Multisoliton solutions of 3,4,6 waves problems connected with semisimple algebras of the second rank $A_3, B_2 = C_2, G_2$

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Abstract

With each semisimple algebra it is possible to connect the system of interacting waves. The number of interacting fields coincides with the number of positive roots of corresponding semisimple algebra. Multisoliton solution of such kind problem are represented in explicit form for the algebras of second rank.

1 Introduction

In the present paper we would like to construct multisoliton solutions for $n$-waves interacting system in explicit form, using technique of discrete transformation theory introduced in [1]. The equation described $n$-waves interacting system are the following [1]

$$[(dh), f_i] + [(ch), f_x] = [[[dh], f], [(ch), f]]$$

(1)

where $f$ is algebra-valued unknown function (taking values in arbitrary semisimple algebra), $(x,t)$ independent arguments of the problem. In component form (1) looks as

$$(c_R \frac{\partial}{\partial t} + d_R \frac{\partial}{\partial x})f_R = \sum_P (c_{R-P}d_P - d_{R-P}c_P)f_{R-P}f_P$$

(2)

where by indexes $P, R$ are defined the set of all roots of semisimple algebra $c_P, d_P$ values of cartan elements $(ch), (dh)$ on these roots. The case of $A_2$ algebra was considered in details in [2] and multisoliton solutions in determinant form for this problem was present in [3]. In [4] this results was rediscovered in the form which allow generalisation on the case of all semisimple algebras of the second rank (see sections below).

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2 Solution of main equations in the case of lower triangular algebra

In this section we find solution of (2) under additional condition $f_R^+ = 0$ for all roots of semisimple algebra.

2.0.1 Simple roots

For each simple root equations (2) looks as

$$(c_\alpha \frac{\partial}{\partial t} + d_\alpha \frac{\partial}{\partial x}) f^{-}_\alpha = 0$$

with the obvious solution $f^{-}_\alpha = \Phi_\alpha(d_\alpha t - c_\alpha x)$. In the problem of the present paper for us it be essential parametrization of $f^{-}_\alpha$ in the form (of Laplace transformation)

$$f^{-}_\alpha = \int d\lambda_\alpha s^\alpha(\lambda_\alpha)e^{\lambda_\alpha(d_\alpha t - c_\alpha x)}$$

Thus arbitrary simple root $\alpha$ of algebra is connected with arbitrary function of one variable $s^\alpha(\lambda_\alpha)$.

2.0.2 Complicate root constructed from two simple ones

The equation for such component is the following

$$((c_\alpha + c_\beta) \frac{\partial}{\partial t} + (d_\alpha + d_\beta) \frac{\partial}{\partial x}) f^{-}_{\alpha,\beta} = (c_\alpha d_\beta - d_\alpha c_\beta)f^{-}_\alpha f^{-}_\beta$$

Without any difficulties it is possible verified that solution of this equation is the following

$$f^{-}_{\alpha,\beta} = \int d\lambda_\alpha d\lambda_\beta \frac{s^\alpha(\lambda_\alpha)s^\beta(\lambda_\beta)}{\lambda_\alpha - \lambda_\beta} e^{t(d_\alpha \lambda_\alpha + d_\beta \lambda_\beta - x(c_\alpha \lambda_\alpha + c_\beta \lambda_\beta))}$$

Plus solution of homogineous equation.

2.0.3 Complicate root constructed from three simple ones

It is possible two cases, when the third root $\gamma$ not equal to one of the previous ones $\alpha$ or $\beta$ or it coincides with one of them. At first let us consider the first possibility. The equation (2) in this case looks as

$$((c_\alpha + c_\beta + c_\gamma) \frac{\partial}{\partial t} + (d_\alpha + d_\beta + d_\gamma) \frac{\partial}{\partial x}) f^{-}_{\alpha,\beta,\gamma} =$$

$$(c_\alpha(d_\beta + d_\gamma) - d_\alpha(c_\beta + c_\gamma))f^{-}_\alpha f^{-}_{\beta,\gamma} + ((c_\alpha + c_\beta)d_\gamma - (d_\alpha + d_\beta)c_\gamma)f^{-}_{\alpha,\beta}f^{-}_{\gamma}$$

(5)
By direct check we find solution of the last equation in the form

$$f_{\alpha,\beta,\gamma}^- = \int d\lambda_\alpha d\lambda_\beta d\lambda_\gamma \frac{s^\alpha(\lambda_\alpha)s^\beta(\lambda_\beta)s^\gamma(\lambda_\gamma)}{(\lambda_\alpha - \lambda_\beta)(\lambda_\beta - \lambda_\gamma)} e^{t(d_\alpha\lambda_\alpha + d_\beta\lambda_\beta + d_\gamma\lambda_\gamma - x(c_\alpha\lambda_\alpha + c_\beta\lambda_\beta + c_\gamma\lambda_\gamma)}$$

In the second case, when for instance $\beta = \gamma$ the defining equation (5) does not contain one of the terms and in this case solution looks as

$$f_{\alpha,\beta,\beta}^- = \int d\lambda_\alpha d\lambda_\beta d\lambda_{\beta_2} \frac{s^\alpha(\lambda_\alpha)s^\beta(\lambda_{\beta_1})s^\beta(\lambda_{\beta_2})}{(\lambda_\alpha - \lambda_{\beta_1})(\lambda_{\alpha} - \lambda_{\beta_2})} e^{t(d_\alpha\lambda_\alpha + d_\beta(\lambda_{\beta_1} + \lambda_{\beta_2}) - x(c_\alpha\lambda_\alpha + c_\beta(\lambda_{\beta_1} + \lambda_{\beta_2}))}$$

### 3 The case of $A_2$ algebra

This section is written only for convinience of the reader and contain necessary for what follows results of [2],[4]. The results concerning $B_2$ and $G_2$ algebras will be presented below in the same form.

Algebra $A_2$ has the following Cartan matrix and basic commutation relations between two generators of the simple roots $X^\pm_{1,2}$ and its Cartan elements $h_{1,2}$

$$k = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad \begin{array}{c} [h_1, X^\pm_1] = \pm 2X^\pm_1 \\ [h_2, X^\pm_2] = \mp X^\pm_2 \end{array}$$

Arbitrary element of the algebra may be represented as (up to Cartan elements which are absent in the problem under consideration)

$$f = f_{1,1}^+ X^+_{\alpha_1+\alpha_2} + f_{0,1}^+ 2X^0_{\alpha_2} + f_{1,0}^+ X^+_{\alpha_1} + f_{0,0}^+ X^-_{\alpha_1} + f_{1,1}^- X^-_{\alpha_1+\alpha_2}$$

$\alpha_{1,2}$ are the indexes of simple roots. $X^+_{\alpha_1+\alpha_2} = [X^+_1, X^+_2]$.

In these notations the system of equations (2) looks as

$$D_{1,0} f_{1,0}^+ = f_{1,1}^+ f_{0,0}^-, \quad D_{1,0} f_{0,0}^- = f_{1,1}^- f_{0,1}^+$$
$$D_{0,1} f_{0,1}^+ = f_{1,1}^+ f_{1,0}^-, \quad D_{0,1} f_{1,0}^- = f_{1,1}^- f_{1,0}^+$$

(6)

$$D_{1,1} f_{1,1}^+ = - f_{0,1}^+ f_{1,0}^-, \quad D_{1,1} f_{1,0}^- = - f_{0,1}^- f_{1,0}^+$$

where operators of differenciation are the following ones $D_{i,j} = \frac{(i\epsilon_1 + j\epsilon_2) \partial}{\partial x^i} + \frac{\partial}{\partial x^j}, \delta \equiv (c_1d_2 - c_2d_1)$.

The discrete transformation of this system are the following ones [2]

#### 3.0.4 $T_3$

The system (6) is invariant with respect to the following transformation $T_3$

$$\tilde{f}_{1,1}^+ = \frac{1}{f_{1,1}}, \quad \tilde{f}_{1,0}^+ = - \frac{f_{0,1}^-}{f_{1,1}}, \quad \tilde{f}_{0,1}^- = \frac{f_{1,0}^-}{f_{1,1}}$$
\[
\tilde{f}_{0,1} = -f_{1,1} D_{0,1} \frac{f_{0,1}}{f_{1,0}}, \quad \tilde{f}_{1,0} = f_{1,0} D_{0,1} \frac{f_{1,0}}{f_{1,1}}
\]

\[
\frac{\tilde{f}_{1,1}}{f_{1,1}} = f_{0,1} f_{1,0} - D_{1,0} D_{0,1} \ln f_{1,1}
\]

3.0.5 \quad T_2

The system (6) is invariant with respect to the following transformation \(T_2\)

\[
\tilde{f}_{0,1}^+ = \frac{1}{f_{0,1}}, \quad \tilde{f}_{1,0}^- = -\frac{f_{1,1}}{f_{0,1}}, \quad \tilde{f}_{1,1}^+ = -\frac{f_{0,1}^+}{f_{0,1}}
\]

\[
\tilde{f}_{1,1}^- = -f_{0,1} D_{1,1} \frac{f_{0,1}^+}{f_{0,1}}, \quad \tilde{f}_{1,1}^- = -f_{0,1} D_{1,0} \frac{f_{1,1}^-}{f_{0,1}}
\]

\[
\frac{\tilde{f}_{0,1}^-}{f_{0,1}^-} = f_{0,1} f_{0,1}^- + D_{1,0} D_{1,1} \ln f_{0,1}^-
\]

3.0.6 \quad T_1

The system (6) is invariant with respect to the following transformation \(T_1\)

\[
\tilde{f}_{1,0}^+ = \frac{1}{f_{1,0}}, \quad \tilde{f}_{1,0}^- = \frac{f_{1,1}}{f_{1,0}}, \quad \tilde{f}_{1,1}^+ = -\frac{f_{0,1}^+}{f_{1,0}}
\]

\[
\tilde{f}_{0,1}^+ = f_{0,1} D_{1,1} \frac{f_{0,1}^+}{f_{1,0}}, \quad \tilde{f}_{1,0}^- = f_{1,0} D_{0,1} \frac{f_{1,0}^-}{f_{1,0}}
\]

\[
\frac{\tilde{f}_{0,1}^-}{f_{0,1}^-} = f_{1,0}^+ f_{1,0}^- + D_{0,1} D_{1,1} \ln f_{1,0}^-
\]

3.0.7 \quad General properties of discrete transformations

Three above transformations are invertable. This means \(f\) may be expressed algebraically in terms of \(\tilde{f}\). Exept of this \(T_3 = T_1 T_2 = T_2 T_1\), what means that all discrete transformation are mutually commutative. This in its turn means that arbitrary discrete transformation may be represented in a form \(T = T_1^{a_1} T_2^{a_2}\) [2].

3.0.8 \quad Result of consequent application of some number of discrete transformations

In determinant form result was found in [2]. In [4] it was rediscovered in the form which can be generalised to the case of arbitrary semi-simple algebra of second rank (see sections below).
Let us rewrite (3) and (4) in a form

\[ f_{1,0}^{-} = \int d\lambda P(\lambda) e^{\lambda(d_1 t - c_1 x)} \equiv \int d\lambda P(\lambda), \quad f_{0,1}^{-} = \int d\mu Q(\mu) e^{\mu(d_2 t - c_2 x)} \equiv \int d\mu Q(\mu), \]

and choose initial condition in a form where

\[ f_{1,1}^{-} = \int d\lambda \int d\mu P(\mu) Q(\lambda) e^{\lambda(d_1 t - c_1 x) + \mu(d_2 t - c_2 x)} \equiv \int d\lambda \int d\mu \frac{P(\lambda)Q(\mu)}{\lambda - \mu} \]

and solution of this kind application of each inverse transformation \( T_i^{-1} \) is meant with zeros in denominators.

Let us introduce determining function \( U(n_1, n_2) \)

\[ U(n_1, n_2) = \prod_{i=1}^{n_1} P(\lambda_i) d\lambda_i \prod_{k=1}^{n_2} Q(\mu_k) d\mu_k \frac{W_{n_1}^2(\lambda)W_{n_2}^2(\mu)}{\prod_{i,k}(\lambda_i - \mu_k)} \tag{10} \]

where \( W_n \) is Vandermond determinant constructed from \( n_{1,2} \)-variables \( \lambda \) or \( \mu \).

Then result of application \( n_1 \) times discrete transformation \( T_1 \) and \( n_2 \) times discrete transformation \( T_2 \) looks as [4]

\[ f_{1,0}^+ = \frac{U(n_1 + 1, n_2)}{U(n_1, n_2)}, \quad f_{0,1}^- = \frac{U(n_1 + 1, n_2)}{U(n_1, n_2)}, \]

\[ f_{1,1}^+ = \frac{U(n_1 - 1, n_2 - 1)}{U(n_1, n_2)}, \quad f_{0,1}^- = \frac{U(n_1 + 1, n_2 + 1)}{U(n_1, n_2)}, \tag{11} \]

In the case of integrable systems connected with \( A_1 \) algebra Vandermond form for soliton solutions was introduced and used in [6].

Let us assume that initial functions has the form

\[ P(\lambda) = \sum_{i=1}^{n_1} \delta(\lambda - l_i) e^{\lambda(d_1 t - c_1 x)} \]

and

\[ Q(\mu) = \sum_{i=1}^{n_2} \delta(\mu - m_i) e^{\mu(d_2 t - c_2 x)}, \]

where \( \delta(x) \) is the usual Dirac \( \delta \) function. In this case as it follows from (11) on the \( n_1, n_2 \) step of discrete transformation \( f_{1,0}^- = f_{0,1}^- = f_{1,1}^- = 0 \) the chain is interrupted on the second end. And from results of [2] conditions of reality lead directly to soliton solutions.

### 4 The case of \( B_2 \) algebra

Algebra \( B_2 \) (equivalent to the algebra of the group of 5-dimensional rotations) has the following Cartan matrix and basics comutation relations

\[ k = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}, \quad \begin{pmatrix} [h_1, X_1^+] = \pm 2X_1^+ & [h_1, X_2^+] = \mp X_2^+ \\ [h_2, X_1^+] = \mp 2X_2^+ & [h_2, X_2^+] = \pm X_2^+ \end{pmatrix} \]
Arbitrary element of this algebra can be parametrised as (up to elements taking values in Cartan subalgebra)

\[ f = f_{1,2}^+ x_{α_1 + 2α_2}^+ + f_{1,1}^+ x_{α_1 + α_2}^+ + f_{0,1}^+ x_{α_1}^+ + f_{1,0}^+ x_{α_1}^+ + f_{1,1}^- x_{α_1}^- + f_{0,1}^- x_{α_1}^- + f_{1,2}^- x_{α_1 + 2α_2}^- \]

\( α_{1,2} \) are the indexes of simple roots. In these notations the system of equations (2) looks as

\[
\begin{align*}
D_{1,0} f_{1,1}^+ &= 2 f_{1,1}^+ f_{0,1}^- + D_{1,0} f_{1,0}^- = 2 f_{1,1}^+ f_{0,1}^- \\
D_{0,1} f_{0,1}^+ &= f_{1,1}^+ f_{1,0}^- + f_{1,2}^+ f_{1,1}^- , & D_{0,1} f_{0,1}^- &= f_{1,1}^+ f_{1,0}^- + f_{1,2}^+ f_{1,1}^- \\
D_{1,1} f_{1,1}^- &= -f_{0,1}^+ f_{1,0}^- + f_{1,2}^+ f_{0,1}^+ , & D_{1,1} f_{1,0}^- &= -f_{0,1}^+ f_{1,0}^- + f_{1,2}^+ f_{0,1}^+ \\
D_{1,2} f_{1,2}^- &= -2 f_{1,1}^+ f_{0,1}^- , & D_{1,2} f_{1,2}^- &= -2 f_{1,1}^+ f_{0,1}^- 
\end{align*}
\]

(12)

where \( D_{i,j} = \left( \frac{c_1 + c_2}{δ} + \frac{c_1 + c_2}{δ} \right) δ \), \( δ \equiv (c_1 d_2 - c_2 d_1) \). The above system of equations (12) is the wide system of interaction of eight fields \( f_{1,0}^+, f_{0,1}^+, f_{1,1}^+, f_{1,2}^- \). This system is obviously allow reducing \((f^-)^H = f^+\) (under additional assumption that all operators of differentiations are real). Exactly such kind solutions of this system is intersting for applications and will be the point of investigation in the present section.

The discrete transformation to the such kind of the systems in the case of arbitrary semisimple algebra was found in [1]. The formulae below are partial case of application general construction for the case of \( B_2 \) algebra. They can be easily checked by direct not combersome calculation.

4.1 Discrete transformation \( T_M \)

The system (12) invariant with respect to following transformation \( T_M \) (discrete transformation of the maximal root of \( B_2 \) algebra)

\[
\begin{align*}
\tilde{f}_{1,2}^+ &= \frac{1}{f_{1,2}^+} , & \tilde{f}_{0,1}^- &= \frac{f_{1,1}^-}{f_{1,2}^-} , & \tilde{f}_{1,1}^- &= -\frac{f_{0,1}^-}{f_{1,2}^-} , & \tilde{f}_{1,0}^+ &= f_{1,0}^+ + \frac{(f_{0,1}^-)^2}{f_{1,2}^-} , & \tilde{f}_{0,1}^- &= f_{0,1}^- - \frac{(f_{1,1}^-)^2}{f_{1,2}^-} \\
\tilde{f}_{1,1}^- &= -D_{1,0} f_{1,1}^+ f_{1,2}^- - f_{1,1}^+ f_{1,2}^+ + \frac{1}{2} f_{0,1}^- D_{1,0} \ln f_{1,2}^- , & \tilde{f}_{1,1}^- &= -D_{1,0} f_{1,1}^- f_{1,2}^+ + f_{1,1}^- f_{1,2}^+ + \frac{1}{2} f_{1,1}^+ D_{1,0} \ln f_{1,2}^- \\
\tilde{f}_{1,2}^- &= \frac{1}{4} D_{1,0} D_{1,0} \ln f_{1,2}^- + \frac{f_{1,1}^+ D_{1,0} f_{0,1}^- f_{1,2}^- - f_{0,1}^- D_{1,0} f_{1,1}^-}{2 f_{1,1}^-} + f_{1,2}^+ f_{1,2}^- + f_{1,1}^- f_{1,1}^+ + f_{0,1}^- f_{0,1}^- 
\end{align*}
\]

By direct not combersome calculations it is possible verified that \( \tilde{f} \) satisfy (12) if \( f \) is solution of this system.
4.2 Discrete transformation $T_{1,0}$

$T_{1,0}$-discrete transformation of the first ($\alpha_1$) root of $B_2$ algebra

$$\tilde{f}_{1,0}^+ = \frac{1}{f_{1,0}}, \quad \tilde{f}_{0,1}^+ = \frac{f_{1,1}}{f_{1,0}}, \quad \tilde{f}_{1,1} = -\frac{f_{1,1}^+}{f_{1,0}}, \quad \tilde{f}_{1,2}^+ = f_{1,2}^+ + \frac{(f_{0,1}^+)^2}{f_{1,0}}, \quad \tilde{f}_{1,2}^- = f_{1,2}^- - \frac{(f_{1,1}^+)^2}{f_{1,0}}$$

$$\tilde{f}_{0,1}^+ = D_{1,2}f_{0,1} - f_{1,1}^+f_{1,0} - \frac{1}{2}f_{0,1}^+D_{1,2}\ln f_{1,0}, \quad \tilde{f}_{1,1} = D_{1,2}f_{1,1}^- + f_{0,1}^-f_{1,0} - \frac{1}{2}f_{1,1}^-D_{1,2}\ln f_{1,0}$$

(13)

$$\frac{\tilde{f}_{1,0}^-}{f_{1,0}} = \frac{1}{4}D_{1,2}D_{1,2}\ln f_{1,0} + \frac{f_{1,1}^-D_{1,2}f_{1,1}^+ - f_{0,1}^-D_{1,2}f_{1,1}^-}{2f_{1,0}} + f_{1,0}^-f_{1,1}^-f_{1,1} + f_{0,1}^-f_{0,1}$$

The most important property of this transformation consists in the fact that if we begin from the initial solution $f_{1,0}^+ = f_{0,1}^+ = f_{1,1}^+ = f_{1,2}^- = 0$ the last three function contains there zero values after discrete transformation $T_{1,0}$ as it follows from (13).

4.3 Inverse discrete transformation $T_{1,0}^{-1}$

To construct inverse $T_{1,0}^{-1}$ it is necessary formulae (13) resolve with respect to untilded functions. Result is the the following one

$$f_{1,0}^- = \frac{1}{f_{1,0}}, \quad f_{0,1}^- = -\frac{f_{1,1}^+}{f_{1,0}}, \quad f_{1,1} = \frac{f_{0,1}^-}{f_{1,0}}, \quad f_{1,2}^+ = \frac{f_{1,2}^+}{f_{1,0}}, \quad f_{1,2}^- = \frac{(f_{1,1}^+)^2}{f_{1,0}}$$

$$f_{0,1}^- = -D_{1,2}\tilde{f}_{0,1}^- - f_{1,1}^-f_{1,0} - \frac{1}{2}\tilde{f}_{0,1}^-D_{1,2}\ln f_{1,0}, \quad f_{1,1}^- = -D_{1,2}\tilde{f}_{1,1}^- + f_{0,1}^-f_{1,0} - \frac{1}{2}\tilde{f}_{1,1}^-D_{1,2}\ln f_{1,0}$$

(14)

$$\frac{f_{1,0}^+}{f_{1,0}} = \frac{1}{4}D_{1,2}D_{1,2}\ln f_{1,0} - \frac{f_{1,1}^-D_{1,2}f_{1,1}^+ - f_{0,1}^-D_{1,2}f_{1,1}^-}{2f_{1,0}} + f_{1,0}^-f_{1,1}^-f_{1,1} + f_{0,1}^-f_{0,1}$$

4.4 Discrete transformation $T_{2\alpha_2}$

In the case of $B_2$ algebra it is not possible to coinide with the help of the Weil group the maximal root with $X_{\alpha_2}^\pm$. But having in mind $T_M$ and $T_1$ it is possible to construct discrete transformation $T_{2\alpha_2} \equiv T_{3\alpha_1+2\alpha_2}T_{\alpha_1} = T_M T_{\alpha_1}^{-1}$ which we will call discrete transformation of the second root and which looks as

$$\tilde{f}_{1,0}^+ = \frac{f_{1,0}^+}{f_{1,0}^+ + \frac{(f_{0,1})^2}{f_{1,0}^+}}, \quad \tilde{f}_{1,1} = \frac{-D_{1,2}f_{0,1} - f_{1,1}^-f_{1,2}^+ + \frac{f_{0,1}^-D_{1,2}f_{1,1}^-}{f_{1,0}^+}}{f_{1,0}^+ + \frac{(f_{0,1})^2}{f_{1,0}^+}}, \quad \tilde{f}_{0,1}^- = \frac{f_{0,1}^-}{f_{1,0}^+ + \frac{(f_{0,1})^2}{f_{1,0}^+}}$$

$$\tilde{f}_{1,2}^+ = \frac{f_{1,0}^+}{f_{1,0}^+f_{1,2}^+ + (f_{0,1})^2}, \quad \tilde{f}_{1,1}^- = \frac{f_{1,0}^-D_{1,2}f_{0,1} - f_{1,1}^-f_{1,2}^- + \frac{f_{0,1}^-D_{1,2}f_{1,2}^-}{f_{1,0}^+}}{f_{1,0}^+ + \frac{(f_{0,1})^2}{f_{1,0}^+}}$$

$$\tilde{f}_{1,2}^- = \frac{f_{1,0}^-D_{1,2}f_{0,1} - f_{1,1}^-f_{1,2}^+ + \frac{f_{0,1}^-D_{1,2}f_{1,2}^-}{f_{1,0}^+}}{f_{1,0}^+ + \frac{(f_{0,1})^2}{f_{1,0}^+}}$$

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From the last formulae it follows that $T_{2n}$ conserve zero values for wave functions $f_{1,0}^+ = f_{1,1}^+ = f_{1,2}^+ = 0$ under initial conditions under consideration. By this reason we present further formulae after substitution in them these values. Because the general formulae are sufficiently combersome.

$$\frac{\dot{f}_{0,1}^-}{f_{0,1}^-} = D_{1,0}^2 \ln f_{0,1}^- + f_{0,1}^+ f_{0,1}^-, \quad \dot{f}_{0,1}^+ = \frac{1}{f_{0,1}^-}$$

This is the equations of the usual Toda lattice solution of which are well known. And at last after some number of manipulations we obtain

$$\dot{f}_{1,2}^- = \frac{1}{4} D_{1,0}^2 f_{1,2}^- + \frac{f_{1,1}^- D_{1,0} f_{0,1}^- - f_{0,1}^- D_{1,0} f_{1,1}^-}{2} + \frac{D_{1,0} f_{0,1}^- D_{1,0} f_{0,1}^-}{(f_{0,1}^-)^2} f_{1,2}^- - \frac{D_{1,0} f_{0,1}^- D_{1,0} f_{1,2}^-}{f_{0,1}^-} + f_{0,1}^- f_{0,1}^+ f_{1,2}^-$$

4.5 Linear integrable chain of the second simple root in the field of Toda lattice

Let us consider consequent action of discrete transformation of the previous subsection on the initial solution $f_{1,0}^+ = f_{0,1}^- = f_{1,1}^+ = f_{1,2}^- = 0$. As it follows from explicit formulae for discrete transformation zero values for $f_{1,0}^+ = f_{1,1}^+ = f_{1,2}^- = 0$ are conserved. Equations for $f_{0,1}^-$, $f_{0,1}^+$ functions are typical one dimension Toda chain with well known solution. Result of the application $n$-times of Toda discrete transformation to initial function $f_{0,1}^- = r$ is the following

$$f_{0,1}^+ = \frac{\text{Det}_{n-1}}{\text{Det}_n}, \quad f_{0,1}^- = \frac{\text{Det}_{n+1}}{\text{Det}_n}$$

(15)

where $\text{Det}_n$ are the the main minores from left upper corner of the matrix

$$\Delta = \begin{pmatrix} r & r_1 & r_11 & \ldots \\ r_1 & r_11 & r_111 & \ldots \\ r_11 & r_111 & r_1111 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

(16)

where $r_1 \equiv D_{1,0} r$ and so on. In these notations equations of Toda latice looks as

$$D_{1,0}^2 (\ln D_n) = \frac{\text{Det}_{n-1} \text{Det}_{n+1}}{\text{Det}_n^2}$$

(17)

Initial function of the second simple root $r = f_{0,1}^- = \int d\mu q(\mu) e^{\mu(d_2 t - c x)} \equiv \int d\mu Q$. In $Q$ all dependence on arguments $x, t$ is included. Obviously $D_{0,1} Q = 0$. In these notations $\text{Det}_n$ may be presented in the form of multidimensional integrale

$$\text{Det}_n = \frac{1}{n!} \int \prod_{i=1}^n d\mu_i Q_{\mu_i} W_n^2(\mu_1, \ldots, \mu_n)$$

where $W_n^2(\mu_1, \ldots, \mu_n)$ is square of Vandermond determinant.
4.5.1 Resolving $T_{2\alpha_2}$ discrete transformation chain

The $T_{2\alpha_2}$ discrete transformation may be presented as linear chain of equations for only two unknown functions $f_{1,1}^-, f_{1,2}^-$. Parameterizing them in the form $(f_{1,1}^-)_n = \frac{A^n}{2^n}$, $(f_{1,2}^-)_n = \frac{B^n}{2^n}$ we pass to the following chain of equations

$$A^{n+1} = \frac{\frac{1}{2}D_{n+1}B^n - B^n(D_{n+1})_1}{D_n}$$

$$ \frac{B^{n+1}}{D_{n+1}^2} = \frac{1}{D_n^2}\left[\frac{1}{4}B^n_1 - (D_{n+1})_1 B^n_1 + \left(\frac{(D_{n+1})_1}{D_{n+1}}\right)^2 B^n_1\right] + \frac{A^n(D_{n+1})_1 - D_{n+1}A^n_1}{2D_n^3} + \frac{D_{n+1}(A^n(D_{n+1})_1 + B^n D_{n-1})}{2D_n^4}$$

The last unknown function in these notations looks as $f_{1,0}^- = \frac{B^{n-1}}{D_n^2}$.

Let us resolve (19) by induction, assuming (see appendix), that

$$A^n = \frac{1}{n!(n+1)!} \int P(\lambda) d\lambda \prod_{i=1}^{(2n+1)} Q(\mu_i)d\mu_i \frac{W^2_n(\mu_1, \ldots \mu_n)W^2_{n+1}(\mu_{n+1}, \ldots \mu_{2n+1})}{\prod_{i=1}^{(2n+1)}(\lambda - \mu_i)}$$

$$B^n = \frac{1}{(n+1)!(n+1)!} \int P(\lambda) d\lambda \prod_{i=1}^{(2n+2)} Q(\mu_i)d\mu_i \frac{W^2_{n+1}(\mu_{1}, \ldots \mu_{n+1})W^2_{n+2}(\mu_{n+2}, \ldots \mu_{2n+2})}{\prod_{i=1}^{(2n+2)}(\lambda - \mu_i)}$$

From this moment all factorial factors are included in definition of Vandermond determinant. Substituting assumed form of solution into (18) we obtain integral function in a form

$$\frac{W^2_{n+1}(\mu_1, \ldots \mu_{n+1})W^2_{n+2}(k_1, \ldots k_{n+1})}{\prod_{i=1}^{(n+1)}(\lambda - \mu_i)(\lambda - k_i)} \left(\frac{1}{2} \sum_{i=1}^{n+1} (\mu_i + k_i) - \sum_{i=1}^{n+1} d_i\right)W^2_{n+1}(d_1, \ldots, d_{n+1})$$

The sum with factor $\frac{1}{2}$ arises after differentiation $B^n$ on $D_{1,0}$. The second sum arises after differentiation of $D_{n+1}$ on the same operator. But domain of integration is symmetrical with respect $(2n + 2)$ parameters $(m, k)$ and $(n + 1)$ parameters $d$. Thus it is possible rewrite the last expression in a form

$$\frac{W^2_{n+1}(\mu_1, \ldots \mu_{n+1})W^2_{n+2}(k_1, \ldots k_{n+1})}{\prod_{i=1}^{(n+1)}(\lambda - \mu_i)(\lambda - k_i)} \left(\sum_{i=1}^{n+1} (\mu_i - d_i)W^2_{n+1}(d_1, \ldots, d_{n+1}) \right)$$

Now let us compare the last expression with solution in the case of $A_2$ arising after application $n$ times discrete transformation $T_2$. For interesting us functions in connection with (11) we have

$$f_{1,0}^- = \frac{V(1, n + 1)}{V(0, n + 1)} \quad f_{1,1}^- = \frac{V(1, n + 2)}{V(0, n + 1)} \quad f_{0,1}^+ = \frac{V(0, n + 1)}{V(0, n + 1)}$$
We rewrite now equation for $f_{1,0}$ (6) conserving only integrant function in both sides
\[
\frac{W_n^2(\mu_1, \ldots, \mu_{n+1})}{\prod_{i=1}^{n+1}(\lambda - \mu_i)} \left( \sum_{i=1}^{n+1} \mu_i - \sum_{i=1}^{n+1} d_i \right) W_{n+1}^2(d_1, \ldots, d_{n+1}) = \\
\frac{W_{n+2}^2(\mu_1, \ldots, \mu_{n+2})}{\prod_{i=1}^{n+2}(\lambda - \mu_i)} W_n^2(d_1, \ldots, d_n)
\]
where $\mu_{n+2} = d_{n+1}$ (to have the same integrable indexes in both sides of the last equality). The last equality after substitution it into (22) finish the proof of recurrent relation for $A^{n+1}$ term of the chain above.

Now we pass to calculation of $B^{n+1}$ defined by (19). The terms in the second row of this equality may be rewritten in a equivalent form
\[
\frac{A^n(D_{n+1}D_n) - D_{n+1}D_n A^n + B^n D_{n+1}D_{n-1}}{2D_n^4}
\]
Substituting into the last expression assuming by reduction form of $A^n$ and using once more (23) we represent it as
\[
\int P(\lambda)d\lambda \prod_{i=1}^{n+2} Q(\mu_i) d\mu_i \prod_{i=1}^n Q(\nu_i) d\nu_i \frac{W_n^2(\nu_1, \ldots, \nu_n) W_{n+2}^2(\mu_1, \ldots, \mu_{n+2})}{\prod_{i=1}^{n+1}(\lambda - \mu_i) \prod_{i=1}^{n+2}(\lambda - \nu_i)}
\]
The summation the terms of the first row of (19) leads to the following result in numerator
\[
\frac{W_{n+1}^2(\mu_1, \ldots, \mu_{n+1}) W_{n+1}^2(\nu_1, \ldots, \nu_{n+1})}{\prod_{i=1}^{n+1}(\lambda - \mu_i) \prod_{i=1}^{n+1}(\lambda - \nu_i)}
\]
\[
\left( \sum_{i=1}^{n+1} \frac{\mu_i + \nu_i}{2} - \sigma_i \right) \left( \sum_{i=1}^{n+1} \frac{\mu_i + \nu_i}{2} - \delta_i \right) W_{n+1}^2(\sigma_1, \ldots, \sigma_{n+1}) W_{n+1}^2(\delta_1, \ldots, \delta_{n+1})
\]
In denumerator we have $D_n^2 D_{n+1}^2$. The origin of the terms in (24) is as follows. Each term contain linear $B^n$ and its derivatives. We parametrised this function by $\nu, \mu$ parameters. Each term contains second degree of $D_n+1$ or its derivatives. We parametrised them by independent parameters $\sigma, \delta$. The first derivatives of $B^n$ function leads to multiplicator $\sum_{i=1}^{n+1}(\mu_i + \nu_i)$. Keeping in mind the factor $\frac{1}{4}$ at the first term in (19) we explain corresponding term in (24). Quadratical in derivates on $(D_{n+1})_1$ terms leads obviously to product of sums of $\sigma$ and $\delta$ parameters and so on.

Now strategy of the further calculations will be the following. We multiply both sides of (19) on $D_n^2$ and use once more (23) in the back direction we come to the following expression in the left side:
\[
\frac{W_{n+2}^2(\mu_1, \ldots, \mu_{n+2})}{\prod_{i=1}^{n+2}(\lambda - \mu_i)} W_n^2(\sigma_1, \ldots, \sigma_n) \frac{W_{n+2}^2(\nu_1, \ldots, \nu_{n+2})}{\prod_{i=1}^{n+2}(\lambda - \nu_i)} W_n^2(\delta_1, \ldots, \delta_n)
\]
where determinant of Vandermond parametrized by \( \mu, \nu \) arised from the assumed form of \( B^{n+1} \) (21), parametrised by \( \delta, \sigma \) ones from \( D^n \). Now let us use (23) twice. We obtain

\[
\frac{W_{n+1}^2(\mu_1, \ldots, \mu_{n+1})}{\prod_{i=1}^{n+1}(\lambda - \mu_i)} \left( \sum_{i=1}^{n+1} \mu_i - \sum_{i=1}^{n+1} \delta_i \right) W_{n+1}^2(\delta_1, \ldots, \delta_{n+1}) \times
\]

\[
\frac{W_{n+1}^2(\nu_1, \ldots, \nu_{n+1})}{\prod_{i=1}^{n+1}(\lambda - \nu_i)} \left( \sum_{i=1}^{n+1} \nu_i - \sum_{i=1}^{n+1} \sigma_i \right) W_{n+1}^2(\sigma_1, \ldots, \sigma_{n+1})
\]

The structure of last expression exactly the same as in (24)- the combination terms in the right side of (21). Different only multiplicators. Difference of them is the following (\( \mu \equiv \sum_{i=1}^{n+1} \mu_i \) and so on)

\[
(\frac{\mu + \nu}{2} - \sigma)(\frac{\mu + \nu}{2} - \delta) - (\mu - \delta)(\nu - \sigma) = \left(\frac{\mu - \nu}{2}\right)^2
\]

In the last expression dependence on parameters \( \delta, \sigma \) is factorised and leads to \( D_{n+1}^2 \) which is canceled with the same term in denominator. The remaining relation, which have been proved is the following

\[
\frac{W_{n+1}^2(\nu_1, \ldots, \nu_{n+1})}{\prod_{i=1}^{n+1}(\lambda - \nu_i)} \left( \sum_{i=1}^{n+1} \nu_i - \sum_{i=1}^{n+1} \mu_i \right)^2 W_{n+1}^2(\mu_1, \ldots, \mu_{n+1}) - \frac{1}{2} \frac{W_{n+1}^2(\nu_1, \ldots, \nu_n) W_{n+2}^2(\mu_1, \ldots, \mu_{n+2})}{\prod_{i=1}^{n+1}(\lambda - \mu_i) \prod_{k=1}^{n}(\lambda - \nu_k)} = 0
\]

But the last equality exactly coincides with equations of the Toda lattice (17). Thus the solution of the linear chain (18) and (19) is proved by induction. Now to this solution it is necessary applicate \( n+1 \) times discrete transformation of the first simple root \( T_1 \). We will not do corresponding calculations but in the next subsection will consider discrete transformation chain of the first simple root.

From this consideration general form of solution will be obvious.

### 4.5.2 Resolving \( T_{1,0} \) discrete transformation chain

As it was mentioned after explicit formulae of discrete transformation (13) after action of this transformation on the initial functions \( f_{1,0}^+ = f_{0,1}^+ = f_{1,1}^+ = f_{1,2}^+ = 0 \) the last three functions conserved there zero values. The remaining equations describing arising chain looks as

\[
f_{1,0}^+ = \frac{1}{f_{1,0}^-}, \quad \tilde{f}_{0,1}^- = \frac{\tilde{f}_{1,1}^-}{f_{1,0}^-}, \quad \tilde{f}_{1,2}^- = f_{1,2}^- - \frac{(\tilde{f}_{1,1}^-)^2}{f_{1,0}^-}, \quad \tilde{f}_{1,1}^- = D_{1,2} f_{1,1}^- + f_{0,1}^- f_{1,0}^- - \frac{1}{2} f_{1,1}^- D_{1,2} \ln f_{1,0}^-
\]

(25)
\[
\frac{\tilde{f}_1}{f_1} = \frac{1}{4} D_{1,2} D_{1,2} \ln f_1 + f_1^+ f_1^- 
\]

Now it is necessary to take into account that \(D_{1,0} f_1^- = 0\) and in connection of equations of motion (12) \(D_{1,1} f_1^- = -f_0 f_1^-\) (the last term \(f_1^- f_0^+ = 0\)) we rewrite the last two equations from (25) in an equivalent form

\[
\frac{\tilde{f}_1}{f_1} = f_1^- D_{0,1} \frac{f_1^-}{f_1} 
\]

\[
\frac{\tilde{f}_2}{f_1} = D_{0,1}^2 \ln f_1 + f_0 f_1^- f_1^-, \quad f_1^- = \frac{1}{f_1} 
\]

The system of equations above for \(f_1^+, f_1^-, f_1^1\) functions exactly coincides with discrete transformation \(T_1\) (9) for the first simple root of \(A_2\) algebra solution of which was found above in the third section. The last function \(f_1^+\) is defined algebraically via \(f_1^-\) from the first row of (25). By reduction using only first equality of Jacobi [2] we obtain

\[
f_1^- = \frac{\tilde{\text{Det}} n+1}{\text{Det} n}, \quad f_1^- = \frac{\tilde{\text{Det}} n+1}{\text{Det} n} 
\]

where \(\text{Det} n\) are the main minors of the matrix

\[
\begin{pmatrix}
  f_1^- & D_{0,1} f_1^- & D_{0,1}^2 f_1^- & \cdots \\
  D_{0,1} f_1^- & D_{0,1}^2 f_1^- & D_{0,1}^3 f_1^- & \cdots \\
  D_{0,1}^2 f_1^- & D_{0,1}^3 f_1^- & D_{0,1}^4 f_1^- & \cdots \\
  \cdots & \cdots & \cdots & \cdots 
\end{pmatrix}
\]

\(\tilde{\text{Det}} n\) determinant of matrix (27) in which last column is constructed from \(f_1^-\) function and its consequent derivatives. And at last \(\tilde{\text{Det}} n\) in matrix (27) the last column and row are exchanged on consequent derivatives of \(f_1^-\) function except of diagonal term which is occupied by \(f_1^-\) function. One example clarify the situation

\[
\tilde{\text{Det}} 3 = \text{Det} 3 \begin{pmatrix}
  f_1^- & D_{0,1} f_1^- & f_1^- \\
  D_{0,1} f_1^- & D_{0,1}^2 f_1^- & D_{0,1} f_1^- \\
  f_1^- & D_{0,1} f_1^- & f_1^- 
\end{pmatrix}
\]

Integrals in the (26) may be calculated directly by methods of [4]. We present calculations for more complicated case of \(\tilde{\text{Det}} n+1\). We parameterize first \(n\) elements of the first column by \(\lambda_1\) (parameter of integration) and the last term \(f_1^-\) by two parameters \(\lambda_1, \mu_1\). Elements of second column are parameterized by \(\lambda_2, \mu_1\) and so on up to \(n-th\) column \(\lambda_n, \mu_1\). Elements of \((n+1)-th\) column \(n\) first one by \(\lambda_{n+1}, \mu_2\) and the last one \(f_1^-\) by three parameters \(\lambda_{n+1}, \mu_1, \mu_2\). As a result under the sign of multidimensional \(((n+3))\) integral we obtain determinant,
which may be easily calculated by the same method as calculated Vandermond determinant. Result is as follows (we present integrand function)

\[
\tilde{D}_{n+1} = \frac{W_{n+1}^2(\lambda)}{\prod_{i=1}^{(n+1)}(\lambda_i - \mu_1)(\lambda_i - \mu_2)}, \quad \tilde{D}_{n+1} = \frac{W_{n+1}^2(\lambda)}{\prod_{i=1}^{(n+1)}(\lambda_i - \mu)}
\]

Keeping in mind that transformation \(T_{2n_2}\) change only \(n_2\) and transformation \(T_n\) changes only \(n_1\) we come to the solution of the next subsection

4.6 General formulae for solution

The result of calculation of discrete transformation \(T_{2n_2}^n T_{1}^n\) may be expressed in terms of the basis function

\[
V(n_1; n_2, n_3) = \frac{1}{n_1! n_2! n_3!} \int \prod_{k=1}^{n_1} P(\lambda_k) d\lambda_k \prod_{i=1}^{n_2} Q(\mu_i) d\mu_i \prod_{j=1}^{n_3} Q(\nu_j) d\nu_j \times \frac{W^2(\lambda)_{n_1} W^2(\mu)_{n_2} W^2(\nu)_{n_3}}{\prod_{k=1}^{n_1} \prod_{i=1}^{n_2} \prod_{j=1}^{n_3} (\lambda_k - \mu_i)(\lambda_k - \nu_j)}
\]

In this notation solution of 4-wave \(B_2\) problem looks similar as it was found before for the case of \(A_2\) algebra

\[
f_{1,0}^+ = \frac{V(n_1 - 1; n_2, n_2)}{V(n_1; n_2, n_2)}, \quad f_{0,1}^+ = \frac{V(n_1; n_2, n_2 - 1)}{V(n_1; n_2, n_2)}
\]

\[
f_{1,1}^+ = \frac{V(n_1 - 1; n_2, n_2 - 1)}{V(n_1; n_2, n_2)}, \quad f_{1,2}^+ = \frac{V(n_1 - 1; n_2 - 1, n_2 - 1)}{V(n_1; n_2, n_2)}
\]

\[
f_{1,0}^- = \frac{V(n_1 + 1; n_2, n_2)}{V(n_1; n_2, n_2)}, \quad f_{0,1}^- = \frac{V(n_1; n_2 + 1, n_2)}{V(n_1; n_2, n_2)}
\]

\[
f_{1,1}^- = \frac{V(n_1 + 1; n_2 + 1, n_2)}{V(n_1; n_2, n_2)}, \quad f_{1,2}^- = \frac{V(n_1 + 1; n_2 + 1, n_2 + 1)}{V(n_1; n_2, n_2)}
\]

(28)

In the case \(n_1 = 0\) this exactly solution of chain of the second root; in the case \(n_2 = 0\) this is solution of the chain of the first root.

5 The case of \(G_2\) algebra

Algebra \(G_2\) has the following Cartan matrix and basic commutation relations between two generators of the simple roots \(X_{1,2}^\pm\) and its Cartan elements \(h_{1,2}\)

\[
k = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}, \quad \begin{pmatrix} \{h_1, X_1^\pm\} = \pm 2X_1^\pm \\ \{h_2, X_1^\pm\} = \mp 3X_2^\pm \end{pmatrix}
\]

\[
\begin{pmatrix} [h_1, X_2^\pm] = \mp X_2^\pm \\ [h_2, X_1^\pm] = \pm 3X_2^\pm \end{pmatrix}
\]

\[
[\{h_1, X_2^\pm\} = \mp 2X_2^\pm
\]

13
Let us rewrite (3) and (4) in a form under condition that all $f^+$.

System of its positive roots contain 6 elements

$$X_1^+ = X_{a_1}^+, X_2^+ = X_{a_2}^+, X_1^+ = [X_2^+, X_1^+] = X_{a_1 + a_2}^+, X_{1;2} = \frac{1}{2} [X_2^+[X_2^+, X_1^+]] = X_{a_1 + 2a_2}^+,$$

$$X_1;222 = \frac{1}{6} [X_2^+[X_2^+, X_1^+]] = X_{a_1 + 3a_2}^+, X_1;1122 = \frac{1}{6} [X_2^+[X_2^+, X_1^+]] = X_{2a_1 + 3a_2}^+.$$

The coefficients of decomposition by these roots will be denoted as $f_{n,m}$ correspondingly. System of equations for 12 functions $f^+_{2,3}, f^+_{1,3}, f^+_{1,2}, f^+_{1,1}, f^+_{1,0}, f^+_{0,1}$ looks as

$$D_{2,3}f^+_{2,3} = 3f^+_{1,0}f_{1,3} - 3f^+_{1,1}f_{1,2}, \quad D_{2,3}f^-_{2,3} = 3f^-_{0,1}f_{1,3} - 3f^-_{1,1}f_{1,2},$$
$$D_{1,3}f^+_{1,3} = -3f^+_{2,3}f_{1,0} - 3f^+_{0,1}f_{1,2}, \quad D_{1,3}f^-_{1,3} = -3f^-_{1,0}f_{2,3} - 3f^-_{0,1}f_{1,2},$$
$$D_{1,2}f^+_{1,2} = f^+_{2,3}f^-_{1,1} + f^+_{1,3}f^-_{0,1} - 2f^+_{0,1}f^+_{1,1}, \quad D_{1,2}f^-_{1,2} = f^-_{2,3}f^+_{1,1} + f^-_{1,3}f^+_{0,1} - 2f^-_{0,1}f^-_{1,1},$$
$$D_{1,1}f^+_{1,1} = f^+_{2,3}f^-_{1,2} + 2f^+_{1,2}f^-_{0,1} - f^+_{0,1}f^-_{1,0}, \quad D_{1,1}f^-_{1,1} = f^-_{2,3}f^+_{1,2} + 2f^-_{1,2}f^+_{0,1} - f^-_{0,1}f^+_{1,0},$$
$$D_{0,1}f^+_{0,1} = f^+_{1,3}f^-_{1,2} + 2f^+_{1,2}f^-_{1,1} + f^+_{1,1}f^-_{1,0}, \quad D_{0,1}f^-_{0,1} = f^-_{1,3}f^+_{1,2} + 2f^-_{1,2}f^+_{1,1} + f^-_{1,1}f^+_{1,0}.$$

5.1 Initial conditions for $G_2$

Let us rewrite (3) and (4) in a form under condition that all $f^+ = 0$

$$f^-_{1,0} = \int d\lambda p(\lambda)e^{\lambda(c_1 x - d_1 t)}, \quad f^-_{0,1} = \int d\mu q(\mu)e^{\mu(c_2 x - d_2 t)},$$
$$f^-_{1,1} = \int d\lambda \int d\mu \frac{p(\lambda)q(\mu)}{\lambda - \mu}e^{\lambda(c_1 x - d_1 t) + \mu(c_2 x - d_2 t)},$$
$$f^-_{1,2} = \int d\lambda \int d\mu_1 d\mu_2 \frac{p(\lambda)q(\mu_1)q(\mu_2)}{(\lambda - \mu_1)(\lambda - \mu_2)}e^{\lambda(c_1 x - d_1 t) + (\mu_1 + \mu_2)(c_2 x - d_2 t)},$$
$$f^-_{1,3} = \int d\lambda \int d\mu_1 d\mu_2 d\mu_3 \frac{p(\lambda)q(\mu_1)q(\mu_2)q(\mu_3)}{(\lambda - \mu_1)(\lambda - \mu_2)(\lambda - \mu_3)}e^{\lambda(c_1 x - d_1 t) + (\mu_1 + \mu_2 + \mu_3)(c_2 x - d_2 t)}e^R$$
$$R = (\lambda_1 + \lambda_2)(c_1 x - d_1 t) + (\mu_1 + \mu_2 + \mu_3)(c_2 x - d_2 t)$$
5.2 Discrete transformation of the first simple root $T_1$

We remind the reader that first seven equations of discrete transformation below is a direct consequence of results of the paper [1]. Of course it is possible consider them as a happy guess. The most simple way of obtaining last five equation consists in direct substitution the 7 previous ones in the system (29).

\[
\begin{align*}
\hat{f}_{1,0}^+ &= \frac{1}{f_{1,0}}, \quad \hat{f}_{1,3}^- = -\frac{f_{2,3}}{f_{1,0}}, \quad \hat{f}_{1,1}^+ = -\frac{\hat{f}_{0,1}^+}{f_{1,0}}, \quad \hat{f}_{1,1}^- = \frac{\hat{f}_{1,1}}{f_{1,0}}, \quad \hat{f}_{2,3}^+ = \frac{\hat{f}_{1,3}}{f_{1,0}}, \\
\hat{f}_{1,2}^+ &= \hat{f}_{1,2}^+ + \frac{\hat{f}_{1,1}^+ f_{1,3}^+ + (f_{0,1}^+)^2}{f_{1,0}}, \quad \hat{f}_{1,2}^- = \hat{f}_{1,2}^- - \frac{f_{2,3} f_{0,1}^- + (f_{1,1}^-)^2}{f_{1,0}}.
\end{align*}
\]

The expressions above now it is possible to substitute into (29) and obtain all other equations of discrete transformation. For instant rewriting equation for $\hat{f}_{1,3}^-$ function for transformed variables

\[
D_{1,3} \hat{f}_{1,3}^- = -3 \hat{f}_{1,0}^+ f_{2,3}^- - 3 f_{0,1}^- \hat{f}_{1,2}^-
\]

and substituting in it transformed functions from above equations after not cumbersome calculations we find $f_{2,3}$ and so on with the result

\[
\begin{align*}
\hat{f}_{2,3}^- &= \frac{f_{1,0}^- f_{1,0}^+ D_{1,2} f_{2,3}^- - \frac{3}{4} f_{2,3}^- D_{1,2} f_{1,0}^- - f_{1,1}^- f_{1,3}^- - (f_{2,3}^+)^2 f_{1,3}^- + 2 (f_{1,1}^+)^3 + 3 f_{2,3} f_{1,1} f_{0,1}^-}{2 f_{1,0}^-} \\
\hat{f}_{0,1}^+ &= \frac{f_{1,0}^- f_{1,0}^+ D_{1,2} f_{0,1}^+ - \frac{3}{4} f_{0,1}^- D_{1,2} f_{1,0}^+ - f_{1,1}^- f_{1,0}^+ + \frac{5}{4} (f_{1,1}^+)^2 f_{1,3}^- + (f_{0,1}^+)^2 f_{1,1}^+ + f_{2,3}^- f_{0,1}^- f_{1,3}^-}{2 f_{1,0}^-} \\
\hat{f}_{1,3}^+ &= \frac{f_{1,0}^- f_{1,0}^+ D_{1,2} f_{1,3}^+ - \frac{3}{4} f_{1,3}^+ D_{1,2} f_{1,0}^- + f_{2,3}^- f_{1,0}^- + \frac{5}{4} (f_{1,3}^+)^2 f_{1,3}^- - 3 f_{1,3}^- f_{1,1} f_{0,1}^- - (f_{0,1}^-)^3}{2 f_{1,0}^-} \\
\hat{f}_{1,1}^- &= \frac{f_{1,0}^- f_{1,0}^+ D_{1,2} f_{1,1}^- - \frac{3}{4} f_{1,1}^- D_{1,2} f_{1,0}^- + f_{0,1}^- f_{1,0}^- + \frac{5}{4} (f_{0,1}^-)^2 f_{1,3}^- + (f_{1,1}^-)^2 f_{0,1}^- + f_{2,3}^- f_{1,1}^- f_{1,3}^-}{2 f_{1,0}^-} \\
\hat{f}_{1,0}^- &= \frac{1}{4} D_{1,2} \ln f_{1,0}^- + f_{1,0}^- f_{1,0}^+ + \frac{1}{2} (f_{0,1}^- f_{1,1}^- + f_{1,1}^- f_{1,1}^+) + \frac{3}{2} (f_{2,3}^- f_{1,2}^- + f_{1,3}^- f_{1,3}^-) + \frac{3}{4} f_{1,0}^- f_{1,0}^+ D_{1,2} f_{1,0}^- - \frac{1}{4} f_{1,0}^- f_{1,0}^+ D_{1,2} f_{0,1}^- - \frac{1}{4} f_{1,3}^- D_{1,2} f_{1,3}^- - \frac{1}{4} f_{2,3}^- D_{1,2} f_{1,3}^- - \frac{f_{0,1}^+}{f_{1,0}^-} (f_{2,3}^- f_{1,2}^- - f_{0,1}^- f_{1,2}^-) - \frac{1}{4} 3 (f_{1,0}^+ f_{0,1}^-)^2 - (f_{2,3}^- f_{1,3}^-)^2 + 6 f_{1,0}^- f_{0,1}^- f_{2,3}^- f_{1,3}^- + 4 (f_{1,1}^-)^3 f_{1,3}^- + 4 (f_{0,1}^-)^3 f_{2,3}^-}{(f_{1,0}^-)^2}.
\end{align*}
\]
5.3 Discrete transformation of the complicate root $T_{01+3\alpha_2}$

The calculations below are not necessary if one pay attention on symmetry of the main system of equations (29) with respect to the following exchange of variables and unknown functions

$$D_{2,3} \rightarrow -D_{2,3}, \quad D_{1,3} \rightarrow -D_{1,0}, \quad D_{1,2} \rightarrow -D_{1,1}, \quad D_{0,1} \rightarrow D_{0,1},$$

$$f^+_{2,3} \rightarrow -f^+_{2,3}, \quad f^+_{1,3} \rightarrow -f^+_{1,0}, \quad f^+_{1,1} \rightarrow -f^+_{1,2}, \quad f^+_{0,1} \rightarrow -f^+_{0,1}$$

Using this substitution or by straightforward calculations for $T_{01+3\alpha_2}$ we obtain

$$\tilde{f}^+_{1,3} = \frac{1}{f^+_{1,3}}, \quad \tilde{f}^+_{0,1} = \frac{f^+_{2,3}}{f^+_{1,3}}, \quad \tilde{f}^+_{1,2} = -\frac{f^+_{0,1}}{f^+_{1,3}}, \quad \tilde{f}^-_{1,0} = \frac{f^-_{2,3}}{f^-_{1,3}}, \quad \tilde{f}^-_{2,3} = -\frac{f^-_{1,0}}{f^-_{1,3}}$$

$$\tilde{f}^+_{1,1} = f^+_{1,1} + \frac{(f^+_{0,1})^2 + f^+_{1,2}f^+_{0,1}}{f^+_{1,3}}, \quad \tilde{f}^-_{1,1} = f^-_{1,1} + \frac{-(f^-_{1,2})^2 + f^-_{2,3}f^-_{0,1}}{f^-_{1,3}}$$

$$\tilde{f}^+_{2,3} = -\frac{f^+_{2,3}D_{1,1}f^+_{2,3} + \frac{1}{2}f^+_{2,3}D_{1,1}f^-_{2,3}}{f^+_{1,3}} + \frac{f^-_{1,0}f^-_{1,3} + (f^-_{2,3})^2f^+_{1,0} - (f^-_{1,2})^3 + 3f^-_{2,3}f^-_{1,1}f^-_{1,2}}{2f^-_{1,3}}$$

$$\tilde{f}^+_{1,0} = -\frac{f^+_{1,0}D_{1,1}f^+_{1,0} + \frac{1}{2}f^+_{1,0}D_{1,1}f^+_{1,3}}{f^+_{1,3}} - \frac{f^+_{2,3}f^+_{1,3} - f^-_{2,3}(f^+_{1,0})^2 + 2(f^+_{0,1})^3 + 3f^+_{0,1}f^+_{1,1}f^+_{1,2}}{2f^+_{1,3}}$$

$$\tilde{f}^-_{1,2} = -\frac{f^-_{2,3}D_{1,1}f^-_{1,2} + \frac{1}{2}f^-_{2,3}D_{1,1}f^-_{1,3}}{f^-_{1,3}} - \frac{f^-_{1,0}f^-_{1,3} + (f^-_{2,3})^2f^-_{0,1} - f^-_{0,1}(f^-_{1,2})^2}{2f^-_{1,3}}$$

$$\tilde{f}^-_{0,1} = -\frac{f^-_{2,3}D_{1,1}f^-_{0,1} + \frac{1}{2}f^-_{2,3}D_{1,1}f^-_{1,3}}{f^-_{1,3}} - \frac{f^-_{1,2}f^-_{1,3} + f^-_{0,1}f^-_{1,3} + 2(f^-_{1,2})^2f^-_{1,0} - f^-_{0,1}f^-_{2,3}f^-_{1,0}}{2f^-_{1,3}}$$

$$\tilde{f}^-_{1,3} = \frac{1}{4}D^2_{1,1}\ln f^-_{1,3} + \frac{3}{4}f^-_{1,3}f^-_{1,3} + \frac{1}{2}(f^-_{0,1}f^-_{0,1} + f^-_{1,2}f^-_{1,2}) + \frac{3}{2}(f^-_{2,3}f^-_{1,3} + f^-_{1,0}f^-_{1,0}) - \frac{3}{4}f^-_{1,3}D_{1,1}f^-_{1,2} - f^-_{1,2}D_{1,1}f^-_{0,1} - \frac{1}{4}f^-_{0,1}D_{1,1}f^-_{2,3} - \frac{1}{4}f^-_{0,1}D_{1,1}f^-_{0,1} + \frac{1}{4}f^-_{0,1}(f^-_{2,3}f^-_{1,1} + f^-_{0,1}f^-_{1,1}) - \frac{1}{4} \left( \frac{f^-_{1,3}(f^-_{0,1})^2}{f^-_{1,3}} - (f^-_{2,3}f^-_{1,0})^2 - 6f^-_{1,3}f^-_{0,1}f^-_{2,3}f^-_{1,0} + 4(f^-_{1,2})^3f^-_{1,0} - 4(f^-_{0,1})^3f^-_{2,3} \right) (f^-_{1,3})^2$$

Of course all formulae above for $T_1$ and $T_{01+3\alpha_2}$ coincide with the general one in the case of arbitrary semisimple algebra of the paper [1].
5.4 General formulae of solution

In spite of comlicate on the first look structure of discrete transformations of the previous two subsections their resolving is observable. We would not like in this paper give the proofs but present only finally result of not simple calculations. Aritary discrete transformation can be represented as consequent applications of two mutual commutative basis transformations $T_1$ and $T_{3\alpha_2} = T_{\alpha_1}^{-1}T_{\alpha_1,3\alpha_2}$ and has the form $T = T_1^{n_1}T_{3\alpha_2}^{n_2}$. The result of calculation of discrete transformation $T_{3\alpha_2}^{n_2}T_1^{n_1}$ may be expessed in terms of the basis function (all factorial factors are included in definition of Wandermond determinants)

$$V(n_1; n_2, n_3, n_4) = \int \prod_{k=1}^{n_1} P(\lambda_k) d\lambda_k \prod_{i=1}^{n_2} Q(\mu_i) d\mu_i \prod_{j=1}^{n_3} Q(\nu_j) d\nu_j \prod_{l=1}^{n_4} Q(\sigma_l) d\sigma_l \times$$

$$\frac{W^2(\lambda_{n_2})W^2(\mu_{n_2})W^2(\nu_{n_2})W^2(\sigma_{n_2})}{\prod_{k=1}^{n_1} \prod_{i=1}^{n_2} \prod_{j=1}^{n_3} \prod_{l=1}^{n_4} (\lambda_k - \mu_i)(\lambda_k - \nu_j)(\lambda_k - \sigma_l)}$$

In this notations solution of 6-wave $G_2$ problem looks similar as it was found before for the case of $A_2$ and $B_2$ algebras

$$f_{1,0}^+ = \frac{V(n_1 - 1; n_2, n_2, n_2)}{V(n_1; n_2, n_2, n_2)}$$
$$f_{0,1}^+ = \frac{V(n_1; n_2 - 1, n_2, n_2)}{V(n_1; n_2, n_2, n_2)}$$
$$f_{1,1}^+ = \frac{V(n_1 - 1; n_2 - 1, n_2 - 1, n_2 - 1)}{V(n_1; n_2, n_2, n_2)}$$
$$f_{1,2}^+ = \frac{V(n_1 - 1; n_2 - 1, n_2 - 1, n_2 - 1)}{V(n_1; n_2, n_2, n_2)}$$
$$f_{1,3}^+ = \frac{V(n_1 - 2; n_2 - 2, n_2 - 2, n_2 - 2)}{V(n_1; n_2, n_2, n_2)}$$

$$f_{1,0}^- = \frac{V(n_1 + 1; n_2, n_2, n_2)}{V(n_1; n_2, n_2, n_2)}$$
$$f_{0,1}^- = \frac{V(n_1; n_2 + 1, n_2, n_2)}{V(n_1; n_2, n_2, n_2)}$$
$$f_{1,1}^- = \frac{V(n_1 + 1; n_2 + 1, n_2 + 1, n_2 + 1)}{V(n_1; n_2, n_2, n_2)}$$
$$f_{1,2}^- = \frac{V(n_1 + 1; n_2 + 1, n_2 + 1, n_2 + 1)}{V(n_1; n_2, n_2, n_2)}$$
$$f_{1,3}^- = \frac{V(n_1 + 2; n_2 + 2, n_2 + 2, n_2 + 2)}{V(n_1; n_2, n_2, n_2)}$$

Of course the formula above on the level of the present paper it is necessary consider as hipotesis and possible happy guess for solution of problem in the case of arbitrary semisimple algebra.

6 Multisoliton solutions

The explicit form of solution in the case of $B_2$ and $G_2$ allow to find conditions of the chain interrupting and construct multisoliton solutions as it was done in the case of $A_2$ algebra [3],[4]

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7 Outlook

The main result of the present paper consists in assurance that calculations of this kind may be done in the case of arbitrary semisimple algebra. It is possible to assume that to this aim is sufficient to resolve the discrete transformation chains of the simple roots of the algebra. The finally result will be obtained by simple multiplications with taking into account only differences of spectral parameters of simple roots connected on the sheam of Dynkin.

The most intruged is the fact that soliton solutions of multidimensional integrable system leads to the same limitations on parameters amplitude-phase (with correction on interaction) as in the case $A_1$ algebra.

The same surprising is existence of universal integrable chains in the field of Toda lattice. Of course there description and understanding of their nature is very interesting problem stated by the present paper.

Usualy all results in the theory of integrable system may be interpreted as some equalities form the theory of representation of semisimple algebras (relations between highest vector of different irreducible representations and so on). What kind problems of representation theory (if any) may be connected with arised equalities of the present paper?

At last the parameters $\lambda, \mu, \nu_i$ (the last after taking into account homogeneous part solutions (section 2) of all elements of lower triangular algebra) in comparison with $A_1$ algebra case play role of spectral parameters. By this reason it is possible to assume that one-dimentional Lax formalism it is necessary to change on multidimensional one. This assumption it is possible to put into connection with the fact that discrete transformation are always commutative and number of basis transformation coincides with the rank of the algebra.

All this problems demand for its solution further investigation.

8 Appendix

In this Appendix we present detail calculations for the most complicate case of discrete transformation $T_{2\alpha_2}$ of $B_2$ algebra. We change greek indexes on latina ones here.

8.1 Initial condition. Zero order case

\[
A^0 = \int \frac{dmMdlL}{l - m}, \quad B^0 = \int \frac{dm_1M_1dm_2M_2dlL}{(l - m_1)(l - m_2)}
\]

8.2 First step of $T_{2\alpha_2}$ transformation

\[
A^1 = \frac{\frac{1}{2}D_1B_1^0 - B^0(D_1)_1}{D_0} = \int \frac{dm_1M_1dm_2M_2dm_3M_3dlL(m_1 + m_2)}{(l - m_1)(l - m_2)} - m_3)
\]
A region of integration with respect to permutations of four parameters

In the process of calculations above we have used only the fact symmetry of

In transformation above it was used only the fact the symmetry of the domain of

And thus for

And thus for \( B^1 \) we obtain

In the process of calculations above we have used only the fact symmetry of region of integration with respect to permutations of four parameters \( m_i \)

Now it is necessary to use equality

with help of which we come to the equality

\[
\int \frac{dm_1 M_1 dm_2 M_2 dm_3 M_3 dl L}{(l-m_1)(l-m_2)(l-m_3)(l-m_4)} (m_1 - m_3)
\]

\[
\frac{1}{2} \int \frac{dm_1 M_1 dm_2 M_2 dm_3 M_3 dl L}{(l-m_1)(l-m_2)} (m_1 - m_3) \frac{1}{l-m_1} - \frac{1}{l-m_3}
\]

\[
\frac{1}{2} \int \frac{dm_1 M_1 dm_2 M_2 dm_3 M_3 dl L}{(l-m_2)(l-m_1)(l-m_3)(l-m_4)} \frac{(m_1 - m_3)^2}{(l-m_1)(l-m_2)(l-m_3)}
\]

In transformation above it was used only the fact the symmetry of the domain of integration with respect to permutation of all indexes \( m_1, m_2, m_3 \). Under calculation of \( B^1 \) from (19) we will not conserve indexes of integrale and differentiales remaining only integrant function. We have consequently

\[
\frac{1}{4} \frac{(m_1 + m_2)^2}{(l-m_1)(l-m_2)} + \frac{m_4 (m_4 - (m_1 + m_2))}{(l-m_1)(l-m_2)} - \frac{1}{2} \frac{(m_1 - m_2)}{(l-m_1)(l-m_2)} = \]

\[
\frac{m_1 (m_2 - m_4) + m_4 (m_3 - m_2)}{(l-m_1)(l-m_2)} = \frac{1}{2} \frac{m_1 (m_2 - m_4)^2}{(l-m_1)(l-m_2)(l-m_3)} - \frac{m_4 (m_2 - m_3)^2}{(l-m_1)(l-m_2)(l-m_3)} = \]

\[
\frac{1}{2} \frac{(m_1 - m_4) (m_2 - m_3)}{(l-m_1)(l-m_2)(l-m_3)} = \frac{1}{4} \frac{(m_1 - m_4)^2 (m_2 - m_3)^2}{(l-m_1)(l-m_2)(l-m_3)(l-m_4)}
\]

\[
A^2 = \frac{1}{8} \frac{(m_5 - m_6)^2 (m_1 - m_4)^2 (m_2 - m_3)^2}{(l-m_1)(l-m_2)(l-m_3)(l-m_4)} [\frac{1}{2} (m_1 + m_2 + m_3 + m_4) - (m_5 + m_6)] = \]

\[
\frac{1}{4} \frac{(m_5 - m_6)^2 (m_1 - m_4)^2 (m_2 - m_3)^2}{(l-m_1)(l-m_2)(l-m_3)(l-m_4)} [m_1 - m_5]
\]

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\[
\frac{1}{2} (m_2 - m_3)^2 (m_5 - m_6)(m_1 - m_6)(l - m_5) W_3(m_1, m_4, m_5) =
\]
\[
\frac{1}{2!3!} (m_2 - m_3)^2 W_3^2(m_1, m_4, m_5)
\]
\[
\frac{1}{(l - m_1)(l - m_2)(l - m_3)(l - m_4)(l - m_5)}
\]

By induction we obtain

\[
A^n = \frac{1}{n!(n+1)!} \int L dl \prod_{i=1}^{(2n+1)} M_i dm_i \frac{W_n^2(m_1,..m_n)W_{n+1}^2(m_{n+1},..m_{2n+1})}{\prod_{i=1}^{(2n+1)}(l - m_i)}
\]

\[
B^n = \frac{1}{(n+1)!(n+1)!} \int L dl \prod_{i=1}^{(2n+2)} M_i dm_i \frac{W_{n+1}^2(m_1,..m_{n+1})W_{n+2}^2(m_{n+2},..m_{2n+2})}{\prod_{i=1}^{(2n+2)}(l - m_i)}
\]

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