SPDEs with fractional noise in space with index $H < 1/2$

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Outline

• Stochastic wave and heat equations with spatially homogeneous noise
• Motivation, objective and strategy
• Stochastic integration
• Picard iteration scheme
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Stochastic heat and wave equations

We consider the stochastic wave equation:

\[
\begin{aligned}
\frac{\partial^2 u}{\partial t^2} (t, x) &= \Delta u(t, x) + \sigma(u(t, x)) \dot{X}(t, x), \quad t \in [0, T], \; x \in \mathbb{R}^d \\
u(0, x) &= u_0(x), \quad \frac{\partial u}{\partial t} (0, x) = v_0(x),
\end{aligned}
\]  
(SWE)

and the stochastic heat equation:

\[
\begin{aligned}
\frac{\partial u}{\partial t} (t, x) &= \frac{1}{2} \Delta u(t, x) + \sigma(u(t, x)) \dot{X}(t, x), \quad t \in [0, T], \; x \in \mathbb{R}^d \\
u(0, x) &= u_0(x)
\end{aligned}
\]  
(SHE)

\- \Delta \text{ denotes the Laplacian operator on } \mathbb{R}^d.
\- \sigma : \mathbb{R} \to \mathbb{R} \text{ is a Lipschitz function},
\- u_0, v_0 : \mathbb{R} \to \mathbb{R} \text{ are bounded and Hölder continuous},
\- \dot{X}(t, x) \text{ is a spatially homogeneous Gaussian noise.}
Spatially homogeneous Gaussian noise

On some probability space \((\Omega, \mathcal{F}, \mathbb{P})\), let \(X = \{X(\varphi), \varphi \in D(\mathbb{R}_+ \times \mathbb{R}^d)\}\) be a zero-mean Gaussian process with

\[
\mathbb{E}[X(\varphi)X(\psi)] = \int_0^\infty \Gamma(\varphi(t, \cdot) \ast \tilde{\psi}(t, \cdot)) \, dt
\]

- \(\mathcal{D}(\mathcal{O})\): functions in \(C^\infty(\mathcal{O})\) with compact support.
- \(\Gamma\) is a non-negative-definite tempered distribution on \(\mathbb{R}^d\).
- \(\tilde{\psi}(t, x) = \psi(t, -x), \ (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d\).
- The Lebesgue integral in \(t\) means that the process is \textit{white} in time.
- There exists a tempered measure \(\mu\) on \(\mathbb{R}^d\) such that \(\mathcal{F}\mu = \Gamma\) in the space \(S'(\mathbb{R}^d)\) of tempered distributions on \(\mathbb{R}^d\).

\[
\mathbb{E}[X(\varphi)X(\psi)] = \int_0^\infty \int_{\mathbb{R}^d} \mathcal{F}\varphi(t, \cdot)(\xi)\overline{\mathcal{F}\psi(t, \cdot)(\xi)} \mu(d\xi) \, dt
\]
The spectral measure $\mu$ satisfies

$$
\int_{\mathbb{R}^d} \left( \prod_{j=1}^{d} \frac{1}{1 + \xi_j^2} \right) \mu(d\xi) < \infty \quad \iff \quad \int_{\mathbb{R}^d} \frac{1}{1 + |\xi|^2} \mu(d\xi) < \infty
$$

In order to solve SPDEs, one aims to construct stochastic integrals with respect to $X$. 

**Remark:** the process $X = \{ X(\varphi), \varphi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R}^d) \}$ defines a stationary random distribution (Itô 1954, Yaglom 1957). That is,

$$
\mathcal{D}(\mathbb{R}_+ \times \mathbb{R}^d) \ni \varphi \mapsto X(\varphi) \in L^2(\Omega)
$$

is linear and continuous, and the covariance is invariant under translations:

$$
\mathbb{E}[X(\tau_h \varphi)X(\tau_h \psi)] = \mathbb{E}[X(\varphi)X(\psi)] \quad \text{for any} \quad h \in \mathbb{R}_+ \times \mathbb{R}^d.
$$
**Mild solutions**

Fix $T > 0$. A random field $\{u(t, x), (t, x) \in [0, T] \times \mathbb{R}^d\}$ is a solution of (SWE) (resp. (SHE)), if it is predictable and, for any $(t, x) \in [0, T] \times \mathbb{R}^d$,

$$u(t, x) = w(t, x) + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y) \sigma(u(s, y)) X(ds, dy) \quad \text{a.s.}$$

where $G_t(x)$ denotes the corresponding fundamental solution: e.g., for $d = 1$ we have

$$G_t(x) = \frac{1}{2} 1_{\{|x| < t\}} \quad \text{(wave)}, \quad G_t(x) = \frac{1}{(2\pi t)^{1/2}} \exp\left(-\frac{|x|^2}{2t}\right) \quad \text{(heat)}$$

and $w(t, x)$ is the contribution of the initial data:

$$w(t, x) = \frac{1}{2} \int_{x-t}^{x+t} v_0(y) dy + \frac{1}{2} \left( u_0(x + t) + u_0(x - t) \right) \quad \text{(wave),}$$

$$w(t, x) = \int_{\mathbb{R}} G_t(x-y) u_0(y) dy \quad \text{(heat)}$$
Motivation and objective

**Recall:** for any $\varphi, \psi \in D(\mathbb{R}_+ \times \mathbb{R}^d)$,

$$E[X(\varphi)X(\psi)] = \int_0^\infty \Gamma(\varphi(t, \cdot) * \tilde{\psi}(t, \cdot)) \, dt = \int_0^\infty \int_{\mathbb{R}^d} \mathcal{F} \varphi(t, \cdot)(\xi) \overline{\mathcal{F} \psi(t, \cdot)(\xi)} \, \mu(d\xi) \, dt$$

Most results in the literature assume the following:

**(A) $\Gamma$ is a non-negative-definite tempered measure** (or in particular, $\Gamma$ is a non-negative locally integrable function $f$).

In this case,

$$E[X(\varphi)X(\psi)] = \int_0^\infty \int_{\mathbb{R}^d} (\varphi(t, \cdot) * \tilde{\psi}(t, \cdot))(x) \Gamma(dx) \, dt$$

$$= \int_0^\infty \int_{\mathbb{R}^d} \varphi(t, x)f(x - y)\psi(t, y) \, dy \, dx \, dt.$$
Under assumptions \((A)\) and
\[
\int_{\mathbb{R}^d} \frac{1}{1 + |\xi|^2} \mu(d\xi) < \infty, \tag{1}
\]
Dalang 1999 (also Dalang and Q-S 2011) proved existence of a unique solution to a general class of SPDEs in \(\mathbb{R}^d\) including:

(i) (SWE) with \(d \in \{1, 2, 3\}\),

(ii) (SHE) for any \(d \geq 1\).

On the other hand, Peszat and Zabczyk 2007 obtained existence and uniqueness of a function-space valued solution to (i) and (ii) under condition (1) and

(B) There exists a constant \(C > 0\) such that \(\Gamma + C\lambda_d\) is a non-negative measure, where \(\lambda_d\) is the Lebesgue measure on \(\mathbb{R}^d\).
As far as (SWE) in any $d \geq 3$ is concerned:

- Peszat 2002 (function-space valued solution): assumption (B) and
  $$\sup_{\eta \in \mathbb{R}} \int_{\mathbb{R}^d} \frac{1}{1 + |\xi - \eta|^2} \mu(d\xi) < \infty. \quad (2)$$
  He proved that, under (B), (2) is equivalent to (1).
- Dalang and Mueller 2003 (hybrid approach): assumption (A) and condition (1).
- Conus and Dalang 2008 (random field solution): assumption (A) and condition (2).

In all these results, the involved stochastic integral can be interpreted as a stochastic integral with respect to a martingale measure (or cylindrical Wiener process): e.g. Walsh 1986, Da Prato and Zabczyk 1992, Dalang 1999, Dalang and Q-S 2011.
From now on, assume \( d = 1 \), and consider the following important example:

- Assume that the space correlation behaves like a fractional Brownian motion with \( H \in (0, 1) \).
- This corresponds to take a spectral measure \( \mu \) of the form

\[
\mu(d\xi) = c_H |\xi|^{1-2H} \, d\xi, \quad \text{with} \quad c_H = \frac{\Gamma(2H + 1) \sin(\pi H)}{2\pi}
\]

- The measure \( \mu \) satisfies (1) for all \( H \in (0, 1) \).
- But condition (2) does not hold for \( H < 1/2 \).
- In fact, if \( H > 1/2 \), \( \Gamma = \mathcal{F}\mu \) is the locally integrable function

\[
f(x) = H(2H - 1)|x|^{2H-2},
\]

which satisfies (A).
- But if \( H < 1/2 \), \( \Gamma = \mathcal{F}\mu \) is a genuine distribution (Jolis 2010):

\[
\Gamma(\varphi) = H(2H - 1) \int_{\mathbb{R}} (\varphi(x) - \varphi(0)) |x|^{2H-2} \, dx, \quad \varphi \in \mathcal{D}(\mathbb{R})
\]
Objective: consider the stochastic wave and heat equations

\[
\begin{cases}
\frac{\partial^2 u}{\partial t^2}(t, x) = \frac{\partial^2 u}{\partial x^2}(t, x) + \sigma(u(t, x))\dot{X}(t, x), & t \in [0, T], \ x \in \mathbb{R} \\
u(0, x) = u_0(x), & \frac{\partial u}{\partial t}(0, x) = v_0(x),
\end{cases}
\]  \tag{SWE}

and

\[
\begin{cases}
\frac{\partial u}{\partial t}(t, x) = \frac{1}{2} \frac{\partial^2 u}{\partial x^2}(t, x) + \sigma(u(t, x))\dot{X}(t, x), & t \in [0, T], \ x \in \mathbb{R} \\
u(0, x) = u_0(x)
\end{cases}
\]  \tag{SHE}

where we assume that

- \(\sigma(z) = az + b\) is an affine function,
- \(\dot{X}(t, x)\) is a spatially homogeneous Gaussian noise with spectral measure \(\mu(\xi) = c_H|\xi|^{1-2H}d\xi\) with \(H \in (\frac{1}{4}, \frac{1}{2})\).
- \(u_0, v_0 : \mathbb{R} \to \mathbb{R}\) are bounded and \(H\)-Hölder continuous,
Under the above hypotheses, we aim to prove the following. Assume that $H \in (\frac{1}{4}, \frac{1}{2})$.

**Theorem**

Equation (SWE) (respectively (SHE)) has a unique solution $u = \{u(t, x), (t, x) \in [0, T] \times \mathbb{R}\}$, which is $L^2(\Omega)$-continuous and satisfies, for any $p \geq 2$,

$$\sup_{(t, x) \in [0, T] \times \mathbb{R}} \mathbb{E}[|u(t, x)|^p] < \infty$$

and

$$\sup_{(t, x) \in [0, T] \times \mathbb{R}} \int_0^t \int_{\mathbb{R}^2} G_{t-s}^2(x - y) \left(\mathbb{E}[|u(s, y) - u(s, z)|^p]\right)^{2/p} \frac{dydzds}{|y - z|^{2-2H}} < \infty.$$ 

The latter condition appears in a natural way, as we apply techniques from the theory of fractional Sobolev spaces.
Strategy

In order to attain our objective, we have developed the following steps:

1. Properly *interpret* the *stochastic integral* with respect to our spatially homogeneous noise:

   \[
   \int_0^t \int_{\mathbb{R}} S(s, y) X(ds, dy) \quad \text{(Basse-O’Connor et al. 2012)}
   \]

2. Obtain a new criterion for integrability, based on tools from the theory of *fractional Sobolev spaces* (Di Nezza et al. 2012).

3. Set a *Picard iteration scheme*, show that it is well-defined and converges, in a convenient topology, to a process which solves our SPDEs.
Our main result covers the cases

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2}(t, x) &= \frac{\partial^2 u}{\partial x^2}(t, x) + u(t, x) \dot{X}(t, x), \quad x \in \mathbb{R} \\
u(0, x) &= c, \quad \frac{\partial u}{\partial t}(0, x) = 0,
\end{align*}
\]

and

\[
\begin{align*}
\frac{\partial u}{\partial t}(t, x) &= \frac{1}{2} \frac{\partial^2 u}{\partial x^2}(t, x) + u(t, x) \dot{X}(t, x), \quad x \in \mathbb{R} \\
u(0, x) &= c
\end{align*}
\]

- These are the **Hyperbolic Anderson Model** and **Parabolic Anderson Model**, resp.
- Study of the series of **multiple stochastic integrals** with respect to \( X \).
- This method has been applied in
  - **Heat equations**: Hu 2001, Hu and Nualart 2009, Balan and Tudor 2010, Hu et al. 2011.
  - **Wave equations**: Dalang et al. 2008, Dalang and Mueller 2009, Balan 2012.
Stochastic integral

**Wiener integral:** Let $\mathcal{H}$ be the completion of $D(\mathbb{R}_+ \times \mathbb{R})$ with respect to

$$
\langle \cdot, \cdot \rangle_{\mathcal{H}} := \mathbb{E}[X(\varphi)X(\psi)] = c_{\mathcal{H}} \int_0^\infty \int_\mathbb{R} \mathcal{F}_\varphi(t, \cdot)(\xi)\mathcal{F}_\psi(t, \cdot)(\xi) |\xi|^{1-2H} d\xi dt
$$

Then, $\varphi \mapsto X(\varphi) \in L^2(\Omega)$ is an isometry which can be extended to $\mathcal{H}$:

$$
X(h) = \int_0^\infty \int_\mathbb{R} h(t, x)X(dt, dx), \quad h \in \mathcal{H}.
$$

For $t \geq 0$ any interval $(x, y] \subset \mathbb{R}$, one proves that $1_{(0, t] \times (x, y]} \in \mathcal{H}$, so we can define the random variable

$$
X_t((x, y]) := X(1_{(0, t] \times (x, y]})
$$

**Problem:** we cannot define $X_t(A)$ for all $A \in \mathcal{B}_b(\mathbb{R})$, since in general the function $1_{(0, t] \times A}$ may not be in $\mathcal{H} (H < 1/2)$. 
But recall that our noise $X = \{X(\varphi), \varphi \in D(\mathbb{R}_+ \times \mathbb{R})\}$ can be viewed as a stationary random distribution (Itô 1954, Yaglom 1957).

Hence, $X$ admits a suitable spectral representation which can be applied to show that

$$X_t((x, y)) := X(1_{(0, t] \times (x, y]}) = \int_{\mathbb{R}} F1_{(x, y]}(\xi) M_t(d\xi),$$

where $\{M_t(A), F_t, t \geq 0, A \in B_b(\mathbb{R})\}$ is a (complex valued) martingale measure with zero mean and covariation

$$\langle M(A), M(B) \rangle_t = t \mu(A \cap B) = t c_H \int_{A \cap B} |\xi|^{1-2H} d\xi, \quad A, B \in B_b(\mathbb{R}).$$

$(F_t)_{t \geq 0}$ denotes the filtration generated by $X$:

$$F_t = \sigma\{X(1_{[0, s]} \varphi), s \in [0, t], \varphi \in D(\mathbb{R})\}.$$
Sketch of the construction of the stochastic integral:

1. \( \mathcal{E} \): linear combinations of processes of the form \( g(\omega, t, x) = Y(\omega)1_{(a,b]}(t)1_{(c,d]}(x) \). Define

\[
\int_0^t \int_{\mathbb{R}} g(s, y)X(ds, dy) := Y(X_{t \wedge b}((u, v]) - X_{t \wedge a}((u, v]))
\]

and extend to \( \mathcal{E} \) by linearity.

2. For any \( g \in \mathcal{E} \), it holds

\[
\int_0^t \int_{\mathbb{R}} g(s, y)X(ds, dy) = \int_0^t \int_{\mathbb{R}} \mathcal{F}g(s, \cdot)(\xi)M(ds, d\xi)
\]

3. Let \( \mathcal{P}_0 \) be the completion of \( \mathcal{E} \) with respect to

\[
\|g\|_0^2 = \mathbb{E} \int_0^T \int_{\mathbb{R}} |\mathcal{F}g(t, \cdot)(\xi)|^2 c_H|\xi|^{1-2H}d\xi dt.
\]

4. By the isometry property of Walsh’s stochastic integral, the map \( \mathcal{E} \ni g \mapsto \left\{ \int_0^t \int_{\mathbb{R}} g(s, y)X(ds, dy) \right\}_{t \in [0, T]} \in \mathcal{M} \) is an isometry, where \( \mathcal{M} \) is a subspace of the space of continuous square-integrable martingales with \( \|N\| = \left\{ \mathbb{E}(N_T^2) \right\}^{1/2} \). This map can be extended to \( \mathcal{P}_0 \).
Identification of integrands:

**Theorem (Basse-O’Connor et al. 2012)**

The elements of $\mathcal{P}_0$ are predictable functions of the form

$$S : \Omega \times [0, T] \rightarrow S'(\mathbb{R})$$

such that $FS(\omega, t, \cdot)$ is a locally integrable function for any $(\omega, t)$ and

$$\mathbb{E} \int_0^T \int_{\mathbb{R}} |FS(t, \cdot)(\xi)|^2 c_H |\xi|^{1-2H} d\xi dt < \infty.$$

In particular, we have the isometry

$$\mathbb{E} \left| \int_0^t \int_{\mathbb{R}} S(s, x)X(ds, dx) \right|^2 = \mathbb{E} \int_0^t \int_{\mathbb{R}} |FS(s, \cdot)(\xi)|^2 c_H |\xi|^{1-2H} d\xi ds,$$

**Remark:** all that we have done is valid for any $H \in (0, 1)$. 
Criterion for integrability:

A measurable function \( g : \mathbb{R} \rightarrow \mathbb{R} \) is *tempered* if there exists a tempered distribution \( T_g \in S'(\mathbb{R}) \) such that \( T_g \varphi = \int_{\mathbb{R}} g(x) \varphi(x) dx \), for all \( \varphi \in S(\mathbb{R}) \).

**Theorem**

Let \( S : \Omega \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R} \) be a predictable function, such that

(a) for almost all \((\omega, t) \in \Omega \times [0, T], S(\omega, t, \cdot)\) is a tempered function,

(b) the Fourier transform \( \mathcal{F} S(\omega, t, \cdot) \) in \( S'(\mathbb{R}) \) is a locally integrable function.

If

\[
I(T) := C_H \mathbb{E} \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|S(t, x) - S(t, y)|^2}{|x - y|^{2 - 2H}} dx dy dt < \infty,
\]

then \( S \in \mathcal{P}_0 \) and

\[
\mathbb{E} \left| \int_0^T \int_{\mathbb{R}} S(s, x) X(ds, dx) \right|^2 = I(T).
\]
The proof of the above criterion is based on the following result, related to the theory of fractional Sobolev spaces (Di Nezza et al. 2012):

**Proposition**

Let $g : \mathbb{R} \to \mathbb{R}$ be a tempered function whose Fourier transform in $S'(\mathbb{R})$ is a locally integrable function. For any $0 < H < 1/2$, 

$$c_H \int_{\mathbb{R}} |\mathcal{F}g(\xi)|^2 |\xi|^{1-2H} d\xi = C_H \int_{\mathbb{R}^2} \frac{|g(x) - g(y)|^2}{|x - y|^{2H-2}} dxdy,$$

when either one of the two integrals above is finite.
Picard iteration scheme

For any \((t, x) \in [0, T] \times \mathbb{R}\), set \(u^0(t, x) = w(t, x)\) and, for \(n \geq 0\),

\[
u^{n+1}(t, x) = w(t, x) + \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)\sigma(u^n(s,y))X(ds, dy)
\]

**Theorem**

Let \(p \geq 2\) and \(\sigma\) be Lipschitz. Then, \(u^n(t, x)\) is well-defined and

\[
\sup_{(t, x) \in [0, T] \times \mathbb{R}} \mathbb{E}[|u^n(t, x)|^p] < \infty,
\]

\[
\sup_{(t, x) \in [0, T] \times \mathbb{R}} \int_0^t \int_{\mathbb{R}^2} G_{t-s}(x-y) \left( \frac{\mathbb{E}[|u^n(s,y) - u^n(s,z)|^p]}{|y-z|^{2-2H}} \right)^{2/p} \, dydzds < \infty
\]

and, for any \(h \in \mathbb{R}\) with \(|h| < 1\),

\[
\sup_{(t, x) \in [0, T] \times \mathbb{R}} \mathbb{E}[|u^n(t, x + h) - u^n(t, x)|^2] \leq C_n|h|^{2H}
\]

\[
\sup_{(t, x) \in [0, T \wedge (T-h)] \times \mathbb{R}} \mathbb{E}[|u^n(t+h, x) - u^n(t, x)|^2] \leq C_n|h|^\beta,
\]

where \(\beta = 2H\) for the wave equation, and \(\beta = H\) for the heat equation.
Remarks on the proof

We start with $n = 0$. Recall that, for the wave equation,

$$w(t, x) = \int_{\mathbb{R}} G_t(x - y)v_0(y)dy + \frac{\partial}{\partial t} \left( \int_{\mathbb{R}} G_t(x - y)u_0(y)dy \right)$$

$$= \frac{1}{2} \int_{x-t}^{x+t} v_0(y)dy + \frac{1}{2} \left( u_0(x + t) + u_0(x - t) \right),$$

$$G_t(x) = \frac{1}{2} 1_{\{|x| < t\}}$$

and, for the heat equation,

$$w(t, x) = \int_{\mathbb{R}} G_t(x - y)u_0(y)dy$$

$$G_t(x) = \frac{1}{(2\pi t)^{1/2}} \exp \left( -\frac{|x|^2}{2t} \right)$$
Using the explicit expression of $G_t(x)$ and that $u_0$, $v_0$ are bounded and $H$-Hölder continuous, one proves

$$\sup_{(t,x)\in[0,T] \times \mathbb{R}} |w(t,x)| < \infty$$

$$\sup_{(t,x)\in[0,T] \times \mathbb{R}} |w(t, x + h) - w(t, x)|^2 \leq C|h|^{2H}$$

$$\sup_{(t,x)\in[0,T \wedge (T-h)] \times \mathbb{R}} |w(t + h, x) - w(t, x)|^2 \leq C|h|^{\beta}$$

It remains to study the expression

$$\int_0^t \int_{\mathbb{R}^2} G_{t-s}^2(x - y) \frac{|w(s,y) - w(s,z)|^2}{|y - z|^{2-2H}} dydzds$$

$$= \int_0^t \int_{\mathbb{R}} G_{t-s}^2(x - y) \left( \int_{\mathbb{R}} \frac{|w(s,y + z) - w(s,y)|^2}{|z|^{2-2H}} dz \right) dyds$$

Decomposing the domain of the $dz$ integral, the latter term is bounded by

$$\int_0^t \int_{\mathbb{R}} G_{t-s}^2(x - y) \left( \int_{|z| \leq 1} |z|^{4H-2} dz + \int_{|z| > 1} |z|^{2H-2} dz \right) dyds,$$

which is uniformly bounded thanks to condition $H \in \left( \frac{1}{4}, \frac{1}{2} \right)$. 
In order to show that $u^{n+1}(t, x)$ is well-defined, one needs to prove that the following stochastic integral is well-defined:

$$\int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)\sigma(u^n(s, y))X(ds, dy)$$

Hence, setting

$$S_n(s, y) := G_{t-s}(x-y)\sigma(u^n(s, y))1_{[0, t]}(s),$$

one proves

(i) $u^n$ has a predictable modification,

(ii) $S_n(\omega, s, \cdot) \in L^1(\mathbb{R})$ for almost all $(\omega, s) \in \Omega \times [0, T],$

(iii) $S_n$ satisfies

$$\sup_{(t,x) \in [0, T] \times \mathbb{R}} \mathbb{E} \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|S_n(s, y) - S_n(s, z)|^2}{|y - z|^{2H-2}} dydzds < \infty.$$
These conditions, together with all remaining estimates for $u^{n+1}$ in the induction hypothesis, can be proved using the following type of techniques:

- $\sigma$ Lipschitz, Burkholder-Davis-Gundy inequality, Jensen inequality, Minkowski inequality (integrals), Fubini theorem, Plancherel theorem, and many changes of variables.

- For all $\alpha \in (-1, 1)$,

\[
\int_0^T \int_\mathbb{R} |\mathcal{F} G_t(\xi)|^2 |\xi|^\alpha \, d\xi \, dt = \begin{cases} 
C_1 T^{2-\alpha} & \text{wave} \\
C_2 T^{(1-\alpha)/2} & \text{heat}
\end{cases}
\]

- For any $\alpha \in (-1, 1)$ and $h \in \mathbb{R}$,

\[
\int_0^T \int_\mathbb{R} |\mathcal{F} G_{t+h}(y) - \mathcal{F} G_t(y)|^2 |\xi|^{\alpha} \, d\xi \, dt \leq \begin{cases} 
CT|h|^{1-\alpha} & \text{wave} \\
C|h|^{(1-\alpha)/2} & \text{heat}
\end{cases}
\]

- For any $\alpha \in (-1, 1)$ and $h \in \mathbb{R}$,

\[
\int_0^T \int_\mathbb{R} (1 - \cos(\xi h)) |\mathcal{F} G_t(\xi)|^2 |\xi|^\alpha \, d\xi \, dt \leq \begin{cases} 
CT|h|^{1-\alpha} & \text{wave} \\
C|h|^{1-\alpha} & \text{heat},
\end{cases}
\]
In fact, in order to show that

\[
\sup_{(t,x) \in [0,T] \times \mathbb{R}} \int_0^t \int_{\mathbb{R}^2} G_{t-s}^2(x-y) \frac{\left( \mathbb{E} \left[ \left| u^{n+1}(s, y) - u^{n+1}(s, z) \right|^p \right] \right)^{2/p}}{|y - z|^{2-2H}} dydzds < \infty,
\]

we are forced to estimate the term

\[
\int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{G_{t-s}^2(x-y)}{|z|^{2-2H}} \left( \int_0^s \int_{\mathbb{R}} |1 - e^{-i\xi z}|^2 |\mathcal{F}G_{s-r}(\xi)|^2 |\xi|^{1-2H} d\xi dr \right) dzdyds
\]
\[
\leq \left( \int_0^t \int_{\mathbb{R}} G_{t-s}^2(x-y) dyds \right) \left( \int_0^T \int_{\mathbb{R}} |\mathcal{F}G_{s-r}(\xi)|^2 |\xi|^{2(1-2H)} d\xi dr \right).
\]

The latter integral is finite if and only if

\[-1 < 2(1 - 2H) < 1 \iff \frac{1}{4} < H < \frac{3}{4}\]
Convergence of Picard iterations

Now we aim to prove that the sequence \( \{u^n(t, x), \ n \geq 0\} \) converges in \( L^p(\Omega) \).

Here we assume that

\[
|\sigma(x) - \sigma(y) - \sigma(u) + \sigma(v)| \leq C|x - y - u + v| \quad \iff \quad \sigma \text{ affine}
\]

In fact, we prove convergence in the Banach space \((\mathcal{X}, \| \cdot \|_\mathcal{X})\): space of \( L^2(\Omega) \)-continuous and adapted processes \( Y = \{Y(t, x), (t, x) \in [0, T] \times \mathbb{R}\} \) such that \( \|Y\|_{\mathcal{X}_1} < \infty \) and \( \|Y\|_{\mathcal{X}_2} < \infty \), where

\[
\|Y\|_{\mathcal{X}_1} = \sup_{(t, x) \in [0, T] \times \mathbb{R}} \left( \mathbb{E}[|Y(t, x)|^p]\right)^{1/p}
\]

and

\[
\|Y\|_{\mathcal{X}_2} = \sup_{(t, x) \in [0, T] \times \mathbb{R}} \left( \int_0^t \int_{\mathbb{R}^2} G^2_{t-s}(x - y) \frac{\mathbb{E}[|Y(s, y) - Y(s, z)|^p]}{|y - z|^{2-2H}} \ dydzds \right)^{1/2}
\]

For any \( Y \in \mathcal{X} \), we define \( \|Y\|_\mathcal{X} := \|Y\|_{\mathcal{X}_1} + \|Y\|_{\mathcal{X}_2} \).
Theorem
The sequence \((u^n)_{n \geq 0}\) converges in \(\mathcal{X}\) to a process \(u\), which is \(L^2(\Omega)\)-continuous, and is the unique solution to equation (SWE) (or (SHE)).

Proof: We have

\[
M_{n+1}(t) \leq \int_0^t (M_n(s) + M_{n-1}(s)) J(t-s) ds
\]

where, setting \(m_n := u^n - u^{n-1}\),

\[
M_n(t) = \sup_{x \in \mathbb{R}} \left( \mathbb{E}\left[ |m_n(t, x)|^p \right] \right)^{2/p}
\]

\[
+ \sup_{x \in \mathbb{R}} \int_0^t \int_{\mathbb{R}^2} G_{t-s}^2(x-y) \frac{\mathbb{E}\left[ |m_n(s, y) - m_n(s, z)|^p \right]}{|y-z|^{2-2H}} dy dz ds,
\]

\[
J(t-s) = \int_{\mathbb{R}} |\mathcal{F}G_{t-s}(\xi)|^2 |\xi|^{1-2H} d\xi + \int_s^t \int_{\mathbb{R}} G_{t-r}^2(z) \int_{\mathbb{R}} |\mathcal{F}G_{r-s}(\xi)|^2 |\xi|^{2(1-2H)} d\xi dz dr
\]
In fact, we have showed that

\[
\int_s^t \int_{\mathbb{R}} G_{t-r}(z) \int_{\mathbb{R}} |\mathcal{F} G_{r-s}(\xi)|^2 |\xi|^{2(1-2H)} d\xi dzdr = \begin{cases} C_1 (t-s)^{4H-1} & \text{wave} \\ C_2 (t-s)^{2H-1} & \text{heat} \end{cases}
\]

We have proved a version of Dalang’s Gronwall lemma in order to treat situations of the form

\[
f_n(t) \leq \int_0^t (f_{n-1}(s) + f_{n-2}(s)) g(t-s) ds
\]

Once we know that there exists \( u = \lim_n u^n \) in \( \mathcal{X} \), we take limits in

\[
u^{n+1}(t, x) = w(t, x) + \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y) \sigma(u^n(s, y)) X(ds, dy)
\]

to deduce that \( u = \{u(t, x), (t, x) \in [0, T] \times \mathbb{R}\} \) solves (SWE) (resp. (SHE)).

Uniqueness has been proved using similar arguments.
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