A generalization of parallelograms involving inscribed ellipses, conjugate diameters, and tangency chords

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Abstract

A convex quadrilateral, \( Q \), is called a midpoint diagonal quadrilateral if the intersection point of the diagonals of \( Q \) coincides with the midpoint of at least one of the diagonals of \( Q \). A parallelogram, \( P \), is a special case of a midpoint diagonal quadrilateral since the diagonals of \( P \) bisect one another. We prove two results about ellipses inscribed in midpoint diagonal quadrilaterals, which generalize properties of ellipses inscribed in parallelograms. First, \( Q \) is a midpoint diagonal quadrilateral if and only if each ellipse inscribed in \( Q \) has tangency chords which are parallel to one of the diagonals of \( Q \). Second, \( Q \) is a midpoint diagonal quadrilateral if and only if each ellipse inscribed in \( Q \) has a pair of conjugate diameters parallel to the diagonals of \( Q \). Finally, we show that there is a unique ellipse, \( E_I \), of minimal eccentricity inscribed in a midpoint diagonal quadrilateral, \( Q \). In addition, we show that the equal conjugate diameters of \( E_I \) are parallel to the diagonals of \( Q \).

1 Introduction

Given a diameter, \( l \), of an ellipse, \( E_0 \), there is a unique diameter, \( m \), of \( E_0 \) such that the midpoints of all chords parallel to \( l \) lie on \( m \); In this case we say that \( l \) and \( m \) are conjugate diameters of \( E_0 \), or that \( m \) is a diameter of \( E \) conjugate to \( l \); We say that \( E_0 \) is inscribed in a convex quadrilateral, \( Q \), if \( E_0 \) lies inside \( Q \) and is tangent to each side of \( Q \). A tangency chord is
any chord connecting two points where $E_0$ is tangent to a side of $Q$. There are two interesting properties (probably mostly known) of ellipses inscribed in parallelograms, $P$, which involve tangency chords and conjugate diameters:

(P1) Each ellipse inscribed in $P$ has tangency chords which are parallel to one of the diagonals of $P$.

(P2) Each ellipse inscribed in $P$ has a pair of conjugate diameters which are parallel to the diagonals of $P$.

This author is not sure if P1 is known at all, while P2 appears to be known only if $E_0$ is the ellipse of maximal area inscribed in $P$. The purpose of this paper is not only to show that P1 and P2 each hold for a larger class of convex quadrilaterals, which we call midpoint diagonal quadrilaterals, but that if $Q$ is not a midpoint diagonal quadrilateral, then no ellipse inscribed in $Q$ satisfies P1 or P2. Hence each of these properties completely characterizes the class of midpoint diagonal quadrilaterals (see Theorems 4 and 3 below), and thus they are a generalization of parallelograms in this sense. Here is the definition: A convex quadrilateral, $Q$, is called a midpoint diagonal quadrilateral if the intersection point of the diagonals of $Q$ coincides with the midpoint of at least one of the diagonals of $Q$. A parallelogram, $P$, is a special case of a midpoint diagonal quadrilateral since the diagonals of $P$ bisect one another. Equivalently, if $Q$ is not a parallelogram, then $Q$ is a midpoint diagonal quadrilateral if and only if the line, $L_Q$, thru the midpoints of the diagonals of $Q$ contains one of the diagonals of $Q$. The line $L_Q$ plays an important role for ellipses inscribed in quadrilaterals due to the following well-known result (see [1] for a proof).

**Theorem 1 (Newton):** Let $M_1$ and $M_2$ be the midpoints of the diagonals of a quadrilateral, $Q$. If $E_0$ is an ellipse inscribed in $Q$, then the center of $E_0$ must lie on the open line segment, $Z$, connecting $M_1$ and $M_2$.

**Remark 1** If $Q$ is a parallelogram, then the diagonals of $Q$ intersect at the midpoints of the diagonals of $Q$, and thus $Z$ is really just one point.

**Remark 2** By Theorem 1, $Q$ is a midpoint diagonal quadrilateral if and only if the center of any ellipse inscribed in $Q$ lies on one of the diagonals of $Q$.

If $E_0$ is an ellipse which is not a circle, then $E_0$ has a unique set of conjugate diameters, $l$ and $m$, where $|l| = |m|$; These are called equal conjugate diameters of $E_0$. By Theorem 4(i), each ellipse inscribed in a midpoint diagonal quadrilateral, $Q$, has conjugate diameters parallel to the diagonals
of $Q$; In particular, Theorem 4(i) applies to the unique ellipse of minimal eccentricity, $E_I$, inscribed in $Q$; However, we prove (Theorem 5) a stronger result: The equal conjugate diameters of $E_I$ are parallel to the diagonals of $Q$.

2 Useful Results on Ellipses and Quadrilaterals

We now state a result, without proof, about when a quadratic equation in $x$ and $y$ yields an ellipse. The first condition ensures that the conic is an ellipse, while the second condition ensures that the conic is nondegenerate.

Lemma 1 The equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, with $A, C > 0$, is the equation of an ellipse if and only if $\Delta > 0$ and $\delta > 0$, where $\Delta = 4AC - B^2$ and $\delta = CD^2 + AE^2 - BDE - F\Delta$.

The following lemma allows us to express the eccentricity of an ellipse as a function of the coefficients of an equation of that ellipse.

Lemma 2 Suppose that $E_0$ is an ellipse with equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$; Let $a$ and $b$ denote the lengths of the semi-major and semi-minor axes, respectively, of $E_0$. Then

$$\frac{b^2}{a^2} = \frac{A + C - \sqrt{(A - C)^2 + B^2}}{A + C + \sqrt{(A - C)^2 + B^2}}.$$  \hspace{1cm} (1)

Proof. Let $\mu = \frac{4\delta}{\Delta^2}$, where $\delta$ and $\Delta$ are given as in Lemma 1. By \[6\],

$$a^2 = \mu \frac{A + C + \sqrt{(A - C)^2 + B^2}}{2},$$  \hspace{1cm} (2)

$$b^2 = \mu \frac{A + C - \sqrt{(A - C)^2 + B^2}}{2}.$$  \hspace{1cm} (2)

Note that $\mu > 0$ by Lemma 1. (1) then follows immediately from (2). \[4\]

Throughout the paper we let $L_Q$ denote the line thru the midpoints of a given quadrilateral, $Q$, and we define an affine transformation, $T : R^2 \to R^2$ to be the map $T(\hat{x}) = A\hat{x} + \hat{b}$, where $A$ is an invertible $2 \times 2$ matrix. Note
that affine transformations map lines to lines, parallel lines to parallel lines, and preserve ratios of lengths along a given line. Also, the family of ellipses, tangent lines to ellipses, and four–sided convex polygons are preserved under affine transformations.

A quadrilateral which has an incircle, i.e., one for which a single circle can be constructed which is tangent to all four sides, is called a tangential quadrilateral. A quadrilateral which has perpendicular diagonals is called an orthodiagonal quadrilateral. The following lemma shows that affine transformations preserve the class of midpoint diagonal quadrilaterals. The proof follows immediately from the properties of affine transformations.

**Lemma 3** Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be an affine transformation and let $Q$ be a midpoint diagonal quadrilateral. Then $Q' = T(Q)$ is also a midpoint diagonal quadrilateral.

The following lemma shows that affine transformations send conjugate diameters to conjugate diameters. Again, the proof follows immediately from the properties of affine transformations.

**Lemma 4** Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be an affine transformation and suppose that $l$ and $m$ are conjugate diameters of an ellipse, $E_0$; Then $l' = T(l)$ and $m' = T(m)$ are conjugate diameters of $E'_0 = T(E_0)$.

The following lemma shows that the scaling transformations preserve the eccentricity of ellipses, as well as the property of the equal conjugate diameters of an ellipse being parallel to the diagonals of $Q$.

**Lemma 5** Let $T$ be the nonsingular affine transformation given by $T(x, y) = (k x, k y), k \neq 0$;

(i) Then $E_0$ and $T(E_0)$ have the same eccentricity for any ellipse, $E_0$.

(ii) If $E_0$ is an ellipse which is not a circle and if the equal conjugate diameters of $E_0$ are parallel to the diagonals of $Q$, then the equal conjugate diameters of $T(E_0)$ are parallel to the diagonals of $T(Q)$.

**Proof.** (i) follows immediately and we omit the proof. To prove (ii), suppose that $l$ and $m$ are equal conjugate diameters of an ellipse, $E_0$, which are parallel to the diagonals of $Q$; By Lemma 4, $l' = T(l)$ and $m' = T(m)$ are conjugate diameters of $E'_0 = T(E_0)$. Since $|T(P_1) T(P_2)| = k |P_1 P_2|$ for any two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, it follows immediately that $l'$ and $m'$ are equal.
conjugate diameters of $T(E_0)$; Since affine transformations take parallel lines to parallel lines, $l'$ and $m'$ are parallel to the diagonals of $T(Q)$. ■

The following lemma shows when a trapezoid can be a midpoint diagonal quadrilateral.

**Lemma 6** Suppose that $Q$ is a midpoint diagonal quadrilateral which is also a trapezoid. Then $Q$ is a parallelogram.

**Proof.** We use proof by contradiction. So suppose that $Q$ is a midpoint diagonal quadrilateral which is a trapezoid, but which is not a parallelogram. By affine invariance, we may assume that $Q$ is the trapezoid with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(1, t)$, $0 < t \neq 1$; The diagonals of $Q$ are then the open line segments $D_1: y = tx$ and $D_2: y = 1 - x$, each with $0 < x < 1$; The midpoints of the diagonals are $M_1 = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $M_2 = \left(\frac{1}{2}, \frac{t}{2}\right)$, and the diagonals intersect at $P = \left(\frac{1}{1+t}, \frac{t}{1+t}\right)$; Now $M_1 = P \iff t = 1$ and $M_2 = P \iff t = 1$; Hence $Q$ is not a midpoint diagonal quadrilateral. ■

**Remark 3** We use the notation $Q(A_1, A_2, A_3, A_4)$ to denote the quadrilateral with vertices $A_1, A_2, A_3,$ and $A_4$, starting with $A_1$ = lower left corner and going clockwise. Denote the sides of $Q(A_1, A_2, A_3, A_4)$ by $s_1, s_2, s_3,$ and $s_4$, going clockwise and starting with the leftmost side, $s_1$; Denote the lengths of the sides of $Q(A_1, A_2, A_3, A_4)$ by $a = |A_1A_4|$, $b = |A_1A_2|$, $c = |A_2A_3|$, and $d = |A_3A_4|$. Finally, denote the diagonals of $Q(A_1, A_2, A_3, A_4)$ by $D_1 = \overline{A_1A_3}$ and $D_2 = \overline{A_2A_4}$.

We note here that there are two types of midpoint diagonal quadrilaterals: Type 1, where $D_1 \subset L$ and Type 2, where $D_2 \subset L$.

Given a convex quadrilateral, $Q = Q(A_1, A_2, A_3, A_4)$, which is not a parallelogram, it will be simpler to work below to consider quadrilaterals with a special set of vertices. In particular, there is an affine transformation which sends $A_1, A_2,$ and $A_4$ to the points $(0, 0), (0, 1),$ and $(1, 0)$, respectively. It then follows that $A_3 = (s, t)$ for some $s, t > 0$; Let $Q_{s,t}$ denote the quadrilateral with vertices $(0, 0), (0, 1), (s, t),$ and $(1, 0)$; Since $Q_{s,t}$ is convex, $s + t > 1$; Also, if $Q$ has a pair of parallel vertical sides, first rotate counterclockwise by 90°, yielding a quadrilateral with parallel horizontal sides. Since we are assuming that $Q$ is not a parallelogram, we may then also assume that $Q_{s,t}$ does not have parallel vertical sides and thus $s \neq 1$. Summarizing, we have
Proposition 1 Suppose that $Q$ is a convex quadrilateral which is not a parallelogram. Then there is an affine transformation which sends $Q$ to the quadrilateral $Q_{s,t} = Q(A_1, A_2, A_3, A_4)$,

\[ A_1 = (0, 0), A_2 = (0, 1), A_3 = (s, t), A_4 = (1, 0), \]

with $(s, t) \in G$, where

\[ G = \{(s, t) : s, t > 0, s + t > 1, s \neq 1\}. \]

Since the midpoints of the diagonals of $Q_{s,t}$ are $M_1 = \left( \frac{1}{2}, \frac{1}{2} \right)$ and $M_2 = \left( \frac{s}{2}, \frac{t}{2} \right)$, by Theorem 4, the center of any ellipse, $E_0$, inscribed in $Q_{s,t}$ must lie on the open line segment \{(h, L_Q(h)) : h \in I\}, where

\[ L_Q(x) = \frac{1}{2} \left( \frac{s}{2} + 2x(t - 1) \right) \quad \text{and} \quad I = \left\{ \begin{array}{ll} \left( \frac{s}{2}, \frac{1}{2} \right) & \text{if } s < 1 \\ \left( \frac{1}{2}, \frac{s}{2} \right) & \text{if } s \geq 1. \end{array} \right. \]

We now answer a very important question: How does one find the equation of an ellipse, $E_0$, inscribed in $Q_{s,t}$ and the points of tangency of $E_0$ with $Q_{s,t}$? We sketch the derivation of the equation and points of tangency now. First, since $E_0$ has center $(h, L_Q(h)), h \in I$, one may write the equation of $E_0$ in the form

\[ (x - h)^2 + B(x - h)(y - L_Q(h)) + C(y - L_Q(h))^2 + F = 0. \] (5)

Throughout we let $J$ denote the open interval $(0, 1)$; Now suppose that $E_0$ is tangent to $Q_{s,t}$ at the points $P_q = (q, 0)$ and $P_v = (0, v)$, where $q, v \in J$; Differentiating (5) with respect to $x$ and plugging in $P_q$ and $P_v$ yields

\[ q - h = \frac{BL_Q(h)}{2}, \] (6)

\[ v - L_Q(h) = \frac{Bh}{2C}. \]

Plugging in $P_q$ and $P_v$ into (5) yields $(q - h)^2 - BL_Q(h)(q - h) + C(L_Q(h))^2 + F = 0$ and $h^2 - Bh(v - L_Q(h)) + C(v - L_Q(h))^2 + F = 0; By (6)$, we have
\[ F = \frac{h^2}{4C} (B^2 - 4C) \text{ and } F = \frac{L_Q^2(h)}{4} (B^2 - 4C); \text{ Using both expressions for } F \text{ gives } \]

\[ C = \frac{h^2}{L_Q^2(h)}. \quad (7) \]

Now by (6) again,

\[ B = \frac{2(q - h)}{L_Q(h)}. \quad (8) \]

(6), (8), and (7) then imply that

\[ v = \frac{qL_Q(h)}{h}. \quad (9) \]

Substituting (8) and (7) into \( F = \frac{h^2}{4C} (B^2 - 4C) \) yields

\[ F = q^2 - 2qh; \quad (5) \]

then becomes

\[ (x-h)^2 + \frac{2(q-h)}{L_Q(h)} (x-h)(y-L_Q(h)) + \frac{h^2}{L_Q^2(h)} (y-L_Q(h))^2 + q^2 - 2qh = 0. \quad (10) \]

**Remark 4** Using Lemma 4 it is not hard to show that (10) defines the equation of an ellipse for any \( h \in I \).

Finally, we want to find \( h \) in terms of \( q \), which makes the final equation simpler than expressing everything in terms of \( h \). One way to do this is to use the following well-known Theorem of Marden (see [?]).

**Theorem 2 (Marden):** Let \( F(z) = \frac{t_1}{z - z_1} + \frac{t_2}{z - z_2} + \frac{t_3}{z - z_3}, \sum_{k=1}^{3} t_k = 1, \) and let \( Z_1 \) and \( Z_2 \) denote the zeros of \( F(z) \). Let \( L_1, L_2, L_3 \) be the line segments connecting \( z_2 \& z_3, z_1 \& z_3, \) and \( z_1 \& z_2, \) respectively. If \( t_1t_2t_3 > 0, \) then \( Z_1 \) and \( Z_2 \) are the foci of an ellipse, \( E_0, \) which is tangent to \( L_1, L_2, \) and \( L_3 \) at the points \( \zeta_1, \zeta_2, \zeta_3, \) where

\[ \zeta_1 = \frac{t_2 z_3 + t_3 z_2}{t_2 + t_3}, \quad \zeta_2 = \frac{t_1 z_3 + t_3 z_1}{t_1 + t_3}, \quad \text{and} \quad \zeta_3 = \frac{t_1 z_2 + t_2 z_1}{t_1 + t_2}, \text{ respectively.} \]
Applying Marden’s Theorem to the triangle $A_2A_3A_5$, where $A_5 = 0, -\frac{t}{s-1}$, one can show that $E_0$ is tangent to $Q_{s,t}$ at the point \( \left( \frac{s-2h}{2(t-1)h+s-t}, 0 \right) \); Many of the details of this can be found in [2]. Hence \( q = \frac{1}{2(t-1)h+s-t} \), which implies that
\[
h = \frac{1}{2} q(t-s) + s \frac{t}{2q(t-1) + 1}. \tag{11}
\]
Substituting for $h$ in (10) using (11) and using $L_{Q,s,t}(x) = \frac{1}{2} s - t + 2x(t-1)$, (10) becomes
\[
t^2x^2 + (4q^2(t-1)t + 2qt(s-t + 2) - 2st)xy + ((1-q)s + qt)^2y^2 - 2qt^2x - 2qt((1-q)s + qt)y + q^2t^2 = 0.\tag{12}
\]
One point of tangency is of course given by \((q,0)\); Using (12), it is then not difficult to find the other points of tangency, which is given in the following proposition (we have relabeled $P_q$ and $P_v$).

**Proposition 2** Suppose that $E_0$ is an ellipse inscribed in $Q_{s,t}$; Then $E_0$ is tangent to the four sides of $Q_{s,t}$ at the points
\[
q_1 = \left( 0, \frac{qt}{(t-s)q+s} \right) \in S_1, \quad q_2 = \left( \frac{(1-q)s^2}{(t-1)(s+t)q+s}, \frac{t(s+q(t-1))}{(t-1)(s+t)q+s} \right) \in S_2, \\
q_3 = \left( \frac{s+q(t-1)}{(s+t-2)q+1}, \frac{(1-q)t}{(s+t-2)q+1} \right) \in S_3, \quad \text{and} \quad q_4 = (q, 0) \in S_4, \quad q \in J = (0, 1).
\]

**Remark 5** It is not hard to show that each of the denominators of the $q_i$ above are non-zero.

Finally we state the analogy of Proposition 2 for parallelograms. A slightly different version was proven in [4]. We omit the details of the proof.

**Proposition 3** Let $P$ be the parallelogram with vertices $A_1 = (-l-d, -k), A_2 = (-l+d, k), A_3 = (l+d, k),$ and $A_4 = (l-d, -k)$, where $l, k > 0, d < l$. Suppose that $E_0$ is an ellipse inscribed in $P$. Then $E_0$ is tangent to the four sides of $P$ at the points
\[
q_1 = (-l+dv, kv) \in S_1, \quad q_2 = (-lv+d, k) \in S_2, \\
q_3 = (l-dv, -kv) \in S_3, \quad \text{and} \quad q_4 = (lv-d, -k) \in S_4.
\]
3 Tangency Chords Parallel to the Diagonals

The following lemma gives necessary and sufficient conditions for the quadrilateral $Q_{s,t}$ given in (3) to be a midpoint diagonal quadrilateral.

**Lemma 7** (i) $Q_{s,t}$ is a type 1 midpoint diagonal quadrilateral if and only if $s = t$.

(ii) $Q_{s,t}$ is a type 2 midpoint diagonal quadrilateral if and only if $s + t = 2$.

**Proof.** The diagonals of $Q_{s,t}$ are $D_1: y = \frac{t}{s}x$ & $D_2: y = 1 - x$, and the diagonals intersect at the point $P = \left(\frac{s}{s+t}, \frac{t}{s+t}\right)$; The midpoints of the diagonals of $Q_{s,t}$ are $M_1 = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $M_2 = \left(\frac{s}{2}, \frac{t}{2}\right)$; Now $M_1 = P \iff \frac{s}{s+t} = \frac{1}{2}$ and $\frac{t}{s+t} = \frac{1}{2}$, both of which hold if and only if $s = t$; That proves (i); $M_2 = P \iff \frac{s}{s+t} = \frac{1}{2}s$ and $\frac{t}{s+t} = \frac{1}{2}t$, both of which hold if and only if $s + t = 2$. That proves (ii) ■

**Theorem 3** Suppose that $E_0$ is an ellipse inscribed in a convex quadrilateral $Q = Q(A_1, A_2, A_3, A_4)$; Let $q_j \in S_j, j = 1, 2, 3, 4$ denote the points of tangency of $E_0$ with $Q$, and let $D_1 = A_1A_3$ and $D_2 = A_2A_4$ denote the diagonals of $Q$.

(i) If $Q$ is a type 1 midpoint diagonal quadrilateral, then $\overrightarrow{q_2q_3}$ and $\overrightarrow{q_1q_4}$ are parallel to $D_2$.

(ii) If $Q$ is a type 2 midpoint diagonal quadrilateral, then $\overrightarrow{q_1q_2}$ and $\overrightarrow{q_3q_4}$ are parallel to $D_1$.

(iii) If $Q$ is not a midpoint diagonal quadrilateral, then neither $\overrightarrow{q_1q_2}$ nor $\overrightarrow{q_3q_4}$ are parallel to $D_1$, and neither $\overrightarrow{q_1q_2}$ nor $\overrightarrow{q_1q_4}$ are parallel to $D_2$.

**Proof. Case 1:** $Q$ is not a parallelogram.

Then by Proposition 1, we may assume that $Q$ equals the quadrilateral $Q_{s,t}$ given in (3), with diagonals $D_1: y = \frac{t}{s}x$ and $D_2: y = 1 - x$. Using Proposition 2 after some simplification we have:

slope of $\overrightarrow{q_1q_2} = \frac{t(2q(t-1) + s)}{s((t-s)q + s)}$, so that the slope of $\overrightarrow{q_1q_2} = \frac{t}{s} \iff \frac{2(t-1)q + s}{(t-s)q + s} = 1 \iff (s + t - 2)q = 0 \iff s + t = 2$ since $q = 0 \notin J$. 

9
Proof. Since Remark 6 we actually prove more–that 

\[
s + t = 2 \quad \text{since } q = 0 \notin J.
\]

slope of \( \overrightarrow{q_2q_3} = -1 \quad \iff \quad t(2t - 1)q + s - t + 1 = (s^2 + s - t) q - s^2 + st + s \iff (s + t - 1)(s - t)(q - 1) = 0 \iff s = t \quad \text{since } q = 1 \notin J \text{ and } s + t \neq 1.
\]

Case 2: \( Q \) is a parallelogram.

As noted in the introduction, Theorem 3 is probably known in this case, and there are undoubtedly other ways to prove it for parallelograms. Using Proposition 3, it follows easily that the slope of \( \overrightarrow{q_2q_3} \) is the slope of \( \overrightarrow{q_3q_4} = \frac{k}{l + d} \) and the slope of \( \overrightarrow{q_2q_3} \) is the slope of \( \overrightarrow{q_1q_4} = \frac{k}{d - l} \); Since the diagonals of \( Q \) are \( D_1: y = k + \frac{k}{l + d}(x - l - d) \) and \( D_2: y = k + \frac{k}{d - l}(x + l - d) \), and a parallelogram is a special case of a midpoint diagonal quadrilateral, that proves Theorem 3 for case 2. Note that one could map \( Q \) to the unit square and then use a simplified version of Proposition 3 but that does not simplify the proof very much.

Recall that the lengths of the sides of \( Q(A_1, A_2, A_3, A_4) \) are denoted by \( a = |A_1A_4|, b = |A_1A_2|, c = |A_2A_3|, \) and \( d = |A_3A_4| \).

Lemma 8 Suppose that \( Q = Q(A_1, A_2, A_3, A_4) \) is both a tangential and a midpoint diagonal quadrilateral. Then \( Q \) is an orthodiagonal quadrilateral.

Remark 6 We actually prove more–that \( Q \) is a kite. That is, that two pairs of adjacent sides of \( Q \) are equal.

Proof. Since \( Q \) is tangential, there is a circle, \( E_0 \), inscribed in \( Q \); Let \( q_j \in S_j, j = 1, 2, 3, 4 \) denote the points of tangency of \( E_0 \) with \( Q \); Define the triangles \( T_1 = \Delta q_4A_1q_1 \) and \( T_2 = \Delta A_4A_1A_2 \), and define the lines \( L_1 = \overrightarrow{q_1q_4} \) and \( L_2 = \overrightarrow{q_2q_3} \); Suppose first that \( Q \) is a type 1 midpoint diagonal
quadrilateral. Then \( L_1 \parallel D_2 \) by Theorem 3(i), which implies that \( T_1 \) and \( T_2 \) are similar triangles. Also, since \( E_0 \) is a circle, \(|A_1q_4| = |A_1q_1|\), which implies that \( T_1 \) is isosceles. Hence \( T_2 \) is also isosceles with \( b = a \); In a similar fashion, one can show that \( c = d \) using the fact that \( L_2 \parallel D_2 \); Thus \( a^2 + c^2 = b^2 + d^2 \), which implies that \( Q \) is an orthodiagonal quadrilateral. The proof when \( Q \) is a type 2 midpoint diagonal quadrilateral is very similar and we omit the details.

We now prove a result somewhat similar to Lemma 8.

**Lemma 9** Suppose that \( Q = Q(A_1, A_2, A_3, A_4) \) is both a tangential and an orthodiagonal quadrilateral. Then \( Q \) is a midpoint diagonal quadrilateral.

**Proof.** Since \( Q \) is tangential, there is a circle, \( E_0 \), inscribed in \( Q \) and \( a + c = b + d \), which implies that \( d = a + c - b \); Since \( Q \) is orthodiagonal, \( a^2 + c^2 = b^2 + d^2 \); Hence \( b^2 + (a + c - b)^2 - a^2 - c^2 = 0 \), which implies that \( 2(b - c)(b - a) = 0 \), and so \( a = b \) and/or \( b = c \); We prove the case when \( a = b \); Let \( q_j \in S_j, j = 1, 2, 3, 4 \) denote the points of tangency of \( E_0 \) with \( Q \); Then the triangle \( T_1 = \Delta q_4A_1q_1 \) is isosceles since \(|A_1q_4| = |A_1q_1|\), and the triangle \( T_2 = \Delta A_4A_1A_2 \) is isosceles since \( a = b \); Thus \( T_1 \) and \( T_2 \) are similar triangles, which implies that the line \( \overrightarrow{q_1q_4} \) is parallel to \( D_2 \); By Theorem 3(iii), \( Q \) is a midpoint diagonal quadrilateral.

4 Conjugate Diameters Parallel to the Diagonals

**Theorem 4** (i) Suppose that \( Q \) is a midpoint diagonal quadrilateral. Then each ellipse inscribed in \( Q \) has conjugate diameters parallel to the diagonals of \( Q \).

(ii) Suppose that \( Q \) is not a midpoint diagonal quadrilateral. Then no ellipse inscribed in \( Q \) has conjugate diameters parallel to the diagonals of \( Q \).

**Proof.** Let \( E_0 \) be an ellipse inscribed in \( Q \) and let \( D_1 \) and \( D_2 \) denote the diagonals of \( Q \); Use an affine transformation, \( T \), to map \( E_0 \) to a circle, \( E'_0 \), inscribed in the tangential quadrilateral, \( Q' = T(Q) \); Let \( L_1 \) be a diameter of \( E_0 \) parallel to \( D_1 \); \( T \) maps \( L_1 \) to a diameter, \( L'_1 \), of \( E'_0 \) parallel to one of the diagonals of \( Q' \), which we call \( D'_1 \); Let \( D'_2 \) be the other diagonal of \( Q' \); Let \( L'_2 \) be the diameter of \( E'_0 \) perpendicular to \( L'_1 \), which implies that
$L_1'$ and $L_2'$ are conjugate diameters since $E_0'$ is a circle. By Lemma 4 $T^{-1}$ maps $L_2'$ to $L_2$, a diameter of $E_0$ conjugate to $L_1$; To prove (i), suppose that $Q$ is a midpoint diagonal quadrilateral. By Lemma 3 $Q'$ is also a midpoint diagonal quadrilateral. By Lemma 8 $Q'$ is an orthodiagonal quadrilateral, which implies that $D_1' \perp D_2'$. Since $L_1' \parallel D_1'$, $L_1' \perp L_2'$, and $D_1' \perp D_2'$, $L_2'$ must be parallel to $D_2'$, which implies that $L_2$ is parallel to $D_2$ since $T^{-1}$ is an affine transformation. That proves (i). To prove (ii), suppose that $Q$ is not a midpoint diagonal quadrilateral. Since $Q'$ is tangential, if $Q'$ were also an orthodiagonal quadrilateral, then by Lemma 9 $Q'$ would be a midpoint diagonal quadrilateral. Hence $Q'$ cannot be an orthodiagonal quadrilateral, which implies that $D_1'$ is not perpendicular to $D_2'$; Now if $L_2'$ were parallel to $D_2'$, then it would follow that $D_1' \perp D_2'$ since $L_1' \parallel D_1'$ and $L_1' \perp L_2'$, a contradiction. Hence $L_2' \parallel D_2'$, which implies that $L_2 \parallel D_2$ since $T$ is an affine transformation. That proves (ii).

5 Equal Conjugate Diameters and the Ellipse of Minimal Eccentricity

By Theorem 4(i), each ellipse inscribed in a midpoint diagonal quadrilateral, $Q$, has conjugate diameters parallel to the diagonals of $Q$. In particular, this holds for the unique ellipse of minimal eccentricity (whose existence we prove below), $E_I$, inscribed in $Q$. However, we have the following stronger result for $E_I$.

**Theorem 5** (i) There is a unique ellipse of minimal eccentricity, $E_I$, inscribed in a midpoint diagonal quadrilateral, $Q$.

(ii) Furthermore, if $E_I$ is not a circle (so that $Q$ is not a tangential quadrilateral), then the equal conjugate diameters of $E_I$ are parallel to the diagonals of $Q$. If $E_I$ is a circle (so that $Q$ is a tangential quadrilateral), then some pair of equal conjugate diameters (that is, radii) of $E_I$ are parallel to the diagonals of $Q$.

**Remark 7** Theorem 3(ii) cannot hold if $Q$ is not a midpoint diagonal quadrilateral, since in that case no ellipse inscribed in $Q$ has conjugate diameters parallel to the diagonals of $Q$ by Theorem 4(ii). But Theorem 3(ii) implies the following weaker result: The smallest nonnegative angle between equal conjugate diameters of $E_I$ equals the smallest nonnegative angle between the
diagonals of $Q$ when $Q$ is a midpoint diagonal quadrilateral. This was proven in [4] for parallelograms. We do not know if this property of $E_I$ can hold if $Q$ is not a midpoint diagonal quadrilateral.

Before proving Theorem 5, we need several preliminary results. We omit the details for the proof of Theorem 5 when $Q$ is a parallelogram. So suppose that $Q$ is a midpoint diagonal quadrilateral which is not a parallelogram. By using an isometry of the plane, we may assume that $Q$ has vertices $(0, 0), (0, u), (s, t),$ and $(v, w),$ and where $s, v, u > 0, t > w;$. To obtain this isometry, first, if $Q$ has a pair of parallel vertical sides, first rotate counterclockwise by $90^\circ,$ yielding a quadrilateral with parallel horizontal sides. Since we are assuming that $Q$ is not a parallelogram, we may then also assume that $Q$ does not have parallel vertical sides. One can now use a translation, if necessary, to map the lower left hand corner vertex of $Q$ to $(0, 0);$ Finally a rotation, if necessary, yields vertices $(0, 0), (0, u), (s, t), \text{ and } (v, w);$ Note that such a rotation leaves $Q$ without parallel vertical sides. In addition, by Lemma 5 with $T(x, y) = \frac{1}{u} (x, y),$ we may also assume that one of the vertices of $Q$ is $(0, 1);$ So we assume now that $Q = Q_z,$ where $Q_z = Q(A_1, A_2, A_3, A_4), A_1 = (0, 0), A_2 = (0, 1), A_3 = (s, t), A_4 = (v, w),$ and $s, v > 0, t > w, s \neq v.$ (13)

By Lemma 3 $Q_z$ is a midpoint diagonal quadrilateral, which implies, by Lemma 6 that $Q_z$ is not a trapezoid since $Q_z$ is not a parallelogram. Now let

$L_1: x = 0, L_2: y = 1 + \frac{t - 1}{s} x, L_3: y = w + \frac{t - w}{s - v} (x - v), \text{ and } L_4: y = \frac{w}{v} x$ denote the lines which make up the boundary of $Q_z.$

• Since $Q$ is convex, $(s, t)$ must lie above $(0, 1) (v, w)$ and $(v, w)$ must lie below $(0, 0) (s, t),$ which implies that

$$v(t - 1) + (1 - w)s > 0, vt - ws > 0.$$ (14)

Since no two sides of $Q$ are parallel, $L_2 \not\parallel L_4,$ which implies that

$$ws - v(t - 1) \neq 0.$$ (15)

$M_1 = \left(\frac{1}{2} v, \frac{1}{2} (w + 1)\right) \text{ and } M_2 = \left(\frac{1}{2} s, \frac{1}{2} t\right)$ are the midpoints of the diagonals of $Q_z$ and the equation of the line thru $M_1$ and $M_2$ is
\[ y = L(x) = \frac{t}{2} + \frac{w + 1 - t}{v - s} \left( x - \frac{s}{2} \right), \quad x \in I, \]  

where

\[ I = \begin{cases} 
\left( \frac{v}{2}, \frac{s}{2} \right) & \text{if } v < s \\
\left( \frac{s}{2}, \frac{v}{2} \right) & \text{if } s < v
\end{cases}. \]  

The diagonals of \( Q_z \) are \( D_1 = (0, 0) (s, t) = \) diagonal from lower left to upper right and \( D_2 = (0, 1) (v, w) = \) diagonal from lower right to upper left.

Now define the following cubic polynomial:

\[ R(h) = (s - 2h)(2h - v)(2(v(t - 1) - ws)h + vs). \tag{18} \]

Note that the three roots of \( R \) are \( h_1 = \frac{1}{2} v, h_2 = \frac{1}{2} s, \) and \( h_3 = \frac{1}{2} \frac{vs}{ws - v(t - 1)}; \) Now \( (s - 2h)(2h - v) > 0 \) for any \( h \in I; \) Also, the linear function of \( h \) given by \( f(h) = 2(v(t - 1) - ws)h + vs \) is positive at the endpoints of \( I \) by (13) and (14) since \( f \left( \frac{v}{2} \right) = v(v(t - 1) + (1 - w)s) \) and \( f \left( \frac{s}{2} \right) = s(vt - ws); \) Thus \( f(h) > 0 \) on \( I, \) which implies that

\[ R(h) > 0, h \in I. \]  

We now state a result somewhat similar to Proposition 2 for the quadrilateral \( Q_z \) below. Proposition 4 gives necessary and sufficient conditions for the general equation of an ellipse inscribed in \( Q_z, \) but we do not require the points of tangency.

**Proposition 4** Let \( I \) be given by (17), let \( L \) be given by (16), and suppose that (13), (14), and (15) hold. Then \( E_0 \) is an ellipse inscribed in \( Q_z \) if and only if the general equation of \( E_0 \) is given by

\[
4(s - v) \left( (s - v)L^2(h) + w(2h - s) \right) (x - h)^2 \\
+ 4(s - v)^2 h^2 (y - L(h))^2 + 4(s - v) \times \\
( -2t + 2 + 2w) h^2 + \\
(vt - s - ws - 2v) h + sv \right) (x - h) (y - L(h)) \\
= R(h), h \in I. 
\]  

14
Proof. Using Lemma 1, it is not hard to show that (20) defines the equation of an ellipse for any \( h \in I \). Using standard calculus techniques, it is also not difficult to show that the ellipse defined by (20) is inscribed in \( Q_z \) for any \( h \in I \); The converse result, that any ellipse inscribed in \( Q_z \) has equation given by (20) for some \( h \in I \) can be proven in a similar fashion to the proof of Proposition 2, but with more somewhat cumbersome details, which we omit here.

By Proposition 4, the general equation of an ellipse inscribed in \( Q_z \) has the form

\[
A(h) (x - h)^2 + B(h) (x - h) (y - L(h)) + C(h) (y - L(h))^2 = R(h),
\]

\[
A(h) = 4 (s - v)^2 \left( \left( \frac{1}{2} t + \frac{w + 1 - t}{s - v} \left( h - \frac{1}{2} s \right) \right)^2 + \frac{w (2h - s)}{s - v} \right),
\]

\[
B(h) = 4 (s - v) \left( 2 (1 + w - t) h^2 + (vt - ws - s - 2v) h + vs \right),
\]

\[
C(h) = 4 (s - v)^2 h^2.
\] (21)

The purpose of the following lemmas is to show that the eccentricity of an inscribed ellipse, as a function of \( h \), has a unique root in \( I \), where \( I \) is given by (17) throughout this section. We also want to find a formula for that root as well. Assume throughout that (13), (14), and (15) hold. First we define the following quadratic polynomial in \( h \):

\[
o(h) = -2 \left( s^2 + t^2 \right) (s - v) h^2 - 2Kh + sK,
\]

\[
K = \left( s^2 + t^2 \right) v^2 - 2wstv + 2s^2 w^2.
\] (22)

Lemma 10 : \( K > 0 \).

Proof. The discriminant of \( K \) as a quadratic in \( v \) is given by \((-2wst)^2 - 4 (s^2 + t^2) (2s^2w^2) = -4s^2w^2(t^2 + 2s^2) < 0 \), which implies that \( K(v) \) has no real roots. Since \( K(0) = 2s^2w^2 > 0 \), it follows that \( K(v) > 0 \) for all real \( v \).

Lemma 11 \( o \) has exactly one root in \( I \).

Proof. Let \( p_1 = (2ws - vt)^2 + v^2 s^2 \); Then it follows that

\[
o \left( \frac{v}{2} \right) = \frac{1}{2} (s - v) p_1, o \left( \frac{s}{2} \right) = -\frac{1}{2} s^2 (s - v) \left( s^2 + t^2 \right).
\] (23)
Since \( p_1 > 0 \), \( o \left( \frac{v}{2} \right) o \left( \frac{s}{2} \right) < 0 \), which implies that \( o \) has an odd number of roots in \( I \); Since \( o \) is a quadratic, \( o \) must have one root in \( I \). Note that since \( o \) has two distinct real roots, the discriminant of \( o \), \( 4K^2 + 8(s^2 + t^2)(s - v) \), is positive. By Lemma 11, \( o \) has two distinct real roots, the discriminant of \( o \), \( 4K(2(s^2 + t^2)s(s - v) + K) \), is positive. By Lemma 11, \( 2(s^2 + t^2)s(s - v) + K > 0 \) and by the quadratic formula, the roots of \( o \) are

\[
h_- = \frac{-\sqrt{K} - \sqrt{2(s^2 + t^2)s(s - v) + K}}{2(s^2 + t^2)(s - v)}, \quad (24)\]

\[
h_+ = \frac{-\sqrt{K} + \sqrt{2(s^2 + t^2)s(s - v) + K}}{2(s^2 + t^2)(s - v)}. \]

Note that \( h_+ - h_- = \frac{\sqrt{K} \sqrt{2(s^2 + t^2)s(s - v) + K}}{(s^2 + t^2)(s - v)} \), which implies that

\[
\begin{cases} 
  h_- > h_+ & \text{if } s < v \\
  h_+ > h_- & \text{if } v < s .
\end{cases}
\]

The following lemma allows us to find the unique root of \( o \) in \( I \).

Lemma 12 \( h_+ \) is the unique root of \( o \) in \( I \).

Proof. Case 1: \( s > v \); Then \( I = \left( \frac{v}{2}, \frac{s}{2} \right) \) and \( h_- < 0 < \frac{v}{2} \), which implies that \( h_- \notin I \); By Lemma 11, \( h_+ \in I \).

Case 2: \( s < v \); Then \( I = \left( \frac{s}{2}, \frac{v}{2} \right) \); Since \( \lim_{h \to \infty} o(h) = \infty \) and \( o \left( \frac{v}{2} \right) < 0 \) by (23), \( o \) has a root in the interval \( \left( \frac{v}{2}, \infty \right) \); Hence the smaller of the two roots of \( o \) must lie in \( I \); Since \( h_+ < h_- \), one must have \( h_+ \in I \). ■

Solving for \( h^2_+ \) in the equation \( o(h_+) = 0 \) yields the following very useful identity:

\[
h^2_+ = \frac{(s - 2h_+)K}{2(s^2 + t^2)(s - v)}, \quad (25)
\]

where \( K \) is given by (22). Now define the polynomials

\[
J(h) = A(h) + C(h), \\
M(h) = (A(h) - C(h))^2 + (B(h))^2,
\]

where \( A(h), B(h), \) and \( C(h) \) are given by (21). Note that ; Some simplification yields

\[
J^2(h) - M(h) = 16(s - v)^2R(h), \quad (27)
\]

where \( R(h) \) is given by (18).
Lemma 13 \( J(h) > 0, h \in I. \)

**Proof.** If \( J(h_0) = 0 \) for some \( h_0 \in I \), then \( R(h_0) \leq 0 \) by (27) and the fact that \( M(h) \geq 0 \), which contradicts (19); Hence \( J(h) \) is nonzero on \( I \); Since 
\[
J\left(\frac{v}{2}\right) = (s - v)^2 ((w - 1)^2 + v^2) > 0,
\]
it follows that \( J(h) > 0, h \in I. \) ■
Note that Lemma 13 implies that

\[
J(h) + \sqrt{M(h)} > 0, h \in I. \tag{28}
\]

In the next section we will adapt the results in this section to midpoint diagonal quadrilaterals. Towards that end, recall the quadrilateral, \( Q_z \), from ??, where \( s, t, v, w \) satisfy (13), (14), and (15). The following lemma gives necessary and sufficient conditions for \( Q_z \) to be a midpoint diagonal quadrilateral.

**Lemma 14** (i) \( Q_z \) is a type 1 midpoint diagonal quadrilateral if and only if
\[
v = \frac{(w + 1)s}{t}. \tag{29}
\]

(ii) \( Q_z \) is a type 2 midpoint diagonal quadrilateral if and only if
\[
(t - 2)v = (w - 1)s. \tag{30}
\]

**Proof.** Recall that \( L(x) = \frac{t}{2} + \frac{w + 1 - t}{v - s} \left( x - \frac{s}{2} \right) \). The line containing \( D_1 \) is \( y = \frac{t}{s} x \), which implies that \( D_1 \subset L \iff \)
\[
\frac{w + 1 - t}{v - s} = \frac{t}{s},
\]
\[
\frac{t}{2} - \frac{2}{2} \frac{w + 1 - t}{v - s} = 0.
\]
It follows easily that (31) holds if and only if (29) holds, which proves (i). The proof of (ii) follows in a similar fashion. ■
5.1 Proof of Theorem 5

Proof. In [2] we proved that there is a unique ellipse of minimal eccentricity inscribed in any convex quadrilateral, \( Q \); The uniqueness for midpoint diagonal quadrilaterals would then follow from that result. However, the proof here, specialized for midpoint diagonal quadrilaterals, is self-contained, uses different methods, and does not require the result from [2]. As noted earlier, it suffices to prove Theorem 5 for the quadrilateral \( Q = Q_z \); To prove (i), let \( E_0 \) be an ellipse inscribed in \( Q_z \), and let \( a = a(h) \) and \( b = b(h) \) denote the lengths of the semi-major and semi-minor axes, respectively, of \( E_0 \). Since the square of the eccentricity of \( E_0 \) equals \( 1 - \frac{b^2}{a^2} \), it suffices to maximize \( \frac{b^2}{a^2} \), which is really a function of \( h \in I \) since we allow \( E_0 \) to vary over all ellipses inscribed in \( Q_z \); Thus we want to maximize \( G(h), h \in I \), where \( G(h) = \frac{b^2(h)}{a^2(h)} \): Using (1), \( G(h) = \frac{J(h) - \sqrt{M(h)}}{J(h) + \sqrt{M(h)}} \), where \( J(h) \) and \( M(h) \) are given by (26). A simple computation yields

\[
G'(h) = \frac{p(h)}{\sqrt{M(h)(J(h) + \sqrt{M(h)})^2}},
\]

where \( p \) is the quartic polynomial given by

\[
p(h) = 2J'(h)M(h) - J(h)M'(h).
\]

We now prove the case when \( Q_z \) is a type 1 midpoint diagonal quadrilaterals, the proof for Type 2 midpoint diagonal quadrilaterals being similar. Thus (29) holds throughout, though we leave the following formulas without substituting for \( v \) since it simplifies the notation somewhat. Using (21), (29), and a computer algebra system to simplify, we have \( p(h) = 256h \left( \frac{s - v}{s} \right)^4 (vt - ws)^2 (s - h) o(h) \), where \( o \) is given by (22). Note that this only holds if one has \( v = \frac{(w + 1)s}{t} \); Hence

\[
G'(h) = \frac{256h \left( \frac{s - v}{s} \right)^4 (vt - ws)^2 (s - h) o(h)}{\sqrt{M(h)(J(h) + \sqrt{M(h)})^2}}.
\]
We assume first that $Q_z$ is a tangential quadrilateral. Then $Q_z$ is an orthodiagonal quadrilateral by Lemma 8, and so the diagonals of $Q_z$ are perpendicular. Also, there is a unique circle, $\Phi$, inscribed in $Q_z$, which implies that $\Phi$ is the unique ellipse of minimal eccentricity inscribed in $Q_z$ since $\Phi$ has eccentricity 0. Since any pair of perpendicular diameters of a circle are equal conjugate diameters, one can choose a pair which is parallel to the diagonals of $Q_z$, and Theorem 5 holds. So assume now that $Q_z$ is not a tangential quadrilateral, which implies that $A(h) \neq C(h)$ for all $h \in I$ and thus $M(h) \neq 0$ for all $h \in I$ by (26); Since $M$ is non–negative we have

$$M(h) > 0, h \in I.$$  

(35)

By (32), (28), and (33), $G$ is differentiable on $I$. By (32) and (33), By Lemma 12 and (34), $h_+$ is the unique root of $G'$ in $I$, where $h_+$ is given by (24). Since $G(h) = \frac{b_2(h)}{a_2(h)}$, it follows that $G(h) > 0$ on $I$; Also, $G\left(\frac{v}{2}\right) = G\left(\frac{s}{2}\right) = 0$; Since $G$ is positive in the interior of $I$ and vanishes at the endpoints of $I$, $h_+$ must yield the global maximum of $G$ on $I$. That proves Theorem 5(i). To prove Theorem 5(ii), by Theorem 4, $E$ has equation $Q \parallel t$; Also, there is a unique circle, $\Phi$, inscribed in $Q$, and so the diagonals of $Q_z$, and Theorem 5 holds. So assume now that $Q_z$ is not a tangential quadrilateral, which implies that $A(h) \neq C(h)$ for all $h \in I$ and thus $M(h) \neq 0$ for all $h \in I$ by (26); Since $M$ is non–negative we have

$$M(h) > 0, h \in I.$$  

(35)

By (32), (28), and (33), $G$ is differentiable on $I$. By (32) and (33), By Lemma 12 and (34), $h_+$ is the unique root of $G'$ in $I$, where $h_+$ is given by (24). Since $G(h) = \frac{b_2(h)}{a_2(h)}$, it follows that $G(h) > 0$ on $I$; Also, $G\left(\frac{v}{2}\right) = G\left(\frac{s}{2}\right) = 0$; Since $G$ is positive in the interior of $I$ and vanishes at the endpoints of $I$, $h_+$ must yield the global maximum of $G$ on $I$. That proves Theorem 5(i). To prove Theorem 5(ii), by Theorem 4, $E$ has conjugate diameters, $L_1$ and $L_2$, parallel to the diagonals, $D_1$ and $D_2$, of $Q_z$; We shall prove that $L_1$ and $L_2$ are equal conjugate diameters of $E_I$; The line containing $D_1$ is $y = \frac{t}{s}x$ and the line containing $D_2$ is $y = 1 + \frac{(w-1)s}{(w+1)s}x$; Since we are assuming that $Q_z$ is type 1, $D_1 \subset L$, the line thru the midpoints of $D_1$ and $D_2$, and thus $L$ has equation $y = \frac{t}{s}x$; Since $E_I$ has center $(h_+, L(h_+)) = (h_+, \frac{zh_+}{s})$ and $L_1$ passes through the center and has slope $\frac{t}{s}$, $L_1 = L$ and thus has equation $y = \frac{t}{s}x$; Since $L_2$ passes through the center and has slope $\frac{(w-1)t}{(w+1)s}$, $L_2$ has equation $y = \frac{t}{s}h_+ + \frac{(w-1)t}{(w+1)s}(x-h_+) = \frac{(w-1)t}{(w+1)s}x + \frac{2th_+}{(w+1)s}$; Now suppose that $L_1$ intersects $E_I$ at the two distinct points $P_1 = (x_1, y_1) = (x_1, \frac{t}{s}x_1)$ and $P_2 = (x_2, y_2) = (x_2, \frac{t}{s}x_2)$; Since $P_1$ and $P_2$ lie on $E_I$, by (21) with $h = h_+$ we have $A(h_+)(x_j-h_+)^2 + B(h_+)(y_j-h_+)(\frac{t}{s}x_j-\frac{t}{s}h_+)+C(h_+)((\frac{t}{s}x_j-\frac{t}{s}h_+)^2 = R(h_+), j = 1, 2$, which simplifies to $A(h_+)(x_j-h_+)^2 + (\frac{t}{s})B(h_+)(x_j-h_+)^2 + (\frac{t}{s})C(h_+)(x_j-h_+)^2 = R(h_+)$, where $R(h)$ is given by (18). Hence

$$A(h_+)(x_j-h_+)^2 = \frac{R(h_+)}{A(h_+)+B(h_+)+C(h_+)}, j = 1, 2.$$  

(36)

Similarly, suppose that $L_2$ intersects $E_I$ at the two distinct points $P_3 =
which implies that \((x_3, y_3) = (x_3, \frac{(w-1)t}{(w+1)s} x_3 + \frac{2h_+}{(w+1)s})\) and \(P_4 = (x_4, y_4) = \left(x_4, \frac{(w-1)t}{(w+1)s} x_4 + \frac{2h_+}{(w+1)s}\right)\);

By (21) again we have
\[ A(h_+) (x_j - h_+)^2 + B(h_+) \left(\frac{(w-1)t}{(w+1)s} x_j + \frac{2h_+}{(w+1)s} - \frac{t}{s} h_+\right) +
C(h_+) \left(\frac{(w-1)t}{(w+1)s} x_j + \frac{2h_+}{(w+1)s} - \frac{t}{s} h_+\right)^2 = R(h_+), j = 3, 4, \text{ which simplifies to}
\[ A(h_+) (x_j - h_+)^2 + B(h_+) \left(\frac{(w-1)t}{(w+1)s} (x_j - h_+)\right)^2 + C(h_+) \left(\frac{(w-1)t}{(w+1)s}\right)^2 (x_j - h_+)^2 = R(h_+); \text{ Hence}
\[ (x_j - h_+)^2 = \frac{R(h_+)}{A(h_+) + \left(\frac{(w-1)t}{(w+1)s}\right) B(h_+) + \left(\frac{(w-1)t}{(w+1)s}\right)^2 C(h_+)}, j = 3, 4. \tag{37}
\]

Now \(L_1\) and \(L_2\) are equal conjugate diameters if and only if \(|P_1 P_2| = |P_3 P_4| \iff \]
\[ (x_2 - x_1)^2 + (y_2 - y_1)^2 = (x_4 - x_3)^2 + (y_4 - y_3)^2. \tag{38}
\]

By (36), \((x_2 - h_+)^2 = (x_1 - h_+)^2\), which implies that \(x_2^2 - 2h_+x_2 - x_1^2 + 2h_+x_1 = 0\), and so \(x_2 + x_1 = 2h_+\); It then follows that \(x_2 - x_1 = 2(h_+ - x_1)\), which implies that \((x_2 - x_1)^2 + (y_2 - y_1)^2 = \left(1 + \left(\frac{t}{s}\right)^2\right) (x_2 - x_1)^2 = 4 \left(1 + \left(\frac{t}{s}\right)^2\right) (h_+ - x_1)^2\); Similarly, by (37), \(x_4 - x_3 = 2(h_+ - x_3)\), which implies that \((x_4 - x_3)^2 + (y_4 - y_3)^2 \leq \left(1 + \left(\frac{(w-1)t}{(w+1)s}\right)^2\right) (x_4 - x_3)^2 \]
\[ = 4 \left(1 + \left(\frac{(w-1)t}{(w+1)s}\right)^2\right) (h_+ - x_3)^2; \text{ Hence (38) holds if and only if} \left(1 + \left(\frac{t}{s}\right)^2\right) (h_+ - x_1)^2 \leq \left(1 + \left(\frac{(w-1)t}{(w+1)s}\right)^2\right) (h_+ - x_3)^2, \text{ which is equivalent to}
\]
\[
\frac{A(h_+) + \left(\frac{t}{s}\right) B(h_+) + \left(\frac{t}{s}\right)^2 C(h_+)}{A(h_+) + \left(\frac{(w-1)t}{(w+1)s}\right) B(h_+) + \left(\frac{(w-1)t}{(w+1)s}\right)^2 C(h_+)} =
\frac{1 + \left(\frac{t}{s}\right)^2}{1 + \left(\frac{(w-1)t}{(w+1)s}\right)^2} \tag{39}
\]

by (36) and (37). Now by (21) and (25), it follows, after some simplification,
Consider the quadrilateral, \( P \), and let
\[
\begin{align*}
A(h_+) &= \frac{2(w-t+1)(t^2K-2ws^2(s^2+t^2))}{s(t^2+2t)} (2h_+ - s), \\
B(h_+) &= -\frac{4w(w-t+1)s^2(w(s^2+t^2)+s^2-t^2)}{t^2(s^2+t^2)} (2h_+ - s), \\
C(h_+) &= \frac{2s(w-t+1)}{t(s^2+t^2)} (2h_+ - s),
\end{align*}
\]
where \( K \) is given by (22). By (10),
\[
\frac{A(h_+) + (\frac{t}{s}) B(h_+) + (\frac{t}{s})^2 C(h_+)}{A(h_+) + (\frac{w-1}{t}) B(h_+) + (\frac{w-1}{t})^2 C(h_+)}
= \frac{-4s t^{-w-1} (w+1)}{-4s (t - w - 1) \frac{2s^2w^2-2w^2w^2+t^2w^2+w^2t^2}{(w+1)^2(s^2+t^2)}} = \frac{(s^2+t^2)(w+1)^2}{(w^2+1)(s^2+t^2)+2w(s^2-t^2)} = 1 + \left(\frac{t}{s}\right)^2,
\]
and thus (39) holds. That proves Theorem 5(ii). □

6 Example

Consider the quadrilateral, \( Q \), with vertices \((0, 0), (0, 1), (8, 4), \) and \((6, 2)\); thus \( Q = Q_z \), with \( s = 8, t = 4, v = 6, \) and \( w = 2, \) which satisfy (13), (14), and (15); \( Q \) is a type 1 midpoint diagonal quadrilateral since \( v = \frac{t}{(w+1)s}; \)
\( I = (3, 4), D_1 \subset y = \frac{1}{2} x, \) which is the equation of \( L, \) and \( D_2 \subset y = 1 + \frac{1}{6} x \)

- Let \( E_0 \) be the ellipse inscribed in \( Q \) corresponding to \( h = h_0 = \frac{7}{2}. \) Then the center of \( E_0 \) is \( \left(\frac{7}{2}, \frac{7}{4}\right) \) and by (21), the equation of \( E_0 \) is \( 33x^2 - 148xy + 196y^2 + 28x - 16y = -36; \)

- Let \( q_1 \) be the ellipse inscribed in \( Q \) with \( q = \left(\frac{3}{7}\right), q_2 = \left(\frac{32}{9}, \frac{7}{3}\right), q_3 = \left(\frac{62}{9}, \frac{26}{9}\right), \) and \( q_4 = \left(\frac{18}{7}, \frac{6}{7}\right) \)

As guaranteed by Theorem 3, the slope of \( \frac{q_2q_3}{q_2q_3} = \) slope of \( \frac{q_4q_4}{q_1q_1} = \frac{1}{6} = \) slope of \( D_2 \)

- Let \( P_1 = \left(\frac{7-\sqrt{13}}{2}, \frac{7-\sqrt{13}}{4}\right), P_2 = \left(\frac{7+\sqrt{13}}{2}, \frac{7+\sqrt{13}}{4}\right), P_3 = \left(\frac{7-3\sqrt{7}}{2}, \frac{7-3\sqrt{7}}{4}\right), \) and \( P_4 = \left(\frac{7+3\sqrt{7}}{2}, \frac{7+3\sqrt{7}}{4}\right) \)
As guaranteed by Theorem 4, \( P_1P_2 \) and \( P_3P_4 \) are conjugate diameters of \( E_0 \) which are parallel to the diagonals of \( Q 

- Let \( E_I \) be the unique ellipse of minimal eccentricity inscribed in \( Q \)

- Let \( m = 41 \); Using (21) with \( h_0 = \frac{-m + 9\sqrt{m}}{5} \), \( E_I \) has equation

\[
(1281 - 189\sqrt{m})x^2 + (-6344 + 936\sqrt{m})xy + (10004 - 1476\sqrt{m})y^2 + (-11644 + 1836\sqrt{m})x + (69864 - 11016\sqrt{m})y = -126756 + 19764\sqrt{m}
\]

- Let \( n = 74 \), \( q = 10 \), \( P_1 = \frac{-82 + 18\sqrt{m} + 9\sqrt{n} - \sqrt{mn}}{20}(2, 1) \), \( P_2 = \frac{-82 + 18\sqrt{m} + 9\sqrt{n} - \sqrt{mn}}{20}(2, 1) \), \( P_3 = \left( \frac{-82 + 18\sqrt{m} + 27\sqrt{q} - 3\sqrt{mq}}{20}, \frac{-82 + 18\sqrt{m} + 27\sqrt{q} - 3\sqrt{mq}}{20} \right) \), and \( P_4 = \left( \frac{-82 + 18\sqrt{m} + 9\sqrt{q} - \sqrt{mq}}{20}, \frac{-82 + 18\sqrt{m} + 9\sqrt{q} - \sqrt{mq}}{20} \right) \);

As guaranteed by Theorem 5, \( P_1P_2 \) and \( P_3P_4 \) are equal conjugate diameters of \( E_I \) which are parallel to the diagonals of \( Q \).

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