AN ENERGY-CAPACITY INEQUALITY FOR LEGENDRIAN SUBMANIFOLDS

GEORGIOS DIMITROGLOU RIZELL AND MICHAEL G. SULLIVAN

ABSTRACT. We prove that the number of Reeb chords between a Legendrian submanifold and its contact Hamiltonian push-off is at least the sum of the $\mathbb{Z}_2$-Betti numbers of the submanifold, provided that the contact isotopy is sufficiently small when compared to the smallest Reeb chord on the Legendrian. Moreover, the established invariance enables us to use two different contact forms: one for the count of Reeb chords and another for the measure of the smallest length, under the assumption that there is a suitable symplectic cobordism from the latter to the former. The size of the contact isotopy is measured in terms of the oscillation of the contact Hamiltonian, together with the maximal factor by which the contact form is shrunk during the isotopy. The main tool used is a Mayer–Vietoris sequence for Lagrangian Floer homology, obtained by “neck-stretching” and “splashing.”

1. Introduction

A by now famous Energy-Capacity inequality for closed Lagrangian submanifolds of tame symplectic manifolds was obtained in [41] by Polterovich. Roughly speaking, this inequality provides a lower bound for the displacement energy of a closed Lagrangian submanifold $L$ (this is an expression involving the Hofer norm [34]) in terms of the minimal area of a pseudoholomorphic disc with boundary on $L \subset (X,\omega)$ of a tame symplectic manifold. In [15] an even stronger statement was established by Chekanov: the number of intersections $L \cap \phi(L)$ is bounded from below by $\dim H_\ast(L;\mathbb{Z}_2)$ whenever $\phi$ is a Hamiltonian diffeomorphism whose Hofer norm is less than this minimal area of a pseudoholomorphic disc, under the additional assumption that the intersection is transverse; see Theorem 1.8 below for a precise formulation. Our goal is to provide a contact geometric analogue of the latter result. There have been previous generalizations of Chekanov’s result in this direction, albeit not in the full generality that is considered here; see Section 1.3 below for an overview.

A contact manifold $(Y,\xi)$ with a contact form is an $(2n + 1)$-dimensional manifold with a maximally non-integrable field of tangent hyperplanes $\xi \subset TY$, and a Legendrian submanifold is an $n$-dimensional submanifold $\Lambda \subset (Y,\xi)$ which is tangent to $\xi$. We will assume that $\xi$ is co-orientable and choose a contact form $\alpha$, i.e. a one-form satisfying $\xi = \ker \alpha$, which gives rise to the Reeb vector field $R_\alpha$ (whose dynamics depends heavily on $\alpha$). A choice of contact form induces a bijective correspondence between contact Hamiltonians $H_t : Y \to \mathbb{R}$ and contact isotopies $\phi^t_{\alpha,H_t} : (Y,\xi) \to (Y,\xi)$ starting at the identity (note that a contact isotopy need not preserve $\alpha$). We refer to Section 2 for more details.

Two Legendrian submanifolds do not generically intersect. However, generically there are integral curves of $R_\alpha$ connecting them: so-called Reeb chords. In fact, if a Legendrian isotopy $\Lambda_t$ is sufficiently $C^0$-small for $t \in [0, 1]$, it follows by the classical result due to Laudenbach–Sikorav [37] and Chekanov [14] that there are at least a number $\dim H_\ast(\Lambda;\mathbb{Z}_2)$ of Reeb chords between $\Lambda_0$ and $\Lambda_1$ for a fixed contact form (in the case when the isotopy is $C^1$-small the bound can be obtained by elementary results from differential topology).

A $C^0$-small Legendrian push-off of $\Lambda \subset (Y,\alpha)$ is contained inside a standard contact neighborhood as in [29], i.e. a neighborhood which can be identified with a neighborhood of the zero-section of $(J^1\Lambda, dz - \lambda_\Lambda)$ under which $\Lambda$ is identified with the zero section. This identification preserves the contact form, and $\lambda_\Lambda$ denotes the so-called Liouville form on $T^*\Lambda$. The aforementioned Reeb chords between $\Lambda$ and the push-off are in bijective correspondence with the intersection points inside $T^*\Lambda$ between the corresponding
images under the canonical projections. The contact-geometric counterpart of Chekanov’s result would therefore answer the following question:

Given a Legendrian submanifold $\Lambda \subset (Y, \alpha)$, how “large” (in the appropriate sense) of a contact Hamiltonian $H_t$ is needed in order for there to be less than a number $\dim H_s(\Lambda, \mathbb{Z}_2)$ of $\alpha$-Reeb chords that go either from $\Lambda$ to $\phi^t_{\alpha,H_t}(\Lambda)$ or vice versa?

Note that the non-relative counterpart of this question concerns the number of so-called translated points of a contactomorphism. This question has received somewhat more attention in comparison with the relative case studied here. We refer to the work of Albers–Frauenfelder [4], Albers–Fuchs–Merry [5, 6], Albers–Hein [7], Sandon [46], and Shelukhin in [47] for related results concerning existence of translated points.

1.1. Results. Given a pair Legendrian submanifold $\Lambda_0, \Lambda_1 \subset (Y, \alpha)$ by

$$Q_\alpha(\Lambda_0, \Lambda_1; a, b) = -\infty \leq a \leq b \leq +\infty,$$

we denote the union of all Reeb chords for the contact form $\alpha$ either

- starting on $\Lambda_0$, ending on $\Lambda_1$, and being of length $0 < \ell \in [a, b]$, or
- starting on $\Lambda_1$, ending on $\Lambda_0$, and being of length $0 < \ell \in [-b, -a]$.

We also write $Q_\alpha(\Lambda_0, \Lambda_1) := Q_\alpha(\Lambda_0, \Lambda_1; -\infty, +\infty)$. Our goal is to obtain a lower bound on these subsets in the case when $\Lambda_0 = \Lambda$ and $\Lambda_1 = \phi^1_{\alpha,H_t}(\Lambda_0)$ whenever $H_t$ is sufficient small in the appropriate sense.

Fix a contact Hamiltonian $H_t : Y \to \mathbb{R}$. Since $\phi^t_{\alpha,H_t}$ preserves the contact distribution $\xi$, we have

$$(\phi^t_{\alpha,H_t})^* \alpha = e^{-\tau^t_{\alpha,H_t}} \alpha,$$

where $e^{-\tau^t_{\alpha,H_t}}$ is called the conformal factor of the contactomorphism. To the contact isotopy $\phi^t_{\alpha,H_t}$ we can then associate the numbers

$$\begin{align*}
A &:= \max_{y \in Y} \min_{t \in [0, 1]} \tau^t_{\alpha,H_t}(y) \geq 0, \\
B &:= -\min_{y \in Y} \max_{t \in [0, 1]} \tau^t_{\alpha,H_t}(y) \geq 0, \\
M_+ &:= -e^A \int_0^1 \min_{y \in Y} H_t dt, \\
M_- &:= -e^A \int_0^1 \max_{y \in Y} H_t dt, \\
\|H\|_{osc} &:= \int_0^1 (\max_{y \in Y} H_t - \min_{y \in Y} H_t) dt \geq 0.
\end{align*}$$

(The signs and the notation $M_\pm$ may look odd at first sight, but this notation will become useful in Section 2 when proving our main theorem.)

Instead of the minimal area of pseudoholomorphic discs, here we consider the following related geometric quantity:

$$\sigma(\alpha, \Lambda) := \text{The minimal } \alpha\text{-length of Reeb chords and periodic Reeb orbits } \gamma \text{ satisfying } [\gamma] = 0 \in \pi_1(Y, L).$$

Note that $\sigma(\alpha, \Lambda) = +\infty$ holds when the set of contractible chords and orbits is empty.

In the following we assume that the Legendrian submanifold $\Lambda \subset (Y, \alpha)$ and the contact manifolds are closed, and that $\alpha$ is generically chosen, making both the periodic Reeb orbits and Reeb chords on $\Lambda$ non-degenerate. We defer the notion of a Lagrangian concordance to Section 2 but we note that $\Lambda \subset (Y, \alpha)$ is Lagrangian concordant to itself.

**Theorem 1.2.** Let $L$ be a Lagrangian concordance from $\Lambda_- \subset (Y_-, \alpha_-)$ to $\Lambda \subset (Y, \alpha)$, and consider a contact Hamiltonian $H_t : Y \to \mathbb{R}$ which satisfies

$$e^A \|H\|_{osc} < \sigma(\alpha_-, \Lambda_-).$$

(1.3)
Then
\[ |Q_\alpha(\Lambda, \phi^1_{\alpha,H_t}(\Lambda); -M_+, -M_-)| \geq \sum_{i=0}^{\dim \Lambda} \dim H_i(\Lambda; \mathbb{Z}_2), \]
assuming that the latter Reeb chords are transverse.

We emphasize that the contact form \( \alpha_- \) for which the obstruction is measured, and the contact form \( \alpha \) for which the count is performed, need not be the same. One nontrivial consequence is that the result can be used as an obstruction to the existence of Lagrangian concordances (cylindrical cobordisms) inside symplectic cobordisms. We refer to Corollary 1.7 below for the case of exact Lagrangian concordances embedded in the symplectization. In addition, note that Lagrangian concordances also arise naturally when we interpolate between two contact forms, as described by Example 2.7. In the case of a “displaceable” Legendrian embedding, we hence immediately conclude that:

**Corollary 1.4.** Assume that
\[ |Q_\alpha(\Lambda, \phi^1_{\alpha,H_t}(\Lambda); -M_+, -M_-)| < \sum_{i=0}^{\dim \Lambda} \dim H_i(\Lambda; \mathbb{Z}_2), \]
is satisfied for the contact Hamiltonian \( H_t \), where all chords are assumed to be transverse. Then for any smooth function \( f: Y \to (0, 1] \), the contact form \( \alpha_- := e^f \alpha \) satisfies
\[ e^A \|H\|_{\text{osc}} > \sigma(\alpha_-, \Lambda_>). \]
In particular, there exists either a contractible chord or orbit for the contact form \( \alpha_- \) satisfying a fixed bound of its length.

With a refined version of the non-bubbling theorem proven in Section 4 it should be possible to replace in the hypothesis of Theorem \( 1.2 \) \( \sigma(\alpha, \Lambda) \) by the minimal \( d\alpha_- \)-energy of either a pseudoholomorphic plane in \( \mathbb{R} \times Y_- \), or a one-punctured pseudoholomorphic disc with boundary on \( \mathbb{R} \times \Lambda_- \). Such a plane or disc is asymptotic to either a periodic Reeb orbit or a Reeb chord on \( \Lambda_- \). Together with the fact that the \( d\alpha_- \)-energy of these pseudoholomorphic curves is equal to the length of the asymptotic orbit by Stokes’ theorem, it now follows that the latter quantity is less than or equal to \( \sigma(\alpha, \Lambda) \).

Theorem \( 1.2 \) also extends to certain non-compact contact manifolds, such as one-jet spaces, if a standard contact form is used outside of a compact subset; we refer to Section 3.7 for more details.

**Corollary 1.5.** Let \( \Lambda \subset (Y, \xi) \) be a Legendrian submanifold and let \( \alpha_0 \) be a (not necessarily generic) contact form on \( (Y, \xi) \) which is relatively hypertight, i.e. for which \( \sigma(\alpha_0, \Lambda) = +\infty \). Suppose that \( \Lambda' \) is Legendrian isotopic to \( \Lambda \). For any choice of contact form \( \alpha \), we then have the bound
\[ |Q_\alpha(\Lambda, \phi^1_{\alpha,H_t}(\Lambda); -M_+, -M_-)| \geq \sum_{i=0}^{\dim \Lambda} \dim H_i(\Lambda; \mathbb{Z}_2), \]
given that the latter chords are transverse.

**Proof.** The contact form \( \alpha \) can be written as \( \alpha = e^f \alpha_0 \) for some real-valued function \( f: Y \to \mathbb{R} \). Choose any constant \( m > -\min_Y f \). It follows that \( \alpha_- := e^{-m} \alpha_0 \) also is a relatively hypertight contact form, while \( \alpha_- = e^{-(f+m)} \alpha \) with \( -(f+m) < -\min f + \min f = 0 \). After a sufficiently small perturbation of the contact form \( \alpha_- \) it may be assumed to be generic, while the inequality \( e^A \|H\|_{\text{osc}} > \sigma(\alpha_-, \Lambda) \) still is satisfied for the contact Hamiltonian generating the isotopy from \( \Lambda \) to \( \phi^1_{\alpha,H_t}(\Lambda) \). Since it is possible to assume that \( \alpha_- = e^g \alpha \) holds for some smooth function \( g: Y \to (0, 1) \) also after the perturbation, the result now follows directly from Corollary 1.4. \( \square \)

**Example 1.6.** The following are well-known examples of Legendrian submanifolds satisfying \( \sigma(\alpha_0, \Lambda) = +\infty \) to which the above corollary can be applied.
Corollary 1.7. Under the above assumptions, we have

\[ S < 0 \]

Suppose \( \Lambda(\alpha) \) \( \in \mathcal{L}(\alpha) \) and \( \Lambda \) or vice versa. (Set this infimum to \( +\infty \) if no such Hamiltonian exists.) Suppose \( \Lambda_-, \Lambda_+ \subset Y \) are two Legendrians which are Lagrangian concordant in the symplectization \((\mathbb{R} \times Y, d(e^t\alpha))\). Without loss of generality, assume that the concordance is cylindrical outside of \([S, 0] \times Y\), where \( S < 0 \).

Corollary 1.7. Under the above assumptions, we have

\[ |S| \geq \ln \left( \frac{\sigma(\alpha, \Lambda_-)}{\text{disp}(\alpha, \Lambda_+)} \right). \]

In particular if the length of the shortest \( \alpha \)-chord of \( \Lambda_- \) is greater than the \( \alpha \)-displacement energy of \( \Lambda_+ \), we achieve a non-trivial lower bound on the length of the Lagrangian concordance.

Proof. Suppose \( H \) displaces \( \Lambda^+ \) is the above sense. Theorem 1.2 implies

\[ \sigma(\alpha, \Lambda_-) \leq \text{disp}(\alpha, \Lambda) \leq e^A\|H\|_{\text{osc}} \]

where \( \alpha_- = e^S\alpha \) and hence \( \sigma(\alpha, \Lambda_-) = e^{-S}\sigma(\alpha, \Lambda_-) \).

\( \square \)

1.2. Previous results in the symplectic setting. In the following we assume that \((X, \omega)\) is a tame symplectic manifold.

1.2.1. Results related to Corollary 1.5. Lower bounds for the number of intersections between a Lagrangian submanifold \( L \) and its image \( \phi^1_{H_0}(L) \) under a Hamiltonian isotopy has been a major topic in symplectic topology. In certain cases it has been shown that

\[ |L \cap \phi^1_{H_0}(L)| \geq \dim H_*(L; \mathbb{Z}_2) \]

given that the intersection is transverse. Here we present the two most classical such results. In [37] Laudenbach–Sikorav used generating family techniques to prove the statement for \((X, \omega) = (T^*M, d\lambda_0)\) and \( L = 0_M \). Floer homology was introduced in [24] by Floer, based upon Gromov’s technique of pseudoholomorphic curves [31]. The most basic version of Floer homology can handle the case \( L \subset (X, \omega) \), under the assumption that there is no non-constant pseudoholomorphic representative of any element in \( \pi_2(X, L) \). The work due to Floer work proves the lower bound in this setting. Finally, we note that a far-reaching generalization of Floer homology has been constructed, which can be used to determine non-trivial such lower bounds in many cases [27, 28].
1.2.2. Results related to Theorem 1.2. In [15] Chekanov provided the following refinement of the Energy-Capacity inequality due to Polterovich [44]. Recall the definition
\[ \| \phi \| := \inf \{ H_t; \phi = \phi_t \} \| H \|_{\text{osc}} \]
\[ \| H \|_{\text{osc}} := \int_0^1 \left( \max_X H_t - \min_X H_t \right) dt, \]
of the Hofer norm of a Hamiltonian diffeomorphism, as well as the definition of the holomorphic disc capacity
\[ 0 \leq \sigma_\omega(L) := \sup_{J \in J(X,\omega)} \inf_{u \in M(L;J)} \int_0^\infty \omega \leq +\infty, \]
where \( M(L;J) \) denotes the moduli space consisting of \( J \)-holomorphic representatives of elements in \( \pi_2(X,L) \) and \( J(X,\omega) \) is the contractible set of tame almost complex structures on \( (X,\omega) \).

**Theorem 1.8 ([15]).** Suppose that \( \phi: (X,\omega) \rightarrow (X,\omega) \) is a Hamiltonian diffeomorphism of a tame symplectic manifold satisfying the inequality \( \| \phi \| < \sigma_\omega(L) \) for a closed Lagrangian submanifold \( L \subset (X,\omega) \). It follows that
\[ |L \cap \phi(L)| \geq \sum_{i=0}^{\dim L} \dim H_i(L;\mathbb{Z}_2), \]
given that the intersection \( L \cap \phi(L) \) is transverse.

1.3. Previous results in the contact setting.

1.3.1. Results related to Corollary 1.5. Chekanov’s refinement [14] of the aforementioned result [37] by Laudenbach–Sikorav establishes the persistence of Reeb chords under general contact isotopies of the zero-section of \( J^1M \) – a set of transformations that obviously is much larger than its subset consisting of the lifts of Hamiltonian isotopies of \( T^*M \). This can be used to deduce Corollary 1.5 in the special case when \( \Lambda = 0_M \subset J^1M \) and when the contact forms all are taken to be standard, i.e. when
\[ \alpha = \alpha_0 = \alpha_{\text{std}}. \]
Recall that this case the Reeb chords are in bijective correspondence with intersection points under the canonical projection to \( T^*M \). Our result can therefore be seen as a generalization of this result to general contact forms. (Recall that we still have to make the requirement that the contact forms coincide with \( \alpha_{\text{std}} \) outside of a compact subset.)

In the more general case when \( \alpha = \alpha_0 \), and \( \sigma(\alpha_0,\Lambda) = +\infty \), but under the additional assumption that
\[ \phi_{\alpha_{\text{std}}}^R(\Lambda) \subset Y \]
is a closed submanifold, the conclusion of Corollary 1.5 also follows from work of Eliashberg–Hofer–Salamon [22, Theorem 2.5.4].

There have also been results inside certain prequantization spaces. Consider the standard Legendrian \( \mathbb{R}P^n \subset \mathbb{R}P^{2n+1} \), i.e. the image of
\[ \Re \mathbb{C}^{n+1} \cap S^{2n+1} \subset \left( S^{2n+1}, \ker \sum_{i=1}^{2n+1} (x_i dy_i - y_i dx_i) \right) \]
under the canonical projection, where the contact structure on \( \mathbb{R}P^{2n+1} \) is induced by the standard contact structure on \( S^{n+1} \). A result by Givental [30] shows that there must exist a Reeb chord between \( \mathbb{R}P^n \) and \( \phi_{\alpha_{H^1}}^R(\mathbb{R}P^n) \) for any choice of contact form \( \alpha \) and contact isotopy; also, see related results [10] by Borman–Zapolsky. Note that this result cannot be deduced from Corollary 1.5 due to the presence of contractible periodic orbits and chords. We expect that the results can be recovered by our methods as well, after a more refined Floer theoretic invariance has been established.
1.3.2. Results related to Theorem 1.2. There have also been previous results along the lines of Theorem 1.2, i.e. taking quantitative properties of the contact Hamiltonian into account. Notably, in the case when $\alpha = \alpha_-$ is the $S^1$-invariant contact form on a prequantization space $S^1 \to (P, \alpha) \to (M, \omega)$, and when $\Lambda = \Lambda_-$ is the lift of an embedded Lagrangian submanifold in $(M, \omega)$, such results were obtained in [33] by Her generalizing previous work by Ono [43]. For a result in contactizations of Liouville domains with the standard contact form we also refer to the more recent result [2] by Akaho.

For a Legendrian submanifold of a general contact manifold (again under the assumption that $\alpha = \alpha_-$ and $\Lambda = \Lambda_-$), Theorem 1.2 can also be seen to follow from a result by Akaho [3], again under the additional assumption that

$$\phi_{R_\alpha}(\Lambda) \subset Y$$

is a closed submanifold satisfying some additional topological constraints. Note that the latter behavior is non-generic and imposes severe restrictions on the contact form.

1.3.3. Results related to Corollary 1.7. Sabloff and Traynor prove a similar result when the Legendrian contact homology DGAs have augmentations [45], which in turn inspired this corollary. Their hypotheses are stronger as many Legendrian contact homology DGAs do not have augmentations. But, on the other hand, they have an improved bound where their numerator is not just a chord of minimal length, but runs over a collection of (possibly longer) chords which represent certain canonical classes in the so-called linearized Legendrian contact homology (this is a chain complex associated to the Legendrian which is generated by its Reeb chords). They also consider general Lagrangian cobordisms in the symplectization, as opposed to just concordances.

1.4. Overview of paper. In Section 2 we review background definitions, although Section 2.6 on actions may be less familiar for some and includes some technical hypotheses. In Section 3 we introduce a Floer theory for Lagrangian cobordisms. Some of this is also not “mainstream” in that we absorb the parameters of the continuation maps into our $\bar{\partial}$-equation. Section 4 proves our main Gromov non-bubbling result. While the result is not surprising, to our knowledge it does not follow from the compactness results which exist in the literature. Sections 5 and 6 discuss several methods to study pseudoholomorphic curves using Reeb chord actions: the version of Usher trick for contact Hamiltonians, as implemented by Shelukhin in [47], together with neck-stretching, and splashing. The splashing construction in this context is analogous to the wrapping considered in [18] by Cieliebak–Oancea in order to prove their Mayer–Vietoris long exact sequence for symplectic homology. We anticipate other applications, as it provides a relatively easy way to decompose a Floer complex into subcomplexes. Up until this point, our constructions and definitions apply to arbitrary exact Lagrangian cobordisms between Legendrians. In Section 7 we provide a technical push-off of our Lagrangian cobordism, assuming it is a concordance. Section 8 proves the main result, putting all the previous sections together to compute various Floer differentials and chain maps.

1.5. Relation to existing techniques. It has come to our attention that the authors in [6], when proving the existence of translated points, use techniques that are similar to some of ours: they use Shelukhin’s contact version of Usher’s trick when computing a certain oscillatory norm; they introduce several actions on the loop space and study continuation maps that relate chords; and they show that similar a priori bounds on energy can prevent certain bubbling in the sense of symplectic field theory. On the other hand, they work with the action functional used in the definition of Rabinowitz Floer homology, instead of the more classical setup of Floer homology used here.

It should be the case that Rabinowitz Floer homology for Lagrangian submanifolds as defined in [40] – this is a Floer complex with differential counting gradient trajectories of the Rabinowitz action functional – is a suitable framework also for studying the questions here. In the case when the obstruction contact form $\alpha_-$ and the contact form $\alpha$ used for counting the orbits are taken to coincide, our results also appear very naturally from this perspective, taking the standard invariance properties of this Floer theory into account; see Remark 8.19. However, we would like to stress that the full result in the
case when $\alpha_-$ and $\alpha$ are related by a symplectic cobordism would require a new form of invariance for Rabinowitz Floer homology, e.g. one which also allows deformations of the contact form while working within some suitable action range. The invariance result here is proven by carefully controlling our continuation maps via “splashing” and “neck-stretching,” in order to produce our Mayer–Vietoris sequence \cite{8,20}. Finally, compared to \cite{6} our analysis of SFT bubbling also has more cases to consider due to the Lagrangian boundary condition, which makes the situation more involved.

In addition, we make the technical remark concerning the relations between the “splashing” and “neck-stretching” that we perform here and the related techniques from \cite{18} that aim at the same results. (For instance, they also obtain a Mayer–Vietoris sequence in a similar setting.) The approach taken here, however, is more basic since we never rely on the full SFT compactness theorem for excluding the existence of Floer strips, but rather attain this by mere means of action computations.

2. Basic definitions

2.1. Symplectic geometry. A symplectic manifold $(X, \omega)$ is a smooth $2n$-dimensional manifold $X$ together with a closed non-degenerate two-form $\omega$. A symplectic manifold is called exact given that the symplectic form exact, i.e. $\omega = d\eta$ for some one-form $\eta$. An $n$-dimensional submanifold $L \subset (X, \omega)$ is called Lagrangian if $\omega|_{TL}$ vanishes and, given that $\omega = d\eta$ is exact with a choice of primitive $\eta$, it is called exact Lagrangian if $\eta|_{TL}$ is an exact one-form on $L$.

A (time-dependent) Hamiltonian $H: X \times [0, 1] \to \mathbb{R}$, usually written as $H_t: X \to \mathbb{R}$ where $t \in [0, 1]$, gives rise to the so-called Hamiltonian vector field $X_{H_t}$ via Hamilton’s equations

$$\omega(\cdot, X_{H_t}) = dH_t(\cdot).$$

The corresponding Hamiltonian flow $\phi^t_{H_t}: (X, \omega) \to (X, \omega)$ with infinitesimal generator $X_{H_t}$ preserves the symplectic form.

For an exact symplectic manifold $(X, d\eta)$ with a choice of primitive $\eta$ of the symplectic form, recall the definition of the Liouville vector field $\zeta$, which is determined uniquely by $t_\zeta d\eta = \eta$.

By a Liouville manifold $(P, d\theta)$ we mean an open exact symplectic manifold satisfying the following properties. The Liouville vector field $\zeta$ is transverse and outward-pointing to the boundary of a smooth compact domain $\overline{P} \subset P$, where $\zeta$ moreover defines a complete (forward-time) flow in the subset $P \setminus \text{int} \overline{P}$. The compact domain $(\overline{P}, d\theta)$ is called a Liouville domain.

2.2. Contact geometry. Recall that a contact manifold is a smooth $(2n+1)$-dimensional manifold $(Y, \xi)$ with a maximally non-degenerate hyperplane distribution $\xi \subset TY$ called the contact distribution. A Legendrian submanifold is a smooth $n$-dimensional submanifold $\Lambda \subset (Y, \xi)$ for which $T \Lambda \subset \xi$.

For us all contact manifolds will be assumed to have coorientable contact distributions, which is equivalent to the existence of a contact form $\alpha \in \Omega^1(Y)$ satisfying $\xi = \ker \alpha$. A choice of contact form determines the Reeb vector field $R_\alpha$ on $Y$ via the equations

$$t_{R_\alpha} d\alpha = 0 \quad \text{and} \quad t_{R_\alpha}(\alpha) = 1.$$  

This vector field then gives rise to the Reeb flow $\phi^t_{R_\alpha}: (Y, \alpha) \to (Y, \alpha)$, which can be seen to preserve $\alpha$.

We assume Legendrian submanifolds are closed unless stated otherwise. We also assume that the contact manifold $(Y, \xi)$ is closed, or that it is the contactization of a Liouville manifold $(P, d\theta)$, i.e.

$$(P \times \mathbb{R}, \ker \{\alpha_{\text{std}} := dz + \theta\})$$

where $z$ denotes the coordinate of the $\mathbb{R}$-factor. Observe that the canonical contact form $\alpha_{\text{std}}$ on the contactization induces the Reeb vector field $R_{\alpha_{\text{std}}} = \partial_z$. When considering a more general contact form $\alpha$ on a contactization, we will always assume that $\alpha = \alpha_{\text{std}}$ holds outside of a compact subset.
Periodic solutions to the Reeb vector field are called **periodic Reeb orbits**. Each periodic Reeb orbit $\gamma$ has a **length** given by

$$\ell(\gamma) := \int_\gamma \alpha > 0.$$ 

A non-trivial integral curve of $R_\alpha$ having end-points on an embedded Legendrian submanifold $\Lambda$ is called a **Reeb chord on** $\Lambda$, and its length is defined as for a periodic Reeb orbit. We denote $Q_\alpha(\Lambda)$ to be the set of Reeb chords on $\Lambda$ and Reeb periodic orbits, and $Q^0_\alpha(\Lambda) \subset Q_\alpha(\Lambda)$ to be the subset consisting of those Reeb chords and periodic orbits which define trivial elements in $\pi_1(Y, \Lambda)$.

**Example 2.1.** The jet space $J^1M = T^*M \times \mathbb{R}$ has a canonical contact form $\alpha_{\text{std}} := dz + \theta_M$, where $\theta_M = pdq$ is the so-called Liouville form on $T^*M$ and $z$ is the standard coordinate on the $\mathbb{R}$-factor. This is also an example of a contactization of a Liouville manifold. The zero-section $0_M \subset J^1M$ is a Legendrian submanifold. Observe that $R_{\alpha_{\text{std}}} = \partial_z$ and, hence, that $Q_{\alpha_{\text{std}}}(0_M) = \emptyset$.

**Remark 2.2.** While two Legendrian submanifolds generically are disjoint, there are typically (a discrete space of) Reeb chords with endpoints on them. In the case of $(J^1M, \alpha_{\text{std}})$, two Reeb chords between $\Lambda_0, \Lambda_1 \subset J^1M$ correspond bijectively to intersection points of the canonical projection $\pi: J^1M \to T^*M$. Observe that $\pi(\Lambda_i) \subset (T^*M, d\theta_M)$, $i = 0, 1$, are exact Lagrangian immersions, i.e. immersions for which the pull-back of the one-form $\theta_M$ is exact.

The above remark relates the phenomenon of intersection points in symplectic geometry with that of Reeb chords in contact geometry. The following passage from a contact manifold to a cylindrical symplectic manifold (and from a Legendrian submanifold to a cylindrical Lagrangian submanifold) will also provide such a correspondence. The exact symplectic manifold

$$(\mathbb{R} \times Y, d(e^r\alpha))$$

associated to a contact manifold is called the **symplectization of** $(Y, \alpha)$, where $r$ denotes the standard coordinate on the $\mathbb{R}$-factor. Observe that the cylinder $\mathbb{R} \times \Lambda \subset (\mathbb{R} \times Y, d(e^r\alpha))$ is (exact) Lagrangian if and only if $\Lambda \subset (Y, \alpha)$ is Legendrian.

2.3. **Contact Hamiltonians.** A **contact Hamiltonian** is a smooth function

$$H: Y \times [0, 1] \to \mathbb{R},$$

which usually will be considered as a family of functions $H_t: Y \to \mathbb{R}$, $t \in [0, 1]$. The Hamiltonian $e^r H_t$ on the symplectization can be seen to have a Hamiltonian flow of the form

$$\phi^t_{e^rH_t}(r, x) = (r + \tau^t_{\alpha,H_t}(x), \phi^t_{\alpha,H_t}(x)), \ (r, x) \in \mathbb{R} \times Y.$$

The translation $\tau^t_{\alpha,H_t}(x)$ is determined by the so-called conformal factor via the formula

$$(\phi^t_{\alpha,H_t})^*\alpha = e^{-\tau^t_{\alpha,H_t}\alpha}.$$

The contact Hamiltonian can be recovered from the formula

$$H_t(\phi^t_{\alpha,H_t}(x)) = \alpha \left( \frac{d}{dt} \phi^t_{\alpha,H_t}(x) \right).$$

Observe that $\tau^t_{\alpha,H_t}: Y \to \mathbb{R}$ is indefinite for each $t \in [0, 1]$, and that it vanishes identically if and only if the contactomorphism preserves the contact form; such contactomorphisms are usually called **strict contactomorphisms**. We have

$$\phi^t_{R_\alpha}(\gamma) = \phi^t_{\alpha,\dot{g}(t)},$$

where $\dot{g}(t): Y \to \mathbb{R}$ is seen as family of constant functions, and where the left hand side is the Reeb flow.

A standard result implies that each one-parameter family of contactomorphisms starting at $I\nu Y$ is induced by a contact Hamiltonian (see e.g. [29]). It is clear that the contact Hamiltonian depends on the choice of contact form.

We will need the following basic fact.
Lemma 2.3. A contact isotopy $\phi^{t}_{\alpha,H}$ can be uniquely factorized as

$$\phi^{t}_{\alpha,H} = \phi^{t}_{\alpha,c_t} \circ \phi^{t}_{\alpha,G_t},$$

for the contact Hamiltonians $c_t, G_t: Y \to \mathbb{R}$ given by

$$c_t := (\max_{Y} H_t + \min_{Y} H_t)/2,$$

$$G_t := H_t \circ \phi^{t}_{\alpha,c_t} - c_t.$$  

In particular, $c_t$ is a family of constant functions and $\phi^{t}_{\alpha,c_t} = \int_{0}^{t} c ds$, while $G_t$ is indefinite for each $t \in [0,1]$ and satisfies $\|G_t\|_{osc} = \|H_t\|_{osc}$.

Proof. The claim that $G_t$ is indefinite, as well as the equivalence between the oscillatory norms, is immediate by construction.

The equality $\phi^{t}_{\alpha,H} = \phi^{t}_{\alpha,c_t} \circ \phi^{t}_{\alpha,G_t}$ of the two flows can be seen by the standard computation

$$\alpha \left( \frac{d}{dt} (\phi^{t}_{\alpha,c_t} \circ \phi^{t}_{\alpha,G_t}) \right) =$$

$$= \alpha(c_t R_{\alpha} (\phi^{t}_{\alpha,c_t} \circ \phi^{t}_{\alpha,G_t})) + \alpha \left( D \phi^{t}_{\alpha,c_t} \left( \frac{d}{dt} \phi^{t}_{\alpha,G_t} \right) \right)$$

$$= c_t + (\phi^{t}_{\alpha,c_t})^* \alpha \left( \frac{d}{dt} (\phi^{t}_{\alpha,G_t}) \right)$$

$$= c_t + G_t \circ \phi^{t}_{\alpha,G_t}$$

$$= H_t \circ \phi^{t}_{\alpha,c_t} \circ \phi^{t}_{\alpha,G_t},$$

where the fourth equality follows since $(\phi^{t}_{\alpha,c_t})^* \alpha = \alpha$, and where the last equality holds by the construction of $G_t = H_t \circ \phi^{t}_{\alpha,c_t} - c_t$. \hfill $\Box$

2.4. Symplectic cobordisms. Consider a compact exact symplectic manifold $(X, d\eta)$ with contact boundary $\partial X = Y_- \sqcup Y_+$, where $\eta$ restricts to the contact form $\alpha_{\pm}$ on $Y_{\pm}$, and where the Liouville vector field determined by $\eta$ is transverse to the boundary, inwards-pointing along $Y_-$, and outwards-pointing along $Y_+$. We will call this a compact exact symplectic cobordism from $(Y_- , \alpha_-)$ to $(Y_+, \alpha_+)$. A compact exact symplectic cobordism can be completed by adjoining half symplectizations of the form

$$((-\infty,0] \times Y_-, d(e^r \alpha_-)) \text{ and } ([0,\infty) \times Y_+, d(e^r \alpha_+)),$$

given that we use appropriate coordinates near $\partial X$ (see e.g. the standard symplectic neighborhood theorem in [39]).

Definition 2.4. The exact symplectic manifold $(X, d\eta)$ together with the choice of embedding $\overline{X} \subset X$, is called a (complete) exact symplectic cobordism. The non-compact subsets $(-\infty,0] \times Y_-$ and $[0,\infty) \times Y_+ \subset X$ are called its concave and convex cylindrical ends, respectively.

Note that the identifications of these cylindrical ends are part of the data of a complete exact symplectic cobordism.

Example 2.5. For any two smooth functions $f_\pm: Y \to \mathbb{R}$ satisfying $f_-(y) < f_+(y)$, the subset

$$\{(r,y): f_-(y) \leq r \leq f_+(y)\} \subset (\mathbb{R} \times Y, d(e^r \alpha))$$

of the symplectization is a compact exact symplectic cobordism from $(Y, e^{f_-} \alpha)$ to $(Y, e^{f_+} \alpha)$. Note that its completion again is symplectomorph to the symplectization $(\mathbb{R} \times Y, d(e^r \alpha))$, but that the canonical cylindrical structure provided by this symplectization differs from the ones induced by the data of our symplectic cobordism.
We will also allow the non-compact case \((\mathring{X}, d\eta) = ([a, b] \times P \times \mathbb{R}, d(e^r \alpha_{\text{std}}))\) when the contact manifold is a contactization of a Liouville manifold.

Let \((X, d\eta)\) be the completion of \(\mathring{X} \subset (X, d\eta)\) as above. Observe that \((X, d\eta)\) equivalently can be obtained as the completion of the domain \(\mathring{X}_{T_-, T_+} := \{T_-, 0\} \times Y_- \cup \mathring{X} \cup \{0, T_+\} \times Y_+ \subset (X, d\eta)\) with smooth contact boundary, given any choices of \(T_- \leq 0 \leq T_+\). The latter domain is a compact exact symplectic cobordism from \((Y_-, e^{T_-} \alpha_-)\) to \((Y_+, e^{T_+} \alpha_+)\).

2.5. **Lagrangian cobordisms and concordances.** Here we develop the notion of both a compact and as well as a complete (typically non-compact) Lagrangian cobordism. First, by a (compact) **Lagrangian cobordism** \(\mathcal{L} \subset (\mathring{X}, d\eta)\) from the Legendrian submanifold \(\Lambda^- \subset (Y_-, \alpha_-)\) to \(\Lambda^+ \subset (Y_+, \alpha_+)\) we mean a Lagrangian embedding \(\mathcal{L} \hookrightarrow (\mathring{X}, d\eta)\) for which the following holds.

- The boundary satisfies \(\partial \mathcal{L} \subset Y_- \cup Y_+ = \partial \mathring{X}\), where moreover
  - \(\mathcal{L} \cap Y_\pm = \Lambda^\pm\), and
  - \(\mathcal{L}\) is invariant under the Liouville flow of \(\eta\) near the boundary,

are satisfied.

- The one-form \(\eta|_{\partial \mathcal{L}}\) has a primitive which is globally constant when restricted to either of \(\Lambda^-; \Lambda^+ \subset \partial \mathcal{L}\). (This condition is empty whenever both of \(\Lambda^\pm\) are connected.)

In the above situation we will also say that \(\Lambda^- \subset (Y, \alpha_-)\) is **Lagrangian cobordant** to \(\Lambda^+ \subset (Y_+, \alpha_+)\).

A Lagrangian cobordism \(\mathcal{L} \subset (\mathring{X}, d\eta)\) can be completed to a properly embedded Lagrangian \(L \subset (X, d\eta)\) inside the completion of the symplectic cobordism, by adjoining the non-compact cylindrical ends

\[
(-\infty, 0] \times \Lambda_- \subset ((-\infty, 0] \times Y_-, d(e^r \alpha_-)), \quad [0, +\infty) \times \Lambda_+ \subset ([0, +\infty) \times Y_+, d(e^r \alpha_+)).
\]

If \(\mathcal{L}\) is an exact Lagrangian, then so too is the resulting submanifold \(L\).

**Definition 2.6.** An exact Lagrangian submanifold \(L \subset (X, d\eta)\) is called a (complete) exact Lagrangian cobordism if there exists \(T_- \leq 0 \leq T_+\) such that \(L\) is obtained by completing a compact exact Lagrangian cobordism \(\mathcal{L} \subset (\mathring{X}_{T_- T_+}, d\eta)\) from \(\Lambda^- \subset (Y_-, e^{T_-} \alpha_-)\) to \(\Lambda^+ \subset (Y_+, e^{T_+} \alpha_+)\). The Legendrian submanifolds \(\Lambda\pm \subset (Y_\pm, \alpha_\pm)\) will be called the \(\pm\)-**ends** of \(L\).

In the case when \(\mathcal{L}\) is diffeomorphic to \(I \times \Lambda\), we call both \(\mathcal{L}\) and \(L\) a **Lagrangian concordance**, and we say that \(\Lambda^- \subset (Y, \alpha_-)\) is **Lagrangian concordant** to \(\Lambda^+ \subset (Y, \alpha_+)\).

**Example 2.7.** Inside the symplectic cobordism \(\{f_- (y) \leq r \leq f_+(y)\} \subset (\mathbb{R} \times Y, d(e^r \alpha))\) from Example 2.3, any Lagrangian cylinder \(\mathbb{R} \times \Lambda\) intersected with this domain is a Lagrangian concordance.

With the above definition there is a Lagrangian cobordism from \(\Lambda \subset (Y, \alpha)\) to \(\Lambda \subset (Y, e^s \alpha)\) if and only if \(s > 0\); this can be seen using the basic fact that no closed exact symplectic manifolds exist. However, by convention, we will also prescribe that any Legendrian submanifold \(\Lambda \subset (Y, \alpha)\) is Lagrangian concordant to itself. It was shown in [12] that a Legendrian isotopy \(\Lambda_t \hookrightarrow (Y, \alpha), t \in [0, 1]\), (i.e. an isotopy through Legendrian submanifolds) induces a Lagrangian concordance from \((\Lambda_0, \alpha)\) to \((\Lambda_1, e^C \alpha)\) for some constant \(C \geq 0\).

2.6. **Primitives and action of Hamiltonian chords.** Assume that we are given an ordered pair of exact Lagrangian cobordisms \(L_0, L_1 \subset (X, \eta)\).

In order to define a Hofer-type symplectic energy and a Hofer-type Floer energy for the pseudoholomorphic strips in Section 3, we will need to specify a one-form on \(X\) which is exact when pulled back to \(L_i, i = 0, 1\). First, note that \(\eta\) pulls back to an exact one-form on \(L_i\) by assumption. It will later
be important to exploit the fact that the one-form $\varphi \eta$ pulls back to an exact one-form also for a large class of piecewise smooth functions $\varphi: X \to \mathbb{R}_{>0}$.

The above energies are important when defining the Floer chain complex for a pair of exact Lagrangian cobordisms. In Section 3 we introduce the Floer complex in this setting, associated to a pair of Lagrangian submanifolds together with a compactly supported time-dependent Hamiltonian $G_t: X \to \mathbb{R}$.

We also need to study the so-called Floer “continuation maps,” between Floer complexes, which provide a morphism from the Floer theory of one set-up to the Floer theory of another. For this reason, we need to generalize out set-up from $G_t$, a time-dependent Hamiltonian on $(X, d\eta)$ as above, to a one-parameter family

$$G_{s,t}: X \to \mathbb{R}, \ (s,t) \in \mathbb{R} \times [0,1],$$

of compactly supported time-dependent Hamiltonians parametrized by $s \in \mathbb{R}$.

We are now ready to define the class of function which will be used for defining the Hofer type energies associated to the triple $(L_0, L_1, G_{s,t})$.

**Definition 2.8.** Let $\mathcal{C}(L_0, L_1, G_{s,t})$ denote the class of piecewise smooth functions $\varphi: X \to \mathbb{R}_{>0}$ satisfying the following properties.

1. $\varphi|_X$ is constant;
2. On the cylindrical ends the function $\varphi|_{X \setminus \overline{X}}$ depends only on $r$, and it satisfies
   $$\varphi'(r) + \varphi(r) \geq 0$$
   wherever it is differentiable (this ensures that $d(\varphi \eta)$ is non-negative on the symplectic two-planes);
3. $\varphi'(r) \equiv 0$ in some neighborhood of the subset
   $$\{ r \in \mathbb{R} \mid (L_0 \cup L_1) \cap (\{r\} \times Y_{±}) \subset (Y_{±}, \alpha_{±}) \} \subset X$$
   of the cylindrical ends;
4. On the concave end we either have $\varphi(r) = e^{-r-r_φ}$ for some $r_φ \geq 0$ or $\varphi(r) \equiv 1$ for all $r \ll 0$ sufficiently small, in which case we set $r_φ := +\infty$; and
5. In the subset
   $$\{ x \in X \mid G_{s,t}(x) \neq 0 \text{ for some } (s,t) \in \mathbb{R} \times [0,1] \}$$
   we have $\varphi(r) \equiv 1$.

Part (5) above is needed in order to prove the energy estimates given by Lemmas 3.5 and 3.6. Note that, in particular, the constant function $1 \in \mathcal{C}(L_0, L_1, G_{s,t})$ is always contained in the above subset. The following lemma is elementary but crucial.

**Lemma 2.9.** The continuous and piecewise smooth one-form $\varphi \eta$ for $\varphi \in \mathcal{C}(L_0, L_1, G_{s,t})$ is exact when pulled back to $L_i$, $i = 0,1$. The primitive on $L_i$ is moreover locally constant in the subset

$$V := \left\{ r \in \mathbb{R} \left| L_i \cap (\{r\} \times Y_{±}) \subset (Y_{±}, \alpha_{±}) \right. \text{ is a Legendrian submanifold} \right\} \subset X$$

of the cylindrical ends.

**Proof.** We establish the existence of primitives on each piece $L_i \cap V$ and $L_i \setminus V$ separately, and then proceed to show that these primitives can be combined to form a globally defined primitive.

The pull-back to $L_i \cap V$ of $\tilde{\varphi}(r)\eta = \tilde{\varphi}(r)e^r \alpha_{±}$ clearly vanishes for any choice of function $\tilde{\varphi}(r)$. In particular, any primitive of the pull-back of $\varphi \eta$ is a locally constant function inside this subset.
The assumptions imply that \( \varphi'(r) \equiv 0 \) holds inside the level-sets of \( r \) contained in \( X \setminus V \). A pull-back of \( \varphi_\eta \) to \( L_i \cap (X \setminus V) \) can thus be taken to be a locally constant rescaling of the primitive of the pull-back of \( \eta \) there (the latter primitives exist since \( L_i \) are assumed to be exact).

\[ \square \]

The above lemma implies that a primitive of \( \varphi_\eta \) pulled back to \( L_i \) is constant outside of a compact subset. The following lemma shows that it is possible to assume that this constant vanishes on both ends for one of the cobordisms in the pair.

**Lemma 2.10.** Assume that the primitive of \( \varphi_\eta \) pulled back to \( L_i \) which vanishes on the negative end, takes the value \( C_i \) on the positive end, \( i = 0, 1 \). After a Hamiltonian isotopy of the union \( \overline{L}_0 \cup \overline{L}_1 \subset (\overline{X},d\eta) \) of Lagrangian cobordisms supported inside the interior \((\overline{X} \setminus \partial \overline{X},d\eta)\), the primitives on the positive end can be assumed to be given by \( C_i + C \), \( i = 0, 1 \), for an arbitrary constant \( C \in \mathbb{R} \), while the primitives still vanish on the negative ends.

**Proof.** The claim follows by considering a suitable Hamiltonian diffeomorphism \( \phi^{t}_0 : (\overline{X},d\eta) \to (\overline{X},d\eta) \), applied to \( \overline{L}_0 \cup \overline{L}_1 \), where the support of \( \sigma(r) \) is a compact neighborhood of the boundary \( \{0\} \times Y_+ \) contained in the collar \(((-\epsilon,0] \times Y_+,d(e^\sigma \alpha_+))\), while \( \sigma'(r) = 0 \) holds in some neighborhood of the boundary.

Fix \( s = s_0 \). A Hamiltonian chord \( p \) of \( \phi^{t}_{G,s_0,t} \) from \( L_0 \) to \( L_1 \) is a path

\[
([0,1],0,1) \to (X,L_0,L_1),
\]

\[
t \mapsto \phi^{t}_{G,s_0,t}(s),
\]

with starting point given by \( x \in L_0 \) and endpoint given by \( \phi^{t}_{G,s_0,t}(s) \in L_1 \). Observe that intersections \( L_0 \cap L_1 \) are Hamiltonian chords from \( L_0 \) to \( L_1 \) of a vanishing Hamiltonian.

Take primitives \( f_i : L_i \to \mathbb{R} \) and \( f^i_0 : L_i \to \mathbb{R} \) of \( \eta \) and \( \varphi_\eta \), respectively, pulled back to \( L_i \), uniquely determined by the requirement that they vanish for all \( r \ll 0 \) sufficiently small on the concave end.

To a Hamiltonian chord \( p \) of \( G_{s_0,t} \) starting on \( x \in L_0 \) and ending on \( \phi^{t}_{G,s_0,t}(s) \in L_1 \), we associate its so-called **action** defined by

\[
a_\varphi(p) := f^0_0(x) - f^0_1(\phi^1_{G,s_0,t}(x)) + \int_0^1 (\eta(\phi^t_{G,s_0,t}(x)) - G_{s_0,t}(\phi^t_{G,s_0,t}(x)))dt.
\]

**Remark 2.11.** The following two claims are straightforward.

1. In the case when \( G_{s_0,t} \equiv 0 \), and hence \( p(t) \equiv x \in L_0 \cap L_1 \) is an intersection point, these actions specialize to the potential differences

\[
a_\varphi(p) := f^0_0(x) - f^0_1(x)
\]

at the intersection. When \( \varphi \equiv 1 \) we also write

\[
a(p) := a_1(p)
\]

for the induced action.

2. If \( p \in L_0 \cap L_1 \) is an intersection point contained outside of the support of \( G_{s_0,t} \), then it can be considered either as a 0-Hamiltonian or a \( G_{s_0,t} \)-Hamiltonian chord. We claim that, for any fixed choice of

\[
\varphi \in \mathcal{C}(L_0,L_1,G_{s,t}) \subset \mathcal{C}(L_0,L_1,0),
\]

the action of this intersection point obtained in the two different cases coincide.
3. The Floer homology for a pair of Lagrangian cobordisms

In this section we introduce the Floer chain complex whose differential is defined with a Hamiltonian perturbation term, and the so-called continuation chain map between such Floer chain complexes needed for proving invariance. Floer homology was originally introduced by Floer in \[24\] for a pair of compact Lagrangian submanifolds, and has since then seen a lot of development. We refer to \[41\] and \[42\] for a thorough and modern treatment, including the incorporation of the Hamiltonian term in Section 14. In this section we describe a version of Floer homology in the present context, i.e. for a pair of non-compact exact Lagrangian cobordisms inside a symplectic cobordism with a concave end. We also refer to \[13\] for a previous construction of Floer homology in a similar setting.

In the following we let \(L_0, L_1 \subset (X, d\eta)\) be complete exact Lagrangian cobordisms of a complete exact symplectic cobordism. We denote by \(\Lambda_i^\pm \subset (Y_\pm, \alpha_\pm), i = 0, 1\), the Legendrian submanifolds being the \(\pm\)-ends of \(L_i\). We also fix a choice of a one-parameter family \(G_{s,t}: X \to \mathbb{R}\) of compactly supported and time-dependent Hamiltonians. The most general case we consider, when defining continuation maps between Floer complexes, is

\[
G_{s,t} = \rho(s)G_t, \text{ where } \rho(s) : \mathbb{R} \to [0, 1] \text{ and } \text{supp}(\rho') \text{ is compact.}
\]

We can thus write \(G_{+,t}\) and \(G_{-,t}\) to denote the Hamiltonian \(G_{s_0,t}\) with \(s_0 \gg 0\) and \(s_0 \ll 0\), respectively.

3.1. Admissible almost complex structures. A \((s,t)\)-dependent almost complex structure \(J = J_{s,t}, (s,t) \in \mathbb{R} \times [0,1]\) on a symplectic manifold \((X, \omega)\), i.e. a smooth one-parameter family of time-dependent almost complex structures parametrised by \(s \in \mathbb{R}\), is said to be tamed by \(\omega\) if \(\omega(v, Jv) > 0\) holds whenever \(v \neq 0\). We assume that for all \(J_{s,t}\) there exists some \(K \geq 0\) such that

\[
J_{s,t} = J_{\pm,t}\quad \text{if } \pm s \geq K.
\]

An almost complex structure \(J\) on a symplectization \((\mathbb{R} \times Y, d(e^r\alpha))\) is said to be cylindrical if

- \(J\partial_r = R_\alpha\),
- \(J\xi = \xi\), where \(\xi := \ker \alpha \subset TY\), and \(J|_\xi\) is tamed by \(d\alpha\), and
- \(J\) is invariant under translations of the coordinate \(r\).

It automatically follows that a cylindrical almost complex structure is tame. Consider a choice of function \(\varphi: X \to \mathbb{R}\), \(\varphi \in C(L_0, L_1, G_{s,t})\), as defined in Section 2.6. We call a tame almost complex structure on \((X, d\eta)\) admissible with respect to \(\varphi\) if it coincides with a cylindrical almost complex structure in each component of

\[
\{r; \varphi'(r) \neq 0\} \subset X \setminus X
\]

in the cylindrical ends. The space of admissible almost complex structures will be denoted by \(\mathcal{J}(X, d\eta, \varphi)\).

Observe that this is a non-empty and contractible space by \[31\].

3.2. Moduli spaces of pseudoholomorphic strips and their energy estimates. We define here the pseudoholomorphic curves which we will need in the next subsection for our maps (differentials and chain maps). We also define and relate a number of different energies of these curves expressed in terms of the actions introduced in Section 2.6. All definitions and results here are standard, with only one minor variation: the energies are induced by the two-form \(d(\varphi\eta)\) which is not necessarily a symplectic form everywhere.

Let \(p_{\pm}(t)\) be \(G_{\pm,t}\)-Hamiltonian chords from \(L_0\) to \(L_1\). (Recall in the case when \(G_{\pm,t} \equiv 0\) these are intersection points \(L_0 \cap L_1\).) Define the moduli-space of pseudoholomorphic strips

\[
\mathcal{M}_{p_+,p_-}(L_0, L_1; G_{s,t})
\]
to be the set of maps \( u : \mathbb{R} \times [0, 1] \to X \) satisfying
\[
\begin{cases}
u(s, 0) \in L_0, & u(s, 1) \in L_1, \\
\lim_{s \to \pm \infty} u(s, t) = p_\pm(t), \\
\partial_s u(s, t) + J_{s,t}(\partial_t u(s, t) - X_{G_{s,t}}(u(s, t))) = 0.
\end{cases}
\] (3.3)

This last condition we state as: \( u \) satisfies the Cauchy-Riemann equation with a Hamiltonian perturbation term. However, for short these solutions will often be referred to simply as pseudoholomorphic strips. The chord \( p_- \) will be called the \textbf{input} while \( p_+ \) will be called the \textbf{output}.

The \textbf{Floer energy} of a strip is the quantity
\[
E_{d(\varphi \eta), J_{s,t}}(u) := \int_{-\infty}^{\infty} \int_0^1 d(\varphi \eta)(\partial_s u(s, t), J_{s,t}(\partial_t u(s, t), X_{u(s, t)})) dt \, ds \geq 0.
\]

This quantity is a priori non-negative on any strip, since \( \varphi \eta \) is non-negative on \( d\eta \)-symplectic planes and since \( J_{s,t} \) is tamed by \( d\eta \).

We define the \textbf{\( d(\varphi \eta) \)}-energy to be given by
\[
E_{d(\varphi \eta)}(u) := \int_u d(\varphi \eta).
\]

Stokes’ theorem together with the exactness of \( L_i, i = 0, 1 \), implies that this quantity only depends on the asymptotics of the strip; more precisely, we have
\[
E_{d(\varphi \eta)}(u) = a_\varphi(p_+) - a_\varphi(p_-) + \int_0^1 G_{+,t}(p_+(t)) dt - \int_0^1 G_{-,t}(p_-(t)) dt.
\] (3.4)

The obvious generalizations of the above formulas also enables us to consider the different energies in the case when the map \( u \) only is defined on a subset of the strip \( \mathbb{R} \times [0, 1] \).

In the case when \( G_{s,t} \equiv G_t \) only depends on the \( t \)-coordinate we have the following precise expressions for the Floer energy which is standard; see e.g. [22, Section 12.3].

\textbf{Lemma 3.5.} Consider \( \varphi \in \mathcal{C}(L_0, L_1, G_t) \) and \( J_{s,t} \in \mathcal{I}(X, d\eta, \varphi) \), and a strip \( u \in \mathcal{M}_{p_+, p_-}(L_0, L_1; G_t) \). The energy can be expressed as
\[
E_{d(\varphi \eta), J_{s,t}}(u) = a_\varphi(p_+) - a_\varphi(p_-) \geq 0,
\]
with equality if and only if \( u \) is contained in a single Hamiltonian chord.

\textbf{Proof.} Since \( u \) satisfies the Cauchy-Riemann equation with Hamiltonian perturbation term, we compute
\[
E_{d(\varphi \eta), J_{s,t}}(u) = \int_u d(\varphi \eta) + \int_{-\infty}^{+\infty} \int_0^1 d(\varphi \eta)(\partial_s u(s, t), -X_{G_t}(u(s, t))) dt \, ds
\]
\[
= \int_u d(\varphi \eta) - \int_0^1 \int_{-\infty}^{+\infty} \partial_s G_t(u(s, t)) ds \, dt,
\]
where we have used the assumption that \( \varphi \in \mathcal{C}(L_0, L_1, G_t) \) in order to infer that
\[
d(\varphi \eta)(\cdot, -X_{G_t}(u(s, t))) = d\eta(\cdot, -X_{G_t}(u(s, t))) = -d_{u(s, t)} G_t(\cdot)
\]
is satisfied.

The expressions of the energies in terms of the actions now follow by elementary applications of Stokes’ theorem together with Equality (3.4).

The assumptions on \( \varphi \) and \( J_{s,t} \) imply that the energies are non-negative; here we have used the assumption that \( d(\varphi \eta) \) is non-negative on \( d\eta \)-symplectic two-planes and that \( J_{s,t} \) is tamed by \( d\eta \). Since, moreover, \( d(\varphi \eta) = C d\eta \) is a \textit{symplectic} form near the Hamiltonian chords by the assumption \( \varphi \in \mathcal{C}(L_0, L_1, G_t) \), it follows that a non-trivial pseudoholomorphic strip in fact must have \textit{positive} energy. \qed
The following lemma gives an action estimate in the case when the family \( G_{s,t} \) of Hamiltonians is allowed to depend on \( s \in \mathbb{R} \) in a very controlled way. These estimates are straightforward adaptations of the estimates in [12, Section 14.4] to the current setting. Take \( G_{s,t} = \rho(s)G_t \) for some smooth \( \rho: \mathbb{R} \to [0,1] \), and let \( p_+, p_- \) denote a Hamiltonian chord of \( G_{s,t} \) and \( G_{-s,t} \), respectively. Moreover, we assume that \( \rho'(s) \) has compact support and satisfies the property that each of the integrals \( \int_{(\rho')^{-1}([0,\infty))} \rho'(s)ds \) and \( \int_{(\rho')^{-1}((-\infty,0])} \rho'(s)ds \) are equal to either 0 or ±1. Again, we take \( \varphi \in \mathcal{C}(L_0, L_1, G_{s,t}) \) and \( J_{s,t} \in \mathcal{J}(X, dq, \varphi) \).

Consider a strip \( u \in \mathcal{M}_{p_+, p_-}(L_0, L_1; G_{s,t}) \) together with a (possibly empty) open subset \( U \subset \mathbb{R} \times [0,1] \) satisfying the assumptions that:

- The subset \( U \) is disjoint from \([T, +\infty) \times [0,1]\) for some number \( T \gg 0 \);
- There is a number \( a_\varphi \) determined as follows: We either require \( U \) to be precompact, in which case we set
  \[
  a_\varphi := a_\varphi(p_-),
  \]
  or that it contains a subset of the form \((-\infty, T] \times [0,1]\) for some \( T \ll 0 \), in which case we set
  \[
  a_- := a_\varphi(p_-) + \int_0^1 G_{-s,t}(p_-(t))dt;
  \]
- \( G_{s,t} \circ u \) vanishes in some neighborhood of \( \overline{U} \setminus U \subset \mathbb{R} \times [0,1] \), where we note that this subset is compact by the previous assumptions.

**Lemma 3.6.** Let the strip \( u \in \mathcal{M}_{p_+, p_-}(L_0, L_1; G_{s,t}) \) be a solution of the perturbed Cauchy-Riemann equation (3.3) and consider a (possibly empty) subset \( U \subset \mathbb{R} \times [0,1] \), where both \( G_{s,t} = \rho(s)G_t \) and \( U \) are assumed to satisfy the properties above.

1. If \( \rho(s) = 1 \) or \( \rho(s) = 0 \) then
   \[
   0 \leq E_{d(\varphi)}(u|_{\mathbb{R} \times [0,1]\setminus U}) \leq a_\varphi(p_+) - a_- - E_{d(\varphi)}(u|_U);
   \]

2. If \( \rho(s) \) is non-constant and \( \rho(s) = 0 \) (resp. \( \rho(s) = 1 \)) whenever \( |s| \gg 0 \) is sufficiently large, i.e. \( G_{-s,t} = G_{s,t} = G_{s,t} \equiv G_t \) then
   \[
   0 \leq E_{d(\varphi)}(u|_{\mathbb{R} \times [0,1]\setminus U}) \leq a_\varphi(p_+) - a_- - E_{d(\varphi)}(u|_U) + \|G_t\|_{osc};
   \]

3. If \( \rho(s) = 0 \) for \( s \ll 0 \), \( \rho(s) = 1 \) for \( s \gg 0 \), and \( \rho'(s) \geq 0 \), then
   \[
   0 \leq E_{d(\varphi)}(u|_{\mathbb{R} \times [0,1]\setminus U}) \leq a_\varphi(p_+) - a_- - E_{d(\varphi)}(u|_U) + \int_0^1 \max_X G_t dt;
   \]

4. If \( \rho(s) = 1 \) for \( s \ll 0 \), \( \rho(s) = 0 \) for \( s \gg 0 \), and \( \rho'(s) \leq 0 \), then
   \[
   0 \leq E_{d(\varphi)}(u|_{\mathbb{R} \times [0,1]\setminus U}) \leq a_\varphi(p_+) - a_- - E_{d(\varphi)}(u|_U) - \int_0^1 \min_X G_t dt;
   \]

**Remark 3.7.** The reason for why we need the energy estimates for the complicated domains in Lemma 3.6 is that we have not established the full symplectic field theory (SFT for short) type compactness result for Floer strips with a Hamiltonian perturbation term as considered here. Ideally, it should be possible to obtain a compactness result which, given that a sequence of strip leaves every compact subset, extracts a limit “building”. These buildings are supposed to consist of several levels of curves, defined with or without Hamiltonian perturbation terms, having punctures (in the interior as well as on the boundary) asymptotic to Reeb chords and orbits. In contrast, here we only establish Corollary 4.7 which, roughly speaking, exhibits the behavior of such a strip just prior to the moment when a breaking of SFT type occurs.
Proof. Similar to the proof of Lemma 3.3 we compute

\[
E_{d(\varphi \eta),J,u}(u \mid_{\mathbb{R} \times [0,1] \setminus U}) = E_{d(\varphi \eta)}(u \mid_{\mathbb{R} \times [0,1]}) - E_{d(\varphi \eta)}(u \mid_{U}) + \int_{u \mid_{\mathbb{R} \times [0,1] \setminus U}} \rho(s)d(\varphi \eta)(\partial_s u(s,t), -X_{G_t}(u(s,t)))ds \, dt.
\]

Using the assumption that \(G_t \circ u\) vanishes in a neighborhood of the boundary of \(U\), the latter term can be computed to be equal to

\[
\int_{-\infty}^{+\infty} \int_{0}^{1} \chi(s,t)\rho(s)d(\varphi \eta)(\partial_s u(s,t), -X_{G_t}(u(s,t)))ds \, dt =
\]

\[
= - \int_{0}^{1} \int_{-\infty}^{+\infty} \chi(s,t)\rho(s)\partial_s G_t(u(s,t))ds \, dt
\]

\[
- \int_{0}^{1} \left[\rho(s)\chi(s,t)G_t(u(s,t))\right]_{s=-\infty}^{+\infty} dt + \]

\[
+ \int_{0}^{1} \int_{-\infty}^{+\infty} \chi(s,t)\rho'(s)G_t(u(s,t))ds \, dt + \]

\[
+ \int_{0}^{1} \int_{-\infty}^{+\infty} \rho(s)G_t(u(s,t))\partial_s \chi(s,t)ds \, dt
\]

for some smooth bump function \(\chi: \mathbb{R} \times [0,1] \rightarrow [0,1]\) that is equal to one when restricted to \(\mathbb{R} \times [0,1] \setminus U\), and which vanishes on \(\text{supp}(G_t \circ u) \cap U\). Here we use the fact that there exists disjoint open neighborhoods of \(\mathbb{R} \times [0,1] \setminus U\) and \(\text{supp}G_t(u(s,t)) \cap U\); this is a consequence of the assumptions made on \(U\).

In particular, since \(\partial_s \chi(s,t)\) vanishes in the subset \(\text{supp}(G_t \circ u) \cap U\), we conclude the vanishing

\[
\int_{0}^{1} \int_{-\infty}^{+\infty} \rho(s)G_t(u(s,t))\partial_s \chi(s,t)ds \, dt = 0
\]

of the last term in the above expression. Also, note that \(\chi(s,t) \equiv 1\) for \(s \gg 0\) by construction.

Finally, we use the fact that

\[
\max_{\mathbb{R} \times [0,1]} \chi(s,t)G_t(u(s,t)) \leq \max_{\mathbb{R} \times [0,1]} G_t(u(s,t)),
\]

\[
\min_{\mathbb{R} \times [0,1]} \chi(s,t)G_t(u(s,t)) \geq \min_{\mathbb{R} \times [0,1]} G_t(u(s,t)).
\]

In combination with the expression of \(E_{d(\varphi \eta)}(u \mid_{\mathbb{R} \times [0,1]})\) given in Equality 3.3, all estimates can now be seen to follow. \(\square\)

3.3. The boundary map and chain maps. We are now ready to define the Floer complexes along with their boundary maps, as well as continuation maps between them. Assume that we are given exact Lagrangian cobordisms \(L_0, L_1 \subset (X, d\eta)\) that are disjoint outside of a compact subset. Furthermore, we consider a compactly supported Hamiltonian diffeomorphism \(\phi^1_{G_t}: (X, \omega) \rightarrow (X, \omega)\) for which the intersection \(\phi^1_{G_t}(L_0) \cap L_1\) is transverse.

For any fixed component \(p \in \pi_0(\Pi(X; L_0, L_1))\) of the space of paths from \(L_0\) and \(L_1\) in \(X\), we define the graded and finite-dimensional vectorspace

\[
CE^p_*(L_0, L_1; G_t) := \mathbb{Z}_2 \left\langle p(t) \mid p(t) = \phi^t_{G_t}(x), \ t \in [0,1], \right. \]

\[
\left. p(0) \in L_0, p(1) \in L_1, [p(t)] \in p, \right\rangle
\]

spanned by chords of \(\phi^t_{G_t}\) from \(L_0\) to \(L_1\) in class \(p\). Up to a global shift, this grading is well-defined modulo the greatest common divisor of the Maslov numbers of the two cobordisms; we refer to [25] for more details. For our purposes the grading will not play any role.
Given two numbers \( m_- \leq m_+ \) we also define
\[
CF^p_*(L_0, L_1; G_t; \varphi)^{m_+}_{m_-} := \mathbb{Z}_2 \left\{ \begin{array}{l}
p(t) = \phi^{t}_G(x), \ t \in [0, 1], \\
p(0) \in L_0, p(1) \in L_1, [p(t)] \in p, \\
m_- \leq a_\varphi(p(t)) \leq m_+,
\end{array} \right\
\]
which is a vector subspace of \( CF^p_*(L_0, L_1; G_t) \). It is important to observe that the subspace depends on the choice of function \( \varphi : X \to \mathbb{R} \) used when defining the action. When the function \( \varphi \) is clear from the context, we will sometimes use the notation
\[
CF^p_*(L_0, L_1; G_t)^{m_+}_{m_-} = CF^p_*(L_0, L_1; G_t; \varphi)^{m_+}_{m_-},
\]
i.e. omitting this choice.

Under certain additional assumptions deferred to Section 3.5 below, we can define the following linear maps.

The differential: This is a linear map
\[
d : CF^p_*(L_0, L_1; G_t) \to CF^{p-1}_*(L_0, L_1; G_t)
\]
that on a generator \( q \) is defined via the \( \mathbb{Z}_2 \)-count
\[
d(q) := \sum_{\dim(M_{p,q}(L_0, L_1; G_t)/\mathbb{R}) = 0} \#_2(M_{p,q}(L_0, L_1; G_t)/\mathbb{R}) p.
\]
Here we have used a fixed almost complex structure \( J = J_t \) only depending on the \( t \)-coordinate, and \( M_{p,q}(L_0, L_1; G_t)/\mathbb{R} \) denotes the solutions up to the action of translation of the \( s \)-coordinate (this action obviously preserves the solutions). Using Lemma 3.5 we see that \( \langle d(q), p \rangle \neq 0 \) being nonzero implies that \( a_\varphi(p) > a_\varphi(q) \). In other words, our differential increases the action.

The continuation maps: Given a one-parameter family of Hamiltonians \( G_{s,t} \) of the form prescribed by Lemma 3.6 the induced continuation map
\[
\Phi_{G_{-t}, G_{+t}} : CF_*(L_0, L_1; G_{-t}) \to CF_*(L_0, L_1; G_{+t})
\]
is defined on a generator \( p_- \) of \( CF_*(L_0, L_1; G_{-t}) \) via the \( \mathbb{Z}_2 \)-count
\[
\Phi_{G_{-t}, G_{+t}}(p_-) := \sum_{\dim(M_{p_-, p_-}(L_0, L_1; G_{s,t}) = 0)} \#_2(M_{p_-, p_-}(L_0, L_1; G_{s,t}) p_+.
\]
The previously mentioned lemma determines its behavior with respect to the action filtration. Observe that a continuation map does not necessarily increase the action.

The following is the transversality result that we need in order to define the above counts, which is standard. The compactness properties will be dealt with in Section 3.5.

**Proposition 3.8.** For \( J_{s,t} \in J(X, d\eta, \varphi) \) generic among those which satisfy (3.2) the moduli spaces
\[
\bigcup_{\kappa \in \mathbb{R}} M_{p_+, p_-}(L_0, L_1; \rho_\kappa(s)G_t),
\]
where \( \rho_\kappa(s) \) depends smoothly on \( \kappa \in \mathbb{R} \) and where \( \rho_\kappa(t) \equiv 0 \) outside of a compact subset (allowed to depend on \( \kappa \), are all transversely cut out. Moreover, the same is true for the moduli spaces
\[
M_{p_+, p_-}(L_0, L_1; G_t)
\]
for generic choices of \( J_t \in J(X, d\eta, \varphi) \) only depending on \( t \in [0, 1] \).

Furthermore, transversality can be achieved by a perturbation of \( J_t \) supported in some arbitrarily small precompact neighborhood of the Hamiltonian chords.

**Proof.** See e.g. [42] Proposition 15.5 for the \( t \)-dependence case, but also [3] Section 8.6 for the analogous result in the closed case (which is similar).

The only note to make is that our almost complex structure must remain admissible after the perturbation (see Section 3.4). To achieve this, note that in Definition 2.8 we can consider arbitrary
time-dependent perturbations of the tame almost complex structure in some neighborhood of the Hamiltonian chords; clearly the strip must enter this neighborhood by the assumption made on its asymptotics.

Finally, observe that the case when \( J \) depends on both \((s,t)\) as opposed to just \( t \) is considerably easier, since the “some-arc-injective” (weaker than “somewhere-injective” which is used when \( J \) domain-independent) condition is not needed.

\[ \square \]

3.4. A condition for non-bubbling. A main technical point of this paper establishes conditions for when \((\CF^B_s(L_0,L_1;J,G_t)_{m-},\partial)\) is a complex and when the continuation map \(\Phi_{G_{-t},G_{+t}}\) is well-defined. To this end, the below result Theorem 3.10 is the only new ingredient needed in our Floer theory set-up, which gives a condition for when strips are confined to some given compact subset.

In this section we will consider a fixed \( \varphi \in \mathcal{C}(L_0,L_1,G_t) \) for action computations, together with an associated constant \( 0 \leq r_{\varphi} \leq +\infty \) as described in Definition 2.8. Let the set \( Q_{\alpha-}^0 \) consist of all contractible periodic \((\alpha-)\)-Reeb orbits and all \((\alpha-)\)-Reeb chords having

- both ends on \( \Lambda_0^- \), while defining the trivial element in \( \pi_1(Y_-,\Lambda_0^-) \),
- both ends on \( \Lambda_1^- \), while defining the trivial element in \( \pi_1(Y_-,\Lambda_1^-) \), or
- start point on \( \Lambda_0^- \) and endpoint on \( \Lambda_1^- \), living in the component \( p \in \pi_0(\Pi(X,L_0,L_1)) \).

We define the quantity

\[
h(\varphi, p, \Lambda_0^-, \Lambda_1^-, \alpha_-) := e^{-r_{\varphi}} \min_{c \in Q_{\alpha-}^0} \int_c \alpha_- > 0,
\]

which for short will be referred to simply as \( h \).

**Remark 3.9.** The above definition of \( h \) does not take the length of any Reeb chord starting at \( \Lambda_1^- \) and ending at \( \Lambda_0^- \) into account. This is important, since in the application that we have in mind, there will be very small such Reeb chords; see the push-off of the Lagrangian cobordism considered in the proof of Theorem 1.2 in Section 8.

In the subsequent Section 4 we prove the following:

**Theorem 3.10** (Non-bubbling for strips). **Fix an admissible function \( \varphi \), a compact family of admissible almost complex structures \( J \in \mathcal{J}(X,dp,\varphi) \) as well as exact Lagrangian submanifolds \( L_i \), and a family \( G_{s,t} = \rho(s)G_t \) of Hamiltonians.** Both the families \( J \) and \( L_i \) are required to be fixed outside of some precompact subset as in Proposition 3.10. Then there is a compact subset \( K \subset X \) containing all \( J \)-holomorphic strips \( u \in \mathcal{M}_{p_+,p_-}(L_0,L_1;G_{s,t}) \) of either of the following types, under the assumption that \( p_{\pm} \in p \).

\( (0) \) In the case when \( \rho \equiv 1 \) or \( \rho \equiv 0 \) we require

\[
\max \{ a_\varphi(p_+) - a_\varphi(p_-) - h, a_\varphi(p_+) - h \} < 0;
\]

\( (1) \) In the case when \( \rho \) is non-constant and \( \rho(s) = 0 \) (resp. \( \rho(s) = 1 \)) whenever \( |s| \gg 0 \) is sufficiently large, i.e. \( G_{-t} \equiv G_{+t} \equiv 0 \) (resp. \( G_{-t} \equiv G_{+t} \equiv G_t \)), we require

\[
\max \{ a_\varphi(p_+) - a_\varphi(p_-) - h, a_\varphi(p_+) - h \} < -\|G_t\|_{\text{osc}};
\]

(2) In the case when \( \rho(s) = 0 \) for \( s \ll 0 \), \( \rho(s) = 1 \) for \( s \gg 0 \), and \( \rho'(s) \geq 0 \), we require

\[
\max \{ a_\varphi(p_+) - a_\varphi(p_-) - h, a_\varphi(p_+) - h \} < -\int_0^1 \max G_{i} dt;
\]

and
(3) In the case when \( \rho(s) = 1 \) for \( s \ll 0 \), \( \rho(s) = 0 \) for \( s \gg 0 \), and \( \rho'(s) \leq 0 \), we require
\[
\max\{a_\varphi(p_+) - a_\varphi(p_-) - \hbar, a_\varphi(p_+) - \hbar\} < \int_0^1 \min_X G_t dt.
\]

3.5. Well-definedness and invariance. We proceed to apply Theorem 3.10 in order to define our Floer complexes, and to show the needed invariance properties.

Using the above non-bubbling theorem we obtain a condition for when the differential of the Floer complex is well-defined. Namely, once the strips have been shown to be confined to a given compact subset, the remaining argument is standard.

Proposition 3.11 (Conditions for a well-defined complex). Under the assumption that
\[
\max\{m_+ - m_- - \hbar, m_+ - \hbar\} < 0
\]
is satisfied with \( h \) and \( \varphi \) as specified above, the map
\[
d: CF^\varphi_*(L_0, L_1; \varphi)^{m_+}_{m_-} \to CF^\varphi_{*-1}(L_0, L_1; \varphi)^{m_+}_{m_-}
\]
is well-defined and satisfies \( d^2 = 0 \) for any generic \( J_t \in J(L_0, L_1, \varphi) \).

Proof. All moduli spaces of strips involved in the definition of \( d \), as well as the glued strips involved in the definition of \( d^2 \), satisfy the energy bounds in Case (0) of the non-bubbling Theorem 3.10. In view of this result, the relation \( d^2 = 0 \) now follows as in [42 Sections 19.2, 19.5] by a gluing argument, taking the one-dimensional moduli spaces into account. \( \square \)

We only establish a limited form of invariance for our Floer complexes with action filtration that will be sufficient for our needs. We work under assumptions for which the above non-bubbling result is satisfied for the relevant strips, and the rest of the argument is then again classical.

Proposition 3.12 (Filtered invariance). Let \( \hbar \) and \( \varphi \) be as specified above. Assume that, for numbers satisfying
\[
m_- + \max\left\{ \int_0^1 \max_X G_t dt, 0 \right\} \leq m'_- \leq m'_+ \leq m_+,
\]
we have an equality
\[
CF^\varphi_*(L_0, L_1; \varphi)^{m'_+}_{m'_-} = CF^\varphi_*(L_0, L_1; \varphi)^{m'_+}_{m'_-}.
\]
Under the additional assumptions that
\[
\max\{m_+ - m_- - \hbar, m_+ - \hbar\} < 0,
\]
\[
\max\{m'_+ - m'_- - \hbar, m'_+ - \hbar\} < -\|G_t\|_{\text{osc}},
\]
\[
\max\{m_+ - m'_- - \hbar, m_+ - \hbar\} < -\int_0^1 \max_X G_t dt,
\]
\[
\max\{m'_+ - m'_- - \hbar, m'_+ - \hbar\} < \int_0^1 \min_X G_t dt,
\]
are satisfied, then there are continuation maps
\[
\Phi_{0,G_t}: CF^\varphi_*(L_0, L_1; \varphi)^{m'_+}_{m'_-} \to CF^\varphi_*(L_0, L_1; \varphi)^{m_+}_{m_-},
\]
\[
\Phi_{G_t,0}: CF^\varphi_*(L_0, L_1; \varphi)^{m'_+}_{m'_-} \to CF^\varphi_*(L_0, L_1; \varphi)^{m_+}_{m_-},
\]
which are chain maps whose composition admits a chain homotopy
\[
\Phi_{G_t,0} \circ \Phi_{0,G_t} \sim \Id_{CF^\varphi_*(L_0, L_1; \varphi)}
\]
making \( \Phi_{G_t,0} \) a left-sided homotopy inverse of \( \Phi_{0,G_t} \).
Proof. The well-definedness of the complexes is implied by Proposition 3.11 above, which shows that the non-bubbling result Theorem 3.10 applies to the Floer strips defining the boundary operators as well as its squares. We need to show that the moduli spaces of strips involved in the chain maps and chain homotopies, as well as the different glued strips appearing in the algebraic relations, all satisfy the corresponding hypotheses of Theorem 3.10. In other words, we need to consider the moduli spaces arising in the definitions of the operations

- \( \Phi_{0,G_t}, \Phi_{0,G_t} \circ \partial, \partial \circ \Phi_{0,G_t} \),
- \( \Phi_{G_t,0}, \Phi_{G_t,0} \circ \partial, \partial \circ \Phi_{G_t,0} \),
- \( K_{0,G_t,0}, \Phi_{G_t,0} \circ \Phi_{0,G_t}, K_{0,G_t,0} \circ \partial, \partial \circ K_{0,G_t,0} \).

Here \( K_{0,G_t,0} \) is the chain homotopy defined by counting rigid Floer strips with a Hamiltonian perturbation term \( \rho_\kappa(s)G_t \) for an appropriate one-parameter family of compactly supported functions \( \rho_\kappa(s) \); see e.g. [42] for more details.

These operations are all solutions of Cauchy-Riemann equations with a perturbation term coming from a family of Hamiltonians of the form \( G_{s,t} := \rho(s)G_t \). Here, \( 0 \leq \rho(s) \leq 1 \) satisfies the assumptions of Lemma 3.6. That the concerned strips satisfy the conditions of Theorem 3.10 is now a consequence of Proposition 3.12 with \( \rho_\kappa(s) \); see e.g. [42] for more details.

As a direct application of Lemma 3.6, the chain map in the above proposition can be shown to satisfy the assumptions of Proposition 3.12 with \( U = \emptyset \) the chain map \( \Phi_{0,G_t} \) can decrease action by at most

\[
\max \left\{ \int_0^1 \max_X G_t dt, 0 \right\},
\]

in combination with Inequalities (3.13) together with the assumption that all the generators of the complex \( CF^\rho_\kappa(L_0, L_1; 0; \varphi) \) have action in the range \([m'_-, m'_+]\).

Since we thus have established that the concerned strips are contained in some fixed compact subset, the rest of the argument is standard. We refer to [32] Sections 19.3, 19.5] for the compactness and gluing argument needed for obtaining the sought algebraic relations satisfied by these maps, which roughly speaking consists of analyzing the boundary of one-dimensional moduli spaces of strips. Here we note that only generators in the specified action ranges are involved in the broken strips arising as boundary points of the relevant one-dimensional moduli spaces. Here it is crucial to use the facts that

- by Lemma 3.5 the differential cannot decrease action, while,
- by Case (2) of Lemma 3.6 (with \( U = \emptyset \)) the chain map \( \Phi_{0,G_t} \) can decrease action by at most

\[
\max \left\{ \int_0^1 \max_X G_t dt, 0 \right\},
\]

As a direct application of Lemma 3.6, the chain map in the above proposition can be shown to satisfy the following behavior with respect to the action filtration.

**Proposition 3.18 (Filtration properties).** Assume that the hypotheses of Proposition 3.12 are satisfied for some choice of \( \varphi \in C(L_0, L_1; G_t) \). Then, for an arbitrary, and possibly different, choice of \( \tilde{\varphi} \in C(L_0, L_1; G_t) \), the following can then be said:

1. If \( \langle \Phi_{0,G_t}(p_-), p_+ \rangle \neq 0 \) holds for generators

\[
p_- \in CF^\rho_\kappa(L_0, L_1; 0; \varphi) \quad \text{and} \quad p_+ \in CF^\rho_\kappa(L_0, L_1; G_t; \varphi)^{m'_+},
\]

then

\[
\alpha_{\tilde{\varphi}}(p_+) > \alpha_{\tilde{\varphi}}(p_-) - \int_0^1 \max_X G_t dt
\]

is satisfied.

2. If \( \langle \Phi_{G_t,0}(q_-), q_+ \rangle \neq 0 \) holds for generators

\[
q_- \in CF^\rho_\kappa(L_0, L_1; G_t; \varphi)^{m'_-} \quad \text{and} \quad q_+ \in CF^\rho_\kappa(L_0, L_1; 0; \varphi)
\]

then

\[
\alpha_{\tilde{\varphi}}(q_+) > \alpha_{\tilde{\varphi}}(q_-) + \int_0^1 \min_X G_t dt
\]

is satisfied.
Consider a vector subspace
\[ C_*^\lambda \subset CF^b_*(L_0, L^\lambda_1; G_t; \varphi) \]
spanned by generators, all which are contained inside the complement of the support of the Hamiltonian isotopy \( L^\lambda_t, \lambda \in [0, 1]. \) (Hence, as vector spaces, \( C_*^\lambda \) are all canonically isomorphic.) Consider a compactly supported family \( J^\lambda_t, \lambda \in [0, 1], \) of admissible \( t \)-dependent almost complex structures. Assume that any two generators \( p, q \in C_*^\lambda \) satisfy
\[ \max \{ a_\varphi(p) - a_\varphi(q) - h, a_\varphi(p) - h \} < 0. \]
Then, by Case (0) of Theorem 3.10, any Floer strip \( u \in \mathcal{M}_{p,q}(L_0, L^\lambda_1; G_t) \) is contained inside a fixed compact subset \( K \subset X. \)

**Proposition 3.19.** (Invariance via bifurcation analysis) In the above setting, under the additional assumption that there exists no Floer strip \( u \in \mathcal{M}_{p,q}(L_0, L^\lambda_1; G_t) \) with output \( p \in C_*^\lambda \) and input \( x \in CF^b_*(L_0, L^\lambda_1; G_t; \varphi) \setminus C_*^\lambda, \) or vice versa, then:

1. Each of \( C_*^0, i = 0, 1, \) is a well-defined Floer when \( J^\lambda_t \) is a generic admissible almost complex structure; and
2. The complexes \( C_*^0 \) and \( C_*^1 \) are chain homotopy equivalent.

**Proof.** (1): This is a direct consequence of Proposition 3.11.

(2): Floer’s original sketch of invariance of Lagrangian Floer theory under Hamiltonian isotopies used bifurcation analysis: as the geometric data changes, the Floer complex changes by stabilizations (births and deaths of pairs of Lagrangian intersections) and handle-slides (the presence of isolated index \(-1\) pseudoholomorphic strips in one-parameter families of such) [24]. This sketch was made rigorous in [40] for Lagrangian Floer theory, as well as in [38] for Hamiltonian Floer theory.

By assumption, there are neither births nor deaths occurring in our family. Moreover, all relevant \((-1)\)-strips involved in the definition of the handle-slide maps are all contained in a fixed compact subset. Together with the assumption on the non-existence of the prescribed strips, the compactness and gluing arguments from [40] again shows that the induced algebraic handle-slide maps are well-defined chain maps on the complexes \( C_*^\lambda. \)

The typical situation when the above proposition can be applied is when an energy computation (possibly using an action \( a_\varphi \) defined with a different \( \varphi \)) prevents the existence of the unwanted Floer strips.

**3.6. Naturality.** It will be useful to switch perspectives between Floer complexes defined in terms of intersection points of Lagrangian submanifolds (e.g. for the neck stretching construction in Section 6.1) and in terms of Hamiltonian chords (e.g. for defining continuation maps). The following naturality property provides a translation between these two definitions.

**Lemma 3.20.** Let \( L_0, L_1 \subset (X, d\eta) \) be exact Lagrangian cobordisms, fix a function \( \varphi \) as in Section 2.1 and let \( G_t: X \to \mathbb{R} \) be a Hamiltonian. Fix a time-dependent tame almost complex structure \( J_t \) on \( (X, d\eta) \). Using action conventions in the same section, there is a canonical action-preserving chain complex isomorphism
\[ CF_*(\phi^1_G, (L_0); L_1; 0) \simeq CF_*(L_0, L_1; G_t), \]
where the former complex is defined using \( J_t \) and the latter complex is defined using
\[ \bar{J}_t := D(\phi^1_G \circ (\phi^t_G)^{-1}); J_t \circ D(\phi^1_G \circ (\phi^t_G)^{-1}), \]
and where an intersection point \( p \in \phi^1_G(L_0) \cap L_1 \) is identified with the Hamiltonian chord \( t \mapsto \phi^1_G((\phi^t_G)^{-1}(p)) \) from \( (\phi^t_G)^{-1}(p) \in L_0 \) to \( p \in L_1. \)
The fact that the two definitions of action coincide follows from Cartan’s formula. Namely, the differential of \( L \) to \( W \) is a simpler result which recovers a certain Reeb chord or orbit, and implies Theorem 3.10.

Due to the (additional) non-compactness in this case, some extra care must be taken when one studies the symplectizations results do not explicitly prove compactness for the set of solutions to Equation (3.3).

In this section we prove Theorem 3.10. Compactness results in symplectizations have been proved in a number of different set-ups, see [11] [11] [17] [23] for example. However, to our knowledge, existing symplectizations results do not explicitly prove compactness for the set of solutions to Equation (3.3).

Rather than extending the full SFT compactness results to our set-up, we prove Proposition 4.6; this is a simpler result which recovers a certain Reeb chord or orbit, and implies Theorem 3.10.

3.7. A note about the non-compact case. There are also interesting cases where the contact manifold is non-compact. We restrict attention to the following situation. Suppose that \((P,d\theta)\) is a Liouville manifold, i.e. a complete exact symplectic null-cobordism with a convex but no concave end. We will consider contact manifolds of the form \((Y,d\eta)\); 3.7. A note about the non-compact case. There are also interesting cases where the contact manifold is non-compact. We restrict attention to the following situation. Suppose that \((P,d\theta)\) is a Liouville manifold, i.e. a complete exact symplectic null-cobordism with a convex but no concave end. We will consider contact manifolds of the form \((Y,d\eta)\);
Our argument to prove Proposition 4.6 is essentially a relative (Lagrangian boundary condition) version of [5] Sections 5 and 6, which is turn almost entirely relies on [17] Section 5. Since [5] is closer to our set-up than [17], we make precise references to [5]. The interested reader can then use [5] to see which specific result is relevant from the original source [17].

We set-up some notation to be used throughout this section. In the proof of Theorem 3.10 below, we will be interested in studying a sequence of pseudoholomorphic strips, $\tilde{u}_k$ with fixed positive and negative punctures at $p_+$ and $p_-$, respectively, both which are assumed to live in the component $p$. We will restrict these to the negative symplectic end $u_k := \tilde{u}_k|_{U_k=(r \leq \kappa) \times Y}$, where $\kappa < -N - 1 < -N - \epsilon$, independent of $k$, is a regular value of the projections in the $\mathbb{R}$-coordinate for $\tilde{u}_k$. The terms $N \gg 1 \gg \epsilon > 0$ will be defined in the construction of our Lagrangians in Section 7. In particular, we will assume for these restrictions that the Hamiltonian term in (3.3) vanishes and that the Lagrangian boundary conditions $L_0, L_1$ are cylindrical $\{r \leq \kappa\} \times \Lambda_0^{-}, \{r \leq \kappa\} \times \Lambda_1^{-}$. Let

$$Z_k := \tilde{u}_k^{-1}(\{r \leq \kappa\} \times Y) \subset \mathbb{R} \times [0, 1]$$

note the domain of $u_k$, which is a Riemann surface due to the regularity assumption on $\kappa$. Write $u_k = (a_k, f_k)$ where $a_k: Z_k \rightarrow \mathbb{R}$ is the projection of the $r$-coordinate.

Throughout this section, $J$ (or $J_k$) denotes an almost complex structures considered in Section 3.1 restricted to this negative end. Since for any $u_k$ that we consider the image is compact, although not a priori uniformly so, the Hofer energy

$$E(u_k) := \sup_{\{v \in C^\infty(\mathbb{R} \times [0, 1]) \mid v \geq 0\}} \int u_k^*(v) < E = E(x, y) < \infty.$$ 

can be seen to be uniformly bounded by a function of the intersection points $x, y \in L_0 \cap L_1$; see e.g. [13], Proposition 3.6.

We begin with relative versions of the Monotonicity Lemma, the Maximum Principle, and the Conformal Modulus, adopting as much as possible the notation of the absolute versions given in [5]. Let $g_J$ be the metric defined by $J$ and the symplectic form. Let $B_{g_J}(\text{center}; \text{radius})$ denote a ball defined with that metric.

**Lemma 4.1** (Monotonicity). There exists constants $C, 1 > 10\epsilon > 0$ such that for any $0 < \delta < \epsilon$ and for any $J$-holomorphic map $v: Z \rightarrow \{r \leq \kappa\} \times Y$, mapping the boundary $\partial Z$ to $\{(\kappa) \times Y\} \cup \{(r \leq \kappa) \times \Lambda_0^{-}\} \cup \{(r \leq \kappa) \times \Lambda_1^{-}\}$,

$$\text{area}_{g_J}(v(Z) \cap B_{g_J}(y; \delta)) \geq C\delta^2$$

whenever there is a $y \in v(Z)$ with $B_{g_J}(y; \delta) \subset \{r < \kappa\} \times Y$ such that $v$ is nonconstant on a component $Z_0 \subset Z$ whose image contains $y$. Moreover, the constants vary continuously with $J$.

**Proof.** This follows from [2] Proposition 4.7.2 after noting that the noncompact Lagrangians are cylindrical, and so the constant $C = C(L_0 \cup L_1)$ in the monotonicity lemma is nonzero. See also [11], Proposition 2.69. 

Denote the closed upper half-plane by

$$\mathbb{H} := \{z \in \mathbb{C}; \; \text{Im}(z) \geq 0\}$$

and its boundary by $\partial \mathbb{H}$.

**Lemma 4.2** (Maximum Principle). Let $U \subset \mathbb{H}$ be a connected neighborhood of $0 \in \mathbb{H}$. If $u: (U, U \cap \partial \mathbb{H}) \rightarrow (\mathbb{R} \times Y, \mathbb{R} \times (\Lambda_0^{-} \cup \Lambda_1^{-}))$ is pseudoholomorphic and non-constant then $r \circ u$ has no local maximum.

**Proof.** This follows from standard properties of subharmonic functions; see e.g. [36], Lemma 5.5. 

□
Lemma 4.3 (Conformal Modulus). Let $X$ denote either the interval $[0,1]$ or the unit circle $S^1 = \mathbb{R}/\mathbb{Z}$. Let $v = (a_v, f_v) : X \times [0,L] \to (-\infty, \kappa] \times Y$ be holomorphic, with Lagrangian boundary conditions $\mathbb{R} \times \Lambda_d^j$ on $\partial X$. Assume $a_v(t,0) \leq R < S \leq a_v(t,L)$ for all $t \in X$. Then $L$, called the conformal modulus of $X \times [0,L]$, is bounded below by $(S-R)/2E(v)$.

Proof. The case when $X = S^1$ is proved as [5, Lemma 6.9]. The proof of the $X = [0,1]$ case can be copied verbatim.

Let $Z$ be a subset of $\mathbb{R} \times [0,1]$ or $\mathbb{R} \times (\mathbb{R}/2\mathbb{Z})$. Let $w = (a_w, f_w) : Z \to (-\infty, \kappa] \times Y$ be a smooth (not necessarily $J$-holomorphic) map, non-constant on all of its components, and where in the case $Z \subset \mathbb{R} \times [0,1]$ we assume Lagrangian conditions

$$w(Z \cap ((-\infty, \kappa] \times \{j\})) \subset \mathbb{R} \times \Lambda_d^j$$

for $j = 0, 1$. Assume $a_w|_{\partial Z} = \kappa$, or in the case of a Lagrangian boundary condition, $a_w|_{\partial Z \setminus \mathbb{R} \times \{0,1\}} = \kappa$. Fix $b \in (\kappa - 2\delta, \kappa - \delta)$ where $b, b \pm \delta$ are regular values of $a_w$. For $R < S < \kappa - 4\delta$ (which we think as “not fixed” compared to $b$) where $R, S, R \pm \delta, S \pm \delta$ are regular values of $a_w$ and $S - R \geq 2\delta$, we will define surfaces $Z_R^S(w), Z_{r+b}(w)$ as done in [5, Section 6]. Let $C_R$ be the connected components of $(a_w)^{-1}([R, R + \delta])$ and of $(a_w)^{-1}([R - \delta, R])$. Begin constructing $C_R^+ \subset C_R$ (resp. $C_R^- \subset C_R$) by including all components that meet $(a_w)^{-1}(\kappa - 4\delta)$ (resp. $(a_w)^{-1}(R + \delta)$) and those in $(a_w)^{-1}([R, R + \delta])$ (resp. $(a_w)^{-1}(R - \delta, R)$) that do not meet $(a_w)^{-1}(\kappa - 4\delta)$. Next extend first $C_R^+$ (resp. second $C_R^-$) to include all components in $C_R$ which connect to the previous $C_R^+$ (resp. $C_R^-$) without passing through $C_R^-$ (resp. the extended $C_R^-$). Repeat this (finite) process as long as $C_R^+$ increases, at which point $C_R^- = C_R^+ \cup C_R$. Set

$$Z_R^S(w) = (a_w)^{-1}([R, R + \delta]) \cup C_R^+ \cup C_R^-,$$

$$Z_{r+b}(w) = (a_w)^{-1}((\kappa - 4\delta, b - \delta)) \cup C_b^-,$$

$$Z_{r+b}^S(w) = Z_{r+b}(w) \setminus Z_R^S(w).$$

Define the subset $P_0 \subset Z$ to be a $\delta$-essential local minimum (resp. maximum) on level $R_0$ of $w$ if $P_0$ is a connected component of $a^{-1}_w((\kappa - 4\delta, R_0 + \delta])$ (resp. $a_w^{-1}([R_0 - \delta, \kappa - \delta)]$) and $R_0 = \min_{R_0} a_w$ (resp. $R_0 = \max_{R_0} a_w$). Note $P_0$ need not be a point and could intersect the boundary of $Z$. Define the function $\chi_w : (-\infty, \kappa] \to \mathbb{Z}$ on the set of regular values of $a_w$ by setting $\chi_w(r) = \chi(Z_R^S(w))$. Recall $b$ is fixed once $u$ is. A value $r \in (-\infty, \kappa]$ is called a jump if there is a non-zero difference in limits (taken over regular values)

$$\lim_{S \to r^+} \sup_{S} \chi_w(S) - \lim_{R \to r^-} \sup_{R} \chi_w(R).$$

Now suppose that $u = (a,f) : Z \to (-\infty, \kappa] \times Y$ is the $J$-holomorphic restriction, to the negative symplectic end $(-\infty, \kappa] \times Y$, of an arbitrary element $\tilde{u} = (\tilde{a}, \tilde{f})$ in the statement of Theorem 3.10. We construct the following “doubles.” Let $\bar{Z} \subset [-1,0] \times \mathbb{R}$ be the complex conjugate of $Z$. Let $Z_d = (\bar{Z} \cup Z)/\{x - i \approx x + i\} \subset \mathbb{R} \times (\mathbb{R}/2\mathbb{Z})$. For any regular values $R \neq S$ of $a$, define $(Z_R^S(u))^d, (Z_R^S(u))^d \subset Z_d$ in the same way. Construct a smooth (but not $J$-holomorphic) map $u^d = (a^d, f^d) : Z_d \to (-\infty, \kappa] \times Y$, such that for all $r \in \mathbb{R}$, $(a^d)^{-1}(r) = (a^{-1}(r))^d$. This implies $(Z_R^S(u))^d = Z_R^S(u)$ for all regular values $R \neq S$. To see more explicitly how $u^d$ can be constructed, fix a constant $\varepsilon > 0$. For $z = x + iy \in Z_d$ with $\varepsilon < |y| < 1 - \varepsilon$, set $u^d(x + iy) = u(x + i|y|)$. For $z = x + iy \in (a^{-1}(r))^d$ with $|y| < \varepsilon/2$ or $|y| > 1 - \varepsilon/2$, set $f^d$ to be locally constant (taking values in $\Lambda_d^j$).

Proposition 4.4. [5, Proposition 6.7] Recall $\delta$ from Lemma 4.4 and the uniform bound $E$ on the Hofer energy of any curve $u (= u_k)$. Let $g$ be a (uniform) bound on the genus of such curves. There exists an $N_0 = N_0(g, E, \delta)$ such that the number of $\delta$-essential minima and the number of jumps of $\chi_a$ is bounded from above by $N_0$. Furthermore, if $R, S, R \pm \delta, S \pm \delta$ are regular values of $a$, if $\chi((Z_R^S(u))^d) = 0$, and if there are no $\delta$-essential minima in $Z_R^S(u)$, then $Z_R^S(u)$ is the union of at most $N_0$ cylinders and/or strips, each running between levels $R$ and $S$.

Proof. The absolute version is proved as [5, Lemmas 6.2-6.6], which establish upper bounds, say $M_j > 0$, on the number of components, $\delta$-essential minima and absolute value of Euler characteristics of various
restrictions of the maps. Here the $M_j = M_j(\delta, E, g)$ and can be thought of “excessively large" since they each contain terms with $\delta^{-2}$. For the lemmas which require holomorphicity of the map, we indicate how the proof of the absolute case extends to the relative case. For the lemmas which only require smoothness of the map, we apply their results to $u^d$. We review each below.

Without any modification, the proof of [5] Lemma 6.2] implies the number of components of $Z^S_{R}(u)$ and $Z^B_{S}(u)$ is at most $M_1$. The relative version of the Monotonicity Lemma [11] supplements the absolute version. Adding Lemma [4.2] to preclude boundary maxima, the proof of [5] Lemma 6.3] shows the number of $\delta$-essential local minima of $u$ is at most $M_2$. Again, the Monotonicity Lemma-based proof carries over verbatim.

Since the proof of [5] Lemma 6.4] uses only smooth topology, we use it to prove
\[
\chi(Z^S_{R}(u^d)), \chi(Z^S_{R}(u^d)) \in [-M_3, M_3], \quad -M_3 \leq \chi(Z^S_{R}(u^d)).
\]

Similarly, the proof of [5] Lemma 6.6] uses only smooth topology and so we apply it (with the $M_3$ bounds above) to prove that the number of jumps of $\chi_{u^d}$, and hence $\chi_u$ as well, is bounded above by some $M_4$. This proof uses $Z^S_{R}(u^d)$ which otherwise appears inessential in our presentation.

Setting $N_0 = \max\{M_1, M_2, M_4\}$ gets us the first claim. It remains to modify [5] Lemma 6.5] to prove the second claim. There is a correspondence between $\delta$-essential minima/maxima of $u$ and of $u^d$ (one-to-one on the boundary and one-to-two in the interior). So if a component of $U$ of $Z^S_{R}(u^d)$ has $\chi(U) > 0$, then $U$ is a disk and $a^d_{\mid U}(\delta U) = S$ since by Lemma [4.2] $u^d$ has no maxima. This implies $u_{\mid U}(\delta U)$ has a $\delta$-essential minimum. So if $\chi(Z^S_{R}(u^d)) = 0$ and if there are no $\delta$-essential minima of $u$ in $Z^S_{R}(u^d)$, then all components of $Z^S_{R}(u^d)$ have Euler characteristic 0 and therefor $u^d_{\mid Z^S_{R}(u^d)}$ is a union of cylinders connecting the $R$ and $S$ levels. $N_0 \geq M_1$ implies the second claim.

\[\square\]

**Proposition 4.5.** [5] Proposition 5.4/6.10] Let
\[u_k = (a_k, f_k) : Z_k \to (-\infty, \kappa] \times Y\]
be the restrictions of $\tilde{u}_k$ introduced at the beginning of this section. Assume $\inf_k \inf_{Z_a} a_k = -\infty$. Then there exists a subsequences of numbers $k_n$, and a subsequence either all of strips or all of cylinders, $C_n \subset Z_{a_n}$ such that an $R$-shift $v_n = (b_n, g_n)$ of the restrictions of $u_{k_n}$ to $C_n$ has the following properties:

(1) $C_n$ is biholomorphic to $[-l_n, l_n] \times [0, 1] \subset \mathbb{C}$ or $[-l_n, l_n] \times S^1$, for all n, and $l_n \to \infty$ as $n \to \infty$. Here and elsewhere, $S^1 = \mathbb{R}/\mathbb{Z}$.

(2) $\int_{C_n} v_n^* d\alpha \to 0$.

(3) There is a sequence $\sigma_n \to \infty$ such that $b_n(\pm l_n, t) \geq \sigma_n$ for each $t \in [0, 1]$.

(4) If the $C_n$ are strips then either $v_n\mid[-l_n, l_n] \times \{0\} \subset L_j$ for each $j \in \{0, 1\}$, or $v_n\mid[-l_n, l_n] \times \{j\} \subset L_j$ for each of $j \in \{0, 1\}$. In the latter case we may moreover require that
\[r(v_n(-l_n, 0)) < r(v_n(l_n, 0)) < 0\]
is satisfied (observe that the continuous path $u_n(s, 0) \in L_0$ has both of its endpoints contained outside of the region $\{r \leq 0\}$).

**Proof.** We review the proof of [5] Proposition 5.4/6.10], modifying it to account for possible cylindrical Lagrangian boundary conditions and for Part (4). A level $\rho \in (\infty, \kappa]$ is defined as essential for $u_k$ if any of the following hold: $\rho = \kappa$; $\rho = \min u_k$; $\rho - \delta$ is a $\delta$-essential minimum; $\chi_{u_k}$ has a jump at $\rho$. Proposition [4.4] implies the number of essential levels of $u_k$ is bounded independent of $k$. Since $\lim_k \min a_k = -\infty$, we can, after possibly passing to a subsequence, find an interval $[\rho_n, s_n] \subset (-\infty, \kappa - \delta)$ of length at least $n$ such that the following hold: $[\rho_n, s_n]$ does not contain any essential levels; $\rho_n, s_n, \rho_n \pm \delta, s_n \pm \delta$ are regular values of $a_n$; and $\int_{a_n^{-1}([\rho_n, s_n])} u_n^* d\alpha \leq E/n$. Proposition [4.4] implies $u_n^{-1}([\rho_n, s_n])$ is a union of cylinders and strips, each running between $\rho_n \times Y$ and $s_n \times Y$. By continuity of each $\tilde{u}_n$ restricted to the boundary $\mathbb{R} \times \{0, 1\}$, at least one component must be a cylinder, a strip with both boundary
conditions on the same Lagrangian $L_j$, or with boundary $Z \cap \{j\} \times \mathbb{R}$) mapping to $L_j$ for $j = 0, 1$. Let $C_n$ denote one such component. The proof now follows from Lemma 4.3.

The final additional condition in Part (4) in the case of a mixed boundary condition follows from elementary topological considerations. □

**Proposition 4.6.** [5, Theorem 5.3] Let $u_k = (a_k, f_k) : Z_k \to (-\infty, \kappa] \times Y$ be the restrictions of $\tilde{u}_k$ introduced at the beginning of this section. Assume that $\inf_k \inf_{Z_k} a_k = -\infty$. Then there exists a subsequence $k_n$ and a subsequence of all strips or of all cylinders $C_n \subset Z_{kn}$ biholomorphically equivalent to the standard strip or cylinder, $[-l_n, l_n] \times [0, 1]$ or $[-l_n, l_n] \times S^1$, such that $l_n \to \infty$ and such that $u_{kn}|C_n$ converges (up to an $\mathbb{R}$-shift in either $C_{\infty, C_0}(\mathbb{R} \times [0, 1]; \mathbb{R} \times Y)$ or $C_{\infty, J}(\mathbb{R} \times S^1; \mathbb{R} \times Y)$) to a trivial strip or cylinder over a Reeb chord or closed orbit in the class $\mathcal{Q}_P$ as defined in Section 7.3.

**Proof.** Recall that because $u_k$ maps to $(-\infty, \kappa] \times Y$, there is no perturbation in the $J$-holomorphic equations and the Lagrangian boundary conditions are cylindrical. This is the case for which Abbas’ book provides a complete set of details for the lemmas used to prove compactness [1].

Aside from the lower bound in the last sentence, the proof of [5, Theorem 5.3], which uses [5, Proposition 5.4/6.10] allows us to deduce this proposition from Proposition 4.5 with the following modifications: replace cylinders with cylinders and/or strips; supplement the absolute Monotonicity Lemma with Lemma 4.1; derive a bound on the gradient not just at interior points but at boundary points as well, which [1, Proposition 2.56] does; supplement [5]’s references to [35] with relative analogues from [1]. For this third modification, [5, Theorem 5.3] uses (1) [35, Lemma 28] and (2) the proof of [35, Theorem 31]. Here (1) states that if $u : \mathcal{C} \to \mathbb{R} \times Y$ is $J$-holomorphic, $E(u) < \infty$ and $\int_{\mathcal{C}} u^* \alpha = 0$, then $u$ is constant. Roughly, (2) states that if $u : \mathcal{C} \to \mathbb{R} \times Y$ is $J$-holomorphic, $E(u) < \infty$, and $u$ is non-constant, then there exists a sequence of $s_k \to \infty$ such that the component of $u(t e^{2\pi i (\kappa + \delta t)})$ in $Y$ converges to a $T$-periodic Reeb orbit. The relative versions (domains are $\mathbb{H}$ instead of $\mathbb{C}$) of (1) and (2) are proved as [1, Proposition 2.55] and [1, Theorem 2.54/2.57], respectively.

The fact that the limiting Reeb chord or closed orbit is an element of $\mathcal{Q}_P$ follows from topology. If it is a chord with endpoints on two different Legendrians, Part (4) of Proposition 4.5 is also required. □

**Corollary 4.7.** Fix $\varphi \in \mathcal{C}(L_0, L_1, G_{s,t})$ such that $r_\varphi < \infty$, as defined in Definition 2.8. Let $u_k = (a_k, f_k) : Z_k \to (-\infty, \kappa] \times Y$ be the restrictions of $\tilde{u}_k$ introduced at the beginning of this section. Assume that $\inf_k \inf_{Z_k} a_k = -\infty$. Then there exist a subsequence $u_{kn}$ and a sequence of connected open $U_n \subset \mathbb{R} \times [0, 1]$ such that one of the following two cases holds.

1. $U_n \subset Z_{kn}$ is precompact and satisfies
   
   (a) $\lim \inf_{n \to \infty} E_{d(\varphi)}(u_{kn}|U_n) \geq h$, and
   
   (b) $\lim \sup_{n \to \infty} a_{kn}(U_n) = -\infty$.

2. $U_n$ contains the cylindrical end $\{s \leq -S\}$ for some $S \gg 0$, while it is disjoint from $\{s \geq S\}$, and satisfies
   
   (a) $\lim \inf_{n \to \infty} E_{d(\varphi)}(\tilde{u}_{kn}|U_n) \geq h - a_{\varphi}(p_-) - \int_0^1 G_{-; t}(p_-(t)) dt$,
   
   (b) $\lim \sup_{n \to \infty} a_{kn}(\partial U_n \setminus (\mathbb{R} \times \{0, 1\})) = -\infty$.

**Proof.** Let $k_n, C_n$ be as in Proposition 4.6.

Case 1: Suppose $u_{kn}|C_n$ converges to either a closed Reeb orbit or to a Reeb chord starting and ending on the same Legendrian component. We wish to show that Case (1) applies. Let $U_n$ be the component of $Z_{kn} \setminus C_n$ which does not intersect $\partial Z_{kn} \setminus (\mathbb{R} \times \{0, 1\})$.

By the Maximum Principle Lemma 4.2, $u_{kn}|U_n$ obtains its maximum on $\gamma_n := \partial U_n \setminus (\mathbb{R} \times \{0, 1\})$. Part (b) of Case (1) then follows, after a possible post-composed shift in the $\mathbb{R}$-direction, from the condition that $\pm l_n \to \pm \infty$ as stated in Proposition 4.6.
Let $\gamma'_n$ be the Reeb chord (resp. orbit) from Proposition 4.1. For arbitrary $\varepsilon > 0$ there exists $N$ such that for all $n \geq N$, there exists a smooth rectangle (resp. annulus) $A_n \subset \mathbb{R} \times Y$ which is a bordism from $u_{k_n}(\gamma_n)$ to $\{a_{k_n}(0,0)\} \times \gamma'_n$ in $((\mathbb{R} \times Y, L_0 \cup L_1)$ for which $\int_{A_n} da_- < \varepsilon$. Stokes’ theorem then implies

$$\int_{U_n} u^*_n \exp(\nu) d\omega_n = e^{-\nu} \int_{\gamma_n} u^*_n \omega_n \geq e^{-\nu} \left( \int_{\gamma_n} \omega_n - \varepsilon \right) \geq h - e^{-\nu} \varepsilon,$$

proving part (a) of Case (1). For the last inequality, we use that $\gamma'_n \in Q_{a_-}$ by Proposition 4.6.

**Case 2:** When $u_{k_n}|C_n$ converges to a Reeb chord running between $\Lambda_0^-$ and $\Lambda_1^-$, we set $U_n$ to be the component of $\mathbb{R} \times [0,1] \setminus C_n$ which includes the cylindrical end $\{s \leq -S\}$, and hence excludes $\{s \geq S\}$.

We again apply the maximum principle (to $u_{k_n}|C_n$) to conclude part (b) of Case (2). Part (a) of Case (2) follows from the same argument in Case (1): appending a smooth rectangle $A_n$ with $\varepsilon$-small $da_-$-area to $\tilde{u}_{k_n}(U_n)$; Stokes’ theorem; together with $e^{-\nu} \int_{\gamma_n} \omega_n \geq h$. The $(\int_0^1 G_{-rt} dt)$-term arises from the use of Stoke’s theorem as in Equality (3.4).

**Proof of Theorem 3.10.** We prove Case (2) of the theorem, as cases (0), (1) and (3) are similar. As discussed at the beginning of this section, we assume the theorem does not hold. Then there exists a sequence of $\tilde{u}_k$ (with restrictions $\tilde{u}_k|Z_k = u_k = (a_k, f_k)$) of solutions to the $G_{s,t}$-perturbed $J_k$-holomorphic equations, such that (after passing to a subsequence and taking $k$ sufficiently large) $\lim_k \min a_k = -\infty$ and the $J_k$ are sufficiently close so that the constants in Lemma 4.1 can be treated as independent of $k \gg 0$.

The hypothesis of Case (2) of Theorem 3.10 implies the inequality

$$0 > \max\{a_{\varphi}(p_+ - a_{\varphi}(p_-) - h, a_{\varphi}(p_+ - h) + \int_0^1 \max G_t dt.$$

We finally reach a contradiction by the below inequalities, derived from the above hypothesis, Cases (1.a) and (2.a) of Corollary 1.11 for $k \gg 0$ sufficiently large, together with Case (2) of Lemma 3.3 (in that precise order):

$$0 > a_{\varphi}(p_+) - a_{\varphi}(p_-) - h + \int_0^1 \max G_t dt$$

$$\geq a_{\varphi}(p_+) - a_{\varphi}^- - E_d(\varphi)|\tilde{u}_k|_{U_k}) + \int_0^1 \max G_t dt$$

$$\geq E_d(\varphi)|J_k(\tilde{u}_k|{[0,1] \times \mathbb{R}} \setminus U_k) > 0.$$



5. **Usher’s trick in the symplectization**

In this section we fix a contact manifold $(Y, \alpha)$ with a contact form $\alpha$. When speaking about the symplectization we will thus always refer to the identification $(\mathbb{R} \times Y, d(e^r \alpha))$. The contact Hamiltonian $H_t: Y \to \mathbb{R}$ has a lift $e^r H_t: \mathbb{R} \times Y \to \mathbb{R}$ to the symplectization which in general, of course, is unbounded. In order to produce a Hamiltonian that we can measure, it will be necessary to cut it off using a smooth bump function. Obviously, this must be done in a way which preserves the Hamiltonian diffeomorphism in some significant subset. In order to perform the cut-off in an efficient manner we will need to use the so-called Usher’s trick, which made its first appearance in the proof of [50, Theorem 1.3] due to Usher.

That this trick is very useful also for contact Hamiltonians was observed by Shelukhin in [17]; see Lemma 5.4. Here we follow his construction in order to prove the following result.
Proposition 5.1. Assume that we are given an indefinite contact Hamiltonian \( H_t \) and any number \( \delta > 0 \). Let \( A \geq 0 \) be as in (1.1). Then there exists a compactly supported Hamiltonian \( G_t: \mathbb{R} \times Y \to \mathbb{R} \) for which

\[
\phi^1_{G_t}(H_t) = \phi^1_{e^{H_t}},
\]

and whose Hofer norm satisfies the bound

\[
e^{A+2\delta} \min_{Y} H_{1-t} < \min_{\mathbb{R} \times Y} G_t \leq \max_{\mathbb{R} \times Y} G_t < e^{A+2\delta} \max_{Y} H_{1-t}.
\]

Moreover, we can take \( G_t \) to be of the form \( \kappa_t \cdot e^r K_t \) for an indefinite contact Hamiltonian \( K_t: Y \to \mathbb{R} \) and a smooth one-parameter family of compactly supported bump-functions \( \kappa_t: \mathbb{R} \times Y \to [0,1] \) satisfying:

1. For \( r \leq 0 \), the functions \( \kappa_t \) depend only on \( r \) and, moreover, they all vanish for \( r \leq -2\delta \);
2. For each fixed \((y,t) \in Y \times [0,1] \), the restriction \( \kappa_t(\cdot,y) \) is a smooth bump-function on \( \mathbb{R} \); and
3. On the subsets

\[
\phi^1_{G_t}((\phi^1_{G_t})^{-1}([A,A+2\delta] \times Y)), t \in [0,1],
\]

we have \( \kappa_t \equiv 1 \).

Proof. We start by fixing the notation

\[
Y_A := (-\infty, A + 2\delta] \times Y \subset \mathbb{R} \times Y.
\]

Recall that we have an equality

\[
\phi^1_{1-t} \circ (\phi^1_{a,F_t})^{-1} = \phi^1_{a,-F_{1-t}}
\]

of contact isotopies for any contact Hamiltonian \( F_t \).

We now consider the lifted contact Hamiltonian \( -e^r H_{1-t} \). Lemma \( \ref{Usher} \) (Usher’s trick) applied to the contact Hamiltonian \( F_t := -H_{1-t} \) produces an indefinite contact Hamiltonian \( \tilde{F}_t \). Denoting this contact Hamiltonian by \( -K_{1-t} := \tilde{F}_t \), we obtain

\[
\phi^1_{-e^r H_{1-t}} = \phi^1_{-e^r K_{1-t}},
\]

(this is Part (1) of that lemma) while for each \( t \in [0,1] \) the equalities

\[
\max_{\phi^1_{-e^r K_{1-t}}(Y_A)} (-e^r K_{1-t}) = \max_{Y_A} (-e^r H_t),
\]

\[
\min_{\phi^1_{-e^r K_{1-t}}(Y_A)} (-e^r K_{1-t}) = \min_{Y_A} (-e^r H_t),
\]

hold (this is Part (2) of that lemma).

We now claim that the corresponding contact Hamiltonian \( K_t \) will do the job, and where the cut-off function \( \kappa_t: \mathbb{R} \times Y \to \mathbb{R}_{\geq 0} \) is constructed to have support inside some arbitrarily small neighborhood of \( \phi^1_{-e^r K_{1-t}}(Y_A) \cap \{r \geq 0\} \) for each \( t \in [0,1] \), while it satisfies \( \kappa_t \equiv 1 \) when restricted to the latter subset.

The statement that \( \phi^1_{G_t} \) has the required behavior in the subset \((\phi^1_{G_t})^{-1}([A,A+2\delta] \times Y) \) will follow from Property (3), which we prove below. Namely, using Formulas \( \ref{phi2} \) and \( \ref{phi1} \), we then get

\[
\phi^1_{e^r K_t} = (\phi^1_{-e^r K_{1-t}})^{-1} = (\phi^1_{-e^r H_{1-t}})^{-1} = \phi^1_{e^r H_t}.
\]

In order to show that Property (3) can be made to hold simultaneously with (2) and (3), we now argue as follows.

First, note that the subsets

\[
\phi^1_{-e^r K_{1-t}}([A,A+2\delta] \times Y), \ t \in [0,1],
\]

all are contained inside \([0,\infty) \times Y\), since Part (4) in Lemma \( \ref{Usher} \) shows that the \( r \)-coordinate of this subset is decreased by at most \( A \) during this isotopy. (Recall that the lemma was applied to \( F_t = -H_{1-t} \) in order to construct \( \tilde{F}_t = -K_{1-t} \).)
Second, observe the equality
\[ \phi_{G_{t}}^{1-t} \circ (\phi_{G_{t}}^{1})^{-1}([A, A + 2\delta] \times Y) = \phi_{G_{t}}^{t}([A, A + 2\delta] \times Y) \]
which holds by the Hamiltonian analogue of Formula (5.2). The right-hand side can here be seen to coincide with \( \phi_{-e^{*}K_{1-t}}^{t}([A, A + 2\delta] \times Y) \) by the construction of the cut-off function \( \kappa_{t} \) together with \( G_{t} = \kappa_{t} \cdot e^{*}K_{t} \). In particular, \( \phi_{G_{t}}^{1-t} \circ (\phi_{G_{t}}^{1})^{-1}([A, A + 2\delta] \times Y) \) is contained in the subset where \( \kappa_{t} \) is equal to one.

The pay-off of this trick is that the Hofer norm for \( G_{t} \) in Proposition [5.1] is controlled with a factor approximately equal to \( e^{A} \) as opposed to \( e^{A+B} \). Such a larger factor would result using the naive approach setting \( G_{t} = \kappa_{t} \cdot e^{*}H_{t} \) instead of \( G_{t} = \kappa_{t} \cdot e^{*}K_{t} \) with a contact Hamiltonian \( K_{t} \) as constructed by Lemma [5.4].

**Lemma 5.4** (Usher). Let \( F_{t}: Y \rightarrow \mathbb{R} \) be a contact Hamiltonian. Then there exists a contact Hamiltonian \( \widetilde{F}_{t}: Y \rightarrow \mathbb{R} \) satisfying the following properties:

1. \( \phi_{\alpha,F_{t}}^{1} = \phi_{\alpha,F_{t}}^{1} \)
2. \( (e^{t} \widetilde{F}_{t}) \circ \phi_{\alpha,F_{t}}^{1} = e^{t} F_{1-t} \)
3. \( \min_{t \in [0,1]} \tau_{\alpha,F_{t}}^{t}(y) = -\max_{t \in [0,1]} \tau_{\alpha,F_{1-t}}^{t}(y) \)
4. \( \max_{t \in [0,1]} \tau_{\alpha,F_{t}}^{t}(y) = -\min_{t \in [0,1]} \tau_{\alpha,F_{1-t}}^{t}(y) \)

In particular, by the second property, \( \widetilde{F}_{t} \) is indefinite if and only if \( F_{t} \) is.

**Proof.** For any two Hamiltonians \( G_{t} \) and \( H_{t} \) on a symplectic manifold, it is a general fact that
\[ \phi_{G_{t}}^{t} \circ \phi_{H_{t}}^{t} = \phi_{G_{t}+H_{t},\circ(\phi_{G_{t}}^{t})^{-1}}^{t} \]
holds. In particular, the Hamiltonian isotopy \( (\phi_{H_{t}}^{t})^{-1} \) is generated by the Hamiltonian \( G_{t} \) satisfying
\[ G_{t} \circ \phi_{G_{t}}^{t} = -H_{t}. \]
The analogous relations also hold for lifts \( e^{*}G_{t} \) and \( e^{*}H_{t} \) of contact Hamiltonians, and hence also for the contact Hamiltonians themselves.

In the case when we apply the above formula for the inverse to the contact Hamiltonian \( H_{t} = -F_{1-t} \), i.e. the contact Hamiltonian that generates
\[ \phi_{\alpha,-F_{1-t}}^{t} = \phi_{\alpha,F_{t}}^{1-t} \circ (\phi_{\alpha,F_{t}}^{1})^{-1} \]
(see Formula (5.2) in the proof of Proposition [5.1] above), the produced contact Hamiltonian \( \widetilde{F}_{t} := G_{t} \) generating \( (\phi_{\alpha,-F_{1-t},\circ(\phi_{\alpha,F_{t}}^{1})^{-1}}^{t})^{-1} \) is the one we seek. Indeed, we have the equalities
\[ \phi_{\alpha,F_{t}}^{1} = (\phi_{\alpha,-F_{1-t},\circ(\phi_{\alpha,F_{t}}^{1})^{-1}}^{1})^{-1} = \phi_{\alpha,F_{t}}^{1} \circ (\phi_{\alpha,F_{t}}^{1})^{-1} = \phi_{\alpha,F_{t}}^{1}. \]

For Part (2) we compute
\[ e^{t} F_{1-t} = -e^{t} H_{t} = (e^{t} G_{t}) \circ \phi_{e^{t} G_{t}}^{t} = (e^{t} \widetilde{F}_{t}) \circ \phi_{e^{t} \widetilde{F}_{t}}^{t}. \]
Parts (3) and (4) are a straightforward consequence of the fact that \( \phi^{*} \alpha = e^{t} \alpha \) if and only if \( (\phi^{-1})^{*} \alpha = e^{-t} \alpha \). \qed
6. Splashing and neck-stretching

Here we use a splitting along a hypersurface of contact type towards two goals. In Section 6.1 we review neck-stretching. In Section 6.2 we wrap a Lagrangian in a certain way that, combined with neck-stretching, ultimately leads to a Mayer–Vietoris long exact sequence in Floer homology. For similar decompositions of complexes defined by counts of pseudoholomorphic curves in different settings, see [18] and [32]. The approach taken here is also closely related to the Mayer–Vietoris decomposition in symplectic homology and Wrapped Floer homology as constructed in [18] by Cieliebak–Oancea.

In this section let $L_0, L_1 \subset (X, d\eta)$ denote two exact Lagrangian submanifolds in an exact symplectic manifold. Consider a dividing hypersurface $(Y, \alpha) \subset (X, d\eta)$ of contact-type, where $\alpha = \eta|_{TY}$, and assume that

$$\Lambda_i := L_i \cap Y \subset (Y, \alpha), \ i = 0, 1,$$

are connected Legendrian submanifolds, where $\Lambda_i \subset L_i$ moreover is dividing and where $\Lambda_0 \cap \Lambda_1 = \emptyset$. We will also take a neighborhood of $Y \subset (X, d\eta)$ exact symplectomorphic to the subset

$$([T - 3\varepsilon, T + 3\varepsilon] \times Y, d(e^\varepsilon\alpha))$$

of the symplectization as part of the data, for some $\varepsilon > 0$ and arbitrary $T \in \mathbb{R}$ (such a neighborhood always exists), where the image of $L_i$ under this identification is given by the cylindrical Lagrangian submanifold

$$[T - 3\varepsilon, T + 3\varepsilon] \times \Lambda_i \subset [T - 3\varepsilon, T + 3\varepsilon] \times Y, \ i = 0, 1.$$

In addition, we here fix a piecewise smooth function $\varphi$ satisfying the properties of Definition 2.8 which moreover is assumed to satisfy

$$\varphi|_{[T - 3\varepsilon, T + 3\varepsilon] \times Y} \equiv 1.$$

Recall that for $i = 0, 1$ we have chosen primitives $f_0^\varphi, f_1^\varphi$ of the pullbacks of $\varphi\eta$ to $L_0, L_1$.

6.1. Neck-stretching. Neck-stretching can be used to prevent certain pseudoholomorphic curves or strips from crossing a hypersurface. In the setting of Floer homology with a Hamiltonian perturbation term, it appears in e.g. [18] Section 2.3, while in the setting of symplectic field theory (SFT for short) we refer to [21], [11]. However, note that the approach taken here is simpler compared to the one in [18], since our method only uses positivity of energy for strips, and does not rely on the SFT compactness theorem.

Denote by $X_L \cup X_R = X \setminus ([T, T + \varepsilon] \times Y)$ the components “to the left” and “to the right” of the cylindrical region, respectively (here we used the property that the hypersurface is dividing). One way to formulate the neck-stretching is as follows. First, excise the cylindrical pair

$$([T, T + \varepsilon] \times Y, [T, T + \varepsilon] \times (\Lambda_0 \cup \Lambda_1)) \subset (X, L_0 \cup L_1),$$

and replace it with the “longer pair”

$$([T, T + \varepsilon + \lambda] \times Y, [T, T + \varepsilon + \lambda] \times (\Lambda_0 \cup \Lambda_1)).$$

Rescale the symplectic form to be $e^\lambda d\eta$ on the subset $X_R$. In this manner we obtain a new pair of exact Lagrangian submanifolds of a new symplectic manifold.

For us we will only be interested in the case when $[T, T + \varepsilon] \times Y$ is a cylindrical subset of a cylindrical end of $X$. By assumption we have $\varphi(r) \equiv 1 \in \mathbb{R}$ for all $r \in [T, T + \varepsilon]$. In this case there is an alternative formulation of the above construction, where instead of replacing the manifold we replace the function $\varphi: X \to \mathbb{R}_{>0}$ with the function

$$\varphi_\lambda: X \to \mathbb{R}_{>0},$$

determined by

$$\varphi_\lambda := \begin{cases} 
\varphi(x), & x \in X_L, \\
ep_{\lambda}(r)\varphi(x) = ep_{\lambda}(r), & x = (r, y) \in [T, T + \varepsilon] \times Y, \\
ep^{\lambda}\varphi(x), & x \in X_R.
\end{cases}$$

(6.1)
where $\rho_\lambda: \mathbb{R} \to [0, \lambda]$ is a smooth bump-function for which $\rho'_\lambda \geq 0$ has compact support, while it satisfies $\rho_\lambda(T) = 0$ and $\rho_\lambda(T + \varepsilon) = \lambda$. Note that $\varphi_\lambda$ again satisfies the requirements of Definition 2.8 (in particular, see Part (2)). Moreover, there is an inclusion

$$\mathcal{J}(L_0, L_1, \varphi_\lambda) \subset \mathcal{J}(L_0, L_1, \varphi)$$

of subsets of admissible almost complex structures.

Let $C_i = f^T_i(T, \cdot)$ denote the value of this primitive at $L_i \cap Y \cap \{r = T\}$, which is constant by the connectedness of $L_i \cap Y$.

**Lemma 6.2.** Consider intersection points $p_L, p_R \in L_0 \cap L_1$ contained inside $X_L$ and $X_R$, respectively. After neck-stretching we have

$$a_{\varphi_\lambda}(p_R) = e^\lambda(a_{\varphi}(p_R) - (C_1 - C_0)) + (C_1 - C_0)$$

while the action of $p_L$ is unaffected, i.e. $a_{\varphi_\lambda}(p_L) = a_{\varphi}(p_L)$.

The effect of replacing $\varphi$ with $\varphi_\lambda$ will be said to be a **neck-stretching with parameter $\lambda$**.

**Proof.** The primitives of $\varphi_\lambda \eta$ pulled-back to $L_i$ are given by

$$f^T_\lambda = \begin{cases} f^T_i(x), & x \in L_i \setminus X_R, \\ e^\lambda(f^T_i(x) - C_i) + C_i, & x \in L_i \cap X_R, \end{cases}$$

as can be seen by a direct computation. \qed

### 6.2. Splashing

Consider the autonomous Hamiltonian

$$h^{\text{Spl}} = \int_0^r e^s \beta^{\text{Spl}}_{\varepsilon, Z_-, Z_+}(s)ds: (X, d\eta) \to \mathbb{R}$$

which only depends on the symplectization coordinate $r \in \mathbb{R}$, and where $\beta^{\text{Spl}}_{\varepsilon, Z_-, Z_+}$ is as shown in Figure [1]. The corresponding Hamiltonian vector field is given by $\beta^{\text{Spl}}_{\varepsilon, Z_-, Z_+} R_\alpha$, whose support inside $[T - 3\varepsilon, T + 3\varepsilon] \times Y$.

We define

$$L_0^{\text{Spl}} := \Theta^{\text{Spl}}_{h^{\text{Spl}}_{\varepsilon, Z_-, Z_+}}(L_0)$$

to be a Hamiltonian deformation of $L_0$, supported in $[T - 3\varepsilon, T + 3\varepsilon] \times Y$. As in [26], we call this a “splash.” We assume $L_0^{\text{Spl}} \cap [T - 3\varepsilon, T + 3\varepsilon] \times Y$ is cylindrical outside of $\{|r - (T \pm 2\varepsilon)| < \varepsilon^2\}$ and $\{|r - T| < \varepsilon^2\}$. The following result is straight forward.

**Lemma 6.3.** For any fixed numbers $Z_- < 0 < Z_+$, we can make the above Hamiltonian $\beta^{\text{Spl}}_{\varepsilon, Z_-, Z_+}$ arbitrarily small in the uniform $C^0$-norm by shrinking $\varepsilon > 0$. After either increasing $Z_+$ or decreasing $Z_-$, we may moreover assume that $h^{\text{Spl}} \equiv 0$ holds outside of $[T - 3\varepsilon, T + 3\varepsilon] \times Y$.

**Lemma 6.4.** The intersection points

$$\{p^-_1\} = L_0^{\text{Spl}} \cap L_1 \cap [T - \varepsilon^2, T) \times Y$$

are in bijective correspondence with a subset of the Reeb chords from $\Lambda_1$ to $\Lambda_0$, while the intersection points

$$\{p^+_1\} = L_0^{\text{Spl}} \cap L_1 \cap (T, T + \varepsilon^2] \times Y$$

are in bijective correspondence with a subset of the Reeb chords from $\Lambda_0$ to $\Lambda_1$. Moreover, the $\varphi$-action of the intersection point $p^+_1$ and the length of the corresponding Reeb chord $c^+_1$ satisfy

$$|\pm \varepsilon^2 \ell(c^+_1) + (C_0 - C_1) - a_{\varphi}(p^+_1)| < K\varepsilon,$$

where the constant $K = K(Z_-, Z_+, |\varphi(T, \cdot)|, T)$ depends continuously on the latter four numbers.
Proof. By the construction of
\[ L^\text{Spl}_0 = \varphi^1_{\beta^\text{Spl}_{\varepsilon,Z_-Z_+}}(L_0), \]
each intersection point \( p \in L^\text{Spl}_0 \cap L_1 \) corresponds to a chord of the time-one flow of the rescaled Reeb vector field \( \beta^\text{Spl}_{\varepsilon,Z_-Z_+} \), where the function \( \beta^\text{Spl}_{\varepsilon,Z_-Z_+} \) only depends on \( r \). The correspondence between intersection points and Reeb chords are hence straightforward. (Also, see Figure 6 for a similar correspondence.) By the same reasons, the pull-back of \( e^r \alpha_+ \) to \( L^\text{Spl}_0 \) takes the form \( e^r \frac{d}{dr} \beta^\text{Spl}_{\varepsilon,Z_-Z_+}(r) dr \).

For \( \{|r - T| \leq \varepsilon|^2\} \) we can assume
\[
\left| e^r \frac{d}{dr} \beta^\text{Spl}_{\varepsilon,Z_-Z_+} - e^T \frac{d}{dr} \beta^\text{Spl}_{\varepsilon,Z_-Z_+} \right| \leq |e^r - e^T| K/\varepsilon^2 \leq K.
\]
The integral of
\[
e^r \frac{d}{dr} \beta^\text{Spl}_{\varepsilon,Z_-Z_+} - e^T \frac{d}{dr} \beta^\text{Spl}_{\varepsilon,Z_-Z_+}
\]
then satisfies the estimate
\[
| \langle f^\varepsilon_0 - C_0 \rangle - (e^T \beta^\text{Spl}_{\varepsilon,Z_-Z_+}(r) - 0) \rangle \leq K \varepsilon.
\]
Since \( f^\varepsilon_0 \) is constantly equal to \( C_1 \) inside this region, the statements concerning the action thus hold. (Recall that \( f^\varepsilon_0(p^\pm_i) - f^\varepsilon_0(p^\mp_i) = a_\varphi(p^\pm_i) \), while \( e^T \beta^\text{Spl}_{\varepsilon,Z_-Z_+}(r(p^\pm_i)) = \pm e^T \ell(c^\pm_i) \).)}

Note there are no intersection points where \( L^\text{Spl}_0 \cap \{T - 3\varepsilon, T + 3\varepsilon\} \times Y \) is cylindrical, which includes the slices \( \{r = T \pm \varepsilon\} \).

**Proposition 6.5.** Fix \( Z_+ > 0 > Z_- \) satisfying
\[
Z_+ + (C_0 - C_1) > \overline{M}_+ > \overline{M}_- > Z_- - (C_0 - C_1).
\]
After possibly shrinking \( \varepsilon > 0 \), increasing \( Z_+ > 0 \) or decreasing \( Z_- < 0 \), the following can be made to hold. Consider the set of intersection points \( p \in L^\text{Spl}_0 \cap L_1 \) such that \( \overline{M}_- \leq a_\varphi(p) \leq \overline{M}_+ \). For any almost complex structure
\[
J(L_0, L_1, \varphi_\lambda) \subset J(L_0, L_1, \varphi),
\]
there exists no pseudoholomorphic strip whose input is an intersection point as above located in \( \{r \leq T + \varepsilon\} \) and whose output is an intersection as above point located in \( \{r \geq T + \varepsilon\} \), or vice versa (i.e. input in \( \{r \geq T - \varepsilon\} \) and output in \( \{r \leq T - \varepsilon\} \)).
Proof. Let $\lambda = \lambda(L_0^{\text{Spl}}, L_1, T, \varphi) > 0$ be some positive constant and perform a neck-stretching with parameter $\lambda$ to one of the slices $\{r = T \pm \varepsilon\}$, with the corresponding functions $\varphi_{\lambda}^\pm : X \to \mathbb{R}_{>0}$. Let $f_0^\pm$, $f_1^\pm$, and denote the primitives of the corresponding pull-backs of $\varphi_{\lambda}^\pm \eta$ to $L_0^{\text{Spl}}$, and $L_1$, respectively, and $f_0^{\text{Spl}}$ denote the pull-back of $\varphi \eta$ to $L_0^{\text{Spl}}$. The induced actions will be denoted by $a_{\varphi_{\lambda}^+}(p) = f_0^+(p) - f_1^+(p)$.

Claim 1: For sufficiently large $\lambda$, any intersection point $p \in L_0^{\text{Spl}} \cap L_1 \cap \{r \geq T + \varepsilon\}$ satisfies $a_{\varphi_{\lambda}^+}(p) < Z_-$.

Proof of Claim 1: Lemma 6.2 implies that
\[
a_{\varphi_{\lambda}^+}(p) = e^\lambda((f_0^{\text{Spl}}(p) - f_1^+(p)) - (f_0^{\text{Spl}}(T + \varepsilon) - f_1^+(T + \varepsilon)))
+ \left((f_0^{\text{Spl}}(T + \varepsilon) - f_1^+(T + \varepsilon)) - \frac{(f_0^{\text{Spl}}(T + 3\varepsilon) - f_1^+(T + 3\varepsilon)) + (C_0 - C_1)}{(C_0 - C_1) + (C_0 - C_1)}\right).
\]
for some constants $\nu_1, \nu_2$ independent of $\lambda$. It suffices to verify $\nu_1 < 0$, that is, $f_0^{\text{Spl}}(p) - f_1^+(p) < (f_0^{\text{Spl}}(T + \varepsilon) - f_1^+(T + \varepsilon))$. Consider first when $p$ is an intersection point in $\{T + \varepsilon \leq r \leq T + 3\varepsilon\}$. In this case, Figure 1 illustrates $f_0^{\text{Spl}}(p) < f_0^{\text{Spl}}(T + \varepsilon)$, while $f_1^+(p) = f_1^+(T + \varepsilon)$. All other intersections points in $\{r \geq T + 3\varepsilon\}$ exist as intersections points of $L_0 \cap L_1$. By hypothesis and the splash construction, we have
\[
f_0^{\text{Spl}}(p) - f_1^+(p) \approx f_0^+(p) - f_1^+(p) < Z_+ + (C_0 - C_1)
\approx f_0^{\text{Spl}}(T + \varepsilon) - f_1^+(T + \varepsilon) - (f_0^{\text{Spl}}(T + 3\varepsilon) - f_1^+(T + 3\varepsilon)) + (C_0 - C_1)
\approx f_0^{\text{Spl}}(T + \varepsilon) - f_1^+(T + \varepsilon) - (C_0 - C_1) + (C_0 - C_1).
\]
The requirement that $\lambda$ satisfies $e^\lambda \nu_1 + \nu_2 < Z_-$, means $\lambda$ can be chosen independent of $\varepsilon$, given that the latter constant $\varepsilon$ is sufficiently small. Also, all approximations are controlled by $e^\lambda$ and the size of the support of the splash Hamiltonian, which is order $\varepsilon^2$; thus, the approximations are absorbed by the $\varepsilon$-error term in the proposition hypothesis.

Claim 2: For sufficiently large $\lambda$, any intersection point $p \in L_0^{\text{Spl}} \cap L_1 \cap \{r \geq T - \varepsilon\}$ satisfies $a_{\varphi_{\lambda}^-}(p) > Z_+$.

Proof of Claim 2: Lemma 6.2 implies that
\[
a_{\varphi_{\lambda}^-}(p) = e^\lambda((f_0^{\text{Spl}}(p) - f_1^-(p)) - (f_0^{\text{Spl}}(T - \varepsilon) + f_1^-(T - \varepsilon)))
+ \left((f_0^{\text{Spl}}(T - \varepsilon) - f_1^-(T - \varepsilon)) - \frac{(f_0^{\text{Spl}}(T + 3\varepsilon) - f_1^-(T + 3\varepsilon)) + (C_0 - C_1)}{(C_0 - C_1) + (C_0 - C_1)}\right).
\]
It suffices to verify $\nu_1 > 0$, that is, $f_0^{\text{Spl}}(p) - f_1^-(p) > f_0^{\text{Spl}}(T - \varepsilon) - f_1^-(T - \varepsilon)$. Consider first when $p$ is an intersection point in $\{T - \varepsilon \leq r \leq T + 3\varepsilon\}$. In this case, Figure 1 illustrates $f_0^{\text{Spl}}(p) > f_0^{\text{Spl}}(T - \varepsilon)$, while $f_1^-(p) = f_1^-(T + \varepsilon)$. All other intersections points in $\{r \geq T + 3\varepsilon\}$ exist as intersections points of $L_0 \cap L_1$. By hypothesis and the splash construction
\[
f_0^{\text{Spl}}(p) - f_1^-(p) \approx f_0^-(p) - f_1^-(p) > Z_+ - (C_0 - C_1)
\approx f_0^{\text{Spl}}(T - \varepsilon) - f_1^-(T - \varepsilon) + (f_0^{\text{Spl}}(T + 3\varepsilon) - f_1^-(T + 3\varepsilon)) + (C_0 - C_1)
\approx f_0^{\text{Spl}}(T - \varepsilon) - f_1^-(T - \varepsilon) + (C_0 - C_1) - (C_0 - C_1).
\]
A pseudoholomorphic strip must increase action (from input to output). Therefore, by Claims 1 and 2, there is no pseudoholomorphic strip with Lagrangian boundary conditions $L_0^{\text{Spl}}$ and $L_1$ with its input (resp. output) in $\{r < T - \varepsilon\}$ and its output (resp. input) in $\{r > T + \varepsilon\}$.
In this section we assume that we are given a complete Lagrangian concordance \( L \subset (X, d\eta) \) inside a complete symplectic cobordism from \((Y_-, \alpha_-) \) to \((Y_+, \alpha_+) \). The \((\pm)\)-ends of \(L\) are denoted by \(\Lambda_\pm \subset (Y_\pm, \alpha_{\pm})\), where \(\Lambda_+ = \Lambda\). The assumption that \(L\) is a concordance means that \(L\) is diffeomorphic to \(\mathbb{R} \times \Lambda\) and that, hence, \(\Lambda_-\) is diffeomorphic to \(\Lambda\).

In this sections we fix two numbers 
\[
0 < \epsilon < \sigma,
\]
where the former is supposed to be sufficiently small and the latter can be arbitrary.

Let \(\Lambda_{\pm}^\epsilon := \phi_{\alpha_{\pm}, t}^{-\epsilon}(\Lambda_{\pm}) \subset (Y_{\pm}, \alpha_{\pm})\) denote the image of the ends of \(L\) under the time-\((-\epsilon)\) Reeb flow. The goal here is to construct a smooth family of Lagrangian cobordisms \(L_{\epsilon, N, s}\) from \(\Lambda_+^\epsilon\) to \(\Lambda_-^\epsilon\) depending smoothly on the parameters \(\epsilon, N, \) and \(s \in [\epsilon, \sigma]\). Each of these Lagrangian cobordisms will be given as the image
\[
L_{\epsilon, N, s} = \phi_{H_{\epsilon, N, s}}^1(L)
\]
of \(L\) under a Hamiltonian diffeomorphism, where \(H_{\epsilon, N, s}\) is a smooth family of autonomous Hamiltonians.

Before we begin, we point out three crucial properties that our construction will be made to satisfy:

- The above Hamiltonian flow is of the form \(\frac{d}{dt}\phi_{H_{\epsilon, N, s}}^t \equiv -\epsilon R_{\alpha_{\pm}}\) outside of the compact subset \(\{|r| \geq N + \epsilon\}\) of the respective cylindrical ends of \((X, d\eta)\). In particular, the cobordisms \(L_{\epsilon, N, s}\) are all cylinders over \(\Lambda_{\pm}^\epsilon\) inside the same subset.

- The above Hamiltonian vector field is of the form \(\beta_{\epsilon, N, s}(r) R_{\alpha_{\pm}}\) inside the complement
\[
X \setminus (\{|r - N| \leq 2\epsilon\} \cup X),
\]
where moreover \(\beta_{\epsilon, N, s}(r) = \beta_{\epsilon, N, \epsilon}(r)\) holds inside the complement
\[
X \setminus (\{|r - N| \leq 2\epsilon\} \cup \{-N + \epsilon \leq r \leq -\epsilon\} \cup X).
\]

- For \(s = \epsilon\) there exists a Weinstein neighborhood of \(L \subset (X, d\eta)\), i.e. a neighborhood symplectically identified with a neighborhood of the zero-section of \((T^*L, d\lambda_L)\), in which all \(\phi_{H_{\epsilon, N, \epsilon}}^t(L)\), \(t \in [0, 1]\), are graphs of the exact one-forms \(t d\eta_{\epsilon, N}\) for a smooth family of functions
\[
g_{\epsilon, N} : L \to \mathbb{R}.
\]

This family of functions are moreover required to be a Morse perturbation of a Morse–Bott function \(\tilde{g}_{\epsilon, N} : L \to \mathbb{R}\) that satisfies:

- The inequality \(d\tilde{g}_{\epsilon, N}(\partial_r) > 0\) holds when restricted to the subsets \(\{|r| \geq N + \epsilon\}\) of the cylindrical ends; and

- The form \(d\tilde{g}_{\epsilon, N}\) is non-vanishing outside of the two critical submanifolds \(\{r = \pm N\} \cap L \subset L\), each of which is non-degenerate in the Bott sense and diffeomorphic to \(\Lambda\);

The function \(g_{\epsilon, N}\) will be taken to be \(C^1\)-close to \(\tilde{g}_{\epsilon, N}\), and to differ with the latter function only in a small neighborhood of the above critical submanifolds.

In order to construct the sought Hamiltonian isotopy, we choose the following strategy. First, we construct an appropriate Weinstein neighborhood of \(L\), second, we construct the above function \(g_{\epsilon, N}\), and, third, we construct the Hamiltonian vector field \(\beta_{\epsilon, N, s}(r) R_{\alpha_{\pm}}\) inside the subset \(\{-N + \epsilon \leq r \leq -\epsilon\}\).

### 7.1. A Weinstein neighborhood of the Lagrangian concordance.

We fix a parametrization \(\psi : \mathbb{R} \times \Lambda \to L\) which maps \((\infty, -1) \times \Lambda\) and \((1, +\infty) \times \Lambda\) into the concave and convex cylindrical ends of \(X\), respectively. More specifically, using \(q\) to denote the standard coordinate on \(\mathbb{R}\), outside of \((-1, 1) \times \Lambda\) we ask that \(\psi\) takes the form
\[
\psi(q, x) = \begin{cases} 
(q + 1, \psi_+(x)) \in (-\infty, 0] \times \Lambda_+ \subset (-\infty, 0] \times Y_+ , & q \geq 1, \\
(q - 1, \psi_-(x)) \in [0, +\infty) \times \Lambda_- \subset [0, +\infty) \times Y_- , & q \leq -1,
\end{cases}
\]
where \( \tilde{\psi}_- \) and \( \tilde{\psi}_+ = \mathrm{Id}_\Lambda \) are parametrizations of \( \Lambda_- \) and \( \Lambda_+ = \Lambda \), respectively.

The Legendrian normal neighborhood theorem (see e.g. [29]) can be used to extend the above Legendrian embeddings \( \tilde{\psi}_\pm \) to contact-form preserving identifications

\[
\tilde{\Psi}_\pm: (V_\pm, dz + \theta_\Lambda) \leftrightarrow (Y_\pm, \alpha_\pm),
\]

defined on neighborhoods

\[
V_\pm \subset (J^1\Lambda = T^*\Lambda \times \mathbb{R}, dz + d\theta_\Lambda)
\]

of the zero-section.

Subsequently, Weinstein’s Lagrangian neighborhood theorem (see e.g. [39]) can be used to extend the above Lagrangian embedding \( \psi \) to a (non-exact) symplectomorphism

\[
\Psi: (U, d\theta_{\mathbb{R} \times \Lambda}) \leftrightarrow (X, d\eta),
\]

defined on a neighborhood

\[
U \subset (T^*(\mathbb{R} \times \Lambda) = T^*\mathbb{R} \times T^*\Lambda, d\theta_{\mathbb{R} \times \Lambda} = d\theta_{\mathbb{R}} \oplus d\theta_\Lambda)
\]

of the zero section.

Again use \( q \) to denote the standard coordinate on the \( \mathbb{R} \)-factor, while \( q \) is used to denote local coordinates on \( \Lambda \). The coordinates \( p \) and \( p \) are then used to denote the corresponding (locally defined) canonical conjugate momenta.

**Lemma 7.1.** The above embedding can be made to coincide with the symplectomorphism

\[
\Psi((q, q), (p, p)) = \begin{cases} 
(q + 1, \bar{\Psi}_-(q, e^{-(q+1)p}, e^{-(q+1)p})) & \in (-\infty, 0] \times Y_- \quad q \leq -1, \\
(q - 1, \bar{\Psi}_+(q, e^{-(q-1)p}, e^{-(q-1)p})) & \in [0, +\infty) \times Y_+ \quad q \geq 1,
\end{cases}
\]

defined on the cylindrical ends of \((X, d\eta)\).

**Proof.** In the proof of Weinstein’s Lagrangian neighborhood theorem, we start with an extension of \( \psi \) to the cotangent bundle by a diffeomorphism (i.e. smooth but not necessarily symplectic), which however is symplectic along the zero-section. After an application of Moser’s trick, this diffeomorphism can then be deformed in order to produce the required symplectomorphism, while fixing it along the zero-section. However, if the original diffeomorphism already is a symplectomorphism above some subset of the base, it is readily seen that the application of Moser’s trick will not deform the map above this subset. \( \square \)

Observe that there is a possibly smaller neighborhood \( U' \subset U \) of the zero-section \( 0_{\mathbb{R} \times \Lambda} \subset T^*(\mathbb{R} \times \Lambda) \) which is preserved by the \( \mathbb{R} \)-action

\[
((q, q), (p, p)) \mapsto ((q + t, q), (e^tp, e^tp)), \quad t \in \mathbb{R},
\]
on \( T^*(\mathbb{R} \times \Lambda) \). This flow rescales the symplectic form by \( e^t \), and corresponds to the Liouville vector field \( \partial_t \) on the convex and concave ends of \((X, d\eta)\)

**7.2. Explicitly defined push-offs along the ends.** First, we consider the function

\[
\beta_{\epsilon,N}^+: \mathbb{R} \to [-\epsilon, \epsilon]
\]

shown in Figure 2 which satisfies

- \(
\frac{d}{dr}\beta_{\epsilon,N}^+(r) \leq 0;
\)
- \( \beta_{\epsilon,N}^+(r) = \epsilon \) for \( r \leq N - \epsilon; \)
- \( \beta_{\epsilon,N}^+(r) = -\epsilon \) for \( r \geq N + \epsilon; \)
• $\beta^+_{\epsilon,N}(N) = 0$ and $\frac{d}{dr} \beta^+_{\epsilon,N}(N) < 0$.

We moreover require this function to satisfy

$$\beta^+_{\epsilon,N}(r) := \epsilon \beta^+((r - N)/\epsilon)$$

for a suitable fixed smooth function

$$\beta^+ : \mathbb{R} \to [-1, 1]$$

satisfying $\beta^+(-r) = -\beta^+(r)$.

We also consider the function $\beta^-_{\epsilon,N,\sigma} : \mathbb{R} \to [-\epsilon, \sigma]$ which coincides with $\beta^-_{\epsilon,N}$ outside of some compact subset of $(-N + \epsilon, 0) \subset \mathbb{R}$, while it satisfies the following properties inside the latter subset

• $\frac{d}{dr} \beta^-_{\epsilon,N,\sigma}(r) \geq 0$ for $-N + \epsilon \leq r \leq -N + 2\epsilon$;

• $\frac{d}{dr} \beta^-_{\epsilon,N,\sigma}(r) \leq 0$ for $-\epsilon \leq r \leq 0$;

• $\beta^-_{\epsilon,N,\sigma}(r) = \sigma$ for $-N + 2\epsilon \leq r \leq -\epsilon$;

For any number $s$ we now define the function

$$\beta^-_{\epsilon,N,s} := \beta^-_{\epsilon,N} + \frac{s - \epsilon}{\sigma - \epsilon} (\beta^-_{\epsilon,N,\sigma} - \beta^-_{\epsilon,N}).$$

In the case $s = \sigma$ the function is shown in Figure 3. In the following, we will only be considering this parameter with values $s \in [\epsilon, \sigma]$.

Finally, note that the vector fields $\beta^+_{\epsilon,N}(r)R_{\alpha_+}$ and $\beta^-_{\epsilon,N,s}(r)R_{\alpha_-}$ constructed above on the convex and concave cylindrical ends of $(X, d\eta)$, respectively, are both Hamiltonian vector fields.
AN ENERGY-CAPACITY INEQUALITY FOR LEGENDRIAN SUBMANIFOLDS

β vector field

In this section we prove our main theorem. This is done by studying the Lagrangian Floer homology of Lagrangians that we start with consists of the concordance of a pair of cobordisms (Section 3) together with the technique of splashing (Section 6). The pair of Lagrangians that we start with consists of the concordance $L$ from $\Lambda_-$ to $\Lambda$, together with its

7.3. A Bott push-off. We now use the coordinates provided by the Weinstein neighborhood defined in Section 7.1 identifying a neighborhood of $L$ with $U \subset (T^*(\mathbb{R} \times \Lambda), d\lambda_{\mathbb{R} \times \Lambda})$. In these coordinates we can find a smooth family of functions

$$g_{\epsilon,N}: (\mathbb{R} \setminus (-1, 1)) \times \Lambda \to \mathbb{R}$$

for which the sections $t d\tilde{g}_{\epsilon,N}$, $t \in [0, 1]$, in the above Weinstein neighborhood are identified with the corresponding time-$t$ flow of $L \setminus X \subset X$ under the vector fields defined in Section 7.2 with $s = \epsilon$. In these coordinates, we may moreover assume that $\tilde{g}_{\epsilon,N}|_{[-1-\delta,-1]} = \epsilon e^{q+1}$, while $\tilde{g}_{\epsilon,N}|_{[1,1+\delta]} = \epsilon e^{q-1} + \epsilon$ for some small $\delta > 0$.

Finally, the above function $\tilde{g}_{\epsilon,N}$ defined on $(\mathbb{R} \setminus \{(-1,1)\}) \times \Lambda$ extends smoothly to a function defined on all of $\mathbb{R} \times \Lambda$, without introducing any critical points inside $[-1,1] \times \Lambda$. After making the latter extension sufficiently $C^1$-small, this finishes the construction of the sought functions $g_{\epsilon,N}: \mathbb{R} \times \Lambda \to \mathbb{R}$. Note that the critical points of these functions consist of the two critical manifolds $\{q = \pm(N+1)\}$, each of which is non-degenerate in the Bott sense and diffeomorphic to $\Lambda$.

7.4. A Morse–Bott perturbation. The push-offs $t d\tilde{g}_{\epsilon,N}$ defined on the cylindrical ends of $(X, d\eta)$ intersect $L$ precisely in the manifolds $\{\pm N\} \times \Lambda_{\pm}$ contained inside the respective cylindrical end. Here we provide a generic perturbation of the above function, making these intersections transverse.

Again consider the coordinates defined in Section 7.1. Fixing a generic Morse function $h: \Lambda \to [0, 1]$. We use $h$ to create a $C^1$-small perturbation of $\tilde{g}_{\epsilon,N}$ which near the critical manifold $\{q = \pm(N+1)\}$ takes the form

$$\tilde{g}_{\epsilon,N} + \epsilon^{10} \rho(q \mp (N+1)) h: \mathbb{R} \times \Lambda \to \mathbb{R}.$$}

Here $\rho: \mathbb{R} \to [0, 1]$ is a fixed smooth bump-function that is equal to 1 in some neighborhood of 0, while its support is contained inside $(-\epsilon^3, \epsilon^3) \subset \mathbb{R}$. It follows that these perturbations have support contained in the subsets $\{q \mp (N+1)\} \leq \epsilon^3\}$. Finally, the produced perturbation of $\tilde{g}_{\epsilon,N}$ is our sought Morse function $g_{\epsilon,N}: \mathbb{R} \times \Lambda \to \mathbb{R}$.

7.5. The “bulge” inside the concave end. Note that the construction of $L_{\epsilon,N,s}$ now is complete in the case when $s = \epsilon$. In order to finish the construction, what remains is thus the cases $s \in [\epsilon, \sigma]$. Recall that these cobordisms all agree outside of the subset $-N + \epsilon \leq r \leq -\epsilon$ by construction. Inside the latter subset, we prescribe $L_{\epsilon,N,s}$ to be equal to the time-1 map of $L$ under the flow of the Hamiltonian vector field $\beta_{\epsilon,N,s} R_{\alpha_-}$ as constructed in Section 7.2 Note that the whole obtained cobordism $L_{\epsilon,N,s}$ is Hamiltonian isotopic to $L$. This finishes the construction.

8. Proof of Theorem 1.2

In this section we prove our main theorem. This is done by studying the Lagrangian Floer homology of a pair of cobordisms (Section 3) together with the technique of splashing (Section 6). The pair of Lagrangians that we start with consists of the concordance $L$ from $\Lambda_-$ to $\Lambda$, together with its

![Figure 4. On the concave end, the exact Lagrangian cobordism $L_{\epsilon,N,s}$ is obtained from $L$ by applying the time-1 flow of the Hamiltonian vector field $\beta_{\epsilon,N,s}(r) R_{\alpha_-}$.](image-url)
Hamiltonian deformation $L_{\epsilon,N,s}$ as constructed in Section 7. The latter is a Lagrangian concordance from $\Lambda^{\epsilon}$ to $\Lambda$, obtained by applying the time-$(-\epsilon)$ Reeb flow to $\Lambda_{-}$ and $\Lambda$, respectively.

8.1. **Passing to an indefinite contact Hamiltonian.** First we must replace the given contact Hamiltonian $H_{t}$ with one which is indefinite, in the sense that $H_{t}$ attains the value zero for all $t \in [0,1]$.

For a given contact isotopy $\phi^{t}_{\alpha,H_{t}}: (Y, \xi) \rightarrow (Y, \xi)$, taking a suitable family $c_{t}$ of constant functions as described Lemma 2.3 this indefiniteness is satisfied for the contact Hamiltonian that generates the composition

$$\phi^{t}_{R_{\alpha}} \circ \phi^{t}_{\alpha,H_{t}} = \phi^{t}_{\alpha,-c_{t}+H_{t} \circ \phi^{t}_{R_{\alpha}}}$$

for $C_{t} := \int_{0}^{t} c_{t} ds$.

In order to see that we may replace $H_{t}$ by the latter indefinite contact Hamiltonian $-c_{t} + H_{t} \circ \phi^{C_{t}}$ when proving Theorem 1.2 we note the following elementary result:

**Lemma 8.1.** (1) Since the Reeb flow $\phi^{t}_{R_{\alpha}}$ preserves the contact form $\alpha$, the numbers $A$ and $B$ associated to the contact isotopies

$$\phi^{-C_{t}}_{R_{\alpha}} \circ \phi^{t}_{\alpha,H_{t}} \text{ and } \phi^{t}_{\alpha,H_{t}},$$

as well as their oscillatory norms coincide; and

(2) For $a \leq b$, there is a bijection between the subsets

$$Q_{\alpha}(\Lambda, \phi^{-C_{1}}_{R_{\alpha}} \circ \phi^{1}_{\alpha,H_{1}}(\Lambda); a - C_{1}, b - C_{1}) \text{ and } Q_{\alpha}(\Lambda, \phi^{1}_{\alpha,H_{1}}(\Lambda); a, b),$$

of Reeb chords.

From now on, we will therefore assume $H_{t}$ to be indefinite in the above sense.

8.2. **Fixing constants.** Recall the strict Inequality (1.3) in the assumptions of the theorem:

$$0 < e^{A}\|H\|_{osc} < \sigma(\Lambda_{-}, \alpha_{-}).$$

The parameters $s \in [\epsilon, \sigma]$ and $N > 0$ in the construction of the Hamiltonian push-offs $L_{\epsilon,N,s}$ will be chosen so that the following properties hold.

First we fix $\sigma \geq \epsilon$ and $\delta > 0$ for which

$$0 < e^{A+2\delta}\|H\|_{osc} < \sigma < \sigma(\Lambda_{-}, \alpha_{-})$$

is satisfied, and where $\delta > 0$ is sufficiently small.

Second, we shrink $\epsilon > 0$, so that it becomes shorter than the length of the smallest Reeb chord on $\Lambda$ (see Lemma 8.4 which requires a bijection between the set of Reeb chords from $\Lambda_{-}$ to itself with the set of Reeb chords from $\Lambda_{-}$ to $\Lambda^{\epsilon}$) and so that we have

$$\sigma < e^{-2\delta}(\sigma(\Lambda_{-}, \alpha_{-}) - \epsilon).$$

8.3. **The homotopy class of “contractible” chords.** We now need to pinpoint a distinguished connected component of the space $\Pi(X;L,L_{\epsilon,N,s})$ of paths in $X$ with starting point in $L$ and endpoint in $L_{\epsilon,N,s}$. Denote by $p_{0} \in \pi_{0}(\Pi(X;L,L_{\epsilon,N,s}))$ the component of paths from $L$ to $L_{\epsilon,N,s}$ containing the Hamiltonian chord

$$[0,\epsilon] \ni t \mapsto (-N - 2\epsilon, \phi^{-t}_{\epsilon}(x)) \in (-\infty,0] \times Y_{-} \subset X, \ x \in \Lambda_{-}.$$ 

The latter Hamiltonian chord lives in the convex end and has endpoints on the cylinder over $\Lambda_{-} \cup \Lambda^{\epsilon}_{-}$; in fact, it corresponds to a very short $\alpha_{-}$-Reeb chord from $\Lambda^{\epsilon}_{-}$ to $\Lambda_{-}$, but traversed in reverse time. The following relationship between Reeb chords on $\Lambda_{-}$ and Reeb chords from $\Lambda_{-}$ to $\Lambda^{\epsilon}_{-}$ holds.
Lemma 8.4. There is a bijective correspondence between the Reeb chords

$$[0, \ell] \ni t \mapsto \phi_{\alpha_{\ell-1}}(t), \ x \in \Lambda_-, \phi_{\alpha_{\ell-1}}(t) \in \Lambda_-,$$

from $\Lambda_-$ to itself of length $\ell > 0$, and the Reeb chords

$$[0, \ell - \varepsilon] \ni t \mapsto \phi_{\alpha_{\ell-1}}(t), \ x \in \Lambda_-, \phi_{\alpha_{\ell-1}}(t) \in \Lambda'_-,$$

from $\Lambda_-$ to $\Lambda'_-$ of length $\ell - \varepsilon$. Furthermore, the former chord is trivial inside $\pi_1(Y_-, \Lambda_-)$ if and only if the latter chord is contained in the class $p_0 \in \pi_0(\Pi(X; L, L_{\varepsilon,N,s}))$ under the canonical embedding $Y_-=\{-N-2\varepsilon\} \times Y_- \subset X$.

The proof is straightforward; see Figure 5 for an illustration.

Lemma 8.5. (1) Any intersection point $L \cap L_{\varepsilon,N,s}$ contained outside of the subset $\{|r \pm N| \geq \varepsilon\}$ is contained in the region $[-N + \varepsilon, -\varepsilon] \times Y$ of the “bulge”, and satisfies the following: There is a one-to-two correspondence between the Reeb chords from $\Lambda'_-$ to $\Lambda_-$ of length $\varepsilon < \ell < \sigma + \varepsilon$ and the latter intersection points. When considered as paths inside $\Pi(X; L_{\varepsilon,N,s}, L)$ (note the order!), these intersection points and corresponding chords moreover live in the same component.

(2) For numbers $s$ satisfying $s \in [\varepsilon, \sigma]$, there is no intersection point $L \cap L_{\varepsilon,N,s} \cap \{|r \pm N| \geq \varepsilon\}$ contained in the component $p_0 \in \pi_0(\Pi(X; L, L_{\varepsilon,N,s}))$.

The correspondence of the above lemma is schematically depicted in Figure 6.

Proof. (1): This follows from the construction of $L_{\varepsilon,N,s}$ in Section 7.2. Namely, the cobordism $L_{\varepsilon,N,s}$ is obtained from $L$ by creating a “bulge” in the region $[-N + \varepsilon, -\varepsilon] \times Y$, where this bulge is created by an application of the positive Reeb flow up to time at most equal to $s$. It can be checked that the constant path at such an intersection point can be homotoped to a Reeb chord inside $\{-N - \varepsilon\} \times Y_-$ from $\{-N - \varepsilon\} \times \Lambda'_-$ to $\{-N - \varepsilon\} \times \Lambda_-$ of length $\varepsilon$ that is traversed in reverse time.

(2): The statement follows from Part (1). Namely, since no Reeb chord on $\Lambda_-$ of length less than or equal to $\sigma$ is contractible by Inequality (8.3), it can be seen that reversed Reeb chords of length $\varepsilon < \ell < \sigma + \varepsilon$ starting from $\Lambda_-$ and ending on $\Lambda'_-$ are not contained inside the component $p_0 \in \pi_0(\Pi(X; L, L_{\varepsilon,N,s}))$.

(This statement is similar to the one in Lemma 8.4.)
2.6, where an action is associated to the primitives of the one-form \( \varphi \eta \) pulled back to the two Lagrangian cobordisms. Here \( \varphi : X \to \mathbb{R}_{>0} \) is a choice of an auxiliary piecewise smooth function. In order to get optimal results, we need to be careful when choosing this function. We will take \( \varphi \) to be constantly equal to 1 on \( X \), while it takes the values

\[
\varphi(r) = \begin{cases} 
  e^{-r-2\epsilon}, & r \leq -N - \epsilon, \\
  e^{N-\epsilon}, & -N - \epsilon \leq r \leq -N + \epsilon, \\
  e^{-r}, & -N + \epsilon \leq r \leq 0, \\
  1, & 0 \leq r \leq N + \epsilon, \\
  e^{-r+N+\epsilon}, & r \geq N + \epsilon,
\end{cases}
\]

(8.6)
on the cylindrical ends. In other words, with the notation from Definition 2.8, we have the associated constant \( r_\varphi = 2\epsilon \). It now follows from Lemma 8.4 that:

**Lemma 8.7.** We have an equality

\[
h = h(\varphi, p_0, \Lambda_-, \Lambda^\epsilon_-, \alpha_-) = e^{-2\epsilon}(\sigma(\Lambda_-, \alpha_-) - \epsilon),
\]

where the left-hand side was defined in Section 3.5.

Although the continuous function \( \varphi \) only is piecewise smooth, such a function is sufficient for our purposes: recall that pseudoholomorphic curves have positive \( d(\varphi \eta) \)-energy by Lemma 3.5. To that end, we must use an admissible almost complex structure in the sense of Section 3.1, which in particular is cylindrical in the complement of the subset \( \{ \varphi'(r) = 0 \} \) of the cylindrical ends.

We will use \( d(\varphi \eta) \)-energy (resp. \( d\eta \)-energy), that is the \( \alpha_\varphi \)-action (resp. \( \alpha \)-action) of Floer generators, in order to obstruct the existence of certain pseudoholomorphic curves when we prove Lemma 8.12 and Proposition 8.22.

Fix \( s \in [\epsilon, \sigma] \) and choose primitives \( f_0 \) and \( f_1 \) of the pull-backs \( \eta \) to \( L \) and \( L_{e,N,s} \), respectively, and \( f_0^\varphi \) and \( f_1^\varphi \) of the pull-backs of \( \varphi \eta \) to \( L \) and \( L_{e,N,s} \), respectively. All primitives here are required to vanish at \( r = -\infty \), and they are thus uniquely determined. In the following we will also assume that

\[
f_0 = f_0^\varphi \equiv 0,
\]

which, in view of Lemma 2.10, causes no restriction. Now we estimate the actions defined in Section 2.6 for the 0-Hamiltonian chords using the above primitives.

**Lemma 8.8** (The main action computation). Let \( O(\epsilon) > o(\epsilon) > 0 \) be fixed, but unspecified, positive functions of \( \epsilon, N \), defined for all \( N, \epsilon \geq 0, \) that tend to 0 as \( \epsilon \to 0 \) for any fixed choice of \( N > 0 \). Let \( p, q \in L \cap L_{e,N,s} \) be 0-Hamiltonian chords. Let \( s \geq \epsilon \). Under the assumption that \( N > 0 \) is sufficiently large and \( \epsilon > 0 \) is sufficiently small, we may assume the following:

![Figure 6. The correspondence between the Reeb chords provided by Lemma 8.5. For every Reeb chord \( c \) from \( \Lambda^\epsilon_- \) to \( \Lambda_- \) of a (suitably) bounded length, there are precisely two corresponding intersection points \( c_1 \) and \( c_2 \) in the bulge region; one is contained near \( r = -N + \epsilon \) while one is contained near \( r = 0 \).](image-url)
Here we use that both primitives $a$ and $\alpha$ for the computations to deduce the inequalities of the lemma. As we will see below, the lower bound for $\epsilon$ follows from $a(p) = 0$ when $p \in [-N - \epsilon, -N + 2\epsilon] \cup [-\epsilon, 0] \cup [N - \epsilon, N + \epsilon]$. Moreover, to within an error bounded by $\epsilon^2$, the bounds for $\epsilon$ follow from $a(p) = 0$ when $p \in [-N - \epsilon, -N + 2\epsilon] \cup [-\epsilon, 0] \cup [N - \epsilon, N + \epsilon]$. Third, on these intervals, all factors $I(r)$ in front of the $\beta'_{\epsilon,N,s}(r)$-term in the integrand are monotonic; thus,

$$-I(r_1)(\beta_{\epsilon,N,s}(r'_0) - \beta_{\epsilon,N,s}(r_0)) \leq -\int_{r_0}^{r'_0} I(r)\beta'_{\epsilon,N,s}(r)dr \leq -I(r_1)(\beta_{\epsilon,N,s}(r'_0) - \beta_{\epsilon,N,s}(r_0))$$

for $\{r_1, r_2\} \subseteq \{r_0, r'_0\}$. We apply these observations (along with the error bounds on the approximations) to several integral computations to deduce the inequalities of the lemma. As we will see below, the lower bound for $a_{\epsilon}(p_2)$ will motivate setting

$$O(\epsilon) = (-e^{-\epsilon} - e^0 + e^{N-3\epsilon})\epsilon - \epsilon^2$$

which we claim is positive because we can assume without loss of generality that $N - 3\epsilon \geq 1$ and $-2 + e^1 > \epsilon$. We omit the precise exponential-polynomial formulation of $O(\epsilon)$ in terms of $\epsilon, N$, since $O(\epsilon)$ only appears as an upper bound (in magnitude). The $(\epsilon^2)$-terms included below will cover the integral approximations made above.

The bounds for $a(p_1)$ follow from

$$-\epsilon^2 > -e^{-N-\epsilon} > -\int_{-N-\epsilon}^{-N} e^r \beta'_{\epsilon,N,s}dr > -e^{-N}\epsilon > -O(\epsilon) + \epsilon^2.$$

The bounds for $a_{\epsilon}(p_1)$ follow from

$$-\epsilon^2 > -e^{-2\epsilon} > -\int_{-N-\epsilon}^{-N} e^r\varphi\beta'_{\epsilon,N,s}dr > -e^{-\epsilon}\epsilon > -O(\epsilon) + \epsilon^2.$$
The bounds for $a(p_2)$ follow from
\[
s - \frac{1}{2} O(\epsilon) - \epsilon^2 > (s - \epsilon)(e^0 - e^{-N+\epsilon}) > -\left( \int_{-N+\epsilon}^{-N+2\epsilon} + \int_{-N}^{-N+\epsilon} \right) e^r \beta'_{\epsilon,N,s} dr > (s - \epsilon)(e^{-\epsilon} - e^{-N+2\epsilon}) > s - O(\epsilon)
\]
and with
\[
\frac{1}{2} O(\epsilon) > (-e^{-N-\epsilon} - e^{-N} + e^N)\epsilon
\]
\[
> -\left( \int_{-N-\epsilon}^{-N} + \int_{-N}^{-N+\epsilon} \right) e^r \beta'_{\epsilon,N,s} dr
\]
\[
> (-e^{-N} - e^{-N+\epsilon} + e^{N-\epsilon})\epsilon > \epsilon^2.
\]

The bounds for $a_{\phi}(p_2)$ follow from
\[
0 = -\left( \int_{-N+\epsilon}^{-N} + \int_{-\epsilon}^{0} \right) e^r \phi'_{\epsilon,N,s} dr
\]
and with
\[
O(\epsilon) - \epsilon^2 > (-e^{-2\epsilon} - e^{-\epsilon} + e^{N-2\epsilon})\epsilon
\]
\[
> -\left( \int_{-N-\epsilon}^{-N} + \int_{-N}^{-N+\epsilon} \right) e^r \phi'_{\epsilon,N,s} dr
\]
\[
> (-e^{-\epsilon} - e^0 + e^{N-3\epsilon})\epsilon = o(\epsilon) + \epsilon^2.
\]

First we note that, since $\beta'|_{[0,N-\epsilon]} = 0$, the claim made in Part (3) that the potentials $f_1$ and $f_1^\epsilon$ are constant in the concerned region holds.

To compute the bounds, we use the same integration estimates as for $a(p_2)$ and $a_{\phi}(p_2)$ above, except that we omit the fifth (last) integral as it occurs when $r > N - \epsilon$. We see that for $0 \leq r \leq N - \epsilon$,
\[
s + O(\epsilon) > (s - \epsilon)(e^0 - e^{-N+\epsilon}) + (e^{-N-\epsilon} - e^{-N})\epsilon + \epsilon^2
\]
\[
> - (f_1 - f_0)(r)
\]
\[
> (s - \epsilon)(e^{-\epsilon} - e^{-N+2\epsilon}) + (e^{-N} - e^{-N+\epsilon})\epsilon - \epsilon^2 > s - O(\epsilon),
\]
as well as
\[
0 > (-e^{-2\epsilon} - e^{-\epsilon})\epsilon + \epsilon^2 > -(f_1^\epsilon - f_0^\epsilon)(r)
\]
\[
> (-e^{-\epsilon} - e^0)\epsilon - \epsilon^2 > -\frac{1}{2} O(\epsilon).
\]

The last inequality holds for sufficiently large $N$ in (8.9) and all arbitrarily small $\epsilon > 0$. Plugging in $f_0^\epsilon = 0 = f_0$ gives the answer. \(\square\)

After possibly shrinking $\epsilon > 0$, we can assume that $O(\epsilon)$ in Lemma 8.8 satisfies
\[
2O(\epsilon) < h,
\]
Further, by using Part (3) of Lemma 8.8 together with Inequality (8.2), and after possibly shrinking $\delta > 0$, we may moreover assume that
\[
e^{A+2\delta}\|H\|_{osc} < \sigma - O(\epsilon) < (f_0 - f_1)|_{[0,N-\epsilon]\times \gamma} < \sigma + O(\epsilon)
\]
is satisfied when $\epsilon > 0$ is sufficiently small and $s = \sigma$ (that is, when $f_1$ is the primitive of the pull back to $L_{\epsilon,N,s} = L_{\epsilon,N,\sigma}$).
8.5. The Floer homology of the push-off. By Lemma 8.5, we conclude that
\[ CF^p_\epsilon(L, L_{\epsilon,N,s}; 0, J_t) = C^-_\epsilon \oplus C^+_\epsilon \]
holds for all \( \epsilon \leq s \leq \sigma \), where \( C^-_\epsilon \), resp. \( C^+_\epsilon \), denotes the vector subspace generated by the intersection points at the convex, resp. concave, end. Recall the superscript introduced in Section 3.3 which indicates that the generating intersection points lie in the same contractible homotopy class. Also, recall the choice of Morse function \( h : \Lambda \to [0, 1] \) made in Section 7.4. By the construction of \( L_{\epsilon,N,\sigma} \) in the same section, there is an isomorphism \( C^-_\epsilon = C^+_\epsilon - 1 = C^\text{Morse}_\epsilon(h) \) of graded vector spaces induced by a canonical identification of the generators.

Lemma 8.12 (Identifying Floer and Morse homology). Using the above canonical identification of generators, the Floer complex
\[ CF^p_\epsilon(L, L_{\epsilon,N,\sigma}; 0, J_t) = C^-_\epsilon \oplus C^+_\epsilon, \quad d = \begin{pmatrix} d_- & 0 \\ \psi & d_+ \end{pmatrix} \]
is well-defined, has homotopy class independent of the choice of admissible almost complex structure, and is given as an acyclic mapping cone of \( \psi \) (the acyclicity is equivalent to \( \psi \) being a quasi-isomorphism). Moreover, there are homotopy equivalences
\[ (C^-_\epsilon, d_-) \sim (C^\text{Morse}_\epsilon(h), \partial_h) \] and \( (C^+_\epsilon - 1, d_+) \sim (C^\text{Morse}_\epsilon(h), \partial_h) \)
of the respective quotient and subcomplexes and the Morse homology complex of \( \Lambda \).

Proof. Inequality (8.10) and Parts (1) and (2) of Lemma 8.8 imply that, for sufficiently small \( \epsilon > 0 \), and for any generators \( p, q \in CF^p_\epsilon(L, L_{\epsilon,N,s}; 0, J_t) \), \( s \in [\epsilon, \sigma] \), the inequality
\[ |a_\varphi(p) - a_\varphi(q)| < h \]
is satisfied. Here we also rely on Lemma 8.5 in order to infer that
\[ CF^p_\epsilon(L, L_{\epsilon,N,s}; 0, J_t) = C^-_\epsilon \oplus C^+_\epsilon \]
holds on the level of vector spaces. Consequently, Part (1) of Proposition 3.19 applies, showing that
\[ CF^p_\epsilon(L, L_{\epsilon,N,s}; 0, J_t), \ s \in [\epsilon, \sigma], \]
are well-defined Floer complexes for generic admissible almost complex structures.

The cone-structure of the complexes now follows from Parts (1) and (2) of Lemma 8.8 combined with Lemma 8.5. Specifically, for any two generators \( p \pm \in C^\pm_\epsilon \), the inequality
\[ a_\varphi(p_+) - a_\varphi(p_-) < 0 \]
holds.

Recall from Section 7.1 for \( s \) sufficiently close to \( \epsilon \), \( L_{\epsilon,N,s} \) is a graphical Lagrangian which sits in a Weinstein tubular neighborhood of \( L \) symplectomorphic to \( T^*L \). So for sufficiently small \( \epsilon \) and \( s \), we can use Floer’s original work which proves that the pseudoholomorphic strips (for a suitable time-dependent \( J_t \)) with boundary on \( L \) and \( L_{\epsilon,N,s} \) converge to gradient flow lines for a suitable metric on \( L \).

To understand the underlying Morse complexes, recall the construction of \( \tilde{g} \) and its close perturbation \( g \).

From standard Morse–Bott analysis (see [19], for example), the negative gradient flows of \( g \) leave the critical level sets \( \{q = \pm (N + 1)\} = \{r = \pm N\} \cap L \) invariant. Moreover, no such negative gradient flows run from \( \{q = -(N + 1)\} \) to \( \{q = +(N + 1)\} \). Thus the result holds for sufficiently small \( \epsilon \) and \( s \) (and suitable \( J \)).

Next consider an arbitrary admissible almost complex structure (see Section 3.1) and an arbitrary \( L_{\epsilon,N,s} \), \( s \in [\epsilon, \sigma] \) as in the statement of the lemma. Floer’s computation is valid after a suitable compactly supported deformation of this almost complex structure and for \( s = \epsilon \). Using the invariance properties of the Floer complex we will deduce the statement for an arbitrary \( s \in [\epsilon, \sigma] \) as well as a general admissible almost complex structure.

The family \( L_{\epsilon,N,s} \), \( s \in [\epsilon, \sigma] \), of Lagrangian cobordisms is generated by a Hamiltonian isotopy by their construction in Section 7. Note that, according to Lemma 8.5, no births/deaths of intersection points \( L \cap L_{\epsilon,N,s} \) corresponding to chords in the homotopy class \( \mathfrak{p} = 0 \) occur during this isotopy. Thus, we
can continue to identify the vector spaces of 0-Hamiltonian chords as $C^+_0$. Part (2) of Proposition 8.19, proven using bifurcation analysis, now shows that the homotopy class of the complexes is invariant. (Here we make use of the fact that the Hamiltonian is compactly supported, and that the Lagrangians are cylindrical outside of a compact subset, in order to obtain a well-defined identification of the “homotopy class” of an intersection point.)

Finally, by Stokes’ theorem together with (1) and (2) of Lemma 8.8, no handle-slide strip in the bifurcation analysis can originate from a generator of $C^+_0$ and terminate at a generator of $C^-_0$. The aforementioned chain homotopy equivalence thus descends to a homotopy equivalence on the sub and quotient complexes, as claimed.

\[ \square \]

8.6. **Action properties when applying Usher’s trick.** Given the contact Hamiltonian $H_t: Y \to \mathbb{R}$ and the constant $A \geq 0$ in the assumption (see Equation (1.1)), Proposition 5.1 produces an associated Hamiltonian $G_t: X \to \mathbb{R}$ for which $\phi^1_{G_t}(L)$ and $\phi^1_{e^r H_t}(L)$ coincide in the subset $[A, A + 2\delta] \times Y$. In particular, both of these images are cylinders over $\phi^1_{\alpha, H_t}(A)$ in that subset.

Let $M_0$ be as defined as in (1.1). Recall the choice of a small number $\delta > 0$ made above (see Inequalities (8.2) and (8.11)) and denote

\[ (8.13) \quad \bar{M}_- := e^{2\delta} M_- < 0 < e^{2\delta} M_+ := \bar{M}_+ \]

The Hamiltonian $G_t: X \to \mathbb{R}$ may be assumed to satisfy

\[ (8.14) \quad \bar{M}_- < -\int_0^1 \max_{x \in X} G_t \, dt \leq -\int_0^1 \min_{x \in X} G_t \, dt < \bar{M}_+ \]

according to Proposition 5.1.

Here we investigate properties of the action of the generators of the Floer complex when the latter Hamiltonian is turned on.

**Lemma 8.15.** The primitives $\mathcal{T}_0, \mathcal{T}^\rho_0: \phi^1_{G_t}(L) \to \mathbb{R}$ of the pull backs of $\eta$ and $\varphi \eta$, respectively, to $\phi^1_{G_t}(L)$, both vanish in the complement of the subset

\[ [0, A] \times Y \cup [A + 2\delta, N - 2\epsilon] \times Y, \]

given that the primitives are chosen to vanish at $r = -\infty$.

**Proof.** First we recall that the corresponding primitives $f_0, f^\rho_0: L \to \mathbb{R}$ both vanish identically.

We compute the change in these primitives under the Hamiltonian isotopy $\phi^1_{G_t}$ in the following manner. By using Cartan’s formula

\[
\frac{d}{dt}(\phi^1_{G_t})^*(e^r \alpha) = d(\iota_X t e^r \alpha) + \iota_X d(e^r \alpha) = d(\iota_X t e^r \alpha) - dG_t,
\]

the new primitives can be obtained by integrating the variable $t$ of the function $\iota_X t e^r \alpha - G_t$.

Outside of $[0, N - \epsilon] \times Y$ we have $G_t \equiv 0$ by construction, and the statement is an easy consequence.

For the image

\[ \phi^1_{G_t}(L) \cap [A, A + 2\delta] \times Y \]

we argue as follows. Any $p \in L$ for which $\phi^1_{G_t}(p) \in [A, A + 2\delta] \times Y$ satisfies the property that

\[ G_t \circ \phi^1_{G_t}(p) = e^r K_t \circ \phi^1_{e^r K_t}(p), \ t \in [0, 1], \]

by the construction in Section 5 (in particular, see Part (2) of Proposition 5.1). At such a point $p \in L$, the above application of Cartan’s formula thus tells us that the infinitesimal change of the primitive is given by

\[ \iota_X t e^r \alpha - e^r K_t = 0 \]

since $\iota_X \alpha = K_t$ (recall that $K_t: Y \to \mathbb{R}$ is a contact Hamiltonian). \[ \square \]
8.7. **Turning on the splashing.** In this section we introduce the splashing construction from Section 6.2. Recall that $\phi_{G_t}(L) \cup L_{\epsilon,N,\sigma}$ both are cylindrical and disjoint inside $[A, A + 2\delta] \times Y$, each intersecting the hypersurface $\{A\} \times Y$ of contact type in a Legendrian submanifold. The splashing construction will be performed inside the cylindrical piece

$$( [A, A + 2\delta] \times Y, d(e^t \alpha) ) \subset (X, d\eta)$$

of the symplectization using the data

$$\begin{cases}
T = A + \delta, \\
3\epsilon = \delta, \\
L_0 = \phi_{G_t}^{1}(L), \\
L_1 = L_{\epsilon,N,\sigma}, \\
Z_+ > \overline{M}_+ > 0, \\
Z_- < \overline{M}_- < 0.
\end{cases}$$

Recall the Hamiltonian $h^{\text{Spl}} := h_{\epsilon,\overline{Z}_{-},Z_{+}} \colon \overline{X} \to \mathbb{R}$ obtained in Section 6.2 which realizes the splashing. We can assume this Hamiltonian to be arbitrarily small in the uniform $C^0$-norm, and supported inside $[A, A + 2\delta] \times Y$.

**Lemma 8.16.** There exists a Hamiltonian $\tilde{G}_t \colon X \to \mathbb{R}$ with support contained in $[0, N - \epsilon] \times Y$ satisfying

$$L^{\text{Spl}} := \phi_{G_t}^{1}(L) = \phi_{h^{\text{Spl}}} \circ \phi_{G_t}^{1}(L),$$

$$\left| \int_0^1 \max \tilde{G}_t dt - \int_0^1 \max G_t dt \right| < \nu/2,$$

$$\left| \int_0^1 \min \tilde{G}_t dt - \int_0^1 \min G_t dt \right| < \nu/2,$$

for any choice of $\nu > 0$. In particular

$$\left| \| \tilde{G}_t \|_{\text{osc}} - \| G_t \|_{\text{osc}} \right| < \nu.$$

**Proof.** The Hamiltonian $\tilde{G}_t$ is constructed as follows. First, it causes no restriction to assume that $G_t$ vanishes equivalently near $t = 0, 1$; this can be done with an arbitrarily small effect on the Hofer norm. Second, we may replace $h^{\text{Spl}}$ by a non-autonomous Hamiltonian $h^{\text{Spl}}_{t}$ also vanishing identically near $t = 0, 1$; this can be done while still having an arbitrarily small Hofer norm. Finally, we consider the concatenation

$$\tilde{G}_t = \begin{cases}
2G_{2t}, & 0 \leq t \leq 1/2, \\
2h^{\text{Spl}}, & 1/2 \leq t \leq 1,
\end{cases}$$

for which

$$L^{\text{Spl}} := \phi_{G_t}^{1}(L) = \phi_{h^{\text{Spl}}} \circ \phi_{G_t}^{1}(L)$$

is satisfied. \hfill $\square$

Although $CF^0_L(L, L_{\epsilon,N,\sigma}; \tilde{G}_t)$ may not be a chain complex due to “unwanted bubbling,” it still defines a vector space whose generators have an associated action. Fix the choice of action induced by $\varphi_{\eta}$, where $\varphi \colon X \to \mathbb{R}_{\geq 0}$ is as defined in Equality (8.6). With such an action one defines the Floer complex

$$C_* := CF^0_L(L, L_{\epsilon,N,\sigma}; \tilde{G}_t \overline{M}_+, \overline{M}_-),$$

with generators in the action range $[\overline{M}_-, \overline{M}_+]$, as described in Section 8.3. Note that this indeed is a well-defined complex by Proposition 8.11 which follows since (8.13), (1.1), (8.2), (8.3), and Lemma 8.7 (in this very order) combine to give

$$(8.17) \quad \overline{M}_+ - \overline{M}_+ = e^{2\delta}(M_+ - M_-) = e^{A+2\delta} ||H_t||_{\text{osc}} < \sigma < h.$$
Since the initial intersection points \( L_{\epsilon,N,\sigma} \cap L \) all lie outside of the support of \( G_t \) (see Proposition \[5.1\]), and hence also outside of the support of \( \tilde{G}_t \) (see Lemma \[8.15\]), these intersections all remain, forming a subset of the generators of \( C F_{\nu}^0(L, L_{\epsilon,N,\sigma}; G_t)_{\mathcal{M}_F} \). In other words, there are inclusions
\[
C^\pm_s \subset C_s
\]
on the level of vector spaces.

We define the “left” \( L_s \), “right” \( R_s \), and “intersection” \( I_s \) vector subspaces as follows.

- \( L_s \subset C_s \) is generated by those chords (in the appropriate action range) corresponding to the intersection points 
  \[
  L^\text{Spl} \cap L_{\epsilon,N,\sigma} \cap \{ r \leq T + \epsilon \},
\]
  and thus in particular \( C^-_s \subset L_s \);
- \( R_s \subset C_s \) is generated by those chords (again in the appropriate action range) corresponding to the intersection points 
  \[
  L^\text{Spl} \cap L_{\epsilon,N,\sigma} \cap \{ r \geq T - \epsilon \},
\]
  and thus in particular \( C^+_s \subset R_s \);
- \( I_s := L_s \cap R_s \subset C_s \).

Here we have made heavy use of the naturality, as described in Section \[3.6\].

\[\text{Lemma 8.18.}\]
\[L_s, R_s \text{ are subcomplexes of } C_s. \text{ For small enough constants } \epsilon, \delta, \nu > 0, \text{ there is a bijective correspondence between the generators of } I_s \text{ and the Reeb chords inside } \]
\[
Q_\alpha(\phi^1_{\alpha,Ht}(\Lambda), \Lambda; M_-, M_+) = Q_\alpha(\Lambda, \phi^1_{\alpha,Ht}(\Lambda); -M_-, -M_+).
\]

\[\text{Remark 8.19.}\]
The intersection complex \( I_s = L_s \cap R_s \) is related to the Rabinowitz–Floer homology complex as defined in \[16\] by Cieliebak–Frauenfelder in the non-relative case.

\[\text{Proof.}\]
Again we will implicitly make use of the naturality property from Section \[3.6\]. Two crucial identities that we need are that
\[
\phi^l_{G_t}(L) \cap [A, A + 2\delta] \times Y = [A, A + 2\delta] \times \phi^1_{\alpha,Ht}(\Lambda),
\]
\[
L_{\epsilon,N,\sigma} \cap [A, A + 2\delta] \times Y = [A, A + 2\delta] \times \Lambda^\epsilon,
\]
both are cylindrical inside \([A, A + 2\delta] \times Y\), the first one being a consequence of Proposition \[5.1\].

Consider the primitives \( \mathcal{T}^0_1 \) and \( \mathcal{T}^1_1 = f^1_{\varphi} \) of \( \varphi \eta \) pulled back to \( \phi^l_{G_t}(L) \) and \( L_{\epsilon,N,\sigma} \), respectively. We have \( \mathcal{T}^0_1 \equiv 0 \text{ inside } [A, A + 2\delta] \times Y \) by Lemma \[8.15\] while Part (3) of Lemma \[8.8\] implies that \( 0 < f^1_{\varphi} < \frac{\epsilon}{2} \).

Recall that we set \( C_1 := \mathcal{T}^1_{\varphi}(T) \) in Section \[6.1\] where \( T = A + \delta \) was specified at the beginning of this subsection. Since \( C_0 = 0 \) and \( 0 < C_1 < \frac{\epsilon}{2} \), the assumptions of Proposition \[6.5\] are satisfied for the constants \( Z_\pm \) chosen at the beginning of this subsection. The claim considering the subcomplex property then follows from Proposition \[6.5\].

The bijection between the concerned generators and the Reeb chords inside \( Q_\alpha(\phi^1_{\alpha,Ht}(\Lambda), \Lambda; M_+, M_-) \) finally follows from Lemma \[6.4\]. (Note that the splash is performed to \( \phi^1_{G_t}(L) \) inside \([A, A + \delta] \times Y\), i.e. to the cylinder over \( \phi^1_{\alpha,Ht}(\Lambda) \)). Here we also need to use the observation that there is an interval \((e^{-A} M_+, e^{-A} M_+ + \mu)\) (resp. \((e^{-A} M_- - \mu, e^{-A} M_-)\),\) for \( \mu > 0 \) sufficiently small (but possibly large compared to \( \delta \)), which contains no length of a Reeb chord from \( \Lambda \) to \( \phi^1_{\alpha,Ht}(\Lambda) \) (resp. from \( \phi^1_{\alpha,Ht}(\Lambda) \) to \( \Lambda \)).

Since Lemma \[8.18\] implies that \( I_s \) is a subcomplex of both \( L_s \) and \( R_s \), in combination with \( L_s + R_s = C_s \), we get the short exact sequence whose maps are induced by inclusions
\[
0 \to I_s \to L_s \oplus R_s \to C_s \to 0.
\]

Recall the definition of \( \Phi_{0,\tilde{G}_t} \) and \( \Phi_{\tilde{G}_t,0} \) induced by the Hamiltonian \( \tilde{G}_t \) in Section \[3.3\].
Lemma 8.21. The maps

\[ \Phi_{0,\tilde{G}_t}: CF_{*}^{p_0}(L, L_{\epsilon,N,\sigma};0) \to CF_{*}^{p_0}(L, L_{\epsilon,N,\sigma};\tilde{G}_t)_{M_{\pm}}, \]

\[ \Phi_{G_t,0}: CF_{*}^{p_0}(L, L_{\epsilon,N,\sigma};\tilde{G}_t)_{M_{\pm}} \to CF_{*}^{p_0}(L, L_{\epsilon,N,\sigma};0), \]

are well-defined chain maps, whose composition \( \Phi_{G_t,0} \circ \Phi_{0,\tilde{G}_t} \) is chain homotopic to the identity.

Proof. Let \( m_{\pm} := \pm O(\epsilon) \) and \( m_{\pm} := M_{\pm} \). With \( \text{(8.14)} \) and \( \text{Lemma 8.16} \) we now see that, for sufficiently small \( \nu > 0 \) (defined in \( \text{Lemma 8.16} \)), there exists a sufficiently small \( \epsilon > 0 \) such that the constants \( m_{-} \leq m_{-} \leq m_{+} \leq m_{+} \) satisfy the hypotheses of \( \text{Proposition 3.12} \). (Here the Hamiltonian \( G_t \) of that proposition is replaced with \( \tilde{G}_t \) of this section.) The claims are now direct consequences of this proposition. \( \square \)

Denote by \( C^{\geq 0}_* \) the vector subspace spanned by all generators of \( C_* \) contained outside of the concave end. I.e. we have a canonical decomposition

\[ C_* = C^-_* \oplus C^{\geq 0}_* \]
as vector spaces.

Proposition 8.22. For a suitable admissible almost complex structure,

\[ C^+_* \subset C^{\geq 0}_* \subset CF_{*}^{p_0}(L, L_{\epsilon,N,\sigma};\tilde{G}_t)_{M_{\pm}} \]
is a sequence of subcomplexes satisfying the following properties:

(1) \( \Phi_{0,\tilde{G}_t}(C^+_*) \subset C^+_* \) and \( \Phi_{G_t,0}(C^+_*) \subset C^+_* \);

(2) \( \Phi_{G_t,0}(C^+_*) \) is a left-sided homotopy inverse of \( \Phi_{0,\tilde{G}_t}|_{C^+_*} \) which induces an isomorphism on the homology level; and

(3) \( \Phi_{G_t,0}(C^{\geq 0}_*) \subset C^+_* \).

Proof. The claim that the vector subspace

\[ C^+_* \subset CF_{*}^{p_0}(L, L_{\epsilon,N,\sigma};\tilde{G}_t)_{M_{\pm}} \]
in fact is a subcomplex will be proven together with Part (1) below, while the claim that the vector subspace

\[ C^{\geq 0}_* \subset CF_{*}^{p_0}(L, L_{\epsilon,N,\sigma};\tilde{G}_t)_{M_{\pm}} \]
is a subcomplex will be proven together with Part (3) below.

In the following we will consider the primitives \( \tilde{f}_0^\epsilon \) and \( \tilde{f}_1^\epsilon = f_1^\epsilon \) of \( \varphi_\eta \) pulled back to \( \phi_{G_t}^\epsilon(L) \) and \( L_{\epsilon,N,\sigma} \), respectively. Note that \( \tilde{f}_0^\epsilon \) coincides with the primitive \( f_0^\epsilon \) of \( \varphi_\eta \) pulled back to \( \phi^\epsilon_{G_t}(L) \) outside of the “splashing region,” i.e. \( [A - \delta, A + \delta] \times Y \).

(1): Let \( p \) be a generator of \( C^+_* \). By Parts (2) and (3) of \( \text{Lemma 8.8} \) together with \( \text{Lemma 8.15} \), the constants \( C_1 := \tilde{f}_1^\epsilon(N - 2\epsilon) < \frac{1}{2} o(\epsilon) \) and \( C_0 := \tilde{f}_0^\epsilon(N - 2\epsilon) = 0 \) satisfy \( a_{\varphi}(p) - (C_1 - C_0) > (1 - \frac{1}{2}) o(\epsilon) \). Since both \( \tilde{f}_i^\epsilon, i = 0, 1 \), are constant inside \( [N - 2\epsilon, N - \epsilon] \), we can stretch the neck at \( \{ N - 2\epsilon \leq r \leq N - \epsilon \} \) to define \( \varphi_\lambda \) as in \( \text{Formula (6.1)} \).

Note that this stretching imposes a specific construction of the almost complex structure in \( \{ N - 2\epsilon \leq r \leq N - \epsilon \} \). However, the transversality arguments of \( \text{Proposition 3.8} \) still hold since all pseudoholomorphic strips that intersect \( \{ N - 2\epsilon \leq r \leq N - \epsilon \} \) also must limit to their Hamiltonian chords contained somewhere outside of this region. The latter subset is where the generic perturbation carried out by \( \text{Proposition 3.8} \) can be taken to occur.

After the neck stretching, \( \text{Lemma 6.2} \) shows that \( a_{\varphi_\lambda}(p) \) can be assumed to be arbitrarily large for the intersection points \( p \) that generate \( C^+_* \) as a vector subspace of either \( CF_{*}^{p_0}(L, L_{\epsilon,N,\sigma};0) \) or
$\text{CF}_s^{\rho_0}(L, L_{\epsilon,N,\sigma}; \tilde{G}_t)$. (Note we first fix $o(\epsilon) > 0$, and hence all the other parameters involved in the construction, such as $\delta$ etc. Then we increase $\lambda$ arbitrarily high.) Let $q$ be an arbitrary generator of the orthogonal complement of $C_s^+$. Note that $a_{\varphi^t}(q) = a_{\varphi}(q)$. So Proposition 3.18 with $\tilde{\varphi} = \varphi^t$ can be applied to show that, for sufficiently large $\lambda$, there are no strips which contribute to either $(\Phi_{0,\tilde{G}_t}(p), q)$ or $(\Phi_{\tilde{G}_t,0}(p), q)$.

(2): Recall Lemma 8.21, which is based upon Proposition 3.12 by which the composition $\Phi_{\tilde{G}_t,0} \circ \Phi_{0,\tilde{G}_t}$ is chain homotopic to the identity. The same stretching argument as in the proof of Part (1) above shows that the Floer continuation homotopies in fact restrict to $C_s^+$. In other words, the composition $\Phi_{\tilde{G}_t,0}|_{C_s^+} \circ \Phi_{0,\tilde{G}_t}|_{C_s^+}$ of restrictions is also homotopic to the identity map $\text{Id}_{C_s^+}$.

The same stretching argument, combined with the invariance result Proposition 3.19 in terms of bifurcation analysis, shows that the homology of the subcomplex

$$C_s^+ \subset \text{CF}_s^{\rho_0}(L, L_{\epsilon,N,\sigma}; \tilde{G}_t) \overline{M}_+$$

is homotopic to the subcomplex

$$C_s^+ \subset \text{CF}_s^{\rho_0}(L, L_{\epsilon,N,\sigma}; 0) \overline{M}_+.$$  

In conclusion, since a left-inverse of a map between equidimensional vector spaces is an inverse, the isomorphism on the level of homology follows.

(3): Now we need to consider the primitives $\tilde{f}_0$ and $\tilde{f}_1$ of $\eta$ pulled back to $\phi^t_{\tilde{G}_t}(L)$ and $L_{\epsilon,N,\sigma}$, respectively. Part (2) of Remark 2.11 allows us to apply Part (3) of Lemma 8.8 and Inequality (8.11) in order to obtain

$$-\tilde{f}_1(N - \epsilon) + \tilde{f}_0(N - \epsilon) \geq e^{A+2\delta} \|H_t\|_{\text{osc}} > -\tilde{f}_1'(N - \epsilon) + \tilde{f}_0'(N - \epsilon) + e^{A+2\delta} \|H_t\|_{\text{osc}}.$$  

Roughly speaking, the “bulge” of $L_{\epsilon,N,\sigma}$ of size $\sigma > 0$ contained inside $[-N + \epsilon, -\epsilon] \times Y_-$ contributes an additional term of

$$\sigma > e^{A+2\delta} \|H_t\|_{\text{osc}}$$

to the primitive $\tilde{f}_1$ of $\eta$ compared to the primitive $\tilde{f}_1^{\rho}$ of $\varphi \eta$. In view of Inequality (8.11), it is important to choose $\epsilon > 0$ sufficiently small here.

Now, fix a generator $p \in C_{s_0}$. If $p$ lies in the support of the Hamiltonian deformation $\tilde{G}_t$, Part (2) of Remark 2.11 does not apply. However, since for any $\tau$, $\tilde{G}_t$ changes $f_0(\tau)$ and $f_0'(\tau)$ by the same (possibly trivial) amount, while leaving $f_1(\tau), f_1'(\tau)$ fixed, we similarly obtain

$$a_p > a_{\varphi}(p) + e^{A+2\delta} \|H_t\|_{\text{osc}}.$$  

By definition of $\cdot$, $a_{\varphi}(p) \leq \overline{M}_+$. Since

$$\overline{M}_+ := e^{2\delta} M_+ \text{ and } e^{A+2\delta} \|H_t\|_{\text{osc}} = e^{2\delta} (M_+ - M_-),$$

we may hence assume that

$$0 < -\int_0^1 \min_{X} \tilde{G}_t dt < e^{2\delta} M_+ \leq a(p),$$

(here we have used Proposition 5.11 and Lemma 8.10).

The claim that

$$C_{s}^{\rho_0} \subset \text{CF}_s^{\rho_0}(L, L_{\epsilon,N,\sigma}; \tilde{G}_t) \overline{M}_+$$

is a subcomplex is now a direct consequence of Lemma 8.8 together with Part (2) of Remark 2.11 (the differential is action increasing). Similarly, the action consideration in Proposition 5.18 using $\varphi = 1$ implies that the inclusion

$$\Phi_{\tilde{G}_t,0}(C_{s_0}^{\rho_0}) \subset C_s^+ \subset \text{CF}_s^{\rho_0}(L, L_{\epsilon,N,\sigma}; 0)$$

is satisfied. Indeed, any $q \in C_{s}^{\rho_0} \subset \text{CF}_s^{\rho_0}(L, L_{\epsilon,N,\sigma}; 0)$ satisfies $a(q) < 0$.\]

\textbf{Corollary 8.23.} For an almost complex structure as used in Proposition 8.22, the following hold.
(1) The homology of the subcomplex
\[ C^+_s \subset CF^p_0(L, L_{\epsilon, N, \sigma}; \tilde{G}_t)_{M^+_{\Sigma}} \]
is isomorphic to the Morse homology complex \( (C^M_{s-1}(h), \partial_h) \) of \( \Lambda \).

(2) Consider the inclusions
\[ C^+_s \subset C^s_{\geq 0} \subset CF^p_0(L, L_{\epsilon, N, \sigma}; \tilde{G}_t)_{M^+_{\Sigma}} \]
of subcomplexes. The first inclusion has full rank in homology, while the composition of inclusions vanishes in homology.

**Proof.** (1): This follows from Part (2) of Proposition 8.22 together with Lemma 8.12.

(2): Parts (2) and (3) of Proposition 8.22 imply the existence of a commutative diagram of the form
\[
\begin{array}{ccc}
H(C^+) & \longrightarrow & H(C_{\geq 0}) \\
\left[\Phi_{\tilde{G}_t, 0}|_{C^+}\right] & \simeq & \left[\Phi_{\tilde{G}_t, 0}|_{C_{\geq 0}}\right] \\
H(C^+) & \longrightarrow & H(C^+),
\end{array}
\]

where the horizontal maps are induced by the canonical inclusions of subcomplexes. This shows the first claim.

Lemma 8.12 and Part (2) of Proposition 8.22 produces a commutative diagram of the form
\[
\begin{array}{ccc}
H(C^+) & \longrightarrow & HF(L, L_{\epsilon, N, \sigma}; 0) = 0 \\
\left[\Phi_{0, \tilde{G}_t}|_{C^+}\right] & \simeq & \left[\Phi_{0, \tilde{G}_t}\right] \\
H(C^+) & \longrightarrow & HF(L, L_{\epsilon, N, \sigma}; \tilde{G}_t)_{M^+_{\Sigma}},
\end{array}
\]

where the horizontal maps are induced by the canonical inclusions of subcomplexes. This implies the vanishing of the bottom map, as sought.

□

8.8. **Finishing the proof of Theorem 1.2.** Recall Lemma 8.18 and Proposition 8.22 which imply that
\[ C^+_s \subset R_s \subset C_s = CF^p_0(L, L_{\epsilon, N, \sigma}; \tilde{G}_t)_{M^+_{\Sigma}} \]
is a sequence of inclusions of subcomplexes. So Part (2) of Corollary 8.23 implies that this composition of inclusions vanishes on the level of homology, while the first inclusion is an inclusion on the homology level. Also, let
\[ L_s \subset C_s = CF^p_0(L, L_{\epsilon, N, \sigma}; \tilde{G}_t)_{M^+_{\Sigma}} \]
denote the inclusion of the subcomplex \( L_s \).

We then have a commutative diagram in homology
\[
\begin{array}{ccc}
\cdots & \longrightarrow & H_*(I) \\
\left[0 \oplus 0\right] & \longrightarrow & H_*(L) \oplus H_*(R) \\
\left[0 \oplus 0\right] & \longrightarrow & H_*(C) \\
\cdots & \longrightarrow & \cdots
\end{array}
\]

where the horizontal sequence is the exact Mayer–Vietoris sequence from (8.20). Combining this diagram with Part (1) of Corollary 8.23 we deduce that
\[
\dim H(I) \geq \dim \ker([\iota_L + \iota_R]) \geq \dim H(C^+) = \dim H_*(\Lambda).
\]

This inequality, combined with Lemmas 8.1 and 8.18 then finishes the proof. □
References

[1] C. Abbas. An introduction to compactness results in symplectic field theory. Springer, Heidelberg, 2014.
[2] M. Akaho. Symplectic displacement energy for exact Lagrangian immersions. Preprint (2015), available at http://arxiv.org/abs/1505.06560.
[3] M. Akaho. Hofer's symplectic energy and Lagrangian intersections in contact geometry. J. Math. Kyoto Univ., 41(3):593–609, 2001.
[4] P. Albers and U. Frauenfelder. Leaf-wise intersections and Rabinowitz Floer homology. J. Topol. Anal., 2(1):77–98, 2010.
[5] P. Albers, U. Fuchs, and W. J. Merry. Orderability and the Weinstein conjecture. Compos. Math., 151(12):2251–2272, 2015.
[6] P. Albers, U. Fuchs, and W. J. Merry. Positive loops and $L^\infty$-contact systolic inequality. Preprint (2016), available at http://arxiv.org/abs/1602.01383, 2016.
[7] P. Albers and D. Hein. Cuplength estimates in Morse cohomology. J. Topol. Anal., 8(2):243–272, 2016.
[8] M. Audin and M. Damian. Morse theory and Floer homology. Universitext. Springer, London; EDP Sciences, Les Ulis, 2014. Translated from the 2010 French original by Reinie Erné.
[9] M. Audin and J. Lafontaine. Introduction: applications of pseudo-holomorphic curves to symplectic topology. In Holomorphic curves in symplectic geometry, volume 117 of Progr. Math., pages 1–14. Birkhäuser, Basel, 1994.
[10] M. S. Borman and F. Zapolsky. Quasi-morphisms on contactomorphism groups and contact rigidity. Geom. Topol.
[11] F. Bourgeois, Y. Eliashberg, H. Hofer, K. Wysocki, and E. Zehtnder. Compactness results in symplectic field theory. Geom. Topol., 7:799–888, 2003.
[12] B. Chantraine. Lagrangian concordance of Legendrian knots. Algebr. Geom. Topol., 10(1):63–85, 2010.
[13] B. Chantraine, G Dimitroglou Rizell, P. Ghiggini, and R. Golovko. Floer theory for Lagrangian cobordisms. Preprint (2015), available at http://arxiv.org/abs/1511.09471.
[14] Yu. V. Chekanov. Critical points of quasifunctions, and generating families of Legendrian manifolds. Funktsional. Anal. i Prilozhen., 30(2):56–69, 96, 1996.
[15] Yu. V. Chekanov. Lagrangian intersections, symplectic energy, and areas of holomorphic curves. Duke Math. J., 95(1):213–226, 1998.
[16] K. Cieliebak and U. A. Frauenfelder. A Floer homology for exact contact embeddings. Pacific J. Math., 239(2):251–316, 2009.
[17] K. Cieliebak and K. Mohnke. Compactness for punctured holomorphic curves. J. Symplectic Geom., 3(4):589–654, 2005. Conference on Symplectic Topology.
[18] K. Cieliebak and A. Oancea. Symplectic homology and the Eilenberg-Steenrod axioms. Preprint (2015), available at http://arxiv.org/abs/1511.00485.
[19] O. Cornea and A. Ranicki. Rigidity and gluing for Morse and Novikov complexes. J. Eur. Math. Soc. (JEMS), 5(4):343–394, 2003.
[20] G. Dimitroglou Rizell and R. Golovko. On homological rigidity and flexibility of exact Lagrangian endocobordisms. Internat. J. Math., 25(10):1450089, 24, 2014.
[21] Y. Eliashberg, A. Givental, and H. Hofer. Introduction to symplectic field theory. Geom. Funct. Anal., (Special Volume, Part II):560–673, 2000. GAFA 2000 (Tel Aviv, 1999).
[22] Y. Eliashberg, H. Hofer, and D. Salamon. Lagrangian intersections in contact geometry. Geom. Funct. Anal., 5(2):244–269, 1995.
[23] J. W. Fish. Target-local Gromov compactness. Geom. Topol., 15(2):765–826, 2011.
[24] A. Floer. Morse theory for Lagrangian intersections. J. Differential Geom., 28(3):513–547, 1988.
[25] A. Floer. A relative Morse index for the symplectic action. Comm. Pure Appl. Math., 41(4):393–407, 1988.
[26] D. Fuchs and D. Rutherford. Generating families and Legendrian contact homology in the standard contact space. J. Topol., 4(1):190–226, 2011.
[27] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono. Lagrangian intersection Floer theory: anomaly and obstruction. Part I, volume 46 of AMS/IP Studies in Advanced Mathematics. American Mathematical Society, Providence, RI, 2009.
[28] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono. Lagrangian intersection Floer theory: anomaly and obstruction. Part II, volume 46 of AMS/IP Studies in Advanced Mathematics. American Mathematical Society, Providence, RI, 2009.
[29] H. Geiges. An introduction to contact topology, volume 109 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2008.
[30] A. B. Givental. The nonlinear Maslov index. In Geometry of low-dimensional manifolds, 2 (Durham, 1989), volume 151 of London Math. Soc. Lecture Note Ser., pages 35–43. Cambridge Univ. Press, Cambridge, 1990.
[31] M. Gromov. Pseudoholomorphic curves in symplectic manifolds. Invent. Math., 82(2):307–347, 1985.
[32] J. G. Harper and M. G. Sullivan. A bordered Legendrian contact algebra. J. Symplectic Geom., 12(2):237–255, 2014.
[33] H.-L. Her. Symplectic energy and Lagrangian intersection under Legendrian deformations. Pacific J. Math., 231(2):417–435, 2007.
[34] H. Hofer. On the topological properties of symplectic maps. Proc. Roy. Soc. Edinburgh Sect. A, 115(1-2):25–38, 1990.
[35] H. Hofer. Pseudoholomorphic curves in symplectizations with applications to the Weinstein conjecture in dimension three. Invent. Math., 114(3):515–563, 1993.
[36] M. Khovanov and P. Seidel. Quivers, Floer cohomology, and braid group actions. J. Amer. Math. Soc., 15(1):203–271, 2002.
[37] F. Laudenbach and J.-C. Sikorav. Persistance d’intersection avec la section nulle au cours d’une isotopie hamiltonienne dans un fibré cotangent. *Invent. Math.*, 82(2):349–357, 1985.

[38] Y.-J. Lee. Reidemeister torsion in Floer-Novikov theory and counting pseudo-holomorphic tori. I. *J. Symplectic Geom.*, 3(2):221–311, 2005.

[39] D. McDuff and D. Salamon. *Introduction to symplectic topology*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1998.

[40] Will J. Merry. Lagrangian Rabinowitz Floer homology and twisted cotangent bundles. *Geom. Dedicata*, 171:345–386, 2014.

[41] Y.-G. Oh. *Symplectic topology and Floer homology*. Vol. 1, volume 28 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2015. Symplectic geometry and pseudoholomorphic curves.

[42] Y.-G. Oh. *Symplectic topology and Floer homology*. Vol. 2, volume 29 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2015. Symplectic geometry and pseudoholomorphic curves.

[43] K. Ono. Lagrangian intersection under Legendrian deformations. *Duke Math. J.*, 85(1):209–225, 1996.

[44] L. Polterovich. Symplectic displacement energy for Lagrangian submanifolds. *Ergodic Theory Dynam. Systems*, 13(2):357–367, 1993.

[45] J. M. Sabloff and L. Traynor. The minimal length of a Lagrangian cobordism between Legendrians. Preprint (2015), available at [http://arxiv.org/abs/1511.03106](http://arxiv.org/abs/1511.03106).

[46] S. Sandon. A Morse estimate for translated points of contactomorphisms of spheres and projective spaces. *Geom. Dedicata*, 165:95–110, 2013.

[47] E. Shelukhin. The Hofer norm of a contactomorphism. Preprint (2014), available at [http://arxiv.org/abs/1411.1457](http://arxiv.org/abs/1411.1457).

[48] S. Sivek. A bordered Chekanov-Eliashberg algebra. *J. Topol.*, 4(1):73–104, 2011.

[49] M. G. Sullivan. $K$-theoretic invariants for Floer homology. *Geom. Funct. Anal.*, 12(4):810–872, 2002.

[50] M. Usher. Observations on the Hofer distance between closed subsets. *Math. Res. Lett.*, 22(6):1805–1820, 2015.

Department of Mathematics, Uppsala University, Box 480, SE-751 06 UPPSALA, SWEDEN

*E-mail address:* georgios.dimitroglou@math.uu.se

Department of Mathematics, University of Massachusetts, Amherst, MA 01002, USA

*E-mail address:* sullivan@math.umass.edu