The 2-type model structure on the category of bicategories

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Abstract

We define a model structure on the category of bicategories closely related to homotopy 2-types. The fibrant objects are bigroupoids. We state that the fibrations satisfy the product formula with respect to the Euler characteristic of bicategories.

Contents

1 Introduction 2
2 Recollection of 2-categories, bicategories, and model categories 3
  2.1 2-categories and bicategories . . . . . . . . . . . . . . . . . . . . . 3
  2.2 Notations of bicategories . . . . . . . . . . . . . . . . . . . . . . 5
  2.3 Nerve of 2-categories and bicategories . . . . . . . . . . . . . . . 7
  2.4 Biequivalences and weak 2-equivalences . . . . . . . . . . . . . . 9
  2.5 Model categories . . . . . . . . . . . . . . . . . . . . . . . . . . . 12
3 The 2-type model structure 14
  3.1 The 2-type model structure on $2\text{Cat}$ . . . . . . . . . . . . . . 14
  3.2 The 2-type model structure on $\text{BiCat}_{s}$ . . . . . . . . . . . . 18
4 The Euler characteristic of bicategories 22
  4.1 The Euler characteristic of small categories . . . . . . . . . . . . 22
  4.2 The Euler characteristic of 2-categories and bicategories . . . . 24
  4.3 The product formula of Euler characteristic for fibrations . . . . 25
1 Introduction

The author defined a model structure on the category of small categories called the 1-type model structure, closely related to covering spaces and homotopy 1-types in the early paper [Tan13]. The weak equivalences are weak 1-equivalences which are functors inducing isomorphisms on the homotopy groups $\pi_i$ of the classifying spaces for $i \leq 1$. The fibrations are categories fibered and cofibered in groupoids, and the fibrant objects are groupoids. This paper develops the 1-type model structure on the category of small categories to the 2-type model structure on the category of 2-categories, and the category of bicategories.

Main Theorem 1. The category of 2-categories (resp. bicategories) admits the following model structure:

1. The weak equivalences are weak 2-equivalences which are 2-functors (resp. strict homomorphisms) inducing isomorphisms on the homotopy groups $\pi_i$ of the classifying spaces for $i \leq 2$.

2. This model structure is a left Bousfield localization of the Lack model structure in [Lac02] and [Lac04].

3. The fibrant objects are pseudogroupoids (resp. bigroupoids).

4. This is Quillen equivalent to the category of 2-groupoids equipped with Moerdijk and Svensson model structure, and the category of simplicial sets equipped with the 2-type model structure.

Another aim of this paper is to show the product formula with respect to the Euler characteristic of bicategories for fibrations in the 2-type model structure. The standard Euler characteristic $\chi$ of topological spaces satisfies the product formula $\chi(E) = \chi(B)\chi(F)$ for a certain fibration $F \hookrightarrow E \rightarrow B$ over a connected base space $B$. On the other hand, Euler characteristic is defined not only for topological spaces but also for posets [Rot64], groupoids [BD01], categories [Lei08], and enriched categories [Lei13, NT] including 2-categories. We extend the definition of Euler characteristic of 2-categories to bicategories, and prove that a fibration in the 2-type model structure also induces the above equation by using the Grothendieck construction for bicategories.

Main Theorem 2. Let $F \hookrightarrow E \rightarrow B$ be a fibration over a connected bicategory $\mathcal{B}$ in the 2-type model structure on the category of bicategories, and let $\chi$ be the Euler characteristic of bicategories. If $F, B$, and $E$ admit Euler characteristic, then we have $\chi(E) = \chi(B)\chi(F)$.

This paper organized as follows. We recall and review of fundamental definitions and properties of 2-categories, bicategories, and model categories in Section 2. We prepare some notations of bicategories which we often use in this paper. Section 3 introduces the 2-type model structure on the category of 2-categories, and the category of bicategories respectively. In the case of 2-categories, it is induced from the 2-type model structure on the category of
simplicial sets through the geometric nerve functor. Section 4 describes the definition and properties of the Euler characteristic of bicategories. We mention the relation between the Euler characteristic and fibrations of the 2-type model structure in the last part of the section.

2 Recollection of 2-categories, bicategories, and model categories

We begin with reviewing fundamental definitions and properties of 2-categories, bicategories, and model categories.

2.1 2-categories and bicategories

Let us first fix notations and terminologies for small categories and functors.

Definition 2.1. A small category $A$ consists of the following data;

1. a set of objects $A_0$,
2. a set of morphisms $A(x, y)$ for each $x, y \in A_0$,
3. a map $\circ \, : \, A(y, z) \times A(x, y) \to A(x, z)$ called the composition for each $x, y, z \in A_0$ satisfying associativity and identity conditions. Sometimes we denote $g \circ f$ simply by $gf$.

A morphism $f \in A(x, y)$ is called invertible or an isomorphism if there exists an inverse $g \in A(y, x)$ such that $gf = 1_x$ and $fg = 1_y$. A category $A$ is called a groupoid when every morphism is invertible.

A functor $f : A \to B$ between small categories $A$ and $B$ consists of a map $f : A_0 \to B_0$ on objects and a map $f_{x, y} : A(x, y) \to B(fx, fy)$ on morphisms for each $x, y \in A_0$ preserving composition and identities. Let $\text{Cat}$ denote the category of small categories and functors.

A bicategory is a generalized notion of a small category.

Definition 2.2. A bicategory $A$ consists of the following data;

1. a set of objects $A_0$,
2. a (small) category of morphisms $A(x, y)$ for each $x, y \in A_0$,
3. a functor $A(y, z) \times A(x, y) \to A(x, z)$ for each $x, y, z \in A_0$,
4. an associator $\alpha_{hgf} : h(gf) \cong (hg)f$ for each composable triple $(h, g, f) \in A(z, w) \times A(y, z) \times A(x, y)$ and an unitor $\eta_k : 1_yk \cong k$, $\theta_k : k1_x \cong k$ for each $k \in A(x, y)_0$ satisfying the coherent condition.
An element of $\mathcal{A}(x, y)_0$ is called a 1-morphism for $x, y \in \mathcal{A}_0$ and denote by a single arrow $x \to y$, and an element of $\mathcal{A}(x, y)((f, g))$ is called a 2-morphism and denote by a double arrow $f \Rightarrow g$. Therefore, a bicategory $\mathcal{A}$ is equipped with three kind of compositions, the composition of 1-morphisms $\mathcal{A}(y, z)_0 \times \mathcal{A}(x, y)_0 \to \mathcal{A}(x, z)_0$, the vertical composition of 2-morphisms $\mathcal{A}(x, y)((g, h)) \times \mathcal{A}(x, y)((f, h)) \to \mathcal{A}(x, y)((f, g))$, and the horizontal composition of 2-morphisms $\mathcal{A}(y, z)((h, i)) \times \mathcal{A}(x, y)((f, g)) \to \mathcal{A}(x, z)((hf, ig))$.

In particular, a bicategory $\mathcal{A}$ is called a 2-category when each associator and unitor is an identity map.

A 1-morphism $f \in \mathcal{A}(x, y)_0$ in a bicategory $\mathcal{A}$ is called an equivalence if there exists a 1-morphism $g \in \mathcal{A}(y, x)_0$ such that $gf \cong 1_x$ and $fg \cong 1_y$.

A bigroupoid is a bicategory in which each 1-morphism is an equivalence and each 2-morphism is invertible, in particular, when a bigroupoid is also a 2-category, we call it pseudogroupoid. A 2-groupoid is a 2-category in which each 1-morphism and 2-morphism is invertible.

**Definition 2.3.** Let $\mathcal{A}$ and $\mathcal{B}$ be bicategories. A lax functor $f : \mathcal{A} \to \mathcal{B}$ consists of the following data;

1. a map $f : \mathcal{A}_0 \to \mathcal{B}_0$,
2. a functor $f_{x,y} : \mathcal{A}(x, y) \to \mathcal{B}(x, y)$ for each $x, y \in \mathcal{A}_0$,
3. a 2-morphism $f_{w,w'} : f(w') \circ f(w) \Rightarrow f(w' \circ w)$ for each $w \in \mathcal{A}(x, y)_0$ and $w' \in \mathcal{A}(y, z)_0$,
4. a 2-morphism $f_x : 1_{f(x)} \Rightarrow f(1_x)$ for each $x \in \mathcal{A}_0$,

satisfying the coherent condition. A lax functor $f$ is called a homomorphism if each $f_{w,w'}$ and $f_x$ is invertible, moreover, called a strict homomorphism if each $f_{w,w'}$ and $f_x$ is an identity map. When both $\mathcal{A}$ and $\mathcal{B}$ are 2-categories, a strict homomorphism between them is called a 2-functor.

**Notation 2.4.** Let $\text{BiCat}_s$ denote the category of bicategories and strict homomorphisms, and $\text{2Cat}$, $\text{PsGrd}$, and $\text{2Grd}$ denote the full subcategories of $\text{BiCat}_s$ consisting of 2-categories, pseudogroupoids, and 2-groupoids, respectively. We can consider canonical inclusions between them and their adjoint functors.

1. The canonical inclusion $\text{PsGrd} \hookrightarrow \text{2Cat}$ has a right adjoint functor $R_{ps} : \text{2Cat} \to \text{PsGrd}$. For a 2-category $\mathcal{A}$, the pseudogroupoid $R_{ps}(\mathcal{A})$ is the maximal pseudogroupoid contained in $\mathcal{A}$.
2. The canonical inclusion $\text{2Grd} \hookrightarrow \text{2Cat}$ has a left adjoint functor $\Pi : \text{2Cat} \to \text{2Grd}$ called the 2-groupoidification. For a 2-category $\mathcal{A}$, the 2-groupoid $\Pi \mathcal{A}$ can be obtained by adding formal inverses for 1-morphisms and 2-morphisms of $\mathcal{A}$.
3. The canonical inclusion $\text{2Cat} \hookrightarrow \text{BiCat}_s$ has a left adjoint functor $S : \text{BiCat}_s \to \text{2Cat}$ called the strictification (see [Lac04] for more details).
Definition 2.5 (Pseudonatural transformation and modification). Let \( f, g : \mathcal{A} \to \mathcal{B} \) be lax functors between bicategories \( \mathcal{A} \) and \( \mathcal{B} \). A lax natural transformation \( \alpha : f \Rightarrow g \) consists of the following data;

1. a 1-morphism \( \alpha_x : f(x) \to g(x) \) for each \( x \in \mathcal{A}_0 \),
2. a 2-morphism \( \alpha_w : \alpha_y \circ f(w) \Rightarrow g(w) \circ \alpha_x \) for each 1-morphism \( w : x \to y \) in \( \mathcal{A} \),

satisfying the coherent condition. A lax natural transformation \( \alpha \) is called a pseudonatural transformation if each \( \alpha_w \) is invertible.

Let \( \alpha, \beta : f \Rightarrow g \) be lax natural transformations between 2-functors \( f, g : \mathcal{A} \to \mathcal{B} \). A modification \( m : \alpha \Rightarrow \beta \) consists of a 2-morphism \( m_x : \alpha_x \Rightarrow \beta_x \) for each \( x \in \mathcal{A}_0 \) satisfying the coherent condition.

For two 2-categories \( \mathcal{A} \) and \( \mathcal{B} \), the 2-category of internal hom \( [\mathcal{A}, \mathcal{B}] \) consists of 2-functors, pseudonatural transformations, and modifications. Similarly, we can define the internal hom in the category of pseudogroupoids, and 2-groupoids.

Theorem 2.6 ([Lac02]). The Gray tensor product and the internal hom give a cartesian closed monoidal structure on the category of 2-categories, pseudogroupoids, and 2-groupoids.

2.2 Notations of bicategories

Here we prepare some notations of bicategories and 2-categories.

Notation 2.7. We use the following notations of small categories.

- \( \emptyset \) is the empty category.
- \( [n] \) denotes the totally ordered set with \( n+1 \) objects and a single morphism \( i \to j \) for each \( i < j \) described as \( 0 \to 1 \to \cdots \to n \).
- \( \mathcal{S}^0 \) is the discrete category with two objects (not having any non-trivial morphism).
- \( \mathcal{S}^1 \) consists of two objects and non-trivial parallel two morphisms described as \( 0 \rightrightarrows 1 \).
- \( \mathcal{L} \) consists of three objects and non-trivial two morphisms with a terminal object described as \( 0 \to 2 \leftarrow 1 \).
- \( \mathcal{C} \mathcal{S}^1 \) consists of three objects and non-trivial four morphisms with a terminal object described as \( 0 \rightrightarrows 1 \to 2 \).
- \( \mathcal{S}^\infty \) is a groupoid consisting of two objects and non-trivial two morphisms described as \( 0 \rightrightarrows 1 \).
Bicategories or 2-categories are closely related to \textbf{Cat}-graphs, which are quivers enriched by the category of small categories. Let denote the category of \textbf{Cat}-graphs by \textbf{Cat-Graph}. The canonical forgetting functor \textbf{BiCat} \to \textbf{Cat-Graph} has a left adjoint functor \( F : \textbf{Cat-Graph} \to \textbf{BiCat} \). The suspension functor \( \Sigma : \textbf{Cat} \to \textbf{Cat-Graph} \) is defined by \( \Sigma A_0 = \{0,1\} \) and \( \Sigma A(0,0) = \Sigma A(1,1) = \Sigma A(1,0) = \phi, \Sigma A(0,1) = A \).

**Notation 2.8.** We use the following notations of \textbf{Cat}-graph.

- \( * \) is the \textbf{Cat}-graph consisting of only single object.
- \( L \) consists of three objects \( 0,1,2 \) with \( L(0,2) = L(1,2) = \{0\} \) and other categories of morphisms are empty.

**Notation 2.9.** We use the following notations of bicategories.

- \( [[2]] \) is a bicategory consisting of three objects \( 0,1,2 \) and 1-morphisms generated by \( f : 0 \to 1, g : 1 \to 2, \) and \( h : 0 \to 2 \) and 2-morphisms generated by \( gf \Rightarrow h \), described as the following form.

\[
\begin{array}{ccc}
0 & \xrightarrow{h} & 2 \\
\downarrow & & \downarrow \\
1 & \xrightarrow{g} & 2 \\
\end{array}
\]

- \( C_1 \) is a bicategory consisting of three objects \( 0,1,2 \) and 1-morphisms generated by \( f : 0 \to 1, g, h : 1 \to 2, \) and \( i : 0 \to 2 \) and 2-morphisms generated by \( \alpha : gf \Rightarrow i, \beta : hf \Rightarrow i \) and \( \gamma_1, \gamma_2 : g \Rightarrow h \) with \( \alpha \circ \gamma_1 f = \alpha \circ \gamma_2 f = \beta \). Define \( C_2 \) to be a bicategory removed 2-morphisms generated by \( \gamma_2 \) from \( C_1 \), and \( C_3 \) to be a bicategory removed 2-morphisms generated by \( \gamma_1 \) and \( \gamma_2 \) from \( C_1 \). These \( C_j \) are described as the following forms for \( j = 1, 2, 3 \).

- \( \mathcal{H} \) is a bigroupoid consists of two objects \( 0,1 \) and 1-morphisms generated by \( s : 0 \to 1 \) and \( t : 1 \to 0 \) and invertible 2-morphisms \( st \to 1 \) and \( st \to 1 \) satisfying the triangle conditions, described as the following form (see \cite{Lac04}).

\[
\begin{array}{ccc}
0 & \xrightarrow{s} & 1 \\
\circlearrowleft & & \circlearrowleft \\
\end{array}
\]

**Remark 2.10.** To give a strict homomorphism from \( [[2]] \) or, \( C_j \) for \( j = 1, 2, 3, \) or \( \mathcal{H} \) in Notation \ref{notation2.9} to a bicategory \( A \) is to choose a diagram in \( A \) described in Notation \ref{notation2.9}.
2.3 Nerve of 2-categories and bicategories

For a small category $A$, its nerve $N(A)$ is a 2-coskeletal simplicial set given by $N_n(A) = \text{Cat}([n], A)$. Although there are several constructions of nerve for a bicategory $A$, one of them is called the geometric nerve or Duskin nerve [Dus01]. This is a 3-coskeletal simplicial set consisting of the objects, 1-morphisms, and 2-morphisms of $A$.

**Definition 2.11** (Geometric nerve). Let $A$ be a bicategory. A simplicial set $N(A)$ is given by the set of lax functors from $[n]$ to $A$ as $N_n(A)$. This is called the geometric nerve, or simply the nerve of $A$. The classifying space $BA$ of a bicategory $A$ is defined as the geometric realization of the nerve $N(A)$.

We can give rise the nerve of bicategories to a functor $N: \text{BiCat} \rightarrow \text{Set}^{\Delta^{op}}$ from the category of bicategories and lax functors to the category of simplicial sets. The followings are important properties of the geometric nerve.

**Proposition 2.12** ([Dus01], [CCG10]). The geometric nerve functor $N: \text{BiCat} \rightarrow \text{Set}^{\Delta^{op}}$ has the following properties:

1. This is full and faithful.
2. $N(A)$ is 3-coskeletal for any bicategory $A$.
3. A lax natural transformation $F \Rightarrow G$ between lax functors $F$ and $G$ induces a homotopy from $N(F)$ to $N(G)$.

The nerve functor of small categories has a left adjoint functor called the categorization. The categorization of a simplicial set $X$ consists of $X_0$ as objects and equivalence classes of sequences of $X_1$ with respect to $X_2$ as morphisms. The geometric nerve functor restricted to $\text{2Cat}$ also has a left adjoint functor.

**Definition 2.13.** For a simplicial set $X$, the 2-categorization $c(X)$ is a 2-category defined as follows:

- The set of objects $c(X)_0 = X_0$.
- The set of 1-morphism from an object $x$ to $y$ is generated by 1-simplices of $X$, i.e.
  \[ c(X)(x, y) = \{ e_n \cdots e_1 | e_i \in X_1, d_1(e_{i-1}) = d_0(e_i), d_1(e_1) = x, d_0(e_n) = y, n \geq 1 \} \]
  with concatenation as the composition. For a 1-morphism $e = e_n \cdots e_1$ in $c(X)$, denote the initial point $d_1(e_1)$ of $e$ by $s(e)$, and the terminal point $d_0(e_n)$ by $t(e)$.
- A 2-morphism from a 1-morphism $f$ to $g$ with $s(f) = s(g)$ and $t(f) = t(g)$ is an equivalence class of sequences of triple with respect to $X_3$
  \[ (h_n \sigma_n h'_n, \ldots, h_1 \sigma_1 h'_1) \]
  where $\sigma_i \in X_2, h_i, h'_i \in c(X)_1$ satisfying...
1. \(d_0d_0(\sigma_i) = s(h_i)\) and \(d_1d_2(\sigma_i) = t(h'_i)\),
2. \(h_1d_1(\sigma_i)h'_i = f\) and \(h_n d_0(\sigma_n) d_2(\sigma_n) h'_n = g\),
3. \(h_{i-1} d_0(\sigma_{i-1}) d_2(\sigma_{i-1}) h'_{i-1} = h_1 d_1(\sigma_i) h'_i\).

Note that the above equivalence relation with respect to 3-simplexes is generated by
\[
(d_0(\tau)(d_2d_3(\tau)), d_2(\tau)) \sim ((d_0d_1(\tau))d_3(\tau), d_1(\tau))
\]
for any 3-simplex \(\tau \in X_3\). The vertical composition is given by the concatenation. On the other hand, horizontal composition
\[
c(X)(f', g') \times c(X)(f, g) \rightarrow c(X)(f'f, g'g)
\]
is given by sending
\[
((h_n \sigma_n h_n', \ldots, h_1 \sigma_1 h_1'), (\ell_m \rho_m \ell_m', \ldots, \ell_1 \rho_1 \ell_1'))
\]
to
\[
(h_n \sigma_n h_n'g, \ldots, h_1 \sigma_1 h_1'g, f' \ell_m \rho_m \ell_m', \ldots, f' \ell_1 \rho_1 \ell_1').
\]

A simplicial map \(f : X \rightarrow Y\) induces a 2-functor \(c(f) : c(X) \rightarrow c(Y)\) such that \(c(f)(x) = f_0(x)\) on objects and \(c(f)(e_n \ldots e_1) = f_1(e_n) \ldots f_1(e_1)\) on 1-morphisms, and
\[
c(f)(h_n \sigma_n h_n', \ldots, h_1 \sigma_1 h_1') = (f_1(h_n) f_2(\sigma_n) f_1(h_n'), \ldots, f_1(h_1) f_2(\sigma_1) f_1(h_1'))
\]
on 2-morphisms. The 2-categorization is a 2-functor \(c : \text{Set}^{\Delta^{op}} \rightarrow \text{2Cat}\).

**Theorem 2.14.** The 2-categorization functor \(c : \text{Set}^{\Delta^{op}} \rightarrow \text{2Cat}\) is left adjoint to the nerve functor \(N : \text{2Cat} \rightarrow \text{Set}^{\Delta^{op}}\).

**Proof.** Let \(\mathcal{A}\) be a 2-category and let \(X\) be a simplicial set. We need to show that the two hom-sets \(\text{Set}^{\Delta^{op}}(X, N(\mathcal{A}))\) and \(\text{2Cat}(c(X), \mathcal{A})\) are naturally isomorphic to each other. Note that \(\text{2Cat}(c(X), \mathcal{A})\) consists of strict 2-functors not lax functors. For a simplicial map \(f : X \rightarrow N(\mathcal{A})\), a 2-functor \(g : c(X) \rightarrow \mathcal{A}\) is defined as follows;

- \(g(x) = f_0(x)\) on objects,
- \(g(e_n \ldots e_1) = f_1(e_n) \circ \ldots \circ f_1(e_1)\) on 1-morphisms,
- \(g(h_n \sigma_n h_n', \ldots, h_1 \sigma_1 h_1') = f_1(h_n) f_2(\sigma_n) f_1(h_n') \circ \ldots \circ f_1(h_1) f_2(\sigma_1) f_1(h_1)\) on 2-morphisms.

On the other hand, a 2-functor \(h : c(X) \rightarrow \mathcal{A}\) gives a simplicial map \(k : X \rightarrow N(\mathcal{A})\) such that \(k_i = h_i\) for \(i = 0, 1, 2\), and for \(\tau \in X_3\), \(k_3(\tau)\) is a lax functor \([3] \rightarrow \mathcal{A}\) induced by \(h_j(\sigma)\) for \(j\)-faces \(\sigma\) of \(\tau\). Since \(N(\mathcal{A})\) is 3-coskeletal, it determines a simplicial map \(k\) uniquely. These correspondences are inverses of each other between \(\text{Set}^{\Delta^{op}}(X, N(\mathcal{A}))\) and \(\text{2Cat}(c(X), \mathcal{A})\).
2.4 Biequivalences and weak 2-equivalences

An equivalence of small categories is essentially surjective and fully faithful functor. One of the notions of equivalence of bicategories or 2-categories is known as biequivalence.

**Definition 2.15.** Let $f : A \to B$ be a homomorphism between bicategories $A$ and $B$.

1. $f$ is called **biessentially surjective** on objects if for any object $b \in B_0$, there exists $a \in A_0$ and an equivalence $f(a) \simeq b$ in $B$.

2. $f$ is called a **locally equivalence** if the functor on the categories of morphisms $A(x, y) \to B(fx, fy)$ is an equivalence of small categories.

3. $f$ is called a **biequivalence** if it is biessentially surjective on objects and a locally equivalence.

Similarly to the case of equivalences of categories, a homomorphism $f : A \to B$ is a biequivalence if and only if there exist a homomorphism $g : B \to A$ and invertible pseudonatural transformations $gf \cong 1_A$ and $fg \cong 1_B$.

The composition $W = \Pi c : \text{Set}^{A^{op}} \to 2\text{Grd}$ of the 2-categorization $c$ and the 2-groupoidification $\Pi$ is called the **Whitehead 2-groupoid**. This is originally defined by paths and homotopies on the geometric realization [MS93]. We can also construct a pseudogroupoid $W_{ps}(X)$ from a simplicial set $X$ in a similar way to the geometric definition of $W$.

**Definition 2.16** (Moore path). A **Moore path** on a space $X$ is pair $(\alpha, r)$ of a path $\alpha : \mathbb{R}_+ \to X$ and its length $r \geq 0$ such that $\alpha(x) = \alpha(r)$ for any $x \geq r$. For a subset $B \subset X$, the **Moore category** $M(X, B)$ consists of $M(X, B)_0 = B$ and Moore paths $M(X, B)(x, y) = \{(\alpha, r) \mid \alpha(0) = x, \alpha(r) = y\}$. The composition $(\beta, s) \circ (\alpha, r) = (\alpha * \beta, r + s)$ is given by

\[
\begin{aligned}
\alpha * \beta(t) &= \alpha(t) & 0 \leq t \leq r \\
\alpha * \beta(t) &= \beta(t - r) & r \leq t \leq r + s \\
\alpha * \beta(t) &= \beta(s) & r + s \leq t.
\end{aligned}
\]

A **reduced Moore category** $\tilde{M}(X, B)$ consists of $\tilde{M}(X, B)_0 = B$ and Moore paths not staying put on a point of $B$, i.e. $\tilde{M}(X, B) = M(X, B)/\sim$ where $\sim$ is the equivalence relation generated by $(\alpha, 0) \sim (\alpha_0, r)$ for the constant path $\alpha_0$ on each $b \in B$ and $r > 0$.

Let $(\alpha, r)$, $(\beta, s)$ be Moore paths from $x$ to $y$. A homotopy from $(\alpha, r)$ to $(\beta, s)$ is a pair $(H, \gamma)$ of a homotopy $H : \mathbb{R}_+ \times I \to X$ from $\alpha$ to $\beta$ and a path $\gamma : I \to \mathbb{R}_+$ from $r$ to $s$ such that $H(0, s) = x$ and $H(v, s) = H(\gamma(s), s) = y$ for any $s \in I$, $v \geq \gamma(s)$. A homotopy from a homotopy $(H, \gamma)$ to $(H', \gamma')$ between Moore paths $(\alpha, t)$ and $(\beta, s)$ is also a pair $(G, \Gamma)$ of a homotopy $G$ between $H$ and $H'$ and a homotopy $\Gamma$ between $\gamma$ and $\gamma'$ such that $(G_u, \Gamma_u)$ is a homotopy between $(\alpha, t)$ and $(\beta, s)$ for each $u \in I$. 

9
Definition 2.17. For a simplicial set $X$, defined $W_{ps}(X)$ as a pseudogroupoid whose underlying category is the reduced Moore category $\tilde{M}(\{X[1],|X[0]\})$ for the pair of 1-skeleton and 0-skeleton of the geometric realization $|X|$ of $X$, and 2-morphisms are homotopy classes of homotopies of Moore paths in $|X|$. We call $W_{ps}(X)$ the Whitehead pseudogroupoid of $X$.

We have the canonical 2-functor $p : W_{ps}(X) \to W(X)$ sending a Moore path $(\alpha, r)$ to the homotopy class $[\alpha/r,1]$ in $|X|^{(1)}$, where $\alpha/r : I \to |X|^{(1)}$ is given by $\alpha/r(t) = \alpha(rt)$.

Proposition 2.18. For any simplicial set $X$, the canonical 2-functor $p : W_{ps}(X) \to W(X)$ is a biequivalence.

Proof. We can see that $p : W_{ps}(X)_0 \to W(X)_0$ on objects is the identity map on $X_0$, and $p : W_{ps}(X)(x,y)_0 \to W(X)(x,y)_0$ on 1-morphisms is the canonical surjection for each $x$ and $y$. Moreover, $p : W_{ps}(X)(x,y)((\alpha, r), (\beta, s)) \to W(X)(x,y)((\alpha, 1), [\beta, 1])$ on 2-morphisms is an isomorphism for each $(\alpha, r)$ and $(\beta, s)$, since $(\alpha, r)$ and $(\alpha/r, 1)$ are homotopic to each other for any Moore path $(\alpha, r)$.

The above $p$ induces a biequivalence $p^\sharp : \Pi W_{ps}(X) \to W(X)$ with $i_{W_{ps}(X)} \circ p^\sharp = p : W_{ps}(X) \to W(X)$.

Definition 2.19. Let $\mathcal{G}$ be a 2-groupoid.

- The set $\pi_0(\mathcal{G})$ of connected components of $\mathcal{G}$ consists of the isomorphism classes of objects.
- The fundamental group $\pi_1(\mathcal{G}, x)$ of $\mathcal{G}$ with an object $x \in G_0$ is defined as the set of connected components $\pi_0(\mathcal{G}(x, x))$.
- For an object $x \in G_0$, define $\pi_2(\mathcal{G}, x)$ as the group of natural transformations from the identity morphism on $x$ to itself.

Lemma 2.20. Let $f : \mathcal{G} \to \mathcal{H}$ be a 2-functor between 2-groupoids. The followings are equivalent.

1. $f$ is a biequivalence.
2. $f$ induces isomorphisms $\pi_0(\mathcal{G}) \to \pi_0(\mathcal{H})$, $\pi_i(\mathcal{G}, x) \to \pi_i(\mathcal{H}, fx)$ for each $x \in G_0$ and $i = 1,2$.

Proof. A 2-functor $f$ is biessentially surjective if and only if it induces an isomorphism $\pi_0(\mathcal{G}) \cong \pi_0(\mathcal{H})$. Note that the category of morphisms $G(x, y)$ is equivalent to $G(x, z)$ if there exists a 1-morphism between $y$ and $z$. Hence, $f$ is locally equivalence if and only if it induces isomorphisms $\pi_i(\mathcal{G}, x) \cong \pi_i(\mathcal{H}, fx)$ for $x \in G_0$ and $i = 1,2$.

Definition 2.21. Let $f : \mathcal{A} \to \mathcal{B}$ be a homomorphism between bicategories. We call $f$ to be a weak 2-equivalence if the induced 2-functor $\Pi S \mathcal{A} \to \Pi S \mathcal{B}$ is a biequivalence of 2-groupoids.
In particular, if a 2-functor \( f : \mathcal{A} \to \mathcal{B} \) is a weak 2-equivalence if and only if \( \Pi \mathcal{A} \to \Pi \mathcal{B} \) is a biequivalence of 2-groupoids. A simplicial map \( X \to Y \) is called a weak 2-equivalence if the induced maps \( \pi_0([X]) \to \pi_0([Y]) \) and \( \pi_i([X], x) \to \pi_i([Y], fx) \) are isomorphisms for each \( x \in X_0 \) and \( i = 1, 2 \).

**Proposition 2.22.** If \( \mathcal{A} \) is a 2-category, then the canonical inclusion \( i_{\mathcal{A}} : \mathcal{A} \to \Pi \mathcal{A} \) is a weak 2-equivalence.

**Proof.** By applying the 2-groupoidification \( \Pi \),

\[
\Pi i_{\mathcal{A}} : \Pi \mathcal{A} \to \Pi(\Pi \mathcal{A}) \cong \Pi \mathcal{A}
\]

is the identity map on \( \Pi \mathcal{A} \). Hence, the canonical inclusion \( i_{\mathcal{A}} \) is a weak 2-equivalence. \( \square \)

**Proposition 2.23** (Theorem 8.5 in [Lac02]). If \( \mathcal{G} \) is a pseudogroupoid, then the canonical inclusion \( i_{\mathcal{G}} : \mathcal{G} \to \Pi \mathcal{G} \) is a biequivalence.

**Lemma 2.24** ([MS93]). For a simplicial set \( X \), \( \pi_0(X) \cong \pi_0(W X) \) and \( \pi_i(X, x) \cong \pi_i(W X, x) \) for each \( x \in X_0 \) and \( i = 1, 2 \).

**Corollary 2.25.** If \( f : X \to Y \) is a map of simplicial sets, then the followings are equivalent.

1. \( f \) is a weak 2-equivalence.
2. \( Wf \) is a biequivalence.
3. \( cf \) is a weak 2-equivalence.

**Proposition 2.26** ([MS93]). A 2-functor \( f : \mathcal{G} \to \mathcal{H} \) between 2-groupoids is biequivalence if and only if \( Nf : N\mathcal{G} \to N\mathcal{H} \) is a weak homotopy equivalence of simplicial sets.

**Proposition 2.27.** The counit map \( \theta : cN\mathcal{A} \to \mathcal{A} \) associated with the adjoint pair \( (c, N) \) is a biequivalence for any 2-category \( \mathcal{A} \).

**Proof.** The counit map \( \theta \) is given by the identity map on objects, and the composition on 1-morphisms and 2-morphisms. Obviously, \( \theta \) is surjective on objects, and locally surjective functor, i.e. the functor on the categories of morphisms \( cN\mathcal{A}(x, y) \to \mathcal{A}(x, y) \) is surjective on objects and full for each \( x, y \in \mathcal{A}_0 \). It remains to show that this functor is faithful, i.e. the map

\[
\theta : cN\mathcal{A}(x, y)(f_n f_{n-1} \cdots f_1, g_m g_{m-1} \cdots g_1) \to \mathcal{A}(x, y)(f_n \circ f_{n-1} \circ \cdots \circ f_1, g_m \circ g_{m-1} \circ \cdots \circ g_1)
\]

is injective. For two 2-morphisms \( \alpha = (h_k \sigma_k h_k', \ldots, h_1 \sigma_1 h_1') \) and \( \beta = (\ell_k \tau_k \ell_k', \ldots, \ell_1 \tau_1 \ell_1') \) in \( cN\mathcal{A}(x, y)(f_n f_{n-1} \cdots f_1, g_m g_{m-1} \cdots g_1) \) with \( \theta(\alpha) = \theta(\beta) \), it implies that the vertical compositions of \( \alpha \) and \( \beta \) are equal to each other in \( \mathcal{A} \). It gives a 3-simplex of \( N\mathcal{A} \) including \( \alpha \) and \( \beta \) as the faces, hence \( \alpha = \beta \) in \( cN\mathcal{A} \). \( \square \)

**Proposition 2.28.** If \( \mathcal{A} \) is a 2-category, then \( \pi_0(\Pi \mathcal{A}) \cong \pi_0(N\mathcal{A}) \) and \( \pi_i(\Pi \mathcal{A}, x) \cong \pi_i(N\mathcal{A}, x) \) for each \( x \in \mathcal{A}_0 \) and \( i = 1, 2 \).
Proof. By Proposition 2.27 and Corollary 2.25, we have
\[ \pi_0(N\mathcal{A}) \cong \pi_0(WN\mathcal{A}) = \pi_0(\Pi cN\mathcal{A}) \cong \pi_0(\Pi \mathcal{A}) \]
and
\[ \pi_i(N\mathcal{A}, x) \cong \pi_i(WN\mathcal{A}, x) = \pi_i(\Pi cN\mathcal{A}, x) \cong \pi_i(\Pi \mathcal{A}, x) \]
for each \( x \in \mathcal{A}_0 \) and \( i = 1, 2 \).

Corollary 2.29. A 2-functor \( f : \mathcal{A} \rightarrow \mathcal{B} \) is a weak 2-equivalence if and only if \( Nf : N\mathcal{A} \rightarrow NB \) is a weak 2-equivalence of simplicial sets.

2.5 Model categories

The notion of model categories was introduced by D. Quillen in [Qui67]. This is a framework to do homotopy theory not only in the category of spaces, but also in a general category.

Definition 2.30. A model structure on a category \( M \) consists of three distinguished classes of morphisms \( W, C, \) and \( F \) closed under retracts and compositions called weak equivalences, cofibrations, and fibrations satisfying the following properties.

1. If \( f \) and \( g \) are morphisms of \( M \) such that \( g \circ f \) is defined, two of \( f, g \) and \( g \circ f \) are weak equivalences, then so is the third.

2. Every morphism in \( W \cap C \) has the right lifting property for \( F \), and every morphism in \( C \) has the right lifting property for \( W \cap F \).

3. For any morphism \( f \) in \( M \) can be factored as \( p \circ i \) by \( p \in F \cap W \) and \( i \in C \), and also \( f \) can be factored as \( q \circ j \) by \( q \in F \) and \( j \in C \cap W \).

A morphism in \( W \cap C \) is called a trivial cofibration, and a morphism in \( W \cap F \) is called a trivial fibration, respectively. A model category is a category \( M \) closed under small limits and colimits together with a model structure on \( M \).

It tends to be quite difficult to prove that a category admits a model structure. The axioms of model structure are always hard to check. The following cofibrantly generated structure helps alleviate such difficulty.

Definition 2.31 ([Hov99], [Hir03]). We say that a model category \( M \) is cofibrantly generated if there exist sets \( I \) and \( J \) of morphisms such that

1. both \( I \) and \( J \) permit the small object argument,

2. \( W \cap F = I\text{-inj} \) and \( F = J\text{-inj} \).

The above set \( I \) is called a generating cofibrations, and \( J \) is called a generating trivial cofibrations.
Example 2.32. Here are some examples of cofibrantly generated model categories closely related to the 2-type model structure on the category of 2-categories and bicategories.

1. The folk model structure on $\text{Cat}$ is introduced in [JT91], [Rez00]. The weak equivalences are equivalence of categories. This is cofibrantly generated with the set of generating cofibrations

$$I_f = \{ \phi \to [0], S^0 \to [1], S^1 \to [1] \}$$

and the set of generating trivial cofibrations $J_f = \{ [0] \to S^\infty \}$.

2. The 1-type model structure on $\text{Cat}$ is a left Bousfield localization of the folk model structure [Tan13]. The weak equivalences are weak 1-equivalences. This is cofibrantly generated with the set of generating cofibrations $I_1 = I_f$ and the set of generating trivial cofibrations $J_1$ consisting of $[0] \to [1], L \to [2], CS^1 \to [2]$ and their opposite functors.

3. The Lack model structure on $\text{BiCat}_s$ is introduced in [Lac04]. The weak equivalences are biequivalences. Let $\text{BiCat}_L$ denote the category of bicategories equipped with the Lack model structure. This is cofibrantly generated with

$$I_L = \{ F(\phi) \to F(\ast), F\Sigma\phi \to F\Sigma[0], F\Sigma S^0 \to F\Sigma[1], F\Sigma S^1 \to F\Sigma[1] \}$$

and $J_L = \{ F(\ast) \to \mathcal{H}, F\Sigma[0] \to F\Sigma S^\infty \}$.

4. The Lack model structure on $\text{2Cat}$ is introduced in [Lac02], [Lac04]. The weak equivalences are biequivalences. Let $\text{2Cat}_L$ denote the category of 2-categories equipped with the Lack model structure. This is cofibrantly generated with $SI_L$ and $SJ_L$.

5. The Lack model structure on $\text{2Cat}$ can be restricted to $\text{PsGrd}$. We always consider this model structure for $\text{PsGrd}$ as long as it is not confusing.

6. The Moerdijk and Svensson model structure on $\text{2Grd}$ is introduced in [MS93]. The weak equivalences and fibrations are biequivalences. We always consider this model structure for $\text{2Grd}$ as long as it is not confusing.

7. The classical model structure on $\text{Set}^{\Delta^{op}}$ is introduced in [Qui67]. The weak equivalences are weak homotopy equivalences. Let $\text{Set}^{\Delta^{op}}_c$ denote the category of simplicial sets equipped with the classical model structure. This is cofibrantly generated with

$$I_c = \{ \partial \Delta[n] \to \Delta[n] \mid n \geq 0 \}$$

and $J_c = \{ \Lambda^k_n \to \Delta[n] \mid n \geq 0, 0 \leq k \leq n \}$. 

13
8. The 2-type model structure on $\text{Set}^{\Delta^{op}}$ is introduced in [EH95]. The weak equivalences are weak 2-equivalences. Let $\text{Set}^{\Delta^{op}}_2$ denote the category of simplicial sets equipped with the 2-type model structure. This is cofibrantly generated with

$$I_2 = \{ \partial \Delta[n] \to \Delta[n] \mid 0 \leq n \leq 3 \}$$

and $J_2 = \{ \wedge_k^n \to \Delta[n], \wedge_j^4 \to \partial \Delta[4] \mid 0 \leq n \leq 3, 0 \leq k \leq n, 0 \leq j \leq 4 \}$.

3 The 2-type model structure

We give the 2-type model structure on the category of 2-categories and bicategories. This model structure is based on the 2-type model structure on the category of simplicial sets [EH95].

3.1 The 2-type model structure on $2\text{Cat}$

We first consider the 2-type model structure on the category of 2-categories. Although the category of bicategories is more general, the category of 2-categories is directly related to the category of simplicial set by the pair of adjoint functors $(c, N)$ in Section 2.3. We can sends the 2-type model structure on the category of simplicial sets [EH95] to the category of 2-categories by using the adjoint functors.

**Theorem 3.1.** The category of 2-categories admits the following model structure:

1. The weak equivalences are weak 2-equivalences.
2. The class of cofibrations is equal to the one of $2\text{Cat}_L$.

**Proof.** We will apply Theorem 13.4.2 in [Hir03] to the pair of adjoint functors $c : \text{Set}^{\Delta^{op}}_2 \rightleftarrows 2\text{Cat} : N$. Let us see $c(I_2)$ and $c(J_2)$ carefully. We can replace $c(I_2)$ to $SI_L$ since $c(I_2)$ can be generated from $SI_L$ by push-outs, and $SI_L$ can be generated from $c(I_2)$ by deformation retractions. Similarly we can replace $c(J_2)$ with the set of 2-functors consisting of the followings:

- $SF(*) \to SF\Sigma[0]$ induced from the inclusion $* \to \Sigma[0]$ sending the unique object to 1,
- $SF\mathcal{L} \to S[[2]]$ induced from the inclusion $F\mathcal{L} \to [[2]]$,
- $SC_3 \to SC_2$ induced from the inclusion $C_3 \to C_2$,
- $SC_1 \to SC_2$ induced from $C_1 \to C_2$ sending $\gamma_2$ to $\gamma_1$,

and their opposite 2-functors with respect to 1-morphisms, 2-morphism, and both of them respectively. For each relative $c(I_2)$-cell complex and relative $c(J_2)$-complex $A \to B$, the codomain $B$ is obtained by repeating to add objects,
1-morphisms and 2-morphisms to $A$ or identifying 2-morphisms of $A$. Since each domain and codomain of $c(I_2)$ and $c(J_2)$ is a finite 2-category, both c($I_2$) and c($J_2$) permit the small object argument.

Let $f : A \to B$ be a relative $c(J_2)$-complex. Note that $f$ is constructed as a transfinite composition of push out of $c(J_2)$. By applying the 2-groupoidification functor $\Pi$, the 2-functor $\Pi f : \Pi A \to \Pi B$ is a trivial cofibration in $2Grd$ since the Whitehead 2-groupoid $W = \Pi c : Set_{\Delta^{op}} \to 2Grd$ is a left Quillen functor preserving trivial cofibrations and colimits, and $W(\Delta^1 \to \partial \Delta^1)$ is also a trivial cofibration for any $0 \leq j \leq 4$. In particular, $\Pi f$ is a biequivalence in $2Grd$, and $f$ is a weak 2-equivalence in $2Cat$. Corollary 2.29 shows that $N$ sends $f$ into a weak 2-equivalence.

Now we can put a model structure on the category of 2-categories by applying Theorem 13.4.2 in [Hir03]. The class of weak equivalences is equal to the class of weak 2-equivalences by Corollary 2.29 and the class of cofibrations is equal to the one of Lack’s model structure.

Let us call the above model structure 2-type and denote the category of 2-categories equipped with the 2-type model structure by $2Cat_2$.

**Theorem 3.2.** The 2-type model category $2Cat_2$ is Quillen equivalent to $Set_{\Delta^{op}}$ and $2Grd$.

**Proof.** By Theorem 13.4.2 in [Hir03], $(c, N)$ is a Quillen pair between $2Cat_2$ and $Set_{\Delta^{op}}$. Let $X$ be a simplicial set and let $A$ be a 2-category. For a morphism $f : X \to N A$, consider the following commutative diagram

$$
\begin{array}{ccc}
\pi_i(X) & \xrightarrow{\cong} & \pi_i(X) \\
\downarrow f_* & & \downarrow \Pi f_* \\
\pi_i(N A) & \cong & \pi_i(N \Pi A) \\
\end{array}
$$

for $i = 0, 1, 2$. We have that the morphism $f$ is a weak 2-equivalence if and only if the corresponding morphism $f^\sharp : cX \to A$ is a weak 2-equivalence. Therefore, $(c, N)$ is a pair of Quillen equivalences. On the other hand, the pair of 2-groupoidification and inclusion $(\Pi, U)$ is also a Quillen pair between $2Cat_2$ and $2Grd$ since $\Pi$ preserves weak equivalences and cofibrations. Let $A$ be a 2-category and let $G$ be a 2-groupoid. The definition of weak 2-equivalences implies that $f : A \to G$ is a weak 2-equivalence if and only if $f^\sharp : \Pi A \to \Pi G \cong G$ is a biequivalence. Therefore, $(\Pi, U)$ is a pair of Quillen equivalences.

The 2-type model structure has the same cofibrations as that of $2Cat_L$ and a weak equivalence in $2Cat_L$ is a weak equivalence in $2Cat_2$. We can see that $2Cat_2$ is a Bousfield localization of $2Cat_L$ (see [Hir03] for more details). The left Bousfield localization of a model category $M$ with respect to a class of morphisms $S$ is another model structure $L_SM$ on $M$ with $S$-local equivalences as the weak equivalences and the same class of cofibrations of $M$. $S$-local equivalences
underlying categories

from A

Note that every object in

Proof. If erated, thus so does the Whitehead pseudogroupoid W

Consider the induced 2-functor W

A morphism (α, r ∈ A

is cofibrant.

Proof. The underlying category of W

A

is weakly equivalent to

is freely generated and W

is cofibrant.

Lemma 3.5. The Whitehead pseudogroupoid W

 preserve cofibrations and weak equivalences.

Proof. The Whitehead 2-groupoid W : Set

preserve weak equivalences, then so does the Whitehead pseudogroupoid W

On the other hand, a cofibration f : X → Y in Set

is a degreewise injection. Consider the induced 2-functor W

: W

→ W

. The functor on underlying categories f

: M(|X

), |X

) → M(|Y

), |Y

) is injective on both objects and morphisms. By Lemma 3.5, M(|Y

), |Y

) is freely generated, thus f

has the left lifting property for any surjective functor. Hence W

is a cofibration in PsGrd.

We can take a cosimplicial resolution ⃗A of a cofibrant 2-category A as ⃗A

= A ⊗ W

(Δ[n]) since W

preserves cofibrations and weak equivalences by Lemma 3.5.

Proposition 3.6. The homotopy function complex Map(A, B) from A to B in 2Cat

is weakly equivalent to NR

[A

, B] where A

is a cofibrant replacement of A. In particular, when B is a pseudogroupoid, the homotopy function complex from A to B is weakly equivalent to N[IA

, IB].

Proof. Note that every object in 2Cat

is fibrant. The (left) homotopy function complex is originally defined by 2Cat(⃗A, B) where ⃗A is a cosimplicial resolution of A. A cosimplicial resolution ⃗A can be chosen as ⃗A

= A

⊗ W

(Δ[n]) where A

is a cofibrant replacement of A. Let Δ denote the cosimplicial simplicial set given as Δ

= Δ[n]. The adjointness relation implies that

2Cat(⃗A, B) ≅ 2Cat(W

(Δ), [A

, B]) ≅ PsGrd(W

(Δ), R

[A

, B]).
The canonical inclusion \( R_{ps}[A', B] \to \Pi R_{ps}[A', B] \) is a weak equivalence between fibrant objects in \( \text{PsGrd} \), then it induces a weak equivalence

\[
\text{PsGrd}(W_{ps}(\Delta), R_{ps}[A', B]) \to \text{PsGrd}(W_{ps}(\Delta), \Pi R_{ps}[A', B])
\]

of simplicial sets. Recall that Proposition 2.18 gives a biequivalence \( \Pi W_{ps}(\Delta) \to W(\Delta) \) between cofibrant objects in \( \text{2Grd} \). Again, the adjointness relation implies that

\[
\text{PsGrd}(W_{ps}(\Delta), \Pi R_{ps}[A', B]) \cong \text{2Grd}(\Pi W_{ps}(\Delta), \Pi R_{ps}[A', B])
\]

\[
\cong \text{2Grd}(W(\Delta), \Pi R_{ps}[A', B])
\]

\[
\cong \text{Set}^{\Delta^{op}}(\Delta, N\Pi R_{ps}[A', B])
\]

\[
\cong N\Pi R_{ps}[A', B] \cong NR_{ps}[A', B].
\]

When \( \mathcal{B} \) is a pseudogroupoid, the canonical inclusion \( \mathcal{B} \to \Pi \mathcal{B} \) being a biequivalence between fibrant objects induces a weak homotopy equivalence \( \text{Map}(\mathcal{A}', \mathcal{B}) \to \text{Map}(\mathcal{A}', \Pi \mathcal{B}) \) on homotopy function complexes. We have

\[
\text{Map}(\mathcal{A}', \Pi \mathcal{B}) \cong NR_{ps}[\mathcal{A}', \Pi \mathcal{B}] \cong N[\mathcal{A}', \Pi \mathcal{B}] \cong N[\Pi \mathcal{A}', \Pi \mathcal{B}]
\]

since \( [\mathcal{A}', \Pi \mathcal{B}] \) is a 2-groupoid.

**Definition 3.7.** Let \( K \) be the set of 2-functors consisting of two inclusions: \( k_1 : SF\Sigma[0] \to \mathcal{H} \) and \( k_2 : SF\Sigma[1] \to SF\Sigma S^\infty \).

**Proposition 3.8.** For the set of 2-functors \( K \) in Definition 3.7 a 2-category is an \( \mathcal{K} \)-local if and only if it is a pseudogroupoid.

**Proof.** Note that each domain and codomain of \( k_1 \) and \( k_2 \) is cofibrant. Hence the homotopy function complex from \( \mathcal{A} \) to \( \mathcal{B} \) in \( \text{2Cat}_L \) is weakly equivalent to \( NR_H[\mathcal{A}, \mathcal{B}] \) when \( \mathcal{A} \) is the domain or codomain of \( k_1 \) or \( k_2 \). Moreover if \( \mathcal{B} \) is a pseudogroupoid, the homotopy function complex is weakly equivalent to \( N[\Pi \mathcal{A}, \Pi \mathcal{B}] \). The 2-functors \( \Pi k_1 \) and \( \Pi k_2 \) are trivial cofibrations in \( \text{2Grd} \), and these induce biequivalences on the internal homs

\[
[\Pi \mathcal{H}, \Pi \mathcal{B}] \to [\Pi SF\Sigma[0], \Pi \mathcal{B}], [\Pi SF\Sigma S^\infty, \Pi \mathcal{B}] \to [\Pi SF\Sigma[1], \Pi \mathcal{B}]
\]

and weak homotopy equivalences on the nerves

\[
N[\Pi \mathcal{H}, \Pi \mathcal{B}] \to N[\Pi SF\Sigma[0], \Pi \mathcal{B}], N[\Pi SF\Sigma S^\infty, \Pi \mathcal{B}] \to N[\Pi SF\Sigma[1], \Pi \mathcal{B}]
\]

respectively. Thus \( \mathcal{B} \) is \( \mathcal{K} \)-local. Conversely, let \( \mathcal{B} \) be \( \mathcal{K} \)-local. Since \( k_1 \) is a cofibration in \( \text{2Cat}_L \), \( k_1^*: \text{Map}(\mathcal{S}\mathcal{H}, \mathcal{B}) \to \text{Map}(SF\Sigma[0], \mathcal{B}) \) is a trivial fibration in \( \text{Set}_{\Delta^{op}} \). Focus on the degree zero, \( k_1^*: [\mathcal{S}\mathcal{H}, \mathcal{B}]_0 \to [SF\Sigma[0], \mathcal{B}]_0 \) is surjective. It implies that any 1-morphism in \( \mathcal{B} \) is an equivalence. Similarly, \( k_2^*: [SF\Sigma S^\infty, \mathcal{B}]_0 \to [SF\Sigma[1], \mathcal{B}]_0 \) is also surjective, and it implies that any 2-morphism in \( \mathcal{B} \) is invertible. Thus, \( \mathcal{B} \) is a pseudogroupoid.
Proposition 3.9. For the set of 2-functors \( K \) in Definition 3.7, a 2-functor is an \( K \)-local equivalence if and only if it is a weak 2-equivalence.

Proof. Let \( f : A \to B \) be a \( K \)-local equivalence, i.e. the induced map on the homotopy function complexes

\[
(\Pi f')^* : N[\Pi B', \Pi G] \to N[\Pi A', \Pi G]
\]

is a weak homotopy equivalence for any pseudogroupoid \( G \). Put \( G = \Pi A' \), then the induced map on the sets of connected components

\[
(\Pi f')^*_0 : \pi_0[\Pi B', \Pi A'] \to \pi_0[\Pi A', \Pi A']
\]

is a bijection. There exists a 2-functor \( g : \Pi B' \to \Pi A' \) and an invertible pseudonatural transformation \( g \circ \Pi f' \sim 1_{\Pi A'} \). On the other hand, we have

\[
(\Pi f')^*_0(\Pi f' \circ g) = \Pi f' = (\Pi f')^*_0(1_{\Pi B'}).
\]

The injectivity of \( (\Pi f')^*_0 \) states that there exists an invertible pseudonatural transformation \( \Pi f' \circ g \sim 1_{\Pi B'} \). Consequently, \( \Pi f' \) is a biequivalence of 2-groupoids, and \( f' \) and \( f \) are weak 2-equivalences.

Conversely, let \( f : A \to B \) be a weak 2-equivalence. The induced map \( f' : A' \to B' \) is also a weak 2-equivalence, and \( \Pi f' : \Pi A' \to \Pi B' \) is a biequivalence between cofibrant objects in \( 2\Grd \). Since the functor \([- , \Pi G] \) takes biequivalences between cofibrant objects to biequivalences for any pseudogroupoid \( G \),

\[
(\Pi f')^* : N[\Pi B', \Pi G] \to N[\Pi A', \Pi G]
\]

is a weak homotopy equivalence. Proposition 3.6 implies that \( f \) is a \( K \)-local equivalence. \( \square \)

Corollary 3.10. The 2-type model structure \( 2\Cat_2 \) on the category of 2-categories is the left Bousfield localization of \( 2\Cat_L \) with respect to \( K \).

3.2 The 2-type model structure on \( \BiCat_s \)

In this subsection, we define the 2-type model structure on the category of bicategories referring to Section 3.1. An important fact about 2-categories and bicategories is called the coherent theorem. It says that any bicategory is biequivalent to a 2-category. We have already seen that as the strictification in Section 2.1. See [Lac02] for a concrete construction of the strictification.

Theorem 3.11 (Coherent theorem). For a bicategory \( A \), the counit homomorphism \( \eta_A : S A \to A \) associated to the adjoint pair \( (S, I) \) is a biequivalence.

Definition 3.12. A strict homomorphism \( f : A \to B \) is called a fibered in bigroupoids if it has the right lifting property in \( \BiCat_s \) for

\[
J' = \{ F(*) \to F\Sigma[0], F\Sigma \to [2], C_3 \to C_2, C_1 \to C_2 \},
\]
introduced in Section 3.1 and Section 2.2.

On the other hand, \( f : A \to B \) is called a cofibered (resp. opfibered, coopfibered) in bigroupoids if it has the right lifting property in \( \text{BiCat} \) for the set \( J^{\text{co}} \) (resp. \( J^{\text{op}}, J^{\text{coop}} \)) consisting of the above opposite morphisms with respect to 1-morphisms (resp. 2-morphism, both 1-morphisms and 2-morphisms).

**Remark 3.13.** [Lac02] A strict homomorphism \( f : A \to B \) between bicategories is a trivial fibration in \( \text{BiCat}_L \) if and only if \( f \) is surjective on objects, and each functor \( \mathcal{A}(x, y) \to \mathcal{B}(fx, fy) \) is surjective on objects and an equivalence.

Recall the definition of weak 2-equivalences of bicategories. It is a homomorphism \( f : A \to B \) inducing a bi-equivalence \( \Pi S A \to \Pi S B \). For a bicategory \( A \), the strictification \( SA \) consists of the same objects of \( A \) and freely generated 1-morphisms and 2-morphisms from that of \( A \), and its 2-groupoidification \( \Pi S A \) is obtained by adding formal inverses of 1-morphisms and 2-morphisms (see [Lac02]). More explicitly, it consists of zigzag sequences of 1-morphisms of \( A \) as 1-morphisms, and zigzag sequences of 2-morphisms of \( A \) as 2-morphisms. The set of 2-morphisms of \( \Pi S A \) also contains a following form \( \alpha : q^{-1}fh^{-1} \Rightarrow i \) for each 2-morphism \( \alpha : f \Rightarrow gih \) in \( A \). Two 1-morphisms \( q^{-1}fh^{-1} \) and \( i \) are identified in \( \Pi A \) if \( f = gih \) in \( A \), and two 2-morphism \( \beta^{-1}\alpha\gamma^{-1} \) and \( \delta \) are identified in \( \Pi A \) if \( \alpha = \beta\delta\gamma \) in \( A \). Moreover, for 2-morphisms \( \alpha, \beta \in \mathcal{A}(x, y)(f, g) \), \( \alpha = \beta \) in \( \Pi A(x, y)(f, g) \) holds if \( h\alpha i = h\beta i \) in \( A \) for some 1-morphisms \( h, i \) of \( A \).

**Proposition 3.14.** A strict homomorphism \( f : A \to B \) between bicategories \( A \) and \( B \) is a trivial fibration in \( \text{BiCat}_L \) if and only if \( f \) is a weak 2-equivalence and fibered, cofibered, opfibered, and coopfibered in bigroupoids.

**Proof.** Let \( f : A \to B \) be a weak 2-equivalence and fibered, cofibered, opfibered, and coopfibered in bigroupoids. For an object \( b \in \mathcal{B}_0 \), there exists an object \( a \in \mathcal{A}_0 \) and a zigzag sequence

\[
f(a) \to b_1 \leftarrow b_2 \to \cdots \leftarrow b_n \to b
\]

of 1-morphisms of \( B \) since the induced 2-functor \( \Pi A \to \Pi B \) is biessentially surjective on objects. By using the lifting property with respect to \( F(*) \to FS[0] \) and its opposite morphism, we can repeat taking the lifts of the above 1-morphisms starting at \( f(a) \). Hence we have \( a_{n+1} \in \mathcal{A}_0 \) such that \( f(a_{n+1}) = b \), and \( f \) is surjective on objects.

For objects \( x, y \in \mathcal{A}_0 \) and a 1-morphism \( w \in \mathcal{B}(fx, fy)_0 \), there exists a 1-morphism in \( \Pi A \), i.e. a zigzag sequence \( z = z_n z_{n-1}^{-1} \cdots z_2^{-1} z_1 \) of 1-morphisms in \( \mathcal{A} \) and a 2-morphism in \( \Pi B \), i.e. a zigzag sequence

\[
\Pi f(z) \Rightarrow w_1 \Leftarrow w_2 \Rightarrow \cdots \Leftarrow w_n \Rightarrow w
\]

of 2-morphisms in \( B \) since the induced 2-functor \( \Pi A \to \Pi B \) is locally essentially surjective. By using the lifting property with respect to \( L \to [2], \ F(*) \to FS[0] \) and their opposite morphisms, we can repeat taking the lifts of the above 2-morphisms starting at \( \Pi f(z) \). Hence we have \( z_{n+1} \in \mathcal{A}(x, y)_0 \) with \( f(z_{n+1}) = w \), and \( f : \mathcal{A}(x, y)_0 \to \mathcal{B}(fx, fy)_0 \) is surjective.
For objects $x, y \in \mathcal{A}_0$ and 1-morphisms $z, w \in \mathcal{A}(x, y)_0$ and a 2-morphism $\beta \in \mathcal{B}(fx, fy)(fz, fw)$, there exists a 2-morphism in $\mathcal{IL}A$, i.e. a zigzag sequence $\alpha = \alpha_0^1 \alpha_1^2 \cdots \alpha_{n-1}^n \alpha_1$ of 2-morphisms in $\mathcal{A}$ with $\Pi f(\alpha) = \beta$ in $\Pi \mathcal{B}(fx, fy)(fz, fw)$ since the induced 2-functor $\mathcal{IL}A \to \Pi \mathcal{B}$ is locally full. By using the lifting property with respect to $F(*) \to F\Sigma[0]$, $L \to [2]$, $C_1 \to C_2$, and their opposite morphisms, we can obtain a 2-morphism $\gamma \in \mathcal{A}(x, y)(z, w)$ such that $\gamma = \alpha$ in $\mathcal{IL}A$, and $f : \mathcal{A}(x, y)(z, w) \to \mathcal{B}(fx, fy)(fz, fw)$ is surjective. Moreover, let $\alpha, \alpha' \in \mathcal{A}(x, y)(z, w)$ be two 2-morphisms in $\mathcal{A}$ with $f(\alpha) = f(\alpha')$ in $\mathcal{B}(fx, fy)(fz, fw)$. It states that $\gamma(f) = f(\alpha')$ in $\Pi \mathcal{B}(fx, fy)(fz, fw)$, and $\alpha = \alpha'$ in $\mathcal{IL}A(x, y)(z, w)$ since the induced 2-functor $\Pi(\mathcal{A}) \to \Pi(\mathcal{B})$ is locally faithful. By using the lifting property with respect to $C_1 \to C_2$ and its opposite morphisms, we have $\alpha = \alpha'$ in $\mathcal{A}(x, y)(fz, fw)$, and $f : \mathcal{A}(x, y)(z, w) \to \mathcal{B}(fx, fy)(fz, fw)$ is injective. Therefore, $f : \mathcal{A} \to \mathcal{B}$ is a trivial fibration in $\mathcal{BiCat}_L$ by Remark 3.13.

Conversely, let $f$ be a trivial fibration in $\mathcal{BiCat}_L$. It suffices to show that $f$ is fibered, cofibered, opfibered, and coopfibered in bigroupoids. We show only the case of fibered in groupoids. Other cases can be shown similarly. The lifting property for $F(*) \to F\Sigma[0]$ is induced from surjectivity on objects and local surjectivity on 1-morphisms of $f$. The lifting property for $L \to [2]$ is induced from locally surjectivity on 1-morphisms and locally fullness of $f$. The lifting property for $C_1 \to C_2$ is induced from locally fullness of $f$. Finally, the lifting property for $C_1 \to C_2$ is induced form locally faithfulness of $f$. $lacksquare$

**Theorem 3.15.** The category of bicategories admits the following model structure:

1. The weak equivalences are weak 2-equivalences.
2. The class of cofibrations is equal to the one of $\mathcal{BiCat}_L$.
3. The fibrations are strict homomorphisms fibered, cofibered, opfibered, and coopfibered in bigroupoids.

Moreover, this model structure denoted by $\mathcal{BiCat}_2$ is cofibrantly generated with the set of generating cofibrations $I = I_L$ and the set of generating trivial cofibration $J = J' \cup J^{op} \cup J^{op} \cup J^{coop}$.

**Proof.** Let us apply Theorem 2.1.29 in [Hov99]. It remains to show that a relative $J$-cell complex is a weak 2-equivalence. This is shown by applying the functor $II S : \mathcal{BiCat}_s \to \mathcal{2Grd}$ for relative $J$-cell complexes. This is left adjoint to the canonical inclusion preserving colimits, and sending trivial cofibrations in $\mathcal{BiCat}_2$ to trivial cofibrations in $\mathcal{2Grd}$. It implies that $II S$ sends a relative $J$-cell complex to a biequivalence, and a relative $J$-cell complex is a weak 2-equivalence. We can apply Theorem 2.1.29 in [Hov99] by Proposition 3.14. $lacksquare$

**Lemma 3.16.** The canonical inclusion $\mathcal{Cat}_L \to \mathcal{BiCat}_L$ preserves cofibrations.

**Proof.** We need to clarify the relation between small categories and compositional graphs (see [Lac04]). A small category can be regarded as a compositional graph. On the other hand, for a compositional graph $X$, we can construct a
small category $TX$ with $TX_0 = X_0$ and $TX(x, y) = X(x, y)/\sim$ where $\sim$ is an
equivalent relation generated by $1_x f \sim f \sim f_1 y$ and $i(hg) \sim (ih)g$ for each mor-
phism $f$ and each composable triple $g, h, i$. Given a morphism of compositional
graph $C \to X$ from a small category $C$ to a compositional graph $X$, we have
the canonical functor $C \to TX$ of small categories with the universal property.
It states that the canonical forgetful functor $V : \text{Cat} \to \text{CGraph}$ is left adjoint
to $T : \text{CGraph} \to \text{Cat}$.
Suppose that $f : A \to B$ is a cofibration in $2\text{Cat}_L$. The functor on under-
lying category $Uf : UA \to UB$ has the lifting property for surjective functors.
Consider the following left-hand commutative diagram in $\text{CGraph}$
\[
\begin{array}{ccc}
VUA & \longrightarrow & X \\
\downarrow & & \downarrow \quad \text{id}_X \\
VUB & \longrightarrow & Y \\
\end{array}
\begin{array}{ccc}
UA & \longrightarrow & TX \\
\downarrow & & \downarrow \quad Tp \\
UB & \longrightarrow & TY \\
\end{array}
\]
it induces right-hand diagram in $\text{Cat}$. The functor $T$ sends surjective morphisms
to surjective functors since $TX$ associates to the natural surjective morphism $X \to TX$. If $p$ is a surjective morphism, there exists a fill-in $UB \to TX$ for
the right-hand diagram, and the adjointness induces a fill-in $VUB \to X$ for
the left-hand diagram. Hence $f$ is a cofibration in $\text{BiCat}_L$ by Proposition 7 in $[\text{Lac04}]$.

The following lemma is induced from the definition of biequivalences. The
pseudogroupoid version is used in $[\text{Lac02}]$.

**Lemma 3.17.** A bicategory which is biequivalent to a bigroupoid is itself a
bigroupoid.

**Proposition 3.18.** The homotopy function complex from $A$ to $B$ in $\text{BiCat}_L$ is
weakly equivalent to $\text{NR}_{ps}[S'A', SB]$ where $A'$ is a cofibrant replacement of $A$.
In particular, when $B$ is a bigroupoid, the homotopy function complex from $A$
to $B$ is weakly equivalent to $\text{N}[\Pi S'A', \Pi B]$.

**Proof.** We can take a cofibrant resolution of $A'$ as $S'A' \otimes W_{ps}(\Delta)$ since the canonical
inclusion $2\text{Cat}_L \to \text{BiCat}_L$ preserves weak equivalences and cofibrations
by Lemma 3.16. Note that the category $\text{BiCat}_s$ has strict homomorphisms as
morphisms, we have

$\text{BiCat}_s(SA' \otimes W_{ps}(\Delta), SB) = 2\text{Cat}(SA' \otimes W_{ps}(\Delta), SB)$.

Similarly to Proposition 3.16 it is weakly equivalent to $\text{NR}_{ps}[S'A', SB]$. Further-
more, $SB$ is a pseudogroupoid if $B$ is a bigroupoid. Therefore, it is weakly
equivalent to $\text{N}[\Pi S'A', \Pi B]$ when $B$ is a bigroupoid. \qed

**Definition 3.19.** Let $K'$ be the set of strict homomorphisms of bicategories
consisting of $k_1' : F\Sigma[0] \to \mathcal{H}$ and $k_2' : F\Sigma[1] \to F\Sigma S^\infty$. 

21
Proposition 3.20. A bicategory is $K'$-local if and only if it is a bigroupoid, and a strict homomorphism is $K'$-equivalence if and only if it is a weak 2-equivalence.

Proof. We have $SK' = K$. Lemma 3.17, Theorem 3.11, and Proposition 3.9 induce the result. □

Corollary 3.21. The 2-type model structure $\text{BiCat}_2$ on the category of bicategories is the left Bousfield localization of $\text{BiCat}_L$ with respect to $K'$.

Theorem 3.22. The pair of adjoint functors $(S, I)$ between $\text{BiCat}_2$ and $\text{2Cat}_2$ is a pair of Quillen equivalences.

Proof. Note that $LS(K') = S(K') = K$ since each domain and codomain of $k'_1$ and $k'_2$ is cofibrant in $\text{BiCat}_L$. The result follows from Theorem 3.3.20 in [Hir03]. □

4 The Euler characteristic of bicategories

The Euler characteristic of spaces is well-known as a classical topological invariant. Thereafter, it is defined not only for such geometric objects, but also for combinatorial and categorical objects [Rot64], [BD01], [Lei08], [Lei13], [NT]. We define the Euler characteristic of bicategories and observe the product formula of the Euler characteristic of bicategories for fibrations in the 2-type model structure, through the Grothendieck construction.

4.1 The Euler characteristic of small categories

Let us recall the definition and properties of the Euler characteristic of finite categories. See [Lei08] for more details.

Definition 4.1. For finite sets $I$ and $J$, an $I \times J$ matrix is a function $I \times J \to \mathbb{Q}$. For an $I \times J$ matrix $\zeta$ and a $J \times H$ matrix $\eta$, the $I \times H$ matrix $\zeta \eta$ is defined by $\zeta \eta(i, h) = \sum_j \zeta(i, j) \eta(j, h)$ for each $i \in I$ and $h \in H$. An $I \times J$ matrix $\zeta$ has a $J \times I$ transpose $\zeta^\text{op}$. Given a finite set $I$, we write $u_I : I \to \mathbb{Q}$ (or simply $u$) for the column vector with $u_I(i) = 1$ for all $i$ in $I$. Let $\zeta$ be an $I \times J$ matrix. A weighting on $\zeta$ is a column vector $k^* : J \to \mathbb{Q}$ such that $\zeta^* = u_I$. A coweighting on $\zeta$ is a row vector $k_* : I \to \mathbb{Q}$ such that $k_* \zeta = u_I^\text{op}$. The matrix $\zeta$ admits Euler characteristic if it has a weighting and a coweighting. Then, its Euler characteristic is defined as

$$|\zeta| = \sum_j k_j^* = \sum_i k_i \in \mathbb{Q}.$$  

Note that this definition does not depend on the choice of a weighting and a coweighting.

Definition 4.2. For a finite category $A$, the similarity matrix $\zeta_A$ of $A$ is defined by $\zeta_A : A_0 \times A_0 \to \mathbb{Q}$ such that $\zeta_A(i, j)$ is the cardinal of the set $A(i, j)$ of morphisms. We say that $A$ admits Euler characteristic if $\zeta_A$ does, and then define the Euler characteristic $\chi(A)$ as $|\zeta_A|$.  

22
We introduce some important properties of the Euler characteristic of small categories, namely, stability for equivalence, relation with the classifying space, and the product formula. Leinster proved these in [Lei08].

**Theorem 4.3.** If \( f : A \to B \) is an equivalence of finite categories, then \( A \) admits Euler characteristic if and only if \( B \) does, and in that case, \( \chi(A) = \chi(B) \).

**Theorem 4.4.** If \( A \) is a finite acyclic category, then \( \chi(BA) = \chi(A) \) where \( BA \) is the classifying space of \( A \).

**Theorem 4.5.** Let \( A \) be a finite category, and let \( X : A \to \text{Cat} \) be a lax functor valued in finite categories. Let \( k^* \) be a weighting on \( \zeta_A \), and suppose the Grothendieck construction \( \text{Gr}(X) \) and each \( X_a \) admit Euler characteristics. Then

\[
\chi(\text{Gr}(X)) = \sum_{a \in A_0} k^a \chi(X_a).
\]

For the topological Euler characteristic, it is well-known that the product formula for fibrations; \( \chi(E) = \chi(B)\chi(F) \) for a fibration \( E \to B \) over connected base \( B \) with fiber \( F \).

**Corollary 4.6.** For a fibered or cofibered category \( E \to B \) between finite categories over a connected base \( B \) with fiber \( F \). Suppose that \( E, B \) and \( F \) admit Euler characteristics, then we have \( \chi(E) = \chi(B)\chi(F) \).

**Proof.** Let \( p : E \to B \) be a fibered category between finite categories. The total category \( E \) is equivalent to the Grothendieck construction \( \text{Gr}(p^*) \) of the fiber functor \( p^* \) of \( p \) given by \( p^*b = p^{-1}(b) \). If \( B \) is connected, any two fibers \( p^*b \) and \( p^*b' \) are equivalent to each other. By applying Theorem 4.5 we obtain

\[
\chi(E) = \chi(\text{Gr}(p^*)) = \sum_{b \in B_0} k^b \chi(p^*b) = \sum_{b \in B_0} k^b \chi(F) = \chi(B)\chi(F).
\]

**Corollary 4.7.** For a fibered or cofibered category \( E \to B \) between finite categories over \( B \) having the connected components \( B_i \) with fiber \( F_i \) over an object in \( B_i \). Suppose that \( E \) and each \( B_i \) and \( F_i \) admit Euler characteristics, then we have \( \chi(E) = \sum_i \chi(B_i)\chi(F_i) \).

Recall that fibrations of the 1-type model structure on the category of small categories [Tan13]. They are categories fibered and cofibered in groupoids as the fibrations. Hence, these fibrations satisfy the product formula of Euler characteristics.

**Corollary 4.8.** For a fibration \( E \to B \) in \( \text{Cat}_1 \) between finite bicategories over \( B \) having the connected components \( B_i \) with fiber \( F_i \) over an object in \( B_i \). Suppose that \( E \) and each \( B_i \) and \( F_i \) admit Euler characteristics, then we have \( \chi(E) = \sum_i \chi(B_i)\chi(F_i) \).
4.2 The Euler characteristic of 2-categories and bicategories

The author and coauthor extended the definition of Euler characteristic of small categories to enriched categories [NT]. It includes 2-categories when we regard them as categories enriched by small categories. We can defined the Euler characteristic of bicategories naturally. We notice that we do not need the data of composition to define that.

**Definition 4.9.** Let $\mathcal{A}$ be a finite $\text{Cat}$-graph, i.e. each set of objects, 1-morphisms, and 2-morphisms is finite. We call that $\mathcal{A}$ admits Euler characteristic if every category of morphism $\mathcal{A}(x, y)$ admits Euler characteristic for $x, y \in \mathcal{A}_0$, and also the similarity matrix $\zeta_{\mathcal{A}}: \mathcal{A}_0 \times \mathcal{A}_0 \to \mathbb{Q}$ given by $\zeta_{\mathcal{A}}(x, y) = \chi(\mathcal{A}(x, y))$ admits Euler characteristic. Then, we define the Euler characteristic $\chi(\mathcal{A})$ as $|\zeta_{\mathcal{A}}|$. If $\mathcal{A}$ is a finite bicategory, we say that $\mathcal{A}$ admits Euler characteristic if its underlying $\text{Cat}$-graph does, and then we define $\chi(\mathcal{A})$ as that of the underlying $\text{Cat}$-graph.

Let us observe the properties of the above Euler characteristic the we have seen in Section 4.1. Some of these properties have been shown for 2-categories in [NT].

**Theorem 4.10.** If a homomorphism $f: \mathcal{A} \to \mathcal{B}$ is a biequivalence of finite bicategories, then $\mathcal{A}$ admits Euler characteristic if and only if $\mathcal{B}$ does, and in that case, $\chi(\mathcal{A}) = \chi(\mathcal{B})$.

**Proof.** A homomorphism $f: \mathcal{A} \to \mathcal{B}$ induces a functor on categories of connected component $\pi_0 f: \pi_0 \mathcal{A} \to \pi_0 \mathcal{B}$. Here, $\pi_0 \mathcal{A}$ is given as $(\pi_0 \mathcal{A})_0 = \mathcal{A}_0$ and $(\pi_0 \mathcal{A})(a, b) = \pi_0(\mathcal{A}(a, b))$. Note that $\pi_0(\mathcal{A}(a, b))$ consists of equivalence classes of 1-morphisms with respect to 2-morphisms including associators and unitors, thus $\pi_0(\mathcal{A})$ satisfies the associativity and unit condition, furthermore, $\pi_0 f$ preserves composition and unit strictly. If $f$ is a biequivalence, $\pi_0 f$ is an equivalence of categories. We can show the result similarly to Proposition 3.10 and Theorem 3.11 in [NT] for 2-category version.

**Definition 4.11.** A $\text{Cat}$-graph $\mathcal{A}$ is called acyclic when $\mathcal{A}(x, y) = \phi$ if $\mathcal{A}(y, x) \neq \phi$ for each $x, y \in \mathcal{A}_0$, and $\mathcal{A}(z, z) = \ast$ for each $z \in \mathcal{A}_0$. Moreover, $\mathcal{A}$ is called biacyclic if $\mathcal{A}$ is acyclic and each $\mathcal{A}(x, y)$ is also acyclic. A bicategory is called acyclic (resp. biacyclic) if its underlying $\text{Cat}$-graph is acyclic (resp. biacyclic).

**Theorem 4.12.** If $\mathcal{A}$ is a finite biacyclic bicategory, then $\chi(B\mathcal{A}) = \chi(\mathcal{A})$.

**Proof.** Recall the strictification $S\mathcal{A}$ of $\mathcal{A}$. It has the same objects as $\mathcal{A}$, sequences of 1-morphisms of $\mathcal{A}$ as 1-morphisms, and sequences of 2-morphisms of $\mathcal{A}$ as 2-morphisms. If $\mathcal{A}$ is finite biacyclic, so is $S\mathcal{A}$. These are biequivalent to each other, hence their similarity matrices are equal. It allows to assume $\mathcal{A}$ to be a 2-category. The classifying space $B\mathcal{A}$ is homotopy equivalent to the classifying space $BT\mathcal{A}$ of the topological category $T\mathcal{A}$ consisting of $(T\mathcal{A})_0 = \mathcal{A}_0$ and $(T\mathcal{A})(x, y) = BA(x, y)$ [CCG10]. Since $\mathcal{A}(x, y)$ is a finite acyclic category, the
classifying space $BA(x, y)$ is a finite cell complex, and $\chi(A(x, y)) = \chi(BA(x, y))$ for each $x, y \in A_0$ from Theorem 4.4. By applying Theorem 4.13 in [NT], we obtain the result.

4.3 The product formula of Euler characteristic for fibrations

In the final part of this paper, we focus on the product formula with respect to the Euler characteristic of bicategories for fibrations in $\text{BiCat}_2$. We have already seen the case of small categories and fibrations in the 1-type model structure by using Grothendieck construction in Section 4.1. Let us recall the Grothendieck construction of bicategories according to the paper [Buc14].

First, we need the notions of tricategories and trihomomorphisms which are weaker notions of 3-categories and 2-functors respectively. Although both are required the complicated higher associativity and unit condition, we do not describe that here. See [Buc14] for the details.

**Definition 4.13.** Let $\mathcal{A}$ be a bicategory and $\text{BiCat}_{tri}$ be the tricategory consisting of bicategories, homomorphisms, lax natural transformations, and modifications. A trihomomorphism $X : \mathcal{A} \to \text{BiCat}_{tri}$ consists of the following data:

1. a bicategory $Xa$ for each $a \in \mathcal{A}_0$,

2. a pseudo functor $\mathcal{A}(a, b) \to \text{BiCat}_{tri}(Xa, Xb)$ for each $a, b \in \mathcal{A}_0$, satisfying the coherent conditions.

The Grothendieck construction is defined as a bicategory for a trihomomorphism $X : \mathcal{A} \to \text{BiCat}_{tri}$ (resp. $\mathcal{A}^{op}$ or $\mathcal{A}^{co}$ or $\mathcal{A}^{coop}$). Here, we describe only its underlying $\text{Cat}$-graph since we do not need the data related to composition in order to calculate the Euler characteristic.

**Definition 4.14.** Let $X : \mathcal{A} \to \text{BiCat}_{tri}$ be a trihomomorphism. The Grothendieck construction $\text{Gr}(X)$ is a bicategory consisting of

1. $\text{Gr}(X)_0 = \{(a, x) \mid a \in \mathcal{A}_0, x \in (Xa)_0\}$,

2. $\text{Gr}(X)((a, x)(b, y)) = \{(f, w) \mid f \in \mathcal{A}(a, b)_0, w \in Xb(Xfx, y)\}$,

3. $\text{Gr}(X)((a, x)(b, y))(f, w)(g, u) = \{(\alpha, \beta) \mid \alpha \in \mathcal{A}(a, b)(f, g), \beta \in Xb(Xfx, y)(u \circ (X\alpha)_x, w)\}$,

with the canonical composition.

**Theorem 4.15.** Let $\mathcal{A}$ be a finite bicategory and $X : \mathcal{A} \to \text{BiCat}_{tri}$ be a trihomomorphism valued in finite bicategories. Let each category of morphisms of $\mathcal{A}$, $\text{Gr}(X)$, and $Xa$ admit Euler characteristics for $a \in \mathcal{A}_0$. If we have weightings on $\zeta_{\mathcal{A}}$ and each $\zeta_{Xa}$ all written $k^a$, then there exists a weighting on $\text{Gr}(X)$ defined by $k^{(a, x)} = k^a k^x$.  

25
Proof. A weighting $k^{(f,w)}$ on the category of morphisms $\text{Gr}(X)((a,x),(b,y))$ can be taken as $k_f k^w$, where $k_f$ is a coweighting and $k^w$ is a weighting for $f \in A(a,b)_0$ and $w \in Xb(Xfx,y)_0$ since
\[
\sum_{(f,w)} \zeta_{\text{Gr}(X)((a,x),(b,y))}((f,w)(g,u)) k_f k^w
= \sum_f \sum_{\alpha \in A(a,b)(f,g)} \sum_w \zeta_{Xb(Xfx,y)}(u \circ (X\alpha)_x, w) k^w k_f
= \sum_f k_f \zeta_{A(a,b)}(f,g) = 1.
\]
The following calculation shows the result;
\[
\sum_{(b,y)} \zeta_{\text{Gr}(X)((a,x),(b,y))} k^b k^y
= \sum_{(b,y)} \chi(\text{Gr}(X)((a,x),(b,y))) k^b k^y
= \sum_{(b,y)} \sum_{(f,w)} k^{(f,w)} k^b k^y
= \sum_{(b,y)} \sum_{f} k_f \chi(Xb(Xfx,y)) k^y k^b
= \sum_{b} \sum_{f} k_f \sum_{y \in (Fb)_0} \chi(Xb(Xfx,y)) k^y k^b
= \sum_{b} \sum_{f} k_f k^b
= \sum_{b} \chi(A(a,b)) k^b
= \sum_{b} \zeta_{A(a,b)} k^b = 1.
\]

\[\square\]

Corollary 4.16. Let $A$ be a finite bicategory and $X : A \to \text{BiCat}_{\text{tri}}$ be a trihomomorphism valued in finite bicategories. Suppose $\text{Gr}(X)$ and each $Xa$ and $A(x,y)$ admit Euler characteristics. If we have a weighting $k^*$ on $\zeta_A$, then we have
\[
\chi(\text{Gr}(X)) = \sum_{a \in A_0} k^a \chi(Xa).
\]

M. Buckley proved the relation between fibered bicategories (or simply called fibration in his paper) and the trihomomorphisms through the Grothendieck construction in [Buc14]. We notice immediately that our fibration in the 2-type model structure is a special case of fibered bicategories.

Theorem 4.17 (Proposition 3.3.11 in [Buc14]). For a fibered bicategory $p : E \to B$, the Grothendieck construction $\text{Gr}(p^*)$ of the fiber trihomomorphism $p^* : B \to \text{BiCat}_{\text{tri}}$ is biequivalent to $E$. 

26
Lemma 4.18. A strict homomorphism fibered in bigroupoids is a fibered bicategory in \([Buc14]\).

Proof. Let \( p : \mathcal{E} \to \mathcal{B} \) be fibered in bigroupoids. The lifting properties of \( p \) shows that every 1-morphism and 2-morphism in \( \mathcal{E} \) is cartesian, and \( p \) is locally fibered. Moreover, \( p \) lifts each 1-morphism \( b \to f(e) \) in \( \mathcal{B} \) to a 1-morphism \( e' \to e \) in \( \mathcal{E} \). Hence, \( p \) is a fibered bicategory. \( \square \)

Let us clarify what the fiber trihomomorphism of a fibration in \( \text{BiCat}_2 \) is.

For a fibration \( p : \mathcal{E} \to \mathcal{B} \) in \( \text{BiCat}_2 \), a trihomomorphism \( p^* : \mathcal{B} \to \text{BiCat}_{tri} \) is given by \( p^*b = \mathcal{E}_b \) for \( b \in \mathcal{B}_0 \), which is a bicategory consisting of \( p^{-1}(b) \) as objects, \( p^{-1}(1_b) \) as 1-morphisms, and \( p^{-1}(1_11_b) \) as 2-morphisms. This is not a subbicategory of \( \mathcal{E} \), since the composition is different from \( \mathcal{E} \). For composable pair \( f, g \) of 1-morphisms in \( \mathcal{E}_b \), the composition \( g \circ f \) is defined as the domain of the lift \( g \circ f \cong gf \) over \( 1_b \cong 1_b1_b \) in \( \mathcal{B} \). For a 1-morphism \( f : b \to b' \) in \( \mathcal{B} \) induces a homomorphism \( \mathcal{E}_b \to \mathcal{E}_{b'} \) by taking the lift of \( f \) stopping at the domain. Conversely, by taking the lift of \( f \) stopping at the codomain, we obtain a reverse direction homomorphism \( \mathcal{E}_{b'} \to \mathcal{E}_b \). These are a pair of biequivalences, and induce the following lemma.

Lemma 4.19. Let \( p : \mathcal{E} \to \mathcal{B} \) be a fibration in \( \text{BiCat}_2 \). Suppose that there exists a 1-morphism between \( b \) and \( b' \) in \( \mathcal{B} \), then \( \mathcal{E}_b \) and \( \mathcal{E}_{b'} \) are biequivalent to each other.

Theorem 4.20. Let \( p : \mathcal{E} \to \mathcal{B} \) be a fibration in \( \text{BiCat}_2 \) between finite bicategories over a connected base \( \mathcal{B} \). Suppose that \( \mathcal{E} \), \( \mathcal{B} \) and the fiber \( \mathcal{G} \) of \( p \) admit Euler characteristics. Then we have \( \chi(\mathcal{E}) = \chi(\mathcal{B})\chi(\mathcal{G}) \).

Proof. Note that \( \chi(\mathcal{E}_b) = \chi(\mathcal{E}_{b'}) = \chi(\mathcal{G}) \) for any \( b, b' \in \mathcal{B}_0 \) if \( \mathcal{B} \) is connected. Theorem 4.17 and Corollary 4.16 follows the result. \( \square \)

Corollary 4.21. Let \( p : \mathcal{E} \to \mathcal{B} \) be a fibration in \( \text{BiCat}_2 \) between finite bicategories over \( \mathcal{B} \) having the connected components \( \mathcal{B}_i \) with fiber \( \mathcal{G}_i \) over an object in \( \mathcal{B}_i \). Suppose that \( \mathcal{E} \) and each \( \mathcal{B}_i \) and \( \mathcal{G}_i \) admit Euler characteristics, then we have \( \chi(\mathcal{E}) = \sum_i \chi(\mathcal{B}_i)\chi(\mathcal{G}_i) \).

Note that a fibration in \( \text{Cat}_2 \) is not a fibered 2-category in the sense of Buckley’s \([\text{Buc14}]\), generally. Hence, a fibration \( \mathcal{E} \to \mathcal{B} \) in \( \text{2Cat}_2 \) gives the fiber functor as a trihomomorphism from \( \mathcal{B} \) to the tricategory of 2-categories, not a 2-functor to the 2-category of 2-categories. Moreover, the Grothendieck construction of the fiber functor associated to a fibration in \( \text{2Cat}_2 \) is a bicategory, not a 2-category.

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