Fidelity and Reversibility in the Three Body Problem

We present two methods to analyze the global effects of a small perturbation in a non-integrable Hamiltonian system, choosing as a paradigmatic example the restricted planar three body problem and focusing on its Poincaré map for the Jacobi invariant. The cumulative effects on the orbit of random or round-off errors leads to a divergence of the perturbed orbit from the exact one. Rather than computing the distance of the perturbed orbit from the reference one after a given number \( n \) of time steps, we measure the distance of the reversed orbit (\( n \) time steps forwards and backwards) from the initial point. This approach does not require the knowledge of the unperturbed map. The asymptotic equivalence of the Reversibility Error Method (REM) with the forward error is proved for noisy linear maps, and it is shown to characterize the phase space stability of the perturbed map just as the Lyapunov Characteristic Exponent. A second indicator of chaos, the Cumulative Orbital Elements (COE) method is also presented. The loss of memory of the perturbed map is quantified by the Fidelity and its decay rate. It is found that Fidelity behaves in a different way for randomly perturbed regular and for chaotic orbits. This property, already known for one-dimensional maps, is confirmed for the considered planar three body problem suggesting a possible validity for generic hyperbolic systems.

When dealing with the study of quasi integrable or non-integrable systems (such as the planar restricted three body problem: R-3bp) one has to rely on numerical integration, except when the canonical perturbation theory can be used. Therefore reliability of the adopted numerical methods is of fundamental importance. Usually fourth or higher order integration schemes are adopted. The use of high accuracy schemes with multiple precision is however computationally very expensive. In several applications, it is of fundamental importance to quantify the behavior of orbits of a given dynamical system when perturbed, for example under the action of random noise or just as a consequence of round-off errors. Ideally, one would compare the perturbed orbit with the exact one (for example in order to determine the Lyapunov spectrum). In this paper we propose to compute the Reversibility Error Method (REM) which consists in evaluating the distance from the initial point of the orbit after a prescribed number of time-steps forward and backward. Even though the asymptotic equivalence of REM with the forward error is only proved for linear symplectic maps, there is strong numerical evidence that is holds much more generally. In the case of the R-3bp we have computed the REM in the Cartesian phase space of position \( x \) and momentum \( \dot{x} \) for a fixed value of the Jacobi constant. The REM maps shows a striking resemblance with the maximum Lyapunov Characteristic Exponent (mLCE) maps computed in the standard way with the forward method. The REM computation is faster and does not require any renormalization procedure or extrapolation. The error on the Cumulative Orbital Elements (COE) using as diagnostics the semi-major axis or eccentricity shows a similar behavior and is also computationally fast. On regular orbits the error remains close to the round-off one, whereas for chaotic orbits it saturates rapidly to one (if normalized to the invariant set diameter). The loss of memory of the initial conditions is measured by the correlations decay which is exponential for chaotic orbits and absent or a power law for regular orbits. When the map is perturbed the perturbation induces a memory loss and the study of this memory loss is provided by the Fidelity decay. The Fidelity is the average of the product of a scalar function evaluated on the unperturbed and perturbed orbit minus the product of averages. For random perturbations of amplitude \( \epsilon \) the decay is exponential in the dynamical steps \((n)\) with decay rate \( \epsilon^2 \) for regular orbits, whereas, for chaotic orbits the Fidelity has a plateau ex-

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tending up to $n_* \sim -\log \epsilon$ followed by an exponential decay. For round-off perturbations of amplitude $\epsilon$, the Fidelity decay is the same as the correlation for regular unperturbed orbits, whereas for chaotic orbits it behaves as for random perturbation. These results, rigorously proved for simple maps and numerically verified for generic maps, is nicely confirmed for the R-3bp. The only limit of the Fidelity analysis is the computational cost of the Monte Carlo sampling for the invariant set since a sampling with $M$ points leads to a $M^{-1/2}$ accuracy. Due to the complementary nature of REM and Fidelity, the first one can be used to explore the entire phase space, whereas the second applied to a few selected orbits. The method is suitable to be extended to the N-body problem and might be relevant for the stability analysis of astrophysical dynamical system such as the planet-planet interaction in our Solar System, or stellar orbits in Elliptical galaxies, or the orbits of star in globular clusters, or finally to the problem of binary Black Holes at the center of galaxies.

I. INTRODUCTION

The dynamic of non-integrable Hamiltonian systems is qualitatively well understood when it reduces to an area preserving map on the Poincaré section. This is the case of the three body problem. However the orbits for generic initial conditions can only be obtained by numerical integration $^{18,19}$. The symplectic integration schemes, preserve the Poincaré invariants $^{20,21}$ but are affected by a local discretization error and by a round-off errors $^{22,23}$ and reference therein for a recent review on the topic). Therefore, it is of extreme importance to be able to quantify the divergence of the numerically integrated orbit with respect to the exact one. If we refer to the perturbed orbit as the one computed with the intrinsic limitations due to numerical precision, the divergence of this from the exact one with same initial conditions can be measured and its asymptotic behavior is characterized by the maximum Lyapunov Characteristic Exponent. Since a symplectic map is reversible, the divergence due to the perturbations can be measured by computing the distance from the initial point after a forward and backward integration of the perturbed map for the same number of times steps. This method is called reversibility test $^{24}$. It is routinely used to test the regularization method for close encounter between two massive object $^{25}$ or in the electromagnetic problems $^{26}$ or to study the collisions in the few body gravitational problem $^{27}$. The REM we propose here is easily applicable since it does not require the computation of the exact map and its asymptotic properties are expected to be the same as for the forward error. This can be analytically proved in the case of a randomly perturbed linear map. Here however we use this method in a different way: not to check the goodness of a regularization method for a N-body integrator but as a new dynamical indicator to investigate the stability of the orbits. The loss of memory of the perturbed orbit is measured by another numerical indicator called the Fidelity. This is defined as the average of the product of an observable computed along the reference and perturbed orbits minus the product of the averages.

The main body of the paper is dedicated to show how the proposed dynamical indicators allow to investigate the effects of small (random or round-off) perturbations in a physically significant model such as the planar circular three body problem extending and confirming the main results already obtained in prototype dynamical models. The three body problem is the paradigm of systems in which both regular and chaotic orbit coexist in a very narrow region of the phase space and can provide a significant insight in astrophysical relevant problems (see reference $^{28}$ for a detailed discussion about the three body problem and its application in astrophysics). Even though in the neighborhood of the equilibria or periodic points in the rotating system the Birkhoff normal forms can be used to approximate the quasi-integrable dynamics with an integrable one and to obtain Nekhoroshev stability estimates, the numerical integration procedure cannot be avoided to explore the whole phase space in the Poincaré section. The tools here proposed to investigate the effect of random and/or round-off errors become essential in order to evaluate the reliability of the numerical results and to have an overall picture of the dynamic behavior in the whole phase space for any chosen value of the Jacobi invariant.

The paper is organized as follows. In section 2 we introduce the notations and the Hamiltonian for the three body problem in the fixed and rotating frame. In section 3 we analyze the error on the first integral for the forth order symplectic and Runge-Kutta integrators. In section 4 we prove analytically, for a linear map with random perturbations, the asymptotic correspondence between the Reversibility Error and Forward Error. Then, in section 5 we compare the Reversibility Error Method (REM) map with the maximum Lyapunov Characteristic Exponent (mLCE) map on a grid of points in the Poincaré section of the Jacobi invariant. We introduce the Cumulative Orbital Error (COE) and compute the corresponding map in section 6. As a concrete application, in section 7 we apply the REM method to the study of stability for prograde and retrograde motion in the R-3bp. In section 8, finally, we analyze the Fidelity for randomly perturbed regular and chaotic orbits. The decay laws are found to be compatible with the results proved for simple prototype maps. Conclusions and perspectives for future works are presented.
II. THREE BODY HAMILTONIAN

We consider the restricted, circular, planar three body problem \(^{(2)}\). There is a primary central body of mass \(m_1\), a secondary body of mass \(m_2 \leq m_1\) describing circular orbits around their center of mass, where we choose the origin of the reference system, and a third body of mass \(m_3\), so small that it does not perturb the motion of the first two bodies. Denoting with \(G\) the gravitational constant, with \(T_\ast\) the period of the second body and with \(r_\ast\) its distance from the first one, the third Kepler law and the energy of the system read

\[
\frac{T_\ast^2}{r_\ast^3} = \frac{4\pi^2}{G(m_1 + m_2)} \quad E_\ast = -\frac{m_1 m_2 G}{r_\ast}. \tag{1}
\]

We scale the space-time coordinates \(t\) and \(r\) and the Lagrangian \(L\) according to

\[
t' = 2\pi \frac{t}{T_\ast} \quad r' = \frac{r}{r_\ast} \quad L' = \frac{r_\ast}{G m_1 m_2} L. \tag{2}
\]

We denote the third body coordinates in the fixed frame with \((x_F, y_F)\) and in the corotating frame with \((x, y)\). Assuming the rotation is anti-clockwise, the relation between the coordinates is given by

\[
\begin{align*}
    r &= \begin{pmatrix} x_F \\ y_F \end{pmatrix} = R(-t') \begin{pmatrix} x \\ y \end{pmatrix} \\
    R(t) &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.
\end{align*} \tag{3}
\]

The Lagrangian of the third body in the scaled coordinates becomes

\[
L'_F = m_3 \frac{m_1 + m_2}{m_1 m_2} \times \left[ \frac{1}{2} \left( \frac{dx'}{dt'} \right)^2 + \frac{m_1}{m_1 + m_2} \frac{1}{r_1'} + \frac{m_2}{m_1 + m_2} \frac{1}{r_2'} \right] \tag{4}
\]

where \(r_1'\) and \(r_2'\) denote the distances of the third body in the fixed frame and the second body. Ignoring the constant factor in front of the Lagrangian, which does not affect the equations of motion, and dropping the primes in the scaled variables, the corresponding Hamiltonian in the extended phase space of the fixed frame reads

\[
H_F = \frac{p_{x_F}^2 + p_{y_F}^2}{2} + p_x + V_F(x_F, y_F, \tau)
\]

\[
V_F(x_F, y_F, \tau) = -\frac{1 - \mu}{r_1} - \frac{\mu}{r_2}
\]

\[
\mu = \frac{m_2}{m_1 + m_2}.
\]

In the comoving frame the first and second body are on the \(x\) axis with coordinates \(x_1c = -\mu\) and \(x_2c = 1 - \mu\). The distances \(r_1, r_2\) of the third body from the first two is given by

\[
r_1 = [(x_F + \mu \cos \tau)^2 + (y_F + \mu \sin \tau)^2]^{1/2}
\]

\[
r_2 = [(x_F - (1 - \mu) \cos \tau)^2 + (y_F - (1 - \mu) \sin \tau)^2]^{1/2}. \tag{6}
\]

In the rotating frame the Hamiltonian is given by

\[
H = \frac{p_x^2 + p_y^2}{2} + y F_x - x p_y + V(x, y)
\]

\[
V(x, y) = -\frac{1 - \mu}{r_1} - \frac{\mu}{r_2},
\]

where the distances are now expressed by

\[
r_1 = [(x + \mu)^2 + y^2]^{1/2}
\]

\[
r_2 = [(x - (1 - \mu)^2 + y^2]^{1/2}. \tag{8}
\]

The Hamiltonian is conserved \(H = E = -J/2\) where \(E\) is the energy and \(J\) the Jacobi integral. As a first integral \(H\) is the sum of the kinetic energy plus the effective potential, given by the sum of the gravitational and the centrifugal potentials

\[
H = \frac{\dot{x}^2 + \dot{y}^2}{2} + V_{eff}(x, y) \tag{9}
\]

\[
V_{eff}(x, y) = V(x, y) - \frac{x^2 + y^2}{2}.
\]

The equilibrium points \(L_4\) and \(L_5\) (maxima of \(V_{eff}\)) are at the vertices of the equilateral triangle \(x_c = 1/2 - \mu\) and \(y_F = \pm \sqrt{3}/2\). The value of the Jacobi integral on these points is \(J_c = -2E_c = V_{eff}(x_c, y_c) = 3 - \mu + \mu^2\). Recalling that \(p_{xc} = -y_c\) and \(p_{yc} = x_c\) we translate the phase space critical point into the origin: \(x' = x - x_c, y' = y - y_c, p'_x = p_x - p_{xc}, p'_y = p_y - p_{yc}\). The quadratic approximation of the Hamiltonian becomes

\[
H = E_c + \frac{p_{x'}^2 + p_{y'}^2}{2} + y'p'x - x'p'_y + \frac{1}{8}x'^2 - \frac{5}{8}y'^2 - \alpha x'y'
\]

\[
\alpha = \frac{3\sqrt{3}}{4}(1 - 2\mu). \tag{10}
\]

Since the equilibrium point in the phase space is a saddle points, the Lyapunov theorem does not apply and the KAM theory is required to prove its stability. Indeed the eigenvalues are

\[
\omega^2_{1,2} = \frac{1}{2} \left[ \left( 1 - 2\mu(1 - \mu) \right) \right]^{1/2} \tag{11}
\]
and after a linear symplectic transformation to the normal phase space coordinates \(X, Y, P_x, P_y\) the Hamiltonian reads

\[
H = \frac{\omega_1 P_x^2 + X^2}{2} + \frac{\omega_2 P_y^2 + Y^2}{2},
\]

(12)

where the sign of \(\omega_1\) is positive whereas the sign of \(\omega_2\) is negative. The higher order expansion of \(H\) in normal coordinates form can be computed though it has an asymptotic character. As a consequence the phase space characterization of the full 3-D manifold for any value of \(J\), even close to \(J_c\), has to be based on a numerical evaluation of the orbits.

The analysis we perform refers to a 3-D manifold \((\mathcal{M}_R)\) in four-dimensional phase space given by a chosen value of the Jacobi constant \((J)\). We choose \(\mu = 0.000954\) which corresponds, approximately, to the Sun-Jupiter-asteroid three body problem and fix the Jacobi constant \(J = 3.07\) close to the value \(J_c\) assumed at the Lagrangian points \(L_4, L_5\) (12). We consider the 2-D sub-manifold \(\mathcal{M}_R\) obtained by intersecting \(\mathcal{M}_J\) with the half hyperplanes \(y = 0\) and \(\dot{y} > 0\). The equations defining \(\mathcal{M}_R\) are given by \(\dot{y}^2 \geq 0\) and \(H(x, y, \dot{x}, \dot{y}) = -J/2\). This implies the follow inequality:

\[
x^2 \leq x^2 + 2 \frac{1 - \mu}{|x + \mu|} + 2 \frac{\mu}{|x - 1 + \mu|} - J_c
\]

(13)

The initial conditions are chosen in this manifold and we examine the intersection of the orbits with this manifold by projecting it on the \(x, \dot{x}\) plane or \(x, p_x\) phase plane. For each orbit the initial conditions are \(x(0) = x_0, y(0) = 0\) and \(\dot{x}(0) = \dot{x}_0\) chosen so that the inequality is satisfied. The missing initial condition \(\dot{y}(0)\) is given by

\[
\dot{y}(0) = \left(\frac{x_0^2 - \dot{x}_0^2 + 2 \frac{1 - \mu}{|x_0 + \mu|} + 2 \frac{\mu}{|x_0 - 1 + \mu|} - J_c}{1/2}\right).
\]

(14)

Because the evolution of the orbits is computed by the symplectic integrator in the fixed frame, at \(t = 0\), we need to transform the initial conditions from the rotating frame to the fixed frame. Whenever needed we transform back the coordinates and velocities from the rotating frame to the fixed frame according to

\[
\begin{pmatrix}
  x_F \\
  y_F \\
  \dot{x}_F \\
  \dot{y}_F
\end{pmatrix} =
\begin{pmatrix}
  R(-t) & 0 \\
  0 & R(-t)
\end{pmatrix}
\begin{pmatrix}
  x \\
  y \\
  \dot{x} \\
  \dot{y}
\end{pmatrix},
\]

(15)

For brevity we denote the previous transformations as

\[
x_F(t) = R(-t)x(t) \quad x(t) = R(t)x_F(t)
\]

(16)

We have chosen to integrate Hamilton’s equations for \(H_F\) using the fourth order symplectic integrator even though higher order integrators are available (15). The local error in this case is \((\Delta t/T)^4\) where \(T = 2\pi\) is the dynamical period. In our computation we typically use \(\Delta t = 10^{-3}T\). For \(\Delta t = 10^{-4}T\) the local error becomes comparable with the round-off error as can be checked by computing the relative error on the Jacobi integral (see the next section).

III. THE NUMERICAL INTEGRATOR

We use a fourth-order symplectic scheme to integrate the equations of motion in the fixed frame and check its accuracy by evaluating the error in the conservation of the Hamiltonian \(H_F\) in the fixed frame, see equation (13), or the Hamiltonian \(H\) in the rotating frame, see equation (9). Since we give our initial conditions in the rotating reference frame and compute in this frame the Poincaré section of the \(H = E\) manifold, a transformation from the fixed to the rotating frame and vice versa is performed whenever it is necessary. In figure 1 is shown the variation \(\Delta H/H\) as function of the number of integration steps \(n_s\) per period. The relative error decreases as \(\Delta H/H \sim n_s^{-4}\) until the saturation when the machine accuracy is reached. We may consider the results obtained with quadruple machine precision as “exact” due to possibility of lowering the relative error orders of magnitude below the accuracy of double precision, by choosing the appropriate time step.

The REM consists in computing the distance from the initial point \(x_0\) in the phase space of the orbits obtained iterating forward the symplectic integrator map up to time \(t = k\Delta t = Tk/n_s\) and backward for the same number of steps. Letting \(x_0\) be the initial point in the rotating frame and \(R(t) = R(k\Delta t)\) the transformation defined by equations (13) (15) the REM error is given by

\[
\Delta_{\text{REM}}(x_0, t) = ||R(t)M_{\Delta t}^k \circ M_{\Delta t}^{-k}(R^{-1}(0)x_0) - x_0||
\]

By \(M_{\Delta t}^k\) we denote the symplectic integrator map \(M_{\Delta t}\) computed with a finite accuracy introduced the round-off error. The exact map is reversible so that \(M_{-\Delta t}M_{\Delta t} = I\) whereas the finite accuracy map is not \(M_{-\Delta t}^kM_{\Delta t}^k \neq I\). The reversibility error \(\Delta_{\text{REM}}(x_0, t)\) is different from zero and grows with time due to round-off error and it accumulate during the iteration of the map. A random error has a similar effect since the perturbed map is no longer reversible. The forward error defined as

\[
\Delta(x_0, t) =
\|
R(t)M_{-\Delta t}^k(R^{-1}(0)x_0) - R(t)M_{\Delta t}^k(R^{-1}(0)x_0)x_0
\|
\]
FIG. 1. Error in the Hamiltonian value. Relative variation \( \Delta H/H \) for the rotating frame Hamiltonian in one period \( T \), changing the number of integration steps \( n_s \). The results for different machine precision are compared: single (pink square), double (green cross), extended (blue asterisks) and quadruple precision (red cross). The integrations are based on a 4-th order symplectic scheme with time step \( \Delta t = T/n_s \). The error saturates when the machine accuracy is reached. The black line is the linear fit in the log-log plane. The initial conditions are for a regular orbit, with \( x = 0.55, y = 0, \dot{x} = 0 \) and \( \dot{y} \) defined by the value of the Jacobi constant set up equal to \( J = 3.07 \).

has a similar behavior and its asymptotic equivalence for linear maps with a random perturbation will be proved in the next section. The behavior of the forward error reference map and the REM error for regular and chaotic orbits is shown in figure 2. Unlike the REM it requires the computation of the “exact” map which in our case is considered to be the one computed using quadruple precision. The two plots are presented in the lin-log scale to show that the error growth in both cases is linear for the regular orbit and exponential for the chaotic one.

Assuming the REM error is at least asymptotically the same as for the forward error, the former can be used to explore the dynamical effects due to the round-off error or small random perturbations. In both cases we observe that the error growth with \( t \) is linear for regular orbits and exponential for chaotic orbits.

Finally we compare our symplectic fourth-order integrator with a fourth-order Runge-Kutta scheme used to integrate the equations in the rotating frame. In figure 3, the relative error \( \Delta H/H \) is plotted. As expected the error for Runge-Kutta present a slowly growth with time whereas it is constant in average for the symplectic scheme, both for regular and chaotic orbits.

FIG. 2. Forward error and REM distance for regular and chaotic orbits. Top panel: Growth with time of the distance \( \Delta(x_0, t) \) for chaotic and regular orbits. Bottom panel: Growth with time of the distance between the initial and final point after a forward and backward integration for chaotic and regular orbits. Initial conditions in the rotating frame Jacobi manifold \( J = 3.07: x_0 = 0.56, \dot{x}_0 = 0, y_0 = 0 \) (chaotic orbit) and \( x_0 = 0.55, \dot{x}_0 = 0, y_0 = 0 \) (regular orbit). The value of \( y_0 \) is calculated according to equation (14).

IV. THEORETICAL REVERSIBILITY ERROR CALCULATION

Let us consider an initial point \( x_0 = (x_0, \dot{x}_0, 0, \dot{y}_0) \) on the two dimensional manifold \( M_P = M_J \times M_y=0, \dot{y}>0 \) intersection of the Jacobi manifold (defined by an assigned value of the Jacobi first integral \( J \)) and the plane \( y = 0 \) with the condition \( \dot{y} > 0 \). Consider the sequence of points \( x(nT) \in M_J \) with initial point \( x(0) = x_0 M_P \). This orbit is approximated by

\[
x_n = R(T)M^n(\mathbf{R}^{-1}(0)x_0)
\]

where \( M = M_{\Delta t}^n \) and \( M_{\Delta t} \) is the symplectic integrator in the rotating frame with time step \( \Delta t = T/n_s \). Since the Hamiltonian flow generated by \( H \) is reversible as well as the symplectic map \( M_{\Delta t} \), namely \( M_{\Delta t}^{-1} = M_{-\Delta t} \), it fol-
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\[ \xi_0 = x_0 - x, \]
\[ \xi_1 = M_\epsilon(x_\epsilon) - M(x_\epsilon), \]
\[ \ldots \]
\[ \xi_n = M_\epsilon(x_{\epsilon,n-1}) - M(x_{\epsilon,n-1}). \]

The global errors \( \xi_n \) are defined by

\[ \xi_0 = \xi_0, \]
\[ \xi_1 = M_\epsilon(x_\epsilon) - M(x), \]
\[ \ldots \]
\[ \xi_n = M^n_\epsilon(x_{\epsilon}) - M^n(x). \]

The local errors due to round-off are correlated. In the case of a stochastic perturbation the \( \xi_n \) are independent random variables with zero mean and unit variance so that the perturbed orbit is defined by the recurrence relation:

\[ x_{\epsilon,n} = M_\epsilon(x_{\epsilon,n-1}) = M(x_{\epsilon,n-1}) + \xi_n \]

The global error is the separation of the perturbed orbit from the reference one and can be expressed as

\[ \xi_n = x_{\epsilon,n} - x_n = M^n_\epsilon(x_\epsilon) - M^n(x) = \]
\[ = M^n_\epsilon(x_\epsilon) - M^n(x_\epsilon) + [M^n_\epsilon(x_\epsilon) - M^n(x)]. \]

where the last term within brackets vanishes if the initial point is the same. In this case the global error can be approximated as

\[ \xi_n = M_\epsilon(x_{\epsilon,n-1}) - M(x_{n-1}) = \]
\[ = M_\epsilon(x_{\epsilon,n-1}) - M(x_{\epsilon,n-1}) + M(x_{\epsilon,n-1}) - M(x_{n-1}) = \]
\[ = \xi_n + \xi M(x_{n-1})\Xi_{n-1} + O(\epsilon^2) = \]
\[ = \xi_n + \xi M(x_{n-1})\Xi_{n-1} + \]
\[ + \xi M(x_{n-2})\Xi_{n-2} + O(\epsilon^2). \]

Where \( DM \) is the tangent map, namely

\[ (DM)_{i,j} = \partial M_i/\partial x_j. \]

Recalling that \( DM^2(x_{n-2}) = DM(x_{n-1})DM(x_{n-2}) \) the final result for a nonzero initial error reads

\[ \xi_n = \xi \sum_{k=1}^{n} DM^{n-k}(x_k)\xi_k + M^n_\epsilon(x_\epsilon) - M^n_\epsilon(x) + O(\epsilon^2) = \]
\[ = \xi \sum_{k=0}^{n} DM^{n-k}(x_k)\xi_k + O(\epsilon^2). \]

If the initial error is zero than the sum start from \( k = 1 \).

We consider now the global error \( \xi_n^{[R]} \) with respect to the initial point when the map \( M_\epsilon \) is iterated forward \( n \)
and backward \( n \) times, denoting with \( \epsilon \xi_{-k} \) the local error at the step \( k \). We start with the relation which can be proved by induction in \( m \leq n \)

\[
M_{\epsilon}^{-m} \circ M_{\epsilon}^{n}(x) = M^{n-m}(x) + \epsilon DM^{-m}(x_n)\Xi_n + \epsilon \sum_{k=1}^{m} DM^{-(m-k)}(x_{n-k})\xi_{-k} + O(\epsilon^2),
\]

where \( x_j = M^j(x) \). Setting \( m = n \) in the previous relation we obtain

\[
\epsilon \Xi^{(R)}_n = M_{\epsilon}^{-n} M_{\epsilon}^n(x) - x = \epsilon DM^{-n}(\xi_n)\Xi_n + \epsilon \sum_{k=1}^{n} DM^{-(n-k)}(x_{n-k})\xi_{-k} + O(\epsilon^2).
\]

We claim that the growth of \( d_n \) is comparable. This would be the case if \( A = DM \) is a constant symplectic matrix indeed in this case we have:

\[
\left\langle (d_n^{(R)})^2 \rightangle = \epsilon^2 \sum_{k=1}^{n} \text{Tr} \left[ DM^{-n}(x_n)DM^{n-k}(x_k) (DM^{-n}(x_n)DM^{n-k}(x_k))^T \right] + \epsilon^2 \sum_{k=1}^{n} \text{Tr} \left[ DM^{-(n-k)}(x_{n-k}) (DM^{-(n-k)}(x_{n-k})^T \right] + \epsilon^2 \sum_{k=1}^{n} \text{Tr} \left[ DM^{-(n-k)}(x_{n-k}) (DM^{-(n-k)}(x_{n-k})^T \right] + \epsilon^2 \sum_{k=1}^{n} \text{Tr} \left[ DM^{-(n-k)}(x_{n-k}) (DM^{-(n-k)}(x_{n-k})^T \right]
\]

where \( \text{mLCE} = \epsilon \Xi^{(R)}_n \) is zero then \( d_n \sim \epsilon n \).

We expect that this result is valid also when the map \( M \) is not linear and that this can be proved for uniformly hyperbolic Hamiltonian systems. We have numerical evidence of the asymptotic equivalence of the forward error and reversibility error for the R-3bp.

\V. POINCARÉ SECTION AND COMPARISON OF REM MAPS WITH LYAPUNOV MAPS

The Poincaré map is useful to visualize the phase space orbits by taking the intersection of the constant energy Jacobi manifold \( M_J \) with the half hyperplane \( y = 0, y > 0 \) and projecting them on the \( (x, \hat{x}) \) plane for visualization. Such a map is useful to investigate the stability and has been used, for instance, in the case of asteroids with low eccentric orbit\( \text{[6]} \) (see also\( \text{[13]} \) and\( \text{[33]} \) for a detailed description and application of the method), showing the existence of chaotic islands surrounded by KAM tori. The orbits in the \( (x, \hat{x}) \) plane belong to a region whose boundary is defined by equation \( \text{[13]} \). The orbits portrait in the \( (x, \hat{x}) \) plane is not symmetric with respect to the origin because of the presence of the second massive body. Resonances appear and their overlap creates zones of irregular motion. The numerical procedure is the following: let \( t_k = k\Delta t \) we consider first the time interval \([t_k, t_{k+1}]\) where the intersection with the \( y = 0 \) occurs and the condition \( y > 0 \) is satisfied.
\[ y(t_k) \cdot y(t_{k+1}) < 0 \]
\[ \dot{y}(t_k) > 0. \]  

This intersection condition is then refined by interpolation. Letting \( y_{\text{int}}(t) \) be the linear interpolation we determine \( \tau \) such that \( y_{\text{int}}(\tau) = 0 \) and subsequently we compute \( x, \dot{x} \) at \( t = \tau \) according to

\[
\begin{align*}
  x_{\text{int}} &= x(t_k) + (x(t_{k+1}) - x(t_k)) \tau \\
  \dot{x}_{\text{int}} &= x(t_k) + (\dot{x}(t_{k+1}) - \dot{x}(t_k)) \tau
\end{align*}
\]

where \( \tau = t_k + \Delta t \frac{y(t_k)}{[y(t_{k+1}) - y(t_k)]} \). The sequence of points so far obtained is then visualized by plotting them on the \((x, \dot{x})\) plane as shown by figure 4. The projection of the Poincaré section manifold \( M_P \) in the \((x, \dot{x})\) is a domain whose boundary is the red curve shown in figure 4. In the same figures we observe the presence of closed orbits, resonant orbits (chains of islands) and chaotic orbits, which cover regions of finite area.

For a regular grid of points in the allowed region of the \((x, \dot{x})\) plane we compute the reversibility error \( d_n(x_0, \dot{x}_0) \) corresponding to the forward and backward evolution after \( n = 1000 \) periods. First we observe that the error \( d_1(x_0, \dot{x}_0) \) for one dynamic time is almost independent on the time step \( \Delta t = T/n_s \) namely on the number of integration steps \( n_s \). The distance \( d_1 \) is of the order of the round-off error as shown in figure 5, where the results for double and quadruple precision are presented. There is no significant difference for a regular and chaotic orbit. The difference appears when \( n \) is increased because the distance grows linearly in \( n \) for a regular orbit
\[
d_n \sim \epsilon n
\]
and exponentially for a chaotic orbit
\[
d_n \sim \epsilon e^{\lambda n T}
\]

where \( \lambda \) is the maximum Lyapunov exponent. This is shown in figure 6 where we compute \( d_n \) for \( n \leq 1200 \) for the same regular and chaotic orbits using double and quadruple precision.

From figure 6 we see that \( d_n \) is proportional to the accuracy and that the law of growth with respect to \( n \) does not change when the accuracy changes. We have computed the reversibility error \( d_n(x_0, \dot{x}_0) \) for a fixed \( n \) on a regular grid within the allowed region. The error varies drastically from the region of regular motion where it is close to the machine accuracy, to the chaotic regions where it is of order 1. In figure 7 the reversibility error map is shown and the comparison with figure 6 shows an excellent correspondence with the precedent maps reproducing the different regions of motion. The behavior of the reversibility error with respect to \( n \) and \( \epsilon \) suggest that for a given accuracy and a fixed value of \( n \) the plot of \( d_n \) for a grid of initial conditions in the \((x, \dot{x})\) plane should provide a picture similar to the one obtained with the mLCE. A similar behavior is found for random perturbations of the same amplitude \( \epsilon \). Compatible and complementary results are obtained from the Fidelity analysis we present in section 7.

The divergence of two orbits with very close initial condi-
FIG. 5. Comparison of the distance after a period using different machine precision and varying the integration steps. The distance value is computed for a chaotic orbit (top panel) changing the number of integration step for a single orbit with double (red) and quadruple (green) precision. Is it possible to see how the distance do not change nor saturate changing the precision. The same for regular orbit (bottom panel).

FIG. 6. Evolution of the distance in time for chaotic and regular orbits changing the machine precision. Top panel show the evolution of the distance is computed for a chaotic (red) and regular (green) orbit with double machine precision. Bottom panel show the same evolution for chaotic (green) and regular (red) orbit using quadruple precision.

In figure 7 we present a color plot of the mLCE for a grid of initial points in two dimensional phase space \((x, \dot{x})\) with \(y = 0\) and the Jacobi integral set to \(J = 3.07\). The mLCE map is very similar to the REM one shown in figure 8. This suggest that the equivalence result proved in the case of linear Poincaré maps very likely holds also for the restricted three body problem R-3bp. In the next section we introduce the cumulative orbital error (COE) whose plot closely resembles the mLCE and REM plots. The results for REM and mLCE do not vary appreciably if 1000 periods rather than 100 periods are used. This holds also for COE which produces the same results if the eccentricity or the semi-major axis are chosen as orbital parameters. Due to the short integration time needed and to the very small \(\epsilon\) parameter, the REM method is faster than mLCE that is based on the shadow particle approach to detect local regions of chaotic and regular motion. The COE method described in the next section is also fast and easy to implement.
Reversibility Error Map for corotating space and prograde orbits. The integration with the symplectic 4th order integrator is computed here to show how the presence of both chaotic and regular orbits near one to another are spotted with the REM method. A zoom (bottom panel) to a particular region of the total phase space is also given to see how regular and chaotic regions overlap together forming a complex web of dynamical stable and unstable orbits.

The advantage of REM is the possibility to avoid the knowledge of the exact orbit. As a consequence of the exact Poincaré section and therefore it is ideally suited to investigate the effect of a small random perturbation or the effect of round-off errors. Moreover REM is a method which do not require any shadow particle nor the integration of the displacement equations.

VI. CUMULATIVE ORBITAL ELEMENT

Another method to detect chaos applied for the first time here for the R-3bp, is based on the different evolution in time of the orbital parameters (e.g.: semi-major axis, eccentricity, mean motion ...). As the Lyapunov Characteristic Exponent, this method is based on the exponential divergence of orbits for two very close initial conditions [21,15] and reference therein). As the time in-
FIG. 9. Lyapunov map for the $x - \dot{x}$ projection of the phase space. In this particular case the mLCE for each of the $10^4$ orbits have been calculated. The result is a density map in which chaotic and regular orbits are well separated and co-exist in a very small portion of the total phase space. The chaotic orbits are due to the overlap of resonances.

crease, if the local region of the phase space in which the motion occur is chaotic, also the difference between the orbital parameters of the two nearby orbits increase.

We define the Cumulative Orbital Element Error method for the eccentricity $COE_{(e)}(n)$ as the sum of errors at each time step until time $t = k \Delta t$. Choosing $t$ as a multiple of the period $t = nT = n \Delta t$ we have

$$COE_{(e)}(n) = \sum_{j=0}^{n,n_e} |e_j \Delta t(x_0 + \epsilon) - e_j \Delta t(x_0)|,$$

The initial conditions are chosen on the manifold $M_P$. Having fixed the value of the Jacobi constant $J = 3.07$ we chose $x_0, \dot{x}_0 = 0$ for the first orbit and $x_0 + \epsilon, \dot{x}_0 = 0$ for the second orbit. The eccentricity up to $t = 200T$ for two close orbits with $\epsilon = 10^{-3}$ is given in figure 10. No difference can be appreciated for the regular orbits whereas for the chaotic orbits the difference is visible after a few tens of periods. In figure 11 the distance between the two orbits for the eccentricity is plotted for a much smaller initial distance $\epsilon = 10^{-10}$. In this case an exponential growth is observed just as for the distance between the orbits. The figure is very similar to figure (2) and figure (6) in which the chaotic orbit diffuse until it saturate but for the regular orbit remain near the initial position and is trend in time is constant. This is the common feature for regular orbit: changing the initial condition slightly in orbital elements change the evolution of the orbit also slightly.

FIG. 10. Orbital parameter evolution in time for regular and chaotic orbit for the R-3bp. Top panel show the eccentricity variation for the regular orbit with initial condition equal to $x = 0.55, \dot{x} = 0$ and $y = 0$. The green line and the red one ($\epsilon = 10^{-3}$) are so similar that only one can be detected from the plot. Bottom panel show the different behavior for the evolution of the eccentricity for two close orbits in a chaotic region of the phase space with initial condition equal to $x = 0.56, \dot{x} = 0$ and $y = 0$. The value of $y$ is calculated fixing the Jacobi constant equal to 3.07.

The same plot could be done for the Cumulative Orbital Element Error parameter for the semi-major axis $COE_{(a)}(n)$:

$$COE_{(a)}(n) = \sum_{j=0}^{n,n_a} |a_j \Delta t(x_0 + \epsilon) - a_j \Delta t(x_0)|,$$

Iterating this process for a grid of different initial conditions is it possible, on a short integration time interval ($t = 100T$), to compute the map with a very small initial distance of the two orbits ($\epsilon = 10^{-10}$). This is shown in figure 12 for the $COE_{(e)}(n)$. Because the variations of the orbital parameters are summed up on all the
Fidelity and Reversibility in the Three Body Problem

FIG. 11. Norm $L_1$ for chaotic and regular orbit computed with eccentricity. Example of the different behavior of the distance of eccentricity for the two nearly orbits for chaotic (green) and regular (red) orbits. The shift between two near orbits is $\epsilon = 10^{-10}$ and it is possible to note how the distance for the chaotic motions reach the value near to 1 after only 120 dynamical times.

In alternative the overflow can be avoided by decreasing sufficiently the initial displacement $\epsilon$.

VII. APPLICATION TO THE PROGRADE AND RETROGRADE MOTION

In this section we present the application of our method to the study of the orbit of prograde and retrograde orbits. With retrograde orbits we define a planet which is inclined by 180° with respect to the orbital plane of motion. The study of the retrograde orbit problem is a recent topic but with applications that goes from the counter-rotating migration of binary black hole (32) and references therein) to star-star interaction in globular cluster and triple star systems (42) and from our Milky Way halo’s stellar components (43) to the studies of stability of retrograde orbits in planetary systems (43 and references therein). Here we focus our attention on the range of parameters such as the distances, the masses and the time-scales that are typical for extrasolar systems. The few body dynamics has experienced a revival of interest after the discovery of the first exoplanets (5,27) and presently several hundreds of multiplanet systems have been observed (http://exoplanetarchive.ipac.caltech.edu/index.html).

Also the increasing interest in astrobiology has motivated the study of the stability of a hearth-like planet in the so called “habitable zone” (the region of the space around the star in which water can be found in the liquid and vapor phases (28)). In particular it is possible to show that, at least, 40% of binary systems can allow the presence of stable earth-like planets (29,3). The stability of these systems depend on the slow diffusion of orbits inside a resonance and due to the overlap of different resonances (9,10,40 and reference therein).

The counter-rotating planets may appear academic, however the very important Kozai mechanism (32) has been suggested as a possible origin for “retrograde” planets that may be 15% of all extrasolar objects (33 and references therein). Also a small number of observations can suggest the presence of highly inclined exoplanet (1, and reference therein).

To show the difference between prograde and retrograde stability we compute the REM map for different planes: the $(x_F, y_F)$-plane and the $(a,e)$-plane. We also present the Poincaré section for both prograde and retrograde orbits in the $(x, \dot{x})$-plane. The figure [13] shows a color plot of the reversibility error choosing the initial conditions in a grid of the $(x_F, y_F)$ plane and initial velocities equal to
In the first case the initial velocity in the rotating frame is zero. Due to the difference in the resonance overlap condition the stability regions change. In the case of retrograde quasi circular orbits the different overlapping of different resonances creates, in the inner part of the system (with respect to the orbit of the planet), a larger stable region for test particles respect to the prograde case. For prograde orbits the stability areas (near the $L_4$ or $L_5$ Lagrangian points) are wider and the test particles can be trapped into resonant orbits. Changing the value of $\mu$, change the width of the stable and unstable regions. In particular this is show in figure 14, in which the outer chaotic regions respect to the orbit of the planet increase.

\[\begin{align*}
\dot{x}_{F0} &= -y_{F0} \quad \text{(prograde)} \\
\dot{y}_{F0} &= x_{F0} \\
\dot{x}_{F0} &= y_{F0} \quad \text{(reroegrade)} \\
\dot{y}_{F0} &= -x_{F0}
\end{align*}\]
increasing the value $\mu$.

We now pass to analyze the REM stability map in the $(a,e)$-plane for the case of retrograde end prograde motion of a test particle. In this case the initial condition for the grid are

$$
\begin{align*}
\dot{x}_0 &= a(1-e) - \mu \quad \text{(prograde)} \\
\dot{y}_0 &= \sqrt{\frac{1 - \mu}{a} \frac{1 + e}{1 - e}} - a(1-e) \\
\dot{x}_0 &= a(1-e) - \mu \quad \text{(retrograde)} \\
\dot{y}_0 &= -\sqrt{\frac{1 - \mu}{a} \frac{1 + e}{1 - e}} - a(1-e),
\end{align*}
$$

where $a$ and $e$ are the semimajor-axis and eccentricity of the test particle and $\mu$ is the reduced mass. The particle start at pericentre position in both prograde and retrograde case. In figure 15 it is possible to see how the prograde and retrograde motion change the amplitude of stability regions due to the different resonance conditions in both cases. We use as an example of prograde orbit a test particle in $8:5$ mean motion resonance with the secondary massive body. Here $\mu = 0.000954$ and the integration time is computed for $10^2$ dynamical times. This map is quit good in agreement with the one found in\textsuperscript{39} using the log($RLI$) as chaotic indicator (figure 2 left in their paper). This is another confirmation of the good agreement between the REM method and other well known stability methods.

The difference in the amplitude of the stable regions are due to the different resonant condition. The stable region for low eccentricity is twice as large for the retrograde orbits respect to the prograde one and show how the retrograde orbits are more stable respect the prograde ones. The width of the stable region for small eccentricity is roughly $1.5 H_{\text{radius}}$ for the prograde orbit and $0.3 H_{\text{radius}}$ for the retrograde case. Finally we make the Poincaré section for both prograde and retrograde orbits in a particular region of the phase space $(x, \dot{x})$-plane. We take 500 random initial points for the values of $(x, \dot{x})$. In figure 16 is it possible to see, in a particular region of interest from the complete phase space, how the prograde and retrograde orbits differ one to another in the Poincaré section map $(x, \dot{x})$. Different resonances are shown for the two cases and different islands of chaotic motion appear.

**VIII. FIDELITY**

The speed at which the dynamic evolution loses memory of the initial condition is measured by the correlations decay rate. Given an orbit $\mathbf{x}_n = M^n(\mathbf{x})$ where $M$ is a symplectic map one defines the correlation as the mean value of the product $f(\mathbf{x}_n)f(\mathbf{x})$ minus the square of the mean value of $f(\mathbf{x})$ where the mean value is defined as the phase space average. More precisely we have two
alternative definitions of the correlation

\[ \hat{C}(n) = \langle f(M^n)f \rangle - \langle f \rangle \langle f \rangle \]
\[ C(n) = \langle f(M^n)f \rangle - \langle f(M^n) \rangle \langle f \rangle \]

where we have the following notation

\[ \langle f \rangle = \int_{\mathcal{E}} f(x) dm(x) \]
\[ \langle f \rangle_\mu = \int_{\mathcal{E}} f(x) d\mu(x) = \lim_{n \to \infty} \int_{\mathcal{E}} f(M^n(x)) dm(x), \]

and \( m(x) \) denotes the normalized Lebesgue measure, \( \mu(x) \) the invariant measure

\[ m(A) = \frac{m_L(A)}{m_L(\mathcal{E})} \]
\[ \mu(M.A) = \mu(A) \quad \text{for any } A \subseteq \mathcal{E}, \]

assuming that \( \mathcal{E} \) is a compact subset of \( \mathbb{R}^d \) invariant with respect to the map \( M \mathcal{E} = \mathcal{E} \). Typically \( \mathcal{E} \) is the closure of an orbit issued from a given point or the union of the images of a given domain. If \( x, x_1, \ldots, x_n \) were random independent variables the correlation would vanish for any \( n \). For a deterministic system the correlation does not decay for regular orbits (for example orbits belonging to KAM tori in a symplectic map) whereas it decays exponentially fast to zero for chaotic orbits. If the system is perturbed deterministically, stochastically or by round-off errors the perturbed orbit may lose memory of the unperturbed one. The de-correlation is measured by the Fidelity defined according to

\[ \hat{F}(n) = \langle f(M^n)f(M^n) \rangle - \langle f \rangle \langle f \rangle \]
\[ F(n) = \langle f(M^n)f \rangle - \langle f(M^n) \rangle \langle f \rangle \]

where \( \mu_e \) is the invariant measure associated to the perturbed map \( M_e \). For regular maps such as translations on the torus \( \mathbb{T}^d \) the correlations and the Fidelity do not decay. For asynchronous maps on the cylinder \( \mathbb{C} = \mathbb{T}^d \times J \) the correlations and the Fidelity have a power law decay \( 1/n \) for observables such that the average on the torus is constant. This behavior is typical of integrable systems. For random perturbations depending on \( \epsilon \xi \) where \( \xi \) is a random variable with zero mean and unit variance, the Fidelity of \( T_e \) with respect to \( T \) decays exponentially if \( T \) is a regular map. For chaotic maps the Fidelity exhibits a plateau extending from \( n = 0 \) to \( n = n_s \) defined by

\[ n_s = \frac{\log \epsilon^{-1}}{\lambda} \]

where \( \lambda \) is the mLCE. After the plateau the Fidelity decays exponentially. These results where proved for linear maps on the torus or the cylinder with additive noise \( M_e = M + \epsilon \xi \). Rigorous results in a more general setting were obtained for chaotic maps with exponentially decaying correlations \( C(n) \).

If the perturbation is due to the round-off error, introduced by the finite precision representation of the reals, then the Fidelity does not decay for regular maps such as the translations on the torus \( \mathbb{T}^d \) and decays as \( 1/n \) for the asynchronous map on the cylinder. As a consequence for regular maps the round-off has no appreciable effect. For a chaotic map, having a positive mLCE and exponential mixing, the Fidelity behaves exactly the same way as for a random perturbation of the same amplitude even though the local error in this case is strongly correlated.

For the R-3bp we compute the Fidelity of the orbit on the Poincaré section defined by the intersection of the \( y = 0 \) plane with the Jacobi manifold \( J = 3.07 \) with the condition \( v_y > 0 \). In our case the symplectic integration is carried out in the fixed system and the perturbation is defined by modifying the order 2 symplectic map according to

\[ x_{k+1} = x_k + (1 + \epsilon \xi) \left[ v_{x,k} \Delta t + f_x(x_k, y_k, \tau_k) \frac{(\Delta t)^2}{2} \right] \]
\[ y_{k+1} = y_k + v_{y,k} \Delta t + f_y(x_k, y_k, \tau_k) \frac{(\Delta t)^2}{2} \]
\[ \tau_{k+1} = \tau_k + \Delta t \]
followed by

\[
\begin{align*}
&v_{x,k+1} = v_{x,k} + (1 + \epsilon \xi) f_x(x_k, y_k, \tau_k) + \frac{f_x(x_{k+1}, y_{k+1}, \tau_{k+1}) \Delta t}{2} \\
&v_{y,k+1} = v_{y,k} + \left[ f_y(x_k, y_k, \tau_k) + f_y(x'_k, y'_k, \tau'_k) \right] \frac{\Delta t}{2} \\
&p_{\tau,k+1} = p_{\tau,k} + \left[ f_x(x_k, y_k, \tau_k) + f_y(x_{k+1}, y_{k+1}, \tau_{k+1}) \right] \frac{\Delta t}{2}
\end{align*}
\]

The integration is carried out with a 4-th order integrator obtained by composing three times the second order one. The random variable \( \xi \) is changed after crossing the section plane and kept constant until the next intersection. Changing the random variable at each time step does not change the results in a significant way.

We analyze the Fidelity for two distinct initial conditions on the Poincaré section at \( x_0 = 0.68, y_0 = 0, v_{y0} = 0 \), which corresponds to a regular non resonant orbit (diffeomorphically to a circle), and \( x_0 = 0.56, y_0 = 0, v_{y0} = 0 \) which corresponds to a chaotic orbit with positive mLCE \( \lambda = 0.025 \). The value of \( v_{x0} \) is uniquely determined by the condition that we start on the Jacobi manifold and that it must be positive. The orbits portraits in the section phase plane \( x, v_x \) for these two orbits are shown in figure 17 for two different values of the noise amplitude.

For the regular orbit we consider a sequence of values of the noise amplitude \( \epsilon = \epsilon_0/2^m \) for \( 0 \leq m \leq 6 \) with \( \epsilon_0 = 2 \cdot 10^{-3} \). The Fidelity shows an exponential decay which can be fitted by

\[
F(n, \epsilon) = F(0) \exp(-c(\epsilon) n^3).
\]

In figure 18 we show the Fidelity for the observable \( x \) and the exponential fit for the above mentioned sequence of values of the noise amplitude \( \epsilon \). The comparison of the fitted values with \( c(\epsilon) = 30\epsilon^2 \) shows that the dependence on \( \epsilon \) is quadratic within a very good accuracy. In a precedent paper, the decay rate \( \exp(-\epsilon^2 n^2) \) was proved to occur for a stochastically perturbed map of the cylinder for an observable \( f(\theta) \) where \( \theta \) is the angle variable. In the R-3bp the unperturbed orbit is close to a circle and the radial diffusion of the perturbed orbit shows that the perturbation affects also the action variable \( j \). As a consequence since \( x \simeq (2j)^{1/2} \cos \theta \) the observed decay law is compatible with the result proven for the stochastically perturbed map of the cylinder.

For the chaotic orbit we compute the Fidelity for the sequence of values of the noise amplitude \( \epsilon = 10^{2m} \) for \( 2 \leq m \leq 7 \). In figure 19 we show the plots of the Fidelity \( F(n, \epsilon) \). For any value of \( \epsilon \) the Fidelity exhibits a plateau up to a value \( n_s(\epsilon) \) followed by an exponential decay with an almost constant decay rate. The plateaus followed by an exponential decay are observed in other chaotic maps and the result was rigorously proved for the Bernoulli maps.

---

**FIG. 17. Phase space unperturbed and noisy orbits**

Top panel: phase space plot in the section plane \( x, v_x \) of the regular orbit with initial points \( x_0 = 0.68 \) (blue dots) and the orbit with a stochastic perturbation of amplitude \( \epsilon = 10^{-3} \) (red dots). Bottom panel: chaotic orbit with initial points \( x_0 = 0.56 \) (blue dots) and orbits with with a stochastic perturbation of amplitude \( \epsilon = 10^{-3} \) (red dots).

**IX. CONCLUSION**

In this paper we have introduced a new method to investigate the stability of orbits for Hamiltonian systems. To this end we consider a small perturbation induced by round-off or by random errors and analyze the divergence of the perturbed orbit from the reference orbit having the same initial condition. The reversibility error provides the same information as the forward error but does not require the knowledge of the exact orbit and is well suited to inspect the effect of round-off errors. The loss of memory of the perturbed orbit is measured by
the Fidelity decay whose computation requires a Monte-Carlo sampling of the unperturbed orbits. This method, first proposed in prototype models of regular and chaotic dynamics, is applied to the restricted planar three body problem which is the simplest non-integrable Hamiltonian system in the field of celestial mechanics.

This paper can be considered as a preliminary work suitable to be extended to particular Hamiltonian systems of astrophysical interest (for example and reference therein). The asymptotic divergence due to a small error in the initial conditions or to a small perturbation of the Hamiltonian flow are quite similar. Rather then computing the divergence of the perturbed orbit with respect to the exact one as for the Lyapunov-type methods, the reversibility error can be evaluated since it is expected to have the same asymptotic behavior. This equivalence is mathematically proved here for linear maps and numerically proved for the R-3bp. The reversibility error is very easy to implement and has a low computational cost. As a consequence one can analyze large phase space regions using a rather fine grid of initial points. The COE method also proves to be effective to analyze the stability of orbits and it is based on the distance between orbital elements for slightly different initial conditions and it is similar to mLCE, though faster to implement.
The divergence and the memory loss were analyzed on the R-3bp for a regular and chaotic orbits. The Fidelity decays exponentially for a regular orbit with random perturbation of amplitude $\epsilon$ and the decay rate is proportional to $\epsilon^2$. For a chaotic map the Fidelity remains constant up to a threshold $n_* \sim \log \epsilon^{-1}$ after which it decays exponentially both for a random perturbation and a round-off error. These results obtained for the translation on the torus and the Bernoulli map on the torus were shown to hold also for the R-3bp computing the phase space averages with Monte-Carlo simulations. For this map the Reversibility Error, computed on a regular grid of points in the phase space, provides a very effective picture of the dynamical properties both for the random and round-off errors.

The Fidelity is computationally rather expensive but provides a relevant additional information. A key difference with respect to REM is that the Fidelity does not decays if the perturbation is due to round-off errors whereas it decays exponentially for random errors. For chaotic orbits the behavior is the same namely a plateau up to a threshold depending logarithmically on the perturbation amplitude for both round-off and random perturbations. If the invariant set is obtained by iterating not a point but a phase space domain the Fidelity provides a averaged estimate of the memory loss rate in this region. As a consequence the Fidelity may be a useful tool to explore the boundary between regular and chaotic motion. Our results confirm that the Reversibility Error and Fidelity provide a reliable estimate of separation of orbits and de-correlation induced by a small perturbation. The present analysis will be extended to N-body systems in a future paper. The Reversibility Error just requires the evaluation of the orbit for $2n$ iterations of the map. The Fidelity computation is based on a Monte-Carlo method on the invariant manifold approximated by iterating the initial point $M$ times with the unperturbed map. Since the statistical error is proportional to $M^{-1/2}$ high accuracy results are computationally expensive. The exact reference orbit may be computed using high precision arithmetic with real numbers represented by $b$ bytes strings using recent compilers having $b$ bytes arithmetic with real numbers represented by $b$ bytes strings using recent compilers having $b$.

As an application of interest for celestial mechanics the prograde and retrograde motion of a test particle was analyzed using the REM and the change in the stability regions observed may be explained in terms of resonances and their overlap, which differs from one case to the other. The results are confirmed and and compatible with the ones using the COE method. The future program is to apply the procedure developed here for the R-3bp to the few bodies problem, which is of more relevant interest in the case of newly observed exoplanets and in many other field of astrophysics such as the characterization of regular and chaotic orbits in elliptical galaxies and the motion of binary black holes at the center of galaxies.

**APPENDIX A**

We consider a symplectic map $M$ and the nearby orbits with initial points $x_0$ and $x_0 + \epsilon w_0$ where $||w_0|| = 1$. The evolution is given by $x_n = M^n(x_0)$ and $x_n + \epsilon w_n = M^n(x_0 + \epsilon w_0)$. The Lyapunov exponent is defined by

$$\lambda = \lim_{n \to \infty} \lim_{\epsilon \to 0} \frac{1}{n} \log d_n$$

where $d_n$ is the orbit divergence at step $n$

$$d_n = ||w_n|| = \| DT^n(x_0)w_0\| + O(\epsilon)$$

Given an invariant ergodic component of the constant energy manifold the sequence convergences to the mLCE $\lambda$ for all the initial directions $w_0$ of the initial perturbation except for a set of measure zero corresponding to the eigenvectors of the smallest Lyapunov exponent. For chaotic orbits the growth of the distance is $d_n \propto \epsilon^{\lambda n}$ and consequently it rapidly reaches the diameter of the invariant sub-manifold. In order to avoid this a renormalization procedure has to be used. The procedure is the following. Letting $y_n$ and $(y_n)_R$ be the nearby orbit and the re-normalized nearby we have starting from $n = 1$

$$x_1 = M(x_0)$$

$$y_1 = x_1 + \epsilon w_1 = M(x_0 + \epsilon w_0) = x_1 + \epsilon w_1 + O(\epsilon)$$

$$w_1 = \epsilon DM(x_0)w_0,$$

and at this step

$$(y_1)_R = y_1$$

$$(d_1)_R = d_1$$

then at the second step

$$x_2 = M(x_1)$$

$$y_2 = x_2 + \epsilon w_2 = M(x_1 + \epsilon w_1) = x_2 + \epsilon DM(x_1)w_1 + O(\epsilon^2)$$

and the renormalized vector $(y_2)_R$ is defined by

$$(y_2)_R = M \left( x_1 + \frac{w_1}{d_1} \right) = x_2 + \epsilon \frac{w_2}{d_1} + O(\epsilon^2)$$

$$(d_2)_R = \frac{1}{\epsilon} \|(y_2)_R - x_2\| = \frac{d_2}{d_1} + O(\epsilon).$$

As a consequence $d_2 = (d_2)_R (d_1)_R$. In general at step $n$
and the mLCE is expressed by

\[ x_n = M(x_{n-1}) \]

\[ y_n = x_n + \epsilon w_n = M(x_{n-1} + \epsilon w_{n-1}) = x_n + \epsilon DM(x_{n-1})w_{n-1} + O(\epsilon^2), \]

and the renormalized vector \((y_n)_R\) is defined by

\[ (y_n)_R = M \left( x_{n-1} + \epsilon \frac{(w_{n-1})_R}{(d_{n-1})_R} \right) = x_n + \epsilon DM(x_{n-1}) \frac{w_{n-1}}{(d_{n-1})_R} + O(\epsilon^2) = x_n + \epsilon (w_n)_R + O(\epsilon^2). \]

It follows that

\[ (w_n)_R = DM(x_{n-1}) \frac{w_{n-1}}{(d_{n-1})_R} + O(\epsilon) = DM(x_{n-1})DM(x_{n-2}) \cdots DM(x_1)DM(x_0)w_0 = \frac{w_n}{(d_{n-1})_R(d_{n-2})_R \cdots (d_1)_R} + O(\epsilon). \]

The final result reads

\[ d_n = \| w_n \| = (d_{n-1})_R(d_{n-2})_R \cdots (d_1)_R + O(\epsilon), \]

and the mLCE is expressed by

\[ \lambda = \lim_{n \to \infty} \lim_{\epsilon \to 0} \frac{1}{n} \sum_{j=1}^{n-1} \log((d_j)_R). \]

The algorithm is extremely simple and is expressed by the recurrence

\[ \epsilon(w_{n-1})_R = (y_{n-1})_R - x_{n-1} \]

\[ (d_{n-1})_R = \| w_{n-1} \| \]

\[ (y_n)_R = M \left( x_{n-1} + \epsilon \frac{w_{n-1}}{(d_{n-1})_R} \right) \]

initialized by

\[ \epsilon(w_1)_R = \epsilon w_1 = M(x_0 + \epsilon w_0) - x_1 \]

\[ (y_1)_R = M(x_0 + \epsilon w_0) \]

\[ (d_1)_R = d_1 = \| w_1 \|. \]
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