On the Markus-Neumann theorem

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Abstract

A well-known result by L. Markus [6], later extended by D. A. Neumann [7], states that two continuous flows on a surface are equivalent if and only if there is a surface homeomorphism preserving orbits and time directions of their separatrix configurations.

In this paper we present several examples showing that, as originally formulated, the Markus-Neumann theorem needs not work. Besides, we point out the gap in its proof and show how to restate it in a correct (and slightly more general) way.

1. The Markus-Neumann theorem

The Markus-Neumann theorem is an often cited result dealing with the topological classification of surface flows. Google Scholar provides 147 explicit references to [6], and 91 to [7], and has been used without explicit mention (mainly in the planar case), as in [11, p. 5], a large number of times. A first planar version was proved by L. Markus [6], under the additional restriction of nonexistence of so-called “limit separatrices”. In [7], D. A. Neumann disposed of this condition and extended the result to arbitrary surfaces. Roughly speaking, the theorem states that two surface flows are equivalent if there is a surface homeomorphism preserving a number of distinguished orbits from both flows. However, Markus missed an important point which, apparently, also passed unnoticed to Neumann and the subsequent readers (see for instance [3, pp. 33–34], [10, p. 294], [9, pp. 245–246] or [8, pp. 225–226]). As a consequence the theorem, as stated in [6, 7], is wrong. In fact, as we will show in the next section, counterexamples can be found in far from pathological settings, even for polynomial plane flows. The good news is that, after appropriately amending the Markus-Neumann notion of separatrix, the theorem works (and can be slightly improved). We should stress that, when using the theorem in the polynomial scenery, researchers typically employ an alternative, easier to handle with, notion of separatrix (see Remark 2.3). Fortunately enough, it turns out to be equivalent to our amended definition (but not to that of Markus-Neumann’!). Therefore, all such papers remain correct without further changes.

In what follows we list a number of notions that will be needed to state the Markus-Neumann theorem (Theorem 1.2 below) or in the next sections. The reader is assumed to be familiar with the basic facts of the qualitative theory of bidimensional flows (although some
of them will be recalled below). Among others, a good general reference on the subject is the book \cite{1}.

Let \( \Phi : \mathbb{R} \times M \to M \) be a continuous flow on a (connected, without boundary) surface \( M \). Note that \( M \) is not assumed to be compact nor orientable. We sometimes refers to is a the couple \((M, \Phi)\).

Some specific flows will be mentioned below. Let \( f_s, f_a, \) and \( f_r \) be the planar vector fields given by \( f_s(x, y) = (1, 0) \), \( f_a(x, y) = (x, 0) \) and \( f_r(x, y) = (-x, -y) \), and associate to them the corresponding planar flows \( \Phi_s, \Phi_a \) and \( \Phi_r \). Also, let \( f_{v_1}(x, y) = (|x|, 0) \), \( f_{v_2}(x, y) = (x, 0) \), \( f_{v_3}(x, y) = (-x, 0) \), \( f_{v_4}(x, y) = (|y|, 0) \) and \( f_{v_5}(x, y) = (y, 0) \), being \( \Phi_{v_1}, \Phi_{v_2}, \Phi_{v_3}, \Phi_{v_4} \) and \( \Phi_{v_5} \), respectively, their associated planar flows. Finally, after identifying points \((x, y)\) and \((x', y')\) in \( \mathbb{R}^2 \) when both \( x - x' \) and \( y - y' \) are integers, \( f_s \) induces a flow \( \Phi_{ss} \) on the torus \( \mathbb{T}^2 \).

Given any \( p \in M \) and any \( t \in \mathbb{R} \), we write indistinguishably \( \Phi(t, p) = \Phi_p(t) = \Phi_t(p) \). Also, for every \( p \in M \), we call the map \( \Phi_p : \mathbb{R} \to M \) the integral curve associated to the point \( p \) and, its image, \( \varphi_p(p) = \{ \Phi_p(t) : t \in \mathbb{R} \} \), the orbit of \( \Phi \) through \( p \). The flow foliates \( M \) as a union of points (the singleton orbits, or singular points, and the orbits containing them), periodic integral curves) and injective copies of the real line. By \( \text{Sing}(\Phi) \) we denote the union set of all singular points; nonsingular points, and the orbits containing them, are called regular. If \( I \subset \mathbb{R} \) is an interval and \( p \in \Omega \), we call \( \Phi_p(I) \) a semi-orbit of \( \varphi_p(p) \).

In the particular cases \( I = [t_1, t_2] \) (with \( \Phi_p(t_1) = p_1, \Phi_p(t_2) = p_2 \), \( I = [0, \infty) \) or \( I = (-\infty, 0] \), we rewrite \( \Phi_p(I) \) as \( \varphi_p(p_1, p_2), \varphi_p(p, +) \) or \( \varphi_p(-, p) \), respectively.

We define the \( \alpha \)-limit set of the orbit \( \varphi_p(p) \) (or the point \( p \)) as the set

\[
\alpha_p(p) = \{ u \in M : \exists t_n \to -\infty; \Phi_p(t_n) \to u \};
\]

the \( \omega \)-limit set, \( \omega_p(p) \), is accordingly defined writing \( t_n \to \infty \) instead. Also, we write \( \alpha'_p(p) = \alpha_p(p) \setminus \varphi_p(p) \) and \( \omega'_p(p) = \omega_p(p) \setminus \varphi_p(p) \). When these sets coincide, that is, the orbit belongs neither to its \( \alpha \)-limit set nor its \( \omega \)-limit set, we call it non-recurrent. Usually, when there is no ambiguity on \( \Phi \), we will omit to mention \( \Phi \) and write, for instance, \( \varphi(p) \) instead of \( \varphi_p(p) \) or \( \omega(p) \) instead of \( \omega_p(p) \).

We say that an orbit \( \varphi(p) \) is stable if for any \( \epsilon > 0 \) there is a number \( \delta > 0 \) such that \( d(p, q) < \delta \) implies that all points from \( \varphi(q) \) stay at a distance less than \( \epsilon \) from \( \varphi(p) \), and it is called unstable otherwise. Contrary to the previous notions, stability depends on the chosen metric; more precisely, problems may arise when \( M \) is noncompact. To avoid them we pass to the one-point compactification \( M_\infty = M \cup \{ \infty \} \), which is metrizable because \( M \) is locally compact and metrizable, and use an arbitrary distance in \( M_\infty \).

If \((M, \Phi)\) is a flow, then we say that a subset \( \Omega \subset M \) is invariant for the flow if it is the union set of some orbits. In such a case \( \Phi \) also defines a flow on \( \Omega \) (\( \Phi \) can be restricted to a map from \( \mathbb{R} \times \Omega \) to \( \Omega \)), the so-called restriction of \( \Phi \) to \( \Omega \), and we write \((\Omega, \Phi)\) to denote it.

We say that two flows \((M_1, \Phi_1)\) and \((M_2, \Phi_2)\) are locally topologically equivalent at the points \( p_1 \in M_1, p_2 \in M_2 \), if there is a homeomorphism \( h : U_1 \to U_2 \) between open neighborhoods of \( p_1 \) and \( p_2 \), with \( h(p_1) = p_2 \), carrying semi-orbits onto semi-orbits and preserving (time) directions. When the homeomorphism maps the whole \( M_1 \) onto \( M_2 \) (hence carrying orbits onto orbits), then we call it a topological equivalence between \((M_1, \Phi_1)\) and \((M_2, \Phi_2)\) and say that the flows \((M_1, \Phi_1)\) and \((M_2, \Phi_2)\) are topologically equivalent. As it is well known,
given a flow \((M, \Phi)\) there is a flow \((M_\infty, \Phi_\infty)\), having \(\infty\) as a singular point, whose restriction \((M, \Phi_\infty)\) is topologically equivalent to \((M, \Phi)\) (in fact, the orbits are the same).

Let \(p\) be a singular point of \(\Phi\). We say that it is \(vertical\) (respectively, \(horizontal\)) if there is a local equivalence between \(\Phi\) and either \(\Phi_{v_1}\), \(\Phi_{v_2}\) or \(\Phi_{v_3}\) (respectively, \(\Phi_{h_1}\) or \(\Phi_{h_2}\)) at \(p\) and \(0\). A singular point which is neither vertical, nor horizontal, is called \(essential\). Among the essential singular points we distinguish the subset of \(trivial\) ones as those points which admit a neighbourhood of singular points.

Let \(p \in M\). If there is a local topological equivalence between \(\Phi\) and \(\Phi_s\) at \(p\) and \(0\), with corresponding homeomorphism \(h: U_1 \to U_2 = (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon)\), then we say that \(U_1\) is a \(tubular\) \(neighbourhood\) of \(p\), and call the regions (by a \(region\) we mean an open connected set) into which \(h^{-1}(\{-\varepsilon, \varepsilon\} \times \{0\})\) decomposes \(U_1\) a \(couple\) of \(lateral\) \(tubular\) \(regions\) at \(p\).

Also, we say that \(h^{-1}(\{0\} \times (-\varepsilon, \varepsilon))\) is a \(transversal\) to \(p\), which is decomposed by \(p\) into a \(couple\) of \(lateral\) \(transversals\) at \(p\). It is a well-known fact that every regular point \(p\) of \(\Phi\) admits a tubular neighbourhood; on the other hand, if \(p\) is a horizontal singular point, couples of lateral tubular regions and of lateral transversals at \(p\) can be analogously defined, just using \(\Phi_{h_1}\) or \(\Phi_{h_2}\) instead of \(\Phi_s\). If \(\mu: \mathbb{R} \to M\) is a continuous injective map with the property that, for any \(s \in \mathbb{R}\), there is \(\epsilon_s > 0\) such that \(\mu((s - \epsilon_s, s + \epsilon_s))\) is a transversal to \(\mu(s)\), then we call \(\mu(\mathbb{R})\) a \(transversal\) to the flow \((M, \Phi)\).

Let \(\Omega\) be an invariant region for \((M, \Phi)\). Following [7], we say that \(\Omega\) is is \(parallel\) when the restriction \((\Omega, \Phi)\) is topologically equivalent to either \((\mathbb{R}^2, \Phi_s)\), \((\mathbb{R}^2 \setminus \{0\}, \Phi_s)\), \((\mathbb{R}^2 \setminus \{0\}, \Phi_r)\) or \((\mathbb{T}^2, \Phi_{ss})\), and use the terms \(strip,\) \(annular,\) \(radial\) and \(toral\), respectively, to distinguish cases. Note that in the toral case \(\Omega = M\) is indeed a torus and all orbits are periodic.

If \(\Omega\) is parallel, and \(T \subset \Omega\) is a transversal, then we say that it is a \(complete\) transversal to \(\Omega\) provided that one the following conditions hold:

- \(\Omega\) is either a strip or an annular region, and \(T\) intersects each orbit from \(\Omega\) at exactly one point. Observe that, in the strip case, \(T\) decomposes \(\Omega\) into two regions, \(\Omega_T^-\) and \(\Omega_T^+\), corresponding to the backward and forward direction of the flow.

- \(\Omega\) is a radial region and, if \(p \in T\), then each of the two transversals into which \(p\) decomposes \(T\) intersects any orbit from \(\Omega\) infinitely many times.

Also, we say that a transversal \(T \subset \Omega\) is \(semi-complete\) when either it is complete, or it is one of the two transversals into which some point decomposes a complete transversal.

Let \((M, \Phi)\) be a flow and let \(p \in M\). In [7] or [6] separatrices are defined as follows:

**Definition 1.1.** We say that an orbit \(\varphi(p)\) of \((M, \Phi)\) is \(ordinary\) if it is neighboured by a parallel region \(\Omega\) such that:

1. \(\alpha'(q) = \alpha'(p)\) and \(\omega'(q) = \omega'(p)\) for any \(q \in \Omega\);
2. \(\text{Bd} \Omega\) is the union of \(\alpha'(p)\), \(\omega'(p)\) and exactly two orbits \(\varphi(a)\) and \(\varphi(b)\) with \(\alpha'(a) = \alpha'(b) = \alpha'(p)\) and \(\omega'(a) = \omega'(b) = \omega'(p)\).

If an orbit is not ordinary, then it is called a \(separatrix\).
Observe that no conditions are imposed on $\alpha'(p)$ and $\omega'(p)$: one or both may be empty (which is to say the infinite point $\infty$ when passing to $\Phi_\infty$), and they may have, or not, empty intersection. On the other hand, $\varphi(a)$ and $\varphi(b)$ must be distinct and disjoint from $\alpha'(p)$ and $\omega'(p)$.

Let $(M, \Phi)$ be a flow and call $S$ the union set of all its separatrices. Note that the set $S$ is closed. The components of $M \setminus S$ are called the canonical regions of $(M, \Phi)$. By a separatrix configuration for $(M, \Phi)$, $S^+$, we mean the union of $S$ together with a representative orbit from each canonical region.

Let $\Phi_1$ and $\Phi_2$ be two flows defined on the same surface $M$ and let $S_1$ and $S_2$ be, respectively, the union sets of their separatrices. We say that their separatrix configurations $S_1^+$ and $S_2^+$ are equivalent if there is a homeomorphism of $M$ onto $M$, carrying orbits of $S_1^+$ onto orbits of $S_2^+$, that preserves directions.

We are ready to state the result we are concerned with in this paper:

**Theorem 1.2** (the Markus-Neumann theorem [6, 7]). Let $M$ be a surface and suppose that $\Phi_1$ and $\Phi_2$ are flows on $M$ whose sets of singular points are discrete. Then $\Phi_1$ and $\Phi_2$ are equivalent if and only if they have equivalent separatrix configurations.

## 2. Counterexamples to the Markus-Neumann theorem

As it turns out, Theorem 1.2 (as presently formulated) is wrong, the problem being that the previous definition of separatrix is too restrictive. A planar counterexample is shown by Figure 1. Both flows share the orbits $\Gamma_1$, $\Gamma_2$, $\Gamma_3$ and $\Gamma_5$ and the singular point $p$, and the separatrices are just, in both cases, $p$, $\Gamma_1$ and $\Gamma_2$. For instance, to show that $\Gamma_3$ is ordinary for the right-hand flow $\Phi_2$, take an orbit $\Gamma$ enclosed by $\Gamma_2$ but not by $\Gamma$. Now the boundary of $\Omega$ consist, as required by Definition 1.1, of the orbits $\Gamma_2$ and $\Gamma$, and the singular point $p$, which is both the $\alpha$-limit and the $\omega$-limit set of all orbits in $\Omega$ and also of $\Gamma_2$ and $\Gamma$. (Here, as in the examples below, there is no need to distinguish between $\alpha(q)$ and $\alpha'(q)$ nor between $\omega(q)$ and $\omega'(q)$, because the only recurrent orbits are the singular points). Likewise, $\Gamma_4$ is ordinary for the left-hand flow $\Phi_1$ (use the strip $\Omega$ consisting of all orbits enclosed by $\Gamma_2$ but not by $\Gamma$).

Now, since the separatrix configurations $S_1^+ = S_2^+ = \{p\} \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_5$ are the same, the flows $\Phi_1$ and $\Phi_2$ should be, according to Theorem 1.2, topologically equivalent. Clearly, they are not: since $\Gamma_2$ is, in both cases, the maximal homoclinic orbit, the topological equivalence should carry it onto itself. However, there are two unstable orbits ($\Gamma_3$ and $\Gamma_4$) inner to $\Gamma_2$ for $\Phi_1$, but just one ($\Gamma_3$) for $\Phi_2$.

**Remark 2.1.** As shown in [4] (see also [12]) flows $\Phi_1$ and $\Phi_2$ can in fact be realized by polynomial vector fields (or, via the Bendixson projection, by analytic sphere flows with just two singular points).

An even cleaner (torus) counterexample is exhibited by Figure 2. Here, the left-hand flow $\Phi_3$ and the right hand flow $\Phi_4$ share the orbits $\Gamma_1$ and $\Gamma_2$ and the singular point $p$, and all orbits are homoclinic. As it happens, $p$ is the only separatrix for both flows. To show,
say, that $\Gamma_1$ is ordinary (for $\Phi_4$), remove from $\mathbb{T}^2$ the closure of the strip delimited by $\Gamma_3$ and $\Gamma_4$ and containing $\Gamma_2$, to get a radial region containing $\Gamma_1$ with boundary $\Gamma_3 \cup \Gamma_4 \cup \{p\}$. In the case of $\Phi_3$, the radial region $\Omega' = \mathbb{T}^2 \setminus (\Gamma_2 \cup \{p\})$ cannot be used (there is just one regular orbit in its boundary), but we take off another orbit $\Gamma$ and use the strip $\Omega = \Omega' \setminus \Gamma$ instead. Once again, the separatrix configurations $S^+_3 = S^+_4 = \{p\} \cup \Gamma_1$ coincide, but $\Phi_3$ and $\Phi_4$ are not equivalent because $\Phi_3$ has three unstable orbits ($p$, $\Gamma_1$ and $\Gamma_2$) and $\Phi_4$ has four ($p$, $\Gamma_1$, $\Gamma_3$ and $\Gamma_4$).

Clearly, the problem with the previous examples is that the neighbouring regions we are using for ordinary orbits are, so to speak, too “big”, and as a consequence the bounding orbits are not what they are “supposed” to be. A way to avoid this is not allowing parallel regions to be radial (trivially they cannot be toral either) in Definition 1.1. Moreover, we can force strips to be “strong”. More precisely, we say that a strip $\Omega$ is strong if there are non-recurrent orbits $\Gamma_1, \Gamma_2$ such that $(\Omega', \Phi)$ is topologically equivalent to the restriction of the flow $\Phi_s$ to $\mathbb{R} \times [-1, 1]$, $\Omega' = \Omega \cup \Gamma_1 \cup \Gamma_2$. We call $\Gamma_1$ and $\Gamma_2$ the border orbits of the strip $\Omega$, and say that a complete transversal to $\Omega$ is strong if it can be extended to an arc (that is, a homeomorphic set to the interval $[0, 1]$) by adding one point from each border orbit. All orbits from a strip are non-recurrent: by requiring that the border orbits of a strong strip also are, we get rid of the annoying distinction between $\alpha_\Phi(p)$ and $\alpha'_\Phi(p)$ or $\omega_\Phi(p)$ and $\omega'_\Phi(p)$ (in the annular case, Definition 1.1(i) and (ii) are quite redundant, anyway).

Unexpectedly, Theorem 1.2 keeps failing even after redefining ordinary orbits as in the paragraph above, see Figure 3. In this torus example, common orbits to $\Phi_5$ and $\Phi_6$ are $\Gamma_1$, $\Gamma_2$, $\Gamma_3$ and $\Gamma_5$ and the singular points $p$ and $q$. Observe that typical orbits of these flows have $\Gamma_1 \cup \{p\}$ as their $\alpha$-limit set and $\{p\}$ as their $\omega$-limit set. Checking that $\Phi_5$ and $\Phi_6$ are not equivalent, while having the same separatrix configurations $S^+_5 = S^+_6 = \{p, q\} \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_5$, is left to the reader’s care.

The underlying problem here is that $\alpha$-limit and $\omega$-limit sets should be separately man-
Figure 2: The phase portraits of flows $\Phi_3$ (left) and $\Phi_4$.

Figure 3: The phase portraits of flows $\Phi_5$ (left) and $\Phi_6$. 
aged by Definition 1.1, but they are not. The orbit $\Gamma_4$ is neighboured by strong strips, as close to it as required, whose boundaries consist of (as prescribed) the border orbits, the $\alpha$-limit set $\Gamma_1 \cup \{p\}$ and, $a$ fortiori, the $\omega$-limit set $\{p\}$. Nevertheless, after removing a strong transversal from the strip, the boundary of the forward semi-strip contains, besides the border semi-orbits, the strong transversal and the $\omega$-limit point $\{p\}$, the “spurious” orbit $\Gamma_1$.

Taking all the above into consideration, we define:

**Definition 2.2.** We say that an orbit $\varphi(p)$ of $(M, \Phi)$ is **almost fine** if it is neighboured by an annular region or by a strong strip $\Omega$ with border orbits $\varphi(a)$ and $\varphi(b)$ such that:

(i) $\alpha(q) = \alpha(p)$ and $\omega(q) = \omega(p)$ for any $q \in \Omega \cup \varphi(a) \cup \varphi(b)$;

(ii) $\text{Bd } \Omega = \varphi(a) \cup \varphi(b) \cup \alpha(p) \cup \omega(p)$.

If $\varphi(p)$ satisfies the analogous conditions, replacing (ii) by

(ii') if $T$ is a strong transversal to $\Omega$ with endpoints $a$ and $b$, then $\text{Bd } \Omega_T^- = T \cup \varphi(-\infty, a) \cup \varphi(-\infty, b) \cup \alpha(p)$ and $\text{Bd } \Omega_T^+ = T \cup \varphi(a, \infty) \cup \varphi(b, \infty) \cup \omega(p)$,

then we say that $\varphi(p)$ is **fine**.

If an orbit is not fine, then it is called a **separator**.

Observe that the union set of all separators is closed as well, when the components of its complementary set will be called **standard regions**. Since all separatrices are separators, every standard region is contained in a canonical region. The notions of separator configuration and of equivalence of separator configurations are accordingly defined.

**Remark 2.3.** Typically, books and papers invoking the Markus-Neumann theorem in the setting of analytic sphere flows (in particular, after carrying polynomial planar flows to the sphere via the Bendixson or the Poincaré projections), use an alternative definition of separatrix, see for instance [10, Section 3.11]. Here, under the additional assumption of finiteness of singular points, an orbit is called a “separatrix” if and only if it is either a singular point, a limit cycle, or an orbit lying in the boundary of an hyperbolic sector. Using the finite sectorial decomposition property for isolated (non-centers) singular points of analytic flows, noting that analyticity excludes the existence of one-sided isolated periodic orbits, and recalling some basic Poincaré-Bendixson theory, it is not difficult to show that this notion is, in fact, equivalent to that of separator (see also Proposition 2.4(b) or (c) below). Of course, as emphasized by our first counterexample (and contrarily to that stated in [10]) there may be orbits bounding hyperbolic sectors which are not separatrices in the Markus-Neumann sense.

Note, finally, that the previous discussion make no sense outside the sphere (just think of the irrational flow on the torus: here all orbits are separatrices).

Trivially, a fine orbit is almost fine. The converse is not true, as shown by the flow $\Phi_6$. Nevertheless, we have:

**Proposition 2.4.** Let $\varphi(p)$ be an almost fine orbit of $(M, \Phi)$. Assume that one of the following conditions holds:
(a) Both \( \alpha(p) \) and \( \omega(p) \) are finite (that is, empty or consisting of one point);

(b) \( M \) has zero genus and the set of essential singular points of \( \Phi \) is totally disconnected;

(c) \( M = \mathbb{R}^2 \) or \( M = SS \).

Then \( \varphi(p) \) is fine.

Proof. In all three cases we must show that if \( \Omega \) satisfies (i) and (ii) in Definition 2.2 then (ii') holds as well.

Assume that (a) holds. We just prove (the other equality is analogous) \( \text{Bd} \Omega_T^− = T \cup \varphi(−\infty, a) \cup \varphi(−\infty, b) \cup \alpha(p) \), when we can assume (otherwise the statement is trivial) \( \alpha(p) = \{u\} \neq \omega(p) \). Then there is a small topological disk \( D \) (that is, a homeomorphic set to the unit disk \( \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \), neighbouring \( u \) (hence not intersecting \( \omega(p) \)), such that \( \varphi(a) \) and \( \varphi(b) \) intersect \( D \) at respective semi-orbits \( \varphi(−, a') \), \( \varphi(−, b') \); moreover, this can be done so that one of the arcs in \( \text{Bd} D \) joining \( a' \) and \( b' \) is the closure of a strong transversal \( T' \) to \( \Omega \). One of the regions into which \( \varphi(−, a') \cup \{u\} \cup \varphi(−, b') \) decomposes \( \text{Int} D \) cannot intersect \( \Omega \), and the other one, call it \( \Omega'_p \), includes \( \Omega_T^− \). By the hypothesis on \( \text{Bd} \Omega \), and the fact that \( \omega(p) \) does not intersect \( D \), both \( \Omega' \) and \( \Omega_T^− \) have the same boundary, hence (because they are connected) \( \Omega' = \Omega_T^− \). This implies the statement.

Now suppose that (b) holds. Because \( M \) has zero genus, there is no loss of generality in assuming that it is a region in \( SS \). Observe that, since \( \varphi(p) \) is almost fine, all non-essential singular points contained in \( \alpha(p) \cup \omega(p) \) must be horizontal. If \( \alpha(p) \cup \omega(p) \) contains a regular point or an horizontal singular point, then a standard Poincaré-Bendixson argument allows to find a semi-orbit \( \varphi(c, d) \) of \( \varphi(p) \), and a transversal joining \( c \) and \( d \), whose union is a circle decomposing \( SS \) into two regions, one including \( \alpha(p) \), the other one including \( \omega(p) \). By the compactness of \( \{a, b\} \cup T \), there is \( t_0 \) such that \( \Omega_{\Phi_{−t_0}(T)}^− \) is included in the first region, while \( \Omega_{\Phi_{−t_0}(T)}^+ \) is included in the second one, which easily implies that \( \varphi(p) \) is fine. In the case when all points from \( \alpha(p) \) and \( \omega(p) \) are singular and essential, total disconnectedness implies finiteness and (a) applies.

Finally, suppose that (c) is true. If suffices to consider the case \( M = SS \), as then the case \( M = \mathbb{R}^2 \) follows by passing to its one-point compactification, which is precisely \( SS \). Moreover, as in (b), we can additionally assume that all points from \( \alpha(p) \cup \omega(p) \) are singular. Let \( S \) denote the set of singular points of \( \Phi \), and let \( U \) be the component of \( SS \setminus S \) including \( \varphi(p) \). Next define the equivalence relation \( \sim \) in \( SS \) by \( u \sim v \) if \( u = v \) or there is a component \( C \) of \( SS \setminus U \) such that \( u, v \in C \). As explained in [2, p. 481], the quotient space \( SS_\sim \) is homeomorphic to \( SS \) and the flow \( \Phi \) collapses, in the natural way, to a flow \( \Phi_\sim \) on \( SS_\sim \), whose set of singular points is totally disconnected. By applying (b) to the collapsed flow, we deduce that the distances \( d(\Phi_t(q), \alpha(p)) \), \( q \in \Omega_T^− \), tend uniformly to zero as \( t \to −\infty \), and the same is true for \( d(\Phi_t(q), \omega(p)) \), \( q \in \Omega_T^+ \) and \( t \to \infty \). Therefore, \( \varphi(p) \) is fine.

After extending the notion of separatrix as described in Definition 2.2 Theorem 1.2 works and, in fact, can be slightly improved, see Theorem 2.3 below. The improvement has to do
with essential singular points. The left-hand flow $\Phi_7$ from Figure 4 is that associated (after deformation to clarify the picture outside the unit circle) to the vector field

$$f_7(x, y) = (1 - x^2 - y^2) \left( -(1 - x^2 - y^2)x - y, x - (1 - x^2 - y^2)y \right),$$

having the origin and the unit circle $S^1$ as its set of singular points. Consecutive points of the semi-orbit starting at $(2, 0)$ and intersecting the positive $x$-semiaxis (respectively, negative $x$-semiaxis, positive $y$-semiaxis) are denoted by $(a_n)_{n=1}^{\infty}$ (respectively, $(b_n)_{n=1}^{\infty}$, $(c_n)_{n=1}^{\infty}$). To construct $\Phi_8$ we modify, as indicated in the picture, the semi-orbits from the regions enclosed by $\varphi_{\Phi_7}(a_n, b_n), \varphi_{\Phi_7}(a_{n+1}, b_{n+1})$, and the segments connecting $a_n$ and $a_{n+1}$ and $b_n$ and $b_{n+1}$. In the lower half-plane, and inside $S^1$, the phase portrait does not change. Thus, for both flows, all regular orbits spiral towards $S^1$ in positive time, and if $\Gamma_1$ denotes the orbit passing through $(2, 0)$ and $\Gamma_2$ is an orbit inside $S^1$, then $\{0\} \cup S^1 \cup \Gamma_1 \cup \Gamma_2$ is a separator configuration for both $\Phi_7$ and $\Phi_8$.

Nevertheless, these flows are not equivalent. The key point is that, while all semi-lines starting from the origin are transversal (except at the unit circle) to $\Phi_7$, transversals connecting the points $(c_n)_n$ for $\Phi_8$ cannot be fully included in the octant $\{(x, y) : y > |x|\}$, hence their diameters are uniformly bounded, from below, by a positive number. Now assume that $h$ is a topological equivalence mapping the orbits of $\Phi_8$ onto those of $\Phi_7$. Then the points $h(c_n)$ converge to a point $d \in S^1$, and there are transversals $T_n$ (for $\Phi_7$) connecting the points $h(c_n)$ and $h(c_{n+1})$ whose diameters tend to zero. Hence the diameters of the transversals $h^{-1}(T_n)$ (for $\Phi_8$) also tend to zero, which is impossible because they connect the points $c_n$.

![Figure 4: The phase portraits of flows $\Phi_7$ (left) and $\Phi_8$.](image-url)
and $c_{n+1}$.

There is no contradiction with Theorem 1.2 here, because the set of singular points is not discrete (by a discrete set we mean a set consisting of isolated points). On the other hand, note that all singular points in the octant are essential for the flow $\Phi_8$, which is really the reason why Theorem 1.2 fails in this case:

**Theorem 2.5.** Let $M$ be a surface and suppose that $\Phi_1$ and $\Phi_2$ are flows on $M$ whose sets of essential singular points are discrete. Then $\Phi_1$ and $\Phi_2$ are equivalent if and only if they have equivalent separator configurations.

**Remark 2.6.** If a continuous flow on $M$ possesses a trivial singular point, then it is clear that its set of essential singular points cannot be discrete.

We outline the proof of Theorem 2.5 in the next section. Let us presently emphasize the usefulness of the improved condition on the singular points. On the one hand, recall that it implies, in the zero genus case, that all almost fine orbits are fine (Proposition 2.4(b)). On the other hand, we have:

**Proposition 2.7.** If $(M, \Phi)$ is associated to an analytic vector field $f$, then either $f$ is identically zero or its set of essential singular points is discrete.

**Proof.** If $\text{Sing}(\Phi)$ has non-empty interior, the analyticity of $f$ and the connectedness of $M$ implies that $\text{Sing}(\Phi) = M$; we discard this trivial case in the rest of the proof.

It is a well-known fact that, for every $p \in M$, there exist a neighbourhood $U_p$ of $p$, an analytic map $\rho_p : U_p \to \mathbb{R}$ and an analytic vector field $g_p$ on $U_p$, such that the restriction of $f$ to $U_p$ equals $\rho_p g_p$ and the vector field $g_p$ has no zeros in $U_p \setminus \{p\}$ (see, e.g., [5, Theorem 4.5]). If $S_1$ denotes the set of singular points with the property that any $g_p$ in such a decomposition vanishes at $p$, then $S_1$ is clearly closed and discrete.

Since $\text{Sing}(\Phi)$ is the set of zeros of the analytic map $f_x^2 + f_y^2$, $f = (f_x, f_y)$, the theory of Puiseux series implies that $\text{Sing}(\Phi)$ is locally, at each of its points $p$, a topological star with finitely many branches ("zero" branches meaning that the point is isolated in $\text{Sing}(\Phi)$); moreover, any such branch $B$ can be parametrized via a bijective analytic map $\varphi : [0, 1] \to B$ with $\varphi(0) = p$ (for a proof see, e.g., [5, Theorem 4.3]). We emphasize that $\varphi$ is analytic in the whole closed interval (that is, it can be analytically extended to a larger open interval containing $[0, 1]$). Clearly, the set $S_2$ of points where $\text{Sing}(\Phi)$ is not locally a 2-star (that is, there is not an arc neighbouring $p$ in $\text{Sing}(\Phi)$) is also closed and discrete.

To finish the proof, it then suffices to show that if $S_3$ is the set of essential singular points not included in $S_1 \cup S_2$, then $S_3$ is closed and discrete as well. Let $p \in S_3$ and assume $U_p$ to be small enough so that it is a tubular neighbourhood of $p$ for the non-vanishing vector field $g_p$. We can also assume that there is an analytic bijection $\lambda_p : (-1, 1) \to \text{Sing}(\Phi) \cap U_p$. Since $p$ is not vertical, $\lambda'_p(0)$ must be parallel to $g_p(p)$, that is, $T_p(s) = \lambda'_{p,x}(s) g_{p,y}(\lambda_p(s)) - \lambda'_{p,y}(s) g_{p,x}(\lambda_p(s))$ vanishes for $s = 0$, and since $p$ is not horizontal, $T_p(s)$ cannot be identically zero. Analyticity then implies that there is $\epsilon > 0$ such that $T_p(s)$ does not vanishes at $(-\epsilon, \epsilon) \setminus \{0\}$, that is, all singular points close enough to $p$ are vertical. In particular, $S_3$ is discrete.

To prove that $S_3$ is closed, it suffices to show that if $(p_n)_n$ is a sequence of pairwise distinct points of $S_3$, then it cannot converge. Assume the opposite and call $p$ its limit,
when we can also assume that all points \( p_n \) belong to the same branch \( B \) of the star of singular points with centre \( p \) and are included in the neighbourhood \( U_p \). Find an analytic parametrization \( \varphi : [0, 1] \rightarrow B \) as previously explained, with \( \varphi(t_n) = p_n \) and \( t_n \rightarrow 0 \), and realize that vectors \( \varphi'(t_n) \) and \( g_p(\varphi(t_n)) \) are parallel for all \( n \). Hence, \( \varphi'(t) \) and \( g_p(\varphi(t)) \) are parallel for all \( t \in [0, 1] \), which is to say that all points \( p_n \) are, in fact, horizontal. This contradiction finishes the proof.

**Corollary 2.8.** Let \( M \) be a surface and suppose that \( \Phi_1 \) and \( \Phi_2 \) are flows on \( M \) associated to analytic vector fields. Then \( \Phi_1 \) and \( \Phi_2 \) are equivalent if and only if they have equivalent separator configurations.

### 3. Why the proof of Theorem 1.2 fails, and how to prove Theorem 2.5

Roughly speaking, the proof of Theorem 1.2 by Markus and Neumann goes as follows. First of all, it is shown that each canonical region for a flow \( (M, \Phi) \) is parallel. (The same reasoning still works, word by word, for standard regions; alternatively, notice that each invariant region in a parallel region is parallel as well.) Here observe that, by a simple connectedness argument, all orbits in a canonical (or a standard) region \( \Omega \) share their \( \alpha \)-limit sets and their \( \omega \)-limit sets. Thus it make sense to write \( \alpha(\Omega) \) and \( \omega(\Omega) \), respectively, to denote them.

Next, under the hypotheses of Theorem 1.2 for \( (M, \Phi_1) \) and \( (M, \Phi_2) \), an easy simplification allows to assume that both separatrix configurations are equal, \( S^+ := S^+_1 = S^+_2 \), hence the canonical regions of \( (M, \Phi_1) \) and \( (M, \Phi_2) \) are also equal and the topological equivalence \( h : M \rightarrow M \) we are looking for should map each canonical region into itself. Note that the existence of a toral canonical region implies that \( M = T^2 \); this trivial case can be discarded, for then both \( (T^2, \Phi_1) \) and \( (T^2, \Phi_2) \) are equivalent to the rational flow \( (T^2, \Phi_{ss}) \).

Now the difficult part of the proof comes (Section 3 in [7] and Section 7 in [6]): starting from assuming that \( h \) is the identity on \( S^+ \), it must be homeomorphically extended to each canonical region \( \Omega \) (mapping orbits from \( (M, \Phi_1) \) into orbits from \( (M, \Phi_2) \) and preserving the directions). After explaining how this extension must be done, the authors first check the continuity from “inside” at the so-called accessible regular points from \( \text{Bd} \Omega \) (by accessible we mean that there is a lateral tubular region at the point which is included in \( \Omega \)), then deduce the continuity from “outside” and at the rest of regular points in \( S^+ \), and finally prove the continuity at the (isolated) singular points. If fact, the argument equally works under the weaker hypothesis that the sets of essential singular points are discrete. Continuity at vertical singular points is guaranteed from the very beginning, because they are interior to \( S^+ \); on the other hand, maximal curves of horizontal singular points can be dealt with exactly as if they were regular orbits.

Unfortunately, in their construction Markus and Neumann take for granted the following intuitively obvious (but, as shown by the counterexamples from the previous section, not necessarily true) fact: if a transversal to a canonical region ends at an accessible point from its boundary, then the transversal must be semi-complete. Using standard regions allows to override this difficulty:
Proposition 3.1. Let $\Omega$ be a strip, annular or radial standard region. If $p \in \partial \Omega$ is a regular or a horizontal singular point, and $L \subset \Omega$ is a transversal ending at $p$ (that is, there is an arc $A$ with endpoint $p$ such that $A' = A \setminus \{p\} \subset L$), then $L$ is semi-complete; more precisely, there is a complete transversal to $\Omega$ including $A'$.

Proof. When $\Omega$ is annular, the result is clear.

Assume now that $\Omega$ is a strip and fix a topological equivalence $h$ between $(\Omega, \Phi)$ and $(\mathbb{R}^2, \Phi_s)$, when there is no loss of generality in assuming that $h$ preserves directions (that is, if $T = h^{-1}\{(0) \times \mathbb{R}\}$, then $\Omega^T_T = h^{-1}\{(0, \infty) \times \mathbb{R}\}$) and $h^{-1}(0,0) = q$ is the other endpoint of the arc $A$.

Let $I = (c,d)$ ($-\infty \leq c < 0 < d \leq \infty$) be the open interval and $\mu : I \to \mathbb{R}$ be the continuous map such that $h(L) = \{(\mu(s), s) : s \in I\}$ when, say, $\lim_{s \to d} h^{-1}((\mu(s), s)) = p$. We argue to a contradiction by assuming that $d < \infty$.

We claim that either $\lim_{s \to d} \mu(s) = \infty$ or $\lim_{s \to d} \mu(s) = -\infty$. Otherwise, there would be a sequence $s_n \to d$ with $\mu(s_n) \to r \in \mathbb{R}$, hence $h^{-1}((s_n, \mu(s_n)))$ would converge both to $h^{-1}(d, r)$, a point in $\Omega$, and to $p$, which belongs to $\partial \Omega$. This is impossible.

We suppose, for instance, $\lim_{s \to d} \mu(s) = \infty$. Moreover, slightly modifying $h$ near $q$ if necessary, we can assume $\mu(s) > 0$ for all $s \in (0,d)$. Hence $T' = h^{-1}\{(0) \times (0, \infty)\}$ does not intersect $A'$.

Let $v = h^{-1}(0,d)$. The orbit $\varphi(v)$ is finite, so there is a strong strip $S \subset \Omega$ neighbouring it, with its border orbits also included in $\Omega$, verifying Definition 2.2(ii'). Since the points in $\text{Cl}(S_{T_1}^+)$ which are not in $\Omega$ belong to $\omega(\Omega) = \omega(v)$, and $p$ is one of such points because $h^{-1}(\mu(s), s)$ is included in $S$ if $s$ is close enough to $d$, we get $p \in \omega(\Omega)$.

Fix now a couple of lateral tubular regions $V$ and $W$ at $p$. We can assume that $V \subset \Omega$ and, moreover, $A \subset V$. Let $B$ be the corresponding lateral transversal at $p$ included in $W$. Since $p \in \omega(\Omega)$, all positive semi-orbits $\varphi(z, +)$, $z \in T$, must intersect $B$ (in fact, infinitely many times). Let $q^*$ be the first point from $\varphi(q, +)$ in $B$, and denote by $B' \subset B$ the transversal with endpoints $p$ and $q^*$. Now let $T_0'$ (respectively, $T_1'$) be the set of points $z \in T'$ such that the first intersection point of $\varphi(z, +)$ with $A' \cup B'$ belongs to $A'$ (respectively, to $B'$). Both sets are disjoint and non-empty ($v \in T_1'$, and all points from $T' \cap V$, in particular those close enough to $q$, belong to $T_0'$), its union is the whole $T'$, and they are clearly open in $T'$ because the orbit $\varphi(q)$ does not intersects $T'$. This contradicts the connectedness of $T'$.

Finally, we assume that $\Omega$ is radial and reason again by way of contradiction, assuming that $A'$ does not intersect all orbits of $\Omega$ infinitely many times. It is clear that, without loss of generality, we can suppose that $A'$ does not meet every single orbit in $\Omega$; with more detail, there is no restriction in assuming that there exist some $z \in \Omega$ and some strip neighbourhood of $\varphi(z)$, $S$, such that $S \cap A' = \emptyset$.

Now consider a new flow $\Phi'$ having exactly the same orbits as $\Phi$ in $M \setminus \varphi(z)$ and having $z$ as a singular point. Then $\varphi(z)$, when seen as a subset of $(M, \Phi')$, consists of three separators for $\Phi'$: the singular point $z$ and two regular orbits given by the components of $\varphi(z) \setminus \{z\}$. Moreover, $\Omega = \Omega \setminus \varphi(z)$ is a strip and, clearly, a standard region for $\Phi'$. The previous argument implies that $A'$ is semi-complete for $\Phi'$, which is impossible because it does not intersect $S$. 

\[\square\]
With the help of Proposition 3.1, Theorem 2.5 can be proved, without further changes, as explained above.

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