Analysis of the error distribution density convergence with its orthogonal decomposition in navigation measurements

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Abstract. Modern trends towards the expansion of online services lead to the need to determine the location of customers, who may also be on a moving object (vessel or aircraft, others vehicle – hereinafter the "Vehicle"). This task is of particular relevance in the fields of medicine – when organizing video conferencing for diagnosis and/or remote rehabilitation, e.g., for post-infarction and post-stroke patients using wireless devices, in education – when organizing distance learning and when taking exams online, etc. For the analysis of statistical materials of the accuracy of determining the location of a moving object, the Gaussian normal distribution is usually used. However, if the histogram of the sample has “heavier tails”, the determination of latitude and longitude's error according to Gaussian function is not correct and requires an alternative approach. To describe the random errors of navigation measurements, mixed laws of a probability distribution of two types can be used: the first type is the generalized Cauchy distribution, the second type is the Pearson distribution, type VII. This paper has shown that it’s possible obtaining the decomposition of the error distribution density using orthogonal Hermite polynomials, without having its analytical expression. Our numerical results show that the approximation of the distribution function using the Gram-Charlier series of type A makes it possible to apply the orthogonal decomposition to describe the density of errors in navigation measurements. To compare the curves of density and its orthogonal decomposition, the density values were calculated. The research results showed that the normalized density and its orthogonal decomposition practically coincide.

Keywords: observation of the moving object location, the error of navigation measurements, mixed distribution laws, orthogonal decomposition of the distribution density, Hermite polynomials, criterion for the coincidence of the density and its decomposition, Gram-Charlier series type A.

1. Introduction
Modern trends towards the expansion of online services lead to the need to determine the location of customers, who may also be on a moving object. Such an object can be, for example, a vessel sailing in constrained waters, an aircraft in an air spatial corridor, a vehicle on a certain type of road [1]. This task is of particular relevance in the fields of medicine – when organizing video conferencing for diagnosis and/or remote rehabilitation, e.g., for post-infarction and post-stroke patients using wireless devices, in education – when organizing distance learning and when taking exams online, etc. [2, 3].
One of the essential aspects of the problem of providing services to a moving object is to improve the accuracy of monitoring the location of such an object (hereinafter the “Vehicle” and “Navigation”) [4].

To ensure the maximum accuracy of observations of the vehicle position, it is necessary to know the law of distribution of errors of navigation measurements. The work of many modern scientists is devoted to the issues of increasing the accuracy of monitoring the location of moving vehicles. So, in [5], the results of the analysis of statistical materials of the accuracy of determining the position of the vessel using a satellite radio navigation system are presented. For the analysis of statistical materials of the accuracy of determining the location of a moving object can be by calculating the local density of data points [6]. In most cases to determine the error distributions and accuracy measures in the navigation Gaussian normal distribution is usually used [7]. However, if the histogram of the sample has “heavier tails”, the determination of latitude and longitude's error according to Gaussian function is not correct and requires an alternative approach [8]. To describe the random errors of navigation measurements, mixed laws of a probability distribution of two types can be used: the first type is the generalized Cauchy distribution, the second type is the Pearson distribution, type VII [7]. It is advisable to use mixed distribution functions, in particular, when observations are stopped at some point. This can be in the absence of a signal from terrestrial or satellite radio navigation systems [9], as well as when it enters the zone of high electromagnetic radiation or electronic warfare equipment (signal jammers). In this case, the determination of the location of a moving object (observation) is carried out using visual observations of fixed external landmarks with known coordinates. This leads to a significant increase in the measurement errors of the navigation parameters of a moving object.

The generalized Poisson distribution was proposed in [10] as an alternative to the normal law. The results of studying the possibility of describing systems of dependent random variables using the generalized Poisson distribution with the basic normal distribution are presented. In [11] the authors used singular value decomposition (SVD) in least square method instead of normal decomposition to enhance the accuracy of ship position. However, in this case, the calculating the observed coordinates of the vehicle does not provide the possibility of obtaining effective coordinates estimates. To obtain effective estimates of the vehicle observed coordinates, it is advisable to use the maximum likelihood method, which takes into account the actual law of error distribution. This circumstance is currently not taken into account, and there are no corresponding analytical expressions for obtaining effective estimates.

The analysis of the considered works shows that the variety of the probability distribution laws of random variables, a feature of which is the presence of “heavier tails”, can be unified by using the orthogonal decomposition with the obtained values of the higher-order central moments. In this case, an essential circumstance is the degree of coincidence of the distribution density with its orthogonal decomposition, which is the subject of this article.

Thus, the purpose of the article is to check the convergence of the distribution density of errors of the mixed law of the first type with density orthogonal decomposition.

2. Orthogonal density decomposition and determination of central moments

Thus, with the specified shortage of statistical materials, it is not possible to use the standard procedure to determine the distribution law of random variables, according to which the random components of the navigation measurement error are distributed. However, the central moments of the distribution can be estimated. We show that it’s possible obtaining the decomposition of the error distribution density using orthogonal Hermite polynomials, without having its analytical expression.

Consider a mixed distribution law of the first type and analyze the convergence of its density with density orthogonal decomposition depending on the number of terms.

Considering that the density curve of the distribution law is symmetric concerning the mathematical expectation, then for the analysis it is advisable to consider only the positive values of the error $\tilde{\xi}$ in the range from 0 to $6\sigma$, – almost the entire interval of error possible values. We divide the indicated interval into $n$ segments of the same length ($6\sigma / n$ each). For each $i$-th segment, we calculate the value of the initial density $f(\tilde{\xi})$ and its orthogonal decomposition $f^s(\tilde{\xi})$, with the value...
of $\xi$ corresponds to the middle of the $i$-th segment. It's chosen the number of segments $n = 24$ with a length of each 0.25$\sigma$.

The following sum $S_n$ was chosen as a coincidence criterion characterizing the conformity of the orthogonal decomposition $f^*(\xi)$ to the original density $f(\xi)$ as in equation (1):

$$S_n = \frac{1}{n} \sum_{i=1}^{n} \left\{ \frac{|f(\xi_i) - f^*(\xi_i)|^2}{f(\xi_i)} \right\}^{1/2},$$

(1)

The sum $S_n$ expresses the relative deviation of the density $f(\xi)$ from its decomposition. As the initial density, we choose the density of the mixed distribution law of the first type.

Let us present an expression for the orthogonal decomposition using Hermite polynomials taking into account the first terms of the decomposition as in equation (2).

$$f^*(y) = (2\pi)^{-1/2} \exp(-y^2/2) \left[ 1 + \sum_{j=2}^{6} \Phi_j \right],$$

(2)

where $\Phi_j = \frac{c_{2j}}{2^j j!} H_{2j}(y)$.

Detailed expressions for $\Phi_j$ are as follows equations (3):

$$\Phi_2 = (\mu_4 - 3)(y^4 - 6y^2 + 3)/4;$$
$$\Phi_3 = (\mu_6 - 15\mu_4 + 30)(y^6 - 15y^4 + 45y^2 - 15)/6;$$
$$\Phi_4 = (\mu_8 - 28\mu_6 + 210\mu_4 - 315)(y^8 - 15y^6 + 45y^4 - 15)/8;$$
$$\Phi_5 = (\mu_{10} - 45\mu_8 + 630\mu_6 - 3150\mu_4 + 3780)(y^{10} - 45y^8 + 630y^6 - 3150y^4 + 4725y^2 - 945)/10!.$$  

(3)

Once you have an expression for the orthogonal decomposition, you can use it instead of the original density. In this case, it is only required to substitute the central moments $\mu_n$ of the original density in the formulas for $\Phi_n$, having previously transformed it to the normalized one.

Let us consider as the initial distribution density of the centered error $\xi$ of the mixed law of the first type, which has the following analytical form [6] as in equation (4):

$$f_1(\xi) = \frac{A_n}{(\xi^2 + \lambda)^{n+1}},$$

(4)

where $A_n$ is a normalizing multiplier; $\lambda$ is a scale parameter; $n$ is an essential parameter.

We transform the obtained density to the normalized form $g(\eta)$, for which it is enough to use the formula (5) given in [11]:

$$g(\eta) = \mu_2^{1/2} f(\frac{\mu_2^{1/2}}{\lambda} \eta),$$

(5)

where $\eta = \xi / \mu_1^{1/2}$ is normalized error with unit variance; $\mu_2$ is variance of the random variable $\xi$.

For the density $f_1(\xi)$ under consideration, the variance is equal $\mu_2 = \frac{2\lambda}{2n-1}$, and the corresponding normalized density $g_1(\eta)$ has the following form as in equation (6):

$$g_1(\eta) = \frac{B_1}{(\eta^2/(2n-1) + 1)^{n+1}},$$

(6)

where $B_1$ is a normalizing multiplier as in equation (7):

$$B_1 = \frac{\lambda^{2n}}{(2n-1)!^{1/2} \pi^{(2n)!}}.$$  

(7)

The central even moments $\mu_{2n}$ of the normalized random variable $\eta$ are determined by the expression (8):
\[ \mu_{2m}^{(1)} = \frac{(2n-1)^{m}n!(2(n-m))!}{(2n)(n-m)m!} \]  

(8)

**3. Results**

To calculate the values of the normalized density \( g_1(\eta) \), it is necessary to calculate the values of the multiplier \( B_1 \).

In this paper, we will consider the normalized density with an essential parameter \( n \), taking values 2, 4, ..., 20, for which table 1 shows the numerical values of the normalizing multiplier.

| \( n \) | 0 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| \( B_1 \) | - | 0.490 | 0.440 | 0.425 | 0.418 | 0.4144 | 0.411 | 0.409 | 0.408 | 0.407 | 0.406 |

When approaching the end of the interval from 0 to \( 6\sigma \), the numerical value of the normalizing multiplier \( B_1 \) tends to 0.4 (figure 1).

![Numerical values of the normalizing multiplier](image)

**Figure 1.** Decrease of the value of the normalizing multiplier \( B_1 \).

Table 2 shows the results of calculating the central even moments of the mixed distribution law of the first type from \( \mu_4 \) to \( \mu_{12} \) for essential parameters \( n = 2, 4, ..., 20 \).

Table 3 shows the values of the normalizing multiplier \( B_1 \) and the fourth central moment \( \mu_4 \) for the essential parameter \( n \).

| \( n \) | \( \mu_4 \) | \( \mu_6 \) | \( \mu_8 \) | \( \mu_{10} \) | \( \mu_{12} \) |
|---|---|---|---|---|---|
| 2 | 9 | – | – | – | – |
| 4 | 4.20 | 49.00 | 2401.00 | – | – |
| 6 | 3.67 | 28.81 | 443.67 | 14641.00 | 1771561.00 |
| 8 | 3.46 | 23.60 | 275.35 | 5310.31 | 175240.30 |
| 10 | 3.35 | 21.24 | 217.25 | 3377.30 | 78428.40 |
| 12 | 3.29 | 19.89 | 188.34 | 2599.14 | 50583.30 |
| 14 | 3.24 | 19.02 | 171.16 | 2189.00 | 38243.10 |
| 16 | 3.21 | 18.41 | 159.80 | 1938.42 | 31476.30 |
| 18 | 3.18 | 17.96 | 151.75 | 1770.38 | 27263.80 |
| 20 | 3.16 | 17.62 | 145.75 | 1650.23 | 24412.00 |
Table 3. Normalizing multiplier $B_1$ and moment $\mu_4$ values.

| n  | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  |
|----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $B_1$ | 0.4903 | 0.4558 | 0.4402 | 0.4314 | 0.4257 | 0.4187 | 0.4147 | 0.4903 | 0.4558 |
| $\mu_4$ | 9.000 | 5.000 | 4.200 | 3.857 | 3.667 | 3.462 | 3.535 | 9.000 | 5.000 |

The values of the central moment $\mu_4$ after $n = 4$ are stabilized; therefore, we present a comparison of the values of the normalized density and its orthogonal decomposition for $n = 3$.

An example of calculating the values of the normalized density of the generalized Poisson distribution $g_3(\eta)$ and its orthogonal decomposition $f^*(n)$ with one (first) decomposition term is shown in table 4 and Figure 2. The green line corresponds to $g_3(\eta)$ and the red line corresponds to $f^*(n)$.

Table 4. Density $g_3(\eta)$ and its orthogonal decomposition $f^*(n)$ values.

| y  | $f^*(n)$ | $g_3(\eta)$ |
|----|----------|-------------|
| 0  | 0.429    | 0.423       |
| 0.25 | 0.412    | 0.407       |
| 0.5 | 0.366    | 0.363       |
| 0.75 | 0.301   | 0.300       |
| 1  | 0.23     | 0.232       |
| 1.25 | 0.165    | 0.169       |
| 1.5 | 0.112    | 0.117       |
| 1.75 | 0.073   | 0.077       |
| 2.00 | 0.047    | 0.049       |
| 2.25 | 0.03     | 0.030       |
| 2.5 | 0.02     | 0.018       |
| 2.75 | 0.012    | 0.011       |
| 3 | 0.0077   | 0.0064      |
| 3.25 | 0.0046   | 0.0037      |
| 3.5 | 0.0026   | 0.0022      |
| 3.75 | 0.0014   | 0.0013      |
| 4  | 0.0007   | 0.0007      |
| 4.25 | 0.0003   | 0.0004      |
| 4.5 | 0.0001   | 0.0002      |
| 4.75 | 0.00005  | 0.00015     |
| 5.00 | 0.00002  | 0.00009     |
| 5.25 | 0.000007 | 0.00006     |
| 5.5 | 0.000002 | 0.00003     |
| 5.75 | 0.000000 | 0.00002     |

From the obtained table 4 it follows that the orthogonal decomposition $f^*(n)$ of the density of the generalized Poisson distribution of navigation measurement errors, containing only the first term, has good convergence with the distribution density itself.

Depending on $n$, the orthogonal decomposition $f_1^*(\hat{\xi})$ contains a different number of components $\Phi_j$, since in the considered type of density the order of the even central moment cannot exceed the essential parameter $n$, i.e., $m < n$. Taking into account the obtained values of the moments $\mu_{2m}$, we
present expressions for the orthogonal decomposition $f_1^*(y)$ depending on the value of the essential parameter $n$ ($n = 2, 4, ..., 20$) as in equation (9):

$$f_1^{*(2)}(y) = (2\pi)^{-1/2} \exp(-y^2/2) [1 + \Phi_1^{*(2)}];$$

$$f_1^{*(4)}(y) = (2\pi)^{-1/2} \exp(-y^2/2) [1 + \Phi_1^{*(4)} + \Phi_2^{*(4)} + \Phi_3^{*(4)}];$$

$$f_1^{*(6)}(y) = (2\pi)^{-1/2} \exp(-y^2/2) [1 + \Phi_1^{*(6)} + \Phi_2^{*(6)} + \Phi_3^{*(6)} + \Phi_4^{*(6)} + \Phi_5^{*(6)}];$$

$$f_1^{*(8)}(y) = (2\pi)^{-1/2} \exp(-y^2/2) [1 + \Phi_1^{*(8)} + \Phi_2^{*(8)} + \Phi_3^{*(8)} + \Phi_4^{*(8)} + \Phi_5^{*(8)}];$$

$$f_1^{*(10)}(y) = (2\pi)^{-1/2} \exp(-y^2/2) [1 + \Phi_1^{*(10)} + \Phi_2^{*(10)} + \Phi_3^{*(10)} + \Phi_4^{*(10)} + \Phi_5^{*(10)}].$$

(9)

To study the properties of the orthogonal decomposition for each density, except for $n = 2$, the efficiency criterion $S_m$ was calculated for a different number of components $\Phi_i$, i.e., at first, only the first term $\Phi_1^{*(n)}$ was taken into account in the orthogonal decomposition $f_1^{*(n)}(y)$ and the efficiency $S_m^{(n)}$ was calculated. Then the value $S_m^{(n)}$ was calculated keeping two terms $\Phi_1^{*(n)}$ and $\Phi_2^{*(n)}$ in the decomposition $f_1^{*(n)}(y)$. The efficiency $S_m^{(n)}$ for decomposition with three, four, and five terms were found similarly. The results of calculating the efficiency $S_m^{(n)}$ of using the decomposition $f_1^{*(n)}(y)$ instead of the initial density $f_1(y)$ for different numbers of terms $\Phi_i^{(n)}$ are given in table 5.

| $n$ | $S_{m1}^{(n)}$ | $S_{m2}^{(n)}$ | $S_{m3}^{(n)}$ | $S_{m4}^{(n)}$ | $S_{m5}^{(n)}$ |
|-----|----------------|----------------|----------------|----------------|----------------|
| 2   | 0.1750         | –              | –              | –              | –              |
| 4   | 0.0216         | 0.0850         | 1.1100         | –              | –              |
| 6   | 0.0094         | 0.0193         | 0.0547         | 0.3340         | 7.0900         |
| 8   | 0.0055         | 0.0085         | 0.0141         | 0.0406         | 0.1880         |
| 20  | 0.0037         | 0.0048         | 0.0057         | 0.0115         | 0.0313         |

As follows from the table above, the best convergence of the orthogonal decomposition is achieved when it contains only one term $\Phi_1^{*(n)}$, and $S_{m1}^{(n)}$ improves with increasing $n$, i.e., the optimal orthogonal decomposition has the following expression (10):

$$f_1^{*(n)}(y) = (2\pi)^{-1/2} \exp(-y^2/2) [1 + \Phi_1^{*(n)}],$$

(10)

where $\Phi_1^{*(n)} = (\mu_4^{(n)} - 3)(y^4 - 6y^2 + 3)/24$.

4. Analysis of the normalized density and its orthogonal decomposition

As these studies have shown, the addition of additional terms in the decomposition impairs its accuracy.

The considered mixed law of distribution of the first type with “heavier tails” can be represented by the orthogonal decomposition of the density in the Gram-Charlier series of type A, containing the first term of the decomposition. This approach makes it possible to use orthogonal decomposition to describe the error density of navigation measurements.

The accuracy of the correspondence of the initial density to its orthogonal decomposition for the considered distribution law increases with an increase in the value of the essential parameter $n$. To compare the density curves and density orthogonal decomposition, their values were calculated. In figure 3 shows the curves of the normalized density $g_1(\eta)$ of the mixed law of the first type for the values of the essential parameter $n = 4, 6$, which are red.
Figure 3. Normalized densities $g_1(\eta)$ and their decompositions $f^*(n)$ for $n = 4, 6$.

Since the density curves are symmetrical, only half of the curve is shown for positive error values that take values in the range of six standard deviations. In the same figure 3, the corresponding curves of the orthogonal decomposition $f^*(n)$ are shown in blue.

The normalized density $g_1(\eta)$ curves for the values of the essential parameter $n = 8, 10$ and the orthogonal decomposition $f^*(n)$ curves are shown in figure 4.

Figure 4. Normalized densities $g_1(\eta)$ and their decompositions $f^*(n)$ for $n = 8, 10$.

Analysis of figure 3 and figure 4 shows that the normalized density $g_1(\eta)$ and its orthogonal decomposition $f^*(n)$ for an essential parameter $n \geq 4$ practically coincide.
5. Conclusions
This study presents an analysis of the error distribution density convergence with its orthogonal decomposition in navigation measurements during determining the location of a moving object.

We analyze the convergence of the distribution density of a mixed law of the first type with its orthogonal decomposition depending on the number of terms. This distribution has reviewed the change of the significant parameter $n$ that responsible for the “heaviness” of the tail. This paper has shown that it’s possible obtaining the decomposition of the error distribution density using orthogonal Hermite polynomials, without having its analytical expression. In the process, the best convergence of the density with its orthogonal decomposition is achieved when it contains only one term.

Our numerical results show that the approximation of the distribution function using the Gram-Charlier series of type A makes it possible to apply the orthogonal decomposition to describe the density of errors in navigation measurements.

To compare the curves of density and its orthogonal decomposition, the density values were calculated, and the results are presented in graphical form.

The research results showed that the normalized density and its orthogonal decomposition practically coincide.

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