Counting of paths and the multiplicity of determinantal rings

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Abstract

In this paper, we derive several formulas of counting families of non-intersecting paths for two-sided ladder-shaped regions. As an application we give a new proof to a combinatorial interpretation of Fibonacci numbers obtained by G. Andrews in 1974.

1 Introduction

Let $X = (x_{ij})$ be an $m \times n$ matrix of indeterminates over a field $K$. Let $I_{r+1}$ be the ideal of $K[X]$ generated by the set of all $(r+1)$-minors of $X$ and set $R_{r+1} = K[X]/I_{r+1}$; then it is well known that the multiplicity of $R_{r+1}$ is given by

$$e(R_{r+1}) = \det[\binom{m+n-i-j}{m-i}]_{i,j=1,...,r}.$$ 

The formula was first found by G.Z. Giambelli in 1909, and a different formula was given by Abhyankar and Galligo later. Recently, Herzog and Trung generalized the formula as follows.

Theorem 1.1 Let $X = (x_{ij})$ be a generic $m \times n$ matrix over a field $K$. Let $a_1 < \cdots < a_r \leq m$, $b_1 < \cdots < b_r \leq n$ be positive integers. Let $D_t$ denote the part of $X$ consisting of the first $a_t - 1$ rows and the first $b_t - 1$ columns. Let

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$G_t$ be the set of all $t$-minors of $D_t$, $t = 1, \ldots, r$, and let $G_{r+1}$ be the set of all $(r + 1)$-minors of $X$. Let $I$ be the ideal of $K[X]$ generated by $G = \bigcup_{t=1}^{r+1} G_t$; then the multiplicity of $K[X]/I$ is

$$\det \left[ \begin{array}{ccc} m + n - a_i - b_j \\ m - a_i \end{array} \right]_{i,j=1,\ldots,r}. $$

The proof of Theorem 1.1 can be sketched as follows. It is well known, from the theory of Gröbner bases, that the Hilbert function of a homogeneous ideal $I$ in a polynomial ring $K[X]$ coincides with the one of the ideal $I^*$ generated by the leading terms of the polynomials of $I$ (with respect to some term order). Therefore we can replace $I$ by $I^*$ which is a monomial ideal. Since $I^*$ is a square free monomial ideal, we can associate with $I^*$ a simplicial complex $\Delta$ such that $K[X]/I^*$ is $K[\Delta]$, the face ring of $\Delta$. By a result of R. Stanley [9], the multiplicity of $K[\Delta]$ is equal to the number of facets of maximal dimensions of $\Delta$ which can be characterized as families of non-intersecting paths. Therefore, the formula can be obtained by a method of I. Gessel and G. Viennot [5] (see also [10, Sect. 2.7]).

The goal of this paper is to compute the multiplicity of certain ladder determinantal rings. Ladder determinantal rings arise for instance in Abhyankar’s study on the singularities of Schubert varieties of flag manifolds. Let’s recall the definition of ladder determinantal rings. Let $X = (x_{ij})$ be an $m \times n$ matrix of indeterminates over a field $K$. A ladder of $X$ is a subset $Y$ which satisfies the condition that whenever $x_{ij}, x_{i'j'} \in Y$ with $i < i'$ and $j < j'$, then $x_{ij'}, x_{i'j} \in Y$. Let $Y$ be a ladder of $X$. Let $G$ be the set of $(r + 1)$-minors of $Y$ and $I$ be the ideal of $K[Y]$ generated by $G$. It is shown in [11, Corollary 4.2] that $G$ is a Gröbner basis for $I$ with respect to some term order of $K[Y]$, therefore $I^*$ is generated by the leading terms of the polynomials of $G$. As a consequence, the multiplicity of $K[Y]/I$ is equal to the number of facets of a simplicial complex $\Delta$ associated to $I^*$. Moreover, we shall see in section 3 that every facet of $\Delta$ can be characterized as a family of non-intersecting paths of certain ladder-shaped region of the plane. Therefore, the computation of the multiplicity for $K[Y]/I$ boils down to counting families of non-intersecting paths. In section 2, we prove a beautiful formula of counting families of non-intersecting paths for two-sided ladder-shaped regions, we state it in the following.
Theorem 1.2 Let $m, n$ and $r$ be positive integers with $r \leq m \leq n$. Let $X = \{(i, j) \mid 1 \leq i \leq m, 1 \leq j \leq n\}$. We define a partial order on $X$ by setting $(i, j) \leq (i', j')$ if $i \geq i'$ and $j \leq j'$. A path $C$ from $P$ to $Q$ is a maximal chain in $X$ with endpoints $P$ and $Q$. We use $w(P, Q)$ for the number of all different paths from $P$ to $Q$. Let $X$ be a two-sided sub-region of $X$ (see Figure 1 for an example of $Y$). Let $P_i, Q_i, i = 1, \ldots, r$, be points of $Y$ with the following properties:

(i) $P_i = (a_i, n)$ and $1 = a_1 < a_2 < \cdots < a_r \leq m$.

(ii) $Q_i = (c_i, d_i)$, $1 \leq c_1 \leq \cdots \leq c_r = m$ and $1 = d_1 < d_2 < \cdots < d_r \leq n$.

(iii) If $(1, 1) \notin Y$, then $(c_1 - 1, 1) \in Y$.

Then the number of non-intersecting paths from $P_i$ to $Q_i$, $i = 1, \ldots, r$, is

$$\det[w(P_i, Q_i)]_{i,j=1,\ldots,r},$$

where $w(P_i, Q_i)$ is the set of all paths from $P_i$ to $Q_i$.

With the help of this formula and Theorem 3.1, we are able to obtain the following results.

Theorem 1.3 Let $m, n, l, k$ and $r$ be non-negative integers with $k, l \leq \min\{m-2, n-2\}$ and $r < \min\{m-k, n-l\}$. Let $K$ be a field and $X = (x_{ij})$ be an $m \times n$ matrix of indeterminates. Let

$$Y = \{x_{ij} \mid 1 \leq i \leq m, \max\{1, l-i+2\} \leq j \leq \min\{n, n+m-m-i\}\}$$

be a ladder of $X$. Let $I = I_{r+1}(Y)$ be the ideal of $K[Y]$ generated by the $(r + 1)$-minors of $Y$ and set $R = K[Y]/I$. Then the multiplicity of $R$ is

$$\det\left[\sum_{(a,b)\in T} (-1)^{a+b}\left(\frac{m+n-i-j}{m-1+(a-b-1)(i-1)+a(k-m)+b(l-n)}\right)\right]_{i,j=1,\ldots,r},$$

where $T = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid a - b = 0 \text{ or } 1\}$.

Theorem 1.4 Let $r, l$ and $n$ be positive integers with $r < l < n$ and $2r \leq n$. Let $X$ be a skew-symmetric $n \times n$ matrix of indeterminates $x_{ij}$, over a field $K$. Let

$$Y = \{x_{ij} \mid \max\{1, i-l\} \leq j \leq \min\{n, l+i\}, 1 \leq i \leq n\}$$
be a diagonal ladder of $X$. Let $Q_r$ be the ideal of $K[Y]$ generated by the set of all $2r$-pfaffians of $Y$ and let $R = K[Y]/Q_r$; then the multiplicity of $R$ is

$$\det[\sum_{(a,b) \in T} (-1)^{a+b} \left( \begin{array}{c} 2n-2r-i-j \\ n-r-i+a(-r+i-1)+b(r-l-i) \end{array} \right) ]_{i,j=1,\ldots,r-1},$$

where $T = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} | a-b = 0 \text{ or } 1\}$.

### 2 Formulas of counting families of non-intersecting paths

Let $m, n$ and $r$ be positive integers with $r \leq m \leq n$. Consider the set of points $X = \{(i, j) \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ of the plane. We define a partial order on $X$ by setting $(i, j) \leq (i', j')$ if $i \geq i'$ and $j \leq j'$. Let $P, Q \in X$ with $P \geq Q$; a path $C$ from $P$ to $Q$ is a maximal chain in $X$ with endpoints $P$ and $Q$. We use $w(P, Q)$ for the number of all different paths from $P$ to $Q$.

Let $P_i, Q_i, i = 1, \ldots, r$, be points of $X$; a subset $W$ of $X$ is called an $r$-tuple of non-intersecting paths from $P_i$ to $Q_i$ ($i = 1, \ldots, r$) if $W = C_1 \cup C_2 \cup \cdots \cup C_r$ where each $C_i$ is a path from $P_i$ to $Q_i$ and where $C_i \cap C_j = \emptyset$ if $i \neq j$. We use $w(P_i, Q_j)$ for the number of all $r$-tuple of non-intersecting paths from $P_i$ to $Q_j$, where $P = \{P_1, \ldots, P_r\}$ and $Q = \{Q_1, \ldots, Q_r\}$. If all the points of $P$ are on the line $y = n$ and all the points of $Q$ are on the line $x = m$, then we have the following result from [10].

**Theorem 2.1 [10, Sect. 2.7]** Let $X = \{(i, j) \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ be a subset of the plane with the partial order given in the beginning. Let $1 \leq a_1 < \cdots < a_r \leq m$ and $1 \leq b_1 < \cdots < b_r \leq n$ be strictly increasing sequences of positive integers. Let $P_i = (a_i, n)$ and $Q_i = (m, b_i)$; then the number of all $r$-tuple of non-intersecting paths from $P$ to $Q$ is

$$\det[w(P_i, Q_j)]_{i,j=1,\ldots,r}.$$ 

The goal of this section is to show that the same equation holds for ladder-shaped regions.

Let $C_{k_1,k_2}$ be a path from $(k_1, n)$ to $(m, k_2)$ with $1 < k_1 \leq m$ and $1 < k_2 \leq n$. Let $\tilde{C}_{k_3,k_4}$ be a path from $(1,k_3)$ to $(k_4,1)$ with $1 \leq k_3 < n$ and
If \( Y \) is the sub-region of \( X \) bounded by \( x = 1, x = m, y = 1, y = n, C_{k_1,k_2} \) and \( \bar{C}_{k_3,k_4} \), then we say \( Y \) is a two-sided sub-ladder of \( X \) (determined by \( C_{k_1,k_2} \) and \( \bar{C}_{k_3,k_4} \)). Let \( P_i, Q_i, i = 1, \ldots, r \), be points of \( Y \) with the following properties:

(i) \( P_i = (a_i, n) \) and \( 1 = a_1 < a_2 < \cdots < a_r \leq m \).

(ii) \( Q_i = (c_i, d_i), 1 \leq c_1 \leq \cdots \leq c_r = m \) and \( 1 = d_1 < d_2 < \cdots < d_r \leq n \).

(iii) \( c_1 > k_2 \) if \( k_2 \geq 2 \).

An example of \( Y \) with \( k_1 = m - 2, k_2 = n - 2, k_3 = 4 \) and \( k_4 = 7 \) is displayed in Figure 1.

![Figure 1](image)

We will show in the following that the same conclusion of Theorem 2.1 holds with respect to \( Y \). In fact, our result is more general as we allow the points of \( Q \) to be in a more general position. However, before doing so, we need a couple of lemmas.

**Lemma 2.2** Let \( Y \) be the region described as above with \((m - 1, k_2) \in C_{k_1,k_2}\). Let \( P = (a, n) \) and \( Q = (m, d) \); then \( w(P, Q) = \sum_{t=d}^{k_2} w(P, Q_t) \), where \( Q_t = (m - 1, t) \).

**Proof.** Let \( S \) be the set of all paths from \( P \) to \( Q \); then \( S \) is the disjoint union of \( S_t \) for \( t = d, \ldots, k_2 \), where

\[
S_t = \{ C \in S \mid (m, t) \in C, (m, t + 1) \notin C \}
\]
if \( t < k_2 \) and
\[
S_{k_2} = \{ C \in S \mid (m, k_2) \in C \}.
\]
For every \( t \), let \( Q_t = (m - 1, t) \) and let \( S_t' \) be the set of all paths from \( P \) to \( Q_t \); then there is a one to one correspondence between \( S_t \) and \( S_t' \) (if \( C \in S_t \) then \( C - \{(m, d), \ldots, (m, t)\} \in S_t' \)), it follows that \( w(P, Q) = \sum_{t=0}^{k_2} w(P, Q_t) \).

\[\textbf{Lemma 2.3} \quad \text{Let } Y \text{ be the region described as above with } k_4 = m - 1. \text{ Assume that } (k_4, h) \in \tilde{C}_{k_3,k_4} \text{ and } (k_4, h + 1) \notin \tilde{C}_{k_3,k_4} \text{ for some } h \geq 2. \text{ If } Q_i = (m, d_i) \text{ with } 1 = d_1 < d_2 < d_3 \leq h + 1; \text{ then}
\]
\[w(P, Q_2) - w(P, Q_3) = (d_3 - d_2)w(P, Q)\]
and
\[w(P, Q_1) - w(P, Q_2) = (d_2 - 1)w(P, Q),\]
where \( P \) is any point on the line \( y = n \) and \( Q = (m - 1, h) = (k_4, h) \).

\[\text{Proof.} \quad \text{We may assume that } (m - 1, k_2) \in C_{k_1,k_2}. \text{ Then by Lemma 2.2}
\]
\[w(P, Q) = \sum_{t=0}^{k_2} w(P, Q_t') \text{ for every } i, \text{ where } Q_t' = (m - 1, t). \text{ However, from the assumptions, we see that } w(P, Q_t') = w(P, Q) \text{ for } 1 \leq t \leq h, \text{ where } Q = (m - 1, h), \text{ therefore}
\]
\[
w(P, Q_2) - w(P, Q_3) = \sum_{t=d_3}^{k_2} w(P, Q_t') - \sum_{t=d_3}^{k_2} w(P, Q'_t)
= \sum_{t=d_3}^{k_2} w(P, Q_t')
= (d_3 - d_2)w(P, Q).
\]
Similarly, \( w(P, Q_1) - w(P, Q_2) = (d_2 - 1)w(P, Q) \).

Now we state the main result of this section as follows.

\[\textbf{Theorem 2.4} \quad \text{Let } m, n \text{ and } r \text{ be positive integers with } r \leq m \leq n. \text{ Let } Y \text{ be the sub-region of } X \text{ determined by some paths } C_{k_1,k_2} \text{ and } \tilde{C}_{k_3,k_4}. \text{ Let } P_i, Q_i, i = 1, \ldots, r, \text{ be points of } Y \text{ with the following properties:}
\]
(i) \( P_i = (a_i, n) \) and \( 1 = a_1 < a_2 < \cdots < a_r \leq m \).
(ii) \( Q_i = (c_i, d_i), 1 \leq c_1 \leq \cdots \leq c_r = m \) and \( 1 = d_1 < d_2 < \cdots < d_r \leq n \).
(iii) \( c_1 > k_2 \) if \( k_2 \geq 2 \).
Then
\[ w(P, Q) = \det[w(P_i, Q_j)]_{i,j=1,\ldots,r}. \]

**Remark 2.5**
1. Condition (ii) in the theorem says that no two points of \( Q \) will be on the same horizontal line.

2. If there is no path from \( P_k \) to \( Q_k \) for some \( k \), then both sides of the equations are 0 since \( w(P, Q) = 0 \) and \( w(P_i, Q_j) = 0 \) for \( i \geq k \) and \( j \leq k \).

**Proof of Theorem 2.4** Assume without loss of generality that \((k_1, n - 1), (m - 1, k_2) \in C_{k_1,k_2} \) and \((2, k_3), (k_4, 2) \in C_{k_3,k_4} \). We proceed the proof by induction on \( r \) and the area of the region \( Y \). We shall consider two situations: one is when \( m - k_4 \geq 2 \) and the other is when \( m - k_4 = 1 \).

Assume that \( m - k_4 \geq 2 \). In this case, we will show the equation by induction on the number of the intersection of \( Q \) with the vertical line \( x = m \). So, let \( l \) be the non-negative integer with the property that \( c_{l+1} = \cdots = c_r = m \) but \( c_l < m \), i.e., the intersection of \( Q \) with the vertical line \( x = m \) is \( \{Q_{l+1}, \ldots, Q_r\} \).

Assume that \( l = r - 1 \). If \( P_r = (m, n) \) then \( k_1 = m \) and \( k_2 = n \). Furthermore, \( w(P_r, Q_r) = 1 \) and \( w(P_r, Q_j) = 0 \) for \( j = 1, \ldots, r - 1 \). Let \( P' = \{P_1, \ldots, P_{r-1}\} \) and \( Q' = \{Q_1, \ldots, Q_{r-1}\} \); then by induction \( w(P', Q') = \det[w(P_i, Q_j)]_{i,j=1,\ldots,r-1} \), it follows that
\[ w(P, Q) = w(P', Q') = \det[w(P_i, Q_j)]_{i,j=1,\ldots,r-1} = \det[w(P_i, Q_j)]_{i,j=1,\ldots,r}. \]

If \( a_r < m \) and \( Q_r = (m, k_2) \), then \( w(P_i, Q_r) = w(P_i, Q'_r) \) for every \( i \), where \( Q'_r = (m - 1, k_2) \). Let \( Q'_j = Q_j \) for \( j = 1, \ldots, r - 1 \) and \( Q' = \{Q'_1, \ldots, Q'_r\} \); then \( P \) and \( Q' \) are all in a proper sub-region of \( Y \), so that by induction
\[ w(P, Q) = w(P, Q') = \det[w(P_i, Q'_j)]_{i,j=1,\ldots,r} = \det[w(P_i, Q_j)]_{i,j=1,\ldots,r}. \]

From the above, we may assume that \( a_r < m \) and \( d_r < k_2 \). Let \( Q_{r,t} = (m, t) \) for \( t = d_r, \ldots, k_2 \). Let \( S_t \) be the set of all \( r \)-tuple of non-intersecting paths from \( P_i \) to \( Q_t \) \((i = 1, \ldots, r)\); then \( S_t \) is the disjoint union of \( S_i \) for \( t = d_r, \ldots, k_2 \), where
\[ S_t = \{C_1 \cup C_2 \cup \cdots \cup C_r : C_t \in S \mid Q_{r,t} \in C_r, Q_{r,t+1} \notin C_r\}. \]
if \( t \neq k_2 \) and 
\[
S_{k_2} = \{ C_1 \cup C_2 \cup \cdots \cup C_r \in S \mid Q_{r,k_2} \in C_r \}.
\]
For \( t = d_r, \ldots, k_2 \), let \( Q'_{r,t} = (m-1,t) \) and let \( S'_t \) be the set of all \( r \)-tuple of non-intersecting paths from \( P \) to \( Q'_t \), where \( Q'_t = \{ Q_1, \ldots, Q_{r-1}, Q_{r,t} \} \). It is clear that there is a one to one correspondence between \( S_t \) and \( S'_t \) (if \( C_1 \cup C_2 \cup \cdots \cup C_r \in S_t \), then \( C_1 \cup C_2 \cup \cdots \cup C_{r-1} \cup (C_r - \{ Q_{r,t}, Q_{r,t-1}, \ldots, Q_{r,d_r} \}) \in S'_t \), therefore
\[
|S_t| = |S'_t| = \det[w(P_1, Q_1) \cdots w(P_1, Q_{r-1}) w(P_1, Q'_{r,t})]_{i=1, \ldots, r}
\]
by induction as \( P \) and \( Q'_t \) are all in a proper sub-region of \( Y \). However, \( \sum_{t=d_r}^{k_2} w(P, Q'_{r,t}) = w(P, Q_r) \) by Lemma 2.2, it follows that
\[
w(P, Q) = \sum_{t=d_r}^{k_2} |S_t| = \det[w(P_1, Q_j)]_{i,j=1, \ldots, r}.
\]
Assume that \( l < r - 1 \). If \( d_{l+2} - d_{l+1} = 1 \), then \( w(P, Q) = w(P, Q') \), where \( Q' = \{ Q_1, \ldots, Q_l, (m-1, d_{l+1}), Q_{l+2}, \ldots, Q_r \} \). Let \( Q'_{l+1} = (m-1, d_{l+1}) \); then it is clear that \( w(P, Q'_{l+1}) + w(P, Q_{l+2}) = w(P, Q_{l+1}) \); it follows by induction that
\[
w(P, Q) = \det[w(P_1, Q_1) \cdots w(P_1, Q'_l) w(P_1, Q_{l+2}) \cdots w(P_1, Q_r)]_{i=1, \ldots, r}
\]
\[
= \det[w(P_1, Q_1) \cdots w(P_1, Q'_l) + w(P_1, Q_{l+2}) w(P_1, Q_{l+2}) \cdots w(P_1, Q_r)]_{i=1, \ldots, r}
\]
\[
= \det[w(P_1, Q_1) \cdots w(P_1, Q_{l+1}) w(P_1, Q_{l+2}) \cdots w(P_1, Q_r)]_{i=1, \ldots, r}.
\]
Assume that \( d_{l+2} - d_{l+1} > 1 \). Let \( Q_{l+1,t} = (m,t) \) for \( t = d_{l+1}, \ldots, d_{l+2} - 1 \) (\( Q_{l+1,d_{l+1}} = Q_{l+1} \)). Let \( S \) be the set of all \( r \)-tuple of non-intersecting paths from \( P_t \) to \( Q_t \) (\( i = 1, \ldots, r \)); then \( S \) is the disjoint union of \( S_t \) for \( t = d_{l+1}, \ldots, d_{l+2} - 1 \), where
\[
S_t = \{ C_1 \cup \cdots C_r \in S \mid (m,t) \in C_{l+1}, (m, t+1) \notin C_{l+1} \}.
\]
For \( t = d_{l+1}, \ldots, d_{l+2} - 1 \), let \( Q'_{l+1,t} = (m-1,t) \) and let \( S'_t \) be the set of all \( r \)-tuple of non-intersecting paths from \( P \) to \( Q'_t \), where
\[
Q'_t = \{ Q_1, \ldots, Q_l, Q'_{l+1,t}, Q_{l+2}, \ldots, Q_r \}.
\]
Lemma 2.3, the first three columns of the matrix \( w \) early dependent, it follows that \( \det w \).

For \( d \in \mathbb{R} \) and \( k \), we have \( d \) \( (k-1) \) \( d \( m \) \( l \) \( S \) \( t \) \( \{ \) \( \} \) \( S \) \( t \) \( \} \) \( S \). Moreover, by Lemma 2.2, it follows that \( \sum_{i=1}^{m-2} \frac{1}{i} \] is linearly dependent, it follows that \( \sum_{i=1}^{m-2} \frac{1}{i} = 1 \). If this is the case, we have \( Q_1 = (m, d) \) and \( Q_1 = (m, 1) \). Moreover, there exists an integer \( h \geq 2 \) such that \( (k, h) \in \tilde{C}_{k_3,k_4} \) and \( (k, h+1) \notin \tilde{C}_{k_3,k_4} \). To show the equation, we consider the following three situations: (1). \( d_3 \leq h + 1 \), (2). \( d_3 > h + 1 \) and \( d_2 \leq h + 1 \), and (3). \( d_2 > h + 1 \).

If \( d_3 \leq h + 1 \), then it is easy to see that \( w(P, Q) = 0 \). Moreover, by Lemma 2.3, the first three columns of the matrix \( \left[ w(P_i, Q_j) \right]_{i,j=1,...,r} \) are linearly dependent, it follows that \( \det \left[ w(P_i, Q_j) \right]_{i,j=1,...,r} = 0 = w(P, Q) \). Suppose that \( d_3 > h + 1 \). Let \( S \) be the set of all \( r \)-tuple of non-intersecting paths from \( P_1 \) to \( Q_i \); then \( S \) is the disjoint union of \( S_t \) for \( t = 1, \ldots, d_2 - 1 \), where

\[ S_t = \{ C_1 \cup \cdots \cup C_r \in S \mid (m, t) \in C_1, (m, t+1) \notin C_1 \}. \]

For \( t = 1, \ldots, d_2 - 1 \), let \( Q_{t,1} = (m-1, t) \) and let \( S_t' \) be the set of all \( r \)-tuple of non-intersecting paths from \( P \) to \( Q_t \), where \( Q_t' = \{ Q_{t,1}, Q_2, \ldots, Q_r \} \), then as the above \( |S_t| = |S_t'| \). Furthermore, let \( Q_1' = (m-1, h), Q_2' = (m, h+1) \) and \( S' \) be the set of all \( r \)-tuple of non-intersecting paths from \( P \) to \( Q' \), where \( Q' = \{ Q_1', Q_2', Q_3, \ldots, Q_r \} \).
Assume \( d_2 \leq h + 1 \). Then it is easy to see that \( |S'_t| = |S'| \) for every \( t \), hence

\[
w(P, Q) = \sum_{t=1}^{d_2-1} |S'_t| = (d_2 - 1)w(P, Q').
\]

Since \( P \) and \( Q' \) are all in a proper sub-region of \( Y \) and the points of \( P \) and \( Q' \) satisfy the conditions (i) to (iii),

\[
|S'| = w(P, Q') = \det[w(P, Q'_1) w(P_1, Q'_2) w(P_2, Q_3) \cdots w(P_r, Q_r)]_{i=1, \ldots, r} \quad (1)
\]

by induction, therefore by Lemma 2.3

\[
w(P, Q_1) - w(P_1, Q_2) = (d_2 - 1)w(P, Q'_1)
\]

and

\[
w(P, Q_1) - w(P_1, Q_2) = (h + 1 - d_2)w(P_1, Q'_1), \quad (2)
\]

we obtain that

\[
det[w(P, Q_1) w(P_1, Q_2) \cdots w(P_r, Q_r)]_{i=1, \ldots, r}
= det[w(P, Q_1) - w(P_1, Q_2) w(P_2, Q_3) \cdots w(P_r, Q_r)]_{i=1, \ldots, r}
= det[(d_2 - 1)w(P_1, Q'_1) w(P_2, Q_3) \cdots w(P_r, Q_r)]_{i=1, \ldots, r}
= det[(d_2 - 1)w(P_1, Q'_1) w(P_2, Q'_2) \cdots w(P_r, Q'_r)]_{i=1, \ldots, r}
= w(P, Q).
\]

Assume \( d_2 > h + 1 \). Then it is easy to see that \( |S'_t| = |S'| \) if \( t \leq h \), hence

\[
w(P, Q) = \sum_{t=1}^{d_2-1} |S'_t| = \sum_{t=h+1}^{d_2-1} |S'_t| + h|S'|. \quad (3)
\]

Notice that \( P \) and \( Q'_t \) are all in a proper sub-region of \( Y \) for \( t \geq h + 1 \), therefore,

\[
w(P, Q'_t) = \det[w(P, Q'_{1,t}) w(P_1, Q'_2) w(P_2, Q_3) \cdots w(P_r, Q_r)]_{i=1, \ldots, r}
\]

by induction. From (1), (2), (3) and the fact that

\[
w(P_1, Q_1) - w(P_1, Q_2) = hw(P, Q'_1) + \sum_{h+1}^{d_2-1} w(P, Q'_t),
\]
we obtain that
\[
\begin{align*}
\det[w(P, Q_1) w(P_1, Q_2) \cdots w(P_r, Q_r)]_{i=1,\ldots,r} &= \det[w(P, Q_1) - w(P_i, Q_2) w(P_i, Q_2) \cdots w(P_r, Q_r)]_{i=1,\ldots,r} \\
&= \det[hw(P_i, Q'_1) w(P_i, Q_2) \cdots w(P_r, Q_r)]_{i=1,\ldots,r} \\
& \quad + \sum_{t=h+1}^{d_2-1} \det[w(P_i, Q'_{1,t}) w(P_i, Q_2) \cdots w(P_r, Q_r)]_{i=1,\ldots,r} \\
&= h \det[w(P_i, Q'_1) w(P_i, Q'_2) w(P_i, Q_3) \cdots w(P_r, Q_r)]_{i=1,\ldots,r} + \sum_{t=h+1}^{d_2-1} |S'_t| \\
&= w(P, Q).
\end{align*}
\]

This completes the proof.

Theorem 2.4 generalized the following result.

**Corollary 2.6** [4]. **Theorem 1** Let \( X = \{(i, j) \mid 0 \leq i \leq j\} \) be a subset of the plane with the partial order given in the beginning. Let \( 0 \leq a_1 < \cdots < a_k \) and \( 0 \leq b_1 < \cdots < b_k \) be strictly increasing sequences of non-negative integers. Let \( A_i = (0, a_i) \) and \( B_i = (b_i, b_i) \); then the number of \( k \)-tuple of non-intersecting paths from \( A_i \) to \( B_i \), \( i = 1, \ldots, k \), is

\[
\det\left[\begin{array}{c}
\begin{array}{c}
\vdots \\
a_i \\
\vdots \\
b_j
\end{array}
\end{array}\right]_{i,j=1,\ldots,k}.
\]

**Proof.** The assertion follows from the fact that \( w(A_i, B_j) = \binom{a_i}{b_j} \). \( \square \)

### 3 Paths of two-sided regions

Let \( X = \{(i, j) \mid 1 \leq i \leq m, 1 \leq j \leq n\} \). We define a partial order on \( X \) by setting \((i, j) \leq (i', j')\) if \( i \geq i' \) and \( j \leq j' \). As before, if \( P \) and \( Q \) are points of \( X \), then a path \( C \) from \( P \) to \( Q \) is a maximal chain in \( X \) with end points \( P \) and \( Q \). A path \( C \) is called a diagonal path if \( C \) satisfies the following two conditions: (1). if \( (i-1, j), (i, j) \in C \) and \( (i, j) \) is not the final point of \( C \), then \( (i, j-1) \in C \). (2). If \( (i, j+1), (i, j) \in C \) and \( (i, j) \) is not the final point of \( C \), then \( (i+1, j) \in C \). For convenience, let \( C_k \) denotes the diagonal path \( \{(m-k+i, n-i) \mid 0 \leq i \leq k\} \cup \{(m-k+i, n-1-i) \mid 0 \leq i \leq k-1\} \), and \( \tilde{C}_l \) denotes the diagonal path \( \{(i, l+2-i) \mid 1 \leq i \leq l+1\} \cup \{(i+1, l+2-i) \mid 1 \leq i \leq l\} \).

Let \( Y \) be a two-sided sub-region of \( X \). If \( Y \) is determined by \( C_k \) and \( \tilde{C}_l \) for some integers \( k \) and \( l \), then we say that \( Y \) is a two-sided diagonal ladder.
(see Figure 2). The goal of this section is to derive the following formula for such regions.

\[
P(1,n) \times (m-k,n) \times (m,n-k) \times Q_t \times Q'_t \times Q(m,1) \times (l+1,1) \times (1,l+1)
\]

**Figure 2**

**Theorem 3.1** Let \( m, n, k \) and \( l \) be non-negative integers with \( k, l \leq \min\{m-2, n-2\} \). Let \( X = \{(i, j) \mid 1 \leq i \leq m, 1 \leq j \leq n\} \) with the partial order given in the beginning. Let \( Y \) be the two-sided diagonal sub-ladder of \( X \) determined by \( C_k \) and \( \tilde{C}_l \). Let \( P = (1, n) \) and \( Q = (m, 1) \); then

\[
\omega(P, Q) = \sum_{(i,j) \in T} (-1)^{i+j} \left( \frac{m+n-2}{m-1+i(k-m)+j(l-n)} \right),
\]

where \( T = \{(i, j) \in \mathbb{Z} \times \mathbb{Z} \mid i-j = 0 \text{ or } 1\} \).

**Remark 3.2** If \( l = 0 \), i.e., \( Y \) is an one-sided diagonal ladder of \( X \) determined by \( C_k \), then

\[
\omega(P, Q) = \left( \frac{m+n-2}{m-1} \right) - \left( \frac{m+n-2}{k-1} \right)
\]

as

\[
\left( \frac{m+n-2}{m-1+i(k-m)+j(n)} \right) = 0
\]

unless \((i, j) = (0, 0) \text{ or } (1, 0)\).

We need several lemmas before proving the formula. The first lemma is fairly easy, we omit the proof.

**Lemma 3.3** Let \( T = \{(i, j) \in \mathbb{Z} \times \mathbb{Z} \mid i-j = 0 \text{ or } 1\} \). Then the following hold.
(i) If $\phi_1 : T \rightarrow T$ is a map given by $\phi_1((i, j)) = (1 - i, -j)$, then $\phi_1$ is a bijection.

(ii) If $\phi_2 : T \rightarrow T$ is a map given by $\phi_2((i, j)) = (-i, -j - 1)$, then $\phi_2$ is a bijection.

**Lemma 3.4** The following identities hold.

(i).
\[
\begin{pmatrix}
    m + k - 2 \\
    m - 1 + i(k - m) + j(l - n)
\end{pmatrix}
= \begin{pmatrix}
    m + k - 2 \\
    m - 1 + (1 - i)(k - m) - j(l - n)
\end{pmatrix}.
\]

(ii). If $m = l + 2$, then
\[
\begin{pmatrix}
    m + n - 4 \\
    m - 3 + i(k - m) + j(l - n)
\end{pmatrix}
= \begin{pmatrix}
    m + n - 4 \\
    m - 3 - i(k - m) - (j + 1)(l - n)
\end{pmatrix}.
\]

**Proof.** The first equality follows from the fact that
\[
[m - 1 + i(k - m) + j(l - n)] + [m - 1 + (1 - i)(k - m) - j(l - n)] = m + k - 2.
\]

Similarly, the second equality follows from the fact that $l = m - 2$ and
\[
[m - 3 + i(k - m) + j(l - n)] + [m - 3 - i(k - m) - (j + 1)(l - n)] = m + n - 4.
\]

\[\square\]

**Lemma 3.5** If $t > l$ and $k \leq n - t$, then
\[
\sum_{(i,j) \in T} (-1)^{i+j} \begin{pmatrix}
    m + n - t - 2 \\
    m - 2 + i(k - m) + j(l - n)
\end{pmatrix}
= \begin{pmatrix}
    m + n - t - 2 \\
    m - 2
\end{pmatrix} - \begin{pmatrix}
    m + n - t - 2 \\
    k - 2
\end{pmatrix}.
\]

**Proof.** To show the equation, it is enough to show that
\[
\begin{pmatrix}
    m + n - t - 2 \\
    m - 2 + i(k - m) + j(l - n)
\end{pmatrix} = 0
\]

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unless $j = 0$. For this, let $(i, j) \in T$. If $j < 0$, then $i \leq 0$, so that

$$m - 2 + i(k - m) + j(l - n) \geq m + n - l - 2 > m + n - t - 2,$$

it follows that

$$\left( \frac{m + n - t - 2}{m - 2 + i(k - m) + j(l - n)} \right) = 0.$$

If $j > 0$, then $i \geq 1$, so that

$$m - 2 + i(k - m) + j(l - n) \leq m - 2 + (k - m) + (l - n) < k + t - n \leq 0,$$

it follows that

$$\left( \frac{m + n - t - 2}{m - 2 + i(k - m) + j(l - n)} \right) = 0.$$

\[\square\]

**Lemma 3.6** Let $a \leq b \leq n$ be positive integers; then

$$\sum_{l=a}^{b} \binom{n-l}{k} = \binom{n-a+1}{k+1} - \binom{n-b}{k+1}.$$

**Proof.** We prove the equality by induction on $b - a$. If $b - a$, then the equality is trivially holds. If $b - a \geq 1$, then $\sum_{l=a}^{b-1} \binom{n-l}{k} = \binom{n-a+1}{k+1} - \binom{n-b+1}{k+1}$ by induction, it follows that

$$\sum_{l=a}^{b} \binom{n-l}{k} = \sum_{l=a}^{b-1} \binom{n-l}{k} + \binom{n-b}{k+1} = \binom{n-a+1}{k+1} - \binom{n-b+1}{k+1} + \binom{n-b}{k+1}.$$

\[\square\]

**Proof of Theorem 3.1** We proceed the proof by induction on $m$. Let $Q_t = (m, t)$ for $t = 1, \ldots, n - k + 1$ (see Figure 2). Let $S$ be the set of all paths from $P$ to $Q$; then $S$ is the disjoint union of $S_t$ for $t = 1, \ldots, n - k$, where

$$S_t = \{C \in S \mid Q_t \in C, Q_{t+1} \notin C\}.$$

For $t = 1, \ldots, n - k$, let $Q_t' = (m-1, t)$ and let $S'_t$ be the set of all paths from $P$ to $Q'_t$. Then there is a one to one correspondence between $S_t$ and $S'_t$ (if $C \in S_t$,}
then $C - \{Q_t, Q_{t-1}, \ldots, Q_1\} \in S'$. Therefore $w(P, Q) = \sum_{t=1}^{n-k} w(P, Q'_t)$. To show the theorem, we consider two situations: (i). $m > l + 2$ and (ii). $m = l + 2$.

Assume first that $m > l + 2$. If $t \leq l$, then

$$w(P, Q'_t) = \sum_{(i,j) \in T} (-1)^{i+j} \left( \frac{m + n - t - 2}{m - 2 + i(k - m) + j(l - n)} \right).$$

(4)

by induction. If $t > l$, then by induction and Remark 3.2

$$w(P, Q'_t) = \left( \frac{m + n - t - 2}{m - 2} \right) - \left( \frac{m + n - t - 2}{k - 2} \right),$$

so that (4) also holds in this case by Lemma 3.3. Furthermore, by Lemma 3.3(i) and Lemma 3.4(i), we see that

$$\sum_{(i,j) \in T} (-1)^{i+j} \left( m + k - 2 \right) = 0.$$ 

(5)

It follows by Lemma 3.3 that

$$w(P, Q) = \sum_{t=1}^{n-k} \sum_{(i,j) \in T} (-1)^{i+j} \left( \frac{m + n - t - 2}{m - 2 + i(k - m) + j(l - n)} \right).$$

(3)

Assume now that $m = l + 2$. If this is the case, then $w(P, Q'_1) = w(P, Q'_2)$, so that $w(P, Q) = w(P, Q'_2) + \sum_{t=2}^{n-k} w(P, Q'_t)$. Again, by induction,

$$w(P, Q'_t) = \sum_{(i,j) \in T} (-1)^{i+j} \left( \frac{m + n - t - 2}{m - 2 + i(k - m) + j(l - n)} \right).$$

Therefore by (3) and Lemma 3.6

$$w(P, Q) = \sum_{t=1}^{n-k} \sum_{(i,j) \in T} (-1)^{i+j} \left( \frac{m + n - 4}{m - 2 + i(k - m) + j(l - n)} \right) + \sum_{t=2}^{n-k} \sum_{(i,j) \in T} (-1)^{i+j} \left( \frac{m + n - t - 2}{m - 2 + i(k - m) + j(l - n)} \right).$$

(3)
However, by Lemma 3.3(ii) and Lemma 3.4(ii) we see that

$$\sum_{(i,j) \in T} (-1)^{i+j} \left( \begin{array}{c} m + n - 4 \\ m - 3 + i(k - m) + j(l - n) \end{array} \right) = 0.$$ 

The formula follows.

There are several consequences of Theorem 3.1:

**Corollary 3.7** Let $m$ be a positive integer. Then

$$\sum_{(i,j) \in T} (-1)^{i+j} \left( \begin{array}{c} 2m \\ m - 2i - 2j \end{array} \right) = 2^m,$$

where $T = \{(i, j) \in \mathbb{Z} \times \mathbb{Z} \mid i - j = 0 \text{ or } 1\}$.

**Proof.** Set $n = m$, $k = m - 2$ and $l = n - 2$ in the formula of Theorem 3.1, we obtain from Figure 3(a) that

$$\sum_{(i,j) \in T} (-1)^{i+j} \left( \begin{array}{c} 2m - 2 \\ m - 1 - 2i - 2j \end{array} \right) = 2^{m-1}.$$ 

Therefore the equation holds. □

![Figure 3(a)](image1)

![Figure 3(b)](image2)

![Figure 3(c)](image3)

**Corollary 3.8** Let $F_i$ be the $i$-th Fibonacci number. Then

$$F_{2m+1} = \sum_{(i,j) \in T} (-1)^{i+j} \left( \begin{array}{c} 2m \\ m - 3i - 2j \end{array} \right)$$
and

\[ F_{2m} = \sum_{(i,j) \in T} (-1)^{i+j} \left( \frac{2m - 1}{m - 3i - 2j} \right), \]

where \( T = \{ (i, j) \in \mathbb{Z} \times \mathbb{Z} \mid i - j = 0 \text{ or } 1 \}. \)

**Proof.** From Figure 3(b), we see that if \( n = m, k = m - 3 \) and \( l = n - 2 \) in the formula of Theorem 3.1, then the number of paths from \((1, m)\) to \((m, 1)\) is

\[ \sum_{(i,j) \in T} (-1)^{i+j} \left( \frac{2m - 2}{m - 1 - 3i - 2j} \right). \]

Moreover, it is easy to see that the path from \((1, m)\) to \((a, m - a + 1)\) is \( F_{2a-1} \) for every \( a \), therefore

\[ F_{2m-1} = \sum_{(i,j) \in T} (-1)^{i+j} \left( \frac{2m - 2}{m - 1 - 3i - 2j} \right). \]

Similarly, from Figure 3(c), we see that if \( n = m - 1, k = m - 3 \) and \( l = n - 2 \) in the formula of Theorem 3.1, then the number of paths from \((1, n)\) to \((m, 1)\) is \( F_{2m-2} \), i.e.,

\[ F_{2m-2} = \sum_{(i,j) \in T} (-1)^{i+j} \left( \frac{2m - 3}{m - 1 - 3i - 2j} \right). \]

\[ \square \]

**Remark 3.9** Corollary 3.1 gives a new proof to a combinatorial interpretation of Fibonacci number obtained by Andrews [2] in 1974.

4 Multiplicity of determinantal ideals

Let \( K \) be a field and \( X = (x_{ij}) \) be an \( m \times n \) matrix of indeterminates over \( K \). Recall that a ladder of \( X \) is a subset \( Y \) which satisfies the condition that whenever \( x_{ij}, x_{i'j'} \in Y \) with \( i < i' \) and \( j < j' \), then \( x_{ij'}, x_{i'j} \in Y \). With the help of the previous section, we can find the multiplicity of certain ladder determinantal ideals.
Theorem 4.1 Let \( m, n, l \) and \( k \) be non-negative integers with \( k, l \leq \min\{m-2, n-2\} \). Let \( K \) be a field and \( X = (x_{ij}) \) be an \( m \times n \) matrix of indeterminants. Let

\[
Y = \{x_{ij} \mid 1 \leq i \leq m, \max\{1, l - i + 2\} \leq j \leq \min\{n, n + m - k - i\}\}
\]

be a ladder of \( X \). Let \( I = I_{r+1}(Y) \) be the ideal of \( K[Y] \) generated by the \((r+1)\)-minors of \( Y \) and set \( R = K[Y]/I \). Then the multiplicity of \( R \) is

\[
\det\left[ \sum_{(a,b) \in T} (-1)^{a+b} \left( m + n - i - j \right) \right]_{i,j=1,\ldots,r},
\]

where \( T = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} \mid a - b = 0 \text{ or } 1\} \).

Proof. Let \( \tau \) be the lexicographical term order of \( K[X] \) induced by the variable order

\[ x_{11} > x_{12} > \cdots > x_{1n} > x_{21} > \cdots > x_{mn}. \]

Let \( I^* \) be the ideal of \( K[Y] \) generated by the leading terms of the polynomials of \( I \) with respect to \( \tau \) and let \( G \) be the set of all \((r+1)\)-minors of \( Y \). Then it is shown in [11, Corollary 4.2] that \( G \) is a Gröbner basis for \( I^* \) with respect to \( \tau \), so that \( I^* \) is generated by the leading terms of the polynomials of \( G \). Moreover, the multiplicity of \( R \) coincides with the one of \( K[Y]/I^* \) and we can associate to \( I^* \) a simplicial complex \( \Delta \) such that \( K[\Delta] = K[Y]/I^* \), where \( K[\Delta] \) is the face ring of \( \Delta \).

In the following we identify \( X \) with the set \( \{(i, j) \mid 1 \leq i \leq m, 1 \leq j \leq n\} \).

Further we introduce the partial order \((i, j) \leq (i', j')\) if \( i \geq i' \) and \( j \leq j' \). As before, if \( P, Q \in X \), then a path from \( P \) to \( Q \) is a chain in \( X \) with endpoints \( P \) and \( Q \). Therefore \( Y \) is a two-sided diagonal sub-ladder of \( X \) determined by \( C_k \) and \( \tilde{C}_i \). For any subset \( Z \) of \( Y \), we use \( \delta(Z) \) for the set of points \((i, j) \in Z \) for which there is no point \((i', j') \in Z \) with \( i' < i \) and \( j' < j \).

Let \( P_i = (i, n) \) and \( Q_i = (m, i) \) for \( i = 1, \ldots, r \). Let \( F \) be a facet of \( \Delta \); then \( F = C_1 \cup \cdots \cup C_r \), where \( C_1 = \delta(F) \) and \( C_i = \delta(F \cup \cup_{j<i} C_j) \) for \( i \geq 2 \). Notice that \( C_i \cap C_j = \emptyset \) if \( i \neq j \) and \( C_i \) is a path from \( P_i \) to \( Q_i \) for every \( i \). Therefore, \( F \) can be decomposed uniquely as a union of non-intersecting paths from \( P_i \)
to \(Q_i, i = 1, \ldots, r\). Since the union of a family of non-intersecting paths from \(P_i\) to \(Q_i, i = 1, \ldots, r\) is a facet of \(\Delta\), we see by [3, Theorem 5.1.7] that the multiplicity of \(K[\Delta]\) is the number of non-intersecting paths from \(P_i\) to \(Q_i, i = 1, \ldots, r\). Now, if we replace \(m, n, l\) by \(m - i + 1, n - j + 1\) and \(l - i - j + 2\), respectively, in the formula of Theorem 3.1, then we obtain

\[
\begin{align*}
  w(P_i, Q_j) &= \sum_{(a,b) \in T} (-1)^{a+b} \left( \frac{m+n-i-j}{m-i+a(k-m+i-1)+b(l-n-i+1)} \right) \\
  &= \sum_{(a,b) \in T} (-1)^{a+b} \left( \frac{m+n-i-j}{m-1+(a-b-1)(i-1)+a(k-m)+b(l-n)} \right),
\end{align*}
\]

therefore the multiplicity of \(R\) is

\[
\det \left[ \sum_{(a,b) \in T} (-1)^{a+b} \left( \frac{m+n-i-j}{m-1+(a-b-1)(i-1)+a(k-m)+b(l-n)} \right) \right]_{i,j=1,\ldots,r}
\]

by Theorem 2.4.

By Corollary 3.2 we have the following result for one-sided ladder.

**Corollary 4.2** Let \(k, m\) and \(n\) be non-negative integers with \(k \leq \min\{m-2, n-2\}\). Let \(K\) be a field and \(X = (x_{ij})\) be an \(m \times n\) matrix of indeterminates. Let \(Y = \{x_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq \min\{n, n+m-k-i\}\}\) be a ladder of \(X\). Let \(I = I_{r+1}(Y)\) be the ideal of \(K[Y]\) generated by the \((r+1)\)-minors of \(Y\) and set \(R = K[Y]/I\). Then the multiplicity of \(R\) is

\[
\det \left[ \left( \frac{m+n-i-j}{m-i} \right) - \left( \frac{m+n-i-j}{k-1} \right) \right]_{i,j=1,\ldots,r}.
\]

Let \(X\) be a skew-symmetric \(n \times n\) matrix of the indeterminates \(x_{ij}\), over a field \(K\). Let \(R = K[X]/Q_r\), where \(Q_r\) is the ideal of \(K[X]\) generated by the set of all \(2r\)-pfaffians of \(X\). Herzog and Trung [7] have given a formula for the multiplicity of \(R\). To end this section, we generalized their result by giving a formula to the multiplicity for (diagonal) ladder pfaffian ideals. For the definition and properties of pfaffian ideals, the reader is referred to [3], [7] and [8].

**Theorem 4.3** Let \(r, l\) and \(n\) be positive integers with \(r < l < n\) and \(2r \leq n\). Let \(X\) be a skew-symmetric matrix of the indeterminates \(x_{ij}\), over a field \(K\). Let

\[
Y = \{x_{ij} \mid \max\{1, i-l\} \leq j \leq \min\{n, l+i\}, 1 \leq i \leq n\}
\]

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be a diagonal ladder of \( X \). Let \( Q_r \) be the ideal of \( K[Y] \) generated by the set of all \( 2r \)-pfaffians of \( Y \) and let \( R = K[Y]/Q_r; \) then the multiplicity of \( R \) is

\[
\det [\sum_{(a,b) \in T} (-1)^{a+b} \left( \begin{array}{c} 2n-2r-i-j \\ n-r-i+a(-r+i-1)+b(r-l-i) \end{array} \right) ]_{i,j=1,...,r-1},
\]

where \( T = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} \mid a-b = 0 \text{ or } 1 \} \).

**Proof.** Let \( \tau \) be the lexicographical term order of \( K[X] \) induced by the variable order

\[
x_{1n} > x_{1n-1} > \cdots x_{12} > x_{2n} > \cdots > x_{n-2,n-1} > x_{n-1,n}.
\]

Let \( J_r \) be the set of all \( 2r \)-pfaffians of \( Y \); then it is shown in [1] that \( J_r \)

is a Gröbner basis for \( Q_r \) with respect to \( \tau \). Let \( Q_r^* \) be the ideal of \( K[Y] \)
generated by the leading terms of the polynomials of \( J_r \); then the multiplicity of \( K[Y]/Q_r \)

coincided with the one of \( K[Y]/Q_r^* \) and we can associate to \( Q_r^* \) a simplicial complex \( \Delta \) such that \( K[\Delta] = K[Y]/Q_r^* \), where \( K[\Delta] \) is the face ring of \( \Delta \).

In the following we identify \( Y \) with the set

\[
\{(i,j) \mid 1 \leq i \leq j \leq \min\{n, l + i\}\}.
\]

We introduce a partial order on \( Y \) by \( (i,j) \leq (i',j') \) if \( i \leq i' \) and \( j \leq j' \). If \( Z \) is a subset of \( Y \), then we use \( \delta(Z) \) for the set of all points \( (i,j) \in Z \) for which there is no point \((i',j') \in Z \) with \( i > i' \) and \( j < j' \).

Let \( P_i = (i, i+1), Q_i = (n-i, n-i+1) \) for \( i = 1, \ldots, r-1 \). Let \( F \) be a facet of \( \Delta \); then \( F = C_1 \cup \cdots \cup C_{r-1} \), where \( C_i = \delta(F) \) and \( C_i = \delta(F \setminus \bigcup_{j < i} C_j) \) for \( i \geq 2 \). Notice that \( C_i \cap C_j = \emptyset \) if \( i \neq j \) and \( C_i \) is a path from \( P_i \) to \( Q_i \) for every \( i \). Therefore, \( F \) can be decomposed uniquely as a union of non-intersecting paths from \( P_i \) to \( Q_i, i = 1, \ldots, r-1 \). Since the union of non-intersecting paths from \( P_i \) to \( Q_i, i = 1, \ldots, r-1 \) is a facet of \( \Delta \), we see from [3, Theorem 5.1.7] that the multiplicity of \( K[\Delta] \) is the number of non-intersecting paths from \( P_i \) to \( Q_i, i = 1, \ldots, r-1 \).

In order to obtain our formula, we replace \( Y \) by its proper sub-region as follows. Let \( T_i = \{(i,j) \mid i+1 \leq j \leq r\} \cup \{(j,n-i+1) \mid n-r+1 \leq j \leq n-i\} \) and let \( Y' = Y \setminus \bigcup_{i=1}^{r-1} T_i \) (see Figure 4 below).
Now, observe that if \( C \) is a path from \( P_i \) to \( Q_i \), then
\[
\{(i, j) \mid i + 1 \leq j \leq r + 1\} \cup \{(j, n - i + 1) \mid n - r \leq j \leq n - i\} \subseteq C,
\]
so that \( C \setminus T_i \) is a path of from \( P_i' \) to \( Q_i' \), where \( P_i' = (i, r + 1) \) and \( Q_i' = (n - r, n - i + 1) \). Furthermore, if \( C' \) is a path from \( P_i' \) to \( Q_i' \); then \( C' \cup T_i \) is a path from \( P_i \) to \( Q_i \). Therefore we see that the multiplicity of \( R \) is equal to the number of non-intersecting paths from \( P_i' \) to \( Q_i' \) \((i = 1, \ldots, r - 1)\). Now, by reflecting the region \( Y \) about \( x \)-axis and by Theorem 3.1 (replace \( m, n, k \) and \( l \) by \( n - r - i + 1, n - r - j + 1, n - 2r \) and \( n - l - i - j + 1 \), respectively), we obtain
\[
w(P_i', Q_i') = \sum_{(a,b) \in T} (-1)^{a+b} \binom{2n - 2r - i - j}{n - r - i + a(-r + i - 1) + b(r - l - i)};
\]
it follows by Theorem 2.4 that the multiplicity of \( R \) is
\[
det[\sum_{(a,b) \in T} (-1)^{a+b} \binom{2n - 2r - i - j}{n - r - i + a(-r + i - 1) + b(r - l - i)}]_{i,j=1,...,r-1}.
\]

\( \square \)

**Corollary 4.4** Let \( X \) be a skew-symmetric \( n \times n \) matrix of the indeterminates \( x_{ij} \), over a field \( K \). Let \( r \) be a positive integer such that \( 2r \leq n \). Let \( Q_r \) be the ideal of \( K[X] \) generated by the set of all \( 2r \)-pfaffians of \( X \) and let \( R = K[X]/Q_r \); then the multiplicity of \( R \) is
\[
det[\binom{2n - 2r - i - j}{n - r - i} - \binom{2n - 2r - i - j}{n - 2r - 1}]_{i,j=1,...,r-1}.
\]
Proof. The formula follows by setting $l = n - 1$ in the formula of Theorem 4.3 and using the fact that

$$\binom{2n - 2r - i - j}{n - r - i + a(-r + i - 1) + b(r - l - i)} = 0$$

unless $(a, b) = (0, 0)$ or $(1, 0)$. □

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