AN OPTIMAL HARDY–LITTLEWOOD–SOBOLEV INEQUALITY ON $\mathbb{R}^{n-k} \times \mathbb{R}^n$ AND ITS CONSEQUENCES

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Dedicated to Professor Bidaut-Véron and Professor Véron on the occasion of their 70th birthday

Abstract. For $n > k \geq 0$, $\lambda > 0$, and $p, r > 1$, we establish the following Hardy–Littlewood–Sobolev type inequality

$$
\int_{\mathbb{R}^{n-k} \times \mathbb{R}^n} \frac{f(x)g(y)}{|x-y|^\lambda} dxdy \leq \|f\|_{L^p(\mathbb{R}^{n-k})} \|g\|_{L^r(\mathbb{R}^n)} 
$$

with $y = (y', y'') \in \mathbb{R}^{n-k} \times \mathbb{R}^k$ under the two necessary conditions

$$
\beta < \begin{cases} 
\frac{k-k/r}{n-\lambda-k/r} & \text{if } 0 < \lambda \leq n-k, \\
\frac{n-\lambda}{n-\lambda-k/r} & \text{if } n-k < \lambda, 
\end{cases}
$$

and

$$
\frac{n-k}{n-p} + \frac{1}{r} + \frac{\beta + \lambda}{n} = 2 - \frac{k}{n}.
$$

We call this the optimal Hardy–Littlewood–Sobolev inequality on $\mathbb{R}^{n-k} \times \mathbb{R}^n$. The existence of an optimal pair for this new inequality is also studied. The motivation of working on the above inequality is to provide a unification of many known Hardy–Littlewood–Sobolev inequalities including the classical Hardy–Littlewood–Sobolev inequality when $k = \beta = 0$, the Hardy–Littlewood–Sobolev inequality on the upper half space $\mathbb{R}^{n-1} \times \mathbb{R}^n_+$ when $k = 1$ and $\beta = 0$, and the Hardy–Littlewood–Sobolev inequality on the upper half space $\mathbb{R}^{n-1} \times \mathbb{R}^n_+$ with extended kernel when $k = 1$ and $\beta \neq 0$. We show that the above condition for $\beta$ is sharp. In the unweighted case, namely $\beta = 0$, our finding immediately leads to the sharp Hardy–Littlewood–Sobolev inequality on $\mathbb{R}^{n-k} \times \mathbb{R}^n$ with the optimal range

$$
0 < \lambda < n-k/r,
$$

which has not been observed before, even in the case $k = 1$. As one of many consequences, we give a short proof of the Stein–Weiss inequality in the context of $\mathbb{R}^{n-k} \times \mathbb{R}^n$.

1. Introduction

In the existing literature, the classical Hardy–Littlewood–Sobolev inequality on $\mathbb{R}^n$, named after Hardy and Littlewood [HL28, HL30] and Sobolev [Sob38], states that for any $n \geq 1$, $p, r > 1$, and $\lambda \in (0, n)$ satisfying the balance condition

$$
1/p + 1/r + \lambda/n = 2, 
$$

there exists a sharp constant $N > 0$ depending on $n$, $\lambda$, and $p$ such that

$$
\int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x-y|^\lambda} dxdy \leq N\|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^r(\mathbb{R}^n)} 
$$

(1.2)

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Relaxing the condition below.

while a detailed discussion and interesting consequences will be exploited in section 4

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optimal functions; see [LL01, Chapter 4]. Here and in what follows, by $\lesssim$ and $\gtrsim$ we mean inequalities up to universal constants such as $n$, $\lambda$, $p$, $r$, etc.

Although the rough form of (1.2) was proved rather earlier, it took quite a long time to find the their sharp form until a seminal work of Lieb in 1983; see [Lie83]. Among other things, Lieb proved the existence of the optimal functions to the inequality (1.2). Apparently, it is impossible for us to make a comprehensive list of all related results developed in the wake of this fundamental achievement. For example, there are new methods to prove the inequality (1.2) and new arguments to prove the existence of the optimal functions; see [Lio84, CL92, LL01, FL10, CCL10, FL11, FL12b, DQZ17]. In addition, one has the sharp HLS inequalities on the upper half space $\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times (0, \infty)$ in [DZ15h, Dou16, Glu20], on bounded domains in [GZ19], on the Heisenberg group in [FL12a, HLZ12], and on compact Riemannian manifolds in [HZ15]. The interaction between the HLS inequality and other important inequalities has also been exploited; see [Bec93, DJ14, JN14]. Besides, many applications have been exploited around the sharp inequality (1.2).

In this work, we look for a possible weighted HLS inequality on $\mathbb{R}^{n-k} \times \mathbb{R}^n$ of the following form

\[
\left| \int_{\mathbb{R}^{n-k} \times \mathbb{R}^n} \frac{f(x)g(y)}{|x-y|^{n-1}y^\beta} \, dx \, dy \right| \lesssim \|f\|_{L^p(\mathbb{R}^{n-k})} \|g\|_{L^r(\mathbb{R}^n)},
\]

where $k$ is a non-negative integer less than $n$, $x \in \mathbb{R}^{n-k}$, $y = (y', y'') \in \mathbb{R}^{n-k} \times \mathbb{R}^k$, and the “distance” $|x-y|$ is being understood as follows

\[
|x-y| = \sqrt{|x-y'|^2 + |y''|^2}.
\]

There is a number of reasons supporting us to work on the weighted HLS inequality (1.3).

For clarity, let us just mention a few connection between (1.3) and some known results, while a detailed discussion and interesting consequences will be exploited in section 4 below.

Clearly, the inequality (1.3) with $\beta = 0$, if true, becomes (1.2) if $k = 0$. In the case $k = 1$, if we let $g$ be such that $g \equiv 0$ on the lower half space $\mathbb{R}^n_- = \mathbb{R}^{n-1} \times (-\infty, 0)$, then (1.2) becomes

\[
\left| \int_{\mathbb{R}^2_+ \times \mathbb{R}^{n-1}} \frac{f(x)g(y)}{|x-y|} \, dx \, dy \right| \lesssim \|f\|_{L^p(\mathbb{R}^{n-1})} \|g\|_{L^r(\mathbb{R}^2_+)}. \tag{1.4}
\]

Inequality (1.4) is known as the HLS inequality on the upper half space $\mathbb{R}^n_+$, which was first proved by Dou and Zhu in [DZ15] under the balance condition

\[
(n-1)(np) + 1/r + \lambda/n = 2 - 1/n. \tag{1.5}
\]

Relaxing the condition $\beta = 0$ gives

\[
\left| \int_{\mathbb{R}^2_+ \times \mathbb{R}^{n-1}} \frac{f(x)g(y)}{|x-y|^{1+\lambda/n}y^\beta} \, dx \, dy \right| \lesssim \|f\|_{L^p(\mathbb{R}^{n-1})} \|g\|_{L^r(\mathbb{R}^2_+)}. \tag{1.6}
\]
which was proved by Gluck in \cite{Glu20} and Liu in \cite{Liu20} under the technical condition \( \beta \leq 0 \) and the balance condition

\[
(n-1)/(np) + 1/r + (\lambda + \beta)/n = 2 - 1/n. \tag{1.7}
\]

On one hand, the restriction \( \beta \leq 0 \) in (1.6) seems to be not natural from the validity of the inequality, indicating that (1.6) could be true for certain \( \beta > 0 \). (See subsection 4.2 below.) On the other hand, since the balance conditions (1.1), (4.3), and (1.7) have a similar form, it is natural to ask whether or not there is a unification of (1.2), (1.4), and (1.6).

It is worth noting that although the two inequalities (1.6) and (1.3) look rather similar, the motivation of working on the inequality (1.3) is not to create an artificial one, but it stems from an intriguing connection between isoperimetric inequalities and HLS inequalities with Poisson-type kernel on \( \mathbb{R}^{n-k} \times \mathbb{R}^k \). For interested readers, we refer to \cite{HWY08} for the intriguing connection between these inequalities. In this paper, we aim to address these points and we are successful leading to the inequality (1.3) above.

That said, in this work, we aim to study (1.3) for arbitrary \( 0 < k < n \). It is worth noting that we do not treat the case \( k = 0 \) because in this case the weight \(|y''|^{-\beta} \) does not appear, hence the inequality (1.10) becomes the classical HLS inequality (1.2) on \( \mathbb{R}^n \). Toward a complete picture of (1.3), our first step concerns the validity of (1.3). We summarize this step as the following theorem.

**Theorem 1.1** (weighted HLS inequality on \( \mathbb{R}^{n-k} \times \mathbb{R}^k \)). Let \( n \geq 2, \ 0 < k < n, \ \lambda > 0, \ p, r > 1, \) and

\[
\beta < \begin{cases} 
\frac{k - \frac{k}{r}}{\lambda} & \text{if } 0 < \lambda \leq n - k, \\
\frac{n - \lambda - \frac{k}{r}}{n} & \text{if } n - k < \lambda,
\end{cases} \tag{1.8}
\]

satisfying the balance condition

\[
\frac{n - k}{n} + \frac{1}{r} + \frac{\beta + \lambda}{n} = 2 - \frac{k}{n}. \tag{1.9}
\]

Then there exists a sharp constant \( N_{n,\lambda,p}^{k,\beta} > 0 \) such that

\[
\left| \iint_{\mathbb{R}^n \times \mathbb{R}^{n-k}} \frac{f(x)}{|x|^\lambda} \frac{g(y)}{|y|^\beta} |dxdy| \right| \leq N_{n,\lambda,p}^{k,\beta} \|f\|_{L^p(\mathbb{R}^{n-k})} \|g\|_{L^r(\mathbb{R}^n)} \tag{1.10}_{k,\beta}
\]

for any functions \( f \in L^p(\mathbb{R}^{n-k}) \) and \( g \in L^r(\mathbb{R}^n) \). Moreover, the condition (1.8) is sharp.

Before moving on, Theorem 1.1 deserves some comments.

- When \( k = 1, \) the inequality (1.10)$_{1,\beta}$ is essentially the same as (1.6) except the fact that the domain of the double integration is no longer \( \mathbb{R}^n \) but \( \mathbb{R}^n \times \mathbb{R}^{n-1} \). As a matter of fact, (1.10)$_{1,\beta}$ deals with a larger class of functions. However, the sharp constant \( N_{n,\lambda,p}^{1,\beta} \) of (1.10)$_{1,\beta}$ and that of (1.6) are related; see subsection 4.3 below.

- In all existing works on \( \mathbb{R}^n \) in the literature, the condition \( 0 < \lambda < n - 1 \) is always assumed. But our optimal inequality (1.10)$_{1,\beta}$ shows that this is not necessary. If we let \( \beta = 0 \) and \( k = 1 \), then (1.10)$_{1,0}$ holds for \( 0 < \lambda < n - 1/r \) but does not if \( n - 1/r \leq \lambda < n \); see subsection 4.1 below.

- An interesting consequence of (1.10)$_{k,\beta}$ is that it holds for any \( \lambda > 0 \), not just \( 0 < \lambda < n \), so long as \( \beta < n - \lambda - k/r \); see subsection 4.3 below.
• Although it is not explicitly stated in Theorem 1.1, there is a lower bound for \( \beta \) because it can be easily seen from (1.9). To be more precise, we must have \( \beta > -\lambda \), thanks to \( p, r > 1 \).

• Our inequality (1.10)\(_{k,\beta}^\gamma\) remains valid if we replace \( |y''|^\beta \) by \( |y'|^{\beta} \). We refer this to the Stein–Weiss inequality on \( \mathbb{R}^{n-k} \times \mathbb{R}^{n} \). Surprisingly, the Stein–Weiss inequality on \( \mathbb{R}^{n-k} \times \mathbb{R}^{n} \) can be quickly obtained from our inequality (1.10)\(_{k,\beta}^\gamma\); see subsection 4.4 below. This illustrates an interesting application of our result.

More interesting applications of (1.10)\(_{k,\beta}^\gamma\) will be discussed in section 4. From now on, we call (1.10)\(_{k,\beta}^\gamma\) the optimal HLS inequality to highlight the fact that all parameters for (1.10)\(_{k,\beta}^\gamma\) are in the optimal range.

As routine, the proof Theorem 1.1 is carried through two steps; see section 2 below. In the first step, we prove a rough form of (1.10)\(_{k,\beta}^\gamma\), namely without the sharp constant \( N_{n,\lambda,p}^\alpha \). Then the existence of the sharp constant \( N_{n,\lambda,p}^\alpha \) is guaranteed through the following variational problem

\[
N_{n,\lambda,p}^\alpha := \sup_{f \geq 0, g \geq 0} \left\{ F_{\alpha,k}^\beta(f, g) : \|f\|_{L^p(\mathbb{R}^{n-k})} = 1, \|g\|_{L^p(\mathbb{R}^n)} = 1 \right\}, \tag{1.11}
\]

where

\[
F_{\alpha,k}^\beta(f, g) = \int_{\mathbb{R}^{n-k} \times \mathbb{R}^n} f(x)g(y) \left| \frac{x}{|x|} \right|^\lambda \left| y'' \right|^{\beta} \, dx \, dy.
\]

As we shall soon see, in the present work, we present two different proofs for the rough inequality. These new proofs do not make use the layer cake representation technique nor the Marcinkiewicz interpolation inequality. Instead, we borrow some ideas from harmonic analysis and the theory of maximal functions.

Once we establish Theorem 1.1, it is natural to ask whether an optimal pair \((f_{\alpha,k}^\beta, g_{\alpha,k}^\beta)\) for the weighted HLS inequality (1.10)\(_{k,\beta}^\gamma\), which consists of non-negative, non-trivial functions, actually exists, namely

\[
\int_{\mathbb{R}^{n-k} \times \mathbb{R}^n} f_{\alpha,k}^\beta(x)g_{\alpha,k}^\beta(y) \left| \frac{x}{|x|} \right|^\lambda \left| y'' \right|^{\beta} \, dx \, dy = N_{n,\lambda,p}^\alpha \|f_{\alpha,k}^\beta\|_{L^p(\mathbb{R}^{n-k})} \|g_{\alpha,k}^\beta\|_{L^p(\mathbb{R}^n)}.
\]

To this purpose, let us first formally introduce an “extension” operator \( \mathcal{E}_{\alpha,k}^\beta \), which turns a function \( f \) on \( \mathbb{R}^{n-k} \) to a function on \( \mathbb{R}^n \) via the following rule

\[
\mathcal{E}_{\alpha,k}^\beta[f](y) = \int_{\mathbb{R}^{n-k}} \frac{f(x)}{|x - y|} \left| y'' \right|^{\beta} \, dx \quad \text{a.e. } y \in \mathbb{R}^n.
\]

Using this operator, we may rewrite (1.10)\(_{k,\beta}^\gamma\) as

\[
\left| \int_{\mathbb{R}^n} (\mathcal{E}_{\alpha,k}^\beta[f])(y)g(y) \, dy \right| \leq N_{n,\lambda,p}^\alpha \|f\|_{L^p(\mathbb{R}^{n-k})} \|g\|_{L^p(\mathbb{R}^n)}.
\]

Then, by duality, the HLS inequality (1.10)\(_{k,\beta}^\gamma\) is equivalent to the following inequality

\[
\|f\|_{L^q(\mathbb{R}^n)} \leq N_{n,\lambda,p}^\alpha \|f\|_{L^p(\mathbb{R}^{n-k})} \tag{1.12}
\]

for any function \( f \in L^p(\mathbb{R}^{n-k}) \) with \( q \) being the number

\[
\frac{1}{q} = 1 - \frac{1}{r} = \frac{n-k}{n} \left( \frac{1}{p} - \frac{n-k-\lambda-\beta}{n-k} \right). \tag{1.13}
\]

It is important to note that \( q > p \), see (3.1) below.
Similarly, one can consider the “restriction” operator $\mathcal{R}^\beta_{\lambda,k}$, which maps a function $g$ on $\mathbb{R}^n$ to a function on $\mathbb{R}^{n-k}$ via the following rule

$$\mathcal{R}^\beta_{\lambda,k}[g](x) = \int_{\mathbb{R}^n} \frac{g(y)dy}{|x-y|^{1+\beta'}} \quad \text{a.e. } x \in \mathbb{R}^{n-k}.$$ 

Clearly, the two operators $\mathcal{E}^\beta_{\lambda,k}$ and $\mathcal{R}^\beta_{\lambda,k}$ are dual in the sense that for any functions $f$ on $\mathbb{R}^{n-k}$ and $g$ on $\mathbb{R}^n$, the following identity

$$\int_{\mathbb{R}^n} (\mathcal{E}^\beta_{\lambda,k}[f])(y)g(y)dy = \int_{\mathbb{R}^{n-k}} f(x)(\mathcal{R}^\beta_{\lambda,k}[g])(x)dx$$

holds, thanks to Tonelli’s theorem. Once we introduce $\mathcal{R}^\beta_{\lambda,k}$, we can easily see that the weighted HLS inequality (1.10) is also equivalent to the following inequality

$$\| \int_{\mathbb{R}^n} g(y)dy \|_{L^q(\mathbb{R}^{n-k})} \leq N^\beta_{n,\lambda,p} \| g \|_{L^r(\mathbb{R}^n)}$$

for any function $g \in L^q(\mathbb{R}^n)$ with $q > 1$ satisfies

$$\frac{1}{q} = 1 - \frac{1}{p} = \frac{n}{n-k}\left(1 - \frac{n-\lambda - \beta}{n}\right).$$

Now we turn our attention to the existence of optimal pairs $(f^\sharp, g^\sharp)$ for the variational problem (1.11). In view of (1.12), to study the existence of optimal pairs for (1.11), we study the following maximizing problem

$$N^\beta_{n,\lambda,p} := \sup_{\|f\|_{L^r(\mathbb{R}^n)} = 1} \left\{ \| \mathcal{E}^\beta_{\lambda,k}[f] \|_{L^q(\mathbb{R}^n)} : \|f\|_{L^r(\mathbb{R}^n)} = 1 \right\}.$$  \hspace{1cm} (1.14)

Clearly, the two maximizing problems (1.11) and (1.14) are actually equivalent; see section 2.1 below. In the next result, we prove that the maximizing problem (1.14) always admits a solution $f^\sharp \in L^p(\mathbb{R}^{n-k})$, thus giving a solution $(f^\sharp, (\mathcal{E}^\beta_{\lambda,k}[f^\sharp])^{\frac{1}{n-k}})$ to the maximizing problem (1.11).

**Theorem 1.2 (existence of optimal functions for (1.14)).** Suppose that all conditions in Theorem 1.1 hold. Let $q$ be given by (1.13). Then, there exists a function $f^\sharp \in L^p(\mathbb{R}^{n-k})$ such that

$$f^\sharp \geq 0, \quad \|f^\sharp\|_{L^p(\mathbb{R}^{n-k})} = 1, \quad \text{and} \quad \| \mathcal{E}^\beta_{\lambda,k}[f^\sharp] \|_{L^q(\mathbb{R}^n)} = N^\beta_{n,\lambda,p}.$$ 

Moreover, the function $f^\sharp$ is strictly decreasing and radially symmetric with respect to some point in $\mathbb{R}^{n-k}$.

We prove Theorem 1.2 in section 3 below. This is done by following Talenti’s proof of the sharp Sobolev inequality by considering (1.14) within the set of symmetric decreasing rearrangements. In view of the constraint in the maximizing problem (1.14), if we denote by $f^\ast$ the symmetric decreasing rearrangement with respect to $\mathbb{R}^{n-k}$ of a function $f \in L^p(\mathbb{R}^{n-k})$, then on one hand, it is well-known that

$$\|f^\ast\|_{L^p(\mathbb{R}^{n-k})} = \|f\|_{L^p(\mathbb{R}^{n-k})}$$

while on the other hand, there holds

$$\| \mathcal{E}^\beta_{\lambda,k}[f] \|_{L^q(\mathbb{R}^n)} \leq \| \mathcal{E}^\beta_{\lambda,k}[f^\ast] \|_{L^q(\mathbb{R}^n)};$$

see (3.2) below. Hence, it suffices to look for an optimal function within the set of symmetric decreasing rearrangements. Due to the presence of the weight $|y'|^{-\beta'}$, it is expected that the condition $\beta \geq 0$ is required in order to apply many inequalities involving symmetrization. However, we should emphasize that we do not need this.
A quick consequence of Theorem 1.2 is the following.

**Proposition 1.3** (existence of optimal pairs for (1.11)). Assume all conditions in Theorem 1.1. Then, the sharp constant $N_{k,\beta}^{n,\lambda,p}$ for the inequality (1.10) is achieved by some optimal pair $(f^\sharp, g^\sharp) \in L^p(R^{n-k}) \times L^r(R^n)$. The functions $f^\sharp$ and $g^\sharp|_{R^{n-k}}$ are radially symmetric with respect to some point in $R^{n-k}$ and monotone decreasing.

To maintain our work in a reasonable length, we leave other questions involving optimal pairs for (1.10) for the future work; see subsection 4.5. We also expect to see applications of the above inequality to partial differential equations. Also, in a future work, we shall study a reverse HLS inequality on $R^{n-k} \times R^n$. The paper is organized as follows:

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References

Before closing Introduction, let us introduce some notation and convention. For a positive integer $\ell$, we denote by $B^\ell_R(0)$ the open ball in $R^\ell$ centered at 0 and radius $R$, namely

$$B^\ell_R(0) = \{\xi \in R^\ell : |\xi| < R\}.$$  

For simplicity, we often write $B^\ell_R(0)$ as $B^\ell_R$.  

2. The weighted HLS inequality on $R^{n-k} \times R^n$

In this section, we give a proof of Theorem 1.1, namely to prove (1.10)$_{k,\beta}$ without the sharp constant $N_{k,\beta}^{n,\lambda,p}$.  

As mentioned in Introduction, this is equivalent to showing that the supremum in (1.11) is finite. Toward this purpose, we first prove in subsection 2.1 below that the
two maximizing problems (1.11) and (1.14) are equivalent. Therefore, to prove the rough inequality \((1.10)_{k,\beta}\), it suffices to prove the rough inequality (1.12), which will be done in subsection 2.2. Finally, we spend subsection 2.3 to verify the necessity of the condition for \(\beta\).

2.1. The equivalence between (1.11) and (1.14). We now prove that the two maximizing problems (1.11) and (1.14) are equivalent. Such a result seems to be foreseeable and standard. We provide a short proof for completeness. Denote by \(N\) the supremum in (1.14), namely

\[
N := \sup_{f \geq 0} \left\{ \|E_{\lambda,k}^{\beta}[f]\|_{L^q(R^n)} ; \|f\|_{L^p(R^{n-k})} = 1 \right\}
\]

with \(q = (1 - 1/r)^{-1}\). (Conventionally, we also write \((1 - 1/r)^{-1} = r'\)) In the first step of the proof, we show that \(N = N_{n,\lambda,p}^{k,\beta}\). Indeed, let \((f^\sharp, g^\sharp)\) be an optimal pair for (1.11). By definition, there holds

\[
\|f^\sharp\|_{L^p(R^{n-k})} = \|g^\sharp\|_{L^{r'}(R^n)} = 1.
\]

As \(1/q + 1/r = 1\), we use Hölder’s inequality to get

\[
N_{n,\lambda,p}^{k,\beta} = \int_{R^n} (E_{\lambda,k}^{\beta}[f^\sharp])(y)g^\sharp(y)dy \leq \|E_{\lambda,k}^{\beta}[f^\sharp]\|_{L^q(R^n)} \|g^\sharp\|_{L^{r'}(R^n)} \leq N.
\]

Hence, we necessarily have \(N_{n,\lambda,p}^{k,\beta} \leq N\). Now let \(h^\sharp\) be an optimal function for (1.14). Obviously, we must have \(\|h^\sharp\|_{L^p(R^{n-k})} = 1\) and

\[
\|(E_{\lambda,k}^{\beta}[h^\sharp])^{q-1}\|_{L^{r'}(R^n)} = \left( \int_{R^n} (E_{\lambda,k}^{\beta}[h^\sharp])^q(y)dy \right)^{1/q} = N^{q-1},
\]

thanks to \((q - 1)r = q\). Then using (1.10)_{k,\beta} applied to \((h^\sharp, (E_{\lambda,k}^{\beta}[h^\sharp])^{q-1})\) we obtain

\[
N^q = \int_{R^n} (E_{\lambda,k}^{\beta}[h^\sharp])(y)(E_{\lambda,k}^{\beta}[h^\sharp])^{q-1}(y)dy \\
\leq N_{n,\lambda,p}^{k,\beta} \|h^\sharp\|_{L^p(R^{n-k})} \|(E_{\lambda,k}^{\beta}[h^\sharp])^{q-1}\|_{L^{r'}(R^n)} \\
= N_{n,\lambda,p}^{k,\beta} N^{q-1}.
\]

Hence, we now get \(N \leq N_{n,\lambda,p}^{k,\beta}\). Thus, we have just shown that \(N_{n,\lambda,p}^{k,\beta} = N\) as claimed.

Now, we show that each optimal pair \((f^\sharp, g^\sharp)\) for (1.11) gives rise to an optimal function \(h^\sharp\) for (1.14) and vice versa. By seeing the above calculation, this fact is quite clear. Obviously, if \((f^\sharp, g^\sharp)\) is an optimal pair for (1.11), then the function \(f^\sharp\) is also an optimal function for (1.14). Conversely, if \(h^\sharp\) is an optimal function for (1.14), then the pair \((h^\sharp, (E_{\lambda,k}^{\beta}[h^\sharp])^{q-1})\) is an optimal pair for (1.11).

2.2. Proof of the weighted HLS inequality (1.12). As mentioned in Introduction, to prove (1.10)_{k,\beta}, it suffices to prove (1.12). As we shall soon see, in the present work, we present two different proofs for the rough inequality (1.12). While the idea of the second proof stems from harmonic analysis and the theory of maximal functions, see Remark 3.3 below, the idea of the first proof, which is presented in this section, demonstrates an intriguing connection between the weighted and unweighted versions of the HLS inequality. In fact, we soon see that (1.12) is equivalent to the classical HLS inequality on \(R^{n-k} \times R^n\); see Lemma 2.2 below. Because of this finding, we expect that there are several ways to prove our inequality (1.10)_{k,\beta}, just like proving the classical HLS inequality (1.2). Ours is simply one of many, which demonstrate the connecting between the weighted one the unweighted one.
To begin, recall that $0 < k < n$, $1 < p, r < +\infty$, $\lambda > 0$, and $\beta$ satisfies

$$\beta < \begin{cases} 
\frac{k - \lambda}{r} & \text{if } 0 < \lambda \leq n - k, \\
-n - \frac{k}{r} & \text{if } n - k < \lambda.
\end{cases}$$

The balance identity (1.9) can be rewritten as follows

$$\frac{1}{p} + \frac{1}{r} + \frac{\lambda - k/q + \beta}{n - k} = 2,$$

where $q$ is given by (1.13). Now we denote

$$\gamma = \lambda - k/q + \beta.$$  \hspace{1cm} (2.2)

As $p, r > 1$ we deduce from (2.1) that $\gamma > 0$. Now we estimate $\gamma$ from the above. As $\beta < k/q$ if $\lambda \leq n - k$, we easily obtain $\gamma < \lambda \leq n - k$ in this range. Now for $\lambda > n - k$, it follows from $\beta < n - k/r - k/q = n - k$. Hence, we obtain the following important estimate

$$0 < \gamma < n - k.$$ for all $\lambda > 0$. For clarity, we split the proof into several steps. First we start with the following simple observation.

**Lemma 2.1.** Let $h$ be a non-negative, non-decreasing function. Then we have

$$\int_0^{+\infty} \left( \frac{1}{\rho^\gamma} h(\rho) \right)^r \frac{d\rho}{\rho} \leq \left( \int_0^{+\infty} \frac{1}{\rho^\gamma} h(\rho) \frac{d\rho}{\rho} \right)^r$$

for any $r \geq 1$.

**Proof.** To see the inequality, we first decompose the left hand side as follows

$$\int_0^{+\infty} \left( \frac{1}{\rho^\gamma} h(\rho) \right)^r \frac{d\rho}{\rho} = \sum_{j \in \mathbb{Z}} \int_{2^j}^{2^{j+1}} \left( \frac{1}{\rho^\gamma} h(\rho) \right)^r \frac{d\rho}{\rho} \leq \sum_{j \in \mathbb{Z}} \int_{2^j}^{2^{j+1}} \left( \frac{1}{\rho^\gamma} h(2^j \rho) \right)^r \frac{d\rho}{\rho} = \sum_{j \in \mathbb{Z}} \left( \frac{1}{2^j \rho^\gamma} h(2^j \rho) \right)^r,$$

thanks to the monotonicity of $h$. Hence, by changing the index of the sum, we arrive at

$$\int_0^{+\infty} \left( \frac{1}{\rho^\gamma} h(\rho) \right)^r \frac{d\rho}{\rho} \leq 2^{r \gamma} \sum_{j \in \mathbb{Z}} \left( \frac{1}{2^j \rho^\gamma} h(2^j) \right)^r \leq 2^{r \gamma} \left( \sum_{j \in \mathbb{Z}} \frac{1}{2^j \rho^\gamma} h(2^j) \right)^r,$$

thanks to $r \geq 1$. Again by the monotonicity of $h$, we see that

$$\frac{1}{2^j \rho^\gamma} h(2^j) = \frac{1}{\log 2} \frac{1}{2^j \rho^\gamma} \int_{2^j}^{2^{j+1}} h(2^j \rho) \frac{d\rho}{\rho} \leq \frac{2^\gamma}{\log 2} \int_{2^j}^{2^{j+1}} \frac{1}{\rho^\gamma} h(\rho) \frac{d\rho}{\rho}.$$ From this we obtain

$$\int_0^{+\infty} \left( \frac{1}{\rho^\gamma} h(\rho) \right)^r \frac{d\rho}{\rho} \leq \left( \frac{4^\gamma}{\log 2} \right)^r \left( \int_0^{+\infty} \frac{1}{\rho^\gamma} h(\rho) \frac{d\rho}{\rho} \right)^r,$$

giving the first inequality. The proof is complete. \qedsymbol

Our next step is the key step to prove (1.12). The idea is to transform a weighted inequality to a suitable unweighted inequality.
Lemma 2.2. Let \( n > k \geq 1, p, r > 1, \) and \( \lambda > 0. \) Suppose that \( \beta \) satisfies (1.8). Then for any non-negative function \( h, \) we have

\[
\int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^{n-k}} \frac{h(x)}{|x-y|^{n-2k}} dy \right)^q dx 
\leq \left( \frac{\lambda 2^{2r+1}}{\gamma} \frac{1}{\lambda - \gamma} \right)^q |S^{k-1}| \int_{\mathbb{R}^{n-k}} \left( \int_{\mathbb{R}^n} \frac{h(x)}{|z-x|^{2k}} dx \right)^q dz,
\]

where \( \gamma \) is given in (2.2) and \( q \) is given in (1.13).

Proof. To see this, first we notice that

\[
\frac{1}{|x-y|^2} = \lambda \int_{|x-y|}^{\infty} \frac{1}{\rho^{k-1}} \frac{d\rho}{\rho}
\]

with \( \lambda > 0, \) which, together with Fubini’s theorem applied for the region

\[
\{ (x, \rho) : \rho > \sqrt{|x-y|^2 + |y'|^2} \} \subset \mathbb{R}_+^{n-k+1},
\]

which lies above the curve in Figure 1, implies

\[
\int_{\mathbb{R}^{n-k}} \frac{h(x)}{|x-y|^{n-2k}} dx = \lambda \int_{|x-y|}^{\infty} \frac{1}{|y'|^{n-2k}} \frac{d\rho}{\rho} \int_{\mathbb{R}^{n-k}} h(x) dx 
\leq \lambda \int_{|y'|}^{\infty} \frac{1}{|y'|^{n-2k}} \frac{d\rho}{\rho} \int_{\mathbb{R}^{n-k}} h(x) dx
\]

as the above region is contained in the ungula

\[
\{ (x, \rho) : \rho > |y'|, |x-y'| < \rho \} \subset \mathbb{R}_+^{n-k+1}.
\]

Keep in mind that the point \( y \in \mathbb{R}^n \) is being fixed. See Figure 1 for details.

![Figure 1](image_url)

Figure 1. This is how we get the inequality in (2.5).
We still need some work on (2.5). For some \( \epsilon > 0 \) to be determined later, we apply Hölder’s inequality to get

\[
\int_{|y|^q}^{+\infty} \frac{1}{|\rho|^{1/q}} \left( \int_{B_{r}^{n-k}(y')} h(x) dx \right) \frac{d\rho}{\rho} \\
\leq \left[ \int_{|y|^q}^{+\infty} \left( \frac{1}{|\rho|^{1/q'}} \int_{B_{r}^{n-k}(y')} h(x) dx \right) \frac{d\rho}{\rho} \right]^{1/q} \left[ \int_{|y|^q}^{+\infty} \rho^{-\epsilon q} \frac{d\rho}{\rho} \right]^{1/q'}
\]

with \( 1/q + 1/q' = 1 \). Hence, combining (2.5) and (2.6) and making use of Fubini’s theorem applied for the 90-degree cone

\[
\{(z, \rho) : \rho \geq |z|\} \subset R^{d+1},
\]

the left hand side of (2.3) can be estimated as follows

\[
\int_{R^d} \left( \int_{R^{d-k}} |x - y|^{\alpha} |y|^\beta d\rho \right)^{q} dy
\]

\[
\leq \lambda^q \int_{R^{d-k}} \left( \int_{|y|^q}^{+\infty} \frac{1}{|\rho|^{1/q}} \left( \int_{B_{r}^{n-k}(y')} h(x) dx \right) \frac{d\rho}{\rho} \right)^{q} \frac{dy}{|y|^\beta} dy'
\]

\[
= \lambda^q \int_{R^{d-k}} \int_{|y|^q}^{+\infty} \left( \frac{1}{|\rho|^{1/q'}} \int_{B_{r}^{n-k}(y')} h(x) dx \right) \frac{d\rho}{\rho} \frac{dy}{|y|^\beta q + \epsilon q} dy'
\]

Notice that under (1.8) the condition \( \beta q < k(r - 1)/r \) always holds. Therefore,

\[
\beta q = \beta(1 - 1/r)^{-1} < k.
\]

This together with \( k \geq 1 \) allows us to choose small \( \epsilon > 0 \) in such a way that \( \beta q + \epsilon q < k \). From this we obtain

\[
\int_{B_{r}^{n-k}(0)} \frac{dy'}{|y'|^{\beta q + \epsilon q}} = \frac{|S^{k-1}|}{k - \beta q - \epsilon q \rho^{k-\beta q - \epsilon q}},
\]

which allows us to further obtain

\[
\int_{R^d} \left( \int_{R^{d-k}} \frac{h(x)}{|x - y|^{\alpha}} |y|^\beta d\rho \right)^{q} dy
\]

\[
\leq \lambda^q (\epsilon q')^{-q/q'} \frac{|S^{k-1}|}{k - \beta q - \epsilon q} \int_{R^{d-k}} \int_{|y|^q}^{+\infty} \left( \frac{1}{|\rho|^{1/q'}} \int_{B_{r}^{n-k}(y')} h(x) dx \right) \frac{d\rho}{\rho} \frac{dy}{dy'}
\]

\[
= \lambda^q (\epsilon q')^{-q/q'} \frac{|S^{k-1}|}{k - \beta q - \epsilon q} \int_{R^{d-k}} \int_{|y|^q}^{+\infty} \left( \frac{1}{|\rho|^{1/q'}} \int_{B_{r}^{n-k}(y')} h(x) dx \right) \frac{d\rho}{\rho} \frac{dy}{dy'}.
\]

Since \( \epsilon > 0 \) is still arbitrary, we may choose one to obtain a rough constant. Observe that

\[
\min_{0 < \epsilon < \beta q / \beta q - \epsilon q} \frac{(\epsilon q')^{-q/q'}}{k - \beta q - \epsilon q} = \left( \frac{q}{k - \beta q} \right)^q
\]
at $\epsilon = (k - \beta q)/(q^\gamma)$. Hence, using this particular choice of $\epsilon$, we arrive at
\[
\int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^{n-k}} \frac{h(x)}{|x-y|^{1+\gamma}} \, dy \right)^{q/\gamma} \, dx \leq \lambda \left( \frac{1}{\beta q - \beta} \right)^q |S^{k-1}| \int_{\mathbb{R}^{n-k}} \left( \int_{B_{\rho}^{-1}(y')} \frac{1}{\rho^{k/q - \beta}} \int_{B_{\rho}^{-1}(y')} h(x) \, dx \right)^q \, d\rho \, dy'.
\] (2.7)

Since $h \geq 0$, the function
\[
\rho \mapsto \int_{B_{\rho}^{-1}(y')} h(x) \, dx
\]
is non-decreasing. This together with $q \geq 1$ allows us to apply Lemma 2.1 with $\tau$ replaced by $q$ and Fubini's theorem applied for the 90-degree cone
\[
\{(x, \rho) : \rho \geq |x-y'| \} \subset \mathbb{R}_+^{n-k+1}
\]
to get
\[
\int_0^{+\infty} \left( \frac{1}{\rho^{\lambda-k/q+\beta}} \int_{B_{\rho}^{-1}(y')} h(x) \, dx \right)^q \, d\rho \leq 2 [2^{(\lambda-k/q+\beta)+1}] |S^{k-1}| \int_{\mathbb{R}^{n-k}} \left( \int_{R^{n-k}} \frac{h(x)}{|x-y|^{1+\gamma}} \, dx \right)^q \, dy'.
\] (2.8)

thanks to (2.4) and $\gamma = \lambda - k/q + \beta > 0$. Putting the above estimates together, we arrive at
\[
\int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^{n-k}} \frac{h(x)}{|x-y|^{1+\gamma}} \, dx \right)^q \, dy \leq \left( \lambda \frac{2^{2^{(\lambda-k/q+\beta)+1}}}{\gamma} \right)^q |S^{k-1}| \int_{\mathbb{R}^{n-k}} \left( \int_{\mathbb{R}^{n-k}} \frac{h(x)}{|x-y|^{1+\gamma}} \, dx \right)^q \, dy'.
\]
This is exactly the inequality (2.3), and the proof of the lemma is complete. \qed

Having all the preparations above, to conclude (1.12), we simply apply Lemma 2.2 and the classical HLS inequality (1.2) on $\mathbb{R}^{n-k} \times \mathbb{R}^{n-k}$, namely
\[
\|e^{\rho}_k[f]\|_{L^q(\mathbb{R}^n)} \leq \left( \int_{\mathbb{R}^{n-k}} \left( \int_{\mathbb{R}^{n-k}} \frac{f(x)}{|z-x|^\gamma} \, dx \right)^q \, dz \right)^{1/q} \leq \|f\|_{L^p(\mathbb{R}^{n-k})},
\] thanks to (2.1) and the key estimate $0 < \gamma < n-k$ for all $\lambda > 0$.

Before closing this part, we prove a reverse version of (2.3), see (2.9) below, which has its own interest. We do not directly use this result in the proof of (1.10)$_{k,\gamma}$.

The strategy of proving (2.9) is similar to that of (2.3), however, instead of using Lemma 2.1, we use a technical result from [PV08], see also [BVNV14], which concerns the series of equivalent norms concerning Radon measures.

**Lemma 2.3.** Let $n > k \geq 1$, $p, r > 1$, and $\lambda > 0$. Suppose that $\beta$ satisfies (1.8). Then for any non-negative function $h$ we have
\[
\int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^{n-k}} \frac{h(x)}{|x-y|^{1+\gamma}} \, dx \right)^q \, dy \leq \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^{n-k}} \frac{h(x)}{|x-y|^{1+\gamma}} \, dx \right)^q \, dy,
\] (2.9)

where $\gamma$ is given in (2.2) and $q$ is given in (1.13).
Proof. Our starting point is the equality in (2.5) together with Fubini’s theorem applied for the 90-degree cone \( \{(x, \rho) : \rho \geq |x - y'| + |y''|\} \subset \mathbb{R}^{n-k+1} \), which lies above the curve in Figure 2, which helps us to write

\[
\int_{\mathbb{R}^{n-k}} \frac{h(x)}{|x - y'|^{\alpha'}|y''|^\beta} dx \geq \frac{\lambda}{|y''|^\beta} \int_{|y''|^\beta}^{+\infty} \frac{1}{\rho^{\alpha}/|y''|^\beta} \left( \int_{B_{||y''|^\beta/\rho}}(y') h(x) dx \right) \frac{d\rho}{\rho}
\]

(2.10)

In the above estimate, the non-decreasing property of the function \( \rho \mapsto \int_{B_{|y''|/\rho}} h(x) dx \) have used once. Again, keep in mind that the point \( y \in \mathbb{R}^n \) is being fixed. See Figure 2 for details. Hence, we arrive at

\[
\int_{\mathbb{R}^{n-k}} \left( \int_{\mathbb{R}^{n-k}} \frac{h(x)}{|x - y'|^{\alpha'}|y''|^\beta} dx \right)^q dy' \geq \lambda^q \int_{\mathbb{R}^{n-k}} \left( \int_{B_{|y''|/\rho}} h(x) dx \right)^q \frac{dy''}{|y''|^{(\beta + \lambda)/q}} dy'.
\]

(2.11)

Figure 2. This is how we get the first inequality in (2.10).

Keep in mind that \( 0 < \lambda - k/q + \beta < n - k \). Arguing as in (2.8) and making use of [PV08, Proposition 5.1], we easily get

\[
\int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^1} \left( \int_{B_{|y''|/\rho}} h(x) dx \right)^q \frac{dy''}{|y''|^{(\beta + \lambda)/q}} dy'.
\]
\[ \int_{\mathbb{R}^{n-k}} \left( \int_{\mathbb{R}^{n-k}} \frac{h(x)}{|z-x|^p} \, dx \right)^q \, dz \sim \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^{n-k}} \frac{h(x)}{|x-y|^{\lambda+q} |y|^\beta} \, dx \right)^q \, dy, \]

where \( \gamma \) is given in (2.2) and \( q \) is given in (1.13).

2.3. Necessity of the condition (1.8). We spend this part to discuss the necessity of the condition

\[ \beta < \begin{cases} k - k/r & \text{if } 0 < \lambda \leq n - k, \\ n - \lambda - k/r & \text{if } n - k < \lambda. \end{cases} \]

The argument performed in this part essentially follows from [Ngo21] in which the weighted \(|y|^{-\beta}\) was considered. This also points out the fundamental difference of the condition for \( \lambda \) between the two cases.

First we establish the necessity of the condition \( \beta < k(r-1)/r \) regardless of the size of \( \lambda \). In this case, we may take \( f \equiv \chi_{B_1^{n-k}} \) the characteristic function of \( B_1^{n-k} \). Then

\[ \|\mathcal{E}_{\lambda,k} f \|_{L^q(\mathbb{R}^n)} \geq \int_{B_1^{n-k}} \left[ \int_{(B_1^{n-k})^2} \frac{dx \, dy''}{(|x-y|^2 + |y''|^2)^{\lambda/2}} \right]^q \, dy'' \]

\[ \geq \int_{B_1^{n-k}} \frac{dy''}{|y''|^\beta} = +\infty, \]

as \( \beta q \geq k \). Here we also use

\[ |x-y|^2 + |y''|^2 \leq 2(|x|^2 + |y|^2) + |y''|^2 \leq 5 \]

to bound the double integral of \(|x-y|^{-\lambda}\) from below.

Notice that the above argument does not cover the range \( \lambda > n - k \) since \( n - \lambda - k/r < k - k/r \) in this range of \( \lambda \). Hence we need extra work to cover the case \( \lambda > n - k \).

Now we rule out the case \( \beta \geq n - \lambda - k/r \) for \( \lambda > n - k \). For some non-negative function \( f \in L^p(\mathbb{R}^{n-1}) \) to be determined later, from (2.11) we write

\[ +\infty > \|f\|_{L^p(\mathbb{R}^{n-1})} \geq \int_{\mathbb{R}^{n-1}} \left( \int_{\mathbb{R}^{n-1}} \frac{f(x)}{|x-y|^{\lambda+q} |y|^\beta} \, dx \right)^q \, dy \]

\[ = \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \frac{f(x)}{|y|^\beta} \left( \int_{\mathbb{R}^{n-1}} \frac{f(x)}{|x-y|^2} \, dx \right)^q \, dy'' \, dy' \]

\[ \geq \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \left( \frac{1}{|y|^\beta} \int_{B_1^{n-1}(y')} f(x) \, dx \right)^q \, dy'' \, dy', \]
with \( q = r/(r-1) \). Hence, by using \( \int_{\mathbb{R}^l} = \| S_k^{l-1} \|_0^{\infty} \), we further obtain
\[
+\infty > \| f \|_{L^q(\mathbb{R}^{n-k})} \geq \int_{\mathbb{R}^{n-k}} \int_0^{\infty} \left( \frac{1}{\rho^{1+\beta}} \int_{B_\rho^{n-k}(y')} f(x)dx \right) dy' = \int_0^{\infty} \left( \frac{1}{\rho^{1+\beta-k/q}} \int_{B_\rho^{n-k}(y')} f(x)dx \right) \frac{d\rho}{\rho} dy'.
\]
For the last line in the above computation, thanks to \( |y'| \geq 2 \) and \( 0 \leq \rho \leq 1 \), we know that the ball
\[ B_1^{n-k} \subset B_\rho^{n-k}(y') \subset B_5^{n-k} \]
and that \( 2 \leq |y'| \leq \sqrt{5} \). Hence, if we choose \( f = \chi_{B_\rho^{n-k}} \), then we can bound
\[
\int_{B_\rho^{n-k}(y')} f(x)dx \geq \rho^{n-k},
\]
which yields
\[
\int_0^{1} \left( \frac{1}{\rho^{1+\beta-k/q}} \int_{B_\rho^{n-k}(y')} f(x)dx \right) \frac{d\rho}{\rho} \geq \int_0^{1} \frac{d\rho}{\rho^{1+\beta-k/q-k-n+1}}.
\]
As \( \lambda - k/q + \beta - (n-k) \geq 0 \) if \( \beta \geq n - \lambda - k/r \), the integral on the right hand side of the preceding inequality always diverges if \( \beta \geq n - \lambda - k/r \). This completes the proof of the necessity of the condition \( \beta < n - \lambda - k/r \) in the range \( \lambda > n-k \).

Finally, we notice that the necessity of the condition \( \lambda > 0 \). This is trivial and we can take it for granted. Otherwise, the inequality (1.10) will be in the opposite direction.

3. Existence of an optimal pair \((f^\sharp, g^\sharp)\) for the HLS inequality

In this section, we prove the existence of an optimal pair \((f^\sharp, g^\sharp)\) for the optimal HLS inequality (1.10) in the full regime of the parameters. Recall that \( n > k \geq 1, p,r > 1, \lambda > 0 \), and \( \beta \) satisfies (1.8). In particular, there holds
\[ 0 < \lambda - k/q + \beta < n-k \]
for all \( \lambda > 0 \). This together with (2.1) helps us to deduce that
\[ 1/p + 1/r > 1 \]
for all \( \lambda > 0 \). Hence, we obtain
\[ q := \left( 1 - \frac{1}{r} \right)^{-1} > p, \quad (3.1) \]
which is very important in the proof; see Step 3 of the proof below.

In this section, we prove Theorem 1.2 and Proposition 1.3, namely there exists an optimal function \( f^\sharp \) to maximizing problem (1.14); see subsection 2.1. This is equivalent to proving that there exists a radially symmetric, strictly decreasing function \( f^\sharp \) such that
\[ \| C_{\lambda,k}^\beta [f^\sharp] \|_{L^q(\mathbb{R}^n)} = N_{n,k,p}^{k\beta} \| f^\sharp \|_{L^p(\mathbb{R}^{n-k})} = 1. \]
This is done within the first three steps of the proof. Finally, to conclude Proposition 1.3 and as any optimal function \( f^\sharp \) for (1.14) gives rise to an optimal pair \((f^\sharp, (C_{\lambda,k}^\beta [f^\sharp])^{q-1})\) for (1.11), we shall show that the function \( C_{\lambda,k}^\beta [f^\sharp] \) is radially symmetric and strictly decreasing, which is the last step in the proof.

As our weighted inequality (1.10) on \( \mathbb{R}^{n-k} \times \mathbb{R}^n \) is reduced to the unweighted inequality on \( \mathbb{R}^{n-k} \times \mathbb{R}^{n-k} \), in the proof of the existence of optimal pairs, it is expect that
the argument for the unweighted case can be used for the weighted case. We shall so that this is true. To be more precise, to prove the existence of optimal pair, we mimic Lieb’s argument in [Lie83] with appropriate changes. For example, for a function \( \eta \), we denote by \( \eta^\ast \) the symmetric decreasing rearrangement of \( \eta \) with respect to the first \( n-k \) coordinates; see [LL01] or [Bur09] for the definition. Since the weight \( |\eta^\prime|^{\frac{\beta}{p}} \) concerns the last \( k \) coordinates, the symmetrization process does not affect the weight \( |\eta^\prime|^{\frac{\beta}{p}} \). Consequently, it is worth noting that we do not require any sign for \( \beta \).

Now we prove the existence of a non-trivial maximizer \( f^\# \) for the problem (1.14). For the sake of clarity, we divide our proof into several steps.

**Step 1. Selecting a suitable minimizing sequence for (1.14).**

We start our proof by letting \( (f_j) \) be a maximizing sequence in \( L^p(\mathbb{R}^{n-k}) \) for the problem (1.14) such that \( f_j \) is non-negative. Keep in mind that

\[
\|f_j\|_{L^p(\mathbb{R}^{n-k})} = \|f_j^\ast\|_{L^p(\mathbb{R}^{n-k})}
\]

Now by using Riesz’s rearrangement inequality, see [LL01, chapter 3], Hölder’s inequality, and \( 1/q + 1/r = 1 \), we know that

\[
\|\mathcal{E}_{\lambda,k}[f_j]\|_{L^q(\mathbb{R}^n)} = \sup_{\|\eta\|_{L^r(\mathbb{R}^n)}=1} \int_{\mathbb{R}^n} \frac{1}{|\eta|^{\frac{\beta}{p}}} \left( \int_{\mathbb{R}^{n-k}} f_j(x)h(y,y')dx dy' \right)^{\frac{1}{p}} dy'' \leq \sup_{\|\eta\|_{L^r(\mathbb{R}^n)}=1} \left\| \mathcal{E}_{\lambda,k}[f_j^\ast]\right\|_{L^q(\mathbb{R}^n)} \|h^\ast\|_{L^1(\mathbb{R}^n)}
\]

Notice that

\[
\int_{\mathbb{R}^n} |h''| dy = \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^{n-k}} |h''(y,y')|dy' \right) dy'' = \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^{n-k}} |h^\ast''| (y,y')dy' \right) dy'' = \int_{\mathbb{R}^n} |h^\ast|^\prime dy.
\]

Hence, as \( \|\eta\|_{L^r(\mathbb{R}^n)} = 1 \) we deduce that \( \|h^\ast\|_{L^1(\mathbb{R}^n)} = 1 \). Thus

\[
\|\mathcal{E}_{\lambda,k}[f_j]\|_{L^q(\mathbb{R}^n)} \leq \|\mathcal{E}_{\lambda,k}[(f_j)^\ast]\|_{L^q(\mathbb{R}^n)}.
\]

Putting the above two estimates between \( f_j \) and \( (f_j)^\ast \) together, we may further assume that \( f_j \) is radially symmetric with respect to the origin and non-increasing. By abusing notations, we shall write \( f_j(x) \) by \( f_j(|x|) \) or even by \( f_j(r) \) where \( r = |x| \). We can normalize \( f_j \) in such a way that \( \|f_j\|_{L^p(\mathbb{R}^{n-k})} = 1 \). From this and the monotonicity of \( f_j \), we have

\[
1 = \left|\mathbb{S}^{n-k-1}\right| \int_0^\infty f_j(r)^p r^{n-k-1} dr \geq \left|\mathbb{S}^{n-k-1}\right| \frac{1}{n-k} f_j(R)^p R^{n-k}
\]

for any \( R > 0 \). From this, we obtain the following estimate

\[
0 \leq f_j(r) \leq \left( \frac{1}{n-k} \right)^{1/p} r^{-(n-k)/p}
\]

for any \( r > 0 \).

**Step 2. Existence of a potential maximizer \( f^\# \) for the problem (1.14).**
For each non-negative function \( h \) on \( \mathbb{R}^{n-k} \), we denote
\[
\|h\| = \sup_{x \in \mathbb{R}^{n-k}, \rho > 0} \left[ \rho^{-\frac{n-k}{p}} \int_{B_{\rho}^{-1}(x)} h(z) dz \right]
\]
with \( p' = p/(p-1) \). Suppose that \( h \in L^p(\mathbb{R}^{n-k}) \). By Hölder’s inequality we have
\[
\rho^{-\frac{n-k}{p'}} \int_{B_{\rho}^{-1}(x)} h(z) dz \leq \|h\| L^p(\mathbb{R}^{n-k}),
\]
for arbitrary \( x \in \mathbb{R}^{n-k} \) and for any \( \rho > 0 \). Hence, by definition we get
\[
\|h\| \leq \|h\| L^p(\mathbb{R}^{n-k}).
\]
To go further, we need an auxiliary result, an analogue of [Lie83, Lemma 2.4] concerning the behavior of \( \|\mathcal{E}_{\lambda,k}^\beta[f]\|_{L^q(\mathbb{R}^n)} \), whose proof is located in Appendix A.

**Lemma 3.1.** Suppose that \( f \in L^p(\mathbb{R}^{n-k}) \) is non-negative. Then there exists a constant \( C_1 > 0 \), independent of \( f \) such that
\[
\int_{\mathbb{R}^n} (\mathcal{E}_{\lambda,k}^\beta[f])^q dy \leq C_1 \|f\|^{q/p} \int_{\mathbb{R}^{n-1}} f^p dx
\]
where \( q = r/(r-1) > p > 1 \).

Going back to the maximizing sequence \((f_j)_j \) in \( L^p(\mathbb{R}^{n-k}) \) for the problem (1.14), for each \( j \) we set
\[
a_j = \sup_{\rho > 0} \left[ \rho^{-\frac{n-k}{p}} f_j(\rho) \right].
\]
In view of (3.3) we know that
\[
0 < a_j \leq \left( \frac{n-k}{S_{n-k-1}} \right)^{1/p}
\]
for all \( j \). Using the monotonicity of \( f_j \), we deduce that
\[
\int_{B_{\rho}^{-1}(x)} f_j(z) dz \leq a_j \int_{B_{\rho}^{-1}(x)} |z|^{-\frac{n-k}{p'}} dz
\]
\[
\leq a_j \int_{B_{\rho}^{-1}(0)} |z|^{-\frac{n-k}{p'}} dz
\]
\[
= \frac{a_j}{(n-k)(1-1/p)} \rho^{(n-k)(1-1/p)}
\]
for arbitrary \( x \in \mathbb{R}^{n-k} \) and for any \( \rho > 0 \). Consequently, there holds
\[
\|f_j\| \leq \frac{a_j}{(n-k)(1-1/p)}
\]
for all \( j \). Recall from the choice of \( f_j \) the following
\[
\|f_j\|_{L^p(\mathbb{R}^{n-k})} = 1, \quad \|\mathcal{E}_{\lambda,k}^\beta[f_j]\|_{L^q(\mathbb{R}^n)} \rightarrow \mathcal{N}_{n,k}^\beta,
\]
Making use of Lemma 3.1 above, we obtain the following estimate
\[
(N_{n,k}^\beta)^q \leq \int_{\mathbb{R}^n} (\mathcal{E}_{\lambda,k}^\beta[f_j])^q dy + o(1) j \rightarrow +\infty
\]
\[
\leq C_1 \|f_j\|^{q/p} \int_{\mathbb{R}^{n-1}} f_j^p dx + o(1) j \rightarrow +\infty
\]
\[
= C_1 \|f_j\|^{q/p} + o(1) j \rightarrow +\infty.
\]
Keep in mind that \( q > p \). Hence, \( \|f_j\|_s \) is bounded from below away from zero. This together with \( \|f_j\|_s \leq a_j \) allows us to assume that \( a_j \geq 2c_0 \) for some \( c_0 > 0 \) and for all \( j \). Consequently, for each \( j \), we can choose \( \lambda_j > 0 \) in such a way that

\[
\lambda_j^\frac{n-k}{p} f_j(\lambda_j) > c_0.
\]

Then we set

\[
g_j(x) = \lambda_j^\frac{n-k}{p} f_j(\lambda_j x).
\]

From this, it is routine to check that \( (g_j)_j \) is also a minimizing sequence for problem (1.14), however, \( g_j(1) > c_0 \) for any \( j \) by our choice for \( \lambda_j \). Consequently, by replacing the sequence \( (f_j)_j \) by the new sequence \( (g_j)_j \), if necessary, we can further assume that our sequence \( (f_j)_j \) obeys

\[
f_j(1) > c_0 \quad \text{for any } j.
\]

Similar to Lieb’s argument in [Lie83], which is based on Helly’s theorem, by passing to a subsequence, we have

\[
f_j \to f^\# \quad \text{a.e. in } \mathbb{R}^{n-k}.
\]

It is now evident that \( f^\# \) is non-negative, radially symmetric, non-increasing, and is in \( L^p(\mathbb{R}^{n-k}) \). Of course, there holds \( f^\# \not\equiv 0 \). The rest of our arguments is to show that \( f^\# \) is indeed the desired minimizer for (1.14).

**Step 3.** The function \( f^\# \) is a maximizer for (1.14).

Recall that \( (f_j)_j \) is a minimizing sequence for the problem (1.14) and \( f_j \to f^\# \) a.e. in \( \mathbb{R}^{n-k} \). The limit function \( f^\# \) satisfies \( \|f^\#\|_{L^p(\mathbb{R}^{n-k})} > 0 \) because \( f_j(x) > c_0 \) for any \( j \) and all \( |x| \leq 1 \). To go further, we need the following auxiliary result.

**Lemma 3.2.** Suppose that \( (f_j)_j \) is a sequence of non-negative functions satisfying

\[
f_j(x) \leq C|x|^{-\frac{n-k}{p}}
\]

for all \( x \in \mathbb{R}^{n-k} \) and for some \( C > 0 \). Then, if \( f_j \to f^\# \) a.e. in \( \mathbb{R}^{n-k} \), then we have

\[
\mathcal{E}_{\lambda,k}^\beta[f_j](y) \to \mathcal{E}_{\lambda,k}^\beta[f^\#](y)
\]

for almost every \( y \in \mathbb{R}^n \).

Lemma 3.2 above simply follows from the dominated convergence theorem. It is worth noting that in order to apply the dominated convergence theorem, we make use of the inequality

\[
\lambda + (n-k)p > n-k,
\]

which always holds true under our assumption (1.9). Hence, we omit the details and its proof is left for interested readers.

Using Lemma 3.2 above, we further know that \( \mathcal{E}_{\lambda,k}^\beta[f_j] \to \mathcal{E}_{\lambda,k}^\beta[f^\#] \) for a.e. in \( \mathbb{R}^n \). The rest of the proof is more or less standard and the condition \( q > p \) plays a role; see [Lie83, Lemma 2.7]. Applying the Brezis–Lieb lemma to get

\[
\int_{\mathbb{R}^{n-k}} |f_j|^p - |f^\#|^p - |f_j - f^\#|^p \, dx \to 0
\]

as \( j \to +\infty \). So, one one hand we have

\[
\|f_j - f^\#\|_{L^p(\mathbb{R}^{n-k})}^p = 1 - \|f^\#\|_{L^p(\mathbb{R}^{n-k})}^p + o(1)_{j \to +\infty}.
\]  

\[ (3.4) \]
However, on the other hand, we can estimate
\[
(N_{n,\lambda,p}^{k,n,q})^q + o(1)_{j \to +\infty} = \|\mathcal{E}_{\lambda,k}[f_j]\|_{L^q(R^n)}^q
\]
\[
= \|\mathcal{E}_{\lambda,k}[f_j]\|_{L^q(R^n)}^q + \|\mathcal{E}_{\lambda,k}[f_j - f^\sharp]\|_{L^q(R^n)}^q + o(1)_{j \to +\infty}
\]
\[
\leq (N_{n,\lambda,p}^{k,n,q})^q \|f^\sharp\|_{L^p(R^n)}^p + o(1)_{j \to +\infty}.
\]

Thus, dividing both sides of the preceding computation by \((N_{n,\lambda,p}^{k,n,q})^q\) gives
\[
1 \leq \|f^\sharp\|_{L^p(R^n)}^p + \|f_j - f^\sharp\|_{L^p(R^n)}^p + o(1)_{j \to +\infty}.
\]

Combining (3.4) and (3.5) and sending \(j \to +\infty\), we arrive at
\[
1 \leq \|f^\sharp\|_{L^p(R^n)}^p + \left(1 - \|f^\sharp\|_{L^p(R^n)}^p\right)^{q/p}.
\]

From the fact that \(q > p\) seeing (3.1) and that \(\|f^\sharp\|_{L^p(R^n)} > 0\), we deduce that
\[
\|f^\sharp\|_{L^p(R^n)} = 1.
\]

This shows that \(f^\sharp\) is a minimizer for (1.14); hence finishing the proof of Step 3.

**Step 4.** The function \(\mathcal{E}_{\lambda,k}^\beta[f^\sharp]\) has two symmetries in \(y'\) and \(y''\) and is strictly decreasing in \(y'\).

This step is for the proof of Proposition 1.3. We show that \(\mathcal{E}_{\lambda,k}^\beta[f^\sharp]\) of variable \(y\) has two symmetries in \(y'\) and \(y''\). While the symmetry with respect to \(y''\) is obvious from the definition of \(\mathcal{E}_{\lambda,k}^\beta[f^\sharp]\), the symmetry with respect to \(y'\) is also clear since \(\mathcal{E}_{\lambda,k}^\beta[f^\sharp]\) is essentially the convolution of two radially symmetric functions \(f^\sharp\) and \((\cdot, |y'|^2)^{-\lambda/2};\) see [Lie83, Lemma 2.2(ii)]. Since the argument is simple and short, we provide a proof for completeness. Indeed, let \(A \in O(n - k)\) be arbitrary. Then
\[
(\mathcal{E}_{\lambda,k}^\beta[f^\sharp])(Ay', y'') = \frac{1}{|y'|^\frac{n\lambda}{2}} \int_{R^{n-k}} \frac{f^\sharp(x)}{|x - Ay'|^2 + |y''|^2} dx
\]
\[
= \frac{1}{|y'|^\frac{n\lambda}{2}} \int_{R^{n-k}} \frac{f^\sharp(A^t x)}{|A^t(A^t x - y')|^2 + |y''|^2} dx
\]
\[
= \frac{1}{|y'|^\frac{n\lambda}{2}} \int_{R^{n-k}} \frac{f^\sharp(A^t x)}{|A^t(A^t x - y')|^2 + |y''|^2} \det A dx,
\]
where \(A^t\) is the transpose of \(A\). Finally, the monotonicity of \(\mathcal{E}_{\lambda,k}^\beta[f^\sharp]\) in \(y'\) follows from [Lie83, Lemma 2.2(ii)].

Notice that the monotonicity of \(\mathcal{E}_{\lambda,k}^\beta[f^\sharp]\) in \(y'\) can also be derived from a general result of Anderson applied to the function \(h = (|\cdot|^2 + |y''|^2)^{-\lambda/2} f^\sharp(- \cdot + y')\) for \(y\) fixed; see [And55, Theorem 1]. This is because
\[
(\mathcal{E}_{\lambda,k}^\beta[f^\sharp])(\tau y', y'') \geq \int_{R^{n-k}} \frac{f^\sharp(x + (1 - \tau)y')}{(|x - \tau y'|^2 + |y''|^2)^{\lambda/2}} dx
\]
\[
= \int_{R^{n-k}} h(x - \tau y') dx
\]
\[ \geq \int_{\mathbb{R}^{n-k}} h(x-y')dx = (E^\beta_{\lambda,k}[f^\beta])(y', y'') \]

for any \( 0 \leq \tau \leq 1 \). Here the radial symmetry and monotonicity of \( f^\beta \) are crucial to get

\[ f^\beta(x) \geq f^\beta(x + (1-\tau)y'). \]

If \( \beta \geq 0 \), then the monotonicity of \( E^\beta_{\lambda,k}[f^\beta] \) in \( y'' \) is clear. But it is not clear if this still holds when \( \beta < 0 \).

Before closing this section, we have the following remark.

**Remark 3.3.** As \( \|f\| \lesssim \|f\|_{L^p(\mathbb{R}^{n-k})} \), Lemma 3.1 gives us another proof of the rough HLS inequality (1.10)\(_{k,\beta}\).

### 4. Discussions

This section is devoted to a number of discussion and application from simple to complex around our main inequality (1.10)\(_{k,\beta}\).

#### 4.1. The HLS inequality on \( \mathbb{R}^{n-k} \times \mathbb{R}^n \) with optimal range \( 0 < \lambda < n-k/r \).

We start this section with a quite surprise application of Theorem 1.1. To be more precise, with \( \beta = 0 \), which is possible because \( \lambda < n-k/r \), we obtain from Theorem 1.1 the following optimal result.

**Theorem 4.1** (optimal HLS inequality on \( \mathbb{R}^{n-k} \times \mathbb{R}^n \)). Let \( n \geq 1, 0 \leq k < n, p, r > 1, \) and \( \lambda \in (0, n-k/r) \) satisfying the balance condition

\[ \frac{n-k}{n} \frac{1}{p} + \frac{1}{r} + \frac{\lambda}{n} = 2 - \frac{k}{n}. \]

Then, there holds

\[ \left| \iint_{\mathbb{R}^{n-k} \times \mathbb{R}^n} \frac{f(x)g(y)}{|x-y|^k}dxdy \right| \leq \|f\|_{L^p(\mathbb{R}^{n-k})}\|g\|_{L^r(\mathbb{R}^n)} \tag{4.1} \]

for any functions \( f \in L^p(\mathbb{R}^{n-k}) \) and \( g \in L^r(\mathbb{R}^n) \).

The sharp constant for the inequality (4.1) is \( N^{k,0}_{n,\lambda,p} \). For arbitrary but fixed \( r > 1 \), Theorem 4.1 is optimal in the sense that it does not hold if \( \lambda \geq n-k/r \) by seeing (1.8). However, we can rephrase Theorem 4.1 in the following way: if we fix \( 0 < \lambda < n \), then by resolving the inequality \( \lambda < n-k/r \) which gives

\[ r > \max \left\{ 1, \frac{k}{n-\lambda} \right\}, \]

we know that Theorem 4.1 holds for the above \( r \). This condition tells us that the closer to \( n \) the parameter \( \lambda \) is, the bigger \( r \) is.

In the special case \( k = 1 \), Theorem 4.1 helps us to revisit the HLS inequality (1.4) on the upper half space \( \mathbb{R}_+^n \) with the optimal range

\[ 0 < \lambda < n - 1/r. \]

This improves the result of Dou and Zhu in [DZ15], which is stated for \( 0 < \lambda < n - 1 \).
4.2. An improvement of the HLS inequality on $\mathbb{R}^n$ with extended kernel (1.6) in the regime $0 < \lambda < n - 1$ and $0 < \beta < 1 - 1/r$. Our motivation of working on this problem also comes from the fact that the HLS inequality (1.6) on $\mathbb{R}^n$ with extended kernel is “weaker” than the HLS inequality (1.4) on $\mathbb{R}^n$. Here by “weaker” we mean we can use (1.4) to derive (1.6). Indeed, as $\beta \leq 0$, we clearly have

$$y_n^{-\beta} \leq |x - y|^{-\beta},$$

giving

$$\int_{\mathbb{R}_+^{n-1} \times \mathbb{R}^n} \frac{f(x)g(y)}{|x - y|^{1+\beta}} dxdy \leq \int_{\mathbb{R}_+^{n-1} \times \mathbb{R}^n} \frac{f(x)g(y)}{|x - y|^{1+\beta}} dxdy,$$

where, for simplicity, all the functions $f$ and $g$ are being non-negative. Notice that the balance condition (1.7) allows us to apply (1.4) with $\lambda$ replaced by $\lambda + \beta$. To be more precise, fixing any $f \in L^p(\partial\mathbb{R}^n_+)$ and any $g \in L^r(\mathbb{R}^n_+)$, we have

$$\int_{\mathbb{R}_+^{n-1} \times \mathbb{R}^n} \frac{f(x)g(y)}{|x - y|^{1+\beta}} dxdy \leq \int_{\mathbb{R}_+^{n-1} \times \mathbb{R}^n} \frac{f(x)g(y)}{|x - y|^{1+\beta}} dxdy \leq \|f\|_{L^p(\partial\mathbb{R}^n_+)} \|g\|_{L^r(\mathbb{R}^n_+)}.$$

provided (1.7) and $\beta \leq 0$ hold. This explains why (1.6) is weaker than (1.4).

From the above discussion, it is natural to ask if (1.6) still holds for suitable $\beta > 0$. If there is such an inequality, then it implies that we will have a “stronger” version of the HLS inequality (1.4) on $\mathbb{R}^n$ in the following sense

$$\int_{\mathbb{R}_+^{n-1} \times \mathbb{R}^n} \frac{f(x)g(y)}{|x - y|^{1+\beta}} dxdy \leq \int_{\mathbb{R}_+^{n-1} \times \mathbb{R}^n} \frac{f(x)g(y)}{|x - y|^{1+\beta}} dxdy \leq \|f\|_{L^p(\partial\mathbb{R}^n_+)} \|g\|_{L^r(\mathbb{R}^n_+)} \leq \|f\|_{L^p(\partial\mathbb{R}^n_+)} \|g\|_{L^r(\mathbb{R}^n_+)}.$$

so long as (1.4) holds.

To be able to compare and for simplicity, we limit ourselves to the case $0 < \lambda < n - 1$, hence the condition for $\beta$ becomes $\beta < 1 - 1/r$. Clearly, under the above setting, our inequality (1.10)$_{1,\beta}$ becomes

$$\left\{ \int_{\mathbb{R}_+^{n-1} \times \mathbb{R}^n} \frac{f(x)g(y)}{|x - y|^{1+\beta}} dxdy \right\} \leq N_{1,\lambda,p}^{1,\beta} \|f\|_{L^p(\partial\mathbb{R}^n_+)} \|g\|_{L^r(\mathbb{R}^n)} \leq N_{1,\lambda,p}^{1,\beta} \|f\|_{L^p(\partial\mathbb{R}^n_+)} \|g\|_{L^r(\mathbb{R}^n)}$$

with $0 < \lambda < n - 1$ and $\beta < 1 - 1/r$; hence providing us an improvement of (1.4) for $\beta$, if we let $g|_{\mathbb{R}^n} \equiv 0$.

In subsection 4.3 below, we show that there is another way, which is quite intriguing, to obtain (1.6) directly from (1.10)$_{k,\beta}$ without assuming $k = 1$. Moreover, it is quite interesting to note from the argument leading to (1.10)$_{k,\beta}$, we can relate the sharp constant $N_{n,\lambda,p}^{k,\beta}$ and that of (1.4).

4.3. From $\mathbb{R}^{n-k} \times \mathbb{R}^n$ to $\mathbb{R}^{n-k} \times \mathbb{R}^{n-k+1}$. An other idea to improve (1.6) for possible $\beta > 0$ is to transform (1.10)$_{k,\beta}$ into (1.6). To fix the idea and for simplicity, we still limit ourselves to the case $0 < \lambda < n - k$, hence $0 < \beta < k(1 - 1/r)$.

To transform (1.10)$_{k,\beta}$ into (1.6), we simply make use of (1.10)$_{k,\beta}$ for function $g$ being radially symmetric in the last $k$ coordinates, namely

$$g(y', y'') = g(y', |y''|).$$

Observe that

$$\int_{\mathbb{R}^n} (g(y)) d\rho = |S^{k-1}| \int_{\mathbb{R}^{k-1}} \int_0^{\infty} (g(y', \rho)) \rho^{k-1} d\rho dy'.$$
Hence, by setting
\[ G(y', |y'\rangle) = |S_{k-1}^{k-1}/r\rangle/|y'\rangle |y'\rangle/(k-1)/r, \]
on one hand, it is easy to verify that
\[ \int_{R^n} g'(y)dy = \int_{R^{n-k+1}} G'(z, \rho)dzd\rho. \]
On the other hand, by using \( \int_{R^k} = \int_{S_{k-1}}^{+\infty} \), we get
\[ F_{\lambda, k}(f, G) = |S_{k-1}^{k-1}/r\rangle \int_{R^{n-k}} \int_{R^{k-1}}^{+\infty} f(x)g(y', \rho) \left[ |x-y'|^2 + \rho^2 \right]^{r/2} \rho^{-k} d\rho dy'dx \]
\[ \quad = |S_{k-1}^{k-1}/r\rangle \int_{R^{n-k}} \int_{R^{k-1}}^{+\infty} f(x)G(y', \rho) \left[ |x-y'|^2 + \rho^2 \right]^{r/2} \rho^{-k-1/(1-1/r)} d\rho dy'dx. \]
Hence, the inequality (1.10) for \((f, G)\) becomes the following inequality for \((f, G)\)
\[ \int_{R^{n-k}} \int_{R^{k-1}}^{+\infty} f(x)G(y', \rho) \left[ |x-y'|^2 + \rho^2 \right]^{r/2} \rho^{-\beta} d\rho dy'dx \]
\[ \quad \leq |S_{k-1}^{k-1}/r\rangle |N_{n, \lambda, p}^{k, \beta}||f||_{L^p(R^{n-k})}||G||_{L^r(R^{k-1})} \]
with
\[ \beta = \beta - (k - 1)(1 - 1/r) < 1 - 1/r. \]
Obviously, if \((f^{\#}, G^{\#})\) is an optimal pair for (1.10)_{k, \beta}, then \((f^{\#}, G^{\#})\) is also an optimal pair for (4.2). However, due to the above transformation, the regularity for \((f^{\#}, G^{\#})\) and \((f^{\#}, G^{\#})\) are quite different. This could shed some light on the problem of classification of all optimal pairs for the inequality on the upper half space.

Nevertheless, from the above derivation and the existence of an optimal pair \((f^{\#}, G^{\#})\) for (1.10)_{k, \beta}, we obtain the following improvement of (1.6), which is optimal.

**Theorem 4.2.** Let \( n \geq 2, \lambda \in (0, n-1), \beta < (r-1)/r, \) and \( p, r > 1 \) satisfying the balance condition
\[ \frac{n-1}{p} + \frac{1}{r} + \frac{\beta + \lambda}{n} = 2 - \frac{1}{n}. \] (4.3)
Then there exists a sharp constant \( p_{\lambda, k, \beta}^{p, n, \lambda, p} > 0 \) such that
\[ \left( \int_{R^{n-k} \times R^{k-1}} f(x)g(y) |x-y|^\alpha \frac{dx}{y} \right)^{\beta} \leq p_{\lambda, k, \beta}^{p, n, \lambda, p} ||f||_{L^p(R^{n-k})} ||g||_{L^r(R^{k-1})} \] (4.4)\beta
for any functions \( f \in L^p(R^{n-k}) \) and \( g \in L^r(R^{k-1}). \)

In the last part of this subsection, we show that the transformed inequality (4.2) reveals a connection between the two sharp constants \( p_{\lambda, k, \beta}^{p, n, \lambda, p} \) and \( N_{n, \lambda, p}^{k, \beta}. \) We turn this observation into a proposition as follows.

**Proposition 4.3.** There holds
\[ p_{n-k+1, \lambda, p}^{n-k, k, \beta} = |S_{k-1}^{k-1}/r\rangle |N_{n, \lambda, p}^{k, \beta+(k-1)\beta}/r\rangle. \]
In particular, with \( k = 1 \) we have
\[ p_{n, \lambda, p}^{n, 1, \beta} = 2^{1/r-1} N_{n, \lambda, p}^{1, \beta}, \]
relating the sharp constant of the HLS inequalities on \( R^{n-1} \times R^n \) and on \( R^{n-1} \times R^n. \)
Proof. We apply (4.2) for \((f^\#, G^\#)\) being an optimal pair for (4.2) to get
\[
P_{n-k+1,\lambda,p}[f^\#|_{L^p(\mathbb{R}^{n-k})}|G^\#|_{L^r(\mathbb{R}^{n-k})}^2] \\
\geq \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^{n-k}} \frac{f^\#(x)G^\#(y')}{|x-y'|^\alpha + |\rho|^\beta} dx dy' \\
= |S^{k-1}|^{1/r-1}N_{n,\lambda,p}[f^\#|_{L^p(\mathbb{R}^{n-k})}|G^\#|_{L^r(\mathbb{R}^{n-k})}]^2.
\]
From this we obtain
\[
P_{n-k+1,\lambda,p}^\# \geq |S^{k-1}|^{1/r-1}N_{n,\lambda,p}^{k}\beta.
\]
Now we use (4.4) with \((f^\#, g^\#)\) being an optimal pair for (4.4) to get
\[
P_{n-k+1,\lambda,p}^\# \geq |S^{k-1}|^{1/r-1}N_{n,\lambda,p}^{k}\beta.
\]
Hence we get
\[
P_{n-k+1,\lambda,p} \leq |S^{k-1}|^{1/r-1}N_{n,\lambda,p}^{k}\beta.
\]
The proof follows by putting the above estimates together.

4.4. An Stein–Weiss type inequality on \(\mathbb{R}^{n-k} \times \mathbb{R}^n\). In the literature, another weighted version of the inequality (1.2), or the doubly weighted HSL inequality, also known as the SW inequality, named after Stein and Weiss, was also obtained in [SW58]. Roughly speaking, for
\[
0 < \lambda < n
\]
and for suitable \(\alpha, \beta\) satisfying
\[
\alpha < n(p-1)/p, \quad \beta < n(r-1)/r, \quad \alpha + \beta \geq 0,
\]
the following rough inequality holds
\[
\left| \int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x-y|^\alpha + |\rho|^\beta} dxdy \right| \leq \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^r(\mathbb{R}^n)} \tag{4.5}
\]
for any \(f \in L^p(\mathbb{R}^n)\) and \(g \in L^r(\mathbb{R}^n)\) together with
\[
1/p + 1/r \geq 1
\]
and the new balance condition
\[
1/p + 1/r + (\alpha + \beta + \lambda)/n = 2.
\]
Inequality (4.5) was extended by Dou to the case of \(\mathbb{R}^n_+\); see [Dou16]. To be more precise, under the condition
\[
0 < \lambda < n - 1
\]
the following sharp inequality was proved
\[
\left| \int_{\mathbb{R}^n_+} \frac{f(x)g(y)}{|x-y|^\alpha + |\rho|^\beta} dxdy \right| \leq \|f\|_{L^p(\mathbb{R}^{n-1})} \|g\|_{L^r(\mathbb{R}^n_+)} \tag{4.6}
\]
for any \(f \in L^p(\mathbb{R}^{n-1})\) and any \(g \in L^r(\mathbb{R}^n_+)\) together with
\[
\alpha < (n-1)(p-1)/p, \quad \beta < n(r-1)/r, \quad \alpha + \beta \geq 0,
\]
and
\[
1/p + 1/r \geq 1
\]
and the new balance condition
\[(n-1)/(np) + 1/r + (\alpha + \beta + \lambda)/n = 2 - 1/n.\]

In the context of \(R^{n-k} \times R^n\), it is expected that the following inequality holds
\[
\left| \left\langle f(x)g(y) \left| \frac{\partial^{|\alpha|} x - y}{|x - y|^n} \right|^\beta \right\rangle \right| \leq ||f||_{L^p(R^{n-k})} ||g||_{L^r(R^n)}
\]
for any functions \(f \in L^p(R^{n-k})\) and \(g \in L^r(R^n)\) under the following balance condition
\[
\frac{n-k}{p} + \frac{1}{r} + \frac{\alpha + \beta + \lambda}{n} = 2 - \frac{k}{n},
\]
where \(k \geq 0\), \(\alpha, \beta, \lambda > 0\), and \(n \geq 1\).

Theorem 4.4 (SW type inequality on \(R^{n-k} \times R^n\)). Let \(n \geq 1\), \(0 \leq k < n\), \(\lambda > 0\),
\[0 \leq \alpha < (n-k)(p-1)/p, \beta \text{ satisfying } (1.8), p, r > 1 \text{ satisfying}
\[1/p + 1/r \geq 1,
\]
and the balance condition (4.8). Then, there holds
\[
\left| \left\langle f(x)g(y) \left| \frac{\partial^{|\alpha|} x - y}{|x - y|^n} \right|^\beta \right\rangle \right| \leq ||f||_{L^p(R^{n-k})} ||g||_{L^r(R^n)}
\]
for any functions \(f \in L^p(R^{n-k})\) and \(g \in L^r(R^n)\).

Proof. It suffices to prove (4.9) for non-negative functions \(f\) and \(g\) and for \(\alpha > 0\). Notice that the balance condition (4.8) can be rewritten as follows:
\[
\frac{1}{p} + \frac{1}{r} + \frac{\lambda - k + \beta + \alpha}{n-k} = 2.
\]

Now we can use Lemma 2.2 with the function \(h(x) = |x|^{-\alpha} f(x)\) and the classical SW inequality (4.5) with \(\beta = 0\) to get
\[
\left| \left\langle f(x)g(y) \left| \frac{\partial^{|\alpha|} x - y}{|x - y|^n} \right|^\beta \right\rangle \right| \leq ||f||_{L^p(R^{n-k})} ||g||_{L^r(R^n)}
\]
for any functions \(f \in L^p(R^{n-k})\) and \(g \in L^r(R^n)\).

This requires some justification due to the presence of \(\alpha\). Indeed, as \(1/p + 1/r \geq 1\) and \(\alpha > 0\), we immediately have \(\lambda - k + \beta < n-k\). To see why \(\lambda - k + \beta > 0\) we make use of the estimate for \(\alpha \leq (n-k)/(1-1/p)\) to get
\[
\frac{2}{p} + \frac{1}{r} + \frac{\lambda - k + \beta}{n-k} + 1 - \frac{1}{p},
\]
which is now enough to see \(\lambda - k + \beta > 0\), thanks to \(r > 1\). Similarly, to be able to apply (4.5) we need
\[
0 < \alpha \leq (n-k)(p-1)/p, \quad 1/p + 1/r \geq 1,
\]
but these conditions are exactly the hypotheses. By duality, we obtain (4.9) as expected. \(\Box\)
The requirement $1/p + 1/r \geq 1$ is necessary for the validity of Theorem 4.4. We do not prove this and refer the reader to [Ngo21] for a possible argument. In the case $\alpha = 0$, it is automatically satisfied thanks to $\lambda - k/q + \beta \leq n - k$.

Using the computation in subsection 4.3 above, it is not hard to see that our SW type inequality $(4.9)_{k,\alpha,\beta}$ also leads us to

$$\iint_{\mathbb{R}^{n-k} \times \mathbb{R}^{k+1}} \frac{f(x)G(y',\rho)d\rho dy'dx}{|x|^a [x - y']^2 + \rho^2} \leq \|f\|_{L^p(\mathbb{R}^{n-k})} \|G\|_{L^r(\mathbb{R}^{n+1})},$$

(4.10)

which is an analogue of (4.6). Clearly, (4.10) is also an improvement of (4.6) if $\tilde{\beta} \geq 0$. We leave the details for interested readers.

Finally, we close this part by proving the SW inequality (4.7) on $\mathbb{R}^{n-k} \times \mathbb{R}^n$, as an application of Theorem 4.4 and Theorem 1.1. In particular, this also provides a new proof of the SW inequality (4.6).

**Theorem 4.5 (SW inequality on $\mathbb{R}^{n-k} \times \mathbb{R}^n$).** Let $n \geq 1$, $0 \leq k < n$, $0 < \lambda < n - k/r$, and $p, r > 1$ satisfying

$$1/p + 1/r \geq 1.$$

We suppose that $\alpha < (n - k)(p - 1)/p$ and $\beta < n(r - 1)/r$ satisfying

$$\alpha + \beta \geq 0$$

and the balance condition (4.8). Then, there holds

$$\left| \iint_{\mathbb{R}^{n-k} \times \mathbb{R}^n} \frac{f(x)g(y)}{|x|^a |x - y|^b} dxdy \right| \leq \|f\|_{L^p(\mathbb{R}^{n-k})} \|g\|_{L^r(\mathbb{R}^n)}$$

for any functions $f \in L^p(\mathbb{R}^{n-k})$ and $g \in L^r(\mathbb{R}^n)$.

**Proof.** It suffices to prove the inequality for non-negative functions $f$ and $g$. Depending on the size of $\alpha$ and $\beta$ we have three possible cases.

**Case 1.** Suppose that both $\alpha \geq 0$ and $\beta \geq 0$. In this case, we always have $|y|^{-\beta} \leq |y''|^{-\beta}$. Hence, we can apply $(4.9)_{k,\alpha,\beta}$ to obtain the desired inequality.

**Case 2.** Suppose that $\beta < 0 \leq \alpha$ In this case we have the estimate

$$|y|^{-\beta} \leq \frac{1}{\lambda} \left( |(x, 0)| + |(x - y', y'')| \right)^{\beta} \leq |x|^{-\beta} + |x - y|^{-\beta}.$$

Hence

$$\iint_{\mathbb{R}^{n-k} \times \mathbb{R}^n} \frac{f(x)g(y)}{|x|^a |x - y|^b} dxdy \leq \iint_{\mathbb{R}^{n-k} \times \mathbb{R}^n} \frac{f(x)g(y)}{|x|^a |x - y|^b} dxdy + \iint_{\mathbb{R}^{n-k} \times \mathbb{R}^n} \frac{f(x)g(y)}{|x|^a |x - y|^a} dxdy.$$

Now the right hand side of the preceding inequality can be estimated from above by using Theorem 4.9. To be more precise, the first integral can be estimated using $(4.9)_{k,\alpha,\beta,0}$ and the second integral can be estimated using $(4.9)_{k,\lambda,0}$ with $\lambda$ replaced by $\lambda + \beta$. Keep in mind that in order to apply $(4.9)_{k,\alpha,\beta}$ with $\beta = 0$, we necessarily have

$$\lambda < n - k/r$$

by seeing (1.8). This clearly indicates the fundamental difference between (4.7) and $(4.9)_{k,\alpha,\beta}$.

**Case 3.** Suppose that $\beta > 0 \geq \alpha$. As in Case 2, we easily get

$$|x|^{-\alpha} \leq \frac{1}{\lambda} \left( |(y', y'')| + |x - y', -y''| \right)^{\alpha} \leq |y|^{-\alpha} + |x - y|^{-\alpha},$$

for some appropriate $\lambda$. Therefore, we have

$$\iint_{\mathbb{R}^{n-k} \times \mathbb{R}^n} \frac{f(x)g(y)}{|x|^a |x - y|^b} dxdy \leq \iint_{\mathbb{R}^{n-k} \times \mathbb{R}^n} \frac{f(x)g(y)}{|x|^a |x - y|^b} dxdy + \iint_{\mathbb{R}^{n-k} \times \mathbb{R}^n} \frac{f(x)g(y)}{|x|^a |x - y|^a} dxdy.$$
which implies
\[
\int_{\mathbb{R}^n \times \mathbb{R}^{n-1}} f(x)g(y) \frac{1}{|x|^\alpha |y|^\beta} \, dxdy \leq \int_{\mathbb{R}^n \times \mathbb{R}^{n-1}} f(x)g(y) \frac{1}{|x|^\alpha |y|^\beta} \, dxdy + \int_{\mathbb{R}^n \times \mathbb{R}^{n-1}} f(x)g(y) \frac{1}{|x-y|^{\alpha+\beta}} \, dxdy.
\]
Keep in mind that we always have \(\beta \geq 0\) and \(\alpha + \beta \geq 0\). Hence, we can further estimate to get
\[
\int_{\mathbb{R}^n \times \mathbb{R}^{n-1}} f(x)g(y) \frac{1}{|x|^\alpha |y|^\beta} \, dxdy \leq \int_{\mathbb{R}^n \times \mathbb{R}^{n-1}} f(x)g(y) \frac{1}{|x|^\alpha |y|^\beta} \, dxdy + \int_{\mathbb{R}^n \times \mathbb{R}^{n-1}} f(x)g(y) \frac{1}{|x-y|^{\alpha+\beta}} \, dxdy.
\]
Now we can apply (1.10) to get
\[
\text{to get the desired inequality.}
\]

It follows from [Ngo21] that all conditions in Theorem 4.5 above are necessary. It is worth noticing that in Theorem 4.5 above, we require \(\lambda < n - k/r\).

4.5. **Characterization of any optimal pair** \((f^\sharp, g^\sharp)\). We now turn our attention to an optimal pair \((f^\sharp, g^\sharp)\) for the maximizing problem (1.11) found by Proposition 1.3 above. To gain further properties on \((f^\sharp, g^\sharp)\), it is often to study the Euler–Lagrange equation together with the constraint
\[
\|f\|_{L^p(\mathbb{R}^n)} = \|g\|_{L^r(\mathbb{R}^n)} = 1.
\]

Clearly, with respect to the function \(f\), the first variation of the functional \(F_{\lambda,k}^\beta\) is
\[
D_f(F_{\lambda,k}^\beta)(f,g)(h) = \int_{\mathbb{R}^{n-1}} \left( \int_{\mathbb{R}^n} \frac{g(y)}{|x-y|^{\alpha+\beta}} \, dy \right) h(x) \, dx
\]
while the first variation of the constraint \(\int_{\mathbb{R}^{n-1}} f(x) \, dx = 1\) is
\[
P \int_{\mathbb{R}^{n-1}} f(x)^{p-2} f(x) h(x) \, dx.
\]

Therefore, by the Lagrange multiplier theorem, there exists some constant \(\alpha\) such that
\[
\int_{\mathbb{R}^{n-1}} \left( \int_{\mathbb{R}^n} \frac{g(y)}{|x-y|^{\alpha+\beta}} \, dy \right) h(x) \, dx = \alpha \int_{\mathbb{R}^{n-1}} f(x)^{p-2} f(x) h(x) \, dx
\]
holds for all function \(h\) defined in \(\mathbb{R}^{n-k}\). From this we know that \(f\) and \(g\) must satisfy the following equation
\[
c_1 |f(x)|^{p-2} f(x) = \int_{\mathbb{R}^n} \frac{g(y)}{|x-y|^{\alpha+\beta}} \, dy
\]
for some constant \(c_1\). Interchanging the role of \(f\) and \(g\), we also know that \(f\) and \(g\) must fulfill the following
\[
c_2 |g(y)|^{r-2} g(y) = \int_{\mathbb{R}^n} \frac{f(x)}{|x-y|^{\alpha+\beta}} \, dx
\]
for some new constant \(c_2\). Using a suitable test function, we know from the constraint that
\[
c_1 = c_2 = F_{\lambda,k}^\beta(f,g).
\]

Hence, up to a constant multiple and simply using the following changes
\[
u = f^{p-1} \quad \text{and} \quad v = g^{r-1},
\]
the two relations above lead us to the following system of integral equations

\[
\begin{align*}
    u(x) &= \int_{\mathbb{R}^n} \frac{v^\kappa(y)}{|x - y|^{1 + \kappa + \beta}} dy, \\
v(y) &= \int_{\mathbb{R}^{n-k}} \frac{u^\theta(x)}{|x - y|^{1 + \kappa + \beta}} dx,
\end{align*}
\] (4.11)_{k,\beta}

with

\[
\kappa = \frac{1}{r - 1} > 0, \quad \theta = \frac{1}{p - 1} > 0.
\]

Using the balance condition (1.9), it is not hard to see that \(\kappa\) and \(\theta\) fulfill

\[
\frac{n - k}{n} \frac{1}{\theta + 1} + \frac{1}{\kappa + 1} = \frac{\lambda + \beta}{n}. \quad (4.12)
\]

In this sense, the condition (4.12) usually refers to the critical condition for (4.11)_{k,\beta}. From the above derivation, any optimal pair \((f^\sharp, g^\sharp)\) for the weighted HLS inequality (1.10)_{k,\beta} must solve the system (4.11)_{k,\beta}. Hence, we can naively ask the following:

- a regularity result for solutions to (4.11)_{k,\beta} with \(\lambda > 0\) and
- a classification for solutions to (4.11)_{k,\beta} with \(\lambda > 0\).

For the question of regularity, the two cases \(\beta \leq 0\) and \(\beta > 0\) could be different. For the question of classification, we expect that in the diagonal case, namely

\[
\frac{n - k}{n} \frac{1}{\theta + 1} + \frac{1}{\kappa + 1} = \frac{\lambda + \beta}{2n},
\]

standard method such as the method of moving spheres/planes could be used. As far as we know, there is no such a result for the above questions, even in the case \(k = 1\).

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Appendix A. Proof of Lemma 3.1

This appendix is devoted to the proof of Lemma 3.1, namely we shall prove the following inequality

\[
\int_{\mathbb{R}^n} (\mathcal{E}_{\lambda, k}^\beta f)(y) dy \leq \|f\|_{L^p}^{p - p} \int_{\mathbb{R}^{n-k}} f^p(x) dx
\]

for any \(f \in L^p(\mathbb{R}^{n-k})\), which can be assumed to be non-negative. As mentioned earlier, Lemma 3.1 also provides us another way to prove (1.10)_{k,\beta}. To prove Lemma 3.1, we mimic the proof of Adams for Riesz potentials; see [Ada75].

Recall from the definition of \(\mathcal{E}_{\lambda, k}^\beta f\) the following

\[
\mathcal{E}_{\lambda, k}^\beta f(y) = \int_{\mathbb{R}^{n-k}} \frac{f(x)}{|x - y|^{1 + \kappa + \beta}} dx.
\]
Hence, our starting point is the following
\[\|\mathcal{L}_{\lambda, k}^\rho [f]\|_{L^q(\mathbb{R}^n)} \leq \int_{\mathbb{R}^{n-k}} \int_0^{+\infty} \left[ \frac{\int_{\mathcal{B}_n^\rho(y')} f(x) dx}{\rho^{\lambda-k/q+\beta}} \right]^q d\rho dy', \quad (A.1)\]
thanks to (2.7). In the sequel, we prove that
\[\int_{\mathbb{R}^{n-k}} \int_0^{+\infty} \left[ \frac{\int_{\mathcal{B}_n^\rho(y')} f(x) dx}{\rho^{\lambda-k/q+\beta}} \right]^q d\rho dy' \leq \|f\|_{L^p(\mathbb{R}^{n-k})}^q \]
To this purpose, recall the following estimate
\[\int_{\mathcal{B}_n^\rho(y')} f(x) dx \leq \rho^{n-k}(M f)(y'), \]
where \(M f\) is the Hardy–Littlewood maximal function of \(f \geq 0\) in \(\mathbb{R}^{n-k}\), defined by
\[(M f)(z) = \sup_{r>0} \frac{1}{|\mathcal{B}_n^{n-k}(z, r)|} \int_{|x-z| \leq r} f(x) dx.\]
In addition, from the definition, we also have
\[\int_{\mathcal{B}_n^\rho(y')} f(x) dx \leq \rho^{\frac{n-k}{p}} \|f\|_p\]
for any \(\rho > 0\) and any \(y' \in \mathbb{R}^{n-k}\). Thus, for some \(\delta > 0\) to be determined later, we can estimate
\[\int_0^{+\infty} \left[ \frac{\int_{\mathcal{B}_n^\rho(y')} f(x) dx}{\rho^{\lambda-k/q+\beta}} \right]^q d\rho \]
\[\leq \int_0^{\delta} \left[ \frac{\rho^{n-k}(M f)(y')}{\rho^{\lambda-k/q+\beta}} \right]^q d\rho + \int_{\delta}^{+\infty} \left[ \frac{\rho^{(n-k)/p} \|f\|_p}{\rho^{\lambda-k/q+\beta}} \right]^q d\rho \]
\[\leq \delta^{(n-k\lambda-q-k\beta)} (M f)(y')^q + \delta^{(n-k\lambda-q-k\beta)} \frac{\|f\|_p^q}{\delta^{n-k\lambda-q-k\beta}}. \]
To obtain the last line in the above estimate, we also note that
\[\lambda - \frac{k}{q} + \beta - \frac{n-k}{p} > 0,\]
thanks to (1.9); otherwise, the integral \(\int_0^{+\infty}\) diverges. The trick is first to select \(\delta\) such that
\[\delta^{(n-k\lambda-q-k\beta)} (M f)(y')^q = \|f\|_p^{q+p} (M f)(y')^p \quad (A.2)\]
and then to select \(q\) such that
\[\delta^{(n-k\lambda-q-k\beta)} \frac{\|f\|_p^{q+p}}{\delta^{n-k\lambda-q-k\beta}} = \|f\|_p^{q+p} (M f)(y')^p. \quad (A.3)\]
Indeed, to fulfill (A.2), we simply choose
\[\delta = \left[ \frac{\|f\|_p}{(M f)(y')} \right]^{\frac{q+p}{n-k\lambda-q-k\beta}}. \]
From this choice of \(\delta\) we deduce
\[\delta^{(n-k\lambda-q-k\beta)} = \left[ \frac{\|f\|_p}{(M f)(y')} \right]^{q+p}.\]
which immediately implies (A.2). Notice that
\[
(n - \lambda - \beta)q - (q-1)k - q(n-k) \frac{p}{p'} = \frac{q}{q' - p'} \left[ (n - \lambda - \beta)q - (q-1)k \right] \frac{p}{p'}
\]
\[
= \frac{p}{q' - p'} \left[ (n - \lambda - \beta)q - (q-1)k \right]
\]
\[
\leq \frac{p}{q' - p'} \left[ (n - \lambda - \beta)q - (q-1)k \right]
\]
thanks to (1.9). Hence, we obtain
\[
\delta^{(n-\lambda-\beta)q - (q-1)k - q(n-k)} \left[ \frac{\|f\|_p}{(Mf)(y')^{p'}} \right]^{p'} = \left[ \frac{\|f\|_p}{(Mf)(y')^{p'}} \right]^{p'}
\]
which yields
\[
\delta^{(n-\lambda)q - (q-1)k - q(N-1)} \left[ \frac{\|f\|_p}{p} \right] = \left[ \frac{\|f\|_p}{p} \right]^{p'} (Mf)(y')^{p'},
\]
which is nothing but (A.3). Thus, we arrive at
\[
\int_0^{+\infty} \left[ \int_{\mathbb{R}^{n-k}} \frac{|f(x)| dx}{\rho^{k/q + \beta}} \right]^{q'} d\rho \leq \left[ \frac{\|f\|_p}{\rho} \right]^{p'} (Mf)(y')^{p'}.
\]
Since \(\|Mf\|_{L^p(\mathbb{R}^{n-k})} \leq \|f\|_{L^p(\mathbb{R}^{n-k})}\), we deduce from the preceding estimate the following
\[
\int_{\mathbb{R}^{n-k}} \int_0^{+\infty} \left[ \int_{\mathbb{R}^{n-k}} \frac{|f(x)| dx}{\rho^{k/q + \beta}} \right]^{q'} d\rho dy' \leq \left[ \frac{\|f\|_p}{\rho} \right]^{p'} \int_{\mathbb{R}^{n-k}} f^p dy'.
\]
Combining (A.1) and (A.4) gives the desired estimate.

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