Deformation of the Wheeler-DeWitt Equation

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Abstract

In this paper, we will analyse the consequences of deforming the canonical commutation relations consistent with the existence of a minimum length and a maximum momentum. We first generalize the deformation of first quantized canonical commutation relation to second quantized canonical commutation relation. Thus, we arrive at a modified version of second quantization. A modified Wheeler-DeWitt equation will be constructed by using this deformed second quantized canonical commutation relation. Finally, we demonstrate that in this modified theory the big bang singularity gets naturally avoided.

1 Introduction

In any approach to quantum gravity, the classical picture of spacetime is expected to break down. This is because at Planck scale the fluctuations in the geometry are expected to become of order unity. Thus, the picture of spacetime as a differential manifold cannot hold below Planck scale. In fact, many studies on physics of the black holes have suggested that all quantum gravity theories should have a minimum measurable length of the order of the Planck length [1]-[2]. The string theory also come naturally equipped with a minimum length [3]-[7]. In fact, in loop quantum gravity the existence of minimum length turns big bang into a big bounce [8]. It may be noted that the existence of a minimum length is not consistent with the conventional Heisenberg uncertainty principle. This is because according to the conventional Heisenberg uncertainty principle, the minimum measurable length is actually zero. To remove this inconsistency, the conventional Heisenberg principle is modified [9]-[23]. The resultant uncertainty principle is called the generalized uncertainty principle. The modification of the uncertainty principle naturally leads to a modification of the Heisenberg algebra. In this new Heisenberg algebra commutation relations between position and momentum operators contain momentum dependent factors. It may be noted that the implications of this modified uncertainty principle for quantum field theory have also been studied [24]-[26].

In doubly special special relativity theories, the Planck energy like the speed of light is an invariant quantity in doubly special special relativity. This is incorporated by a modification of the Heisenberg algebra [27]-[29]. A consequence of this modification is that, doubly special relativity come naturally equipped with a maximum measurable momentum. A modified version of general rel-
ativity called the Gravity’s Rainbow has been constructed [30]-[31]. Gravity’s Rainbow both velocity of light and the Planck energy are again invariant quantities. Thus, two different deformations of Heisenberg algebra have been studied. In fact, a algebra has also been constructed which is consistent with the existence of both a minimum length and a maximum momentum [33]-[40]. The transition rate of ultra cold neutrons in gravitational field has been analysed using this deformed algebra [11]. In fact, the modification to the Lamb shift and Landau levels have also been analysed in this deformed algebra [42].

The relation between the quantum mechanical commutators and the classical Poisson brackets has also been used to study modification to the Friedman-Robertson-Walker cosmology [34]. In this paper, we will deform the second quantized commutating relations and study the effect of the deformation on Wheeler-DeWitt equation. The solutions to the Wheeler-DeWitt equation gives us the wave function of the universes [35]-[36]. As the wave function of the universe describes the quantum state of the universe, so all the physical information about the universe can be extracted from it [37]-[38]. This wave function can also be obtained by taking a sum over all geometries and field configurations which match with a particular field configuration at a spatial section of the spacetime. This approach is called the Hartle-Hawking no-boundary proposal. In this approach a Wick rotation to Euclidean time makes this integral well defined. This wave function of the universe can also be viewed as a solution of the Wheeler-DeWitt equation [35]-[36]. The Wheeler-DeWitt equation is a second quantized equation which can be interpreted as the Schroedinger’s equation for gravity. However, there is no time in the Wheeler-DeWitt equation because it has to satisfy the time invariance which is required by general relativity [39]. In this paper we will analyse the implications of modifying the Wheeler-DeWitt equation, by using a deformed version of second quantized canonical commutator. Usually, the generalized uncertainty is studied as a implication of some quantum gravitational effect. However, in this paper, we will reverse this and study the implications of generalized uncertainty on quantum gravity.

2 Deformed Heisenberg Algebra

In this section, we will first review the deformation of the Heisenberg consistent with the existence of minimum length and maximum momentum. Then we will construct a second quantized version of this algebra. We let \( \ell_{Pl} \approx 10^{-35} \) m be the Planck length, \( M_{Pl} \) be the Planck mass, \( M_{Pl}c^2 \approx 10^{19} \) GeV be the Planck energy, and \( \beta_0 \) be a constant of order unity. Now the existence of a minimum length is consistent with the following algebra \([x^i, p^j] = i[\delta^i_j + \beta|p|\delta^i_j + 2\beta p^i p^j], \) and the existence of a maximum momentum is consistent with the following algebra \([x^i, p^j] = i[1 - \beta|p|\delta^i_j + \beta p^i p^j], \) where \( \beta = \beta_0 \ell_{Pl}/\hbar \) and \( \beta = \ell_{Pl}. \) Both these deformations of Heisenberg algebra have been combined into a single algebra [33]-[40]

\[
[x^i, p^j] = i \left[ \delta^i_j - \alpha|p|\delta^i_j + \alpha|p|^{-1} p^i p^j + \alpha^2 p^2 \delta^i_j + 3\alpha^2 p^i p^j \right],
\]

with \( \alpha = \alpha_0/M_{Pl}c = \alpha_0 \ell_{Pl}/\hbar. \) It may be noted that the norm of the momentum is defined as

\[
||p|| = \sqrt{p^i p_i}.
\]
In the one dimensional case this corresponds to the uncertainty relation given by
\[ \Delta x \Delta p = [1 - 2\alpha p + 4\alpha^2 p^2]. \]
These imply the existence of a minimum length \( \Delta x \geq \alpha_0 \ell_P \), and a maximum momentum \( \Delta p \leq \Delta p_{\text{max}} \leq \alpha_0^{-1} M_P \). Now the momentum in the coordinate representation can be written as
\[ p_i = \tilde{p}_i (1 - \alpha ||\tilde{p}|| + 2\alpha^2 ||\tilde{p}||^2), \]
where \([x^i, \tilde{p}_j] = i \delta^i_j\), and so, \( \tilde{p}_i = -\partial_i \). Thus, we can write
\[ p_i = -i \left( 1 + \alpha \sqrt{-\partial\partial_j - 2\alpha^2 \partial_i \partial_j} \right) \partial_i. \] (3)

Now we will analyse deform the second quantized commutator similar to this deformation of the first quantized commutator. Thus, we can write the commutator of a scalar field theory as
\[ [\phi(x), \pi(y)] = i\delta(x - y) + i\alpha A(x, y) + i\alpha^2 B(x - y), \] (4)
where
\[ A(x, y) = ||\pi||\delta(x - y) + ||\pi||^{-1}\pi(x)\pi(y), \]
\[ B(x, y) = ||\pi||^2\delta(x - y) + 3\pi(x)\pi(y). \] (5)

Here we have defined the norm of \( ||\pi|| \) as follows,
\[ ||\pi|| = \sqrt{\int dx\delta(x - y)\pi(x)\pi(y)}. \] (6)
This corresponds to taking the deformation for \( \pi(x) \)
\[ \pi(x) = (1 - \alpha ||\tilde{\pi}|| + 2\alpha^2 ||\tilde{\pi}||^2) \tilde{\pi}(x). \] (7)
where
\[ [\phi(x), \tilde{\pi}(y)] = i\delta(x - y). \] (8)
So, we have
\[ \tilde{\pi}(x) = -i \frac{\delta}{\delta\phi(x)}. \] (9)
Thus, we get
\[ \pi(x) = -i \left( 1 + \alpha \sqrt{-\int dzdy\delta(z - y) \frac{\delta}{\delta\phi(z)} \frac{\delta}{\delta\phi(y)}} - 2\alpha^2 \int dzdy\delta(z - y) \frac{\delta}{\delta\phi(z)} \frac{\delta}{\delta\phi(y)} \right) \frac{\delta}{\delta\phi(x)}. \] (10)
Thus, we observe that the deformation of the second quantized canonical commutation relation induces non-locality in the quantum field theory.

### 3 Deformed Wheeler-DeWitt Equation

In this section, we will deform the Wheeler-DeWitt equation and analyse its consequences. The line element in the Arnowitt-Deser-Misner 3 + 1 decomposition of general relativity is given by
\[ ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu = (-N^2 + N_i N^i) dt^2 + 2N_j dt dx^j + h_{ij} dx^i dx^j. \] (11)
where \( N \) is the shift function and \( N \) is the lapse function. Thus, the Lagrangian for gravity in the Arnowitt-Deser-Misner 3 + 1 decomposition of general relativity, can be written as

\[
\mathcal{L}[N, N_i, h_{ij}] = \sqrt{-g} R = \frac{N}{2\kappa} \left( K_{ij} K^{ij} - K^2 + \left( \frac{3}{2} R - 2\Lambda \right) \right),
\]

(12)

Here \( \Lambda \) is the cosmological constant, \( \frac{3}{2} R \) is the three dimensional scalar curvature, \( K_{ij} \) is the second fundamental form, and \( K = h^{ij} K_{ij} \) is the trace of the second fundamental form. The Hamiltonian is obtained via Legendre transformation,

\[
\tilde{H} = dx \left[ NH + N_i H^i \right],
\]

(13)

where

\[
H = (2\kappa) G_{ijkl}(\pi^j_{\pi^k_{\pi^l}} - \sqrt{\frac{\kappa}{h}} \left( \frac{3}{2} R - 2\Lambda \right)),
\]

(14)

\[
H^i = -2\nabla_j \pi^{ji},
\]

and \( \pi^{ij} \) is the momentum conjugate to \( h_{ij} \). Here \( G_{ijkl} \) is defined by

\[
G_{ijkl} = \frac{1}{2\kappa} (h_{ik} h_{jl} + h_{il} h_{jk} - h_{ij} h_{kl}).
\]

(15)

The two classical constraints \( H = 0, \) and \( H^i = 0, \) are obtained through the equation of motion. At the quantum level the constraints \( H = 0, \) becomes the Wheeler-DeWitt equation, \( \mathcal{H} \psi[h] = 0. \) Here \( \psi[h] \) is the wave function of the universe. We will now use the modified canonical commutation relation to obtain a deformed version of this Wheeler-DeWitt equation. The modified canonical commutation relation are

\[
[h_{ij}(x), \pi^{kl}(y)] = (\delta_i^k \delta_j^l + \delta_i^l \delta_j^k) \delta(x - y) + i\alpha \mathcal{A}(x, y) + i\alpha^2 \mathcal{B}(x - y),
\]

(16)

where

\[
\mathcal{A}(x, y) = ||\pi|| \delta(x - y) + ||\pi||^{-1} \pi(x) \pi(y),
\]

\[
\mathcal{B}(x, y) = ||\pi||^2 \delta(x - y) + 3\pi(x) \pi(y).
\]

(17)

Here we have defined the norm of \( ||\pi|| \) as follows,

\[
||\pi|| = \sqrt{\int dxdy \delta(x - y) G_{ijkl}(x, y) \pi^{ij}(x) \pi^{kl}(y)}.
\]

(18)

The momentum operator corresponding to this deformed canonical commutation relation is given by

\[
\pi_{ij}(x) = (1 - \alpha ||\pi|| + 2\alpha^2 ||\pi||^2) \tilde{\pi}_{ij}(x).
\]

(19)

where

\[
[h_{ij}(x), \tilde{\pi}^{kl}(y)] = i(\delta_i^k \delta_j^l + \delta_i^l \delta_j^k) \delta(x - y).
\]

(20)
So, we have
\[ \tilde{\pi}_{ij}(x) = -i \frac{\delta}{\delta h_{ij}(x)}. \]  
(21)

Thus, we get
\[
\pi_{ij}(x) = -i \left(1 + \alpha \right) \sqrt{-\int dxdy \delta(x-y)G_{ijkl}(x,y) \frac{\delta}{\delta h_{ij}(x)} \frac{\delta}{\delta h_{kl}(y)} \right) \times \frac{\delta}{\delta h_{ij}(x)}. 
\]  
(22)

Now we can write the deformed Wheeler-DeWitt equation
\[ H \psi[h] = 0 \]  
(23)

where
\[ H = -(2\kappa) G_{ijkl} D \frac{\delta}{\delta h_{ij}} D \frac{\delta}{\delta h_{kl}} = \frac{\sqrt{2}h}{2\kappa} \left(3R - 2\Lambda \right), \]  
(24)

and
\[
D = 1 + \alpha \sqrt{-\int dxdy \delta(x-y)G_{ijkl}(x,y) \frac{\delta}{\delta h_{ij}(x)} \frac{\delta}{\delta h_{kl}(y)} \right) \times \frac{\delta}{\delta h_{ij}(x)}. 
\]  
(25)

Now we consider a minisuperspace approximation to the Wheeler-DeWitt equation. So, will now consider a closed universe filled with a vacuum of constant energy density and the radiation, \( \rho(a) = \rho_v + \epsilon/a^4 \), where \( \rho_v \) is the vacuum energy density, \( a \) is the scale factor and \( \epsilon \) is a constant characterizing the amount of radiation. The Friedman-Robertson-Walker metric for \( k = 1 \) is given by
\[ ds^2 = -N^2 dt^2 + a^2(t) d\Omega_3^2, \]  
(26)

where \( d\Omega_3^2 \) is the usual line element on the three sphere. Thus, we obtain the following result,
\[ -\frac{3\pi}{4G} \dot{a}^2 - \frac{3\pi}{4G} a + 2\pi^2 a^3 \rho(a) = 0. \]  
(27)

If we assume \( 256 \pi^2 G^2 \rho_v \epsilon/9 < 1 \), then a big bang occurs at \( a = 0 \) and the universe expands to a maximum radius before tunneling into a phase of unbounded expansion. Now we can write the following Lagrangian for this system
\[ \mathcal{L} = -\frac{3\pi}{4G} \dot{a}^2 + \frac{3\pi a}{4G} - 2\pi^2 a^3 \rho(a). \]  
(28)

We can obtain a Hamiltonian from this Lagrangian as follows,
\[ H = -\frac{G}{3\pi} \frac{p^2}{a} - \frac{3\pi}{4G} a + 2\pi^2 a^3 \rho(a). \]  
(29)
Now we deform the momentum operator for this minisuperspace model as
\[ p = -i \left( 1 + i\alpha \frac{d}{da} - 2\alpha^2 \frac{d^2}{d^2a} \right) \frac{d}{da}. \]  
(30)

The Wheeler-DeWitt equation corresponding to this representation of the momentum is given by
\[ \mathcal{H}\psi(a) = 0, \]  
(31)

where
\[ \mathcal{H} = \left( \frac{G}{3\pi} \frac{d^2\psi}{dx^2} - 2\alpha^2 \frac{G}{3\pi} \frac{d^3\psi}{dx^3} + 5\alpha^2 \frac{G}{3\pi} \frac{d^4\psi}{dx^4} + \frac{3\pi}{4G} a^2 + 2\pi^2 a^4 \rho(a) \right). \]  
(32)

This modified quantization is consistent with the following
\[ \Delta a \Delta p = 1 - 2\alpha < p > + 4\alpha^2 < p^2 >. \]  
(33)

These imply the existence of a minimum length \( \Delta a \geq \Delta a_{\text{min}} \). It also implies the existence of a maximum momentum \( \Delta p \leq \Delta p_{\text{max}} \). Thus, in the radius of the universe according to this modified Wheeler-DeWitt equation cannot shrink to zero. This way we can avoid the big bang singularity using this modified quantization.

4 Conclusion

In this paper we generalized the deformation of first quantized commutating relations to second quantized commutating relations. Then we analysed the Wheeler-DeWitt equation using this formalism and demonstrated that in this formalism the big bang singularity is naturally avoided. It may be noted that various other boundary conditions have been used for obtaining the wave function of the universe. The wave function of the universe has also been constructed using a quantum tunneling transition \[ \text{[13]-[14].} \] In fact, it is possible for a baby universe to be created by a quantum fluctuation of the vacuum. This universe can eventually jump into an inflationary period and undergo a period of rapid expansion till its Hubble length becomes very large. This way a universe can be created by a quantum fluctuation of the vacuum. It will be interesting to study this mechanism for the creation of the universe using the modified Wheeler-DeWitt equation.

It may be noted that if a single universe can be created from the quantum fluctuation of the vacuum, there is no reason why other universes cannot be similarly created. Thus, this model naturally predicts the existence of the multiverse. In fact, this model of inflation is called the chaotic inflationary multiverse \[ \text{[45].} \] In this model, the total number of distinguishable locally Friedman universes generated by eternal inflation is proportional to the exponent of the entropy of inflationary perturbations \[ \text{[46].} \] The multiverse also appears naturally in the Multiverse can also be used to explain the landscape in the string theory \[ \text{[47].} \] This is because all the \( 10^{500} \) different string theory vacuum states \[ \text{[48],} \] can be viewed as real vacuum states of different universes \[ \text{[49].} \] The multiverse is most naturally analysed using a third quantized formalism of quantum
This is because the Wheeler-DeWitt equation can be viewed as the Schroedinger’s equation for a single universe. Just as a single particle wave equation has to be second quantized to account for the creation and annihilation of particles, the Wheeler-DeWitt equation has to be third quantized to account for the creation and annihilation of universes. It will be interesting to analyse the third quantization of this deformed Wheeler-DeWitt equation.

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