ON THE REVERSIBILITY AND THE CLOSED IMAGE PROPERTY OF LINEAR CELLULAR AUTOMATA

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Abstract. When \( G \) is an arbitrary group and \( V \) is a finite-dimensional vector space, it is known that every bijective linear cellular automaton \( \tau: V^G \rightarrow V^G \) is reversible and that the image of every linear cellular automaton \( \tau: V^G \rightarrow V^G \) is closed in \( V^G \) for the prodiscrete topology. In this paper, we present a new proof of these two results which is based on the Mittag-Leffler lemma for projective sequences of sets. We also show that if \( G \) is a non-periodic group and \( V \) is an infinite-dimensional vector space, then there exist a linear cellular automaton \( \tau_1: V^G \rightarrow V^G \) which is bijective but not reversible and a linear cellular automaton \( \tau_2: V^G \rightarrow V^G \) whose image is not closed in \( V^G \) for the prodiscrete topology.

1. Introduction

Let \( G \) be a group and let \( A \) be a set. The set \( A^G \), which consists of all maps \( x: G \rightarrow A \) is called the set of \textit{configurations} over the group \( G \) and the \textit{alphabet} \( A \). There is a natural left action of the group \( G \) on \( A^G \) defined by \( gx(h) = x(g^{-1}h) \) for all \( g, h \in G \) and \( x \in A^G \). This action is called the \textit{G-shift} on \( A^G \) and is the fundamental object of study in the branch of mathematics known as \textit{symbolic dynamics}. Although classical symbolic dynamics often restricts to the case when the alphabet set \( A \) is finite, it is clear from recent developments in the theory, such as the ones contained in the influential work of M. Gromov [6], [7], that the study of shift systems with infinite alphabet \( A \) also deserves attention, especially when \( A \) is equipped with some additional (linear, algebraic, symplectic, etc.) structure.

A \textit{cellular automaton} over the group \( G \) and the alphabet \( A \) is a map \( \tau: A^G \rightarrow A^G \) satisfying the following property: there exist a finite subset \( M \subset G \) and a map \( \mu: A^M \rightarrow A \) such that

\[
\tau(x)(g) = \mu(\pi_M(g^{-1}x)) \quad \text{for all } x \in A^G \text{ and } g \in G,
\]

where \( \pi_M: A^G \rightarrow A^M \) denotes the restriction map. Such a set \( M \) is then called a \textit{memory set} for \( \tau \) and \( \mu \) is called the \textit{local defining map} for \( \tau \) associated with \( M \). It follows from this definition that every
cellular automaton $\tau: A^G \to A^G$ commutes with the shift action, i.e., it satisfies
\[ \tau(gx) = g\tau(x) \quad g \in G \text{ and } x \in A^G. \]

In other words, $\tau$ is $G$-equivariant.

A cellular automaton $\tau: A^G \to A^G$ is said to be reversible if $\tau$ is bijective and the inverse map $\tau^{-1}: A^G \to A^G$ is also a cellular automaton.

We equip $A^G$ with its prodiscrete topology, that is, with the product topology obtained by taking the discrete topology on each factor $A$ of $A^G$. This turns out $A^G$ into a totally disconnected Hausdorff topological space. Every cellular automaton $\tau: A^G \to A^G$ is continuous with respect to the prodiscrete topology. Conversely, when the alphabet $A$ is finite, it follows from the Curtis-Hedlund theorem [9] that every continuous map $f: A^G \to A^G$ which commutes with the shift action is a cellular automaton.

When $A$ is finite, the space $A^G$ is compact by the Tychonoff product theorem and one immediately deduces from the Curtis-Hedlund theorem that every bijective cellular automaton $\tau: A^G \to A^G$ is reversible.

Suppose now that $V$ is a vector space over a field $\mathbb{K}$. A cellular automaton $\tau: V^G \to V^G$ over the alphabet $V$ is said to be linear if $\tau$ is $\mathbb{K}$-linear with respect to the natural $\mathbb{K}$-vector space structure on $V^G$. In this setting we have the following:

**Theorem 1.1.** Let $G$ be a group and let $V$ be a finite-dimensional vector space over a field $\mathbb{K}$. Then every bijective linear cellular automaton $\tau: V^G \to V^G$ is reversible.

This result was proved in [3] under the assumption that the group $G$ is countable and then extended to any group $G$ in [4]. In this paper, we use the Mittag-Leffler lemma for projective sequences of sets to derive a new proof of Theorem 1.1. We then provide examples of bijective linear cellular automata which are not reversible. More precisely, we shall prove the following (recall that a group is said to be periodic if all its elements have finite order):

**Theorem 1.2.** Let $G$ be a non-periodic group and let $\mathbb{K}$ be a field. Let $V$ be an infinite-dimensional vector space over $\mathbb{K}$. Then there exists a bijective linear cellular automaton $\tau: V^G \to V^G$ which is not reversible.

If $A$ is an infinite set, it is always possible to find a vector space with the same cardinality as $A$. One can take for example the $\mathbb{K}$-vector space based on $A$ for an arbitrary finite field $\mathbb{K}$. Therefore, as an immediate consequence of Theorem 1.2, we get:

**Corollary 1.3.** Let $G$ be a non-periodic group and let $A$ be an infinite set. Then there exists a bijective cellular automaton $\tau: A^G \to A^G$ which is not reversible.
Given a set $X$ and a topological space $Y$, one says that a map $f: X \to Y$ has the closed image property if the set $f(X)$ is closed in $Y$. The closed image property is often used to establish surjectivity results. Indeed, to prove that a map $f: X \to Y$ with the closed image property is surjective, it suffices to show that $f(X)$ is dense in $Y$. When the alphabet $A$ is finite, every cellular automaton $\tau: A^G \to A^G$ has the closed image property (with respect to the prodiscrete topology) by compactness of $A^G$. In the linear setting one has:

**Theorem 1.4.** Let $G$ be a group and let $V$ be a finite-dimensional vector space over a field $\mathbb{K}$. Then every linear cellular automaton $\tau: V^G \to V^G$ has the closed image property with respect to the prodiscrete topology on $V^G$.

This was proved in [2] when the group $G$ is countable, and then extended to any group $G$ in [4] (see also [6, Section 4.D] for more general results). As for the reversibility result mentioned above (cf. Theorem 1.1), we shall present here a new proof of Theorem 1.4 based on the Mittag-Leffler lemma for projective sequences of sets. Moreover, we shall also establish the following:

**Theorem 1.5.** Let $G$ be a non-periodic group and let $\mathbb{K}$ be a field. Let $V$ be an infinite-dimensional vector space over $\mathbb{K}$. Then there exists a linear cellular automaton $\tau': V^G \to V^G$ such that $\tau'(V^G)$ is not closed in $V^G$ with respect to the prodiscrete topology.

As above, this gives us:

**Corollary 1.6.** Let $G$ be a non-periodic group and let $A$ be an infinite set. Then there exists a cellular automaton $\tau': A^G \to A^G$ such that $\tau'(A^G)$ is not closed in $A^G$ with respect to the prodiscrete topology. □

The remainder of the paper is organized as follows. In Section 2, for the sake of completeness and the convenience of the reader, we briefly recall from [4] the definitions of induction and restriction of cellular automata and list some of their properties. Section 3 is devoted to the Mittag-Leffler lemma for projective sequences of sets. This set-theoretic version of the Mittag-Leffler lemma may be easily deduced from Theorem 1 in [1, TG II. Section 5] (see also [8, Section I.3]) but we present here a self-contained proof for the convenience of the reader. The proofs of Theorem 1.1 and Theorem 1.4 which are based on the Mittag-Leffler lemma are given in Section 4. Each proof is divided into two steps. We first establish the result in the case when $G$ is countable by means of the Mittag-Leffler lemma and then extend it to the general case by applying the first step to the cellular automaton obtained by restriction to the subgroup generated by a memory set. In Section 5, we give the proofs of Theorem 1.2 and Theorem 1.3. These are also divided into two steps: we first treat the case $G = \mathbb{Z}$ and then we use
the technique of induction to extend them to any non-periodic group. In the final section, we discuss the question whether the results from Section 5 extend to periodic groups which are not locally finite.

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2. Induction and restriction of cellular automata

In this section we recall the notions of induction and restriction for cellular automata (cf. [4]).

Let $G$ be a group, $A$ a set, and $H$ a subgroup of $G$.

Suppose that a cellular automaton $\tau: A^G \rightarrow A^G$ admits a memory set $M$ such that $M \subset H$. Let $\mu: A^M \rightarrow A$ denote the associated local defining map. Then the map $\tau_H: A^H \rightarrow A^H$ defined by

$$\tau_H(y)(h) = \mu(\pi_M(h^{-1}y)) \quad \text{for all } y \in A^H, h \in H,$$

is a cellular automaton over the group $H$ and the alphabet $A$ with memory set $M$ and local defining map $\mu$. One says that $\tau_H$ is the cellular automaton obtained by restriction of $\tau$ to $H$.

Conversely, let $\sigma: A^H \rightarrow A^H$ be a cellular automaton with memory set $M \subset H$ and local defining map $\mu: A^M \rightarrow A$. Then the map $\sigma^G: A^G \rightarrow A^G$ defined by

$$\sigma^G(x)(g) = \mu(\pi_M(g^{-1}x)) \quad \text{for all } x \in A^G, g \in G,$$

is a cellular automaton over the group $G$ and the alphabet $A$ with memory set $M$ and local defining map $\mu$. One says that $\sigma^G$ is the cellular automaton obtained by induction of $\sigma$ to $G$.

It immediately follows from their definitions that induction and restriction are operations one inverse to the other in the sense that one has $(\tau_H)^G = \tau$ and $(\sigma^G)_H = \sigma$ for every cellular automaton $\tau: A^G \rightarrow A^G$ over $G$ admitting a memory set contained in $H$ and every cellular automaton $\sigma: A^H \rightarrow A^H$ over $H$. We shall use the following results (see [H] Theorem 1.2 for proofs):

**Theorem 2.1.** Let $G$ be a group, $A$ a set, and $H$ a subgroup of $G$. Let $\tau: A^G \rightarrow A^G$ be a cellular automaton admitting a memory set contained in $H$. Then the following holds.

(i) $\tau$ is bijective if and only if $\tau_H$ is bijective;

(ii) $\tau$ is reversible if and only if $\tau_H$ is reversible;

(iii) $\tau(A^G)$ is closed in $A^G$ if and only if $\tau_H(A^H)$ is closed in $A^H$ (for the prodiscrete topology on $A^H$);

(iv) when $A$ is a vector space, $\tau$ is linear if and only if $\tau_H$ is linear.
3. The Mittag-Leffler lemma

In this section, we give the proof of the version of the Mittag-Leffler lemma that we shall use in the next section in order to establish Theorem 1.1 and Theorem 1.4. Let us first recall a few facts about projective limits of projective sequences in the category of sets.

Denote by \( \mathbb{N} \) the set of nonnegative integers. A projective sequence of sets is a sequence \((X_n)_{n \in \mathbb{N}}\) of sets equipped with maps \( f_{nm} : X_m \to X_n \), defined for all \( n, m \in \mathbb{N} \) with \( m \geq n \), satisfying the following conditions:

(PS-1) \( f_{nm} \) is the identity map on \( X_n \) for all \( n \in \mathbb{N} \);
(PS-2) \( f_{nk} = f_{nm} \circ f_{mk} \) for all \( n, m, k \in \mathbb{N} \) such that \( k \geq m \geq n \).

We denote such a projective sequence by \((X_n, f_{nm})\) or simply by \((X_n)\).

The projective limit \( \lim \lim X_n \) of the projective sequence \((X_n, f_{nm})\) is the subset of \( \prod_{n \in \mathbb{N}} X_n \) consisting of the sequences \((x_n)_{n \in \mathbb{N}}\) satisfying \( x_n = f_{nm}(x_m) \) for all \( n, m \in \mathbb{N} \) such that \( m \geq n \).

One says that the projective sequence \((X_n)\) satisfies the Mittag-Leffler condition if the following holds:

(ML) for each \( n \in \mathbb{N} \), there exists \( m \in \mathbb{N} \) with \( m \geq n \) such that \( f_{nk}(X_k) = f_{nm}(X_m) \) for all \( k \geq m \).

Lemma 3.1 (Mittag-Leffler). Let \((X_n, f_{nm})\) be a projective sequence of nonempty sets satisfying the Mittag-Leffler condition. Then its projective limit \( X = \lim X_n \) is not empty.

Proof. We first observe that given an arbitrary projective sequence of sets \((X_n, f_{nm})\), then Property (PS-2) implies that, for each \( n \in \mathbb{N} \), the sequence of sets \( f_{nm}(X_m) \), \( m \geq n \), is non-increasing. Let us set \( X'_n = \bigcap_{m \geq n} f_{nm}(X_m) \) (this is called the set of universal elements in \( X_n \), cf. [2]). The map \( f_{nm} \) clearly induces by restriction a map \( g_{nm} : X'_m \to X'_n \) for all \( m \geq n \). Then \((X'_n, g_{nm})\) is a projective sequence having the same projective limit as the projective sequence \((X_n, f_{nm})\).

Suppose now that all the sets \( X_n \) are nonempty and that the projective sequence \((X_n, f_{nm})\) satisfies the Mittag-Leffler condition. This means that, for each \( n \in \mathbb{N} \), there is an integer \( m \geq n \) such that \( f_{nk}(X_k) = f_{nm}(X_m) \) for all \( k \geq m \). This implies \( X'_n = f_{nm}(X_m) \) so that, in particular, the set \( X'_n \) is not empty. We claim that the map \( g_{n,n+1} : X'_{n+1} \to X'_n \) is surjective for every \( n \in \mathbb{N} \). To see this, let \( n \in \mathbb{N} \) and \( x'_n \in X'_n \). By the Mittag-Leffler condition, we can find an integer \( p \geq n + 1 \) such that \( f_{nk}(X_k) = f_{np}(X_p) \) and \( f_{n+1,k}(X_k) = f_{n+1,p}(X_p) \) for all \( k \geq p \). It follows that \( X'_n = f_{np}(X_p) \) and \( X'_{n+1} = f_{n+1,p}(X_p) \). Consequently, we can find \( x_p \in X_p \) such that \( x'_n = f_{np}(x_p) \). Setting \( x'_{n+1} = f_{n+1,p}(x_p) \), we have \( x'_{n+1} \in X'_{n+1} \) and

\[ g_{n,n+1}(x'_{n+1}) = f_{n,n+1}(x'_{n+1}) = f_{n,n+1} \circ f_{n+1,p}(x_p) = f_{np}(x_p) = x'_n. \]

This proves our claim that \( g_{n,n+1} \) is onto. As the sets \( X'_n \) are nonempty, we can now construct by induction a sequence \((x'_n)_{n \in \mathbb{N}}\) such that \( x'_n =
Proof of Theorem 1.1. Let $g_{n,n+1}(x'_{n+1})$ for all $n \in \mathbb{N}$. This sequence is in the projective limit $\lim_{n} X'_{n} = \lim_{n} X_{n}$. This shows that $\lim_{n} X_{n}$ is not empty. \qed

4. Reversibility and the Closed Image Property in the Finite-Dimensional Case

This section contains the proofs of Theorem 1.1 and Theorem 1.4 based on the Mittag-Leffler lemma for projective sequence of sets.

Proof of Theorem 1.1. Let $\tau : V^{G} \to V^{G}$ be a bijective linear cellular automaton. We have to show that $\tau$ is reversible. We split the proof into two steps.

Suppose first that the group $G$ is countable. Since $\tau$ is linear and $G$-equivariant (cf. (1.2)), the inverse map $\tau^{-1} : V^{G} \to V^{G}$ is linear and $G$-equivariant as well. Let us show that the following local property is satisfied by $\tau^{-1}$: there exists a finite subset $N \subseteq G$ such that

\[ (*) \text{ for } y \in V^{G}, \text{ the element } \tau^{-1}(y)(1_G) \text{ only depends on the restriction of } y \text{ to } N. \]

This will show that $\tau$ is reversible. Indeed, if (*) holds for some finite subset $N \subseteq G$, then there exists a (unique) map $\nu : V^{N} \to V$ satisfying

\[ \tau^{-1}(y)(1_G) = \nu(\pi_{N}(y)). \]

From the $G$-equivariance of $\tau^{-1}$ we then deduce

\[ \tau^{-1}(y)(g) = g^{-1}\tau^{-1}(y)(1_G) = \tau^{-1}(g^{-1}y)(1_G) = \nu(\pi_{N}(g^{-1}y)). \]

for all $y \in V^{G}$, which implies that $\tau^{-1}$ is the cellular automaton with memory set $N$ and local defining map $\nu$.

Let us assume by contradiction that there exists no finite subset $N \subseteq G$ satisfying condition (*). Let $M$ be a memory set for $\tau$ such that $1_G \in M$. Since $G$ is countable, we can find a sequence $(A_{n})_{n \in \mathbb{N}}$ of finite subsets of $G$ such that $G = \bigcup_{n \in \mathbb{N}} A_{n}$, $M \subseteq A_{0}$ and $A_{n} \subseteq A_{n+1}$ for all $n \in \mathbb{N}$. Let $B_{n} = \{ g \in G : gM \subseteq A_{n} \}$. Note that $G = \bigcup_{n \in \mathbb{N}} B_{n}$, $1_G \in B_{0}$, and $B_{n} \subseteq B_{n+1}$ for all $n \in \mathbb{N}$.

Since there exists no finite subset $N \subseteq G$ satisfying condition (*), we can find, for each $n \in \mathbb{N}$, two configurations $y'_{n}, y''_{n} \in V^{G}$ such that $y'_{n}|_{B_{n}} = y''_{n}|_{B_{n}}$ and $\tau^{-1}(y'_{n})(1_G) \neq \tau^{-1}(y''_{n})(1_G)$. By linearity of $\tau^{-1}$, the configuration $y_{n} = y'_{n} - y''_{n} \in V^{G}$ satisfies

\[ y_{n}|_{B_{n}} = 0 \quad \text{and} \quad \tau^{-1}(y_{n})(1_G) \neq 0. \]

(4.1)

It follows from (1.1) that if $x$ and $x'$ are elements in $V^{G}$ such that $x$ and $x'$ coincide on $A_{n}$ then the configurations $\tau(x)$ and $\tau(x')$ coincide on $B_{n}$. Therefore, given $x_{n} \in V^{A_{n}}$ and denoting by $\tilde{x}_{n} \in V^{G}$ a configuration extending $x_{n}$, the element

\[ u_{n} = \tau(\tilde{x}_{n})|_{B_{n}} \in V^{B_{n}} \]
does not depend on the particular choice of the extension \( \tilde{x}_n \) of \( x_n \). Thus we can define a map \( \tau_n : V^{A_n} \to V^{B_n} \) by setting \( \tau_n(x_n) = u_n \). It is clear that \( \tau_n \) is \( \mathbb{K} \)-linear.

Consider, for each \( n \in \mathbb{N} \), the subset \( X_n \subset V^{A_n} \) consisting of all \( x_n \in V^{A_n} \) such that \( x_n \in \text{Ker}(\tau_n) \) and \( x_n(1_G) \neq 0 \). Note that \( X_n \) is not empty since \( (\tau^{-1}(y_n))|_{A_n} \subset X_n \) by (4.1). Now observe that, for \( m \geq n \), the restriction map \( \rho_{nm} : V^{A_m} \to V^{A_n} \) is \( \mathbb{K} \)-linear and induces a map \( f_{nm} : X_m \to X_n \). Indeed, if \( u \in X_m \), then we have \( u|_{A_n} \in X_n \) since \( \tau_n(u|_{A_n}) = (\tau_m(u))|_{B_n} = 0 \) and \( (u|_{A_n})(1_G) = u(1_G) \neq 0 \). Conditions (PS-1) and (PS-2) are trivially satisfied so that \( (X_n, f_{nm}) \) is a projective sequence of nonempty sets. Let us show that \( (\tilde{x}_n, f_{nm}) \) satisfies the Mittag-Leffler condition (ML). Consider, for all \( m \geq n \), the set \( f_{nm}(X_m) \subset X_n \subset V^{A_n} \). By definition, we have that \( f_{nm}(X_m) = \rho_{nm}(\text{Ker}(\tau_m)) \cap X_n \). Observe now that, if \( n \leq m \leq m' \), then \( \rho_{nm}(\text{Ker}(\tau_m)) \subset \rho_{nm}(\text{Ker}(\tau_{m'})) \). Therefore, if we fix \( n \), the sequence \( \rho_{nm}(\text{Ker}(\tau_m)) \), where \( m = n, n + 1, \ldots, \) is a non-increasing sequence of vector subspaces of \( V^{A_n} \). As the vector space \( V^{A_n} \) is finite-dimensional, this sequence stabilizes, and it follows that, for each \( n \in \mathbb{N} \), there exists an integer \( m \geq n \) such that \( f_{nk}(X_k) = f_{nm}(X_m) \) if \( k \geq m \). This shows that condition (ML) is satisfied. We then deduce from Lemma [5.1] that the projective limit \( X = \lim X_n \) is not empty. Let \( (z_n) \in X \). Then there exists a unique \( z \in V^G \) such that \( \pi_{A_n}(z) = z_n \) for all \( n \in \mathbb{N} \). But \( \tau(z) = 0 \) since \( \pi_{B_n}(\tau(z)) = \tau_n(z_n) = 0 \) for all \( n \) and \( z(1_G) = z_0(1_G) \neq 0 \). This contradicts the injectivity of \( \tau \).

This shows, that there exists a finite subset \( N \subset G \) satisfying (*) and therefore that \( \tau \) is reversible.

We now drop the countability assumption on \( G \) and prove the theorem in the general case. Choose a memory set \( M \subset G \) for \( \tau \) and denote by \( H \) the subgroup of \( G \) generated by \( M \). Observe that \( H \) is countable since \( M \) is finite. By assertions (i) and (iv) of Theorem [2.1], the restriction cellular automaton \( \tau_H : V^H \to V^H \) is linear and bijective. It then follows from the previous step that \( \tau_H \) is reversible (that is, the inverse map \( (\tau_H)^{-1} : V^H \to V^H \) is a cellular automaton). By applying assertion (ii) of Theorem [2.1], we conclude that \( \tau \) is also reversible. \( \square \)

**Proof of Theorem [1.4]** Let \( \tau : V^G \to V^G \) be a linear cellular automaton. We have to show that \( \tau \) has the closed image property with respect to the prodiscrete topology on \( V^G \). We split the proof into two steps as in the preceding proof.

Suppose first that the group \( G \) is countable. Choose, as in the first step of the preceding proof, a sequence \( (A_n)_{n \in \mathbb{N}} \) of finite subsets of \( G \) such that \( G = \bigcup_{n \in \mathbb{N}} A_n \), \( M \subset A_0 \) and \( A_n \subset A_{n+1} \), and consider, for each \( n \in \mathbb{N} \), the \( \mathbb{K} \)-linear map \( \tau_n : V^{A_n} \to V^{B_n} \), where \( B_n = \{ g \in G : gM \subset A_n \} \) and \( \tau_n \) is defined by \( \tau_n(x_n) = (\tau(x_n))|_{B_n} \) for all \( x_n \in V^{A_n} \) and \( \tilde{x}_n \in V^G \) extending \( x_n \).
Let now \( y \in V^G \) and suppose that \( y \) is in the closure of \( \tau(V^G) \). Then, for all \( n \in \mathbb{N} \), there exists \( z_n \in V^G \) such that
\[
\pi_{B_m}(y) = \pi_{B_m}(\tau(z_n)).
\]
Consider, for each \( n \in \mathbb{N} \), the affine subspace \( X_n \subset V^{A_n} \) defined by \( X_n = \tau_n^{-1}(\pi_{B_m}(y)) \). We have \( X_n \neq \emptyset \) for all \( n \) by (4.2). For \( m \geq n \), the restriction map \( V^{A_m} \to V^{A_n} \) induces an affine map \( f_{nm} : X_m \to X_n \). Conditions (PS-1) and (PS-2) are trivially satisfied so that \( (X_n, f_{nm}) \) is a projective sequence. We claim that \( (X_n, f_{nm}) \) also satisfies the Mittag-Leffler condition (ML). Indeed, consider, for all \( m \geq n \), the affine subspace \( f_{nm}(X_m) \subset X_n \). We have \( f_{nm'}(X_{m'}) \subset f_{nm}(X_m) \) for all \( n \leq m \leq m' \) since \( f_{nm'} = f_{nm} \circ f_{nm'} \). As the sequence \( f_{nm}(X_m) \) \((m = n, n + 1, \ldots )\) is a non-increasing sequence of finite-dimensional affine subspaces, it stabilizes, i.e., for each \( n \in \mathbb{N} \) there exists an integer \( m \geq n \) such that \( f_{nk}(X_k) = f_{nm}(X_m) \) if \( k \geq m \). Thus, condition (ML) is satisfied. It follows from Lemma [3.1] that the projective limit \( \lim_{\leftarrow} X_n \) is nonempty. Choose an element \( (x_n)_{n \in \mathbb{N}} \in \lim_{\leftarrow} X_n \). We have that \( x_{n+1} \) coincides with \( x_n \) on \( A_n \) and that \( x_n \in V^{A_n} \) for all \( n \in \mathbb{N} \). As \( G = \bigcup_{n \in \mathbb{N}} A_n \), we deduce that there exists a (unique) configuration \( x \in V^G \) such that \( x|_{A_n} = x_n \) for all \( n \). We have \( \tau(x)|_{B_n} = \tau_n(x_n) = y_n = y|_{B_n} \) for all \( n \). Since \( G = \bigcup_{n \in \mathbb{N}} B_n \), this shows that \( \tau(x) = y \). This completes the proof in the case that \( G \) is countable.

Let us treat now the case of an arbitrary (possibly uncountable) group \( G \). As in the second step of the preceding proof, choose a memory set \( M \subset G \) for \( \tau \) and consider the countable subgroup of \( G \) generated by \( M \). By the previous step, we have that the restriction cellular automaton \( \tau_H : V^H \to V^H \) has the closed image property, that is, \( \tau_H(V^H) \) is closed in \( V^H \) for the prodiscrete topology. By applying Theorem [2.1](iii), we deduce that \( \tau(V^G) \) is also closed in \( V^G \) for the prodiscrete topology. Thus \( \tau \) satisfies the closed image property. \( \square \)

5. Proofs of Theorem 1.2 and of Theorem 1.5

In this section we present the proofs of Theorem 1.2 and Theorem 1.5. In both cases, we divide the proofs into two steps: we first treat the case \( G = \mathbb{Z} \) and then we use the technique of induction to extend them to any non-periodic group.

Let \( K \) be a field and let \( V \) be an infinite-dimensional vector space over \( K \).

Since \( V \) is infinite-dimensional, we can find an infinite sequence \( (v_i)_{i \geq 1} \) of linearly independent vectors in \( V \). Let \( E \) denote the vector subspace spanned by all the vectors \( v_i \) and let \( F \) be a vector subspace of \( V \) such that \( V = E \oplus F \).

Let us first construct a bijective linear cellular automaton \( \sigma : V^{\mathbb{Z}} \to V^{\mathbb{Z}} \) which is not reversible.
For each $j \geq 1$, denote by $E_j$ the vector subspace spanned by the vectors $v_i$, where $(j-1)j/2+1 \leq i \leq j(j+1)/2$. We have $E = \bigoplus_{j \geq 1} E_j$ and $\dim_K(E_j) = j$ for all $j \geq 1$.

For each $j \geq 1$, let $\varphi_j : E_j \to E_j$ denote the unique $K$-linear map such that

$$\varphi_j(v_i) = \begin{cases} 0 & \text{if } i = (j-1)j/2 + 1, \\ v_{i-1} & \text{if } (j-1)j/2 + 2 \leq i \leq j(j+1)/2. \end{cases}$$

Observe that $\varphi_j$ is nilpotent of degree $j$.

Consider now the maps $\sigma_j : E_j^Z \to E_j^Z$ defined by

$$\sigma_j(x_j)(n) = x_j(n) - \varphi_j(x_j(n+1))$$

for all $x_j \in E_j^Z$ and $n \in \mathbb{Z}$.

We define the map $\sigma : V^Z \to V^Z$ by

$$\sigma = \left( \bigoplus_{j \geq 1} \sigma_j \right) \oplus \text{Id}_{F^Z},$$

where we use the natural identification $V^Z = \left( \bigoplus_{j \geq 1} E_j^Z \right) \oplus F^Z$ and denote by $\text{Id}_X$ the identity map on a set $X$.

**Lemma 5.1.** The map $\sigma : V^Z \to V^Z$ is a linear cellular automaton which is bijective but not reversible.

**Proof.** The $K$-linearity of the maps $\sigma_j$ and hence of $\sigma$ is straightforward from their definition. It is also clear that $\sigma$ is a cellular automaton admitting $S = \{0, 1\}$ as a memory set. The associated local defining map is the map $\mu : V^S \to V$ given by

$$\mu = \left( \bigoplus_{j \geq 1} \mu_j \right) \oplus \text{Id}_{F^S},$$

where $\mu_j : E_j^S \to E_j^S$ is defined by $\mu_j(u_0, u_1) = u_0 - \varphi_j(u_1)$ for all $(u_0, u_1) \in E_j^S = E_j \times E_j$.

Consider the map $\nu_j : E_j^Z \to E_j^Z$ defined by

$$(5.1) \quad \nu_j(x_j)(n) = \sum_{k=0}^{j-1} \varphi_j^k(x_j(n+k)) \quad \text{for all } x_j \in E_j^Z \text{ and } n \in \mathbb{Z},$$

where $\varphi_j^0 = \text{Id}_{E_j}$ and $\varphi_j^{k+1} = \varphi_j \circ \varphi_j^k$ for all $k \geq 0$. Using the fact that $\varphi_j$ is nilpotent of degree $j$, one immediately checks that

$$\sigma_j \circ \nu_j = \nu_j \circ \sigma_j = \text{Id}_{E_j^Z}.$$
Thus \( \sigma_j \) is bijective with inverse map \( \nu_j \). This implies that \( \sigma \) is bijective with inverse map

\[
\sigma^{-1} = \left( \bigoplus_{j \geq 1} \nu_j \right) \oplus \text{Id}_{\mathbb{Z}}.
\]

If \( \sigma^{-1} \) was a cellular automaton, there would be a finite subset \( M \subset \mathbb{Z} \) such that \( \sigma^{-1}(x)(0) \) would only depend of the restriction of \( x \in V^\mathbb{Z} \) to \( M \). To see that this is impossible, suppose that there is such an \( M \) and choose an integer \( j_0 \geq 1 \) large enough so that \( M \subset (-\infty; j_0 - 2] \).

Consider the configurations \( y, z \in V^\mathbb{Z} \) defined by \( y(n) = 0 \) for all \( n \in \mathbb{Z} \) and \( z(n) = \begin{cases} v_{j_0(j_0-1)/2} & \text{if } n = j_0 - 1, \\ 0 & \text{if } n \in \mathbb{Z} \setminus \{j_0 - 1\}. \end{cases} \)

Then \( y \) and \( z \) coincide on \( (-\infty; j_0 - 2] \) and hence on \( M \). However, \( \sigma^{-1}(y)(0) \neq \sigma^{-1}(z)(0) \) since \( \sigma^{-1}(y) = 0 \) while, by (5.2) and (5.1), we have

\[
\sigma^{-1}(z)(0) = \psi_{j_0}^{-1}(v_{j_0(j_0+1)/2}) = v_{(j_0-1)j_0/2+1} \neq 0.
\]

This shows that \( \sigma^{-1} \) is not a cellular automaton. Therefore \( \sigma \) is not reversible. \( \square \)

**Proof of Theorem 1.2.** Since \( G \) is non-periodic, we can find an element \( g_0 \in G \) with infinite order. Denote by \( H \) the infinite cyclic subgroup of \( G \) generated by \( g_0 \). We identify \( \mathbb{Z} \) with \( H \) via the map \( n \mapsto g_0^n \). Consider the cellular automaton \( \sigma : V^H \to V^H \) studied in Lemma 5.1. Then, by virtue of Lemma 5.1 and assertions (iv), (i) and (ii) in Theorem 2.1, the induced cellular automaton \( \tau = \sigma^G : V^G \to V^G \) is linear, bijective and not reversible. \( \square \)

We now construct a linear cellular automaton \( \sigma' : V^\mathbb{Z} \to V^\mathbb{Z} \) such that \( \sigma'(V^\mathbb{Z}) \) is not closed in \( V^\mathbb{Z} \) with respect to the prodiscrete topology.

Let \( \psi : V \to V \) denote the unique \( \mathbb{K} \)-linear map satisfying \( \psi(v_i) = v_{i+1} \) for all \( i \geq 1 \), and whose restriction to \( F \) is identically 0. Define the map \( \sigma' : V^\mathbb{Z} \to V^\mathbb{Z} \) by

\[
\sigma'(x)(n) = x(n+1) - \psi(x(n))
\]

for all \( x \in V^\mathbb{Z} \) and \( n \in \mathbb{Z} \).

**Lemma 5.2.** The map \( \sigma' : V^\mathbb{Z} \to V^\mathbb{Z} \) is a linear cellular automaton and \( \sigma'(V^\mathbb{Z}) \) is not closed in \( V^\mathbb{Z} \).

**Proof.** The \( \mathbb{K} \)-linearity of \( \sigma' \) immediately follows from the \( \mathbb{K} \)-linearity of \( \psi \). It is also clear that \( \sigma' \) is a cellular automaton admitting \( S = \{0, 1\} \) as a memory set. The associated local defining map is the map \( \mu' : V^S \to V \) given by \( \mu'(u_0, u_1) = u_1 - \psi(u_0) \) for all \( (u_0, u_1) \in V^S = V \times V \).

Let \( c \in V^\mathbb{Z} \) denote the constant configuration defined by \( c(n) = v_1 \) for all \( n \in \mathbb{Z} \). Let us show that \( c \) is in the closure of \( \sigma'(V^\mathbb{Z}) \). By
definition of the prodiscrete topology, it suffices to show that for each finite subset $F \subset \mathbb{Z}$, there exists a configuration $x_F \in V^Z$ such that $c$ and $\sigma'(x_F)$ coincide on $F$. To see this, choose an integer $n_0 \in \mathbb{Z}$ such that $F \subset [n_0, \infty)$. Consider the configuration $x_F \in V^Z$ defined by

$$x_F(n) = \begin{cases} 0 & \text{if } n \leq n_0 - 1, \\ v_1 + v_2 + \cdots + v_{n-n_0+1} & \text{if } n \geq n_0. \end{cases}$$

Observe that $\sigma'(x_F)(n) = v_1$ for all $n \geq n_0$, so that the configurations $\sigma'(x_F)$ and $c$ coincide on $[n_0, \infty)$ and hence on $F$. This shows that $c$ is in the closure of $\sigma'(V^Z)$.

However, $c$ is not in the image of $\sigma'$. Indeed, suppose on the contrary that $c = \sigma'(x)$ for some $x \in V^Z$. This means that $x(n+1) = v_1 + \psi(x(n))$ for all $n \in \mathbb{Z}$. By induction, we get

$$x(n) = v_1 + v_2 + \cdots + v_i + \psi^i(x(n-i))$$

for all $n \in \mathbb{Z}$ and $i \geq 1$. As the image of $\psi^i$ is the vector subspace of $V$ spanned by the vectors $v_{i+1}, v_{i+2}, \ldots$, this implies that $x(n) \in E$ and that the $i$-th coordinate of $x(n)$ in the basis of $E$ formed by the sequence $(v_i)_{i \geq 1}$ is equal to 1. This gives us a contradiction since the number of nonzero coordinates of a vector with respect to a given basis must always be finite. Therefore, $c$ does not belong to the image of $\sigma'$. This shows that $\sigma'(V^Z)$ is not closed in $V^Z$. \hfill \Box

Proof of Theorem 1.5. Suppose that $G$ is not periodic and let $H \subset G$ be an infinite cyclic subgroup. Consider the cellular automaton $\sigma': V^H \to V^H$ studied in Lemma 5.2. By virtue of this lemma and assertions (iv) and (iii) in Theorem 2.1, the cellular automaton $\tau': V^G \to V^G$ induced by $\sigma'$ is linear and its image $\tau'(V^G)$ is not closed in $V^G$. \hfill \Box

6. Concluding remarks and questions

Recall that a group is called locally finite if all its finitely generated subgroups are finite. It is clear that every locally finite group is periodic.

Proposition 6.1. Let $G$ be a locally finite group and let $A$ be a set. Then:

(i) every bijective cellular automaton $\tau: A^G \to A^G$ is reversible;

(ii) every cellular automaton $\tau: A^G \to A^G$ has the closed image property.

Proof. Let $\tau: A^G \to A^G$ be a cellular automaton and let $M \subset G$ be a memory set for $\tau$. Denote by $H$ the subgroup of $G$ generated by $M$ and consider the cellular automaton $\tau_H: A^H \to A^H$ over $H$ obtained by restriction of $\tau$.

Since $H$ is finite, the prodiscrete topology on $A^H$ coincides with the discrete topology. As every subset of a discrete topological space is
closed, it follows that $\tau_H(A^H)$ is closed in $A^H$. We deduce that $\tau(A^G)$ is closed in $A^G$ by using assertion (iii) in Theorem 2.1. This shows (ii).

Suppose now that $\tau$ is bijective. Then $\tau_H$ is bijective by assertion (i) in Theorem 2.1. As $H$ is finite, every map $f : A^H \to A^H$ which commutes with the $H$-shift is a cellular automaton over $H$ (with memory set $H$ and local defining map $f$). It follows that $\tau_H^{-1}$ is a cellular automaton and hence $\tau_H$ is reversible. By applying Theorem 2.1(ii), we conclude that $\tau$ is reversible. This shows (i). □

In view of the preceding proposition, it is natural to ask whether the results presented in Section 1 extend to periodic groups which are not locally finite. Note that, by Theorem 2.1, it suffices to consider finitely-generated infinite periodic groups. Examples of such groups are provided by the free Burnside groups $B(m,n)$ on $m \geq 2$ generators with large odd exponent $n$ and the celebrated Grigorchuk group [5] which is a infinite 2-group generated by 3 involutions.

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