SPATIAL-TEMPORAL SWITCHING ESTIMATORS FOR IMAGING LOCALLY
CONCENTRATED DYNAMICS

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ABSTRACT

The evolution of images with physics-based dynamics is often
spatially localized and nonlinear. A switching linear dynamic
system (SLDS) is a natural model under which to pose such
problems when the system’s evolution randomly switches
over the observation interval. Because of the high parameter
space dimensionality, efficient and accurate recovery of the
underlying state is challenging. The work presented in this
paper focuses on the common cases where the dynamic evo-
lution may be adequately modeled as a collection of decou-
pled, locally concentrated dynamic operators. Patch-based
hybrid estimators are proposed for real-time reconstruction
of images from noisy measurements given perfect or partial
information about the underlying system dynamics. Numer-
ical results demonstrate the effectiveness of the proposed
approach for denoising in a realistic data-driven simulation
of remotely sensed cloud dynamics.

Index Terms— localized dynamics, Kalman filter, recon-
struction, denoising.

1. INTRODUCTION

The image formation of time-varying phenomena is common-
place in astronomy [1], remote sensing [2], and many other
disciplines, e.g., [3]. The challenge is to recover spatio-
temporal parameters of a physical phenomenon given noisy,
ex situ measurements. Such time-dependent inverse problems

The remainder of the paper is organized as follows. Section 2
introduces the SLDS model which captures locally con-
centrated dynamics. The patch-based estimator is proposed in
section 3 and its computational complexity is studied in Sec-
section 4. Section 5 compares the performance of the proposed
filter to a SKF in terms of efficiency and accuracy.

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2. LOCALLY CONCENTRATED HYBRID STATE-SPACE MODEL

A spatial-temporal multi-modal LDS may be represented as

\[
\begin{bmatrix}
    x_n(1) \\
    \vdots \\
    x_n(d)
\end{bmatrix}
= \begin{bmatrix}
    A_{s_1,1}(1,1) & \ldots & A_{s_1,d}(1,d) \\
    A_{s_2,1}(2,1) & \ldots & A_{s_2,d}(2,d) \\
    \vdots & \ddots & \vdots \\
    A_{s_d,1}(d,1) & \ldots & A_{s_d,d}(d,d)
\end{bmatrix}
\begin{bmatrix}
    x_{n-1}(1) \\
    \vdots \\
    x_{n-1}(d)
\end{bmatrix}
+ \nu_n,
\]

\[y_n = H_n x_n + \omega_n.
\]

In the above, the state vector is \(x_n(i)\), where subscript \(n\) and \(i\) represent the time and the state index, respectively, \(d\) is the dimension of the state, \(A\) is the evolution operator, and \(s_{i,j}\) refers to a hidden switching random variable that determines the correlation between dynamic evolution of pixel \(i\) with pixel \(j\). The given information are \(y_n\), the measurement, \(H_n\), the measurement operator, and \(\nu_n \sim N(0, Q_n)\), \(\omega_n \sim N(0, R_n)\) \([N(\mu, \Sigma)\) refers to the Gaussian distribution with mean and covariance \(\mu, \Sigma\)] are the evolution and measurement noise.

When the observed state variable is a function of variables that are locally concentrated in space, the evolution model in the neighboring pixels depend only on a single hidden random variable. It is assumed that each pixel in the image is correlated with its neighboring pixels with distance of at most \(r\) pixels (which means \(A_{s_{i,j}}(i, j) = 0\) if \(|i - j| > r\)). The state dynamic equation is then

\[
\begin{bmatrix}
    x_n^{(1)} \\
    \vdots \\
    x_n^{(B)}
\end{bmatrix}
= \begin{bmatrix}
    A_{s_1,1}^{(1)} & \ldots & 0 \\
    \vdots & \ddots & \vdots \\
    0 & \ldots & A_{s_d,d}^{(B)}
\end{bmatrix}
\begin{bmatrix}
    x_{n-1}^{(1)} \\
    \vdots \\
    x_{n-1}^{(B)}
\end{bmatrix}
+ \nu_n,
\]

where \(B\) is the total number of patches, \(x_n^{(j)}\) refers to the pixels at time \(n\) in the \(j^{th}\) patch that follow the same dynamic behavior, \(A_{s_{i,j}}^{(i)}\) is the evolution model for patch \(i\), and \(A_{s_{i,j}}^{(i,j)}\) is the correlating term between the dynamic model of patch \(i\) with its neighboring patch \(j \in B_i\) where \(B_i\) is the set of patches neighboring of patch \(i\).

3. PATCH-BASED ESTIMATION

For an LDS, the optimal Bayesian estimator is the Kalman filter (KF). In this section, we first review the KF/SKF procedure, as they are the building blocks of the proposed algorithm. Patch estimators using multiple low-dimensional SKFs are then proposed for efficient estimation of a spatial-temporal, locally concentrated evolving dynamic system.

3.1. Kalman filter

With \(y_n^i\) defined as the set of all samples \(y_1, y_2, \ldots, y_n\), the initial state is \(x_0 \sim N(x_0|0, P_0|0)\), \(x_n|n = E[x_n|y_n^i]\) and \(P_n|n = C(x_n|y_n^i)\) are the conditional mean and covariance, and \(L_n = p(y_n|y_n^{n-1})\) the likelihood, one step of the KF is

\[(x_n|n, P_n|n, L_n) = \text{Filter}(A_n, H_n, x_{n-1|n-1}, P_{n-1|n-1}, Q_n, R_n, y_n^i),
\]

which involves the following two recursive steps:

**Step 1: Time update equations**

\[x_n|n-1 = A_n x_{n-1|n-1},
\]

\[P_n|n-1 = A_n P_{n-1|n-1} A_n^T + Q_n;
\]

**Step 2: Measurement Update equations**

\[e_n = y_n - H_n x_{n|n-1},
\]

\[B_n = H_n P_{n|n-1} A_n^T + R_n,
\]

\[K_n = P_{n|n-1} H_n^T B_n^{-1},
\]

\[L_n = N(e_n, 0, B_n),
\]

\[x_n|n = x_{n|n-1} + K_n e_n,
\]

\[P_n|n = (I - K_n H_n) P_{n|n-1}.
\]

For an SLDS, \(A_n\) and \(Q_n\) can change with time and they must be detected as a component of the SKF using the obtained likelihoods \(L_n\) using a Bayesian approach [4].

3.2. Patch estimator

The state-space equations for patch \(i\) may be written as

\[x_n^{(i)} = A^{(i)} x_{n-1}^{(i)} + \sum_{j \in B_i} A^{(i,j)} x_{n-1}^{(j)} + \nu_n^{(i)},
\]

\[\Gamma_i y_n = \Gamma_i H x_n + \Gamma_i \omega_n,
\]

where \(A^{(i)}, A^{(i,j)},\) and \(Q^{(i)}\) are functions of a hidden switching random variable corresponding to patch \(i\) and neighboring patch \(j\), and \(\Gamma_i\) is an operator applied to the measurements in order to find the localized measurements for process \(i\). For localization, it is sufficient to have \(\Gamma_i H \in T = \begin{bmatrix} 0, \ldots, 0 \\ 0, \Theta^{(i)}, 0 \\ 0, \ldots, 0 \end{bmatrix}\), if such \(\Gamma_i, \Theta\) exist. Otherwise, one must solve the constrained minimization problem \(\text{argmin}_{\Gamma_i} ||\Gamma_i y_n - \Gamma_i H x_n||^2\) s.t. \(\Gamma_i H \in T\). A special case is when local measurements are calculated for each pixel. In this case, the minimization problem to obtain local measurements is equivalent to solving an inverse problem, and the patch estimator may then be applied to the inverse problem’s solution for denoising.

The estimation of a locally concentrated SLDS when the modes’ spatial extent is unknown can be formulated as the following optimization problem for the set of possible dynamic models, patch sizes, shapes, and state variables:

\[\arg\min_{x_n|n, U_i, U_{ij}, B} L(x_n, s_i, U_i, U_{ij}, B),
\]

(2)
of size $2r \times 2r$, it is also possible to use windows of size $(\alpha + 2r) \times (\alpha + 2r)$, slide the window over the whole image, and estimate each pixel using the centered window; we refer to this approach as the sliding windowed SKF (swSKF). The swSKF helps to reduce the boundary effects of windowing. It is notable that the window size must be greater than $2r \times 2r$ and that $\alpha > 0$ must be chosen such that the windows are not too small (overfitting) or large (underfitting).

The decoupled estimation covariance will be larger compared the full SKF with knowledge of the mode spatial extent, since some information propagates through the image when the patches are dependent. Since running a SKF for every possible mode spatial extent is intractable, patch-based processing will have some information loss.

**Algorithm 1:** Patch-based estimation

| Result: $\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n$ |
| --- |
| **Input:** $\mathcal{F} = \{F_1, F_2, \ldots, F_B\}$ s.t. $F_i \in \mathcal{F}, H, R, \hat{x}_0, P_0, r$ |
| $Q = \{Q_1, \ldots, Q_L\}$ s.t. $Q_i \in \mathcal{Q}$ |
| for $n = 1 : T$ do |
| for $i \in \{1, \ldots, B\}$ do |
| $\hat{x}_n(i) = SKF(\mathcal{F}, Q, \Gamma_i y_n, R, x_n(i), P_0(i))$ |
| end |
| $\hat{x}_n = [\hat{x}_n; \hat{x}_n(i)]$ |
| end |
4. COMPUTATIONAL COST

Consider the estimation of a $\sqrt{d} \times \sqrt{d}$ image with $K$ non-overlapping processes that switch between $l$ modes over time. Thus, an SKF that has perfect knowledge of mode spatial extent must consider $K^l$ different modes at each time step with computational complexity $O(K^l d^3)$. On the other hand, decoupling the image into $K$ windows where each window switches between $l$ modes only requires $O(K^l r^3 w)$ where $\sqrt{r^w} \times \sqrt{r^w}$ is the window size. (For an illustration, see Fig. 1). Similarly, running an swSKF requires $O(K'^l r^w)$, where $K'$ is the number of sliding windows and $K' > K$. Thus, the computational requirement of the patch-based SKF is much smaller than that of the full SKF, even given perfect knowledge of the spatial extent of the modes, and this difference becomes more significant as $d, K$, and $l$ increase.

5. SIMULATION RESULTS

The performance of the proposed patch-based estimator for denoising reconstructed video frames from tomographic measurements is studied in this section. A sequence of $32 \times 32$ images are reconstructed in time using a nonlinear, locally switching dynamic model, and a set of measurement with an average SNR of 11dB is generated accordingly; these images are meant to represent the evolution of cloud density obtained by infrared imaging in which clear local concentration is known to exist. The data are generated such that the structures’ elements have random movements with slow/fast velocities (movement velocities are 0.01, 0.94 pixels per time step) in each quarter of the image (as in Fig. 1), and the velocity can switch randomly in each quarter over time.

The goal is to denoise the set of tomographically reconstructed images. Because the system dynamics are complicated, we use a purely stochastic model with two evolution covariance matrices corresponding to slow and fast velocities such that the variances are roughly set to the velocity of movements in the two cases. Assuming knowledge of the variances only, we apply a bi-modal full SKF, wSKF, and swSKF with window size $8 \times 8$ to the sequence of images. Fig. 2a shows realizations of the ground truth, noisy reconstructed image, and the estimates for three representative time steps. The full SKF clearly cannot recover the ground truth details as well as the wSKF/swSKF. This conclusion is even more pronounced in Fig. 2b, where the error for each filter is averaged over 100 realizations and the full SKF MSE using the bi-modal model tends to diverge since the model used is mismatched. It is also notable that the swSKF error is slightly smaller compared to the wSKF. The (s)wSKF can find the estimates without the knowledge of the spatial extent of the modes and by detecting it instead while requiring fewer computations using the locality assumption. The run time of the filters are presented in the table below.

|        | full SKF | wSKF  | swSKF |
|--------|----------|-------|-------|
| Run time(sec) | 68.52    | 1.26  | 4.18  |

6. CONCLUSION

A patch-based filtering framework is proposed for the estimation of image sequences governed by locally concentrated dynamics and shown to have superior performance with respect to the full filter in terms of computation and accuracy, when provided perfect/partial information about the evolution model. Solving the reconstruction problem for a general measurement operator, fusing the local estimates to obtain global estimates, and online learning of the evolution models/statistics of the patches locally are future work.
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