\textbf{Abstract.} We study parabolically induced representations for $GSpin_m(F)$ with $F$ a $p$–adic field of characteristic zero. The Knapp-Stein $R$–groups are described and shown to be elementary two groups. We show the associated cocycle is trivial proving multiplicity one for induced representations. We classify the elliptic tempered spectrum. For $GSpin_{2n+1}(F)$, we describe the Arthur (Endoscopic) $R$–group attached to Langlands parameters, and show these are isomorphic to the corresponding Knapp-Stein $R$–groups.

\section*{Introduction}

We continue our study of parabolically induced representations for $p$–adic groups of classical type. Here we turn our attention to the group $GSpin_m(F)$, as defined by Asgari \cite{4}. These are groups of type $B_{[m/2]}$ if $m$ is odd and type $D_{m/2}$ if $m$ is even. A long term goal is to study the group $Spin_m(F)$, which is the simply connected split group of type $B$ or $D$, depending on whether $m$ is odd or even, respectively. The advantage of studying $GSpin$ groups is their Levi subgroups are nicer, making the problem more tractable. We hope to apply the information derived here to $Spin$ groups, and we leave this to further study.

Let $F$ be a nonarchimedean field of characteristic zero, and suppose $G$ is a connected reductive quasi-split group defined over $F$. We denote the $F$–points, $G(F)$, by $G$ and use this notational convention throughout this manuscript. The admissible dual of $G$ can be be studied through the theory of parabolically induced representations, as described in Harish-Chandra’s philosophy of cusp forms \cite{16}. Moreover, the discrete, tempered, and admissible spectra are classified through parabolic induction from supercuspidal, discrete series, and tempered representations (via the Langlands Classification) \cite{30}. One also wishes to divide the tempered spectrum into the elliptic classes, \cite{2}, which are those which contribute to the Placherel Formula, and the non-elliptic classes. For this purpose, we let $\mathcal{E}_c(G), \mathcal{E}_t(G), \mathcal{E}_2(G)$, and $^0\mathcal{E}(G)$ be the classes of irreducible admissible, tempered, discrete series, and unitary supercuspidal representations, respectively, of $G$. We make no distinction between a representation $\pi$ and its class $[\pi] \in \mathcal{E}_c(G)$. 

Let $\mathbf{P} = \mathbf{MN}$ be a parabolic subgroup of $\mathbf{G}$, and suppose $\sigma \in \mathcal{E}_2(M)$. We let $\text{Ind}_\mathbf{P}^\mathbf{G}(\sigma)$ or $i_{\mathbf{G},M}(\sigma)$ denote the representation of $\mathbf{G}$ obtained through normalized induction from $\mathbf{P}$, with $\sigma$ extended trivially from $M$ to $\mathbf{P}$. In the case of archimedean groups, Knapp and Stein developed the theory of standard and normalized intertwining operators (see [21], for example). Through a combinatorial study of the inductive properties of these normalized intertwining operators they were able to describe a finite group, $R(\sigma)$, whose representation theory classifies the components of $i_{\mathbf{G},M}(\sigma)$, in that there is a bijection $\rho \mapsto \pi_\rho$ from the irreducible representations $\hat{R}(\sigma)$ to the inequivalent components of $i_{\mathbf{G},M}(\sigma)$, in that there is a bijection $\rho \mapsto \pi_\rho$ from the irreducible representations $\hat{R}(\sigma)$ to the inequivalent components of $i_{\mathbf{G},M}(\sigma)$. More precisely, the intertwining algebra $\mathcal{C}(\sigma)$ of $i_{\mathbf{G},M}(\sigma)$ is isomorphic to the twisted group algebra $\mathbb{C}[R(\sigma)]_\eta$, with $\eta$ a particular 2–cocycle of $R(\sigma)$ arising from composition of intertwining operators [2, 20]. In the archimedean case, $R(\sigma)$ is always abelian (in fact an elementary 2–group), so each $\rho$ is a character and $\pi_\rho$ appears in $i_{\mathbf{G},M}(\sigma)$ with multiplicity one. Silberger [28, 29] extended the theory of $R$–groups to $p$–adic fields. Knapp and Zuckerman [22] showed there are cases when $R(\sigma)$ would be non-abelian, and hence the multiplicity of $\pi_\rho$ could be greater than one.

If $\mathbf{G} = \mathbf{G}_n = G\text{Spin}_{2n}$, or $G\text{Spin}_{2n+1}$, then any Levi subgroup is of the form

$$
\mathbf{M} \simeq GL_{n_1} \times \cdots \times GL_{n_r} \times G_m,
$$

with $n_1 + \cdots + n_r + m = n$. So, for any $\sigma \in \mathcal{E}_2(M)$ we have

$$
\sigma \simeq \sigma_1 \otimes \cdots \otimes \sigma_r \otimes \tau,
$$

with $\sigma_i \in \mathcal{E}_2(GL_{n_i}(F))$, and $\tau \in \mathcal{E}_2(G_m)$. The similarity between this situation and that of the classical groups $Sp_{2n}(F)$ and $SO_n(F)$, makes it amenable to the techniques of [12]. In fact we prove the $R$–groups have the same structure as these classical groups. Thus, our first main results can be phrased as $R$–groups for $G\text{Spin}$ groups mirror those for split classical groups (cf. Theorems 2.5 and 2.7). In particular, $R(\sigma) \simeq \mathbb{Z}^d$, for some $0 \leq d \leq r$.

Arthur [2] undertook the study of the elliptic spectrum, and was able to use the extension of $R(\sigma)$ defined by $\eta$ to characterize when components of $i_{\mathbf{G},M}(\sigma)$ have elliptic components. Herb [18] used this characterization, along with the description of the $R$–groups in [12], to determine the elliptic tempered spectrum of $Sp_{2n}(F)$ and $SO_n(F)$. Because the description of $R$–groups in our case is similar to that of [12], the techniques of [18] can be applied, and again the results are similar. To be more precise, the cocycle $\eta$ always splits and $i_{\mathbf{G},M}(\sigma)$ has elliptic components if and only if $d$ is as large as possible (this turns out to be $d = r$ or $r - 1$ cf. Lemma 3.1 and Theorems 3.3 and 3.4).

On the other hand, the local Langlands conjecture predicts a canonical bijection $\varphi \to \Pi_\varphi(G)$ between admissible homomorphisms $\varphi : W' \rightarrow L^G$ and $L$–packets $\Pi_\varphi(G)$ of $G$. Here, $W'_F$ is the
Weil-Deligne group, \( L^G = \hat{G} \times W_F \) is the Langlands \( L \)-group, with \( \hat{G} \) the connected Langlands dual group, and \( W_F \) is the Weil group. The \( L \)-packets \( \Pi_\varphi(G) \) are finite sets which partition \( \mathcal{E}_c(G) \), and the members of \( \Pi_\varphi(G) \) are to be \( L \)-indistinguishable, in the sense that the Langlands \( L \)-functions and \( \varepsilon \)-factors are to be constant on \( \Pi_\varphi(G) \).

If \( \sigma \in \mathcal{E}_2(M, \varphi) \) and \( \varphi : W'_F \to \mathcal{L}M \) is its Langlands parameter (i.e., \( \sigma \in \Pi_\varphi(M) \)), then composing with the inclusion \( \mathcal{L}M \hookrightarrow L^G \) gives an \( L \)-packet \( \Pi_\varphi(G) \), and the elements of this \( L \)-packet should be all components of \( i_{G,M}(\sigma') \), with \( \sigma' \in \Pi_\varphi(M) \). Langlands predicted the \( R \)-group, \( R(\sigma) \) should be encoded in this arithmetic information, and Arthur made this more precise in [1]. In particular, Arthur defined a finite group \( R_{\varphi,\sigma} \) attached to \( \varphi \) and \( \sigma \), and predicts \( R(\sigma) \simeq R_{\varphi,\sigma} \). This conjecture has been confirmed in many cases [6, 7, 9, 10, 14, 20, 27]. Here we are able to prove \( R(\sigma) \simeq R_{\varphi,\sigma} \) for \( \text{GSpin}_{2n+1} \) in several steps. The first is to reduce the isomorphism to the case where \( M = \text{maximal} \), and this we do in the wider context of split groups (cf. Lemma 4.1). Arthur identifies the stabilizer \( W(\sigma) \) of \( \sigma \) in the Weyl group with a subgroup \( W_{\varphi,\sigma} \) of a certain Weyl group in \( \hat{M} \). \( R(\sigma) \) can be realized as a quotient \( W(\sigma)/W' \) of \( W(\sigma) \), with \( W' \) the subgroup of \( W(\sigma) \) generated by root reflections in the zeros of the rank 1 Plancherel measures.

On the other hand \( R_{\varphi,\sigma} = W_{\varphi,\sigma}/W'_{\varphi,\sigma} \) is a quotient of \( W_{\varphi,\sigma} \), where \( W'_{\varphi,\sigma} \) is the intersection of \( W_{\varphi,\sigma} \) with another, smaller Weyl group. Thus, it is enough to show, under the isomorphism of \( W(\sigma) \) and \( W_{\varphi,\sigma} \) that \( W' \) is identified with \( W'_{\varphi,\sigma} \). Hence it is enough to show \( W'_{\varphi,\sigma} \) is generated by co-root reflections coming from the roots for which the Plancherel measures are zero. Shahidi [26] showed, in the generic case, the zeros of the rank 1 Plancherel measures are equivalent to poles of Langlands \( L \)-functions, \( L(s, \sigma, r_i) \), (where \( i = 1, 2 \) is determined in a particular way [25] and \( r_i \) is a representation of \( L^M \) coming from its adjoint representation). The local Langlands conjecture predicts \( L(s, \sigma, r_i) = L(s, r_i \circ \varphi) \), where the right hand side is the Artin \( L \)-function. We separate the proof of the isomorphism of Knapp-Stein and Arthur \( R \)-groups into two (maximal) cases, the Siegel parabolic subgroup, i.e \( M \simeq GL_n \times GL_1 \), and the non-Siegel maximal parabolic subgroups, \( M \simeq GL_k \times G_m \), with \( m \geq 2 \). The final results in these two cases can be found in Corollary 4.6 and Theorem 4.12. For the latter we need conjecture 9.4 of [26] (otherwise known as the Tempered \( L \)-packet Conjecture).

The structure and isomorphism of Knapp-Stein and Arthur \( R \)-groups plays a crucial role in the transfer of automorphic forms from classical to general linear groups in [3], and among the important results therein is a proof of the Tempered \( L \)-packet Conjecture in the case of classical groups. We expect if the methods of [3] can be extended to \( \text{GSpin} \) groups, then the isomorphism of \( R(\sigma) \) and \( R_{\varphi,\sigma} \) would play a similar role.
In Section 1 we recall the basic facts about the \( GSpin \) groups. In Section 2 we work to determine the zeros of the Plancherel measures and compute the \( R \)-groups for \( GSpin \) groups. In Section 3 we show the cocycle which, along with the \( R \)-group, determines the structure of \( i_{G,M}(\sigma) \) is a coboundary. We then use the results of Section 2 to classify the elliptic tempered spectra of \( GSpin \) groups. In Section 4 we prove the isomorphism of the Knapp-Stein and Arthur \( R \)-groups for the \( GSpin_{2n+1} \) groups.

1. Preliminaries

Let \( F \) be a local nonarchimedean field of characteristic zero. Let \( G = G_n = GSpin_{2n} \), or \( GSpin_{2n+1} \). We adopt the convention that \( G_0 = GL_1 \). Let \( H = Spin_{2n} \) or \( Spin_{2n+1} \). We recall the exact sequence

\[
1 \to \mathbb{Z}_2 \to H \to H' \to 1,
\]

where \( H' = SO_{2n} \) or \( SO_{2n+1} \). We have \( G \) and \( H \) are of type \( D_n \) in the first case and type \( B_n \) is the second case. Let \( \hat{G} \) be the connected component of the Langlands \( L \)-group. Then \( \hat{G} = GSO_{2n}(\mathbb{C}) \) if \( G = GSpin_{2n} \) and is \( GSp_{2n} \) if \( G = GSpin_{2n+1} \). Then since \( G \) is split, \( L^* = \hat{G} \times W_F \), with \( W_F \) the Weil group of \( F \). We fix \( B \) to be the Borel subgroup in \( G \) lying over the upper triangular Borel subgroup in \( H' \). Let \( B = TU \) be the Levi decomposition of \( B \). Let \( \Phi = \Phi(G, T) \) be the roots of \( T \) in \( G \), and let \( \Delta \) be the simple roots determined by \( B \). Then \( \Delta = \{ \alpha_1, \alpha_2, \ldots, \alpha_n \} \), where \( \alpha_i = e_i - e_{i+1} \), for \( i = 1, 2, \ldots, n - 1 \), and

\[
\alpha_n = \begin{cases} 
\varepsilon_{n-1} + \varepsilon_n & \text{if } G = GSpin_{2n}, \\
\varepsilon_n & \text{if } G = GSpin_{2n+1}.
\end{cases}
\]

Recall the Weyl group is \( W = W(G, T) = N_G(T)/T \). Note, if \( G = GSpin_{2n+1} \), then \( W \cong S_n \rtimes \mathbb{Z}_2 \), while if \( G = GSpin_{2n} \), we have \( W \cong S_n \rtimes \mathbb{Z}_2^{-1} \). One can compute this directly from the description in [5], or one can note that \( W(\hat{G}, \hat{T}) \) is of this form, and use duality. Taking this last approach, the description of these Weyl groups given in [13], which we summarize. Note

\[
\hat{T} = \{ \text{diag} \{ a_1, a_2, \ldots, a_n, \lambda a_n^{-1}, \ldots, \lambda a_2^{-1}, \lambda a_1^{-1} \} | a_i, \lambda \in \mathbb{C}^\times \}
\]

in either case. We may denote an element of \( \hat{T} \) by \( t(a_1, a_2, \ldots, a_n, \lambda) \). If \( s \in S_n \), then we also denote by \( \hat{s} \) a representative of the element of \( W(\hat{G}, \hat{T}) \) such that \( \hat{s}t(a_1, a_2, \ldots, a_n, \lambda)\hat{s}^{-1} = t(a_{s(1)}, a_{s(2)}, \ldots, a_{s(n)}, \lambda) \). If \( G = GSpin_{2n+1} \), then denote by \( \hat{c}_i \) a representative of the element of \( W(\hat{G}, \hat{T}) \) such that \( \hat{c}_i t(a_1, \ldots, a_i, \ldots, a_n, \lambda)\hat{c}_i^{-1} = t(a_1, \ldots, \lambda a_i^{-1}, \ldots, a_n, \lambda) \). Then \( W(\hat{G}, \hat{T}) \) is generated by \( \{ \hat{s} | s \in S_n \} \) and \( \{ \hat{c}_i | 1 \leq i \leq n \} \). If \( G = GSpin_{2n} \), then \( W(\hat{G}, \hat{T}) \) is generated by \( \{ \hat{s} | s \in S_n \} \) and \( \{ c_i c_j | 1 \leq i, j \leq n \} \).
Let $P = MN \supset B$ be a standard parabolic subgroup of $G$. Then, for some $\theta \subset \Delta$ we have $P = P_{\theta} = M_0 N_{\theta}$. Then there is a partition $n = n_1 + n_2 + \cdots + n_r + m$, so that $\theta = \Delta \setminus \{\alpha_{n_1}, \alpha_{n_1+n_2}, \ldots, \alpha_{n_1+n_2+\cdots+n_r} \}$, if $m = 0$, and $\theta = \Delta \setminus \{\alpha_{n_1}, \alpha_{n_1+n_2}, \ldots, \alpha_{n_1+n_2+\cdots+n_r} \}$, if $m > 0$. Then

\begin{equation}
M \simeq GL_{n_1} \times GL_{n_2} \times \cdots \times GL_{n_r} \times G_m.
\end{equation}

Let $A$ be the split component of $P$, and let $\Phi(P, A)$ be the reduced roots of $A$ in $P$. For $i = 1, 2, \ldots, r$, we set $k_i = n_1 + \cdots + n_i$. Then, for $1 \leq i < j \leq r$, set $\alpha_{ij} = e_{k_i} - e_{k_{j-1} + 1}$, and $\beta_{ij} = e_{k_i} + e_{k_{j-1} + 1}$, and

\[ \gamma_i = \begin{cases} 
  e_{k_i} + e_n & \text{if } G = GSpin_{2n}; \\
  e_{k_i} & \text{if } G = GSpin_{2n+1}.
\end{cases} \]

We describe the relative Weyl group $W_M = N_G(A_M)/Z_G(A_M) = N_G(A_M)/M$. Suppose $M$ is as above. As in the case of other groups of classical type, $W_M \subset S_r \ltimes \mathbb{Z}_2^r$. If $G$ is of type $B_n$, then $W_M \simeq S \ltimes \mathbb{Z}_2^r$, for some subgroup $S$ of $S_r$. In fact $S = \langle (ij) | i < j, n_i = n_j \rangle$. More precisely, let $k_0 = 0$, and for $i = 1, 2, \ldots, r - 1$, let $k_i$ be as above. If $n_i = n_j$, let $[ij] \in W(G, T)$ be the element $\prod_{k=1}^{n_i} (k_{i-1} + k)$. Then $[ij] \mapsto (ij)$ gives an isomorphism of $W_M \cap S_n$ to $S$. We generally denote these elements by the more standard $(ij)$. For $1 \leq i < r$, we set $C_i = \prod_{k=1}^{n_i} c_{k_i + 1 + k}$. We call $C_i$ a block sign change, and $\langle C_i | i = 1, \ldots, r \rangle \simeq \mathbb{Z}_2^r$ is the sign change subgroup of $W_M$. The action of $S$ on $M$ is given by

\[ (ij) : (g_1, \ldots, g_r, h) = (g_1, \ldots, g_{i-1}, g_j, g_{i+1}, \ldots, g_{j-1}, g_i, \ldots, g_r, h). \]

Also, from the action of $C_i$ on the root datum of $G$ (see [4]) we have $C_i \cdot (g_1, \ldots, g_i, \ldots, g_r, h) = (g_1, \ldots, g_i^{-1}, \ldots, g_r, e_0(\det g_i)h)$. If $G$ is of type $D_n$, then $W_M \simeq S \ltimes C$, where $S$ is as above for type $B_n$, and $C \subset \mathbb{Z}_2^r$. If $m = 0$, then we have $C = C_1 \times C_2$, where $C_1 = \langle C_i | n_i \text{ is even} \rangle$, and $C_2 = \langle C_i C_j | n_i, n_j \text{ are odd} \rangle$. If $m > 0$, then $C \simeq \mathbb{Z}_2^r$, and

\[ C = \langle C_i | n_i \text{ is even} \rangle \times \langle C_i c_n | n_i \text{ is odd} \rangle. \]

We note that $S$ and each $C_i$ acts as in the case of type $B_n$, (and of course $C_i C_j$ acts as the product in type $D_n$). In the case of $m > 0$ and $n_i$ odd, we have $C_i c_n \cdot (g_1, \ldots, g_i, \ldots, g_r, h) = \ldots$
\((g_1, \ldots, g_i^{-1}, \ldots, g_r, (\det g_i)(c_n \cdot h))\), where \(c_n\) is given by the outer automorphism on the Dynkin diagram of \(G_m\).

### 2. R-groups for \(G\text{Spin}\)

We continue with the notation of the previous section. Let \(M\) be a Levi subgroup of \(G = G_n\) and assume \(M\) is of the form \([1.1]\). Let \(\sigma \in \mathcal{E}_2(M)\). Then \(\sigma \simeq \sigma_1 \otimes \sigma_2 \cdots \otimes \sigma_r \otimes \tau\), where \(\sigma_i \in \mathcal{E}_2(GL_{n_i}(F))\), and \(\tau \in \mathcal{E}_2(G_m)\). For \(\alpha \in \Phi(P, A)\), we set \(A_\alpha = (A \cap \ker \alpha)^\circ\), and \(M_\alpha = Z_G(A_\alpha)\).

Then \(\star P_\alpha = P \cap M_\alpha = MN_\alpha\), where \(N_\alpha = N \cap M_\alpha\) is a maximal parabolic subgroup of \(M_\alpha\) with Levi component \(M\). We let \(W_\alpha = W(M_\alpha, A)\). If \(W_\alpha \neq \{1\}\), we let \(w_\alpha\) be the unique nontrivial element of \(W_\alpha\). We recall the Plancherel measure, \(\mu_\alpha(\sigma)\) is determined by the standard intertwining operator attached to \(\text{Ind}_{M_\alpha}^{\star P_\alpha}(\sigma)\), and in particular, \(\mu_\alpha(\sigma) = 0\) if and only if \(w_\alpha \sigma \simeq \sigma\) and \(\text{Ind}_{M_\alpha}^{\star P_\alpha}(\sigma)\) is irreducible.

We note if \(\alpha = \alpha_{ij}\), then

\[
(2.1) \quad M_\alpha \simeq \prod_{k \neq i, j} GL_{n_k} \times GL_{n_i + n_j} \times G_m,
\]

and

\[
W_\alpha = \begin{cases} 
1 & \text{if } n_i \neq n_j; \\
\{1, (ij)\} & \text{if } n_i = n_j.
\end{cases}
\]

If \(\alpha = \beta_{ij}\), then \(M_\alpha \simeq M_{\alpha_{ij}}\) is again given by \((2.1)\), and

\[
W_\alpha = \begin{cases} 
1 & \text{if } n_i \neq n_j; \\
\{1, (ij)C_iC_j\} & \text{if } n_i = n_j.
\end{cases}
\]

Finally, for \(\alpha = \gamma_i\), we have

\[
M_\alpha \simeq \prod_{k \neq i} GL_{n_k} \times G_{n_i + m}.
\]

If \(G\) is of type \(B_n\), or \(n_i\) is even, then \(W_\alpha = \{1, C_i\}\). If \(G\) is of type \(D_n\), and \(n_i\) is odd, then

\[
W_\alpha = \begin{cases} 
C_iC_n & \text{if } m > 0; \\
1 & \text{if } m = 0.
\end{cases}
\]
We note, for $G$ of type $B_n$,

$$w \alpha \sigma \simeq \begin{cases} 
\sigma_1 \otimes \cdots \otimes \sigma_{i-1} \otimes \sigma_j \otimes \sigma_{i+1} \cdots \otimes \sigma_{j-1} \otimes \sigma_i \otimes \cdots \otimes \sigma_r \otimes \tau; \\
\sigma_1 \otimes \cdots \otimes \sigma_{i-1} \otimes (\tilde{\sigma}_j \otimes \omega_r) \otimes \sigma_{i+1} \cdots \otimes \sigma_{j-1} \otimes (\tilde{\sigma}_i \otimes \omega_r) \otimes \sigma_{j+1} \cdots \otimes \sigma_r \otimes \tau; \\
\sigma_1 \otimes \cdots \otimes \sigma_{i-1} \otimes (\tilde{\sigma}_i \otimes \omega_r) \otimes \cdots \otimes \sigma_r \otimes \tau,
\end{cases}$$

if $\sigma = \alpha_{ij}, \beta_{ij},$ or $\gamma_i$, respectively. Here $\omega_r$ is the central character of $\tau$ restricted to the identity component of the center of $G_m$. For type $D_n$, the result is as above, except in the case where $\alpha = \gamma_i$, $n_i$ is odd and $m > 0$, in which case

$$w \alpha \sigma \simeq \sigma_1 \otimes \cdots \otimes (\tilde{\sigma}_i \otimes \omega_r) \otimes \cdots \otimes \sigma_r \otimes (c_n \cdot \tau).$$

**Lemma 2.1.** For $1 \leq i < j \leq r - 1$ we have $\text{Ind}_{P_{n_{ij}}}^M(\sigma)$ is irreducible. Similarly $\text{Ind}_{P_{\beta_{ij}}}^M(\sigma)$ is irreducible.

**Proof.** Let $\alpha = \alpha_{ij}$. In this case $M_\alpha$ is given by (2.1). Let $Q_{ij}$ be the standard $GL_{n_i} \times GL_{n_j}$–parabolic subgroup of $GL_{n_i+n_j}$. Then,

$$\text{Ind}_{P_{n_{ij}}}^M(\sigma) \simeq \left( \bigotimes_{\ell \neq i,j+1} \sigma_\ell \right) \otimes \left( \text{Ind}_{Q_{ij}}^{GL_{n_i+n_j}(F)}(\sigma_i \otimes \sigma_j) \right) \otimes \tau,$$

and the result now follows from Olsanskii or Bernstein and Zelevinski [8, 24].

If $\alpha = \beta_{ij}$, then we again have $M_\alpha$ is given by (2.1), and in this case

$$\text{Ind}_{P_{\beta_{ij}}}^M(\sigma) \simeq \left( \bigotimes_{\ell \neq i,j} \sigma_\ell \right) \otimes \left( \text{Ind}_{Q_{ij}}^{GL_{n_i+n_j}(F)}(\sigma_i \otimes (\tilde{\sigma}_j \otimes \omega_r)) \right) \otimes \tau.$$

Thus, the result again follows from [8, 24]. \qed

From this we derive the following result.

**Corollary 2.2.** If $\alpha = \alpha_{ij}$, then $\mu_\alpha(\sigma) = 0$ if and only if $n_i = n_j$ and $\sigma_i \simeq \sigma_j$. If $\alpha = \beta_{ij}$, then $\mu_\alpha(\sigma) = 0$ if and only if $n_i = n_j$ and $\sigma_i \simeq \tilde{\sigma}_j \otimes \omega_r$.

**Lemma 2.3.** Let $\sigma = \sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_r \otimes \tau \in E_2(M)$, and let $R = R(\sigma)$. Suppose $w \in R$ and $w = sc$, with $s \in S_r$ and $c \in \mathbb{Z}_r^\times$. Then $s = 1$.

**Proof.** This is a Keys argument as defined in [12] and introduced in [19]. Since the sign changes act independently on the disjoint cycles of $s$, we may suppose, without loss of generality, that $s = (12 \cdots j)$. Furthermore, if $G$ is of type $B_n$, then up to conjugation by sign changes we may
assume \( c = C_j c' \), or \( c = c' \), with \( c' \) not changing signs among \( 1, 2, \ldots, j \). If \( G \) is of type \( D_n \), then we may assume \( c \) is either of the same form, or of the form \( C_{j-1} C_j c' \), with \( c' \) changing no signs among \( 1, 2, \ldots, j \). If \( c \) changes no (block) signs among \( 1, 2, \ldots, j \), then we note that \( \sigma_1 \simeq \sigma_2 \simeq \cdots \simeq \sigma_j \). So, in particular \( \alpha_{ij} \in \Delta' \) and \( w(\alpha_{ij}) = -\alpha_{12} < 0 \), so \( w \notin R(\sigma) \). If \( c = C_j c' \), then \( \sigma_1 \simeq \sigma_2 \simeq \cdots \simeq \sigma_{j-1} \simeq \sigma_j \simeq (\tilde{\alpha}_1 \otimes \omega_{\tau}) \) and thus \( \beta_{1j} \in \Delta' \). However, \( w\beta_{1j} = -\alpha_{12} < 0 \), so \( w \notin R \). Finally, if \( c = C_i C_j c' \), then \( w\sigma \simeq \sigma \) implies \( \sigma_1 \simeq \sigma_2 \simeq \cdots \simeq \sigma_{j-1} \simeq (\tilde{\alpha}_j \otimes \omega_{\tau}) \), and therefore, again, \( \beta_{1j} \in \Delta' \), with \( w\beta_{1j} = -\alpha_{12} < 0 \), showing \( w \notin R \).

\[ \square \]

**Corollary 2.4.** For \( G = GSpin_{2n} \) or \( GSpin_{2n+1} \), we have \( R \subset \mathbb{Z}_2^\circ \).

We let \( W(\sigma) = \{ w \in W(G, A_M) \mid w\sigma \simeq \sigma \} \). If \( G \) is of type \( B_n \), and \( W(\sigma) \neq 1 \), then one of the following holds:

\begin{align*}
(2.2) & \quad \sigma_i \simeq \sigma_j \text{ for some } i \neq j; \\
(2.3) & \quad \sigma_i \simeq \tilde{\sigma}_j \otimes \omega_{\tau} \text{ for some } i \neq j; \text{ and} \\
(2.4) & \quad \sigma_i \simeq \tilde{\sigma}_i \otimes \omega_{\tau}.
\end{align*}

Note that (2.2) holds if \((ij) \in W(\sigma)\), (2.3) holds if \((ij) C_i C_j \in W(\sigma)\), while (2.4) holds if \( C_i \in W(\sigma) \). Also notice if \( w = (ij) C_i \in W(\sigma) \), then \( w^2 = C_i C_j \in W(\sigma) \), so this case is covered by (2.4).

For \( w \in W(G, A) \), we let \( R(w) = \{ \alpha \in \Phi(P, A) \mid w\alpha < 0 \} \).

For \( B \subset \{ 1, 2, \ldots, r \} \), we let \( C_B = \prod_{i \in B} C_i \). If \( C_B \in R(\sigma) \), then \( R(C_B) \cap \Delta' = \emptyset \). Note that

\[ R(C_B) = \{ \alpha_{ij}, \beta_{ij} \mid i \in B, i < j \} \cup \{ \gamma_i \mid i \in B \} \] .

We let \( Q_i \) be the standard \( GL_{n_i} \times G_m \) parabolic subgroup of \( G_{n_i+m} \).

**Theorem 2.5.** Let \( G = GSpin_{2n+1} \) and \( M \simeq GL_{n_1} \times \cdots \times GL_{n_r} \times G_m \), with \( \sum_{i} n_i = n \). Let \( \sigma \simeq \sigma_1 \otimes \cdots \otimes \sigma_r \otimes \tau \in E_2(M) \), with each \( \sigma_i \in E_2(GL_{n_i}(F)) \) and \( \tau \in E_2(G_m) \). Let \( d \) be the number of nonequivalent classes among \( \{ \sigma_1, \ldots, \sigma_r \} \) for which \( \text{Ind}_{Q_i}^{G_{n_i+m}}(\sigma_i \otimes \tau) \) is reducible. Then \( R(\sigma) \simeq \mathbb{Z}_2^d \).

More precisely, let

\[ \Omega(\sigma) = \left\{ i \mid \text{Ind}_{Q_i}^{G_{n_i+m}}(\sigma_i \otimes \tau) \text{ is reducible, and } \sigma_j \not\simeq \sigma_i \text{ for all } j > i \right\} . \]

Then \( R(\sigma) = \langle C_i \rangle_{i \in \Omega(\sigma)} \).
Remark 2.6. By Bruhat Theory we know if $\text{Ind}_{Q_i}^{G_{n_i}+m}(\sigma_i \otimes \tau)$ is reducible implies $C_i \in W(\sigma)$, so $\sigma_i \simeq \bar{\sigma}_i \otimes \omega_\tau$.

**Proof.** From Corollary 2.4 we know $R \subseteq \{C_1, \ldots, C_r\} \simeq \mathbb{Z}_2^r$. Suppose $B \subseteq \{1, 2, \ldots, r\}$, with $C_B \in R(\sigma)$. Then $C_B \in W(\sigma)$, so $\sigma_i \simeq \bar{\sigma}_i \otimes \omega_\tau$, for all $i \in B$. Thus, for each $i \in B$, we have $C_i \in W(\sigma)$. Since $R(C_i) \subset R(C_B)$, and $R(C_B) \cap \Delta' = \emptyset$, we have $R(C_i) \cap \Delta' = \emptyset$. So $C_i \in R(\sigma)$. Therefore, for some subset, $\Omega$, of $\{1, 2, \ldots, r\}$ we have $R(\sigma) = \langle C_i \mid i \in \Omega \rangle$. Now suppose $C_i \in R(\sigma)$. For each $j > i$, we have $\bar{\sigma}_i \otimes \omega_\tau$. However, since $\sigma_i \simeq \bar{\sigma}_i \otimes \omega_\tau$, we see $\bar{\sigma}_i \otimes \omega_\tau$. Thus, finally, since $\gamma_i \in R(C_i)$, we must have $\gamma_i \notin \Delta'$. We note

$$M_{\gamma_i} \simeq \prod_{j \neq i} GL_{n_j} \times G_{n_i+n_j},$$

and

$$\text{Ind}_{M_{\gamma_i}} \sigma \simeq \bigotimes_{j \neq i} \sigma_j \otimes \text{Ind}_{Q_i}^{G_{n_i}+m}(\sigma_i \otimes \tau).$$

Since $C_i \in W(\sigma) \cap W_{\gamma_i}$, we have $\gamma_i \notin \Delta'$ if and only if $\text{Ind}_{Q_i}^{G_{n_i}+m}(\sigma \otimes \tau)$ is reducible. Thus, $i \in \Omega(\sigma)$, so $\Omega \subset \Omega(\sigma)$. Conversely, if $i \in \Omega(\sigma)$, then $C_i \sigma \simeq \sigma$, and $R(C_i) \cap \Delta' = \emptyset$, so $C_i \in \Omega$. Thus $\Omega = \Omega(\sigma)$, and $R(\sigma)$ has the form we claim. □

Now suppose $G$ is of type $D_n$. Let $M \simeq GL_{n_1} \times \cdots \times GL_{n_t} \times G_m$. We may assume $n_i$ is even for $i = 1, 2, \ldots, t$, and $n_i$ is odd for $i = t+1, \ldots, r$. If $m = 0$, then

$$C \simeq \begin{cases} \mathbb{Z}_2^{r-1} & \text{if } t < r; \\ \mathbb{Z}_2 & \text{otherwise.} \end{cases}$$

If $m > 0$, then $C \simeq \mathbb{Z}_2^r$, as described above. If $m = 0$ or $c_\tau \neq \tau$, then the following describes the conditions under which $W(\sigma) \neq \{1\}$:

(2.5) $\sigma_i \simeq \sigma_j$ for some $i \neq j$;

(2.6) $\sigma_i \simeq \bar{\sigma}_j \otimes \omega_\tau$ for some $i \neq j$;

(2.7) $\sigma_i \simeq \bar{\sigma}_i \otimes \omega_\tau$ for some $i$ with $n_i$ even;

(2.8) $\sigma_i \simeq \bar{\sigma}_i \otimes \omega_\tau$ and $\sigma_j \simeq \bar{\sigma}_j \otimes \omega_\tau$ for some $i \neq j$ with $n_i, n_j$ odd.

We have (2.5) holding if and only if $(ij) \in W(\sigma)$, (2.6) holds if and only if $(ij)C_iC_j \in W(\sigma)$, while (2.7) and (2.8) are the conditions for either $C_i$ (for $n_i$ even) or $C_iC_j$ to be in $W(\sigma)$. If $m > 0$ and
$c_n \tau \simeq \tau$, then (2.5), (2.6), and (2.7) are the conditions, with the restriction on parity removed from (2.7).

**Theorem 2.7.** Let $G = G_{\text{spin}}_{2n}$, and $M \simeq GL_{n_1} \times \cdots \times GL_{n_r} \times G_m$, with $m + \sum n_i = n$. Let $\sigma \in \mathcal{E}_2(M)$, with each $\sigma_i \in \mathcal{E}_2(GL_{n_i}(F))$, and $\tau \in \mathcal{E}_2(G_m)$.

(i) If $m = 0$ or $c_n \tau \not\simeq \tau$, then we let

$$\Omega_1(\sigma) = \{1 \leq i \leq r | n_i \text{ is even, } \text{Ind}_{Q_i}^{G_{n_i}+m}(\sigma_i \otimes \tau) \text{ is reducible, and } \sigma_j \not\simeq \sigma_i \text{ for all } j > i\},$$

and

$$\Omega_2(\sigma) = \{1 \leq i \leq r | n_i \text{ is odd, } \sigma_i \simeq \tilde{\sigma}_i \otimes \omega_\tau, \text{ and } \sigma_j \not\simeq \sigma_i \text{ for all } j > i\}.$$

We set $d_i = |\Omega_i(\sigma)|$, for $i = 1, 2$. Then $R(\sigma) \simeq \mathbb{Z}_2^{d_1 + d_2 - 1}$, unless $d_2 = 0$, in which case $R(\sigma) \simeq \mathbb{Z}_2^{d_1}$. More precisely,

$$R(\sigma) = \langle C_i | i \in \Omega_1(\sigma) \rangle \times \langle C_i C_j | i, j \in \Omega_2(\sigma) \rangle.$$

(ii) If $m > 0$ and $c_n \tau \simeq \tau$, we let

$$\Omega(\sigma) = \{1 \leq i \leq r | \text{Ind}_{Q_i}^{G_{n_i}+m}(\sigma_i \otimes \tau) \text{ is reducible, and } \sigma_j \not\simeq \sigma_i \text{ for all } j > i\}.$$

Let $d = |\Omega(\sigma)|$. Then $R(\sigma) \simeq \mathbb{Z}_2^d$, and in particular,

$$R(\sigma) = \langle C_i | i \in \Omega(\sigma) \text{ and } n_i \text{ is even} \rangle \times \langle C_i c_n | i \in \Omega(\sigma) \text{ and } n_i \text{ is odd} \rangle.$$

**Proof.** We assume $n_i$ is even for $i = 1, 2, \ldots, t$, and $n_i$ is odd for $i = t + 1, \ldots, r$. Suppose $m = 0$. Then $W_M = S \ltimes C$, where

$$C = \langle C_i | 1 \leq i \leq t \rangle \times \langle C_i C_j | t + 1 \leq i, j \leq r \rangle.$$

By Corollary 2.3, $R(\sigma) \subset C$. Suppose $B \subset \{1, 2, \ldots, r\}$. Then we let $B_1 = B \cap \{1, 2, \ldots, t\}$, and $B_2 = B \setminus B_1$. Suppose $C_B = \prod_{i \in B} C_i \subset R(\sigma)$. Then $\sigma_i \simeq \tilde{\sigma}_i \otimes \omega_\tau$, for each $i \in B$. Thus, $C_i \in W(\sigma)$, for each $i \in B_1$, and $C_i C_j \in W(\sigma)$, for each $i, j \in B_2$. As in the case of type $B_n$, we have, for each $i \in B$, $R(C_i) \subset R(C_B)$, and thus $C_i \in R(\sigma)$, for each $i \in B_1$, and $C_i C_j \in R(\sigma)$ for each $i, j \in B_2$. Thus, there is some $\Omega \subset \{1, \ldots, r\}$, for which

$$R(\sigma) = \langle C_i | i \in \Omega_1 \rangle \times \langle C_i C_j | i, j \in \Omega_2 \rangle.$$

For $1 \leq i \leq t$, we have

$$R(C_i) = \{\gamma_i\} \cup \{\alpha_{ij}, \beta_{ij} \}_{j>i}.$$
We have \( R(C_i) \cap \Delta' = \emptyset \), so by Corollary 2.2 \( \sigma_j \not\simeq \sigma_i \) for all \( j > i \), as in the case of type \( B_n \). Further note, since \( C_i \in W(\sigma) \), we have \( \gamma_i \in \Delta' \) if and only if \( \text{Ind}_{P_{\gamma_i}}^{M_{\beta_i}} \sigma \) is irreducible. Since

\[
\text{Ind}_{P_{\gamma_i}}^{M_{\beta_i}} \sigma \simeq \left( \bigotimes_{j \neq i} \sigma_i \right) \otimes \text{Ind}_{Q_{\gamma_i}}^{G_{\gamma_i} \oplus m} (\sigma_i \otimes \tau),
\]

we see \( C_i \in R(\sigma) \) implies \( \text{Ind}_{Q_{\gamma_i}}^{G_{\gamma_i} \oplus m} (\sigma_i \otimes \tau) \) is reducible. Thus, \( i \in \Omega_1(\sigma) \). Therefore, we have \( \Omega_1 \subset \Omega_1(\sigma) \). However, the reverse inclusion is now obvious.

Now suppose \( i, j \geq t + 1 \), and \( C_iC_j \in R(\sigma) \). Then we have noted \( \sigma_i \simeq \tilde{\sigma}_i \otimes \omega_\tau \), and \( \sigma_j \simeq \tilde{\sigma}_j \otimes \omega_\tau \). Note further,

\[
R(C_iC_j) = \{ \gamma_i, \gamma_j \} \cup \{ \alpha_{ik}, \beta_{ik} \}_{k > i} \cup \{ \alpha_{jt}, \beta_{jt} \}_{t > j}.
\]

As above, this now implies \( \sigma_i \not\simeq \sigma_k \), for \( k > i \), and \( \sigma_j \not\simeq \sigma_\ell \), for \( \ell > j \). Thus, we see \( i, j \in \Omega_2(\sigma) \), so \( \Omega_2 \subset \Omega_2(\sigma) \). For the opposite inclusion we note, \( W_{M_{\gamma_i}} = \{ 1 \} = W_{M_{\gamma_j}} \), and hence \( \gamma_i, \gamma_j \not\in \Delta' \). Thus, if \( i, j \in \Omega_2(\sigma) \), then \( C_iC_j \in R(\sigma) \). Therefore, \( R(\sigma) \) has the form we claim.

If \( m > 0 \) and \( c_n \tau \not\simeq \tau \), then the argument above is essentially valid with the following adjustments. We note \( W_M = S \ltimes C \), with

\[
(2.9) \quad C = (C_i | 1 \leq i \leq t) \times (C_i c_n | i > t),
\]

and since \( c_n \tau \not\simeq \tau \), we have \( C_i c_n \not\in W(\sigma) \), for \( i > t \). Also, we note for \( i > t \), \( W_{M_{\gamma_i}} = \{ 1, C_i c_n \} \), so \( W_{M_{\gamma_i}} \cap W(\sigma) = \{ 1 \} \), and again we must have \( \gamma_i \not\in \Delta' \).

(ii) Now suppose \( m > 0 \) and \( c_n \tau \simeq \tau \). We still have \( W_M = S \ltimes C \), with \( C \) given by (2.9). For \( i = 1, 2, \ldots, r \), we let

\[
\bar{C}_i = \begin{cases} C_i & \text{if } i \leq t; \\ C_i c_n & \text{if } i > t. \end{cases}
\]

If \( B \subset \{ 1, 2, \ldots, r \} \), and \( \bar{C}_B = \prod_{i \in B} \bar{C}_i \in R(\sigma) \), then \( \sigma_i \simeq \tilde{\sigma}_i \otimes \omega_\tau \), for each \( i \in B \). So \( \bar{C}_i \in W(\sigma) \), for each \( i \in B \). Further,

\[
R(\bar{C}_B) = \bigcup_{i \in B} R(\bar{C}_i),
\]

so \( \bar{C}_i \in R(\sigma) \) for each \( i \in B \). Thus, there is some \( \Omega \subset \{ 1, 2, \ldots, r \} \) such that \( R(\sigma) = \langle \bar{C}_i | i \in \Omega \rangle \). Since

\[
R(\bar{C}_i) = \{ \alpha_{ij}, \beta_{ij} \}_{j > i} \cup \{ \gamma_i \},
\]
and, given $\tilde{C}_i \in W(\sigma)$, we have $\alpha_{ij}, \beta_{ij} \in \Delta'$ if and only if $\sigma_i \simeq \sigma_j$. Further, as above, $\gamma_i \in \Delta$ if and only if $\tilde{C}_i \in W(\sigma)$, and $\text{Ind}_{Q_i}^{G_{n+m}}(\sigma \otimes \tau)$ is irreducible. Thus,

$$\Omega = \{ i | \text{Ind}_{Q_i}^{G_{n+m}}(\sigma \otimes \tau) \text{ is reducible, and } \sigma_j \not\simeq \sigma_i, \text{ for all } j > i \} = \Omega(\sigma),$$

as claimed. \hfill \qed

3. Elliptic Representations for $GSpin$ Groups

We now consider the question of which tempered representations of $G = GSpin_n(F)$ are elliptic. We can adapt the arguments of [15] to our current situation. We let $G_e$ be the set of regular elliptic elements of $G$. If $\pi$ is an irreducible representation of $G$, then we denote by $\Theta_\pi$ its character. By Harish-Chandra [15] we know $\Theta_\pi$ is given by a locally integrable function, also denoted $\Theta_\pi$, on the regular set. We let $\Theta_{\pi e}$ be the restriction of $\Theta_\pi$ to $G_e$. Then $\pi \in \mathcal{E}_e(G)$ is elliptic if $\Theta_{\pi e} \neq 0$.

We begin by showing the 2-cocyle arising from constructing self intertwining operators in $\mathcal{C}(\sigma)$ is a coboundary. Let $G_n = GSpin_{2n}$ or $GSpin_{2n+1}$. Suppose $M \simeq GL_{n_1} \times \cdots \times GL_{n_r} \times G_m$ is a proper Levi subgroup of $G$. Let $\sigma \simeq \sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_r \otimes \tau$ be an irreducible discrete series of $M$.

Let $V$ be the space of the representation $\sigma$. For each $w \in R(\sigma)$, we choose an intertwining operator $T_w : V \to V$ so that $T_w \circ w\sigma = \sigma \circ T_w$.

**Lemma 3.1.** We can choose the operators $T_w$ so that $T_{w_1 w_2} = T_{w_1} T_{w_2}$.

**Proof.** For each $i$, we let $V_i$ be the space of the representation $\sigma_i$. So $V = V_1 \otimes \cdots \otimes V_r \otimes V_\tau$. Denote by $\sigma_i^*$ the representation on $V_i$ given by $\sigma_i^*(g) = \sigma_i((g)^{-1})$. By the work of Gelfand and Kazhdan [11], we know $\sigma_i^* \simeq \tilde{\sigma}_i$. Let $\mathcal{B}(\sigma) = \{ i | \sigma_i \simeq \tilde{\sigma}_i \otimes \omega_\tau \}$. For each $i \in \mathcal{B}(\sigma)$, we choose an intertwining operator $T_i : V_i \to V_i$, with $T_i(\sigma_i^* \otimes \omega_\tau) = \sigma T_i$. We note $T_i^2$ is a scalar on $V_i$, and so we can choose $T_i$ so that $T_i^2 = 1$. Extend this to an operator on $V$, by setting $T_i^V$ to be trivial on each factor, except for $V_i$, where it is $T_i$. Now $T_i^V \circ C_i \sigma = \sigma T_i^V$, and $(T_i^V)^2 = \mathrm{Id}$. Also note, for $i \neq j$, we have $T_i^V T_j^V = T_j^V T_i^V$.

If $G$ is of type $D_n$, and $c_n \tau \simeq \tau$, we choose $T_\tau$ intertwining $\tau$ and $c_n \tau$, again with $T_\tau^2 = \mathrm{Id}$. Extend $T_\tau$ to $V$ by setting $T_\tau^V$ to be trivial on each $V_i$ and to be $T_\tau$ on $V_\tau$. Suppose $B \subset \mathcal{B}(\sigma)$, and that

$$w = C_B = \prod_{i \in B} C_i \in R(\sigma).$$

Then, we set

$$T_w = \prod_{i \in B} T_i^V.$$
In the case where $G$ is of type $D_n$ and $c_n^\tau \simeq \tau$, we may have
\[ w = \bar{C}_B = \left( \prod_B C_i \right) c_n \in R(\sigma), \]
in which case we set
\[ T_w = \left( \prod_{i \in B} T^V_i \right) T^V_\tau. \]
We then see that for $C_B, C_{B'} \in R(\sigma)$, we have
\[ T_{C_B} T_{C_{B'}} = \prod_B T^V_i \prod_{B'} T^V_j = \prod_{B \land B'} T^V_i, \]
where $B \land B'$ is the symmetric difference. Since $C_B C_{B'} = C_{B \land B'}$, we have the result in this case. A similar argument shows, in the case where $G = D_n$ and $c_n^\tau \simeq \tau$, that
\[ T_{\bar{C}_B} T_{\bar{C}_{B'}} = T_{\bar{C}_{B \land B'}}, \]
and
\[ T_{\bar{C}_B} T_{\bar{C}_{B'}} = T_{\bar{C}_B \land B'} = T_{\bar{C}_{B'} \bar{C}_B}. \]
Thus, we have the claim. \qed

Since the cocycle $\eta : R(\sigma) \times R(\sigma) \to \mathbb{C}$ is determined by $T_{w_1 w_2} = \eta(w_1, w_2) T_{w_1} T_{w_2}$ we have $\eta$ is a coboundary, and immediately get the following result.

**Corollary 3.2.** For any Levi subgroup $M$ of $G_n$ and any $\sigma \in \mathcal{E}_2(M)$, we have $\mathcal{C}(\sigma) \simeq \mathbb{C}[R(\sigma)]$, so $i_{G,M}(\sigma)$ decomposes with multiplicity one.

Now, let $A = A_\theta$ be the split component of $M$, and let $a = a_\theta$ be its real Lie algebra. If $\sigma \in \mathcal{E}_2(M)$, and $w \in R(\sigma)$, then we let $a_w = \{ H \in a | w \cdot H = H \}$. We let $Z$ be the split component of $G$ and $\mathfrak{z}$ be its real Lie algebra. Now, by Theorem 1.1 of [9], we know $i_{G,M}(\sigma)$ has elliptic components if and only if there is a $w \in R(\sigma)$ with $a_w = \mathfrak{z}$. Further, if $a_{R(\sigma)} = \bigcap_{w \in R(\sigma)} a_w$, then each component of $i_{G,M}(\sigma)$ is irreducibly induced from an elliptic tempered representation if there is some $w \in R(\sigma)$ so that $a_R = a_w$.

**Theorem 3.3.** Let $G = GSpin_{2n+1}$, and suppose $M \simeq GL_{n_1} \times \cdots \times GL_{n_r} \times G_m$, and $\sigma \in \mathcal{E}_2(M)$. Then $\text{Ind}_{P}^{G}(\sigma)$ has elliptic constituents if and only if $R(\sigma) \simeq \mathbb{Z}_2^r$. Any $\pi \in \mathcal{E}(G)$, is either elliptic, or there is a choice of $M'$ and an irreducible elliptic tempered representation $\sigma$ of $M'$ with $\pi = \text{Ind}_{P'}^{G'}(\sigma)$. 
Proof. We will use the explicit realization of $R(\sigma)$ we developed in Theorem 2.7. Suppose $R \simeq \mathbb{Z}_2^d$. Let $a = a_M$. Then we can identify $a$ with $\{(x_1, x_2, \ldots, x_r, y)|x_i, y \in \mathbb{R}\}$, and note, under this identification $\mathcal{J} = \{(y/2, \ldots, y/2, y)|y \in \mathbb{R}\}$. $C$ acts on $a$ by
\[ C_i(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_r, y) = (x_1, \ldots, x_{i-1}, y - x_i, x_{i+1}, \ldots, x_r, y). \]

Thus, if $C = C_B$, as above, then $a_C = \{(x_1, \ldots, x_r, y)|x_i = y/2, \forall i \in B\}$ Without loss of generality, we may assume $R(\sigma) = (C_r, C_{r-1}, \ldots, C_{r-d+1})$. Let $w_0 = C_{r-d+1}C_{r-d+2} \cdots C_r$. Note, for each $w \in R(\sigma)$, we have $a_w \subset a_{w_0}$, and thus $a_{R(\sigma)} = a_{w_0}$. Now, $a_{w_0} = \mathcal{J}$ if and only if $w_0 = C_1C_2 \cdots C_r$, and thus, by [2][13] Ind$^G\sigma$ has elliptic constituents if and only if $R(\sigma) \simeq \mathbb{Z}_2^d$. In this case, every component of Ind$^G\sigma$ is elliptic. The last statement of the claim follows from the fact $a_{R(\sigma)} = a_{w_0}$, and Lemma 1.3 of [13].

Theorem 3.4. Let $G = GSpin_{2n}$, and $M \simeq GL_{n_1} \times \cdots \times GL_{n_r} \times G_m$. Let $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_r \otimes \tau \in \mathcal{E}_2(M)$.

(i) Suppose $m = 0$ or $c_\tau \neq \tau$. We let $\Omega_1(\sigma), \Omega_2(\sigma), d_1, d_2$, and $d$ be defined as in Theorem 2.7. Then Ind$^G\sigma$ has elliptic components if and only if $d = r$ and $d_2$ is even, in which case every component is elliptic. If $\pi \subset$ Ind$^G\sigma$ is not elliptic, then $\pi \simeq$ Ind$^M\sigma'$ for some elliptic representation $\sigma'$ of a Levi subgroup $M'$ of $G$ if and only if $d_2$ is even or $d_2 = 1$.

(ii) Suppose $m > 0$ and $c_\tau \simeq \tau$. Let $R(\sigma) \simeq \mathbb{Z}_2^d$. Then Ind$^G\sigma$ has elliptic components if and only if $d = r$, in which case all components are elliptic. Furthermore, for any $\pi \in \mathcal{E}_2(G)$ there is a Levi subgroup $M'$ of $G$, and an irreducible elliptic tempered representation $\sigma'$ of $M'$ so that $\pi \simeq$ Ind$^M\sigma'$.

Proof. (i) As in Theorem 2.7 we assume $\Omega_1(\sigma) = \{r - d_1 + 1, r - d_1 + 2, \ldots, r\}$, and $\Omega_2(\sigma) = \{r - d + 1, r - d + 2, \ldots, r - d_1\}$. Then
\[ R(\sigma) = \langle C_iC_j|i, j \in \Omega_2(\sigma)\rangle \times \langle C_i|i \in \Omega_1(\sigma)\rangle. \]

We note $a = a_M$ can be identified with $\{(x_1, \ldots, x_r, y)|x_i, y \in \mathbb{R}\}$, in such a way so $C_i \cdot (x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_r, y) = (x_1, \ldots, x_{i-1}, y - x_i, x_{i+1}, \ldots, x_r, y)$. So, if $d_2 \neq 1$, we have
\[ a_{R(\sigma)} = \{(x_1, \ldots, x_r, y)|x_i = \frac{y}{2} \text{ for all } r - d + 1 \leq i \leq r\}, \]
while if $d_2 = 1$, then
\[ a_{R(\sigma)} = \{(x_1, \ldots, x_r, y)|x_i = \frac{y}{2} \text{ for all } r - d_1 + 1 \leq i \leq r\}. \]
If $d_2$ is even then $w_0 = C_{r-d+1}C_{r-d+2}\cdots C_r \in R(\sigma)$, and $a_{w_0} = a_{R(\sigma)}$. If $d_2 = 1$, then $w_0 = C_{r-d+1}C_{r-d+2}\cdots C_r \in R(\sigma)$, and again $a_{w_0} = a_{R(\sigma)}$. Thus, in either of these cases, we have each component must be irreducibly induced from an elliptic tempered representation of some Levi subgroup [18]. On the other hand, if $d_2 \geq 3$ and $d_2$ is odd, then, for any $w \in R(\sigma)$ we have $a_{R(\sigma)} \subseteq a_w$, so components of these induced representations are not irreducibly induced from elliptic representations. Finally, since $\mathfrak{z}$ is identified with $\{(y/2,y/2,\ldots,y/2)|y \in \mathbb{R}\}$, then we see $\text{Ind}_{G}^G(\sigma)$ has elliptic components if and only if $C_1C_2\ldots C_r \in R(\sigma)$, which occurs if and only if $d = r$ and $d_2$ is even.

(ii) Now suppose $m > 0$ and $c_n\tau \simeq \tau$. We let $\Omega(\sigma)$ be defined as in Theorem [2.7]. We assume, without loss of generality, $\Omega(\sigma) = \{r-d+1,\ldots,r\}$. Then

$$R(\sigma) = \langle C_i | r-d+1 \leq i \leq r \text{ and } n_i \text{ is even} \rangle \times \langle C_i c_n | r-d+1 \leq i \leq r \text{ and } n_i \text{ is odd} \rangle.$$

Let $d_2 = \{i| r-d+1 \leq i \leq r \text{ and } n_i \text{ is odd}\}$, and

$$w_0 = \begin{cases} C_{r-d+1}C_{r-d+2}\cdots C_r & \text{if } n_i \text{ is even}; \\ C_{r-d+1}C_{r-d+2}\cdots C_r c_n & \text{if } n_i \text{ is odd}. \end{cases}$$

With the identification of $a = a_M$ with $\mathbb{R}^{r+1}$ as above, we have

$$a_{w_0} = \{(x_1,\ldots,x_r,y) | x_i = y/2 \text{ for all } r-d+1 \leq i \leq r\}.$$

Note, for any $w \in R(\sigma)$ we have $a_{w_0} \subseteq a_w$, so $a_{R(\sigma)} = a_{w_0}$. Now, $a_{w_0} = \mathfrak{z}$ if and only if $d = r$. Thus, the elliptic spectrum is as claimed, and the tempered spectrum is irreducibly induced from the elliptic spectra of the Levi subgroups.

Now we assume $G = G_n = GSpin_{2n}$ or $GSpin_{2n+1}$. Denote $R = R(\sigma)$, and let $\hat{R}$ be the set of irreducible characters of $R$. We let $\kappa \leftrightarrow \pi_\kappa$ be the correspondence between $\hat{R}$ and the (equivalence classes of) irreducible components of $\text{Ind}_{G}^G(\sigma)$ described by Keys [20] (see also Arthur [2] and Herb [18]). Suppose $\text{Ind}_{G}^G(\sigma)$ has elliptic components, as described in Theorems [3.3] and [3.4]. Then either $C_1C_2\ldots C_r \in R$ or $C_1C_2\ldots C_r c_n \in R$. Let

$$C_0 = \begin{cases} C_1C_2\ldots C_r c_n & \text{if } G = GSpin_{2n}, d_2 \text{ is odd }, \text{ and } c_n\tau \simeq \tau; \\ C_1C_2\ldots C_r & \text{otherwise}. \end{cases}$$

For $\kappa \in \hat{R}$ we let $\varepsilon(\kappa) = \kappa(C_0)$.
Theorem 3.5. Suppose \( G = GSpin_{2n} \) or \( GSpin_{2n+1} \). Let \( M \cong GL_{n_1} \times \cdots \times GL_{n_r} \times G_m \) be a Levi subgroup and suppose \( \sigma = \sigma_1 \otimes \cdots \otimes \sigma_r \otimes \tau \in \mathcal{E}_2(M) \). Suppose \( \text{Ind}^G_M(\sigma) \) has elliptic components. Let \( \kappa \in \hat{R} \). Then \( \Theta^\kappa_{\pi_+} = \kappa(C_0) \Theta^\kappa_{\pi_1} \).

Proof. First suppose \( G = GSpin_{2n+1} \), or \( c_n \tau \simeq \tau \). For \( 1 \leq i \leq r \), we let \( M_i \) be the Levi subgroup of \( G \) of the form \( GL_{n_1} \times G_{n-n_i} \). Let \( N_i = M_i \cap N \), and let \( P_i = MN_i \). Let \( R_i = R_i(\sigma) \) be the \( R \)-group attached to \( \text{Ind}^M_{P_i}(\sigma) \). Suppose \( R_i = R_i(\sigma) \) is generated by \( \{ C_j | 1 \leq j \leq r, j \neq i \} \), or \( \{ C_j | 1 \leq j \leq r, j \neq i \} \), where \( C_i \) is defined as in the proof of Theorem 2.7. We now combine these situations by letting \( R = (D_i | 1 \leq i \leq r) \), where \( D_1 = C_1 \) or \( C_i \), in the obvious way. Let \( \eta \leftrightarrow \rho_\eta \) be the correspondence between \( \hat{R} \) and \( \text{Ind}^M_{P_i}(\sigma) \). If \( \eta \in \hat{R} \), then \( \hat{R}(\eta) = \{ \kappa \in \hat{R} | R_i \kappa R_i = \eta \} \). Then \( \hat{R}(\eta) = \{ \eta^+, \eta^- \} \), where \( \eta^+(D_j) = \eta(D_j) \), for \( i \neq j \), and \( \eta^+(D_i) = \pm 1 \). By Arthur [2] we have \( \text{Ind}^M_{P_i}(\rho_\eta) = \pi_{\eta^+} \oplus \pi_{\eta^-} \). Moreover, since the character of this induced representation vanishes on \( G_\tau \), we have \( \Theta^\kappa_{\pi_{\eta^+}} = -\Theta^\kappa_{\pi_{\eta^-}} \).

For \( \kappa \in \hat{R} \), we let \( s(\kappa) = \{ 1 \leq i \leq r | \kappa(D_i) = -1 \} \). Note, if \( s(\kappa) = 0 \), then \( \kappa = 1 \), and the claim is trivially true. Suppose \( s \geq 0 \) and the claim holds for any \( \kappa \in \hat{R} \) with \( s(\kappa) = s \). Suppose \( s(\kappa) = s + 1 \). Then we fix some \( 1 \leq i \leq r \) for which \( \kappa(D_i) = -1 \). Then consider \( M_i \) and \( R_i \) as above. Let \( \eta = \kappa | R_i \), and suppose \( \rho_\eta \) is the corresponding component of \( \text{Ind}^M_{P_i}(\sigma) \). Then \( \eta = \eta^- \), so by our discussion above, we have \( \Theta^\kappa_{\pi_{\eta^-}} = -\Theta^\kappa_{\pi_{\eta^+}} \). Moreover \( s(\eta^+) = s \), so, by our hypothesis, \( \Theta^\kappa(\pi_{\eta^+}) = \eta^+(C_0) \Theta^\kappa_{\pi_1} \). Now, \( \Theta^\kappa_{\pi_+} = -\Theta^\kappa_{\pi_{\eta^+}} = -\eta^+(C_0) \Theta^\kappa_{\pi_1} = \kappa(C_0) \Theta^\kappa_{\pi_1} \). So the claim holds for all \( \kappa \) with \( s(\kappa) = s + 1 \), and by induction the claim holds for all \( \kappa \in \hat{R} \).

Now consider the case where \( G = GSpin_{2n} \) and \( c_n \tau \not\simeq \tau \). The proof is essentially the same as above, but we give some details for completeness. Let \( \Omega_1(\sigma), \Omega_2(\sigma), d_1, d_2, d \) be as in Theorem 2.7(i). If \( d_2 = 0 \), then the proof is identical to the one above, so we assume \( d_2 > 0 \) is even. From Theorem 3.4 we know \( d = r \). Then, we again see \( \Delta' = \emptyset \), so we easily apply the results of Arthur [2] and Herb [18]. Without loss of generality, we assume \( \Omega_1(\sigma) = \{ 1, \ldots, d_1 \} \), and \( \Omega_2(\sigma) = \{ d_1 + 1, \ldots, r \} \). Then, \( R \cong \mathbb{Z}_{2}^{r-1} \), with generators \( D_1, \ldots, D_{r-1} \), where \( D_1 = C_1 \), for \( 1 \leq i \leq d_1 \), and \( D_i = C_i \) for \( d_1 + 1 \leq i \leq r - 1 \). For each \( 1 \leq i \leq r - 1 \), we let \( M_i \) and \( R_i \) be defined as in the previous cases. We again let \( \eta \leftrightarrow \rho_\eta \) be the correspondence between \( \hat{R} \) and the components of \( \text{Ind}^M_{P_i}(\sigma) \). Then, we again have \( \hat{R}(\eta) = \{ \eta^+, \eta^- \} \), and so \( \Theta^\kappa_{\pi_{\eta^-}} = -\Theta^\kappa_{\pi_{\eta^+}} \). Let \( \kappa \in \hat{R} \) and let \( s(\kappa) = \{ |D_i| | \kappa(D_i) = -1 \} \). Then \( s(1) = 0 \), so the claim holds for the case with \( s(\kappa) = 0 \). If we assume the result when \( s(\kappa) = s \),
then the same argument as above shows it holds when \( s(\kappa) = s + 1 \), and so the claim holds by induction. \( \square \)

4. Parameters and \( R \)-groups for \( GSpin \) groups

In this section we discuss the computation of Arthur’s \( R \)-group associated to a parameter \( \varphi : W_G \to L G \), in the case when \( G = GSpin_m(F) \). We begin with a lemma which applies to split reductive groups in general.

**Lemma 4.1.** Suppose \( R_{\varphi, \pi} \simeq R(\pi) \), whenever \( \psi : W'_\varphi \to L L \to L H \), with \( L \) a maximal proper Levi subgroup of a quasi-split connected group \( H \), and \( \psi \) is an elliptic parameter for the \( L \)-packet \( \Pi_\psi(L) \), containing the square integrable representation \( \pi \). Let \( M \) be an arbitrary Levi subgroup of \( G \), and \( \varphi : W'_\varphi \to L M \) an elliptic parameter for an \( L \)-packet \( \Pi_\varphi(M) \) containing a square integrable representation \( \sigma \). Then \( R_{\varphi, \sigma} \simeq R(\sigma) \).

**Proof.** The proof of this relies on the following result.

**Lemma 4.2.** Suppose \( M \subset L \) are Levi subgroups of \( G \). Suppose \( \varphi : W'_\varphi \to L M \) is a parameter. Let \( S_\varphi = Z_\hat{G}(\varphi) \) and \( S_{L, \varphi} = Z_\hat{L}(\varphi) \). Then \( S_{L, \varphi}^* = S_\varphi^* \cap \hat{L}^* \).

Since \( S_\varphi \) is reductive and \( S_{L, \varphi} \) is a reductive (Levi subgroup (e.g., by [7] Lemma 2.1) this is a standard result. \( \square \)

Now we have \( W(G, A_M) \simeq W(\hat{G}, A_{\hat{M}}) \), with the isomorphism given by \( s_\alpha \mapsto \hat{s}_\alpha \). We let \( M_\alpha \) be the Levi subgroup of \( G \) generated by \( M \) and \( \alpha \). Let \( R_\alpha(\sigma) \) be the \( R \)-group attached to \( i_{M_\alpha, M}(\sigma) \).

Considering \( \varphi : W'_\varphi \to L M \to L M_\alpha \), we let \( S_{\varphi, \alpha} = Z_\hat{M}_\alpha(\varphi) = S_\varphi \cap \hat{M}_\alpha \). By Lemma 4.2, \( S_{\varphi, \alpha}^* = S_\varphi^* \cap \hat{M}_\alpha \).

We know from Lemma 2.2 of [7] that \( (A_{\hat{M}} \cap S_\varphi)^* \) is a maximal torus of \( S_\varphi^* \), so we may take \( T_\varphi = (A_{\hat{M}} \cap S_\varphi)^* \). Then \( L M = Z_\hat{G}(T_\varphi) \) ([7], Lemma 2.1). Since \( T_\varphi \subseteq \hat{M} \subseteq \hat{M}_\alpha \), it follows \( T_\varphi \subseteq S_{\varphi, \alpha} \), so \( T_\varphi \) is a maximal torus in \( S_{\varphi, \alpha}^* \).

Let \( W_{\varphi, \alpha} = N_{S_{\varphi, \alpha}}(T_\varphi)/Z_{S_{\varphi, \alpha}}(T_\varphi) \) and \( W_{\varphi, \alpha}^* = N_{S_{\varphi, \alpha}^*}(T_\varphi)/Z_{S_{\varphi, \alpha}^*}(T_\varphi) \). Lemma 2.2 of [7] tells us that \( W_{\varphi} \) (respectively, \( W_{\varphi, \alpha} \)) can be identified with the subgroup of \( W(\hat{G}, A_{\hat{M}}) \) (respectively, \( W(\hat{M}_\alpha, A_{\hat{M}}) \)) consisting of the elements that can be represented by elements of \( S_\varphi \) (respectively, \( S_{\varphi, \alpha}^* \)). Under these identifications, we have \( W_{\varphi, \alpha, \sigma} \cap \hat{M}_\alpha = W_{\varphi, \alpha, \sigma} \).

Now let \( R_{\varphi, \alpha, \sigma} = W_{\varphi, \alpha, \sigma} \). The hypothesis implies \( R_\alpha(\sigma) \simeq R_{\varphi, \alpha, \sigma} \). Let \( \alpha \in \Delta' \). Then \( \mu_\alpha(\sigma) = 0 \). Thus, \( s_\alpha \in W(\sigma) \), and \( R_\alpha(\sigma) = 1 \). Note \( s_\alpha \in W(M_\alpha, A_M) \simeq W(\hat{M}_\alpha, A_{\hat{M}}) \), so \( s_\alpha \in
\[ W_{\phi, \sigma} \cap \hat{M}_\alpha = W_{\phi, \alpha, \sigma}. \] Since \( R_{\phi, \alpha, \sigma} \simeq R_\alpha(\sigma) = 1 \), we have \( s_\alpha \in W_{\phi, \alpha, \sigma} \), as claimed. Conversely, assume \( s_\alpha \in W_{\phi, \sigma} \). As \( s_\alpha \in W_{\phi, \sigma} \), we have \( s_\alpha \in W(\sigma) \). Again, considering \( M_\alpha \), we have \( R_{\phi, \alpha, \sigma} = 1 \), so \( R_\alpha(\sigma) = 1 \), which implies \( s_\alpha \in W' \). Therefore, \( \alpha \in \Delta' \), as claimed.

\[ \square \]

We now return to the setting where \( G = GSO_{2n} \). Then \( \hat{G} = GSO_{2n}(\mathbb{C}) \), if \( m = 2n \), and \( \hat{G} = GSp_{2n}(\mathbb{C}) \), if \( m = 2n + 1 \). Since \( G \) is split, we have \( \ell(G) = \hat{G} \times W_F \). We consider a parameter \( \varphi : W_F \to \ell(G) \). Let us describe matrix realizations of \( GSO_{2n}(\mathbb{C}) \) and \( GSp_{2n}(\mathbb{C}) \). Let

\[
\mu = \begin{cases} 
1, & \text{if } \hat{G} = GSO_{2n}(\mathbb{C}), \\
-1, & \text{if } \hat{G} = GSp_{2n}(\mathbb{C}),
\end{cases} \quad \hat{w}_n = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \quad \hat{J}_{2n} = \begin{pmatrix} 0 & \hat{w}_n \\ \mu \hat{w}_n & 0 \end{pmatrix},
\]

and

\[
\mathcal{G} = \{ g \in GL_{2n}(\mathbb{C}) |^t g \hat{J}_{2n} g = \lambda(g) \hat{J}_{2n} \text{ for some } \lambda(g) \in \mathbb{C}^\times \}.
\]

If \( \mu = -1 \), then \( \mathcal{G} \) is a connected algebraic group denoted by \( GSp_{2n}(\mathbb{C}) \). If \( \mu = 1 \), then \( \mathcal{G} = GO_{2n}(\mathbb{C}) \) has two connected components. In this case, we can define the similitude norm

\[ \nu : GO_{2n}(\mathbb{C}) \to \{ \pm 1 \}, \quad g \mapsto \lambda(g)^{-n} \det(g). \]

The kernel of this map, denoted by \( GSO_{2n}(\mathbb{C}) \), is the connected component of \( GO_{2n}(\mathbb{C}) \).

We let \( \hat{M} \) be the Siegel parabolic subgroup of \( \hat{G} \), so \( \hat{M} \simeq GL_n(\mathbb{C}) \times GL_1(\mathbb{C}) \). More precisely, for \( g \in GL_n(\mathbb{C}) \) we let \( \hat{\varepsilon}(g) = \hat{w}_n \begin{pmatrix} g & 0 \\ 0 & \lambda(g) \end{pmatrix} \hat{w}_n^{-1} \). Then

\[
\hat{M} = \left\{ \begin{pmatrix} g & 0 \\ 0 & \lambda(\varepsilon(g)) \end{pmatrix} \left| \begin{array}{l} g \in GL_n(\mathbb{C}), \lambda \in GL_1(\mathbb{C}) \end{array} \right. \right\}.
\]

Let \( \hat{A}_M \) be the split component of \( \hat{M} \), so \( \hat{M} = \{ \text{diag} \{ a I_n, \lambda a^{-1} I_n \} \} \). If \( \hat{G} = GSO_{2n} \), and \( n \) is odd, then \( W_{\hat{M}} = \{ 1 \} \). Otherwise, \( W_{\hat{M}} = W(\hat{G}, \hat{A}_M) = \{ 1, \hat{w} \} \), where \( \hat{w} : (g, \lambda) \mapsto (\lambda \hat{\varepsilon}(g), \lambda) \), and is represented by \( \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \).

Thus, we know \( M \simeq GL_n \times GL_1, A_M \simeq GL_1 \times GL_1, \) and

\[
W(G, A_M) = \begin{cases} 
\{ 1 \} & \text{if } G = GSpin_{2n} \text{ and } n \text{ is odd;} \\
\{ 1, w \} & \text{otherwise},
\end{cases}
\]

where \( w : (g, \lambda) \mapsto (\lambda \hat{\varepsilon}(g), \lambda) \), and \( \varepsilon \) is the dual involution given by \( \hat{\varepsilon} \).
Now let $\sigma$ be an irreducible unitary supercuspidal representation of $M$, so $\sigma \simeq \sigma_0 \otimes \psi$, with $\sigma_0$ an irreducible unitary supercuspidal representation of $GL_n(F)$ and $\psi$ a unitary character of $F^\times$.

So, if $\varphi : W_F \to \tilde{M}$ is the corresponding Langlands parameter, then $\varphi = \varphi_0 \times \hat{\psi}$, where $\varphi_0$ is the Langlands parameter of $\sigma_0$ and $\hat{\psi}$ is the character of $C^\times$ associated to $\psi$ by local class field theory.

Since $\sigma_0$ is irreducible and supercuspidal, we know $\varphi_0$ is irreducible. We abuse notation to write

$$\varphi(w) = \begin{pmatrix} \varphi_0(w) & 0 \\ 0 & \hat{\psi}(w) \hat{\epsilon}(\varphi_0(w)) \end{pmatrix}.$$  

4.1. **Reducibility and poles of $L$-functions.** Let $\mathfrak{n}$ denote the Lie algebra of the unipotent radical of $\tilde{M}$. Let $\rho_n$ denote the standard representation of $GL_n(C)$. The adjoint action $r$ of $\tilde{M}$ on $\mathfrak{n}$ is given as follows:

$$r = \begin{cases} \wedge^2 \rho_n \otimes \rho_1^{-1}, & \text{if } \tilde{G} = GSO_{2n}(C), \\
\text{Sym}^2 \rho_n \otimes \rho_1^{-1}, & \text{if } \tilde{G} = GSp_{2n}(C). \end{cases}$$

More precisely, let $V = \{ X \in \mathfrak{gl}_{2n}(C) \mid \, ^t X = -\mu X \}$. Then $(g, \lambda) \in \tilde{M}$ acts on $X \in V$ by $(g, \lambda) \cdot X = \lambda^{-1} gX^t g$.

Suppose $L(s, \wedge^2 \varphi_0 \otimes \hat{\psi}^{-1})$ has a pole at $s = 0$. Then $\wedge^2 \varphi_0 \otimes \hat{\psi}^{-1}$ contains the trivial representation, so there exists a nonzero $X \in M_n(C)$ such that $^t X = -X$ and $(\wedge^2 \varphi_0 \otimes \hat{\psi}^{-1})(w) \cdot X = X$, for all $w \in W_F$. We have

$$\hat{\psi}(w)^{-1} \varphi_0(w)X^t \varphi_0(w) = X, \quad \forall w \in W_F.$$  

It follows that $X$ is a nonzero intertwining operator between $^t \varphi_0^{-1}$ and $\hat{\psi}^{-1} \otimes \varphi_0$. Since $\varphi_0$ is irreducible, $X$ is invertible. Observe that this can happen only if $n$ is even (every antisymmetric odd dimensional matrix is singular). In addition, it follows from (4.1) that $\varphi_0$ factors through $GSp_n(C)$.

Similarly, if we assume that $L(s, \text{Sym}^2 \varphi_0 \otimes \hat{\psi}^{-1})$ has a pole at $s = 0$, we obtain that $^t \varphi_0^{-1} \simeq \hat{\psi}^{-1} \otimes \varphi_0$ and $\varphi_0$ factors through $GO_n(C)$.

On the other hand, if $^t \varphi_0^{-1} \simeq \hat{\psi}^{-1} \otimes \varphi_0$, then (4.1) holds for some $X \in GL_n(C)$. By standard arguments, $X$ is symmetric or antisymmetric. It follows that one of the $L$-functions $L(s, \wedge^2 \varphi_0 \otimes \hat{\psi}^{-1})$ or $L(s, \text{Sym}^2 \varphi_0 \otimes \hat{\psi}^{-1})$ has a pole at $s = 0$.

We summarize the above considerations in the following lemma:

**Lemma 4.3.** Let $\varphi_0 : W_F \to GL_n(C)$ and $\hat{\psi} : W_F \to GL_1(C)$ be irreducible $L$-parameters. If $\varphi_0 \simeq \hat{\psi}^{-1} \otimes \varphi_0$, then precisely one of the $L$-functions $L(s, \wedge^2 \varphi_0 \otimes \hat{\psi}^{-1})$ or $L(s, \text{Sym}^2 \varphi_0 \otimes \hat{\psi}^{-1})$ has a pole at $s = 0$. 

(1) If \( n \) is odd, then \( L(s, \wedge^2 \varphi_0 \otimes \hat{\psi}^{-1}) \) is always holomorphic at \( s = 0 \) and \( \varphi_0 \) factors through \( GO_n(\mathbb{C}) \).

(2) If \( n \) is even, then \( L(s, \wedge^2 \varphi_0 \otimes \hat{\psi}^{-1}) \) has a pole at \( s = 0 \) if and only if \( \varphi_0 \) factors through \( GSp_n(\mathbb{C}) \).

**Proposition 4.4.** Let \( G = GSpin_{2n} \), \( G' = GSpin_{2n+1} \), and consider the Siegel Levi subgroup \( M \simeq GL_n \times GL_1 \). Let \( \sigma \simeq \sigma_0 \otimes \psi \) be an irreducible unitary supercuspidal representation of \( M = M(F) \) with corresponding Langlands parameter \( \varphi = \varphi_0 \times \hat{\psi} \). Assume \( \varphi_0 \simeq \hat{\psi}^{-1} \otimes \varphi_0 \). Let \( \pi = \text{Ind}^G_M(\sigma) \) and \( \pi' = \text{Ind}^G_M(\sigma) \).

(1) If \( n \) is odd, then \( \pi \) and \( \pi' \) are both irreducible and \( \varphi_0 \) factors through \( GO_n(\mathbb{C}) \).

(2) If \( n \) is even, then \( \pi \) is irreducible if and only if \( \pi' \) is reducible. Moreover, \( \pi \) is irreducible if and only if \( \varphi_0 \) factors through \( GSp_n(\mathbb{C}) \).

**Proof.** (1) is clear. For (2), assume \( n \) is even and consider \( G = GSpin_{2n} \). Then \( \pi = \text{Ind}^G_M(\sigma) \) is irreducible if and only if \( L(s, \sigma_0 \otimes \psi, \wedge^2 \rho_n \otimes \rho_1^{-1}) \) has a pole at \( s = 0 \) \cite{26, 30}. We know from \cite{17}, Theorem 1.4 that

\[
L(s, \sigma_0 \otimes \psi, \wedge^2 \rho_n \otimes \rho_1^{-1}) = L(s, \wedge^2 \varphi_0 \otimes \hat{\psi}^{-1}).
\]

The statement follows from Lemma 4.3. Similar arguments work for \( G' = GSpin_{2n+1} \). \( \square \)

4.2. **Centralizers for the Siegel Parabolic.** We wish to compute \( S_\varphi = Z_G(\text{Im} \varphi) \). First, we will compute \( Z_G(\text{Im} \varphi) \), where \( G = GSp_{2n}(\mathbb{C}) \) or \( GO_{2n}(\mathbb{C}) \). Suppose \( X \in G \) centralizes \( \varphi \), and write

\[
X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \text{ with } A, B, C, D \in M_n(\mathbb{C}).
\]

Computing directly we have, for all \( w \in W_F \),

\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \varphi_0(w) \\ 0 \end{pmatrix} = \begin{pmatrix} \varphi_0(w) \\ 0 \end{pmatrix},
\]

which gives

\[
A \varphi_0(w) = \varphi_0(w)A, \quad D \hat{\psi}(\varphi_0(w)) = \hat{\psi}(\varphi_0(w))D, \quad B \hat{\psi}(\varphi_0(w)) = \varphi_0(w)B,
\]

and \( C \varphi_0(w) = \hat{\psi}(\varphi_0(w))C \). The irreducibility of \( \varphi_0 \) tells us \( A \) and \( D \) are scalars (denoted \( a_{11} I_n \) and \( a_{22} I_n \), respectively) and also shows \( C = B = 0 \), unless \( \varphi_0 \simeq (\hat{\psi} \varphi_0) \otimes \hat{\psi} \). Thus, if \( \sigma_0 \not\simeq \sigma_0 \otimes \psi \otimes \text{det} \), then \( Z_G(\varphi) = \left\{ \begin{pmatrix} aI_n \\ \lambda a^{-1} I_n \end{pmatrix} \right\} \sim \text{M} \simeq \mathbb{C}^\times \times \mathbb{C}^\times \), and clearly, \( Z_G(\varphi) = Z_G(\varphi) \). So, suppose
\[ \sigma_0 \simeq \tilde{\sigma}_0 \otimes \psi \circ \det. \] Fix an equivalence, \( B \) between these two representations, i.e., take \( B \) so that \( B^{-1} \varphi_0(w)B = \hat{\psi}(w)\hat{\varphi}(\varphi_0(w)) \). By Schur’s Lemma, \( B \) is unique up to scalar. We note

\[
(B\hat{\varphi}(B))^{-1}\varphi_0(w)(B\hat{\varphi}(B)) = \hat{\psi}(B)^{-1}(\hat{\psi}(w)\hat{\varphi}(\varphi_0(w)))\hat{\psi}(B) = \hat{\psi}(B^{-1}\varphi_0(w)B)\hat{\psi}(w) = \varphi_0(w),
\]

and thus \( B\hat{\varphi}(B) = cI_n \), for some \( c \in \mathbb{C}^\times \). We write this as \( B\hat{w}_n = c\hat{w}_n I_n \). Note that if \( J = B\hat{w}_n \), then we have \( ^tJ = c^{-1}J \), so \( c = \pm 1 \), and \( J \) is a symmetric or symplectic form fixed by \( \varphi_0 \) up to the multiplier \( \hat{\psi} \).

Now, we have \( X = \begin{pmatrix} a_{11}I_n & a_{12}B \\ a_{21}B^{-1} & a_{22}I_n \end{pmatrix}, \) and since \( X \in \mathcal{G} \), we have

\[
\begin{pmatrix} \hat{w}_n \\ \mu \hat{w}_n \end{pmatrix} X = \begin{pmatrix} \lambda \hat{w}_n \\ \lambda \mu \hat{w}_n \end{pmatrix}
\]
or,

\[
\begin{pmatrix} a_{11}a_{21}(1+\mu c)\hat{w}_nB^{-1} & (a_{11}a_{22} + a_{21}a_{12}\mu c)\hat{w}_n \\ (a_{11}a_{22} + a_{21}a_{12}\mu c)\mu \hat{w}_n & a_{12}a_{22}(1+\mu c)B\hat{w}_n \end{pmatrix} = \begin{pmatrix} \lambda \hat{w}_n \\ \lambda \mu \hat{w}_n \end{pmatrix}.
\]

We see this is equivalent to the \( 2 \times 2 \) complex matrix \( Y = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \) satisfying \( ^tY \begin{pmatrix} 1 \\ \mu c \end{pmatrix} Y = \begin{pmatrix} \lambda \\ \lambda \mu c \end{pmatrix} \). Thus \( X \mapsto Y \) is an isomorphism,

\[
(4.2) \quad Z_\varphi(\varphi) \simeq \begin{cases} 
GSp_2(\mathbb{C}) \simeq GL_2(\mathbb{C}) & \text{if } \mu c = -1; \\
GO_{1,1}(\mathbb{C}) & \text{if } \mu c = 1.
\end{cases}
\]

This is equal to \( S_\varphi \) if \( \hat{G} = GSp_{2n}(\mathbb{C}) \).

Now, let \( \hat{G} = GSO_{2n}(\mathbb{C}) \), so \( \mu = 1 \). Let \( X = \begin{pmatrix} a_{11}I_n & a_{12}B \\ a_{21}B^{-1} & a_{22}I_n \end{pmatrix} \in Z_\varphi(\varphi) \). We have to determine whether \( X \in \hat{G} \). Assume first \( c = -1 \). Then

\[
\begin{pmatrix} (a_{11}a_{22} - a_{12}a_{21})\hat{w}_n \\ (a_{11}a_{22} - a_{12}a_{21})\hat{w}_n \end{pmatrix} = \begin{pmatrix} \lambda \hat{w}_n \\ \lambda \hat{w}_n \end{pmatrix},
\]
so \( \lambda = a_{11}a_{22} - a_{12}a_{21} \). We use the formula \( \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det A \det (D - CA^{-1}B) \), if \( A \) is invertible. Therefore, if \( a_{11} \neq 0 \), we have 

\[
\det X = a_{11}^2 \det(a_{22}I_n - a_{21}a_{11}^{-1}a_{12}B^{-1}B) = \det(a_{11}a_{22} - a_{12}a_{21})I_n = \lambda^n.
\]

The similitude norm \( \nu(X) = \lambda^{-n} \det X = 1 \), so \( X \in GSO_{2n}(\mathbb{C}) \). If \( a_{11} = 0 \), then

\[
\det X = \det \begin{pmatrix} 0 & a_{12}B \\ a_{21}B^{-1} & a_{22}I_n \end{pmatrix} = (-1)^n \det \begin{pmatrix} a_{21}B^{-1} & a_{22}I_n \\ 0 & a_{12}B \end{pmatrix} = \lambda^n,
\]

and again \( X \in GSO_{2n}(\mathbb{C}) \).

Assume \( c = 1 \). Then we have

\[
\begin{pmatrix} 2a_{11}a_{21}\hat{w}_nB^{-1} & (a_{21}a_{12} + a_{11}a_{22})\hat{w}_n \\ (a_{11}a_{22} + a_{12}a_{21})\hat{w}_n & 2a_{12}a_{22}I^t B\hat{w}_n \end{pmatrix} = \begin{pmatrix} \lambda\hat{w}_n \\ \lambda\hat{w}_n \end{pmatrix}.
\]

It follows \( a_{12} = a_{21} = 0 \) or \( a_{11} = a_{22} = 0 \). If \( a_{12} = a_{21} = 0 \), then \( a_{22} = \lambda a_{11}^{-1} \) and \( X = \begin{pmatrix} a_{11}I_n \\ \lambda a_{11}^{-1}I_n \end{pmatrix} \). The similitude norm \( \nu(X) = \lambda^{-n} \det(X) = 1 \), so \( X \in GSO_{2n}(\mathbb{C}) \). If \( a_{11} = a_{22} = 0 \), then \( X = \begin{pmatrix} a_{12}B \\ \lambda a_{12}^{-1}B^{-1} \end{pmatrix} \) and

\[
\nu(X) = \lambda^{-n} \det(X) = (-1)^n \lambda^{-n} \lambda^n = (-1)^n.
\]

It follows that \( X \in GSO_{2n}(\mathbb{C}) \) if \( n \) is even and \( X \not\in GSO_{2n}(\mathbb{C}) \) if \( n \) is odd. Therefore,

\[
S_\varphi = Z_G(\varphi) \simeq \begin{cases} 
GSp_2(\mathbb{C}) \simeq GL_2(\mathbb{C}) & \text{if } c = -1; \\
GO_{1,1}(\mathbb{C}) & \text{if } c = 1, n \text{ even}; \\
\mathbb{C}^\times & \text{if } c = 1, n \text{ odd}.
\end{cases}
\]

4.3. The Arthur \( R \)-group. Now we can compute \( R_\varphi \), the Arthur \( R \)-group of \( \varphi \). We summarize the above computation as follows.

**Theorem 4.5.** Let \( G = GSpin_{2n+1} \) or \( GSpin_{2n} \) and consider the Siegel Levi subgroup \( M \simeq GL_n \times GL_1 \). Let \( \sigma \simeq \sigma_0 \otimes \psi \) be an irreducible unitary supercuspidal representation of \( M = M(F) \) with corresponding Langlands parameter \( \varphi = \varphi_0 \otimes \psi \).

1. If \( \varphi_0 \not\simeq \bar{\varphi}_0 \otimes \bar{\psi} \), then \( R_{\varphi,\sigma} = R_\varphi = 1 \).
(2) Assume $\varphi_0 \simeq \hat{\varphi}_0 \otimes \hat{\psi}$. If $G = \text{GSpin}_{2n+1}$, then

$$R_{\varphi, \sigma} = R_{\varphi} = \begin{cases} 
1, & \text{if } \varphi_0 \text{ factors through } GO_n(\mathbb{C}); \\
\mathbb{Z}_2, & \text{if } \varphi_0 \text{ factors through } \text{GSp}_n(\mathbb{C}).
\end{cases}$$

If $G = \text{GSpin}_{2n}$, then

$$R_{\varphi, \sigma} = R_{\varphi} = \begin{cases} 
1, & \text{if } \varphi_0 \text{ factors through } \text{GSp}_n(\mathbb{C}); \\
\mathbb{Z}_2, & \text{if } \varphi_0 \text{ factors through } GO_n(\mathbb{C}) \text{ and } n \text{ is even}, \\
1, & \text{if } \varphi_0 \text{ factors through } GO_n(\mathbb{C}) \text{ and } n \text{ is odd}.
\end{cases}$$

**Corollary 4.6.** For $G = \text{GSpin}_{2n+1}$ or $\text{GSpin}_{2n}$, and $M \simeq GL_n \times GL_1$, we have $R(\sigma) \simeq R_{\varphi, \sigma}$, as conjectured by Arthur.

**Proof.** This follows from the theorem and Proposition 4.4. \hfill $\Box$

4.4. **Centralizers (The General Case).** Let $V$ be a finite dimensional complex vector space. Let $B$ be a non-degenerate bilinear form on $V$ and

$$\mathcal{G}_B = \{g \in GL_n(V) \mid B(gu, gv) = \lambda(g)B(u, v), \text{ for some } \lambda(g) \in \mathbb{C}^\times, \forall u, v \in V\}.$$ 

**Lemma 4.7.** Let $\varphi : W'_F \to GL_n(V)$ be an irreducible parameter and let $B$ be a non-degenerate bilinear form on $V$. Then $\varphi$ factors through $\mathcal{G}_B$ if and only if $\varphi \simeq \chi \otimes {}^t\varphi^{-1}$, where $\chi = \lambda \circ \varphi$. If $\varphi$ factors through $\mathcal{G}_B$, then $B$ is unique up to a scalar multiple.

**Proof.** Suppose that $\varphi$ factors through $\mathcal{G}_B$. Let $A$ be the matrix corresponding to $B$, $B(u, v) = {}^t u A v$. Then for all $w \in W'_F$, ${}^t \varphi(w) A \varphi(w) = \lambda(\varphi(w)) A$. It follows

$$\varphi(w) = \chi(w) A^{-1} \varphi(w)^{-1} A^{-1}, \quad \forall w \in W'_F,$$

where $\chi = \lambda \circ \varphi$. Hence, $\varphi \simeq \chi \otimes {}^t\varphi^{-1}$. If $B'$ is another non-degenerate bilinear form on $V$ such that $\varphi$ factors through $\mathcal{G}_{B'}$, and if $A'$ is the corresponding matrix, we have

$$\varphi(w) = \chi(w) A' \varphi(w)^{-1} (A')^{-1}, \quad \forall w \in W'_F,$$

By transposing and taking inverses, equation (4.3) gives us $^t \varphi(w)^{-1} = \chi(w)^{-1} A^{-1} \varphi(w) A$. We substitute this in equation (4.4) and we obtain

$$\varphi(w) = A' A^{-1} \varphi(w) A(A')^{-1}, \quad \forall w \in W'_F.$$ 

Since $\varphi$ is irreducible, it follows $A' A^{-1} = c I$ and $A' = c A$. 

Next, suppose $\varphi \simeq \chi \otimes t^\varphi^{-1}$ for a character $\chi$. Let $A$ be a matrix such that

$$\varphi(w) = \chi(w)A^t\varphi(w)^{-1}A^{-1},$$

for all $w \in W'_F$. Standard arguments show that $A^tA^{-1} = cI$ and $c = \pm 1$. It follows that $B(u, v) = t^uAv$ is a non-degenerate bilinear form such that $\varphi$ factors through $G_B$. \hfill \Box

**Lemma 4.8.** Let $\varphi : W'_F \to G_B$ be a parameter. Suppose $\varphi = \varphi_0 \oplus \cdots \oplus \varphi_m$, where $\varphi_0$ is an irreducible parameter such that $\varphi_0$ factors through $G_{B_0}$ for some non-degenerate bilinear form $B_0$. Then

$$Z_{G_B}(\text{Im } \varphi) \simeq \begin{cases} 
GO(m, \mathbb{C}), & \text{if } B \text{ and } B_0 \text{ are both symmetric or both symplectic}, \\
GSp(m, \mathbb{C}), & \text{otherwise}.
\end{cases}$$

**Proof.** Let $V_0$ denote the space of the representation $\varphi_0$. Then $V \simeq W \otimes V_0$, where $W = \text{Hom}_{W'_F}(V_0, V)$ with trivial $W'_F$-action. The map $W \otimes V_0 \to V$ is given by

$$(4.5) \quad f \otimes v \mapsto f(v), \quad f \in W, v \in V_0.$$ 

For $f, g \in W$, we define a bilinear form $B_{f,g}$ on $V_0$ by $B_{f,g}(u, v) = B(f(u), g(v)).$ Then

$$B_{f,g}(\varphi_0(w)u, \varphi_0(w)v) = B(f(\varphi_0(w)u), g(\varphi_0(w)v)) = B(\varphi(w)f(u), \varphi(w)g(v)) = \lambda \circ \varphi(w)B_{f,g}(u, v).$$

It follows from Lemma 4.7 that $B_{f,g}$ is a scalar multiple of $B_0$; denote that scalar by $(f, g)$. The map $(f, g) \mapsto \langle f, g \rangle$ defines a bilinear form $\langle \cdot, \cdot \rangle$ on $W$. The form $\langle \cdot, \cdot \rangle$ is symmetric if $B$ and $B_0$ are both symmetric or both symplectic, and symplectic otherwise. Moreover, if we identify $W \otimes V_0$ and $V$ using equation (4.5), we have

$$B(f \otimes u, g \otimes v) = B(f(u), g(v)) = B_{f,g}(u, v) = (f, g)B_0(u, v),$$

for all $f, g \in W, u, v \in V_0$.

Now, $\text{Im } \varphi = \{ I_W \otimes g \mid g \in \text{Im } \varphi_0 \}$ and

$$Z_{GL(V)}(\text{Im } \varphi) = \{ g \otimes z \mid g \in GL(W), z = cI_{V_0}, c \in \mathbb{C}^\times \} = \{ g \otimes I_{V_0} \mid g \in GL(W) \}.$$ 

The element $g \otimes I_{V_0}$ belongs to $G_B$ if for some $\lambda \in \mathbb{C}^\times$,

$$B((g \otimes I_{V_0})(f \otimes u), (g \otimes I_{V_0})(h \otimes v)) = \lambda B(f \otimes u, h \otimes v), \quad \forall f, h \in W, \forall u, v \in V_0,$$
Proposition 5.6 of [4]. In particular, we have
that is,
\[(gf, gh) = \lambda(f, h), \quad \forall f, h \in W.\]
It follows \(Z_{G_n}(\text{Im} \varphi) \simeq G_{(.)}.\)

4.5. Reducibility for generic representations. Let \(G = \text{GSpin}_m(F)\) and let \(P = MN\) be a maximal Levi subgroup. Then \(M \simeq \text{GL}_k(F) \times \text{GSpin}_\ell(F)\), where \(2k + \ell = m\). In the case \(\ell = 0\) or \(1\), \(P\) is the Siegel parabolic subgroup and that case was considered earlier. We assume \(\ell > 2\). Let \(\pi = \sigma \otimes \tau\) be an irreducible unitary generic supercuspidal representation of \(M\). Let \(\alpha \in \Delta\) be the unique reduced root of \(\Theta\) in \(N\) and set \(\tilde{\alpha} = (\rho, \alpha)^{-1}\alpha\), where \(\rho\) is half the sum of positive roots in \(N\). We have \(\tilde{\alpha}/i \otimes \pi = \nu^{1/2} \alpha \otimes \tau\). Assume \(\sigma \simeq \tilde{\sigma} \otimes \omega_\tau\). According to [26], exactly one of the following representations is reducible: \(\text{Ind}_{\ell}^G(\sigma \otimes \tau), \text{Ind}_{\ell}^G(\nu^{1/2} \sigma \otimes \tau),\) or \(\text{Ind}_{\ell}^G(\nu \sigma \otimes \tau)\).

**Lemma 4.9.** Let \(G = \text{GSpin}_m(F)\) and \(M \simeq \text{GL}_k(F) \times \text{GSpin}_\ell(F)\), where \(2k + \ell = m\), \(\ell > 2\). Let \(\pi = \sigma \otimes \tau\) be an irreducible unitary generic supercuspidal representation of \(M\). Assume \(\sigma \simeq \tilde{\sigma} \otimes \omega_\tau\). Let \(\varphi_0\) denote the Langlands parameter of \(\sigma\).

(1) Suppose \(G = \text{GSpin}_{2n+1}(F)\). If \(\text{Ind}_{\ell}^G(\nu^{1/2} \sigma \otimes \tau)\) reduces, then \(\varphi_0\) factors through \(\text{GO}_k(\mathbb{C})\). Otherwise, \(\varphi_0\) factors through \(\text{GSp}_k(\mathbb{C})\).

(2) Suppose \(G = \text{GSpin}_{2n}(F)\). If \(\text{Ind}_{\ell}^G(\nu^{1/2} \sigma \otimes \tau)\) reduces, then \(\varphi_0\) factors through \(\text{GSp}_k(\mathbb{C})\). Otherwise, \(\varphi_0\) factors through \(\text{GO}_k(\mathbb{C})\).

**Proof.** Let \(\hat{n}\) denote the Lie algebra of the unipotent radical of \(\hat{M}\). Denote the standard representations of the groups \(\text{GL}_k(\mathbb{C}), \text{GSp}_{2\ell}(\mathbb{C})\) and \(\text{GSO}_{2\ell}(\mathbb{C})\) by \(\rho_k, R_{2\ell}^1\) and \(R_{2\ell}^2\), respectively. Let \(\mu\) be the similitude character of \(\text{GSp}_{2\ell}(\mathbb{C})\) or \(\text{GSO}_{2\ell}(\mathbb{C})\). The adjoint action \(r\) of \(\hat{M}\) on \(\hat{n}\) is described in Proposition 5.6 of [4]. In particular, we have

(a) If \(G = \text{GSpin}_{2n+1}(F)\), then \(r = r_1 \oplus r_2\), where
\[r_1 = \rho_k \otimes \widetilde{R}_{\ell-1}^1, \quad r_2 = \text{Sym}^2 \rho_k \otimes \mu^{-1}.\]

(b) If \(G = \text{GSpin}_{2n}(F)\), then \(r = r_1 \oplus r_2\), where
\[r_1 = \rho_k \otimes \widetilde{R}_{\ell}^2, \quad r_2 = \wedge^2 \rho_k \otimes \mu^{-1}.\]

Let \(P_{\pi, 1}\) and \(P_{\pi, 2}\) be the polynomials defined in [26]. The Langlands-Shahidi \(L\)-function attached to \(\pi\) and \(r_i\) is defined as
\[L(s, \pi, r_i) = P_{\pi, i}(q^{-s})^{-1}.\]
Assume $G = GSpin_{2n}(F)$. Theorem 8.1 of [26] tells us that $\text{Ind}_F^G(\nu^{1/2}\sigma \otimes \tau)$ is reducible if and only if $P_{\tau,2}(1) = 0$. Equivalently, $L(s, \pi, r_2)$ has a pole at $s = 0$. In order to complete the proof, we need the following result.

**Lemma 4.10.** Let $G = GSpin_m$ and $M \simeq GL_k \times GSpin_\ell$. Let $\pi = \sigma \otimes \tau$ be an irreducible admissible generic representation of $M$. Let $\varphi = (\varphi_0, \varphi_\tau)$ be the Langlands parameter attached to $\pi$.

a) If $m = 2n$ is even, then

$$L(s, \pi, r_2) = L(s, \sigma \otimes \tau, \wedge^2 \rho_k \otimes \mu^{-1}) = L(s, \sigma \otimes \psi, \wedge^2 \rho_k \otimes \mu_1^{-1}) = L(s, \varphi_0 \otimes \hat{\psi}^{-1}).$$

b) If $m = 2n + 1$ is odd, then

$$L(s, \pi, r_2) = L(s, \sigma \otimes \tau, \text{Sym}^2 \rho_k \otimes \mu^{-1}) = L(s, \sigma \otimes \psi, \text{Sym}^2 \rho_k \otimes \mu_1^{-1}) = L(s, \varphi_0 \otimes \hat{\psi}^{-1}).$$

**Proof.** We continue with the notation of the proof of Lemma 4.9. Suppose $m = 2n$. Then $\varphi_\tau : W_F \to GSO_{2\ell}(\mathbb{C})$. First, we prove (4.6) holds for any unramified generic $\pi$. By Prop. 2.3(a) of [5] we know $Z(GSp_{2\ell}(F)) = \{\varepsilon_0^*(\lambda) | \lambda \in F^\times\}$. So, the central character of $\tau$ is given by

$$\omega_\tau(\lambda) \text{Id}_{V_\tau} = \tau(e_0^*(\lambda)).$$

Let $\hat{\psi} : W_F \to \mathbb{C}^\times$ be the character attached to $\omega_\tau$ by Class Field Theory. In particular, $\omega_\tau(\varphi_F) = \hat{\psi}(\text{Fr}_F)$, where $\text{Fr}_F$ is the Frobenius class of $F$. Let $\hat{T}$ be the maximal torus of $GSO_{2m}(\mathbb{C})$. Then $\mu(t) = e_0^*(t)$, (by [5] pg. 149). Now, we have

$$L(s, \pi, r_2) = L(s, \sigma \otimes \tau, \wedge^2 \rho_k \otimes \mu^{-1}) = L(s, \wedge^2 \rho_k \otimes \mu^{-1}(\varphi_0, \varphi_\tau)).$$

Note, for $w \in W_F$, we have $\wedge^2 \rho_k \otimes \mu^{-1}(\varphi_0, \varphi_\tau)(w) = \wedge^2(\varphi_0(w))\mu^{-1}(\varphi_\tau(w))$. Now

$$\mu^{-1}(\varphi_\tau(\text{Fr}_F)) = (e_0^*(\varphi_\tau(\text{Fr}_F)))^{-1} = \tau(e_0^*(\varphi_F))^{-1} = \omega_\tau(\varphi_F).$$

So

$$L(s, \pi, r_2) = L(s, \wedge^2 \rho_k \varphi_0 \otimes \hat{\psi}^{-1}) = L(s, \sigma \otimes \omega_\tau, \wedge^2 \rho_k \otimes \mu_1^{-1}).$$

If $S_n$ denotes the $n$-dimensional complex representation of $SL(2, \mathbb{C})$, then $\text{Im}(S_n)$ is orthogonal or symplectic. Therefore, $\mu(\varphi \otimes S_n) = \mu(\varphi)$. We conclude that equation (4.6) holds if $\pi$ has an Iwahori fixed vector. In addition, for $\varphi$ unramified, the Artin $\varepsilon$–factor associated to $\mu(\varphi \otimes S_n)$ is equal to 1.

Now, we apply Theorem 3.5 of [26] to $\pi = \sigma \otimes \tau$ and independently we apply the same theorem to $\sigma \otimes \omega_\tau$. The theorem guarantees existence of the $\gamma$–factors $\gamma_2(s, \sigma \otimes \tau, \psi_F, \hat{w})$ and $\gamma_1(s, \sigma \otimes \omega_\tau, \psi_F, \hat{w})$. 
with the subscripts determined by the ordering of the components of the adjoint representations of the $L$-groups of the Levi subgroups in two distinct situations. Moreover, conditions 1, 3, and 4 from the theorem determine these $\gamma$–factors uniquely. These conditions are satisfied by $\gamma_2(s, \sigma \otimes \tau, \psi_F, \tilde{w})$ and independently by $\gamma_1(s, \sigma \otimes \omega_\tau, \psi_F, \tilde{w})$. In the inductive property 3 for $\gamma_1(s, \sigma \otimes \omega_\tau, \psi_F, \tilde{w})$, only $\sigma$ can be induced from a smaller parabolic subgroup, not $\omega_\tau$. Therefore, if we look at the inductive property for $\gamma_1(s, \sigma \otimes \omega_\tau, \psi_F, \tilde{w})$, the same conditions are satisfied for $\gamma_2(s, \sigma \otimes \tau, \psi_F, \tilde{w})$. Even though we can have additional conditions for $\gamma_2(s, \sigma \otimes \tau, \psi_F, \tilde{w})$, the conditions for $\gamma_1(s, \sigma \otimes \omega_\tau, \psi_F, \tilde{w})$ are enough to guarantee uniqueness. Since we have equality of $\gamma$–factors for representations with Iwahori fixed vectors, we conclude that $\gamma_2(s, \sigma \otimes \tau, \psi_F, \tilde{w}) = \gamma_1(s, \sigma \otimes \omega_\tau, \psi_F, \tilde{w})$. The definition of $L$-functions from [26] then implies (4.6).

The proof of the case $G = GSpin_{2n+1}$ is similar.

We return to the proof of Lemma 4.9. It follows from Lemma 4.3 and Lemma 4.10 that Ind$_P^G(\psi^{1/2} \otimes \tau)$ is reducible if and only if $\psi_0$ factors through $GSp_k(C)$. Finally, we remark the claim will follow in general from the generic $L$–packet conjecture of Shahidi [24].

The proof for $G = GSpin_{2n+1}(F)$ is similar.

Let $G = GSpin_{2n+1}(F)$ and let $\tau$ be a generic discrete series representation of $G$. As in [24], let $\text{Jord}(\tau)$ denote the set of pairs $(\rho, a)$, where $\rho \in 0^\epsilon(GL(d_\rho, F))$ and $a \in \mathbb{Z}^+$ such that $\delta(\rho, a) \propto \tau$ is irreducible and there exists an integer $a'$ of the same parity as $a$ such that $\delta(\rho, a') \propto \tau$ is reducible. Here $\delta \propto \tau = \text{Ind}^{G\ell(d+\ell)}_{GL_d(F) \times G(\ell)}(\delta \otimes \tau)$. The $L$-parameter of $\tau$ is given by

$$\varphi_\tau = \bigoplus_{(\rho, a) \in \text{Jord}(\tau)} \varphi_\rho \otimes S_a,$$

where $\varphi_\rho$ is the $L$-parameter of $\rho$.

**Theorem 4.11.** Let $G = GSpin_{2n+1}$ and consider the Levi subgroup $M \cong GL_k(F) \times GSpin_{2\ell+1}(F)$. Let $\pi = \sigma \otimes \tau$ be a generic discrete series representation of $M$. Let $\varphi$ be the $L$-parameter of $\pi$. Then $R_{\varphi_\pi} \cong R(\pi)$.

**Proof.** The parameter $\varphi$ can be written as $\varphi \simeq \varphi_\sigma \oplus \varphi_\tau \oplus (\hat{\varepsilon}(\varphi_\sigma) \otimes \hat{\psi})$, where $\hat{\psi}$ is the character corresponding to the central character of $\tau$, (restricted to the connected component of the center) by Class Field Theory. The representation $\sigma$ is of the form $\sigma \simeq \delta(\rho, a)$, where $\rho \in 0^\epsilon(GL(d, F))$ and $a \in \mathbb{Z}^+$, $da = k$. Then $\varphi_\sigma = \varphi_\rho \otimes S_a$.

If $\sigma \neq \hat{\sigma} \otimes \omega_\tau$, it is easy to show $R_{\varphi_\pi} = 1$ and $R(\pi) = 1$. Assume $\sigma \simeq \hat{\sigma} \otimes \omega_\tau$. Then $\hat{\varepsilon}(\varphi_\sigma) \otimes \hat{\psi} \simeq \varphi_\sigma$, so $\varphi \simeq \varphi_\sigma \oplus \varphi_\tau \oplus \varphi_\sigma$. 


Theorem 4.12. If \((\rho, a) \in \text{Jord}(\tau)\), then the multiplicity of \(\varphi_\sigma\) in \(\varphi \simeq \varphi_\sigma \oplus \varphi_\tau \oplus \varphi_\sigma\) is three. Lemma 4.8 implies \(R_\varphi \equiv 1\). On the other hand, since \((\rho, a) \notin \text{Jord}(\tau)\), we have \(\sigma \times \tau\) is irreducible, so \(R(\tau) = 1\).

Now, consider the case \(\sigma \simeq \tilde{\sigma} \otimes \omega_\tau\) and \((\rho, a) \notin \text{Jord}(\tau)\). There exist a supercuspidal generic representation \(\tau_{\text{cusp}}\) of \(GSpin_{2m+1}(F)\) and an irreducible generic representation \(\theta\) of \(GL_r(F)\) such that \(\tau\) is a subrepresentation of

\[\theta \ltimes \tau_{\text{cusp}} = i_{GSpin_{2m+1}(F),GL_r(F) \times GSpin_{2m+1}(F)}(\theta \otimes \tau_{\text{cusp}}).\]

We apply the Langlands classification for \(GL_r(F)\) in the subrepresentation setting. It follows that there exist \(\delta(\rho_1, a_1), \delta(\rho_2, a_2), \ldots, \delta(\rho_s, a_s)\) and real numbers \(b_1 < b_2 < \cdots < b_s\) such that \(\theta\) is the unique subrepresentation of the induced representation

\[\nu^{b_1}(\rho_1, a_1) \times \nu^{b_2}(\rho_2, a_2) \times \cdots \times \nu^{b_s}(\rho_s, a_s).\]

For \(i \in \{1, \ldots, s\}\), define \([i] = \{j \in \{1, \ldots, s\} \mid \rho_i \simeq \rho_j\}\). The Casselman square integrability criterion for \(\tau\) implies that for \(i = 1, \ldots, s\), there exists \(j \in [i]\) such that the representation

\[\nu^{b_i}(\rho_j, a_j) \ltimes \tau_{\text{cusp}}\]

is reducible, and \(b_i - b_j \in \mathbb{Z}\). This implies \(b_j \in \frac{1}{2}\mathbb{Z}\) and therefore \(b_i \in \frac{1}{2}\mathbb{Z}\).

Assume first \(\sigma \times \tau\) is reducible. Then \(R(\pi) \simeq \mathbb{Z}_2\). It can be shown, taking into account the structure of \(\theta\), that reducibility of \(\sigma \times \tau\) implies reducibility of \(\sigma \times \tau_{\text{cusp}}\). Then there exists \(b \geq 0, b \in \left\{-\frac{(a-1)}{2}, -\frac{(a-1)}{2} + 1, \ldots, \frac{(a-1)}{2}\right\}\) such that \(\nu^b \rho \ltimes \tau_{\text{cusp}}\) is reducible. Since \(\tau_{\text{cusp}}\) is supercuspidal and generic, we have \(b = 0, 1/2\) or \(1\). If \(b = 1/2\), then \(a\) is even. In addition, Lemma 4.9 implies that \(\varphi_{\rho}\) factors through \(GO_d(\mathbb{C})\). Then \(\varphi_\sigma = \varphi_\rho \otimes S_a\) factors through \(GSp_k(\mathbb{C})\). Now Lemma 4.8 tells us that \(S_\varphi \simeq GO(2, \mathbb{C})\). It follows \(R_\varphi = R_{\varphi, \pi} \simeq \mathbb{Z}_2\). If \(b = 0\) or \(1\), then \(a\) is odd. In addition, Lemma 4.9 implies that \(\varphi_{\rho}\) factors through \(GSp_d(\mathbb{C})\). Then \(\varphi_\sigma = \varphi_\rho \otimes S_a\) factors through \(GSp_k(\mathbb{C})\).

As before, we obtain \(R_{\varphi, \pi} \simeq \mathbb{Z}_2\).

It remains to consider the case when \(\sigma \times \tau\) is irreducible, \(\sigma \simeq \tilde{\sigma} \otimes \omega_\tau\) and \((\rho, a) \notin \text{Jord}(\tau)\). Irreducibility of \(\sigma \times \tau\) implies \(R(\pi) = 1\). Let \(b \in \{0, 1/2, 1\}\) such that \(\nu^b \rho \ltimes \tau_{\text{cusp}}\) is reducible. Since \((\rho, a) \notin \text{Jord}(\tau)\), \(a\) and \(2b + 1\) are not of the same parity. Therefore, if \(b = 1/2\), then \(a\) is odd. Then \(\varphi_{\rho}\) factors through \(GO_d(\mathbb{C})\) and \(\varphi_\sigma = \varphi_\rho \otimes S_a\) factors through \(GO_k(\mathbb{C})\). It follows \(R_\varphi = R_{\varphi, \pi} = 1\).

Similarly, if \(b = 0\) or \(1\), then \(a\) is even, \(\varphi_{\rho}\) factors through \(GSp_d(\mathbb{C})\) and \(\varphi_\sigma = \varphi_\rho \otimes S_a\) factors through \(GO_k(\mathbb{C})\), implying \(R_\varphi = R_{\varphi, \pi} = 1\).

\(\square\)

**Theorem 4.12.** Let \(G = GSpin_{2n+1}\) and \(P = MN\) be an arbitrary parabolic subgroup of \(G\). Suppose \(\pi\) is a discrete series representation of \(M\) and \(\varphi = \varphi_\pi : W_F \to L^M\) is the corresponding
Langlands parameter for the $L$-packet $\Pi_M(\varphi)$ containing $\pi$. Let $R(\pi)$ be the Knapp-Stein $R$-group of $\pi$ and $R_{\varphi,\pi}$ the Arthur $R$-group attached to $\varphi$ and $\pi$. Then $R(\pi) \simeq R_{\varphi,\pi}$, and this isomorphism is realized by the map $\alpha \mapsto \hat{\alpha}$ between roots and coroots.

**Proof.** By Lemma 4.1 it is enough to prove this isomorphism in the case $P$ is maximal. This, however, is exactly the content of Corollary 4.6 and Theorem 4.11.

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