A NOTE ON LYAPUNOV INSTABILITY IN NEWTONIAN DYNAMICS

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Abstract. In the class of analytic potentials, we give a new sufficient condition for the Lyapunov instability of a local minimum of the potential. In contrast with similar analytical results concerning the first non zero jet of the potential, this new condition called the geodesic property is geometric and covers certain potentials that the previous techniques do not. In particular, we show that an analytic counterexample to the converse of the Lagrange-Dirichlet stability Theorem cannot have the geodesic property.

1. Introduction

In Arnold’s Problems book [ArI] we find the following problem:

“1971-4. Prove the instability of the equilibrium 0 of analytic system \( \ddot{x} = -\partial U/\partial x \) in the case where the isolated critical point 0 of the potential \( U \) is not a minimum”

For the bibliography and comments related to the problem we highly recommend the comment section in [ArI], pages 250-253. In that section, the authors recognize the fact that “...The problem on the converse of the Lagrange-Dirichlet theorem makes therefore sense only under one or another additional assumptions (e.g., that of analyticity of the potential).”

In this note we study the case of a nonnegative analytic potential such that the critical locus of zero is degenerated for an arbitrary number of degrees of freedom. This is the first open problem described in section 3, Open problems and a conjecture, in [Pa] and as far we know it is still open. The case of two degrees of freedom was treated in [LP], [Br] and [GT]. The case of a non positive potential is trivial by a Jacobi metric argument. We prove the Lyapunov instability of a critical point in the locus under the additional assumption of having the so called geodesic property: There is a smooth geodesic contained in the smooth region of the critical locus converging to the critical point. It is interesting that while most of the additional assumptions considered in the literature are analytic concerning the first nonzero jet of the potential at the critical point [GT], [KoI], [KoII], [Ku], [KP], [MN], [Ta], ours is geometric and covers certain potentials that the previous techniques do not.

We show that in the case that the critical locus is a manifold, the zero energy limit of the Newtonian trajectories converge pointwise to a smooth geodesic in the critical locus. However, without additional hypothesis on the equilibrium point, this approach doesn’t work if the critical locus is not a smooth manifold. With this respect, we show an example of a polynomial potential with an equilibrium point such that there is no smooth geodesic converging to it through the smooth region of the locus and every shortest path has a billiard like behavior.

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The geodesic property is a sufficient but not a necessary condition; i.e. There are potentials with Lyapunov unstable critical points not having the geodesic property. The moral of our Theorem is that an analytic counterexample to the converse of the Lagrange-Dirichlet stability Theorem cannot have the geodesic property.

2. Geodesic Property

Looking forward to making the notation easier, we use the usual abuse of notation of not writing explicitly the evaluations and the Einstein summation convention for repeated indices.

Lemma 2.1. Consider a real function $f$ defined on a domain $\Omega$ of the plane. Define the following real function on $\Omega$:

$$K_f := \frac{||\nabla f||}{1 + ||\nabla f||^2}||\text{Hess } f||$$

Consider a geodesic in the graph of $f$ and its projection $\alpha$ on $\Omega$. Then:

$$K_\alpha(t) \leq K_f(\alpha(t))$$

where $K_\alpha(t)$ is the curvature of the plane curve $\alpha$ at $t$.

Proof: Consider the Monge coordinate chart $(\Omega, \phi)$ such that $\phi(x, y) := (x, y, f(x, y))$. The pullback of the euclidean metric tensor is the following:

$$g := \phi^*(\eta_3) = \begin{pmatrix} 1 + (\partial_x f)^2 & \partial_x f \partial_y f \\ \partial_y f \partial_x f & 1 + (\partial_y f)^2 \end{pmatrix} = 1 + \nabla f \otimes \nabla f$$

The determinant is $\det(g) = 1 + ||\nabla f||^2$ and its inverse is the following:

$$g^{-1} = \frac{1}{1 + ||\nabla f||^2} (1 + J(\nabla f) \otimes J(\nabla f))$$

where $J$ is the usual complex structure; i.e. $J(a, b) = (-b, a)$. The Christoffel symbols are:

$$\Gamma^k_{ab} = \frac{1}{1 + ||\nabla f||^2} ((\nabla f)^k (\text{Hess } f)_{ab})$$

Consider a geodesic $\phi \circ \gamma$ with arclength $s$ in the graph of $f$:

$$(1) \quad \frac{d^2 \gamma}{ds^2} = -\frac{1}{1 + ||\nabla f||^2} \left( \frac{d\gamma}{ds} (\text{Hess } f) \frac{d\gamma}{ds} \right) \nabla f$$

Now consider the projection of the geodesic over $\Omega$ with arclength $l$; i.e. A reparametrization of $\gamma$ such that:

$$\left\| \frac{d\gamma}{dl} \right\| = 1$$

Because of the fact:

$$1 = \left\| \frac{d\gamma}{ds} \right\|^2_{g} = \left\| \frac{d\gamma}{ds} \right\|^2 + \left( \nabla f \cdot \frac{d\gamma}{ds} \right)^2$$

we have:

$$\frac{dl}{ds} = \left( 1 - \left( \nabla f \cdot \frac{d\gamma}{ds} \right)^2 \right)^{1/2}$$
and its second derivative reads as follows:
\[
\frac{d^2 l}{ds^2} = - \left( \nabla f \cdot \frac{d\gamma}{ds} \right) \left( \nabla f \cdot \frac{d^2 \gamma}{ds^2} + \frac{d\gamma^t}{ds} \left( \text{Hess } f \right) \frac{d\gamma}{ds} \right) \left( \frac{dl}{ds} \right)^{-1}
\]
\[
= - \left( \nabla f \cdot \frac{d\gamma}{ds} \right) \left( \frac{d\gamma^t}{ds} \left( \text{Hess } f \right) \frac{d\gamma}{ds} \right) \left( \frac{1}{1 + \|\nabla f\|^2} \right) \left( \frac{dl}{ds} \right)^{-1}
\]
\[
= - \left( \nabla f \cdot \frac{d\gamma}{ds} \right) \left( \frac{d\gamma^t}{ds} \left( \text{Hess } f \right) \frac{d\gamma}{ds} \right) \frac{1}{1 + \|\nabla f\|^2} \left( \frac{dl}{ds} \right)^{-1}
\]
where we have substituted equation (1) in the first line. In particular we have:

(2) \[
\frac{d^2 l}{ds^2} \left( \frac{dl}{ds} \right)^{-2} = - \left( \nabla f \cdot \frac{d\gamma}{dl} \right) \left( \frac{d\gamma^t}{dl} \left( \text{Hess } f \right) \frac{d\gamma}{dl} \right) \frac{1}{1 + \|\nabla f\|^2}
\]

By equations (1) and (2) we have the curvature of the projected curve:
\[
\frac{d^2 \gamma}{dl^2} = \frac{d^2 \gamma}{ds^2} \left( \frac{dl}{ds} \right)^{-2} - \frac{d\gamma}{dl} \frac{d^2 l}{dl^2} \left( \frac{dl}{ds} \right)^{-2}
\]
\[
= - \frac{1}{1 + \|\nabla f\|^2} \left( \frac{d\gamma^t}{dl} \left( \text{Hess } f \right) \frac{d\gamma}{dl} \right) \left( \nabla f - \left( \nabla f \cdot \frac{d\gamma}{dl} \right) \frac{d\gamma}{dl} \right)
\]
\[
= - \frac{1}{1 + \|\nabla f\|^2} \left( \frac{d\gamma^t}{dl} \left( \text{Hess } f \right) \frac{d\gamma}{dl} \right) P_{\nabla f} (\nabla f)^\perp
\]
where \( P_V \) is the orthogonal projection onto the vector space \( V \). In particular, the norm of the curvature is bounded as follows:

\[
K \leq \frac{\|\nabla f\|}{1 + \|\nabla f\|^2} \|\text{Hess } f\| = K_f
\]
and we have the result. \( \square \)

Consider the Christoffel symbols as the components of an operator \( \Gamma \) respect to the canonical basis; i.e. The following is the Christoffel operator:
\[
\Gamma = \frac{1}{1 + \|\nabla f\|^2} \nabla f \otimes \text{Hess } f
\]

Then, the curvature bound function \( K_f \) is the norm of the Christoffel operator \( \Gamma \); i.e \( K_f = \|\Gamma\| \).

Consider the critical locus of a smooth function. Every point where the critical locus is locally a submanifold will be called a smooth point. We define the smooth region as the set of smooth points. If the function is analytic, by the Whitney stratification Theorem, the smooth region is the union of the maximal strata (respect to the incidence relation). In particular, in the analytic case the smooth region of a critical locus is a manifold (possibly the union of different dimension submanifolds) and a dense topological subspace of the critical locus.

**Proposition 2.2.** Consider the following non negative polynomial potential:

(3) \[
U(x, y, z) = (z^2 + (y - x^2)^5(y - 2x^2)^5)^2
\]

There is no smooth geodesic contained in the smooth region of the critical locus and converging to the origin.
Proof: Zero is the only critical value and its critical locus is the real singular algebraic variety:
\[ z^2 + (y - x^2)^5(y - 2x^2)^5 = 0 \]
The smooth points are those in the graph of the functions \( f_\pm \) over the region \( \Omega \) such that:
\[ f_\pm(x, y) = \pm(y - x^2)^{5/2}(2x^2 - y)^{5/2} \]
and the region \( \Omega \) is the set of points \((x, y)\) such that \( x^2 < y < 2x^2 \).

Suppose there is a smooth geodesic \( \gamma \) in the smooth region of the critical locus and converging to the origin. Then, it must be contained in the graph of one of the functions \( f_+ \) or \( f_- \). Consider its projection \( \alpha \) onto the region \( \Omega \). By Lemma 2.1, the limit of the curvature of \( \alpha \) at the origin exists and is zero. But if this limit exists, then it must be greater than or equal to two and the result follows. \( \square \)

An interesting consequence is that every shortest path (in general non smooth, it exists by the Hopf-Rinow-Cohn-Vossen Theorem, the non smooth version of the Hopf-Rinow Theorem, Theorem 2.5.28 [BBI]) from a smooth point in the critical locus to the origin has an infinite number of non smooth points; i.e. The path infinitely bounces between the curves \( z = 0, y = x^2 \) and \( z = 0, y = 2x^2 \) in a billiard like way.

See that the curve \( z = 0, y = x^2, x > 0 \) is a smooth geodesic contained in the critical locus converging to the origin. However, it is not contained in the smooth region of the critical locus. The previous result motivates the following definition:

Definition 2.1. A critical point of a potential has the geodesic property if there is some smooth geodesic converging to it contained in the smooth region of the critical locus.

Is clear that every smooth point has the geodesic property. By Proposition 2.2, respect to the potential \( \mathcal{U} \) the origin does not have the geodesic property.

3. Stability of the critical point

As it was mentioned in the previous section, to make the notation easier we use the usual abuse of notation of not writing explicitly the evaluations and the Einstein summation convention for repeated indices.

Lemma 3.1. Consider a non negative analytic potential with nonempty zero potential critical locus. Consider a smooth geodesic contained in the smooth region of the zero potential critical locus; i.e. \( \gamma : [0, s_\ast] \to \text{Smooth} ([\mathcal{U} = 0]) \). For every \( v > 0 \) consider the Newtonian trajectory \( x_v \) with initial values \( x_v(0) = \gamma(0) \) and \( \dot{x}_v(0) = v\dot{\gamma}(0) \). Then:
\[ \gamma(s) = \lim_{v \to 0^+} x_v(v^{-1}s) \]
for every \( s \) where \( \gamma \) is defined.

Proof: Because the result is local, without loss of generality, we suppose that the critical locus is a manifold and because the potential is analytic, by the Souček-Souček Theorem (the analytic version of the Sard-Brown Theorem [SS]), we also suppose that the corresponding critical value is the unique critical value.

The idea of the proof is to define a modified set of polar coordinates along the critical locus manifold and decompose Newton’s second law in the radial and longitudinal equation. Because the radial coordinate is controlled by the fact that
with expression (7) we have the equations of motion:

\[ \ddot{\psi}(r, \theta^a, y^a) = \rho(r) \frac{\nabla U}{||\nabla U||} (\psi(r, \theta^a, y^a)) \]

(4)

\[ \lim_{r \to 0^+} \psi(r, \theta^a, y^a) = \phi(y^a) \]

(5)

where \( \rho(r) = \exp(-r^{-1}) \) for \( r > 0 \). The parameter \( r \) is a reparametrization of the radial coordinate and \( \theta^a \) denotes a set of angular coordinates.

The map \( \psi \) is a continuous extension of \( \phi \) differentiable outside \( r = 0 \). It defines curvilinear coordinates with scale factors \( h_r, h_a, h_\alpha \) and normal vectors \( e_r, e_\alpha, e_a \):

\[ \frac{\partial \psi}{\partial r} = h_r e_r, \quad \frac{\partial \psi}{\partial \theta^a} = h_a e_\alpha, \quad \frac{\partial \psi}{\partial y^a} = h_a e_a \]

Denote by \( e^r, e^\alpha, e^a \) the respective dual vectors of \( e_r, e_\alpha, e_a \). These dual vectors are not normal in general.

Instead of considering \( x_v \), we will consider the reparameterized solution \( \zeta_v(s) = x_v(v^{-1}s) \) with the following motion equation:

\[ \ddot{\zeta}_v(s) = \frac{1}{v^2} \nabla U (\zeta_v(s)) = -\frac{\|\nabla U (\zeta_v(s))\|}{v^2} e_r (\zeta_v(s)) \]

(6)

For simplicity we will suppose there are no angular coordinates; the introduction of them in the argument is straightforward. In terms of the \( \psi \) coordinate chart, we have:

\[ \ddot{\xi}_v = \frac{\partial^2 \psi}{\partial r^2} \dot{r}^2 + \frac{\partial^2 \psi}{\partial y^a \partial y^b} \dot{y}^a \dot{y}^b + 2 \frac{\partial^2 \psi}{\partial y^a \partial \theta^b} \dot{y}^a \dot{\theta}^b + \frac{\partial \psi}{\partial \theta^a} \dot{\theta}^a \]

(7)

In terms of the scale factors the expression (7) reads as follows:

\[ \ddot{\xi}_v = \left( \frac{\partial h_r}{\partial r} \dot{r}^2 + h_r \left( e^r, \frac{\partial e_r}{\partial r} \right) \dot{r}^2 + \left( e^r, \frac{\partial^2 \psi}{\partial y^a \partial y^b} \right) \dot{y}^a \dot{y}^b + 2 h_r \left( e^r, \frac{\partial \psi}{\partial y^a} \right) \dot{y}^a \dot{\theta}^b + 2 h_r \left( e^r, \frac{\partial \psi}{\partial \theta^a} \right) \dot{\theta}^a \right) e_r \]

where we have used the fact \( \partial \theta^a h_r = 0 \). Then, combining the motion equation (4) with expression (7) we have the equations of motion:

\[ \frac{h_r}{h_k} \left( e^k, \frac{\partial e_r}{\partial r} \right) \dot{r}^2 + \frac{1}{h_k} \left( e^k, \frac{\partial^2 \psi}{\partial y^a \partial y^b} \right) \dot{y}^a \dot{y}^b + 2 \frac{h_r}{h_k} \left( e^k, \frac{\partial \psi}{\partial y^a} \right) \dot{y}^a \dot{\theta}^b + \dot{y}^k = 0 \]

(8)

The solution lies in the region \( U \leq v^2/2 \) where \( ||\dot{\xi}_v|| \leq 1 \) and because the factor scales \( h_r \) and \( h_k \) are bounded over compacts sets, \( \dot{r} \) and \( \dot{y}^a \) are also bounded over compacts sets. By definition of the coordinate chart \( \psi \), because \( \dot{U} \) is analytic, \( \partial_r e_r \) and \( \partial_\theta e_r \) are also bounded over compacts sets. Moreover, because \( \bigcap_{v > 0} U \leq v^2/2 \), \( ||\dot{\xi}_v||/v^2 = v^2/2 - U \) hence \( ||\dot{\xi}_v||^2/2 = 1/2 - U/v^2 \leq 1/2 \); i.e. \( ||\dot{\xi}_v|| \leq 1 \).

\( ^1 \)In the region \( U \leq v^2/2 \), \( ||\dot{\xi}_v||^2/2 = v^2/2 - U \) hence \( ||\dot{\xi}_v||^2/2 = 1/2 - U/v^2 \leq 1/2 \); i.e. \( ||\dot{\xi}_v|| \leq 1 \).
\( v^2/2 = M \) when \( v \) tends to zero, \( r \) tends to zero hence \( h_r(r) = \rho(r) \) tends to zero, \( h_a \) tends to \( h_a^0 > 0 \) (the scale factors of the map \( \phi \)) and \( \partial_r e_r \) tends to zero.

In particular, we conclude that when \( v \) tends to zero the differential equation \( (\text{8}) \) doesn’t have singularities and converges uniformly over compact sets to the equation:

\[
\frac{1}{h_k} \left( e^k, \frac{\partial^2 \phi}{\partial y^a \partial y^b} \right) y^a y^b + \ddot{y}^k = 0
\]

In particular, assuming that the initial conditions agree, when \( v \) tends to zero the solution \( \zeta_v(s) \) converges pointwise to \( \phi(y(s)) \) where \( y(s) \) is the solution of equation \( (\text{9}) \).

We claim that equation \( (\text{9}) \) is the geodesic equation; i.e. The coefficients are the Christoffel symbols. In fact, by definition we have:

\[
\partial_a (h_b e_b) = \frac{\partial^2 \phi}{\partial y^a \partial y^b} = \partial_b (h_a e_a)
\]

Hence:

\[
\frac{\partial^2 \phi}{\partial y^a \partial y^b} = \frac{1}{2} (\partial_a (h_b e_b) + \partial_b (h_a e_a)) = \frac{1}{2} \sum_l \left\langle \partial_a (h_b e_b) + \partial_b (h_a e_a), e_l \right\rangle e^l
\]

and finally:

\[
\frac{1}{h_k} \left( e^k, \frac{\partial^2 \phi}{\partial y^a \partial y^b} \right) = \frac{1}{2} \sum_l \left\langle \frac{e^k}{h_k}, e^l \right\rangle \left\langle \partial_b (h_a e_b) + \partial_b (h_a e_a), h_l e_l \right\rangle
\]

\[
= \frac{1}{2} \sum_l g^{kl} (g_{bl,b} + g_{bl,a} - g_{ab,l}) = \Gamma^k_{ab}
\]

where we have used the identities \( g_{ab} = h_a h_b \langle e_a, e_b \rangle \) and \( g^{ab} = (h_a h_b)^{-1} \langle e_a, e_b \rangle \).

Because the initial conditions of \( \zeta_v \) and \( \gamma \) agree, we have proved that the solution \( \zeta_v \) converges pointwise to the geodesic \( \gamma \) over the manifold \( M \) and the result follows. \( \square \)

Lemma \( 3.1 \) is a heuristically explanation of the variational Mapertuis principle \( (\text{A11}) \), Section D, Chapter 9): Consider a submanifold \( M \subset \mathbb{R}^n \) and a smooth geodesic \( \gamma \) in \( M \). Consider a nonnegative potential \( U \) such that \( M := \{ U = 0 \} \) and for every \( v > 0 \) define the renormalized potential \( U/v^2 \) and the Newtonian trajectory \( y_v \) with initial conditions \( y_v(0) = \gamma(0) \) and \( y_v(0) = \dot{\gamma}(0) \). The trajectory \( y_v \) is a geodesic of the Jacobi metric \( ds^2 = (1 - 2U/v^2) dy^2 \) and in the limit when \( v \) tends to zero, the trajectory \( y_v \) tends to the geodesic in \( \gamma \) in \( M \). This is the content of Lemma \( 3.1 \).

**Theorem 3.2.** Consider an analytical potential with a local minimum \( \mathbf{p} \) having the geodesic property. Then \( \mathbf{p} \) is a Lyapunov unstable equilibrium point of the Newtonian dynamics.

**Proof:** By hypothesis, there is a smooth geodesic \( \gamma \) contained in the smooth region of the critical locus and converging to \( p \); i.e. \( \gamma : (0, s_{\ast}) \to \text{Smooth}[U = 0] \) such that \( \lim_{r \to 0^+} \gamma(s) = p \). Without loss of generality, we suppose that \( q := \gamma(s_{\ast}) \neq p \). Define \( \varepsilon := d(p, q)/2 \).

By Lemma \( 3.1 \) for every natural \( n \) there is a real number \( v_n > 0 \) such that the sequence \( (v_n) \) tends to zero and \( d(x_n(T_n), q) < \varepsilon \) where \( T_n := v_n^{-1}(s_{\ast} - n^{-1}) \) and \( x_n \) is the Newtonian trajectory with initial conditions \( x_n(0) = \gamma(n^{-1}) \) and
\[ \dot{x}_n(0) = v_n \gamma(n^{-1}). \]

In particular, for every \( n \) we have \( d(x_n(T_n), p) > \varepsilon \); i.e. \( p \) is not Lyapunov stable.

Because the first nonzero jet of the potential \( \mathbf{8} \) at the origin is degenerate, none of the analytical results mentioned at the introduction applies here for the study of its stability. However, as a Corollary of Theorem \( \mathbf{3.2} \) we conclude that the origin is not Lyapunov stable. This is because \( [z = 0] \) is an invariant subspace of the Newtonian dynamics and restricted to this subspace the origin has the geodesic property hence by Theorem \( \mathbf{3.2} \) it is not Lyapunov stable.

In particular, the geodesic property is a sufficient but not a necessary condition and the potential \( \mathbf{8} \) is an example; i.e. Respect to this potential the origin is not Lyapunov stable and does not have the geodesic property. The moral of Theorem \( \mathbf{3.2} \) is the following: An analytic counterexample to the converse of the Lagrange-Dirichlet stability Theorem cannot have the geodesic property.

A completely analog argument to Proposition \( \mathbf{2.2} \) shows that the origin respect to the following polynomial potential does not the geodesic property:

\[ U(x, y, z) = \left((z - x^3)^2 + (y - x^2)^5(y - 2x^2)^5 \right)^2 \]

Its first nonzero jet is degenerate. In particular, we cannot infer nothing about the Lyapunov stability of the origin neither from Theorem \( \mathbf{3.2} \) nor from the results mentioned at the the introduction involving the positivity of the first non zero jet.

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