Cramér-type Large deviation and non-uniform central limit theorems in high dimensions

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Abstract

Central limit theorems (CLTs) for high-dimensional random vectors with dimension possibly growing with the sample size have received a lot of attention in the recent times. Chernozhukov et al. (2017) proved a Berry–Esseen type result for high-dimensional averages for the class of hyperrectangles and they proved that the rate of convergence can be upper bounded by $n^{-1/6}$ up to a polynomial factor of $\log p$ (where $n$ represents the sample size and $p$ denotes the dimension). In the classical literature on central limit theorem, various non-uniform extensions of the Berry–Esseen bound are available. Similar extensions, however, have not appeared in the context of high-dimensional CLT. This is the main focus of our paper. Based on the classical large deviation and non-uniform CLT results for random variables in a Banach space by Bentkus, Račkauskas, and Paulauskas, we prove three non-uniform variants of high-dimensional CLT. In addition, we prove a dimension-free anti-concentration inequality for the absolute supremum of a Gaussian process on a compact metric space.

1 Introduction

In modern statistical applications like high dimensional estimation and multiple hypothesis testing problems, the dimension of the data is often much larger than the sample size. As can be expected from the classical asymptotic theory, the central limit theorem plays a very pivotal role for inference. In this paper, we prove three variants of the high-dimensional central limit theorem. The setting we use is as follows. Consider independent and identically distributed (i.i.d.) mean zero random vectors $X_1, \ldots, X_n \in \mathbb{R}^p$ with covariance matrix $\Sigma := \mathbb{E}[X_1X_1^\top]$. Here $p$ is allowed to be larger than $n$. Define the scaled average

$$S_n := \frac{X_1 + \cdots + X_n}{\sqrt{n}} \in \mathbb{R}^p.$$
Let $Y$ represent a multivariate Gaussian random vector with mean zero and variance-covariance matrix $\Sigma$. The problem of central limit theorem is the comparison of the probabilities $\mathbb{P}[S_n \in A]$ and $\mathbb{P}[Y \in A]$ for $A \subseteq \mathbb{R}^p$. This problem has received significant interest in the recent times. The series of papers Chernozhukov et al. (2013), Chernozhukov et al. (2015) and Chernozhukov et al. (2017) have studied this problem extensively under general conditions on the random vectors when the sets $A$ are sparsely convex sets and in particular hyperrectangles. The main result of Chernozhukov et al. (2017) bounds the difference $|\mathbb{P}[S_n \in A] - \mathbb{P}[Y \in A]|$ uniformly over $A \in \mathcal{A}^e$ as a function of $n$ and $p$. Here $\mathcal{A}^e$ is the class of all hyperrectangles. Proposition 2.1 of Chernozhukov et al. (2017) implies that

$$\sup_{A \in \mathcal{A}^e} |\mathbb{P}[S_n \in A] - \mathbb{P}[Y \in A]| \leq C \left( \frac{\log^7(pn)}{n} \right)^{1/6},$$

under certain moment assumptions and a constant $C$ depending on the distribution of the random vectors $X_i$, $1 \leq i \leq n$. In their earlier papers, Chernozhukov et al. (2015) and Chernozhukov et al. (2013) a special sub-class of sets $\mathcal{A}^m \subset \mathcal{A}^e$ were considered, where $\mathcal{A}^m$ is the class of all sets $A$ of the form $A = \{ x \in \mathbb{R}^p : x(j) \leq a \text{ for all } 1 \leq j \leq p \}$. Here and throughout, we use the notation $x(j)$ for a vector $x \in \mathbb{R}^p$ to represent the $j$-th coordinate of $x$. Since the dependence on the sample size, $n^{-1/6}$, in bound (1) is larger than dependence $n^{-1/2}$ that appears in multivariate Berry–Esseen bounds (Bentkus, 2004), the result (1) does not provide useful information when the probability $\mathbb{P}[Y \in A]$ is smaller. This leads naturally to the question of non-uniform version of (1). In particular, an interesting question is to find quantitative upper bound on

$$\left| \frac{\mathbb{P}[S_n \in A^c]}{\mathbb{P}[Y \in A]} - 1 \right|,$$

as a function of $p$ and $n$. In this paper we consider a special class of sets of the form

$$A = \{ x \in \mathbb{R}^p : -a \leq x(j) \leq a \text{ for all } 1 \leq j \leq p \},$$

and find an upper bound on (2). Note that sets of the form (3) are $l_\infty$ balls. Another variant of non-uniform CLT is to consider how the difference $|\mathbb{P}[\|S_n\|_\infty \leq r] - \mathbb{P}[\|Y\|_\infty \leq r]|$ scales with $r$ for $r > 0$. To this end we prove upper bounds on

$$\sup_{r \geq 0} r^m |\mathbb{P}[\|S_n\|_\infty \leq r] - \mathbb{P}[\|Y\|_\infty \leq r]|,$$

for some $m \geq 3$.

The first problem (2) is well-studied in the classical central limit theorem under the name “Cramér-type large deviation”. We refer to the encyclopedic work Saulis and Statulevičius (1991) for a review of the extensive literature on Cramér-type large deviation for sums of
independent random variables along with extensions to multivariate random vectors. The classical result for univariate \((p = 1)\) random variables is of the form

\[
\frac{\mathbb{P}[S_n \geq x]}{\mathbb{P}[Y \geq x]} = \exp \left( \frac{C x^3}{6 \Sigma^{3/2} n^{1/2}} \right) \left[ 1 + C \left( \frac{x + 1}{\sqrt{n}} \right) \right],
\]

(5)

for \(0 \leq x = O(n^{-1/6})\). (Here \(C\) is a constant depending on the distribution of \(X_1\).) See Theorem 5.23 (and Section 5.8) of Petrov (1995) for a precise statement. The second problem (4) is usually referred to as a “non-uniform CLT”. A result of this kind is also useful in proving convergence of moments. The classical result for univariate random variables of type (4) is given by

\[
|\mathbb{P}[S_n \leq x] - \mathbb{P}[Y \leq x]| \leq C(r)(1 + |x|)^{-r} \left( \frac{\mathbb{E}[|X_1|^3]}{\Sigma^{3/2} n^{1/2}} + \frac{\mathbb{E}[|X_1|^r]}{\Sigma^{r/2} n^{(r-2)/2}} \right),
\]

(6)

for all \(x \in \mathbb{R}, r \geq 3\) and for some constant \(C(r)\) depending only on \(r\). See Theorem 5.15 of Petrov (1995) for a precise statement. Section 5.5 of Petrov (1995) provides various results in this direction. Also, see Sazonov (1981) for results related to random vectors. The classical results (5) and (6) provide rates that scale like \(n^{-1/2}\) (as a function of \(n\)) in the non-uniform versions of CLT as does the classical Berry–Esseen bound. In light of the fact that the Berry–Esseen type result (1) for high-dimensions is available only with the rate \(n^{-1/6}\), we only derive rates in large deviation and non-uniform CLT with a scaling of order \(n^{-1/6}\) as a function of \(n\).

It is well-known (Bentkus, 1985) that the rate \(n^{-1/6}\) is optimal in the central limit theorem for Banach spaces and the space \((\mathbb{R}^p, ||\cdot||_{\infty})\) with \(p\) diverging behaves like an infinite-dimensional space. In this respect it is of particular interest to look back at the rich literature on the CLTs for Banach space valued random variables. These old and well-known large deviation and non-uniform CLTs for Banach space play a central role in the derivation of ours presented here. The basic setting for these results is as follows: Suppose \(X_1, \ldots, X_n\) are i.i.d. random variables taking values in a Banach space \(B\) such that \(\mathbb{E}[X_1] = 0\) and \(Y\) is a mean zero \(B\)-valued Gaussian random variable with same the covariance (operator) as \(X_1\). The problem as before is the study of closeness of the distributions of \(|Y|\) and \(|S_n|\) where \(|\cdot|\) is a Banach space norm. Several results on this problem are available in Bentkus et al. (2000), Paulauskas and Račkauskas (2012). The papers Bentkus (1987), Bentkus and Račkauskas (1990) and Paulauskas and Račkauskas (1991) are of particular interest to us since they provide bounds on (2) and (4) for Banach space valued random variables. One of the main contributions of our work is to make the constants explicit in terms of the dimension. In Bentkus (1987) and Bentkus and Račkauskas (1990) the problem of the convergence of ratio \(\mathbb{P}[||S_n|| > r]/\mathbb{P}[||Y|| > r]\) to 1 was considered. In Bentkus (1987) it is proved that under some conditions

\[
\mathbb{P}[||S_n|| > r] = (1 + \theta M_2(r + 1)n^{-1/6}) \mathbb{P}[||Y|| > r]
\]
for \(0 \leq r \leq -1 + M_1 n^{1/6}\) where \(|\theta| \leq 1\), \(M_1\) and \(M_2\) are constants depending on the distribution of \(X_1\). The non-uniform version of central limit theorem is also available for the Banach spaces from Paulauskas and Račkauskas (1991). Understanding the implication of these results for the high-dimensional case is the main focus of our paper.

1.1 Our contributions

As described in the introduction, we study the non-uniform variants of high dimensional central limit theorem. We assume the \(X_1, \ldots, X_n \in \mathbb{R}^p\) are i.i.d. random vectors with mean 0 and covariance matrix \(\Sigma\). Let \(Y\) be a centered Gaussian random vector in \(\mathbb{R}^p\) with the same covariance matrix \(\Sigma\).

(i) We derive Cramér-type large deviation for the \(l_\infty\) norm in the high dimensional CLT set up. We assume \(\mathbb{E} [\exp \{H \|X_1\|_\infty\}] < \infty\) for some \(H > 0\) and prove that
\[
P [\|S_n\|_\infty > r] = (1 + 2M_1 (r + 1)n^{-1/6})P [\|Y\|_\infty > r]
\]
for any \(r \leq -1 + M_2 n^{1/6}\). Here \(M_1\) and \(M_2\) are dimension free constants depending on the distributions of \(X_1\) and \(Y\) which can be bounded by a polynomial of \(\log p\), under certain tail assumptions on the coordinates of \(X_1\). The proof is motivated by the techniques of Bentkus (1987) and is modified for the high dimensional set up. The constants \(M_1\) and \(M_2\) are also made explicit in Theorem 3.1.

(ii) We derive upper bounds to (4). We prove in Theorem 3.2 that for \(m \geq 3\) and \(n\) large enough,
\[
\sup_{r \geq 0} r^m |P [\|S_n\|_\infty \leq r] - P [\|Y\|_\infty \leq r]| \leq \frac{2\nu_m}{2^n} + \frac{\Theta_m \Phi_2 \Gamma_m \nu_m^{1/m}}{n^{1/2 - 1/m}} + \frac{\Theta_m \Phi_2 \Gamma_m \nu_m^{1/3}}{n^{1/6}}.
\]
Here \(\nu_m\) is the \(m\)-th pseudo-moment defined as
\[
\nu_m := \int_{\mathbb{R}^p} \|x\|_\infty^m |L(X) - L(Y)| (dx), \quad \Gamma_m := \mathbb{E} \left[\left(\|Y\|_\infty + 1\right)^{m+1}\right],
\]
and \(\Phi_2, \Theta_m\) are constants depending on the distribution of \(X_1\) which can be bounded by a polynomial of \(\log p\) under certain tail assumptions on the coordinates of \(X_1\); see Theorem 3.2 for a detailed description of the constants. The proof of Theorem 3.2 is motivated by techniques of Paulauskas and Račkauskas (1991) with appropriate modification for the high dimensional CLT setup. In Theorem 3.3, we extend the above bound by replacing \(r^m\) by a general non-decreasing function \(\phi(\cdot)\).

(iii) In Theorem 3.5, we provide a refinement of Lemma 3.1 of Bentkus and Račkauskas (1990) with exact constants. Lemma 3.1 of Bentkus and Račkauskas (1990) is about
inequalities related to probabilities of $\varepsilon$-strips of Gaussian process. In Corollary 3.3 of Theorem 3.5, we derive a dimension-free anti-concentration inequality for Gaussian processes indexed by a compact metric space which might be of independent interest.

1.2 Organization of the paper

The paper is organized as follows. In Section 2, we define some useful notations and state the assumptions used. Section 3 is dedicated for our main results. It is divided into three parts. In the first part of the section we state the main results in Theorems 3.1, 3.2 and 3.3. Theorem 3.1 is about the Cramér-type large deviation and Theorems 3.2 and 3.3 are about non uniform estimates for polynomials and general function decay. In the second part of the section, we provide specific corollaries for the high-dimensional case with exponential tail assumptions to show the dependence on $\log p$. In the third part of the section we give two results which verify the smoothing assumption (A1) and the density assumption (A2) in our set up. The section ends with a corollary about anti-concentration. In section 4, the sketches of the proofs of the main theorems are given. Finally, we conclude with a brief summary and future directions in Section 5. Proofs of all the results are given in the Appendices A, B, C and D.

2 Preliminaries

2.1 Notations

As discussed earlier, we shall consider i.i.d. random vectors $X_1, \ldots, X_n \in \mathbb{R}^p$ with mean zero and covariance $\Sigma$. Let $Y \in \mathbb{R}^p$ denote a Gaussian random vector with mean zero and covariance $\Sigma$. We let $Y_1, Y_2, \ldots, Y_n$ denote $n$ i.i.d. copies of the random vector $Y$. The $l_\infty$ norm on $\mathbb{R}^p$ is denoted by $\|\cdot\|$. We also use $\mathcal{L}(X)$ and $\mathcal{L}(Y)$ for the laws of $X_1$ and $Y$ respectively. We also need the following notations throughout the paper.

\[
S_n := n^{-1/2} (X_1 + X_2 + \ldots + X_n),
\]
\[
\nu_m := \int_{\mathbb{R}^p} \|x\|^m |\mathcal{L}(X) - \mathcal{L}(Y)| (dx), \quad m \geq 3,
\]
\[
\Delta_n(r) := |\mathbb{P}(\|S_n\| \leq r) - \mathbb{P}(\|Y\| \leq r)|,
\]
\[
\zeta := \mathcal{L}(X) - \mathcal{L}(Y).
\]

2.2 Assumptions

In this paper we make the following assumptions which will be justified later.

(A1) **Approximating Functions:** For any $r, \varepsilon > 0$, there exists a function $\varphi_{r, \varepsilon} : \mathbb{R}^p \to [0, 1]$
and constants $C_1, C_2, C_3 \geq 1$ such that $\varphi_{r,\varepsilon}(\cdot)$ is three times differentiable,

$$
\varphi_{r,\varepsilon}(x) := \begin{cases} 
1, & \text{if } \|x\| \leq r, \\
0, & \text{if } \|x\| > r + \varepsilon,
\end{cases}
$$

and for $1 \leq j \leq 3$,

$$
\sup_{r,\varepsilon > 0} \sup_{x \in \mathbb{R}^p} \|D^j \varphi_{r,\varepsilon}(x)\|_1 \leq C_j,
$$

where for $k \geq 1$ and any function $g$ that is $k$ times differentiable,

$$
\left\| D^k g(x) \right\|_1 := \sum_{i_1=1}^{p} \cdots \sum_{i_k=1}^{p} \left| \frac{\partial^k g(x)}{\partial x(i_1) \cdots \partial x(i_k)} \right|.
$$

**(A2)** Density Assumptions: There exists constants $\Phi_0, \Phi_1, \Phi_3 > 0, \Phi_2, \Phi_4 \geq 1$ such that for all $r > 0$ and $\varepsilon > 0$,

$$
\mathbb{P}(\|Y\| \geq r) \geq \Phi_0 \exp(-\Phi_1 r^2),
$$

$$
\mathbb{P}(r - \varepsilon \leq \|Y\| \leq r + \varepsilon) \leq \Phi_2 \varepsilon (1 + r) \mathbb{P}(\|Y\| \geq r - \varepsilon)
$$

$$
\mathbb{P}(\|Y\| \geq r - \varepsilon) \leq \Phi_3 \exp(\Phi_4 (r + 1) \varepsilon) \mathbb{P}(\|Y\| \geq r).
$$

The requirements $C_1, C_2, C_3 \geq 1$ in **(A1)** and $\Phi_2, \Phi_4 \geq 1$ in **(A2)** are made only for simplifying the bounds to be discussed in the forthcoming sections.

### 3 Main Results

#### 3.1 Dimension-free Large Deviation and Non-uniform CLTs

We are now ready to state the main results of this paper. The proofs of all the results in this section are given in Appendices B and C. The sketches of these proofs are presented, for readers’ convenience, in Section 4. Theorems 3.1, 3.2 and 3.3 below are stated and proved only under assumptions **(A1)** and **(A2)**. These results do not require the random vectors to be finite-dimensional. We do use the Taylor series expansion in the proofs which works even for Banach spaces under Fréchet differentiability. All the results in this section are dimension-free and only depend on pseudo-moments. The importance of dependence on pseudo-moments can be seen from the fact that if the random variables are Gaussian then the probability $\mathbb{P}(\|S_n\| \geq r)$ is exactly equal to $\mathbb{P}(\|Y\| \geq r)$ which implies that all the pseudo-moments are zero. This is reflected in our bounds below.

The following theorem proves a Cramér-type large deviation result for $\|S_n\|$. A version of this result appeared in Bentkus (1987, Theorem 1) for the case of Banach space valued random variables. In the following result, we make the dependences on distributional constants precise.
Theorem 3.1. (Cramér-type large deviation) Suppose that assumptions (A1) and (A2) hold and that there exists $H > 0$ such that

\[ \beta := \int_{\mathbb{R}^p} \exp \left( H \|x\| \right) |\mathcal{L}(X) - \mathcal{L}(Y)|(dx) < \infty. \]  

(9)

Then for all $n \geq n^*$ and $r + 1 \leq n^{1/6}/R$ we have

\[ \left| \frac{\mathbb{P}(\|S_n\| \geq r)}{\mathbb{P}(\|Y\| \geq r)} - 1 \right| \leq 2M(r + 1)n^{-1/6}. \]

Here $n^* := (4e \log(n))^6(MG)^2$, $R := 2eG^{4/3}M^{1/3}$, with

\[ M := \max \left\{ \sqrt{2}, \left( 3e\Phi_3\Phi_2 + \sqrt{2}e\nu_2C_3\Phi_3 \right)^{4/3}, \left( 2\sqrt{2}C_3\Phi_2\nu_3 \right)^{2/3}, \left( \frac{80\beta\nu_3}{\Phi_0H^3} + 5\sqrt{2}C_3\nu_3B \right)^{4/7} \right\}, \]

and $G := \Phi_4(4BM^{-1/4} + 1)$ with $B := 2(\Phi_1 + 1)/H$.

Theorem 3.1 is written specifically for $n \geq n^*$ and the result of this type actually holds for all $n \geq 1$ as shown in Lemma B.2.1 in Section B.2, but with $n^{-1/8}$.

The following two theorems prove two non-uniform versions of the central limit theorem. Recall the definition of $\zeta$ from (8). Define for any $n \geq 1$ and $m \geq 0$,

\[ V_m(t) := \int_{\|x\| > t} \|x\|^m |\zeta| \, (dx), \quad \text{for} \quad t \geq 0, \]

\[ \Delta_n^{(m)} := \sup_{r \geq 0} r^m \left| \mathbb{P}(\|S_n\| \leq r) - \mathbb{P}(\|Y\| \leq r) \right|. \]  

(10)

Theorem 3.2. (Non-uniform CLT: polynomial decay) For any $m \geq 1$, let $T_m$ be any positive number $T$ such that

\[ V_m \left( 2^{1/m}(4\sqrt{2}/3)(C_1 + C_2)^{1/m}T^{1/m} \right) \leq \frac{T}{4}. \]

If assumptions (A1) and (A2) hold true, then for any $m \geq 3$ and for all $n \geq 1$ we have

\[ \Delta_n^{(m)} \leq \frac{2\nu_m}{2^n} + a(m) \left( \frac{C_1 + C_2}{n^{(m-2)/2}} \right) + b(m) \frac{\Phi_2\Gamma_m(C_1 + C_2)^{1/m}T^{1/m}}{n^{1/2 - 1/m}} + c(m) \frac{\Phi_2\Gamma_m(C_2 + C_3)^{1/3}\nu_3^{1/3}}{n^{1/6}} + d(m) \frac{(C_2 + C_3)^{m/3}\nu_3^{m/3}}{n^{m/6}}, \]

where $\Gamma_m$ is as defined in (7), and

\[ a(m) := 8 \left( \frac{8\sqrt{2}}{3} \right)^m, \quad b(m) := 8\sqrt{2} \left( \frac{4}{3} \right)^m, \quad c(m) := 17 \times 3^m, \quad d(m) := 2(8\sqrt{2})^m(4\sqrt{2})^{m^2/3}. \]
Since $V_m(t) \leq \nu_m$ for all $t$, one choice of $T_m$ is $4\nu_m$. With lighter tailed random variables, $T_m$ can be chosen to be of smaller order. Theorem 3.2 is an improved version of Theorem 1 in Paulauskas and Račkauskas (1991), since in Theorem 1 of Paulauskas and Račkauskas (1991), $T_m$ is always chosen to be $4\nu_m$.

Theorem 3.2 proves a non-uniform CLT specifically with polynomial growth and the following result allows for general non-decreasing functions $\phi(\cdot)$. It should, however, be mentioned here that for any specific function $\phi(\cdot)$, the proof can be used to get a better bound by utilizing the structure of $\phi(\cdot)$. To state this result, define for any function $\phi: \mathbb{R}^+ \mapsto \mathbb{R}^+$:

\[
U_{n,k} := n^{-1/2}(X_1 + \ldots + X_k + Y_{k+1} + \ldots + Y_n),
\]

\[
\Delta_{n,\phi} := \sup_{r > 0} \phi(r)|\mathbb{P}(\|S_n\| \leq r) - \mathbb{P}(\|Y\| \leq r)|,
\]

\[
\delta_{n,\phi} := \sup_{1 \leq k \leq n} \sup_{r > 0} \phi(r)|\mathbb{P}(\|U_{n,k}\| \leq r) - \mathbb{P}(\|Y\| \leq r)|.
\]

**Theorem 3.3. (Non-uniform CLT: general function decay)** Let $\phi: \mathbb{R}^+ \mapsto \mathbb{R}^+$ be a non-decreasing function satisfying

\[
\phi(x + y) \leq C_\phi \phi(x)\phi(y), \quad \text{for all } x, y \in \mathbb{R}^+,
\]  

for some constant $C_\phi \geq 1$. Suppose that assumptions (A1) and (A2) are satisfied. Also, suppose that there exists a constant $0 \leq \Xi_{\phi,X} < \infty$ depending only on $\phi(\cdot)$ and the distribution of $X$, such that

\[
\sup_{n \geq 1} \delta_{n,\phi} \leq \Xi_{\phi,X}.
\]  

Then for all $n \geq 1$,

\[
\Delta_{n,\phi} \leq \frac{\Xi_{\phi,X}}{2n^{1/2}} + \frac{5 [A_\phi \Phi(1 + 2C_\phi \phi(1)) + \Xi_{\phi,X} \left(C_\phi^2 \phi(1)(C_2 + C_3)\nu_{\phi,3}\right)]^{1/3}}{n^{1/6}},
\]

where

\[
\nu_{\phi,3} := \int \phi(\|x\|) \|x\|^3 |\zeta| (dx), \quad \text{and} \quad A_\phi := \max \left\{1, \sup_{r > 0} (1 + r)\phi(r)|\mathbb{P}(\|Y\| > r)\right\}.
\]

Theorem 3.3 is applicable to function $\phi(x) = (1+x^m)$ (as shown below) but the conclusion of Theorem 3.3 is worse than the one attained from Theorem 3.2 in terms of the dependence on distributional constants. The reason, as mentioned before, is that taking into account the exact form of $\phi(\cdot)$ makes a difference.

**Remark 3.1** Examples of functions satisfying condition (11) are $\phi(x) = 1 + x^m$, $m \geq 0$ and $\phi(x) = \exp(\lambda x^\alpha)$, $0 \leq \alpha \leq 1$. For the proof, note that for $x, y \in \mathbb{R}^+$,

\[
(1 + (x+y)^m) \leq 1 + 2^m x^m + 2^m y^m \leq 2^m(1 + x^m + y^m) \leq 2^m(1 + x^m)(1 + y^m).
\]
Also, note that for $0 \leq \alpha \leq 1$,
\[
\exp(\lambda(x + y)\alpha) \leq \exp(\lambda(x^\alpha + y^\alpha)) \leq \exp(\lambda x^\alpha) \exp(\lambda y^\alpha).
\]

It is interesting to note that $\phi(x) = \exp(\lambda x^\alpha)$ for $\alpha > 1$ does not satisfy condition (11).  

**Remark 3.2** Condition (12) is trivially satisfied with $\Xi_{\phi, X} = 1$, if $\phi$ is bounded by 1, in particular if $\phi$ is identically equal to 1. More precisely from Theorem 3.3 with $\phi$ identically equal to 1, we get that

\[
\sup_{r \geq 0} \Delta_n(r) \leq \frac{\Theta \phi_2 \Gamma_0 (C_2 + C_3)^{1/3} \nu^{1/3}}{n^{1/6}},
\]

for some universal constant $\Theta > 0$. For a proof of this, note that the constant function $\phi(\cdot)$ satisfies condition (11) with $C_\phi = 1$. Also, in this case, $\Xi_{\phi, X} = 1$,

\[
\nu_{\phi, 3} = \int \|x\|^3 \mathcal{L}(X) - \mathcal{L}(Y)(dx) = \nu_3,
\]

and by Markov's inequality,

\[
\sup_{r \geq 0} (1 + r)\phi(r)\mathbb{P}(\|Y\| > r) = \sup_{r \geq 0} (1 + r)\mathbb{P}(\|Y\| > r) \leq \mathbb{E}(1 + \|Y\|) = \Gamma_0.
\]

Substituting these in the bound of Theorem 3.3 implies (13).  

**Remark 3.3** (Verification of Condition (12)) Even though the constant $\Xi_{\phi, X}$ in condition (12) is implicit and unclear, note that if $\delta_{n, \phi}$ is not bounded uniformly in $n$, then the conclusion of Theorem 3.3 cannot be expected. Condition (12) can be easily verified for a large class of random variables using Theorem 6.21 of Ledoux and Talagrand (2011). Suppose $X_1, X_2, \ldots, X_n$ are independent Banach space valued random variables satisfying $\|X_i\|_\psi_\alpha < \infty$, for some $0 < \alpha \leq 1$, where $\|\cdot\|_B$ represents the norm on the Banach space and $\|\cdot\|_\psi_\alpha$ represents the $\psi_\alpha$-Orlicz norm (see Section 3.2). Then Theorem 6.21 of Ledoux and Talagrand (2011) implies that

\[
\left\|\frac{1}{\sqrt{n}} \sum_{i=1}^n X_i\right\|_B \leq K_\alpha \left(\mathbb{E}\left[\left\|\frac{1}{\sqrt{n}} \sum_{i=1}^n X_i\right\|_B\right] + \frac{1}{\sqrt{n}} \max_{1 \leq i \leq n} \left\|X_i\right\|_B \right)^{\psi_\alpha},
\]

for some constant $K_\alpha$ depending only on $\alpha$. This implies that

\[
\mathbb{E}\left[\exp\left(\frac{\|S_n\|^\alpha_B}{(K_\alpha \mathbb{E}[\|S_n\|_B] + K_\alpha (\log n)^{1/\alpha} n^{-1/2} \|X\|_B \|\psi_\alpha\|^\alpha)}\right)\right] \leq 2.
\]

In particular, we get that if $\mathbb{E}[\|S_n\|_B]$ is bounded uniformly in $n$, then one can take $\Xi_{\phi, X} = 2$ in condition (12) for any non-decreasing function $\phi(\cdot)$ satisfying (11) and for all $r \geq 0$,

\[
\phi(r) \leq \exp\left(\frac{r^\alpha}{\tau^{\alpha}}\right), \quad \text{where} \quad \tau := K_\alpha \left(\sup_{n \geq 1} \mathbb{E}[\|S_n\|_B] + \|\|X\|_B\|_{\psi_\alpha}\right).
\]
In particular, all polynomial type functions are allowed. It is easy to see that Gaussian random variable $Y$ satisfies $\|Y\|_\psi_0 \leq K_\alpha \|Y\|_\psi_2 < \infty$, for some constant $K_\alpha$ depending only on $0 < \alpha \leq 1$.

**Remark 3.4 (Convergence of Moments)** Theorem 3.2 along with Remark 3.2 are useful in proving convergence of $m$-th moment of $\|S_n\|$ to that $\|Y\|$ at an $n^{-1/6}$ rate (up to factors depending on $\nu_{m+1}$). To prove an explicit bound, note that for $m \geq 1$:

$$
|E[\|S_n\|^m] - E[\|Y\|^m]| = \left| \int_0^\infty mr^{m-1} (P(\|S_n\| \geq r) - P(\|Y\| \geq r)) \, dr \right|
\leq m \int_0^1 \Delta_n(r) \, dr + \int_1^\infty \frac{m}{r} \Delta_n^{(m+1)}(r) \, dr
\leq m \sup_{r \geq 0} \Delta_n(r) + m \Delta_n^{(m+1)}
\leq m \frac{\Theta \Phi_2 \Gamma_0 (C_2 + C_3)^{1/3} \nu_3^{1/3}}{n^{1/6}} + m \Delta_n^{(m+1)}, \quad (14)
$$

Here the last step follows from Remark 3.2. The second term in the right hand side above can be bounded using Theorem 3.3 for $m \geq 2$. To bound $\Delta_n^{(m+1)}$ for $m \leq 2$, observe that

$$
\Delta_n^{(m)} \leq \sup_{0 \leq r \leq 1} r^m \Delta_n(r) + \sup_{r \geq 1} r^m \Delta_n(r) \leq \sup_{r \geq 0} \Delta_n(r) + \sup_{r \geq 1} r^3 \Delta_n(r)
\leq \frac{\Theta \Phi_2 \Gamma_0 (C_2 + C_3)^{1/3} \nu_3^{1/3}}{n^{1/6}} + \Delta_n^{(3)}.
$$

Combining this with (14), Theorem 3.2 proves convergence of moments for all $m \geq 0$.

### 3.2 Specific Corollaries for High Dimensions

As mentioned before, the results of the previous section apply to general Banach spaces and does not require finite-dimensional ones. In this section, we provide explicit results in terms of the dimension $p$ using Theorems 3.1 and 3.2. The benchmark results available in the literature for comparison are those of Chernozhukov et al. (2017). Our specific corollaries in this section are different from those of Chernozhukov et al. (2017) in many respects as listed below:

1. We mainly focus on non-uniform version of the central limit theorem either through the ratio or the difference but Chernozhukov et al. (2017) consider the uniform difference problem. However, it is important to note that they consider more general sets like rectangles or sparse convex sets and we restrict our attention to $l_\infty$-balls. Also, they allow for non-identically distributed random vectors and we restrict to i.i.d. random vectors.
2. All our results only depend on pseudo-moments and not directly on the moments of the random variables. This dependence is useful in case the random variables \( X_1, X_2, \ldots, X_n \) have a distribution either close to or is exactly Gaussian. The result of Chernozhukov et al. (2017) on the other hand depends directly on the moments of the random variables.

3. Our results always show the dependence of \( n^{-1/6} \) on the sample size under finiteness of certain pseudo-moments. In this respect the dependence on the sample size can be worse from the results of Chernozhukov et al. (2017). For instance, Proposition 2.1 there with \( q = 3 \) in assumption (E.2) provides a rate of \( n^{-1/9} \) whereas under the same assumption our result provides a rate of \( n^{-1/6} \). However, the dependence on \( \log p \) of our results is worse when compared to their results. See Remark 3.6 for further details.

The proofs of the results in this section are deferred to Appendix D. To state these, we at first introduce Orlicz norms.

**Definition 3.1.** Let \( X \) be a real-valued random variable and \( \psi : [0, \infty) \to [0, \infty) \) be a non-decreasing function with \( \psi(0) = 0 \). Then, we define

\[
\|X\|_\psi = \inf \left\{ c > 0 : \mathbb{E} \psi \left( \frac{|X|}{c} \right) \leq 1 \right\},
\]

where the infimum over the empty set is taken to be \( \infty \).

It follows from Jensen’s inequality, that when \( \psi \) is a non-decreasing, convex function, \( \|\cdot\|_\psi \) is a norm on the set of random variables \( X \) for which \( \|X\|_\psi < \infty \). Such norms are referred to as Orlicz norms. The commonly used Orlicz norms are derived from

\[
\psi_\alpha(x) := \exp(x^\alpha) - 1,
\]

for \( \alpha \geq 1 \), which are obviously increasing and convex. For \( 0 < \alpha < 1 \), \( \psi_\alpha \) is not convex, and \( \|X\|_{\psi_\alpha} \) is not a norm, but a quasinorm. A random variable \( X \) is called sub-exponential if \( \|X\|_{\psi_1} < \infty \), and a random variable \( X \) is called sub-Gaussian if \( \|X\|_{\psi_2} < \infty \).

Finally, define

\[
\sigma^2_{\text{max}} := \max_{1 \leq j \leq p} \text{Var}(X(j)), \quad \text{and} \quad \sigma^2_{\text{min}} := \min_{1 \leq j \leq p} \text{Var}(X(j)).
\]

Recall that \( X(j) \) represents the \( j \)-th coordinate of \( X \in \mathbb{R}^p \). The following is a corollary of Theorem 3.1.

**Corollary 3.1.** Suppose that there exists a constant \( 1 \leq K_p < \infty \), such that

\[
\max_{1 \leq j \leq p} \|X(j)\|_{\psi_\alpha} \leq K_p \tag{15}
\]
for some $1 \leq \alpha \leq 2$, where $X = (X(1), \ldots, X(p))$. Then, there exist positive constants $\Theta_1, \Theta_2$ and $\Theta_3$ possibly depending on $\alpha, \sigma_{\min}$ and $\sigma_{\max}$, such that

$$\left| \frac{\mathbb{P} (\|S_n\| \geq r)}{\mathbb{P} (\|Y\| \geq r)} - 1 \right| \leq \Theta_1 K_p^4 (\log p)^{4/\alpha + 8/3} (r + 1) n^{-1/6}$$

for all $n \geq \Theta_2 K_p^8 (\log p)^{28/\alpha + 8/3} (\log n)^6$ and $r + 1 \leq \Theta_3 K_p^{-4/3} (\log p)^{-32/\alpha (3 \alpha) n^{1/3}}$.

Since the pseudo-moments depend on the Gaussian distribution, it is not possible take advantage of (15) even if $\alpha \geq 2$.

**Remark 3.5** In particular, Corollary 3.1 implies the following bounds for sub-Gaussian and sub-exponential random vectors. Under the assumption (15) with $\alpha = 2$, we get for all $n \geq \Theta_2 K_p^8 (\log p)^{40/3} (\log n)^6$ and $r + 1 \leq \Theta_3 K_p^{-4/3} (\log p)^{-38/9} n^{1/6}$,

$$\left| \frac{\mathbb{P} (\|S_n\| \geq r)}{\mathbb{P} (\|Y\| \geq r)} - 1 \right| \leq \Theta_1 K_p^4 (\log p)^{14/3} (r + 1) n^{1/6}.$$

Similarly, under the assumption (15) with $\alpha = 1$, we get for all $n \geq \Theta_2 K_p^8 (\log p)^{52/3} (\log n)^6$ and $r + 1 \leq \Theta_3 K_p^{-4/3} (\log p)^{-44/9} n^{1/6}$,

$$\left| \frac{\mathbb{P} (\|S_n\| \geq r)}{\mathbb{P} (\|Y\| \geq r)} - 1 \right| \leq \Theta_1 K_p^4 (\log p)^{20/3} (r + 1) n^{1/6}.$$

The following is a corollary of Theorem 3.2 with $m = 3$ and is useful in proving consistency of expectation and variance of $\|S_n\|$ to $\|Y\|$.

**Corollary 3.2.** Suppose that there exists a constant $1 \leq K_p < \infty$, such that

$$\max_{1 \leq j \leq p} \|X(j)\|_{\psi_\alpha} \leq K_p$$

for some $0 < \alpha \leq 2$, where $X = (X(1), \ldots, X(p))$. Then, there exists a constant $\Theta > 0$ possibly depending on $\alpha, \sigma_{\min}$ and $\sigma_{\max}$, such that

$$\Delta_n^{(3)} \leq \Theta \left[ K_p^3 (\log p)^{2 + 3/\alpha} n^{-1/2} + K_p^5 (\log p)^{10 + 3 + 1/\alpha} n^{-1/6} \right],$$

for all $n \geq 1$.

**Remark 3.6** (Convergence of Moments: High-dimensional case) Using the calculations from Remark 3.4, it follows under the assumption (16) of Corollary 3.2 that

$$\sup_{r \geq 0} \Delta_n(r) \leq \Theta n^{-1/6} K_p (\log p)^{13/6 + 1/\alpha},$$

$$\left| \mathbb{E} \|S_n\| - \mathbb{E} \|Y\| \right| \leq \Theta n^{-1/6} \left[ K_p^3 (\log p)^{2 + 3/\alpha} n^{-1/3} + K_p^5 (\log p)^{11/3 + 1/\alpha} \right],$$

$$\left| \mathbb{E} (\|S_n\|^2) - \mathbb{E} (\|Y\|^2) \right| \leq \Theta n^{-1/6} \left[ K_p^3 (\log p)^{2 + 3/\alpha} n^{-1/3} + K_p^5 (\log p)^{11/3 + 1/\alpha} \right].$$
Here $\Theta > 0$ represents a constant depending only on $\alpha, \sigma_{\max}$ and $\sigma_{\min}$. The first of these bounds compares poorly with the results of Chernozhukov et al. (2017, Proposition 2.1) in the sense that our bound here has worse dependence on $\log p$ than the result from the aforementioned paper. It should, however, be noted that Remark 3.2 always provides a rate of $n^{-1/6}$ for $\sup \{\Delta_n(r) : r \geq 0\}$ as long as $\nu_3$ is finite. Observe that under assumption (E.2) with $q = 3$ of Chernozhukov et al. (2017), Proposition 2.1 there provides a much worse dependence $n^{-1/9}$ on the sample size as the rate. To the best of our limited knowledge, the moment convergence bounds in (17) are the first of the kind available in the high-dimensional case.

3.3 Verification of the assumptions

Now we state two results which verifies assumptions (A1) and (A2).

**Theorem 3.4.** (Theorem 1 in Bentkus (1990)) Consider the space $\mathbb{R}^p$ and let $\|\cdot\|$ be the $l_\infty$ norm. Then for any $r \geq 0$ and $\varepsilon > 0$ there exists a function $0 \leq \varphi_{r, \varepsilon}(x) \leq 1$, $\varphi_{r, \varepsilon}(x) \in \mathbb{C}_\infty$ and

$$
\varphi_{r, \varepsilon}(x) := \begin{cases} 
1, & \text{if } \|x\| \leq r, \\
0, & \text{if } \|x\| > r + \varepsilon,
\end{cases}
$$

such that for any $j \geq 1$, 

$$
\sup_{x \in \mathbb{R}^p} \|D^j \varphi_{r, \varepsilon}(x)\|_1 \leq c(j) \varepsilon^{-j} \log^{-1} (p + 1)
$$

where the constant $c(j)$ depends only on $j$.

Theorem 3.4 in particular verifies (A1) for $\mathbb{R}^p$ with $l_\infty$ norm. Verification of assumption (A1) for more general Hilbert/Banach spaces is discussed in Paulauskas and Račkauskas (2012).

Now we state a result which verifies (A2) for a Gaussian process on a compact metric space. This result is an improved version of Lemma 3.1 in Bentkus and Račkauskas (1990) where implicit constants were used. In the proof, we make use of a result from Giné (1976). In the following version the constants are made explicit and dimension-free which can be bounded by polynomials in $\log p$.

**Theorem 3.5.** Let $Y$ be a sample continuous centered Gaussian process on a compact metric space $S$ such that $\sigma_{\min}^2 \leq \mathbb{E}[Y^2(s)] \leq \sigma_{\max}^2$ for all $s \in S$. Let $\mu$ denote the median of $\|Y\|$. Then the following are true:

1. For all $r \geq 0$,

$$
\mathbb{P} (\|Y\| \geq r) \geq \frac{1}{6} \exp \left( - \frac{r^2}{\sigma_{\max}^2} \right).
$$
2. For all \( r, \varepsilon \geq 0 \),
\[
\mathbb{P} (\|Y\| \geq r - \varepsilon) \leq 20 \exp (\Phi_4 (r + 1) \varepsilon) \mathbb{P} (\|Y\| \geq r),
\]  
(18)
where \( \Phi_4 \) is given by
\[
\Phi_4 := \max \left\{ 1, \frac{56(\mu + 1.5\sigma_{\text{max}})(\mu + 4.1\sigma_{\text{max}})}{\sigma_{\text{max}}^2\sigma_{\text{min}}^2}, \frac{32\pi (2.6\sigma_{\text{min}} + \mu)^2 (\sigma_{\text{min}}^2 + 32\sigma_{\text{min}}\mu + 12\mu^2)}{\sigma_{\text{min}}^4} \right\}.
\]

3. For all \( r \geq 0 \) and \( \varepsilon > 0 \),
\[
\mathbb{P} (r - \varepsilon \leq \|Y\| \leq r + \varepsilon) \leq \Phi_2 \varepsilon (r + 1) \mathbb{P} (\|Y\| \geq r - \varepsilon),
\]  
(19)
where
\[
\Phi_2 := \max \left\{ \frac{51(\mu + 4.1\sigma_{\text{max}})}{\sigma_{\text{min}}^2}, \frac{32\pi (\mu + 2.6\sigma_{\text{min}})^2}{\sigma_{\text{min}}^4} \right\}
\]

Observe that the set \( \{1, \ldots, p\} \) is compact. So the result can be used in the high dimension case. One of the advantages of Theorem 3.5 is that it is dimension-free and the dependence on the “complexity” of \( S \) appears only through the median of \( \|Y\| \). It is not hard to derive the following anti-concentration inequality for \( \|Y\| \) as an immediate corollary of Theorem 3.5.

**Corollary 3.3.** Under the assumptions of Theorem 3.5,
\[
\limsup_{\varepsilon \to 0} \frac{\mathbb{P} (r - \varepsilon \leq \|Y\| \leq r + \varepsilon)}{2\varepsilon} \leq \begin{cases} 2\sigma_{\text{min}}^{-2} (2.6\sigma_{\text{min}} + \xi) & \text{when } r \leq \xi, \\ \Phi_2 \exp (-r^2 \sigma_{\text{max}}^{-2}/144) & \text{otherwise.} \end{cases}
\]  
(20)
where \( \xi := \max \{144\sigma_{\text{max}}^2, 3\mu/2\} \).

**Remark 3.7** Observe that one can bound the left hand side of (20) by \( (\alpha_1 + \alpha_2\xi) \) for some constants \( \alpha_1 \) and \( \alpha_2 \) depending only on \( \sigma_{\text{max}}, \sigma_{\text{min}} \). When \( r \leq \xi \) the right hand side of (20) is a linear function of \( \xi \). When \( r > \xi \), \( \exp (-r^2/144\sigma_{\text{max}}^2) \leq \exp (-\mu^2/64\sigma_{\text{max}}^2) \). However, \( \Phi_2 \) is just a polynomial in \( \mu \). As a consequence,
\[
\sup_{\mu} \frac{\Phi_2 \exp (-\mu^2/64\sigma_{\text{max}}^2)}{\alpha_1 + \alpha_2\xi} < \infty,
\]
for any fixed \( \alpha_1 \) and \( \alpha_2 \).

**Remark 3.8** (Comparison with Chernozhukov et al. (2015)) Possibly the first work proving the anti-concentration inequality for the maximum of Gaussian random variables is Chernozhukov et al. (2015). Theorem 3 of Chernozhukov et al. (2015) implies the anti-concentration for \( \|Y\| \) and provides a dimension-free bound depending on the median of
only under the additional assumption of $\sigma_{\text{max}} = \sigma_{\text{min}}$. For the general case, their bound has an additional term of $\log(1/\varepsilon)$ and so makes their bound weaker than the one from Corollary 3.3. In terms of the proof technique, we note that the techniques of both works are the same for $r \leq \xi$. For the case $r \geq \xi$, Chernozhukov et al. (2015) apply the Gaussian concentration inequality which leads to the extra $\log(1/\varepsilon)$ factor while we apply inequality (19) which leads to the sharper version above. Corollary 3.3 is the first result on dimension-free anti-concentration inequality for $\|Y\|$ and answers the open question raised in Remark A.1 of Chernozhukov et al. (2017). Additionally our result readily applies to Gaussian processes on compact metric spaces. The results of Nazarov (2003) imply an anti-concentration inequality that explicitly depends on the dimension as $\sqrt{\log p}$.

4 Sketch of proofs of Theorems 3.1, 3.2 and 3.3

We now give the sketch of the proofs of Theorem 3.1, 3.2 and 3.3, the main results of the paper. All the proofs rely on Lindeberg method and we first provide a description of this method. Let $Y_1, \ldots, Y_n$ be i.i.d. copies of $Y$. Recall the definition $\Delta_n(r)$ from (8). The fundamental idea is to bound $\Delta_n(r)$ by a Lindeberg replacement scheme. The first step is to relate $\Delta_n(r)$, which involves expectations of non-smooth (indicator) functions to expectations of smooth functions which is done by the following smoothing lemma from Paulauskas and Račkauskas (2012).

Lemma 4.1. (Lemma 5.1.1 in Paulauskas and Račkauskas (2012)) Let $\varphi_1(x) := \varphi_{r,\varepsilon}(x)$ and $\varphi_2(x) = \varphi_{r-\varepsilon,\varepsilon}(x)$ (these functions exist by assumption (A1)). Then for any $n \geq 1$,

$$\Delta_n(r) \leq \max_{j=1,2} \|E[\varphi_j(S_n) - \varphi_j(Y)]\| + P(r - \varepsilon \leq \|Y\| \leq r + \varepsilon).$$

Now the task of bounding $\Delta_n(r)$ is reduced to bounding $\max_{j=1,2} |E[\varphi_j(S_n) - \varphi_j(Y)]|$. This done by a replacement scheme as follows. To begin with, we introduce the following notations.

$$U_{n,k} := n^{-1/2}(X_1 + X_2 + \ldots + X_k + Y_{k+1} + Y_{k+2} + \ldots + Y_n),$$
$$W_{n,k} := n^{-1/2}(X_1 + X_2 + \ldots + X_{k-1} + Y_{k+1} + \ldots + Y_n),$$
$$\Delta_{n,k}(r) := |P(\|U_{n,k}\| \leq r) - P(\|Y\| \leq r)|.$$ (21)

Now the task of bounding $\Delta_n(r)$ is reduced to bounding $\max_{1 \leq k \leq n} \Delta_{n,k}(r)$. We write from Lemma 4.1,

$$\Delta_{n,k}(r) \leq \max_{j=1,2} |E[\varphi_j(U_{n,k}) - \varphi_j(Y)]| + P(r - \varepsilon \leq \|Y\| \leq r + \varepsilon).$$ (22)
For convenience, we use \( \varphi \) for \( \varphi_1 \) and \( \varphi_2 \) both. Observe that \( Y \overset{d}{=} U_{n,0} \) which implies,

\[
|\mathbb{E} [\varphi(U_{n,k}) - \varphi(Y)]| \leq \sum_{j=1}^{k} |\mathbb{E} [\varphi(U_{n,j-1}) - \varphi(U_{n,j})]| \leq \sum_{i=1}^{k} \left| \int \mathbb{E} \left[ \varphi \left( W_{n,j} + n^{-1/2}x \right) \right] \zeta(dx) \right|.
\] (23)

The last step follows from the following two observations: \( U_{n,j} = W_{n,j} + n^{-1/2}X_j \) and \( U_{n,j-1} = W_{n,j} + n^{-1/2}Y_j \). Now by Taylor series expansion we have

\[
\varphi \left( W_{n,j} + n^{-1/2}x \right) = \varphi(W_{n,j}) + n^{-1/2}x^\top \nabla \varphi(W_{n,j}) + \frac{1}{2n} x^\top \nabla^2 \varphi(W_{n,j}) x + \text{Rem}_n(W_{n,j}, x)
\]

where \( \text{Rem}_n(W_{n,j}, x) \) is a remainder term. Since \( \mathbb{E}[X_j] = \mathbb{E}[Y_j] = 0 \) and \( \mathbb{E}[X_jX_j^\top] = \mathbb{E}[Y_jY_j^\top] \), we have

\[
\int \mathbb{E} [\varphi(W_{n,j})] \zeta(dx) = \int \mathbb{E} \left[ n^{-1/2}x^\top \nabla \varphi(W_{n,j}) \right] \zeta(dx) = \int \mathbb{E} \left[ \frac{1}{2n} x^\top \nabla^2 \varphi(W_{n,j}) x \right] \zeta(dx) = 0.
\]

Now observe that

\[
\left| \int \mathbb{E} [\text{Rem}_n(W_{n,j}, x)] \zeta(dx) \right| \leq \int \mathbb{E} [||\text{Rem}_n(W_{n,j}, x)||] |\zeta| dx. \tag{24}
\]

### 4.1 Sketch of the proof of Theorem 3.1

The proof of Theorem 3.1 relies upon Lindeberg method and a refined induction argument.

The starting point for the proof of the large deviation result is (24). To control the right hand side of (24) we split it into two parts depending on \( ||x|| \) is large or small. The case when \( ||x|| \) is large the integral is bounded using (9) and Markov’s inequality by the following quantity:

\[
\frac{C_3 n^{-1/2} \varepsilon^{-3} \nu_3}{6} \frac{48 \beta}{\Phi_0 H^3 n} \mathbb{P} \left[ ||Y|| \geq r \right]. \tag{25}
\]

The integral corresponding to the case when \( ||x|| \) is small is bounded by

\[
\frac{C_3 n^{-1/2} \varepsilon^{-3} \nu_3}{6} \mathbb{P} \left[ a_n(r) \leq ||W_{n,j}|| \leq b_n(r) \right] \tag{26}
\]

for some suitably chosen \( a_n(r) \) and \( b_n(r) \). Here \( C_3 \) and \( \nu_3 \) are defined in (A1) and (8), respectively. Observe that \( \mathbb{P} \left[ a_n(r) \leq ||W_{n,j}|| \leq b_n(r) \right] \leq \mathbb{P} \left[ ||W_{n,j}|| \geq a_n(r) \right] \) and \( W_{n,j} \overset{d}{=} \left( (n-1)/n \right)^{1/2} U_{n-1,j} \). Thus we get

\[
\Delta_n(r) \leq \mathbb{P} \left[ r - \varepsilon \leq ||Y|| \leq r + \varepsilon \right] + \frac{C_3 \varepsilon^{-3} n^{-1/2}}{6} \mathbb{P} \left[ a_n(r) \leq ||U_{n-1,k}|| \leq b_n(r) \right] + \frac{48 \beta}{\Phi_0 H^3 n} \mathbb{P} \left( ||Y|| \geq r \right), \tag{27}
\]
for some \( a_n(r) \) and \( b_n(r) \). Our next task is to bound the right hand side of (27) in term of \( \mathbb{P} [\|Y\| \geq r] \). We now inductively use a bound on \( \mathbb{P} [U_{n-1,j} \geq a_n(r)] \) in terms of \( \mathbb{P} [\|Y\| \geq r] \) to get a bound for (27). Finally bounding \( \mathbb{P} [r - \varepsilon \leq \|Y\| \leq r + \varepsilon] \) in terms of \( \mathbb{P} [\|Y\| \geq r] \) using (A2) and summing this with the bounds for (25) and (26) we obtain the following result. For any \( 1 \leq k \leq n \),

\[
\mathbb{P} (\|U_{n,k}\| > r) = \mathbb{P} (\|Y\| > r) \left( 1 + \theta \Pi_{n,r}^{1/4} \right),
\]

for all \( r \in \mathbb{R} \) satisfying \( \Phi_4 (2B + \Pi^{1/4}) T_{n,r} \leq 1 \). Here \(|\theta| < 1\), \( T_{n,r} = (r + 1)^3 n^{-1/2} \), \( B = 2(\Phi_1 + 1)/H \) and

\[
\Pi = \max \left\{ 1, \left( \frac{16 e \beta C_3}{\Phi_0 H^3} \right)^{4/7}, 2e \left[ \Phi_2 \Phi_3 + \frac{\nu_3 C_3 \Phi_3}{3} \right] \right\}.
\]

One can find the details in Lemma B.2.1. Since \( T_{n,r}^{1/4} = (r + 1)^3 n^{-1/8} \), Lemma B.2.1 gives a large deviation with rate \( n^{-1/8} \). However this rate can be modified to rate \( n^{-1/6} \) by a more refined induction argument. This is done in detail in the final part of the proof of Theorem 3.1.

4.2 Sketch of the proof of Theorem 3.2

We now give a sketch of the proof of Theorem 3.2. Like Theorem 3.1, this proof is also based on the Lindeberg method. To begin with, we define

\[
\delta_{n,m} := \sup_{1 \leq k \leq n} \sup_{r > 0} r^m |\mathbb{P} (\|U_{n,k}\| \leq r) - \mathbb{P} (\|Y\| \leq r)| = \max_{1 \leq k \leq n} \sup_{r \geq 0} r^m \Delta_{n,k}(r).
\]

Recall the definition of \( \Delta_n^{(m)} \) from (10). The the main task of Theorem 3.2 is to bound \( \Delta_n^{(m)} \). We shall bound \( \delta_{n,m} \) instead of \( \Delta_n^{(m)} \). Recall from (22) that

\[
\Delta_{n,k}(r) \leq \max_{j=1,2} E[\varphi_j(U_{n,k}) - \varphi_j(Y)] + \mathbb{P} [r - \varepsilon \leq \|Y\| \leq r + \varepsilon]
\]

where \( \varphi_1(\cdot) \) and \( \varphi_2(\cdot) \) have been defined in the sketch of proof of Theorem 3.1. Here also with slight abuse of notation, we shall refer both \( \varphi_1(\cdot) \) and \( \varphi_2(\cdot) \) by \( \varphi(\cdot) \).

We at first bound \( |E[\varphi(U_{n,k}) - \varphi(Y)]| \). Similar to (23), we at first write

\[
|E[\varphi(U_{n,k}) - \varphi(Y)]| \leq \sum_{j=1}^{k} \left| \int E \left[ \varphi \left( W_{n,j} + n^{-1/2}x \right) \right] \zeta(x) dx \right| =: \sum_{j=1}^{k} |I_j|.
\]

The analysis from now on is different from that of proof of Theorem 3.1 although the main idea is to do Taylor series expansion of \( \varphi(W_{n,j} + n^{-1/2}x) \) and analyze the error terms. Here
we divide the integral $I_j$ in the following two parts

$$I_j^{(1)} := \int_{\|x\| \leq rn^{1/2}} \mathbb{E} \left[ \varphi \left( W_{n,j} + n^{-1/2}x \right) \right] \zeta(dx),$$

$$I_j^{(2)} := \int_{\|x\| > rn^{1/2}} \mathbb{E} \left[ \varphi \left( W_{n,j} + n^{-1/2}x \right) \right] \zeta(dx).$$

Write

$$\varphi \left( W_{n,j} + xn^{-1/2} \right) := \varphi (W_{n,j}) + n^{-1/2}x^\top \nabla \varphi (W_{n,j}) + \frac{1}{2n}x^\top \nabla^2 \varphi (W_{n,j}) x + \text{Rem}_n (x, W_{n,j}).$$

Using (29), we write

$$I_j^{(1)} = I_j^{(1)} + I_j^{(2)} + \frac{1}{2} I_j^{(1)} + I_j^{(4)} + I_j^{(5)},$$

where

$$I_j^{(1)} := \int_{\|x\| \leq rn^{1/2}} \mathbb{E} \left[ \varphi (W_{n,j}) \right] \zeta(dx), \quad I_j^{(2)} := \int_{\|x\| \leq rn^{1/2}} \mathbb{E} \left[ n^{-1/2}x^\top \nabla \varphi (W_{n,j}) \right] \zeta(dx),$$

$$I_j^{(3)} := \int_{\|x\| \leq rn^{1/2}} \mathbb{E} \left[ n^{-1}x^\top \nabla^2 \varphi (W_{n,j}) x \right] \zeta(dx),$$

$$I_j^{(4)} := \int_{\|x\| \leq \varepsilon n^{1/2}} \mathbb{E} \left[ \text{Rem}_n (W_{n,j}, x) \right] \zeta(dx), \quad I_j^{(5)} := \int_{\varepsilon n^{1/2} \leq \|x\| \leq rn^{1/2}} \mathbb{E} \left[ \text{Rem}_n (W_{n,j}, x) \right] \zeta(dx).$$

The quantities $|I_j^{(1)}|, |I_j^{(2)}|, |I_j^{(3)}|$ and $|I_j^{(4)}|$ are bounded using Markov’s inequality. However in bounding $|I_j^{(2)}|$ and $|I_j^{(3)}|$, one gets a term involving $\mathbb{P} (r - \varepsilon \leq \|W_{n,j}\| \leq r + \varepsilon)$. Observing $W_{n,j} \overset{d}{=} ((n-1)/n)^{1/2} U_{n-1,j}$ allows us to bound $|I_j^{(2)}|$ and $|I_j^{(3)}|$ in terms of $\delta_{n-1,m}$. $|I_j^{(4)}|$ and $|I_j^{(5)}|$ are also bounded by Taylor series expansion. One might look at the proof of Theorem 3.2 for details. The bounds here also depend on $\delta_{n-1,m}$. Using all these bounds, one obtains the following recursive inequality (stated in (89))

$$\delta_{n,m} \leq \frac{2^{m+1}V_m(2\varepsilon n^{1/2})}{n^{(m-2)/2}} + \delta_{n-1,m} F_m(\varepsilon) + \Phi_2 \Gamma_m \varepsilon \left[ 2(4/3)^m + F_m(\varepsilon) \right] + 4^m \varepsilon^m.$$

where

$$F_m(\varepsilon) := \varepsilon^{-m} \left\{ \frac{4(2/3)^m (C_1 + C_2) V_m(2\varepsilon n^{1/2})}{n^{(m-2)/2}} \right\} + \varepsilon^{-3} \left\{ \frac{(8\sqrt{2}) 4^m (C_2 + C_3) \nu_3}{n^{1/2}} \right\}.$$

The proof is now completed by solving the recursive inequality with a proper choice of $\varepsilon$. 

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4.3 Sketch of the proof of Theorem 3.3

The technique of the proof of Theorem 3.3 is same as that of the proof of Theorem 3.2. Since $\Delta_{n,\phi} \leq \delta_{n,\phi}$, we aim at bounding $\delta_{n,\phi}$. In view of the smoothing lemma, this boils down to controlling the quantities:

$$I_j := \left| \mathbb{E} \left[ \varphi(W_{n,j} + n^{-1/2}X_j) - \varphi(W_{n,j} + n^{-1/2}Y_j) \right] \right|.$$ 

for $1 \leq j \leq n$. Using a Taylor series expansion, and the fact that the first and the second moments of $X$ and $Y$ are equal, we get:

$$I_j \leq I_j^{(1)} + I_j^{(2)},$$

where for some $\varepsilon > 0$,

$$I_j^{(1)} := \int_{\|x\| \leq \varepsilon n^{1/2}} |\mathbb{E} [\text{Rem}_n(W_{n,j}, x)]| |\zeta| (dx),$$

$$I_j^{(2)} := \int_{\|x\| > \varepsilon n^{1/2}} |\mathbb{E} [\text{Rem}_n(W_{n,j}, x)]| |\zeta| (dx),$$

The quantities $I_j^{(1)}$ and $I_j^{(2)}$ are then bounded using Markov’s inequality, to get:

$$I_j \leq \frac{2C_2^2\phi(\varepsilon)(C_2 + C_3)\varepsilon^{-3} \nu_{\phi,3}}{(n-1)\sqrt{r}} \{2A_\phi \Phi_2\varepsilon + \delta_{n-1,\phi} \}.$$

for $1 \leq j \leq n$. This gives the following recursive inequality:

$$\delta_{n,\phi} \leq F_\phi(\varepsilon)\delta_{n-1,\phi} + \Upsilon_\phi(\varepsilon), \tag{30}$$

where

$$F_\phi(\varepsilon) := 4C_2^2\phi(\varepsilon)(C_2 + C_3)\varepsilon^{-3} \nu_{\phi,3}n^{-1/2},$$

$$\Upsilon_\phi(\varepsilon) := 2F_\phi(\varepsilon)A_\phi \Phi_2\varepsilon + 2C_\phi A_\phi \Phi_2\varepsilon \phi(\varepsilon).$$

The proof is now completed by solving (30) using a proper choice of $\varepsilon$.

5 Summary and Future Directions

In this paper, we proved large deviation and non-uniform central limit theorems for scaled averages of independent high-dimensional random vectors. These results are based on the well-known results from the literature on infinite-dimensional central limit and large deviation theorems. The primary focus of this work is to understand the implications of infinite-dimensional results for the high-dimensional case. It should be mentioned here that we
credit Bentkus (1987), Paulauskas and Račkauskas (1991) for the proofs of Theorems 3.1, 3.2 and 3.3. These proofs are much more elementary than the ones from Chernozhukov et al. (2017) where Slepian interpolation was used. The main tools of the proofs here are Taylor series expansion and induction. In comparison to Chernozhukov et al. (2017), we mention that our setting is restrictive in the sense we consider i.i.d. random vectors and \( l_\infty \) balls while Chernozhukov et al. (2017) consider independent random vectors and sparsely convex sets.

The present form of the work is a preliminary version with basic probability results. Extensions of these to non-identically distributed and dependent random vectors along with applications of these inequalities for statistical methods will be considered in an extended version. Primary areas of interest in terms of applications are bootstrap and high-dimensional vectors with a specified group structure. Since our main results apply for general Banach spaces, consideration of problems related to empirical processes form an interesting direction. See Norvaiša and Paulauskas (1991) and Chernozhukov et al. (2014) for some results.

**Acknowledgment**

We would like to thank Prof. Jian Ding for comments that led to an improved presentation.

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### A Proof of Theorem 3.5 and Corollary 3.3

#### A.1 A preliminary lemma

We start with a lemma about anti-concentration. A similar version of this lemma can be found in Giné (1976).

**Lemma A.1** (Lemma 2.5 of Giné (1976)). Let $Z$ be a centered sample continuous Gaussian process on a compact metric space $S$ such that $\mathbb{E}[\{Z(s)\}^2] \geq \sigma^2 > 0$ for every $s \in S$. Then, for $\varepsilon \leq \sigma/2$ and $\lambda > 0$,

$$
\mathbb{P}\left(\lambda - \varepsilon \leq \sup_{s \in S} |Z(s)| \leq \lambda + \varepsilon\right) \leq 2\varepsilon K(\lambda).
$$

where

$$K(\lambda) := 2\sigma^{-1}(2.6 + \lambda/\sigma).$$

**Proof.** We shall actually prove that

$$
\mathbb{P}\left(\lambda - \varepsilon \leq \sup_{s \in S} Z(s) \leq \lambda + \varepsilon\right) \leq \varepsilon K(\lambda).
$$

After proving (31) the result follows since

$$
\left\{\lambda - \varepsilon \leq \sup_{s \in S} |Z(s)| \leq \lambda + \varepsilon\right\} \subseteq \left\{\lambda - \varepsilon \leq \sup_{s \in S} Z(s) \leq \lambda + \varepsilon\right\} \cup \left\{\lambda - \varepsilon \leq \sup_{s \in S} -Z(s) \leq \lambda + \varepsilon\right\}.
$$
Now observe that

\[
P \left( \lambda - \varepsilon \leq \sup_{s \in S} Z(s) \leq \lambda + \varepsilon \right) = P \left( -\varepsilon \leq \sup_{s \in S} (Z(s) - \lambda) \leq \varepsilon \right)
\]

\[
= P \left( \bigcup_{s \in S} \{ -\varepsilon \leq Z(s) - \lambda \} \cap \bigcap_{s \in S} \{ (Z(s) - \lambda) \leq \varepsilon \} \right)
\]

\[
\leq P \left( \bigcup_{s \in S} \left\{ -\varepsilon \leq \frac{Z(s) - \lambda}{\sigma(s)} \leq \varepsilon \right\} \cap \bigcap_{s \in S} \left\{ \frac{Z(s) - \lambda}{\sigma(s)} \leq \frac{\varepsilon}{\sigma} \right\} \right)
\]

\[
= P \left( \frac{\lambda - \varepsilon}{\sigma} \leq \sup_{s \in S} \frac{Z(s) - \lambda}{\sigma(s)} \leq \frac{\varepsilon}{\sigma} \right)
\]

\[
= P \left( \frac{\lambda - \varepsilon}{\sigma} \leq \sup_{s \in S} \frac{Z(s) - \lambda}{\sigma(s)} + \frac{\lambda}{\sigma} \leq \frac{\varepsilon + \lambda}{\sigma} \right)
\]

Observe that the process \((Z(s) - \lambda)/\sigma(s) + \lambda/\sigma\) has non-negative mean and variance identical to 1. So from the arguments from Giné (1976), we have

\[
P \left( \lambda - \varepsilon \leq \sup_{s \in S} Z(s) \leq \lambda + \varepsilon \right) \leq \int_{\sigma^{-1}(\lambda-\varepsilon)}^{\sigma^{-1}(\lambda+\varepsilon)} f(x) dx
\]

where

\[
f(x) \leq \begin{cases} 
2.6, & \text{if } x \leq 1, \\
x + x^{-1}, & \text{otherwise}.
\end{cases}
\]

Now consider the following two cases (1) \(\lambda \leq \sigma/2\), (2) \(\sigma/2 \leq \lambda\). In case (1), \(\sigma^{-1}(\lambda + \varepsilon) \leq 1\) so, using \(f(x) \leq 2.6\), we have

\[
\int_{\sigma^{-1}(\lambda-\varepsilon)}^{\sigma^{-1}(\lambda+\varepsilon)} f(x) dx \leq 5.2\sigma^{-1}\varepsilon.
\]

In case (2), we have \(f(x) \leq 2.6 + x\) and so,

\[
\int_{\sigma^{-1}(\lambda-\varepsilon)}^{\sigma^{-1}(\lambda+\varepsilon)} f(x) dx \leq 5.2\sigma^{-1}\varepsilon + \frac{1}{2\sigma^2} \left( (\lambda + \varepsilon)^2 - (\lambda - \varepsilon)^2 \right) = 5.2\sigma^{-1}\varepsilon + \frac{2\lambda\varepsilon}{\sigma^2} = \varepsilon K(\lambda).
\]

This completes the proof. \(\square\)

A.2 Proof of Theorem 3.5

Proof of part (1): Let \(t_0 \in S\) be the index such that

\[
\sigma_{\text{max}}^2 = \text{Var}(Y(t_0)) = \sup_{t \in S} \text{Var}(Y(t)).
\]

It is clear that

\[
P (\|Y\| \geq r) \geq P (|Y(t_0)| \geq r).
\]
Since \( Y(t_0) \sim N(0, \sigma_{\text{max}}^2) \), by Mill’s ratio (Gordon, 1941) we get that:

\[
\mathbb{P}( |Y(t_0)| \geq r ) = \mathbb{P} \left( \frac{|Y(t_0)|}{\sigma_{\text{max}}} \geq \frac{r}{\sigma_{\text{max}}} \right) \geq \frac{(r/\sigma_{\text{max}})}{1 + (r/\sigma_{\text{max}})^2} \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{r^2}{2\sigma_{\text{max}}^2} \right).
\]

If \( r \geq \sigma_{\text{max}} \), then we claim that

\[
\frac{(r/\sigma_{\text{max}})}{1 + (r/\sigma_{\text{max}})^2} \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{r^2}{2\sigma_{\text{max}}^2} \right) \geq \frac{1}{2\sqrt{2\pi}} \exp \left( -\frac{r^2}{2\sigma_{\text{max}}^2} \right).
\]

Since \( r/\sigma_{\text{max}} \geq 1 \), we have:

\[
\exp \left( \frac{r^2}{2\sigma_{\text{max}}^2} \right) \geq 1 + \frac{r^2}{2\sigma_{\text{max}}^2} \geq 1 + \frac{(r/\sigma_{\text{max}})^2}{2r/\sigma_{\text{max}}},
\]

proving (32). Thus, for \( r \geq \sigma_{\text{max}} \),

\[
\mathbb{P}( |Y(t_0)| \geq r ) \geq \frac{1}{2\sqrt{2\pi}} \exp \left( -\frac{r^2}{2\sigma_{\text{max}}^2} \right).
\]

If \( r \leq \sigma_{\text{max}} \), then

\[
\mathbb{P}( |Y(t_0)| \geq r ) \geq \mathbb{P}( |Y(t_0)| \geq \sigma_{\text{max}} ) = 2Q(-1) > \frac{1}{4} \geq \frac{1}{4} \exp \left( -\frac{r^2}{\sigma_{\text{max}}^2} \right)
\]

where \( Q(\cdot) \) is the distribution function of a standard Gaussian random variable. Therefore, for all \( r \geq 0 \),

\[
\mathbb{P}( \|Y\| \geq r ) \geq \min \left\{ \frac{1}{4}, \frac{1}{2\sqrt{2\pi}} \right\} \exp \left( -\frac{r^2}{2\sigma_{\text{max}}^2} \right) = \frac{1}{2\sqrt{2\pi}} \exp \left( -\frac{r^2}{\sigma_{\text{max}}^2} \right).
\]

Since \( 1/(2\sqrt{2\pi}) > 1/6 \), the result follows. \( \square \)

**Proof of part (2):** Let \( Q(x) = \mathbb{P}[Z \leq x] \) where \( Z \) is a standard Gaussian random variable and we define \( \Psi(x) = 1 - Q(x) \). Ehrhard’s inequality implies that for all convex Borel sets \( A, C \subset B \),

\[
Q^{-1} \left[ \mathbb{P}[Y \in \alpha A + \beta C] \right] \geq \alpha Q^{-1} \left[ \mathbb{P}[Y \in A] \right] + \beta Q^{-1} \left[ \mathbb{P}[Y \in C] \right]
\]

when \( \alpha, \beta \geq 0 \) and \( \alpha + \beta = 1 \). See, for example, Latała (2002, Theorem 3.1) and Borell (2003, Section 3). Denote \( q(r) := Q^{-1} \left[ \mathbb{P}[\|Y\| \leq r] \right] \). We now show that \( q(\cdot) \) is a concave function. Fix \( \alpha, \beta > 0 \) such that \( \alpha + \beta = 1 \) and any \( r, s \in \mathbb{R}^+ \). From the definition of \( q(\cdot) \)

\[
q(\alpha r + \beta s) = Q^{-1} \left[ \mathbb{P}[\|Y\| \leq \alpha r + \beta s] \right].
\]

Observe that if \( x \in \alpha B_{\|\cdot\|}(0, r) + \beta B_{\|\cdot\|}(0, s) \), then \( \|x\| \leq \alpha r + \beta s \) from triangle inequality. Hence

\[
q(\alpha r + \beta s) \geq Q^{-1} \left[ \mathbb{P}[Y \in \alpha B_{\|\cdot\|}(0, r) + \beta B_{\|\cdot\|}(0, s)] \right].
\]

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Now the concavity of \( q(\cdot) \) follows from (33). Let \( \mu \) be the median of \( \|Y\| \) or equivalently \( q(\mu) = 0 \). Set \( d_1 = 4\sigma_{\min}^{-1}(2\mu + \mu\sigma_{\min}^{-1}) \). From Lemma A.1, we get that,

\[
\mathbb{P}(\mu - \delta \leq \|Y\| \leq \mu + \delta) \leq d_1\delta, \quad \text{for all } \delta \leq \sigma_{\min}/2. \tag{34}
\]

In order to prove (18) we at first prove that it is enough to assume the following five conditions:

(i) \( \varepsilon \leq r/4 \). (ii) \( \mu + \varepsilon \leq r - \varepsilon \). (iii) \( \mu \leq r - \varepsilon \). (iv) \( q(r - \varepsilon) \geq 1 \). (v) \( \mu + 1/(3d_1) \leq r \).

From the proof of part (1), we have \( \mathbb{P}(\|Y\| \geq r) \geq (2\sqrt{2\pi})^{-1} \exp \left(-\frac{r^2}{2\sigma^2_{\max}}\right) \) for all \( r > 0 \).

Here \( \sigma_{\max} \) is the maximum variance of the coordinates. Now we verify the conditions one by one.

**Condition (i):** Suppose \( \varepsilon > r/4 \). Then

\[
\mathbb{P}(\|Y\| \geq r - \varepsilon) \leq 1 \leq 2\sqrt{2\pi} \exp \left(\frac{r^2}{2\sigma^2_{\max}}\right) \mathbb{P}(\|Y\| \geq r) \leq 2\sqrt{2\pi} \exp \left(\frac{4r\varepsilon}{\sigma^2_{\max}}\right) \mathbb{P}(\|Y\| \geq r). \tag{35}
\]

**Condition (ii):** Suppose \( \varepsilon \leq r/4 \) and \( \mu + \varepsilon > r - \varepsilon \). We divide this case in the following two sub cases. Observe that in this case \( r \leq 2\mu \).

**Sub case (a):** Here we assume \( \varepsilon \geq 1/(6d_1) \). In this case, we have

\[
\varepsilon r \geq \frac{r}{6d_1} \geq \frac{r^2}{12d_1\mu} \quad \Rightarrow \quad \frac{r^2}{\sigma^2_{\max}} \leq \left(\frac{12d_1\mu}{\sigma^2_{\max}}\right) \varepsilon r \leq c_2\varepsilon(r + 1),
\]

for \( c_2 = 12d_1\mu\sigma_{\max}^{-2} \). So,

\[
\mathbb{P}(\|Y\| \geq r - \varepsilon) \leq 1 \leq 2\sqrt{2\pi} \exp \left(\frac{r^2}{2\sigma^2_{\max}}\right) \mathbb{P}(\|Y\| \geq r) \leq 2\sqrt{2\pi} \exp \left(c_2\varepsilon(r + 1)\right) \mathbb{P}(\|Y\| \geq r). \tag{36}
\]

**Sub case (b):** Here we assume \( \varepsilon < 1/(6d_1) \). As a consequence, we have

\[
r \leq \mu + 2\varepsilon \leq \mu + \frac{1}{3d_1}.
\]

Now noting that \( 2\sigma^{-1}_{\min} \leq 3d_1 \), we get from (34) that

\[
\mathbb{P} \left[ \mu \leq \|Y\| \leq \mu + \frac{1}{3d_1} \right] \leq \frac{1}{3}.
\]

As a consequence,

\[
\mathbb{P}(\|Y\| \geq r) = 1 - \mathbb{P}(\|Y\| \leq r)
\]

\[
\geq 1 - \mathbb{P}(\|Y\| \leq \mu) - \mathbb{P}(\mu \leq \|Y\| \leq \mu + 1/(3d_1)) \geq 1 - \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.
\]

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So
\[ P(\|Y\| \geq r - \varepsilon) \leq 1 \leq 6P(\|Y\| \geq r) \leq 6\exp(r(1))P(\|Y\| \geq r). \] (37)

**Condition (iii):** If (iii) fails then (ii) fails which is covered above.

**Condition (iv):** If (iv) fails then, \( q(r - \varepsilon) \leq 1 \). Let \( \tilde{r} := q^{-1}(1) \). We have \( r \leq \tilde{r} + \varepsilon \) and using \( \varepsilon \leq r/4 \), we get \( r \leq 4\tilde{r}/3 \). From Lemma 3.1 of Ledoux and Talagrand (2011), we get

\[ P(\|Y\| \geq \mu + \sigma_{\max}\Psi^{-1}(\Psi(1)/2)) \leq \Psi(1) = P(\|Y\| \geq \tilde{r}). \]

The last equality above follows from the definition of \( \tilde{r} \). Thus, \( \tilde{r} \leq \mu + 1.5\sigma_{\max} \). We now divide the case in two sub cases.

**Sub case (a):** Here we assume \( \varepsilon \geq 1/(10\tilde{d}) \), where \( \tilde{d} := 4\sigma^{-1}_{\min}(2.6 + \tilde{r}\sigma^{-1}_{\min}) \). In this case we have

\[
\frac{r^2}{\sigma_{\max}^2} \leq \frac{r}{\sigma_{\max}^2} \left(\frac{4\tilde{r}}{3}\right) \leq r\varepsilon \left(\frac{14\tilde{r}\tilde{d}}{\sigma_{\max}^2}\right) \leq c_3\varepsilon(r + 1),
\]

for \( c_3 := 14\sigma^{-2}_{\max}\tilde{d} \). So

\[ P(\|Y\| \geq r - \varepsilon) \leq 1 \leq 2\sqrt{2\pi} \exp\left(\frac{r^2}{\sigma_{\max}^2}\right)P(\|Y\| \geq r) \leq 2\sqrt{2\pi} \exp\left(c_3\varepsilon(r + 1)\right)P(\|Y\| \geq r). \] (38)

**Sub case (b):** If \( \varepsilon \leq 1/(10\tilde{d}) \), then we have

\[ P(\|Y\| \geq r) \geq 1 - P(\|Y\| \leq \tilde{r}) - P\left(\tilde{r} \leq \|Y\| \leq \tilde{r} + 1/(10\tilde{d})\right) \geq \Psi(1) - 0.1 \geq 0.05. \]

So

\[ P(\|Y\| \geq r - \varepsilon) \leq 1 \leq 20P(\|Y\| \geq r) \leq 20\exp(r\varepsilon)P(\|Y\| \geq r). \] (39)

**Condition (v):** If condition (v) fails, then

\[ r \leq \mu + 1/(3d_1). \]

Since \( 2\sigma^{-1}_{\min} \leq 3d_1 \), we get \( 1/(3d_1) \leq \sigma_{\min}/2 \) and so, by the bound (34), we get

\[ P(\|Y\| \leq r) \leq P\left(\|Y\| \leq \mu + \frac{1}{3d_1}\right) = P(\|Y\| \leq \mu) + P\left(\mu \leq \|Y\| \leq \mu + \frac{1}{3d_1}\right) \leq \frac{1}{2} + \frac{1}{3}. \]

Thus, \( 6P(\|Y\| \geq r) \geq 1 \) and so,

\[ P(\|Y\| \geq r - \varepsilon) \leq 1 \leq 6P(\|Y\| \geq r) \leq 6\exp(r\varepsilon)P(\|Y\| \geq r). \] (40)
Combining inequalities (35), (36), (37), (38), (39) and (40), we get
\[
P \left( \| Y \| \geq r - \varepsilon \right) \leq 20 \exp \left( \Phi_4 \varepsilon (r + 1) \right) P \left( \| Y \| \geq r \right),
\] (41)
where
\[
\Phi_4 := \max \left\{ 1, \frac{56(\mu + 1.5\sigma_{\text{max}})(\mu + 4.1\sigma_{\text{max}})}{\sigma_{\text{max}}^2 \sigma_{\text{min}}}, \frac{4}{\sigma_{\text{max}}^2} \right\}
\]
Now that the result is proved if one of conditions (i) – (v) fail, we proceed to proving the result under all the conditions (i) – (v). Take \( \alpha = \varepsilon (r - \mu)^{-1} \) and \( \beta = 1 - \alpha \). From condition (iii) we have \( 0 \leq \alpha \leq 1 \). Observe that \( \alpha \mu + \beta r = r - \varepsilon \). Since \( q \) is concave, we have
\[
q(r - \varepsilon) = q(\alpha \mu + \beta r) \geq \alpha q(\mu) + \beta q(r) = \beta q(r) \implies q(r) \leq \beta^{-1} q(r - \varepsilon).
\]
Using this inequality along with the definition of \( q(\cdot) \), we get that
\[
P \left( \| Y \| \geq r - \varepsilon \right) = \Psi(q(r - \varepsilon))
\]
\[
= \frac{P \left( \| Y \| \geq r \right)}{\Psi(q(r))} \Psi(q(r - \varepsilon)) \leq \frac{\Psi(q(r - \varepsilon))}{\Psi(\beta^{-1} q((r - \varepsilon)))} P \left( \| Y \| \geq r \right).
\]
Note from Mill’s ratio (Gordon, 1941) that for \( t \geq 1 \),
\[
\frac{1}{2t} \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{t^2}{2} \right) \leq \frac{t}{1 + t^2} \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{t^2}{2} \right) \leq \Psi(t) \leq \frac{1}{2} \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{t^2}{2} \right)
\]
From condition (iv), \( q(r - \varepsilon) \geq 1 \) and so, \( \beta^{-1} q(r - \varepsilon) \geq 1 \). As a consequence
\[
\frac{\Psi(q(r - \varepsilon))}{\Psi(\beta^{-1} q((r - \varepsilon)))} \leq 2 \beta \exp \left( -\frac{q^2(r - \varepsilon)}{2} \right) \leq \frac{2}{\beta} \exp \left( \frac{q^2(r - \varepsilon)}{2} \left( \frac{1}{\beta^2} - 1 \right) \right).
\] (42)
Now we prove that \( \beta \geq 1/2 \). We have
\[
\beta - \frac{1}{2} = \frac{1}{2} - \frac{\varepsilon}{r - \mu} = \frac{r - \mu - 2\varepsilon}{2(r - \mu)} \geq 0
\]
from condition (ii) and (iii). Plugging this in (42) we have
\[
\frac{\Psi(q(r - \varepsilon))}{\Psi(\beta^{-1} q((r - \varepsilon)))} \leq 4 \exp \left( \frac{q^2(r - \varepsilon)}{2} \left( \frac{1}{\beta^2} - 1 \right) \right).
\] (43)
So it is enough to bound
\[
\exp \left( \frac{q^2(r - \varepsilon)}{2} \left( \frac{1}{\beta^2} - 1 \right) \right).
\]
From condition (iii) we know \( r - \varepsilon \geq \mu \). Since \( q \) is a concave function, \( \mu > 0 \) and \( q(\mu) = 0 \), we have
\[
q(r - \varepsilon) \leq q'(\mu)(r - \varepsilon) \leq q'(\mu)r.
\] (44)
Since \( Q(q(r)) = \mathbb{P} (\|Y\| \leq r) \), we get that

\[
Q'(q(r))q'(r) = \frac{d}{dr} \mathbb{P} (\|Y\| \leq r)
\]

Substituting \( r = \mu \) in this equation, we get

\[
\frac{1}{\sqrt{2\pi}} q'(\mu) = \lim_{\delta \to 0} \frac{1}{2\delta} \mathbb{P} (\mu - \delta \leq \|Y\| \leq \mu + \delta) \leq \frac{d_1}{2},
\]

where the last inequality follows from inequality (34). This implies that \( q'(\mu) \leq \sqrt{2\pi}d_1/2 \).

Thus using (44), we obtain

\[
q^2(r - \varepsilon) \left( \frac{1}{\beta^2} - 1 \right) \leq \frac{1}{2} \pi d_1^2 r^2 \left( \frac{1 - \beta^2}{\beta^2} \right) \leq 2\pi d_1^2 r^2 (1 - \beta^2)
\]

\[
= 2\pi d_1^2 r^2 \left( 1 - 1 - \frac{\varepsilon^2}{(r - \mu)^2} + 2\frac{\varepsilon}{r - \mu} \right) \leq 4\pi d_1^2 r\varepsilon \left( \frac{r}{r - \mu} \right).
\]

Note from condition (v) that

\[
\frac{r}{r - \mu} = 1 + \frac{\mu}{r - \mu} \leq 1 + 3d_1\mu.
\]

Thus,

\[
\frac{q^2(r - \varepsilon)}{2} \left( \frac{1}{\beta^2} - 1 \right) \leq 2\pi d_1^2 (1 + 3d_1\mu)\varepsilon (r + 1).
\]

Plugging this in (43), we have

\[
\mathbb{P} (\|Y\| \geq r - \varepsilon) \leq 4 \exp \left( 2\pi d_1^2 (1 + 3d_1\mu)(r + 1)\varepsilon \right) \mathbb{P} (\|Y\| \geq r).
\]

Combining this with inequality (41) that holds if one of conditions (i) – (v) fail, we get

\[
\mathbb{P} (\|Y\| \geq r - \varepsilon) \leq 20 \exp (\Phi_4 (r + 1)\varepsilon) \mathbb{P} (\|Y\| \geq r),
\]

where \( \Phi_4 \) is redefined as

\[
\Phi_4 := \max \left\{ 1, \frac{56(\mu + 1.5\sigma_{\max})(\mu + 4.1\sigma_{\max})}{\sigma_{\max}^2\sigma_{\min}^2}, 2\pi d_1^2 (1 + 3d_1\mu), \frac{4}{\sigma_{\max}^2} \right\}.
\]

Since

\[
1 + 3d_1\mu = 1 + 12\sigma_{\min}^{-1}(2.6 + \mu\sigma_{\min}^{-1})\mu \leq 1 + 32\sigma_{\min}^{-1}\mu + 12\mu^2\sigma_{\min}^{-2},
\]

the result follows.

**Proof of part (3):** We follow the notation from the proof of part (2) and consider two cases:

(i) \( q(r - \varepsilon) \leq 1 \), and (ii) \( q(r - \varepsilon) \geq 1 \).
Under case \((i)\), \(\mathbb{P}(|Y| \geq r - \varepsilon) \geq \Psi(1)\) and \(r - \varepsilon \leq \tilde{r} \leq \mu + 1.5\sigma_{\text{max}}\). Here \(\tilde{r} = q^{-1}(1)\). Recall the function \(K(\cdot)\) from Lemma A.1. Also, from Lemma A.1, it follows that for \(\varepsilon \leq \sigma_{\text{min}}/4\),

\[
\mathbb{P}(r - \varepsilon \leq |Y| \leq r + \varepsilon) \leq \mathbb{P}((r - \varepsilon) - 2\varepsilon \leq |Y| \leq (r - \varepsilon) + 2\varepsilon) \\
\leq K(r - \varepsilon)4\varepsilon \leq 4K(\mu + 1.5\sigma_{\text{max}})\varepsilon(r + 1) \\
\leq 4K(\mu + 1.5\sigma_{\text{max}})\varepsilon(r + 1)\frac{\mathbb{P}(|Y| \geq r - \varepsilon)}{\Psi(1)}.
\]

Thus, for \(r\) satisfying \(q(r - \varepsilon) \leq 1\) and \(\varepsilon \leq \sigma_{\text{min}}/4\),

\[
\mathbb{P}(r - \varepsilon \leq |Y| \leq r + \varepsilon) \leq \frac{4K(\mu + 1.5\sigma_{\text{max}})}{\Psi(1)}\varepsilon(r + 1)\mathbb{P}(|Y| \geq r - \varepsilon). \tag{45}
\]

If \(\varepsilon \geq \sigma_{\text{min}}/4\), then

\[
\mathbb{P}(r - \varepsilon \leq |Y| \leq r + \varepsilon) \leq \mathbb{P}(|Y| \geq r - \varepsilon) \leq \frac{4\varepsilon(r + 1)}{\sigma_{\text{min}}}\mathbb{P}(|Y| \geq r - \varepsilon). \tag{46}
\]

In order to verify \((19)\) under case \((ii)\), note that for any \(z \geq 0\),

\[Q(q(z)) = \mathbb{P}(|Y| \leq z) \Rightarrow Q'(q(z))q'(z) = p(z),\]

where \(p(z)\) represents the density of \(|Y|\). Since \(Q\) represents the distribution function of a standard normal random variable, we get

\[p(z) = \frac{q'(z)}{\sqrt{2\pi}}\exp\left(-\frac{q^2(z)}{2}\right).\]

So,

\[
\mathbb{P}(|Y| \geq r - \varepsilon) = \int_{r-\varepsilon}^{\infty} \frac{q'(z)}{\sqrt{2\pi}}\exp\left(-\frac{q^2(z)}{2}\right) dz \\
= \int_{q(r-\varepsilon)}^{\infty} \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y^2}{2}\right) dy \\
\geq \frac{1}{2\sqrt{2\pi}q(r-\varepsilon)}\exp\left(-\frac{q^2(r-\varepsilon)}{2}\right),
\]

where the last inequality follows from Mill’s ratio and the fact that under case \((ii)\), \(q(r - \varepsilon) \geq 1\). Since \(q(\cdot)\) is increasing,

\[q(r - \varepsilon) \geq 1 \Rightarrow r - \varepsilon \geq q^{-1}(1) \geq q^{-1}(0) = \mu.
\]

Since \(q(\cdot)\) is concave, this implies that \(q'(r - \varepsilon) \leq q'(\mu)\). Thus, for all \(z \geq r - \varepsilon\),

\[
\frac{p(z)}{q'(\mu)} = \frac{q'(z)}{\sqrt{2\pi}q'(\mu)}\exp\left(-\frac{q^2(z)}{2}\right) \\
\leq \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{q^2(z)}{2}\right) \leq 2q(r - \varepsilon)\mathbb{P}(|Y| \geq r - \varepsilon).
\]

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Summarizing the inequalities above, we obtain
\[ p(z) \leq 2q'(\mu)q(r-\varepsilon)\mathbb{P}(\|Y\| \geq r - \varepsilon). \]

Using concavity of \( q(\cdot) \) and the fact \( q(\mu) = 0 \), we get \( q(z) \leq q'(\mu)(z - \mu) \leq q'(\mu)z \) and so,
\[ p(z) \leq 2 \left( q'(\mu) \right)^2 z \mathbb{P}(\|Y\| \geq r - \varepsilon), \quad \text{for all} \quad z \geq r - \varepsilon. \]

Observe now that
\[
\mathbb{P}(r - \varepsilon \leq \|Y\| \leq r + \varepsilon) = \int_{r-\varepsilon}^{r+\varepsilon} p(z)dz \leq 2 \left( q'(\mu) \right)^2 \mathbb{P}(\|Y\| \geq r - \varepsilon)(2r\varepsilon)
\]
\[
\leq 4 \left( q'(\mu) \right)^2 \varepsilon(r + 1)\mathbb{P}(\|Y\| \geq r - \varepsilon)
\]
Combining inequalities (45), (46) and (47), we get for all \( r \geq 0 \) and \( \varepsilon \geq 0 \),
\[
\mathbb{P}(r - \varepsilon \leq \|Y\| \leq r + \varepsilon) \leq \max \left\{ \frac{4K(\mu + 1.5\sigma_{\max})}{\Psi(1)}, \frac{4\sigma_{\min}}{\sigma_{\min}}, \frac{4(q'(\mu))^2}{\sigma_{\min}} \right\} \varepsilon(r + 1)\mathbb{P}(\|Y\| \geq r - \varepsilon).
\]
Since \( q'(\mu) = \sqrt{2\pi}p(\mu) \leq \sqrt{2\pi}K(\mu) \), and using the form of \( K(\lambda) \) from Lemma A.1 the result follows.

A.3 Proof of Corollary 3.3

Proof. We consider two cases as follows: (1) \( r \leq \xi \) and (2) \( r > \xi \). Since we are only concerned with limsup as \( \varepsilon \to 0 \), we also take \( \varepsilon \leq \min\{\sigma_{\min}/2, \mu/4\} \).

For case (1) we can apply Lemma A.1 to get that
\[
\mathbb{P}(r - \varepsilon \leq \|Y\| \leq r + \varepsilon) \leq 4\varepsilon \left( \frac{1}{\sigma_{\min}} \left( 2.6 + \frac{r}{\sigma_{\min}} \right) \right) \leq 4\varepsilon \left( \frac{1}{\sigma_{\min}} \left( 2.6 + \frac{\xi}{\sigma_{\min}} \right) \right).
\]

For case (2), using Theorem 3.5 part (3), it boils down to estimate \( \mathbb{P}(\|Y\| \geq r - \varepsilon) \).
\[
\mathbb{P}(\|Y\| \geq r - \varepsilon) = \mathbb{P}(\|Y\| - \mu \geq r - \varepsilon - \mu) \\
\leq \mathbb{P}(\|Y\| - \mu \geq r - \varepsilon - \mu) \\
\leq 2 \exp \left( -\frac{1}{2\sigma_{\max}^2} (r - \varepsilon - \mu)^2 \right) \\
\leq 2 \exp \left( -\frac{1}{2\sigma_{\max}^2} \left( \frac{r}{6} \right)^2 \right) = 2 \exp \left( -\frac{r^2}{72\sigma_{\max}^2} \right).
\]

Here the third step follows from Lemma 3.1 of Ledoux and Talagrand (2011) and the fourth step follows from the fact \( r/6 \leq (r - \varepsilon - \mu) \). We know from Theorem 3.5 part (3) that
\[
\mathbb{P}(r - \varepsilon \leq \|Y\| \leq r + \varepsilon) \leq \Phi_2\varepsilon(r + 1)\mathbb{P}(\|Y\| \geq r - \varepsilon).
\]
Therefore,
\[
\mathbb{P}(r - \varepsilon \leq \|Y\| \leq r + \varepsilon) \leq 2\Phi_2 \varepsilon (r + 1) \exp \left( -\frac{r^2}{72\sigma_{\max}^2} \right)
\]
\[
\leq 2\Phi_2 \exp (r) \exp \left( -\frac{r^2}{72\sigma_{\max}^2} \right)
\]
\[
\leq 2\Phi_2 \varepsilon \exp \left( -\frac{r^2}{144\sigma_{\max}^2} \right)
\]
\[
= 2\Phi_2 \varepsilon \exp \left( -\frac{r^2}{144\sigma_{\max}^2} \right). \tag{27}
\]
This completes the proof. \qed

\section{Proof of Theorem 3.1}

We will require the following notations in the proof of Theorem 3.1.
\[
U_{n,k} := n^{-1/2} (X_1 + X_2 + \ldots + X_k + Y_{k+1} + Y_{k+2} + \ldots + Y_n),
\]
\[
W_{n,k} := n^{-1/2} (X_1 + X_2 + \ldots + X_{k-1} + Y_{k+1} + \ldots + Y_n),
\]
\[
\Delta_{n,k}(r) := |\mathbb{P}(\|U_{n,k}\| \leq r) - \mathbb{P}(\|Y\| \leq r)|.
\]

Below, we state an elementary fact about Taylor series expansion.

\begin{lemma}
For any thrice differentiable function \( f : \mathbb{R}^p \to \mathbb{R} \), we have
\[
f(y + xn^{-1/2}) - f(y) - n^{-1/2}x^\top \nabla f(y) - \frac{1}{2n}x^\top \nabla^2 f(y)x = \text{Rem}_n(x, y),
\]
where
\[
|\text{Rem}_n(x, y)| \leq \min \left\{ \frac{\|x\|^3}{6n^{3/2}} \sup_{0 \leq \theta \leq 1} \left\| D^3 f(y + x\theta n^{-1/2}) \right\|_1, \frac{\|x\|^2}{n} \sup_{0 \leq \theta \leq 1} \left\| D^2 f(y + x\theta n^{-1/2}) \right\|_1 \right\}.
\]
\end{lemma}

\begin{proof}
The proof follows directly from the mean value theorem and the definition of \( D^2, D^3 \).
\end{proof}

As sketched in Section 4, the proof of Theorem 3.1 proceeds through several steps, the details of which are described in this section.

\subsection{First Step: Proving (27)}

\begin{lemma}
Set \( B = 2(\Phi_1 + 1)/H \) and \( f_n(r) = B(r^2 + \log n)n^{-1/2} \). Then, for all \( \varepsilon > 0, r \geq 0 \) and \( n \geq 2 \), we have:
\[
\Delta_n(r) \leq \mathbb{P}(r - \varepsilon \leq \|Y\| \leq r + \varepsilon)
\]
\[
+ \frac{C_3 \varepsilon^3 n^{-1/2}}{6} \left[ \nu_3 \max_{0 \leq k \leq n} \mathbb{P}(a_n(r) \leq \|U_{n-k,j}\| \leq b_n(r)) + \frac{48\beta}{\Phi_0 H^3 n} \mathbb{P}(\|Y\| \geq r) \right].
\]
\end{lemma}
Here,

\[ a_n(r) = r - \varepsilon - f_n(r), \quad \text{and} \quad b_n(r) = e^{1/n}(r + \varepsilon + f_n(r)). \]

**Proof.** Note that

\[ \Delta_n(r) \leq \max_{0 \leq k \leq n} \Delta_{n,k}(r), \]

and so, it suffices to control \( \Delta_{n,k}(r) \). Towards this, let \( \varphi_1(x) := \varphi_{r,\varepsilon}(x) \) and \( \varphi_2(x) = \varphi_{r-\varepsilon,\varepsilon}(x) \). Note that, these functions exist by assumption (A1). By Lemma 4.1, we get:

\[ \Delta_{n,k}(r) \leq \max_{j=1,2} \| \mathbb{E}[\varphi_j(U_{n,k}) - \varphi_j(Y)] \| + \mathbb{P}(r - \varepsilon \leq \|Y\| \leq r + \varepsilon). \] (48)

For convenience, let \( \varphi \) denote either \( \varphi_1 \) or \( \varphi_2 \). Also, for \( 1 \leq k \leq n \), define

\[ I_k := \mathbb{E}\left[ \varphi(W_{n,k} + n^{-1/2}X_k) - \varphi(W_{n,k} + n^{-1/2}Y_k) \right], \quad G := \mathcal{L}(X) - \mathcal{L}(Y). \]

It is clear that

\[ I := \| \mathbb{E}[\varphi(U_{n,k}) - \varphi(Y)] \| \leq \sum_{j=1}^{k} I_j, \quad \text{and} \quad I_j = \int \mathbb{E} \left[ \varphi(W_{n,j} + n^{-1/2}x) \right] \zeta(dx). \]

By a Taylor series expansion, we have

\[ \varphi(W_{n,j} + n^{-1/2}x) = \varphi(W_{n,j}) + n^{-1/2}x^\top \nabla \varphi(W_{n,j}) + \frac{1}{2n} x^\top \nabla^2 \varphi(W_{n,j}) x + \text{Rem}_n(W_{n,j}, x), \]

where the remainder \( \text{Rem}_n(W_{n,j}, x) \) satisfies the bounds

\[ |\text{Rem}_n(W_{n,j}, x)| \leq \frac{C_3 n^{-3/2}}{6} \|x\|^3 1\{r - \varepsilon - n^{-1/2} \|x\| \leq \|W_{n,j}\| \leq r + \varepsilon + n^{-1/2} \|x\|\} \]

\[ |\text{Rem}_n(W_{n,j}, x)| \leq C_2 n^{-1} \|x\|^2 1\{r - \varepsilon - n^{-1/2} \|x\| \leq \|W_{n,j}\| \leq r + \varepsilon + n^{-1/2} \|x\|\}. \]

Noting that \( \mathbb{E}[X] = \mathbb{E}[Y] = 0 \) and \( \mathbb{E}[XX^\top] = \mathbb{E}[YY^\top] \) along with the Taylor series expansion, we get

\[ |I_j| = \left| \int \mathbb{E} [\text{Rem}_n(W_{n,j}, x)] \zeta(dx) \right| \]

\[ \leq \int_{\|x\| \leq n^{1/2}f_n(r)} \mathbb{E} [\|\text{Rem}_n(W_{n,j}, x)\|] \zeta(dx) + \int_{\|x\| \geq n^{1/2}f_n(r)} \mathbb{E} [\|\text{Rem}_n(W_{n,j}, x)\|] \zeta(dx) \]

\[ \leq \frac{1}{6} \int_{\|x\| \leq n^{1/2}f_n(r)} C_3 \varepsilon^{-3} n^{-3/2} \|x\|^3 \mathbb{P}(r - \varepsilon - f_n(r) \leq \|W_{n,j}\| \leq r + \varepsilon + f_n(r)) \zeta(dx) \]

\[ + \frac{1}{6} \int_{\|x\| > n^{1/2}f_n(r)} C_3 \varepsilon^{-3} n^{-3/2} \|x\|^3 \zeta(dx) \]

\[ \leq \frac{C_3 \varepsilon^{-3} n^{-3/2} \|x\|^3}{6} \mathbb{P}(a_n(r) \leq \|W_{n,j}\| \leq r + \varepsilon + f_n(r)) \]

\[ + \frac{C_3 \varepsilon^{-3} n^{-3/2}}{6} \int_{\|x\| > n^{1/2}f_n(r)} \|x\|^3 \frac{\exp(H \|x\|/2)}{\exp(H n^{1/2}f_n(r)/2)} \zeta(dx). \]
For the second term above, note that
\[
\frac{(H \|x\|/2)^3}{3!} \leq \exp \left( H \|x\| / 2 \right) \quad \Rightarrow \quad \|x\|^3 \leq \frac{8(3!)}{H^3} \exp \left( H \|x\| / 2 \right).
\]
Also note that
\[
\frac{Hn^{1/2}f_n(r)}{2} = \frac{n^{1/2}(\Phi_1 + 1)(r^2 + \log n)}{\sqrt{n}} = (\Phi_1 + 1)(r^2 + \log n).
\]
This implies that
\[
\int_{\|x\| > n^{1/2}f_n(r)} \|x\|^3 \frac{\exp(H \|x\|/2)}{\exp(Hn^{1/2}f_n(r)/2)} \zeta(\|x\|)(dx)
\]
\[
\leq \frac{8(3!)}{H^3} \exp \left( -\frac{Hn^{1/2}f_n(r)}{2} \right) \int \exp (H \|x\|) \zeta(\|x\|)(dx)
\]
\[
\leq \frac{48\beta}{H^3} \exp \left( -(\Phi_1 + 1)(r^2 + \log n) \right) \leq \frac{48\beta}{\Phi_0 H^3 n^{1+\Phi_1}} \mathbb{P}(\|Y\| \geq r).
\]
Thus,
\[
\Delta_n(r) \leq \mathbb{P}(r - \varepsilon \leq \|Y\| \leq r + \varepsilon)
\]
\[
+ \frac{C_\beta \varepsilon^{-3} n^{-1/2}}{6} \left[ \nu_3 \max_{0 \leq k \leq n} \mathbb{P}(a_n(r) \leq \|W_{n,k}\| \leq r + \varepsilon + f_n(r)) + \frac{48\beta}{\Phi_0 H^3 n^{1+\Phi_1}} \mathbb{P}(\|Y\| \geq r) \right].
\]
Note that $W_{n,k}$ is identically distributed as $((n-1)/n)^{1/2}U_{n-1,k}$. Therefore, for every $0 \leq k \leq n - 1$, we have
\[
\mathbb{P}(a_n(r) \leq \|W_{n,k}\| \leq r + \varepsilon + f_n(r))
\]
\[
= \mathbb{P}\left( \left( \frac{n-1}{n} \right)^{1/2} a_n(r) \leq \|U_{n-1,k}\| \leq \left( \frac{n-1}{n} \right)^{1/2} (r + \varepsilon + f_n(r)) \right)
\]
\[
\leq \mathbb{P}\left( a_n(r) \leq \|U_{n-1,k}\| \leq \left( \frac{n-1}{n} \right)^{1/2} (r + \varepsilon + f_n(r)) \right)
\]
\[
\leq \mathbb{P}\left( a_n(r) \leq \|U_{n-1,k}\| \leq e^{1/n}(r + \varepsilon + f_n(r)) \right).
\]
The last step holds since
\[
\left( \frac{n}{n-1} \right)^{1/2} = \left( 1 - \frac{1}{n} \right)^{-1/2} \leq \exp(1/n).
\]
Lemma B.1.1 now follows.
B.2 Second Step: Proving (28)

Lemma B.2.1. Let $T_{n,r} = (r + 1)^3 n^{-1/2}$ and

$$
\Pi = \max \left\{ 1, \left( \frac{16eB_3}{F_0 H^3} \right)^{4/7}, \left( 2e \left[ \Phi_2 \Phi_3 + \frac{\nu_3 C_3 \Phi_3}{3} \right] \right)^{4/3} \right\}.
$$

Then for all $n \geq 1$ and $0 \leq k \leq n$,

$$
\mathbb{P} \left( \|U_{n,k}\| > r \right) = \mathbb{P} \left( \|Y\| > r \right) \left( 1 + \theta \Pi T_{n,r}^{1/4} \right)
$$

holds for all $r \in \mathbb{R}$ satisfying $\Phi_4^4 \left( 2B + \Pi^{1/4} \right)^4 T_{n,r} \leq 1$. Here $\theta$ is a number bounded in absolute value by 1.

Proof. The proof for $r < 0$ is trivial so we assume that $r > 0$. The proof is done by induction on $n$.

The case $n = 1$: Here $T_{1,r} = (r + 1)^3$ and $T_{1,r} \Pi \leq 1$, which implies that $(r + 1)^3 \Pi \leq 1$. Hence $r < 0$, and we are done.

We now assume that the result is true for all $l \leq n - 1$ and our aim is to prove the bound for $l = n$.

The induction step: We have for any $u$ satisfying $\Phi_4^4 \left( 2B + \Pi^{1/4} \right)^4 (u + 1)^3 \leq (n - 1)^{1/2}$,

$$
\mathbb{P} \left( \|U_{n-1,k}\| > u \right) = \mathbb{P} \left( \|Y\| > u \right) \left( 1 + \theta \Pi T_{n-1,u}^{1/4} \right)
$$

$$
\leq 2\Pi^{3/4} \mathbb{P} \left( \|Y\| > u \right). \quad (49)
$$

The last step follows due to the fact $|\theta| < 1$, and $\Pi T_{n-1,u} \leq 1$, hence $\theta \Pi T_{n-1,u}^{1/4} \leq \Pi^{3/4}$.

Now fix $r$ and let $\varepsilon := (T_{n,r} \Pi)^{1/4} (r + 1)^{-1}$. Then we know from Lemma B.1.1 that

$$
\Delta_{n,r} \leq \mathbb{P} \left( r - \varepsilon < \|Y\| < r + \varepsilon \right)
$$

$$
+ \frac{C_3 \varepsilon ^{-3} n^{-1/2}}{6} \left[ 48B_3 \frac{\Pi^{1/4} \mathbb{P} \left( \|Y\| > r \right) + \nu_3 \max _{0 \leq k \leq n-1} \mathbb{P} \left( \|U_{n-1,k}\| > a_n(r) \right) \right]. \quad (50)
$$

From Lemma B.4.1 with $s = 0$ we have $f_n(r) \leq 2BT_{n,r}^{1/4} / (r + 1)$. Since

$$
\frac{T_{n,r}^{1/4}}{r + 1} = \frac{(T_{n,r} \Pi)^{1/4}}{\Pi^{1/4} (r + 1)} = \varepsilon \Pi^{-1/4},
$$

we have $r - \varepsilon - f_n(r) \geq r - c' \varepsilon$ where $c' = 2B \Pi^{-1/4} + 1$. Now

$$
\mathbb{P} \left( \|U_{n-1,k}\| > a_n(r) \right) \leq \mathbb{P} \left( \|U_{n-1,k}\| > r - c' \varepsilon \right). \quad (51)
$$

By Lemma B.4.2, $\Phi_4^4 \left( 2B + \Pi^{1/4} \right)^4 (r - c' \varepsilon + 1)^3 \leq (n - 1)^{1/2}$ and so, from the induction hypothesis (49), we have:

$$
\mathbb{P} \left( \|U_{n-1,k}\| > r - c' \varepsilon \right) \leq 2\Pi^{3/4} \mathbb{P} \left( \|Y\| > r - c' \varepsilon \right). \quad (52)
$$
Now from assumption (A2), we know
\[ P(r - \varepsilon < \|Y\| < r + \varepsilon) \leq \Phi_2 \varepsilon (r + 1) P(\|Y\| > r - \varepsilon) \]
and
\[ P(\|Y\| > r - \varepsilon) \leq \Phi_3 \exp \{ \Phi_4 (r + 1) \varepsilon \} P(\|Y\| > r). \]  
Combining these bounds, we have
\[ P(r - \varepsilon < \|Y\| < r + \varepsilon) \leq \Phi_2 \Phi_3 \varepsilon (r + 1) \exp \{ \Phi_4 (r + 1) \varepsilon \} P(\|Y\| > r). \]  
Also, note that from (53), we get
\[ P(\|Y\| \geq r - c' \varepsilon) \leq \Phi_3 \exp \{ \Phi_4 c' \varepsilon (r + 1) \} P(\|Y\| \geq r). \]
This implies using (51) and (52)
\[ P(\|U_{n-1,k}\| > a_n(r)) \leq 2 \Pi^{3/4} \Phi_3 \exp \{ \Phi_4 c' \varepsilon (r + 1) \} P(\|Y\| \geq r). \]
Plugging this bound in (50), we have
\[ \Delta_n(r) \leq \Phi_2 \Phi_3 \varepsilon (r + 1) \exp \{ \Phi_4 (r + 1) \varepsilon \} P(\|Y\| \geq r) \]
\[ + \frac{C_3 \varepsilon^{-3} n^{-2}}{6} \left[ \frac{48 \beta}{\Phi_0 H^3 n} P(\|Y\| \geq r) + 2 \nu_3 \Pi^{3/4} \Phi_3 \exp \{ \Phi_4 c' \varepsilon (r + 1) \} P(\|Y\| \geq r) \right] \]
\[ = P(\|Y\| \geq r) \exp \{ \Phi_4 c' \varepsilon (r + 1) \} \left[ \Phi_2 \Phi_3 \varepsilon (r + 1) + \frac{8 \beta C_3 \varepsilon^{-3}}{\Phi_0 H^3 n^{3/2}} + \frac{\nu_3 C_3 \Phi_3 \Pi^{3/4} \varepsilon^{-3}}{3 n^{1/2}} \right]. \]
Substituting the definition of \( \varepsilon \), we get
\[ \varepsilon (r + 1) = (T_{n,r} \Pi)^{1/4}, \quad \text{and} \quad \varepsilon^{-3} n^{-1/2} = \frac{(r + 1)^3 n^{-1/2}}{(T_{n,r} \Pi)^{3/4}} = \frac{T_{n,r}}{(T_{n,r} \Pi)^{3/4}} = \frac{T_{n,r}^{1/4}}{\Pi^{3/4}}. \]
Thus,
\[ \Phi_2 \Phi_3 \varepsilon (r + 1) + \frac{8 \beta C_3 \varepsilon^{-3}}{\Phi_0 H^3 n^{3/2}} + \frac{\nu_3 C_3 \Phi_3 \Pi^{3/4} \varepsilon^{-3}}{3 n^{1/2}} \]
\[ = \Phi_2 \Phi_3 (T_{n,r} \Pi)^{1/4} + \frac{8 \beta C_3 T_{n,r}^{1/4}}{\Phi_0 H^3 \Pi^{3/4} n} + \frac{\nu_3 C_3 \Phi_3 (T_{n,r} \Pi)^{1/4}}{3 \Pi^{1/4}}, \]
and so, using the assumption \( \Phi_4 c'(T_{n,r} \Pi)^{1/4} \leq 1 \), we have:
\[ \Delta_n(r) \leq P(\|Y\| \geq r) \exp \{ \Phi_4 c'(T_{n,r} \Pi)^{1/4} \} \left[ \Phi_2 \Phi_3 (T_{n,r} \Pi)^{1/4} + \frac{8 \beta C_3 (T_{n,r} \Pi)^{1/4}}{\Phi_0 H^3 \Pi n} + \frac{\nu_3 C_3 \Phi_3 (T_{n,r} \Pi)^{1/4}}{3 \Pi^{1/4}} \right] \]
\[ \leq e (T_{n,r} \Pi)^{1/4} P(\|Y\| \geq r) \left[ \Phi_2 \Phi_3 + \frac{8 \beta C_3}{\Phi_0 H^3 \Pi} + \frac{\nu_3 C_3 \Phi_3}{3 \Pi^{1/4}} \right] \]
\[ = \Pi T_{n,r}^{1/4} P(\|Y\| \geq r) \frac{e}{\Pi^{1/4}} \left[ \Phi_2 \Phi_3 + \frac{8 \beta C_3}{\Phi_0 H^3 \Pi} + \frac{\nu_3 C_3 \Phi_3}{3} \right]. \]
From the definition of \( \Pi \), we get
\[
\frac{8e^3 \beta C_3}{\Pi^{3/4} \Phi_0 H^3} \leq 1/2, \quad \text{and} \quad \frac{e}{\Pi^{3/4}} \left[ \Phi_2 \Phi_3 + \frac{\nu_3 C_3 \Phi_3}{3} \right] \leq 1/2.
\]
This implies that for all \( n \geq 1 \) and \( r \geq 0 \) satisfying \( \Phi_3^4(2B + \Pi^{1/4})T_{n,r} \leq 1 \),
\[
\Delta_n(r) \leq \Pi T_{n,r}^{1/4} \mathbb{P}(\|Y\| \geq r).
\]
The proof of Lemma B.2.1 is now complete. \( \square \)

### B.3 Third Step: Proving Theorem 3.1

We are now ready to prove Theorem 3.1. To start with, let \( l = [\log n] \). Clearly, \( 0 \leq l \leq \frac{n}{2} \) for all \( n \geq 1 \). Hence, by Lemma B.1.1, we have for \( n \geq 3 \), \( 0 \leq s \leq l \), \( \varepsilon > 0 \) and \( r \geq 0 \):
\[
\Delta_{n-l+s}(r) \leq \mathbb{P}(r - \varepsilon \leq \|Y\| \leq r + \varepsilon)
+ \frac{8C_3 \beta \varepsilon^{-3}}{\Phi_0 H^3(n - l + s)^{3/2}} \mathbb{P}(\|Y\| \geq r)
+ \frac{\nu_3 C_3 \varepsilon^{-3}}{6(n - l + s)^{1/2}} \max_{0 \leq k \leq n-l+s-1} \mathbb{P}(a_{n-l+s}(r) \leq \|U_{n-l+s-1,k}\| \leq b_{n-l+s}(r)).
\]
(54)

Since \( n - l + s \geq n - l \geq \frac{n}{2} \) and \( n^{-1} \leq T_{n,r}^2 \), we have:
\[
\frac{8C_3 \beta \varepsilon^{-3}}{\Phi_0 H^3(n - l + s)^{3/2}} \mathbb{P}(\|Y\| \geq r) \leq \frac{16 \sqrt{2} C_3 \beta \varepsilon^{-3} T_{n,r}^2}{\Phi_0 H^3 n^{1/2}} \mathbb{P}(\|Y\| \geq r).
\]
(55)
To deal with the last term in (54), define
\[
A_n(r) := r - \varepsilon - 2f_n(r) \quad \text{and} \quad B_n(r) := e^{2/n}(r + \varepsilon + 2f_n(r)).
\]
(56)

By Lemma B.4.1, we have \( A_n(r) \leq a_{n-l+s}(r) \) and \( B_n(r) \geq b_{n-l+s}(r) \). Hence, using \( n-l+s \geq n/2 \), we get:
\[
\frac{\nu_3 C_3 \varepsilon^{-3}}{6(n - l + s)^{1/2}} \max_{0 \leq k \leq n-l+s-1} \mathbb{P}(a_{n-l+s}(r) \leq \|U_{n-l+s-1,k}\| \leq b_{n-l+s}(r))
= \frac{\sqrt{2} \nu_3 C_3 \varepsilon^{-3}}{6n^{1/2}} \max_{0 \leq k \leq n-l+s-1} \mathbb{P}(A_n(r) \leq \|U_{n-l+s-1,k}\| \leq B_n(r)).
\]
(57)
Combining (55) and (57) into (54), we get:
\[
\Delta_{n-l+s}(r) \leq \mathbb{P}(r - \varepsilon \leq \|Y\| \leq r + \varepsilon)
+ \frac{16 \sqrt{2} C_3 \beta \varepsilon^{-3} T_{n,r}^2}{\Phi_0 H^3 n^{1/2}} \mathbb{P}(\|Y\| \geq r)
+ \frac{\sqrt{2} \nu_3 C_3 \varepsilon^{-3}}{6n^{1/2}} \max_{0 \leq k \leq n-l+s-1} \mathbb{P}(A_n(r) \leq \|U_{n-l+s-1,k}\| \leq B_n(r)).
\]
(58)
for all $\varepsilon > 0$. Next, for each $\bar{M}$, define:

$$
\alpha(s) = \frac{1 - 4^{-s-1}}{3}, \quad \varepsilon_{s,\bar{M}} = \frac{\bar{M}^{1/4} T_{n,r}^\alpha(s)}{r + 1},
$$

(59)

and the interval

$$
I_{s,\bar{M}} = \left( -\infty, -1 + G_M^{-4/3} \bar{M}^{-1/3} n^{1/6} e^{-2s/n} - 2s f_n(r) - \sum_{i=1}^s \varepsilon_{i,\bar{M}} \right),
$$

where $G_M := \Phi_4(4BM^{-1/4} + 1)$. Now, note that by Lemma B.3.1, we have:

$$
\Delta_n(r) \leq MT_{n,r}^{\alpha(l)} \mathbb{P}(\|Y\| \geq r)
$$

for all $n \geq 3$ and $r \in I_{l,M}$. Now, note that:

$$
T_{n,r}^{-1/3} T_{n,r}^{\alpha(l)} = T_{n,r}^{-1/3} \leq T_{n,r}^{-1} \leq \left( n^{1/2} \right)^{4 - \log(n)/3} = \left( n^{1/6} \right)^{n - \log(4)} \leq n^{1/6n} \leq 2.
$$

This shows that $T_{n,r}^{\alpha(l)} \leq 2T_{n,r}^{1/3}$. Hence, we have for all $n \geq 3$ and $r \in I_{l,M}$:

$$
\Delta_n(r) \leq 2MT_{n,r}^{1/3} \mathbb{P}(\|Y\| \geq r) = 2M(r + 1)n^{-1/6} \mathbb{P}(\|Y\| \geq r)
$$

Next, we aim at finding $c$ such that $(-\infty, -1 + cG_M^{-4/3} M^{-1/3} n^{1/6}) \subseteq I_{l,M}$ for all large $n$ and $r + 1 \leq G_M^{-4/3} M^{-1/3} n^{1/6}$. Note that, $(-\infty, -1 + cG_M^{-4/3} M^{-1/3} n^{1/6}) \subseteq I_{l,M}$ if and only if

$$
c \leq e^{-2l/n} - 2l f_n(r) n^{-1/6} G_M^{-4/3} M^{1/3} - n^{-1/6} G_M^{-4/3} M^{1/3} \sum_{i=1}^l \varepsilon_{i,M}.
$$

Since $l \leq n/2$, we have $e^{-2l/n} \geq e^{-1}$. Next,

$$
2l f_n(r) n^{-1/6} G_M^{-4/3} M^{1/3} = 2lB(r^2 + \log(n)) M^{1/3} G_M^{-4/3} n^{-2/3}
\leq 2lB \log(n)(r + 1)^2 M^{1/3} G_M^{-4/3} n^{-2/3}
\leq 2B \log^2(n)(r + 1)^2 M^{1/3} G_M^{-4/3} n^{-2/3}
\leq 2B \log^2(n) G_M^{-4/3} M^{-1/3} n^{-1/3}
\leq 2BM^{-1/4} \log^2(n) G_M^{-1} n^{-1/4} \leq \log^2(n) n^{-1/4}/2.
$$

Note that in the last step, we used the fact that $G_M \geq 4BM^{-1/4}$. Finally, we have:

$$
n^{-1/6} G_M^{-4/3} M^{1/3} \sum_{i=1}^l \varepsilon_{i,M} = \frac{G_M^{4/3} M^{7/12} n^{-1/6}}{r + 1} \sum_{i=1}^l T_{n,r}^{\alpha(i)} \leq \frac{G_M^{4/3} M^{7/12} n^{-1/6}}{r + 1} \log(n) T_{n,r}^{1/4}
\leq \frac{G_M^{4/3} M^{7/12} n^{-1/6}}{r + 1} \log(n) M^{1/3} \leq G_M^{1/3} n^{-1/6} \log(n) M^{1/3}.
$$
In the last step, we used the fact \((MT_{n,r})^{1/4} \leq G_M^{-1}\). Thus, we have:

\[ e^{-2l/n} - 2lf_n(r)n^{-1/6}G_M^{4/3}M^{1/3} - n^{-1/6}G_M^{4/3}M^{1/3} \sum_{i=1}^{l} \varepsilon_{i,M} \geq e^{-1} - \frac{\log^2(n)}{2n^{1/3}} - \frac{G_M^{1/3} \log(n)M^{1/3}}{n^{1/6}}, \]

whenever \(r + 1 \leq G_M^{-4/3}M^{-1/3}n^{1/6}\). It turns out that if \(n \geq (4e \log(n))^6(MG_M)^2\), then

\[ e^{-1} - \frac{\log^2(n)}{2n^{1/3}} - G_M^{1/3}n^{-1/6} \log(n)M^{1/3} \geq (2e)^{-1}. \]

Hence, we conclude that if \(n \geq (4e \log(n))^6(MG_M)^2\) and \(r + 1 \leq G_M^{-4/3}M^{-1/3}n^{1/6}\), then

\[ \left(-\infty, -1 + (2e)^{-1}G_M^{-4/3}M^{-1/3}n^{1/6}\right) \subseteq I_{l,M}. \]

Hence, for \(n \geq (4e \log(n))^6(MG_M)^2\) and \(r + 1 \leq (2e)^{-1}G_M^{-4/3}M^{-1/3}n^{1/6}\), we have \(r \in I_{l,M}\), which implies that:

\[ \Delta_n(r) \leq 2M(r + 1)n^{-1/6}\mathbb{P}(\|Y\| \geq r). \]

This implies that:

\[ \left| \frac{\mathbb{P}(\|S_n\| \geq r)}{\mathbb{P}(\|Y\| \geq r)} - 1 \right| \leq 2M(r + 1)n^{-1/6} \]

for all \(n \geq (4e \log(n))^6(MG_2)^2\) and \(r + 1 \leq (2e)^{-1}G_2^{-4/3}M^{-1/3}n^{1/6}\). Theorem 3.1 now follows.

**Lemma B.3.1.** For all \(n \geq 3\) and \(0 \leq s \leq l\), we have:

\[ \Delta_{n-l+s}(r) \leq MT_{n,r}^{0(s)}\mathbb{P}(\|Y\| \geq r) \]

for all \(r \in I_{s,M}\).

**Proof.** We prove Lemma B.3.1 by induction on \(s\). For \(s = 0\), the claim becomes:

\[ \Delta_{n-l}(r) \leq MT_{n,r}^{1/4}\mathbb{P}(\|Y\| \geq r) \]

for all \(n \geq 3\), \(l \geq 0\), \(r \leq -1 + G_M^{-4/3}M^{-1/3}n^{1/6}\). Towards showing this, let \(M_0 := \sqrt{2}\Pi\), where recall that

\[ \Pi = \max \left\{ 1, \left(\frac{16e\beta C_3}{\Phi_0 H^3}\right)^{4/7}, \left(2e \left[ \Phi_2 \Phi_3 + \frac{\nu \Phi_3}{3}\right]\right)^{4/3} \right\}. \]

Now, suppose that we have \(r + 1 \leq G_M^{-4/3}M^{-1/3}n^{1/6}\). It is clear that

\[ M_0 \leq M \quad \Rightarrow \quad \Phi_4(4BM_0^{-1/4} + 1)M_0^{1/4} \leq \Phi_4(4BM^{-1/4} + 1)M^{1/4} \]

\[ \Rightarrow \quad G_M^{-4/3}M^{-1/3} \leq G_M^{-4/3}M_0^{-1/3}. \]
Thus, \( r + 1 \leq G_M^{-4/3} M_0^{-1/3} n^{1/6} \). This implies that \( T_{n-l,r} \leq \sqrt{2} \left( \Phi_4(4BM_0^{-1/4} + 1) \right)^{-4} M_0^{-1} \), and hence,

\[
\Phi_4^4(4BM_0^{-1/4} + 1)^4 T_{n-l,r} M_0 2^{-5/4} = \Phi_4^4(4BM_0^{-1/4} + 1)^4 T_{n-l,r} \Pi \leq 1.
\]

Since \( 2\Pi^{-1/4} \leq 4BM_0^{-1/4} \), we get \( \Phi_4^4(2\Pi^{-1/4} + 1)^4 T_{n-l,r} \Pi \leq 1 \). Hence, from the last step in the proof of Lemma B.2.1 we have:

\[
\Delta_{n-l}(r) \leq \Pi T_{n-l,r}^{1/4} \mathbb{P}(\|Y\| \geq r)
= \Pi (r + 1)^{3/4} (n - l)^{-1/8} \mathbb{P}(\|Y\| \geq r)
\leq 2^{1/8} \Pi (r + 1)^{3/4} n^{-1/8} \mathbb{P}(\|Y\| \geq r)
\leq MT_{n-r}^{1/4} \mathbb{P}(\|Y\| \geq r).
\]

This completes the base case \( s = 0 \).

We now assume that the claim holds for \( 0, \ldots, s - 1 \), and aim at proving the claim for \( s \). We set \( \varepsilon = \varepsilon_{s,M} \) in (58). Whenever we write expressions like \( A_n(r) \) or \( B_n(r) \), we will implicitly assume that \( \varepsilon = \varepsilon_{s,M} \). Let \( n \geq 3, l \geq s \) and \( r \in I_{s,M} \). Recall \( A_n(r) \) and \( B_n(r) \) from (56). Now, note that:

\[
\begin{align*}
\mathbb{P}(A_n(r) \leq \|U_{n-l+s-1,k}\| \leq B_n(r))
&= \mathbb{P}(\|U_{n-l+s-1,k}\| \geq A_n(r)) - \mathbb{P}(\|Y\| \geq A_n(r)) + \mathbb{P}(A_n(r) \leq \|Y\| \leq B_n(r))
- \mathbb{P}(\|U_{n-l+s-1,k}\| > B_n(r)) - \mathbb{P}(\|Y\| > B_n(r))
\leq \Delta_{n-l+s-1}(A_n(r)) + \Delta_{n-l+s-1}(B_n(r)) + \mathbb{P}(A_n(r) \leq \|Y\| \leq B_n(r)).
\end{align*}
\]

Let us now prove that \( B_n(r) \in I_{s-1,M} \). Since \( r \in I_{s,M} \), we have:

\[
\begin{align*}
r &\leq -1 + G_M^{-4/3} M^{-1/3} n^{1/6} e^{-2s/n} - 2sf_n(r) - \sum_{i=1}^{s} \varepsilon_{i,M}
\implies r + \varepsilon_{s,M} + 2f_n(r) &\leq -1 + G_M^{-4/3} M^{-1/3} n^{1/6} e^{-2s/n} - 2(s - 1)f_n(r) - \sum_{i=1}^{s-1} \varepsilon_{i,M}
\implies e^{2/n} (r + \varepsilon_{s,M} + 2f_n(r)) &\leq -e^{2/n} + G_M^{-4/3} M^{-1/3} n^{1/6} e^{-2(s-1)/n} - 2(s - 1)f_n(r) e^{2/n} - e^{2/n} \sum_{i=1}^{s-1} \varepsilon_{i,M}.
\end{align*}
\]

Since \( e^{2/n} > 1 \), we have:

\[
B_n(r) \leq -1 + G_M^{-4/3} M^{-1/3} n^{1/6} e^{-2(s-1)/n} - 2(s - 1)f_n(r) - \sum_{i=1}^{s-1} \varepsilon_{i,M}.
\]

This shows that \( B_n(r) \in I_{s-1,M} \). Hence, \( A_n(r) \in I_{s-1,M} \) too.
Hence, by induction hypothesis, we have:

\[ \Delta_{n-l+s-1}(A_n(r)) + \Delta_{n-l+s-1}(B_n(r)) \leq 2MT_{n,r}^{\alpha(s-1)}\mathbb{P}(\|Y\| \geq A_n(r)), \]

and hence, we have:

\[ \mathbb{P}(A_n(r) \leq \|U_{n-l+s-1,k}\| \leq B_n(r)) \leq \mathbb{P}(A_n(r) \leq \|Y\| \leq B_n(r)) + 2MT_{n,r}^{\alpha(s-1)}\mathbb{P}(\|Y\| \geq A_n(r)). \tag{60} \]

Now, we have:

\[
A_n(r) = r - \varepsilon_{s,M} - 2f_n(r) \geq r - \varepsilon_{s,M} - \frac{4BT_{n,r}^{\alpha(s)}}{r + 1} = r - (4BM^{-1/4} + 1)\varepsilon_{s,M} =: \alpha_n(r), \tag{61}
\]

\[
B_n(r) = e^{2/n}(r + \varepsilon_{s,M} + 2f_n(r)) \leq e^{2/n} \left( r + \varepsilon_{s,M} + \frac{4BT_{n,r}^{\alpha(s)}}{r + 1} \right) = e^{2/n} \left( r + (4BM^{-1/4} + 1)\varepsilon_{s,M} \right) =: \beta_n(r). \tag{62}
\]

Hence, we have from (60):

\[ \mathbb{P}(A_n(r) \leq \|U_{n-l+s-1,k}\| \leq B_n(r)) \leq \mathbb{P}(\alpha_n(r) \leq \|Y\| \leq \beta_n(r)) + 2MT_{n,r}^{\alpha(s-1)}\mathbb{P}(\|Y\| \geq \alpha_n(r)). \tag{63} \]

We now prove that \( \beta_n(r) - \alpha_n(r) \leq R_M\varepsilon_{s,M} \), where \( R_M := 5 + 12BM^{-1/4} \). For this, first observe that for \( n \geq 3 \),

\[ \beta_n(r) - \alpha_n(r) \leq r \left( e^{2/n} - 1 \right) + \left( e^{2/3} + 1 \right) \left( 4BM^{-1/4} + 1 \right)\varepsilon_{s,M}. \]

Now, we have using \( \alpha(s) \leq 1/3 \) and \( r \leq r + 1 \leq (n^{1/2}/M)^{1/3} \):

\[
\frac{r \left( e^{2/n} - 1 \right)}{\varepsilon_{s,M}} = \frac{r(r + 1) \left( e^{2/n} - 1 \right)}{M^{1/4}T_{n,r}^{\alpha(s)}} = \frac{r(r + 1) \left( e^{2/n} - 1 \right)}{M^{1/4}(r + 1)^{3\alpha(s) - \alpha(s)/2}} 
\]

\[
= \frac{r \left( e^{2/n} - 1 \right)}{M^{1/4}(r + 1)^{3\alpha(s) - \alpha(s)/2}} = n^{\alpha(s)/2} \left( e^{2/n} - 1 \right) \left( \frac{r(r + 1)^{4-s-1}}{M^{1/4}} \right) 
\]

\[
\leq n^{1/6} \left( e^{2/n} - 1 \right) \left( \frac{r(r + 1)^{4-s-1}}{M^{1/4}} \right) 
\]

\[
\leq n^{1/6} \left( e^{2/n} - 1 \right) \left( \frac{n^{1/2}}{M^{1/4}} \right) \left( \frac{(1+4-s-1)/3}{M^{1/4}} \right) 
\]

\[
\leq M^{-1/4}n^{1/6} \left( e^{2/n} - 1 \right) n^{5/24} 
\]

\[
\leq n^{9/24} \left( e^{2/n} - 1 \right) \leq 2 \quad \text{(since } n \geq 3). \]

Thus, we proved that: \( \beta_n(r) - \alpha_n(r) \leq (2 + 3(4BM^{-1/4} + 1))\varepsilon_{s,M} = R_M\varepsilon_{s,M} \). Hence, by
Assumption (A2), we have:

\[
\mathbb{P}(\alpha_n(r) \leq \|Y\| \leq \beta_n(r)) \\
\leq \Phi_2 \left( \frac{\beta_n(r) - \alpha_n(r)}{2} \right) \left( \frac{\alpha_n(r) + \beta_n(r)}{2} + 1 \right) \mathbb{P}(\|Y\| \geq \alpha_n(r)) \\
\leq \frac{1}{2} \Phi_2 \mathcal{R}_M \varepsilon_{s,M} \left( \frac{\alpha_n(r) + \beta_n(r)}{2} + 1 \right) \mathbb{P}(\|Y\| \geq \alpha_n(r)).
\]

Now, from Lemma B.4.3, we have for \( n \geq 3, \)

\[
\frac{\alpha_n(r) + \beta_n(r)}{2} + 1 \leq \frac{3}{2}(r + 1).
\]

Hence, we have from (63) and (64):

\[
\mathbb{P}(A_n(r) \leq \|U_{n-l+s-1,k}\| \leq B_n(r)) \leq \left( \frac{3}{4} \Phi_2 \mathcal{R}_M \varepsilon_{s,M}(r + 1 + 2MT_{n,r}^{\alpha(s-1)}) \right) \mathbb{P}(\|Y\| \geq \alpha_n(r)),
\]

for all \( 0 \leq k \leq n - l + s - 1. \) Plugging this in (58), we get:

\[
\Delta_{n-l+s}(r) \leq \mathbb{P}(r - \varepsilon_{s,M} \leq \|Y\| \leq r + \varepsilon_{s,M}) \\
+ \frac{16 \sqrt{2} C_3 \beta \varepsilon_{s,M}^3 T_{n,r}^2}{\Phi_0 H^3 n^{1/2}} \mathbb{P}(\|Y\| \geq r) \\
+ \frac{\sqrt{2} \nu_3 C_3 \varepsilon_{s,M}^3}{6 n^{1/2}} \left( \frac{3}{4} \Phi_2 \mathcal{R}_M \varepsilon_{s,M}(r + 1 + 2MT_{n,r}^{\alpha(s-1)}) \right) \mathbb{P}(\|Y\| \geq \alpha_n(r)).
\]

Now, from the Assumption (A2), we have:

\[
\mathbb{P}(r - \varepsilon_{s,M} \leq \|Y\| \leq r + \varepsilon_{s,M}) \leq \Phi_2 \mathcal{R}_M \varepsilon_{s,M}(r + 1) \exp(\Phi_4(r + 1)\varepsilon_{s,M}) \mathbb{P}(\|Y\| \geq r),
\]

\[
\mathbb{P}(\|Y\| \geq \alpha_n(r)) \leq \Phi_3 \exp(\Phi_4(r + 1)(4BM^{-1/4} + 1)\varepsilon_{s,M}) \mathbb{P}(\|Y\| \geq r).
\]

Hence, we have:

\[
\Delta_{n-l+s}(r) \leq \mathbb{P}(\|Y\| \geq r) A(r),
\]

where

\[
A(r) := \Phi_2 \mathcal{R}_M \varepsilon_{s,M}(r + 1) \exp(\Phi_4(r + 1)\varepsilon_{s,M}) + \frac{16 \sqrt{2} C_3 \beta \varepsilon_{s,M}^3 T_{n,r}^2}{\Phi_0 H^3 n^{1/2}} \\
+ \frac{\sqrt{2} \nu_3 C_3 \varepsilon_{s,M}^3}{6 n^{1/2}} \left( \frac{3}{4} \Phi_2 \mathcal{R}_M \varepsilon_{s,M}(r + 1 + 2MT_{n,r}^{\alpha(s-1)}) \right) \Phi_3 \exp(\Phi_4(r + 1)(4BM^{-1/4} + 1)\varepsilon_{s,M}).
\]

Now, recall that \( \varepsilon_{s,M}(r + 1) = M^{3/4} T_{n,r}^{\alpha(s)}. \) Also, we have:

\[
\varepsilon_{s,M}^{-3} t^{-1/2} = \frac{(r + 1)^{3/2} n^{-1/2}}{M^{3/4} T_{n,r}^{\alpha(s)}} \leq \frac{(r + 1)^{3/2} n^{-1/2}}{M^{3/4} T_{n,r}} = M^{-3/4}.
\]
Hence using the definition of $\alpha(s)$ from (59),

$$
\left(\varepsilon_{s,M}^{-3}n^{-1/2}\right)MT_{n,r}^{(s-1)} = M^{1/4}T_{n,r}^{(s)} = M^{1/4}T_{n,r}^{\alpha(s)}.
$$

Thus, we get from (65)

$$
\Lambda(r) \leq \Phi_2 \Phi_3 M^{1/4}T_{n,r}^{\alpha(s)} \exp \left(\Phi_4(r + 1)\varepsilon_{s,M}\right) + \frac{16\sqrt{2}C_3\beta M^{-3/4}T_{n,r}^{\alpha(s)}}{\Phi_0 H^3} \\
+ \frac{\sqrt{2} \nu_3 C_3}{6} \left(\frac{3}{4} M^{-1/2} \Phi_2 R M^{\alpha(s)} + 2 M^{1/4} T_{n,r}^{\alpha(s)}\right) \Phi_3 \exp \left(\Phi_4(r + 1)(4BM^{-1/4} + 1)\varepsilon_{s,M}\right) \\
= M^{1/4}T_{n,r}^{\alpha(s)} \left[\Phi_2 \Phi_3 \exp \left(\Phi_4(r + 1)\varepsilon_{s,M}\right) + \frac{16\sqrt{2}C_3\beta M^{-1}}{\Phi_0 H^3} \\
+ \frac{\sqrt{2} \nu_3 C_3}{6} \left(\frac{3}{4} M^{-3/4} \Phi_2 R M + 2\right) \Phi_3 \exp \left(\Phi_4(r + 1)(4BM^{-1/4} + 1)\varepsilon_{s,M}\right)\right].
$$

(66)

Note that $r \in I_{s,M}$ implies $r + 1 \leq G_M^{-4} M^{-1/4} n^{1/6}$ and so $(r + 1)\varepsilon_{s,M} \leq G_M^{-1} M^{-1/4} n^{1/6}$. Hence, we have:

$$
G_M \varepsilon_{s,M}(r + 1) = G_M M^{1/4} T_{n,r}^{\alpha(s)} \leq G_M M^{1/4} T_{n,r}^{1/4} = G_M M^{1/2} (r + 1)^{1/2} n^{-1/8} \leq 1.
$$

This shows that $\exp \left(\Phi_4(r + 1)(4BM^{-1/4} + 1)\varepsilon_{s,M}\right) \leq e$, and so, $\exp \left(\Phi_4(r + 1)\varepsilon_{s,M}\right) \leq e$. Thus, we have from (65) and (66):

$$
\Delta_{n-I_{s,M}}(r) \leq M^{1/4} T_{n,r}^{\alpha(s)} \mathbb{P}(\|Y\| \geq r) \left[\Phi_2 \Phi_3 e + \frac{16\sqrt{2}C_3\beta}{\Phi_0 H^3 M} + \frac{\sqrt{2} \nu_3 C_3}{6} \left(\frac{3}{4} \Phi_2 R M + 2\right) \Phi_3 e\right] \\
\leq M^{1/4} T_{n,r}^{\alpha(s)} \mathbb{P}(\|Y\| \geq r).
$$

Now note that

$$
\Phi_2 \Phi_3 e + \frac{16\sqrt{2}C_3\beta}{\Phi_0 H^3 M} + \frac{\sqrt{2} \nu_3 C_3}{6} \left(\frac{3}{4} \Phi_2 R M + 2\right) \Phi_3 e \\
= \left(\Phi_2 \Phi_3 e + \frac{\sqrt{2} \nu_3 C_3 \Phi_3 e}{3}\right) + \frac{5\sqrt{2} \nu_3 C_3 \Phi_2}{8} M^{-3/4} + \left(\frac{16\sqrt{2}C_3\beta}{\Phi_0 H^3} + \frac{3\nu_3 C_3 \Phi_2}{\sqrt{2}}\right) M^{-1}.
$$

So, from the definition of $M$,

$$
M^{1/4} \left[\Phi_2 \Phi_3 e + \frac{16\sqrt{2}C_3\beta}{\Phi_0 H^3 M} + \frac{\sqrt{2} \nu_3 C_3}{6} \left(\frac{3}{4} \Phi_2 R M + 2\right) \Phi_3 e\right] \leq M.
$$

This proves the Lemma B.3.1. 

□
Some preliminary facts used in the proof of Theorem 3.1

In this subsection, we prove some preliminary facts used in the proof of Theorem 3.1.

Lemma B.4.1. Let $B := 2(\Phi_1 + 1)/H$, $f_n(r) := B(r^2 + \log n)n^{-1/2}$, $T_{n,r} := (r + 1)^3 n^{-1/2}$ and $\alpha(s) = (1 - 4^{-s-1})/3$. Then, for all $n \geq 1$ and $s \geq 0$, we have

$$f_n(r) \leq 2BT_{n,r}^{\alpha(s)}/(r + 1).$$

Moreover, letting $l = \lfloor \log n \rfloor$, we have for any $n \geq 1$ and $0 \leq s \leq l$:

$$f_{n-l+s}(r) \leq 2f_n(r).$$

Proof. Note that

$$\frac{(r + 1)f_n(r)}{T_{n,r}^{\alpha(s)}} = \frac{B(r + 1)(r^2 + \log n)}{n^{1/2}T_{n,r}^{\alpha(s)}} = \frac{Bn^{(1-4^{-s-1})/6}(r + 1)(r^2 + \log n)}{n^{1/2}(r + 1)\{1-4^{-s-1}\}}$$

$$= \frac{B(r^2 + \log n)(r + 1)^{1-s-1}}{n^{(2+4^{-s-1})/6}} = \frac{B(r^2 + \log n)}{n^{1/3}} \left(\frac{r + 1}{n^{1/6}}\right)^{4^{-s-1}}$$

$$\leq 2B,$$

since $(r + 1)n^{-1/6} = T_{n,r}^{1/3} \leq 1$. This proves the first part of Lemma B.4.1.

For the second part of Lemma B.4.1, note that

$$f_{n-l+s}(r) = B(n - l + s)^{-1/2}(r^2 + \log(n - l + s))$$

$$\leq B\sqrt{2}n^{-1/2}(r^2 + \log n)$$

$$\leq 2f_n(r).$$

The proof of Lemma B.4.1 is now complete. \qed

Lemma B.4.2. If $\Phi_4^4 \left(2B + \Pi^{1/4}\right)^4 T_{n,r} \leq 1$, then $\Phi_4^4 \left(2B + \Pi^{1/4}\right)^4 (r - c\varepsilon + 1)^3 \leq (n - 1)^{1/2}$ for any $c \geq 1$ and $\varepsilon = (T_{n,r}\Pi)^{1/4}(r + 1)^{-1}$.

Proof. By our assumption, we have

$$\Phi_4^4 \left(2B + \Pi^{1/4}\right)^4 (r + 1)^3 \leq n^{1/2}.$$
Now, note that
\[
\Phi_4^4 \left( 2B + \Pi^{1/4} \right)^4 (r - c\varepsilon + 1)^3(n - 1)^{-1/2}
\]
\[
= \frac{1}{(r + 1)^{3n-1/2}} \Phi_4 \left( 2B + \Pi^{1/4} \right)^4 (r + 1)^3n^{-1/2}(r - c\varepsilon + 1)^3(n - 1)^{-1/2}
\]
\[
\leq \frac{1}{(r + 1)^{3n-1/2}} (r - c\varepsilon + 1)^3(n - 1)^{-1/2}
\]
\[
= \left( \frac{r - c\varepsilon + 1}{r + 1} \right)^3 \left( \frac{n}{n - 1} \right)^{1/2}
\]
\[
= \left( 1 - \frac{c\varepsilon}{r + 1} \right)^3 \left( 1 + \frac{1}{n - 1} \right)^{1/2}.
\]
In order to prove Lemma B.4.2, it is thus, enough to prove that
\[
\left( 1 - \frac{c\varepsilon}{r + 1} \right) \leq \left( 1 - \frac{1}{n} \right)^{1/6}.
\]
Now,
\[
\left[ 1 - \left( 1 - \frac{1}{n} \right)^{1/6} \right] \leq \frac{1}{n} \forall n \in \mathbb{N}.
\]
So it is enough to prove that
\[
\frac{1}{n} \leq \frac{c\varepsilon}{r + 1} = \frac{c(T_{n,r}\Pi)^{1/4}}{(r + 1)^{2}} = \frac{c(r + 1)^{3/4}n^{-1/8}}{(r + 1)^{2}} \Pi^{1/4},
\]
which is equivalent to showing that
\[
(r + 1)^{5/4} \leq n^{7/8} c\Pi^{1/4}.
\]
Now, we know that
\[
(r + 1) \leq n^{1/6} \implies (r + 1)^{5/4} \leq n^{5/24} \leq n^{21/24} \leq n^{7/8} c\Pi^{1/4} \forall n \geq 1.
\]
Here the last inequality follows as \( \Pi \geq 1 \). This completes the proof of Lemma B.4.2. \( \square \)

**Lemma B.4.3.** For all \( n \geq 3 \), \( s \geq 0 \) and \( r \in I_{s,M} \), we have:
\[
\frac{\alpha_n(r) + \beta_n(r)}{2} \leq \frac{3r + 1}{2},
\]
where \( \alpha_n(r) \) and \( \beta_n(r) \) are as in (61) and (62) respectively.

**Proof.** Since \( r \in I_{s,M} \), \( r + 1 \leq G_{M}^{-4/3} M^{-1/3} n^{-1/6} \). Hence, we have:
\[
\varepsilon_{s,M} = \frac{T_{n,r}^{\alpha(s)} M^{1/4}}{r + 1} \leq \frac{(r + 1)^{3\alpha(s)} n^{-\alpha(s)/2} M^{1/4}}{r + 1} \leq \frac{G_{M}^{-4\alpha(s)} M^{-\alpha(s)+1/4}}{r + 1} \leq \frac{1}{(r + 1)G_{M}}.
\]
Hence,
\[
\left(4BM^{-1/4} + 1\right)\varepsilon_{s,M} = \frac{G_M\varepsilon_{s,M}}{\Phi_4} \leq \frac{1}{r+1} \leq 1.
\] (73)

Therefore, we have from (73):
\[
\frac{\alpha_n(r) + \beta_n(r)}{2} = \frac{r}{2} \left(e^{2/n} + 1\right) + \frac{1}{2} \left(4BM^{-1/4} + 1\right)\varepsilon_{s,M} \left(e^{2/n} - 1\right)
\leq \frac{3r}{2} + \frac{e^{2/n} - 1}{2} \leq \frac{3r + 1}{2}.
\]

Lemma B.4.3 is thus proved. \(\square\)

C  Proofs of Theorems 3.2 and 3.3

We start with a few preliminary lemmas for proving Theorems 3.2 and 3.3.

C.1 Preliminary Lemmas

At first, we state and prove some preliminary lemmas for proving Theorem 3.2. Our first lemma gives a bound to \(r_m \mathbb{P}(r - \varepsilon \leq \|Y\| \leq r + \varepsilon)\) which will be useful in the proof.

**Lemma C.1.1.** Under assumption (A2), for any \(m \geq 1\) and for \(0 < a \leq 1\),
\[
\sup_{r \geq 0: r - \varepsilon > ar} r_m \mathbb{P}(r - \varepsilon \leq \|Y\| \leq r + \varepsilon) \leq \Phi_2 \varepsilon a^{-m-1} \Gamma_m,
\]
where \(\Gamma_m\) is as defined in (7).

**Proof.** Observe that by assumption (A2),
\[
\mathbb{P}(r - \varepsilon \leq \|Y\| \leq r + \varepsilon) \leq \Phi_2 \varepsilon (1 + r) \mathbb{P}(\|Y\| \geq (r - \varepsilon))
\leq \Phi_2 \varepsilon (1 + r) \mathbb{P}(\|Y\| \geq ar)
= \Phi_2 \varepsilon (1 + r) \mathbb{P}(\|Y\| + a \geq ar + a)
\leq \Phi_2 \varepsilon (1 + r) \mathbb{E} \left[\frac{(||Y|| + a)^{m+1}1\{||Y|| \geq ar\}}{(r + 1)^{m+1}a^{m+1}}\right]
\leq \Phi_2 \varepsilon (1 + r)^{-m} a^{-m-1} \Gamma_m \leq \Phi_2 \varepsilon r^{-m} a^{-m-1} \Gamma_m.
\]

This implies that
\[
\sup_{r \geq 0: r - \varepsilon \geq ar} r_m \mathbb{P}(r - \varepsilon \leq \|Y\| \leq r + \varepsilon) \leq \Phi_2 \varepsilon a^{-m-1} \Gamma_m.
\]

Recall the definition of \(W_{n,j}\) from (21). The following lemma gives an upper bound to \(r_m \mathbb{P}(r - \varepsilon \leq \|W_{n,j}\| \leq r + \varepsilon)\).
Lemma C.1.2. For \( n \geq 2 \) and \( 0 < a \leq 1 \),
\[
\sup_{r > 0 : r - \varepsilon \geq ar} r^m \mathbb{P} (r - \varepsilon \leq \|W_{n,j}\| \leq r + \varepsilon) \leq a^{m-1} \left[ \sqrt{2} \Phi_2 \varepsilon \Gamma_m + 2a \delta_{n-1,m} \right].
\]

Proof. Set \( t_n = (n/(n - 1))^{1/2} \) and observe that using \( t_n \geq 1 \),
\[
\mathbb{P} (r - \varepsilon \leq \|W_{n,j}\| \leq r + \varepsilon) = \mathbb{P} (t_n (r - \varepsilon) \leq t_n \|W_{n,j}\| \leq t_n (r + \varepsilon)) \leq \mathbb{P} (t_n (r - \varepsilon) \leq \|Y\| \leq t_n (r + \varepsilon)) \leq 2 (r - \varepsilon)^m \delta_{n-1,m}.
\]

Combining the above inequalities, we get
\[
r^m \mathbb{P} (r - \varepsilon \leq \|W_{n,j}\| \leq r + \varepsilon) \leq \sqrt{2} \Phi_2 \varepsilon a^{m-1} \Gamma_m + 2a \delta_{n-1,m}.
\]

Taking the supremum proves the result.

The following lemma proves the result when \( n = 1 \).

Lemma C.1.3. For any \( m \geq 1 \), \( \delta_{1,m} \leq \nu_m \).

Proof. Note that
\[
\delta_{1,m} = \sup_{r \geq 0} r^m |\mathbb{P} (\|X\| \geq r) - \mathbb{P} (\|Y\| \geq r)|
\]
\[
= \sup_{r \geq 0} r^m \int_{\|x\| \geq r} |\mathcal{L}(X)(dx) - \mathcal{L}(Y)(dx)|
\]
\[
\leq \sup_{r > 0} r^m \int_{\|x\| \geq r} (\|x\|^m |\zeta|(dx)
\]
\[
\leq \sup_{r > 0} \int_{\|x\| \geq r} (\|x\|^m |\zeta|(dx) = \nu_m.
\]

We now state and prove some preliminary lemmas for proving Theorem 3.3. The following lemma is similar to Lemma C.1.1:
**Lemma C.1.4.** For any $\alpha \geq 1$ and $r, \varepsilon > 0$,

$$P(\alpha(r - \varepsilon) \leq \|Y\| \leq \alpha(r + \varepsilon)) \leq \frac{2C_\phi A_\phi \Phi_2 \alpha^2 \varepsilon \phi(\varepsilon)}{\phi(r)},$$

where

$$A_\phi := \max \left\{ 1, \sup_{r > 0} (1 + r)\phi(r)P(\|Y\| > r) \right\}.$$ 

**Proof.** If $0 < \varepsilon \leq 1$ and $r \geq \varepsilon$, then

$$P(\alpha(r - \varepsilon) \leq \|Y\| \leq \alpha(r + \varepsilon)) \leq \Phi_2 \alpha \varepsilon (1 + \alpha r)P(\|Y\| \geq \alpha(r - \varepsilon)) \leq \Phi_2 \alpha^2 \varepsilon (1 + r)P(\|Y\| \geq r - \varepsilon).$$

From the definition of $A_\phi$,

$$P(\|Y\| \geq r - \varepsilon) \leq \frac{A_\phi}{(1 + r - \varepsilon)\phi(r - \varepsilon)}.$$  \hfill (74)

Also from property (11),

$$\phi(r) = \phi(r - \varepsilon) \leq C_\phi \phi(r - \varepsilon) \phi(\varepsilon) \Rightarrow \phi(r - \varepsilon) \geq \frac{\phi(r)}{C_\phi \phi(\varepsilon)}.$$  \hfill (75)

Thus, for $0 < \varepsilon \leq 1$ and $r \geq \varepsilon$,

$$P(\alpha(r - \varepsilon) \leq \|Y\| \leq \alpha(r + \varepsilon)) \leq \Phi_2 \alpha^2 \varepsilon (1 + r) \frac{A_\phi C_\phi \phi(\varepsilon)}{(1 + r - \varepsilon)\phi(r)} \leq 2C_\phi A_\phi \alpha^2 \Phi_2 \frac{\varepsilon \phi(\varepsilon)}{\phi(r)},$$  \hfill (76)

since $(1 + r) \leq 2(1 + r - \varepsilon)$. If $\varepsilon > 1$ and $r \geq \varepsilon$, then using (74) and (75),

$$P(\alpha(r - \varepsilon) \leq \|Y\| \leq \alpha(r + \varepsilon)) \leq P(\|Y\| \geq r - \varepsilon) \leq \frac{C_\phi A_\phi \phi(\varepsilon)}{(1 + r - \varepsilon)\phi(r)} \leq \frac{C_\phi A_\phi \varepsilon \phi(\varepsilon)}{\phi(r)}.$$  \hfill (77)

The last inequality above follows since $\varepsilon > 1$. If $\varepsilon > 0$ and $r < \varepsilon$, then

$$P(\alpha(r - \varepsilon) \leq \|Y\| \leq \alpha(r + \varepsilon)) \leq P(0 \leq \|Y\| \leq 2\alpha \varepsilon) = P(0 - 2\alpha \varepsilon \leq \|Y\| \leq 0 + 2\alpha \varepsilon) \leq 2P_2 \alpha \varepsilon \leq 2\Phi_2 \alpha \varepsilon \frac{\varepsilon \phi(\varepsilon)}{\phi(r)},$$  \hfill (78)

since $\phi(\cdot)$ is increasing implies $\phi(r) \leq \phi(\varepsilon)$. Combining the bounds (76), (77) and (78), we get for all $r, \varepsilon > 0$

$$P(\alpha(r - \varepsilon) \leq \|Y\| \leq \alpha(r + \varepsilon)) \leq \frac{2C_\phi A_\phi \Phi_2 \alpha^2 \varepsilon \phi(\varepsilon)}{\phi(r)}.$$  

The result follows.  \hfill \qed
Recall that
\[ \delta_{n,\phi} = \sup_{1 \leq k \leq n} \sup_{r > 0} \phi(r) \mathbb{P}(\|U_{n,k}\| \leq r) - \mathbb{P}(\|Y\| \leq r) . \]

The following lemma is similar to Lemma C.1.2:

**Lemma C.1.5.** For \( n \geq 2 \), and \( r, \varepsilon > 0 \),
\[ \mathbb{P}(r - \varepsilon \leq \|W_{n,j}\| \leq r + \varepsilon) \leq \frac{2t_n^2 C_{\phi} \phi(\varepsilon)}{\phi(r)} \left[ A_{\phi} \Phi_{2\varepsilon} + \delta_{n-1,\phi} \right] . \]

**Proof.** Recall that
\[ W_{n,j} = n^{-1/2}(X_1 + \ldots + X_{j-1} + Y_{j+1} + \ldots + Y_n) = t_n^{-1}(n-1)^{-1/2}(X_1 + \ldots + X_{j-1} + Y_{j+1} + \ldots + Y_n), \]
where \( t_n = (n/(n-1))^{1/2} \). Thus, using Lemma C.1.4,
\[ \mathbb{P}(r - \varepsilon \leq \|W_{n,j}\| \leq r + \varepsilon) = \mathbb{P}(t_n(r - \varepsilon) \leq \|t_n W_{n,j}\| \leq t_n(r + \varepsilon)) \leq \mathbb{P}(t_n(r - \varepsilon) \leq \|Y\| \leq t_n(r + \varepsilon)) \]
\[ + \delta_{n-1,\phi} \left(\{\phi(t_n(r - \varepsilon))\}^{-1} + \{\phi(t_n(r + \varepsilon))\}^{-1}\right) \]
\[ \leq \mathbb{P}(t_n(r - \varepsilon) \leq \|Y\| \leq t_n(r + \varepsilon)) + 2\delta_{n-1,\phi} \left(\{\phi(r - \varepsilon)\}^{-1}\right) \]
\[ \leq \mathbb{P}(t_n(r - \varepsilon) \leq \|Y\| \leq t_n(r + \varepsilon)) + 2C_{\phi} \delta_{n-1,\phi} \phi(\varepsilon) / \phi(r) \leq 2t_n^2 C_{\phi} A_{\phi} \Phi_{2\varepsilon} \phi(\varepsilon) / \phi(r) + 2C_{\phi} \delta_{n-1,\phi} \phi(\varepsilon) / \phi(r) \]
\[ \leq \frac{2t_n^2 C_{\phi} \phi(\varepsilon)}{\phi(r)} \left[ A_{\phi} \Phi_{2\varepsilon} + \delta_{n-1,\phi} \right] , \]

since \( t_n \geq 1 \). \( \Box \)

**C.2 Proof of Theorem 3.2**

With Lemmas B.1, C.1.1, C.1.2 and C.1.3, we now give a proof of Theorem 3.2. Since \( \Delta_n^{(m)} \leq \delta_{n,m} \), it is enough to bound \( \delta_{n,m} \). If \( r < 4\varepsilon \), then trivially
\[ \delta_{n,m} \leq 4^m \varepsilon^m . \tag{79} \]

Now consider the case \( r \geq 4\varepsilon \). Let \( \varphi_1(x) := \varphi_{r,\varepsilon}(x) \) and \( \varphi_2(x) = \varphi_{r-\varepsilon,\varepsilon}(x) \) as before. Recall (48) to get that
\[ \mathbb{P}(\|U_{n,k}\| \leq r) - \mathbb{P}(\|Y\| \leq r) \leq \max_{j=1,2} \mathbb{E}[\varphi_j(U_{n,k}) - \varphi_j(Y)] + \mathbb{P}(r - \varepsilon \leq \|Y\| \leq r + \varepsilon) . \tag{80} \]
For convenience, let $\varphi$ denote either $\varphi_1$ or $\varphi_2$. Also, define for $1 \leq k \leq n$

$$I_k := \left| \mathbb{E} \left[ \varphi(W_{n,k} + n^{-1/2}X_k) - \varphi(W_{n,k} + n^{-1/2}Y_k) \right] \right|,$$

$$H := \mathcal{L}(X) - \mathcal{L}(Y).$$

It is clear that

$$I := \left| \mathbb{E} \left[ \varphi(U_{n,k}) - \varphi(Y) \right] \right| \leq \sum_{j=1}^{k} I_j, \quad \text{and} \quad I_j \leq |I_j^{(1)}| + |I_j^{(2)}|, \quad (81)$$

where

$$I_j^{(1)} := \int_{\|x\| \leq r_{n1/2} / 2} \mathbb{E} \left[ \varphi \left( W_{n,j} + n^{-1/2}x \right) \right] \zeta(dx),$$

$$I_j^{(2)} := \int_{\|x\| > r_{n1/2} / 2} \mathbb{E} \left[ \varphi \left( W_{n,j} + n^{-1/2}x \right) \right] \zeta(dx).$$

By Lemma B.1,

$$\varphi \left( W_{n,j} + n^{-1/2}x \right) = \varphi(W_{n,j}) + n^{-1/2}x^\top \nabla \varphi(W_{n,j}) + \frac{1}{2n} x^\top \nabla_2 \varphi(W_{n,j}) x + \text{Rem}_n(W_{n,j}, x).$$

Therefore, we get

$$I_j^{(1)} = I_j^{(1)} + I_j^{(1)} + \frac{1}{2} I_j^{(1)} + I_j^{(1)} + I_j^{(1)},$$

where

$$I_j^{(1)} := \int_{\|x\| \leq r_{n1/2} / 2} \mathbb{E} \left[ \varphi(W_{n,j}) \right] \zeta(dx),$$

$$I_j^{(2)} := \int_{\|x\| \leq r_{n1/2} / 2} \mathbb{E} \left[ n^{-1/2}x^\top \nabla \varphi(W_{n,j}) \right] \zeta(dx),$$

$$I_j^{(3)} := \int_{\|x\| \leq r_{n1/2} / 2} \mathbb{E} \left[ n^{-1} x^\top \nabla_2 \varphi(W_{n,j}) x \right] \zeta(dx),$$

$$I_j^{(4)} := \int_{\|x\| \leq r_{n1/2} / 2} \mathbb{E} \left[ \text{Rem}_n(W_{n,j}, x) \right] \zeta(dx),$$

$$I_j^{(5)} := \int_{\|x\| \leq r_{n1/2} / 2} \mathbb{E} \left[ \text{Rem}_n(W_{n,j}, x) \right] \zeta(dx).$$

For the first three equalities above, we have used the fact that

$$\int \mathbb{E} \left[ \varphi(W_{n,j}) \right] \zeta(dx) = \int \mathbb{E} \left[ x^\top \nabla \varphi(W_{n,j}) \right] \zeta(dx) = \int \mathbb{E} \left[ x^\top \nabla_2 \varphi(W_{n,j}) x \right] \zeta(dx) = 0.$$

Since $\|\varphi\|_\infty \leq 1$, we get

$$\max \left\{ \left| I_j^{(1)} \right|, \left| I_j^{(2)} \right| \right\} \leq \int_{\|x\| > r_{n1/2} / 2} |\zeta|(dx) \leq \int_{\|x\| > r_{n1/2} / 2} \left( \frac{2 \|x\|}{r_{n1/2} / 2} \right)^m |\zeta|(dx) \leq 2^m r^{-m} n^{-m/2} V_m(r_{n1/2} / 2).$$
Thus,

\[ |I_{j1}^{(1)}| + |I_{j1}^{(2)}| \leq 2^{m+1} r^{-m} n^{-m/2} V_m \left( \frac{rn^{1/2}}{2} \right). \]  

(82)

For \( I_{j2}^{(1)} \), note from assumption (A1) that

\[ x^T \nabla \varphi(W_{n,j}) \leq \|x\| \|\nabla \varphi(W_{n,j})\|_1 \leq \|x\| C_1 \varepsilon^{-1} \mathbb{1} \{ r - \varepsilon \leq \|W_{n,j}\| \leq r + \varepsilon \}, \]

since \( \varphi(x) \) is possibly non-constant only on the set \( \{ r - \varepsilon \leq \|x\| \leq r + \varepsilon \} \). Thus

\[ |I_{j2}^{(1)}| \leq C_1 \varepsilon^{-1} \int_{\|x\| > r^{n^{1/2}/2}} n^{-1/2} \|x\| |\zeta|(dx) \mathbb{P} (r - \varepsilon \leq \|W_{n,j}\| \leq r + \varepsilon) \]

\[ \leq C_1 \varepsilon^{-1} \int_{\|x\| > r^{n^{1/2}/2}} \left( \frac{2n^{-1/2} \|x\|}{r} \right)^{m-1} n^{-1/2} \|x\| |\zeta|(dx) \mathbb{P} (r - \varepsilon \leq \|W_{n,j}\| \leq r + \varepsilon) \]

\[ \leq \frac{2^{m-1} C_1 \varepsilon^{-1}}{r^{m-1} n^{m/2}} V_m \left( \frac{rn^{1/2}}{2} \right) \mathbb{P} (r - \varepsilon \leq \|W_{n,j}\| \leq r + \varepsilon) \]

\[ \leq \frac{C_1 \varepsilon^{-m}}{2^{m-1} n^{m/2}} V_m \left( \frac{rn^{1/2}}{2} \right) \mathbb{P} (r - \varepsilon \leq \|W_{n,j}\| \leq r + \varepsilon). \]  

(83)

The last inequality above follows from the fact that \( r > 4 \varepsilon \). For \( I_{j3}^{(1)} \), a similar calculation holds. Observe that

\[ x^T \nabla_2 \varphi(W_{n,j}) x \leq \|x\|^2 \|\nabla_2 \varphi(W_{n,j})\|_1 \leq \|x\|^2 C_2 \varepsilon^{-2} \mathbb{1} \{ r - \varepsilon \leq \|W_{n,j}\| \leq r + \varepsilon \}. \]

Thus,

\[ |I_{j3}^{(1)}| \leq \int_{\|x\| > r^{n^{1/2}/2}} n^{-1} \|x\|^2 C_2 \varepsilon^{-2} |\zeta|(dx) \mathbb{P} (r - \varepsilon \leq \|W_{n,j}\| \leq r + \varepsilon) \]

\[ \leq C_2 \varepsilon^{-2} \int_{\|x\| > r^{n^{1/2}/2}} \left( \frac{2n^{-1/2} \|x\|}{r} \right)^{m-2} n^{-1} \|x\|^2 |\zeta|(dx) \mathbb{P} (r - \varepsilon \leq \|W_{n,j}\| \leq r + \varepsilon) \]

\[ \leq \frac{C_2 \varepsilon^{-m}}{2^{m-2} n^{m/2}} V_m \left( \frac{rn^{1/2}}{2} \right) \mathbb{P} (r - \varepsilon \leq \|W_{n,j}\| \leq r + \varepsilon). \]  

(84)

Combining inequalities (83) and (84), we get

\[ |I_{j2}^{(1)}| + \frac{1}{2} |I_{j3}^{(1)}| \leq \frac{(C_1 + C_2) \varepsilon^{-m}}{2^{m-1} n^{m/2}} V_m \left( \frac{rn^{1/2}}{2} \right) \mathbb{P} (r - \varepsilon \leq \|W_{n,j}\| \leq r + \varepsilon). \]

Note that in the steps above we have used the fact that \( m \geq 2 \). Since \( r \geq 4 \varepsilon \), by Lemma C.1.2 with \( a = 3/4 \), we get

\[ r^m \mathbb{P} \left( r - \varepsilon \leq \|W_{n,j}\| \leq r + \varepsilon \right) \leq \left( \frac{4}{3} \right)^{m+1} \left[ \sqrt{2} \Phi_2 \varepsilon \Gamma_m + \frac{3}{2} \delta_{n-1,m} \right] \]

\[ \leq 2 \left( \frac{4}{3} \right)^m \left[ \Phi_2 \varepsilon \Gamma_m + \delta_{n-1,m} \right]. \]

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Therefore,
\[
\left| I_{j}^{(1)} \right| + \frac{1}{2} \left| I_{j}^{(3)} \right| \leq \frac{4(2/3)^m(C_1 + C_2)\varepsilon^{-m}r^{-m} \Phi_2 \varepsilon \Gamma_m + \delta_{n-1,m}}{n^{m/2}} V_m \left( \frac{rn^{1/2}}{2} \right), \tag{85}
\]

For \( I_{j}^{(1)} \), we use the first bound on \( \text{Rem}_n \) from Lemma B.1,
\[
\left| \text{Rem}_n(W_{n,j}, x) \right| \leq \frac{C_3 \varepsilon^{-3} \|x\|^3}{6n^{3/2}} \left\{ r - 2\varepsilon \leq \|W_{n,j}\| \leq r + 2\varepsilon \right\}, \quad \text{for} \quad \|x\| \leq \varepsilon n^{1/2}.
\]
Here we used the fact
\[
\|W_{n,j}\| - n^{-1/2} \|x\| \leq \|W_{n,j} + n^{-1/2}x\| \leq \|W_{n,j}\| + n^{-1/2} \|x\|.
\]
This implies that
\[
\left| I_{j}^{(1)} \right| \leq \int_{\|x\| \leq \varepsilon n^{1/2}} \frac{C_3 \varepsilon^{-3} \|x\|^3}{6n^{3/2}} |\zeta|(dx) \mathbb{P}(r - 2\varepsilon \leq \|W_{n,j}\| \leq r + 2\varepsilon)
\leq \frac{C_3 \varepsilon^{-3} \|x\|^3}{6n^{3/2}} \int_{\varepsilon n^{1/2} \leq \|x\| \leq \varepsilon n^{1/2}} \left\{ 2\sqrt{2\Phi_2 \varepsilon \Gamma_m + \delta_{n-1,m}} \right\} \leq \frac{2m C_3 \varepsilon^{-3} r^{-m} \nu_3}{n^{3/2}} \left\{ \Phi_2 \varepsilon \Gamma_m + \delta_{n-1,m} \right\}.
\]
Here Lemma C.1.2 is applied by using the fact \( r - 2\varepsilon \geq r/2 \). Therefore,
\[
\left| I_{j}^{(1)} \right| \leq \frac{2m C_3 \varepsilon^{-3} r^{-m} \nu_3}{n^{3/2}} \left\{ \Phi_2 \varepsilon \Gamma_m + \delta_{n-1,m} \right\}. \tag{86}
\]

For \( I_{j}^{(2)} \), we use the second bound on \( \text{Rem}_n \) from Lemma B.1,
\[
\left| \text{Rem}_n(W_{n,j}, x) \right| \leq \frac{C_3 \varepsilon^{-2} \|x\|^2}{n} \left\{ r - \varepsilon - \|x\| n^{-1/2} \leq \|W_{n,j}\| \leq r + \varepsilon + \|x\| n^{-1/2} \right\}.
\]
This implies that
\[
\left| I_{j}^{(2)} \right| \leq \int_{\varepsilon n^{1/2} \leq \|x\| \leq \varepsilon n^{1/2}} \frac{C_3 \varepsilon^{-2} \|x\|^2}{n} \mathbb{P}(r - \varepsilon - \|x\| n^{-1/2} \leq \|W\|_{n,j} \leq r + \varepsilon + \|x\| n^{-1/2}) |\zeta|(dx)
\]
Since \( r \geq 4\varepsilon \), on the set \( \|x\| n^{-1/2} \leq r/2 \),
\[
r - \varepsilon - \|x\| n^{-1/2} \geq r/4.
\]
Now applying Lemma C.1.2 and using \( \varepsilon \leq n^{-1/2} \|x\| \), we get
\[
\mathbb{P}(r - \varepsilon - \|x\| n^{-1/2} \leq \|W\|_{n,j} \leq r + \varepsilon + \|x\| n^{-1/2}) \leq r^{-m} \frac{4^{m+1}}{4^m} \left\{ \sqrt{2\Phi_2 \Gamma_m} (\varepsilon + n^{-1/2} \|x\|) + \frac{1}{2} \delta_{n-1,m} \right\}
\leq r^{-m} \frac{4^{m+1}}{4^m} \left\{ 2\sqrt{2\Phi_2 \Gamma_m} n^{-1/2} \|x\| + \frac{1}{2} \delta_{n-1,m} \right\}.
\]
Therefore,
\[
\left| I_{j5}^{(1)} \right| \leq \int_{\varepsilon n^{1/2} \leq \|x\| \leq n^{1/2}} \frac{C_2\varepsilon^{-2} \|x\|^2}{n^{m+1}} \left[ 2\sqrt{2} \Phi_2 \Gamma_m n^{m-1/2} \|x\| + \frac{1}{2} \delta_{n-1,m} \right] |\zeta| (dx)
\]
\[
\leq \frac{2\sqrt{2} C_2 \varepsilon^{-2} \Phi_2 \Gamma_m n^{m-3/2}}{n^{3/2}} V_3 \left( \varepsilon n^{1/2} \right) + \frac{2(4)\varepsilon^{-2} \Phi_2 \varepsilon \Gamma_m n^{m-1/2}}{n^{3/2}} V_3 \left( \varepsilon n^{1/2} \right)
\]
\[
= \frac{C_2 \varepsilon^{-3} \Phi_2 \varepsilon \Gamma_m n^{m-3/2}}{n^{3/2}} \left[ 8\sqrt{2} \Phi_2 \varepsilon \Gamma_m n + 2\delta_{n-1,m} \right] V_3 \left( \varepsilon n^{1/2} \right).
\]
Hence, we get
\[
\left| I_{j5}^{(1)} \right| \leq \frac{C_2 \varepsilon^{-3} \Phi_2 \varepsilon \Gamma_m n^{m-3/2}}{n^{3/2}} \left[ 8\sqrt{2} \Phi_2 \varepsilon \Gamma_m n + 2\delta_{n-1,m} \right].
\] (87)

Combining the bounds (86) and (87), we get
\[
\left| I_{j4}^{(1)} \right| + \left| I_{j5}^{(1)} \right| \leq \frac{\varepsilon^{-3} \Phi_2 \varepsilon \Gamma_m n^{m-3/2}}{n^{3/2}} \left[ 2^m \Gamma_m + (8\sqrt{2})^4 \Gamma_m \right].
\] (88)

Now combining inequalities (82), (85), and (88), we get for any \( 1 \leq k \leq n \) that
\[
r^m I \leq \frac{2^{m+1}}{n^{(m-2)/2}} V_m \left( \frac{r n^{1/2}}{2} \right) + \frac{4(2/3)^m (C_1 + C_2) \varepsilon^{-m} \Phi_2 \varepsilon \Gamma_m n + \delta_{n-1,m}}{n^{(m-2)/2}} V_m \left( \frac{r n^{1/2}}{2} \right)
\]
\[
+ \frac{(8\sqrt{2})^4 \varepsilon^{-3} \nu_3 (C_3 + C_2) \Phi_2 \varepsilon \Gamma_m n + \delta_{n-1,m}}{n^{1/2}}
\]
\[
= \frac{2^{m+1}}{n^{(m-2)/2}} V_m \left( \frac{r n^{1/2}}{2} \right)
\]
\[
+ \delta_{n-1,m} \left[ \varepsilon^{-m} \left\{ \frac{4(2/3)^m (C_1 + C_2) V_m (r n^{1/2}/2)}{n^{(m-2)/2}} \right\} + \varepsilon^{-3} \left\{ \frac{(8\sqrt{2})^4 (C_2 + C_3) \nu_3}{n^{1/2}} \right\} \right]
\]
\[
+ \Phi_2 \Gamma_m \varepsilon \left[ \varepsilon^{-m} \left\{ \frac{4(2/3)^m (C_1 + C_2) V_m (r n^{1/2}/2)}{n^{(m-2)/2}} \right\} + \varepsilon^{-3} \left\{ \frac{(8\sqrt{2})^4 (C_2 + C_3) \nu_3}{n^{1/2}} \right\} \right]
\]
\[
\leq \frac{2^{m+1} V_m (2\varepsilon n^{1/2})}{n^{(m-2)/2}} + \delta_{n-1,m} F_m (\varepsilon) + \Phi_2 \Gamma_m \varepsilon F_m (\varepsilon),
\]
where
\[
F_m (\varepsilon) := \varepsilon^{-m} \left\{ \frac{4(2/3)^m (C_1 + C_2) V_m (2\varepsilon n^{1/2})}{n^{(m-2)/2}} \right\} + \varepsilon^{-3} \left\{ \frac{(8\sqrt{2})^4 (C_2 + C_3) \nu_3}{n^{1/2}} \right\}.
\]
Thus, for \( r \geq 4\varepsilon \),
\[
r^m I + r^m P (r - \varepsilon \leq \|Y\| \leq r + \varepsilon)
\]
\[
\leq \frac{2^{m+1} V_m (2\varepsilon n^{1/2})}{n^{(m-2)/2}} + \delta_{n-1,m} F_m (\varepsilon) + \Phi_2 \Gamma_m \varepsilon F_m (\varepsilon) + (4/3)^m \Phi_2 \Gamma_m \varepsilon
\]
\[
\leq \frac{2^{m+1} V_m (2\varepsilon n^{1/2})}{n^{(m-2)/2}} + \delta_{n-1,m} F_m (\varepsilon) + \Phi_2 \Gamma_m [2(4/3)^m + F_m (\varepsilon)].
\]

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Combining this with the bound (79) for \( r \leq 4\varepsilon \) and substituting in (48), we get,
\[
\delta_{n,m} \leq \frac{2^{m+1}V_m(2\varepsilon n^{1/2})}{n^{(m-2)/2}} + \delta_{n-1,m}F_m(\varepsilon) + \Phi_2 \Gamma_m\varepsilon \left[2(4/3)^m + F_m(\varepsilon)\right] + 4^m\varepsilon^m.
\] (89)

Set for \( \varepsilon > 0 \),
\[
\Upsilon_m(\varepsilon) := \frac{2^{m+1}V_m(2\varepsilon n^{1/2})}{n^{(m-2)/2}} + \Phi_2 \Gamma_m\varepsilon \left[2(4/3)^m + F_m(\varepsilon)\right] + 4^m\varepsilon^m.
\]

Using this notation, the recursive inequality (89) can be written as
\[
\delta_{n,m} \leq \delta_{n-1,m}F_m(\varepsilon) + \Upsilon_m(\varepsilon) \quad \text{for all} \quad n \geq 2 \quad \text{and for all} \quad \varepsilon > 0.
\] (90)

Now take
\[
\varepsilon = \varepsilon^* := \max \left\{ \frac{2^{1/m}(2\sqrt{3}/3)(C_1 + C_2)^{1/m}T_m^{1/m}}{n^{1/2-1/m}}, \frac{16^3}{n^1/6} \right\}.
\]

The quantity \( \varepsilon^* \) is chosen such that \( V_m(2\varepsilon^* n^{1/2}) \leq T_m/4 \) and \( F_m(\varepsilon^*) \leq 2^{-m/2} \). Substituting \( \varepsilon = \varepsilon^* \) in the definition of \( \Upsilon_m(\varepsilon) \), we get for all \( n \geq 2 \),
\[
\Upsilon_m(\varepsilon^*) \leq \frac{2^{m+1}V_m(2\varepsilon^* n^{1/2})}{n^{(m-2)/2}} + 3(4/3)^m\Phi_2 \Gamma_m\varepsilon^* + 4^m(\varepsilon^*)^m,
\]
\[
\leq \frac{2^{m-1}T_m}{n^{(m-2)/2}} + 3(4/3)^m\Phi_2 \Gamma_m\frac{(4\sqrt{2}/3)(C_1 + C_2)^{1/m}T_m^{1/m}}{n^{1/2-1/m}}
\]
\[
+ 3(4/3)^m\Phi_2 \Gamma_m\frac{(16\sqrt{2})^{1/3}(4\sqrt{2})^{m/3}(C_2 + C_3)^{1/3}\nu_3^{1/3}}{n^{1/6}}
\]
\[
+ 2(8\sqrt{2}/3)^m\frac{(C_1 + C_2)T_m}{n^{(m-2)/2}} + \frac{(8\sqrt{2})^m(4\sqrt{2})^{m/3}(C_2 + C_3)^{m/3}\nu_3^{m/3}}{n^{m/6}}.
\]

Therefore, from (90), we get for all \( n \geq 2 \),
\[
\delta_{n,m} \leq \frac{\delta_{n-1,m}}{2^{m/2}} + 4(8\sqrt{2}/3)^m\frac{(C_1 + C_2)T_m}{n^{(m-2)/2}} + 3(4/3)^m\Phi_2 \Gamma_m\frac{(4\sqrt{2}/3)(C_1 + C_2)^{1/m}T_m^{1/m}}{n^{1/2-1/m}}
\]
\[
+ 3(4/3)^m\Phi_2 \Gamma_m\frac{(16\sqrt{2})^{1/3}(4\sqrt{2})^{m/3}(C_2 + C_3)^{1/3}\nu_3^{1/3}}{n^{1/6}}
\]
\[
+ \frac{(8\sqrt{2})^m(4\sqrt{2})^{m/3}(C_2 + C_3)^{m/3}\nu_3^{m/3}}{n^{m/6}}.
\] (91)

To solve this recursive inequality, we hypothesize that for all \( n \geq 1 \),
\[
\delta_{n,m} \leq \frac{2\nu_m}{2^n} + 8(8\sqrt{2}/3)^m\frac{(C_1 + C_2)T_m}{n^{(m-2)/2}} + 6(4/3)^m\Phi_2 \Gamma_m\frac{(4\sqrt{2}/3)(C_1 + C_2)^{1/m}T_m^{1/m}}{n^{1/2-1/m}}
\]
\[
+ 6(4/3)^m\Phi_2 \Gamma_m\frac{(16\sqrt{2})^{1/3}(4\sqrt{2})^{m/3}(C_2 + C_3)^{1/3}\nu_3^{1/3}}{n^{1/6}}
\]
\[
+ 2 \times \frac{(8\sqrt{2})^m(4\sqrt{2})^{m/3}(C_2 + C_3)^{m/3}\nu_3^{m/3}}{n^{m/6}}.
\] (92)
(Note that the hypothesized value \((92)\) has the same terms as that in \((91)\) except for \(\delta_{n-1,m}/2^{m/2}\) replaced by \(\nu_m/2^n\) and for an additional factor of 2 on all other terms.) To prove \((92)\), we use induction. First note that from Lemma \((C.1.2)\), \(\delta_{1,m} \leq \nu_m\) and the right hand side of \((92)\) is larger than \(\nu_m\) for \(n = 1\). So, the claim is true for \(n = 1\). Suppose now that the claim is true up to \(n - 1\). This implies that

\[
\delta_{n-1,m} \leq \frac{2\nu_m}{2^{n-1}} + 8(8\sqrt{2}/3)^m \frac{(C_1 + C_2)T_m}{(n-1)(m-2)/2} + 6(4/3)^m \Phi_2 \Gamma_m \frac{(4\sqrt{2}/3)(C_1 + C_2)^{1/m} T_m^{1/m}}{(n-1)^{1/2-1/m}} \\
+ 6(4/3)^m \Phi_2 \Gamma_m \frac{(16\sqrt{2})^{1/3}(4\sqrt{2})^{m/3}(C_2 + C_3)^{1/3} \nu_3^{1/3}}{(n-1)^{1/6}} \\
+ 2 \times \frac{(8\sqrt{2})^m (4\sqrt{2})^{m^2/3}(C_2 + C_3)^{m/3} \nu_3^{m/3}}{(n-1)^{m/6}}.
\]

We substitute this in the recursive inequality \((91)\). Note (for simplifying \(\delta_{n-1,m}/2^{m/2}\)) that

\[
\frac{2}{2^{m/2}(n-1)(m-2)/2} = \frac{1}{(2(n-1))(m-2)/2} \leq \frac{1}{n^{(m-2)/2}}, \\
\frac{2}{2^{m/2}(n-1)^{1/2-1/m}} \leq \frac{1}{(2(n-1))^{1/2-1/m}} \leq \frac{1}{n^{1/2-1/m}}, \\
\frac{2}{2^{m/2}(n-1)^{1/6}} \leq \frac{1}{(2(n-1))^{1/6}} \leq \frac{1}{n^{1/6}}, \\
\frac{2}{2^{m/2}(n-1)^{m/6}} \leq \frac{1}{(2(n-1))^{m/6}} \leq \frac{1}{n^{m/6}}.
\]

(This simplification is the reason for making \(F_m(\delta^*) \leq 2^{-m/2}\) and hypothesizing the bound on \(\delta_{n,m}\) with a factor of 2.) Substituting these bounds in \((91)\), we get \((92)\). This completes the proof of Theorem \(3.2\).

### C.3 Proof of Theorem 3.3

To begin with, note that, making use of \((80)\) and \((81)\), we need to control

\[
I_j := \left| \mathbb{E} \left[ \varphi(W_{n,j} + n^{-1/2}X_j) - \varphi(W_{n,j} + n^{-1/2}Y_j) \right] \right|.
\]

Using the fact that \(\mathbb{E}[X] = \mathbb{E}[Y]\) and \(\mathbb{E}[XX^\top] = \mathbb{E}[YY^\top]\), we get

\[
I_j \leq I_j^{(1)} + I_j^{(2)},
\]

where

\[
I_j^{(1)} := \int_{\|x\| \leq \epsilon n^{1/2}} |\mathbb{E} [\text{Rem}_n(W_{n,j}, x)]| |\zeta| (dx), \\
I_j^{(2)} := \int_{\|x\| > \epsilon n^{1/2}} |\mathbb{E} [\text{Rem}_n(W_{n,j}, x)]| |\zeta| (dx).
\]

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Note that \( \varepsilon > 0 \) will be chosen suitably, later. We know that for all \( x \),

\[
|\text{Rem}_n(W_{n,j}, x)| \leq \frac{C_3 \varepsilon^{-3}}{6n^{3/2}} \|x\|^3 \left\{ r - \varepsilon - n^{-1/2} \|x\| \leq \|W_{n,j}\| \leq r + \varepsilon + n^{-1/2} \|x\| \right\}.
\]

Thus by Lemma C.1.5, we get

\[
I_j^{(1)} \leq \frac{C_3 \varepsilon^{-3}}{6n^{3/2}} \int_{\|x\| \leq \varepsilon n^{1/2}} \mathbb{P} \left( r - \varepsilon - n^{-1/2} \|x\| \leq \|W_{n,j}\| \leq r + \varepsilon + n^{-1/2} \|x\| \right) \|x\|^3 |\zeta|(dx)
\]

\[
\leq \frac{C_3 \varepsilon^{-3}}{6n^{3/2}} \int_{\|x\| \leq \varepsilon n^{1/2}} \frac{2t_2^2 \frac{C_2 \phi(\varepsilon + n^{-1/2} \|x\|)}{\phi(r)}}{\phi(r)} \left[ A_\phi \Phi_2 \left( \varepsilon + n^{-1/2} \|x\| \right) + \delta_{n-1,\phi} \right] \|x\|^3 |\zeta|(dx)
\]

\[
\leq \frac{C_3 \varepsilon^{-3}}{6n^{3/2}} \int_{\|x\| \leq \varepsilon n^{1/2}} \frac{2t_2^2 \frac{C_2 \phi(\varepsilon + n^{-1/2} \|x\|)}{\phi(r)}}{\phi(r)} \left[ 2A_\phi \Phi_2 \varepsilon + \delta_{n-1,\phi} \right] \|x\|^3 |\zeta|(dx)
\]

\[
\leq \frac{2t_2^2 C_2^2 \phi(\varepsilon) C_3 \varepsilon^{-3} \{2A_\phi \Phi_2 \varepsilon + \delta_{n-1,\phi}\}}{6\phi(r)n^{3/2}} \int_{\|x\| \leq \varepsilon n^{1/2}} \phi(n^{-1/2} \|x\|) \|x\|^3 |\zeta|(dx)
\]

For \( I_j^{(2)} \), note that for all \( x \),

\[
|\text{Rem}_n(W_{n,j}, x)| \leq \frac{C_2 \varepsilon^{-2}}{n} \|x\|^2 \left\{ r - \varepsilon - n^{-1/2} \|x\| \leq \|W_{n,j}\| \leq r + \varepsilon + n^{-1/2} \|x\| \right\}.
\]

Thus by Lemma C.1.5, we get

\[
I_j^{(2)} \leq \frac{C_2 \varepsilon^{-2}}{n} \int_{\|x\| > \varepsilon n^{1/2}} \|x\|^2 \mathbb{P} \left( r - \varepsilon - n^{-1/2} \|x\| \leq \|W_{n,j}\| \leq r + \varepsilon + n^{-1/2} \|x\| \right) |\zeta|(dx)
\]

\[
\leq \frac{C_2 \varepsilon^{-2}}{n} \int_{\|x\| > \varepsilon n^{1/2}} \frac{2t_2^2 \frac{C_2 \phi(\varepsilon + n^{-1/2} \|x\|)}{\phi(r)}}{\phi(r)} \left[ A_\phi \Phi_2 \left( \varepsilon + n^{-1/2} \|x\| \right) + \delta_{n-1,\phi} \right] \|x\|^2 |\zeta|(dx)
\]

\[
\leq \frac{2t_2^2 C_2^2 \phi(\varepsilon) C_2 \varepsilon^{-2}}{n\phi(r)} \int_{\|x\| > \varepsilon n^{1/2}} \phi(n^{-1/2} \|x\|) \left[ 2n^{-1/2} A_\phi \Phi_2 \varepsilon + \delta_{n-1,\phi} \right] \|x\|^2 |\zeta|(dx)
\]

\[
\leq \frac{2t_2^2 C_2^2 \phi(\varepsilon) C_2 \varepsilon^{-2}}{n\phi(r)} \int_{\|x\| > \varepsilon n^{1/2}} \phi(n^{-1/2} \|x\|) \left[ 2A_\phi \Phi_2 + \delta_{n-1,\phi} \varepsilon^{-1} \right] n^{-1/2} \|x\|^3 |\zeta|(dx)
\]

\[
\leq \frac{2t_2^2 C_2^2 \phi(\varepsilon) C_2 \varepsilon^{-3} \{2A_\phi \Phi_2 \varepsilon + \delta_{n-1,\phi}\}}{n^{3/2}\phi(r)} \int_{\|x\| > \varepsilon n^{1/2}} \phi(n^{-1/2} \|x\|) \|x\|^3 |\zeta|(dx).
\]

Combining the bounds on \( I_j^{(1)} \) and \( I_j^{(2)} \), we get

\[
I_j \leq \frac{2t_2^2 C_2^2 \phi(\varepsilon)(C_2 + C_3) \varepsilon^{-3} \nu_{\varepsilon,3}}{n^{3/2}\phi(r)} \{2A_\phi \Phi_2 \varepsilon + \delta_{n-1,\phi}\}.
\]

This implies that for \( 1 \leq k \leq n \),

\[
I \leq \sum_{j=1}^{k} I_j \leq \frac{2t_2^2 C_2^2 \phi(\varepsilon)(C_2 + C_3) \varepsilon^{-3} \nu_{\varepsilon,3}}{n^{1/2}\phi(r)} \{2A_\phi \Phi_2 \varepsilon + \delta_{n-1,\phi}\}.
\]

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Note by Lemma C.1.4,
\[
\mathbb{P}(r - \varepsilon \leq \|Y\| \leq r + \varepsilon) \leq \frac{2C_\phi A_\phi \Phi_2 \varepsilon \phi(\varepsilon)}{\phi(r)}.
\]
Therefore,
\[
\delta_{n,\phi} \leq 2t^2 n C^2_\phi \phi(\varepsilon)(C_2 + C_3) \varepsilon^{-3} \nu_{\phi,3} n^{-1/2} \{2A_\phi \Phi_2 \varepsilon + \delta_{n-1,\phi} \} + 2C_\phi A_\phi \Phi_2 \varepsilon \phi(\varepsilon)
\]
\[
\leq 4C^2_\phi \phi(\varepsilon)(C_2 + C_3) \varepsilon^{-3} \nu_{\phi,3} n^{-1/2} \{2A_\phi \Phi_2 \varepsilon + \delta_{n-1,\phi} \} + 2C_\phi A_\phi \Phi_2 \varepsilon \phi(\varepsilon)
\]
Define:
\[
F_{n,\phi}(\varepsilon) := 4C^2_\phi \phi(\varepsilon)(C_2 + C_3) \varepsilon^{-3} \nu_{\phi,3} n^{-1/2},
\]
\[
\Upsilon_{n,\phi}(\varepsilon) := 2F_{n,\phi}(\varepsilon) A_\phi \Phi_2 \varepsilon + 2C_\phi A_\phi \Phi_2 \varepsilon \phi(\varepsilon).
\]
The recursive inequality, thus, can be written as
\[
\delta_{n,\phi} \leq F_{n,\phi}(\varepsilon) \delta_{n-1,\phi} + \Upsilon_{n,\phi}(\varepsilon).
\]
For each \(k\), define:
\[
\varepsilon_k = (8C^2_\phi(1)(C_2 + C_3) \nu_{\phi,3})^{1/3} k^{-1/6}, \quad \text{and} \quad n^*_\phi := 2 \left(8C^2_\phi(1)(C_2 + C_3) \nu_{\phi,3}\right)^2.
\]
Then, \(\varepsilon_k \leq 1\) for all \(k \geq n^*_\phi/2\). Consequently, \(F_{k,\phi}(\varepsilon_k) \leq 1/2\) and \(\Upsilon_{k,\phi}(\varepsilon_k) \leq K_\phi \varepsilon_k\) for all \(k \geq n^*_\phi/2\), where \(K_\phi := (1 + 2C_\phi(1)) A_\phi \Phi_2\). Hence, we get
\[
\delta_{k,\phi} \leq \frac{1}{2} \delta_{k-1,\phi} + K_\phi \varepsilon_k,
\]
for all \(k \geq n^*_\phi/2\). Thus, for \(n \geq n^*_\phi\), we have:
\[
\delta_{n,\phi} \leq \frac{1}{2^{[n/2]+1}} \delta_{n-[n/2]-1,\phi} + K_\phi \varepsilon_{n-[n/2]} \sum_{j=0}^{[n/2]} 2^{-j} \leq \frac{\Xi_{\phi,\varepsilon n_{\phi}}}{2^{n/2}} + 2K_\phi \varepsilon_{n-[n/2]}.
\]
The proof of Theorem 3.3 now follows from the observation that \(\Delta_{n,\phi} \leq \delta_{n,\phi}\) and the fact that for all \(n \leq n^*_\phi\), \(\delta_{n,\phi} \leq \Xi_{\phi} n_{\phi}^{1/6}/n^{1/6}\).

D Proofs of Corollaries 3.1 and 3.2

The following preliminary lemma is required for the proofs.
Lemma D.1. Let $X = (X(1), \ldots, X(p))$ be an $\mathbb{R}^p$ valued random variable. Suppose that there exists a constant $0 \leq K_p < \infty$, such that

$$\max_{1 \leq j \leq p} \|X(j)\|_{\psi_\alpha} \leq K_p$$

for some $\alpha \geq 0$. Then, for all $q \geq 1$, we have:

$$\mathbb{E} \|X\|_\infty^q \leq K_p^q \left(2^{1/q} \left(\frac{6q}{e\alpha}\right)^{1/\alpha} + 2^{1/\alpha} (\log p)^{1/\alpha}\right)^q. \quad (95)$$

Moreover, for $\alpha \geq 1$, we have

$$\mathbb{E} \exp \left[\frac{\|X\|_\infty}{3^{1/\alpha} (\log 2)^{(1-\alpha)/\alpha} K_p (1 + (\log p)^{1/\alpha})}\right] \leq 2. \quad (96)$$

Proof. We prove (96) first. Fix $1 \leq j \leq p$. Since $\|X(j)\|_{\psi_\alpha} \leq K_p$ and $\psi_\alpha$ is increasing, we have

$$\mathbb{E} \psi_\alpha \left(\frac{|X(j)|}{K_p}\right) \leq 1.$$

Hence, by an application of Markov’s inequality, we have for all $t \geq 0$:

$$\mathbb{P} \left(|X(j)| \geq K_p t^{1/\alpha}\right) \leq \frac{\mathbb{E} \psi_\alpha \left(|X(j)|/K_p\right) + 1}{\psi_\alpha (t^{1/\alpha}) + 1} \leq 2 e^{-t}. \quad (97)$$

It follows from (97) and an union bound, that for all $t \geq 0$,

$$\mathbb{P} \left(\|X\|_\infty \geq K_p (t + \log p)^{1/\alpha}\right) \leq 2 e^{-t}.$$

Since $\alpha \geq 1$, $(t + \log p)^{1/\alpha} \leq t^{1/\alpha} + (\log p)^{1/\alpha}$. Hence, for all $t \geq 0$,

$$\mathbb{P} \left(\|X\|_\infty - K_p (\log p)^{1/\alpha} \geq K_p t^{1/\alpha}\right) \leq 2 e^{-t}. \quad (98)$$

Define

$$W := \left(\|X\|_\infty - K_p (\log p)^{1/\alpha}\right)_+. \quad$$

Then it follows from (98), that for all $t \geq 0$,

$$\mathbb{P} \left(W \geq K_p t^{1/\alpha}\right) \leq 2 e^{-t}. \quad (99)$$

Hence, we have from (99),

$$\mathbb{E} \left[\exp \left(\frac{W^{\alpha}}{3 K_p^{\alpha}}\right) - 1\right] = \int_0^\infty \mathbb{P} \left(\exp \left(\frac{W^{\alpha}}{3 K_p^{\alpha}}\right) - 1 \geq t\right) dt$$

$$= \int_0^\infty \mathbb{P} \left(W \geq K_p (3 \log(1 + t))^{1/\alpha}\right) dt$$

$$\leq 2 \int_0^\infty (1 + t)^{-3} dt = 1.$$
Hence, \( \|W\|_{\psi_\alpha} \leq 3^{1/\alpha}K_p \). Consequently,

\[
\|X\|_{\infty, \psi_\alpha} \leq \|W\|_{\psi_\alpha} + \|K_p(\log p)^{1/\alpha}\|_{\psi_\alpha} \\
\leq 3^{1/\alpha}K_p + K_p(\log p)^{1/\alpha}(\log 2)^{-1/\alpha} \\
\leq 3^{1/\alpha}K_p \left(1 + (\log p)^{1/\alpha}\right).
\]

Hence, from Problem 5 of Chapter 2.2 of van der Vaart and Wellner (1996),

\[
\|X\|_{\infty, \psi_1} \leq \|X\|_{\infty, \psi_\alpha}(\log 2)^{(1-\alpha)/\alpha} \leq 3^{1/\alpha}(\log 2)^{(1-\alpha)/\alpha}K_p \left(1 + (\log p)^{1/\alpha}\right).
\]

This proves (96). For proving (95), note that we have by a similar argument as before, but using the additional fact that \((t + \log p)^{1/\alpha} \leq 2^{1/\alpha}t^{1/\alpha} + 2^{1/\alpha}(\log p)^{1/\alpha}\) for \(\alpha \geq 0\),

\[
P\left(\tilde{W} \geq 2^{1/\alpha}K_p t^{1/\alpha}\right) \leq 2e^{-t},
\]

where \(\tilde{W} := (\|X\|_{\infty} - 2^{1/\alpha}K_p(\log p)^{1/\alpha})_{+}\). Hence, we have:

\[
E\left[\exp\left(\frac{\tilde{W}^\alpha}{6K_p^\alpha}\right) - 1\right] \leq 1,
\]

(106)

It is easy to see that for all \(x \geq 0\) and \(v_1, v_2 \geq 0\), we have:

\[
x^{v_1} \exp\left(-\frac{x}{v_2}\right) \leq v_1^{v_1}v_2^{v_2} \exp(-v_1).
\]

Using (D) with \(x = (\tilde{W}/(6^{1/\alpha}K_p))^\alpha\), \(v_1 = q/\alpha\) and \(v_2 = 1\), we have:

\[
\frac{\tilde{W}^q}{6^{q/\alpha}K_p^q} \exp\left(-\frac{\tilde{W}^\alpha}{6K_p^\alpha}\right) \leq \left(\frac{q}{e\alpha}\right)^{q/\alpha}.
\]

(107)

It follows from (107) and (106), that

\[
E\tilde{W}^q \leq 2K_p^q \left(\frac{6q}{e\alpha}\right)^{q/\alpha}.
\]

Consequently, we have

\[
(E \|X\|_{\infty}^q)^{1/q} \leq \left(E\tilde{W}^q\right)^{1/q} + 2^{1/\alpha}K_p(\log p)^{1/\alpha} \\
\leq K_p \left(2^{1/q} \left(\frac{6q}{e\alpha}\right)^{1/\alpha} + 2^{1/\alpha}(\log p)^{1/\alpha}\right).
\]

This proves (95) and completes the proof of Lemma D.1. 

\[\square\]
The notation $\Theta$ from now on denotes a constant depending only on $\sigma_{\min}$ and $\sigma_{\max}$ and can be different in different lines. The proofs of Corollaries 3.1 and 3.2 follow by noting the following:

$$\max\{\Phi_0, \Phi_1, \Phi_3, \beta\} \leq \Theta,$$
$$\Phi_2 \leq \Theta \log p,$$
$$\Phi_4 \leq \Theta (\log p)^2,$$
$$\mu \leq \Theta \sqrt{\log p},$$
$$H \geq \Theta K_p^{-1}(\log p)^{-1/\alpha},$$
$$C_j \leq \Theta (\log p)^{j-1}, \quad j = 1, 2, 3,$$
$$\nu_3 \leq \Theta K_p^3(\log p)^{3/\alpha}.$$

Substituting these in Theorems 3.1 and 3.2 the results follow.