Cosmic vorticity on the brane

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We study vector perturbations about four-dimensional brane-world cosmologies embedded in a five-dimensional vacuum bulk. Even in the absence of matter perturbations, vector perturbations in the bulk metric can support vector metric perturbations on the brane. We show that during de Sitter inflation on the brane vector perturbations in the bulk obey the same wave equation for a massless five-dimensional field as found for tensor perturbations. However, we present the second-order effective action for vector perturbations and find no normalisable zero-mode in the absence of matter sources. The spectrum of normalisable states is a continuum of massive modes that remain in the vacuum state during inflation.

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I. INTRODUCTION

Recently a new approach to dimensional reduction has been proposed in which matter fields are restricted to a hypersurface, or brane, while gravity propagates in the higher-dimensional bulk \cite{1-5}. In particular Randall and Sundrum \cite{6} have proposed a scenario in which four-dimensional gravity may be recovered at low energies on a brane embedded in five dimensional anti-de Sitter space \((\text{AdS}_5)\). To test this scenario we must find distinct experimental signatures of such a model and in this paper we take up this challenge in the context of inhomogeneous perturbations about a homogeneous and isotropic cosmology on the brane.

We shall consider linear perturbations about a four-dimensional Friedmann-Robertson-Walker (FRW) spacetime embedded in a five-dimensional vacuum bulk. Matter perturbations couple not only to metric perturbations on the brane, but also to perturbations in the bulk. The gauge-invariant formalism to describe metric perturbations in brane-world models is still being developed \cite{7-10} (see also \cite{11} for a covariant approach). As in the more familiar four-dimensional cosmological context, metric perturbations can be decomposed into scalar, vector and tensor perturbations with respect to their properties on maximally symmetric spacelike 3-surfaces \cite{12-14} which are defined throughout the bulk. Solutions to the metric perturbation equations have been presented for tensor modes on a de Sitter brane, or in the long-wavelength limit \cite{16}. Solutions have also been found for some scalar modes, specifically long-wavelength curvature perturbations defined with respect to worldlines comoving with the matter \cite{17,18}, and at low-energy \cite{19}. Unlike tensor modes, vector modes can couple to linear matter perturbations on the brane, but compared with scalar modes which also couple to matter, present us with a smaller number of equations to deal with.

In this paper we will focus on vector perturbations, which are solenoidal 3-vectors, describing cosmic vorticity \cite{12-14}. Even in the absence of matter on the brane, they offer a much richer phenomenology in a brane-world than is possible in ordinary four-dimensional general relativity where the gauge-invariant vector perturbations are constrained to vanish in the absence of any matter vorticity. In a 4D brane-world the metric vorticity on the brane need not vanish as it may be supported by vorticity in the bulk gravitational field. Vector metric perturbations in the bulk are part of a 4D gravi-photon obtained from dimensional reduction of five-dimensional gravity \cite{20}. We give the second-order perturbed effective action in terms of the gauge-invariant vector perturbations in the bulk and derive their equations of motion.

We study the particular case of de Sitter cosmology on the brane. This is the archetypal model for slow-roll inflation in the early universe which produces primordial spectra of perturbations on large (super-horizon) scales at late cosmic times from an initial vacuum state of small scale fluctuations in light fields. We shall show that in this special case of de Sitter inflation on the brane, the vector perturbations obey the same wave equation found for tensor perturbations, permitting a massless zero-mode. This raises the possibility that long-wavelength vector modes, entirely absent in conventional models of inflation, could be generated from vacuum fluctuations, just as happens for tensor modes \cite{16}. However, we will show that these effectively massless modes are non-normalisable in the bulk, corresponding to a divergent action and hence are absent in the spectrum of quantum fluctuations. This is complements results previously obtained for bulk vector fields with a Minkowski brane embedded in AdS$_5$ \cite{15}.
II. VECTOR PERTURBATIONS

A. Background model

We will assume that the gravitational field in the bulk obeys the five-dimensional vacuum Einstein equations

\[
(5) G_{AB} = -\kappa_5^2 (5) \Lambda (5) g_{AB},
\]

where \(\kappa_5^2\) is the five-dimensional gravitational constant and \(\Lambda\) the vacuum energy in the bulk. The gravitational field is also subject to appropriate boundary conditions at the brane. The energy-momentum tensor for matter, \(T_{\mu\nu}\), restricted to an infinitesimally thin hypersurface, and the brane tension, \(\lambda\), causes a discontinuity in the extrinsic curvature, \(K_{\mu\nu}\), given by the junction conditions \[21–23\]

\[
[K_{\mu\nu}]_b^t = -\kappa_5^2 \left( \frac{\Lambda}{3} g_{\mu\nu} + T_{\mu\nu} - \frac{1}{3} g_{\mu\nu} T \right),
\]

where \(K_{\mu\nu} = g^A_\mu g^B_\nu \nabla_A n_B n^A\), \(n^A\) is the spacelike unit-vector normal to the brane, and the projected metric on the brane is given by

\[
g_{AB} = (5) g_{AB} - n_A n_B.
\]

If we assume that the brane is located at a \(Z_2\)-symmetric orbifold fixed point at \(y = 0\), then the energy-momentum tensor for matter, \(T_{\mu\nu}\), and the brane tension, \(\lambda\), determine the extrinsic curvature close to the brane

\[
K_{\mu\nu} = -\kappa_5^2 \left( \frac{\lambda}{3} g_{\mu\nu} + T_{\mu\nu} - \frac{1}{3} g_{\mu\nu} T \right).
\]

In order to study inhomogeneous perturbations we will pick a specific form for the unperturbed 5-d spacetime metric that accommodates spatially flat FRW cosmological solutions on the brane (at any constant-\(y\) hypersurface),

\[
ds_5^2 = -n^2(t, y) dt^2 + a^2(t, y) \delta_{ij} dx^i dx^j + dy^2,
\]

which includes anti-de Sitter spacetime as a special case. Cosmological solutions of this form have been extensively studied in the literature \[2, 22–25\].

B. Metric and matter perturbations

We consider arbitrary linear vector perturbations, \(V^i\), about the background metric given in Eq. \([23]\) that are solenoidal 3-vectors on spatial hypersurfaces of constant \(t\) and \(y\), i.e, \(\partial_j V^i = 0\). The reason for splitting generic metric perturbations into scalar, vector and tensor modes is that they are decoupled in the first-order perturbation equations and hence their solutions can be derived independently of one another.

We can write the most general vector metric perturbation to first-order as

\[
(5) g_{AB} = \begin{pmatrix}
-n^2 & -a^2 S_i & 0 \\
-a^2 S_j & a^2 \left[ \delta_{ij} + F_{ij} + F_{j|i} \right] & -a^2 S_{yi} \\
0 & -a^2 S_{yi} & 1
\end{pmatrix},
\]

where \(S_i, F_i,\) and \(S_{yi}\) are solenoidal (divergence-free) 3-vectors on hypersurfaces of constant \(t\) and \(y\). In a four-dimensional FRW metric we have only \(S_i\) and \(F_i\) \[2, 14\].

We will also decompose the vector perturbations into Fourier modes on the 3-space, with time and bulk dependent amplitudes

\[
S_i = S(t, y) \hat{e}_i(x),
\]

\[
S_{yi} = S_y(t, y) \hat{e}_i(x),
\]

\[
F_i = F(t, y) \hat{e}_i(x),
\]

where \(\hat{e}_i(x)\) is a solenoidal (\(\partial^i \hat{e}_i = 0\)) unit eigenvector of the spatial Laplacian such that \(\partial^i \partial_j \hat{e}_i = -k^2 \hat{e}_j\).

Under an arbitrary vector gauge transformation \[12\] \(x^i \rightarrow x^i + \delta x^i\) (where \(\partial^i \delta x_i = 0\)) we find
\[ S_i \rightarrow S_i + \delta x_i, \quad (2.10) \]
\[ S_{yi} \rightarrow S_{yi} + \delta x'_i, \quad (2.11) \]
\[ F_i \rightarrow F_i - \delta x_i. \quad (2.12) \]

Thus there are essentially only two gauge-invariant variables in the bulk which can be written as

\[ \tau \equiv S + \dot{F}, \quad (2.13) \]
\[ \sigma \equiv S_y + F', \quad (2.14) \]

though we will also find it useful to introduce a third gauge-invariant combination

\[ \Delta \equiv S' - \dot{S}_y = \tau' - \dot{\sigma}. \quad (2.15) \]

In four-dimensional general relativity there is only one gauge-invariant vector metric perturbation, \( \tau [13] \). The perturbed energy-momentum tensor for matter on the brane can be given as

\[ T^\mu_\nu = \left( \frac{\rho}{(\rho + P)\nu^i} \cdot (\nu_j - S_j|_{y=0}) \right) \cdot \left( \frac{\rho + P}{P\delta^i_j + \delta \pi^i_j} \right). \quad (2.16) \]

Under a 4D gauge transformation on the brane, \( x^i \rightarrow x^i + \delta x^i|_{y=0} \), the velocity perturbation transforms as \( v^i \rightarrow v^i + \delta x^i|_{y=0} \) so that both the momentum and anisotropic pressure perturbations,

\[ (\rho + P)(\nu_i - S_i|_{y=0}) \equiv \dot{e}_i \delta \rho(t), \quad (2.17) \]
\[ \delta \pi^i_j \equiv (\dot{e}^i_j + \partial_j \dot{e}^i) \delta \pi(t), \quad (2.18) \]

are gauge-invariant.

C. Equations of motion

The second-order perturbed Einstein-Hilbert action for gravity in the bulk yields the effective action for the metric perturbations

\[ \delta S = \int dt dy \left\{ -\frac{1}{2n} \frac{a^5}{n} \Delta^2 + \frac{1}{2} \frac{a^3}{n} k^2 \sigma^2 - \frac{1}{2} \frac{a^3}{n} k^2 \tau^2 \right\}. \quad (2.19) \]

The three equations of motion can be derived either from first-order perturbations of the five-dimensional Einstein equation [13] [14], or by extremising \( \delta S \) with respect to variations in \( F, S, \) and \( S_y \). In either case we obtain

\[ \frac{1}{n^2} \left\{ \ddot{\tau} + \left( 3 \frac{\dot{a}}{a} - \frac{\dot{n}}{n} \right) \dot{\tau} + \left( 3 \frac{\dot{a}^2}{a^2} - 6 \frac{\dot{a} \dot{n}}{a n} - \frac{\ddot{n}}{n} + \frac{3 \dot{n}^2}{n^2} \right) \sigma \right\} + \frac{k^2}{a^2} \tau = \Delta', \quad (2.20) \]
\[ \frac{k^2}{a^2} \sigma = \Delta + \left( 5 \frac{\dot{a}}{a} - \frac{n'}{n} \right) \Delta. \quad (2.21) \]

From these we can obtain coupled second-order evolution equations for \( \tau \) or \( \sigma \):

\[ \frac{1}{n^2} \left\{ \dddot{\tau} + 3 \left( \frac{\dot{a}}{a} - \frac{\dot{n}}{n} \right) \ddot{\tau} + \left( 3 \frac{\dddot{a}}{a^2} - 6 \frac{\dot{a} \dddot{n}}{a n} - \frac{\ddot{n}}{n} + 3 \frac{\dot{n}^2}{n^2} \right) \dot{\tau} \right\} + \frac{k^2}{a^2} \tau = \tau'' + \left( 5 \frac{\dot{a}}{a} - \frac{n'}{n} \right) \tau' + 2 \left( \frac{n'}{n} - \frac{\dot{a}}{a} \right) \sigma + \left( \frac{3 \dddot{a}}{a^2} - \frac{\dot{n}'}{n} + \dddot{n} \right) \sigma, \quad (2.23) \]
\[ \frac{1}{n^2} \left\{ \dddot{\sigma} + \left( 5 \frac{\dot{a}}{a} - \frac{n'}{n} \right) \ddot{\sigma} - 2 \frac{\dddot{a}}{a} \tau' + \left( 3 \frac{\dddot{a}}{a^2} - 6 \frac{\dot{a} \dddot{n}}{a n} - \frac{\ddot{n}}{n} + 3 \frac{\dot{n}^2}{n^2} \right) \dot{\sigma} \right\} + \frac{k^2}{a^2} \sigma = \sigma'' + \left( \frac{3 \dddot{a}}{a^2} - \frac{\dot{n}'}{n} + \dddot{n} \right) \sigma' + \left( \frac{3 \dddot{a}}{a^2} - 3 \frac{\dot{a}^2}{a^2} + \frac{\dot{n}'}{n} + \frac{\dddot{n}}{n^2} + \frac{n'}{n} + 6 \frac{n' a'}{n a} \right) \sigma, \quad (2.24) \]
and a decoupled evolution equation for $\Delta$:
\[
\frac{1}{n^2} \left\{ \dddot{\Delta} + \left( \frac{\dot{a}}{a} - \frac{3 \dot{n}}{n} \frac{\dot{\Delta}}{\Delta} \right) \right\} - \left\{ \Delta'' + \left( \frac{a'}{a} - \frac{n'}{n} \Delta' \right) \right\} + \frac{k^2}{a^2} \Delta = \left\{ 5 \frac{a''}{a} + \frac{5 a'^2}{a^2} - 2 \frac{a' n'}{a n} - \frac{n''}{n^2} - \frac{1}{n^2} \left( 5 \frac{\dddot{a}}{a} + \frac{5 \dot{a}^2}{a^2} - 12 \frac{\dot{a} \dot{n}}{a n} - \frac{\dot{n}}{n} + \frac{3 \dot{n}^2}{n^2} \right) \right\} \Delta. \tag{2.25}
\]

### D. Master variable

The form of the constraint equation (2.20) implies that there exists a function $\Omega$ such that
\[
\tau = \frac{n}{a^3} \dot{\Omega}, \quad \sigma = \frac{1}{n a^3} \ddot{\Omega}. \tag{2.26}
\]
Substituting this into the evolution equations (2.21) and (2.22) they can be integrated to yield
\[
\Delta = \frac{n}{a^3} k^2 \Omega, \tag{2.27}
\]
(whose variation with respect to $\Omega$ gives the equation of motion (2.28).)

### E. Junction conditions

The first-order perturbations in the extrinsic curvature of constant-$y$ hypersurfaces due to vector perturbations are
\[
\delta K_i^0 = \frac{a^2}{2 n^2} \Delta \dot{e}_i, \tag{2.30}
\]
\[
\delta K_i^j = \frac{1}{2} \sigma \left( \partial^i \dot{e}_j + \partial_j \dot{e}_i \right). \tag{2.31}
\]
Equations (2.4), (2.16), (2.17) and (2.18) thus yield
\[
\Delta_b = -\kappa^2 \frac{n^2}{a^2} \delta p, \tag{2.32}
\]
\[
\sigma_b = -\kappa^2 \frac{n^2}{a^2} \delta \pi, \tag{2.33}
\]
where the subscript "b" denotes quantities evaluated on the brane. Thus we find that the gauge-invariant metric perturbation, $\sigma$, vanishes on the brane for a perfect fluid with vanishing anisotropic pressure. $\Delta$, defined in Eq. (2.15), is constrained to vanish on the brane in the absence of any solenoidal 3-momentum. However the gauge-invariant vector perturbation, $\tau$, remains non-vanishing in general on the brane, even when the matter on the brane possesses no vorticity. Its first derivative normal to the brane, $\tau_n$, is zero if both the momentum and anisotropic pressure perturbations vanish, but $\tau_i$ may still be non-zero at the brane and the evolution of $\tau$ on the brane cannot in general be decoupled from the evolution of $\tau$ in the bulk.

In terms of the master variable $\Omega$ we have
\[ k^2 \Omega_b = -\kappa_b^2 n_b a_b^3 \delta p, \quad (2.34) \]
\[ \dot{\Omega}_b = -\kappa_b^2 n_b a_b^3 \delta \pi. \quad (2.35) \]

Taken together these are consistent with usual momentum conservation equation for matter on the brane:
\[ \dot{\delta p} + \left( 3 \frac{\dot{a}_b}{a_b} + \frac{\dot{n}_b}{n_b} \right) \delta p = k^2 \delta \pi. \quad (2.36) \]

### III. DE SITTER INFLATION ON THE BRANE

The only separable solution for the background metric
\[ n = \mathcal{A}(y), \quad a = a_b(t)\mathcal{A}(y), \quad (3.1) \]
describes de Sitter expansion on the brane (\( \dot{a}_b/a_b = H = \text{constant} \)) in an AdS\(_5\) bulk. Solving for the metric in the bulk yields \([24,16]\)
\[ \mathcal{A}(y) = \frac{H}{\mu} \sinh [\mu (y_h - |y|)], \quad (3.2) \]
where \( \mu \) is the anti-de Sitter curvature scale, and there is a Cauchy horizon at \( |y| = y_h \) where \( \mathcal{A}(y) = 0 \) \([24]\).

#### A. Metric perturbation, \( \tau \)

In this case the equation of motion \([2,23]\) for \( \tau \) decouples from \( \sigma \) in the bulk and we have the equation of motion
\[ \ddot{\tau} + 3H \dot{\tau} + \frac{k^2}{a_b^2} \tau = \mathcal{A}^2 \tau'' + 4A^2 \tau'. \quad (3.3) \]

This has exactly the same form as the equation for tensor perturbations with de Sitter expansion on the brane. Following Refs. \([28,16]\) we separate
\[ \tau(t, y) = \int dm \varphi_m(t) \mathcal{E}_m(y), \quad (3.4) \]
and obtain two ordinary differential equations
\[ \ddot{\varphi}_m + 3H \dot{\varphi}_m + \left[ m^2 + \frac{k^2}{a_b^2} \right] \varphi_m = 0, \quad (3.5) \]
\[ \mathcal{E}_m'' + 4A^2 \mathcal{E}_m' + \frac{m^2}{A^2} \mathcal{E}_m = 0. \quad (3.6) \]

Each bulk eigenmode, characterised by the eigenvalue \(-m^2\), becomes a 4D effective field with mass \(m^2\). The general solution for the time-dependence of a massive field in de Sitter is given by \([24]\)
\[ \varphi_m(t) = \left( \frac{k}{a_b H} \right)^{3/2} B_\nu \left( \frac{k}{a_b H} \right), \quad (3.7) \]
where \(B_\nu\) is a linear combination of Bessel functions of order \(\nu = \sqrt{(9/4) - (m^2/H^2)}\). Fields with \(m \leq 3H/2\) are light during inflation and vacuum fluctuations on small scales \((k \gg a_b H)\) can give rise to a spectrum of perturbations on large scales \((k \ll a_b H)\) at late times. By contrast heavy modes (with \(m > 3H/2\)) remain in the vacuum state with essentially no fluctuations on large scales.

In Ref. \([16]\) the amplitude of vacuum fluctuations for tensor modes was calculated by finding the spectrum of modes with finite perturbed effective action in the bulk. In that case a discrete massless \((m = 0)\) mode and a continuum of massive modes \((m > 3H/2)\) was found, so that during inflation the massive modes remain in their vacuum state and only the massless mode acquires a spectrum of fluctuations on large scales \([16]\).
For \( m = 0 \) our bulk eigenmode has the general solution

\[
\mathcal{E}_0(y) = C_1 + C_2 \int \frac{dy}{A} \, .
\] (3.8)

Although this is divergent for \( C_2 \neq 0 \) at the horizon where \( A \to 0 \), the boundary conditions at the brane, given in Eqs. (2.32) and (2.33) require \( \tau'_b \) and hence \( C_2 = 0 \) in the absence of solenoidal momentum and anisotropic pressure on the brane (\( \delta p = 0, \delta \pi = 0 \)), as in the case of slow-roll inflation driven by scalar fields.

In the case of tensor perturbations studied in Ref. [16] this was sufficient to keep the perturbed effective action finite, leading to the generation of large-scale tensor perturbations in this massless mode from small-scale vacuum fluctuations during inflation on the brane. However in the present case of vector perturbations, the effective action (2.19) includes contributions from the other gauge-invariant metric perturbation, \( \sigma \). To evaluate the corresponding solution for \( \sigma \) we turn to the master variable \( \Omega \) introduced in subsection II.D.

\section*{B. Master variable}

For a de Sitter brane we are able to separate

\[
\Omega(t, y) = \int dm \, \omega_m(t) W_m(y) \, ,
\] (3.9)

where the equation of motion (2.28) requires

\[
\ddot{\omega}_m - 3H \dot{\omega}_m + \left[ m^2 + \frac{k^2}{a^2} \right] \omega_m = 0 \, ,
\] (3.10)

\[
W''_m - 2 \frac{A'}{A} W'_m + \frac{m^2}{A^2} W_m = 0 \, .
\] (3.11)

From the definition of \( \tau \) in Eq. (2.26) we see that \( \omega_m = a^3 \phi_m \), which is consistent with Eqs. (3.5) and (3.10).

Following the original paper of Randall and Sundrum [6] for a Minkowski brane, and the analysis of tensor modes for a de Sitter brane [28,16], we define

\[
\Psi_m = A^{-3/2} W_m \, , \quad dz = \frac{dy}{A} \, ,
\] (3.12)

and hence \( z \to \infty \) as \( y \to y_h \). The canonical field \( \Psi_m \) then obeys the Schrödinger-like equation

\[
\frac{d^2 \Psi_m}{dz^2} - V(z) \Psi_m = -m^2 \Psi_m \, ,
\] (3.13)

with the effective potential

\[
V(z) = \frac{3}{4} \mu^2 A^2 + \frac{9}{4} H^2 + 3H \delta(z) \, .
\] (3.14)

Like the original ‘Volcano potential’ of Ref. [6], \( V(z) \) decreases as the ‘warp factor’ \( A(y) \) decreases away from the brane, and like its cosmological generalisation for tensor modes [28,16], it approaches a constant value, \( V \to 9H^2/4 \), as \( z \to \infty \). However a crucial difference is that the Dirac \( \delta \)-function, which has a negative coefficient and yields the 4D graviton localised on the brane in Refs. [6,28,16], here has a positive coefficient so that there is no bound zero-mode for vector perturbations.

The general solution for the zero-mode, \( m = 0 \), is given by

\[
W_0(y) = \tilde{C}_2 + C_1 \int_0^y A^2(y') dy' \, .
\] (3.15)

Note that substituting this expression into Eq. (2.26) in order to determine \( \mathcal{E}_0(y) \) yields only the homogeneous solution, \( \mathcal{E}_0 = C_1 \) in Eq. (3.8).

The junction condition (2.34) at the brane yields
\[ \Omega_b = a_b^3 \varphi_0 \dot{C}_2 = -\frac{\kappa_5^2}{k^2} a_b^3 \delta p. \] (3.16)

In order to calculate the spectrum of vacuum fluctuations on the brane we need to evaluate the effective action for each bulk eigenmode. For each mode \( m \) we obtain from Eqs. (2.23) and (3.14)

\[ \delta S_m = C_m^2 \int dt \frac{k^2 a_0^3}{2} \left[ \dot{\varphi}_m^2 - \left( \frac{k^2}{a_0^2} + m^2 \right) \varphi_m^2 \right], \] (3.17)

where the normalisation constant for each mode is given by

\[ C_m^2 = \int_{-\infty}^{\infty} |\Psi_m|^2 \, dz. \] (3.18)

This yields an effective action which has the standard form for a four-dimensional field, \( \Phi_m = k C_m \varphi_m \), with mass \( m \) in a FRW spacetime with scale factor \( a_b \), for any normalisable modes, i.e., modes with finite \( C_m^2 \).

For modes with \( m^2 > 9H^2/4 \) their asymptotic solutions for \( \Psi_m \) at large \( z \gg H^{-1} \) are plane waves \( \Psi \sim e^{\pm ikz} \), where \( \tilde{k}^2 = m^2 - 9H^2/4 \), and these modes are thus normalisable.

In the absence of solenoidal momentum or pressure perturbations, the junction conditions at the brane in Eqs. (2.34) and (2.35) yields \( \dot{C}_2 = 0 \), while allowing \( C_1 \neq 0 \) in Eq. (3.11). Although this leaves \( W_0 \) finite, we have \( \Psi_0 = \Psi_0 / A^3/2 \), and this leads to a divergent integral in Eq. (3.18) for \( C_0 \). Thus there is no normalisable zero-mode in the absence of vector matter perturbations on the brane.

Thus, even though the gauge-invariant metric perturbation, \( \tau \), obeys the same wave equation (3.3) for a massless 5D field as previously found for tensor perturbations \( 28,16 \), we find no ‘light’ normalisable vector modes \( m \leq 3H/2 \) that could be excited by de Sitter expansion on the brane in order to generate large scale (super-horizon) vector metric perturbations on the brane from an initial vacuum state on small scales.

We can only construct a normalisable zero-mode if \( W_0 \) vanishes at the Cauchy horizon, corresponding to \( \dot{C}_2 = -C_1 f_0^{(n)} A^2dy. \) In this case the junction condition (3.16) then requires that the momentum of matter on the brane is \((\tilde{k}^2/\kappa_5^2)\tilde{\gamma}_0\), which appears to have no physical motivation. Similarly, if we were to introduce a ‘regulator brane’ just within the Cauchy horizon in order to keep the normalisation constant, \( C_m^2 \), finite with \( \dot{C}_2 = 0 \), the regulator brane would have to possess just the right matter perturbation in order to satisfy the junction condition. This is in contrast to the original Randall-Sundrum zero-mode \( 1 \) where the regulator brane has constant brane tension, and the normalisation of the zero-mode has a well-defined limit as the regulator brane approaches the Cauchy horizon.

IV. THE VIEW FROM THE BRANE

Shiromizu, Maeda and Sasaki \( 23 \) showed that the effective four-dimensional Einstein equations on the brane can be obtained by projecting the five-dimensional variables onto the 4D brane world. If our 4D-world is described by a domain wall (3-brane) \( (M, g_{\mu\nu}) \) in five-dimensional spacetime \( (\mathcal{M}, (^{(5)}g_{\alpha\beta})) \), where the induced metric, \( g_{\mu\nu} \), was defined in Eq. (2.3), then, using the Gauss equation \( 24 \), one obtains the 4-dimensional effective equations as

\[ (4)G_{\mu\nu} = -\frac{1}{2} \kappa_5^2 (5)\Lambda + K K_{\mu\nu} - K_{\mu}^\sigma K_{\nu\sigma} - \frac{1}{2} g_{\mu\nu} (K^2 - K^{\alpha\beta} K_{\alpha\beta}) - E_{\mu\nu}, \] (4.1)

where \( K_{\mu\nu} \) is the extrinsic curvature of the brane, \( K = K^\mu_\mu \) is its trace, and the effect of the non-local bulk gravitational field is described by the projected five-dimensional Weyl tensor

\[ E_{\mu\nu} = (5)C^{E}_{\alpha\beta} m^E g_{\mu}^\alpha g_{\nu}^\beta. \] (4.2)

Using the junction conditions given in Eq. (2.4), we can give the extrinsic curvature in terms of the energy-momentum tensor on the brane and we obtain

\[ (4)G_{\mu\nu} = -(4)\Lambda g_{\mu\nu} + \kappa_5^2 T_{\mu\nu} + \kappa_5^4 \Pi_{\mu\nu} - E_{\mu\nu}, \] (4.3)

where
\[ (4) \Lambda = \frac{\kappa_b^2}{2} \left[ (5) \Lambda + \frac{1}{6} \kappa_b^2 \lambda^2 \right], \quad (4.4) \]
\[ \kappa_4^2 = 8\pi G_N = \frac{\kappa_b^2}{6} \lambda, \quad (4.5) \]
\[ \Pi_{\mu\nu} = -\frac{1}{4} T_{\mu\nu} T^\alpha_{\nu} + \frac{1}{12} \mathcal{T} T_{\mu\nu} + \frac{1}{8} g_{\mu\nu} T_{\alpha\beta} T^{\alpha\beta} - \frac{1}{24} g_{\mu\nu} T^2. \quad (4.6) \]

The power of this approach is that the above form of the four-dimensional effective equations of motion is independent of the evolution of the bulk spacetime, being given entirely in terms of quantities defined at the brane. Thus these equations apply to brane-world scenarios with infinite or finite bulk, stabilised or evolving.

The perturbed 4D Einstein tensor, derived for the perturbed metric induced on the brane, is
\[ \delta^{(4)} G^0_i = \frac{1}{2n_b^2} k^2 \tau_b \dot{e}_i, \quad (4.7) \]
\[ \delta^{(4)} G^j_i = \frac{1}{2n_b^2} \left[ \dot{\tau}_b + \left( \frac{3a_b'}{a_b} - \frac{n_b'}{n_b} \right) \tau_b \right] (\partial^i \dot{e}_j + \partial^j \dot{e}_i). \quad (4.8) \]

Note that only \( \tau_b \) appears in these expressions as it is the only gauge-invariant vector metric perturbation of the 4D metric \([3]\).

Substituting the perturbed energy-momentum tensor as given in Eq. (2.16) into Eq. (4.3) gives the contribution of the matter on the brane
\[ \kappa_b^2 \delta T^0_i = \kappa_4^2 \left( 1 + \frac{\rho}{\lambda} \right) \delta p \dot{e}_i, \quad (4.9) \]
\[ \kappa_b^2 \delta T^j_i + \kappa_4^2 \delta \Pi^j_i = \kappa_4^2 \left( 1 - \frac{\rho + 3P}{2\lambda} \right) \delta \pi^j_i. \quad (4.10) \]

where the contribution from \( \delta \Pi^0_i \) becomes negligible for \( \rho \ll \lambda \) and \( P \ll \lambda \). The junction condition at the brane, given in Eq. (2.2), relates these matter perturbations to metric perturbations at the brane:
\[ \kappa_4^2 \left( 1 + \frac{\rho}{\lambda} \right) \delta p \dot{e}_i = a_b^2 \left( \frac{a_b'}{a_b} \Delta_b \right) \dot{e}_i, \quad (4.11) \]
\[ \kappa_4^2 \left( 1 - \frac{\rho + 3P}{2\lambda} \right) \delta \pi^j_i = \frac{1}{2} \left( \frac{n_b'}{n_b} + \frac{a_b'}{a_b} \right) \sigma_b \left( \partial^i \dot{e}_j + \partial^j \dot{e}_i \right). \quad (4.12) \]

Finally the contribution of metric perturbations in the bulk to the modified Einstein equations on the brane is given by the projected Weyl tensor \( \delta E^\mu_i \). For vector perturbations, in terms of our gauge invariant variables, we have
\[ \delta E^0_i = -\frac{1}{6n_b^2} \left\{ 2a_b^2 \left[ \Delta'_b + \left( \frac{2a_b'}{a_b} - \frac{n_b'}{n_b} \right) \Delta_b \right] + k^2 \tau_b \right\} \dot{e}_i, \quad (4.13) \]
\[ \delta E^j_i = -\frac{1}{6n_b^2} \left\{ \dot{\tau}_b + \left( 3 \frac{a_b'}{a_b} - \frac{n_b'}{n_b} \right) \tau_b + n_b^2 \left[ 2\sigma'_b + \left( \frac{3a_b'}{a_b} - \frac{n_b'}{n_b} \right) \sigma_b \right] \left( \partial^i \dot{e}_j + \partial^j \dot{e}_i \right) \right\}. \quad (4.14) \]

Substituting Eqs. (4.7), (4.14) into the four-dimensional Einstein equations (4.3), yields the first two five-dimensional field equations (2.20) and (2.21). The remaining five-dimensional field equation (2.22) can be obtained directly from the conservation of the matter momentum (2.30), on the brane using Eqs. (2.32) and (2.33). However the equations at the brane do not in general yield a closed set of equations on the brane. Although \( \sigma_b \) and \( \Delta_b \) are determined by the junction conditions (2.32) and (2.33), the behaviour of their y-derivatives, \( \sigma'_b \) and \( \Delta'_b \), must be determined by solving the five-dimensional equations in the bulk.

The projected Weyl tensor acts as an effective energy-momentum tensor on the brane, whose effective momentum can be written, using Eqs. (4.13) and (2.21), as
\[ \kappa_4^2 \delta p = \frac{a_b^2}{2n_b^2} \left\{ \Delta'_b + \left( 3 \frac{a_b'}{a_b} - \frac{n_b'}{n_b} \right) \Delta_b \right\}, \quad (4.15) \]
and the effective anisotropic pressure can be written, using Eqs. (4.14) and (2.20), as
\[ \kappa_4^2 \delta \pi = \frac{1}{2} \left\{ \sigma'_b + 2a_b^2 \sigma_b \right\}. \quad (4.16) \]
Even when there are no vector perturbations in the matter energy-momentum tensor on the brane, the projected 5D Weyl tensor can supply an effective anisotropic momentum and stress and hence support vector perturbations in the induced 4D metric.

The contracted Bianchi identities $(\nabla_\mu G^{\mu}_\nu = 0)$ and energy-momentum conservation for matter on the brane $(\nabla_\mu T^{\mu}_\nu = 0)$ ensures, from Eq. (1.3), that $\nabla_\mu E^{\mu}_\nu = \kappa^2 \nabla_\mu \Pi^{\mu}_\nu$. The interaction with the quadratic energy-momentum tensor, $\Pi^{\mu}_\nu$, gives rise to the momentum conservation equation for the Weyl-fluid

$$\dot{\tilde{\delta}p} + \left(3 \frac{\dot{a}_b}{a_b} + \frac{\dot{n}_b}{n_b}\right) \tilde{\delta}p = k^2 \tilde{\delta}\pi + 6 \left(\frac{2 + P}{\lambda}\right) \left(\frac{\dot{a}_b}{a_b} \delta p - \frac{k^2}{2} \delta\pi\right). \quad (4.17)$$

Thus the Weyl-momentum is conserved in the absence of vorticity in the matter, or when $(\rho + P)/\lambda$ is negligible.

V. CONCLUSIONS

We have studied the nature of vector perturbations of brane-world cosmologies embedded in a five-dimensional bulk described by vacuum Einstein gravity. By a simple extension of the standard four-dimensional cosmological studies vector perturbations are described by divergence-free 3-vectors on Euclidean spatial hypersurfaces of fixed time $t$ and bulk coordinate $y$ which foliate the five-dimensional spacetime.

There is only one gauge-invariant vector perturbation, $\tau$, of the four-dimensional FRW metric induced on the brane. However in the five-dimensional bulk we can define a second gauge-invariant vector metric perturbation, $\sigma$, and in general the evolution of the vector perturbation on the brane cannot be decoupled from the bulk vector perturbation leading to coupled equations of motion. Even in the case of vanishing matter perturbations, the vector metric perturbations in the bulk can support vector metric perturbations on the brane, in contrast to four-dimensional Einstein gravity where the vector metric perturbations are constrained to vanish in the limit of vanishing matter vorticity.

In the case of de Sitter expansion on the brane $\tau$ decouples from $\sigma$ and obeys the equation of motion for a massless five-dimensional field, as previously found for tensor perturbations. In the case of tensor perturbations this leads to a spectrum of large-scale (super-horizon) perturbations being generated from an initial quantum vacuum state on sub-horizon scales. The prediction of a spectrum of vector perturbations from cosmological inflation on the brane would be a distinctive observational prediction of the brane-world cosmology, but we have shown that there is no normalisable zero-mode for the vector perturbations in the bulk that respect the vacuum junction conditions at the brane. The spectrum of normalisable modes (with finite 4D effective action) is a continuum of modes with masses $m > 3H/2$, where $H$ is the Hubble expansion rate. This includes the case of a Minkowski brane in the limit $H \to 0$.

The effective action for vector perturbations is most concisely written in terms of the “master variable”, $\Omega$, from which both $\tau$ and $\sigma$ can be derived. Following the approach of Randall and Sundrum we derive the Schrödinger-like equation for the canonically normalised variable with a modified “Volcano” potential. The absence of a normalisable zero-mode is due to the change in sign of the Dirac delta-function at the brane. The only way to obtain a normalisable vector zero-mode seems to be to have a matter source on the brane whose momentum cancels out this effect.

Except in the case of a de Sitter brane, the equations of motion in the bulk are not separable, and we cannot decompose the five-dimensional perturbations into Kaluza-Klein modes. The effect of the bulk metric perturbations on the brane can be described by the projection of the five-dimensional Weyl tensor on the brane. This is seen by the brane-world observer as a form of dark matter on the brane with trace-free energy-momentum tensor which may include anisotropic stresses. We have shown how the bulk vector perturbations would be interpreted by an observer in the brane-world as an effective momentum and anisotropic stress on the brane, arising from this projected bulk Weyl ‘fluid’. The contracted Bianchi identities on the brane yield an effective momentum conservation equation for the Weyl fluid, but the effective anisotropic stress cannot be determined from quantities locally on the brane, a consequence of the five-dimensional origin of the perturbations. Although the anisotropic stress decouples from the momentum evolution on large scales, it is needed in order to determine the gauge-invariant metric perturbation, $\tau$, which enters the Sachs-Wolfe formula for anisotropies in the cosmic microwave background. A similar effect was found recently for scalar metric perturbations.

It has been suggested that bulk metric perturbations might thus provide ‘active seeds’ for cosmological perturbations, similar to topological defects. Such a scenario could be realised by large-scale fluctuations of the master variable, $\Omega$. While spatial gradients on the brane are negligible, $\Omega = \text{constant}$ throughout the bulk is an approximate solution of the bulk equation of motion which yields no vector perturbations on the brane ($\tau$ and $\sigma$ both vanish). However, when spatial gradients become important (e.g., after horizon entry during the radiation or matter
dominated eras) a non-zero constant $\Omega$ is no longer a solution of the bulk equation of motion and the resulting time and bulk variation of $\Omega$ would generate vector perturbations on the brane.

Although such vector perturbations can appear apparently from nothing on the brane, five-dimensional gravity is a causal system, and any large-scale variations of $\Omega$ can be traced back to some initial conditions in the bulk. Such a scenario seems possible for scalar or tensor perturbations which possess a normalisable zero-mode which may be excited during a period of inflation in the early universe. However we have shown that the absence of a normalisable zero-mode for vector perturbations in the bulk means that a period of de Sitter inflation in the early universe leaves effectively no large-scale vector perturbations in the bulk. Massive modes remain in their vacuum state, and $\Omega = 0$ remains a solution to the bulk equation of motion at all subsequent times.

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[1] I. Antoniadis, Phys. Lett. B246, 377 (1990).
[2] J. Polchinski, Phys. Rev. Lett. 75, 4724 (1995) [hep-th/9510017].
[3] P. Horava and E. Witten, Nucl. Phys. B460, 506 (1996) [hep-th/9510200].
[4] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. B429, 263 (1998) [hep-ph/9803315].
[5] A. Lukas, B. A. Ovrut and D. Waldram, Phys. Rev. D60, 086001 (1999) [hep-th/9806022].
[6] L. Randall and R. Sundrum, Phys. Rev. Lett. 83, 4690 (1999) [hep-th/9906064].
[7] S. Mukohyama, Phys. Rev. D62, 084015 (2000) [hep-th/0004067].
[8] D. Langlois, Phys. Rev. D62, 126012 (2000) [hep-th/0005023; hep-th/0010063].
[9] H. Kodama, A. Ishibashi and O. Seto, Phys. Rev. D62, 064022 (2000) [hep-th/0004160].
[10] C. van de Bruck, M. Dorca, R. H. Brandenberger and A. Lukas, Phys. Rev. D62, 123515 (2000) [hep-th/0005032; C. van de Bruck, M. Dorca, C. J. A. P. Martins and M. Parry, Phys. Lett. B495, 183 (2000) [hep-th/0009050].
[11] R. Maartens, Phys. Rev. D62, 084023 (2000) [hep-th/0004166].
[12] J. M. Bardeen, Phys. Rev. D 22, 1882 (1980).
[13] H. Kodama and M. Sasaki, Prog. Theor. Phys. Supp. 78, 1 (1984).
[14] V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger, Phys. Rep. 215, 203 (1992).
[15] A. Pomarol, Phys. Lett. B486, 153 (2000) [hep-ph/9911294]; B. Bajc and G. Gabadadze, Phys. Lett. B474, 282 (2000) [hep-th/9912232; I. Oda, [hep-th/0009074].
[16] D. Langlois, R. Maartens and D. Wands, Phys. Lett. B489, 250 (2000) [hep-th/0006007].
[17] D. Wands, K. A. Malik, D. H. Lyth and A. R. Liddle, Phys. Rev. D62, 043527 (2000) [astro-ph/0003278].
[18] C. Gordon and R. Maartens, hep-th/0009010.
[19] K. Koyama and J. Soda, hep-th/0005239.
[20] Y. S. Myung, [hep-th/0009114].
[21] W. Israel, Nuovo Cim. 44B, 1 (1966).
[22] P. Binetruy, C. Deffayet and D. Langlois, Nucl. Phys. B565, 269 (2000) [hep-th/9905012].
[23] T. Shiromizu, K. Maeda and M. Sasaki, Phys. Rev. D62, 024012 (2000) [gr-qc/9910076].
[24] N. Kaloper, Phys. Rev. D60, 123506 (1999) [hep-th/9905211].
[25] C. Csaki, M. Graesser, C. Kolda and J. Terning, Phys. Lett. B462, 34 (1999) [hep-ph/9906513; J. M. Cline, C. Grojean and G. Servant, Phys. Rev. Lett. 83, 2495 (1999) [hep-ph/9906523].
[26] P. Binetruy, C. Deffayet, U. Ellwanger, and D. Langlois, Phys. Lett. B477, 285 (2000) [hep-th/9910121].
[27] E. E. Flanagan, S. H. Tye and I. Wasserman, Phys. Rev. D62, 044039 (2000) [hep-ph/9910498].
[28] J. Garriga and M. Sasaki, Phys. Rev. D62, 043523 (2000) [hep-th/9912115].
[29] N. D. Birrell and P. C. W. Davies, Quantum fields in curved space, Cambridge University Press (1982).
[30] R. Durrer, Fundamentals of cosmic physics 15, 209 (1994) [astro-ph/9311041].
[31] D. Langlois, R. Maartens, M. Sasaki and D. Wands, [hep-th/001204].