THE THRESHOLD DYNAMICS OF A DISCRETE-TIME ECHINOCOCCOsis TRANSMISSION MODEL

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Abstract. In this paper, based on the transmission mechanism of echinococcosis in China, we propose a discrete-time dynamical model for the transmission of echinococcosis. The research results indicate that transmission dynamics of this discrete-time model are determined by basic reproduction number $R_0$. It is shown that when $R_0 \leq 1$ then the disease-free equilibrium is globally asymptotically stable and when $R_0 > 1$ then the model is permanent while the disease-free equilibrium is unstable. Finally, on the basis of the theoretical results established in this paper, we come up with some specific measures to control the transmission of echinococcosis.

1. Introduction. Echinococcosis is mainly a zoonotic parasitic disease caused by Echinococcus granulosus and Echinococcus multilocularis. The tapeworm is a double host parasite and requires two species of mammals: definitive hosts (carnivorous animals, such as dogs, wolves and foxes)(see [1, 4, 9, 33]) and intermediate hosts (unhoofed and wild rodents/sheep, such as goats, cattle and camels)(see [17, 35]). The development stage of echinococcus granulosus is usually divided into three parts. The adult of echinococcus parasitize the viscus definitive hosts which include dogs, wolves, and other small canines. The eggs are released into the wild through the feces of the definitive host. The life cycle of the eggs in the field varies from several weeks to several months. Sheep, cattle and horses etc. are defined as intermediate hosts which eat the worm eggs into the body by eating grass and drinking water. In the small bowels of the intermediate host, the eggs hatch into larvae which is called oncosphere and penetrates the intestinal wall into the circulatory system of the intermediate host, and enter various organs, especially the

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liver and lungs. A cyst is developed by the oncosphere and grow slowly in these organs, which can product a few of protoscolex and daughter cyst that fill the mass with hydatids in the intermediate host. After a couple years, the diameter of the mass can be up to 20 cm. Then, when the definitive host eats the intermediate host’s organs which contain the mass, each of the protoscolex and daughter cyst in the mass develops into an adult in the digestive system of the definitive host in about 7-8 weeks, the adults then release the eggs and release them into the wild with the feces of the definitive host. During the transmission of hydatid disease above, humans usually feed eggs into the body by eating food and drinking water as intermediate host and product protoscolex and daughter cyst in body, especially the liver, lung, and other organs. Then the people are infected with echinococcosis (see [19, 37]).

Since the first human alveolar echinococcosis in 1852 was discovered in Munich (Germany) by BuHL, echinococcosis has been widely spread all over the world (see [6, 13]). Cystic Echinococcosis and Alveolar Echinococcosis are two main forms of Echinococcosis in humans. It is mainly distributed in the Eastern and Middle Asian, the Northern African, the North America and the Europe countries (see [3, 10, 19, 27, 34]). It is well known that mathematical model plays an important role in quantitative analysis of the echinococcosis and in the dynamic propagation of time and space, at the same time it can also provide effective control measures of infectious diseases. From the research of Roberts et al. in [26], different kinds of echinococcosis transmission dynamics models have been established, it can be used to study the transmission of echinococcosis at different levels (see [2, 3, 17, 18, 31, 32, 34, 35, 38]).

In China, the prevalence scope of echinococcosis to both animals and humans in alpine meadows, pastoral areas and agro pastoral regions, such as Xinjiang, Ningxia, Sichuan, Tibet, Gansu, Qinghai and Inner Mongolia, are very extensive in cold weather, drought and less rain (see [5, 8, 11, 21, 22, 24, 29]). Scientists have done a lot of research on echinococcosis. In [28, 38], the authors studied a dynamical transmission model which described the dog-livestock-egg-human of echinococcosis in Xinjiang and Inner Mongolia, respectively. [23] proposed a time-delayed dynamical transmission of echinococcosis model.

In addition, we see that some new research works on the population and epidemic dynamical models are published. For example, in [39], a new SIS epidemic model considering the infection rates of multiple edges interfere with each other on complex networks was proposed. In [40], a class of fractional-order prey-predator models with Holling III type functional response and discontinuous harvest is investigated.

It is generally known that the selection of time scale plays an important role in the study of biology and ecology. At the present time, there are various discretization methods to discretize a continuous model, such as in [7] is using exponentiation and discretization. The advantage of exponentiation and discretization of the population model is that it can not only obtain the positivity and boundedness of the solution, the stability of the equilibrium, but also obtain more complex dynamic behaviors, such as various bifurcation and chaos phenomena, etc. The discretization method of exponentiation and discretization is mainly using for population model, but for epidemic models, we usually use directly discretized instead of exponentiation and discretization (See [14, 15, 16, 30, 36]). The advantage of Euler backward direct difference discretization of infectious disease model is that the discretization model can obtain the threshold criterion of persistence or extinction of infectious disease.
And in the process of infectious disease research, the data of infectious diseases are collected by day, week, month or year, that is to say, the collected infectious disease data are discrete. At the same time, using the discrete time method is easier for us to use the advantages of computers for programming and iteration, so as to solve physical problems.

In this paper, motivated by the results given in [2, 35, 34, 38, 39, 40], we use the compartment modelling method (see [12, 25]) to establish a discrete-time dynamical model for the transmission of echinococcosis. The technical contribution in this paper is that we use Euler backward difference method to transform the partial differential equations into discrete equations, thus simplifying the solution process. It can provide a new idea for us to solve complex partial differential equations, so as to replace complex analytical solutions with high-precision numerical solutions. Afterwards, as the scientific contribution of this paper is by using the stability theory of difference equations and the persistence theory of dynamic systems (see [20, 41]) to investigate detailedly the positivity and ultimate boundedness of solutions, the existence and global stability of equilibrium, the permanence of the positive solutions, and further to establish a series of effective criteria.

The organization of this paper is as follows. In Section 2, we propose a discrete-time dynamical model for the transmission of echinococcosis. In Section 3, we prove the positivity and ultimate boundedness of solutions. We also obtain the basic reproduction number $R_0$ and the existence of the endemic equilibrium. In Section 4, we prove the local and global asymptotic stability of the disease-free equilibrium. In Section 5, we establish a theorem on the permanence of all positive solutions. In Section 6, we give a discussion. As application of theoretical results established in this paper, we discuss some effective measures to control the spread of echinococcosis.

2. Model formulation. Based on the transmission dynamics of echinococcosis given in the previous section, we consider the definitive host dog, the intermediate host livestock, the secondary intermediate hosts human and eggs in the environment. We provide the following assumption:

1. We divide the dogs population into two compartments: susceptible and infected populations which quantity or density in the $n$ year are represented by $S_D(n)$ and $I_D(n)$, respectively. The dog infected with echinococcosis by swallowing the infected cyst-containing organs of the intermediate host. Whether or not the dog gets sick depends on the life expectancy of the eggs, the immunocompetence of dogs and the frequency how often the deworming drugs are used.

2. We divide the livestock into two compartments: susceptible and infected populations which quantity or density in the $n$ year are represented by $S_L(n)$ and $I_L(n)$, respectively. Livestock infected with echinococcosis by swallowing field echinococcus ovale. After infection, eggs hatch into larvae. It can form cystic masses in the organs of livestock and become bigger and bigger.

3. Through the feces of the definitive host, echinococcus eggs are released into the wild, the quantity or density in the $n$ year is represented by $x(n)$. The number of sick dogs and the death rate of the eggs mainly depend on the density of the eggs. The consumption of the egg in the middle host has little effect on the change of the density of the eggs.

4. People are also infected with echinococcosis by ingestion of echinococcus ovale eggs in the wild. The initial stage of the person’s infection of the egg is
without clinical symptoms. Small cysts do not cause serious symptoms and may last for years of latency. Hatching period of eggs may last for months to years. Therefore, we can divide the human into three compartments: susceptible, exposed and infected populations which quantity or density in the \( n \) year are represented by \( S_H(n), E_H(n) \) and \( I_H(n) \), respectively.

According to the above assumptions, by using Euler’s backward difference method (see [14, 15, 16, 30, 36]) we establish the following discrete-time model of echinococcosis transmission containing eight equations

\[
\begin{align*}
S_D(n+1) &= S_D(n) + A_1 - \beta_1 S_D(n+1) I_L(n+1) - d_1 S_D(n+1) + \sigma I_D(n+1) \\
I_D(n+1) &= I_D(n) + \beta_1 S_D(n+1) I_L(n+1) - (d_1 + \sigma) I_D(n+1) \\
S_L(n+1) &= S_L(n) + A_2 - \beta_2 S_L(n+1) x(n+1) - d_2 S_L(n+1) \\
I_L(n+1) &= I_L(n) + \beta_2 S_L(n+1) x(n+1) - d_2 I_L(n+1) \\
x(n+1) &= x(n) + a I_D(n+1) - \sigma(x(n+1)) \\
S_H(n+1) &= S_H(n) + A_3 - \beta_3 S_H(n+1) x(n+1) - d_3 S_H(n+1) + \gamma I_H(n+1) \\
E_H(n+1) &= E_H(n) + \beta_3 S_H(n+1) x(n+1) - (d_3 + \sigma) E_H(n+1) \\
I_H(n+1) &= I_H(n) + w E_H(n+1) - (d_3 + \mu + \gamma) I_H(n+1)
\end{align*}
\]

In model (1), \( A_1 \) and \( d_1 \) are the annual average growth rate and natural mortality rate of dogs, respectively, \( \sigma \) is the recovery rate of sick dogs, the incidence rate of dogs is \( \beta_1 S_D(n+1) I_L(n+1) \), \( A_2 \) and \( a_2 \) are the average annual growth rate and natural mortality rate of livestock, respectively, the incidence rate of livestock is \( \beta_2 S_L(n+1) x(n+1) \), \( a \) is the release rate of eggs released by dogs into the environment, \( d \) is the mortality rate of eggs, \( A_3 \) and \( d_3 \) are the annual human growth rate and natural mortality rate, respectively, \( \mu \) is the mortality rate of patients due to sickness, \( \frac{1}{w} \) is the latency of the disease in the human, \( \gamma \) is the recovery rate of the patient, the incidence rate of people who are from susceptible population to latent population through swallowing the eggs is \( \beta_3 S_H(n+1) x(n+1) \). According to the background of epidemiology, we assume that all the parameters in model (1) are positive constants.

3. Basic properties. We first study the positivity and ultimate boundedness of solutions of model (1), we have the following results.

**Theorem 3.1.** For any positive initial value \((S_D(0), I_D(0), S_L(0), I_L(0), x(0), S_H(0), E_H(0), I_H(0))\), model (1) has a unique positive solution \((S_D(n), I_D(n), S_L(n), I_L(n), x(n), S_H(n), E_H(n), I_H(n))\) defined for all \( n = 1, 2, \cdots \), and this solution is also ultimately bounded.

**Proof.** It is clear that \( S_H(n), E_H(n) \) and \( I_H(n) \) do not appear in the first five equations in model (1). Then, we can study model (1) in two parts. Firstly, we consider the sub-model of first five equations of model (1)
The initial values for sub-model (2) are \( S_D(0) > 0, I_D(0) > 0, S_L(0) > 0, I_L(0) > 0 \) and \( x(0) > 0 \).

From sub-model (2), we have

\[
x(n+1) = \frac{1}{1+\frac{d_1}{d_2}} \left[ x(n) + aI_D(n+1) \right],
\]

(3)

\[
S_L(n+1) = \frac{1}{1+\frac{d_1}{d_2} + \frac{\beta_2}{1+\frac{d_1}{d_2}}} \left[ S_L(n) + A_2 \right]
\]

= \frac{1}{1+\frac{d_1}{d_2} + \frac{\beta_2}{1+\frac{d_1}{d_2}}} \left[ x(n) + aI_D(n+1) \right] \left[ S_L(n) + A_2 \right],
\]

(4)

\[
I_L(n+1) = \frac{1}{1+\frac{d_1}{d_2}} \left[ I_L(n) + \beta_2 S_L(n+1)x(n+1) \right]
\]

= \frac{1}{1+\frac{d_1}{d_2}} \left[ I_L(n) + \frac{\beta_2}{1+\frac{d_1}{d_2}} \left[ x(n) + aI_D(n+1) \right] \left[ S_L(n) + A_2 \right] \right],
\]

(5)

and

\[
S_D(n+1) = \frac{1}{1+\frac{d_1}{d_2} + \frac{\beta_1 I_L(n+1)}{1+\frac{d_1}{d_2}}} \left[ S_D(n) + A_1 + \sigma I_D(n+1) \right].
\]

(6)

Obviously, \( x(n+1), S_L(n+1), I_L(n+1) \) and \( S_D(n+1) \) are increasing with respect to \( I_D(n+1) \). Let \( y = I_D(n+1) \) and \( \Phi(y) = y - \frac{1}{1+\frac{d_1}{d_2} + \sigma} [I_D(n) + \beta_1 S_D(n) + \beta_1 I_L(n+1)] \), then one has

\[
\Phi(y) = y - \frac{1}{1+\frac{d_1}{d_2} + \sigma} \left[ I_D(n) + \beta_1 S_D(n) + A_1 + \sigma y \right]
\]

= \frac{1}{1+\frac{d_1}{d_2} + \sigma} \left[ -I_D(n) + (1+\frac{d_1}{d_2} + \frac{\beta_1}{1+\frac{d_1}{d_2}}) y - \beta_1 I_L(n+1) \right]

= \frac{1}{1+\frac{d_1}{d_2} + \sigma} \left[ -I_D(n) + \frac{(1+\frac{d_1}{d_2})y + \beta_1}{1+\frac{d_1}{d_2} + \beta_1} y - \frac{(1+\frac{d_1}{d_2})y + \beta_1}{1+\frac{d_1}{d_2} + \beta_1} \right]

= \frac{1}{1+\frac{d_1}{d_2} + \sigma} \left[ -I_D(n) + \frac{(1+\frac{d_1}{d_2})y + \beta_1}{1+\frac{d_1}{d_2} + \beta_1} y - \frac{(1+\frac{d_1}{d_2})y + \beta_1}{1+\frac{d_1}{d_2} + \beta_1} \right].
By (3)-(5), we get
\[ \frac{I_L(n+1)}{y} = \frac{1}{1 + d_2} \left[ \frac{I_L(n)}{y} + \beta_2 x(n+1) \frac{x(n+1)}{y} \right] \]
\[ = \frac{1}{1 + d_2} \left[ \frac{I_L(n)}{y} + \beta_2 \left( 1 + d_2 + \beta_2 x(n+1) \right) S_L(n) + A_2 \right], \]
and
\[ \frac{(1 + d_2 + \beta_2 x(n+1)) y}{x(n+1)} = \frac{(1 + d_2)(1 + d)y}{x(n) + ay} + \beta_2 y, \]
it shows that \( \frac{y}{I_L(n+1)} \) is increasing with respect to \( y \). Owing to \( I_L(n+1) \) is increasing with respect to \( y \), we further have \( \Phi(y) \) is also increasing with respect to \( y \).

When \( n = 0 \), we have \( y = I_D(1) \). From (3) and (5), one has
\[ \lim_{y \to 0} I_L(1) = \frac{1}{1 + d_2} \left[ I_L(0) + \frac{\beta_2}{1 + d_2 + \beta_2 y} S_L(0) + A_2 \right] : = u(0). \]
It follows that
\[ \lim_{y \to 0} \Phi(y) = - \frac{1}{1 + d_2 + \sigma} \left[ I_D(0) + \beta_1 S_D(0) + A_1 \right] u(0) < 0. \]

On the other hand, we have
\[ \lim_{y \to \infty} I_L(1) = \frac{1}{1 + d_2} \left[ I_L(0) + S_L(0) + A_2 \right]. \]
It follows that
\[ \lim_{y \to \infty} \Phi(y) = \lim_{y \to \infty} \frac{1}{1 + d_2 + \sigma} \left[ - I_D(0) \right. \]
\[ + \frac{(1 + d_2 + \sigma)(1 + d_1) y}{I_L(1)} \left. + (1 + d_2) \beta_1 y - \beta_1 [S_D(0) + A_1] \right] = + \infty. \]
This indicates that equation \( \Phi(y) = 0 \) has a unique solution \( y^* \in (0, +\infty) \). Hence, \( I_D(1) = y^* \) uniquely exists and is positive. From the equations (3)-(6), we further obtain that \( x(1), S_L(1), I_L(1) \) and \( S_D(1) \) also uniquely exist and are positive. Therefore, we obtain the existence, uniqueness and positivity of \( (S_D(1), L_D(1), S_L(1), I_L(1), x(1)) \). By using mathematical induction, we further obtain that \( (S_D(n), L_D(n), S_L(n), I_L(n), x(n)) \) uniquely exists and is positive for all \( n \). Therefore, for any initial value \( S_D(0) > 0, I_D(0) > 0, S_L(0) > 0, I_L(0) > 0 \) and \( x(0) > 0 \), sub-model (2) has a unique positive solution.

Secondly, we consider the sub-model of the last three equations of model (1).
\[ \begin{align*}
S_H(n+1) &= S_H(n) + \frac{A_3 - \beta_3 S_H(n+1) x(n+1) - d_3 S_H(n+1)}{1 + \gamma I_H(n+1)} \\
E_H(n+1) &= E_H(n) + \beta_3 S_H(n+1) x(n+1) - (d_3 + w) E_H(n+1) \\
I_H(n+1) &= I_H(n) + w E_H(n+1) - (d_3 + w + \gamma) I_H(n+1)
\end{align*} \]
(7)
The initial values for sub-model (7) are \( S_H(0) > 0, E_H(0) > 0 \) and \( I_H(0) > 0 \). We can rewrite sub-model (7) in the following form
\[ \begin{align*}
I_H(n+1) &= \frac{1}{1 + d_3 + \mu + \gamma} [I_H(n) + w E_H(n+1)] \\
S_H(n+1) &= \frac{1}{1 + d_3 + \beta_3 x(n+1)} [S_H(n) + A_3 + \gamma I_H(n+1)] \\
E_H(n+1) &= \frac{1}{1 + d_3 + w} [E_H(n) + \beta_3 S_H(n+1) x(n+1)]
\end{align*} \]
(8)
Let $z = E_H(n+1)$ and 
\[ \varphi(z) = z - \frac{1}{1+d_3+w}[E_H(n) + \beta_3S_H(n+1)x(n+1)]. \]

Then from first and second equations of (8) we can further obtain 
\[ \varphi(z) = z - \frac{\beta_3x(n+1)\gamma w}{(1+d_3+w)(1+d_3+\beta_3x(n+1))(1+d_3+\mu+\gamma)}z - \tilde{C}(n), \]
where 
\[ \tilde{C}(n) = \frac{1}{1+d_3+w}[E_H(n) + \frac{\beta_3x(n+1)}{1+d_3+\beta_3x(n+1)}(S_H(n) + A_3 + \frac{\gamma I_H(n)}{1+d_3+\mu+\gamma})]. \]

It is clear that $\varphi(z)$ for $z \in (0, +\infty)$ is increasing. When $n = 0$, we have $\varphi(0) = -\tilde{C}(0) < 0$ and 
\[ \lim_{z \to \infty} \varphi(z) = \lim_{z \to \infty} [z - \frac{\beta_3x(1)\gamma w}{(1+d_3+w)(1+d_3+\beta_3x(1))(1+d_3+\mu+\gamma)}z - \tilde{C}(0)] = \infty. \]

Hence, equation $\varphi(z) = 0$ has a unique solution $z^* \in (0, +\infty)$. This implies that $E_H(1)$ uniquely exists and is positive. Furthermore, from the first and second equations of (8) we can obtain that $I_H(1)$ and $S_H(1)$ also uniquely exist and are positive, respectively. Thus, $(S_H(1), E_H(1), I_H(1))$ uniquely exists and is positive. The existence, uniqueness and positivity of $(S_H(n), E_H(n), I_H(n))$ for all $n$ can be obtained by using the mathematical induction. Therefore, for any initial value $S_H(0) > 0$, $E_H(0) > 0$ and $I_H(0) > 0$, sub-model (7) has a unique positive solution.

Next, we prove the ultimate boundedness of model (1). Let $N_D(n) = S_D(n) + I_D(n)$, $N_L(n) = S_L(n) + I_L(n)$ and $N_H(n) = S_H(n) + E_H(n) + I_H(n)$, by using the iteration method, one has 
\[ N_D(n+1) = \frac{1}{1+d_2}(A_2 + N_D(n)) \leq \frac{A_1}{d_1} + \frac{N_D(0)}{(1+d_1)^{n+1}}. \]

Hence 
\[ \limsup_{n \to \infty} [S_D(n) + I_D(n)] \leq \frac{A_1}{d_1}. \]

By the same way, we get 
\[ N_L(n+1) = \frac{1}{1+d_2}(A_2 + N_L(n)) \leq \frac{A_2}{d_2}(1 - \frac{1}{(1+d_2)^{n+1}}) + \frac{N_L(0)}{(1+d_2)^{n+1}}, \]
\[ x(n+1) = \frac{1}{1+d_1}[x(n) + A_1] \leq \frac{aA_1}{d_1}(1 - \frac{1}{(1+d_1)^{n+1}}) + \frac{x(0)}{(1+d_1)^{n+1}} \]
and 
\[ N_H(n+1) = \frac{1}{1+d_3}[N_H(n) + A_3] \leq \frac{A_3}{d_3}(1 - \frac{1}{(1+d_3)^{n+1}}) + \frac{N_H(0)}{(1+d_3)^{n+1}}. \]

Hence, we can obtain 
\[ \limsup_{n \to \infty} [S_L(n) + I_L(n)] \leq \frac{A_2}{d_2}, \quad \limsup_{n \to \infty} x(n) \leq \frac{aA_1}{d_1}, \]
\[ \limsup_{n \to \infty} [S_H(n) + E_H(n) + I_H(n)] \leq \frac{A_3}{d_3}. \]

This shows that all nonnegative solutions of model (1) are ultimately bounded. This completes the proof. \qed
Now, we define the basic reproduction number of model (1) as follows

$$R_0 = \frac{aA_1A_2\beta_1\beta_2}{(d_1 + \sigma)d_2^2d_1}.$$  

On the existence of equilibrium of model (1), we have the following result.

**Theorem 3.2.** (1) If $R_0 \leq 1$, then model (1) has a unique disease-free equilibrium $E_0 = (\frac{A_1}{d_1}, 0, \frac{A_2}{d_2}, 0, 0, \frac{A_3}{d_3}, 0, 0, 0, 0)$.

(2) If $R_0 > 1$, then model (1) has a unique endemic equilibrium $E^* = (S^*_D, I^*_D, S^*_L, I^*_L, x^*, S^*_H, E^*_H, I^*_H)$, except for $E_0$.

**Proof.** Obviously, equilibrium $(S_D, I_D, S_L, I_L, x, S_H, E_H, I_H)$ of model (1) satisfies the following equations

$$
\begin{cases}
A_1 - \beta_1S_D I_L - d_1S_D + \sigma I_D = 0, \\
A_2 - \beta_2S_L x - d_2S_L = 0, \\
A_I_D - dx = 0, \\
\beta_3S_H x - (d_3 + w)E_H = 0,
\end{cases}
$$

(9)

Firstly, equation (9) has solution $E_0 = (\frac{A_1}{d_1}, 0, \frac{A_2}{d_2}, 0, 0, \frac{A_3}{d_3}, 0, 0, 0)$, it is the disease-free equilibrium of model (1). From the first five equations of (9) we can obtain

$$S_D = \frac{A_1}{d_1} - I_D, \quad S_L = \frac{A_2}{d_2} - I_L, \quad x = \frac{a}{d} I_D. \quad (10)$$

Further we have

$$\beta_1(\frac{A_1}{d_1} - I_D)I_L - (d_1 + \sigma)I_D = 0, \quad \beta_2(\frac{A_2}{d_2} - I_L)\frac{a}{d}I_D - d_2I_L = 0. \quad (11)$$

This implies that

$$I_L = \frac{\beta_2 a(d_1 + \sigma) + \beta_1 \beta_2 \frac{A_2}{d_2} a}{\beta_1 \beta_2 \frac{A_1}{d_1} a + \beta_1 d_2 d} I_D. \quad (12)$$

Substituting (12) into the first equation of (11), we finally get

$$I_D = \frac{(d_1 + \sigma)d_2 d}{a \beta_2 (d_1 d_2 + d_2 \sigma + \beta_1 A_2)} (R_0 - 1). \quad (13)$$

From the last three equations of (9) we can obtain

$$E_H = \frac{d_3 + \mu + \gamma}{w} I_H, \quad S_H = \frac{(d_3 + w)(d_3 + \mu + \gamma)}{w \beta_3 x} I_H,$$

$$I_H = \frac{w \beta_3 x A_3}{(\beta_3 x + d_3)(d_3 + w)(d_3 + \mu + \gamma) - w \beta_3 \gamma x}. \quad (14)$$

It is clear that $(\beta_3 x + d_3)(d_3 + w)(d_3 + \mu + \gamma) - w \beta_3 \gamma x > 0.$
From (10)-(14), we finally obtain that when \( R_0 > 1 \) model (1) has a unique endemic equilibrium \( E^* = (S^*_D, I^*_D, S^*_L, I^*_L, x^*, S^*_H, E^*_H, I^*_H) \), where

\[
S^*_D = \frac{d_2(d_1 + \sigma)(A_1\beta_2a + d_1d_2d)}{\alpha\beta_2d_1(A_2\beta_1 + d_1d_2 + d_2\sigma)}, \quad I^*_D = \frac{(d_1 + \sigma)d_1^2d}{\alpha\beta_2(d_1d_2 + d_2\sigma + \beta_1A_2)}(R_0 - 1),
\]

\[
S^*_L = \frac{d\beta_1(A_1\beta_2 + d_1d_2\sigma)}{\beta_1(aA_1\beta_2 + dd_1d_2)}, \quad I^*_L = \frac{(d_1 + \sigma)d_2dd_1}{\beta_1(A_1\beta_2a + d_1d_2d)}(R_0 - 1),
\]

\[
x^* = \frac{(d_1 + \sigma)d_2}{\beta_2(d_1d_2 + d_2\sigma + \beta_1A_2)}(R_0 - 1), \quad S^*_H = \frac{(d_3 + w)(d_3 + \mu + \gamma)}{w\beta_3x^*}I^*_H,
\]

\[
E^*_H = \frac{d_3 + \mu + \gamma}{w}I^*_H, \quad I^*_H = \frac{w\beta_3x^*A_3}{(\beta_3x^* + d_3 + w)(d_3 + \mu + \gamma)} - w\beta_3\gamma x^*.
\]

This completes the proof. \( \square \)

4. Stability of a disease-free equilibrium. On the local stability of disease-free equilibrium \( E_0 \) of model (1), we have the following results.

**Theorem 4.1.** (1) If \( R_0 < 1 \), then disease-free equilibrium \( E_0 \) of model (1) is locally asymptotically stable.

(2) If \( R_0 > 1 \), then \( E_0 \) is unstable.

**Proof.** Firstly, we investigate sub-model (2). When \( R_0 < 1 \), it has a unique disease-free equilibrium \( \left( \frac{4A_1}{d_1}, 0, \frac{4A_2}{d_2}, 0, 0 \right) \). Taking the transformation for sub-model (2): \( U_n = S_D(n) - \frac{4A_1}{d_1}, \quad V_n = I_D(n), \quad W_n = S_L(n) - \frac{4A_2}{d_2}, \quad Y_n = I_L(n) \) and \( Z_n = x(n) \), we can obtain the linearization system as follows

\[
\begin{align*}
U_{n+1} &= U_n - \beta_1 \frac{A_1}{d_1} Y_{n+1} - d_1 U_{n+1} + \sigma V_{n+1} \\
V_{n+1} &= V_n + \beta_1 \frac{A_1}{d_1} Y_{n+1} - (d_1 + \sigma)V_{n+1} \\
W_{n+1} &= W_n - \beta_2 \frac{A_2}{d_2} Z_{n+1} - d_2 W_{n+1} \\
Y_{n+1} &= Y_n + \beta_2 \frac{A_2}{d_2} Z_{n+1} - d_2 Y_{n+1} \\
Z_{n+1} &= Z_n + a V_{n+1} - d Z_{n+1}
\end{align*}
\]

(15)

Define matrix \( A \) as follows

\[
A = \begin{pmatrix}
1 + d_1 & -\sigma & 0 & \beta_1 \frac{A_1}{d_1} & 0 \\
0 & 1 + \sigma + d_1 & -\beta_1 \frac{A_1}{d_1} & 0 & 0 \\
0 & 0 & 1 + d_2 & 0 & \beta_2 \frac{A_2}{d_2} \\
0 & 0 & 0 & 1 + d_2 & -\beta_2 \frac{A_2}{d_2} \\
0 & -a & 0 & 0 & 1 + d_1
\end{pmatrix},
\]

then system (15) becomes into the following form

\[
\begin{pmatrix}
U_{n+1} \\
V_{n+1} \\
W_{n+1} \\
Y_{n+1} \\
Z_{n+1}
\end{pmatrix} = A^{-1}\begin{pmatrix}
U_n \\
V_n \\
W_n \\
Y_n \\
Z_n
\end{pmatrix}.
\]

The characteristic equation of matrix \(-A\) is

\[
F(\lambda) = |\lambda E + A| = (1 + d_1 + \lambda)(1 + d_2 + \lambda)f(\lambda),
\]
where
\[ f(\lambda) = (1 + \sigma + d_1 + \lambda)(1 + d_2 + \lambda)(1 + d + \lambda) - a\beta_1\beta_2 \frac{A_1 A_2}{d_1 d_2}. \]
Let \( r = \lambda + 1 \) and \( g(r) = f(r - 1) \), then we have
\[ g(r) = r^3 + mr^2 + nr + p, \]
where \( m = d_2 + d + \sigma + d_1 \), \( n = d_2 d + (\sigma + d_1)(d_2 + d) \) and \( p = (\sigma + d_1)d_2 d - a\beta_1\beta_2 \frac{A_1 A_2}{d_1 d_2} \).

When \( R_0 < 1 \), we further have \( m > 0, n > 0, p > 0 \), and
\[
m - p = (d_2 + d + \sigma + d_1)(d_2 d + (\sigma + d_1)(d_2 + d)) - ((\sigma + d_1)d_2 d - a\beta_1\beta_2 \frac{A_1 A_2}{d_1 d_2}) \]
\[
= (d_2 + d)d_2 d + (d_2 + d + \sigma + d_1)(\sigma + d_1)(d_2 + d) + a\beta_1\beta_2 \frac{A_1 A_2}{d_1 d_2} > 0.
\]

Hence, all roots \( r^* \) of equation \( g(r) = 0 \) have negative real parts. Let \( \lambda^* = r^* - 1 \), then matrix \(-A\) has characteristic roots \( \lambda^* \). Obviously, \( |\lambda^*| > 1 \). This implies that the modules of all characteristic roots for matrix \( A^{-1} \) are less than 1. Therefore, equilibrium \((\frac{A_1}{d_1}, 0, \frac{A_2}{d_2}, 0, 0)\) is locally asymptotically stable, if \( R_0 < 1 \).

When \( R_0 > 1 \), we prove that equilibrium \((\frac{A_1}{d_1}, 0, \frac{A_2}{d_2}, 0, 0)\) is unstable. Suppose that it is stable, then it is also the stable equilibrium for the linearization system (15). By the definition of stability of equilibrium, for any constant \( \varepsilon > 0 \) small enough, there exists a \( \delta > 0 \) such that for any initial point \((U_0, V_0, W_0, Y_0, Z_0)\)

satisfying
\[ |U_0 - \frac{A_1}{d_1}| + |V_0 - \frac{A_2}{d_2}| + Y_0 + Z_0 < \delta, \]
then we have
\[ |U_n - \frac{A_1}{d_1}| + |V_n - \frac{A_2}{d_2}| + Y_n + Z_n < \varepsilon, \]
for any \( n \geq 1 \). Notice that \( V_n = I_D(n), Y_n = I_L(n) \) and \( Z_n = x(n) \), we also have \( V_n \geq 0, Y_n \geq 0 \) and \( Z_n \geq 0 \) for any \( n \geq 0 \).

Choosing the Lyapunov function
\[ W_n = C_1 V_n + C_2 Y_n + C_3 Z_n, \]
where \( C_1, C_2 \) and \( C_3 \) are positive constants which will be determined below. Calculating \( \Delta = W_{n+1} - W_n \), we have
\[ \Delta = C_1(V_{n+1} - V_n) + C_2(Y_{n+1} - Y_n) + C_3(Z_{n+1} - Z_n) \]
\[ = (C_1\beta_1 \frac{A_1}{d_1} - C_2d_2)Y_{n+1} + (C_3a - C_1(d_1 + \sigma))V_{n+1} + (C_2\beta_2 \frac{A_2}{d_2} - C_3d)Z_{n+1}. \]

If \( R_0 > 1 \), namely \( \frac{aA_1A_2\beta_1\beta_2}{(d_1 + \sigma)d_2d_1} > 1 \). Choosing \( C_2 = 1 \) and \( C_3 < \frac{\beta_3 A_2}{d_2 d_1} \) such that \( \frac{\beta_3 A_2}{d_2 d_1} C_3 > 1 \). Then we further choose \( C_1 > \frac{d_2 A_1}{\beta_1 A_1} \) such that \( aC_3 > (d_1 + \sigma)C_1 \). This shows that we can choose the positive constants \( C_1, C_2 \) and \( C_3 \) such that
\[ C_1\beta_1 \frac{A_1}{d_1} > C_2d_2, C_3a > C_1(d_1 + \sigma), C_2\beta_2 \frac{A_2}{d_2} > C_3d. \]

For the constants \( C_1, C_2 \) and \( C_3 \), we obtain
\[ \Delta \geq mW_{n+1}, \ n = 0, 1, 2, \cdots, \]
where
\[ 0 \leq m < \min\{1, \frac{C_1\beta_1 \frac{A_1}{d_1} - C_2d_2}{C_2}, \frac{C_3a - C_1(d_1 + \sigma)}{C_1}, \frac{C_2\beta_2 \frac{A_2}{d_2} - C_3d}{C_3} \}. \]
Thus, for all \( n \geq 0 \) we can obtain
\[
W_n \geq (\frac{1}{1-m})^n W_0.
\]
We have \( \lim_{n \to \infty} W_n = \infty \) when \( W_0 > 0 \), which leads to a contradiction with the boundedness of the \( W_n \). Therefore, equilibrium \((\frac{A_1}{A_3}, 0, \frac{A_2}{A_3}, 0, 0)\) is unstable. It follows that disease-free equilibrium \( E_0 \) of model (1) is unstable if \( R_0 > 1 \).

Next, we consider sub-model (7). When \( R_0 < 1 \), it has only unique disease-free equilibrium \((\frac{A_1}{A_3}, 0, 0)\). The linearization of sub-model (7) is given by
\[
\begin{align*}
O_{n+1} &= O_n - \beta_3 \frac{A_3}{A_5} Z_{n+1} - d_3 O_{n+1} + \gamma Q_{n+1} \\
P_{n+1} &= P_n + \beta_3 \frac{A_3}{A_5} Z_{n+1} - (d_3 + w) P_{n+1} \\
Q_{n+1} &= Q_n + w P_{n+1} - (d_3 + \mu + \gamma) Q_{n+1}
\end{align*}
\]
(16)

Obviously, system (16) can be rewrote in the following form
\[
\begin{pmatrix}
O_{n+1} \\
P_{n+1} \\
Q_{n+1}
\end{pmatrix} = B^{-1} \begin{pmatrix}
Q_n \\
P_n \\
Q_n
\end{pmatrix} + B^{-1} \begin{pmatrix}
-\beta_3 \frac{A_3}{A_5} Z_{n+1} \\
\beta_3 \frac{A_3}{A_5} Z_{n+1}
\end{pmatrix},
\]
where
\[
B = \begin{pmatrix}
1 + d_3 & 0 & -\gamma \\
0 & 1 + w + d_3 & 0 \\
0 & -w & 1 + d_3 + \mu + \gamma
\end{pmatrix}
\]

From the discussion of above, we know that \( \lim_{n \to \infty} Z_n = 0 \) if \( R_0 < 1 \). Therefore,
\[
\lim_{n \to \infty} \begin{pmatrix}
-\beta_3 \frac{A_3}{A_5} Z_{n+1} \\
\beta_3 \frac{A_3}{A_5} Z_{n+1}
\end{pmatrix} = 0.
\]
The characteristic equation of matrix \( B \) is:
\[
K(\lambda) = (1 + d_3 - \lambda)(1 + w + d_3 - \lambda)(1 + d_3 + \mu + \gamma - \lambda) = 0,
\]
which has the characteristic roots \( \lambda_1 = 1 + d_3,  \lambda_2 = 1 + w + d_3 \) and \( \lambda_3 = 1 + d_3 + \mu + \gamma \). Obviously, \( |\lambda_i| > 1 \) for \( i = 1, 2, 3 \). Hence, the modules of all characteristic roots of \( B^{-1} \) are less than 1. This implies that disease-free equilibrium \((\frac{A_1}{A_3}, 0, 0)\) is locally asymptotically stable. Thus, we finally obtain that disease-free equilibrium \( E_0 \) of model (1) is locally stable if \( R_0 < 1 \). This completes the proof.

\[\square\]

**Theorem 4.2.** If \( R_0 \leq 1 \), then disease-free equilibrium \( E_0 \) of model (1) is globally asymptotically stable.

**Proof.** Firstly, from the ultimate boundedness of solutions of model (1), without loss of generality, we can assume \( S_D(n) + I_D(n) \leq \frac{A_3}{d_1}, S_L(n) + I_L(n) \leq \frac{A_2}{d_2} \) and \( S_H(n) + E_H(n) + I_H(n) \leq \frac{A_1}{d_1} \) for any \( n \geq 1 \).

Consider sub-model (2). It can be rewrote as the following form
\[
\begin{align*}
S_D(n+1) &= S_D(n) - \beta_1 S_D(n+1) I_L(n+1) - d_1 (S_D(n+1) - \frac{A_1}{d_1}) + \sigma I_D(n+1) \\
I_D(n+1) &= I_D(n) + \beta_1 S_D(n+1) I_L(n+1) - (d_1 + \sigma) I_D(n+1) \\
S_L(n+1) &= S_L(n) - \beta_2 S_L(n+1) x(n+1) - d_2 (S_L(n+1) - \frac{A_2}{d_2}) \\
I_L(n+1) &= I_L(n) + \beta_2 S_L(n+1) x(n+1) - d_2 I_L(n+1) \\
x(n+1) &= x(n) + a I_D(n+1) - d x(n+1)
\end{align*}
\]
(17)
Define the Lyapunov function

\[ W_n = C_1 I_D(n) + C_2 I_L(n) + C_3 x(n), \]

where \( C_1, C_2 \) and \( C_3 \) are positive constants which will be determined below. Calculating \( \Delta = W_{n+1} - W_n \), we obtain

\[
\Delta = C_1 (I_D(n+1) - I_D(n)) + C_2 (I_L(n+1) - I_L(n)) + C_3 (x(n+1) - x(n))
\]

\[
= C_1 (\beta_1 S_D(n+1) I_L(n+1) - (d_1 + \sigma) I_D(n+1)) + C_2 (\beta_2 S_L(n+1) x(n+1)
\]

\[
- d_2 I_L(n+1)) + C_3 (a I_D(n+1) - dx(n+1))
\]

\[
= (C_1 \beta_1 S_D(n+1) - C_2 d_2 I_L(n+1) + (C_3 a - C_1 (d_1 + \sigma)) I_D(n+1)
\]

\[
+ (C_2 \beta_2 S_L(n+1) - C_3 d) x(n+1)
\]

\[
\leq (C_1 \beta_1 \frac{A_1}{d_1} - C_2 d_2) I_L(n+1) + (C_3 a - C_1 (d_1 + \sigma)) I_D(n+1)
\]

\[
+ (C_2 \beta_2 \frac{A_2}{d_2} - C_3 d) x(n+1).
\]

If \( R_0 < 1 \), we have \( \frac{A_1 A_2 \beta_1}{d_1 d_2} \frac{\beta_2}{\sigma d_1} < 1 \). Choosing firstly \( C_2 = 1 \) and \( C_3 > \frac{\beta_2 A_2}{d_2} \) such that \( \frac{\beta_1}{\sigma d_1} A_1 < 1 \). Further, we choose \( C_1 < \frac{d_2 d_1}{\beta_1 A_1} \) such that \( C_3 < \frac{d_1 + \sigma}{\sigma d_1} C_1 \). Therefore, we obtain the following inequalities

\[
C_1 \beta_1 \frac{A_1}{d_1} < C_2 d_2, \ C_3 a < C_1 (d_1 + \sigma), \ C_2 \beta_2 \frac{A_2}{d_2} < C_3 d.
\]

Thus, we have \( \Delta \leq 0 \) for any \( S_D(n+1) \geq 0, I_D(n+1) \geq 0, S_L(n+1) \geq 0, I_L(n+1) \geq 0 \) and \( x(n+1) \geq 0 \). If \( \Delta = 0 \), we get \( I_D(n+1) = 0, I_L(n+1) = 0 \) and \( x(n+1) = 0 \) for any \( n \geq 1 \). Then, from system (17), we further get \( S_D(n+1) = \frac{A_1}{d_1} \) and \( S_L(n+1) = \frac{A_2}{d_2} \) for \( n \geq 1 \).

If \( R_0 = 1 \), we have \( \frac{A_1 A_2 \beta_1}{d_1 d_2} \frac{\beta_2}{\sigma d_1} = 1 \). Choosing \( C_1 = \frac{d_2 d_1}{\beta_1 A_1} \), \( C_2 = 1 \) and \( C_3 = \frac{\beta_2 A_2}{d_2} \), then

\[
C_1 \beta_1 \frac{A_1}{d_1} = C_2 d_2, \ C_3 a = C_1 (d_1 + \sigma), \ C_2 \beta_2 \frac{A_2}{d_2} = C_3 d.
\]

We further obtain

\[
\Delta = C_1 (\beta_1 S_D(n+1) - \beta_1 \frac{A_1}{d_1}) I_L(n+1) + C_2 (\beta_2 S_L(n+1) - \beta_2 \frac{A_2}{d_2}) x(n+1) \leq 0,
\]

for any \( S_D(n+1) \geq 0, I_D(n+1) \geq 0, S_L(n+1) \geq 0, I_L(n+1) \geq 0, x(n+1) \geq 0 \). When \( \Delta = 0 \), we have \( S_D(n+1) = \frac{A_1}{d_1} \) or \( I_L(n+1) = 0 \) and \( S_L(n+1) = \frac{A_2}{d_2} \) or \( x(n+1) = 0 \) for all \( n \geq 1 \). Then from (17) we can further obtain \( S_D(n+1) = \frac{A_1}{d_1}, I_D(n+1) = 0, S_L(n+1) = \frac{A_2}{d_2}, I_L(n+1) = 0 \) and \( x(n+1) = 0 \) for any \( n \geq 1 \).

Therefore, by the LaSalle’s invariance principle of the stability of discrete dynamical systems (see [20]), we finally obtain that disease-free equilibrium \((\frac{A_1}{d_1}, 0, \frac{A_2}{d_2}, 0, 0)\) of sub-model (2) is globally attractive.

Secondly, consider sub-model (10). We define Lyapunov function

\[ V_n = E_H(n) + I_H(n), \]
5. Permanence of model.

We obtain for any $n \geq 0$
\[ V_{n+1} - V_n = E_H(n+1) - E_H(n) + I_H(n+1) - I_H(n) \]
\[ = (\beta_3 S_H(n+1)x(n+1) - (d_3 + w) E_H(n+1)) \]
\[ + (wE_H(n+1) - (d_3 + \mu + \gamma)I_H(n+1)) \]
\[ < \beta_3 \frac{A_3}{d_3} x(n+1) - d_3 E_H(n+1) - d_3 I_H(n+1) \]
\[ = \beta_3 \frac{A_3}{d_3} x(n+1) - d_3 V_{n+1}. \]

Hence, $(1 + d_3)V_{n+1} < V_n + \beta_3 \frac{A_3}{d_3} x(n+1)$ for any $n \geq 0$. When $R_0 \leq 1$, from the above discussion we have $\lim_{n \to \infty} x(n) = 0$. Hence, for any constant $\varepsilon > 0$ there exists a positive integer $N$ such that $x(n) < \varepsilon$ for any $n \geq N$. Thus, when $n \geq N$ we have $V_{n+1} < \frac{1}{1 + d_3} (V_n + \beta_3 \frac{A_3}{d_3} \varepsilon)$. It follows that
\[ V_n < \left( \frac{1}{1 + d_3} \right)^n V_N - \beta_3 \frac{A_3}{d_3} \varepsilon \left( \frac{1}{1 + d_3} \right)^n + \beta_3 \frac{A_3}{d_3} \varepsilon. \]

Hence, $\limsup_{n \to \infty} V_n \leq \beta_3 \frac{A_3}{d_3} \varepsilon$. From the arbitrariness of $\varepsilon$, we further have $\lim_{n \to \infty} V_n = 0$. This shows that $\lim_{n \to \infty} E_H(n) = 0$, $\lim_{n \to \infty} I_H(n) = 0$ and $\lim_{n \to \infty} S_H(n) = \frac{A_3}{\mu}$. Therefore, disease-free equilibrium $(\frac{A_3}{\mu}, 0, 0)$ of sub-model (2) is globally attractive, if $R_0 \leq 1$.

Summarize the above discussion, we finally get that disease-free equilibrium $E_0$ of model (1) is globally attractive. According to Theorem 4.1, we know that $E_0$ is also globally asymptotically stable. This completes the proof.

\[ \square \]

\textbf{Remark 1.} When $R_0 > 1$, from the above discussions we only obtain that model (1) has a unique endemic equilibrium $E^*$, and $E^*$ is also stable. An evident problem is whether equilibrium $E^*$ also is globally asymptotically stable. Unfortunately, in this paper we do not get it. However, it is gratified that we can obtain the permanence of model (1) when $R_0 > 1$, which will be discussed in next section.

\textbf{5. Permanence of model.}

\textbf{Theorem 5.1.} If $R_0 > 1$, then model (1) is permanent. That is, there exists a constant $m > 0$ such that for any positive solution $(S_D(n), I_D(n), S_L(n), I_L(n), x(n), S_H(n), E_H(n), I_H(n))$ of model (1),
\[ \liminf_{n \to \infty} \min \{S_D(n), I_D(n), S_L(n), I_L(n), x(n), S_H(n), E_H(n), I_H(n)\} \geq m. \]

\textbf{Proof.} We firstly study the permanence of sub-model (2). Define the sets as follows
\[ X = \left\{ (S_L, I_L, S_D, I_D, x) : S_L > 0, I_L \geq 0, S_D > 0, I_D \geq 0, x \geq 0 \right\}, \]
\[ X_0 = \left\{ (S_L, I_L, S_D, I_D, x) \in X : I_L > 0, I_D > 0, x > 0 \right\}, \]
\[ \partial X_0 = \left\{ (S_L, I_L, S_D, I_D, x) \in X : I_L I_D x = 1 \right\}. \]

Let $P(n) = (S_L(n), I_L(n), S_D(n), I_D(n), x(n))$ be the solution of sub-model (2) with initial value $P(0) = (S_L, I_L, S_D, I_D, x)$. Define
\[ M_3 = \{ (S_D, I_D, S_L, I_L, x) \in \partial X_0 : P(n) \in \partial X_0, n = 0, 1, 2, \cdots \}. \]

Since the solution of sub-model (2) with initial condition $P(0) = (S_L, 0, S_D, 0, 0)$ is $(S_L(n), 0, S_D(n), 0, 0)$, we have $\{(S_L, 0, S_D, 0, 0) : S_D > 0, S_L > 0\} \subset M_3$. 

Therefore, model (1) is permanent.
Suppose that there exists \((S_L, I_L, S_D, I_D, x) \in M_\beta\) such that 
\[(S_L, I_L, S_D, I_D, x) \notin \{(S_L, 0, S_D, 0, 0) : S_L > 0, S_D > 0\}.
\]
Then we have \(I_L > 0\) or \(I_D > 0\) or \(x > 0\).

Consider solution \(P(n) = (S_L(n), I_L(n), S_D(n), I_D(n), x(n))\) of sub-model (2) with initial value \(P(0) = (S_L, I_L, S_D, I_D, x)\). Let \(I_L(0) = I_L > 0\). Owing to 
\[I_L(n + 1) \geq I_L(n) - d_3I_L(n + 1),\]
we obtain \(I_L(n) \geq (\frac{1}{1 + d_2})^nI_L(0) > 0\) for any \(n \geq 1\). Owing to 
\[S_D(n + 1) > S_D(n) - (\beta_1I_L(n + 1) + d_1)S_D(n + 1),\]
we get \(S_D(n) > \prod_{i=1}^{n} \frac{1}{1 + d_i + \beta_1S_D(i)}S_D(0) \geq 0\) for any \(n \geq 1\). Since 
\[I_D(n + 1) > I_D(n) - (d_1 + \sigma)I_D(n + 1),\]
we obtain \(I_D(n) > (\frac{1}{1 + d_1 + \sigma})^nI_D(0) \geq 0\) for any \(n \geq 1\). Lastly, since 
\[x(n + 1) > x(n) - dx(n + 1),\]
we get \(x(n) > (\frac{1}{1 + d})^nx(0) \geq 0\) for any \(n \geq 1\). So, we finally get \(I_D(n) > 0, I_L(n) > 0\) and \(x(n) > 0\) for any \(n \geq 1\). Similarly, when \(I_D(0) = I_D > 0\) or \(x(0) = x > 0\), we can also get \(I_D(n) > 0, I_L(n) > 0\) and \(x(n) > 0\) for any \(n \geq 1\). This indicates \(P(n) = (S_L(n), I_L(n), S_D(n), I_D(n), x(n)) \notin \partial X_0\) for any \(n \geq 1\), which leads to a contradiction. Thus, we have 
\[M_\beta \subset \{(S_L, 0, S_D, 0, 0) : S_L > 0, S_D > 0\},\]
and hence, 
\[M_\beta = \{(S_L, 0, S_D, 0, 0) : S_L > 0, S_D > 0\}.
\]

It is clear that in set \(M_\beta\) sub-model (2) has a globally attractive equilibrium \(P_0(\frac{A_1}{d_1}, 0, \frac{A_2}{d_2}, 0, 0)\). Hence, \(P_0\) is also isolated invariable and acyclic in \(M_\beta\).

Now we prove that \(W^*(P_0) \cap X_0 = \emptyset\), where \(W^*(P_0)\) is the stable set of equilibrium \(P_0\), which is defined as 
\[W^*(P_0) = \{(S_L(0), I_L(0), S_D(0), I_D(0), x(0)) : \lim_{n \to \infty} (S_L(n), I_L(n), S_D(n), I_D(n), x(n)) = P_0\}.
\]
Assume that there is a point \((S_L(0), I_L(0), S_D(0), I_D(0), x(0)) \in X_0\) such that 
\[\lim_{n \to \infty} (S_L(n), I_L(n), S_D(n), I_D(n), x(n)) = P_0.
\]
Consider the Lyapunov function 
\[W_n = C_1I_D(n) + C_2I_L(n) + C_3x(n),\]
where \(C_1, C_2\) and \(C_3\) are positive constants which will be determined below. Calculating \(\Delta = W_{n+1} - W_n\), we obtain 
\[\Delta = C_1(I_D(n + 1) - I_D(n)) + C_2(I_L(n + 1) - I_L(n)) + C_3(x(n + 1) - x(n))\]
\[= C_1(\beta_1S_D(n + 1)I_L(n + 1) - (d_1 + \sigma)I_D(n + 1)) + C_2(\beta_2S_L(n + 1)x(n + 1) - d_2I_L(n + 1)) + C_3(aI_D(n + 1) - dx(n + 1))\]
\[= (C_1\beta_1S_D(n + 1) - C_2d_2)I_L(n + 1) + (C_3a - C_1(d_1 + \sigma))I_D(n + 1) + (C_2\beta_2S_L(n + 1) - C_3d)x(n + 1).
\]
For any $\varepsilon > 0$, there exists $N$, for all $n > N$ we have $S_L(n) > \frac{A_2}{d_2} - \varepsilon$ and $S_D(n) > \frac{A_1}{d_1} - \varepsilon$. Hence,

$$
\Delta > (C_1\beta_1(\frac{A_1}{d_1} - \varepsilon) - C_2d(\varepsilon))I_L(n + 1) + (C_3a - C_1(d_1 + \sigma))I_D(n + 1)
$$

$$
+ (C_2\beta_2(\frac{A_2}{d_2} - \varepsilon) - C_3d)x(n + 1).
$$

When $R_0 > 1$, according to the proof of Theorem 4.1, there are positive constants $C_1$, $C_2$ and $C_3$ such that

$$
C_1\beta_1(\frac{A_1}{d_1}) > C_2d, \quad C_3a > C_1(d_1 + \sigma), \quad C_2\beta_2(\frac{A_2}{d_2}) > C_3d.
$$

Choosing $\varepsilon > 0$ small enough such that

$$
C_1\beta_1(\frac{A_1}{d_1} - \varepsilon) - C_2d > 0, \quad C_3a - C_1(d_1 + \sigma) > 0, \quad C_2\beta_2(\frac{A_2}{d_2} - \varepsilon) - C_3d > 0.
$$

Thus, we can obtain $\Delta \geq mW(n)$ for any $n \geq N$, where

$$
0 < m < \min\{\frac{C_1\beta_1(\frac{A_1}{d_1} - \varepsilon) - C_2d}{C_2}, \frac{C_3a - C_1(d_1 + \sigma)}{C_1}, \frac{C_2\beta_2(\frac{A_2}{d_2} - \varepsilon) - C_3d}{C_3}\}.
$$

Hence, we further get for all $n \geq N$

$$
W_n \geq (\frac{1}{1 - m})^{n-N}W_N.
$$

By the positivity of the solution, we have $W_N > 0$, it follows that $\lim_{n \to \infty} W_n = \infty$, which leads a contradiction with the boundedness of $W_n$. Hence, $W^*(P_0) \cap X_0 = \emptyset$.

From the persistence theory of dynamical systems (see [41]), we finally obtain that sub-model (2) is permanent.

Secondly, we consider sub-model (7). From the above discussion, we know that $x(n)$ is permanent. Without loss of generality, assume $\bar{x} \leq x(n) \leq \tilde{x}$ for all $n \geq 0$, where $\tilde{x}$ and $\bar{x}$ are two positive constants. From the first equation of sub-model (7), we obtain

$$
S_H(n + 1) = \frac{1}{1 + d_3 + \beta_3\bar{x}(n + 1)}[S_H(n) + A_3 + \gamma I_H(n + 1)]
$$

$$
\geq \frac{1}{1 + d_3 + \beta_3\bar{x}}[S_H(n) + A_3] \geq \frac{A_3}{1 + d_3 + \beta_3\bar{x}} \triangleq \bar{S}_H.
$$

From the second equation of sub-model (7), we further obtain

$$
E_H(n + 1) = \frac{1}{1 + d_3 + w}[E_H(n) + \beta_3S_H(n + 1)x(n + 1)]
$$

$$
\geq \frac{1}{1 + d_3 + w}\beta_3\bar{S}_H\tilde{x} \triangleq \bar{E}_H
$$

and

$$
E_H(n + 1) = \frac{1}{1 + d_3 + w}[E_H(n) + \beta_3S_H(n + 1)x(n + 1)]
$$

$$
\leq \frac{1}{1 + d_3 + w}[E_H(n) + \beta_3A_3\bar{x}].
$$

It follows that

$$
E_H(n) \leq (\frac{1}{1 + d_3 + \omega})^nE_H(0) + [(\frac{1}{1 + d_3 + \omega})^n + \cdots + \frac{1}{1 + d_3 + \omega}]\beta_3A_3\bar{x}.
$$
Hence, we have
\[ \limsup_{n \to \infty} E_H(n) \leq \frac{\beta_3 A_3 \bar{E}}{d_3(d_3 + \omega)} \equiv \bar{E}_H. \]

Without loss of generality, we assume that \( E_H(n) \leq \bar{E}_H \) for any \( n \geq 1 \). From the third equation of sub-model (7)
\[ I_H(n + 1) = \frac{1}{1 + d_3 + \mu + \gamma}[I_H(n) + wE_H(n + 1)] \geq \frac{1}{1 + d_3 + \mu + \gamma} w\bar{E}_H \equiv \bar{I}_H \]
and
\[ I_H(n + 1) = \frac{1}{1 + d_3 + \mu + \gamma}[I_H(n) + wE_H(n + 1)] \leq \frac{1}{1 + d_3 + \mu + \gamma} [I_H(n) + w\bar{E}_H]. \]
It follows that
\[ I_H(n) \leq \left( \frac{1}{1 + d_3 + \mu + \gamma} \right)^n I_H(0) + \left[ \left( \frac{1}{1 + d_3 + \mu + \gamma} \right)^n + \cdots + \frac{1}{1 + d_3 + \mu + \gamma} \right] w\bar{E}_H. \]
Hence, we have
\[ \limsup_{n \to \infty} I_H(n) \leq \frac{\omega\bar{E}_H}{d_3 + \mu + \gamma} = \frac{\omega \beta_3 A_3 \bar{E}}{d_3(d_3 + \omega)(d_3 + \mu + \gamma)} \equiv \bar{I}_H. \]
This shows that sub-model (7) is permanent. We finally obtain that model (1) is permanent. This completes the proof. \( \square \)

6. Numerical example. In this section, we give the following numerical example for model (1) to indicate that when \( R_0 > 1 \) endemic equilibrium \( E^* \) may be globally asymptotically stable.

The selection of parameters in model (1) is mainly in [38]. That is, we choose parameters \( d = 10.42, d_1 = 0.08, \sigma = 2, A_2 = 1.05 \times 10^8, d_3 = 0.0141, A_3 = 2.4 \times 10^5, w = 1/15, \mu = 9.52 \times 10^{-4}, A_1 = 2 \times 10^5, d_2 = 0.33, \gamma = 0.75, \beta_1 = 4.8 \times 10^{-8}, a = 51, \beta_2 = 7.4 \times 10^{-8} \) and \( \beta_3 = 4.2 \times 10^{-11} \). By calculating, we have the endemic equilibrium \( E^* \) and basic reproduction number \( R_0 \) as follows
\[ E^* = (408880, 2091100, 9.6562 \times 10^7, 2.2162 \times 10^8, 1.0235 \times 10^7, 1.6923 \times 10^7, 90068, 7848.6), \]
\[ R_0 = \frac{aA_1A_2\beta_1\beta_2}{(d_1 + \sigma)d_2dd_1} = 20.1471 > 1. \]
The numerical simulations are given in Figure 1.

From the above numerical example we can guess that endemic equilibrium \( E^* \) of model (1) is globally asymptotically stable when \( R_0 > 1 \).

7. Discussion. In terms of mathematical theory, under some specific conditions, the discrete model shows more abundant dynamic behavior than the continuous model. Especially when some continuous models are difficult to carry out or can not get the expected results through theoretical analysis, it is more conducive for us to obtain the research results by transforming the continuous model into a discrete model and then analyzing it.

In this paper we propose a discrete-time echinococcosis transmission model. From the main results obtained in this paper, we see that echinococcosis is persistent or extinct in a region depends entirely on basic reproduction number \( R_0 \). When \( R_0 \leq 1 \), the disease-free equilibrium is globally asymptotically stable, this shows that echinococcosis will die out. Otherwise, when \( R_0 > 1 \), the model is permanent, this shows that echinococcosis will be epidemic. Therefore, in order to control the
Figure 1. Numerical simulations of solution \((S_D(t), I_D(t), S_L(t), I_L(t), x(t), S_H(t), E_H(t), I_H(t))\) with initial value \((S_D(0), I_D(0), S_L(0), I_L(0), x(0), S_H(0), E_H(0), I_H(0)) = (2 \times 10^6, 8 \times 10^5, 8.4 \times 10^5, 5.7 \times 10^7, 1.44 \times 10^7, 3 \times 10^7, 9 \times 10^3, 8 \times 10^4), (5 \times 10^4, 4 \times 10^6, 0.1 \times 10^8, 1 \times 10^8, 3 \times 10^7, 5 \times 10^6, 1 \times 10^4, 1 \times 10^3), (0.4 \times 10^6, 2 \times 10^6, 1.2 \times 10^8, 2.2 \times 10^8, 1 \times 10^7, 1.5 \times 10^7, 7 \times 10^4, 6 \times 10^4),\) respectively.
echinococcosis transmission, a key factor is to take effective measures to reduce the size of $R_0$, and finally make it less than 1. Here we propose four specific control measures as follows.

**Measures 1.** Killing the final host dogs of echinococcus, thus achieve to reduce the number of sick dogs. This is equivalent to increase the dog’s death rate $d_1$ and decrease the supplement rate $A_1$, and finally make $R_0 < 1$.

**Measures 2.** Burying or burning the internal organs of livestock with echinococcus and reducing the infection rate of the definitive host dog infected with echinococcus, which is equivalent to making the infection rate $\beta_1$ of dog is smaller, resulting in $R_0 < 1$.

**Measures 3.** Regularly make the definitive host dog eat anthelmintic drugs, so as to improve the cure rate of sick dogs, which means that the cure rate $\sigma$ of dogs is bigger, which ultimately leads to $R_0 < 1$.

**Measures 4.** Use the pesticides or high temperature sterilization. This method can effectively kill eggs of echinococcus in the environment and lower the contact rate of human with eggs. The result is to increase the death rate $d$ of eggs and decrease infection rate $\beta_2$, and ultimately can make $R_0 < 1$.

It is a pity that in this paper we do not obtain the local and global stability of endemic equilibrium when basic reproduction number $R_0 > 1$. However, from above numerical example, we see that the endemic equilibrium may be globally asymptotically stable when $R_0 > 1$. We will investigate this problem in the future.

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