Correctness of the initial-boundary problem of the compressible fluid filtration in a viscous porous medium

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Abstract. The local solvability of initial-boundary value problem for the system of the equations of non stationary fluid motion in a viscous deformable medium in the field of gravity is proved.

1. Introduction
The process of filtration compressible fluid in a deformable viscous porous medium is described by a system of equations which includes the laws of conservation of mass for each phase, Darcy’s law for fluid phase, taking into account the motion of a solid skeleton, the rheological law and the equation of conservation of momentum for system [1]–[3]. The solvability of this problem in some particular case is proved in [4]. The localization of solution for the filtration problem in an elastic medium is established in [5]. Similar problems with variable porosity were considered in [6]–[12].

2. Statement of the problem and formulation of the main result
A quasi-linear system of equations of composite type is considered:

\[
\begin{align*}
\frac{\partial (1-\phi)\rho_s}{\partial t} + \frac{\partial}{\partial x}((1-\phi)\rho_s v_s) &= 0, \\
\frac{\partial (\rho_f\phi)}{\partial t} + \frac{\partial}{\partial x}(\rho_f\phi v_f) &= 0, \\
\phi(v_f - v_s) &= -k(\phi)(\frac{\partial p_f}{\partial x} - \rho_f g), \\
\frac{\partial v_s}{\partial x} &= -\frac{1}{\xi(\phi)} p_e, \\
\frac{\partial p_{tot}}{\partial x} &= -\rho_{tot} g, \\
\rho_{tot} &= \phi \rho_f + (1-\phi) \rho_s.
\end{align*}
\]

We seek a solution of this system in the domain \((x, t) \in Q_T = \Omega \times (0, T), \Omega = (0, 1)\), under the boundary and initial conditions

\[
v_s \mid_{x=0, x=1} = v_f \mid_{x=0, x=1} = 0, \quad \phi \mid_{t=0} = \phi^0(x), \quad \rho_f \mid_{t=0} = \rho^0(x), \quad p_{tot} \mid_{x=0} = p^0(t).
\]
This initial-boundary problem describes 1D non-stationary isothermal motion of a compressible fluid in a viscous porous medium. Here $\rho_f$, $p_s$, $v_f$, $v_s$ are, respectively, real density and velocity of solid and fluid phases, $\phi$ is the porosity, $p_f, p_s$ are, respectively, pressures of the fluid and solid phases; $p_c$ is the effective pressure, $p_{tot}$ is the total pressure, $\rho_{tot}$ is the density of the two-phase medium, $g$ is the density of the mass forces; $k(\phi)$ is the coefficient of filtration, $\xi(\phi)$ is the coefficient of rock shear viscosity (specified function). The problem is written in the Eulerian coordinates $x, t$. The real density of the solid particles $\rho_s$ is assumed constant. The unknown quantities are $\phi, \rho_f, v_f, v_s, p_f, p_s$. The system of equations (1)–(4) is closed by using the equation of state of the fluid phase $p_f = p_f(\rho_f)$.

We use the notation of functional spaces and the relevant norms adopted in [13]. In this paper by a solution of problem (1)–(5) we mean the set of functions $v_s \in C^{3+\alpha,1+\alpha/2}(Q_T) \ (\phi, p_f, p_s) \in C^{2+\alpha,1+\alpha/2}(Q_T), v_f \in C^{1+\alpha,1+\alpha/2}(Q_T)$ such that $0 < \phi < 1$, $p_f > 0$, $p_f > 0$. These functions satisfy the equations (1)–(4), the initial and boundary conditions (5) and regarded as continuous functions in $\overline{Q_T}$.

Let us state the main results of the paper.

**Theorem.** Suppose that the data of problem (1)–(5) satisfies the following conditions:

1) the functions $k(\phi), \xi(\phi)$ and their derivatives up to the second order are continuous for $\phi \in (0, 1)$ and satisfy the conditions

$$k_0^{-1} \phi^{a_1}(1-\phi)^{a_2} \leq k(\phi) \leq k_0 \phi^{a_1}(1-\phi)^{a_2}, 1/\xi(\phi) = a_0(\phi)\phi^{a_1}(1-\phi)^{a_2 - 1}, \ 0 < R_1 \leq a_0(\phi) \leq R_2,$$

where $k_0, a_i, R_i, i = 1, 2$ are positive constants, $q_1, ..., q_4$ are fixed real parameters, $p_f = R(\rho_f), \ R = \text{const} > 0$.

2) the functions $g$, the initial and boundary functions $\phi^0, \rho^0, p^0(t)$ satisfy the following smoothness conditions: $g \in C^{1+\alpha,1+\alpha/2}(Q_T), \phi^0 \in C^{2+\alpha}(\Omega), \rho^0 \in C^{2+\alpha}(\Omega), p^0(t) \in C^{1+\alpha/2}(0, T)$ and the matching conditions

$$((1 - \phi^0) \frac{dp^0}{dx} - \rho^0 g(x, 0)) |_{x=0,x=1} = 0,$$

as well as satisfy the inequalities

$$0 < m_0 \leq \phi^0(x) \leq M_0 < 1, 0 < m_1 \leq \rho^0(x) \leq M_1 < \infty, 0 < g(x, t) \leq g_0 < \infty, \ x \in \overline{\Omega},$$

where $m_0, M_0, m_1, M_1, g_0$ are given positive constants.

Then problem (1)–(5) has a local solution, i.e. there exists a value of $t_0 \in (0, T)$ such that

$$v_s(x, t) \in C^{3+\alpha,1+\alpha/2}(\overline{Q}_{t_0}), (\phi(x, t), p_s(x, t), p_f(x, t), \rho_f(x, t)) \in C^{2+\alpha,1+\alpha/2}(\overline{Q}_{t_0}),$$

$$v_f(x, t) \in C^{1+\alpha,1+\alpha/2}(\overline{Q}_{t_0}).$$

Moreover, $0 < \phi(x, t) < 1, \rho_f(x, t) > 0$ in $\overline{Q}_{t_0}$.

3. Local solvability

Following [5], [14], we rewrite the system (1)–(5) in the mass Lagrangian variables. We obtain the following problem for unknown functions $\rho_f, \phi$:

$$\frac{\partial}{\partial t}(a(\phi)\rho_f) - \frac{\partial}{\partial x}(K(\phi)b(\rho_f)\frac{\partial \rho_f}{\partial x} - \frac{K(\phi)}{1-\phi} \rho_f^2 g) = 0,$$

$$\frac{\partial G}{\partial t} = p_f - p_{tot}, \ (1 - \phi) \frac{\partial p_{tot}}{\partial x} = -\rho_{tot} g,$$

$$((1 - \phi) \frac{\partial \rho_f}{\partial x} - \rho_f g) |_{x=0,x=1} = 0, \ \rho_f |_{t=0} = \rho^0(x), \ \phi |_{t=0} = \phi^0(x),$$

where

$$a(\phi) = (1 - \phi) \frac{\rho_{tot}}{1 + \rho_{tot}}, \ \ K(x, t) = \rho_{tot}(\Phi(x, t)).$$
We note that on the set $V$ where (6)–(8) has a unique local solution, i.e. there exists a value of $t_0$ such that

$$(\phi(x,t), \rho_f(x,t), p_{tot}(x,t)) \in C^{2+\alpha,1+\alpha/2}(\overline{Q}_{t_0}).$$

**Lemma 1.** Let the data of problem (6)–(8) satisfy the conditions of the theorem. Then problem (6)–(8) has a unique local solution, i.e. there exists a value of $t_0$ such that

$$(\phi(x,t), \rho_f(x,t), p_{tot}(x,t)) \in C^{2+\alpha,1+\alpha/2}(\overline{Q}_{t_0}).$$

Furthermore, $0 < \phi(x,t) < 1$, $\rho_f(x,t) > 0$ $\overline{Q}_{t_0}$.

The solvability of problem (6)–(8) is established by using the Tikhonov-Schauder fixed-point theorem [15].

Since the function $\psi = G(\phi)$ is strictly monotone, at $\phi \in (0,1)$, than the inverse function exists: $\phi = G^{-1}(\psi)$. Assuming that $\rho(x,t) = \rho_f(x,t) - \rho^0(x)$, $\omega(x,t) = G(\phi) - G(\phi^0)$. We represent the equations (6),(7) in the form

$$\frac{\partial}{\partial t} \left( a(\omega)(\rho + \rho^0) \right) = \frac{\partial}{\partial x} \left( K(\omega) b(\rho + \rho^0) \frac{\partial(\rho + \rho^0)}{\partial x} - \frac{K(\omega)}{1-\phi(\omega)} \rho(\rho + \rho^0)^2 g \right),$$

$$\frac{\partial \omega}{\partial t} = p_f(\rho + \rho^0) - p_{tot}(1 - \phi(\omega)) \frac{\partial p_{tot}}{\partial x} = -\rho_{tot} g.$$  \hspace{1cm} (10)

Here $a(\omega) = \frac{\phi(\omega)}{1-\phi(\omega)}, K(\omega) = k(\phi)(1 - \phi(\omega)), \phi(\omega) = G^{-1}(\omega + G(\phi^0)).$ Moreover,

$$\rho|_{t=0} = \omega |_{t=0} = ((1 - \phi(\omega)) \frac{\partial(\rho + \rho^0)}{\partial x} - (\rho + \rho^0)g) |_{x=0,x=1} = 0, p_{tot}|_{x=0} = \rho^0(t).$$

For the Banach space, we choose the space $C^{2+\beta,1+\beta/2}(\overline{Q}_{t_0})$, where $\beta$ is any number from the interval $(0, \alpha)$, $\alpha \in [0,1)$. Let

$$V = \{ \bar{\rho}(x,t), \bar{\omega}(x,t) \in C^{2+\alpha,1+\alpha/2}(\overline{Q}_{t_0}) | \bar{\rho} |_{t=0} = \bar{\omega} |_{t=0} = ((1 - \phi(\omega)) \frac{\partial(\rho + \rho^0)}{\partial x} - (\rho + \rho^0)g) |_{x=0,x=1} = 0,$$

$$\hat{m}_1 - \rho^0(x) \leq \bar{\rho}(x,t) \leq \hat{M}_1 - \rho^0(x) < \infty, \quad \hat{m}_1 = \hat{m}_1(2+\frac{4g_0k_0}{R(1-M_0)})^{-1}, \quad \hat{M}_1 = \hat{m}_1(2+\frac{4g_0k_0}{R(1-M_0)}),$$

$$G(\frac{m_0}{2}) - G(\phi^0) \leq \hat{\omega}(x,t) \leq G(\frac{M_0+1}{2}) - G(\phi^0) < \infty, \quad (x,t) \in Q_{t_0},$$

$$\{ \hat{\omega} |_{x=1}, |\rho|_{x=1} - (\rho^0(x) - \rho^0(x)) \leq K_1, \quad |\hat{\omega} |_{x=2}, |\rho|_{x=2} - (\rho^0(x) - \rho^0(x)) \leq K_1 + K_2 \},$$

where $K_1$ is an arbitrary positive constant, while the positive constant $K_2$ will be given later. We note that on the set $V$ following inequalities hold: $0 < \frac{m_0}{2} \leq \phi(\bar{\omega}) \leq \frac{M_0+1}{2} < 1, \quad a(\bar{\omega}) > 0, \quad K(\hat{\omega}) > 0.$

Let us construct an operator $\Lambda$ mapping $V$ in $V$. Suppose that $\bar{\omega}, \bar{\rho} \in V$. Using (10), we define the functions $p_{tot}, \omega$ by the equalities

$$p_{tot} = \rho^0(t) - \int_0^x g(\rho_s + (\bar{\rho} + \rho^0(\xi)) \frac{\phi(\bar{\omega})}{1-\phi(\bar{\omega})}) d\xi,$$

$$\omega = \int_0^t \left( R(\bar{\rho}(x,t) + \rho^0(\xi)) - p^0(\tau) + \int_0^x g(\rho_s + (\bar{\rho} + \rho^0(\xi)) \frac{\phi(\bar{\omega})}{1-\phi(\bar{\omega})}) d\xi \right) d\tau. \hspace{1cm} (11)$$
From the representation (11) it follows that smoothness \( \omega \) and \( p_{\text{tot}} \) is determined by the smoothness of functions \( \tilde{\rho}, \tilde{\omega}, \rho^0, p^0 \) and \( g \), and there exists a value \( t_1 = t_1(m_0, M_0, m_1, M_1) \), such that for all \( t_0 \leq t_1 \) the following inequality holds

\[
0 < \frac{m_0}{2} \leq \phi(x, t) \leq \frac{M_0 + 1}{2}, \quad (x, t) \in Q_{t_0},
\]

(12)

In particular, we have an estimate

\[
|\omega|_{2 + a, 1 + \alpha/2, Q_{t_0}} = C_1(m_0, M_0, m_1, M_1, K_1, T, |g|_{1 + a, \Omega}, |\rho^0|_{2 + a, \Omega}, |p^0|_{a/2, [0, T]}) (1 + t_0) |\rho_{xx}|_{a, \alpha/2, \Omega}.
\]

Taking into account (12) we also have the estimate for function \( \omega(x, t) : G\left(\frac{m_0}{2}\right) \leq \omega(x, t) + G(\phi^0) \leq G\left(\frac{M_0 + 1}{2}\right) \).

Using (9), \( \tilde{\rho} \) and \( \tilde{\omega}(x, t) \) we find the function \( \rho(x, t) \) as a solution of the problem (here and elsewhere, we assume that the initial and boundary conditions are matched):

\[
\frac{\partial}{\partial t} (a(\omega)(\rho + \rho^0)) = \frac{\partial}{\partial x} \left( K(\omega) b(\tilde{\rho}) \frac{\partial (\rho + \rho^0)}{\partial x} - \frac{K(\omega)}{1 - \phi(\omega)} (\tilde{\rho} + \rho^0)(\rho + \rho^0) g \right),
\]

(13)

\[
\rho \big|_{t=0} = 0, \quad ((1 - \phi(\omega)) \frac{\partial (\rho + \rho^0)}{\partial x} - (\rho + \rho^0) g) \big|_{x=0, x=1} = 0.
\]

The equation for \( \rho(x, t) \) is uniformly parabolic. In view of the properties of \( \tilde{\omega}(x, t) \) and \( \rho^0(x) \) the problem (13) has a classical solution [16]. In addition, we have the following estimate:

\[
\left| \frac{1}{a(\omega)} \frac{\partial a(\omega)}{\partial t} \right| \leq C_0(m_0, M_0, m_1, M_1, \max_{0 \leq t \leq T} |p^0(t)|).
\]

Imposing the additional smallness condition on the quantity \( t_0 \), we can state the following lemma.

**Lemma 2.** There exists a value \( t_2 \), such that, when \( t_0 \leq \min(t_1, t_2) \), the classical solution of problem (13) satisfies the following inequality in \( Q_{t_0} \)

\[
0 < \hat{m}_1 \leq \rho(x, t) + \rho^0(x) \leq \hat{M}_1 < \infty.
\]

**Proof.** Further, setting \( U(x, t) = \rho(x, t) + \rho^0(x) \), we can express problem (13) in the form

\[
\frac{\partial}{\partial t} (a(\omega)U) = \frac{\partial}{\partial x} \left( K(\omega) b(\tilde{\rho}) \frac{\partial U}{\partial x} - \frac{K(\omega)}{1 - \phi(\omega)} (\tilde{\rho} + \rho^0) U g \right), \quad \left( \frac{\partial U}{\partial x} - \tilde{d} U \right) |_{x=0, x=1} = 0, \quad U |_{t=0} = \rho^0,
\]

(14)

where \( \tilde{d} = g \frac{1 - \phi(\omega)}{2(1 - \phi(\omega))} \). First, we show that \( U(x, t) \geq 0, (x, t) \in Q_{t_0} \). In equation (14), let us make the change \( U(x, t) = -z(x, t) \). Then

\[
z \frac{\partial a}{\partial t} + a \frac{\partial z}{\partial t} = \frac{\partial}{\partial x} \left( K b \frac{\partial z}{\partial x} - \frac{K}{1 - \phi(\omega)} (\tilde{\rho} + \rho^0) g z \right).
\]

Let

\[
z^{(0)}(x, t) = \max\{z, 0\}, \quad z^{(0)}(x, t) |_{t=0} = \max\{-\rho^0, 0\} = 0,
\]

\[
\sigma_e(x, t) = z^{(0)}(x, t)(|z^{(0)}(x, t)|^2 + \varepsilon)^{-1/2}, \varepsilon > 0.
\]

Let us multiply the equation for the function \( z \) by \( \sigma_e \) and then integrate over \( \Omega \). Following [4], we obtain the estimate

\[
\int_0^1 a z^{(0)}(x) dx \leq \varepsilon^{1/2} \int_0^t \int_0^1 \left| \frac{\partial a}{\partial \tau} \right| dx d\tau + \varepsilon^{1/2} \int_0^1 a |_{t=0} dx.
\]
Passing to the limit as \( \varepsilon \to 0 \), we find that \( z^{(0)} = 0 \), i.e. \( U \geq 0 \).

The problem (14) can be represented as:

\[
U_t - \tilde{a}_{11}U_{xx} + \tilde{a}_1U_x + \tilde{a}U = 0, \quad (U_x - \tilde{d}U)_{|x=0,1} = 0,
\]

where

\[
\tilde{a}_{11} = \frac{Kb}{a}, \quad \tilde{a}_1 = \frac{d - (Kb)_x}{a}, \quad \tilde{a} = \frac{a_t + d_x}{a}, \quad \tilde{d} = \frac{g}{(1-\phi)R}.
\]

Following [16], we pass from the function \( U(x,t) \) to the new function \( v(x,t) \) associated with the last one by the equality \( U(x,t) = v(x,t)e^{\lambda t} \), while the constant \( \lambda \) will be given later. The function \( v \) satisfies the equation

\[
v_t - \tilde{a}_{11}v_{xx} + \tilde{a}_1v_x + (\tilde{a} + \lambda)v = 0.
\]

Suppose that \( \lambda > \max_{Q_0}[-\tilde{a}] \). Since \( U \geq 0, g > 0 \), it follows from the boundary conditions (15), that \( U_{x|x=0,1} = (\tilde{d}U)_{|x=0,1} \geq 0 \), i.e. at \( x = 0 \) the maximum of the function can not be achieved. Suppose that \( \omega(x,t) = v(x,t)\varphi(x) \), where

\[
\varphi = -mx^2 + mx + 1 > 0, \quad m \equiv 2 \max |\tilde{d}| = \frac{4g_0k_0}{(1-M_0)R}.
\]

The function \( \omega \) is a solution of the problem

\[
\omega_t - \tilde{a}_{11}\omega_{xx} + (\tilde{a}_1 + \frac{2a_{11}\varphi_x}{\varphi})\omega_x + (-2\tilde{a}_{11}\frac{\varphi_x^2}{\varphi^2} + \tilde{a}_{11}\frac{\varphi_{xx}}{\varphi} - \tilde{a}_1\frac{\varphi_x}{\varphi} + \tilde{a} + \lambda)\omega = 0,
\]

\[
\omega_x|_{x=0,1} = ((\frac{\varphi_x}{\varphi} + \tilde{d})\omega)|_{x=0,1}, \quad \omega_x|_{x=0} > 0, \quad \omega_x|_{x=1} < 0.
\]

We choose

\[
\lambda > \max_{Q_0}\{\max\{2\tilde{a}_{11}\frac{\varphi_x^2}{\varphi^2} - \tilde{a}_{11}\frac{\varphi_{xx}}{\varphi} + \tilde{a}_1\frac{\varphi_x}{\varphi} - \tilde{a}\}, \max_{Q_0}[-\tilde{a}]\},
\]

then the function \( \omega \) reaches a positive maximum for \( t = 0 \). Therefore, we have an upper bound for \( U \):

\[
U \leq e^{\lambda t}M_1(1 + \frac{g_0k_0}{(1-M_0)R}).
\]

Then there exists a value \( t_2 = \ln 2^{1/\lambda} \), such that for all \( t \leq t_2 \) we have the estimate from above for \( \rho \) from Lemma 2.

To obtain a lower estimate we represent equation (14) in the form \( (z(x,t) = 1/U(x,t)) \)

\[
z_t - \tilde{a}_{11}z_{xx} + \frac{2\tilde{a}_{11}}{z}z_x + \tilde{a}_1z_x - \tilde{a}z = 0.
\]

Applying a similar approach, we obtain the required estimate from below for \( \rho \) for any \( t \leq t_2 \).

Lemma 2 is proved.

In view of Lemma 2 and the properties of \( \bar{\omega} \) we have the following estimates [16]:

\[
|\rho|_{a,\alpha/2,Q_0} \leq C_2,
\]

\[
|\rho|_{2+a,1+a/2,Q_0} \leq C_3 \left( 1 + |\rho_0|_{2+a,\alpha} + |\bar{\rho}_z|_{a,\alpha/2,Q_0} + |\bar{\omega}_t|_{a,\alpha/2,Q_0} + |\bar{\omega}_x|_{a,\alpha/2,Q_0} \right),
\]
in which the constants $C_2, C_3$ depends on $K_1, m_0, m_1, M_0, M_1$. Therefore
\[ |\rho|_{2+\alpha,1+\alpha/2,Q_{0}} \leq C_4(K_1, m_0, m_1, M_0, M_1). \]

Let $C_5 = \max\{C_1, C_4\}$. Choose $K_2$ so that $C_5 \leq \frac{K_1 + K_2}{2}$. Then, for $t_0 < \min(t_1, t_2, (K_1 + K_2)^{-1})$ we obtain
\[ |\rho|_{2+\alpha,1+\alpha/2,Q_{0}} \leq K_1 + K_2, \quad |\omega|_{2+\alpha,1+\alpha/2,Q_{0}} \leq K_1 + K_2. \]

It remains to verify to conditions
\[ |\rho|_{1+\alpha,(1+\alpha)/2,Q_{0}} \leq K_1, \quad |\omega|_{1+\alpha,(1+\alpha)/2,Q_{0}} \leq K_1. \]

Integrating equation (13) with respect to time, we obtain $|\rho|_{0,Q_{0}} \leq C_0 t_0$. From the equation (11) we obtain $|\omega|_{0,Q_{0}} \leq C_7 t_0$. Further, using for $\rho, \omega$ an inequality of the form [13]
\[ |u|_{1+\alpha,(1+\alpha)/2,Q_{0}} \leq C |u|_{2+\alpha,1+\alpha/2,Q_{0}}^{1-c}, \quad c = (1 + \alpha)(2 + \alpha)^{-1}, \]
we find that there exists a sufficiently small value of $t_0$, depending on $K_1$ and $K_2$, such that the required estimates hold: $|\rho|_{1+\alpha,(1+\alpha)/2,Q_{0}} \leq K_1, |\omega|_{1+\alpha,(1+\alpha)/2,Q_{0}} \leq K_1$.

Thus, the operator $\Lambda$ maps the set $V$ into itself for sufficiently small values of $t_0$. Using the estimates obtained above, we can easily show the continuity of the operator $\Lambda$ in the norm of the space $C^{2+\beta,1+\beta/2}(Q_{t_0})$. By the Tikhonov-Schauder theorem, there exists a fixed point $(\rho, \omega) \in V$ of the operator $\Lambda$. Uniqueness is established in the standard way [13].

Lemma 1 is proved.

Since $(\phi, p_f) \in C^{2+\alpha,1+\frac{\alpha}{2}}(Q_{t_0})$, we have:
\[ v_\varepsilon \in C^{3+\alpha,1+\frac{\alpha}{2}}(Q_{t_0}), \quad (p_f, p_\varepsilon) \in C^{2+\alpha,1+\frac{\alpha}{2}}(Q_{t_0}), \quad v_f \in C^{1+\alpha,1+\frac{\alpha}{2}}(Q_{t_0}). \]

Theorem is proved.

4. Conclusion

We have studied the model of filtration of a compressible fluid in a viscous porous medium in the field of gravity. We establish the local solvability of the problem for the case in which the real densities of the solid phase are constant.

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