The Galaxy and Mass N-Point Correlation Functions: a Blast from the Past

P. J. E. Peebles

Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544

Abstract. Correlation functions and related statistics have been favorite measures of the distributions of extragalactic objects ever since people started analyzing the clustering of the galaxies in the 1930s. I review the evolving reasons for this choice, and comment on some of the present issues in the application and interpretation of these statistics, with particular attention to the question of how closely galaxies trace mass.

1. Introduction

The choice of statistical measures of the distributions of galaxies and other extragalactic objects depends on the questions one wants to address, and it is influenced by tradition and personal taste. Both have favored correlation functions and related measures, such as the moments of counts in cells, for reasons that have evolved with the subject. This review of the role of N-point correlation functions in analyses of the distributions of galaxies and mass is a folk history, that reflects my experiences. I offer it as raw material for a real history of the development of modern cosmology, and, I hope, as a useful perspective on present directions of research.

With the exception of some basic definitions and relations, presented in §2.1, I refer to references for all technical details. Almost all I know is in Peebles (1973, hereafter SAC) and Peebles (1980, hereafter LSS). It used to be good form to refer to a selection of recent papers through which the reader could trace earlier contributions. But now even the more computer-challenged, such as me, can as easily trace forward in time through citation links, and the weight of selection of references may be shifted to earlier papers. This informs my choice of references to examples of current research.

2. Two-Point Correlation Functions and Power Spectra

2.1. Definitions

Modern cosmology starts with Einstein’s (1917) assumption that the observable universe is homogeneous and isotropic in the large-scale average. I believe Jerzy Neyman and Elizabeth Scott (commencing with Neyman & Scott 1952) were the first to translate this to the approximation of the galaxy distribution as a realization of a stationary random point process. At the time the evidence for
homogeneity was sparse; it now seems compelling (Peebles 1993, §3, §7; Davis 1997).

The number density, \( n \), in a stationary point process in three dimensions may be defined by the probability a point appears in the randomly placed volume element \( dV \),

\[
dP = n
dV,
\]

and the two-point correlation function by the probability points appear in each of the volume elements \( dV_1 \) and \( dV_2 \) at separation \( r_{12} \),

\[
dP = n^2
dV_1
dV_2\left[1 + \xi(r_{12})\right].
\]

The analog for a continuous function, \( f(\vec{r}) \), is the autocorrelation function,

\[
\xi_c(r) = \langle f(\vec{x} + \vec{r})f(\vec{x})\rangle / \langle f \rangle^2 - 1.
\]

If \( f \) is constant, \( \xi_c \) vanishes. In a homogenous Poisson point process, the probability a point appears in a given volume element is statistically independent of what happens everywhere else, so the reduced function, \( \xi_c \), vanishes.

The Fourier amplitude of the distribution of points at positions \( \vec{r}_i \), in flat space periodic in a box of volume \( V_u \), is

\[
\delta_{\vec{k}} = \sum \epsilon^{i\vec{k}\cdot\vec{r}_i}.
\]

The expectation value of the square of the amplitude defines the power spectrum,

\[
P(k) = \int d^3r \xi(r)e^{i\vec{k}\cdot\vec{r}} = \frac{\langle |\delta_{\vec{k}}|^2 \rangle}{n^2V_u} - \frac{1}{n}.
\]

The \( \delta_{\vec{k}} \) are uncorrelated,

\[
\langle \delta_{\vec{k}}\delta_{\vec{k}'} \rangle = 0 \quad \text{if} \quad \vec{k} \neq \vec{k}',
\]

though not in general statistically independent. In a fair sample — size large compared to the scale over which the process is statistically related — the probability distribution of the power spectrum for each \( \vec{k} \) is exponential, a consequence of the central limit theorem.

The variance — the second central moment — of the count of points in a randomly placed volume \( V \) is

\[
\sigma^2 = \langle (N - \langle N \rangle)^2 \rangle = nV + n^2 \int_V dV_1 dV_2 \xi(r_{12}),
\]

in terms of the correlation function, and

\[
\sigma^2 = nV + n^2 \int d^3k P(k)|W_{\vec{k}}|^2, \quad W_{\vec{k}} = \int_V d^3re^{i\vec{k}\cdot\vec{r}}/(2\pi)^{3/2},
\]

in terms of the power spectrum.
2.2. Power Spectra and Correlation Functions

The Princeton program of \( N \)-point correlation function analyses of the distributions of extragalactic objects grew out of a question Sydney van den Bergh, then at the David Dunlap Observatory, the University of Toronto, put to me in 1966. Does the distribution of Abell’s (1958) rich clusters of galaxies exhibit convergence to the large-scale homogeneity assumed in most cosmologies? At the time the best evidence for large-scale homogeneity came from Hubble’s deep galaxy counts and the near isotropy of the radio, microwave, and X-ray radiation backgrounds (Peebles 1971, chapter II). Abell’s catalog was the deepest large sample of objects with good distance estimates. There are two parts to van den Bergh’s question. First, might the large fluctuations in the cluster distribution mean Abell’s catalog samples only part of a clustering hierarchy that extends to larger scales (Kiang 1967; de Vaucouleurs 1970)? Second, might the fluctuations be only apparent, a result of patchy obscuration (Neyman & Scott 1952; Limber 1953)? As will be discussed, both are tested by the scaling of statistical measures of the clustering with depth.

Our first choice of statistic was driven by Blackman & Tukey’s (1959) eloquent demonstrations of the advantages of power spectra over correlation functions in detecting a weak signal. For a distribution on the sphere one replaces the Fourier transform with a spherical harmonic transform, \( a_{lm} \), with power spectrum \( |a_{lm}|^2 \) (Yu & Peebles 1969). In a fair sample, where the coherence length is small compared to the catalog depth, the distribution of \( |a_{lm}|^2 \) is close to exponential, as noted in §2.1, and the slope of the exponential distribution is a useful measure of the number of objects per independent clump (Yu & Peebles 1969, eq. [15]). The coherence length in the Abell sample is not very much smaller than the sample depth, but the distributions of \( |a_{lm}|^2 \) are strikingly close to exponential (Hauser & Peebles 1973, Fig. 4). The key result, that answered van den Bergh’s question, is that the power spectrum of the angular distribution of the clusters scales with depth in the way expected if Abell’s catalog is a fair sample of a stationary process, as opposed to a sample from a clustering hierarchy that extends to larger scales (Hauser & Peebles 1973, Fig. 9). This analysis is continued in Bahcall & Soneira (1983).

The advantages of power spectra over correlation functions for the measurement of small fluctuations make the spectrum the statistic of choice for analyses of the anisotropy of the 3K thermal cosmic radiation (the CBR; Hu et al. 2000), and measurements of large-scale fluctuations in the galaxy distribution (Peacock & Dodds 1994). These analyses usually remove the part of sky at low galactic latitude, where interference by the Milky Way is most serious. This makes the measured spherical harmonic amplitudes convolutions of the true \( a_{lm} \) (SAC eq. [42]). Examples of the resulting correlation in the power spectrum are in Hauser & Peebles (1973). The convolution is not a problem for analyses of catalogs, where the true \( |a_{lm}|^2 \) varies slowly with \( l \). But the relatively large dipole term in the CBR anisotropy, from our peculiar velocity, interferes with estimates of higher order spherical harmonic amplitudes. Wright’s (1993) remedy is to define new expansion functions that are orthogonal to the dipole terms in the observed part of the sky. Górski (1994) discusses expansion functions that are orthogonal in the observed sky, so all the expansion coefficients are uncorrelated (though not statistically independent).
The statistics of choice for the measurement of the small-scale strongly non-linear fluctuations in the galaxy distribution are correlation functions, for they are easier to measure and interpret than their Fourier or spherical harmonic transforms. This follows the first studies of fluctuations in the galaxy distribution, that used the variance of galaxy counts in cells (Bok 1934; Zwicky 1953; Rubin 1954; Neyman, Scott & Shane 1954; Kiang 1967) The natural step is to the mean lagged product, or two-point correlation function (Limber 1953; Rubin 1954; Neyman, Scott & Shane 1954; Irvine 1961; Layzer 1963; Kiang 1967; Kiang & Saslaw 1969; Totsuji & Kihara 1969). Further comments on the use of galaxy correlation functions and power spectra are in LSS §29.

2.3. Scaling

In the first galaxy catalogs large enough for statistical analyses of clustering, the only distance measure was the apparent magnitude or magnitude limit, and the luminosity function was not well known. But the following considerations allowed useful measures of clustering and tests for reliability of the results.

The (ensemble average) reduced angular correlation function in a catalog of galaxies selected by apparent magnitude is a linear integral over the reduced spatial function. This very useful relation was derived by Limber (1953). The integral relation depends on a selection function — the fraction of galaxies selected for the catalog as a function of distance — and the selection function depends on the galaxy luminosity function.

If the catalog is close to a fair sample the reduced angular correlation functions are substantially different from zero only at angles \( \theta \ll 1 \). Then the scaling of the angular statistic with apparent magnitude is independent of the selection function (provided the selection function scales with apparent magnitude according to the inverse square law; SAC eqs. [67] and [69]). If the estimates agree with this scaling it argues the catalog is close to a fair sample, not seriously affected by variable obscuration.

Scaling in Abell’s (1958) catalog of clusters of galaxies is demonstrated in Hauser & Peebles (1973), as noted above. Scaling for galaxies is demonstrated in Peebles & Hauser (1974) and Groth & Peebles (1977) for the Zwicky et al. (1961-68) catalog, the deeper Lick Catalog (Shane & Wirtanen 1967), and the still deeper Jagellonian Field (Rudnicki et al. 1973). Maddox et al. (1990) extend the demonstration to the APM sample, at about the same depth as the Jagellonian sample, but in a much larger field. The scaling is persuasive evidence that we have a fair measure of the galaxy distribution.

The present issue of interest is the departure from scaling, at depths large enough to reveal the evolution of the clustering of the galaxies (Hogg, Cohen & Blandford 2000, Fig. 7).

2.4. The Galaxy Correlation Function

In the scaling limit a power law spatial function, \( \xi(r) \propto r^{-\gamma} \), with \( \gamma > 1 \), produces a power law angular function, \( w(\theta) \propto r^{1-\gamma} \), independent of the selection function. Totsuji & Kihara (1969) found the first evidence that the galaxy function is close to a power law. The evidence is extended, and scaling demonstrated, in Groth & Peebles (1977). The power law behavior is extended to still smaller scales in Gott & Turner (1979), and to larger scales in Maddox et al. (1990),
Correlation Functions

showing the spatial function is well approximated as

$$\xi(r) = \left(\frac{r_o}{r}\right)^\gamma, \quad \gamma = 1.77 \pm 0.04, \quad hr_o = 5 \pm 0.5 \text{ Mpc},$$

at the range of separations

$$10 \text{ kpc} \lesssim hr \lesssim 10 \text{ Mpc}.$$  \hspace{1cm} (9)

At the lower bound the luminous parts of the galaxies nearly overlap. An extension to still smaller scales, in the galaxy-mass cross correlation function, is discussed in §4.1. At $hr \sim 20$ Mpc the galaxy correlation function breaks below the power law. This appears in the Lick and Jagellonian samples (Groth & Peebles 1977; Fry & Seldner 1982), but perhaps too close to the systematic errors to be convincing. The break is well established in the APM sample. It is thought that at larger separations the galaxy distribution is anticorrelated, $\xi < 0$. This would mean that at small wavenumber, $k$, the power spectrum, $P(k)$, increases with increasing $k$. Detecting this requires a relatively deep sample and good control of the selection function as a function of position in the sky. The effect likely is seen (Sutherland et al. 1999, Fig. 9), in measurements of $P(k)$ that extend to $hr \sim 100$ Mpc.

Catalogs in progress will tighten bounds on departures from a small-scale power law, including the evidence that $\xi(r)$ rises above the power law at $r \sim r_o$ before the break down at $r \sim 3r_o$ (Soneira & Peebles 1978, Fig. 6). Modern catalogs have distance measures from redshifts and predictors of luminosities, but neither is accurate enough for a direct reconstruction of the small-scale spatial distribution. Analyses will follow Limber (1953) and Rubin (1954) in deriving spatial functions from projected functions (as in Davis & Peebles 1983).

2.5. Peculiar Velocities

One does need redshifts to measure galaxy peculiar velocities. In the space of galaxy redshift and angular position the two-point correlation function is a function of two variables, the transverse and radial separation, the former probing spatial clustering and the latter relative peculiar velocities (Davis & Peebles 1983). Large-scale flows are usefully measured by the power spectrum in redshift space (Kaiser 1987), and by the peculiar velocity autocorrelation function derived from distance predictors (Groth, Juszkiewicz & Ostriker 1989). On relatively small scales the rms relative peculiar velocity is dominated by the rich clusters (Marzke et al. 1995); here moments are not the best measure of the distribution of relative velocities (Davis, Miller & White 1997).

In §4.2 I comment on early advances in the measurements of galaxy peculiar velocities, in connection with the biased galaxy formation picture. For the state of the art, see Courteau, Strauss & Willick (2000).

3. Higher Moments and Clustering Models

One uses statistics to reduce a lot of information to a more interesting and understandable quantity. It can be useful to have a sequence of statistics, that

---

1Hubble’s constant is $H_o = 100h$ km s$^{-1}$ Mpc$^{-1}$. 

---
allows recovery of progressively more detail. Thus one may characterize a single random number by its mean and second central moment, then the third moment, and on to higher moments, and one may characterize the galaxy distribution by the mean number density, then the two-point correlation function, then the three-point function, and so on.

3.1. Higher Order Correlation Functions

Following the definition of the two-point correlation function for a stationary point process in three dimensions (eq. 2), one writes the probability of finding points in the three volume elements \(dV_1, dV_2\) and \(dV_3\) that define a triangle with sides \(r_{12}, r_{23}\) and \(r_{31}\) as

\[
dP = n^3 dV_1 dV_2 dV_3 [1 + \xi(r_{12}) + \xi(r_{23}) + \xi(r_{31}) + \zeta(r_{12}, r_{23}, r_{31})].
\]

In an idealized ensemble of catalogs of angular positions the probability of finding points in the elements of solid angle \(d\Omega_1, d\Omega_2\) and \(d\Omega_3\) is similarly written

\[
dP = N^3 d\Omega_1 d\Omega_2 d\Omega_3 [1 + w(\theta_{12}) + w(\theta_{23}) + w(\theta_{31}) + z(\theta_{12}, \theta_{23}, \theta_{31})],
\]

where \(N\) is the mean surface number density.

The linear combination in brackets in equation (12) has a simple interpretation: the two-point angular function \(w(\theta_{12})\) represents the probability points 1 and 2 are correlated in space, with point 3 accidentally close in projection, and so on, so that the reduced angular function, \(z(\theta_{12}, \theta_{23}, \theta_{31})\), is an integral over the reduced spatial function, \(\zeta(r_{12}, r_{23}, r_{31})\). Since galaxy distances never will be known well enough to untangle galaxies seen close in projection, it remains very useful to have this simple way to take account of projections.

At least five considerations motivated the introduction of the three-point correlation function to the analysis of the galaxy distribution (Peebles & Groth 1975). First, Saslaw's (1972) discussion of the theory of gravitational dynamics demonstrated to us the usefulness of methods from the treatment of a non-ideal gas. This includes the hierarchy of \(N\)-point correlation functions in position and momentum. Second, Munk (1966) showed a useful application to data, in the behavior of ocean waves. Munk discussed the third moment of the time series of wave height at a fixed position, and the bispectrum — the Fourier transform of the three-point correlation function — of ocean wave trains. Fry & Seldner (1982) gave the first estimates of the bispectrum of the galaxy distribution. The remarkable advances in the analysis of the bispectrum on the scale of weakly nonlinear fluctuations are noted in §4.3. Third, the quite stable estimates of the galaxy two-point function, evidenced by the success of the scaling test, showed us the catalogs contain other useful information, that could help distinguish different clustering models with the same two-point function. Fourth, I suspected the gravitational growth of galaxy clustering would make the reduced three-point function vary with the size of the triangle defined by the three points as the square of the two-point function (Peebles 1974a). Measurements of the galaxy three-point function thus might test the gravitational instability picture. The status of this idea is discussed in §4. And finally, the first estimates of the angular three-point function, in Zwicky’s (1961-68) catalog, were seen to be consistent with this conjecture, in the very convenient form

\[
z = q [w(\theta_{12})w(\theta_{23}) + w(\theta_{23})w(\theta_{31}) + w(\theta_{31})w(\theta_{12})],
\]
Correlation Functions

where \( q \) is a constant. This expression approaches zero when two points are close and the other far away. The other simple form constructed from \( w(\theta) \) with this wanted property, \( z \propto w(\theta_{12})w(\theta_{23})w(\theta_{31}) \), is quite inconsistent with the measurements. The great convenience for the early study of galaxy clustering is that equation (13) follows from the spatial function

\[
\zeta = Q[\xi(r_{12})\xi(r_{23}) + \xi(r_{23})\xi(r_{31}) + \xi(r_{31})\xi(r_{12})].
\] (14)

One only needs the luminosity function or selection function to translate \( q \) to \( Q \).

We introduced equation (14) to model the gravitational growth of clustering, and, at least as important, because it offers a wonderfully easy way to translate from the angular to the spatial function. The success of the scaling test applied to the three-point angular functions from the Zwicky, Lick, and Jagellonian samples convinced us the spatial function is reasonably well measured, and equation (14) is a remarkably good approximation. With the extension to larger scales, from moments of counts of APM galaxies in cells (Gaztanaga 1994; Szapudi et al. 1995), the galaxy three-point function is measured to better than a factor of two at separations in the range

\[
0.1 \lesssim hr \lesssim 10 \text{ Mpc}.
\] (15)

Equation (14) fits the measured variation of the three-point correlation function with triangle size, and the measured variation with triangle shape at \( r \lesssim r_o \). In a remarkable advance, Szapudi et al. (2000b) find that equation (14) fits the measured three-point function back to redshift \( z \sim 1 \).

The four-point function also conveniently approximates products of two-point functions (Fry & Peebles 1978). The extension to the five-point function is an even greater effort (Sharp, Bonometto & Lucchin 1984). The sensible consensus is that these higher order correlation functions are best probed by the moments of counts in cells. But the full three-point function has proved to be useful, and I would not be surprised to see a return of interest in the full \( N \)-point functions at \( N = 4 \) and 5.

3.2. The Galaxy Clustering Hierarchy Model

Neyman and Scott (1952; Neyman 1962) pioneered the development of analytically prescribed models of the galaxy distribution. The Neyman-Scott model prescription places galaxies in clusters. This model, and a close relative, a halo model that usefully represents predictions of the adiabatic cold dark matter (CDM) model for the mass distribution, are discussed in §4.4, in connection with the issue of how closely galaxies trace mass.

The measured two- through four-point galaxy correlation functions on scales \( \lesssim r_o \) are simply fit by a scale-invariant clustering hierarchy, a fractal with dimension \( D = 3 - \gamma = 1.23 \pm 0.04 \). The fractal model does not fit current ideas on the character of the mass distribution, but these ideas are debatable enough to leave open the possibility that the small-scale clustering hierarchy is of more than historical interest.

Synthetic maps constructed by the clustering hierarchy prescription look reasonably like the Lick galaxy map, provided the hierarchy is cut off at \( hr \sim 20 \text{ Mpc} \) (Soniera & Peebles 1978). If instead the hierarchy extends to much
larger scales, so $\xi(r) \propto r^{-\gamma}$ to $\xi \ll 1$, the large scale of statistically related fluctuations has a numerically small effect on the correlation functions but a distinct effect on the visual impressions of the synthetic map, making it look "blotchy." Our first check that this is remedied by truncating the hierarchy was to cut a "blotchy" map into strips $\sim 20$ Mpc wide, randomly reassemble, and repeat in the orthogonal direction. To match the evidence for a rather sharp break in the angular two-point function, we had to arrange the truncation so the spatial function, $\xi(r)$, rises above the power law at $r \sim r_o$, and then breaks sharply downward.

The clustering hierarchy model produces voids of reasonable size (Vettolani et al. 1985). I don't know how well it might agree with the striking tendency to smooth walls around voids (de Lapparent, Geller & Huchra 1986). Rich clusters of galaxies are not represented in the hierarchy model; one has to picture them as places where dynamical relaxation has erased the hierarchy. I don't know whether a hierarchical model fixed to make realistic clusters might agree with the Bahcall-Soneira (1983) scaling of the cluster richness with the cluster-cluster clustering length.

4. How Well do Galaxies Trace Mass?

The complicated history of ideas on the relation between the distributions of galaxies and mass has given us a rich suite of observations and theory that can be used to argue for or against the proposition that galaxies are useful mass tracers. We do know that, if conventional gravity physics is a good approximation on the scale of galaxies, most of the mass of a spiral or large elliptical galaxy is in a dark halo outside most of the starlight. On this scale starlight certainly is not a good tracer of mass. But the issue is whether there are length scales on which galaxies are good tracers in the sense that the galaxy $N$-point correlation functions, or the galaxy-mass cross correlation functions, are close approximations to the mass autocorrelation functions. I begin with a review of results from the 1970s that, in my reading of the evidence, argue for close relations between galaxies and mass.

4.1. Suggestions from the Statistics

I was — and am — taken with the following three aspects of the galaxy and mass $N$-point correlation functions.

First, the galaxy two-point function, $\xi(r)$, is large and positive at $r \lesssim r_o$, and quite close to zero but likely negative at larger separations. The negative part is thought to be a result of initial conditions; $\xi(r)$ need not be negative at any separation $r$. To see this, recall that in one version of the Neyman-Scott (1952) prescription all galaxies are in clusters, and the clusters are placed as a stationary homogeneous Poisson process. In this construction, the probability a galaxy is found in the volume element $dV$ at distance $r$ from a galaxy is the sum of $ndV$, the contribution of all the other clusters, because their positions are statistically independent, and the contribution from the cluster to which the galaxy belongs. That is, $\xi(r) = 0$ at $r$ greater than the largest cluster diameter, and $\xi(r) > 0$ at smaller $r$, from the chance of encountering a galaxy belonging to the same cluster. It is equally easy to construct locally compensated density
fluctuations: make an initially smooth galaxy distribution clustered by drawing the galaxies in regions of space of size $\sim r_o$ into tighter clumps. This local rearrangement makes $\xi(r)$ positive at small $r$, and $\xi(r)$ negative, representing anticorrelation, at $r \sim r_o$, in such a way that the integral

$$J(r) = \int_0^r \xi(r)d^3r$$

rapidly converges to zero at $r \gtrsim r_o$.

This rapid convergence is not observed. One might have expected to have seen it if the galaxy clustering were a result of local rearrangement, as by explosions (Peebles 1974b). Our folk theorem was that, if the strongly nonlinear small-scale clustering grew by gravity out of a primeval power spectrum that is close to flat, the mass autocorrelation function would be positive or zero everywhere. This is because gravitational flow has shear but no divergence in linear perturbation theory. In the 1970s this seemed to be a pretty good reason to suspect that gravity rather than explosions produced the clustering of the galaxies. Now other evidence for the gravitational instability picture includes large-scale flows (Courteau, Strauss & Willick 2000) and the CBR anisotropy (Hu et al. 2000).

The galaxies would be expected to have followed the growth of clustering of the mass if galaxies had been around long enough to be drawn into clumps with the mass (Peebles 1986; Tegmark & Peebles 1998).

The second aspect is a numerical coincidence between the small-scale galaxy correlation function and the dark mass halos typical of large galaxies. The mean number density of galaxies at distance $r \ll r_o$ from a galaxy (in the sense of the coadded density of neighbors averaged across a fair sample of galaxies) is $n_r = n\xi(r) = n(r_o/r)^7$, where $n$ is the cosmic mean number density. The mean mass density at distance $r \ll 100$ kpc from a large spiral galaxy is reasonably well approximated as $\rho_r \sim v_c^2/(4\pi G r^2)$, where the circular velocity, $v_c$, varies only slowly with radius. The cosmic mean mass density, $\rho$, is represented as $\Omega H_o^2 = 8\pi G \rho/3$, where $\Omega$ is the density parameter (in matter capable of clustering). The ratio is

$$\frac{\rho_r}{\rho} \sim \frac{2}{3\Omega} \left( \frac{v_c}{H_o r} \right)^2.$$  

Most of the cosmic luminosity density comes from galaxies like the Milky Way, where $v_c \sim 200$ km s$^{-1}$. With this number, and at radius $hr = 10$ kpc, where the dark halo is becoming dominant, the ratios of local to cosmic number and mass densities are

$$\frac{n_r}{n} \sim 6 \times 10^4, \quad \frac{\rho_r}{\rho} \sim \frac{3 \times 10^4}{\Omega}.$$  

These numbers agree to a factor of two (for the formerly popular value, $\Omega = 1$, or the current favorite, $\Omega \sim 0.25$).

The hierarchical model (§3.2) applied to the mass distribution, with the fractal dimension of the galaxy distribution, has to fail on small scales, because the massive halos of galaxies are not fractal. If the small-scale part of the mass autocorrelation function, $\xi_{\rho\rho}(r)$, is dominated by massive halos with the density

\begin{align*}
\text{Correlation Functions} & \\
9 & \\
\text{fluctuations: make an initially smooth galaxy distribution clustered by drawing the galaxies in regions of space of size $\sim r_o$ into tighter clumps. This local rearrangement makes $\xi(r)$ positive at small $r$, and $\xi(r)$ negative, representing anticorrelation, at $r \sim r_o$, in such a way that the integral} \\
& \quad J(r) = \int_0^r \xi(r)d^3r \text{ rapidly converges to zero at $r \gtrsim r_o$.} \\
& \text{This rapid convergence is not observed. One might have expected to have seen it if the galaxy clustering were a result of local rearrangement, as by explosions (Peebles 1974b). Our folk theorem was that, if the strongly nonlinear small-scale clustering grew by gravity out of a primeval power spectrum that is close to flat, the mass autocorrelation function would be positive or zero everywhere. This is because gravitational flow has shear but no divergence in linear perturbation theory. In the 1970s this seemed to be a pretty good reason to suspect that gravity rather than explosions produced the clustering of the galaxies. Now other evidence for the gravitational instability picture includes large-scale flows (Courteau, Strauss & Willick 2000) and the CBR anisotropy (Hu et al. 2000).} \\
& \text{The galaxies would be expected to have followed the growth of clustering of the mass if galaxies had been around long enough to be drawn into clumps with the mass (Peebles 1986; Tegmark & Peebles 1998).} \\
& \text{The second aspect is a numerical coincidence between the small-scale galaxy correlation function and the dark mass halos typical of large galaxies. The mean number density of galaxies at distance $r \ll r_o$ from a galaxy (in the sense of the coadded density of neighbors averaged across a fair sample of galaxies) is $n_r = n\xi(r) = n(r_o/r)^7$, where $n$ is the cosmic mean number density. The mean mass density at distance $r \ll 100$ kpc from a large spiral galaxy is reasonably well approximated as $\rho_r \sim v_c^2/(4\pi G r^2)$, where the circular velocity, $v_c$, varies only slowly with radius. The cosmic mean mass density, $\rho$, is represented as $\Omega H_o^2 = 8\pi G \rho/3$, where $\Omega$ is the density parameter (in matter capable of clustering). The ratio is} \\
& \quad \frac{\rho_r}{\rho} \sim \frac{2}{3\Omega} \left( \frac{v_c}{H_o r} \right)^2. \\
& \text{Most of the cosmic luminosity density comes from galaxies like the Milky Way, where $v_c \sim 200$ km s$^{-1}$. With this number, and at radius $hr = 10$ kpc, where the dark halo is becoming dominant, the ratios of local to cosmic number and mass densities are} \\
& \quad \frac{n_r}{n} \sim 6 \times 10^4, \quad \frac{\rho_r}{\rho} \sim \frac{3 \times 10^4}{\Omega}. \\
& \text{These numbers agree to a factor of two (for the formerly popular value, $\Omega = 1$, or the current favorite, $\Omega \sim 0.25$).} \\
& \text{The hierarchical model (§3.2) applied to the mass distribution, with the fractal dimension of the galaxy distribution, has to fail on small scales, because the massive halos of galaxies are not fractal. If the small-scale part of the mass autocorrelation function, $\xi_{\rho\rho}(r)$, is dominated by massive halos with the density}
run in equation (17), it gives

$$\xi_{\rho\rho}(r) = \frac{\pi^3}{2} \left( \frac{\rho_c}{\rho} \right)^2 n_g r^3 \sim 2 \times 10^3,$$

at $hr = 10$ kpc, galaxy number density $n_g \sim 0.01 h^3 \text{ Mpc}^{-3}$, and $\Omega = 0.25$. This is a factor of 30 below $\xi(r)$. As for clusters, one might think of massive halos as regions where the hierarchy is erased, maybe leaving a signature in the similarities of the galaxy-galaxy correlation function and the galaxy-mass cross correlation function in equations (17) and (18).

The third aspect is the remarkable regularity of the low order galaxy spatial correlation functions. The two-point function, $\xi(r)$, is a good approximation to a power law over three orders of magnitude in the separation, $r$, and five orders of magnitude in the value of $\xi$ (eq. (3)). The measured values of the three-point function span some seven orders of magnitude (eq. (15)). The simple model in equation (14) fits, to the accuracy of the measurements, apart from the shape-dependence at the large-scale end.

Are these regularities physically significant? As remarked, I used to think the gravitational growth of clustering could produce mass correlation functions with the same regularities. If so, it would be good evidence galaxies trace mass. This now seems questionable (Ma & Fry 2000a). The issues are reviewed in §4, but a general point is worth noting.

If structure on the scale of galaxies and larger grew by gravity, and the galaxy correlation functions were not good approximations to the mass functions, the situation would be curious. The mass functions, that matter for the operation of gravity, would not have the striking regularities of the galaxy functions. The mass functions could have regularities of their own, that are related to the galaxy functions in some subtly elegant way. Or maybe the regularities of the galaxy functions are only accidents. But both seem contrived. On the face of it, the reasonable conclusion would be that the galaxy and mass functions are closely related. There are other considerations, of course.

4.2. The Biased Galaxy Formation Picture

In this picture for structure formation galaxies are more strongly clustered than mass. This naturally follows from the adiabatic cold dark matter (CDM) model for galaxy formation, as will be discussed, and it offers an elegant way to reconcile the small relative peculiar velocities of galaxies outside the rich clusters with the mass density of the Einstein-de Sitter model. The cosmic mean mass density is now thought to be well below the Einstein-de Sitter value, but the history of ideas is worth remembering. I begin with the issue of the mass density.

Early redshift surveys revealed that the only galaxies with negative redshifts (corrected for the rotation of the Milky Way) are members of the Local Group or the Virgo Cluster (Humason, Mayall & Sandage 1956). If this were

---

2 It will be recalled that the Einstein-de Sitter cosmological model has negligible curvature of space sections at fixed world time, and a negligible cosmological constant $\Lambda$. The ratio of the cosmic mean mass density to the Einstein-de Sitter value is $\Omega$
representative it would mean galaxy peculiar velocities outside rich clusters are no more than a few hundred km s$^{-1}$, well below velocities within clusters. Consistent with this, a statistical analysis of redshifts in the Reference Catalogue of Bright Galaxies (de Vaucouleurs & de Vaucouleurs 1964) indicated that the small-scale one-dimensional rms relative velocity dispersion is only about 200 km s$^{-1}$ (Geller & Peebles 1973).

Fall (1975) introduced an important application of the relation between the rms mass peculiar velocity and the gravitational potential energy measured by the mass autocorrelation function (Irvine 1961; further discussed by Layzer 1963). Fall naturally used the galaxy correlation function. He found that if $\Omega = 1$, as in the Einstein-de Sitter model, the rms peculiar velocity would have to exceed 1000 km s$^{-1}$, well above what was suggested by the observations.

In the mid 1970s Zwicky’s (1933) missing mass problem was under discussion, albeit muted, and people generally considered the value of the cosmological density parameter, $\Omega$, a number to be measured rather than predicted. An example of the former is the remark in Geller & Peebles (1973), that the relative velocity dispersion is larger than would be expected from the mass in the luminous parts of the galaxies, an indication of missing mass. An example of the latter is Fall’s (1975) conclusion that $\Omega$ likely is well below unity. He expressed no regret; that was the straightforward reading of the evidence (Gott et al. 1974). I regretted Fall’s result, because the scale-invariant Einstein-de Sitter model seemed the best way to accommodate the gravitational growth of a near scale-invariant nonlinear galaxy clustering hierarchy (Peebles 1974a; Davis, Groth & Peebles 1977).

It is easier to measure relative velocities of galaxies than the peculiar velocities that enter Irvine’s (1961) relation. The rms relative velocities are related to $\Omega$ through the mass two- and three-point correlation functions (Peebles 1976, which generalizes Geller & Peebles 1973). A first application gave $\Omega = 0.4 \pm 0.1$, close enough to unity to keep alive my hopes for the Einstein-de Sitter model (Peebles 1979). But the result from the application to the Center for Astrophysics (CfA) redshift sample (Huchra et al. 1983) was quite inconsistent with the Einstein-de Sitter model, if galaxies trace mass (Davis & Peebles 1983).

We now know the peculiar velocity of the Local Group is large, $\sim 600$ km s$^{-1}$, but that is thought to be a result of the large coherence length of the mass fluctuations, not large $\Omega$. Also, the CfA relative velocity dispersion is biased low, because rich clusters of galaxies are under-represented (Marzke et al. 1995). But the mass within the rich clusters certainly is well below the Einstein-de Sitter value; it is the low relative velocity dispersion outside clusters that would be so difficult to reconcile with the Einstein-de Sitter mass, if it were concentrated with the field galaxies, as was pretty clearly evident in the early 1980s.

In the early 1980s many had accepted the inflation picture of the very early universe as a compelling argument for the Einstein-de Sitter model. That led to an intense interest in the nature and amount of the missing mass, under its new name, dark matter. Inflation and dark matter were the leading talks

---

3The peculiar velocity is measured relative to the homogeneous expansion of the Hubble flow.
at the 1982 Texas Symposium on Relativistic Astrophysics (Guth 1984; Pagels 1984). A crisis was the evidence that the mass density is less than the critical Einstein-de Sitter value.

A resolution, biased galaxy formation, grew out of an excellent remark by Kaiser (1984). The two-point correlation function of positions of the rich clusters of galaxies is much larger than the galaxy correlation function (Peebles & Hauser 1974; Bahcall & Soneira 1983). Kaiser (1984) showed that if clusters were the rare peaks in the mass distribution from the gravitational growth of clustering out of Gaussian fluctuations with a broad coherence length, clusters would be more strongly clustered than mass. Kaiser (1986) and Bardeen (1986) extended the thought: if most galaxies formed at less rare density peaks, they would be less clustered than the clusters, but more strongly clustered than the mass, as wanted.

These considerations led Davis et al. (1985) to explore biased galaxy formation in numerical simulations of the CDM model. Giant galaxies would form preferentially in high density regions, in tight concentrations. The assembly of less massive galaxies would tend to be completed later, in less dense regions, in looser concentrations. Voids between the concentrations of ordinary \( L \sim L_* \) galaxies\(^4\) would contain most of the mass. The void mass would be seeded for galaxy formation, but seeds for \( L \sim L_* \) galaxies in voids would tend to germinate late, under conditions that could be unfavorable for the development of observable galaxies.

Biased galaxy formation agrees with the observation that the most massive galaxies, such as the cD galaxies that prefer to be in clusters, are more strongly clustered than \( L \sim L_* \) galaxies (Hamilton 1988; Valotto & Lambas 1997), though the effect is small (Szapudi et al. 2000a). Biasing is not so clearly consistent with the similar distributions of bright and faint galaxies outside clusters, an early example of which is the strikingly similar maps of bright and faint galaxies in the CfA sample (Davis et al. 1982, Figs. 2a and 2d). This phenomenon led me to learn to like a low density universe (Peebles 1984, 1986). Other early reasons to consider a low density spatially flat universe are discussed by Turner, Steigman & Krauss (1984), Vittorio & Silk (1985), and Efstathiou, Sutherland & Maddox (1990).

Recent advances in applications of the cosmological tests indicate the density parameter in matter capable of clustering is (Hu et al. 2000; Bahcall et al. 2000)

\[
\Omega = 0.25 \pm 0.1. \quad (20)
\]

This would mean there is no need to sequester mass in the voids: galaxy relative peculiar velocities outside clusters, on scales

\[
100 \text{ kpc} \lesssim hr \lesssim 10 \text{ Mpc}, \quad (21)
\]

are consistent with the assumption that galaxies trace mass.

Though biased galaxy formation is no longer thought to be needed, numerical simulations of the CDM model still predict the presence of low mass halos

\(^4L_*\) is the luminosity at the knee of the luminosity function, and the characteristic luminosity of the galaxies that produce most of the starlight.
between the concentrations of dark matter halos that would seem to be suitable homes for $L \sim L^*$ galaxies. This does not violate equation (21), because little mass is involved, but it is quite at odds with the observations. A vivid illustration comes from the extensions of the CfA redshift survey (de Lapparent, Geller & Huchra 1986; Thorstensen et al. 1995), that add many low luminosity galaxies to the low redshift part of the first CfA survey. These faint galaxies more sharply define the voids, rather than spilling into them in the way suggested by CDM simulations. This applies to a broad range of objects, including dwarf and irregular galaxies, star-forming galaxies, low surface brightness galaxies, and high surface density gas clouds. The observational literature is reviewed in Peebles (2001a). I consider it a serious crisis for the low density CDM model.

4.3. Correlation Dynamics

Since the use of correlation functions to describe the galaxy distribution was inspired in part by the example of nonideal gases, it was natural to consider the application of a dynamical method from the analysis of nonideal gases, the BBGKY hierarchy of relations among correlation functions in position and momentum. This was first done by Saslaw (1972), Fall & Saslaw (1976), Fall & Severne (1976) and Davis & Peebles (1977). Perhaps the key lesson from the last reference is that the problem is exceedingly difficult with available methods.

The sensible response is to develop the perturbation theory of the growth of initially small departures from homogeneity. Fry (1984) led this approach. Recent applications (Frieman & Gaztañaga 1999; Feldman et al. 2001) show that if (1) the galaxy correlation functions are good approximations to the mass functions on the scale of weakly nonlinear departures from homogeneity, and (2) the clustering grew out of Gaussian initial fluctuations, then perturbation theory is in remarkably good agreement with the galaxy three-point functions derived from optical and infrared catalogs. The first assumption agrees with other evidence for a low density universe in which galaxies trace mass on scales $r \sim r_o$ (eq. [20]). The situation on smaller scales is discussed next.

4.4. The Halo Clustering Model

The Neyman-Scott prescription (§3.2) places all galaxies in clusters. The probability a galaxy is placed in the volume element $d^3r$ at position $\vec{r}$ relative to the cluster center is

$$dP \propto \rho(r) d^3r.$$  \hspace{1cm} (22)

The constant of proportionality may be a random number, independently assigned to each cluster. If the cluster centers are uniformly distributed one can use the free function, the cluster number density run $\rho(r)$, to fit the small-scale galaxy correlation function, $\xi(r)$. One can anticorrelate the cluster positions to fit the weak anticorrelation of galaxies at large separations. Variants of the Neyman-Scott prescription appear in McClelland & Silk (1978), Scargle (1981), Scherrer & Bertschinger (1991), and in the halo model (Sheth & Jain 1997; Ma & Fry 2000a, 2000b; Seljak 2000; Peacock & Smith 2000). The halo model is suggested by numerical simulations of the CDM model for structure formation. The ideas in earlier references are at most only loosely related; the prescription is broadly appealing.
I used to think that adjusting the prescription to fit the higher order galaxy correlation functions is problematic. If the cluster number density run were taken to be a power law, \( \rho(r) \propto r^{-\epsilon} \), to fit the small-scale galaxy two-point function, the predicted three- and four-point functions would be quite unacceptable (Peebles & Groth 1975; McClelland & Silk 1978; LSS §61). Ma & Fry (2000a) present a numerical demonstration. With two free functions, \( \rho(r) \) and the frequency distribution of cluster richness as a function of cluster mass, one can fit the galaxy two-point function and the scaling of the three-point function \( \zeta \) with triangle size at fixed triangle shape. That would leave the problem of fitting the variation of \( \zeta \) with triangle shape (eq. [14]), and of fitting the galaxy four-point function. But the halo cluster model takes this approach.

Generalizations of the Neyman-Scott prescription to fit the higher order correlation functions were considered by Neyman, Scott & Shane (1956) — who had in mind the evidence for hierarchical clustering — and McClelland & Silk (1978). The present state of ideas, in the context of the halo model, is reviewed in Ma & Fry (2000b) and Scoccimarro et al. (2001). To my mind the main point of principle emerging from these considerations is that the CDM model seems to predict that at \( r \sim < r_o \) the mass three-point autocorrelation function is quite different from the galaxy three-point function: galaxies are not good tracers of nonlinear mass fluctuations.

4.5. So How Well Do Galaxies Trace Mass?

The theoretical situation seems clear: the adiabatic CDM model predicts significant differences between the small-scale distributions of galaxies and mass, in two aspects. First, biased galaxy formation is a natural consequence of the model, in particular the prediction that objects with relatively low mass dark halos spread into the voids defined by the halos of \( L \sim L_* \) galaxies. Second, the CDM halo model, with a suitable prescription for the assignment of numbers of galaxies to halos, predicts a fit to the galaxy three-point function at \( r < r_o \) that is quite different from the mass three-point function.

I distrust the first point because it doesn’t agree with the observations: all known galaxy types avoid the voids. I dislike the second point because it reminds me of epicycles; I regret the loss of simplicity of a scale-invariant fractal.

The evidence that we live at a special time in the evolution of the universe, at the transition to \( \Lambda \)-dominated expansion, is a cautionary demonstration that Nature does not always choose the apparently simplest way. Are the simple regularities of the galaxy correlation functions at \( r \lesssim r_o \) physically significant, or only a sequence of accidents? The adiabatic CDM model led us to this conundrum; should we trust the model?

The dramatic success of the low density cosmologically flat CDM model in correlating the measurements of the CBR anisotropy with astrophysically reasonable parameters (Hu et al. 2000) argues this is a good approximation to how structure started forming, at redshift \( z \sim 1000 \), on the length scales probed by the CBR measurements. The CDM model also successfully coordinates observations of cosmic structure on the smaller scales of superclusters down to groups of galaxies (Bahcall et al. 2000). But Sellwood & Kosowsky (2000) list deep challenges to the CDM model on the scale of galaxies. I would add the issue
Correlation Functions

of the epoch of galaxy formation (Peebles 2001b), and, on intermediate length scales, the challenge of the void phenomenon.

In short, I can see good arguments for and against the CDM prediction that the galaxy correlation functions are not useful approximations to the mass functions on scales $\lesssim r_o$.

5. Other Histories

The $N$-point correlation functions and related statistics that are the subject of this review certainly are not always the most useful. Nearest neighbor statistics are a better probe for voids. The topology of large-scale structure (Gott et al. 1989) reveals effects not seen in the correlation functions. And one should not underestimate the importance of visual comparisons of data and synthetic maps (Scott, Shane & Swanson 1954; Kiang 1967; Soneira & Peebles 1978). It would be fascinating to see a map at the Lick depth based on the CDM halo model.

I have touched on some aspects of the history of ideas of how the galaxies formed, and the establishment of the gravitational instability picture in the form of the low density CDM model. Many other ideas for structure formation were considered in the last two decades; they employ excellent physics that could reappear in theories of the astrophysics. The universe is complicated; maybe structure formation is too. Explosions caused by superconducting cosmic strings (Ostriker, Thompson & Witten 1986) could make admirable voids, for example, provided they were subdominant enough not to be detectable on the scale of the present weakly nonlinear clustering.

Acknowledgments. I am grateful to Marc Davis, Jim Fry, Ed Groth, Chung-Pei Ma, Román Scoccimarro and Uros Seljak for helpful discussions. This work was supported in part by the USA National Science Foundation.

References

Abell, G. O. 1958, ApJS, 3, 211
Bahcall, N. A., Cen, R., Davé, R., Ostriker, J. P. & Yu, Q. 2000, ApJ, 541, 1
Bahcall, N. A. & Soneira, R. M. 1983, ApJ, 270, 20
Bardeen, J. M. 1986, in Inner Space Outer Space, eds. E. W. Kolb, M. S. Turner, D. Lindley, K. Olive & D. Seckel (Chicago: the University of Chicago Press), p. 212
Blackman, R. B. & Tukey, J. W. 1959, The Measurement of Power Spectra (New York: Dover)
Bok, B. J. 1934, Bull Harvard Obs, 895, 1
Courteau, S., Strauss, M. A. & Willick, J. A. 2000, Cosmic Flows 1999, ASP Conference Series 201
Davis, M. 1997, in Critical Dialogues in Cosmology, ed. N. Turok (Singapore: World Scientific)
Davis, M., Efstathiou, G., Frenk, C. S. & White, S. D. M. 1985, ApJ, 292, 371
Davis, M., Groth, E. J. & Peebles, P. J. E. 1977, ApJ, 212, L107
Davis, M., Huchra, J., Latham, D. W. & Toury, J. 1982, ApJ, 253, 423
Davis, M., Miller, A. & White, S. D. M. 1997, ApJ, 490, 63
Davis, M. & Peebles, P. J. E. 1977, ApJS, 34, 425
Davis, M. & Peebles, P. J. E. 1983, ApJ, 267, 465
de Lapparent, V., Geller, M. J. & Huchra, J. P. 1986, ApJ, 302, L1
de Vaucouleurs, G. 1970, Science, 167, 1203
de Vaucouleurs, G. & de Vaucouleurs, A. 1964, Reference Catalogue of Bright Galaxies (Austin: University of Texas Press)
Efstathiou, G., Sutherland, W. J. & Maddox, S. J 1990, Nature, 348, 705
Einstein, A. 1917, S-B Preuss Akad Wiss 142
Fall, S. M. 1975, MNRAS, 172, 23P
Fall, S. M. & Saslaw, W. C. 1976, ApJ, 204, 631
Fall, S. M. & Severne, G. 1976, MNRAS, 174, 241
Feldman, H. A., Frieman, J. A., Fry, J. N. & Scoccimarro, R. 2001, ApJ, 546, 652
Frieman, J. A. & Gaztañaga, E. 1999, ApJ, 521, L83
Fry, J. N. 1984, ApJ, 279, 499
Fry, J. N. & Peebles, P. J. E. 1978, ApJ, 221, 19
Fry, J. N. & Seldner, M. 1982, ApJ, 259, 474
Gaztañaga, E. 1994, MNRAS, 268, 913
Geller, M. J. & Peebles, P. J. E. 1973, ApJ, 184, 329
Górski, K. M. 1994, ApJ, 430, L85
Gott, J. R., Gunn, J. E., Schramm, D. N. & Turner, E. L. 1974, ApJ, 194, 543
Gott, J. R. & Turner, E. L. 1979, ApJ, 232, L79
Gott, J. R. et al. 1989, ApJ, 340, 625
Groth, E. J., Juszkiewicz, R. & Ostriker, J. P. 1989, ApJ, 346, 558
Groth, E. J. & Peebles, P. J. E. 1977, ApJ, 217, 385
Guth, A. H. 1984, Ann NY Acad Sci, 422, 1
Hamilton, A. J. S. 1988, ApJ, 331, L59
Hauser, M. G. & Peebles, P. J. E. 1973, ApJ, 185, 757
Hogg, D. W., Cohen, J. G. & Blandford, R. 2000, ApJ, 545, 32
Hu, W., Fukugita, M., Zaldarriaga, M., & Tegmark, M. 2000, astro-ph/0006436
Huchra, J., Davis, M., Latham, D., Tonry, J. 1983, ApJS, 52, 89
Humason, M. L., Mayall, N. U. & Sandage, A. R. 1956, AJ, 61, 97
Irvine, W. M. 1961, doctoral dissertation, Harvard University
Kaiser, N. 1984, ApJ, 284, L9
Kaiser, N. 1986, in Inner Space Outer Space, eds. E. W. Kolb, M. S. Turner, D. Lindley, K. Olive & D. Seckel (Chicago: the University of Chicago Press), p. 258
Kaiser, N. 1987, MNRAS, 227, 1
Kiag, T. 1967, MNRAS, 135, 1
Kiang, T. & Saslaw, W. C. 1969, MNRAS, 143, 129
Layzer, D. 1963, ApJ, 138, 174
Limber, D. N. 1953, ApJ, 117, 134
Ma, C.-P. & Fry, J. N. 2000a, ApJ, 531, L87
Ma, C.-P. & Fry, J. N. 2000b, ApJ, 543, 503
Maddox, S. J., Efstathiou, G., Sutherland, W. J. & Loveday, J. 1990, MNRAS, 242, 43P
Marzke, R. O., Geller, M. J. da Costa, L. N. & Huchra, J. P. 1995, AJ, 110, 477
McClelland, J. & Silk, J. 1978, ApJS, 36, 389
Munk, W. H. 1966, in Advances in Earth Science, ed. P. M. Hurley (Cambridge: MIT Press), p. 185
Neyman, J. 1962, in Problems of Extragalactic Research, ed. G. C. McVittie (New York: Macmillan) p. 294
Neyman, J. & Scott, E. L. 1952, ApJ, 116, 144
Neyman, J., Scott, E. L. & Shane, C. D. 1954, ApJS, 1, 269
Neyman, J., Scott, E. L. & Shane, C. D. 1956, Proc Third Berkeley Symposium on Mathematical Statistics and Probability, p. 75, §11
Ostriker, J. P., Thompson, C. & Witten, E. 1986, Phys Lett B, 280, 321
Pagels, H. R. 1984, Ann NY Acad Sci, 422, 15
Peacock, J. A. & Dodds, S. J. 1994, MNRAS, 267, 1020
Peacock, J. A. & Smith, R. E. 2000, MNRAS, 318, 1144
Peebles, P. J. E. 1971, Physical Cosmology (Princeton: Princeton University Press)
Peebles, P. J. E. 1973, ApJ, 185, 413
Peebles, P. J. E. 1974a, ApJ, 189, L52
Peebles, P. J. E. 1974b, AA, 32, 197
Peebles, P. J. E. 1976, ApJ, 205, L109
Peebles, P. J. E. 1979, AJ, 84, 730
Peebles, P. J. E. 1980, The Large-Scale Structure of the Universe (Princeton: Princeton University Press)
Peebles, P. J. E. 1984, ApJ, 284, 439
Peebles, P. J. E. 1986, Nature, 321, 27
Peebles, P. J. E. 1993, Principles of Physical Cosmology (Princeton: Princeton University Press)
Peebles, P. J. E. 2001a, astro-ph/0101127
Peebles, P. J. E. 2001b, astro-ph/0102327
Peebles, P. J. E. & Groth, E. J. 1975, ApJ, 196, 1
Peebles, P. J. E. & Hauser, M. G. 1974, ApJS, 28, 19
Rubin, V. C. 1954, Proc NAS, 40, 541
Rudnicki, K., Dworak, T. Z., Flin, P. Baranowski, B & Sendrakowski, A. 1973, Acta Cosmologica, 1, 7
Saslaw, W. C. 1972, ApJ, 177, 17
Scargle, J. D. 1981, ApJS, 45, 1
Scherrer, R. J. & Bertschinger, E. 1991, ApJ, 381, 349
Scoccimarro, R., Sheth, R. K., Hui, L. & Jain, B. 2001, ApJ, 546, 20
Scott, E. L., Shane, C. D. & Swanson, M. D. 1954, ApJ, 119, 91
Seljak, U. 2000, MNRAS, 318, 203
Sellwood, J. A. & Kosowsky, A. 2000, in Gas and Galaxy Evolution, eds. J. E. Hibbard, M. P. Rupen & J. H. van Gorkum, ASP Conference Series, astro-ph/0009074
Sheth, R. K. & Jain, B. 1997, MNRAS, 285, 231
Shane, C. D. & Wirtanen, C. A. 1967, Pub Lick Obs, 22, Part 1
Sharp, N. A., Bonometto, S. A. & Lucchin, F. 1984, AA, 130, 79
Soneira, R. M. & Peebles, P. J. E. 1978, AJ, 83, 845
Sutherland, W. et al. 1999, MNRAS, 308, 289
Szapudi, I., Branchini, E., Frenk, C. S., Maddox, S. & Saunders, W. 2000a, MNRAS, 318, L45
Szapudi, I., Dalton, G. B., Efstathiou, G. & Szalay, A. S. 1995, ApJ, 444, 520
Szapudi, I., Postman, M., Lauer, T. L. & Oegerle, W. 2000b, astro-ph/008131
Tegmark, M. & Peebles, P. J. E. 1998, ApJ, 500, L79
Thorstensen, J. R., Kurtz, M. J., Geller, M. J., Ringwald, F. A. & Wegner, G. 1995, AJ, 109, 2368
Totsuji, H. & Kihara, T. 1969, Publ Astron Soc Japan, 21, 221
Turner, M. S., Steigman, G. & Krauss, L. M. 1984, Phys Rev Lett, 52, 2090
Valotto, C. A. & Lambas, D. G. 1997, ApJ, 481, 594
Vettolani, G., de Souza, R. E., Marano, B. & Chincarini, G. 1985, AA, 144, 506
Vittorio, N. & Silk, J. 1985, ApJ, 297, L1
Wright, E. L. 1993, Ann NY Acad Sci, 688, 836
Yu, J. T. & Peebles, P. J. E. 1969, ApJ, 158, 103
Zwicky, F. 1933, Helv. Phys. Acta, 6, 110
Zwicky, F. 1953, Helv Phys Acta, 26, 241
Zwicky, F., Herzog, E., Wild, P., Karpowicz, M. & Kowal, C. T. 1961-68, Catalogue of Galaxies and Clusters of Galaxies, in 6 vols. (Pasadena: California Institute of Technology)