]

If \(A\) is divisible, this induces an isomorphism

\[K^d_f (VarF^n)[\hat{q}^{-1}, (1 - q^{-m})^{-1}]_{m} \cong K^d_f (VarF)[\hat{q}^{-1}, (1 - q^{-m})^{-1}]_{m} \otimes T_A[\mathbb{Q}^A]_A[(1 - q^{-m})^{-1}]_{m} \]

**Remark 3.25.** (1) The inverted \(1 - \hat{q}^{-m}\) on the \(VarF\) seems to correspond to nothing on the \(VolF\)-side; see Lemma 2.10 and Remark 5.14 for an explanation.

(2) We took \(K^d_f (VarF)[[V]/[G_m^*] : V \in VarF, \dim(V) \leq n\}. The localization is by \([G_a]/[G_m]\) and \([G^k_a - [1]k]/G^k_m, k = 1, 2, \ldots\)

**Proof of Theorem 3.24** Let \(sp\) be the semiring congruence on \(K_+ volRV\) generated by \(([1]_1, [RV^{>0}]_1), \) with the constant \(\Gamma\)-form \(0 \in \Gamma\). The restriction to \(K_+ volRV^{bdd}\) is denoted by the same letter, as is the corresponding ideal of \(K_0 volRV^{bdd}\). (The proof of Lemma 8.20 never goes out of the semi-bounded category.)

By \([1]\) Theorem 8.29,

\[K_+(volVF^{bdd}[n]) \cong K_+(volVF^{bdd})/sp\]

Restricting to \(\Gamma\)-valued measures as in (8.5), we obtain an isomorphism

\[K_+(volVF^{bdd}[\ast]) \cong K_+(volVF^{bdd}[\ast])/sp\]

If \(b = [1]_k - [RV^{>0}]_1\), this induces a ring isomorphism

\[K(volVF^{bdd}[\ast]) \cong K(volVF^{bdd}[\ast])/b\]
We take \([G_m(k)]_1\) as the distinguished element of \(K(\text{volRV}^{\text{bdd}})[1]\), and correspondingly the class \([U_0]\) of the annulus \(U_0 = \{ x : \text{val}(x) = 0 \in K(\text{Vol}_T[1]) \}; i.e. \(K^{df}(\text{volRV}^{\text{bdd}}) = K^{df}_{[G_m(k)]_1}(\text{volRV}^{\text{bdd}}), K^{df}(\text{Vol}_T) = K^{df}_{[U_0]}(\text{Vol}_T)\).

Let \(\xi = \frac{b}{[G_m(k)]_1} \). By Lemma \(22\) and \(3.23\)

\[
K^{df}_Q(\text{Vol}_T) = K^{df}_Q(\text{volVF}^{\text{bdd}}) \cong K^{df}_Q(\text{volRV}^{\text{bdd}})/\xi
\]

Thus it suffices to find the canonical homomorphism on \(K^{df}_Q(\text{volRV}^{\text{bdd}})/\xi\). This involves work with \(\text{RV}\) alone. At this point we may assume \(A\) is divisible ; the homomorphism in the general case can then be obtained by composing with the canonical homomorphism \(K^{df}_Q(\text{volRV}^{\text{bdd}})_A \rightarrow K^{df}_Q(\text{volRV}^{\text{bdd}})_Q\).

By Proposition \(3.20\)

\[
K^{df}_Q(\text{var}\_F) \cong K^{df}_Q(\text{var}_F) \otimes K^{df}_Q(\text{vol}\Gamma^{\text{bdd}})
\]

Under this isomorphism, \(\xi\) corresponds to

\[
\xi_{VF} = \left[ \frac{1}{[G_m(k)]} \right] \otimes 1 - 1 \otimes \left[ \frac{\text{volRV}^{\geq 0}}{q_0} \right] = \left[ \frac{1}{[G_m(k)]} \right] \otimes 1 - 1 \otimes (\theta - 1)
\]

while \(q(ma_0)\) corresponds under the composition of \((25), (26)\) to \(q_{VF}(ma_0) = q^{-m}\), and \(\theta\) to \(\theta_{VF}\).  

Hence by Proposition \(3.15\) using \(1 - \hat{q}^{-1} = \theta_{VF}^{-1}\),

\[
K^{df}_Q(\text{Vol}_T)[(1 - q^{-m})^{-1}, \hat{q}^{-1}]_{m=1,2,...} \cong
\]

\[
K^{df}_Q(\text{var}_F) \otimes T_A[q^\theta][\theta, \theta^{-1}, (1 - q(ma_0))^{-1}, (1 - (1 - \theta^{-1})^m)^{-1}]_{m=1,2,...}/\xi_{VF}
\]

We can view the relation \(\xi_{VF}\) as defining \(1 \otimes (\theta - 1) = (\hat{q} - 1)^{-1} \otimes 1\) where \((\hat{q} - 1)^{-1} := \frac{[1]}{[G_m(k)]}\).

Then \(27\) becomes:

\[
K^{df}_Q(\text{var}_F)[(\hat{q} - 1)^{-1}, \hat{q}^{-1}, (1 - \hat{q}^{-m})^{-1}]_{m=1,2,...} \otimes T_A[q^\theta][(1 - q(ma_0))^{-1}]_{m=1,2,...}
\]

As \((\hat{q} - 1)^{-1} = \hat{q}^{-1}(1 - \hat{q}^{-1})^{-1}\), this term is redundant, so

\[
K^{df}_Q(\text{Vol}_T)[(1 - q^{-m})^{-1}, \hat{q}^{-1}]_{m=1,2,...} \cong
K^{df}_Q(\text{var}_F)[\hat{q}^{-1}, (1 - \hat{q}^{-m})^{-1}]_{m=1,2,...} \otimes T_A[q^\theta][(1 - q(ma_0))^{-1}]_{m=1,2,...}
\]

\(\square\)

So far, we always used \([0_T]_1\) as a dehomogenizing element. An alternative choice is \([1_T]_1\); it goes along with \(M\) in \(K(\text{Vol}_T[1])\) and \([G^{>0}]\) in \(K(\text{vol}\Gamma^{\text{bdd}})\). There appears to be a deep duality transposing these choices. With the latter choice too one has an analogue of Theorem \(??\), of which we indicate the beginning.

Let

\[
K^{df'}(\text{volRV}^{\text{bdd}}) = K(\text{volRV}^{\text{bdd}}[\ast])^{df}_1
\]

\[
K^{df'}(\text{Vol}_T) = K(\text{Vol}_T)[\ast]^{df}_{\{\lambda\}}
\]

\[
K^{df'}(\text{vol}\Gamma^{\text{bdd}}) = K(\text{vol}\Gamma^{\text{bdd}})^{df}_{[\ast_{\text{RV}^{>0}}]}
\]

Also let \(\alpha' = \frac{[G_m]}{[1]} \otimes 1 - 1 \otimes \frac{[0]}{[\text{RV}^{>0}]}\).
Lemma 3.26.  
\[ K(\text{volRV}_A^{\text{bdd}}[s]/b)_{[1]}^{df} \cong K^{df}(\text{Var}_F[s]) \otimes K^{df}(\text{volR}^{\text{bdd}}[s])/a' \]

Proof.  
(28)  
\[ K^{df}(\text{Vol}_F) = K^{df}(\text{volVR}^{\text{bdd}}) \cong K(\text{volRV}_A^{\text{bdd}}[s]/b)_{[1]}^{df} \]

In the ring \( K(\text{volRV}_A^{\text{bdd}}[s]/b) \), we have \([1] = [Rv_0] \), so Lemma 2.1 applies. As in Proposition 3.20 letting \( a \) be the ideal generated by \([G_m] \otimes 1 - 1 \otimes [0] \), we have from 22.

\[ K(\text{volRV}^{\text{bdd}}[s]) \cong K(\text{Var}_F[s]) \otimes K(\text{volR}^{\text{bdd}}[s])/a \]

and so

\[ K(\text{volRV}^{\text{bdd}}[s])/b \cong K(\text{Var}_F[s]) \otimes K(\text{volR}^{\text{bdd}}[s])/(a, b) \]

The statement of the lemma follows from Lemma 2.1.

If \( V \) is a definable subset of a variety over \( F \) and \( \omega \) a definable volume form, call \((V, \omega)\) strictly absolutely integrable if there exists \((V', \omega') \in \text{Ob Vol}_F\) and a definable bijection \( g : V \to V' \) (up to a smaller dimensional set), such that \( \text{val}\gamma' = \text{val}\omega \). Define \( \int_V \omega \) to be the image of \([V', \omega']\) under the homomorphism of Theorem 3.24. This clearly does not depend on the choice of \((V', \omega')\).

Let \( \mathcal{R} \) be the target ring of Theorem 3.24 and \( \mathcal{I} \) the homomorphism. \( \mathcal{R} \) admits a natural decreasing \( \Gamma \)-filtration:

\[ F_\gamma \mathcal{R} = K^{df}(\text{Var}_F)[q^{-1}, (1 - q^{-m})^{-1}]_{m=1,2,...} \otimes T_A[q^{\Delta^{-\gamma}}][(1 - q(\text{vol}_A))]_{m=1,2,...} \]

where \( A^{\Delta^{-\gamma}} = \{c \in A : c > \gamma\} \).

Remark 3.27. Any \((V, \omega)\) admits a definable map \( c : V \to \Gamma^{\geq 0} \), such that each fiber is strictly absolutely integrable. Hence so is the inverse image of any bounded subset of \( \Gamma \). Moreover if \( V_\gamma = c^{-1}(\gamma) \), then for large \( \gamma \int_{V_\gamma} \omega = \sum_{i=1}^n r_i P_i(\alpha_i \beta_i)q^{\alpha_i \gamma} \), with \( r_i, P_i \in \mathcal{R}, P_i \in \mathcal{Q}[X], \beta_i \in \mathcal{Q}^m, \alpha_i \in \mathcal{Q} \). If all \( \alpha_i \geq 0 \), and \( \alpha_i = 0 \) implies \( P_i \) is constant, we call \((V, \omega)\) absolutely integrable and define \( \int_V \omega = \sum_{\alpha_i = 0} r_i P_i \). This does not depend on the choice of \( c \), but it is not clear if it is really more general than strict absolute integrability.

Remark 3.28. In 11 more general volume forms are considered. \( \mu_1^{\text{bdd}} \mathcal{V} \) is equivalent to the category of pairs \((V, \theta)\) with \( \theta \) a bounded, bounded support section of the \( \Gamma \)-bundle \( \text{val}_A \Lambda^{\dim(V)}TV \) induced from the top form bundle via the valuation map. If \((V, \omega) \in \text{Vol}_F\) then \((V, \text{val}\omega) \in \mu_1^{\text{bdd}} \mathcal{V} \), but the converse need not be true.

It is possible to define an integral \( \int(V, \theta) \) with values in \( K^{df} \text{Vol}_F \). One can easily find definable functions \( c : V \to \Gamma \) such that with \( V_\gamma = c^{-1}(\gamma) \), \((V_\gamma, \omega|_{V_\gamma}) \) lies in the image of \( \text{Vol}_F \). Then define \( \int(V, \theta) = \int_{V_\gamma} \int_{V_\gamma} \omega|_{V_\gamma} \). The expression is well-defined. However, the dimension-free Grothendieck ring \( K^{df}_1(\mu_1^{\text{bdd}} \mathcal{V}) \) is not identical with \( K^{df}(\text{Vol}_F) \).

For instance \( q \) has a square root in \( K^{df}_1(\mu_1^{\text{bdd}} \mathcal{V}) \), namely \( d = [(0), (\frac{1}{2})]/[(0), 0] \). We have \( d^2 = q \), as opposed to the conditional square root \( d' = q(\frac{1}{2}) \in K^{df}(\text{Vol}_F) \) which only satisfies \((d')^2 = q e(1/2) \). Equivalently, the idempotent \( e(1/2) \) has a nontrivial square root \( \frac{d'}{d} \).
4. Appendix

In this appendix we define the Iwahori Hecke algebra of $SL_2$ over an algebraically closed valued field. We continue to denote by $F$ a valuation field with value group $\Gamma$, ring of integers $\mathcal{O}$ and residue field $\mathbf{F}$. We denote by $\mathcal{O}^\times_\mathbf{F}, \mathcal{O}^\times_{cl}, \mathcal{A}^\times$ the (classes of the) open ball, closed ball and annulus of radius $\gamma \in \Gamma$. We also denote $q = \mathcal{O}^\times_{cl}/\mathcal{O}^\times_\mathbf{F}$. In particular, $\mathcal{A}^\times = (q - 1)\mathcal{O}^\times_\mathbf{F}$. To ease notation, we choose a section $\Gamma \to F$ denoted by $\gamma \mapsto t^\gamma$. Note, however, that this is never used in an essential way.

We denote by $G$ the group $SL_2(F)$, by $B$ the subgroup of upper triangular matrices, by $N$ the subgroup of unipotent upper triangular matrices and by $A$ the subgroup of diagonal matrices. We will abuse notations and write $G(\mathcal{O}), G(\mathbf{F})$ etc. for the groups of points of the corresponding algebraic groups. We have a residue map $\text{res} : G(\mathcal{O}) \to G(\mathbf{F})$. All integrals over $G$ will be taken with respect to the Haar form on $G$, which is

$$dg \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \frac{1}{a} da \wedge db \wedge dc$$

So, for example, the measure of the set of matrices such that $\text{val}(a) = \gamma_a, \text{val}(b) = \gamma_b, \text{val}(c) = \gamma_c$ is $t^{-\gamma_a} \mathcal{A}^{\gamma_a} \mathcal{A}^{\gamma_b} \mathcal{A}^{\gamma_c} = \mathcal{A}^0 \mathcal{A}^{\gamma_a} \mathcal{A}^{\gamma_b} \mathcal{A}^{\gamma_c}$.

In order that the convolution makes sense, the field of coefficients will be taken to be a field $E$ together with a ring homomorphism $K^{bdd}(\text{Vol}_F) \to E$. By Proposition 3.14, there is such a field with nontrivial homomorphism.

**Definition 4.1.** A definable function from $G(F)$ to $E$ is a function of the form $f(g) = \sum_{i=0}^N c_i \phi_i(g)$ where $\phi_i$ are definable functions from $G(F)$ to $K(\text{Var}_F)$ and $c_i \in E$. A definable function is called bounded if there is $\gamma \in \Gamma$ such that $f(g) = 0$ unless all entries of $g$ have valuation less than $\gamma$.

**Definition 4.2.** The convolution of two bounded definable functions $f_1, f_2$ from $G(F)$ to $K(\text{Var}_F)$ is the function $f_1 \ast f_2(g) = \int_{h \in G(F)} f_1(gh^{-1}) f_2(h) dh$, which is easily seen to be a bounded definable function. This definition extends to convolution of bounded definable functions from $G(F)$ to $E$.

**Remark 4.3.** We can similarly define bounded definable functions from $\Gamma$ to $E$ and convolution of them.

**Definition 4.4.** The Iwahori subgroup $I \subset G(\mathcal{O})$ is the inverse image of $B(\mathbf{F})$ under the map $\text{res}$. As a vector space, the Iwahori Hecke algebra $\mathcal{H}$ is the $E$ vector space of bounded definable functions from $G(F)$ to $E$ that are invariant under left and right multiplication by $I$. This is an algebra where the multiplication is convolution of functions.

A special role will be played by the following $\mathcal{H}$ module:

**Definition 4.5.** Let $M$ be the right $\mathcal{H}$ module consisting of bounded definable functions from $G(F)$ to $E$ that are invariant under the left multiplication by $A(\mathcal{O})N(F)$ and under the right multiplication by $I$.

The proof of the following lemmas is standard:

**Lemma 4.6.** Let $g = \left( \begin{array}{cc} x & y \\ z & w \end{array} \right)$ and $\gamma \in \Gamma$ be negative. Then

1. $g \in I \left( \begin{array}{cc} t^\gamma & 0 \\ 0 & t^{-\gamma} \end{array} \right)$ if and only if $\text{val}(x) = \gamma, \text{val}(y) \geq \gamma, \text{val}(z) > \gamma, \text{val}(w) \geq \gamma$.
2. $g \in I \left( \begin{array}{cc} t^{-\gamma} & 0 \\ 0 & t^\gamma \end{array} \right)$ if and only if $\text{val}(x) > \gamma, \text{val}(y) \geq \gamma, \text{val}(z) > \gamma, \text{val}(w) = \gamma$. 

Lemma 4.7. Let 

\[
\begin{align*}
30 & \quad \text{EHUD HRUSHOVSKI, DAVID KAZHDAN} \\
& \quad \text{Lemma 4.7. Let } g = \left(\begin{array}{cc} x & y \\ z & w \end{array}\right). \text{ Then} \\
& \quad (1) \text{ If } \text{val}(z) \leq \text{val}(w) \text{ then } g \in A(O)N \left(\begin{array}{cc} 0 & z^{-1} \\ z & 0 \end{array}\right) I. \\
& \quad (2) \text{ If } \text{val}(z) > \text{val}(w) \text{ then } g \in A(O)N \left(\begin{array}{cc} w^{-1} & 0 \\ 0 & w \end{array}\right) I.
\end{align*}
\]

We show 2. for example. We first find the coefficients of the convolution equals the value of the convolution at the point \(t^{-\delta} 0 0 t^\delta\). This, in turn, equals to the measure of the set of elements \(g \in S_\gamma\) for which there is \(h \in v_0\) such that \(gh = \left(\begin{array}{cc} t^{-\delta} & 0 \\ 0 & t^\delta \end{array}\right)\). Suppose \(\left(\begin{array}{cc} x & y \\ z & w \end{array}\right) \in v_0\) and \(\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in S_\gamma\). If their product is \(\left(\begin{array}{cc} t^{-\delta} & 0 \\ 0 & t^\delta \end{array}\right)\) then

\[
\left(\begin{array}{cc} x & y \\ z & w \end{array}\right) = \left(\begin{array}{cc} t^{-\delta} & 0 \\ 0 & t^\delta \end{array}\right) \left(\begin{array}{cc} d & -b \\ -c & a \end{array}\right) = \left(\begin{array}{cc} t^{-\delta}d & -t^{-\delta}b \\ -t^\delta c & t^\delta a \end{array}\right)
\]

So \(\text{val}(t^\delta a) = 0\) and \(\text{val}(a) = \gamma\), hence \(\delta = -\gamma\). The constraints are \(\text{val}(a) = \gamma\), \(\text{val}(b), \text{val}(d) \geq \gamma\), \(\text{val}(c) > \gamma\), so the coefficient is \(t^{-\gamma}A^\gamma O^0 O^d I\). To compute the coefficient of \(u_\delta\), we proceed similarly. Suppose that the product is \(\left(\begin{array}{cc} 0 & t^\delta \\ t^{-\delta} & 0 \end{array}\right)\). Then

\[
\left(\begin{array}{cc} x & y \\ z & w \end{array}\right) = \left(\begin{array}{cc} 0 & t^\delta \\ t^{-\delta} & 0 \end{array}\right) \left(\begin{array}{cc} d & -b \\ -c & a \end{array}\right) = \left(\begin{array}{cc} t^\delta d & -t^\delta a \\ -t^{-\delta} c & t^{-\delta} b \end{array}\right)
\]

The conditions are \(\text{val}(a) = \gamma\), \(\text{val}(b) \geq \gamma\), \(\text{val}(c) > \gamma\), \(\text{val}(d) \geq \gamma\), \(\text{val}(t^\delta d) > \text{val}(t^\delta b) = 0\). This implies \(\gamma \leq \text{val}(b) = -\delta\) and \(\text{val}(d) > \text{val}(b)\). We should also have \(ad - bc = 1\), hence \(0 = \text{val}(ad - bc) \geq \min\{\text{val}(ad), \text{val}(bc)\} > \gamma - \delta\). Hence it is neccessary that \(\gamma < \delta \leq -\gamma\).
Under this assumption, the conditions are \( \text{val}(a) = \gamma, \text{val}(b) = -\delta, \text{val}(c) > \gamma \) (since \( \text{val}(d) = \text{val}\left(\frac{\gamma}{a} + \frac{\delta}{x}\right) \geq \min\{-\gamma, -\delta\} > \gamma \)) and the coefficient is \( t^{-\gamma}A^\gamma A^{-\delta}O^\gamma_\theta \).

We make the following change of base:

\[
e_{\gamma} = \frac{1}{A^{-\gamma}}v_{-\gamma} \quad f_{\gamma} = \frac{1}{A^{\gamma}}u_{\gamma}
\]

for all \( \gamma \) and

\[
S_{\gamma} = O_{\delta}^0O_{\delta}^0A^\gamma R_{\gamma}qS_{-\gamma} = O_{\delta}^0O_{\theta}^0A^\gamma R_{\gamma}qS_{-\gamma} = O_{\delta}^0O_{\delta}^0A^\gamma R_{-\gamma}qS_{-\gamma} = O_{\delta}^0O_{\delta}^0A^\gamma R_{-\gamma}qS_{-\gamma}
\]

for \( \gamma < 0 \). \( R_{0}^- \) is defined using the third equality and not the forth. we get

**Corollary 4.9.** Let \( \gamma < 0 \). Then

1. \( e_0R_0 = e_0 \)
2. \( e_0R_{\gamma} = e_\gamma + \int_{\gamma < \delta < -\gamma} \frac{q-1}{q} f_{\delta} \)
3. \( e_0R_{-\gamma} = e_{-\gamma} \)
4. \( e_0R_0 = f_0 \)
5. \( e_0R_{-\gamma} = f_{\gamma} \)
6. \( e_0R_{-\gamma} = f_{-\gamma} + \int_{\gamma < \delta < -\gamma} (q-1)e_{\delta} \)
7. \( f_0R_0 = qe_0 + (q-1)f_0 \)

So the transformation \( h \mapsto v_0h \) from \( H \) to \( M \) is given by the following block matrix:

\[
\begin{pmatrix}
1d & 0 & X & A \\
0 & 1d & 0 & 0 \\
0 & 0 & 1d & 0 \\
B & Y & 0 & 1d
\end{pmatrix}
\]

where the blocks correspond to the partition \( R_{<}, R_{\geq}, R_{<}, R_{\geq} \) and \( e_{<}, e_{\geq}, f_{<}, f_{\geq} \). Here, for example, \( A \) is the transformation between two spaces with bases \( \{E_{\gamma}\}_{\gamma < 0} \) and \( \{E_\gamma\}_{\gamma > 0} \) which equals

\[
AE_{\gamma} = \frac{q-1}{q} \int_{\gamma < \delta < 0} E_{\delta}
\]

This transformation is invertible iff \( Id - AB \) is invertible. Now,

\[
(Id - AB)(E_{\gamma}) = E_{\gamma} - \frac{(q-1)^2}{q} \int_{\gamma < \eta < 0} \int_{\eta < \gamma} \Gamma_{\eta, \gamma}E_{\eta} \]

We look for inverse to \( Id - AB \) of the form

\[
E_{\gamma}' \mapsto E_{\gamma} + \int_{\gamma < \delta < 0} G(\gamma - \delta)E_{\delta}
\]

The condition on \( G \) is that it satisfies

\[
G(z) - \frac{(q-1)^2}{q} \int_{\gamma < \eta < 0} \Gamma_{\eta, \gamma}E_{\eta} = 0
\]

for every \( z < 0 \). The condition is the same for left and right inverse. There is such a function.

\[
\int_{x \in \gamma, 0} O^{\gamma - x}1_{[x, 0]} = \int_{x \in \gamma, 0} \int_{y \in [x, 0]} \int_{y \in \gamma, 0} \int_{x \in \gamma, 0} \int_{y \in \gamma, 0} A^{\gamma - x} = \frac{1}{q-1} \int_{y \in \gamma, 0} \int_{x \in \gamma, 0} O^{\gamma - y} - O^0_{\delta} = \frac{1}{q-1} \int_{y \in \gamma, 0} \int_{x \in \gamma, 0} A^{\gamma - y} - O^0_{\delta} = \]

Proof.\bigskip
by computing the action of both sides on \( \gamma \):

\[
\frac{q}{(q-1)^2} \int_{y \in (\gamma,0)} A^{\gamma - y} - \frac{q}{q-1} 1_{(\gamma,0)} \mathcal{O}_o^0 = \frac{q}{(q-1)^2} (\mathcal{O}_o^\gamma - \mathcal{O}_o^0) - \frac{q}{q-1} 1_{(\gamma,0)} \mathcal{O}_o^0 = 0.
\]

Similarly,

\[
\int_{x \in (\gamma,0)} \mathcal{O}^{\gamma - x} 1_{[x,0)} = \frac{1}{q-1} 1_{(\gamma,0)} \mathcal{O}_o^0 - \frac{1}{(q-1)^2} \mathcal{O}_o^0 + \frac{q}{(q-1)^2} \mathcal{O}_o^\gamma
\]

From which we see that

\[
G(\gamma) = \frac{q}{q^2-1} \left( \frac{1}{q} \mathcal{O}^\gamma - q \mathcal{O}_o^\gamma \right)
\]

satisfies the equation.

It follows from the above discussion that \( M \) is a rank one free module over \( \mathcal{H} \). In particular, \( \mathcal{H} = \text{End}_H(M) \).

**Corollary 4.10.** There is an embedding \( \Gamma \to \mathcal{H} \), denoted by \( \gamma \mapsto \tau_\gamma \) such that

\[
\tau_\gamma(v_\delta) = v_{\delta - \gamma} \quad \tau_\gamma(u_\delta) = u_{\delta - \gamma}
\]

**Proof.** \( \Gamma \) acts on \( A(\mathcal{O})N \setminus G/I = \{ \pm 1 \} \times (X_*(A) \otimes \Gamma) \) by translations, and hence acts on \( M \). The action is \( \tau_\gamma f(g) = f \left( \begin{pmatrix} t^{-\gamma} & 0 \\ 0 & t^\gamma \end{pmatrix} g \right) \). These transformations are endomorphisms (as left translation commutes with right convolution) so for any \( \gamma \) there is an unique element \( \tau_\gamma \) acting as the translation. Finally,

\[
(\tau_\gamma v_\delta)(g) = v_\delta \left( \begin{pmatrix} t^{-\gamma} & 0 \\ 0 & t^\gamma \end{pmatrix} g \right) = 1
\]

iff

\[
\left( \begin{pmatrix} t^{-\gamma} & 0 \\ 0 & t^\gamma \end{pmatrix} g \right) A(\mathcal{O})N \left( \begin{pmatrix} t^{-\delta} & 0 \\ 0 & t^\delta \end{pmatrix} I \right)
\]

iff

\[
g \in \left( \begin{pmatrix} t^{-\gamma} & 0 \\ 0 & t^\gamma \end{pmatrix} A(\mathcal{O})N \left( \begin{pmatrix} t^{-\delta} & 0 \\ 0 & t^\delta \end{pmatrix} I \right) = A(\mathcal{O})N \left( \begin{pmatrix} t^{-\gamma} & 0 \\ 0 & t^\delta \end{pmatrix} \right) \right)
\]

iff

\[
v_{\delta - \gamma}(g) = 1
\]

We have that

\[
\tau_\gamma(e_\delta) = \frac{A^{-\gamma}}{A^0} e_{\delta + \gamma} \quad \tau_\gamma(f_\delta) = \frac{A^{-\gamma}}{A^0} f_{\delta - \gamma}.
\]

This map extends to an embedding of \( F_n(\Gamma) \) into \( \mathcal{H} \) which is clearly an algebra homomorphism. Denote \( T_\gamma = \frac{A^{-\gamma}}{A^0} \tau_\gamma \). The \( T_\gamma \) act as translations on the \( e_\gamma, f_\gamma \)'s: \( T_\gamma e_\delta = e_{\gamma + \delta}, T_\gamma f_\delta = f_{\delta - \gamma} \). We also have \( T_\gamma T_\delta = T_{\gamma + \delta} \).

**Corollary 4.11.** \( (R_0^-)^2 = (q-1)R_0^- + qI \).

**Proof.** by computing the action of both sides on \( v_0 \). \( \square \)
We let $\hat{M}$ be the set of (definable) functions from $\mathcal{A}(O)N \backslash G/I$ that vanish on $v_\gamma, u_\gamma$ for $\gamma$ critical enough. It is clear that $\hat{M}$ is an $\mathcal{H}$ module but it is also a $\hat{H}$ module, where $\hat{H}$ is the obvious completion of $\mathcal{H}$. We define $\mathcal{I} : M \to \hat{M}$ by

$$\mathcal{I} \varphi(g) = \int_N \varphi(wng)dn$$

Where $N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right\}$ is the unipotent upper triangular matrices and $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the nontrivial element of the Weyl group.

**Lemma 4.12.** $\mathcal{I}$ is a well defined homomorphism of $\mathcal{H}$ modules. We have

1. $\mathcal{I}T_\gamma = T_{-\gamma} \mathcal{I}$
2. $\mathcal{I}e_0 = O_0^{\mathcal{A}}f_0 + \mathcal{A}^0 \int_{(0, \infty)} T_\gamma e_0$.
3. $\mathcal{I}f_0 = O_0^{\mathcal{A}}f_0 + \mathcal{A}^0 \int_{(0, \infty)} T_\gamma f_0$
4. $\mathcal{I}(e_0 + f_0) = (O_0^{\mathcal{A}}T_0 + \mathcal{A}^0 \int_{\gamma \in (0, \infty)} T_\gamma)(e_0 + f_0)$

**Proof.** (1) Denote $W_\gamma = \begin{pmatrix} t^{-\gamma} & 0 \\ 0 & t^\gamma \end{pmatrix}$. Then $wW_\gamma = W_{-\gamma}w$ and $W_\gamma \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}W_{-\gamma} = \begin{pmatrix} 1 & t^{-2}\gamma x \\ 0 & 1 \end{pmatrix}$.

$$\mathcal{I} \tau_\gamma f(g) = \int_N (\tau_\gamma f)(wng)dn = \int_N f(W_\gamma wng)dn = \int_N f(wW_{-\gamma}nW_\gamma W_{-\gamma}g)dn$$

The change of coordinates $m = W_{-\gamma}nW_\gamma$ satisfies $dm = A^{-2\gamma}_0 \mathcal{A}^0dn$

$$= A^{-2\gamma}_0 \int_N f(wmW_{-\gamma}g)dn = A^{\mathcal{A}}_{-2\gamma} \tau_{-\gamma} \mathcal{I} f(g)$$

So

$$\mathcal{I}T_\gamma = \frac{A^\tau}{\mathcal{A}} \mathcal{I} \tau_\gamma = \frac{A^\tau A^{-2\gamma}_0}{\mathcal{A}^0} \tau_{-\gamma} \mathcal{I} = T_{-\gamma} \mathcal{I}$$

(2) Let $n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$. Then $wn = \begin{pmatrix} 0 & 1 \\ -1 & -x \end{pmatrix}$. To compute the coefficient of $v_\gamma$, suppose

$$wn \begin{pmatrix} t^{-\gamma} & 0 \\ 0 & t^\gamma \end{pmatrix} = \begin{pmatrix} 0 & t^\gamma \\ -t^{-\gamma} & -xt^\gamma \end{pmatrix} \in AN I$$

Then $-\gamma = val(t^{-\gamma}) > val(xt^\gamma) = 0$. Hence $\gamma < 0$ and the measure of $x$'s that contribute is $\mathcal{A}^{-\gamma}$. If, on the other hand,

$$wn \begin{pmatrix} 0 & t^{-\gamma} \\ -t^\gamma & 0 \end{pmatrix} = \begin{pmatrix} t^\gamma & 0 \\ -t^{-\gamma} & -xt^\gamma \end{pmatrix} \in AN I$$

Then $val(xt^\gamma) > val(t^{-\gamma}) = 0$, hence $\gamma = 0$ and the measure of $x$'s that contributes is $O_0^{\mathcal{A}}$. Hence $\mathcal{I}(v_\gamma) = O_0^{\mathcal{A}}v_\gamma + \mathcal{I}_{(\gamma, 0)}^{\mathcal{A}}, A^{-\gamma}v_\gamma$, so $\mathcal{I}e_0 = O_0^{\mathcal{A}}f_0 + \mathcal{A}^0 \int_{(0, \infty)} T_\gamma e_0$.

(3) Similarly, assume

$$wn \begin{pmatrix} t^{-\gamma} & 0 \\ 0 & t^\gamma \end{pmatrix} = \begin{pmatrix} 0 & t^\gamma \\ -t^{-\gamma} & -xt^\gamma \end{pmatrix} \in AN wI$$

Then $0 = val(t^{-\gamma}) \leq val(xt^\gamma)$, so $\gamma = 0$ and the measure of $x$'s is $O_0^{\mathcal{A}}$. If, on the other hand,

$$wn \begin{pmatrix} 0 & t^{-\gamma} \\ -t^\gamma & 0 \end{pmatrix} = \begin{pmatrix} t^\gamma & 0 \\ -xt^\gamma & -t^{-\gamma} \end{pmatrix} \in AN wI$$
Then $0 = \text{val}(xt^\gamma) \leq \text{val}(t^{-\gamma}) = -\gamma$, so $\gamma \leq 0$ and the measure of $x$’s is $A^{-\gamma}$. Hence $\mathcal{I}u_0 = O^0_\mathcal{I}u_0 + \int_{(-\infty,0]} A^{-\gamma}u_0$. This implies that $\mathcal{I}f_0 = O^0_\mathcal{I}e_0 + A^0 \int_{(0,\infty)} T_\gamma f_0$. (4) follows from (2) and (3).

Note that $\mathcal{I}$ does not preserve $M$. However, we claim that the operator $J_b = (1 - T_b)\mathcal{I}$ preserves $M$ for every $b \in \Gamma$. Take for example $b > 0$. By computing the action on $e_0$ we see that

$$J_b = O_a(1 - T_b)R_0^+ + A^0 \int_{(0,b]} T_\gamma.$$

Fix $a \in \Gamma$ and let $b > 0$ be smaller in absolute value. Using $J_b T_a = T_{-a}J_b$ and the last equality we get

$$(1 - T_b)O^0_a R_0^- T_a + A^0 \int_{(a,a+b]} T_\gamma = (1 - T_b)O^0 T_{-a}R_0^- + A^0 \int_{(-a,-a+b]} T_\gamma$$

and so if $a > 0$,

$$(1 - T_b)O_a(R_0^- T_a - T_{-a}R_0^-) = A^0 \left( \int_{(-a,-a+b]} T_\gamma - \int_{(a,a+b]} T_\gamma \right) = (1 - T_b)A^0 \int_{(-a,a]} T_\gamma$$

and if $a < 0$,

$$(1 - T_b)O_a(R_0^- T_a - T_{-a}R_0^-) = -(1 - T_b)A^0 \int_{(a,-a]} T_\gamma$$

**Lemma 4.13.** The element $1 - T_b$ does not annihilate non zero elements of $H$.

**Proof.** Suppose $X \in H$ is non zero. We can view $X$ as a definable function from $\{\pm 1\} \times \Gamma$ to $E$. The support of $X$ is a definable set, hence there is a supremum $\gamma$ for it. Let $\epsilon \in \Gamma$ be positive and smaller than $b$ such that $X(\gamma - \epsilon) \neq 0$. Then $(1 - T_b)X(\gamma + b - \epsilon) \neq 0$, so $(1 - T_b)X \neq 0$. □

**Corollary 4.14.** (Bernstein’s presentation) Every element in $H$ is of the form $\int_{\Gamma} f_1(\gamma) T_\gamma + \int_{\Gamma} f_w(\gamma)T_aR_0^-$. Multiplication is defined by being $\Gamma$-additive and the relations

$$R_0^- T_a = T_{-a}R_0^- + (q - 1) \int_{(-a,a]} T_\gamma$$

and

$$(R_0^- - q)(R_0^- + 1) = 0$$

**Proposition 4.15.** The center of $H$ consists of all elements of the form $\int_{\Gamma} f(\gamma)(T_\gamma + T_{-\gamma})$.

**Proof.** Denote by $L$ the algebra (or space) generated by the $T_\gamma + T_{-\gamma}$. Clearly, $L$ is contained in the center. On the other hand, every element in $H$ can be uniquely written as a combination of elements of the form $T_\gamma + T_{-\gamma}$, $T_\gamma - T_{-\gamma}$, $(T_\gamma + T_{-\gamma})R_0^-$, $(T_\gamma - T_{-\gamma})R_0^-$ (note that $T_\gamma e_0 = e_\gamma$ and $R_0^- e_0 = f_0$). Every one of those subspaces is $L$ invariant and they are linearly independent. □

**Corollary 4.16.** The algebra $H$ is finite over its center.

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