BOUNDRED REDUCTIVE SUBALGEBRAS OF $\text{sl}_n$

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Abstract. Let $\mathfrak{g}$ be a reductive Lie algebra and $\mathfrak{k} \subset \mathfrak{g}$ be a reductive in $\mathfrak{g}$ subalgebra. A $(\mathfrak{g}, \mathfrak{k})$-module $M$ is a $\mathfrak{g}$-module for which any element $m \in M$ is contained in a finite-dimensional $\mathfrak{k}$-submodule of $M$. We say that a $(\mathfrak{g}, \mathfrak{k})$-module $M$ is bounded if there exists a constant $C_M$ such that the Jordan-Hölder multiplicities of any simple finite-dimensional $\mathfrak{g}$-module in every finite-dimensional $\mathfrak{k}$-submodule of $M$ are bounded by $C_M$. In the present paper we describe explicitly all reductive in $\mathfrak{sl}_n$ subalgebras $\mathfrak{k}$ which admit a bounded simple infinite-dimensional $(\mathfrak{sl}_n, \mathfrak{k})$-module. Our technique is based on symplectic geometry and the notion of spherical variety. We also characterize the irreducible components of the associated varieties of simple bounded $(\mathfrak{g}, \mathfrak{k})$-modules.

1. Introduction

Throughout this paper $\mathfrak{g}$ will be a reductive Lie algebra and $\mathfrak{k} \subset \mathfrak{g}$ will be a reductive in $\mathfrak{g}$ subalgebra. Recall that a $(\mathfrak{g}, \mathfrak{k})$-module is a $\mathfrak{g}$-module which is locally finite as a $\mathfrak{k}$-module, i.e. the space generated by $m, km, k^2m, ...$ is finite-dimensional for all $m \in M$ and $k \in \mathfrak{k}$. If a $(\mathfrak{g}, \mathfrak{k})$-module is simple, it is a direct sum of simple finite-dimensional $\mathfrak{k}$-modules. There are two well-known categories of $(\mathfrak{g}, \mathfrak{k})$-modules: the category of Harish-Chandra modules and the category $\mathcal{O}$. In the first case $\mathfrak{k}$ is a symmetric subalgebra of $\mathfrak{g}$ (i.e. $\mathfrak{k}$ coincides with the fixed points of an involution of $\mathfrak{g}$), and in the second case $\mathfrak{k}$ is a Cartan subalgebra $\mathfrak{h}_g$ of $\mathfrak{g}$. In both cases the $(\mathfrak{g}, \mathfrak{k})$-modules in question have the important additional property that they have finite $\mathfrak{k}$-multiplicities, i.e. have finite-dimensional $\mathfrak{k}$-isotypic components.

I. Penkov, V. Serganova, and G. Zuckerman have proposed to study, and attempt to classify, simple $(\mathfrak{g}, \mathfrak{k})$-modules with finite $\mathfrak{k}$-multiplicities for arbitrary reductive in $\mathfrak{g}$ subalgebras $\mathfrak{k}$. \cite{16}, \cite{15}. Such classifications are known for Harish-Chandra modules, see \cite{8} and references therein. In the classification of simple $(\mathfrak{g}, \mathfrak{h}_g)$-modules of finite type the bounded simple modules play a crucial role. Based on this, and on the experience with Harish-Chandra modules, I. Penkov and V. Serganova have proposed to study bounded $(\mathfrak{g}, \mathfrak{k})$-modules for general reductive subalgebras $\mathfrak{k}$, i.e. $(\mathfrak{g}, \mathfrak{k})$-modules whose multiplicities are uniformly bounded. A question arising in this context is, given $\mathfrak{g}$, to describe all reductive in $\mathfrak{g}$ bounded subalgebras, i.e. reductive in $\mathfrak{g}$ subalgebras $\mathfrak{k}$ for which at least one infinite-dimensional bounded $(\mathfrak{g}, \mathfrak{k})$-module exists. In \cite{14} I. Penkov and V. Serganova gave a partial answer to this problem, and in particular proved an important inequality which restricts severely the class of possible $\mathfrak{k}$. They also gave the complete list of bounded reductive subalgebras of $\mathfrak{g} = \mathfrak{sl}_n$ which are maximal subalgebras.

In the present paper we describe explicitly all reductive in $\mathfrak{sl}_n$ bounded subalgebras. Our technique is based on symplectic geometry and the notion of spherical variety. We also characterize the irreducible components of the support varieties of simple bounded $(\mathfrak{g}, \mathfrak{k})$-modules.

2. Basic definitions and statement of results

The ground field $\mathbb{F}$ is algebraically closed and of characteristic 0. We work in the category of algebraic varieties over $\mathbb{F}$. All Lie algebras considered are finite-dimensional. In what follows we use the term $\mathfrak{k}$-type for a simple finite-dimensional $\mathfrak{k}$-module. By $TX$ we denote the total space of the tangent bundle $\mathcal{T}X$ of a smooth variety $X$, and by $T_xX$ the tangent space to $X$ at a point $x \in X$.

Definition 1. For a $\mathfrak{k}$-module $M$ and a $\mathfrak{k}$-type $W_\lambda$, we define the $\mathfrak{k}$-multiplicity of $W_\lambda$ in $M$ as the supremum of the Jordan-Hölder multiplicities of $W_\lambda$ in all finite-dimensional $\mathfrak{k}$-submodules of $M$.

Definition 2. A $(\mathfrak{g}, \mathfrak{k})$-module $M$ is called bounded if $M$ is a bounded $\mathfrak{k}$-module, i.e. $\mathfrak{k}$-multiplicities of all $\mathfrak{k}$-types in $M$ are bounded by the same constant $C_M > 0$. A bounded $(\mathfrak{g}, \mathfrak{k})$-module $M$ is multiplicity-free if $C_M$ can be chosen as 1.
Definition 3. We say that a variety $X$ is a $\mathfrak{t}$-variety if a homomorphism of Lie algebras $\tau_X : \mathfrak{t} \to \mathfrak{X}$ is given. We say that $\mathfrak{t}$ has an open orbit on $X$ if there exists a point $x \in X$ such that the homomorphism $\tau_X|_x : \mathfrak{t} \to T_xX$ is surjective.

Definition 4. Let $X$ be a $\mathfrak{t}$-variety. Then $X$ is $\mathfrak{t}$-spherical if and only if there exists a multiplicity-free simple infinite-dimensional ($\mathfrak{t}$-module if and only if the projective space $\mathbb{P}(\mathfrak{t}^*)$ is a spherical $\mathfrak{t}$-variety.

Any finitely generated $\mathfrak{g}$-module $M$ has an associated graded $\mathfrak{S}(\mathfrak{g})$-module $\mathfrak{g}M$. We denote by $V(M)$ the support of $\mathfrak{g}M$ (for the precise definitions see the subsequent two sections). As a step towards the proof of Theorem 1 we establish the following result which might be of interest on its own.

Proposition 1. a) A finitely generated $(\mathfrak{g}, \mathfrak{t})$-module $M$ is bounded if and only if its support variety $V(M)$ is $\mathfrak{t}$-spherical.

b) If the equivalent conditions of a) are satisfied any irreducible component $\tilde{V}$ of $V(M)$ is a conical Lagrangian subvariety of

$$G\tilde{V} := \{x \in \mathfrak{g}^* \mid x = gv \text{ for some } g \in G \text{ and } v \in \tilde{V}\}.$$ 

All finite-dimensional $\mathfrak{t}$-modules $W$ such that $\mathbb{P}(W)$ is a $\mathfrak{t}$-spherical variety are known from the works of V. Kac [2], C. Benson and G. Ratcliff [1], A. Leahy [9]. The list of respective pairs $(\mathfrak{t}, W)$ is reproduced in the Appendix. Theorem 1 implies the following.

Corollary 1. The list of pairs $(\mathfrak{sl}(W), \mathfrak{t})$ for which $\mathfrak{t}$ is reductive and bounded in $\mathfrak{sl}(W)$ coincides with the list of Benson-Ratcliff and Leahy reproduced in the Appendix.

We also prove the Conjecture 6.6 of [14]: there exists an infinite-dimensional bounded $(\mathfrak{sl}_n, \mathfrak{t})$-module if and only if there exists a multiplicity-free simple infinite-dimensional $(\mathfrak{sl}_n, \mathfrak{t})$-module.

3. PRELIMINARIES ON SYMPLECTIC GEOMETRY

In what follows we denote by $G$ the adjoint group of $[\mathfrak{g}, \mathfrak{g}]$, and by $K$ the connected subgroup of $G$ with Lie algebra $\mathfrak{k} = \mathfrak{g} \cap [\mathfrak{g}, \mathfrak{g}]$. By $T^*X$ we denote the total space of the cotangent bundle of $X$ and by $T^*_xX$ — the space dual to $T_xX$.

Definition 5. Suppose that $X$ is a smooth variety which admits a closed nondegenerate 2-form $\omega$. Such a pair $(X, \omega)$ is called a symplectic variety. If $X$ is a $G$-variety and $\omega$ is $G$-invariant, $(X, \omega)$ is called a symplectic $G$-variety.

Example 1. Let $X$ be a smooth $G$-variety. Then $T^*X$ has a one-form $\alpha_X$ defined at a point $(l, x) (l \in T^*_xX)$ by the equality $\alpha_X(\xi) = l(\pi_\xi x)$ for any $\xi \in T_{(l, x)}(T^*X)$, where $\pi : T^*X \to X$ is the projection. The differential $d\alpha_X$ is a nondegenerate $G$-invariant two-form on $T^*X$ and therefore $(T^*X, d\alpha_X)$ is a symplectic $G$-variety.

Example 2. Let $\mathfrak{o}$ be a $G$-orbit in $\mathfrak{g}^*$. Then $\mathfrak{o}$ has a Kostant-Kirillov 2-form $\omega(\cdot, \cdot)$ defined at a point $x \in \mathfrak{g}^*$ by the equality $\omega_x(\tau_{\mathfrak{o}} p|_x, \tau_{\mathfrak{o}} q|_x) = x([p, q])$ for $p, q \in \mathfrak{g}$.

Definition 6. Let $(X, \omega)$ be a symplectic variety. We call a subvariety $Y \subset X$

a) isotropic if $\omega|_{T_{y}Y} = 0$ for a generic point $y \in Y$;

b) coisotropic if $\omega|_{(T_{y}Y)_{\perp \omega}} = 0$ for a generic point $y \in Y$;

c) Lagrangian if $T_{y}Y = (T_{y}Y)^{\perp _{\omega}}$ for a generic point $y \in Y$ or equivalently if it is both isotropic and coisotropic.

Example 3. Let $X$ be a smooth variety and $Y \subset X$ be a smooth subvariety. Then the total space $N^*_Y/X$ of the conormal bundle to $Y$ in $X$ is Lagrangian in $T^*X$.

Proposition 2 (see for example [2, Lemma 1.3.27]). Any closed irreducible conical (i.e. $\mathbb{P}^*$-stable) Lagrangian $G$-subvariety of $T^*X$ is the closure of the total space $N^*_Y/X$ of the conormal bundle to a $G$-subvariety $Y \subset X$. 
Let $X$ be a $G$-variety. The map

$$T^*X \times \mathfrak{g} \to \mathbb{F} \quad (x, l) \to l(\tau_X g|_x),$$

where $g \in \mathfrak{g}, x \in X, l \in T_x^*X$, induces a map $\phi : T^*X \to \mathfrak{g}^*$ called the moment map. This map provides the following description of an orbit $Gu \subset \mathfrak{g}^*$ such that $0 \in Gu$ (in what follows we call such orbits nilpotent). Suppose $P$ is a parabolic subgroup of $G$.

**Theorem 2** (R. Richardson [17]). The moment map $\phi_P : T^*(G/P) \to \mathfrak{g}^*$ is a proper morphism to the closure of some nilpotent orbit $Gu$ and is a finite morphism over $Gu$.

**Example 4.** Let $G = SL_n$ and $G/P = \mathbb{F}(\mathbb{F}^n)$. Then $\phi_P(T^*(G/P))$ (considered as a subset of $\mathfrak{sl}_n$) coincides with the set of nilpotent matrices of rank $\leq 1$. The $SL_n$-orbit open in $\phi_P(T^*(G/P))$ is the set of $SL_n$-highest weight vectors in $\mathfrak{sl}_n^*$ and is contained in the closure of any nonzero nilpotent $SL_n$-orbit in $\mathfrak{sl}_n^*$.

For $G = SL_n$, each moment map $\phi_P$ corresponding to a parabolic subgroup $P$ is a birational isomorphism of $T^*(G/P)$ with the image of $\phi_P$, and one can obtain the closure of any nilpotent orbit $Gu$ as the image of a suitable moment map $\phi_M$. Moreover the canonical symplectic form of $T^*(G/P)$ coincides on an open set with the pullback of the Kostant-Kirillov form of $\phi_P(T^*(G/P))$.

### 4. Preliminaries on $\mathfrak{g}$-modules

Let $\{U_i\}_{i \in \mathbb{Z}_{>0}}$ be the standard filtration on $U := U(\mathfrak{g})$. Suppose that $M$ is a $U(\mathfrak{g})$-module and that a filtration $\cup_{i \in \mathbb{Z}_{\geq 0}} M_i$ of vector spaces is given. We say that this filtration is good if

1. $U_i M_j = M_{i+j}$;
2. $\dim M_i < \infty$ for all $i \in \mathbb{Z}_{\geq 0}$.

Such a filtration arises from any finite-dimensional space of generators $M_0$. The corresponding associated graded object $\text{gr}M = \oplus_{i \in \mathbb{Z}_{\geq 0}} M_{i+1}/M_i$ is a module over $\text{gr} U(\mathfrak{g}) \cong S(\mathfrak{g})$, and we set

$$J_M := \{ s \in S(\mathfrak{g}) \mid \text{there exists } k \in \mathbb{Z}_{\geq 0} \text{ such that } s^k m = 0 \text{ for all } m \in \text{gr}M \}.$$

In this way we associate to any $\mathfrak{g}$-module $M$ the variety

$$V(M) := \{ x \in \mathfrak{g}^* \mid f(x) = 0 \text{ for all } f \in J_M \}.$$

It is easy to check that the module $\text{gr}M$ depends on the choice of good filtration, but the ideal $J_M$ and the variety $V(M)$ does not (see for instance [2]). Let $N_G(\mathfrak{g}^*) \subset \mathfrak{g}^*$ be the union of all nilpotent orbits.

**Lemma 1** ([2]). Let $M$ be a simple $\mathfrak{g}$-module. Then $V(M) \subset N_G(\mathfrak{g}^*)$.

**Theorem 3** (O. Gabber [5]). Any irreducible component $\tilde{V}$ of $V(M)$ is coisotropic inside a unique open $G$-orbit of $GV := \{ x \in \mathfrak{g}^* \mid x = gv \text{ for some } g \in G, v \in \tilde{V} \}$.

**Corollary 2** (S. Fernando [4]). Set $V(M)^\perp := \{ g \in \mathfrak{g} \mid v(g) = 0 \text{ for all } v \in V(M) \subset \mathfrak{g}^* \}$. Then $V(M)^\perp$ is a Lie algebra and $V(M)$ is a $V(M)^\perp$-variety.

**Theorem 4** (S. Fernando [4 Cor. 2.7], V. Kac [7]). Set $\mathfrak{g}[M] := \{ g \in \mathfrak{g} \mid \text{dim}(\text{span}_{v \in \mathbb{Z}_{\geq 0}} \{ g^m \}) < \infty \text{ for all } m \in M \}$. Then $\mathfrak{g}[M]$ is a Lie algebra and $\mathfrak{g}[M] \subset V(M)^\perp$.

**Corollary 3.** Let $M$ be a $(\mathfrak{g}, \mathfrak{t})$-module. Then $V(M) \subset \mathfrak{t}^\perp$ and $V(M)$ is a $\mathfrak{t}$-variety.

### 5. Proof of Theorem 1

Let $M$ be a $(\mathfrak{g}, \mathfrak{t})$-module and $M_0$ be a $\mathfrak{t}$-stable finite-dimensional space of generators of $M$; $J_M$, $\text{gr}M$ be the corresponding objects constructed as in Section 4. Consider the $S(\mathfrak{g})$-modules

$$J^{-1}_M \{ 0 \} := \{ m \in \text{gr}M \mid j_1 ... j_k m = 0 \text{ for all } j_1, ..., j_k \in J_M \}.$$

One can easily see that these modules form an ascending filtration of $\text{gr}M$ such that

$$\cup_{i=1}^{\infty} J^{-1}_M \{ 0 \} = \text{gr}M.$$

Since $S(\mathfrak{g})$ is a Nötherian ring, the filtration stabilizes, i.e. $J^{-1}_M \{ 0 \} = \text{gr}M$ for some $i$. By $\text{gr}M$ we denote the corresponding graded object. By definition, $\text{gr}M$ is an $S(\mathfrak{g})/J_M$-module. Suppose that $f\text{gr}M = 0$ for some $f \in S(\mathfrak{g})$. Then $f\text{gr}M = 0$ and hence $f \in J_M$. This proves that the annihilator of $\text{gr}M$ in $S(\mathfrak{g})/J_M$ equals zero.

The following lemma is a reformulation in the terms of the present paper of a result of Œ. Vinberg and B. Kimelfeld [19 Thm. 2].
Lemma 2. A quasiaffine algebraic $K$-variety $X$ is $\mathfrak{g}$-spherical if and only if the space of regular functions $\mathbb{F}[X]$ is a bounded $\mathfrak{g}$-module. If the variety $X$ is irreducible, then $\mathbb{F}[X]$ is a multiplicity-free $\mathfrak{g}$-module.

Theorem 5 (D. Panyushev [13 Thm 2.1]). Let $X$ be a smooth $G$-variety and $M$ a smooth locally closed $G$-stable subvariety. Then, for any Borel subgroup $B \subset G$, the generic stabilizers of the actions of $B$ on $X, N_{M/X}$ and $N_{M/X}^*$ are isomorphic.

Proposition 1. a) The module $M$ is bounded if and only if its support variety $V(M)$ is $\mathfrak{g}$-spherical.

b) If the equivalent conditions of a) are satisfied any irreducible component $\tilde{V}$ of $V(M)$ is a conical Lagrangian subvariety of $GV$.

Proof. a) As $M$ is a finitely generated $\mathfrak{g}$-module, the $S(\mathfrak{g})$-modules $gr M$ and $\mathfrak{g} \mathfrak{g} M$ are finitely generated. Let $M_0$ be a $\mathfrak{t}$-stable finite-dimensional space of generators of $\mathfrak{g} \mathfrak{g} M$. Then there is a surjective homomorphism $\psi : M_0 \otimes (S(\mathfrak{g})/J_M) \rightarrow \mathfrak{g} \mathfrak{g} M$.

Suppose that the variety $V(M)$ is $\mathfrak{g}$-spherical. Then $M_0 \otimes (S(\mathfrak{g})/J_M)$ is a bounded $\mathfrak{t}$-module. Therefore $\mathfrak{g} \mathfrak{g} M$ is bounded, which implies that $M$ is a bounded $\mathfrak{t}$-module too.

Assume now that a $\mathfrak{g}$-module $M$ is $\mathfrak{t}$-bounded. Set $RadM = \{m \in \mathfrak{g} M \mid fm = 0 \text{ and } f \neq 0 \text{ for some } f \in S(\mathfrak{g})/J_M\}$.

Then $RadM$ is a proper $\mathfrak{t}$-stable submodule of $\mathfrak{g} \mathfrak{g} M$ and $M_0 \not\subseteq RadM$. The homomorphism $\psi$ induces the injective homomorphism $\tilde{\psi} : S(\mathfrak{g})/J_M \rightarrow M_0^* \otimes \mathfrak{g} \mathfrak{g} M$. Therefore $S(\mathfrak{g})/J_M$ is a $\mathfrak{t}$-bounded module and $V(M)$ is a $\mathfrak{t}$-spherical variety. This completes the proof of a).

b) Let $\tilde{V} \subset V(M)$ be an irreducible component and $x \in \tilde{V}$ be a generic point. As $x \in \mathfrak{t}^\perp$ we have

$$x([k_1, k_2]) = \omega_x(r_\mathfrak{g}^* k_1|_x, r_\mathfrak{g}^* k_2|_x) = 0$$

for all $k_1, k_2 \in \mathfrak{t}$. Therefore any $K$-orbit in $\mathfrak{t}^\perp$ is isotropic. As $\tilde{V}$ is a $\mathfrak{t}$-spherical variety, $\tilde{V}$ has an open $K$-orbit. Therefore $\tilde{V}$ is Lagrangian in $GV$ and this completes the proof of b).

□

Theorem 6. Assume that $g = \mathfrak{sl}_n$. If there exists a simple infinite-dimensional bounded $(\mathfrak{sl}_n, \mathfrak{t})$-module, then $Gr(r, \mathbb{F}^n)$ is a spherical $\mathfrak{g}$-variety for some $r$.

Proof. Let $\tilde{V}$ be an irreducible component of $V(M)$. By the discussion succeeding after Example 4 the variety $\tilde{G} \tilde{V}$ (here $G \cong \mathbb{SL}_n$) is birationally isomorphic to $T^*(\mathbb{SL}_n/P)$ for some parabolic subgroup $P \subset \mathbb{SL}_n$, and the subvariety $\tilde{V} \subset G \tilde{V}$ is isomorphic to a conical Lagrangian subvariety $Y$ of $T^*(G/P)$. Any conical Lagrangian subvariety of $T^*(G/P)$ is the closure of the total space of the conormal bundle $N_{Z/G}(P)$ to some smooth subvariety $Z \subset G/P$, see Proposition 2 above. Therefore $\tilde{V}$ is birationally isomorphic to the total space of the conormal bundle to a smooth subvariety $Z \subset G/P$.

Obviously $Z$ is $t$-stable. Then, by Theorem 5 the variety $G/P$ has an open orbit of a Borel subalgebra of $\mathfrak{t}$, i.e. is $\mathfrak{t}$-spherical. This shows that $G/P$ is $\mathfrak{t}$-spherical for any maximal parabolic subgroup $P$ with $P \supset P$. Since $G/P$ is a Grassmannian, the proof is complete. □

Theorem 7. Assume that $G = \mathbb{SL}(V)$. Let $P \subset G$ be a parabolic subgroup such that $G/P$ is $\mathfrak{t}$-spherical. Then there exists a simple infinite-dimensional multiplicity-free $(\mathfrak{g}, \mathfrak{t})$-module $M$.

Proof. Theorem 6.3 in [14] proves the existence of a simple infinite-dimensional multiplicity-free $(\mathfrak{g}, \mathfrak{t})$-module under the assumption that there exists a parabolic subgroup $P \subset G$ for which $K$ has a proper closed orbit on $G/P$ such that the total space of its conormal bundle is $K$-spherical. By Theorem 5 this latter condition is equivalent to the $K$-sphericity of $G/P$. It remains to consider the case when $G/P$ has no proper closed $K$-orbits on $G/P$, i.e. $K$ has only one orbit on $G/P$. This happens only if $g \cong \mathfrak{sl}_2n, t \cong \mathfrak{sp}_{2n}$. However, in this last case the existence of a simple infinite-dimensional multiplicity-free $(\mathfrak{g}, \mathfrak{t})$-module is well known, see for instance [14].

We have thus proved the following weaker version of Theorem 1.

Corollary 4. Assume that $g = \mathfrak{sl}_n$. A pair $(\mathfrak{sl}_n, \mathfrak{t})$ admits an infinite-dimensional bounded simple $(\mathfrak{sl}_n, \mathfrak{t})$-module if and only if $Gr(r, \mathbb{F}^n)$ is a spherical $\mathfrak{t}$-variety for some $r$.

Proof. The statement follows directly from Theorems 7 and 6. □
Corollary 5 (see also [14], Conjecture 6.6). Assume that $g = sl_n$. If there exists a bounded simple infinite-dimensional $(sl_n, \mathfrak{t})$-module, then there exists a multiplicity-free simple infinite-dimensional $(sl_n, \mathfrak{t})$-module.

Proof. The statement follows directly from Corollary 4.

In order to prove Theorem 1 it remains to show that $r$ in Corollary 4 can be chosen to equal 1. For this we need the following result.

Theorem 8 (I. Losev [18]). Suppose $X$ is a strongly equidefectinal [18, Def. 1.2.5] normal affine irreducible Hamiltonian $G$-variety. Then $F(X)^G = \text{Quot}(\mathbb{F}[X]^G)$, where $\text{Quot}(\mathbb{F}[X]^G)$ is the field of fractions of $\mathbb{F}[X]^G$.

We say that a $K$-symplectic variety $(X, \omega)$ is $K$-coisotropic if the generic $K$-orbit on $X$ is $K$-coisotropic in $X$. We are now ready to prove the following.

Theorem 9. Let $V$ be a $K$-module. Suppose $Gr(r, V)$ is a $K$-spherical variety for some $r < \dim V$. Then $\mathbb{P}(V)$ is a $K$-spherical variety.

Proof. The variety $Gr(r, V)$ is $K$-spherical if and only if the variety $T^*Gr(r, V)$ is $K$-coisotropic [18, Ch. II, Cor. 1]. As $T^*Gr(r, V)$ is $K$-birationally isomorphic to some nilpotent orbit $\mathcal{O} \subset sl(V)^*$, this orbit $\mathcal{O} \subset sl(V)^*$ is $K$-coisotropic. Therefore $F(O)^K$ is a Poisson-commutative subfield of $F(V)^K$ [18, Ch. II, Prop. 5]. Since the closure of any $SL_2$(-orbit in $sl(V)^*$ is a strongly equidefectinal normal affine irreducible Hamiltonian $K$-variety [18, Cor. 3.4.1], $F(O)^K$ is Poisson-commutative.

Let $\mathcal{O}_{min}$ be the nonzero nilpotent orbit in $sl(V)$ of minimal dimension. It is well known that $\mathcal{O}_{min} \subset \overline{\mathcal{O}}$. Hence $F(\overline{\mathcal{O}})^K$ is a quotient of $F(O)^K$ and $F(\overline{\mathcal{O}})^K$ is Poisson-commutative. As $\mathcal{O}_{min}$ is $K$-birationally isomorphic to $T^*\mathbb{P}(V)$ (see Example 4 above), the variety $\mathbb{P}(V)$ is $K$-spherical.

6. Appendix. Results of C. Benson, G. Ratcliff, A. Leahy, V. Kac

The classification of the spherical modules has been worked out in several steps. V. Kac has classified the simple spherical modules in [6], C. Benson and G. Ratcliff have classified all spherical modules in [11]. The classification is contained also in paper [11] of A. Leahy. Below we reproduce their list.

Let $W$ be a $K$-module. Then $W$ is a spherical $\mathfrak{t}$-variety if and only if the pair $(\mathfrak{t}, W)$ is a direct sum of pairs $(\mathfrak{t}, W_i)$ listed below and in addition $\mathfrak{t} + \oplus \mathfrak{c}_i = N_{\mathfrak{g}(W)}(\mathfrak{t} + \oplus \mathfrak{c}_i)$ for certain abelian Lie algebras $\mathfrak{c}_i$ attached to $(\mathfrak{t}_i, W_i)$. Here $N_{\mathfrak{g}(W)}$ stands for the normalizer of $\mathfrak{t}$ in $\mathfrak{g}(W)$ and $\mathfrak{c}_i$ is a 0-, 1- or 2-dimensional Lie algebra listed in square brackets after the pair $(\mathfrak{t}_i, W_i)$. This subalgebra is generated by linear operators $h_1$ and $h_{m,n}$. By definition, $h_1 = \text{id}$. The notation $h_{m,n}$ is used only when $W = W_1 \oplus W_2$: in this case $h_{m,n} |_{W_1} = m \cdot \text{id}$ and $h_{m,n} |_{W_2} = n \cdot \text{id}$. The notation $(\mathfrak{t}_i, \{W_i, W_i')\})$ is shorthand for $(\mathfrak{t}_i, W_i)$ and $(\mathfrak{t}_i, W_i')$.

Finally, $\omega_i$ stands for the $i$-th fundamental weight and the corresponding fundamental representation. We follow the enumeration convention for fundamental weights of [20].

Table 6.1: Indecomposable spherical representations.
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