A NOTE ABOUT EC-(s, t)-WEAK TRACTABILITY OF MULTIVARIATE APPROXIMATION WITH ANALYTIC KOROBOV KERNELS

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Abstract. This note is devoted to discussing multivariate approximation of continuous functions on [0, 1]^d with analytic Korobov kernels in the worst and average case settings. We only consider algorithms that use finitely many evaluations of arbitrary continuous linear functionals. We study EC-(s, t)-weak tractability under the absolute or normalized error criterion, and obtain necessary and sufficient conditions for 0 < \min(s, t) < 1 and \max(s, t) \leq 1 in the worst case setting and for s, t > 0 in the average case setting.

1. Introduction and main results

We approximate multivariate problems \( S = \{S_d\}_{d \in \mathbb{N}} \) by algorithms that use finitely many linear functionals. The information complexity \( n(\varepsilon, S_d) \) is defined as the minimal number of linear functionals which are needed to find an approximation to within an error threshold \( \varepsilon \).

We consider exponentially-convergent tractability (EC-tractability) of the multivariate problems \( S = \{S_d\} \). There are two kinds of tractability based on polynomial-convergence and exponential-convergence. The classical tractability describes how the information complexity behaves as a function of \( d \) and \( \varepsilon^{-1} \), while the exponentially-convergent tractability (EC-tractability) does as one of \( d \) and \( (1 + \ln \varepsilon^{-1}) \). Nowadays study of tractability and EC-tractability has become one of the busiest areas of research in information-based complexity (see [16, 17, 18, 4, 6, 19, 22] and the references therein).

We briefly recall the basic EC-tractability notions. Let \( S = \{S_d\}_{d \in \mathbb{N}} \). We say \( S \) is

- **Exponential convergent and strong polynomial tractable (EC-SPT)** iff there exist non-negative numbers \( C \) and \( p \) such that for all \( d \in \mathbb{N}, \varepsilon \in (0, 1) \),
  \[ n(\varepsilon, S_d) \leq C(1 + \ln \varepsilon^{-1})^p; \]

- **Exponential convergent and polynomial tractable (EC-PT)** iff there exist non-negative numbers \( C, p \) and \( q \) such that for all \( d \in \mathbb{N}, \varepsilon \in (0, 1) \),
  \[ n(\varepsilon, S_d) \leq Cd^q(1 + \ln \varepsilon^{-1})^p; \]

- **Exponential convergent and quasi-polynomial tractable (EC-QPT)** iff there exist two constants \( C, t > 0 \) such that for all \( d \in \mathbb{N}, \varepsilon \in (0, 1) \),
  \[ n(\varepsilon, S_d) \leq C \exp\{t[1 + \ln(1 + \ln \varepsilon^{-1})](1 + \ln d)\}; \]
• Exponential convergent and uniformly weakly tractable (EC-UWT) iff for all \( \alpha, \beta > 0 \),
  \[
  \lim_{\varepsilon^{-1} + d \to \infty} \frac{\ln n(\varepsilon, S_d)}{(\ln \varepsilon - 1)^\alpha + d^\beta} = 0;
  \]

• Exponential convergent and weakly tractable (EC-WT) iff
  \[
  \lim_{\varepsilon^{-1} + d \to \infty} \frac{\ln n(\varepsilon, S_d)}{\ln \varepsilon - 1 + d} = 0;
  \]

• Exponential convergent and \((s, t)\)-weakly tractable (EC-\((s, t)\)-WT) for positive \( s \) and \( t \) iff
  \[
  \lim_{\varepsilon^{-1} + d \to \infty} \frac{\ln n(\varepsilon, S_d)}{(\ln \varepsilon - 1)^s + d^t} = 0.
  \]

Clearly, EC-(1,1)-WT is the same as EC-WT, and for \( 0 < s_1 < s, 0 < t_1 < t \),
EC-(\(s_1, t_1\))-WT \implies EC-(s,t)-WT. We also have

EC-SPT \implies EC-PT \implies EC-QPT \implies EC-UWT \implies EC-WT.

In the definitions of EC-SPT, EC-PT, EC-QPT, EC-UWT, EC-WT, and EC-(s,t)-WT, if we replace \((1 + \ln \varepsilon^{-1})\) by \(\varepsilon^{-1}\), we get the definitions of strong polynomial tractability (SPT), polynomial tractability (PT), quasi-polynomial tractability (QPT), uniform weak tractability (UWT), weak tractability (WT), and \((s,t)\)-weak tractability \((s,t)\)-WT), respectively.

This note is devoted to discussing EC-(s,t)-WT of multivariate approximation with analytic Korobov kernels in the worst and average case settings.

Let \( a = \{a_k\}_{k \geq 1} \) be a non-decreasing sequence of positive numbers, and let \( b = \{b_k\}_{k \geq 1} \) be a sequence of positive numbers having a positive infimum \( b_\ast \) so that
\[
(1.1) \quad 0 < a_1 \leq a_2 \leq \cdots \leq a_k \leq \cdots, \quad \text{and} \quad b_\ast := \inf_{k \geq 1} b_k > 0.
\]

Assume that the analytic Korobov kernel \( K_{d,a,b} \) is of product form,
\[
(1.2) \quad K_{d,a,b}(x, y) = \prod_{k=1}^d K_{1,a_k,b_k}(x_k, y_k), \quad x, y \in [0,1]^d,
\]
where \( K_{1,a,b} \) are univariate analytic Korobov kernels,
\[
K_{1,a,b}(x, y) = \sum_{h \in \mathbb{Z}} \omega^{a|h|^b} \exp(2\pi i h(x - y)), \quad x, y \in [0,1].
\]

Here \( \omega \in (0,1) \) is a fixed positive number, \( i = \sqrt{-1}, \quad a, b > 0 \). Hence, we have
\[
(1.3) \quad K_{d,a,b}(x, y) = \sum_{h \in \mathbb{Z}^d} \omega_h \exp(2\pi i h \cdot (x - y)), \quad x, y \in [0,1]^d,
\]
where
\[
(1.4) \quad \omega_h = \omega^{\sum_{k=1}^d a_k |h_k|^b_k},
\]
for fixed \( \omega \in (0,1) \) and all \( h = (h_1, h_2, \ldots, h_d) \in \mathbb{Z}^d \), and
\[
x \cdot y = \sum_{k=1}^d x_k y_k, \quad x = (x_1, x_2, \ldots, x_d), \quad y = (y_1, y_2, \ldots, y_d) \in \mathbb{R}^d
\]
denotes the usual Euclidean inner product.
First we consider the worst case setting. Denote by $H(K_{d,a,b})$ the analytic Korobov space which is a reproducing kernel Hilbert space with the reproducing kernel $K_{d,a,b}$ given by (1.3). Such space $H(K_{d,a,b})$ has been widely used in tractability study (see [3, 4, 6, 7, 8, 11]).

We consider multivariate approximation problem $\text{APP} = \{\text{APP}_d\}_{d \in \mathbb{N}}$ which is defined via the embedding operator
\begin{equation}
\text{APP}_d : H(K_{d,a,b}) \rightarrow L_2([0,1]^d) \quad \text{with} \quad \text{APP}_d f = f.
\end{equation}

We approximate $\text{APP}_d$ by algorithms that use only finitely many continuous linear functionals on $H(K_{d,a,b})$. A function $f \in H(K_{d,a,b})$ is approximated by an algorithm
\begin{equation}
A_{n,d}f = \phi_{n,d}(L_1(f), L_2(f), \ldots, L_n(f)),
\end{equation}
where $L_1, L_2, \ldots, L_n$ are continuous linear functionals on $H(K_{d,a,b})$, and $\phi_{n,d} : \mathbb{R}^n \rightarrow L_2([0,1]^d)$ is an arbitrary measurable mapping. The worst case error of approximation by an algorithm $A_{n,d}$ of the form (1.6) is defined as
\[ e_{\text{wor}}(A_{n,d}) = \sup_{\|f\|_{H(K_{d,a,b})} \leq 1} \|\text{APP}_d f - A_{n,d} f\|_{L_2([0,1]^d)}. \]

The $n$th minimal worst case error, for $n \geq 1$, is defined by
\[ e_{\text{wor}}(n,d) = \inf_{A_{n,d}} e_{\text{wor}}(A_{n,d}), \]
where the infimum is taken over all algorithms of the form (1.6).

For $n = 0$, we use $A_{0,d} = 0$. We remark that the so-called initial error $e_{\text{wor}}(0,d)$, defined by
\[ e_{\text{wor}}(0,d) = \sup_{\|f\|_{H(K_{d,a,b})} \leq 1} \|\text{APP}_d f\|_{L_2([0,1]^d)}, \]
is equal to 1. In other words, the normalized error criterion and the absolute error criterion coincide.

The information complexity $n(\varepsilon, d)$ is defined by
\[ n(\varepsilon, d) = \min\{n : e_{\text{wor}}(n,d) \leq \varepsilon\}. \]

The classical tractability of the multivariate problem $\text{APP}$ has been investigated and solved completely in [8, 11, 6]. For the EC-tractability of $\text{APP}$, the sufficient and necessary conditions for EC-SPT, EC-PT, EC-QPT, EC-UWT, EC-WT, and EC-$(s, t)$-WT with $\max(s, t) > 1$ were given in [6]. See the following EC-tractability results of $\text{APP}$:

- EC-SPT holds iff EC-PT holds iff
  \[ \sum_{k=1}^{\infty} b_k^{-1} < \infty \quad \text{and} \quad \lim_{k \to \infty} \frac{\ln a_k}{k} > 0. \]

- EC-QTP holds iff
  \[ \sup_{d \in \mathbb{N}} \sum_{k=1}^{d} \frac{b_k^{-1}}{1 + \ln d} < \infty \quad \text{and} \quad \lim_{k \to \infty} \frac{(1 + \ln k) \ln a_k}{k} > 0. \]

- EC-UWT holds iff
  \[ \lim_{k \to \infty} \frac{\ln a_k}{\ln k} = \infty. \]

- EC-$(s, t)$-WT with $\max(s, t) > 1$ always holds.
• EC-WT holds iff WT holds iff
\[ \lim_{k \to \infty} a_k = \infty. \]

However, the authors did not find out the conditions on EC-(s, t)-WT with \( \max(s, t) \leq 1 \) and \( \min(s, t) < 1 \) in [6]. In this note, we fill the gap and obtain the sufficient and necessary conditions for EC-(s, t)-WT with \( \max(s, t) \leq 1 \) and \( \min(s, t) < 1 \). We use estimates of entropy numbers and technique in [21] to obtain the sufficient conditions for EC-(s, t)-WT. Such method is first used in [10].

**Theorem 1.1.** Consider the approximation problem APP in the worst case setting with the sequences \( a \) and \( b \) satisfying (1.1). Then

(i) EC-(1, t)-WT with \( t < 1 \) holds iff
\[ \lim_{j \to \infty} \frac{\ln j}{a_j} = 0. \]

(ii) EC-(s, t)-WT with \( s < 1 \) and \( t \leq 1 \) holds iff
\[ \lim_{j \to \infty} \frac{j^{(1-s)/s}}{a_j} = 0. \]

Next we discuss the average case setting. Consider the approximation problem
\[ I = \{ I_d \}_{d \in \mathbb{N}}, \]
\[ I_d : C([0,1]^d) \to L_2([0,1]^d) \quad \text{with} \quad I_d f = f. \]

The space \( C([0,1]^d) \) of continuous real functions is equipped with a zero-mean Gaussian measure \( \mu_d \) whose covariance kernel is given by the analytic Korobov kernel \( K_{d,a,b} \). We approximate \( I_d f \) by algorithms \( A_{n,d} f \) of the form (1.6) that use \( n \) continuous linear functionals on \( C([0,1]^d) \). The average case error for \( A_{n,d} \) is defined by
\[ e_{\text{avg}}(A_{n,d}) = \left( \int_{C([0,1]^d)} \left\| I_d f - A_{n,d} f \right\|_{L_2([0,1]^d)}^2 \mu_d(\text{d} f) \right)^{1/2}. \]

The \( n \)th minimal average case error, for \( n \geq 1 \), is defined by
\[ e_{\text{avg}}(n,d) = \inf_{A_{n,d}} e(A_{n,d}), \]
where the infimum is taken over all algorithms of the form (1.6).

For \( n = 0 \), we use \( A_{0,d} = 0 \). We obtain the so-called initial error
\[ e_{\text{avg}}(0,d) = e_{\text{avg}}(A_{0,d}). \]

The information complexity for \( I_d \) can be studied using either the absolute error criterion (ABS), or the normalized error criterion (NOR). Then we define the information complexity \( n_{\text{avg}-X}(\varepsilon,d) \) for \( X \in \{ \text{ABS}, \text{NOR} \} \) as
\[ n_{\text{avg}-X}(\varepsilon,d) = \min \{ n : e_{\text{avg}}(n,d) \leq \varepsilon CRI_d \}, \]
where
\[ CRI_d = \begin{cases} 1, & \text{for } X = \text{ABS}, \\ e_{\text{avg}}(0,d), & \text{for } X = \text{NOR}. \end{cases} \]

The classical tractability of the multivariate problem \( I = \{ I_d \} \) has been investigated in [12, 13, 2]. For the EC-tractability of \( I \), the sufficient and necessary conditions for EC-SPT, EC-PT, EC-UWT, EC-WT under ABS or NOR were given in [12], see the following EC-tractability results of \( I \):
• For ABS or NOR, EC-SPT holds iff EC-PT holds iff
\[ \sum_{k=1}^{\infty} b_k^{-1} < \infty \quad \text{and} \quad \lim_{k \to \infty} \frac{\ln a_k}{k} > 0. \]

• For ABS or NOR, EC-UWT holds iff
\[ \lim_{k \to \infty} \frac{\ln a_k}{\ln k} = \infty. \]

• For ABS or NOR, EC-WT holds iff
\[ \lim_{k \to \infty} a_k = \infty. \]

In this note, we obtain the sufficient and necessary conditions for EC-(s,t)-WT. We use the connection about EC-tractability in the worst and average case settings (see [22][14]). Such connection was used to study the EC-tractability of multivariate approximation with Gaussian kernel in the average case setting (see [1]). Specially, according to [22, Theorems 3.2 and 4.2] and [14, Theorem 3.2], we have the same results in the worst and average case settings concerning EC-WT, EC-UWT, and EC-(s,t)-WT for 0 < s ≤ 1 and t > 0 under ABS.

**Theorem 1.2.** Consider the above approximation problem \( I = \{ I_d \} \) with the sequences \( a \) and \( b \) satisfying (1.1). Then

(i) for ABS or NOR, if \( s > 0 \) and \( t > 1 \) then EC-(s,t)-WT always holds;

(ii) for ABS or NOR, EC-(s,1)-WT with \( s \geq 1 \) holds iff EC-WT holds iff
\[ \lim_{j \to \infty} a_j = \infty; \]

(iii) for ABS, EC-(1,t)-WT with \( t < 1 \) holds iff
\[ \lim_{j \to \infty} \frac{\ln j}{a_j} = 0; \] (1.10)

(iv) for ABS or NOR, EC-(s,t)-WT with \( s < 1 \) and \( t \leq 1 \) holds iff
\[ \lim_{j \to \infty} \frac{j^{(1-s)/s}}{a_j} = 0; \] (1.11)

(v) for ABS or NOR, EC-(s,t)-WT with \( s > 1 \) and \( t < 1 \) holds iff
\[ \lim_{j \to \infty} j^{1-t} a_j = 0. \] (1.12)

The paper is organized as follows. In Section 2 we give some necessary preliminaries in the worst and average case settings. In Section 3, we give the proofs of Theorems 1.1 and 1.2.

2. Preliminaries

For a fixed \( \omega \in (0,1) \), let \( K_{d,a,b} \) be the analytic Korobov kernel given by (1.3) with \( a, b \) satisfying (1.1). By (1.2) we know that the reproducing kernel Hilbert space \( H(K_{d,a,b}) \) is a tensor product of the univariate reproducing kernel Hilbert spaces \( H(K_{1,a_i,b_j}) \), \( i = 1, \ldots, d \) with reproducing kernels \( K_{1,a_i,b_j} \), i.e.,
\[ H(K_{d,a,b}) = H(K_{1,a_1,b_1}) \otimes H(K_{1,a_2,b_2}) \otimes H(K_{1,a_d,b_d}). \]
From [16] we know that \( \varepsilon^\text{wor}(n, d) \) depends on the eigenpairs of the operator

\[
W_d = \text{APP}_d^* \text{APP}_d : H(K_{d,a,b}) \mapsto H(K_{d,a,b}),
\]

where \( \text{APP}_d \) is given by (1.5). We have

\[
W_d f = \sum_{\mathbf{h} \in \mathbb{Z}^d} \omega_h \langle f, e_h \rangle_{H(K_{d,a,b})} e_h
\]

with

\[
e_h(x) = (\omega_h)^{1/2} \exp(2\pi i \mathbf{h} \cdot x).
\]

This means that \( \{\omega_h, e_h\}_{\mathbf{h} \in \mathbb{Z}^d} \) are the eigenpairs of \( W_d \), i.e.,

\[
W_d e_h = \omega_h e_h, \quad \text{for all } \mathbf{h} \in \mathbb{Z}^d,
\]

and \( \{e_h\}_{\mathbf{h} \in \mathbb{Z}^d} \) is an orthonormal basis for \( H(K_{d,a,b}) \).

Let \( \{(\lambda_{d,j}, \eta_{d,j})\}_{j \in \mathbb{N}} \) be the rearrangement of the eigenpairs \( \{\omega_h, e_h\}_{\mathbf{h} \in \mathbb{Z}^d} \), such that the eigenvalues \( \omega_{\mathbf{h}} \), \( \mathbf{h} \in \mathbb{Z}^d \) are arranged in decreasing order, i.e.,

\[
\lambda_{d,1} \geq \lambda_{d,2} \geq \cdots \lambda_{d,k} \geq \cdots \geq 0.
\]

From [16, p. 118] we get that the \( n \)th minimal worst case error is

\[
\varepsilon^\text{wor}(n, d) = (\lambda_{d,n+1})^{1/2},
\]

and it is achieved by the algorithm

\[
A_{n,d}^* f = \sum_{k=1}^{n} \lambda_{d,k} \langle f, \eta_{d,k} \rangle_{H(K_{d,a,b})} \eta_{d,k}.
\]

Since \( \lambda_{d,1} = \omega_0 = \prod_{k=1}^{d} \lambda(k,1) = 1 \), we get that the normalized error criterion and the absolute error criterion coincide. Then the information complexity \( n(\varepsilon, d) \) of \( \text{APP} \) satisfies

\[
n(\varepsilon, d) = \min \{ n \in \mathbb{N} \mid \varepsilon^\text{wor}(n, d) \leq \varepsilon \} = \min \{ n \in \mathbb{N} \mid \lambda_{d,n+1} \leq \varepsilon^2 \},
\]

or equivalently, the number of eigenvalues \( \{\lambda_{d,j}\}_{j \in \mathbb{N}} = \{\omega_h\}_{\mathbf{h} \in \mathbb{Z}^d} \) of the operator \( W_d \) greater than \( \varepsilon^2 \). Due to (1.4), we can rewrite the information complexity as

\[
n(\varepsilon, d) = \# \{ \mathbf{h} \in \mathbb{Z}^d \mid \omega_{\mathbf{h}} = \omega \sum_{k=1}^{d} a_k |h_k|^b_k > \varepsilon^2 \}
\]

(2.1)

where \#A represents the number of elements in a set A.

Next we can give explicit formulas for the \( n \)th minimal average case error \( e^{avg}(n, d) \) and the corresponding \( n \)th optimal algorithm, see [16, Section 4.3]. We recall that the space \( C([0,1]^d) \) is equipped with a zero-mean Gaussian measure \( \mu_d \) whose covariance kernel is given by the analytic Korobov kernel \( K_{d,a,b} \). Let

\[
C_{\mu_d} : (C([0,1]^d))^\ast \mapsto C([0,1]^d)
\]

denote the covariance operator of \( \mu_d \), as defined in [16, Appendix B]. Then the induced measure \( \nu_d = \mu_d(I_d)^{-1} \) is a zero-mean Gaussian measure on the Borel sets of \( L_2([0,1]^d) \), with covariance operator \( C_{\nu_d} : L_2([0,1]^d) \mapsto C([0,1]^d) \) given by

\[
C_{\nu_d} = I_d C_{\mu_d} (I_d)^\ast,
\]
where $I_d$ is defined by (1.3), $(I_d)^\ast : L_2([0, 1]^d) \rightarrow (C([0, 1]^d))^\ast$ is the operator dual to $I_d$. It is well-known that $C_{\nu_d}$ is a self-adjoint nonnegative-definite operator with finite trace on $L_2([0, 1]^d)$ and for any $f \in L_2([0, 1]^d)$,

$$C_{\nu_d} f(x) = \int_{[0,1]^d} K_{d,a,b}(x,y) f(y)dy.$$  

Then $\{(\omega_h^, \tilde{e}_h)\}_{h \in \mathbb{Z}^d}$ are the eigenpairs of $C_{\nu_d}$ with $\tilde{e}_h(x) = \exp(2\pi i h \cdot x)$, i.e.,

$$C_{\nu_d} \tilde{e}_h = \omega_h \tilde{e}_h, \quad \text{for all } h \in \mathbb{Z}^d,$$

and $\{\tilde{e}_h\}_{h \in \mathbb{Z}^d}$ is an orthonormal basis for $L_2([0, 1]^d)$.

Let $\{\lambda_{d,j}\}_{j \in \mathbb{N}}$ be the non-increasing rearrangement of $\{(\omega_h)_{h \in \mathbb{Z}^d}$ just as in the worst case setting. Then the eigenvalues of the covariance operator $C_{\nu_d}$ are just $\lambda_{d,j}$, $j \in \mathbb{N}$. Denote by $\xi_{d,j}$ the corresponding eigenvector of $C_{\nu_d}$ with respect to the eigenvalue $\lambda_{d,j}$. Then the $n$th minimal average case error $e^{\text{avg}}(n,d)$ is (see (16))

$$e^{\text{avg}}(n,d) = \left( \sum_{k=n+1}^{\infty} \lambda_{d,k} \right)^{1/2} \geq e^{\text{wor}}(n,d).$$

and it is achieved by the algorithm

$$A_{n,d}^\ast f = \sum_{k=1}^{n} \langle I_d f, \xi_{d,k} \rangle_{L_2([0,1]^d)} \xi_{d,k}.$$  

The average case information complexity can be studied using either ABS or NOR. Then we define the worst case information complexity $n^{\text{wor, X}}(\varepsilon,d)$ for $X \in \{\text{ABS, NOR}\}$ as

$$n^{\text{avg, X}}(\varepsilon,d) = \min \{ n : e^{\text{avg}}(n,d) \leq \varepsilon CRI_d \},$$

where

$$CRI_d = \begin{cases} 1, & \text{for } X=\text{ABS,} \\ e^{\text{avg}}(0,d), & \text{for } X=\text{NOR} \\ \left( \sum_{j=1}^{\infty} \lambda_{d,j} \right)^{1/2}, & \text{for } X=\text{NOR.} \end{cases}$$

Obviously, we have

(2.2)  \quad n^{\text{avg, NOR}}(\varepsilon,d) \leq n^{\text{avg, ABS}}(\varepsilon,d) = n^{\text{avg, NOR}}((e^{\text{avg}}(0,d))^{-1} \varepsilon,d).$$

We remark that the eigenvalues of the operator $W_d$ or $C_{\nu_d}$ are given by

$$\{\lambda_{d,j}\}_{j \in \mathbb{N}} = \{\omega_h\}_{h \in \mathbb{Z}^d} = \{\lambda(1,j_1)\lambda(2,j_2)\ldots \lambda(d,j_d)\}_{(j_1,\ldots,j_d) \in \mathbb{N}^d},$$

where $\lambda(k,1) = 1$, and

$$\lambda(k,2j) = \lambda(k,2j+1) = \omega^{a_k j^k}, \quad j \in \mathbb{N}, \; 1 \leq k \leq d.$$  

This implies that for any $\tau_0 > 0$ and $\tau > \tau_0$,

$$\sum_{j \in \mathbb{N}} \lambda_{d,j} = \prod_{k=1}^{d} \sum_{j=1}^{\infty} \lambda(k,j)^\tau = \prod_{k=1}^{d} \left( 1 + 2 \sum_{j=1}^{\infty} \omega^{\tau a_k j^k} \right)$$

$$= \prod_{k=1}^{d} \left( 1 + \omega^{\tau a_k} \sum_{j=1}^{\infty} \omega^{\tau a_k (j^k-1)} \right) = \prod_{k=1}^{d} \left( 1 + \omega^{\tau a_k} H(k,\tau) \right),$$
where
\[ 1 \leq H(k, \tau) = 2 \sum_{j=1}^{\infty} \omega^{\tau a_k(j^{k^*} - 1)} \leq 2 \sum_{j=1}^{\infty} \omega^{\tau a_1(j^{k^*} - 1)} \leq 2 \sum_{j=1}^{\infty} \omega^{\tau a_1(j^{k^*} - 1)}. \]

Since
\[ \omega^{\tau a_1(j^{k^*} - 1)} = j^{\frac{\tau a_1(j^{k^*} - 1) \ln \frac{1}{\omega}}{\ln j}}, \]
and
\[ \lim_{j \to \infty} \frac{\tau a_1(j^{k^*} - 1) \ln \frac{1}{\omega}}{\ln j} = \infty, \]
we get that
\[ M_{\tau_0} := 2 \sum_{j=1}^{\infty} \omega^{\tau a_1(j^{k^*} - 1)} < \infty. \]

It follows that for any \( \tau > \tau_0 > 0, \)
\[ \ln \frac{2}{2} \sum_{k=1}^{d} \omega^{a_k} \leq \sum_{k=1}^{d} \ln(1 + \omega^{a_k}) \leq \ln \left(\sum_{j \in \mathbb{N}} \lambda_{d,j}^\tau\right) \]
\[ = \sum_{k=1}^{d} \ln(1 + \omega^{a_k} H(k, \tau)) \leq \ln(1 + M_{\tau_0} \omega^{a_k}) \leq M_{\tau_0} \sum_{k=1}^{d} \omega^{a_k}, \]
where in the first inequality we used the inequality \( \ln(1 + x) \geq x \ln 2, \ x \in [0, 1], \)
and in the last inequality we used the inequality \( \ln(1 + x) \leq x, \ x > 0. \)

By (2.3) we have
\[ \frac{\omega^{a_1} \ln 2}{2} \leq \frac{\ln 2}{2} \sum_{k=1}^{d} \omega^{a_k} \leq \ln(e^{a_{\infty}}(0, d)) = \frac{1}{2} \ln \left(\sum_{j \in \mathbb{N}} \lambda_{d,j}^\tau\right) \leq \frac{M_1}{2} \sum_{k=1}^{d} \omega^{a_k} \leq \frac{d M_1 \omega^{a_1}}{2}. \]

3. Proofs of Theorems 1.1 and 1.2

In order to prove Theorem 1.1, we shall use the estimates of entropy numbers of \( \ell_p^d \)-unit balls with \( \ell_{\infty}^d \)-balls. Such method is firstly used in [10].

Let \( \ell_p^d (0 < p \leq \infty) \) denote the space \( \mathbb{R}^d \) equipped with the \( \ell_p^d \)-norm defined by
\[ \|x\|_{\ell_p^d} := \left\{ \begin{array}{ll} \left( \sum_{i=1}^{d} |x_i|^p \right)^{\frac{1}{p}}, & 0 < p < \infty; \\ \max_{1 \leq i \leq d} |x_i|, & p = \infty. \end{array} \right. \]

The unit ball of \( \ell_p^d \) is denoted by \( B_{\ell_p^d} \).

Let \( A \subset \mathbb{R}^d \). An \( \varepsilon \)-net for \( A \) is a discrete set of points \( x_1, x_2, \ldots, x_n \) in \( \mathbb{R}^d \) such that
\[ A \subset \bigcup_{i=1}^{n} (x_i + \varepsilon B_{\ell_\infty^d}). \]

The covering number \( N_\varepsilon(A) \) is the minimal natural number \( n \) such that there is an \( \varepsilon \)-net for \( A \) consisting of \( n \) points. Inverse to the covering numbers \( N_\varepsilon(A) \) are the (nondyadic) entropy numbers
\[ \varepsilon_n(A, \ell_\infty^d) := \inf \{ \varepsilon > 0 \mid N_\varepsilon(A) \leq n \}. \]

Points \( y_1, y_2, \ldots, y_m \) in \( \mathbb{R}^d \) are called \( \varepsilon \)-distinguishable if the \( \ell_\infty \) distances between any two of them exceeds \( \varepsilon \), i.e.,
\[ \|y_i - y_k\|_{\ell_\infty^d} > \varepsilon \quad \text{for all } i \neq k, \ 1 \leq i, k \leq m. \]
Let $M_\varepsilon(A)$ be the maximal natural number $m$ such that there is an $\varepsilon$-distinguishable set in $A$ consisting of $m$ points. Then we have (see [15, Chapter 15, Proposition 1.1])

$$M_{2\varepsilon}(A) \leq N_\varepsilon(A) \leq M_\varepsilon(A).$$

For $A \subset \mathbb{R}^d$, let $G(A)$ be the grid number of points in $A$ that lie on the grid $\mathbb{Z}^d$, i.e.,

$$G(A) = \#(A \cap \mathbb{Z}^d).$$

In the case $A = B_{\ell_p^d}$, $0 < p < \infty$, the behavior in $n$ and $d$ of the entropy numbers $\varepsilon_n(B_{\ell_p^d}, \ell_\infty^d)$ is completely understood (see [5, 9, 15, 20]). It follows that for $0 < p < \infty$ and $\varepsilon \in (0, 1)$,

$$\ln(N_\varepsilon(B_{\ell_p^d})) \leq C_p \left\{ \varepsilon^{-p} \ln(2d\varepsilon^p), \quad \varepsilon^{-p} \leq 1, \\
\ln(2(d\varepsilon^p)^{-1}), \quad \varepsilon^{-p} \leq 1, \right. \right.$$

where $C(p)$ is depending only on $p$, but independent of $d$ and $\varepsilon$.

**Lemma 3.1.** For $0 < p < \infty$ and $m \geq 1$, we have

$$\ln\left(\#\{h \in \mathbb{Z}^d \mid \sum_{k=1}^{d} |h_k|^p \leq m\}\right) \leq C_p \left\{ \frac{m \ln(\frac{2d}{m})}{d \ln(2d)}, \quad d \geq m, \\
\frac{d \ln(\frac{2d}{m})}{\ln(2d)}, \quad m \geq d, \right. \right.$$

where $C_p$ is a constant depending only on $p$, but independent of $d$ and $m$.

**Proof.** We set $A = m^{1/p}B_{\ell_p^d}$. Then

$$G(A) = \#(A \cap \mathbb{Z}^d) = \#\{h \in \mathbb{Z}^d \mid \sum_{k=1}^{d} |h_k|^p \leq m\}.$$ 

For $m \geq 1$, $A \cap \mathbb{Z}^d$ is $\rho$-indistinguishable for any $\rho \in (1/2, 1)$ in $A$. This means that

$$G(A) \leq M_\rho(A) \leq N_{\rho/2}(A) \leq N_{1/4}(m^{1/p}B_{\ell_p^d}) = N_{m^{-1/p}/4}(B_{\ell_p^d}).$$

By (3.2) we obtain that

$$\ln G(A) \leq \ln\left(N_{m^{-1/p}/4}(B_{\ell_p^d})\right) \leq C_p \left\{ \frac{m \ln(\frac{2d}{m})}{d \ln(2d)}, \quad d \geq m, \\
\frac{d \ln(\frac{2d}{m})}{\ln(2d)}, \quad m \geq d. \right. \right.$$

Lemma 3.1 is proved. \qed

**Corollary 3.2.** For $0 < p < \infty$ and $m \geq 1$, we have

$$\ln\left(\#\{h \in \mathbb{Z}^d \mid \sum_{k=1}^{d} |h_k|^p \leq m\}\right) \leq C_p d \left(\ln(2d) + \ln(2m)\right).$$

**Proof of Theorem 1.1.**
(i) Suppose that EC-(1, t)-WT with $t < 1$ holds for APP. We want to show (1.7). It follows that EC-WT holds also and hence $\lim_{j \to \infty} a_j = \infty$. By (2.1) we have

$$n(\varepsilon, d) = \# \left\{ h \in \mathbb{Z}^d \mid \sum_{k=1}^{d} a_k |h_k| < \frac{\ln \varepsilon^{-2}}{\ln \omega^{-1}} \right\}$$

$$\geq \# \left\{ h \in \{-1, 0, 1\}^d \mid \sum_{k=1}^{d} a_k |h_k| < \frac{\ln \varepsilon^{-2}}{a_d \ln \omega^{-1}} \right\}$$

$$\geq \# \left\{ h \in \{-1, 0, 1\}^d \mid \sum_{k=1}^{d} |h_k| < \frac{\ln \varepsilon^{-2}}{a_d \ln \omega^{-1}} \right\}$$

$$= \# \left\{ h \in \{-1, 0, 1\}^d \mid \sum_{k=1}^{d} |h_k| \leq m \right\}$$

$$= \sum_{j=0}^{m} \binom{3^d}{2^j}, \quad 0 \leq m \leq d,$$

where

$$m = \left\lceil \frac{\ln \varepsilon^{-2}}{a_d \ln \omega^{-1}} \right\rceil - 1.$$

It follows by the inequality

$$\binom{m + d}{m} \geq \max \left\{ \left(1 + \frac{m}{d}\right)^d, \left(1 + \frac{d}{m}\right)^m \right\}$$

that for $1 \leq m < d$,

$$n(\varepsilon, d) \geq \binom{d}{m} \geq \left(\frac{d}{m}\right)^m.$$

Set $\varepsilon = \varepsilon_d \in (0, 1)$ such that

$$\frac{\ln \varepsilon^{-2}}{a_d \ln \omega^{-1}} = d^t$$

for sufficiently large $d \in \mathbb{N}$. Then we have

$$m \leq \frac{\ln \varepsilon^{-2}}{a_d \ln \omega^{-1}} = d^t \leq m + 1.$$

This yields

$$\ln \frac{d}{m} \geq \ln d^{1-t} = (1 - t) \ln d,$$

and

$$\ln \varepsilon^{-1} \leq \frac{1}{2} \ln \frac{1}{\omega} a_d (m + 1).$$
Since EC-(1, t)-WT with \( t < 1 \) holds, we have
\[
0 = \lim_{\varepsilon^{-1} + d \to \infty} \frac{\ln n(\varepsilon, d)}{\ln \varepsilon^{-1} + d} \\
\geq \lim_{d \to \infty} \frac{m \ln \frac{d}{m}}{\frac{1}{2} \ln \frac{1}{m} a_d (m + 1) + (m + 1)} \\
\geq \lim_{d \to \infty} \frac{(1 - t) \ln d}{\frac{1}{2} \ln \frac{1}{m} a_d (1 + \frac{1}{m}) + (1 + \frac{1}{m})} \\
= \lim_{d \to \infty} \frac{(1 - t) \ln d}{\frac{1}{2} \ln \frac{1}{m} a_d} \geq 0,
\]
which implies that
\[
\lim_{d \to \infty} \frac{\ln d}{a_d} = 0,
\]
and hence (17).

Next we suppose that (17) holds. We want to show that EC-(1, t)-WT with \( t < 1 \) holds. By (21) we have
\[
n(\varepsilon, d) = \#\{h \in \mathbb{Z}^d \mid \sum_{k=1}^{d} a_k |h_k|^b_k < \frac{\ln \varepsilon^{-2}}{\ln \omega^{-1}}\} \leq \#\{h \in \mathbb{Z}^d \mid \sum_{k=1}^{d} a_k |h_k|^b_k \leq \frac{\ln \varepsilon^{-2}}{\ln \omega^{-1}}\} \\
\leq \#\{h \in \mathbb{Z}^{i-1} \mid \sum_{k=1}^{i-1} a_k |h_k|^b_k \leq \frac{\ln \varepsilon^{-2}}{a_1 \ln \omega^{-1}}\} \cdot \#\{h \in \mathbb{Z}^{d-i+1} \mid \sum_{k=i}^{d} a_k |h_k|^b_k \leq \frac{\ln \varepsilon^{-2}}{a_i \ln \omega^{-1}}\}
\]
It follows that
\[
\ln n(\varepsilon, d) \leq \ln \left(\#\{h \in \mathbb{Z}^{i-1} \mid \sum_{k=1}^{i-1} |h_k|^b_k \leq \frac{\ln \varepsilon^{-2}}{a_1 \ln \omega^{-1}}\}\right) \\
+ \ln \left(\#\{h \in \mathbb{Z}^{d-i+1} \mid \sum_{k=i}^{d} |h_k|^b_k \leq \frac{\ln \varepsilon^{-2}}{a_i \ln \omega^{-1}}\}\right) \\
=: \text{term}_1 + \text{term}_2.
\]
By (22) we have
\[
\text{term}_1 \leq (i - 1)\left\{\ln \left[2(i - 1)\right] + \ln \left(\frac{2 \ln \varepsilon^{-2}}{a_i \ln \omega^{-1}}\right)\right\}.
\]
We set
\[
y = \max(d', \ln \varepsilon^{-1}), \quad \delta \in (0, 1), \quad \text{and} \quad i = \min(d + 1, 1 + |y^{1-\delta}|).
\]
Then we have
\[
i - 1 \leq y^{1-\delta}, \quad \ln \varepsilon^{-1} \leq y \leq \ln \varepsilon^{-1} + d',
\]
and \( y \to \infty \) as \( \varepsilon^{-1} + d \to \infty \). It follows that
\[
(3.5) \quad \frac{\text{term}_1}{\ln \varepsilon^{-1} + d'} \leq \frac{\ln (2y^{1-\delta}) + \left[\ln(4y) - \ln(a_1 \ln \omega^{-1})\right]}{y^\delta} \to 0,
\]
as \( y \to \infty \).
Now we deal with term 2. Note that if \( d \leq \lfloor y^{1-\delta} \rfloor \), then \( i = d + 1 \) and then \( \text{term}_2 = 0 \). Hence we can assume that \( d > \lfloor y^{1-\delta} \rfloor \). Then \( i \leq d \) and both \( d \) and \( i \) go to infinity with \( y \), and hence \( a_i \to \infty \).

If \( m = \frac{\ln \varepsilon^{-2}}{a_i \ln \omega^{-1}} \geq (d - i + 1) \), then by (3.2) we get

\[
\frac{\text{term}_2}{\ln \varepsilon^{-1} + d^t} = \frac{C(d - i + 1) \ln(2t)}{y} \leq \frac{C \ln \varepsilon^{-2} \ln(2t)}{ya_i \ln \omega^{-1}} \leq \frac{2C}{a_i \ln \omega^{-1}} \to 0, \tag{3.6}
\]

as \( \varepsilon^{-1} + d \to \infty \), where \( t = \frac{m}{y - d + 1} \geq 1 \), and in the last inequality we used \( \ln(2t) \leq t \) for \( t \geq 1 \).

If \( m = \frac{\ln \varepsilon^{-2}}{a_i \ln \omega^{-1}} < 1 \), then \( \text{term}_2 = 0 \). We omit this case. If \( 1 \leq m = \frac{\ln \varepsilon^{-2}}{a_i \ln \omega^{-1}} \leq (d - i + 1) \), then by (3.2) we get

\[
\frac{\text{term}_2}{\ln \varepsilon^{-1} + d^t} \leq \frac{Cm \ln(2(d - i + 1))}{y} \leq \frac{2C \ln(2da_i \ln \omega^{-1})}{a_i \ln \omega^{-1}} \leq \frac{2C}{a_i \ln \omega^{-1}} \ln 2 + \ln d + \ln a_i + \ln(\ln \omega^{-1}),
\]

Note that

\[
i = 1 + \lfloor y^{1-\delta} \rfloor \geq y^{1-\delta} \geq d^{(1-\delta)}.
\]

It follows by (1.7) that

\[
\lim_{i \to \infty} \frac{\ln d}{a_i} \leq \frac{1}{t(1-\delta)} \lim_{i \to \infty} \frac{\ln i}{a_i} = 0.
\]

We continue to obtain that

\[
\frac{\text{term}_2}{\ln \varepsilon^{-1} + d^t} \leq \frac{2C}{\ln \omega^{-1}} \frac{\ln 2 + \ln d + \ln a_i + \ln(\ln \omega^{-1})}{a_i} \to 0, \tag{3.7}
\]

as \( i \to \infty \). By (3.5), (3.6), and (3.7), we obtain

\[
\frac{\ln n(\varepsilon, d)}{\ln \varepsilon^{-1} + d^t} \leq \frac{\text{term}_1 + \text{term}_2}{\ln \varepsilon^{-1} + d^t} \to 0,
\]

as \( \varepsilon^{-1} + d \to \infty \). This means that EC-(1, t)-WT with \( t < 1 \) holds for APP if (1.8) holds. Theorem 1.1 (i) is proved.

(ii) Suppose that EC-(s, t)-WT with \( s < 1 \) and \( t \leq 1 \) holds for APP. We want to prove (1.8). Set \( \varepsilon = \varepsilon_d \in (0, 1) \) for sufficiently large \( d \in \mathbb{N} \) such that

\[
m \leq \frac{\ln \varepsilon^{-2}}{a_d \ln \omega^{-1}} = \frac{d}{2} \leq m + 1.
\]

This gives that

\[
\frac{d}{m} \geq 2, \quad \text{and} \quad \ln \varepsilon^{-1} \leq \frac{1}{2} \ln(\ln \omega^{-1} a_d (m + 1)).
\]
Since EC-(s, t)-WT with s < 1 and t ≤ 1 holds, by (3.4) we have

\[
0 = \lim_{\varepsilon^{-1} + d \to \infty} \frac{\ln n(\varepsilon, d)}{m \ln \frac{d}{m}} + d^t
\]
\[
\geq \lim_{d \to \infty} \frac{\frac{1}{2} \ln \frac{1}{a_d} (m + 1)^s}{m^{1-s} \ln 2}
\]
\[
\geq \lim_{d \to \infty} \frac{\frac{d}{2} \ln \frac{1}{a_d} (1 + \frac{1}{m})^s}{m^{1-s} \ln 2}
\]
\[
= \lim_{d \to \infty} \frac{\frac{d}{2} \ln \frac{1}{a_d} (1 + \frac{1}{m})^s}{m^{1-s} \ln 2} \geq 0,
\]

which yields that

\[
\lim_{d \to \infty} \frac{d^{1-s}}{a_d^s} = 0,
\]

and hence (1.8).

Next we suppose that (1.8) holds. We want to show that EC-(s, t)-WT with s < 1 and t ≤ 1 holds. Set

\[ a_k = k^{\frac{1-s}{s}} \hat{h}(k) \quad \text{and} \quad \tilde{h}(k) = \inf_{j \geq k} \hat{h}(j). \]

Then the sequence \{\tilde{h}(k)\}_{k \in \mathbb{N}} is non-decreasing and satisfies \hat{h}(k) ≥ \tilde{h}(k) and

\[
\lim_{k \to \infty} \tilde{h}(k) = \lim_{k \to \infty} \hat{h}(k) = \infty.
\]

We put

\[ h(1) = \tilde{h}(1), \quad h(k + 1) = \min\left\{ (1 + 1/k)\hat{h}(k), \tilde{h}(k + 1) \right\}, \quad k = 1, \ldots. \]

Clearly, we have

\[ h(k) ≤ (1 + 1/k)\hat{h}(k) \quad \text{and} \quad h(k) ≤ \tilde{h}(k) ≤ \hat{h}(k + 1), \]

which yields that the sequence \{h(k)\}_{k \in \mathbb{N}} is non-decreasing. We also note that

\[ \hat{h}(k) ≥ \tilde{h}(k) ≥ h(k) \]

and

\[ h(2k) ≤ \frac{2k}{2k - 1} h(2k - 1) ≤ \cdots ≤ \frac{2k}{2k - 1} \frac{2k - 1}{2k - 2} \cdots \frac{k + 1}{k} h(k) = 2h(k). \]

If \varepsilon^{-1} is bounded by a constant \( M \), then by (2.1) and (3.2) we have

\[
\frac{\ln n(\varepsilon, d)}{(\ln \varepsilon^{-1})^s + d^t} \leq \frac{\ln \left( \# \{ h \in \mathbb{Z}^d \mid \sum_{k=1}^d |h_k|^{s'} \leq \frac{\ln M^2}{a_1 \ln \omega^{-1}} \} \right)}{d^t} \leq C M_0 \frac{2^d}{\ln M_0} \to 0,
\]
as \( d \to \infty \), where \( M_0 = \frac{\ln M^2}{a_1 \ln \omega^{-1}} \). In this case, EC-(s, t)-WT with s < 1 and t ≤ 1 holds.
Therefore without loss of generality, we may assume that \( \varepsilon^{-1} \) tends to infinity. By (2.1) we get

\[
n(\varepsilon, d) \leq \# \left\{ h \in \mathbb{Z}^d \mid \sum_{k=1}^d a_k |h_k|^{b_*} \leq \frac{\ln \varepsilon^{-2}}{\ln \omega^{-1}} \right\}
\]

By (2.1) we get

\[
\leq \# \left\{ h \in \mathbb{Z}^d \mid \sum_{k=1}^4 |h_k|^{b_*} \leq \frac{\ln \varepsilon^{-2}}{a_1 \ln \omega^{-1}} \right\}
\]

\[
\cdot \# \left\{ h \in \mathbb{Z}^{[\log_2 d]} \mid \sum_{l=2}^{[\log_2 d]-1} \left( \sum_{k=2l+1}^{2l+1} |h_k|^{b_*} \right) a_{2l} \leq \frac{\ln \varepsilon^{-2}}{a_1 \ln \omega^{-1}} \right\}
\]

\[
\leq \# \left\{ h \in \mathbb{Z}^d \mid \sum_{k=1}^4 |h_k|^{b_*} \leq \frac{\ln \varepsilon^{-2}}{a_1 \ln \omega^{-1}} \right\}
\]

\[
\cdot \prod_{l=2}^{[\log_2 d]-1} \# \left\{ h \in \mathbb{Z}^{2^l} \mid \sum_{k=2^l+1}^{2^{l+1}} |h_k|^{b_*} \leq \frac{\ln \varepsilon^{-2}}{a_{2^l} \ln \omega^{-1}} \right\}
\]

It follows that

\[
\ln n(\varepsilon, d) \leq \ln \left( \# \left\{ h \in \mathbb{Z}^d \mid \sum_{k=1}^4 |h_k|^{b_*} \leq \frac{\ln \varepsilon^{-2}}{a_1 \ln \omega^{-1}} \right\} \right)
\]

\[
+ \sum_{l=2}^{[\log_2 d]-1} \ln \left( \# \left\{ h \in \mathbb{Z}^{2^l} \mid \sum_{k=2^l+1}^{2^{l+1}} |h_k|^{b_*} \leq \frac{\ln \varepsilon^{-2}}{a_{2^l} \ln \omega^{-1}} \right\} \right)
\]

\[
\leq \ln \left( \# \left\{ h \in \mathbb{Z}^d \mid \sum_{k=1}^4 |h_k|^{b_*} \leq \frac{\ln \varepsilon^{-2}}{a_1 \ln \omega^{-1}} \right\} \right)
\]

\[
+ \sum_{l=2}^{[\log_2 d]-1} \ln \left( \# \left\{ h \in \mathbb{Z}^{2^l} \mid \sum_{k=2^l+1}^{2^{l+1}} |h_k|^{b_*} \leq \frac{\ln \varepsilon^{-2}}{a_{2^l} \ln \omega^{-1}} \right\} \right)
\]

\[
=: I_{1, \varepsilon} + \sum_{l=2}^{[\log_2 d]-1} I_{l, \varepsilon}.
\]

By (3.2) we have

\[
\frac{I_{1, \varepsilon}}{(\ln \varepsilon^{-1})^s + d^t} \leq \frac{4C \ln \left( \frac{2 \ln \varepsilon^{-2}}{a_1 \ln \omega^{-1}} \right)}{(\ln \varepsilon^{-1})^s + d^t} \rightarrow 0
\]

as \( \varepsilon^{-1} + d \rightarrow \infty \).

We set

\[
m_{l, \varepsilon} = \frac{\ln \varepsilon^{-2}}{2^{l-1} h(2^l) \ln \omega^{-1}}.
\]

It is easy to see that the sequence

\[
\{d_{l, \varepsilon}\} = \left\{ \frac{\ln \varepsilon^{-2}}{2^{l-1} h(2^l) \ln \omega^{-1}} \right\}
\]
satisfies
\begin{equation}
2^{-(1+1/s)} \leq \frac{d_{l+1,\varepsilon}}{d_{l,\varepsilon}} = \frac{h(2^l)}{2^{l/s} h(2^{l+1})} \leq 2^{-1/s} < 1, \\
\lim_{l \to \infty} d_{l,\varepsilon} = \lim_{l \to \infty} \frac{m_{l,\varepsilon}}{2^{l/2}} = 0, \quad \text{and} \quad m_{2,\varepsilon} \geq 4 \text{ for sufficiently large } \varepsilon^{-1}.
\end{equation}

Then there exists an \( l_0 \geq 2 \) such that
\begin{equation}
d_{l_0,\varepsilon} = \frac{m_{l_0,\varepsilon}}{2^{l_0}} \geq 1 \quad \text{and} \quad d_{l_0+1,\varepsilon} = \frac{m_{l_0+1,\varepsilon}}{2^{l_0+1}} < 1.
\end{equation}

It follows that
\begin{equation}
d_{l,\varepsilon} \leq 2^{(1+1/s)(l_0+1-l)} d_{l_0+1,\varepsilon} \leq 2^{(1+1/s)(l_0+1-l)} \quad \text{for } l \leq l_0,
\end{equation}
\begin{equation}
\frac{1}{d_{l,\varepsilon}} \leq 2^{(1+1/s)(l-l_0)} \frac{1}{d_{l_0,\varepsilon}} \leq 2^{(1+1/s)(l-l_0)} \quad \text{for } l > l_0,
\end{equation}
and
\begin{equation}
1 \leq (d_{l_0,\varepsilon})^{-s} = \frac{(\ln \varepsilon^{-2})^s}{2^{l_0} h(2^{l_0})^s (\ln \omega^{-1})^s} \leq 2^{1+s}.
\end{equation}

It follows that
\begin{equation}
2^{l_0} \leq \frac{(2 \ln \varepsilon^{-1})^s}{(h(2^{l_0}))^s (\ln \omega^{-1})^s},
\end{equation}
and \( h(2^{l_0}) \) tends to \( \infty \) as \( \varepsilon^{-1} \to \infty \).

We note that \( I_{l,\varepsilon} = 0 \) if \( m_{l,\varepsilon} < 1 \). By (3.2) we have
\begin{equation}
I_{l,\varepsilon} \leq C \left\{ \begin{array}{ll}
m_{l,\varepsilon} \ln \left( \frac{2}{d_{l,\varepsilon}} \right), & d_{l,\varepsilon} \leq 1, \\
2^{4} \ln (2d_{l,\varepsilon}), & d_{l,\varepsilon} \geq 1.
\end{array} \right.
\end{equation}

Hence, by (3.11), (3.12), (3.13), and (3.14) we have
\begin{align*}
\sum_{l=2}^{[\log_2 d]-1} I_{l,\varepsilon} & \leq \sum_{l=2}^{l_0} C 2^l \ln (2d_{l,\varepsilon}) + \sum_{l=l_0+1}^{\infty} C m_{l,\varepsilon} \ln \left( \frac{2}{d_{l,\varepsilon}} \right) \\
& \leq C \sum_{l=2}^{l_0} 2^l [(1 + 1/s)(l_0 + 1 - l) + 1] \ln 2 \\
& \quad + C \sum_{l=l_0+1}^{\infty} \frac{\ln \varepsilon^{-2}}{2^{l_0} h(2^{l_0}) \ln \omega^{-1}} [(1 + 1/s)(l - l_0) + 1] \ln 2 \\
& \leq C_1 2^{l_0} + C_2 \frac{\ln \varepsilon^{-2}}{2^{l_0} h(2^{l_0}) \ln \omega^{-1}} \leq C_2 2^{l_0}.
\end{align*}

Hence, by (3.14) we have
\begin{align*}
\sum_{l=2}^{[\log_2 d]-1} I_{l,\varepsilon} \frac{\ln \varepsilon^{-1}}{d^t} + \frac{d^t}{(\ln \varepsilon^{-1})^s} & \leq \frac{C_3 2^{l_0}}{\ln \varepsilon^{-1}} \frac{\ln \varepsilon^{-2}}{2^{l_0} h(2^{l_0}) \ln \omega^{-1}} \leq \frac{C_4 2^{t}}{(\ln \varepsilon^{-1})^s} \frac{\ln \varepsilon^{-1}}{h(2^{l_0})^s (\ln \omega^{-1})^s} \to 0
\end{align*}
as \( \varepsilon^{-1} \to \infty \). This, combining with (3.8) and (3.9) means that EC-(s, t)-WT with t \leq 1 holds for APP if (1.8) holds. Theorem 1.1 is proved. \( \square \)
Proof of Theorem 1.2

According to [14, Theorem 3.1], we know that we have the same results in the worst and average case settings under ABS concerning EC-(s,t)-WT for 0 < s ≤ 1 and t > 0.

(i) It follows that EC-(s,t)-WT always holds for 0 < s ≤ 1 and t > 1 for ABS. This yields that EC-(s,t)-WT holds for s > 1 and t > 1 for ABS, and by (2.2) also for NOR. Hence (i) holds.

(ii) If EC-(s,1)-WT with s ≥ 1 holds for ABS or NOR, then (s,1)-WT with s ≥ 1 holds also for ABS or NOR. It follows from [13, Theorem 5.1] that lim j→∞ a_j = ∞.

On the other hand, if lim j→∞ a_j = ∞, then EC-WT holds for ABS or NOR and hence, EC-(s,1)-WT with s ≥ 1 also holds for ABS or NOR. This completes the proof of (ii).

(iii) EC-(1,t)-WT with t < 1 holds for ABS iff (1.10) holds. (iii) is proved.

(iv) If (1.11) holds, then EC-(s,t)-WT with s < 1, t ≤ 1 holds for ABS, and also for NOR by (2.2).

On the other hand, assume that EC-(s,t)-WT with s < 1, t = 1 holds for ABS or NOR. By (2.2), we know that EC-(s,t)-WT with s < 1, t = 1 holds also for NOR. Also by (2.2), we have

\[ (3.16) \quad \frac{\ln n_{\text{avg, ABS}}(\varepsilon, d)}{(\ln \varepsilon)^s + d} = \frac{[\ln(e_{\text{avg}}(0,d)\varepsilon^{-1})]^s + d}{(\ln \varepsilon)^s + d} \frac{\ln n_{\text{avg, NOR}}((e_{\text{avg}}(0,d))^{-1}\varepsilon, d)}{(\ln(e_{\text{avg}}(0,d)\varepsilon^{-1}))^s + d}. \]

By (2.2) we have e_{avg}(0,d)\varepsilon^{-1} + d → ∞ iff ε^{-1} + d → ∞, and

\[ (3.17) \quad \frac{[\ln(e_{\text{avg}}(0,d)\varepsilon^{-1})]^s + d}{(\ln \varepsilon)^s + d} \leq 2s[\ln(e(0,d))]^s + \frac{2s(\ln \varepsilon^{-1})^s + d}{(\ln \varepsilon)^s + d} \leq M_1^s \omega^{s+1} + 2^s. \]

Since EC-(s,t)-WT with s < 1, t = 1 holds for NOR, we get

\[ \lim_{\varepsilon^{-1} + d \to \infty} \frac{\ln n_{\text{avg, NOR}}((e_{\text{avg}}(0,d))^{-1}\varepsilon, d)}{(\ln(e_{\text{avg}}(0,d)\varepsilon^{-1}))^s + d} = 0, \]

which combining with (3.16) and (3.17), yields that

\[ \lim_{\varepsilon^{-1} + d \to \infty} \frac{\ln n_{\text{avg, ABS}}(\varepsilon, d)}{(\ln \varepsilon)^s + d} = 0. \]

It follows that EC-(s,t)-WT with s < 1, t = 1 holds for ABS. Hence (1.11) holds. (iv) is proved.

(v) If EC-(s,t)-WT with s > 1 and t < 1 holds, then (s,t)-WT with s > 1 and t < 1 holds. It follows from [11, Theorem 4.7], we have (1.12).

On the other hand, suppose that (1.12) holds. We want to show that (s,t)-WT with s > 1 and t < 1 holds under ABS or NOR. By (2.2) it suffices to prove that for s > 1 and t < 1,

\[ (3.18) \quad \lim_{\varepsilon^{-1} + d \to \infty} \frac{\ln n_{\text{avg, ABS}}(\varepsilon, d)}{(\ln \varepsilon)^s + d^s} = 0. \]
It follows from [2, Equation (3.12)] that for any \( s_d \in (0, 1/2] \)
\[
n_{\text{avg}, \text{ABS}}(\varepsilon, d) \leq \varepsilon^{-2(1-s_d)/s_d} \left( \sum_{k=1}^{\infty} \lambda_{d,k}^{-1} \right)^{1/s_d},
\]
where
\[
(3.19) \quad u_d := \max(\omega^{a_d}, \frac{1}{2d}), \quad \text{and} \quad s_d := \frac{1}{2} (\ln^+ \frac{1}{u_d})^{-1}, \quad d \in \mathbb{N}.
\]
Furthermore, if (1.12) holds, then it follows from [2, Equations (3.13) and (3.14)]
\[
\ln n_{\text{avg}, \text{ABS}}(\varepsilon, d) \leq \frac{2}{s_d} \ln \varepsilon^{-1} + \frac{e^{1/2} M_{1/2}}{s_d} \sum_{k=1}^{d} u_k,
\]
and
\[
\lim_{d \to \infty} d^{1/s_d} \sum_{k=1}^{d} u_k = 0,
\]
which means that
\[
\lim_{\varepsilon^{-1+d} \to \infty} s_d (\ln \varepsilon^{-1})^{s_d} + d^{s_d} \leq e^{1/2} M_{1/2} \lim_{d \to \infty} \sum_{k=1}^{d} u_k = 0.
\]
In order to prove (3.18), by (3.20) it suffices to prove that for \( s > 1 \),
\[
(3.21) \quad \lim_{\varepsilon^{-1+d} \to \infty} \frac{2 \ln \varepsilon^{-1}}{s_d (\ln \varepsilon^{-1})^{s_d} + d^{s_d}} = 0.
\]
By (3.19) we have
\[
(3.22) \quad \frac{1}{s_d} = 2 \ln^+ \frac{1}{u_d} \leq 2 \ln^+(2d).
\]
For \( s > 1 \), by the Young inequality \( ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}, a, b \geq 0, 1/p + 1/p' = 1 \) with \( p = \frac{1+s}{s}, p' = \frac{s+1}{s} \) we have
\[
\lim_{\varepsilon^{-1+d} \to \infty} \frac{\ln^+(2d) \ln(\varepsilon^{-1})}{(1 + \ln \varepsilon^{-1})^{s_d} + d^{s_d}} = \lim_{\varepsilon^{-1+d} \to \infty} \frac{(\ln \varepsilon^{-1})^{\frac{1+s}{s}} + (\ln^+(2d))^{\frac{s+1}{s}}}{(\ln \varepsilon^{-1})^{s_d} + d^{s_d}} = 0,
\]
which combining (3.22), gives (3.21). This finishes the proof of (v).

The proof of Theorem 1.2 is completed. \( \square \)

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