Projectively self-concordant barriers on convex sets

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Abstract

Self-concordance is the most important property required for barriers in convex program-
mong. We introduce an alternative, stronger notion, which we call \textit{projective self-concordance},
define the corresponding Dikin sets by a quadratic inequality, and develop a corresponding
duality theory. Our notion is equivariant with respect to the group of projective transfor-
mations, which is larger than the affine group corresponding to the classical notion. Our Dikin
sets are larger than the classical Dikin ellipsoids, depend on the gradient of the barrier at the
center point, are non-symmetric, and may even be unbounded. From the derivatives of the
barrier at a given point we construct a quadratic set which overbounds the underlying con-
vex set, which is not possible for the classical notion of self-concordance. This opens the way
to design algorithms which make larger steps and hence have a faster convergence rate than
traditional interior-point methods. We give many examples of convex sets with projectively
self-concordant barriers.

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1 Introduction

A wide-spread class of algorithms used for solving convex programming problems are the interior-
point methods [5, 7]. These methods employ barrier functions on the feasible set having a special
property named \textit{self-concordance}, which was introduced by Y.E. Nesterov and A.S. Nemirovski
[5].

Definition 1.1. Let $C \subset \mathbb{R}^n$ be a regular convex set (a closed convex set with non-empty interior
and containing no lines). A \textit{self-concordant barrier} on $C$ with parameter $\vartheta$ is a $C^3$ function
$F : C^o \to \mathbb{R}$ satisfying the conditions

- $F''(x) > 0$ for all $x \in C^o$,
- $\lim_{x \to \partial C} F(x) = +\infty$,
- $|F'''(x)[u, u, u]| \leq 2(F''(x)[u, u])^{3/2}$ for all $x \in C^o$, $u \in T_x C^o$,
- $|F'(x)[u]| \leq \sqrt{\vartheta} (F''(x)[u, u])^{1/2}$ for all $x \in C^o$, $u \in T_x C^o$.

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Short-step interior-point methods produce a sequence of iterates converging to the solution of the problem such that at each step the next iterate $x_{k+1}$ is contained in the Dikin ellipsoid

$$E_{x_k} = \{ x \mid (x - x_k)^T F''(x_k)(x - x_k) < 1 \}$$

(1)

centered on the previous iterate $x_k$. The self-concordance property of the barrier implies that the algorithm produces only interior points of the feasible convex set $C$, by guaranteeing that the Dikin ellipsoid centered on any interior point of $C$ entirely lies in $C^o$.

Definition (1) of the Dikin ellipsoid depends only on the Hessian of the barrier at the center point and hence disregards important information on the gradient of the barrier. In particular, the Dikin ellipsoid is centrally symmetric and hence insensitive to the position of the center point with respect to the boundary of the set, which is highly asymmetric if the sequence of iterates approaches a solution on the boundary.

In this paper we introduce an alternative notion of self-concordance involving the gradient of the barrier and define a corresponding Dikin set by a quadratic inequality. We show that our notion has superior theoretical properties, while at the same time it is applicable to many classical convex sets.

**Definition 1.2.** Let $C \subseteq \mathbb{R}^n$ be a regular convex set. A **projectively self-concordant barrier** on $C$ with parameter $\gamma$ is a $C^3$ function $f : C^o \rightarrow \mathbb{R}$ satisfying

- $f''(x) - f'(x) \otimes f'(x) > 0$ for all $x \in C^o$,
- $\lim_{x \rightarrow \partial C} f(x) = +\infty$,
- $|f''(x)[u, u, u] - 6f''(x)[u, u]f'(x)[u] + 4(f'(x)[u])^3| \leq 2\gamma(f''(x)[u, u] - (f'(x)[u])^2)^{3/2}$ for all $x \in C^o$, $u \in T_x C^o$.

The condition in Definition (1.2) is stronger than that in Definition (1.1) in the sense that for a projectively self-concordant barrier $f$ on $C$ with parameter $\gamma = \frac{\vartheta - 2}{\sqrt{\vartheta - 1}}$, the function $F = \vartheta \cdot f$ is a self-concordant barrier on $C$ with parameter $\vartheta$ (Lemma 3.1).

The name of the introduced modified notion of self-concordance is motivated by another property, namely that a projectively self-concordant barrier on some set gives rise to such barriers on projective images of this set (Section 5), and hence corresponds to a larger symmetry group than classical self-concordance.

**Definition 1.3.** Let $C \subseteq \mathbb{R}^n$ be a regular convex set and $f$ a projectively self-concordant barrier on $C$ with parameter $\gamma = \frac{\vartheta - 2}{\sqrt{\vartheta - 1}}$. The Dikin set $E_{x_0}$ of $f$ around the point $x_0 \in C^o$ is defined as the connected component of the set

$$\left\{ x \in \mathbb{R}^n \mid \left( \begin{array}{c} 1 \\ x - x_0 \end{array} \right)^T \left( \begin{array}{c} 1 \\ -f'(x_0) \\ \vartheta - 1 \end{array} \right) \left( \begin{array}{c} 1 \\ -f'(x_0) \\ \vartheta - 1 \end{array} \right)^T \left( \begin{array}{c} -f'(x_0) \\ \vartheta - 1 \end{array} \right) \left( \begin{array}{c} 1 \\ x - x_0 \end{array} \right) > 0 \right\}$$

(2)

containing the point $x_0$.

The Dikin set is always contained in $C^o$, but overbounds the corresponding Dikin ellipsoid defined by (1) (Lemma 3.3). Note that the Dikin set is not necessarily an ellipsoid, it may likewise be unbounded, with its boundary given by a paraboloid or a convex hyperboloid.

In a similar way we may define an **overbounding set** $\Gamma_{x_0}$ as the connected component of the set

$$\left\{ x \in \mathbb{R}^n \mid \left( \begin{array}{c} 1 \\ x - x_0 \end{array} \right)^T \left( \begin{array}{c} 1 \\ -f'(x_0) \\ \frac{\vartheta - 1}{\vartheta - 1} \end{array} \right) \left( \begin{array}{c} 1 \\ -f'(x_0) \\ \frac{\vartheta - 1}{\vartheta - 1} \end{array} \right)^T \left( \begin{array}{c} -f'(x_0) \\ \vartheta - 1 \end{array} \right) \left( \begin{array}{c} 1 \\ x - x_0 \end{array} \right) > 0 \right\}$$

(3)
containing the point \( x_0 \) (Lemma 3.3). Note that for a traditional self-concordant barrier no overbounding set can be constructed from the derivatives of the barrier at a given point.

The fact that the Dikin set approximates the underlying convex set more tightly from inside than the Dikin ellipsoid opens the possibility to design interior-point methods which make larger steps and hence converge faster. The availability of a quadratic overbounding set gives qualitatively new information which can be used to design new classes of interior-point methods.

In order to exploit the larger capabilities of projectively self-concordant barriers it is necessary in the first place to have them available. In Section 2 we shall provide many examples of convex sets with efficiently computable projectively self-concordant barriers, among them polyhedra and spectrahedra, while in Section 3 we provide tools to construct such barriers on more complex sets from known ones on simpler sets.

In the remaining sections we prove the claims made above (Section 3), introduce a notion of duality for projectively self-concordant barriers (Section 4), and prove an auxiliary theorem from which many of our results easily follow (Section 2).

2 Conic extensions

In this section we link projectively self-concordant functions on convex sets to self-concordant functions on the conic extensions of the sets.

**Theorem 2.1.** Let \( D \subset \mathbb{R}^n \) be an open convex set, and let \( K = \{(t, x) \in \mathbb{R}^{n+1} \mid t > 0, t^{-1}x \in D\} \) be its conic extension. Let \( f : D \to \mathbb{R} \) be a \( C^3 \) function, let \( \vartheta \geq 2 \) be a number, and define \( F : K \to \mathbb{R} \) by \( F(t, x) = \vartheta(-\log t + f(t^{-1}x)) \). Set further \( \gamma = \frac{\vartheta - 2}{\sqrt{\vartheta - 1}} \). Then the following are equivalent:

1. the function \( f \) satisfies conditions 1 and 3 of Definition 1.2 on \( D \) with parameter \( \gamma \),
2. the function \( F \) satisfies conditions 1 and 3 of Definition 1.2 on \( K \) with parameter \( \vartheta \).

Note that condition 4 of Definition 1.1 follows from condition 1 and the logarithmic homogeneity of \( F \).

**Proof.** We first show that the second item in the theorem implies the first one.

Let \( \bar{x} \in D \) and \( \bar{v} \in T_\bar{x}D \setminus \{0\} \) be arbitrary, and define \( x = (1, \bar{x}) \in K \) and \( v = (0, \bar{v}) \in T_xK \). Clearly the vector \( v \) is linearly independent from \( x \). We have [5 Prop. 2.3.4]

\[
F'(x)[x] = -\vartheta, \quad F''(x)[x, x] = \vartheta, \quad F''(x)[x, v] = -\vartheta f'(\bar{x})[\bar{v}],
\]

and by further differentiation

\[
F'''(x)[x, x, x] = -2\vartheta, \quad F'''(x)[x, x, v] = -2F''(x)[x, v] = 2\vartheta f'(\bar{x})[\bar{v}],
\]

\[
F'''(x)[x, v, v] = -2F''(x)[v, v] = -2\vartheta f''(\bar{x})[\bar{v}, \bar{v}].
\]

Consider \( u = v + \alpha x \in T_xK \) for \( \alpha \in \mathbb{R} \). We obtain

\[
F'''(x)[u, u, u] = \alpha^3 F'''(x)[x, x, x] + 3\alpha^2 F'''(x)[x, x, v] + 3\alpha F'''(x)[x, v, v] + F'''(x)[v, v, v]
\]

\[
= \vartheta(-2\alpha^3 + 6\alpha^2 f'(\bar{x})[\bar{v}] - 6\alpha f''(\bar{x})[\bar{v}, \bar{v}] + f'''(\bar{x})[\bar{v}, \bar{v}, \bar{v}]),
\]

\[
F''(x)[u, u] = \alpha^2 F''(x)[x, x] + 2\alpha F''(x)[x, v] + F''(x)[v, v] = \vartheta(\alpha^2 - 2\alpha f'(\bar{x})[\bar{v}] + f''(\bar{x})[\bar{v}, \bar{v}]).
\]
Now $F''(x)[u, u] > 0$ for every $\alpha$, which implies that the discriminant $(f'(\tilde{x})[v])^2 - f''(\tilde{x})[\tilde{v}, \tilde{v}]$ of the quadratic polynomial $\alpha^2 - 2\alpha f'(\tilde{x})[v] + f''(\tilde{x})[\tilde{v}, \tilde{v}]$ is negative. It follows that $f''(\tilde{x}) - f'(\tilde{x}) \otimes f'(\tilde{x}) > 0$. This proves the first property in Definition 1.2.

Set $c_3 = f'''(\tilde{x})[\tilde{v}, \tilde{v}] - 6f''(\tilde{x})[\tilde{v}, \tilde{v}]f'(\tilde{x})[v] + 4(f'(\tilde{x})[v])^3$, $c_2 = f''(\tilde{x})[\tilde{v}, \tilde{v}] - (f'(\tilde{x})[v])^2 > 0$, $t = c_2^{-1/2}(\alpha - f'(\tilde{x})[\tilde{v}])$, $\mu = c_2^{-3/2}c_3$. The self-concordance condition 3 in Definition 1.1 implies

$$(-2\alpha^3 + 6\alpha^2 f'(\tilde{x})[v] - 6\alpha f''(\tilde{x})[\tilde{v}, \tilde{v}] + f'''(\tilde{x})[\tilde{v}, \tilde{v})[v, \tilde{v}]^2 \leq 4\delta(\alpha^2 - 2\alpha f'(\tilde{x})[v] + f''(\tilde{x})[\tilde{v}, \tilde{v})] \quad \forall \alpha \in \mathbb{R}$$

or equivalently

$$p_{\mu}(t) = 4(\delta - 1)t^6 + 12(\delta - 2)t^4 + 4\mu t^3 + 12(\delta - 3)t^2 + 12\mu t + 4\delta - \mu \geq 0 \quad \forall t \in \mathbb{R}.$$ 

Let us show that this condition implies $|\mu| \leq 2\gamma$, which is the third property in Definition 1.2. Set $\vartheta = \kappa^2 + 1, \kappa \geq 1$, then

$$p_{\mu}(-\kappa^{-1}) = \left(\frac{2(\kappa^4 + 4\kappa^2 + 2)}{\kappa^3} + \mu\right)(\frac{2(\kappa^2 - 1)}{\kappa} - \mu),$$

$$p_{\mu}(\kappa^{-1}) = \left(\frac{2(\kappa^4 + 4\kappa^2 + 2)}{\kappa^3} - \mu\right)(\frac{2(\kappa^2 - 1)}{\kappa} + \mu).$$

For $\kappa \geq 1$ we have $\frac{2(\kappa^2 - 1)}{\kappa} \leq \frac{2(\kappa^4 + 4\kappa^2 + 2)}{\kappa^3}$, and hence both expressions $p_{\mu}(\pm \kappa^{-1})$ are simultaneously nonnegative if and only if $|\mu| \leq 2\gamma = 2\frac{\vartheta - 2}{\sqrt{\vartheta} - 1} = 2\gamma$.

Let us now show the reverse implication. By logarithmic homogeneity of $F$ we have

$$F''(\tau x)[\tau u, \tau u] = F''(x)[u, u], \quad F'''(\tau x)[\tau u, \tau u, \tau u] = F'''(x)[u, u, u]$$

for all $x \in K, u \in T_x K, \tau > 0$. Therefore we need to show conditions 1 and 3 of Definition 1.1 only at $x = (1, \tilde{x})$ with $\tilde{x} \in D$. Let $u \in T_x D \setminus \{0\}$ be arbitrary. Then there exists a unique decomposition $u = (0, \tilde{v}) + \alpha x$, where $\tilde{v} \in T_{\tilde{x}} D$.

Let us first consider the case $\tilde{v} = 0$. Then $F''(x)[u, u] = \alpha^2 \vartheta > 0$, and $|F'''(x)[u, u, u]| = 2|\alpha|^3 \vartheta = 2\vartheta - 1/2(F''(x)[u, u])^{3/2} < 2(F''(x)[u, u])^{3/2}$, which proves our claim.

Now consider the case $\tilde{v} \neq 0$. By condition 3 in Definition 1.2, the discriminant of the polynomial $\alpha^2 - 2\alpha f'(\tilde{x})[v] + f''(\tilde{x})[\tilde{v}, \tilde{v}]$ is negative, and hence $F''(x)[u, u] > 0$, which proves the first condition in Definition 1.1. Assume above notations, and set $\mu = \pm 2\gamma$. Then we get

$$p_{\mu_+}(t) = 4\kappa^{-2}(\kappa t + 1)^2(\kappa(t^2 + 2)(t - 1)^2 + (\kappa - 1)((t^4 + 3t^2 + 3)\kappa + 1)),$$

$$p_{\mu_-}(t) = 4\kappa^{-2}(\kappa t - 1)^2(\kappa(t^2 + 2)(t + 1)^2 + (\kappa - 1)((t^4 + 3t^2 + 3)\kappa + 1)).$$

These polynomials are hence nonnegative by virtue of $\kappa \geq 1$. Since $p_{\mu}(t)$ is concave in $\mu$, and $\mu \in [-\mu_-, \mu_+]$ by condition 3 of Definition 1.2, the polynomial $p_{\mu}(t)$ will also be nonnegative. Reversing the chain of equivalences, we obtain the third condition in Definition 1.1.

This completes the proof. \qed
3 Inner and outer approximations

In this section we formalize and prove the inclusion results announced in the introduction.

**Lemma 3.1.** Let $C \subset \mathbb{R}^n$ be a regular convex set and $f$ a projectively self-concordant barrier on $C$ with parameter $\gamma$. Then the function $F = \vartheta \cdot f$ is a self-concordant barrier on $C$ with parameter $\vartheta = (\gamma + \sqrt{\gamma^2 + 4})\sqrt{\frac{\gamma^2}{4} + 1}$.

**Proof.** Let $F$ be as defined in the lemma. The first two properties in Definition 1.1 follow from the corresponding properties in Definition 1.2.

The first property in Definition 1.2 can also be rewritten as

$$\begin{pmatrix} 1 & f'(x)^T \\ f'(x) & f''(x) \end{pmatrix} > 0 \iff \begin{pmatrix} \vartheta & F'(x)^T \\ F'(x) & F''(x) \end{pmatrix} > 0,$$

from which we get

$$\begin{pmatrix} \vartheta & F'(x)[u] \\ F'(x)[u] & F''(x)[u,u] \end{pmatrix} > 0.$$ The fourth property in Definition 1.1 readily follows.

The third property in Definition 1.1 follows from Theorem 2.1 by restricting $F$ constructed on the conic extension back to the set $C^o$. \qed

In order to prove the inclusion of the Dikin set in the underlying convex set $C$ we need to estimate projectively self-concordant functions on intervals. The key idea to obtain these bounds is to consider the self-concordance condition as a differential inclusion giving rise to a controlled dynamical system.

**Lemma 3.2.** Let $I \subset \mathbb{R}$ be an open interval, let $p : I \to \mathbb{R}$ be a $C^2$ function satisfying the conditions

$$p' - p^2 > 0, \quad p'' = 6p'p - 4p^3 + 2u\gamma(p' - p^2)^{3/2}, \quad u \in [-1, 1]$$

for some $\gamma \geq 0$, let $x_0 \in I$ be a point, and set $p_0 = p(x_0)$, $h_0 = p'(x_0)$, $g_0 = \sqrt{h_0 - p_0^2} > 0$. Let further

$$p_{\pm}(t; p_0, h_0) = \frac{p_0 + t(g_0^2 - p_0^2 \mp \gamma g_0 p_0)}{-g_0^2 t^2 + (p_0 t - 1)^2 \pm \gamma g_0 t(p_0 t - 1)},$$

such that $p_{\pm}(x - x_0; p_0, h_0)$ are the solutions of (4) with the above initial conditions and control $u \equiv \pm 1$, respectively. Let $I_{\pm} \subset \mathbb{R}$ be the intervals of definition of these solutions.

Then for every $x \in I \cap I_-$ we have $p_-(x - x_0; p_0, h_0) \leq p(x)$ and for every $x \in I \cap I_+$ we have $p(x) \leq p_+(x - x_0; p_0, h_0)$.

**Proof.** It is verified by direct calculation that $p_{\pm}(0; p_0, h_0) = p_0$, $p'(0; p_0, h_0) = h_0$, and $p_{\pm}$ satisfy the above differential equation with $u \equiv \pm 1$.

Let us prove the inequality $p(x) \leq p_+(x - x_0; p_0, h_0)$. At $t = 0$ the denominator of $p_+$ equals 1, but at $t_+ = \frac{p_0 + \frac{\gamma}{2} g_0}{p_0 + \frac{\gamma}{2} g_0}$, if this value is finite, it equals $\frac{(\gamma^2 + 4)g_0^2}{(2p_0 + \gamma g_0)^2} < 0$. Hence $x = x_0 + t_+ \notin I_+$, and for all $x = x_0 + t \in I_+$ we have $1 - (p_0 + \frac{\gamma}{2} g_0) t > 0$. For every such $t$ we have

$$\frac{\partial p_+}{\partial h_0} = \frac{t(1 - (p_0 + \frac{\gamma}{2} g_0) t)}{((p_0 t - 1)^2 - t^2 g_0^2 + \gamma t(p_0 t - 1) g_0)^2}.$$
Lemma 3.4. Let $I \subset \mathbb{R}$ be an open interval and $f : I \rightarrow \mathbb{R}$ a projectively self-concordant barrier on $I$ with parameter $\gamma = \frac{\vartheta - 2}{\sqrt{\vartheta - 1}}$. Let $x_0 \in I$ be a point and $f(x_0) = f_0$, $f'(x_0) = p_0$, $f''(x_0) = h_0$, $g_0 = \sqrt{h_0 - p_0^2}$. Let

$$I_{\pm} = \left\{ x \in \mathbb{R} \mid (x - x_0)(p_0 + \frac{g_0}{\sqrt{\vartheta - 1}}) < 1, (x - x_0)(p_0 \pm g_0\sqrt{\vartheta - 1}) < 1 \right\}$$

be the domains of definition of the functions

$$f_{\pm}(x) = -\frac{\vartheta - 1}{\vartheta} \log(1 - (x - x_0)(p_0 \mp \frac{g_0}{\sqrt{\vartheta - 1}})) - \frac{1}{\vartheta} \log(1 - (x - x_0)(p_0 \pm g_0\sqrt{\vartheta - 1})) + f_0.$$

Then for every $x \in I \cap I_-$ we have $\sigma f_-(x) \leq \sigma f(x)$ and for every $x \in I \cap I_+$ we have $\sigma f(x) \leq \sigma f_+(x)$, where $\sigma = \text{sgn}(x - x_0)$.

Proof. The derivative $p = f'$ satisfies the conditions of Lemma 3.2. It is not hard to check that

$$f_{\pm}(x) = f_0 + \int_{0}^{x-x_0} p_{\pm}(t; p_0, h_0) \, dt,$$

where $p_{\pm}$ are defined in Lemma 3.2. Since $f(x) = f_0 + \int_{x_0}^{x} p(\tau) \, d\tau$, the estimates on $f$ follow from the estimates on $p$ in this lemma. \qed

The bounds $f_{\pm}$ may escape to $+\infty$ at finite points. The domain $I$ of definition of the projectively self-concordant barrier $f$ must then contain the domain of definition of the upper bound and be contained in the domain of definition of the lower bound. This implies the following constraints on the interval $I$.

If $p_0 + g_0\sqrt{\vartheta - 1} \leq 0$, or equivalently $p_0 \leq -\sqrt{\frac{\vartheta - 1}{\vartheta}}h_0$, then also $p_0 - \frac{g_0}{\sqrt{\vartheta - 1}} \leq 0$, and $I_+$ is unbounded to the right. It follows that $[x_0, +\infty) \subset I$. In the opposite case the right end-point of $I_+$ is given by $x_0 + \frac{1}{p_0 + g_0\sqrt{\vartheta - 1}}$, and hence $[x_0, x_0 + \frac{1}{p_0 + g_0\sqrt{\vartheta - 1}}) \subset I$.

If $p_0 + \frac{g_0}{\sqrt{\vartheta - 1}} \leq 0$, or equivalently $p_0 \leq -\sqrt{\frac{\vartheta - 1}{\vartheta}}h_0$, then also $p_0 - g_0\sqrt{\vartheta - 1} \leq 0$, and $I_-$ is unbounded to the right. In the opposite case the right end-point of $I_-$ is given by $x_0 + (p_0 + \frac{g_0}{\sqrt{\vartheta - 1}})^{-1}$, and $[x_0 + (p_0 + \frac{g_0}{\sqrt{\vartheta - 1}})^{-1}, +\infty) \cap I = \emptyset$.

We are now in a position to prove the inclusion relations claimed in the introduction.

Lemma 3.4. Let $C \subset \mathbb{R}^n$ be a regular convex set, $f$ a projectively self-concordant barrier on $C$ with parameter $\gamma = \frac{\vartheta - 2}{\sqrt{\vartheta - 1}}$, and $x_0 \in C^\circ$ a point. Then the Dikin set from Definition 1.3 is convex, contained in $C^\circ$, and contains the Dikin ellipsoid $\{ x \mid (x - x_0)^T f''(x_0)(x - x_0) \leq \vartheta^{-1} \}$. 


Proof. Let \( u \neq 0 \) be an arbitrary vector, and let \( l \) be the line through \( x_0 \) parallel to \( u \). Define \( p_0 = \langle f'(x_0), u \rangle \), \( h_0 = \langle f''(x_0)u, u \rangle \), and \( g_0 = \sqrt{h_0 - p_0} \). The restriction of \( f \) to the intersection \( C^o \cap l \) satisfies the conditions of Corollary 3.3. From the consideration of the upper bound \( f^+ \) from this corollary we obtain the following.

If \( p_0 \leq -\frac{1}{\vartheta} h_0 \), then \( C^o \) contains the whole ray \( r = \{ x_0 + tu \mid t \geq 0 \} \). If \( p_0 > -\frac{1}{\vartheta} h_0 \), then \( C^o \) contains the interval \( \left[ x_0, x_0 + \frac{1}{p_0 + g_0\sqrt{\vartheta - 1}}u \right] \).

The matrix in (2) can be written as
\[
\begin{pmatrix}
-1 & 0 \\
-1 & 0 \\
-(\vartheta - 1) & -1 & 0 & 0 & 0
\end{pmatrix} - \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix} \begin{pmatrix}
f'(x_0) \\
f''(x_0) + f'(x_0)f'(x_0)^T
\end{pmatrix},
\]
and hence is a sum of a positive semi-definite rank 1 matrix and a negative semi-definite rank \( n \) matrix. Therefore the connected components of set (2) are convex.

Let now \( x = x_0 + tu \), \( t > 0 \). The quadratic inequality defining set (2) can then be rewritten as
\[
1 - 2tp_0 + \vartheta t^2 p_0^2 - (\vartheta - 1)t^2 h_0 > 0,
\]
or equivalently \( |t^{-1} - p_0| > \sqrt{\vartheta - 1}g_0 \). Hence the intersection of the ray \( r \) with the Dikin set equals the whole ray if and only if \( p_0 + g_0\sqrt{\vartheta - 1} \leq 0 \). In the opposite case the ray \( r \) leaves the Dikin set at the point \( x_0 + (p_0 + g_0\sqrt{\vartheta - 1})^{-1}u \). Therefore the Dikin set is contained in \( C^o \).

Finally, the Dikin ellipsoid (1) centered on \( x_0 \) is given by
\[
\left\{ x \left| \begin{pmatrix}
1 \\
1 \\
0
\end{pmatrix} \begin{pmatrix}
(x - x_0)^T \\
(x - x_0)^T \\
(x - x_0)^T
\end{pmatrix} > 0 \right. \right\}.
\]
The set of points on which the involved quadratic form is positive is contained in the set of points on which the form from (2) is positive, as is evidenced by the easily verifiable matrix inequality
\[
\begin{pmatrix}
1 & -(\vartheta - 1) & 0 \\
-1 & -\vartheta f''(x_0) & f'(x_0)f'(x_0)^T
\end{pmatrix} \geq \frac{1}{\vartheta - 1} \begin{pmatrix}
1 & 0 \\
0 & -\vartheta f''(x_0)
\end{pmatrix}.
\]
Hence the Dikin ellipsoid must be contained in the Dikin set. This completes the proof. \( \square \)

In a similar manner we may prove the following result on the outer approximation of \( C^o \), using the lower bound \( f^- \) from Corollary 3.3.

**Lemma 3.5.** Let \( C \subseteq \mathbb{R}^n \) be a regular convex set, \( f \) a projectively self-concordant barrier on \( C \) with parameter \( \gamma = \frac{\vartheta - 2}{\sqrt{\vartheta - 1} \vartheta} \), and \( x_0 \in C^o \) a point. Then the set defined in (3) is convex and contains \( C^o \). \( \square \)

For a self-concordant barrier on a set \( C \) in the sense of Definition 1.1 the construction of a non-trivial outer approximating set to \( C \) from the first two derivatives of the barrier at a given interior point only is not possible, because the quadratic approximation at this point is defined on the whole space. Hence the notion of projective self-concordance allows for qualitatively different, tighter approximations than classical self-concordance and as a consequence makes possible the design of more efficient interior-point algorithms.

### 4 Duality

In this section we develop a duality theory for projectively self-concordant barriers. We need the following technical result.

**Lemma 4.1.** Let \( D \subseteq \mathbb{R}^n \) be an open convex set, and let \( f : D \to \mathbb{R} \) be a function of class \( C^2 \). Suppose there exists a co-vector field \( w \) on \( D \) such that \( f''(x) - f'(x) \otimes w(x) - w(x) \otimes f'(x) > 0 \) for all \( x \in D \). Then \( f \) is quasi-convex.
Proof. For the sake of contradiction, let $I \subset D$ be a closed finite interval such that $f_I$ assumes its maximum at some point $x$ in the relative interior of $I$. Let $u \in T_x D$ be a non-zero vector in the direction of $I$. Then $f'(x)[u] = 0$, $f''(x)[u, u] \leq 0$, and hence $f''(x)[u, u] - 2f'(x)[u] \cdot w(x)[u] \leq 0$, which contradicts the assumption on $f$.

Thus for every $x, y \in D$ and every $\lambda \in (0, 1)$ we have $f(\lambda x + (1 - \lambda)y) \leq \max(f(x), f(y))$, i.e., $f$ is quasi-convex.

**Definition 4.2.** Let $C \subset \mathbb{R}^n$ be a regular convex set, and let $f : C^* \to \mathbb{R}$ be a projectively self-concordant barrier with parameter $\gamma$ on $C$. The dual function is defined by

$$f_*(p) = -\min_{x \in C^*} (f(x) + \log(1 + \langle x, p \rangle)).$$

Here the domain of definition of $f_*$ consists of all points $p$ such that the above function has a (local) minimizer in the interior of $C$.

Let us now establish some properties of the dual function. We shall concentrate on local properties, as global properties such as convexity of the domain of definition of the dual function do not hold without further assumptions on the behaviour of $f$ at infinity.

**Lemma 4.3.** The minimizer of $h_p(x) = f(x) + \log(1 + \langle x, p \rangle)$ is unique and the Hessian of $h_p$ is positive definite at this point.

Proof. The gradient of $h_p$ is given by $f'(x) + \frac{p}{1 + \langle x, p \rangle}$ and the Hessian is given by

$$\frac{\partial^2 h_p}{\partial x^2} = f''(x) - \frac{pp^T}{(1 + \langle p, x \rangle)^2} = f''(x) - f'(x) \otimes f'(x) + h'_p(x) \otimes w(x) + w(x) \otimes h'_p(x)$$

with $w(x) = \frac{1}{2}(f'(x) - \frac{p}{1 + \langle x, p \rangle})$. By virtue of the first property in Definition 4.2 the function $h_p$ satisfies the conditions of Lemma 4.1 and is hence quasi-convex, and $h''_p \succ 0$ whenever $h'_p = 0$. Our claim now easily follows.

**Lemma 4.4.** The map $x \mapsto p = \frac{f'(x)}{1 + \langle f'(x), x \rangle}$ is a bijection between the domains of definition of $f$ and $f_*$. It takes the positive definite symmetric form $f'' - f' \otimes f'$ to the form $f''_* - f'_* \otimes f'_*$, which as a consequence is also positive definite. It also takes the symmetric 3-form $f''_{ijk} - 2f''_{ik}f'_j - 2f''_{ij}f'_k - 2f''_{jk}f'_i + 4f''_{ij}f'_k$ to the 3-form $-f''_{ijk} - 2f''_{ij}f'_{ik} - 2f''_{ik}f'_{ij} - 2f''_{jk}f'_{ij} + 4f''_{ij}f'_{ik}$.

Here we denoted $f'_i = \frac{\partial f}{\partial x_i}$, $f''_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$ etc.

Proof. By Lemma 4.3 the minimizer is a function of the point $p$ in the domain of definition of $f_*$. On the other hand, setting $h'_p(x) = 0$ we obtain $f'(x) = -\frac{p}{1 + \langle f'(x), x \rangle}$, or equivalently $p = -\frac{f'(x)}{1 + \langle f'(x), x \rangle}$. Hence $p$ is also a function of the minimizer, and the first claim follows.

Differentiating the expression for $p$ we obtain

$$\frac{\partial p}{\partial x} = \frac{-(1 + \langle f'(x), x \rangle)I + f'(x)x^T(f''(x) - f'(x)f'(x)^T)}{(1 + \langle f'(x), x \rangle)^2},$$

and by inversion

$$\frac{\partial x}{\partial p} = -(1 + \langle f'(x), x \rangle)(f''(x) - f'(x)f'(x)^T)^{-1}(I + f'(x)x^T).$$
The expression for $p$ also yields $f'_*(p) = -f(x) + \log(1 + \langle f'(x), x \rangle)$, and by differentiation
\[
f'_*(p) = \left( \frac{\partial x}{\partial p} \right)^T \frac{f''(x) - f'(x)f'(x)^T}{1 + \langle f'(x), x \rangle} = -(1 + \langle f'(x), x \rangle)x.
\]

Differentiating $f'_*$ further, we get
\[
f''_*(p) = -((1 + \langle f'(x), x \rangle)I + xf'(x)^T + xx^Tf''(x)) \frac{\partial x}{\partial p}
\]
\[
= -((1 + \langle f'(x), x \rangle)(I + xf'(x)^T) + xx^T(f''(x) - f'(x)f'(x)^T)) \frac{\partial x}{\partial p}
\]
\[
= (1 + \langle f'(x), x \rangle)^2 ((I + xf'(x)^T)(f''(x) - f'(x)f'(x)^T)^{-1}(I + f'(x)x^T) + xx^T),
\]
and therefore
\[
f''_*(p) - f'_*(p)f'_*(p)^T = (1 + \langle f'(x), x \rangle)^2((I + xf'(x)^T)(f''(x) - f'(x)f'(x)^T)^{-1}(I + f'(x)x^T)
\]
\[
= \left( \frac{\partial x}{\partial p} \right)^T (f''(x) - f'(x)f'(x)^T) \frac{\partial x}{\partial p}.
\]

The second claim of the lemma follows.

Further we have
\[
\frac{\partial x^k}{\partial p_m} \frac{\partial^2 p_m}{\partial x^i \partial x^j} = \frac{\partial x^k}{\partial p_m} \frac{\partial}{\partial x^j} (-(1 + \langle f', x \rangle)\delta^j_m + f''_{ij}x^i)(f''_{ik} - f''_{j}f''_{i})
\]
\[
= g^{km} \left(-g_{mj}g_{i}x^i - g_{mj}g_{ij}x^i \right) \frac{1 + \langle f', x \rangle}{1 + \langle f', x \rangle} + f''_{mj} - g_{mj}f''_{i} - g_{mj}f''_{i} - 2g_{i}f''_{m} - 2f''_{j}f''_{i}.
\]
\[
(f'_*)^T \frac{\partial p}{\partial x} = \frac{x^T g}{1 + \langle f'(x), x \rangle}.
\]

Here we denoted $g = f'' - f' \otimes f'$ and used the summation convention over repeating indices. The quantities $g_{ij}$ are the elements of the second order tensor $g$, while $g^{ij}$ are the elements of its inverse. The symbol $\delta^i_m$ is the Kronecker symbol. Differentiating the identity
\[
\left( \frac{\partial p}{\partial x} \right)^T (f'_*(p) - f'_*(p)f'_*(p)^T) \frac{\partial p}{\partial x} = f''(x) - f'(x)f'(x)^T
\]
with respect to $x$ and replacing the derivatives of $p$ by the derivatives of $f$ according to the above relations, we obtain after some calculations that
\[
\frac{\partial p_r}{\partial x^i} \frac{\partial p_s}{\partial x^j} \frac{\partial p_t}{\partial x^k} (f''_{rst} - 2f''_{rs}f''_{st} - 2f''_{rs}f''_{sr} - 2f''_{rs}f''_{ss}f''_{st} + f''_{rst}f''_{sr} + 4f''_{rs}f''_{ss}f''_{st}) + f''_{ijk} - 2f''_{ijk}f''_{k} - 2f''_{ik}f''_{j}f''_{k} - 2f''_{ik}f''_{j}f''_{k} + g^{ijk}f''_{i}f''_{j}f''_{k} = 0.
\]

This proves the last claim. \hfill \square

Lemma 4.4 easily yields the following result.

**Corollary 4.5.** The projective self-concordance condition on $f$ implies a similar condition on the dual function $f_*$ with the same parameter $\gamma$. \hfill \square

The exists a symmetry between $f$ and $f_*$ which justifies the notion of duality. We have $f'_*(p) = -\frac{x}{1 + \langle f'(p), p \rangle}$, or equivalently $x = -\frac{f'_*(p)}{1 + \langle f'(p), p \rangle}$, which is similar to the expression for $p$ as a function of $x$. It is also easily verified that the functions $F, F_*$ constructed from $f, f_*$, respectively, as in Theorem 2.1 are the Legendre duals of each other.
5 Construction of projectively self-concordant barriers

In this section we show how to construct projectively self-concordant barriers on convex sets from such barriers on simpler sets.

Affine sections: Let $C \subset \mathbb{R}^n$ be a regular convex set and $f$ a projectively self-concordant barrier on $C$ with parameter $\gamma$. Let $A \subset \mathbb{R}^n$ be an affine subspace intersecting the interior of $C$, and define $\tilde{C} = C \cap A$. From Definition 1.2 it follows in a straightforward manner that $\tilde{f} = f|_{\tilde{C}}$ is a projectively self-concordant barrier on $\tilde{C}$ with parameter $\gamma$.

Projective images: Let $C \subset \mathbb{R}^n$ be a regular convex set and $f$ a projectively self-concordant barrier on $C$ with parameter $\gamma$. Let $q$ be an affine-linear function on $\mathbb{R}^n$ and $A : \mathbb{R}^n \to \mathbb{R}^n$ an affine-linear isomorphism, such that there is no point at which $q$ and $A$ vanish simultaneously. Define

$$\tilde{C} = \left\{ \frac{A(x)}{q(x)} \mid x \in C, \, q(x) > 0 \right\}$$

and assume this set is regular. Then the function $\tilde{f}$ defined by

$$\tilde{f} \left( \frac{A(x)}{q(x)} \right) = f(x) + \log q(x)$$

is a projectively self-concordant barrier on $\tilde{C}$ with parameter $\gamma$.

Indeed, let $y^*$ be a boundary point of $\tilde{C}$ and $y_k \in \tilde{C}^o$, $k \in \mathbb{N}$, a sequence of points tending to $y^*$. Let $x^* \in \partial \tilde{C}$ and $x_k \in C^o$ be such that $\frac{A(x^*)}{q(x^*)} = y^*$ and $\frac{A(x_k)}{q(x_k)} = y_k$. Then $q(x_k) \to q(x^*) > 0$ as $k \to +\infty$, and hence $\log q(x_k) \to \log q(x^*)$. On the other hand, $f(x_k) \to +\infty$, and hence also $\tilde{f}(y_k) \to +\infty$. This proves the second condition in Definition 1.2.

The other two conditions follow from the following lemma.

Lemma 5.1. The map $x \mapsto y = \frac{A(x)}{q(x)}$ carries the symmetric second order tensor $G = f'' - f' \otimes f'$ on $C^o$ to $\tilde{G} = \tilde{f}'' - \tilde{f}' \otimes \tilde{f}'$ on $\tilde{C}^o$ and the symmetric third order tensor $T_{ijk} = f'''_{ij}f_k' - 2f''_{ij}f_k'f'_i' - 2f''_{ik}f_j'f'_i' + 4f''_{ij}f'_i'f'_k' - 2f''_{ij}f'_i'f'_k' + 4f''_{ik}f'_i'f'_j' - 2f''_{ik}f'_i'f'_j' + 4f''_{jk}f'_i'f'_j' - 2f''_{jk}f'_i'f'_j' + 4f''_{jk}f'_i'f'_j' - 2f''_{jk}f'_i'f'_j' + 4f''_{jk}f'_i'f'_j' - 2f''_{jk}f'_i'f'_j' + 4f''_{jk}f'_i'f'_j' - 2f''_{jk}f'_i'f'_j'$ on $C^o$ to $\tilde{T}_{ijk} = \tilde{f}'''_{ij} \tilde{f}_k' - 2\tilde{f}''_{ij} \tilde{f}_k' \tilde{f}'_i' - 2\tilde{f}''_{ik} \tilde{f}_j' \tilde{f}'_i' + 4\tilde{f}''_{ij} \tilde{f}'_i' \tilde{f}'_k' - 2\tilde{f}''_{ij} \tilde{f}'_i' \tilde{f}'_k' + 4\tilde{f}''_{ik} \tilde{f}'_i' \tilde{f}'_j' - 2\tilde{f}''_{ik} \tilde{f}'_i' \tilde{f}'_j' + 4\tilde{f}''_{jk} \tilde{f}'_i' \tilde{f}'_j' - 2\tilde{f}''_{jk} \tilde{f}'_i' \tilde{f}'_j' + 4\tilde{f}''_{jk} \tilde{f}'_i' \tilde{f}'_j' - 2\tilde{f}''_{jk} \tilde{f}'_i' \tilde{f}'_j' + 4\tilde{f}''_{jk} \tilde{f}'_i' \tilde{f}'_j' - 2\tilde{f}''_{jk} \tilde{f}'_i' \tilde{f}'_j'$ on $\tilde{C}^o$.

The lemma can be proven by direct calculation. Instead of reproducing this here we shall rather use a geometric result which yields an interpretation of these tensors.

Proof. Define the domain $D = \{(t,x) \mid t > 0, \, t^{-1}x \in C^o\} \subset \mathbb{R}^{n+1}$ and a function $F : D \to \mathbb{R}$ by $F(t,x) = \log t - f(t^{-1}x)$. Let $\Gamma \subset \mathbb{R}^{n+1}$ be the level hypersurface $F = 0$ and let $\iota : C^o \to \Gamma$ be the bijection defined by $\iota(x) = (e^f(x), e^{f(x)}x)$, i.e., the map taking $x \in C^o$ to the unique point in $\Gamma$ which lies on the same ray as $(1,x)$. Then by [4, Lemma 2.3] the tensors $-G$ and $-T$ on $C^o$ are taken by $\iota$ to the centro-affine metric and the centro-affine cubic form of the hypersurface $\Gamma$, respectively.

Let $\bar{D}, \bar{F}, \bar{\Gamma}, \bar{\iota}$ be similar objects defined by means of the function $\tilde{f}$.

Consider now the linear map $L : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ which takes $(1,x)$ to $(q(x), A(x))$ for every $x \in \mathbb{R}^n$. This map is a bijection because $q$ and $A$ do not vanish simultaneously. Moreover, the domain $\bar{D} = \{(t,y) \mid t > 0, \, t^{-1}y \in \tilde{C}^o\} \subset \mathbb{R}^{n+1}$ is a subset of the linear image $L[D]$. Moreover, for every $y = \frac{A(x)}{q(x)} \in \tilde{C}^o$ and every $\lambda > 0$ we have

$$\bar{F}(\lambda q(x), \lambda A(x)) = \log \lambda + \log q(x) - \tilde{f}(q(x)^{-1}A(x)) = \log \lambda + \log q(x) - f(x) - \log q(x) = F(\lambda, \lambda x).$$

Hence $\bar{F} = F \circ L^{-1}$, and the level hypersurface $\bar{\Gamma}$ is a subset of $L[\Gamma]$. But linear isomorphisms leave the centro-affine metric and the centro-affine cubic form invariant by construction [6]. Thus
third property in Definition 1.2 follow from the corresponding conditions on \( p \). We have self-concordant barriers on these sets with parameters \( \gamma \).

The second property in Definition 1.2 follows from the corresponding property in Definition 1.1.

**Proof.** The second property in Definition 1.2 follows from the corresponding property in Definition 1.1.

Introduce a coordinate system \((t, x_1, \ldots, x_{n-1})\) in \( \mathbb{R}^n \) such that the affine section \( C \) lies in the hyperplane given by \( t = 1 \), and assume the notations in the first part of the proof of Theorem 2.1. We have \( p_1(-1) + p_2(1) = 64(\nu - 2) - 2\nu^2 \geq 0 \), and therefore \( \nu \geq 2 \). By Theorem 2.1 the first and third property in Definition 1.2 follow from the corresponding conditions on \( F \) in Definition 1.1.

This completes the proof. \( \square \)

**Remark 6.2.** If the section \( C \) of \( K \) is not proper, i.e., contains the origin, then \( f = \nu^{-1}F|_C \) still satisfies the second and third condition in Definition 1.2, but the matrix inequality in the first condition becomes non-strict.

Now we are in a position to construct projectively self-concordant barriers on different sets.

**Polyhedra:** Let \( P = \{ x \in \mathbb{R}^n \mid Ax \leq b \} \) be a polyhedron given by \( m \) linear inequalities, with \( b \neq 0 \) and linearly independent columns of \( A \). Then \( f(x) = -\frac{1}{m} \sum_{i=1}^{m} \log(b - Ax)_i \) is a projectively self-concordant barrier on \( P \) with parameter \( \gamma = \frac{m - 2}{\sqrt{m - 1}} \).

Indeed, \( P \) can be represented as a proper affine section of the cone \( \mathbb{R}^m_+ \), on which the standard logarithmic barrier with parameter \( \nu = m \) gives rise to the above function.\[ \nu^{-1} \circ L^{-1} \circ L \] maps the tensors \(-G, -T\) to \(-G, -T\), respectively. But this is exactly the map which takes \( y = \frac{A(x)}{q(x)} \) to \( x \), which completes the proof. \( \square \)

It follows that a projectively self-concordant barrier \( f \) on a convex set \( C \) defines a projectively invariant metric \( G = f'' - f' \otimes f' \) on the interior \( C^0 \).

**Direct products:** Let \( C_i \subset \mathbb{R}^{n_i} \), \( i = 1, 2 \), be regular convex sets, and let \( f_i \) be projectively self-concordant barriers on these sets with parameters \( \gamma_i = \frac{\nu - 2}{\sqrt{\nu - 1}} \), respectively. Then \( f(x, y) = \frac{f_1(x) + f_2(y)}{\sqrt{\nu_1 + \nu_2 - 2}} \) is a projectively self-concordant barrier on the set \( C = C_1 \times C_2 \) with parameter \( \gamma = \frac{\nu - 2}{\sqrt{\nu - 1}} \).

Indeed, the second condition in Definition 1.2 follows from the corresponding conditions on the barriers \( f_i \). The first and third condition follow from Theorem 2.1 and the fact that a barrier on a direct product of cones can be constructed as a sum of barriers on the individual factor cones, its parameter being the sum of the parameters of the barriers on the factor cones.

Note that it is possible to construct self-concordant barriers in the sense of Definition 1.1 on affine sections and images and direct products from a barrier on the original set. However, the equivariance with respect to projective transformations is a qualitatively new property of projective self-concordance.

6 Examples

In this section we construct projectively self-concordant barriers on different sets by virtue of the following result.

**Lemma 6.1.** Let \( n \geq 2 \), let \( K \subset \mathbb{R}^n \) be a regular convex cone, and \( F : K^0 \rightarrow \mathbb{R} \) a logarithmically homogeneous self-concordant barrier on \( K \) with parameter \( \nu \). Then \( \nu \geq 2 \), and \( f = \nu^{-1}F|_{C^0} \) is a projectively self-concordant barrier on every proper affine section \( C \) of \( K \) with parameter \( \gamma = \frac{\nu - 2}{\sqrt{\nu - 1}} \).

**Proof.** The second property in Definition 1.2 follows from the corresponding property in Definition 1.1.

Introduce a coordinate system \((t, x_1, \ldots, x_{n-1})\) in \( \mathbb{R}^n \) such that the affine section \( C \) lies in the hyperplane given by \( t = 1 \), and assume the notations in the first part of the proof of Theorem 2.1.

We have \( p_\mu(-1) + p_\mu(1) = 64(\nu - 2) - 2\nu^2 \geq 0 \), and therefore \( \nu \geq 2 \). By Theorem 2.1 the first and third property in Definition 1.2 follow from the corresponding conditions on \( F \) in Definition 1.1.

This completes the proof. \( \square \)

Remark 6.2. If the section \( C \) of \( K \) is not proper, i.e., contains the origin, then \( f = \nu^{-1}F|_{C^0} \) still satisfies the second and third condition in Definition 1.2, but the matrix inequality in the first condition becomes non-strict.

Now we are in a position to construct projectively self-concordant barriers on different sets.

**Polyhedra:** Let \( P = \{ x \in \mathbb{R}^n \mid Ax \leq b \} \) be a polyhedron given by \( m \) linear inequalities, with \( b \neq 0 \) and linearly independent columns of \( A \). Then \( f(x) = -\frac{1}{m} \sum_{i=1}^{m} \log(b - Ax)_i \) is a projectively self-concordant barrier on \( P \) with parameter \( \gamma = \frac{m - 2}{\sqrt{m - 1}} \).

Indeed, \( P \) can be represented as a proper affine section of the cone \( \mathbb{R}^m_+ \), on which the standard logarithmic barrier with parameter \( \nu = m \) gives rise to the above function.
Spectrahedra: Let $S = \{ x \mid A(x) \succeq 0 \}$ be a spectrahedron given by a linear matrix inequality of size $m \times m$, with $A$ an inhomogeneous affine map. Then $f(x) = -\frac{1}{m} \log \det A(x)$ is a projectively self-concordant barrier on $P$ with parameter $\gamma = \frac{m-2}{\sqrt{m-1}}$.

Epigraph of exponential function: Consider the set $C_{\exp} = \{ (x, y) \mid y \geq e^x \} \subset \mathbb{R}^2$. On this set we have the projectively self-concordant barrier

$$f(x, y) = -\frac{1}{3} (\log(\log y - x) + \log y)$$

with parameter $\gamma = \frac{\sqrt{2}}{3}$, which comes from the barrier on the exponential cone with parameter $\nu = 3$ defined in [2].

Epigraph of power functions: For $p > 1$, consider the set $C_p = \{ (x, y) \mid y \geq |x|^p \}$. This set can be represented as an affine slice of the power cone

$$K_p = \{ (x, y, z) \mid |x| \leq y^{1/p} z^{1/q} \},$$

where $\frac{1}{q} = 1 - \frac{1}{p}$. The canonical barrier on this cone [3] leads to the projectively self-concordant barrier

$$f(x, y) = -\frac{p+1}{3p} \log y + \frac{1}{3} \phi(y^{-1/p} |x|),$$

with the function $\phi : [0, 1) \to \mathbb{R}$ given implicitly by the relations

$$\log t = -\frac{1}{2p} \log \left( 1 + \frac{p+1}{\rho} \right) - \frac{1}{2q} \log \left( 1 + \frac{q+1}{\rho} \right),$$

$$2\phi(t) = \left( 1 + \frac{1}{p} \right) \log (\rho + p + 1) + \left( 1 + \frac{1}{q} \right) \log (\rho + q + 1),$$

with $\rho$ ranging from 0 to $+\infty$. The parameter of this barrier is given by $\gamma = \frac{\max(p,q)-2}{\sqrt{(2 \max(p,q)-1)(\max(p,q)+1)}}$.

We have the following general existence result.

**Corollary 6.3.** Let $C \subset \mathbb{R}^n$ be a regular convex set. Then there exists a projectively self-concordant barrier with parameter $\gamma \leq \frac{n-1}{\sqrt{n}}$ on $C$.

**Proof.** The set $K = \text{cl} \{ (t, x) \mid t^{-1} x \in C, \ t > 0 \} \subset \mathbb{R}^{n+1}$ is a regular convex cone, and $C$ can be represented as a proper affine section of $K$. But on $K$ there exist logarithmically homogeneous self-concordant barriers with parameter $\nu \leq n + 1$ [3],[1]. The claim now follows from Lemma 6.1.

7 Outlook

In this contribution we presented a new class of barrier functions for convex optimization with a modified self-concordance property, which we called projective self-concordance. It has superior theoretical properties in comparison to the class of classical self-concordant barrier functions. In particular, the Dikin sets are larger and there exists also an outer approximation of the underlying convex set centered on an arbitrary interior point which can be constructed from the derivatives of the barrier at this point.
The elementary iteration in interior-point methods consists in a Newton step for the problem of minimization of the sum of the barrier function and a linear function. The barrier is replaced by its quadratic approximation around the current iterate and the minimum of the approximated sum is chosen as the next iterate.

In order to implement this scheme for a projectively self-concordant barrier \( f \) on a set \( C \) we need to define an analog of the quadratic approximation around an interior point \( x_0 \in C^o \). This approximation may be defined by the function

\[
q(x) = f(x_0) - \frac{1}{2} \log \left( (1 - \langle f'(x_0), x - x_0 \rangle)^2 - (x - x_0)^T (f''(x_0) - f'(x_0)f'(x_0)^T)(x - x_0) \right),
\]

which shares the function values and the first two derivatives at \( x_0 \) with \( f \) and is itself projectively self-concordant with the lowest possible parameter value \( \gamma = 0 \). The minimizer of the sum of \( q \) and a linear function can be computed analytically.

The domain of definition of the quadratic approximation \( q \) is given by the connection component of the set

\[
\left\{ x \in \mathbb{R}^n \mid \begin{pmatrix} 1 \\ x - x_0 \end{pmatrix}^T \begin{pmatrix} 1 & -f'(x_0)^T \\ f'(x_0) & -f''(x_0) + 2f'(x_0)f'(x_0)^T \end{pmatrix} \begin{pmatrix} 1 \\ x - x_0 \end{pmatrix} > 0 \right\}
\]

which contains \( x_0 \). Note that the domain of definition is a regular convex set, delineated by a quadric, which at the same time serves as an approximation of the original set. Thus projectively self-concordant barriers admit not only a quadratic approximation around each interior point, but also furnish a quadratic approximation of the underlying set. This additional feature might also open possibilities for new classes of interior-point methods.

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