\textbf{K-THEORETIC EXCEPTIONAL COLLECTIONS AT ROOTS OF UNITY}

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\textbf{Abstract.} Using cyclotomic specializations of the equivariant $K$-theory with respect to a torus action we derive congruences for discrete invariants of exceptional objects in derived categories of coherent sheaves on a class of varieties that includes Grassmannians and smooth quadrics. For example, we prove that if $X = \mathbb{P}^{n_1-1} \times \ldots \times \mathbb{P}^{n_k-1}$, where $n_i$'s are powers of a fixed prime number $p$, then the rank of an exceptional object on $X$ is congruent to $\pm 1$ modulo $p$.

1. Introduction

This paper is concerned with $K$-theory classes of exceptional objects in the derived category of coherent sheaves $D(X) := D^b(\text{Coh} X)$ on a smooth projective variety $X$. Recall that an object $E$ of a $k$-linear triangulated category is called \textit{exceptional} if $R\text{Hom}(E, E) = k$. An \textit{exceptional collection} is a collection of exceptional objects $(E_1, \ldots, E_n)$ such that $R\text{Hom}(E_i, E_j) = 0$ for $i > j$. An exceptional collection is called \textit{full} if it generates the entire triangulated category. For example, $(\mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n))$ is a full exceptional collection in $D(\mathbb{P}^n)$ (see [1]). We refer to [5] for more background on exceptional collections (see also section 3.1 of [4] for a brief introduction). There is a naturally defined action of the braid group $B_n$ on the set of exceptional collections of length $n$ given by mutations. It is conjectured that in the case of full exceptional collections this action is transitive (where each object is considered up to a shift $E \mapsto E[m]$). We refer to this property as \textit{constructibility}. This property is known only in some low-dimensional cases (e.g., it is checked for Del Pezzo surfaces in [11]). Note that if $X$ is a smooth projective variety then for a full exceptional collection $(E_1, \ldots, E_n)$ in $D(X)$ the classes $([E_i])$ in the Grothendieck group $K_0(X)$ form a basis over $\mathbb{Z}$. Furthermore, this basis is semiorthogonal with respect to the Euler bilinear form $\chi([V], [W]) := \chi(X, V^* \otimes W)$ on $K_0(X)$, i.e., we have $\chi([E_i], [E_j]) = 0$ for $i > j$, $\chi([E_i], [E_i]) = 1$. One still has an action of the braid group on the set of semiorthogonal bases in $K_0(X)$, so the problem of constructibility can be formulated at the level of $K_0(X)$. Even for this question very little is known ([17] seems to be the only work dealing with 3-dimensional examples). For example, this problem is open for projective spaces of dimension $\geq 4$.

In the case when $X$ admits an action by an algebraic torus $T = \mathbb{G}_m^r$, every exceptional object in $D(X)$ can be equipped with a $T$-equivariant structure, i.e., comes from an object of $D^b(\text{Coh}^T(X))$ (see Lemma 2.2). Thus, the constructibility question can also be asked for bases in the $T$-equivariant $K$-group $K^T_0(X)$, semiorthogonal with respect to the equivariant Euler form. Note that $K^T_0(X)$ is a module over the representation ring $R = R(T) \simeq \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. The main observation of this paper is that in some situations one can choose carefully an element $t_0 \in T$ of finite order $N$ such that after the specialization with respect to the homomorphism $\text{Tr}(t_0, ?) : R \rightarrow \mathbb{Z}[\sqrt[N]{\mathbb{T}}]$ the equivariant Euler form becomes Hermitian (and positive-definite). This means that every full exceptional collection provides an orthonormal basis of $K^T_0(X) \otimes_R \mathbb{Z}[\sqrt[N]{\mathbb{T}}]$ with respect to the specialization of the Euler form. The action of the braid group in this specialization reduces to the action by permutations of basis vectors (up to a sign). Hence, the analog of constructibility at this level should assert that the obtained orthonormal basis does not depend

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on a full exceptional collection up to rescaling. We observe that this is indeed the case because of the following simple consequence of Kronecker’s theorem (see Proposition 2.6): if \( z_1, \ldots, z_n \) is a set of cyclotomic integers such that \( |z_1|^2 + \ldots + |z_n|^2 = 1 \) then all \( z_i \)'s are zero except one which is a root of unity.

Examples of the above situation include some homogeneous spaces (e.g., Grassmannians and quadrics), Hirzebruch surfaces \( F_n \) for even \( n \), as well as products of such varieties. In the case when \( N \) is a power of a prime \( p \) we can make further specialization to \( K_0(X) \otimes \mathbb{Z}/p\mathbb{Z} \). This way we get congruences modulo \( p \) for classes of exceptional objects (see Corollary 1.2 and Theorem 1.3 for the case of projective spaces and their products).

Here is a precise statement of our main result.

**Theorem 1.1.** Let \( X \) be a smooth projective variety over \( \mathbb{C} \) equipped with an action of an algebraic torus \( T \). Assume that the set of invariant points \( X^T \) is finite and there exists an element of finite order \( t_0 \in T \) such that

\[
(\ast) \text{ for every } p \in X^T \text{ one has } \det(1 - t_0, T_pX) \neq 0 \text{ and } \det(t_0, T_pX) = (-1)^{\dim X}, \text{ where } T_pX \text{ is the tangent space at } p.
\]

Consider the homomorphism

\[
\text{Tr}(t_0, \cdot) : R \to \mathbb{Z}[\sqrt[\nu]{\tau}],
\]

where \( N \) is the order of \( t_0 \), and set \( K_{t_0} = K_0^T(X) \otimes_R \mathbb{Z}[\sqrt[\nu]{\tau}] \). Then

(i) the equivariant Euler form \( \chi^T(\cdot, \cdot) \) on \( K_0^T(X) \) specializes to a Hermitian form on \( K_{t_0} \) with values in \( \mathbb{Z}[\sqrt[\nu]{\tau}] \).

(ii) If \( E \) is an exceptional object of \( D(X) \) equipped with a \( T \)-equivariant structure (i.e., a lift to an object of \( D^b(\text{Coh}^T X) \)) then the class \( v(E) \) of \( E \) in \( K_{t_0} \) has length 1 with respect to the Hermitian form in (i). If \( (E_1, E_2) \) is an exceptional pair in \( D(X) \), where both \( E_1 \) and \( E_2 \) are equipped with a \( T \)-equivariant structure, then the vectors \( v(E_1) \) and \( v(E_2) \) are orthogonal with respect to this form.

(iii) Assume \( D(X) \) admits a full exceptional collection \( (E_1, \ldots, E_n) \) where each \( E_i \)'s is equipped with a \( T \)-equivariant structure, and let \( (v_1 = v(E_1), \ldots, v_n = v(E_n)) \) be the corresponding orthonormal \( \mathbb{Z}[\sqrt[\nu]{\tau}] \)-basis of \( K_{t_0} \). Then every unit vector in \( K_{t_0} \) is of the form \( \pm \zeta v_i \) for some \( i \) and some \( N \)th root of unity \( \zeta \).

(iv) In the situation of (iii) assume in addition that the action of \( T \) on \( X \) extends to an action of an algebraic group \( N \supset T \), such that \( T \) is a normal subgroup in \( N \) and for some element \( w \in N \) one has \( wt_0w^{-1} = t_0^{-1} \). Assume in addition that all the objects \( E_i \) admit a \( N \)-equivariant structure. Then for every exceptional object \( E \) of \( D(X) \) admitting an \( N \)-equivariant structure one has \( v(E) = \pm v_i \) for some \( i \).

**Remarks.**

1. We will determine for which generalized Grassmannians \( G/P \) (where \( P \) is a maximal parabolic subgroup) there exists an element \( t_0 \) satisfying \((\ast)\) in Proposition 3.6, leaving out only several cases with \( G \) of type \( E_7 \) and \( E_8 \). In particular, for \( G \) of classical type the spaces we get are either Grassmannians, or smooth quadrics or (connected components) of maximal isotropic Grassmannians, orthogonal or symplectic.

2. The assumptions of part (iv) are often easy to check when \( T \) is a maximal torus of a simply connected semisimple group \( G \) acting on \( X \): it suffices to find an element of the Weyl group \( W \) sending \( t_0 \) to \( t_0^{-1} \) (since in this case every exceptional object admits a \( G \)-equivariant structure by Lemma 2.2). In almost all of the cases considered in Proposition 3.6 this holds for \( wt_0 \), the element of maximal length in \( W \) (the exception is the case of type \( A_n \) where one should take a different element of \( W \)).

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1 Rescaling by a root of unity corresponds to changing a \( T \)-equivariant structure on an exceptional object. An additional sign may appear as an effect of mutations.
3. Assume that \( (X', T', t'_0 \in T') \) is another data such that the assumption (\( \ast \)) of Theorem 1.1 is satisfied. Let equip \( X \times X' \) with the natural action of the torus \( T \times T' \). Then the element \((t_0, t'_0) \in T \times T' \) will still satisfy the assumption (\( \ast \)).

In the case when the order of \( t_0 \) in the above Theorem is a power of prime we can derive some congruences in the usual Grothendieck group of \( X \).

**Corollary 1.2.** In the situation of Theorem 1.1(iii) assume in addition that the order of \( t_0 \) equals \( p^k \), where \( p \) is a prime. Then the reduction of the Euler form \( \chi(\cdot, \cdot) \) modulo \( p \) is symmetric. Furthermore, for every exceptional object \( E \) of \( D(X) \) one has the following congruence in \( K_0(X) \otimes \mathbb{Z}/p\mathbb{Z} \):

\[
[E] \equiv \pm [E_i]
\]

for some \( i \in [1, n] \). Also, for such an object one has

\[
\sum_{i=1}^{n} \chi(E_i, E) \equiv \pm 1 \mod(p).
\]

**Remark.** Note that in the situation of the above Corollary there are typically more vectors of length 1 in \( K_0(X) \otimes \mathbb{Z}/p\mathbb{Z} \) than just those coming from exceptional objects, so the congruence (1.1) cannot be obtained by just looking at \( K_0(X) \otimes \mathbb{Z}/p\mathbb{Z} \). On the other hand, in the case \( p = 2 \) the fact that the ranks of exceptional objects are odd can often be checked only with the help of \( K_0(X) \otimes \mathbb{Z}/2\mathbb{Z} \) — see Remark after Corollary 3.17.

In most of our examples the torus \( T \) is a maximal torus in a connected reductive group \( G \) acting on \( X \) in such a way that the center \( Z_G \subset G \) acts trivially. In this situation we have a decomposition of the category \( \text{Coh}^G(X) \) of \( G \)-equivariant coherent sheaves (and of its derived category) into the direct sum of subcategories indexed by characters of \( Z_G \). We say that an object \( V \in D(X/G) := D^b(\text{Coh}^G(X)) \) is central if it belongs to one of these subcategories, and we call the corresponding character \( \chi : Z_G \to \mathbb{G}_m \) the central character of \( V \) (when \( V \) is a \( G \)-equivariant coherent sheaf this means that \( Z_G \) acts on \( V \) through \( \chi \)). For example, anyindecomposable object in \( D(X/G) \) is central. Note that in the case when the commutator of \( G \) is simply connected every exceptional object in \( D(X) \) admits a \( G \)-equivariant structure (see Lemma 2.2), and hence can be viewed as a central object of \( D(X/G) \). Using a \( G \)-equivariant structure often allows to extract more precise information on the class of an exceptional object in \( K_0^G(X) \) (see Propositions 3.2, 3.3).

In section 3 we will consider some concrete examples when the conditions of Theorem 1.1 are satisfied. In the case of projective spaces we will prove the following result.

**Theorem 1.3.** (i) Let \( p \) be a prime, \( n = p^k \), and let \( V \) be a central object in \( D(\mathbb{P}^{n-1}/\text{GL}_n) \). Then one has the following congruence in \( K_0(\mathbb{P}^{n-1}) \otimes \mathbb{Z}/p\mathbb{Z} \):

\[
[V] \equiv \begin{cases} 
0, & \text{rk}(V) \equiv 0 \mod(p), \\
\text{rk}(V)[\mathcal{O}(\text{deg}(V))/\text{rk}(V)], & \text{rk}(V) \not\equiv 0 \mod(p),
\end{cases}
\]

where we use the fact that the class of \( \mathcal{O}(m) \) in \( K_0(\mathbb{P}^{n-1}) \otimes \mathbb{Z}/p\mathbb{Z} \) depends only on \( m \mod(n) \).

(ii) If in the above situation \( E \) is an exceptional object of \( D(\mathbb{P}^{n-1}) \) then \( \text{rk}(E) \equiv \pm 1 \mod(p) \). If \((E_1, E_2)\) is an exceptional pair then

\[
\frac{\text{deg}(E_1)}{\text{rk}(E_1)} \not\equiv \frac{\text{deg}(E_2)}{\text{rk}(E_2)} \mod(p^k),
\]

and \( \chi(E_1, E_2) \equiv 0 \mod(p) \). For a full exceptional collection \((E_1, \ldots, E_n)\) in \( D(\mathbb{P}^{n-1}) \) the slopes modulo \( p^k \) \( (\text{deg}(E_i)/\text{rk}(E_i) \mod(p^k)) \) form a complete system of remainders modulo \( p^k \).
(iii) Let \( p \) be a prime and let \( X = \mathbb{P}^{n_1-1} \times \ldots \times \mathbb{P}^{n_k-1} \), where \( n_i = p^{k_i} \). Then for an exceptional object \( E \in D(X) \) the class of \( E \) in \( K_0(X) \otimes \mathbb{Z}/p\mathbb{Z} \) coincides up to a sign with the class of some line bundle. In particular, \( \text{rk}(E) \equiv \pm 1 \mod(p) \).

There is a relative version of this result for the product of a projective space with a smooth projective variety (see Theorem 3.18). We will also consider other situations where Theorem 1.1 can be applied. For example, we will prove that if \( p \) is a prime then the rank of an exceptional object on the product of Grassmannians \( G(k_1, p) \times \ldots \times G(k_s, p) \) is not divisible by \( p \). (see Theorem 3.9). In the case of Grassmannians and smooth quadrics we derive some relations between the rank and the central character of an exceptional object. For example, we prove that an exceptional object on the smooth quadric of dimension \( 2^r \) (with \( r \geq 2 \)) can be equipped with an \( SO(2^r + 2) \)-equivariant structure if and only if it has an odd rank (see Proposition 3.11). For maximal isotropic Grassmannians (orthogonal and symplectic) full exceptional collections have been constructed only in few cases, so our results are more limited. For example, we show that all exceptional objects in the derived category of \( OG(5, 10) \) (resp., \( SG(3, 6) \)) have an odd rank (see Theorem 3.16). As examples of non-homogeneous varieties we consider Hirzebruch surfaces \( F_n \) with even \( n \). We show that Theorem 1.1 applies in this case with \( N = 4 \), and hence the class of every exceptional object in \( K_0(F_n) \otimes \mathbb{Z}/2\mathbb{Z} \) coincides with one of the 4 classes coming from line bundles. Similar result holds for products of such surfaces, as well as for their products with projective spaces of dimension \( 2^k - 1 \) (see Corollary 3.17). In particular, the rank of every exceptional object on such products is odd.

Notations and conventions. We work with smooth projective varieties over \( \mathbb{C} \). We denote by \( \zeta_n \) a primitive \( n \)-th root of unity in \( \mathbb{C} \) and by \( \mathbb{Z}[\sqrt{f}] \subset \mathbb{C} \) the subring generated by \( \zeta_n \). \( R(G) \) denotes the representation ring (over \( \mathbb{Z} \)) of an algebraic group \( G \). For an algebraic group \( G \) acting on a variety \( X \) we always assume the existence of a \( G \)-equivariant ample line bundle on \( X \). We denote by \( \text{Coh}^G(X) \) (resp., \( \text{Coh}(X) \)) the category of \( G \)-equivariant (resp., usual) coherent sheaves on \( X \) and by \( D(X/G) \) (resp., \( D(X) \)) its bounded derived category.

2. Lefschetz formula in equivariant \( K \)-theory and the proof of Theorem 1.1

Let \( X \) be a smooth projective variety equipped with an action of an algebraic group \( G \). We denote by \( K^G_0(X) \) the Grothendieck group of the category of \( G \)-equivariant coherent sheaves on \( X \). We always assume that \( X \) admits a \( G \)-equivariant ample line bundle (this is automatic if \( G \) is linear algebraic, see [7]). In this case every \( G \)-equivariant coherent sheaf admits a finite resolution by \( G \)-equivariant bundles, so \( K^G_0(X) \) can also be defined using \( G \)-equivariant bundles. We can view the group \( K^G_0(X) \) as a module over \( R(G) \) in a natural way. It is also equipped with the commutative product induced by the tensor product and with the involution \( [V] \mapsto [V^*] \), where \( V^* \) is the dual vector bundle to \( V \). Since cohomology of a \( G \)-equivariant coherent sheaf is equipped with \( G \)-action, we obtain the \( G \)-equivariant Euler characteristic \( \chi^G(V) = \chi^G(X,V) = \sum_{i \geq 0} (-1)^i[H^i(X,V)] \) with values in \( R(G) \). Similarly, we have an equivariant version of the Euler bilinear form with values in \( R(G) \) defined by \( \chi^G(V,W) = \chi^G([V^* \otimes W]) \).

An object \( V \) of \( D(X) = D^b(\text{Coh} X) \) (resp., of \( D(X/G) = D^b(\text{Coh}^G X) \)) has an associated class \( [V] \) in \( K_0(X) \) (resp., \( K^G_0(X) \)). By a \( G \)-equivariant structure on an object \( V \in D(X) \) we mean an object \( \tilde{V} \in D(X/G) \) such that the image of \( \tilde{V} \) under the forgetful functor \( D(X/G) \to D(X) \) is isomorphic to \( V \).

Lemma 2.1. Let \( (E_1, \ldots, E_n) \) be a full exceptional collection in \( D(X) \). Assume that each \( E_i \) is equipped with a \( G \)-equivariant structure. Then the classes \( ([E_i]) \) form a basis of \( K^G_0(X) \) as \( R(G) \)-module.
Proof. Recall that with every exceptional object \( E \) in \( D(X) \) one associates a functor \( L_E \) from \( D(X) \) to itself, so that there is a distinguished triangle (functorial in \( F \))

\[
L_E(F) \to R\text{Hom}(E,F) \otimes E \to F \to L_E(F)[1].
\]

Moreover, we have the distinguished triangle of the corresponding kernels in \( D(X \times X) \):

\[
K \to E^* \boxtimes E \to \Delta_* \mathcal{O}_X \to K[1],
\]

where \( E^* = R\text{Hom}(E,\mathcal{O}_X) \). Now assume that \( E \) is equipped with a \( G \)-equivariant structure. Then we can represent \( E \) by a complex of \( G \)-equivariant vector bundles. Note that the canonical morphism \( E^* \boxtimes E \to \Delta_* \mathcal{O}_X \) of complexes of sheaves on \( X \times X \) is compatible with the diagonal action of \( G \). Hence, there is an analog of (2.1) in \( D(X/G) \) with \( R\text{Hom}(E,F) \) replaced by \( R\pi_*(E^* \otimes F) \in D(R(G) - \text{mod}) \), where \( \pi : X \to \text{Spec}(\mathbb{C}) \) is the structure morphism. The operator on \( K_0^G(X) \) corresponding to the functor \( F \mapsto L_E(F) \) is given by \( [F] \mapsto \chi^G([E],[F]) \cdot [E] - [F] \). Hence, the fact that \( L_{E_1} \ldots L_{E_n} F = 0 \) for every \( F \) translates into the assertion that the classes \( ([E_i]) \) span \( K_0^G(X) \) over \( R(G) \). Using the semiorthogonality condition \( \chi^G(E_i,E_j) = 0 \) for \( i > j \), \( \chi^G(E_i,E_i) = 1 \) for all \( i \), one easily checks linear independence of these classes over \( R(G) \). \( \square \)

Thus, it is important to know when exceptional objects can be equipped with equivariant structures. The following result shows that this is always possible for connected reductive groups \( G \) with simply connected commutant.

**Lemma 2.2.** (i) Let \( X \) be a smooth projective variety equipped with an action of a linear algebraic group \( G \), and let \( E \) be an exceptional object of \( D(X) \). Assume that \( G \) has trivial Picard group and has no nontrivial central extensions by \( \mathbb{G}_m \) in the category of algebraic groups. Then \( E \) admits a \( G \)-equivariant structure, unique up to tensoring with a character of \( G \).

(ii) The above assumptions on \( G \) are satisfied if \( G \) is a connected reductive group with \( \pi_1(G) \) torsion free, or equivalently, with simply connected commutant \( D\mathcal{G} \).

**Proof.** (i) We will use the algebraic stack \( \mathcal{M} \) “parametrizing” objects \( E \in D(X) \) with \( \text{Hom}^i(E,E) = 0 \) for \( i < 0 \) (see [13], we could also use the stack defined in [6]). More precisely, in the terminology of [13] \( \mathcal{M} \) represents the functor of universally glueable relatively perfect complexes. Consider the pull-back \( a^* E \) of \( E \) via the action map \( a : G \times X \to X \). By Proposition 2.19 of [13], \( a^* E \) is a family over \( G \) of the above type. Hence, we have the corresponding morphism \( G \to \mathcal{M} \). Since the tangent space to \( \mathcal{M} \) at \( E \) is \( \text{Hom}^1(E,E) = 0 \), it follows that this morphism is constant, so the objects \( a^* E \) and \( p_2^* E \) on \( G \times X \) become isomorphic over a covering of \( G \) in flat topology. Note that if \( U \to G \) is one of the elements of this cover then automorphisms of \( p_2^*E \) over \( U \) reduce to multiplication by invertible functions on \( U \). Now the triviality of \( \text{Pic}(G) \) implies that we can choose a global isomorphism \( \alpha : a^* E \to p_2^* E \). Next, we should try to check the cocycle condition for \( \alpha \) on \( G \times G \times X \). The obstacle will be some group 2-cocycle of \( G \) with values in \( \mathbb{G}_m \). By the assumptions \( G \) has no central extensions by \( \mathbb{G}_m \), hence we can adjust \( \alpha \) so that the cocycle condition will be satisfied. It remains to use the argument of Theorem 3.2.4 of [2] to deduce that \( E \) can be represented by a complex of \( G \)-equivariant sheaves. More precisely, we observe that \( G \)-equivariant sheaves of \( \mathcal{O} \)-modules on \( X \) can be viewed as strict simplicial systems of sheaves of \( \mathcal{O} \)-modules over the simplicial system \( (G^n \times X) \) associated with the action of \( G \) on \( X \). The above construction extends \( E \) to a strict simplicial system in the corresponding derived category. Now we can apply the argument of Theorem 3.2.4 of [2] observing that the vanishing condition used in Proposition 2.9 of loc. cit. boils down to the vanishing of \( \text{Hom}^i(E,E) \) for \( i < 0 \) (similar vanishing for the pull-back of \( E \) to \( G^n \times X \) follows by the Künneth formula).

The uniqueness part is checked as follows. Suppose \( E \) and \( E' \) are objects of \( D(X/G) \) that become isomorphic in \( D(X) \) and assume that \( \text{Hom}_{D(X)}(E,E) = \mathbb{C} \). Then \( \text{Hom}_{D(X)}(E,E') \) is a one-dimensional representation of \( G \). Hence, after tensoring \( E' \) with a character of \( G \) we will get a morphism \( f : E \to E' \) in \( D(X/G) \) such that \( f \) induces an isomorphism in \( D(X) \), so \( f \) itself is an isomorphism.
(ii) The equivalence of two conditions on \( G \) follows from Corollary 1.7 of [15]. The triviality of the Picard group follows from Proposition 1.10 of [15] (see also [18]). Now let \( 1 \to \mathbb{G}_m \to \tilde{G} \to G \to 1 \) be a central extension of \( G \) by \( \mathbb{G}_m \). Then the derived group \( \mathcal{D}G \) of \( \tilde{G} \) is a connected semisimple group and we have an isogeny \( \mathcal{D}G \to DG \). Since \( DG \) is simply connected this implies that the above extension splits over \( \mathcal{D}G \subset G \). Therefore, it is induced by a central extension of the torus \( G/DG \) by \( \mathbb{G}_m \). Hence, we are reduced to the case when \( G \) is a torus. In this case \( \tilde{G} \) is a connected solvable algebraic group with trivial unipotent radical, hence, \( \tilde{G} \) is itself a torus. Therefore, the above sequence splits. \( \square \)

Let \( X \) be a smooth projective variety equipped with an action of an algebraic torus \( T \simeq \mathbb{G}_m^r \) that has a finite number of stable points \( X^T \), and let \( R = R(T) \). The usual (non-equivariant) \( K \)-group can be recovered from the equivariant one due to the following result of Merkurjev.

**Lemma 2.3.** ([15], Cor.4.4) The natural map

\[
K_0^T(X) \otimes_R \mathbb{Z} \to K_0(X),
\]

induced by the homomorphism \( \text{Tr}(1,?) : R \to \mathbb{Z} \), is an isomorphism.

Consider the natural map of \( R \)-modules

\[
K_0^T(X) \to \bigoplus_{p \in X^T} R
\]

given by the restriction to \( X^T \). It is well known that it becomes an isomorphism after tensoring over \( R \) with the quotient field of \( R \) (see [16] Theorem 3.2, generalizing [20] Prop. 4.1 in topological case). Now let \( t_0 \in T \) be an element of finite order \( N \). Specializing (2.2) with respect to the homomorphism \( \text{Tr}(t_0, ?) : R \to \mathbb{Z}[\sqrt[N]{1}] \) we get a map

\[
K_{t_0} = K_0^T(X) \otimes_R \mathbb{Z}[\sqrt[N]{1}] \to \bigoplus_{p \in X^T} \mathbb{Z}[\sqrt[N]{1}]
\]

**Lemma 2.4.** Assume that \( K_0^T(X) \) is a free \( R \)-module, and \( \det(1-t_0, T_pX) \neq 0 \) for every \( p \in X^T \). Then the map (2.3) is injective and becomes an isomorphism after tensoring with \( \mathbb{Q} \).

**Proof.** First, Theorem 3.2 of [16] easily implies that in the case when \( K_0^T(X) \) is a free \( R \)-module the map (2.2) becomes an isomorphism after inverting in \( R \) all elements \( \chi - 1 \), where \( \chi \) is a character occuring in the \( T \)-action on one of the tangent spaces \( T_pX \) for \( p \in X^T \). Therefore, under our assumptions on \( t_0 \) the map (2.3) becomes an isomorphism after tensoring with \( \mathbb{Q} \). Since it is a map of free \( \mathbb{Z}[\sqrt[N]{1}] \)-modules, the injectivity follows. \( \square \)

For a class \( F \in K_0^T(X) \) we denote by \( v(F) \) the corresponding class in \( K_{t_0} \). We also set \( v(F)_p = \text{Tr}(t_0, F|_p) \in \mathbb{Z}[\sqrt[N]{1}] \) for \( p \in X^T \), so that the map (2.3) sends \( v(F) \) to \( (v(F)_p)_{p \in X^T} \).

**Lemma 2.5.** Under the assumptions of Lemma 2.4 for a \( T \)-equivariant line bundle \( L \) on \( X \) the class \( v(L^m) \in K_{t_0} \) depends only on the remainder \( m \mod (N) \).

**Proof.** For each \( p \in X^T \) let \( \chi_p \) be the character of the torus \( T \) corresponding to its action on \( L|_p \). Then \( v(L^m)_p = \chi_p(t_0)^m \) depends only on the remainder \( m \mod (N) \). Now the assertion follows from the injectivity of (2.3) in this case. \( \square \)

For a \( T \)-equivariant vector bundle \( V \) on \( X \) (and hence, for an object of \( D(X/T) \)) we have the following Lefschetz type formula for the equivariant Euler characteristic:

\[
\chi^T(X,V) = \sum_{p \in X^T} \det(1-t, T_pX)^{-1}[V|_p],
\]

(2.4)
where \([V_p] \in R\) (see [16], 4.9). Hence, for \(t_0\) such that \(\det(1 - t_0, T_p(X)) \neq 0\) for all \(p \in X^T\), for classes \([V], [W] \in K^0_f(X)\) one has

\[
\text{Tr}(t_0, \chi^T(V, W)) = \sum_{p \in X^T} c_p v(V^*)_p v(W)_p,
\]

(2.5)

where \(c_p = \det(1 - t_0, T_pX)^{-1}\). Since the eigenvalues of \(t_0\) are roots of unity, we have \(v(V^*)_p = \overline{v(V)_p}\)
(where bar denotes complex conjugation). Let us equip the free \(\mathbb{Z}[\sqrt{T}]\)-module \(\oplus_{p \in X^T} \mathbb{Z}[\sqrt{T}]\) with the sesquilinear form

\[
h(v, w) = \sum_{p \in X^T} c_p \overline{v_p} w_p,
\]

and let \(H\) denote the pull-back of this form to \(K_{t_0}\) via (2.3). Then we can rewrite (2.5) as

\[
\text{Tr}(t_0, \chi^T(V, W)) = H(v(V), v(W)).
\]

(2.6)

Recall that by Serre duality we have

\[
\chi^T(V, W)^* = (-1)^{\dim X} \chi^T(W, V \otimes \omega_X).
\]

Taking traces of \(t_0\) and using (2.6) we obtain

\[
H(v(V), v(W)) = (-1)^{\dim X} H(v(V), v(V \otimes \omega_X)).
\]

The condition (*) implies that \(t_0\) acts as \((-1)^{\dim X}\) on the fibers of \(\omega_X\) at all points \(p \in X^T\). Hence, \(v(V \otimes \omega_X)_p = (-1)^{\dim X} v(V)_p\) for all \(p \in X^T\), and the above equation becomes

\[
H(v(V), v(W)) = H(v(V), v(V)),
\]

which implies part (i) of Theorem 1.1 because of (2.6).

Now part (ii) follows from the observation that if \(E\) is exceptional then \(\chi^T(E, E) = 1\) (resp, if \((E_1, E_2)\) is an exceptional pair then \(\chi^T(E_2, E_1) = 0\)).

Part (iii) is an immediate consequence of the following number theoretic result (with \(r = 1\)).

**Proposition 2.6.** Let \(z_1, \ldots, z_n \in \mathbb{Z}[\sqrt{T}]\) be such that \(|z_1|^2 + \cdots + |z_n|^2 = r\), where \(r\) is a rational number, \(0 < r \leq 1\). Then for some \(i_0\) we have \(z_i = 0\) for \(i \neq i_0\), and \(z_{i_0}\) is a root of unity (so \(r = 1\)).

**Proof.** Since the Galois group of \(\mathbb{Q}(\sqrt{T})\) over \(\mathbb{Q}\) is abelian, for every element \(\sigma\) in this group we have \(\sigma(|z_i|^2) = |\sigma(z_i)|^2\). Hence, applying \(\sigma\) to the equality \(|z_1|^2 + \cdots + |z_n|^2 = r\) we get that all conjugates of \(|z_i|^2\) are positive real numbers \(\leq 1\) for each \(i\). Assume that \(z_1 \neq 0\). Then the fact that the norm of \(|z_1|^2\) is an integer implies that \(|z_1|^2 = 1\). Thus, all conjugates of \(z_1\) have absolute value 1. By Kronecker’s theorem, it follows that \(z_1\) is a root of unity. \(\square\)

**Remark.** The above proof works also in a more general situation when \(z_i\)’s are integers in some Galois extension of \(\mathbb{Q}\) such that the complex conjugation induces a central element of the corresponding Galois group.

To prove part (iv) of Theorem 1.1 we start by observing that the action of \(N\) on \(X\) preserves \(X^T\). Furthermore, for \(p \in X^T\) and an \(N\)-equivariant object \(E\) we have

\[
v(E)_{wp} = \text{Tr}(t_0^{-1}, E|_{wp}) = \text{Tr}(w t_0 w^{-1}, E|_{wp}) = \text{Tr}(t_0, E|_{p}) = v(E)_p.
\]

(2.7)

From (iii) we derive that if \(E\) is exceptional then \(v(E) = \pm \zeta v(E_i)\) for some \(i\). This implies that \(v(E)_p = \pm \zeta v(E_i)_p\) for all \(p \in X^T\). It remains to observe that (2.7) applies both to \(E\) and \(E_i\) and that the map (2.3) is injective by Lemma 2.4 (recall that by Lemma 2.1, \(K^0_f(X)\) is a free \(R\)-module). Hence, the coefficient of proportionality between \(v(E)\) and \(v(E_i)\) should be real, so \(v(E) = \pm v(E_i)\).

**Proof of Corollary 1.2.** Let \(\Phi_N\) denote the \(N\)th cyclotomic polynomial. Since for \(N = p^k\) one has \(\Phi_N(1) \equiv 0 \mod(p)\), there is a well-defined ring homomorphism \(\rho : \mathbb{Z}[\sqrt{T}] \to \mathbb{Z}/p\mathbb{Z}\) sending \(\zeta_N\) to 1.
The composition $\rho \circ \text{Tr}(t_0, ?) : R \to \mathbb{Z}/p\mathbb{Z}$ coincides with the reduction modulo $p$ of the homomorphism $\text{Tr}(1, ?) : R \to \mathbb{Z}$. Therefore, from Lemma 2.3 we deduce an isomorphism

$$K_{t_0} \otimes_{\mathbb{Z}} [\sqrt{p}] \mathbb{Z}/p\mathbb{Z} \simeq K_0(X) \otimes \mathbb{Z}/p\mathbb{Z}$$

compatible with the Euler forms. It follows that $\chi \mod(p)$ is symmetric, and for any object $F$ of $D(X)$ equipped with $T$-equivariant structure, the class associated with $F$ in $K_0(X) \otimes \mathbb{Z}/p\mathbb{Z}$ is obtained from the class $v(F) \in K_{t_0}$ using the specialization with respect to $\rho$. This immediately implies the first assertion. Note that every exceptional object $E \in D(X)$ admits a $T$-equivariant structure (see Lemma 2.2). Hence, by part (iii) of Theorem 1.1, we have $v(E) = \pm \zeta^n v_i$ for some $i$. Applying the homomorphism $\rho$ we deduce (1.1), which in turn implies (1.2).

\[\square\]

3. CENTRAL EQUIVARIANT OBJECTS AND APPLICATIONS

The following result due to Merkurjev is explained as Lemma 2.9 in [21] (it is a combination of Cor. 2.15 and Prop. 4.1 of [15]).

**Lemma 3.1.** Assume that $T$ is a maximal torus in a connected reductive group $G$ such that the commutant of $G$ is simply connected, and let $X$ be a smooth projective variety with an action of $G$. Then the natural morphism

$$K^G(X) \otimes_{R(G)} R(T) \to K^T(X)$$

is an isomorphism.

**Definition.** Let $X$ be a variety equipped with an action of an algebraic group $G$. Assume that the center $Z_G \subset G$ acts trivially on $X$. Then we have a decomposition of the category $\text{Coh}^G(X)$ into the direct sum of subcategories $\text{Coh}^G(X)_\chi$, where $\chi$ runs through characters of $Z_G$ and $Z_G$ acts via $\chi$ on $G$-equivariant coherent sheaves in $\text{Coh}^G(X)_\chi$. We say that an object $V \in D(X/G)$ is central if it belongs to $D(\text{Coh}^G(X)_\chi)$ for some $\chi : Z_G \to \mathbb{G}_m$ (called the central character of $V$).

For example, any indecomposable object in $D(X/G)$ is central. Hence, every exceptional object equipped with a $G$-equivariant structure is central. The tensor product of central objects is again central (and the central characters get multiplied).

Using central $G$-equivariant bundles we get a simple way of decomposing $K_{t_0} = K^G_0(X) \otimes_R \mathbb{Z}[\sqrt{p}]$ into $\mathbb{Z}[\sqrt{p}]$-submodules, mutually orthogonal with respect to the specialization of the equivariant Euler form $\text{Tr}(t_0, \chi^T(\cdot, \cdot))$.

**Proposition 3.2.** Let $G$ be a connected reductive group with simply connected commutant, and let $X$ be a smooth projective variety with an action of $G/Z_G$. Let $t_0 \in T$ be an element of order $N$ in a maximal torus in $G$. Assume that $K^T_0(X)$ is a free $R$-module, and $\det(1 - t_0 T_p X) \neq 0$ for every $p \in X^T$. Assume also that for some element $z_0 \in Z_G$ and some element $w_1 \in W$ in the Weyl group of $G$ one has $w_1(t_0) = z_0 t_0$. Then we have an orthogonal (with respect to the Euler form $\text{Tr}(t_0, \chi^T(\cdot, \cdot))$) direct sum decomposition:

$$K_{t_0} = \bigoplus_{\zeta^n = 1} K_{t_0}(\zeta),$$

where $n$ is the order of $z_0$, and for each $n$-th root of unity $\zeta$ we denote by $K_{t_0}(\zeta)$ the subgroup in $K_{t_0}$ generated by the classes of $G$-equivariant bundles on which $z_0$ acts by $\zeta$. If in addition, $N$ is a power of a prime $p$ then we have a similar decomposition of $K_0(X) \otimes \mathbb{Z}/p\mathbb{Z}$, orthogonal with respect to the reduction of the Euler form.

**Proof.** By Lemma 3.1, the classes of $G$-equivariant bundles generate $K^T_0(X)$ over $R$. Also, Lemma 2.4 together with (2.6) imply that the bilinear form $\text{Tr}(t_0, \chi^T(\cdot, \cdot))$ on $K_{t_0}$ is nondegenerate. Hence, it suffices to check that the pieces $K_{t_0}(\zeta)$ and $K_{t_0}(\zeta')$ are orthogonal for $\zeta \neq \zeta'$. It is enough to check that if $V$ is
a $G$-equivariant vector bundle such that $z_0$ acts on $V$ by a scalar $\zeta \neq 1$ then $\text{Tr}(t_0, \chi^T(V)) = 0$. To this end we observe that $\chi^T(V)$ is a $W$-invariant element in $R$. Furthermore, $\chi^T(V)$ belongs to the $\mathbb{Z}$-span of characters $\chi$ of $T$ such that $\chi(z_0) = \zeta$. Therefore, we get

$$\text{Tr}(t_0, \chi^T(V)) = \text{Tr}(w_1(t_0), \chi^T(V)) = \text{Tr}(z_0t_0, \chi^T(V)) = \zeta \text{ Tr}(t_0, \chi^T(V))$$

which implies the required vanishing. \qed

We will combine Proposition 3.2 with the following result employing the same idea as in part (iv) of Theorem 1.1.

**Proposition 3.3.** Let $G$ be a connected reductive group with simply connected commutant, and let $X$ be a smooth projective variety with an action of $G/Z_G$. Let $t_0 \in T$ be an element of order $N$ in a maximal torus in $G$. We assume that $T$ comes from a split torus over $\mathbb{Q}$ and we use the corresponding Galois action on the elements of finite order in $T$. Assume that for every $\sigma \in \text{Gal}(\mathbb{Q}(\sqrt{T})/\mathbb{Q})$ there exists $w_\sigma \in W$ such that $\sigma(t_0) = w_\sigma(t_0)$. Let us denote by $M \subset K_{t_0}$ the $\mathbb{Z}$-span of the classes of $G$-equivariant bundles on $X$. Then the natural ring homomorphism

$$M \otimes_{\mathbb{Z}} \mathbb{Z}[\sqrt{T}] \rightarrow K_{t_0}$$

is an isomorphism. If $(E_1, \ldots, E_k)$ is a full exceptional collection in $D(X)$, where each $E_i$ can be equipped with $G$-equivariant structure, then $M$ coincides with the $\mathbb{Z}$-span of the classes of $E_i$’s.

**Proof.** By Lemma 3.1, we have

$$K_{t_0} \cong K_0^G(X) \otimes_{R(G)} \mathbb{Z}[\sqrt{T}],$$

where we use the ring homomorphism $R(G) = R(T)^W \rightarrow \mathbb{Z}[\sqrt{T}]$ induced by $\text{Tr}(t_0, ?) : R(T) \rightarrow \mathbb{Z}[\sqrt{T}]$. It remains to note that our assumptions imply that for every $W$-invariant element $f \in R(T)$ the element $\text{Tr}(t_0, f) \in \mathbb{Z}[\sqrt{T}]$ will be invariant with respect to $\text{Gal}(\mathbb{Q}(\sqrt{T})/\mathbb{Q})$, hence, an integer. For the last assertion one has to use the fact that the classes of $E_i$’s form a basis of $K^G_0(X)$ over $R(G)$ (see Lemma 2.1). \qed

Now let us specialize to the particular case $X = G/P$, where $G$ is a simply connected semisimple group, $P$ is a maximal parabolic subgroup of $G$. Let $\Delta$ denote the set of roots associated with $G$, and let $\Pi = (\alpha_1, \ldots, \alpha_n)$ denote the set of simple roots. Assume that $P$ is the standard maximal parabolic in $G$ associated with a simple root $\alpha_i$ (or rather, with the subset $\Pi \setminus \{\alpha_i\} \subset \Pi$). Then the weights of the maximal torus $T$ on $g/p$ are exactly $-\alpha$, where $\alpha$ is a positive root in which $\alpha_i$ enters with nonzero multiplicity, i.e., $(\alpha, \omega_i) > 0$, where $\omega_i$ is the fundamental weight associated with $\alpha_i$. The set of $T$-stable points $X^T$ is in bijection with $W/W_P$, where $W = N/T$ is the Weyl group of $G$, and $W_P \subset W$ is the subgroup generated by the reflections with respect to $\Pi \setminus \{\alpha_i\}$. If $p(w) \in X^T$ denotes the point corresponding to $wW_P \in W/W_P$ then the weights of $T$ on the tangent space $T_{p(w)}X$ are $-\omega_a$, where $\alpha$ is as above. Thus, the assumption $(\star)$ in Theorem 1.1 can be reformulated in the following form.

**Proposition 3.4.** Let $G$ be a simply connected semisimple group, and let $X = G/P$, where $P$ is the standard maximal parabolic subgroup associated with a simple root $\alpha_i$. Set

$$N = N_i = \sum_{\alpha \in \Delta \setminus \{\alpha, \omega_1\} > 0} \frac{(\alpha, \omega_1)}{(\omega_i, \omega_i)},$$

Then an element $t_0$ of the maximal torus $T$ satisfies the assumption $(\star)$ of Theorem 1.1 if and only if the following two conditions are satisfied:

(a) for every root $\alpha$ one has $\alpha(t_0) \neq 1$;
(b) $\omega_i(t_0)^N = (-1)^{\text{dim}X}$ for every $\alpha \in \Delta$ such that $||\alpha|| = ||\alpha_i||$ one has $\alpha(t_0)^N = 1$. 

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Proof. The set of weights of $T$ on $g/p$ is invariant with respect to $W_P \subset W$. Hence, the sum of these weights is proportional to $\omega_i$. It follows that this sum equals $-N\omega_i$, so we have
\[
\det(t, g/p) = \omega_i(t)^{-N}.
\]
Thus, the condition $(\ast)$ can be restated as follows:
\[
(w\omega_i)(t_0)^N = (-1)^{\dim X}t
\]
and $(w\omega)(t_0) \neq 1$ for all $w \in W$ and all roots $\alpha$ such that $(\alpha, \omega_i) \neq 0$. It is easy to see that in fact every root can be written in the form $w\omega$ with $(\alpha, \omega_i) \neq 0$ for some $w \in W$. Hence, the inequalities are equivalent to the condition $\alpha(t_0) \neq 1$ for all roots $\alpha$. On the other hand, since $s_i\omega_i = \omega_i - \alpha_i$, the equalities (3.1) imply that $\lambda(t_0)^N = 1$ for all $\lambda$ in the lattice $Q_1$ spanned by $(w\alpha_i)_{w \in W}$ (i.e., by the set of roots of the same length as $\alpha_i$). Note that since $s_j\omega_i = \omega_i - \delta_{ij}\alpha_j$, the $Q_i$-coset $\omega_i + Q_i$ is stable under $W$. This implies our statement. \hfill \Box

Corollary 3.5. In the situation of the above Proposition, assume that $\dim X$ is odd. Then the necessary condition for the existence of an element $t_0$ satisfying $(\ast)$ is that the class of $\omega_i \mod(Q_1)$ has an even order, where $Q_1$ is the sublattice of the weight lattice spanned by all roots of the same length as $\alpha_i$.

Using Proposition 3.4 we will be able to check for almost all of the spaces $G/P$ (where $P$ is a maximal parabolic) whether an element $t_0$ satisfying $(\ast)$ exists, leaving out several cases of types $E_7$ and $E_8$. In the next Proposition we use notations from the Tables I-IX in [3]. Recall that since $G$ is simply connected, the character lattice of the maximal torus $T$ coincides with the weight lattice of the corresponding root system. For example, for type $C_n$ this gives an identification of $T$ with $G_m^n$. For types $B_n$ and $D_n$ the weight lattice is spanned by the standard lattice $\mathbb{Z}^n$ together with the vector $(\sum_{i=1}^n \varepsilon_i)/2$. Hence, we obtain
\[
T = \{(x_1, \ldots, x_n; x) \in G_m^{n+1} \mid x^2 = \prod x_i\}
\]
in these cases. The maximal torus for type $F_4$ has the same description with $n = 4$.

Proposition 3.6. Let $X = G/P$, where $G$ is a simply connected simple algebraic group, and $P$ is the standard maximal parabolic subgroup associated with the simple root $\alpha_i$.

(a) If $G$ is of classical type then $t_0$ satisfying $(\ast)$ exists only in the following cases:
(i) $G$ is of type $A_n$, arbitrary $i$;
(ii) $G$ is of type $B_n$, $i = 1$ or $i = n$;
(iii) $G$ is of type $C_n$, $i = 1$ or $i = n$;
(iv) $G$ is of type $D_n$, $i = 1, n - 1$, or $n$.

In the cases (ii)-(iv) $t_0$ can be chosen in such a way that
\[
w_0(t_0) = t_0^{-1},
\]
where $w_0 \in W$ is the element of maximal length.

(b) $t_0$ satisfying $(\ast)$ does not exist if $G$ is of type $G_2$ or $F_4$.

(c) If $G$ is of type $E_6$ then $t_0$ satisfying $(\ast)$ exists if and only if either $i = 1$ or $i = 6$. In these cases it can be chosen in such a way that (3.2) holds.

(d) If $G$ is of type $E_7$ and $i = 1, 3$, or 4 (resp., $G$ is of type $E_8$ and $i = 6, 7$ or 8) then $t_0$ satisfying $(\ast)$ does not exist.

(e) if $G$ is of type $E_7$ and $i = 7$ then $t_0$ satisfying $(\ast)$ exists and (3.2) holds.

Proof. (a) For type $A_n$ the maximal torus is $T = \{(x_1, \ldots, x_{n+1}) \in G_m^{n+1} \mid \prod x_i = 1\}$. Also, with the notations of Proposition 3.4 we have $N_i = n + 1$. Thus, the conditions of this Proposition for the element $t_0 = (x_1, \ldots, x_{n+1})$ become
\[
(x_1 \ldots x_i)^{(n+1)} = (-1)^{i(n+1-i)},
\]
\[
x_k^{n+1} = x_i^{n+1} \text{ and } x_k \neq x_i \text{ for } k < l.
\]
Thus, these conditions are satisfied when \( \{x_1, \ldots, x_{n+1}\} \) is the set of all \((n+1)\)th roots of \((-1)^{(n+1-i)}\).

For type \(B_n\) the element \(t_0 = (x_1, \ldots, x_n; x)\) should satisfy \(x_k \neq 1\) for all \(i < n\) and \(x_k \neq x_i^{\pm 1}\) for \(k < l\).

For \(i < n\) we have \(\omega_i = \varepsilon_i + \ldots + \varepsilon_i, N_i = 2n - i\), dim \(X = i(i+1)/2 + 2i(n-i)\), and the lattice \(Q_i \subset \mathbb{Z}^n\)

\(Q_i \subset \mathbb{Z}^n\) consists of all vectors with even sums of coordinates. Thus, we should have \(x_k^{2N_i} = 1\) for all \(k\), \(x_k^{N_i} = x_i^{N_i}\)

for \(k < l\). In other words, \(x_k^{N_i} = \pm 1\) does not depend on \(k\), which is impossible for \(i > 1\). In the case \(i = 1\) all \(x_k\)'s should be \((2n-1)\)th roots of \(-1\), and we can set \(x_1 = -1\) and choose \(\{x_2, \ldots, x_n\}\) to contain one from each conjugate pair of the remaining \(2n-2\) roots (and let \(x = \) a square root of \(\prod x_k\)). In the case \(i = n\) we have \(\omega_n = (\sum \varepsilon_k)/2, N_n = 2n\), dim \(X = n(n+1)/2\), and \(Q_i = \mathbb{Z}^n\). Thus, \(x_k\)'s should be \((2n)\)th roots of \(1\), and \(x\) should satisfy \(x^{2n} = (-1)^{(n+1)/2}\). Hence, we can set \(x_k = 2n\), for \(k = 1, \ldots, n\), and let \(x\) be a square root of \(\prod x_k\). The condition (3.2) for types \(B_n\) and \(C_n\) is automatic since \(w_0\) sends every \(t \in T\) to \(t^{-1}\).

For type \(C_n\) the element \(t_0 = (x_1, \ldots, x_n)\) should satisfy \(x_k \neq \pm 1\) for all \(k\) and \(x_k \neq x_i^{\pm 1}\) for \(k < l\).

We have \(\omega_i = \varepsilon_i + \ldots + \varepsilon_i, N_i = 2n - i - 1\), dim \(X = i(i+1)/2 + 2i(n-i)\), for \(i < n\) the lattice \(Q_i \subset \mathbb{Z}^n\) consists of all vectors with even sums of coordinates. Hence, as in the case of \(B_n\) we deduce that \(x_k^{N_i} = \pm 1\) does not depend on \(k\) which is impossible for \(i > 1\) (recall that now we have the condition \(x_k \neq \pm 1\)). In the case \(i = 1\) we can choose \(\{x_1, \ldots, x_n\}\) to contain one from each conjugate pair of \((2n)\)th roots of \(-1\). In the case \(i = n\) the lattice \(Q_n \subset \mathbb{Z}^n\) consists of all vectors with even coordinates. Hence, we can set \(x_k = 2n+2\) for \(k = 1, \ldots, n\).

For type \(D_n\) the element \(t_0 = (x_1, \ldots, x_n; x)\) should satisfy \(x_k \neq x_i^{\pm 1}\) for \(k < l\).

For \(i < n-1\) we have \(\omega_i = \varepsilon_i + \ldots + \varepsilon_i, N_i = 2n - i - 1\), dim \(X = i(i+1)/2 + 2i(n-i)\), the lattice \(Q_i = Q_i \subset \mathbb{Z}^n\) consists of all vectors with even sums of coordinates. As above we can rule out the cases \(1 < i < n - 1\). For \(i = 1\) we can take \(x_k = 2n+2\), \(k = 1, \ldots, n\) (and let \(x\) be a square root of \(\prod x_k\)). For \(i = n\) we have \(\omega_n = (\sum \varepsilon_k)/2, N_n = 2n - 2\), dim \(X = n(n-1)/2\). Hence, we can use the same element \(t_0\) for as \(i = 1\). If \(n\) is even then the condition (3.2) is automatic. If \(n\) is odd then

\[ w_0(x_1, \ldots, x_n; x) = (x_1^{-1}, \ldots, x_{n-1}^{-1}, x_n; x^{-1}x_n), \]

so the condition (3.2) holds since \(x_n = 1\). The case \(i = n - 1\) follows by the symmetry of the Dynkin diagram.

(b) The case of type \(G_2\) and the cases \(i = 1, 4\) of type \(F_4\) follow immediately from Corollary 3.5. In the remaining two cases \(i = 2, 3\) for type \(F_4\) we have \(N_2 = 5, N_3 = 7\), and the element \(t_0 = (x_1, \ldots, x_4; x)\) should satisfy \(x_k \neq 1, x_k \neq x_i^{\pm 1}\) for \(k < l,\) and \(x_k^{N_i} = \pm 1\) does not depend on \(k\), which is impossible.

(cd) The non-existence of \(t_0\) in all the relevant cases follows Corollary 3.5. It remains to consider the case of type \(E_6\) and \(i = 1\) (the case \(i = 6\) will follow by the symmetry of the Dynkin diagram). We have \(N_1 = 12\). Thus, the element \(t_0\) should satisfy \(t_0^{12} = 1\) and \(\alpha(t_0) \neq 1\) for every root \(\alpha\). Let \(\Lambda\) denote the weight lattice. Then the group of elements of order 12 in \(T\) is canonically dual to the finite group \(\Lambda \otimes \mathbb{Z}/12\mathbb{Z}\). Thus, to give \(t_0\) it is enough to specify an element of order 12 in \(\Lambda^* \otimes \mathbb{Q}/\mathbb{Z}\). Equivalently, we have to produce an element \(\lambda \in \Lambda \otimes \mathbb{Q}\) such that \(12(\lambda, \omega) \in \mathbb{Z}\) for every weight \(\omega\) and \((\lambda, \alpha) \notin \mathbb{Z}\) for every root \(\alpha\). Set \(12\lambda = \sum_{i=1}^{5} a_i \varepsilon_i + b_8 \varepsilon_8 - \varepsilon_7 - \varepsilon_6\). Then the conditions can be rewritten in terms of these coordinates as follows:

\[ a_i \in \frac{1}{2}\mathbb{Z}, a_i \equiv a_j \bmod \mathbb{Z} \text{ for all } i, j, c := \frac{1}{2}\sum a_i \in \mathbb{Z} \text{ and } \]

\[ a_i \neq \pm a_j \bmod \mathbb{Z} \text{ for } i \neq j, \]

\[ c \neq \sum_{i \in S} a_i \bmod \mathbb{Z} \text{ for } S \subset [1, 5] \text{ with } |S| \text{ even.} \]

If we want in addition to have \(w_0(t_0) = t_0^{-1}\) then we should impose two relations: \(a_2 - a_1 = a_4 - a_3\) and \(c = a_2 + a_3 + 2a_5\). It is easy to check that

\[ (a_1, \ldots, a_5; c) = (0, 1, 2, 3, 4; 11) \]

is a solution.
(c) The condition (3.2) holds automatically in this case since \( w_0 \) sends every \( t \in T \) to \( t^{-1} \). We have \( N_7 = 18 \) and \( \dim X = 27 \), so to give \( t_0 \) satisfying (⋆) is equivalent to finding a rational weight \( \lambda \) such that \( 18(\lambda, \alpha) \in \mathbb{Z} \), \( (\lambda, \alpha) \not\equiv 0 \) for every root \( \alpha \), while \( 18(\lambda, \omega T) - 1/2 \in \mathbb{Z} \). Set \( 18 \lambda = \sum_{i=1}^6 a_i \varepsilon_i + b(\varepsilon_8 - \varepsilon_7) \). Then we should have \( a_i \in \frac{1}{2} \mathbb{Z} \), \( a_i \equiv a_j \mod(\mathbb{Z}) \) for all \( i, j \), \( \sum_{i=1}^6 a_i \equiv 1 \mod(2\mathbb{Z}) \), \( c := b + \frac{1}{2} \sum_{i=1}^6 a_i \equiv 6 \mod(\mathbb{Z}) \), and

\[
a_i \not\equiv \pm a_j \mod(18\mathbb{Z}) \quad \text{for} \ i \neq j, \quad 2c \neq \sum_{i=1}^6 a_i \mod(18\mathbb{Z}),
\]

\[
c \not\equiv \sum_{i \in S} a_i \mod(18\mathbb{Z}) \quad \text{for} \ S \subset [1, 6] \text{ with } |S| \text{ odd}.
\]

We can take

\[
(a_1, \ldots, a_6; c) = (0, 1, 2, 3, 4, 5; 16) \text{ or } (0, 1, 2, 3, 4, 5; 17)
\]
as solutions. \( \square \)

Now let us consider applications of Theorem 1.1 (and of Propositions 3.2 and 3.3) to concrete varieties.

**Projective spaces**

In this case it is more convenient to work with the action of \( \text{GL}_n \) on \( \mathbb{P}^{n-1} \), rather than \( \text{SL}_n \), so \( T \) will denote the set of diagonal matrices in \( \text{GL}_n \). Let \( p_1, \ldots, p_n \subset \mathbb{P}^{n-1} \) denote the \( T \)-fixed points where the action of \( T \) on the fiber of \( \mathcal{O}_{\mathbb{P}^{n-1}}(1) \) at \( p_i \) corresponds to the \( i \)-th coordinate character \( \varepsilon_i : T \rightarrow \mathbb{G}_m \).

As in Proposition 3.6 one checks that the element

\[
t_0 = (1, \zeta_n, \zeta_n^2, \ldots, \zeta_n^{n-1}) \quad (3.3)
\]
satisfies the assumption (⋆) of Theorem 1.1 (where \( \zeta_n \) is a primitive \( n \)-th root of unity).

Let \( V \) be a central object in \( D(\mathbb{P}^{n-1}/\text{GL}_n) \). The action of the center \( \mathbb{Z}_{\text{GL}_n} = \mathbb{G}_m \) on \( V \) is given by the character \( \mathbb{G}_m \rightarrow \mathbb{G}_m : \lambda \mapsto \lambda^m \) for some \( m \in \mathbb{Z} \). Let us denote by \( m(V) \in \mathbb{Z}/n\mathbb{Z} \) the remainder of \( m \) modulo \( n \). Note that if we tensor \( V \) with a character of \( \text{GL}_n \) then \( m(V) \) will not change. In particular, for an exceptional object \( V \) in \( D(\mathbb{P}^{n-1}) \) we can define \( m(V) \in \mathbb{Z}/n\mathbb{Z} \) by choosing any \( \text{GL}_n \)-equivariant structure on \( V \) (see Lemma 2.2). Note that the center of \( \text{GL}_n \) acts on \( \mathcal{O}_{\mathbb{P}^{n-1}}(1) \) through the identity character. This implies that for a central object \( V \) on \( D(\mathbb{P}^{n-1}/\text{GL}_n) \) one has

\[
m(V) \text{rk}(V) \equiv \deg(V) \mod(n).
\]

Let \( w_1 \in W = S_n \subset \text{GL}_n \) be the cyclic permutation such that \( w_1(p_i) = p_{i-1} \). Then we have

\[
w_1(t_0) = \zeta_n \cdot t_0,
\]

where \( \zeta_n \) is viewed a scalar matrix in \( T \). Thus, the assumptions of Propositions 3.2 and Proposition 3.3 are satisfied in this case: for the latter we can take \( w_m \in S_n \) to be the permutation of the set of \( n \)th roots of unity induced by \( \sigma \in \text{Gal}(\mathbb{Q}(\sqrt{m})/\mathbb{Q}) \). Hence, we derive the following corollary from these Propositions.

**Corollary 3.7.** Let \( V \) be a central object in \( D(\mathbb{P}^{n-1}/\text{GL}_n) \). Then for some integer \( a(V) \) one has

\[
v(V) = a(V)v(\mathcal{O}(m(V))) \quad (3.4)
\]
in \( K_0^T(\mathbb{P}^{n-1}) \otimes R \mathbb{Z}[\sqrt{m}] \) (recall that \( v(\mathcal{O}(m)) \) depends only on \( m \mod(n) \) by Lemma 2.5).

**Proof of Theorem 1.3.** In the case when \( n = p^k \), where \( p \) is prime, we can apply the homomorphism \( \rho : \mathbb{Z}[\sqrt{m}] \rightarrow \mathbb{Z}/p\mathbb{Z} \) that sends \( p^k \)th roots of unity to 1. Then we get from Corollary 3.7 the following congruence in \( K_0(\mathbb{P}^{n-1}) \otimes \mathbb{Z}/p\mathbb{Z} \):

\[
[V] \equiv a(V)[\mathcal{O}(m(V))].
\]
Taking ranks of both sides we see that \( a(V) \equiv \text{rk}(V) \mod(p) \). This immediately implies part (i) of Theorem 1.3. For part (ii) we have to recall that by Lemma 2.2 an exceptional object \( E \) admits a \( GL_n \)-equivariant structure, unique up to tensoring with a character of \( GL_n \). Also, by Theorem 1.1(iii) we see that \( a(E) = \pm 1 \). Finally, part (iii) follows immediately from Corollary 1.2.

\[ \square \]

**Remark.** The fact that \( a(E) = \pm 1 \) for an exceptional object \( E \) can also be proven by calculating 
\[ 1 = \text{Tr}(t_0, \chi^T(E,E)) \] using (3.4). Also, part (iii) of Theorem 1.3 can be deduced from a version of (3.4) that holds for products of projective spaces. This would make a proof of Theorem 1.3 completely independent from Theorem 1.1. Because of this it is possible to generalize this argument to relative projective spaces, see Theorem 3.18 below.

**Grassmannians**

Let \( T \) be the maximal torus of \( GL_n \) acting on the Grassmannian \( G(k,n) \) of \( k \)-planes in the \( n \)-dimensional space in the standard way. It is easy to see that the element (3.3) still satisfies the condition \((\ast)\) of Theorem 1.1 (cf. Proposition 3.6).

As before, for a central object \( V \) in \( D(G(k,n)/GL_n) \) with the central character \( \lambda \mapsto \lambda^m \) we set 
\[ m(V) = m \mod(n). \]
This remainder does not change under tensoring \( V \) with a character of \( GL_n \). We denote by \( \mathcal{O}(1) \) the ample generator of \( \text{Pic}(G(k,n)) \). Note that it has the central character \( \lambda \mapsto \lambda^k \).

Therefore, one has 
\[ m(V) \text{rk}(V) \equiv k \deg(V) \mod(n). \tag{3.5} \]

Recall that if \( \mathcal{U} \) is the tautological rank \( k \) bundle on \( G(k,n) \) then the vector bundles \( \Sigma^k \mathcal{U} \), where \( \lambda \) runs over partitions \( n - k \geq \lambda_1 \geq \ldots \geq \lambda_k \geq 0 \), can be ordered to form a full exceptional collection on \( G(k,n) \) (see [8]). Hence, the classes \( v(\Sigma^k \mathcal{U}) \) with \( \lambda \) as above form a basis of \( K^T_G(G(k,n)) \otimes_R \mathbb{Z}[\sqrt{k}] \) over \( \mathbb{Z}[\sqrt{k}] \).

As in the case of projective spaces, using Propositions 3.2 and 3.3 we derive the following.

**Corollary 3.8.** (i) For a central object \( V \) in \( D(G(k,n)/GL_n) \) the class \( v(V) \) in \( K^T_G(G(k,n)) \otimes_R \mathbb{Z}[\sqrt{k}] \) is a linear combination with integer coefficients of the classes \( v(\Sigma^k \mathcal{U}) \), where \( \lambda \) runs over partitions \( n - k \geq \lambda_1 \geq \ldots \geq \lambda_k \geq 0 \) such that \( \lambda_1 + \ldots + \lambda_k \equiv m(V) \mod(n) \).

(ii) Let \( V \) and \( V' \) be central objects in \( D(G(k,n)/GL_n) \) with \( m(V) \neq m(V') \mod(n) \). Assume that \( n = p^r \), where \( p \) is a prime. Then \( \chi(V,V') \equiv 0 \mod(p) \).

**Remark.** The (integer) structure constants of the multiplication on \( K^T_G(G(k,n)) \otimes_R \mathbb{Z}[\sqrt{k}] \) can be easily computed from the Littlewood-Richardson rule. One just has to observe (looking at the definition of the Schur functions) that for an arbitrary partition \( \lambda_1 \geq \ldots \geq \lambda_k \geq 0 \) the class \( v(\Sigma^k \mathcal{U}) \) up to a sign depends only on the residues modulo \( n \) of the numbers \( \lambda_1 + k - 1, \lambda_2 + k - 2, \ldots, \lambda_k \). More precisely, if for some \( i \neq j \) we have \( \lambda_i - i \equiv \lambda_j - j \mod(n) \) then \( v(\Sigma^k \mathcal{U}) = 0 \). Otherwise, the class \( v(\Sigma^k \mathcal{U}) \) coincides up to a sign with one of the basis classes.

**Theorem 3.9.** (i) Let \( (E_1, \ldots, E_s) \) be a full exceptional collection in \( D(G(k,n)) \) (so that \( s = \binom{n}{k} \)). Assume that \( k \) is relatively prime to \( n \). Then for each \( m \in \mathbb{Z}/n\mathbb{Z} \) exactly \( s/n \) objects \( E_i \) from the collection have \( m(E_i) \equiv m \mod(n) \).

(ii) Let \( p \) be a prime, and let \( 1 \leq k \leq p - 1 \). For every exceptional object \( E \) in \( D(G(k,p)) \) one has \( \text{rk}(E) \neq 0 \mod(p) \). The same congruence holds for exceptional objects on the products \( G(k_1,p) \times \ldots \times G(k_1,p) \). If \( (E_1, \ldots, E_s) \) is a full exceptional collection in \( D(G(k,p)) \) then for every \( \mu \in \mathbb{Z}/p\mathbb{Z} \) exactly \( s/p \) of the objects \( E_i \) have \( \deg(E_i)/\text{rk}(E_i) \equiv \mu \mod(p) \).

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(iii) Now let \( n = p^r \), where \( p \) is prime, and let \( V \) be a central object in \( D(G(p, n)/GL_n) \) with \( m(V) \neq 0 \mod(p) \). Then \( \text{rk}(V) \equiv 0 \mod(p) \).

(iv) Let \( E \) be an exceptional object in \( D(G(2, 2^r)) \). Then \( \text{rk}(E) \equiv m(V) + 1 \mod(2) \).

Proof. (i) By Proposition 3.2, we have an orthogonal decomposition
\[
K_{t_0} = \bigoplus_{m \in \mathbb{Z}/n\mathbb{Z}} K_{t_0}(m),
\]
where \( K_{t_0} = K_0^T(G(k, n)) \otimes_R \mathbb{Z}[\sqrt[4]{-1}] \), and \( K_{t_0}(m) \) is the \( \mathbb{Z}[\sqrt[4]{-1}] \)-span of the classes of \( E_i \) such that \( m(E_i) \equiv m \mod(n) \). Tensoring with \( \mathcal{O}(1) \) gives an isomorphism \( K_{t_0}(m) \rightarrow K_{t_0}(m + k) \) of \( \mathbb{Z}[\sqrt[4]{-1}] \)-modules. Since \( k \) is relatively prime to \( n \), this implies that each \( K_{t_0}(m) \) has rank \( s/n \) over \( \mathbb{Z}[\sqrt[4]{1}] \).

(ii) By Corollary 1.2, it is enough to check that every bundle from the exceptional collection \((\Sigma^\lambda \mathcal{U})\) described above has rank prime to \( p \). To this end we can use the hook-content formula to check that the dimension of the irreducible representation of \( GL_k \) associated with partitions \( p - k \geq \lambda_1 \geq \ldots \geq \lambda_k \geq 0 \) is not divisible by \( p \). Indeed, this dimension equals
\[
s_\lambda = \prod_{x \in \Lambda} \frac{k + c(x)}{h(x)},
\]
where for every point \( x = (i, j) \) of the Young diagram of \( \lambda \), \( h(x) \) is the hook length of \( x \) and \( c(x) = j - i \) is the content of \( x \) (see [14],L.3, Ex.4). But our conditions on \( \lambda \) imply that \( h(x) \leq p - 1 \) and \( k - i \leq c(x) + k \leq p - i \) for \( x = (i, j) \), so all these numbers are relatively prime to \( p \). The last assertion follows from part (i) together with (3.5).

(iii) This follows immediately from (3.5).

(iv) By Corollary 1.2, it is enough to check this for bundles from the exceptional collection \((\Sigma^\lambda \mathcal{U})\) which in this case consists of the symmetric powers of \( \mathcal{U} \) tensored with some line bundles.

\[\square\]

Quadrics

Let \( Q^n \) denote the smooth quadric of dimension \( n \), where \( n \geq 3 \). It is a homogeneous space of the form \( G/P \), where \( G = \text{Spin}(n+2) \). Recall that we have a surjective homomorphism \( G = \text{Spin}(n+2) \rightarrow \text{SO}(n+2) \) with the kernel of order 2. Let us denote by \( Z_0 \subset G \) this kernel.

First, let us consider the case when \( n \) is even. Then the group \( G \) is simply connected of type \( D_k \), where \( n + 2 = 2k \) (\( k \geq 3 \)), \( P \) is the maximal parabolic associated with the root \( \alpha_1 \). As in Proposition 3.6, we identify the standard maximal torus \( T \subset G \) with the group
\[
\{(x_1, \ldots, x_k; x) \in \mathbb{G}_m^{k+1} | x^2 = \prod x_i \}
\]
in such a way that the projection to the coordinate \( x_i \) corresponds to the character \( \varepsilon_i \), while the projection to the coordinate \( x \) corresponds to the character \( (\sum \varepsilon_i)/2 \) of \( T \). Under this identification the projection to the first \( k \) coordinates is exactly the map from \( T \) to the maximal torus in \( \text{SO}(2k) \). Hence, \( Z_0 \) is generated by the element \( z_0 \in T \) that has all \( x_i = 1 \) and \( x = -1 \). The Weyl group action on the weight lattice can permute \( \varepsilon_i \)'s and can multiply an even number of them by \( -1 \). Hence, its action on \( T \) is generated by permutations of coordinates \( x_i \) together with the operator
\[
(x_1, \ldots, x_k, x) \mapsto (x_1^{-1}, x_2^{-1}, x_3, \ldots, x_k, x x_1^{-1} x_2^{-1}).
\]

We denote by \( \mathcal{O}(1) \) the \( G \)-equivariant line bundle on \( G/P \) associated with the character \( \varepsilon_1 \) of \( T \) (it is an ample generator of the Picard group). Note that \( Z_0 \) acts trivially on the line bundle \( \mathcal{O}(1) \).

We will use a slightly different element \( t_0 \) than in Proposition 3.6. Namely, we set
\[
t_0 = (1, \zeta_n, \ldots, \zeta_n^{k-1}; x_0),
\]

where $x_0$ is a square root of $\zeta_n^{k(k-1)/2}$ (recall that $n = 2k - 2$). The assumption $(\ast)$ is still satisfied for this element.

Let us denote by $w_1 \in W$ the element such that

$$w_1(x_1, \ldots, x_k; x) = (x_1^{-1}, x_2, \ldots, x_k^{-1}, x_1 x_k^{-1}).$$

Then we have

$$w_1(t_0) = z_0 t_0. \quad (3.6)$$

On the other hand, consider the element $w_2 \in W$ such that

$$w_2(x_1, \ldots, x_k; x) = (x_k, x_k^{-1}, x_2^{-1}, x_1 x_k^{-1} x_2^{-1} x_k),$$

where $\epsilon = (-1)^k$. Then we have

$$w_2(t_0) = z t_0, \quad (3.7)$$

where $z \in T$ is an element in the center of $\text{Spin}(2k)$:

$$z = (-1, \ldots, -1 : -\zeta_n^{k(k-1)/2}).$$

Note that $z^2 = z_0^k$.

Let $N$ be the order of $t_0$ (where $N|2n$). Then $t_0$ is a $\mathbb{Q}(\sqrt{\chi})$-point of the torus $T$. Even though the set $S = \{\zeta_n^2, \ldots, \zeta_n^{-2}\}$ is not invariant under the action of $\text{Gal}(\mathbb{Q}(\sqrt{\chi})/\mathbb{Q})$, we claim that the conditions of Proposition 3.3 are still satisfied for $t_0$. Namely, for every $\sigma \in \text{Gal}(\mathbb{Q}(\sqrt{\chi})/\mathbb{Q})$ and every $s \in S$ we have either $\sigma(s) \in S$ or $\sigma(s)^{-1} \in S$. Thus, we have a well defined set of signs $s_i = \pm 1$, $i = 2, \ldots, k - 1$, and a permutation $w$ of $2, \ldots, k - 1$, such that

$$\sigma(t_0) = (1, \zeta_n^{s_2 w(2)}, \ldots, \zeta_n^{s_{k-1} w(k-1)}, -1; \sigma(x_0)).$$

Thus, there exists a unique element $\tilde{w}_\sigma \in W$ such that $\tilde{w}_\sigma(\zeta_k) = \zeta_k$ and

$$\sigma(t_0) = \tilde{w}_\sigma(t_0) z_0^\pi(\sigma)$$

for some $\pi(\sigma) \in \mathbb{Z}/2\mathbb{Z}$. It follows that $\pi : \text{Gal}(\mathbb{Q}(\sqrt{\chi})/\mathbb{Q}) \to \mathbb{Z}/2\mathbb{Z}$ and $\sigma \mapsto \tilde{w}_\sigma$ are group homomorphisms.

Now we set $w_\sigma = \tilde{w}_\sigma w_1^{\pi(\sigma)}$. Since the elements $\tilde{w}_\sigma$ commute with $w_1$, we get that the map $\sigma \mapsto w_\sigma$ is a group homomorphism. From the above equation and from (3.6) we get

$$\sigma(t_0) = w_\sigma(t_0).$$

Recall (see [9]) that we have a full exceptional collection on $Q^{2k-2} = G/P$ consisting of the $2k$ bundles $(\mathcal{O}, S_+, S_-, \mathcal{O}(1), \ldots, \mathcal{O}(2k - 3))$, where $S^\pm$ are the spinor bundles. The center $Z_G$ of $G = \text{Spin}(2k)$ contains a subgroup $Z_0$ (the kernel of the homomorphism to $\text{SO}(2k)$), and $Z_G/Z_0 \cong \mathbb{Z}/2\mathbb{Z}$. Let $\chi_0 : Z_G \to \{-1\}$ denote the unique nontrivial character of $Z_G$, that has trivial restriction to $Z_0$. Note that $Z_G$ acts on $\mathcal{O}(1)$ through $\chi_0$. Let $\chi_\pm : Z_G \to \mathbb{G}_m$ denote the characters with which the center acts on the spinor bundles $S_\pm$. These characters are nontrivial on $Z_0$ and we have $\chi_+ = \chi_0 \chi_-$. The characters $\chi_0$ and $\chi_\pm$ are all nontrivial characters of $Z_G$. Now Propositions 3.2 and 3.3 give the following result.

**Corollary 3.10.** Let $V$ be a central object in $D(Q^{2k-2}/\text{Spin}(2k))$, where $k \geq 3$.

(i) If $V$ has trivial central character (resp., central character $\chi_0$) then the class $v(V) \in K_0^T(Q^{2k-2}) \otimes_R \mathbb{Z}[\sqrt{\chi}]$ is a linear combination with integer coefficients of the classes $v(\mathcal{O}), v(\mathcal{O}(2)), \ldots, v(\mathcal{O}(2k - 4))$ (resp., $v(\mathcal{O}(1)), v(\mathcal{O}(3)), \ldots, v(\mathcal{O}(2k - 3)))$.

(ii) If the central character of $V$ is $\chi_+$ (resp., $\chi_-$) then $v(V)$ is an integer multiple of $v(S_+)$ (resp., $v(S_-)$).

(iii) Assume now that $2k - 2 = 2^r$ and the central character of $V$ is nontrivial. Then $\chi(V) \equiv 0 \text{mod}(2)$. If the central character of $V$ is $\chi_\pm$ then $\text{rk}(V) \equiv 0 \text{mod}(2)$. 

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Note that the last assertion in (iii) follows from (ii) and the fact that the rank of the spinor bundles \( S_k \) is \( 2k-2 \), which is even since \( k \geq 3 \).

As in the case of Grassmannians, using the concrete full exceptional collection on \( Q^{2k} \) we derive some information about arbitrary exceptional collections.

**Proposition 3.11.** (i) Let \( (E_1, \ldots , E_{2k}) \) be a full exceptional collection in \( D(Q^{2k-2}) \), where \( k \geq 3 \).

Then exactly \( k-1 \) objects from the collection have trivial central character, \( k-1 \) objects have central character \( \chi_0 \), and the remaining two have central characters \( \chi_+ \) and \( \chi_- \).

(ii) Let \( E \) be an exceptional object in \( D(Q^{2r}) \), where \( r \geq 2 \). Let us equip \( E \) with a \( \text{Spin}(2k) \)-equivariant structure, where \( k = 2^r - 1 \). Then \( \text{rk} E \) is odd iff the action of \( Z_0 \) on \( E \) is trivial.

**Proof.** (i) Using equations (3.6),(3.7) and Proposition 3.2, we see that \( K^T_0(Q^{2n}) \otimes_R \mathbb{Q}[\sqrt{1}] \) has a decomposition into 4 summands corresponding to different characters of \( Z_G \). Now the assertion follows from the form of the full exceptional collection on \( Q^{2k} \).

(ii) By Corollary 1.2, it is enough to check that this is true for the bundles of our exceptional collection. \( \square \)

Now let us consider odd-dimensional quadrics. These are homogeneous spaces of the form \( G/P \), where \( G = \text{Spin}(2k+1) \) is the simply connected group of type \( B_k \) \((k \geq 2)\), \( P \) is the maximal parabolic associated with the root \( \alpha_1 \) (the dimension of the quadric equals \( 2k-1 \)). The maximal torus \( T \subset G \) has the same description as in the case of type \( D_k \). However, the Weyl group is now bigger: it is generated by the \( \text{Weyl group is now bigger: it is generated by the} \)

By Proposition 3.6, the following element satisfies the condition (\(*\)):

\[
t_0 = (-1, -\zeta_{2k-1}, -\zeta^2_{2k-1}, \ldots, -\zeta^{k-1}_{2k-1}; x_0),
\]

where \( x_0 = i^{k, k^2(k-1)/2} \).

The center of \( \text{Spin}(2k+1) \) coincides with \( Z_0 = \ker(\text{Spin}(2k+1) \to \text{SO}(2k+1)) \). Its only nontrivial element is \( z_0 = (1, \ldots , 1; -1) \in T \). Let \( w_1 \in W \) be the element acting on \( T \) by the involution (3.8). Then

\[
w_1(t_0) = z_0 t_0.
\]

As in the case of even-dimensional quadrics, we can check that the conditions of Proposition 3.3 are satisfied for \( t_0 \). Namely, let \( N \) be the order of \( t_0 \). Then for every \( \sigma \in \text{Gal}(\mathbb{Q}(\sqrt{1})/\mathbb{Q}) \) we have a well defined set of signs \( s_i = \pm 1 \), \( i = 2, \ldots, k \), and a permutation \( w \) of \( 2, \ldots, k \), such that

\[
\sigma(t_0) = (-1, \zeta^{s_2 w(2)}_{2k-1}, \ldots, \zeta^{s_k w(k)}_{2k-1}; \sigma(x_0)).
\]

Thus, as before we have group homomorphisms \( \pi : \text{Gal}(\mathbb{Q}(\sqrt{1})/\mathbb{Q}) \to \mathbb{Z} \) and \( \sigma \mapsto \bar{w}_{\sigma} \), where \( \bar{w}_{\sigma}(\varepsilon_1) = \varepsilon_1 \) and

\[
\sigma(t_0) = \bar{w}_{\sigma}(t_0) z_0^\pi(\sigma).
\]

Hence, setting \( w_{\sigma} = \bar{w}_{\sigma} w_1^{\pi(\sigma)} \) we get

\[
\sigma(t_0) = w_{\sigma}(t_0).
\]

We have a full exceptional collection on \( Q^{2k-1} = G/P \) consisting of the \( 2k \) bundles \( (O, S, O(1), \ldots, O(2k-2)) \), where \( S \) is the spinor bundle (see [9]). The center of \( \text{Spin}(2k+1) \) acts trivially on line bundles, and nontrivially on \( S \). Thus, Propositions 3.2 and 3.3 give the following result.

**Corollary 3.12.** Let \( V \) be a central object in \( D(Q^{2k-1}/\text{Spin}(2k+1)) \).

(i) If \( V \) has trivial central character then the class \( v(V) \in K^T_0(Q^{2k-1}) \otimes_R \mathbb{Z}[\sqrt{1}] \) is a linear combination with integer coefficients of the classes \( v(O), v(O(1)), \ldots, v(O(2k-2)) \).

(ii) If \( V \) has a nontrivial central character then \( v(V) \) is an integer multiple of \( v(S) \).

As in the case of even-dimensional quadrics, we also obtain some information on central characters of objects in an arbitrary full exceptional collection.
Proposition 3.13. Let \((E_1, \ldots, E_{2k})\) be a full exceptional collection in \(D(Q^{2k-1})\). Then one of the objects \(E_i\) has a nontrivial central character, while the remaining \(2k-1\) have trivial central character.

Maximal isotropic Grassmannians

Let us denote by \(OG(k, n)\) (resp., \(SG(k, n)\)) the (connected component of) the Grassmannian of isotropic \(k\)-dimensional subspaces in an \(n\)-dimensional orthogonal (resp., symplectic) vector space. The maximal isotropic Grassmannians are \(OG(k, 2k)\), \(OG(k, 2k + 1)\) and \(SG(k, 2k)\). Note that there is an isomorphism \(OG(k, 2k + 1) \simeq OG(k + 1, 2k + 1)\). At present, the existence of full exceptional collections is not known in these cases (except in small dimensions). However, it is known that \(K_0^T(X)\) is a free \(R(T)\)-module (in fact, this is always true for generalized flag varieties \(X = G/P\), see [10]).

Let us first consider the orthogonal case: \(X = OG(k, 2k)\). Then \(X = G/P\), where \(G = \text{Spin}(2k)\), \(P\) is the maximal parabolic subgroup associated with \(\alpha_k\). Let us consider the same element \(t_0 = (1, \zeta_{2k-2}, \ldots, \zeta_{2k-2}^{-1}; x_0)\) as for the even-dimensional quadric. By Proposition 3.6, it satisfies the condition \((\ast)\). Hence, Propositions 3.2 and 3.3 are still applicable in this case. Note that the generator of the Picard group \(\mathcal{O}(1)\) in this case corresponds to the character \(\omega_k = (\sum \varepsilon_i)/2\), so \(Z_0\) acts on it nontrivially. Thus, from Proposition 3.2 we derive the following result.

Corollary 3.14. (i) Let \(N\) be the order of \(t_0\) (note that \(N|4(k-1)\)). Then we have an orthogonal decomposition

\[K_0^T(OG(k, 2k)) \otimes_R \mathbb{Z}[\sqrt{k}] = M \oplus M \cdot [\mathcal{O}(1)],\]

where \(M \subset K_0^T(OG(k, 2k)) \otimes_R \mathbb{Z}[\sqrt{k}]\) is the \(\mathbb{Z}[\sqrt{k}]\)-span of the classes of \(SO(2k)\)-equivariant bundles. If in addition \(k\) is odd then there is an orthogonal decomposition

\[M = M_0 \oplus M_0 \cdot [\mathcal{O}(2)],\]

where \(M_0 \subset M\) is the \(\mathbb{Z}[\sqrt{k}]\)-span of the classes of \(SO(2k)/\{\pm 1\}\)-equivariant bundles.

(ii) Let \(V\) be a central object in \(D(OG(k, 2k)/\text{Spin}(2k))\) with a nontrivial central character. Assume that \(k = 2^r + 1\). Then \(\chi(V) \equiv 0 \mod(2)\).

In the symplectic case we have \(X = SG(k, 2k) = \text{Sp}(2k)/P\), where \(P\) is the maximal parabolic associated with \(\alpha_k\). As in Proposition 3.6, let us consider the element

\[t_0 = (\zeta_{2k+2}, \zeta_{2k+2}^2, \ldots, \zeta_{2k+2}^{k})\]

satisfying the condition \((\ast)\). Let also \(w_1\) be the element of the Weyl group sending \((x_1, x_2, \ldots, x_k)\) to \((x_k^{-1}, \ldots, x_2^{-1}, x_1^{-1})\). Then we have

\[w_1(t_0) = zt_0,\]

where \(z = (-1, \ldots, -1) \in T\) is the generator of the center of \(\text{Sp}(2k)\). The generator of the Picard group \(\mathcal{O}(1)\) corresponds to the character \(\sum \varepsilon_i\). Hence, the element \(z\) acts on \(\mathcal{O}(1)\) as \((-1)^k\). As before, Proposition 3.2 gives the following result.

Corollary 3.15. (i) Let \(N = 2k + 2\). We have an orthogonal decomposition

\[K_0^T(SG(k, 2k)) \otimes_R \mathbb{Z}[\sqrt{k}] = M_+ \oplus M_-\]

where \(M_+\) (resp., \(M_-\)) is the \(\mathbb{Z}[\sqrt{k}]\)-span of the classes of \(\text{Sp}(2k)\)-equivariant bundles with trivial (resp., nontrivial) central character. If \(k\) is odd then \(M_- = M_+ \cdot [\mathcal{O}(1)]\).

(ii) Let \(V\) be a central object in \(D(SG(k, 2k)/\text{Sp}(2k))\) with a nontrivial central character. Assume that \(k = 2^r - 1\). Then \(\chi(V) \equiv 0 \mod(2)\).

The first interesting examples of maximal isotropic Grassmannians are \(SG(3, 6)\) and \(OG(5, 10)\) (note that \(SG(2, 4)\) and \(OG(4, 8)\) are smooth quadrics). There exists a full exceptional collection on \(SG(3, 6)\) (resp., \(OG(5, 10)\)) consisting of line bundles and vector bundles of rank 3 (resp., 5), see [12], 6.2 and 6.3 (also [19] in the case of \(SG(3, 6)\)). Hence, Corollary 1.2 leads to the following result.
Theorem 3.16. Every exceptional object on $SG(3,6)$ (resp., $OG(5,10)$) has an odd rank.

Hirzebruch surfaces and products

For an integer $n$ let us consider the ruled surface

$$F = F_n = \mathbb{P}(O_{p_1} \oplus O_{p_2}(n)) \rightarrow \mathbb{P}^1.$$  

We have a natural $GL_2$-action on $F$ induced by its action on $\mathbb{P}^1$. In addition, let us equi it with the fiberwise action of $\mathbb{G}_m$ that acts trivially on $O_{p_1}$ and by the identity character on $O_{p_2}(n)$. The maximal torus $T = \mathbb{G}_m^2 \subset GL_2$ acts on $F$ with 4 stable points: two on the fiber $\pi^{-1}(p_1)$ and two on $\pi^{-1}(p_2)$ (where $p_1$ and $p_2$ are $T$-stable points on $\mathbb{P}^1$). The tangent bundle $T_F$ to $F$ fits into the exact sequence

$$0 \rightarrow T_\pi \rightarrow T_F \rightarrow \pi^* T_{\mathbb{P}^1} \rightarrow 0,$$

where $T_\pi \cong O_F(2) \otimes \pi^* O_{p_2}(n)$. Hence, the weights of the action of $(x_1, x_2, u) \in T \times \mathbb{G}_m$ on the tangent spaces to the $T$-stable points are: (i) for two points in $\pi^{-1}(p_1)$: $(x_1^n u, x_1/x_2)$ and $(x_1^{-n} u^{-1}, x_1/x_2)$; (ii) for two points in $\pi^{-1}(p_2)$: $(x_2^n u, x_2/x_1)$ and $(x_2^{-n} u^{-1}, x_2/x_1)$. Thus, in the case when $n$ is even the condition (*) of Theorem 1.1 is satisfied for an element

$$t_0 = \begin{cases} (i, i^{n+1}, 1) & \text{if } n \equiv 2(4), \\ (i, i^{n-1}, -1) & \text{if } n \equiv 0(4) \end{cases}$$

of order 4. Thus, we deduce from Corollary 1.2 the following result.

Corollary 3.17. Let $X = F_{2n_1} \times \ldots \times F_{2n_r} \times \mathbb{P}^{2k_1-1} \times \ldots \times \mathbb{P}^{2k_r-1}$. Then for every exceptional object $E$ in $D(X)$ the class of $E$ in $K_0(X) \otimes \mathbb{Z}/2\mathbb{Z}$ coincides with the class of one of the line bundles. In particular, $\text{rk}(E)$ is odd.

Remark. Let $X$ be a smooth projective variety that admits a full exceptional collection $(E_i)$ (where $E_i \in D(X)$) consisting of objects of odd rank. Assume also that

$$\chi(x, y) \equiv \chi(y, x) \mod 2$$

for all $x, y \in K_0(X)$. Then expressing $[V] \in K_0(X)$ in terms of the basis $([E_i])$ one immediately checks that

$$\chi(V, V) \equiv \text{rk}(V) \mod 2.$$  

In particular, every exceptional object on $X$ has odd rank. The class of varieties with above properties is closed under products and includes projective spaces of dimension $2^k - 1$, Hirzebruch surfaces $F_n$ for even $n$, and varieties $OG(5,10)$ and $SG(3,6)$.

Relative projective spaces

Let $S$ be a smooth projective variety. We consider the relative projective space $\mathbb{P}_S^{n-1} = \mathbb{P}^{n-1} \times S$. Recall that $p_1, \ldots, p_n \in \mathbb{P}^{n-1}$ denote the $T$-stable points, where $T \subset GL_n$ is the standard maximal torus, and the action of $T \cong \mathbb{G}_m^n$ on the fiber of $\mathcal{O}(1)$ at $p_i$ is given by the projection to the $i$-th coordinate. Consider the element $t_0 \in T$ given by (3.3) and the corresponding homomorphism $R = R(T) \rightarrow \mathbb{Z}[\sqrt{T}]$. For an object $V \in D(\mathbb{P}_S^{n-1}/T)$ let us denote by $v(V)$ the corresponding class in $K_0^R(\mathbb{P}_S^{n-1}) \otimes R \mathbb{Z}[\sqrt{T}]$.

As before, for a central object in $D(\mathbb{P}_S^{n-1}/GL_n)$ with the central character $\lambda \mapsto \lambda^n$ we set $m(V) = m \mod(n)$. By Lemma 2.2, every exceptional object $E$ of $D(\mathbb{P}_S^{n-1})$ has a $GL_n$-equivariant structure, unique up to tensoring with a character. Hence, $m(E) \in \mathbb{Z}/n\mathbb{Z}$ is well defined.
Theorem 3.18. (i) For a central object $V \in D(\mathbb{P}^{n-1}_S / \text{GL}_n)$ set
\[ \tau(V) := \text{tr}(t_0, V|_{p_1 \times S}) \in K_0(S) \otimes \mathbb{Z}[\sqrt{n}] . \]
Then $\tau(V)$ belongs to $K_0(S) \subset K_0(S) \otimes \mathbb{Z}[\sqrt{n}]$, and
\[ v(V) = \tau(V) \cdot v(\mathcal{O}(m(V))) \] (3.9)
in $K_0^T(\mathbb{P}^{n-1}_S) \otimes_R \mathbb{Z}[\sqrt{n}]$.

(ii) Let $E$ be an exceptional object of $D(\mathbb{P}^{n-1}_S)$. Let us equip $E$ with a $\text{GL}_n$-equivariant structure and consider the element $\tau(E) \in K_0(S)$. Then
\[ \chi_S(\tau(E), \tau(E)) = 1 , \]
where $\chi_S$ denotes the Euler form on $K_0(S)$. If $(E_1, E_2)$ is an exceptional pair then either $m(E_1) \not\equiv m(E_2)$ or $\chi_S(\tau(E_2), \tau(E_1)) = 0$. If $(E_1, \ldots, E_N)$ is a full exceptional collection in $D(\mathbb{P}^{n-1}_S)$, then for each $m \in \mathbb{Z}/n\mathbb{Z}$ exactly $N/n$ objects $E_i$ have $m(E_i) \equiv m$, and the corresponding $N/n$ elements $\tau(E_i)$ form a semiorthogonal basis of $K_0(S)$.

(iii) Assume now that $n = p^r$, where $p$ is prime. Then for a central object $V \in D(\mathbb{P}^{n-1}_S / \text{GL}_n)$ one has the following congruence in $K_0(\mathbb{P}^{n-1}_S) \otimes \mathbb{Z}/p\mathbb{Z}$:
\[ [V] \equiv [V|_{p_1 \times S}] \cdot [\mathcal{O}(m(V))] . \]

If $E$ is an exceptional object of $D(\mathbb{P}^{n-1}_S)$ then
\[ \chi_S(E|_{p_1 \times S}, E|_{p_1 \times S}) \equiv 1 \mod(p) . \]

If $(E_1, E_2)$ is an exceptional pair with $m(E_1) \not\equiv m(E_2) \mod(n)$ then $\chi(E_1, E_2) \equiv 0 \mod(p)$. In the case $m(E_1) \equiv m(E_2) \mod(n)$ we have $\chi_S(E_2|_{p_1 \times S}, E_1|_{p_1 \times S}) \equiv 0 \mod(p)$.

Proof. (i) By the projective bundle theorem, we have a decomposition
\[ K_0^T(\mathbb{P}^{n-1}_S) = \bigoplus_{i=0}^{n-1} [\mathcal{O}(i)] \cdot K_0(S) \otimes R . \]
Hence,
\[ K_0^T(\mathbb{P}^{n-1}_S) \otimes_R \mathbb{Z}[\sqrt{n}] = \bigoplus_{i=0}^{n-1} M_i , \]
where $M_i$ is the $\mathbb{Z}[\sqrt{n}]$-submodule generated by $K_0(S) \cdot v(\mathcal{O}(i))$. Applying Proposition 3.2 we see that $M_i$ coincides with the submodule generated by the classes of $\text{GL}_n$-equivariant bundles $V$ with $m(V) \equiv i \mod(n)$. In particular, the submodules $M_i$ and $M_j$ are orthogonal with respect to the form $\text{Tr}(t_0, \chi^T(\cdot, \cdot))$ for $i \not= j$. Furthermore, the restriction to $p_1 \times S$ gives the inverse of the natural isomorphism
\[ K_0(S) \otimes \mathbb{Z}[\sqrt{n}] \xrightarrow{\sim} M_i : x \mapsto x \cdot [\mathcal{O}(i)] \]
(since $t_0$ acts trivially on the fiber of $\mathcal{O}(1)$ at $p_1$). This immediately implies (3.9). Note that $\tau(V)$ is the specialization at $t_0$ of the class of $V|_{p_1 \times S}$ in $K_0^T(S) = K_0(S) \otimes R$. Since $V$ is $\text{GL}_n$-equivariant, the class $V|_{p_1 \times S}$ is preserved under the action of $S_{n-1} \subset S_n$ that stabilizes $p_1$. Therefore, its specialization at $t_0$ is invariant under the Galois action on $\mathbb{Z}[\sqrt{n}]$ (permuting the nontrivial $n$-th roots of unity), so $\tau(V) \in K_0(S)$.

(ii) If $E$ is exceptional then we have $1 = \text{Tr}(t_0, \chi^T(E, E))$. Hence, using (3.9) we immediately deduce that $\chi_S(\tau(E), \tau(E)) = 1$. In the same way we get the vanishing of $\chi_S(\tau(E_2), \tau(E_1))$ in the case when $(E_1, E_2)$ is an exceptional pair with $m(E_1) \equiv m(E_2)$. This easily implies the last assertion.

(iii) This follows from (i) and (ii) using the homomorphism $\rho : \mathbb{Z}[\sqrt{n}] \rightarrow \mathbb{Z}/p\mathbb{Z}$. \hfill $\square$
References

[1] A. Beilinson, Coherent sheaves on $P^n$ and problems in linear algebra, Functional Anal. Appl. 12 (1978), no. 3, 214–216.
[2] A. Beilinson, J. Bernstein, P. Deligne, Faisceaux pervers, Astérisque, 100, Soc. Math. France, Paris, 1982.
[3] N. Bourbaki, Éléments de mathématique. Groupes et algèbres de Lie. Ch. IV–VI. Hermann, Paris, 1968.
[4] T. Bridgeland, t-structures on some local Calabi-Yau varieties, J. Algebra 289 (2005), no. 2, 453–483.
[5] A. Gorodentsev, S. Kuleshov, Helix theory, Mosc. Math. J. 4 (2004), no. 2, 377–440.
[6] M. Inaba, Toward a definition of moduli of complexes of coherent sheaves on a projective scheme, J. Math. Kyoto Univ. 42 (2002), no. 2, 317–329.
[7] T. Kambayashi, Projective representation of algebraic linear groups of transformations, American Journal of Math. 88, no. 1, 199–205.
[8] M. Kapranov, On the derived category of coherent sheaves on Grassmann manifolds, Math USSR Izvestiya 24 (1985), 183–192.
[9] M. Kapranov, Derived category of coherent bundles on a quadric, Functional Anal. Appl. 20 (1986), no. 2, 141–142.
[10] B. Kostant, S. Kumar, T-equivariant K-theory of generalized flag varieties, J. Diff. Geom. 32 (1990), 549–603.
[11] S. Kuleshov, D. Orlov, Exceptional sheaves on del Pezzo surfaces, Izvestiya Math. 44 (1995), no. 3, 479–513.
[12] A. Kuznetsov, Hyperplane sections and derived categories, Izvestiya Math. 70 (2006), no. 3, 447–547.
[13] M. Lieblich, Moduli of complexes on a proper morphism, J. Algebraic Geom. 15 (2006), no. 1, 175–206.
[14] I. M. Macdonald, Symmetric functions and Hall polynomials, Oxford University Press, New York, 1979.
[15] A. S. Merkurjev, Comparison of the equivariant and the standard K-theory of algebraic varieties, St. Petersburg Math. J. 9 (1998), no. 4, 815–850.
[16] H. A. Nielsen, Diagonalizable linearized coherent sheaves, Bull. Soc. Math. France 102 (1974), 85–97.
[17] D. Nogin, Helices on some Fano threefolds: constructivity of semiorthogonal bases of $K_0$, Ann. Sci. cole Norm. Sup. (4) 27 (1994), no. 2, 129–172.
[18] V. L. Popov, Picard groups of homogeneous spaces of linear algebraic groups and one-dimensional homogeneous vector fibrings, Math. USSR Izvestiya 8 (1974), 301–327.
[19] A. Samokhin, The derived category of coherent sheaves on $\text{LC}^2_3$, Russian Math. Surveys 56 (2001), 592–594.
[20] G. Segal, Equivariant K-theory, IHES Publ. Math. 34 (1968) 129–151.
[21] G. Vezzosi, A. Vistoli, Higher algebraic K-theory of group actions with finite stabilizers, Duke Math. J. 113 (2002), no. 1, 1–55.

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