Modeling the bounce of a gas-filled ball

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The coefficient of restitution $\epsilon$ characterizes the energy retained when a ball bounces, and can easily be measured in an “at home” experiment. For thin-walled gas-filled balls such as basketballs, we construct a simple two parameter model to describe how $\epsilon$ changes as a function of the ball’s internal pressure. A comparison with data shows good agreement. With additional assumptions, the model also predicts how $\epsilon$ changes as a function of temperature. A comparison with tennis ball data shows surprisingly good agreement.

I. INTRODUCTION

The bouncing of a ball demonstrates important features of mechanics, including the conservation of energy and momentum, the conversion of energy between potential and kinetic, and the loss of mechanical energy through conversion to thermal energy and sound. An interesting experiment is to drop a ball from a fixed height and to measure how high it bounces. The less energy is lost during the bounce, the closer the ball returns to its starting height. Accurate measurements are possible using a mobile phone to video the bounce, and then examining individual frames.

By making simplifying assumptions, one can construct physical models to describe such behavior. While these oversimplify or ignore complex phenomena, they can still provide a remarkably accurate description. Here, starting from the principle of energy conservation, we derive a model to describe the bounce, and compare this to data collected while varying the pressure inside a basketball. We then extend the model to describe how the bounce is affected by temperature. This is checked against data collected while varying the temperature of a tennis ball [1].

Others [2–6] have done such experiments with a variety of balls. Often, the data is fit to an ad-hoc model which is not derived from physical laws, a point also noted in [6]. While our model is an oversimplification, it nevertheless provides an excellent description of the behavior.

II. BOUNCING BALLS

The Internet offers many slow-motion videos of bouncing balls. Some are remarkable, for example, a solid rubber golf ball bouncing off a steel plate at very high speeds [7, 8]. Gas-filled balls such as tennis balls, soccer balls, and basketballs exhibit similar behavior [9].

These images show that balls are compressed and deformed when they bounce. Their kinetic energy is converted into potential energy and stored: the deformed ball behaves like a compressed spring. It then “jumps back” to a round shape, converting stored potential energy back into kinetic energy, and leaping into the air. In this process, some energy is converted into thermal energy and sound, and consequently each bounce of the ball is lower than the previous one. A detailed discussion of the mechanisms at work, and citations to the literature, can be found in a highly cited paper by Cross [10] and work by Bridge [2, 3].

How can we build a simple model to describe the bounce of a gas-filled ball? Here, we do this by starting from the principle of conservation of energy. The physical system we model consists of the ball and the gas inside it. Some energy is lost from this system (i.e., transferred to the surroundings) when the ball bounces, since otherwise the ball would return to its starting height. In our model, we assume that the energy which is not lost is stored in the compressed gas [11] and in the wall of the ball, and that the energy loss is proportional to the amount by which the ball is deformed during impact. The model predicts how changing the internal gas pressure affects the bounce.

To test our model, we first experiment with a basketball. We change the pressure inside the basketball by pumping air in or letting air out through the valve. We later extend our model to describe the behavior of a gas-filled ball as the temperature varies. We test it with a tennis ball heated and cooled to a range of temperatures.

Related work models how tennis balls grip and are spun up when they strike the ground at different angles [12, 13], measures the room-temperature coefficient of restitution of tennis balls and Superballs [14], measures the temperature variation of the coefficient of restitution of pressureless and pressurized tennis balls between 0°C and 40°C [4, 15], describes a testing method for tennis ball bounce quality [16], and models the impact of the

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Consider a ball of mass \( m \), which is released at rest from height \( h_0 \) above a level surface and moves up and down along a vertical line. The subscript “\( b \)” indicates “before the bounce”. After release, the ball moves downwards under the influence of gravity, accelerating until it hits the floor. Just before that impact, denote its vertical velocity by \( v_a \), as shown in Fig. 1. We neglect air resistance, so equating the potential and kinetic energy gives \( E_b = \frac{1}{2}mv_a^2 = mgh_a \), where \( g = 9.8 \text{ m/s}^2 \) is the acceleration of gravity, and \( E_b \) is the energy of the ball before it hits the ground.

One way to quantify the loss of energy during the bounce is via the “coefficient of restitution” \( \epsilon \). This dimensionless number is the ratio of the vertical speed of the ball just after the bounce \( |v_a| \) to the vertical speed of the ball just before the bounce

\[
\epsilon = \frac{|v_a|}{|v_b|},
\]

where subscript “\( a \)” means “after the bounce”. Since the speed after the bounce is smaller than before, the coefficient of restitution lies in the range \( 0 \leq \epsilon < 1 \).

While it is not given this name, the coefficient of restitution is described by Newton in discussing the relative velocities before and after bouncing impacts (reflection) for balls made of wool, steel, cork, and glass. Cook [18] provides a detailed discussion of Newton’s approach, which is often called “Newton’s experimental law of impacts”. The literature uses three symbols to denote the coefficient of restitution: COR, \( \epsilon \), and \( \epsilon \). To avoid confusion with Euler’s number \( e = 2.71828 \cdots \) we follow the third convention.

The ratio of the ball’s energy \( E_a \) after the bounce to its energy \( E_b \) before the bounce can be expressed in terms of \( \epsilon \). Just before and just after the bounce, the potential energy vanishes, and all of the energy is kinetic, given by \( \frac{1}{2}mv^2 \), where \( v \) is the vertical velocity. The ratio of energies is then

\[
\frac{E_a}{E_b} = \frac{mv_a^2/2}{mv_b^2/2} = \left( \frac{v_a}{v_b} \right)^2 = \epsilon^2,
\]

where the final equality follows from Eq. (1).

The ratio of heights after and before the bounce may also be expressed in terms of the coefficient of restitution \( \epsilon \). After the bounce, the ball moves upwards, slowing down, and transforming its kinetic energy into potential energy. It reaches the maximum height \( h_a \) at the moment when the velocity drops to zero, so (neglecting air resistance) all of the kinetic energy is converted to potential energy \( mgh_a \). Thus

\[
\frac{E_a}{E_b} = \frac{mgh_a}{mgh_b} = \frac{h_a}{h_b} = \epsilon^2,
\]

where we have used Eq. (2) to obtain the final equality.

It is hard to measure the speed of the ball just before and just after the bounce to compute \( \epsilon \) directly from the definition in Eq. (1). It is easier to measure the release height and the bounce height. So in an experiment, the coefficient of restitution may be determined from Eq. (3) via

\[
\epsilon = \sqrt{\frac{h_a}{h_b}},
\]

where \( h_a \) and \( h_b \) are measured from the ground to the bottom of the ball. For a very bouncy ball, \( \epsilon \) is close to 1, and for a very “dead” ball, \( \epsilon \) is close to 0.

IV. MODELING THE ENERGY STORED DURING THE BOUNCE

Our model neglects air resistance, so the energy of the ball is constant \( E_b \) before the bounce, and has a smaller constant value \( E_a \) after the bounce. The difference between these is the energy which is lost (e.g., converted to thermal energy and sound) during the bounce. The remaining energy, which is stored, is \( E_a \).

To model the energy stored in the bounce, first we treat the ball as a bag containing gas under pressure. When the ball hits the ground and comes to a stop, the internal volume \( V \) of the bag decreases by a small amount \( \Delta V \geq 0 \) to \( V - \Delta V \), as shown in Fig. 1. This decrease in volume requires energy, since a force must be applied against the pressure of the gas inside the ball. For now, assume that the material of the bag does not store or release any energy.

To compute the energy needed, consider a “cubical box” model of a ball, as shown in Figure 2, where one wall is free to slide in and out, but sealed against gas loss. This is like a bicycle pump, in which a sliding piston compresses air, or an internal combustion engine, where an air-fuel explosion creates pressure that moves a piston.

Suppose that the box is filled with gas at pressure \( P_{in} \), and surrounded by gas at pressure \( P_{out} \). (At sea level \( P_{out} = 1 \text{ atmosphere} \approx 1 \text{ bar} = 100 \text{kPa} \approx 14.7 \text{ psi} \).) The net outwards force acting on the moving wall is \( F = PA \), where \( A \) is the area of the wall, and \( P = P_{in} - P_{out} \) is the pressure difference. To push the wall in by a small distance \( d \), we must do work \( W = Fd = PAd \). This...
reduces the volume of the box from $V = LA$ to $V - \Delta V = (L - d)A = LA - Ad$, so the decrease in the box volume is $\Delta V = Ad$. Thus, the work we have done to change the volume (hence the subscript “V”) is

$$W_V = P\Delta V.$$  \hfill (5)

This is the same for any sealed volume containing a gas, regardless of its shape.

Now, let’s look more closely at the wall of the ball. Because the wall is under tension from the pressure of the gas inside, increasing its area requires energy, and decreasing its area releases energy. To see this, suppose that the ball wall is a thin spherical rubber shell of thickness $h$. Consider a bit of that material, as shown in Fig. 3, which is small enough to be treated as a flat slab. In the directions tangent to the surface of the ball, this has an internal stress (force per unit area) $\sigma$. To stretch the slab a small distance $d$ requires work $W = Fd$. Since $hL$ is the cross-sectional area of the slab, the force is $F = \sigma hL$. Thus, the work done to change the area of the wall (hence the subscript “A”) is

$$W_A = \sigma hLd = \sigma h\Delta A,$$  \hfill (6)

where $\Delta A = Ld$ is the increase in the surface area of the ball.

For a round ball of uniform thickness, symmetry implies that the internal stress is the same in all directions tangent to the surface, so the work done is independent of how the area is changed. Since the wall of the ball is under tension, decreasing its area by an amount $\Delta A$ releases energy $\sigma h\Delta A$.

We now show that in the ball’s equilibrium, the internal stress $\sigma$ in the ball wall is proportional to the difference $P$ between the inside and outside pressure. In equilibrium, the forces acting on any bit of the wall must sum to zero. As shown in Fig. 3, the pressure difference creates an upwards force on the top half of the ball, because the horizontal components sum to zero. The vertical components sum to $F_{up} = \pi R^2 P$ upwards, where $R$ is the radius of the ball. This is canceled by an equal magnitude downwards force from the bottom half of the ball, carried by an annular ring of area $2\pi Rh$. Hence, the tangential stress $\sigma$ in the ball wall is related to the pressure difference $P$ by $\sigma = F/A = \pi R^2 P/2\pi Rh = (R/2h)P$. (The ratio $R/2h$ can be very large, i.e., several thousand in an inflated balloon.)

The energy needed to compress the ball is the difference of $W_V$ in Eq. (5) and $W_A$ in Eq. (6). The first is the work done to compress the gas inside. The second is energy released from the reduction in wall area. Thus, the energy stored when the ball is compressed slightly away from its round equilibrium is

$$E_a = W_V - W_A - P\Delta V - \sigma h\Delta A = P(\Delta V - \Delta A/2), \hfill (7)$$

where the final equality uses $\sigma = (R/2h)P$. Here, $P = P_{in} - P_{out} > 0$ is the pressure difference, $\Delta V > 0$ is the decrease in the ball volume, $\Delta A > 0$ is the decrease in the wall area, and $h$ is the thickness of the (thin) wall.

Throughout this analysis, we assume that the decrease in volume $\Delta V$ is small compared to the total volume. This means that the internal pressure does not increase by much, so we can treat the pressure difference $P$ as a constant in Eq. (7). We also assume that the decrease in the area of the wall is small compared to the total area so we can treat the internal stress $\sigma$ as a constant.
The bottom part of the ball (height $\Delta z \geq 0$) is compressed so that part of the wall becomes a section of a plane rather than a section of a sphere, as shown in Fig. 1. The reduction in ball volume (volume of the spherical cap) is [21 Sec. 4.8.4]

$$\Delta V = \pi R \Delta z^2 - (\pi/3) \Delta z^3. \quad (8)$$

The reduction in the area of the ball wall (area of the spherical cap minus the area of the bounding circle) is [21 Sec. 4.8.4]

$$\Delta A = \pi \Delta z^2. \quad (9)$$

For small $\Delta z$, consistent with our assumption that the volume and area changes are small, the stored energy Eq. (7) is

$$E_a = P(\Delta V - R \Delta A/2) \approx \pi R P \Delta z^2/2 \approx (P/2) \Delta V, \quad (10)$$

where $E_a \geq 0$. This shows that energy is needed to produce the distortion shown in Fig. 1 the energy released by the ball wall provides only half of the amount needed to decrease the volume of the ball.

V. MODELING THE ENERGY LOST DURING THE BOUNCE

When a ball bounces, energy is lost in several ways. The work $P \Delta V$ done to compress the gas slightly increases the average squared velocity of the gas molecules inside the ball. This increases their kinetic energy, and hence the gas temperature and pressure. That energy is mostly recovered when the ball re-expands and the gas cools (a “reversible” or “adiabatic” process) but a bit of the thermal energy is lost to the environment. Further energy is lost because the rubber wall is deformed and “squished” during the bounce. This heats up the rubber, transforming some kinetic energy into thermal energy. The collision also creates vibrations, which are damped by internal friction and thus converted to thermal energy. (In tennis balls, most of the energy loss is due to hysteresis [23] and other energy loss mechanisms: each one contributes to the thermal energy. Balls with a large $\mu$ generate a lot of thermal energy for a small change in volume, resulting in a weak bounce. The model of Eq. (11) includes all energy loss mechanisms: each one contributes to the loss coefficient $\mu$.)

VI. PRESSURE DEPENDENCE OF THE COEFFICIENT OF RESTITUTION

We can now determine the pressure dependence of the bounce. Conservation of energy implies that

$$E_b = E_a + E_{\text{lost}} = (P/2) \Delta V + (\mu/2) \Delta V, \quad (12)$$

where we have used Eqs. (10) and (11). If we divide Eq. (10) by Eq. (12), $\Delta V$ cancels, yielding a simple relationship for the coefficient of restitution

$$\epsilon^2 = \frac{E_a}{E_b} = \frac{P_{\text{in}} - P_{\text{out}}}{P_{\text{in}} - P_{\text{out}} + \mu}, \quad (13)$$

where the first equality follows from Eq. (2) and the pressure difference $P = P_{\text{in}} - P_{\text{out}}$.

A nice feature of this model: it predicts that the coefficient of restitution $\epsilon$ is independent of the ball’s velocity. This is typically the case if the velocities are not too large [2 Fig. 8]. Such smaller velocities are also consistent with our assumption that $\Delta V$ (the change in ball volume during the bounce) is small.

VII. EXPERIMENTAL TESTING AND REVISED MODEL

To test our model, we drop a basketball from the top of a door ($h_b = 2.05m$) and video the bounce with a mobile phone. This is repeated three times for 15 different internal pressures. We measure the images with a vernier caliper, scale them to determine the bounce height, then infer $\epsilon$ using Eq. (4) and estimate its uncertainty $\Delta \epsilon$. We then compare to the model.

The basketball pressure is measured using a dial-type pressure gauge reading in pounds per square inch (psi) which displays the difference $P_{\text{in}} - P_{\text{out}}$ between the pressure inside and outside the ball. The results of the bounce tests are shown in Table I for pressure differences up to 13 psi, taking $P_{\text{out}} = 14.7$ psi. Out of fear that it might explode, we did not inflate the basketball to higher pressures [25].

As can be seen from the first row of Table I, even with the ball deflated (inside and outside pressures equal) it still bounces weakly (34cm, 17% of its starting height). In contrast, our model incorrectly predicts that if $P_{\text{in}} = P_{\text{out}}$, then there is no bounce, since Eq. (13) gives $\epsilon = 0$. 

Clearly, our model needs correction, so we revisit its assumptions. Videos show that the deflated ball does not deform much when it bounces, so our assumption that the change in volume is small is satisfied. We conclude that the thin rubber shell of the ball is storing energy, and hence adjust our model, treating this stored energy as if it were due to an additional fictitious gas pressure \( \mu_1 \) inside the ball. If this is a faithful model, then Eq. (13) is still valid, but with an additive offset \( \mu_1 \) to the internal pressure: \( P_{\text{in}} \to P_{\text{in}} + \mu_1 \). Thus, the revised model takes the form

\[
\epsilon = \sqrt{\frac{P_{\text{in}} - P_{\text{out}} + \mu_1}{P_{\text{in}} - P_{\text{out}} + \mu_2}},
\]

(14)

where \( \mu_2 = \mu + \mu_1 \). The constant \( \mu_1 \) characterizes the energy storage in the rubber, whereas \( \mu_2 \) is determined by both the energy storage in the rubber and by its loss coefficient.

To test the model, we fit the data of Table I to Eq. (14). We find the values of the unknown constants \( \mu_1 \) and \( \mu_2 \) with a standard \( \chi^2 \) statistic, which quantifies the deviations between the model and the data. \( \chi^2 \) is the sum of the squared differences between the model and data, weighted by the uncertainty in the measurements,

\[
\chi^2 = \sum_{i=1}^{15} \frac{(\epsilon(P_{\text{in},i}) - \epsilon_i)^2}{\Delta \epsilon_i^2}.
\]

(15)

The sum is over the 15 data points in Table I and the function \( \epsilon(P_{\text{in}}) \) is given by Eq. (14). The best-fit values of \( \mu_1 \) and \( \mu_2 \) are the ones which minimize \( \chi^2 \).

To find \( \mu_1 \) and \( \mu_2 \), we wrote a short Python notebook which computes \( \chi^2 \) on a dense rectangular grid of \( \mu_1 \) and \( \mu_2 \) values. The minimum of \( \chi^2 \) is 13.4 is found at \( \mu_1 = 0.80 \, \text{psi} \) and \( \mu_2 = 4.67 \, \text{psi} \). The data and the best-fit model are plotted in Fig. 6 and are in good agreement.

Since \( \mu_1 \) is smaller than typical values of \( P_{\text{in}} - P_{\text{out}} \), the bounce arising from the fictitious gas pressure is small.

If the model were exact and the experimental errors were distributed as independent Gaussian random variables, then with 90\% confidence a \( \chi^2 \) value (13 degrees of freedom) of less than 19.8 would be obtained. So, the \( \chi^2 \) value of 13.4 indicates a good fit, consistent with the estimated uncertainties.

### VIII. Temperature Dependence of the Coefficient of Restitution

We were curious if our model also describes the temperature-dependence of the coefficient of restitution for a sealed ball. In this case, the ideal gas law implies that the pressure of a fixed number of gas molecules (those inside the ball) is proportional to the absolute gas temperature. Thus, changing the temperature of the gas inside the ball changes \( P_{\text{in}} \).

If we assume that the loss coefficient \( \mu \) does not change with temperature, then we can show how \( \epsilon \) depends upon the temperature \( T \) (in Kelvin) of the gas in the ball. From the ideal gas law, \( P_{\text{in}} = kT \). Here, \( k \) is a constant: the product of Boltzmann’s constant and the number of gas molecules per unit volume inside the ball. Substitute this into Eq. (14), removing the common factor of \( k \) from numerator and denominator. One obtains the dependence of \( \epsilon \) on temperature \( T \):

\[
\epsilon(T) = \sqrt{\frac{T - T_1}{T - T_2}}.
\]

(16)

Here \( T_1 = P_{\text{out}}/k \) and \( T_2 = (P_{\text{out}} - \mu)/k \) are constants with units of Kelvin. These constants depend upon the outside pressure, \( k \), and \( \mu \). Although we don’t know these values, we can measure the coefficient of restitution \( \epsilon \)

| Index i | \( P_{\text{in}} \) (psi) | \( \epsilon \)  | \( \Delta \epsilon \) |
|---------|------------------|----------------|------------------|
| 1       | 14.7             | 0.407          | 0.009            |
| 2       | 15.7             | 0.582          | 0.012            |
| 3       | 16.7             | 0.674          | 0.014            |
| 4       | 17.7             | 0.718          | 0.015            |
| 5       | 18.45            | 0.727          | 0.015            |
| 6       | 18.7             | 0.755          | 0.016            |
| 7       | 19.7             | 0.768          | 0.016            |
| 8       | 20.7             | 0.783          | 0.016            |
| 9       | 21.7             | 0.806          | 0.016            |
| 10      | 22.7             | 0.824          | 0.017            |
| 11      | 23.7             | 0.836          | 0.017            |
| 12      | 24.45            | 0.834          | 0.019            |
| 13      | 25.7             | 0.856          | 0.017            |
| 14      | 26.45            | 0.865          | 0.017            |
| 15      | 27.7             | 0.869          | 0.018            |

TABLE I. The coefficient of restitution \( \epsilon \) and its uncertainty \( \Delta \epsilon \) for a basketball at 15 different internal pressures.
experimentally at several different temperatures and fit Eq. (16) to the data to obtain the values of $T_1$ and $T_2$. To test Eq. (16), we drop a tennis ball at different temperatures from the top of a door ($h_b = 2.05\text{m}$) and video the bounce with a mobile phone. This is repeated three times for each of seven different temperatures spanning a 102K range from $-22^\circ\text{C}$ to $80^\circ\text{C}$. A storage freezer, a refrigerator and an oven are used to cool and heat the ball. The data are shown in Table II.

| Index $i$ | $T_\text{(Kelvin)}$ | $\epsilon$ | $\Delta \epsilon$ |
|-----------|---------------------|------------|-----------------|
| 1         | 251                 | 0.447      | 0.023           |
| 2         | 274                 | 0.621      | 0.014           |
| 3         | 278                 | 0.629      | 0.005           |
| 4         | 293                 | 0.704      | 0.009           |
| 5         | 323                 | 0.766      | 0.008           |
| 6         | 343                 | 0.801      | 0.021           |
| 7         | 353                 | 0.813      | 0.004           |

TABLE II. The coefficient of restitution $\epsilon$ for a tennis ball at different temperatures, and the estimated uncertainty $\Delta \epsilon$.

After those trials were complete, we punctured the tennis ball (drilling a 4mm diameter hole) to release all of the internal pressure. At room temperature the ball still bounced, showing that (as for the basketball) the rubber body stores some energy. We again model this as an offset to the internal pressure. This does not change the form of Eq. (16), but modifies the interpretation of the constants $T_1 = (P_\text{out} - \mu_1)/k$ and $T_2 = (P_\text{out} - \mu_2)/k$.

Some manufacturers produce “pressureless” tennis balls [26], which get their bounce from a spherical rubber core. Downing [27] and Rose and Coe [4] show that their $\epsilon$ behaves differently than that of pressurized balls, so our model is unlikely to apply. One use of pressureless balls is at high altitudes, where normal tennis balls bounce higher than at sea level. This follows from Eq. (14): increasing $P_\text{in} - P_\text{out}$ increases $\epsilon$. Indeed, there are “high altitude” pressurized balls, which correct for this by having smaller $P_\text{in}$ than normal balls.

Again, the model of Eq. (16) is fit to the data of Table II using a $\chi^2$ statistic. The sum in Eq. (15) is now over the seven data points in Table II and the modeled $\epsilon$ is the function of temperature $T_i$, given in Eq. (16). The minimum value of $\chi^2$ is at $T_1 = 237.3\text{K}$ and $T_2 = 177.3\text{K}$, for which $\chi^2 = 3.3$.

The data and the best-fit model are plotted in Fig. 7 and are in good agreement. If the model were exact and the experimental errors were distributed as independent Gaussian random variables, then with 90% confidence a $\chi^2$ value (five degrees of freedom) of less than 9.2 would be obtained.

Our experiment used an old, “dead” tennis ball. A fresh ball, which will have a higher internal pressure, should have a smaller value of $T_1$.

Since the model and data agree well, one might think that the assumptions used to model the temperature dependence of tennis balls are correct. Previous studies of tennis balls indicate that this is not the case. For example, we assume that the rubber properties, such as the loss coefficient $\mu$, are temperature independent. Consequently, our model predicts a decreasing contact time as the temperature (and hence internal pressure) increase. However, more detailed studies of tennis balls [28] show that the properties of the rubber change significantly and the contact time increases [15, 27] with temperature over the range 10-40°C. Indeed, the International Tennis Federation states that “high temperatures can affect the ball rubber, thus increasing bounce height” [29].

Since some of its assumptions are not satisfied, it is remarkable that the model fits the temperature data so accurately. This may be because the model’s functional form applies more broadly. For example, we have assumed that the gas stores the energy during the bounce. However, the same functional form is obtained regardless of where energy is stored, provided that the energy stored and the energy lost are proportional to the change in volume $\Delta V$ and are linear functions of the pressure and/or temperature.

**IX. COMPARISON WITH OTHER DATA/MODELS**

The published literature contains data showing how the coefficient of restitution $\epsilon$ varies with pressure for several types of thin gas-filled balls, and some analytic and numerical models. These include data for play balls [2, 3], soccer balls [6], basketballs [6], and volleyballs [6]. We checked our model against these.

Our model is a good fit to play ball data taken by Bridge [2]. This shows the “bounce efficiency” $\epsilon^2$ for pressure differences $P = P_\text{in} - P_\text{out}$ ranging from 3 to 146 kPa, with the ball dropped from 1 m height. Bridge fits the data [2, Fig. 9] to a two-parameter $(P_\text{in}, n)$ power-law model $\epsilon^2(P) = 1/(1 + (P/P_0)^n)$; the best fit has $n = -0.62$. However, this model is ad-hoc, and not de-
rived from physical principles. Our model fits the data about as well as the power law model, although the lack of error bars in Fig. 9 of Bridge [2] makes it impossible to quantify if the fit of our model is consistent with the observational/experimental uncertainties. The same data is shown in Fig. 6 of a later paper by Bridge [3], where it is compared to a cylindrically-symmetric numerical model. Our model fits the data somewhat better than that numerical model.

We also compared our model to basketball, soccer ball and volleyball data taken by Georgallas and Landry (G&L) for drop heights of 0.75m and 1.5m [6, Fig. 3]. Note that while G&L derive a model for $\epsilon$, and test it against their data, they exclude data points with “gauge pressure” $P = P_{in} - P_{out} = 0$ from their fits. Indeed, the lowest-pressure points deviate significantly from their model predictions. G&L argue that this is expected, since the balls are significantly deformed at low pressure. Their model, like ours, assumes small deformations.

**Basketball:** The G&L fits are good, apart from three data points with $P < 10$ kPa; the bottom two points differ from the model by many standard deviations. Our model is a comparable fit for $P > 10$ kPa and is a better fit at smaller $P$, either including or excluding the $P = 0$ data point from the fit.

**Soccer ball:** The G&L fits are good for the larger $P$ values but are very poor for the three points with $P < 10$ kPa; we suspect that these were ignored or excluded by their fitting procedure. If we also exclude these points, then our model fits about as well as their model.

**Volleyball:** The G&L fits are poor for the two points with $P < 6$ kPa but good for higher pressures. Even when we drop these points (as G&L appear to have done) our model does not fit the data quite as well as theirs. Our model predicts $\epsilon$ values that are systematically about half of a standard deviation too low for $20 < P/kPa < 60$ and that are about the same amount too high for $80 < P/kPa$.

It is interesting to compare the G&L analytic model [6] to ours. Other work analyzes forces, see [30, 51] and references therein. G&L’s is the only model we have found which is also derived using conservation of energy. The key difference is the assumed model for energy loss. G&L model this as due to a “constant dissipative force” $F_D$, whose direction is opposite to the ball velocity. Thus, the energy loss $2F_D\Delta z$ is proportional to the deformation $\Delta z$ at maximum compression (note: G&L denote $\Delta z$ by $x$ and $\epsilon$ by $\epsilon$). This energy loss is proportional to $\sqrt{\Delta V}$, since Eq. (6) shows that the reduction of the ball volume at maximum compression $\Delta z \propto \Delta z^2$ for small $\Delta z$. In contrast, our model assumes that the energy loss is proportional to $\Delta V$. If G&L modeled energy loss as we do, then the denominator of [6 Eq. (6)] becomes $1 - \epsilon^2$ rather than $(1 - \epsilon^2)^2$, and the G&L model reduces to ours.

The G&L model has a troubling feature. All but one of the “constants” that appear in their energy transformation equations ($m$, $g$, $R$, $G$, $D_{LV}$, $F_D$) depend upon the ball shape and material properties: they are indeed constants. The exception is $F_D$, which is not a constant: It has an unspecified velocity dependence. This means that their model is incomplete. Unlike our model, which predicts the same $\epsilon$ for all velocities, their model contains a parameter $F_D$ which must be determined experimentally for each different drop height/incoming velocity [6, Table 2]. The model is not predictive because it does not specify how $F_D$ varies with drop height or incoming velocity. For example, we can not determine if their model predicts a velocity-independent $\epsilon$ for small velocities.

The literature also contains other models for the energy loss. One is based on “momentum flux” in the ball. This gives rise to a dissipative force proportional to the square of the instantaneous velocity during the bounce. The force only acts when the ball is moving upwards $\dot{x}$ giving rise to an energy loss proportional to $\Delta V^{3/2}$. In contrast, the G&L model corresponds to a force of constant magnitude oriented opposite to the instantaneous velocity leading to an energy loss proportional to $\Delta V^{1/2}$. Our model corresponds to a dissipative force, acting in both directions, proportional to the instantaneous velocity, leading to an energy loss proportional to $\Delta V$. The resulting equations of motion are those of a standard damped harmonic oscillator (dissipative force linear in velocity). This model was also adopted in [5], and used to characterize tennis balls, superballs, soccer balls, squash balls and table-tennis balls.

### X. CONCLUSION

We have derived a simple two-parameter model which describes how the coefficient of restitution $\epsilon$ of a gas-filled ball varies with pressure. Our model, which we have not found in the literature, is a good fit to data from many types of gas-filled balls. We expect that it will work well for any type of thin-wall gas-filled ball, provided that the volume deformations are small.

A fundamental assumption of our model is that the energy loss is proportional to the maximum change in the volume of the ball during a bounce. We expect that this would break down when the height is large enough that the ball is significantly deformed. Note that our test velocity (around 6 m/s) is an order of magnitude smaller than the velocity reached during many competitive sports. For example, see Fig. 3.9 of [52] which shows the maximum deformation of a tennis ball expected at 30 m/s. The assumptions of our model could be explored using such finite element simulation models [52, 53].

When we extended our model to describe how $\epsilon$ changes as a function of temperature, we assumed that the loss coefficient $\mu$ is a temperature independent constant. While this is incorrect, the model agrees surprisingly well with the data. Indeed, Feynman remarks that simple models sometimes work better and apply more broadly than expected. In that spirit, it would be interesting to further examine our model. For example, to see if the model (which was developed for gas-filled
balls) correctly describes the behavior of other types of balls such as solid rubber Superballs. We anticipate that it won’t work well, because these balls store most energy in the compression of their rubber bodies rather than in the compression of gas. Then, the behavior of the rubber would govern the balance between energy stored and energy lost.

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CONFLICT OF INTEREST STATEMENT

The authors have no conflicts to disclose.

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[20] If $E_u = 0$ in Eq. [7], then it costs no energy to compress or expand the ball. That would mean that the ball is unstable, since it could spontaneously shrink in size or expand. At first glance, our analysis suggests that a round uncompressed ball is not stable if its radius is slightly reduced from $R$ to $R - \Delta R$, while maintaining the round shape. This change decreases the volume by $\Delta V = 4\pi (R^3 - (R - \Delta R)^3)/3 \approx 4\pi R^2 \Delta R$ and decreases the area (similar calculation) by $\Delta A = 8\pi R \Delta R$. Thus, the two terms $\Delta V - R \Delta A/2$ which appear in Eq. [7] cancel: in this constant-pressure constant-tension model, energy is neither required nor released. What maintains the stability of the round ball are “second-order effects” that model neglects: the decrease in radius slightly increases the internal pressure, and slightly reduces the wall tension. These create a nonzero energy cost and thus provide stability. Fortunately, Eq. [7] is sufficient to model/describe all other possible deformations of the ball (which necessarily break spherical symmetry). This is because a sphere has the smallest area of all surfaces that enclose a given volume.
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In [9] the unnumbered formulas for $k_i$ and $k_f$ contain an $F_D x$ term; linearity in $x$ implies that $F_D$ is a constant opposing force.

Warning: bursting a basketball via overpressure is dangerous and can lead to serious injuries such as permanent loss of sight or hearing.

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See [15, 32], but note that this data is mainly collected for the larger ball velocities and smaller temperature range found in the sport. For example, in our experiment, the ball velocity at the ground is a bit less than 6 m/s; in competitive tennis, it can reach 65 m/s.

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