POLYNOMIAL BLOW-UP UPPER BOUNDS FOR
THE EINSTEIN-SCALAR FIELD SYSTEM
UNDER SPHERICAL SYMMETRY

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Abstract. For general gravitational collapse, inside the black-hole region, singularities \((r = 0)\) may arise. In this article, we aim to answer how strong these singularities could be. We analyze the behaviors of various geometric quantities. In particular, we show that in the most singular scenario, the Kretschmann scalar obeys polynomial blow-up upper bounds \(O(1/r^N)\). This improves previously best-known double-exponential upper bounds \(O(\exp\exp(1/r))\). Our result is sharp in the sense that there are known examples showing that no sub-polynomial upper bound could hold. Finally we do a case study on perturbations of the Schwarzschild solution.

1. Introduction

1.1. Motivation. In [4], Christodoulou studied the dynamical evolution of Einstein-scalar field system:

\[
\text{Ric}_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 2 T_{\mu\nu},
\]

\[
T_{\mu\nu} = \partial_{\mu} \phi \partial_{\nu} \phi - \frac{1}{2} g_{\mu\nu} \partial^\sigma \phi \partial_\sigma \phi.
\]

(1.1)

Since \(\nabla^\mu (\text{Ric}_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}) = 0\), the scalar field satisfies \(\Box g \phi = 0\).

Under spherical symmetry, Christodoulou first established a sharp trapped surface formation criterion. Consider the characteristic initial value problem for (1.1) in the rectangle region of a Penrose diagram blow:

We use a double-null foliation. Here \(u\) and \(v\) are optical functions: \(u = \text{constant}\) stands for the outgoing null hypersurface; \(v = \text{constant}\) stands for the incoming null hypersurface.

Under spherical symmetry, axial \(\Gamma\) is the center (invariant under \(SO(3)\)). Initial data are prescribed along outgoing cone \(u = u_0\) and incoming cone \(v = v_1\).

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\footnote{A trapped surface is a two-dimensional sphere, with both incoming and outgoing null expansions negative.}
Under the above assumption, we have the following ansatz for the metric of the 3 + 1-dimensional spacetime:

\[ g_{\mu\nu}dx^\mu dx^\nu = -\Omega^2(u, v)du dv + r^2(u, v)(d\theta^2 + \sin^2 \theta d\phi^2). \]  

(1.2)

Each point \((u, v)\) in above diagram stands for a 2-sphere \(S_{u,v}\). We define its Hawking mass as

\[ m(u, v) = \frac{r}{2}(1 + 4\Omega^2 \partial_u r \partial_v r). \]  

(1.3)

For initial mass input along \(u = u_0\), we define

\[ \eta_0 := \frac{m(u_0, v_2) - m(u_0, v_1)}{r(u_0, v_2)}, \]  

and denote \(\delta_0 := \frac{r(u_0, v_2) - r(u_0, v_1)}{r(u_0, v_2)}\).

**Theorem 1.1.** Let

\[ E(x) := \frac{x}{(1 + x)^2 \left[ \ln \left( \frac{1}{2x} \right) + 5 - x \right]} \]

We prescribe characteristic initial data along \(u = u_0\) and \(v = v_1\) for solving \((1.1)\). For initial mass input along \(u = u_0\), suppose that the following lower bound holds for \(\eta_0\):

\[ \eta_0 > E(\delta_0) \]

Then there exist a trapped surface i.e. \(\partial_v r < 0\) in \(D\).

**Remark 1.** By comparing the order of the lower bounds of \(\eta_0\), for \(0 < \delta_0 \ll 1\) we have: If \(\eta_0 > \delta_0 \ln \left( \frac{1}{\delta_0} \right)\), then a trapped surface is guaranteed to form in \(D\).

**Remark 2.** To prove Theorem \(\text{I.1}\) Christodoulou didn’t impose any assumption along incoming cone \(v = v_1\). And his original proof was based on a geometric Bondi coordinate together with a null frame. In a forth coming paper \([3]\) we reprove Theorem \(\text{I.1}\) with double null foliations and generalize this result to Einstein-Maxwell-scalar field system.

Once a trapped surface is formed, in \([4]\) Christodoulou further showed that the Penrose diagram for such spacetimes is as follows:

![Penrose diagram](https://via.placeholder.com/150)

Here \(\Gamma\) is the center (invariant under \(SO(3)\)). \(B_0\) is the first singular point along \(\Gamma\). \(A\) stands for an apparent horizon. Under spherical symmetry, \(A = \{(u, v)| \text{ where } \partial_v r(u, v) = 0\}\). The spacetime region between \(A\) and \(B\) is called the trapped region \(\mathcal{T}\), where \(\partial_v r(u, v) < 0\) and \(r(u, v) > 0\). The
hypersurface $\mathcal{B}$ is the future boundary of this spacetime; it is singular. In [4], Christodoulou also proved that at any point $(u, v)$ of the singular boundary $\mathcal{B}$, we have $r(u, v) = 0$.

A natural question to ask is: how singular are the curvatures at this future boundary $\mathcal{B}$? In [4], Christodoulou showed that $\mathcal{B}$ is spacelike. And at any point $(u, v) \in \mathcal{T}$ and $(u, v)$ is close to $\mathcal{B}$, a lower bound of Kretschmann scalar holds:

$$R^\alpha\beta\gamma\delta R_{\alpha\beta\gamma\delta}(u, v) \geq \frac{1}{r(u, v)^6}.$$  

To prove the lower bounds, Christodoulou used an ODE type estimates: in [4], by algebraic calculations, it can be showed that at $(u, v)$

$$R^\alpha\beta\gamma\delta R_{\alpha\beta\gamma\delta}(u, v) \geq \frac{32 m(u, v)^2}{r(u, v)^6}, \quad (1.4)$$

where $m(u, v)$ is the Hawking mass of $S_{u,v}$ defined in (1.3). Remarkably, $m(u, v)$ satisfies an ODE type monotone property: in the trapped region $\mathcal{T}$, it holds that $\partial_u m(u, v) \geq 0$.

![Diagram](https://via.placeholder.com/150)

Fix $\tilde{b}_0 \in \mathcal{T}$ and $\tilde{b}_0$ close to $\mathcal{B}$. Assume $\tilde{b}_0$ has coordinate $(\tilde{u}_0, \tilde{v}_0)$ and $b_1 \in \mathcal{A}$ has coordinate $(\tilde{u}_1, \tilde{v}_0)$. Then at $\tilde{b}_0$ we have

$$R^\alpha\beta\gamma\delta R_{\alpha\beta\gamma\delta}(\tilde{u}_0, \tilde{v}_0) \geq \frac{32 m(\tilde{u}_0, \tilde{v}_0)^2}{r(\tilde{u}_0, \tilde{v}_0)^6} \geq \frac{32 m(\tilde{u}_1, \tilde{v}_0)^2}{r(\tilde{u}_0, \tilde{v}_0)^6} = \frac{8 r(\tilde{u}_1, \tilde{v}_0)^2}{r(\tilde{u}_0, \tilde{v}_0)^6}.$$  

For the second inequality, we use $\partial_u m(u, v) \geq 0$. And for the last identity, we use that along apparent horizon $\mathcal{A}$ it holds that $\partial_v r(\tilde{u}_1, \tilde{v}_0) = 0$ and thus $m(\tilde{u}_1, \tilde{v}_0) = r(\tilde{u}_1, \tilde{v}_0)/2$. Hence we get at $\tilde{b}_0 \in \mathcal{T}$ near $\mathcal{B}$

$$R^\alpha\beta\gamma\delta R_{\alpha\beta\gamma\delta}(\tilde{u}_0, \tilde{v}_0) \geq \frac{1}{r(\tilde{u}_0, \tilde{v}_0)^6}.$$  

This derives the lower bounds of $R^\alpha\beta\gamma\delta R_{\alpha\beta\gamma\delta}$ close to $\mathcal{B}$.

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2See Theorem 5.1 (j) in [4].

3See Theorem 5.1 (l) in [4].
How about the upper bound? Following the qualitative extension principle established by Christodoulou in [5], it can be proved that at any point \((u, v) \in \mathcal{T}\) near \(\mathcal{B}\)
\[
R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}(u, v) \lesssim \exp \left( \exp \left( \frac{1}{r(u, v)} \right) \right).
\]
To get a better upper bound, we need to give a different proof and we need to improve all the estimates into quantitative sharp estimates. In this article, we improve the double-exponential upper bounds to polynomial rates.

**Theorem 1.2.** With the same characteristic initial data Christodoulou used in [4], for the dynamical spacetime solutions of (1.1) under spherical symmetry, inside a trapped region, at any point \((u, v) \in \mathcal{T}\) and \((u, v)\) is close to \(\mathcal{B}\), there exists a positive number \(N\) (depending on the initial data at an earlier time), such that
\[
R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}(u, v) \lesssim \frac{1}{r(u, v)^N}.
\]

**Remark 3.** With the previously mentioned lower bound, we have
\[
\frac{1}{r(u, v)^6} \lesssim R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}(u, v) \lesssim \frac{1}{r(u, v)^N}.
\]
Hence polynomial blow-up upper bounds are sharp.

**Remark 4.** By using (1.4), we also bound the Hawking mass. And it holds
\[
m(u, v) \lesssim \frac{1}{r(u, v)^{\frac{N}{2} - 3}},
\]
where \(N \geq 6\) is a constant depends on initial data.

To obtain Theorem 1.2, a crucial step is to prove sharp blow-up rates for \(\partial_u \phi\) and \(\partial_v \phi\). Here we have

**Theorem 1.3.** Under the same assumptions as in Theorem 1.2 at any point \((u, v) \in \mathcal{T}\) and \((u, v)\) is close to \(\mathcal{B}\), there exist positive numbers \(D_1\) and \(D_2\) (depending on the initial data), such that
\[
|\partial_u \phi(u, v)| \leq \frac{D_1}{r(u, v)^2}, \quad |\partial_v \phi(u, v)| \leq \frac{D_2}{r(u, v)^2}.
\]

**Remark 5.** Here the exponent 2 is sharp.

**Remark 6.** These estimates further imply \(|\phi| \lesssim |\log r|\). And the \(N\) in Theorem 1.2 depends on the values of \(D_1\) and \(D_2\).

As a case study, in Section 9 we also provide more precise upper bounds for spacetimes close to Schwarzschild metric:

**Theorem 9.2.** We consider the trapezoid region \(T_0\) below.

\footnote{It states that for characteristic initial data prescribed on initial incoming and outgoing hypersurfaces \(\{(u, v)\}\), where \(r(u, v) \geq \epsilon > 0\), then for (1.1) the local existence towards the future can be proved.}
For $l_0$ being a large positive constant, we prescribe initial data along $r = 1/2l_0$: requiring

$$
|\partial_r r + \frac{M}{r}| \leq o_0(1) \cdot \frac{M}{r}, \quad |\partial_v r + \frac{M}{r}| \leq o_0(1) \cdot \frac{M}{r},
$$

$$
|\Omega^2 - \frac{2M}{r}| \leq o_0(1) \cdot \frac{M}{r},
$$

$$
|\partial_v \phi| \leq o_0(1) \cdot \frac{1}{r^{2}}, \quad |\partial_v \phi| \leq o_0(1) \cdot \frac{1}{r^{2}},
$$

where $o_0(1)$ is a small positive number depending on initial data. Then for the dynamical spacetime solutions of (1.1) under spherical symmetry, under the prescribed initial data, in the open trapezoid region above, we have

$$
|R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta}| \lesssim \frac{1}{r^{6+o_0(1)^2}}. \quad (1.5)
$$

Remark 7. From the above theorem, we can also conclude that as the initial perturbation $o_0(1) \to 0$, the upper bound of blow-up rate $6 + o_0(1)^2 \to 6$.

1.2. New Ingredients.

(1) In this paper we study the blow-up mechanism, which is NOT ODE type. And we find an interesting log structure.

To derive the blow-up upper bounds, we use the full expression of $R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}$. See (8.1). And the sharp upper bounds of $\Omega^{-2}(u,v)$ are crucial. To bound $\Omega^{-2}(u,v)$, we need to use a wave-type equation for $\log \Omega(u,v)$:

$$
r^2 \partial_u \partial_v \log \Omega = \partial_u r \partial_v r + \frac{1}{4} \Omega^2 - r^2 \partial_u \phi \partial_v \phi. \quad (1.6)
$$

The log structure here will play a very important role. From the above equation, we also see that to bound $\Omega^{-2}$ the sharp bounds for $\partial_u \phi, \partial_v \phi$ are also required. This requires a thorough analysis of the following wave equation as well

$$
r \partial_u \partial_v \phi = -\partial_u r \partial_v \phi - \partial_v r \partial_u \phi.
$$

In this paper, we explore the log structure and study the above wave equations. We derive the sharp upper bounds for $\partial_u \phi, \partial_v \phi$ and $\Omega^{-2}$. For Einstein-scalar field system, these bounds are new.

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5 More details of this log structure will be explained in next page.
(2) Our blow-up upper bounds are optimal. The log structure in (1.6) is crucially used.

We proceed to derive the bounds for $\Omega^2(u,v)$. Unlike $\Omega^{-2}(u,v)$, via a monotonic property (see Section 3.2), we can prove $\Omega^2(u,v) \lesssim 1/r(u,v)$. The lower bound of $\Omega^2(u,v)$, that is the upper bound of $\Omega^{-2}(u,v)$ is much harder and it is the key for the polynomial blow-up upper bounds.

In [4], Christodoulou showed that at each $b_0 \in B$, it holds that $|r\partial_u r|$ and $|r\partial_r r|$ are bounded and are close to some non-zero constant depending on $b_0$. Let’s first pretend to ignore $-r^2\partial_u \phi \partial_v \phi$ term in (1.6). From

$$|\partial_u r \partial_r r + \frac{1}{4} \Omega^2| \lesssim \frac{1}{r^2} + \frac{1}{r} \lesssim \frac{1}{r^2},$$

we have

$$r^2 \partial_u \partial_v \log \Omega(u,v) | \leq 1/r(u,v)^2,$$ i.e. $|\partial_u \partial_v \log \Omega(u,v)| \leq 1/r(u,v)^4$.

With the fact $\{r\partial_r r, r\partial_u r\}$ are close to non-zero constants, last inequality above implies

$$|\partial_u \log \Omega(u,v)| \lesssim \text{initial data} + \int_{u_1}^{u} \frac{1}{r(u',v)^2} du' \lesssim \text{initial data} + \int_{u_1}^{u} \frac{\partial_u r}{r(u',v)^3} du'$$

$$\lesssim \text{initial data} + \int_{r(u_1,v)}^{r(u,v)} \frac{1}{r(u',v)^3} dr \lesssim \text{initial data} + \frac{1}{r(u,v)^2}.$$ And

$$\lesssim \text{initial data} + \int_{v_1}^{v} \frac{1}{r(u,v)^2} dv' \lesssim \text{initial data} + \int_{v_1}^{v} \frac{\partial_v r}{r(u,v')^3} dv'$$

$$\lesssim \text{initial data} + \int_{r(v_1,u)}^{r(u,v)} \frac{1}{r(u,v')^3} dr \lesssim \text{initial data} + \log r(u,v).$$

Thus $\log \frac{1}{\Omega(u,v)^2} \lesssim \text{initial data} + \log \frac{1}{r(u,v)}$. Using the log structure, this means that there exists a positive constant $C$ such that

$$|\Omega(u,v)^{-2}| \lesssim r(u,v)^C.$$ And $C$ depends on the constants in above inequalities.

Now we take the term $-r^2\partial_u \phi \partial_v \phi$ into account. Our goal is to show

$$|\partial_u \phi(u,v)| \lesssim \frac{1}{r(u,v)^2}, \quad |\partial_v \phi(u,v)| \lesssim \frac{1}{r(u,v)^2}.$$ Then we would have $| -r^2\partial_u \phi \partial_v \phi | \lesssim \frac{1}{r^2}$, and it would be the same blow-up rates to $|\partial_u r \partial_v r + \frac{1}{4} \Omega^2| \lesssim \frac{1}{r^2}$. Repeat the calculation above,
\[ |\Omega(u, v)^{-2}| \leq r(u, v)^C \] still holds for some positive constant \( C \).

On the other hand, if we cannot obtain the optimal exponent 2. For \( 0 < \epsilon \ll 1 \), assume that we could only prove

\[ |\partial_u \phi(u, v)| \lesssim \frac{1}{r(u, v)^{2+\epsilon}}, \quad |\partial_v \phi(u, v)| \lesssim \frac{1}{r(u, v)^{2+\epsilon}}. \]

For this case, we get

\[ |\partial_u \partial_v \log \Omega(u, v)| \leq \frac{1}{r(u, v)^{4+2\epsilon}}. \]

And it implies

\[ |\partial_v \log \Omega(u, v)| \leq 1/(r(u, v)^{4+2\epsilon}). \]

(1.7)

And we achieve this goal in our Theorem 1.3. The proof is based on a crucial improved estimates for \( r\partial_u r \) and \( r\partial_v r \). These improved estimates may have other applications. And we highlight them in next paragraph.

(3) In this paper, we found crucial geometric improved estimates for \( r\partial_u r(u, v) \) and \( r\partial_v r(u, v) \). Our result is general and non-perturbative. We do not require our spacetimes to be close to Schwarzschild metric.

To prove (1.7), we take two steps. For the first step, we employ an important observation by Christodoulou and we reprove it with double null foliation in Proposition 3.1.
Given the same characteristic initial value problem for (1.1) as above. Assume \( b_0 \in B \) and \( b_0 \) has coordinate \((\tilde{u}_0, \tilde{v}_0)\), then as \((\tilde{u}, \tilde{v}) \to (\tilde{u}_0, \tilde{v}_0)\) we have \(- (r \partial_r u)(\tilde{u}, \tilde{v}_0) \to E(\tilde{v}_0)\) as \( u \to \tilde{u}_0^-\), where \( E \) is a positive continuous function. Similarly, as \((\tilde{u}_0, \tilde{v}) \to (\tilde{u}_0, \tilde{v}_0)\) we have \(- (r \partial_r v)(\tilde{u}_0, \tilde{v}) \to E^*(\tilde{u}_0)\), as \( v \to \tilde{v}_0^-\), where \( E^* \) is a positive continuous function.

Proposition 3.1 shows that near \( b_0 \in B \), there exists positive constants \( C_1 \) and \( C_2 \), and by the continuity of \( E(v) \) and \( E^*(u) \), for points close to \( b_0 \), the followings hold

\[ |r \partial_u r + C_1| = o(1), \quad |r \partial_v r + C_2| = o(1). \]

With these and an energy estimate, we first obtain Proposition 4.1: for \( 0 < \alpha \ll 1 \), in the region of interest, we have

\[ |\partial_u \phi(u, v)| \lesssim \frac{1}{r(u, v)^{3+\alpha}}, \quad |\partial_v \phi(u, v)| \lesssim \frac{1}{r(u, v)^{3+\alpha}}. \]

The next is one of the key points in this paper. In Proposition 5.1 together with a novel geometric argument and by applying bounds in Proposition 4.1 for \( \partial_u \phi, \partial_v \phi \), we have a crucial quantitive improvement of the estimates for \( r \partial_u r \) and \( r \partial_v r \): we obtain that for any \((u, v)\) close to \( b_0 \), it holds

\[ |r \partial_u r + C_1|(u, v) \leq 2r(u, v)^{1/10}, \quad |r \partial_v r + C_1|(u, v) \leq 2r(u, v)^{1/10}. \]

This crucial improvement enables us to correct a potential divergent \( \log r(u, v) \) term with a finite constant. (See the proof in Theorem 1.3.)

The conclusion and the argument in Proposition 5.1 will lead to future applications. With these crucial improvements, in Section 6 via using a constant \( r(u, v) \) foliation, we prove Theorem 1.3:

\[ |\partial_u \phi(u, v)| \lesssim \frac{1}{r(u, v)^2}, \quad |\partial_v \phi(u, v)| \lesssim \frac{1}{r(u, v)^2}. \]

Note that our proof in Proposition 5.1 is very general. We don’t need our spacetimes to be close to Schwarzschild metric. Hence, the blow-up upper bounds we derived are also general. And our proof is not perturbative.

1.3. Background. In a series of celebrated papers [4]-[7], Christodoulou proved weak cosmic censorship for (1.1) under spherical symmetry. He showed that for generic initial data, the singularities formed in the evolution of (1.1) are hidden inside black hole regions.

One could further ask: inside black holes, what are the future boundaries like? This question is related to strong cosmic censorship. For spacetimes like Kerr and Reissner-Nordström black holes, their future boundaries are null hypersurfaces, called Cauchy horizons. In recent breakthrough papers [26] by
Luk and Oh and [14] by Dafermos and Luk, the regularities of Cauchy horizons are studied in detail. Interested readers are also referred to [10]-[15], [18]-[28] and [30, 31].

For spacetimes close to a Schwarzschild black hole, their future boundaries could be more singular than the spacetimes near Kerr or Reissner-Nordström black holes. In [29] Sbierski proved the $C^0$-inextendibility of Schwarzschild spacetime. In [16] Fournodavlos studied the backward stability of the Schwarzschild singularity for Einstein vacuum equations; Alexakis and Fournodavlos [1] are exploring the forward stability problem under axial symmetry.

Fournodavlos and Sbierski [17] also studied the asymptotic behaviours of linear waves in the interior region of Schwarzschild spacetime. For linear wave equation in Schwarzschild background

$$\Box g_{\text{Sch}} \phi = 0,$$

close to spacelike singularity $r(u, v) = 0$, they proved that $|\phi(u, v)| \lesssim |\log r(u, v)|$ and also gave the leading order asymptotic behaviours. Note that their bounds are consistent with the upper bounds we derive in Theorem 1.3. Their analysis is for linear wave equation in precise Schwarzschild background and they don’t impose symmetry assumption. For our results, we impose spherical symmetry, but our theorem is for the full Einstein-scalar field system and our spacetime metric could be far away from Schwarzschild metric.

The future boundary $B$ in [4] and Schwarzschild singularities share some common properties: in both spacetimes, the singular boundaries are spacelike. And for any point $(u, v)$ along $B$, we have $r(u, v) = 0$. But spacetimes in [4] are much more general. The future boundaries $B$ in [4] are beyond the perturbative regimes of Schwarzschild singularities. In this following, we will explore how singular $B$ could be.

2. Acknowlegements

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3. Settings and Basic Control of Geometric Quantities

Under spherical symmetry, with double null foliations, we have the following ansatz for metric of the 3 + 1-dimensional spacetime:

$$g_{\mu \nu}dx^\mu dx^\nu = -\Omega^2(u, v)du dv + r^2(u, v)\left(d\theta^2 + \sin^2 \theta d\phi^2\right).$$

With this ansatz, the Einstein scalar field system

$$\text{Ric}_{\mu \nu} - \frac{1}{2}R g_{\mu \nu} = 2T_{\mu \nu},$$

$$T_{\mu \nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2}g_{\mu \nu} \partial^\rho \phi \partial_\rho \phi,$$
can be rewritten as

\[ r \partial_u \partial_v r = -\partial_u r \partial_v r - \frac{1}{4} \Omega^2, \]  
(3.1)

\[ r^2 \partial_u \partial_v \log \Omega = \partial_u r \partial_v r + \frac{1}{4} \Omega^2 - r^2 \partial_u \phi \partial_v \phi, \]  
(3.2)

\[ r \partial_u \partial_v \phi = -\partial_u r \partial_v \phi - \partial_v r \partial_u \phi, \]  
(3.3)

\[ \partial_u (\Omega^{-2} \partial_u r) = -r \Omega^{-2} (\partial_u \phi)^2, \]  
(3.4)

\[ \partial_v (\Omega^{-2} \partial_v r) = -r \Omega^{-2} (\partial_v \phi)^2. \]  
(3.5)

For later use, we define Hawking mass \( m(u, v) \) for a two-sphere \( S_{u,v} \) implicitly by

\[ 1 - \frac{2m}{r} = -4\Omega^{-2} \partial_u r \partial_v r. \]

We further introduce the dimensionless quantity

\[ \mu = \frac{2m}{r}. \]

Note that along the apparent horizon \( A \), we have \( \partial_v r = 0 \). This implies

\[ 1 - \frac{2m}{r} = 0, \quad \mu = 1, \quad 2m = r, \quad \text{along } A. \]

And inside the trapped region \( T \), we have \( \partial_v r < 0, \partial_u r < 0 \). It follows that

\[ 1 - \frac{2m}{r} < 0, \quad \mu > 1, \quad 2m > r, \quad \text{in } T. \]

With \( m \) and \( \mu \), we could rewrite (3.1)-(3.5) and further have

\[ \partial_u (\partial_u r) = \frac{\mu}{(1 - \mu)r} \partial_v r \partial_u r, \]  
(3.6)

\[ \partial_v (\partial_v r) = \frac{\mu}{(1 - \mu)r} \partial_v r \partial_u r, \]  
(3.7)

\[ 2\partial_u r \partial_u m = (1 - \mu) r^2 (\partial_u \phi)^2, \]  
(3.8)

\[ 2\partial_v r \partial_v m = (1 - \mu) r^2 (\partial_v \phi)^2. \]  
(3.9)
3.1. **Estimates for $\partial_u r$ and $\partial_v r$.** Let us recall a proposition by Christodoulou in [5] and give a proof in double null foliation:

**Proposition 3.1.** (Proposition 8.2 in [5]): Given the same characteristic initial data in a double-null foliation as above. Assume $b_0 \in B$ has coordinate $(u^*(v), v)$, we have

$$-(r \partial_v r)(u, v) \to E(v)$$

as $u \to u^*(v)$, where $E$ is a positive continuous function of $v$. Similarly, assume each $b_0 \in B$ has coordinate $(u, v^*(u))$. Then it also holds

$$-(r \partial_u r)(u, v) \to E^*(u)$$

as $v \to v^*(u)$, where $E^*$ is a positive continuous function of $u$. Also, the non-central component $B \setminus B_0$ of the singular boundary $B$ is a $C^1$ strictly spacelike curve, i.e., the functions $v^*$ and $u^*$ are strictly decreasing $C^1$ functions.

Proof. In the trapped region from (3.8), we have $\partial_u m \geq 0$. For each $v$, we assume the 2-sphere $b_1 = (u_A(v), v)$ lays on $A$ with radius $r_A(v)$. Hence for any $(u, v) \in T$, we have

$$2m(u, v) \geq 2m(u_A(v), v) = r_A(v).$$

Thus in $T$, we have

$$\mu(u, v) \geq \frac{r_A(v)}{r(u, v)}.$$

From (3.6), we obtain

$$\frac{1}{|\partial_v r|} \cdot \frac{\partial |\partial_v r|}{\partial u} = \frac{\mu}{\mu - 1} \cdot \frac{\partial \log r}{\partial u},$$

and hence

$$\frac{-\partial \log |\partial_v r|/\partial u}{\partial \log r/\partial u} = \frac{\mu}{\mu - 1}.$$

We conclude that in $T$

$$1 < \frac{-\partial \log |\partial_v r|/\partial u}{\partial \log r/\partial u} \leq \frac{r_A}{r_A - r}.$$
we have

\[ 0 < \frac{\partial \log r | \partial_v r |}{\partial u}. \]

Taking \( v \) fixed and integrating with respect to \( u \) we obtain

\[ r | \partial_{u,v} r |(u,v) > r | \partial_{u,v} r |(u_1,v), \]

where we require \( u_A(v) < u_1 \leq u \leq u^*(v) \), and \( (u^*(v),v) \) is according to the singular boundary along constant \( v \).

Using

\[ -\frac{\partial \log | \partial_v r |}{\partial u} \leq \frac{r_A}{r_A - r}, \]

we have

\[
\frac{\partial \log r | \partial_v r |}{\partial u} \leq -\frac{\partial \log r}{\partial u} \cdot \frac{r}{r_A - r} = \frac{\partial r}{\partial u} \cdot \frac{1}{r_A - r} = \frac{\log(r_A - r)}{\partial u}.
\]

This implies

\[ \log \left( \frac{r | \partial_{u,v} r |(u,v)}{r | \partial_{u,v} r |(u_1,v)} \right) \leq \log \left( \frac{r_A(v) - r(u,v)}{r_A(v) - r(u_1,v)} \right), \]

and hence

\[ \frac{r | \partial_{u,v} r |(u,v)}{r | \partial_{u,v} r |(u_1,v)} \leq \frac{r_A(v) - r(u,v)}{r_A(v) - r(u_1,v)} \leq \frac{r_A(v)}{r_A(v) - r(u_1,v)}. \]

Therefore, we conclude

\[ \frac{r_A(v)}{r_A(v) - r(u_1,v)} (r | \partial_{u,v} r |(u_1,v)) \leq (r | \partial_{u,v} r |(u,v)) < (r | \partial_{u,v} r |(u_1,v)). \]

A direct checking gives

\[ \limsup_{u \to u^*(v)} (-(r | \partial_{u,v} r |(u,v))) - \liminf_{u \to u^*(v)} (-(r | \partial_{u,v} r |(u,v))) \leq \frac{r(u_1,v)}{r_A - r(u_1,v)} (-(r | \partial_{u,v} r |(u_1,v))). \]

Taking \( u_1 \to u^*(v) - \), we have that \( r(u_1,v) \to 0 \). Therefore, we conclude

\[ -(r | \partial_{u,v} r |)(u,v) \text{ tends to a positive limit } E(v) \text{ as } u \to u^*(v) - . \]

Similar arguments work for \( -(r | \partial_{u,v} r |)(u,v) \). And we have

\[ -(r | \partial_{u,v} r |)(u,v) \text{ tends to a positive limit } E^*(u) \text{ as } v \to v^*(u) - . \]

Last, since

\[ \frac{\partial r^2}{\partial u^2} = 2r \partial_{v,r}, \quad \frac{\partial r^2}{\partial v^2} = 2r \partial_{u,r}, \]

the above yields that function \( r^2(u,v) \) extends to a \( C^1 \) function to \( Q \cup (B \setminus B_0) \). And thus \( B \setminus B_0 \) is a \( C^1 \) curve.

\[ \square \]
3.2. Upper Bounds for $\Omega^2$. From (3.1) we have

$$\partial_u(\Omega^{-2} \cdot -\partial_u r) = r\Omega^{-2}(\partial_u \phi)^2 \geq 0.$$ 

Thus, for $(u, v) \in \mathcal{T}$ and $(u_0(v), v) \in \mathcal{A}$ we have

$$\frac{-\partial_u r(u, v)}{\Omega^2(u, v)} \geq \frac{-\partial_v r(u_0(v), v)}{\Omega^2(u_0(v), v)} = c_0,$$

for some positive constant $c_0$. Hence

$$\Omega^2(u, v) \leq c_0^{-1} \cdot [-\partial_u r(u, v)] \leq \frac{D}{r(u, v)},$$

where $D$ is a uniform number depending on initial data.

4. Preliminary Bounds for $\partial_u \phi$ and $\partial_v \phi$

The aim of this and the next two sections is to prove

$$|r^2 \partial_u \phi| \leq D_1, \text{ and } |r^2 \partial_v \phi| \leq D_2,$$

with $D_1, D_2$ being uniform numbers depending on initial data. To achieve this goal, we first derive some preliminary estimates.

**Proposition 4.1.** For $0 < \alpha \leq 1$, in the region of interest, we have

$$|\partial_u \phi(u, v)| \leq \frac{I_0}{r^{3+\alpha}(u, v)}, \text{ and } |\partial_v \phi(u, v)| \leq \frac{I_0}{r^{3+\alpha}(u, v)},$$

where $I_0$ is a uniform number depending on initial data.

**Proof.** Assume the whole diamond region below is in trapped region $\mathcal{T}$.

We denote the rectangular region on the left to be $D_0$. Choose $D_0$ to be small enough. And we will focus on this region.

From (3.3), we have

$$\partial_v [r^{2\alpha}(r^2 \partial_v \phi)^2] = (2\alpha + 2)\partial_u r \cdot r^{2\alpha} \cdot r^3(\partial_u \phi)^2 - 2\partial_u r \cdot r^{2\alpha} \cdot r^3 \partial_u \phi \partial_v \phi,$$

$$\partial_u [r^{2\alpha}(r^2 \partial_v \phi)^2] = (2\alpha + 2)\partial_u r \cdot r^{2\alpha} \cdot r^3(\partial_u \phi)^2 - 2\partial_u r \cdot r^{2\alpha} \cdot r^3 \partial_u \phi \partial_v \phi.$$ 

For $0 < \alpha \leq 1$, we have

$$\partial_v [r^{2\alpha}(r^2 \partial_v \phi)^2] = (2\alpha + 2)\partial_u r \cdot r^{2\alpha} \cdot r^3(\partial_u \phi)^2 - 2\partial_u r \cdot r^{2\alpha} \cdot r^3 \partial_u \phi \partial_v \phi,$$

$$\partial_u [r^{2\alpha}(r^2 \partial_v \phi)^2] = (2\alpha + 2)\partial_u r \cdot r^{2\alpha} \cdot r^3(\partial_u \phi)^2 - 2\partial_u r \cdot r^{2\alpha} \cdot r^3 \partial_u \phi \partial_v \phi.$$ 

By Proposition 3.1 inside a sufficiently small $D_0$ we have:

$$\partial_v r + \frac{C_2}{r} = \frac{o(1)}{r}, \quad \partial_u r + \frac{C_1}{r} = \frac{o(1)}{r}.$$ 

We obtain

$$\partial_v [r^{2\alpha}(r^2 \partial_v \phi)^2] = (2\alpha+2)[-C_2+o(1)] \cdot r^{2\alpha+2} \cdot (\partial_u \phi)^2 - 2[-C_1+o(1)] \cdot r^{2\alpha+2} \cdot \partial_u \phi \partial_v \phi,$$

$$\partial_u [r^{2\alpha}(r^2 \partial_v \phi)^2] = (2\alpha+2)[-C_1+o(1)] \cdot r^{2\alpha+2} \cdot (\partial_v \phi)^2 - 2[-C_2+o(1)] \cdot r^{2\alpha+2} \cdot \partial_u \phi \partial_v \phi.$$
Adding these two expression together, we have
\[ \partial_v[C_2 \cdot r^{2\alpha}(r^2 \partial_v \phi)^2] + \partial_u[C_1 \cdot r^{2\alpha}(r^2 \partial_v \phi)^2] \]
\[= 2\alpha \cdot C_2 \cdot [-C_2 + o(1)]r^{2\alpha+2} (\partial_u \phi)^2 + 2\alpha \cdot C_1 \cdot [-C_1 + o(1)]r^{2\alpha+2} (\partial_u \phi)^2 \]
\[+ 2 \cdot C_2 \cdot [-C_2 + o(1)]r^{2\alpha+2} (\partial_v \phi)^2 + 2 \cdot C_1 \cdot [-C_1 + o(1)]r^{2\alpha+2} (\partial_v \phi)^2 \]
\[= 2\alpha \cdot C_2 \cdot [-C_2 + o(1)]r^{2\alpha+2} (\partial_u \phi)^2 + 2\alpha \cdot C_1 \cdot [-C_1 + o(1)]r^{2\alpha+2} (\partial_u \phi)^2 \]
\[+ 2 \cdot C_2 \cdot [-C_2 + o(1)]r^{2\alpha+2} (\partial_v \phi)^2 + 2 \cdot C_1 \cdot [-C_2 + o(1)]r^{2\alpha+2} (\partial_v \phi)^2 \]
\[= 2\alpha \cdot C_2 \cdot [-C_2 + o(1)]r^{2\alpha+2} (\partial_u \phi)^2 + 2\alpha \cdot C_1 \cdot [-C_1 + o(1)]r^{2\alpha+2} (\partial_u \phi)^2 \]
\[+ 2 \cdot C_2 \cdot [-C_2 + o(1)]r^{2\alpha+2} (\partial_v \phi)^2 + 2 \cdot C_1 \cdot [-C_2 + o(1)]r^{2\alpha+2} (\partial_v \phi)^2 \]
\[\leq 2\alpha \cdot C_2 \cdot [-C_2 + o(1)]r^{2\alpha+2} (\partial_u \phi)^2 + 2\alpha \cdot C_1 \cdot [-C_1 + o(1)]r^{2\alpha+2} (\partial_u \phi)^2 \]
\[+ 2 \cdot o(1) \cdot C_2 \cdot [-C_2 + o(1)]r^{2\alpha+2} (\partial_v \phi)^2 + 2 \cdot o(1) \cdot C_1 \cdot [-C_1 + o(1)]r^{2\alpha+2} (\partial_v \phi)^2 \]
\[+ 2 \cdot o(1) \cdot C_1 \cdot [-C_2 + o(1)]r^{2\alpha+2} (\partial_v \phi)^2 + 2 \cdot o(1) \cdot C_2 \cdot [-C_2 + o(1)]r^{2\alpha+2} (\partial_v \phi)^2 \]
\[\leq 0, \text{ for sufficiently small } o(1) \text{ and any fixed } \alpha > 0. \]

Rewrite
\[\iint_{D_0} \left( \partial_v[C_2 \cdot r^{2\alpha}(r^2 \partial_v \phi)^2] + \partial_u[C_1 \cdot r^{2\alpha}(r^2 \partial_v \phi)^2] \right) dudv \leq 0, \]
and we have
\[\int_{v=V_0}^{v=V_U} C_1 \cdot (r^{2+\alpha} \partial_u \phi)^2(U_0, v)dv + \int_{u=U_0}^{u=U_U} C_2 \cdot (r^{2+\alpha} \partial_v \phi)^2(u, V_0)du \]
\[\leq \int_{v=V_0}^{v=V_U} C_1 \cdot (r^{2+\alpha} \partial_u \phi)^2(U', v)dv + \int_{u=U_0}^{u=U_U} C_2 \cdot (r^{2+\alpha} \partial_v \phi)^2(u, V')du \]
=initial data = \( I_0^2 \).

Note that equation (3.34) is equivalent to
\[\partial_u(r \partial_v \phi) = -\partial_v r \partial_u \phi, \quad (4.3)\]
and
\[\partial_v(r \partial_u \phi) = -\partial_u r \partial_v \phi. \quad (4.4)\]
At $P = (U_0, V_0)$, we have
\[
 r \partial_v \phi(U_0, V_0) = r \partial_v \phi(U', V_0) - \int_{U'}^{U_0} \partial_v r \partial_u \phi(u, V_0) du. \tag{4.5}
\]

This gives
\[
| r \partial_v \phi(U_0, V_0) | 
\leq | r \partial_v \phi(U', V_0) | + \left| \int_{U'}^{U_0} \partial_v r \partial_u \phi(u, V_0) du \right| 
\leq | r \partial_v \phi(U', V_0) | + \left( \int_{U'}^{U_0} (2^{2+\alpha} \partial_u \phi)^2(u, V_0) du \right)^{\frac{1}{2}} \left( \int_{U'}^{U_0} \left( \frac{\partial_v r}{r^{2+\alpha}} \right)^2(u, V_0) du \right)^{\frac{1}{2}} 
\leq \text{const} + \frac{I_0}{\sqrt{C_2}} \cdot \left( \int_{U'}^{U_0} \left( \frac{\partial_v r}{r^{2+\alpha}} \right)^2(u, V_0) du \right)^{\frac{1}{2}} 
\lesssim \text{const} + \frac{I_0}{r^{2+\alpha}(U_0, V_0)}. \tag{4.6}
\]

For the last step, we use
\[
\frac{I_0}{\sqrt{C_2}} \cdot \left( \int_{U'}^{U_0} \left( \frac{\partial_v r}{r^{2+\alpha}} \right)^2(u, V_0) du \right)^{\frac{1}{2}} 
\leq \frac{I_0}{\sqrt{C_2}} \cdot \left( \int_{U'}^{U_0} \left( \frac{\partial_u r}{r^{4+2\alpha}} \right)^2 r_u(u, V_0) du \right)^{\frac{1}{2}} 
= \frac{I_0}{\sqrt{C_2}} \cdot \left( \int_{r(U', V_0)}^{r(U_0, V_0)} \left( \frac{\partial_u r}{r^{4+2\alpha}} \right)^2 dr \right)^{\frac{1}{2}} 
\approx \frac{I_0}{\sqrt{C_2}} \cdot \left( \int_{r(U', V_0)}^{r(U_0, V_0)} \left( \frac{-C_2^2}{C_1 r^{5+2\alpha}} \right)^2 dr \right)^{\frac{1}{2}} 
\approx \frac{I_0}{r^{2+\alpha}(U_0, V_0)}. \tag{4.7}
\]

Hence, we have for $0 < \alpha < 1$
\[
| \partial_v \phi | \leq \frac{I_0}{r^{3+\alpha}}.
\]

Similarly,
\[
| \partial_u \phi | \leq \frac{I_0}{r^{3+\alpha}}.
\]

□
5. Refined Estimates of $r\partial_u r$ and $r\partial_v r$

In [4], for general gravitational collapse, we have the Penrose diagram on the left. The curve marked with $\mathcal{A}$ is the apparent horizon.

To study the singular boundary $\mathcal{B}$ (where $r = 0$), we consider the diamond region (in $\mathcal{T}$) below. The rectangular region is called $D_0$.

Take $D_0$ to be sufficiently small. By Proposition 3.1 and continuity, in $D_0$ we have

$$r\partial_u r + C_1 = o(1), \quad r\partial_v r + C_2 = o(1).$$

To derive sharp blow-up rates for $\partial_u \phi$ and $\partial_v \phi$, we will need improved estimates for $r\partial_u r$ and $r\partial_v r$.

We then zoom in $D_0$ and assume $U_0 = 0, V_0 = 0$.

In the image above, we have $U, V, \tilde{U}, \tilde{V}, U', V' < 0$. We are now ready to state and prove

**Proposition 5.1.** For $Q \in D_0$ sufficiently close to $P$, we have improved estimates

$$|r\partial_u r + C_1|(Q) \leq 2r(Q)^{1\over 100}, \quad |r\partial_v r + C_2|(Q) \leq 2r(Q)^{1\over 100}. \quad (5.1)$$

**Proof.** Denote $r(Q) = r_0$. Along $u = 0$, we first find $A$ in the past of $P$ and satisfying

$$r(A) = r(Q)^{1\over 100} = r_0^{1\over 100}.$$
Polynomial Blow-up Upper Bounds

Assume $B$ is the intersection of $v = V'$ and $u = \tilde{U}$. When $Q$ is sufficiently close to $P$. $A$ and $B$ are still in the region $D_0$. And in $D_0$, we have

$$r \partial_u r + C_1 = o(1), \quad r \partial_v r + C_2 = o(1).$$

This implies

$$r^2(u, 0) = r^2(0, 0) + \int_0^u \partial_u (r^2(u', 0)) du' = \int_0^u (2r \partial_u r(u', 0)) du'$$

$$= \int_0^u [-2C_1 + o(1)] du' = [-2C_1 + o(1)] u.$$

Similarly, we further have

$$r^2(u, v) = r^2(u, 0) + \int_0^v \partial_v (r^2(u, v')) dv' = r^2(u, 0) + \int_0^v (2r \partial_v r(u, v')) dv'$$

$$= [-2C_1 + o(1)] u + \int_0^v [-2C_2 + o(1)] dv'$$

$$= [-2C_1 + o(1)] u + [-2C_2 + o(1)] v.$$

(5.2)

In particular, at $Q$, where $u = \tilde{U}$, $v = \tilde{V}$, it holds

$$[-2C_1 + o(1)] \tilde{U} + [-2C_2 + o(1)] \tilde{V} = r_0^2.$$

This implies

$$|\tilde{U}| \leq \frac{r_0^2}{C_1}, \quad |\tilde{V}| \leq \frac{r_0^2}{C_2}.$$

Along $AP$, we use equation

$$\partial_v (r \partial_u r) = -\frac{1}{4} \Omega^2$$

and taking (3.10) into account, we obtain

$$|(r \partial_u r)(P) - (r \partial_u r)(A)| = \int_{v'}^0 \frac{\Omega^2}{4} (0, v') dv' \lesssim \int_{v'}^0 \frac{D}{r(0, v')} dv'$$

$$\approx \int_{v'}^0 -\partial_u r(0, v') dv' = r(A) = r_0^{\frac{10}{10}}.$$

(5.3)

Along $BA$, from (3.4) we have

$$\partial_u (\partial_u r) = -r(\partial_u \phi)^2 + 2\partial_u \log \Omega \cdot \partial_u r.$$

Hence

$$\partial_u (r \partial_u r) = r \partial_u (\partial_u r) + \partial_u r \cdot \partial_u r$$

$$= -r^2(\partial_u \phi)^2 + 2\partial_u \log \Omega \cdot r \cdot \partial_u r + \partial_u r \partial_u r.$$

(5.4)

Using Proposition 4.1, we have $r^2(\partial_u \phi)^2 + |r^2 \partial_u \phi \partial_u \phi| \leq I_0^2/r^{4+2\alpha}$. Proposition 3.1 gives $|\partial_u r|, |\partial_v r| \lesssim 1/r$. Hence via (3.2):

$$r^2 \partial_u \partial_v \log \Omega = \partial_u r \partial_v r + \frac{1}{4} \Omega^2 - r^2 \partial_u \phi \partial_u \phi.$$
and (3.10), taking (5.2) into account and integrating (3.2) we obtain $|\partial_u \log \Omega| \leq I_0^2 / r^{4+2\alpha}$. With these estimates, we bound the RHS of (5.4) and obtain

$$\left| (r \partial_u r)(A) - (r \partial_u r)(B) \right| \leq \int_0^r \frac{I_0^2}{r^{4+2\alpha}}(u', V') du' \lesssim \frac{1}{r^{4+2\alpha}} \cdot |\tilde{U}|,$$

$$\leq \frac{r_0^{\frac{1}{600}}}{C_1} \cdot \frac{r_0^{\alpha}}{r_0} \leq r_0. \tag{5.5}$$

Combining (5.3) and (5.5), we get

$$\left| (r \partial_u r)(P) - (r \partial_u r)(B) \right| \leq r_0^{\frac{1}{600}}. \tag{5.6}$$

Along $AB$, using $r \partial_u r \sim \text{const}$ we have

$$|\partial_u r| \lesssim \frac{1}{r(A)} = \frac{1}{r_0^{\frac{1}{600}}}.$$ 

Thus, it follows

$$|r(B) - r(A)| \leq \int_0^r |\partial_u r(u', V')| du' \leq \frac{1}{r_0^{\frac{1}{600}}} \cdot \frac{r_0^{\alpha}}{C_1} \leq r_0.$$

Hence, for $r(B)$ we have

$$r_0^{\frac{1}{600}} \leq r(A) \leq r(B) \leq r(A) + r_0 \leq 2r_0^{\frac{1}{600}}.$$

Lastly, along $BQ$, we use equation $\partial_v(r \partial_u r) = -\frac{1}{4} \Omega^2$ and obtain

$$\left| (r \partial_v r)(B) - (r \partial_v r)(Q) \right| = \int_{V'} \frac{1}{4} \Omega^2(\tilde{U}, v') dv' \lesssim \int_{V'} \frac{D}{r(\tilde{U}, v')} dv' \approx \int_{V'} -\partial_v r(\tilde{U}, v') dv' = r(B) - r(Q) \leq 2r_0^{\frac{1}{600}}. \tag{5.7}$$

Combining (5.6) and (5.7), we then obtain

$$\left| (r \partial_u r)(P) - (r \partial_u r)(Q) \right| \leq 2r_0^{\frac{1}{600}}. \tag{5.8}$$

This gives

$$|r \partial_u r + C_1|(Q) \leq 2r(Q)^{\frac{1}{600}}.$$ 

Similarly, by using $A'$ and $B'$, we have

$$|r \partial_v r + C_2|(Q) \leq 2r(Q)^{\frac{1}{600}}.$$ 

Note that this conclusion holds for all the points $Q$ sufficiently close to $P$. \(\square\)
6. Sharp Estimates for $\partial_u \phi$ and $\partial_v \phi$

We are now ready to prove:

**Theorem 1.3** Under the same assumptions as in Theorem 1.2, at any point $(u, v) \in \mathcal{T}$ near $B$, there exists positive number $D_1$ and $D_2$ (depending on the initial data), such that

$$|\partial_u \phi(u, v)| \leq \frac{D_1}{r(u, v)^2}, \quad |\partial_v \phi(u, v)| \leq \frac{D_2}{r(u, v)^2}.$$ 

**Proof.** We consider the spacetime region (in $D_0$) below. Fix $l \gg 1$ so that the entire figure below is in our region of interest. Let $n \gg l$ be arbitrary.

In $D_0$, we consider different constant $r$-level sets $\{L_r\}$. Let

$$\Psi(r) = \max \left\{ \sup_{P \in L_r} |C \cdot r \partial_u \phi|(P), \sup_{Q \in L_r} |C_1 \cdot r \partial_v \phi|(Q) \right\}.$$ 

At $P$, we have

$$-r \partial_u r(P) = C_1 > 0, \quad -r \partial_v r(P) = C_2 > 0.$$ 

Then from (3.3), i.e.

$$\partial_u (r \partial_v \phi) = -r_v \partial_u \phi$$
we have
\[ |C_1 \cdot r \partial_v \phi|(P_n) \leq I.D. + \int_{u(Q_{r-1})}^{u(P_n)} -r_u |C_1 \cdot \partial_u \phi| \, du \]
\[ \leq I.D. + \int_{u(Q_{r-1})}^{u(P_n)} -r_u \cdot \frac{r \partial_v r}{\partial_u r} |C_1 \cdot \partial_u \phi| \, du \]
\[ = I.D. + \int_{r(Q_{r-1})}^{r(P_n)} \frac{r \partial_v r}{\partial_u r} \cdot \frac{C_1}{C_2} \cdot \frac{1}{r} |C_2 \cdot r \partial_u \phi| \, dr \]
\[ = I.D. + \int_{r(Q_{r-1})}^{r(P_n)} \frac{1 + O(r^{1/100})}{r} \cdot |C_2 \cdot r \partial_u \phi| \, dr \] (use Proposition 5.1)

Similarly, we have
\[ |C_2 \cdot r \partial_u \phi|(P_n) \leq I.D. + \int_{r(Q_{r-1})}^{r(P_n)} - \frac{1 + O(r^{1/100})}{r} \cdot |C_1 \cdot r \partial_v \phi| \, dr. \]

Combining these two inequality together, we have
\[ \Psi(2^{-n}) \leq I.D. + \int_{r=2^{-n-1}}^{r=2^{-n}} \frac{1 + O(r^{1/100})}{r} \cdot \Psi(r) \, dr. \]
Here $2^{-n}$ could be replaced by any small positive number. Hence it is true that for any small enough $\tilde{r} > 0$
\[ \Psi(\tilde{r}) \leq I.D. + \int_{r=2^{-n-1}}^{\tilde{r}} \frac{1 + O(r^{1/100})}{r} \cdot \Psi(r) \, dr = I.D. + \int_{\tilde{r}}^{2^{-n-1}} \frac{1 + O(r^{1/100})}{r} \cdot \Psi(r) \, dr \]
By Grönwall’s inequality, we have
\[ \Psi(\tilde{r}) \leq I.D. \times e^{\int_{\tilde{r}}^{2^{-n-1}} \frac{1 + O(r^{1/100})}{r} \, dr} = I.D. \times e^{-\ln \tilde{r} + O(1)} \leq \frac{C}{\tilde{r}}, \]
where $C$ is a uniform number depending on initial data. This gives
\[ \tilde{r} \Psi(\tilde{r}) \leq C \text{ for any } \tilde{r} > 0, \]
which further implies
\[ r^2 |\partial_u \phi| \leq D_1, \quad r^2 |\partial_v \phi| \leq D_2 \]
for any $r \geq 0$, where $D_1, D_2$ are uniform numbers depending only on initial data. □

7. Higher Order Estimates

For the purpose of future use, we first state several useful estimates.

**Proposition 7.1.** For $\Omega^2(u, v)$, we have
\[ |\Omega^2(u, v)| \lesssim \frac{1}{r(u, v)}, \quad |\partial_u \log |\Omega|(u, v) | \lesssim \frac{1}{r^2(u, v)}, \quad |\partial_v \log |\Omega|(u, v) | \lesssim \frac{1}{r^2(u, v)}. \]
\[ |\partial_u (\Omega^2(u, v))| \lesssim \frac{1}{r^3(u, v)}, \quad |\partial_v (\Omega^2(u, v))(u, v)| \lesssim \frac{1}{r^3(u, v)}. \]
**Proof.** For $\Omega^2(u, v)$, we already proved $|\Omega^2(u, v)| \lesssim \frac{1}{r(u, v)}$ in (3.10). From (3.2) we have
\[
\frac{1}{\Omega^2} \frac{\partial_u \Omega^2(u, v)}{\partial_v} = \frac{\partial_u \log(\Omega^2)}{\partial_v} = \frac{1}{2r^2} (\partial_u r \partial_v + \frac{1}{4} \Omega^2 - r^2 \partial_u \phi \partial_v \phi).
\]
This implies
\[
\frac{1}{\Omega^2} \frac{\partial_u \Omega^2(u, v)}{\partial_v} = \frac{1}{\Omega^2} \frac{\partial_u \log(\Omega^2)}{\partial_v} = \frac{1}{2r^2} (\partial_u r \partial_v + \frac{1}{4} \Omega^2 - r^2 \partial_u \phi \partial_v \phi)(u, v') dv'.
\]
\[
\lesssim \text{I.D.} + \frac{1}{r^2(u, v)}.
\]
In the above inequality chain we used Theorem 1.3 and (3.10). Note that the above I.D. is uniformly bounded. Together with $\Omega^2(u, v) \lesssim \frac{1}{r(u, v)}$ by (3.10), we conclude
\[
|\partial_u \Omega^2(u, v)| \lesssim \Omega^2(u, v) \left[ \text{I.D.} + \frac{1}{r^2(u, v)} \right] \lesssim \frac{1}{r^3(u, v)}.
\]
Similarly, we have
\[
|\partial_v \Omega^2(u, v)| \lesssim \frac{1}{r^3(u, v)}.
\]
□

**Proposition 7.2.** For $r(u, v)$, we have
\[
|\partial_u \partial_v (r^3)| \lesssim \frac{1}{r} \lesssim \frac{1}{r^2}, \quad |\partial_u \partial_u (r^2)| \lesssim \frac{1}{r^2}, \quad |\partial_v \partial_v (r^2)| \lesssim \frac{1}{r^2},
\]
\[
|\partial_u \partial_v r| \lesssim \frac{1}{r^3}, \quad |\partial_u \partial_u r| \lesssim \frac{1}{r^3}, \quad |\partial_v \partial_v r| \lesssim \frac{1}{r^3}.
\]

Proof. For $\partial_u(\partial_v (r^2))$, we have from (3.1):
\[
\partial_u(\partial_v (r^2)) = -\frac{1}{2} \Omega^2,
\]
the desired estimate follows from the derived estimate $\Omega^2 \lesssim \frac{1}{r} \lesssim \frac{1}{r^2}$.

For $\partial_u(\partial_u (r^2))$, we first recall from (3.1):
\[
-2 \partial_u \log \Omega \cdot \partial_u r + \partial_u (\partial_u r) = -r (\partial_u \phi)^2.
\]
This implies
\[
-4r \partial_u \log \Omega \cdot \partial_u r + \partial_u (2r \partial_u r) - 2 \partial_u r \partial_u r = -2r^2 (\partial_u \phi)^2,
\]
that is
\[
\partial_u(\partial_u (r^2)) = 4r \partial_u \log \Omega \cdot \partial_u r + 2 \partial_u r \partial_u r - 2r^2 (\partial_u \phi)^2.
\]
By the estimates in Proposition 7.1 (3.10) and Proposition 1.3 we have
\[
|\partial_u(\partial_u r^2)| \lesssim \frac{1}{r^2}.
\]
Similarly, we also have
\[ |\partial_v(\partial_v r^2)| \lesssim \frac{1}{r^2}. \]
Hence we have every inequality in the second line of the statement. \qed

**Proposition 7.3.** For \( \phi(u, v) \), we have
\[ |\partial_u \partial_v \phi| \lesssim \frac{1}{r^4}, \quad |\partial_u \partial_u \phi| \lesssim \frac{1}{r^4}, \quad |\partial_v \partial_v \phi| \lesssim \frac{1}{r^4}. \]

**Proof.** The first estimate follows from (3.3):
\[ r \partial_u \partial_v \phi = - \partial_u r \partial_v \phi - \partial_v r \partial_u \phi \]
and Proposition 1.3. We then differentiate the above equation with respect to \( u \). Rewrite it. We then get
\[ \partial_v (r \partial^2 u u \phi) = - \partial^2 u u r \cdot \partial_v \phi - \partial_u r \partial^2 v \phi. \]
By previous bounds, the right hand side of the above inequality is \( \lesssim \frac{1}{r^7} \). Integrate both sides with respect to \( v \), we derive
\[ |r \partial^2 u u \phi| \leq \frac{1}{r^3}, \]
which implies
\[ |\partial^2 u u \phi| \leq \frac{1}{r^4}. \]
We similarly have the other desired estimate on \( \partial^2 v v \phi \). \qed

**Proposition 7.4.** For \( \log (\Omega^2)(u, v) \) we have
\[ |\partial_u \partial_v (\log \Omega^2)| \lesssim \frac{1}{r^4}, \quad |\partial_u \partial_u (\log \Omega^2)| \lesssim \frac{1}{r^4}, \quad |\partial_v \partial_v (\log \Omega^2)| \lesssim \frac{1}{r^4}, \]
\[ |\partial_v \partial_u (\log \Omega^2)| \lesssim \frac{1}{r^5}, \quad |\partial_u \partial_u (\log \Omega^2)| \lesssim \frac{1}{r^5}, \quad |\partial_v \partial_v (\log \Omega^2)| \lesssim \frac{1}{r^5}. \]

**Proof.** This first estimate is easily obtained from
\[ \partial_v \partial_u (\log \Omega^2) = \frac{1}{2r^2} (\partial_u r \partial_v r + \frac{1}{4} \Omega^2 - r^2 \partial_u \phi \partial_v \phi). \]
Differentiate this equation with respect to \( u \) and integrate the result with respect to \( v \), with the help of derived estimates, we arrive at
\[ |\partial_u \partial_u (\log \Omega^2)| \lesssim \frac{1}{r^4}. \]
Since
\[ \frac{\partial_u \partial_u (\log \Omega^2)}{\Omega^2} - \frac{\partial_u (\log \Omega^2) \cdot \partial_u (\log \Omega^2)}{\Omega^2} = \partial_u \frac{\partial_u (\log \Omega^2)}{\Omega^2} - \partial_u \partial_u (\log \Omega^2). \]
That is
\[ \partial_u \partial_u (\log \Omega^2) = \partial_u (\log \Omega^2) \cdot \partial_u (\log \Omega^2) + \Omega^2 \cdot \partial_u \partial_u (\log \Omega^2). \]
By the estimates derived above, we have
\[ |\partial_u \partial_u (\log \Omega^2)| \lesssim \frac{1}{r^5}. \]
Similarly, we also have
\[ |\partial_u \partial_v \log (\Omega^2)| \lesssim \frac{1}{r^4}, \quad \text{and} \quad |\partial_u \partial_v (\Omega^2)| \lesssim \frac{1}{r^5}. \]

From
\[ \frac{\partial_u \partial_v (\Omega^2)}{\Omega^2} - \frac{\partial_u (\Omega^2) \cdot \partial_v \log (\Omega^2)}{\Omega^2} = \partial_u \frac{\partial_v \Omega^2}{\Omega^2} = \partial_v \partial_u \log (\Omega^2), \]
we have
\[ \partial_v \partial_u \Omega^2 = \partial_u \Omega^2 \cdot \partial_v \log \Omega^2 + \Omega^2 \cdot \partial_v \partial_u \log \Omega^2. \]

With derived estimates, we have
\[ |\partial_v \partial_u (\Omega^2)| \lesssim \frac{1}{r^5}. \]

With these estimates, in the same fashion, with equations
\[ \partial_u (\partial_u (r^2)) = 4r \partial_u \log \Omega \cdot \partial_u r + 2 \partial_u r \partial_u r - 2r^2 (\partial_u \phi)^2, \]
we then get
\[ |\partial_u \partial_u \partial_u (r^2)| \lesssim \frac{1}{r^4}, \quad |\partial_u \partial_v \partial_u (r^2)| \lesssim \frac{1}{r^4}. \]

Similarly, we also have
\[ |\partial_v \partial_v \partial_v (r^2)| \lesssim \frac{1}{r^4}, \quad |\partial_v \partial_u \partial_v (r^2)| \lesssim \frac{1}{r^4}. \]

Repeatedly, we can derive all desired estimates through the following order
\[ \partial_u \log (\Omega^2), \partial_v \log (\Omega^2), \partial_u (\Omega^2), \partial_v (\Omega^2) \]
\[ \rightarrow r_{uu}, r_{vv}, r_{uv}, (r^2)_{uu}, (r^2)_{vv}, (r^2)_{uv} \]
\[ \rightarrow \partial^2_u \phi, \partial^2_v \phi, \partial^2_{uv} \phi \]
\[ \rightarrow \partial_u \log (\Omega^2), \partial_v \log (\Omega^2), \partial_u (\Omega^2), \partial_v (\Omega^2) \]  \hspace{1cm} (7.1)
\[ \rightarrow r_{uu}, r_{uv}, r_{uv}, (r^2)_{uu}, (r^2)_{uv}, (r^2)_{uv} \]
\[ \rightarrow \partial^3_{uu} \phi, \partial^3_{uv} \phi, \partial^3_{uv} \phi, \partial^3 {vv} \phi \]
\[ \rightarrow \cdots \]

In particular, in \( D_0 \) we get

**Proposition 7.5.** For any \( m, n \in \mathbb{N} \), we have
\[ |(\partial_u)^m (\partial_v)^n (r^2)(u,v)| \leq \frac{1}{r^{2m+2n-2}(u,v)} \]
and
\[ |(\partial_u)^m (\partial_v)^n \log (\Omega^2)(u,v)| \leq \frac{1}{r^{2m+2n}(u,v)}. \]
Recall in $D_0$, we have

$$r^2(u, v) = [-2C_1 + o(1)]u + [-2C_2 + o(1)]v.$$  

This further implies, in $D_0$ when it is close to a singular boundary point with coordinate $(u, v) = (0, 0)$

**Proposition 7.6.** For any $m, n \in \mathbb{N}$, we have

$$|(u \partial_u)^m (v \partial_v)^n r^2(u, v)| \lesssim r^2.$$ 

8. *Estimates of Kretschmann Scalar*

By direct calculation, for Christoffel symbols of metric (1.2) we have

$$\Gamma^u_{uu} = \frac{2\partial_u \Omega}{\Omega}, \quad \Gamma^u_{\theta\theta} = \frac{2r \partial_u r}{\Omega^2}, \quad \Gamma^u_{\phi\phi} = \frac{2r \sin^2 \theta \cdot \partial_u r}{\Omega^2},$$

$$\Gamma^v_{vv} = \frac{2\partial_v \Omega}{\Omega}, \quad \Gamma^v_{\theta\theta} = \frac{2r \partial_v r}{\Omega^2}, \quad \Gamma^v_{\phi\phi} = \frac{2r \sin^2 \theta \cdot \partial_v r}{\Omega^2},$$

$$\Gamma^\theta_{u\theta} = \frac{\partial_u r}{r}, \quad \Gamma^\theta_{v\theta} = \frac{\partial_v r}{r}, \quad \Gamma^\theta_{\phi\phi} = -\sin \theta \cdot \cos \theta,$$

$$\Gamma^\phi_{u\phi} = \frac{\partial_u r}{r}, \quad \Gamma^\phi_{v\phi} = \frac{\partial_v r}{r}, \quad \Gamma^\phi_{\theta\phi} = \frac{\cos \theta}{\sin \theta}.$$
We then write down the expression for the Kretschmann scalar:

\[
R^{\alpha \beta \rho \sigma} R_{\alpha \beta \rho \sigma} = \frac{4}{r^4 \Omega^8} \left( 16 \cdot \left( \frac{\partial^2 r}{\partial u \partial v} \right)^2 \cdot r^2 \cdot \Omega^4 + 16 \cdot \frac{\partial^2 r}{\partial u^2} \cdot \frac{\partial^2 r}{\partial v^2} \cdot r^2 \cdot \Omega^4 \right)
\]

\[
+ \frac{4}{r^4 \Omega^8} \left( -32 \cdot \frac{\partial^2 r}{\partial u^2} \cdot \partial_v r \cdot r^2 \cdot \Omega^3 \cdot \partial_v \Omega - 32 \cdot \frac{\partial^2 r}{\partial v^2} \cdot r^2 \cdot \partial_v r \cdot \Omega^3 \cdot \partial_u \Omega \right)
\]

\[
+ \frac{4}{r^4 \Omega^8} \left( 16 \cdot (\partial_v r)^2 \cdot (\partial_u r)^2 \cdot \Omega^4 + 64 \cdot \partial_v r \cdot r^2 \cdot \partial_u r \cdot \Omega^2 \cdot \partial_v \Omega \cdot \partial_u \Omega + 8 \cdot \partial_v r \cdot \partial_u r \cdot \Omega^6 \right)
\]

\[
+ \frac{4}{r^4 \Omega^8} \left( 16 \cdot r^4 \cdot \frac{\partial^2 \Omega}{\partial v \partial u} \cdot \frac{\partial^2 \Omega}{\partial v \partial u} \cdot \Omega \cdot \partial_v \Omega \cdot \partial_u \Omega \right)
\]

\[
+ \frac{4}{r^4 \Omega^8} \left( 16 \cdot r^4 \cdot (\partial_v \Omega)^2 \cdot (\partial_u \Omega)^2 + \Omega^8 \right).
\]

(8.1)

By Proposition \ref{polynomial blow-up upper bounds} and Proposition \ref{polynomial blow-up upper bounds 2}, we obtain polynomial upper bounds for \(|\partial_v r|, |\partial_u r|, |\partial_v \partial_u r|, |\partial_v \partial_r r|, |\partial_u \partial_r r|\). Through (8.10) and Proposition \ref{polynomial blow-up upper bounds}, we bound \(|\Omega|, |\partial_v \log \Omega|, |\partial_u \log \Omega|\).

To control \(\partial_v \partial_u \Omega\), we use

\[
\partial_v \partial_u (\Omega^2) = 2 \partial_v (\Omega \cdot \partial_u \Omega) = 2 \Omega \cdot \partial_v \partial_u \Omega + 2 \partial_u \Omega \cdot \partial_v \Omega,
\]

which implies

\[
\Omega \cdot \partial_v \partial_u \Omega = \frac{1}{2} \cdot \partial_v \partial_u (\Omega^2) - \partial_u \Omega \cdot \partial_v \Omega = \frac{1}{2} \cdot \partial_v \partial_u (\Omega^2) - \partial_v \log \Omega \cdot \partial_u \log \Omega. \quad (8.2)
\]

For \(\partial_v \partial_u (\Omega^2)\), we have

\[
\frac{\partial_v \partial_u (\Omega^2)}{\Omega^2} - \frac{\partial_u (\Omega^2) \cdot \partial_v \log (\Omega^2)}{\Omega^2} = \partial_v \frac{\partial_u \Omega^2}{\Omega^2} = \partial_v \partial_u \log (\Omega^2),
\]

which gives

\[
\frac{1}{2} \cdot \partial_v \partial_u (\Omega^2) - \frac{1}{2} \cdot \partial_u (\Omega^2) \cdot \partial_v \log (\Omega^2) = \frac{1}{2} \cdot \Omega^2 \cdot \partial_v \partial_u \log (\Omega^2). \quad (8.3)
\]
Combining (8.2) and (8.3), we have
\[ \Omega \cdot \partial_u \partial_u \Omega = \frac{1}{2} \cdot \partial_u (\Omega^2) \cdot \partial_v \log(\Omega^2) - \Omega^2 \cdot \partial_v \log \Omega \cdot \partial_u \log \Omega + \frac{1}{2} \cdot \Omega^2 \cdot \partial_v \partial_u \log (\Omega^2). \] (8.4)

With estimates derived in (3.10), Propositions 7.1 and 7.4, we hence obtain polynomial upper bound for \(|\Omega \cdot \partial_u \partial_u \Omega|\).

The last step is to derive upper bound for \(1/\Omega^2\). This is equivalent to deriving lower bound for \(\Omega^2\). Here we appeal to Theorem 1.3. From (3.2):
\[ r^2 \partial_u \partial_v \log \Omega = \partial_u r \partial_v r + \frac{1}{4} \Omega^2 - r^2 \partial_u \phi \partial_v \phi, \]
we have
\[ \log \Omega(U_0, V_0) = \log \Omega(U_0, V_0') + \log \Omega(U', V_0) - \log \Omega(U', V') + \int_{U'}^{U_0} \int_{V'}^{V_0} \left( \frac{\partial_u r \partial_v r}{r^2} + \frac{\Omega^2}{4r^2} - \partial_u \phi \partial_v \phi \right) (u, v) dudv. \]

With the bounds in Proposition 3.1 for \(\partial_u r, \partial_v r\), and the estimate in (3.10) for \(\Omega^2\), we have
\[ |\log \Omega(U_0, V_0)| \leq |\log \Omega(U_0, V_0')| + |\log \Omega(U', V_0)| + |\log \Omega(U', V')| + \int_{U'}^{U_0} \int_{V'}^{V_0} \tilde{D}^2 \frac{1}{r^4} (u, v) dudv \]
\[ \leq \tilde{C} \log \frac{1}{r}(U_0, V_0) \leq \log \frac{1}{r\tilde{C}}(U_0, V_0), \]
where \(\tilde{D}\) and \(\tilde{C}\) are some uniform positive constants depending on initial data. This implies
\[ \log \Omega(U_0, V_0) \geq \log r\tilde{C}(U_0, V_0), \]
and together with (3.10) we have
\[ r^{2\tilde{C}}(U_0, V_0) \leq \Omega^2(U_0, V_0) \leq \frac{D}{r(U_0, V_0)}, \quad \text{and} \]
\[ \frac{r(U_0, V_0)}{D} \leq \Omega^{-2}(U_0, V_0) \leq r^{-2\tilde{C}}(U_0, V_0). \]

Putting all these estimates together, we hence conclude that:

Theorem 1.2. For spacetime solutions to (1.1) under spherical symmetry, at any point \((u, v) \in T\) and \((u, v)\) close to \(B\), there exists a positive number \(N\) (depending on the initial data at an earlier time), such that
\[ \frac{1}{r(u, v)^6} \lesssim R^{\alpha \beta \gamma \delta} R_{\alpha \beta \gamma \delta} \lesssim \frac{1}{r(u, v)^N}. \]

Remark 8. Integrating \(\partial_u \phi\) respect to \(u\), using Proposition 3.1, we also conclude \(|\phi| \lesssim |\log r|\) uniformly in the region \(\{u \leq U_0, v \leq V_0\}\) close enough to \(P\).
9. A Case Study: Perturbations of the Schwarzschild Solution

9.1. Initial Data. Denote $o_0(1)$ to be a small number depending only on initial data. We consider the open trapezoid region $T_0$ below. For initial data along $r = 1/2^{l_0}$, we prescribe

$$\partial_v r + \frac{M}{r} = o_0(1) \cdot \frac{M}{r}, \quad \partial_u r + \frac{M}{r} = o_0(1) \cdot \frac{M}{r},$$

$$\Omega^2 = \frac{2M}{r} + o_0(1) \cdot \frac{M}{r},$$

$$|\partial_u \phi| \leq o_0(1) \cdot \frac{1}{r^2}, \quad |\partial_v \phi| \leq o_0(1) \cdot \frac{1}{r^2}.$$ 

9.2. Bootstrap Assumptions. Fix a positive parameter $M$. We choose the following bootstrap assumptions

$$\partial_v r + \frac{M}{r} = o(1) \cdot \frac{M}{r}, \quad \partial_u r + \frac{M}{r} = o(1) \cdot \frac{M}{r},$$

$$\Omega^2 \leq \frac{3M}{r},$$

9.3. Improving the Estimates.

9.3.1. Estimates for $\partial_u r$ and $\partial_v r$. With (8.1), we have

$$\partial_u (r \partial_v r) = -\frac{1}{4} \Omega^2.$$ 

This gives

$$(r \partial_v r)(U, v) - (r \partial_v r)(U', v) = \int_{U'}^{U} -\frac{1}{4} \Omega^2(u, v)du,$$

and

$$|(r \partial_v r)(U, v) - (r \partial_v r)(U', v)| \leq \int_{U'}^{U} \frac{1}{4} \Omega^2(u, v)du \leq \int_{r(U', v)}^{r(U, v)} \frac{1}{4} \Omega^2(u, v)dr \leq r(U', v) - r(U, v).$$

Together with $(r \partial_v r)(U', v) = -M + o_0(1)$, we have

$$|(r \partial_v r)(U, v) + M| \leq r(U', v) - r(U, v) + o_0(1).$$

Pick $r(U', v)$ and $o_0(1)$ sufficient small, we have

$$|(r \partial_v r)(U, v) + M| \leq r(U', v) - r(U, v) + o_0(1) \leq 2^{-l_0} + o_0(1) < \frac{1}{2} o(1).$$

This improves (9.1). Similarly, we also could improve (9.2).
9.3.2. Estimates for $\Omega^2$. With (3.4) we have
\[
\partial_u \left[ \frac{\Omega^2}{-4\partial_u r} \right] = \left[ \frac{\Omega^2}{-4\partial_u r} \right] \cdot \frac{r(\partial_u \phi)^2}{\partial_u r}
\]
This implies
\[
\frac{\Omega^2}{-4\partial_u r}(U, v) = \frac{\Omega^2}{-4\partial_u r}(U', v) \cdot \exp \left[ \int_{U'}^U \frac{r(\partial_u \phi)^2}{\partial_u r}(u, v) du \right].
\]
Hence,
\[
\frac{\Omega^2}{-4\partial_u r}(U, v) \leq \frac{\Omega^2}{-4\partial_u r}(U', v).
\]
With the information of the initial data and estimates for $\partial_u r$, we have derived
\[
\Omega^2(U, v) \leq \frac{2.5M}{r}.
\]
This improves (9.3).

9.3.3. Estimates for $\partial_u \phi$ and $\partial_v \phi$. We consider the spacetime region (in $T_0$) below. Let $1 \ll l \ll n$.

For different constant $r$-level sets $\{L_r\}$. Let
\[
\Psi(r) = \max \left\{ \sup_{P \in L_r} |C_2 \cdot r\partial_u \phi|(P), \sup_{Q \in L_r} |C_1 \cdot r\partial_v \phi|(Q) \right\}.
\]
At $P$, we have
\[
-r\partial_u r(P) = C_1 > 0, \quad -r\partial_v r(P) = C_2 > 0.
\]
By (9.2) and (9.1), we have
\[
|C_1 - M| \leq o(1), \quad |C_2 - M| \leq o(1).
\]
Then from
\[ \partial_u (r \partial_v \phi) = -r_v \partial_u \phi \]
we have
\[ |C_1 \cdot r \partial_v \phi| (P_n) \leq o_0(1) + \int_{u(Q_{l-1})}^{u(P_n)} -r_v |C_1 \cdot \partial_u \phi| \, du \]
\[ = o_0(1) + \int_{r(Q_{l-1})}^{r(P_n)} r \partial_u r \cdot \frac{C_1}{C_2} \cdot \frac{1}{r} \cdot |C_2 \cdot r \partial_u \phi| \, dr \]
\[ = o_0(1) + \int_{r(Q_{l-1})}^{r(P_n)} \left( \frac{1}{r} + O\left(\frac{1}{r^{\frac{1}{100}}} \right) \right) \cdot |C_2 \cdot r \partial_u \phi| \, dr \]
(9.4)

Similarly, we have
\[ |C_2 \cdot r \partial_u \phi| (P_n) \leq o_0(1) + \int_{r(Q_{l-1})}^{r(P_n)} \left( \frac{1}{r} + O\left(\frac{1}{r^{\frac{1}{100}}} \right) \right) \cdot |C_1 \cdot r \partial_v \phi| \, dr. \]

Combining these two inequality together, we have
\[ \Psi(2^{-n}) \leq o_0(1) + \int_{r=2^{-l+1}}^{r=2^{-n}} \left( \frac{1}{r} + O\left(\frac{1}{r^{\frac{1}{100}}} \right) \right) \cdot \Psi(r) \, dr. \]

Here \( 2^{-n} \) could be replaced by any small positive number. Hence it is true that for any small enough \( \tilde{r} > 0 \)
\[ \Psi(\tilde{r}) \leq o_0(1) + \int_{r=2^{-l+1}}^{\tilde{r}} \left( \frac{1}{r} + O\left(\frac{1}{r^{\frac{1}{100}}} \right) \right) \cdot \Psi(r) \, dr = o_0(1) + \int_{\tilde{r}}^{2^{-l+1}} \left( \frac{1}{r} + O\left(\frac{1}{r^{\frac{1}{100}}} \right) \right) \cdot \Psi(r) \, dr \]

By Grönwall’s inequality, we have
\[ \Psi(\tilde{r}) \leq o_0(1) \times e^{\int_{\tilde{r}}^{2^{-l+1}} \frac{1}{r} + O\left(\frac{1}{r^{\frac{1}{100}}} \right) \, dr} = o_0(1) \times e^{-\ln \tilde{r} + O(1)} \leq \frac{o_0(1)}{\tilde{r}}. \]

This gives
\[ \tilde{r} \Psi(\tilde{r}) \leq o_0(1) \text{ for any } \tilde{r} > 0, \]
which further implies
\[ r^2 |\partial_u \phi| \leq o_0(1), \quad r^2 |\partial_v \phi| \leq o_0(1). \]

9.4. Upper Bounds for Kretschmann Scalar. We now start to derive an upper bound for Kretschmann scalar. Recall from (8.1) in Section 8, we have
\[ R^{\alpha\beta\rho\sigma} R_{\alpha\beta\rho\sigma} \]
\[ = \frac{4}{r^4 \Omega^8} \left( 16 \cdot \left( \frac{\partial^2 r}{\partial u \partial v} \right)^2 \cdot r^4 \cdot \Omega^4 + 16 \cdot \frac{\partial^2 r}{\partial u^2} \cdot \frac{\partial^2 r}{\partial v^2} \cdot r^2 \cdot \Omega^4 \right) \]
\[ + \frac{4}{r^4 \Omega^8} \left( -32 \cdot \frac{\partial^2 r}{\partial u^2} \cdot \partial_v r \cdot r^2 \cdot \Omega^3 \cdot \partial_v \Omega - 32 \cdot \frac{\partial^2 r}{\partial v^2} \cdot r^2 \cdot \partial_u r \cdot \Omega^3 \cdot \partial_u \Omega \right) \]
\[ + \frac{4}{r^4 \Omega^8} \left( 16 \cdot (\partial_v r)^2 \cdot (\partial_u r)^2 \cdot \Omega^4 + 64 \cdot \partial_v r \cdot r^2 \cdot \partial_u r \cdot \Omega^2 \cdot \partial_u \Omega \cdot \partial_v \Omega + 8 \cdot \partial_v r \cdot \partial_u r \cdot \Omega^6 \right) \]
\[ + \frac{4}{r^4 \Omega^8} \left( 16 \cdot r^4 \cdot (\partial_v \Omega)^2 \cdot \Omega^2 - 32 \cdot r^4 \cdot \left( \frac{\partial^2 \Omega}{\partial v \partial u} \right) \cdot \Omega \cdot \partial_v \Omega \cdot \partial_u \Omega \right) \]
\[ + \frac{4}{r^4 \Omega^8} \left( 16 \cdot r^4 \cdot (\partial_u \Omega)^2 \cdot (\partial_v \Omega)^2 + \Omega^8 \right). \] (9.5)

Thus, to obtain an upper bound for \( R^{\alpha\beta\rho\sigma} R_{\alpha\beta\rho\sigma} \), it is crucial to derive an upper bound for \( \Omega^{-2} \).

**Proposition 9.1.** Give the prescribed initial data, in the diamond region, we have
\[ \Omega^{-1}(U, v) \leq C \cdot r(U, v)^{\frac{1}{2} - o(1)^2}, \]
for some positive constant \( C \).

**Proof.** We revisit subsection 9.3.2. From (3.4) we have
\[ \partial_u \left[ \frac{\Omega^2}{-4 \partial_u r} \right] = \left[ \frac{\Omega^2}{-4 \partial_u r} \right] \cdot \frac{r(\partial_u \phi)^2}{\partial_u r}. \]

This implies
\[ \frac{\Omega^2}{-4 \partial_u r}(U, v) = \frac{\Omega^2}{-4 \partial_u r}(U', v) \cdot \exp \left[ \int_{U}^{U'} \frac{r(\partial_u \phi)^2}{\partial_u r}(u, v) du \right]. \] (9.6)

Since
\[ |\partial_u \phi| \leq o_0(1)/r^2, \quad \text{and} \quad \partial_u r + \frac{M}{r} = o(1) \cdot \frac{M}{r}, \]
we have
\[ \int_{U'}^{U} \frac{r(\partial_u \phi)^2}{\partial_u r}(u, v) du \geq -o_0(1)^2 \cdot \int_{U'}^{U} \frac{1}{M r^2} (u, v) du \geq -o_0(1)^2 \cdot \int_{r(U, v)}^{r(U', v)} \frac{1}{M^2 r^2} (u, v) dr(u, v) \]
\[ \geq o_0(1)^2 \cdot \log r(U, v) \geq \log r(U, v)^{o_0(1)^2}. \]

Using (9.6), we have
\[ \frac{\Omega^2}{-4 \partial_u r}(U, v) \geq \frac{\Omega^2}{-4 \partial_u r}(U', v) \cdot \exp \left[ \int_{U'}^{U} \frac{r(\partial_u \phi)^2}{\partial_u r}(u, v) du \right] \]
\[ \geq \frac{\Omega^2}{-4 \partial_u r}(U', v) \cdot r(U, v)^{o_0(1)^2} \]
\[ \geq \frac{1}{2} \cdot [1 + o(1)] \cdot r(U, v)^{o_0(1)^2}. \]
This gives
\[ \Omega^2(U, v) \geq -4\partial_u r(U, v) \cdot \frac{1}{2} [1 + o(1)] \cdot r(U, v)^{o(1)^2} \leq 2M \cdot [1 + o(1)] \cdot r(U, v)^{o(1)^2 - 1}. \]

Therefore, we obtain
\[ \Omega^{-2}(U, v) \leq \frac{1 + o(1)}{2M} \cdot r(U, v)^{1 - o(1)^2}, \]
and
\[ \Omega^{-1}(U, v) \leq C \cdot r(U, v)^{\frac{1}{2} - o(1)^2}, \]
for some positive constant \( C \).

We now move on to prove the main theorem of this section.

**Theorem 9.2.** We consider the open trapezoid region \( T_0 \) lying in \( T \) below.

For \( l_0 \) being a large positive constant, we prescribe initial data along \( r = 1/l_0 \): requiring
\[
|\partial_v r + \frac{M}{r}| \leq o_0(1) \cdot \frac{M}{r}, \quad |\partial_u r + \frac{M}{r}| \leq o_0(1) \cdot \frac{M}{r},
\]
\[
|\Omega^2 - \frac{2M}{r}| \leq o_0(1) \cdot \frac{M}{r},
\]
\[
|\partial_v \phi| \leq o_0(1) \cdot \frac{1}{r^2}, \quad |\partial_u \phi| \leq o_0(1) \cdot \frac{1}{r^2},
\]
where \( o_0(1) \) is a small positive number depending on initial data. Then for the dynamical spacetime solutions of (1.1) under spherical symmetry, under the prescribed initial data, in the region above, we have
\[ |R^\alpha\beta\rho\sigma R_{\alpha\beta\rho\sigma}| \lesssim \frac{1}{r^{6 + o_0(1)^2}}. \tag{9.7} \]

**Proof.** Use (3.2), we have
\[ |\partial_v \frac{\Omega}{\Omega}| = |\partial_v \log \Omega| \leq \frac{C}{r^2}. \]

This implies
\[ |\partial_v \Omega| \leq \frac{C}{r^2} \cdot |\Omega| \leq \frac{C}{r^3}. \]

Similarly, we have
\[ |\partial_u \Omega| \leq \frac{C}{r^3}. \]

Since
\[ \partial_u \partial_v \log \Omega = \partial_u \left( \frac{\partial_v \Omega}{\Omega} \right) = \frac{\partial_u \partial_v \Omega}{\Omega} - \frac{\partial_u \Omega \cdot \partial_v \Omega}{\Omega^2}, \]
we have
\[ \frac{\partial^2 \Omega}{\partial v \partial u} \cdot \Omega = \partial_v \Omega \cdot \partial_u \Omega = \Omega^2 \partial_u \partial_v \log \Omega. \]
We then bound the last two lines in (9.5)

\[
\frac{4}{r^4 \Omega^8} \left( 16 \cdot r^4 \cdot (\frac{\partial^2 \Omega}{\partial v \partial u})^2 \cdot \Omega^2 - 32 \cdot r^4 \cdot \frac{\partial^2 \Omega}{\partial v \partial u} \cdot \Omega \cdot \partial_v \Omega \cdot \partial_u \Omega \right) \\
+ \frac{4}{r^4 \Omega^8} \left( 16 \cdot r^4 \cdot (\partial_v \Omega)^2 \cdot (\partial_u \Omega)^2 + \Omega^8 \right) \\
= \frac{4}{r^4 \Omega^8} \cdot 16 \cdot r^4 \cdot \left( \frac{\partial^2 \Omega}{\partial v \partial u} \cdot \Omega - \partial_v \Omega \cdot \partial_u \Omega \right)^2 + \frac{4}{r^4} \\
= \frac{64}{r^4} \cdot (\partial_v \partial_u \log \Omega)^2 + \frac{4}{r^4} \\
\leq \frac{C}{r^8} \cdot r^{2-o(1)^2} + \frac{4}{r^4} \\
\leq \frac{C}{r^{6+o(1)^2}}.
\]

We control the third line in (9.5)

\[
\left| \frac{4}{r^4 \Omega^8} \left( 16 \cdot (\partial_v r)^2 \cdot (\partial_u r)^2 \cdot \Omega^4 + 64 \cdot \partial_v r \cdot r^2 \cdot \partial_u r \cdot \Omega^2 \cdot \partial_v \Omega \cdot \partial_u \Omega + 8 \cdot \partial_v r \cdot \partial_u r \cdot \Omega^8 \right) \right| \\
\leq \frac{64}{r^4 \Omega^4} \cdot (\partial_v r)^2 \cdot (\partial_u r)^2 + \frac{256}{r^2 \Omega^4} \cdot \partial_v r \cdot \partial_u r \cdot \log \Omega \cdot \partial_v \log \Omega + \frac{32}{r^4 \Omega^2} \cdot \partial_v r \cdot \partial_u r \\
\leq \frac{C}{r^{6+o(1)^2}}.
\]

Finally, we bound the first two lines in (9.5).

Note that (3.4) is equivalent to

\[
-\frac{2}{\Omega^3} \cdot \partial_u \log \Omega \cdot \partial_u r + \frac{\partial^2 r}{\partial u^2} = -r(\partial_u \phi)^2.
\]

Hence,

\[
\frac{\partial^2 r}{\partial u^2} = -r(\partial_u \phi)^2 + \frac{2}{\Omega^3} \cdot \partial_u \log \Omega \cdot \partial_u r.
\]

Similarly, we have

\[
\frac{\partial^2 r}{\partial v^2} = -r(\partial_v \phi)^2 + \frac{2}{\Omega^3} \cdot \partial_v \log \Omega \cdot \partial_v r.
\]

From (3.1), we have

\[
\partial_u \partial_v r = -\frac{\partial_u r \partial_v r}{r} - \frac{1}{4r} \Omega^2.
\]

Using all the estimates derived above, we thus conclude

\[
\left| \frac{\partial^2 r}{\partial u^2} \right| \leq \frac{C}{r^3}, \quad \left| \frac{\partial^2 r}{\partial v^2} \right| \leq \frac{C}{r^3}, \quad \left| \partial_u \partial_v r \right| \leq \frac{C}{r^3}.
\]

We have for the first line in (9.5):

\[
\left| \frac{4}{r^4 \Omega^8} \left( 16 \cdot \left( \frac{\partial^2 r}{\partial u \partial v} \right)^2 \cdot r^2 \cdot \Omega^4 + 16 \cdot \frac{\partial^2 r}{\partial u^2} \cdot \Omega^4 \right) \right| \leq \frac{C}{r^{6+o(1)^2}}.
\]
The second line in (9.5) obeys
\[
\left| \frac{4}{r^4 \Omega^8} \left[ -32 \frac{\partial^2 r}{\partial u^2} \cdot \partial_r r \cdot r^2 \cdot \Omega^3 \cdot \partial_u \Omega - 32 \frac{\partial^2 r}{\partial v^2} \cdot r^2 \cdot \partial_u r \cdot \Omega^3 \cdot \partial_v \Omega \right] \right|
\]
\[
= \left| \frac{4}{r^4 \Omega^8} \left[ 32 r \frac{\partial^2 r}{\partial u^2} \cdot \partial_r r \cdot r^2 \cdot \Omega^4 \cdot \partial_u \log \Omega - 32 \frac{\partial^2 r}{\partial v^2} \cdot r^2 \cdot \partial_u r \cdot \Omega^4 \cdot \partial_v \log \Omega \right] \right|
\]
\[
\leq \frac{C}{r^{6+o_0(1)^2}}.
\]
Therefore, we have proved the main theorem of this section. □

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