THE SLOW DIVERGENCE INTEGRAL AND TORUS KNOTS

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Abstract. The goal of this paper is to study global dynamics of $C^\infty$-smooth slow-fast systems on the 2-torus of class $C^\infty$ using geometric singular perturbation theory and the notion of slow divergence integral. Given any $m \in \mathbb{N}$ and two relatively prime integers $k$ and $l$, we show that there exists a slow-fast system $Y_\epsilon$ on the 2-torus that has a $2m$-link of type $(k, l)$, i.e. a (disjoint finite) union of $2m$ slow-fast limit cycles each of $(k, l)$-torus knot type, for all small $\epsilon > 0$. The $(k, l)$-torus knot turns around the 2-torus $k$ times meridionally and $l$ times longitudinally. There are exactly $m$ repelling canard limit cycles and $m$ attracting non-canard limit cycles.

1. Introduction

Singularly perturbed systems on the 2-torus $\mathbb{T}^2$ have been studied in [7] and [9, 10, 11, 12]. In [7] the authors constructed a slow-fast system on $\mathbb{T}^2$ (depending only on a singular parameter $\epsilon$) with the following property: there is a sequence of $\epsilon$-intervals accumulating at 0 such that the system has exactly 2 limit cycles (one is a stable canard and the other one is an unstable canard) for each $\epsilon > 0$ from these intervals. A limit cycle is called a canard\footnote{There exists a more classical definition of canard: a canard limit cycle passes near both attracting and repelling portions of the critical manifold.} limit cycle if it contains a part passing near repelling portions of the critical or slow curve. In [9, 10, 11, 12], I. V. Schurov generalized the results of [7] (the main focus has been directed towards the existence of (attracting) canard cycles on $\mathbb{T}^2$ under more general conditions). The main tool is the Poincaré map from $S^1$ to itself. If the rotation number of the Poincaré map is an integer and the critical manifold is connected, then the number of canard limit cycles is bounded by the number of fold points of the critical manifold (10).

The papers mentioned above mainly deal with “unknotted” limit cycles on the 2-torus (the limit cycles make one pass along the slow direction, i.e. the case of integer rotation number) and the connected critical curve with jump contact points does not turn around the 2-torus along the slow direction and the fast direction. An exception is [9] where a result was proved for non-integer rotation number (more precisely, canard limit cycles make two passes along the slow direction). The main purpose of our paper is to show the existence of slow-fast systems, with one parameter $\epsilon$, with an arbitrary finite number of repelling canard cycles on $\mathbb{T}^2$ that make $k$ passes along the slow direction and $l$ passes along the fast direction ($k$ and $l$ are relatively prime). Such limit cycles are called $(k, l)$-torus knots and occur for each small $\epsilon > 0$ (for more details about the torus knots see the rest of this section and Appendix A). They are generated by an unconnected normally hyperbolic critical curve (each component is a $(k, l)$-torus knot). In order to prove this and to find the global dynamics on $\mathbb{T}^2$, we will use a fixed point theorem for
segments, the notion of slow divergence integral defined along the critical curve and a generalization of Poincaré-Bendixson theorem to $\mathbb{T}^2$ due to Schwartz (see [13]). In our slow-fast setting, it is more convenient to not use the Poincaré map (see Remark 1).

To start fixing ideas, let us consider a slow-fast system

$$X_\epsilon: \begin{cases} \dot{x} &= \sin(y-x) \\ \dot{y} &= \epsilon \end{cases}$$

where $(x, y) \in \mathbb{T}^2 = \mathbb{R}^2 \setminus (2\pi \mathbb{Z}^2)$ and $\epsilon \geq 0$ is a (small) singular parameter. The vector field $X_\epsilon$ is $2\pi$-periodic in both variables and we restrict our attention to the dynamics of $X_\epsilon$, with $\epsilon \geq 0$, on the two-dimensional torus $\mathbb{T}^2$ (we keep $x, y$ in $[0, 2\pi]$ and glue together the opposite segments $x = 0$ and $x = 2\pi$, and $y = 0$ and $y = 2\pi$).

In the limit $\epsilon = 0$, system (1) has horizontal fast orbits and two disjoint simple closed curves of singularities given by $C_-=\{y=x\}$ and $C_+ = \{y = x + \pi\}$. These two closed curves pass through $\mathbb{T}^2$ horizontally and vertically only once (see Figure 1). All the singularities on $C_-$ (resp. $C_+$) are normally attracting (resp. normally repelling). When $\epsilon > 0$, these singularities disappear and the dynamics near $C_\pm$ are given by the reduced flow $x' = 1$ (or equivalently, $y' = 1$).

![Figure 1. Dynamics of $X_0$.](image)

We are interested in the dynamics of the regular system $X_\epsilon$ on $\mathbb{T}^2$ for each small and positive parameter $\epsilon$. Any orbit $O_\epsilon$ of $X_\epsilon$ with initial point located away from $C_\pm$ is attracted to $C_-$ and stays close to $C_-$ forever. Such orbit $O_\epsilon$ cannot therefore be closed. It is clear now that closed orbits of $X_\epsilon$ may appear only in a tubular neighborhood of $C_-$ or $C_+$. Notice that $C_-$ and $C_+$ are limit periodic sets at level $\epsilon = 0$ without fast segments ($C_-$ consists of one attracting slow part while $C_+$ has one repelling slow part).

We show that $C_-$ (resp. $C_+$) generates precisely one limit cycle which is hyperbolic and attracting (resp. repelling and canard), for each small and positive $\epsilon$. This result will be true in an $\epsilon$-uniform neighborhood of $C_-$ (resp. $C_+$). It means that the neighborhood does not shrink to $C_\pm$ as $\epsilon \to 0$. The $\omega$-limit set (resp. the $\alpha$-limit set) of all other orbits (different from the two limit cycles) is the attracting (resp. repelling) limit cycle Hausdorff close to $C_-$ (resp. $C_+$).

This global result will be proved not only for system (1), but more general $C^\infty$-smooth slow-fast systems on $\mathbb{T}^2$ as well. Instead of two critical curves one of which
is normally attracting and the other one is normally repelling, we can have $m$ normally attracting closed critical curves and $m$ normally repelling closed critical curves where $m \in \mathbb{N}$ is arbitrary and fixed. More precisely, let us look at the following generalization of system (1):

\[
Y_\epsilon : \begin{cases}
\dot{x} = \sin m(ly - kx) \\
\dot{y} = \epsilon
\end{cases}
\]

where $m \in \mathbb{N}$, $k \in \mathbb{N}$, $l$ is a non-negative integer and the pair $(k, l)$ is relatively prime. It is not difficult to see that $Y_0$ has $2m$ disjoint simple closed curves of singularities on $\mathbb{T}^2$ and that each closed curve wraps $k$ times vertically around $\mathbb{T}^2$ and $l$ times horizontally. This means that each closed curve crosses the horizontal interval $[0, 2\pi] \times \{0\}$ exactly $k$ times and the vertical interval $\{0\} \times [0, 2\pi]$ $l$ times (see Figure 2). Therefore, we have $m$ normally attracting curves denoted by $C_1^-, \ldots, C_m^-$ and $m$ normally repelling curves denoted by $C_1^+, \ldots, C_m^+$ (note that $k, m > 0$). We shall show that the slow-fast system (2) has exactly $m$ hyperbolically attracting (non-canard) limit cycles and $m$ hyperbolically repelling (canard) limit cycles for each $\epsilon > 0$ and $\epsilon \sim 0$.

**Theorem 1.1.** There exist $\epsilon_0 > 0$ and tubular neighborhoods $\mathcal{U}_\pm^i$ of $C^i_\pm$ inside $\mathbb{T}^2$, with $i = 1, \ldots, m$, such that system $Y_\epsilon$ produces exactly 1 limit cycle in $\mathcal{U}_\pm^i$, denoted by $\mathcal{O}_\pm^i$, for each $\epsilon \in [0, \epsilon_0]$. Each limit cycle $\mathcal{O}_\pm^i$ is hyperbolically attracting and non-canard ($-$) or repelling and canard ($+$), turns around the unknotted torus $\mathbb{T}^2$ $k$ times vertically and $l$ times horizontally, and tends in Hausdorff sense to $C^i_\pm$ in the limit $\epsilon \to 0$. Moreover, fixing $\epsilon \in [0, \epsilon_0]$, for any $\tau \in \mathbb{T}^2$ such that $\tau \notin \mathcal{O}_\pm^i$, we have that $\alpha(\tau)$ is one of the repelling cycles $\mathcal{O}_+^i$ and $\omega(\tau)$ is one of the attracting cycles $\mathcal{O}_-^i$.

Theorem 1.1 will follow from Theorem 2.2 of Section 2.2 stated in a more general framework.

**Remark 1.**
• For \( k, l > 0 \) relatively prime, notice that the slow flow of \( Y_\epsilon \), that is \( \lim_{\epsilon \to 0} \frac{1}{\epsilon} Y_\epsilon |_{C^1} \), is a translation on \( \mathbb{T}^2 \). It then follows from Theorem 1.1 (and e.g. 2) that the limit cycles of \( Y_\epsilon \) have rotation number \( \frac{1}{k} \) and that \( Y_\epsilon \) has no dense orbits.

• Related to the previous point, let \( J \) denote the horizontal interval \( J = [0, 2\pi] \times \{0\} \) and let \( \Pi : J \to J \) be the Poincaré map induced by the flow of \( Y_\epsilon \) for \( \epsilon > 0 \) sufficiently small. It follows that \( \Pi \) is well-defined, particularly since \( \dot{y} = \epsilon \). This observation, in principle, would allow us to instead consider \( \epsilon \)-perturbations of translations over \( \frac{1}{\epsilon} \) on \( \mathbb{T}^1 = \mathbb{S}^1 \). We prefer to not consider this route because in our general result of Theorem 2.2, this Poincaré map is not necessarily well-defined as we do not restrict the sign of the slow flow. For example, for a \((k, l)\)-torus knot, a Poincaré map on the torus would need to be defined as the first return map after \( k \) vertical rotations. So, if the flow on the critical manifolds have distinct directions, there would be points on \( J \) that return to \( J \) before turning vertically along the torus.

A simple closed curve (i.e. an embedding \( \mathbb{S}^1 \to \mathbb{T}^2 \)) that turns around the torus \( k \) times vertically (meridionally) and \( l \) times horizontally (longitudinally) is often called a \((k, l)\)-torus knot. We call a disjoint finite union of such torus knots a torus link (see Appendix A and 11 [6, 8]). Using this terminology and Theorem 1.1 we can say that for each small \( \epsilon > 0 \) system \( Y_\epsilon \) has a link consisting of \( 2m \) limit cycles (each limit cycle is a \((k, l)\)-torus knot). When \( k = 1 \), \( l = 0 \) or \( l = 1 \), we deal with trivial knots or unknotted circles (see, for example, system \( X_\epsilon \)). If \( (k, l) = (3, 2) \) or \( (k, l) = (2, 3) \), then the limit cycles of \( Y_\epsilon \) are the trefoil knots (see Figure 3). It is not possible to untangle the trefoil knot into the unknotted circle through \( \mathbb{R} \)-space without “cutting” or “gluing”. If \( (k, l) = (5, 2) \) or \( (k, l) = (2, 5) \), then \( 2m \) Solomon’s seal knots occur in \( Y_\epsilon \) for each \( \epsilon > 0 \) (see Figure 3). For more torus knots see a list in [8].

In [4] the cyclicity of planar common slow-fast cycles has been studied. The common slow-fast cycles contain either attracting or repelling portions of the critical manifold. The authors were focused on the local study of a single common slow-fast cycle in the plane. In our paper we focus on the global study of slow-fast vector fields defined on \( \mathbb{T}^2 \) in the presence of a disjoint collection of common limit periodic sets of torus-knot type.

In Section 2 we define our slow-fast model on the 2-torus and state the main result (Theorem 2.2). Section 3 is devoted to the proof of Theorem 2.2. In Section 4 we explain how we obtain a similar result to Theorem 2.2 in the presence of regular nilpotent contact points. In Section A we give some basic definitions and results about torus knots.

2. Assumptions and statement of results

2.1. Definition of a slow-fast model on \( \mathbb{T}^2 \) and assumptions. Suppose that \( X_{\epsilon, \rho} : \mathbb{T}^2 \to \mathbb{T} \mathbb{T}^2 \) is a \((C^\infty)\) smooth \((\epsilon, \rho)\)-family of vector fields defined on the torus \( \mathbb{T}^2 \) of class \( C^\infty \) where \( \epsilon \geq 0 \) is the singular parameter and \( \rho \sim \rho_0 \in \mathbb{R}^p \). The tangent bundle of \( \mathbb{T}^2 \) is denoted by \( \mathbb{T} \mathbb{T}^2 \). The parameter \( \rho \) is included for the sake of generality. The first assumption deals with the dynamics of the fast subsystem \( X_{0, \rho} \).

Assumption 1. The system \( X_{0, \rho} \) has a smooth \( \rho \)-family of \( 2m \) disjoint embedded closed curves \( C_{\rho, -}^1, C_{\rho, +}^1, \ldots, C_{\rho, -}^m, C_{\rho, +}^m \) of singularities of \( X_{0, \rho} \), for some \( m \in \mathbb{N} \).
Figure 3. Examples of knotted critical manifolds of $T^2$ (left) and on $[0, 2\pi]^2$ (right) for $m = 1$: first row: $(3, 2)$-knot (trefoil); second row: $(2, 3)$-knot; third row $(5, 2)$-knot (Solomon’s seal); and fourth row: $(2, 5)$-knot. Notice that $(k, l)$-knots are ambient isotopic (see Appendix A) to $(l, k)$-knots.
N. Each curve wraps \( k \) times vertically around the torus and \( l \) times horizontally before it comes back to its initial point, for some fixed relatively prime non-negative integers \( k \) and \( l \) with \( k + l > 0 \). Moreover, \( C^i_{\rho,-} \) (resp. \( C^i_{\rho,+} \)) consists of normally attracting (resp. repelling) singularities for each \( i = 1, \ldots, m \).

We call the critical manifold of \( X_{0,\rho} \) (i.e., the disjoint collection of \( 2m \) \((k, l)\)-torus knots from Assumption 1) the \( 2m \)-link and denote it by \( C_\rho \).

**Remark 2.** Notice that for \( m \geq 2 \), the vector field \( X_{0,\rho} \) is well-defined only if the critical manifolds \( C_{\rho, \pm}^i \), \( i = 1, \ldots, m \), of \( X_{0,\rho} \) alternate each other stability-wise along the fast fibers.

**Remark 3.** It is worth noting that torus knots and slow-fast dynamics interact in a non-trivial way. To be more precise, while the two pairs of torus knots of Figure 4 are equivalent (up to homeomorphism and even ambient isotopy) on \( T^2 \), they lead to nonequivalent slow-fast systems, as in Figure 5. In our main Theorem 2.2 we restrict to normally hyperbolic critical manifolds. This has the advantage of making the proof more concise. However, as we argue in Section 4, it is possible to extend the results of Theorem 2.2 to some cases where the critical manifold is not normally hyperbolic.

![Figure 4](image-url)  

**Figure 4.** An example of a pair of equivalent (up to homeomorphism and even ambient isotopy) knots, that is, \( C_i \sim C'_i \), \( i = 1, 2 \). Compare with Figure 5

A simple topological argument on \( T^2 \) implies that disjoint torus knots have to be of the same type (see [8]). More precisely, we have

**Lemma 2.1.** If \( C \) and \( D \) are two disjoint knots on \( T^2 \) of types \((k_1, l_1)\) and \((k_2, l_2)\) where \((k_1, l_1)\) and \((k_2, l_2)\) have the property given in Assumption 1, then \( k_1 = k_2 \) and \( l_1 = l_2 \).

In order to ensure that the slow divergence integral of \( X_{\epsilon, \rho} \) is finite (i.e. well-defined), we have to assume that the slow dynamics of \( X_{\epsilon, \rho} \) along the \( 2m \)-link \( C_\rho \) is regular. Let us recall that the slow dynamics or the slow flow, denoted by \( \tilde{X}_\rho \), is a \( C^\infty \)-smooth \( \rho \)-family of one-dimensional vector fields that describes the passage near \( C^i_{\rho,-} \) or \( C^i_{\rho,+} \), with \( i = 1, \ldots, m \), when \( \epsilon \) is positive and small. We have

\[
\tilde{X}_\rho(\tau) = \lim_{\epsilon \to 0} \frac{(X_{\epsilon, \rho} + 0 \frac{d}{d\epsilon})(\tau)|_{M_{\epsilon}}}{\epsilon}.
\]
Figure 5. Even if the critical manifolds $C_i$ on the left and $C'_i$ on the right are equivalent as torus knots (compare with Figure 4), they lead to completely different, and in fact nonequivalent, slow-fast dynamics. In the main part of this paper we assume that every knot of the $2m$-link critical manifold is normally hyperbolic, but refer also to Section 4.

where $M_\tau$ is any $C^n$ center manifold of $X_{\epsilon,\rho} + 0 \frac{\partial}{\partial \epsilon}$ at the point $\tau \in C_\rho$ ($n \in \mathbb{N}$ can be as large as we want). For more details about the definition of the slow dynamics see e.g. [4].

Assumption 2. The slow dynamics $\tilde{X}_{\rho_0}$ is nonzero at each point $\tau \in C_{\rho_0}$.

Remark 4. From Assumption 2 follows that along each of the $2m$ $(k,l)$-torus knots defined in Assumption 1 the slow dynamics $\tilde{X}_{\rho_0}$ is regular (thus, without singularities). The slow dynamics may have different directions along the torus knots. For example, if we replace $\dot{y} = \epsilon$ in (1) with $\dot{y} = \epsilon \cos(y - x)$, then the slow flow along $C_-$ points upwards and along $C_+$ downwards.

Now we can define the slow divergence integral of $X_{\epsilon,\rho}$ along $C_{\rho,-}^i$ (resp. $C_{\rho,+}^i$) as:

$$I_{\pm}^i (\rho) = \int_{C_{\rho,-}^i} \text{div} X_{0,\rho} ds$$

where $i = 1, \ldots, m$ and $s$ is the slow time of $\tilde{X}_\rho$. The slow divergence integral $I_{\pm}^i$ is independent of the local chart and the chosen volume form on $T^2$, and represents the leading order term of the integral of divergence along orbits of $X_{\epsilon,\rho}$, multiplied by $\epsilon$ (see Section 3). We typically compute $I_{\pm}^i$ by dividing the compact curve $C_{\rho,-}^i$ into a finite number of segments $[\tau_1, \tau_2], [\tau_2, \tau_3], \ldots, [\tau_{r-1}, \tau_r]$, where $r \in \mathbb{N}$, $\tau_1, \ldots, \tau_r \in C_{\rho,-}^i$ and $\tau_1 = \tau_r$ ($C_{\rho,-}^i$ is closed for all $\rho \sim \rho_0$), and by calculating the integral on each segment in suitable normal form coordinates:

$$I_{\pm}^i (\rho) = \int_{\tau_1}^{\tau_2} \text{div} X_{0,\rho} ds + \int_{\tau_2}^{\tau_3} \text{div} X_{0,\rho} ds + \cdots + \int_{\tau_{r-1}}^{\tau_r} \text{div} X_{0,\rho} ds.$$

We assume that the points $\tau_1, \tau_2, \ldots$ follow the direction of the slow flow along $C_{\rho,-}^i$ and that the above normally attracting segments are small enough such that
on each segment we can use a Takens normal form for $C^n$-equivalence

$$\begin{aligned}
\dot{x} &= -x \\
\dot{y} &= \epsilon
\end{aligned}$$

(see [5]). The slow segment is lying inside $\{x = 0\}$. Since the divergence of the vector field in (4) for $\epsilon = 0$ is $-1$, it is clear that each integral in (3) is negative. Thus, the slow divergence integral $I_i^-$ is negative for all $\rho \sim \rho_0$ and $i = 1, \ldots, m$. Similarly, we see that $I_i^+$ is positive, for all $\rho \sim \rho_0$ and $i = 1, \ldots, m$, because each $C_{\rho,\pm}^i$ is repelling.

2.2. Statement of results. In this section we state our main result. Let $X_{\epsilon,\rho}$ satisfy Assumptions 1–2. Then for all $\epsilon \sim 0$ and $\epsilon > 0$ a 2m-link of type $(k,l)$ occurs inside $X_{\epsilon,\rho}$. More precisely,

Theorem 2.2. Suppose that system $X_{\epsilon,\rho}$ satisfies Assumptions 1–2. Then there exist $\epsilon_0 > 0$, a neighborhood $\mathcal{V}$ of $\rho_0$ and tubular neighborhoods $\mathcal{U}_{\epsilon,\rho}^i$ of $C_{\rho_0,\pm}^i$, for each $i = 1, \ldots, m$, such that $X_{\epsilon,\rho}$ has exactly one limit cycle in $\mathcal{U}_{\epsilon,\rho}^i$, denoted by $O_{\epsilon,\rho,\pm}^i$, for all $(\epsilon, \rho) \in [0, \epsilon_0] \times \mathcal{V}$ and $i = 1, \ldots, m$. The limit cycle $O_{\epsilon,\rho,-}^i$ (resp. $O_{\epsilon,\rho,+}^i$) is a hyperbolic and attracting non-canard (resp. repelling canard) limit cycle. Moreover, each $O_{\epsilon,\rho,\pm}^i$ tends in Hausdorff sense to the $(k,l)$-torus knot $C_{\rho_0,\pm}^i$ as $(\epsilon, \rho) \to (0, \rho_0)$. If $\tau$ is any point on $T^2$ not lying in $O_{\epsilon,\rho,\pm}^i$, then the $\omega$-limit (resp. the $\alpha$-limit) of $\tau$ is one of the attracting (resp. repelling) limit cycles $O_{\epsilon,\rho,-}^1, \ldots, O_{\epsilon,\rho,-}^m$ (resp. $O_{\epsilon,\rho,+}^1, \ldots, O_{\epsilon,\rho,+}^m$).

We prove Theorem 2.2 in Section 3.

3. Proof of Theorem 2.2

This section is devoted to the proof of Theorem 2.2. Let $X_{\epsilon,\rho}$ satisfy Assumptions 1–2. We divide the proof of Theorem 2.2 into three parts:

![Figure 6. A flow box neighborhood with the inset (red) and the outset (blue). (a) $\epsilon = 0$ (b) $\epsilon > 0$.](image-url)
(1) **Flow-box neighborhoods near** $C^{\iota}_{0,\pm}$ Let $i \in \{1, \ldots, m\}$. We construct a succession of flow box neighborhoods along the attracting closed curve $C^{\iota}_{0,\pm}$ (we can do the same with the repelling slow curve $C^{\iota}_{0,\pm}$ by reversing time). Since $X_{\epsilon,\rho}$ has no singularities on $\mathbb{T}^2$ for all small $\epsilon > 0$ and $\rho \sim \rho_0$ (Assumptions 1, 2 are satisfied), we can cover $C^{\iota}_{0,\pm}$ with a finite number of flow boxes that are uniform in $\epsilon > 0$ and $\rho \sim \rho_0$, i.e. their size is fixed as $(\epsilon, \rho) \to (0, \rho_0)$. (Such flow box neighborhoods don’t exist in the limit $\epsilon = 0$.) This together with a fixed point theorem for segments will enable us to show the existence of a closed orbit in an $(\epsilon, \rho)$-uniform tubular neighborhood $U^\iota_\epsilon$ of $C^{\iota}_{0,\pm}$, for each small $\epsilon > 0$ and $\rho \sim \rho_0$. Our approach is based on the techniques from [4].

(2) **The slow divergence integral along** $C^{\iota}_{0,\pm}$ We relate the slow divergence integral defined along $C^{\iota}_{0,\pm}$ (see Section 3) to the integral of divergence along orbits of $X_{\epsilon,\rho}$ inside $U^\iota_\epsilon$. Using a well-known connection between the derivative of the Poincaré map and the divergence integral, and the fact that the slow divergence integral is negative, we show that $X_{\epsilon,\rho}$ has at most one limit cycle in $U^\iota_\epsilon$ up to shrinking $U^\iota_\epsilon$ if needed. The limit cycle is a hyperbolically attracting $(k, l)$-torus knot. We also use a generalization of Poincaré-Bendixon Theorem to $\mathbb{T}^2$ due to Schwartz [13].

(3) **Global dynamics of** $X_{\epsilon,\rho}$ on $\mathbb{T}^2$ To study the global dynamics on $\mathbb{T}^2$, we use the same result due to Schwartz [13].

1. Flow-box neighborhoods near $C^{\iota}_{0,\pm}$ Let $\tau \subset \mathbb{T}^2$ and let $X$ be a $C^\infty$-smooth vector field on $\mathbb{T}^2$. We say that a neighborhood $U$ of $\tau$ is a flow box neighborhood of $\tau$ for $X$ if $U = F([0, 1] \times [0, 1])$ where $F : [0, 1] \times [0, 1] \to \mathbb{T}^2$ is a $C^\infty$-smooth diffeomorphism with the following property: the vector field $X$ is directed from outside to inside $U$ along $F([0, 1] \times \{0\})$ (the inset), from inside to outside $U$ along $F([0, 1] \times \{1\})$ (the outset), $F(\{0, 1\} \times [0, 1])$ are parts of orbits of $X$ and $X$ has no singularities in $U$. Thus, all orbits of $X$ starting at the inset reach the outset. The set $\tau$ can be a point, a segment, etc. We use this definition to introduce a slow-fast family of flow box neighborhoods of $\tau$ for $X_{\epsilon,\rho}$, i.e. a family $\{U_{\epsilon,\rho} : (\epsilon, \rho) \in [0, \epsilon_0] \times V\}$ of neighborhoods of $\tau$ where $\epsilon_0 > 0$, $V$ is a neighborhood of $\rho_0$, $U_{\epsilon,\rho} = F_{\epsilon,\rho}(\{0, 1\} \times [0, 1])$, $F_{\epsilon,\rho} : [0, 1] \times [0, 1] \to \mathbb{T}^2$ is an $(\epsilon, \rho)$-family of smooth diffeomorphisms such that $U_{\epsilon,\rho}$ is a flow box neighborhood of $\tau$ for $X_{\epsilon,\rho}$ for each $(\epsilon, \rho) \in [0, \epsilon_0] \times V$ and such that the intersection of the sets $U_{\epsilon,\rho}$ with $(\epsilon, \rho) \in [0, \epsilon_0] \times V$, is a neighborhood of $\tau$. For more details see Definition 9 in [4].

Fix $i \in \{1, \ldots, m\}$. We divide the compact curve $C^{\iota}_{0,\pm}$ into the segments $[\tau_1, \tau_2], [\tau_2, \tau_3], \ldots, [\tau_{r-1}, \tau_r]$ such that $\tau_j \in C^{\iota}_{0,\pm}$ for all $j = 1, \ldots, r$, $\tau_1 = \tau_r$ and such that the points $\tau_1, \tau_2, \ldots, \tau_r$ follow the direction of the slow flow along $C^{\iota}_{0,\pm}$ (i.e. the slow flow of $X_{0,\rho_0}$ goes from $\tau_1$ to $\tau_2$, from $\tau_2$ to $\tau_3$, etc.). For each segment $[\tau_j, \tau_{j+1}] \subset C^{\iota}_{0,\pm}$, $j = 1, \ldots, r - 1$, we define a slow-fast family of flow box neighborhoods of $[\tau_j, \tau_{j+1}]$ inside $X_{\epsilon,\rho}$ (see Figure 6). We describe it using equivalence normal form coordinates. There exists a local chart on $\mathbb{T}^2$ around $[\tau_j, \tau_{j+1}]$ in which $X_{\epsilon,\rho}$ is locally given by the Takens normal form (4) (the segment $[\tau_j, \tau_{j+1}]$ can be as small as we need). In the normal form coordinates, the segment is given by $[-1, 1]$ on the $y$-axis. The outset is a parabola like segment above the line $\{y = 1\}$. From the end points of the outset we have two parts of orbits of $X_{\epsilon,\rho}$ (they are horizontal in the limit $\epsilon = 0$). The inset consists of two lines and a convex segment between them as indicated in Figure 6. Note that the vector
field (4) is directed from outside to inside the neighborhood along the inset for each small \( \epsilon > 0 \) because the slow dynamics \( y' = 1 \) points upwards near the \( y \)-axis. A detailed description of the slow-fast family of flow box neighborhoods can be found in Section 5 of [4].

Thus, we have constructed a finite number of flow box neighborhoods \( U_{\epsilon, \rho}^j \), \( j = 1, \ldots, r - 1 \), along \( C_{i, \rho_0}^j \) (see Figure 7). Since each flow box neighborhood is contracted along its segment, we may assume that the outset of \( U_{\epsilon, \rho}^j \) is completely inside \( U_{\epsilon, \rho}^{j+1} \), for each \( j = 1, \ldots, r - 2 \), and that the outset of \( U_{\epsilon, \rho}^{r-1} \) lies inside \( U_{\epsilon, \rho}^1 \) (see Proposition 4 of [4]). This implies that the Poincaré map from the inset of \( U_{\epsilon, \rho}^1 \) to itself is well defined and smooth for \( \epsilon > 0 \) small enough. Since the inset is a segment, there is a fixed point by Brouwer’s Fixed Point Theorem. Thus, there exists a closed orbit near \( C_{i, \rho_0}^j \) for each small \( \epsilon > 0 \) and \( \rho \sim \rho_0 \).

2. The slow divergence integral along \( C_{i, \rho_0, \pm}^j \). Since \( C_{i, \rho_0, -}^j \) is normally attracting and compact, it is well known (see e.g. [5]) that for any small \( \kappa > 0 \) we can find \( \epsilon_0 > 0 \), a neighborhood \( \mathcal{V} \) of \( \rho_0 \) and a tubular neighborhood \( U_{\epsilon, \rho}^\ell \) of \( C_{i, \rho_0, -}^j \) such that

\[
\frac{I_{\ell}^-(\rho_0) - \kappa}{\epsilon} \leq \int_{O_{\epsilon, \rho}} \text{div} X_{\epsilon, \rho} dt \leq \frac{I_{\ell}^+(\rho_0) + \kappa}{\epsilon}
\]

where \( I_{\ell}^\pm(\rho_0) < 0 \) is the slow divergence integral defined in Section 2 and \( O_{\epsilon, \rho} \) is an arbitrary closed orbit of \( X_{\epsilon, \rho} \) in \( U_{\epsilon, \rho}^\ell \) with \( \epsilon \in [0, \epsilon_0] \) and \( \rho \in \mathcal{V} \). We use now (5) to show that any closed orbit \( O_{\epsilon, \rho} \) of \( X_{\epsilon, \rho} \) in \( U_{\epsilon, \rho}^\ell \) is hyperbolically attracting. It suffices to take a \( \kappa > 0 \) such that \( \kappa < -I_{\ell}^-(\rho_0) \) and to use the Poincaré formula

\[
P'(s_0) = e^{\int_{O_{\epsilon, \rho}} \text{div} X_{\epsilon, \rho} dt}
\]

where \( P \) is the Poincaré map defined on a transverse section near \( O_{\epsilon, \rho} \) parametrized by a regular parameter \( s \) (\( O_{\epsilon, \rho} \) corresponds to \( s_0 \)). Note that the torus is orientable. Since the integral of divergence in (5) is negative (and of order \( O(1/\epsilon) \)), we have \(|P'(s_0)| < 1\) and the closed orbit \( O_{\epsilon, \rho} \) is a hyperbolically attracting limit cycle.

To prove that \( C_{i, \rho_0, -}^j \) can produce at most one limit cycle inside \( U_{\epsilon, \rho}^\ell \), we use the following result due to Schwartz [13] (see also Theorem 6.6 in [3]):

**Figure 7.** A trefoil knot covered with flow box neighborhoods.
Theorem 3.1. Assume that \( x' = X(x) \) is an autonomous system of class \( C^2 \) on a compact, connected two-dimensional orientable manifold \( M \) of class \( C^2 \). Assume that this system defines a flow, and that the manifold \( M \) is not a minimal set. Then, if the \( \omega \)-limit set \( \omega(\tau) \) of a point \( \tau \in M \) does not contain any critical point, \( \omega(\tau) \) must be homeomorphic to the circle \( \mathbb{S}^1 \) (i.e. is a periodic trajectory).

We apply Theorem 3.1 to \( M = \mathbb{T}^2 \) and \( X = -X_{\epsilon,\rho} \) for each fixed \( \epsilon > 0 \) and \( \rho \sim \rho_0 \) (in this paper, both \( \mathbb{T}^2 \) and \( X_{\epsilon,\rho} \) are of class \( C^\infty \)). Let us recall that a set \( S \subset M \) is minimal if it is nonempty, invariant, closed and there is no proper subset of \( S \) that has all these properties. Since \( X_{\epsilon,\rho} \) has at least one closed (or periodic) orbit for each \( \epsilon > 0 \) small enough (see Step 1), \( \mathbb{T}^2 \) is not minimal. As already mentioned, \( X_{\epsilon,\rho} \) has no critical points on \( \mathbb{T}^2 \) for \( \rho \sim \rho_0 \) and \( \epsilon > 0 \) small enough.

Suppose now that \( X_{\epsilon,\rho} \) has at least two limit cycles in \( U_\epsilon^\omega(\mathcal{O}_1 \cup \mathcal{O}_2) \). Then \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \) are \( (k,l) \)-torus knots and bound an invariant set \( S \). Take any \( \tau \in S \) not lying on a closed orbit of \( X_{\epsilon,\rho} \) (such \( \tau \) exists if we are close enough to the isolated closed orbits \( \mathcal{O}_1 \) or \( \mathcal{O}_2 \)). Following Theorem 3.1, \( \omega(\tau) \subset S \) w.r.t. \( -X_{\epsilon,\rho} \) is a periodic trajectory different from \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \) (\( \mathcal{O}_1 \) and \( \mathcal{O}_2 \) are repelling for \( -X_{\epsilon,\rho} \)).

This leads to a contradiction because the periodic trajectory \( \omega(\tau) \) is not repelling (all closed orbits generated by \( C_{\rho_0,-}^\infty \) are hyperbolic and repelling limit cycles w.r.t. \( -X_{\epsilon,\rho} \)).

From Step 1 and Step 2 follows that there exists \( \epsilon_0 > 0 \), a small neighborhood \( V \) of \( \rho_0 \) and a tubular neighborhood \( U_\epsilon^\omega \) of \( C_{\rho_0,-}^\infty \) such that for each \( (\epsilon,\rho) \in [0, \epsilon_0] \times V \) system \( X_{\epsilon,\rho} \) has precisely one limit cycle in \( U_\epsilon^\omega \). It is hyperbolic and attracting and clearly of \( (k,l) \)-torus knot type. By reversing time, we can prove a similar result near \( C_{\rho_0,+}^i \) for each \( i = 1, \ldots, m \) (\( C_{\rho_0,+}^i \) generates one hyperbolic and repelling limit cycle).

3. Global dynamics of \( X_{\epsilon,\rho} \) on \( \mathbb{T}^2 \). First note that the \( 2m \) limit cycles obtained in Step 1 and Step 2 are the only possible periodic trajectories of \( X_{\epsilon,\rho} \). Indeed, any orbit with the initial point \( \tau \in \mathbb{T}^2 \setminus \cup_{i=1}^m U_\epsilon^\pm \) (\( \tau \) is thus uniformly away from the critical manifold of \( X_{0,\rho_0} \)) cannot be periodic (it is attracted to an attracting closed part \( C_{\rho_0,-}^\infty \) and stays there forever). Theorem 3.1 implies now the rest of Theorem 2.2.

4. Regular nilpotent contact points of finite order

In this section we briefly argue that a similar result to Theorem 2.2 can be obtained, without much effort, in some cases where the critical manifold \( C_{\rho} \) is not normally hyperbolic. Instead of providing all the technicalities, we refer to [4], and only indicate the main ideas.

Definition 1. A point \( \tau \in C_{\rho_0} \) is called a nilpotent contact point if the linear part of \( X_{0,\rho_0} \) at \( \tau \) (computed in local coordinates) is nilpotent. Near such a point, with \( (\epsilon,\rho) \sim (0,\rho_0) \), system \( X_{\epsilon,\rho} \) is \( C^\infty \)-equivalent to \( \{ \dot{x} = y - f(x,\rho), \dot{y} = \epsilon(g(x,\epsilon,\rho) + O(y - f(x,\rho))) \} \) where \( (x,y) \) are local normal form coordinates, \( \tau \) is given by \( (x,y) = (0,0) \), \( f,g,O \) are \( C^\infty \)-smooth and the 1-jet \( j_1 f(x,\rho_0) \) at \( x = 0 \) is zero (see [3]). A nilpotent contact point \( \tau \) is further called regular if \( g(0,0,\rho_0) \neq 0 \). Finally, a nilpotent contact point \( \tau \) is called of finite order, if there exists an integer \( n \geq 2 \) such that the vanishing order of \( f(x,\rho_0) \) at \( x = 0 \) is equal to \( n \).
It is not difficult to see that the above definition is independent of the chosen normal form.

It turns out that the main ideas of Theorem 2.2 also hold to the case where the critical manifold may contain isolated regular nilpotent contact points of finite order. To be more precise, Assumption 1 can be relaxed in the following sense:

**Assumption 1’.** The system $X_{\theta, \rho}$ has a smooth $\rho$-family of $2m$ disjoint embedded curves $C_{\rho, -}^1, C_{\rho, +}^1, \ldots, C_{\rho, -}^m, C_{\rho, +}^m$ of singularities of $X_{\theta, \rho}$, for some $m \in \mathbb{N}$. Each curve wraps $k$ times vertically around the torus and $l$ times horizontally before it comes back to its initial point, for some fixed relatively prime non-negative integers $k$ and $l$ with $k + l > 0$. Moreover, $C_{\rho, \pm}^i$ consists of normally hyperbolic singularities for each $i = 1, \ldots, m$ except at a finite number of points where $C_{\rho, -}^i$ (resp. $C_{\rho, +}^i$) may have isolated regular nilpotent contact points of finite order.

Then, one can show that under Assumptions 1’ and 2 (Assumption 2 is valid at normally hyperbolic singularities and the slow flow is unbounded at the contact points), $X_{\theta, \rho}$ has exactly $2m$ limit cycles from which $m$ are attracting and $m$ are repelling. The proof would follow the same arguments as in Section 3 and [4]. Instead of describing these technicalities, we simply recall that the passage near regular nilpotent contact points of finite order is a local contraction, see the details in [4] where also Figure 5 is insightful. Examples of knotted critical manifolds with regular nilpotent contact points of finite order are provided in Figures 8 and 9.

**Figure 8.** Example of a 2-link critical manifold consisting of two $(1, 1)$-knots with an odd regular contact point. An odd contact point is locally given by the singularity $y = x^{2n+1}$, $n \in \mathbb{N}$. On the left we sketch the singular limit, observe that the critical manifold has an odd contact with the fast foliation at $x = \pi$. The red knot corresponds to the attracting critical manifold, while the blue knot is repelling one. On the right we sketch possible orbits for $\epsilon > 0$ sufficiently small, where the red knot indicates the attracting non-canard limit cycle and the blue knot the repelling canard limit cycle. Orbits away from the canard limit cycle approach the non-canard limit cycle as $t \to \infty$.

Although we do not provide a specific model of a smooth slow-fast system on $T^2$ with knotted critical manifold with contact points, we point out that it is possible to construct such systems using, for example, Hermite interpolation [14].
Figure 9. Example of a 2-link critical manifold consisting of two Trefoils (\((3, 2)\)-knots) with regular contact points. On the left we sketch the singular limit, observe that the critical manifolds (thick red and blue knots) contain, for example, jump points. All flows are shown in forward time. The shaded red path indicates a singular slow-fast cycle, that is a knot consisting of slow and fast orbits of the singular limit. As sketched on the right, the singular slow-fast knot perturbs to a non-canard attracting limit cycle, which in fact corresponds to a relaxation oscillation. On the other hand, the blue knot perturbs to a repelling canard limit cycle, also of slow-fast type (singular limit not shown). The main argument to show that a unique limit cycle bifurcates from each singular slow-fast knot for \(\epsilon > 0\) sufficiently small, is that the local passage through the regular contact points is a contraction.

Appendix A. Basic notions of Knot Theory

In this section we present some basic notions, definitions, and results about knot theory and, in particular, torus knots. For more details see, for example, [1, 8]. We start with the definition of a knot:

**Definition 2.** A knot is an embedding \(K : S^1 \to \mathbb{R}^3\) of a 1-sphere into \(\mathbb{R}^3\). A link \(L : \bigcup S^1 \to \mathbb{R}^3\) is a disjoint, finite union of knot\(^2\).

In order to introduce a notion of equivalence between knots, we have the following:

**Definition 3.** A homotopy \(h_t : \mathbb{R}^3 \to \mathbb{R}^3\) is called an ambient isotopy if \(h_0\) is the identity and every \(h_t\) is a homeomorphism.

There are several notions of equivalence for knots. For our purposes it will suffice to keep in mind the following two:

**Definition 4.**

1. Two knots (or links) \(K\) and \(K'\) are equivalent, if there is a homeomorphism \(h : \mathbb{R}^3 \to \mathbb{R}^3\) such that \(h(K) = K'\).

\(^2\)More generally, one can define a knot as being an embedding \(K : S^r \to X\), where \(X\) is a topological space (usually \(S^n\) or \(\mathbb{R}^n\)). Alternatively, one can consider a knot as being a subset homeomorphic to \(S^r\), \(r \geq 1\).
(2) Two knots (or links) $K$ and $K'$ are ambient isotopic, if there is an ambient isotopy such that $h_1(K) = K'$.

In this paper we are concerned with torus knots, that is, knots that lie in the 2-torus $T^2 = S^1 \times S^1 \subset \mathbb{R}^3$. In this regard, we say that a torus knot $K$ is of class $(k, l)$ if the knot winds the torus $k$-times vertically and $l$-times horizontally, and where $k$ and $l$ are coprime (see e.g. Figure 3). It turns out that the only nonequivalent knots (up to homeomorphism) are those presented in Figure 10.

![Figure 10. Representatives of the two classes of nonequivalent torus knots (up to homeomorphism).](image)

We see that equivalence up to homeomorphism is too coarse. In contrast, ambient isotopies preserve orientation, which leads to a finer classification.

**Theorem A.1** ([6, 8]). Let $K_i$, $i = 1, 2$, be torus knots of class $(k_i, l_i)$ respectively. Then $K_1$ and $K_2$ are ambient isotopic if and only if $(k_1, l_1) = \pm (k_2, l_2)$. If one considers only the case where $k_i$ and $l_i$ are positive, then $K_1$ and $K_2$ are ambient isotopic if and only if $k_1 = k_2$ and $l_1 = l_2$ or $k_1 = l_2$ and $l_1 = k_2$.

Examples of equivalent knots modulo ambient isotopy can be seen in Figure 3.

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