ON THE ASYMPTOTIC PLATEAU’S PROBLEM FOR CMC HYPERSURFACES ON RANK 1 SYMMETRIC SPACES OF NONCOMPACT TYPE

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Abstract. Let $M^n$, $n \geq 3$, be a Hadamard manifold with strictly negative sectional curvature $K_M \leq -\alpha$, $\alpha > 0$. Assume that $M$ satisfies the strict convexity condition at infinity according to [18] (see also the definition below) and, additionally, that $M$ admits a helicoidal one parameter subgroup $\{\varphi_t\}$ of isometries (i.e. there exists a geodesic $\gamma$ of $M$ such that $\varphi_t(\gamma(s)) = \gamma(t + s)$ for all $s, t \in \mathbb{R}$). We then prove that, given a compact topological $\{\varphi_t\}$-shaped hypersurface $\Gamma$ in the asymptotic boundary $\partial_\infty M$ of $M$ (that is, the orbits of the extended action of $\{\varphi_t\}$ to $\partial_\infty M$ intersect $\Gamma$ at one and only one point), and given $H \in \mathbb{R}$, $|H| < \sqrt{\alpha}$, there exists a complete properly embedded constant mean curvature (CMC) $H$ hypersurface $S$ of $M$ such that $\partial_\infty S = \Gamma$.

This result extends Theorem 1.8 of B. Guan and J. Spruck [11] to more general ambient spaces, as rank 1 symmetric spaces of noncompact type, and allows $\Gamma$ to be $\{\varphi_t\}$-shaped with respect to more general one parameter subgroup of isometries $\{\varphi_t\}$ of the ambient space. For example, in $\mathbb{H}^n$, $\Gamma$ can be loxodromic -shaped, where loxodromic is a curve in $S^{n-1} = \partial_\infty \mathbb{H}^n$ that makes a constant angle with a family of circles connecting two points of $S^{n-1}$.

A fundamental result used to prove our main theorem, which has interest on its own, is the extension of the interior gradient estimates for CMC Killing graphs proved in Theorem 1 of [7] to CMC graphs of Killing submersions.

1. Introduction

Let $M^n$ be a Cartan-Hadamard manifold (namely a simply connected, complete Riemannian manifold with nonpositive sectional curvature) of dimension $n \geq 3$.

The asymptotic boundary $\partial_\infty M$ of $M$ is defined as the set of all equivalence classes of unit speed geodesic rays in $M$; two such rays $\gamma_1, \gamma_2 : [0, \infty) \to M$ are equivalent if $\sup_{t \geq 0} d(\gamma_1(t), \gamma_2(t)) < \infty$, where

\[ d(\gamma_1(t), \gamma_2(t)) = \inf\{d(\eta(t), \gamma_2(t)) \mid \eta \text{ is a minimizing geodesic in } M \text{ joining } \gamma_1(t) \text{ and } \gamma_2(t) \}. \]
$d$ is the Riemannian distance in $M$. The so called geometric compactification $\overline{M}$ of $M$ is then given by $\overline{M} := M \cup \partial_\infty M$, endowed with the cone topology (see [9] or [19], Ch. 2). For any subset $S \subset M$, we define $\partial_\infty S = \partial_\infty M \cap S$.

The asymptotic Plateau problem for $k \geq 2$ dimensional area minimizing submanifolds in $M$ consists in finding, for a given a $k-1$ dimensional, closed, topological submanifold $\Gamma$ of $\partial_\infty M$, a locally area minimizing, complete submanifold $S^k$ of $M$ such that $\partial_\infty S = \Gamma$.

By using methods from the Geometric Measure Theory, this problem was first studied in the hyperbolic space by M.T. Anderson [2] and his results extended to Gromov hyperbolic manifolds by U. Lang and V. Bangert ([3], [13], [14]).

Within the framework of the classical Plateau problem, the second author of the present paper with F. Tomi [20] study the asymptotic problem for minimal disk type surfaces in a general Hadamard manifold $M$.

In codimension 1, given $H \in \mathbb{R}$, we may consider the asymptotic Plateau’s problem for the constant mean curvature (CMC) $H$ hypersurface in $M$, namely, given a compact topological hypersurface $\Gamma \subset \partial_\infty M$, find a complete CMC $H$ hypersurface $S$ of $M$ ($H-$hypersurface, for short) such that $\partial_\infty S = \Gamma$. This problem has also attracted the attention of many mathematicians more recently. The results of M.T. Anderson [2] have been extended to the CMC case by Y. Tonegawa [21] and H. Alencar and H. Rosenberg in [1].

Both Geometric Measure theory and Plateau’s technique are methods that lead, in general, to the existence of hypersurfaces with singularities. Thus, a natural question, raised by B. Guan and J. Spruck in [11], asks about the existence of a smooth constant mean curvature hypersurface asymptotic to $\Gamma$ at infinity in $\mathbb{H}^n$. This problem in fact had already been studied earlier in the minimal case by F. H. Lin [15].

A way to obtain smooth solutions is by finding a suitable system of coordinates in order to write the hypersurface as a graph, and then to use standard elliptic PDE methods. In [15], F.H. Lin represented the hypersurfaces in the half space model of $\mathbb{H}^n$ as vertical graphs, that is, in the usual way of $\mathbb{R}^n_+$ when using the cartesian system of coordinates.

The results of F.H. Lin [15] were extended to the CMC case by B. Nelli and J. Spruck in [16] where they proved the existence of a smooth CMC $|H| < 1$ hypersurface in the hyperbolic space $\mathbb{H}^n$ with sectional curvature $-1$ if $\Gamma$ is assumed to be convex and compact. Later, also using PDE’s techniques, B. Guan and J. Spruck [11] (see also [8] for a different approach based on a variational method) improved
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the convexity condition by requiring a starshaped property of \( \Gamma \). We refer the reader to the nice survey of B. Coskunuzer [5], where the references of many other closely related papers to this subject can be found.

In both papers [16] and [11] the authors used the underlying Euclidean structure of the half space model for \( \mathbb{H}^n \) to state the convexity and starshaped properties of \( \Gamma \). However, although the convexity is not an intrinsic notion of the hyperbolic geometry, the starshapness of \( \Gamma \) is. It can be formulated in intrinsic terms using the conformal structure of \( \mathbb{H}^n \) by requiring \( \Gamma \) to be “circle shaped”, meaning that there are two points \( p_1, p_2 \in S^{n-1} = \partial_\infty \mathbb{H}^n \) such that any arc of circle from \( p_1 \) to \( p_2 \) intersects \( \Gamma \) at one and only one point. A limit circle shaped condition, where \( p_1 = p_2 \), was also introduced and used by the second author in [17] to ensure the existence of a smooth CMC hypersurface having \( \Gamma \) as asymptotic boundary (see the Introduction and Theorem 6 of [17] for a detailed description of this case).

In the present paper we extend Theorem 1.8 of [11] in two directions. First, we allow \( \Gamma \) to be “shaped” with respect to a more general one parameter subgroup of conformal diffeomorphisms of \( S^{n-1} = \partial_\infty \mathbb{H}^n \). Secondly, we allow the ambient space to be any rank 1 symmetric space of noncompact type. Both results are consequences of a more general theorem that holds in a Hadamard manifold endowed with some special Killing field.

As we shall see in the proof ahead, the Killing field allows to introduce a special system of coordinates which is quite suitable for using standard elliptic PDE techniques. To write down precise statements we first introduce some general notions and terminology.

Let \( \gamma : (-\infty, \infty) \to M \) be an arc length geodesic. We say that a one parameter subgroup of isometries \( \{ \varphi_t^\gamma \} \) of \( M \) associated to \( \gamma \) is helicoidal if \( \varphi_t^\gamma (\gamma(s)) = \gamma(t+s) \) for all \( s, t \in \mathbb{R} \). In the sequel, since there is no possibility of confusion, we shall omit the dependance of \( \{ \varphi_t^\gamma \} \) with respect to the geodesic \( \gamma \).

Let us illustrate the previous definition with a simple case that justifies this terminology: If \( M = \mathbb{R}^3 \) then any helicoidal one parameter subgroup of isometries, up to a conjugation, is of the form

\[
\varphi_t^\gamma (x, y, z) = \left( \begin{array}{ccc} \cos at & \sin at & x \\ -\sin at & \cos at & y \\ 0 & 0 & z + t \end{array} \right)
\]

for some \( a \in \mathbb{R} \). When \( a = 0 \), \( \{ \varphi_t \} \) is a one parameter subgroup of transvections along the \( z \)-axis. More generally, a one parameter subgroup of transvections along a geodesic in a symmetric space (see [12]) is a particular case of helicoidal one parameter subgroup of isometries.
Since the equivalence relation between geodesics and convergent sequences are preserved under isometries, the action of \( \{ \varphi_t \} \) on \( M \) extends to the compactification \( \overline{M} \) of \( M \) and the extended action is continuous. The orbits of \( \{ \varphi_t \} \) are the curves \( O(x) := \{ \varphi_t(x) \mid t \in \mathbb{R} \} \) where \( x \in \overline{M} \). Observe that \( \{ \varphi_t \} \) has two singular orbits in \( M \), namely, \( O(\gamma(\pm \infty)) \), where \( \gamma \) is the geodesic translated by \( \{ \varphi_t \} \).

Finally, we will also need to use the Strictly Convexity Condition ("SC condition") introduced in [20]. We say that \( M \) satisfies the SC condition if, given \( x \in \partial_{\infty} M \) and a relatively open subset \( W \subset \partial_{\infty} M \) containing \( x \), there exists a \( C^2 \) open set \( \Omega \subset M \) such that \( x \in \text{Int}(\partial_{\infty} \Omega) \subset W \) and \( M\setminus \Omega \) is convex, where \( \text{Int}(\partial_{\infty} \Omega) \) stands for the interior of \( \partial_{\infty} \Omega \) in \( \partial_{\infty} M \).

We are now in position to state our main result:

**Theorem 1.** Let \( M \) be a Hadamard manifold with sectional curvature \( K_M \leq -\alpha \), for some \( \alpha > 0 \), satisfying the SC condition. Let \( \{ \varphi_t \} \) be a helicoidal one parameter subgroup of isometries of \( M \). Let \( \Gamma \subset \partial_{\infty} M \) be a compact topological embedded \( \{ \varphi_t \} \)–shaped hypersurface of \( \partial_{\infty} M \), that is, any nonsingular orbit of \( \{ \varphi_t \} \) in \( \partial_{\infty} M \) intersects \( \Gamma \) at one and only one point. Then, given \( H \in \mathbb{R} \), \( |H| < \sqrt{\alpha} \), there exists a complete, properly embedded \( H \)–hypersurface \( S \) of \( M \) such that \( \partial_{\infty} S = \Gamma \). Moreover any orbit of \( \{ \varphi_t \} \) intersects \( S \) at one and only one point.

We point out that the SC condition is satisfied by a large class of manifolds. For example, if the sectional curvature is bounded from above by a strictly negative constant and decreases at most exponentially (see Theorem 14 of [18]) then the SC condition is satisfied. In particular, it is satisfied by any rank 1 symmetric spaces of noncompact type. Therefore, as an immediate consequence of the previous theorem, we obtain:

**Corollary 2.** Assume that \( M \) is a rank 1 symmetric space of noncompact type and assume that the sectional curvature of \( M \) is bounded by \( -\alpha \), \( \alpha > 0 \). Let \( \{ \varphi_t \} \) be a one parameter of transvections of \( M \). Let \( \Gamma \subset \partial_{\infty} M \) be a compact embedded topological \( \{ \varphi_t \} \)–shaped hypersurface of \( \partial_{\infty} M \). Then, given \( H \in \mathbb{R} \), \( |H| < \sqrt{\alpha} \), there exists a complete, properly embedded \( H \)–hypersurface \( S \) of \( M \) such that \( \partial_{\infty} S = \Gamma \). Moreover any orbit of \( \{ \varphi_t \} \) intersects \( S \) at one and only one point.

Finally we point out an interesting corollary of Theorem 1 in the case where \( M = \mathbb{H}^n \), the hyperbolic space of constant sectional curvature \( -1 \). Recalling that a loxodromic curve is a curve in \( S^{n-1} \) that intersects with a constant angle \( \theta \) any arc of circle of \( S^{n-1} \) connecting two fixed points of \( S^{n-1} \) (see [22]). These curves are induced by one-parameter
subgroups of isometries of $\mathbb{H}^n$ of heliodidal type. For example, in the
half space model $z > 0$ of $\mathbb{H}^3$, up to conjugation, they are of the form
$$\varphi_t(x, y, z) = e^{t\left(\begin{array}{cc} \cos \theta t & \sin \theta t \\ -\sin \theta t & \cos \theta t \end{array}\right)} \left(\begin{array}{c} x \\ z \end{array}\right).$$

**Corollary 3.** Let $0 \leq \theta < \pi/2$ and $p_1, p_2 \in S^{n-1} = \partial_\infty \mathbb{H}^n$ be two
distinct points of $S^{n-1}$. Let $L_\theta$ be the family of loxodromic curves that
intersects any arc of circle from $p_1$ to $p_2$ with a constant angle $\theta$. Let
$\Gamma \subset S^{n-1}$ be a compact embedded topological $L_\theta$-shaped hypersurface
of $S^{n-1}$. Then, given $H \in \mathbb{R}$, $|H| < 1$, there exists a complete, properly
embedded $H$-hypersurface $S$ of $\mathbb{H}^n$ such that $\partial_\infty S = \Gamma$.

We notice that, taking $\theta = 0$ in the previous corollary, we recover
Theorem 1.8 of [11]. Theorem 1.8 also follows from Corollary 2 since
radial graphs (considered in [11]) are transvections along a geodesic of $\mathbb{H}^n$.

A fundamental result for proving the above theorems, which has
interest on its own, are the interior gradient estimates of the solutions
of the CMC $H$ graph PDE for Killing submersions (see Theorem 4
below). It extends Theorem 1 of [7].

2. Proofs of the results

In what follows we use most of the nomenclature and the results
proved by M. Dajczer and J. H. de Lira in [6]. However, we introduce
the notion of a Killing graph on a manner slightly different from the
one considered in [6].

For the next result we allow $M$ be any Riemannian manifold and $Y$
a Killing field in $M$ without singularities. Denote by $O(x)$ the integral
curve (which we also call orbit) of $Y$ through a point $x \in M$. By a
complete $Y-$ Killing section (we shall refer only to a Killing section
because $Y$ will be fixed throughout the text) we mean a complete up
to the boundary (possibly empty) hypersurface $P$ of $M$ such that any
orbit $O(p)$ of $Y$ through a point $p$ of $P$ intersects $P$ only at $p$ and the
intersection is transversal. We call $\Omega := P \setminus \partial P$ a Killing domain. If
$P = \Omega$ is a hypersurface of class $C^{2,\alpha}$ in $M$ then we say that $\Omega$ is a
$C^{2,\alpha}$ Killing domain.

If $u$ is a function defined on a subset $T$ of $P$, the Killing graph of $u$
is given by
$$\text{Gr}(u) = \{\varphi(u(p), p) \mid p \in T\}$$
where $\varphi(s, x) = \varphi_s(x)$ is the flow of $Y$. In the sequel, $s$ will stand for
the flow parameter. We also set
$$\Gamma_T = \{\varphi(s, x) \mid x \in T \text{ and } s \in \mathbb{R}\}.$$
Next, we denote by $\Pi : M \rightarrow P$ the projection defined by $\Pi(x) = O(x) \cap P$. In all the sequel, we endow $P$ with the Riemannian metric $\langle \cdot, \cdot \rangle_P$ such that $\Pi$ becomes a Riemannian submersion.

Assume that $\Omega$ is a $C^{2,\alpha}$ Killing domain. Given $H \in \mathbb{R}$, it is not difficult to show that $\text{Gr}(u)$ has CMC $H$ with respect to the unit normal vector field $\eta$ to $\text{Gr}(u)$ such that $\langle Y, \eta \rangle \leq 0$ if and only if $u$ satisfies a certain second order quasi-linear elliptic PDE $Q_H[u] = 0$ on $M$ in terms of the metric $\langle \cdot, \cdot \rangle_\Pi$ in $P$ (for details, including an explicit expression of $Q_H$, see Section 2.1 of [6] or the short revision done below).

We may then refer to the CMC $H$ Dirichlet problem in a Killing domain $\Omega \subset M$ and for a given boundary data $\phi \in C^0(\partial\Omega)$ as the PDE boundary problem

\[ \begin{cases} 
Q_H[u] = 0 \text{ in } \Omega \\
|u| = \phi. 
\end{cases} \tag{1} \]

We begin by obtaining interior gradient estimates for the solutions of (1). Our result generalizes Theorem 1 of [7] to the case of CMC $H$ graph PDE of Killing submersions.

Fix a point $o \in \Omega$ and let $r > 0$ be such that $r < i(o)$, the injectivity radius of $M$ at $o$. We obtain the following result:

**Theorem 4.** Let $\Omega$ be a Killing domain in $M$. Let $o \in \Omega$ and $r > 0$ such that the open geodesic ball $B_r(o)$ is contained in $\Omega$. Let $u \in C^3(B_r(o))$ be a negative solution of $Q[u] = 0$ in $B_r(o)$. Then there is a constant $L$ depending only on $u(o)$, $r$, $|Y|$ and $H$ such that $|\nabla u(o)| \leq L$.

Before proving the above theorem, we review the nomenclature and some facts of [6].

We fix a local reference frame $v_1, \ldots, v_n$ on $\overline{\Omega}$ and we set $\sigma_{ij} = \langle v_i, v_j \rangle_\Pi$. We will now define a local frame in $M$. We denote by $D_1, \ldots, D_n$ the basic vector fields $\Pi$-related to $v_1, \ldots, v_n$. The frame $D_0, \ldots, D_n$, we considered on $M$, is defined by $D_0 = f^{1/2}\partial_s$, where $f = \frac{1}{|Y|^2}$, $(\partial_s(q) = \varphi_\ast(s, p)\partial_s(p))$, and $D_i(q) = \varphi_\ast(s, p)D_i(p)$, where $q = \varphi(s, p), p \in P$. We point out that the unit normal vector field to $\text{Gr}(u)$ pointing upward is given by

\[ N = \frac{1}{W}(f^{1/2}D_0 - \hat{u}^jD_j), \tag{2} \]

where $\hat{u}^j = \sigma^{ij}D_i(u - s)$ and $W^2 = f + \hat{u}^i\hat{u}_i = f + \sigma_{ij}\hat{u}^i\hat{u}^j$. We notice that $\hat{u}_i$ and $W$ are not depending on $s$ and therefore can be seen as function defined on $P$. Finally, using the previous notation,
the operator $Q$ (defined in (1)) can be written as

$$Q_H[u] = \frac{1}{W}(A^{ij}\hat{u}_{j;i} - \frac{(f + W^2)}{W^2} \langle \Pi_s \nabla D_0, D_0 \nabla D_0 \rangle) - nH,$$

where $\hat{u}_{i;j} = \langle \nabla D_i, \nabla (u - s), D_j \rangle$, $D_0 = \Pi_s \nabla (u - s)$ and $A^{ik} = \sigma^{ik} - \hat{u}^i \hat{u}^k W^2$.

Proof of Theorem 4. The proof will follow closely the one of Theorem 1 in [7]. Let $p \in B_r(o)$ be an interior point where $h = \eta W$ attains its maximum, where $\eta$ is a smooth function with support in $B_r(o)$ which will be determined in the sequel. In all this section, the computations will be done at the point $p$. Let $v_1, \ldots, v_n$ be an orthonormal tangent frame at $p \in B_r(o)$. Then we have $h_i = 0$ (where the derivative is taken with respect to $v_i$). This implies that

$$\eta_i W = -\eta W_i.$$

We also have, since $\frac{A^{ij}}{W}$ is definite positive, that

$$0 \geq \frac{1}{W} A^{ij} h_{ij} = \frac{1}{W} A^{ij} (W \eta_{i;j} + 2\eta_i W_j + \eta W_{i;j}).$$

Using (3), the previous inequality can be rewritten as

$$A^{ij} \eta_{i;j} + \frac{\eta}{W^2} A^{ij} (W W_{i;j} - 2W_i W_j) \leq 0.$$  

From (2), we have

$$N^k = -\frac{\hat{u}^k}{W}.$$  

Derivating $W$, we find

$$W_i = \frac{f_i}{2W} + \frac{\hat{u}^k \hat{u}_{k;i}}{W} = \frac{f_i}{2W} - N^k \hat{u}_{k;i}.$$

From (5), we get

$$N^k_{ij} = -\frac{\hat{u}^i_{j}}{W} + \frac{\hat{u}^k W_j}{W^2}.$$
Using the previous inequalities, we have

\[
W_{i;j} = \frac{f_{ij}}{2W} - \frac{f_i W_j}{2W^2} - N^k \hat{u}_{k;i} - N^k \hat{u}_{k;ij} \\
= \frac{f_{ij}}{2W} + \frac{\hat{u}_{j;ik} \hat{u}_{k;i}}{W} - \frac{f_i W_j}{2W^2} - N^k \hat{u}_{k;i} \\
= \frac{f_{ij}}{2W} + \frac{\hat{u}_{j;ik} \hat{u}_{k;i}}{W} - W_i W_j - N^k \hat{u}_{k;i} \\
= \frac{f_{ij}}{2W} + \frac{A_{kl}}{W} \hat{u}_{l;ij} \hat{u}_{k;i} + \frac{f_i f_j}{4W^3} - \frac{1}{2W^2} (W_i f_j + W_j f_i) - N^k \hat{u}_{k;ij}.
\]

Multiplying by \( A_{ij} \) the above equation and using (3), we find (6)

\[
A_{ij} W_{i;j} = \frac{A_{ij} f_{ij}}{2W} + \frac{A_{ij} A_{kl}}{W} \hat{u}_{l;ij} \hat{u}_{k;i} + \frac{A_{ij} f_i f_j}{4W^3} + \frac{1}{\eta W} A_{ij} \eta_i f_j - A_{ij} N^k \hat{u}_{k;ij}.
\]

In order to get rid of the term involving three derivatives of \( u \) in (6), we want to find a commutation formula for \( \hat{u}_{k;ij} \). We recall (see equation (11) of [6]) that

\[
\hat{u}_{k;i} = u_{k;i} - s_{k;i} + \frac{1}{2} \gamma_{ki},
\]

where \( \gamma_{ki} = f^k \langle [D_k, D_i], D_0 \rangle \). We deduce from the previous equality that

\[
\hat{u}_{k;ij} = u_{k;ij} - s_{k;ij} + \frac{1}{2} (\gamma_{ki})_j = u_{i;jk} + R^l_{kji} u_l - s_{k;ij} + \frac{1}{2} (\gamma_{ki})_j \\
= (\hat{u}_{j;ik} + s_{j;ik} - \frac{1}{2} \gamma_{ji})_k + R^l_{kji} u_l - s_{k;ij} + \frac{1}{2} (\gamma_{ki})_j \\
= \hat{u}_{j;ik} + s_{j;ik} - \frac{1}{2} (\gamma_{ji})_k + R^l_{kji} u_l - s_{k;ij} + \frac{1}{2} (\gamma_{ki})_j \\
= \hat{u}_{j;ik} + R^l_{kji} \hat{u}_l + R^l_{kji} s_l + s_{j;ik} - s_{k;ij} + \frac{1}{2} ((\gamma_{ki})_j - (\gamma_{ji})_k) \\
= \hat{u}_{j;ik} + R^l_{kji} \hat{u}_l + C_{ijk},
\]

where \( C_{ijk} = R^l_{kji} s_l + s_{j;ik} - s_{k;ij} + \frac{1}{2} ((\gamma_{ki})_j - (\gamma_{ji})_k) \) is not depending on \( u \). Using (11) and the commutation formula, the last term of (6)
Rewrites as
\[ A^{ij} N^k \hat{u}_{k;ij} = A^{ij} N^k \hat{u}_{j;ik} - \frac{A^{ij} R_{kji} \hat{u}_l \hat{u}^k}{W} - \frac{\hat{u}^k A^{ij} C_{ijk}}{W} \]
\[ = N^k (A^{ij} \hat{u}_{j;ik})_k - N^k A^{ij} \hat{u}_{i;ij} - \frac{A^{ij} R_{kji} \hat{u}_l \hat{u}^k}{W} - \frac{\hat{u}^k A^{ij} C_{ijk}}{W} \]
\[ = nN^k (WH)_k + N^k \left( \frac{(f + W^2)}{W^2} \langle \Pi_s \nabla D_0, Du \rangle \right)_k \]
\[-N^k A^{ij} \hat{u}_{i;ij} - \frac{A^{ij} R_{kji} \hat{u}_l \hat{u}^k}{W} - \frac{\hat{u}^k A^{ij} C_{ijk}}{W}.\]
Straightforward computations using (3) give
\[ (WH)_k = W_k H + WH_k = \frac{W}{\eta} (-\eta_k H + \eta H_k), \]
and
\[ (\frac{f + W^2}{W^2})_k = \frac{f_k}{W^2} - \frac{f}{W^4} (f_k + 2 \hat{u}^l \hat{u}_{l;k}) = \frac{1}{W^2} (f_k + 2 \frac{f \eta_k}{\eta}). \]
We also have, using (7),
\[ \left( \frac{(f + W^2)}{W^2} \langle \Pi_s \nabla D_0, Du \rangle \right)_k = \frac{(f + W^2)}{2W^2} \left[ (\frac{f_k f_k}{f} - f_k f_l) W N^l + f^l \hat{u}_{l;k} \right] \]
\[ + \langle \Pi_s \nabla D_0, Du \rangle \frac{1}{W^2} (f_k + 2 \frac{f \eta_k}{\eta}), \]
and
\[ A^{ij}_{k;k} = -\frac{1}{W^2} (\hat{u}_j \hat{u}^j + \hat{u}^j \hat{u}_j) + \frac{1}{W^4} (f_k - 2W \hat{u}^l \hat{u}_{l;k}) \hat{u}^i \hat{u}^i \]
\[ = \frac{1}{W^2} (\hat{u}_k N^i N^j + \hat{u}_j N^i N^j) + \frac{1}{W^2} (\hat{u}_k N^i N^j + \hat{u}_j N^i N^j) \]
\[ = \frac{1}{W^2} A^{il} \hat{u}_{l;k} N^j + \frac{1}{W^2} A^{jl} \hat{u}_{l;k} N^i + \frac{1}{W^2} f_k N^i N^j. \]
Multiplying the previous equality by \( N^k \hat{u}_{j;ij} \), we find
\[ N^k A^{ij}_{k;k} \hat{u}_{j;ij} = \frac{1}{W} N^k \hat{u}_{j;ij} \hat{u}_{l;k} (A^{il} N^j + A^{lj} N^i) + \frac{1}{W^2} f_k N^i N^j N^k \hat{u}_{j;ij}. \]
Recalling that
\[ N^k \hat{u}_{k;i} = \frac{f_i}{2W} + \frac{W \eta_i}{\eta}, \]
and
\[ \hat{u}_{i;j} = \hat{u}_{j;i} + \gamma_{ij}, \]
we have
\[ N^k A_{ij}^k \dot{u}_{ij} = \frac{1}{W^2} f_k N^k N^i \left( \frac{f_i}{2W} + \frac{W \eta_i}{\eta} \right) + \frac{1}{W} A^i (\frac{f_i}{2W} + \frac{W \eta_i}{\eta}) N^k (\dot{u}_{ki} + \gamma_{kl}) \]
\[ + \frac{1}{W} A^j N^k N^j (\dot{u}_{ij} + \gamma_{ji})(\dot{u}_{ki} + \gamma_{kl}) \]
\[ = \frac{1}{W^2} f_k N^k N^i \left( \frac{f_i}{2W} + \frac{W \eta_i}{\eta} \right) + \frac{1}{W} A^i (\frac{f_i}{2W} + \frac{W \eta_i}{\eta}) \left( \frac{f_i}{2W} + \frac{W \eta_i}{\eta} \right) \]
\[ + \frac{1}{W} A^j N^k N^j \gamma_{ji} \gamma_{kl} + \frac{3}{W} A^i (\frac{f_i}{2W} + \frac{W \eta_i}{\eta}) N^k \gamma_{lk}, \]
and
\[ N^k f_i \dot{u}_{tk} = N^k f_i (\dot{u}_{kt} - \gamma_{kl}) = \frac{f}{2W} \sigma_{kl} f_k f_i + \frac{W}{\eta} f_i \gamma_{kl} - \gamma_{kl} N^k f_i. \]

Using the previous computations, we deduce that the last term of (9) can be rewritten as
\[ A_{ij}^k N^k \dot{u}_{ij} = n N^k \frac{W}{\eta} (-\eta_k H + \eta H_k) - 2 \frac{W}{\eta} A^i (\frac{f_i}{2W} + \frac{W \eta_i}{\eta}) \left( \frac{f_i}{2W} + \frac{W \eta_i}{\eta} \right) \]
\[ - \frac{1}{W} f_k N^k N^i \left( \frac{f_i}{2W} + \frac{W \eta_i}{\eta} \right) - \frac{1}{W} A^j N^k N^j \gamma_{ji} \gamma_{lk} \]
\[ - \frac{3}{W} A^i (\frac{f_i}{2W} + \frac{W \eta_i}{\eta}) N^k \gamma_{lk} - \frac{A^i R_{kji}^l \dot{u}_{il} \dot{u}^k}{W} \]
\[ - \frac{\dot{u}^k A^i C_{ij} \dot{u}_{jk}}{W} - \left( \frac{f}{2W} \sigma_{kl} + W N^k N^l \right) f_k f_i f - W N^k N^l f_k i + \frac{W}{\eta} f_i \gamma_{kl} - \gamma_{kl} N^k f_i. \]

Thus, from (9), we obtain
\[ A^i W j - \frac{2}{W} A^i W j \]
\[ = \frac{3}{4W^3} A^i f_j f_j + \frac{1}{W} A^i A^k \dot{u}_{ij} \dot{u}_{kl} + \frac{3}{W \eta} A^i f_i \eta_j + \frac{1}{2W} A^i f_i j \]
\[ - n N^k \frac{W}{\eta} (-\eta_k H + \eta H_k) + \frac{1}{W} f_k N^k N^i \left( \frac{f_i}{2W} + \frac{W \eta_i}{\eta} \right) + \frac{1}{W} A^i N^k N^j \gamma_{ji} \gamma_{lk} \]
\[ + \frac{3}{W} A^i (\frac{f_i}{2W} + \frac{W \eta_i}{\eta}) N^k \gamma_{lk} + \frac{A^i R_{kji}^l \dot{u}_{il} \dot{u}^k}{W} \]
\[ + \frac{\dot{u}^k A^i C_{ij} \dot{u}_{jk}}{W} - \left( \frac{f}{2W} \sigma_{kl} + W N^k N^l \right) f_k f_i f - W N^k N^l f_k i + \frac{W}{\eta} f_i \gamma_{kl} - \gamma_{kl} N^k f_i. \]
Multiplying by $\frac{\eta}{W}$, we have

$$\frac{\eta}{W}(A^{ij}W_{ij} - \frac{2}{W}A^{ij}W_iW_j) \geq \left[-nN^kH_k - \frac{f_k}{W^3}N^k\langle \Pi_\ast \hat{\nabla} D_0, Du \rangle + \frac{1}{2W^2}A^{ij}f_{ij} \right.$$ 

$$+ \frac{1}{W^2}A^{il}N^k\gamma_{ji}\gamma_{lk} + \frac{3}{2W^3}A^{il}f_iN^k\gamma_{lk} + \frac{\hat{u}^kA^{ij}C_{ijk} + A^{ij}R^l_{kji}\hat{u}_i\hat{u}^k}{W^2}$$

$$- \frac{(f + W^2)}{W^2} \frac{1}{2f} \left[ \frac{f}{2W^2}\sigma^{kl} + N^k N^l \right] \frac{f_k f_l}{f} - N^k N^l f_{k;l} - \frac{1}{W} \gamma_{kl}N^k f_l \right] \eta \right.$$ 

$$+ \left[ \left( nH + \frac{1}{W^2}N^k f_k - \frac{2f}{W^3} \langle \Pi_\ast \hat{\nabla} D_0, Du \rangle \right)N^i \right.$$ 

$$\left.+ \frac{3}{W^2}A^{il}N^k\gamma_{ik} + \left( \frac{3}{W^2}A^{ij} - \frac{(f + W^2)}{W^2} \frac{1}{2f}\sigma^j \right)f_j \right] \eta_i. \right]$$

Thus it is easy to see that there exists a constant $M > 0$, not depending on $u$, such that

$$\frac{\eta}{W^2}(W^2A^{ij}W_{ij} - 2A^{ij}W_iW_j) \geq -M\eta - A^i\eta_i,$$ 

where $A^i$ is the coefficient of $\eta_i$. From (4), we deduce that

$$(8) \quad A^{ij}\eta_{ij} - M\eta - A^i\eta_i \leq 0.$$

We are now ready to choose an explicit $\eta$. We take

$$\eta(x) = g(\phi(x)) = eC_1\phi(x) - 1 = eC_1\left(1 - \frac{d^2(x)}{r^2} + \frac{u(x)}{C}\right)^+ - 1,$$

where $C = -\frac{1}{2u(o)}$. Straightforward computations give

$$\eta_i = g'(\frac{-r^{-2}(d^2)}{i} + C(u_i - s_i)) = g'(\frac{-r^{-2}(d^2)}{i} + C\hat{u}_i),$$

and

$$\eta_{ij} = g'(\frac{-r^{-2}(d^2)}{ij} + C\hat{u}_{ij}) + g''(\frac{-r^{-2}(d^2)}{i} + C\hat{u}_i)(\frac{-r^{-2}(d^2)}{j} + C\hat{u}_j).$$

We deduce from the two previous lines that

$$A^{ij}(\frac{-r^{-2}(d^2)}{i} + C\hat{u}_i)(\frac{-r^{-2}(d^2)}{j} + C\hat{u}_j) \geq \frac{C^2f}{W^2} \left( |Du|^2 - \frac{2}{Cr^2}\langle Du, \nabla d^2 \rangle \right),$$

and

$$A^{ij}(\frac{-r^{-2}(d^2)}{ij} + C\hat{u}_{ij}) = \frac{-r^{-2}A^{ij}(d^2)_{ij}}{ij}$$

$$+ C \left( nWH + \frac{f + W^2}{W^2} \langle \Pi_\ast \hat{\nabla} D_0, Du \rangle + A^{ij}\gamma_{ij} \right),$$
where
\[ A^{ij}(d^2)_{ij} = \Delta(d^2) - \frac{1}{W^2} \langle \nabla D_\nu \nabla d^2, D_\nu \rangle. \]

Inserting the previous expressions into (8), we have
\[
\frac{C^2 f}{W^2} \left( |Du|^2 - \frac{2}{Cr^2} \langle Du, \nabla d^2 \rangle \right) g''
+ \left[ C \left( nWH + \frac{f + W^2}{W^2} \langle \Pi_x \nabla D_0, Du \rangle + A^{ij} \gamma_{ij} \right)
- r^{-2} \left( \Delta(d^2) - \frac{1}{W^2} \langle \nabla D_\nu \nabla d^2, D_\nu \rangle \right) \right] g'
\leq Mg + A^i (-r^2 (d^2)_i + C \hat{u}_i) g'.
\]

Using the explicit expression of \( A^i \), it is easy to see that \( CA^i \hat{u}_i \) contains bounded terms and the term
\[
C \left( nWH + \frac{f + W^2}{W^2} \langle \Pi_x \nabla D_0, Du \rangle \right).
\]

Therefore, we conclude that
\[
\frac{C^2 f}{W^2} (|Du|^2 - \frac{2}{Cr^2} \langle Du, \nabla d^2 \rangle) g'' + Pg' - Mg \leq 0,
\]
where \( P \) and \( M \) do not depend on \( u \). Finally, it is easy to check that the coefficient of \( g'' \) is strictly positive if we assume that \( |Du| \geq \frac{16u_0}{r} \).

It implies that
\[
W(p) \leq C_2 = \sup_{B_r(o)} f + \frac{16u_0}{r}.
\]

Since \( p \) is the maximum point of \( h \), this implies that
\[
(e^{\frac{Cr^2}{4}} - 1)W(0) \leq C_2 e^{C_1}.
\]

For the proof of Theorem 1 we make use of the following lemma, which shows that the SC condition implies an explicit mean convexity condition. Precisely:

**Lemma 5.** Assume \( M \) is a Hadamard manifold satisfying the strict convexity condition and such that \( K_M \leq -\alpha \), for some constant \( \alpha > 0 \). Then \( M \) satisfies the \( h \)-mean convexity condition for \( h < \sqrt{\alpha} \), that is, given \( x \in \partial_\infty M \), a relatively open subset \( W \subset \partial_\infty M \) containing \( x \) and \( h < \sqrt{\alpha} \), there exists a \( C^2 \) open set \( \Lambda \subset M \) such that \( x \in \text{Int}(\partial_\infty \Lambda) \subset W \) and the mean curvature of \( M \setminus \Lambda \) with respect to the normal vector pointing to \( M \setminus \Lambda \) is bigger than or equal to \( h \).
Proof. Given $x \in \partial_\infty M$ and a relatively open subset $W \subset \partial_\infty M$ containing $x$, let $\Omega$ be a convex unbounded domain in $M$, given by the SC condition such that $x \in \text{Int}(\partial_\infty \Omega) \subset W$. Denote by $d : \Omega \to \mathbb{R}$ the distance function to $\partial \Omega$. Then the hessian comparison theorem (see [4]) yields

$$\Delta d \geq (n - 1) \sqrt{\alpha \tanh(\sqrt{\alpha}d)},$$

i.e. the equidistant hypersurface $\Omega_d$ of $\Omega$ is $\sqrt{\alpha \tanh(\sqrt{\alpha}d)}$-convex. Since $\tanh(\sqrt{\alpha}d) \to 1$ as $d \to \infty$, we deduce that $M$ also satisfies the $h$-mean convexity condition for $h < \sqrt{\alpha}$.

□

Proof of Theorem 1. Let $\gamma : (-\infty, \infty) \to M$ be the geodesic translated by $Y$. Set $P = \exp \{ Y(o) \} \perp'_o$ where $o = \gamma(0)$. Let $p \in P$ and $t \in \mathbb{R}$ be given. We may write $p = \exp_{\gamma(s)} u$ for some $s \in \mathbb{R}$ and $u \in \gamma'(s) \perp$. Since $\tilde{\gamma}(r) = \exp_{\gamma(s)}(ru)$, $r \in [0, 1]$, is a geodesic and $\varphi_t$ an isometry, $\beta(r) := \varphi_t(\tilde{\gamma}(r))$ is also a geodesic which, moreover, satisfies the initial conditions

$$\beta(0) = \varphi_t(\gamma(0)) = \varphi_t(\gamma(s)) = \gamma(s + t)$$

$$\beta'(0) = d(\varphi_t)_{\gamma(s)} u =: v,$$

we have $\beta(r) = \exp_{\gamma(s+t)}(rv)$ by uniqueness. It follows that

$$\varphi_t(p) = \beta(1) = \exp_{\gamma(s+t)} v.$$

Moreover, since

$$0 = \langle u, \gamma'(s) \rangle = \left( d(\varphi_t)_{\gamma(s)} u, d(\varphi_t)_{\gamma(s)} \gamma'(s) \right) = \langle v, \gamma'(s + t) \rangle$$

we have $v \in \gamma'(s + t) \perp$ and, as the normal exponential map of a geodesic in Hadamard manifold is a diffeomorphism from the normal bundle of the geodesic onto $M$, we have $\varphi_t(p) \cap P \neq \emptyset$ if and only if $t = 0$.

We now observe that $Y$ is everywhere transversal to $P$. Indeed, assume by contradiction that $Y$ is not transversal to $P$ at some point $p \in P$. Let $d : N \to \mathbb{R}$ be the distance function to $P$. We set $f(t) = d(\varphi(t,p))$ and observe that $f(0) = 0$. Moreover, since $\varphi(t, \cdot)$ is an isometry of $N$, we have, for any fixed $t$,

$$\text{grad} d(\varphi(t,p)) = d\varphi(t,p)_p (\text{grad} d(p)).$$
Therefore, we obtain
\[
f'(t) = \left\langle \text{grad} \, d, \frac{\partial \varphi(s,p)}{\partial s} \right|_{s=t} \right\rangle = \left\langle \text{grad} \, d(\varphi(t,p)), Y(\varphi(t,p)) \right\rangle
\]
\[
= \left\langle d\varphi(t,p)_p \left( \text{grad} \, d(p) \right), d\varphi(t,p)_p (Y(p)) \right\rangle
\]
\[
= \langle \text{grad} \, d(p), Y(p) \rangle = 0.
\]
This implies that \( f \equiv 0 \) and, in return, that \( \varphi(t,p) \in P \) for all \( t \), which yields to a contradiction. This proves that \( P \) is a Killing section.

Since any orbit of \( \varphi \) at \( \partial_{\infty} N \) intersects \( \Gamma \) at one and exactly one point, \( \Gamma \) is the Killing graph of a function \( \phi \in C^0(\partial_{\infty} P) \). Let \( F \in C^{2,\alpha}(P) \cap C^0(\bar{P}) \) (\( \bar{P} = P \cup \partial_{\infty} P \)) be such that \( F|_{\partial_{\infty} P} = \phi \).

Let \( \rho \) be the geodesic distance in \( P \) to a fixed point \( o \in P \). We denote by \( B_k \), for \( k = 2, 3, \ldots \), the geodesic ball in \( P \) centered in \( o \) and of radius \( k \). We first show that, for any \( k = 2, 3, \ldots \), there is a solution \( u_k \in C^{2,\alpha}(\bar{B}_k) \) of
\[
\begin{align*}
Q_H[u_k] &= 0, & \text{on } B_k \\
\left. u_k \right|_{\partial B_k} &= F_k = F|_{\partial B_k}.
\end{align*}
\]
In order to prove the existence of the \( u_k \)'s, we will need some a priori height estimate. More precisely, we claim that given some \( k \geq 2 \), there is a constant \( C_j \) depending only on \( j \) such that if \( u_k \) is a solution of (9) and \( j \leq k \) then \( \sup_{B_j} |u_k| \leq C_j \). Let us prove the claim. We choose two open subsets \( U_{\pm} \) of \( \gamma(\pm \infty) \) in \( \partial_{\infty} M \). Using the SC condition, we obtain the existence of two \( C^2 \) convex subsets \( W_{\pm} \) of \( M \) such that \( \partial_{\infty} W_{\pm} \subset U_{K_{\pm}} \). Denote by \( K_{\pm} \) the hypersurfaces \( K_{\pm} = \partial W_{\pm} \). As observed in Lemma 4 and since \( |H| < \sqrt{\alpha} \), we may assume that \( K_{\pm} \) are \( H_0 \) mean convex with \( H_0 \geq H \). We then choose \( C_j \) such that the orbit of \( \{ \varphi_t \} \) through a point of \( B_j \) intersects \( W_{\pm} \) for some \( t \geq C_j \). It is clear that we may assume that the Killing graph of \( F \) does not intersect \( K_{\pm} \). The claim then follows from the tangency principle.

We have two important consequences of the previous height estimate. The first one is that problem (9) is solvable for any \( k \geq 2 \). In fact, the only missing hypothesis to apply Theorem 1 of [6] to guarantee the solvability of (9) are on the Ricci curvature of \( M \) and on the mean curvature of the Killing cylinder over the boundary of \( B_k \) (see [6], Theorem 1). Concerning the hypothesis on the mean curvature of the Killing cylinder over the boundary of \( B_k \), we claim that it holds true in our setting. Indeed, since the orbits of \( \varphi_t \) are equidistant curves of \( \gamma \), it follows that the Killing cylinder \( K_k \) over \( \partial B_k \) is an equidistant hypersurface of \( \gamma \). Therefore the mean curvature \( H_{K_k} \) of \( K_k \) with respect to the inner normal vector field of \( K_k \) coincides with the Laplacian of the
distance to $\gamma$. One may then apply the hessian comparison theorem to obtain

$$H_{K_k} \geq \sqrt{\alpha} \tanh (k \sqrt{\alpha}) \geq H.$$

Moreover, a direct inspection on the proof of Theorem 1 of [6] shows that the hypothesis on the Ricci curvature is only used to obtain a priori height estimates, which we just obtained above.

Secondly, the a priori height estimates we obtained above, Theorem 4 and classical Schauder estimate for linear elliptic PDE (see [10]) guarantee the compactness of the sequence of solutions that the hypothesis on the Ricci curvature is only used to obtain a priori height estimates, which we just obtained above.

Let $(x_k)_{k}$ be a sequence of points of $P$ converging to $x \in \partial_\infty P$. Since $\overline{P}$ is compact, there exists a subsequence $\varphi(u(x_k), x_k)$ of $\varphi(u(x_k), x_k)$ which converges to $z \in \overline{P}$. Since $x_k$ diverges and $\varphi(u(x_k), x_k) \in Gr(u)$, we have that $z \in \partial_\infty Gr(u)$. We claim that $z \in Gr(\phi)$. To prove the claim, we will show that if $z \in \partial_\infty M\setminus Gr(\phi)$ then $z \notin \partial_\infty Gr(u)$. Let $z \in \partial_\infty M\setminus Gr(\phi)$. Since $Gr(\phi)$ is compact and $z \notin Gr(\phi)$, using the SC condition, we can find an hypersurface $E \subset M$ such that $\partial_\infty E$ separates $z$ and $Gr(\phi)$. Moreover, using Lemma 5 the mean curvature of $E$ with respect to the unit normal vector field pointing to the connected component $U$ of $M\setminus E$ whose asymptotic boundary contains $Gr(\phi)$, is larger or equal to $h$ for $h < \sqrt{\alpha}$. Since $u_k|_{\partial B_k} \to \phi$, there exists $k_0$ such that, for all $k \geq k_0$, $\partial Gr(u_k) \subset U$ and $\partial Gr(u_k) \cap E = \emptyset$. By the tangency principle and using that $|H| < \sqrt{\alpha}$, we deduce that, for all $k \geq k_0$, $Gr(u_k) \subset U$. It follows that $z \notin \partial_\infty Gr(u)$. This proves the claim i.e. $z \in Gr(\phi)$. In particular, it follows that $u$ is bounded.

Now, since $\partial_\infty Gr(u) \subset Gr(\phi)$, there exists $x_0 \in \partial_\infty P$ such that $z = \varphi(u(x_0), x_0)$. Using that $u$ is bounded, we deduce there exists a subsequence $\{u(x_{k_j})\}$ which converges to some $t_0 \in \mathbb{R}$. It follows from the fact that the extension of $\varphi_{t_0}$ to $\overline{P}$ is continuous that

$$z = \lim_{i \to \infty} \varphi(u(x_{k_{j_i}}), x_{k_{j_i}}) = \varphi(t_0, x).$$

Since $\varphi: \mathbb{R} \times \partial_\infty P \to \partial_\infty M$ is injective, we deduce that $t_0 = \phi(x_0)$ and $x_0 = x$. Since this last fact holds true for every converging subsequences, we have proved that $u(x_k) \to \phi(x)$. This concludes the proof of Theorem 1.
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