SOME CHARACTERIZATIONS FOR COMPOSITION OPERATORS ON THE FOCK SPACE

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ABSTRACT. We study composition operators on the Fock spaces $F_α^2(C^n)$, problems considered include the essential norm, normality, spectra, cyclicity and membership in the Schatten classes. We give perfect answers for these basic properties, which present lots of different characterizations with the composition operators on the Hardy space or the weighted Bergman spaces.

1. INTRODUCTION AND PRELIMINARIES

Let $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ be the points in $C^n$, the inner product is

$$\langle z, w \rangle = \sum_{j=1}^{n} z_j \overline{w_j}$$

and $|z| = \sqrt{\langle z, z \rangle}$.

For any $\alpha > 0$, consider the Gaussian probability measure $dv_\alpha(z) = \left(\frac{\alpha}{\pi}\right)^n e^{-\alpha |z|^2} dv(z)$ on $C^n$, where $dv$ is the Lebesgue volume measure on $C^n$. The Fock space $F_\alpha^2(C^n)$ consists of all holomorphic functions $f$ on $C^n$ with

$$\|f\|_\alpha^2 = \int_{C^n} |f(z)|^2 dv_\alpha(z) < \infty.$$

Thus, $F_\alpha^2(C^n)$ is a Hilbert space with the following inner product

$$\langle f, g \rangle_\alpha = \int_{C^n} f(z)\overline{g(z)} dv_\alpha(z).$$

Its reproducing kernels are given by

$$K(z, w) = K_w(z) = e^{\alpha \langle z, w \rangle}$$

with $\|K_w\|_\alpha^2 = \exp(\alpha |z|^2)$.

For a given holomorphic mapping $\varphi : C^n \to C^n$, the composition operator $C_\varphi$ on the Fock spaces $F_\alpha^2(C^n)$ is defined by $C_\varphi(f) = f \circ \varphi$.

In 2003, Carswell et al. [5] first studied composition operators on the classical Fock space $F_1^2(C^n)$ when $\alpha = 1/2$, usually denoted by $F^2(C^n)$. They found the following information.

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Theorem A. Suppose \( \varphi : \mathbb{C}^n \to \mathbb{C}^n \) is a holomorphic mapping.

(a) \( C_\varphi \) is bounded on \( \mathcal{F}^2(\mathbb{C}^n) \) if and only if \( \varphi(z) = Az + B \), where \( A \) is an \( n \times n \) matrix and \( B \) is an \( n \times 1 \) vector. Furthermore, \( \|A\| \leq 1 \), and \( \langle A\zeta, B \rangle = 0 \) if \( |A\zeta| = |\zeta| \) for some \( \zeta \in \mathbb{C}^n \).

(b) \( C_\varphi \) is compact on \( \mathcal{F}^2(\mathbb{C}^n) \) if and only if \( \varphi(z) = Az + B \), where \( \|A\| < 1 \).

From this, we know that only a class of linear mappings of \( \mathbb{C}^n \) can induce bounded composition operators on \( \mathcal{F}^2(\mathbb{C}^n) \). In fact, this is the same on the Fock space \( \mathcal{F}_\alpha^2(\mathbb{C}^n) \) for any \( \alpha > 0 \). So, it is natural to look forward that more properties of bounded composition operators on \( \mathcal{F}_\alpha^2(\mathbb{C}^n) \) could be completely characterized.

In recent years, the study of composition operators and weighted composition operators on Fock spaces has attracted a lot of attention ([6], [7], [19], [21], [24]). However, some basic problems about composition operators on the Fock spaces \( \mathcal{F}_\alpha^2(\mathbb{C}^n) \) are still open. Only in the setting of complex plane \( \mathbb{C} \), Guo and Izuchi [12] described some properties of composition operators on Fock type spaces, including spectra, cyclicity and connected components of the set of composition operators.

In this work, we try to investigate the basic operator properties of composition operators on \( \mathcal{F}_\alpha^2(\mathbb{C}^n) \), and find that some behaviors are so distinctive. For simplicity, we will discuss our results on the classical Fock space \( \mathcal{F}^2(\mathbb{C}^n) \). But all results hold on Fock spaces \( \mathcal{F}_\alpha^2(\mathbb{C}^n) \) for all \( \alpha > 0 \).

Let \( \varphi \) be a holomorphic mapping of \( \mathbb{C}^n \), which induces a bounded composition operator \( C_\varphi \) on \( \mathcal{F}^2(\mathbb{C}^n) \). First, we will calculate the essential norm \( \|C_\varphi\|_e \) in Section 2. When \( C_\varphi \) is not compact, we find that \( \|C_\varphi\|_e = \|C_\varphi\| \). In this section, we also discuss the membership in the Schatten classes and obtain that all compact composition operators belong to the Schatten \( p \)-class \( S_p \) for all \( 0 < p < \infty \). Section 3 is devoted to describe the normality of \( C_\varphi \). It is interesting that when \( C_\varphi \) is hyponormal on \( \mathcal{F}^2(\mathbb{C}^n) \), then \( C_\varphi \) must be normal. Moreover, there are no non-trivial essentially normal composition operators on \( \mathcal{F}^2(\mathbb{C}^n) \). All above results are different to that when \( C_\varphi \) acting on other classical function spaces, such as the Hardy space and the weighted Bergman spaces.

In Section 4, we completely give the spectrum of \( C_\varphi \) on \( \mathcal{F}^2(\mathbb{C}^n) \), that is,

\[
\sigma(C_\varphi) = \{\lambda_1^{\gamma_1} \cdots \lambda_n^{\gamma_n} : (\gamma_1, \ldots, \gamma_n) \in \mathbb{N}^n\},
\]

where \( \lambda_1, \ldots, \lambda_n \) are the eigenvalues of the matrix \( A \) (which described in Theorem A). Finally, the cyclicity of composition operators will be studied in Section 5. We observe that all bounded composition operators are not supercyclic on \( \mathcal{F}^2(\mathbb{C}^n) \), and give a necessary and sufficient condition for \( C_\varphi \) to be cyclic when \( \varphi \) is unitary. Thus, it remains that, whether \( C_\varphi \) is cyclic is unknown if \( \varphi \) is univalent, not unitary.

2. Essential Norm and Schatten Class

When \( C_\varphi \) is bounded on the Fock space \( \mathcal{F}^2(\mathbb{C}^n) \), by Theorem A, we have \( \varphi(z) = Az + B \). The norm of \( C_\varphi \) has been obtained by Carswell et al. [5] as the following:

\[
\|C_\varphi\| = \exp\left(\frac{1}{4}(|z_0|^2 - |Az_0|^2 + |B|^2)\right),
\]

where \( z_0 \) is any solution to \( (I - A^*A)z = A^*B \).

In this section, we first determine the essential norm of \( C_\varphi \), which defined as:

\[
\|C_\varphi\|_e = \|C_\varphi^*\|_e = \inf\{\|C_\varphi^* - F\| : F \text{ is compact on } \mathcal{F}^2(\mathbb{C}^n)\}.
\]
Theorem 2.1. Let \( \varphi(z) = Az + B \) and \( C_\varphi \) be bounded, not compact, on \( \mathcal{F}^2(\mathbb{C}^n) \). Then the essential norm of \( C_\varphi \) is

\[
\|C_\varphi\|_e = \|C_\varphi\| = \exp \left( \frac{1}{4} \langle \varphi(z_0), B \rangle \right),
\]

where \( z_0 \) is any solution to \( (I - A^*A)z = A^*B \).

Proof. Since \( C_\varphi \) is not compact, we get that \( \|A\| = 1 \) from Theorem A. Let \( A = V \Sigma W \) be the singular value decomposition of \( A \), where \( V, W \) are unitary and \( \Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n) \) with \( \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0 \) being the singular values of \( A \). Set \( j = \max\{r : \sigma_r = 1\} \).

It is clear that \( j \geq 1 \) from \( \sigma_1 = \|A\| = 1 \).

Let \( \psi(z) = \Sigma^j z + B' \) be the normalization of \( \varphi \) (see Proposition 1 in [5]), that is, \( \varphi = V \circ \psi \circ W \), where \( B' = V^*B \). Denote \( C_W \) and \( C_V \) the unitary operators respectively given by \( C_W(f) = f \circ W \) and \( C_V(f) = f \circ V \). This means \( C_\varphi = C_W C_\psi C_V \). Using Corollary 1 of [5], we get that \( \|C_\varphi\| = \|C_\psi\| \) and \( \|C_\varphi\|_e = \|C_\psi\|_e \). So it suffices to calculate the essential norm of \( C_\psi \).

Let \( K_w \) be the reproducing kernels for \( \mathcal{F}^2(\mathbb{C}^n) \). The proof of Lemma 3 in [5] together with (11) of [5] gives

\[
\|C_\psi\| = \|C_\psi^*\| = \sup_{w \in \mathbb{C}^n} \frac{\|C_\psi^*(K_w)\|}{\|K_w\|} = \sup_{w \in \mathbb{C}^n} \frac{\|K_w\|}{\|K_w\|},
\]

and \( \|K_w\|/\|K_w\| \) attains its maximum at points \( w = (w_1, w_2, \ldots, w_n) \in \mathbb{C}^n \) which satisfy, for \( j + 1 \leq m \leq n \),

\[
|w_m| = \frac{\sigma_m |b_m'|}{1 - \sigma_m^2} \quad \text{and} \quad \arg w_m \text{ chosen so that } w_m b_m' \geq 0,
\]

and arbitrary \( w_m \) for \( m \leq j \), where \( b_m' \) is the \( m^{th} \) coordinate of \( B' \).

On the other hand, the normalized reproducing kernel functions \( k_w = K_w/\|K_w\| \) tend to 0 weakly on \( \mathcal{F}^2 \) as \( |w| \to \infty \), so that

\[
\|C_\psi\|_e = \sup_{w \in \mathbb{C}^n} \frac{\|C_\psi^*(K_w)\|}{\|K_w\|} = \sup_{w \in \mathbb{C}^n} \frac{\|K_w\|}{\|K_w\|}.
\]

However, we know that \( \|C_\psi\|_e \leq \|C_\psi\| \). This together with the above discussion gives

\[
\|C_\psi\| = \|C_\psi\| = \sup_{w \in \mathbb{C}^n} \frac{\|C_\psi^*(K_w)\|}{\|K_w\|}.
\]

Therefore, \( \|C_\varphi\| = \|C_\psi\| = \|C_\psi\| = \|C_\varphi\| \).

Now, Theorem 4 of [5] gives that

\[
\|C_\varphi\| = \exp \left( \frac{1}{4} (|z_0|^2 - |A z_0|^2 + |B|^2) \right),
\]
where \( z_0 \) is any solution to \((I - A^*A)z = A^*B\). In fact, \((I - A^*A)z_0 = A^*B\) yields that
\[
|z_0|^2 - |Az_0|^2 = \langle z_0, (I - A^*A)z_0 \rangle = \langle z_0, A^*B \rangle = \langle Az_0, B \rangle.
\]
It follows that
\[
\|C_\varphi\|_e = \|C_\varphi\| = \exp \left( \frac{1}{4}(|z_0|^2 - |Az_0|^2 + |B|^2) \right)
= \exp \left( \frac{1}{4}(\langle Az_0, B \rangle + |B|^2) \right)
= \exp \left( \frac{1}{4}\langle \varphi(z_0), B \rangle \right).
\]

Next, we will discuss Schatten class composition operators on the Fock space \( \mathcal{F}^2(\mathbb{C}^n) \). Recall that if \( T \) is a compact operator on a separable Hilbert space \( H \), then there exist orthonormal sets \( \{e_n\} \) and \( \{\sigma_n\} \) in \( H \) such that
\[
Tx = \sum_n \lambda_n \langle x, e_n \rangle \sigma_n, \quad x \in H,
\]
where \( \lambda_n \) is the singular value of \( T \), i.e. it is the eigenvalue of \(|T| = (T^*T)^{1/2}\).

Given \( 0 < p < \infty \), if the sequence \( \{\lambda_n\} \) belongs to \( l^p \), we say that \( T \) belongs to the Schatten \( p \)-class of \( H \), denoted \( S_p(H) \) or \( S_p \), and define the norm
\[
\|T\|_p = \left[ \sum_n |\lambda_n|^p \right]^{1/p}.
\]

Usually, \( S_1 \) is also called the trace class and \( S_2 \) is called the Hilbert-Schmidt class.

It is well known that if \( T \) is compact on \( H \), then \( T \in S_p \) if and only if \( T^*T \in S_p \) with \( \|T\|_p = \|T^*T\|^{1/2} \). We refer to [25] for more information about the Schatten classes.

For a Borel measure \( \mu \) on \( \mathbb{C}^n \), we define the Toeplitz operator \( T_\mu \) on \( \mathcal{F}^2(\mathbb{C}^n) \) as follows:
\[
T_\mu(f)(z) = \left( \frac{1}{2\pi} \right)^n \int_{\mathbb{C}^n} f(w)K(z,w)e^{-\frac{|w|^2}{2}}d\mu(w).
\]

The Berezin transform of \( \mu \) is defined:
\[
\tilde{\mu}(z) = \langle T_\mu k_z, k_z \rangle = \left( \frac{1}{2\pi} \right)^n \int_{\mathbb{C}^n} |k_z(w)|^2 e^{-\frac{|w|^2}{2}}d\mu(w)
= \left( \frac{1}{2\pi} \right)^n \int_{\mathbb{C}^n} e^{-\frac{1}{2}|z-w|^2}d\mu(w),
\]
where \( k_z(w) = K(w,z)/\sqrt{K(z,z)} = e^{\frac{1}{2}(w,z) - \frac{1}{4}|z|^2} \) are the normalized reproducing kernels of \( \mathcal{F}^2(\mathbb{C}^n) \).

**Theorem 2.2.** Suppose that \( C_\varphi \) is compact on \( \mathcal{F}^2(\mathbb{C}^n) \), then \( C_\varphi \) belongs to the Schatten \( p \)-class \( S_p \) for all \( 0 < p < \infty \).
Proof. First, for a function \( \varphi \) in \( F^2(\mathbb{C}^n) \), it is easy to check that
\[
C_\varphi^* C_\varphi f(z) = \langle C_\varphi^* C_\varphi f, K_z \rangle = \langle C_\varphi f, C_\varphi K_z \rangle
\]
\[
= \left( \frac{1}{2\pi} \right)^n \int_{\mathbb{C}^n} f(\varphi(u)) K_z(\varphi(u)) e^{-\frac{|u|^2}{2}} \, dv(u)
\]
\[
= \left( \frac{1}{2\pi} \right)^n \int_{\mathbb{C}^n} f(\varphi(u)) K_z(\varphi(u)) e^{-\frac{|u|^2}{2}} e^{\frac{1}{2}(|\varphi(u)|^2 - |u|^2)} \, dv(u)
\]
\[
= \left( \frac{1}{2\pi} \right)^n \int_{\mathbb{C}^n} f(u) K_z(u) e^{-\frac{|u|^2}{2}} \, d\mu(u) = T_\mu(f)(z),
\]
where \( d\mu = dv \circ \varphi^{-1} \) with \( dv(z) = e^{\frac{1}{2}(|\varphi(z)|^2 - |z|^2)} \, dv(z) \).

Thus, for \( 0 < p < \infty \), the composition operator \( C_\varphi \) belongs to the Schatten \( p \)-class \( S_p \) if and only if \( T_\mu = C_\varphi^* C_\varphi \) belongs to \( S_{p/2} \). By Theorem 2.7 and Theorem 3.2 in [13], it is equivalent to that \( \bar{\mu}(z) \) is in \( L^{p/2}(\mathbb{C}^n, dv) \).

Now, we calculate that
\[
\int_{\mathbb{C}^n} \bar{\mu}(z)^{p/2} \, dv(z) = \int_{\mathbb{C}^n} \left( \left( \frac{1}{2\pi} \right)^n \int_{\mathbb{C}^n} e^{-\frac{1}{2}|z-w|^2} \, d\mu(w) \right)^{p/2} \, dv(z)
\]
\[
= \int_{\mathbb{C}^n} \left( \left( \frac{1}{2\pi} \right)^n \int_{\mathbb{C}^n} e^{-\frac{1}{2}|z-\varphi(u)|^2} e^{\frac{1}{2}(|\varphi(u)|^2 - |u|^2)} \, dv(u) \right)^{p/2} \, dv(z)
\]
\[
= \int_{\mathbb{C}^n} e^{-\frac{1}{4}|z|^2} \left( \left( \frac{1}{2\pi} \right)^n \int_{\mathbb{C}^n} e^{Re(\varphi(u), z)} e^{-\frac{1}{2}|u|^2} \, dv(u) \right)^{p/2} \, dv(z).
\]

If \( C_\varphi \) is compact, then \( \varphi(z) = Az + B \) with \( \|A\| < 1 \) from Theorem A. On the other hand, Lemma 3 in [9] gives that
\[
\int_{\mathbb{C}^n} |e^{s(z, A)}| \, dv_{\alpha}(z) = e^{s^2|\alpha|^2/4n}
\]
for all \( \alpha \in \mathbb{C}^n \), where \( s \) is real. Therefore, using the formula (2.1) twice,
\[
\int_{\mathbb{C}^n} \bar{\mu}(z)^{p/2} \, dv(z)
\]
\[
= \int_{\mathbb{C}^n} e^{-\frac{1}{4}|z|^2} \left( \left( \frac{1}{2\pi} \right)^n \int_{\mathbb{C}^n} e^{Re(Au+B, z)} e^{-\frac{1}{2}|u|^2} \, dv(u) \right)^{p/2} \, dv(z)
\]
\[
= \int_{\mathbb{C}^n} |e^{\frac{1}{2}(z, B)}| |e^{\frac{1}{4}|z|^2}| \left( \left( \frac{1}{2\pi} \right)^n \int_{\mathbb{C}^n} |e^{(u, A^* z)}| e^{-\frac{1}{2}|u|^2} \, dv(u) \right)^{p/2} \, dv(z)
\]
\[
= \int_{\mathbb{C}^n} |e^{\frac{1}{4}(z, B)}| |e^{\frac{1}{4}|z|^2}| (e^{\|A^* z|^2/2})^{p/2} \, dv(z)
\]
\[
\leq \int_{\mathbb{C}^n} |e^{\frac{1}{4}(z, B)}| |e^{\frac{1}{4}|z|^2}| e^{\frac{1}{2}\|A\|^2 |z|^2} \, dv(z)
\]
\[
= \int_{\mathbb{C}^n} |e^{\frac{1}{4}(z, B)}| |e^{\frac{1}{4}|z|^2}| (1-\|A\|^2 |z|^2) \, dv(z)
\]
\[
\leq C e^{\frac{1}{2}\|A\|^2 |z|^2} < \infty,
\]
where \( C > 0 \) is a constant depending on \( \|A\| \). Thus, \( \bar{\mu}(z) \in L^{p/2}(\mathbb{C}^n, dv) \) and \( C_\varphi \) is in the Schatten \( p \)-class \( S_p \) for any \( 0 < p < \infty \).
In fact, this result has been obtained by Du [10] and Schatten class weighted composition operators on the Fock spaces have been studied by many authors (see [11, 16, 17, 22]). However, our different proof leads to the following interest fact: If $C_{\varphi}$ is compact on $F^2(\mathbb{C}^n)$, the conditions
\[
\int_{\mathbb{C}^n} \|C_{\varphi} k_z\|^p dv(z) < \infty \quad \text{and} \quad \int_{\mathbb{C}^n} \|C_{\varphi}^* k_z\|^p dv(z) < \infty
\] (2.2)
are equivalent. While, in the setting of the Bergman space, Xia [23] found a self-mapping $\varphi$ of the unit disc $D$, which induces a composition operator satisfying
\[
\int_D \|C_{\varphi} k_z\|^p \frac{dv(z)}{1-|z|^2} < \infty \quad \text{but} \quad \int_D \|C_{\varphi}^* k_z\|^p \frac{dv(z)}{1-|z|^2} = \infty.
\]

Theorem 2.2 and the above discussion reflect that there are lots of different characteristics for composition operators on the Fock spaces. As we know, many examples show that compact composition operators on the Bergman space maybe are not in any of the Schatten classes (see [4, 15, 26]).

Next, we will prove how the conditions in (2.2) are equivalent when $C_{\varphi}$ is compact on $F^2(\mathbb{C}^n)$. The proof of Theorem 2.2 gives that
\[
\int_{\mathbb{C}^n} \|C_{\varphi} k_z\|^p dv(z) = \int_{\mathbb{C}^n} \langle C_{\varphi}^* C_{\varphi} k_z, k_z \rangle z^{p+1} dv(z)
\]
\[
= \int_{\mathbb{C}^n} \langle T_{\mu} k_z, k_z \rangle z^{p+1} dv(z) = \int_{\mathbb{C}^n} \bar{\mu}(z) \langle z, k_z \rangle \bar{z} dv(z)
\]
\[
\leq C e^{\frac{\|B\|^2}{1-\|A\|^2}} < \infty.
\]

On the other hand, using the fact $C_{\varphi}^* K_z = K_{\varphi}(z)$ and the equation (2.1), we obtain
\[
\int_{\mathbb{C}^n} \|C_{\varphi} k_z\|^p dv(z) = \int_{\mathbb{C}^n} \left( \frac{\|K_{\varphi}(z)\|}{\|K_z\|} \right)^p z^{p+1} dv(z)
\]
\[
= \int_{\mathbb{C}^n} e^{\frac{\|z^{\varphi}(z) - |z|^2\|}{1-\|A\|^2}} dv(z) = \int_{\mathbb{C}^n} e^{\frac{|z^{\varphi}(z) - |z|^2|}{1-\|A\|^2}} dv(z)
\]
\[
\leq e^{\frac{|B|^2}{1-\|A\|^2}} \int_{\mathbb{C}^n} e^{\frac{|z^{\varphi}(z,A^* B)|}{1-\|A\|^2}|z|^2} dv(z)
\]
\[
\leq C e^{\frac{|B|^2}{1-\|A\|^2}} e^{\frac{|A^* B|^2}{1-\|A\|^2}} < \infty.
\]

3. Normal Composition Operators

In this section, we will characterize normal composition operators on the Fock space $F^2(\mathbb{C}^n)$. This property for weighted composition operators has been investigated in some articles [18, 20]. But we will try to reveal a perfect nature of normal composition operators on the Fock spaces.

First, we give a complete characterization for normal composition operators. Using this, we find an interesting result, that is, hypernormal composition operators must be normal. Furthermore, we prove that if $C_{\varphi}$ is essentially normal then $C_{\varphi}$ must be compact or normal. Here, $C_{\varphi}$ is called essentially normal if the commutator $[C_{\varphi}, C_{\varphi}] = C_{\varphi}^* C_{\varphi} - C_{\varphi} C_{\varphi}^*$ is compact.

**Theorem 3.1.** Let $\varphi$ be a holomorphic mapping of $\mathbb{C}^n$. Assume that $C_{\varphi}$ is bounded on $F^2(\mathbb{C}^n)$. Then $C_{\varphi}$ is normal if and only if $\varphi(z) = A z$ with $A^* A = AA^*$. 
Theorem 3.3. Position operators can be essentially normal.

Proof. This result can be shown using the same idea as Theorem 8.2 of [8]. Here, we give a simple argument.

First, since $C_\varphi$ is bounded on $\mathcal{F}^2(\mathbb{C}^n)$, by Theorem A, we have $\varphi(z) = Az + B$. Now, assume that $C_\varphi$ is normal (then it is hyponormal) on $\mathcal{F}^2$. Then

$$1 = \|C_\varphi 1\|^2 \geq \|C^*_\varphi 1\|^2 = \|C^*_\varphi K_0\|^2 = \|K_\varphi(0)\|^2 = r(\varphi(0))^2/2,$$

which implies $\varphi(0) = 0$. So we get $B = 0$ and $\varphi(z) = Az$.

Lemma 2 in [5] shows that the adjoint $C^*_\varphi = C_\tau$ with $\tau(z) = A^*z$. Since $C_\varphi$ is normal, we have

$$C_{\tau \circ \varphi} = C_\tau C_\varphi = C^*_\varphi C_\varphi = C_\varphi C_\tau = C_{\tau \circ \varphi}.$$

It follows that $AA^*z = \varphi \circ \tau(z) = \tau \circ \varphi(z) = A^*Az$ for any $z \in \mathbb{C}^n$. Therefore, we deduce $AA^* = A^*A$. The other direction is obvious, so we complete the proof. \(\square\)

Remark. From this result, we may find that Proposition 2.4 in [20] is not true, because the authors deduced that if $C_\varphi$ is normal on $\mathcal{F}^2(\mathbb{C}^n)$, then $\varphi$ is univalent.

Next, we also present the following interesting result.

Proposition 3.2. Assume that $\varphi : \mathbb{C}^n \to \mathbb{C}^n$ is a holomorphic mapping and $C_\varphi$ is bounded on $\mathcal{F}^2(\mathbb{C}^n)$. If $C_\varphi$ is hyponormal, then it is normal.

Proof. Suppose that $C_\varphi$ is hyponormal on $\mathcal{F}^2(\mathbb{C}^n)$. The preceding argument in the proof of Theorem 3.1 shows that $\varphi(z) = Az$ for some $n \times n$ matrix $A$ with $\|A\| \leq 1$.

Let $v_1$ be an eigenvector of $A^*$ with corresponding eigenvalue $\lambda_1$. Define $f \in \mathcal{F}^2(\mathbb{C}^n)$ by $f(z) = \langle z, v_1 \rangle$. Because $C_\varphi - \lambda_1 I$ is also hyponormal, we have

$$\|(C_\varphi - \lambda_1 I)f\| \geq \|(C_\varphi - \lambda_1 I)^*f\|. \tag{3.1}$$

However, for each $z$,

$$(C_\varphi - \lambda_1 I)f)(z) = f(\varphi(z)) - \lambda_1 f(z) = \langle (A - \lambda_1 I)z, v_1 \rangle = (z, 0) = 0;$$

that is, $(C_\varphi - \lambda_1 I)f$ is the zero function. Hence, by (3.1), $(C_\varphi - \lambda_1 I)^*f$ is also the zero function. This means that for each $z$,

$$0 = ((C_\varphi - \lambda_1 I)^*f)(z) = \langle (A^* - \lambda_1 I)z, v_1 \rangle = \langle z, (A - \lambda_1)v_1 \rangle,$$

so that $(A - \lambda_1)v_1 = 0$, where we have used $C^*_\varphi = C_\tau$ with $\tau(z) = A^*z$. Thus $v_1$ is also an eigenvector for $A$ (with eigenvalue $\lambda_1$).

We see that the subspace $W_1$ of $\mathbb{C}^n$ spanned by $v_1$ is reducing for $A^*$. This gives $A^* W_1^+ \subseteq W_1^+$. Now, we apply the argument of the preceding paragraph a second time, starting with an eigenvector $v_2 \in W_1^+$ for $A^*$ with corresponding eigenvalue $\lambda_2$. Then, we obtain $A^*v_2 = \lambda_2 v_2$. If $n = 2$, we are done, $A^*$ and $A$ commute on the basis $\{v_1, v_2\}$ of $\mathbb{C}^n$. Otherwise, notice that the subspace $W_2$ of $\mathbb{C}^n$ spanned by $v_1$ and $v_2$ is reducing for $A^*$ and $A^* W_2^+ \subseteq W_2^+$. Thus, we can again apply the argument of the preceding paragraph to obtain a vector $v_3 \in W_2^+$, which is both an eigenvector for $A^*$ and for $A$. We can continue this process until we get a basis of $\mathbb{C}^n$ on which $A^*$ and $A$ commute. Hence, by Theorem 3.1, we know that $C_\varphi$ is normal on $\mathcal{F}^2(\mathbb{C}^n)$. \(\square\)

Finally, on the Fock space $\mathcal{F}^2(\mathbb{C}^n)$, we observe that only compact and normal composition operators can be essentially normal.

Theorem 3.3. Suppose that $\varphi$ is a holomorphic mapping of $\mathbb{C}^n$ and $C_\varphi$ is bounded on $\mathcal{F}^2(\mathbb{C}^n)$. Then $C_\varphi$ is essentially normal if and only if $C_\varphi$ is either compact or normal.
Proof.} Compact and normal operators are trivially essentially normal. We only need to prove the other direction.

Assume that \( C_\varphi \) is essentially normal, i.e. the commutator \([C_\varphi^*, C_\varphi]\) is compact. Let \( k_p = K_p/\|K_p\| \) be the normalized reproducing kernel at \( p \in \mathbb{C}^n \). Note that the sequence \( \{k_p\} \) tends to zero weakly on \( \mathcal{F}^2(\mathbb{C}^n) \) as \( |p| \to \infty \), so we have

\[
\limsup_{|p|\to\infty} ||[C_\varphi^*, C_\varphi]k_p|| = 0.
\]

Since \( C_\varphi \) is bounded on \( \mathcal{F}^2(\mathbb{C}^n) \), we get \( \varphi(z) = Az + B \), where \( A \) and \( B \) are described as Theorem A. Moreover, Lemma 2 of [5] gives \( C_\varphi^* = M_{K_B}C_\tau \) with \( \tau(z) = A^*z \). Thus, for each \( p \in \mathbb{C}^n \),

\[
C_\varphi K_p = (M_{K_B}C_\tau)^*K_p = K_B^*(p)K_{\varphi(p)}.
\]

It follows that

\[
||[C_\varphi^*, C_\varphi]k_p|| \geq ||[C_\varphi^*, C_\varphi]k_p, k_p|| = \frac{||C_\varphi^*K_p||^2 - ||C_\varphi^*K_p||^2}{||K_p||^2}
\]

\[
= \frac{||K_B(p)K_{\varphi(p)}||^2 - ||K_{\varphi(p)}||^2}{||K_p||^2}
\]

\[
= \frac{||K_B(p)||^2}{||K_p||^2} ||K_{\varphi(p)}||^2 \left| 1 - \frac{||K_B(p)||^2}{||K_{\varphi(p)}||^2} \right|.
\]

If \( ||A|| < 1 \), then \( C_\varphi \) is compact and the result obviously holds. Thus, it suffices to prove that \( C_\varphi \) is normal in the case of \( ||A|| = 1 \). This means, we should prove that \( B = 0 \) and \( A^*A = AA^* \) according to Theorem 3.1. Now, assume \( ||A|| = 1 \), there exists a \( \zeta \in \mathbb{C}^n \) such that \( |A\zeta| = |\zeta| \). Choosing \( p = t\zeta \) with \( t \to \infty \), using Theorem A, we find that

\[
\langle Ap, B \rangle = \langle A(t\zeta), B \rangle = t\langle A\zeta, B \rangle = 0
\]

and \( |Ap|^2 = |t\zeta|^2 = |p|^2 \). This implies

\[
\frac{||K_{\varphi(p)}||^2}{||K_p||^2} = \exp\left( \frac{|\varphi(p)|^2 - |p|^2}{2} \right) = \exp\left( \frac{|Ap + B|^2 - |p|^2}{2} \right)
\]

\[
= \exp\left( \frac{|Ap|^2 + 2Re\langle Ap, B \rangle + |B|^2 - |p|^2}{2} \right)
\]

\[
= \exp(|B|^2/2),
\]

and

\[
\frac{||K_B(p)||^2}{||K_{\varphi(p)}||^2} = \exp\left( \frac{||\tau(p)||^2 - |\varphi(p)|^2 + 2Re\langle p, B \rangle}{2} \right)
\]

\[
= \exp\left( \frac{|A^*p|^2 - |Ap + B|^2 + 2Re\langle p, B \rangle}{2} \right)
\]

\[
= \exp\left( \frac{|A^*p|^2 - |p|^2 - |B|^2 + 2Re\langle p, B \rangle}{2} \right).
\]

Therefore,

\[
||[C_\varphi^*, C_\varphi]k_p|| \geq \frac{||K_{\varphi(p)}||^2}{||K_p||^2} \left| 1 - \frac{||K_B(p)||^2}{||K_{\varphi(p)}||^2} \right|
\]

\[
= \exp(|B|^2/2) \left| 1 - \exp\left( \frac{|A^*p|^2 - |p|^2 - |B|^2 + 2Re\langle p, B \rangle}{2} \right) \right|.
\]
Because \( \limsup_{t \to \infty} \| [C_{\varphi}^*, C_{\varphi}] k_t \| = 0 \), we deduce that
\[
\lim_{t \to \infty} \exp(|B|^2/2) \left| 1 - \exp \left( \frac{t^2|A^* |^2 - t^2| \zeta |^2 - |B|^2 + 2t \text{Re}(\zeta, B)}{2} \right) \right| = 0,
\]
that is,
\[
\lim_{t \to \infty} \exp \left( \frac{t^2|A^* |^2 - t^2| \zeta |^2 - |B|^2 + 2t \text{Re}(\zeta, B)}{2} \right) = 1.
\]
As a consequence, we must have \( B = 0 \) and \( |A^* \zeta| = |\zeta| \). This yields that \( \varphi(z) = Az \) and \( C_{\varphi}^* = C_{\tau} \).

Now, we have
\[
[C_{\varphi}^*, C_{\varphi}] = C_{\varphi}^* C_{\varphi} - C_{\varphi} C_{\varphi}^* = C_{\tau} C_{\varphi} - C_{\varphi} C_{\tau} = C_{\varphi \tau} - C_{\tau \varphi}.
\]
Since \( \|A\| = 1 \) implies that \( \|A^* A\| = 1 \) and \( \|A A^*\| = 1 \), by Theorem A, both \( C_{\varphi \tau} \) and \( C_{\tau \varphi} \) are not compact. On the other hand, Choe et al. [6] have shown that for holomorphic mappings \( \psi \neq \phi \), the operator \( C_{\phi} \phi - C_{\psi} \psi \) is compact on \( F^2(\mathbb{C}^n) \) if and only if both \( C_{\phi} \) and \( C_{\psi} \) are compact. Thus, \( [C_{\varphi}^*, C_{\varphi}] = C_{\varphi \tau} - C_{\tau \varphi} \) is compact must give that \( \varphi \circ \tau = \tau \circ \varphi \).

It follows then that \( AA^* = A^* A \) and \( C_{\varphi} \) is normal by Theorem 3.1. So we obtain the desired result. □

4. SPECTRA OF COMPOSITION OPERATORS

When \( \varphi \) is a holomorphic mapping of the complex plane \( \mathbb{C} \), Guo and Izuchi [12] have described the spectrum of \( C_{\varphi} \) on \( F^2(\mathbb{C}) \) as follows: Let \( \varphi(z) = az + b, |a| \leq 1 \) and \( a \neq 1 \), which inducing a bounded composition operator \( C_{\varphi} \) on \( F^2(\mathbb{C}) \). Then \( \sigma(C_{\varphi}) = \{a^n, n \in \mathbb{Z}_+ \} \), where \( \sigma(C_{\varphi}) \) denotes the spectrum of \( C_{\varphi} \).

In higher dimensions, we also want to know the spectral structure of \( C_{\varphi} \) on \( F^2(\mathbb{C}^n) \). In fact, when \( \varphi \) is a unitary map, the spectrum of \( C_{\varphi} \) is clear.

**Theorem B.** [24] Let \( \varphi(z) = Uz \) with \( \{\lambda_1, \ldots, \lambda_n\} \) being eigenvalues of the unitary matrix \( U \). Then the spectrum of \( C_{\varphi} \) on \( F^2(\mathbb{C}^n) \) is the closure of the set \( \{\lambda_1^n, \ldots, \lambda_n^n : (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \} \).

In this section, we will completely give the spectrum of \( C_{\varphi} \) for any bounded composition operator \( C_{\varphi} \) on \( F^2(\mathbb{C}^n) \). The idea comes from Bayart [2]. First, we need the following lemma.

**Lemma 4.1.** Assume that \( A \) is an arbitrary \( n \times n \) matrix with \( \|A\| \leq 1 \). Then there exists a unitary matrix \( U \in \mathbb{C}^{n \times n} \) such that \( UAU^* = M \) with
\[
M = \begin{pmatrix}
D & 0 \\
0 & A_1
\end{pmatrix},
\]
where \( D = \text{diag}(e^{i\theta_1}, \ldots, e^{i\theta_s}) \) and \( A_1 \in \mathbb{C}^{(n-s) \times (n-s)} \) is upper-triangular with all diagonal elements less than 1.

**Proof.** For any \( A \in \mathbb{C}^{n \times n} \), by Schur Decomposition, there exist a unitary matrix \( U \in \mathbb{C}^{n \times n} \) and an upper triangular matrix \( M \in \mathbb{C}^{n \times n} \) such that
\[
UAU^* = M.
\]
Assume that \( a_{11}, a_{22}, \ldots, a_{nn} \) are the diagonal elements of \( M \) with \( |a_{11}| \geq |a_{22}| \geq \ldots \geq |a_{nn}| \). It is clear that \( a_{11}, a_{22}, \ldots, a_{nn} \) are the eigenvalues of \( M \). Moreover, \( \|M\| = \|A\| \leq 1 \) gives \( |a_{11}| \leq 1 \). If \( \|M\| < 1 \), then \( |a_{11}| < 1 \) and \( s = 0 \). The result is true.
If \( \|M\| = 1 \), assume that \( |a_{11}| = |a_{22}| = \cdots = |a_{ss}| = 1 \) for some \( 1 \leq s \leq n \) and \( M = (a_{jk}) \). We will show that \( a_{ik} = 0 \) for \( 1 \leq i \leq s, i < k \leq n \). Let \( e_i = (0, \ldots, 1, \ldots, 0)^t, i = 1, \ldots, s \), the unit vectors of \( \mathbb{C}^n \). Since \( \|M^*\| = \|M\| = 1 \), we have
\[
|M^*e_i|^2 = |a_{ii}|^2 + \sum_{i<k \leq n} |a_{ik}|^2 \leq 1.
\]
Combining this with \( |a_{ii}| = 1 (i = 1, \ldots, s) \), we get \( a_{ik} = 0 \) for \( i < k \leq n \). Therefore, \( M \) has the form
\[
M = \begin{pmatrix} D & 0 \\ 0 & A_1 \end{pmatrix},
\]
where \( D = \text{diag}(a_{11}, a_{22}, \ldots, a_{ss}) \) and \( A_1 \) is an upper-triangular matrix with the diagonal elements satisfying \( \max_{s<i\leq n} |a_{ii}| < 1 \).

**Theorem 4.2.** Let \( \varphi \) be a holomorphic self-mapping of \( \mathbb{C}^n \). Suppose that \( \varphi \) is bounded on \( \mathcal{F}^2(\mathbb{C}^n) \). Then \( \varphi(z) = Az + B \) and the spectrum \( \mathcal{C}_\varphi \) is the closure of the set
\[
\{\lambda_1^{\alpha_1} \cdots \lambda_n^{\alpha_n} : (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \},
\]
where \( \lambda_1, \ldots, \lambda_n \) are the eigenvalues of the matrix \( A \).

**Proof.** Since \( \mathcal{C}_\varphi \) is bounded on \( \mathcal{F}^2(\mathbb{C}^n) \), by Theorem A, we have \( \varphi(z) = Az + B \) with \( \|A\| \leq 1 \). Applying Lemma 4.1, there exists unitary \( U \) such that \( UAU^* = M \) with
\[
M = \begin{pmatrix} D & 0 \\ 0 & A_1 \end{pmatrix},
\]
where \( D = \text{diag}(e^{i\theta_1}, \ldots, e^{i\theta_s}) \) and \( A_1 \in \mathbb{C}^{(n-s) \times (n-s)} \) is upper-triangular. Let \( \lambda_1, \ldots, \lambda_{n-s} \) be the eigenvalues of \( A_1 \), then \( \lambda = \max_{1 \leq i \leq n-s} |\lambda_i| < 1 \).

Now, we compute that
\[
C_\varphi^* C_\varphi C_U f(z) = f(U \circ \varphi \circ U^*(z)) = f(UAU^*z + UB) = f(Mz + B') = C_\varphi f(z).
\]
That is, \( C_\varphi \) is similar to \( C_\psi \) with \( \psi(z) = Mz + B' \) and \( B' = UB \).

Note that the boundedness of \( C_\varphi \) means that \( C_\psi \) is also bounded. Using Theorem A, we have \( \langle M\zeta, B' \rangle = 0 \) for any \( \zeta \in \mathbb{C}^n \) with \( |M\zeta| = |\zeta| \). Choosing \( \zeta = e_i, i = 1, \ldots, s \), we obtain that
\[
b'_1 = \cdots = b'_s = 0,
\]
where \( b'_i \) is the \( i^{th} \) coordinate of \( B' \). Thus,
\[
\psi(z) = \psi(w, v) = (Dw, A_1v + B_1)
\]
for \( z = (w, v) \in \mathbb{C}^n \times \mathbb{C}^{n-s} \).

Because the spectrum is similarly invariant, we will compute the spectrum of \( C_\psi \). Let \( v(i) \in \mathbb{C}^{(n-s) \times 1} \) be a non-zero eigenvector of \( A_1^T \) associated to \( \lambda_i \), i.e. \( A_1^T v(i) = \lambda_i v(i) \), \( i = 1, \ldots, n-s \). Since \( \lambda < 1 \), we see that \( I - A_1 \) is invertible. Choose the vector \( C \) such that \( C = (I - A_1)^{-1}B \), that is, \( B_1 - C = -A_1C \). Set the function
\[
F(z) = F(w, v) = w^\beta [(v - C)^T v(1)]^{\gamma_1} \cdots [(v - C)^T v(n-s)]^{\gamma_{n-s}},
\]
where \( \beta = (\beta_1, \ldots, \beta_s) \in \mathbb{N}^s \) and \( \gamma = (\gamma_1, \ldots, \gamma_{n-s}) \in \mathbb{N}^{n-s} \). Note that \( [(v-C)^Tv(i)]s \) is a polynomial of \( v \) with degree \( \gamma_0 \) and the Fock space \( \mathcal{F}^2(\mathbb{C}^n) \) contains all polynomials, then the function \( F(w,v) \) is in \( \mathcal{F}^2(\mathbb{C}^n) \). Next, we compute that

\[
C_\psi F(w,v) = F \circ \psi(w,v) = F(Dw, A_1 v + B_1)
\]

\[
= (Dw)^\beta [(A_1 v + B_1 - C)^Tv(1)]^{\gamma_1} \cdots [(A_1 v + B_1 - C)^Tv(n-s)]^{\gamma_{n-s}}
\]

\[
= e^{i(\beta_1, \ldots, \beta_s, \theta_s)} (A_1 v + B_1 - C)^Tv(1) \cdots (A_1 v + B_1 - C)^Tv(n-s)
\]

\[
= e^{i(\beta_1, \ldots, \beta_s, \theta_s)} \lambda_1^{\gamma_1} \cdots \lambda_{n-s}^{\gamma_{n-s}} F(w,v),
\]

Thus, for any multi-index \((\beta, \gamma) \in \mathbb{N}^n\), the function \( F(z, w) \) is an eigenvector of \( C_\psi \) associated to the eigenvalue \( e^{i(\beta_1, \ldots, \beta_s, \theta_s)} \lambda_1^{\gamma_1} \cdots \lambda_{n-s}^{\gamma_{n-s}} \). Hence, the spectrum of \( C_\psi \) contains the closure of the set

\[\{ e^{i(\beta_1, \ldots, \beta_s, \theta_s)} \gamma_1 \cdots \gamma_{n-s} : \alpha = (\beta, \gamma) \in \mathbb{N}^n \}, \]

where \( e^{i\theta_1}, \ldots, e^{i\theta_s}, \lambda_1, \ldots, \lambda_{n-s} \) are the eigenvalues of \( M \).

In fact, we find that the spectrum of \( C_\psi \) is exactly the closure of the above set. Next, we will prove the other direction.

Let \( \psi_N = \psi \circ \cdots \circ \psi (N \text{ times}) \). Note that \( \psi(w,v) = (Dw, A_1 v + B_1) \) gives

\[ \psi_N(w,v) = (D^{Nw}, A_1^{Nv} + B_N), \]

where \( B_N = (A_1^{N-1} + \cdots + A_1 + I)B_1 = (I - A_1)(I - A_1)^{-1}B_1 \). Moreover, \( D_N = \text{diag}(e^{iN\theta_1}, \ldots, e^{iN\theta_s}) \) and \( A_1^{Nv} \) is still an upper-triangular matrix with the diagonal elements \( \lambda_1^N, \ldots, \lambda_{n-s}^N \). Since \( \lambda = \max\{\lambda_1, \ldots, \lambda_{n-s}\} < 1 \), it is easy to check that \( A_1^N \to O \) (here, \( O \) denotes the zero matrix) and \( B_N \to (I - A_1)^{-1}B_1 \) as \( N \to \infty \). Thus, we may choose an integer \( N > 0 \) large enough such that \( \|A_1^N\| < 1 \).

Using the spectral mapping theorem, we know that \( \|\sigma(C_\psi)\|^N = \sigma(\|C_\psi\|^N) = \sigma(\|C_\psi\|^N) \).

This leads to that we may still use \( \psi(w,v) = (Dw, A_1 v + B_1) \) with \( \|A_1\| < 1 \) instead of \( \psi_N(w,v) = (D^{Nw}, A_1^{Nv} + B_N) \) with \( \|A_1^N\| < 1 \) to compute the spectrum of \( \sigma(C_\psi) \).

Now, we introduce a decomposition for the Fock space \( \mathcal{F}^2(\mathbb{C}^n) \). For \( \gamma \in \mathbb{N}^{n-s} \), let \( H_\gamma \) be the set of all functions \( F \) in \( \mathcal{F}^2(\mathbb{C}^n) \), where \( F \) may be written as \( F(z, w) \). Set \( K_m = \bigoplus_{|\gamma| < m} H_\gamma \) for any integer \( m \geq 0 \). Thus, we have a finite orthogonal decomposition

\[ \mathcal{F}^2(\mathbb{C}^n) = \bigoplus_{|\gamma| < m} H_\gamma \bigoplus K_m. \]

On the other hand, for the set \( \mathbb{N}^{n-s} \), we need an order (see [2]); for \( \alpha, \beta \in \mathbb{N}^{n-s} \),

\[ \alpha < \beta \iff \begin{cases} |\alpha| < |\beta|, \\ |\alpha| \leq |\beta|, \alpha_j = \beta_j \text{ for } j < j_0 \text{ and } \alpha_{j_0} < \beta_{j_0}. \end{cases} \]

Since \( A_1 \) is upper-triangular, we have

\[ C_\psi F(w,v) = F(\psi(w,v) = F_\gamma(Dw)(A_1 v + B_1)^\gamma \]

\[ = F_\gamma(Dw) \prod_{j=1}^{n-s} \left( \lambda_j v_j + \sum_{k > j} a_{jk} v_k + b_j \right)^{\gamma_j}. \]

This implies that the representation matrix of \( C_\psi \) will be upper-triangular when using the above order for the decomposition \( \mathcal{F}^2(\mathbb{C}^n) = \bigoplus_{|\gamma| < m} H_\gamma \bigoplus K_m \).
Set \( \rho(w, v) = (Dw, v) \) and \( \tilde{\psi} = (Dw, \tilde{A}_1 v) \) with \( \tilde{A}_1 = \text{diag}(\lambda_1, \ldots, \lambda_{n-s}) \). It is easy to check that the composition operator \( C_{\tilde{\psi}} \) is bounded and has a diagonal matrix in the decomposition of \( F^2(C^n) \). Let \( T_\gamma \) and \( S_\gamma \) respectively denote the diagonal blocks of \( C_{\tilde{\psi}} \) and \( C_\rho \) corresponding to \( H_\gamma \). Then

\[
T_\gamma = \lambda_1^{\gamma_1} \cdots \lambda_{n-s}^{\gamma_{n-s}} S_\gamma.
\]

Note that \( C_\rho \) is unitary on \( F^2(C^n) \), by Theorem B, the spectrum of \( C_\rho \) is the closure of the set

\[
\{ e^{i(\beta_1 \theta_1 + \cdots + \beta_s \theta_s)} : (\beta_1, \ldots, \beta_s) \in \mathbb{N}^s \}.
\]

Using Lemma 7.17 of [8], we get \( \sigma(S_\gamma) \subseteq \sigma(C_\rho) \) and

\[
\sigma(C_{\tilde{\psi}}) \subseteq \bigcup_{|\gamma| < m} \sigma(T_\gamma) \bigcup \sigma(C_{\tilde{\psi}|K_m})
\]

\[
\subseteq \bigcup_{|\gamma| < m} \lambda_1^{\gamma_1} \cdots \lambda_{n-s}^{\gamma_{n-s}} \sigma(S_\gamma) \bigcup \sigma(C_{\tilde{\psi}|K_m}).
\]

Thus, we obtain that the spectrum of \( C_{\tilde{\psi}} \) is in the closure of

\[
\{ e^{i(\beta_1 \theta_1 + \cdots + \beta_s \theta_s)} \lambda_1^{\gamma_1} \cdots \lambda_{n-s}^{\gamma_{n-s}} : \alpha = (\beta, \gamma) \in \mathbb{N}^n \}.
\]

On the other hand, since the matrix of \( C_{\tilde{\psi}} \) becomes upper-triangular in the decomposition of \( F^2(C^n) \) we have designed before, the spectra for the diagonal locks of \( C_{\tilde{\psi}} \) corresponding to the subspace \( H_\gamma \) are equal to those of \( C_{\tilde{\psi}} \). We also let \( T_\gamma \) denote the diagonal blocks of \( C_{\tilde{\psi}} \) corresponding to \( H_\gamma \). Using Lemma 7.17 of [8] again,

\[
\sigma(C_{\tilde{\psi}}) \subseteq \bigcup_{|\gamma| < m} \sigma(T_\gamma) \bigcup \sigma(C_{\tilde{\psi}|K_m})
\]

\[
\subseteq \bigcup_{|\gamma| < m} \lambda_1^{\gamma_1} \cdots \lambda_{n-s}^{\gamma_{n-s}} \sigma(S_\gamma) \bigcup \sigma(C_{\tilde{\psi}|K_m}),
\]

where \( D(0, ||C_{\tilde{\psi}|K_m}||) \) denotes the disk with the radius \( ||C_{\tilde{\psi}|K_m}|| \). Therefore, if we show that \( ||C_{\tilde{\psi}|K_m}|| \) tends to zero as \( m \to \infty \), then \( C_{\tilde{\psi}} \) and \( C_{\rho} \) have the same spectrum.

Let \( A_1 = T \Sigma \Psi \) be a singular value decomposition of \( A_1 \), where \( T, \Psi \) are unitary matrices of \( C^{(n-s)\times(n-s)} \) and \( \Sigma = \text{diag}(\mu_1, \ldots, \mu_{n-s}) \) with \( \mu_1 \geq \cdots \geq \mu_{n-s} \), the non-negative square roots of the eigenvalues of \( A_1^* A_1 \). Now \( ||A_1|| < 1 \) yields that \( \mu_1 = ||A_1|| < 1 \). Assume \( \mu_1 < \mu < 1 \) for a positive constant \( \mu \). Let \( \phi_1(w, v) = (Dw, \Sigma v) \) and \( \phi_2(w, v) = (w, \Sigma v + T^*B_1) \), then \( C_{\phi_1} = C_{\phi_2} \) are unitary operators when acting on \( F^2(C^n) \). Furthermore, \( C_{\phi_2} \) preserves both \( K_m \) and \( K_m^{\perp} \). Hence, it is sufficient to prove that \( ||C_{\phi_2}|K_m|| \to 0 \) as \( m \to \infty \).

For \( F(z) = \sum_{|\gamma| \geq m} F_\gamma(w)v^\gamma \in K_m \), we compute that

\[
||C_{\phi_2} F||^2 = \int_{C^n} |F \circ \psi_{\Sigma}(w, v)|^2 e^{-\frac{1}{2}(||w||^2 + ||v||^2)} dw dv
\]

\[
\leq \sum_{|\gamma| \geq m} \int_{C^n} |F_\gamma(w)(\Sigma v + T^*B_1)^\gamma|^2 e^{-\frac{1}{2}(||w||^2 + ||v||^2)} dw dv
\]

\[
= \sum_{|\gamma| \geq m} \int_{C^n} |F_\gamma|^{2} e^{-\frac{1}{2}||w||^2} dw \int_{C_{n-s}} |(\Sigma v + T^*B)^\gamma|^{2} e^{-\frac{1}{2}||v||^2} dv
\]
Let $T^*B = (b_1, \ldots, b_{n-s})$, then $\Sigma v + T^*B = (\mu_1 v_1 + b_1, \ldots, \mu_{n-s} v_{n-s} + b_{n-s})$. Since $\mu_1 < \mu < 1$, for $i = 1, \ldots, n-s$, it is easy to see $|\mu_j v_j + b_j| \leq |v_j|$ when $|v_j| \geq M$ for large enough $M > 0$. It follows that $|(|\Sigma v + T^*B|)| \leq \mu^m |v_j|$ off a compact subset of $\mathbb{C}^{n-s}$. Therefore, using the orthogonality of $F_\gamma(w)v^\gamma$ and $F_{\gamma'}(w)v^{\gamma'}$ for $\gamma \neq \gamma'$,

$$\|C_{\psi_2}F\|^2 \leq C \mu^{2m} \sum_{|\gamma| \geq m} \int_{\mathbb{C}^n} |F_\gamma(w)|^2 e^{-\frac{1}{2} |w|^2} dw \int_{\mathbb{C}^{n-s}} |v^{\gamma}|^2 e^{-\frac{1}{2} |v|^2} dv$$

$$= C \mu^{2m} \int_{\mathbb{C}^n} \left( \sum_{|\gamma| \geq m} F_\gamma(w)v^\gamma \right)^2 e^{-\frac{1}{2} |w|^2} dw dv$$

$$= C \mu^{2m} \|F\|^2,$$

where $C$ is a sufficiently large constant. Now $\mu < 1$ gives

$$\lim_{m \to \infty} \|C_{\psi_2}K_m\| \leq \lim_{m \to \infty} C \mu^{2m} = 0.$$ 

The desired result holds. \qed

5. **Cyclicity of composition operators**

A bounded linear operator $T$ on a linear metric space $\mathcal{H}$ is said to be cyclic if there exists a vector $x \in \mathcal{H}$ such that

$$\text{span}\{T^m x : m = 0, 1, \ldots\} = \mathcal{H}.$$

If there exists a vector $x \in \mathcal{H}$ such that the orbit

$$\{T^m x : m = 0, 1, \ldots\}$$

is dense, then $T$ is said to be hypercyclic. If there exists a vector $x \in \mathcal{H}$ such that the projective orbit

$$\{\lambda T^m x : m = 0, 1, \ldots \text{ and } \lambda \in \mathbb{C}\}$$

is dense, then $T$ is said to be supercyclic. See [3] for more information.

In this section, we study the dynamics of composition operators on the Fock space $\mathcal{F}^2(\mathbb{C}^n)$. On the complex plane, the following result has been proved by Guo and Izuchi [12].

**Proposition 5.1.** (a) If $\varphi(z) = az$ with $|a| = 1$, then $C_\varphi$ is cyclic on the Fock space $\mathcal{F}^2(\mathbb{C})$ if and only if $a^n \neq a$ for every $n > 1$.

(b) If $\varphi(z) = az + b$ with $|a| < 1$ and $a \neq 0$, then $C_\varphi$ is cyclic on $\mathcal{F}^2(\mathbb{C})$.

**Proof.** (a) This result is Proposition 2.3(i) of [12].

(b) This is an immediate result of Theorem 4.2 in [12]. Here, we present a different proof. Let $K_z(w) = \exp(\langle w, z \rangle/2)$ be the reproducing kernels for $\mathcal{F}^2(\mathbb{C})$. We will prove that $K_z$ is a cyclic vector of $C_\varphi$ for any $z \neq 0$. We see that

$$C^m_\varphi K_z(w) = K_z(\varphi_m(w)) = K_z\left(aw + \frac{1 - a^m}{1 - a}b\right)$$

$$= \exp\left(\left\langle aw + \frac{1 - a^m}{1 - a}b, z \right\rangle/2\right)$$

$$= \exp(\langle w, \frac{a^m}{1 - a}z \rangle/2) \exp\left(\frac{1 - a^m}{1 - a}b, z \right\rangle/2\right)$$

$$= c_m K_{\varphi_m z}(w),$$
Lemma 5.2. If \( \phi \) induces bounded \( \phi \) is cyclic on \( F^2(\mathbb{C}) \). Applying the same technique as Theorem 5.2 in [11], we obtain that \( \phi \) is not supercyclic on \( F^2(\mathbb{C}) \). Therefore, the cyclicity of composition operators on \( F^2(\mathbb{C}) \) is very simple.

In order to describe the dynamics of composition operators on the Fock space \( F^2(\mathbb{C}^n) \), we first give the following characterization for those symbols which inducing bounded composition operators on \( F^2(\mathbb{C}^n) \).

**Lemma 5.2.** If \( \varphi(z) = Az + B \) induces a bounded composition operator on \( F^2(\mathbb{C}^n) \), then \( \varphi \) fixes a point in \( \mathbb{C}^n \).

**Proof.** It suffices to show \( B \) belongs to the orthogonal complement of \( \ker(I - A^*) \). Suppose that \( v \) is a unit vector in \( \ker(I - A^*) \). Then \( A^*v = v \) and since \( \|A\| \leq 1 \), we get \( Av = v \) as well:

\[
1 \geq |Av| = |Av||v| \geq |\langle Av, v \rangle| = |\langle A A^*v, v \rangle| = |\langle A^*v, A^*v \rangle| = |\langle v, v \rangle| = 1;
\]

thus, \( |Av| = |v| = |\langle Av, v \rangle| \) and we conclude that \( Av = \lambda v \) for some constant \( \lambda \). Finally, \( \lambda = \langle Av, v \rangle = \langle A^*v, v \rangle = \langle v, A^*v \rangle = \langle v, v \rangle = 1 \). In particular, we have \( |Av| = |v| \) and hence,

\[
\langle B, v \rangle = \langle B, Av \rangle = 0
\]

since \( C_\varphi \) is bounded. \( \square \)

First, similar to the proof of Theorem 3.3 in [14], we obtain a necessary and sufficient condition for unitary composition operators to be cyclic on \( F^2(\mathbb{C}^n) \). Here, we omit its proof.

**Theorem 5.3.** If \( \varphi(z) = Uz \) and \( U \) is unitary with the eigenvalues \( e^{i\theta_1}, \ldots, e^{i\theta_n} \), then \( C_\varphi \) is cyclic on \( F^2(\mathbb{C}^n) \) if and only if \( \theta_1, \ldots, \theta_n, \pi \) are rationally linearly independent.

For all bounded composition operators on the Fock space \( F^2(\mathbb{C}^n) \), the following result shows that their are not supercyclic.

**Theorem 5.4.** Let \( \varphi : \mathbb{C}^n \to \mathbb{C}^n \) be a holomorphic mapping. If \( C_\varphi \) is bounded on \( F^2(\mathbb{C}^n) \), then \( C_\varphi \) is not supercyclic.

**Proof.** We will use the similar idea as Theorem 5.2 in [11] to obtain our result.

As shown in Lemma 5.2, \( \varphi(z) = Az + B \) must have a fixed point \( p \). Suppose that \( f \) is a supercyclic vector for \( C_\varphi \). It is clear that we must have \( f(p) \neq 0 \). Assume that \( f(p) = 1 \). If the function \( g \in F^2(\mathbb{C}^n) \) is in the projective orbit of \( f \) under \( C_\varphi \), then there exists a
sequence \( \{\lambda_{nk}\} \) such that \( \{\lambda_{nk} C_{\varphi_{nk}} f\} \) tends to \( g \) as \( k \to \infty \). Since norm convergence implies pointwise convergence, we get

\[
g(p) = \lim_{k \to \infty} \lambda_{nk} C_{\varphi_{nk}} f(p) = \lim_{k \to \infty} \lambda_{nk} f(\varphi_{nk}(p)) = \lim_{k \to \infty} \lambda_{nk} f(p) = \lim_{k \to \infty} \lambda_{nk}.
\]

If \( \varphi(z) = Uz \). Let \( e^{i\theta_1}, \ldots, e^{i\theta_n} \) be the eigenvalues of \( U \), then \( \theta_1, \ldots, \theta_n, \pi \) are rationally linearly independent. Otherwise, using Theorem 5.3, \( C_\varphi \) is not cyclic, and then not supercyclic on \( F^2(\mathbb{C}^n) \). Therefore, by extracting a subsequence, we may assume that the sequence \( \{\varphi_{nk}(z)\} = \{U^k z\} \) converges to a map \( \varphi(z) = Vz \) with unitary \( V \). Now, we choose a univalent function \( g \) with \( g(p) \neq 0 \). For each \( z \in \mathbb{C}^n \), we have

\[
g(z) = \lim_{k \to \infty} \lambda_{nk} C_{\varphi_{nk}} f(z) = \lim_{k \to \infty} \lambda_{nk} f(\varphi_{nk}(z)) = (\varphi(p), V) = f(Vz).
\]

This yields that \( f(z) = g \varphi V^{-1}(z)/g(p) \) is univalent. It follows that all the scalar multiplies of the \( C_\varphi \) orbit of \( f \) are univalent functions. This means that \( C_\varphi \) can not be supercyclic.

For more general \( \varphi(z) = Az + B \), it is enough to prove that \( C_\psi \) is not supercyclic, where \( \psi(w, v) = (Dw, A_1 v + B_1) \) is described in the proof of Theorem 4.2. Note that \( \psi_0(w, v) = (D^m w, A_1^m v + (A_1^{m-1} + \cdots + I) B_1) \). Let \( E \) be any compact subset of \( \mathbb{C}^{n-s} \) and write \( 0 \times E = \{(0, v) \in \mathbb{C}^s \times \mathbb{C}^{n-s}, v \in E\} \). Since \( \lambda = \max\{\lambda_1, \ldots, \lambda_{n-s}\} < 1 \), as pointed in the proof of Theorem 4.2, the sequence of iterates of \( \psi \) tends to the point \( (0, (I - A_1)^{-1} B_1) \) uniformly on the set \( 0 \times E \) of \( \mathbb{C}^n \). Moreover, it is clear that \( (0, (I - A_1)^{-1} B_1) \) is exactly the fixed point \( p \) of \( \psi \). It follows that

\[
g(0, v) = \lim_{k \to \infty} \lambda_{nk} C_{\psi_{nk}} f(0, v) = \lim_{k \to \infty} \lambda_{nk} f(\psi_{nk}(0, v)) = g(p)/f(p) = g(p)
\]

for any \( (0, v) \in 0 \times E \). Consequently, only functions which are independent with the last \( n-s \) coordinates can be in the closure of the projective \( C_\psi \) orbit of \( f \). Therefore, \( C_\psi \) is not supercyclic and the proof is complete.

Finally, it is clear that \( C_\varphi \) is not cyclic on the Fock space \( F^2(\mathbb{C}^n) \), if \( \varphi(z) = Az + B \) and the matrix \( A \) is not invertible. Now, we almost have known the dynamics of bounded composition operators on \( F^2(\mathbb{C}^n) \), only the following problem still open.

**Problem:** For the mapping \( \varphi(z) = Az + B \), if \( A \) is invertible with \( \|A\| \leq 1 \) and \( C_\varphi \) bounded on \( F^2(\mathbb{C}^n) \), is it cyclic?

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