Protecting Time Series Data with Minimal Forecast Loss

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Forecasting could be negatively impacted due to anonymization requirements in data protection legislation. To measure the potential severity of this problem, we derive theoretical bounds for the loss to forecasts from additive exponential smoothing models using protected data. Following the guidelines of anonymization from the General Data Protection Regulation (GDPR) and California Consumer Privacy Act (CCPA), we develop the $k$-nearest Time Series ($k$-nTS) Swapping and $k$-means Time Series ($k$-mTS) Shuffling methods to create protected time series data that minimizes the loss to forecasts while preventing a data intruder from detecting privacy issues. For efficient and effective decision making, we formally model an integer programming problem for a perfect matching for simultaneous data swapping in each cluster. We call it a two-party data privacy framework since our optimization model includes the utilities of a data provider and data intruder. We apply our data protection methods to thousands of time series and find that it maintains the forecasts and patterns (level, trend, and seasonality) of time series well compared to standard data protection methods suggested in legislation. Substantively, our paper addresses the challenge of protecting time series data when used for forecasting. Our findings suggest the managerial importance of incorporating the concerns of forecasters into the data protection itself.

Key words: optimal data swapping, data privacy, forecast loss, exponential smoothing, data protection methods, anonymization, GDPR

1. Introduction

Protecting data for intended use cases involving forecasting is challenging due to the repeated nature of time series data. First, a data provider needs to protect time series data at every time $i$ in order to prevent a data intruder from exploiting privacy issues in the confidential time series. Second, the data provider needs to ensure that a forecaster can use the protected data to make similar forecasts to those using the confidential data for time $i+1$. These two challenges compete with one another – that is, better data privacy often results in worse forecasts.

Recent data privacy legislation such as the General Data Protection Regulation (GDPR) (European Parliament and Council of European Union (2016)) in the European Union and the California Consumer Privacy Act (CCPA) (California State Legislature (2018)) requires the protection of personal data, such as time series data related to a natural person. To protect personal data, a data provider can use data security measures or anonymization methods. Data security measures
might include encryption or restricting access to users which lowers the chance of a data breach. However, these measures are considered pseudonymization because they are reversible (Article 29 Data Protection Working Party (2014)). They do not randomize or generalize the personal data directly and offer no privacy in the event of a data breach of the confidential data.

On the other hand, anonymization methods permanently and irreversibly alter the personal data which reduces the identifiability of natural persons. Table 1 summarizes standard data protection methods reported from the Article 29 Data Protection Working Party (2014) which was a precursor to GDPR. Standard anonymization methods or data protection methods (DPMs) include suppression (removal or redaction of personal data), generalization (“generalizing, or diluting, the attributes of data subjects by modifying the respective scale or order of magnitude,” p.16), or randomization (altering “the veracity of the data in order to remove the strong link between the data and the individual,” p.12). Anonymization methods may also include more sophisticated DPMs using local differential privacy (Cheu et al. (2021)) or synthetic data regression models with time series data (Schneider and Abowd (2015)). However, a gap in the literature exists on using DPMs to protect time series data intended for forecasting (Boone et al. (2019)). For a further review on other anonymization mechanisms, see Xu and Zhang (2021).

| Standard DPM | Description | Category |
|--------------|-------------|----------|
| Pseudonymization, Encryption, Hash function, Tokenization | Replacing one attribute in a record with another | Neither Generalization or Randomization |
| Aggregating, Averaging | Grouping attribute values so that each individual shares the same value | Generalization |
| Bottom-(or Top)-Coding | Grouping the bottom (or top) percent of attribute values to the same value | Generalization |
| Noise Addition | Adding a random number to a data point | Randomization |
| Swapping or Permutation | Swapping the values of data points | Randomization |

The main issue with standard DPMs is that they do not consider the perspective of the forecaster when choosing how to protect time series data. In practice, data protection is usually chosen by non-forecasters, such as IT or legal counsel. For example, California’s “15/15” rule (Harvey (2013)) requires all time series data on household energy usage to be aggregated into groups of 15 households or more. This generalization is problematic because a DPM using the “15/15” rule is arbitrary with respect to its effects on forecasting. Other organizations also use generalization such as the United States Department of Agriculture which releases public data from the National Household Food Acquisition and Purchase Survey. They require annual income and expenses to
be top-coded at the 99% quantile, liquid assets to be aggregated into three categories (<$2000, $2000 to $3000, ≥ $3000), and number of job changes to be aggregated into two categories (1, more than 1) (Economic Research Service USDA (2016)). Despite the rule-based simplicity of standard DPMs, they are not tailored to minimize forecast loss.

Generalization of personal data also has managerial implications. For example, research shows that aggregating or averaging time series data across time series (Christen et al. (1997)) or time periods (Bass and Leone (1983)) causes an aggregation bias when estimating demand functions. Consequently, debiasing techniques are still being proposed (Wang et al. (2021)) to improve the estimation accuracy from aggregated time series data, but the bias is difficult to eliminate. Similarly, noise addition results in an uncorrectable bias when estimating the correlations between non-temporal data (see Muralidhar et al. (1999)), and swapping methods (Muralidhar and Sarathy (2006)) have been proposed as an alternative. With respect to forecasting, Luo et al. (2018) empirically simulated a hacker on a power grid and found that forecasts are no longer accurate when the underlying time series data are aggregated or randomized with additive Gaussian white noise. An open question that we address from these papers is how to protect time series data that requires updating every time period. In our problem, the data provider has the ability to choose how to anonymize the confidential data every time period before it is shared.

Protecting time series data every time period is also an operational issue and there is a paucity of research on how this affects forecasting practice and management. Good forecast planning involves systematically balancing benefits and costs within a decision-theoretic framework (Hogarth and Makridakis (1981)) which suggests a balance between low forecast loss and good data privacy. Also, protection for time series data requires a data provider to newly share protected data at time period $t$ with all prior data being unprotected. This is an important managerial issue because Taylor and Xiao (2010) found that manufacturers’ profitabilities are altered when their retailers demand forecasting capabilities change in the future. Thus, it is desirable to keep a consistent DPM and maximum forecast loss across all time periods. To address these challenges, we design a two-party framework between the utilities of a data provider and data intruder. The data provider protects data and forecasts on rolling horizon basis, bounding the maximum forecast loss as time rolls forward. Management chooses the amount of forecast loss they are willing to accept in tradeoff with the utility of a data intruder (inverse of data privacy), which is a social choice (Abowd and Schmutte (2019)).

This paper provides three contributions to the literature. First, we develop mathematical models and a decision-making framework for forecasting between a data provider and data intruder. Our goal is to minimize forecast loss while keeping data privacy at a desirable level. When it comes to the information loss in general, this topic has long been studied (e.g., see Menon and Sarkar...
Recently, Baena et al. (2020) studied optimization models with applications for data privacy. Using a large scale discrete optimization, they solved nicely a cell suppression problem in order to remove vulnerable data. In our case, however, such optimization problem formulation would not be suitable since we also want to minimize the forecast loss of time series data, not only the privacy issues. Substantively, our model solves the challenge of protecting time series data for forecasting applications which is a known gap in the literature (Boone et al. 2019). Our two-party framework is twofold: (i) individual decision making of a data provider to trade-off protecting privacy with maintaining reasonable forecasts (ii) individual decision making of a data intruder to exploit privacy issues in time series data. We characterize the two decision makers’ judgements based on their own preferences with utility functions.

Second, we provide theoretical bounds of forecast loss for any DPM, which is defined as the difference between forecasts made using the confidential data and forecasts made using the protected data. We provide these bounds for the additive Holt-Winters exponential smoothing models in Section 2 by decomposing the forecasts. This decomposition also extends into insights about how the underlying time series structures such as levels, trends, and seasonalities, also degrade with data protection. We provide empirical results for the DPMs in Section 6.

Third, we create a DPM that decreases the maximum forecast loss based on a multidimensional Euclidean space of similar time series. The data provider replaces confidential data with the protected data by randomly swapping it with values from $k$-nearest time series or randomly permuting it within $k$ groups of similar time series. Protected data are chosen so that they balance the trade-off between (a) forecast loss and (b) a data intruder’s ability to target a privacy issue in a time series (represented by a utility function). We use similar time series for the randomization within our DPMs because past research found improvements in forecast accuracy when similar time series or past time sequences were explicitly included in the forecasting model. For example, Arroyo and Maté (2009) used $k$-nearest neighbors on past time sequences in order to forecast future patterns of the same time series. Also, Wang et al. (2016) and Sun et al. (2019) used $k$-means to group time series together before fitting multivariate forecasting models separately on each of the groups.

To the best of our knowledge, similarity-based approaches have not been used for protecting time series data in the context of forecasting. Importantly, our approach differs from previous research because we focus on forecast loss at time $i$ rather than forecast accuracy at time $i+1$. This allows the data provider to evaluate and optimize the DPM simultaneously while choosing the DPM in time $i$. The data provider can provide a quality guarantee to a forecaster before the future is realized.

The structure of the paper is as follows. Section 2 establishes the theoretical bounds of forecast loss for any DPM including the standard DPMs in legislation. Section 3 defines the standard DPMs
and introduces our $k$-nearest Time Series (k-nTS) Swapping method to improve the theoretical bound on forecast loss. It also introduces the $k$-means Time Series (k-mTS) Shuffling method to replace actual values with a protected value based on a group of time series rather than an individual time series. Section 4 introduces the data intruder’s utility function and decision analysis. In Section 5, we present the two-party data privacy framework and its related mathematical models for forecasting between the data provider and data intruder, and find its related optimal data swapping strategy. Our main optimization model is presented in Section 5.2. Finally, Section 6 applies our DPM to sensitive time series data and measures the trade-off between forecast loss and data privacy. We also measure how DPMs change time series structures and conclude in Section 7.

2. Forecast Loss from Data Protection

We define the forecast loss as $F_{i+1} - F_{i+1}^*$, where $F_{i+1}$ is a forecast using the confidential data and $F_{i+1}^*$ is a forecast using the protected data. Framing the problem in terms of forecast loss provides us with two key advantages. First, unlike traditional forecast accuracy measures such as mean squared error, forecast loss can be computed at time $i$ (for time $i+1$) without the knowledge of the confidential value, $A_{i+1}$, at time $i+1$. Thus, the data provider can alter the protected data, $P_i$, at time $i$ if the forecast loss is too large. Second, we can establish a priori bounds of forecast loss with the commonly used exponential smoothing models in subsection 2.1. These bounds also imply a bound for the change in forecast accuracy at time $i+1$ in subsection 2.2.

2.1. Forecast Loss for Exponential Smoothing

We use the additive Holt-Winters’ equations to establish tractable bounds on forecast loss for any data protection method. The Holt-Winters’ forecasts are a function of the level ($l_t$), trend($b_t$), and seasonality ($s_t$) of each time series. These components differ depending on whether the confidential data, $A_t$, or the protected data, $P_t$, was input into the equations for single exponential smoothing (SES), double exponential smoothing (DES), and triple exponential smoothing (TES). With data protection starting at time $t$, if $|A_t - P_t|$ is bounded by some value, say $M \in \mathbb{R}_+$ (i.e., $|A_t - P_t| \leq M$), for any integer $i \geq t$, then the forecast loss at any time points after $t$ can be written up in terms of smoothing factors and $M$. Specifically, we discovered a functional form of inequalities, i.e., the bounds for the forecast loss at time $T+1$ for any $T \geq t+1$, which is a geometric series $M \sum_{n=1}^{T-t} a(\cdot)r(\cdot)^{n-1}$ where $a(\cdot)$, the initial term and $r(\cdot)$, the common ratio are both the functions of related smoothing factors $\alpha, \beta, \gamma$.

2.1.1. Single Exponential Smoothing (SES) Let us begin with single exponential smoothing. We choose exponential smoothing because it is commonly used by practitioners and robust
under distributional assumptions which will be required for arbitrary data protection methods (see, e.g., Gelper et al. (2009)). The forecasting equation is the following,

\[ F_{t+1} = \alpha A_t + (1 - \alpha) F_t, \quad 0 \leq \alpha \leq 1, \quad t \in \mathbb{Z}_+, \quad (1) \]

where \( \alpha \) is the smoothing factor for the level \( l \). We assume that the data provider begins data protection starting at time \( t \) and therefore, we let \( F_{t+1} = l_t \). Then, given confidential data up to time \( t-1 \), we can write the level for the confidential data, \( l_t \) and the level for protected data, \( l_t^* \) as

Confidential: \[ l_t = \alpha A_t + (1 - \alpha) l_{t-1} \]

Protected: \[ l_t^* = \alpha P_t + (1 - \alpha) l_{t-1}. \quad (2) \]

**Theorem 1** (Bounded forecast loss in case of SES). Suppose that data protection starts at time \( t \). For any integer \( i \geq t \), if \( |A_i - P_i| \) is bounded by \( M \in \mathbb{R}_+ \) (i.e., \( |A_i - P_i| \leq M \)), then we have a recursive formula for the forecast loss bound only in terms of \( M \) and \( \alpha \). For any \( T \geq t+1 \), the bound for the forecast loss \( F - F^* \) can be written up as:

\[ |F_T - F_T^*| \leq \alpha M + (1 - \alpha) |F_{t+1} - F_t^*| \quad (3) \]

with the initial value \( F_{t+1} - F_{t+1}^* = \alpha (A_t - P_t) \). Equivalently, we have the following

\[ |F_T - F_T^*| \leq \sum_{n=1}^{T-t} (1 - \alpha)^{n-1} (2\alpha - \alpha^2) M, \quad (4) \]

which is a geometric series \( M \sum_{n=1}^{T-t} a^{(r)^n} \) where \( a \), the initial term and \( r \), the common ratio are the functions of related smoothing factor \( \alpha \).

**Proof.** Suppose that data protection starts at time \( t \). Then \( F_{t+1} = l_t \) and \( F_{t+1}^* = l_t^* \). Subtraction of \( l_t \) from \( l_t^* \) (in (2)) gives us the following:

\[ l_t - l_t^* = \alpha (A_t - P_t) = \alpha \Delta_t, \quad (5) \]

where \( \Delta_t = A_t - P_t \) and this is equivalent to the forecast loss for time \( t+1 \):

\[ F_{t+1} - F_{t+1}^* = \alpha \Delta_t. \quad (6) \]

Next, we consider the case of data protection in both time period \( t \) and \( t+1 \). If we roll forward data protection to \( t+1 \), then we have for the confidential data:

\[ l_{t+1} = \alpha A_{t+1} + (1 - \alpha) l_t = \alpha A_{t+1} + (1 - \alpha) (\alpha A_t + (1 - \alpha) l_{t-1}), \quad (7) \]

and in case of the protected data:

\[ l_{t+1}^* = \alpha P_{t+1} + (1 - \alpha) l_t^* = \alpha P_{t+1} + (1 - \alpha) (\alpha P_t + (1 - \alpha) l_{t-1}). \quad (8) \]
By subtraction of (8) from (7) we get:

\[ l_{t+1} - l^*_{t+1} = \alpha(A_{t+1} - P_{t+1}) + \alpha(1 - \alpha)(A_t - P_t) = \alpha \Delta_{t+1} + \alpha(1 - \alpha) \Delta_t, \]

which is equivalent to the forecast loss for \( t + 2 \):

\[ F_{t+2} - F^*_{t+2} = \alpha \Delta_{t+1} + \alpha(1 - \alpha) \Delta_t = \alpha \Delta_{t+1} + (1 - \alpha)(F_{t+1} - F^*_{t+1}) \]  

(9)

Since data protection starts at time \( t \) (confidential data up to time \( t - 1 \)), for any time \( T \geq t + 1 \) we have the same form as in the equation (10) only with different subscripts. Therefore, we can write the following recursion:

\[ F_{T+1} - F^*_{T+1} = \alpha \Delta_T + (1 - \alpha)(F_T - F^*_T). \]  

(11)

For any integer \( i \geq t \), if \( |A_i - P_i| \) is bounded by \( M \in \mathbb{R}_+ \) (i.e., \( |A_i - P_i| = |\Delta_i| \leq M \)), we have the following

\[ |F_{T+1} - F^*_{T+1}| \leq \alpha M + (1 - \alpha)|F_T - F^*_T|. \]  

(12)

Using the initial value \( F_{t+1} - F^*_{t+1} = \alpha \Delta_t \) and the recursion, we have the following

\[ |F_{T+1} - F^*_{T+1}| \leq \sum_{n=1}^{T-t} (1 - \alpha)^{n-1} (\alpha M + (1 - \alpha)|F_{t+1} - F^*_{t+1}|) \]

\[ \leq \sum_{n=1}^{T-t} (1 - \alpha)^{n-1} (2\alpha - \alpha^2) M, \]  

(13)

which completes the proof. \( \square \)

**Remark 1 (Data protection method for bounding \( A_n - P_n \)).** Note that a sequence \( \Delta_n = A_n - P_n \) for all \( n \in \mathbb{Z}_+ \) can be bounded if we use our proposed data protection methods (in Sections 3.1, 3.2). Given the number of time periods for protection is \( T - t + 1 \), we establish that the maximal forecast loss for \( T + 1 \) is bounded by the maximal difference \( (M \in \mathbb{R}_+) \) between confidential and protected data over all time periods. This implies that the bounds in forecast loss from standard DPMs in Table 1 are larger if they have a single time period where the difference between the confidential and protected data is large. For example, top-coding will have a large bound in forecast loss if the confidential data was much larger than the p% upper quantile limit at any time period. Additive noise will have a bound that depends on the largest generated random noise in any time period. Since \( P_n \) plays a significant role, the data provider can limit the maximal forecast loss in \( T + 1 \) by carefully choosing \( P_n \) with respect to \( A_n \).
2.1.2. Double Exponential Smoothing (DES) We turn our attention to forecasting with DES. In addition to the smoothing factor $\alpha$ for the level $l$, let us consider a smoothing factor $\beta$ for the trend $b$ (assuming there is a trend). The following is the DES formula for the confidential data without data protection:

\[
\begin{align*}
  l_t &= \alpha A_t + (1 - \alpha) F_t \\
  b_t &= \beta (l_t - l_{t-1}) + (1 - \beta) b_{t-1} \\
  F_{t+1} &= l_t + b_t,
\end{align*}
\]

(14)

where $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$, and the forecast horizon is 1. Suppose that time series data up to time $t - 1$ are not protected. Then, with protected data at time $t$ we have the following equations

\[
\begin{align*}
  l_t^* &= \alpha P_t + (1 - \alpha)(l_{t-1} + b_{t-1}) \\
  b_t^* &= \beta (l_t^* - l_{t-1}) + (1 - \beta) b_{t-1} \\
  F_{t+1}^* &= l_t^* + b_t^*.
\end{align*}
\]

(15)

\textbf{Theorem 2} (Bounded forecast loss in case of DES). Suppose that data protection starts at time $t$. For any integer $i \geq t$, if $|A_i - P_i|$ is bounded by $M \in \mathbb{R}_+$ (i.e., $|A_i - P_i| \leq M$), then we have a recursive formula for the forecast loss bound in terms of $M$, $\alpha$ and $\beta$. For any $T \geq t + 1$, the bound for the forecast loss $F - F^*$ can be written up as:

\[
|F_{T+1} - F^*_{T+1}| \leq (\alpha + 2\alpha \beta) M + (1 - \alpha(1 + \beta))|F_T - F^*_T|,
\]

(16)

with the initial value $F_{t+1} - F^*_{t+1} = \alpha(1 + \beta)(A_t - P_t)$. Equivalently, we have the following

\[
|F_{T+1} - F^*_{T+1}| \leq \sum_{n=1}^{T-t} (1 - \alpha(1 + \beta))^{n-1}(2\alpha + 3\alpha \beta - \alpha^2(1 + \beta)^2) M,
\]

(17)

which is a geometric series $M \sum_{n=1}^{T-t} a(\cdot)r(\cdot)^{n-1}$ where $a$, the initial term and $r$, the common ratio are the functions of related smoothing factors $\alpha$ and $\beta$.

\textbf{Proof.} Subtracting the corresponding equations of (15) from those of (14) gives us

\[
\begin{align*}
  l_t - l_t^* &= \alpha(A_t - P_t) = \alpha \Delta_t, \\
  b_t - b_t^* &= \beta(l_t - l_t^*) = \beta \alpha(A_t - P_t) = \alpha \beta \Delta_t,
\end{align*}
\]

(18) \hspace{1cm} (19)

where $\Delta_t = A_t - P_t$. Since $F_{t+1} = l_t + b_t$ and $F^*_{t+1} = l_t^* + b_t^*$, we can write the forecast loss $F_{t+1} - F^*_{t+1}$ for time $t + 1$ by adding (18) to (19) as the following:

\[
F_{t+1} - F^*_{t+1} = (\alpha + \alpha \beta) \Delta_t = \alpha(1 + \beta) \Delta_t.
\]

(20)
For $F_{t+2}$ and $F_{t+2}^*$ we roll forward to protect data at time $t + 1$, provided that all time series up to time $t - 1$ are not protected as in case of $F_{t+1}$. Since $F_{t+2} - F_{t+2}^* = l_{t+1} + b_{t+1} - (l_{t+1}^* + b_{t+1}^*)$, let us present equation for each term:

\[
\begin{align*}
l_{t+1} &= \alpha A_{t+1} + (1 - \alpha)(l_t + b_t) \\
b_{t+1} &= \beta(l_{t+1} - l_t) + (1 - \beta)b_t, \\
l_{t+1}^* &= \alpha P_{t+1} + (1 - \alpha)(l_t^* + b_t^*) \\
b_{t+1}^* &= \beta(l_{t+1}^* - l_t^*) + (1 - \beta)b_t^*,
\end{align*}
\]

(21)

Together with (18) and (19), subtraction of corresponding equations of (22) from those of (21) gives us:

\[
\begin{align*}
l_{t+1} - l_{t+1}^* &= \alpha(A_{t+1} - P_{t+1}) + (1 - \alpha)[(l_t - l_t^*) + (b_t - b_t^*)] \\
&= \alpha\Delta_{t+1} + (1 - \alpha)(F_{t+1} - F_{t+1}^*). \\
b_{t+1} - b_{t+1}^* &= \beta[(l_{t+1} - l_{t+1}^*) - (l_t - l_t^*)] + (1 - \beta)(b_t - b_t^*) \\
&= \beta(l_{t+1} - l_{t+1}^*) - \beta[(l_t - l_t^*) + (b_t - b_t^*)] + (b_t - b_t^*) \\
&= \beta \{\alpha\Delta_{t+1} + (1 - \alpha)(F_{t+1} - F_{t+1}^*)\} - \beta(F_{t+1} - F_{t+1}^*) + \alpha\beta\Delta_t \\
&= \alpha\beta\Delta_{t+1} - \alpha\beta(F_{t+1} - F_{t+1}^*) + \alpha\beta\Delta_t \\
&= \alpha\beta\{\Delta_{t+1} + \Delta_t - (F_{t+1} - F_{t+1}^*)\}.
\end{align*}
\]

(23)

By adding up (23) and (24) we have the forecast loss for $t + 2$ as

\[
F_{t+2} - F_{t+2}^* = \alpha\Delta_{t+1} + \alpha\beta\Delta_{t+1} + \alpha\beta\Delta_t + (1 - \alpha - \alpha\beta)(F_{t+1} - F_{t+1}^*).
\]

(25)

Since data protection starts at time $t$ (confidential data up to time $t - 1$), for any time $T \geq t + 1$ we have the same form as in the equation (25) only with different subscripts. Therefore, we can write the following recursion:

\[
F_{T+1} - F_{T+1}^* = \alpha\Delta_T + \alpha\beta\Delta_T + \alpha\beta\Delta_{T-1} + (1 - \alpha - \alpha\beta)(F_T - F_T^*).
\]

(26)

Let $M \in \mathbb{R}_+$ such that $|A_n - P_n| = |\Delta_n| \leq M$ for all $n \in \mathbb{Z}_+$. Then (26) can be written up as

\[
|F_{T+1} - F_{T+1}^*| \leq (\alpha + 2\alpha\beta)M + (1 - \alpha(1 + \beta))|F_T - F_T^*|.
\]

(27)

Using the initial value $F_{t+1} - F_{t+1}^* = \alpha(1 + \beta)\Delta_t$ and recursion, we have the following

\[
|F_{T+1} - F_{T+1}^*| \leq \sum_{n=1}^{T-t} (1 - \alpha(1 + \beta))^{n-1} \left[(\alpha + 2\alpha\beta)M + (1 - \alpha(1 + \beta))|F_{t+1} - F_{t+1}^*|\right]
\]

\[
\leq \sum_{n=1}^{T-t} (1 - \alpha(1 + \beta))^{n-1}(2\alpha + 3\alpha\beta - \alpha^2(1 + \beta)^2)M,
\]

(28)

which completes the proof. \(\square\)
Note that from (26), we can remove some terms to express the bound only with $M$ and $\alpha$:

$$|F_{T+1} - F_{T+1}^*| \leq \alpha M + \alpha \beta M + \alpha \beta M + (1 - \alpha - \alpha \beta)|F_T - F_T^*|$$

$$\leq 3\alpha M + (1 - \alpha)|F_T - F_T^*|.$$  \hspace{1cm} (29)

The second inequality of (29) holds because $0 \leq \alpha \beta \leq \alpha$. Using $|F_{t+1} - F_{t+1}^*| \leq \alpha(1 + \beta)M \leq 2\alpha M$, it follows that for any $T \geq t + 1$, we have the following

$$|F_{T+1} - F_{T+1}^*| \leq \frac{T - t}{\alpha} \sum_{n=1}^{T-t} (1 - \alpha)^{n-1} [3\alpha M + 2\alpha(1 - \alpha)M]$$

$$= \frac{T - t}{\alpha} \sum_{n=1}^{T-t} (1 - \alpha)^{n-1} (5\alpha M - 2\alpha^2 M)$$

$$\leq \frac{T - t}{\alpha} \sum_{n=1}^{T-t} 5\alpha(1 - \alpha)^{n-1}M,$$

where the last inequality can be used in practice when $\alpha^2$ and $\alpha \beta$ are negligible.

2.1.3. Triple Exponential Smoothing (TES) The standard Holt-Winters’ triple exponential smoothing (TES) equations are as follows:

$$F_{t+1} = l_t + b_t + s_{t+1-m},$$ \hspace{1cm} (31)

where

$$l_t = \alpha(A_t - s_{t-m}) + (1 - \alpha)(l_{t-1} + b_{t-1})$$

$$b_t = \beta(l_t - l_{t-1}) + (1 - \beta)b_{t-1}$$

$$s_t = \gamma(A_t - l_{t-1} - b_{t-1}) + (1 - \gamma)s_{t-m},$$ \hspace{1cm} (32)

where the forecast horizon is 1, $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$, $0 \leq \gamma \leq (1 - \alpha)$, $F_{t+1}$ is the forecast for time $t + 1$ made at time $t$, $l_t$ is the smoothed level, $b_t$ is the trend, $m$ is the frequency of the time series, and $s_t$ is the seasonal adjustment.

However, unlike SES and DES, forecasts using TES depend on $m$ seasonality adjustments computed from the time period before to ensure that they sum to zero. To do this continuous normalization, McKenzie (1986) (and later Archibald and Koehler (2003)) first suggests reparameterizing the above equations as follows

$$l_t = l_{t-1} + b_{t-1} + \alpha(A_t - F_t)$$ \hspace{1cm} (33)

$$b_t = b_{t-1} + \alpha \beta(A_t - F_t)$$ \hspace{1cm} (34)

$$s_t = s_{t-m} + \gamma(1 - \alpha)(A_t - F_t),$$ \hspace{1cm} (35)

where the initial coefficients $\alpha$, $\beta$, and $\gamma$ are multiplied together and only the latest seasonality adjustment is recomputed. Then, to maintain the same forecasts, we subtract

$$r_t = \frac{\gamma(1 - \alpha)}{m}(A_t - F_t)$$ \hspace{1cm} (36)
from all seasonality adjustments (including the latest in equation (35)) and add this \( r_t \) to \( l_t \). Without loss of generality, \( s(t-1) \) are the seasonality adjustments computed at time \( t-1 \) and \( s \) are the seasonality adjustments computed at time \( t \). The final equations with continuous normalization for TES are as follows. The equations become

\[
l_t = l_{t-1} + b_{t-1} + \alpha(A_t - F_t) + r_t \tag{37}
\]

\[
b_t = b_{t-1} + \alpha \beta(A_t - F_t) \tag{38}
\]

\[
s_t = s_{t-m}(t-1) + \gamma(1-\alpha)(A_t - F_t) - r_t
\]

\[
= s_{t-m}(t-1) + (m-1)r_t \tag{39}
\]

followed by

\[
s_{t+k-m} = s_{t+k-m}(t-1) - r_t, \text{ for } k = 1, \ldots, m-1. \tag{40}
\]

Note that this reparameterization implies \( F_{t+1} = l_t + b_t + s_{t+1-m} \) is unchanged since \( r_t \) is both added and subtracted to \( l_t \) and \( s_t \), respectively. Also, the seasonality factors computed at time \( t \) sum to 0 as long as the seasonality factors at time \( t-1 \) do. Also note that the following

\[
\sum_{k=1}^{m} s_{t+k-m}(t-1) = 0 \tag{41}
\]

implies

\[
\sum_{k=1}^{m} s_{t+k-m} = \sum_{k=1}^{m} s_{t+k-m}(t-1) - (m+1)r_t + (m-1)r_t = 0. \tag{42}
\]

Suppose that time series data up to time \( t-1 \) are not protected. Then, by substituting \( P_t \) we have the following subtractions with \( k = 1 \):

\[
l_t - l_t^* = \alpha(A_t - P_t) + \frac{\gamma(1-\alpha)}{m}(A_t - P_t) \tag{43}
\]

\[
b_t - b_t^* = \alpha \beta(A_t - P_t) \tag{44}
\]

\[
s_{t+1-m} - s_{t+1-m}^* = \frac{\gamma(1-\alpha)}{m}(P_t - A_t) \tag{45}
\]

Using \( F_{t+1} = l_t + b_t + s_{t+1-m} \), the last two terms for (43) and (45) sum to 0 and the forecast loss at time \( t+1 \) becomes

\[
F_{t+1} - F_{t+1}^* = (\alpha + \alpha \beta)(A_t - P_t) = \alpha (1+\beta) \Delta_t, \tag{46}
\]

which is the same as DES in equation (20) and only depends on \( \Delta_t \). Note that, for any time point \( i \), all of \( l_i - l_i^*, b_i - b_i^*, s_{i+1-m} - s_{i+1-m}^* \) are in terms of \( \Delta_i = A_i - P_i \) with given constants \( \alpha, \beta, \gamma \). We also know that rolling forward to later time periods, forecast loss comes from that of the previous time periods. Therefore, it is not hard to see that the forecast loss \( F - F^* \) at time \( T+1 \) (for any \( T \geq t+1 \)) can be written up in terms of forecast losses \( F_i - F_i^* \) and \( \Delta_i = A_i - P_i \) for \( i = t, t+1, \ldots, T \) up to time \( T \).
Theorem 3 (Bounded forecast loss in case of TES). Suppose that data protection starts at time $t$. For any integer $i \geq t$, if $|A_i - P_i|$ is bounded by $M \in \mathbb{R}_+$ (i.e., $|A_i - P_i| \leq M$), then we have a recursive formula for the forecast loss bound only in terms of $M, \alpha, \beta, \gamma$ and $m$. For any $T \geq t + 1$, the bound for the forecast loss $F - F^*$ can be written up as:

$$|F_{t+1} - F^*_{t+1}| \leq \left(2\alpha + 3\alpha\beta + \frac{\gamma(1-\alpha)(m+1)}{m}\right) M + (\alpha + \alpha\beta + \gamma(1 - \alpha))|F_T - F^*_T|$$

(47)

with the initial value $F_{t+1} - F^*_{t+1} = \alpha(1 + \beta)(A_i - P_i)$. Equivalently, for any $T \geq t + 1$, we have the following

$$|F_T - F^*_{T+1}| \leq \sum_{n=1}^{T-t} r^{n-1} \left(2\alpha + 3\alpha\beta + (\alpha + \alpha\beta)^2 + \gamma(1 - \alpha) \left(\frac{m+1}{m} + \alpha + \alpha\beta\right)\right) M,$$  

(48)

where $r = \alpha + \alpha\beta + \gamma(1 - \alpha)$ and $m$ designates the frequency of the time series. Note that this is a geometric series $M \sum_{n=1}^{T-t} a(\cdot)r(\cdot)^{n-1}$ where $a$, the initial term and $r$, the common ratio are the functions of related smoothing factors $\alpha, \beta$ and $\gamma$.

Proof. Data protection starts at time $t$ and let $\Delta_t = A_t - P_t$. Then for the forecast loss at time $t + 1$, by (46), we can write $F_{t+1} - F^*_{t+1} = \alpha(1 + \beta)\Delta_t$. For $F_{t+2}$ and $F^*_{t+2}$ we roll forward to protect data at time $t + 1$. Since $F_{t+2} - F^*_{t+2} = l_{t+1} + b_{t+1} + s_{t+2-m} - (l^*_{t+1} + b^*_{t+1} + s^*_{t+2-m})$, let us present the components for both $F_{t+2}$ and $F^*_{t+2}$. The components of $F_{t+2}$ are:

$$l_{t+1} = l_t + b_t + \alpha(A_{t+1} - F_{t+1}) + r_{t+1}$$
$$b_{t+1} = b_t + \alpha\beta(A_{t+1} - F_{t+1})$$
$$s_{t+2-m} = s_{t+2-m}(t + 1) + (m - 1)r_{t+1}$$

(49)

and for $F^*_{t+2}$ we have:

$$l^*_{t+1} = l^*_t + b^*_t + \alpha(P_{t+1} - F^*_{t+1}) + r^*_{t+1}$$
$$b^*_{t+1} = b^*_t + \alpha\beta(P_{t+1} - F^*_{t+1})$$
$$s^*_{t+2-m} = s^*_{t+2-m}(t + 1) + (m - 1)r^*_{t+1}.$$  

(50)

Subtraction of corresponding equations of (50) from those of (49) gives us:

$$l_{t+1} - l^*_{t+1} = (l_t - l^*_t) + (b_t - b^*_t) + \alpha\Delta + \alpha(\Delta_{t+1} - \alpha(F_{t+1} - F^*_{t+1})) + r_{t+1} = r^*_{t+1}$$
$$b_{t+1} - b^*_{t+1} = (b_t - b^*_t) + \alpha\beta\Delta_{t+1} - \alpha\beta(F_{t+1} - F^*_{t+1})$$
$$s_{t+2-m} - s^*_{t+2-m} = (m - 1)(r_{t+1} - r^*_{t+1}),$$

(51)

which is followed by

$$F_{t+2} - F^*_{t+2} = \left(l_{t+1} - l^*_{t+1}\right) + (b_{t+1} - b^*_{t+1}) + \left(s_{t+2-m} - s^*_{t+2-m}\right)$$
$$= (l_t - l^*_t) + 2(b_t - b^*_t) + (\alpha + \alpha\beta)\Delta + \alpha\beta\Delta_{t+1} - \alpha\beta(F_{t+1} - F^*_{t+1}) + m(r_{t+1} - r^*_{t+1})$$
$$= \left(\alpha + 2\alpha\beta + \frac{\gamma(1-\alpha)}{m}\right)\Delta + (\alpha + \alpha\beta)\Delta_{t+1} - (\alpha + \alpha\beta)(F_{t+1} - F^*_{t+1})$$
$$+ m\left(\frac{\gamma(1-\alpha)}{m}(A_{t+1} - F_{t+1}) - \frac{\gamma(1-\alpha)}{m}(P_{t+1} - F^*_{t+1})\right)$$
$$= \left(\alpha + 2\alpha\beta + \frac{\gamma(1-\alpha)}{m}\right)\Delta + (\alpha + \alpha\beta)\Delta_{t+1} - (\alpha + \alpha\beta)(F_{t+1} - F^*_{t+1})$$
$$+ \gamma(1 - \alpha)(\Delta_{t+1} - (F_{t+1} - F^*_{t+1}))$$
$$= \left(\alpha + 2\alpha\beta + \frac{\gamma(1-\alpha)}{m}\right)\Delta + (\alpha + \alpha\beta + \gamma(1 - \alpha))(\Delta_{t+1} - (F_{t+1} - F^*_{t+1})).$$

(52)
Since data protection starts at time $t$ (confidential data up to time $t-1$), for any time $T \geq t+1$ we have the same form as in the above equation only with different subscripts. Therefore, we can write the following recursion:

$$F_{T+1} - F^*_T = \left(\alpha + 2\alpha \beta + \frac{\gamma(1-\alpha)}{m}\right) \Delta_{T-1} + (\alpha + \alpha \beta + \gamma(1-\alpha)) (\Delta_T - (F_T - F^*_T)).$$ \hspace{1cm} (53)

Let $M \in \mathbb{R}_+$ such that $|A_n - P_n| = |\Delta_n| \leq M$ for all $n \in \mathbb{Z}_+$. Then (53) can be written up as

$$|F_{T+1} - F^*_T| \leq \left(\alpha + 2\alpha \beta + \frac{\gamma(1-\alpha)}{m}\right) M + (\alpha + \alpha \beta + \gamma(1-\alpha)) (M + |F_T - F^*_T|)$$

$$= \left(2\alpha + 3\alpha \beta + \frac{\gamma(1-\alpha)(m+1)}{m}\right) M + (\alpha + \alpha \beta + \gamma(1-\alpha)) |F_T - F^*_T|$$ \hspace{1cm} (54)

with the initial value $F_{t+1} - F^*_t = \alpha (1 + \beta) \Delta_t$ as in (46). If we let $r = \alpha + \alpha \beta + \gamma(1-\alpha)$, then using the initial value and recursion, we can write the following

$$|F_{T+1} - F^*_T| \leq \sum_{n=1}^{T-t} r^{n-1} \left[ \left(2\alpha + 3\alpha \beta + \frac{\gamma(1-\alpha)(m+1)}{m}\right) M + r |F_{t+1} - F^*_t| \right]$$

$$\leq \sum_{n=1}^{T-t} r^{n-1} \left(2\alpha + 3\alpha \beta + (\alpha + \alpha \beta)^2 + \gamma(1-\alpha) \left(\frac{m+1}{m} + \alpha + \alpha \beta\right)\right) M,$$ \hspace{1cm} (55)

which completes the proof. \hspace{1cm} \textcircled{□}

### 2.2. Changes in Actual Forecast Accuracy and Perceived Forecast Accuracy

The Mean Absolute Error (MAE) and Mean Absolute Scaled Error (MASE, \textcite{Hyndman and Koehler (2006)}) measure the actual forecast accuracy using the confidential data at a future time after the data has been protected. We average over $T - t$ time periods from time $t+1$ to $T$ to compute forecast accuracy. Intuitively, the MASE is less than one when, on average, the MAE of a forecasting model is less than the MAE of the one-step Naïve model, where the one-step Naïve model is defined by using the last confidential value at time $t$ to forecast the confidential value at time $t+1$. The MAE and MASE are defined as

$$\text{MAE} = \frac{1}{T-t} \sum_{i=t+1}^{T} |F_i - A_i|,$$ \hspace{1cm} (56)

$$\text{MASE} = \frac{\frac{1}{T-t} \sum_{i=t+1}^{T} |F_i - A_i|}{\frac{1}{T-t} \sum_{i=t+1}^{T} |A_{i-1} - A_i|},$$ \hspace{1cm} (57)

where $F_i$ is the forecast for time $i$ made at time $i-1$ using the confidential data, $A_i$ is the confidential value realized at time $i$, $t+1$ is the start of the forecast period, and $T$ is the end of the forecast period. Note that the MAE and MASE can only be computed by the data provider, since $A_i$ is confidential and never released to a forecaster. Also, we change the denominator of MASE slightly from \textcite{Hyndman and Koehler (2006)} since data protection does not start until time $t$. 
To evaluate the change in forecast accuracy due to data protection, the data provider measures the change in MAE (and MASE) between forecasts using the protected data and forecasts using the confidential data. The data provider substitutes $F^*_i$ for $F_i$ to compute $\Delta MAE$ and $\Delta MASE$ using the results in Section 2.1:

$$\Delta MAE = \frac{1}{T-t} \sum_{i=t+1}^{T} |F_i - A_i| - \frac{1}{T-t} \sum_{i=t+1}^{T} |F^*_i - A_i|$$ (58)

and

$$\Delta MASE = \frac{1}{\sum_{i=t+1}^{T} |A_{i-1} - A_i|} \left( \sum_{i=t+1}^{T} |F_i - A_i| - \sum_{i=t+1}^{T} |F^*_i - A_i| \right)$$ (59)

**Theorem 4 (Change in Mean Absolute Error).** The change in MAE (Mean Absolute Error) is always bounded above by the average of forecast loss, $\Delta MAE \leq \frac{1}{T-t} \sum_{i=t+1}^{T} |F_i - F^*_i|$. 

**Proof.**

$$\Delta MAE = \frac{1}{T-t} \sum_{i=t+1}^{T} |F_i - A_i| - \frac{1}{T-t} \sum_{i=t+1}^{T} |F^*_i - A_i|$$

$$= \frac{1}{T-t} \left( \sum_{i=t+1}^{T} |F_i - F^*_i + F^*_i - A_i| - \sum_{i=t+1}^{T} |F^*_i - A_i| \right)$$

$$\leq \frac{1}{T-t} \left( \sum_{i=t+1}^{T} |F_i - F^*_i| + \sum_{i=t+1}^{T} |F^*_i - A_i| - \sum_{i=t+1}^{T} |F^*_i - A_i| \right)$$

$$= \frac{1}{T-t} \sum_{i=t+1}^{T} |F_i - F^*_i|,$$

which completes the proof. □

The above theorem implies that for additive exponential smoothing models, the change in forecast accuracy is bounded by the choice of smoothing parameters $\alpha, \beta, \gamma$, frequency $m$, and $M$, the upper bound of $|A_i - P_i|$ for all $i$ ($M$ can be considered the quality guarantee provided by the data provider). Equations (13), (30) and (55) can be used in time $i$ to predetermine the exact bound for the change in MAE for a future time period $i+1$.

So far, we focused only on the change in forecast accuracy without considering the experience of the end user, a forecaster. Unlike the data provider, a forecaster only has access to the protected data $P_i$ and the forecasts using the protected data, $F^*_i$. Hence, the forecaster only perceives a forecast accuracy instead of measuring an actual forecast accuracy. We define the forecaster’s Perceived Mean Absolute Error (PMAE) and the Perceived Mean Absolute Scaled Error (PMASE) similarly as follows:

$$PMAE = \frac{1}{T-t} \sum_{i=t+1}^{T} |F^*_i - P_i|$$ (61)
Theorem 5 (Perceived Mean Absolute Error (PMAE)). The forecaster’s Perceived Mean Absolute Error (PMAE) is always bounded as
\[ PMAE \leq MAE - \Delta MAE + M, \]
where \( M \) is the upper bound of \( |A_i - P_i| \) for all \( i \), which can be considered the quality guarantee provided by the data provider. PMAE becomes smaller as \( \Delta MAE \) in equation (60) increases, which acts in opposition to \( M \).

Proof. Using equations (56) and (58), we can write equation (61) as
\[
PMAE = \frac{1}{T-t} \sum_{i=t+1}^{T} |F_i^* - P_i| \\
= \frac{1}{T-t} \sum_{i=t+1}^{T} |F_i^* - A_i + A_i - P_i| \\
\leq \frac{1}{T-t} \sum_{i=t+1}^{T} \{ |F_i^* - A_i| + |A_i - P_i| \} \\
= MAE - \Delta MAE + \frac{1}{T-t} \sum_{i=t+1}^{T} |A_i - P_i| \\
\leq MAE - \Delta MAE + M,
\]
which completes the proof. \( \Box \)

Remark 2. Note that we find that the forecaster’s perceived forecast accuracy is bounded upward by the actual forecast accuracy and bound in forecast loss, and downward by the change in forecast accuracy due to data protection (\( \Delta MAE \)). Thus, it is important for \( \Delta MAE \) to be small so that the forecaster does not perceive their forecast accuracy to be less than it actually is. Data protection methods that minimize the average forecast loss in all time periods decrease \( \Delta MAE \) and lead to a more truthful representation of PMAE.

Corollary 1 (Perceived Mean Absolute Error (PMAE)). The forecaster’s Perceived Mean Absolute Error (PMAE) is always bounded as the following:
\[ PMAE \leq MAE + \text{Average of } |A_i - P_i| + \text{Average Forecast Loss}. \]

Proof. Equation (61) can be written up as
\[
PMAE = \frac{1}{T-t} \sum_{i=t+1}^{T} |F_i^* - P_i| \\
= \frac{1}{T-t} \sum_{i=t+1}^{T} |F_i^* - A_i + A_i - P_i| \\
\leq \frac{1}{T-t} \sum_{i=t+1}^{T} \{ |F_i^* - A_i| + |A_i - P_i| \} \\
= \frac{1}{T-t} \sum_{i=t+1}^{T} |F_i^* - A_i| + \frac{1}{T-t} \sum_{i=t+1}^{T} |A_i - P_i| \\
\leq \frac{1}{T-t} \sum_{i=t+1}^{T} |F_i^* - F_i| + \frac{1}{T-t} \sum_{i=t+1}^{T} |F_i - A_i| + \frac{1}{T-t} \sum_{i=t+1}^{T} |A_i - P_i| \\
= \text{Average Forecast Loss} + MAE + \text{Average of } |A_i - P_i|,
\]
which completes the proof. \( \Box \)
Section 3 will show that an outlier time series with unusual patterns require stronger levels of data protection than others, which comes at a trade-off with forecast loss (and by extension, forecast accuracy due to the bounds established above). For illustration, consider a time series with an $A_t$ larger than all other time series at time $t$. DPMs will likely choose $P_t < A_t$ and as a result, $F^*_t < F_{t+1}$ because the equations (6), (20) and (46) all have a positive value on the right-hand side. For example, suppose $P_t < A_t$ with $F^*_t = 3$ when $F_{t+1} = 4$ and in the subsequent time period $t+1$, again $P_{t+1} < A_{t+1}$ with $P_{t+1} = 3$ when $A_{t+1} = 5$. In this case, MAE = $|4 - 5| = 1$, but PMAE = $|3 - 3| = 0$, which implies that a forecaster perceives the forecast accuracy to be better than it actually is. We expect this to occur only for time series that truly require a lot of protection due to their unusual patterns. Note that PMAE $\leq$ MAE $- \Delta$MAE $+ M = 1 - 1 + 2 = 2$ per Theorem 5 and PMAE $\leq$ Average Forecast Loss + MAE + Average of $|A_i - P_i| = 1 + 1 + 2 = 4$ per Corollary 1. For our proposed $k$-nTS Swapping method, most time series will have $k$ other time series with values both above and below $A_t$.

3. Data Provider: Bounding Forecast Loss with Data Protection

At time $t$, the data provider uses a data protection method to release protected data $P_{j,t}$ for each time series $j$ based on the confidential values of all time series up until time $t$. Standard DPMs previously mentioned in the introduction are formulated in Table 2. As Table 2 shows, the main issue with these methods is that they choose protected values based on predefined rules and not forecast loss. However, the goal of the data provider should be to change $A_{j,t}$ to $P_{j,t}$ with the minimal forecast loss while also increasing privacy to an acceptable threshold.

| Data Protection Method | Description | Formulation |
|------------------------|-------------|-------------|
| None                   | Release confidential observation | $P_{j,t} = A_{j,t}$ |
| Bottom-Coding          | Bottom $p$ percent of observations are replaced with the $p$ quantile | $P_{j,t} = \begin{cases} B & \text{if } A_{j,t} \leq B \\ A_{j,t} & \text{if } A_{j,t} > B \end{cases}$ where $B = \inf\{x_j \in \mathbb{R} \mid F(x_j) \geq p\}$ |
| Top-Coding             | Top $p$ percent of observations are replaced with the $1-p$ quantile | $P_{j,t} = \begin{cases} T & \text{if } A_{j,t} \geq T \\ A_{j,t} & \text{if } A_{j,t} < T \end{cases}$ where $T = \sup\{x_j \in \mathbb{R} \mid F(x_j) \leq p\}$ |
| Additive Noise         | Add a normal random number with mean zero and standard deviation $\sigma$ | $P_{j,t} = A_{j,t} + r$, where $r \sim N(0, \sigma^2)$ and $\sigma^2 = E[x_j - E[x_j]^2]$ |
In this section, we design the k-nTS (k-nearest time series) Swapping method to alter the confidential data using randomization to balances the trade-off between forecast loss and privacy. Depending on the quantity of available data, both k-nTS Swapping and standard DPMs can use rolling windows of data. Rolling windows adjust for dynamic changes in the relationships between time series. For example, if we choose a rolling window size, say \( n \), then \( x_j = (A_{j,t-n+1}, A_{j,t-n+2}, \ldots, A_{j,t-1}, A_{j,t})^T \) where \( x_j \in \mathbb{R}^n \). Also, define \( F(x_j) \) as the empirical cumulative distribution function of \( x_j \). Protection in subsequent time periods from \( t + 1 \) to \( T \) rolls \( x_j \) forward from \( x_j = (A_{j,t-n+2}, A_{j,t-n+3}, \ldots, A_{j,t}, A_{j,t+1})^T \) to \( (A_{j,T-n+1}, A_{j,T-n+2}, \ldots, A_{j,T-1}, A_{j,T})^T \), respectively. Compared to standard DPMs, k-nTS Swapping matches similarly patterned time series together based on the distance of a rolling window of their past values. Then, it uses randomization to replace \( A_{j,t} \) with a confidential value from a similar time series. Additionally, we design the k-means Time Series (k-mTS) Shuffling method in order to simultaneously swap multiple time series data. Subsections 3.1 and 3.2 develop the k-nTS Swapping method and k-mTS Shuffling methods, respectively.

### 3.1. The k-nearest Time Series (nTS) Swapping method

Let \( X = \{x_1, \ldots, x_J\} \) be a given set of time series data (\( n \)-vectors). For each time series \( x_j \in \mathbb{R}^n \), the data provider computes a set of squared distances of the elements of the set \( X \). Let us define \( \text{dist}(x_j, x_i) = d_{j,i} \) as the distance between \( x_j \) and \( x_i \), i.e., two distinct time series data from a given set \( X \). Without loss of generality, we use the Euclidean norm, or \( \ell^2 \)-norm for this paper (all norms on \( \mathbb{R}^n \) are equivalent to the Euclidean norm) as a distance metric. Since our case is multivariate and partially ordered, we can get a totally ordered set based on the Euclidean distance.

Let us define \( x_j^{(k)} \) as the kth nearest neighbor of \( x_j \). Then, for a time series \( x_j \), we have \( \{d_{j,(1)}, d_{j,(2)}, \ldots, d_{j,(J-1)}\} \) such that \( d_{j,(k)} \leq d_{j,(l)} \) for any integers \( k < l \) where \( d_{j,(k)} = \|x_j - x_j^{(k)}\| \).

Note that \( x_j^{(i)} \in X \setminus \{x_j\} \) and the superscript \((i)\) means the \( i^{th} \) order statistic of the related Euclidean distances from \( x_j \). Thus, for a given time series vector \( x_j \), its k-nTS (k-nearest time series) can be represented as the set \( K_j = \{x_j^{(1)}, \ldots, x_j^{(k)}\} \) based on \( \|x_j - x_j^{(1)}\| \leq \cdots \leq \|x_j - x_j^{(k)}\| \) or an ordered set \( \{d_{j,(1)}, d_{j,(2)}, \ldots, d_{j,(k)}\} \).

For more efficient computation for such ordering, let us introduce a distance matrix \( D \) using the squared distances. The squared distance between \( x_i \) and \( x_j \) is given by \( d_{i,j} = \|x_i - x_j\|^2 \), and \( d_{i,j} \) is the \((i,j)\)th entry of \( D \). Hence \( D \) is symmetric. (Also note that \( \text{rank}(D) \leq n + 2 \).) For our applications, the vector \( x_j \in \mathbb{R}^n \), \( j = 1, \ldots, J \) where \( n \ll J \), typically. Suppose that we are given a data matrix \( X = [x_1, x_2, \ldots, x_J] \), \( x_j \in \mathbb{R}^n \) (i.e., \( X \in \mathbb{R}^{n \times J} \)). We can write such data matrix \( X \), i.e., a confidential data matrix, as the following.
\[ X = [x_1, x_2, \ldots, x_J] = \begin{pmatrix}
A_{1,t-n+1} & A_{1,t-n+2} & \cdots & A_{1,t-1} & A_{1,t} \\
A_{2,t-n+1} & A_{2,t-n+2} & \cdots & A_{2,t-1} & A_{2,t} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
A_{J,t-n+1} & A_{J,t-n+2} & \cdots & A_{J,t-1} & A_{J,t}
\end{pmatrix}^T, \]  

where \( x_j = (A_{j,t-n+1}, A_{j,t-n+2}, \ldots, A_{j,t-1}, A_{j,t})^T \) where \( x_j \in \mathbb{R}^n \) and also \( x_j \in \mathbb{X} \).

Note that we can find the distance matrix \( D \) using the fact that \( \|x_i - x_j\|^2 = (x_i - x_j)^T(x_i - x_j) = x_i^T x_i - x_i^T x_j - x_j^T x_i + x_j^T x_j \), which can be written up as the following:

\[ D = \mathbf{1}diag(X^T X)^T - 2X^T X + diag(X^T X)\mathbf{1}^T, \]

where the symbol \( \mathbf{1} \) denotes a column vector of \( J \) ones. It is easy to see that the column vector \( diag(X^T X) = (\|x_1\|^2, \ldots, \|x_J\|^2)^T \). Let \( d_j \) denote the \( j \)th column of \( D \). Then we can write the \( J \times J \) distance matrix \( D = [d_1, \ldots, d_J] \), where \( d_j \in \mathbb{R}^J \).

In the general case \( k \ll J \), for each time series \( x_j \) we sort \( d_j \), the \( j \)th column of \( D \) from the smallest to largest components and find the \( k \)th smallest component so that we have

\[ K_j = \{x_j^{(1)}, \ldots, x_j^{(k)}\}. \]

That is, the data provider then selects a value of \( k \) from 1 to a maximum of \( J - 1 \) and selects the \( k \)-nearest time series. In case of \( k = J \), random swapping is simply done by rearranging the components in the last row of matrix \( X \). Let the \( i \)th nearest time series from \( x_j \) be \( x_j^{(i)} = (A_{j,t-n+1}^{(i)}, A_{j,t-n+2}^{(i)}, \ldots, A_{j,t}^{(i)})^T \) where \( n \) is the length of the rolling window of past data. Then the swap of the last component of \( x_j \) with the last component of one of its \( k \)-nearest time series \( x_j^{(i)} \), \( i = 1, \ldots, k \) can simply be written as the following random swapping:

\[ k - \text{nTS Swapping}: \quad P_{j,t} = \begin{cases} \frac{1}{k} \quad A_{j,t}^{(1)} \\ \vdots \\ \frac{1}{k} \quad A_{j,t}^{(k)} \end{cases}, \]

which is equivalent to the following: the last component of \( x_j \) is randomly replaced by the last component of \( x_j^{(i)} \in K_j \) with probability \( \frac{1}{k} \) for \( i = 1, \ldots, k \).

By Algorithm 1, we can obtain \( X' \): a matrix of protected time series data at time point \( t \) for all \( J \) time series for a given rolling window size \( n \). The \( k \)-nearest time series data protection method can be written up as the following protected data matrix

\[ X' = [x'_1, x'_2, \ldots, x'_J] = \begin{pmatrix}
A_{1,t-n+1} & A_{1,t-n+2} & \cdots & A_{1,t-1} & P_{1,t} \\
A_{2,t-n+1} & A_{2,t-n+2} & \cdots & A_{2,t-1} & P_{2,t} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
A_{J,t-n+1} & A_{J,t-n+2} & \cdots & A_{J,t-1} & P_{J,t}
\end{pmatrix}^T. \]
Algorithm 1 The $k$-nTS Swapping method

**Require:** [Initialization]

(i) [Time Series Matrix $X \in \mathbb{R}^{n \times J}$] $X = [x_1, x_2, \ldots, x_J]$, $x_j \in \mathbb{R}^n$ for $j = 1, \ldots, J$ as in (65).

(ii) [Distance Matrix $D$] $D = \mathbf{1}_{diag}(X^T X)^T - 2X^T X + diag(X^T X)1^T$ as in (66).

for $j = 1, 2, \ldots, J$ do

[Finding a set $K_j$ for $x_j$] Let $d_j$ denote the $j$th column of $D$. Sort $d_j$ from the smallest to largest components and find the $k$th smallest component, followed by $K_j$ as in (67).

[Random swapping] $x_j \leftarrow x_j^{(i)}$ (last components only) for some $i \in \{1, \ldots, k\}$ as in (68).

end for

**Remark 3 (Time complexity).** Let $\mathbb{X} = \{x_1, \ldots, x_J\}$ be a given set of time series data ($n$-vectors). The time complexity to find $k$-nearest neighbors of each one of $x_j$’s can be calculated by the following. First, the cost for the computation of distances $\|x_i - x_j\|$ for $j = 1, \ldots, J$, $j \neq i$ is $3(J - 1)n$ ($2(J - 1)n$ if we use squares of distances), in either case it would be $O(Jn)$. In order to find the $k$-nearest neighbors we need to sort the calculated distances and it would be $O(J \log J)$.

Since we typically have $J \gg n$, the cost of computation is $O(J \log J)$ for each $x_j$. If we use $k$-d tree algorithm (e.g., see Mahajan et al. (2009) and references therein), the whole process can be done in $O(J \log J)$.

Figure 1 Random edge selection for $k$-nTS Swapping

Note. For a given time series $x_3$, its last component is replaced by the last component of $x_9$, that is randomly selected among $x_3$’s 10-nearest time series data. $K_3 = \{x_1, x_2, x_4, x_6, x_7, x_9, x_{14}, x_{16}, x_{21}, x_{27}\}$.

We can represent each time series $x_j$, $j = 1, \ldots, J$ as a vector, and then put them in the graph $G = (V, E)$, which consists of a set $V$ of vertices (or nodes) and a set $E$ of undirected edges. In our
case, we can use weighted edges to represent the Euclidean distance between associated time series vectors: \( w_{i,j} = \text{dist}(x_i, x_j) \). If we put all the nodes on the graph and assign weight on every edge (every pair of nodes), e.g., \( w_{i,j} = \text{dist}(x_i, x_j) \) for all \( i \neq j \), then we will have a complete graph. As described in Figure 1, the \( k \)-nearest time series swapping method can be considered as a random edge selection problem of the graph. Figure 1 depicts the case of \( k = 10 \) for the \( k \)-nearest time series of \( x_3 \), where the last component of \( x_3 \) is swapped with that of \( x_9 \).

### 3.2. The \( k \)-means Time Series (mTS) Shuffling method

In this subsection, we treat the protection of each time series in terms of a group of time series rather than individuals. In doing so, we simultaneously swap multiple data for more efficient data protection. For effective protection, we use a clustering algorithm to partition given time series into multiple groups of similar ones. There are clustering algorithms widely used in practice, e.g., \( k \)-means ([Lloyd (1982)])14, hierarchical ([Corpet (1988)]), isolation forest ([Liu et al. (2008)])15 clustering methods, etc. Clustering and shuffling for data privacy has also been studied, e.g., [Muralidhar and Sarathy (2006), Li and Sarkar (2013) and references therein].

There are \( J \) points in \( n \)-dimensional space, i.e., \( x_j \in \mathbb{R}^n, j = 1, \ldots, J \). Our confidential data matrix is \( X = [x_1, x_2, \ldots, x_J] \) as in (65). Our goal is to partition those points into \( k \) means clusters ([Lloyd (1982)], first proposed in 1957), where there are \( k \) centroids \( c_1, \ldots, c_k \). Each centroid \( c \) can be found by the optimization problem: \( \min \sum \| c - x_j \|_2 \), which minimizes the total distance to points \( x_j = (A_{j,t-n+1}, \ldots, A_{j,t-1}, A_{j,t})^T \) in its cluster.

For \( k \)-means clustering, we minimize the following:

\[
\min E = \sum_{i=1}^{k} \sum_{j \in Q_i} \| x_j - c_{Q_i} \|^2,
\]

where the clusters \( Q_i \) have centers \( c_{Q_i} \) for \( i = 1, \ldots, k \). The centroid \( c_i \) is the average \( c_{Q_i} = \left( \sum_{j \in Q_i} x_j \right) / |Q_i| \) of the vectors in cluster \( Q_i \).

It is well-known that the optimal clustering is an NP-hard problem even for smallest problems (see [Mahajan et al. (2009)]). In our applications, we start from a given set of vectors: \( k \) centroids (representatives), selected by a data provider or by random sampling, which is followed by partitioning into \( k \) clusters (as in Algorithm 2). Suppose that, by the \( k \)-representative partitioning, we can obtain \( n_i \) points for each cluster \( Q_i, i = 1, \ldots, k \) such as

\[
Q_i = \{x_{1_i}, \ldots, x_{n_i}\},
\]

where \( x_{1_i}, \ldots, x_{n_i} \) are the elements of the cluster \( i \). Note that \( \sum_{i=1}^{k} n_i = J \). If we put the points in a matrix as columns, then the matrix \( Q_i = [x_{1_i}, x_{2_i}, \ldots, x_{n_i}], x_{ji} \in \mathbb{R}^n \) and the matrix \( Q_i \in \mathbb{R}^{n \times n_i} \).
Algorithm 2 Clustering using $k$-representatives

Require: [Initialization] A list of $J$ vectors $x_1, \ldots, x_J$ and an initial list of $k$ group representative vectors $c_1, \ldots, c_k$ out of the given $J$ time series (given by a data provider or random sampling).

for the fixed representatives $c_i, i = 1, 2, \ldots, k$ do

[Partitioning vectors]

for all $x_j, j = 1, \ldots, J$ do

[Assignment] $\min_{i=1, \ldots, k} \|x_j - c_i\|^2 + \cdots + \min_{i=1, \ldots, k} \|x_J - c_i\|^2$

end for

[Finding $k$ clusters $Q_i, i = 1, \ldots, k$]

Based on the vector assignment by partitioning, we find the corresponding clusters by

$Q_i = \{x_j \mid \min_{j=1, \ldots, J} \|x_j - c_i\|^2\}$

end for

Algorithm 3 the $k$-mTS Shuffling method

Require: [Initialization] $k$ clusters $Q_i, i = 1, \ldots, k$. (by Algorithm 2)

for all $x_j \in Q_i$ do

[Random permutation] Construct a data matrix $Q_i \in \mathbb{R}^{n \times n_i}$ using $x_j \in Q_i$ as columns. ($|Q_i| = n_i, i = 1, \ldots, k$.) For each cluster $Q_i$, create an array between 1 and $n_i$ and then generate a random permutation, followed by an $n_i$ by $n_i$ permutation matrix $P_i$.

[Random shuffle] Random shuffling can be done by $Q_i P_i$, from which we use the last row for a protected data matrix $X'_i$ for $i = 1, \ldots, k$.

end for

For each cluster $Q_i$, there is an array between 1 and $n_i$ (columns of matrix $Q_i$). For random shuffling, we generate a random permutation, from which we get an $n_i$ by $n_i$ permutation matrix $P_i$ (simply by permuting columns of the $n_i$ by $n_i$ identity matrix.) By Algorithm 3 we can obtain a protected data matrix $X'_i$ for cluster $i$ on a given rolling window size $n$. The $k$-means time series clustering and shuffling data protection method can be written up using the following protected data matrix for cluster $i$:

$$X'_i = [x_{1,i}', x_{2,i}', \ldots, x_{n_i,i}'] = \begin{pmatrix} A_{1,i,t-n+1} & A_{1,i,t-n+2} & A_{1,i,t-n+3} & \cdots & A_{1,i,t-1} & P_{1,i,t} \\ A_{2,i,t-n+1} & A_{2,i,t-n+2} & A_{2,i,t-n+3} & \cdots & A_{2,i,t-1} & P_{2,i,t} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ A_{n_i,i,t-n+1} & A_{n_i,i,t-n+2} & A_{n_i,i,t-n+3} & \cdots & A_{n_i,i,t-1} & P_{n_i,t} \end{pmatrix}^T,$$  \hspace{1cm} (72)

where the row vector $(P_{1,t}, P_{2,t}, \ldots, P_{n_i,t})$ is the last row vector of $Q_i P_i$.

According to the partition by Algorithm 2 let us rearrange the original data matrix $X = [x_1, x_2, \ldots, x_J]$ defined in (65) into $X = [Q_1, Q_2, \ldots, Q_k]$, where each data matrix $Q_i \in \mathbb{R}^{n \times n_i}$ con-
sists of time series vectors in the corresponding cluster \( i \) for \( i = 1, \ldots, k \). Then we can write the following.

\[
X'' = XP = [Q_1, Q_2, \ldots, Q_k] \begin{pmatrix}
P_1 & & \\
& P_2 & \\
& & \ddots \\
& & & P_k
\end{pmatrix} = [Q_1P_1, Q_2P_2, \ldots, Q_kP_k],
\]

(73)

where \( P \) is a block diagonal matrix with “randomly generated” permutation matrices \( P_i \) for \( i = 1, \ldots, k \) (with all off-diagonal blocks being zero matrices). If we take the last row (i.e., the \( n \)-th row) of \( X'' \), with which replace that of \( X \), then a new matrix \( X' \in \mathbb{R}^{n \times j} \) can be constructed as a protected data matrix:

\[
X' = [X'_1, X'_2, \ldots, X'_k] = \left( \begin{array}{cccc}
A_{11, t-n+1} & A_{11, t-n+2} & \cdots & A_{11, t-1} \\
A_{12, t-n+1} & A_{12, t-n+2} & \cdots & A_{12, t-1} \\
\vdots & \vdots & \ddots & \vdots \\
A_{nk, t-n+1} & A_{nk, t-n+2} & \cdots & A_{nk, t-1}
\end{array} \right) \begin{pmatrix}
P_{1, t} \\
P_{2, t} \\
\vdots \\
P_{k, t}
\end{pmatrix}^T,
\]

(74)

where the row vector \((P_{1, t}, \ldots, P_{n, t}) \in \mathbb{R}^j\) is from the last row vector (the \( n \)-th row) of \( X'' \), and by shuffling of time series data in the same cluster, we obtain the protected data matrices \( X'_i \)'s for clusters \( i = 1, \ldots, k \).

In this section, we studied two DPMs (data protection methods) for data provider. Using these new DPMs, we develop novel mathematical models and decision making framework in Section 5, incorporating data intruder’s utilities in Section 4.

4. Data Privacy: Anomaly Detection and Targeting Extreme Values

\( k \)-nTS Swapping and \( k \)-mTS Shuffling use generalization (\( k \)) and randomization (swapping and shuffling, respectively) to protect time series from a data intruder. We quantify the privacy of protected data by defining a data intruder’s utility function which represents the satisfaction the data intruder receives from discovering privacy issues at time \( i \). As noted in the previous sections, we assume that data protection begins at time \( t \), and therefore, we can write confidential time series \( j \) as an \( n \)-vector \( x_j = (A_{j,i-n+1}, \ldots, A_{j,t-1}, A_{j,t}, \ldots, A_{j,i})^T \) and protected time series \( j \) as an \( n \)-vector \( x'_j = (A_{j,i-n+1}, \ldots, A_{j,t-1}, P_{j,t}, \ldots, P_{j,i})^T \).

There are serious privacy concerns due to a potential attack from data intruders. Data protection against such threats has long been studied. We refer the readers to the literature, e.g., [Garfinkel].
et al. (2002), Sweeney (2002), Li and Sarkar (2009), etc. For this paper, we consider two basic types of utility for a data intruder at time $i$: first, anomaly detection of the protected data $P_{j,i}$ within a time series $j$ and second, detection of anomalous time series compared to all time series $j = 1, \ldots, J$. We refer the readers to relevant literature for a thorough review on anomaly detection and defining extreme values for time series, e.g., Teng (2010), Gorr and Schneider (2013), Laptev et al. (2015), Lee et al. (2017), Munir et al. (2019) and references therein.

**Defining Privacy Issues.** The data provider uses anomaly pattern detection at time $i$ on the confidential data $x_j = (A_{j,i-n+1}, \ldots, A_{j,t}, \ldots, A_{j,i})^T$ to define privacy issues within a time series. Privacy issues exist at time $i$ if $u(A_{j,i}) > \tau^*$ for $j = 1, \ldots, J$ where $u(\cdot)$ is a utility function of data intruders, defined in Sections 4.1 and 4.2. The threshold $\tau^*$ can be chosen by the data provider. The threshold is based on a quantile $q = \hat{F}_{x_j}^{-1}(p)$, where $\hat{F}_{x_j}$ denotes an estimated CDF and $p$ is 0.95, for example. When defining a privacy issue, we set $g_{j,i} = 1$ and otherwise, $g_{j,i} = 0$. Thus, the total number of privacy issues at time $i$ is $G_i = \sum_{j=1}^J g_{j,i}$.

### 4.1. Data Intruder 1’s Utility: Anomaly Detection Within Individual Time Series

Anomaly pattern detection within a protected time series $x_j' = (A_{j,i-n+1}, \ldots, A_{j,t}, P_{j,t}, \ldots, P_{j,i})^T$ at time $i$ can be thought of as the surprise of the new value $P_{j,i}$ compared to the past patterns (e.g., seasonality adjustments, levels) within the same times series. To define surprise for series $j$, we fit a kernel density $\hat{f}_{x_j'}$ on the past $n-1$ observations up until time $i-1$ (each time series as an $n$-vector). We then estimate the density of the new protected value at time $i$ to get a density value for $P_{j,i}$. For kernel density estimation and related topics we refer the readers to relevant literature, e.g., Azzalini (1981), Jones et al. (1996), Chen (2017) and references therein. If the density $\hat{f}_{x_j'}(P_{j,i})$ is small, this is defined as an anomaly and a privacy issue which increases the data intruder’s utility. We can define the data intruder’s utility for time series $j$ at time $i$ as the following:

$$u(P_{j,i}) = \sqrt{\frac{1}{\hat{f}_{x_j'}(P_{j,i})}}. \quad (75)$$

With some positive constant $\tau$ as a threshold, data intruders may flag $x_j$ as anomalous if the utility function value in (75) is above the threshold: time series $x_j'$ is marked as anomaly if $u(P_{j,i}) > \tau$ for $j = 1, \ldots, J$. This means, it looks like a good target for data intruder to attack. Such threshold can be based on a quantile $q = \hat{F}_{x_j'}^{-1}(p)$, where $\hat{F}_{x_j'}$ denotes an estimated CDF and $p$ is large, e.g., 0.95, 0.99, etc. In this regards, for the unimodal case, a reasonable confidence interval would also be useful to detect anomalous data since points outside such interval can be identified as anomalies.

### 4.2. Data Intruder 2’s Utility: Anomaly Detection for Multiple Time Series

A data intruder may also want to simultaneously find anomalous time series, instead of investigating individual data points within time series. In this case, it would be helpful to partition all the time
series into multiple groups of similar time series, followed by anomaly detection on each group. Let us consider $k$-means clustering here. Suppose that there are $k$ groups of the data, characterized by group representatives such as centroids, $c_1, \ldots, c_k$. For each cluster $Q_h \subseteq \mathbb{R}^n, h = 1, \ldots, k$, the following metric may be useful for anomaly detection

$$g_h(x) = e^{-\|x-c_h\|^2/\sigma_h^2},$$

(76)

where $x, c \in \mathbb{R}^n$ and $\sigma_h$ denotes standard deviation of data points in cluster $Q_h$. Note that the metric of (76) is always between 0 and 1, and its value gets closer to zero when the distance between $x$ and $c_h$ gets larger. Therefore, data intruders may want to use some small positive constant $0 < \epsilon \ll 1$ as a threshold, in order to detect the following anomaly:

$$\{y \in Q_h \mid g_h(y) < \epsilon\}.$$  

(77)

Note that (76) and (77) can be used for the general case, and it can easily be modified, e.g., the vectors $x$ and $c$ are both replaced by $\lambda^T x$ and $\lambda^T c$, $\lambda = (\lambda_1, \ldots, \lambda_n)^T \in \mathbb{R}_+^n, \|\lambda\|_1 = 1$. We can write the following

$$g_h(x) = e^{-\|\lambda^T x - \lambda^T c_h\|^2/\sigma_h^2},$$

(78)

where $\sigma_h$ denotes a standard deviation of $\lambda^T x, x \in Q_h$. In order to detect any surprise of the new value $P_{j,i}$ at time $i$, we can simply use $\lambda_i = 1$ with all other components being zero.

**Figure 2** Description of targeting across time series

*Note.* Seven clusters. Each edge has a value, e.g., distance or weighted distance between nodes for each pair.
4.3. Targeting Decisions of the Data Intruder

Define a decision at time $i$ for time series $j$ as

$$\text{Decision}_{i,j} = \begin{cases} 1 & \text{if } u(P_{j,i}) > \tau \\ 0 & \text{if } u(P_{j,i}) \leq \tau, \end{cases}$$

where $u(\cdot)$ is defined as in (75) for targeting anomalous values within time series, and for targeting anomalous time series we have the following:

$$\text{Decision}_{i,j} = \begin{cases} 0 & \text{if } g_h(x_j) \geq \epsilon \\ 1 & \text{if } g_h(x_j) < \epsilon, \end{cases}$$

where $g_h(\cdot)$ is defined in (78), $x_j \in Q_h$ for $h = 1, \ldots, k$. Let us define the likelihood ratio for threshold $\tau$ as

$$\text{LR}_\tau = \frac{\text{TPR}_\tau}{\text{FPR}_\tau},$$

where TPR$_\tau$ is the true positive rate and FPR$_\tau$ is the false positive rate. If a set of targeting decisions is better than random guessing, then LR$_\tau > 1$. The data provider attempts to decrease the maximum value of LR$_\tau$ over all values of $\tau$ subject to a minimum FPR$_\tau$ to avoid division by zero.

Table 3 illustrates all possible decisions over the values of $\tau$. The left side of the figure represents the targeting of no time series when $\tau$ is greater than or equal to all values of $u(P_{j,i})$. The right side of the figure represents the targeting of all time series when $\tau$ is lower than all values of $u(P_{j,i})$, as in (79).

| Time Series | Protected Data | Decision($\tau_1$) | Decision($\tau_2$) | Decision($\tau_3$) | Decision($\tau_4$) | ⋯ | Decision($\tau_{\text{min}}$) |
|-------------|----------------|-------------------|-------------------|-------------------|-------------------|---|-------------------|
| 1           | $P_{1,i}$      | Target            | Target            | Target            | Target            | ⋯ | Target            |
| 2           | $P_{2,i}$      | Target            | Target            | Target            | Target            | ⋯ | Target            |
| 3           | $P_{3,i}$      | Target            | Target            | Target            | Target            | ⋯ | Target            |
| 4           | $P_{4,i}$      | Target            | Target            | Target            | Target            | ⋯ | Target            |
| 5           | $P_{5,i}$      | Target            | Target            | Target            | Target            | ⋯ | Target            |
| 6           | $P_{6,i}$      | Target            | Target            | Target            | Target            | ⋯ | Target            |
| 7           | $P_{7,i}$      | Target            | Target            | Target            | Target            | ⋯ | Target            |
| ⋮           | ⋮              | ⋮                 | ⋮                 | ⋮                 | ⋮                 | ⋯ | ⋮                 |
| $J$         | $P_{J,i}$      |                   |                   |                   |                   | ⋯ | Target            |

For each value of $\tau$, there is a true positive $\text{TP}_{i,j} = 1$ when $u(P_{j,i}) > \tau$ and $g_{j,i} = 1$, and there is a false positive $\text{FP}_{i,j} = 1$ if $u(P_{j,i}) \leq \tau$ and $g_{j,i} = 0$. The corresponding $\text{TPR}_\tau$ and $\text{FPR}_\tau$ are:

$$\text{TPR}_\tau = \frac{\sum_{j=1}^{J} \text{TP}_{i,j}}{G},$$

$$\text{FPR}_\tau = \frac{\sum_{j=1}^{J} \text{FP}_{i,j}}{J - G}.$$
For a decision analysis of the data intruder, the above equations can be used to compute the \( LR_\tau \) in order to limit the maximum value, as previously mentioned.

For a more comprehensive decision analysis, the combination of all values of \( FPR_\tau \) and \( TPR_\tau \) can be represented in a Receiver Operating Characteristic (ROC) Curve. The area under the ROC Curve is the probability that a randomly chosen privacy issue \( (g_{j,i} = 1) \) has a higher \( u(P_{j,i}) \) than a randomly chosen non-privacy issue \( (g_{j,i} = 0) \). The data provider may also assume the data intruder has a limited budget and only targets anomalies up to a predetermined FPR limit=\( r \), resulting in a Partial Area Under the ROC Curve (PAUC). \( PAUC_i \) is piece-wise linear and \( 0 \leq \frac{PAUC_i}{r} \leq 1 \). Best-case privacy (random targeting) occurs when \( TPR_\tau = FPR_\tau \) for all values of \( \tau \) and \( PAUC_i = \frac{1}{2} r^2 \). Worst-case privacy occurs when \( FPR_\tau = 0 \) for all decisions and \( PAUC_i = r \).

For example, Figure 3 and Table 4 show the case of \( J = 4 \) time series where \( 0.01 < u(P_{1,i}) < 0.40 < u(P_{2,i}) < 0.50 < u(P_{3,i}) = u(P_{4,i}) < 0.75 \) which results in 4 possible values of \( \tau \). Only one set of decisions with \( \tau = 0.40 \) beats random targeting and has \( TPR_\tau > FPR_\tau \). When \( r = 1 \), \( PAUC_i = \frac{5}{8} \) which has worse privacy than random targeting with \( PAUC_i = \frac{1}{2} r^2 = \frac{1}{2} \). When \( \tau = 0.40 \), the set of decisions also has the highest \( LR_\tau \) (worst privacy, but the best utility for data intruder) with \( LR_\tau = \frac{1}{0.5} = 2 \).

**Figure 3   **ROC Curve at time \( i \)

![](ROC_Curve.png)

**Table 4   **Decisions at time \( i \)

| Time Series \( j \) | Protected Data | Decision | Privacy Issue |
|---------------------|----------------|----------|---------------|
| 1                   | \( P_{1,i} \)  | No       | No            |
| 2                   | \( P_{2,i} \)  | Target   | Yes           |
| 3                   | \( P_{3,i} \)  | Target   | Yes           |
| 4                   | \( P_{4,i} \)  | Target   | No            |

Decisions are made based on \( \tau \) with the lowest \( LR_\tau \).
5. Two-Party Framework: Optimality for Effective Data Protection

The involved parties in this framework are: the data provider who chooses how to protect the data prior to release; the data intruder who seeks to exploit the protected data at time \( i \). The data intruder can be a data provider’s employee, a contracted third-party vendor, or a hacker. The primary goals of the data provider are to (a) ensure a forecaster can achieve similar forecasts with the protected data at time \( i \), and (b) prevent the data intruder from targeting time series \( j \) at time \( i \). The two-party data privacy framework should be used by data providers to decide the optimal protection \( k \) while engaged in the practice of forecasting.

5.1. Optimal time series selection for the \( k \)-nTS swapping using fixed-radius \( \delta \)

As described in Figure 1 in Section 3.1, the \( k \)-nTS swapping can be considered a random edge selection problem for given \( k \) neighbors of each time series \( j = 1, \ldots, J \). Since the selection of \( k \) plays an important role in both forecasting loss and data privacy perspectives, we need to be able to find an optimal \( k \) (i.e., optimal set \( K \)) before random swapping. For each time series \( j \in J \), we can find the optimal \( k \) by the following:

\[
\begin{align*}
\max_{k} & \quad k \\
\text{subject to} & \quad \|x_j - x_{(k)}\| \leq \delta \\
& \quad \|x_j - x_{(k)}\|_\infty \leq M,
\end{align*}
\]

where \( \delta > 0 \) and \( M > 0 \) are given constants determined by a data provider. The first constraint is to find \( k \) neighboring time series of \( x_j \in \mathbb{R}^n \) within a given radius \( \delta \) from \( x_j \), and the second constraint requires to find time series in such a way that maximum of absolute values of components of a vector \( x_j - x_{(k)} \) is bounded above by \( M \). Let \( x^{(jk)} = x_j - x_{(k)} \). Then the second constraint can be written up as \( \|x^{(jk)}\|_\infty = \max_i |x^{(jk)}_i| \leq M \), where \( x^{(jk)}_i \) denotes the \( i \)th component of a vector \( x_j - x_{(k)} \). This means that, using (84), data providers can put an upper bound \( M \) for forecast loss because the second constraint is equivalent to \( |A_t - P_t| \leq M \) for all \( t = 1, \ldots, n \).

This problem formulation always provides us with the best \( k \) with a given \( \delta \). This problem formulation gives us different \( k \) for each time series \( j = 1, \ldots, J \), which is ideal, since \( k \)-nearest neighbors of \( x_j \) may include points too far from most of neighbors of \( x_j \). This is because the set \( K_j \) for \( x_j \) (as defined in (67)) is based on the ordering, no matter how far from the given data point. Swapping with such irrelevant time series for data protection may result in an unacceptable forecast loss level. For this reason, it would be useful to set an upper bound on the distance from \( x_j \) only to include neighbors with acceptable forecast loss levels.

Using the best \( k \) from (84), we do a random edge selection from a fixed-radius \( \delta \)-near time series (as in Algorithm 4). Let us propose to topologize \( n \)-dimensional Euclidean space using open ball as the following:

\[
\text{Fixed-radius } \delta \text{-near time series } x_i \in \mathcal{B}_\delta(x_j) = \{ y \in \mathbb{R}^n \mid \text{dist}(x_j, y) < \delta \},
\]

(85)
Algorithm 4 The fixed-radius $\delta$-nTS Swapping method

**Require:** [Initialization] Same as in Algorithm 1.

for $j = 1, 2, \ldots, J$ do

[Component sorting] Sort $d_j$ from the smallest to largest, and let $D_j = \{d_j^{(1)}, \ldots, d_j^{(J-1)}\}$.

[Finding $\delta$-near time series] Find $d_j^{(k)} = \sup \{d \in d_j : d < \delta^2\}$ and let $B_j = \{x_j^{(1)}, \ldots, x_j^{(k)}\}$.

[Random swapping] (last components) $x_j \leftarrow x_j^{(i)} \in B_j = \{x_j^{(1)}, \ldots, x_j^{(k)}\}$ with probability $1/k$.

end for

where $x_i, x_j \in \mathbb{R}^n$ but $1 \leq i \neq j \leq J$. There is no problem also using subspace topology in case of weighted time series data only on certain time points. Figure 4 depicts the time series data within $B_\delta(x_3) = \{y \in \mathbb{R}^n : \text{dist}(x_3, y) < \delta\} = \{x_4, x_6, x_9, x_{14}, x_{16}\}$, with which $x_3$ can be randomly swapped (summarized in Algorithm 4).

**Figure 4** Fixed-radius $\delta$-near time series

![Fixed-radius $\delta$-near time series](image)

Note. The set $\{x_4, x_6, x_9, x_{14}, x_{16}\}$ consists of the time series within $B_\delta(x_3) = \{y \in \mathbb{R}^n : \text{dist}(x_3, y) < \delta\}$. Note that we use the same example of Figure 1 that is the case of $k = 10$ for the $k$-nearest neighbors of $x_3$.

It is also reasonable to swap with artificial time series data instead of a confidential data. Let $K_j$ denote the set of $k$-nearest neighbors of $x_j$, i.e., $|K_j| = k$. We can use a centroid of the set of vectors, say $c_j$, for swapping with $x_j$ at time $i$. That is, we have the protected data matrix $X' = [x'_1, \ldots, x'_J]$, where $x'_j = (x_{j,1}, \ldots, x_{j,n-1}, c_{j,n})^T$ for $j = 1, \ldots, J$ for the confidential data matrix $X = [x_1, \ldots, x_J]$.

For description (see Figure 5), we consider, among others, three different representatives regarding $K_j$, i.e., the set of $k$-nearest neighbors of $x_j$: (i) centroid of the vertices of $\text{conv}(K_j)$, the convex hull of $K_j$; (ii) centroid of the vectors in the $K_j$; (iii) centroid of the vectors in the $B_\delta(x_j)$, the
set of $\delta$-near time series of $x_j$. Note that there are a few other representatives for swapping, e.g., Chebyshev center, weighted mean center (based on the distances from the given point), etc. Swapping of the given point with an artificial data point such as a centroid removes the possibility of poor selection of a node which locates farther than most neighbors. This keeps forecast loss at an improved bound while protecting the given time series with randomization. In Section 5.2, we present an integer programming model (matching problem) for optimal swapping, where centroids are used in case of odd-numbered time series data in a cluster.

Figure 5 Alternative representatives for the group of neighbors for swapping

Note. $k = 10$ for the $k$-nearest neighbors of $x_3$. $K_3 = \{x_1, x_2, x_4, x_6, x_7, x_9, x_{14}, x_{16}, x_{21}, x_{27}\}$ (i) $x'_3 = \text{centroid of conv}(K_3)$ (ii) $x''_3 = \text{centroid of } K_3 \cup \{x_3\}$ (iii) $x^*_3 = \text{centroid of } B_\delta(x_3) \cup \{x_3\}$.

5.2. Minimum weight perfect matching for the $k$-mTS shuffling

Our goal is to minimize forecast loss while keeping data privacy issues at a desirable level. In this section, we present optimal shuffling by the use of an integer programming model for a matching problem, where simultaneous data-swapping takes place within clusters. For each cluster $Q_h, h = 1, \ldots, k$ with $|Q_h| = n_h$, we put the points in a matrix as columns, then the matrix $Q_h = [x_{1h}, x_{2h}, \ldots, x_{nh}] \in \mathbb{R}^{n \times n_h}$. By Algorithm 3 in Section 3.2, we shuffle time series data within the cluster in order to obtain a protected data matrix $X'_h$:

$$X'_h = [x'_{1h}, x'_{2h}, \ldots, x'_{nh}] = \begin{pmatrix} A_{1h,t-n+1} & A_{1h,t-n+2} & A_{1h,t-n+3} & \cdots & A_{1h,t-1} & P_{1h,t} \\ A_{2h,t-n+1} & A_{2h,t-n+2} & A_{2h,t-n+3} & \cdots & A_{2h,t-1} & P_{2h,t} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ A_{nh,t-n+1} & A_{nh,t-n+2} & A_{nh,t-n+3} & \cdots & A_{nh,t-1} & P_{nh,t} \end{pmatrix}^T.$$  \hspace{1cm} (86)
For a perfect matching problem formulation, we put the column vectors (time series) of the matrix \( Q_h \) on the graph \( G \), where every pair of distinct nodes is connected by a unique edge. We then transform such complete graph into bipartite graph with edge weights in terms of the data intruder’s utility, where weights \( \infty \) are assigned to the loops (i.e., \( w(\{j, j\}) = \infty \)). For the weight on edge \( \{j, j'\} \), for \( j' \neq j \) we use the following:

\[
\text{(Data intruder’s utility)} \quad w_{j, j'} = w(\{j, j'\}) = u(P_{j', i}) = \sqrt{1 \hat{f}_x(j)}(P_{j', i}), \quad (87)
\]

where \( \hat{f}_x(j) \) denotes a kernel density based on the past \( n - 1 \) observations up until time \( i - 1 \) for time series \( j \).

Since the forecast loss \( |F - F'| \) is based on \( |A_t - P_t| \) (as in Theorems 1, 2, 3), we define another edge weight on \( \{j, j'\} \), for \( j' \neq j \) as follows.

\[
\text{(Forecast loss base)} \quad f_{j, j'} = f(\{j, j'\}) = |A_{j, t} - A_{j', t}|. \quad (88)
\]

Let us consider graph \( G \) with two partitions \( \mathbb{A} \) and \( \mathbb{B} \), that is, \( G = (V, E) \) and \( V = \mathbb{A} \cup \mathbb{B} \) and \( E \subseteq \{ab : a \in \mathbb{A}, b \in \mathbb{B}\} \). Then, using (87) and (88), we can write the following min-weight perfect matching problem:

\[
\begin{align*}
\min \sum_{e \in E} (w_e + f_e)x_e \\
\text{subject to} \\
x(\delta(a)) = 1, \text{ for all } a \in \mathbb{A} \\
x_e \in \{0, 1\}, \text{ for all } e \in E,
\end{align*} \quad (89)
\]

where \( \delta(a) = \{ab : ab \in E, a \in \mathbb{A}, b \notin \mathbb{A}\} \). Equivalently, (89) can be written up as the following.

\[
\begin{align*}
\min \sum_{a, b} (w_{ab} + f_{ab})x_{ab} \\
\text{subject to} \\
\sum_{a \in \mathbb{A}} x_{ab} = 1 \\
\sum_{b \in \mathbb{B}} x_{ab} = 1 \\
x_{ab} \in \{0, 1\},
\end{align*} \quad (90)
\]

where \( w_{ab} \) denotes the data intruder’s utility in (87) when time series data of \( a \in \mathbb{A} \) at time \( i \) is replaced by one of the time series \( b \in \mathbb{B} \). Note that \( \sum f_{ab}x_{ab} \) denotes the total forecast loss base from the time series in \( Q_h \).

Using \( 0 \leq \lambda \leq 1 \), we can write its variation as the following.

\[
\begin{align*}
\min \sum_{a, b} \lambda w_{ab}x_{ab} + (1 - \lambda)f_{ab}x_{ab} \\
\text{subject to} \\
\sum_{a \in \mathbb{A}} x_{ab} = 1 \\
\sum_{b \in \mathbb{B}} x_{ab} = 1 \\
x_{ab} \in \{0, 1\},
\end{align*} \quad (91)
\]
which means minimum intruder utility perfect matching when $\lambda = 1$, and minimum forecast loss perfect matching when $\lambda = 0$.

Figure 6  Optimal shuffling to simultaneously minimize data intruder’s utility and forecast loss

Note. Taking each cluster as a complete graph, we then transform it into a bipartite graph for the min-weight perfect matching problem. $G = (V, E)$ and $V = A \cup B$ and $E \subseteq \{ab : a \in A, b \in B\}$, where each time series $j$ appears in both sets $A$ and $B$ as $a_j$ and $b_j$. The centroid (yellow node) in Figure 7 is depicted as $a_c$ on the left and $b_c$ on the right.

Figure 7  Optimal shuffling as a min-weight perfect matching problem to minimize data intruder’s utility

Note. Cluster with an odd number of nodes. $Q = \{x_1, x_2, x_3, x_4, x_6, x_7, x_9, x_{14}, x_{16}, x_{21}, x_{27}\}$. Inclusion of a centroid (yellow node) for perfect matching. Blue edges are the solution of the optimization problem (89), (90) or (91).
The above problem formulations provide data providers with optimal solution to minimize data intruder’s utility. Since perfect matching contains $|V|/2$ edges, there is no perfect matching for a graph with an odd number of vertices. Therefore, for a cluster with an odd number of time series, we include a centroid in the set of nodes for matching problem as described in Figures 6 and 7, which are equivalent. The case in the figures can be written up as the ordered sets in Table 5.

| Time Series $j$ at time $i$ | $A_{1,i}$ | $A_{2,i}$ | $A_{3,i}$ | $A_{4,i}$ | $A_{6,i}$ | $A_{7,i}$ | $A_{9,i}$ | $A_{14,i}$ | $A_{16,i}$ | $A_{21,i}$ | $A_{27,i}$ | $A_{c,i}$ |
|-----------------------------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|

For data protection, swapped by $A_{3,i}$ $A_{6,i}$ $A_{1,i}$ $A_{14,i}$ $A_{2,i}$ $A_{27,i}$ $A_{4,i}$ $A_{21,i}$ $A_{4,i}$ $A_{c,i}$ $A_{9,i}$ $A_{7,i}$ $A_{16,i}$

Optimal shuffle (simultaneous swapping for multiple time series data for each cluster) of the example in Figures 6 and 7.

In Section 6, we do the $k$-mTS shuffling by solving (91) with various $k$ and $\lambda$’s over the dataset of 1,572 time series of JCRs from Ohio in 2001 Q2 to 2010 Q1 ($T = 35$ total quarters) at the US County and NAICS two-digit code aggregation level. Section 6 presents that the numerical solutions of (91) provide us effective data protection while keeping forecast loss at a desirable level.

6. Experiments

We use an employee-employer dataset from the US Census Bureau’s Longitudinal Employer-Household Dynamics (LEHD) program to test our two-party data privacy framework. The dataset contains time series of the number of job creations relating to individual employees and their employers in specific industries and geographies. Identifiable information can result when a quarterly job creation has a count of 1 because it relates to a single individual. Hence, these data require protection because Title 13 of the U.S. Code states that US Census Bureau data released to the public cannot contain “any identifiable information about individuals, households, or businesses.” For data protection reasons, these data are usually generalized (e.g., aggregated) to US county and North American Industry Classification System (NAICS) industry classifications, which results in over 50,000 time series nationwide that are available for download (https://lehd.ces.census.gov/data/). Furthermore, job creations are transformed into Job Creation Rates (JCRs) because rates are bounded (between 0 and 2) and normalize the counts across small and large geographies. JCRs are computed by dividing each job creation by the county-industry average of employment at the beginning and end of that quarter. A JCR of 0 implies no new jobs and a JCR of 2 implies the creation of a new county-industry.

1 https://www.census.gov/content/dam/Census/library/factsheets/2019/comm/2020-confidentiality-factsheet.pdf
2 We use sanitized job creations in this paper because they have the same underlying structure as the confidential data behind the Census Bureau’s secure access servers.
6.1. Data Protection Process

To illustrate our data protection methods, we use 1,572 time series of JCRs from Ohio in 2001 Q2 to 2010 Q1 \((T = 35\) total quarters) at the US County and NAICS two-digit code aggregation level. Of these, we eliminated 334 time series because disclosure avoidance rules at the US Census Bureau required data suppression in at least one time period. Of the 1,238 remaining time series, we use the data from time \(i = 1\) to \(i = 25\) to initialize the DPMs. Protected data from each DPM are released on a rolling basis from time \(i = 26, \ldots, 35\) by using the actual data up until time \(i = 25, \ldots, 34\), respectively. The DPMs include bottom-coding (5, 10, and 20 %), top-coding (5, 10, and 20 %), additive noise (standard deviation parameter equal to 1 and 2 standard deviations of the actual time series data), and \(k\)-mTS Shuffling \((k = 10, 15, 20, 25, 30, 35, 40, 45\) and \(\lambda = 0.1, 0.3, 0.5, 0.7, 0.9\)). We calculate forecast loss and privacy at times \(i = 26, \ldots, 35\) after the data is protected at time \(i\). Note that the protected data only affects the forecasts at times \(i \geq 27\) since the data protection begins at \(i = 26\). Thus, in our application, \(t + 1 = 27\) and \(T = 35\).

6.2. Data Privacy

Data privacy can be measured irrespective of the forecasts as it only depends on \(x_j = (A_{j,i-n+1}, \ldots, A_{j,t-1}, A_{j,t}, \ldots, A_{j,i})^T\) and \(x_j' = (A_{j,i-n+1}, \ldots, A_{j,t-1}, P_{j,t}, \ldots, P_{j,i})^T\). To define the privacy issues in the actual data per Section 4.1, we set \(\tau^*\) equal to the 97% quantile of all \(J\) values of \(u(A_{j,i})\). This results in a binary outcome, \(g_{i,j}\), for each time series \(j\) at time \(i\). We then calculate \(u(P_{j,i})\) per Equation (75) and compare the intruder’s utility on each time series \(j\) to \(g_{i,j}\).

Figure 8 plots the ROC Curves for additive noise with 1 standard deviation (AdditiveNoise1SD), top-coding 20% (top20), and \(k\)-mTS Shuffling solution of our optimization model (91) (i.e., simultaneous swapping of multiple data within each cluster) for \(k = 40, \lambda = 0.3\) and \(k = 45, \lambda = 0.3\). The figure shows that a data intruder can successfully target more privacy issues when the protected data is generated with AdditiveNoise1SD or top-coding 20% compared to \(k\)-mTS Shuffling. Also, the data intruder’s ability for \(k\)-mTS Shuffling is similar to random chance over the entire range of false positive rates.

Table 6 presents the Area Under the ROC Curve (AUC) and the associated 95% confidence intervals(CIs) for the DPMs. The third row shows the maximum likelihood ratio per Equation 81 when the false positive rate (FPR) is greater than 5% (ensures the data intruder will target at least some time series incorrectly). As described previously, a \(LR_{\tau}\) greater than 1 implies that the intruder’s targeting decision is better than random chance. For all DPMs, this situation occurs most commonly at low FPR ranges. However, as the data intruder incurs more false positives, their performance is similar to random guessing for \(k\)-mTS Shuffling.
6.3. Forecast Loss

To compute forecast loss, we first consider using the R package forecast (Hyndman et al. 2008) for the exponential smoothing models, however, this package re-optimizes the initial level, trends, and seasonalities at every time period. This poses a problem for calculating forecast loss because forecast loss assumes the same levels, trends, and seasonalities are updated sequentially at each time period per Section 2.1. Hence, we code the forecasts using Equations (2) for SES, Equations (15) for DES, and Equations (37) to (40) for TES.

Tables 7, 8, and 9 present the maximum forecast loss for all exponential smoothing models across varying time periods, broken by the corresponding levels, trends, and seasonalities. The results show that additive noise has the most forecast loss and bottom-coding has the least. Also, the maximum forecast loss of top-coding seems comparable to k-mTS Shuffling. With regard to the time series structures, the maximum forecast loss seems to be driven by the maximum loss in level first, then seasonality, and lastly trend. Our results indicate that the maximum forecast loss
generally increases as more of the data was protected from \( i = t + 1 \) to \( i = T \) which corresponds to our findings in Theorem 1. For clarity reasons, we do not present the maximum forecast loss for DES and TES time periods earlier than \( i = T \), but our findings are similar to SES and in accordance with Theorems 2 and 3. Finally, we note that maximum forecast loss increases as the additive exponential smoothing models have more components (as expected since \( A_t - P_t \) shows up in all three Equations (43) to (45) in Subsection 2.1.3).

### Table 7  Maximum Forecast Loss for SES

| Time  | None | AdditiveNoise1SD | AdditiveNoise2SD | bottom5 | bottom20 | top5 | top20 | k = 40 | k = 45 |
|-------|------|-----------------|----------------|--------|---------|------|------|-------|-------|
| \( i = t + 1 \) | 0 | 0.032 | 0.070 | 0.001 | 0.011 | 0.011 | 0.032 | 0.041 | 0.018 |
| \( i = t + 5 \) | 0 | 0.070 | 0.128 | 0.002 | 0.005 | 0.044 | 0.045 | 0.054 | 0.046 |
| \( i = T \) | 0 | 0.133 | 0.212 | 0.002 | 0.010 | 0.046 | 0.053 | 0.067 | 0.047 |

Last two columns are based on the \( k\)-mTS shuffling solutions of (91) in the cases of \( k = 40, \lambda = 0.3 \) and \( k = 45, \lambda = 0.3 \).

### Table 8  Maximum Losses for Forecasts, Levels, and Trends of DES at \( i = T \)

| Max loss | None | AdditiveNoise1SD | AdditiveNoise2SD | bottom5 | bottom20 | top5 | top20 | k = 40 | k = 45 |
|----------|------|-----------------|----------------|--------|---------|------|------|-------|-------|
| \( \max |F_T - F_T^*| \) | 0 | 0.152 | 0.262 | 0.003 | 0.012 | 0.053 | 0.067 | 0.085 | 0.056 |
| \( \max |l_T - l_T^*| \) | 0 | 0.146 | 0.252 | 0.002 | 0.012 | 0.051 | 0.064 | 0.082 | 0.054 |
| \( \max |b_T - b_T^*| \) | 0 | 0.007 | 0.010 | 0.000 | 0.000 | 0.002 | 0.003 | 0.003 | 0.002 |

Last two columns are based on the \( k\)-mTS shuffling solutions of (91) in the cases of \( k = 40, \lambda = 0.3 \) and \( k = 45, \lambda = 0.3 \).

### Table 9  Maximum Losses for Forecasts, Levels, Trends, and Seasonalities of TES at \( i = T \)

| Max loss | None | AdditiveNoise1SD | AdditiveNoise2SD | bottom5 | bottom20 | top5 | top20 | k = 40 | k = 45 |
|----------|------|-----------------|----------------|--------|---------|------|------|-------|-------|
| \( \max |F_T - F_T^*| \) | 0.000 | 0.173 | 0.367 | 0.004 | 0.017 | 0.091 | 0.125 | 0.109 | 0.091 |
| \( \max |l_T - l_T^*| \) | 0.000 | 0.173 | 0.293 | 0.003 | 0.014 | 0.062 | 0.074 | 0.093 | 0.063 |
| \( \max |b_T - b_T^*| \) | 0.000 | 0.006 | 0.010 | 0.000 | 0.000 | 0.002 | 0.003 | 0.003 | 0.002 |
| \( \max |s_T - s_T^*| \) | 0.000 | 0.031 | 0.065 | 0.002 | 0.004 | 0.037 | 0.049 | 0.058 | 0.036 |

Last two columns are based on the \( k\)-mTS shuffling solutions of (91) in the cases of \( k = 40, \lambda = 0.3 \) and \( k = 45, \lambda = 0.3 \).

### 6.4. Tradeoff Between Data Intruder Utility and Maximum Forecast Loss

In this subsection, we compare the tradeoff between the utility of a data intruder and maximum forecast loss. Figures 9 and 10 show that the \( k\)-mTS Shuffling DPMs of (91) for higher values of \( k \) (40 to 45) and moderate values of \( \lambda \) (0.3 to 0.7) have a lower maximum forecast loss per unit of privacy. Importantly, a privacy level (AUC) of 0.5 is equivalent a data intruder randomly guessing which time series have privacy issues. Comparatively, the bottom-coding and additive noise DPMs do not seem to protect the data well. Top-coding protects the data better, but top-coding is not on the efficient frontier on the tradeoff.
6.5. Protected Time Series Structures

We also compare the patterns from the protected data to the actual data across all \( J \) time series. Figure 11 plots the empirical densities of the \( J \) spectral entropies (Kang et al. (2017)) of the protected data from time \( i = t + 1 \) to \( i = T \). We can see that the \( k \)-mTS Shuffling of (91) with \( k = 45 \) and \( \lambda = 0.3 \) produces more protected time series with higher spectral entropies, indicating that more time series are less forecastable (and have more disorder) for privacy protection reasons.
This illustrates the purposeful tradeoff ($\lambda$) that $k$-mTS Shuffling made between data privacy and forecast loss to protect the time series with privacy issues.

**Figure 11** Spectral Entropy of Protected Data from $i = t + 1$ to $i = T$

*Note.* The standard deviations of the smoothing kernels for density estimation are 0.01 for all DPMs. $k$-mTS with $k = 45, \lambda = 0.3$ has more protected time series with spectral entropies closer to 1.

### 6.6. Managerial and Policy Implications

In our application, we found that standard DPMs mentioned in legislation are not the best choice for forecasters; instead, organizations should consider using DPMs that minimize forecast loss while preserving privacy, such as $k$-mTS Shuffling which chooses the protected data based on the ability to maintain forecasts. These findings suggest the importance of explicitly incorporating the concerns of forecasters (i.e., forecast loss and by extension, $A_t - P_t$) into the data protection itself, and represent a contribution to the literature. It also encourages forecasters, legal counsel, and IT to have a seat at the same table.

Besides forecast loss, using the additive exponential smoothing models allowed data providers to also see how time series structures were affected by data protection. Thus, our research also has implications for managers concerned with how data protection might bias the structures within time series. Based on our findings with only 10 time periods for data protection, we found that
levels were most affected and then seasonality coefficients. Trends were not greatly affected at low (but typical) values for $\alpha$, $\beta$, and $\gamma$. Additive noise altered the time series structures the most while not providing much data privacy. For data protection in more than 10 time periods, we provide the reader with theory in Section 2.

Our solution considered the operational requirement of starting data protection at time period $t$ with all the prior data being unprotected. We found that the maximal forecast loss was lower in earlier time periods when the data protection first started. As opposed standard DPMs like additive noise, the $k$-mTS method of (91) stabilized maximal forecast loss for all forecasting models (SES, DES, and TES) in later time periods when $k$ was higher, indicating that more ($k = 45$) means were better for the consistency of maximal forecast loss. Importantly, the data provider could choose whether or not to release the protected data (and which DPM to use) during time $i$, providing management with a ability to make social choice of the tradeoff between forecast loss and the utility of the data intruder in time $i$.

7. Conclusions
In this paper, the aim of the data privacy framework was to alter time series data for privacy reasons while ensuring that a forecaster could still maintain the quality of their forecasts. We provide theoretical guarantees for the maximum forecast loss as each additional time period is protected. Operationally, our framework allows data providers to make a social choice of the tradeoff between data utility and data privacy while they release protected time series data.

For the decision-making process, we explicitly incorporate the perspective of the forecaster and data intruder into the protected time series data. We treat each time series as a vector and construct a data matrix, together with a distance matrix based on Euclidean geometry. We then turn it into a bipartite graph, where vertices are a time series vector, and their associated edge-weights are calculated by the data intruders’ utility and forecast loss on every pair of given time series vectors. Using such bipartite graph we solve the minimum-weight perfect matching problem for optimal decision making (i.e., optimal swapping or shuffling).

Our theoretical findings are generalizable to not only forecast loss, but also forecast accuracy, perceived forecast accuracy, and time series structures such as level, trend, and seasonality. Section 2.2 details how forecast loss at a current time $i$ translates to forecast accuracy and perceived forecast accuracy at the future time $i+1$. However, we focused on maximum forecast loss because our goal was to provide a guarantee of forecast quality to managers for every time series. To protect time series data when using additive exponential smoothing models, we recommend data providers use our proposed DPM of $k$-mTS Shuffling over standard DPMs found in legislation.

In the future, we plan to expand our research to include other forecasting models. We did not apply the framework or theory to multivariate forecasting models such as neural networks.
(Januschowski et al. 2020; Fry and Brundage 2020), gradient boosted models (Ke et al. 2017), Vector autoregression models, or hybrid machine learning-exponential smoothing models (Smyl 2020). We also plan to study related optimization models using stochastic programming techniques (Prékopa 1995; Birge and Louveaux 1997), especially from the set theoretical approach using related probability bounds (Boros and Prékopa 1989; Boros et al. 2014). An open question for future research is whether complex forecasting models with related probability bounds will improve forecast loss by capturing patterns of related time series or further degrade forecast loss by over fitting on the protected data. We leave this area to future research.

References
Abowd J, Schmutte IM (2019) An economic analysis of privacy protection and statistical accuracy as social choices. American Economic Review 109(1):171–202.

Archibald BC, Koehler A (2003) Normalization of seasonal factors in winters’ methods. International Journal of Forecasting 19:143–148.

Arroyo J, Maté C (2009) Forecasting histogram time series with k-nearest neighbours methods. International Journal of Forecasting 25(1):192–207.

Article 29 Data Protection Working Party (2014) Opinion 05/2014 on anonymisation techniques URL https://ec.europa.eu/justice/article-29/documentation/opinion-recommendation/files/2014/wp216_en.pdf.

Azzalini A (1981) A note on the estimation of a distribution function and quantiles by a kernel method. Biometrika 68:326–328.

Baena D, Castro J, Frangioni A (2020) Stabilized benders methods for large-scale combinatorial optimization, with application to data privacy. Manag. Sci. 66:3051–3068.

Bass FM, Leone RP (1983) Temporal aggregation, the data interval bias, and empirical estimation of bimonthly relations from annual data. Manag. Sci. 29(1):1–11.

Birge J, Louveaux F (1997) Introduction to Stochastic Programming (Springer), URL http://dx.doi.org/10.1007/978-1-4614-0237-4.

Boone T, Ganeshan R, Jain A, Sanders NR (2019) Forecasting sales in the supply chain: Consumer analytics in the big data era. International Journal of Forecasting 35:170–180.

Boros E, Prékopa A (1989) Closed form two-sided bounds for probabilities that exactly $r$ and at least $r$ out of $n$ events occur. Math. Oper. Res. 14:317–342.

Boros E, Scozzari A, Tardella F, Veneziani P (2014) Polynomially computable bounds for the probability of the union of events. Math. Oper. Res. 39(4):1311–1329.

California State Legislature (2018) California consumer privacy act of 2018 URL https://oag.ca.gov/privacy/ccpa.
Chen Y (2017) A tutorial on kernel density estimation and recent advances. *Biostatistics and Epidemiology* 1:161 – 187.

Cheu A, Smith A, Ullman JR (2021) Manipulation attacks in local differential privacy. *J. Priv. Confidentiality* 11.

Christen M, Gupta S, Porter JC, Staelin R, Wittink DR (1997) Using market-level data to understand promotion effects in a nonlinear model. *Journal of Marketing Research* 34(3):322–334.

Corpet F (1988) Multiple sequence alignment with hierarchical clustering. *Nucleic acids research* 16 22:10881–10890.

Economic Research Service USDA (2016) National Household Food Acquisition and Purchase Survey (FoodAPS): Users Guide to Survey Design, Data Collection, and Overview of Datasets.

European Parliament and Council of European Union (2016) Regulation (eu) 2016/679 of the european parliament and of the council URL [https://eur-lex.europa.eu/eli/reg/2016/679/oj](https://eur-lex.europa.eu/eli/reg/2016/679/oj).

Fry C, Brundage M (2020) The m4 forecasting competition – a practitioner’s view. *International Journal of Forecasting* 36:156–160.

Garfinkel R, Gopal R, Góes P (2002) Privacy protection of binary confidential data against deterministic, stochastic, and insider threat. *Manag. Sci.* 48:749–764.

Gelper S, Fried R, Croux C (2009) Robust forecasting with exponential and holt-winters smoothing. *Journal of Forecasting* 29:285–300.

Gorr WL, Schneider MJ (2013) Large-change forecast accuracy: Reanalysis of m3-competition data using receiver operating characteristic analysis. *International Journal of Forecasting* 29(2):274–281.

Harvey SJ (2013) Smart meters, smarter regulation: Balancing privacy and innovation in the electric grid. *UCLA L. Rev.* 61:2068.

Hogarth RM, Makridakis S (1981) Forecasting and planning: An evaluation. *Manag. Sci.* 27(2):115–138.

Hyndman R, Koehler A (2006) Another look at measures of forecast accuracy. *International Journal of Forecasting* 22:679–688.

Hyndman RJ, Khandakar Y, et al. (2008) Automatic time series forecasting: the forecast package for r. *Journal of statistical software* 27(3):1–22.

Januschowski T, Gasthaus J, Wang Y, Salinas D, Flunkert V, Bohlke-Schneider M, Callot L (2020) Criteria for classifying forecasting methods. *International Journal of Forecasting* 36:167–177.

Jones MC, Marron JS, Sheather S (1996) A brief survey of bandwidth selection for density estimation. *Journal of the American Statistical Association* 91:401–407.

Kang Y, Hyndman RJ, Smith-Miles K (2017) Visualising forecasting algorithm performance using time series instance spaces. *International Journal of Forecasting* 33(2):345–358.
Ke G, Meng Q, Finley T, Wang T, Chen W, Ma W, Ye Q, Liu T (2017) Lightgbm: A highly efficient gradient boosting decision tree. NIPS.

Laptev N, Amizadeh S, Flint I (2015) Generic and scalable framework for automated time-series anomaly detection. Proceedings of the 21th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining.

Lee J, Kim J, Prékopa A (2017) Extreme value estimation for a function of a random sample using binomial moments scheme and boolean functions of events. Discret. Appl. Math. 219:210–218.

Li XB, Sarkar S (2009) Against classification attacks: A decision tree pruning approach to privacy protection in data mining. Oper. Res. 57:1496–1509.

Li XB, Sarkar S (2013) Class-restricted clustering and microperturbation for data privacy. Manag. Sci. 59(4):796–812.

Liu F, Ting K, Zhou Z (2008) Isolation forest. 2008 Eighth IEEE International Conference on Data Mining 413–422.

Lloyd ST (1982) Least Square Quantization in PCM. IEEE Transactions on Information Theory 28:129–137.

Luo J, Hong T, Fang SC (2018) Benchmarking robustness of load forecasting models under data integrity attacks. International Journal of Forecasting 34(1):89–104.

Mahajan M, Nimbhorkar P, Varadarajan K (2009) The Planar k-Means Problem is NP-Hard (In: Das S., Uehara R. (eds) WALCOM: Algorithms and Computation. WALCOM 2009. Lecture Notes in Computer Science, vol 5431. Springer, Berlin, Heidelberg.).

McKenzie E (1986) Technical note - renormalization of seasonals in winters’ forecasting systems: Is it necessary? Oper. Res. 34:174–176.

Menon S, Sarkar S (2007) Minimizing information loss and preserving privacy. Manag. Sci. 53:101–116.

Munir M, Siddiqui S, Dengel A, Ahmed S (2019) Deepant: A deep learning approach for unsupervised anomaly detection in time series. IEEE Access 7:1991–2005.

Muralidhar K, Parsa R, Sarathy R (1999) A general additive data perturbation method for database security. Manag. Sci. 45(10):1399–1415.

Muralidhar K, Sarathy R (2006) Data shuffling - a new masking approach for numerical data. Manag. Sci. 52:658–670.

Prékopa A (1995) Stochastic programming (Kluwer Academic Publishers).

Schneider MJ, Abowd J (2015) A new method for protecting interrelated time series with bayesian prior distributions and synthetic data. Journal of The Royal Statistical Society Series A-statistics in Society 178:963–975.

Smyl S (2020) A hybrid method of exponential smoothing and recurrent neural networks for time series forecasting. International Journal of Forecasting 36:75–85.
Sun Z, Zhao S, Zhang J (2019) Short-term wind power forecasting on multiple scales using vmd decomposition, k-means clustering and lstm principal computing. IEEE Access 7:166917–166929.

Sweeney L (2002) k-anonymity: A model for protecting privacy. Int. J. Uncertain. Fuzziness Knowl. Based Syst. 10:557–570.

Taylor TA, Xiao W (2010) Does a manufacturer benefit from selling to a better-forecasting retailer? Manag. Sci. 56(9):1584–1598.

Teng M (2010) Anomaly detection on time series. 2010 IEEE International Conference on Progress in Informatics and Computing 1:603–608.

Wang X, Lee WJ, Huang H, Szabados RL, Wang DY, Van Olinda P (2016) Factors that impact the accuracy of clustering-based load forecasting. IEEE Transactions on Industry Applications 52(5):3625–3630.

Wang Z, Yang C, Yuan H, Zhang Y (2021) Aggregation bias in estimating log-log demand function. Production and Operations Management URL http://dx.doi.org//10.1111/poms.13488

Xu H, Zhang N (2021) Implications of data anonymization on the statistical evidence of disparity. Manag. Sci. .