A LOCAL DISCONTINUOUS GALERKIN APPROXIMATION FOR THE $p$-NAVIER–STOKES SYSTEM, PART III: CONVERGENCE RATES FOR THE PRESSURE

ALEX KALTENBACH† AND MICHAEL RŮŽIČKA‡

Abstract. In the present paper, we prove convergence rates for the pressure of the Local Discontinuous Galerkin (LDG) approximation, proposed in Part I of the paper (cf. [20]), of systems of $p$-Navier–Stokes type and $p$-Stokes type with $p \in (2, \infty)$. The results are supported by numerical experiments.

Key words. discontinuous Galerkin, $p$-Navier–Stokes system, error bounds, pressure

MSC codes. 76A05, 35Q35, 65N30, 65N12, 65N15

1. Introduction. In this paper, we continue our study of the Local Discontinuous Galerkin (LDG) scheme, proposed in Part I of the paper (cf. [20]), of steady systems of $p$-Navier–Stokes type. In this paper, as we already did in Part II of the paper (cf. [21]), we restrict ourselves to the homogeneous problem, i.e.,

$$
\begin{align*}
-\text{div} \mathbf{S}(Dv) + [\nabla v]v + \nabla q &= g \\
\text{div} v &= 0 \\
\text{div} v &= 0 \\
v &= 0 \\
\end{align*}
$$

in $\Omega$, \quad in $\partial\Omega$.

This system describes the steady motion of a homogeneous, incompressible fluid with shear-dependent viscosity. More precisely, for a given vector field $\mathbf{g} : \Omega \rightarrow \mathbb{R}^d$ describing external body forces and a homogeneous Dirichlet boundary condition (1.1) 3, we seek for a velocity vector field $v = (v_1, \ldots, v_d)^T : \Omega \rightarrow \mathbb{R}^d$ and a scalar kinematic pressure $q : \Omega \rightarrow \mathbb{R}$ solving (1.1). Here, $\Omega \subseteq \mathbb{R}^d$, $d \in \{2, 3\}$, is a bounded polyhedral domain having a Lipschitz continuous boundary $\partial\Omega$. The extra stress tensor $\mathbf{S}(Dv) : \Omega \rightarrow \mathbb{R}^{d \times d}_{\text{sym}}$ depends on the strain rate tensor $Dv := \frac{1}{2}(\nabla v + \nabla v^T) : \Omega \rightarrow \mathbb{R}^{d \times d}_{\text{sym}}$, i.e., the symmetric part of the velocity tensor $\mathbf{L} := \nabla v : \Omega \rightarrow \mathbb{R}^{d \times d}$. The convective term $[\nabla v]v : \Omega \rightarrow \mathbb{R}^d$ is defined via $([\nabla v]v)_i := \sum_{j=1}^d v_j \partial_j v_i$ for all $i = 1, \ldots, d$.

Throughout the paper, we assume that the extra stress tensor $\mathbf{S}$ has $(p, \delta)$-structure (cf. Assumption 2.1). The relevant example falling into this class is

$$
\mathbf{S}(Dv) = \mu (\delta + |Dv|)^{p-2}Dv,
$$

where $p \in (1, \infty)$, $\delta \geq 0$, and $\mu > 0$.

For a discussion of the model and the state of the art, we refer to Part I of the paper (cf. [20]). As already pointed out, to the best of the authors’ knowledge, there are no investigations using DG methods for the $p$-Navier–Stokes problem (1.1). In this paper, we continue the investigations of Part I and Part II of the paper (cf. [20, 21]), and prove convergence rates for the pressure of the homogeneous $p$-Navier–Stokes problem (1.1) under the assumption that the velocity and $\mathbf{g}$ satisfy natural regularity conditions and a smallness condition for the velocity in the energy norm. In doing so, we
restrict ourselves to the case $p \in (2, \infty)$. Our approach is inspired by the results in [15], [19], and [4]. The same results are obtained for the $p$-Stokes problem without the smallness condition. We would like to point out that there are no results in the literature proving convergence rates for the pressure for the $p$-Navier–Stokes equations (1.1) ($p \neq 2$) neither for DG methods nor for FE methods. Even in the case of the $p$-Stokes problem ($p \neq 2$) there is only one result for DG methods and some for FE methods (cf. Remark 4.7).

This paper is organized as follows: In Section 2, we introduce the employed notation, define relevant function spaces, basic assumptions on the extra stress tensor $\mathcal{S}$ and its consequences, the weak formulations Problem (Q) and Problem (P) of the system (1.1), and the discrete operators. In Section 3, we define our numerical fluxes and derive the flux and the primal formulation, i.e. Problem (Q$_h$) and Problem (P$_h$), of the system (1.1). In Section 4, we derive error estimates for our problem (cf. Theorem 4.1, Corollary 4.2). These are the first convergence rates for a DG-method for systems of $p$-Navier–Stokes type. In Section 5, we present numerical experiments.

2. Preliminaries.

2.1. Function spaces. We use the same notation as in Part I of the paper (cf. [20]). For the convenience of the reader, we repeat some of it.

We employ $c, C > 0$ to denote generic constants, that may change from line to line, but are not depending on the crucial quantities. For $k \in \mathbb{N}$ and $p \in [1, \infty]$, we employ the customary Lebesgue spaces $(L^p(\Omega), \|\cdot\|_p)$ and Sobolev spaces $(W^{k,p}(\Omega), \|\cdot\|_{k,p})$, where $\Omega \subseteq \mathbb{R}^d$, $d \in \{2, 3\}$, is a bounded, polyhedral Lipschitz domain. The space $W^{1,p}_0(\Omega)$ is defined as the space of functions from $W^{1,p}(\Omega)$ whose trace vanishes on $\partial \Omega$. We equip $W^{1,p}_0(\Omega)$ with the norm $\|\nabla \cdot \|_p$.

We do not distinguish between spaces for scalar, vector- or tensor-valued functions. However, we always denote vector-valued functions by boldface letters and tensor-valued functions by capital boldface letters. The mean value of a locally integrable function $f$ over a measurable set $M \subseteq \Omega$ is denoted by $(f)_M := \frac{1}{|M|} \int_M f \, dx$. Moreover, we employ the notation $(f, g) := \int_{\Omega} f g \, dx$, whenever the right-hand side is well-defined.

From the theory of Orlicz spaces (cf. [29]) and generalized Orlicz spaces (cf. [17]), we use N-functions $\psi : \mathbb{R}^{0+} \to \mathbb{R}^{0+}$ and generalized N-functions $\psi : \Omega \times \mathbb{R}^{0+} \to \mathbb{R}^{0+}$, i.e., $\psi$ is a Carathéodory function such that $\psi(x, \cdot)$ is an N-function for a.e. $x \in \Omega$, respectively. For $f \in L^0(\Omega)^1$, the modular is defined via $\rho_\psi(f) := \rho_{\psi, \Omega}(f) := \int_{\Omega} \psi(|f|) \, dx$ if $\psi$ is an N-function and $\rho_\psi(f) := \rho_{\psi, \Omega}(f) := \int_{\Omega} \psi^*(|f|) \, dx$ if $\psi$ is a generalized N-function. Then, for a (generalized) N-function $\psi$, we denote by $L^\psi(\Omega) := \{ f \in L^0(\Omega) \mid \rho_\psi(f) < \infty \}$, the (generalized) Orlicz space. Equipped with the induced Luxembourg norm, i.e., $\|f\|_\psi := \inf \{ \lambda > 0 \mid \rho_\psi(f/\lambda) \leq 1 \}$, the space (generalized) Orlicz space $L^\psi(\Omega)$ is a Banach space. If $\psi$ is a generalized N-function, then, for every $f \in L^\psi(\Omega)$ and $g \in L^{\psi^*}(\Omega)$, there holds the generalized Hölder inequality

$$ (f, g) \leq 2\|f\|_\psi\|g\|_{\psi^*}. \tag{2.1} $$

An N-function $\psi$ satisfies the $\Delta_2$-condition, if there exists $K > 2$ such that for all $t \geq 0$, it holds $\psi(2t) \leq K \psi(t)$. We denote the smallest such constant by $\Delta_2(\psi) > 0$. We need the following version of the $\varepsilon$-Young inequality: for every $\varepsilon > 0$, there exits a constant $c_\varepsilon > 0$, depending only on $\Delta_2(\psi), \Delta_2(\psi^*) < \infty$, such that for every $s, t \geq 0$, it holds

$$ ts \leq \varepsilon \psi(t) + c_\varepsilon \psi^*(s). \tag{2.2} $$

$^3$Here, $L^0(\Omega)$ denotes the set of Lebesgue measurable scalar function defined on $\Omega$.  

[25]
2.2. Basic properties of the extra stress tensor. Throughout the entire paper, we always assume that the extra stress tensor $S$ has $(p, \delta)$-structure, which is defined here in a more stringent way compared to Part I of the paper (cf. [20]). A detailed discussion and full proofs can be found, e.g., in [13, 26]. For a given tensor $\mathbf{A} \in \mathbb{R}^{d \times d}$, we denote its symmetric part by $\mathbf{A}^{\text{sym}} := \frac{1}{2}(\mathbf{A} + \mathbf{A}^\top) \in \mathbb{R}^{d \times d}_{\text{sym}} := \{ \mathbf{A} \in \mathbb{R}^{d \times d} \mid \mathbf{A} = \mathbf{A}^\top \}$.

For $p \in (1, \infty)$ and $\delta \geq 0$, we define a special $N$-function $\varphi := \varphi_{p,\delta} : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ by

$$\varphi(t) := \int_0^t \varphi'(s) \, ds, \quad \text{where } \varphi'(t) := (\delta + t)^{p-2} t, \quad \text{for all } t \geq 0. \quad (2.3)$$

The properties of $\varphi$ are discussed in detail in [13, 26, 20].

An important tool in our analysis play shifted $N$-functions $\{\psi_a\}_{a \geq 0}$, cf. [14, 26]. For a given $N$-function $\psi : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$, we define the family of shifted $N$-functions $\psi_a : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$, $a \geq 0$, via

$$\psi_a(t) := \int_0^t \psi'_a(s) \, ds, \quad \text{where } \psi'_a(t) := \psi'(a + t) \frac{t}{a + t}, \quad \text{for all } t \geq 0. \quad (2.4)$$

**Assumption 2.1** (Extra stress tensor). We assume that the extra stress tensor $S : \mathbb{R}^{d \times d} \to \mathbb{R}^{d \times d}_{\text{sym}}$ belongs to $C^0(\mathbb{R}^{d \times d}; \mathbb{R}^{d \times d}_{\text{sym}}) \cap C^1(\mathbb{R}^{d \times d}; \mathbb{R}^{d \times d}_{\text{sym}})$ and satisfies $S(A) = S(A^{\text{sym}})$ for all $A \in \mathbb{R}^{d \times d}$ and $S(0) = 0$. Moreover, we assume that the tensor $S = (S_{ij})_{i,j=1,\ldots,d}$ has $(p, \delta)$-structure, i.e., for some $p \in (1, \infty)$, $\delta \in [0, \infty)$, and the $N$-function $\varphi = \varphi_{p,\delta}$ (cf. (2.3)), there exist constants $C_0, C_1 > 0$ such that

$$\sum_{i,j,k,l=1}^d \partial_{kl} S_{ij}(A) B_{ij} B_{kl} \geq C_0 \frac{\varphi'(\|A^{\text{sym}}\|)}{\|A^{\text{sym}}\|} \|B^{\text{sym}}\|^2, \quad (2.5)$$

$$|\partial_{kl} S_{ij}(A)| \leq C_1 \frac{\varphi'(\|A^{\text{sym}}\|)}{\|A^{\text{sym}}\|} \quad (2.6)$$

are satisfied for all $A, B \in \mathbb{R}^{d \times d}$ with $A^{\text{sym}} \neq 0$ and all $i, j, k, l = 1, \ldots, d$. The constants $C_0, C_1$ and $p \in (1, \infty)$ are called the characteristics of $S$.

**Remark 2.2.** (i) It is well-known (cf. [26]) that the conditions (2.5), (2.6) imply the conditions in the definition of the $(p, \delta)$-structure in Part I of the paper (cf. [20]).

(ii) Assume that $S$ satisfies Assumption 2.1 for some $\delta \in [0, \delta_0]$. Then, if not otherwise stated, the constants in the estimates depend only on the characteristics of $S$ and on $\delta_0 \geq 0$, but are independent of $\delta \geq 0$.

(iii) Let $\varphi$ and $\{\varphi_a\}_{a \geq 0}$ be defined in (2.3) and (2.4), respectively. Then, the shifted operators $S_a : \mathbb{R}^{d \times d} \to \mathbb{R}^{d \times d}_{\text{sym}}$, $a \geq 0$, defined, for every $a \geq 0$ and $A \in \mathbb{R}^{d \times d}$, via

$$S_a(A) := \frac{\varphi_a(\|A^{\text{sym}}\|)}{\|A^{\text{sym}}\|} A^{\text{sym}}, \quad (2.7)$$

have $(p, \delta + a)$-structure. In this case, the characteristics of $S_a$ depend only on $p \in (1, \infty)$ and are independent of $\delta \geq 0$ and $a \geq 0$.

Closely related to the extra stress tensor $S$ with $(p, \delta)$-structure is the non-linear function $F : \mathbb{R}^{d \times d} \to \mathbb{R}^{d \times d}_{\text{sym}}$, for every $A \in \mathbb{R}^{d \times d}$, defined via

$$F(A) := (\delta + \|A^{\text{sym}}\|) \frac{p-2}{p} A^{\text{sym}}. \quad (2.8)$$

The connections between $S, F : \mathbb{R}^{d \times d} \to \mathbb{R}^{d \times d}_{\text{sym}}$ and $\varphi_a, (\varphi_a)^* : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$, $a \geq 0$, are best explained by the following result (cf. [13, 26, 15]).
The constants in (2.9) depend only on the characteristics of \( \mathcal{S} \).

Remark 2.4. For the operators \( \mathcal{S}_a : \mathbb{R}^{d \times d} \to \mathbb{R}^{d \times d} \), \( a \geq 0 \), defined in (2.7), the assertions of Proposition 2.3 hold with \( \varphi : \mathbb{R}^0 \to \mathbb{R}^0 \) replaced by \( \varphi_a : \mathbb{R}^0 \to \mathbb{R}^0 \), \( a \geq 0 \).

The following results can be found in [14, 26].

Lemma 2.5 (Change of Shift). Let \( \varphi \) be defined in (2.3) and let \( F \) be defined in (2.8). Then, for each \( \varepsilon > 0 \), there exists \( c_\varepsilon \geq 1 \) (depending only on \( \varepsilon > 0 \) and the characteristics of \( \varphi \)) such that for every \( A, B \in \mathbb{R}^{d \times d} \) and \( t \geq 0 \), it holds

\[
\varphi_{[A]}(t) \leq c_\varepsilon \varphi_{[A]}(t) + \varepsilon |F(B) - F(A)|^2,
\]

\[
\varphi_{[A]}(t) \leq c_\varepsilon \varphi_{[A]}(t) + \varepsilon |F(A)|^2.
\]

2.3. The p-Navier–Stokes system. Let us briefly recall some well-known facts about the p-Navier–Stokes system (1.1). For \( p \in (1, \infty) \), we define the function spaces

\[
\hat{V} := (W_0^1(p)(\Omega))^d, \quad \hat{Q} := L_0^p(\Omega) := \{ f \in L^p(\Omega) \mid \langle f \rangle_\Omega = 0 \}.
\]

With this particular notation, the weak formulation of problem (1.1) is the following:

Problem (Q). For given \( g \in L^p(\Omega) \), find \( (v, q) \in \hat{V} \times \hat{Q} \) such that for all \( (z, z^T) \in \hat{V} \times \hat{Q} \), it holds

\[
(\mathcal{S}(Dv), Dz) + (|\nabla v| v, z) - (q, \text{div } z) = (g, z),
\]

(2.10)

\[
\text{(div } v, z) = 0.
\]

(2.11)

Alternatively, we can reformulate Problem (Q) “hidding” the pressure.

Problem (P). For given \( g \in L^p(\Omega) \), find \( v \in \hat{V}(0) \) such that for all \( z \in \hat{V}(0) \), it holds

\[
(\mathcal{S}(Dv), Dz) + (|\nabla v| v, z) = (g, z),
\]

(2.12)

where \( \hat{V}(0) := \{ z \in \hat{V} \mid \text{div } z = 0 \} \).

The theory of pseudo-monotone operators yields the existence of a weak solution of Problem (P) for \( p > \frac{3d}{d+2} \) (cf. [22]). DeRham’s lemma, the solvability of the divergence equation, and the negative norm theorem, then, ensure the solvability of Problem (Q).

2.4. DG spaces, jumps and averages.

2.4.1. Triangulations. We always denote by \( \mathcal{T}_h \), \( h > 0 \), a family of uniformly shape regular and conforming triangulations of \( \Omega \subseteq \mathbb{R}^d \), \( d \in \{2, 3\} \), cf. [7], each consisting of \( d \) dimensional simplices \( K \). The parameter \( h > 0 \), refers to the maximal mesh-size of \( \mathcal{T}_h \), for which we assume for simplicity that \( h \leq 1 \). Moreover, we assume that the chunkiness is bounded by some constant \( \omega_0 > 0 \), independent on \( h \). By \( \Gamma_h^1 \), we denote the interior faces,
and put \( \Gamma_h := \Gamma_h^i \cup \partial \Omega \). We assume that each simplex \( K \in \mathcal{T}_h \) has at most one face from \( \partial \Omega \). We introduce the following scalar product on \( \Gamma_h \)

\[
\langle f, g \rangle_{\Gamma_h} := \sum_{\gamma \in \Gamma_h} \langle f, g \rangle_{\gamma}, \quad \text{where} \quad \langle f, g \rangle_{\gamma} := \int_{\gamma} fg \, ds \quad \text{for all} \quad \gamma \in \Gamma_h,
\]

if all the integrals are well-defined. Similarly, we define the products \( \langle \cdot, \cdot \rangle_{\partial \Omega} \) and \( \langle \cdot, \cdot \rangle_{\Gamma_h} \). We extend the notation of modulars to the sets \( \Gamma_h^i, \partial \Omega, \) and \( \Gamma_h \), i.e., we define the modulars \( \rho_{\nu \cdot f}(f) := \int_B \psi(|f|) \, ds \) for every \( f \in L^p(B) \), where \( B = \Gamma_h^i \) or \( B = \partial \Omega \) or \( B = \Gamma_h \).

### 2.4.2. Broken function spaces and projectors

For every \( m \in \mathbb{N}_0 \) and \( K \in \mathcal{T}_h \), we denote by \( \mathcal{P}_m(K) \), the space of polynomials of degree at most \( m \) on \( K \). Then, for given \( k \in \mathbb{N}_0 \) and \( p \in (1, \infty) \), we define the spaces

\[
Q^k_h := \{ q_h \in L^1(\Omega) \mid q_h|_K \in \mathcal{P}_k(K) \text{ for all } K \in \mathcal{T}_h \},
\]

\[
V^k_h := \{ v_h \in L^1(\Omega)^d \mid v_h|_K \in \mathcal{P}_k(K)^d \text{ for all } K \in \mathcal{T}_h \},
\]

\[
X^k_h := \{ X_h \in L^1(\Omega)^{d\times d} \mid X_h|_K \in \mathcal{P}_k(K)^{d\times d} \text{ for all } K \in \mathcal{T}_h \},
\]

\[
W^{1,p}(\mathcal{T}_h) := \{ w_h \in L^1(\Omega)^d \mid w_h|_K \in W^{1,p}(K)^d \text{ for all } K \in \mathcal{T}_h \}.
\]

In addition, for given \( k \in \mathbb{N}_0 \), we set \( Q^k_T := Q^k_h \cap C^0(\Omega) \). Note that \( W^{1,p}(\Omega) \subseteq W^{1,p}(\mathcal{T}_h) \) and \( V^k_T \subseteq W^{1,p}(\mathcal{T}_h) \). We denote by \( \Pi^k_h : L^2(\Omega) \rightarrow V^k_h \) the (local) \( L^2 \)-projection into \( V^k_h \), which for every \( v \in L^2(\Omega) \) and \( z_h \in V^k_h \) is defined via \( (\Pi^k_h v, z_h) = (v, z_h) \). Analogously, we define the (local) \( L^2 \)-projection into \( X^k_h \), i.e., \( \Pi^k_h : L^2(\Omega)^{d\times d} \rightarrow X^k_h \).

For every \( w_h \in W^{1,p}(\mathcal{T}_h) \), we denote by \( \nabla_h w_h \in \mathcal{L}(\Omega) \), the local gradient, defined via \( (\nabla_h w_h)|_K := \nabla(w_h|_K) \) for all \( K \in \mathcal{T}_h \). For every \( w_h \in W^{1,p}(\mathcal{T}_h) \) and interior faces \( \gamma \in \Gamma_h^i \) shared by adjacent elements \( K^-_\gamma, K^+_\gamma \in \mathcal{T}_h \), we denote by

\[
\{ w_h \}_\gamma := \frac{1}{2}(\text{tr}^+_\gamma(w_h) + \text{tr}^-_\gamma(w_h)) \in L^p(\gamma),
\]

\[
\lbrack w_h \otimes n \rbrack_\gamma := \text{tr}^+_\gamma(w_h) \otimes n^+_\gamma + \text{tr}^-_\gamma(w_h) \otimes n^-_\gamma \in L^p(\gamma),
\]

the average and normal jump, resp., of \( w_h \) on \( \gamma \). Moreover, for boundary faces \( \gamma \in \partial \Omega \), we define boundary averages and boundary jumps, resp., via

\[
\{ w_h \}_\gamma := \text{tr}^+_\gamma(w_h) \in L^p(\gamma),
\]

\[
\lbrack w_h \otimes n \rbrack_\gamma := \text{tr}^+_\gamma(w_h) \otimes n \in L^p(\gamma),
\]

where \( n : \partial \Omega \rightarrow \mathbb{R}^{d-1} \) denotes the unit normal vector field to \( \Omega \) pointing outward. Analogously, we define \( \{ X_h \}_\gamma \) and \( \lbrack X_h \otimes n \rbrack_\gamma \) for all \( X_h \in X^k_h \) and \( \gamma \in \Gamma_h \). Furthermore, if there is no danger of confusion, then we will omit the index \( \gamma \in \Gamma_h \), in particular, when we interpret jumps and averages as global functions defined on whole \( \Gamma_h \).

### 2.4.3. DG gradient and jump operators

For every \( k \in \mathbb{N}_0 \) and face \( \gamma \in \Gamma_h \), we define the (local) jump operator \( \mathcal{R}^k_{h,\gamma} : W^{1,p}(\mathcal{T}_h) \rightarrow X^k_h \) for every \( w_h \in W^{1,p}(\mathcal{T}_h) \) (using Riesz representation) via \( (\mathcal{R}^k_{h,\gamma} w_h, X_h) := \langle \lbrack w_h \otimes n \rbrack_\gamma, \{ X_h \}_\gamma \rangle \) for all \( X_h \in X^k_h \). For every \( k \in \mathbb{N}_0 \), the (global) jump operator \( \mathcal{R}^k_h := \sum_{\gamma \in \Gamma_h} \mathcal{R}^k_{h,\gamma} : W^{1,p}(\mathcal{T}_h) \rightarrow X^k_h \) by definition, for every \( w_h \in W^{1,p}(\mathcal{T}_h) \) and \( X_h \in X^k_h \) satisfies

\[
(\mathcal{R}^k_h w_h, X_h) = \langle \lbrack w_h \otimes n \rbrack, \{ X_h \}_\gamma \rangle_{\Gamma_h}.
\]

Then, for every \( k \in \mathbb{N}_0 \), the DG gradient operator \( \mathcal{G}^k_h := \nabla_h - \mathcal{R}^k_h : W^{1,p}(\mathcal{T}_h) \rightarrow L^p(\Omega) \), for every \( w_h \in W^{1,p}(\mathcal{T}_h) \) and \( X_h \in X^k_h \) satisfies

\[
(\mathcal{G}^k_h w_h, X_h) = \langle \nabla_h w_h, X_h \rangle - \langle \lbrack w_h \otimes n \rbrack, \{ X_h \}_\gamma \rangle_{\Gamma_h}.
\]
Apart from that, for every \( w_h \in W^{1,p}(\mathcal{T}_h) \), we introduce the DG norm as
\[
\| w_h \|_{\nabla, p, h} := \| \nabla_h w_h \|_p + h^{\frac{1}{2}} \| h^{-1} [w_h \otimes n] \|_{p, \Gamma_h} .
\] (2.20)

There exists a constant \( c > 0 \) (cf. [15, (A.26)–(A.28)]) such that for every \( w_h \in W^{1,p}(\mathcal{T}_h) \), it holds
\[
c^{-1} \| w_h \|_{\nabla, p, h} \leq \| \mathcal{G}_h^k w_h \|_p + h^{\frac{1}{2}} \| h^{-1} [w_h \otimes n] \|_{p, \Gamma_h} \leq c \| w_h \|_{\nabla, p, h} .
\] (2.21)

The following result extends the embedding results for classical Sobolev spaces \( W^{1,p}(\Omega) \) and broken polynomial spaces \( V_h^p \) to DG Sobolev spaces \( W^{1,p}(\mathcal{T}_h) \).

**Proposition 2.6.** Let \( p, q \in [1, \infty) \) be such that \( W^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \). If \( p > q \), then we additionally assume that \( h \sim h_K \) uniformly with respect to \( K \in \mathcal{T}_h \). Then, there exists a constant \( c = c(p, q, \omega_0) > 0 \) such that for every \( w_h \in W^{1,p}(\mathcal{T}_h) \), it holds
\[
\| w_h \|_q \leq c \| w_h \|_{\nabla, p, h} ,
\] (2.22)
i.e., \( W^{1,p}(\mathcal{T}_h) \hookrightarrow L^q(\Omega) \).

**Proof.** Note that \( \Pi_h^p : L^1(\Omega) \to V_h^p \) satisfies [5, Assumption A.1] (with \( S_K \) replaced by \( K \) and \( r_0 = 0 \)). Therefore, proceeding (with some simplifications) as in the proof of [5, Proposition A.2], we deduce that there exists a constant \( c = c(p, q, \omega_0) > 0 \) such that for every \( w_h \in W^{1,p}(\mathcal{T}_h) \), it holds
\[
\| w_h - \Pi_h^p w_h \|_q \leq c h^{1+d \min\{0, \frac{1}{q} - \frac{1}{p}\}} \| \nabla_h w_h \|_p .
\] (2.23)

Using in (2.23) that \( 1+d \min\{0, \frac{1}{q} - \frac{1}{p}\} \geq 0 \), the discrete embedding [11, Theorem 5.3] for functions from \( V_h^p \), and the approximation properties of \( \Pi_h^p \) (cf. [15, Appendix A.1], [19, Corollary A.8, Corollary A.19]), we obtain
\[
\| w_h \|_q \leq \| w_h - \Pi_h^p w_h \|_q + \| \Pi_h^p w_h \|_q \\
\leq c \| \nabla_h w_h \|_p + c \| \Pi_h^p w_h \|_{p, \nabla,h} \\
\leq c \| \nabla_h w_h \|_p + c \| \Pi_h^p w_h - w_h \|_{p, \nabla,h} + c \| w_h \|_{p, \nabla,h} \\
\leq c \| w_h \|_{p, \nabla,h} .
\]

For an N-function \( \psi \), we define the pseudo-modular\(^2\) \( m_{\psi,h} : W^{1,\psi}(\mathcal{T}_h) \to \mathbb{R}^+ \) for every \( w_h \in W^{1,\psi}(\mathcal{T}_h) \) via
\[
m_{\psi,h}(w_h) := h \rho_{\psi,\Gamma_h} (h^{-1} [w_h \otimes n]) .
\] (2.24)

For \( \psi = \varphi_{p,0} \), we have that \( m_{\psi,h}(w_h) = h \| h^{-1} [w_h \otimes n] \|_{p, \Gamma_h} \) for all \( w_h \in W^{1,\psi}(\mathcal{T}_h) \).

**2.4.4. Symmetric DG gradient and symmetric jump operators.** For every \( w_h \in W^{1,p}(\mathcal{T}_h) \), we denote by \( D_h w_h := [\nabla_h w_h]_{\text{sym}} \in L^p(\Omega; \mathbb{R}^{d \times d}) \) the local symmetric gradient. In addition, for every \( k \in \mathbb{N}_0 \) and \( X_h^{k,\text{sym}} := X_h^k \cap L^p(\Omega; \mathbb{R}^{d \times d}) \), we define the symmetric DG gradient operator \( \mathcal{D}_h^k : W^{1,p}(\mathcal{T}_h) \to L^p(\Omega; \mathbb{R}^{d \times d}) \), for every \( w_h \in W^{1,p}(\mathcal{T}_h) \), via \( \mathcal{D}_h^k w_h := [\mathcal{G}_h^k w_h]_{\text{sym}} \in L^p(\Omega; \mathbb{R}^{d \times d}) \), i.e., for every \( X_h \in X_h^{k,\text{sym}} \), we have that
\[
(\mathcal{D}_h^k w_h, X_h) = (D_h w_h, X_h) - (\| w_h \otimes n \|, X_h)_{\Gamma_h} .
\] (2.25)

\(^2\)The definition of an pseudo-modular can be found in [24]. We extend the notion of DG Sobolev spaces to DG Sobolev-Orlicz spaces \( W^{1,\psi}(\mathcal{T}_h) := \{ w_h \in L^1(\Omega) | w_h|_K \in W^{1,\psi}(K) \} \) for all \( K \in \mathcal{T}_h \).
Apart from that, for every \( w_h \in W^{1,p}(\mathcal{T}_h) \), we introduce the symmetric DG norm as
\[
\|w_h\|_{D,p,h} := \|D_h w_h\|_p + h^\frac p2 \|h^{-1}\|w_h \otimes n\|_{p,\Gamma_h} .
\] (2.26)

The following discrete Korn type inequalities play an important role in the numerical analysis of the \( p \)-Navier–Stokes system (1.1).

**Proposition 2.7** (Discrete Korn inequality). For every \( p \in (1, \infty) \) and \( k \in \mathbb{N} \), there exists a constant \( c_{\text{Korn}} > 0 \) such that for every \( v_h \in V^k_h \), it holds
\[
\|v_h\|_{V,p,h} \leq c_{\text{Korn}} \|v_h\|_{D,p,h} .
\] (2.27)

*Proof.* See [20, Proposition 2.4]. \( \square \)

**Proposition 2.8** (Korn type inequality). For every \( p \in (1, \infty) \) and \( k \in \mathbb{N} \), there exists a constant \( c > 0 \) such that for every \( v_h \in V^k_h \) and every \( w_h \in W^{2,p}(\mathcal{T}_h) \), it holds
\[
\|w_h - v_h\|_{V,p,h} \leq c \|w_h - v_h\|_{D,p,h} + c h^p \|\nabla w_h\|_p .
\] (2.28)

*Proof.* See [21, Proposition 2.8]. \( \square \)

For the symmetric DG norm, there holds a similar relation like (2.21).

**Proposition 2.9.** For every \( p \in (1, \infty) \) and \( k \in \mathbb{N} \), there exists a constant \( c > 0 \) such that for every \( w_h \in W^{1,p}(\mathcal{T}_h) \), it holds
\[
c^{-1} \|w_h\|_{D,p,h} \leq \|D_h^s w_h\|_p + h^\frac p2 \|h^{-1}\|w_h \otimes n\|_{p,\Gamma_h} \leq c \|w_h\|_{D,p,h} .
\] (2.29)

*Proof.* See [20, Proposition 2.5]. \( \square \)

**2.4.5. DG divergence operator.** For every \( w_h \in W^{1,p}(\mathcal{T}_h) \), we denote by \( \text{div}_h w_h := \text{tr}(\nabla_h w_h) \in L^p(\Omega) \), the local divergence. In addition, for every \( k \in \mathbb{N}_0 \), the DG divergence operator \( \text{Div}_h^k : W^{1,p}(\mathcal{T}_h) \to L^p(\Omega) \), for every \( w_h \in W^{1,p}(\mathcal{T}_h) \), is defined via \( \text{Div}_h^k w_h := \text{tr}(\mathcal{G}_h^k w_h) = \text{tr}(D_h^k w_h) \in Q^k_h \), i.e., for every \( z_h \in Q^k_h \), we have that
\[
\langle \text{Div}_h^k w_h, z_h \rangle = \langle \text{div}_h w_h, z_h \rangle = \langle \|w_h \cdot n\|, \{z_h\} \rangle_{\Gamma_h} .
\] Therefore, for every \( v \in W^{1,p}_0(\Omega) \) and \( z_h \in Q^k_h \), we have that
\[
\langle \text{Div}_h^k \Pi_h^k v, z_h \rangle = -\langle v, \nabla z_h \rangle = \langle \text{div} v, z_h \rangle .
\] (2.29)

**3. Fluxes and LDG formulations.** To obtain the LDG formulation of (1.1) for \( k \in \mathbb{N} \), we proceed as in Part I [20, Sec. 3] to get the discrete counterpart of Problem (Q). Recall that, restricting ourselves to the case that \( q_h \in Q^k_h \), the numerical fluxes are, for every stabilization parameter \( \alpha > 0 \), defined via
\[
\tilde{v}_{h,\sigma}(v_h) := \begin{cases} \{v_h\} & \text{on } \Gamma_h^i, \\ \{v^*\} & \text{on } \partial \Omega \end{cases}, \quad \tilde{v}_{h,\sigma}(v^*) := \begin{cases} \{v_h\} & \text{on } \Gamma_h^i, \\ \{v^*\} & \text{on } \partial \Omega \end{cases},
\] (3.1)
\[
\tilde{q}(q_h) := q_h \quad \text{on } \Gamma_h ,
\] (3.2)
\[
\tilde{S}(v_h, S_h, t_h) := \{S_h\} - \alpha S(\|\Pi_h^k \mathcal{D}_{t_h}^k v_h\|) \langle h^{-1}\|\mathcal{D}_{t_h}^k (v_h - v_0^*) \otimes n\| \rangle \quad \text{on } \Gamma_h^i ,
\] (3.3)
\[
\tilde{K}(v_h) := \{K_h\} \quad \text{on } \Gamma_h ,
\] (3.4)
\[
\tilde{G}(\Pi_h^k G) := \{\Pi_h^k G\} \quad \text{on } \Gamma_h ,
\] (3.5)
where the operator \( S(\|\Pi_h^k \mathcal{D}_{t_h}^k v_h\|) \) is defined as in (2.7). Thus we arrive at an inf-sup stable system without using a pressure stabilization. It is also possible to work
with a discontinuous pressure. In this case one has to modify the fluxes as follows: 
\( \hat{q}(q_h) := \{q_h\} \) on \( \Gamma_h \) and \( \hat{v}_{h,q}(v_h) := \{v_h\} + h\|q_h\|_n \) on \( \Gamma_h \), \( \hat{v}_{h,q}(v_h) = v^* \) on \( \partial \Omega \).

As in Part I of the paper (cf. [20]), we arrive at the flux formulation of (1.1), which reads: For given \( g \in L^p(\Omega) \), find \( (L_h, S_h, K_h, v_h, q_h)^\top \in X_h^k \times X_h^k \times X_h^k \times V_h^k \times Q_{h,c}^k \) such that for all \( (X_h, Y_h, Z_h, \zeta_h, \eta_h)^\top \in X_h^k \times X_h^k \times X_h^k \times V_h^k \times Q_{h,c}^k \), it holds

\[
(L_h, X_h) = (\mathcal{G}_h^k v_h, X_h), \\
(S_h, Y_h) = (\mathcal{S}(L_h^\text{sym}), Y_h), \\
(K_h, Z_h) = (v_h \otimes v_h, Z_h), \\
\left( D\partial v_h, \zeta_h \right) = 0. 
\]

Now, we eliminate in the system (3.6) the variables \( L_h \in X_h^k, S_h \in X_h^k \) and \( K_h \in X_h^k \) to derive a system only expressed in terms of the two variables \( v_h \in V_h^k \) and \( q_h \in Q_{h,c}^k \). To this end, we observe that it follows from (3.6) that \( L_h = \mathcal{G}_h^k v_h + \mathcal{R}_h^k v^* \), \( L_h^\text{sym} = D\partial v_h, S_h = \Pi_h^k \mathcal{S}(L_h^\text{sym}), K_h = \Pi_h^k (v_h \otimes v_h) \). If we insert this into (3.6)4, we get the discrete counterpart of Problem (Q):

**Problem (Q)**. For given \( g \in L^p(\Omega) \), find \( (v_h, q_h)^\top \in V_h^k \times Q_{h,c}^k \) such that for all \( (z_h, \eta_h)^\top \in V_h^k \times Q_{h,c}^k \), it holds

\[
\left( \mathcal{S}(D_h^k v_h) - \frac{1}{2} v_h \otimes v_h - q_h \text{Id}, D_h^k z_h \right) = \left( g - \frac{1}{2} \mathcal{G}_h^k v_h | v_h, z_h \right) - \alpha \left( \mathcal{S}([\Pi_h^k \mathcal{G}_h^k | v_h] (h^{-1}[v_h \otimes n]), [z_h \otimes n]) \right)_{\Gamma_h}, \\
\left( D\partial v_h, \eta_h \right) = 0. 
\]

Next, we eliminate in the system (3.7), the variable \( q_h \in Q_{h,c}^k \) to derive a system only expressed in terms of the single variable \( v_h \in V_h^k \). To this end, we introduce the space

\[
V_h(k) := \{v_h \in V_h^k \mid D\partial v_h, \eta_h = 0 \text{ for all } \eta_h \in Q_{h,c}^k \}. 
\]

Consequently, since \( (z_h \text{Id}, D_h^k z_h) = (z_h, D\partial v_h, \eta_h) = 0 \) for all \( z_h \in Q_{h,c}^k \) and \( v_h \in V_h^k(0) \), we get the discrete counterpart of Problem (P):

**Problem (P)**. For given \( g \in L^p(\Omega) \), find \( v_h \in V_h^k(0) \) such that for all \( z_h \in V_h^k(0) \), it holds

\[
\left( \mathcal{S}(D_h^k v_h) - \frac{1}{2} v_h \otimes v_h, D_h^k z_h \right) = \left( g - \frac{1}{2} \mathcal{G}_h^k v_h | v_h, z_h \right) - \alpha \left( \mathcal{S}([\Pi_h^k \mathcal{G}_h^k | v_h] (h^{-1}[v_h \otimes n]), [z_h \otimes n]) \right)_{\Gamma_h}. 
\]

Problem (Q) and Problem (P) are called primal formulations of the system (1.1).

Well-posedness (i.e., solvability), stability (i.e., a priori estimates), and (weak) convergence of Problem (Q) and Problem (P) are proved in Part I of the paper (cf. [20]).

4. Convergence rates for the pressure. Let us start with the main result of this paper:

**Theorem 4.1.** Let \( \mathcal{S} \) satisfy Assumption 2.1 with \( p \in (2, \infty) \) and \( \delta > 0 \), let \( k \in \mathbb{N} \), and let \( g \in L^p(\Omega) \). Moreover, let \( (v, q)^\top \in \hat{V}(0) \times \hat{Q} \) be a solution of Problem (Q) (cf. (2.10), (2.11)) with \( F(Dv) \in W^{1,2}(\Omega) \) and let \( (v_h, q_h)^\top \in V_h^k(0) \times Q_{h,c}^k \) be a solution of Problem (Q) (cf. (3.7)) for \( \alpha > 0 \). Then, there exists a constant \( c_0 > 0 \), depending only on the characteristics of \( \mathcal{S} \), \( \delta^{-1}, \omega_0, \alpha^{-1} \), and \( k \), such that if \( \|\nabla v\|_2 \leq c_0 \), then it holds

\[
\|q_h - q\|_{p'} \leq c h + c (\rho(\hat{\varphi} Dv))^\gamma (h \nabla q)^{\frac{1}{2}} 
\]
with a constant \( c > 0 \) depending only on the characteristics of \( \mathcal{S} \), \( \| F(Dv) \|_{1,2} \), \( \| v \|_{p'} \), \( \| g \|_{p'} \), \( \delta^p | \Omega | \), \( \delta^{-1} \), \( \omega_0 \), \( \alpha^{-1} \), \( k \), and \( c_0 \).

**Corollary 4.2.** Let the assumptions of Theorem 4.1 be satisfied. Then, it holds

\[
\| q_h - q \|_{p'} \leq c h^{\frac{1}{2}}
\]

with a constant \( c > 0 \) depending only on the characteristics of \( \mathcal{S} \), \( \| F(Dv) \|_{1,2} \), \( \| v \|_{p'} \), \( \| g \|_{p'} \), \( \delta^p | \Omega | \), \( \delta^{-1} > 0 \), \( \omega_0 \), \( \alpha^{-1} \), \( k \), and \( c_0 \). If, in addition, \( g \in L^2(\Omega) \), then

\[
\| q_h - q \|_{p'} \leq c h
\]

with a constant \( c > 0 \) depending only on the characteristics of \( \mathcal{S} \), \( \| F(Dv) \|_{1,2} \), \( \| g \|_{p'} \), \( \| (\delta + |Dv|)^{2/p} v \|_{2} \), \( \| v \|_{p'} \), \( \delta^p | \Omega | \), \( \delta^{-1} \), \( \omega_0 \), \( \alpha^{-1} \), \( k \), and \( c_0 \).

**Remark 4.3.** (i) Note that, in view of [21, Lemma 2.6], in Theorem 4.1 the assumption \( g \in L^p(\Omega) \) is equivalent to \( \nabla q \in L^p(\Omega) \) and in Corollary 4.2, the assumption \( g \in L^2(\Omega) \) can be replaced by \( (\delta + |Dv|)^{2/p} |v|^2 \in L^1(\Omega) \).

(ii) To the best of the authors knowledge the results in Theorem 4.1 and Corollary 4.2 are the first convergence results for the pressure for the \( p \)-Navier–Stokes equations (1.1) both for DG methods and FE methods.

Due to the modified numerical flux (3.3), the error analysis can no longer be performed in terms of modulars as in [4], but in terms of Luxembourg norms only. The reason for that is that estimates in modulars are usually proved by the additive Young’s inequality, while for estimates in norms we can use the multiplicative Hölder’s inequality. Thus, certain terms which can not be absorbed in the modular appear as a bounded factor in the corresponding norm estimate. Since all error estimates proved in [21] are formulated in terms of modulors, we need to translate them in terms of Luxembourg norms. The following lemma helps us to do this.

**Lemma 4.4.** Let \( \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \) be defined by (2.3) for \( p \in [2, \infty) \) and \( \delta \geq 0 \) and let \( \alpha \in L^p(\Omega) \) with \( \alpha \geq 0 \) a.e. in \( \Omega \). Then, the following statements apply:

(i) For every \( c_0 \geq 1 \), \( \gamma \leq 1 \), and \( f \in L^p(\Omega) \) from \( \rho_{\varphi, \alpha}(f) \leq c_0 \gamma \), it follows that

\[
\| f \|_{\rho_{\varphi, \alpha}} \leq (2c_0)^{\frac{1}{2}} \Delta_2(\varphi) \gamma^{\frac{1}{2}}.
\]

(ii) For every \( c_0 \geq 1 \), \( \gamma \leq 1 \), and \( g \in L^p(\Omega) \) from \( \rho_{\varphi, \alpha}(g) \leq c_0 \gamma \), it follows that

\[
\| g \|_{\rho_{\varphi, \alpha}} \leq (c_0 c(p))^{\frac{1}{p}} \gamma^{\frac{1}{2}}.
\]

**Proof.** Ad (i). Observing that, owing to [26, Lemma 5.1, (5.11)], (2.4), (2.3), \( p \geq 2 \), the \( \Delta_2 \)-condition of \( \varphi(x) : \mathbb{R}^+ \to \mathbb{R}^+ \) for a.e. \( x \in \Omega \) and [26, Lemma 5.3], for every \( \lambda \leq 1 \), \( c \geq 1 \), \( t \geq 0 \) and for a.e. \( x \in \Omega \), it holds

\[
\varphi(x) \left( \frac{t}{c} \right) \leq (\varphi(x))' \left( \frac{t}{c} \right) \frac{t}{c} = \left( \delta + a(x) + \frac{t}{c} \right)^{p-2} \left( \frac{t}{c} \right)^2.
\]

Choosing \( \lambda = \gamma^{\frac{1}{2}} \) and \( c = (2c_0)^{\frac{1}{2}} \Delta_2(\varphi) \), we find that

\[
\rho_{\varphi, \alpha}(x) \left( \frac{f}{(c_0 \Delta_2(\varphi)^{\frac{1}{2}} \gamma^{\frac{1}{2}})} \right) \leq \frac{1}{c_0 \gamma} \rho_{\varphi, \alpha}(f) \leq 1,
\]
so that, from the definition of the Luxembourg norm, we conclude the assertion.

*ad (ii).* Observing that, due to \( p \geq 2 \), for every \( \lambda \leq 1, \ c \geq 1, \) and \( t \geq 0 \), it holds

\[
(\varphi_{a(x)})^* \left( \frac{t}{c \lambda} \right) \leq c(p) \left( \frac{\delta^{p-1} + a(x)^{p-1} + \frac{t}{c \lambda}}{c} \right)^{p' - 2} \left( \frac{t}{c} \right)^2 \leq \frac{c(p)}{\lambda} \left( \frac{\delta^{p-1} + a(x)^{p-1} + \frac{t}{c \lambda}}{c} \right) \leq \frac{c(p)}{\lambda c^p} (\varphi_{a(x)})^*(t),
\]

where \( c(p) > 0 \) depends only on \( p \geq 2, \) choosing \( \lambda = \gamma^2 \) and \( c = (c_0 c(p))^\frac{1}{p} \), we find that

\[
\rho(\varphi_{a*}, \Omega) \left( \frac{g}{(c_0 c(p))^\frac{1}{p} \gamma^2} \right) \leq \frac{1}{c_0 \gamma} \rho(\varphi_{a*}, \Omega)(g) \leq 1,
\]

so that, from the definition of the Luxembourg norm, we conclude the assertion. \( \square \)

In order to prove the results in Theorem 4.1 and Corollary 4.2, we need to derive a system similar to (3.7), which is satisfied by a solution of our original problem (1.1). Using the notation \( L = \nabla v, S = S(L_{sym}), K = v \otimes v \), we find that \( (v, L, S, K)^T \in W^{1,p}(\Omega) \times L^{p'}(\Omega) \times L^{p'}(\Omega) \times L^{p'}(\Omega) \). If, in addition, \( S, K, q \in W^{1,1}(\Omega) \), we observe as in [15], i.e., using integration-by-parts, the projection property of \( \Pi_h^k \), the definition of the discrete gradient and jump functional, that

\[
(L, X_h) = (\nabla v, X_h),
(S, Y_h) = (S(L_{sym}), Y_h),
(K, Z_h) = (v \otimes v, Z_h),
(S - \frac{1}{2} K - q I_d, D_h^k z_h) = (g - \frac{1}{2} L v, z_h) + \langle \{S\} - \{\Pi_h^k S\}, [z_h \otimes n] \rangle_{\Gamma_h} + \frac{1}{2} \langle \{\Pi_h^k K\} - \{K\}, [z_h \otimes n] \rangle_{\Gamma_h} + \langle \{\Pi_h^k (q I_d)\} - \{q I_d\}, [z_h \otimes n] \rangle_{\Gamma_h},
\]

\[
(Div_h^k v, z_h) = 0
\]

is satisfied for all \( (X_h, Y_h, Z_h, z_h)^T \in X_h^k \times X_h^k \times X_h^k \times V_h^k \). As a result, using (4.3), (3.7) and (2.29), we arrive at

\[
(S(D_h^k v_h) - S(D v), D_h^k z_h) + \alpha \langle S([\Pi_h^k D_h^k v_{h\lambda}]) (h^{-1} [v_h \otimes n]), [z_h \otimes n] \rangle_{\Gamma_h} (4.4)
\]

\[
= (q_h - q, Div_h^k z_h) + b_h(v, v, z_h) - b_h(v_h, v, z_h) + \langle \{S\} - \{\Pi_h^k S\}, [z_h \otimes n] \rangle_{\Gamma_h} + \frac{1}{2} \langle \{\Pi_h^k K\} - \{K\}, [z_h \otimes n] \rangle_{\Gamma_h} + \langle \{\Pi_h^k (q I_d)\} - \{q I_d\}, [z_h \otimes n] \rangle_{\Gamma_h},
\]

which is satisfied for all \( (z_h, z_h)^T \in V_h^k \times Q_{h, c}^k \). Here, we denoted the discrete convective term by \( b_h: W^{1,1}(\bar{T}_h) \times W^{1,1}(\bar{T}_h) \times W^{1,1}(\bar{T}_h) \rightarrow \mathbb{R} \), which is defined via

\[
b_h(x_h, y_h, z_h) := \frac{1}{2} (z_h \otimes x_h, g_{h h}^k y_h) - \frac{1}{2} (y_h \otimes x_h, g_{h h}^k z_h)
\]

for all \( (x_h, y_h, z_h)^T \in W^{1,1}(\bar{T}_h) \times W^{1,1}(\bar{T}_h) \times W^{1,1}(\bar{T}_h) \).

Now we have prepared everything to prove our main result Theorem 4.1.
Proof of Theorem 4.1. From our assumptions, resorting to [21, Lemma 2.6], follows that \( \nabla q \in L^p(\Omega) \), which together with \((\varphi(Dv))^+(t) \leq c t^p\), valid for every \( t \geq 0 \), yields \( \rho_{(\varphi(Dv))^+,\Omega}(\nabla q) \leq \|\nabla q\|_{L^p}^p \). Moreover, appealing to [11, Lemma 6.10], we deduce the existence of a constant \( \beta > 0 \) such that for every \( z_h \in Q_h \), it holds the LBB condition

\[
\beta \|z_h\|_{L^p} \leq \sup_{z_h \in V_h, \|z_h\|_{V,p,h} \leq 1} (z_h, D\varphi_h^k z_h). \tag{4.5}
\]

On the other hand, due to \((4.4)\), for every \( z_h \in V_h \), we have that

\[
(q_h - q, D\varphi_h^k z_h) = (S(D_h^k v_h) - S(Dv), D_h^k z_h)
+ \alpha (S([n_h^k D_h^k v_h]) h^{-1} [v_h \otimes n], [z_h \otimes n])_{\Gamma_h}
+ b_h(v, v, z_h) + b_h(v, v_h, z_h)
+ \langle \{\Pi_h^k S\} - \{S\}, [z_h \otimes n] \rangle_{\Gamma_h}
+ \frac{1}{2} \langle \{K\} - \{z_h \otimes n\} \rangle_{\Gamma_h}
+ \langle \{qI_d\} - \{\Pi_h(qI_d)\}, [z_h \otimes n] \rangle_{\Gamma_h}
= I_1 + \alpha I_2 + I_3 + \cdots + I_6.
\]

So, let us next estimate \( I_1, \ldots, I_6 \) for some arbitrary \( z_h \in V_h \) with \( \|z_h\|_{V,p,h} \leq 1 \): (ad \( I_1 \)). Using the generalized H"older inequality \((2.1)\), we find that

\[
|I_1| \leq 2 \left\| S(D_h^k v_h) - S(Dv) \right\|_{(\varphi(Dv))^+,1} \left\| D_h^k z_h \right\|_{(\varphi(Dv))^+,1}
= 2 I_{1.1} \cdot I_{1.2}. \tag{4.7}
\]

Appealing to \((2.9)\) and [21, Theorem 4.1], we have that

\[
\rho_{(\varphi(Dv))^+,\Omega}(S(D_h^k v_h) - S(Dv)) \leq c \left\| F(D_h^k v_h) - F(Dv) \right\|_{2,1}
\leq ch^2 \left\| F(Dv) \right\|_{2,1}^2 + c \rho_{(\varphi(Dv))^+,\Omega} h \nabla q. \tag{4.8}
\]

Since, by assumption, we have that \( h \leq 1 \), for

\[
c_0 := \max \left\{ 1, c \left\| F(Dv) \right\|_{2,1}^2 + c \rho_{(\varphi(Dv))^+,\Omega} \nabla q \right\} \geq 1,
\gamma := c_0^{-1} \left( c h^2 \left\| F(Dv) \right\|_{2,1}^2 + c \rho_{(\varphi(Dv))^+,\Omega} h \nabla q \right) \leq 1,
\]

Lemma 4.4 yields a constant \( c(p) > 0 \), depending only on \( p \in (2, \infty) \), such that

\[
I_{1.1} \leq c_0^{\frac{1}{2} - \frac{1}{p}} c(p)^{\frac{1}{2}} \left( c h^2 \left\| F(Dv) \right\|_{2,1}^2 + c \rho_{(\varphi(Dv))^+,\Omega} h \nabla q \right)^{\frac{1}{2}}. \tag{4.9}
\]

Using the shift change in Lemma 2.5, that \( \varphi(t) \leq c(p) (\delta^p + t^p) \) for all \( t \geq 0 \), and \((2.28)\), we find that

\[
\rho_{(\varphi(Dv))^+,\Omega}(D_h^k z_h) \leq c \rho_{\varphi,\Omega}(D_h^k z_h) + c \rho_{\varphi,\Omega}(Dv)
\leq c \|D_h^k z_h\|_p + c \delta^p |\Omega| + c \rho_{\varphi,\Omega}(Dv)
\leq c + c \delta^p |\Omega| + c \rho_{\varphi,\Omega}(Dv)
\leq \max \left\{ 1, c + c \delta^p |\Omega| + c \rho_{\varphi,\Omega}(Dv) \right\},
\]

LDG APPROXIMATION FOR THE \( p \)-NAVIER–STOKES SYSTEM

11
Appealing to (2.9) and [21, Theorem 4.1], we have that
\[ I_{1,2} \leq \left( 2 \max \left\{ 1, c + c \delta p |\Omega| + c \rho_{\varphi, \Omega}(Dv) \right\} \right)^{\frac{1}{2}} \Delta_2(\varphi). \] (4.10)

Then, combining (4.9) and (4.10) in (4.8), we deduce that
\[ |I_1| \leq ch + c \left( \rho(\varphi, Dv)^{\gamma}, \Omega(h \nabla q) \right)^{\frac{1}{2}}. \] (4.11)

(ad I_2). Using the generalized Hölder inequality (2.1), we find that
\[ |I_2| \leq 2 h^{1 - \frac{1}{p}} \left\| S_{\mathcal{H}_h^p \mathcal{F}^k \mathcal{V}_h} \right\|_{\mathcal{F}(\mathcal{H}_h^p \mathcal{F}^k \mathcal{V}_h), \Gamma_h} \times \left\| h^{h - 1} \left[ \mathbf{z}_h \otimes \mathbf{n} \right] \right\|_{\mathcal{F}(\mathcal{H}_h^p \mathcal{F}^k \mathcal{V}_h), \Gamma_h} \] (4.12)
\[ =: 2 h^{1 - \frac{1}{p}} I_{2,1} \cdot I_{2,2}. \]

Appealing to (2.9) and [21, Theorem 4.1], we have that
\[ \rho_{\varphi(\mathcal{H}_h^p \mathcal{F}^k \mathcal{V}_h)), \Gamma_h} \left( S_{\mathcal{H}_h^p \mathcal{F}^k \mathcal{V}_h} \right) \left( h^{-1} \left[ \mathbf{v}_h \otimes \mathbf{n} \right] \right) \leq ch^{-1} \mathbb{F}_{\mathcal{H}_h^p \mathcal{F}^k \mathcal{V}_h}, h \left( \mathbf{v}_h \right) \] (4.13)
\[ \leq ch \left\| \mathbf{F}(Dv) \right\|_{1,2}^2 + ch^{-1} \rho_{\varphi(\mathcal{H}_h^p \mathcal{F}^k \mathcal{V}_h)), \Omega(h \nabla q) \right\|_{1,2}. \]

Since, by assumption, we have that \( h \leq 1 \), for
\[ c_0 := \max \left\{ 1, c \left\| \mathbf{F}(Dv) \right\|_{1,2}^2 + c \rho_{\varphi(\mathcal{H}_h^p \mathcal{F}^k \mathcal{V}_h)), \Omega(h \nabla q) \right\|_{1,2} \right\} \geq 1, \]
\[ \gamma := c_0^{-1} \left( c \left\| \mathbf{F}(Dv) \right\|_{1,2}^2 + ch^{-1} \rho_{\varphi(\mathcal{H}_h^p \mathcal{F}^k \mathcal{V}_h)), \Omega(h \nabla q) \right\|_{1,2} \right\} \leq 1, \]

Lemma 4.4 yields a constant \( c(p) > 0 \), depending only on \( p \in (2, \infty) \), such that
\[ I_{2,1} \leq c_0^{\frac{1}{p} - \frac{1}{2}} c(p)^{\frac{1}{2}} \left( ch \left\| \mathbf{F}(Dv) \right\|_{1,2}^2 + ch^{-1} \rho_{\varphi(\mathcal{H}_h^p \mathcal{F}^k \mathcal{V}_h)), \Omega(h \nabla q) \right\|_{1,2} \right)^{\frac{1}{2}}. \] (4.14)

Using that \( \varphi(\mathcal{H}_h^p \mathcal{F}^k \mathcal{V}_h)), h(t) \leq ch \varphi(\mathcal{H}_h^p \mathcal{F}^k \mathcal{V}_h)), (t) \) for all \( t \geq 0 \) and \( h \in [0,1] \), the shift change in Lemma 2.5, that \( \varphi(t) \leq c(p)(\delta p + t^p) \) for all \( t \geq 0 \), \( h \mathcal{H}^{d-1}(\Gamma_h) \leq c |\Omega| \), the discrete trace inequality [19, (A.23)], the Orlicz-stability properties of \( \mathcal{H}_h^p \) [19, (A.12)], and the a priori estimate [20, Proposition 5.7], we find that
\[ \rho_{\varphi(\mathcal{H}_h^p \mathcal{F}^k \mathcal{V}_h)), \Gamma_h} \left( h^{h - 1} \left[ \mathbf{z}_h \otimes \mathbf{n} \right] \right) \leq ch \rho_{\varphi(\mathcal{H}_h^p \mathcal{F}^k \mathcal{V}_h)), \Gamma_h} \left( h^{-1} \left[ \mathbf{z}_h \otimes \mathbf{n} \right] \right) \]
\[ \leq ch \rho_{\varphi, \Gamma_h} \left( h^{-1} \left[ \mathbf{z}_h \otimes \mathbf{n} \right] \right) + c \rho_{\varphi, \Gamma_h} \left( \left\| \mathcal{H}_h^p \mathcal{F}^k \mathcal{V}_h \right\|_p \right) \]
\[ \leq ch \left\| \mathbf{z}_h \otimes \mathbf{n} \right\|_{\mathcal{F}(\mathcal{H}_h^p \mathcal{F}^k \mathcal{V}_h), \Gamma_h} \]
\[ + c \rho_{\varphi, \Omega} \left( \mathcal{H}_h^p \mathcal{F}^k \mathcal{V}_h \right) \]
\[ \leq \max \left\{ 1, c + c \delta p |\Omega| + c \left\| \mathbf{g} \right\|_{p'}^p \right\}, \]
so that Lemma 4.4, where \( \lambda = 1 \) and \( c_0 = \max \left\{ 1, c + c \delta p |\Omega| + c \left\| \mathbf{g} \right\|_{p'}^p \right\} \), yields that
\[ I_{2,2} \leq \left( 2 \max \left\{ 1, c + c \delta p |\Omega| + c \left\| \mathbf{g} \right\|_{p'}^p \right\} \right)^{\frac{1}{2}} \Delta_2(\varphi). \] (4.16)

Then, combining (4.14) and (4.16) in (4.12), we deduce that
\[ |I_2| \leq ch + c \left( \rho_{\varphi(\mathcal{H}_h^p \mathcal{F}^k \mathcal{V}_h)), \Omega(h \nabla q) \right\|_{1,2} \right)^{\frac{1}{2}}. \] (4.17)
(ad $I_3$). Introducing the notation $e_h := v_h - v \in W^{1,p}(T_h)$, $I_3$ can be re-written as
\[
I_3 = b_h(v, v - \Pi_h^k v, z_h) - b_h(e_h, \Pi_h^k v, z_h) + b_h(v_h, \Pi_h^k e_h, z_h)
\]
\[
= I_{3,1} + I_{3,2} + I_{3,3}. \tag{4.18}
\]
So, we have to estimate $I_{3,i}$, $i = 1, 2, 3$:

(ad $I_{3,1}$). The definition of $b_h : W^{1,p}(T_h) \times W^{1,p}(T_h) \times W^{1,p}(T_h) \to \mathbb{R}$ yields:
\[
2I_{3,1} = (z_h \otimes v, \mathcal{G}_h^k(v - \Pi_h^k v)) - ((v - \Pi_h^k v) \otimes v, \mathcal{G}_h^k z_h) =: I_{3,1}^1 + I_{3,1}^2. \tag{4.19}
\]
As already observed in [21, Proof of Lemma 2.6], we have that $v \in W^{2,2}(\Omega)$, where
\[
\|v\|_{2,2} \leq c \delta^{2-p} \|F(Dv)\|_{1,2}. \tag{4.20}
\]
Thus, exploiting that, by the Sobolev embedding theorem, $v \in W^{2,2}(\Omega) \to L^\infty(\Omega)$, the discrete Sobolev embedding theorem (cf. [12, Theorem 5.3]), the identities $\mathcal{G}^k_h = \nabla_h - \mathcal{R}^k_h$ and $v - \Pi_h^k v = v - \Pi_h^k(v - \Pi_h^k v)$, the approximation properties of $\Pi_h^k$ (cf. [19, Corollary A.8, Lemma A.1, Corollary A.19]), and (4.20), we find that
\[
|I_{3,1}^1| \leq \|v\|_{\infty} \|z_h\|_2 \|\mathcal{G}^k_h(v - \Pi_h^k v)\|_2
\]
\[
\leq c h \|z_h\|_{V,p,h}\|\nabla^2 v\|_2
\]
\[
\leq c h \|\nabla F(Dv)\|_2. \tag{4.21}
\]
Similarly, we get, also using (2.21) and $p > 2$, that
\[
|I_{3,1}^2| \leq \|v\|_{\infty} \|v - \Pi_h^k v\|_2 \|\mathcal{G}^k_h z_h\|_2
\]
\[
\leq c h^2 \|\nabla^2 v\|_2 \|z_h\|_{V,p,h}
\]
\[
\leq c h^2 \|\nabla F(Dv)\|_2. \tag{4.22}
\]
(ad $I_{3,2}$). The definition of $b_h : W^{1,p}(T_h) \times W^{1,p}(T_h) \times W^{1,p}(T_h) \to \mathbb{R}$ yields:
\[
2I_{3,2} = (z_h \otimes e_h, \mathcal{G}_h^k \Pi_h^k v) + (\Pi_h^k v \otimes e_h, \mathcal{G}_h^k z_h) =: I_{3,2}^1 + I_{3,2}^2. \tag{4.23}
\]
Then, exploiting (2.21), the Sobolev embedding theorem (cf. Proposition 2.6), the DG-stability property of $\Pi_h^k$ (cf. [15, (A.19)]), the Korn type inequality in Proposition 2.8, the estimates (4.20), [21, (4.50)], and [21, Theorem 4.1], we find that
\[
|I_{3,2}^1| \leq \|\mathcal{G}^k_h \Pi_h^k v\|_2 \|e_h\|_4 \|z_h\|_4
\]
\[
\leq c \|\Pi_h^k v\|_{V,2,h} \|e_h\|_{V,2,h} \|z_h\|_{V,p,h}
\]
\[
\leq c \|\nabla v\|_2 (\|e_h\|_{D,2,h} + h \|\nabla^2 v\|_2) \|z_h\|_{V,p,h}
\]
\[
\leq c (\|F(Dv) - F(D_h^k v_h)\|_2 + (m_{\varphi}(m_{\varphi}^k)^h \varphi)^h (v_h - v)) \frac{1}{2} + h \|\nabla^2 v\|_2)
\]
\[
\leq c (h \|F(Dv)\|_{1,2} + (\rho(\varphi,Dv))\cdot\Omega(h \nabla q)) \frac{1}{2}) \tag{4.24}
\]
Similarly, we find that
\[
|I_{3,2}^2| \leq \|\mathcal{G}^k_h z_h\|_2 \|\Pi_h^k v\|_4 \|e_h\|_4
\]
\[
\leq c \|z_h\|_{V,p,h} \|\Pi_h^k v\|_{V,2,h} \|e_h\|_{V,2,h}
\]
\[
\leq c (h \|F(Dv)\|_{1,2} + (\rho(\varphi,Dv))\cdot\Omega(h \nabla q)) \frac{1}{2}) \tag{4.25}
\]
(ad $I_{3,3}$). The definition of $b_h : W^{1,p}(T_h) \times W^{1,p}(T_h) \to \mathbb{R}$ yields:

$$2I_{3,3} = (\mathbf{z}_h \otimes \mathbf{v}_h, \mathbf{g}^h_\Pi \mathbf{e}_h) + (\Pi_h^\Pi \mathbf{e}_h \otimes \mathbf{v}_h, \mathbf{g}^h_\Pi \mathbf{z}_h) =: I_{3,3}^1 + I_{3,3}^2. \quad (4.26)$$

Then, exploiting (2.21), the discrete Sobolev embedding theorem (cf. [12, Theorem 5.3]), the DG-stability properties of $\Pi_h^\Pi$ (cf. [15, (A.18)]), the apriori estimate in [20, Proposition 5.7], the Korn type inequality in Proposition 2.8, the estimates (4.20), [21, (4.50)], and [21, Theorem 4.1], we find that

$$|I_{3,3}^1| \leq \|\mathbf{g}^h_\Pi \mathbf{e}_h\|_2 \|\mathbf{z}_h\|_4 \|\mathbf{v}_h\|_4$$

$$\leq c \|\mathbf{e}_h\|_{H^2,T_h} \|\mathbf{z}_h\|_{H^2,T_h} \|\mathbf{v}_h\|_{H^2,T_h} \quad (4.27)$$

Similarly, we find that

$$|I_{3,3}^2| \leq \|\mathbf{g}^h_\Pi \mathbf{z}_h\|_2 \|\Pi_h^\Pi \mathbf{e}_h\|_4 \|\mathbf{v}_h\|_4$$

$$\leq c \|\mathbf{z}_h\|_{H^2,T_h} \|\mathbf{e}_h\|_{H^2,T_h} \|\mathbf{v}_h\|_{H^2,T_h} \quad (4.28)$$

Eventually, combining (4.18)–(4.28), we conclude that

$$|I_3| \leq c h + c (\rho(\varphi, \mathbf{Dv})^*, \Omega(h \nabla q))^\frac{1}{2} \quad (4.29)$$

(ad $I_4$). Using the generalized H"older inequality (2.1), we find that

$$|I_4| \leq 2 h^{1 - \frac{1}{2}} \|\{ \Pi_h^\Pi \mathbf{S}(\mathbf{Dv}) \} - \{ \mathbf{S}(\mathbf{Dv}) \}\|_{(\varphi, \mathbf{Dv})^*, \Gamma_h} \|h^{\frac{1}{2} - 1}\|\mathbf{z}_h \otimes \mathbf{n}\|_{(\varphi, \mathbf{Dv}), \Gamma_h} \quad (4.30)$$

Appealing to [21, (4.43), (4.45)], we have that

$$\rho(\varphi, \mathbf{Dv})^*, \Gamma_h (\{ \Pi_h^\Pi \mathbf{S}(\mathbf{Dv}) \} - \{ \mathbf{S}(\mathbf{Dv}) \}) \leq c h^2 \|\nabla \mathbf{F}(\mathbf{Dv})\|_2^2 \quad (4.31)$$

Since, by assumption, we have that $h \leq 1$, for

$$c_0 := \max \{1, c \|\nabla \mathbf{F}(\mathbf{Dv})\|_2^2\}, \quad \gamma := c_0^{-1} c h^2 \|\nabla \mathbf{F}(\mathbf{Dv})\|_2^2,$$

Lemma 4.4 yields a constant $c(p) > 0$, depending only on $p \in (2, \infty)$, such that

$$I_{4,1} \leq c_0^{\frac{1}{2}} c^{\frac{1}{2}} c(p) h \|\nabla \mathbf{F}(\mathbf{Dv})\|_2 \quad (4.32)$$

Using a shift change in Lemma 2.5, [21, Lemma 4.11], (4.15), [21, Theorem 4.1], the convexity of $(\varphi(\mathbf{Dv}(x)))^*: \mathbb{R}^2 \to \mathbb{R}^2$ for a.e. $x \in \Omega$ to together with $\sup_{\alpha \geq 0} \Delta_2((\varphi_\alpha)^*) < \infty$ and $h \leq 1$, we find that

$$\rho(\varphi, \mathbf{Dv}, \Gamma_h (h^{\frac{1}{2} - 1}\|\mathbf{z}_h \otimes \mathbf{n}\|) \leq \rho_{\varphi, (\mathbf{Dv}_h, \Gamma_h (h^{\frac{1}{2} - 1}\|\mathbf{z}_h \otimes \mathbf{n}\|)}$$

$$+ \rho(\varphi, \mathbf{Dv}(x)), \Gamma_h (\|\mathbf{S}(\mathbf{Dv}) - \{ \{ \Pi_h^\Pi \mathbf{Dv}_h \} \| \mathbf{n}\|)$$

$$\leq \rho_{\varphi, (\mathbf{Dv}_h, \Gamma_h (h^{\frac{1}{2} - 1}\|\mathbf{z}_h \otimes \mathbf{n}\|)}$$

$$+ c \|\nabla \mathbf{F}(\mathbf{Dv})\|_2^2 + c h^{-1} \|\mathbf{F}(\mathbf{Dv}) - \mathbf{F}(\mathbf{Dv})\|_2^2 \quad (4.33)$$

$$\leq \max \{1, c + c \delta^2 |\Omega| + c \|\mathbf{Dv}_0\|_{\mathbf{Dv}} + c h \|\mathbf{F}(\mathbf{Dv})\|_{\mathbf{Dv}} \} \quad (4.34)$$
Thus, Lemma 4.4, where \( \gamma = 1 \) and \( c_0 = \max \{ 1, c + c\delta p |\Omega| + c \|g\|_{L^p} + c \|F(Dv)\|_{L^2} + c\rho(\rho(\varphi_{\{Dv\}})\cdot \Omega(\nabla q)) \} \), yields that

\[
I_{4,2} \leq (2 \max \{ 1, c + c\delta p |\Omega| + c \|g\|_{L^p} + c \|F(Dv)\|_{L^2} + c\rho(\rho(\varphi_{\{Dv\}})\cdot \Omega(\nabla q)) \})^\frac{1}{2} \Delta_2(\varphi). \tag{4.34}
\]

Combining (4.32) and (4.34), we deduce that \( |I_4| \leq \varepsilon h \). \tag{4.35}

(ad \( I_5 \)). Using Hölder’s inequality, we find that

\[
|I_5| \leq h^{1-\frac{1}{2}} \|\{K\} - \{\Pi_h^k K\}\|_{2,\Gamma_h} \|h^{\frac{1}{2}-1}[z_h \otimes n]\|_{2,\Gamma_h}. \tag{4.36}
\]

Taking into account \( v \in L^\infty(\Omega) \cap W^{1,2}_0(\Omega) \), we have that \( K = v \otimes v \in W^{1,2}_0(\Omega) \), where \( \|\nabla K\|_2 \leq c \|v\|_\infty \delta^{2-p} \|F(Dv)\|_2 \). Thus, [19, Corollary A.19] yields

\[
\|\{K\} - \{\Pi_h^k K\}\|_{2,\Gamma_h} \leq c h^{\frac{1}{2}} \|\nabla K\|_2 \leq c h^{\frac{1}{2}} \|F(Dv)\|_2, \tag{4.37}
\]

which together with \( \|h^{\frac{1}{2}-1}[z_h \otimes n]\|_{2,\Gamma_h} \leq c |\Omega|^{\frac{1}{2}-\frac{1}{2}} \|h^{\frac{1}{2}-1}[z_h \otimes n]\|_{p,\Gamma_h} \leq c \) yields

\[
|I_5| \leq \varepsilon h. \tag{4.38}
\]

(ad \( I_6 \)). Using the generalized Hölder inequality (2.1), we find that

\[
|I_6| \leq 2 h^{1-\frac{1}{2}} \|\{qL_d\} - \{\Pi_h^k (qL_d)\}\|_{(\varphi_{[Dv]}),\Gamma_h} \|h^{\frac{1}{2}-1}[z_h \otimes n]\|_{\varphi_{[Dv]},\Gamma_h}. \tag{4.39}
\]

\[
= 2 h^{1-\frac{1}{2}} I_{6,1} \cdot I_{4,2}. \tag{4.40}
\]

Appealing to [21, Proposition 4.9], we have that

\[
\rho(\rho(\varphi_{\{Dv\}})\cdot \Gamma_h (\{qL_d\} - \{\Pi_h^k (qL_d)\})) \leq c h \|\nabla F(Dv)\|_2^2 + c h^{-1} \rho(\rho(\varphi_{\{Dv\}})\cdot \Omega(h \nabla q)).
\]

Since, by assumption, we have that \( h \leq 1 \), for

\[
c_0 := \max \{ 1, c \|F(Dv)\|_{L^2}^2 + c \rho(\rho(\varphi_{\{Dv\}})\cdot \Omega(\nabla q)) \},
\]

\[
\gamma := c_0^{-1} (c h \|\nabla F(Dv)\|_{L^2}^2 + c h^{-1} \rho(\rho(\varphi_{\{Dv\}})\cdot \Omega(h \nabla q)))^{\frac{1}{2}}.
\]

Lemma 4.4 yields a constant \( c(p) > 0 \), depending only on \( p \in (1, \infty) \), such that

\[
\|\{qL_d\} - \{\Pi_h^k (qL_d)\}\|_{(\varphi_{[Dv]}),\Gamma_h} \leq c_0^{\frac{1}{2}-\frac{1}{2}} c (c h \|\nabla F(Dv)\|_2^2 + c h^{-1} \rho(\rho(\varphi_{\{Dv\}})\cdot \Omega(h \nabla q)))^{\frac{1}{2}}.
\]

As a result, also using (4.34), we deduce that

\[
|I_6| \leq \varepsilon h + c (\rho(\rho(\varphi_{\{Dv\}})\cdot \Omega(h \nabla q)))^{\frac{1}{2}}. \tag{4.41}
\]

Putting it all together, for every \( z_h \in V_h^k \) with \( \|z_h\|_{V^k,h} \leq 1 \), we conclude that

\[
(q_h - q, \text{Div}_h^k z_h) \leq c h + c (\rho(\rho(\varphi_{\{Dv\}})\cdot \Omega(h \nabla q)))^{\frac{1}{2}}. \tag{4.42}
\]
Therefore, for every \( z_h \in \bar{Q}_h^{k,c} \), we find that
\[
\|q_h - q\|_{p'} \leq \sup_{z_h \in V_h^k} (q_h - q_h, D\nu_h^k z_h) + \|z_h - q\|_{p'}
\leq \sup_{z_h \in V_h^k} (q_h - q_h, D\nu_h^k z_h) + c \|z_h - q\|_{p'}
\leq c h + c (\rho(\bar{\varphi} Dv), \omega (h \nabla q))^\frac{1}{2} + c \|z_h - q\|_{p'}.
\]
\[
(4.43)
\]
Next, denote by \( \Pi_h^{Q,k} : L^p(\Omega) \to Q_h^{k,c} \), the Céline quasi-interpolation operator (cf. [8]), for which we have that
\[
\|q - \Pi_h^{Q,k} q\|_{p'} \leq c h \|\nabla q\|_{p'}.
\]
\[
(4.44)
\]
Noting that the infimum of \( \|z_h - q\|_{p'} \) over \( Q_h^{k,c} \) and \( \bar{Q}_h^{k,c} \) are comparable for \( q \in \bar{Q}_h^{k,c} \), the assertion of Theorem 4.1 follows from (4.43) and (4.44), if we choose \( z_h = \Pi_h^{Q,k} q \).

**Proof of Corollary 4.2.** Using that \( \varphi^*(h t) \leq c h \varphi^*(t) \) for all \( t \geq 0 \), valid for \( p > 2 \) (cf. [4]), we deduce from Theorem 4.1 that
\[
\|q_h - q\|_{p'} \leq c h + c h \|\nabla q\|_{p'}.
\]

If, in addition, \( g \in L^2(\Omega) \), then [21, Lemma 2.6] implies \( (\delta + |Dv|)^{2-p}|\nabla q|^2 \in L^1(\Omega) \). Moreover, it holds \( (\varphi_n)^*(h t) \sim (\delta + a)^{2-p} h^{2} t^2 \leq (\delta + a)^{2-p} h^{2} t^2 \) for all \( t \geq 0 \), since \( p > 2 \). Therefore, from Theorem 4.1, we deduce that
\[
\|q_h - q\|_{p'} \leq c h + c h \|\nabla q\|_{p'},
\]
which is the assertion.

The same method of proof of course also works for the \( p \)-Stokes problem, i.e., we neglect the convective term in (1.1), Problem (P), Problem \((P_h)\), Problem (Q), and Problem \((Q_h)\). Note that the dependence on \( \delta^{-1} > 0 \) comes solely from the convective term. Thus, we obtain for the \( p \)-Stokes problem a better dependence on the constants.

**Theorem 4.5.** Let \( \mathcal{S} \) satisfy Assumption 2.1 with \( p \in (2, \infty) \) and \( \delta \geq 0 \), let \( k \in \mathbb{N} \), and let \( g \in L^p(\Omega) \). Moreover, let \( (v, q)^T \in V(0) \times \bar{Q} \) be a solution of Problem (Q) without the convective term (cf. (2.10), (2.11)) with \( F(Dv) \in W^{1,2}(\Omega) \) and let \( (v_h, q_h)^T \in V_h^k(0) \times \bar{Q}_h^{k,c} \) be a solution of Problem \((Q_h)\) without the terms coming from the convective term (cf. (3.7)) for \( \alpha > 0 \). Then, there exists a constant \( c > 0 \), depending only on the characteristics of \( \mathcal{S} \), \( \|F(Dv)\|_{1,2}, \|\nabla q\|_{p'}, \|g\|_{p'}, \delta^p(\Omega), \omega_0, \alpha^{-1}, \) and \( k \), such that
\[
\|q_h - q\|_{p'} \leq c h + c (\rho(\bar{\varphi} Dv), \omega (h \nabla q))^\frac{1}{2}.
\]

**Corollary 4.6.** Let the assumptions of Theorem 4.5 be satisfied. Then, it holds
\[
\|q_h - q\|_{p'} \leq c h \|\nabla q\|_{p'}
\]
(4.45)
with a constant \( c > 0 \) depending only on the characteristics of \( \mathcal{S} \), \( \|F(Dv)\|_{1,2}, \|\nabla q\|_{p'}, \|g\|_{p'}, \delta^p(\Omega), \omega_0, \alpha^{-1}, \) and \( k \). If, in addition, \( g \in L^2(\Omega) \), then
\[
\|q_h - q\|_{p'} \leq c h
\]
(4.46)
with a constant \( c > 0 \) depending only on the characteristics of \( \mathcal{S} \), \( \|F(Dv)\|_{1,2}, \|g\|_{p'}, \|\delta + |Dv|\|^{2-p} \|\nabla q\|_{2}, \delta^p(\Omega), \omega_0, \alpha^{-1}, \) and \( k \).
Remark 4.7. It seems that the only other result for DG methods proving for the $p$-Stokes problem ($p \neq 2$) an error estimate for the pressure is [6]. There it is proved that \( \|q_h - q\|_{p'} \leq c h^{\frac{p}{p} - 1} \) if \( v \in W^{k+2,p}(\Omega) \), \( q \in W^{k+1,p'}(\Omega) \), \( k \in \mathbb{N} \), which requires at least one order higher regularity compared to our results. In the special case that the extra stress tensor has \((2, \delta)\)-structure, i.e., it is non-linear and possesses linear growth, it is proved in [9, 10, 16] that the pressure has linear order of convergence, which agrees with our results. The results in Corollary 4.2 can also be found in the context of FE methods. In particular, estimate (4.45) is proved in [2, 4, 18], while estimate (4.46) is proved in [27]. Even for FE methods there are no theoretical results proving the experimentally observed convergence rates in the case \( p > 2 \) (cf. Section 5).

Proof of Theorem 4.5. We proceed analogously to the proof of Theorem 4.1. In view of the absence of the convective term, the equality (4.6) now reads

\[
(q_h - q, \text{Div}_h \bar{z}_h) = I_1 + \alpha I_2 + I_4 + I_6, \tag{4.47}
\]

where \( I_i, i = 1, 2, 4, 6 \) are defined in (4.6). Then, resorting in (4.47) to (4.11), (4.17), (4.35), (4.41), we conclude that for every \( \bar{z}_h \in V^k_h \) with \( \|z_h\|_{\nabla, p, h} \leq 1 \), we have that

\[
(q_h - q, \text{Div}_h \bar{z}_h) \leq c h + c \left( \rho(\varepsilon(Dv)) \cdot (h^{-1} \nabla q) \right) \frac{1}{2} \tag{4.48}
\]

with a constant \( c > 0 \) depending only on the characteristics of \( S \), \( \|F(Dv)\|_{1,2} \), \( \|\nabla q\|_{p'} \), \( \|g\|_{p'} \), \( \delta p|\Omega| \), \( \omega_0 \), \( \alpha^{-1} \), and \( k \). Having at our disposal (4.48), we conclude the proof as in the proof of Theorem 4.1. \( \square \)

Proof of Corollary 4.6. We follow the arguments in the proof of Corollary 4.2, but now resort to Theorem 4.5. \( \square \)

5. Numerical experiments. In this section, we apply the LDG scheme (3.6) (or (3.7) and (3.8)) to solve numerically the system (1.1) with \( S : \mathbb{R}^{d \times d} \to \mathbb{R}^{d \times d} \), for every \( \mathbf{A} \in \mathbb{R}^{d \times d} \) defined via \( S(\mathbf{A}) := (\delta + |\mathbf{A}^{\text{sym}}|)^{p-2} \mathbf{A}^{\text{sym}} \), where \( \delta := 1e-4 \) and \( p > 2 \). We approximate the discrete solution \( v_h \in V^k_h \) of the non-linear problem (3.6) by deploying the Newton solver from PETSc (version 3.17.3), cf. [23], with an absolute tolerance of \( \tau_{\text{abs}} = 1e-8 \) and a relative tolerance of \( \tau_{\text{rel}} = 1e-10 \). The linear system emerging in each Newton step is solved using a sparse direct solver from MUMPS (version 5.5.0), cf. [1]. For the numerical flux (3.3), we choose the fixed parameter \( \alpha = 2.5 \). This choice is in accordance with the choice in [15, Table 1]. In the implementation, the uniqueness of the solution is enforced via a zero mean condition.

All experiments were carried out using the finite element software package FEniCS (version 2019.1.0), cf. [23].

For our numerical experiments, we choose \( \Omega = (-1,1)^2 \) and linear elements, i.e., \( k = 1 \). We choose \( g \in L^{p'}(\Omega) \) and boundary data \( v_0 \in W^{1,1-\frac{1}{4}}(\partial \Omega)^{3} \) such that \( v \in W^{1,p}(\Omega) \) and \( q \in Q \), for every \( x := (x_1, x_2) \in \Omega \) defined by

\[
v(x) := |x|^\beta (x_2, -x_1)\,^T, \quad q(x) := \eta \left( |x|^\gamma - (|x|^\gamma)_{\Omega} \right) \tag{5.1}
\]

are a solutions of (1.1). Here, we choose \( \beta = 1e-2 \), which implies \( F(Dv) \in W^{1,2}(\Omega) \). Concerning the pressure regularity, we consider two cases: Namely, we choose either \( \gamma = 1 - \frac{2}{p} + 1e-4 \) and \( \eta = 25 \), which just yields \( q \in W^{1,p'}(\Omega) \) (case 1), or we choose

\[\text{The exact solution is not zero on the boundary of the computational domain. However, the error is clearly concentrated around the singularity and, thus, this small inconsistency with the setup of the theory does not have any influence on the results of this paper. In particular, note that Part I of the paper (cf. [20]) already established at least the weak convergence of the method also for the fully non-homogeneous case.}\]
\[ \gamma = \alpha \frac{p-2}{2} + 1e-4 \text{ and } \eta = 1e+3, \text{ which just yields } (\delta + |\nabla v|)^{\frac{2}{p-2}} \nabla q \in L^2(\Omega) \text{ (case 2).} \]

Thus, for \( \gamma = 1 - \frac{2}{p} + 1e-4 \) and \( \eta = 25 \) (case 1), we can expect the convergence rate \( \frac{p}{2} \), while for \( \gamma = \alpha \frac{p-2}{2} + 1e-4 \) and \( \eta = 1e+3 \) (case 2), we can expect the convergence rate 1. (cf. Corollary 4.2).

We construct an initial triangulation \( T_{h_0} \), where \( h_0 = \frac{\sqrt{2}}{2} \), by subdividing a rectangular cartesian grid into regular triangles with different orientations. Finer triangulations \( T_h \), \( i = 1, \ldots , 5 \), where \( h_{i+1} = \frac{h_i}{\sqrt{2}} \) for all \( i = 1, \ldots , 5 \), are obtained by regular subdivision of the previous grid: Each triangle is subdivided into four equal triangles by connecting the midpoints of the edges, i.e., applying the red-refinement rule, cf. [3, Definition 4.8 (i)].

Then, for the resulting series of triangulations \( T_h \), \( i = 1, \ldots , 5 \), we apply the above Newton scheme to compute the corresponding numerical solutions \((v_h, L_h, S_h)^\top \in V_{h_i}^k \times X_{h}^k \times X_{h}^k, i = 1, \ldots , 5\), and the error quantities

\[ e_{q,i} := ||q_i - q||_{p'}. \quad i = 1, \ldots , 5. \]

As estimation of the convergence rates, the experimental order of convergence (EOC)

\[ \text{EOC}(e_{q,i}) := \frac{\log(e_{q,i}/e_{q,i-1})}{\log(h_i/h_{i-1})}, \quad i = 1, \ldots , 5, \]

is recorded. For different values of \( p \in \{2.25, 2.5, 2.75, 3.25, 3.5\} \) and a series of triangulations \( T_h \), \( i = 1, \ldots , 5 \), obtained by regular, global refinement as described above, the EOC is computed and presented in Table 1. In it, we observe for case 1 a convergence ratio of about \( \text{EOC}(e_{q,i}) \approx 1, i = 1, \ldots , 5 \), and in case 2 a convergence ratio of about \( \text{EOC}(e_{q,i}) \approx \frac{2}{3}, i = 1, \ldots , 5 \). Both are higher than the proved convergence rates (4.1), (4.2) in Corollary 4.2. The same convergence rate as in case 1 for the \( p \)-Stokes problem is observed in the analogous numerical experiment in [4] in the context of FE methods. This indicates that the error estimates in Corollary 4.2 and Corollary 4.6 are yet sub-optimal and it might, therefore, be possible to improve the estimates (4.1), (4.45) to \( ||q_i - q||_{p'} \leq c h \) and the estimates (4.2), (4.46) to \( ||q_i - q||_{p'} \leq c h^{2/p'} \). However, without additional regularity assumptions, such a result seems to be out of reach at the present time due to the disbalance of the shifts in the various terms.

| \( \gamma \) | \( p \) | \( \text{case 1} \) | \( \text{case 2} \) |
|---|---|---|---|
| 1 | 2.25 | 0.988 | 0.984 | 0.983 | 0.982 | 1.096 | 1.179 | 1.237 | 1.285 | 1.324 | 1.357 |
| 2 | 0.997 | 0.995 | 0.994 | 0.992 | 0.991 | 1.107 | 1.191 | 1.258 | 1.312 | 1.356 | 1.392 |
| 3 | 0.999 | 0.999 | 0.998 | 0.997 | 0.996 | 1.111 | 1.198 | 1.267 | 1.323 | 1.370 | 1.410 |
| 4 | 1.000 | 1.000 | 0.999 | 0.999 | 0.999 | 1.112 | 1.201 | 1.272 | 1.322 | 1.364 | 1.403 |
| 5 | 1.000 | 1.000 | 0.999 | 0.998 | 0.998 | 1.112 | 1.202 | 1.277 | 1.324 | 1.373 | 1.434 |
| expected | 0.900 | 0.833 | 0.786 | 0.750 | 0.722 | 0.700 | 1.000 | 1.000 | 1.000 | 1.000 |

Table 1: Experimental order of convergence: \( \text{EOC}(e_{q,i}), i = 1, \ldots , 5 \).

REFERENCES

[1] P. R. Amestoy, I. S. Duff, J. Koster, and J.-Y. L’Excellent, A fully asynchronous multifrontal solver using distributed dynamic scheduling, SIAM Journal on Matrix Analysis and Applications, 23 (2001), pp. 15–41.

[2] J. W. Barrett and W. B. Liu, Quasi-norm error bounds for the finite element approximation of a non-Newtonian flow, Numer. Math., 68 (1994), pp. 437–456.
[3] S. Bartels, *Numerical approximation of partial differential equations*, vol. 64 of Texts in Applied Mathematics, Springer, 2016, doi:10.1007/978-3-319-32354-1, https://doi.org/10.1007/978-3-319-32354-1.

[4] L. BeLENKI, L. C. BERSELLI, L. DIENING, AND M. RůŽIČKA, *On the Finite Element approximation of p-Stokes systems*, SIAM J. Numer. Anal., 50 (2012), pp. 373–397.

[5] L. C. BERSELLI AND M. RŮZIČKA, *Space-time discretization for nonlinear parabolic systems with p-structure*, IMA J. Numerical Analysis, 42 (2022), pp. 260–299, doi:10.1093/imaman/draa079, http://arxiv.org/abs/2001.09888.

[6] M. BOTTI, D. CASTANO quarantine, D. A. DI PIETRO, AND A. HARNIST, *A hybrid high-order method for creeping flows of non-Newtonian fluids*, ESAIM Math. Model. Numer. Anal., 55 (2021), pp. 2045–2073, doi:10.1051/m2an/2021051, http://arxiv.org/abs/arXiv:2003.13467.

[7] S. BRENNER AND L. SCOTT, *The mathematical theory of finite element methods*, vol. 15 of Texts in Applied Mathematics, Springer, New York, third ed., 2008, doi:10.1007/978-0-387-75934-0, https://doi.org/10.1007/978-0-387-75933-3.

[8] F. BREZZI AND M. FORTIN, *Mixed and hybrid finite element methods*, vol. 15 of Springer Series in Computational Mathematics, Springer-Verlag, New York, 1991.

[9] R. BÜSTINZA AND G. GATICA, *A mixed local discontinuous Galerkin method for a class of nonlinear problems in fluid mechanics*, J. Comput. Phys., 207 (2005), pp. 427–456.

[10] S. CONGREVE, P. HOUSTON, E. SÜLI, AND T. P. WHILER, *Discontinuous Galerkin finite element approximation of quasilinear elliptic boundary value problems II: strongly monotone quasi-Newtonian flows*, IMA J. Numer. Anal., 33 (2013), pp. 1386–1415, doi:10.1093/imanum/drs046, https://doi.org/10.1093/imanum/drs046.

[11] D. DI PIETRO AND A. ERN, *Mathematical aspects of discontinuous Galerkin methods*, vol. 69 of Mathématiques & Applications, Springer, Berlin, 2012.

[12] D. A. DI PIETRO AND A. ERN, *Mathematical aspects of discontinuous Galerkin methods*, vol. 69 of Mathématiques & Applications (Berlin) [Mathematics & Applications], Springer, Heidelberg, 2012, doi:10.1007/978-3-642-22980-0.

[13] L. DIENING AND F. ETTWIEIN, *Fractional estimates for non-differentiable elliptic systems with general growth*, Forum Math., 20 (2008), pp. 523–556.

[14] L. DIENING AND C. KREUZER, *Linear convergence of an adaptive finite element method for the p-Laplace equation*, SIAM J. Numer. Anal., 46 (2008), pp. 614–638, doi:10.1137/070681508, https://doi.org/10.1137/070681508.

[15] L. DIENING, D. KRÖNER, M. RŮZIČKA, AND I. TOULOUPOULOS, *A Local Discontinuous Galerkin approximation for systems with p-structure*, IMA J. Numer. Anal., 34 (2014), pp. 1447–1488, doi:10.1093/imanum/drt040.

[16] G. N. GATICA AND F. A. SEQUEIRA, *Analysis of an augmented HDG method for a class of quasi-Newtonian Stokes flows*, J. Sci. Comput., 65 (2015), pp. 1270–1308, doi:10.1007/s10915-015-0008-5, https://doi.org/10.1007/s10915-015-0008-5.

[17] P. Harjulehto and P. HäSTO, *Orlicz spaces and generalized Orlicz spaces*, vol. 2236 of Lecture Notes in Mathematics, Springer, 2019, doi:10.1007/978-3-030-15100-3, https://doi.org/10.1007/978-3-030-15100-3.

[18] A. HIHN, *Approximation of the p-Stokes equations with equal-order finite elements*, J. Math. Fluid Mech., 15 (2013), pp. 65–88.

[19] A. KALTENBACH AND M. RŮZIČKA, *Convergence analysis of a Local Discontinuous Galerkin approximation for nonlinear systems with balanced Orlicz-structure*, ESAIM Math. Model. Numer. Anal., (2022), http://arxiv.org/abs/2204.09984. accepted.

[20] A. KALTENBACH AND M. RŮZIČKA, *A Local Discontinuous Galerkin approximation for the p-Navier-Stokes system, Part I: Convergence analysis*, SIAM J. Numer. Anal., (2023), https://doi.org/10.1137/2204.09984. accepted.

[21] A. KALTENBACH AND M. RŮZIČKA, *A Local Discontinuous Galerkin approximation for the p-Navier-Stokes system, Part II: Convergence rates for the velocity*, SIAM J. Numer. Anal., (2023), https://doi.org/10.1137/2204.04107. accepted.

[22] J. LIONS, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Dunod, Paris, 1969.

[23] A. LOGG AND G. N. WELLS, *Dolfin: Automated finite element computing*, ACM Transactions on Mathematical Software, 37 (2010), pp. 1–28, doi:10.1145/1731022.1731030.

[24] J. Musielak, *Orlicz Spaces and Modular Spaces*, Springer, Berlin, 1983.

[25] M. M. Rao AND Z. D. REN, *Theory of Orlicz spaces*, vol. 146 of Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker Inc., New York, 1991.

[26] M. RŮZIČKA AND L. DIENING, *Non–Newtonian fluids and function spaces*, in *Nonlinear Analysis, Function Spaces and Applications*, Proceedings of NAFSA 2006 Prague, vol. 8, 2007, pp. 95–144.

[27] D. Sandri, *Sur l'approximation numérique des écoulements quasi-newtoniens dont la viscosité suit la loi puissance ou la loi de Carreau*, RAIRO Modél. Math. Anal. Numér., 27 (1993), pp. 131–155.