FaVeST: Fast Vector Spherical Harmonic Transforms

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Vector spherical harmonics on the unit sphere of $\mathbb{R}^3$ have wide applications in geophysics, quantum mechanics and astrophysics. In the representation of a tangent vector field, one needs to evaluate the expansion and the Fourier coefficients of vector spherical harmonics. In this paper, we develop fast algorithms (FaVeST) for vector spherical harmonic transforms on these evaluations. The forward FaVeST evaluates the Fourier coefficients and has computational cost proportional to $N \log \sqrt{N}$ for $N$ number of evaluation points. The adjoint FaVeST which evaluates a linear combination of vector spherical harmonics with degree up to $\sqrt{M}$ for $M$ evaluation points has cost proportional to $M \log \sqrt{M}$. Numerical examples of simulated tangent fields illustrate the accuracy and efficiency of FaVeST.

CCS Concepts: • Mathematics of computing → Mathematical analysis; Numerical analysis; Computations of transforms;

Additional Key Words and Phrases: Vector spherical harmonics, tangent vector fields, FFT

1 INTRODUCTION

Vector spherical harmonics on the unit sphere $S^2$ in $\mathbb{R}^3$ are widely used in many areas such as astrophysics [1–3], quantum mechanics [29, 34, 66], geophysics and geomagnetics [5, 15, 21, 37, 38], 3D fluid mechanics [60, 61], global atmospherical modelling [13, 19, 30] and climate change modelling [53–55]. For example, in constructing the numerical solution to the Navier–Stokes equations on the unit sphere [17], divergence-free vector spherical harmonics are used. In simulating scattering waves by single or multiple spherical scatterers [43, 62] modelled by the 3-dimensional Helmholtz equation, both divergence-free and curl-free vector spherical harmonics are used. In these problems, the solutions which are vector fields are represented by an expansion of vector spherical harmonics. One then needs to evaluate the coefficients of vector spherical harmonics for a vector field and also a linear combination of vector spherical harmonics. They can be evaluated by the (discrete) forward vector spherical harmonic transform (FwdVSHT) and adjoint vector spherical harmonic transform (AdjVSHT) respectively. In this paper, we develop fast algorithms for the forward and adjoint vector spherical harmonic transforms for tangent (vector) fields on $S^2$ and their software implementation.

Let $\{ (y_{\ell, m}, z_{\ell, m}) : \ell = 1, 2, \ldots, m = -\ell, \ldots, \ell \}$ be a set of pairs of the (complex-valued) divergence-free and curl-free vector spherical harmonics on $S^2$. The coefficients for divergence-free and curl-free vector spherical harmonics are given by, for $\ell = 1, 2, \ldots, m = -\ell, \ldots, \ell$,

$\bar{T}_{\ell, m} = \int_{S^2} T(x) y_{\ell, m}^* (x) d\sigma(x), \quad \bar{T}_{\ell, m} = \int_{S^2} T(x) z_{\ell, m}^* (x) d\sigma(x),$

where the $V^*$ is the complex conjugate transpose of the tangent field $V$. The FwdVSHT for a spherical tangent field $T : S^2 \to \mathbb{C}^3$ evaluates these coefficients by approximating the integrals of the coefficients with a quadrature rule which
is a set of $N$ pairs of weights $w_i$ and points $x_i$ on $S^2$:

$$\widehat{T}_{\ell, m} \simeq \sum_{i=1}^{N} w_i T(x_i) y_{\ell, m}^i(x_i), \quad \overline{\widehat{T}}_{\ell, m} \simeq \sum_{i=1}^{N} w_i T(x_i) z_{\ell, m}^i(x_i). \quad (1)$$

The AdjVSHT evaluates the expansion of $y_{\ell, m}, z_{\ell, m}$ with two complex sequences $a_{\ell, m}, b_{\ell, m}, \ell = 1, \ldots, M \geq 1$:

$$\sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} (a_{\ell, m} y_{\ell, m}(x_i) + b_{\ell, m} z_{\ell, m}(x_i)), \quad i = 1, \ldots, M. \quad (2)$$

To directly compute the summation of (1) for the coefficients with degree up to $L$ for $L \geq 1$, the computational cost for FwdVSHT is $O(NL^2)$. Here the quadrature rule should be properly chosen to minimize the approximation error. The optimal-order number of nodes for this purpose is $N = O(L^2)$. (See Section 4.3.) Thus, the cost of direct computation for forward vector spherical harmonic transform is $O(N^2)$. On the other hand, to directly compute the expansion (2) with truncation degree $L$ for $\ell$ incurs $O(ML^2)$ computational steps. Then, to evaluate $M = O(L^2)$ points, the computational complexity for direct evaluation of AdjVSHT is $O(M^2)$.

In this paper, we develop fast computational strategies for FwdVSHT and AdjVSHT by explicit representations for the divergence-free and curl-free vector spherical harmonics in terms of scalar spherical harmonics. Here the close representation formula exploit Clebsch-Gordan coefficients in quantum mechanics. By this way, the fast scalar spherical harmonic transforms can be applied to speed up computation. The resulting algorithm reduces the computational cost to $O(N \log N)$ and $O(M \log M)$, both of which are nearly linear. We thus call the algorithms the Fast Vector Spherical Harmonic Transform or FaVeST. A software package in Matlab is provided for implementing FaVeST. We then validate the FaVeST algorithm by the numerical examples of simulated tangent fields on the sphere. The algorithm accompanied by its software implementation fills the blank of fast algorithms for vector spherical harmonic transforms.

The rest of this paper is organized as follows. In Section 2, we review the fast Fourier transforms for scalar spherical harmonics on $S^2$ and related works on computation of vector spherical harmonic transforms. In Section 3, we introduce definitions and notation on scalar and vector spherical harmonics. In Sections 4.1 and 4.2, we give the representations for FwdVSHT and AdjVSHT in terms of forward and adjoint scalar spherical harmonic transforms. From these representations, in Section 4.3, we describe the fast algorithms (FaVeST) for the evaluation of FwdVSHT and AdjVSHT, and show that the computational complexity of the proposed FaVeST is nearly linear. We also estimate the approximation error for forward and adjoint FaVeSTs. Section 4.4 describes the software package in Matlab developed for FaVeST. In Section 5, we give numerical examples of simulated spherical tangent fields to test FaVeST.

2 RELATED WORKS

Fast Fourier Transform (FFT) for $\mathbb{R}^d$ is one of the most influential algorithms in science and engineering [8, 45, 46, 56]. On the sphere, fast transforms for scalar spherical harmonics have been extensively studied by many researchers [4, 11, 20, 22, 23, 31, 32, 39, 42, 48, 50–52, 57, 58]. In particular, Keiner et al. provide the software library NFFT [32] which implements the fast forward and adjoint FFT algorithms for scalar spherical harmonics based on the non-equispaced FFT [31, 33, 35]. Their package is easy to use in Matlab environment and has been applied in many areas. Suda and Takami in [52] propose a fast scalar spherical harmonic transform algorithm with computational complexity $O(N \log \sqrt{N})$ based on the divide-and-conquer approach with split Legendre functions (where $N$ is the number of nodes in the discretization for integral on $S^2$), and the algorithm is used to solve the shallow water equation [51].
Rokhlin and Tygert [48] develop the fast algorithms for scalar spherical harmonic expansion with computational time proportional to $N \log \sqrt{N} \log(1/\epsilon)$ for a given precision $\epsilon > 0$. Later, Tygert [57, 58] improves their algorithm to achieve computational cost proportional to $N \log \sqrt{N}$ at any given precision. Reinecke and Seljebotn and Gorski et al. [20, 47] develop the FFTs for spherical harmonics to evaluate Hierarchical Equal Area isoLatitude Pixelation (HEALPix) points [20] in various programming languages. Their algorithm can work on millions of evaluation points with spherical harmonic degree up to 6, 143.

In contrast, fast transforms for vector spherical harmonics receive less attention. To the best of our knowledge, there are no existing fast algorithms for the forward and adjoint vector spherical harmonic transforms. Ganesh et al. [17] simply use FFTs to speed up their algorithms for solving Navier-Stokes PDEs on the unit sphere. However, their method is based on the idea that applies conventional FFTs to evaluate complex azimuthal exponential terms involved in the formulation of vector spherical harmonics. As fast Legendre transforms are not implemented, their method is not a fast transform. Wang et al. [62] evaluate vector spherical harmonic expansions via spectral element grids, which is neither fast computation.

3 VECTOR SPHERICAL HARMONICS

In this section, we present definitions and properties about spherical tangent fields, scalar and vector spherical harmonics and Clebsch-Gordan coefficients, see e.g. [9, 12], which we will use in the representation of FwdVSHT and AdjVSHT in the next section. A tangent (vector) field $T$ is a mapping from $S^2$ to $\mathbb{C}^3$ satisfying the normal component $(T \cdot x)x$ of $T$ is zero, here $T \cdot x := \sum_{i=1}^3 T^{(i)}(x^{(i)})$ is the inner product of $\mathbb{C}^3$ for column vectors $T := (T^{(1)}, T^{(2)}, T^{(3)})'$ and $x := (x^{(1)}, x^{(2)}, x^{(3)})'$, and the $'$ denotes the transpose of a vector (or matrix). Let $L_2(S^2)$ be $L_2$ space of tangent fields on the sphere $S^2$ with inner product

$$\langle T, V \rangle = \int_{S^2} T^*(x)V(x)d\sigma(x)$$

and $L_2$ norm $\|T\|_2 = \sqrt{\langle T, T \rangle}$, where $T^*(x)$ is the complex conjugate transpose of $T(x)$. Using spherical coordinates, the scalar spherical harmonics can be explicitly written as, for $\ell = 0, 1, \ldots$

$$Y_{\ell,m}(x) := Y_{\ell,m}(\theta, \varphi) := \sqrt{\frac{2\ell + 1}{4\pi} \frac{\ell!}{(\ell - m)!}} Y_{\ell,m}(\cos \theta) e^{im\varphi}, \quad m = 0, 1, \ldots, \ell,$$

$$Y_{\ell,-m}(x), \quad m = -\ell, \ldots, -1.$$  

We would suppress the variable $x$ in $Y_{\ell,m}(x)$ if no confusion arises.

In the following, we introduce vector spherical harmonics, see e.g. [12, 59]. For each $\ell = 1, 2, \ldots, m = -\ell, \ldots, \ell$, and integers $j_1, j_2, m_1, m_2$ satisfying $j_2 \geq j_1 \geq 0, j_2 - j_1 \leq \ell \leq j_2 + j_1$ and $-j_1 \leq m_1 \leq j_1, i = 1, 2$, the Clebsch-Gordan (CG) coefficients are

$$C_{j_1, j_2, m_1, m_2}^{\ell, m} := (-1)^{m+j_1-j_2}2\ell + 1 \begin{pmatrix} j_1 & j_2 & \ell \\ m_1 & m_2 & -m \end{pmatrix},$$

see e.g. [40, Chapter 3.5]. The covariant spherical basis vectors are

$$e_+ = -\frac{1}{\sqrt{2}} ( [1, 0, 0] + i[0, 1, 0])^T, \quad e_0 = [0, 0, 1]^T, \quad e_- = \frac{1}{\sqrt{2}} ( [1, 0, 0] - i[0, 1, 0])^T. \quad (3)$$

Let

$$c_{\ell} := \sqrt{\frac{\ell + 1}{2\ell + 1}}, \quad d_{\ell} := \sqrt{\frac{\ell}{2\ell + 1}}. \quad (4)$$
We define the coefficients
\[
B_{+1, \ell, m} = c_{\ell} C_{\ell-1, m-1, 1, 1}^\ell Y_{\ell-1, m-1}^2 + d_{\ell} C_{\ell+1, m-1, 1, 1}^\ell Y_{\ell+1, m-1}^2,
\]
\[
B_{0, \ell, m} = c_{\ell} C_{\ell-1, m, 1, 0}^\ell Y_{\ell-1, m}^2 + d_{\ell} C_{\ell+1, m, 1, 0}^\ell Y_{\ell+1, m}^2,
\]
\[
B_{-1, \ell, m} = c_{\ell} C_{\ell+1, m+1, 1, -1}^\ell Y_{\ell-1, m+1}^2 + d_{\ell} C_{\ell+1, m+1, 1, -1}^\ell Y_{\ell+1, m+1}^2,
\]
and
\[
D_{+1, \ell, m} = i C_{\ell, m-1, 1, 1}^\ell Y_{\ell, m-1}^2,
\]
\[
D_{0, \ell, m} = i C_{\ell, m, 1, 0}^\ell Y_{\ell, m}^2,
\]
\[
D_{-1, \ell, m} = i C_{\ell, m+1, 1, -1}^\ell Y_{\ell, m+1}^2.
\]

**Definition 3.1 (Vector Spherical Harmonics).** For \(\ell = 1, 2, \ldots, m = -\ell, \ldots, \ell\), using the notation of (3), (5) and (6), the divergence-free and curl-free vector spherical harmonics are defined by
\[
y_{\ell, m} = B_{+1, \ell, m} e_{+1} + B_{0, \ell, m} e_0 + B_{-1, \ell, m} e_{-1}.
\]
\[
z_{\ell, m} = D_{+1, \ell, m} e_{+1} + D_{0, \ell, m} e_0 + D_{-1, \ell, m} e_{-1}.
\]

Or equivalently, by (3) and (7),
\[
y_{\ell, m} = \left( -\frac{1}{\sqrt{2}} \begin{bmatrix} B_{+1, \ell, m} - B_{-1, \ell, m} \\ B_{+1, \ell, m} + B_{-1, \ell, m} \end{bmatrix} \right),
\]
\[
z_{\ell, m} = \left( -\frac{1}{\sqrt{2}} \begin{bmatrix} D_{+1, \ell, m} - D_{-1, \ell, m} \\ D_{+1, \ell, m} + D_{-1, \ell, m} \end{bmatrix} \right).
\]

The set of vector spherical harmonics \(\{y_{\ell, m}, z_{\ell, m}: \ell = 1, 2, \ldots, m = -\ell, \ldots, \ell\}\) in (7) or (8) forms an orthonormal basis for \(L_2(S^2)\). Using the property of 3-j symbols in [10], the CG coefficients in (5) and (6) have the following explicit formula.
\[
C_{\ell-1, m-1, 1}^\ell = \sqrt{\frac{(\ell + m)(\ell + m - 1)}{(2\ell)(2\ell - 1)}},
\]
\[
C_{\ell-1, m, 1, 0}^\ell = \frac{\ell + m}{\ell(2\ell - 1)},
\]
\[
C_{\ell-1, m+1, 1, -1}^\ell = \frac{\ell - m}{\ell(2\ell - 1)},
\]
\[
C_{\ell, m-1, 1, 1}^\ell = \frac{m}{\ell(2\ell + 2)},
\]
\[
C_{\ell, m, 1, 0}^\ell = \frac{m}{\ell},
\]
\[
C_{\ell, m+1, 1, -1}^\ell = \frac{(\ell + m)(\ell + m + 1)}{(2\ell + 2)}.
\]

The expression of these CG coefficients in (9) will simplify computations in the proposed fast algorithms below, and also help with the interested readers follow the routines in our software.

**4 Fast Vector Spherical Harmonic Transforms**

In Subsections 4.1 and 4.2 below, we prove two theorems to represent the vector spherical harmonic transforms by scalar spherical harmonics and Clebsch-Gordan coefficients. From the representation formula, we obtain a computational
strategy for fast evaluation of FwdVSHT and AdjVSHT. Subsection 4.3 shows the analysis of computational complexity and approximation error for the proposed FaVeST. In Subsection 4.4, we provide the “user guide” of the software implementation for FaVeST in Matlab environment.

4.1 Fast Computation for FwdVSHT

In this section, we provide an efficient way to evaluate the Fourier coefficients for vector spherical harmonics. The evaluation is based on the connection between the coefficients for vector and scalar spherical harmonics. This connection together with FFTs for scalar spherical harmonics allows fast computation of FwdVSHT and AdjVSHT.

The divergence-free and curl-free coefficients of a tangent field $T$ on $S^2$ are, for $\ell = 1, 2, \ldots, m = -\ell, \ldots, \ell$,

$$\hat{T}_{\ell,m} := \langle T, y_{\ell,m} \rangle, \quad \hat{\ell}_{\ell,m} := \langle T, x_{\ell,m} \rangle.$$ 

To numerically evaluate them, one needs to discretize the integrals of the coefficients by a quadrature rule [27], which is a set $Q_N := \{(w_i, x_i)\}_{i=1}^N$ of $N, N \geq 2$, pairs of real numbers and points on $S^2$.

**Definition 4.1 (FwdVSHT).** For a sequence of column vectors $\{T_k\}_{k=1}^N$ in $\mathbb{R}^3$, discrete forward divergence-free and curl-free transforms for $\{T_k\}_{k=1}^N$ associated with $Q_N$, or simply FwdVSHT, are the weighted sums

$$\tilde{F}_{\ell,m}(T_k) := \tilde{F}_{\ell,m}(\langle T_k \rangle_{k=1}^N, Q_N) := \sum_{k=1}^N w_k y_{\ell,m}(x_k) T_k, \quad \tilde{F}_{\ell,m}(T) := \tilde{F}_{\ell,m}(\langle T_k \rangle_{k=1}^N) := \sum_{k=1}^N w_k y_{\ell,m}(x_k) T(T_k).$$

(10)

For a tangent field $T$, the divergence-free and curl-free coefficients can be approximated by FwdVSHT for the sequence of values $\{T(x_i)\}_{i=1}^N$ of the tangent field at quadrature nodes $\{x_i\}_{i=1}^N$:

$$\tilde{T}_{\ell,m} \approx \tilde{F}_{\ell,m}(\langle T(x_i)\rangle_{i=1}^N, Q_N) := \sum_{k=1}^N w_k y_{\ell,m}(x_k) T(x_k), \quad \tilde{\ell}_{\ell,m} \approx \tilde{F}_{\ell,m}(\langle T(x_i)\rangle_{i=1}^N) := \sum_{k=1}^N w_k z_{\ell,m}(x_k) T(x_k).$$

(11)

As mentioned, we call (10) forward vector spherical harmonic transform, or FwdVSHT for tangent field $T$.

Now, we design fast computation for FwdVSHT in Definition 4.1 using the scalar version of FwdVSHT. Let $\{f_k\}_{k=1}^N$ be a real sequence. For $\ell \geq 1, m = -\ell, \ldots, \ell$, the discrete forward scalar spherical harmonic transform (FwDSHT) is

$$F_{\ell,m}(f) := F_{\ell,m}(f_k) := F_{\ell,m}(\langle f_k \rangle_{k=1}^N, Q_N) := \sum_{k=1}^N w_k f_k y_{\ell,m}(x_k).$$

(12)

They are approximations of spherical harmonic coefficients $\{f, y_{\ell,m}\}$ for $f$ by the quadrature rule $Q_N$. The following theorem shows a representation of the FwdVSHT by FwdVSHT for a sequence $\{T_k\}_{k=1}^N$, which would allow us to efficiently compute $\tilde{F}_{\ell,m}(T_k)$ and $\tilde{F}_{\ell,m}(T)$. For each $k = 1, \ldots, N$, let $(t^{(1)}_k, t^{(2)}_k, t^{(3)}_k)$ be the components of the vector $T_k$. For $\ell = 1, \ldots, m = -\ell, \ldots, \ell$, using the notation of (4) and (9), we define the coefficients

$$s^{(1)}_{\ell,m} := c_{\ell+1} C_{\ell+1,m+1}, \quad s^{(2)}_{\ell,m} := d_{\ell-1} C_{\ell-1,m+1},$$
$$s^{(3)}_{\ell,m} := c_{\ell+1} C_{\ell+1,m-1}, \quad s^{(4)}_{\ell,m} := d_{\ell-1} C_{\ell-1,m-1},$$
$$s^{(5)}_{\ell,m} := c_{\ell+1} C_{\ell+1,0}, \quad s^{(6)}_{\ell,m} := d_{\ell-1} C_{\ell-1,0},$$

and

$$p^{(1)}_{\ell,m} := C_{\ell,m+1}, \quad p^{(2)}_{\ell,m} := C_{\ell,m+1}, \quad p^{(3)}_{\ell,m} := C_{\ell,m-1}.$$ (13)
Theorem 4.2. Let \( \{T_k\}_{k=1}^N \) be a sequence of column vectors in \( \mathbb{R}^3 \) and \( Q_N := \{(w_i, x_i)\}_{i=1}^N \) a quadrature rule on \( S^2 \). Then, for \( \ell = 1, 2, \ldots, m = -\ell, \ldots, \ell \), the FowdVSHT can be represented by its scalar version FowdSHT, as follows.

\[
\begin{align*}
\hat{F}_{\ell,m}(T) &= \frac{1}{\sqrt{2}} \left[ \xi^{(1)}_{\ell-1,m-1} \left[-F_{\ell-1,m-1} \left(T^{(1)}\right) + i F_{\ell-1,m-1} \left(T^{(2)}\right)\right] + \xi^{(2)}_{\ell-1,m-1} \left[-F_{\ell+1,m-1} \left(T^{(1)}\right) + i F_{\ell+1,m-1} \left(T^{(2)}\right)\right] \\
&\quad + \xi^{(3)}_{\ell-1,m+1} \left[F_{\ell-1,m+1} \left(T^{(1)}\right) + i F_{\ell-1,m+1} \left(T^{(2)}\right)\right] + \xi^{(4)}_{\ell+1,m+1} \left[F_{\ell+1,m+1} \left(T^{(1)}\right) + i F_{\ell+1,m+1} \left(T^{(2)}\right)\right] \\
&\quad + \xi^{(5)}_{\ell+1,m} F_{\ell+1,m} \left(T^{(3)}\right) + \xi^{(6)}_{\ell+1,m} F_{\ell+1,m} \left(T^{(3)}\right),
\end{align*}
\]

\[
\widetilde{F}_{\ell,m}(T) = -\frac{1}{\sqrt{2}} \left[ \mu^{(1)}_{\ell,m-1} \left[-F_{\ell,m-1} \left(T^{(1)}\right) + i F_{\ell,m-1} \left(T^{(2)}\right)\right] + \mu^{(2)}_{\ell,m+1} \left[F_{\ell,m+1} \left(T^{(1)}\right) + i F_{\ell,m+1} \left(T^{(2)}\right)\right] \\
= -i\mu^{(2)}_{\ell,m} F_{\ell,m} \left(T^{(3)}\right),
\]

(15)

where we use the notation of (10), (12), (13) and (14).

Proof. Let \( \ell = 1, 2, \ldots, m = -\ell, \ldots, \ell \). By (5) and (13), the discrete forward divergence-free transform

\[
\begin{align*}
\hat{F}_{\ell,m}(T) &= \sum_{k=1}^{N} w_k Y^*_{\ell,m}(x_k)T_k \\
&= \sum_{k=1}^{N} w_k \left( \frac{1}{\sqrt{2}} T^{(1)}_k \left(B^*_+\ell,m(x_k) - B^*_-\ell,m(x_k)\right) + \frac{1}{\sqrt{2}} i T^{(2)}_k \left(B^*_+\ell,m(x_k) + B^*_-\ell,m(x_k)\right) \right) \\
&\quad + T^{(3)}_k B^*_{0,\ell,m}(x_k)
\end{align*}
\]

\[
= \sum_{k=1}^{N} w_k \left( \frac{1}{\sqrt{2}} \left(-T^{(1)}_k + iT^{(2)}_k\right) c_{\ell,m} \xi^{(1)}_{\ell-1,m-1,1} Y^*_{\ell-1,m-1}(x_k) \\
+ \left( \frac{1}{\sqrt{2}} \left(-T^{(1)}_k + iT^{(2)}_k\right) d_{\ell,m} \xi^{(2)}_{\ell-1,m-1,1} Y^*_{\ell-1,m-1}(x_k) \right) \\
+ \left( \frac{1}{\sqrt{2}} \left(T^{(1)}_k + iT^{(2)}_k\right) c_{\ell,m} \xi^{(3)}_{\ell+1,m+1,1} Y^*_{\ell+1,m+1}(x_k) \right) \\
+ \left( \frac{1}{\sqrt{2}} \left(T^{(1)}_k + iT^{(2)}_k\right) d_{\ell,m} \xi^{(4)}_{\ell+1,m+1,1} Y^*_{\ell+1,m+1}(x_k) \right) \\
+ \left( \xi^{(5)}_{\ell+1,m} F_{\ell+1,m} \left(T^{(3)}\right) + \xi^{(6)}_{\ell+1,m} F_{\ell+1,m} \left(T^{(3)}\right) \right)
\right)
\]

where we use the notation of (10), (12), (13) and (14).
In a similar way, for the curl-free case, we use (6) and (14) to obtain

\[
\bar{F}_{\ell,m}(T) := \sum_{k=1}^{N} w_k \bar{z}_{\ell,m}(x_k) T(x_k)
\]

\[
= \sum_{k=1}^{N} w_k \left\{ -\frac{1}{\sqrt{2}} i \left[ \mu^{(1)}_{\ell,m-1} Y_{\ell,m-1}(x_k) \left( -T^{(1)}_k + iT^{(2)}_k \right) + \mu^{(3)}_{\ell,m+1} Y^*_{\ell,m+1}(x_k) \left( T^{(1)}_k + iT^{(2)}_k \right) \right] \\
- \mu^{(2)}_{\ell,m} Y_{\ell,m}(x_k) T^{(3)}_k \right\}
\]

\[
= -\frac{1}{\sqrt{2}} i \left[ \mu^{(1)}_{\ell,m-1} \left( -T^{(1)} + iT^{(2)} \right) + \mu^{(3)}_{\ell,m+1} \left( T^{(1)} + iT^{(2)} \right) \right] \\
- \mu^{(2)}_{\ell,m} F_{\ell,m} T^{(3)}
\]

thus completing the proof. \(\Box\)

4.2 Fast Computation for AdjVSHT

In this section, we study the adjoint vector spherical harmonic transforms for a tangent field on the sphere.

**Definition 4.3 (AdjVSHT).** For \(L \geq 1\) and two complex sequences \((a_{\ell,m}, b_{\ell,m} : \ell = 1, 2, \ldots, m = -\ell, \ldots, \ell)\), the adjoint vector spherical harmonic transform or AdjVSHT of degree \(L\) is the Fourier partial sum

\[
S_L(a_{\ell,m}, b_{\ell,m}; x) := \sum_{\ell=1}^{L} \sum_{m=-\ell}^{\ell} \left( a_{\ell,m} Y_{\ell,m}(x) + b_{\ell,m} z_{\ell,m}(x) \right), \quad x \in \mathbb{S}^2.
\]

(16)

Or equivalently, by (3) and (7),

\[
S_L(a_{\ell,m}, b_{\ell,m}) := \sum_{\ell=1}^{L} \sum_{m=-\ell}^{\ell} \left( -\frac{1}{\sqrt{2}} \left( a_{\ell,m} B_{\ell-1,m+1} + a_{\ell,m} B_{\ell-1,m-1} - a_{\ell,m} B_{\ell+1,m-1} \right) \right) + \sum_{\ell=1}^{L} \sum_{m=-\ell}^{\ell} \left( -\frac{1}{\sqrt{2}} \left( b_{\ell,m} B_{\ell+1,m} + b_{\ell,m} B_{\ell-1,m} \right) \right).
\]

(17)

For \(L \geq 0\) and a finite complex sequence \(g_{\ell,m}, \ell = 0, 1, \ldots, L, m = -\ell, \ldots, \ell\), the adjoint scalar spherical harmonic transform or AdjSHT of degree \(L\) is the Fourier partial sum of scalar spherical harmonics \(Y_{\ell,m}\):

\[
S_L(g_{\ell,m}) := \sum_{\ell=0}^{L} \sum_{m=-\ell}^{\ell} g_{\ell,m} Y_{\ell,m}.
\]

(18)

The AdjVSHT can be represented by its scalar version with CG coefficients, as shown by the following theorem.

With the notation of (4) and (9), we define the coefficients

\[
v^{(1)}_{\ell,m} := c_{\ell+1} \left( a_{\ell+1,m+1} C_{\ell,m+1}^{\ell+1,m+1} - a_{\ell+1,m-1} C_{\ell,m+1}^{\ell+1,m-1} \right), \quad \ell = 0, \ldots, L - 1, \ m = -\ell, \ldots, \ell,
\]

\[
v^{(2)}_{\ell,m} := \left\{ \begin{array}{ll} d_{\ell-1} \left( a_{\ell-1,m+1} C_{\ell,m+1}^{\ell-1,m+1} - a_{\ell-1,m-1} C_{\ell,m+1}^{\ell-1,m-1} \right), & \ell = 2, \ldots, L + 1, \ |m| = 0, 1, \ldots, \ell - 2, \\
0, & \ell = 0, 1 \text{ or } |m| = \ell - 1, \ell, \end{array} \right.
\]

\[
v^{(3)}_{\ell,m} := i d_{\ell+1} \left( a_{\ell+1,m+1} C_{\ell,m+1}^{\ell+1,m+1} + a_{\ell+1,m-1} C_{\ell,m+1}^{\ell+1,m-1} \right), \quad \ell = 0, \ldots, L - 1, \ m = -\ell, \ldots, \ell,
\]

\[
v^{(4)}_{\ell,m} := \left\{ \begin{array}{ll} id_{\ell-1} \left( a_{\ell-1,m+1} C_{\ell,m+1}^{\ell-1,m+1} + a_{\ell-1,m-1} C_{\ell,m+1}^{\ell-1,m-1} \right), & \ell = 2, \ldots, L + 1, \ m = -\ell, \ldots, \ell, \\
0, & \ell = 0, 1 \text{ or } |m| = \ell - 2, \ell - 1, \end{array} \right.
\]

(19)
\[
\begin{align*}
    v_{\ell,m}^{(5)} &:= a_{\ell+1,m} c_{\ell+1,m}^{\ell+1,m} \quad \ell = 0, \ldots, L - 1, \ m = -\ell, \ldots, \ell, \\
    v_{\ell,m}^{(6)} &:= \begin{cases} 
        a_{\ell-1,m} d_{\ell-1,m}^{\ell-1,m} & \ell = 2, \ldots, L + 1, \ |m| = 0, 1, \ldots, \ell - 1, \\
        0, & \ell = 0 \text{ or } |m| = \ell,
    \end{cases}
\end{align*}
\]  

and

\[
\begin{align*}
    \eta_{\ell,m}^{(1)} &:= \begin{cases} 
        i \left( b_{\ell,m}^{\ell,m+1} - b_{\ell,m-1}^{\ell,m-1} \right) & \ell = 1, \ldots, L, \ m = -\ell, \ldots, \ell, \\
        0, & \ell = 0, \ m = 0,
    \end{cases}
\end{align*}
\]

\[
\begin{align*}
    \eta_{\ell,m}^{(2)} &:= \begin{cases} 
        b_{\ell,m}^{\ell,m+1} c_{\ell,m}^{\ell,m+1} + b_{\ell,m-1}^{\ell,m-1} c_{\ell,m}^{\ell,m-1} & \ell = 1, \ldots, L, \ m = -\ell, \ldots, \ell, \\
        0, & \ell = 0, \ m = 0,
    \end{cases}
\end{align*}
\]

\[
\begin{align*}
    \eta_{\ell,m}^{(3)} &:= \begin{cases} 
        ib_{\ell,m}^{\ell,m} c_{\ell,m}^{\ell,m} & \ell = 1, \ldots, L, \ m = -\ell, \ldots, \ell, \\
        0, & \ell = 0, \ m = 0.
    \end{cases}
\end{align*}
\]

**Theorem 4.4.** Let \(\{a_{\ell,m}, b_{\ell,m} : \ell = 1, 2, \ldots, m = -\ell, \ldots, \ell\}\) be two complex sequences. For \(L \geq 1\), the AdjVSHT for \(a_{\ell,m}, b_{\ell,m}\) can be represented by its scalar version AdjSHT, as follows.

\[
S_L(a_{\ell,m}, b_{\ell,m}) = \begin{cases} 
    -\frac{1}{\sqrt{2}} \left( S_{L-1}(v_{\ell,m}^{(1)} + S_{L+1}(v_{\ell,m}^{(2)}) + S_L(\eta_{\ell,m}^{(3)}) \right) \\
    -\frac{1}{\sqrt{2}} \left( S_{L-1}(v_{\ell,m}^{(3)} + S_{L+1}(v_{\ell,m}^{(2)}) - S_L(\eta_{\ell,m}^{(2)}) \right) \\
    -\frac{1}{\sqrt{2}} \left( S_{L-1}(v_{\ell,m}^{(5)} + S_{L+1}(v_{\ell,m}^{(6)}) + S_L(\eta_{\ell,m}^{(3)}) \right)
\end{cases}
\]

where we use the notation of (18), (19), (20) and (21).

**Proof.** By (5), we write each component of (17) by AdjSHT, as follows. For the divergence-free term in (17),

\[
-\frac{1}{\sqrt{2}} \sum_{\ell=1}^{L} \sum_{m=-\ell}^{\ell} \left( a_{\ell,m} B_{\ell+1} - a_{\ell,m} B_{\ell-1} \right) = \frac{1}{\sqrt{2}} \sum_{\ell=1}^{L} \sum_{m=-\ell}^{\ell} a_{\ell,m} \left( c_{\ell,m} c_{\ell-1,m-1,1,1} \right) Y_{\ell-1,m-1}
\]

\[
+ \sum_{\ell=1}^{L} \sum_{m=-\ell}^{\ell} a_{\ell,m} \left( d_{\ell+1}^{\ell,m} - d_{\ell-1}^{\ell,m} \right) Y_{\ell+1,m-1}
\]

\[
- \sum_{\ell=1}^{L} \sum_{m=-\ell}^{\ell} a_{\ell,m} \left( e_{\ell}^{\ell,m} c_{\ell-1,m+1,1,1} \right) Y_{\ell-1,m+1}
\]

\[
- \sum_{\ell=1}^{L} \sum_{m=-\ell}^{\ell} a_{\ell,m} \left( d_{\ell+1}^{\ell,m} - d_{\ell-1}^{\ell,m} \right) Y_{\ell+1,m+1}
\]

\[
= \frac{1}{\sqrt{2}} \sum_{\ell=0}^{L} \sum_{m=-\ell-2}^{\ell+2} a_{\ell+1,m+1} \left( c_{\ell+1}^{\ell+1,m+1} \right) Y_{\ell,m}
\]

\[
+ \sum_{\ell=0}^{L+1} \sum_{m=-\ell}^{\ell+2} a_{\ell-1,m+1} \left( d_{\ell-1}^{\ell-1,m+1} \right) Y_{\ell,m}
\]

\[
- \sum_{\ell=0}^{L} \sum_{m=-\ell}^{\ell+2} a_{\ell+1,m+1} \left( c_{\ell+1}^{\ell+1,m+1} \right) Y_{\ell,m}
\]

\[
- \sum_{\ell=0}^{L} \sum_{m=-\ell}^{\ell+2} a_{\ell-1,m+1} \left( d_{\ell-1}^{\ell-1,m+1} \right) Y_{\ell,m}
\]

\[
- \sum_{\ell=0}^{L} \sum_{m=-\ell}^{\ell+2} a_{\ell+1,m+1} \left( c_{\ell+1}^{\ell+1,m+1} \right) Y_{\ell,m}
\]
This and (9) give

\[-\frac{1}{\sqrt{2}} \sum_{\ell=1}^{L} \sum_{m=-\ell}^{\ell} \left( a_{\ell,m} B_{\ell+1,\ell,m} - a_{\ell,m} B_{\ell-1,\ell,m} \right) \]

\[= -\frac{1}{\sqrt{2}} \left\{ \sum_{\ell=0}^{L-1} \sum_{m=-\ell}^{\ell} c_{\ell+1} \left( a_{\ell+1,m+1} f_{\ell+1,m+1} - a_{\ell+1,m-1} f_{\ell+1,m-1} \right) Y_{\ell,m} \right. \]

\[+ \sum_{\ell=2}^{L} \sum_{m=-\ell-2}^{\ell-2} d_{\ell-1} \left( a_{\ell-1,m+1} f_{\ell-1,m+1} - a_{\ell-1,m-1} f_{\ell-1,m-1} \right) Y_{\ell,m} \}\]

(23)

where for the $|m| > \ell$ (which exceeds the range of $m$ for spherical harmonics), $Y_{\ell,m} = 0$. We then let

\[v_{\ell,m}^{(1)} := c_{\ell+1} \left( a_{\ell+1,m+1} f_{\ell+1,m+1} - a_{\ell+1,m-1} f_{\ell+1,m-1} \right), \quad \ell = 0, \ldots, L - 1, \quad m = -\ell, \ldots, \ell,\]

\[v_{\ell,m}^{(2)} := \left\{ \begin{array}{ll}
d_{\ell-1} \left( a_{\ell-1,m+1} f_{\ell-1,m+1} - a_{\ell-1,m-1} f_{\ell-1,m-1} \right), & \ell = 2, \ldots, L + 1, \quad |m| = 0, \ldots, \ell - 2, \\
0, & \ell = 0, 1 \text{ or } |m| = \ell - 1, \ell, \end{array} \right.\]

by which and (23),

\[-\frac{1}{\sqrt{2}} \sum_{\ell=1}^{L} \sum_{m=-\ell}^{\ell} \left( a_{\ell,m} B_{\ell+1,\ell,m} - a_{\ell,m} B_{\ell-1,\ell,m} \right) = -\frac{1}{\sqrt{2}} \left( S_{L-1} (v_{\ell,m}^{(1)}) + S_{L+1} (v_{\ell,m}^{(2)}) \right).\]

Similarly,

\[-\frac{1}{\sqrt{2}} \sum_{\ell=1}^{L} \sum_{m=-\ell}^{\ell} \left( a_{\ell,m} B_{\ell+1,\ell,m} + a_{\ell,m} B_{\ell-1,\ell,m} \right) \]

\[= -\frac{1}{\sqrt{2}} \left\{ \sum_{\ell=0}^{L-1} \sum_{m=-\ell}^{\ell} i c_{\ell+1} \left( a_{\ell+1,m+1} f_{\ell+1,m+1} + a_{\ell+1,m-1} f_{\ell+1,m-1} \right) Y_{\ell,m} \right. \]

\[+ \sum_{\ell=2}^{L} \sum_{m=-\ell-2}^{\ell-2} i d_{\ell-1} \left( a_{\ell-1,m+1} f_{\ell-1,m+1} + a_{\ell-1,m-1} f_{\ell-1,m-1} \right) Y_{\ell,m} \}\]

Let

\[v_{\ell,m}^{(3)} := i c_{\ell+1} \left( a_{\ell+1,m+1} f_{\ell+1,m+1} + a_{\ell+1,m-1} f_{\ell+1,m-1} \right), \quad \ell = 0, \ldots, L - 1, \quad m = -\ell, \ldots, \ell,\]

\[v_{\ell,m}^{(4)} := \left\{ \begin{array}{ll}
d_{\ell-1} \left( a_{\ell-1,m+1} f_{\ell-1,m+1} + a_{\ell-1,m-1} f_{\ell-1,m-1} \right), & \ell = 2, \ldots, L + 1, \quad m = -\ell, \ldots, \ell, \\
0, & \ell = 0, 1 \text{ or } |m| = \ell - 2, \ell - 1, \end{array} \right.\]

then

\[-\frac{1}{\sqrt{2}} \sum_{\ell=1}^{L} \sum_{m=-\ell}^{\ell} \left( a_{\ell,m} B_{\ell+1,\ell,m} + a_{\ell,m} B_{\ell-1,\ell,m} \right) = -\frac{1}{\sqrt{2}} \left( S_{L-1} (v_{\ell,m}^{(3)}) + S_{L+1} (v_{\ell,m}^{(4)}) \right).\]

As

\[\sum_{\ell=1}^{L} \sum_{m=-\ell}^{\ell} a_{\ell,m} B_0, \ell,m = \sum_{\ell=0}^{L-1} \sum_{m=-\ell}^{\ell} a_{\ell+1,m} c_{\ell+1,m,1} f_{\ell+1,m,1,0} Y_{\ell,m} + \sum_{\ell=2}^{L} \sum_{m=-\ell}^{\ell} a_{\ell-1,m} d_{\ell-1,m,1,0} Y_{\ell,m},\]

we let

\[v_{\ell,m}^{(5)} := a_{\ell+1,m} c_{\ell+1,m,1} f_{\ell+1,m,1,0}, \quad \ell = 0, \ldots, L - 1, \quad m = -\ell, \ldots, \ell,\]
\begin{align*}
v^{(6)}_{\ell, m} := \begin{cases} 
a_{\ell-1, m} d_{\ell-1} c_{\ell, m, 1, 0} & \ell = 2, \ldots, L + 1, \ |m| = 0, 1, \ldots, \ell - 1, \\
0, & \ell = 0, 1 \text{ or } |m| = \ell,
\end{cases}
\end{align*}

then,
\begin{align*}
\sum_{\ell=1}^L \sum_{m=-\ell}^{\ell} a_{\ell, m} B_{0, \ell, m} &= S_{L-1}(v_{\ell, m}^{(5)}) + S_{L+1}(v_{\ell, m}^{(6)}).
\end{align*}

For the curl-free term in (17),
\begin{align*}
-\frac{1}{\sqrt{2}} \sum_{\ell=1}^L \sum_{m=-\ell}^{\ell} (b_{\ell, m} D_{+1, \ell, m} - b_{\ell, m} D_{-1, \ell, m})
&= -\frac{1}{\sqrt{2}} \left( \sum_{\ell=1}^L \sum_{m=-\ell}^{\ell} i b_{\ell, m} c_{\ell, m, 1, 1} Y_{\ell, m-1} - \sum_{\ell=1}^L \sum_{m=-\ell}^{\ell} i b_{\ell, m} c_{\ell, m+1, 1, 1} Y_{\ell, m+1} \right) \\
&= -\frac{1}{\sqrt{2}} \left( \sum_{\ell=1}^L \sum_{m=-\ell}^{\ell-1} i b_{\ell, m+1} c_{\ell, m+1, 1} Y_{\ell, m} - \sum_{\ell=1}^L \sum_{m=-\ell+1}^{\ell} i b_{\ell, m-1} c_{\ell, m-1, 1} Y_{\ell, m} \right) \\
&= -\frac{1}{\sqrt{2}} \sum_{\ell=1}^L \sum_{m=-\ell}^{\ell} i \left( b_{\ell, m+1} c_{\ell, m+1, 1} - b_{\ell, m-1} c_{\ell, m-1, 1} \right) Y_{\ell, m},
\end{align*}

where we use (4) and (9). Let
\begin{align*}
\eta^{(1)}_{\ell, m} := \begin{cases} 
i (b_{\ell, m+1} c_{\ell, m+1, 1} - b_{\ell, m-1} c_{\ell, m-1, 1}), & \ell = 1, \ldots, L, \ m = -\ell, \ldots, \ell, \\
0, & \ell = 0, \ m = 0,
\end{cases}
\end{align*}

we then obtain the equation
\begin{align*}
-\frac{1}{\sqrt{2}} \sum_{\ell=1}^L \sum_{m=-\ell}^{\ell} (b_{\ell, m} D_{+1, \ell, m} - b_{\ell, m} D_{-1, \ell, m}) &= -\frac{1}{\sqrt{2}} S_{L}(\eta^{(1)}_{\ell, m}).
\end{align*}

In a similar way,
\begin{align*}
-\frac{1}{\sqrt{2}} \sum_{\ell=1}^L \sum_{m=-\ell}^{\ell} (b_{\ell, m} D_{+1, \ell, m} + b_{\ell, m} D_{-1, \ell, m}) &= \frac{1}{\sqrt{2}} \sum_{\ell=1}^L \sum_{m=-\ell}^{\ell} \left( b_{\ell, m+1} c_{\ell, m+1, 1} + b_{\ell, m-1} c_{\ell, m-1, 1} \right) Y_{\ell, m}.
\end{align*}

If letting
\begin{align*}
\eta^{(2)}_{\ell, m} := \begin{cases} 
b_{\ell, m+1} c_{\ell, m+1, 1} + b_{\ell, m-1} c_{\ell, m-1, 1}, & \ell = 1, \ldots, L, \ m = -\ell, \ldots, \ell, \\
0, & \ell = 0, \ m = 0,
\end{cases}
\end{align*}

we then obtain
\begin{align*}
-\frac{1}{\sqrt{2}} \sum_{\ell=1}^L \sum_{m=-\ell}^{\ell} (b_{\ell, m} D_{+1, \ell, m} + b_{\ell, m} D_{-1, \ell, m}) &= \frac{1}{\sqrt{2}} S_{L}(\eta^{(2)}_{\ell, m}).
\end{align*}

As
\begin{align*}
\sum_{\ell=1}^L \sum_{m=-\ell}^{\ell} b_{\ell, m} D_{0, \ell, m} &= \sum_{\ell=1}^L \sum_{m=-\ell}^{\ell} i b_{\ell, m} c_{\ell, m, 1, 0} Y_{\ell, m},
\end{align*}

we let
\begin{align*}
\eta^{(3)}_{\ell, m} := \begin{cases} 
i b_{\ell, m} c_{\ell, m, 1, 0}, & \ell = 1, \ldots, L, \ m = -\ell, \ldots, \ell, \\
0, & \ell = 0, \ m = 0,
\end{cases}
\end{align*}
one then has the representation

\[ \sum_{\ell=1}^{L} \sum_{m=-\ell}^{\ell} b_{\ell, m} D_{0, \ell, m} = S_L(\eta^{(3)}_{\ell, m}). \]

Thus,

\[
S_L(a_{\ell, m}, b_{\ell, m}) = \left( -\frac{1}{\sqrt{S}} \right) \begin{pmatrix}
S_{L-1}(v_{1, \ell, m}^{(1)}) + S_{L+1}(v_{1, \ell, m}^{(2)}) \\
S_{L-1}(v_{3, \ell, m}^{(1)}) + S_{L+1}(v_{3, \ell, m}^{(4)}) \\
S_{L-1}(v_{5, \ell, m}^{(1)}) + S_{L+1}(v_{5, \ell, m}^{(6)}) \\
\end{pmatrix} + \left[ -\frac{1}{\sqrt{S}} S_L(\eta^{(1)}_{\ell, m}) \\
\frac{1}{\sqrt{S}} S_L(\eta^{(2)}_{\ell, m}) \\
\frac{1}{\sqrt{S}} S_L(\eta^{(3)}_{\ell, m}) \\
\right],
\]

thus completing the proof. □

### 4.3 Algorithms and Errors

#### 4.3.1 Fast algorithms

In Algorithms 1 and 2, we write down the algorithms for FwdVSHT and AdjVSHT from Theorems 4.2 and 4.4. They achieve fast computation by FFTs for scalar spherical harmonics. In this paper, we use the NFFT package [32] for fast scalar spherical harmonic transforms on \( S^2 \) which have the computational cost \( O \left( N \log \sqrt{N} \right) \) and \( O \left( M \log \sqrt{M} \right) \) for \( N \) evaluation points and \( M \) (Fourier) coefficients.

**Algorithm 1:** Forward FaVeST

**Input:** A sequence \( \{T_i, \ldots, T_N\} \subset \mathbb{R}^3, N \geq 2 \) and a quadrature rule \( Q_N := \{(w_i, x_i)\}_{i=1}^N \) on \( S^2 \), and maximal degree \( L, L \geq 1 \).

**Output:** Complex sequences of AdjVSHT \( \tilde{F}_{\ell, m}(T_k) \) and \( \tilde{F}_{\ell, m}(T_k) \) in (10), \( \ell = 1, \ldots, L \), \( m = -\ell, \ldots, \ell \).

**Step 1** Compute the AdjSHT \( F_{\ell, m} \left( -T^{(1)} + i T^{(2)} \right), F_{\ell, m} \left( T^{(1)} + i T^{(2)} \right) \) and \( F_{\ell, m} \left( T^{(3)} \right) \) for \( \ell = 0, \ldots, L + 1, m = -\ell, \ldots, \ell \), by forward FFT for scalar spherical harmonics.

**Step 2** Compute \( \xi^{(1)}_{\ell, m}, \ell = 1, 2, \ldots, 6 \) and \( \mu^{(j)}_{\ell, m}, j = 1, 2, 3, \) for \( \ell = 0, \ldots, L + 1, m = -\ell, \ldots, \ell \).

**Step 3** Compute \( \tilde{F}_{\ell, m}(T_k) \) and \( \tilde{F}_{\ell, m}(T_k) \) for \( \ell = 1, \ldots, L \), \( m = -\ell, \ldots, \ell \), by (15).

**Algorithm 2:** Adjoint FaVeST

**Input:** Two complex sequences of coefficients \( \{(a_{\ell, m}, b_{\ell, m}) : \ell = 1, \ldots, L, m = -\ell, \ldots, \ell \} \), and \( M \) evaluation points \( \{x_i\}_{i=1}^M \).

**Output:** Complex sequence of AdjointVSHT \( S_L(a_{\ell, m}, b_{\ell, m}; x_i) \) in (16), \( i = 1, \ldots, M \). For each point \( x_i \), \( i = 1, \ldots, M \), do the following steps.

**Step 1** Compute \( \xi^{(1)}_{\ell, m}, \eta^{(j)}_{\ell, m}, \ell = 1, \ldots, 6, j = 1, 2, 3 \), by (19), (20) and (21).

**Step 2** Evaluate \( S_L(v_{1, \ell, m}^{(1)}), S_L(v_{1, \ell, m}^{(2)}), S_L(v_{3, \ell, m}^{(1)}), S_L(v_{3, \ell, m}^{(4)}), S_L(v_{5, \ell, m}^{(1)}), S_L(v_{5, \ell, m}^{(6)}), S_L(\eta^{(1)}_{\ell, m}), S_L(\eta^{(2)}_{\ell, m}) \) and \( S_L(\eta^{(3)}_{\ell, m}) \), by adjoint FFT for scalar spherical harmonics.

**Step 3** Compute \( S_L(a_{\ell, m}, b_{\ell, m}; x_i) \), by (22).

In Algorithm 1, using the forward FFT for scalar spherical harmonics, Step 1 uses 3 times scalar FFTs with degree up to \( L + 1 \) and then has cost \( O \left( N \log \sqrt{N} \right) \) (assuming using \( N = O \left( L^2 \right) \) points); Step 2 is the direct computation of \( \xi^{(1)}_{\ell, m} \) and \( \mu^{(j)}_{\ell, m} \) by (13), (14) and (9) for degree \( \ell \) at most \( L + 1 \) and then has cost \( O \left( \sqrt{N} \right) \); Step 3 evaluates (15) using the results of Steps 1 and 2 and has cost \( O \left( 1 \right) \). Thus, evaluating the \( \tilde{F}_{\ell, m}(T_k)_{k=1}^N; Q_N \) and \( \tilde{F}_{\ell, m}(T_k)_{k=1}^N; Q_N \) incurs
computational cost proportional to $N \log \sqrt{N}$. In Algorithm 2, with $M = O(L^2)$ evaluation points, Step 1 is computed by (19), (20), (21) and (4), (9) for degree up to $L + 1$ has computational cost $O(\sqrt{M})$; Step 2 uses 8 times adjoint FFTs for scalar spherical harmonics and then has computational cost $O(M \log \sqrt{M})$; Step 3 computes (22) by the results of the previous steps has cost $O(1)$. Thus, the FaVeST for the adjoint case has computational cost proportional to $M \log \sqrt{M}$. These analyses show that the computational complexity of the proposed algorithms is nearly linear. We then call the

These analyses show that the computational complexity of the proposed algorithms is nearly linear. We then call the

4.3.2 Errors. Let $T$ be a tangent field in $L_2(S^2)$. The approximation error of FwdVSHT $\tilde{F}_{\ell,m}(\{T(x_k)\})_{k=1}^N, Q_N)$ and $\tilde{F}_{\ell,m}(\{T(x_k)\})_{k=1}^N, Q_N$ for the divergence-free and curl-free coefficients $\tilde{t}_{\ell,m}$ and $\tilde{t}_{\ell,m}$ in (11) depends on the approximation quality of quadrature rule $\{(w_k, x_k)\}_{k=1}^N$ for integrals on the sphere and the smoothness of the tangent field $T$. Given a tangent field, the choice of quadrature rule is the key to reducing the approximation error. A quadrature rule $\{(w_i, x_i)\}_{i=1}^N$ is called exact for polynomials of degree $L$, $L \geq 0$ if for all spherical polynomials $p$ of degree at most $L$,

$$\sum_{i=1}^N w_ip(x_i) = \int_{S^2} p(x) d\sigma(x),$$

see e.g. [27]. Algorithm 1 using a quadrature rule that is exact for scalar spherical polynomial of degree $L$ has the approximation error $CL^{-s}$ for $y_{\ell,m}^\alpha T$ lying in Sobolev space $H^s(S^2)$ (of scalar spherical functions) and $s > 1$, where the constant $C$ depends only on the Sobolev norm of the function $y_{\ell,m}^\alpha T$. The order $L^{-s}$ is optimal, as a consequence of [24–26]. On the other hand, the error of Algorithm 2 can evaluate FwdVSHT with zero-loss. Its approximation for the expansion (2) is equal to the truncation error for the vector spherical harmonic expansion, as determined by

$$\left\| \sum_{\ell=L+1}^{\infty} \sum_{m=-\ell}^{\ell} (a_{\ell,m} y_{\ell,m} + b_{\ell,m} z_{\ell,m}) \right\|_{L_2}^2 = \sum_{\ell=L+1}^{\infty} \sum_{m=-\ell}^{\ell} \left( |a_{\ell,m}|^2 + |b_{\ell,m}|^2 \right),$$

where we have used the orthogonality of vector spherical harmonics.

4.4 Software Description

We provide the software package in Matlab for FaVeST which includes Matlab demo and routines for FaVeST and the routines for numerical examples in the next section. The FaVeST package can be downloaded from GitHub at https://github.com/mingli-ai/FaVeST. It has been tested in Matlab environment in operating systems including Ubuntu 16.04.6, macOS High Sierra, macOS Mojave, Windows 7, Windows 8 and Windows 10. In the Matlab library repository, the main m-files include FaVeST_fwd.m and FaVeST_adj.m, corresponding to Algorithms 1 and 2 respectively. Inside these two functions, the package NFFT\(^1\) [32] is used to run the scalar FFTs on the sphere. (The NFFT is the only package requested by the FaVeST package.) There are four inputs for the function FaVeST_fwd.m: $T, L, X, w$, where $T$ is a tangent field sampled at the point set $X$, $L$ is the highest degree of spherical harmonics, $X$ is the set of quadrature nodes and $w$ is the set of quadrature weights. The output includes the two sequences of divergence-free and curl-free coefficients $\tilde{F}_{\ell,m}(T)$ and $\tilde{F}_{\ell,m}(T)$ for degree $\ell = 1, \ldots, L$ and $m = -\ell, \ldots, \ell$ as given in (10). The input of FaVeST_adj.m contains two finite sequences $a_{\ell,m}$ and $b_{\ell,m}$ (with $\ell \leq L$ for some $L \geq 1$ and $m = -\ell, \ldots, \ell$) as the coefficients for divergence-free and curl-free parts, and a point set $X$ for evaluation. The output is the AdjVSHT in (16) for $a_{\ell,m}$ and $b_{\ell,m}$.

\(^{1}\)https://www-user.tu-chemnitz.de/~potts/nfft/
5 NUMERICAL EXAMPLES

In this section, we show numerical examples for verification of the performance of the proposed FaVeST algorithm. We start from the description of two types of polynomial-exact quadrature rules used in the experiments, and then present the examples of tangent fields on the sphere. We show the reconstruction and error in orthographic projection on the sphere for FaVeST using both kinds of point sets. We also show the CPU computational time of FaVeST which evaluates a tangent field for degree up to 2,250 and at 10 million spherical points.

5.1 Quadrature Rules

We use two types of point sets on $S^2$ in the experiments, as follows.

(1) Gauss-Legendre tensor product rule (GL) [28]. The Gauss-Legendre tensor (product) rule is a (polynomial-exact but not equal area) quadrature rule $Q_N := \{(w_i, x_i)\}_{i=1}^N, i = 0, \ldots, N$ on the sphere generated by the tensor product of the Gauss-Legendre nodes on the interval $[-1, 1]$ and equi-spaced nodes on the longitude with non-equal weights. To be exact for polynomials of degree $L$, one needs to use around $2L^2$ GL points. Figure 1(a) shows $N = 512$ GL points for degree $L = 16$.

(2) Symmetric spherical $t$-designs (SD) [68]. The symmetric spherical $t$-design is a (polynomial-exact) quadrature rule $Q_N := \{(w_i, x_i)\}_{i=1}^N, i = 0, \ldots, N$ on the sphere $S^2$ with equal weights $w_i = 1/N$. The points are almost uniformly (or in formal definition, quasi-uniform) distributed on the sphere. To be exact for polynomials of degree $L$, one needs to use the symmetric spherical $t$-design for $t = L$ with around $L^2/2$ points. Figure 1(b) shows $N = 498$ SD points for degree $L = 31$.

5.2 Tangent Fields

To verify our theoretical results in Section 3, we use three types of simulated tangent fields as provided in [16]. All these tangent fields are generated using stream function and velocity potential so that we can easily split the divergence-free and curl-free parts of the field. Let $s$ and $v$ be the stream function and velocity potential, then, each of the tangent fields can be represented by

$$T = \frac{\mathbf{L}_s}{f_{\text{div}}} + \frac{\nabla \times v}{f_{\text{curl}}}.$$
Recall from Section 3 that \( \mathbf{L} \) and \( \nabla_s \) denote the surface curl and surface gradient, and \( \mathbf{L} \)s and \( \nabla_s \) are divergence-free and curl-free. We define the three tangent fields, as follows.

**Tangent Field A.** The stream function and velocity potential for this field are linear combinations of spherical harmonics. They can be used to generate realistic synoptic scale meteorological wind fields. The stream function is defined by

\[
s_1(x) = -\frac{1}{\sqrt{3}} Y_{1,0}(x) + \frac{8\sqrt{2}}{3\sqrt{385}} Y_{5,4}(x),
\]

which is known as a Rosby–Haurwitz wave and is an analytic solution to the nonlinear barotropic vorticity equation on the sphere [30, pp. 433–454]. In [67], \( s_1 \) was used as the initial condition for the shallow water wave equations on the sphere. The velocity potential is given by

\[
v_1(x) = \frac{1}{25} (Y_{4,0}(x) + Y_{0,3}(x)).
\]

Note that we can choose spherical harmonics with different degrees and different coefficients in the above formula. Here, we have used the same setting as [16].

**Tangent Field B.** This field still uses the Rosby–Haurwitz wave (24) as the stream function. But the velocity potential is a linear combination of compactly supported functions:

\[
v_2(x) = \frac{1}{8} f(x; 5, \pi/6, 0) - \frac{1}{7} f(x; 3, \pi/5, -\pi/7) + \frac{1}{5} f(x; 5, -\pi/6, \pi/2) - \frac{1}{8} f(x; 3, -\pi/5, \pi/3),
\]

where

\[
f(x; \sigma, \theta_c, \lambda_c) = \frac{\sigma^3}{12} \sum_{j=0}^{4} (-1)^j \binom{4}{j} r - \frac{(j - 2)!}{\sigma}.
\]

**Tangent Field C.** Let \( \mathbf{x}_c \in S^2 \) in spherical coordinates \( (\theta_c, \lambda_c) \), and \( t = \mathbf{x} \cdot \mathbf{x}_c \) and \( a = 1 - t \). Define

\[
g(x; \theta_c, \lambda_c) = -\frac{1}{2} ((3t + 3\sqrt{2}a^{3/2} - 4) + (3t^2 - 4t + 1) \log(a) + (3t - 1)a \log(\sqrt{2}a + a)).
\]

The stream function for this tangent field is given by

\[
s_3(x) = \int_{-\pi/2}^\theta \sin^{14}(2\xi) d\xi - 3g(x; \pi/4, -\pi/12),
\]

where \( \theta \) denotes the latitudinal coordinate of \( \mathbf{x} \). With \( g \), the velocity potential is given by

\[
v_3(x) = \frac{5}{2} g(x; \pi/4, 0) - \frac{7}{4} g(x; \pi/6, \pi/9) - \frac{3}{2} g(x; 5\pi/16, \pi/10).
\]

### 5.3 Reconstruction and Errors

The left columns of Figures 2 and 3 present the tangent fields sampled at \( N = 1922 \) GL points and \( N = 1894 \) SD points, respectively. The middle columns of Figure 2 and 3 show the reconstructed field \( T^{\text{rec}} \) for the tangent field \( T \) with \( N = 1922 \) GL points and \( N = 1849 \) SD points for evaluation. The corresponding point-wise error \( T - T^{\text{rec}} \) (at the sampling points) is displayed in the right columns. Here, the length and arrow indicate the scalar value and direction of the field at a spherical point. We observe that the reconstruction performance of the FaVeST is excellent: the relative error for the reconstruction is small compared with the magnitude of the original field in both GL and SD cases. The reconstructed field restores the direction and scalar value of the original field in high precision, especially when the field is sufficiently smooth. The error fields for Tangent Fields B and C show some local features of the evaluation of
**FaVeST**: the smoother part has smaller error, which maybe partially interpreted by the localization approximation behaviour of spherical polynomials, see e.g. [18, 36, 63–65].

Table 1 reports the relative $L_2$-errors $\|T - T_{rec}\|_2/\|T\|_2$ for the reconstruction with degree $L$ up to 150 in the cases of GL and SD. We observe that the FaVeSTs with GL and SD exhibit similar error orders for each tangent field. These results illustrate that the FaVeST with a polynomial-exact quadrature rule is precise. In both GL and SD cases, the error increases for each degree as the smoothness of the field (from A to C) decreases. This means that the accuracy of the algorithm improves as the smoothness of the tangent field reduces. The error here is the superposition of the errors for forward FaVeST and adjoint FaVeST. They are determined by the approximation error of numerical integration by quadrature rule and the truncation error of vector spherical harmonic expansion, both of which increase as the smoothness of the tangent field reduces, see e.g. [6, 7, 14].

| Quadrature | $L = 10$ | $L = 30$ | $L = 50$ | $L = 100$ | $L = 120$ | $L = 150$ |
|------------|---------|---------|---------|---------|---------|---------|
| Tangent Field A | GL 8.6133e-12 4.3287e-12 3.1993e-12 2.6626e-12 2.5678e-12 2.4932e-12 | SD 5.3367e-12 3.2721e-12 2.9385e-12 2.5873e-12 4.0959e-12 1.4713e-10 |
| Tangent Field B | GL 7.5102e-02 2.9206e-03 6.5037e-04 1.0389e-04 7.1028e-05 3.5907e-05 | SD 7.1155e-02 3.0262e-03 6.9277e-04 1.1280e-04 7.6203e-05 4.0761e-05 |
| Tangent Field C | GL 2.6919e-01 5.7499e-03 1.9874e-03 4.7176e-04 3.3860e-04 2.5335e-04 | SD 1.7225e-01 5.6764e-03 2.1336e-03 5.5568e-04 4.1357e-04 2.4919e-04 |

Table 1. Relative $L_2$-errors $\|T - T_{rec}\|_2/\|T\|_2$ in GL and SD cases for various numbers of nodes.

### 5.4 Computational Complexity

To test the time complexity of FaVeST, we carry out experiments with different numbers of GL points: let $L = 250k + 500$, $k = 0, 1, \ldots, 8$, which corresponds to $N_k \approx 2L^2_k$ GL nodes for FwdVSHT, and $M_k \approx L^4_k + L_k$ coefficients for AdjVSHT. The CPU time consumption by FaVeSTs is reported in Table 2. As indicated by the quotient ratios (in the round brackets) of the CPU times for degrees $L_k$ and $L_{k-1}$, the computational time is almost proportional to $N_k$ and $M_k$ for forward and adjoint FaVeSTs. This is also confirmed by Figure 4. Figure 4(a) shows the trend of CPU time changing with the increase of the number of GL points for forward FaVeST. Figure 4(b) shows the trend of CPU time varying with the increase of coefficient number for adjoint FaVeST. The fitting curves for the forward and adjoint FaVeST cases have the powers (up to a constant) $N^1.2$ and $M^1.2$, which means that both the forward and adjoint FaVeSTs have near linear computational complexity. This is consistent with the analysis in Section 4.3.

### 6 CONCLUSIONS AND DISCUSSION

This work proposes the first concrete fast algorithm which evaluates the forward and adjoint transforms of vector spherical harmonics for tangent fields. The fast algorithm (which we call FaVeST) is made possible from the representation of FwdVSHT and AdjVSHT by scalar spherical harmonics and Clebsch-Gordan coefficients. By scalar FFTs on the sphere, the proposed algorithms for FwdVSHT and AdjVSHT both achieve the near linear computational complexity. The accuracy and computational speed of FaVeST are validated by the numerical examples of simulated tangent fields. We develop a software package in Matlab for FaVeST, which works for polynomial-exact quadrature rules. This package has been used in the fast computation for tensor needlet transform — a fast multiresolution analysis — for tangent field on the sphere in [37]. In future work, we will use the package for solving partial differential equations on the sphere.
(a) Reconstruction of Tangent Field A with GL

(b) Reconstruction of Tangent Field B with GL

(c) Reconstruction of Tangent Field C with GL

**Fig. 2.** Visualization of reconstructed tangent field and error by FaVeST with quadrature rule of Gauss-Legendre tensor (GL). The first and the second columns show the target tangent field $T$ and reconstructed field $T_{\text{rec}}$. The third column shows the error $T - T_{\text{rec}}$. All plots are the orthographic projections of the fields evaluated at $N = 1922$ GL nodes. The normalized maximum norms for $T$, $T_{\text{rec}}$ and $T - T_{\text{rec}}$ are displayed in the titles.
(a) Reconstruction of Tangent Field A with SD

(b) Reconstruction of Tangent Field B with SD

(c) Reconstruction of Tangent Field C with SD

Fig. 3. Visualization of reconstructed tangent field and error by FaVeST with quadrature rule of symmetric spherical $t$-designs (SD). The first and the second columns show the target tangent field $T$ and reconstructed field $T_{\text{rec}}$. The third column shows the error $T - T_{\text{rec}}$. All plots are the orthographic projections of the fields evaluated at $N = 1,849$ SD nodes. The normalized maximum norms for $T$, $T_{\text{rec}}$ and $T - T_{\text{rec}}$ are displayed in the titles.
Table 2. Forward FaVeST CPU time $t^\text{fwd}$ v.s. number of points and adjoint FaVeST CPU time $t^\text{adj}$ v.s. number of coefficients. For $L = L_k = 250k + 500$, $k = 0, 1, \ldots, 8$, forward FaVeST uses Gauss-Legendre tensor rule which has $N = N_k \approx 2L_k^2$ nodes, and adjoint FaVeST uses $M = M_k = L_k^2 + L_k$ coefficients. The numbers inside the brackets are the ratios $t^\text{fwd}(N_k)$ and $t^\text{adj}(M_k)$. The numerical test is run under Intel Core i7-6700 CPU @ 3.40GHz with 16GB RAM in Windows 10.

![Graph](image)

Fig. 4. CPU time of Forward and Adjoint FaVeSTs.

such as Stokes or Navier-Stokes equations on $\mathbb{S}^2$. As full sky maps in cosmology are usually evaluated at HEALPix points [20], we will develop a software package of FaVeST for HEALPix in Python in future.

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