INVARIANT SOLUTIONS OF GRADIENT $k$-YAMABE SOLITONS

TOKURA, W. 1,*, BARBOZA, M. 2, BATISTA, E. 3, AND KAI, P. 4

Abstract. The purpose of this paper is to study gradient $k$-Yamabe solitons conformal to pseudo-Euclidean space. We characterize all such solitons invariant under the action of an $(n - 1)$-dimensional translation group. For rotational invariant solutions, we provide the classification of solitons with null curvatures. As an application, we construct infinitely many explicit examples of geodesically complete steady gradient $k$-Yamabe solitons conformal to the Lorentzian space.

1. Introduction and main results

The concept of gradient $k$-Yamabe soliton, introduced in the celebrated work [7], corresponds to a natural generalization of gradient Yamabe solitons. We recall that a pseudo-Riemannian manifold $(M^n, g)$ is a gradient $k$-Yamabe soliton if it admits a constant $\lambda \in \mathbb{R}$ and a function $f \in C^\infty(M)$ satisfying the equation

$$\nabla^2 f = 2(n-1)(\sigma_k - \lambda)g,$$

where $\nabla^2 f$ and $\sigma_k$ stand, respectively, for the Hessian of $f$ and the $\sigma_k$-curvature of $g$. Recall that, if we denote by $\lambda_1, \lambda_2, \ldots, \lambda_n$ the eigenvalues of the symmetric endomorphism $g^{-1}A_g$, where $A_g$ is the Schouten tensor defined by

$$A_g = \frac{1}{n-2} \left[ Ric_g - \frac{scal_g}{2(n-1)}g \right],$$

then the $\sigma_k$-curvature of $g$ is defined as the $k$-th symmetric elementary function of $\lambda_1, \ldots, \lambda_n$, namely

$$\sigma_k = \sigma_k(g^{-1}A_g) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}, \quad \text{for} \quad 1 \leq k \leq n.$$ 

Since $\sigma_1$ is the trace of $g^{-1}A_g$, the gradient 1-Yamabe solitons simply correspond to gradient Yamabe solitons [3, 6, 11, 12, 16, 19]. As usual, the quadruple $(M^n, g, f, \lambda)$ is classified into three types according to the sign of $\lambda$: expanding if $\lambda < 0$, steady if $\lambda = 0$ and shrinking if $\lambda > 0$. Moreover, when $f$ is a constant function the soliton is called trivial.

In recent years, many efforts have been devoted to study the geometry of Riemannian Yamabe solitons and their generalizations. For instance, Hsu in [13] shown that any compact gradient 1-Yamabe soliton is trivial. For $k > 1$, the extension of the previous result was recently investigated. Catino et al. [7] proved that any compact gradient $k$-Yamabe soliton with nonnegative Ricci curvature is trivial. Bo et al. [5] also proved that any compact gradient $k$-Yamabe soliton with negative constant scalar curvature necessarily has constant $\sigma_k$-curvature. The previous results were generalized in [20], where was shown that any compact gradient $k$-Yamabe soliton must be trivial.

For the noncompact case, Catino et al. [7] provide an important relation between gradient $k$-Yamabe solitons and conformally flat spaces; its result establishes that any complete, noncompact gradient $k$–Yamabe soliton with nonnegative Ricci tensor and positive at some point is rotationally symmetric and globally conformally equivalent to Euclidean space. In the context of conformally flat spaces, Neto and Tenenblat [17] study invariant by translation solutions for gradient 1-Yamabe solitons. They reduced a system of PDEs, that comes from the corresponding 1-Yamabe soliton equation, to a system of ODEs by considering a function invariant under translations and, as a result, a complete classification and infinitely many examples are obtained. It is considered this approach in other works like [1, 2, 3, 6, 10, 14, 15, 19]. In general, the technique of transforming a PDE system into an ODE or a PDE with less independent variables is known as an ansatz and

2010 Mathematics Subject Classification. 53C21, 53C50, 53C25.

Key words and phrases. gradient $k$-Yamabe solitons, Yamabe solitons, $\sigma_k$-curvature, $k$-Yamabe problem, invariant solutions, complete examples.

* Corresponding author.
an important method for generating ansatz is based in the theory of Lie point symmetry groups for PDE\cite{18}.

In this paper, we focus our attention on gradient $k$-Yamabe solitons conformal to pseudo-Euclidean space whose solutions are invariant under the action of an $(n-1)$-dimensional translation group or invariant under the action of a pseudo-orthogonal group. First, we work with the invariant by translation case. More precisely, we consider the pseudo-Riemannian metric

$$
\delta = \sum_{i=1}^{n} \varepsilon_{i}dx_{i} \otimes dx_{i},
$$

in coordinates $x = (x_{1}, \ldots, x_{n})$ of $\mathbb{R}^{n}$, where $n \geq 2$, $\varepsilon_{i} = \pm 1$. For an arbitrary choice of non zero vector $\alpha = (\alpha_{1}, \ldots, \alpha_{n})$ we define the translation function $\xi : \mathbb{R}^{n} \to \mathbb{R}$ by

$$
\xi(x_{1}, \ldots, x_{n}) = \alpha_{1}x_{1} + \cdots + \alpha_{n}x_{n}.
$$

Next, we assume that $\mathbb{R}^{n}$ admits a group of symmetries consisting of translations\cite{18} and we then look for smooth functions $\varphi, f : (a, b) \subset \mathbb{R} \to \mathbb{R}$, with $\varphi > 0$, such that the compositions

$$
f = f \circ \xi : \xi^{-1}(a, b) \to \mathbb{R}, \quad \varphi = \varphi \circ \xi : \xi^{-1}(a, b) \to \mathbb{R},
$$

satisfies the gradient $k$-Yamabe soliton equation

$$
\nabla^{2}f = 2(n-1)(\sigma_{k} - \lambda) \frac{\delta}{\varphi^{2}},
$$

or equivalently,

$$
f_{x_{i}, x_{j}} = \sum_{k=1}^{n} \Gamma^{k}_{ij}f_{x_{k}} = 2(n-1) \left[ \sum_{1 \leq i_{1} \cdots \leq i_{k}} \lambda_{i_{1}} \cdots \lambda_{i_{k}} - \lambda \right] \frac{\delta_{ij}}{\varphi^{2}}, \quad i, j \in \{1, \ldots, n\},
$$

where $\varphi_{x_{i}}, \varphi_{x_{i} x_{j}}$ denote the derivatives of $\varphi$ with respect the variables $x_{i}$ and $x_{i} x_{j}$ respectively, and $\Gamma^{k}_{ij}$ correspond to Christoffel symbols on the conformal metric $\delta \varphi^{-2}$. What has been said above is summed up in the next results.

**Theorem 1.1.** With $(\mathbb{R}^{n}, \delta)$ and $f = f \circ \xi$, $\varphi = \varphi \circ \xi$ as above, the manifold $\xi^{-1}(a, b) \subset \mathbb{R}^{n}$ furnished with the metric tensor

$$
g = \frac{\delta}{\varphi(x)^{2}},
$$

is a gradient $k$-Yamabe soliton if, and only if,

$$
f'' + 2 \frac{f' \varphi'}{\varphi} = 0,
$$

$$
\frac{b_{n,k}}{2} \left( k \varphi \varphi'' - \frac{n}{2} (\varphi')^{2} \right) (\varphi')^{2(k-1)} ||\alpha||^{2k} + \frac{\varphi \varphi'' f'}{2(n-1)} ||\alpha||^{2} = \lambda,
$$

where

$$
b_{n,k} = \frac{(n-1)!}{k!(n-k)!} \left( -1 \right)^{k-1} \frac{1}{2^{k-1}}.
$$

**Remark 1.2.** The previous theorem can be view as an extension of Theorem 2 of \cite{17}, which may be obtained for the particular case $k = 1$.

In the following results we provide the solutions of systems (2) and (3) when $\lambda = 0$, i.e., in the steady case.

**Theorem 1.3.** With the considerations of Theorem (17) if $\alpha = \sum_{i=1}^{n} \alpha_{i} \frac{\partial}{\partial x_{i}}$ is light-like vector, i.e., $||\alpha||^{2} = 0$, then the gradient $k$-Yamabe soliton is steady. Moreover, given the conformal function $\varphi$, the potential function is given by

$$
f(\xi) = \int \frac{c}{\varphi^{2}(\xi)} d\xi + d, \quad c, d \in \mathbb{R}.
$$

In the case $||\alpha||^{2} \neq 0$, the behavior of $\varphi$ is classified according to the relation between $2k$ and the dimension $n$. First, we consider the case $n \neq 2k$.

**Theorem 1.4.** Let $(\mathbb{R}^{n}, \delta \varphi^{-2}), n \neq 2k$, be a steady gradient $k$-Yamabe soliton with $f$ as potential function. Then $\varphi$ and $f$ are invariant under an $(n-1)$-dimensional translation group whose basic invariant is $\xi = \sum \alpha_{i}x_{i}$ and $\alpha = \sum_{i=1}^{n} \alpha_{i} \frac{\partial}{\partial x_{i}}$ is a space-like or time-like vector if, and only if, $\varphi$ and $f$ verify
(a) In the case $\varphi' = 0$,

$$\varphi(\xi) = b, \quad f(\xi) = c\xi + d, \quad c, d \in \mathbb{R}, \quad b \in (0, \infty).$$

(b) In the case $\varphi' \neq 0$,

\[ f(\xi) = \int \frac{c}{\varphi'^2(\xi)} \, d\xi + d, \quad c, d \in \mathbb{R}, \quad \text{(4)} \]

and

\[ \int \frac{d\varphi}{\varphi^2 \left[ \left( n + 2 \right) - e^{(1 - 2)\varphi} - e^{2k - 2} - 2k - n + 2k \right] + c_1} = \frac{2k}{2k - n} \xi + c_2, \quad c_1, c_2 \in \mathbb{R}, \quad \text{(5)} \]

where

\[ p = \frac{c}{2(n - 1)kB_{n,k}} \left( \frac{2k - n}{2k} \right)^{2(k - 1)} \]

Now, for $n = 2k$, we have the following result.

**Theorem 1.5.** Let $(\mathbb{R}^n, \delta_{ij}\varphi^{-2})$, $n = 2k$, be a steady gradient $k$-Yamabe soliton with $f$ as potential function. Then $\varphi$ and $f$ are invariant under an $(n - 1)$-dimensional translation group whose basic invariant is $\xi = \sum \alpha_i x_i$ and $\alpha = \sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial x_i}$ is a space-like or time-like vector if, and only if, $\varphi$ and $f$ satisfy

(a) In the case $\varphi' = 0$,

$$\varphi(\xi) = c_1, \quad f(\xi) = c\xi + d, \quad c, d \in \mathbb{R}, \quad c_1 \in (0, \infty).$$

(b) In the case $\varphi' \neq 0$,

\[ f(\xi) = \int \frac{c}{\varphi'^2(\xi)} \, d\xi + d, \quad c, d \in \mathbb{R}, \quad \text{(6)} \]

and

\[ \int \frac{d\varphi}{\sqrt{b_{n,k}n^2\varphi + c_1\varphi^{n-1}}} = \xi + c_2, \quad c_1, c_2 \in \mathbb{R}. \quad \text{(7)} \]

Following, we deal with the rotational case. In the same way as in the invariant by translation group, we consider $\mathbb{R}^n$ endowed with the pseudo-Euclidean metric $\delta = \sum_{i=1}^{n} \varepsilon_i dx_i \otimes dx_i$ and we define the rotational function $r : \mathbb{R}^n \rightarrow \mathbb{R}$ by

\[ r(x_1, \ldots, x_n) = \varepsilon_1 x_1^2 + \cdots + \varepsilon_n x_n^2. \]

Next, we look for smooth functions $\varphi, f : (a, b) \subset \mathbb{R} \rightarrow \mathbb{R}$, with $\varphi > 0$, such that the compositions

\[ f = f \circ r : r^{-1}(a, b) \rightarrow \mathbb{R}, \quad \varphi = \varphi \circ r : r^{-1}(a, b) \rightarrow \mathbb{R}, \]

satisfies the gradient $k$-Yamabe soliton equation.

**Theorem 1.6.** Let $(\mathbb{R}^n, \delta_{ij}\varphi^{-2})$ be a conformal to pseudo-Euclidean space. Consider smooth functions $\varphi(r)$, $f(r)$, where $r = \sum \varepsilon_i x_i^2$. Then $\delta \varphi^{-2}$ is a gradient $k$-Yamabe soliton with $f$ as a potential function if, and only if, $\varphi$ and $f$ satisfy

\[ f'' + 2f'\varphi' = 0, \quad \text{(8)} \]

\[ c_{n,k} \left[ (n - 1)! \right]^{k-1} \left( 2n\varphi' - 2nr(\varphi')^2 + 4k^2\varphi \varphi'' \right) - \frac{\varphi^2 f'}{n - 1} + \frac{2rf'\varphi'}{n - 1} = \lambda, \quad \text{(9)} \]

where

\[ c_{n,k} = \frac{(n - 1)!}{k!(n - k)!}^2. \]

The next result provide the solutions of (5) and (6) for gradient $k$-Yamabe solitons with null curvatures.

**Theorem 1.7.** Let $(\mathbb{R}^n, \delta_{ij}\varphi^{-2})$ be an invariant by rotation gradient $k$-Yamabe soliton with $f$ as potential function and null $\sigma_1, \sigma_s$, for $s \in \{2, \ldots, n\}$. Then $(\mathbb{R}^n, \delta_{ij}\varphi^{-2})$ is one of the following solitons:
(a) The Gaussian soliton \((\mathbb{R}^n, \delta_{ij}\varphi^{-2})\) with potential function and conformal factor given, respectively, by
\[
f(r) = \frac{(n-1)\lambda}{c_2^2} + r + c_1, \quad \varphi(r) = c_2, \quad c_2 \in \mathbb{R}, \quad c_1 \in (0, \infty).
\]

(b) The soliton \((\mathbb{R}^n, \delta_{ij}\varphi^{-2})\) with potential function and conformal factor given, respectively, by
\[
f(r) = -\frac{(n-1)\lambda}{c_0^2} + r + c_1, \quad \varphi(r) = c_0r, \quad c_0 \in (0, \infty), \quad c_1 \in \mathbb{R}.
\]

We finalize the session with a sufficient condition for the conformal factor \(\varphi\) in Theorem 1.1 and Theorem 1.6 to produce complete metrics in the Riemannian context.

**Theorem 1.8.** Let \((\mathbb{R}^n, \delta_{ij}\varphi^{-2})\), be a Riemannian gradient \(k\)-Yamabe soliton with \(f\) as potential function. If \(f\) and \(\varphi\) are invariant solutions in Theorem 1.1 or Theorem 1.6 with conformal factor satisfying \(0 < |\varphi| \leq L\) for some constant \(L\), then the gradient \(k\)-Yamabe soliton metric \(\delta_{ij}\varphi^{-2}\) is complete.

### 2. Examples

Before proving our main results, we provide examples illustrating the above theorems. It is worth pointing out that geodesic completeness in pseudo-Riemannian manifolds is an essential concept for studying singularities in general relativity. However, obtaining the geodesics and their singularities explicitly in a pseudo-Riemannian manifold is a difficult task.

Our first example provides geodesically complete steady gradient \(k\)-Yamabe solitons conformal to the Lorentzian space (see section 3 for more details).

**Example 2.1.** Consider the Lorentzian space \((\mathbb{R}^n, g)\) with coordinates \((x_1, \ldots, x_n)\) and signature \(\varepsilon_1 = -1, \varepsilon_i = 1\) for all \(i \in \{2, \ldots, n\}\). Let \(\xi = x_1 + x_2\) and choose \(\theta \in \mathbb{N}\). Then, from Theorem 1.3, the functions
\[
f(\xi) = c_\xi + 2c_2\frac{e^{2\theta+1}}{2\theta + 1} + c_4\frac{e^{4\theta+1}}{4\theta + 1}, \quad \varphi(\xi) = \frac{1}{1 + \xi^2}, \quad c \in \mathbb{R},
\]
defines a family of geodesically complete steady gradient \(k\)-Yamabe soliton with potential function \(f\) (see section 3).

**Example 2.2.** In Theorem 1.4 consider \((\mathbb{R}^n, \delta_{ij}\varphi^{-2})\), \(n \neq 2k\), with \(c = 0\), then the functions
\[
f(\xi) = c_0, \quad \varphi(\xi) = \frac{2k-n}{2k} = \xi + c_1, \quad c_0, c_1 \in \mathbb{R},
\]
provide a steady gradient \(k\)-Yamabe soliton in the semi-space \(\xi + c_1 > 0\).

**Example 2.3.** In Theorem 1.5 consider \((\mathbb{R}^n, \delta_{ij}\varphi^{-2})\) with \(n = 2k\), \(k > 1\), \(c_1 = 0\) and \(c = b_n k n^2\), then the functions
\[
f(\xi) = \frac{c(n-1)}{n-2} \left( \frac{n}{n-1} \xi + c_4 \right)^{-1} + c_3, \quad \varphi(\xi) = \frac{n \xi + c_4}{\sqrt{n-1}}, \quad c_3, c_4 \in \mathbb{R},
\]
provide a steady gradient \(k\)-Yamabe soliton in the semi-space \(n \xi + (n-1)c_4 > 0\).

**Example 2.4.** (Gaussian soliton) The Gaussian soliton on \(\mathbb{R}^n\) is given by
\[
g_{ij} = \delta_{ij}, \quad f(x) = \frac{\lambda}{2} |x|^2 + c_1, \quad c_1 \in \mathbb{R}.
\]
Since \(\sigma_k(g) = 0\) for \(k \in \{1, \ldots, n\}\), the fundamental equation turns out
\[
\nabla^2 f = \lambda g.
\]
It is worth noting that we can get the Gaussian soliton from Theorem 1.6. In fact, in Theorem 1.6 consider \(\varepsilon_i = 1\) for all \(i \in \{1, 2, \ldots, n\}\), then the functions
\[
f(r) = \frac{\lambda}{2} r, \quad \varphi(r) = 1,
\]
provide a complete gradient \(k\)-Yamabe soliton with soliton constant \(\lambda\) and potential function \(f\).
Example 2.5. In Theorem 1.6 consider \((\mathbb{R}^n \setminus \{0\}, \delta_{ij} \varphi^{-2})\) with \(\epsilon_i = 1, \forall i \in \{1,2,\ldots,n\}\) and \(k \neq 1\), then the functions

\[
f(r) = -\frac{(n-1)\lambda}{c^2 r}, \quad \varphi(r) = c_0 r, \quad c_0 \in (0, \infty).
\]

provide a family of Riemannian gradient \(k\)-Yamabe soliton with soliton constant \(\lambda\).

Example 2.6. In Theorem 1.6 consider \((\mathbb{R}^n \setminus \{0\}, \delta_{ij} \varphi^{-2})\) with \(\epsilon_i = 1, \forall i \in \{1,2,\ldots,n\}\) and \(k = 1\), then the functions

\[
f(r) = \frac{(n-1)(n+2)}{2} \log(1+r) + c_0, \quad \varphi(r) = \sqrt{1+r}, \quad c_0 \in \mathbb{R}.
\]

provide a family of Riemannian gradient \(k\)-Yamabe soliton with soliton constant \(\lambda = \frac{n-2}{n-1}\). In the particular case in which \(n = 2\), this soliton is known as Hamilton’s cigar soliton [12].

3. Proofs

Proof of Theorem 1.1: It is well known that for the conformal metric \(\bar{g} = \varphi^{-2} \delta\), the Ricci curvature is given by (4):

\[
Ric_{\bar{g}} = \frac{1}{\varphi^2} \left\{ (n-2) \varphi \nabla^2 \varphi + \left| \varphi \Delta \varphi - (n-1) |\nabla \varphi|^2 \right| \delta \right\}.
\]  

(10)

So, we easily see that the scalar curvature on conformal metric is given by

\[
scal_{\bar{g}} = (n-1)(\varphi \Delta \varphi - n |\nabla \varphi|^2).
\]  

(11)

Now, in order to compute the Schouten Tensor on the conformal geometry \(A_\bar{g}\) we evoke the expression

\[
A_{\bar{g}} = \frac{1}{n-2} \left( Ric_{\bar{g}} - \frac{scal_{\bar{g}}}{2(n-1)} \bar{g} \right).
\]

Therefore, from (10) and (11) we deduce that

\[
A_{\bar{g}} = \frac{\nabla^2 \varphi}{\varphi} - \frac{|\nabla \varphi|^2}{2 \varphi^2} \delta.
\]

Throughout this work we will denote by \(\varphi_{x_i}, \varphi_{x_i x_j}\), the derivatives of \(\varphi\) with respect the variables \(x_i\) and \(x_i x_j\), respectively. That being said, since we are assuming that \(\varphi(\xi)\) and \(f(\xi)\) are functions of \(\xi = a_1 x_1 + \cdots + a_n x_n\), we get

\[
\varphi_{x_i} = \varphi' a_i, \quad f_{x_i} = f' a_i, \quad \varphi_{x_i x_j} = \varphi'' a_i a_j, \quad f_{x_i x_j} = f'' a_i a_j.
\]

Hence

\[
(\bar{g}^{-1} A_{\bar{g}})_{ij} = \varepsilon_j \varphi'' \alpha_i a_j - \frac{1}{2} (\varphi')^2 |a_i|^2 \delta_{ij}.
\]

The eigenvalues of \(\bar{g}^{-1} A_{\bar{g}}\) are \(\theta = -\frac{1}{2} (\varphi')^2 |a_i|^2\) with multiplicity \((n-1)\), and \(\mu = (\varphi'' - \frac{1}{2} (\varphi')^2 |a_i|^2\) with multiplicity 1. The formula for \(\sigma_k\) can be found easily by the binomial expansion of \((x-\theta)^{n-1}(x-\mu)\)

\[
\sigma_k = \frac{(n-1)!}{k!(n-k)!} \left( (n-k) \theta + k \mu \right) \theta^{k-1}
\]

= \frac{(n-1)!}{k!(n-k)!} (1-k) \cdot \frac{1}{2^{k-1}} \left[ k \varphi'' - \frac{n}{2} (\varphi')^2 \right] (\varphi'')^{2(k-1)} |a_i|^{2k}.
\]

(12)

Now, in order to compute the Hessian \(\nabla^2 \varphi f\) of \(f\) relatively to \(\bar{g}\) we evoke the expression

\[
(\nabla^2 \varphi f)_{ij} = f_{x_i x_j} - \sum_{k=1}^n \Gamma^k_{ij} f_{x_k},
\]

where the Christoffel symbol \(\Gamma^k_{ij}\) for distinct \(i, j, k\) are given by

\[
\Gamma^k_{ij} = 0, \quad \Gamma^i_{ij} = -\frac{\varphi_{x_i}}{\varphi}, \quad \Gamma^k_{ii} = \varepsilon_{i j k} \frac{\varphi_{x_k}}{\varphi}, \quad \text{and} \quad \Gamma^i_{ii} = -\frac{\varphi_{x_i}}{\varphi}.
\]
where

\[ v \]

Considering and then

\[ w \]

we have from (2) that

\[ f''(\zeta) = k \]

if there exist \( i, j \) such that \( \alpha_i \alpha_j \neq 0 \), then we get

\[ f''(\zeta) + 2(\zeta')^2 = 0, \]

which provides equation (2). And for \( i = j \), substituting (12) and (13) into (1) we obtain (3).

Now, we need to consider case \( \alpha_k = 1 \), \( \alpha = 0 \) for \( k \neq k_0 \). In this case, substituting (13) into (1) we obtain

\[ 2(n - 1)(\sigma_k - \lambda) \frac{\varepsilon_i}{\varphi^2} = -\varepsilon_i \frac{\varphi f'}{\varphi}, \]

for \( i \neq k_0 \), that is, \( \alpha_i = 0 \), and

\[ 2(n - 1)(\sigma_k - \lambda) \frac{\varepsilon_k}{\varphi^2} = f'' + (2 - \varepsilon_k) \frac{\varphi f'}{\varphi}, \]

for \( i = k_0 \), that is, \( \alpha_k = 1 \). However, this equations are equivalent to (2) and (3). This completes the demonstration. \( \square \)

**Proof of Theorem 1.3**: Since \( ||\alpha||^2 = 0 \), we have by equation (3) of Theorem 1.1 that \( \lambda = 0 \). On the other hand, given \( \varphi \) we have from (2) that

\[ f'' + 2(\varphi')^2 = 0. \]

Integrating we get

\[ f(\xi) = \int \frac{c}{\varphi^2(\xi)} d\xi + d, \quad c, d \in \mathbb{R}. \]

**Proof of Theorem 1.4**: Item (a): Since \( \varphi' = 0 \) we have that (3) is trivially satisfied and from (2) we conclude that \( f(\xi) = c\xi + d \) for some \( c, d \in \mathbb{R} \).

Item (b): From equation (2) of Theorem 1.1 we deduce that

\[ f'(\xi) = \frac{c}{\varphi^2(\xi)}, \quad c \in \mathbb{R}, \]

and then

\[ f(\xi) = \int \frac{c}{\varphi^2(\xi)} d\xi + d, \quad c, d \in \mathbb{R}, \]

which provide equation (4).

Next, without loss of generality, we may consider \( ||\alpha||^2 = \pm 1 \). Then, substituting (14) into (3) we deduce

\[ \left( \varphi \varphi'' - \frac{n}{2k} (\varphi')^2 \right) (\varphi')^{2(k-1)} + \frac{c}{2k b_{n,k} (n - 1)} \frac{\varphi'}{\varphi} = 0. \]

(15)

Considering \( v(\xi) = \varphi(\xi) \frac{1}{\varphi'} \), we obtain from (15) the following equivalent condition

\[ v'' + p(v')^{3-2k} v^{2n-2k+4k} = 0. \]

(16)

where

\[ p = \frac{c}{2(n - 1) b_{n,k}} \left( \frac{2k - n}{2k} \right)^{2(k-1)} \]

Now, from one more change \( w(v) = (v')^{3-2k} \), we obtain that (16) is equivalent to the following first order differential equation

\[ w'(v) + (3 - 2k) p w(v) \frac{d}{dv} v^{2n-2k+4k} = 0, \]
whose solution is
\[ w(v) = \left( \frac{(n - 2k)p(1 - 2k)}{2n - n + 2k} \right)^{\frac{2n - n + 2k}{2n - 1}}\left(c_1 + \frac{2k}{n - 2k}\right), \quad c_1 \in \mathbb{R}. \tag{17} \]

Replacing (17) back into \( w(v) = (v')^{3 - 2k} \) we deduce that
\[ v' = \left( \frac{(n - 2k)p(1 - 2k)}{2n - n + 2k} \right)^{\frac{2n - n + 2k}{2n - 1}}\left(c_1 + \frac{2k}{n - 2k}\right). \]

This implies that
\[ \int \frac{dv}{\left( \frac{(n - 2k)p(1 - 2k)}{2n - n + 2k} \right)^{\frac{2n - n + 2k}{2n - 1}}\left(c_1 + \frac{2k}{n - 2k}\right)} = \xi + c_2, \quad c_2 \in \mathbb{R}. \]

Therefore, it follows from \( v(\xi) = \varphi(\xi)^{1 - \frac{2k}{n}} \) that
\[ \int \frac{d\varphi}{\varphi^{\frac{2n - n + 2k}{2n - 1}}\left(c_1 + \frac{2k}{n - 2k}\right)} = \frac{2k}{2k - n} \xi + c_3, \quad c_3 \in \mathbb{R}. \]

which provide equation (5).

\[ \square \]

**Proof of Theorem 1.5:** Item (a): The proof is analogous to the proof of item (a) in Theorem 1.4.

Item (b): From equation (2) of Theorem 1.1 we deduce that
\[ f'(\xi) = c \varphi^{2}(\xi), \quad c \in \mathbb{R}, \tag{18} \]

and then
\[ f(\xi) = \int \frac{c}{\varphi^{2}(\xi)} d\xi + d, \quad c, d \in \mathbb{R}, \]

which provide equation (6).

Substituting (18) into (3) and considering \( n = 2k \), we deduce that
\[ \left[ \varphi \varphi'' - (\varphi')^2 \right] (\varphi')^{(n - 2)} + \varphi' \frac{c}{\varphi b_{n,k}n(n - 1)} = 0. \tag{19} \]

Considering
\[ w(\varphi) = \varphi', \tag{20} \]

we obtain \( w' = \frac{\varphi''}{\varphi} \). So, equation (19) is equivalent to
\[ (w'(w(\varphi))\varphi - w(\varphi)^2)w(\varphi)^{n - 2} + \frac{w(\varphi)}{\varphi} \frac{c}{b_{n,k}n(n - 1)} = 0, \]

whose solution is
\[ w(\varphi) = \frac{1}{\sqrt{b_{n,k}n^2\varphi}} + c_1 \varphi^{n - 1}, \quad c_1 \in \mathbb{R}. \tag{21} \]

Replacing (21) back into (20) we deduce that
\[ \varphi' = \frac{1}{\sqrt{b_{n,k}n^2\varphi}} + c_1 \varphi^{n - 1}, \quad c_1 \in \mathbb{R}. \]

This implies that
\[ \int \frac{d\varphi}{\sqrt{b_{n,k}n^2\varphi} + c_1 \varphi^{n - 1}} = \xi + c_2, \quad c_2 \in \mathbb{R}, \]

which provide equation (7).

\[ \square \]
whose general solution is given by 
\[ \varphi_{x_1} = 2\varepsilon_1 x_1 \varphi', \quad \varphi_{x_1,x_1} = 4\varepsilon_1 \varepsilon_2 x_1 x_2 \varphi'' + 2\varepsilon_1 \delta_{ij} \varphi', \] (22)

Substituting (23) and (24) into (1) and considering on the conformal geometry
\[ 2r = \varepsilon \varphi'' - 2r(\varphi')^2 \delta_{ij}. \]

The eigenvalues of \( \bar{\varphi} = \varepsilon \varphi'' - 2r(\varphi')^2 \delta_{ij} \).

Proof of Theorem 1.6: Since we are assuming that \( \varphi(r) \) and \( f(r) \) are functions of \( r \), where \( r = \varepsilon_1 x_1^2 + \cdots + \varepsilon_n x_n^2 \), we get
\[ \varphi_{x_1} = 2\varepsilon_1 x_1 \varphi', \quad \varphi_{x_1,x_1} = 4\varepsilon_1 \varepsilon_2 x_1 x_2 \varphi'' + 2\varepsilon_1 \delta_{ij} \varphi', \]

Substituting (22) into expression of the Schouten tensor on conformal geometry
\[ A_2 = \nabla^2 \nabla \varphi - \frac{[\nabla \delta \varphi]^2}{2\varphi^2} \delta, \]
we deduce that
\[ (\bar{g}^{-1} A_2)_{ij} = 4\varepsilon_1 \varepsilon_2 x_1 x_2 \varphi'' - 2r(\varphi')^2 \delta_{ij}. \]

The eigenvalues of \( \bar{g}^{-1} A_2 \) are \( \lambda = 2\varepsilon_1 \varphi'' - 2r(\varphi')^2 \) with multiplicity \( n-1 \), and \( \mu = 4\varepsilon_1 \varphi'' + 2r(\varphi')^2 \) with multiplicity 1. The formula for \( \sigma_k \) can be found easily by the binomial expansion of \( (x - \lambda)^{n-1}(x - \mu)^1 \)
\[ \sigma_k = \frac{(n - 1)!}{k!(n - k)!} 2^{k-1} [\varphi'' - r(\varphi')^2]^{k-1} [2n\varphi'' - 2n(\varphi')^2 + 4kr\varphi'']. \] (23)

Proceeding in a similar way as in the proof of Theorem 1.1, we obtain the hessian expression on the conformal geometry
\[ (\nabla^2 f)_{ij} = f_{x_1,x_1} + \varphi^{-1}(x) f_{x_1} f_{x_1} + \varphi_{x_1} f_{x_1} - \delta_{ij} \sum_k \varepsilon_k \varphi^{-1} f_{x_k} f_{x_k} \]
\[ = 4\varepsilon_1 \varepsilon_2 x_1 x_2 \left( f'' + 2\frac{f'}{\varphi} \right) + 2\varepsilon_1 \delta_{ij} \left( f' - 2\frac{f'}{\varphi} \right). \] (24)

Substituting (23) and (24) into (1) and considering \( i \neq j \) we obtain
\[ f'' + 2\frac{f'}{\varphi} = 0, \]
which provides equation (8), and for \( i = j \), substituting (23) and (24) into (1) we obtain (9).

Proof of Theorem 1.7: The hypothesis \( \sigma_1 = \sigma_s = 0, s \in \{2, \ldots, n\} \) implies that
\[ \sigma_1 = [2n\varphi'' - 2n(\varphi')^2 + 4r\varphi''] = 0, \] (25)
and
\[ \sigma_s = \frac{(n - 1)!}{s!(n - s)!} 2^{s-1} [\varphi'' - r(\varphi')^2]^{s-1} [2n\varphi'' - 2n(\varphi')^2 + 4rs\varphi''] = 0. \] (26)

Therefore, from (25) and (26), we conclude that
\[ [\varphi'' - r(\varphi')^2]^{s-1} [4(s - 1)r\varphi''] = 0, \]
whose general solution is given by \( \varphi(r) = ar + b, a, b \in \mathbb{R} \). We claim that \( a = 0 \) or \( b = 0 \). In fact, replacing \( \varphi(r) = ar + b \) into (25), we deduce that
\[ 2nab = 2n(ar + b) - 2nra^2 = 0 \]
which proves the assertion.

Now, suppose that \( a = 0 \), then the conformal factor is constant \( \varphi(r) = b > 0 \). Hence, from (8) and (11) we conclude that
\[ f(r) = \frac{(n - 1)\lambda}{b^2} r + c_1, \quad c_1 \in \mathbb{R}, \]
which provide the proof of item (a).

On the other hand, suppose that \( a \neq 0 \), then \( b = 0 \) and the conformal factor is given by \( \varphi(r) = ar, a \in (0, \infty) \). Hence, from (8) and (11) we conclude that
\[ f(r) = -\frac{(n - 1)\lambda}{a^2 r} + c_1, \quad c_1 \in \mathbb{R}, \]
which provide the proof of item (b).

\[ \square \]
Proof of Theorem 1.8: Let \((\mathbb{R}^n, g = \delta \varphi^{-2})\) be the gradient k-Yamabe soliton with invariant solutions \(f\) and \(\varphi\). Since \(0 < |\varphi| \leq L\), we have that
\[
0 < N \leq \frac{1}{L^2} \leq \frac{1}{\varphi^2},
\]
for some \(N \in (0, \infty)\). This implies that \(|v|_g \geq |v|_d\) for any vector \(v \in \mathbb{R}^n\). Since \((\mathbb{R}^n, \delta)\) is complete, it follows that \((\mathbb{R}^n, \delta \varphi^{-2})\) is complete.

\(\square\)

Proof of completeness of Example 2.1: Let \((\mathbb{R}^n, \delta)\) be the standard pseudo-Euclidean space where \(\delta = -dx_1^2 + \sum_{i=2}^n dx_i^2\). Take \(\theta \in \mathbb{N}\) and consider the functions
\[
f(\xi) = c\xi + 2\xi \frac{\xi^{2\theta+1}}{2\theta + 1} + \frac{\xi^{4\theta+1}}{4\theta + 1}, \quad \varphi(\xi) = \frac{1}{1 + \xi^{2\theta}}, \quad \xi = x_1 + x_2, \quad \theta \in \mathbb{N}, \quad c \in \mathbb{R}.
\]
We will now prove that \((\mathbb{R}^n, \delta \varphi^{-2})\) is geodesically complete by showing that any geodesic \(\gamma(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \in \mathbb{R}^n\) is defined for all \(t \in \mathbb{R}\). From the fundamental geodesic equations
\[
x''_i(t) = -\sum_{i,j} \Gamma^i_{ij} x'_i(t) x'_j(t)
\]
and the Christoffel symbol \(\Gamma^i_{ij}\) on the conformal metric
\[
\Gamma^i_{ij} = 0, \quad \Gamma^i_{ij} = -\frac{\varphi x_i}{\varphi}, \quad \Gamma^i_{ii} = \varepsilon_i \varepsilon_j \frac{\varphi x_i}{\varphi} \quad \text{and} \quad \Gamma^i_{ii} = -\frac{\varphi x_i}{\varphi},
\]
we deduce, for \(l = 1, 2, \ldots, n\), that
\[
x''_l(t) = \frac{1}{\varphi} \left[ 2(\varphi \circ \gamma)'(t) x'_l(t) - \varepsilon_l \alpha_l \varphi' \circ \xi \circ \gamma(t) \sum_{i=1}^n \varepsilon_i (x'_i(t))^2 \right]. \tag{27}
\]
Therefore, since \(\varepsilon_1 = -1, \varepsilon_i = 1, i \geq 2, \alpha_1 = \alpha_2 = 1\) and \(\alpha_i = 0, i \geq 3\), we get
\[
x''_l(t) = \frac{1}{\varphi} [2(\varphi \circ \gamma)'(t) x'_l(t)] = \frac{4\theta (\xi \circ \gamma(t))^{2\theta-1}}{1 + (\xi \circ \gamma(t))^{2\theta}} (\xi \circ \gamma)'(t) x'_l(t), \quad l \geq 3.
\]
This implies that
\[
[x''_l(t)(1 + (\xi \circ \gamma(t))^{2\theta})^2]' = 0,
\]
and hence
\[
x'_l(t) = \frac{c_l}{(1 + (\xi \circ \gamma(t))^{2\theta})^2}, \quad c_l \in \mathbb{R}, \quad l \geq 3. \tag{28}
\]
Now, for \(l = 1\), we have from (27) that
\[
x''_1(t) = \frac{1}{\varphi} \left[ 2(\varphi \circ \gamma)'(t) x'_1(t) - \varepsilon_1 \alpha_1 \varphi' \circ \xi \circ \gamma(t) \sum_{i=1}^n \varepsilon_i (x'_i(t))^2 \right].
\]
\[
= -\frac{2\theta (\xi \circ \gamma(t))^{2\theta-1}}{1 + (\xi \circ \gamma(t))^{2\theta}} (2x'_1(t)(\xi \circ \gamma)'(t) - (x'_1(t))^2 + (x'_2(t))^2) + \frac{2\theta (\xi \circ \gamma(t))^{2\theta-1}}{1 + (\xi \circ \gamma(t))^{2\theta}} \sum_{i=3}^n c_i
\]
\[
= -\frac{2\theta (\xi \circ \gamma(t))^{2\theta-1}}{1 + (\xi \circ \gamma(t))^{2\theta}} [(\xi \circ \gamma)'(t))^2 - \frac{2\theta (\xi \circ \gamma(t))^{2\theta-1}}{1 + (\xi \circ \gamma(t))^{2\theta}} \sum_{i=3}^n c_i. \tag{29}
\]
Similarly, for \(l = 2\), we get
\[
x''_2(t) = -\frac{2\theta (\xi \circ \gamma(t))^{2\theta-1}}{1 + (\xi \circ \gamma(t))^{2\theta}} [(\xi \circ \gamma)'(t))^2 + \frac{2\theta (\xi \circ \gamma(t))^{2\theta-1}}{1 + (\xi \circ \gamma(t))^{2\theta}} \sum_{i=3}^n c_i. \tag{30}
\]
Therefore,
\[
(\xi \circ \gamma)''(t) = x''_1(t) + x''_2(t) = -\frac{4\theta (\xi \circ \gamma(t))^{2\theta-1}}{1 + (\xi \circ \gamma(t))^{2\theta}} [(\xi \circ \gamma)'(t))^2,
\]
which implies that
\[
(\xi \circ \gamma)'(t) = \frac{k_1}{(1 + (\xi \circ \gamma(t))^{2\theta})^2}. \tag{31}
\]
It follows from (28), (29), (30) and (31) that we are looking for the solutions of the system

\[
\begin{align*}
x_1'(t) &= y_1(t), \\
x_2'(t) &= y_2(t), \\
y_1'(t) &= -\frac{2\theta(x_1(t) + x_2(t))^{2\theta - 1}}{[1 + (x_1(t) + x_2(t))^{2\theta}]^2} (k_1^2 + \sum_{i=3}^{n} c_i), \\
y_2'(t) &= \frac{2\theta}{[1 + (x_1(t) + x_2(t))^{2\theta}]^2} (-k_1^2 + \sum_{i=3}^{n} c_i), \\
x_l'(t) &= \frac{c_l}{(1 + (\xi \circ \gamma(t))^{2\theta})^2}, \\
\end{align*}
\]

Since the functions

\[
p(x_1, x_2) = \frac{(x_1 + x_2)^{2\theta - 1}}{[1 + (x_1 + x_2)^{2\theta}]^2}, \quad q(x_1, x_2) = \frac{1}{[1 + (x_1 + x_2)^{2\theta}]^2},
\]

are of bounded derivative, we conclude that \( p \) and \( q \) are Lipschitz. Therefore, the solutions of above system exists for all \( t \in \mathbb{R} \) and hence all geodesic \( \gamma(t) \) are defined for the entire real line, which means that \( (\mathbb{R}^n, \varphi^{-2} \delta) \) is geodesically complete.

\[
\square
\]

References

[1] E. Barbosa, R. Pina, and K. Tenenblat. On gradient ricci solitons conformal to a pseudo-euclidean space. Israel Journal of Mathematics, 200(1):213–224, 2014.
[2] M. Barboza, B. Leandro, and R. Pina. Invariant solutions for the einstein field equation. Journal of Mathematical Physics, 59(6):062501, 2018.
[3] E. Batista, L. Adriano, and W. Tokura. On warped product gradient ricci-harmonic soliton. arXiv preprint arXiv:1905.02006 2019.
[4] A. L. Besse. Einstein manifolds. Springer Science & Business Media, 2007.
[5] L. Bo, P. T. Ho, and W. Sheng. The k-yamabe solitons and the quotient yamabe solitons. Nonlinear Analysis, 166:181–195, 2018.
[6] P. G. C. Bonfim and R. Pina. Quasi-einstein manifolds with structure of warped product. arXiv preprint arXiv:1906.11933 2019.
[7] G. Catino, C. Mantegazza, and L. Mazzieri. On the global structure of conformal gradient solitons with nonnegative ricci tensor. Communications in Contemporary Mathematics, 14(06):1250045, 2012.
[8] B. Chow. The yamabe flow on locally conformally flat manifolds with positive ricci curvature. Communications on pure and applied mathematics, 45(8):1003–1014, 1992.
[9] P. Daskalopoulos and N. Sesum. The classification of locally conformally flat yamabe solitons. Advances in Mathematics, 240:346–369, 2013.
[10] M. L. de Sousa and R. Pina. Gradient ricci solitons with structure of warped product. Results in Mathematics, 71(3-4):825–840, 2017.
[11] L. F. Di Cerbo and M. M. Disconzi. Yamabe solitons, determinant of the laplacian and the uniformization theorem for riemann surfaces. Letters in Mathematical Physics, 83(1):13–18, 2008.
[12] R. S. Hamilton. The ricci flow on surfaces, mathematics and general relativity (santa cruz, ca, 1986), 237–262. Contemp. Math, 71:301–307, 1988.
[13] S.-Y. Hsu. A note on compact gradient yamabe solitons. Journal of Mathematical Analysis and Applications, 388(2):725–726, 2012.
[14] B. Leandro and R. Pina. Invariant solutions for the static vacuum equation. Journal of Mathematical Physics, 58(7):072502, 2017.
[15] B. Leandro, R. Pina, and T. P. F. Bezerra. Invariant solutions for gradient ricci almost solitons. São Paulo Journal of Mathematical Sciences, pages 1–16, 2020.
[16] L. Ma and V. Miquel. Remarks on scalar curvature of yamabe solitons. Annals of Global Analysis and Geometry, 42(2):195–205, 2012.
[17] B. L. Neto and K. Tenenblat. On gradient yamabe solitons conformal to a pseudo-euclidian space. Journal of Geometry and Physics, 123:284–291, 2018.
[18] P. J. Olver. Applications of Lie groups to differential equations, volume 107. Springer Science & Business Media, 2000.
[19] W. Tokura, L. Adriano, R. Pina, and M. Barboza. On warped product gradient yamabe solitons. *Journal of Mathematical Analysis and Applications*, 2018.

[20] W. Tokura and E. Batista. Triviality results for compact k-yamabe solitons. *Journal of Mathematical Analysis and Applications*, 502(2):125274, 2021.

1 Universidade Estadual de Mato Grosso do Sul, 79150-000, Av. João Pedro Fernandes, 2101 - Centro, Maracaju, MS, Brazil.

   Email address: williamisotokura@hotmail.com

2 Instituto Federal Goiano, 75790-000, Rodovia Geraldo Silva Nascimento Km 2,5, Urutai, GO, Brazil.

   Email address: marcelo.barboza@ifgoiano.edu.br

3 Universidade Federal de Goiás, IME, 131, 74001-970, Goiânia, GO, Brazil.

   Email address: edbatista@gmail.com.br

4 Universidade Federal de Goiás, INF, s/n, 74900-900, Goiânia, GO, Brazil.

   Email address: priscila.kai@hotmail.com