On the Tate Conjecture in Codimension One for Varieties with \( h^{2,0} = 1 \) over Finite Fields

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Abstract

We prove that the Tate conjecture over finite fields is “generically true” for mod \( p \) reductions of complex projective varieties with \( h^{2,0} = 1 \), under a mild assumption on moduli. By refining this general result, we prove in characteristic \( p \geq 5 \) the BSD conjecture for a height 1 elliptic curve \( \mathcal{E} \) over a function field of genus 1, under the generic assumption that the singular fibers in its minimal compactification are all irreducible. We also prove the Tate conjecture over finite fields for admissible algebraic surfaces with \( p_g = K^2 = 1 \) in characteristic \( p \geq 5 \) and even dimensional Gushel-Mukai varieties in characteristic \( p \geq 3 \).

Contents

1 Introduction  2

2 Preliminaries  7

2.1 Basic Definitions on Motives  7

2.2 Norm Functors  8

2.3 Motives of Varieties with \( h^{2,0} = 1 \)  11

2.4 Monodromy Groups  12

3 Shimura varieties  13

3.1 Moduli interpretation  14

3.2 Union of conjugate Shimura varieties  16

3.3 Orthogonal Shimura Varieties over Totally Real Fields  18

4 \( p \)-adic Riemann-Hilbert correspondence and relative de Rham cycles  19

5 Period Morphisms  22

5.1 The case when \( p^2(\mathcal{X}/S) \) is abelian  23

5.2 Totally Real Case  23

5.3 The Integral Period Morphism  25

6 Proof of the Generic Case  27

6.1 A specialization lemma for monodromy  27

6.2 Proof of Thm 1.3  29

7 Deforming Curves on Parameter Spaces  32

7.1 Families of Curves which homogeneously dominate a Variety  32

7.2 Applications of the Baire category theorem  34

8 Elliptic Surfaces with \( p_g = q = 1 \)  35
1 Introduction

The past decade witnessed great progress on the Tate conjecture in codimension 1 for varieties with $h^{2,0} = 1$. For K3 surfaces over finite fields, Maulik (42), Madapusi-Pera (40), and Charles (15) settled the remaining supersingular case left open by Nygaard-Ogus (50). In characteristic $0$, Moonen proved the Tate (and Mumford-Tate) conjecture for more or less arbitrary smooth projective varieties with $h^{2,0} = 1$, as long as they satisfy a mild property $(\heartsuit)$, which we recall below. The goal of the paper is to achieve this over finite fields.

Definition 1.1. Let $K$ be a field of characteristic 0. Let $f: \mathcal{X} \to S$ be a smooth projective family of varieties with $h^{2,0} = 1$ over a connected smooth $K$-variety $S$. It is said to satisfy property $(\heartsuit)$ if the Kodaira-Spencer map

$$\nabla: T_{S/K} \to \text{Hom}(\mathbb{R}f_1^!(\Omega^1_{\mathcal{X}/S}), \mathbb{R}f_2^*\mathcal{O}_{\mathcal{X}/S})$$

(1.2)

is nontrivial. A smooth projective variety $X$ over $K$ with $h^{2,0} = 1$ is said to satisfy property $(\heartsuit)$ if it is a fiber of a family as above which satisfies property $(\heartsuit)$.

We first prove that the finite field analogue of Moonen’s main theorem in 47 is “generically true”.

Theorem 1.3. (Genericity Theorem) Let $M$ be a connected scheme which is smooth, separated, and of finite type over Spec($\mathbb{Z}$) and let $f: \mathcal{X} \to M$ be a smooth projective family of varieties. If the restriction of $\mathcal{X}$ to $M_{\mathbb{Q}}$ satisfies property $(\heartsuit)$, and the $\mathbb{C}$-fibers of $\mathcal{X}$ have Hodge number $h^{2,0} = 1$, then for $p \gg 0$, every fiber of $\mathcal{X}$ over a closed point of residue characteristic $p$ satisfies the Tate conjecture in codimension 1.

The base $M$ in Thm 1.3 should be thought of as the moduli of varieties of a certain type, or obtained from a complex $h^{2,0} = 1$ variety by spreading out.

Next, we illustrate through examples that for concrete families we can often give explicit lower bounds on $p$ in the theorem above for the Tate conjecture to hold.

New case of BSD The Tate conjecture for elliptic surfaces is particularly interesting, as it is equivalent to the BSD conjecture for elliptic curves over function fields. Let $C$ be a smooth proper curve over a field $k$ with algebraic closure $\bar{k}$. The minimal non-singular compactification of an elliptic curve $E$ over $k(C)$ is
an elliptic surface $\pi : X \to C$. Let $L$ be the fundamental line bundle (i.e., the dual of $R^1\pi_*\mathcal{O}_X$) and let $\text{ht}(E) := \text{deg}(L)$ denote the height of $E$. Consider the irreducible components of the singular fibers of $\pi$ and let $\epsilon(E)$ be the number of those components which do not meet the zero section. In particular, $\epsilon(E) = 0$ if and only if all singular fibers are irreducible (and hence also geometrically irreducible). The Shioda-Tate formula tells us that if $k = \bar{k}$, then the Mordell-Weil rank $\text{MW}(E)$ of $E$ and the Picard number $\rho(X)$ of $X$ are related by

$$\rho(X) - 2 = \text{MW}(E) + \epsilon(E).$$

Our paper has the following contribution:

**Theorem 1.4.** Assume that $k$ is a finite field of characteristic $p \geq 5$. If $g(C) = \text{ht}(E) = 1$, and $\epsilon(E) = 0$, then the Tate conjecture holds for $X$, or equivalently, the BSD conjecture holds for $E$.

We remark that $\epsilon = 0$ is an open dense condition on the moduli stack $\mathcal{M}$ of elliptic surfaces with $g(C) = \text{ht}(E) = 1$, so this condition is generically satisfied. The current evidence for function field BSD seems to be extremely meager: It is known when the elliptic surface is isotrivial, rational, or is a K3 surface (34), or when the $L$-function vanishes to order 1 (63). As far as the authors are aware of, there are no more systematic results on the conjecture only conditional on some simple invariants.

**Surfaces with $p_g = K^2_X = 1$** To illustrate our method, we also analyze a class of surfaces of general type. For a proper algebraic surface $X$ over a field, we use $K_X$ to denote the canonical divisor and $p_g$ the geometric genus $H^0(X, K_X)$.

**Theorem 1.5.** Assume that $k$ is a finite field of characteristic $p \geq 5$. Let $X$ be a projective algebraic surface over $k$ with $p_g = K^2_X = 1$. If $K_X$ is ample, then $X$ satisfies the Tate conjecture.

Over $\mathbb{C}$, surfaces with invariants $p_g = K^2_X = 1$ were classified by Catanese (14). They are simply connected, have a coarse moduli space of dimension 18 and were among the first examples for which both the local and the global Torelli theorem fail (13).

**Gushel-Mukai Varieties** There are no Fano surfaces for which the Tate conjecture is interesting because they are all rational. However, we can test our method on a class of higher dimensional Fano varieties, called Gushel-Mukai varieties. These varieties come in dimensions 3, 4, 5, 6, and we may view the even dimensional ones as $h^{2,0} = 1$ varieties by applying a Tate twist to their middle cohomology.

**Theorem 1.6.** Assume that $k$ is a finite field of characteristic $p \geq 3$. Then every Gushel-Mukai variety $X$ of dimension $2n$ for $n = 2$ or 3 over $k$ satisfies the Tate conjecture. That is, for every $i$ and $\ell \neq p$, the cycle class map

$$\text{cl} : \text{CH}^i(X) \otimes \mathbb{Q}_\ell \to H^i_{\text{et}}(X_{\bar{k}}, \mathbb{Q}_\ell)^{\text{Gal}_k}$$

is surjective, where $\text{CH}^i(X)$ denotes the Chow group of codimension $i$ algebraic cycles on $X$.

We remark that the theorem above is only nontrivial when $i = n$. Recently, the theorem above was also obtained by Fu-Moonen (25) for $p \geq 5$ but for all $k$ finitely generated over its prime field. Their methods are very different from ours, and we refer the reader to (10) for a comparison.

**Discriminants** Some of the techniques in our paper lead to the following byproduct, which a priori does not seem to be related to the Tate conjecture:

$$\rho(X) - 2 = \text{MW}(E) + \epsilon(E).$$
Theorem 1.7. Let $r \geq 1, d \geq 2$ be positive integers and assume that $r$ is odd. Let $\mathbb{P} := |O_{\mathbb{P}^r}(d)|$ be the projective space over $\mathbb{Z}$ parametrizing degree $d$ hypersurfaces of $\mathbb{P}^r := \text{Proj}(\mathbb{Z}[X_0, X_1, \ldots, X_r])$. Let $F$ be the universal polynomial which defines the universal hypersurface in $\mathbb{P}^r \times _{\mathbb{Z}} \mathbb{P}$. Then the divided discriminant $\text{disc}_d(F)$ of $F$ is congruent to the square of an irreducible polynomial modulo 2.

For example, the $r = 1$ and $d = 2$ case of theorem simply says that the discriminant $b^2 - 4ac$ for the quadratic equation $ax^2 + bxy + cy^2 = 0$ is a square of an irreducible polynomial modulo 2. In [55], Saito proved that for general $d \geq 2$ and odd $r \geq 1$, $\text{disc}_d(F)$ is always a square modulo 4 up to a sign. The theorem above provides a refinement which says that $\text{disc}_d(F)$ modulo 2 can not be further factorized. Interestingly, our proof of this simple statement involves a quite recent theorem of Beilinson ([6]).

Sketch of Proofs We take the overall strategy of [40] as a starting point and interpret it as follows: Suppose that $M$ is the moduli space over $\text{Spec}(\mathbb{Z})$ for some type of smooth projective varieties and let $f : \mathcal{X} \to M$ be the universal family. Let $k$ be a finite extension of $\mathbb{F}_p$, and let $s \in M$ be a closed point with residue field $k$. Suppose that the fiber $X := \mathcal{X}_s$ has the following property:

For any $\xi \in \text{NS}(X)$, $X$ can be lifted to a $\mathbb{C}$-fiber of $\mathcal{X}$ together with $\xi$. \hfill (\ast)

By the Lefschetz (1, 1)-theorem and the specialization of line bundles, the Tate conjecture for $X$ is equivalent to:

The image of $\bigcup_{\tilde{s} \to s} H^2(\mathcal{X}_{\tilde{s}}, \mathbb{Q})^{(1,1)}$ under specialization spans $H^2_{\text{ét}}(X_k, \mathbb{Q})^{\text{Gal}_k}$; \hfill (\ast')

where $\tilde{s}$ runs through all $\mathbb{C}$-points which specialize to $s$. Note that (\ast') is a condition which only concerns cohomology. So ideally, if we know enough about the cohomology sheaves of $\mathcal{X}$ over $M$, we obtain the Tate conjecture for $X$ under assumption [5].

We now explain the difficulties in generalizing the above strategy to general $h^{2,0} = 1$ varieties and how to overcome them.

Step I: Prove (\ast) The $h^{2,0} = 1$ condition implies that the obstruction to deforming $\xi$ is given by a power series $\hat{f}_\xi$ on $\tilde{M}_s$, the formal completion of $M$ at $s$. It is enough to show that $p \nmid \hat{f}_\xi$. For K3 surfaces, this was proved by Deligne ([19]). However, Deligne’s method relies crucially on the fact that K3 surfaces have unobstructed deformation and the Kodaira-Spencer map is an isomorphism. This method is not possible in our case: In general, deformations of $h^{2,0} = 1$ varieties may well be obstructed, and understanding the Kodaira-Spencer map is essentially a local Schottky problem, which is famously hard. In fact, the map can indeed be quite chaotic (e.g., not flat) for natural families of complex $p_g = 1$ surfaces.

To circumvent the local Schottky problem, we take instead a topological approach towards the lifting question. The key observation is that $\hat{f}_\xi$ algebraizes to a local section $f_\xi$ of $O_M$. For simplicity, assume that $M_{\mathbb{Q}}$ and $M_{\mathbb{Q}}$ are geometrically irreducible, and let $\eta_{\mathbb{Q}}$ and $\eta$ be their geometric generic points respectively. If $p \nmid f_\xi$ and $\xi$ does not lift to characteristic 0, then $\text{NS}(\mathcal{X}_{\tilde{\eta}})$ is strictly bigger than $\text{NS}(\mathcal{X}_\tilde{\eta})$, and we have an inequality

$$\dim \left( \lim_{U \subseteq \pi_{\mathbb{Q}}^{-1}(M_{\mathbb{Q}}, \eta)} H^2_{\text{ét}}(\mathcal{X}_{\tilde{\eta}}, \mathbb{Q}_\ell)^U \right) > \dim \left( \lim_{U' \subseteq \pi_{\mathbb{Q}}^{-1}(M_{\mathbb{Q}}, \eta)} H^2_{\text{ét}}(\mathcal{X}_{\tilde{\eta}}, \mathbb{Q}_\ell)^{U'} \right)$$

(1.8)

where the limits are taken as $U, U'$ run through compact open subgroups. The crucial fact we are using is
that the right hand side is equal to \( \text{rank } \text{NS}(\mathcal{X}^\circ_p) \), which follows from a suitable version of the theorem of the fixed part due to André and the classification of possible Mumford-Tate groups due to Zarhin. Hence to show \((\ast)\) for a prime \( p \) it is sufficient to show that the above inequality is impossible.

Our main tool to compare \( \pi^t_!(\mathcal{M}_{\mathbb{Q}}, \tilde{\eta}) \) with \( \pi^t_!(\mathcal{M}_{\mathbb{Q}}, \tilde{\eta}) \) is Grothendieck’s specialization theorem on tame fundamental groups, provided that \( \mathcal{M} \) admits a good compactification \( \mathcal{M} \) relative to \( \mathbb{Z}_p \). By “good” we mean that the boundary is a (relative) simple normal crossing divisor. By a spreading out argument, such a compactification can always be found for \( p > 0 \). In particular, \((1.9)\) fails for \( p > 0 \). Combined with step II, this gives Thm 1.3

However, for any fixed \( p \), it is easy to construct counterexamples where the inequality \((1.8)\) does hold, and some consequences of the theorem of the fixed part in characteristic 0 fails in characteristic \( p \) (Ex. 6.11 and 6.12). Therefore, for concretely given \( \mathcal{M} \), it is a subtle problem to effectively determine a range for \( p \), away from which \((1.8)\) cannot possibly hold.

**Step I+: Analyze Discriminant Varieties** Interestingly, for all the examples we consider, the moduli \( \mathcal{M} \) admits a natural relative compactification. More precisely, \( \mathcal{M} \) admits a morphism to a certain base \( B \) such that \( \mathcal{M} \) embeds as an open subscheme into a projective bundle \( \mathcal{P} \) over \( B \). Moreover, the boundary \( \mathcal{D} := \mathcal{P} - \mathcal{M} \) is a discriminant variety. That is, there exists an extension \( \mathcal{X} \) of the family \( \mathcal{X} \) to \( \mathcal{P} \) such that \( \mathcal{D} \) is the image of the singular locus of the morphism \( \mathcal{X} \to \mathcal{P} \).

We may disprove inequality \((1.8)\) if we find a smooth proper curve \( C \) over \( W := W(\overline{\mathbb{F}}_p) \) with a morphism \( C \to \mathcal{P}_W \) such that (a) the geometric special fiber \( C := C_{\overline{\mathbb{F}}_p} \) intersects \( \mathcal{D}_{\overline{\mathbb{F}}_p} \) transversely, and (b) \( (C - C \cap \mathcal{D}_W)_C \) is not contracted by the period morphism and does not lie in the Noether-Lefschetz loci, for some (and hence any) embedding \( W \hookrightarrow C \).

We construct \( C \) backwards. Namely, we construct the \( \overline{\mathbb{F}}_p \)-curve \( C \) first and then lift it appropriately to \( W \). To start, we construct a family of proper rational curves on \( \mathcal{M} \) parametrized by a smooth \( W \)-scheme \( T \), i.e., a morphism \( \varphi: \mathbb{P}^1_T \to \mathcal{M} \), such that the special and generic fibers of \( \varphi \) homogeneously dominate those of \( \mathcal{M} \) (see Def. 7.1). This means that there is the same amount of curves in this family which pass through any point on \( \mathcal{M} \) in any given tangent direction. By Bertini-type arguments, a general \( \overline{\mathbb{F}}_p \)-point \( t \) on \( T \) gives a curve which satisfies (a) as long as

\[
\text{the codimension 1 irreducible components of } \mathcal{D}_{\overline{\mathbb{F}}_p} \text{ are generically reduced modulo } p. \quad (\diamond)
\]

Then we apply the Baire category theorem to find an appropriate lifting which satisfies (b). The key is that the “bad” locus on the generic fiber of \( T \) to be avoided is contained in countably many proper subvarieties.

The most intricate part of the proof is to find a sufficient condition on \( p \) for \((\diamond)\) to hold, as sometimes even natural discriminant varieties may become nonreduced modulo certain \( p \) (see Appendix A). We draw ideas from enumerative geometry to detect possibly problematic \( p \). By applying the Grothendieck-Ogg-Shafarevich (GOS) formula, we show that \((\diamond)\) holds as long as the singular fibers of the morphism \( \mathcal{X} \to \mathcal{P} \) lying over \( \mathcal{D}_{\overline{\mathbb{F}}_p} \) generically have the same total dimension of vanishing cycles as those over \( \mathcal{D}_C \). Note that the total dimension of vanishing cycles incorporates both the “topological” vanishing cycles and the Swan conductors (\cite{[2]}, §XVI).

- For elliptic surfaces of height \( h \) over a base curve of genus \( g \), we show that \((\diamond)\) holds if \( p \geq 5 \) and \( 2h \geq g + 1 \). More precisely, with the aid of Kodaira’s classification of singular fibers in an elliptic fibration, we prove the key intermediate statement that the generic singular fiber over \( \mathcal{D}_{\overline{\mathbb{F}}_p} \) is the Weierstrass normal form of an elliptic surface with a single \( I_2 \)-fiber (in Kodaira’s table), so that this
singular fiber is smooth away from an ordinary double point (ODP). The proof suggests that this statement may well fail if \( h \) were not sufficiently big relative to \( g \), but luckily \( h = g = 1 \) is right at the boundary for which we can verify the statement, so we obtain Thm 1.4.

- For Thm 1.5 and 1.6, \( \mathcal{D} \) is the discriminant variety of certain linear system and we may use linear pencils for \( \varphi \). We may simply analyze the singularities by mildly extending Katz’ existence criterion for Lefshetz pencils.

In the appendix, we show the effectiveness of our enumerative approach to study the mod \( p \) behavior of discriminant varieties through examples where \( \Box \) fails. We prove Thm 1.7 and prove a similar result for Katz’ nonexample of Lefschetz embedding.

**Step II : Prove \( \star' \)** This is where we make use of the Kuga-Satake (KS) construction, which is a long tradition in the field. For a suitable lattice \( L \), the orthogonal Shimura variety \( \text{Sh}(L)_\mathbb{C} \) parametrizes Hodge structures of K3-type on \( L \), such that there is a complex period map \( \rho_C : M_\mathbb{C} \to \text{Sh}(L)_\mathbb{C} \). The Shimura variety \( \text{Sh}(L) \) has reflex field \( \mathbb{Q} \) and admits a canonical integral model \( \mathcal{X}(L) \) over \( \mathbb{Z}_{(p)} \). As in [40], we proceed in 3 broad steps: (i) Show that \( \rho_C \) descends to \( \mathbb{Q} \) and extends to \( \mathbb{Z}_{(p)} \). (ii) Show that the cohomology sheaves on \( M \) given by \( \mathcal{X} \) can be recovered by certain automorphic sheaves on \( \mathcal{X}(L) \). (iii) Convert \( \star' \) to a question about automorphic sheaves, and prove it by appealing to Kuga-Satake abelian varieties, whose cohomology gives a different realization of these automorphic sheaves.

Step (iii) will work verbatim as in loc. cit. for us once (i) and (ii) are done, but (i) and (ii) require new ideas. For K3 surfaces, these two steps were built upon the fact that the moduli of K3 surfaces have maximal monodromy, and the motive of a K3 surface is abelian in Deligne’s category of motives with absolute Hodge cycles. These inputs are not available to us for general \( h_{2,0} = 1 \) varieties. Problems arise when that the motives in question have a totally real endomorphism field \( E \) bigger than \( \mathbb{Q} \). However, we extract from Moonen’s work [47] that certain “\( E/\mathbb{Q} \)-norm” of these motives is still abelian. This allows us to show that \( \text{Sh}(L) \) contains some Shimura subvarieties \( \text{Sh}' \), whose reflex field is \( E \), such that \( \rho_C \) factors as

\[
M_\mathbb{C} \to \prod_{E \to \mathbb{C}} \text{Sh}' \otimes_{\mathbb{C}} E \to \text{Sh}(L)_\mathbb{C},
\]

and both morphisms descend to \( \mathbb{Q} \). This gives (i). Morally speaking, \( \text{Sh}' \) should be viewed as moduli spaces of these “norm” motives. We make crucial use of the theory of conjugate Shimura varieties, and show that the coproduct of a Shimura variety with all its conjugates admits a natural moduli interpretation (see §3) under some mild assumptions.

After we do step (i), we may identify the norm of cohomology sheaves with that of the automorphic sheaves, so step (ii) is essentially about salvaging a comparison of the original sheaves from that of their norms. This is relatively straightforward for the Betti and \( \ell \)-adic sheaves (including \( p \)-adic sheaves whenever in characteristic 0), but gets subtle for the de Rham sheaves—one needs to prove that the \( p \)-adic Riemann-Hilbert (RH) correspondence and the classical one gives the same identification of de Rham sheaves. In fact, this would follow from the general conjecture that “Hodge cycles are de Rham”. To this end, we extend Blasius’ notion of de Rham cycles to a relative setting, and apply a trick of deforming transcendental classes in Hodge structures to algebraic classes.

**Remark 1.9.** Every class of general type surface in Moonen’s list [47, Thm 9.4], except possibly type (b), has a parameter space with a natural relative compactification structure as described in Step I+;
hence is amenable to similar analysis. We picked a class where the defining equations of the varieties are manageable to illustrate ideas, but will provide more examples in future work.

**Organization of the Paper** For technical reasons, the paper is written in the order (Step II ⇒ Step I ⇒ Step I+), which is different from the sketch of proofs. In §2 we recall results on the motives and monodromy of \( k^{2,0} = 1 \) varieties and extract the geometric input we need from Moonen’s paper. In §3, we recall the moduli interpretation of conjugate Shimura varieties, so that we may set up period morphisms. In §4, we define relative de Rham cycles in terms of Riemann-Hilbert correspondences, and give a criterion for an isomorphism between weight 2 Hodge structures coming from motives to be de Rham. In §5, we set up the integral period morphisms and compare the cohomology sheaves with automorphic sheaves. The genericity theorem is proved in §6. In §7, we prove some lemmas about families of curves that homogeneously dominate a variety, and in §8 and §9, we prove Thm 1.4-1.6.

**Notations** Let \( f : X \to S \) be a morphism between schemes. If \( T \to S \) is another morphism, denote by \( X_T \) the base change \( X \times_S T \). We say that \( X \) is an \( S \)-variety if it is separated and of finite type. We use \( H^i_{\text{ét}}(X/S, A) \) (resp. \( H^i_{\text{dR}}(X/S) \)) to denote \( R^if_{\text{ét}*}A \), whenever \( A = \mathbb{Z}_\ell \) or \( \mathbb{Q}_\ell \) for \( \ell \in \mathcal{O}_K^\times \) (resp. \( R^if_*\Omega^\bullet_{X/S} \)). Given \( T \to S \), we may simply write \( H^i_{\text{ét}}(X|_T, A) \) (resp. \( H^i_{\text{dR}}(X|_T) \)) for \( H^i_{\text{ét}}(X_T/T, A) \) (resp. \( H^i_{\text{dR}}(X_T/T) \)). If \( S \) is a \( \mathbb{C} \)-variety, apply the same conventions to the relative Betti cohomology.

For an irreducible scheme \( X \), we write \( k(X) \) for its fraction field and \( \eta(X) \) or \( \eta_X \) for its generic point. In particular, if \( x \) is a point, we write \( k(x) \) for its residue field. If \( k \) is an algebraically closed field, and \( X \) is a (not necessarily irreducible) \( k \)-variety, by “a general \( k \)-point has property \( P \)” we mean “there exists an open dense subscheme \( U \subseteq X \) such that every \( k \)-point on \( U \) has property \( P \).

Unless otherwise noted, letters \( p \) and \( \ell \) will always denote some prime numbers and \( \ell \neq p \). We write \( \hat{\mathbb{Z}} \) for the profinite completion of \( \mathbb{Z} \) and \( \hat{\mathbb{Q}}_p \) for its prime-to-\( p \) part. Define \( \mathbb{A}_f := \hat{\mathbb{Z}} \otimes \mathbb{Q} \) and \( \mathbb{A}_p := \hat{\mathbb{Q}}_p \otimes \mathbb{Q} \). If \( k \) is a perfect field of characteristic \( p \), we write \( W(k) \) for its ring of integers and or simply \( W \) when \( k \) is understood. For any field \( k \), denote by \( \bar{k} \) a chosen algebraic closure when such a choice does not need to be specified. For any object \( M \) over \( \mathbb{C} \) and \( \sigma \in \text{Aut}(\mathbb{C}) \), denote \( M \otimes_\sigma \mathbb{C} \) by \( M^\sigma \) whenever applicable.

We use the following abbreviations: VHS for “variations of Hodge structures”, LHS (resp. RHS) for “left (resp. right) hand side”, and ODP for “ordinary double point”.

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## 2 Preliminaries

### 2.1 Basic Definitions on Motives

Let \( K \) be a field of characteristic 0. Let \( \text{smPr}(K) \) denote the category of smooth projective varieties over \( K \). Let \( \text{Mot}_{\text{AH}}(K) \) denote Deligne’s category of motives with absolute Hodge cycles. We use the following
definition: If $K$ has finite transcendence degree over $\mathbb{Q}$, then for $X \in \text{smPr}(K)$, an absolute Hodge cycle of degree $d$ is a pair

$$(z_{A, f}, z_{\text{dR}}) \in H^d_{\text{et}}(X_K, A_f) \times H^d_{\text{dR}}(X/K)$$

such that for every $\sigma : \overline{K} \hookrightarrow \mathbb{C}$, there exists a Betti class $z_B \in H^d(X_\sigma(\mathbb{C}), \mathbb{Q})$ which is compatible with $(z_{A, f}, z_{\text{dR}})$ in the obvious sense. This definition can be extended to an arbitrary field of characteristic 0 by [22, §1 Prop. 2.9]. For $X, Y \in \text{smPr}(K)$, let $\text{Corr}_{\text{AH}}(X, Y)$ denote the $\mathbb{Q}$-module of absolute Hodge correspondences of degree $r$. An object in $\text{Mot}_{\text{AH}}(K)$ is given by a tuple $(X, e, n)$ where $X$ is a smooth projective variety over $K$, $e \in \text{Corr}_{\text{AH}}(X, X)$ is an idempotent projector, and $n \in \mathbb{Z}$. For every $X \in \text{smPr}(K)$, we denote by $h^r X$ the object $(X, \pi_X^r, 0)$, where $\pi_X^r$ is the $r$th Künneth projector.

Let $\text{Mot}(K)$ denote the category of André’s motives over $K$. Usually André’s motives are defined with the aid of a reference Weil cohomology theory, but the resulting category does not depend on this choice. For our purposes, we view $\text{Mot}(K)$ as a subcategory of $\text{Mot}_{\text{AH}}(K)$ which is defined in much the same way except that the absolute Hodge cycles are replaced by motivated cycles, and we write $\text{Corr}'(X, Y)$ for the $\mathbb{Q}$-module of motivated correspondences of degree $r$. Recall that André showed that $\pi_X^r$ is indeed always motivated ([3, Prop. 2.2]), so $h^r X$ as in the paragraph above lies in $\text{Mot}(K)$.

Let $\text{Mot}_{\text{AB}}(K)$ (resp. $\text{Mot}_{\text{AB}}^r(K)$) denote the full Tannakian subcategory of $\text{Mot}(K)$ (resp. $\text{Mot}_{\text{AH}}(K)$) generated by Artin motives and the motives attached to abelian varieties. Let $\text{Hdg}_K$ denote the category of polarizable Hodge structures over $\mathbb{Q}$. Recall the following fundamental result, due to Deligne and André ([22, I Thm 2.11], [3, Thm 0.6.2]):

**Theorem 2.1.** If $K$ is embedded into $\mathbb{C}$, then the natural functors from $\text{Mot}_{\text{AH}}^r(K)$ and $\text{Mod}_{\text{AB}}^r(K)$ to $\text{Hdg}_K$ are both faithful, and are fully faithful if $K$ is in addition algebraically closed.

**Families of Motives** Let $S$ be a smooth connected $K$-variety with generic point $\eta_S$. Currently there does not seem to be a unanimous definition for a family of motives over $S$. We recall two of them: According to [44, Def. 2.37], a motive $\mathfrak{M}$ over $S$ is a motive $M$ in $\text{Mot}_{\text{AH}}(\eta_S)$ such that the action of $\pi_1^\text{et}(\eta_S, \bar{\eta}_S)$ on the $\mathbb{A}_f$-realization $\omega_{\mathfrak{M}}(\mathbb{A}_f)$ factors through $\pi_1^\text{et}(S, \bar{\eta}_S)$. We may view such objects as constituting a subcategory $\text{Mot}_\eta(S)$ of $\text{Mot}(\eta_S)$, with morphisms defined in the obvious way. We define $\text{Mot}_\eta(S)$, $\text{Mot}_{\text{AB}}^r(S)$ and $\text{Mot}_{\text{AB}}^r(S)$ when $\text{Mot}_{\text{AH}}(\eta_S)$ is replaced by $\text{Mot}(S)$, $\text{Mot}_{\text{AB}}(S)$ and $\text{Mot}_{\text{AB}}^r(S)$ respectively.

We may also define a motive over $S$ by a tuple $(X, e, n)$ where $f : X \to S$ is a smooth projective morphism with connected fibers, $e$ is a global section of $R^{2d}(f \times_S f)_*Q_\ell(d)$ ($d = \dim(X/S)$), and $n$ is an integer such that for every $s \in S$, the fiber of the object $(X, e, n)$ over $s$ defines an object of $\text{Mot}(k(s))$. This is Moonen’s definition ([40, Def. 4.3.3]), except that we do not require $S$ to be geometrically connected. Denote the category of such objects by $\text{Mot}(S)$. Define the $\ell$-adic realization functor $\omega_\ell$ of $M = (X, e, n) \in \text{Mot}(S)$ as:

$$\omega_\ell(M) = e \cdot \left( \bigoplus_{i \geq 0} \mathbb{R}^i f_* Q_\ell(n) \right)$$

and define the de Rham, Betti and Hodge realization functors $\omega_{\text{dR}}, \omega_B, \omega_{\text{Hdg}}$ by adapting the above definition in the obvious way (of course, $\omega_B$ and $\omega_{\text{Hdg}}$ are only applicable when $K = \mathbb{C}$). Set $\omega_{A_f}$ to be the restricted product of all $\omega_\ell$’s. If $S = \text{Spec}(K)$ is a single point, we customarily view $\omega_\ell(M)$ as a $\text{Gal}_K$-module over $Q_\ell$.

There is certainly a functor $\xi_\eta : \text{Mot}(S) \to \text{Mot}_\eta(S)$ given by taking the generic fibers. As a “good reduction theory” for motives is currently unclear, we do not know whether this functor is essentially surjective. However, there are some useful partial results in this direction. Suppose now that $K = \mathbb{C}$. Let
Hdg_Q(S) denote the category of polarizable Q-VHS on S. Then the extension properties of polarizable Q-VHS ([44, Thm 2.40]) allows us to define a functor \( \omega^\eta_{Hdg} : \text{Mot}_\eta(S) \to \text{Hdg}_Q(S) \). This gives us a diagram

\[
\begin{align*}
\text{Mot}(S) & \xrightarrow{\xi_\eta} \text{Mot}_\eta(S) \\
\omega_{Hdg} & \downarrow \quad \downarrow \omega^\eta_{Hdg} \\
\text{Hdg}_Q(S) & \end{align*}
\]

such that for every \( M \in \text{Mot}(S) \), \( \omega_{Hdg}(M) \) is canonically isomorphic to \( \omega^\eta_{Hdg}(\xi_\eta(M)) \). The functor \( \omega^\eta_{Hdg} \) is fully faithful when restricted to \( \text{Mot}^{Ab}_{\eta, AH}(S) \) and we say that an object in the essential image, which we denote by \( \text{Hdg}_{Q}^{\eta}(S) \), is \textit{abelian-motivic} (cf. Def. 2.30 and Prop. 2.42 in loc. cit.). For an object \( M \in \text{Mot}^{Ab}_{\eta, AH}(S) \) and \( s \in S(\mathbb{C}) \), we define the fiber \( M_s \) to be an object in \( \text{Mot}^{Ab}_{\eta, AH}(\mathbb{C}) \) such that \( \omega_{Hdg}(M_s) \) is isomorphic to the fiber of \( \omega^\eta_{Hdg}(M) \) over \( s \). The object \( M_s \) is well defined up to isomorphism.

**Remark 2.2.** Again assume that \( K = \mathbb{C} \). For any \( \sigma \in \text{Aut}(\mathbb{C}) \), there is an obvious equivalence of categories \( \varphi_\sigma : \text{Mot}(S) \xrightarrow{\sim} \text{Mot}(S^\sigma) \) given by \( - \otimes_{\sigma} \mathbb{C} \). Since a motive \( M_{\eta} \) over \( \eta \mathbb{S} \) extends to an object in \( \text{Mot}(U) \) for some open dense \( U \subseteq S \), we may similarly define \( \varphi_\sigma : \text{Mot}(\eta^S) \to \text{Mot}(\eta^S) \), which clearly restricts to an isomorphism \( \text{Mot}_\eta(S) \xrightarrow{\sim} \text{Mot}_\eta(S^\sigma) \). If we fix a quasi-inverse to \( \text{Mot}^{Ab}_{\eta, AH}(S) \to \text{Hdg}_{Q}^{\eta}(S) \), then this also gives to \( \text{Hdg}^{\eta}_{Q}(S) \xrightarrow{\sim} \text{Hdg}^{\eta}_{Q}(S^\sigma) \).

**Remark 2.3.** The \( \ell \)-adic and Betti realization functors \( \omega_{\ell} \) and \( \omega_B \) are naturally defined for \( \text{Mot}_\eta(S) \). Moreover, \( \omega_B \) on \( \text{Mot}_\eta(S) \) is compatible with \( \omega^\eta_{Hdg} \). However, there does not seem to be a de Rham realization functor \( \omega_{dR} \) from \( \text{Mot}_\eta(S) \) to the category of filtered vector bundles with integrable connection over \( S \) in literature when \( K = \mathbb{C} \) or has finite transcendence degree over \( \mathbb{Q} \). Let \( M \in \text{Mot}_\eta(S) \). If \( K = \mathbb{C} \), we write \( \omega_{dR}(M) \) for the underlying filtered vector bundle with integrable connection of \( \omega^\eta_{Hdg}(M) \). Now suppose that \( K \) is of finite transcendence degree over \( \mathbb{Q} \). Recall that any two embeddings \( K \rightarrow \mathbb{C} \) differ by an element of \( \text{Aut}(\mathbb{C}) \). We choose an embedding \( \iota : K \hookrightarrow \mathbb{C} \) and take \( U \subseteq S \) to be an open dense subvariety such that \( M \) is defined by an object \( M_U \in \text{Mot}(U) \). Denote \( - \otimes_{\mathbb{C}} \mathbb{C} \) with a superscript \( \sim \mathbb{C} \). Let \( V \) be the filtered vector bundle with integrable connection \( \omega_{dR}(M_U) \) over \( U \). The \( K \)-model \( U \) of \( U_{\mathbb{C}} \) defines a Galois descent datum \( \{ g_\sigma : U_{\mathbb{C}}^\sigma \xrightarrow{\sim} U_{\mathbb{C}} \}_{\sigma \in \text{Aut}(\mathbb{C}/K)} \), and the \( K \)-model \( V \) of \( V_{\mathbb{C}} \) defines a Galois descent datum \( \{ \psi_\sigma : g_\sigma^* V_{\mathbb{C}} \xrightarrow{\sim} V_{\mathbb{C}}^{\sigma} \}_{\sigma \in \text{Aut}(\mathbb{C}/K)} \). However, \( \psi_\sigma \) comes from an isomorphism of polarizable \( \mathbb{Q} \)-VHS, so \( K \) can be extended uniquely to a Galois descent datum for the vector bundle \( \omega_{dR}(M_U) \) over \( S_{\mathbb{C}} \). We define \( \omega_{dR}(M_U) \) to be \( \omega_{dR}(M_C) \) together with the descent data \( \{ \psi_\sigma \}_{\sigma \in \text{Aut}(\mathbb{C}/K)} \). If in addition the descent data \( \{ \psi_\sigma \} \) is known to be effective\(^1\), then we identify \( \omega_{dR}(M_U) \) with the descent.

**2.2 Norm Functors**

We recall the basics of norm functors. The reader may refer to [47, §3] for more details. Let \( k \) be a field in characteristic zero and \( E \) be an étale \( k \)-algebra. Let \( \mathcal{C} \) be any Tannakian \( k \)-linear category and \( \mathcal{C}(E) \) be the category of \( E \)-modules in \( \mathcal{C} \). Then we may consider a norm functor \( \text{Nm}_{E/k} : \mathcal{C}(E) \to \mathcal{C} \) ([47, §3.6]). For any object \( M \in \mathcal{C}(E) \), we write \( M_{(k)} \) for the underlying object in \( \mathcal{C} \) when we forget the \( E \)-linear structure.

We now specialize to the case when \( \mathcal{C} \) is the category of \( k \)-modules \( \text{Mod}_k \). For any \( M \in \text{Mod}_E \), there is a functorial polynomial map \( \nu_M : M \to \text{Nm}_{E/k}(M) \) such that \( \nu_M(em) = \text{Nm}_{E/k}(e)\nu_M(m) \) for any \( e \in E \).

\(^1\) We expect that this is always true, but we have not checked the details.
and \( m \in M \). The norm functor \( \text{Nm}_{E/k} \) is a \( \otimes \)-functor and is non-additive (unless \( E = k \)). However, for any \( M_1, M_2 \in \text{Mod}_E \), there is an identification
\[
\text{Nm}_{E/k}(\text{Hom}_E(M_1, M_2)) = \text{Hom}_k(\text{Nm}_{E/k}(M_1), \text{Nm}_{E/k}(M_2)).
\]
Let us write \( N(\cdot) \) for \( \text{Nm}_{E/k}(\cdot) \) when no ambiguity arises. The above identification gives us a structural map
\[
\eta: \text{Res}_{E/k} \text{GL}(V) \to \text{GL}(N(V))
\]
for any \( V \in \text{Mod}_E \), which sends an \( E \)-linear automorphism \( f \) to \( \text{Nm}_{E/k}(f) \).

Let \( T_E \) denote the torus \( \text{Res}_{E/k} \text{G}_m \) and \( T^1_E \) denote the kernel of the norm map \( T_E \to T_k \). If \( V \) is a faithful \( E \)-module, then \( \ker(\eta) = T^1_E \). Given an algebraic group \( G \) over \( E \) we denote the Weil restriction by \( G_{E/Q} \). For \( V \in \text{Mod}_E \) and \( G \subseteq \text{GL}(V_{(k)}) \), we denote the intersection of \( G \) with the centralizer of \( T_E \) in \( \text{GL}(V) \) by \( C_E(G) \).

We now further specialize to the situation when \( k = \mathbb{Q} \) and \( E \) is a totally real field. Recall that \((V, \tilde{\phi}) \mapsto (V_{\mathbb{Q}}, \phi := \text{tr}_{E/\mathbb{Q}} \circ \tilde{\phi}) \) defines an equivalence of categories between quadratic forms over \( E \) and quadratic forms over \( \mathbb{Q} \) with a self-adjoint \( E \)-action ([32, Ch 1, Thm 7.4.1]). We call \( \phi \) the transfer of \( \tilde{\phi} \). This equivalence identifies \( O_{E/\mathbb{Q}}(V, \tilde{\phi}) \) with the centralizer \( C_E(O(V_{\mathbb{Q}}, \phi)) \). Let \( Z = T^1_E \cap (\mu_2)_E/\mathbb{Q} \). Then we have a commutative diagram

\[
\begin{array}{ccc}
1 & 1 \\
\downarrow & & \downarrow \\
\text{SO}_{E/\mathbb{Q}}(V) & \longrightarrow & \text{SO}_{E/\mathbb{Q}}(V) \\
\downarrow & & \downarrow \\
1 & \longrightarrow & C_E(\text{SO}(V_{\mathbb{Q}})) \\
\downarrow & & \downarrow^{(\text{det})_{E/\mathbb{Q}}} \\
1 & \longrightarrow & \text{O}_{E/\mathbb{Q}}(V) \\
\downarrow & & \downarrow \\
1 & \longrightarrow & (\mu_2)_E/\mathbb{Q} \\
\downarrow & & \downarrow^{\text{Nm}} \\
1 & \longrightarrow & \mu_2 \\
\downarrow & & \downarrow \\
1 & \longrightarrow & 1 \\
\end{array}
\]

(2.4)

with exact rows and columns. The key observation we will use is that if \( \dim_E V \) is odd, then the vertical exact sequences are naturally split by the maps induced by the inclusion \( \mu_2, E \hookrightarrow O(V) \), and the composition

\[
\text{SO}_{E/\mathbb{Q}}(V) \hookrightarrow \text{GL}_{E/\mathbb{Q}}(V) \xrightarrow{\eta} \text{GL}(N(V))
\]

is injective.

Finally, we recall some facts about spin groups from [47, §4.2, 4.3]. The transfer of quadratic forms induces a morphism \( \text{CSpin}_{E/\mathbb{Q}}(V) \to \text{CSpin}(V_{\mathbb{Q}}) \) with coimage \( \overline{\text{CSpin}}_{E/\mathbb{Q}}(V) := \text{CSpin}_{E/\mathbb{Q}}(V)/T^1_E \). Moreover, we have an identification
\[
\overline{\text{CSpin}}_{E/\mathbb{Q}}(V) = \text{SO}_{E/\mathbb{Q}}(V) \times_{\text{SO}(V_{\mathbb{Q}})} \text{CSpin}(V_{\mathbb{Q}}).
\]

(2.6)
2.3 Motives of Varieties with $h^{2,0} = 1$

**Definition 2.7.** A polarized Hodge structure $V$ is said to be of K3-type if $h^{-1,1} = h^{1,-1} = 1$, and $h^{i,j} = 0$ when $|i - j| > 2$. The transcendental part $T(V)$ of a Hodge structure of K3 type on $V$ is the orthogonal complement of $V \cap V^{(0,0)}_C$.

We recall the following fundamental result of Zarhin, which tells us all the possibilities for Mumford-Tate groups.

**Theorem 2.8 ([66, §2]).** Let $V$ be a Hodge structure of K3-type with polarization $\phi$ such that $V = T(V)$.

(a) The endomorphism algebra $E := \text{End}_{Hdg} V$ is either a totally real field or a CM field.

(b) The adjoint map $e \mapsto \bar{e}$ defined by $\phi(ex, y) = \phi(x, \bar{y})$ is the identity map when $E$ is totally real and is complex conjugation when $E$ is CM.

(c) Moreover, writing $\bar{\phi}$ for the transfer of $\phi$, we have that $\text{MT}(V) = \text{SO}_{E/\bar{E}}(V, \bar{\phi})$ when $E$ is totally real, and $\text{MT}(V) = \text{U}_{E/\bar{E}}(V, \bar{\phi})$ when $E$ is CM.

**Definition 2.9.** Let $S$ be an integral normal scheme of finite type with generic point $\eta$. Let $f : \mathcal{X} \to S$ be a smooth proper family of varieties. Let $(\Lambda, \lambda \in \Lambda)$ be a pointed $\mathbb{Z}$-lattice. A $(\Lambda, \lambda)$-polarization on $\mathcal{X}/S$ is an embedding $\Lambda \hookrightarrow \text{NS}(\mathcal{X}_\eta)$ such that the image of $\lambda$ comes from a relatively ample line bundle on $\mathcal{X}/S$. The family $\mathcal{X}/S$ is said to be maximally polarized if $\Lambda$ spans $\text{NS}(\mathcal{X}_\eta)_\mathbb{Q}$ and $\text{NS}(\mathcal{X}_\eta)_\mathbb{Q} = \text{NS}(\mathcal{X}_0)_\mathbb{Q}$.

Note that the condition $\text{NS}(\mathcal{X}_\eta)_\mathbb{Q} = \text{NS}(\mathcal{X}_0)_\mathbb{Q}$ can always be achieved by replacing $S$ by a connected finite étale cover (use e.g., [3, Scholie 2.5] and [60, 0BQJ]) if $k(\eta)$ is a perfect field.

**Lemma 2.10.** For a smooth proper family of varieties $f : \mathcal{X} \to S$, every line bundle on $\mathcal{X}_\eta$ extends to a unique class in $\text{Pic}_{\mathcal{X}/S}(S)$.

**Proof.** We may always extend $\xi$ on $\mathcal{X}_\eta$ to $\mathcal{X}_U$ for some open subscheme $U \subseteq S$, and then to a line bundle on $\mathcal{X}$ (see e.g. [28, Prop. II.6.5]). But any two extensions give the same relative line bundle over $S$ by [28, ErrIV Cor. 21.4.13].

For a lattice-polarized family $\mathcal{X}/S$ we denote by $p^2(\mathcal{X}/S) \in \text{Mot}(S)$ the second primitive motive (if $S$ is defined over a characteristic 0 field) and use $\text{PH}^2(\mathcal{X}/S)$ for the primitive cohomology modules. We note that for a maximally polarized family $\mathcal{X}$ over a connected smooth $\mathbb{C}$-variety $S$, there is no algebraic class in $\text{PH}^2(\mathcal{X}_s, \mathbb{Q})$ when $s$ is a Hodge-generic point (cf. [46, p. 31]).

The following statements are the key inputs we will use from Moonen’s paper.

**Theorem 2.11.** Let $\mathcal{X}$ be a maximally polarized family of $h^{2,0} = 1$ varieties over a connected smooth $\mathbb{C}$-variety $S$ which satisfies $(\Diamond)$. Let $E$ be the endomorphism field of $p^2(\mathcal{X}/S)$. Let $s \in S$ be any $\mathbb{C}$-point. Write $X$ for $\mathcal{X}_s$ and $p$ for $p^2(X)(1)$.

(a) The action of $E$ on $p$ is given by motivated cycles.

(b) If $E$ is totally real, $\dim E \omega_B(p) \neq 4$, and $\dim E \omega_B(p)$ is odd, then $\text{Nm}_{E/\mathbb{Q}}(p_s) \otimes \text{det}(p_{s,(\mathbb{Q})})$ is an object of $\text{Mot}^{\text{Ab}}(\mathbb{C})$.

(c) If $E$ is totally real, and $\dim E T(X) = 4$, $\text{Nm}_{E/\mathbb{Q}}(p_s)$ is an object of $\text{Mot}^{\text{Ab}}(\mathbb{C})$.

(d) If $E$ is CM, then $p_s$ is an object of $\text{Mot}^{\text{Ab}}_{\text{Aht}}(\mathbb{C})$.  

11
Proof. The first three statements follow directly from results in [4]. (a) is Prop. 6.6, (b) is stated in the last sentence of §5.4 and (c) is a direct consequence of Prop. 8.2. We only need to explain (d). As explained in §7.4 of loc. cit., there exists a motive \( u \) equipped with an \( E \)-action such that \( \omega_B(u) \cong E \) and \( p \otimes_E u = h \) for some \( h \in \text{Mot}_{ab} \). Note that the motive \( u \) is independent of \( s \). It suffices to show that \( \omega_B(u) \) is spanned by absolute Hodge classes. By the proof of [4, Lem. 7.5], for some \( s \in S \), the algebraic part \( \omega_B(p(0,0)) \) of \( \omega_B(p) \) is nonempty. Since every class in \( \omega_B(u) \) is of type \((0,0)\), we have \( \omega_B(p(0,0)) \otimes E \omega_B(u) = \omega_B(h(0,0)) \). As every class in \( \omega_B(p(0,0)) \) and \( \omega_B(h(0,0)) \) is absolute Hodge, we may now conclude by Prop. 2.12 below. □

Proposition 2.12. Let \( E \) be a number field and let \( m,n \) be objects of \( \text{Mot}(\mathbb{C}) \) with \( E \)-action. Let \( h \coloneqq m \otimes_E n \). Let \( m \in \omega_B(m), n \in \omega_B(n) \) be nonzero Hodge cycles and define \( h \in \omega_B(h) \) to be \( m \otimes_E n \). If \( h \) and \( m \) are both absolute Hodge, then so is \( n \).

Proof. We need to show that for every \( \sigma \in \text{Aut}(\mathbb{C}) \), the image \( n^\sigma \) of \( n \) under the canonical isomorphism

\[
\omega_B(n) \otimes (\mathbb{C} \otimes A_f) \cong \omega_{dR}(n) \times \omega_{H}(n) \cong \omega_{dR}(n^\sigma) \times \omega_{H}(n^\sigma) \cong \omega_B(n^\sigma) \otimes (\mathbb{C} \otimes A_f).
\]

is contained in \( \omega_B(h^\sigma) \). Since \( m \) and \( h \) are absolute Hodge, we know that \( m^\sigma \in \omega_B(m^\sigma) \) and \( h^\sigma = m^\sigma \otimes n^\sigma \in \omega_B(h^\sigma) \). We conclude by Lemma 2.13 below. □

Lemma 2.13. Let \( E/k \) be a field extension, \( R \) be a \( E \)-algebra and \( M,N \) be finite dimensional \( E \)-vector spaces. Let \( H \coloneqq M \otimes_E N \) and suppose \( h \in H \otimes_k R \) is a nonzero element of the form \( m \otimes n \) under the canonical isomorphism

\[
H \otimes_k R \cong (M \otimes_k R) \otimes_{E \otimes_k R} (N \otimes_k R),
\]

where \( m \in M \otimes_k R \) and \( n \in N \otimes_k R \). If \( h \in H \) and \( m \in M \), then \( n \in N \).

Proof. By choosing bases of \( M \) and \( N \), we may assume \( M = E^m \) and \( N = E^n \) and identify \( H \) with \( E^{m \times n} \) and \( h \) with \( m \cdot n^T \). We denote by \((m_i),(n_j),(h_{i,j})\) the respective coordinates of \( m,n \) and \( h \) and fix \( i \) such that \( m_i \neq 0 \). Then \( h_{i,j} = m_i \cdot n_j \) and hence \( n_j = m_i^{-1} h_{i,j} \in E \). □

2.4 Monodromy Groups

Definition 2.14. Let \( S \) be a smooth connected \( \mathbb{C} \)-variety with a point \( s \in S \) and let \( V \) be a \( \mathbb{Q} \)-local system.

We define the monodromy group \( \text{Mon}(V,s) \) of \( V \) at \( s \) as the Zariski closure of the image of the monodromy action \( \pi_1(S,s) \to \text{GL}(V_s) \). When \( S \) is a normal scheme of finite type with a geometric point \( s \), and \( V_{\ell} \) is a lisse-étale \( \ell \)-adic sheaf on \( S \), define \( \text{Mon}(V_{\ell},s) \subseteq \text{GL}(V_{\ell,s}) \) analogously. The identity component of \( \text{Mon}(-,-) \) is denoted by \( \text{Mon}^0(-,-) \).

We recall the following fundamental theorem due to André, who said the following is essentially a consequence of the theorem of the fixed part:

Theorem 2.15 ([1 Thm 1]). Let \( S \) be a smooth connected \( \mathbb{C} \)-variety and \( V \) be a polarizable \( \mathbb{Q} \)-VHS on \( S \). Let \( s \) be a Hodge-generic point. Then \( \text{Mon}^0(V,s) \) is a normal subgroup of the derived subgroup of \( \text{MT}(V_s) \).

Using the theorem above, we immediately deduce a “geometric Tate conjecture” for \( h^{2,0} = 1 \) varieties.

Lemma 2.16. Let \( S \) be a smooth connected \( \mathbb{C} \)-variety and \( \mathcal{F} \to S \) be a projective family of \( h^{2,0} = 1 \) varieties which satisfies (\( \heartsuit \)). Let \( \eta \) be the generic point of \( S \). Then the natural morphism

\[
\text{NS}(\mathcal{F}_\eta) \otimes \mathbb{Q}_\ell \xrightarrow{\sim} \lim_{U \subseteq \pi_1^0(s,\overline{\eta})} H^2_{et}(\mathcal{F}_\eta,\mathbb{Q}_\ell)^U,
\]

12
where \( U \) runs through compact open subgroups, is an isomorphism.

**Proof.** We recall that as \( S \) is normal, the natural map \( \pi^\ell_1(\eta, \bar{\eta}) \to \pi^\ell_1(S, \bar{\eta}) \) is surjective and open ([60, 0BQJ]). The elements of \( \text{NS}(\mathcal{X}_\eta) \) are hence stabilized by an open subgroup of \( \pi^\ell_1(S, \bar{\eta}) \). This gives rise to the map in the lemma. Moreover, it is clearly injective.

To see the surjectivity, choose a polarization on \( \mathcal{X}/S \) and let \( V \) be the \( \mathbb{Q} \)-VHS \( \text{PH}^2(\mathcal{X}/S, \mathbb{Q})(1) \). It suffices to show that for a very general point \( s \in S \), we have

\[
\text{PH}^2(\mathcal{X}_s, \mathbb{Q})(1,1) = \lim_{U \subseteq \pi_1(S, s)} \text{PH}^2(\mathcal{X}_s, \mathbb{Q})^U,
\]

as \( U \) runs through finite index subgroups. The RHS can be identified with the subspace of \( \text{PH}^2(\mathcal{X}_s, \mathbb{Q}) \) fixed by the identity component \( \text{Mon}^\circ(V, s) \) of \( \text{Mon}(V, s) \). Let \( T \) be the transcendental part of \( V_s \). We reduce to showing that \( \text{Mon}^\circ(V, s) \) cannot fix any element in \( T \).

Let \( E := \text{End}_{\text{Hdg}} T \). By Thm 2.8 and the above, there are only three possibilities for \( \text{Mon}^\circ(V, s) \) (cf. [47, Prop. 6.4(iii)] and its proof):

(a) If \( E \) is totally real and \( \dim_E T \neq 4 \), \( \text{Mon}^\circ(V, s) = \text{SO}_{E/\mathbb{Q}}(T) \).

(b) If \( E \) is totally real, \( \dim_E T = 4 \), and \( \text{Mon}^\circ(V, s) \) is a proper subgroup of \( \text{SO}_{E/\mathbb{Q}}(T) \), then \( \text{Mon}^\circ(V, s) = H_{E/\mathbb{Q}} \), where \( H \) is a nontrivial normal subgroup of the \( E \)-group \( \text{SO}(T) \).

(c) If \( E \) is CM, then \( \text{Mon}^\circ(V, s) = \text{SU}_{E/\mathbb{Q}}(T) \).

The statements (a) and (c) are straightforward as in those cases \( \text{SO}_{E/\mathbb{Q}}(T) \) and \( \text{SU}_{E/\mathbb{Q}}(T) \) are simple. (b) was mentioned in [47, §8.1] and follows from [11, Prop. 6.18].

We first prove the lemma in case (a). Since the set of rational points in a connected reductive group is Zariski dense, it suffices to show that the action of \( \text{SO}_{E/\mathbb{Q}}(T) \otimes_{\mathbb{Q}} \mathbb{C} \) on \( T \otimes_{\mathbb{Q}} \mathbb{C} \) has no fixed elements. But then the statement is clear, as

\[
\text{SO}_{E/\mathbb{Q}}(T) \otimes_{\mathbb{Q}} \mathbb{C} = \prod_{\sigma : E \to \mathbb{C}} \text{SO}(T) \otimes_{\sigma} \mathbb{C}.
\]

Case (c) is entirely similar. Now we consider (b). For any embedding \( \sigma : E \to \mathbb{C} \), \( H \otimes_{\sigma} \mathbb{C} \) is one of the two normal subgroups of \( \text{SO}(T) \otimes_{\sigma} \mathbb{C} \) isomorphic to \( \text{SL}_2 \). Recall that over \( \mathbb{C} \) we have \( (\text{SL}_2 \times \text{SL}_2)/\mu_2 \cong \text{SO}_4 \). If we view \( \text{SO}_4 \) as defined by the determinant form on the space of \( 2 \times 2 \) matrices, then the two copies of \( \text{SL}_2 \) act via left or right multiplication. It is clear then neither of them fixes any nonzero element. \( \square \)

## 3 Shimura varieties

Let \((G, \Omega)\) be a Shimura datum. As usual, we say that it is of Hodge type if there exists an embedding \((G, \Omega) \to (\text{GSp}(H, \psi), \mathcal{H}^\pm)\) where \((H, \psi)\) is a symplectic space over \( \mathbb{Q} \) and \( \mathcal{H}^\pm \) is the associated Siegel double space. Let \( w_\Omega \) denote the weight and \( E := E(G, \Omega) \) denote the reflex field. Let \( \tau_0 : E \hookrightarrow \mathbb{C} \) be the embedding which restricts to the identity. Let \( \chi_\infty \) be the set of all embeddings of \( E \) into \( \mathbb{C} \) and for each \( \tau \in \chi_\infty \) let \( \mathbb{C}_\tau \) denote \( E \otimes_{E} \mathbb{C}_\sigma \), i.e., the field \( \mathbb{C} \) but with \( E \) acting through \( \tau \). In particular, \( \mathbb{C} = \mathbb{C}_{\tau_0} \). For \( \sigma \in \text{Aut}(\mathbb{C}) \), we set \( \sigma(\tau) := \sigma^{-1} \circ \tau \).
For any compact open subgroup $K \subset G(\mathbb{A}_f)$ we denote by $\text{Sh}_K(G, \Omega)$ the canonical model of the Shimura variety of level $K$. Moreover, we set

$$\text{Sh}(G, \Omega) := \lim_{\leftarrow K} \text{Sh}_K(G, \Omega).$$

Let $Z \subseteq G$ denote the center. In this section, we put the standing assumption that either $(G, \Omega)$ is of Hodge type and satisfies the additional axiom (SV5) (i.e., $Z(\mathbb{Q})$ is discrete in $Z(\mathbb{A}_f)$), or is the adjoint Shimura datum to some $(\widetilde{G}, \widetilde{\Omega})$ of Hodge type and satisfies (SV5).

**Notation** For any vector space $M$, let $M^\oplus$ denote the direct sum of all the vector spaces that can be obtained from $M$ by taking duals, tensor products, symmetric powers and exterior powers. We use the same notation also for any object $M$ for which $M^\oplus$ makes an obvious sense.

### 3.1 Moduli interpretation

In this subsection we recall and mildly extend Milne’s construction of the canonical model as moduli spaces from [44, §3]. We note that Milne put a simplifying assumption that there exists a morphism $t : G \to \mathbb{G}_m$ such that $t \circ w_{\Omega} = 2$. If $(G, \Omega)$ is of Hodge type, this is automatically satisfied, but may not or may not be satisfied by the adjoint Shimura datum of $(G, \Omega)$.

Let $\xi : G \to \text{GL}(V)$ be a faithful representation which realizes $G$ as the stabilizer of a collection of tensors $(s_\alpha)_{\alpha \in t} \subseteq V^\otimes$.

**Proposition 3.1** ([44, Prop. 3.10]). If $K \subseteq G(\mathbb{A}_f)$ is neat, then $\text{Sh}_K(G, \Omega)_C$ is a solution to the following moduli problem: For every smooth connected $C$-variety $T$, the set $\text{Sh}_K(G, \Omega)_C(T)$ is in canonical bijection with isomorphism classes of triples $(\mathbb{W}, (t_\alpha), [\eta])$ where

- $\mathbb{W}$ is a polarizable $\mathbb{Q}$-VHS on $T$,
- $(t_\alpha)$ is a family of global Hodge tensors in $\mathbb{W}^\oplus$ and
- $[\eta]$ is a level-$K$-structure on $\mathbb{W}(\mathbb{A}_f)$, i.e. for some base point $t \in T(\mathbb{C})$ a $\pi_1(T, t)$-stable $K$-orbit of isomorphisms $\eta : V_{\mathbb{A}_f} \sim \mathbb{W} \otimes \mathbb{A}_f$ mapping $(s_\alpha \otimes 1)$ to $(t_\alpha)$

such that for every $t \in T(\mathbb{C})$ there exists an $h \in \Omega$ isomorphism $(\mathbb{W}_t, (t_\alpha)) \cong ((V, h), (s_\alpha))$.

**Proof.** The proof of [44, Prop. 3.10] still works as it is based on [44, Thm. 2.12] which does not require the simplifying assumption that $t \circ w_{\Omega} = 2$ for some $t : G \to \mathbb{G}_m$. Note that (SV5) implies that the above objects have no non-trivial automorphisms ensuring that $\text{Sh}_K(G, \Omega)_C$ is a fine moduli space. \qed

**Remark 3.2.** Prop. 3.1 remains valid if one in addition requires $\mathbb{W}$ to be an object of $\text{Hdg}_{\mathbb{Q}}^{\text{Ab}}(T)$. If $(G, \Omega)$ is of Hodge type, then this is a special case of [44, Thm. 3.13]. Recall that the natural functor $\omega_{\text{Hdg}}^\eta : \text{Mot}_{\text{Ab}}^\eta(T) \to \text{Hdg}_{\mathbb{Q}}^{\text{Ab}}(T)$ is an equivalence of categories. The adjoint-to-Hodge type case follows from the Hodge type case by Lem. 2.33 of loc. cit.

**Construction 3.3.** For any field $k$ containing $E$ and $\tau \in \chi_{\infty}$, let $\mathcal{M}_{\tau, K}(k)$ denote the groupoid of tuples $(M, (t_\alpha), [\eta])$ where $M \in \text{Mot}_{\text{Ab}}^\eta(k)$, $(t_\alpha)$ is a collection of tensors on $M$, and $[\eta]$ is a level-$K$-structure on $\omega_{\mathbb{A}_f}(M)$ which satisfy the following conditions:
(i) For every $E$-embedding $i: k \hookrightarrow \mathbb{C}$, there is an isomorphism $\beta : \omega_B(M \otimes_\mathbb{C} \mathbb{C}) \sim V$ sending each $t_\alpha$ to $s_\alpha$ and $\omega_{\text{Hdg}}(M \otimes_\mathbb{C})$ is realized by a point on $\Omega$ via $\beta$.

(ii) Some (and hence every) representative of $[\eta]$ sends $s_\alpha \otimes 1$ to $t_\alpha \otimes 1$.

For a smooth connected $E$-variety $T$, let $\mathcal{M}_{\tau,K}(T)$ be the groupoid of isomorphism classes of triples $(M \in \text{Mot}_\eta^\text{Ab}(S), (t_\alpha), [\eta])$ such that for each $t \in T(C)$, the fiber $(M, (t_\alpha), [\eta])_t$ is an object of $\mathcal{M}_{\tau,K}(C)$. When $\tau = \tau_0$, we simply write $\mathcal{M}_{\tau,K}(-)$ for $\mathcal{M}_{\tau,K}(-)$.

**Remark 3.4.** The above definition admits the following useful variant: Let $\mathcal{M}'_{K}(k)$ be the groupoid defined in the same way as $\mathcal{M}_{K}(k)$ except that we only require that $M$ is an object of $\text{Mot}_\eta^\text{Ab}(k)$, i.e., $M$ is isomorphic to an object in $\text{Mot}_\eta^\text{Ab}(k)$ only in the category $\text{Mot}_\eta^\text{Ab}(k)$, and define $\mathcal{M}'_{K}(S)$ accordingly. Then the natural inclusion $\mathcal{M}_{K}(S) \rightarrow \mathcal{M}'_{K}(S)$ is an equivalence of categories. In short, the above moduli problem remains unchanged if we replace “$\text{Mot}_\eta^\text{Ab}_1$” by “$\text{Mot}_\eta^\text{Ab}$” in the definition.

**Proposition 3.5.** If $K \subseteq G(\mathbb{A}_f)$ is neat, then the canonical model $\text{Sh}_K(G, \Omega)$ of $\text{Sh}_K(G, \Omega)$ over $E$ is a solution to the fine moduli problem $S \rightarrow \mathcal{M}_{K}(S)$.

**Proof.** When $(G, \Omega)$ is of Hodge type, this is a special case of [44, Thm 3.31].

Now assume that $(G, \Omega)$ is the adjoint Shimura datum to some $(\tilde{G}, \tilde{\Omega})$ of Hodge type which satisfies (SV5). Let $K \subseteq \tilde{G}(\mathbb{A}_f)$ be a neat subgroup which lies in the preimage of $K$ under the adjoint morphism $\tilde{G} \rightarrow G$. Then we have a morphism $\varpi : \text{Sh}_K(\tilde{G}, \tilde{\Omega})_C \rightarrow \text{Sh}_K(G, \Omega)_C$ which surjects to some connected components of the target. Let $\sigma \in \text{Aut}(\mathbb{C}/E)$ be any element and let $f_\sigma : \text{Sh}_K(\tilde{G}, \tilde{\Omega})^\sigma \sim \text{Sh}_K(G, \Omega)_C$ be the isomorphism defined by the canonical model.

Let us denote by $\mathcal{M}^\sigma_{K}(\mathbb{C})$ the set of tuples defined just as $\mathcal{M}_{K}(\mathbb{C})$ but with the $\sigma$ in (i) fixed to be the identity. Then Prop. 3.4 says that the $\mathbb{C}$-points of $\text{Sh}_K(G, \Omega)_C$ are in canonical bijection with $\mathcal{M}^\sigma_{K}(\mathbb{C})$. Let $x \in \mathcal{M}^\sigma_{K}(\mathbb{C})$ be a point given by tuple $(M, (t_\alpha), [\eta])$.

Since the canonical model of $\text{Sh}_K(G, \Omega)$ is uniquely characterized by the property that $\varpi$ is defined over $E$, it would be enough to show the following:

(a) $(M^\sigma, (t^\sigma_\alpha), [\eta^\sigma])$ defines a new point $\sigma(x) \in \mathcal{M}^\sigma_{K}(\mathbb{C})$.

(b) For any $\tilde{x} \in \text{Sh}_K(\tilde{G}, \tilde{\Omega})_C$ lifting $x$, $\varpi(f_\sigma(\tilde{x})) = \sigma(x)$.

Note that up to changing the level structure on $x$, we may always make sure that $x$ lies in the image of $\text{Sh}_K(\tilde{G}, \tilde{\Omega}) \rightarrow \text{Sh}_K(G, \Omega)$.

Choose a faithful representation $\tilde{\xi} : \tilde{G} \rightarrow \text{GL}(\tilde{V})$ which realizes $\tilde{G}$ as the stabilizer of $\tilde{s}_\beta$’s. View $V$ as a $\tilde{G}$-representation via $\tilde{G} \rightarrow G$. Since the category of $\tilde{G}$-representations is semisimple and $V$ is faithful, for some finite direct sum $H$ of modules of the form $\tilde{V}^m \otimes \tilde{V}^n$, $(m, n) \in \mathbb{Z}_{\geq 0}^2$, there exists some embedding $i : V \hookrightarrow H$ of $\tilde{G}$-representations together with a projection $\pi : H \rightarrow V$ such that $\pi \circ i = \text{id}$. One may extend the pair $(i, \pi)$ to $(i_* : V^\otimes \hookrightarrow H^\otimes, \pi_* : H^\otimes \rightarrow V^\otimes)$ such that $\pi_* \circ i_* = \text{id}$, and view $i_* \circ \pi_*$ as an element of $V^\otimes$.

Now we view $\tilde{G}$ as the stabilizer of the collection of tensors $\{\tilde{s}_\beta, \{i \circ \pi, i_*(s_\alpha), i_* \circ \pi_*\}\}$ in $\text{GL}(\tilde{V})$ and endow $\text{Sh}_K(\tilde{G}, \tilde{\Omega})$ with a moduli interpretation using $\tilde{\xi}$ and this group of tensors. Let $\tilde{x}$ be a lifting of $x$. Then $\tilde{x}$ corresponds to a tuple of the form $(\tilde{M}, \{\{i_\beta, \{i \circ \pi, i_*(t_\alpha), i_* \circ \pi_*\}\}, [\tilde{\eta}])$. Let $H(\tilde{M})$ be the submodule of $\tilde{M}^\otimes$ formed out of $\tilde{M}$ in the same way as $H$ out of $\tilde{V}$. Then there is an isomorphism

$$(\omega_{\text{Hdg}}(M), t_\alpha, [\eta]) \sim (\text{im}(i \circ \pi) \subseteq \omega_{\text{Hdg}}(H(\tilde{M})), i_*(t_\alpha), \pi(H([\tilde{\eta}]))).$$
where $H(\tilde{\eta})$ is defined in the obvious way. The RHS is part of the data of the tuple $(\tilde{M}, \{(\tilde{t}_\alpha), \{(i \circ \pi, i_\pi(t_\alpha), i_\pi \circ \pi_s)\}, [\tilde{\eta}])$ and the map $\varpi : \tilde{x} \mapsto x$ is simply given by forgetting extra data. Since the pair $(i, \pi)$ is absolute Hodge, we obtain an isomorphism

$$(\omega_{\text{Hdg}}(M^\sigma), t^\sigma_\alpha, [\eta^\sigma]) \overset{\sim}{\rightarrow} \text{im}((i \circ \pi)^\sigma) \subseteq \omega_{\text{Hdg}}(H(\tilde{M}^\sigma)), i_\pi(t^\sigma_\alpha, \pi(H([\eta^\sigma]))).$$

The RHS is a part of the data of the tuple $(\tilde{M}^\sigma, \{(\tilde{t}^\sigma_\alpha), \{(i \circ \pi)^\sigma, (i_\pi \circ \pi_s)^\sigma\}, [\eta^\sigma])$, which defines $f_\sigma(\tilde{x})$ as the proposition is known in the Hodge type case. This allows one to easily check (a) and (b).

**Proposition 3.6.** If $(G, \Omega)$ is of Hodge type and $K$ is neat, the object $M \in \text{Mot}_{\eta}^\text{Ab}(\text{Sh}_K(G, \Omega))$ in the universal object given by $\mathcal{M}_K(\text{Sh}_K(G, \Omega))$ lies in the essential image of $\text{Mot}_{\eta}^\text{Ab}(\text{Sh}_K(G, \Omega))$.

**Proof.** Choose an embedding $(G, \Omega) \hookrightarrow (\text{GSp}(H, \psi), \mathcal{H}^\pm)$ where $(H, \psi)$ is a symplectic space over $\mathbb{Q}$ and $\mathcal{H}^\pm$ is the associated Siegel double space. Up to replacing $K$ by a smaller compact open subgroup, we may assume that for some neat $K \subseteq \text{GSp}(\mathfrak{A}_f)$, the embedding of Shimura data induces an embedding $\text{Sh}_K(G, \Omega) \to \text{Sh}_K(\text{GSp}, \mathcal{H}^\pm)_E$ of Shimura varieties.

Let $\mathfrak{A}$ be the pullback of the universal abelian scheme on $\text{Sh}_K(\text{GSp}, \mathcal{H}^\pm)_E$ to $\text{Sh}_K(G, \Omega)$. Since the symplectic representation $H$ is faithful, $V$ is isomorphic to a subrepresentation in some finite direct sum of modules of the form $H^m \otimes (H^\vee)^n$, $(m, n) \in \mathbb{Z}_+^2$. Then one easily adapts the argument for the preceding proposition and that $M$ is isomorphic to a submotive of $h^1(\mathfrak{A}/\text{Sh}_K(G, \Omega))$. Finally, we conclude using that $h^1(\mathfrak{A}/\text{Sh}_K(G, \Omega))$ is an object of $\text{Mot}_{\eta}^\text{Ab}$, not just $\text{Mot}_{\eta}^\text{Ab}$.

3.2 Union of conjugate Shimura varieties

In this paper we will need a slightly more general terminology. The reason is that we want to construct a period morphism over $\mathbb{Q}$, but $\text{Sh}_K(G, \Omega)$ is only defined over $E$. Therefore, we construct a $\mathbb{Q}$-model for the union of Shimura varieties conjugate to $\text{Sh}_K(G, \Omega)$.

First, we briefly recall how to define conjugate Shimura data in our case. Let $K \subseteq G(\mathfrak{A}_f)$ be a neat compact open subgroup. Choose a faithful representation $\xi : G \to \text{GL}(V)$ and let $(M, (t_\alpha), [\eta])$ be the universal object over $\text{Sh}_K(G, \Omega)$ given by the moduli problem $\mathcal{M}_K$. Let $\sigma \in \text{Aut}(\mathbb{C})$ be any automorphism. Take a point $s \in \text{Sh}_K(G, \Omega)(\mathbb{C})$. We define $G^{\sigma,s}$ as the stabilizer of $(t^\sigma_\alpha, s)$ in $\omega_B(M^\sigma_s)$. The canonical isomorphism of étale cohomology $\omega_{\text{Hdg}}(M^\sigma_s) \cong \omega_{\text{Hdg}}(M^\sigma_s)$ gives us an isomorphism $G(\mathfrak{A}_f) \cong G^{\sigma,s}(\mathfrak{A}_f)$ through which we may transport $K$ to $G^{\sigma,s}(\mathfrak{A}_f)$. Let $h^s : S \to \text{Aut}(\omega_B(M^\sigma_s) \otimes \mathbb{R}, (t^\sigma_\alpha, s))$ be the morphism defined by the Hodge structure $\omega_{\text{Hdg}}(M^\sigma_s, (t^\sigma_\alpha, s))$ and $h^s_{\sigma}$ be the one defined by $(\omega_{\text{Hdg}}(M^\sigma_s, (t^\sigma_\alpha, s)))$. Let $\Omega^{\sigma,s}$ be the $G^{\sigma,s}(\mathbb{R})$-conjugacy class of $h^s_{\sigma}$. Then we have

**Proposition 3.7.** The isomorphism class of the pair $(G^{\sigma,s}, \Omega^{\sigma,s})$ does not depend on $s$ and $\xi$, and depends on $\sigma$ only up to $\sigma|_E$. Moreover, $(G^{\sigma,s}, \Omega^{\sigma,s})$ defines a Shimura datum such that $\text{Sh}_K(G^{\sigma,s}, \Omega^{\sigma,s})$ represents the functor $\mathcal{M}_{\tau,K}$ for $\tau = \sigma^{-1}(\tau_0)$.

**Proof.** We first explain that the isomorphism class of the pair $(G^{\sigma,s}, \Omega^{\sigma,s})$ does not depend on the choice of $s$ and depends only on $\sigma$ up to the action of $\text{Aut}(\mathbb{C})/E$. Let $s'$ be another point on $\text{Sh}_K(G, \Omega)(\mathbb{C})$. Up to changing the level structure of the defining tuple of $s'$, we may assume that $s$ and $s'$ lie on the same connected component. By choosing a topological path from $\sigma(s)$ to $\sigma(s')$ on $\text{Sh}_K(G, \Omega)(\mathbb{C})$, we obtain by parallel transport an isomorphism $\omega_B(M^\sigma_{s'}, (t^\sigma_{\alpha, s'})) \cong \omega_B(M^\sigma_s, (t^\sigma_{\alpha, s})))$, through which $h^\sigma_{s'}$ defines the same $G^{\sigma,s}(\mathbb{R})$-conjugacy class as $h^\sigma_s$. This shows the independence of $s$. Next, if $\sigma' = \sigma \circ \xi$
for $\zeta \in \text{Aut}(\mathbb{C}/E)$, we have by Prop. 3.3

$$\left( M^\text{univ}_s, (t_{\alpha,s}) \right)^{\sigma'} = \left( (M_s, (t_{\alpha,s}))^{\sigma} \right) \cong \left( (M_{\zeta,s}, (t_{\alpha,s}))^{\sigma} \right);$$

thus the isomorphism class of $(G^{\sigma,s}, \Omega^{\sigma,s})$ depends only on the restriction $\sigma|_E$ rather than $\sigma$.

When $s$ is a special point, Langlands defined a Shimura datum in [35, §6], which he denoted by $(G^{\sigma,\mu}, \Omega^{\sigma,\mu})$, where $\mu$ is the cocharacter defined by $h$. The pair $(G^{\sigma,\mu}, \Omega^{\sigma,\mu})$ is isomorphic to our $(G^{\sigma,s}, \Omega^{\sigma,s})$ $(43, \text{Rmk } 4.1(a))$. This implies that $(G^{\sigma,s}, \Omega^{\sigma,s})$ is a Shimura datum and is moreover independent of the choice of the representation $\xi$ as Langlands’ construction does not involve choosing $\xi$. If $(G, \Omega)$ is of Hodge type, then clearly so is $(G^{\sigma,s}, \Omega^{\sigma,s})$; moreover, as $G^{\sigma,s}$ is an inner form of $G$, they have the same center. Hence our standing assumptions on $(G, \Omega)$ also apply to $(G^{\sigma,s}, \Omega^{\sigma,s})$. Finally, let $\xi^{\sigma,s} : G^{\sigma,s} \to \text{GL}(\omega_B(M^\sigma))$ be the tautological representation through which we view $G^{\sigma,s}$ as the stabilizer of $(t^{\sigma,s}_a)$’s. Let $\mathcal{M}^{\sigma,s}_K$ be the moduli problem which is represented by $\text{Sh}_K(G^{\sigma,s}, \Omega^{\sigma,s})$, as given by Prop. 3.5. One easily checks using the first paragraph that the moduli problem $\mathcal{M}^{\sigma,s}_K$ is isomorphic to $\mathcal{M}_{\tau,K}$ for $\tau := \sigma^{-1}(\tau_0)$.

The above proposition allows us to write $\text{Sh}_K(G^{\sigma,s}, \Omega^{\sigma,s})$ as $\text{Sh}_K(G^\tau, \Omega^\tau)$. As a solution to a fine moduli problem, $\text{Sh}_K(G^\tau, \Omega^\tau)$ is well defined up to unique isomorphism. Note also that for any $\sigma$ sending $\tau$ to $\tau_0$, $\mathcal{M}_{\tau,K}$ is also represented by $\text{Sh}_K(G, \Omega)^\sigma$, so we obtain a canonical isomorphism

$$\varphi_{\sigma,K} : \text{Sh}_K(G, \Omega)^\sigma \sim \text{Sh}_K(G^\tau, \Omega^\tau).$$

This construction is compatible with varying $K$ such that it defines an isomorphism

$$\varphi_\sigma : \text{Sh}(G, \Omega)^\sigma \sim \text{Sh}(G^\tau, \Omega^\tau).$$

In [35, §6], Langlands also constructed such an isomorphism in more generality (e.g., without assuming (SV5)). Our construction construction agrees with Langlands’ (see e.g., [43, Thm 4.2]).

**Lemma 3.8.** If $\tau \neq \tau'$, then an object of $\mathcal{M}_{\tau,K}(\mathbb{C})$ cannot be isomorphic to an object in $\mathcal{M}_{\tau',K}(\mathbb{C})$.

**Proof.** Let $(M, (t_a), [\eta])$ be an object of $\mathcal{M}_{\tau,K}(\mathbb{C})$. Without loss of generality, we may assume that $\tau = \tau_0$. Let $\sigma$ be an element of $\text{Aut}(\mathbb{C})$ such that $\sigma(\tau') = \tau$. Let us choose an isomorphism $G \cong \text{Aut}(\omega_B(M), (t_a))$ and denote $\text{Aut}(\omega_B(M^\sigma), (t^\sigma_a))$ by $G'$. Let $\mu$ be the cocharacter of $G$ defined by $\omega_{\text{Hdg}}(M)$, and let $C(\mu)$ be the $G(\mathbb{C})$-conjugacy class of $\mu$. Define $\mu'$ and $C(\mu')$ analogously. It suffices to explain that if there exists an isomorphism

$$\gamma : (\omega_B(M) \otimes \mathbb{C}, (t_a)) \sim (\omega_B(M^\sigma) \otimes \mathbb{C}, (t^\sigma_a))$$

which sends $C(\mu)$ to $C(\mu')$, then $\sigma$ has to fix $E$. Note that such $\gamma$ is unique up to $G(\mathbb{C})$-conjugacy if it exists, so the image of $C(\mu)$ does not depend on $\gamma$.

Note that the canonical $\sigma$-linear isomorphism

$$\sim (\omega_{\text{dR}}(M), (t_a)) \sim (\omega_{\text{dR}}(M^\sigma), (t^\sigma_a))$$

always sends $C(\mu)$ to $C(\mu')$. Therefore, as $\gamma$ also sends $C(\mu)$ to $C(\mu')$, $\sigma$ has to fix $C(\mu)$. As the reflex field $E$ is defined to be the field of definition of $C(\mu)$, the conclusion follows. 

17
Now we define the coproduct
\[
\text{Sh}_K(G^\bullet, \Omega^\bullet)_C := \prod_{\tau \in \chi_{\infty}} \text{Sh}_K(G^\tau, \Omega^\tau)_C
\]
to be the extended Shimura variety defined by \((G, \Omega)\).

**Theorem 3.9.** The collection \(\varphi := \{\prod \varphi_{\tau, K}\}_{\tau \in \text{Aut}(\C)}\) defines an effective descent data which gives a \(\Q\)-model \(\text{Sh}_K(G^\bullet, \Omega^\bullet)\) for \(\text{Sh}_K(G^\bullet, \Omega^\bullet)_C\). Moreover, the restriction of \(\varphi\) to \(\text{Aut}_r(E)(\C)\) and \(\text{Sh}_K(G^\tau, \Omega^\tau)\) is split by the canonical model of \(\text{Sh}_K(G^\tau, \Omega^\tau)\).

**Proof.** The second claim is a direct consequence of Prop. 3.8. In particular, \(\varphi\) defines an effective descent of \(\text{Sh}(G^\bullet, \Omega^\bullet)_C\) to the normal closure \(\bar{E}\) of \(E\). Since \(\bar{E}/\Q\) is finitely generated, it follows that \(\varphi\) must be effective by a theorem of Weil (see e.g. [45, Thm 1.1]).

We call the above \(\Q\)-model \(\text{Sh}_K(G^\bullet, \Omega^\bullet)_C\) the canonical model of \(\text{Sh}_K(G^\bullet, \Omega^\bullet)_C\).

**Proposition 3.10.** The canonical model \(\text{Sh}_K(G, \Omega)\) represents the moduli problem \(\mathcal{M}_K\) over \(\Q\) defined as follows: For every smooth connected \(\Q\)-variety \(S\), \(\mathcal{M}_K(S)\) is the groupoid of tuples \((M, (t_\alpha), [\eta])\) where \(M \in \text{Mot}_{\text{Ab}}(S)\), \((t_\alpha)\) is a collection of tensors on \(M\), and \(\eta\) is a level-\(K\)-structure on \(\omega_{K/}(M)\), such that for every \(s \in S(\C)\), the fiber \((M_s, (t_\alpha)_s, [\eta_s])\) is isomorphic to an object of \(\mathcal{M}_r(K)(\C)\) for some \(r \in \chi_{\infty}\).

**Proof.** By Lem. 3.8, for each \(s \in S(\C), \tau\) as above is unique. Therefore, by applying Prop. 3.1 to each connected component of \(S_C\), we obtain a well defined morphism \(S_C \to \text{Sh}_K(G^\bullet, \Omega^\bullet)_C\). Then one readily checks that the morphism is defined over \(\Q\).

**Remark 3.11.** The formation of \(\text{Sh}(G^\bullet, \Omega^\bullet)\) is somewhat dual to that of \(\text{Res}_{E/\Q}\text{Sh}(G, \Omega)\). Both define natural \(\Q\)-varieties out of \((G, \Omega)\). The descent data for
\[
\text{Res}_{E/\Q}\text{Sh}(G, \Omega)_C = \prod \text{Sh}(G^\tau, \Omega^\tau)_C
\]
is given by \(\{\prod \varphi_{\tau}\}_{\tau \in C}\) (cf. [43, Cor. 5.6]). So \(\text{Sh}(G^\bullet, \Omega^\bullet)\) and \(\text{Res}_{E/\Q}\text{Sh}(G, \Omega)\) are defined by essentially the same descent data. It will be interesting to have a moduli interpretation for \(\text{Res}_{E/\Q}\text{Sh}(G, \Omega)\) as well.

### 3.3 Orthogonal Shimura Varieties over Totally Real Fields

In this section we provide a direct construction of the Shimura data \((H^\tau, \Omega^\tau)\) for the cases in which the theory will be applied. Let \(E\) be a totally real field. Let \(\chi_f\) and \(\chi_{\infty}\) be the set of its finite and real places, respectively. Let \(V_f\) be a quadratic form over \(E \otimes \A_f\) of dimension \(n + 2\) and discriminant \(d \in E_f^{\times}/E_f^{\times, 2}\) such that the Hasse-Witt invariant of \(V_w := V_f \otimes E_w\) equals 1 for all but finitely many places \(w\) of \(E\) and satisfies
\[
\prod_{w \in \chi_f} \epsilon(V_w) = -1.
\]
Furthermore, we fix real quadratic forms \(V_{\infty, 1}, V_{\infty, 2}\) of signature \((0, n + 2)\) and \((2, n)\), respectively.

By the theorem of Hasse-Minkowski, there exists a unique (up to isomorphism) quadratic form \(V^\tau\) over
Applying the theorem of Hasse-Minkowski to the underlying $\mathbb{Q}$-quadratic form $V_\tau$ of $V_\tau$, we see that its isomorphism class is independent of $\tau$. We fix a quadratic form $V_0$ over $\mathbb{Q}$ and isomorphisms $V^\tau(\mathbb{Q}) \cong V_0$ and $V^\tau \otimes \mathbb{Q} \mathbb{A}_f \cong V_f$. We write $H^\tau$ for $\text{SO}_{E/\mathbb{Q}}(V_\tau)$ and $\Omega^\tau$ for the (unique) conjugacy class of morphisms $h: S \to G^\tau_\mathbb{R}$ defining a polarised Hodge structure of $\text{K3}$-type on $V^\tau$. By (2.6), $(H^\tau, \Omega^\tau)$ is the adjoint Shimura datum to $(\overline{\text{CSpin}}(V^\tau), \Omega^\tau)$, which is of Hodge type and satisfies (SV5). We will not explicitly make use of $(\overline{\text{CSpin}}(V^\tau), \Omega^\tau)$, but its existence allows us to apply results in this section to $(H^\tau, \Omega^\tau)$.

4 p-adic Riemann-Hilbert correspondence and relative de Rham cycles

Let $K$ denote a finite extension of $\mathbb{Q}_p$. Let $\hat{S}$ be a smooth rigid analytic variety over $K$. For a suitable object $\ast$, $\text{Loc}_{\mathbb{Q}_p}(\ast)$ (resp. $\text{FIC}(\ast)$) denote the category of (lisse) $p$-adic étale local systems (resp. filtered vector bundle with integrable connection) over $\ast$. In [38], Liu and Zhu defined a tensor functor

$$\mathbb{D}_{\text{dr}}: \text{Loc}_{\mathbb{Q}_p}(\hat{S}) \to \text{FIC}(\hat{S}).$$

Over a point, it recovers the classical Fontaine’s functor denoted by the same symbol. When $\hat{S}$ comes from the $p$-adic analytification of a smooth variety $S$ over $K$, $\mathbb{D}_{\text{dr}}$ can also be algebraized in the sense that there exists a functor $\mathbb{D}_{\text{alg}}^{\text{dr}}: \text{Loc}_{\mathbb{Q}_p}(S) \to \text{FIC}(S)$ such that $\mathbb{D}_{\text{dr}}$ is the composition of $\mathbb{D}_{\text{alg}}^{\text{dr}}$ with the analytification functor ([23, Thm 1.1]). Fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. Next, we prepare to formulate an analogue of Conjecture 1.4 in loc. cit. for cycle classes.

**Definition 4.1.** Let $S$ be a smooth $\mathbb{C}$-variety and $\mathfrak{M} \in \text{Mot}(S)$ be a family of motives. Let $(\mathfrak{z}_B, \mathfrak{z}_{\text{dR}, \mathfrak{c}}) \in \Gamma(\text{Mot}(S))$ be a Hodge cycle. Then $(\mathfrak{z}_B, \mathfrak{z}_{\text{dR}, \mathfrak{c}})$ is said to be de Rham with respect to an isomorphism $\iota: \overline{\mathbb{Q}}_p \sim \mathbb{C}$ if it satisfies the following two properties:

(a) For some field $K \subseteq \overline{\mathbb{Q}}_p$ finite over $\mathbb{Q}_p$ such that $\mathfrak{M}/S$ descends to a family $\mathfrak{M}/\overline{S}$ over $K$, $\mathfrak{z}_B \otimes 1 \in \Gamma(\omega_B(\mathfrak{M}/S) \otimes \overline{\mathbb{Q}}_p)$ descends to an element $\mathfrak{z}_p \in \Gamma(\omega_p(\mathfrak{M}/S))$.

(b) The canonical isomorphism\footnote{This is given by applying $\mathbb{D}_{\text{alg}}^{\text{dr}}$ to the isomorphism provided by [55, Thm 8.8].}

$$c_{\text{dR}}: \mathbb{D}_{\text{alg}}^{\text{dr}}(\omega_p(\mathfrak{M}/S)) \xrightarrow{\sim} \omega_{\text{dR}}(\mathfrak{M}/S)$$

sends $\mathbb{D}_{\text{alg}}^{\text{dr}}(\mathfrak{z}_p)$ to $\mathfrak{z}_{\text{dR}, \mathfrak{c}}$ when composed with

$$\omega_{\text{dR}}(\mathfrak{M}/S) \to \omega_{\text{dR}}(\mathfrak{M}/S) \otimes \mathbb{C} = \omega_{\text{dR}}(\mathfrak{M}/S).$$

We say that $(\mathfrak{z}_B, \mathfrak{z}_{\text{dR}, \mathfrak{c}})$ is absolute de Rham at $p$ if it is de Rham with respect to every $\iota$. 
Lemma 4.5. Let assume that there is no local section of $V_B$ for readers’ convenience. Up to replacing $V_{B,t}$ has big variation at a point $t$. Theorem 4.6. Similarly, one checks that the choice of $\iota$ is only important up to precomposing with an element of $G_{Q_p}$.

By the variational Hodge conjecture, relative Hodge cycles are expected to be algebraic, so we propose:

Conjecture 4.3. Every relative Hodge cycle $(j_B, j_{\text{HR},C})$ as in Def. 4.1 is absolute de Rham at every $p$.

We remark that this conjecture is essentially a compatibility statement about the classical Riemann-Hilbert correspondence (which produces $j_{\text{HR},C}$ out of $j_B$) and the $p$-adic version. Therefore, it is an analogue of [23, Conj. 1.4], except that the question is not about local systems themselves but about global sections.

We now prepare to prove Thm 4.6, which is a special case of Conj. 4.3. Let $B$ be a smooth connected $\mathbb{C}$-variety. We say that a VHS $V$ over $B$ of pure weight 2 has big variation at a point $t \in T$ if for $\lambda \in \bar{V}^{(1,1)}$ the Kodaira-Spencer map $\nabla_{t,\lambda}: T_{B,t} \rightarrow \bar{V}^{0,2}$ is surjective.

Lemma 4.5. Let $B$ be a smooth connected $\mathbb{C}$-variety and $V$ be a $\mathbb{Q}$-VHS of pure weight 2 with big variation at a point $t_0 \in B$. Let the underlying $\mathbb{Q}$-local system of $V$ be denoted by $V_B$. Let $U \subseteq B$ be any complex analytic disk containing $t_0$. Then some non-empty open ball $\mathcal{B}$ of $V_{B,t_0}$ has the following property: For every $\beta \in \mathcal{B}$, there exists a point $t \in U$ such that $\beta_t \in V_{B,t}$ obtained by $\beta$ via parallel transport within $U$ is of Hodge type $(1, 1)$.

Proof. This is implied by the proof of the density of Hodge loci [64, Thm 3.5]. We sketch the argument for readers’ convenience. Up to replacing $B$ by an étale cover and splitting off a direct summand, we may assume that there is no local section of $V$ which is constant of type $(1,1)$. Denote by $V_R$ the real bundle $V_{B} \otimes \mathbb{R}$ over $S$. Let $F^1 V$ (resp. $V$, $V_R$) denote the total space of the vector bundle $\text{Fil}^1 V_{\text{dr}}$ (resp. $V_{\text{dr}}$, $V_R$) restricted to $U$. The connection on $V$ defines a canonical trivialization of holomorphic vector bundle $V|_U = V_{t_0} \times U$, so there is a natural projection $V \rightarrow V_{t_0}$. Let $\Phi$ denote its restriction to $F^1 V$.

It follows from the assumption that for some $\tilde{\lambda} \in (V_R \cap F^1 V)_{t_0}$, $\nabla_{t_0} \tilde{\lambda}: T_{B,t_0} \rightarrow \bar{V}^{0,2}_{t_0}$ is surjective. By Lem. 3.6 in loc. cit., $\Phi_R: F^1 V \cap V_R \rightarrow V_{R,t_0}$ is a submersion, in particular an open map, near $\tilde{\lambda}$. Take $\mathcal{B}$ to be the intersection of $V_{B,t_0}$ with a small open neighborhood of $\tilde{\lambda}$. Then the projection of $\phi_{\mathcal{B}}^{-1}(\mathcal{B})$ to $U$ is the Hodge locus.

Theorem 4.6. Let $K \subseteq \bar{Q}_p$ be a finite extension of $\mathbb{Q}_p$ and choose an isomorphism $\iota: \bar{Q}_p \cong \mathbb{C}$. Let $S$ be a smooth $K$-variety. Let $\mathfrak{h} \in \text{Mot}(S)$ be a family of motives over $S$ such that:

- The $\mathbb{Q}$-VHS $V(\mathfrak{h}_C/S_C)$ is pure of weight 2 and has big variation somewhere on every connected component of $S_C$.

- For every $s \in S_C$, every $(1,1)$-class in $\omega_B(\mathfrak{h}_s)$ is algebraic, i.e., comes from a submotive isomorphic to the Lefschetz motive $l$. 

20
Let \( h' \) be another family with the same properties and let \((f_B, f_{\text{dR}})\): \( V(h_C/S_C) \rightarrow V(h'_C/S_C) \) be an isomorphism such that \( f_B \otimes_{\mathbb{Q}_p} \) descends to an isomorphism \( f_p: \omega_p(h/S) \rightarrow \omega_p(h'/S) \). Then \((f_B, f_{\text{dR}})\) satisfies Conj. 4.3 as a relative Hodge cycle on \( \text{Hom}(h_C, h'_C) \) over \( S_C \).

Proof. Let \( \alpha := f_{\text{dR}}^{-1} \circ \mathbb{D}^{\text{alg}}(f_p) \otimes \mathbb{C} \). Then \( \alpha \) is an automorphism of \( \omega_{\text{dR}}(h_C/S_C) \) as an object of \( \text{FIC}(S_C) \). Since the choice of \( \iota \) is arbitrary, it suffices to show that \( \alpha = \text{id} \). Let \( S_C^c \) be any connected component of \( S_C \) and choose a base point \( s \in S_C^c \). To show that \( \alpha|_{S_C^c} = \text{id} \) it suffices to show that \( \alpha_s = \text{id} \). Let \( s \) be another point \( s' \) of \( S_C^c \) and \( \gamma \) be a topological path from \( s \) to \( s' \). Denote the parallel transport operators induced by \( \gamma \) by the same letter. Then \( \alpha_s \) can be viewed as \( \gamma^{-1}\alpha_{s'}\gamma \), which is independent of the choice of \( \gamma \). This implies that to check \( z \in \omega_{\text{dR}}(h_s) \) is invariant under \( \alpha_s \), it suffices to check that \( \gamma(z) \in \omega_{\text{dR}}(h_{s'}) \) is invariant under \( \alpha_{s'} \).

Claim: If \( z \) lies in \( \omega_B(h_s)^{(1,1)} \), then \( z \) is \( \alpha_s \)-invariant. By assumption, \( z = \text{cl}(\xi) \) for some submotive \( \xi \cong 1 \) of \( h \). By Rmk 4.2 we may replace \( K \) by a finite extension and assume that \( (s, \xi) \) are defined over \( K \) and denote the descent by \( (s, \xi) \). The we have a commutative diagram:

\[
\begin{align*}
\omega_B(h_s) & \rightarrow \omega_B(h'_s) \\
\omega_{\text{dR}}(h_s) & \rightarrow \omega_{\text{dR}}(h'_s)
\end{align*}
\]

In the diagram above, every solid arrow becomes an isomorphism when the source is tensored with the natural coefficient of the target (i.e., \( \mathbb{Q}_p, K \) or \( \mathbb{C} \)), and all the squares with only solid arrows commute except possibly the bottom one, whose commutativity is precisely what we want to show. The main obstacle is that the dashed arrows do not literally exist. Instead, they are only drawn to suggest the de Rham comparison isomorphisms, which only exist when the modules are tensored with \( B_{\text{dR}} \).

This problem goes away, however, when we restrict to considering the Galois invariant part of \( \omega_p(-) \): For any trivial \( G_K \)-representation \( M \), the isomorphism \( M \otimes_{\mathbb{Q}_p} B_{\text{dR}} \sim \mathbb{D}_{\text{dR}}(M) \otimes_K B_{\text{dR}} \) comes from the natural \( \mathbb{Q}_p \)-linear map \( M = M \otimes_{\mathbb{Q}_p} K = \mathbb{D}_{\text{dR}}(M) \). Now we set \( \omega_p^G(-) = \omega_B(h_s(1))^{(0,0)} \cap \omega_p(-1)^{G_K} \). Then we obtain a diagram:

\[
\begin{align*}
\omega^G_B(h_s) & \rightarrow \omega^G_B(h'_s) \\
\omega_{\text{dR}}(h_s) & \rightarrow \omega_{\text{dR}}(h'_s)
\end{align*}
\]

Again, every square commutes, except possibly the bottom one. In particular, the squares on the top, left,
right and front commute because the classes in $\omega_B^s(\mathfrak{h}_s)$ are precisely given by submotives of $\mathfrak{h}_s$ which are defined over $K$. This implies that, if we denote the image of $\omega_B^s(\mathfrak{h}_s)$ in $\omega_{\text{dR}}(\mathfrak{h}_s)$ as $\omega_{\text{dR}}(\mathfrak{h}_s)^\dagger$, then restriction of the bottom diagram

$$\begin{array}{c}
\omega_{\text{dR}}(\mathfrak{h}_s)^\dagger \\
\downarrow \\
\omega_{\text{dR}}(\mathfrak{h}_s)
\end{array} \quad \begin{array}{c}
\omega_{\text{dR}}(\mathfrak{h}_s) \\
\downarrow \\
\omega_{\text{dR}}(\mathfrak{h}_s')
\end{array}$$

commutes. In particular, $\alpha_s(z) = z$ as desired.

Now we return to the proof of the theorem. By the preceding lemma, there exists a basis $\{z_i\}_{i \in I}$ of $\omega_B(\mathfrak{h}_s)$ such that for each $z_i$ has the following property: There exists another point $s'$ and a topological path $\gamma$ from $s$ to $s'$ such that $\gamma(s')$ is of type $(1, 1)$, and hence invariant under $\alpha_{s'}$ by the claim. This implies, by the first paragraph, that $z_i$ is invariant under $\alpha_s$. Since $\{z_i\}_{i \in I}$ is also a $C$-basis of $\omega_{\text{dR}}(\mathfrak{h}_s)$, $\alpha_s$ has to be the identity. \hfill $\Box$

**Remark 4.7.** The above proof is inspired by that of [47, Prop. 6.6], which also uses non-emptyness properties of Hodge loci. However, there are some key differences. Unlike the proof in loc. cit., we do not prove that the Hodge cycle in question $(f_B, f_{\text{dR},C})$ has the desired properties by finding a point where it is known to be algebraic (i.e., motivated), although we do leverage the algebraicity of certain classes in the domain of $(f_B, f_{\text{dR},C})$ when it viewed as a morphism. As a result, we are not able to use the above argument to show that $(f_B, f_{\text{dR},C})$ remains rational under $\text{Aut}(C)$. It seems to us that for a Hodge cycle, being absolute Hodge is a condition which is more transcendental in nature than being absolute de Rham, due to this requirement of rationality.

## 5 Period Morphisms

Let $\mathcal{X}/S$ be a polarized family of $h^{2,0} = 1$ varieties over a smooth connected $\mathbb{Q}$-variety $S$ which satisfies condition (\textcircled{\textit{C}}). Let $\eta$ be the generic point of $S$ and choose an embedding $k(\bar{\eta}) \hookrightarrow \mathbb{C}$. We denote by $b \in S(\mathbb{C})$ the resulting point and denote by $S_b^0$ the connected component of $S_b^0$ containing $b$. We call it the distinguished component. Let $T(\mathcal{X}_b, \mathbb{Q})$ be the transcendental part of $H^2(\mathcal{X}_b, \mathbb{Q}(1))$ and define $T(\mathcal{X}_b, \mathbb{Q}_\ell) := T(\mathcal{X}_b, \mathbb{Q}) \otimes \mathbb{Q}_\ell$ for every prime $\ell$. Up to replacing $S$ by a connected étale cover, we assume that $\text{NS}(\mathcal{X}_\eta)_{\mathbb{Q}} = \text{NS}(\mathcal{X}_\eta)_{\mathbb{Q}} = \text{NS}(\mathcal{X}_b)_{\mathbb{Q}}$ and the monodromy group $\text{Mon}(H^2_{\text{dR}}(\mathcal{X}/S, \mathbb{Q}_2), b)$ is a connected subgroup of $O(T(\mathcal{X}_b, \mathbb{Q}_2))$. Note that $O(T(\mathcal{X}_b, \mathbb{Q}_2))$ can be naturally viewed as the subgroup of $O(H^2_{\text{dR}}(\mathcal{X}_b, \mathbb{Q}_2))$ which stabilizes $\text{NS}(\mathcal{X}_b)$. By [36, Prop. 6.14], $\text{Mon}(H^2_{\text{dR}}(\mathcal{X}/S, \mathbb{Q}_2), b)$ is connected for every other $\ell$. Recall that by Lem. 2.10 representatives of elements of $\text{NS}(\mathcal{X}_\eta)_{\mathbb{Q}}$ extend uniquely to relative line bundles of $\mathcal{X}/S$.

Let $E$ be the endomorphism algebra $\text{End}_{\text{Hdg}}(T(\mathcal{X}_b, \mathbb{Q}))$. Recall that $E$ come from motivated cycles (Thm 2.11). These cycles are defined over $k(\bar{\eta})$ ([18, Scolie 2.5]); moreover, as their $\ell$-adic realizations are $\pi_{\text{et}}^* (S,b)$-invariant, they are defined over $S$. We hence obtain an $E$-action on the entire family $p^2(\mathcal{X}/S)$. Note that for every prime $\ell$, the connectedness of $\text{Mon}(H^2_{\text{dR}}(\mathcal{X}/S, \mathbb{Q}_2), b)$ implies that it is a subgroup of $\text{SO}_{E/\mathbb{Q}}(T(\mathcal{X}_b, \mathbb{Q}))_{\mathbb{Q}_\ell}$, which is the identity component of $C_E(\text{O}(T(\mathcal{X}_b, \mathbb{Q})))_{\mathbb{Q}_\ell}$.

Let $V$ be a quadratic form over $\mathbb{Q}$ which is isomorphic to $\text{PH}^2(\mathcal{X}_b, \mathbb{Q})$ and choose an isomorphism $\psi: V \rightarrow \text{PH}^2(\mathcal{X}_b, \mathbb{Q})$. Set $n := \text{dim} V$. Let $G := \text{SO}(V)$ and consider the Shimura variety $\text{Sh}(G)$ defined by $V$. All the Shimura varieties used here are the orthogonal ones recalled in [33, 34] so we omit the
Hermitian symmetric domains from the notation. Recall that the reflex field of $\text{Sh}(G)$ is $\mathbb{Q}$. The standard representation of $G$ on $V$ endows $\text{Sh}(G)$ with a family of abelian motives $\mathcal{M} \in \text{Mot}_{\eta, \text{AH}}^{\text{Ab}}(\text{Sh}(G))$. Set $V_B := \omega_B(\mathcal{M}|_{\text{Sh}(G)_c})$, $V_\ell := \omega_\ell(\mathcal{M})$ and $V_{\text{dR}} := \omega_{\text{dR}}(\mathcal{M})$. Note that $V_{\text{dR}}$ is the automorphic vector bundle defined by the standard representation, so the descent data of $\omega_{\text{dR}}(\mathcal{M})$ is effective (Rmk 2.3). Similarly, let us simply write $P_B, P_\ell, P_{\text{dR}}$ for $\text{PH}^2(\mathcal{X}/_{\mathbb{S},C}, \mathbb{Q}(1)), \text{PH}^2_{\text{dR}}(\mathcal{X}/_{S, \mathbb{Q}_\ell}(1))$ and $\text{PH}^2_{\text{dR}}(\mathcal{X}/_S)(1)$ respectively. The key fact we want to prove is the following.

**Theorem 5.1.** For any neat compact open subgroup $K \subseteq G(\mathbb{A}_f)$ such that the monodromy representation $\pi^\dagger_1(S, b) \to \text{GL}(P_{\mathbb{A}_f, b})$ factors through $K$ via $\psi$, there exists a $\mathbb{Q}$-morphism $\rho: S \to \text{Sh}_K(G)$ such that there is an isomorphism of $\mathbb{Q}$-VHS

$$(\alpha^\dagger_{\mathcal{M}}(\mathcal{X}), \alpha^\dagger_{\text{dR}, C}(\mathcal{X})) : (\rho|_{\mathbb{S}_C})^*(V_B, V_{\text{dR}, C}) \cong (P_B, P_{\text{dR}, C})|_{\mathbb{S}_C}$$

and an isomorphism of $\mathbb{Q}$-lisse étale sheaves $\alpha_{\ell}: \rho^*V_\ell \cong P_\ell$ for each $\ell$ such that $\alpha^\dagger_{\mathcal{M}}(\mathcal{X}) \otimes \mathbb{Q}_\ell = \alpha_{\ell}|_{\mathbb{S}_C}$ under the Artin comparison isomorphism.

Note that for any $K$, the hypothesis can be satisfied by replacing $S$ by a connected finite étale cover, and by étale descent arguments, we may always assume that $K$ is sufficiently small.

### 5.1 The case when $p^2(\mathcal{X}/S)$ is abelian

We first treat the simple case when the family of motives $p^2(\mathcal{X}/S)$ is an object of $\text{Mot}_{\text{AH}}^{\text{Ab}}(S)$. Note that by Thm 2.11 this is automatically satisfied when $E$ is a CM field. Let $\delta \in \text{det}(V)$ be any nonzero element. Note that the standard representation of $G$ realizes $G$ as the stabilizer of the quadratic form on $V$ as well as $\delta$. As $p^2(\mathcal{X}_b)$ is an object of $\text{Mot}_{\text{AH}}^{\text{Ab}}(\mathbb{C}, \psi_\ell(\delta))$ defines an element of $\text{det}(p^2(\mathcal{X}_b))$. Moreover, as this element is $\pi^\dagger_1(S, b)$-invariant, it globalizes to a relative cycle $\delta$ in $\wedge^n p^2(\mathcal{X}/S)$. Since the monodromy representation of $\pi^\dagger_1(S, b)$ factors through $K$ via $\psi$, we may extend $\varphi$ to a $K$-level structure $\widetilde{\varphi}$ on $P_{\mathbb{A}_f}$. Let $\eta_{\text{std}}$ be the tautological level structure on $V_{\mathbb{A}_f}$.

By Prop. 3.3 there exists a period morphism $\rho: S \to \text{Sh}(G)$ characterized by the property that there is an isometry $\rho^*(\mathcal{M}) \to p^2(\mathcal{X}/S)$ which sends $\delta$ to $\delta$ and $\eta_{\text{std}}$ to $\widetilde{\varphi}$. Then clearly this $\rho$ satisfies the conclusion of Thm 5.1. In fact, the isomorphisms $\alpha^\dagger_{\mathcal{M}}(\mathcal{X})$ and $\alpha^\dagger_{\text{dR}, C}$ are given by restrictions to $S_C$ of isomorphisms $\rho^*(\mathcal{M})(\mathcal{X}) \cong (P_B, P_{\text{dR}, C})$ which are defined over the entire $S_C$.

### 5.2 Totally Real Case

Let $\mathcal{V}(E)$ be a quadratic form over $E$ which is isomorphic to $\text{PH}^2(\mathcal{X}_b, \mathbb{Q})$ equipped with the $E$-action. Fix an $E$-isometry $\psi: \mathcal{V} \cong \text{PH}^2(\mathcal{X}_b, \mathbb{Q})$ such that $V = \mathcal{V}(\mathbb{Q})$. Write $H$ for $\text{SO}_{E/\mathbb{Q}}(\mathcal{V})$. Let $K \subseteq \text{SO}_{E/\mathbb{Q}}(\mathcal{V})(\mathbb{A}_f)$ be a neat compact open subgroup. Assume that $K \subseteq K$ and $\text{Mon}(\mathcal{H}^2_{\text{dR}}(\mathcal{X}/S, \mathbb{Q}_\ell), b) \subseteq \psi^{-1}(K)$, which can always be achieved by replacing $S$ by a further connected étale cover. Then the isomorphism $\psi$ globalizes to a $K$-level structure $\psi$ on $\text{PH}^2_{\text{dR}}(\mathcal{X}/S, \mathbb{A}_f)$.

**The odd case** Assume for now that $\text{dim}_E \mathcal{V}$ is odd. When $|E : \mathbb{Q}| \neq 4$, we define $\mathcal{N} = \mathcal{N}(V)$ to be $\text{Nm}(V) \otimes \text{det}(V)$ and when $|E : \mathbb{Q}| = 4$, we define $\mathcal{N}$ to be $\text{Nm}(V)$. We consider two faithful representations of $H$. One is the standard representation $r_{\text{std}}: H \to \text{O}(V)$ and the other is the composition

$$r_{\text{Nm}}: H \to \text{GL}_{E/\mathbb{Q}}(\mathcal{V}) \to \text{GL}(\mathcal{N})$$

23
Let $\lambda \in V^\otimes$ be a tensor such that $H$ is the stabilizer of the $\mathbb{Q}$-pairing, $E$-action and $\lambda$, and let $\{s_\alpha\} \subseteq \text{Nm}(V)^\otimes$ be tensors such that $\text{SO}_{E/\mathbb{Q}}(V)$ is the stabilizer of $s_\alpha$’s via $\text{Nm}$. Moreover, the automorphic sheaves on $\text{Sh}_K(H)$ are given by $r_{\text{std}}$ and $\eta_{\text{std}}$. The following trick: Let $\mathcal{N}_s := \mathcal{N}(\mathcal{V}_s)$ be the global sections of $\mathcal{V}_s^\otimes$ and $\mathcal{N}(\mathcal{V})^\otimes$ given by $\lambda$ and $s_\alpha$’s respectively.

Define $\lambda_\alpha := \lambda(\alpha)$. As $\text{Mon}^2(\mathcal{X}|_{S_\mathbb{C}}, \mathbb{Q})$ is a subgroup of $\text{SO}_{E/\mathbb{Q}}(PH^2(\mathcal{X}_B, \mathbb{Q}))$, $\lambda_\alpha$ spreads to a global section $\lambda$ of $(P_B|_{S_\mathbb{C}})^\otimes$. The moduli interpretation of $\text{Sh}_K(H)$ given by $r_{\text{std}}$ then gives us a period morphism

$$\rho_{\text{std}}^\circ : S_\mathbb{C}^\circ \to \text{Sh}_K(H)_\mathbb{C},$$

characterized by the property that there is (necessarily unique) isomorphism $\rho^*(V_B, V_{\text{dR}}) \cong (P_B, P_{\text{dR}})|_{S_\mathbb{C}}$ which preserves the $E$-actions, $\mathbb{Q}$-pairing, $\lambda_\alpha$ to $\lambda$, and respects the $K$-level structures $\psi$ and $\eta_{\text{std}}$.

Similarly, consider $s_{\alpha, b} := \psi(s_\alpha)$. Note that as $\mathcal{N}(p^2(\mathcal{X}/S))$ is an object of $\text{Mot}^{\text{Ab}}(S)$ (Thm. 2.11), $s_{\alpha, b}$ is given by a absolute Hodge cycle. Moreover, as its $E$-level structures are $p_1^1(S, b)$-invariant, $s_{\alpha, b}$ extends to a relative cycle $s_\alpha$ on $\mathcal{N}(p^2(\mathcal{X}/S))^\otimes$. The moduli interpretation of $\text{Sh}_K(H)$ given by $r_{\text{Nm}}$ then gives another period morphism

$$\rho_{\text{Nm}} : S \to \text{Sh}_K(H^\bullet)$$

characterized by the property that there is (necessarily unique) isomorphism $\rho^*(\mathcal{V}_B, \mathcal{V}_{\text{dR}}) \cong \mathcal{N}(p^2(\mathcal{X}/S))$ which sends $s_\alpha$ to $s_\alpha$ and $\eta_{\text{Nm}}$ to $\mathcal{N}(\psi)$. It is easy to check that by construction $\rho_{\text{Nm}|_{S_\mathbb{C}}} = \rho_{\text{std}}^\circ$. If we know that $p^2(\mathcal{X}/S)$ is a family of abelian motives, then in fact $\rho_{\text{std}}^\circ$ can also be extended to $\rho_{\text{std}} : S \to \text{Sh}_K(H^\bullet)$ and $\rho_{\text{Nm}} = \rho_{\text{std}}$ on the entire $S$.

**The even case** When $\dim_E V$ is even, one can still construct $\rho_{\text{std}}$ verbatim. However, $r_{\text{Nm}}$ is no longer faithful. We overcome this difficulty by the following trick: Let $d := [E : \mathbb{Q}]$. Define $\mathcal{Y}_0 := \mathcal{X}$ and iteratively define $\mathcal{Y}_{i+1}$ to be the relative Hilbert scheme of $2$ points on $\mathcal{Y}_i$. Then the morphism $\mathcal{Y}_{i+1} \to \mathcal{Y}_i$ is smooth since it satisfies the formal lifting criterion by [34, top of p. 34]; note that we can apply Koller’s construction since any length-$2$ subscheme is lci and can be embedded into an affine open subscheme. Set $\mathcal{Y}$ to be $\mathcal{Y}_d$. Then we have an isomorphism of motives $h^2(\mathcal{X}/S) \oplus \mathbb{Q}(-1)^{d}\mathbb{C}_S \cong h^2(\mathcal{Y}/S)$. We denote by $p^2(\mathcal{Y}/S)$ the image of $p^2(\mathcal{X}/S) \oplus \mathbb{Q}(-1)^{\otimes d}\mathbb{C}_S$. By choosing an isomorphism $\mathbb{Q}^{\otimes d} \cong E$, we equip $\mathbb{Q}(-1)^{\otimes d}$ with an $E$-action and the trace pairing. This way we have equipped $p^2(\mathcal{Y}/S)$ with an $E$-action and an $E$-symmetric pairing

$$p^2(\mathcal{Y}/S)(1) \times p^2(\mathcal{Y}/S)(1) \to \mathbb{Q}_S.$$

Moreover, the $E$-action and the pairing are both given by motivated cycles.

Set $\tilde{V} := V \oplus E$, $\tilde{V} = V \oplus \mathbb{Q}^{\otimes d}$, and $\tilde{H} := \text{SO}_{E/\mathbb{Q}}(\tilde{V})$. Note that $\dim_E \tilde{V}$ now becomes odd. Suppose now we are given a neat compact open subgroup $K \subseteq \tilde{H}(\mathbb{A}_f)$ such that $K \subseteq \tilde{K} \cap H(\mathbb{A}_f)$, and the inclusion $V \hookrightarrow \tilde{V}$ induces an embedding $\iota : \text{Sh}_K(H) \hookrightarrow \text{Sh}_{\tilde{K}}(\tilde{H})$. By applying the constructions in the odd case to the family $p^2(\mathcal{Y}/S)$, we obtain morphisms $\tilde{\rho}_{\text{std}}$ and $\tilde{\rho}_{\text{Nm}}$ which fit into a commutative diagram.
Since $\tilde{\rho}_{Nm}$ is defined over $\mathbb{Q}$ and $\text{Aut}(\mathbb{C})$ acts transitively on the set of connected components of $S_\mathbb{C}$, $\tilde{\rho}_{Nm}$ factors through $\iota^*$, so that we can fill in the dashed arrow and denote it suggestively as $\rho_{Nm}$.

We denote the automorphic sheaves associated to $\tilde{\mathcal{V}}$ by $\tilde{\mathcal{V}}_*$ and the cohomology sheaves associated to $P^2(\mathcal{Y}/S)(1)$ by $\tilde{P}_*$.

**Proposition 5.2.** If $E$ is totally real, then for the morphism $\rho_{Nm}: S \to \text{Sh}_k(H^\bullet)$ constructed above there is an isomorphism of VHS

$$(\alpha_B^0, \alpha_{dR,C}^0): (\rho_{Nm}|_{S^0})^*(\mathcal{V}_B, \mathcal{V}_{dR,C}) \sim (P_B, P_{dR,C})|_{S^0}$$

and an isomorphism of $\mathbb{Q}_\ell$-lisse étale sheaves $\alpha_\ell : \rho_{Nm}^*\mathcal{V}_\ell \sim P_\ell$ for each $\ell$ such that $\alpha_B^0 \otimes \mathbb{Q}_\ell = \alpha_\ell|_{S^0}$ under the Artin comparison isomorphism.

**Proof.** Take $\rho$ to be the morphism $\rho_{Nm}$ constructed above. The existence of $(\alpha_B^0, \alpha_{dR,C}^0)$ follows from the fact that $\rho|_{S^0}$ is given by $\rho_{std}^0$. Note that in particular $(\alpha_B^0, \alpha_{dR,C}^0)$ is an extension of an isomorphism of Hodge structures $(\mathcal{V}_{B,\rho(b)}, \mathcal{V}_{dR,\rho(b)}) \sim (PH^2(\mathcal{Z}_b, \mathbb{Q}), PH^2_{dR}(\mathcal{Z}_b/\mathbb{C}))$. We only need to show that $\alpha_B^0 \otimes \mathbb{Q}_\ell$ is $\pi_1^\text{et}(S, b)$-equivariant, so that it extends to the $\alpha_\ell$’s as desired.

We first treat the case when $\dim_E \mathcal{V}$ is odd. By construction of $\rho_{Nm}$, we have that $\mathcal{N}(\alpha_B^0 \otimes \mathbb{Q}_\ell)$ globalizes to an isomorphism $\mathcal{N}_{\mathcal{V}_\ell} \sim \mathcal{N}(P_\ell)$, i.e., $\mathcal{N}(\alpha_B^0 \otimes \mathbb{Q}_\ell)$ is $\pi_1^\text{et}(S, b)$-equivariant. As the image of the monodromy representation $\pi_1^\text{et}(S, b) \to \text{GL}(P_{\ell,b})$ lies in $H(\mathbb{Q}_\ell)$ via $\psi$, and $r_{Nm}$ is a faithful representation of $H$, the $\pi_1^\text{et}(S, b)$-equivariance of $\mathcal{N}(\alpha_B^0 \otimes \mathbb{Q}_\ell)$ implies that of $\alpha_B^0 \otimes \mathbb{Q}_\ell$.

Now we treat the case when $\dim_E \mathcal{V}$ is even. Then we apply the argument in the above paragraph to the family $\mathcal{Y}/S$ to conclude that there is an isomorphism

$$(\tilde{\alpha}_B^0, \tilde{\alpha}_{dR,C}^0): (\tilde{\rho}_{Nm}|_{S^0})^*(\tilde{\mathcal{V}}_B, \tilde{\mathcal{V}}_{dR,C}) \sim (\tilde{P}_B, \tilde{P}_{dR,C})|_{S^0}$$

such that $\tilde{\alpha}_{B,b} \otimes \mathbb{Q}_\ell$ is $\pi_1^\text{et}(S, b)$-equivariant. But this implies the $\pi_1^\text{et}(S, b)$-equivariance of $\alpha_{B,b} \otimes \mathbb{Q}_\ell$, because the $\ell$-adic realizations of the exceptional divisors are certainly $\pi_1^\text{et}(S, b)$-invariant.

Now the proof of Thm 5.1 is complete: We just need to take $\rho$ to be the composition of $\rho_{Nm}$ with the natural morphism $\text{Sh}_k(H^\bullet) \to \text{Sh}_K(G)$.

### 5.3 The Integral Period Morphism

Let $\hat{L} := \psi^{-1}(PH^2_\mathbb{Z}(\mathcal{Z}_b, \hat{\mathbb{Z}}))$. For each prime $\ell$, write $L_\ell$ for the $\ell$-component of $\hat{L}$. We assume that $L_{p}$ is self-dual and write $L_{(p)}$ for the $\mathbb{Z}_{(p)}$-lattice $V \cap L_{p}$. Then $G$ extends to the reductive $\mathbb{Z}_{(p)}$-group $\text{SO}(L_{(p)})$, which we still write as $G$ by abuse of notation.
Let $K \subseteq \text{SO}(\mathcal{L} \otimes \mathbb{A}_f)$ be a neat compact open subgroup which stabilizes $\hat{L}$. The lattice $\hat{L}$ defines a $\mathbb{Z}_\ell$-integral (resp. $\mathbb{Z}$-integral) structure for each $V_\ell$ (resp. $V_B$) and we denote the resulting $\mathbb{Z}_\ell$-sheaf by $L_\ell$ (resp. $L_B$). Let us assume that $K$ is of the form $K_p K^\prime$, where $K_p = \text{SO}(L_p)$ and $K^\prime \subseteq \text{SO}(L(\mathcal{P}) \otimes \mathbb{A}_f^\prime)$.

By [31] and [41], $\text{Sh}_K(G)$ admits a canonical integral model $\mathcal{S}_K(G)$ over $\mathbb{Z}_p$. Moreover, for every $\ell \neq p$ étale sheaf $L_\ell$ extends to $\mathcal{S}_K(G)$; the vector bundle $V_{\text{dr}}$ extends to a vector bundle $L_{\text{dr}}$ over $\mathcal{S}_K(G)$; and there exists an $F$-crystal $L_{\text{cris}}$ on $\mathcal{S}_K(G)_{\mathbb{Z}_p}$ such that its evaluation on $\mathcal{S}_K(G)_{\mathbb{Z}_p}$ gives a vector bundle with integrable connection which is naturally identified with $L_{\text{dr}}$.

We recall the concrete construction of the $L_\ast$ sheaves given in [41]. Let $\hat{G} := \text{CSpin}(L(p))$ and $\mathcal{C} \subseteq \hat{G}(\mathbb{A}_f)$ be the preimage of $K$ under the natural morphism $\hat{G} \to G$. Then there is a Shimura variety $\text{Sh}_K(G)$ defined by the group $\hat{G}$ and the same Hermitian symmetric domain as $\text{Sh}_K(G)$. The reflex field of $\text{Sh}_K(G)$ is $\mathbb{Q}$ and by [31] there is an integral canonical model $\mathcal{S}_K(\hat{G})$ over $\mathbb{Z}_p$. It is equipped with a universal abelian scheme $\mathcal{A}$. Let $h : \mathcal{A} \to \mathcal{S}_K(\hat{G})$ be the structural morphism. Define the sheaves $H_B := \mathbb{R}^1\mathcal{A}_c \mathcal{A}' \mathbb{Q}$, $H_\ell := \mathbb{R}^1\mathcal{A}_c \mathcal{A}' \mathbb{Q}_p$ ($\ell \neq p$), $H_p := \mathbb{R}^1\mathcal{A}_c \mathcal{A}' \mathbb{Q}_p$, $H_{\text{dr}} := \mathbb{R}^1\mathcal{A}_c \mathcal{A}' \mathcal{A}_c \mathcal{A}' \mathbb{Q}_p$ and $H_{\text{cris}} := \mathbb{R}^1\mathcal{A}_c \mathcal{A}' \mathbb{Q}_p$ ($\tilde{a} := a \otimes \mathbb{F}_p$). The abelian scheme $\mathcal{A}$ is equipped with a “CSpin-structure” a $\mathbb{Z}/2\mathbb{Z}$-action and an idempotent projector $\pi : \text{End}(\mathcal{A}) \to \text{End}(\mathcal{A})$. The dual of the images of $\pi_a$ gives rise to sheaves $L_\ast$, which descend to (various applicable fibers of) $\mathcal{S}_K(G)$. These descent give the sheaves denoted by the same letters in the previous paragraph. We refer the reader to [41, §4] for the details or [65, §3.1.3] for a quick summary.

Suppose now that the family $S$ extends to a smooth $\mathbb{Z}_p$-variety and $\mathcal{X}$ also extends to a smooth projective family, which we denote by the same letters. Up to replacing $S$ by a connected étale cover, we may assume that the monodromy representation $\pi_1^\dR(S, b) \to \text{GL}(P_{\mathbb{A}_f, b})$ factors through $K$ via $\psi$. Then we have:

**Theorem 5.3.** The morphism $\rho$ given by Thm [5.4] extends to a $\mathbb{Z}_p$-morphism $S \to \mathcal{S}_K(G)$. Moreover,

(a) $\alpha_\ell$ extends to an isomorphism of $\mathbb{Z}_\ell$-lisse étale sheaves $\rho^\ast L_\ell \cong P_\ell$ over $S$ for every $\ell \neq p$, where we use the natural $\mathbb{Z}_\ell$-integral structure of $P_\ell$;

(b) there is an isomorphism of filtered vector bundles $\alpha_{\text{dr}, \mathbb{Q}_p} : \rho_{\mathbb{Q}_p}^\ast L_{\text{dr}} \cong P_{\text{dr}}|_{S \otimes \mathbb{Q}_p}$ over $S \otimes \mathbb{Q}_p$ such that for any chosen isomorphism $\mathbb{Q}_p \cong \mathbb{C}$, $\alpha_{\text{dr}, \mathbb{Q}_p} = \alpha_{\text{dr}, \mathbb{C}}$ and $(\alpha_{\text{dr}, \mathbb{Q}_p})_p = \mathbb{D}_{\text{dr}}(\alpha_p)$.

Assume in addition that $p \geq 5$, $P_{\text{dr}} := \text{PH}^2_{\text{dr}}(\mathcal{X}/S)(1)$ is locally free, and for every $S$-scheme $T$, the natural morphism $H^2_{\text{dr}}(\mathcal{X}/S) \otimes \mathcal{O}_T \to H^2_{\text{dr}}(\mathcal{X}_T/T)$ is an isomorphism. Moreover, assume that for every closed point $s \in M$, $H^3_{\text{cris}}(\mathcal{X}_s/W(k(s)))$ and $H^3_{\text{cris}}(\mathcal{X}_s/W(k(s)))$ are torsion free.

(c) $\alpha_{\text{dr}, \mathbb{Q}_p}$ extends to an isomorphism $\alpha_{\text{dr}} : \rho^\ast L_{\text{dr}} \cong P_{\text{dr}}$ over $S$;

(d) there is an isomorphism $\alpha_{\text{cris}} : \rho_{\mathbb{Q}_p}^\ast L_{\text{cris}}(-1) \cong P_{\text{cris}}(-1)$ of $F$-crystals whose evaluation on $S_{\mathbb{Z}_p}$ agrees with $\alpha_{\text{dr}}$ via the crystalline-de Rham comparison isomorphism.

**Proof.** That $\rho$ can be extended over $\mathbb{Z}_p$ follows from the extension property of canonical integral models.

(a) follows simply from the fact that the natural map $\pi_1^\dR(\eta, b) \to \pi_1^\dR(S, b)$ is surjective ([66, 0BOJ]). (b) follows from Thm [4.9]. Technically, we do not know if $L_{\text{dr}}$ comes from an object of $\text{Mot}(\text{Sh}_K(G))$. However, we can check this by passing to $S \times_p \text{Sh}_K(\hat{G})$ and then use Prop. [3.6].

For (c), we only need to show that $\mathbb{D}_{\text{dr}}(\alpha_p) : \rho_{\mathbb{Q}_p}^\ast L_{\text{dr}} \otimes \mathbb{Q}_p \cong P_{\text{dr}} \otimes \mathbb{Q}_p$ respects the integral structure. By [42, Lem. 6.15], it suffices to show that for every closed point $s \in S$ and $W := W(k(s))$-liftings $\tilde{s}$, $\mathbb{D}_{\text{dr}}(\alpha_p)$ restricts to an isomorphism $L_{\text{dr}, \mathbb{Q}_p}(\tilde{s}) \cong P_{\text{dr}, \tilde{s}}$. Let $K_0 := W[1/p]$ and choose an algebraic closure $\overline{K_0}$. Let $\tilde{s}$ be the $\overline{K_0}$-point over $\tilde{s}$. By [8, Thm 1.3], the $W$-module $P_{\text{dr}, \tilde{s}}$ is recovered from $P_{p, \tilde{s}} = \text{PH}^2_{\text{cr}}(\mathcal{X}_{\tilde{s}}, \mathbb{Z}_p)$.
as $\mathfrak{M}(\mathcal{P}_{p,\overline{s}}) \otimes_{\mathcal{O}} \mathcal{W}$, which is naturally a $W$-submodule of $\mathcal{D}_{dR}(\mathcal{P}_{p,\overline{s}}) \otimes \mathbb{Q}_p$. Here $\mathfrak{M}(\cdot)$ denote the Breuil-Kisin functor, which takes a crystalline $\mathbb{Z}_p$-representation of $\text{Gal}_{K_0}$ to a $\mathfrak{G} := W[u]$-module with extra structures. The $W$-module $\mathcal{L}_{dR,\alpha}(\overline{s})$ is recovered from $\mathcal{L}_{p,\alpha}(\overline{s})$ is the same way. As $\alpha_{p,\overline{s}}$ maps the $\mathbb{Z}_p$-lattice $\mathcal{P}_{p,\overline{s}}$ isomorphically onto $\mathcal{L}_{p,\overline{s}}$, $\mathcal{D}_{dR}(\alpha_{p,\overline{s}})$ maps the $W$-module $\mathcal{L}_{dR,\alpha}(\overline{s})$ isomorphically onto $\mathcal{P}_{dR,\overline{s}}$.

(d) is a direct consequence of (c). The crystalline-de Rham comparison isomorphism always gives us an isomorphism of crystals $\alpha_{\text{cris}}$. We only need to check that it is $F$-equivariant. We may transport the Frobenius action to $\mathcal{L}_{dR}$ and $\mathcal{P}_{dR}$ and check that $\alpha_{dR}$ is $F$-equivariant. It suffices to show this over the generic fiber, i.e., $\alpha_{dR} \otimes \mathbb{Q}_p$ is $F$-equivariant. By the preceding paragraph, this is true for a Zariski dense set of points on $S_{\mathbb{Q}_p}$. Then we are done because $F$-equivariance is clearly a Zariski closed condition. □

**Remark 5.4.** Parts (b), (c) and (d) are written for the sake of completeness. For the sake of proving the Tate conjecture, we will only make use of (a).

### 6 Proof of the Generic Case

#### 6.1 A specialization lemma for monodromy

For any profinite group $G$, we denote by $G^{(p)}$ the maximal prime-to-$p$ quotient. We recall some simple facts:

**Proposition 6.1.** Let $G$ be a profinite group, $N \subseteq G$ be a closed normal subgroup, and $P$ by any $p$-Sylow subgroup of $G$.

(a) $NP/N$ is a $p$-Sylow subgroup of $G/N$.

(b) $P \subseteq N$ if and only if $G/N$ is prime-to-$p$, i.e., $p \nmid [G : N]$, where $[G : N]$ is interpreted as a supernatural number.

(c) If $N$ is the minimal closed normal subgroup $p(G)$ of $G$ which contains one (and hence all) $p$-Sylow subgroups, then $G/N = G^{(p)}$.

(d) If $p \nmid [G : N]$, where $[G : N]$ is interpreted as a supernatural number, then $p(G) \subseteq N$, and the natural morphism $N^{(p)} \to G^{(p)}$ is injective.

**Proof.** (a) is well known for finite groups. For a reference in the generality of profinite groups, see [53, Ex. 2.3.3]. The other statements are all easy consequences of (a). □

**Definition 6.2.** Let $S$ be a scheme over a base scheme $T$. We say that $\overline{S}$ is a *good relative compactification* of $S$ if $\overline{S}$ is a proper $T$-scheme such that $S$ is an open subvariety of $\overline{S}$ and $\overline{S} - S$ is a relative normal crossing divisor relative to $T$.

For Lem. 6.3 and 6.6 below, let $R$ be a strictly Henselian DVR with fraction field $K$ and residue field $k$ of characteristic $p > 0$. Let $\overline{K}$ be an algebraic closure of $K$. Let $S$ be a smooth $R$-variety which admits a good relative compatification $\overline{S}$ with geometrically connected fibers over $R$. Let $a$ and $b$ be geometric points on $\overline{S}$ and $S_{\overline{K}}$ with a chosen étale path from $b$ to $a$.

**Lemma 6.3.** Let $\mathbb{V}_{\ell}$ be a rank $n$ free $\mathbb{Z}_{\ell}$-local system over $S$. Consider the monodromy representations

$$
\rho_K : \pi_1^{\text{ét}}(S_{\overline{K}}, b) \to \text{GL}(\mathbb{V}_{\ell}|_b) \quad \text{and} \quad \rho_k : \pi_1^{\text{ét}}(S_k, a) \to \text{GL}(\mathbb{V}_{\ell}|_a).
$$

27
If \( p \mid |\text{GL}_n(\mathbb{F}_\ell)| \), or \( \mathcal{V}_\ell \) is tamely ramified over \( \overline{S} \), then the identity component of the Zariski closure of the image of \( \rho_K \) is equal to that of \( \rho_k \) under the identification \( \mathcal{V}_{\ell|a} = \mathcal{V}_{\ell|b} \) along the chosen étale path.

By abuse of notation, we may often ignore the base points. The reader can take conjugation by the chosen étale path whenever needed.

**Proof.** Note that \( \rho_K \) and \( \rho_k \) are both obtained by the restricting the monodromy representation \( \rho : \pi^\text{ét}_1(S, a) \to \text{GL}(\mathcal{V}_{\ell|a}) \) of the entire \( S \). If \( \mathcal{V}_\ell \) is tamely ramified along \( \overline{S} \), then \( \rho \) factors through the tame fundamental group \( \pi^\text{ét}_1(S, a) \), and the lemma follows simply from the fact that the natural map \( \pi^\text{étt}_1(S_k, a) \to \pi^\text{ét}_1(S, a) \) is an isomorphism ([26, §XIII 2.10]).

Now we prove the lemma under the assumption \( p \nmid |\text{GL}_n(\mathbb{F}_\ell)| \). We pick a basis and identify \( \mathcal{V}_{\ell|a} \) with \( \mathbb{Z}^n_{\ell} \). For a subgroup \( G \) of \( \text{GL}_n(\mathbb{Q}_\ell) \), let use denote by \( \overline{G} \) the Zariski closure in \( \text{GL}_n(\mathbb{Q}_\ell) \) and \( \overline{G}^\circ \) the identity component. We will repeatedly apply the following observation: \( \overline{G}^\circ \) remains unchanged if we replace \( G \) by a finite index subgroup.

Let \( U_\ell := \ker(\text{GL}_n(\mathbb{Z}_\ell) \to \text{GL}_n(\mathbb{F}_\ell)) \). Then \( U_\ell \) is a pro-\( \ell \) open normal subgroup of \( \text{GL}_n(\mathbb{Z}_\ell) \) of index \( |\text{GL}_n(\mathbb{F}_\ell)| \). We use subscript \( \ell \) to denote the pullback of \( U_\ell \) under (various restrictions of) \( \rho \), i.e., for \( (*) = (\emptyset, k, \overline{K}) \) we have a fiber diagram

\[
\begin{array}{ccc}
\pi^\text{ét}_1(S_*)_{\ell} & \longrightarrow & \pi^\text{ét}_1(S) \\
\downarrow^\rho & & \downarrow^\rho \\
U_\ell & \longrightarrow & \text{GL}_n(\mathbb{Z}_\ell).
\end{array}
\]

Moreover, we have \( \pi^\text{ét}_1(S_k)_{\ell} \to \pi^\text{ét}_1(S)_{\ell} \leftarrow \pi^\text{ét}_1(S_K)_{\ell} \) which is compatible with \( \pi^\text{ét}_1(S_k) \to \pi^\text{ét}_1(S) \leftarrow \pi^\text{ét}_1(S_K) \) via the above diagrams. This implies that we have two sequences of inclusions which are compatible in the obvious way:

\[
\text{im}(\rho_{K, \ell}) \subseteq \text{im}(\rho_k)^\circ \supseteq \text{im}(\rho_{k, \ell}), \quad \text{and \ \im}(\rho_K)^\circ \subseteq \im(\rho)^\circ \supseteq \im(\rho_k)^\circ.
\]

In particular, as \( \pi^\text{ét}_1(S_*)_{\ell} \) is of finite index in \( \pi^\text{ét}_1(S_*) \), we have \( \text{im}(\rho_*) = \text{im}(\rho_{*, \ell}) \). Then it suffices to show that the inclusion \( \text{im}(\rho)^\circ \supseteq \text{im}(\rho_{k, \ell}) \) is an equality. Note that since \( U_\ell \) is pro-\( \ell \), any morphism \( G \to U_\ell \) from a profinite group \( G \) factors through \( G \to G^{(p)} \). In particular, the morphisms \( \rho_{*, \ell} \) factors through \( \rho^{(p)}_{*, \ell} : [\pi^\text{ét}_1(S_*)_{\ell}]^{(p)} \to U_\ell \). For simplicity, write \( \beta_\ast \) for \( \rho^{(p)}_{*, \ell} \). The question now reduces to showing that \( \text{im}(\beta_k) \subseteq \text{im}(\beta) \) is an equality.

We have a commutative diagram:

\[
\begin{array}{ccc}
[\pi^\text{ét}_1(S_k)]^{(p)} & \longrightarrow & [\pi^\text{ét}_1(S)]^{(p)} \\
\downarrow & & \downarrow \\
[\pi^\text{ét}_1(S)]^{(p)} & \longrightarrow & [\pi^\text{ét}_1(S)]^{(p)}
\end{array}
\]

In the above diagram, the horizontal arrows have finite cokernels by construction and are injective by Prop. [6.1]. Grothendieck’s theorem [26, §XIII 2.10] tells us that the right vertical arrow is an isomorphism, so the left vertical arrow has finite cokernel. By the diagram

\[
\begin{array}{ccc}
[\pi^\text{ét}_1(S_k)]^{(p)} & \xrightarrow{\beta_{k, \ell}} & U_\ell \\
\downarrow \quad \beta_k \quad \downarrow \\
[\pi^\text{ét}_1(S)]^{(p)} & \xrightarrow{\beta} & U_\ell
\end{array}
\]

28
\[ \text{im}(\beta_p) = \text{im}(\beta) \] as desired.

**Lemma 6.4.** Let \( p \) be a prime number and \( n \geq 1 \) be any natural number. Then there exists a prime \( \ell \) such that \( p \nmid |\GL_n(\mathbb{F}_\ell)| \) if and only if \( p - 1 > n \).

**Proof.** Recall the well known formula:

\[ |\GL_n(\mathbb{F}_\ell)| = \prod_{j=0}^{n-1} (\ell^n - \ell^j). \]

Then it is easy to see that \( p \nmid |\GL_n(\mathbb{F}_\ell)| \) if and only if \( \{1, \ell^2, \ldots, \ell^n\} \) all have different residues modulo \( p \), i.e., the order of \( \ell \) in \( \mathbb{F}_\ell^\times \) is greater than \( n \). Therefore, if \( p - 1 > n \), by Dirichlet’s theorem there are infinitely many such \( \ell \), but otherwise such \( \ell \) does not exist by the pigeonhole principle.

**Example 6.5.** One may naturally wonder in Prop. 6.1(d), if we replace the assumption \( p \nmid |G : N| \) by the assumption that \( |G : N| \) is finite, we still have that the natural morphism \( N^{(p)} \rightarrow G^{(p)} \) has finite kernel. If so, then the restriction on \( p \) in the above lemma can be removed. Unfortunately, this is not true. Here is a counterexample, which is kindly provided to us by Mark Shusterman: Let \( p = 2 \), \( G' = \text{the additive group of } \mathbb{Z}_3 \) and \( G \) be the semidirect product \( G' \rtimes \mathbb{Z}/2\mathbb{Z} \), where \( \mathbb{Z}/2\mathbb{Z} \) acts on \( G' \) by negation. Then \( G' \leq G \) and \( |G : G'| = 2 \). Clearly \( (G')^{(2)} = G' \). On the other hand, \( G \) can be viewed as the inverse limit of the dihedral group \( D_{3^n} \) (the symmetry group of \( (3^n) \)-gons), with \( G' \) being the subgroup generated by rotations. This implies that \( G^{(2)} \) is trivial, as the reflections generate the entire \( G \).

**Lemma 6.6.** Let \( f : X \rightarrow S \) be a maximally polarized family of smooth proper varieties such that the characteristic 0 fibers of \( X \) have Hodge number \( h^{2,0} = 1 \) and the \( (\mathcal{X}/S)_K \) satisfies (\( \bigvee \)). Let \( \mathcal{V}_0 := \PH^2(\mathcal{X}/S, \mathbb{Q}_\ell) \). Assume that \( \overline{K} \) can be embedded into \( \mathbb{C} \).

If \( p - 1 > \text{rank} \mathcal{V}_0 \), or the local system \( \mathcal{V}_0 \) is tamely ramified over \( S \), then there does not exist a connected finite étale cover \( \overline{S}_K \) of \( S_K \) such that \( \mathcal{V}_0 \) admits a nonzero global section.

**Proof.** This follows easily from Lem. 2.15, Lem. 6.3 and 6.4. Note that by [36, Prop. 6.14] we are allowed to replace \( \ell \) by a different prime \( \neq p \), as the conclusion is independent of \( \ell \).

**6.2 Proof of Thm 1.3**

Before proving the genericity theorem, we first give a precise condition for the Tate conjecture to hold:

**Theorem 6.7.** Let \( M \) be a connected smooth variety over \( \mathbb{Z}_p \), and let \( f : X \rightarrow M \) be a smooth projective family of varieties whose characteristic 0 fibers have Hodge number \( h^{2,0} = 1 \). Let \( \eta \) be the generic point of \( M \) and assume that \( \text{NS}(X_\eta)_\mathbb{Q} = \text{NS}(X_\eta)_\mathbb{Q} \). Let \( \ell \) be any prime \( \neq p \) and let \( \mathcal{V}_0 := \PH^2(\mathcal{X}/M, \mathbb{Q}_\ell) \). Assume the following conditions:

(a) The \( \mathbb{Q} \)-family \( X|_{M_\mathbb{Q}} \) satisfies property (\( \bigvee \));

(b) For some (and hence every) point \( b \in M(\mathbb{C}) \), the quadratic \( \mathbb{Z}_p \)-lattice \( \PH^2(X_0(\mathbb{C}), \mathbb{Z}_p) \) is self-dual;

(c) Let \( S \) be any irreducible component of \( M_{\overline{\mathbb{F}_p}} \) and let \( \overline{\eta} \) be a geometric generic point. Then the \( \text{Mon}^\circ(\mathcal{V}_0|_S, \overline{\eta}_S) \)-invariant part of \( \mathcal{V}_0|_{\overline{\eta}_S} \) is spanned by the image of the specialization map \( \text{NS}(X_\eta)_\mathbb{Q} \rightarrow \text{NS}(X_{\overline{\eta}_S})_\mathbb{Q} \).

Then for every finite field \( k \) over \( \mathbb{F}_p \) and \( s \in M(k) \), the fiber \( X_s \) satisfies the Tate conjecture.
Note that the specialization map \( \text{NS}(\mathcal{X}_\eta)_\mathbb{Q} \to \text{NS}(\mathcal{X}_{\overline{\eta}})_\mathbb{Q} \) is well defined by Lem. 2.10 and up to replacing \( M \) by a finite connected étale cover, \( \text{NS}(\mathcal{X}_\eta)_\mathbb{Q} = \text{NS}(\mathcal{X}_{\overline{\eta}})_\mathbb{Q} \) can always be satisfied.

**Proof.** Fix the point \( b \in M(\mathbb{C}) \) lying above \( k(\bar{\eta}) \) as a base point and let \( L \) be the quadratic \( \mathbb{Z}_p \)-lattice \( \text{PH}^2(\mathcal{X}_\eta(\mathbb{C}), \mathbb{Z}_p) \). Let \( G := \text{SO}(L(p)) \) and \( \overline{G} := \text{CSpin}(L(p)) \). Let \( K \subseteq G(\mathbb{A}_f) \) be a neat compact open subgroup of the form \( K_pK^p \) where \( K_p = G(\mathbb{Z}_p) \) and \( K^p \subseteq G(\mathbb{A}_f^p) \) and let \( \bar{K} \subseteq \overline{G}(\mathbb{A}_f) \) be the preimage of \( K \). Up to replacing \( M \) by a connected étale cover, Thm 5.3 gives us a morphism \( \rho : M \to \mathcal{X}_K(G) \). Since the natural morphism \( \mathcal{X}_K(\overline{G}) \to \mathcal{X}_K(G) \) is étale, up to replacing \( M \) by a further connected étale cover, we may assume that \( \rho \) lifts to a morphism \( \overline{\rho} : M \to \mathcal{X}_K(\overline{G}) \). Let the abelian scheme \( \mathcal{A} \to \mathcal{X}_K(\overline{G}) \) and sheaves \( H_s, L_s \) on \( \mathcal{X}_K(\overline{G}) \) be as in §5.3. Let \( \kappa \) be a field and let \( s, t \) be \( \kappa \)-points of \( \mathcal{X}_K(\overline{G}) \), and \( M_\kappa \) respectively such that \( \overline{\rho}(t) = s \).

We first observe that if \( \text{char } \kappa = 0 \), then there is an isomorphism \( \theta : \text{LEnd}(\mathcal{A}_s)_\kappa \to \text{PNS}(\mathcal{X}_t)_\kappa \) such that for every \( \zeta \in \text{LEnd}(\mathcal{A}_s)_\kappa \), \( \zeta \) and \( \theta(\zeta) \) have the same cohomological realization in \( \mathbb{P}_t, \mathbb{P}_s \). Indeed, by the Galois descent properties of \( \text{NS}(\mathcal{X}_t)_\kappa \) and \( \text{End}(\mathcal{A}_s)_\kappa \) we may reduce to considering the case when \( \kappa = \mathbb{C} \). Then the statement follows from the fact that both \( \text{PNS}(\mathcal{X}_t)_\kappa \) and \( \text{LEnd}(\mathcal{A}_s)_\kappa \) can be identified with the \((0,0)\)-classes in \( \mathbb{L}_{B,s} \). Indeed, although Thm 5.3 a priori guarantees this only when \( t \) lies in the same connected component of \( M_\kappa \) as \( b \), we may still conclude using that \( \text{Aut}(\mathbb{C}) \) acts transitively on the set of \( \mathbb{C} \)-connected components of \( M_\kappa \) and that \((0,0)\)-classes are algebraic.

Now consider the case when \( \kappa \) is a perfect field of characteristic \( p \). Let \( \widehat{U}_s \) be the formal completion of \( \mathcal{X}_K(\overline{G})_{W(\kappa)} \) at \( s \). For a special endomorphism \( \zeta \in \text{LEnd}(\mathcal{A}_s) \) consider the following functor:

\[
\text{Def}(\zeta, s) : R \to \{ \bar{s} \in \widehat{U}_s(R) \mid \zeta \text{ deforms to } \text{LEnd}(\mathcal{A}_s) \}
\]

(6.8) where \( R \) runs through all Artin \( W(\kappa) \)-algebras. By [44, §5.14], \( \text{Def}(\zeta, s) \) is represented by a closed formal subscheme of \( \widehat{U}_s \) cut out by a single formal power series \( \bar{f}_\zeta \in \mathcal{O}_{\widehat{U}_s} \). Now suppose that \( t \in M_\kappa \) is a point such that \( \overline{\rho}(t) = s \). Let \( \widehat{M}_t \) be the formal completion of \( M_{W(\kappa)} \) at \( t \). Then \( \overline{\rho} \) restricts to a morphism \( \widehat{M}_t \to \widehat{U}_s \). Let \( \text{Def}(\zeta, t) \) be the functor defined by (6.8) with \( \widehat{U}_s \) replaced by \( \widehat{M}_t \). Then we have a fiber diagram:

\[
\begin{array}{ccc}
\text{Def}(\zeta, t) & \to & \text{Def}(\zeta, s) \\
\downarrow & & \downarrow \\
\widehat{M}_t & \to & \widehat{U}_s
\end{array}
\]

In particular, \( \text{Def}(\zeta, t) \) is a closed formal subscheme of \( \widehat{M}_t \) cut out by the pullback \( \overline{\rho}^*(\bar{f}_\zeta) \).

We prove the following claim:

\[
\text{Def}(\zeta, t) \text{ is flat over } \mathbb{Z}_p.
\]

(6.9) It suffices to show that \( \overline{\rho}^*(\bar{f}_\zeta) \) is identically 0 if and only if it is divisible by \( p \). For the proof of the claim we may assume that \( \kappa \) is algebraically closed. Suppose that it is indeed divisible by \( p \). This implies that \( \widehat{M}_t \otimes \mathbb{F}_p \subseteq \text{Def}(\zeta, t) \), i.e., \( \zeta \in \mathcal{A}_s \) deforms to the entire disk \( \mathcal{D} := \widehat{M}_t \otimes \mathbb{F}_p \). Let \( S \) be the irreducible component of \( M_\kappa \) which contains \( s \). Note that the geometric point \( \bar{\eta}(\mathcal{D}) \) \( \to M_\kappa \) factors through \( \bar{\eta}_S \), so we obtain an element \( \zeta \in \text{LEnd}(\mathcal{A}_s)_\kappa \) which specializes to \( \zeta \). By assumption (c), \( \zeta \) comes from the specialization of an element of \( \text{LEnd}(\mathcal{A}_s)_\kappa \) \( = \text{PNS}(\mathcal{X}_s)_\kappa \). But then \( \bar{f}_\zeta \) has to vanish identically on \( \widehat{M}_t \).

Now assume that \( \kappa \) is an algebraic extension of \( \mathbb{F}_p \). We construct a morphism \( \theta : \text{LEnd}(\mathcal{A}_s)_\kappa \to \text{PNS}(\mathcal{X}_s)_\kappa \) which respects the cohomological realizations just as in the second paragraph. Again by Galois descent it suffices to construct the morphism when \( \kappa = \mathbb{F}_p \). Take any \( \zeta \in \text{LEnd}(\mathcal{A}_s)_\kappa \). By (6.9) there exists
a characteristic $0$ point $\tilde{t}$ which specializes to $t$ such that $\zeta$ comes from the specialization of an element $\tilde{\zeta} \in \mathcal{A}_g$, where $\tilde{A} := \hat{\rho}(\tilde{t})$. By the second paragraph we obtain an element of $\tilde{\zeta} \in \text{PNS}(\mathcal{X}_t)_\mathbb{Q}$ with the same cohomological realization as $\zeta$. Then we specialize $\tilde{\zeta}$ to an element $\xi \in \text{PNS}(\mathcal{X}_t)$, which we define to be $\theta(\zeta)$. It is clear that the definition does not depend on the choice of $\tilde{t}$.

Finally assume that $\kappa = k$. By the above paragraph we have a commutative diagram

$$
\begin{array}{c}
\text{LEnd}(\mathcal{A}_g) \otimes \mathbb{Q}_\ell \\
\downarrow \theta \\
\text{PNS}(\mathcal{X}_t) \otimes \mathbb{Q}_\ell \\
\end{array} \xrightarrow{\cong} 
\begin{array}{c}
\mathbf{L}^\text{Galg} \\
\text{P} \mathbf{H}^2_{\text{ét}}(\mathcal{X}_t, \mathbb{Q}_\ell(1))
\end{array}
$$

By the Tate conjecture for special endomorphisms [40, Thm 6.4], the top arrow is an isomorphism, and hence so is the bottom arrow. This affirms the Tate conjecture for $\mathcal{X}_t$.

**Remark 6.10.** The reason that our proof does not depend on any good local property of $\rho$ is that $\hat{\rho}^*(\tilde{f}_\xi)$ is always a single equation, no matter how bad $\rho$ is on the formal completions. The domain $\mathcal{M}$ itself is assumed to be smooth over $\mathbb{Z}_p$, so as along as $\hat{\rho}^*(\tilde{f}_\xi)$ is not divisible by $p$, one may find a characteristic $0$ point on its vanishing.

We are now ready to prove Thm 1.3.

**Proof.** By combining Nagata’s compactification and Hironaka’s resolution of singularities in characteristic zero, we can find a compactification $\mathcal{M}$ of the generic fiber $\mathcal{M}_0$ such that the boundary $\mathcal{D} := \mathcal{M} - \mathcal{M}_0$, equipped with the reduced scheme structure, is a normal crossing divisor. For some open subscheme $U$ of $\text{Spec}(\mathbb{Z})$, $\mathcal{M}$ and $\mathcal{D}$ are defined over $U$, and $\mathcal{D}$ becomes a relative normal crossing divisor over $U$. Now we conclude by Lem. 6.6 and Thm 6.7.

**Example 6.11.** Let $S$ be a smooth $k$-variety for some algebraically closed field $k$, and let $f : \mathcal{X} \to S$ be a smooth and proper family of $h^{2,0} = 1$ varieties. Let $b \in S$ be a $k$-point and let $\mathbb{V}_\ell := \mathbb{R}^2 f_* \mathbb{Q}_\ell$. If $\text{char} k = 0$, then by the theory of the fixed part we know that $\text{Mon}^\text{g}(\mathbb{V}_\ell, b)$ is non-trivial if and only if the family $\mathcal{X}/S$ satisfies $(\forall)$ (cf. [17, Prop. 6.4(i)]). If $\text{char} k = p > 0$ however, the connection breaks down: Let $\mathcal{X}/S$ be any non-isotrivial family of supersingular K3 surface (or abelian surface). Then $\mathcal{X}/S$ satisfies $(\forall)$, but $\text{Mon}^\text{g}(\mathbb{V}_\ell, b)$ is trivial. One can easily deduce this from a theorem of Artin [51, Prop. 5.5]: Up to replacing $S$ be a finite étale cover, we may assume that $\mathcal{P} \text{ic}_{\mathcal{X}/S}(S) \to \text{NS}(\mathcal{X}_s)$ has a finite cokernel of $p$-power length. Then since $\text{Pic}(\mathcal{X}_s) \otimes \mathbb{Z}_p \to H^2_{\text{ét}}(\mathcal{X}_s, \mathbb{Z}_p)$ is an isomorphism for every $s$, $\mathbb{V}_\ell$ becomes trivial.

**Example 6.12.** \footnote{We thank Ananth Shankar and Boya Wen for informing us this example.} We extract a well-structured non-example for Thm 6.7 from [12, §III]. For readers’ convenience we follow the notations in loc. cit. Let $\Delta$ be an indefinite quaternion division algebra over $\mathbb{Q}$ and let $\Delta^*$ denote the reductive group over $\mathbb{Q}$ defined by $\Delta^*(R) = (\Delta \otimes \mathbb{Q} R)^\times$ for every $\mathbb{Q}$-algebra $R$. Let $\mathcal{H}^+$ be the double upper-half plane $\mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R})$. Then for every compact open subgroup $U \subseteq \Delta^*(\mathcal{A}_f)$ there is a Shimura curve $S_U$, whose complex uniformization is $S_U(\mathbb{C}) = \Delta^*(\mathbb{Q}) \backslash \mathcal{H}^+ \times \Delta^*(\mathcal{A}_f)/U$.

Let $p$ be a prime such that $\Delta$ ramifies at $p$ and suppose that $U = U^p \cap U_p$ where $U_p \subseteq \Delta^*(\mathcal{A}_f^p)$ and $U^p$ is the unique maximal compact open subgroup of $\Delta^*(\mathbb{Q}_p)$. There is a natural $\mathbb{Z}_p$-model $M_U$ of $S_U$ which admits a moduli interpretation of abelian surfaces with action by (the ring of integers of) $\Delta$ ([12, §III 3.1]). Let $h : \mathcal{A}_g \to M_U$ be the universal family, which can be principally polarized. Honda-Tate theory
implies that for every \( s \in M_U(\overline{\mathbb{F}}_p) \), the fiber \( \mathcal{A}_s \) is supersingular ([12, §III Prop. 2]), i.e., \( \mathcal{A}_p \) is an entirely supersingular family. By the preceeding example, even after one remove the singular points \( M_U \otimes \mathbb{F}_p \), \( M_U \) does not satisfy assumption (c) of Thm 6.7; moreover, one may construct a period morphism for \( M_U \) just as in the proof of Thm 6.7, but the key flatness claim (6.9) will no longer be true for every \( \zeta \).

7 Deforming Curves on Parameter Spaces

7.1 Families of Curves which homogeneously dominate a Variety

Let \( k \) be an algebraically closed field. For a morphism \( f : X \to Y \) between \( k \)-varieties, we say that \( f \) has equi-dimensional fibers if for every two points \( y, y' \in Y \), \( \dim X_y = \dim X_{y'} \). If \( f : X \to Y \) is a smooth morphism between \( k \)-varieties, denote by \( T(X/Y) \) the relative tangent bundle, i.e., the dual of \( \Omega^1_{X/Y} \). If \( Y = \text{Spec}(k) \), then we simply write \( T_X \) for \( T(X/Y) \), and for a \( k \)-point \( x \in X \), we write \( T_xX \) for the tangent space \( T_xX \) to emphasize that it is a fiber of \( T_X \).

**Definition 7.1.** Let \( S \) and \( T \) be two smooth irreducible \( k \)-varieties, \( g : \mathcal{C} \to T \) be a smooth family of connected curves, and \( \varphi : \mathcal{C} \to S \) be a morphism with equi-dimensional fibers. Let \( U \) be the maximal open subvariety of \( \mathcal{C} \) on which the composition \( \mathcal{T}(\mathcal{C}/T) \hookrightarrow \mathcal{T} \to \varphi^*(\mathcal{T}S) \) does not vanish. Suppose that

(a) the induced morphism \( \mathcal{C} \to T \times S \) is quasi-finite,

(b) for every \( k \)-point \( s \in S \), \( U \cap \varphi^{-1}(s) \) is dense in \( \varphi^{-1}(s) \);

(c) the morphism \( U \to \mathbb{P}(\mathcal{T}S) \) has equi-dimensional fibers.

Then we say that the family of curves \( \mathcal{C}/T \) homogeneously dominates \( S \) (via the morphism \( \varphi \)). If there exists an open dense subvariety \( T' \subseteq T \) such that the restriction \( \mathcal{C}|_{T'} \) homogeneously dominates \( S \), then we say that \( \mathcal{C}/T \) strongly dominates \( S \).

The natural morphism \( U \to \mathbb{P}(\mathcal{T}S) \) is induced by the identification \( \mathcal{C} \cong \mathbb{P}(\mathcal{T}(\mathcal{C}/T)) \). Roughly speaking, the family \( \mathcal{C}/T \) homogeneously dominates \( S \) if there are curves passing through every given point on \( S \) in any given direction, and the sub-family of such curves has a fixed dimension. Below is a picture for the morphism \( U \to \mathbb{P}(\mathcal{T}S) \) when restricted to \( \varphi^{-1}(s) \cap U \):

The notion “\( \mathcal{C}/T \) strongly dominates \( S \)” is only defined for convenience, as sometimes the natural families of curves have some bad locus on \( T \) of smaller dimension which does not affect applications.
Lemma 7.2. Let $S$ be a smooth $k$-variety and $\mathcal{P} \to S$ be a smooth family of varieties over $S$. Let $\mathcal{X} \subseteq \mathcal{P}$ be a relative effective Cartier divisor whose total space is smooth. If $s \in S(k)$ is a point such that $\mathcal{X}_s$ has isolated singularities, then there exists an open dense subvariety $U \subseteq \mathcal{P}(T_s)S$ with the following property:

For every unramified morphism $\varphi: C \to S$ from a smooth curve $C$ which sends a point $c \in C$ to $s$, the total space of the pullback family $\mathcal{X}|_C$ has no singularity on $\mathcal{X}_s$ if $d\varphi(T_cC) \subset U$.

Proof. Since the question is étale-local in nature, we may assume that $S = \mathbb{P}^m_k$ for $m = \dim S$, $s = 0$, $\mathcal{X}_s$ has a single isolated singularity at a $k$-point $P \in \mathcal{X}_s \subseteq \mathcal{P}_s$, and $\mathcal{P}$ is isomorphic to $\mathbb{P}^n_k$ near $P$. Suppose that $\mathcal{X}$ is locally cut out by an equation $F(x_1, \ldots, x_n, s_1, \ldots, s_m)$ near $P$. That $P$ is a singularity of the fiber $\mathcal{X}_s$ but not of the total space $\mathcal{X}$ implies that $\partial F/\partial s_j \neq 0$ at $P$ for some $j$. One may simply take $U$ to be the open subscheme of $\mathcal{T}_s S \cong \mathbb{P}^{m-1}$ where the coordinate of the basis vector $\partial s_j$ is nonzero. \qed

Definition 7.3. Let $f: X \to S$ be a morphism of schemes of finite type.

(a) Let the singular locus $\text{Sing}(f)$ be the reduced closed subscheme of $X$ whose support consists of all points where $f$ fails to be smooth.

(b) If $f$ is in addition proper and flat, we say that the scheme-theoretic image of $\text{Sing}(f)$ is the (generalized) discriminant locus of $f$, and denote it by $\text{Disc}(f)$.

(c) In the above situation, we say that $\text{Disc}(f)$ is mild if it has codimension at least 1 in $S$ and there exists a dense open subscheme $V \subseteq \text{Disc}(f)$ such that for every geometric point $s$ on $V$, the fiber $X_s$ has only isolated singularities.

Note that the properness assumption on $f$ implies that $\text{Disc}(f)$ is closed in $S$. Moreover, since it is defined to be the scheme-theoretic image of a reduced scheme, it is also reduced. Its formation commutes with flat base change but not arbitrary base change: For any morphism $T \to S$, $\text{Disc}(f_T)$ is always the reduced subscheme of $\text{Disc}(f)_T$, so they are equal if and only if the latter is reduced.

Proposition 7.4. Let $\mathcal{X}$ and $S$ be as in Lem. 7.2. Suppose that

- the generalized discriminant variety $D := \text{Disc}(f) \subseteq S$ is mild;
- there is a family of smooth curves $\mathcal{C}/T$ which homogeneously dominates $S$ through a morphism $\varphi$.

Then for a general $k$-point $t \in T$, the total space of the pullback family $\mathcal{X}|_{\varphi_t}$ is smooth.

Proof. By assumption we have a diagram

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\varphi} & \mathcal{X} \\
\downarrow{g} & & \downarrow{f} \\
T & & S
\end{array}$$

Let $Z \subseteq S \times T$ be the subset of points $(s, t)$ such that $\mathcal{C}_t$ passes through $t$ and the total space of $\mathcal{X}|_{\varphi_t}$ has a singularity lying above $\varphi_t^{-1}(s)$. It is easy to see that $Z$ is constructible: Let $\mathcal{X}|_{\varphi_t}$ be the pullback of $\mathcal{X}$ along $\varphi$. Then we have a natural morphism $\mathcal{X}|_{\varphi} \to T \times_k S$ and $Z$ is the set-theoretic image of $\text{Sing}(\mathcal{X}|_{\varphi} \to T)$. Endow $Z$ with the structure of a reduced scheme.

It suffices to show that $\dim Z < \dim T$. Let $V$ be as in Def. 7.3(c) and for $s \in V(k)$, let $U_s \subseteq \varphi^{-1}(s)$ be as in Def. 7.3(b). Let $t = g(s)$. By Lem. 7.2 there exists a proper closed subvariety $U_{s, \text{bad}} \subseteq U$ such

\[\text{Note that this definition is different from the one given in \cite[Tag 0C3H]{33}.}\]
that the total space of $\mathcal{E}|_{\mathcal{U}}$ is not smooth near the fiber $\mathcal{U}_u$ only if $u \in U_{s,\text{bad}}$. Let $\tilde{Z}_s \subseteq \varphi^{-1}(s)$ be the union of the complement of $U_s$ and the Zariski closure of $U_{s,\text{bad}}$. Then $\tilde{Z}_s$ is a proper closed subvariety of $\varphi^{-1}(s)$. Since the morphism $\varphi^{-1}(s) \to T$ is quasi-finite and the fiber $\tilde{Z}_s \subseteq T$ is contained in the image of $\tilde{Z}_s$, we have

$$\dim \tilde{Z}_s < \dim \varphi^{-1}(s) = \dim \mathcal{E} - \dim S.$$ 

Since the image of $\mathcal{Z}$ in $S$ is contained in $D$, and $s$ was taken as a general point of $D$, we have that $\dim \mathcal{Z} \leq \dim \mathcal{E} - 2 = \dim T - 1$ as desired.

**Example 7.5.** Here is a prototypical example which motivates the above proposition: Consider the complete linear system $|\mathcal{O}(d)|$ of degree $d \geq 1$ hypersurfaces in a projective space $\mathbb{P}^{r+1}$ over $k$. Let $G := G(2,H^0(\mathcal{O}(d)))$ be the Grassmannian of lines in the projective space $|\mathcal{O}(d)|$. It is well known that a general element $\gamma: \mathbb{P}^1 \to |\mathcal{O}(d)|$ in $G$ is a Lefschetz pencil, and the the total space of the pencil is simply the blow up of $\mathbb{P}^{r+1}$ along a smooth subvariety, and hence is smooth. The above proposition shows that this statement about the smoothness of the total space holds in much more generality.

**Example 7.6.** The hypothesis that $D \subseteq S$ is a proper subvariety, i.e., the morphism $f$ is generically smooth, is necessary when char $k = p > 0$. Moret-Bailly constructed the following nice example: When $p = 3$, the surface $\mathcal{X} \subseteq \mathbb{P}^2 \times \mathbb{A}^1$ given by the equation $y^2z = x^3 - tz^3$ ($t$ is the coordinate on $\mathbb{A}^1$ and $x, y, z$ are homogeneous coordinates on $\mathbb{P}^2$) is smooth itself, but its projection to $\mathbb{A}^1$ does not have a smooth general fiber ([18]). However, note that for Lem. [7.2] this hypothesis is unnecessary.

### 7.2 Applications of the Baire category theorem

Let $k$ be an algebraically closed field of characteristic $p > 0$, $W := W(k)$ be its ring of Witt vectors and $\hat{K} := W[1/p]$. Choose an algebraic closure $\hat{K}$ of $K$.

**Lemma 7.7.** Let $S$ be a smooth $W$-variety and let $N \subseteq S_{\hat{K}}$ be a countable union of closed proper subvarieties. Suppose that $S$ admits a flat morphism to another smooth $W$-variety $B$. Let $\hat{B}_b$ be the formal completion of $B$ at $b$. Then for any $b \in B(k)$, the subset of points $\tilde{b} \in \hat{B}_b(W)$ such that $\text{supp} (\tilde{S}_{\hat{b},\hat{K}}) \subseteq N$ is analytically dense in the disk $\hat{B}_b(W)$.

**Proof.** Let $U := \hat{B}_b(W)$. By taking the union of $N$ with all its Galois conjugates, we may assume that $N$ is defined over $K$. Let $N_1, N_2, \cdots$ be the irreducible components of $N$. By the flatness of $S \to B$, for each $i$ there exists a proper closed subscheme $Z_i \subseteq B$ such that every $z \in B(K)$ satisfies $\text{supp} (\tilde{S}_{z}) \subseteq N$ only if $z \in Z_i$. Indeed, one may simply take $Z_i$ to be the complement of the image of $S - N_i$. Since each $U - Z_i$ is open dense in analytic topology, we may conclude by the Baire category theorem for complete metric spaces that $U - \cup_{i=1}^{\infty} Z_i$ is analytically dense.

**Lemma 7.8.** Let $S$ be a smooth $W$-variety and $N \subseteq S_{\hat{K}}$ be a countable union of closed proper subvarieties. Suppose that $\mathcal{E}$ is a smooth family of connected curves over a smooth $W$-variety $T$, and there is a morphism $\varphi: \mathcal{E} \to S$ such that the geometric generic fiber $(\mathcal{E}/T)_{\hat{K}}$ strongly dominates $S_{\hat{K}}$. Then for any $t \in T(k)$, and the set of points $\tilde{t} \in \hat{T}_t(W)$ such that $\varphi(\text{supp} (\mathcal{C}_{\tilde{t},\hat{K}})) \subseteq N$ is analytically dense.

**Proof.** Again we may assume that $N$ is defined over $K$ with irreducible components $N_i$. Let $M_i \subseteq \mathcal{E}_{\hat{K}}$ be the complements of $\varphi^{-1}(S - N_i)$ and $M = \cup_{i=1}^{\infty} M_i$. The assumption that $(\mathcal{E}/T)_{\hat{K}}$ strongly dominates $S_{\hat{K}}$ implies that each $M_i$ is a proper subvariety. Now we apply the above lemma with $(S \to B, N)$ replaced by $(\mathcal{E} \to T, M)$.
8 Elliptic Surfaces with $p_g = q = 1$

8.1 Generalities on Elliptic Surfaces

In this section we recall some basic facts about elliptic surfaces and describe their moduli. Let $k$ be an algebraically closed field of characteristic $\neq 2, 3$. Let $C$ be a smooth projective curve over $k$ and $\pi: X \to C$ be an elliptic surface over $C$ with a zero section $\sigma: C \to X$ through which we also view $C$ as a curve on $X$. The fundamental line bundle $L$ of $X/C$ is defined to be the dual of the normal bundle $N_{C/X}$, or equivalently that of $\mathbb{R}^1 \pi_* \mathcal{O}_X$. The degree of $L$ is defined to be the height of $X$, which we denote by $\text{ht}(X)$. Set $V_r = H^0(L^r)$. There exists a pair $(a_4, a_6) \in V_4 \times V_6 - \{0\}$, which is unique up to the action of $\lambda \in k^*$ by $\lambda \cdot (a_4, a_6) = (\lambda^4 a_4, \lambda^6 a_6)$, such that $X$ is the minimal resolution of the hypersurface $X' \subseteq \mathbb{P}(L^2 \oplus L^3 \oplus \mathcal{O}_C)$ defined by the Weierstrass equation (\cite[Thm 1]{35})

$$y^2 z = x^3 - a_4 x z^2 - a_6 z^3,$$

(8.1)

where $x, y, z$ as homogeneous coordinates on $L^2, L^3, \mathcal{O}_C$ respectively. The hypersurface $X'$ has at most rational double point singularities and is called the \textit{Weierstrass normal form} of the original surface $X$. If $X'$ is smooth, then of course $X = X'$.

Next, we recall that Kodaira classified all the possible singular fibers in the elliptic fibration $\pi: X \to C$ when $k = \mathbb{C}$ in \cite[33]{35}, and his classification is well known to hold verbatim in characteristic $\neq 2, 3$ as well. We refer the reader to \cite[\S 4]{35} for a summary. Set $\Delta := 4a_4^3 - 27a_6^2$. Let $c \in C$ be a point and denote by $\text{val}_c$ the valuation defined by a uniformizer at $c$. The only facts we shall need are the following:

**Proposition 8.2.** (a) The fiber $X_c$ is singular if and only if $\Delta$ vanishes at $c$, i.e., $\text{val}_c(\Delta) \geq 1$.

(b) $X_c$ is of $I_n$ type ($n > 0$) if and only if $\text{val}_c(a_4) = \text{val}_c(a_6) = 0$, and $n = \text{val}_c(\Delta)$.

(c) $X_c$ is of $II$-type if and only if $\text{val}_c(a_4) \geq 1$ and $\text{val}_c(a_6) = 1$.

(d) If $X_c$ is a singular fiber of any other type, $\text{val}_c(\Delta) \geq 3$.

(e) If $X_c$ is of $I_1$-type or $II$-type, then $X_c = X'_c$. In other words, the singularity on $X'_c$ is not a surface singularity.

(f) If $X_c$ is of $II_2$-type, then $X'_c$ has a unique ordinary double point singularity given by contracting the irreducible component not meeting the zero section.

The degree of the discriminant $\text{disc}(x^3 + Ax + B)$ is $12n = e(X)$. Recall that the genus $g(C)$ is equal to the irregularity $q(X)$ and we have $p_g(X) = n - 1 + g(C)$. Therefore, elliptic surfaces with $p_g = 1$ fall into two types:

- $n = 2$ and $g(C) = 0$. These are elliptic K3 surfaces.
- $n = g(C) = 1$. These surfaces have Kodaira dimension 1.

We are interested in the latter class. Note that although these surfaces are elliptic fibrations over genus 1 curves, one should not confuse them with \textit{bielliptic surfaces}, which are of Kodaira dimension 0.

For future reference we introduce some notation. Let $S$ be a base scheme and $V$ be a vector bundle over $S$. We denote by $\mathbb{A}(V)$ the relative affine space over $S$ defined by $V$ and $\mathbb{A}(V)^*$ the open part of $\mathbb{A}(V)$.

35
minus the zero section. Given a sequence of numbers \( q = (q_0, \cdots, q_m) \) and vector bundles \( V_0, \cdots, V_m \) such that \( V = \bigoplus_{i=0}^{m} V_i \), we denote by \( \mathcal{P}_q(V) \) the resulting weighted projective stack, i.e., the quotient stack of \( \mathbb{G}_m \)-action on \( V \) given by

\[
\lambda: (v_0, \cdots, v_m) \mapsto (\lambda^{q_0} v_0, \cdots, \lambda^{q_m} v_m) \quad \text{for} \quad \lambda \in \mathbb{G}_m,
\]

and by \( \mathcal{P}_q(V) \) the coarse moduli space of \( \mathcal{P}_q(V) \). It is well known that this coarse moduli space can be constructed explicitly by applying the relative Proj functor to a sheaf of graded algebras over \( S \). We omit the details. If \( q \) is not specified, then it is assumed to be \( (1, \cdots, 1) \).

**Construction 8.3.** Let \( S \) be a \( \mathbb{Z}[1/6] \)-scheme, \( \varpi: C \to S \) be a family of smooth projective curves over \( S \) of genus \( g \) and \( \mathcal{L} \) be a relative line bundle on \( C \) of degree \( h \). Assume that \( 4h \geq 2g - 1 \) and \( h \geq 1 \). Let \( \mathcal{V}_r \) denote the vector bundle \( \varpi_* \mathcal{L}^r \) for \( r \geq 4 \). Let \( \mathcal{X} \) be the subscheme of \( \text{Spf}(\mathcal{V}_r) \) minus the zero section. Given a sequence of numbers \( q \) let \( W \) be a \( \mathbb{G}_m \)-vector bundle over \( P \) over the scheme \( V \). We remark that these stacks are only "stacky" because of the base.

Let \( \mathcal{Q}(\mu) \) denote the quotient stack of the \( \mathbb{G}_m \)-action \( \mu \). Then \( \mathcal{X} \) descends to an algebraic substack \( \mathcal{X}' \) of \( \mathcal{Q}(\mu) \). Note that \( \mathcal{Q}(\mu) \), and hence \( \mathcal{X} \), admit natural morphisms to \( \mathcal{P}_q(\mathcal{V}_4 \oplus \mathcal{V}_6) \).

We remark that these stacks are only "stacky" because of the base.

**Lemma 8.5.** Let \( T \) be an \( S \)-scheme and \( \psi: T \to \mathcal{P}_q(\mathcal{V}_4) \) be a morphism. Then the pullback \( \psi^* \mathcal{Q}(\mu) \), and hence \( \psi^* \mathcal{X} \), are projective schemes over \( T \).

**Proof.** The quotient stack \( \mathcal{Q}(\mu) \) can be alternatively constructed as follows: View \( \mathbb{A}(\mathcal{V}_4 \oplus \mathcal{V}_6)^* \times \mathbb{A}(\mathcal{L}^2 \oplus \mathcal{L}^3 \oplus \mathcal{O}_C) \) as the total space of a rank 3 vector bundle \( \mathcal{V} \) over \( \mathbb{A}(\mathcal{V}_4 \oplus \mathcal{V}_6)^* \times C \). View \( (x, y, z) \) in (8.3) as affine coordinates on \( \mathcal{V} \), then the \( \mu \)-action can be viewed as one on \( \mathcal{V} \) and defines a descent \( \mathcal{V}' \), which is a rank 3 vector bundle over \( \mathcal{P}_q(\mathcal{V}_4 \oplus \mathcal{V}_6) \times C \). Then \( \mathcal{Q}(\mu) \) is canonically isomorphic to the projectivization \( \mathbb{P}(\mathcal{V}) \). In particular, it is a \( \mathbb{P}^2 \)-bundle over \( \mathcal{P}_q(\mathcal{V}_4 \oplus \mathcal{V}_6) \times C \). Therefore, the pullback \( \psi^* \mathcal{Q}(\mu) \) is a \( \mathbb{P}^2 \)-bundle over the scheme \( T \times C \). Since \( \mathbb{G}_m \) is flat, the quotient stack \( \psi^* \mathcal{X} \) is algebraic ([60, Tag 06DC]). Being a closed substack of the scheme \( \psi^* \mathcal{Q}(\mu) \), it has to be a projective scheme. ☐

**Proposition 8.6.** The morphism \( \mathcal{X} \to S \) is smooth.

**Proof.** We may assume that \( S = \text{Spec}(k) \), where \( k \) is an algebraically closed field of characteristic \( \neq 2, 3 \). Let us simply write \( \mathcal{X} \) for \( \mathbb{A}(\mathcal{V}_4 \oplus \mathcal{V}_6)^* \). Choose a point \( u \in \mathbb{A}(k) \) and \( c \in C(k) \). Choose a uniformizer \( t \) of \( C \) at \( c \) and bases \( \{\sigma_i\}, \{\theta_j\} \) for \( \mathcal{V}_4, \mathcal{V}_6 \) respectively. Then the formal completion of \( \mathbb{A}^* \times C \) at \( (u, c) \) can be identified with \( \text{Spf}(R) \), where \( R = k[[t, \alpha_0, \cdots, \alpha_{4h-g}, \beta_0, \cdots, \beta_{6h-g}]] \).

By choosing a local \( \mathcal{O}_C \)-generator of \( L \) at \( c \), we turn \( \sigma_i \)'s and \( \theta_j \)'s into elements in \( k[t] \). Let \( \{a_i\}, \{b_j\} \in k^{10h-2g} \) be the affine coordinates of \( u \) in \( \mathbb{A}(\mathcal{V}_4 \oplus \mathcal{V}_6) \). Then the restriction of \( \mathcal{X} \) to \( \text{Spf}(R) \) can be identified with the subscheme of \( \text{Proj} R[t, y, z] \) defined by the equation

\[
W := y^2 z - x^3 + \sum_{i=0}^{4h-g} (a_i + \alpha_i)\sigma_i(t)xz^2 + \sum_{j=0}^{6h-g} (b_j + \beta_j)\theta_j(t)z^3 \quad (8.7)
\]

Let \( r \) be the special point of \( \text{Spf}(R) \). The singularity of the (generalized) elliptic curve \( \mathcal{X}' \) defined by the above equation when \( t, \alpha_i \)'s and \( \beta_j \)'s all vanish cannot appear on the \( z = 0 \) chart. So we may set \( z = 1 \) in
the above equation and consider the resulting scheme in Spec\(R[x, y]\). Since \(\theta_0(0) \neq 0\), we deduce that the partial derivative \(\partial W/\partial \beta_0\) remains nonzero on the special fiber. This implies that the total space of the restriction of \(\mathcal{X}\) to Spf\(R)\) is smooth. But the choice of \((u, c)\) is arbitrary, so \(\mathcal{X}\) is smooth.

**Definition 8.8.** Let \(\overline{D} \subseteq \mathcal{P}_{(4,6)}(\mathcal{V}_4 \oplus \mathcal{V}_6)\) be the reduced closed subscheme such that a geometric point on \(\mathcal{P}_{(4,6)}(\mathcal{V}_4 \oplus \mathcal{V}_6)\) lies in \(\overline{D}\) if and only if the fiber of \(\mathcal{X}\) over it is singular. Let the corresponding affine quasi-cone \(\mathbb{A}(\mathcal{V}_4 \oplus \mathcal{V}_6)\) be denoted by \(\mathcal{D}\).

**Remark 8.9.** \(\overline{D}\) is the natural discriminant locus to be considered for elliptic surfaces. However, unlike the usual discriminants in linear systems, \(\overline{D}\) naturally resides in a *weighted* projective space (or stack). This is because \(\overline{D}\) is fundamentally non-linear.

### 8.2 Nonlinear Bertini Theorems for Families of Elliptic Surfaces

**Proposition 8.10.** Let \(C\) be a smooth projective curve of genus \(g\) over \(k\) and let \(L\) be a line bundle on \(C\) with degree \(h\). Set \(V_r := H^0(L^r)\) for every \(r \in \mathbb{N}\). For every \(d \in \mathbb{N}\), consider the closed subset of \(\mathbb{A}(\mathcal{V}_4 \oplus \mathcal{V}_6) \times C\) defined by

\[
\mathcal{K}_d := \{(a_4, a_6, c) \in V_4 \times V_6 \times C \mid \text{val}_c(\Delta) \geq d\}
\]

and endow it with the reduced subscheme structure. Likewise, let \(D \subseteq C \times C\) be the diagonal and define a closed subscheme in \(\mathbb{A}(\mathcal{V}_4 \oplus \mathcal{V}_6) \times (C \times C - D)\) by

\[
\mathcal{K}^+_2 := \{(a_4, a_6, c) \in V_4 \times V_6 \times (C \times C - D) \mid \text{val}_c(\Delta) \text{ and val}_{c'}(\Delta) \text{ are both } \geq 2\}.
\]

If \(2h \geq g + 1\), then we have the following:

(a) \(\mathcal{K}_d\) has codimension \(d\) for \(d \leq 3\).

(b) \(\mathcal{K}_2\) has two irreducible components \(\mathcal{K}_2(I)\) and \(\mathcal{K}_2(\Pi)\) characterized by conditions \(\text{val}_c(a_6) = 0\) and \(\text{val}_c(a_6) \geq 1\) respectively.

(c) \(\mathcal{K}^+_2\) has codimension 4.

**Proof.** We will repeatedly use the following simple consequence of the Riemann-Roch theorem: For any line bundle \(M\) on \(C\), if \(\deg(M) \geq 2g - 1\), then \(h^0(M) = \deg(M) - g + 1\); if \(\deg(M) = 2g - 2\), then \(h^0(M) = \deg(M) - g + 1\) unless \(M \cong \omega_C\).

Fix any point \(c \in C\) and consider the projection \(\mathcal{K}_d \rightarrow C\). It suffices to show that the fiber \(\mathcal{K}_{d,c}\) over \(c\), viewed naturally as a closed subscheme of \(\mathbb{A}(\mathcal{V}_4 \oplus \mathcal{V}_6)\), has codimension \(d\). We identify the completion of \(C\) along \(c\) with \(\text{Spf}(k[[t]])\) by choosing a uniformizer \(t\). After choosing a local generator of \(L\), we may consider the Taylor series of any \(\sigma \in H^0(L^r)\), which is a power series \(\sigma(t) \in k[[t]]\). By the first paragraph, for \(r \geq 4\) and \(d \leq 3\), we have

\[
h^0(L^r((1 - d)c)) = h^0(L^r(-dc)) + 1.
\]

Therefore, we may choose a basis \(\sigma_0, \cdots, \sigma_{4h-g}\) for \(V_4\) such that \(\text{val}_c(\sigma_i) = 0\) for \(i = 0, 1, 2\) and \(\{\sigma_i\}_{3 \leq i \leq 4h-g}\) forms a basis for \(H^0(L^4(-3c))\). We may assume that \(\sigma_0(t) \equiv 1, \sigma_1(t) \equiv t\) and \(\sigma_2(t) \equiv t^2\) modulo \(t^3\). We choose a basis \(\{\theta_0, \cdots, \theta_{6h-g}\}\) in an entirely similar way.
With the given choices of bases, we use \( \{\alpha_i\} \) and \( \{\beta_j\} \) for the coordinates of \( V_4 \) and \( V_6 \) respectively, so that \( \Delta \) can be expressed as

\[
\Delta = 4 \left( \sum_{i=0}^{4h-g} \alpha_i \sigma_i \right)^3 - 27 \left( \sum_{j=0}^{6h-g} \beta_j \theta_j \right)^2.
\] (8.11)

Then the fiber \( K_{d,c} \) \((d \leq 3)\) is cut out in \( \mathbb{A}(V_4 \oplus V_6) \) by the first \( d \) equations from below:

\[
\begin{align*}
\Delta(0) &= 4\alpha_0^3 - 27\beta_0^2 \\
\Delta'(0) &= 3(4\alpha_0^2\alpha_1 - 18\beta_0\beta_1) \\
\Delta''(0) &= 24(\alpha_0^2\alpha_2 + \alpha_0^2\alpha_2) - 54(\beta_1^2 + 2\beta_0\beta_2)
\end{align*}
\] (8.12)

The statement (a) is clear for \( d = 0, 1 \). For \( d = 2 \), it is clear that \( K_2 \) contains the following subscheme

\[
K_2(\Pi) := \{(a_4, a_6, c) \in V_4 \times V_6 \times C \mid \text{val}_c(a_4, c) \geq 1, \text{val}_c(a_6) \geq 1\},
\]

such that the fiber of \( K_2(\Pi) \) over \( c \) is simply cut out by \( \alpha_0 = \beta_0 = 0 \). Let \( K^2 := \mathbb{A}(k(\sigma_0, \theta_0)) \) be the affine space with coordinates \((\alpha_0, \beta_0)\) and \( C^1 \subseteq K^2 \) be the cuspidal curve defined by \( \Delta(0) = 0 \). Then the fiber of \( K_2 - K_2(\Pi) \) over a point in \( C' - \{(0, 0)\} \) is given by a codimension 1 hyperplane in \( \mathbb{A}(k(\sigma_1, \beta_1)) \). This implies that \( K_2 - K_2(\Pi) \) is irreducible of codimension 2 in \( \mathbb{A}(V_4 \oplus V_6) \), and we denote this component by \( K_2(I_2) \). Note that this implies (b). To see the \( d = 3 \) case for (a), just note that \( \Delta''(0) \) does not vanish identically on both \( K_2(I_2) \) and \( K_2(\Pi) \).

Finally we treat (c). We consider the projection \( \Phi: K^2_2 \rightarrow (C \times C - D) \) and take a point \((c, c') \in (C \times C - D) \). Denote the fiber of \( \Phi \) over \((c, c')\) by \( \Phi_{(c, c')} \). We assume first that \( L^{4} \not\cong \omega_{C}(2c + 2c') \). This condition is automatically satisfied when \( 2h > g + 1 \) and ensures that \( h^0(L^{r}(-2c - 2c')) = rh + g - 5 \) for \( r \geq 4 \). Then we may choose \( \sigma_0, \cdots, \sigma_3 \in V_4 \) with the following vanishing orders:

\[
\begin{array}{c|ccccc}
\text{val}_c & \sigma_0 & \sigma_1 & \sigma_2 & \sigma_3 \\
\hline
\text{val}_c & 0 & 1 & \geq 2 & \geq 2 & \geq 2 \\
\end{array}
\] (8.13)

Then we complete \( \{\sigma_0, \cdots, \sigma_3\} \) to a basis \( \{\sigma_i\} \) of \( V_4 \) by adjoining a basis for \( \mathbb{H}^0(L^{4}(-2c - 2c')) \). Let \( t, s \) be uniformizers of the completions of \( C \) along \( c \) and \( c' \) respectively. After choosing local generators of \( L \), we may consider Taylor series \( \sigma_i(t) \in k[[t]] \) and \( \sigma_i(s) \in k[[s]] \), and assume that \( \sigma_0(t) \equiv 1, \sigma_1(t) \equiv t \mod t^2 \) and \( \sigma_2(s) \equiv 1, \sigma_3(s) \equiv s \mod s^2 \). Choose an entirely similar basis \( \{\theta_j\} \) for \( V_6 \) and express \( \Delta \) again as in (8.11). Then the defining equations for \( \Phi_{(c, c')} \) in \( \mathbb{A}(V_4 \oplus V_6) \) are

\[
\begin{align*}
4\alpha_0^3 - 27\beta_0^2 &= 4\alpha_3^2 - 27\beta_3^2 = 0 \\
3(4\alpha_0^2\alpha_1 - 18\beta_0\beta_1) &= 3(4\alpha_3^2\alpha_2 - 18\beta_2\beta_3) = 0
\end{align*}
\] (8.14)

By the same argument for the \( d = 2 \) case in (a), the above equations define a codimension 4 subscheme.

The point is that the variables with indices 0, 1 do not interfere with those with 2, 3.

It remains to deal with the case when \( 2h = g + 1 \) and \( L^{4} \cong \omega_{C}(2c + 2c') \). Note that in this case \( g \geq 1 \), so the condition \( L^{4} \cong \omega_{C}(2c + 2c') \) defines a closed subscheme of \( (C \times C - D) \) of codimension at least 1.

Therefore, it is enough to show that the codimension of \( \Phi_{(c, c')} \) is at least 3. Note that we are able to choose
a basis \( \{ \theta_j \} \) just as before, but this time choose \( \{ \sigma_0, \sigma_1, \sigma_2 \} \) with the following vanishing orders:

| \( \sigma_0 \) | \( \sigma_1 \) | \( \sigma_2 \) |
|---|---|---|
| val_\(c \) | 0 | 1 |
| val_\(c^* \) | 2 | 0 |

and complete it to a basis of \( V_4 \) by adjoining a basis of \( H^0(L^4(-2c-2c')) \). Assume that \( \sigma_0(t) = 1, \sigma_1(t) = t \mod t^3 \) and \( \sigma_2(s) = 1 \mod s \). Then the conditions \( \text{val}_c(\Delta) \geq 2 \) and \( \text{val}_{c^*}(\Delta) \geq 1 \) give us 3 equations which are necessarily satisfied by \( \Phi_{(c,c^*)} \):

\[
\begin{align*}
4\alpha_0^2 - 27\beta_0^2 &= 4\alpha_3^2 - 27\beta_3^2 = 0 \\
3(4\alpha_0^2\alpha_1 - 18\beta_0\beta_1) &= 0
\end{align*}
\]

(8.15)

It is clear that these indeed cut out a subscheme of codimension 3.

**Proposition 8.17.** Apply Construction 8.3 to \( S : = \text{Spec} (k) \). The resulting discriminant \( \mathfrak{D} \) is a proper subvariety of \( \mathbb{A}(V_4 \oplus V_6) \). If \( \text{codim} \mathfrak{D} = 1 \), then \( \mathfrak{D} \) has a unique irreducible component \( \mathfrak{D}_0 \) of maximal dimension; moreover, for a general point \( a \) on \( \mathfrak{D}_0 \), \( \tilde{X}_a \) is smooth away from a single ordinary double point.

**Proof.** Note that for \( a \in \mathbb{A}(V_4 \oplus V_6) \), \( \tilde{X}_a \) is singular if and only if it has a reducible singular fiber. By Prop. 8.2 it is clear that \( \mathfrak{D} \) is contained in the image of \( K_2 \), and hence has codimension at least 1. If \( \text{codim} \mathfrak{D} = 1 \), then by Prop. 8.10(c), there exists an open dense subset \( U \subseteq \mathfrak{D} \) such that if \( a \in U \), \( \tilde{X}_a \) has at most one singular fiber not of \( I_1 \)-type. If moreover this singular fiber is of \( II \)-type, then \( \tilde{X}_a \) is smooth and \( a \notin \mathfrak{D} \). Therefore, the only possible irreducible component of maximal dimension is the Zariski closure of the image of \( K_2(1_2) \). \( \square \)

### 8.3 Mod \( p \) Behavior of Discriminants

**Construction 8.18.** Suppose that \( O_S \) is a local ring, so that the vector bundles \( V_4 \) and \( V_6 \) are trivial \( O_S \)-bundles. By choosing \( O_S \)-generators for \( V_4 \) and \( V_6 \), we identify \( \mathbb{A}(V_4 \oplus V_6) \) with \( \mathbb{A}^{d_1} \oplus \mathbb{A}^{d_2} \), where \( d_i = 2hi - g + 1 \). Consider \( \mathbb{P}^1 = \text{Proj} O_S[u,v] \) and let \( \mathbb{A}^1 = \text{Spec} (O_S[u]) \) be the \( v = 1 \) chart on \( \mathbb{P}^1 \). Let \( W_r \) be the \( O_S \)-module of degree \( r \) homogenous polynomials in \( O_S[u,v] \) or equivalently the module of degree \( \leq r \) polynomials in \( O_S[u] \). Consider the open subscheme \( T \subseteq \mathbb{A}(V_4^{d_1} \oplus V_6^{d_2}) \) consisting of the points of the form

\[
\{(f_1, \cdots, f_{d_1}, g_1, \cdots, g_{d_2}) \mid \text{the common vanishing locus } V(\{ f_i, g_j \}) = \emptyset \}.
\]

Then it is clear that there is a natural morphism \( \mathbb{P}^1_S \times S T = \mathbb{P}^1_T \to \mathbb{P}(4,6) \). By setting \( v = 1 \) in the polynomials \( f_i \)’s and \( g_j \)’s, we also obtain an \( S \)-morphism \( \mathbb{A}^1_S \times S T = \mathbb{A}^1_T \to \mathbb{A}(V_4 \oplus V_6)^* \), which fits into a commutative diagram

\[
\begin{array}{ccc}
\mathbb{A}^1_T & \xrightarrow{\varphi} & \mathbb{A}(V_4 \oplus V_6)^* \\
\uparrow & & \downarrow \\
\mathbb{P}^1_T & \xrightarrow{\tilde{\varphi}} & \mathbb{P}(4,6)
\end{array}
\]

(8.19)

**Proposition 8.20.** When \( O_S = k \) for some algebraically closed field \( k \), the family \( \mathbb{A}^1_T/T \) strongly dominates \( \mathbb{A}(V_4 \oplus V_6)^* \) via \( \varphi \). Moreover, for a general point \( t \in T \), \( \tilde{\varphi}_t(\infty) \notin \mathfrak{D} \).
Proof. Let us denote $A(V_d^1 + V_d^2)$ temporarily by $T^+$ and $A(V_1 + V_0)$ by $A$. Note that $T^+ \cong A^{5d_1+7d_2}$, as it simply parametrizes the coefficients of $f_i$ and $g_j$'s. We also view $T^+$ as a $k$-vector space of dimension $5d_1+7d_2$.

There is a natural morphism $\varphi^+: A \to A$ which extends $\varphi$. With the chosen coordinates, we may naturally identify $T \times A$ with $A \times A$. Consider the natural morphism $\Gamma: A^2 \times_k T^+ \to T \times A$ defined by

$$\Gamma(u,(f_i,g_j)) = ((f_i(u),g_j(u)),(f'_i(u),g'_j(u))).$$

For any $u_0 \in A^1(k)$, the fiber $\Gamma_{u_0}$ is simply a morphism of vector spaces $T^+ \to k^2$. Moreover, it is clearly surjective, which readily implies that the morphism $\Gamma$ has equi-dimensional fibers. Then one readily checks that $A^{1+}$ strongly dominates $A$ via $\varphi^+$.

To see that $A^{1+}$ strongly dominates $A^*$ it suffices to show that for each point $(x,v) \in T_xA$, a general point $(f_i,g_j)$ in $T^*((x,v))$ satisfies the property that the common vanishing locus of (re-homogenized) $f_i$ and $g_j$'s on $\mathbb{P}^1$ is empty. One easily checks this by dimension counting.

For the second statement, consider the $\infty$-section $\sigma_{\infty}: T \to \mathbb{P}^1_k$ which sends every $t \in T$ to $\infty$. It suffices to show that the closed subscheme $\bar{\tau}^*_t(\mathcal{D})$ is not the entire $T$, for which we only need to construct a point on $T$ not in $\bar{\tau}^*_t(\mathcal{D})$. Let $t \in T$ be any point such that for some point $w \in A^1$, $\varphi_t(w) \notin \mathcal{D}$. Then we can always apply a linear change of variables (i.e., an automorphism of $\mathbb{P}^1$) to switch $w$ and $\infty$. This gives us another point $t' \in T$ and by construction $\varphi_t(\infty) \notin \mathcal{D}$.

Lemma 8.21. Suppose that $\mathcal{O}_S = W$ in Construction 8.18. For a general $k$-point $t \in T_k$, $\bar{\tau}_t$ has the following properties:

(a) $\bar{\tau}_t(\infty) \notin \mathcal{D}_k$, the total space of the family $\bar{\tau}_t(X) \to \mathbb{P}^1_k$ is smooth and every fiber has at most a single ODP singularity.

(b) $\bar{\tau}_t(\mathcal{D}_k)$ is reduced. Moreover, there exists a dense analytically open subset $Q$ of the disk of all $W$-liftings of $t$ on $T$ such that for any $\tilde{t} \in Q$, $\bar{\tau}_{\tilde{t}}(\mathcal{D})$ is finite étale over $S$.

Proof. If $\text{codim } \mathcal{D}_k \leq 2$, then for a general $t \in T(k)$, we in fact have $\varphi_t^*(\mathcal{D}_k) = \emptyset$ and the family $\bar{\tau}_t(X) \to \mathbb{P}^1_k$ is smooth. So we only need to treat the case when $\text{codim } \mathcal{D}_k = 1$. For a general $k$-point $t \in T_k$, (a) follows from Prop. 7.4 and Prop. 8.20, so we only need to prove (b).

Let $\mathcal{D}_k^0$ and $\mathcal{D}_k^0$ be the irreducible components of maximal dimension of $\mathcal{D}_k$ and $(\mathcal{D}_k)_{\text{red}}$ respectively. Let $\mathcal{D}_0, \ldots, \mathcal{D}_\ell$ be the irreducible components of $\mathcal{D}$ such that $\mathcal{D}_0$ is the component which contains $\mathcal{D}_k^0$. We claim that $\mathcal{D}_0$ contains $\mathcal{D}_k^0$ as well. Let us write $k(V_0 + V_0)$ as $\mathcal{A}_1$. Then $\mathcal{D}_{0,k} \subseteq \mathcal{A}_k^*$ is cut out by a single polynomial $F$ in the coordinates of the affine space $\mathcal{A}_K$. By minimally clearing denominators, we may assume that the coefficients of $F$ are defined in $W$ and generate $W$. Using the fact that $\mathcal{A}$ is affine and $\mathcal{O}_X$ is a UFD, one checks that the Zariski closure of $\mathcal{D}_{0,k}$ in $\mathcal{A}$ contains the vanishing locus $V(F)$ of $F \in \mathcal{O}\mathcal{A}$. Note that $F$ is weighted-homogeneous, so $V(F)_k \subseteq \mathcal{A}_k$ at least contains the origin. In particular, $V(F) \to S$ is surjective. By \[60\] Tag 0B2J, $\dim V(F)_k = \dim V(F)_K$. This implies that $\dim \mathcal{D}_{0,k} = \dim \mathcal{D}_{0,k}$. By the uniqueness of $\mathcal{D}_k^0$ as an irreducible component of maximal dimension, we conclude that $\mathcal{D}_k^0 \subseteq \mathcal{D}_0$. By applying \[60\] Tag 0B2J again, we also conclude that for any $i > 0$, $\mathcal{D}_{i,k}$ has codimension $\geq 2$ in $\mathcal{A}_k^*$.

Set $U \subseteq \mathcal{D}_0$ be the complement of the closed subscheme $\bigcup_{i>1} (\mathcal{D}_0 \cap \mathcal{D}_i)$. Then $U_{\text{red}}$ is dense in $\mathcal{D}_k^0$. As $t$ is general, we may assume that the intersection $\text{im}(\varphi_t) \cap (\mathcal{D}_k)_{\text{red}}$ is transverse and lies in $U_{\text{red}}$. For any $\tilde{t} \in T(W)$ lifting $t$, we claim that $\varphi_\tilde{t}^*(\mathcal{D})$ is flat over $W$. Indeed, note that $\mathcal{D}_0$ is a Weil divisor, and
hence also a Cartier divisor of $\mathcal{O}^*$, as $\mathcal{O}^*$ is regular. This implies that $Z := \varphi_1^*(\mathcal{O}) = \varphi_1^*(U) = \varphi_1^*(\mathcal{O}_0)$ is everywhere locally cut out in $\mathbb{P}^1_W$ by a single equation. Since $Z_k \subseteq \mathbb{P}^1_k$ is of codimension 1, $Z$ is flat over $W$ by [60, Tag 00MF].

The flatness of $Z/W$ implies that $Z_k$ is reduced if and only if we have an equality of number of points $\#Z_k(k) = \#Z_{\bar{K}}(\bar{K})$. Note that it suffices to check this for any $\bar{t}$ lifting $t$. By Lem. 7.8, we may choose $\bar{t}$ such that properties (a) and (b) are true for $\bar{t} \otimes \bar{K}$ as well. By combining the Leray spectral sequence and the Grothendieck-Ogg-Shafarevich formula, we have

$$\chi(\mathcal{F}_t(\mathcal{X})) = \chi(\mathcal{A}_{\mathcal{P}(\mathbb{P}^1)})\chi(\mathbb{P}^1_k) + \sum_{z \in Z_k(k)} \left( \chi(\mathcal{X}_z) - \chi(\mathcal{A}_{\mathcal{P}(\mathbb{P}^1)}) - \text{sw}_z(H^*_\text{et}(\mathcal{A}_{\mathcal{P}(\mathbb{P}^1)}, \mathbb{Q}_\ell)) \right) \tag{8.22}$$

where $\text{sw}_z(H^*_\text{et}(\mathcal{A}_{\mathcal{P}(\mathbb{P}^1)}, \mathbb{Q}_\ell))$ denotes the alternating sum of Swan conductors

$$\text{sw}_z(H^*_\text{et}(\mathcal{A}_{\mathcal{P}(\mathbb{P}^1)}, \mathbb{Q}_\ell)) := \sum_{i=0}^4 (-1)^i \text{sw}_z(H^*_\text{et}(\mathcal{A}_{\mathcal{P}(\mathbb{P}^1)}, \mathbb{Q}_\ell)).$$

Since for every $z \in Z_k(k)$, $\mathcal{X}_z$ is smooth away from an ODP, [20, §4.2] tells us that the local monodromy action on $H^*_\text{et}(\mathcal{A}_{\mathcal{P}(\mathbb{P}^1)}, \mathbb{Q}_\ell)$ factors through $\mathbb{Z}/2\mathbb{Z}$ and hence is tamely ramified as $p \neq 2$. This implies that the Swan conductors all vanish. Moreover, by loc. cit. we also know that $\chi(\mathcal{X}_z) - \chi(\mathcal{A}_{\mathcal{P}(\mathbb{P}^1)}) = -1$, so

$$\#Z_k(k) = \chi(\mathcal{A}_{\mathcal{P}(\mathbb{P}^1)}) - \chi(\mathcal{F}_t(\mathcal{X})). \tag{8.23}$$

The above is true verbatim when $t$ (resp. $k$) is replaced by $\bar{t} \otimes \bar{K}$ (resp. $\bar{K}$), and the RHS of the above equation remains unchanged by the smooth and proper base change theorem for étale cohomology. We may then conclude that $\#Z_k(k) = \#Z_{\bar{K}}(\bar{K})$ as desired. \qed

**Remark 8.24.** The statement (b) in the above lemma is in fact equivalent to the reducedness of $\mathcal{O}_{0,k}$. Note that the discussion of Swan conductors fundamentally uses the $p \neq 2$ assumption. This is irrelevant here because we are working with the $p \geq 5$ assumption such that Kodaira’s classification of singular fibers works verbatim as in characteristic 0. However, we refer the reader to the appendix for an example of what goes wrong in characterisitic 2.

### 8.4 Proof of Thm 1.4

Let $\mathcal{M}_{1,1}$ be the moduli stack of the pair of a genus 1 curve together with a degree 1 line bundle (i.e., an elliptic curve) and the universal family be denoted by $(\mathcal{E}, \mathcal{L})$. Let $S$ be the $\mathbb{Z}[1/6]$-scheme defined by

$$\{(a, b) \in \text{Spec}(\mathbb{Z}[\frac{1}{6}]) | 4a^3 - 27b^2 \neq 0\}.$$

Then the Weierstrass equation equips $S$ with a surjective morphism $S \to \mathcal{M}_{1,1}$. Apply Construction 8.18 to $S$ and denote the resulting $P_{(a, b)}(V_4 \oplus V_6)$ and $\mathcal{D}$ over $\mathcal{M}_{1,1}$ by $P$ and $D$ respectively, and set $M := P - D$. Let the family $\mathcal{X}$ over $P$ be denoted by $\mathcal{F}$ and its restriction over $M$ by $\mathcal{X}$.

Note that $\mathcal{X}$ is smooth over $M$ and for any algebraically closed field $k$ over $\mathbb{Z}[1/6]$ and elliptic surface $X$ over $k$ with $p_g = q = 1$, there exists a point $s \in P(k)$ such that $X$ is the minimal model of $\mathcal{F}_s$.

**Remark 8.25.** The point $z$ is not unique only due to the redundancy of $S$ in parametrizing elliptic curves, and we work with $S$ instead of $\mathcal{M}_{1,1}$ directly only because of the convenience of talking about schemes as
We need a lower bound on the rank of the Kodaira-Spencer map, or equivalently the image of the period morphism over \( \mathbb{C} \).

**Proposition 8.26.** For a general \( z \in M_\mathbb{C} \) and \( X := \mathcal{X}_z \), the Kodaira-Spencer map

\[
T_z M_\mathbb{C} \to \text{Hom}(H^1(\Omega^1_X), H^2(O_X))
\]

has rank at least 3.

*Proof.* This follows from a construction of Ikeda [29] that we now briefly recall. In loc. cit., Ikeda constructed a subfamily of elliptic surfaces over \( \mathbb{C} \) of genus 3 with rank at least \( p_g = q = 1 \) using bielliptic curves and bielliptic surfaces. Let \( \tilde{C} \) be a bielliptic curve of genus 3, equipped with an involution \( \sigma \) such that \( \tilde{C}/\sigma \) is a smooth genus 1 curve \( C \). On the symmetric square \( \tilde{C}^{(2)} \) of \( \tilde{C} \), \( \sigma \) also lifts to an involution \( \sigma^{(2)} \). Consider the surface \( Y' = \tilde{C}^{(2)}/\sigma^{(2)} \), which is shown to be a projective surface of Kodaira dimension 1 with 6 ODPs. Its minimal resolution \( Y \) is an elliptic surface with \( p_g = q = 1 \). By Prop. 2.9 in loc. cit., the morphism \( \tilde{C} \to C \) can be recovered from \( Y' \). Note however the Weierstrass model of \( Y' \) is singular, so \( Y \) is not given by a point on \( M_\mathbb{C} \).

By applying the Artin-Brieskorn resolution of singularities [4] to \( \mathcal{X}_{\mathcal{M}} \), we obtain a smooth and proper algebraic space \( \mathcal{X}_{\mathcal{M}} \to \mathcal{X}_{\mathcal{M}}, \) where \( \mathcal{M} \) is an algebraic space which admits a morphism to \( \mathcal{P}_\mathbb{C} \) which is bijective on geometric points. Moreover, since the fibers of \( \mathcal{X}_{\mathcal{M}} \) have at most rational double point singularities [30, Thm 1], [4, Thm 2] tells us that for any \( \mathcal{C} \)-point \( z \) of \( \mathcal{P}_\mathbb{C}^{\#} \) which maps to a point \( s \) of \( \mathcal{P}_\mathbb{C} \), the Henselianization (and hence also the completion) of \( \mathcal{P}_\mathbb{C}^{\#} \) at \( z \) maps surjectively to that of \( \mathcal{P}_\mathbb{C} \) at \( s \). Let \( \mathcal{P}_\mathbb{C}^{\#} \) be a resolution of singularities of \( \mathcal{P}_\mathbb{C}^{\#} \) and pullback the family \( \mathcal{X}_{\mathcal{M}} \) to \( \mathcal{P}_\mathbb{C}^{\#} \).

Now let \( \mathcal{B} \) be the moduli space of bielliptic curves of genus 3 over \( \mathbb{C} \). Then there is a family of elliptic surfaces \( \mathcal{X} \) over \( \mathcal{B} \). Let \( \Omega \) be the period domain parametrizing Hodge structures of K3-type on the integral lattice \( \Lambda \) given by the singular cohomology of any complex elliptic surface with \( p_g = q = 1 \). Let \( \tilde{\mathcal{B}} \) be the universal cover of \( \mathcal{B} \). Then there is a period morphism \( \tilde{\mathcal{B}} \to \Omega \) which is well defined up to \( \text{O}(\Lambda) \). In [29, §5], the author showed that the period image has dimension at least 3.

One quickly obtains the conclusion using the observation that the period image of the universal cover of \( \mathcal{P}_\mathbb{C}^{\#} \) contains the period image of \( \tilde{\mathcal{B}} \) up to the action of \( \text{O}(\Lambda) \) and that \( M_\mathbb{C} \) is connected. Indeed, the pullback of \( M_\mathbb{C} \) to \( \mathcal{P}_\mathbb{C}^{\#} \) is open and dense. By a continuity argument, we know that the period image of \( M_\mathbb{C} \) has the same dimension as that of \( \mathcal{P}_\mathbb{C}^{\#} \).

**Remark 8.27.** In a recent work [59], Shepherd-Barron proved a generic local Torelli theorem (i.e., the injectivity of the differential of the period morphism) for elliptic surfaces with \( 4p_g \geq 5(q-1) \) and \( p_g \geq q+3 \). The latter condition is not satisfied in our case, so we are not able to use his results. Saito’s main theorem in [54] does apply to our case. However, Ikeda pointed out a gap in Saito’s work and in fact the main results of [29] contradicts Saito’s theorem. We remark that Shepherd-Barron restricted to considering general elliptic surfaces (i.e., all singular fibers are nodal), whereas Ikeda’s construction cannot give general elliptic surfaces. Therefore, the generic local Torelli theorem might still be true in our case.

We are now ready to prove Thm 1.4.

*Proof.* Let \( X \) be any elliptic surface with \( p_g = q = 1 \) defined over a field. Let \( f \) and \( s \) be the elements in \( NS(X) \) given by a fiber and the zero section respectively. Then the quadratic lattice \( \mathbb{Z}\langle f, s \rangle \) generated by \( f \)
and s is determined by \((f, f) = 0, (f, s) = 1\) and \((s, s) = -1\). It is clear that the family \(X \rightarrow M\) is polarized by the lattice \(\mathbb{Z}(f, s)\), whose discriminant is prime to \(p\) for any \(p \geq 5\). Let \(V_{\ell}\) be the \(\mathbb{Q}_{\ell}\)-local system \(\text{PH}^2(M \otimes_{\mathbb{Z}} \mathbb{Q}_{\ell})\). By Thm 6.7, it suffices to show that for some (and hence any) geometric point \(z\) on \(M_{\mathbb{Q}_{\ell}}\), \(\text{Mon}^s(V_{\ell}|_{M_{\mathbb{Q}_{\ell}}}, z)\) does not fix any element of \(\text{PH}^2(X_{\mathbb{Q}}, \mathbb{Q}_{\ell})\) that does not come from the specialization of an element of \(\text{PNS}(X_{\mathbb{Q}}(M))\).

Let \(k\) denote \(\mathbb{P}_p\) and write \(W\) for \(W(k)\). Choose an embedding \(W \hookrightarrow \mathbb{C}\). Let \(V\) denote the \(\mathbb{Q}\)-VHS given by \(\text{PH}^2(X|_{M_{\mathbb{C}}}, \mathbb{Q})\). Let \(s \in M_k\) by any \(k\)-point. By Lem. 6.6 and Thm 6.7 there exists a \(W\)-point such that the restriction of \(V\) to the fiber \(M_{k \otimes \mathbb{C}}\) is non-isotrivial (or equivalently, the restriction of \(X\) to \(M_{k \otimes \mathbb{C}}\) satisfies \(\langle \bigotimes \rangle\)) and \(M_{k \otimes \mathbb{C}}\) does not lie in the Noether-Lefschetz loci of \(M_{\mathbb{C}}\).

We now restrict our attention to \((C, L) := (\mathcal{X}, \mathcal{L})\) over \(W\). Apply construction 8.18 to \((C, L)\) with \(S = \text{Spec}(W)\) and let \(T, \varphi, \mathcal{H}(\mathcal{V}_1 \oplus \mathcal{V}_6)^*\) be as in (8.19). Writ \(\mathcal{D} \subseteq \mathcal{H}^*\) be \(\text{Disc}(\mathcal{X}|_{\mathcal{H}^*})\). Let \(t \in T_k\) be a \(k\)-point such that \(\varphi_t(\mathcal{L})\) satisfies the conclusion of Lem. 8.21. In particular, we may choose a \(W\)-lifting \(\tilde{t}\) of \(t\) such that \(\varphi_{\tilde{t}}(\mathcal{L})\) is finite étale over \(W\). Since the choice of such \(\tilde{t}\) is open dense in the set of all \(W\)-liftings of \(t\), Lem. 7.8 allows us to choose \(\tilde{t}\) such that, after we choose an embedding \(W \hookrightarrow \mathbb{C}\), the image \(\varphi_{\tilde{t}}(\mathcal{P}_1)\) has the following properties: The restriction of \(\mathcal{X}\) to \(\mathcal{U}_C := \varphi_{\tilde{t}}(\mathcal{P}_1) - \mathcal{D}_C\) satisfies \(\langle \bigotimes \rangle\) and \(\mathcal{U}_C\) does not lie in the Noether-Lefschetz loci of \(M_C\). Now we conclude by Lem. 6.9 and Thm 6.7.

9 Examples of General or Fano Type

9.1 Surfaces with \(p_g = K^2 = 1\) and \(q = 0\)

In this section, we consider algebraic surfaces of general type with invariants \(p_g = K^2 = 1\) and irregularity \(q = 0\) over an algebraic closed field of characteristic 0 or \(p > 3\). We recall the construction of such surfaces and their period mapping.

Over complex number field, such surfaces are classified in [13, 62]. The upshot is that the canonical model of such a surface \(S\) can be realized as a \((6, 6)\) complete intersection in the weighted projective space \(\mathbb{P}(1, 2, 2, 3, 3)\). This canonical model has at most rational double points as singularities. Let \((x_0, x_1, x_2, x_3, x_4)\) be the coordinates of \(\mathbb{P} := \mathbb{P}(1, 2, 2, 3, 3)\). As proved in [13, Prop. 1.8], the canonical model can be given by two equations of the following form:

\[
x_3^2 + x_0x_4\alpha(x_0^2, x_1, x_2) + F(x_0^2, x_1, x_2) = 0
\]

(9.1)

where \(\alpha\) is a linear form, and \(F\) is a cubic form. For generic choice of two equations as above, the resulting canonical model is smooth and hence isomorphic to \(S\). For convenience, we introduce the following definition.

**Definition 9.2.** A smooth projective surface \(S\) of general type is called admissible if its canonical bundle is ample. Equivalently, the canonical model of \(S\) is smooth and isomorphic to \(S\).

In characteristic \(p > 3\), the classification result still holds. In fact, the only ingredient in [13] that requires characteristic 0 is [10], which states that the linear series \(|4K_S|\) is base point free. This is shown to hold in positive characteristic by [24].

\[\text{We expect that PNS}(X_{\mathbb{Q}}(M)) = 0.\]
Note the following facts over $\mathbb{C}$: first, these surfaces are known to be simply connected $\mathbb{C}$, hence the second integral cohomology group is torsion-free. Also, the second Betti number

$$b_2 = 22 - 8q - K^2 = 21.$$

The primitive Hodge structure is polarized by $K$, hence the intersection pairing has discriminant 1.

**Lemma 9.3.** Let $Q = (q_0, \cdots, q_r)$ be a tuple of natural numbers such that $\{q_i : 0 \leq i \leq r\} = \{1, 2, 3\}$. Then $\mathcal{O}(6)$ is a very ample line bundle on the weighted projective space $\mathbb{P}(Q)$.

**Proof.** By [7, Thm 4B.7(c)], it suffices to show that for any positive integer $h$, and every tuple of natural numbers $(A, B, C)$ such that $A + 2B + 3C = 6 + 6h$, there exist a tuple of natural numbers $(a, b, c)$ such that $a \leq A$, $b \leq B$, $c \leq C$ and $a + 2b + 3c = 6h$.

We first show this for $h = 1$. If we have $A \geq 6$, $B \geq 3$, or $C \geq 2$, then $(a, b, c)$ certainly exists. Now suppose $A < 6$, $B < 3$, and $C < 2$. However, in that case $A + 2B + 3C \leq 5 + 2 \cdot 2 + 3 = 12$, so that we must have $(A, B, C) = (5, 2, 1)$. Then we may take $(a, b, c)$ to be, e.g., $(2, 2, 0)$. Now the lemma follows from a simple inductive argument. \hfill $\Box$

Now we view the weighted projective space $\mathbb{P}$ as being defined over $\text{Spec}(\mathbb{Z}/[1/6])$. Consider the (relative) complete linear system $\mathcal{O}_\mathbb{P}(6)$, which is a projective space over $S$. Note that by the preceeding lemma, $\mathcal{O}_\mathbb{P}(6)$ is very ample.

**Proposition 9.4.** Let $k$ be an algebraically closed field of characteristic $p > 3$. Let $Y \subseteq \mathbb{P}_k$ be a proper smooth hypersurface. Then embedding $\iota : Y \hookrightarrow \mathbb{P} := |\mathcal{O}_\mathbb{P}(6)|$ is a Lefschetz embedding.

The definition of a Lefschetz embedding is taken from [27, §XVII, 2.3]. The proposition is an analogue of Thm 2.5.1 in loc. cit. for the particular weighted projective space of our interest. The proof is essentially the same, except that one has to pay attention to the degree of variables.

**Proof.** Since the coordinate $x_0$ has degree 1, the $x_0 \neq 0$ locus of $\mathbb{P}_k$ is an affine chart isomorphic to $\mathbb{A}^4_k$ with coordinates $y_i := x_i/x_0^{\deg x_i}$. We view the affine variable $y_i$ as having degree $\deg x_i$. Let $P$ be a point on $Y$ and assume that the projection of $\mathbb{A}^4_k$ to the $y_4$-axis is etale at $P$. Up to a translation, we further assume that $y_i = 0$ at $P$ for $i = 1, 2, 3$. By Prop. 3.3 and Cor. 3.5.0 in loc. cit., it suffices to show that for some $H \in |\mathcal{O}_{\mathbb{P}_k}(6)|$, the intersection $H \cap Y$ has an ODP at $P$. To this end, it suffices to present a polynomial in variables $y_1, y_2, y_3$ of weighted degree $\leq 6$ such that its vanishing locus has an odp singularity at the origin, for which one may simply take $y_1^2 + y_1y_2 = 0$. The role of $y_4$ could be replaced by any $y_i$'s, since we have $\deg y_i + \deg y_j \leq 6$ for any $1 \leq i, j \leq 4$. \hfill $\Box$

Let $B \subseteq \mathbb{A}^{13}$ be the open subscheme which parametrizes $(\alpha, F)$ such that the hypersurface defined by $\alpha$ has isolated singularities. Let $\mathcal{P}$ be the trivial projective bundle $|\mathcal{O}_\mathbb{P}(6)| \times B$ over $B$. Let $M \subseteq \mathcal{P}$ be the open subscheme consisting of points $(H, H_0) \in |\mathcal{O}_\mathbb{P}(6)| \times B$ such that $H \cap H_0$ is non-singular. Let $\mathcal{M} \rightarrow \mathcal{P}$ be the natural family given by $H \cap H_0$. Then every admissible surface of general type with $p_g = K^2 = 1$ and $q = 0$ defined over an algebraically closed field of characteristic $p > 3$ can be found as the fiber over a geometric point on $M$. Let $\mathcal{D}$ be the complement of $M$ in $\mathcal{P}$, equipped with a reduced scheme structure, i.e., $\text{Disc}(\mathcal{M} / \mathcal{P})$.

Over $\mathbb{C}$, the second primitive cohomology is polarized by $K$. As the intersection pairing on the second singular cohomology is unimodular, and $K^2 = 1$, we know the discriminant of the second primitive Betti
cohomology is 1. Also since the period morphism is dominant to the period domain, we know that for a very general surface, its primitive cohomology does not contain any nonzero Hodge classes. Hence there are no more line bundles on a very general surface other than the polarization. Therefore, we polarize the family \( \mathcal{X}/M \) only by the canonical bundle and let \( \nabla_t \) denote \( \text{PH}_0^2(\mathcal{X}/M, \mathbb{Q}_\ell) \).

**Proposition 9.5.** Let \( k \) be an algebraically closed field of characteristic \( p > 3 \). Let \( s \) be any \( k \)-point on \( B_k \) which corresponds to a smooth degree 6 hypersurface \( H_0 \) in \( \mathbb{P}_k \).

(a) For a general \( k \)-point \( t \) on \( D_s \), the fiber \( \overline{\mathcal{X}}_t \) is smooth away from an ODP. Moreover, the fiber \( D_s \) is generically reduced.

(b) There does not exist a finite connected étale cover over \( M_s \) such that the pullback of \( \nabla_t \) has a nonzero global section.

**Proof.** (a) The first statement is a straightforward consequence of Prop. 9.4. For the second statement, it suffices to show that the degree of the hypersurface \((D_s)_{\text{red}} \) in the projective space \(|O_{\mathbb{P}_k}(6)| \) is independent of \( s \). Let \( H_0 \) be the degree 6 hypersurface of \( \mathbb{P}_k \) given by \( s \). Let \( \gamma : \mathbb{P}_k^1 \hookrightarrow |O_{\mathbb{P}_k}(6)| \) be a general pencil, which by Prop. 9.4 is necessarily a Lefschetz pencil for \( H_0 \) in the sense of [27, §XVII, 2.2]. Then \( d_s := \deg(D_s)_{\text{red}} \) is simply the number of singular fibers on \( \overline{\mathcal{X}}_t \). Note that the total space of \( \overline{\mathcal{X}}_t \) is smooth and every singular fiber is smooth away from an ODP. Let \( \tilde{\eta} \) be the geometric generic point of \( \mathbb{P}_k^1 \). We apply the key argument for Lem. 8.21 again: By the Leray spectral sequence and the Grothendieck-Ogg-Shafarevich formula, we have

\[
d_s = \chi(\overline{\mathcal{X}}_t) - \chi(\mathbb{P}^1)\chi(\overline{\mathcal{X}}_\tilde{\eta}) = \chi(\mathbb{P}^1)\chi(\overline{\mathcal{X}}_\tilde{\eta})
\]

as the total dimension of vanishing cycles of each singular fiber is 1. By a simple deformation argument, one sees that the RHS of the above equation is independent of \( s \), and hence so is the LHS.

(b) First, we argue that for a general \( \mathbb{C} \)-point \( b \in B_C \), the period morphism does not contract the fiber \( M_s \). This follows from a dimension count. By [13, Thm 3.3], the period morphism is dominant to the 18-dimensional period domain. As \( B_C \) is 13 dimensional, we conclude that the period morphism does not contract the generic fiber.

Next, we take \( s \) to be a \( k \)-point on \( M \). Let \( W := W(k) \) and choose an embedding \( W \hookrightarrow \mathbb{C} \). By Lem. 7.7, we can find a \( W \)-lifting \( \tilde{s} \) of such that the fiber \( M_{\tilde{s} \otimes \mathbb{C}} \) is not contracted by the period morphism and does not lie in the Noether-Lefschetz loci. Let \( T \) be the Grassmanian of lines in \(|O_{W}(6)| \) and \( G \) be the universal family of \( T \). Then we have a natural morphism \( \varphi : G \rightarrow |O_{\mathbb{P}_k}(6)| \). Let \( t \in T_k \) be a general \( k \)-point and again let \( \gamma : \mathbb{P}_k^1 \hookrightarrow |O_{\mathbb{P}_k}(6)| \) be the resulting pencil. By Lem. 7.8, we may choose a \( W \)-lifting \( \tilde{t} \) of \( t \) such that the \( \mathbb{C} \)-fiber of the resulting \( \tilde{\gamma} : \mathbb{P}^1_{\tilde{t}} \rightarrow |O_{\mathbb{P}_k}(6)| \) is not contracted by the period morphism and is not contained in the Noether-Lefschetz loci. Now we conclude by Lem. 6.6.

Now Thm 17.5 is a direct consequence of the proposition above and Thm 6.7.

### 9.2 Gushel-Mukai Varieties of Dimension 4 and 6

We take the following definition of Gushel-Mukai varieties over an arbitrary base, which is a natural extension of the corresponding definition over \( \mathbb{C} \) (cf. [17, §1], [23, Def. 2.2]).

**Definition 9.7.** Let \( S \) be a \( \mathbb{Z}[1/2] \)-scheme and \( n \in \{3, 4, 5, 6\} \). Let \( \mathcal{V} := O_S^0 \). Let \( \text{CGr}(2, \mathcal{V}) \subseteq \mathbb{P}(O_S \oplus \wedge^2 \mathcal{V}) \) be the cone over \( \text{Gr}(2, \mathcal{V}) \), viewed as embedded into \( \mathbb{P}(\wedge^2 \mathcal{V}) \) via the Plücker embedding. For simplicity, we
Let \( \mathcal{F} := \mathcal{O}_S \oplus \wedge^2 \mathcal{V} \). Let \( \mathcal{H} \subseteq \mathcal{F} \) be a submodule which is locally a direct summand of rank \( n + 5 \), and let \( \mathcal{Q} \subseteq \mathbb{P}(\mathcal{F}) \) be a relative quadric. Note that when \( n = 6 \), we necessarily have \( \mathcal{H} = \mathcal{F} \).

We say that a smooth projective family of varieties \( f : \mathcal{X} \to S \) is a relative Gushel-Mukai variety of dimension \( n \) if it is obtained as the intersection \( \text{CGr}(2, \mathcal{V}) \cap \mathbb{P}(\mathcal{H}) \cap \mathbb{P}(\mathcal{Q}) \) for some \( \mathcal{H} \) and \( \mathcal{Q} \).

From Def. 9.7, it is easy to construct a natural parameter space \( M \) for Gushel-Mukai varieties over \( S \). Let \( B := \text{Gr}(n + 5, \mathcal{F}) \) be the Grassmannian parametrizing direct summands of rank \( n + 5 \) in \( \mathcal{F} \) and let \( |\mathcal{O}_{\mathcal{F}(F)}(2)| \) be the complete linear system parametrizing quadrics in \( \mathbb{P}(\mathcal{F}) \). Let \( \mathcal{P} := |\mathcal{O}_{\mathcal{F}(F)}(2)| \times_S B \). We view \( \mathcal{P} \) primarily as a \( B \)-scheme with fiber \( |\mathcal{O}_{\mathcal{F}(F)}(2)| \). Let \( \mathcal{F} \subseteq \mathcal{P} \times_S \mathbb{P}(\mathcal{F}) \) be the intersection of the trivial family \( \mathcal{P} \times_S \text{CGr}(2, \mathcal{V}) \) with the restrictions of the universal families over \( B \) and \( |\mathcal{O}_{\mathcal{F}(F)}(2)| \) over \( \mathcal{P} \). Take \( M \subseteq \mathcal{P} \) to be the complement of the discriminant locus of \( \mathcal{F} \to \mathcal{P} \) and let the restriction of \( \mathcal{F} \) over \( M \) by \( \mathcal{X} \).

Note that \( \mathcal{X} \) is naturally polarized by the restriction of the relative line bundle \( \mathcal{O}_{\mathcal{F}(F)}(1) \) restricted to \( M \). It has degree 10 for any \( n \). For \( n = 4, 6 \), let \( L_n \) be the integral lattice \( \text{PH}^n(X, \mathbb{Z}) \) given by the vanishing cohomology of the Gushel-Mukai variety of dimension \( n \), i.e. the orthogonal of \( \gamma(H^n(\text{Gr}(2, \mathcal{V}), \mathbb{Z})) \subset H^n(X, \mathbb{Z}) \). The intersection matrix of Schubert cycles \( \sigma_1, \sigma_2 \) is given by \( \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix} \), of discriminant 4.

Since the Poincaré pairing on the middle cohomology \( H^n(X, \mathbb{Z}) \) is unimodular, \( L_4 \) also has discriminant 4. The discriminant when \( n = 6 \) is also 4 and can be computed similarly. Moreover, this lattice is torsion free, see [18, Prop. 3.9].

Let \( \mathcal{V}_B := \text{PH}^n(\mathcal{X}/M, \mathbb{Q}) \) and \( \mathcal{V}_\ell := \text{PH}^n(\mathcal{X}/M, \mathbb{Q}_\ell) \).

**Lemma 9.8.** The period morphism defined by the \( \mathbb{Q} \)-VHS on \( \mathcal{V}_B \) over \( M_C \) is dominant. For a general \( b \in B_C \), the period morphism does not contract the fiber \( M_b \).

**Proof.** The fact that the period morphism is dominant can be seen in [18]. Note that \( B_C \cong \text{Gr}(n + 5, 11) \), hence is of dimension 18 when \( n = 4 \) and is a point when \( n = 6 \). On the other hand, the period domain is 20 dimensional. So a generic fiber cannot be contracted by the period morphism.

**Proposition 9.9.** Let \( k \) be an algebraically closed field of characteristic \( p \geq 3 \). There does not exist a connected étale cover over \( M_k \) such that the pullback of \( \mathcal{V}_\ell \) has a nonzero global section.

**Proof.** Let \( b \in B_k \) be a \( k \)-point such that the corresponding rank \( n + 4 \) subspace \( \mathbb{P}(\mathcal{H}_b) \) of \( \mathbb{P}(\mathcal{F}_b) \) has smooth intersection with \( \text{CGr}(2, \mathcal{V}_b) \). Denote this intersection by \( Y \).

Fix an embedding \( W := W(k) \hookrightarrow \mathbb{C} \). Choose a \( W \)-point \( \tilde{b} \) on \( B_W \) such that the restriction of the \( \mathbb{Q} \)-VHS \( \mathcal{V} \) on \( M_{\mathbb{C}} \) is non-isotrivial, and \( M_{\mathbb{C}} \) does not lie in the Noether-Lefschetz loci. Let \( \mathcal{Y} \subseteq \mathbb{P}(\mathcal{F}_W) \) be the lifting of \( Y \) over \( W \) induced by \( \tilde{b} \). Let the restriction of \( \mathcal{O}_{\mathcal{F}_W}(2) \) to \( \mathcal{Y} \) (resp. \( Y \)) be denoted by \( \mathcal{O}_{\mathcal{Y}}(2) \) (resp. \( \mathcal{O}_Y(2) \)). Let \( \mathcal{D} \subseteq |\mathcal{O}_{\mathcal{Y}}(2)| \) be the discriminant variety defined by the universal hypersurface. Then the open subscheme \( M_{\mathbb{C}} \subseteq \mathcal{D} \) is the complement of \( \mathcal{D} \).

Let \( T \) be the relative Grassmannian of lines in \( |\mathcal{O}_{\mathcal{Y}}(2)| \) and let \( \mathcal{C} \) be the universal family over \( T \). Then we have a natural morphism \( \varphi : \mathcal{C} \to |\mathcal{O}_{\mathcal{Y}}(2)| \). By [27, §XVII Thm 2.5.1], the embedding \( \mathcal{Y} \to |\mathcal{O}_{\mathcal{Y}}(2)| \) is a Lefschetz embedding. Therefore, for a general \( k \)-point \( t \in T_k \), the fiber \( \varphi_t : \mathcal{C}_t \cong \mathbb{P}_k \to |\mathcal{O}_{\mathcal{Y}}(2)| \) defines a Lefschetz pencil. Choose a \( W \)-lifting \( \tilde{t} \) of \( t \) such that the \( C \)-fiber of \( \varphi_{\tilde{t}} : \mathbb{P}_{\tilde{t}}^1 \to |\mathcal{O}_{\mathcal{Y}}(2)| \) is also a Lefschetz pencil, is not contracted by the period morphism, and does not lie in the Noether-Lefschetz loci. By the same argument as in Lem. 6.21, the intersection \( \mathbb{P}_{\tilde{t}}^1 \cap \mathcal{D} \) is étale over \( W \), and the restriction of \( \mathcal{V}_{\tilde{t}} \) to \( \mathbb{P}_{\tilde{t}}^1 \setminus (\mathbb{P}_{\tilde{t}}^1 \cap \mathcal{D}) \) is tamely ramified. We may now conclude by Lem. 6.6.
For complex Gushel-Mukai fourfolds and sixfolds, the Hodge conjecture is proved. Here we refer to \[52\] Prop. 8.2, Cor. 8.4] and the reference therein.

**Proposition 9.10.** The Hodge conjecture holds for complex Gushel-Mukai fourfolds and sixfolds.

*Proof of Thm 1.6.* This follows from Thm 6.7 and Prop. 9.9. Set \[n = 2, 3\]. We only explain the necessary adaptations. First of all, one always uses \(\text{PH}^{2n}(-, -(n))\) of 2\(n\)-dimensional Gushel-Mukai’s to take the role of \(\text{PH}^{2}(-, -(1))\) of \(h^{2.0} = 1\) varieties.

Note that \(V\) is of even rank, but we can no longer apply the trick of taking Hilbert squares to add in a trivial direct summand to \(\text{PH}^{2n}(\mathcal{X}|_{\mathcal{M}_{C}}, \mathbb{Q})\) as in \[52\]. Luckily, for a Gushel-Mukai varieties \(X\) of dimension \(2n\) \((n = 2, 3)\), one deduces directly that \(\text{p}^{2n}(X)\) is an object of \(\text{Mot}^{\text{Ab}}(\mathbb{C})\) from André’s theorem \[2\] Thm 1.5.1]. Let \(G := \text{SO}(L_{2n} \otimes \mathbb{Z}_{p})\), \(K_p := G(\mathbb{Z}_p)\) and \(K^p \subseteq G(\mathbb{A}_f^p)\) be a neat compact open subgroup of \(\text{SO}(L_{2n} \otimes \mathbb{Z}_{p})\). Then for \(K := K^p \rho\), one constructs a period morphism \(\rho : M \otimes \mathbb{Z}_p \to \mathcal{J}_K(G)\) as in \[52\].

Note that for a Hodge-generic \(C\)-point \(s \in \mathcal{M}_{C}\), \(\text{PH}^{2n}(\mathcal{X}_s, \mathbb{Q}(n))\) has no Hodge classes, so the natural analogue of the condition \(\text{NS}(\mathcal{X}_s, \mathbb{Q}) = \text{NS}(\mathcal{X}_s)\) in Thm 6.7 holds. The proof of Thm 6.7 now goes through verbatim if we let the Hodge conjecture for Gushel-Mukai’s to take the role played by the Lefschetz (1, 1)-theorem.

\[\blacksquare\]

10 Further Remarks

**Comparison to Previous Work**  The method of proving the Tate conjecture for K3’s over finite fields by reducing to the Lefschetz (1, 1)-theorem via the Kuga-Satake construction first appeared in \[49\] and was subsequently extended by \[50\]. The basic idea of both papers is that for ordinary or more generally finite height K3’s, by an appropriate (canonical or quasi-canonical) lifting, both (⋆) and (⋆′) (in the sketch of proofs) can be achieved. This method was not able to treat the supersingular case partially because it is impossible to lift line bundles on a supersingular K3 simultaneously. By using canonical integral models of Shimura varieties, Madapusi-Pera was able to prove (⋆′) by lifting one divisor at each time, so that even the supersingular case can be treated uniformly. Our method relies crucially on lifting one divisor at each time—even for ordinary \(h^{2.0} = 1\) varieties, we do not expect that simultaneously lifting all line bundles is always possible, because the period image may not have big enough dimension.

Our idea of studying monodromy by looking at certain curves in moduli spaces was initially inspired by Lyons’ thesis \[39\]. Lyons proved that a class of surfaces discovered by Catanese-Ciliberto has the biggest possible monodromy group (the full orthogonal group). However, unlike Lyons, we do not make use of global Picard-Lefschetz theory. This is also not possible for elliptic surfaces for reasons explained below. We also remark that Lyons also made use of the signature of Betti cohomology towards the end of his proof of Thm A. This argument, just like ours involving \[2\], is fundamentally transcendental. However, note that \[39\] and \[47\] both use monodromy to prove that the Kuga-Satake construction is motivated, whereas we use monodromy primarily to lift line bundles (once Kuga-Satake is set up).

Finally, we discuss how our approach is compared to that of the recent paper \[25\] by Fu and Moonen. The key difference is that they do solve the “local Schottky problem”. In particular, they showed that there are no global vector fields on these varieties in characteristic \(p\), thereby showing that the integral period morphism is smooth. By contrast, our argument does not depend on the integral period morphism having good local properties (cf. Rmk \[6.10\]). It also does not use crystalline cohomology, so we may circumvent certain small characteristic subtleties and treat the \(p = 3\) case. However, the smoothness of the integral
period morphism that Fu and Moonen proved is very strong and has many potential arithmetic applications beyond the Tate conjecture.

**Influence of Birational Type** For the examples of general or Fano type considered by us or Lyons, the variety \( X \) in question is naturally a divisor in another variety \( Y \) of one-dimension bigger. Moreover, this divisor is ample. This ampleness was important for Lyons to apply global Picard-Lefschetz theory and the hard Lefschetz theorem, and for us to use Katz’ criterion for Lefschetz embeddings. In these examples, the ampleness of \( O_Y(X) \) is related to that of the canonical bundle (or its dual) of \( X \). It would be interesting to investigate if this is a general phenomenon for \( h^{2,0} = 1 \) varieties of general or Fano type.

For elliptic surfaces the picture is very different. Let \( X \) be an elliptic surface over a base curve \( C \) with fundamental line bundle \( L \). The natural threefold to be considered here is the \( \mathbb{P}^2 \)-bundle \( \mathbb{P}(L^2 \oplus L^3 \oplus O_C) \) over \( C \). Our analysis of the monodromy works under the assumption that \( \deg(L) \) is sufficiently big with respect to \( g(C) \), i.e., \( L \) is sufficiently positive. However, \( X \) is not an ample divisor of \( Y \). Let us view \( C \) as a curve in \( Y \) via the zero section. Then there is a relation

\[
X.C = \deg O_Y(X)|_C = -3 \deg(L).
\]

(10.1)

To see this, note that \( Y \cong \mathbb{P}(L^{-1} \oplus O_C \oplus L^{-3}) \), and under this isomorphism, \( C \) is identified with the section \([0, 1, 0]\). Now the equation \([8.1]\) is homogeneous of degree \(-3\), hence the computation. As a consequence, the line bundle \( O_Y(X) \) is not only not ample, it is in fact further from being ample as \( L \) becomes more positive. This partially explains why the argument for elliptic surfaces gets much more involved (see also Rmk \([8.3]\)).

Interestingly, from the above perspective, the case of K3 surfaces is yet more different. If \( X \) is a K3, then there is no natural choice of a polarization on \( X \) and there is no natural threefold \( Y \) which contains \( X \) as a divisor. In fact, as the work of Lieblich-Maulik-Snowden \([37]\) illustrates, the Tate conjecture is equivalent to being able to find polarizations of bounded degree on any K3 surfaces defined over a fixed finite field. This was taken up by Charles \([16]\) to give a proof of the Tate conjecture for K3’s of a different flavor compared to Madapusi-Pera’s.

### A Discriminants which become nonreduced modulo \( p \)

In this appendix, we study examples of discriminant varieties defined over a DVR of mixed characteristic \((0, p)\) which become nonreduced modulo \( p \) by applying the Grothendieck-Ogg-Shafarevich (GOS) formula. We heartily thank Daichi Takeuchi for generously sharing his insights. In particular, the observations on how to apply Saito’s lemma \([57]\ Lem. 5.4]\) and Beilinson’s theorem \([3]\ Thm 1.7]\) are due to him.

Let \( k \) be an algebraically closed field. Let us first introduce two notations:

(a) Let \( X, Y \) be subvarieties of a smooth \( k \)-variety \( Z \) that intersect properly at a \( k \)-point \( z \in Z \). Denote the intersection multiplicity of \( X, Y \) at \( z \) by \( (X, Y)_{Z, z} \).

(b) Let \( S \) be a smooth \( k \)-curve and let \( f : X \to S \) be a morphism from another smooth \( k \)-variety \( X \). Let \( x \in X \) be an isolated singularity on the fiber \( X_{f(x)} \), and assume that there exists an open neighborhood \( U \) of \( x \) in \( X \) such that \( U \to S \) is smooth away from \( x \). We denote the Milnor number of \( f \) at \( x \) by \( \mu(f, x) \).
We remark that both definitions are obviously local, so to apply these definitions one may replace the varieties by open neighborhoods of the points in question.

As usual, by a line in a projective space over \( k \) we mean an one-dimensional linear subspace.

We recall the following well known result in local Lefschetz theory:

A.1 Milnor Number as Intersection Multiplicity

In this subsection, we work over an algebraically closed base field \( k \). For convenience, we pick notations that closely follow those of [56].

Let \( \mathbb{P} \) denote the projective space of dimension \( \geq 1 \) over \( k \) and let \( \mathbb{P}^* \) denote the dual space, viewed as the complete linear system \(|\mathcal{O}_\mathbb{P}(1)|\). Let \( Q \subseteq \mathbb{P} \times \mathbb{P}^* \) denote the universal hyperplane. Let \( X \subseteq \mathbb{P} \) be a smooth closed subvariety and let \( \mathcal{X} \) be the universal hyperplane section. That is, we have a fiber diagram

\[
\begin{array}{ccc}
\mathcal{X} & \longrightarrow & Q \\
\downarrow & & \downarrow \\
X \times \mathbb{P}^* & \longrightarrow & \mathbb{P} \times \mathbb{P}^*
\end{array}
\]

Let \( N \) be the conormal bundle of \( X \) in \( \mathbb{P} \), so that there is an exact sequence

\[
0 \rightarrow N \rightarrow T^*\mathbb{P}|_X \rightarrow T^*X \rightarrow 0
\]

of vector bundles on \( X \). Recall that \( \mathbb{P}(N) \) can be identified with the closed subvariety of \( \mathbb{P} \times \mathbb{P}^* \) such that the closed points on \( \mathbb{P}(N) \) are given by pairs \((x, H) \in X \times \mathbb{P} Q\), such that the hyperplane \( H \) intersects \( X \) at \( x \), but not transversely ([27, \S XVII 3.1]). This implies:

Lemma A.1. The discriminant locus \( D \subseteq \mathbb{P}^* \) of the morphism \( \mathcal{X} \to \mathbb{P}^* \) is simply the image of the projection \( \mathbb{P}(N) \to \mathbb{P}^* \).

Let \( \mathcal{X} \) be a line and denote the corresponding pencil of hypersurfaces by \( p_L : \mathcal{X}_L \to L \). Let \( A_L \) be the intersection of all hyperplanes parametrized by \( L \). We will always assume that \( X \) intersects \( A_L \) transversely. In this case, the total space of \( \mathcal{X}_L \) is obtained as the blow up of \( X \) along \( X \cap A_L \). Let \( \mathcal{X}_L^\circ \) be the complement of the exceptional locus. Note that we may view \( \mathcal{X}_L^\circ \) as an open subvariety of \( X \).

Lemma A.2. Let \( x \in \mathcal{X}_L^\circ \) be a \( k \)-point which is an isolated singularity on the fiber \( \mathcal{X}_{p_L(x)} \). Then the Milnor number \( \mu(p_L, x) \) is computed by the intersection multiplicity \( (\mathbb{P}(N), \mathcal{X}_L^\circ, x) \).

Proof. This follows from [56, Lem. 5.4(1)], with \( C \) being the 0-section of \( T^*X \) and \( \mathbb{P}(C) \) being \( \mathbb{P}(N) \). Unless otherwise noted, all references below come from loc. cit.

First, we explain that \( x \) is an isolated characteristic point of \( p_L^\circ : X_L^\circ \to L \), as defined in Def. 5.3(1), for which we need to check that

(a) the pair \( X \leftarrow X_L^\circ \setminus \{x\} \to L \) is \( C \)-transversal at a punctured neighborhood of \( x \);

(b) \( X \leftarrow X_L^\circ \to L \) is not \( C \)-transversal at \( x \).

We refer the reader to Def. 3.3 and 3.5 for the definition of \( C \)-transversality in different contexts. Since \( X_L^\circ \to X \) is an isomorphism locally at \( x \) and \( C \) is the 0-section, this morphism is \( C \)-transversal at an open neighborhood of \( x \) by Lem. 3.4(2). Let \( dp_L^{-1}(C) \) denote the pullback of \( C \) along the natural morphism
$dp_L : T^*L|_{X_L} \to T^*X_L$. We need to show that for a small open neighborhood $U$ of $x$ in $X^r_L$, $dp_L^{-1}(C)|_{U-\{x\}}$ is contained in the 0-section, but the fiber $dp_L^{-1}(C)|_x$ is not. This follows from the assumption that $p_L$ is a submersion on the punctured neighborhood $U-\{x\}$ but not at $x$.

Now Lem. 5.4(1) tells us that $(\mathbb{P}(N),X_L,x) = (C,dp_L^{-1}(C))$. The number $(C,dp_L^{-1}(C))$ is defined as follows (cf. Def. 5.3(2)): Let $V \subseteq L$ be an open neighborhood of $p_L(x)$ and $\omega \in \Gamma(V,\Omega^1_L)$ be a nowhere vanishing section. Then $p_L^{-1}(\omega)$ defines an embedding $p_L^{-1}(V) \to T^*X_L$. The number $(C,dp_L^{-1}(C))$ is defined to be $(C,p_L^{-1}(V))_{T^*X_L}$, which is independent of the choice of $(V,\omega)$. It follows from [27, §XVI Ex. 1.3] that $(C,p_L^{-1}(V))_{T^*X_L}$ is precisely the Milnor number $\mu(p_L,x)$.

**A.2 Discriminant of Even Dimensional Hypersurfaces**

Let $X := \mathbb{P}^r_Z$ be the $r$-dimensional projective space over $\mathbb{Z}$. Let $d$ be a natural number $\geq 2$ and $i : \mathbb{P}^r_Z \to \mathbb{P}^d_Z := |O_{\mathbb{P}^r_Z}^d|$ be the degree $d$ Veronese embedding ($N_d = \binom{n+d}{d} - 1$). Let $X \subseteq \mathbb{P}^r_Z \times \mathbb{P}^d_Z$ be the universal degree $d$ hypersurface in $\mathbb{P}^r_Z$ and $D \subseteq \mathbb{P}^r_Z$ be the discriminant locus of $X \to \mathbb{P}^r_Z$. Thm 1.7 is equivalent to the following:

**Theorem A.3.** Let $p$ be a prime and let $\bar{F}_p$ be an algebraic closure of $\mathbb{F}_p$. Then the fiber $D \times \bar{F}_p$ is irreducible unless $r$ is odd and $p = 2$, in which case the defining equation of $D \times \bar{F}_p$ in $\mathbb{P}^r_Z \times \bar{F}_p$ is the square of an irreducible polynomial.

The theorem above recovers Prop. 2.12 and partially recovers Thm 4.3 in [55]. Note that in addition implies that the polynomial $A$ in [55, Thm 4.3] is actually irreducible over $\bar{F}_2$. As far as the authors are aware of, this statement is new in literature.

**Proof.** Let $k$ denote $\bar{F}_p$ and write $W$ for $W(k)$. Let $\bar{K}$ be an algebraic closure of $K := W[1/p]$. Note that by Lem. A.1 $D_{k,\text{red}}$ is irreducible, so it suffices to show that

(a) when $p \neq 2$ or $r$ is even, $\deg(D_{k,\text{red}}) = \deg(D_{\bar{K}})$;

(b) when $p = 2$ and $r$ is odd, $2\deg(D_{k,\text{red}}) = \deg(D_{\bar{K}})$.

Here by degree we just mean the usual notion of the degree of the hypersurface $D_{k,\text{red}}$ (resp. $D_{\bar{K}}$) in the projective space $\mathbb{P}^r_Z$ (resp. $\mathbb{P}^d_Z$).

By Katz’ criterion, the embedding $i_k : \mathbb{P}^r_Z \to \mathbb{P}^d_Z$ is a Lefschetz embedding, so we may choose a Lefschetz pencil $L \subseteq \mathbb{P}^d_Z$. We may choose a $W$-lifting $\bar{L} \subseteq \mathbb{P}^d_{\bar{K}}$ such that the geometric generic fiber $\bar{L}_{\bar{K}}$ is a Lefschetz pencil over $\bar{K}$. We may assume that $L$ and $\bar{L}_{\bar{K}}$ intersects $D_{k,\text{red}}$ and $D_{\bar{K}}$ transversely. For $\kappa = k$ or $\bar{K}$, let $x'$ denote the unique singularity on the fiber of $X$ over a point $x \in (\bar{L} \cap D)_\kappa$ (cf. Def. 5.3(2)). Then we have a formula (cf. [27, §XVI Prop. 2.1])

$$\chi(X^r_{\bar{L}_{\bar{K}}}) = \chi(\mathbb{P}^1_k)\chi(X^r_{\bar{L}_{\bar{K}}}) - (-1)^{r-1}\sum_{x \in (\bar{L} \cap D)_\kappa} \mu(X^r_L/L,x').$$

(A.4)

One may easily check by the smooth and proper base change theorem that the numbers $\chi(X^r_{\bar{L}_{\bar{K}}})$ and $\chi(X^r_{\bar{L}_{\bar{K}}})$ do not depend on whether $\kappa = k$ or $\bar{K}$. As $x'$ in the above equation is always an ordinary quadratic singularity, the same is true for $\mu(X^r_{\bar{L}_{\bar{K}}}/L,x')$ as well for every $x'$ when $p \neq 2$ or $r$ is even ([27, §XVI Cas 1, 2]). In fact, these Milnor numbers are all equal to 1. Therefore, we have $\#(\bar{L} \cap D)_\kappa = \#(\bar{L} \cap D)_{\bar{K}}$. This implies (a).
For (b) we apply a theorem of Beilinson \[6\], Thm 1.7] with \( \mathcal{F} := \mathbb{F}_\ell \) being the constant sheaf on \( \mathbb{P}^r \). Let \( N \) denote the conormal bundle of \( X_k \) in \( \mathbb{P}^N_k \). Below we follow the notations of loc. cit., except that his \( \mathbb{P} \) is our \( X_k \) and his \( \mathbb{P} \) is our \( \mathbb{P}^{N_d}_k \). Let \( \tilde{Q} \subseteq \mathbb{P}^{N_d}_k \times \mathbb{P}^{N_d}_k \) be the universal hyperplane section and let \( D_\mathcal{F} \subseteq \mathbb{P}^{N_d} \) be the smallest closed subvariety such that on the complement \( \tilde{\mathcal{F}} = \mathbb{P}^{N_d} - D_\mathcal{F} \) the Radon transform \( \tilde{\mathcal{R}}(i_*\mathcal{F}) \) is locally constant. Since \( X_k = X_k \times_{\mathbb{P}^{N_d}_k} \tilde{Q} \), the proper base change theorem tells that \( \tilde{\mathcal{R}}(i_*\mathcal{F}) \cong (f_*\mathcal{F})[\mathcal{F}_d - 1] \) as objects in the bounded derived category of constructible sheaves on \( \mathbb{P}_k \) with \( \mathbb{F}_\ell \)-coefficients, where \( f \) is the composition \( X \rightarrow Q \rightarrow \mathbb{P}^N \). Now the smooth and proper base change theorem implies that \( D_\mathcal{F} \subseteq D_{k, \text{red}} \). However, by Beilinson’s theorem, \( D_\mathcal{F} \) is a divisor. As \( D_{k, \text{red}} \) is irreducible, we must in fact have \( D_\mathcal{F} = D_{k, \text{red}} \). Then Beilinson’s theorem further tells us that the morphism \( \mathbb{P}(N) \rightarrow D_{k, \text{red}} \) is generically radicial and has generic degree at most 2. Indeed, it is not hard to see that \( D_\mathcal{F} = \mathbb{P}(i_*\mathcal{C}) \) for \( C \) being the 0-section of \( T^*X_k \) (the notation \( i_0 \) is explained in the last paragraph of \(6\), §1.2).

Therefore, we may take an open dense subset \( U \subseteq D_{k, \text{red}} \) such that \( \mathbb{P}(N)|_U \rightarrow U \) is finite flat of degree \( \leq 2 \). We may assume that the our Lefschetz pencil \( L \) intersects \( U \) transversely, which is a generic property. By Lem. A.2 we have that for every \( x \in (L \cap D_{k})(k) \),

\[
\mu(x_L/L, x') = (\mathbb{P}(N), X_L)_{x,x'} \leq 2.
\]

By \[6\], Prop. 3.24], \( \mu(x_L/L, x') \) is always an even number. Since it has to be positive, it is equal to 2. Together with (A.4), this implies (b).

\[\square\]

Remark A.5. The usage of the formula in \[27\], §XVI Prop. 2.1] is equivalent to the usage of the Grothendieck-Ogg-Shafarevich (GOS) formula. Indeed, the GOS formula states (A.4) when the Milnor number \( \mu(x_L/L, x') \) is replaced by the total dimension \( n(x') \) of vanishing cycles at \( x' \), but \( \mu(x_L/L, x') = n(x') \) by Milnor’s formula (see Thm 2.4 in loc. cit.), whose proof in fact proceeds by comparing the GOS formula with \[27\], §XVI Prop. 2.1].

A.3 Katz’ non-example of Lefschetz embeddding

Katz provided the following non-example of Lefschetz embedding: Let \( k \) be an algebraically closed field of characteristic \( p > 2 \). Let \( X \) be the smooth hypersurface of \( \mathbb{P}^r = \text{Proj} k[x_0, \ldots, x_r] \) defined by the equation

\[
F = x_0^m + x_1^m + \cdots + x_r^m,
\]

for some natural number \( m \geq 1 \). Katz took \( m = r \), but this is not important. Again let \( N \) denote the conormal bundle of \( X \) in \( \mathbb{P}^r \). Since \( X \) is a smooth hypersurface, it is naturally identified with \( \mathbb{P}(N) \). The natural morphism \( \mathbb{P}(N) \rightarrow \mathbb{P}^r \) can be identified with the Gauss map

\[
(x_0, \ldots, x_r) \mapsto \left( \frac{\partial F}{\partial x_0}, \ldots, \frac{\partial F}{\partial x_r} \right) = (x_0^m, \ldots, x_r^m),
\]

which in this case is simply the \( m \)-power of the absolute Frobenius morphism (if we identify \( \mathbb{P}^r \) with \( \tilde{\mathbb{P}}^r \) using the chosen coordinates in the natural way). By \[27\], §XVII Cor. 3.5.0], the embedding \( X \subseteq \mathbb{P}^r \) is not a Lefschetz embedding.

Proposition A.6. View \( X \) and \( \mathbb{P}^r \) as being defined over \( W := W(k) \) by the same equation. Let \( D \subseteq \mathbb{P}^r \) be the discriminant locus of the universal hyperplane section. Then \( D_k \subseteq \mathbb{P}^r_k \) is defined by the \( p^m(r-1) \)-th power of an irreducible polynomial. In particular, \( D_k \) is not reduced.
Proof. Take a line $L \subseteq \tilde{\mathbb{P}}^r$ that intersects transversely with the (scheme-theoretic) image $\tilde{X}_k$ of $X_k$ under the Gauss map. By Lem. 7.2, we may choose the line $L$ such that $\mathcal{X}_L$ has a smooth total space. Choose a $W$-lifting $\tilde{L}$ of $L$ such that $\tilde{L}_k$ is a Lefschetz pencil.

Let $x \in X$ be a $k$-point and $\tilde{x} \in \tilde{X}_k$ be its image under the Gauss map. Note that the association of $x$ to $\tilde{x}$ is bijective. If $\tilde{x} \notin L \cap \tilde{X}_k$, then the fiber $X_{\tilde{x}}$ is smooth; otherwise, the fiber $X_{\tilde{x}}$ has a unique singularity at $x$. Therefore, $\mathcal{D}_{k, \text{red}}$ is nothing but $\tilde{X}_k$. By applying [27, §XVI Prop. 2.1] and a lifting argument as in the proof of Thm A.3, we reduce to showing that $\mu(\mathcal{X}_L/L, x) = p^m$ whenever $\tilde{x} \in L \cap \tilde{X}_k$. By Lem. A.2, $\mu(\mathcal{X}_L/L, x)$ is computed by $(\mathbb{P}(N), \mathcal{X}_L)_{X,x}$. Now note that

$$\mathbb{P}(N) \cap \mathcal{X}_L = \mathbb{P}(N) \times_X \mathcal{X}_L = \mathbb{P}(N) \times_X X \times_\mathbb{P} L = \mathbb{P}(N) \times_\mathbb{P} L.$$ 

Together with the fact that $L$ intersects $\tilde{X}_k$ transversely, this implies that $(\mathbb{P}(N), \mathcal{X}_L)_{X,x}$ is the degree of the preimage of $\tilde{x}$ under the Gauss map $X = \mathbb{P}(N) \to \tilde{\mathbb{P}}$, and this degree is clearly $p^m(r-1)$ as desired. 

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