C\(^{1+\alpha}\)-REGULARITY OF VISCOSITY SOLUTIONS OF GENERAL NONLINEAR PARABOLIC EQUATIONS

N.V. KRYLOV

Abstract. We investigate the C\(^{1+\alpha}\)-regularity of solutions of parabolic equations \(\partial_t v + H(v, Dv, D^2v, t, x) = 0\). Our main result says that under rather general assumptions there exist C-viscosity and \(L^p\)-viscosity solutions which are in \(C^{1+\alpha}_{loc}\). We allow \(H\) to be just measurable in \(t\) and for its principal part to have sufficiently small discontinuities as a function of \(x\). No Lipschitz continuity of \(H\) with respect to \(v, Dv\) is required.

1. Introduction

For a real-valued measurable function \(H(u, t, x)\),

\[ u = (u', u''), \quad u' = (u'_0, u'_1, ..., u'_d) \in \mathbb{R}^{d+1}, \quad u'' \in S, \quad (t, x) \in \mathbb{R}^{d+1}, \]

where \(S\) is the set of symmetric \(d \times d\) matrices, and sufficiently regular functions \(v(t, x)\) we set

\[ H[v](t, x) = H(v(t, x), Dv(t, x), D^2v(t, x), t, x), \]

and we will be dealing with the parabolic equations

\[ \partial_t v(t, x) + H[v](t, x) = 0 \quad (1.1) \]

in subsets of \([0, T) \times \mathbb{R}^d\), where \(T \in (0, \infty)\) is fixed. Above

\[ \mathbb{R}^d = \{x = (x^1, ..., x^d) : x^1, ..., x^d \in \mathbb{R}\}, \]

\[ \partial_t = \frac{\partial}{\partial t}, \quad D^2u = (D_{ij}u), \quad Du = (D_iu), \quad D_i = \frac{\partial}{\partial x^i}, \quad D_{ij} = D_i D_j. \]

If \(R \in (0, \infty)\) and \((t, x) \in \mathbb{R}^{d+1}\), then

\[ B_R = \{x \in \mathbb{R}^d : |x| < R\}, \quad B_R(x) = x + B_R, \]
\[ C_R = [0, R^2) \times B_R, \quad C_R(t, x) = (t, x) + C_R. \]

We also take a bounded domain \(\Omega \subset \mathbb{R}^d\) of class \(C^{1,1}\) and set

2010 Mathematics Subject Classification. 35K55, 35B65.

Key words and phrases. Fully nonlinear equations, viscosity solutions, regularity of solutions.
\[ \Pi = [0, T) \times \Omega, \quad \partial'\Pi = \bar{\Pi} \setminus \{0\} \times \bar{\Omega} \]

We will be dealing with viscosity solutions of (1.1) in \( \Pi \). The following definition is taken from [3] and has the same spirit as in [2].

**Definition 1.1.** For each choice of “regularity” class \( R = C \) or \( R = L^p \) we say that \( u \) is an \( R \)-viscosity subsolution of (1.1) in \( \Pi \) provided that \( u \) is continuous in \( \Pi \) and, for any \( C_r(t_0, x_0) \subset \Pi \) and any function \( \phi \), that is continuous in \( C_r(t_0, x_0) \) and whose generalized derivatives satisfy \( \partial_t \phi, D\phi, D^2\phi \in R(C_r(t_0, x_0)) \), and is such that \( u - \phi \) attains its maximum over \( C_r(t_0, x_0) \) at \( (t_0, x_0) \), we have

\[
\lim_{\rho \downarrow 0} \text{ess sup}_{C_r(t_0, x_0)} \left[ \partial_t \phi(t, x) + H(u(t, x), D\phi(t, x), D^2\phi(t, x), t, x) \right] \geq 0. \quad (1.2)
\]

In a natural way one defines \( R \)-viscosity supersolution and calls a function an \( R \)-viscosity solution if it is an \( R \)-viscosity supersolution and an \( R \)-viscosity subsolution.

Note that \( C_r(t_0, x_0) \) contains \( \{ (t, x) : t = t_0, |x - x_0| < r \} \), which is part of its boundary. Therefore, the conditions like \( D^2 \phi \in C(C_r(t_0, x_0)) \) mean that the second-order derivatives of \( \phi \) are continuous up to this part of the boundary.

**Remark 1.1.** If \( H(u, t, x) \) is a continuous function of \( (u, t, x) \) and we are talking about the \( C \)-viscosity subsolutions, then (1.2) becomes, of course, just

\[
\partial_t \phi(t_0, x_0) + H(u(t_0, x_0), D\phi(t_0, x_0), D^2\phi(t_0, x_0), t_0, x_0) \geq 0.
\]

Also note that \( L^p \)-viscosity solutions are automatically \( C \)-viscosity solutions.

The reader is referred to [3] for numerous properties of \( L^p \)-viscosity solutions and to [2] for those of \( C \)-viscosity solutions.

The notion of \( L^p \)-viscosity solution generalizes that of \( W^{1,2}_p \)-solution, which is seen from the following well-known fact (see [3]). Set

\[
[u'] = (u'_1, \ldots, u'_d).
\]

**Theorem 1.1.** Suppose that \( H \) is a nonincreasing function of \( u'_0 \), Lipschitz continuous with respect to \( ([u'], [u'']) \) with constant independent of \( u'_0 \) and \( (t, x) \), and at all points of its differentiability with respect to \( u'' \) we have \( D_{u''}H \in S_\delta \), where the constant \( \delta \in (0, 1] \). Suppose that \( p \geq d + 1 \), \( v \) is a continuous in \( \bar{\Pi} \), \( L^p \)-viscosity solution of (1.1), and \( w \in W^{1,2}_{p,\text{loc}}(\bar{\Pi}) \cap C(\bar{\Pi}) \) is a function satisfying (1.1) (a.e.) in \( \Pi \). Finally, let \( v = w \) on \( \partial'\Pi \). Then \( v = w \) in \( \Pi \).
Our main result says that under rather general assumptions there exist $C$-viscosity and $L^p$-viscosity solutions which are in $C^{1+\alpha}_{loc}(\Pi)$. We allow $H$ to be just measurable in $t$ and for its principal part to have sufficiently small discontinuities as a function of $x$.

Wang in Theorem 1.3 of [17] assumes a structure condition on $H$ which in the case of linear equations implies that the coefficients of $Du$ are independent of $(t,x)$. On the other hand, he proves the result for any $C$-viscosity solution.

Our Theorem 2.1 contains Proposition 5.4 of [3] proved there for equations $\partial_t u + H(D^2 u) = 0$ apart from the fact that Proposition 5.4 of [3] is valid for any $C$-viscosity solution. Theorem 2.1 is also close to Theorem 7.3 of [3], that is proved for any $L^p$-viscosity solution and not for the $C$-viscosity or $L^p$-viscosity solution we construct. The most significant difference in the assumptions is that $H$ in [3] is supposed to be Lipschitz in $[u]' := (u_1',\ldots,u_d')$ and for any $u$ be uniformly in $(t,x)$ close to a function that is uniformly continuous in $(t,x)$ (and not only in $x$).

In [4] the $C^{1+\alpha}$-regularity is investigated when $\bar{G}$ is summable to different powers in $t$ and $x$. Again $H$ is Lipschitz in $u$ and satisfies the continuity condition like in $(t,x)$ like in [3].

In what concerns the fully nonlinear elliptic equations, Caffarelli [1] and Trudinger [15], [16] were the first authors who proved $C^{1+\alpha}$ regularity for $C$-viscosity solutions of equations

$$H[u] = f$$

(1.3)

without convexity assumptions on $H$. The assumptions in these papers are different. In [1] the function $H(u,x)$ is independent of $u'$ and, for each $u''$, is uniformly sufficiently close to a function which is continuous with respect to $x$. In [15] and [16] the function $H$ depends on all arguments but is Hölder continuous in $x$. If we ignore the difference between $C$-viscosity and $L^p$-viscosity solutions, the next step in what concerns $C^{1+\alpha}$-estimates was done by Świȩch [14], who considered general $H$ and imposed the same condition as in [1] on the $x$-dependence, which is much weaker than in [15] and [16] (but also imposed the Lipschitz condition on the dependence of $H$ on $u'$, whereas in [15] and [16] only the continuity with respect to $u'$ is assumed). It is worth emphasizing that these results are about any continuous viscosity solution and not only about the ones we construct. The same bears on the results in [12], where the boundary $C^{1+\alpha}$ regularity is obtained.

The best value of $\alpha$ is largely unknown. However, it is proved in [13] in case $H = H(u'')$ that the solutions of $H[u] = 0$ are almost $C^{1+\alpha}$ regular if the solutions of $\hat{H}[u] = 0$ are $C^{1+\alpha}$ regular, where

$$\hat{H}(u'') := \lim_{\tau \to \infty} \frac{1}{\tau} H(\tau u'')$$

assuming that the limit exists.
By the way, it follows from a local version of Theorem 2.1 of [7] that, if \( \hat{H} \) is convex, then the solutions of \( H[u] = 0 \) are in \( W^{2,\text{loc}}_p \) for any \( p > 1 \), and thus in \( C^{1+\alpha} \) for any \( \alpha < 1 \). This covers Corollary 1.2 of [13].

2. Main results

Recall some definitions. For \( \kappa \in (0,1] \) and functions \( \phi(t, x) \) on \( \Pi \) set

\[
[\phi]_{C^{\kappa}(\Pi)} = \sup_{(t,x),(s,y)\in\Pi} \frac{|\phi(t, x) - \phi(s, y)|}{|t - s|^{\kappa/2} + |x - y|^{\kappa}}, \quad \|\phi\|_{C(\Pi)} = \sup_{\Pi} |\phi|,
\]

\[
\|\phi\|_{C^{\kappa}(\Pi)} = \|\phi\|_{C(\Pi)} + [\phi]_{C^{\kappa}(\Pi)}.
\]

For \( \kappa \in (1,2) \) and sufficiently regular \( \phi \) set

\[
[\phi]_{C^{\kappa}(\Pi)} = \sup_{t,s\in[0,T),x\in\Omega} \frac{|\phi(t, x) - \phi(s, x)|}{|t - s|^{\kappa/2}} + \sup_{x,y\in\Omega,t\in[0,T)} \frac{|D\phi(t, x) - D\phi(t, y)|}{|x - y|^{\kappa-1}}, \quad \|\phi\|_{C^{\kappa}(\Pi)} = \|\phi\|_{C^1(\Pi)} + [\phi]_{C^{\kappa}(\Pi)}.
\]

The set of functions with finite norm \( \| \cdot \|_{C^{\kappa}(\Pi)} \) is denoted by \( C^{\kappa}(\Pi) \).

Observe that any \( u \in C^{\kappa}(\Pi) \) admits a unique extension to \( \bar{\Pi} \) and we will always consider this extension while dealing with \( u(T, x) \). For \( \kappa = 2 \) we prefer a less ambiguous notation \( W^{1,2}_\infty(\Pi) \) instead of \( C^2(\Pi) \). As usual we write \( u \in C_{\text{loc}}^\kappa(\Pi) \) if \( u \in C^\kappa(\bar{C}_R(t, x)) \) for any \( C^1_R(t, x) \) such that \( \bar{C}_R(t, x) \subset \Pi \).

For \( \kappa \in (1,2) \) we are also going to use the spaces \( C^{1+\kappa}(\Pi) \) of functions \( u \in W^{1,2}_\infty(\Pi) \) with finite norm

\[
\|u\|_{C^{1+\kappa}(\Pi)} = \|v\|_{C^2(\Pi)} + [u]_{C^{1+\kappa}(\Pi)},
\]

where

\[
[u]_{C^{1+\kappa}(\Pi)} = [\partial_t u]_{C^{\kappa-1}(\Pi)} + [D^2 u]_{C^{\kappa-1}(\Pi)}.
\]

Remark 2.1. Sometimes it is useful to invoke the well-known embedding theorem according to which, if \( \kappa \in (1,2) \) and \( \phi \in C^{\kappa}(C_R) \), then

\[
|D\phi(t, x) - D\phi(s, x)| \leq N(d)|t - s|^{(\kappa-1)/2}[\phi]_{C^{\kappa}(C_R)} \quad (2.1)
\]

whenever \( (t, x), (s, x) \in C_R \).

For \( \delta \in (0,1] \) denote

\[
S_\delta = \{ a \in \mathbb{S} : \delta^{-1}|\lambda|^2 \geq a^{ij}\lambda^i\lambda^j \geq \delta|\lambda|^2, \forall \lambda \in \mathbb{R}^d \}.
\]
Assumption 2.1. (i) The function $H(u, t, x)$ is Lipschitz continuous in $u''$ for every $u', \langle t, x \rangle \in \mathbb{R}^{d+1}$ and at all points of differentiability of $H(u, t, x)$ with respect to $u''$, we have $D_{u''}H \in S_\delta$, where $\delta$ is a fixed constant in $(0, 1]$.

(ii) Either

(a) for any $u_0 \in \mathbb{R}^{d+1} \times S$ and $(t_0, x_0) \in \Pi$ the function $H(u, t, x)$ is continuous at $u_0$ as a function of $u$ uniformly with respect to $(t, x)$ belonging to a neighborhood of $(t_0, x_0)$;

or

(b) For any $M \in (0, \infty)$ the function $H(u, t, x)$ is continuous with respect to $|u'| = (u'_1, \ldots, u'_d)$ uniformly with respect to $|u'_0| \leq M$, $u'' \in S$, $(t, x) \in \Pi$.

We fix some constants

$p \in (d + 2, \infty), \; K_0, K_1, \in [0, \infty), \; R_0 \in (0, 1], \; \hat{R}_0 \in [0, \infty],$

a nonnegative function

$\bar{G} \in L_p(\Pi),$

and in the following assumption also use $\theta_0 = \theta_0(d, \delta) \in (0, 1]$, whose value is specified in the proof of Lemma 6.3.

Assumption 2.2. We have a representation

$H(u, t, x) = F(u'_0, u'', t, x) + G(u, t, x).$

(i) The functions $F$ and $G$ are measurable functions of their arguments, continuous with respect to $u$ for any $(t, x)$.

(ii) For all values of the arguments

$|G(u, t, x)| \leq K_0|u'| + \bar{G}(t, x).$

(iii) The function $F$ is Lipschitz continuous with respect to $u''$ and at all points of differentiability of $F$ with respect to $u''$ and any $\varepsilon \in S$ such that $|\varepsilon| = 1$ we have

$D_{u''}F + \theta_0\varepsilon \in S_{\delta/2}.$

(iv) For any $u'_0, v'_0 \in \mathbb{R}, \; (t, x_i) \in \mathbb{R}^{d+1}, \; i = 1, 2$, and $u'' \in S$ we have

$|F(u'_0, u'', t, x_1) - F(v'_0, u'', t, x_2)| \leq K_1 + \theta_0|u''|$

as long as $|x_1 - x_2| \leq R_0$ and $|u'_0 - v'_0| < \hat{R}_0$.

(v) We have $F(u'_0, 0, t, x) = 0$ for all $u'_0 \in \mathbb{R}, \; (t, x) \in \mathbb{R}^{d+1}$.

Remark 2.2. The value $\infty$ is allowed for $\hat{R}_0$ with the purpose to cover the cases in which $F$ is almost independent of $u'_0$ when conditions like (2.2) below are automatically satisfied.

Remark 2.3. Observe that Assumption 2.2 (iv) is automatically satisfied if $F$ is a function of only $u''$ and $t$.

Assumption 2.3. We are given $g \in C(\Pi)$. 
Theorem 2.1. There is a constant $\theta_0 = \theta_0(d, \delta) \in (0, 1]$ such that, if the above assumptions are satisfied with this $\theta_0$, then there exist a $\kappa = \kappa(d, \delta, p) \in (1, 2)$ and a function $v \in C_{\text{loc}}^\kappa(\Pi) \cap C(\Pi)$ that is a $C$-viscosity or an $L_p$-viscosity solution of equation (1.1) in $\Pi$ with boundary condition $v = g$ on $\partial'\Pi$ according as requirements (a) or (b) in Assumption 2.1 (ii) are satisfied.

Furthermore, for any $r, R \in (0, R_0]$ satisfying $r < R$ and $(t, x) \in \Pi$ such that $C_R(t, x) \subset \Pi$ and
\[
\text{osc}_{C_R(t,x)} v < \hat{R}_0
\]
we have
\[
[v]_{C_{\kappa}(C_R(t,x))} \leq N(R - r)^{-\kappa} \sup_{C_R(t,x)} |v| + N \left( K_1 + \|G\|_{L_p(C_R(t,x))} \right),
\]
where $N$ depend only on $d, \delta,$ and $K_0$.

This theorem is proved in Section 6.

Remark 2.4. We assumed that $\Omega \in C^{1,1}$ just to be able to refer to the results available at this moment, but actually much less is needed for Theorem 2.1 to hold. For instance the exterior cone condition would suffice.

Example 2.1. Let $A$ and $B$ be countable sets and suppose that for any $\alpha \in A$ and $\beta \in B$ we are given an $S_{\beta}$-valued functions $a_{ij}^{\alpha\beta}(u_0', t, x)$ and a real-valued function $b_{ij}^{\alpha\beta}(u', t, x)$ defined for $u', (t, x) \in \mathbb{R}^{d+1}$. Assume that these functions are measurable as functions of $(u', t, x)$, continuous with respect to $u'$ uniformly with respect to $(\alpha, \beta, (t, x)) \in A \times B \times \Pi$, and $a_{ij}^{\alpha\beta}(u_0', t, x)$ is continuous with respect to $x$ uniformly with respect to $(\alpha, \beta, u_0', t) \in A \times B \times \mathbb{R} \times \mathbb{R}$. Finally, suppose that for all values of indices and arguments
\[
|b_{ij}^{\alpha\beta}(u', t, x)| \leq K_0 |u'| + G(t, x).
\]
Then the following equation
\[
\partial_t v + \inf_{\alpha \in A} \sup_{\beta \in B} \left[ a_{ij}^{\alpha\beta}(v, t, x) D_{ij} v + b_{ij}^{\alpha\beta}(v, Dv, t, x) \right] = 0
\]
in $\Pi$ with boundary condition $v = g$ on $\partial'\Pi$ has an $L_p$-viscosity solution which belongs to $C_{\text{loc}}^\kappa(\Pi) \cap C(\Pi)$.

This follows immediately from Theorem 2.1 if one sets
\[
F(u_0', u'', t, x) = \inf_{\alpha \in A} \sup_{\beta \in B} a_{ij}^{\alpha\beta}(u_0', t, x) u_{ij}''
\]
and observes that, for any $\theta_0$, in particular, for the one from Theorem 2.1, one can find $R_0$ and $\hat{R}_0$, for which Assumption 2.2 (iv) is satisfied with an appropriate $K_1$.

One can also see that the continuity of $a_{ij}^{\alpha\beta}(u_0', t, x)$ with respect to $x$ can be relaxed allowing sufficiently small discontinuities. It is also worth noting that in [3] in case of the Isaacs equations $a$ is independent of $u_0'$ and $b$ is an affine function of $u'$. 
We explain the main rough ideas in the proof of Theorem 2.1 in case $F$ is independent of $u_0'$ and $\bar{G}$ is bounded. It consists of establishing a priori estimates of the type

$$\sup_{C_r(t_0,x_0)} |v - l| \leq Nr^\kappa$$

(2.4)

for any $C_r(t_0,x_0)$ which is strictly inside $\Pi$ and any small $r > 0$ as long as an affine function $l = l(x)$ is chosen appropriately. This turns out to be enough to get an estimate of the $C^\kappa$-norm of $v$ in small cylinders which are strictly inside $\Pi$ (see Lemma 6.2). Then, to obtain (2.4) we represent $v$ as $h + w$, where $h$ is a solution of

$$\partial_t h + F[h] = 0$$

(2.5)

in $C_r(t_0,x_0)$ with boundary data $v$ and $w = v - h$ is found from

$$0 = \partial_t v + H[v] - \partial_t h - F[h] = \partial_t w + a^{ij}D_{ij}w + G[v],$$

where $(a^{ij})$ is a certain $\mathcal{S}_\delta$-valued function. Since $w = 0$ on $\partial' C_r(t_0,x_0)$ by the maximum principle $|w| \leq Nr^2(1 + \sup(|Dv|, C_r(t_0,x_0)))$. The latter supremum is irrelevant because we are going to estimate the $C^\kappa$-norm of $v$ and $\kappa > 1$.

Then we see that, to get (2.4), it suffices to prove that (2.4) holds with $h$ in place of $v$. To do this step we want to replace $F$ with the one independent of $(t,x)$. Freezing the coefficients does not help because there is no hope to control the second-order derivatives of solutions of such equations. Therefore, we just replace $F$ with

$$F_0(\pm) = F(u'',t,x_0) \pm (K_1 + \theta_0|u''|)$$

and introduce $v^{(\pm)}$ as solutions of

$$\partial_t v^{(\pm)} + F_0^{(\pm)}[v^{(\pm)}] = 0,$$  

(2.6)

$$\partial_t v^{(-)} + F_0^{(-)}[v^{(-)}] = 0$$  

(2.7)

in $C_r(t_0,x_0)$ with boundary condition $v$. Since $F_0^{(-)} \leq F \leq F_0^{(+)}$ in $C_r(t_0,x_0)$ if $r$ is small enough, we have $v^{(-)} \leq h \leq v^{(+)}$ by the maximum principle. Furthermore, since $F_0^{(\pm)}$ is independent of $x$, one can differentiate the equations (2.6) and (2.7) with respect to $x$ and get estimates of the H"older constants of $Dv^{(\pm)}$ by the Krylov-Safonov theorem. These estimates guarantee that $v^{(\pm)}$ can be approximated by affine functions of $x$ as in (2.4).

Then the only thing which remains is to estimate $|v^{(\pm)} - v^{(-)}|$. To this end, for $\theta \in [-\theta_0, \theta_0]$ we introduce

$$F_0(u'',t,\theta) = F_0(u'',t) + \theta|u''| + K_1\theta/\theta_0$$

and define $v^\theta$ from the equation

$$\partial_t v^\theta + F_0(D^2v^\theta, t, \theta) = 0$$

(2.8)

in $C_r(t_0,x_0)$ with boundary condition $v$. Since $v^{\pm\theta_0} = v^{(\pm)}$ to estimate $|v^{(\pm)} - v^{(-)}|$, it suffices to estimate $D_{\theta}v^\theta$. 
Now comes an idea originated in the theory of diffusion processes. We look at (2.8) as an equation in variables \((t, x, \theta)\). There is no derivatives with respect to \(\theta\), so that it is a degenerate equation, but this was never a problem in such matters in that theory, that suggests that the function

\[
V(t, \theta, x, \tau, \xi) := \tau D_\theta v^\theta(t, x) + \xi^i D_{x^i} v^\theta(t, x)
\]
satisfies a parabolic equation with respect to the variables \((t, \theta, x, \tau, \xi)\). Indeed

\[
\partial_t V + a^{ij} D_{x^i x^j} V + \tau \varepsilon^{ij} D_{x^i \xi^j} V + N \tau^2 \delta^{ij} D_{\xi^i \xi^j} V + K_1 \tau / \theta_0 = 0,
\]

where \((a^{ij})\) is an \(S_{d/2}\)-valued function (see Assumption 2.2 (iii)),

\[
\varepsilon^{ij} = D_{x^i x^j} v^\theta / |D_x^2 v^\theta|,
\]

and \(N\) is any constant. Since there is no derivatives with respect to \(\tau\), it is just a parameter and for \(W(t, \theta, x, \xi) := V(t, \theta, x, 1, \xi)\) we obtain the equation

\[
\partial_t W + a^{ij} D_{x^i x^j} W + \varepsilon^{ij} D_{x^i \xi^j} W + N \delta^{ij} D_{\xi^i \xi^j} W + K_1 / \theta_0 = 0,
\]

which is parabolic if \(N\) is sufficiently large. We consider this equation in \((t, x, \xi) \in C_r(t_0, x_0) \times \mathbb{R}^d\) and see that to estimate \(W\), it suffices to have a good control of it on the parabolic boundary of this set, where \(D_\theta v^\theta(t, x) = 0\) by construction. Thus we see that we need to estimate \(|D_x v^\theta|\) in \(C_r(t_0, x_0)\). This will be done by differentiating (2.8) with respect to \(x\) and using the maximum principle, which reduces the matter to estimating \(|D_x v^\theta|\) in \(\partial' C_r(t_0, x_0)\).

As a general comment we point out that, since we have to have sufficiently smooth solutions in the above argument, we use cut-off equations and use finite-differences to avoid using third-order derivatives.

### 3. Auxiliary results about linear equations

We will be using a common way of approximating functions in \(C^\kappa(C_1)\) by infinitely differentiable ones.

**Lemma 3.1.** Let \(\kappa \in (0, 2)\), \(r \in (0, \infty)\), \(g, h \in C^\kappa(C_r)\). Then for any \(\varepsilon > 0\) there exists an infinitely differentiable functions \(g^\varepsilon\) and \(h^\varepsilon\) on \(\mathbb{R}^{d+1}\) such that in \(C_r\)

\[
|g - g^\varepsilon| \leq N[g]_{C^\kappa(C_r)}(r\varepsilon)^\kappa, \quad |Dg - Dg^\varepsilon| \leq N[g]_{C^\kappa(C_r)}(r\varepsilon)^{\kappa-1},
\]

\[
|\partial_t g^\varepsilon| + |D^2 g^\varepsilon| + r\varepsilon |D^3 g^\varepsilon| + r\varepsilon |D\partial_t g^\varepsilon| \leq N[g]_{C^\kappa(C_r)}(r\varepsilon)^{\kappa-2}, \quad (3.1)
\]

\[
[h^\varepsilon]_{C^1(C_r)} \leq N[h]_{C^\kappa(C_r)}(r\varepsilon)^{-1}.
\]
Proof. Parabolic scalings reduce the general situation to the one in which \( r = 1 \). Then this well-known result is obtained by first continuing \( g(t, x) \), \( h(t, x) \) as functions of \( t \) to \( \mathbb{R} \) to become even, 2-periodic functions, then continuing thus obtained functions across \( |x| = 1 \) almost preserving \([g]_{C^{\kappa}(C_1)}\), \([h]_{C^{\kappa}(C_1)}\) in the whole space and then taking convolutions with \( \delta \)-like kernels. The lemma is proved. \( \Box \)

**Theorem 3.2.** Let \( \kappa \in (1, 2) \) and let \( g \in C^{\kappa}(C_1) \). Then there exists a unique \( u \in C^{\kappa}_{\text{loc}}(C_1) \cap C^{\kappa}(C_1) \) which satisfies the heat equation

\[
\partial_t u + \Delta u = 0
\]

in \( C_1 \) and equals \( g \) on \( \partial' C_1 \). Furthermore, there exists a constant \( N = N(d, \kappa) \) such that

\[
[u]_{C^{\kappa}(C_1)} \leq N[g]_{C^{\kappa}(C_1)}, \quad (3.3)
\]

\[
|D^2u(t, x)| \leq N\left((1 - |x|) \land \sqrt{1-t}\right)^{\kappa/2-2}[g]_{C^{\kappa}(C_1)}. \quad (3.4)
\]

Proof. One can subtract an affine function of \( x \) from \( g \) and reduce the general situation to the one where

\[
g(1, 0) = 0, \quad Dg(1, 0) = 0. \quad (3.5)
\]

In that case take \( g^\varepsilon \) from Lemma 3.1. Then by a classical result (see, for instance, Theorem 5.14 in [10] or Theorems 10.3.3 and 10.2.2 in [6]) there exists a unique \( u^\varepsilon \in C^{1+\kappa}(C_1) \) satisfying (3.2) in \( C_1 \) and equal to \( g \) on \( \partial' C_1 \). In addition,

\[
\|u^\varepsilon\|_{C^{1+\kappa}(C_1)} \leq N\|g^\varepsilon\|_{C^{1+\kappa}(C_1)}. \quad (3.6)
\]

Furthermore, by classical results (see, for instance, Theorem 8.12.1 in [6]) the functions \( u^\varepsilon \) are infinitely differentiable with respect to \( x \) in \( \overline{C_r} \) for any \( r < 1 \) and by Theorem 8.4.4 in [6] and the maximum principle any derivative of \( u^\varepsilon \) of any order with respect to \( x \) is bounded in \( \overline{C_r} \) for any \( r < 1 \) by a constant independent of \( \varepsilon \). As it follows from equation (3.2) itself, the same holds for derivatives with respect to \( t \) of any derivative of any order with respect to \( x \) (cf. Exercise 8.12.4 in [6]).

In addition, by the maximum principle and (3.1),

\[
|u^{\varepsilon_1} - u^{\varepsilon_2}| \leq |g^{\varepsilon_1} - g^{\varepsilon_2}| \leq N[g]_{C^{\kappa}(C_1)}(\varepsilon_1^\kappa + \varepsilon_2^\kappa). \quad (3.7)
\]

It follows that, as \( \varepsilon \downarrow 0 \), \( u^\varepsilon \) converges uniformly on \( \overline{C_1} \) to a continuous function, which is equal to \( g \) on \( \partial' C_1 \), is infinitely differentiable in \( C_1 \) and satisfies (3.2). In light of (3.7) we have

\[
|u - u^\varepsilon| \leq N[g]_{C^{\kappa}(C_1)}\varepsilon^\kappa. \quad (3.8)
\]

Also (3.6) and (3.5) along with (3.1) imply that

\[
[u^\varepsilon]_{C^{1+\kappa}(C_1)} \leq N[g^\varepsilon]_{C^{1+\kappa}(C_1)} \leq N[g]_{C^{\kappa}(C_1)} \varepsilon^{-1}.
\]
Now take \( x_0 \in \mathbb{R}^d \), unit \( l \in \mathbb{R}^d \) and \( 0 < h_1 \leq h \) such that
\[
x_0, x_0 + h_1 l_1, x_0 + 2 h l, x_0 + 2 h l + h_1 l_1 \in \bar{B}_1.
\]
Observe that for any \( t \in [0,1] \)
\[
u(t, x) = [u(t, x) - u^\varepsilon(t, x)] + u^\varepsilon(t, x_0) + (x^i - x_0^i) D_i u^\varepsilon(t, x_0)
\]
\[+ (1/2)(x^i - x_0^i)(x^i - x_0^i) D_{ij} u^\varepsilon(t, x_0) + v(t, x),\]
where
\[
|v(t, x)| \leq N|x - x_0|^{1+\kappa [D^2 u^\varepsilon]_{C^{\kappa-1}(C_1)}} \leq N|x - x_0|^{1+\kappa [g]_{C^{\kappa}(C_1)}} \varepsilon^{-1}.
\]
Since the third-order finite difference of any quadratic polynomial is zero
and
\[
|u(t, x) - u^\varepsilon(t, x)| \leq N[g]_{C^{\kappa}(C_1)} \varepsilon^\kappa,
\]
we have
\[
|(T_{h,l} - 1)^2(T_{h,l_1} - 1)u(t, x_0)| \leq N[g]_{C^{\kappa}(C_1)} \varepsilon^\kappa + Nh^{1+\kappa [g]_{C^{\kappa}(C_1)}} \varepsilon^{-1}.
\]
By taking \( \varepsilon = h \) we arrive at
\[
|(T_{h,l} - 1)^2(T_{h,l_1} - 1)u(t, x_0)| \leq N[g]_{C^{\kappa}(C_1)} h^\kappa.
\]
As it can be shown (or extracted from [5]) that, the arbitrariness of \( x_0, h, h_1, l, l_1 \) in the above inequality implies that for any \( t \in [0,1] \)
\[
[u(t, \cdot)]_{C^\kappa(B_1)} \leq N([g]_{C^{\kappa}(C_1)} + \text{osc } u),
\]
which along with the maximum principle show that
\[
[u(t, \cdot)]_{C^\kappa(B_1)} \leq N([g]_{C^{\kappa}(C_1)} + \text{osc } g),
\]
where \( \text{osc}_{\partial C} g \) can be replaced with \([g]_{C^{\kappa}(C_1)}\) in light of (3.5).
Next, fix \( x \in B_1 \) and take \( t_0 \in (0,1) \), \( h > 0 \), such that \( t_0 + 2 h^2 \in (0,1) \).
Observe that
\[
u(t, x) = [u(t, x) - u^\varepsilon(t, x)] + u^\varepsilon(t_0, x) + (t - t_0) \partial_t u^\varepsilon(t_0, x) + w(t, x),
\]
where
\[
|w(t, x)| \leq |t - t_0|^{(\kappa+1)/2 [\partial_t u^\varepsilon]_{C^{\kappa-1}(C_1)}} \leq N|t - t_0|^{(\kappa+1)/2 [g]_{C^{\kappa}(C_1)}} \varepsilon^{-1}.
\]
Since the second-order differences of linear function are equal to zero,
\[
|u(t_0 + 2 h^2, x) - 2 u(t_0 + h^2, x) + u(t_0, x)|
\]
\[\leq N[g]_{C^{\kappa}(C_1)} \varepsilon^\kappa + Nh^{1+\kappa [g]_{C^{\kappa}(C_1)}} \varepsilon^{-1}.
\]
Here, for \( \varepsilon = h \), the right-hand side becomes
which implies that

\[
|u(t, x) - u(s, x)| \leq |t - s|^\kappa/2 N \left( [g]_{C^\kappa(C_1)} + \text{osc } u, x \right)
\]

if \( s, t \in [0, 1] \). Again the last oscillation can be replaced by \([g]_{C^\kappa(C_1)}\), and this proves (3.3).

By Theorem 8.4.4 in [6] for any \((t_0, x_0) \in C_1\)

\[
|\partial_t u(t_0, x_0)| + |D^2 u(t_0, x_0)| \leq NR^{-2} \sup_{C_R(t_0, x_0)} |u|,
\]

where \( R = (1 - |x_0|) \wedge \sqrt{1 - t_0} \). This holds for any sufficiently regular solution of (3.2) and not only for the one constructed above. Therefore \( u \) on the right can be replaced with \( u - \hat{u} \), where \( \hat{u} \) is any affine function of \( x \). By taking \( \hat{u} \) as the first-order Taylor polynomial of \( u(t_0, x) \) with respect to \( x \) at \( x_0 \) and using (3.3) we come to (3.4). The theorem is proved. \( \square \)

Lemma 3.3. For \( \kappa \in (1, 2) \) there is a function \( \Phi \in C^{1,2}_{\text{loc}}(C_1) \cap C(\bar{C}_1) \) and a constant \( N = N(\kappa, \delta) \) such that

\[
\partial_t \Phi \leq a^{ij} D_{ij} \Phi \leq -\left[ (1 - |x|) \wedge \sqrt{1 - t} \right]^{\kappa - 2}
\]

(3.9) in \( C_1 \) for any \( (a^{ij}) \in \mathcal{S}_\delta \) and

\[
0 \leq \Phi(t, x) \leq N(1 - |x|).
\]

Proof. Set \( \rho(x) = 1 - |x|^2 \), \( \beta = (\kappa + 1)/2 \), and, for a constant \( N_0 \) to be determined later, define

\[
\Phi(t, x) = N_0 \left[ \rho(x) - (1/\kappa) \rho^\kappa(x) \right] + \beta^{-1} (1 - t) \rho(x).
\]

We have

\[
D_i \rho = -2x^i, \quad D_{ij} \rho = -2 \delta_{ij},
\]

\[
D_{ij} \Phi = -2N_0[1 - \rho^{\kappa - 1}] \delta_{ij} - 4N_0(\kappa - 1) \rho^{\kappa - 2} x^i x^j - 2\beta^{-1} (1 - t) \delta_{ij}.
\]

Hence,

\[
\partial_t \Phi + a^{ij} D_{ij} \Phi \leq -\delta I \rho^{\kappa - 2} - (1 - t)^{\beta - 1} \rho,
\]

where

\[
I := 2N_0[\rho^{\kappa-}\rho] + 4N_0(\kappa - 1)|x|^2.
\]

Obviously, there exists \( N_0 = N_0(\delta, \kappa) \) such that in \( C_1 \) we have \( I \geq 2/\delta \), in which case

\[
-\delta I \rho^{\kappa - 2} \leq -2\rho^{\kappa - 2} = -2(1 + |x|)\kappa - 2(1 - |x|)\kappa - 2 \leq -(1 - |x|)^{\kappa - 2}
\]
and (3.9) holds if \(1 - |x| \leq \sqrt{1 - t}\). In case \(1 - |x| \geq \sqrt{1 - t}\) we have \(\rho \geq \sqrt{1 - t}\) and
\[
-(1 - t)^{\beta - 1} \rho \leq -(1 - t)^{\beta - 1/2} = -(1 - t)^{\kappa/2 - 1},
\]
so that (3.9) holds again. The lemma is proved. \(\square\)

4. Estimating the difference of solutions of two different equations

We need the following Theorem 6.1 of [8].

**Theorem 4.1.** Suppose that Assumption 2.1 (i) is satisfied, \(\Omega \in C^{1,1}\), \(H\) is a continuous function of \(u\), the number \(\bar{H} := \sup_{u',t,x} (|H(u',0,t,x)| - K_0|u'|) \geq 0\) is finite, and \(g \in W^{1,2}(\mathbb{R}^{d+1})\). Then there exists a convex positive homogeneous of degree one function \(P(u'')\) such that at all points of its differentiability \(D_{uu'} P \in S_{\hat{\delta}}\), where \(\hat{\delta} = \delta(d,\delta) \in (0,\delta)\), and for \(P[u] = P(D^2 u)\) and any \(K > 0\) the equation
\[
\partial_t v + \max(H[v], P[v] - K) = 0 \tag{4.1}
\]
in \(\Pi\) with boundary condition \(v = g\) on \(\partial\Pi\) has a solution \(v \in W^{1,2}(\Pi)\) for any \(p \geq 1\).

Let \(F_0(u'',t)\) be a function satisfying Assumption 2.2 (iii), measurable in \(t\), and such that \(F_0(0, t) = 0\). Fix a constant \(K_1 \geq 0\) and set
\[
F^{(\pm)}(u'',t) = F_0(u'',t) \pm (K_1 + \theta_0|u''|),
\]
take \(P(u'')\) from Theorem 4.1 with \(\delta/2\) in place of \(\delta\) and for fixed \(R \in (0,\infty), K > 0\) consider the equations
\[
\partial_t v + \max \left(F^{(\pm)}[v], P[v] - K\right) = 0
\]
in \(C_R\) with boundary data \(v = g\) on \(\partial C_R\), where \(g \in W^{1,2}_\infty(C_R)\) is a given function. By \(v^{(\pm)}\) we denote their solutions that exist by Theorem 4.1 and belong to \(W^{1,2}_p(C_R)\) for any \(p \in [1,\infty)\).

**Theorem 4.2.** For any \(\kappa \in (1,2)\) there exists a constant \(N = N(\kappa,\delta,d)\) such that in \(C_R\) we have
\[
|v^{(+)} - v^{(-)}| \leq NR^2 K_1 + NR^\kappa [g]_{C^\kappa(C_R)}.
\]

The proof of this theorem is based on the following two auxiliary results.
Lemma 4.3. Let $a$ be a measurable $S_δ$-valued function, $p > d + 2$, and let $v \in W_p^{1,2}(C_1)$ be a solution of
\[
\partial_t v + a^{ij}D_{ij}v + f = 0
\]
in $C_1$ (a.e.) with boundary condition $v = g$ on $\partial' C_1$, where $|f| \leq \bar{f}$ for a constant $\bar{f} \in [0, \infty)$. Assume that (3.5) holds. Then, for any $\kappa \in (1, 2)$, there exists $N = N(\delta, d, \kappa)$ such that
\[
\sup_{(0,1) \times \partial B_1} |Dv| \leq N|Dg|_{C^\kappa(C_1)} + N\bar{f}.
\]

Proof. By the embedding Lemma 2.3.3 of [9], the function $Dv$ is continuous in $\bar{C}_1$. Then take the function $u$ from Theorem 3.2 and set $w(t, x) = v(t, x) - u(t, x)$. We have
\[
\partial_tw + a^{ij}D_{ij}w + h = 0,
\]
where $h := f + \partial_t u + a^{ij}D_{ij}u$. By using Lemma 3.3 we conclude
\[
|\partial_tw + a^{ij}D_{ij}w| \leq \bar{f} + N\left((1 - |x|) \wedge \sqrt{1 - t}\right)^{\kappa - 2}|g|_{C^\kappa(C_1)}
\]
\[
\leq -\partial_t \Psi - a^{ij}D_{ij}\Psi,
\]
where $\Psi = N|g|_{C^\kappa(C_1)}\Phi + \bar{f}\delta^{-1}(1 - |x|^2)$. By the maximum principle
\[
|w(t, x)| \leq \Psi(t, x) \leq N_0\left(|g|_{C^\kappa(C_1)} + \bar{f}\right)(1 - |x|).
\]
Consequently, the normal derivative of $v(t, x)$ at a point $x_0 \in \partial B_1$ by magnitude is less than the absolute value of the normal derivative of $u(t, x)$ plus $N_0\left(|g|_{C^\kappa(C_1)} + \bar{f}\right)$. By interpolation inequalities $|Du|$ is estimated by $|u|_{C^\kappa(C_1)}$ and $\text{osc}_{C_1} u \leq \text{osc}_{C_1} g$, where the former is estimated in (3.3) by $|g|_{C^\kappa(C_1)}$ and the latter is estimated by the same quantity due to (3.5). Thus, the normal derivative of $v(t, x)$ admits the estimate we are after. By noting that the tangential derivatives of $v(t, x)$ coincide with those of $g(t, x)$, we finally come to (4.2). The lemma is proved.

Lemma 4.4. Let $R, \chi \in (0, \infty)$ and $\kappa \in (1, 2)$ be constants and let $F(\theta, u''', t)$ be a measurable function on $[-\theta_0, \theta_0] \times S \times \mathbb{R}$, which is Lipschitz continuous in $(\theta, u''')$ for any $t$ and such that at all points of its differentiability
\[
|D_\theta F| \leq \chi + |u'''|, \quad D_w F \in S_\delta.
\]
Also suppose that $F(\theta, 0, t)$ is bounded. Take $g \in W_\infty^{1,2}(C_R)$, and assume that for any $\theta \in [-\theta_0, \theta_0]$ the equation
\[
\partial_v F(\theta, D^2 v, t) = 0
\]
in $C_R$ (a.e.) with boundary condition $v = g$ on $\partial' C_R$ has a solution $v = v(\theta, \cdot) \in W_\infty^{1,2}(C_R)$, where $p > d + 2$, $p \geq 2d + 1$. Then for any $(t, x) \in C_R$
the function \(v(\theta, t, x)\) is Lipschitz continuous on \([-\theta_0, \theta_0]\) and at all points of its differentiability with respect to \(\theta\)

\[
|D_\theta v| \leq R^2(\chi + N \sup_t |F(\theta, 0, t)|) + NR^\kappa[g]_{C^\kappa(C_R)}, \tag{4.4}
\]

where the constant \(N\) depends only on \(\delta, d, \kappa\).

Proof. The idea of the proof comes from the theory of diffusion processes and is explained at the end of Section 2. The parabolic equation for the directional derivative of \(v(\theta, t, x)\) with respect to \((\theta, x)\) in appropriate directions will be what we are interested in. Since in our situation there is no guarantee that \(v\) is smooth enough, we follow Trudinger’s method (see [15]) based on finite-differences.

As usual, parabolic scalings reduce the general situation to the one in which \(R = 1\). Also by subtracting from \(g\) and \(v\) the same affine function of \(x\) we may assume that (3.5) holds.

In that case fix \(\theta_0 \in (-\theta_0, \theta_0)\) and for sufficiently small \(h\) introduce

\[
w(t, x, \xi) = v(\theta_0 + h, t, x + \xi) - v(\theta_0, t, x),
\]

where \(t, x, \xi \in \bar{Q}\) with

\[
Q := [0, 1) \times \{(x, \xi) : x, x + \xi \in B_1\}.
\]

Note for the future that, by embedding theorems \(D_x v(\theta, t, x)\) is a continuous function in \(C_1\) for any \(\theta\).

Next, observe that

\[
F(\theta_0 + h, D_x^2 v(\theta_0 + h, t, x + \xi), t) - F(\theta_0, D_x^2 v(\theta_0, t, x), t)
\]

\[
= [F(\theta_0, D_x^2 v(\theta_0 + h, t, x + \xi), t) - F(\theta_0, D_x^2 v(\theta_0, t, x), t)] + I
\]

\[
= a^{ij} D_{x^i x^j} w + I,
\]

in \(Q\) (a.e.) where \((a^{ij})\) is a measurable \(\mathbb{S}_\delta\)-valued function and

\[
I = F(\theta_0 + h, D_x^2 v(\theta_0 + h, t, x + \xi), t) - F(\theta_0, D_x^2 v(\theta_0 + h, t, x + \xi), t).
\]

Since \(|D_\theta F| \leq \chi + |u''|\) by assumption, we have

\[
I = \tau h + h \varepsilon^{ij} D_{x^i x^j} v(\theta_0 + h, t, x + \xi) = \tau h + h \varepsilon^{ij} D_{x^i \xi^j} w(t, x, \xi),
\]

where \(|\tau| \leq \chi\) and \((\varepsilon^{ij})\) is an \(\mathbb{S}\)-valued function with norm majorated by one. Hence, in \(Q\) (a.e.) we have

\[
\partial_t w + \mathcal{L} w + \tau h = 0,
\]

where

\[
\mathcal{L} = [a^{ij} - N_0 h^2 \delta^{ij}] D_{x^i x^j} + h \varepsilon^{ij} D_{x^i \xi^j} + N_0 h^2 \delta^{ij} D_{\xi^i \xi^j}
\]

is a uniformly elliptic operator for an appropriate \(N_0 = N_0(\delta, d)\) and all sufficiently small \(h \neq 0\).
Notice that on $\partial'Q$ either $t = 1$, and then
\[|w| \leq \sup_{|x| \leq 1} |Dg| |\xi| = \sup_{|x| \leq 1} |Dg - Dg(1,0)| |\xi|,\]
or $|x| \leq 1$ and $|x + \xi| = 1$, in which case $v(\theta^0 + h, t, x + \xi) = v(\theta^0, t, x + \xi)$ and
\[|w(t, x, \xi)| \leq \sup_{C_1} |D_x v(\theta^0, \cdot)| |\xi|,\]
or else $|x + \xi| \leq 1$ and $|x| = 1$, in which case $v(\theta^0, t, x) = v(\theta^0 + h, t, x)$ and
\[|w(t, x, \xi)| \leq \sup_{C_1} |D_x v(\theta^0 + h, \cdot)| |\xi|.\]

In all cases on $\partial'Q$ we have
\[|w(t, x, \xi)| \leq N_1|\xi| \leq N_1 h + N_1 h^{-1}|\xi|^2,\]
where
\[N_1 = [Dg]_{C^\alpha(C_1)} + \max_{\theta = \theta^0, \theta^0 + h \in C_1} \sup_{C_1} |D_x v(\theta, \cdot)|.\]

As is easy to see, there is a constant $N_2 = N_2(d, \delta)$ such that, for the function
\[\phi(t, x, \xi) = h\chi(1 - t) + N_1 h + N_1 h^{-1}[|\xi|^2 + N_2 h^2(1 - |x|^2)]\]
we have
\[\partial_t \phi + L\phi + \tau h \leq 0\]
in $Q$ and, of course, $|w| \leq \phi$ on $\partial'Q$. By the maximum principle $|w| \leq \phi$ in $\bar{Q}$, in particular, (take $\xi = 0$) in $C_1$
\[|v(\theta^0 + h, t, x) - v(\theta^0, t, x)| \leq h\chi + NN_1 h,\]
where $N = N(\delta, d)$.

It follows that to prove the lemma, it suffices to show that for any $\theta \in [-\theta_0, \theta_0]$, with a constant $N = N(\delta, d, \kappa)$, we have
\[\sup_{C_1} |D_x v(\theta, \cdot)| \leq N[Dg]_{C^\alpha(C_1)} + N \sup_t |F(\theta, 0, t)|. \tag{4.5}\]

By applying finite-difference operators with respect to $x$ to (4.3), we see that, for small $h$ and unit $l \in \mathbb{R}^d$, the function \[\left[\frac{v(\theta, t, x + hl) - v(\theta, t, x)}{h}\right]\]
satisfies a parabolic equation with zero free term in a domain slightly smaller than $C_1$. Hence, its sup over the domain is achieved on the parabolic boundary. By letting $h \to 0$ we conclude that
\[\sup_{C_1} |D_x v(\theta, \cdot)| = \sup_{\partial' C_1} |D_x v(\theta, \cdot)|,\]
and since $v(\theta, 1, x) = g(1, x)$ for $|x| \leq 1$, to prove (4.5), it suffices to prove that
\[\sup_{(0,1) \times \partial B_1} |D_x v(\theta, \cdot)| \leq N[Dg]_{C^\alpha(C_1)} + N \sup_t |F(\theta, 0, t)|. \tag{4.6}\]
We fix $\theta$ and observe that
\[
0 = \partial_t v(\theta, t, x) + \left[ F(\theta, D^2 v(\theta, t, x), t) - F(\theta, 0, t) \right] + f \\
= \partial_t v(\theta, t, x) + a^{ij} D_{ij} v(\theta, t, x) + f,
\]
(a.e.), where $(a^{ij})$ is a measurable $\mathbb{S}_3$-valued function and $f = F(\theta, 0, t)$. After that (4.6) immediately follows from Lemma 4.3. The lemma is proved. 

**Proof of Theorem 4.2.** For $\theta \in [-\theta_0, \theta_0]$ introduce
\[
F(\theta, u'', t) = \max \left( F_0(u'', t) + \theta (K_1/\theta_0 + |u''|), P(u'') - K \right).
\]

By Theorem 4.1 equation (4.3) in $C_R$ with boundary condition $v = g$ on $\partial' C_R$ admits a solution $v = v(\theta, \cdot) \in W^{1,2}_p(C_R)$ for any $p \geq 1$. By the maximum principle the solution is unique. Obviously, $|D_0 F(\theta, u'')| \leq K_1/\theta_0 + |u''|$ whenever the left-hand side is well defined. Also $|F(\theta, 0, t)| \leq K_1$. After that, it only remains to observe that, in light of Lemma 4.4,
\[
|v^+ - v^-| \leq \int_{-\theta_0}^{\theta_0} \left( R^2 (K_1/\theta_0 + NK_1) + NR^2[g]_{C^\infty(C_r)} \right) d\theta \\
\leq NR^2 K_1 + NR^2 \theta_0[g]_{C^\infty(C_r)}.
\]

The theorem is proved. 

Later on we will need one more piece of information about $v^{(\pm)}$, before which we prove the following two auxiliary facts.

**Lemma 4.5.** Take $k \in \{1, \ldots, d\}$, $\rho \in \mathbb{R}$, $\lambda \in (0, \infty)$, and introduce
\[
V(t, x) = \exp \left( - \lambda |x^k - \rho|^2/(r^2 - t) \right).
\]

Let $(a^{ij}) \in \mathbb{S}$ be such that $0 \leq a^{kk} \leq 1/(4\lambda)$. Then for $t < r^2$ it holds that
\[
\partial_\nu V(t, x) + a^{ij} D_{ij} V(t, x) \leq 0.
\] (4.7)

The proof is achieved by a straightforward computation showing that the left-hand side of (4.7) equals
\[
V(t, x) \left[ - \lambda |x^k - \rho|^2/(r^2 - t)^2 + 4\lambda^2 a^{kk} |x^k - \rho|^2/(r^2 - t)^2 - 2\lambda a^{kk} \frac{1}{r^2 - t} \right] \leq 0.
\]

**Lemma 4.6.** Let $r \in (0, \infty)$, $v \in W^{1,2}_{d+1,\text{loc}}(C_r) \cap C(C_r)$, and assume that, for constants $\gamma \in (0, 2]$, $M \geq 0$, we have
\[
|v(r^2, x)| \leq M|x|^\gamma \quad \text{if} \quad |x| \leq r.
\]

Also assume that $|\partial_t v + a^{ij} D_{ij} v| \leq \theta$ in $C_r$ for an $\mathbb{S}_3$-valued function $a = (a^{ij})$ and a constant $\theta \in [0, \infty)$. Then there exists a constant $N = N(d, \delta, \gamma)$ such that for $t \in [0, r^2]$
\[ |v(t,0)| \leq N (M + r^{-\gamma} \sup_{C_r} |v|)(r^2 - t)^{\gamma/2} + \theta(r^2 - t). \quad (4.8) \]

Proof. Take \( \varepsilon > 0 \), set \( \rho = r/\sqrt{d} \), \( \lambda = \delta/4 \), and consider the function

\[ V(t,x) = M \varepsilon + M \varepsilon^{1-2/\gamma} (|x|^2 + 2(r^2 - t)d/\delta) + \theta(r^2 - t) \]

\[ + \sup_{C_r} \sum_{k=1}^{d} (V^k_{(+)}(t,x) + V^k_{(-)}(t,x)), \]

where

\[ V^k_{(\pm)}(t,x) = \exp\left(-\lambda |x^k \pm \rho|^2/(r^2 - t)\right). \]

Observe that, by Lemma 4.5, we have

\[ \partial_t V(t,x) + a_{ij} D_{ij} V \leq -\theta \]

in \( C_r \), in particular, in the cylinder \((0,r^2) \times (-\rho, \rho)^d\). On the parabolic boundary of this cylinder either \( t = r^2 \) and \( V \geq M|x|^{\gamma} \geq |v| \) by Young’s inequality (\( \gamma \leq 2 \)), or one of \( x^i \) is equal to either \( \rho \) or \(-\rho\), when the corresponding \( V^i \) equals 1, and again \( V \geq |v| \). By the maximum principle, for \( t \in [0,r^2] \) we have

\[ v(t,0) \leq V(t,0) \leq NM(\varepsilon + \varepsilon^{1-2/\gamma}(r^2 - t)) \]

\[ + \theta(r^2 - t) + N \sup_{C_r} |v| \exp\left(-\lambda \rho^2/(r^2 - t)\right). \]

After that, to estimate \( v(t,0) \) by the right-hand side of (4.8), it only remains to take the inf with respect to \( \varepsilon > 0 \) and observe that

\[ \exp\left(-\lambda \rho^2/(r^2 - t)\right) \leq N (r^2 - t)^{\gamma/2}/r^\gamma. \]

Similarly \(-v(t,0)\) is estimated from above. The lemma is proved. \( \square \)

Recall that \( v^{(\pm)} \) are introduced before Theorem 4.2.

**Theorem 4.7.** There exist constants \( \kappa_0 = \kappa_0(d, \delta) \in (1,2) \) and \( N \in (0, \infty) \) depending only on \( d \) and \( \delta \) such that for any \( r \in (0,R) \)

\[ [v^{(\pm)}]_{C^{\kappa_0}(C_r)} \leq N (R - r)^{-\kappa_0} \left[ \sup_{\partial C_r} (g - \hat{g}) + K_1 R^2 \right], \]

where \( \hat{g} = \hat{g}(x) \) is any affine function of \( x \).
Proof. As usual we may take \( \hat{g} = 0 \) and recall that \( Dv(\pm) \) is bounded and even Hölder continuous in \( CR \) since \( v(\pm) \in W^{1,2}_p(CR) \) for any \( p \geq 1 \). Then observe that for any \( \gamma \in (0,1) \) and function \( f(x) \) of one variable \( x \in [0,\varepsilon] \), \( \varepsilon > 0 \), we have

\[
|f'(0)| \leq |f'(0) - (f(\varepsilon) - f(0))/\varepsilon| + \varepsilon^{-1} \text{osc } f \leq \varepsilon\gamma |f'|_{C^\gamma[0,\varepsilon]} + \varepsilon^{-1} \text{osc } f.
\]

By applying this fact to functions \( v(x) \) given in \( BR \) we obtain that for any \( r_n + 1 < r_{n+2} \leq R \) and any \( \varepsilon \in (0,1) \) and smooth \( v = u(x) \)

\[
|Dv| \leq \varepsilon\gamma (r_{n+2} - r_{n+1}) |Dv|_{C^\gamma(B_{r_{n+2}})} + \varepsilon^{-1}(r_{n+2} - r_{n+1})^{-1} \text{osc } v
\]

(4.9)

in \( B_{r_{n+1}} \).

Next, for unit \( l \in \mathbb{R}^d \) and \( h > 0 \) define

\[
\delta_{h,l}u(t,x) = h^{-1}[u(t,x + hl) - u(t,x)]
\]

and note that for any \( r_1 \in (0,R) \) the function \( \delta_{h,l}v(+) \) satisfies an equation of the type

\[
\partial_t \delta_{h,l}v(+) + a^{ij}D_{ij}\delta_{h,l}v(+) = 0
\]

in \( Cr_1 \) (a.s.) with some measurable \( (a^{ij}) \) taking values in \( S_\delta \) if \( h \) is sufficiently small. By the Krylov-Safonov theorem, for \( r_0 \in (0,r_1) \) and perhaps even smaller \( h \) we have

\[
[\delta_{h,l}v(+) ]_{C^\gamma(C_{r_0})} \leq N(r_1 - r_0)^{-\gamma} \sup_{C_{r_1}} |\delta_{h,l}v(+) |
\]

where \( \gamma \in (0,1) \) and \( N \) depend only on \( \delta \) and \( d \). By letting \( h \to 0 \) we conclude

\[
[Dv(+) ]_{C^\gamma(C_{r_0})} \leq N(r_1 - r_0)^{-\gamma} \sup_{C_{r_1}} |Dv(+) |\]

(4.10)

By using (4.9) and (4.10) and setting

\[
r_0 = r, \quad r_n = r + (R-r) \sum_{k=1}^{n} 2^{-k}, \quad n \geq 1,
\]

we obtain

\[
A_n := \sup_{[0,r_n^2]} [Dv(+) (t,\cdot)]_{C^\gamma(B_{r_n})} \leq N(r_{n+1} - r_n)^{-\gamma} \sup_{C_{r_{n+1}}} |Dv(+) |
\]

\[
\leq N_1 \varepsilon^\gamma A_{n+2} + N_2 (R-r)^{-(1+\gamma)} \varepsilon^{-1} 2^{(1+\gamma)n} \text{osc } v(+) ,
\]

(4.11)

where the constants \( N_i \) are different from the one in (4.10) but still depend only on \( \delta \) and \( d \). Without losing generality we may assume that \( N_1 \geq 1 \) and we first take \( \varepsilon \) so that

\[
N_1 \varepsilon^\gamma = 2^{-5}
\]
then take \( n = 2k, \ k = 0, 1, \ldots, \) multiply both parts of (4.11) by \( 2^{-5k} \) and sum up with respect to \( k. \) Then upon observing that \((1 + \gamma)2k \leq 4k\) we get

\[
\sum_{k=0}^{\infty} A_{2k} 2^{-5k} \leq \sum_{k=1}^{\infty} A_{2k} 2^{-5k} + N(R-r)^{-(1+\gamma)} \sum_{k=0}^{\infty} 2^{-k} \sup_{C_R} \text{osc} v^+(+).
\]

By canceling (finite) like terms we find

\[
\sup_{[0,r^2]} \left[ DV^+(t, \cdot) \right] C^\gamma(B_r) \leq N(R-r)^{-\gamma} \text{osc} v^+(+). \tag{4.12}
\]

Note for the future that (4.12) and the second inequality in (4.11) also imply that

\[
\sup_{C_R} |DV^+(+) | \leq N(R-r)^{-1} \text{osc} v^+(+). \tag{4.13}
\]

Next, we use the fact that \( v^+(+) \) itself satisfies the equation

\[
0 = \partial_t v^+(+) + \max \left( F^+(+) [v^+(+)], P[v^+(+)] - K \right) - \max(K_1, -K) + K_1 \tag{4.14}
\]

\[
= \partial_t v^+(+) + a_{ij} D_{ij} v^+(+) + K_1
\]

in \( C_R \) (a.e.) with some measurable \((a_{ij})\) taking values in \( S_\beta. \) Furthermore, for any \((t_0, x_0) \in C_R\) the function

\[
v(t, x) := v^+(+) (t, x) - v^+(+) (t_0, x_0) - (x^i - x_0^i) D_i v^+(+) (t_0, x_0)
\]

satisfies the same equation and, owing to (4.12),

\[
|v(t, x)| \leq |Dv^+(+) (t_0, \cdot)| C^\gamma(B_r) |x - x_0|^{1+\gamma} \leq N(R-r)^{-(1+\gamma)} |x - x_0|^{1+\gamma} \text{osc} v^+(+)
\]

if \((t_0, x_0) \in C_r\) and \(|x - x_0| \leq \rho := (R-r)/2.\)

Also, for \( t_1 = 0 \vee (t_0 - \rho^2), \) due to (4.13), we have that

\[
\sup_{C_{t_0-t_1, \rho}(t_1, x_0)} |v| \leq \sup_{C_{t_0-t_1, \rho}(t_1, x_0)} \text{osc} v^+(+) + N \rho |Dv^+(+) (t_0, x_0)| \leq N \text{osc} v^+(+) C_R.
\]

Then we apply Lemma 4.6 with \( C_{t_0-t_1, \rho}(t_1, x_0) \) in place of \( C_r \) and take into account that \( \rho^{-(1+\gamma)} \leq N(R-r)^{-1+\gamma}. \) Then for \( t \in [t_1, t_0] \) we obtain

\[
\frac{|v^+(+) (t_0, x_0) - v^+(+) (t, x_0)|}{(t_0 - t) \gamma/2} \leq \frac{N}{(R-r)^{1+\gamma}} \text{osc} v^+(+) + K_1 (t_0 - t)^{(1-\gamma)/2}.
\]

Here \( t_0 - t \leq \rho^2 \) and \((t_0 - t)^{(1-\gamma)/2} \leq R^2 (R-r)^{-1+\gamma}. \) Therefore

\[
\frac{|v^+(+) (t_0, x_0) - v^+(+) (t, x_0)|}{(t_0 - t)^{(1+\gamma)/2}} \leq \frac{N \text{osc} v^+(+) + K_1 R^2}{(R-r)^{1+\gamma}}. \tag{4.15}
\]
If $x_0, t_0$ are as above but $0 \leq t < t_1$, then

$$t_0 - t \geq \rho^2 = (R - r)^2/4$$

so that (4.15) holds again.

This provides the necessary estimate of the oscillation of $v^+$ in the time variable and along with (4.12) shows that

$$[v^+]_{C^{1+\gamma}(C_R)} \leq N(R - r)^{-(1+\gamma)}[\frac{\text{osc} v^+}{C_R} + K_1 R^2].$$

Now the assertion of the theorem about $v^+$ with $\kappa_0 = 1 + \gamma$ follows from the fact that

$$\frac{\text{osc} v^+}{C_R} \leq \frac{\text{osc} g}{\partial^\gamma C_R} + N K_1 R^2.$$

The function $v^-$ is considered similarly with the only difference that it satisfies an equation similar to (4.14) with $-\max(-K_1, -K) + \max(-K_1, -K)$ in place of $-\max(K_1, -K) + K_1$. Since $|\max(-K_1, -K)| \leq K_1$ this difference is irrelevant. The theorem is proved. \hfill \Box

5. **Main estimate for solutions of an auxiliary cut-off equation**

Let $F(u', u'', t, x)$ satisfy Assumptions 2.2 (i), (iii), (iv), and (v). Take $K \in (0, \infty)$, $R \in (0, R_0]$, and $g \in W^{1,2}_{\infty}(C_R)$ and take $P[u]$ as in the beginning of Section 4.

By Theorem 4.1 there exists $u \in W^{1,2}_{p}(C_R)$ for all $p \geq 1$, such that $u = g$ on $\partial^\gamma C_R$ and the equation

$$\partial_t u + \max(F[u], P[u] - K) = 0. \quad (5.1)$$

holds (a.e.) in $C_R$.

Here is the result of this section.

**Theorem 5.1.** Suppose that

$$\frac{\text{osc} g}{\partial^\gamma C_R} < \hat{R}_0.$$ 

Then there exist a constant $\kappa_0 = \kappa_0(d, \delta) \in (1, 2]$ such that, if $r < R \leq R_0$, one can find an affine function $\hat{u} = \hat{u}(x)$ for which

$$|u - \hat{u}| \leq N K_1 R^2 + N \theta_0 [g]_{C^{\kappa}(C_R)} R^\kappa$$

$$+ N r^{-\kappa_0} (R - r)^{-\kappa_0} \left[\frac{\text{osc} (g - \hat{g})}{\partial^\gamma C_R} + K_1 R^2\right]$$

in $\hat{C}_r$ for any $\kappa \in (0, 2)$, where $\hat{g} = \hat{g}(x)$ is any affine function of $x$ and $N = N(d, \delta, \kappa)$.
Proof. Set

\[ F_0(u'', t) = F(g(R^2, 0), u'', t, 0), \]

and take \( v^{(\pm)} \) from Section 4.

Observe that since \( \max (F(u'_0, 0, t, x), P(0) - K) = 0 \), as we have seen many times in the past,

\[ \partial_t u + a^{ij} D_{ij} u = 0 \]

for an appropriate \( S_\delta \)-valued \( (a^{ij}) \). It follows that

\[ \text{osc} \frac{C_R}{\partial C_R} u = \text{osc} \frac{C_R}{\partial C_R} g < \tilde{R}_0, \]

and in \( C_R \) by Assumption 2.2 (iv)

\[ F(u(t, x), D^2 u(t, x), t, x) \leq F_0(D^2 u(t, x), t) + K_1 + \theta_0 |D^2 u(t, x)|, \]

\[ F_0(D^2 u(t, x), t) - K_1 - \theta_0 |D^2 u(t, x)| \leq F(u(t, x), D^2 u(t, x), t, x). \]

Since the operators \( F^{(\pm)} \) satisfy the maximum principle, the above inequalities imply that \( v^{(-)} \leq u \leq v^{(+)} \).

Next, set \( p^{(\pm)}(x) = v^{(\pm)}(0) + x^i D_i v^{(\pm)}(0) \) and observe that by Theorem 4.7 in \( \tilde{C}_R \) we have

\[ |p^{(+)} - v^{(+)}| + |p^{(-)} - v^{(-)}| \leq N r^{\kappa_0} N (R-r)^{-\kappa_0} \left[ \text{osc} \frac{C_R}{\partial C_R} (g - \hat{g}) + K_1 R^2 \right], \]

where \( \kappa_0 \) is taken from Theorem 4.7. Furthermore by Theorem 4.2 in \( \tilde{C}_R \)

\[ |p^{(+)} - p^{(-)}| \leq |p^{(+)} - v^{(+)}| + |p^{(-)} - v^{(-)}| \]

\[ + N R^2 K_1 + N R^{\kappa_0} \theta_0 [g]_{C^0(C_R)}. \]

After that it only remains to note that, since \( v^{(-)} \leq u \leq v^{(+)} \), we have

\[ p^{(+)} - v^{(+)} \leq p^{(+)} - u \leq [p^{(+)} - p^{(-)}] + [p^{(-)} - v^{(-)}] + v^{(-)} - u \]

\[ \leq [p^{(+)} - p^{(-)}] + [p^{(-)} - v^{(-)}]. \]

The theorem is proved. \( \square \)
6. Proof of Theorem 2.1

We first take a constant \( K > 0 \), assume that \( g \in W^{1,2}_\infty(\mathbb{R}^{d+1}) \), \( \bar{G} \) is bounded, and investigate solutions \( v_K \) of the cut-off equation (4.1). Take \( \kappa_0 = \kappa_0(d, \delta) \in (1, 2] \) from Theorem 5.1. Naturally, we suppose that the assumptions of Theorem 2.1: Assumptions 2.1, 2.2, and 2.3, are satisfied with \( \theta_0 \) yet to be specified.

Lemma 6.1. Let \( R \in (0, R_0] \) and let \( v \in W^{1,2}_\infty(C_R) \) be a solution of (4.1) in \( \bar{C}_R \) such that

\[
\text{osc}_{\partial \nu^C_R} v < \hat{R}_0.
\]

Then for each \( r \in (0, R) \) one can find an affine function \( \hat{v}(x) \) such that in \( C_r \) for any \( \kappa \in (1, 2) \)

\[
|v - \hat{v}| \leq N\theta_0[v]_{C^\kappa(C_R)} R^\kappa + NR^{\kappa_0}(R - r)^{-\kappa_0}[R^\kappa[v]_{C^\kappa(C_R)} + K_1 R^2]
\]

\[
+ NK_0 R^2 \sup_{C_R} (|v| + |Dv|) + NK_1 R^2 + NR^{2-(d+2)/p}\|\bar{G}\|_{L_p(C_R)},
\]

where the constants \( N \) depend only on \( d, \delta, \) and \( \kappa \).

Proof. Observe that

\[
\max (H[v], P[v] - K) = \max (F[v], P[v] - K) + h
\]

where \( h \), defined by the above equality, satisfies

\[
|h| \leq |H[v] - F[v]| \leq K_0 (|v| + |Dv|) + \bar{G}.
\]

Next define \( u \in W^{1,2}_{d+1}(C_R) \) as a solution of

\[
\partial_t u + \max (F[u], P[u] - K) = 0
\]

with boundary data \( u = v \) on \( \partial^\nu C_R \), which exists by Theorem 4.1. Then, in light of the fact that

\[
\max (F(u(t, x), 0, t, x), P(0) - K) = 0,
\]

there exists an \( \mathbb{S}_\beta \)-valued measurable function \( a \) such that in \( C_R \) (a.e.) we have

\[
\partial_t (v - u) + a^{ij} D_{ij} (v - u) + h = 0.
\]

By the parabolic Aleksandrov estimate

\[
|v - u| \leq NR^{d/d+1} \|h\|_{L^{d+1}(C_R)} = NR^2 \left( \int_{C_R} |h|^{d+1} dx dt \right)^{1/(d+1)}
\]

\[
\leq NK_0 R^2 \sup_{C_R} (|v| + |Dv|) + NR^{2-(d+2)/p}\|\bar{G}\|_{L_p(C_R)}.
\]
After that our assertion follows from Theorem 5.1 and the lemma is proved. □

We need a characterization of $C^{1+\alpha}$-functions.

**Lemma 6.2.** Let $r_0 \in (0, \infty)$, $\kappa \in (1, 2)$, $\phi \in C^{\kappa}(C_{r_0})$ and assume that there is a constant $N_0$ such that for any $(t, x) \in C_{r_0}$ and $r \in (0, 2r_0]$ there exists an affine function $\hat{\phi}(x)$ such that
\[
\sup_{C_r(t, x) \cap C_{r_0}} |\phi - \hat{\phi}| \leq N_0 r^\kappa.
\]
Then
\[
[\phi]_{C^{\kappa}(C_{r_0})} \leq N N_0,
\]
where $N$ depends only on $d$ and $\kappa$.

**Proof.** The fact that, for any $t \in (0, r^2)$, we have
\[
[D\phi(t, \cdot)]_{C^{\kappa-1}(B_{r_0})} \leq N N_0
\]
follows from Theorem 2.1 of [11]. To estimate $|\phi(t, x) - \phi(s, x)|$ we may assume that $t > s$, so that $(t, x), (s, x) \in \bar{C}_r(s, x)$, where $r = \sqrt{t - s}$. Then, for an appropriate $\hat{\phi}(x)$
\[
|\phi(t, x) - \phi(s, x)| \leq |\phi(t, x) - \hat{\phi}(x)| + |\phi(t, x) - \hat{\phi}(x)| \leq 2 N_0 r^\kappa = 2 N_0 (t - s)^{\kappa/2}.
\]
The lemma is proved. □

**Lemma 6.3.** Take $r_1 \in (0, R_0)$, $r_0 \in (0, r_1)$, and define
\[
\kappa = \kappa(d, \delta, p) = \frac{1 + \kappa_0}{2} \land \left(2 - \frac{d + 2}{p}\right).
\]

Let $v \in W^{1,2}_{\infty}(C_{r_1})$ be a solution of (4.1) in $C_{r_1}$. Then there exists $\theta_0 = \theta_0(d, \delta) \in (0, 1]$ such that, if Assumption 2.2 (iv) is satisfied with this $\theta_0$ and
\[
\osc_{C_{r_1}} v < \hat{R}_0,
\]
then
\[
[v]_{C^{\kappa}(C_{r_0})} \leq (1/2)[v]_{C^{\kappa}(C_{r_1})} + N(K_0 + 1)(r_1 - r_0)^{-\kappa} \sup_{C_{r_1}} |v|
\]
\[+ N(K_0 + 1)(r_1 - r_0)^{-\kappa-1} \sup_{C_{r_1}} |Dv| + N(K_1 + \|G\|_{L_p(C_{r_1})})],
\]
where $N = N(d, \delta)$. (6.3)
Proof. Take \((t_0, x_0) \in C_{r_0}, \varepsilon \in (0, 1)\) to be specified later, define
\[
r'_0 = \frac{\varepsilon}{3}(r_1 - r_0),
\]
and notice that for any \((t, x) \in C'_{r_0}(t_0, x_0), r \in (0, 2r'_0),\) and \(R = \varepsilon^{-1}r,\) we have \(R \leq r_1 \leq R_0 \leq 1\) and
\[
C_R(t, x) \subset C_{r_1}.
\]
Therefore, by Lemma 6.1 we can find an affine function \(\hat{v}(x)\) such that
\[
\sup_{C_r(t, x) \cap C'_{r_0}(t_0, x_0)} |v - \hat{v}| \leq \sup_{C_r(t, x)} |v - \hat{v}| \leq N\theta_0[v]_{C^\kappa(C_R(t, x))}\varepsilon^{-\kappa}r^\kappa
\]
\[
+N\varepsilon^{\kappa_0 - \kappa}(1 - \varepsilon)^{-\kappa_0}\varepsilon^{\kappa}[v]_{C^\kappa(C_R(t, x))} + N\varepsilon^{\kappa_0 - 2}(1 - \varepsilon)^{-\kappa_0}K_1r^2
\]
\[
+N\varepsilon^{-\kappa}r^\kappa\|\bar{G}\|_{L^p(C_{r_1})} \leq N\varepsilon I(\theta_0, \varepsilon, r_1),
\]
where the constants \(N\) depend only on \(d\) and \(\delta\) and
\[
I(\theta_0, \varepsilon, r_1) := \left(\theta_0 \varepsilon^{-\kappa} + \varepsilon^{\kappa_0 - \kappa}(1 - \varepsilon)^{-\kappa_0}\varepsilon^{\kappa_0 - 2}(1 - \varepsilon)^{-\kappa_0}K_1
\]
\[
+\varepsilon^{-2}K_0\sup_{C_{r_1}} (|v| + |Dv|) + K_1\varepsilon^{-2} + \varepsilon^{-\kappa}\|\bar{G}\|_{L^p(C_{r_1})}.
\]
It follows by Lemma 6.2 that
\[
[v]_{C^\kappa(C'_{r_0}(t_0, x_0))} \leq N_1 I(\theta_0, \varepsilon, r_1),
\]
where \(N_1\) depends only on \(d\) and \(\delta.\) We can now specify \(\theta_0\) and \(\varepsilon.\) First we chose \(\varepsilon \in (0, 1)\) so that
\[
N_1\varepsilon^{\kappa_0 - \kappa}(1 - \varepsilon)^{-\kappa_0} = 1/4.
\]
Since \(\kappa_0 - \kappa \geq (\kappa_0 - 1)/2 > 0\) and \(\kappa_0\) depends only on \(d\) and \(\delta\) and \(N_1\) depends only on \(d\) and \(\delta,\) \(\varepsilon\) also depends only on \(d\) and \(\delta.\) After that we take \(\theta_0 = \theta_0(d, \delta) \in (0, 1)\) so that \(N_1\theta_0\varepsilon^{-\kappa} \leq 1/4.\)

Then
\[
[v]_{C^\kappa(C'_{r_0}(t_0, x_0))} \leq \left(1/2\right)[v]_{C^\kappa(C_{r_1})} + NJ, \tag{6.4}
\]
where \(N = N(d, \delta)\) and
\[
J = K_0\sup_{C_{r_1}} (|v| + |Dv|) + K_1 + \|\bar{G}\|_{L^p(C_{r_1})}.
\]

Now observe that if \((t, x), (s, x) \in C_{r_0}\) and \(t > s,\) then either \(|t - s| < (r'_0)^2,\) in which case \((t, x) \in C'_{r_0}(s, x)\) and
and \((4.9)\) allow us to find constants \(N_{c}\) where 

\[ K_{N} \in \mathcal{N} \]

Theorem 6.4. Take \(\bigtriangleup\) This proves \((6.3)\) and the lemma.

\[(6.3)\] and suppose that Assumption 2.2 \((iv)\) is satisfied with this \(\theta_{0}\). Let \(v \in W^{1,2}_{\infty}(C_{R})\) be a solution of \((4.1)\) in \(C_{R}\) such that 

\[ \osc_{C_{R}} v < \bar{R}_{0}, \]

Then 

\[ [v]_{C^{\kappa}(C_{r})} \leq N(R - r)^{-\kappa} \sup_{C_{R}} |v| + N\left(K_{1} + \|\bar{G}\|_{L_{p}(C_{R})}\right), \tag{6.5} \]

where \(N\) depends only on \(d, \delta, \) and \(K_{0}\).

Proof. Fix a number \(c \in (0, 1)\) such that \(c^{4} > 3/4\) and introduce

\[ r_{0} = r, \quad r_{n} = r + c_{0}(R - r) \sum_{k=1}^{n} c^{k}, \quad n \geq 1, \]

where \(c_{0}\) is chosen in such a way that \(r_{n} \to R\) as \(n \to \infty\). Then Lemma 6.3 and \((4.9)\) allow us to find constants \(N_{1}\) and \(N\) depending only on \(d, \delta, \) and \(K_{0}\), such that for all \(n\) and \(\varepsilon \in (0, 1)\)

\[ A_{n} := [v]_{C^{\kappa}(C_{r_{n}})} \leq \left(2^{-1} + N_{1}\varepsilon^{\kappa-1}\right)A_{n+2} + N\left(K_{1} + \|\bar{G}\|_{L_{p}(C_{R})}\right)
+ N(R - r)^{-\kappa}c^{-n\kappa}(1 + \varepsilon^{-1}) \sup_{C_{R}} |v|. \]
We choose $\varepsilon < 1$ so that $2^{-1} + N_1 \varepsilon^{\kappa - 1} \leq 3/4$ and then recalling that $\kappa \leq 2$ conclude that

$$
\sum_{m=0}^{\infty} (3/4)^m A_{2m} \leq \sum_{k=1}^{\infty} (3/4)^m A_{2m} + N(K_1 + \|\bar{G}\|_{L_p(C_R)})
$$

$$+ N(R - r)^{-\kappa} \sup_{C_R} \sum_{m=0}^{\infty} (3/4)^m c^{-4m},$$

where the last series converges since $3c^{-4}/4 < 1$. By canceling like terms we come to (6.5) and the theorem is proved. □

**Proof of Theorem 2.1.** We keep assuming that that $g \in W_{\infty}^{1,2}(\mathbb{R}^{d+1})$ and $\bar{G}$ is bounded.

Since

$$|H(u', 0, t, x)| = |G(u', 0, t, x)| \leq K_0 |u'| + \bar{G}(t, x),$$

for $H_K = \max(H, P - K)$ we have

$$|H_K(u', 0, t, x)| \leq |H(u', 0, t, x)| \leq K_0 |u'| + \bar{G}(t, x).$$

It follows by Lemma 3.2 of [8] that for any $K$ there exist measurable $\mathbb{S}_\delta$-valued $a$, $\mathbb{R}^d$-valued $b$, and real-valued $f$ such that in $\Pi$ (a.s.)

$$\partial_tv_K + a^{ij} D_{ij}v_K + b^i D_i v_K + f = 0 \quad (6.6)$$

and $|b| \leq K_0$, $|f| \leq \bar{G} + K_0 |v_K|$. 

Then by the parabolic Aleksandrov estimates

$$|v_K| \leq N(\|g\|_{C(\Pi)} + \|\bar{G}\|_{L^{d+1}(\Pi)}), \quad (6.7)$$

where $N$ depends only on $d$, $\delta$, $K_0$, $T$, and the diameter of $\Omega$.

Then, since (6.6) has form of a linear equation, by the well-known results from the linear theory we estimate not only $|v_K|$ but also the modulus of continuity of $v_K$ through that of $g$, $\sup |g|$, and $\|\bar{G}\|_{L^{d+1}(\Pi)}$ with constants independent of $K$.

Hence, the family $\{v_K; K \geq 1\}$ is equicontinuous on $\bar{\Pi}$. More precisely there exists a function $\bar{\omega}(\varepsilon)$, depending depending only on $\varepsilon$, $\|\bar{G}\|_{L^{d+1}(\Pi)}$, $\delta$, $d$, $K_0$, $\rho_{\text{ext}}(\Omega)$, $\|g\|_{C(\Pi)}$, and the modulus of continuity of $g$ on $\partial\Pi$, such that $\bar{\omega}(\varepsilon) \to 0$ as $\varepsilon \downarrow 0$ and

$$|v_K(t, x) - v_K(s, y)| \leq \bar{\omega} \left( \rho((t, x), (s, y)) \right) \quad (6.8)$$

for any $x, y \in \Omega$ and $s, t \in [0, T]$.

It follows that there is a sequence $K_n \to \infty$ and a function $v$ such that $v^n := v_{K_n} \to v$ uniformly in $\bar{\Pi}$. Of course, (2.3) holds, owing to Theorem 6.4, which also implies that $Dv^n \to Dv$ locally uniformly in $\Pi$.

The following lemma, in which the boundedness of $G$ is not used, will allow us to prove that $v$ is a viscosity solution. Introduce
Lemma 6.5. There is a constant $\alpha$, depending only on $d$ and $\delta$, such that for any $C_r(t,x)$ satisfying $C_r(t,x) \subset \Pi$ and $\phi \in W^{1,2}_{d+1}(C_r(t,x))$ we have on $C_r(t,x)$ that

$$v \leq \phi + Nr^{d/(d+1)}\left\| \partial_t \phi + H^0[\phi]\right\|_{L_{d+1}(C_r(t,x))} + \max_{\partial^r C_r(t,x)} (v - \phi)_+,$$

(6.9)

$$v \geq \phi - Nr^{d/(d+1)}\left\| \partial_t \phi + H^0[\phi]\right\|_{L_{d+1}(C_r(t,x))} - \max_{\partial^r C_r(t,x)} (v - \phi)_-..$$

(6.10)

Proof. For $m = 1, 2, \ldots$ introduce

$$H^m(u^n, t, x) = \sup_{n \geq m} H(v^n(t, x), Du^n(t, x), u^n, t, x)$$

and observe that for $n \geq m$

$$\partial_t v_n + \max (H^m[v^n], P[v^n] - K_n) \geq 0,$$

implying that

$$\partial_t v_n + \max (H^m[v^n], P[v^n] - K_n) \geq \partial_t \phi - \max (H^m[\phi], P[\phi] - K_n)$$

$$+ \partial_t v^n + \max (H^m[v^n], P[v^n] - K_n)$$

$$= \partial_t (v^n - \phi) + a^{ij} D_{ij}(v^n - \phi),$$

where $a = (a^{ij})$ is an $S_\delta$-valued function.

It follows by the parabolic Aleksandrov estimates that

$$v^n \leq \phi + \max_{\partial^r C_r(t,x)} (v^n - \phi)_+ + Nr^{d/(d+1)}\left\| \partial_t \phi + H^m[\phi]\right\|_{L_{d+1}(C_r(t,x))},$$

where $N = N(d, \delta)$. By sending $n \to \infty$ and using the dominated convergence theorem, we obviously get

$$v \leq \phi + \max_{\partial^r C_r(t,x)} (v - \phi)_+ + Nr^{d/(d+1)}\left\| \partial_t \phi + H^m[\phi]\right\|_{L_{d+1}(C_r(t,x))}. (6.11)$$

Since $v^n \to v$ and $Du^n \to Du$ uniformly in $C_r(t,x)$ we have that $|H^m[\phi]| \leq |H^m[0]| + N |D^2 \phi|$, where $|H^m[0]|$ is dominated by an $L_{d+1}$-function independent of $m$. Furthermore, the continuity of $H(u, t, x)$ with respect to $u$ implies that $H^m[\phi] \to H^0[\phi]$ as $m \to \infty$ at any point in $C_r(t,x)$. By using the dominated convergence theorem once more and sending $m \to \infty$ in (6.11) we come to (6.9).

Similarly (6.10) is established. The lemma is proved.

Now we prove that $v$ is a $C$-viscosity solution of (1.1) if part (a) of Assumption 2.1 (ii) is satisfied. Let $(t_0, x_0) \in \Pi, r > 0$, and $\phi \in C^{1,2}(C_r(t_0, x_0))$
be such that $\bar{C}_r(t_0, x_0) \subset \Pi$ and $v - \phi$ attains a local maximum at $(t_0, x_0)$. Then for $\varepsilon > 0$ and all small $r > 0$ for

$$\phi_{\varepsilon, r}(t, x) = \phi(t, x) - \phi(t_0, x_0) + v(t_0, x_0) + \varepsilon(|x - x_0|^2 + t - t_0 - r^2)$$

we have that

$$\max_{\partial^r C_r(t_0, x_0)} (v - \phi_{\varepsilon, r})_+ = 0.$$  

Hence, by Lemma 6.5

$$\varepsilon r^2 = (v - \phi_{\varepsilon, r})(t_0, x_0) \leq N r^d/(d+1) \left\| \left( \partial_t \phi_{\varepsilon, r} + H^0[\phi_{\varepsilon, r}] \right)_+ \right\|_{L^d+1(C_r(t_0, x_0))}$$

$$= N r^d/(d+1) \left\| \left( \partial_t \phi_{\varepsilon} + H^0[\phi_{\varepsilon}] \right)_+ \right\|_{L^d+1(C_r(t_0, x_0))},$$

where $\phi_{\varepsilon} = \phi + \varepsilon(|x|^2 + t)$. It follows that

$$N \text{ess sup}_{C_r(t_0, x_0)} \left( \partial_t \phi_{\varepsilon} + H^0[\phi_{\varepsilon}] \right)_+ \geq \varepsilon,$$

$$N \lim \text{ess sup}_{r \downarrow 0} \left( \partial_t \phi_{\varepsilon} + H^0[\phi_{\varepsilon}] \right)_+ \geq \varepsilon.$$

By letting $\varepsilon \downarrow 0$ we conclude that

$$\lim \text{ess sup}_{r \downarrow 0} \left[ \partial_t \phi(t, x) + H(v(t, x), Dv(t, x), D^2 \phi(t, x), t, x) \right] \geq 0. \quad (6.12)$$

Now, note that, as $(t, x) \to (t_0, x_0)$, we have $v(t, x) \to v(t_0, x_0)$ and (see Remark 2.1) $Dv(t, x) \to Dv(t_0, x_0)$. Also $D\phi(t, x) \to D\phi(t_0, x_0) = Dv(t_0, x_0)$ and $D^2 \phi(t, x) \to D^2 \phi(t_0, x_0)$. It follows by Assumption 2.1 (ii) (a) that one can replace $Dv(t, x)$ in (6.12) with $D\phi(t, x)$. Then, so modified (6.12) implies (1.2) meaning that $v$ is a $C$-viscosity supersolution.

The fact that it is also a $C$-viscosity supersolution is proved similarly on the basis of (6.10).

In case Assumption 2.1 (ii) (b) is satisfied, we still come to (6.12) and can replace $Dv(t, x)$ with $D\phi(t, x)$ just because of the continuity of $H(u, t, x)$ in $[u']$ uniform with respect to $u_0'$, $u''$, and $(t, x)$.

This proves Theorem 2.1 in our particular case that $g \in W^{1,2}_\infty(\mathbb{R}^d+1)$ and $\bar{G}$ is bounded. In the same case we also have estimates (6.7), (6.8) with $v$ in place of $v_K$ and the same $N$ and $\bar{w}$. Also (2.3) holds by the above.

In the case of general $\bar{G}$, for $n = 1, 2, \ldots$, we replace $H(u, t, x)$ in (1.1) with

$$H(u, t, x)I_{\bar{G}(t, x) \leq n} + F(u_0', u''', t, x)I_{\bar{G}(t, x) > n}$$

$$= F(u_0', u''', t, x) + G(u, t, x)I_{\bar{G}(t, x) \leq n}$$
and apply the already proved version of Theorem 2.1 to introduce $u_n$ as the $C$-viscosity or $L_p$-viscosity solutions in $\Pi$ of so obtained equations with the same boundary condition $u_n = g$ on $\partial \Pi$.

From the above we see that the estimates (6.7) and (6.8) with $u_n$ in place of $v_K$ and the estimates of $[u_n]_{C^\alpha(C_r(t,x))}$ are uniform with respect to $n$. This and the fact that the boundedness of $G$ is not used in Lemma 6.5 allow us to repeat what was said about $v^n$ with obvious changes and proves the theorem for general $H$ but still assuming that $g \in W^{1,2}_{\infty}({\mathbb R}^{d+1})$.

One drops this assumption by using uniform approximations of $g$ by smooth ones preserving the modulus of continuity on $\partial \Pi$. This guarantees that for the approximating solutions the estimates like (6.7) and (6.8) will hold and referring to the argument in the previous paragraph brings the proof of Theorem 2.1 to an end. □

References

[1] L.A. Caffarelli, *Interior a priori estimates for solutions of fully non-linear equations*, Ann. Math., Vol. 130 (1989), 189–213.
[2] M.G. Crandall, H. Ishii, and P.-L. Lions, *User’s guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. (N.S.), Vol. 27 (1992), 1–67.
[3] M. G. Crandall, M. Kocan, and A. Świȩch, *$L^n$-theory for fully nonlinear uniformly parabolic equations*, Comm. Partial Differential Equations, Vol. 25 (2000), No. 11-12, 1997–2053.
[4] J.V. da Silva and E.V. Teixeira, *Sharp regularity estimates for second order fully nonlinear parabolic equations*, Math. Ann. DOI 10.1007/s00208-016-1506-y
[5] K. K. Golovkin, *On equivalent normalizations of fractional spaces*, Automatic programming, numerical methods and functional analysis, Trudy Mat. Inst. Steklov., Vol. 66, Acad. Sci. USSR, Moscow–Leningrad, 1962, 364–383.
[6] N.V. Krylov, “Lectures on elliptic and parabolic equations in Hölder spaces”, Amer. Math. Soc., Providence, RI, 1996; Russian translation, “Nauchnaya kniga”, Novosibirsk, 1998.
[7] N.V. Krylov, *On the existence of $W^2_\alpha$ solutions for fully nonlinear elliptic equations under relaxed convexity assumptions*, Comm. Partial Differential Equations, Vol. 38 (2013), No. 4, 687–710.
[8] N.V. Krylov, *On the existence of $W^{1,2}_p$ solutions for fully nonlinear parabolic equations under either relaxed or no convexity assumptions*, Harvard University, Center of Mathematical Sciences and Applications, Nonlinear Equation Publication, http://arxiv.org/abs/1705.02400
[9] O.A. Ladyzhenskaya, V.A. Solonnikov, and N.N. Ural’tseva, “Linear and quasi-linear parabolic equations”, Nauka, Moscow, 1967, in Russian; English translation: Amer. Math. Soc., Providence, RI, 1968.
[10] G.M. Lieberman, “Second order parabolic differential equations”, World Scientific, Singapore, 1996.
[11] M. V. Safonov, *On the classical solutions of nonlinear elliptic equations of second order*, Izvestija Acad. Nauk SSSR, ser. matemat., Vol. 52 (1988), No. 6, 1272–1287 in Russian; English translation in Math. USSR Izvestiya, Vol. 33 (1989), No. 3, 597–612.
[12] L. Silvestre and B. Sirakov, *Boundary regularity for viscosity solutions of fully nonlinear elliptic equations*, Comm. Partial Differential Equations, Vol. 39 (2014), No. 9, 1694–1717.
[13] L. Silvestre and E.V. Teixeira, *Regularity estimates for fully non linear elliptic equations which are asymptotically convex*, Contributions to nonlinear elliptic equations and systems, 425–438, Progr. Nonlinear Differential Equations Appl., Vol. 86, Birkhäuser/Springer, Cham, 2015.

[14] A. Świȩch, *$W^{1,p}_0$-interior estimates for solutions of fully nonlinear, uniformly elliptic equations*, Adv. Differential Equations, Vol. 2 (1997), 1005–1027.

[15] N.S. Trudinger, *Comparison principles and pointwise estimates for viscosity solutions of nonlinear elliptic equations*, Rev. Mat. Iberoamericana, Vol. 4 (1988), No. 3–4, 453–468.

[16] N.S. Trudinger, *On regularity and existence of viscosity solutions of nonlinear second order, elliptic equations*, Partial differential equations and the calculus of variations, Vol. II, 939–957, Progr. Nonlinear Differential Equations Appl., Vol. 2, Birkhäuser Boston, Boston, MA, 1989.

[17] L. Wang, *On the regularity of fully nonlinear parabolic equations: II*, Comm. Pure Appl. Math., Vol. 45 (1992), 141–178.

E-mail address: nkrylov@umn.edu

127 Vincent Hall, University of Minnesota, Minneapolis, MN, 55455