SOME PROPERTIES OF THE SPACE OF COMPACT OPERATORS

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ABSTRACT. Let $X$ be a separable Banach space, $Y$ be a Banach space and $\Lambda$ be a subset of the dual group of a given compact metrizable abelian group. We prove that if $X^*$ and $Y$ have the type I-$\Lambda$-RNP (resp. type II-$\Lambda$-RNP) then $K(X,Y)$ has the type I-$\Lambda$-RNP (resp. type II-$\Lambda$-RNP) provided $L(X,Y) = K(X,Y)$. Some corollaries are then presented as well as results concerning the separability assumption on $X$. Similar results for the NearRNP and the WeakRNP are also presented.

1. INTRODUCTION

Let $X$ and $Y$ be two Banach spaces. We denote by $L(X,Y)$ (resp. $K(X,Y)$) the Banach space of all bounded (resp. compact bounded) linear operators from $X$ into $Y$.

This note is devoted to study different types of Radon-Nikodym Property (RNP) of the space $K(X,Y)$. Recall that in [4] Diestel and Morrison proved the following theorem:

Theorem 1. Suppose that $X$ is a separable Banach space such that $X^*$ has the RNP and $Y$ is a Banach space with the RNP. Suppose in addition that $L(X,Y) = K(X,Y)$. Then the space $K(X,Y)$ has the RNP.

Later Andrews [1] showed that one can remove the separability assumption on $X$ if either $Y$ is a dual space or if $X$ satisfies the following topological property: (*) The weak$^*$-closure of every bounded norm separable subset of $X^*$ is weak$^*$-metrizable.

In is natural to ask if the same type of result holds for different types of Radon-Nikodym properties such as Analytic Radon-Nikodym property (ARNP), Weak Radon-Nikodym property (WeakRNP), Near Radon-Nikodym property (NearRNP) (a weakening of the RNP introduced by Kaufman, Petrakis, Riddle and Uhl [11]),...

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In the first part of this note we will present some conditions on $X$ and $Y$ so that the separability of the space $X$ is no longer needed. Then we consider some types of Radon-Nikodym properties associated with subsets of countable discrete abelian group (type I-$\Lambda$-RNP and type II-$\Lambda$-RNP) which generalize the usual Radon-Nikodym property and the Analytic Radon-Nikodym property. These properties were introduced by Dowling \[6\] and Edgar \[8\]. We show that the theorem above still holds if one replace the usual RNP by the type I-$\Lambda$-RNP or the type II-$\Lambda$-RNP. In particular Theorem 1. is valid for the Analytic Radon-Nikodym property. We prove also that similar result can be obtained for the NearRNP and finally we discuss when a $K(X, Y)$-valued measure has a Pettis-integrable density. We will show that the above theorem holds for the weakRNP if the range space $Y$ is a dual space.

All unexplained terminologies can be found in \[2\] and \[3\].

2. THE RADON-NIKODYM PROPERTY FOR THE SPACE OF OPERATORS

In this section we will provide some sufficient conditions on the Banach spaces $X$ and $Y$ so that the above theorem is still valid for $X$ non separable.

We say that a series $\sum_{n=1}^{\infty} x_n$ is a **weakly unconditionally Cauchy (w.u.c)** in $X$ if it satisfies one of the following equivalent statements:

(a) $\sum_{n=1}^{\infty} |x^*(x_n)| < \infty$, for every $x^* \in X^*$;

(b) $\sup\{|| \sum_{n \in \sigma} x_n ||; \; \sigma \text{ finite subset of } \mathbb{N} \} < \infty$;

(c) $\sup_{n} \sup_{\epsilon_i = \pm 1} || \sum_{i=1}^{n} \epsilon_i x_i || < \infty$.

Some more equivalent formulations can be found in \[2\].

Let us begin by defining the following property introduced by Pełczynski in \[12\].

**Definition 1.** (Pełczynski) A Banach space $X$ has **property (u)** if for any weakly Cauchy sequence $(e_n)_n$ in $X$, there exists a weakly unconditionally Cauchy series $\sum n x_n$ in $X$ such that the sequence $(e_n - \sum_{i=1}^{n} x_i)$ converges weakly to zero in $X$.
Theorem 2. Let $X$ and $Y$ be Banach spaces such that $X^*$ and $Y$ have the RNP and $L(X,Y) = K(X,Y)$. If either $X$ has property (u) or $Y$ is weakly sequentially complete then $K(X,Y)$ has the RNP.

The main ingredient for the proof is the following proposition due to Heinrich and Mankiewicz (see Proposition 3.4 of [10]).

Proposition 1. (Heinrich and Mankiewicz) [10] Let $X$ be a Banach space and $X_0$ be a separable subspace of $X$. Then there exists a separable subspace $X_1$ of $X$ containing $X_0$ and an isometric embedding $J : X_1^* \to X^*$ with the property that $\langle z, Jz^* \rangle = \langle z, z^* \rangle$ for every $z \in X_1$ and $z^* \in X_1^*$. In particular, $J(X_1^*)$ is 1-complemented in $X^*$.

Proposition 2. Let $Z$ be a separable subspace of $X^*$, then there exist a 1-complemented subspace $Z_1$ of $X^*$ with $Z \subset Z_1 \subset X^*$ and a separable subspace $X_1$ of $X$ such that $Z_1$ is isometric to $X_1^*$.

Proof. If $\{ f_n, n \geq 1 \}$ is a countable dense subset of the unit ball of $Z$ and $\{ x_{n,j}; n, j \geq 1 \}$ is a sequence in $X$ such that for every $n \in \mathbb{N}$, $\lim_{j \to \infty} f_n(x_{n,j}) = ||f_n||$. Let $X_0$ be the separable subspace of $X$ spanned by the sequence $(x_{n,j})_{n,j}$. By Proposition [4], there exist a separable subspace $X_1$ of $X$ with $X_0 \subset X_1 \subset X$ and $J : X_1^* \to X^*$ as in Proposition [4]. We will show that $Z$ is isometrically isomorphic to the subspace $J(X_1^*)$ of $X^*$. For that let $Q : X^* \to X_1^*$ be the restriction map and $i : Z \to X^*$ the inclusion. We claim that $Q \circ i : Z \to X_1^*$ is an isometry. In fact since $X_0$ is norming for $Z$, we have for every $f \in Z$,

$$||Q \circ i(f)|| = \sup_{||z|| \leq 1} (Q(f), z) = \sup_{||z|| \leq 1} (f, z) = ||f||$$

which shows that $Q \circ i$ is an isometry. Now since $J$ is an isometry, we have $Z$ embedded isometrically into $J(X_1^*)$ and the proposition is proved.

We are now ready to provide the proof of the theorem: It is enough to show that every separable subspace $S$ of $K(X,Y)$ is isometric to a subspace of $K(X_1, Y_1)$ where $X_1$ is a separable subspace of $X$ and $Y_1$ is a separable subspace of $Y$ and such that $L(X_1,Y_1) = K(X_1,Y_1)$. 

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Let $S$ be a separable subspace of $K(X,Y)$. It is clear by compactness that the space
\[ \{ Tx; \ T \in S, \ \xi \in \mathcal{X} \} \] is separable. The space $Y_1 = \overline{\operatorname{span}} \{ Tx; \ T \in S, \ \xi \in \mathcal{X} \}$ is separable and $S \subset K(X,Y_\infty)$. Using similar argument with the adjoints, we get that the space
\[ Z = \overline{\operatorname{span}} \{ T^*y^*; \ T \in S, \ \xi^* \in Y^* \} \] is a separable subspace of $X^*$. Let $Z_1$ and $X_1$ as in Proposition 2. It is clear that the restriction map from $S$ into $K(X_1,Y_1)$ is an isometry and we claim that $L(X_1,Y_1) = K(X_1,Y_1)$. For that let $J : X_1^* \rightarrow X^*$ and $Q : X^* \rightarrow X_1^*$ as before and fix $\theta \in L(X_1,Y_1)$. Assume that $X$ has property (u); since $X_1$ does not contain $\ell^1$ and $Y_1$ does not contain $c_0$, by [12], the operator $\theta$ is weakly compact. Similarly if $Y_1$ is weakly sequentially complete, the operator $\theta$ is weakly compact. So in both cases, the operator $\theta$ is weakly compact. Consider $J \circ \theta^* : Y_1^* \rightarrow X^*$; by the weak compactness of $\theta$, we have
\[ (J \circ \theta^*)(X^{**}) = \theta^{**} \circ J^*(X^{**}) \subset Y_1 \]
and since $L(X,Y) = K(X,Y)$, we get that $(J \circ \theta^*)(X^{**})$ is compact and therefore $(J \circ \theta^*)(B_X)$ is relatively compact in $Y_1$.
\[ (J \circ \theta^*)(B_X^{**}) \subset (J \circ \theta^*)(B_X)^{**} \]
which shows that $J \circ \theta^*(B_X^{**})$ is relatively compact. Hence $J \circ \theta^*$ is compact. To complete the proof, let $\pi : X^* \rightarrow J(X_1^*)$ be the norm 1- projection; it can be easily checked that
\[ \theta^* = J^{-1} \circ \pi \circ (J \circ \theta^*) \] which proves that $\theta^*$ (and hence $\theta$) is compact. The theorem is proved.

3. SOME VARIANTS OF THE RADON-NYKODYM PROPERTY FOR THE SPACE OF OPERATORS

Throughout the remaining of this paper $G$ will denote a compact metrizable abelian group, $\mathcal{B}(G)$ is the $\sigma$-algebra of the Borel subsets of $G$, and $\lambda$ the normalized Haar measure on $G$. We will denote by $\Gamma$ the dual group of $G$ i.e the set of continuous homomorphisms $\gamma : G \rightarrow \mathbb{C}$ ($\Gamma$ is a countable discrete abelian group).

Let $X$ be a Banach space and $1 \leq p \leq \infty$, we will denote by $L^p(G,X)$ the usual Bochner function spaces for the measure space $(G,\mathcal{B}(G),\lambda)$, $M(G,X)$ the space of $X$-valued countably additive measure of bounded variation, $C(G,X)$ the space of $X$-valued continuous functions and $M^\infty(G,X) = \{ \mu \in M(G,X), |\mu| \leq C\lambda \text{ for some } C > 0 \}$. 


(i) If \( f \in L^1(G, X) \), we denote by \( \hat{f} \) the Fourier transform of \( f \) which is the map from \( \Gamma \) to \( X \) defined by \( \hat{f}(\gamma) = \int_G \bar{\gamma} f \ d\lambda \).

(ii) If \( \mu \in M(G, X) \), we denote by \( \hat{\mu} \) the Fourier transform of \( \mu \) which is the map from \( \Gamma \) to \( X \) defined by \( \hat{\mu}(\gamma) = \int_G \bar{\gamma} d\mu \).

If \( \Lambda \subset \Gamma \) is a set of characters, let

\[
L^p_\Lambda(G, X) = \{ f \in L^p(G, X), \hat{f}(\gamma) = 0 \text{ for all } \gamma \notin \Lambda \}
\]

\[
C_\Lambda(G, X) = \{ f \in C(G, X), \hat{f}(\gamma) = 0 \text{ for all } \gamma \notin \Lambda \}
\]

\[
M_\Lambda(G, X) = \{ \mu \in M(G, X), \hat{\mu}(\gamma) = 0 \text{ for all } \gamma \notin \Lambda \}
\]

\[
M^\infty_\Lambda(G, X) = \{ \mu \in M^\infty(G, X), \hat{\mu}(\gamma) = 0 \text{ for all } \gamma \notin \Lambda \}
\]

**Definition 2.** (i) A subset \( \Lambda \) of \( \Gamma \) is a Riesz set if and only if \( M_\Lambda(G, \mathbb{C}) = L^1_\Lambda(G, \mathbb{C}) \)

(ii) A subset \( \Lambda \) of \( \Gamma \) is a Rosenthal set if and only if \( C_\Lambda(G) = L^\infty_\Lambda(G) \).

The following properties were introduced by Edgar [8], and Dowling [6].

**Definition 3.** (i) A Banach space \( X \) is said to have type I-\( \Lambda \)-Radon Nikodym Property (type I-\( \Lambda \)-RNP) if and only if \( M^\infty_\Lambda(G, X) = L^\infty_\Lambda(G, X) \).

(ii) A Banach space \( X \) is said to have type II-\( \Lambda \)-Radon Nikodym Property (type II-\( \Lambda \)-RNP) if and only if \( M_{\Lambda,ac}(G, X) = L^1_\Lambda(G, X) \) where

\[
M_{\Lambda,ac}(G, X) = \{ \mu \in M_\Lambda(G, X), \mu \text{ is absolutely continuous with respect to } \lambda \}.
\]

**Remarks:**

(a) It is obvious that type II-\( \Lambda \)-RNP implies type I-\( \Lambda \)-RNP.

(b) Since \( \mathcal{B}(G) \) is countably generated, one can see that these two properties are separably determined.

(c) If \( G = \mathbb{T} \) then \( \Gamma = \mathbb{Z} \). Then type I-\( \mathbb{Z} \)-RNP is equivalent to type II-\( \mathbb{Z} \)-RNP which is also equivalent to the usual RNP. Similarly, type I-\( \mathbb{N} \)-RNP is equivalent to type II-\( \mathbb{N} \)-RNP and is equivalent to the Analytic Radon Nikodym Property (see [8]).

(d) If \( \Lambda \) is a Riesz subset, then \( M_{\Lambda,ac}(G, X) = M_\Lambda(G, X) \).

We are now ready to present our results.
Theorem 3. Let $X$ and $Y$ be Banach spaces such that:

(i) $X$ is separable;
(ii) $X^*$ and $Y$ have type $I$-$\Lambda$-RNP (resp. type $II$-$\Lambda$-RNP);
(iii) $L(X,Y) = K(X,Y)$;

Then $K(X,Y)$ has type $I$-$\Lambda$-RNP (resp. type $II$-$\Lambda$-RNP).

We will present the proof for type $I$-$\Lambda$-RNP case (the type $II$-$\Lambda$-RNP case can be done with minor changes).

Consider $\mathcal{B}(G)$ the $\sigma$-Algebra generated by the Borel subsets of $G$ and fix a measure $F : \mathcal{B}(G) \to K(X,Y)$ such that

a) $|F| \leq \lambda$;

b) $\hat{F}^{*}(\gamma) = 0$ for $\gamma \notin \Lambda$.

Our main goal is to show that the measure $F$ has a Bochner integrable density.

For $x \in X$, we will denote by $F(\cdot)x$ the $Y$-valued measure $A \to F(A)x$ and similarly, for $y^* \in Y^*$, $F(\cdot)y^*$ will be the $X^*$-valued measure $A \to F(A)^*y^*$.

Let us begin with the following simple observation:

For every $x \in X$, $y^* \in Y^*$ and $\gamma \in \hat{G} = \Gamma$, we have

$$\langle \hat{F}(\gamma), x \otimes y^* \rangle = \langle F(\cdot)x(\gamma), y^* \rangle = \langle \hat{F}(\cdot)^*y^*(\gamma), x \rangle$$

These equalities imply that for any $x \in X$ and $y^* \in Y^*$, $F(\cdot)x \in M^{\infty}(G,Y)$ and $F(\cdot)y^* \in M^{\infty}_{\Lambda}(G,X^*)$.

Notice also that without loss of generality we can and do assume that $Y$ is separable.

We need the following definition for the next Proposition.

Definition 4. Let $E$ and $F$ be Banach spaces and $(G,(\mathcal{B})(G),\lambda)$ be a measure space. A map $T : G \to L(E,F)$ is said to be strongly measurable if $\omega \to T(\omega)e$ is $\lambda$-measurable for every $e \in E$.

Proposition 3. There exists a strongly measurable map $\omega \to T(\omega) (G \to K(X,Y))$ such that:
(a) \( F(A)x = \text{Bochner} - \int_AT(\omega)x \, d\lambda(\omega) \) for every \( A \in \mathcal{B}(G) \) and \( x \in X \);

(b) For every \( y^* \in Y^* \), the map \( \omega \mapsto T(\omega)^*y^* \) is norm-measurable and \( F(A)^*y^* = \text{Bochner} - \int_A T(\omega)^*y^* \, d\lambda(\omega) \) for every \( A \in \mathcal{B}(G) \).

**Proof.** Using similar argument as in [4], one can construct a strongly measurable map \( \omega \mapsto T(\omega) \) such that \( F(A)x = \text{Bochner} - \int_AT(\omega)x \, d\lambda(\omega) \) for every \( x \in X \) and \( A \in \mathcal{B}(G) \) so the first part is proved.

For the second part, notice from the strong measurability of \( T(\omega) \) that the map \( \omega \mapsto T(\omega)^*y^* \) is weak*-scalarly measurable and \( F(A)^*y^* = \text{Bochner} - \int_A T(\omega)^*y^* \, d\lambda(\omega) \) for every \( A \in \mathcal{B}(G) \).

Since the measure \( F(.)^*y^* \) belongs to \( M_\infty^\infty(G, X^*) \) and \( X^* \) has the type I-\( \Lambda \)-RNP, there exists a Bochner integrable map \( h_{y^*} : G \rightarrow X^* \) such that \( F(A)^*y^* = \text{Bochner} - \int_A h_{y^*}(\omega) \, d\lambda(\omega) \) for every \( A \in \mathcal{B}(G) \) so we get that

\[
\int_A \langle T(\omega)^*y^*, x \rangle \, d\lambda(\omega) = \int_A \langle h_{y^*}(\omega), x \rangle \, d\lambda(\omega)
\]

for every \( x \in X \) and \( A \in \mathcal{B}(G) \). Now fix \( \{x_n, n \in \mathbb{N}\} \) a countable dense subset of \( X \). There exists a measurable subset \( G' \) of \( G \) with \( \lambda(G \setminus G') = 0 \) and such that for \( \omega \in G' \) and \( n \in \mathbb{N} \), we have \( \langle T(\omega)^*y^*, x_n \rangle = \langle h_{y^*}(\omega), x_n \rangle \) which of course implies that

\[
T(\omega)^*y^* = h_{y^*}(\omega) \quad \text{for} \ \omega \in G'
\]

so \( T(.)^*y^* \) is norm measurable and satisfies the required property. \( \square \)

**Proposition 4.** Let \( X \) and \( Y \) be separable Banach spaces and \( Z \) be a separable subspace of \( X^* \) then the set

\[
\mathcal{A} = \{T \in K(X, Y); T^*\hat{\imath}^* \in Z, \hat{\imath}^* \in Y^*\}
\]

is separable in \( K(X, Y) \).

**Proof.** Let \( \Delta \) be the unit ball of \( Y^* \) with the weak*-topology. Since \( Y \) is separable \( \Delta \) is a compact metric space. Let \( C(\Delta) \) be the Banach space of all continuous functions on \( \Delta \) with the usual sup norm. It is well known that the Banach space \( Y \) embeds isometrically into
C(∆). Let J : Y → C(∆) be the natural isometry. Consider an operator J# : K(X, Y) → K(X, C(∆)) defined as follows:

\[ J#(T) = J \circ T. \]

It is clear that J# is an isometry and we will show that J#(A) is separable. For that notice that J#(A) is a subset of

\[ \mathcal{M} = \{ S \in \mathcal{K}(X, C(\Delta)); S^* \cap^* \in Z, \cap^* \in C(\Delta)^* \}. \]

In fact for every \( u^* \in C(\Delta)^* \) and \( T \in A \), we have \( J#(T)^*u^* = (J \circ T)^*u^* = T^*(J^*u^*) \in Z \).

Now since \( M(\Delta) = C(\Delta)^* \) has the approximation property, we have

\[ K(X, C(\Delta)) = K_{w^*}(X^{**}, C(\Delta)) = X^* \hat{\otimes} \epsilon C(\Delta) \]

where \( K_{w^*}(X^{**}, C(\Delta)) \) denotes the space of compact operators from \( X^{**} \) into \( C(\Delta) \) that are weak* to weak continuous and \( \hat{\otimes} \) is the injective tensor product. Let \( S \in \mathcal{M} \); since \( S^* \in K_{w^*}(M(\Delta), Z) = C(\Delta) \hat{\otimes}_\epsilon Z \), it can be approximated by elements of \( C(\Delta) \otimes Z \) and by duality \( S \) can be approximated by elements of \( Z \otimes C(\Delta) \) and therefore \( \mathcal{M} \subset Z \hat{\otimes}_\epsilon C(\Delta) \) which is a separable space. We are done \( \square \)

To complete the proof of the theorem, let us choose a sequence \( (A_n)_n \subset \mathcal{B}(\mathcal{G}) \) such that \( \{ F(A_n), n \geq 1, A_n \in \mathcal{B}(\mathcal{G}) \} \) is dense in the range of the measure \( F \) (this is possible because \( \mathcal{B}(\mathcal{G}) \) is countably generated).

For each \( n \geq 1 \), the operator \( F(A_n) \) is compact so the set \( F(A_n)^*(B_{Y^*}) \) is compact in \( X^* \) and therefore \( F(A_n)^*(Y^*) \) is separable in \( X^* \). Define a subspace \( Z \) of \( X^* \) as follows:

\[ Z = \text{span} \left\{ \bigcup_{n \geq 1} F(A_n)^*(Y^*) \right\}. \]

The space \( Z \) is obviously a separable subspace of \( X^* \) and for every \( A \in \mathcal{B}(\mathcal{G}) \) and \( y^* \in Y^* \), we have \( F(A)^*y^* \in Z \). We will need the following lemma.

**Lemma 1.** Let \( A \) be the separable subspace of \( K(X, Y) \) as in Proposition 4. For a.e \( \omega \in G \), \( T(\omega) \in A \).
**Proof of Lemma** Let \( \{ y_n^*, n \geq 1 \} \) be a countable weak*-dense subset of \( B_{Y^*} \). Let \( n \in \mathbb{N} \) fixed. Since \( F(A)^* \} y_n^* \in Z \) and \( F(A)^* \} y_n^* = \text{Bochner} - \int_A T(\omega)^* \} y_n^* \ d\lambda(\omega) \) for every \( A \in \mathcal{B}(G) \), we get that \( T(\omega)^* \} y_n^* \in Z \) for almost every \( \omega \in G \). There exists a measurable subset \( O_n \) of \( G \) with \( \lambda(O_n) = 0 \) and for \( \omega \notin O_n \), \( T(\omega)^* \} y_n^* \in Z \). Let \( O = \bigcup_{n=1}^{\infty} O_n \); \( \lambda(O) = 0 \) and if \( \omega \notin O \), we have

\[
T(\omega)^* \} y_n^* \in Z \quad \forall n \in \mathbb{N}.
\]

Now for \( y^* \in B_{Y^*} \), choose a sequence \( (y_{n_j}^*)_{j \in \mathbb{N}} \) that converges to \( y^* \) for the weak*-topology. Since \( T(\omega)^* \) is weak* to norm continuous, the sequence \( \{ T(\omega)^* \} (y_{n_j}^*) \}_{j \in \mathbb{N}} \) converges to \( T(\omega)^* \} y^* \) for the norm-topology which implies that \( T(\omega)^* \} y^* \in Z \) for every \( \omega \in G \setminus O \) (independent of \( y^* \)) hence \( T(\omega) \in \mathcal{A} \) for every \( \omega \in G \setminus O \) and the lemma is proved. \( \square \)

We complete the proof by noticing that the map \( T : G \rightarrow K(X, Y) \) that takes \( \omega \) to \( T(\omega) \) is \( \lambda \)-essentially separably valued and there exists a norming subset of \( K(X, Y)^* \) (namely \( X \otimes Y^* \)) such that the map \( \omega \rightarrow \langle T(\omega), \phi \rangle \) is \( \lambda \)-measurable for every \( \phi \in X \otimes Y^* \) and by the Pettis-measurability Theorem (see [3], Theorem II-2), the map \( \omega \rightarrow T(\omega) \) is norm-measurable and it is now clear that the measure \( F \) is represented by the map \( T : G \rightarrow K(X, Y) \). The proof is complete. \( \square \)

**Remark:** The argument used by Andrews in [1] can be adjusted to show that for the case where \( Y \) is a dual space, the assumption that \( X \) is separable may be dropped.

Some corollaries are now in order

**Corollary 1.** If \( X \) is a Banach space with the Schur property and \( \Lambda \) is a Riesz subset then \( L^1(\lambda) \otimes_c X \) (the completion of the space of Pettis representable measures with the semivariation norm) has type I-\( \Lambda \)-RNP (resp. type II-\( \Lambda \)-RNP) if and only if \( X \) has type I-\( \Lambda \)-RNP (resp. type II-\( \Lambda \)-RNP).

**Proof.** Notice that \( L^1(\lambda) \otimes_c X \) is a subspace of \( M(G) \otimes_c X \) which is isometrically isomorphic to the space \( K(C(G), X) \). Now if \( X \) has the Schur property, \( L(C(G), X) = K(C(G), X) \). We are done. \( \square \)
Corollary 2. Let $X$ be a Banach space having the type I-$\Lambda$-RNP (resp. type II-$\Lambda$-RNP) and denote by $\ell^1[X]$ the Banach space of all W.U.C. series in $X$ normed by

$$||| (x_n) ||| = \sup \left\{ \sum_{n=1}^{\infty} |x^*(x_n)| ; \ x^* \in X^*, \ ||x^*|| \leq 1 \right\}.$$ 

Then $\ell^1[X]$ has the type I-$\Lambda$-RNP (resp. type II-$\Lambda$-RNP).

Proof. This is due to the well known fact that $\ell^1[X] = L(c_0, X)$ and since $X$ does not contain any copy of $c_0$, we have $L(c_0, X) = K(c_0, X)$. An appeal to Theorem 2 completes the proof. \qed

For the next corollary, let us introduce a new compact metrizable abelian group $\tilde{G}$ which is not necessarily the same as $G$. We will denote by $\tilde{\Gamma}$ its dual and $\tilde{\lambda}$ its normalized Haar measure. The following result was first obtained by Dowling [7] for the usual RNP. It was also proved in [14] (see also [13]) under the assumption that $\Lambda$ is a Riesz subset.

Corollary 3. Assume that

1. $\tilde{\Lambda}$ is a Rosenthal subset of $\tilde{\Gamma}$;
2. $X$ is a Banach space with the Schur property and has the type II-$\Lambda$-RNP.

Then the space $C_{\tilde{\Lambda}}(\tilde{G}, X)$ has the type II-$\Lambda$-RNP.

Proof. If $\tilde{\Lambda}$ is a Rosenthal subset, then $C_{\tilde{\Lambda}}(\tilde{G})$ has the RNP and $C_{\tilde{\Lambda}}(\tilde{G}) = L^\infty_{\tilde{\Lambda}}(\tilde{G}) = \left( L^1(\tilde{G})/L^1_{\tilde{\Lambda}}(\tilde{G}) \right)^*$ is a dual space. Now since $X$ has the Schur property and $L^1(\tilde{G})/L^1_{\tilde{\Lambda}}(\tilde{G})$ does not contain any copy of $\ell^1$, we have

$$L(L^1(\tilde{G})/L^1_{\tilde{\Lambda}}(\tilde{G}), X) = K(L^1(\tilde{G})/L^1_{\tilde{\Lambda}}(\tilde{G}), X)$$

and since $K(L^1(\tilde{G})/L^1_{\tilde{\Lambda}}(\tilde{G}), X) = C_{\tilde{\Lambda}}(\tilde{G}, X)$, the proof is complete. \qed

For the next result, we need to recall some definitions.

Definition 5. A bounded linear operator $D : L^1[0,1] \longrightarrow X$ is called a Dunford-Pettis operator if $D$ sends weakly compact sets into norm compact sets.

Definition 6. An operator $T : L^1[0,1] \longrightarrow X$ is said to be nearly representable if $T \circ D$ is (Bochner) representable for every Dunford-Pettis operator $D : L^1[0,1] \longrightarrow L^1[0,1]$. 
The following class of Banach spaces was introduced by Kaufman, Petrakis, Riddle and Uhl in [11]:

**Definition 7.** A Banach space $X$ is said to have the Near Radon-Nikodym Property (Near-RNP) if every nearly representable operator from $L^1[0, 1]$ into $X$ is representable.

**Theorem 4.** Let $X$ and $Y$ be Banach spaces such that:

(i) $X$ is separable;
(ii) $X^*$ and $Y$ has the NearRNP;
(iii) $L(X, Y) = K(X,Y)$.

Then $K(X,Y)$ has the NearRNP.

**Proof.** Let $T : L^1[0,1] \to K(X,Y)$ be a nearly representable operator and let $F : \Sigma_{[0,1]} \to K(X,Y)$ be the representing measure of the operator $T$. For each $x \in X$, define $T_x : L^1[0,1] \to Y$ as follows:

$$T_x(f) = \langle T(f), x \rangle, \quad \forall f \in L^1[0,1].$$

$T_x$ is clearly nearly representable and therefore representable (since $Y$ has the NearRNP). Hence the measure $F(.)(x)$ which can be easily checked to be the representing measure of $T_x$ has Bochner integrable density. Similarly for each $y^* \in Y^*$, the operator $T^y^* : L^1[0,1] \to X^*$ given by

$$T^y^*(f) = \langle (Tf)^*, y^* \rangle, \quad \forall f \in L^1[0,1]$$

is nearly representable and therefore representable (since $X^*$ has the NearRNP) and since the measure $F(.)(y^*)$ is the representing measure of $T^y^*$, it has Bochner integrable density. Now we can proceed as in the proof of Theorem 2 to conclude that the measure $F$ has Bochner integrable density which shows that the operator $T$ is representable.

**Remark:** Corollary 1, Corollary 2 and Corollary 3 are still valid if we replace the Λ-RNP by the NearRNP.

Let us now turn our attention to measures that can be represented by Pettis-integrable functions. Recall that for a Banach space $X$, a function $f : G \to X$ is Pettis-integrable if $f$
is weakly scalarly measurable i.e. for each \( x^* \in X^* \) the function \( \langle f(\cdot), x^* \rangle \) is measurable and for each \( A \in \mathcal{B}(\mathcal{G}) \), there exists \( x_A \in X \) such that

\[
\langle x_A, x^* \rangle = \int_A \langle f(\omega), x^* \rangle \, d\lambda(\omega).
\]

For more details about Pettis integral we refer to [15].

**Definition 8.** Using the same notation as before, we say that a Banach space \( X \) has type I-\( \Lambda \)-WeakRNP (resp. type II-\( \Lambda \)-WeakRNP) if every measure \( F \) in \( M_\infty^\Lambda(G, X) \) (resp. \( M_\Lambda,ac(G, X) \)) has Pettis-integrable density.

**Theorem 5.** Let \( X \) and \( Y \) be Banach spaces such that:

(i) \( Y \) is a dual Banach space;
(ii) \( X^* \) and \( Y \) have the type I-\( \Lambda \)-WeakRNP (resp. type II-\( \Lambda \)-WeakRNP);
(iii) \( L(X, Y) = K(X, Y) \).

Then \( K(X, Y) \) has the type I-\( \Lambda \)-WeakRNP (resp. type II-\( \Lambda \)-WeakRNP).

**Proof.** Again we will present the type I case: Let \( F : \mathcal{B}(\mathcal{G}) \to K(X, Y) \) be a measure such that:

a) \( |F| \leq \lambda \);

b) \( \hat{F}(\gamma) = 0 \) for \( \gamma \notin \Lambda \).

We will show that the measure \( F \) has Pettis integrable density. Notice first that the range of \( F \) is separable. Using similar argument as in section 1, we can assume without loss of generality that the predual of \( Y \) is separable.

Let \( \rho \) be a lifting of \( L^\infty(\lambda) \) (see [1] or [16] for the definition). Since \( L(X, Y) = L(X, Z^*) \) is a dual space, we can apply Theorem 4 of [3] (P. 263) to get a unique bounded function \( T : G \to L(X, Z^*) \) such that:

1. \( \langle z, F(A)x \rangle = \int_A \langle z, T(\omega)x \rangle \, d\lambda(\omega) \) for \( z \in Z, x \in X \) and \( A \in \mathcal{B}(\mathcal{G}) \);
2. \( \rho(\langle z, T(\omega)x \rangle) = \langle z, T(\omega)x \rangle \) for all \( z \in Z \) and \( x \in X \).

We claim that \( \omega \to T(\omega) \) is Pettis integrable. We need several steps.
Lemma 2. For every $x \in X$ and $y^* \in Y^*$, the maps $\omega \mapsto T(\omega)x$ and $\omega \mapsto T(\omega)^*y^*$ are Pettis-integrable and for every $A \in \mathcal{B}(G)$ we have:

(a) $F(A)x = \text{Pettis} - \int_A T(\omega) \, d\lambda(\omega)$;
(b) $F(A)^*y^* = \text{Pettis} - \int_A T(\omega)^*y^* \, d\lambda(\omega)$.

The proof can be done with essentially the same idea as in the proof of Proposition 3, so we will omit the detail.

Let us now consider $\Delta = \text{the unit ball of } Y^*$ with the weak*-topology. As in the proof of Theorem 3, we denote by $J$ the natural isometry from $Y$ into $C(\Delta)$ and $J^#$ the isometry from $K(X,Y)$ into $K(X,C(\Delta))$. Consider $\phi(\omega) = J^#(T(\omega)) = J \circ T(\omega) \in K(X,C(\Delta))$.

Since $M(\Delta) = C(\Delta)^*$ has the metric approximation property, there exists a sequence of finite rank operators $\theta_n$ in $L(C(\Delta), C(\Delta))$ such that $\sup_{n \in \mathbb{N}} ||\theta_n|| \leq 1$ and $\theta_n$ converges to the identity operator uniformly on every compact subset of $C(\Delta)$. Since the $\phi(\omega)$’s are compacts, we have

$$\lim_{n \to \infty} ||\theta_n \circ \phi(\omega) - \phi(\omega)|| = 0$$

for every $\omega \in G$.

Let $\theta_n = \sum_{k=1}^{k_n} \mu_{k,n} \otimes f_{n,k}$ where $\mu_{k,n}$ and $f_{k,n}$ belong to $M(\Delta)$ and $C(\Delta)$ respectively. We have for every $\omega \in G$,

$$\theta_n \circ \phi(\omega)x = \sum_{k=1}^{k_n} \langle \phi(\omega)^* \mu_{k,n}, x \rangle f_{k,n}$$

and if $I \in K(X,C(\Delta))^* = (X^* \hat{\otimes}_c C(\Delta))^* = C(\Delta, X^*)^* = I(C(\Delta), X^*)$; here $I(C(\Delta), X^*)$ denotes the space of integral operators from $C(\Delta)$ into $X^*$ (see [3] P. 232). We get

$$\langle \theta_n \circ \phi(\omega), I \rangle = \sum_{k=1}^{k_n} \langle \phi(\omega)^* \mu_{k,n}, I(f_{k,n}) \rangle$$

which is measurable and therefore the map $\omega \mapsto \theta_n \circ \phi(\omega)$ is weakly scalarly measurable and for every $A \in \mathcal{B}(G)$ and $I \in K(X,C(\Delta))$, we have

$$\langle \theta_n \circ J \circ F(A), I \rangle = \int_A \langle \theta_n \circ \phi(\omega), I \rangle \, d\lambda(\omega).$$
And now since $||\theta_n|| \leq 1$, the set $\{\langle \theta_n \circ \phi(\cdot), I \rangle; \ n \in \mathbb{N}; \ ||I|| \leq 1\}$ is uniformly integrable so by taking the limit as $n$ tends to $\infty$, we get (see [15] Theorem 5-3-1) that

$$J^\#(F(A)) = \text{Pettis} - \int_A J^\#(T(\omega)) \ d\lambda(\omega)$$

and since $J^\#$ is an isometry, the adjoint $(J^\#)^*$ is onto and therefore the map $\omega \mapsto T(\omega)(G \rightarrow K(X,Y))$ is weakly scalarly measurable and it is now clear that $\omega \mapsto T(\omega)$ is Pettis integrable with $F(A) = \text{Pettis} - \int_A T(\omega) \ d\lambda(\omega)$. The proof is complete.

Remark: In [9], Emmanuele obtained the usual WeakRNP case of Theorem 4 but his method of proof is quite different and cannot be extended to the general case of type I-$\Lambda$-WeakRNP or type II-$\Lambda$-WeakRNP.

Let us finish by asking the following question

Question: If $X$ and $Y$ have type I-$\Lambda$-RNP (resp. type II-$\Lambda$-RNP, resp. NearRNP, resp. WeakRNP) and $L_{w^*}(X^*,Y) = K_{w^*}(X^*,Y)$; does $K_{w^*}(X^*,Y)$ have the same property? (here $L_{w^*}(X^*,Y)$ (resp. $K_{w^*}(X^*,Y)$) denotes the space of bounded (resp. compact bounded) operators from $X^*$ into $Y$ that are weak* to weak continuous).

The answer to this question is still unknown even for the usual RNP.

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