The Weyl–Brandt groupoid of a Nichols algebra of diagonal type

I. Heckenberger

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Abstract

The theory of Nichols algebras of diagonal type is known to be closely related to that of semisimple Lie algebras. In this paper the connection between both theories is made closer. For any Nichols algebra of diagonal type invertible transformations are introduced, which remind one of the action of the Weyl group on the root system associated to a semisimple Lie algebra. They give rise to the definition of a Brandt groupoid. As an application an alternative proof of classification results of Rosso, Andruskiewitsch, and Schneider is obtained without using any technical assumptions on the braiding.

Key Words: Brandt groupoid, Hopf algebra, pseudo-reflections, Weyl group

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1 Introduction

A method of Andruskiewitsch and Schneider for the classification of pointed Hopf algebras contains as an essential step the determination of

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Nichols algebras with certain growth conditions. However even in the simplest case, when the Nichols algebra is of diagonal type, there exists up to now no complete answer to the latter problem. General assertions which relate Nichols algebras to semisimple Lie algebras were proved for example by Rosso [10] and Andruskiewitsch and Schneider [4]. In both cases the introduction of additional technical assumptions was necessary.

Kharchenko [9] proved that any Nichols algebra $B(V)$ of diagonal type has a (restricted) Poincaré–Birkhoff–Witt basis consisting of iterated skew-commutators of basis vectors of $V$. Further, $B(V)$ has a natural $\mathbb{Z}^n$-grading, where $n$ is the rank of $B(V)$. This gives rise to the definition of the “root system” of $B(V)$, see Section 3. It is natural to look for a structure which plays here a similar role as the Weyl group for ordinary root systems. In Section 4 transformations of Nichols algebras of diagonal type are introduced. They naturally give rise to a Brandt groupoid structure, see Section 5, associated to $B(V)$. It will be called the Weyl–Brandt groupoid of $B(V)$ and denoted by $W(V)$. If the Nichols algebra is of Cartan type then $W(V)$ is isomorphic to $G \times B$ for a group $G$ and a set $B$. If the corresponding Cartan matrix $C$ is symmetrizable then $G$ is isomorphic to the Weyl group associated to $C$ and $B$ is the orbit of the ordered standard basis of $\mathbb{Z}^n$ under the action of the Weyl group. In the last section of this paper $W(V)$ is used to give a new proof of the classification results of Rosso and Andruskiewitsch and Schneider without using technical conditions on the braiding. The relative simplicity of the proof stems from the fact that Kharchenko’s results and the use of $W(V)$ allow to determine the degrees and heights of the (restricted) Poincaré–Birkhoff–Witt generators without knowing the defining relations of $B(V)$ explicitly.

Throughout this paper $k$ denotes a field of characteristic zero and tensor products $\otimes$ are taken over this field. Given an algebra $B$, let $B^{op}$ denote $B$ with the opposite product. Similarly, if $C$ is a coalgebra then $C^{cop}$ denotes $C$ with the opposite coproduct. For the coproduct and the antipode of a Hopf algebra the symbols $\Delta$ and $\kappa$ are used. The coproduct of elements $a$ of a
coalgebra is written in the Sweedler notation $\Delta(a) = a_{(1)} \otimes a_{(2)}$. The set of natural numbers not including 0 is denoted by $\mathbb{N}$ and we write $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

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2 Left and right skew-differential operators

Let $k$ be a field of characteristic zero, $G$ an abelian group, and $V \in \text{kG} \text{YD}$ a finite dimensional Yetter–Drinfel’d module with completely reducible $kG$-action. Let $\delta : V \to kG \otimes V$ and $\cdot : kG \otimes V \to V$ denote the left coaction and left action of $kG$ on $V$, respectively. Set $n := \text{dim}_k V$. By assumption there exist nonzero numbers $q_{ij} \in k$, a basis $\{x_i \mid 1 \leq i \leq n\}$ of $V$, and for each $i$ with $1 \leq i \leq n$ an element $g_i \in G$ such that

$$g_i.x_j = q_{ij}x_j, \quad \delta(x_j) = g_j \otimes x_j$$

(1)

for all $i, j \in \{1, 2, \ldots, n\}$. Then the braiding $\sigma \in \text{End}_k(V \otimes V)$ of $V$ where

$$\sigma(v \otimes w) = (v_{(-1)}w) \otimes v_{(0)}, \quad \sigma^{-1}(v \otimes w) = w_{(0)} \otimes (\kappa^{-1}(w_{(-1)})v),$$

and $\delta(v) = v_{(-1)} \otimes v_{(0)}$ for $v \in V$, is of diagonal type. Let $\mathcal{B}(V)$ denote the Nichols algebra generated by $V$. More precisely, as proved in [1] and noted in [2, Prop. 2.11],

$$\mathcal{B}(V) = k \oplus V \oplus \bigoplus_{m=2}^{\infty} V^\otimes m / \ker S_m$$

where $S_m \in \text{End}_k(V^\otimes m)$, $S_{1,j} \in \text{End}_k(V^\otimes j+1)$,

$$S_m = \prod_{j=1}^{m-1} (\text{id}^\otimes m-j-1 \otimes S_{1,j}) = \prod_{j=1}^{m-1} (S_{j,1} \otimes \text{id}^\otimes m-j-1),$$

$$S_{1,j} = \text{id} + \sigma_{12}^{-1} + \sigma_{12}^{-1}\sigma_{23}^{-1} + \cdots + \sigma_{12}^{-1}\sigma_{23}^{-1} \cdots \sigma_{j,j+1}^{-1},$$

$$S_{j,1} = \text{id} + \sigma_{j,j+1}^{-1} + \sigma_{j,j+1}^{-1}\sigma_{j-1,j}^{-1} + \cdots + \sigma_{j,j+1}^{-1} \cdots \sigma_{2,3}^{-1}\sigma_{12}^{-1}.$$
(in leg notation) for $m \geq 2$ and $j \in \mathbb{N}_0$. The maps $S_m$ are called braided symmetrizer. They have analogues where the braiding $\sigma$ is used instead of $\sigma^{-1}$. Set $\mathcal{B}(V)_i := V^{\otimes i}/\ker S_i \subset \mathcal{B}(V)$. The algebra $\mathcal{B}(V)$ is $\mathbb{Z}^n$-graded, where the degrees of the generators $x_i$ are $\deg x_i = e_i$, and $\{e_i | 1 \leq i \leq n\}$ is a basis of the $\mathbb{Z}$-module $\mathbb{Z}^n$. 

Note that in our setting $V$ is additionally a Yetter–Drinfel’d module over $k\mathbb{Z}^n$.

$$e_i \triangleright x_j = q_{ij} x_j, \quad \gamma(x_j) = e_j \otimes x_j. \tag{2}$$

Here $\{e_i | 1 \leq i \leq n\}$ is a fixed basis of the $\mathbb{Z}$-module $\mathbb{Z}^n$, and $\triangleright$ and $\gamma$ denote the left action and left coaction of $k\mathbb{Z}^n$ on $V$. In order to avoid misunderstandings we will use the exponential notation for elements of $\mathbb{Z}^n$; thus $e_i^{-1}$ is the inverse of $e_i$ in the group $\mathbb{Z}^n$. The braiding $\sigma$ of $V$ commutes both with the action of $g_i$ and the action of $e_i$: for all $i, j, m \in \{1, 2, \ldots, n\}$ one has

$$\sigma(g_i (x_j \otimes x_m)) = g_i \sigma(x_j \otimes x_m), \quad \sigma(e_i \triangleright (x_j \otimes x_m)) = e_i \triangleright \sigma(x_j \otimes x_m). \tag{3}$$

The dual vector space $V^*$ gives rise to left and right skew-differential operators $y^L_l$ and $y^R_l$ on $\mathcal{B}(V)$, where $1 \leq i \leq n$, in the following way. Let $\{y_l | 1 \leq i \leq n\}$ denote the basis of $V^*$ dual to $\{x_i | 1 \leq i \leq n\}$. For $m \in \mathbb{N}$, $\rho \in \mathcal{B}(V)_m$ and $i \in \{1, 2, \ldots, n\}$ set

$$y^L_l(1) = y^R_l(1) = 0, \quad y^L_l(\rho) = \rho'_l, \quad y^R_l(\rho) = \rho''_l. \tag{4}$$

where

$$S_{1,m-1}(\rho) = \sum_{l=1}^n x_l \otimes \rho'_l, \quad S_{m-1,1}(\rho) = \sum_{l=1}^n \rho''_l \otimes x_l.$$ 

Note that the antipode $\kappa$ of group algebras satisfies $\kappa^2 = \text{id}$ and hence left and right duals of Yetter–Drinfel’d modules coincide. One can consider the vector space $\text{Lin}_k \{y^L_i | 1 \leq i \leq n\}$ as a Yetter–Drinfel’d module over $kG$ dual to $V$, that is for $i, j \in \{1, 2, \ldots, n\}$ one has

$$g_i \cdot y^L_j = q_{ij}^{-1} y^L_j, \quad \delta(y^L_j) = g_j^{-1} \otimes y^L_j. \tag{5}$$
Similarly, the vector space \( \text{Lin}_k \{ y_i^k \mid 1 \leq i \leq n \} \) becomes a Yetter–Drinfel’d module over \( k \mathbb{Z}^n \) dual to \( V \), that is one has
\[
  e_i \triangleright y_j^R = q_{ji}^{-1} y_j^R, \quad \gamma(y_j^R) = e_j^{-1} \otimes y_j^R
\]  
for all \( i, j \in \{1, 2, \ldots, n\} \). Moreover, the skew-differential operators \( y_i^L \) and \( y_i^R \), where \( 1 \leq i \leq n \), and the braiding \( y \) satisfy the equations
\[
\begin{align*}
  (y_i^L \otimes \text{id})(\sigma^{-1}(x_j \otimes x_m)) &= (g_i^{-1} x_j) y_i^L(x_m), \\
  (\text{id} \otimes y_i^R)(\sigma^{-1}(x_j \otimes x_m)) &= y_i^R(x_j) e_i^{-1} \triangleright x_m
\end{align*}
\]
for all \( j, m \in \{1, 2, \ldots, n\} \). Therefore equations (6) give that
\[
\begin{align*}
  y_i^L(\rho_1 \rho_2) &= y_i^L(\rho_1) \rho_2 + g_i^{-1} \rho_1 y_i^L(\rho_2), \\
  y_i^R(\rho_1 \rho_2) &= \rho_1 y_i^R(\rho_2) + y_i^R(\rho_1) e_i^{-1} \triangleright \rho_2
\end{align*}
\]  
for all \( \rho_1, \rho_2 \in \mathcal{B}(V) \) and \( i \in \{1, 2, \ldots, n\} \). Note that by definition the skew-differential operators \( y_i^L \) and \( y_j^R \) commute:
\[
y_i^L(y_j^R(\rho)) = y_j^R(y_i^L(\rho)) \quad \text{for all } i, j \in \{1, 2, \ldots, n\}, \rho \in \mathcal{B}(V).
\]  
Moreover, the skew-differential operators \( y_i^L \) and \( y_i^R \) and the group-like elements \( g_j, e_j \), where \( i, j \in \{1, 2, \ldots, n\} \), satisfy the equations
\[
\begin{align*}
  g_j.(y_i^L(g_j^{-1} \rho)) &= g_j^{-1} y_i^L(\rho), \quad e_j \triangleright (y_i^L(e_j^{-1} \triangleright \rho)) &= g_j^{-1} y_i^L(\rho), \\
  g_j.(y_i^R(g_j^{-1} \rho)) &= g_j^{-1} y_i^R(\rho), \quad e_j \triangleright (y_i^R(e_j^{-1} \triangleright \rho)) &= g_j^{-1} y_i^R(\rho)
\end{align*}
\]  
for all \( \rho \in \mathcal{B}(V) \). It is a standard fact that the algebra generated by the skew-differential operators \( y_i^L \), where \( i \in \{1, 2, \ldots, n\} \), is isomorphic to \( \mathcal{B}(V^*) \). Further, as noted in [5, Sect. 2.1], the assignment \( x_i \mapsto y_i^L \) extends uniquely to an algebra isomorphism \( \iota : \mathcal{B}(V) \to \mathcal{B}(V^*) \). Finally, by Lemma 1 and Corollary 2 in [5] there exists a unique bilinear map \( \langle \cdot, \cdot \rangle : (\mathcal{B}(V^*) \# kG) \otimes \mathcal{B}(V) \to \mathcal{B}(V) \) which defines a \( \mathcal{B}(V^*) \# kG \)-module algebra structure on \( \mathcal{B}(V) \) extending the action of \( y_i^L \) and \( g_i \) for all \( i \in \{1, 2, \ldots, n\} \).
3 Finiteness conditions

Retain the notation from the previous section. By a theorem of Kharchenko [9, Theorem 2] the algebra \(\mathcal{B}(V)\) has a (restricted) Poincaré–Birkhoff–Witt basis consisting of homogeneous elements with respect to the \(\mathbb{Z}^n\)-grading of \(\mathcal{B}(V)\). Let \(\Delta^+(\mathcal{B}(V)) \subset \mathbb{Z}^n\) denote the set of degrees of the (restricted) Poincaré–Birkhoff–Witt generators of \(\mathcal{B}(V)\), counted with multiplicities. By the definition of the \(\mathbb{Z}^n\)-degree of \(\mathcal{B}(V)\) one clearly has \(\Delta^+(\mathcal{B}(V)) \subset \mathbb{N}_0^n\). Set \(\Delta(\mathcal{B}(V)) = \Delta^+(\mathcal{B}(V)) \cup -\Delta^+(\mathcal{B}(V))\).

Let’s consider the following finiteness conditions on \(\mathcal{B}(V)\).

(F1) \(\dim_k \mathcal{B}(V) < \infty\),

(F2) the set \(\Delta^+(\mathcal{B}(V))\) is finite,

(F3) \(\dim_k \mathcal{B}(V) < \infty\),

where \(\dim_k\) denotes Gel’fand–Kirillov dimension. Obviously one has the implications \((F1) \Rightarrow (F2) \Rightarrow (F3)\). Further, the condition \((F1)\) holds if and only if \((F2)\) is satisfied and the heights of all restricted Poincaré–Birkhoff–Witt generators of \(\mathcal{B}(V)\) are finite. It is not known whether \((F3)\) implies \((F2)\).

As in [2] define

\[
\text{ad}_{\sigma} x_i(\rho) := x_i \rho - (g_i, \rho) x_i
\]

for \(\rho \in \mathcal{B}(V)\) and \(i \in \{1, 2, \ldots, n\}\). Consider the sets

\[
M_{i,j} := \{(\text{ad}_{\sigma} x_i)^m (x_j) | m \in \mathbb{N}_0\}
\]

for \(i, j \in \{1, 2, \ldots, n\}, i \neq j\). By [10, Lemma 20], if one assumes that \((F3)\) holds then all \(M_{i,j}\) are finite sets. More generally, by [2, Lemma 3.7] or [6, Sect. 4.1] for given \(i, j \in \{1, 2, \ldots, n\}\) with \(i \neq j\) the number

\[
m_{ij} := \min\{m \in \mathbb{N}_0 | (m + 1) q_i (q_i q_{ij} q_{ji} - 1) = 0\}
\]

is well-defined if and only if \(M_{i,j}\) is finite. In this case one has the relations \((\text{ad}_{\sigma} x_i)^{m_{ij}+1}(x_j) = 0\) and \((\text{ad}_{\sigma} x_i)^{m_{ij}}(x_j) \neq 0\).
Fix $i \in \{1,2,\ldots, n\}$. By the above paragraph, if $M_{i,j}$ is finite for all $j \in \{1,2,\ldots, n\}$ with $j \neq i$ (for example if (F3) holds) then one can introduce a $\mathbb{Z}$-linear mapping $s_i : \mathbb{Z}^n \to \mathbb{Z}^n$ as follows:

$$s_i(e_j) := \begin{cases} -e_i & \text{if } j = i, \\ e_j + m_{ij} e_i & \text{if } j \neq i. \end{cases}$$

(13)

In [3, Ch. 5, §2] such maps are called pseudo-reflections. Note that $s_i^2 = \text{id}$.

### 4 Transformations of Nichols algebras

Assume for the whole section that $B(V)$ is a rank $n$ Nichols algebra of diagonal type. Further, suppose that $i \in \{1,2,\ldots, n\}$ is chosen such that $M_{i,j}$ is finite for all $j \in \{1,2,\ldots, n\}$ with $j \neq i$. We describe the construction of a Nichols algebra associated to $i$.

Since $k y_i^R \in k \mathcal{YD}$ one can construct the smash product

$$H_i := k[y_i^R]#k[e_i, e_i^{-1}].$$

It has a unique Hopf algebra structure satisfying the formulas

$$e_i y_i^R = q_{ii}^{-1} y_i^R e_i, \quad \Delta(e_i) = e_i \otimes e_i, \quad \Delta(y_i^R) = 1 \otimes y_i^R + y_i^R \otimes e_i^{-1}.$$  \hspace{1cm} (14)

By Equations (10) and (8) one obtains that $B(V)$ is an $H_i$-module algebra, where $e_i$ and $e_i^{-1}$ act via $\triangleright$ and $y_i^R$ acts by evaluation. By slight abuse of notation the symbol $\triangleright$ will also be used for the left action of $H_i$ on $B(V)$. Let $(B(V)^{\text{op}}#H_i^{\text{cop}})^{\text{op}}$ denote the opposite algebra of the smash product of $B(V)^{\text{op}}$ and $H_i^{\text{cop}}$. Recall that it contains $B(V)$ and $H_i^{\text{op,cop}}$ as subalgebras and one has

$$\rho h = h(1)(h(2) \triangleright \rho),$$

(15)

and in particular

$$\rho y_i^R = y_i^R \cdot (e_i^{-1} \triangleright \rho) + y_i^R(\rho),$$

(16)
for all \( \rho \in \mathcal{B}(V) \) and \( h \in H_i \), where \( \Delta(h) = h^{(1)} \otimes h^{(2)} \) denotes the coproduct of \( h \in H_i \). Further, \( (\mathcal{B}(V)^{\mathrm{op}} \# H_i^\mathrm{cop})^{\mathrm{op}} \) is a Yetter–Drinfel’d module over \( kG \) where the left action . and left coaction \( \delta \) on \( (\mathcal{B}(V)^{\mathrm{op}} \# H_i^\mathrm{cop})^{\mathrm{op}} \) are given by

\[
\begin{align*}
\delta(x_j) &= g_j \otimes x_j, \\
\delta(y_i^R) &= g_i^{-1} \otimes y_i^R, \\
\delta(e_i) &= 1 \otimes e_i,
\end{align*}
\]

for all \( j, m \in \{1, 2, \ldots, n\} \).

**Proposition 1.** Assume that \( i \in \{1, 2, \ldots, n\} \) such that \( M_{i,j} \) is finite for all \( j \in \{1, 2, \ldots, n\} \) with \( j \neq i \). Let \( V_i \) denote the subspace of the algebra \( (\mathcal{B}(V)^{\mathrm{op}} \# H_i^\mathrm{cop})^{\mathrm{op}} \) generated by the set \( \{(\mathrm{ad}_\sigma x_i)^{m,j}(x_j) \mid j \neq i\} \cup \{y_i^R\} \). The subalgebra \( \mathcal{B}_i \) of \( (\mathcal{B}(V)^{\mathrm{op}} \# H_i^\mathrm{cop})^{\mathrm{op}} \) generated by \( V_i \) is isomorphic to the Nichols algebra \( \mathcal{B}(V_i) \), and the relation \( \Delta^+(\mathcal{B}_i) = (s_i(\Delta^+(\mathcal{B}(V))) \setminus \{-e_i\}) \cup \{e_i\} \) holds.

**Remark.** It will be clear from the construction that the transformation in Proposition 1 is invertible. This fact is in accord with the relation \( s_i^2 = \text{id.} \)

**Proof.** One easily checks that \( V_i \) is a Yetter–Drinfel’d submodule of \( (\mathcal{B}(V)^{\mathrm{op}} \# H_i^\mathrm{cop})^{\mathrm{op}} \) over \( kG \) where the Yetter–Drinfel’d module structure of \( (\mathcal{B}(V)^{\mathrm{op}} \# H_i^\mathrm{cop})^{\mathrm{op}} \) is described above the proposition.

In the following we will sometimes consider \( (\mathcal{B}(V)^{\mathrm{op}} \# H_i^\mathrm{cop})^{\mathrm{op}} \) as a vector space and as such we identify it with \( \mathcal{B}(V) \otimes H_i \).

We show that

\[
\mathcal{B}_i = \ker y_i^k \otimes k[y_i^R] (\subset \mathcal{B}(V) \otimes H_i).
\]

Recall that \( y_i^R \) may have finite order, but it is always the same as the order of \( x_i \). Note also that by Equations (14), (15), and (9) the space \( \ker y_i^k \otimes k[y_i^R] \) is a subalgebra of \( (\mathcal{B}(V)^{\mathrm{op}} \# H_i^\mathrm{cop})^{\mathrm{op}} \). The inclusion \( \mathcal{B}_i \subset \ker y_i^k \otimes k[y_i^R] \) holds by definition of \( \mathcal{B}_i \). In order to prove that \( \mathcal{B}_i \supset \ker y_i^k \otimes k[y_i^R] \) one first checks that \( \mathcal{B}(V) \cong \ker y_i^k \otimes k[x_i] \) (as graded vector spaces) using methods similar
to [Lemma 2.2]. Further, \( y_i^\ell \) is generated as an algebra by \( \bigcup_{j \neq i} M_{i,j} \). Now one proves by induction that

\[
y_i^\ell ((\text{ad} \sigma x_i)^m(x_j)) = q_{ji}^{-1}(m) q_{ji}^{-1}(1 - q_{ji}^{m-1} q_{ij} q_{ji})(\text{ad} \sigma x_i)^{m-1}(x_j)
\]

(19)

for all \( m \in \mathbb{N}_0 \) and all \( j \in \{1, 2, \ldots, n\} \). Hence by Equation (16) one obtains that \( \ker y_i^\ell \otimes k[y_i^\ell] \) is generated by \( V_i \).

In order to prove that the subalgebra \( B_i \) of \( (B(V)^{\text{op}} \# H_i^{\text{cop}})^{\text{op}} \) is a Nichols algebra it suffices to find \( n \) appropriate skew-differential operators on \( B_i \). More precisely, according to the fact that the Yetter–Drinfel’d module \( V_i \) generates \( B_i \), one has to find for all \( m \in \{1, 2, \ldots, n\} \) a map \( Y_m \in \text{End}(B_i) \) such that

\[
Y_j(\rho_1 \rho_2) = Y_j(\rho_1) \rho_2 + (g_i^{-m_{ij}} g_j^{-1} \cdot \rho_1) Y_j(\rho_2),
\]

\[
Y_i(\rho_1 \rho_2) = Y_i(\rho_1) \rho_2 + (g_i \cdot \rho_1) Y_i(\rho_2),
\]

(20)

\[
Y_l((\text{ad} \sigma x_i)^{m_{ij}}(x_j)) = \delta_{lj}, \quad Y_l(y_i^\ell) = \delta_{li} \quad \text{(Kronecker’s delta)}
\]

for all \( \rho_1, \rho_2 \in B_i \), and \( j, l \in \{1, 2, \ldots, n\}, j \neq i \). For \( \rho \in B_i \) set

\[
Y_i(\rho) := \text{ad} \sigma x_i(\rho) = x_i \rho - (g_i \cdot \rho)x_i.
\]

By definition of \( \text{ad} \sigma \), \( Y_i \) satisfies the second equation of (20) for all \( \rho_1, \rho_2 \in (B(V)^{\text{op}} \# H_i^{\text{cop}})^{\text{op}} \). Moreover, by Equation (8) one obtains for all \( \rho \in \ker y_i^\ell \) the formula

\[
y_i^\ell(x_i \rho - (g_i \cdot \rho)x_i) = \rho - g_i^{-1} (g_i \cdot \rho) = 0.
\]

Hence \( Y_i \) maps \( \ker y_i^\ell \subset B(V) \) onto itself. By Equations (17) and (16) one gets also

\[
Y_i(y_i^\ell) = x_i y_i^\ell - (g_i \cdot y_i^\ell)x_i = x_i y_i^\ell - q_{ii}^{-1} y_i^\ell x_i = y_i^\ell(x_i) = 1.
\]

Finally, the equation \( Y_i((\text{ad} \sigma x_i)^{m_{ij}}(x_j)) = (\text{ad} \sigma x_i)^{m_{ij}+1}(x_j) = 0 \) holds for all \( j \in \{1, 2, \ldots, n\}, j \neq i \), by the definition of \( m_{ij} \).

Let’s now turn to the construction of \( Y_j \), where \( j \in \{1, 2, \ldots, n\}, j \neq i \). Set \( \lambda_j := \langle \iota((\text{ad} \sigma x_i)^{m_{ij}}(x_j)), (\text{ad} \sigma x_i)^{m_{ij}}(x_j) \rangle \) (for the notation see the last
paragraph of Section [2]. By [6, Sect. 4.1] and the definition of $m_{ij}$ one obtains that $\lambda_j \in k \setminus \{0\}$. Define

$$Y_j(\rho \otimes (y_i^R)^m) := \frac{1}{\lambda_j}(\iota((\text{ad} \sigma x_i)^{m_{ij}}(x_j)), \rho) \otimes (y_i^R)^m$$

for all $\rho \in \ker y_i^l$, $m \in \mathbb{N}_0$, and $j \in \{1, 2, \ldots, n\}$, $j \neq i$. By [6, Eqn. (2)] the coproduct of $\iota((\text{ad} \sigma x_i)^{m_{ij}}(x_j)) \in \mathcal{B}(V^*)^\#kG$ takes the form

$$\iota((\text{ad} \sigma x_i)^{m_{ij}}(x_j)) \otimes 1 + \sum_{m=0}^{m_{ij}} c_i(y_i^l)^{m_{ij}-m} g_i^{-m} g_j^{-1} \otimes \iota((\text{ad} \sigma x_i)^{m}(x_j))$$

for certain $c_i \in k$, where $c_{m_{ij}} = 1$. Since $y_i^l$ vanishes on $\ker y_i^l$, the first equation of (20) holds for all $\rho_1 \in \ker y_i^l$, $\rho_2 \in \mathcal{B}$. It remains to show that the first equation of (20) is valid for $\rho_1 = y_i^R$ and $\rho_2 \in \ker y_i^l$. By Equations (10) and (9) one obtains that

$$Y_j(y_i^R \rho_2) = Y_j((e_i \triangleright \rho_2) \otimes y_i^R - y_i^R(e_i \triangleright \rho_2)) = Y_j(e_i \triangleright \rho_2) \otimes y_i^R - y_i^R(Y_j(e_i \triangleright \rho_2)). \quad (21)$$

On the other hand, by Equations (10) and (9) one gets

$$(g_i^{-m_{ij}} g_j^{-1} y_i^R) Y_j(\rho_2) = g_i^{-m_{ij}} g_j y_i^R (e_i \triangleright Y_j(\rho_2)) \otimes y_i^R - y_i^R(e_i \triangleright Y_j(\rho_2)).$$

By Equation (10) the latter coincides with (21).

Finally, it has to be shown that $\Delta^+(\mathcal{B}_i) = (s_i(\Delta^+(\mathcal{B}(V))) \setminus \{-e_i\}) \cup \{e_i\}$. In the $\mathbb{Z}^n$-graded algebra $(\mathcal{B}(V)^{\text{op}} \# H_i^{\text{cop}})^{\text{op}}$ the elements $x_j$ and $y_i^R$ have degree $e_j$ and $-e_i$, respectively, for all $j \in \{1, 2, \ldots, n\}$. Hence the elements $(\text{ad}_x)^{m_{ij}}(x_j)$ have degree $e_j + m_{ij} e_i = s_i(e_j)$ for all $j \in \{1, 2, \ldots, n\}$, $j \neq i$. Fix the $\mathbb{Z}^n$-degrees of the generators of $\mathcal{B}_i$ by

$$\deg y_i^R := e_i, \quad \deg (\text{ad}_x)^{m_{ij}}(x_j) := e_j \quad (22)$$

for $j \in \{1, 2, \ldots, n\}$, $j \neq i$. Then $s_i(\Delta^+(\mathcal{B}_i))$ is exactly the set of degrees of the (restricted) Poincaré–Birkhoff–Witt generators of $\mathcal{B}_i$ in $(\mathcal{B}(V)^{\text{op}} \# H_i^{\text{cop}})^{\text{op}}$.  

10
Recall that any Nichols algebra $\mathcal{B}$ of rank $n$ is isomorphic as a $\mathbb{Z}^n$-graded vector space to the algebra $k[x_r | r \in \Delta^+(\mathcal{B})]/(x_r^{h_r} | r \in \Delta^+(\mathcal{B}), h_r < \infty)$. Here $h_r$ denotes the height of the (restricted) Poincaré–Birkhoff-Witt generator corresponding to $r \in \Delta^+(\mathcal{B})$, and by Kharchenko’s theorem it is uniquely determined by the $\mathbb{Z}^n$-degree of $x_r$. Therefore it suffices to know the multiplicities of the $\mathbb{Z}^n$-homogeneous components of $\mathcal{B}$ in order to determine $\Delta^+(\ker y_i \otimes k[y_i])$. This fact and the equation $\mathcal{B}(V) \cong \ker y_i \otimes k[x_i]$ allows us to conclude that (with obvious interpretation) $\Delta^+(\ker y_i \otimes k[y_i]) = (\Delta^+(\mathcal{B}(V)) \setminus \{e_i\}) \cup \{-e_i\}$. Since the determination of $\Delta^+$ can be performed using different total orderings of $\mathbb{Z}^n$, we conclude from Equation (18) that $s_i(\Delta^+(\mathcal{B}_i)) = (\Delta^+(\mathcal{B}(V)) \setminus \{e_i\}) \cup \{-e_i\}$. This proves the proposition. 

**Remark.** 1. Proposition 11 has the following interpretation. Set $\Delta := \Delta(\mathcal{B}(V)) \subset \mathbb{Z}^n$. Then the set $\Delta(\mathcal{B}_i)$ coincides with $\Delta$ with respect to the basis $\{s_i(e_j) | 1 \leq j \leq n\}$ of $\mathbb{Z}^n$. With other words, the transformation doesn’t change $\Delta(\mathcal{B}(V))$, it changes only the basis of $\mathbb{Z}^n$. Additionally, this base change is performed in such a way that the new basis is a subset of $\Delta(\mathcal{B}(V))$.

2. The constants $m_{ij}$ appearing in the definition of the map $s_i$, where $j \in \{1, 2, \ldots, n\}$ and $j \neq i$, depend on the structure constants of the braiding. The latter usually change if one performs a transformation. However in a special case, namely if the braiding is of Cartan type, this change is not essential. This situation is the one which is best understood, and it will also be analyzed in more detail in Section 6. 

5 **The Weyl–Brandt groupoid associated to a Nichols algebra**

Let $\mathcal{G}$ be a nonempty set, $D \subset \mathcal{G} \times \mathcal{G}$ a nonempty subset, and $\circ : D \to \mathcal{G}$ a map of sets. The pair $(\mathcal{G}, \circ)$ is called Brandt groupoid if it satisfies the following conditions (see for example [4, Sect. 3.3]).
• If \((x, y) \in D\) then each of the three elements \(x, y, x \circ y\) is uniquely determined by the other two.

• If \((x, y), (y, z) \in D\) then \((x \circ y, z) \in D\) and \((x \circ y) \circ z = x \circ (y \circ z)\).

• If \((x, y), (x \circ y, z) \in D\) then \((y, z) \in D\) and \((x \circ y) \circ z = x \circ (y \circ z)\).

• For all \(x \in \mathcal{G}\) there exist unique elements \(e, f, y \in \mathcal{G}\) such that \((e, x), (x, f), (y, x) \in D\), \(e \circ x = x \circ f = x\), and \(y \circ x = f\).

• If \(e \circ e = e, f \circ f = f\) for certain \(e, f \in \mathcal{G}\) then there exists \(x \in \mathcal{G}\) such that \(e \circ x = x \circ f = x\).

Let \(B(\mathcal{V})\) be a rank \(n\) Nichols algebra of diagonal type. Let \(E_0 := (e_1, \ldots, e_n)\) denote the (ordered) standard basis of \(\mathbb{Z}^n\). Define

\[
W(\mathcal{V}) := \{ (s, E) \mid s \in \text{Aut}(\mathbb{Z}^n), E \text{ is an ordered basis of } \mathbb{Z}^n, \\
\text{there exist } m_1, m_2 \in \mathbb{N}_0, m_1 \leq m_2, \\
\text{and } i_1, \ldots, i_{m_2} \in \{1, 2, \ldots, n\}, \text{ such that} \\
s_{i_{m_2}} \ldots s_{i_1}(E_0) \text{ is well defined for all } m \leq m_2 \text{ and} \\
s_{i_{m_1}} \ldots s_{i_1}(E_0) = E, s = s_{i_{m_2}} \ldots s_{i_{m_1+1}} \}. 
\]

Note that \(W(\mathcal{V})\) is not empty since \((\text{id}, E_0) \in W(\mathcal{V})\). Recall that \(s_i^2 = \text{id}\) whenever \(s_i\) is defined. Hence there is a natural Brandt groupoid structure on \(\mathcal{G} = W(\mathcal{V})\) such that \((s, E) \circ (t, F)\) is defined (and is then equal to \((st, F)\)) if and only if \(t(F) = E\). We call \(W(\mathcal{V})\) the Weyl–Brandt groupoid of \(B(\mathcal{V})\).

The definition of \(W(\mathcal{V})\) gives an important consequence of Proposition \(\mathbb{H}\).

**Corollary 2.** If \(B(\mathcal{V})\) is a rank \(n\) Nichols algebra of diagonal type satisfying \((F2)\) then \(W(\mathcal{V})\) is finite. In particular, the orbit of any element and any ordered basis of \(\mathbb{Z}^n\) under the action of \(W(\mathcal{V})\) is finite.
6 Nichols algebras of Cartan type

If the braiding $\sigma$ of $V$ is of diagonal type and the structure constants $q_{ij}$ satisfy the equations

$$q_{ij}q_{ji} = q_{ii}^{a_{ij}}, \quad i, j \in \{1, 2, \ldots, n\},$$

where $a_{ii} = 2$ and $a_{ij}$ is a nonpositive integer for all $i \neq j$, then one says that $\sigma$ (and $V$ and $B(V)$) is of Cartan type. It is then always assumed that the $a_{ij}$ are maximal with the above properties. Note that if $a_{ij} > 0$ for some $j \neq i$ and $q_{ii}$ is not a root of unity then by Rosso [10, Lemma 20] the Gel’fand–Kirillov dimension of $B(V)$ is infinite.

**Lemma 3.** Assume that $V$ is an $n$-dimensional Yetter–Drinfel’d module of Cartan type and let $(a_{ij})_{i,j\in\{1,2,\ldots,n\}}$ denote the corresponding Cartan matrix.

(i) Suppose that $i \in \{1, 2, \ldots, n\}$ and $m \in \mathbb{N}_0$ such that $m < -a_{ij}$ for at least one $j \in \{1, 2, \ldots, n\}$. Then $q_{ii}^{m+1} \neq 1$.

(ii) For the numbers in Equation (23) one obtains $m_{ij} = -a_{ij}$ for all $i, j \in \{1, 2, \ldots, n\}$.

**Proof.** To (i). Since $V$ is of Cartan type, one has $q_{ii}^{a_{ij}} = q_{ij}q_{ji}$. Assume that $m + 1 \leq -a_{ij}$ and $q_{ii}^{m+1} = 1$. One obtains that $q_{ii}^{m+1+a_{ij}} = q_{ij}q_{ji}$, and $m + 1 + a_{ij} \leq 0$. This is a contradiction to the choice of $a_{ij}$.

To (ii). This follows from the definition of $m_{ij}$ and from (i).

One says that $\sigma$ is of finite type if the Cartan matrix $(a_{ij})_{i,j\in\{1,2,\ldots,n\}}$ is of finite type. There exist classification results of Rosso [10, Theorem 21] and Andruskiewitsch and Schneider [11, Theorem 1.1] on Nichols algebras of Cartan type with finite Gel’fand–Kirillov dimension (F3) and finite dimension (F1), respectively. The introduction of the Weyl–Brandt groupoid $W(V)$ in the previous section allows to state a theorem without technical assumptions on the numbers $q_{ij}$.

**Theorem 4.** Let $V$ be an $n$-dimensional Yetter–Drinfel’d module of Cartan type.
tan type with corresponding Cartan matrix $C := (a_{ij})_{i,j \in \{1,2,\ldots,n\}}$.

(i) If $C$ is not of finite type then $\Delta(B(V))$ is infinite.
(ii) If $C$ is of finite type then $\Delta(B(V))$ can be identified with the set of roots of the semisimple Lie algebra corresponding to $C$. Moreover, in any connected component the heights of the (restricted) Poincaré–Birkhoff–Witt generators depend only on the lengths of the roots corresponding to them.

**Proof.** To (i). Assume that $\Delta(B(V))$ is finite. By Proposition 1 for each $i \in \{1,2,\ldots,n\}$ there exists a Nichols algebra $B_i \cong B(V_i)$ of rank $n$. By Lemma 3(ii) one has $m_{ij} = -a_{ij}$. Choose the basis of $V_i$ in such a way that the $j$th basis vector is $(\text{ad}_{\sigma x_i})^{-a_{ij}}(x_j)$ if $j \neq i$ and $y_i^l$ if $j = i$, respectively. Let $\{q(i)_{jm} \mid j, m \in \{1,2,\ldots,n\}\}$ denote the set of structure constants of the braiding of $V_i$ with respect to this basis. By Equations (17) one gets

$$q(i)_{jm} = \begin{cases} q_{ii} & \text{if } j = m = i, \\ q_{ii} q_{im}^{-1} q_{mi} & \text{if } j = i, m \neq i, \\ q_{ii} q_{ji}^{-1} = q_{ij} & \text{if } j \neq i, m = i, \\ q_{ii} a_{ij} a_{im} q_{im}^{-a_{ij}} q_{jm} = q_{ij} q_{im}^{-a_{ij}} & \text{if } j \neq i, m \neq i. \end{cases}$$

In particular, one obtains that

$$q(i)_{jj} = q_{jj}, \quad q(i)_{jm} q(i)_{mj} = q(i)_{jj} a_{jm}^{a_{jm}}$$

for all $j, m \in \{1,2,\ldots,n\}$. Hence $B_i$ is of the same Cartan type as $B(V)$. This means that the linear maps $s$ in the elements $(s, E) \in W(V)$ don’t depend on the basis $E$ of $Z^n$. By (F2) and Corollary 2 the Weyl–Brandt groupoid $W(V)$ is finite. If $C$ is symmetrizable then $W(V)$ is canonically isomorphic to $W \times B$ where $W$ is the Weyl group associated to $C$ and $B$ is the orbit of $E_0$ under $W$. Here $E_0 := (e_1,\ldots,e_n)$ denotes the ordered standard basis of $Z^n$. It is well-known [7, Ch. 1, Theorem 4.8] that $W$ is finite if and only if the symmetrizable Cartan matrix is of finite type. Thus for (i) it suffices to show that the group $W(V)$ is infinite whenever $C$ is not symmetrizable. Note that if $W(V')$ corresponding to a Yetter–Drinfel’d submodule $V'$ of $V$ is infinite
then $W(V)$ is itself infinite. Further, if the Dynkin diagram associated to a Cartan matrix is simply-laced or has no cycles then it is symmetrizable. Thus we only have to show that $W(V)$ is infinite if the corresponding Dynkin diagram is a cycle which is not simply-laced. Further, if we remove a node from a cycle, then we come again to the symmetrizable case. Thus it is sufficient to consider cycles (which are not symmetrizable) such that after removing an arbitrary node the resulting diagram is of finite type. Using the classification result of Cartan matrices of finite type one obtains easily (for similar argumentations confer also [1, Sect. 4.4]) that such cycles have three nodes, or the corresponding Cartan matrix is one of the following:

$$
\begin{pmatrix}
2 & -2 & 0 & -1 \\
-1 & 2 & -1 & 0 \\
0 & -1 & 2 & -2 \\
-1 & 0 & -1 & 2
\end{pmatrix},
\begin{pmatrix}
2 & -2 & 0 & 0 & -1 \\
-1 & 2 & -1 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 \\
0 & 0 & -1 & 2 & -1 \\
-1 & 0 & 0 & -1 & 2
\end{pmatrix}.
$$

In the last case Equations (23) imply that $q_{11}^2 = q_{22} = q_{33} = q_{44} = q_{55} = q_{11}$ and hence $q_{12}q_{21} = q_{11} = 1$. By the maximality assumption on $a_{ij}$ the latter means that this type of Cartan matrix does not appear.

Consider cycles with three nodes. The matrices $t_i$ of $s_i, i \in \{1, 2, 3\}$, with respect to the basis $\{e_1, e_2, e_3\}$ take the form

$$
t_1 = \begin{pmatrix}
-1 & -a_{12} - a_{13} \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\quad t_2 = \begin{pmatrix}
1 & 0 & 0 \\
-a_{21} & -1 & -a_{23} \\
0 & 0 & 1
\end{pmatrix},
\quad t_3 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-a_{31} - a_{32} & -1
\end{pmatrix}.
$$

Without loss of generality one can suppose that $a_{12} < -1$. The element $s_1s_2s_3$ (recall the independence of the $s_i$ from the basis of $\mathbb{Z}^n$) has the matrix

$$
t_1t_2t_3 = \begin{pmatrix}
\begin{array}{ccc}
-1 & a_{12}a_{21} + a_{13}a_{31} - a_{12}a_{23}a_{31} & a_{12} + a_{13}a_{32} - a_{12}a_{23}a_{32} & a_{13} - a_{12}a_{23} \\
0 & a_{23}a_{31} - a_{21} & a_{23}a_{32} - 1 & a_{23} \\
0 & -a_{31} & -a_{32} & -1
\end{array}
\end{pmatrix}.
$$
Since we assumed (F2), by Corollary 2 this matrix has to have finite order with respect to multiplication. This means in particular that all eigenvalues of the (invertible) matrix $t_1t_2t_3$ have to have absolute value 1, and hence its trace is not bigger than 3. Further, if the trace is 3 then the matrix has to be the identity. Since $a_{12} \leq -2$, for the trace of $t_1t_2t_3$ one obtains the relation

$$\text{tr}(t_1t_2t_3) = a_{12}a_{21} + a_{13}a_{31} + a_{23}a_{32} - a_{12}a_{23}a_{31} - 3 \geq 2 + 1 + 1 + 2 - 3 = 3.$$  

As $t_1t_2t_3$ is obviously not the identity this yields that $s_1s_2s_3$ doesn’t have finite order and hence $W(V)$ is infinite. An analogous conclusion holds for the remaining cycle with 4 nodes, where the matrix of $s_1s_2s_3s_4$ and its trace are

$$t_1t_2t_3t_4 = \begin{pmatrix} 6 & 0 & 3 & -5 \\ 3 & 0 & 1 & -2 \\ 2 & 1 & 1 & -2 \\ 1 & 0 & 1 & -1 \end{pmatrix}, \quad \text{tr}(t_1t_2t_3t_4) = 6 > 4.$$  

For the proof of (ii) we need the following lemma.

**Lemma 5.** Let $C$ be a symmetrizable Cartan matrix of finite type with corresponding Weyl group $W$. Let $\Delta_C$ denote the root system corresponding to $C$. Let $\pi$ be a fixed set of simple roots generating $\Delta_C$. Then

$$\{m\alpha \mid m \in \mathbb{Z}, \alpha \in \Delta_C\} = \{\alpha \in \mathbb{Z}\pi \mid W\alpha \subset \mathbb{N}_0\pi \cup -\mathbb{N}_0\pi\}.$$  

**Proof.** The inclusion $\subset$ in the equation of the lemma is well-known. Further, if the rank $n$ of $C$ is one then the inclusion $\supset$ is trivial.

Suppose that the inclusion $\supset$ in the lemma doesn’t hold. Without loss of generality the rank $n$ of $C$ is minimal with this property. By the above remark one has $n \geq 2$. Let $\alpha \in \mathbb{Z}\pi$ with $W\alpha \subset \mathbb{N}_0\pi \cup -\mathbb{N}_0\pi$. Without loss of generality one can take $\alpha \in \mathbb{N}_0\pi$. If $\alpha \notin \mathbb{N}\pi$ then $\alpha = m\beta$ for some $\beta \in \Delta_C$, $m \in \mathbb{Z}$, by minimality of $n$. Otherwise, since application of a simple reflection $s_i$ onto $\alpha$ changes only 1 of its coefficients, $s_i(\alpha) \in \mathbb{N}_0\pi$ by assumption on $\alpha$. Again, if $s_i(\alpha) \notin \mathbb{N}\pi$ then $s_i(\alpha) = m\beta$ for some $\beta \in \Delta_C$, $m \in \mathbb{Z}$, by
minimality of $n$, and hence $\alpha = ms_i(\beta)$. Finally, the case $W\alpha \subset \mathbb{N}\pi$ can not appear since $w_0\alpha \in -\mathbb{N}\pi$ where $w_0$ is the longest element of $W$.

To (ii). If $V$ is of finite type then the Cartan matrix $C = (a_{ij})_{i,j \in \{1,2,\ldots,n\}}$ is symmetrizable and hence $W(V)$ is isomorphic to $W \times B$. Define a $\mathbb{Z}$-linear map $\phi : \mathbb{Z}\pi \to \mathbb{Z}^n$, where $\pi = \{\alpha_1, \ldots, \alpha_n\}$ is a fixed set of simple roots of the root system $\Delta_C$ associated to $C$, by the formula

$$\phi(\alpha_i) := e_i.$$  

Note that $\phi$ commutes with the action of the maps $s_i$, where $s_i$ are also interpreted as reflections on the set $\Delta_C$ with respect to simple roots. Since all elements of $\Delta_C$ may be obtained from simple roots by application of an element of the Weyl group, one obtains $\Delta(B(V)) \supset \phi(\Delta_C)$. Since the multiplicities of the degrees of the generators of $B(V)$ are one, the multiplicity of $\phi(\alpha)$ is one for all $\alpha \in \Delta_C$.

Assume now that $\alpha \in \Delta(B(V)) \setminus \phi(\Delta_C)$. It is well-known that $\alpha \notin \mathbb{Z}e_i$ for all $i \in \{1,2,\ldots,n\}$. By application of elements in $W$ one obtains that $\alpha \notin \{m\phi(\beta) \mid m \in \mathbb{Z}, \beta \in \Delta_C\}$. By Lemma 5 there exists $w \in W$ such that $w\alpha \notin \mathbb{N}\pi \cup -\mathbb{N}\pi$. This is a contradiction to $\Delta^+(B(V)) \cup -\Delta^+(B(V)) = \Delta(B(V)) = w(\Delta(B(V)))$.

Recall that $q_{ii} = q_{jj}$ whenever $i,j \in \{1,2,\ldots,n\}$ are in the same connected component, and any root $\alpha \in \Delta_C$ can be written as $\alpha = w(\alpha_i)$ for some simple root $\alpha_i$ and an element $w$ of the Weyl group. Thus the last statement of the theorem follows from Equation 24.

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