LINEAR STABILITY ANALYSIS FOR 2D SHEAR FLOWS NEAR COUETTE IN THE ISENTROPIC COMPRESSIBLE EULER EQUATIONS

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Abstract. In this paper, we investigate linear stability properties of the 2D isentropic compressible Euler equations linearized around a shear flow given by a monotone profile, close to the Couette flow, with constant density, in the domain $\mathbb{T} \times \mathbb{R}$. We begin by directly investigating the Couette shear flow, where we characterize the linear growth of the compressible part of the fluid while proving time decay for the incompressible part (inviscid damping with slower rates).

Then we extend the analysis to monotone shear flows near Couette, where we are able to give an upper bound, superlinear in time, for the compressible part of the fluid. The incompressible part enjoys an inviscid damping property, analogous to the Couette case. In the pure Couette case, we exploit the presence of an additional conservation law (which connects the vorticity and the density on the moving frame) in order to reduce the number of degrees of freedom of the system. The result then follows by using weighted energy estimates.

In the general case, unfortunately, this conservation law no longer holds. Therefore we define a suitable weighted energy functional for the whole system, which can be used to estimate the irrotational component of the velocity but does not provide sharp bounds on the solenoidal component. However, even in the absence of the aforementioned additional conservation law, we are still able to show the existence of a functional relation which allows us to recover somehow the vorticity from the density, on the moving frame. By combining the weighted energy estimates with the functional relation we also recover the inviscid damping for the solenoidal component of the velocity.

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1. Introduction

Let us consider the isentropic compressible Euler system

\begin{align}
\partial_t \rho + \text{div} (\rho u) &= 0, \quad \text{in } \mathbb{T} \times \mathbb{R} \\
\partial_t (\rho u) + \text{div} (\rho u \otimes u) + \frac{1}{M^2} \nabla p(\rho) &= 0,
\end{align}

in the periodic strip $\mathbb{T} \times \mathbb{R}$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Here $\rho$ is the density of the fluid, $v$ the velocity, $p(\rho)$ the pressure and $M$ is the usual Mach number.

It is well known that a class of divergence-free stationary solutions to the Euler system (1.1)-(1.2) is given by horizontal shear flows, namely velocity profiles $u_E = (U(y), 0)$ with constant density $\rho_E = 1$. We are interested in studying the linear stability properties of monotone shear flows in a perturbative regime close to the Couette flow, i.e. $U(y) = y$.

If we consider a perturbation around this shear flows solution, $\tilde{\rho} = \rho + \rho_E$, $u = v + u_E$, then the linearized system for $\tilde{\rho}$, $u$ around a shear flow, is given by the following equations

\begin{align}
\partial_t \rho + U(y) \partial_x \rho + \text{div} v &= 0, \quad \text{in } \mathbb{T} \times \mathbb{R} \\
\partial_t v + U(y) \partial_x v + (U'(y)v_2, 0)^T + \frac{1}{M^2} \nabla \rho &= 0,
\end{align}

where $p'(1) = 1$.

The interest in the analysis of shear flows dates back to the times of Newton and subsequently of Stokes, in relation to the study of the behaviour of a liquid between two rotating cylinders. Later also important scientists such as Kelvin [19], Rayleigh [61], Couette [26] and Taylor [65] have been attracted by this problem. We refer to the interesting article by Donnelly [30] for an historical review on these topics, which includes a fairly complete list of historical references on these subjects. At the end of the XIX century, Couette and Mallock realized the physical experiment of a fluid contained between two rotating cylinders, showing the onset of turbulence in some condition. A closely related to this physical configuration, concerns the motion of an incompressible fluid at constant pressure between two parallel planes, one kept moving and the other fixed, known in the literature as plane Couette flow.

Different from the Couette flow, which is a drag-induced motion, there is another classical shear flow which is pressure-induced, named after Pousielle. Indeed at high Reynolds number regime $Re \gg 1$, if we assume that the pressure around a submerged cylindrical body (say of radius 1) is constant in the radial direction and that the axial velocity is radially symmetric, under suitable boundary conditions, in the 2–D representation we get $U(y) \sim Re (-\frac{dp}{dx})(1 - y^2)$.

We have a fundamental interest in understanding the role of shear flows in the general picture of boundary layer analysis [3]. When a fluid laps a submerged cylindrical body, for sufficiently large $Re$, the flow lines that pass near the front of the body, detach in the rear, enclosing vortices of fluid in slow motion inside them. As $Re$ increases, the fluid detaches from the rear (in two points of detachment in the case of a 2–D representation), enclosing within it a region in which the flow is reversed, namely it goes from right to left creating a pair of vortices (Von Kármán vortex street). When the number of $Re$ is very high, the formation of vortices involves the detachment of the boundary layer. This is relevant since downstream of the detachment there is a strong decrease in both pressure and shear stress, with a decrease in strength and lift. For example, in the case of a body hit by a laminar flow, the unperturbed speed increases along the longitudinal coordinate with a decrease in pressure. Therefore, in the boundary layer the speed is equal to the sum
of a Couette type profile, due to the speed of the upper layer, and to a Pousile type, due to the negative pressure gradient. At the rear, the unperturbed speed decreases, so the pressure gradient is positive, with consequent deceleration of the fluid. Hence, at the detachment point, the speed gradient on the wall is cancelled and the adhesion is lacking, that is, the shear stress is zero, so that the boundary layer detaches. The understanding of these complex phenomena requires the development of a very accurate stability analysis in significant model problems.

Linear stability properties for the incompressible Couette flow were already considered by Kelvin [49]. For more general incompressible shear flows, Rayleigh [61] investigated the linear stability via the normal mode method, which yields to the Rayleigh inflection point Theorem, namely a necessary condition for spectral instability to hold is that $U'' = 0$ at least in one point. Fjørtoft in [34] gave a stronger criterion. As noted in the classical book of Landau-Lifschitz [51, pg 171]: "Physically, this instability is due to the resonance-type interaction between the oscillations of the medium and the movement of its particles in the main stream; in this sense, it is analogous to the Landau damping (or amplification in the case of instability) of oscillations in a collisionless plasma." For a useful overview concerning spectral stability properties and other stability problems, we refer also to the books [32, 62, 73, 19] and the references therein. The above mentioned classical stability analysis in general does not agree with numerical and physical observations. For instance, the Couette flow, which mathematically is spectrally stable for all Reynolds numbers, exhibit instabilities observed in various experiments. In particular, they become turbulent at large Reynolds numbers. This phenomenon leads to what is commonly referred in the literature as the Sommerfeld paradox, see for example [53] and references therein for a mathematical treatment. A key observation was made by Trefethen et al. [67], where it has been shown that a pure eigenvalues analysis could hide several problems. These are due to the fact that the operators involved in the linearization around a shear flow are in general non normal, even in the simpler cases. Non normality leads to highly non trivial pseudospectral properties, see [66].

A different approach was taken in 1966 by Arnold [1], who investigated the Lyapunov stability of a perturbation around a shear flow. By using a variational approach, Arnold proved, in the incompressible setting, the nonlinear Lyapunov stability for a class of strictly concave shear flows (hence not including the Couette case), see also [2, Ch. 1-8]. Holm et al. [42] used Arnold’s approach also to investigate the isentropic compressible case, the MHD equations and other related systems.

Regarding the inviscid and incompressible plane Couette flow, Orr [60] studied the linear stability directly by considering the initial value problem. This particular case reduces to just a simple transport equation for the vorticity. By recovering the velocity via the Biot-Savart law, Orr observed that the velocity may experience a transient growth, suggesting that this phenomenon may be a possible source of instabilities in the nonlinear problem. One can see that, assuming regularity on the initial perturbation ($H^2$ for instance), it is possible to obtain the so called inviscid damping, namely the perturbed velocity converges strongly to zero in $L^2$ with suitable polynomial rates. Hence linear stability properties depends on regularity rather than smallness of the perturbation. We refer to [7, Sec. 2.3] for more details about this linear mechanism.

To prove inviscid damping at a nonlinear level, the picture is dramatically different. For example Lin-Zeng [53] provided an evidence of instability for perturbations in $H^s$, $s < 3/2$. There are also numerical [63, 68, 69] and physical [75, 76] evidences of instabilities related to the phenomenon of echoes, roughly speaking the nonlinearity can sustain a cascade of transient growths leading to a measurable instability after a long time. This problem shares
analogies with the *Landau damping*, see the work of Mohout-Villani [59] and references therein.

Bedrossian-Masmoudi in [9], by considering an initial perturbation in the Gevrey-$s$ class, for $s > 1/2$, proved asymptotic stability for the 2D incompressible Couette flow in $\mathbb{T} \times \mathbb{R}$. To deal with the aforementioned problem of echoes, they construct suitable weights in order to infer energy estimates on the vorticity. The techniques used in [9] turned out to be useful also for the Landau damping [10]. Deng-Masmoudi [27] proved that the Gevrey regularity requirement are essentially sharp for the nonlinear asymptotic stability of the vorticity. Ionescu-Jia [45] proved that nonlinear asymptotic stability holds true also for perturbations around the Couette flow, with compactly supported vorticity, in $\mathbb{T} \times [-1,1]$. In a recent paper, Zillinger-Deng [28], proposed a model which suggests the possibility that inviscid damping for the velocity may hold even in the presence of echo chains that leads to instabilities for the vorticity.

Linear inviscid damping was also proved by Wei-Zhang-Zhao [71] for a more general class of shear flows in 2D. Other available results consider linear inviscid damping for classes of monotone shear flows [40, 47, 70, 77, 78, 79, 3]. See [29] for a case where linear inviscid damping does not hold for a monotone shear flow. See [39] for a spectral instability result. Nonlinear asymptotic stability for perturbations around a class of monotone shear flows, with compactly supported vorticity in $\mathbb{T} \times [-1,1]$, was recently proved by Ionescu-Jia [46] and Masmoudi-Zhao [58].

A strictly related problem to the stability of shear flows is the stability of vortices. In this direction, Bedrossian-Coti Zelati-Vicol [14] investigated linear stability properties for a rather general class of vortex states. Ionescu-Jia in [44] proved the nonlinear asymptotic stability of perturbations around a point vortex, with vorticity concentrated in an annular region away from the point vortex.

In the viscous case, more informations are available. The 2D Couette flow in $\mathbb{T} \times \mathbb{R}$, in the Gevrey-$s$ regularity class, $s > 1/2$, was considered by Bedrossian-Masmoudi-Vicol in [11]. Due to the presence of viscosity, it is also possible to look for stability thresholds in Sobolev spaces, see [12, 59, 57]. The 3D viscous Couette case was studied by Bedrossian-Germain-Masmoudi [6]. Bedrossian-He [8] considered the linear viscous 2D Couette case in a periodic channel, where one has to deal with boundary layers.

In general, the presence of a background shear can *enhance the dissipation*. This means that the time scales of dissipation for the vorticity, or a passive tracer in the fluid, may be faster with respect to the standard one given by the heat kernel. For works in this direction, see [13, 22, 23, 24, 25, 40, 43, 72, 54].

Also in the viscous case, stability properties of vortices are considered, see the works of Gallay [35], Gallay-Wayne [36] and Li-Wei-Zhang [52].

For a 2D incompressible fluid with non constant density, for an instability result see [21]. Linear inviscid damping for a perturbation of the exponentially stratified Couette flow was proven by Yang-Lin [74].

In the context of compressible flow dynamics the literature about shear flows is considerably less developed. Highly incomplete list of works concerning the study of spectral properties for the linearization around shear flows in the compressible case, includes [15, 16, 20, 31, 33, 48, 51, 63].

As already said, Lyapunov type stability properties for the isentropic compressible case can be found in [42], where the case of strictly concave shear flows is covered.

The linearization around the Couette flow in the 2D isentropic compressible Euler dynamics was considered in the physics literature, both from the numerical point of view and from the theoretical one, a partial literature includes [41, 17, 18, 37, 38, 41, 48, 51].
We also mention two interesting papers concerning the formation of spiral arms in a rotating disk galaxy by Goldreich and Lynden-Bell \cite{GoldreichLyndenBell1965,GoldreichLyndenBell1970}, where they investigate the stability of the Couette flow for 2D isentropic compressible Euler system in presence of a Coriolis force. Their formal analysis is interesting since they directly consider the linearized initial value problem, performing the Fourier analysis on the moving frame, closely related to what was done by Orr in the incompressible case. Their analysis provides some instability phenomena which appears specifically due to the compressibility of the flow. In particular they show that the density, in the frequency space, can grow linearly in time.

The problem without Coriolis force, with analogous computations was considered in \cite{GoldreichLyndenBell1970,Majda1974}. More recently a more refined analysis, supplemented also with numerical simulations, can be found in \cite{GoldreichLyndenBell1970,Majda1974}.

The linear instability phenomena found in the above mentioned literature, will be caught in a more rigorous and precise way in our subsequent analysis.

1.1. Statement of the results. The first result in our paper concerns the linear dynamics in the Couette case. For any velocity field $v$, let

$$
\alpha = \text{div} v, \quad \omega = \text{curl} v,
$$

let $\nabla^\perp = (-\partial_y, \partial_x)^T$, we define the Helmholtz projection operators in the usual way, namely

$$
\begin{align}
\v = (v_1, v_2)^T &= \nabla \Delta^{-1} \alpha + \nabla^\perp \Delta^{-1} \omega := Q(v) + P(v). \\
\end{align}
$$

We now begin with the system (1.3)-(1.4) in the case of the classical Couette flow, namely $U(y) = y$. By taking div and curl to (1.4), we rewrite the system (1.3)-(1.4) as follows

$$
\begin{align}
\frac{\partial}{\partial t} \rho + y \partial_x \rho + \alpha &= 0, \quad \text{in } T \times \mathbb{R}, \\
\frac{\partial}{\partial t} \alpha + y \partial_x \alpha + 2 \partial_x v_2 + \frac{1}{M^2} \Delta \rho &= 0, \\
\frac{\partial}{\partial t} \omega + y \partial_x \omega - \alpha &= 0.
\end{align}
$$

Notice that, by the Helmholtz decomposition, it holds

$$
(\alpha) \quad v_2 = \partial_y (\Delta^{-1}) \alpha + \partial_x (\Delta^{-1}) \omega,
$$

hence (1.6)-(1.8) is a closed system in terms of the variables $(\rho, \alpha, \omega)$.

By adding (1.6) to (1.8), one immediately infers that $\rho + \omega$ is a conserved quantity transported by the flow.

We now state the main result related to the linear stability in the particular case of the Couette flow, the more general case will be presented later in Theorem 1.6. A more precise statement will be given in Section 3, see Theorems 3.4 and 3.6.

We emphasize that a more complete result, where we do not restrict to zero $x$-average, will be given in Section 3.

**Theorem 1.1.** Assume that $\int_T \rho_{in} dx = \int_T \omega_{in} dx = \int_T \alpha_{in} dx = 0$. Let $(\rho_{in}, \alpha_{in}, \omega_{in}) \in H^2(T \times \mathbb{R})$ be the initial data of (1.6)-(1.8). Let $Q(v)$ and $P(v)$ as defined in (1.5) and $(t) = \sqrt{1 + t^2}$. Therefore

$$
(1.10) \quad \langle t \rangle C_1 (\rho_{in}, \alpha_{in}, \omega_{in}) \leq \|Q(v)\|^2_{L^2} + \frac{1}{M^2} \|\rho\|^2_{L^2} \leq \langle t \rangle C_2 (\rho_{in}, \alpha_{in}, \omega_{in}),
$$

where the lower bound in (1.10) holds true up to a nowhere dense set of initial data.
Moreover, for the incompressible part of the fluid $P(v)$, we have the following decay estimates,

\begin{align}
\|P_1(v)\|_{L^2} & \leq \frac{M}{\langle t \rangle^{1/2}} C(\rho_{in}, \alpha_{in}, \omega_{in}) + \frac{1}{\langle t \rangle} C(\rho_{in}, \omega_{in}), \\
\|P_2(v)\|_{L^2} & \leq \frac{M}{\langle t \rangle^{3/2}} C(\rho_{in}, \alpha_{in}, \omega_{in}) + \frac{1}{\langle t \rangle} C(\rho_{in}, \omega_{in}).
\end{align}

**Remark 1.2.** The estimate (1.10) gives a rigorous justification to the linear growth predicted in [4, 17, 18, 41], where the authors had to restrict to a small Mach number regime required for to implement a WKB asymptotic analysis. We emphasize that the above result is more precise, since in particular it removes smallness restrictions and provides the explicit dependence of the asymptotic time scale on the Mach number.

**Remark 1.3.** It is interesting to remark that Theorem 1.1 implies that the density and the irrotational part of the velocity exhibit a growth in time even when the initial perturbation satisfies $\rho_{in} = \alpha_{in} = 0$. This can be straightforwardly seen from the linearized equations (1.6)-(1.8), where the identity (1.9) for $v_2$ yields a source term in the equation for the divergence (1.7). This phenomenon underlines that the linear stability analysis for the Couette flow in the compressible case is remarkably different from the incompressible one.

**Remark 1.4.** To obtain the lower bound in (1.10), we need to exclude a nowhere dense set of initial data in a proper Sobolev space, see Proposition 3.5 where we characterize this set.

**Remark 1.5.** Notice that when $M = 0$ and $\rho_{in} = 0$, formally the estimates (1.11)-(1.12) give the same result on the inviscid damping as in the incompressible case.

Let us briefly discuss the strategy of proof for Theorem 1.1. By using the conservation of the quantity $\rho + \omega$, we are able to reduce the degrees of freedom for system (1.6)-(1.8) and write a 2 X 2 system, see (3.7)-(3.8) below, only involving the density and the divergence in the moving frame. Taking its Fourier transform in all the space variables, it can be studied as a 2 X 2 non-autonomous dynamical system at any fixed frequency $(k, \eta)$. By using particular Fourier multipliers in the moving frame we are able to obtain suitable energy estimates. Lemma 3.2, which appears later one, provides a slightly more general result. Finally, Theorem 1.1 can proved by going back to the original variables.

We now turn our attention to the equations for a general background shear $U(y)$,

\begin{align}
\partial_t \rho + U \partial_x \rho + \alpha & = 0, \quad \text{in } \mathbb{T} \times \mathbb{R}, \\
\partial_t \alpha + U \partial_x \alpha + 2U' \partial_x (\partial_y \Delta^{-1} \alpha + \partial_x \Delta^{-1} \omega) + \frac{1}{M^2} \Delta \rho & = 0, \\
\partial_t \omega + U \partial_x \omega - U' \alpha & = U'' (\partial_y \Delta^{-1} \alpha + \partial_x \Delta^{-1} \omega).
\end{align}

In our paper we study monotone shear flows which are close to Couette, namely we assume $U'(y) \approx 1$ and $U''(y) \approx 0$. The study of the linear stability for such particular flows can be motivated by the (considerably more difficult) study of the nonlinear stability for the Couette case. That is also the case for the incompressible dynamics, see [78], where the problem can be treated with techniques which share analogies with the nonlinear problem 9.

The main difference between the Couette case discussed in Theorem 1.1 and (1.13)-(1.15) is that, while in the former we can somehow exploit an explicit computation in the Fourier space, in the latter one we need to make use of a perturbative approach. More precisely, in order to follow the background shear, we introduce the variables $X = x - U(y)t$, $Y = U(y)$. 
Consequently, when writing equations (1.13) - (1.15) in the moving frame, the following quantities appear,

\[ g(Y) = U'(U^{-1}(Y)), \quad b(Y) = U''(U^{-1}(Y)). \]

In this way, we say that \( U(y) \) is a shear flow close to Couette if

\[ \|g - 1\|_{H^s} \ll 1, \quad \|b\|_{H^s} \ll 1, \]

for some \( s > 1 \) which will be specified below. The main result of our paper proves upper bounds analogous to the ones inferred in (1.10), (1.11), (1.12), with an arbitrarily small loss in the time asymptotic rate (which will be discussed below).

**Theorem 1.6.** Let \( \epsilon \ll 1 \) and assume that \( \|g^2 - 1\|_{H^{s_0}} \leq \epsilon \) and \( \|b\|_{H^{s_0}} \leq \epsilon \) for a fixed \( s_0 \in \mathbb{R} \). Let \( \rho_{in}, \alpha_{in}, \omega_{in} \in H^b(T \times \mathbb{R}) \) be the initial data of (1.13) - (1.15) such that \( \int_T \rho_{in} dx = \int_T \omega_{in} dx = \int_T v_{in}^3 dx = 0 \).

Then, there is a constant \( C > 1 \) such that for \( \bar{\epsilon} = C\epsilon < 1/16 \), it holds that

\[ \|Q(v)\|_{L^2}^2 + \frac{1}{M^2} \|\rho\|_{L^2}^2 \leq \frac{M}{\langle t \rangle^{1/2}} C(\rho_{in}, \alpha_{in}, \omega_{in}), \]

\[ \|P_1(v)\|_{L^2} \leq \frac{M}{\langle t \rangle^{1/2}} C(\rho_{in}, \alpha_{in}, \omega_{in}) + \frac{1}{\langle t \rangle} C(\rho_{in}, \omega_{in}), \]

\[ \|P_2(v)\|_{L^2} \leq \frac{M}{\langle t \rangle^{3/2}} C(\rho_{in}, \alpha_{in}, \omega_{in}) + \frac{1}{\langle t \rangle^2} C(\rho_{in}, \omega_{in}). \]

In what follows we sketch the strategy of our proof, also in comparison with Theorem 1.1. In the Couette case, a key property is given by the conserved quantity \( \rho + \omega \), along the characteristics. It reduces the number of degrees of freedom in the problem and leads to the study of a 2 \( \times \) 2 dynamical system. Therefore it is immediate to define explicit weights, which allow us to deduce suitable energy estimates. On the other hand, for the general case under consideration in the Theorem 1.6 this conservation law is no longer true. The most natural extension of this conserved quantity is given by \( U' \rho + \omega \), which may be considered an almost conserved quantity since in our case \( U'' \ll 1 \). Indeed, although it involves the study of a more complicate \( 3 \times 3 \) system, see (5.14) - (5.16), we are still able to construct an energy functional in the quantities \( (\rho, \alpha, U' \rho + \omega) \), defined in (5.21). A non-trivial choice of the weights in (5.21) allows us to obtain the bound (1.16).

However, the upper bounds obtained in this way for \( U' \rho + \omega \) cannot provide the estimate (1.17), (1.18) on the incompressible part \( P(v) \). To overcome this problem and to prove (1.17), (1.18), we show the existence of a functional relation, which expresses the vorticity in terms of the density and the initial data. Namely by denoting

\[ \Omega = \omega(t, X + U(y)t, U(y)), \quad R = \rho(t, X + U(y)t, U(y)), \]

we are able to show, in Section 5.1, that an identity of the following type holds

\[ \Omega + \Phi_1 R = \Xi_{in} + \int_0^t \Phi_2 R \, d\tau, \]

where \( \Xi_{in} \) only depends on the initial data and \( \Phi_1 \) are suitable pseudodifferential operators, see Section 5.1 for more details.

Therefore, combining the relation in (1.19) and the energy estimates obtained for \( R \), we are able to obtain the bounds in (1.17), (1.18).

**Remark 1.7.** The regularity assumptions on the background shear in the previous Theorem are by no means sharp. However, they are sufficient (far from necessary) conditions to be able to control all the necessary commutator estimates. We are not interested in
computing the optimal $s_0$, which in any case must satisfy $s_0 \geq 8$ (see for example Theorem 4.5).

**Remark 1.8.** One immediately sees that instead of a linear growth, we obtain a superlinear one which deteriorates the exponent in the bound (1.16) by a factor $\tilde{\epsilon}$. We expect this loss to be natural, as we explain in Remark 4.7.

**Outline of the paper.** The paper is organized as follows.

In Section 2 we show how to remove the zero $x$-average ($k=0$ mode) assumptions made in the previous Theorems. In particular, we show that the dynamics of the $k = 0$ is completely decoupled from the dynamics of any $k \neq 0$.

In Section 3 we describe the dynamics of the Couette case in the frequency space and we prove Theorem 1.1.

In Section 4 we deal with problems related to the shear near Couette that will be used as building blocks for the proof of Theorem 1.6.

In Section 5 we first present the functional relation between $\Omega$ and $R$. Then we set up the weighted energy functional that we are able to control. By combining the energy bounds with the functional relation, we finally prove (1.6).

**Notations.** With the symbols $\lesssim, \gtrsim$, we intend less or greater up to some multiplicative constant and analogously we use $\approx$.

Define the japanese bracket as follows, for any $a, b \in \mathbb{R}$, let

$$\langle a, b \rangle = (1 + (|a| + |b|)^2)^{1/2}$$

We take the Fourier transform in $x$ and $y$, where we define

$$\hat{f}(k, \eta) = \frac{1}{2\pi} \int_{T \times \mathbb{R}} e^{-i(kx + \eta y)} f(x, y) dx dy,$$

with $k \in \mathbb{Z}$ and $\eta \in \mathbb{R}$. The inverse Fourier transform is given by

$$f(x, y) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} e^{i(kx + \eta y)} \hat{f}(k, \eta) d\eta.$$

Given two real functions $f, g \in H^s(\mathbb{T} \times \mathbb{R})$ we denote the scalar product as

$$\langle f, g \rangle_s = \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} \langle k, \eta \rangle^{2s} \hat{f}(k, \eta) \bar{\hat{g}}(k, \eta) d\eta,$$

and when $s = 0$ we omit the subscript.

When $k \neq 0$, the inverse Laplacian is well defined on the Fourier side as

$$- (\Delta^{-1} f)(k, \eta) = \frac{1}{k^2 + \eta^2} \hat{f}(k, \eta).$$

We say that $f \in H^s_x H^s_y$ whenever

$$\|f\|_{H^s_x H^s_y}^2 = \sum_k \langle k \rangle^{2s_1} \langle \eta \rangle^{2s_2} |\hat{f}|^2(k, \eta) d\eta < +\infty.$$
We define the standard Picard iteration
\[
\Phi_L(t,t_0) = 1 + \sum_{n=1}^{\infty} J_n(t,t_0),
\]
(1.21)
\[
J_{n+1}(t,t_0) = \int_{t_0}^{t} \mathcal{L}(\tau) J_n(\tau,t_0) d\tau, \quad J_1(t,t_0) = \int_{t_0}^{t} \mathcal{L}(\tau) d\tau.
\]
\(\Phi_L\) is the evolution operator associated to \(\mathcal{L}\). In particular it has the flow property
\[
\Phi_L(t,t_0) = \Phi_L(t,s)\Phi_L(s,t_0).
\]
If \(\mathcal{L}(t)\mathcal{L}(s) = \mathcal{L}(s)\mathcal{L}(t)\) then \(\Phi_L(t,t_0) = \exp(\int_{t_0}^{t} \mathcal{L}(\tau) d\tau)\). If \(\mathcal{L}(t)\) is continuous in any induced operator norm, then the flow map \(\Phi_L\) is well defined.

2. Acoustic propagation of the \(k = 0\) modes

In this section we investigate in more detail the dynamics of the \(x\)-averages of the perturbations. Due to the structure of the shear and the fact that the equations are linear, it is clear that the zero mode in \(x\) has an independent dynamic with respect to other modes. From a mathematical point of view, the filtering of the \(x\)-average out of the dynamics is a necessary condition in order to have a good definition for the inverse of some differential operators, naturally appearing in the equations, see (1.14)-(1.15).

We will make use of the following notation
\[
\mathcal{F} = \int_T f dx.
\]

We directly deal with the case of a general shear flow, namely (1.13)-(1.15). Integrating in \(x\) equations (1.13)-(1.14), thanks to periodic boundary conditions, we get that
\[
\begin{align*}
\partial_t \rho + \alpha & = 0, \\
\partial_t \alpha + \frac{1}{M^2} \partial_y p & = 0.
\end{align*}
\]
(2.1) (2.2)
Clearly, the system (2.1)-(2.2) displays a dynamics which is completely decoupled from the rest of the perturbation. More precisely, given \(\overline{\rho}_m, \overline{\alpha}_m\) the solution is given by the linear 1–D wave equation in \(\mathbb{R}\).

Integrating in \(x\) equation (1.15), we obtain that
\[
\partial_t \omega - U' \overline{\alpha} = U'' v_2.
\]
(2.3)
Let us remark that \(v_2\) can be recovered from the definition of the divergence, namely \(\partial_y v_2 = \overline{\alpha}\). In particular, if the initial data satisfies the zero \(x\)-average condition, then this is satisfied by the solution of (1.13)-(1.15) for all \(t \geq 0\).

**Proposition 2.1.** Let \(\rho_{in}, \alpha_{in}, \omega_{in}\) be the initial data of (1.13)-(1.15). Then the solution \(\rho, \alpha, \omega\) can be decomposed as \(\rho = \overline{\rho} + \rho, \alpha = \overline{\alpha} + \alpha, \omega = \overline{\omega} + \omega\), where \(\overline{\rho}, \overline{\alpha}, \overline{\omega}\) satisfies (2.1)-(2.3) and \(\rho, \alpha, \omega\) satisfies (1.13)-(1.15).

In particular, it holds that
\[
\overline{\rho}_{in} = \overline{\alpha}_{in} = \overline{\omega}_{in} = 0 \implies \overline{\rho} = \overline{\alpha} = \overline{\omega} = 0.
\]

The proof of Proposition (2.1) follows straightforwardly by using (2.1)-(2.3) and the fact that \(\partial_x \mathcal{F} = 0\).

Hence, any perturbation around a shear flow, can be decoupled in two separate dynamics, one given by the \(k = 0\) mode satisfying (2.1)-(2.3), the other by the system (1.13)-(1.15) with initial data with zero \(x\)-average.
3. The Couette case

In this section we investigate in detail the Couette case. More precisely, we are going to prove the results stated in Theorem 1.1. In what follows is convenient to treat separately the analysis for the lower and upper bounds, respectively. For this reason, in Theorem 3.4 below we will show the lower bound whereas Theorem 3.6 provides us the upper bounds.

As observed in Section 2, we can remove the $x$-average from the dynamics, so we directly deal with initial perturbations without the $k = 0$ mode, namely $\bar{\rho}_{in} = \bar{\alpha}_{in} = \bar{\omega}_{in} = 0$.

Recall that we denote the divergence and the vorticity of the velocity as

$$\alpha = \text{div} \, v, \quad \omega = \text{curl} \, v = -\text{div} \, v^\perp,$$

where $v^\perp = (-v_2, v_1)^T$. Then, by the Helmholtz decomposition we get

$$v = \nabla \Delta^{-1} \alpha + \nabla \nabla^\perp \Delta^{-1} \omega := Q(v) + P(v).$$

The system \([1.3]-[1.4]\) written in terms of $\rho, \alpha, \omega$, becomes

$$\begin{align*}
\partial_t \rho + y \partial_x \rho + \alpha &= 0, \\
\partial_t \alpha + y \partial_x \alpha + 2 \partial_x (\partial_y (\Delta^{-1}) \alpha + \partial_x (\Delta^{-1}) \omega) + \frac{1}{M^2} \Delta \rho &= 0, \\
\partial_t \omega + y \partial_x \omega - \alpha &= 0.
\end{align*}$$

To proceed with the analysis of the system \(\begin{align*} [3.2] \end{align*}\)-\(\begin{align*} [3.3] \end{align*}\), we make a change of variables in order to eliminate the transport term. The natural choice for the new variables is the following

$$X = x - yt, \quad Y = y,$$

and the operators will change as follows

$$\begin{align*}
\partial_x &= \partial_X, \\
\partial_y &= \partial_Y - t \partial_X, \\
\Delta &= \Delta_L := \partial_{XX} + (\partial_Y - t \partial_X)^2.
\end{align*}$$

We also define the functions on the moving frame as

$$\begin{align*}
R(t, X, Y) &= \rho(t, X + tY, Y), \\
A(t, X, Y) &= \alpha(t, X + tY, Y), \\
\Omega(t, X, Y) &= \omega(t, X + tY, Y).
\end{align*}$$

In addition, by summing \(\begin{align*} [3.2] \end{align*}\) with \(\begin{align*} [3.3] \end{align*}\), we notice that $\rho + \omega$ is transported by the Couette flow. By defining

$$\Xi(t, X, Y) := R(t, X, Y) + \Omega(t, X, Y),$$

we have that $\partial_t \Xi = 0$, and consequently

$$\begin{align*}
\Omega(t, X, Y) &= \Xi_{in}(X, Y) - R(t, X, Y),
\end{align*}$$

where $\Xi_{in} = \omega_{in} + \rho_{in}$.

Thus, we can rewrite the system \(\begin{align*} [3.2] \end{align*}\)-\(\begin{align*} [3.3] \end{align*}\) on the moving frame in terms only of $A$ and $R$, as follows

$$\begin{align*}
\partial_t R &= -A, \\
\partial_t A &= -2 \partial_X (\partial_Y - t \partial_X) (\Delta^{-1}) A + \left[ - \frac{1}{M^2} \Delta + 2 \partial_{XX} (\Delta^{-1}) \right] R \\
&\quad - 2 \partial_{XX} (\Delta^{-1}) \Xi_{in}.
\end{align*}$$
3.1. Fourier analysis. In this section, we study the problem in the frequency space. For the sake of notational convenience, we introduce

\[ p(t, k, \eta) = -\Delta L = k^2 + (\eta - kt)^2, \]

\[ p'(t, k, \eta) = -2k(\eta - kt). \]

By taking the Fourier transform of the system (3.7)-(3.8), we get that

\[ \partial_t \hat{R} = -\hat{A}, \]

\[ \partial_t \hat{A} = \frac{p'}{p} \hat{A} + \left( \frac{p}{M^2} + \frac{2k^2}{p^2} \right) \hat{R} - \frac{2k^2}{p^2} \hat{\Xi}_{in}. \]

Remark 3.1 (Transient decay). Just looking at (3.9), one notice that, for \( t < \eta/k \), the first term in the r.h.s. of (3.9) acts as a damping for \( \hat{A} \), since \( p' < 0 \). For \( t > \eta/k \), the fact that \( p' > 0 \) introduces a growth on \( \hat{A} \).

Now just divide (3.9) by \( p \) to have that

\[ \frac{\partial_t \hat{R}}{p} = -\hat{A}, \]

\[ \frac{\partial_t \hat{A}}{p} = \left( \frac{1}{M^2} + \frac{2k^2}{p^2} \right) \hat{R} - \frac{2k^2}{p^2} \hat{\Xi}_{in}. \]

We want to write the system (3.10)-(3.11) as a dynamical system, or equivalently we are looking for a proper symmetrization. So define

\[ \hat{Z}(t) = \left( \frac{\hat{R}}{Mp^{1/4}(t)}, \frac{\hat{A}}{p^{3/4}(t)} \right)^T, \]

where the dependence on \( k, \eta \) is omitted, since from now on we will think on \( k, \eta \) as fixed parameters. Now we have reduced everything to provide lower and upper bounds for \( |\hat{Z}(t)| \).

We divide (3.10) by \( Mp^{1/4} \) and multiply (3.11) by \( p^{1/4} \), to find that \( \hat{Z}(t) \) solves

\[ \frac{d}{dt} \hat{Z}(t) = L(t)\hat{Z}(t) + F(t)\hat{\Xi}_{in}, \]

where

\[ L(t) = \begin{bmatrix} -\frac{p'}{4p} & -\frac{\sqrt{p}}{M} \\ \frac{\sqrt{p}}{M} & \frac{2Mk^2}{p^{3/2}} \end{bmatrix}, \quad F(t) = \begin{bmatrix} 0 \\ -\frac{2k^2}{p^{3/2}} \end{bmatrix}. \]

Now we have to deal with a nonautonomous 2D dynamical system. The solution of (3.13) is given by Duhamel’s formula as

\[ \hat{Z}(t) = \Phi_L(t, 0) \left( \hat{Z}_{in} + \int_0^t \Phi_L(0, s)F(s)\hat{\Xi}_{in}ds \right), \]

where \( \Phi_L \) is the evolution operator defined in (1.21). Notice that \( \Phi_L \neq \exp(L) \) since \( L(t)L(s) \neq L(s)L(t) \).

We present the properties of \( \Phi_L \) in a slightly more general setting for a 2D nonautonomous dynamical system.

Lemma 3.2. Let \( V(t) = (V_1(t), V_2(t))^T : [t_0, +\infty) \times \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) such that

\[ \frac{d}{dt} V(t) = \mathcal{L}(t)V(t), \]

\[ V(t_0) = V_{in}, \]

where

\[ \mathcal{L}(t) = \begin{bmatrix} 0 & -\frac{\sqrt{p}}{M} \\ \frac{\sqrt{p}}{M} & \frac{2Mk^2}{p^{3/2}} \end{bmatrix}. \]
where
\begin{equation}
L(t) = \begin{bmatrix}
-a(t) & -b(t) \\
b(t) + c(t) & a(t)
\end{bmatrix}.
\end{equation}

Assume that \( a, b, c \geq 0 \) and \( c \leq Cb \) for some constant \( C > 0 \). Let \( \zeta^2 = (b + c)/b \) and \( \beta^2 = (b + c)b \). Consider coefficients which satisfies
\begin{equation}
|a| \leq \frac{1}{2}, \quad \int_{t_0}^{\infty} \left| \frac{d}{dt} \log(\zeta(\tau)) \right| d\tau < +\infty, \quad \int_{t_0}^{\infty} \left| \frac{d}{dt} \left( \frac{a}{\beta} \right) \right| d\tau < +\infty
\end{equation}

Let \( \Phi_{L} \) be the flow map for the problem (3.15), as defined in (1.21).

Then, there exists two constants \( c, C > 0 \) such that
\begin{equation}
c|V_{in}| \leq |\Phi_{L}(t, t_0)V_{in}| \leq C|V_{in}|.
\end{equation}

In addition, let \( V_1(t) = r(t) \cos(\theta(t)) \) and \( V_2(t) = r(t) \sin(\theta(t)) \). Then it holds that
\begin{equation}
\frac{d}{dt}\theta(t) = b(t) + c(t) \cos(\theta(t)) + a(t) \sin(2\theta(t)).
\end{equation}

Let us just stress that it is a key Lemma which tells us that the evolution of the dynamical system associated to (3.16) is contained in an annular region of the plane and rotates with an angular velocity given by \( \theta \).

The hypothesis on the coefficients (3.17) are by no means sharp, but for us it is important that are satisfied by the matrix \( L(t) \) uniformly with respect to \( k, \eta, M \). In our case, we have that
\begin{equation}
a(t) = \frac{1}{4} p', \quad b(t) = \sqrt{p}/M, \quad c(t) = \frac{2Mk^2}{p^{3/2}},
\end{equation}
so it is straightforward to see that (3.17) is fulfilled.

**Remark 3.3.** A direct consequence of (3.19) in our case is the following. Consider \( \hat{Z}(t) \) as defined in (3.12), by writing it in polar coordinates, namely \( \hat{Z}(t) = r(t)(\cos(\theta(t)), \sin(\theta(t))) \), it holds that
\begin{equation}
\frac{d}{dt}\theta(t) = \frac{\sqrt{p}}{M} + \frac{2Mk^2}{p^{3/2}} \cos(\theta(t)) + \frac{p'}{4p} \sin(2\theta(t)).
\end{equation}

Hence, by integrating (3.20) and retaining leading order terms one may infer a dispersion relation like \( M^{-1}\sqrt{k^2 + (\eta - kt)^2} \), which was also observed in [4][11]. However, dispersive properties requires a more delicate analysis and we will deal with it in a forthcoming paper.

We now present the proof of Lemma 3.2.

**Proof.** We have to prove a bound for
\begin{equation}
V(t) = \Phi_{L}(t, 0)V_{in},
\end{equation}
where \( \Phi_{L}(t, 0) \) is defined in (1.20). It will be useful to write explicitly the dynamical system, so let \( V = (V_1, V_2) \). Then we are dealing with
\begin{equation}
\frac{d}{dt}V_1 = -a(t)V_1 - b(t)V_2,
\end{equation}
\begin{equation}
\frac{d}{dt}V_2 = (b(t) + c(t))V_1 + a(t)V_2.
\end{equation}
First of all we want to bound $|V(t)|$. For this purpose, define $\zeta(t) = \sqrt{(b + c)/b}$ and $\beta(t) = (\sqrt{b + c})\sqrt{b}$. Then multiply (3.21) by $\zeta$ and divide (3.22) by $\zeta$, to obtain that

$$\zeta \frac{d}{dt} V_1 = -a \zeta V_1 - \beta V_2,$$

(3.23)

$$\frac{1}{\zeta} \frac{d}{dt} V_2 = \beta V_1 + \frac{a}{\zeta} V_2.$$ 

(3.24)

Now multiply (3.23) by $V_1$, (3.24) by $V_2$ and add the two equations to have that

$$\frac{\zeta}{2} \frac{d}{dt} V_1^2 + \frac{1}{2\zeta} \frac{d}{dt} V_2^2 = -a \left( \zeta V_1^2 - \frac{1}{\zeta} V_2^2 \right).$$

(3.25)

Then, thanks to (3.21)-(3.22), just observe the following

$$\frac{d}{dt} (V_1 V_2) = \beta \left( \zeta V_1^2 - \frac{1}{\zeta} V_2^2 \right).$$

(3.26)

So using (3.26) into (3.25) we get that

$$\frac{\zeta}{2} \frac{d}{dt} V_1^2 + \frac{1}{2\zeta} \frac{d}{dt} V_2^2 + \frac{a}{\beta} \frac{d}{dt} (V_1 V_2) = 0.$$ 

(3.27)

Defining $\tilde{E}(t) = \zeta V_1^2 + \frac{1}{\zeta} V_2^2 + 2\frac{a}{\beta} V_1 V_2$, we have that

$$\frac{d}{dt} \tilde{E}(t) = \frac{d}{dt} \left( \log(\zeta) \right) \zeta V_1^2 + \frac{1}{\zeta} \frac{d}{dt} V_2^2 + \frac{a}{\beta} \frac{d}{dt} (V_1 V_2).$$

(3.28)

Calling $E(t) = \zeta V_1^2 + \zeta^{-1} V_2^2$, since $|a|/\beta < 1/2$, it holds that

$$-\frac{1}{2} E(t) \leq 2\frac{|a|}{\beta} V_1 V_2 \leq \frac{1}{2} E(t).$$

From (3.28), it follows that

$$\frac{1}{2} E(t) \leq \tilde{E}(t) \leq \frac{3}{2} E(t).$$

(3.29)

By combining (3.27) with (3.29), we get

$$\frac{d}{dt} \tilde{E}(t) \leq \frac{3}{2} \left( \left| \frac{d}{dt} \log(\zeta) \right| + \left| \frac{d}{dt} \log \left( \frac{a}{\beta} \right) \right| \right) \tilde{E}(t),$$

(3.30)

$$\leq \frac{9}{4} \left( \left| \frac{d}{dt} \log(\zeta) \right| + \left| \frac{d}{dt} \log \left( \frac{a}{\beta} \right) \right| \right) \tilde{E}(t).$$

Analogously, we get that

$$\frac{d}{dt} \tilde{E}(t) \geq -\frac{1}{4} \left( \left| \frac{d}{dt} \log(\zeta) \right| + \left| \frac{d}{dt} \log \left( \frac{a}{\beta} \right) \right| \right) \tilde{E}(t).$$

(3.31)

By combining (3.30), (3.31) with the hypothesis in (3.17), applying Grönwall’s inequality we infer that

$$c_0 \tilde{E}_{in} \leq \tilde{E}(t) \leq c_1 \tilde{E}_{in},$$

(3.32)

where $c_0, c_1$ can be explicitly computed. In view of (3.29), from (3.32) we get that

$$cE_{in} \leq E(t) \leq CE_{in},$$

where
hence proving (3.18) since $1 \leq \zeta \leq C$ thanks to our hypothesis on $b, c$. Observe also that if $\zeta \to 1$ and $a/\beta \to 0$ for $t \to \infty$, then $\tilde{E}(t) \to E(t)$, which means that $(V_1(t), V_2(t))$ will converge to a circular orbit for $t$ large enough.

Then compute the angular velocity, namely let

$$V_1(t) = r(t) \cos(\theta(t)), 
V_2(t) = r(t) \sin(\theta(t)).$$

Then it holds that $r^2 \dot{\theta} = xy - \dot{x}y$. So by (3.21) and (3.22) we infer that

$$\dot{\theta} = b(t) + c(t) \cos(\theta) + a(t) \sin(2\theta).$$

Now we are ready to restate Theorem 1.1, where we treat lower and upper bounds separately. Recall that we are denoting $\bar{f} = \int_x f \, dx$.

**Theorem 3.4.** Let $\rho_{in}, \omega_{in} \in L^2_x H_y^{-1/2}$ and $\alpha_{in} \in H_x^{-3/2} H_y^{-2}$. Then the solution of (3.2)-(3.3) with initial data $\rho_{in}, \alpha_{in}, \omega_{in}$ can be decomposed into its $x$-average satisfying (2.1)-(2.3) and the fluctuation around the average satisfies the following estimate

$$(3.33) \quad \|Q(v) - \bar{Q}(v)\|^2_{L^2_x} + \frac{1}{M^2} \|\rho - \bar{\rho}\|^2_{L^2_x} \gtrsim \|Z_{in} + \int_0^t \Phi_L(0, s) F(s) \Xi_{in} \, ds\|^2_{L^2_x H_y^{-1/2}}.$$ 

Clearly, looking at (3.33), if $\Xi_{in} = 0$, namely $\rho_{in} = -\omega_{in}$, we immediately have a linear growth in time for non-trivial initial conditions. When $\Xi_{in} \neq 0$, it may happen that the r.h.s. of (3.33) becomes zero for some $t$. For this reason, in the following Proposition we show that the set of initial data for which the r.h.s. of (3.33) vanishes at some time has empty in interior in an appropriate Sobolev space.

**Proposition 3.5.** Given $\rho_{in}, \omega_{in} \in L^2_x H_y^{-1/2}$ and $\alpha_{in} \in H_x^{-3/2} H_y^{-2}$, let

$$(3.34) \quad \Gamma(t, Z_{in}, \Xi_{in}) = Z_{in} + \int_0^t \Phi_L(0, s) F(s) \Xi_{in} \, ds$$

where $Z_{in}$ is defined as in (3.12) and $\Xi_{in} = \rho_{in} + \omega_{in}$.

Then for any $\epsilon > 0$, sufficiently small, there exist $(\rho_{in}^\epsilon, \alpha_{in}^\epsilon, \omega_{in}^\epsilon)$ such that

$$\|\rho_{in} - \rho_{in}^\epsilon\|^2_{L^2_x H_y^{-1/2}} + \|\omega_{in} - \omega_{in}^\epsilon\|^2_{L^2_x H_y^{-1/2}} + \|\alpha_{in} - \alpha_{in}^\epsilon\|^2_{H_x^{-3/2} H_y^{-2}} \leq 2\epsilon,$$

and by defining $Z_{in}^\epsilon, \Xi_{in}^\epsilon$ accordingly, the following inequality holds

$$(3.35) \quad \inf_{t \geq 0} \|\Gamma(t, Z_{in}^\epsilon, \Xi_{in}^\epsilon)\|^2_{L^2_x H_y^{-1/2}} \geq \frac{\epsilon}{2}.$$ 

We prove Proposition 3.5 in Subsection 3.2 where we construct the perturbation $(\rho_{in}^\epsilon, \alpha_{in}^\epsilon, \omega_{in}^\epsilon)$ explicitly, which satisfies a non-degeneracy condition analogue to (3.35) at fixed frequency.

Now we state the upper bounds. Notice that even in the Theorem below, we decouple our dynamics into its $x$-average and fluctuations around it, as done in Theorem 3.4.

**Theorem 3.6.** Let $\rho_{in}, \omega_{in} \in H_x^1 H_y^2$ and $\alpha_{in} \in H_x^{-1/2} H_y^3$ be the initial data of (3.2)-(3.3). Then it holds that
\[\|Q(v) - \overline{Q(v)}\|_{L^2}^2 + \frac{1}{M^2} \|\rho - \overline{\rho}\|_{L^2}^2 \lesssim \langle t \rangle \left( \frac{1}{M^2} \|\rho_{in}\|_{H^{1,\frac{1}{2}}} + \|\alpha_{in}\|_{H^{-\frac{1}{2}}} \right) + \|\rho_{in}\|_{H^1} + \|\omega_{in}\|_{H^{1,2}}. \tag{3.36}\]

For the incompressible part of the fluid, we have the following estimates,

\[\|P_1(v) - \overline{P_1(v)}\|_{L^2} \lesssim \langle t \rangle^{1/2} \left( \frac{1}{M} \|\rho_{in}\|_{H_x^{1/2} H_y^1} + \|\alpha_{in}\|_{H_x^{-1/2} H_y^{1/2}} \right) + \|\rho_{in}\|_{L^2 H_y^{1/2}} + \|\omega_{in}\|_{L^2 H_y^1}. \tag{3.37}\]

\[\|P_2(v)\|_{L^2} \lesssim \langle t \rangle^{3/2} \left( \frac{1}{M} \|\rho_{in}\|_{H_x^{1/2} H_y^{3/2}} + \|\alpha_{in}\|_{H_x^{-1/2} H_y^{3/2}} \right) + \|\rho_{in}\|_{H^1 H_y^{3/2}} + \|\omega_{in}\|_{H^1 H_y^{3/2}}. \tag{3.38}\]

As it will be clear from the proof of the previous Theorem, the upper bounds inferred at any fixed frequencies yields also a control of any Sobolev norms. Consequently, we could choose a suitable Sobolev space where also the acoustic part decays. This is an evidence of a weak type convergence.

**Theorem 3.7.** Assume the additional smoothness required to let the quantities on the r.h.s. to be finite.

For \(s_1 > 0, s_2 > 1/2\) it holds that

\[\|Q(v) - \overline{Q(v)}\|_{H_x^{s_1} H_y^{s_2}}^2 + \frac{1}{M^2} \|\rho - \overline{\rho}\|_{H_x^{s_1} H_y^{s_2}}^2 \leq \frac{1}{\langle t \rangle^{2s_2 - 1}} C(\rho_{in}, \alpha_{in}, \omega_{in}),\]

where the constant involves Sobolev norms of the initial data.

In the following we prove Theorems 3.4, 3.6 where we assume that \(\overline{\rho}_{in} = \overline{\alpha}_{in} = \overline{\omega}_{in} = 0\), to recover the general case see Proposition 2.1.

**Proof of Theorem 3.4.** We know that the elements of \(L(t)\), defined in (3.14), satisfies the hypothesis (3.17) required to apply Lemma 3.2. Recall that the solution of (3.13) is given by Duhamel’s formula as

\[\hat{Z}(t) = \Phi_L(t, 0) \left( \hat{Z}_{in} + \int_0^t \Phi_L(0, s) \hat{F}(s) \hat{\xi}_{in} ds \right) = \Phi_L(t, 0) \Gamma(t, \hat{Z}_{in}, \hat{\xi}_{in}), \tag{3.39}\]

where we have also used the definition of \(\Gamma\) given in (3.34). By Lemma 3.2 we infer that

\[|\hat{Z}(t)| \geq c |\Gamma(t, \hat{Z}_{in}, \hat{\xi}_{in})|. \tag{3.40}\]

Then we state the following basic inequalities

\[p = k^2 + (\eta - kt)^2 \geq (\eta - kt)^2, \quad (\eta - kt)/(\eta) \geq (kt). \tag{3.41}\]
Recalling the definitions given in (3.5) and (3.12), we are ready to prove (3.33). In fact, by the Helmholtz decomposition, see (3.1), it holds that

\[ \|Q(v)\|_{L^2}^2 + \frac{1}{M^2} \|\rho\|_{L^2}^2 = \|\partial_x \Delta^{-1} \alpha\|_{L^2}^2 + \|\partial_y \Delta^{-1} \alpha\|_{L^2}^2 + \frac{1}{M^2} \|\rho\|_{L^2}^2 = \sum_k \int \frac{\hat{\alpha}^2(t,k,\eta)}{k^2 + \eta^2} + \frac{1}{M^2} \hat{\rho}_2^2(t,k,\eta) d\eta \]

\[ = \sum_k \int \frac{\hat{\alpha}^2(t,k,\eta)}{k^2 + \eta^2} + \frac{1}{M^2} \hat{\rho}_2^2(t,k,\eta) d\eta \]

where in the last line we have just performed a change of variables. Thanks to (3.40) we have that

\[ \|Q(v)\|_{L^2}^2 + \frac{1}{M^2} \|\rho\|_{L^2}^2 = \sum_k \int \sqrt{p} \left( \frac{\hat{A}}{p^{3/4}} \right)^2 + \left( \frac{\hat{R}}{M p^{1/4}} \right)^2 d\eta \]

\[ \geq \sum_k \int \sqrt{p} |\Gamma(t, \hat{Z}_{in}, \hat{\Xi}_{in})|^2 d\eta \]

\[ \geq \sum_k \int \langle \eta - kt \rangle |\Gamma(t, \hat{Z}_{in}, \hat{\Xi}_{in})|^2 d\eta \]

\[ \geq \langle t \rangle \sum_k \int \frac{1}{\langle \eta \rangle} |\Gamma(t, \hat{Z}_{in}, \hat{\Xi}_{in})|^2 d\eta, \]

where in the last two lines we have used (3.41). Hence we get that

\[ \|Q(v)\|_{L^2}^2 + \frac{1}{M^2} \|\rho\|_{L^2}^2 \geq \langle t \rangle \|\Gamma(t, \hat{Z}_{in}, \hat{\Xi}_{in})\|_{L^2, H^{-1/2}_0}^2 \]

proving (3.33).

Proof of Theorem 3.6. We first prove the bounds for the incompressible part, namely (3.37) and (3.38). Observe that, by (3.6), it holds

\[ |\hat{\Omega}(t,k,\eta)| \leq |\hat{R}(t,k,\eta)| + |\hat{\Xi}_{in}(t,k,\eta)|. \]

Then we prove (3.37) as follows,

\[ \|P_1(v)\|_{L^2}^2 = \|\partial_y \Delta^{-1} \omega\|_{L^2}^2 \]

\[ = \sum_k \int \frac{(\eta - kt)^2 \hat{\Omega}^2}{p^2} d\eta \]

\[ \leq \sum_k \int M^2 \frac{(\eta - kt)^2}{p^{3/2}} \left( \frac{\hat{R}}{M p^{1/4}} \right)^2 + \frac{(\eta - kt)^2}{p^2} \hat{\Xi}_{in}^2 d\eta \]

\[ \leq \sum_k \int \frac{M^2}{\sqrt{p}} (|\hat{Z}_{in}|^2 + |\hat{\Xi}_{in}|^2) + \frac{1}{p} \hat{\Xi}_{in}^2 d\eta. \]
Arguing similarly as in the proof of Theorem 3.4, by Helmholtz decomposition we have

\[
\|P_1(v)\|^2_{L^2} \lesssim \frac{M^2}{\langle t \rangle} \left( \left\| \frac{\rho_{\infty}}{M} \right\|^2_{H^\frac{1}{2} H^\frac{1}{2}_v} + \left\| \alpha_{\infty} \right\|^2_{H^\frac{1}{2} H^\frac{1}{2}_v} + \left\| \omega_{\infty} \right\|^2_{L^2 H^\frac{1}{2}_v} \right) + \frac{1}{\langle t \rangle^2} \left( \left\| \rho_{\infty} \right\|^2_{L^2 H^1_v} + \left\| \omega_{\infty} \right\|^2_{L^2 H^1_v} \right),
\]

Similarly for \(P_2(v)\), we prove (3.38) by the following computation,

\[
\|P_2(v)\|^2_{L^2} = \left\| \partial_x \Delta^{-1} \omega \right\|^2_{L^2} \lesssim \sum_k \int M^2 \left( \frac{k^2}{\rho^{1/2}} \left\| \hat{R}_{mp} \right\|^2 (t, k, \eta) + \frac{k^2}{\rho^2} \hat{\omega}_{\infty}(t, k, \eta) d\eta \right) \lesssim \frac{M^2}{\langle t \rangle^3} \left( \left\| \rho_{\infty} \right\|^2_{H^\frac{1}{2} H^\frac{1}{2}_v} + \left\| \alpha_{\infty} \right\|^2_{H^\frac{1}{2} H^\frac{1}{2}_v} + \left\| \omega_{\infty} \right\|^2_{H^1 H^\frac{1}{2}_v} \right) + \frac{1}{\langle t \rangle^4} \left( \left\| \rho_{\infty} \right\|^2_{H^1 H^1_v} + \left\| \omega_{\infty} \right\|^2_{H^1 H^1_v} \right).
\]

To prove (3.36) first of all observe that

\[
\int_0^\infty |\Phi_L(t, s)F(s)| ds \leq C \int_0^\infty |F(s)| ds = C \int_0^\infty \frac{ds}{k^{3/2} (1 + (\eta/k - s)^2)^{7/4}} < \infty.
\]

Hence, recalling the definition of \(Z\), see (3.39), by combining Lemma 3.2 with (3.42) we infer that

\[
|\tilde{Z}(t, k, \eta)| \lesssim |\hat{Z}_{in}(k, \eta)| + |\hat{\Xi}_{in}(k, \eta)| \quad \text{for any } t \geq 0.
\]

Arguing similarly as in the proof of Theorem 3.4 by Helmholtz decomposition we have that

\[
\|Q(v)\|^2_{L^2} + \frac{1}{M^2} \left\| \rho \right\|^2_{L^2} \lesssim \sum_k \int \sqrt{p} \left( |\tilde{Z}_{in}|^2 + |\hat{\Xi}_{in}|^2 \right) d\eta.
\]

Then (3.36) directly follows from the previous inequality. \(\square\)

Finally, we present the proof of Proposition 3.5.

### 3.2. Proof of Proposition 3.5

Recall that

\[
\Gamma(t, Z_{in}, \Xi_{in}) = Z_{in} + \int_0^t \Phi_L(0, s)F(s)\Xi_{in} ds.
\]

By taking the Fourier transform in (3.43), with a slight abuse of notations, we have that

\[
\hat{\Gamma}(t, k, \eta) = \hat{Z}_{in}(k, \eta) + \int_0^t \Phi_L(0, s)F(s)\hat{\Xi}_{in}(k, \eta) ds.
\]

Now, let us fix the frequencies \(k, \eta\). In this way, \(t \mapsto \hat{\Gamma}(t)\) is a regular curve in \(\mathbb{R}^2\). We now want to construct a perturbation of the initial data.
First of all, by a computation similar to (3.42), we know that \( \lim_{t \to \infty} \Gamma(t, k, \eta) = \Gamma^\infty(k, \eta) \). Then, let us first consider the case \( \Gamma^\infty \neq 0 \).

We claim that in this case \( \Gamma(t, k, \eta) \) vanishes at most in a finite number of times \( t_i \) for \( i = 0, \ldots, n \).

Indeed, since \( |\Gamma^\infty| > 0 \), there is a \( T(\Gamma^\infty, k, \eta) > 0 \) such that

\[
|\Gamma(t, k, \eta)| > \frac{1}{2}|\Gamma^\infty(k, \eta)| \quad \text{for} \quad t \geq T(\Gamma^\infty, k, \eta).
\]

Hence, we know that \( \Gamma \) may vanish only for \( t \in [0, T(\Gamma^\infty, k, \eta)] \). Observe that, by (3.18), we have

\[
|\partial_t \Gamma(t, k, \eta)| = |\Phi_L(0, t)F(t)\hat{Z}_{\eta}(k, \eta)| \geq C|F(t)\hat{Z}_{\eta}(k, \eta)| > C(T)|\hat{Z}_{\eta}(k, \eta)|.
\]

Consequently, by continuity of \( \partial_t \Gamma \), we have that \( \Gamma \) vanishes at most in a finite number of times in the interval \( [0, T(\Gamma^\infty)] \).

Now we can construct the perturbation of the initial data. Consider the set of vectors tangential to the zeros of \( \Gamma \), namely \( \partial_t \Gamma(t_i, k, \eta) \) for \( i = 0, \ldots, n \). For any \( \epsilon < \min\{|\Gamma^\infty|/2, 1\} \), there is at least one unit vector \( \nu_t(k, \eta) \) which is not parallel to any \( \partial_t \Gamma(t_i, k, \eta) \) and such that

\[
|\Gamma(t, k, \eta) + \epsilon e^{-(k^2 + \eta^2)}\nu_t(k, \eta)| > \frac{1}{2}\epsilon e^{-(k^2 + \eta^2)}.
\]

By choosing

\[
\alpha^\ell_{\eta}(k, \eta) = \alpha_{\eta}(k, \eta) + \epsilon(k^2 + \eta^2)\frac{2}{3} e^{-(k^2 + \eta^2)}\nu_{\eta}(k, \eta),
\]

\[
\rho^\ell_{\eta}(k, \eta) = \rho_{\eta}(k, \eta) + \epsilon \frac{1}{M}(k^2 + \eta^2)^2 e^{-(k^2 + \eta^2)}\nu_{\eta}(k, \eta),
\]

\[
\omega^\ell_{\eta}(k, \eta) = \omega_{\eta}(k, \eta) - \epsilon \frac{1}{M}(k^2 + \eta^2)^2 e^{-(k^2 + \eta^2)}\nu_{\eta}(k, \eta),
\]

we have that in particular

\[
\hat{Z}_{\eta}^\ell(k, \eta) = \hat{Z}_{\eta}(k, \eta) + \epsilon e^{-(k^2 + \eta^2)}\nu_{\eta}(k, \eta),
\]

\[
\hat{Z}_{\eta}(k, \eta) = \hat{Z}_{\eta}(k, \eta).
\]

Consequently

\[
\hat{\Gamma}(t, k, \eta) = \hat{Z}_{\eta}^\ell(k, \eta) + \int_0^t \Phi_L(0, s)F(s)\hat{Z}_{\eta}(k, \eta)ds = \Gamma(t, k, \eta) + \epsilon e^{-(k^2 + \eta^2)}\nu_{\eta}(k, \eta).
\]

By combining (3.44) with (3.45), and our choice of \( \epsilon \), we get that

\[
|\Gamma^\ell(t, k, \eta)| \geq \frac{1}{2} \min\left(|\Gamma^\infty(k, \eta)|, \epsilon e^{-(k^2 + \eta^2)}\right) \quad \text{for any} \quad t > 0.
\]

Let us now turn to the case \( \Gamma^\infty(k, \eta) = 0 \). First we choose

\[
\alpha_{\eta}^1 = \alpha_{\eta} + \epsilon(k^2 + \eta^2)\frac{2}{3} e^{-(k^2 + \eta^2)},
\]

so that for the corresponding \( \Gamma^1 \) we get \( |\Gamma^1(k, \eta)| = \epsilon e^{-(k^2 + \eta^2)} \). At this point, we can repeat the previous argument.

Resuming, by using Plancherel’s Theorem, from the bound (3.46) we obtain (3.35). \( \square \)
4. Building blocks for shear flow near Couette

In this Section we deal with some technical difficulties that we need to overcome before proving Theorem 1.6. The problems that we face in this Section are the building blocks for Section 5 where we prove Theorem 1.6.

Recall the equations for the linearization of a perturbation around a general shear flow \((U(y), 0)\), given by

\begin{align*}
\partial_t \rho + U \partial_x \rho + \alpha &= 0, & \text{in } \mathbb{T} \times \mathbb{R}, \\
\partial_t \alpha + U \partial_x \alpha + 2U' \partial_x \left( \partial_y \Delta^{-1} \alpha + \partial_x \Delta^{-1} \omega \right) + \frac{1}{M^2} \Delta \rho &= 0, \\
\partial_t \omega + U \partial_x \omega - U' \alpha &= U'' \left( \partial_y \Delta^{-1} \alpha + \partial_x \Delta^{-1} \omega \right).
\end{align*}

From now on we consider monotone shear flows. To follow the background shear, we perform the following change of variables

\[ X = x - U(y)t, \quad Y = U(y). \]

Now let

\begin{align*}
R(t, X, Y) &= \rho(t, X + tY, Y), \\
A(t, X, Y) &= \alpha(t, X + tY, Y), \\
\Omega(t, X, Y) &= \omega(t, X + tY, Y).
\end{align*}

By the monotonicity of \(U\), we can define the following quantities

\[ g(Y) := U'(U^{-1}(Y)), \quad b(Y) := U''(U^{-1}(Y)), \]
so that the differential operators change as follows
\begin{align*}
\partial_x &= \partial_X, \\
\partial_y &= g(Y)(\partial_Y - t\partial_X),
\end{align*}
(4.4)
\[ \Delta = \Delta_t := \partial_{XX} + g^2(Y)(\partial_Y - t\partial_X)^2 + b(Y)(\partial_Y - t\partial_X). \]

In what follows we will also need the following operator defined as
\[ \partial_t \Delta_L = -2\partial_X(\partial_Y - t\partial_X), \]
(4.5)
where \( \Delta_L \) is as in (3.4), so (4.5) is the natural definition that comes from the Fourier side.

We consider shear flows \( U(y) \) near Couette in the sense that we will assume
\[ \|g^2 - 1\|_{H^s} \leq \epsilon, \quad \|b\|_{H^s} \leq \epsilon, \]
for some \( s \geq 1 \), where \( \epsilon \ll 1 \).

Let us notice that, by equations (4.2)-(4.3), we need to give a proper definition to \( \Delta_t^{-1} \) in the moving frame. By assuming for the moment we are able to define \( \Delta_t^{-1} \), the equations (4.1)-(4.3) in the new reference frame become
\begin{align*}
\partial_t R &= -A, \quad \text{in } T \times \mathbb{R}, \\
\partial_t A &= g^2[\partial_t \Delta_L] \Delta_t^{-1} A - 2g\partial_{XX} \Delta_t^{-1} \Omega - \frac{1}{M^2} \Delta_t R, \\
\partial_t \Omega &= [g + bg(\partial_Y - t\partial_X)] A + b\partial_X \Delta_t^{-1} \Omega.
\end{align*}
(4.6)-(4.8)

In the following Subsections we consider three necessary steps in order to prove Theorem 1.6.

In Subsection 4.1 we define \( \Delta_t^{-1} \) and we present useful properties of operators associated to it.

In Subsection 4.2 we deal with the following problem
\[ \partial_t f = b\partial_X \Delta_t^{-1} f. \]

In Subsection 4.3 we consider
\[ \partial_t f = g^2[\partial_t \Delta_L] \Delta_t^{-1} f. \]

At the beginning of each Subsection we comment more about the problems under consideration.

4.1. Definition of \( \Delta_t^{-1} \). The purpose of this Subsection is to define the operator \( \Delta_t^{-1} \). Usually it is enough to define the stream function associated to it, see [12, 47, 77, 78]. Instead, here we define \( \Delta_t^{-1} \) through a perturbative argument from \( \Delta_L^{-1} \). Moreover, we will also provide some useful properties associated to \( \Delta_t^{-1} \).

Recall that
\[ \Delta_L = \partial_{XX} + (\partial_Y - t\partial_X)^2, \]
and we know that, for \( k \neq 0 \), \( \Delta_L^{-1} \) is well defined as a time dependent Fourier multiplier.

First of all, since \( \Delta_L \) is a well defined negative self-adjoint operator, we define the space \( \tilde{H}^2 \) as follows
\[ \|f\|^2_{\tilde{H}^2} = \|(I - \Delta_L)f\|^2_{L^2} = \langle f, f \rangle + \langle \nabla_L f, \nabla_L f \rangle, \]
where \( \nabla_L = (\partial_X, \partial_Y - t\partial_X)^T \). Then we have the following.
Proposition 4.1. Let $\|g^2 - 1\|_{H^k} \leq \epsilon$, $\|b\|_{H^k} \leq \epsilon$. Then $\Delta_i^{-1} : L^2(\mathbb{T} \times \mathbb{R}) \to \tilde{H}^2(\mathbb{T} \times \mathbb{R})$ is well defined and bounded for $k \neq 0$. In addition one has that

$$(4.9) \quad \Delta_i^{-1} = \Delta_L^{-1} T_2 = T_1 \Delta_L^{-1},$$

where $T_2 : L^2(\mathbb{T} \times \mathbb{R}) \to L^2(\mathbb{T} \times \mathbb{R})$ and $T_1 : \tilde{H}^2(\mathbb{T} \times \mathbb{R}) \to \tilde{H}^2(\mathbb{T} \times \mathbb{R})$ are operators bounded by the identity. In particular it holds that

$$(4.10) \quad T_2 = \sum_{n=0}^{+\infty} \tilde{T}_2^n, \quad T_1 = \sum_{n=0}^{+\infty} \tilde{T}_1^n,$$

where

$$(4.11) \quad \tilde{T}_2 = [(g^2 - 1)(\partial_Y - t\partial_X)^2 + b(\partial_Y - t\partial_X)] (-\Delta_L^{-1}),$$

$$\tilde{T}_1 = (-\Delta_L^{-1}) [(g^2 - 1)(\partial_Y - t\partial_X)^2 + b(\partial_Y - t\partial_X)].$$

In addition, we have the following bounds

$$(4.12) \quad \|\tilde{T}_2\|_{L^2 \to L^2} \leq C\epsilon, \quad \|\tilde{T}_1\|_{\tilde{H}^2 \to \tilde{H}^2} \leq C\epsilon,$$

for a suitable $C > 1$ and $C\epsilon < 1$.

Before proving Proposition 4.1, we state a useful Corollary to list some properties of operators related to $\Delta_i^{-1}$.

Corollary 4.2. Let $B(\nabla)$ be a Fourier multiplier, $f \in H^s(\mathbb{T} \times \mathbb{R})$ and $T_2$ as defined in (4.10). Assume that there is a $\beta \geq 0$ such that

$$|B(k, \eta)| \lesssim \langle \eta - \xi \rangle^\beta |B(k, \xi)|.$$  

Then, if $\|g^2 - 1\|_{H^s} \leq \epsilon$ and $\|b\|_{H^s} \leq \epsilon$, we have that

$$(4.13) \quad \frac{1}{1 + C\epsilon} \|Bf\|_{H^s} \leq \|BT_2 f\|_{H^s} \leq \frac{1}{1 - C\epsilon} \|Bf\|_{H^s},$$

for a suitable $C > 1$ and $C\epsilon < 1$.

Let $\tilde{T}_2$ as defined in (4.11), it holds that

$$(4.14) \quad \partial_t \tilde{T}_2 = \tilde{T}_2 [\partial_t \Delta_L] \Delta_L^{-1} + [(g^2 - 1)\partial_t \Delta_L - b\partial_X] (-\Delta_L^{-1})$$

$$(4.15) \quad \partial_t T_2 = -T_2 [\partial_t \tilde{T}_2] T_2,$$

$$(4.16) \quad \partial_t \Delta_L^{-1} = -\Delta_L^{-1} \partial_t \Delta_L \Delta_L^{-1} = -[\partial_t \Delta_L] \Delta_L^{-1} \Delta_L^{-1} + \Delta_L^{-1} \partial_t T_2.$$  

Remark 4.3. Regarding the operators defined in (4.14)-(4.16), following the strategy to prove (4.13), one can prove that

$$(4.17) \quad \|B\partial_t T_2 f\|_{H^s} \leq C\epsilon \left( \|\partial_t \Delta_L [(-\Delta_L^{-1}) B] f\|_{H^s} + \|\partial_X [(-\Delta_L^{-1}) B] f\|_{H^s} \right),$$

$$(4.18) \quad \|B\partial_t T_2 f\|_{H^s} \leq C\epsilon \left( \|\partial_t \Delta_L [(-\Delta_L^{-1}) B] f\|_{H^s} + \|\partial_X [(-\Delta_L^{-1}) B] f\|_{H^s} \right),$$

where $\partial_t \Delta_L^{-1} f \leq (1 + C\epsilon) \left( \|\partial_t \Delta_L [(-\Delta_L^{-2}) B] f\|_{H^s} + C\epsilon \right) \left( \|\partial_X [(-\Delta_L^{-2}) B] f\|_{H^s} \right).$

We will not detail the proof of (4.17)-(4.18) since we never explicitly use one of these bounds.

Now we present the proof of Proposition 4.1.
Proof of Proposition 4.1. Notice that
\[
\Delta_t = \Delta_L + (g^2 - 1)(\partial_Y - t\partial_X)^2 + b(\partial_Y - t\partial_X)
\]
\[
= \left[I - ((g^2 - 1)(\partial_Y - t\partial_X)^2 + b(\partial_Y - t\partial_X))(-\Delta_L^{-1})\right]\Delta_L
\]
\[
:= [I - \tilde{T}_2] \Delta_L.
\]
Let \(f, g \in L^2(\mathbb{T} \times \mathbb{R})\). Then we have that
\[
\left|\langle \tilde{T}_2 f, h \rangle\right| \leq |\langle (g^2 - 1)(\partial_Y - t\partial_X)^2(-\Delta_L^{-1})f, h \rangle| + |\langle b(\partial_Y - t\partial_X)(-\Delta_L^{-1})f, h \rangle|
\]
\[
:= S_1 + S_2.
\]
Let us bound \(S_1\). By Plancherel Theorem, it holds that
\[
S_1 \leq \sum_{k \neq 0} \int |\hat{h}(k, \eta)| \int_{\xi} \left|\frac{(\xi - kt)^2}{k^2 + (\xi - kt)^2}\right| f(k, \eta) d\xi d\eta
\]
\[
\leq \left|\langle \hat{h}, |g^2 - 1| * |\hat{f}| \rangle\right|
\]
\[
\leq \|h\|_{L^2} \|f\|_{L^2} \left\|\left(g^2 - 1\right)\right\|_{L^1_b}
\]
\[
\leq C \|h\|_{L^2} \|f\|_{L^2}
\]
where the last two lines follow from Cauchy-Schwarz and Young’s inequality, see (B.2).

Proceeding analogously for \(S_2\), we prove (4.12). So define
\[
T_2 := [I - \tilde{T}_2]^{-1} = \sum_{n=0}^{+\infty} \tilde{T}_2^n,
\]
where the last equality holds since the Neumann series is well defined if \(\epsilon\) is small enough. Hence, the first equality of (4.9) follows.

For the other characterization, similarly we have that
\[
\Delta_t = \Delta_L \left[I - (-\Delta_L^{-1})(g^2 - 1)(\partial_Y - t\partial_X)^2 + b(\partial_Y - t\partial_X)\right]
\]
\[
:= \Delta_L [I - \tilde{T}_1].
\]
Proceeding analogously as done for \(\tilde{T}_2\), we infer that \(\|\tilde{T}_1\|_{H^{2} \rightarrow H^2} < C\epsilon\). Then define
\[
T_1 := (I - \tilde{T}_1)^{-1} = \sum_{n=0}^{\infty} \tilde{T}_1^n,
\]
proving also the second equality of (4.9). \(\square\)

Now we pass to the proof of Corollary 4.2 which we detail since we will use it several times in what follows.

Proof of Corollary 4.2. To prove (4.13), from the expression of \(T_2\) as Neumann series and triangular inequality, we have that
\[
\left\|Bf\right\|_{H^s} - \left\|B\tilde{T}_2 T_2 f\right\|_{H^s} \leq \left\|BT_2 f\right\|_{H^s} \leq \left\|Bf\right\|_{H^s} + \left\|B\tilde{T}_2 T_2 f\right\|_{H^s}.
\]
It is not hard to show that (4.13) is proved if we are able to show
\[
\left\|B\tilde{T}_2 T_2 f\right\|_{H^s} \leq C\epsilon \left\|BT_2 f\right\|_{H^s}.
\]
Then, we proceed analogously as done in (4.19). For convenience, recalling the definition of \(\tilde{T}_2\) given in (4.11), let
\[
\tilde{T}_2 := \tilde{T}_2^g + \tilde{T}_2^b.
\]
Now we have that
\[
\left\|BT_2^*T_2f\right\|_{H^s}^2 \leq \sum_k \int_{\mathbb{R}} \langle k, \eta \rangle^{2s} B^2(k, \eta) d\eta \left( \int_{\mathbb{R}} (g^2 - 1)(\eta - \xi) T_2 f(k, \xi) d\xi \right)^2,
\]
where we have used the hypothesis on $B$ and the fact that $\langle k, \eta \rangle \lesssim \langle \eta - \xi \rangle \langle k, \xi \rangle$. Then, thanks to Young’s convolution inequality, see (B.1), we have that
\[
\left\|BT_2^*T_2f\right\|_{H^s}^2 \lesssim \left\|\langle \cdot \rangle^{\beta+s}(g^2 - 1) \ast \langle \cdot \rangle^s BT_2 f\right\|_{L^2}^2 \lesssim \left\|g^2 - 1\right\|_{H^{s+\beta+1}}^2 \left\|BT_2 f\right\|_{H^s}^2.
\]
Arguing analogously for $\tilde{T}_2$, we prove that
\[
\left\|BT_2^*T_2f\right\|_{H^s} \lesssim \|b\|_{H^{s+\beta+1}} \left\|BT_2 f\right\|_{H^s}.
\]
By combining (4.21) and (4.22) we prove (4.20). So the bounds in (1.13) are proved.

The equality (4.14) is just the Leibniz rule applied to $\tilde{T}_2$ on the Fourier side, namely for any $f \in L^2(\mathbb{T} \times \mathbb{R})$, one has that
\[
\left\|\partial_i \tilde{T}_2 f\right\|_{H^s} = \mathcal{F}^{-1} \left(\left[\partial_i \mathcal{F}(\tilde{T}_2)\right] \ast \hat{f}\right).
\]
By an explicit computation of (4.23) one infer (4.14).

Since $T_2$ and $T_2^{-1}$ are defined through $\tilde{T}_2$, we can apply the Leibniz rule also to them. So observe that
\[
0 = \partial_i (T_2 T_2^{-1}) = T_2 (\partial_i T_2^{-1}) + (\partial_i T_2) T_2^{-1}.
\]
By the definition of $T_2$, we know that $T_2^{-1} = (I - \tilde{T}_2)$, hence (4.15) is proved.

The proof of the first equality in (4.16) is exactly the same performed in (4.24). The second equality it is just the Leibniz rule applied to the first definition in (4.9) but can be proven also in the following way
\[
-\Delta_l^{-1} \partial_i \Delta_l \Delta_l^{-1} f = -\Delta_L^{-1} T_2 \partial_i \left[\Delta_L - \tilde{T}_2 \Delta_L\right] \Delta_L^{-1} T_2
\]
\[
= -\Delta_L^{-1} T_2 \partial_i \Delta_L \Delta_L^{-1} T_2 + \Delta_L^{-1} T_2 \partial_i \tilde{T}_2 T_2 + \Delta_L^{-1} T_2 \partial_i \Delta_L \Delta_L^{-1} T_2.
\]
Since $T_2 = I + T_2 \tilde{T}_2$, by combining the previous identity with (4.15) we have proved (4.16).

4.2. Building block 1. In this section, we consider the following problem
\[
\partial_t f = b \partial_X \Delta_l^{-1} f, \quad \text{in } \mathbb{T} \times \mathbb{R},
\]
\[
f|_{t=0} = f_{in}.
\]
We are interested in considering (4.25) because one of the extra terms with respect to the Couette case (3.7)-(3.8), which appears in (4.6)-(4.8) and needs to be treated separately, is the last term in the r.h.s. of (4.8). This is also one of the terms that prevents the conservation of $R + \Omega$.

Observe that (4.25) is the equation of a monotone shear flow near Couette in the incompressible setting, see [77, 78]. In particular, in [78, Sec. 4] the problem (4.25) is treated via weighted energy estimates. Then we need the following result.
Theorem 4.4. Let \( \epsilon \ll 1 \) and assume \( \|g^2 - 1\|_{H^{s+1}_T} \leq \epsilon, \|b\|_{H^{s+2}_T} \leq \epsilon \). Let \( f_{in} \in H^s(\mathbb{T} \times \mathbb{R}) \) the initial datum of (4.25). Then it holds that
\[
\|f(t)\|_{H^s} \lesssim \|f_{in}\|_{H^s}.
\]

We present the proof of Theorem 4.4 since it will be propedeutical to our analysis of the other building block considered in Subsection 4.3. We will follow essentially the same line of the proof for [78, Theorem 4.1], one difference is that here we make use of the definition of \( \Delta_t^{-1} \) given in (4.9).

Proof. The idea is that we have to properly weight the function \( f \), since if we try to directly estimate \( \|f\|_{H^s} \) using (4.25), in principle we should control \( \|f\|_{H^{s+2}} \) to recover integrability in time from the operator \( \Delta_t^{-1} \). Instead, define the following Fourier multiplier
\[
\partial_t z(t, k, \eta) = \frac{|k|}{k^2 + (\eta - k t)^2} z(t, k, \eta),
\]
\[
z(0, k, \eta) = 1.
\]
Notice that \( z \) is a bounded Fourier multiplier, in fact we have that
\[
z(t, k, \eta) = \exp \left( \frac{1}{|k|} \left[ \arctan \left( \frac{\eta}{k} - t \right) - \arctan \left( \frac{\eta}{k} \right) \right] \right) \leq C,
\]
where \( C \) does not depends on \( k, \eta, t \). Notice that \( z^{-1} \) is a bounded Fourier multiplier too and it holds
\[
\|z^{-1} f\|_{H^s} \approx \|f\|_{H^s}.
\]
The latter equivalence means that if we are able to bound \( \|z^{-1} f\|_{H^s} \), we prove Theorem 4.4.

We have the following properties that allows us to exchange frequencies,
\[
(4.26) \quad z(t, k, \xi) \approx z(t, k, \eta),
\]
\[
\left| \frac{\partial_t z}{z} \right| (t, k, \xi) = \frac{|k|}{k^2 + (\xi - k t)^2} \lesssim |\eta - \xi|^2 \left| \frac{\partial_t z}{z} \right| (t, k, \eta),
\]
\[
(4.27) \quad \langle k, \eta \rangle \lesssim |\eta - \xi| \langle k, \xi \rangle.
\]
Using (4.25), we directly compute the following
\[
\frac{1}{2} \frac{d}{dt} \|z^{-1} f\|_{H^s}^2 = - \left( z^{-1} f, \frac{\partial_t z}{z} z^{-1} f \right)_s + \langle z^{-1} f, z^{-1} b \partial_X \Delta_t^{-1} f \rangle_s
\]
\[
:= - \left\| \sqrt{\frac{\partial_t z}{z}} z^{-1} f \right\|_{H^s}^2 + S_b.
\]
(4.28)

We are reduced to provide a bound for \( S_b \). By Plancherel Theorem, we have that
\[
|S_b| \leq \left( z^{-1} |f|, z^{-1} |b \ast \Delta_t^{-1} f| \right)_s
\]
\[
= \left( z^{-1} |f|, z^{-1} |b \ast \Delta_t^{-1} T_2 f| \right)_s,
\]
where the last equality follows by Proposition (4.1). Notice that, by definition, we have
\[ \partial_t z/z = |\partial_X \Delta^{-1}_L|, \]
which implies that
\[ |S_b| \lesssim \left( \langle \cdot \rangle \hat{z}^{-1} f, \langle \cdot \rangle \hat{S}_b \right) \lesssim \left( \langle \cdot \rangle \hat{z}^{-1} f, \langle \cdot \rangle \hat{S}_b \right), \]
In the last line we have used properties (4.26)-(4.27) to exchange frequencies in order to
commute the multipliers with the convolution, see also Lemma B.2.

Now we can apply Cauchy-Schwarz plus Young’s inequality, see (B.2), to obtain that
\[ |I_z| \lesssim \|b\|_{H^{s+2}} \left\| \sqrt{\partial_t z} \right\| \left\| \sqrt{\partial_t z} \right\| \lesssim \epsilon \left\| \sqrt{\partial_t z} \right\|^2, \]
where we have used the assumption on \( b \) and the Corollary 4.2, since \( \sqrt{\partial_t z} \) satisfies the required hypothesis to apply
Corollary 4.2.

By combining (4.28) with (4.29), for \( \epsilon \) small enough, we get
\[ \frac{d}{dt} \|z^{-1} f\|_{H^s}^2 \leq 0, \]
hence proving Theorem 4.4.

4.3. Building block 2. In this section, we consider the problem given by
\[ \partial_t f = g^2 \partial_t \Delta_L \Delta^{-1}_L f, \quad \text{in } T \times \mathbb{R}, \]
\[ f|_{t=0} = f_{in}. \]

We need to deal with the problem (4.30) since another main difference with respect to
the Couette case comes from the first term on the r.h.s. of (4.7). While the analogous term
in the Couette case can be treated explicitly, here we need to study in a different way the
problem. More precisely, we have the following.

**Theorem 4.5.** Let \( f_{in} \in H^s(T \times \mathbb{R}) \) the initial datum of (4.30). Let \( \epsilon \ll 1 \) and assume
that \( \|g^2 - 1\|_{H^{s+8}} \leq \epsilon, \|b\|_{H^{s+8}} \leq \epsilon. \)

Then there is a constant \( C > 1 \), such that for \( \epsilon = 2C \epsilon < 1 \) it holds
\[ \left\| \Delta_L^{-1+\epsilon} f \right\|_{H^s} \lesssim \|f_{in}\|_{H^s}. \]

**Remark 4.6.** As done in the proof of Theorem 4.4, to obtain (4.31) we will perform a
weighted energy estimate. Namely we need two weights \( w, m \), see (4.32) and (4.36) below,
to prove that
\[ \frac{d}{dt} \left\| w^{-1} m^{-1} f \right\|_{H^s}^2 \leq 0. \]
Looking at the bound (4.31), we are losing derivatives on \( f_{in} \). Anyway, one could prove that
\[ \left\| \Delta_L^{-1+\epsilon} f \right\|_{H^s} \lesssim \left\| \Delta_L^{-1+\epsilon} f_{in} \right\|_{H^s}, \]
just by changing the definition of \( w(t = 0) = 1 \) with \( w(t = 0) = (-\Delta)^{1+\epsilon}. \)
We prefer to use \( w(t = 0) = 1 \) since in Section 5 with the other definition we would have to pay regularity on other initial quantities.

**Remark 4.7.** The \( \tilde{\epsilon} \)-correction present in (4.31), is due to the fact that \( g^2 \) is a function close to one. In fact, if instead of (4.30) we have

\[
\partial_t f = c[\partial_t \Delta_L] \Delta_L^{-1} f,
\]

by explicit computation on the Fourier side we get \( \Delta_L^{-\epsilon} f \) bounded in any \( H^s \).

Now we present the proof of Theorem 4.5.

**Proof.** The strategy of the proof is similar to the one of Theorem 4.4, namely we want to control a weighted norm of \( f \). Recall the following definitions

\[
p(t, k, \eta) = -\hat{\Delta}_L = k^2 + (\eta - kt)^2,
\]

\[
p'(t, k, \eta) = -2k(\eta - kt).
\]

Define the following weight

\[
\partial_t w(t, k, \eta) = (1 + \tilde{\epsilon}) \left| \frac{p'}{p} \right| w(t, k, \eta),
\]

\[
w(0, k, \eta) = 1,
\]

where \( \tilde{\epsilon} > 0 \) is a fixed parameter to be chosen later. The explicit solution of (4.32) is given by

\[
w(t, k, \eta) = \begin{cases} 
\left( \frac{k^2 + \eta^2}{p(t, k, \eta)} \right)^{(1+\tilde{\epsilon})} & \text{for } \eta k > 0 \text{ and } t < \frac{\eta}{k}, \\
\frac{1}{k} \left( \frac{p(t, k, \eta)}{k^4} \right)^{(1+\tilde{\epsilon})} & \text{for } t \geq \frac{\eta}{k}.
\end{cases}
\]

Let us remark that the weight \( w \) has to mimic the behaviour of \( p \). We need to define it as in (4.32) since we want \( \partial_t w / w \) to have positive sign for technical reasons.

We now claim that

\[
w^{-1}(t, k, \eta) \lesssim (\eta - \xi)^{4(1+\tilde{\epsilon})} w^{-1}(t, k, \xi),
\]

\[
\frac{\partial_t w}{w}(t, k, \eta) \lesssim (\eta - \xi)^3 \frac{k^2}{p(t, k, \xi)} + (\eta - \xi)^2 \frac{\partial_t w}{w}(t, k, \xi).
\]

The proof is given in the Appendix, see Lemma B.3. It is natural that in order to exchange frequency for \( \partial_t w / w \) we need a correction term, since in principle it may happen that \( p'(t, k, \eta) = 0 \) while \( p'(t, k, \xi) \neq 0 \). Due to the extra term in (4.35), we need to introduce also the following weight

\[
\partial_t m(t, k, \eta) = \frac{k^2}{p} m(t, k, \eta),
\]

\[
m(0, k, \eta) = 1.
\]

Notice that \( m \) can be given explicitly by

\[
m(t, k, \eta) = \exp \left( \arctan(\frac{\eta}{k} - t) - \arctan(\frac{\eta}{k}) \right).
\]
To exchange frequencies, by using the estimates in Lemma \[\text{B.3}\] we have that
\[m(t, k, \eta) \approx m(t, k, \xi),\]
\[
\frac{\partial_t m}{m} (t, k, \eta) \lesssim (\eta - \xi)^2 \frac{\partial_t m}{m} (t, k, \xi),
\]
Now our purpose is to control \(m^{-1}w^{-1}f\) in \(H^s\). From \[(4.30),\] we directly compute that
\[
\frac{1}{2} \frac{d}{dt} \left\| m^{-1}w^{-1}f \right\|_{H^s}^2 = - \left\| \frac{\partial_t m}{m} m^{-1}w^{-1}f \right\|_{H^s}^2 - \left\| \frac{\partial_t w}{w} m^{-1}w^{-1}f \right\|_{H^s}^2 + \left\langle m^{-1}w^{-1} (g^2[\partial_t \Delta_L] \Delta^{-1}_t) f, m^{-1}w^{-1}f \right\rangle_s.
\]

Then, we need to bound the scalar product in \[(4.38),\] so we define
\[
S_{g^2} = \left\langle m^{-1}w^{-1} (g^2[\partial_t \Delta_L] \Delta^{-1}_t) f, m^{-1}w^{-1}f \right\rangle_s
\]
Rewrite \(S_{g^2}\) as follows
\[
S_{g^2} = \left\langle m^{-1}w^{-1} ((g^2 - 1)[\partial_t \Delta_L] \Delta^{-1}_t T_2 f), m^{-1}w^{-1}f \right\rangle_s
\]
\[
+ \left\langle m^{-1}w^{-1} ([\partial_t \Delta_L] \Delta^{-1}_t T_2 f), m^{-1}w^{-1}f \right\rangle_s
\]
\[
= S_{g^2-1} + S_1,
\]
where we have used \[(4.3)\] to expand \(\Delta^{-1}_t\).

Let us start with \(S_{g^2-1}\). By Plancherel Theorem we infer that
\[
|S_{g^2-1}| \leq \left\langle m^{-1}w^{-1} \left( (g^2 - 1) * \frac{\partial_t w}{w} T_2 f \right), m^{-1}w^{-1}f \right\rangle_s
\]
\[
\leq \frac{C}{1 + \epsilon} \left\langle m^{-1}w^{-1} \left( |(g^2 - 1)| * \frac{\partial_t w}{w} \right), m^{-1}w^{-1}f \right\rangle_s,
\]
where in the last inequality we used the definition of \(w\) given in \[(4.33).\]

By using \[(4.35)\] to commute \(\sqrt{\partial_t w/w}\) with \(g^2 - 1\), we get that
\[
|S_{g^2-1}| \leq \frac{C}{1 + \epsilon} \left\langle \langle \gamma \rangle^{s} m^{-1}w^{-1} \left( \langle \gamma \rangle^{g^2 - 1} \right) * \left( \frac{\partial_t w}{w} \right) \right\rangle_s \langle \gamma \rangle^{s} \left\langle \frac{\partial_t w}{w} m^{-1}w^{-1} |f| \right\rangle_s
\]
\[
+ \frac{C}{1 + \epsilon} \left\langle \langle \gamma \rangle^{s} m^{-1}w^{-1} \left( \langle \gamma \rangle^{3/2} g^2 - 1 \right) * \left( \frac{\partial_t w}{w} \right) \right\rangle_s \langle \gamma \rangle^{s} \left\langle \frac{\partial_t w}{w} m^{-1}w^{-1} |f| \right\rangle_s,
\]
where in the last line we used the fact that \(k^2/p = \partial_t m/m\). Now, by \[(4.34),\] \[(4.37)\] we have that
\[
|S_{g^2-1}| \leq C \left\langle \langle \gamma \rangle^{s+1+4(1+\epsilon)} |g^2 - 1| \left( \frac{\partial_t w}{w} \right) \right\rangle_s \langle \gamma \rangle^{s} \left\langle \frac{\partial_t w}{w} m^{-1}w^{-1} |f| \right\rangle_s
\]
\[
+ C \left\langle \langle \gamma \rangle^{s+3/2+4(1+\epsilon)} g^2 - 1 \right\rangle_s \left( \frac{\partial_t w}{w} \right) \right\rangle_s \langle \gamma \rangle^{s} \left\langle \frac{\partial_t w}{w} m^{-1}w^{-1} |f| \right\rangle_s,
\]
Now apply Cauchy-Schwarz and Young’s inequality, see \[(B.2),\] to have that
\[
|S_{g^2-1}| \leq C \left\| g^2 - 1 \right\|_{H^{s+s+4\epsilon}} \left\| \frac{\partial_t w}{w} m^{-1}w^{-1} T_2 f \right\|_{H^s} \left\| \frac{\partial_t w}{w} m^{-1}w^{-1} f \right\|_{H^s}
\]
\[
+ C \left\| g^2 - 1 \right\|_{H^{s+7+4\epsilon}} \left\| \frac{\partial_t w}{w} m^{-1}w^{-1} T_2 f \right\|_{H^s} \left\| \frac{\partial_t w}{w} m^{-1}w^{-1} f \right\|_{H^s},
\]
Since we have enough regularity on the background shear and we know how to exchange frequencies, we can apply Corollary 4.2. Then from (4.40) we get that

\[
|S_{g^2-1}| \leq C \epsilon \frac{1}{\sqrt{t}} \left\|rac{\partial w}{w} w^{-1} f \right\|_{H^s}^2 + C \epsilon \left\|rac{\partial m}{m} m^{-1} w^{-1} f \right\|_{H^s}^2.
\]

(4.41)

To bound \(S_1\), from the definition (4.39) and thanks to Corollary 4.2, we directly have that

\[
|S_1| \leq (1 + C \tilde{\epsilon}) \left\|rac{\partial w}{w} w^{-1} f \right\|_{H^s}^2.
\]

(4.42)

By combining (4.38) with (4.41) and (4.42) we infer that

\[
\frac{1}{2} \frac{d}{dt} \left\|(m^{-1} w^{-1} f(t) \right\|_{H^s}^2 \leq \frac{1}{1 + \tilde{\epsilon}} \left(1 - \tilde{\epsilon} + C \epsilon + 1\right) \left\|rac{\partial w}{w} w^{-1} f \right\|_{H^s}^2
\]

\[
- (1 - C \epsilon) \left\|rac{\partial m}{m} m^{-1} w^{-1} f \right\|_{H^s}^2.
\]

(4.43)

Then it is enough to choose \(\tilde{\epsilon}\) such that the right hand side of (4.43) is negative. A possible choice is

\[
\tilde{\epsilon} = 2C\epsilon.
\]

So we have proved that

\[
\left\|(m^{-1} w^{-1} f(t) \right\|_{H^s} \leq \left\|(m^{-1} w^{-1} f(0) \right\|_{H^s} \leq \|f_{in}\|_{H^s}.
\]

(4.44)

Finally, to prove (4.31), just observe that by the definition of \(w\), see (4.33), the bound-endness of \(m\) and (4.44) it holds that

\[
\left\|\Delta_L^{-(1+s)} f \right\|_{H^s} \lesssim \left\|(m^{-1} w^{-1} f \right\|_{H^s} \leq \|f_{in}\|_{H^s}.
\]

Hence Theorem 4.5 is proved.

\[\square\]

5. Stability analysis for shear flows near Couette

Recall that we are considering the system (4.6)-(4.8), namely

\[
\partial_t R = -A, \quad \text{in } T \times \mathbb{R},
\]

\[
\partial_t A = g^2 [\partial_t \Delta_L] \Delta_t^{-1} A - 2g \partial X \Delta_t^{-1} \Omega - \frac{1}{M^2} \Delta_t R,
\]

\[
\partial_t \Omega = \left[g + b g (\partial_Y - t \partial X) \Delta_t^{-1}\right] A + b \partial X \Delta_t^{-1} \Omega.
\]

(5.2)

In Section 4 we have studied the building blocks to prove the upper bounds claimed in Theorem 1.6 which we recall here.

Theorem 5.1. Let \(\epsilon \ll 1\) and assume that \(\|g^2 - 1\|_{H^{s_0}_Y} \leq \epsilon\) and \(\|b\|_{H^{s_0}_Y} \leq \epsilon\), for a fixed \(s_0 \in \mathbb{R}\). Let \(\rho_{in}, \alpha_{in}, \omega_{in} \in H^6(T \times \mathbb{R})\) be the initial data of (5.1) and (5.2).
Then, there is a constant $C > 1$, such that for $\tilde{\epsilon} = C\epsilon < 1/16$ it holds that

$$\|Q(v) - \overline{Q(v)}\|_{L^2}^2 + \frac{1}{M^2} \|\rho - \overline{\rho}\|_{L^2}^2 \leq \langle t \rangle^{1+\tilde{\epsilon}} C(\rho_{in}, \alpha_{in}, \omega_{in}),$$

(5.3)

$$\|P_1(v) - \overline{P_1(v)}\|_{L^2} \leq \frac{M}{\langle t \rangle^{1/2-\epsilon}} C(\rho_{in}, \alpha_{in}, \omega_{in}) + \frac{1}{\langle t \rangle} C(\rho_{in}, \omega_{in}),$$

(5.4)

$$\|P_2(v)\|_{L^2} \leq \frac{M}{\langle t \rangle^{3/2-\epsilon}} C(\rho_{in}, \alpha_{in}, \omega_{in}) + \frac{1}{\langle t \rangle^2} C(\rho_{in}, \omega_{in}),$$

(5.5)

where the constants in the r.h.s. of (5.3)-(5.5) depend on suitable Sobolev norms of the initial data (e.g. $H^6$) and they can be explicitly computed.

Remark 5.2 (On regularity requirements). Theorem 5.1 is the analogous of Theorem 3.6 for shear close to Couette. We need to require more regularity on the initial data with respect to the Couette case, since to recover $\Omega$ from $R$ we need to pay regularity on the initial data, as we explain in Section 5.1 and in Corollary 5.10.

The regularity requirements on the background shear, namely $H^{s_0}$, are far from being sharp. We need at least that $s_0 \geq 8$, in fact as we have seen in Section 4, to perform weighted energy estimates we need to pay regularity on $g^2 - 1$ and $b$ for commutator type bounds.

Remark 5.3. As already done for the Couette case in Section 3, we decompose the dynamics into its $x$-average and fluctuations around it. For this reason, we only deal with the fluctuations, hence from now on we will assume that $\bar{\rho}_{in} = \bar{\alpha}_{in} = \bar{v}_{2in} = \bar{\omega}_{in} = 0$. To recover the general case see Proposition 2.1.

To prove Theorem 5.1, the main idea is to deduce a weighted energy estimate on the moving frame. In the Couette case, thanks to the exact conservation of $R + \Omega$, the first step was to properly weight $R$ and $A$, see (3.12). Then, it was easy to obtain an energy equality, encoded in (3.27), that we were able to handle with a Grönwall inequality.

For the system (5.1)-(5.2), it is not straightforward to infer an exact conservation for a combination of $R$ and $\Omega$. For this reason, in Subsection 5.1 we present a functional relation, see (5.12), that connects $R$ and $\Omega$.

In Subsection 5.2 we present the weighted energy estimate needed to prove Theorem 5.1. Due to the loss of regularity in the functional relation, see Remark 5.5, we cannot directly use it in an energy estimate. Then we need to exploit the auxiliary quantity $\Xi = \Omega + gR$, for which its time derivative involves terms multiplied by $b$, see (5.13). By defining a properly weighted energy functional involving $R, A, \Xi$, see (5.21), we are able to prove an energy estimate, see Proposition 5.9.

5.1. Functional relation between $\Omega$ and $R$. First of all, we need Theorem (4.4), which tells us that the evolution operator associated to (4.25) is well defined. We need to be somehow more explicit in order to isolate some lower order term. So we express the evolution operator associated to (4.25) as a Picard’s iteration as follows

$$\Phi_b(t, s) = I + \sum_{n=0}^{\infty} b \int_s^t \partial_X \Delta_{\tau}^{-1} \Phi_n^b(\tau, s) d\tau,$$

(5.6)

$$\Phi_n^b(t, s) = b \int_s^t \partial_X \Delta_{\tau} \Phi_{n-1}^b(\tau, s) d\tau, \quad \Phi_0^b(t, s) = I.$$
We will also denote the inverse as
\[
\Phi_b^{-1}(t,s) := \Phi_b(s,t) = I + \sum_{n=0}^{\infty} b(-1)^{n+1} \int_s^t \partial_X \Delta_e^{-1} \Phi^n_b(\tau,s) d\tau
\]
(5.7)
\[= I + b\Phi(t,s),\]
where we isolate the terms that are multiplied by \(b\).

**Remark 5.4.** The Picard’s iteration (5.6), without knowing Theorem 4.4, can be defined thanks to the definition of \(\Delta_t^{-1}\) given in (4.9). In that case, an a priori bound on its operatorial norm is given by
\[
\|\Phi_b(t,s)\|_{H^s \to H^s} \leq e^{C(t-s)}.
\]
To get a uniform bound in time, a priori would be possible from \(H^{s+2} \to H^s\) for example.

Instead, Theorem 4.4 tells us that, assuming enough regularity on the background shear,
\[
\|\Phi_b(t,s)\|_{H^s \to H^s} \leq C,
\]
where \(C\) does not depend on time.

The operator \(\tilde{\Phi}\) defined in (5.7) enjoys analogous properties by definition.

Since we will have to commute Fourier multipliers with \(\Phi_b\), in Lemma C.1 we show the bound that we need. Essentially, for shear close to Couette \(\Phi_b\) is a perturbation of the identity.

Now, define \(\tilde{\Omega} = \Phi_b^{-1}\Omega\) and, thanks to Theorem 4.4 rewrite (5.2) as follows
\[
\partial_t \tilde{\Omega} = \Phi_b^{-1}[g + b g(\partial_Y - t \partial_X) \Delta_t^{-1}]]A,
\]
(5.8)
\[= [g + b \left( g(\partial_Y - t \partial_X) \Delta_t^{-1} + \Phi_b + b g(\partial_Y - t \partial_X) \Delta_t^{-1} \right)]] A,
\]
\[= [g + b G]A.
\]
The operator \(G\) is given by
\[
G := g(\partial_Y - t \partial_X) \Delta_t^{-1} + \Phi_b + b g(\partial_Y - t \partial_X) \Delta_t^{-1}.
\]
(5.9)
By combining (5.1) and (5.8), we infer that
\[
\tilde{\Omega} = \tilde{\Omega}_m - \int_0^t [g + b G] \partial_t R d\tau
\]
(5.10)
\[= \tilde{\Omega}_m + [g + b G_0] R_m - [g + b G] R + b \int_0^t [\partial_t G] R d\tau,
\]
where the last one follows by an integration by parts. In Lemma C.2 we show that
\[
\partial_t G = \sum_{i=1}^8 F^1_i \partial_X \Delta_t^{-1} F^2_i,
\]
(5.11)
where \(F^1_i, F^2_i\) are bounded operators in any \(H^s\), provided that one has sufficient regularity and smallness of the background shear. Essentially, besides the fact that can be properly defined, (5.11) is all that we need to know about \(\partial_t G\).

Finally, apply \(\Phi_b\) to (5.10) to infer that
\[
\Omega = \Omega_m + [g + b G_m] R_m - \Phi_b [g + b G] R + \Phi_b b \int_0^t [\partial_t G] R d\tau
\]
(5.12)
Remark 5.5. Clearly, if $R$ is given, then also $\Omega$ is well defined, since each operator involved is defined. We immediately see that the integral in time can cause some problem to infer estimates. Heuristically, notice that in view of (5.11), if one can pay regularity on $R$, then we can take out integrable factors in time from $\partial_t G$. In an energy type estimate, we do not want to pay regularity besides initial quantities. The functional relation (5.12) is crucial to prove that $\Omega$ enjoys the same bounds as $R$, once we know that we can pay regularity only on the initial data. But we cannot use (5.12) in the energy estimates that we need to perform.

5.2. The weighted energy functional. In order to simplify the notation, from now on we set $M = 1$ and in the end we explain how to properly scale some quantity in order to recover the bounds with any Mach number.

We now want to perform a weighted energy estimate. In order to do that, we use the following quantity

$$\Xi = \Omega + gR,$$

which is somehow the analogue of (3.5). Notice that

$$\partial_t \Xi = bg(\partial_Y - t\partial_X)\Delta_t^{-1}A - b\partial_X \Delta_t^{-1}gR + b\partial_X \Delta_t^{-1}\Xi.$$

In the r.h.s. of (5.13), everything is multiplied by a factor $b$, which has to be thought as a smallness parameter. We will see that $\Xi$ is the right quantity to infer the weighted estimate but not to infer the sharp bound on $\Omega$.

By writing (5.1)-(5.2) in terms of $\Xi$, we get that

$$\partial_t R = -A,$$

$$\partial_t A = g^2[\partial_t \Delta_t] \Delta_t^{-1} A + (-\Delta_t + 2g\partial_X \Delta_t^{-1}g) - 2g\partial_X \Delta_t^{-1}A,$$

$$\partial_t \Xi = bg(\partial_Y - t\partial_X)\Delta_t^{-1}A - b\partial_X \Delta_t^{-1}gR + b\partial_X \Delta_t^{-1}\Xi.$$

Now, we have to properly design the weights to catch the bounds claimed for $R, A$, and somehow to absorb all the contributions which comes from the auxiliary variable $\Xi$ and other technical error terms. In Remark 5.8 we comment more on the choice of the weights.

Let $0 < c < 1$ and $N > 1$ to be chosen later, by recalling that

$$p = k^2 + (\eta - kt)^2,$$

$$p' = -2k(\eta - kt),$$

the weights are defined as follows:

$$\partial_t w = (1 + \tilde{c})\frac{|p'|}{p}w, \quad w|_{t=0} = 1,$$

$$\partial_t m = \frac{Nk^2}{p}m, \quad m|_{t=0} = 1,$$

$$v^2 = (-\Delta_t)^{-1}w^{2(1-c)},$$

$$h = \sqrt{c}\sqrt{\frac{\partial_t w}{w}m^{-1}w^{-(1-c)}}.$$  

Now we are ready to introduce the energy functional that we want to control. Let $s \geq 0$, then we define

$$E_s(t) := \frac{1}{2} \left( \|m^{-1}v^{-1}R\|^2_s + \|m^{-1}w^{-(1-c)}A\|^2_s + \|m^{-1}w^{-(1-c)}\Xi\|^2_s + 2 \langle hR, hA \rangle_s \right),$$
where the subscript $s$ denotes the Sobolev regularity and we have omitted the dependence on time in the r.h.s. of (5.21).

**Remark 5.6.** Thanks to the choice of the weights, for $c \leq 1/4$, by Lemma B.4 we have that

$$E_s(t) \approx \|m^{-1}w^{-1}R\|_s^2 + \|m^{-1}w^{-1-c}A\|_s^2 + \|m^{-1}w^{-1-c}\Xi\|_s^2.$$  

In addition, it holds also that

$$E_s(0) \lesssim \|R_{in}\|_{s+1}^2 + \|A_{in}\|_s^2 + \|\Xi_{in}\|_s^2.$$  

The extra regularity for $R_{in}$ it is required because the weight $v$ is not defined through a differential equation, so its value at time $t = 0$ cannot be chosen. One could think to choose $w|_{t=0}$ such that $v|_{t=0} = 1$, but in this case one would pay regularity on $A_{in}$ and $\Xi_{in}$, see also Remark 4.6.

**Remark 5.7 (About the choice of the energy functional).** Notice that the mixed scalar product term present in the energy functional given in (5.21) is crucial in order to infer the almost optimal upper bound. This can also be seen in Appendix A, where we present a toy model retaining an analogous structure. In fact the natural choice of defining the energy functional by summing up suitable weighted norms would lead to a non-optimal upper bound growing like $(t)^{2+\tilde{\epsilon}}$. Essentially we mimic the energy estimate performed in the Couette case, see the proof of Lemma 3.2, which underline the fact that the dynamic lives on an ellipse in the phase space and not on a circle.

The auxiliary variable $\Xi$ instead, is taking care of the fact that the dynamics of the system is not 2D as in the Couette case, which is possible thanks to the conservation of $R + \Omega$. In the case of shear close to Couette, a priori we are a 3D system due to the lack of an exact conservation law for a linear combination of $R$ and $\Omega$.

**Remark 5.8 (About the choice of the weights).** During the proof of the energy estimates, we stress where each weight plays a role. Anyway, let us comment about them.

The weight of $w$ is exactly the same one introduced in the second building block, see (4.32). In particular, it has to mimic the behaviour of $p$ but we also need that $\partial_tw/w > 0$ for technical convenience.

The weight $m$ is essentially the same, up to the constant $N$, also defined for (4.36). A similar weight is familiar in the incompressible literature of monotone shear near Couette, see [12, 77, 78]. We need $m$ also to control the other error terms. For this reason we include the constant $N > 1$ to be chosen during the proof. It will be enough $N = 32$.

Regarding the choice of the weight $v$, see (5.19), this is constructed in such a way to have a crucial cancellation in the derivative of the energy functional, see (5.40). This is important because the second term on the l.h.s. of (5.40) cannot be controlled with terms with a negative sign in the time derivative of the energy functional. We just stress that, thanks to the characterization of $\Delta_t^{-1}$ given in Proposition 4.1, it is easy to check that $(-\Delta_t)^{-1}$ is still positive. It is also possible to take $\partial_tv^{-2}$, which involves $\partial_t\Delta_t^{-1}$ that we have defined in (4.16).

Then, $h$ for the moment may appear obscure but will play a crucial role in choosing the constant $c$ that will determine the time rates. Notice that $h$ is the weight of the mixed scalar product that will allows us to make use of the weakened norm of $R$, see also Appendix A.
Finally, heuristically we have that $w \approx p$ and $m \approx 1$, so we are trying to bound for example $\|p^{-(1-c)}A\|^2_{s}$. In the Couette case we are able to bound exactly $\|p^{-3/4}A\|^2_{s}$, see (3.12). This means that we expect to get $c = 1/4$ up to some $\epsilon$ correction.

The main result of this Section is the following weighted energy estimate.

Proposition 5.9. Let $s > 0$, $\epsilon \ll 1$ and assume that $\|g^2 - 1\|_{H^{s+10}} \leq \epsilon$ and $\|b\|_{H^{s+10}} \leq \epsilon$. Let $(w, v, m, h)$ the weights defined in (5.17)-(5.20). Assume that $R_{in} \in H^{s+1}(\mathbb{T} \times \mathbb{R})$ and $A_{in}, \Xi_{in} \in H^{s}(\mathbb{T} \times \mathbb{R})$ are the initial data of (5.14)-(5.16).

Then, for $N = 32$ there is a suitable $C > 1$ such that for $\tilde{c} = C\epsilon < 1/16$, choosing $c = 1/4 - \tilde{c}$, it holds that

$$E_{s}(t) \leq E_{s}(0).$$

From the definition of the energy functional (5.21), we see that $\Xi$ needs to be weighted with same multiplier of the divergence. In particular, $\Xi$ grows in time with the same rate as $A$. For this reason, the energy estimate (5.22) would not provide the optimal bounds for the vorticity needed to prove (5.4)-(5.5). To overcome this difficulty, we need to exploit the functional relation (5.12).

Corollary 5.10. Let $\Phi_{b}$ be the evolution operator defined in (5.6). Let $G$ be the operator defined in (5.9). Under the hypothesis of Proposition 5.9 we infer that

$$\|m^{-1}v^{-1}\Omega\|_{L^{2}} \lesssim E_{5}(0)$$

In the following, we present the proof of Corollary 5.10. Then, by having the energy estimate (5.22) and (5.25) at hand we are able to prove Theorem 5.1. The proof of Proposition 5.9 will be given at the end of this Section since it is the most technical one.

Proof. From the functional relation (5.12), we have that

$$\|m^{-1}v^{-1}\Omega\|_{L^{2}} \lesssim \|m^{-1}v^{-1}(\Omega_{in} + [g + bG_{in}]R_{in})\|_{L^{2}} + \|m^{-1}v^{-1}\Phi_{b}[g + bG]R\|_{L^{2}}$$

(5.24)

The bound on the first term of (5.24) can be easily given in terms of the initial data. The second term can controlled by a commutator estimate and (5.22). Thanks to Lemma C.1 we have that

$$\|m^{-1}v^{-1}\Phi_{b}[g + bG]R\|_{L^{2}} \lesssim \|m^{-1}v^{-1}[g + bG]R\|_{L^{2}} \lesssim \|m^{-1}v^{-1}R\|_{L^{2}},$$

(5.25)

where the last inequality follows by the fact that $g + bG = 1 + (g - 1) + bG$, which allows us to apply Lemma B.2. We use also the boundedness of $G$, see Lemma C.2.

Thanks to Proposition 5.9 we have that

$$\|m^{-1}v^{-1}\Phi_{b}[g + bG]R\|_{L^{2}} \lesssim E_{0}(0),$$

(5.26)

Now we analyse the last term in the r.h.s. in (5.24). Since $m^{-1}v^{-1}\Phi_{b}$ is bounded in $L^{2}$, we get that

$$\|m^{-1}v^{-1}\Phi_{b}b\int_{0}^{t}[[\partial_{\tau}G]Rd\tau\|_{L^{2}} \lesssim \|b\|_{H^{1}}\int_{0}^{t}\|[[\partial_{\tau}G]R\|_{L^{2}} d\tau,$$

(5.27)

where in the last inequality we have also used Cauchy-Schwarz plus Young’s inequalities, see (B.2), and Minkowski inequality to take out the integral in time. Let us now focus on the integrand in time. From Lemma C.2 recall that

$$\partial_{\tau}G = \sum_{i=1}^{8} F_{i}^{1} \partial_{X} \Delta_{L}^{-1} F_{i}^{2},$$
with $F_i^1, F_i^2$ bounded in $H^s$. From the definition of $v$, see (5.19), it holds that
\[
\| [\partial_r G] R \|_{L^2} = \sum_{i=1}^{8} \| F_i^1 \partial_x \Delta_L^{-1} F_i^2 (vm)m^{-1}v^{-1}R \|_{L^2} \leq \frac{1}{\langle \tau \rangle^2} \| ((-\Delta_r)^{-1/2}w^{(1-c)}m)m^{-1}v^{-1}R \|_{H^2} \leq \frac{\langle \tau \rangle^{-1+2(1-c)+\tilde{\varepsilon}}}{\langle \tau \rangle^2} \| m^{-1}v^{-1}R \|_{H^5},
\]
(5.28)
where the last two inequalities follow from Lemma [B.5] to take out time factors from $\Delta_L^{-1}$, Corollary 4.2 and Lemma [B.6] to take out time factors from $w$. Since $\varepsilon = 1/4 - \tilde{\varepsilon}$, by combining (5.28) with Proposition 5.9, we get that
\[
\| [\partial_r G] R \|_{L^2} \leq \frac{1}{\langle \tau \rangle^{3/2-\varepsilon}} E_5(t) \leq \frac{1}{\langle \tau \rangle^{3/2-\varepsilon}} E_5(0).
\]
(5.29)
Putting together (5.27) with (5.29) we obtain that
\[
\| m^{-1}v^{-1}b \|_{L^2} \leq \| b \|_{H^1} \int_0^t \frac{1}{\langle \tau \rangle^{3/2-\varepsilon}} E_5(0) d\tau \leq \| b \|_{H^1} E_5(0),
\]
and the last inequality holds upon choosing $\varepsilon$ small enough such that $\tilde{\varepsilon} < 1/2$. By combining (5.24) with (5.26) and (5.30), the Corollary is proved.

In what follows we prove Theorem 5.1 by assuming Proposition 5.9.

**Proof of Theorem 5.1 assuming Proposition 5.9** To prove (5.3), we proceed as follows: by writing the acoustic part on the moving frame and by using the Helmholtz decomposition, we obtain that
\[
\| \rho \|_{L^2}^2 + \| Q(v) \|_{L^2}^2 = \| R \|_{L^2}^2 + \| p^{-1/2}A \|_{L^2}^2 \\
= \| (vm)m^{-1}v^{-1}R \|_{L^2}^2 + \| (p^{-1/2}w^{(1-c)}m)m^{-1}w^{-(1-c)}A \|_{L^2}^2 \\
\leq \| (p^{-1/2}w^{(1-c)}m)m^{-1}v^{-1}R \|_{L^2}^2 + \| (p^{-1/2}w^{(1-c)}m)m^{-1}w^{-1}A \|_{L^2}^2,
\]
where the last inequality follows by the definition of $v$, see (5.19), the boundedness of $m$, and Corollary 4.2. Then, by the definition of $w$, see (5.17) and taking out factor of time thanks to (B.23), (B.24), it holds that
\[
\| \rho \|_{L^2}^2 + \| Q(v) \|_{L^2}^2 \leq \langle t \rangle^{1+\varepsilon} E_2(t) \leq \langle t \rangle^{1+\varepsilon} E_2(0),
\]
where the last inequality follows by Proposition 5.9.

To bound the incompressible part, we use the functional relation (5.12). So define
\[
\tilde{\Xi}_{in} = \Omega_{in} + [g + bG_0]R_{in}.
\]
By the Helmholtz decomposition, and the basic property of $p$ given in (B.23), it holds that
\[
\| P_1(v) \|_{L^2} \leq \| p^{-1/2} \Omega \|_{L^2} \leq \frac{1}{\langle t \rangle} \| \tilde{\Xi}_{in} \|_{H^2} + \| p^{-1/2} \Phi_{b}[g + bG]R \|_{L^2} \\
+ \| p^{-1/2} \Phi_{b} \int_0^t [\partial_r G] R d\tau \|_{L^2} \leq \frac{1}{\langle t \rangle} \| \tilde{\Xi}_{in} \|_{H^2} + N_R^1 + N_R^2,
\]
(5.31)
To bound $N^1_R$, thanks to Lemma C.1 analogously as done for (5.25), we get that

$$N^1_R \lesssim \left\| p^{-1/2} \Phi_b[g + bG] R \right\|_{L^2} \lesssim \left\| p^{-1/2} R \right\|_{L^2}.$$  

Then, by applying Corollary 4.2 to bound $v$, see (5.19), we have that

$$N^1_R \lesssim \left\| p^{-1/2}(vm)^{-1} v^{-1} R \right\|_{L^2} \lesssim \left\| p^{-1}(w^{-1} - c) m^{-1} v^{-1} R \right\|_{L^2} \lesssim \frac{1}{(t)^{1/2 - \tilde{\epsilon}}} E_2(0),$$  

where we have used also Lemma B.5 and Lemma B.6 to take out factors of time. Notice that the last bound is the main contribution in (5.4).

The bound for $N^2_R$ follows in a similar way as done in the proof of Corollary (5.10). In particular, one proves that

$$N^2_R \leq \frac{\tilde{\epsilon}}{t} E_3(0) \int_0^t \left( \frac{\tau}{t} \right)^{1+\tilde{\epsilon}} d\tau \lesssim \frac{\tilde{\epsilon}}{t^{1-4\tilde{\epsilon}}} E_3(0).$$  

The bound for (5.5) follows the same line of the previous estimates, just notice that in (5.31) one has to substitute $p^{-1/2}$ with $p^{-1}$. □

The rest of the paper is dedicated to the proof of Proposition 5.9

**Proof of Proposition 5.9**

Recall the definition of the energy functional given in (5.21), namely

$$E_s(t) := \frac{1}{2} \left( \left\| m^{-1} v^{-1} R \right\|_s^2 + \left\| m^{-1} w^{-1} (1-c) A \right\|_s^2 + \left\| m^{-1} w^{-1} (1-c) \Xi \right\|_s^2 + 2 \langle hR, hA \rangle_s \right).$$  

Then we have the following.

**Lemma 5.11.** Let $E_s(t)$ as defined in (5.32). Then it holds that

$$\left(5.33\right) \frac{d}{dt} E_s(t) = -N^w_A - N^m_A - N^m_R - N^w_\Xi - N^m_\Xi + S_{A,A} + S_{R,R} + S_{R,A} + S_{\Xi,R} + S_{\Xi,A} + S_{\Xi,\Xi}.$$  

where we define

$$N^w_A := \left\| \frac{\partial w}{w} m^{-1} w^{-1} (1-c) A \right\|_s^2,$$

$$N^m_A := \left\| \frac{\partial m}{m} m^{-1} w^{-1} (1-c) A \right\|_s^2,$$

$$N^m_R := \left\| \frac{\partial m}{m} m^{-1} (w^c v)^{-1} R \right\|_s^2,$$

$$N^m_\Xi := \left\| \frac{\partial m}{m} m^{-1} w^{-1} (1-c) \Xi \right\|_s^2,$$

$$N^w_\Xi := (1-c) \left\| \frac{\partial w}{w} m^{-1} w^{-1} (1-c) \Xi \right\|_s^2.$$  

As regards the divergence, from (5.15) we have that
\[
S_{A,A} := \left\langle m^{-1} w^{-(1-c)} \left( g^2 [\partial_t \Delta_L] \Delta_t^{-1} A \right), m^{-1} w^{-(1-c)} A \right\rangle_s,
\]
\[
S_{R,R} := \frac{1}{2} \left\langle m^{-2} \partial_t v^{-2} R, R \right\rangle_s + \left\langle h(-\Delta_t) R, hR \right\rangle_s,
\]
\[
S_{R,A} := \left\langle m^{-1} w^{-(1-c)} \left( 2g \partial_X \Delta_t^{-1} g \right) R, m^{-1} w^{-(1-c)} A \right\rangle_s
+ \left\langle h g^2 [\partial_t \Delta_L] \Delta_t^{-1} A, hR \right\rangle_s + \left\langle \partial_t (h^2) R, A \right\rangle_s,
\]
\[
S_{\Xi,R} := - \left\langle h(2g \partial_X \Delta_t^{-1}) \Xi, hR \right\rangle_s
- \left\langle m^{-1} w^{-(1-c)} (b \partial_X \Delta_t^{-1}) R, m^{-1} w^{-(1-c)} \Xi \right\rangle_s,
\]
\[
S_{\Xi,A} := \left\langle m^{-1} w^{-(1-c)} b g (\partial_Y - t \partial_X) \Delta_t^{-1} A, m^{-1} w^{-(1-c)} \Xi \right\rangle_s
- \left\langle m^{-1} w^{-(1-c)} \left( 2g \partial_X \Delta_t^{-1} \right) \Xi, m^{-1} w^{-(1-c)} A \right\rangle_s,
\]
\[
S_{\Xi,\Xi} := \left\langle m^{-1} w^{-(1-c)} (b \partial_X \Delta_t^{-1}) \Xi, m^{-1} w^{-(1-c)} \Xi \right\rangle_s.
\]

Proof. In order to prove (5.33), first of all we need a direct computation of energy equalities, which we include for convenience of the reader.

By (5.14), we compute that
\[
\frac{1}{2} \frac{d}{dt} \left\| m^{-1} v^{-1} R \right\|^2_s = \frac{1}{2} \frac{d}{dt} \left\langle m^{-2} v^{-2} R, R \right\rangle_s = \left\langle \frac{\partial m}{m} m^{-2} v^{-2} R, R \right\rangle_s + \frac{1}{2} \left\langle m^{-2} \partial_t v^{-2} R, R \right\rangle_s
- \left\langle m^{-2} v^{-2} R, A \right\rangle_s
- \left\| \frac{\partial m}{m} m^{-1} v^{-1} R \right\|^2_s + \frac{1}{2} \left\langle m^{-2} \partial_t v^{-2} R, R \right\rangle_s,
\]
(5.34)

As regards the divergence, from (5.15) we have that
\[
\frac{1}{2} \frac{d}{dt} \left\| m^{-1} w^{-(1-c)} A \right\|^2_s = - (1 - c) \left\| \frac{\partial_t w}{w} m^{-1} w^{-(1-c)} A \right\|^2_s
- \left\| \frac{\partial m}{m} m^{-1} w^{-(1-c)} A \right\|^2_s
+ \left\langle m^{-1} w^{-(1-c)} g^2 [\partial_t \Delta_L] \Delta_t^{-1} A, m^{-1} w^{-(1-c)} A \right\rangle_s
+ \left\langle m^{-1} w^{-(1-c)} (-\Delta_t) R, m^{-1} w^{-(1-c)} A \right\rangle_s
+ \left\langle m^{-1} w^{-(1-c)} (2g \partial_X \Delta_t^{-1}) g R, m^{-1} w^{-(1-c)} A \right\rangle_s
- \left\langle m^{-1} w^{-(1-c)} (2g \partial_X \Delta_t^{-1}) \Xi, m^{-1} w^{-(1-c)} A \right\rangle_s.
\]
(5.36)
From (5.16), we infer the following
\[
\frac{1}{2} \frac{d}{dt} \left\| m^{-1} w^{-(1-c)} \Xi \right\|_s^2 = - \left\| \frac{\partial_t m}{m} m^{-1} w^{-(1-c)} \Xi \right\|_s^2 - (1 - c) \left\| \frac{\partial_t w}{w} m^{-1} w^{-(1-c)} \Xi \right\|_s^2 \\
+ \left\langle m^{-1} w^{-(1-c)} bg(\partial_Y - t \partial_X) \Delta \Xi, m^{-1} w^{-(1-c)} \Xi \right\rangle_s \\
+ \left\langle m^{-1} w^{-(1-c)} (b \partial_X \Delta^{-1} \Xi), m^{-1} w^{-(1-c)} \Xi \right\rangle_s \\
- \left\langle m^{-1} w^{-(1-c)} (b \partial_X \Delta^{-1} g) R, m^{-1} w^{-(1-c)} \Xi \right\rangle_s.
\]

Finally, from (5.14) and (5.15), observe that
\[
\frac{d}{dt} \left\langle h R, h A \right\rangle_s = - \| h A \|_s^2 + \left\langle \partial_t (h^2) R, A \right\rangle_s + \left\langle h g^2 [\partial_t \Delta_L] \Delta^{-1} A, h R \right\rangle_s \\
+ \left\langle h(-\Delta_t) R, h R \right\rangle_s + \left\langle h(2\partial_X \Delta^{-1} g) R, h R \right\rangle_s \\
- \left\langle h(2g \partial_X \Delta^{-1}) \Xi, h R \right\rangle_s.
\]

Then, we exploit some cancellation which comes when we sum up the energy identities. First of all, recall the definition of \( h \) given in (5.20), namely
\[
h = \sqrt{c} \sqrt{\frac{\partial_t w}{w} m^{-1} w^{-(1-c)}}.
\]

Considering the sum of the term in the r.h.s. of (5.35) with (5.37), we have that
\[
\frac{d}{dt} \left\langle h R, h A \right\rangle_s = - \| h A \|_s^2 - (1 - c) \left\| \frac{\partial_t w}{w} m^{-1} w^{-(1-c)} A \right\|_s^2 - \left\| \frac{\partial_t w}{w} m^{-1} w^{-(1-c)} A \right\|_s^2,
\]
where the last equality directly comes from the definition of \( h \).

Then, rewrite the term (5.36) as follows
\[
\left\langle m^{-1} w^{-(1-c)} (-\Delta_t) R, m^{-1} w^{-(1-c)} A \right\rangle_s = \left\langle m^{-2} w^{-2(1-c)} (-\Delta_t) R, A \right\rangle_s.
\]

By the choice of \( v \), see (5.19), we know that
\[
m^{-2} v^{-2} = m^{-2} w^{-2(1-c)} (-\Delta_t).
\]

Hence, by adding (5.34) with (5.39), we have that
\[
- \left\langle m^{-2} v^{-2} R, A \right\rangle_s + \left\langle m^{-2} w^{-2(1-c)} (-\Delta_t) R, A \right\rangle_s = 0,
\]

Finally, by using (5.38), (5.40) and rearranging the remaining terms in the previous energy equalities we get (5.33). \( \square \)

Now that we have proved
\[
\frac{d}{dt} E_s(t) = - N^v_A - N^w_A - N^m_A - N^w_R - N^m_R + S_{A,A} + S_{R,R} + S_{R,A} + S_{\Xi, R} + S_{\Xi, A} + S_{\Xi, \Xi},
\]
we want to control the terms without a definite sign in terms of the negative ones in order to obtain that \( \frac{d}{dt} E_s(t) \leq 0 \).

Now we divide the proof in the bounds for each term.
5.3. **Bound on** $S_{A,A}$. Recall that

$$S_{A,A} := \left\langle m^{-1}w^{-(1-c)}\left( g^2[\partial_t \Delta L] \Delta_t^{-1}A \right), m^{-1}w^{-(1-c)}A \right\rangle_s.$$  

As done in the proof of Theorem 4.5 thanks to the choice of $w$ and $m$, see (5.17) and (5.18) respectively, we infer that

$$-N_A^w - N_A^m + |S_{A,A}| \leq -\tilde{\epsilon}N_A^w - (1 - \tilde{\epsilon})N_A^m.$$  

The proof of (5.42) is the same as Theorem 4.5, up to minor changes on the commutation prices to pay on the background shear. In fact, in Theorem 4.5 we have to commute $m^{-1}w^{-1}$ with $g^2[\partial_t \Delta L] \Delta_t^{-1}$. For $S_{A,A}$, we have to commute $m^{-1}w^{-1}$ with exactly the same term. We stress that the cancellation (5.38) is crucial to have the term $N_A^w$, which is exactly the one that appears in (4.38).

By combining (5.41) with (5.42) we get that

$$\frac{d}{dt} E_s(t) \leq -\tilde{\epsilon}N_A^w - (1 - \tilde{\epsilon})N_A^m - N_R^w - N_R^m + S_{R,R} + S_{R,A} + S_{\Xi, R} + S_{\Xi, A} + S_{\Xi, \Xi}.$$  

5.4. **Bound on** $S_{R,R}$. The bound on this term, is the crucial one in order to choose the constant $c$ that will give the time rates in Theorem 5.1.

Recall that

$$S_{R,R} := \frac{1}{2} \left\langle m^{-2}\partial_t v^{-2}R, R \right\rangle_s + \langle h(-\Delta_t)R, hR \rangle_s$$  

$$+ \langle h(2\partial_{XX} \Delta_t^{-1})gR, hR \rangle_s := S_{R,R}^1 + S_{R,R}^2.$$  

We proceed in controlling separately the terms previously defined.

5.4.1. **Bound on** $S_{R,R}^1$. We begin with the term $S_{R,R}^1$, namely

$$S_{R,R}^1 = \frac{1}{2} \left\langle m^{-2}\partial_t v^{-2}R, R \right\rangle_s + \langle h(-\Delta_t)R, hR \rangle_s.$$  

This term is the one that will determine the constant $c$. Notice that, thanks to (4.16), we can apply the Leibniz rule to differentiate $v$, hence we have that

$$0 = \partial_t \left( v^2v^{-2} \right) = v^2 \left( \partial_t v^{-2} \right) + \left( \partial_t v^2 \right) v^{-2},$$  

which implies that

$$S_{R,R}^1 = -\frac{1}{2} \left\langle m^{-2}v^{-2}[\partial_t v^2]v^{-2}R, R \right\rangle_s + \langle h(-\Delta_t)R, hR \rangle_s.$$  

Before proceeding with the bound, let us give a heuristic idea. By the definition of $w$ and $v$, see (5.17), (5.19) respectively, formally we get

$$v \approx w^{1/2-c-\tilde{\epsilon}}.$$  

Looking at the first term in the r.h.s. of (5.44), we expect that

$$\frac{1}{2} v^{-2} \partial_t v^2 v^{-2} \approx \left( \frac{1}{2} - c - \tilde{\epsilon} \right) \frac{\partial_t w}{w} v^{-1}.$$  

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More precisely, we claim that

\[
\frac{1}{2} \left\langle m^{-2}v^2 \left[ \partial_tv^2 \right] v^{-2}R, R \right\rangle_s \geq \left( \frac{1}{2} - c - \bar{\epsilon} \right) \left\| \sqrt{\frac{\partial_tw}{w}} m^{-1}v^{-1}R \right\|_{H^s}^2 - \bar{\epsilon} \left\| \sqrt{\frac{\partial_tm}{m}} m^{-1}v^{-1}R \right\|_{H^s}^2.
\]

(5.45)

To prove (5.45), by the definition of \( v \), see (5.19), we have that

\[
v^2 = (-\Delta_t)^{-1}w^{2(1-c)}.
\]

By (4.16), we know how to define \( \partial_t \Delta_t^{-1} \), so we infer that

\[
\partial_tv^2 = \partial_t \left( (-\Delta_t)^{-1}w^{2(1-c)} \right)
\]

\[
= 2(1-c)(-\Delta_t)^{-1}w^{2(1-c)}\frac{\partial_tw}{w} - (-\Delta_t^{-1})(-\partial_t\Delta_t) \left( (-\Delta_t)^{-1}w^{2(1-c)} \right),
\]

(5.46)

where the last inequality on the operators follows since \( \partial_t\Delta_t \) is not positive a priori. So far, thanks to (5.46), since the other operators are positive, we have that

\[
\frac{1}{2} \left\langle m^{-2}v^2 \left[ \partial_tv^2 \right] v^{-2}R, R \right\rangle_s \geq (1-c) \left\| \sqrt{\frac{\partial_tw}{w}} m^{-1}v^{-1}R \right\|_{H^s}^2 - \frac{1}{2} \left\langle m^{-1}v^{-1}(-\Delta_t)^{-1}|\partial_t\Delta_t| R, m^{-1}v^{-1}R \right\rangle_s
\]

(5.47)

Thanks to Lemma B.7, we have that

\[
\frac{1}{2} \left\langle m^{-1}v^{-1}(-\Delta_t)^{-1}|\partial_t\Delta_t| R, m^{-1}v^{-1}R \right\rangle_s \leq \left( \frac{1}{2} + \bar{\epsilon} \right) \left\| \sqrt{\frac{\partial_tw}{w}} m^{-1}v^{-1}R \right\|_{H^s}^2 + \bar{\epsilon} \left\| \sqrt{\frac{\partial_tm}{m}} m^{-1}v^{-1}R \right\|_{H^s}^2.
\]

(5.48)

The bound (5.45) follows by combining (5.47) with (5.48). In particular, (5.45) allows us to infer that

\[
S_{R,R}^{1} \leq \bar{\epsilon} \left\| \sqrt{\frac{\partial_tm}{m}} m^{-1}v^{-1}f \right\|_{H^s} - \left( \frac{1}{2} - c - \bar{\epsilon} \right) \left\| \sqrt{\frac{\partial_tw}{w}} m^{-1}v^{-1}R \right\|_{H^s}^2 + \left\langle h (-\Delta_t) R, hR \right\rangle_s
\]

(5.49)

\[
:= \bar{\epsilon}N_{R,R}^{1} - \left( \frac{1}{2} - c - \bar{\epsilon} \right) N_{R,R}^{1} + \left\langle h (-\Delta_t) R, hR \right\rangle_s
\]

By the definition of \( v \) and \( h \), see (5.19) and (5.20) respectively, notice that

\[
h^2(-\Delta_t) = h^2(-\Delta_t)(-\Delta_t)^{-1}w^{2(1-c)}v^{-2} = c \frac{\partial_tw}{w} m^{-2}w^{-2(1-c)}w^{2(1-c)}v^{-2}.
\]
Hence we infer that

\[(5.50) \quad <h(-\Delta_t)R, hR>_s = c \left\| \frac{\partial w}{w} m^{-1} v^{-1} R \right\|_s^2.\]

By combining (5.49) with (5.50) we get that

\[(5.51) \quad S_{R,R}^1 \leq \tilde{\epsilon} N_R^m - \left( \frac{1}{2} - 2c - \tilde{\epsilon} \right) \left\| \frac{\partial w}{m} m^{-1} v^{-1} R \right\|_s^2 := \tilde{\epsilon} N_R^m - \tilde{\epsilon} N_R^m,\]

where the last equality follows upon choosing

\[c = \frac{1}{4} - \tilde{\epsilon}.\]

5.4.2. Bound on \(S_{R,R}^2\). It remains to bound \(S_{R,R}^2\), that we recall is given by

\[(5.52) \quad S_{R,R}^2 = <h \left( 2 \partial_{XX} \Delta t^{-1} \right) g R, h R>_s.\]

By Plancherel’s Theorem and the definition of \(\Delta t^{-1}\), see (4.9), we have that

\[|S_{R,R}^2| \leq \left\{ \frac{2}{p} \left\| \hat{T}_2 g R \right\|_s, \left\| \hat{T}_1 g R \right\|_s \right\} \leq \frac{2}{N} \left\{ \left\| \frac{\partial m}{m} h \right\|_s, \left\| \frac{\partial m}{m} h \right\|_s \right\}.\]

Then, by Cauchy-Schwarz inequality and Corollary 4.2 we infer that

\[(5.53) \quad |S_{R,R}^2| \leq \left( \frac{32}{25N} + \tilde{\epsilon} \right) \left\| \frac{\partial m}{m} m^{-1} v^{-1} R \right\|_s^2 = \left( \frac{32}{25N} + \tilde{\epsilon} \right) N_R^m,\]

where the last inequality follows by Lemma B.4. Hence, from (5.51) and (5.53) we get that

\[(5.54) \quad S_{R,R} \leq -\tilde{\epsilon} N_R^w + \left( \tilde{\epsilon} + \frac{32}{25N} \right) N_R^m.\]

By using (5.54) in (5.43), we obtain the following bound

\[(5.55) \quad \frac{d}{dt} E_s(t) \leq -\tilde{\epsilon} N_A^w - (1 - \tilde{\epsilon}) N_A^m - \tilde{\epsilon} N_R^w - \left( 1 - \frac{32}{25N} - \tilde{\epsilon} \right) N_R^m - N_R^w - N_R^w + S_{R,A} + S_{\Xi,R} + S_{\Xi,A} + S_{\Xi,\Xi}.\]

5.5. Bound on \(S_{R,A}\). Recall the definition of \(S_{R,A}\), namely

\[(5.56) \quad S_{R,A} = <m^{-1} w^{-(1-c)} (2g \partial_{XX} \Delta t^{-1} g) R, m^{-1} w^{-(1-c)} A>_s \]

\[+ <h g^2 [\partial_t \Delta_L] \Delta t^{-1} A, h R>_s + <\partial_t (h^2) R, A>_s, \]

\[:= S_{R,A}^1 + S_{R,A}^2 + S_{R,A}^3.\]

We claim that

\[(5.57) \quad |S_{R,A}| \leq \left( \frac{16}{25} + \frac{6}{N} + \tilde{\epsilon} \right) (N_A^w + N_R^w).\]

We will prove (5.57) by controlling the terms defined in (5.56).
5.5.1. Bound on $S_{R,A}^1$. The bound for the term $S_{R,A}^1$ is analogous to the one performed for $S_{R,R}^2$; see (5.52). In fact, let us rewrite it as follows

$$S_{R,A}^1 := \left\langle m^{-1}w^{-(1-c)}(\partial_{XX}\Delta_L^{-1}T_2 g) R, m^{-1}w^{-(1-c)} A \right\rangle_s + \left\langle m^{-1}w^{-(1-c)}(2(g - 1)\partial_{XX}\Delta_L^{-1}T_2 g) R, m^{-1}w^{-(1-c)} A \right\rangle_s.$$  

By applying Lemma B.2 and Corollary 4.2, we get that

$$|S_{R,A}^1| \leq \left( \frac{2}{N} + \tilde{\epsilon} \right) \left\| \frac{\partial h}{m} m^{-1}w^{-(1-c)} R \right\|_{H^s} \left\| \frac{\partial h}{m} m^{-1}w^{-(1-c)} A \right\|_{H^s}.  

Then, from the definition of $v$, see (5.19), since $w^{-(1-c)} \leq v^{-1}$, from (5.58) we get that

$$|S_{R,A}^1| \leq \left( \frac{2}{N} + \tilde{\epsilon} \right) (N_A^m + N_R^m).$$

5.5.2. Bound on $S_{R,A}^2$. Rewrite the term $S_{R,A}^2$ as follows,

$$S_{R,A}^2 = \left\langle h^2 g^2[\partial L]\Delta^{-1} A, hR \right\rangle_s = \left\langle h^2 \left( \frac{p'}{p} \hat{T}_2 A, \hat{R} \right) \right\rangle_s + \left\langle h^2 \left( (g^2 - 1) \frac{p'}{p} \hat{T}_2 A \right), \hat{R} \right\rangle_s$$

Observe that, from the definition of $m$, see (5.18), we have that

$$\left\| \frac{p'}{p} \right\| \leq \frac{2k}{\sqrt{p}} = \frac{2}{\sqrt{N}} \sqrt{\frac{\partial h}{m}}.$$  

In particular, thanks to the definition of $h$, see (5.20), from (5.60) we get that

$$h^2 = c(1 + \tilde{\epsilon}) \left\| \frac{p'}{p} \right\| \leq \frac{2c(1 + \tilde{\epsilon})}{\sqrt{N}} \sqrt{\frac{\partial h}{m} m^{-2}w^{-2(1-c)}}.$$  

Then, thanks to (5.60) and (5.61), we bound $S_{R,A}^2$ as follows

$$|S_{R,A}^2| \leq \frac{4c(1 + \tilde{\epsilon})}{N} \left\langle \frac{\partial h}{m} m^{-2}w^{-2(1-c)} |\hat{T}_2 A|, |\hat{R}| \right\rangle_s + \frac{4c(1 + \tilde{\epsilon})}{N} \left\langle \sqrt{\frac{\partial h}{m} m^{-2}w^{-2(1-c)}} \left( g^2 - 1 \right) \frac{p'}{p} |\hat{T}_2 A|, |\hat{R}| \right\rangle_s.$$  

Thanks to Lemma B.2 and Corollary 4.2 from (5.62) we infer that

$$|S_{R,A}^2| \leq \left( \frac{2}{N} + \tilde{\epsilon} \right) \left\| \frac{\partial h}{m} m^{-1}w^{-(1-c)} R \right\|_{H^s} \left\| \frac{\partial h}{m} m^{-1}w^{-(1-c)} A \right\|_{H^s}.$$  

By using again that $w^{-(1-c)} \leq v^{-1}$ and $c \leq 1/4$, we conclude that

$$|S_{R,A}^2| \leq \left( \frac{2}{N} + \tilde{\epsilon} \right) (N_A^m + N_R^m).$$
5.5.3. Bound on $S_{R,A}^3$. Recall that

$$S_{R,A}^3 = \langle \partial_t (h^2) R, A \rangle_s.$$ 

By the definition of $h$, see (5.20), we can explicitly compute $\partial_t h^2$. In particular, thanks to Lemma B.8 we have that

$$|S_{R,A}^3| \leq \frac{4(1 + \tilde{\epsilon})^2}{N} \left\| \frac{\partial_t m}{m} m^{-1} w^{-(1-c)} R \right\|_s \left\| \frac{\partial_t m}{m} m^{-1} w^{-(1-c)} A \right\|_s + 2 \left\| \frac{\partial_t m}{m} h R \right\|_s \left\| \frac{\partial_t m}{m} h A \right\|_s ,$$

In view of Lemma B.4 and the fact that $w^{-(1-c)} \leq v^{-1}$, we have that

$$|S_{R,A}^3| \leq \left( \frac{16}{25} + \frac{2}{N} + \tilde{\epsilon} \right) (N^w_A + N^w_R).$$

Putting together (5.59), (5.63) with (5.64), we prove (5.57), namely

$$|S_{R,A}| \leq \left( \frac{16}{25} + \frac{6}{N} + \tilde{\epsilon} \right) (N^w_A + N^w_R).$$

Then, the bound (5.55), becomes

$$\frac{d}{dt} E_s(t) \leq -\tilde{\epsilon} N^w_A - \left( \frac{9}{25} - \frac{6}{N} - \tilde{\epsilon} \right) N^w_A - \tilde{\epsilon} N^w_R - \left( \frac{9}{25} - \frac{182}{25N} - \tilde{\epsilon} \right) N^w_R - N^w_{\Xi} - N^w_{\Xi} + S_{\Xi,R} + S_{\Xi,A} + S_{\Xi,\Xi}.$$

5.6. Bound on $S_{\Xi,R}$. Recall that

$$S_{\Xi,R} := -\langle h (2g \partial_X \Delta^{-1}_t) \Xi, h R \rangle_s - \langle m^{-1} w^{-(1-c)} (b \partial_X \Delta^{-1}_t g) R, m^{-1} w^{-(1-c)} \Xi \rangle_s := S^1_{\Xi,R} + S^2_{\Xi,R}.$$ 

The bound for $S^1_{\Xi,R}$ is analogous to the one performed for the term defined in (5.52). In particular, thanks to Corollary 4.2 Lemma B.4 and Lemma B.2 we get that

$$|S^1_{\Xi,R}| \leq \left( \frac{32}{25N} + \tilde{\epsilon} \right) \left\| \frac{\partial_t m}{m} m^{-1} v^{-1} R \right\|_s \left\| \frac{\partial_t m}{m} m^{-1} w^{-(1-c)} \Xi \right\|_s = \left( \frac{16}{25N} + \tilde{\epsilon} \right) (N^w_R + N^w_{\Xi}).$$

Analogously, for the term $S^2_{\Xi,R}$ thanks to Corollary 4.2 Lemma B.4 and Lemma B.2 we get that

$$|S^2_{\Xi,R}| \leq \tilde{\epsilon} \left\| \frac{\partial_t m}{m} m^{-1} w^{-(1-c)} R \right\|_s \left\| \frac{\partial_t m}{m} m^{-1} w^{-(1-c)} \Xi \right\|_s \leq \tilde{\epsilon} \left\| \frac{\partial_t m}{m} m^{-1} v^{-1} R \right\|_s \left\| \frac{\partial_t m}{m} m^{-1} w^{-(1-c)} \Xi \right\|_s = \tilde{\epsilon} (N^w_R + N^w_{\Xi}),$$

where the last one follows since $w^{-(1-c)} \leq v^{-1}$. 
By combining (5.66) and (5.67) we infer that

\[ |S_{\Xi,R}| \leq \left( \frac{16}{25N} + \tilde{\epsilon} \right) (N_R^m + N_{\Xi}^m). \]

By using (5.68) in (5.65), we have that

\[ \frac{d}{dt} E_s(t) \leq -\tilde{\epsilon} N_A^w - \left( \frac{9}{25} - \frac{6}{N} - \tilde{\epsilon} \right) N_A^m - \tilde{\epsilon} N_R^m - \left( \frac{9}{25} - \frac{198}{25N} - \tilde{\epsilon} \right) N_R^m \]

\[ - N_{\Xi}^w - \left( 1 - \frac{16}{25N} - \tilde{\epsilon} \right) N_{\Xi}^m + S_{\Xi,A} + S_{\Xi,\Xi}. \]

5.7. Bound on $S_{\Xi,A}$. Recall that

\[ S_{\Xi,A} = \left\langle m^{-1}w^{-(1-c)}bg(\partial_Y - t\partial_X)\Delta^{-1}A, m^{-1}w^{-(1-c)}\Xi \right\rangle_s \]

\[ - \left\langle m^{-1}w^{-(1-c)}(2g\partial_X\Delta^{-1})\Xi, m^{-1}w^{-(1-c)}A \right\rangle_s \]

\[ := S_{\Xi,A}^1 + S_{\Xi,A}^2. \]

To control the term $S_{\Xi,A}^1$, thanks to Plancherel’s Theorem and the definition of $\Delta^{-1}$, see (4.9), we get that

\[ |S_{\Xi,A}^1| \lesssim \left\langle m^{-1}w^{-(1-c)} \left( |\hat{g}| * \frac{|p'|}{2k^2p^2} T_{2A} \right), m^{-1}w^{-(1-c)}|\hat{\Xi}| \right\rangle_s. \]

Then, since $|p'|/p \leq \partial w/w$, by using (B.15) to commute $\sqrt{\partial_t w/w}$ with $bg$, we have that

\[ |S_{\Xi,A}^1| \lesssim \left\langle m^{-1}w^{-(1-c)} \left( \langle \cdot \rangle |\hat{g}| * \sqrt{\frac{\partial w}{w} T_{2A}} \right), \sqrt{m^{-1}w^{-1-c}|\hat{\Xi}|} \right\rangle_s \]

\[ + \left\langle m^{-1}w^{-(1-c)} \left( \langle \cdot \rangle^{3/2} |\hat{g}| * \sqrt{\frac{\partial w}{w} T_{2A}} \right), \sqrt{m^{-1}w^{-1-c}|\hat{\Xi}|} \right\rangle_s. \]

Hence, by applying Lemma B.2 estimate (B.2) and Corollary 4.2 we obtain that

\[ |S_{\Xi,A}^1| \leq \tilde{\epsilon} \left\| \sqrt{\frac{\partial w}{w} m^{-1}w^{-(1-c)}A} \right\| \left( \left\| \sqrt{\frac{\partial w}{w} m^{-1}w^{-(1-c)}\Xi} \right\| + \left\| \sqrt{m^{-1}w^{-1-c}\Xi} \right\| \right) \]

\[ \leq \frac{\tilde{\epsilon}}{2} N_A^w + 4\tilde{\epsilon}(N_{\Xi}^w + N_R^m). \]

To bound $S_{\Xi,A}^2$, thanks to Corollary 4.2 and Lemma B.2 we have that

\[ |S_{\Xi,A}^2| \leq \left( \frac{2}{N} + \tilde{\epsilon} \right) \left\| \sqrt{\frac{\partial m}{m} m^{-1}w^{-(1-c)}\Xi} \right\| \left\| \sqrt{\frac{\partial m}{m} m^{-1}w^{-(1-c)}A} \right\| \]

\[ = \left( \frac{2}{N} + \tilde{\epsilon} \right) (N_{\Xi}^m + N_A^m). \]

By combining (5.70), (5.71) with (5.69), we get that

\[ \frac{d}{dt} E_s(t) \leq -\frac{\tilde{\epsilon}}{2} N_A^w - \left( \frac{9}{25} - \frac{8}{N} - \tilde{\epsilon} \right) N_A^m - \tilde{\epsilon} N_R^m - \left( \frac{9}{25} - \frac{198}{25N} - \tilde{\epsilon} \right) N_R^m \]

\[ - (1 - \tilde{\epsilon}) N_{\Xi}^w - \left( 1 - \frac{66}{25N} - \tilde{\epsilon} \right) N_{\Xi}^m + S_{\Xi,\Xi}. \]
5.8. Bound on $S_{\Xi,\Xi}$. Recall that

$$S_{\Xi,\Xi} := \left\langle m^{-1} w^{-(1-c)} (b\partial_X \Delta_t^{-1}) \Xi, m^{-1} w^{-(1-c)} \Xi \right\rangle_s.$$  

Thanks to Corollary 4.2 and Lemma B.2 we get that

(5.73) \[ |S_{\Xi,\Xi}| \leq \hat{\epsilon} \left\| \frac{\partial h_m}{m} m^{-1} w^{-(1-c)} \Xi \right\|_s^2 = \hat{\epsilon} N_{\Xi}^w. \]

Combining (5.73) with (5.72), we have that

$$\frac{d}{dt} E_s(t) \leq -\frac{\hat{\epsilon}}{2} N_{s,M}^w - \left( \frac{9}{25} - \frac{8}{N} - \hat{\epsilon} \right) N_{A}^m - \hat{\epsilon} N_{w}^m - \left( \frac{9}{25} - \frac{198}{25N} - \hat{\epsilon} \right) N_{\Xi}^m - (1 - \hat{\epsilon}) N_{s,M}^w.$$  

Finally, by choosing $N = 32$ and $\epsilon$ small enough, we have that

$$\frac{d}{dt} E_s(t) \leq 0,$$

hence proving Proposition 5.9. \[ \square \]

**Remark 5.12.** To recover the bounds for a general Mach number, it is enough to consider

$$E_{s,M}(t) = \frac{1}{2} \left( \left\| m^{-1} v^{-1} \frac{1}{M} R \right\|_s^2 + \left\| m^{-1} w^{-(1-c)} A \right\|_s^2 + \left\| m^{-1} w^{-(1-c)} \Xi \right\|_s^2 + 2 \langle hR, hA \rangle_s \right),$$

and perform exactly the same estimates as before.

**Appendix A. A toy model with non-optimal upper bounds**

In this section, we explain why we have to optimize the choice of the weights involved in the energy estimates performed to prove Proposition 5.9.

Consider instead of (5.14)-(5.16) the following system

$$\partial_t r = -a,$$

$$\partial_t a = g^2 [\partial_t \Delta_L] \Delta_t^{-1} a - \Delta_L r,$$

Now, assume that one has two weights $\tilde{v}, \tilde{w}$, to be defined and compute that

$$\frac{1}{2} \frac{d}{dt} \left( \| \tilde{v}^{-1} r \|_s^2 + \| \tilde{w}^{-1} a \|_s^2 \right) = -\left( \left\| \sqrt{\frac{\partial_t \tilde{v}}{\tilde{v}} \tilde{v}^{-1} r \right\|_s^2 - \left\| \sqrt{\frac{\partial_t \tilde{w}}{\tilde{w}} \tilde{w}^{-1} a \right\|_s^2 \right)$$

(A.1) \[ + \langle \tilde{w}^{-1} \left( g^2 [\partial_t \Delta_L] \Delta_t^{-1} a \right), \tilde{w}^{-1} a \rangle_s \]

$$- \langle \tilde{v}^{-1} a, \tilde{v}^{-1} r \rangle_s + \langle \tilde{w}^{-1} a, \tilde{w}^{-1} r \rangle_s.$$  

Then, to control the mixed scalar product in the last line of (A.1), we can choose

$$\tilde{v}^{-1} = \tilde{w}^{-1}(-\Delta_L).$$

In particular, equation (A.1) becomes

$$\frac{1}{2} \frac{d}{dt} \left( \| v^{-1} r \|_s^2 + \| \tilde{w}^{-1} a \|_s^2 \right) = -\left( \left\| \sqrt{\frac{\partial_t \tilde{v}}{v} v^{-1} r \right\|_s^2 - \left\| \sqrt{\frac{\partial_t \tilde{w}}{w} \tilde{w}^{-1} a \right\|_s^2 \right)$$

+ \langle \tilde{w}^{-1} \left( g^2 [\partial_t \Delta_L] \Delta_t^{-1} a \right), \tilde{w}^{-1} a \rangle_s.$$
Choosing $\tilde{w} = wm$, for $w, m$ defined as in (5.17)-(5.18), as done in Section 4.3 we can control the remaining scalar product.

Then, since $v = (-\Delta_L)^{-1/2}\tilde{w}$ and $\tilde{w} \approx \Delta_L^{1+\tau}$, assuming regularity on the initial data, one would obtain the following bound

$$\|r\|_{L^2}^2 + \left\|(-\Delta_L)^{-1/2} a\right\|_{L^2}^2 \lesssim \tau^2 C_{\eta w},$$

The $(-\Delta_L)^{-1/2} a$ is to mimic $Q(v)$, but this bound of course is not the expected one when looking at the Couette case.

The toy model gives us an important drawback. We are not using at all the dissipation created by the weakening of the norm on $r$. We should also recall that in the proof of Lemma 3.2 namely the energy estimate in the Couette case, it was crucial to include the mixed scalar product in the energy functional.

### Appendix B. Properties of the weights

In the following we show some standard inequalities that we need to use several times throughout the paper.

**Lemma B.1.** Let $f, g \in L^2(\mathbb{T} \times \mathbb{R})$, $h \in H^s(\mathbb{R})$, for $s > 1$. Then

(B.1) \[ \|\hat{f} \ast \hat{g}\|_{L^2} \lesssim \|f\|_{L^2} \|h\|_{H^s}, \]

(B.2) \[ |\langle \hat{f}, \hat{g} \ast \hat{h} \rangle| \lesssim \|f\|_{L^2} \|g\|_{L^2} \|h\|_{H^s}, \]

**Proof.** Inequality (B.1) is just a standard Young’s inequality followed by Cauchy-Schwarz, since

$$\|\hat{h}\|_{L^1} = \int_\mathbb{R} \frac{1}{s} \langle \eta \rangle^s \hat{h}(\eta) d\eta \lesssim \|h\|_{H^s},$$

and the last follows since $s > 1$. The inequality (B.2) is Cauchy-Schwarz plus (B.1). \qed

Then we state a useful Lemma which tells us how commute operators when we know how to exchange frequencies.

**Lemma B.2.** Let $B(\nabla)$, $T(\nabla)$ be two Fourier multipliers such that, for a given $\beta, \gamma > 0$, it holds

(B.3) \[ |B(k, \eta)| \lesssim |\eta - \xi|^{\gamma} |B(k, \xi)| \]

(B.4) \[ |T(k, \eta)| \lesssim |\eta - \xi|^{\gamma} |T(k, \xi)|. \]

Consider $f \in H^{s+\beta+\gamma+1}(\mathbb{R})$ and $g, h \in L^2(\mathbb{T} \times \mathbb{R})$ such that $Tg, Bh \in H^s(\mathbb{T} \times \mathbb{R})$.

Then it holds that

$$|\langle T(\hat{f} \ast B\hat{g}), \hat{h} \rangle_s| \lesssim \left\langle \left( \langle \cdot \rangle^{s+\gamma+\beta} \hat{f} \ast \langle \cdot \rangle^s [T\hat{g}] \right) \ast \langle \cdot \rangle^s [B\hat{h}] \right\rangle \lesssim \|f\|_{H^{s+\gamma+\beta+1}} \|Tg\|_{H^s} \|Bh\|_{H^s}.$$

**Proof.** Writing down explicitly the scalar product, we have that

$$|\langle T(\hat{f} \ast B\hat{g}), \hat{h} \rangle_s| = \sum_{k \in \mathbb{Z}} \int_\mathbb{R} (k, \eta)^s T(k, \eta) \left( \int_\mathbb{R} \hat{f}(\eta - \xi) B(k, \xi) \hat{g}(k, \xi) d\xi \right) \langle k, \eta \rangle^s \hat{h}(k, \eta) d\eta \]

$$\lesssim \sum_{k \in \mathbb{Z}} \int_\mathbb{R} \left( \int_\mathbb{R} (\eta - \xi)^{s+\gamma+\beta} |\hat{f}|(\eta - \xi) \langle k, \xi \rangle^s \hat{g}(k, \xi) d\xi \right) \times \langle k, \eta \rangle^s |B(k, \eta)\hat{h}(k, \eta) d\eta,$$

where in the last line we have used (B.3), (B.4) and the fact that $\langle k, \eta \rangle \lesssim \langle \eta - \xi \rangle \langle k, \xi \rangle$. 
Rewriting the term in the r.h.s. of the last inequality, we have that
\[
\langle T(\hat{f} * B\hat{g}), \hat{h} \rangle_s \lesssim \left( \langle \cdot \rangle^{s+\gamma+\beta} \hat{f} * \langle \cdot \rangle^s T\hat{g} \right), \langle \cdot \rangle^s |B\hat{h}| \right) 
\lesssim \|f\|_{H^{s+\gamma+\beta+1}} \|Tg\|_{H^s} \|Bh\|_{H^s},
\]
and the last bound follows from (B.2).

Now we recall here the definitions of all the weights used in Section 5, namely
(B.5) \[ p = k^2 + (\eta - kt)^2 \]
(B.6) \[ p' = -2k(\eta - kt) \]
(B.7) \[ \partial_t w = (1 + \tilde{\epsilon}) \frac{|p'|}{p} w, \quad w|_{t=0} = 1, \]
(B.8) \[ v^2 = (-\Delta_t)^{-1} w^{2(1-c)}, \]
(B.9) \[ h = \sqrt{c} \sqrt{\partial_t w} w m^{-1} w^{-(1-c)}. \]

Recall also that \( w \) is explicitly given by
(B.10) \[
w(t, k, \eta) = \begin{cases} 
\left( k^2 + \eta^2 \right)^{1+\tilde{\epsilon}} \left( \frac{k^2}{p(t, k, \eta)} \right)^{1+\tilde{\epsilon}} & \text{for } \eta k > 0 \text{ and } t < \frac{\eta}{k}, \\
\left( k^2 + \eta^2 \right)^{1+\tilde{\epsilon}} \left( \frac{k^2}{p(t, k, \eta)} \right)^{1+\tilde{\epsilon}} & \text{for } t \geq \frac{\eta}{k}.
\end{cases}
\]

The weight \( m \), it is given by
(B.11) \[
m(t, k, \eta) = \exp \left( N[\arctan(\frac{\eta}{k} - t) - \arctan(\frac{\eta}{k})] \right).
\]

In order to make use of Lemma [B.2] for multipliers which involves the weights previously defined, we need to know how them exchanges frequencies.

**Lemma B.3.** Let \( p, p', w, m \) the weights defined in (B.5)–(B.7). Then it holds that
(B.12) \[
p^{-1}(t, k, \eta) \lesssim \langle \eta - \xi \rangle^2 p^{-1}(t, k, \xi),
\]
(B.13) \[
\frac{|p'|}{p}(t, k, \eta) \lesssim \langle \eta - \xi \rangle^3 \frac{k^2}{p(t, k, \eta)} + \langle \eta - \xi \rangle^2 |p'| p(t, k, \xi).
\]

For the weight \( w \) we have that
(B.14) \[
w^{-1}(t, k, \eta) \lesssim \langle \eta - \xi \rangle^{4(1+\tilde{\epsilon})} w^{-1}(t, k, \xi),
\]
(B.15) \[
\frac{\partial_t w}{w}(t, k, \eta) \lesssim \langle \eta - \xi \rangle^3 \frac{k^2}{p(t, k, \xi)} + \langle \eta - \xi \rangle^2 \frac{\partial_t w}{w}(t, k, \xi).
\]

Regarding the weight \( m \), we get that
(B.16) \[
m(t, k, \eta) \lesssim m(t, k, \xi),
\]
(B.17) \[
\frac{\partial_t m}{m}(t, k, \eta) \lesssim \langle \eta - \xi \rangle^2 \frac{\partial_t m}{m}(t, k, \xi).
\]

**Proof.** To get (B.12), simply observe that
(B.18) \[
\frac{1}{k^2 + (\eta - kt)^2} = \frac{1}{k^2(\frac{\eta}{k} - t)^2} \leq \langle \eta - \xi \rangle^2 \frac{1}{k^2(\frac{\eta}{k} - t)^2},
\]

\( \blacksquare \)
Lemma B.4. \(|\langle a - b \rangle \rangle \geq \langle a \rangle \rangle\) for any \(a, b \in \mathbb{R}\). Since \(k > 1\), (B.12) directly follows from (B.18).

The bound (B.13) we proceed as follows

\[
\left| \frac{p'}{p} \right| (t, k, \eta) = \frac{2|k|^2|p - t|}{k^2(1 + (\frac{2}{k} - t)^2)} \lesssim \frac{2|\eta - \xi| + |\xi - t|}{(1 + (\frac{2}{k} - t)^2)} \lesssim \langle \eta - \xi \rangle^3 \frac{k^2}{p(t, k; \xi)} + \langle \eta - \xi \rangle^2 \left| \frac{p'}{p} \right| (t, k, \xi).
\]

To prove (B.14), recalling (B.10), observe that

\[
(B.19) \quad \frac{p(t, k, \eta)}{k^2 + \eta^2} \lesssim \langle \eta - \xi \rangle^2 \frac{(\eta - \xi)^2 + p(t, k, \xi)}{k^2 + \xi^2} \leq \langle \eta - \xi \rangle^4 \frac{p(t, k, \xi)}{k^2 + \xi^2},
\]

where the last one follows since \(p > 1\). Proceeding analogously, we have also that

\[
(B.20) \quad \frac{k^4}{(k^2 + \eta^2)p(t, k, \eta)} \leq \langle \eta - \xi \rangle^4 \frac{k^4}{(k^2 + \xi^2)p(t, k, \xi)}.
\]

By using (B.19), (B.20) in (B.10) we get (B.14).

The estimate (B.15) it is exactly the same as (B.13).

Then, (B.16) follows since \(m(t, k, \eta)\) is uniformly bounded in \(t, k, \eta\), see (B.11). The bound (B.17) it is exactly the same as (B.12). \(\square\)

Next we prove the Lemma that guarantees that the scalar product in the definition of \(E_s\), see (B.21), can be controlled with the other terms.

Lemma B.4. Let \(f \in H^s(\mathbb{T} \times \mathbb{R})\). Consider the weights defined in (B.6)-(B.9). Then, for \(c < \frac{8}{25(1+c)}\), it holds that

\[
(B.21) \quad \|hf\|_s \leq \frac{4}{5} \|m^{-1}v^{-1}f\|_s, \quad \|hf\|_s \leq \frac{4}{5} \|m^{-1}w^{-(1-c)}f\|_s.
\]

Proof. To prove (B.21), first of all notice that \(|p'|/p \leq 2\). Then, thanks to the restriction on \(c\), from the definition of \(h\), see (B.9), it directly follows that

\[
(B.22) \quad h = c(1 + \bar{c}) \sqrt{\frac{|p'|}{p} m^{-1}w^{-(1-c)}} \leq \frac{4}{5} m^{-1}w^{-(1-c)} \leq \frac{4}{5} m^{-1}v^{-1},
\]

where the last inequality follows by the definition of \(v\), see (B.8). In particular, (B.22) implies (B.21). Observe that for \(\bar{c} \leq 1/4\), then \(\frac{1}{4} \leq \frac{8}{25(1+c)}\). \(\square\)

Then we state a rough version of an inequality for the Laplacian in the moving frame in the Couette case.

Lemma B.5. Let \(p = -\Delta_L = k^2 + (\eta - kt)^2\), then for any function \(f \in H^{s+2\beta}(\mathbb{T} \times \mathbb{R})\), it holds that

\[
(B.23) \quad \left\| p^{-\beta} f \right\|_s \lesssim \frac{1}{(t)^{2\beta}} \|f\|_{s+2\beta}, \quad \left\| p^\beta f \right\|_s \lesssim (t)^{2\beta} \|f\|_{s+2\beta},
\]

for any \(\beta > 0\).

Proof. The bounds (B.23) follows just by Plancherel Theorem and the basic inequalities for japanese brackets \(\langle k, \eta \rangle \leq \langle \eta - \xi \rangle \langle k, \xi \rangle\). \(\square\)

Now we want to see how to take out time factors from \(w\), in analogy with Lemma B.3.
Lemma B.6. For any function \( f \in H^s(\mathbb{T} \times \mathbb{R}) \) and any \( \beta > 0 \), it holds that
\[
(B.24) \quad \left\| w^{-\beta} f \right\|_s \leq \frac{1}{(2(t+1))^{\frac{3}{2}}} \left\| f \right\|_{s+2\beta(1+t)}, \quad \left\| w^\beta f \right\|_s \leq (t)^{2\beta(1+t)} \left\| f \right\|_{s+2\beta(1+t)}.
\]

Proof. By using (B.10), the first inequality of (B.24) is obtained by the following basic inequalities
\[
(B.25) \quad \begin{cases}
\frac{p}{k^2 + \eta^2} & = \frac{(\eta/k - t)^2}{(\eta/k)^2} \leq 1 \leq \frac{(\eta/k)^2}{(t)^2} \quad \text{for } \eta k > 0 \text{ and } 0 \leq t < \frac{\eta}{k}, \\
\frac{k^4}{(k^2 + \eta^2)p} & = \frac{k^4(\eta/k)^2(\eta/k - t)^2}{(t)^2} \leq 1 \quad \text{for } t \geq \frac{\eta}{k}.
\end{cases}
\]
The last inequality of (B.24) follows by a similar computation performed in (B.25). \( \square \)

Now we see how to bound a term which is crucial in the bound of \( S^1_{R,R} \), see (5.48), which determines the optimal constant \( c \) in the energy functional.

Lemma B.7. Let \( \Delta_t \) as defined in (4.4). Consider \( B(\nabla) \) be a Fourier multiplier and \( f \in H^s(\mathbb{T} \times \mathbb{R}) \). Assume that there is a \( \beta \geq 0 \) such that
\[
|B(k, \eta)| \lesssim (\eta - \xi)^\beta |B(k, \xi)|.
\]
Then, if \( \|g^2 - 1\|_{H^{s+\beta+3}} \leq \epsilon \) and \( \|b\|_{H^{s+\beta+3}} \leq \epsilon \), we have that
\[
\langle B(-\Delta_t)^{-1}\partial_t \Delta_t |f, Bf\rangle_s \leq (1 + \epsilon) \left\| \sqrt{\frac{\partial_tw}{w} Bf} \right\|^2_s + \epsilon \left\| \sqrt{\frac{\partial_t \mathcal{M}}{\mathcal{M}} Bf} \right\|^2_s.
\]

Proof. First of all, by the definition of \( \Delta_t \), see (4.4), it follows that
\[
\partial_t \Delta_t = \partial_t \Delta_L + (g^2 - 1) \partial_t \Delta_L - b \partial_X.
\]
Then, thanks to the explicit expression of \( \Delta_t^{-1} \), see (4.9), we get that
\[
\left| \langle B(-\Delta_t)^{-1}\partial_t \Delta_t |f, Bf\rangle_s \right| \leq \left| \langle B(\Delta_L^{-1}T_2)\partial_t \Delta_L |f, Bf\rangle_s \right|
\]
\[
+ \left| \langle B(\Delta_L^{-1}T_2)(g^2 - 1) \partial_t \Delta_L f, Bf\rangle_s \right|
\]
\[
+ \left| \langle B(\Delta_L^{-1}T_2) b \partial_X f, Bf\rangle_s \right|
\]
\[
= S_1 + S_2 + S_3.
\]
To bound \( S_1 \), observe that
\[
(B.26) \quad \left| \langle B(\Delta_L^{-1}T_2)|\partial_t \Delta_L |f, Bf\rangle_s \right| = \left\| \sqrt{B^2(-\Delta_L^{-1})T_2|\partial_t \Delta_L |f} \right\|_{H^s}^2,
\]
and the operator under the square root is positive. Combining (B.26) with Corollary 4.2, we have that
\[
\left\| \sqrt{B^2(-\Delta_L^{-1})T_2|\partial_t \Delta_L |f} \right\|_{H^s}^2 \leq \frac{1}{1 - C\epsilon} \left\| \sqrt{\frac{\partial_tw}{w} Bf} \right\|_{H^s}^2.
\]
The bound on \( S_2 \) it is analogous to the previous one.
To bound \( S_3 \) just notice that instead of \( \partial_tw/w \) one has to use \( \partial_t \mathcal{M}/\mathcal{M} \). \( \square \)

Finally, we state the bound used to treat the term \( S^3_{R,A} \), see Section 5.5.3. It is a direct computation involving explicit Fourier multipliers.
Lemma B.8. Let \( f, g \in H^s(\mathbb{T} \times \mathbb{R}) \), then it holds that

\[
|\langle \partial_t h^2 f, g \rangle_s| \leq \frac{4(1 + \tilde{\varepsilon})^2}{N} \left\| \partial_t m \right\|_{s-1} \left\| \partial_t w \right\|_{s-1} f \left\| \partial_t m \right\|_{s-1} \left\| \partial_t w \right\|_{s-1} g
\]

(B.27)

\[
+ 2 \left\| \partial_t m \right\|_{s} \left\| \partial_t m \right\|_{s} h f \left\| \partial_t m \right\|_{s} \left\| \partial_t m \right\|_{s} h g.
\]

\]

Proof. Since the definition of \( h \) involves explicit operators, the proof is just a bound on the Fourier multipliers involved. By the definition of \( h \), see (5.20), we directly compute that

\[
\partial_t h^2 = \partial_t \left( c \frac{\partial_t w}{w} m^{-2} w^{-2(1-c)} \right)
\]

\[
= -2(1-c) c \frac{\partial_t w}{w} m^{-2} \frac{\partial_t w}{w} m^{-2} w^{-2(1-c)} - 2 \frac{\partial_t w}{w} m^{-2} w^{-2(1-c)}
\]

\[
+ c \partial_t \left( \frac{\partial_t w}{w} \right) m^{-2} w^{-2(1-c)}
\]

\[
= -2c(1-c) \left( \frac{\partial_t w}{w} \right)^2 m^{-2} w^{-2(1-c)} - 2 \frac{\partial_t m}{m} h^2
\]

\[
+ c \left( \frac{\partial_t w}{w} - \left( \frac{\partial_t w}{w} \right)^2 \right) m^{-2} w^{-2(1-c)}.
\]

Now observe that

\[
\partial_t w = (1 + \tilde{\varepsilon}) \partial_t \left( \frac{|p'|}{p} w \right) = (\partial_t w)^2 w + (1 + \tilde{\varepsilon}) \partial_t \left( \frac{|p'|}{p} \right) w.
\]

Then, since \(|p'| \leq 2|k| \sqrt{p}\), by the definition of \( v \) and \( m \), it holds that

\[
\left( \frac{\partial_t w}{w} \right)^2 = (1 + c)^2 \left( \frac{p''}{p} \right)^2 \leq \frac{4(1 + \tilde{\varepsilon})^2}{N} \partial_t m
\]

Hence, combining (B.28), (B.29), and (B.30), we get that

\[
|\partial_t h^2| \leq 2c(1-c) \frac{4(1 + \tilde{\varepsilon})^2}{N} \frac{\partial_t m}{m} m^{-2} w^{-2(1-c)} + 2 \frac{\partial_t m}{m} h^2
\]

(B.31)

\[
+ c(1 + \tilde{\varepsilon}) \frac{p''}{p} m^{-2} w^{-2(1-c)} + c \partial_t \left( \frac{\partial_t m}{m} w^{-2(1-c)} \right) \leq 2c(2-c) \frac{4(1 + \tilde{\varepsilon})^2}{N} \frac{\partial_t m}{m} m^{-2} w^{-2(1-c)} + 2 \frac{\partial_t m}{m} h^2,
\]

where the last follows since \( p'' = 2k^2 \). Finally, since \( c \leq 1/4 \), we have that

\[
|\partial_t h^2| \leq \frac{4(1 + \tilde{\varepsilon})^2}{N} \frac{\partial_t m}{m} m^{-2} w^{-2(1-c)} + 2 \frac{\partial_t m}{m} h^2.
\]

Then (B.31) follows by (B.32) and Cauchy-Schwarz’s inequality. \( \square \)
Appendix C. Properties of some operator

We now present commutation properties of the operator $\Phi_b$ defined in (5.6), that we have implicitly used in Section 5. Recall that $\Phi_b$ is given by

$$
\Phi_b(t, s) = I + \sum_{n=0}^{\infty} b \int_s^t \partial_X \Delta^{-1} \Phi^n_b(\tau, s) d\tau,
$$

$$
\Phi^n_b(t, s) = b \int_s^t \partial_X \Delta^{-1} \Phi^{n-1}_b(\tau, s) d\tau,
$$

$$
\Phi^0_b(t, s) = I.
$$

Then we state the following.

**Lemma C.1.** Let $\Phi_b$ the operator defined in (C.1). Consider $B(\nabla)$ be a Fourier multiplier and $f \in H^s(T \times \mathbb{R})$. Assume that there is a $\beta \geq 0$ such that

$$
|B(k, \eta)| \lesssim \langle \eta - \xi \rangle^\beta |B(k, \xi)|.
$$

Then, if $\|g^2 - 1\|_{H^{s+\beta+3}} \leq \varepsilon$ and $\|b\|_{H^{s+\beta+3}} \leq \varepsilon$, it holds that

$$
\|B\Phi_b f\|_{H^s} \leq \frac{1}{1 - C\varepsilon} \|Bf\|_{H^s},
$$

for a proper $C > 1$ and $\varepsilon$ such that $C\varepsilon \leq 1/2$.

**Proof.** Since $\Phi_b$ it is defined recursively, we will prove (C.2) just for $\Phi^1_b$. We proceed in an analogous way as done in the proof of Corollary 4.2. In particular, thanks to the definition of $\Delta^{-1}_T = \Delta_L T_2^*$ given in (4.9) and the expression of $T_2$ as a Neumann series, see (4.10), we have that

$$
\|B\Phi^1_b f\|_{H^s} = \left\|Bb \left( \int_s^t \partial_X \Delta_L^{-1} T_2 d\tau \right) f \right\|_{H^s},
$$

$$
\leq \left\|Bb \left( \int_s^t \partial_X \Delta_L^{-1} d\tau \right) f \right\|_{H^s} + \left\|Bb \left( \int_s^t \partial_X \Delta_L^{-1} \tilde{T}_2 T_2 d\tau \right) f \right\|_{H^s},
$$

$$
:= N_1 + N_2.
$$

To control $N_1$, we know that

$$
\int_s^t \partial_X \Delta_L^{-1} d\tau = -i \int_s^t \frac{k}{k^2 + (\eta - k\tau)^2} d\tau,
$$

so the latter operator it is a bounded Fourier multiplier. By Plancherel Theorem we infer that

$$
N_1 \leq \left\| \left( \langle \cdot \rangle^{s+\beta} \tilde{b} \right) * \left( \int_s^t \partial_X \Delta_L^{-1} d\tau \right) \langle \cdot \rangle^s B\tilde{f} \right\|_{L^2} \leq \|b\|_{H^{s+\beta+1}} \|Bf\|_{H^s},
$$

where we have used (B.2) in the last inequality.

As regards $N_2$, by Plancherel Theorem we have that

$$
N_2 = \left\| \left( \langle \cdot \rangle^s B \right) \left( \tilde{b} * \left( \int_s^t \partial_X \Delta_L^{-1} \tilde{T}_2 \tilde{T}_2 d\tau \right) * \tilde{f} \right) \right\|_{H^s}.
$$
By the explicit definition of $\tilde{T}_2$ given in \[4.11\], from \[C.4\], after rearranging the convolutions, we infer that

$$N_2 \lesssim \left\| \langle \cdot \rangle^{s+\beta} (g^2 - 1) \ast \left( \langle \cdot \rangle^s B \ast \left( \int_s^t \partial_X \Delta^{-1}_L \ast \tilde{T}_2 d\tau \right) \ast \hat{f} \right) \right\|_{L^2}$$

(C.5)

$$+ \left\| \langle \cdot \rangle^{s+\beta} \hat{b} \ast \left( \langle \cdot \rangle^s B \ast \left( \int_s^t \partial_X \Delta^{-1}_L \ast \tilde{T}_2 d\tau \right) \ast \hat{f} \right) \right\|_{L^2}$$

$$\lesssim \left( \|g^2 - 1\|_{H^{s+\beta+1}} + \|b\|_{H^{s+\beta+1}} \right) \left\| Bb \left( \int_s^t \partial_X \Delta^{-1}_L T_2 d\tau \right) f \right\|_{H^s}. \tag{C.6}$$

In particular, from (C.5) and initial hypothesis on $g^2 - 1$ and $b$, we get that

(C.6)

$$N_2 \leq C\varepsilon \|B\Phi_b f\|_{H^s}.$$ 

Hence by combining (C.3) with (C.6) we prove that

$$\|B\Phi_b f\|_{H^s} \leq \frac{1}{1 - C\varepsilon} \|B f\|_{H^s}.$$ 

As regards other iterates of order $n$, in the bound (C.6) one has $(C\varepsilon)^n$ instead of $C\varepsilon$. \qed

In the next, we show the equality claimed in \[5.11\]. The regularity on the background shear is just to be sure that we have enough regularity to commute the operators involved.

**Lemma C.2.** Assume that $\|g^2 - 1\|_{H^{s+10}} \leq \epsilon$ and $\|b\|_{H^{s+10}} \leq \epsilon$. Consider the following operator

$$G = g(\partial_Y - t\partial_X)\Delta^{-1}_L + \tilde{\Phi} \left( g + bg(\partial_Y - t\partial_X)\Delta^{-1}_L \right). \tag{C.7}$$

Then it holds that

$$\partial_t G = \sum_{i=1}^{8} F_i^1 \partial_X \Delta^{-1}_L F_i^2, \tag{C.8}$$

where $F_i^1, F_i^2$ are bounded operators from $H^s$ to $H^s$.

**Proof.** Notice that, by \[5.7\], we know that

$$\partial_t \tilde{\Phi} = \partial_X \Delta^{-1}_L \tilde{\Phi}_b^{-1}. \tag{C.9}$$

Then, just by the definition \[C.7\] compute that

$$\partial_t G = -g\partial_X \Delta^{-1}_L + \partial_X \Delta^{-1}_L \tilde{\Phi}_b^{-1} g - \tilde{\Phi} bg \partial_X \Delta^{-1}_L$$

$$+ \partial_X \Delta^{-1}_L \tilde{\Phi}_b^{-1} \left[ bg(\partial_Y - t\partial_X)\Delta^{-1}_L \right]$$

$$- \left[ \tilde{\Phi} g + (g + \tilde{\Phi} bg) (\partial_Y - t\partial_X) \right] \partial_t \Delta^{-1}_L.$$ 

Thanks to Corollary 4.2, we know that $\partial_t \Delta^{-1}_L$ is defined, see \[4.16\]. By using the explicit characterization of $\Delta^{-1}_L$, boundedness and regularity of the background shear, it is an exercise to check that actually $\partial_t G$ satisfies \[C.8\]. \qed

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