Lattice realization of compact $U(1)$ Chern-Simons theory with exact 1-symmetries

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We propose a bosonic $U(1)$ rotor model on a three dimensional space-time lattice. With the inclusion of a Maxwell term, we show that the low-energy semi-classical behavior of our model is consistent with the Chern-Simons field theory, $S = \int d^3x \frac{1}{4} F_{\mu\nu} A^\mu A^\nu$ at low energies. We require that the action be periodic in the lattice variables, which enforces the quantization of the $K$-matrix as a symmetric integer matrix with even diagonals. We also show that our lattice model has the exact 1-symmetries. In particular, some of those 1-symmetries are anomalous (i.e. non-on-site) in the expected way. The anomaly can be probed via the breaking of those 1-symmetries by the boundaries of space-time.

Introduction: Chern-Simons (CS) field theory is a very important field theory with myriad applications from condensed matter to quantum gravity. Though well studied in the continuum as field theory, defining CS field theory on the lattice presents an opportunity to better tame the field integration measure as well as allowing us to consider non-smooth gauge fields with singularities. Furthermore, it is well known that it is quite non-trivial to define the action of CS field theory if the fiber bundle described by the gauge field is non-trivial on the space-time, which leads to an obstruction to have a globally defined gauge fields (the connection 1-forms). One way to address this problem is to define CS theory on a space-time lattice where the lattice gauge fields for “distinct fiber bundles” are continuously connected. Then the lattice gauge field with monopoles/flux-lines is continuously connected to gauge field without monopoles/flux-lines. So once we have a space-time lattice description of CS theory, the theory is automatically well defined for gauge fields of topologically non-trivial fiber bundle, as well as for gauge fields of monopoles and flux-lines. Certainly, the lattice description of CS field theory also remove the infinity problem of the field theory.

One natural way to put CS theory on spatial lattice with continuous time is to consider lattice boson or fermion systems. If the bosons or fermions are in a quantum Hall state, then the topological order of the quantum Hall state is described by a low energy effective CS field theory of a compact gauge group. However, the boson or fermion lattice models are usually not solvable. Given a boson or fermion lattice model, we usually do not know if it is in a quantum-Hall topologically ordered phase. We usually do not know if the lattice model produces a CS theory at low energy or not, and we do not know which CS theory it produces. So here we are looking for a better result, where we can derive, under a controlled approximation, the low energy effective CS field theory from the lattice model.

A related lattice formulation of the CS action has been proposed in Ref. [2] for spatial lattice and continuous time. The proposed lattice model is not a local bosonic model, but a lattice gauge theory with a Hilbert space that does not have a tensor product decomposition (i.e. the total many-body Hilbert space of the lattice model is not a tensor product of local Hilbert space on each site). This is because the construction in Ref. [2] impose the $U(1)$ gauge equivalence generated by the “small” and “large” $U(1)$ gauge transformations, and thus the path integration is not the product of integrals over local variables (and hence the non-locality of the lattice model). The constructed (non-local) lattice model describes a CS theory of a compact $U(1)$ on a space with no boundary. Furthermore, since the gauge invariance is broken on space with boundary, it remains to figure out how to define a CS theory of compact $U(1)$ on a space with boundary. In comparison, this paper presents a different approach by starting from a local space-time lattice model (i.e. the path integration is the product of integrals over independent local variables), which also allows us to define a CS theory of compact $U(1)$ on a space with boundary. In short, the model constructed in Ref. [2] is a lattice gauge theory which is non-local, while the model constructed in this paper is a local bosonic model on space-time lattice.

Ref. [2] tried to put CS theory of non-compact $U(1)$ on space-time lattice as a quadratic (non-interacting) theory. Another lattice formulations of the CS action have been proposed in Ref. [3] for space-time lattice. Such a theory also has a non-compact $U(1)$. Furthermore, as a lattice gauge theory, the space-time lattice model is not local. For a CS theory with non-compact $U(1)$, its ground state degeneracy (as a quantized theory) on a torus will be infinity, suggesting that the lattice model, as a quantum theory, is not well defined since no well-defined local lattice quantum theory can produce non-compact $U(1)$ CS theory as low energy effective field theory.

We see that people tried to put CS theory on lattice for almost 30 years. Despite those efforts, putting an arbitrary compact-$U(1)$ CS theory on space-time lattice remain an unsolved problem, if we require that that we can reliably determine the low energy effective compact $U(1)$ CS field theory from a local lattice model (i.e. a model where the field integral is a product of integrals of local variables). In this paper, we will propose a solution to this problem.
We will propose a well-defined local bosonic model on space-time lattice eqn. 3. Under a controlled semi-classical approximation for small $g$ in eqn. 3, we show that our space-time lattice model can produce any even-$K$-matrix CS field theory of compact $U(1)$’s in continuum limit (see eqn. 8). We will rely on the cochain theory familiar from algebraic topology to construct our lattice model. While formulating “ada” on the lattice is straightforward, the key to this approach is insuring that the lattice variables remain periodic and that the action is invariant under the period (i.e. the $U(1)$’s are compact), even for space-time with boundary.

A striking character of lattice model eqn. 3 is that the Lagrangian density is not a continuous function of the field values. Also the lattice model is defined for $\mathbb{R}/\mathbb{Z}$-valued fields and is not quadratic (i.e. corresponds to an interacting theory). But the weak fluctuations in small $g$ limit are described by a quadratic action. This is why we can reliably obtain the low energy effective CS field theory in small $g$ limit.

However, being able to reliably obtain the low energy effective theory is not the most important character of our constructed model eqn. 3. What really special of a generic $U(1)$ CS theory has many 1-symmetries on lattice.8–13 A generic $U(1)$ CS theory also has many 1-form symmetries in continuum.14,15 Our lattice realization of a generic $U(1)$ CS theory is a very special one, that those 1-form symmetries in continuum become the exact 1-symmetries in our lattice model. In contrast, the quantum Hall realization of CS theory does not have those 1-symmetries. Therefore, we can state our result more precisely as the following:

We construct a local bosonic rotor model on space-time lattice that, at low energies, realizes most general compact $U(1)$ CS field theory characterized by $K$-matrix, where the 1-form symmetries in the CS field theory are realized as the exact $\mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2} \times \cdots$ 1-symmetries in our lattice model.16 Here $k_i$ are diagonal entries of the Smith normal form of the $K$-matrix. Some of the 1-symmetries are anomalous.15,17–19 Different lattice realizations of the same $U(1)$ CS field theory may lead to different anomalous 1-symmetries.

We like to remark that our model is not a gauge theory, since it is defined via a path integral that is a product of integrals of local variables. We build our model to be periodic in the lattice variables, and take the field integral over one period of each lattice variable. This periodic redundancy provides level quantization, without relying on the underlying topology of the manifold as “large gauge transformations” do. Since our lattice model is not a lattice gauge theory, its action does not has to be gauge invariant. Indeed, the action eqn. 3, on space-time with boundary, is not invariant under the usual gauge redundancy $a \rightarrow a + d\theta$, with $\theta$ a 0-cochain.

We also like to remark that our lattice model eqn. 3 is actually a tensor network path integral in space-time.20 Thus, we have found a tensor network path integral that realize a topologically ordered phase described by CS field theory. We note that the tensor in the tensor network is indexed by a $\mathbb{R}/\mathbb{Z}$-value. In other words, the dimension of the tensor is infinity.

Chern Simons Theory on Lattice: To construct our local bosonic space-time lattice model, we will use a cochain formalism on a space-time complex. A space-time complex (lattice) is a triangulation of the three-dimensional space-time with a branching structure.21–24 which is denoted as $\mathcal{M}^3$. The space-time complex is formed by simplices – the vertices, links, triangles, etc., We will use $i,j, \cdots$ to label vertices of the space-time complex. The links of the complex (the 1-simplices) will be labeled by $\langle ij \rangle, \cdots$. Similarly, the triangles of the complex (the 2-simplices) will be labeled by $\langle ijk \rangle, \cdots$. The degrees of freedom of lattice model live on the links of the space-time complex: $(a_{I}^{\mathbb{R}/\mathbb{Z}})_{ij}$ on link $\langle ij \rangle$, $I = 1,2,\cdots,\kappa$. $(a_{I}^{\mathbb{R}/\mathbb{Z}})_{ij}$ is $\mathbb{R}/\mathbb{Z}$-valued, i.e. $(a_{I}^{\mathbb{R}/\mathbb{Z}})_{ij}$ and $(a_{I}^{\mathbb{R}/\mathbb{Z}})_{ij}$ are equivalent if $(a_{I}^{\mathbb{R}/\mathbb{Z}})_{ij} - (a_{I}^{\mathbb{R}/\mathbb{Z}})_{ij} = 0$ mod 1. Such $\mathbb{R}/\mathbb{Z}$-valued fields on the links are simply the so called 1-cochains $a_{I}^{\mathbb{R}/\mathbb{Z}}$ on the space-time complex $\mathcal{M}^3$. Here we have $\kappa$ different 1-cochains $a_{I}^{\mathbb{R}/\mathbb{Z}}$ labeled by $I$. The lattice action of our bosonic model will be constructed from those 1-cochains using cup product and cochain derivative. For a more detailed introduction to the cochain formalism for defining local bosonic space-time lattice models, see Ref. 24 and Appendix A.

We want to construct our lattice bosonic model in such a way that it is very similar to a CS theory. Hopefully, the resulting lattice bosonic model realizes a topologically ordered state described the CS topological quantum field theory. Due to the similarity between 1-cochains and differential 1-form, between the cup product for cochains and wedge product for differential forms, as well as the derivative $d$ acting on them, naively, we would write the partition function for a bosonic lattice as:

$$Z = \int \prod_{I} da_{I}^{\mathbb{R}/\mathbb{Z}} e^{i2\pi \sum_{I,J} k_{IJ} \int_{M^3} \langle a_{I}^{\mathbb{R}/\mathbb{Z}} \rangle \cdot d(a_{J}^{\mathbb{R}/\mathbb{Z}})} , \quad (1)$$

which formally looks like the continuum CS field theory written in terms of differential 1-forms. Here $k_{IJ}$ are integers, and $\int_{M^3}$ means the sum over all 3-simplices in $\mathcal{M}^3$. Also $\int \prod da_{I}^{\mathbb{R}/\mathbb{Z}} \equiv \prod_{ij} \prod_{I,J} \int_{\frac{1}{2} \mathbb{Z}} d(a_{I}^{\mathbb{R}/\mathbb{Z}})_{ij}$ gives rise to the path integral, where $\prod_{ij}$ is a product over all the links. Here we have lifted the $\mathbb{R}/\mathbb{Z}$-valued $(a_{I}^{\mathbb{R}/\mathbb{Z}})_{ij}$ to a $\mathbb{R}$-valued $(a_{I}^{\mathbb{R}/\mathbb{Z}})_{ij} \in (-\frac{1}{2}, \frac{1}{2})$ before we do the path integral.

Since $a_{I}^{\mathbb{R}/\mathbb{Z}}$ is $\mathbb{R}/\mathbb{Z}$-valued, we require the action amplitude in eqn. 1 to be invariant under the following “gauge” transformation

$$a_{I}^{\mathbb{R}/\mathbb{Z}} \rightarrow a_{I}^{\mathbb{R}/\mathbb{Z}} + n_{I}, \quad (2)$$
where $n^I$ are arbitrary $\mathbb{Z}$-valued 1-cochains. But, the action amplitude $e^{i2\pi\sum_{i\leq j} k_{ij} f_{M^3} a_i^{R/Z} da_j^{R/Z}}$ is not gauge invariant and we need to fix it. For a bosonic system, with $k_{ij} \in \mathbb{Z}$, one way to fix this problem is to consider the following modified partition function (which is the main result of this paper):

$$Z = \int \prod_{i} da_i^{R/Z} \ e^{i2\pi\sum_{i\leq j} k_{ij} f_{M^3} d\left(a_i^{R/Z}(a_j^{R/Z} - |a_j^{R/Z}|)\right)}$$
$$e^{i2\pi\sum_{i\leq j} k_{ij} f_{M^3} a_i^{R/Z} \left(da_j^{R/Z} - |da_j^{R/Z}| - da_j^{R/Z}a_j^{R/Z}\right)}$$
$$e^{-i2\pi\sum_{i\leq j} k_{ij} f_{M^3} a_j^{R/Z} \left(-da_i^{R/Z} - |da_i^{R/Z}| - da_i^{R/Z}a_i^{R/Z}\right)}$$
$$e^{-i2\pi\sum_{i\leq j} k_{ij} f_{M^3} a_j^{R/Z} \left(da_i^{R/Z} - |da_i^{R/Z}| - da_i^{R/Z}a_i^{R/Z}\right)}$$

(3)

Here $\lfloor x \rfloor$ denotes the nearest integer to $x$ and $|da_j^{R/Z}|$ is the 2-cochain whose value on the triangle $(ijk)$ is given by $\langle (da_j^{R/Z})_{ijk} \rangle$. The 1-cup product $\cup$ is defined in Appendix A. We note that when $da_i^{R/Z} \approx 0$, eqn. 3 reduces to eqn. 1. The Maxwell term $e^{-\int f_{M^3} \left(da_i^{R/Z} - |da_i^{R/Z}|\right)}$ is included to make $da_i^{R/Z}$ nearly an integer if we choose $g$ to be small.

To see that the path integral eqn. 3 is invariant under gauge transformation eqn. 2 for $M^3$ with boundary, we first note that $e^{-i2\pi\sum_{i\leq j} k_{ij} f_{M^3} a_j^{R/Z} \left(-da_i^{R/Z} - |da_i^{R/Z}| - da_i^{R/Z}a_i^{R/Z}\right)}$ and $da_i^{R/Z} - |da_i^{R/Z}|$ are invariant under eqn. 2. Under eqn. 2, the term $e^{i2\pi\sum_{i\leq j} k_{ij} f_{M^3} a_j^{R/Z} \left(da_i^{R/Z} - |da_i^{R/Z}| - da_i^{R/Z}a_i^{R/Z}\right)}$ changes by a factor

$$e^{i2\pi\sum_{i\leq j} k_{ij} f_{M^3} n^I d\left(a_j^{R/Z} - da_j^{R/Z}\right) - d\left(a_j^{R/Z}a_j^{R/Z}\right)}$$
$$= e^{-i2\pi\sum_{i\leq j} k_{ij} f_{M^3} n^I d\left(a_j^{R/Z}\right)}$$

(4)

Such a factor is cancelled by the change of the term

$$e^{i2\pi\sum_{i\leq j} k_{ij} f_{M^3} d\left(a_j^{R/Z} - |a_j^{R/Z}| - da_j^{R/Z}a_j^{R/Z}\right)}$$
$$= e^{i2\pi\sum_{i\leq j} k_{ij} f_{M^3} \left(a_j^{R/Z} - |a_j^{R/Z}| - da_j^{R/Z}a_j^{R/Z}\right)}$$

(5)

So the action amplitude of the above path integral is indeed invariant under eqn 2 even when $M^3$ has boundary.

Now, we like to argue that the bosonic lattice model eqn. 3 realizes a topological order described by $U^n(1)$ CS topological quantum field theory, in the small $g$ limit. In such a limit, $da_j^{R/Z}$ is close to an $\mathbb{Z}$-valued cocycle. On a local patch of space-time, we use the gauge transformation eqn. 2 to make $da_j^{R/Z}$ to be near zero on the patch. In this case, the action amplitude in the path integral eqn. 3 becomes quadratic (i.e. non-interacting)

$$e^{i2\pi\sum_{i\leq j} k_{ij} f_{M^3} a_j^{R/Z} \cup da_j^{R/Z}}$$

(6)

Since $da_j^{R/Z}$ is close to zero, we can use a 1-form $A^I$ to describe the 1-cochain $a_j^{R/Z}$:

$$\int_{J} A^I = 2\pi (a_j^{R/Z})_{ij}$$

(7)

Then the above action amplitude can be rewritten as

$$e^{i2\pi\sum_{i\leq j} k_{ij} f_{M^3} a_j^{R/Z} \cup da_j^{R/Z}} \approx e^{i\sum_{ij} K_{ij} f_{M^3} A^I dA^J}$$

$$K_{ij} = K_{ji} \equiv \langle 2k_{ij}, \text{ if } I = J, k_{ij}, \text{ if } I < J \rangle$$

(8)

in the small $da_j^{R/Z}$ limit when $A^I$ is nearly constant on the lattice scale. Hence the low energy dynamics of our lattice bosonic model are described by a $U^n(1)$ CS field theory eqn. 5 at low energies.

We like to remark that, when $da_i^{R/Z}$ is near integers, $|da_i^{R/Z}|$ is a $\mathbb{Z}$-valued 2-cocycle. This is because if $da_i^{R/Z} = \epsilon + \lfloor da_i^{R/Z} \rfloor$ where $\epsilon$ is small, then

$$d(da_i^{R/Z}) = -de + dda_i^{R/Z} = -de.$$  

(9)

Since $d(da_i^{R/Z})$ is quantized as integer, we have

$$d(da_i^{R/Z}) = 0.$$  

(10)

Such a $\mathbb{Z}$-valued 2-cocycle $[da_i^{R/Z}]$ characterize the $U^n(1)$ principle bundle on the space-time, since

$$\int_{M^2} \left(da_i^{R/Z} - [da_i^{R/Z}]\right) = -\int_{M^2} [da_i^{R/Z}]$$

(11)

for any closed $M^2$. Note that $\int_{M^2} [da_i^{R/Z}]$ is the magnetic flux through $M^2$ which is always quantized to be an integer. In other words, $-\int_{M^2} [da_i^{R/Z}]$ is the Chern number.

The above discussion of dynamics only apply when $da_i^{R/Z}$ is near integers, i.e. when $g$ is small. When $g$ is large, the large quantum fluctuations of $a_i^{R/Z}$ in the lattice bosonic model can go between configurations representing different $U^n(1)$ principle bundles. The large $g$ ground state of our model eqn. 3 may have a different topological order from the one described by the K-matrix CS theory.

What is really special about our constructed action is that it has many 1-symmetries. First, consider the model on a closed manifold, so that we may ignore the surface term. Then under the shift

$$a_i^{R/Z} \rightarrow a_i^{R/Z} + \beta_i^{R/Z}, \quad \sum_i \beta_i^{R/Z} K_{IJ} \in \mathbb{Z}$$

(12)

where $\beta_i^{R/Z}$ are $\mathbb{R}$-$\mathbb{Z}$-valued 1-cocycles, the exponentiated action is invariant so long as $\sum_i \beta_i^{R/Z} K_{IJ}$ are $\mathbb{Z}$-valued 1-cochain. Such the transformations eqn. 12 are the 1-symmetries of lattice model eqn. 3.

To see the above result, we first note that, under the transformation eqn. 12, the action amplitude in eqn. 3 on a closed manifold changes by a factor

$$e^{i2\pi\sum_{i\leq j} k_{ij} f_{M^3} \beta_i^{R/Z} \cup da_j^{R/Z} - \left(da_j^{R/Z}a_j^{R/Z}\right)}$$
$$e^{-i2\pi\sum_{i\leq j} k_{ij} f_{M^3} \beta_j^{R/Z} \cup da_j^{R/Z} - \left(da_j^{R/Z}a_j^{R/Z}\right)}$$

(13)
Because we may integrate by parts on a closed manifold and $d\beta_{J}^{R/Z} = 0$, the change is of the form (see eqn. A10):

$$e^{-i2\pi \sum_{I \leq J} k_{I,J} f_{\partial M^{3}} \beta_{I}^{R/Z} [da_{J}^{R/Z}] + [da_{J}^{R/Z}] \beta_{J}^{R/Z} + \beta_{J}^{R/Z} - d[da_{J}^{R/Z}]} = e^{-i2\pi \sum_{I \leq J} k_{I,J} f_{\partial M^{3}} \beta_{I}^{R/Z} [da_{J}^{R/Z}]}$$

which remains unity for all $[da_{J}^{R/Z}]$ iff $\sum_{I \leq J} k_{I,J} \beta_{I}^{R/Z}$ are $\mathbb{Z}$-valued cochains. We see that on, a fixed link $ij$, the allowed values $(\beta_{I}^{R/Z})_{ij}$ form the rational lattice $K^{-1}$. The 1-symmetries are given by the rational lattice $K^{-1}$ mod out the integer lattice, which is same as integer lattice mod out lattice $K$. In other words, the 1-symmetries are $Z_{k_{1}} \times Z_{k_{2}} \times \cdots$ 1-symmetries with $k_{i}$ being the diagonal entries of the Smith normal form of $K$.

For example, for $U(1)$ Chern Simons theory with $\kappa = 1$ and $K_{11} = 2k_{11} = k$, we have a $Z_{k}$ 1-symmetry. For mutual CS theory (that describes a $Z_{n}$ gauge theory), with $(K_{I,J}) = \begin{pmatrix} 0 & n \\ n & 0 \end{pmatrix}$ we have a $Z_{n} \times Z_{n}$ 1-symmetry.

Some of the above 1-symmetries are anomalous. To see which 1-symmetries are anomalous, we need check which of the transformations in eqn. 12 changes the action amplitude when the space-time has a boundary. Under the transformation eqn. 12, the action amplitude in eqn. 8 only changes by a factor defined on the boundary $\partial M^{3}$:

$$e^{i2\pi \sum_{I \leq J} k_{I,J} f_{\partial M^{3}} \beta_{I}^{R/Z} [da_{J}^{R/Z}]} = e^{i2\pi \sum_{I \leq J} k_{I,J} f_{\partial M^{3}} \beta_{I}^{R/Z} - [da_{J}^{R/Z}] - \beta_{I}^{R/Z} a_{J}^{R/Z}}$$

We see that the transformations leave the action amplitude invariant if $\sum_{I \leq J} k_{I,J} \beta_{I}^{R/Z} = 0$, $\sum_{I \leq J} k_{I,J} \beta_{I}^{R/Z} =$ integer. We note that $\beta_{I}^{R/Z}$ satisfy the condition $\sum_{I,J} K_{I,J} \beta_{I}^{R/Z} =$ integer. Thus the first equation implies the second one. We find that the 1-symmetry transformations in eqn. 12 are anomaly-free if

$$\sum_{I \leq J} k_{I,J} \beta_{J}^{R/Z} = 0$$

(16)

For the level $k = K_{11}$ CS theory with a single $U(1)$ gauge field, this is simply the fact that the only $Z_{k}$ 1-symmetry must break at the boundary and is anomalous. For the case of mutual CS theory (ie the $Z_{n}$ gauge theory) with $Z_{n} \times Z_{n}$ 1-symmetry, this implies that one of the $Z_{n}$ 1-symmetry must break at the boundary and is anomalous. The other $Z_{n}$ 1-symmetry is anomaly-free. Note that the choice of lattice model automatically selects which of the $Z_{n}$ 1-symmetry is anomalous; one can select the opposite by replacing all $\sum_{I \leq J} k_{I,J}$ with $\sum_{I \geq J}$.

**Strong coupling limit:** We have argued that the weak coupling limit ($g \to 0$) of our model eqn. 8 gives rise to a topological order described by Abelian CS field theory eqn. 8. For invertible $K$-matrix, our model always has exactly anomalous 1-symmetry on lattice. Therefore, in the strong coupling limit $g \to \infty$, our model is either in a phase where the anomalous 1-symmetries are spontaneously broken, or a gapless phase. Symmetric trivial product state cannot be the ground state for our model. Since spontaneously broken higher symmetry is nothing but topological order, in the strong coupling limit $g \to \infty$, our model must be in a topologically ordered phase or a gapless phase.

**Framing anomaly:** It is well known that the CS theory has a framing anomaly. In other words, after integrating out the physical degrees of freedom $a_{I}^{R/Z}$ in eqn. 8 in small $g$ limit, we should get a partition function given by the 2+1D gravitational CS term:

$$Z(M^{3}, g_{\mu\nu}) \propto e^{\frac{i}{4\pi} \int_{M^{3}} \omega_{3}}$$

where the 3-form $\omega_{3}$ satisfies $d\omega_{3} = p_{1}$ and $p_{1}$ is the first Pontryagin class for the tangent bundle. Here $c$ is the chiral central charge — the difference between the numbers of positive and negative eigenvalues of the $K$-matrix. In particular,

$$\text{if we choose } K \text{ in eqn. 9 to be the } E_{8} \text{ matrix (i.e. the integer matrix with even diagonal and } \det(K) = 1), \text{ then eqn. 8 realizes the } 2+1\text{D invertible topological order with chiral central charge } c = 8.\text{ 28}$$

One may wonder, if the framing anomaly prevents us to have a lattice realization of chiral CS theory with a non-zero central charge $c \neq 0$. Our construction shows that chiral $U(1)$ CS theory can always be realized on any 2+1D space-time lattice. We think that this is possible because our space-time lattice has an extra structure — the branching structure. It is possible that for the same space-time lattice, if we choose different branching structures, the resulting partition function $Z$ may be different. This branching structure dependence of partition function may represent the framing anomaly.

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Appendix A: Cohomologies and Cohomology

Let us first set some notation. We consider a three-dimensional simplicial complex $M$, which we take to contain 0-simplices (vertices), 1-simplices (links), 2-simplices (faces), and 3-simplices (faces). In this paper, we consider our complex (also referred as space-time lattice in this paper) to have matter fields living on the vertices, and gauge fields living on the links. Moreover, we assume that these fields take values in an Abelian group.

The matter fields, denoted $g_i$, form a map from the 0-simplices of $M$ (or more formally, form a map from the space of 0-chains of $M$) to the target space. This map is called a 0-cochain, as it defines a linear map from the free abelian group on the 0-simplices of $M$ to the target space. Similarly, a gauge field $a_{ij}$ living on the links of the lattice defines a 1-cochain. One can continue this, with $n$-cochains defining maps from the $n$-simplices to the target space.

To see the explicit action of a $n$-cochain, let us label simplices by their vertices, so that an $n$-simplex is given by $[v_0, ..., v_n]$. Then an $n$-cochain $a$ assigns a target space element $a([v_0, ..., v_n])$ to any $n$-simplex. Furthermore, this map is multilinear over a formal sum of $n$-simplices with coefficients in $\mathbb{Z}$. For example, let $\sigma_1 = [v_0, ..., v_n]$ and $\sigma_2 = [w_0, ..., w_n]$. Then

$$a(2\sigma_1 - \sigma_2) = 2a(\sigma_1) - a(\sigma_2) \quad (A1)$$

Given any $n$-cochain $a$, we can create an $n+1$ cochain $da$ via:

$$da([v_0, ..., v_{n+1}]) = \sum_{i=0}^{n+1} (-1)^i a([v_0, ..., \hat{v}_i, ..., v_{n+1}]) \quad (A2)$$

where $\hat{v}_i$ means that we omit that index. One can then see that $d(da) = 0$. This allows us to construct cohomology groups as the cohomology of the complex of $n$-chains:

$$\Leftarrow C^n \Leftarrow C^{n-1} \Leftarrow \cdots \Leftarrow C^1 \Leftarrow C^0 \Leftarrow 0 \quad (A3)$$

Here the maps are just given by $d$, eg $d$ acting on elements of $C^m$, and the cohomology groups are just $\mathbb{Z}/d_n/k\text{er}d_n$.

In the case that the target space is a ring, we have an additional structure called the cup product. The cup product takes an $m$-form $a_m$ and an $n$-form $b_n$ and returns an $m+n$ form $a \cup b$, defined by:

$$a_m \cup b_n([v_0, ..., v_{m+n}]) = a_m([v_0, ..., v_m]) b_n([v_m, ..., v_{m+n}]) \quad (A4)$$

Furthermore, when considered on cohomology classes, this cup product is graded anticommutative. This means that there is a $n + m - 1$ cochain $c_{m+n-1}$ such that:

$$a_m \cup b_n = (-1)^{nm} b_n \cup a_m + dc_{m+n-1} \quad (A5)$$

To get an explicit expression for $c_{m+n-1}$, we need to introduce higher cup product $a_m \cup_k b_n$ which gives rise to a $(m + n - k)$-cochain:

$$a_m \cup_k b_n([0, 1, \cdots, m + n - k]) = \sum_{0 \leq i_0 < \cdots < i_k < n} \left(-1\right)^{i_0} a_{i_0}([0 \to i_0, i_1 \to i_2, \cdots]) \times b_n([i_0 \to i_1, i_2 \to i_3, \cdots]) \quad (A6)$$

and $a_m \cup_k b_n = 0$ for $k < 0$ or for $k > m$ or $n$. Here $i \to j$ is the sequence $i, i+1, \cdots, j-1, j$, and $p$ is the number of permutations to bring the sequence $0 \to i_0, i_1 \to i_2, \cdots; i_0 + 1 \to i_1 - 1, i_2 + 1 \to i_3 - 1, \cdots$ to the sequence $0 \to m + n - k$. (A7)

For example

$$a_m \cup_k b_n([0 \to m + n - 1]) = \sum_{i=0}^{m-1} (-)^{i(m-i)(n+1)} \times a_m([0 \to i, i + n \to m + n - 1]) b_n([i \to i + n]). \quad (A9)$$

We can see that $a \cup_0 \equiv \omega$. Unlike cup product at $k = 0$, the higher cup product of two cocycles may not be a cocycle.

For cochains $a_m, b_n$, we have:

$$d(a_m \cup_k b_n) = d a_m \cup_k b_n + (-)^m a_m \cup_k d b_n + (-)^{m+n-k} a_m \cup_k b_n \cup_k d b_n \cup_k a_m \quad (A10)$$

The above result also allows us to see that the cup product interacts with $d$ in the familiar way:

$$d(a_m \cup_k b_n) = (d a_m) \cup_k b_n + (-)^m a_m \cup_k d b_n \quad (A11)$$

which we can interpret as the Leibniz rule. In the case that there is no boundary, we can interpret this as yielding a form of integration by parts, so that $d a_m \cup_k b_n = (-1)^m a_m \cup_k d b_n$. We will abbreviate the cup product with using `$\cup$' so that $ab \equiv a \cup b$.

Appendix B: Another form of $U(1)$ CS theory on lattice

In the following, we are going to present another form of $U(1)$ CS theory on lattice:

$$Z_1 = \int [\prod \frac{R/Z}{a_I} \Theta_0[a_I] \Theta_1[a_I] \Theta_2[a_I]] \quad (B1)$$

where

$$\Theta_0[a_I] = e^{i\pi K_{ij} a_{ji}} e^{\frac{\partial}{\partial a_{ij}}} e^{-\frac{\partial}{\partial a_{ij}}}, \quad \Theta_1[a_I] = e^{-i\pi K_{ij} a_{ji}} e^{\frac{\partial}{\partial a_{ij}}} = e^{\frac{\partial}{\partial a_{ij}}} \Theta_0[a_I]$$

$$\Theta_2[a_I] = e^{-i\pi K_{ij} a_{ji}} e^{\frac{\partial}{\partial a_{ij}}} = e^{\frac{\partial}{\partial a_{ij}}}, \quad \Theta_2[a_I] = e^{-\frac{\partial}{\partial a_{ij}}} e^{\frac{\partial}{\partial a_{ij}}},$$

$$\Theta_3[a_I] = e^{-\frac{\partial}{\partial a_{ij}}} e^{\frac{\partial}{\partial a_{ij}}}$$
and $K_{ij} = \text{even integer}, K_{ij} = \text{integer}$. We note that when $d_{ij}^R/\mathbb{Z} \approx 0$, eqn. [B1] reduces to eqn. [A] (up to a surface term). The Maxwell term $e^{-\int_{M^3} \frac{\sqrt{-g}}{2} \left( F^a_{ij} \right)^2}$ is included to make $d_{ij}^R/\mathbb{Z}$ nearly an integer if we choose $g$ to be small. In the following, we will also assume that $\mathcal{M}^3$ has no boundary.

To see that the path integral eqn. [B1] is invariant under eqn. [2], we proceed term-by-term. Under eqn. [2] the term $\Theta_0$ changes by a factor

$$e^{i\pi K_{ij} \int_{M^3} n^i \left( \left[ d_{ij}^R \right] - \left[ d_{ij}^R \right] \right) n^j}$$

Using eqn. [A10], we can rewrite the terms involving $d_{ij}^R/\mathbb{Z}$:

$$e^{i\pi K_{ij} \int_{M^3} n^i \left[ d_{ij}^R \right] + \left[ d_{ij}^R \right] n^j}$$

$$= e^{i\pi \sum_{i<j} K_{ij} \int_{M^3} n^i \left[ d_{ij}^R \right] + \left[ d_{ij}^R \right] n^j}$$

$$= e^{i\pi \sum_{i<j} K_{ij} \int_{M^3} n^i \left[ d_{ij}^R \right]}$$

In the first equality, we used equation eqn. [A10] with $k = 1$, while in the third we used eqn. [A10] with $k = 2$. We have also noted that since all quantities in the exponential are integers times $i\pi$, we may dispense with minus signs at will. In the second equality, we have used the fact that the diagonal elements of the $K$-matrix are even: $K_{ij} = \text{even}$. Using the same approach, we also have that:

$$e^{i\pi K_{ij} \int_{M^3} n^i \left[ d_{ij}^R \right] n^j} = e^{i\pi \sum_{i<j} K_{ij} \int_{M^3} n^i n^j}$$

Combining the above results, we see that $\Theta_0$ changes by:

$$e^{i\pi \sum_{i<j} K_{ij} \int_{M^3} n^i \left[ d_{ij}^R \right] + \left[ d_{ij}^R \right] n^j}$$

$$= e^{i\pi \sum_{i<j} K_{ij} \int_{M^3} n^i \left[ d_{ij}^R \right] + \left[ d_{ij}^R \right] n^j}$$

Similarly, under the gauge transformation eqn. [2], the term $i\pi \sum_{i<j} K_{ij} \int_{M^3} \frac{[d_{ij}^R/\mathbb{Z}]}{2} [d_{ij}^R/\mathbb{Z}]$ changes by a factor

$$e^{i\pi \sum_{i<j} K_{ij} \int_{M^3} \frac{[d_{ij}^R/\mathbb{Z}]}{2} [d_{ij}^R/\mathbb{Z}]}$$

and so the product $\Theta_0 \Theta_1 \Theta_2$ is indeed invariant under eqn. [2] but only when $\mathcal{M}^3$ has no boundary.

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There is a small difference between 1-form symmetry transformations, although they both act on closed sub-manifold with codimension-1. The 1-form symmetry transformation acting contractable sub-
manifold is a trivial transformation (i.e. an identity operator), while the 1-symmetry transformation acting contractable sub-manifold is a non-trivial transformation not equal to identity operator.

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