Second-Order Symmetric Lorentzian Manifolds

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Abstract. Spacetimes with vanishing second covariant derivative of the Riemann tensor are studied. Their existence, classification and explicit local expression are considered. Related issues and open questions are briefly commented.

INTRODUCTION

Our aim is to characterize, as well as to give a full list of, the $n$-dimensional manifolds $\mathcal{V}$ with a metric $g$ of Lorentzian signature such that the Riemann tensor $R^\alpha_{\beta\gamma\delta}$ of $(\mathcal{V}, g)$ locally satisfies the second-order condition

$$\nabla_\mu \nabla_\nu R^\alpha_{\beta\gamma\delta} = 0. \quad (1)$$

It is quite surprising that, hitherto, despite their simple definition, this type of Lorentzian manifolds have been hardly considered in the literature. Probably this is due to the classical results concerning these manifolds in the proper Riemannian case, to the difficulties arising in other signatures, and to the little reward: only very special cases survive.

Apart from their obvious mathematical interest, from a physical point of view they are relevant in several respects: as a second local approximation to any spacetime (using for instance expansions in normal coordinates); as examples with a finite number of terms in Lagrangians; as interesting exact solutions for supergravity/superstring or M-theories; for invariant classifications; for solutions with parallel vector fields or spinors.

A more complete treatment, with a full list of references, is given in [23].

SYMMETRIC SPACES AND ITS GENERALIZATIONS

Semi-Riemannian manifolds satisfying (1) are a direct generalization of the classical locally symmetric spaces which satisfy

$$\nabla_\mu R^\alpha_{\beta\gamma\delta} = 0. \quad (2)$$

These were introduced, studied and classified by E. Cartan [10] in the proper Riemannian case\footnote{With a positive-definite metric.}, see e.g. [11, 16, 15], and later in [6, 9, 7] for the Lorentzian and general semi-Riemannian cases—see e.g. [8, 19] and references therein. They are themselves generalizations of the constant curvature spaces and, actually, there is a hierarchy of conditions,
shown in Table 1 that can be placed on the curvature tensor. In the table, the restrictions on the curvature tensor decrease towards the right and each class is strictly contained in the following ones. The table has been stopped at the level of semi-symmetric spaces, defined by the condition $\nabla_\mu \nabla_\nu R^\alpha_{\beta \gamma \delta} = 0$, which were introduced also by Cartan [11] and studied in [24, 25] as the natural generalization of symmetric spaces for the proper Riemannian case—see also [4] and references therein.

Why was semisymmetry considered to be the natural generalization of local symmetry? And, why not going further on to higher derivatives of the Riemann tensor? The answer to both questions is actually the same: a classical theorem [17, 18, 27] states that in any proper Riemannian manifold

$$\nabla_{\mu_1} \ldots \nabla_{\mu_k} R^\alpha_{\beta \gamma \delta} = 0 \iff \nabla_\mu R^\alpha_{\beta \gamma \delta} = 0 \quad (3)$$

for any $k \geq 1$ so that, in particular, (1) is strictly equivalent to (2) in proper Riemannian spaces. This may well be the reason why there seems to be no name for the condition (1) in the literature. However, an analogous condition has certainly been used for the so-called $k$-recurrent spaces [26, 12]; thus, I will call the spaces satisfying (1) second-order symmetric, or in short 2-symmetric, and more generally $k$-symmetric when the left condition in (3) holds—see [23] for further details.

### Results at generic points

As a matter of fact, the equivalence (3) holds as well in “generic” cases of semi-Riemannian manifolds of any signature. For some results on this one can consult [27, 12]. By “generic point” the following is meant: any point $p \in \mathcal{V}$ where the matrix $(R^\alpha_{\beta \gamma \delta})|_p$ of the Riemann tensor, considered as an endomorphism on the space of 2-forms $\Lambda_2(p)$, is non-singular. Then, for instance one can prove the following general result, see [23] for a proof.

**Proposition 1** For any tensor field $T$, and at generic points, one has

$$\nabla_{\mu_1} \ldots \nabla_{\mu_k} T = 0 \iff \nabla T = 0$$

for any $k \geq 1$.

Of course, these results apply in particular to the Riemann tensor, and in fact sometimes even stronger results can be proven. For instance, one can prove a conjecture in [12].

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1. (Square) round brackets enclosing indices indicate (anti-)symmetrization, respectively.
namely, that all $k$-symmetric (and also all $k$-recurrent) spaces are necessarily of constant curvature on a neighbourhood of any generic $p \in \mathcal{V}$. As a matter of fact, a slightly more general result is proven in [23]:

**Theorem 1** All semi-symmetric spaces are of constant curvature at generic points.

Therefore, there is little room for spaces (necessarily of non-Euclidean signature) which are $k$-symmetric but not symmetric nor of constant curvature. It is remarkable that there have been many studies on 2-recurrent spaces, but surprisingly enough the assumption that they are not 2-symmetric has always been, either implicitly or explicitly, made. The paper [23] tries to fill in this gap for the case of 2-symmetry and Lorentzian signature.

**LORENTZIAN 2-SYMMETRY**

To deal with the problem of $k$-symmetric and $k$-recurrent spaces one needs to combine several different techniques. Among them (i) pure classical standard tensor calculus by using the Ricci and Bianchi identities; (ii) study of parallel (also called covariantly constant) tensor and vector fields, and their implications on the manifold holonomy structure; and (iii) consequences on the curvature invariants. I now present the main points and results needed to reach the sought results. It turns out that the so-called "superenergy" and causal tensors [22, 3] are very useful, providing positive quantities associated to tensors that can be used to replace the ordinary positive-definite metric available in proper Riemannian cases.

**Identities in 2-symmetric semi-Riemannian manifolds**

Of course, some tensor calculation is obviously needed, mainly to prove some helpful quadratic identities. To start with, one needs a generalization of Proposition 1 to the case of non-generic points.

**Lemma 1** Let $(\mathcal{V}, g)$ be an $n$-dimensional 2-symmetric semi-Riemannian manifold of any signature. If $\nabla_\nu R^\rho_{\lambda\mu\nu} T^\mu_{\mu_1...\mu_q} = 0$ then

$$
\sum_{i=1}^{q} \nabla_\nu R^\rho_{\alpha_i\lambda\mu} T^\alpha_{\lambda_1...\alpha_{i-1}\rho\alpha_{i+1}...\alpha_q} - R^\rho_{\nu\lambda\mu} \nabla_\rho T^\alpha_{\lambda_1...\alpha_q} = 0,
$$

(4)

$$
(\nabla_\nu R^\rho_{\tau\lambda\mu} + \nabla_\tau R^\rho_{\nu\lambda\mu}) \nabla_\rho T^\mu_{\mu_1...\mu_q} = 0,
$$

(5)

$$
(\nabla_\nu R^\rho_{\mu\nu} - \nabla_\mu R^\rho_{\nu\nu}) \nabla_\rho T^\mu_{\mu_1...\mu_q} = 0,
$$

(6)

By using the decomposition of the Riemann tensor,

$$
R_{\alpha\beta\lambda\mu} = C_{\alpha\beta\lambda\mu} + \frac{2}{n-2} \left( R_{\alpha[\lambda \gamma \beta \mu]} - R_{\beta[\lambda \gamma \beta \mu]} \right) - \frac{R}{(n-1)(n-2)} \left( g_{\alpha\lambda} g_{\beta\mu} - g_{\alpha\mu} g_{\beta\lambda} \right)
$$

(7)
a selection of the formulas satisfied in 2-symmetric manifolds are given next

**Lemma 2** The Riemann, Ricci and Weyl tensors of any \( n \)-dimensional 2-symmetric semi-Riemannian manifold of any signature satisfy

\[
R^\rho_{\alpha \lambda \mu} R^\rho_{\beta \gamma \delta} + R^\rho_{\beta \lambda \mu} R^\rho_{\alpha \rho \gamma \delta} + R^\rho_{\lambda \alpha \mu} R^\rho_{\beta \rho \gamma \delta} + R^\rho_{\lambda \beta \mu} R^\rho_{\alpha \rho \gamma \delta} = 0 \tag{8}
\]

\[
R^\rho_{\lambda \mu \nu} \nabla^{\rho} R_{\alpha \beta \gamma \delta} + R^\rho_{\lambda \mu \nu} \nabla^{\nu} R_{\rho \beta \gamma \delta} + R^\rho_{\lambda \mu \nu} \nabla^{\nu} R_{\rho \alpha \beta \gamma \delta} + R^\rho_{\lambda \mu \nu} \nabla^{\nu} R_{\rho \alpha \beta \gamma \delta} = 0 \tag{9}
\]

\[
\nabla_{(\tau} R^\rho_{\nu)} \lambda \mu \nabla^{\rho} R_{\alpha \beta \gamma \delta} = 0, \quad \nabla_{(\tau} R^\rho_{\nu)} \lambda \mu \nabla^{\nu} C_{\alpha \beta \gamma \delta} = 0, \quad \nabla_{(\tau} R^\rho_{\nu)} \lambda \mu \nabla^{\rho} R_{\alpha \beta \gamma \delta} = 0, \tag{10}
\]

\[
R^\rho_{\lambda \mu} C_{\beta \gamma \delta} = 0, \quad R^\rho_{\lambda \mu} C_{\beta \gamma \delta} = 0, \quad C^\rho_{\lambda \mu} C_{\beta \gamma \delta} = 0, \quad R^\rho_{\lambda \mu} R^\rho_{\beta \gamma \delta} = 0, \quad R^\rho_{\lambda \mu} R^\rho_{\beta \gamma \delta} = 0, \quad R^\rho_{\lambda \mu} R^\rho_{\beta \gamma \delta} = 0, \quad R^\rho_{\lambda \mu} R^\rho_{\beta \gamma \delta} = 0, \tag{11}
\]

\[
(n - 2) \left( C_{\rho \lambda \mu} C_{\beta \gamma \delta} + C_{\rho \lambda \mu} C_{\beta \gamma \delta} \right) - 2 \left( R_{\lambda \mu \gamma \delta} + R_{\gamma \delta \lambda \mu} \right) - 2 \left( R_{\lambda \mu \gamma \delta} + R_{\gamma \delta \lambda \mu} \right) = 0 \tag{12}
\]

\[
-2 \left( R_{\lambda \mu \gamma \delta} + R_{\gamma \delta \lambda \mu} \right) + 2 \frac{R}{n - 1} \left( \delta_{\lambda \mu \gamma \delta} + \delta_{\gamma \delta \lambda \mu} \right) = 0 \tag{13}
\]

and their non-written traces, such as the appropriate specializations of \( \delta \). Actually, \( \delta \) and \( \lambda \) are valid in arbitrary semi-symmetric spaces.

**Holonomy and reducibility in Lorentzian manifolds**

Some basic lemmas on local holonomy structure are also essential. The classical result here is the de Rham decomposition theorem \([23, 16]\) for positive-definite metrics. However, this theorem does not hold as such for other signatures, and one has to introduce the so-called non-degenerate reducibility \([23, 29, 30]\). See also \([1]\) for the particular case of Lorentzian signature. To fix ideas, recall that the holonomy group \([16]\) of \( (\mathcal{V}, g) \) is called reducible (when acting on the tangent spaces) if it leaves a non-trivial subspace of \( T_p \mathcal{V} \) invariant. And it is called non-degenerately reducible if it leaves a non-degenerate subspace (that is, such that the restriction of the metric is non-degenerate) invariant.

Only a simple result is needed. This relates the existence of parallel tensor fields to the holonomy group of the manifold in the case of Lorentzian signature. It is a synthesis (adapted to our purposes) of the results in \([14]\) but generalized to arbitrary dimension \( n \) (see \([23]\) for a proof):

**Lemma 3** Let \( D \subset \mathcal{V} \) be a simply connected domain of an \( n \)-dimensional Lorentzian manifold \( (\mathcal{V}, g) \) and assume that there exists a non-zero parallel symmetric tensor field \( h_{\mu \nu} \) not proportional to the metric. Then \( (D, g) \) is reducible, and further it is not non-degenerately reducible only if there exists a null parallel vector field which is the unique parallel vector field (up to a constant of proportionality).

Some important remarks are in order here:

1. If there is a parallel 1-form \( v_\mu \), then so is obviously \( h_{\mu \nu} = v_\mu v_\nu \) and the manifold (arbitrary signature) is reducible, the Span of \( v_\mu \) being invariant by the holonomy
group. If \( v_\mu \) is not null, then \((\mathcal{V}, g)\) is actually non-degenerately reducible. In this case, the metric can be decomposed into two orthogonal parts as \( g_{\mu\nu} = c v_\mu v_\nu + (g_{\mu\nu} - c v_\mu v_\nu) \), where \( c = 1/(v_\mu v_\mu) \) is constant. Thus, necessarily \( g_{\mu\nu} \) is a flat extension \([21]\) of a \((n-1)\)-dimensional non-degenerate metric \( g_{\mu\nu} - c v_\mu v_\nu \).

2. If there is a parallel non-symmetric tensor \( H_{\mu\nu} \), then its symmetric part is also parallel, so that one can put \( h_{\mu\nu} = H_{(\mu\nu)} \) in the lemma. In the case that \( H_{\mu\nu} = H_{[\mu\nu]} \neq 0 \) is antisymmetric, then in fact one can define \( H_{\mu\rho} H_{\nu\rho} = h_{\mu\nu} \), which is symmetric, parallel, non-zero and not proportional to the metric if \( n > 2 \). For these last two statements, see e.g. \([3]\).

3. Actually, the above can also be generalized to an arbitrary parallel \( p\)-form \( \Sigma_{\mu_1...\mu_p} \) by defining \( h_{\mu\nu} = \Sigma_{\mu_\rho_2...\rho_p} \Sigma_{\nu_\rho_2...\rho_p} \).

Curvature invariants in 2-symmetric Lorentzian manifolds

Recall that a curvature scalar invariant \([13]\) is a scalar constructed polynomially from the Riemann tensor, the metric, the covariant derivative and possibly the volume element \( n\)-form of \((\mathcal{V}, g)\). They are called linear, quadratic, cubic, etcetera if they are linear, quadratic, cubic, and so on, on the Riemann tensor. This defines its degree. The order can be defined for homogeneous invariants, that is, so that they have the same number of covariant derivatives in all its terms. This number is the order of the scalar invariant. Of course, all non-homogeneous invariants can be broken into their respective homogeneous pieces, and therefore in what follows only the homogeneous ones will be considered. Similarly, one can define curvature 1-form invariants, or more generally, curvature rank \(-r\) invariants in the same way but leaving 1, \ldots, \( r \) free indices \([13]\).

A simple but very useful lemma is the following \([23]\).

**Lemma 4** Let \((D, g)\) be as before with arbitrary signature. Any 1-form curvature invariant which is parallel must be necessarily null (possibly zero).

It follows that, in 2-symmetric spaces, either \( R \) is constant or \( \nabla_\mu R \) is null and parallel. This is a particular example of the following general important result \([23]\).

**Proposition 2** Let \( D \subset \mathcal{V} \) be a simply connected domain of an \( n\)-dimensional 2-symmetric Lorentzian manifold \((\mathcal{V}, g)\). Then either

- all (homogeneous) scalar invariants of the Riemann tensor of order \( m \) and degree up to \( m + 2 \) are constant on \( D \); or
- there is a parallel null vector field on \( D \).

(Observe also that there will be no non-zero invariants involving derivatives of order higher than one. Then, the degree is necessarily greater or equal than the order.)

The previous proposition has immediate consequences providing more information about curvature invariants. For instance \([23]\).

**Corollary 1** Under the conditions of Proposition \([2]\) either there is a parallel null vector field on \( D \) or the following statements hold

1. All curvature scalar invariants of any order and degree formed as functions of the homogeneous ones of order \( m \) and degree up to \( m + 2 \) are constant on \( D \);
2. All 1-form curvature invariants of order \( m \) and degree up to \( m + 1 \) are zero.
3. All scalar invariants with order equal to degree vanish.
4. All rank-2 tensor invariants with order equal to degree are zero.

**Remark:** Of course, it can happen that the mentioned curvature invariants vanish and there is a parallel null vector field too.

There is a very long list of vanishing curvature invariants as a result of this Corollary—if there is no null parallel vector field—. The list of the quadratic ones is (only an independent set [13] is given, omitting those containing \( \nabla \mu R = 0 \):

\[
R^\mu_\nu \nabla_\alpha R^\nu_\mu = 0, \quad R^\mu_\nu \nabla_\mu R^\nu_\nu = 0, \quad R^\mu_\nu \nabla_\nu R^\nu_\alpha \nabla_\alpha R^\mu_\nu = 0 = R^\mu_\nu \nabla_\alpha R^\nu_\mu \nabla_\mu R^\nu_\sigma \nabla_\sigma R^\mu_\nu, \quad (14)
\]

\[
\nabla_\alpha R^\mu_\nu \nabla_\beta R^\nu_\mu = \nabla_\mu R^\nu_\beta \nabla_\alpha R^\mu_\nu = \nabla_\mu R^\nu_\alpha \nabla_\nu R^\nu_\beta = \nabla_\mu R^\nu_\nu \nabla_\alpha R^\mu_\nu = 0, \quad (15)
\]

\[
\nabla_\mu R^\nu_\rho \nabla_\alpha R^\rho_\mu \nabla_\alpha R^\nu_\nu = \nabla_\mu R^\nu_\nu \nabla_\alpha R^\nu_\rho \nabla_\rho R^\mu_\mu = 0, \quad (16)
\]

\[
\nabla_\alpha R^\mu_\nu \nabla_\nu \rho R^\rho_\rho = \nabla_\sigma R^\mu_\nu \nabla_\nu \rho R^\rho_\sigma = 0, \quad (17)
\]

where of course the traces of (16-18) vanish, and one could also write the same expressions using the Weyl tensor instead of the Riemann tensor.

**MAIN RESULTS**

All necessary results to prove the main theorems have now been gathered. Then, by using the so-called future tensors and "superenergy" techniques [22, 3] one can prove the following 3:

**Theorem 2** Let \( D \subset \mathcal{V} \) be a simply connected domain of an \( n \)-dimensional 2-symmetric Lorentzian manifold \((\mathcal{V}, g)\). Then, if there is no null parallel vector field on \( D \), \((D, g)\) is either Ricci-flat (i.e. \( R^\mu_\nu = 0 \)) or locally symmetric.

Finally, one can at last prove that the narrow space left between locally symmetric and 2-symmetric Lorentzian manifolds can only be filled by spaces with a parallel null vector field.

**Theorem 3** Let \( D \subset \mathcal{V} \) be a simply connected domain of an \( n \)-dimensional 2-symmetric Lorentzian manifold \((\mathcal{V}, g)\). Then, if there is no null parallel vector field on \( D \), \((D, g)\) is in fact locally symmetric.

Thus we have arrived at

**Theorem 4** Let \( D \subset \mathcal{V} \) be a simply connected domain of an \( n \)-dimensional 2-symmetric Lorentzian manifold \((\mathcal{V}, g)\). Then, the line element on \( D \) is (possibly a flat extension

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3 It must be stressed that this proof is only valid for Lorentzian manifolds, as the definition of future tensors requires this signature.
of) the direct product of a certain number of locally symmetric proper Riemannian manifolds times either

1. a Lorentzian locally symmetric spacetime (in which case the whole \((D, g)\) is locally symmetric), or

2. a Lorentzian manifold with a parallel null vector field so that its metric tensor can be expressed locally as an appropriately restricted case of formula (19) below.

Again, the following remarks are important:

1. Of course, the number of proper Riemannian symmetric manifolds can be zero, so that the whole 2-symmetric spacetime, if not locally symmetric, is given just by a line-element of the form (19) restricted to be 2-symmetric.

2. Although mentioned explicitly for the sake of clarity, it is obvious that the block added in any flat extension can also be considered as a particular case of a locally symmetric part building up the whole space.

3. This theorem provides a full characterization of the 2-symmetric spaces using the classical results on the symmetric ones: their original classification (for the semisimple case) was given in [2], and the general problem was solved for Lorentzian signature in [9]. Combining these results with those for proper Riemannian metrics [10, 11, 15], a complete classification is achieved.

Thus, the only 2-symmetric non-symmetric Lorentzian manifolds contain a parallel null vector field. The most general local line-element for such a spacetime was discovered by Brinkmann [5] by studying the Einstein spaces which can be mapped conformally to each other. In appropriate local coordinates \(\{x^0, x^1, x^i\} = \{u, v, x^i\}\) \((i, j, k, \ldots = 2, \ldots, n - 1)\) the line-element reads

\[
d s^2 = -2dudv + Hdu + W_i dx^i \tag{19}
\]

where the functions \(H, W_i\) and \(g_{ij} = g_{ji}\) are independent of \(v\), otherwise arbitrary, and the parallel null vector field is given by

\[
k_\mu dx^\mu = -du, \quad k^\mu \partial_\mu = \partial_v. \tag{20}
\]

It is now a simple matter of calculation to identify which manifolds among (19) are actually 2-symmetric. Using Theorem 4 and its remarks, this will provide —by direct product with proper Riemannian symmetric manifolds if adequate— all possible non-symmetric 2-symmetric spacetimes. By doing so [23] one finds, among other results, that (i) the \(g_{ij}\) are a one-parameter family, depending on \(u\), of locally symmetric proper Riemannian metrics\(^4\); (ii) for a given choice of \(g_{ij}\) in agreement with the previous point, the integrability conditions provide the explicit form of the functions \(H\) and \(W_i\); (iii) finally, the scalar curvature coincides with the corresponding scalar curvature \(\bar{R}\) of \(g_{ij}\):

\(^4\) As these are classified in e.g. [11, 15], the part \(g_{ij}\) of the metric is completely determined. For an explicit formula, one only has to take any of them from the list and let any arbitrary constants appearing there to be functions of \(u\).
\( R = \bar{R} \). Due to (i), the function \( \bar{R} \) depends only on \( u \), and thus the 2-symmetry implies

\[
R = \bar{R}(u) = au + b
\]

(21)

where \( a \) and \( b \) are constants. In particular, \( \nabla_{\mu} R = -ak_{\mu} \). Thus, we see that given any locally symmetric proper Riemannian \( g_{ij} \) and letting the constants appearing there to be functions of \( u \) is too general, and these functions are restricted by the 2-symmetry so that, for example, (21) holds.

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