Special-relativistic model flows of viscous fluid

A. D. Rogava

Department of Theoretical Astrophysics, Abastumani Astrophysical Observatory, Kazbegi str. N. 2*, Tbilisi 380060, Republic of Georgia and Department of Physics, Tbilisi State University, Chavchavadze ave. 2, 380028 Tbilisi, Republic of Georgia

(January 6, 2022)

Two, the most simple cases of special-relativistic flows of a viscous, incompressible fluid are considered: plane Couette flow and plane Poiseuille flow. Considering only the regular motion of the fluid we found the distribution of velocity in the fluid (velocity profiles) and the friction force, acting on immovable wall. The results are expressed through simple analytical functions for the Couette flow, while for the Poiseuille flow they are expressed by higher transcendental functions (Jacobi’s elliptic functions).

I. INTRODUCTION

A recent development of theoretical astrophysics revealed a wide class of hydrodynamic and hydromagnetic flows having the close relevance to a number of astrophysical objects such as stellar winds [1], accretion flows [2], outflows of matter in Active Galactic Nuclei (jets) [3] etc. The spatial symmetry of these flows is different, thermodynamic state is sometimes unusual and velocities are often relativistic. Relativity definitely influences the physical processes that take place in such flows and leads to various outstanding appearances of these objects.

The subject of the present paper is to study how the relativistic velocities affect physical conditions and dynamics of the most simple model hydrodynamic flows. For the present consideration we chose such examples of flows, which being simple enough are, at the same time, relevant to the most astrophysical cases due to their simple geometry.

In particular, we are going to study the special-relativistic generalization of a couple of examples of the well-known classic hydrodynamic flows [4]: plane Couette flow and plane Poiseuille flow. We consider only regular motion in these flows. In both cases we find exact solutions of the problem.

In particular, for the plane Couette flow we determined the velocity distribution throughout the flow. The velocity profile is expressed analytically and appears to be nonlinear. The nonlinearity noticeably increases with the increase of the velocity of the moving plane and becomes almost "steplike" for $v_0 \approx 1$. We have found the average velocity of the flow that in nonrelativistic limit tends to $v_0/2$ (as it should be), while for $v_0 \approx 1$ it tends to unity. We have also determined the friction force acting on the immovable wall, which is exactly equal to the corresponding classic (nonrelativistic) value.

For the plane Poiseuille flow the velocity distribution was derived by the expressing of the solution of the corresponding differential equation through the higher transcendental functions. In particular, the velocity profile is expressed by Jacobi’s elliptical functions. The general property of this solution is that with the increase of the pressure gradient value (the very gradient that induces the whole flow) the profile becomes more and more "flattened."

In the concluding section we discuss the obtained results in the context of their possible relevance to the future modeling of real relativistic astrophysical flows.

II. MAIN CONSIDERATION

Henceforth, we shall use the following notations: greek indices will represent spacetime components, while latin ones will be used for spatial components. We shall use geometrical units, so that $G = c = 1$. The signature of the spacetime (Minkowskian) metric:

$$ds^2 = -d\tau^2 = -dt^2 + dx^2 + dy^2 + dz^2,$$  

is chosen as $(- + + +)$.

We shall consider motion of the incompressible, viscous fluid with the following stress-energy tensor $\mathbf{2}$:

$$T^{\alpha\beta} = \epsilon U^\alpha U^\beta + P h^{\alpha\beta} - 2\eta \sigma^{\alpha\beta} - \xi \theta h^{\alpha\beta},$$  

(2.2)
where all notations are standard [1]. In particular, \( e \) is the density of the mass-energy, \( P \)—the pressure, \( U^\alpha \) is the 4-velocity field of the flow, \( h^{\alpha \beta} \) is the projection tensor, defined as:

\[
h^{\alpha \beta} \equiv \eta^{\alpha \beta} + U^\alpha U^\beta, \tag{2.3a}
\]

\( \eta \) and \( \xi \) are the coefficients of the shear and bulk viscosity respectively. \( \theta \) is the "expansion" of the fluid world lines:

\[
\theta \equiv U_{;\alpha}^\alpha, \tag{2.3b}
\]

and \( \sigma^{\alpha \beta} \) is the shear tensor:

\[
\sigma_{\alpha \beta} \equiv \frac{1}{2} \left[ U_{\alpha ; \mu} h_{\beta}^{\mu} + U_{\beta ; \mu} h_{\alpha}^{\mu} \right] - \frac{1}{3} \theta h_{\alpha \beta}. \tag{2.3c}
\]

For the fluid with nonrelativistic temperature, density of the mass-energy \( e \) is equal to the "usual" density of matter \( \rho \) in the proper frame of reference. Since a fluid is thought to be incompressible: \( \rho = \text{const} \) and the continuity equation: \( (\rho U^\alpha)_{;\alpha} = 0 \), clearly leads to \( \theta = 0 \) ("expansion" of the incompressible fluid is equal to zero).

Under mentioned circumstances the stress-energy conservation law may be generally written as:

\[
T_{\alpha \beta} = 0. \tag{2.4}
\]

A. Plane Couette flow

Let us consider the plane flow of the incompressible viscous fluid, which is situated between two infinite parallel planes one of which is moving with arbitrary velocity \( v_0 \). Let an \( X \) axis be parallel to the planes and an \( Y \) axis be normal to them so that \( y = 0 \) in the plane at rest, while \( y = L \) for the plane in motion. The symmetry of the problem implies that the sole nonzero component of the fluid velocity vector field \( \vec{v}(\vec{r}, t) \) is \( v_x = \text{const}(x) \). Naturally, the 4-velocity nonzero components should be equal to:

\[
U^t = (1 - v_x^2)^{-1/2} = \gamma, \tag{2.5a}
\]

\[
U^x = \gamma v_x. \tag{2.5b}
\]

Taking into account (2.5) one easily finds that the \( x \)-th, nonzero component of the stress-energy conservation equation (2.4) reduces to:

\[
\frac{\partial^2 U_x}{\partial y^2} = 0, \tag{2.6}
\]

and its general solution, evidently, is: \( U_x = Ay + B \) where \( A \) and \( B \) are some constants of integration. Since the fluid at the immovable plane \( (y = 0) \) should be at rest, \( B = 0 \) and then from (2.5b) we can obtain expression for the \( x \)-th component of the 3-velocity, which is:

\[
v_x = \frac{Ay}{\sqrt{1 + A^2 y^2}}. \tag{2.7}
\]

Another boundary condition: \( v_x(y = L) = v_0 \) helps to find the remaining integration constant \( A \):

\[
A = \gamma_0 v_0 / L, \tag{2.8}
\]

and (2.7) may be written explicitly as:

\[
v_x(y) = \frac{v_0 y}{\sqrt{L^2 (1 - v_0^2) + v_0^2 y^2}}, \tag{2.9}
\]

as for \( \gamma(y) \) we get:

\[
\gamma(y) = \sqrt{1 + \left( \frac{v_0 \gamma_0 y}{L} \right)^2}. \tag{2.10}
\]
Now, let us calculate an average velocity of this flow \( \bar{v} \). According to the definition we should have:

\[
\bar{v} \equiv \frac{1}{L} \int_0^L v_x(y) dy
\]

Taking into account (2.9) and performing necessary integration we get the following result:

\[
\bar{v} = \frac{1}{v_0} - \frac{\sqrt{1 - v_0^2}}{v_0}.
\]  

(2.11)

When \( v_0 \ll 1 \) (2.11) leads to the self-evident asymptotic result \( \bar{v} \approx v_0/2 \). For the case \( v_0 \to 1 \) the average velocity \( \bar{v} \to 1 \) as it, certainly, should be.

It must be noted that the tangential friction force, acting on the immovable \((y = 0)\) plane, defined as:

\[
f \equiv 2\eta |\sigma_{xy}| = \eta |\partial U_x/\partial y| = \eta v_0/L,
\]

(2.12)

coincides with the classic, nonrelativistic result \[4\].

**B. Plane Poiseuille flow**

Let us consider, now, a flow of the viscous, incompressible fluid between two parallel infinite immovable planes, induced by the existence of the pressure gradient along the flow axis. Let \( X \) axis be coincided with the symmetry axis of the flow. Therefore, the planes are situated at \( y = \pm L \). The symmetry of the problem implies that:

\[
U^\alpha = [U^t, U^x \equiv U = f(y), U^y = 0].
\]

(2.13)

Note that the y-th component of the stress-energy conservation equation leads to \( P = \text{const}(y) \). The x-th component of the same equation reduces to the following expression:

\[
(1 + U^2) \frac{\partial P}{\partial x} - \eta \frac{\partial U}{\partial y} \frac{\partial^2 U}{\partial y^2} = 0.
\]

(2.14)

As far as \( U = \text{const}(x) \) the gradient of the pressure along the flow should be constant: \( \partial P/\partial x = \text{const} \). Furthermore, this constant should be negative if \( U \) is taken to be positive. Expressing the second derivative appearing in (2.14) as:

\[
\frac{\partial^2 U}{\partial y^2} = \frac{\partial U}{\partial y} \frac{\partial U}{\partial U} \left( \frac{\partial U}{\partial y} \right),
\]

(2.15)

we can rewrite the equation as:

\[
\frac{1}{2} \frac{\partial}{\partial U} \left[ \left( \frac{\partial U}{\partial y} \right)^2 \right] = -\frac{1}{\eta} \frac{|\partial P|}{\partial x} \left( 1 + U^2 \right).
\]

(2.16)

Introducing new dimensionless variable: \( \varepsilon \equiv (2|\partial P/\partial x|/3\eta)^{1/2}y \) we reduce (2.16) to the following first order differential equation:

\[
\frac{\partial U}{\partial \varepsilon} = \sqrt{-\left(U^3 + 3U + \text{const} \right)},
\]

(2.17a)

where \( \text{const} \) is some constant of integration. It may be determined through the obvious condition, arising from the flow symmetry: \( \partial U/\partial \varepsilon \bigg|_{x=0} = 0 \) and \( U(\varepsilon = 0) = U_{\text{max}} \). These conditions together imply that \( \text{const} = -U_{\text{max}}^3 - 3U_{\text{max}} \).

Thus (2.17a) may also be written as:

\[
\frac{\partial U}{\partial \varepsilon} = \sqrt{(U_{\text{max}} - U)(U_{\text{max}}^2 + U_{\text{max}}U + U^2 + 3)},
\]

(2.17b)

This equation is solved in higher transcendental functions—Jacobi’s elliptical functions (see, for details \[8\] and Appendix of this paper). Taking into account that according to (A.5):

\[
-\lambda \varepsilon = F(\varphi|m),
\]
where

\[ \lambda^2 = \sqrt{3(1 + U_{\text{max}}^2)}, \quad (2.20a) \]

\[ m = \frac{1}{2} \left[ 1 + \frac{\sqrt{3}}{2} V_{\text{max}} \right], \quad (2.20b) \]

\[ \cos \varphi = \frac{\lambda^2 - (U_{\text{max}} - U)}{\lambda^2 + (U_{\text{max}} - U)}, \quad (2.20c) \]

for \( U(\varepsilon) \) we find the following expression:

\[ U(\varepsilon) = U_{\text{max}} - \left[ \frac{\lambda sn(\lambda \varepsilon)}{1 + cn(\lambda \varepsilon)} \right]^2. \quad (2.21) \]

As for the \( x \)-th component of 3-velocity \( V(y) \) we can simply derive it through (2.20) via the obvious relation:

\[ V(\varepsilon) = \frac{U(\varepsilon)}{\sqrt{1 + U^2(\varepsilon)}}. \quad (2.22) \]

We see that with increasing of \( V_{\text{max}} \) velocity profile becomes more and more ”flattened out”. The similar tendency is clearly seen also for Couette flow, considered in the previous section.

### III. CONCLUSION

In this study we consider two kinds of the special-relativistic flows of incompressible, viscous fluid. In particular, we examine the most simple two-dimensional (plane) model flows between two parallel infinite planes: Couette flow and Poiseuille flow. In both cases we find distribution of the velocity throughout the flow (velocity profiles) and in the former case we have also found the average velocity of the flow and the friction force, acting on the immovable wall. The obtained solutions are exact: for Couette flow it is analytic, while for the Poiseuille flow it is expressed by Jacobi’s elliptic functions.

As the further step of the investigation one may consider a study of hydrodynamic instability of these flows. It is presumable that the relativity may introduce new qualitative features in the instability dynamics as well. However, such a study is beyond the scope of the present paper and will be considered elsewhere.

Since the geometrical properties of considered model flows are the most simple we think that the results obtained in the present study and the mathematical methods of research that we are here developing, may be relevant and useful in some kinds of prototype astrophysical relativistic flows with comparable properties. Here we mean the relativistic regions of accretion discs of compact objects, accretion flows and stellar winds with spherical or quasispherical symmetry, outflows of matter in Active Galactic Nuclei, etc. Certainly, in some of these real astrophysical flows special-relativistic effects will be intricately interlaced with general-relativistic effects evoked by strong gravitational fields and electromagnetic effects (for plasma flows) arising due to the existence of superstrong magnetic fields. We hope that the present study may be of some use as the standing point in the future investigations of these complicated types of relativistic astrophysical flows.

### IV. ACKNOWLEDGEMENTS

My research was supported, in part, by International Science Foundation (ISF) long-term research grant RVO 300.

### V. APPENDIX

In the general theory of Jacobi’s elliptical functions it is known \[ \text{[1]} \] that the integral:

\[ \int \frac{dt}{\sqrt{-P(t)}}. \quad (A.1) \]
(where $P(t) = t^3 + a_1 t^2 + a_2 t + a_3$ is the third order polynomial of $t$, having only one real root $t = \beta$) may be reduced to the Jacobi’s elliptical integral of the first kind. For this purpose one must introduce the following coefficients:

$$\lambda^2 \equiv \sqrt{P'(\beta)},$$  \hspace{1cm} (A.3a)

$$m \equiv \sin^2 \alpha = \frac{1}{2} - \frac{1}{8} \frac{P''(\beta)}{\sqrt{P'(\beta)}},$$  \hspace{1cm} (A.3b)

and use, also, the following substitution:

$$\cos \phi = \frac{\lambda^2 - (\beta - t)}{\lambda^2 + (\beta - t)}.$$  \hspace{1cm} (A.4)

In these notations the initial integral may be written as:

$$\int \frac{dt}{\sqrt{-P(t)}} = -\frac{1}{\lambda} \int \frac{d\theta}{\sqrt{1 - m \sin^2 \theta}}.$$  \hspace{1cm} (A.5)

The latter integral is equal to the Jacobi’s elliptical integral of the first kind: $F(\phi \mid m)$.

In our case $a_1 = 3; a_2 = 0$ and $a_3 = \text{const}$. Despite that $a_3$ is an arbitrary parameter, the sole real root of the polynomial appearing in (2.17) should be the value of $U(\varepsilon)$ on the flow axis, since the symmetry of the flow implies that at $\varepsilon = 0$, $\partial U/\partial \varepsilon = 0$ and at this point $U = U_{\max}$.

[1] C. F. Kennel, F. S. Fujimura and I. Okamoto, (1983) Geophys. Ap. Fluid Dynamics, 26, 147.
[2] N. Straumann, General Relativity and Relativistic Astrophysics (Springer-Verlag, New-York, 1984).
[3] A. H. Bridle and R. A. Perley, (1984) Ann. Rev. Astron. Astrophys., 22, 319.
[4] L. D. Landau, and E. M. Lifshitz, Hydrodynamics (Third Edition) (Pergamon, Oxford, 1976).
[5] A. P. Lightman, W. H. Press, R. H. Price and S. A. Teukolsky, Problem Book in Relativity and Gravitation (Princeton University Press, Princeton, New Jersey, 1975).
[6] M. Abramowitz, and I. A. Stegun, Handbook of Mathematical Functions (National Bureau of Standards, 1964).

Author’s E-mail address is: andro@dtapha.kheta.georgia.su