Twisted Superspace on a Lattice

ALESSANDRO D’ADDA¹, ISSAKU KANAMORI² NOBORU KAWAMOTO³ and KAZUHIRO NAGATA⁴

INFN sezione di Torino, and Dipartimento di Fisica Teorica, Universita di Torino,
I-10125 Torino, Italy
and
Department of Physics, Hokkaido University
Sapporo, 060-0810, Japan

Abstract

We propose a new formulation which realizes exact twisted supersymmetry for all the supercharges on a lattice by twisted superspace formalism. We show explicit examples of $N = 2$ twisted supersymmetry invariant BF and Wess-Zumino models in two dimensions. We introduce mild lattice noncommutativity to preserve Leibniz rule on the lattice. The formulation is based on the twisted superspace formalism for $N = D = 2$ supersymmetry which was proposed recently. From the consistency condition of the noncommutativity of superspace, we find an unexpected three-dimensional lattice structure which may reduce into two dimensional lattice where the superspace describes semilocally scattered fermions and bosons within a double size square lattice.

¹dadda@to.infn.it
²kanamori@particle.sci.hokudai.ac.jp
³kawamoto@particle.sci.hokudai.ac.jp
⁴nagata@particle.sci.hokudai.ac.jp
1 Introduction

We know by now that the lattice regularization is the existing most successful and systematic regularization scheme of field theory which covers not only perturbative but also non-perturbative regime. This is most successfully shown in lattice QCD. It was also shown numerically and analytically that dynamical lattice triangulation is the very powerful regularization scheme of two dimensional quantum gravity even with matter degrees of freedom for the observable like the fractal dimension of quantum surface[1].

It is thus very natural to expect that the lattice regularization may play an important role even as a regularization scheme of unified theory of all interactions including gravity. If we, however, try to consider to accommodate fermions and bosons on the lattice we face with the notorious chiral fermion difficulty. If we stick to obtain a chiral gauge theory, the difficulty may still remain or can be avoided by the new definition of chiral symmetry on the lattice à la Ginsparg-Wilson[2]. We claim that there is an alternative approach on this problem.

The simplest version of the chiral fermion problem[3, 4] could be phrased in the following: If we formulate a chiral invariant free Dirac Lagrangian on a lattice by simply replacing the derivative operator to a difference operator, we cannot avoid of having species doublers. This naive fermion formulation is equivalently transformed to the staggered fermion by the spin diagonalization[5], which is equivalently transformed to the Kogut-Susskind fermion[6] with an extra second derivative term[7, 8]. In fact all these formulations are equivalent each other and furthermore they are obtained from Dirac-Kähler fermion formalism which is formulated by differential form[9]. In these frameworks the species doublers of the chiral fermion formulation on the lattice can turn into “flavor” degrees of freedom. It was then proposed that this “flavor” degrees of freedom is fundamentally related to the extended SUSY degrees of freedom, where the twisting procedure in the quantization of topological field theories is essentially equivalent to the Dirac-Kähler fermion mechanism[10]. It has then led to the proposal of a twisted superspace formalism[11].

It was shown that the quantized topological Yang-Mills action with instanton gauge fixing led twisted $N = 2$ super Yang-Mills action where the BRST charge is equivalent to the scalar component of twisted supercharge[12]. It was also shown that the two-dimensional version of quantized topological Yang-Mills action of the generalized gauge theory[13] with a two-dimensional instanton gauge fixing led to $N = 2$ super Yang-Mills action[10]. In this investigation it was recognized that the twisting is nothing but the Dirac-Kähler fermion mechanism. Here is the explicit realization of the “flavor” suffixes of the Dirac-Kähler fermion as the $N = 2$ extended SUSY suffixes. The $R$-symmetry of the $N = 2$ twisted SUSY algebra is the “flavor” symmetry of Dirac-Kähler fermion. It turned out that this Dirac-Kähler twisting mechanism works universally in the quantization of topological field theory and an extended SUSY is generated. Furthermore the twisted superspace formulation is hidden behind the formulation. This was explicitly shown for $N = 2$ twisted SUSY invariant BF action and Wess-Zumino action[11]. In this context we can
now recognize that the vector SUSY which was discovered in the quantization of topological fields theories is the vector counterpart of the twisted SUSY[14]. In this paper we formulate this $N = D = 2$ twisted superspace on the lattice as a particular example of general formulation.

Trials of formulating SUSY on the lattice has a long history which includes analyses of Wess-Zumino model with the ideas of Nicolai mapping, stochastic processes and others [15]. In those early investigations the difficulty of lattice SUSY formulation was recognized since SUSY does not exist on the lattice due to the lack of Poincaré symmetry. The domain wall fermion idea triggered not only the investigations of chiral fermion problem on the lattice but also lattice SUSY investigations [16]. There are deconstruction approaches of SUSY invariant lattices where the spatial lattice is created by “orbifolding” [17]. The lattice Wess-Zumino model was investigated with the recent contexts by several authors [18]. There are some twisted SUSY approaches on the lattice where the scalar part of the nilpotent super charge is identified as the BRST charge[19]. A new type of fermionic symmetry was proposed for super Yang-Mills theory [20]. There are many numerical lattice investigations to try to answer the dynamical questions of supersymmetric gauge theories, which are nicely reviewed in [21] and references are therein.

Here we propose a new formulation of lattice SUSY which is different from any of past formulations in the following two points: Firstly we formulate a twisted superspace on the lattice. Secondly we introduce mild noncommutativity to preserve the lattice Leibniz rule. And then as a consequence we obtain exact lattice SUSY for all twisted supercharges.

There are two important well-known obstacles in formulating lattice SUSY: Firstly Poincaré invariance is lost on the lattice, thus SUSY is lost as well. Secondly Leibniz rule does not hold on the lattice. In the current approach superfields satisfy the algebraic relations of twisted SUSY with a difference operator. In other words SUSY holds exactly on the lattice with the help of superspace.

The importance of the Leibniz rule for lattice SUSY is stressed in [22]. It has been recognized that the lattice Leibniz rule can be satisfied if we introduce mild noncommutativity between the difference operator and a function. This type of noncommutativity has been already investigated by several authors in the context of noncommutative differential form on a lattice[23]. Since differential form can be formulated on the lattice by the noncommutative geometry à la Connes[24], it is possible to formulate staggered fermion and Dirac-Kähler fermion formulation on the lattice in this framework [25]. In particular Clifford product was formulated on the lattice and thus Dirac-Kähler fermion can be successfully formulated with this noncommutative framework[26]. In this paper we use this noncommutative framework to establish twisted SUSY algebra.

This paper is organized as follows: In Section 2 we first discuss the Leibniz rule and noncommutativity on the lattice. Then we summarize the continuum formulation of twisted $N = D = 2$ SUSY and the corresponding superspace formulation in Section 3. We then reformulate the twisted superspace on the lattice in Section 4. We construct explicit examples of twisted SUSY invariant BF and Wess-Zumino
actions in Section 5. We then summarize the results and discuss future problems in the last section.

2 A new definition of the Leibniz rule on the lattice

One of the difficulties in formulating exact SUSY on a regular lattice lies in the fact that the continuum Poincaré group is broken by the lattice structure. The Lorentz group on a hypercubic lattice is reduced to the finite group of rotations by multiples of $\pi/2$ around the fundamental axis and translations are discretized to integer multiples of the lattice spacing. As a consequence derivatives are replaced on the lattice by finite differences, and these do not satisfy the Leibniz rule. This is an important point, and defining lattice derivatives that satisfy the Leibniz rule is a crucial step in establishing exact SUSY on a lattice. In view of that, we shall devote the present section to a detailed discussion of this problem and show how the Leibniz rule can be preserved on the lattice at the price of introducing a mild noncommutativity in the definition of the derivative operator.

The derivative $\partial_{\mu}$ is replaced on the lattice by the finite difference $\partial_{+\mu}$ defined by:

$$\partial_{+\mu} \Phi(x) = \Phi(x + 2\hat{n}_{\mu}) - \Phi(x),$$

(2.1)

where $\hat{n}_{\mu}$ corresponds to the shift of one lattice spacing in the $\mu$ direction. The definition is taken on two lattice spacings for reasons that will be later clarified.

The definition (2.1) does not satisfy the Leibniz rule. In fact it is immediate to verify that

$$\partial_{+\mu} (\Phi_1(x)\Phi_2(x)) \neq (\partial_{+\mu} \Phi_1(x)) \Phi_2(x) + \Phi_1(x) (\partial_{+\mu} \Phi_2(x)),$$

(2.2)

and that we have instead the following relation:

$$\partial_{+\mu} (\Phi_1(x)\Phi_2(x)) = (\partial_{+\mu} \Phi_1(x)) \Phi_2(x) + \Phi_1(x + 2\hat{n}_{\mu}) (\partial_{+\mu} \Phi_2(x)).$$

(2.3)

Eq. (2.3) defines a modified Leibniz rule satisfied by the finite difference $\partial_{+\mu}$ on the lattice. The new rule involves a certain degree of noncommutativity, in the sense that as $\partial_{+\mu}$ moves to the right of a function (for instance $\Phi_1(x)$ in (2.3)) it produces a corresponding shift in the argument. We may then define the lattice difference operator which carries the shift $2\hat{n}_{\mu}$ as follows:

$$\partial_{+\mu} \Phi(x) = \Delta_{+\mu} \Phi(x) - \Phi(x + 2\hat{n}_{\mu}) \Delta_{+\mu},$$

(2.4)

which reproduces the lattice version of modified Leibniz rule by the successive operation of $\Delta_{+\mu}$ to the product of two functions. As we shall see in the following sections this is a general feature shared by all symmetry operators consistently defined on the lattice, including obviously the SUSY ones.
The violation of the Leibniz rule, and the modified rule (2.3) can be easily understood. In fact the infinitesimal generator $P_\mu = -i\partial_\mu$ is associated on the lattice with a finite translation generated by the shift operator

$$T(\hat{n}_\mu) = e^{-im_\mu \cdot \vec{P}},$$

(2.5)

where $\hat{n}_\mu$ is a unit vector in the $\mu$ direction: $(\hat{n}_\mu)\nu = \delta_\mu^\nu$. The shift operator generates a corresponding shift in the argument of a field $\Phi(x)$ on the lattice:

$$T(\hat{n}_\mu)\Phi(x) = \Phi(x - \hat{n}_\mu)T(\hat{n}_\mu).$$

(2.6)

and can be used to define the difference operator. In fact if we define:

$$\Delta_+\mu = -T(2\hat{n}_\mu),$$

(2.7)

we have:

$$[\Delta_+\mu, \Phi(x)] = T(2\hat{n}_\mu) \partial_+\mu \Phi(x).$$

(2.8)

Eqs (2.4) and (2.7) are two distinct ways of introducing the finite difference $\partial_+\mu \Phi(x)$ as a result of some type of commutator. In that respect they are both the lattice equivalent of the commutator defining the partial derivative in the continuum:

$$[P_\mu, \Phi(x)] = -i\partial_\mu \Phi.$$ 

(2.9)

The two operators $\Delta_+\mu$ and $\Delta_+\mu$ are obviously related. In general, if an operator $O$ satisfies the commutation relations

$$[O, \Phi(x)] = T(\hat{a}_O) \Delta_O \Phi(x),$$

(2.10)

it follows that the operator $\Delta_O$ defined by

$$\mathcal{O} = T(\hat{a}_O) \Delta_O,$$

(2.11)

obeys the following “shifted” commutator:

$$\delta_O \Phi(x) = \Delta_O \Phi(x) - \Phi(x + \hat{a}_O) \Delta_O \Phi(x).$$

(2.12)

In the last equations $\hat{a}_O$ is a shift of the two dimensional lattice associated to the operator $\mathcal{O}$. In this article we discriminate the operators and fields which carry the “shift” from normal ones by underlining the corresponding operators and fields. All symmetry operators on the lattice that we shall consider in the rest of the paper will satisfy commutation relations of the form (2.10) or, in their “arowed” version, of the form (2.12). Needless to say, the two formalism are entirely equivalent and the use of one or the other will be dictated by convenience. The modified Leibniz rule for a general operator on the lattice reads

$$\delta_O (\Phi_1(x)\Phi_2(x)) = (\delta_O \Phi_1(x)) \Phi_2(x) + \Phi_1(x + \hat{a}_O) (\delta_O \Phi_2(x)).$$

(2.13)
This is a straightforward consequence of (2.12) or equivalently, if we take the definition (2.8), of the Leibniz rule for the commutator.

In addition to the finite difference operator $\Delta_+\mu$ that carries a shift in the positive direction of the $\mu$-axis one can introduce a negative difference operator $\Delta_-\mu$ defined by

$$\Delta_-\mu = T(-2\hat{n}_\mu).$$

(2.14)

Correspondingly we introduce a finite difference

$$\partial_-\mu \Phi(x) = \Phi(x) - \Phi(x - 2\hat{n}_\mu),$$

(2.15)

which is obtained from the commutator

$$[\Delta_-\mu, \Phi(x)] = T(-2\hat{n}_\mu) \partial_-\mu \Phi(x),$$

(2.16)

or from the “shifted commutator” using the operator $\tilde{\Delta}_-\mu$:

$$\partial_-\mu \Phi(x) = \tilde{\Delta}_-\mu \Phi(x) - \Phi(x - 2\hat{n}_\mu) \tilde{\Delta}_-\mu = \Phi(x) - \Phi(x - 2\hat{n}_\mu).$$

(2.17)

Notice that from the definition of the “arrowed” operators (2.11) and Eqs (2.14) and (2.7) we find:

$$\tilde{\Delta}_\pm\mu = \mp 1,$$

(2.18)

consistently with (2.4) and (2.17).

In the following sections we shall also use instead of the variation $\delta \mathcal{O}$ of (2.10) a symmetric variation $\delta^{(s)} \mathcal{O}$ defined by:

$$[\mathcal{O}, \Phi(x)] = T(\hat{a}_\mathcal{O}/2) \delta^{(s)} \mathcal{O} \Phi(x) T(\hat{a}_\mathcal{O}/2),$$

(2.19)

or equivalently by

$$\delta^{(s)} \mathcal{O} \Phi(x) = \tilde{\mathcal{O}} \Phi(x - \hat{a}_\mathcal{O}/2) - \Phi(x + \hat{a}_\mathcal{O}/2).$$

(2.20)

Clearly $\delta^{(s)} \mathcal{O} \Phi(x)$ and $\delta \mathcal{O} \Phi(x)$ differ simply by a shift in the argument:

$$\delta^{(s)} \mathcal{O} \Phi(x) = \delta \mathcal{O} \Phi(x - \hat{a}_\mathcal{O}/2).$$

(2.21)

The advantage of $\delta^{(s)} \mathcal{O}$ is to make all formulas left-right symmetric, for instance the modified Leibniz rule becomes:

$$\delta^{(s)} \mathcal{O} (\Phi_1(x)\Phi_2(x)) = \left(\delta^{(s)} \mathcal{O} \Phi_1(x)\right) \Phi_2(x - \hat{a}_\mathcal{O}/2) + \Phi_1(x + \hat{a}_\mathcal{O}/2) \left(\delta \mathcal{O} \Phi_2(x)\right).$$

(2.22)

Besides, with the definition (2.19) the positive and negative finite difference operators $\Delta_+\mu$ and $\Delta_-\mu$ generate the same symmetric finite difference $\partial^{(s)}_\pm$:

$$[\Delta_{\pm\mu}, \Phi(x)] = T(\pm\hat{n}_\mu) \partial^{(s)}_\mu \Phi(x) T(\pm\hat{n}_\mu),$$

(2.23)

where

$$\partial^{(s)}_\mu \Phi(x) = \Phi(x + \hat{n}_\mu) - \Phi(x - \hat{n}_\mu).$$

(2.24)
3 \( N = 2 \) Twisted SUSY in continuum two dimensions

In this section, we summarize the continuum formulation of \( N = 2 \) twisted superspace which was proposed recently[11]. Throughout this paper, we consider two-dimensional Euclidean space-time.

We first introduce two-dimensional \( N = D = 2 \) SUSY algebra:

\[ \{ Q_{\alpha i}, Q_{\beta j} \} = 2 \delta_{ij} \gamma_{\alpha \beta}^\mu P_\mu, \] (3.1)

where \( Q_{\alpha i} \) is supercharge, where the left-indices \( \alpha (= 1, 2) \) and the right-indices \( i (= 1, 2) \) are Lorentz spinor and internal spinor suffixes labeling two different \( N = 2 \) supercharges, respectively. We can take these operators to be Majorana in two dimensions. \( P_\mu \) is generator of translation.

Since the above supercharges have double spinor indices, we can decompose them into the following scalar, vector and pseudo-scalar components which we call twisted supercharges:

\[ Q_{\alpha i} = (1 s + \gamma^\mu s_\mu + \gamma^5 \tilde{s})_{\alpha i}. \] (3.2)

Then the relations (3.1) can be rewritten by the twisted generators:

\[ \{ s, s_\mu \} = P_\mu, \quad \{ \tilde{s}, s_\mu \} = -\epsilon_{\mu \nu} P^\nu, \]
\[ s^2 = \tilde{s}^2 = \{ s, \tilde{s} \} = \{ s_\mu, s_\nu \} = 0. \] (3.3)

This is the twisted \( N = D = 2 \) SUSY algebra. In this paper we do not consider \( J \) and \( R \) symmetry of the \( N = 2 \) algebra. The notations of gamma matrices are given in Appendix A.

Similar to the supercharges we can introduce twisted super parameters as

\[ \theta_{\alpha i} = \frac{1}{2} \left( 1 \theta + \gamma^\mu \theta_\mu + \gamma^5 \tilde{\theta} \right)_{\alpha i}, \] (3.4)

then we can define \( N = 2 \) super symmetry transformation as

\[ \delta_{\theta} = \theta_{\alpha i} Q_{\alpha i} = \theta s + \theta^\mu s_\mu + \tilde{\theta} \tilde{s}. \] (3.5)

3.1 Twisted superspace and superfield

We consider the following super group element:

\[ G(x^\mu, \theta, \tilde{\theta}) = e^{i(-x^\mu P_\mu + \theta s + \theta^\mu s_\mu + \tilde{\theta} \tilde{s})}, \] (3.6)

where \( \theta \)'s are anticommuting parameters. Twisted \( N = D = 2 \) superspace is defined in the parameter space of \( (x^\mu, \theta, \theta^\mu, \tilde{\theta}) \).
By using the relations (3.3), we can show the following relation:

\[ G(0, \xi, \xi^\mu, \tilde{\xi})G(x^\mu, \theta, \theta^\mu, \tilde{\theta}) = G(x^\mu + a^\mu, \theta + \xi, \theta^\mu + \xi^\mu, \tilde{\theta} + \tilde{\xi}), \]  

(3.7)

where \( a^\mu = \frac{i}{2}\xi^\nu \theta^\mu + \frac{i}{2}\xi^\mu \theta^\nu + \frac{i}{2}\xi^\nu \tilde{\xi} \theta^\nu \). This multiplication induces a shift transformation in superspace \( (x^\mu, \theta, \theta^\mu, \tilde{\theta}) \):

\[ (x^\mu, \theta, \theta^\mu, \tilde{\theta}) \rightarrow (x^\mu + a^\mu, \theta + \xi, \theta^\mu + \xi^\mu, \tilde{\theta} + \tilde{\xi}), \]  

(3.8)

which is generated by the following differential operators \( Q, Q_\mu, \) and \( \tilde{Q} \):

\[ Q = \frac{\partial}{\partial \theta} + \frac{i}{2} \theta^\mu \partial_{\mu}, \]
\[ Q_\mu = \frac{\partial}{\partial \theta^\mu} + \frac{i}{2} \theta \partial_{\mu} - \frac{i}{2} \tilde{\theta} \epsilon_{\mu \nu} \partial^\nu, \]
\[ \tilde{Q} = \frac{\partial}{\partial \tilde{\theta}} - \frac{i}{2} \theta^\mu \epsilon_{\mu \nu} \partial^\nu. \]

Indeed we find

\[ \delta_x \begin{pmatrix} x^\mu \\ \theta \\ \theta^\mu \\ \tilde{\theta} \end{pmatrix} = (\xi Q + \xi^\mu Q_\mu + \tilde{\xi} \tilde{Q}) \begin{pmatrix} x^\mu \\ \theta \\ \theta^\mu \\ \tilde{\theta} \end{pmatrix} = \begin{pmatrix} a^\mu \\ \xi \\ \xi^\mu \\ \tilde{\xi} \end{pmatrix}. \]  

(3.10)

These operators satisfy the following relations:

\[ \{Q, Q_\mu\} = i \partial_{\mu}, \quad \{\tilde{Q}, Q_\mu\} = -i \epsilon_{\mu \nu} \partial^\nu, \]
\[ Q^2 = \tilde{Q}^2 = \{Q, \tilde{Q}\} = \{Q_\mu, Q_\nu\} = 0. \]  

(3.11)

The general scalar superfields in twisted \( N = D = 2 \) superspace are defined as the functions of \((x^\mu, \theta, \theta^\mu, \tilde{\theta})\), and can be expanded as follows:

\[ F(x^\mu, \theta, \theta^\mu, \tilde{\theta}) = \phi(x) + \theta^\mu \phi_\mu(x) + \theta^2 \tilde{\phi}(x) \]
\[ + \theta \left( \psi(x) + \theta^\mu \psi_\mu(x) + \theta^2 \tilde{\psi}(x) \right) \]
\[ + \tilde{\theta} \left( \chi(x) + \theta^\mu \chi_\mu(x) + \theta^2 \tilde{\chi}(x) \right) \]
\[ + \tilde{\theta} \tilde{\theta} \left( \lambda(x) + \theta^\mu \lambda_\mu(x) + \theta^2 \tilde{\lambda}(x) \right), \]  

(3.12)

where the leading component \( \phi(x) \) can be taken to be not only bosonic but also fermionic.

The transformation law of the superfield \( F \) is defined as follows:

\[ \delta_x F(x^\mu, \theta, \theta^\mu, \tilde{\theta}) = \delta_x \phi(x) + \theta^\mu \delta_x \phi_\mu(x) + \theta^2 \delta_x \tilde{\phi}(x) \]
\[ + \theta \left( \delta_x \psi(x) + \theta^\mu \delta_x \psi_\mu(x) + \theta^2 \delta_x \tilde{\psi}(x) \right) \]
\[ + \tilde{\theta} \left( \delta_x \chi(x) + \theta^\mu \delta_x \chi_\mu(x) + \theta^2 \delta_x \tilde{\chi}(x) \right) \]
\[ + \tilde{\theta} \tilde{\theta} \left( \delta_x \lambda(x) + \theta^\mu \delta_x \lambda_\mu(x) + \theta^2 \delta_x \tilde{\lambda}(x) \right) \]
\[ = (\xi Q + \xi^\mu Q_\mu + \tilde{\xi} \tilde{Q}) F(x^\mu, \theta, \theta^\mu, \tilde{\theta}), \]  

(3.13)
where $Q$, $Q_{\mu}$ and $\tilde{Q}$ are the differential operators (3.9). The transformation laws of the component fields $\phi^A(x) = (\phi(x), \phi_\mu(x), \tilde{\phi}(x), \ldots)$ are obtained by comparing coefficients of the same superspace parameters in (3.13) (see Table 1). Those transformation laws lead to the following supercharge algebra:

$$\{s, s_{\mu}\} = -i \partial_{\mu}, \quad \{\tilde{s}, s_{\mu}\} = i \epsilon_{\mu\nu} \partial^\nu,$$

$$s^2 = \tilde{s}^2 = \{s, \tilde{s}\} = \{s_{\mu}, s_{\nu}\} = 0,$$  \hspace{1cm} (3.14)

which are the same of (3.3) with $P_{\mu} = -i \partial_{\mu}$. It should be noted that the only difference between the supercharge algebra and the corresponding differential operator algebra is a sign difference for the derivative.

Table 1: $N = 2$ twisted SUSY transformation laws of the component fields of $F$.

| $\phi^A$ | $s\phi^A$ | $s_{\mu}\phi^A$ | $\tilde{s}\phi^A$ |
|--------|-----------|----------------|----------------|
| $\phi$ | $\psi$ | $\phi_{\mu}$ | $\chi$ |
| $\phi_\rho$ | $-\psi_\rho - \frac{i}{2} \partial_\rho \phi$ | $-\epsilon_{\mu\rho} \tilde{\phi}$ | $-\chi_\rho + \frac{i}{2} \epsilon_{\rho\sigma} \partial^\sigma \phi$ |
| $\tilde{\phi}$ | $\tilde{\psi} + \frac{i}{2} \epsilon_{\rho\sigma} \partial_\rho \phi_\sigma$ | $0$ | $\tilde{\chi} - \frac{i}{2} \partial^\rho \phi_\rho$ |
| $\psi$ | $0$ | $\psi_{\mu} - \frac{i}{2} \partial_{\mu} \phi$ | $\lambda$ |
| $\psi_\rho$ | $-\frac{i}{2} \partial_{\rho} \psi$ | $-\epsilon_{\rho\mu} \tilde{\psi} + \frac{i}{2} \partial_{\mu} \phi_\rho$ | $-\lambda_{\rho} + \frac{i}{2} \epsilon_{\rho\sigma} \partial^\sigma \psi$ |
| $\tilde{\psi}$ | $\frac{i}{2} \epsilon_{\rho\sigma} \partial_\rho \psi_\sigma$ | $-\frac{i}{2} \partial_{\mu} \tilde{\phi}$ | $\tilde{\lambda} - \frac{i}{2} \partial^\rho \psi_\rho$ |
| $\chi$ | $-\lambda$ | $\chi_{\mu} + \frac{i}{2} \epsilon_{\mu\nu} \partial^\nu \phi$ | $0$ |
| $\chi_\rho$ | $-\frac{i}{2} \partial_{\mu} \chi$ | $-\epsilon_{\rho\mu} \tilde{\chi} - \frac{i}{2} \epsilon_{\rho\mu} \partial^\nu \phi_\rho$ | $+\frac{i}{2} \epsilon_{\rho\sigma} \partial^\sigma \chi$ |
| $\tilde{\chi}$ | $-\tilde{\lambda} + \frac{i}{2} \epsilon_{\rho\sigma} \partial_\rho \chi_\sigma$ | $\frac{i}{2} \epsilon_{\mu\nu} \partial^\nu \tilde{\phi}$ | $-\frac{i}{2} \partial^\rho \chi_\rho$ |
| $\lambda$ | $0$ | $\lambda_{\mu} + \frac{i}{2} \partial_{\mu} \chi + \frac{i}{2} \epsilon_{\mu\nu} \partial^\nu \psi_\rho$ | $0$ |
| $\lambda_\rho$ | $-\frac{i}{2} \partial_{\rho} \lambda$ | $-\epsilon_{\rho\mu} \tilde{\lambda} - \frac{i}{2} \partial_{\mu} \chi_\rho - \frac{i}{2} \epsilon_{\mu\nu} \partial^\nu \psi_\rho$ | $\frac{i}{2} \epsilon_{\rho\sigma} \partial^\sigma \lambda$ |
| $\tilde{\lambda}$ | $\frac{i}{2} \epsilon_{\rho\sigma} \partial_\rho \lambda_\sigma$ | $\frac{i}{2} \partial_{\mu} \tilde{\chi} + \frac{i}{2} \epsilon_{\mu\nu} \partial^\nu \tilde{\psi}$ | $-\frac{i}{2} \partial^\rho \lambda_\rho$ |

Given the transformation laws of the component fields, we can expand the superfield $F$ as follows:

$$F(x^\mu, \theta, \theta^\mu, \tilde{\theta}) = e^{\delta_\theta \phi}(x)$$

$$= \phi(x) + \delta_\theta \phi(x) + \frac{1}{2} \delta_\theta^2 \phi(x) + \frac{1}{3!} \delta_\theta^3 \phi(x) + \frac{1}{4!} \delta_\theta^4 \phi(x),$$  \hspace{1cm} (3.15)

where $\delta_\theta$ is defined in (3.5).

As we have seen, the differential operators in (3.9) generate the shift transformation of superspace induced by left multiplication $G(0, \xi, \xi^\mu, \tilde{\xi})G(x^\mu, \theta, \theta^\mu, \tilde{\theta})$. On the other hand, there exist the differential operators which generate the shift trans-
formation induced by right multiplication $G(x^\mu, \theta, \bar{\theta}^\mu)G(0, \xi, \bar{\theta}^\mu, \bar{\xi})$:

$$
D = \frac{\partial}{\partial \theta} - \frac{i}{2} \theta^\mu \partial_\mu, \\
D_\mu = \frac{\partial}{\partial \theta^\mu} - \frac{i}{2} \theta \partial_\mu + \frac{i}{2} \bar{\theta} \epsilon_{\mu\nu} \partial^\nu, \\
\tilde{D} = \frac{\partial}{\partial \bar{\theta}} + \frac{i}{2} \theta^\mu \epsilon_{\mu\nu} \partial^\nu, \\
$$

(3.16)

which satisfy the relations:

$$
\{D, D_\mu\} = -i \partial_\mu, \quad \{\tilde{D}, D_\mu\} = i \epsilon_{\mu\nu} \partial^\nu, \\
D^2 = \tilde{D}^2 = \{D, \tilde{D}\} = \{D_\mu, D_\nu\} = 0,
$$

(3.17)

where only the sign of $\partial_\mu$ is changed from the left operator algebra (3.11). $Q^A = (Q, Q_\mu, \tilde{Q})$ and $D^A = (D, D_\mu, \tilde{D})$ anticommute:

$$
\{Q^A, D^B\} = 0.
$$

(3.18)

### 3.2 Chiral and anti-chiral superfields

The chiral conditions for the twisted chiral superfield can be given by

$$
D \Psi(x^\mu, \theta, \bar{\theta}^\mu) = 0, \quad \tilde{D} \Psi(x^\mu, \theta, \bar{\theta}^\mu) = 0.
$$

(3.19)

The details of the twisted chiral superfield formulation can be found in [11]. It is convenient to rewrite the chiral conditions by using the operator which satisfy the following relations for the differential operators:

$$
UDU^{-1} = \frac{\partial}{\partial \theta}, \quad U\tilde{D}U^{-1} = \frac{\partial}{\partial \bar{\theta}}, \\
U^{-1}D_\mu U = \frac{\partial}{\partial \theta^\mu}, \\
$$

(3.20)

where

$$
U = e^{-\frac{i}{2}(\theta^\mu \partial_\mu - \bar{\theta} \epsilon_{\mu\nu} \partial^\nu)}.
$$

(3.21)

Then the chiral conditions (3.19) can be transformed into

$$
\frac{\partial}{\partial \theta} U \Psi(x^\mu, \theta, \bar{\theta}^\mu) U^{-1} = 0, \quad \frac{\partial}{\partial \bar{\theta}} U \Psi(x^\mu, \theta, \bar{\theta}^\mu) U^{-1} = 0,
$$

(3.22)

which leads

$$
U \Psi(x^\mu, \theta, \bar{\theta}^\mu) U^{-1} = \Psi'(x^\mu, \theta^\mu).
$$

(3.23)

Then the solution for the original chiral condition (3.19) is obtained as

$$
\Psi(x^\mu, \theta, \bar{\theta}^\mu) = U^{-1} \Psi'(x^\mu, \theta^\mu) U = \Psi'(z^\mu, \theta^\mu),
$$

(3.24)
where
\[ z^\mu = x^\mu + \frac{i}{2} \theta \theta^\mu - \frac{i}{2} \epsilon^\mu_\nu \theta^\nu \tilde{\theta}. \]  \( (3.25) \)

This solution can be expanded as follows:
\[
\Psi(x^\mu, \theta, \tilde{\theta}, \theta^\mu) = \Psi'(z^\mu, \theta^\mu) = \phi(z) + \theta^\mu \psi_\mu(z) + \theta^2 \phi(z) + \frac{1}{2} \theta^\mu \partial_\mu \phi(x) + \frac{1}{2} \theta^2 \partial^\mu \phi(x) + \frac{1}{2} \theta^2 \psi_\mu(x) + \frac{1}{4} \theta^4 \partial^2 \phi(x),
\]  \( (3.26) \)

where \( \theta^2 = \frac{1}{2} \epsilon_{\mu\nu} \theta^\mu \theta^\nu \) and \( \theta^4 = \theta \tilde{\theta} \theta^2 \).

Anti-chiral conditions for the anti-chiral superfield are given by
\[
D_\mu \overline{\Psi}(x^\mu, \theta, \tilde{\theta}, \theta^\mu) = 0.
\]  \( (3.27) \)

Similar to the chiral condition (3.22), we can transform the original anti-chiral condition (3.27) into the following form:
\[
U^{-1} D_\mu U U^{-1} \overline{\Psi}(x^\mu, \theta, \tilde{\theta}, \theta^\mu) U = \frac{\partial}{\partial \theta^\mu} U^{-1} \overline{\Psi}(x^\mu, \theta, \tilde{\theta}, \theta^\mu) U = 0,
\]  \( (3.28) \)

which leads
\[
U^{-1} \overline{\Psi}(x^\mu, \theta, \tilde{\theta}, \theta^\mu) U = \overline{\Psi}(x^\mu, \theta, \tilde{\theta}).
\]  \( (3.29) \)

Then the solution for the original anti-chiral condition (3.27) is obtained as
\[
\overline{\Psi}(x^\mu, \theta, \tilde{\theta}, \theta^\mu) = U \overline{\Psi}(x^\mu, \theta, \tilde{\theta}) U^{-1} = \overline{\Psi}(z^\mu, \theta, \tilde{\theta}),
\]  \( (3.30) \)

where
\[ z^\mu = x^\mu - \frac{i}{2} \theta \theta^\mu + \frac{i}{2} \epsilon^\mu_\nu \theta^\nu \tilde{\theta}. \]  \( (3.31) \)

This solution can be expanded as follows:
\[
\overline{\Psi}(x^\mu, \theta, \tilde{\theta}, \theta^\mu) = \overline{\Psi}(z^\mu, \theta, \tilde{\theta}) = \varphi(\tilde{z}) + \theta \chi(\tilde{z}) + \tilde{\theta} \tilde{\chi}(\tilde{z}) + \theta \tilde{\theta} \tilde{\varphi}(\tilde{z}) + \varphi(x) + \theta \chi(x) + \tilde{\theta} \tilde{\chi}(x)
\]  \( (3.32) \)

The SUSY transformations of the chiral and anti-chiral superfields are given by
\[
\begin{align*}
\slashed{s}_A \Psi(x^\mu, \theta, \tilde{\theta}, \theta^\mu) &= Q_A \Psi(x^\mu, \theta, \tilde{\theta}, \theta^\mu), \\
\slashed{s}_A \overline{\Psi}(x^\mu, \theta, \tilde{\theta}, \theta^\mu) &= Q_A \overline{\Psi}(x^\mu, \theta, \tilde{\theta}, \theta^\mu),
\end{align*}
\]  \( (3.33) \)
where \( s_A = (s, \tilde{s}, s_\mu) \) and \( Q_A = (Q, \tilde{Q}, Q_\mu) \). These SUSY transformation can be transformed into the following form by using the operator (3.21):

\[
\begin{align*}
    s_A U \Psi(x^\mu, \theta, \tilde{\theta}, \theta^\mu) U^{-1} & = UQ_A U^{-1} U \Psi(x^\mu, \theta, \tilde{\theta}, \theta^\mu) U^{-1}, \\
    s_A U^{-1} \Phi(x^\mu, \theta, \tilde{\theta}, \theta^\mu) U & = U^{-1} Q_A U U^{-1} \Phi(x^\mu, \theta, \tilde{\theta}, \theta^\mu) U,
\end{align*}
\]

(3.34)

which can be equivalently written as

\[
\begin{align*}
    s_A U \Psi(x^\mu, \theta, \tilde{\theta}, \theta^\mu) U^{-1} & = UQ_A U^{-1} U \Psi(x^\mu, \theta, \tilde{\theta}, \theta^\mu) U^{-1}, \\
    s_A U^{-1} \Phi(x^\mu, \theta, \tilde{\theta}, \theta^\mu) U & = U^{-1} Q_A U U^{-1} \Phi(x^\mu, \theta, \tilde{\theta}, \theta^\mu) U.
\end{align*}
\]

(3.35)

After obtaining the twisted \( N = 2 \) SUSY transformation, we find the following natural relations:

\[
\begin{align*}
    \Psi'(x^\mu, \theta^\mu) & = e^{\theta^\mu s_\nu} \phi(x) \\
                    & = \phi(x) + \theta^\mu s_\nu \phi(x) + \theta^2 s_2 s_1 \phi(x) \\
                    & = \phi(x) + \theta^\mu \psi_\mu(x) + \theta^2 \phi(x), \\
    \Phi'(x^\mu, \tilde{\theta}, \theta^\mu) & = e^{\theta^\mu \tilde{s}_\nu} \phi(x) \\
                       & = \phi(x) + \theta^\mu \tilde{s}_\nu \phi(x) + \theta^2 \phi(x), \\
\end{align*}
\]

(3.38)

which shows that the components of the chiral and anti-chiral fields are defined by operating the twisted supercharges to the leading field.
3.3 \( N = 2 \) twisted supersymmetric BF and Wess-Zumino actions

We now introduce off-shell \( N = 2 \) twisted supersymmetric action:

\[
S = \int d^2x \int d^4\theta \left( i\epsilon\Psi(x^\mu, \theta, \theta^\mu)\bar{\Psi}(x^\mu, \theta, \theta^\mu) \right). \tag{3.39}
\]

We can take the chiral superfields to be not only bosonic but also fermionic. \( \epsilon_\Psi \) should be taken 0 or 1 for bosonic or fermionic (\( \Psi, \bar{\Psi} \)), respectively.

For fermionic (\( \Psi, \bar{\Psi} \)), the fields in the expansion of the superfield (3.26) and (3.32) can be renamed as:

\[
\Psi(x^\mu, \theta, \tilde{\theta}, \theta^\mu) = \Psi'(z^\mu, \theta^s) = i\epsilon\theta^s \gamma^\mu c(z) = i\epsilon \theta^2 \lambda(z) \tag{3.40}
\]

\[
\bar{\Psi}(x^\mu, \theta, \tilde{\theta}, \theta^\mu) = \bar{\Psi}'(\tilde{z}^\mu, \bar{\theta}, \bar{\theta}) = i\epsilon\theta^s \tilde{\gamma}^\mu \tilde{c}(\tilde{z}) + \theta b(\tilde{z}) + \tilde{\theta} \phi(\tilde{z}) - i\partial \bar{\theta} \rho(\tilde{z}),
\]

where we have the correspondence of the fields; (\( \varphi, \chi, \bar{\chi}, \bar{\varphi} \)) \( \rightarrow \) (\( i\epsilon, \bar{b}, \phi, -i\rho \)) and (\( \psi^\mu, \tilde{\phi} \)) \( \rightarrow \) (\( i\epsilon, \omega^\mu, i\lambda \)). \( N = 2 \) twisted SUSY transformations of the renamed fields can be read off from Table 2. Then the action (3.39) leads

\[
S_f = \int d^2x \int d^4\theta \left( i\epsilon\Psi(x^\mu, \theta, \theta^\mu)\bar{\Psi}(x^\mu, \theta, \theta^\mu) \right) = \int d^2x \int d^4\theta \ e^{\delta\theta}(\xi \bar{c})
\]

\[
= \int d^2x \ s\tilde{s} \frac{1}{2} \epsilon^{\mu\nu} s_\mu s_\nu (\xi \bar{c})
\]

\[
= \int d^2x \left( \phi \epsilon^{\mu\nu} \partial_\mu \omega_\nu + b \partial^\mu \omega_\mu + i \partial_\mu \bar{\tau} \partial^\mu c + i \rho \lambda \right), \tag{3.41}
\]

where \( \delta\theta \) is defined in (3.5). This action can be identified as the quantized BF action with auxiliary fields and has off-shell \( N = 2 \) twisted SUSY up to the surface terms by construction.

For bosonic (\( \Psi, \bar{\Psi} \)) the action (3.39) can be written as follows:

\[
S_b = \int d^2x \int d^4\theta \left( \bar{\Psi}(\xi c) \right) = \int d^2x \int d^4\theta \ e^{\delta\theta} (\varphi \phi)
\]

\[
= \int d^2x \ s\tilde{s} \frac{1}{2} \epsilon^{\mu\nu} s_\mu s_\nu (\varphi \phi)
\]

\[
= \int d^2x \left( \bar{\varphi} \epsilon^{\mu\nu} \partial_\mu \psi_\nu + i \bar{\chi} \partial^\mu \psi_\mu - \partial^\mu \varphi \partial_\mu \phi + \bar{\phi} \bar{\varphi} \right), \tag{3.42}
\]

where (\( \phi, \varphi, \bar{\varphi}, \bar{\phi} \)) and (\( \psi^\mu, \chi, \bar{\chi} \)) are bosonic and fermionic fields, respectively. The fermionic terms in (3.42) change into matter fermions via Dirac-Kähler fermion mechanism:

\[
\int d^2x \ (i\bar{\chi} \epsilon^{\mu\nu} \partial_\mu \psi_\nu + i\bar{\chi} \partial^\mu \psi_\mu) = \int d^2x \ Tr (i\bar{\xi} \gamma^\mu \partial_\mu \xi), \tag{3.43}
\]
where the Dirac-Kähler fermion $\xi$ is defined as

$$\xi_{\alpha\beta} = \frac{1}{2} (1 \chi + \gamma^\mu \psi_\mu + \gamma^5 \bar{\chi})_{\alpha\beta},$$  

(3.44)

with $\xi = C\xi^T C^{-1} = \xi^T$. It is natural to identify the fermionic antisymmetric tensor fields, $\chi, \psi_\mu, \bar{\chi}$, as Dirac-Kähler fields of 0-, 1-, 2-forms, respectively. We can recognize that each spinor suffix of this Dirac-Kähler fermion has the Majorana Weyl fermion nature. We now redefine the bosonic fields as follows:

$$\phi_0 = \frac{1}{2} (\phi + \varphi), \phi_1 = \frac{1}{2} (\phi - \varphi),$$

$$F_0 = \frac{1}{2} (\bar{\phi} + \bar{\varphi}), F_1 = \frac{1}{2} (\bar{\phi} - \bar{\varphi}).$$

(3.45)

Then the action (3.42) can be rewritten by the new fields:

$$S_b = \int d^2x \sum_{i=1}^2 \left( i\xi_i \gamma^\mu \partial_\mu \xi_i + \partial_\mu \phi^i \partial^\mu \phi^i - F_i F_i \right),$$

(3.46)

where we further redefine $i\phi_0 \to \phi_2$ and $iF_0 \to F_2$. This is the 2-dimensional version of $N = 2$ Wess-Zumino action which has off-shell $N = 2$ SUSY invariance of standard $N = 2$ SUSY algebra (3.1). It is important to recognize at this stage that the “flavor” suffix of the Dirac-Kähler matter fermion in the action is $N = 2$ extended SUSY suffix.

### 3.4 Non-Abelian Extension

The Abelian version of the chiral and anti-chiral conditions (3.19) and (3.27) can be covariantly extended to the following non-Abelian conditions:

$$D \Phi(x^\mu, \theta, \bar{\theta}, \theta^\mu) - i \Phi^2(x^\mu, \theta, \bar{\theta}, \theta^\mu) = 0, \quad \bar{D} \Phi(x^\mu, \theta, \bar{\theta}, \theta^\mu) = 0,$$

$$D_\mu \overline{\Phi}(x^\mu, \theta, \bar{\theta}, \theta^\mu) = 0,$$

(3.47)

(3.48)

where $\Phi(x^\mu, \theta, \bar{\theta}, \theta^\mu)$ and $\overline{\Phi}(x^\mu, \theta, \bar{\theta}, \theta^\mu)$ are, respectively, non-Abelian chiral and anti-chiral superfields. Notice that the above extension makes sense only in the case of $\Phi$ to be fermionic. Similar to the Abelian case we can transform the non-Abelian chiral condition (3.47) into the following form:

$$\frac{\partial}{\partial \theta} \Phi' - i(\Phi')^2 = 0,$$

(3.49)

$$\frac{\partial}{\partial \bar{\theta}} \Phi' = 0,$$

(3.50)

where $\Phi' \equiv U \Phi(x^\mu, \theta, \bar{\theta}, \theta^\mu) U^{-1}$ and $U$ is given by (3.21). Unlike the Abelian case, $\Phi'$ is still a function of $(x^\mu, \theta, \bar{\theta}, \theta^\mu)$ due to inhomogeneous component in the non-Abelian chiral condition (3.47). In order to find solutions for these chiral conditions, we expand $\Phi'$ as

$$\Phi'(x^\mu, \theta, \bar{\theta}, \theta^\mu) = F_1(x^\mu, \theta^\mu) + \theta B_1(x^\mu, \theta^\mu) + \bar{\theta} B_2(x^\mu, \theta^\mu) + \theta \bar{\theta} F_2(x^\mu, \theta^\mu).$$

(3.51)
Then the condition (3.49) leads

\begin{align}
B_1 - i(F_1)^2 &= 0, \\
F_2 + i [F_1, B_2] &= 0, \\
[F_1, B_1] &= 0, \\
\{F_1, F_2\} + [B_1, B_2] &= 0,
\end{align}

while the other condition (3.50) leads

\begin{align}
B_2 = 0, \quad F_2 = 0.
\end{align}

These conditions (3.52)~(3.56) are solved as

\begin{align}
B_1 = i(F_1)^2, \quad B_2 = 0, \quad F_2 = 0.
\end{align}

Therefore the possible expression for $\Phi'$ turns out to be

\begin{align}
\Phi'(x^\mu, \theta, \tilde{\theta}, \theta^\mu) = \Psi'(x^\mu, \theta^\mu) + i\theta(\Psi'(x^\mu, \theta^\mu))^2,
\end{align}

where we have renamed $F_1(x^\mu, \theta^\mu)$ as $\Psi'(x^\mu, \theta^\mu)$ which satisfies the same form of the Abelian chiral condition (3.22) by definition,

\begin{align}
\frac{\partial}{\partial \theta} \Psi'(x^\mu, \theta^\mu) = 0, \quad \frac{\partial}{\partial \tilde{\theta}} \Psi'(x^\mu, \theta^\mu) = 0.
\end{align}

We can then obtain the non-Abelian version of chiral superfield as:

\begin{align}
\Phi(x^\mu, \theta, \tilde{\theta}, \theta^\mu) = \Psi(x^\mu, \theta, \tilde{\theta}, \theta^\mu) + U^{-1} \Psi'(x^\mu, \theta^\mu)U = \Psi'(z^\mu, \theta^\mu),
\end{align}

where $\Psi(x^\mu, \theta, \tilde{\theta}, \theta^\mu) \equiv U^{-1} \Psi'(x^\mu, \theta^\mu)U$ satisfies the same form of the Abelian chiral conditions as (3.19),

\begin{align}
D\Psi(x^\mu, \theta, \tilde{\theta}, \theta^\mu) = 0, \quad \bar{D}\Psi(x^\mu, \theta, \tilde{\theta}, \theta^\mu) = 0.
\end{align}

Since the non-Abelian version of the anti-chiral condition (3.48) has the same form as the Abelian condition (3.27), the solution of the anti-chiral superfield should have the similar form as Abelian case:

\begin{align}
\overline{\Psi}(x^\mu, \theta, \tilde{\theta}, \theta^\mu) = U \overline{\Psi}(x^\mu, \theta, \tilde{\theta})U^{-1} = \overline{\Psi}(z^\mu, \theta, \tilde{\theta}).
\end{align}

Similar to the Abelian case, the super transformations for the component fields can be read off from

\begin{align}
s_A \Phi'(x^\mu, \theta, \tilde{\theta}, \theta^\mu) &= Q'_A \Phi'(x^\mu, \theta, \tilde{\theta}, \theta^\mu) \\
s_A \overline{\Psi}(x^\mu, \theta, \tilde{\theta}) &= Q''_A \overline{\Psi}(x^\mu, \theta, \tilde{\theta}),
\end{align}
\[
\begin{align*}
\text{fields} & \quad s & \quad s_\rho & \quad \tilde{s} \\
\rho & -i\omega & -i\epsilon_{\mu\lambda} & 0 \\
\omega_\mu & \partial_\mu c + [\omega_\mu, c] & -i\epsilon_{\mu\rho} & -\epsilon_{\mu\nu}\partial_\nu c \\
\lambda & \epsilon_{\mu\nu}\partial_\mu\omega_\nu + \epsilon_{\mu\nu}\omega_\mu\omega_\nu - \{c, \lambda\} & 0 & -\partial_\mu\omega_\mu \\
c & -c^2 & 0 & -i\phi \\
b & -ib & \partial_\rho c & -i\rho \\
\phi & i\rho & -\epsilon_{\mu\nu}\partial_\nu c & 0 \\
\rho & 0 & -\partial_\mu\partial_\nu b & 0 \\
\end{align*}
\]

Table 3: \(N = 2\) twisted SUSY transformation of component fields for non-Abelian BF.

where \(Q'_A\) and \(Q''_A\) are, respectively, given by (3.36) and (3.37). More explicitly we can equivalently write as

\[
\begin{align*}
\Psi'(x^\mu, \theta^\mu) & = Q'_A \Psi'(x^\mu, \theta^\mu) + i \Psi'^2(x^\mu, \theta^\mu), \\
\tilde{s}_A & = Q'_A \tilde{s}, \\
\end{align*}
\]

(3.65)

\[
\begin{align*}
\tilde{s}_A (x^\mu, \theta, \tilde{\theta}) & = Q''_A \tilde{s}, \\
\tilde{s}_A (x^\mu, \theta, \tilde{\theta}) & = Q''_A \tilde{s}, \\
\end{align*}
\]

(3.66)

where \(\Psi'(x^\mu, \theta^\mu)\) and \(\tilde{\Psi}(x^\mu, \theta, \tilde{\theta})\) has the similar expansion form as the Abelian case (3.40).

The non-Abelian version of \(N = 2\) twisted SUSY transformation is given in Table 3. The non-Abelian version of \(N = 2\) twisted supersymmetric action can be constructed in the similar way as the Abelian case:

\[
S_{BF} = \int d^2x \int d^4\theta \text{Tr}(i\tilde{s} s) = \int d^2x \ s s^\dagger \frac{1}{2} \epsilon_{\mu\nu} s_\mu s_\nu \text{Tr}(-i\tilde{c}c) \\
= \int d^2x \text{Tr} \left[ \phi(\epsilon_{\mu\nu}\partial_\mu\omega_\nu + \epsilon_{\mu\nu}\omega_\mu\omega_\nu - \{c, \lambda\}) \\
+ b\partial_\mu\omega_\mu - i\tilde{c}\partial_\mu D_\mu c + i\rho\lambda \right],
\]

(3.67)

with \(D_\mu c \equiv \partial_\mu c + [\omega_\mu, c]\). This action can be identified as the quantized non-Abelian BF model with Landau gauge fixing accompanied by auxiliary fields.

4 Exact twisted SUSY on a lattice

The \(N = 2\) twisted SUSY algebra in two continuum dimensions has been reviewed in Section 3, and fully discussed in Ref [11]. In this section we formulate the twisted superspace on a lattice parallel to the continuum formulation. We introduce a mild noncommutativity to preserve the lattice Leibniz rule as we discussed in Section 2.
4.1 $N = D = 2$ twisted superspace on a lattice

Twisted $N = D = 2$ superspace is defined by the parameter space $(x^\mu, \theta^A)$ where the label $A$ can take four values. The differential operators $Q_A$ that generate infinitesimal SUSY transformations in superspace were given in Eq. (3.9) and the algebra they satisfy in Eq. (3.11). It will sometimes be convenient in this section, rather than writing the individual equations, to use a compact notation and write the differential operators as

$$Q_A = \frac{\partial}{\partial \theta^A} + \frac{i}{2} f^\mu_{AB} \theta^B \partial_\mu,$$

and the corresponding algebra as

$$\{Q_A, Q_B\} = -i f^\mu_{AB} \partial_\mu,$$

where the constants $f^\mu_{AB}$ are symmetric in $AB$ and can be read from (3.11).

As discussed in Section 2 the derivative operator $\partial_\mu$ is replaced on a square lattice by one of the finite difference operators $\Delta^+\mu$ and $\Delta^-\mu$ defined in Eqs (2.7) and (2.14). So if we denote the generators of the SUSY algebra (4.2) on the lattice by the same $Q_A$ then the algebra becomes:

$$\{Q_A, Q_B\} = -i f^\mu_{AB} \Delta^\pm_\mu.$$

The ambiguity at the r.h.s. of (4.3) is inherent to the lattice formulation and has to be removed by choosing, for each pair of values of $A$ and $B$ corresponding to a non zero value of $f^\mu_{AB}$, either $\Delta^+\mu$ or $\Delta^-\mu$. As we shall see shortly this can be done on consistency grounds. In view of the algebra (4.3) and of the commutators (2.8) and (2.16) it is natural to assume that the supercharges $Q_A$ act on a superfield $\mathcal{F}(x, \theta)$ according to Eq. (2.10), namely:

$$\{Q_A, \mathcal{F}(x, \theta)\} = T(2\hat{a}_A) s_A \mathcal{F}(x, \theta) = T(\hat{a}_A) s_A^{(s)} \mathcal{F}(x, \theta) T(\hat{a}_A),$$

where $s_A$ and $s_A^{(s)}$ are SUSY transformations (the latter symmetrized in the sense of Section 2) and $T(2\hat{a}_A)$ the shift operator associated to $Q_A$. The mixed bracket notation at the l.h.s. denotes a commutator or an anticommutator according to the Grassmann grading of $\mathcal{F}(x, \theta)$.

Consistency of Eq. (4.4) with the graded algebra (4.3) leads to a set of equations for the shifts $\hat{a}_A$ associated to the SUSY transformations. In fact, consider the Jacobi identity:\n
$$[[Q_A, Q_B], \mathcal{F}(x, \theta)] - \{[Q_B, \mathcal{F}(x, \theta)], Q_A\} + \{[\mathcal{F}(x, \theta), Q_A], Q_B\} = 0. \quad (4.5)$$

From the super algebra (4.3) and the commutators (4.4) and (2.8) we obtain:

$$-i f^\mu_{AB} T(\pm 2\hat{a}_\mu) \mathcal{Q}_\pm^\mu \mathcal{F}(x, \theta) + T(2\hat{a}_B) T(2\hat{a}_A) s_A s_A^{(s)} \mathcal{F}(x, \theta)$$

$$+ T(2\hat{a}_A) T(2\hat{a}_B) s_A s_B \mathcal{F}(x, \theta) = 0. \quad (4.6)$$

\footnote{We take $\mathcal{F}(x, \theta)$ here to be a bosonic superfield. Obvious changes in the signs would apply in the fermionic case.}
As shifts are additive this implies the following relations:

\[ \hat{a}_A + \hat{a}_B = \pm \hat{n}_\mu \quad \text{iff} \quad f_{AB}^\mu \neq 0, \quad (4.7) \]

and

\[ (\mathbf{s}_B \mathbf{s}_A + \mathbf{s}_A \mathbf{s}_B) \mathcal{F}(x, \theta) = i f_{AB}^\mu \partial_{\pm \mu} \mathcal{F}(x, \theta). \quad (4.8) \]

In both (4.7) and (4.8) the plus (resp. minus) sign at the r.h.s. is chosen if \( \Delta_{+\mu} \) (resp. \( \Delta_{-\mu} \)) appears at the r.h.s. of (4.3). Consider now the shift equations (4.7) in the specific case of the super algebra (3.11). By setting in (4.7) the values of \( A \) and \( B \) for which \( f_{AB}^\mu \neq 0 \) we obtain the following equations:

\[ i) \quad \hat{a} + \hat{a}_1 = \pm \hat{n}_1, \quad ii) \quad \hat{a} + \hat{a}_2 = \pm \hat{n}_2, \]

\[ iii) \quad \hat{a} + \hat{a}_1 = \pm \hat{n}_2, \quad iv) \quad \hat{a} + \hat{a}_2 = \pm \hat{n}_1. \quad (4.9) \]

In principle the signs at the r.h.s. of Eqs (4.9) can be chosen in an arbitrary way. However by comparing the linear combinations \( i) + iv) \) and \( ii) + iii) \) one finds that the resulting equations are compatible only if the signs in front of \( \hat{n}_1 \) in \( i) \) and \( iv) \) are opposite and at the same time the signs in front of \( \hat{n}_2 \) in \( ii) \) and \( iii) \) are also opposite. If these conditions are satisfied, the linear combinations \( i) + iv) \) and \( ii) + iii) \) are compatible and are equivalent to a unique condition, namely the vanishing of the sum of all shifts:

\[ \hat{a} + \hat{a}_1 + \hat{a}_2 + \hat{a} = 0. \quad (4.10) \]

The system (4.9) is then replaced by the condition (4.10) plus two of the equations (4.9), for instance \( i) \) and \( ii) \), that form with (4.10) a system of linearly independent equations. As the linearly independent equations are now only three, the solution will depend on an arbitrary vector. In other words, one of the shifts, for instance \( \hat{a}_A \), is not determined and can be chosen arbitrarily. Besides this arbitrariness, there are four possible sign choices in Eqs \( i) \) and \( ii) \) that will give four distinct solutions. Let us postpone the discussion about the meaning of such multiplicity of solutions and concentrate for the moment on a specific sign choice, choosing for instance the plus sign at the r.h.s. of both \( i) \) and \( ii) \). The linearly independent shift equations are then given by Eq. (4.10) and by

\[ i) \quad \hat{a} + \hat{a}_1 = +\hat{n}_1, \quad ii) \quad \hat{a} + \hat{a}_2 = +\hat{n}_2. \quad (4.11) \]

As already remarked this system does not determine \( \hat{a}_A \) completely and one shift, say \( \hat{a} \) can be treated as a free parameter. The whole formalism can be developed without ever specifying this free parameter and in what follows, unless otherwise specified, the formulas (such as for instance the different superfield expansions) will be valid for an arbitrary \( \hat{a} \). On the other hand it will be convenient, in order to represent the different component fields on a two dimensional square lattice, to make a particular choice for \( \hat{a} \). In this respect two choices appear to be particularly convenient: the first one is the most symmetric, in the sense that all vectors \( \hat{a}, \hat{a}_1, \hat{a}, \hat{a}_2 \)
have the same length and can be obtained from any of them by successive rotations of $\frac{\pi}{2}$:

\[ A) \quad \hat{a} \equiv (1/2, 1/2), \quad \hat{a} \equiv (-1/2, -1/2), \quad \hat{a}_1 \equiv (1/2, -1/2), \quad \hat{a}_2 \equiv (-1/2, 1/2). \tag{4.12} \]

The second choice, which we shall call “asymmetric”, corresponds to putting simply $\hat{a} = 0$ and gives:

\[ B) \quad \hat{a} \equiv (0, 0), \quad \hat{a} \equiv (-1, -1), \quad \hat{a}_1 = \hat{n}_1 \equiv (1, 0), \quad \hat{a}_2 = \hat{n}_2 \equiv (0, 1). \tag{4.13} \]

Let us consider now the three extra solutions that correspond to the three remaining choices of the signs in i) and ii). It is easy to check that these can be obtained from (4.12) by the mirror image reflection with respect to 1- and/or 2-axis. Now the problem is: why four distinct solutions? As long as one restricts oneself to the SUSY algebra (3.11), then one can consistently choose one solution, say (4.12), and forget about the others. However if one wants to implement on the lattice the $R$-symmetry and the discrete Lorentz rotations of multiples of $\pi/2$ (which is beyond the scope of the present paper) then all four solutions will have to be considered. In fact $R$-symmetry and discrete Lorentz rotations would mix the SUSY algebras on the lattice that correspond to the four different solutions. With the sign choice (4.11) all ambiguity in (4.3) is removed and we can explicitly write the non trivial anticommutators of the super algebra (3.11) on the lattice:

\[ \{Q, Q_\mu\} = i \Delta_{+\mu}, \quad \{\tilde{Q}, Q_\mu\} = -i \epsilon_{\mu\nu} \Delta_{-\nu}. \tag{4.14} \]

An explicit representation of the algebra (4.14) in terms of the $\theta^A$ variables can easily be found:

\[ Q = \frac{\partial}{\partial \theta} + \frac{i}{2} \theta^\mu \Delta_{+\mu}, \quad \tilde{Q} = \frac{\partial}{\partial \tilde{\theta}} - \frac{i}{2} \epsilon_{\mu\nu} \theta^\mu \Delta_{-\nu}, \]
\[ Q_\mu = \frac{\partial}{\partial \theta^\mu} + \frac{i}{2} \left( \theta^\lambda \Delta_{+\mu} - \tilde{\theta}^\lambda \epsilon_{\mu\nu} \Delta_{-\lambda} \right). \tag{4.15} \]

The supercharges (4.15) are consistent with the SUSY variations (4.4) only if the superfield $F(x, \theta)$ itself depends on the shift operators $T(2\hat{a}_A)$. It is not difficult to check that the correct dependence is obtained by simply replacing in $F(x, \theta)$ the Grassmann variables $\theta^A$ with “dressed” Grassmann variables $\tilde{\theta}^A$ defined as:

\[ \tilde{\theta}^A = T(2\hat{a}_A)\theta^A. \tag{4.16} \]

Notice that the new variables $\tilde{\theta}_A$ do not (anti)commute anymore with a field $\Psi(x)$:

\[ \tilde{\theta}^A \Psi(x + \hat{a}_A) = (-1)^{|\Psi|} \Psi(x - \hat{a}_A) \tilde{\theta}^A. \tag{4.17} \]

The expansion of a superfield $F(x, \tilde{\theta}_A)$ into its component fields has to now take into account the noncommutativity of the Grassmann variables $\tilde{\theta}^A$ shown in (4.17). As
a result the arguments of the component fields are sensitive to their relative order with respect to the Grassmann \( \theta \) variables:

\[
\mathcal{F}(x, \theta_A) = \varphi(x) + \theta_A \varphi_A(x + \hat{a}_A) + \frac{1}{2} \theta_A \theta_B \varphi_{AB}(x + \hat{a}_A + \hat{a}_B) + \cdots
\]

\[
= \varphi(x) + (-1)^{|x|} \varphi_A(x - \hat{a}_A) \theta_A + \frac{1}{2} \varphi_{AB}(x - \hat{a}_A - \hat{a}_B) \theta_A \theta_B + \cdots.
\]

(4.18)

The superfield \( \mathcal{F}(x, \theta_A) \) and its leading component \( \varphi(x) \) are defined on a lattice \( L(x \in L) \) which will be specified later. It is clear from (4.18) that the support of the other components are lattices shifted with respect to \( L \). For instance the field \( \varphi_A \) is defined on a lattice \( L_A \) which is shifted of \( \hat{a}_A \) with respect to \( L \).

So it is apparent from (4.18) that a superfield on the lattice is a non local object since the component fields of \( F(x, \theta_A) \) are spread on a cluster of points around \( x \). It is also true the vice versa in the sense that a component field on a definite site, for instance \( \varphi_A(x - \hat{a}_A) \), is not linked to a definite superfield but can appear as a component field either in the expansion of \( \mathcal{F}(x, \theta_A) \) or of \( \mathcal{F}(x - 2\hat{a}_A, \theta_A) \) according to which of the two expansions in (4.18) is used. Let us now determine the lattice \( L \). Consider first that the multiplication of a superfield by \( \theta_A \) has to be well defined in order to have a well defined multiplication between two superfields. Hence in the relation

\[
\theta_A \mathcal{F}(x, \theta) = (-1)^{|x|} \mathcal{F}(x - 2\hat{a}_A, \theta) \theta_A,
\]

(4.19)

both sides have to be well defined, and the r.h.s. makes sense only if from \( x \in L \) it follows \( x - 2\hat{a}_A \in L \) for any \( A \). Suppose now that the origin of the axis is a point of on \( L \), then all points of \( L \) are obtained as linear combinations, with integer even coefficients, of the fundamental shifts \( \hat{a}_A \). As a matter of fact, because of (4.10), these are not linearly independent and any three of them, for instance \( \hat{a} \) and \( \hat{a}_\mu \), can be chosen as basis:

\[
x \in L \quad \text{iff} \quad x = 2k\hat{a} + 2k_1\hat{a}_1 + 2k_2\hat{a}_2,
\]

(4.20)

which can also be written as

\[
x = 2(k - k_1 - k_2)\hat{a} + 2k_1\hat{n}_1 + 2k_2\hat{n}_2.
\]

(4.21)

Eq. (4.20) shows an interesting and unexpected feature. Although the original theory is two-dimensional the resulting lattice, corresponding to the most general choice of the shifts \( \hat{a}_A \), naturally lives in three dimensions as it is parameterized by the three integers \( \{k, k_1, k_2\} \). See Fig. 1. Of course only two of the dimensions are dynamical, in the sense that only finite differences in the \( \hat{n}_1 \) and \( \hat{n}_2 \) directions appear in the algebra and eventually in the Lagrangian. The third dimension, defined by the \( \hat{a} \) direction in (4.21), appears to be essentially related to the realization of SUSY although a deeper understanding of this point is probably needed. The lattices \( L_A, L_{AB} \) etc. on which the different components of the superfield are defined can be easily
obtained from (4.18) by inserting $x$ given in (4.20). It is convenient to relabel the component fields in the expansion of $\mathcal{F}(x, \theta)$ according to the following labeling of the expansion parameters: $\theta_A = (\theta_S, \theta_1, \theta_2) = (\theta, \theta_1, \theta_2)$. For instance the coefficient of $\hat{a}_A$ in the expansion of $\mathcal{F}(x, \theta)$ will be denoted by $\varphi_{\theta_1}$ and so on. We find then that:

i) Due to the vanishing sum of $\hat{a}_A$, Eq. (4.10), component fields with complementary sets of labels are defined on the same lattice. For instance $\varphi_S$ has the same support $L$ as $\varphi_1$, $\varphi_2$ the same as $\varphi_1$, and so on. Without loss of generality we can then restrict ourselves to component fields whose labels do not contain the index $P$.

ii) Consider then the field $\varphi_i$ with $i \in \{\emptyset, \{S\}, \{1\}, \{2\}, \{S1\}, \{S2\}, \{12\}, \{S12\}\}$. It is defined on the lattice $L_i$ whose points $y$ are of the form

$$y = m_S \hat{a} + m_1 \hat{a}_1 + m_2 \hat{a}_2,$$

(4.22)

where $m_S, m_1, m_2$ are odd integers if the label $i$ of the field contains the corresponding index and even integers otherwise. For instance if $i \equiv \{S1\}$ then $m_S$ and $m_1$ are odd and $m_2$ even. The union $L_0$ of all $L_i$’s, namely $L_0 = \bigcup_i L_i$ is made of all the points given in (4.22) with arbitrary integers $m_S, m_1, m_2$. A particular superfield can be associated with a fundamental cell of $L_0$, its component fields being associated in pairs its vertices. Moving of one unit in any of the three directions of the lattice (namely increasing of one unit any of the integers $m_S, m_1, m_2$) leads from a site occupied by a bosonic component field to one occupied by a fermionic one or vice versa. See Fig. 2.

The symmetric and asymmetric choices A) and B) of the shift parameters amount to impose an extra linear relation (respectively $\hat{a} = \frac{1}{2}(\hat{n}_1 + \hat{n}_2)$ and $\hat{a} = 0$) among the shifts, and hence to effectively reduce the dimension of the lattice from three to two. Consider for instance the symmetric parameter choice A). If we insert in (4.21) the relation $\hat{a} = \frac{1}{2}(\hat{n}_1 + \hat{n}_2)$ we find for the points $x$ of the lattice $L$ on which the superfield is defined:

$$x = (2k_1 + h)\hat{n}_1 + (2k_2 + h)\hat{n}_2,$$

(4.23)

where $h = k - k_1 - k_2$ is an arbitrary integer. Hence the coordinates of $x$ on the two dimensional plane are given by either two even integers (if $h$ is even) or two odd integers (if $h$ is odd). In this choice all shifts $\hat{a}_A$ have half integer coordinates.
Figure 3: Component fields on the lattice with the symmetric choice of shift parameter A) for $\mathcal{F}(x, \theta_A)$ where $\theta_A$ is located on (a) the left and (b) the right of the component fields.

(see Eq. (4.12)), it is then clear from (4.18) that the field $\varphi_A$ is defined on a lattice $L_A$ which is shifted with respect to $L$ and whose sites have half-integer coordinates. In general, as clearly shown in the expansion (4.18), with the symmetric choice A) if bosonic fields are defined on sites with integer coordinates fermionic fields have half-integer coordinates and vice versa. The position on the lattice of the different component fields of $\mathcal{F}(x, \theta_A)$ for the symmetric parameter choice A) is shown in Fig.3, where (a) and (b) refer to the two expansions of Eq. (4.18).

In the asymmetric parameter choice B) ($\hat{a} = 0$) we have simply from (4.21):

$$x = 2k_1 \hat{n}_1 + 2k_2 \hat{n}_2,$$

hence the sites of $L$ are all of the (even,even) type. As all shifts $\hat{a}_A$ have integer coordinates bosonic and fermionic component fields are all defined on integer sites and the same site has in general both bosonic and fermionic fields. This is shown in Fig.4 where the position on the lattice of the different component fields of $\mathcal{F}(x, \theta_A)$ is shown.

Following the pattern already introduced in Section 2, Eq. (2.11), it is convenient to define the "arrowed" supercharges $\vec{Q}_A$:

$$Q_A = T(2\hat{a}_A) \vec{Q}_A,$$

which have an expression in terms of $\theta^A$:

$$\vec{Q} = \frac{\partial}{\partial \theta^A} + i \frac{\partial}{\partial \bar{\theta}} \vec{\Delta}_{+}, \quad \vec{Q} = \frac{\partial}{\partial \bar{\theta}} + i \frac{\partial}{\partial \theta^A} \vec{\Delta}_{-},$$

$$\vec{Q}_\mu = \frac{\partial}{\partial \theta^A} + i \frac{\partial}{\partial \bar{\theta}} \left( \bar{\theta} \vec{\Delta}_{+} - \epsilon_{\mu \nu} \theta^A \vec{\Delta}_{-} \right).$$

$$\text{(4.26)}$$
while for the "symmetrized" variations $s$ in by calculating both side of the equation and equating the coefficients of the expansion

Using the $Q_A$ charges SUSY transformations $s_A$ on a lattice can be written as shifted commutators as in (2.12):

$$s_A \mathcal{F}(x, \theta) = Q_A \mathcal{F}(x, \theta) - (-1)^{|\mathcal{F}|} \mathcal{F}(x + 2\hat{a}_A, \theta) \overline{Q}_A,$$

while for the "symmetrized" variations $s_A^{(s)}$ we have:

$$s_A^{(s)} \mathcal{F}(x, \theta) = Q_A \mathcal{F}(x - \hat{a}_A, \theta) - (-1)^{|\mathcal{F}|} \mathcal{F}(x + \hat{a}_A, \theta) \overline{Q}_A.$$

It is worth to note here that the location of the lattice position of the component fields in $s_A^{(s)} \mathcal{F}(x, \theta)$ is not necessary on $x$. Eqs (4.27) or (4.28) can be used to calculate the SUSY transformations of the component fields of the superfield $\mathcal{F}(x, \theta)$ by calculating both side of the equation and equating the coefficients of the expansion in $\theta_A$. Let us write the expansion of $\mathcal{F}(x, \theta)$ using the same notations as in the continuum case (Eq. (3.12)):

$$\mathcal{F}(x, \theta) = \phi(x) + \theta^\mu \dot{\phi}_\mu(x + \hat{a}_\mu) + \theta^2 \ddot{\phi}(x + \hat{a}_1 + \hat{a}_2) + \theta(\psi(x + \hat{a}) + \theta^\mu \dot{\psi}_\mu(x + \hat{n}_\mu) + \theta^2 \ddot{\psi}(x - \hat{a})) + \theta(\chi(x + \hat{a}) + \theta^\mu \dot{\chi}_\mu(x - |\epsilon_{\mu\nu}|\hat{\nu}_\mu) + \theta^2 \ddot{\chi}(x - \hat{a})) + \theta(\bar{\lambda}(x + \hat{a} + \hat{\bar{a}}) + \theta^\mu \bar{\lambda}_\mu(x + \hat{n}_\mu + \hat{\bar{a}}) + \theta^2 \bar{\lambda}(x)).$$

The position on the lattice of the different component fields of $\mathcal{F}(x, \theta)$ can be recognized by the same figures for (4.18): Figs: 3, 4 with the identifications: $(\varphi) = (\bar{\phi})$, $(\varphi_{S_1}, \varphi_{S_1^P}, \varphi_{S_2}, \varphi_{S_2^P}) = (\psi, \dot{\psi}_1, \dot{\psi}_2, \ddot{\psi}), (\varphi_{S_1}, \varphi_{S_2}, \varphi_{S_{1P}}, \varphi_{S_{2P}}, \varphi_{S_{12P}}) = (\psi_1, \psi_2, \lambda, \dot{\phi}_1, \dot{\phi}_2, \dot{\phi}_3, \ddot{\lambda}, \ddot{\phi}_1, \ddot{\phi}_2, \ddot{\phi}_3), (\varphi_{S_{12}}, \varphi_{S_{12P}}, \varphi_{S_{12P}}, \varphi_{S_{12P}}) = (\psi, \lambda_1, \lambda_2, \lambda_3), (\varphi_{S_{12P}}) = (\bar{\lambda}).$
By inserting the expansion above into Eq. (4.28) and using the explicit form of the supercharges $\overrightarrow{Q}_A$, one obtains the SUSY transformations of the different components of $\Phi(x, \theta)$. These are given in Table 4. The arguments of the fields in Table 4 should be the appropriate ones corresponding to the expansion (4.29).

| $\phi^A$ | $s_0^A \phi^A$ | $s_\mu^A \phi^A$ | $\tilde{s}_\mu^A \phi^A$ |
|----------|-----------------|-----------------|-----------------|
| $\phi$ | $\bar{\phi}$ | $\bar{\phi}_\mu$ | $\bar{\chi}$ |
| $\phi_\rho$ | $-\bar{\psi}_\rho - \frac{i}{2} \partial^{(s)} \overrightarrow{\phi}_\rho$ | $-\epsilon_{\mu\rho} \bar{\phi}$ | $-\chi_\rho + \frac{i}{2} \epsilon_{\mu\rho} \partial^{(s)} \overrightarrow{\phi}_\rho$ |
| $\bar{\phi}$ | $\bar{\psi} + \frac{i}{2} \epsilon_{\rho\sigma} \partial^{(s)} \overrightarrow{\phi}_\rho \bar{\phi}_\sigma$ | 0 | $\lambda - \frac{i}{2} \partial^{(s)} \overrightarrow{\phi}_\rho$ |
| $\bar{\psi}$ | 0 | $\bar{\psi}_\mu - \frac{i}{2} \partial^{(s)} \overrightarrow{\phi}_\mu$ | $\lambda$ |
| $\phi_\rho$ | $-\frac{i}{2} \partial^{(s)} \overrightarrow{\phi}_\rho \bar{\psi}$ | $-\epsilon_{\mu\rho} \bar{\psi} + \frac{i}{2} \partial^{(s)} \overrightarrow{\phi}_\rho \bar{\phi}_\mu$ | $-\lambda + \frac{i}{2} \epsilon_{\mu\rho} \partial^{(s)} \overrightarrow{\phi}_\rho \bar{\psi}$ |
| $\overrightarrow{\lambda}$ | $-\lambda$ | $\lambda_\mu + \frac{i}{2} \epsilon_{\mu\rho} \partial^{(s)} \overrightarrow{\phi}_\rho \bar{\phi}_\mu$ | 0 |
| $\lambda_\rho$ | $\lambda_\rho - \frac{i}{2} \partial^{(s)} \overrightarrow{\phi}_\rho \overrightarrow{\lambda}$ | $-\epsilon_{\mu\rho} \lambda_\mu - \frac{i}{2} \epsilon_{\mu\rho} \partial^{(s)} \overrightarrow{\phi}_\rho \bar{\phi}_\mu + \frac{i}{2} \epsilon_{\mu\rho} \partial^{(s)} \overrightarrow{\phi}_\rho \bar{\psi} \overrightarrow{\phi}_\mu$ | 0 |
| $\overrightarrow{\lambda}$ | $\lambda + \frac{i}{2} \epsilon_{\rho\sigma} \partial^{(s)} \overrightarrow{\phi}_\rho \overrightarrow{\lambda}_\sigma$ | $\partial^{(s)} \lambda_\mu + \frac{i}{2} \epsilon_{\mu\rho} \partial^{(s)} \overrightarrow{\phi}_\rho \bar{\phi}_\mu \overrightarrow{\lambda}_\mu + \frac{i}{2} \epsilon_{\mu\rho} \partial^{(s)} \overrightarrow{\phi}_\rho \overrightarrow{\phi}_\mu \overrightarrow{\lambda}_\mu$ | 0 |

Table 4: $N = 2$ twisted SUSY transformation of the component fields of the superfield $\Phi(x, \theta)$ on a lattice.

The SUSY transformations on the lattice of Table 4 coincide with the ones obtained in the continuum (see Table 1) by defining $s_A F(x^\mu, \theta, \theta^\mu, \bar{\theta}) = \{Q_A, F(x^\mu, \theta, \theta^\mu, \bar{\theta})\}$ provided we replace the fields in the continuum with the corresponding underlined fields on the lattice, we make the appropriate shifts in the fields’ arguments and we replace the partial derivative $\partial_\mu$ with the symmetrized finite difference $\partial^{(s)}_\mu$.

(4.30)
Finally let us consider the Leibniz rule, which follows from the Leibniz rule for the (anti)commutator (4.4) or more directly from Eqs (4.27) or (4.28):

\[
\begin{align*}
\mathcal{L}_A(F_1(x,\theta) \ F_2(x,\theta)) &= (\mathcal{L}_A F_1(x,\theta)) \ F_2(x,\theta) \\
&\quad + (-1)^{\bar{F}_1} F_1(x + 2\hat{a}_A,\theta) \ (\mathcal{L}_A F_2(x,\theta)), \quad (4.31) \\
\mathcal{L}_A^{(s)}(F_1(x,\theta) \ F_2(x,\theta)) &= (\mathcal{L}_A^{(s)} F_1(x,\theta)) \ F_2(x - \hat{a}_A,\theta) \\
&\quad + (-1)^{\bar{F}_1} F_1(x + \hat{a}_A,\theta) \ (\mathcal{L}_A^{(s)} F_2(x,\theta)). \quad (4.32)
\end{align*}
\]

4.2 Chiral superfields

Chiral and anti-chiral conditions were defined in the continuum using the differential operators which generate the shift transformation induced by right multiplication in superspace. They are given explicitly in Eq. (3.16) which can be summarized in the compact notation of Eq. (4.1) as:

\[
D_A = \frac{\partial}{\partial \theta} - \frac{i}{2} f^\mu_{AB} \theta^B \partial_\mu.
\]

They anticommute with the SUSY generators \(Q_A\) and satisfy the same algebra, up to a sign: \(\{D_A, D_B\} = -\{Q_A, Q_B\} = i f^\mu_{AB} \partial_\mu\). The definition of \(D_A\) on a square lattice proceeds in the same way as for \(Q_A\). If we denote by \(D_A\) the corresponding lattice operators we have, in analogy with (4.3)

\[
\{D_A, D_B\} = i f^\mu_{AB} \Delta_{\pm\mu}.
\]

(4.33)

The sign ambiguity is resolved in exactly the same way as for the SUSY generators \(Q_A\), and in analogy with Eq. (4.14) we find:

\[
\{D, D_\mu\} = -i \epsilon_{\mu\nu} \Delta_{-\nu}, \quad \{\bar{D}, D_\mu\} = i \epsilon_{\mu\nu} \Delta_{-\nu},
\]

(4.34)

with all the other anticommutators vanishing. Again an explicit expression for \(D_A\) satisfying (4.35) can easily be found:

\[
\begin{align*}
D &= \frac{\partial}{\partial \theta} - \frac{i}{2} \theta^\mu \Delta_{+\mu}, \\
\bar{D} &= \frac{\partial}{\partial \theta} + \frac{i}{2} \epsilon_{\mu\nu} \theta^\mu \Delta_{-\nu}, \\
D_\mu &= \frac{\partial}{\partial \theta^\mu} - \frac{i}{2} \left( \theta \Delta_{+\mu} - \theta \epsilon_{\mu\nu} \Delta_{-\nu} \right).
\end{align*}
\]

(4.35)

Alternatively, as in the case of the supercharges \(Q_A\) it is possible to introduce the arrowed differential operators \(\vec{D}_A\) defined by

\[
D_A = T(2\hat{a}_A) \vec{D}_A.
\]

(4.36)

Their explicit form is quite similar to the one given for \(Q_A\) in (4.26), namely:

\[
\begin{align*}
\vec{D} &= \frac{\partial}{\partial \theta} - \frac{i}{2} \theta^\mu \Delta_{+\mu}, \\
\vec{\bar{D}} &= \frac{\partial}{\partial \theta} + \frac{i}{2} \epsilon_{\mu\nu} \theta^\mu \Delta_{-\nu}, \\
\vec{D}_\mu &= \frac{\partial}{\partial \theta^\mu} - \frac{i}{2} \left( \theta \Delta_{+\mu} - \epsilon_{\mu\nu} \Delta_{-\nu} \right).
\end{align*}
\]

(4.37)
It is convenient, as in the continuum case, to write the differential operators $D_A$ as

$UDU^{-1} = \frac{\partial}{\partial \theta}$, $U\bar{D}U^{-1} = \frac{\partial}{\partial \bar{\theta}}$, $U^{-1}D_\mu U = \frac{\partial}{\partial \theta^\mu}$, \hspace{1cm} (4.39)

where the operator $U$ is given by

$U = e^{-\frac{\imath}{2}(\theta^\mu \Delta_{\mu\nu} - \bar{\theta}^\mu \theta^\nu \Delta_{\mu\nu})} = e^{-\frac{\imath}{2}(\theta^\mu \bar{\Delta}_{\mu\nu} - \bar{\theta}^\mu \theta^\nu \bar{\Delta}_{\mu\nu})}$. \hspace{1cm} (4.40)

We shall now consider chiral and anti-chiral superfields on the lattice, which we shall denote respectively $\Psi(x, \theta)$ and $\bar{\Psi}(x, \bar{\theta})$. They are defined by imposing on a generic superfield $F(x, \theta)$ the chiral and anti-chiral conditions, namely:

$D \Psi(x, \theta) \equiv [D, \Psi(x, \theta)] = 0$, \hspace{1cm} $\bar{D} \bar{\Psi}(x, \bar{\theta}) \equiv [\bar{D}, \bar{\Psi}(x, \bar{\theta})] = 0$, \hspace{1cm} (4.41)

$D^\mu \bar{\Psi} \equiv [D^\mu, \bar{\Psi}(x, \bar{\theta})] = 0$. \hspace{1cm} (4.42)

The general solution of the chiral and anti-chiral conditions (4.41) and (4.42) can be found by using the form (4.39) of the differential operators $D_A$. In fact if we define

$U \Psi(x, \theta) U^{-1} = \Psi'(x, \theta)$, \hspace{1cm} (4.43)

$U^{-1} \overline{\Psi}(x, \bar{\theta}) U = \overline{\Psi}'(x, \bar{\theta})$, \hspace{1cm} (4.44)

the conditions (4.41) and (4.42) respectively become:

$\frac{\partial}{\partial \theta} \Psi'(x, \theta) = 0$, $\frac{\partial}{\partial \bar{\theta}} \overline{\Psi}'(x, \bar{\theta}) = 0$, \hspace{1cm} (4.45)

and

$\frac{\partial}{\partial \theta^\mu} \overline{\Psi}'(x, \bar{\theta}) = 0$. \hspace{1cm} (4.46)

Hence the reduced superfields $\Psi'(x, \theta)$ and $\overline{\Psi}'(x, \bar{\theta})$ are functions only of $(x, \theta^\mu)$ and of $(x, \bar{\theta}, \bar{\theta})$ respectively:

$\Psi'(x, \theta) = \phi(x) + \theta^\mu \psi(x + \hat{a}_\mu) + \frac{1}{2} \epsilon^\mu_\nu \theta^{\nu} \tilde{\phi}(x + \hat{a}_1 + \hat{a}_2)$, \hspace{1cm} (4.47)

$\overline{\Psi}'(x, \bar{\theta}) = \varphi(x) + \bar{\theta} \chi(x + \hat{a}) + \tilde{\bar{\theta}} \tilde{\chi}(x + \hat{a}) + \bar{\theta} \theta \bar{\phi}(x + \hat{a} + \hat{\bar{a}})$. \hspace{1cm} (4.48)

Finally, by inserting (4.47) and (4.48) respectively into (4.43) and (4.44) a straightforward calculation gives the explicit expansion of a chiral and anti-chiral superfield on the lattice:

$\Psi(x, \theta) = \phi(x) + \theta^\mu \psi(x + \hat{a}_\mu) + \frac{1}{2} \epsilon_{\mu\nu} \theta^\nu \tilde{\phi}(x + \hat{a}_1 + \hat{a}_2) + \frac{i}{2} \theta \theta^{\nu} \partial^{(s)}_{\nu} \phi(x + \hat{n}_\nu)$

$- \frac{i}{2} \epsilon_{\mu\nu} \theta^\mu \partial^{(s)}_{\nu} \phi(x + \hat{n}_\nu) + \frac{i}{4} \epsilon_{\mu\nu} \theta^\mu \epsilon^{\nu}_{\rho\sigma} \partial^{(s)}_{\rho} \partial^{(s)}_{\sigma} \psi(x + \hat{n}_1 + \hat{a}_2)$

$\frac{i}{4} \epsilon_{\mu\nu} \theta^\mu \epsilon^{\nu}_{\rho\sigma} \partial^{(s)}_{\rho} \partial^{(s)}_{\sigma} \phi(x)$, \hspace{1cm} (4.49)
\[ \Psi(x, \theta) = \varphi(x) + \theta \chi(x + \hat{a}) + \tilde{\theta} \tilde{\chi}(x + \tilde{a}) + \partial \tilde{\theta} \tilde{\varphi}(x + \hat{a} + \tilde{a}) \]

\[ - \frac{i}{2} \bar{\theta} \partial^{(s)} \varphi(x + \hat{n}_\mu) + \frac{i}{2} \bar{\theta} \epsilon_{\mu\nu} \partial^{(s)} \varphi(x - \hat{n}_\nu) + \frac{i}{2} \bar{\theta} \epsilon_{\mu\nu} \partial^{(s)} \chi(x + \hat{n}_\mu + \hat{a}) \]

\[ + \frac{i}{2} \bar{\theta} \partial^{(s)} \tilde{\chi}(x + \hat{n}_\mu + \hat{a}) + \frac{1}{4} \bar{\theta} \epsilon_{\mu\nu} \partial^{(s)} \tilde{\varphi}(x). \]  

(4.50)

Again the above expansion of chiral and anti-chiral superfields are exactly the same as one would obtain in the continuum, with the partial derivative replaced by the symmetric finite difference on the lattice and suitable shifts in the arguments of the fields.

5 \textbf{N = 2 SUSY invariant BF and Wess-Zumino actions on the lattice}

Based on the formulation of \( N = 2 \) twisted superspace formalism on the lattice, we can explicitly construct SUSY invariant BF and Wess-Zumino actions on the lattice. The construction proceeds quite parallel to the continuum formulation with the introduction of suitable noncommutative shifts for the Abelian case. In the case of non-Abelian extension we need to take care of the nontrivial shift dependence of the superfield.

5.1 Abelian super BF in two dimensions

We first consider the case of fermionic chiral and anti-chiral superfield which leads to the twisted \( N = 2 \) SUSY invariant quantized BF action on the lattice. We expand the reduced superfield \( \Psi'(x, \theta, \bar{\theta}) \) and \( \bar{\Psi}'(x, \bar{\theta}) \) given in (4.47) and (4.48) with the same renaming of the fields as the continuum expression (3.40):

\[ \Psi'(x^\mu, \theta^\mu) = U \Psi(x^\mu, \theta^\mu, \bar{\theta}^\mu) U^{-1} = c_{\theta^\mu} \bar{\phi}(x) \]

\[ = i \bar{\phi}(x) + i \theta^\mu \bar{\phi}(x) + i \partial^\mu \bar{\phi}(x) \]

\[ \bar{\Psi}'(x^\mu, \bar{\theta}, \bar{\bar{\theta}}) = U^{-1} \bar{\Psi}(x^\mu, \theta, \theta^\mu, \bar{\bar{\theta}}) U = \bar{c}_{\theta^\mu} \phi(x) \]

\[ = i \phi(x) + i \theta \phi(x) + i \partial \phi(x) \]

\[ = i \phi(x) + i \theta \phi(x) + i \partial \phi(x) + i \partial \phi(x) - i \partial \phi(x) + i \partial \phi(x). \]  

(5.1)

These are the lattice counterpart of the relations in (3.38) and provide additional reasonings why the component fields in the chiral and anti-chiral superfields need the corresponding shifts. Since the supercharge \( s_A \) carries the noncommutative shift \( 2\hat{a}_A \), the lattice coordinate of \( s_A \varphi(x) \) is naturally correlated with the \( \hat{a}_A^\mu \) direction. In order to keep the symmetric nature of the formulation as we have shown in the last section of (4.18), we define the new field \( \varphi(x + \hat{a}_A) = s_A \varphi(x) \) on the center of the coordinate between \( x^\mu \) and \( x^\mu + 2\hat{a}_A^\mu \). See Fig. 5.
Compared with the general superfield of (4.29), the chiral and anti-chiral superfields have smaller number of elementary components which are again nonlocally scattered within the double size lattice. For the cases of the symmetric choice of the shift parameter A) in (4.12) and the asymmetric choice B) in (4.13), we show how the component fields of the chiral and anti-chiral superfields are scattered on the lattice in Fig. 6 and Fig. 7, respectively. For the case of the symmetric choice of the parameter A), the fermionic fields are on the integer lattice sites or equivalently the original lattice sites while the bosonic fields are on the half integer lattice namely on the dual lattice sites. For the case of the asymmetric parameter choice B), fermions and bosons are mixed up on the integer lattice sites. It is interesting to note that the nonlocal extension of the fields is limited in the double size lattice for both cases. This structure of the locations of component fields on the lattice is expected from the general arguments of Section 4.

The SUSY transformation of the component fields of the chiral and anti-chiral
superfields on the lattice can be obtained from

\[
\begin{align*}
\mathcal{S}_A \Psi'(x^\mu, \theta^\mu) &= \overrightarrow{Q}'_A \overrightarrow{\Psi}'(x^\mu, \theta^\mu) + \overrightarrow{\Psi}'(x^\mu + 2 \hat{a}_A, \theta^\mu) \overrightarrow{Q}'_A, \\
\mathcal{S}_A \overrightarrow{\Psi}'(x^\mu, \theta, \bar{\theta}) &= \overrightarrow{Q}''_A \overrightarrow{\Psi}'(x^\mu, \theta, \bar{\theta}) + \overrightarrow{\Psi}'(x^\mu + 2 \hat{a}_A, \theta, \bar{\theta}) \overrightarrow{Q}''_A,
\end{align*}
\]

(5.2)

where the lattice version of \( \overrightarrow{Q}'_A \) and \( \overrightarrow{Q}''_A \) in (3.37) are given by

\[
\begin{align*}
\overrightarrow{Q}' &= \frac{\partial}{\partial \theta^\mu} + i \theta^\mu \overrightarrow{\Delta}_{+\mu}, \\
\overrightarrow{Q}' &= \frac{\partial}{\partial \bar{\theta}^\mu} - i \theta^\mu \epsilon_{\mu\nu} \overrightarrow{\Delta}_{-\nu}, \\
\overrightarrow{Q}'' &= \frac{\partial}{\partial \theta^\mu}, \\
\overrightarrow{Q}'' &= \frac{\partial}{\partial \bar{\theta}^\mu} + i \theta^\mu \overrightarrow{\Delta}_{+\mu} - i \bar{\theta} \epsilon_{\mu\nu} \overrightarrow{\Delta}_{-\nu}.
\end{align*}
\]

(5.3)

(5.4)

\( N = 2 \) twisted SUSY transformation for the component fields of chiral and anti-chiral fields on the lattice is given in Table 5, where the symmetric difference operator defined in (2.24) is used. Each of the term in the table has natural geometrical meaning. For example \( \mathcal{S}_A \mathcal{C}(x) \) defines a new field \(-i \omega_{\mu}(x + \hat{a}_\mu)\) on the lattice site \(x + \hat{a}_\mu\) as we can see in Fig.6. Then \( \mathcal{S}_A \omega_{\mu}(x + \hat{a}_\mu) \) defines the symmetric difference of the field \( \mathcal{C} \) on the site \(x + \hat{a}_\mu + \hat{a} = x + \hat{n}_\nu\) in \(\nu\) direction.

The twisted \( N = 2 \) SUSY invariant BF action is obtained from a bilinear form
of the lattice version of the chiral and anti-chiral fields as in the continuum:

$$S_{BF}^{lat} = \int d^4\theta \sum_x \bar{\Psi}(x, \theta, \theta_\mu, \tilde{\theta}) \Psi(x, \theta, \theta_\mu, \tilde{\theta})$$

$$= \sum_x s\bar{s}s_1s_2(-i\bar{c}(x)c(x))$$

$$= \sum_x \left[ \bar{\phi}(x - \hat{a})\epsilon_{\mu\nu}\partial_{\mu}^{(s)}\omega_{\nu}(x - \hat{a}) + \bar{b}(x - \hat{a})\partial_{\mu}^{(s)}\omega_{\mu}(x - \hat{a}) \\
- i\bar{c}(x)\partial_{\mu}^{(s)}\partial_{\nu}^{(s)}c(x) + i\bar{\rho}(x - \hat{a})\Delta(x - \hat{a} - \hat{a}) \right]$$

$$= \sum_x \left[ \bar{\phi}(x - \hat{a})\epsilon_{\mu\nu}\partial_{\mu+\nu}\omega_{\nu}(x + \hat{a}) + \bar{b}(x - \hat{a})\partial_{\mu-\nu}\omega_{\mu}(x + \hat{a}) \\
- i\bar{c}(x)\partial_{\mu+\nu}\partial_{\mu-\nu}c(x) + i\bar{\rho}(x - \hat{a})\Delta(x - \hat{a} - \hat{a}) \right],$$

where two equivalent expressions with symmetric difference operators and with forward and backward difference operators are given. In the case of the symmetric difference operator the assigned field coordinate is located in the middle of the coordinate of the difference fields while in the case of the forward and backward difference operator the assigned field coordinate is the ending and starting site of the corresponding difference. The twisted $N = 2$ SUSY invariance up to the surface terms is obvious since the action has an exact form with respect to the twisted lattice supercharges. We show the component fields of super BF action on the lattice for the symmetric choice of the shift parameters $A$ in Fig.8.

Component fields except for the leading one are generated by operating supercharges on the leading and carry the same noncommutativity as supercharges. For example, $\omega_{\mu}(\hat{a}_\mu) = i\bar{c}(x)$ carries the noncommutative shift $2\hat{a}_\mu$ in addition to a possible shift carried by $c(x)$. Here we assume $c(x)$ and $\bar{c}(x)$ do not carry a shift. Then we should be careful when interchanging the order of product. In the action, however, we can freely interchange the field cyclicly without shift of the arguments:

$$S_{BF}^{lat} \sim \sum_x \left[ s\bar{c}(x + 2\hat{a} + 2\hat{a}_1 + 2\hat{a}_2)\bar{s}s_1s_2c(x) + \cdots \right]$$

$$= \sum_x \left[ \bar{s}s_1s_2c(x + 2\hat{a})s\bar{c}(x) + \cdots \right]$$

$$= \sum_x \left[ \bar{s}s_1s_2c(x)s\bar{c}(x - 2\hat{a}) + \cdots \right].$$

In the first equality, we change the order of the product by taking into account the noncommutativity. Next we shift the argument in the summation. Since the shift parameters satisfy eq.(4.10),

$$\hat{a} + \hat{a}_1 + \hat{a}_2 + \hat{a} = 0,$$

$-2\hat{a}$ in eq.(5.11) can be replaced by $2\hat{a} + 2\hat{a}_1 + 2\hat{a}_2$ and thus we recover the original argument of $\bar{c}$. It is straightforward to generalize this argument into a cyclic interchange of three or more fields.
Figure 8: Component fields of Abelian super BF action on the lattice for the symmetric choice of the shift parameters \( A \).

It is interesting to note that each term in the action has geometrical meaning. For example the field \( \phi(x - \hat{a}) \) is located in the middle of the surrounding fields \( \omega_2(x - \hat{a} \pm \hat{n}_1) \) and \( \omega_1(x - \hat{a} \pm \hat{n}_2) \) which define the rotation \( \epsilon_{\mu\nu} \partial_\mu \omega_\nu(x - \hat{a}) \). The field \( b(x - \hat{a}) \) is located at the center of a cross composed by \( \omega_1(x - \hat{a} \pm \hat{n}_1) \) and \( \omega_2(x - \hat{a} \pm \hat{n}_2) \) which define the divergence \( \partial_- \omega_\mu(x + \hat{a}_\mu) \). In the case of the symmetric parameter choice \( A \), \( \bar{c}(x), \phi(x) \) and \( b(x - \hat{a} - \hat{\bar{a}}), \Delta(x - \hat{a} - \hat{\bar{a}}) \) are located in the same lattice site \( x \) since \( \hat{a} + \hat{\bar{a}} = 0 \).

It is important to recognize that \( \bar{c}(x) \) and \( (\phi(x - \hat{a}), b(x - \hat{a})) \) are in the same coordinate sum of the action (5.8) while \((\phi(x - \hat{a}), b(x - \hat{a}))\) are not included in the component expansion of the anti-chiral superfields \( \bar{\Psi}(x, \theta, \tilde{\theta}) \) in (5.1) where \( \bar{c}(x) \) is the leading component. Instead \( \phi(x - \hat{\bar{a}}) \) and \( b(x - \hat{\bar{a}}) \) belong to the component expansion of \( \bar{\Psi}(x^\mu + \hat{n}_1 + \hat{n}_2, \theta, \tilde{\theta}) \) and \( \bar{\Psi}(x^\mu - \hat{n}_1 - \hat{n}_2, \theta, \tilde{\theta}) \), respectively. Therefore for the symmetric choice \( A \) the summation of the action should cover the coordinates of \((\text{even,even})=(2n_1,2n_2)\) and \((\text{odd,odd})=(2n_1+1,2n_2+1)\) with \( n_1, n_2 \in \mathbb{Z} \):

\[
\sum_x = \sum_{(\text{even,even})} + \sum_{(\text{odd,odd})} . \quad (5.13)
\]

As we have explained in Section 4, this structure of summation region of lattice sites naturally appears from the general arguments as well.

We also show the component fields of super BF action on the lattice for the
Figure 9: Component fields of Abelian super BF action on the lattice for the asymmetric choice of the shift parameters B).

asymmetric choice of the shift parameters B) in Fig.9 where only the summation of (even,even) sites appears for coordinate sum of the action, which can be understood from the general arguments in the last section.

5.2 $N = 2$ supersymmetric Wess-Zumino action in two dimensions

We next consider the bosonic version of the chiral and anti-chiral superfields (4.47) and (4.48):

$$
\overline{\Phi}'(x^\mu, \theta^\mu) = U \Phi(x^\mu, \theta, \bar{\theta}, \tilde{\theta}) U^{-1} = \phi(x) + \theta^\mu \psi(x + \hat{a}_\mu) + \bar{\theta}^2 \tilde{\phi}(x + \hat{a}_1 + \hat{a}_2)
$$

$$
\Phi'(x^\mu, \theta, \tilde{\theta}) = U^{-1} \overline{\Phi}(x^\mu, \theta, \theta^\mu, \bar{\theta}) U = \varphi(x) + \theta \chi(x + \hat{a}) + \bar{\theta} \tilde{\chi}(x + \hat{a}) + \theta \bar{\theta} \tilde{\varphi}(x + \hat{a} + \hat{a}),
$$

(5.14)

where we rename $\Psi' = \Phi'$, $\overline{\Psi'} = \overline{\Phi}'$ to show the bosonic nature of the superfields. Compared with the fermionic superfields, the Grassmann nature of the component fields in superfields is interchanged. Similar to the fermionic superfields, bosonic chiral and anti-chiral superfields include fermionic and bosonic component fields scattered within a double size lattice as shown in Fig.10 for the symmetric shift parameter choice A), and in Fig.11 for the asymmetric shift parameter choice B). In contrast with the component fields of the fermionic superfields the bosonic fields,
Figure 10: Component fields of bosonic chiral and anti-chiral superfields for the symmetric parameter choice A).

Figure 11: Component fields of bosonic chiral and anti-chiral superfields for the asymmetric choice of shift parameter B).

\( \phi(x), \bar{\phi}(x), \varphi(x) \) and \( \bar{\varphi}(x) \) are located on the integer sites or equivalently on the original lattice sites while the fermionic fields are on the dual sites for the symmetric shift parameter choice A). The interchange of relative position of fermionic and bosonic fields on the lattice for the asymmetric shift parameter choice B) works similar to the symmetric case.

The \( N = 2 \) twisted SUSY transformation of component fields of superfields can be obtained similar to the fermionic version by (5.2) with the same lattice supercharge operators as (5.3) and (5.4). We list the SUSY transformation of the component fields on the lattice in Table 6. The relative location between the original field and the transformed field has a natural geometrical interpretation due to the shifting nature of the twisted supercharge operators like the fermionic superfields.

We can now obtain \( N = 2 \) twisted super symmetric action by the bilinear product of the chiral and anti-chiral bosonic superfields which has an exact form with respect
Table 6: N=2 twisted SUSY transformation of component fields for the bosonic superfield.

\[
\begin{array}{|c|c|c|c|}
\hline
\phi^A & s\phi^A & s_{\mu}\phi^A & \tilde{s}\phi^A \\
\hline
\phi(x) & 0 & \psi_\mu(x + \hat{a}_\mu) & 0 \\
\psi_\nu(x + \hat{a}_\nu) & -i\partial_{\nu}\phi(x) & -\epsilon_{\mu\nu}\phi(x + \hat{a}_1 + \hat{a}_2) & i\epsilon_{\nu\mu}\partial^-\phi(x) \\
\tilde{\phi}(x + \hat{a}_1 + \hat{a}_2) & i\epsilon^{\mu\nu}\partial_{\nu}\psi_\nu(x + \hat{a}_\nu) & 0 & -i\partial^-\psi^\mu(x + a_\mu) \\
\varphi(x) & \chi(x + \hat{a}) & 0 & \tilde{\chi}(x + \hat{a}) \\
\chi(x + \hat{a}) & \tilde{\chi}(x + \hat{a}) & -i\partial_{\nu}\varphi(x) & i\epsilon_{\nu\mu}\partial^-\varphi(x) \\
\tilde{\varphi}(x + \hat{a} + \hat{a}) & 0 & i\epsilon_{\mu\nu}\partial^-\chi(x + \hat{a}) + i\partial^\nu\tilde{\chi}(x + \hat{a}) & 0 \\
\hline
\end{array}
\]

to all \( N = 2 \) twisted supercharges:

\[
S_{WZ}^{\text{lat}} = \int d^4\theta \sum_x \overline{\Phi}(x, \theta, \theta_\mu, \tilde{\theta}) \Phi(x, \theta, \theta_\mu, \tilde{\theta}) \\
= \frac{s\tilde{s}s_1s_2}{s} \sum_x \varphi(x)\phi(x) \\
= \sum_x \left[ \varphi(x)\partial^\mu\partial^-\phi(x) + \tilde{\varphi}(x + \hat{a}_1 + \hat{a}_2)\tilde{\phi}(x + \hat{a}_1 + \hat{a}_2) \\
+ i\chi(x - \hat{a})\partial^-\psi(x + \hat{a}_\mu) + i\epsilon_{\mu\nu}\tilde{\chi}(x - \hat{a})\partial^\nu\psi(x + \hat{a}_\mu) \right] \\
= \sum_x \left[ \phi(x)\partial^\mu\partial^-\tilde{\phi}(x) - F_\mu(x + \hat{a}_1 + \hat{a}_2)F^-\mu(x + \hat{a}_1 + \hat{a}_2) \\
+ i\tilde{\chi}(x)(\gamma_\mu)_{\alpha\beta} \frac{\partial_{\nu\mu}}{2} \delta_i \xi_\mu(x) - i\tilde{\chi}(x)(\gamma_5)_{\alpha\beta} \frac{\partial_{\nu\mu}}{2} \delta_i \xi_\mu(x)(\gamma_5\gamma_\mu)_{ji} \right].
\] (5.15)
mechanism as follows:

\[
F_1(x + \hat{a}_1 + \hat{a}_2) = \frac{1}{2} \left( \tilde{\phi}(x + \hat{a}_1 + \hat{a}_2) - \tilde{\varphi}(x + \hat{a}_1 + \hat{a}_2) \right),
\]

\[
F_2(x + \hat{a}_1 + \hat{a}_2) = \frac{i}{2} \left( \tilde{\varphi}(x + \hat{a}_1 + \hat{a}_2) + \tilde{\varphi}(x + \hat{a}_1 + \hat{a}_2) \right),
\]

\[
\tilde{\phi}_1(x) = \frac{1}{2} \left( \tilde{\phi}(x) - \tilde{\varphi}(x) \right),
\]

\[
\tilde{\varphi}_1(x) = \frac{i}{2} \left( \tilde{\phi}(x) + \tilde{\varphi}(x) \right),
\]

\[
\tilde{\xi}_{\alpha i}(x) = \frac{1}{2} \left( \chi(x - \hat{a}) + \gamma_\mu \psi_\mu(x - \hat{a} + \hat{n}_\mu) + \gamma_5 \bar{\chi}(x - \hat{a} + \hat{n}_1 + \hat{n}_2) \right)_{\alpha i}.
\]

(5.16)

We can recognize that the first three terms in the last action are the two copies of the Wess-Zumino action in two dimensions. On the other hand the last term in the action mixes the components of Dirac fermions by \(N = 2\) SUSY transformation, which is necessary to keep the exact \(N = 2\) SUSY on the lattice for the whole action. We thus recognize that this action is the Wess-Zumino action on the lattice. We show the configuration of the component fields for the Wess-Zumino action on the lattice in Fig.12 for the symmetric shift parameter choice A), where the summation of the action should again cover the coordinates of (even,even) and (odd,odd) sites:

\[
\sum_x = \sum_{(\text{even,even})} + \sum_{(\text{odd,odd})}.
\]

(5.17)

We show the case of the asymmetric shift parameter choice B) in Fig. 13, where the coordinate sum of the action covers (even,even) sites.

The third term in the action is the standard kinetic term of Dirac fermion with first derivative while the last term includes second derivative for the fermionic fields and thus higher order with respect to the lattice constant. Those terms stem from the well-known relations between the Dirac-Kähler fermion, staggered fermion and Kogut-Susskind fermion formulation on the lattice \([7, 8, 26]\). It is interesting to recognize that to keep the exact SUSY invariance for the Wess-Zumino model on the lattice naive lattice version of the Wess-Zumino action is not enough to ensure the SUSY invariance on the lattice. The Dirac-Kähler fermion mechanism should be fundamentally introduced with extended SUSY of \(N = 2\), which is twisted \(N = 2\) SUSY in two dimensions. In the current formulation the component fields: \(\chi(x - \hat{a}), \psi_\mu(x - \hat{a} + \hat{n}_\mu)\) and \(\bar{\chi}(x - \hat{a} + \hat{n}_1 + \hat{n}_2)\) can be recognized from (5.16) as the differential form of 0-, 1- and 2-form, respectively, and are defined on the site, link and plaquette of the dual lattice for the symmetric shift parameter choice A) which can be seen in Fig.12, while it has similar structure on the double size original lattice for the asymmetric shift parameter choice B), which we can recognize in Fig.13.

In the above expression of the Wess-Zumino action we have used forward and backward difference operators to avoid unnecessary confusion instead of using the
Figure 12: Component fields of Wess-Zumino action for the symmetric choice of the shift parameters A).

Figure 13: Component fields of Wess-Zumino action for the asymmetric choice of the shift parameters B).
symmetric difference operator. In using the symmetric difference operator the last term of the action includes the difference of the symmetric difference operator which superficially vanishes. We need some care to translate $\partial_+ \mu - \partial_\mu$ into the difference of symmetric difference operator. In defining the new bosonic fields we have introduced imaginary component in the definitions of $F_2(x + a_1 + a_2)$ and $\phi_2(x)$ to make the final form the same as the Wess-Zumino action. This identification is related to do with the indefinite metric nature of the quantized ghost, which is more clearly stated in the formulation of $N = 2$ twisted superspace formalism in the continuum[11].

The standard $N = 2$ SUSY transformation for the bosonic fields, auxiliary fields and fermion fields for the Wess-Zumino model on the lattice can be obtained from the corresponding twisted SUSY transformation of the component fields. The result is given in Appendix B.

### 5.3 Non-Abelian extension of super BF

The Abelian BF model constructed in the previous subsection has no requirement for the shifting properties of $\Psi(x, \theta)$ and $\overline{\Psi}(x, \theta)$, which means that one can simply take $\Psi(x, \theta)$ and $\overline{\Psi}(x, \theta)$ as shift-less superfields $\Phi(x, \theta)$ and $\overline{\Phi}(x, \theta)$, respectively. On the other hand, if one wishes to construct a non-Abelian BF model on the lattice, a more careful treatment would be needed and we will actually see that the distinction between $\Psi(x, \theta)$ and $\Phi(x, \theta)$ is crucial to construct a non-Abelian model in a consistent way on the lattice.

Keeping the above points in mind, one may begin with the following lattice counterparts of non-Abelian chiral and anti-chiral conditions (3.47) and (3.48),

\[
D\Phi(x, \theta) - i\Phi(x, \theta)^2 \equiv \frac{i}{2} \{D - i\Phi(x, \theta), D - i\Phi(x, \theta)\} = 0, \quad (5.18)
\]

\[
\overline{D}\Phi(x, \theta) \equiv \{\overline{D}, \Phi(x, \theta)\} = 0, \quad (5.19)
\]

\[
D_\mu \Psi(x, \theta) \equiv \{D_\mu, \Psi(x, \theta)\} = 0, \quad (5.20)
\]

which are expressed with $D_A$’s defined in (4.36) and shift-less superfields $\Phi(x, \theta)$ and $\overline{\Phi}(x, \theta)$. The crucial point here is to notify from (5.18) that if one wishes to rewrite the chiral and anti-chiral conditions with the arrowed operators as in (4.37), the shifting property of $D$ and $\overline{D}$ should coincide. In other words the shift operator to relate $D$ and $\overline{D}$ or $\Phi(x, \theta)$ and $\overline{\Phi}(x, \theta)$ should be matched and thus one needs to introduce $\Phi(x, \theta)$ as

\[
\Phi(x, \theta) = T(\hat{a})\overline{\Phi}(x, \theta)T(\hat{a}), \quad (5.21)
\]

and in a similar way for $\Psi(x, \theta)$,

\[
\Psi(x, \theta) = T(\hat{a})\overline{\Psi}(x, \theta)T(\hat{a}), \quad (5.22)
\]

where $\theta_A$’s are defined in(4.16). Furthermore, in order to ensure the shift-less property of resulting action, one also needs to introduce $\overline{\Psi}(x, \theta)$ with a opposite sign of shifting parameter $\hat{a}$,

\[
\overline{\Psi}(x, \theta) = T(-\hat{a})\overline{\Psi}(x, \theta)T(-\hat{a}), \quad (5.23)
\]
so that the shift operator for the combination $\Psi(x,\theta)\Phi(x,\theta)$ vanishes. Note that in the above definitions the underlined superfields are introduced symmetrically w.r.t. $T(\hat{a}_A)$ which reproduces the similar coordinate dependence as the component fields of the superfields in the previous subsection.

With using the above definitions, the non-Abelian chiral conditions (5.18), (5.19) and (5.20) now can be rewritten as

\begin{align}
\overrightarrow{D} \Phi(x - \hat{a}, \theta) &+ \Phi(x + \hat{a}, \theta) \overrightarrow{D} - i \Phi(x + \hat{a}, \theta) \Phi(x - \hat{a}, \theta) = 0, \\
\overrightarrow{D} \Phi(x + \hat{a}, \theta) &+ \Phi(x - \hat{a}, \theta) \overrightarrow{D} = 0, \\
\overrightarrow{D}_\mu \Psi(x - \hat{a}_\mu, \theta) &+ \Psi(x + \hat{a}_\mu, \theta) \overrightarrow{D}_\mu = 0,
\end{align}

where the lattice coordinate of the underlined superfields are shifted to $x \pm \hat{a}_A$ by taking off the shift operators $T(\hat{a}_A)$ symmetrically from (5.18), (5.19) and (5.20).

This symmetric coordinate dependence of the above chiral conditions has the same structure as that of the symmetric operator acting on the superfield as in (4.28).

The solution for $\Phi(x)$ can be found in a parallel way as in the continuum case,

\begin{equation}
\Phi(x - \hat{a}, \theta) = \Psi(x - \hat{a}, \theta) + i \theta \Psi(x + \hat{a}, \theta) \Psi(x - \hat{a}, \theta),
\end{equation}

where $\Psi(x \pm \hat{a}, \theta)$ satisfies

\begin{align}
\overrightarrow{D} \Psi(x - \hat{a}, \theta) &+ \Psi(x + \hat{a}, \theta) \overrightarrow{D} = 0, \\
\overrightarrow{D} \Psi(x - \hat{a}, \theta) &+ \Psi(x + \hat{a}, \theta) \overrightarrow{D} = 0.
\end{align}

The component-wise expansions for $\Psi(x,\theta)$ and $\Psi(x,\theta)$ can be expressed using the operator $U$ defined in (4.40) as

\begin{equation}
\Psi(x + \hat{a}, \theta) = U^{-1} \Psi'(x + \hat{a}, \theta) U, \quad \Psi(x + \hat{a}, \theta) = U \Psi'(x + \hat{a}, \theta) U^{-1},
\end{equation}

with

\begin{align}
\Psi'(x + \hat{a}, \theta) &= i \mathcal{L}(x + \hat{a}) + \theta_\mu \omega_\mu(x + \hat{n}_\mu) + i \theta_1 \theta_2 \lambda(x + \hat{n}_1 + \hat{n}_2), \\
\Psi'(x + \hat{a}, \theta) &= i \mathcal{L}(x + \hat{a}) + \theta_1 \phi(x + 2\hat{a}) + \theta_2 \phi(x + \hat{a} + \hat{\theta}) - i \theta \theta \rho(x + 2\hat{a} + \hat{\theta}).
\end{align}

The locations of the component fields appeared in the above superfields are depicted in Fig.14 for the symmetric choice of the shift parameters A) while for the asymmetric choice B) one can show that the configurations coincide with the Abelian case (Fig.7). It is important to note here that in the symmetric choice A), the fermionic fields are located on the half-integer sites namely dual sites while the bosonic fields are on the integer sites or equivalently the original sites, which is in contrast with the situation in the Abelian BF model. It is also interesting to point out that $\omega_\mu$ now lives on the site $x + \hat{n}_\mu$ and carries the shift $2\hat{n}_\mu$ and thus has a nature of link gauge variable, which is in contrast with the Abelian case where $\omega_\mu$ lives on the site $x + \hat{a}_\mu$ and carries the shift $2\hat{a}_\mu$. This difference stems from the fact that the superfields themselves now carry the shift for the non-Abelian case.
Table 7: $N = 2$ twisted SUSY non-Abelian transformation for the component fields.

The SUSY transformation can now be obtained in a straightforward way from

$$
\mathcal{S}_\mu \Phi (x + \hat{a}, \theta) = \mathcal{Q}_A \Phi (x + \hat{a}, \theta) + \Phi (x + \hat{a} + 2\hat{a}, \theta) \mathcal{Q}_A,
$$

$$
\mathcal{S}_\mu \Psi (x + \hat{a}, \theta) = \mathcal{Q}_A \Psi (x + \hat{a}, \theta) + \Psi (x + \hat{a} + 2\hat{a}, \theta) \mathcal{Q}_A,
$$

whose results are summarized in Table 7 with the following notations:

$$
\mathcal{D}_i (x) = \varphi (x + \hat{n}_i) - \varphi (x - \hat{n}_i),
$$

$$
\omega_i (x + \hat{a}) \equiv \omega_i (x + \hat{a}) \mathcal{E} (x - \hat{n}_i) - \mathcal{E} (x + \hat{a}) \omega_i (x - \hat{a}),
$$

$$
(\epsilon_{\mu\nu} \omega_i \omega_j) (x) \equiv \epsilon_{\mu\nu} \omega_i (x + \hat{n}_j) \omega_j (x - \hat{n}_i),
$$

$$
\mathcal{L}_1 (x + \hat{a} + \hat{a}_2) \mathcal{L} (x - \hat{a}) + \mathcal{L}_1 (x + \hat{a}) \mathcal{L} (x - \hat{n}_1 - \hat{a}_2).
$$

Since we use the symmetric difference operator, the coordinate dependence in the SUSY transformation table is cooperated to the SUSY operation as in the Table 4.

We can construct a super invariant action in the similar way as in the Abelian
where the second line (5.40) is obtained from the first line by inserting Eqs (5.21) and (5.23) and moving the shift operator $T(\pm \hat{a})$ to cancel each other and thus no redundant shift operator appears in the action. The covariant difference operators in (5.42) and (5.43) are introduced as,

$$D_{\mu}c(x) \equiv \partial_{\mu}c(x) + [\omega_{\mu}, c](x),$$

$$D_{+\mu}c(x) \equiv \partial_{+\mu}c(x) + [\omega_{\mu}, \bar{c}](x + \hat{n}_{\mu}),$$

respectively, and again the notation (5.35)~(5.38) are used. Even with the interaction terms, the exactness of the action with respect to all twisted supercharges ensures the $N = 2$ twisted SUSY invariance by construction. The field configurations appeared in the action on the lattice are depicted in Fig. 15 for the symmetric choice of the shift parameters A), while for the asymmetric choice B), one can again show that the locations of the component fields coincides with the Abelian case (Fig.9). For the symmetric choice A) the summation of the lattice coordinates for the interaction terms, the exactness of the action with respect to the all twisted supercharges ensures the $N = 2$ twisted SUSY invariance by construction. The field configurations appeared in the action on the lattice are depicted in Fig. 15 for the symmetric choice A), while for the asymmetric choice B), one can again show that the locations of the component fields coincides with the Abelian case (Fig.9). For the symmetric choice A) the summation of the lattice coordinates for the action should cover the same coordinate sites as the Abelian case:

$$\sum_{x} = \sum_{(even, even)} + \sum_{(odd, odd)},$$

while it covers (even, even) sites for the asymmetric choice B) similar to the Abelian case again.
Figure 14: Non-Abelian field configurations in the superfield for the symmetric choice of the shift parameters A).

Figure 15: Component fields of non-Abelian super BF action on the lattice for the symmetric choice of the shift parameters A).
6 Summary and Discussions

We have proposed a new formulation of a twisted superspace on a lattice based on the continuum formulation. The most important difference from the previous trials of formulating SUSY on the lattice is that we have introduced mild noncommutativity between the difference operators and fields to preserve the Leibniz rule on the lattice. As a result an exact extended SUSY for all the twisted supercharges is realized on the lattice.

The origin of the noncommutative nature of the difference operator stems from the fact that the shift operator itself plays a role of difference operator in the representation space of a commutator. Since the twisted super algebra includes the difference operator, the twisted supercharges carry the noncommutative shifts correspondingly. We found the consistency conditions of the shift parameters explicitly for $N = 2$ twisted SUSY algebra in two dimensions. Those consistency conditions naturally led to define the supercharge differential operators in a consistent way, where the twisted super parameters need to carry the opposite shifts to the corresponding supercharges.

Parallel to the continuum twisted superspace formulation, we have introduced a twisted superspace on the lattice. The crucial difference from the continuum formulation is that each of the component fields of the superspace carries the shifting nature since the supercharges and parameters both carry the shifts. We have found a new type of unexpected three-dimensional lattice structure which is generated by the consistency condition of the noncommutativity between a $N = D = 2$ twisted superfield and super parameters. This three-dimensional lattice has a cell structure where unit cell contains all the component fields of a superfield with chiral pairs per site. When the three-dimensional lattice is reduced into two dimension, the lattice super field still has an interesting geometrical interpretation about how the component fields of the superfield are scattered semilocally within a double size lattice in a systematic way. The reason why the component fields are confined within a double size lattice is due to the Grassmann odd nature of the twisted super parameters which carry the noncommutative shifts.

Based on the formulation of twisted superspace on the lattice we have constructed explicit examples of super BF models and Wess-Zumino models in two dimensions. Those actions have exact $N = 2$ twisted SUSY invariance on the lattice for all the twisted supercharges. Each term in the action has a geometrical correspondence with the location of the fields. The lattice counterpart of Wess-Zumino model which has the standard $N = 2$ SUSY invariance on the lattice has a second derivative Dirac fermion term which is necessary to preserve exact SUSY and originally appeared from the Dirac-Kähler fermion mechanism on the lattice[7, 8, 26]. In the non-Abelian extension of super BF model the chiral condition required the same shifting property between the super covariant derivative and the superfield which led to enforce a nontrivial shift to the superfield itself in contrast with the Abelian case. Exact $N = 2$ twisted SUSY invariance is assured even with the non-Abelian interaction terms.
As we show in Section 5, the component fields in the action are cyclically permutable due to the vanishing property of total sum of the shift parameters. It is then effectively similar as if the underlined fields do not carry the noncommutative shifts and have the similar feature as the standard shift-less fields. It is, however, an open question if this feature remains true in the quantum level and in the continuum limit of the lattice.

In this paper we have given a general formulation of twisted superspace formalism on the lattice. As a specific example, however, we have chosen particular models: super BF and Wess-Zumino models which are constructed by the bilinear form of chiral and anti-chiral superfields. It is natural to ask if we can formulate twisted super Yang-Mills action on the lattice which was already formulated in the continuum twisted superspace in two dimensions[10, 11]. The answer is affirmative and it will be given elsewhere[27]. The next immediate question could be if we can extend $N = D = 2$ twisted superspace formalism on the lattice into four dimensions. We first need continuum $N = D = 4$ twisted superspace formalism which will be published soon[28].

Another important question could be to understand the twisting mechanism from algebraic point of view on the lattice. As it was pointed out in [10, 11] that the twisting mechanism is essentially the Dirac-Kähler fermion mechanism and the $R$-rotation of the twisted super algebra is equivalent to the rotation of the flavor suffixes of the Dirac-Kähler fermion. We need to clear up how the ghost related fermionic fields having an integer spin leads to construct Dirac or Majorana half integer spin fermion on the lattice. This problem is essentially related to the chiral fermion problem as well.

Acknowledgments

A. D’Adda thanks kind hospitality for his stay in Sapporo where this collaboration started while N. Kawamoto thanks warm hospitality for his stay in Torino where intensive discussions of this collaboration continued. This work is supported in part by Japanese Ministry of Education, Science, Sports and Culture under the grant number 13640250 and 13135201.

Appendix

A Notations

We use the following notation for $\gamma$ matrices:

$$\gamma_1 = \sigma_3 \quad \gamma_2 = \sigma_1 \quad \gamma_5 = \gamma_1\gamma_2.$$  \hspace{1cm} (A.1)

$$\gamma_\mu^T = \gamma_\mu (\mu = 1, 2) \quad \text{and} \quad \gamma_5^T = -\gamma_5.$$  \hspace{1cm} (A.2)

They satisfy

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$$  \hspace{1cm} (A.2)

$$C\gamma_\mu C^{-1} = \gamma_\mu^T.$$  \hspace{1cm} (A.3)
where $C$ is a charge conjugation matrix and can be taken as $C = 1$ in Euclidean two dimensions.

The following relations are useful in the formulation of Dirac-Kähler fermion mechanism:

\begin{align}
1_{ij}1_{kl} + (\gamma_\mu)_{ij}(\gamma_\mu)_{kl} + (\gamma_5)_{ij}(\gamma_5)_{kl} &= 2\delta_{ik}\delta_{jl}, \\
1_{ij}1_{kl} - (\gamma_\mu)_{ij}(\gamma_\mu)_{kl} - (\gamma_5)_{ij}(\gamma_5)_{kl} &= 2(\gamma_5)_{ik}(\gamma_5)_{jl}.
\end{align}

(A.4) (A.5)

B \quad N = 2 SUSY transformation for the component fields of Wess-Zumino action

Here we explicitly give the $N = 2$ SUSY transformation of fields for the Wess-Zumino model in two dimensions. These SUSY transformations can be derived by the twisted SUSY transformation of the original component fields from superfields. Dirac-Kähler fermion mechanism is essentially used to transform from the tensor suffixes to the spinor suffixes.

For the scalar field $N = 2$ SUSY transformation is given by

\begin{align}
\mathcal{S}_\alpha \phi_1(x) &= (\gamma_5)_{\alpha\beta} \xi_{\beta j}(x)(\gamma_5)_{ji} \\
&\quad - \frac{1}{2}\delta_{\alpha\beta}(\xi_{\beta j}(x + 2\hat{a}) - \xi_{\beta j}(x)) + \frac{1}{2}(\gamma_5)_{\alpha i}(\gamma_5)_{ji}(\xi_{\beta j}(x + 2\hat{a}) - \xi_{\beta j}(x)) \\
\mathcal{S}_{\alpha i} \phi_2(x) &= i\xi_{\alpha i}(x) \\
&\quad + \frac{i}{2}\delta_{\alpha i}(\xi_{\alpha j}(x + 2\hat{a}) - \xi_{\alpha j}(x)) - \frac{i}{2}(\gamma_5)_{\alpha i}(\gamma_5)_{ji}(\xi_{\beta j}(x + 2\hat{a}) - \xi_{\beta j}(x)).
\end{align}

(B.1)
For Dirac-Kähler fermion fields the $N = 2$ SUSY transformation is given by

\[
\begin{align*}
\mathcal{S}_{\alpha i}^{\xi_{ij}}(x) &= -(\gamma_5)_{\alpha i} \delta_{ij} E_1(x + \hat{a}_1 + \hat{a}_2) + i \delta_{\alpha i}(\gamma_5)_{ij} E_2(x + \hat{a}_1 + \hat{a}_2) \nonumber \\
&\quad + i(\gamma^\mu \gamma_5)_{\alpha i} \frac{\partial^+_{\mu} + \partial^-_{\mu}}{2} \phi_1(x) - (\gamma_\mu)_{\alpha i} \delta_{ij} \frac{\partial^+_{\mu} + \partial^-_{\mu}}{2} \phi_2(x) \nonumber \\
&\quad - \frac{1}{2}(\gamma_5)_{\alpha i} \delta_{ij} (F_1(x + \hat{a}_1 + \hat{a}_2 + 2\hat{a}) - F_1(x + \hat{a}_1 + \hat{a}_2)) \nonumber \\
&\quad + \frac{1}{2} \delta_{\alpha i}(\gamma_5)_{ij} (F_1(x + \hat{a}_1 + \hat{a}_2 + 2\hat{a}) - F_1(x + \hat{a}_1 + \hat{a}_2)) \nonumber \\
&\quad - \frac{i}{2}(\gamma_\mu)_{\alpha i} \delta_{ij} \frac{\partial^+_{\mu} - \partial^-_{\mu}}{2} \phi_1(x) \nonumber \\
&\quad - \frac{i}{2}(\gamma_5)_{\alpha i} \delta_{ij} \frac{\partial^+_{\mu} - \partial^-_{\mu}}{2} \phi_2(x) \nonumber \\
&\quad + \frac{1}{2}(\gamma_\mu)_{\alpha i} \delta_{ij} \frac{\partial^+_{\mu} - \partial^-_{\mu}}{2} \phi_1(x) - \phi_1(x - 2\hat{a}) \nonumber \\
&\quad + \frac{1}{2}(\gamma_5)_{\alpha i} \delta_{ij} \frac{\partial^+_{\mu} - \partial^-_{\mu}}{2} \phi_2(x) - \phi_2(x - 2\hat{a}). 
\end{align*}
\]

(B.2)

For the auxiliary fields we obtain the following SUSY transformation:

\[
\begin{align*}
\mathcal{S}_{\alpha i} E_1(x + \hat{a}_1 + \hat{a}_2) &= i \frac{\partial^+_{\mu} + \partial^-_{\mu}}{2} (\gamma_\mu \gamma_5)_{\alpha i} \xi_{\beta j}\xi_{\beta j}(x) \nonumber \\
&\quad + i \frac{\partial^+_{\mu} - \partial^-_{\mu}}{2} (\gamma_\mu)_{\alpha i} \xi_{\beta j}(\gamma_5)_{ji} \xi_{\beta j}(x) - \frac{1}{2}(\gamma_5)_{\alpha i} \delta_{ij} \xi_{\beta j}(x - 2\hat{a}) - \xi_{\beta j}(x) \nonumber \\
&\quad + \frac{i}{2}(\gamma_\mu)_{\alpha i} \delta_{ij} \frac{\partial^+_{\mu} - \partial^-_{\mu}}{2} \xi_{\beta j}(x - 2\hat{a}) - \xi_{\beta j}(x). 
\end{align*}
\]

(B.3)

\[
\begin{align*}
\mathcal{S}_{\alpha i} E_2(x + \hat{a}_1 + \hat{a}_2) &= \frac{\partial^+_{\mu} + \partial^-_{\mu}}{2} (\gamma_\mu)_{\alpha i} \xi_{\beta j}\xi_{\beta j}(x) (\gamma_5)_{ji} \nonumber \\
&\quad + \frac{\partial^+_{\mu} - \partial^-_{\mu}}{2} \xi_{\beta j}(x)(\gamma_5 \gamma_\mu) (\gamma_5)_{ji} \nonumber \\
&\quad - \frac{1}{2}(\gamma_5 \gamma_\mu)_{\alpha i} \delta_{ij} \frac{\partial^+_{\mu} - \partial^-_{\mu}}{2} \xi_{\beta j}(x - 2\hat{a}) - \xi_{\beta j}(x) \nonumber \\
&\quad + \frac{1}{2}(\gamma_\mu)_{\alpha i} \delta_{ij} \frac{\partial^+_{\mu} - \partial^-_{\mu}}{2} \xi_{\beta j}(x - 2\hat{a}) - \xi_{\beta j}(x). 
\end{align*}
\]

(B.4)
References

[1] N. Kawamoto, V.A. Kazakov, Y. Saeki and Y. Watabiki, Phys. Rev. Lett. 68 (1992) 2113; Nucl.Phys.B(Proc. Suppl.) 26 (1992) 584.
H. Kawai, N. Kawamoto, T. Mogami and Y. Watabiki, Phys. Lett. B306 (1993) 19 [hep-th/9302133].
J. Ambjørn, K.N. Anagnostopoulos, T. Ichihara, L. Jensen, N. Kawamoto, Y. Watabiki and K. Yotsuji, Phys.Lett.B397 (1997) 177 [hep-lat/9611032]; Nucl.Phys.B511 (1998) 673 [hep-lat/9706009].
N. Kawamoto and K. Yotsuji, Nucl. Phys. B644 [FS] (2002) 533, [hep-lat/0207007], and related references are therein.

[2] P. H. Ginsparg and K. G. Wilson, Phys. Rev. D25 (1982) 2649.
H. Neuberger, Phys. Lett. B417 (1998) 141 [hep-lat/9707022]; Phys. Lett. B427 (1998) 353 [hep-lat/9801031].
M. Lüsher, Phys. Lett. B428 (1998) 342 [hep-lat/9802011].
P. Hasenfratz, Nucl. Phys. B525 (1998) 401 [hep-lat/9802007].

[3] H. B. Nielsen and M. Ninomiya, Nucl. Phys. B185 (1981) 20 [Erratum-ibid. B195 (1982) 541]; Nucl. Phys. B193 (1981) 173.

[4] L.H. Karsten and J. Smit, Nucl.Phys.B183 (1981) 103.

[5] N. Kawamoto and J. Smit, Nucl. Phys. B192 (1981), 100.

[6] J. B. Kogut and L. Susskind, Phys. Rev. D11 (1975) 395.
L. Susskind, Phys. Rev. D16 (1977) 3031.

[7] F. Gliozzi, Nucl. Phys. B204 (1982), 419.

[8] H. Kluberg-Stern, A. Morel, O. Napoly, and B. Petersson, Nucl. Phys. B220 (1983), 447.

[9] D. Ivanenko and L. Landau, Z. Phys. 48 (1928) 340.
E. Kähler, Rend. Math. 21 (1962), 425.
W. Graf, Annales Poincare Phys. Theor. 29 (1978), 85.
P. Becher and H. Joos, Zeit. Phys. C15 (1982), 343.
T. Banks, Y. Dothan, and D. Horn, Phys. Lett. B117 (1982), 413.
J. M. Rabin, Nucl. Phys. B201 (1982), 315.
I. M. Benn and R. W. Tucker, Commun. Math. Phys. 89 (1983), 341.
P. Mitra, Nucl. Phys. B227 (1983) 349.
H. Aratyn and A. H. Zimerman, Phys. Rev. D33 (1986) 2999.
J. A. Bullinaria, Ann. Phys. 168 (1986), 301.
[10] N. Kawamoto and T. Tsukioka, Phys. Rev. D61 (2000) 105009 [hep-th/9905222].

[11] J. Kato, N. Kawamoto and Y. Uchida, Int. J. Mod. Phys. (2004) (in press) [hep-th/0310242].

[12] E. Witten, Commun. Math. Phys. 117, 353 (1988).

L. Baulieu and I. Singer, Nucl. Phys. B(Proc. Suppl.) 5, 12 (1988).

R. Brooks, D. Montano and J. Sonnenschein, Phys. Lett. 214B (1988) 91.

J.M.F. Labastida and M. Pernici, Phys. Lett. 212B (1988) 56; Phys. Lett. 213B (1988) 319.

D. Birmingham, M. Rakowski and G. Thompson, Phys. Lett. 214B (1988) 381; Phys. Lett. 212B (1988) 187; Nucl. Phys. B315 (1989) 577.

[13] N. Kawamoto and Y. Watabiki, Commun. Math. Phys. 144, 641 (1992); Mod. Phys. Lett. A 7, 1137 (1992).

[14] D. Birmingham, M. Rakowski, and G. Thompson, Nucl. Phys. B329, 83 (1990).

D. Birmingham and M. Rakowski, Mod. Phys. Lett. A 4, 1753 (1989); Phys. Lett. B 269, 103 (1991); ibid B 272, 217 (1991).

F. Delduc, F. Gieres, and S.P. Sorella, Phys. Lett. B225, 367 (1989).

F. Delduc, C. Lucchesi, O. Piguet and S.P. Sorella, Nucl. Phys. B 346 (1990) 313.

[15] P. Dondi and H. Nicolai, Nuovo Cim. A 41 (1977) 1.

S. Elitzur, E. Rabinovici and A. Schwimmer, Phys. Lett. B 199 (1982) 165.

T. Banks and P. Windey, Nucl. Phys. B198 (1982) 226.

S. Cecotti and L. Girardello, Nucl. Phys. B226 (1983) 417.

N. Sakai and M. Sakamoto, Nucl. Phys. B229 (1983) 173.

S. Elitzur and A. Schwimmer, Nucl. Phys. B226 (1983) 109.

I. Ichinose, Phys. Lett. B122 (1983) 68.

J. Bartels and J. B. Bronzan, Phys. Rev. D28 (1983) 818.

J. Bartels and G. Kramer, Z. Phys. C20 (1983) 159.

D. B. Kaplan, Phys. Lett. B136 (1984) 162.

R. Nakayama and Y. Okada, Phys. Lett. B134 (1984) 241.

S. Nojiri, Prog. Theor. Phys. 74 (1985) 819.

G. Curci and G. Veneziano, Nucl. Phys. B292 (1987) 555.

M. Golterman and D. Petcher, Nucl. Phys. B319 (1989) 307.
[16] J. Nishimura, Phys. Lett. B406 (1997) 215 [hep-lat/9701013].
N. Maru and J. Nishimura, Int. J. Mod. Phys. A13 (1998) 2841 [hep-th/9705152].
H. Neuberger, Phys. Rev. D57 (1998) 5417 [hep-lat/9710089].
D. B. Kaplan and M. Schmaltz, Chin. J. Phys. 38 (2000) 543 [hep-lat/0002030].
G. T. Fleming, J. B. Kogut and P. M. Vranas, Phys. Rev. D64 (2001) 034510 [hep-lat/0008009].
I. Montvay, Int. J. Mod. Phys. A17 (2002) 2377 [hep-lat/0112007].

[17] D. B. Kaplan, E. Katz and M. Ünsal, JHEP 0305 (2003) 037 [hep-lat/0206019].
A. G. Cohen, D. B. Kaplan, E. Katz and M. Ünsal, JHEP 0308 (2003) 024 [hep-lat/0302017]; [hep-lat/0307012].
J. Nishimura, S. J. Rey and F. Sugino, JHEP 0302 (2003) 032 [hep-lat/030125].
J. Giedt, E. Poppitz and M. Rozali, JHEP 0303 (2003) 035 [hep-th/0301048].
J. Giedt, Nucl. Phys. B668 (2003) 138 [hep-lat/0304006]; Nucl. Phys. B674 (2003) 259 [hep-lat/0307024].

[18] W. Bietenholz, Mod. Phys. Lett. A14 (1999) 51 [hep-lat/9807010].
K. Fujikawa and M. Ishibashi, Nucl. Phys. B622 (2002) 115 [hep-th/0109156],
Phys. Lett. B528 (2002) 295 [hep-lat/0112025].
Y. Kikukawa and Y. Nakayama, Phys. Rev. D66 (2002) 094508 [hep-lat/0207013].
K. Fujikawa, Phys. Rev. D66 (2002) 074510 [hep-lat/0208015].
M. Bonini and A. Feo, [hep-lat/0402034].

[19] S. Catterall and S. Karamov, Phys. Rev. D65 (2002) 094501 [hep-lat/0108024]; Phys. Rev. D68 (2003) 014503 [hep-lat/0305002].
S. Catterall, JHEP 0305 (2003) 038 [hep-lat/0301028].
S. Catterall and S. Ghadab, [hep-lat/0311042].
F. Sugino, JHEP 0403 (2004) 067 [hep-lat/0411021], JHEP 0401 (2004) 015 [hep-lat/0401017].

[20] K. Itoh, M. Kato, H. Sawanaka, H. So and N. Ukita, JHEP 0302 (2003) 033 [hep-lat/0210049], Prog. Theor. Phys. 108 (2002) 363 [hep-lat/0112052].

[21] A. Feo, Nucl. Phys. Proc. Suppl. 119 (2003) 198 [hep-lat/0112052], [hep-lat/0311037], and references therein.
[22] K. Fujikawa, Nucl. Phys. B636 (2002) 80 [hep-th/0205095].

[23] A. Dimakis, F. Müller-Hoissen, and T. Striker, J. Phys. A26 (1993) 1927; Phys. Lett. B300 (1993) 141.
A. Dimakis and F. Müller-Hoissen, J. Math. Phys. 35 (1994) 6703; J. Phys. A27 (1994) 3159 [hep-th/9401149]; J. Math. Phys. 40 (1999) 1518 [gr-qc/9808023].
P. Aschieri and L. Castellani, Int. J. Mod. Phys. A8 (1993), 1667 [hep-th/9207084].
A. P. Balachandran, G. Bimonte, G. Landi, F. Lizzi, and P. Teotonio-Sobrinho, J. Geom. Phys. 24 (1998), 353 [hep-lat/9604012].
J. Dai and X. C. Song, [hep-th/0101184].

[24] A. Connes, Noncommutative Geometry, (Academic Press, New York, 1994); A. Connes and J. Lott, Nucl. Phys. B (Proc. Suppl.) 18B (1990) 29 and in The Proceedings of the 1991 Cargese Summer Conference, eds. J. Fröhlich et al. (Plenum, 1992).

[25] J. Dai and X. C. Song, Phys. Lett. B508 (2001), 385 [hep-th/0101130]; Commun. Theor. Phys. 39 (2003) 519 [hep-lat/0105017].
J. J. Vaz, Advances in Applied Clifford algebras, 7 (1997) 37; [hep-th/9706189].
T. Striker, [hep-lat/9507017].

[26] I. Kanamori and N. Kawamoto, Int. J. Mod. Phys. A19 (2004) 695 [hep-th/0305094], Nucl. Phys. Proc. Suppl. 129 (2004) 877 [hep-lat/0309120].

[27] A. D’Adda, I. Kanamori, N. Kawamoto and K. Nagata, in preparation.

[28] J. Kato, N. Kawamoto and A. Miyake, to appear.