On the comparison of volumes of quantum states

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Abstract

This paper aims to study the $\alpha$-volume of $K$, an arbitrary subset of the set of $N \times N$ density matrices. The $\alpha$-volume is a generalization of the Hilbert–Schmidt volume and the volume induced by partial trace. We obtain two-side estimates for the $\alpha$-volume of $K$ in terms of its Hilbert–Schmidt volume. The analogous estimates between the Bures volume and the $\alpha$-volume are also established. We employ our results to obtain bounds for the $\alpha$-volume of the sets of separable quantum states and of states with positive partial transpose. Hence, our asymptotic results provide answers for questions listed in Zyczkowski et al (1998 Phys. Rev. A 58 883) for large $N$ in the sense of $\alpha$-volume.

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1. Introduction

Recent development of quantum information and quantum computation theory has attracted considerable attention on the geometry of quantum states, which studies the geometry of the set and/or subsets of quantum states on the (composite) Hilbert space $H$ (of complex dimension $N$). Among those subsets, the set of all quantum states ($D$), the set of separable quantum states ($S$), the set of entangled quantum states ($E := D \setminus S$) and the set of quantum states with positive partial transpose ($\text{PPT}$) are of particular interest. In applications (see [2, 3]), entangled quantum states play fundamental roles because of the Einstein–Podolsky–Rosen (EPR) correlations [4] (see also [5]). Hence it is important to study (1) the probability of finding separable quantum states within $D$ and (2) the necessary and/or sufficient conditions of a quantum state being separable (or entangled). Unfortunately, it turns out that the second problem is difficult [6]. One important and powerful necessary condition for separability is the well-known Peres–Horodecki positive partial transpose (PPT) criterion [7]. However, it is known that the Peres–Horodecki PPT criterion is not sufficient in general (except in very special cases; for instance, [8–10]), and entangled quantum states with PPT have been constructed [11]. Thus, a natural question is (3) how precise is the Peres–Horodecki PPT criterion as tools to detect the separability? Another question regarding the Peres–Horodecki PPT criterion is
(4) is the Peres–Horodecki PPT criterion precise as a tool to detect entanglement? To answer questions (1), (3) and (4), one often needs to study the size of $E$, $S$, $\mathcal{PPT}$ and $D$ (for various relevant measures of size).

In the literature, measures on $D$ have one common feature: unitary invariance. These measures can be written as the product of measures on simplex (of eigenvalues) and the measure on the manifold (of eigenvectors). Important measures on $D$ include, for instance, the Hilbert–Schmidt measure ($V_{HS}$), the Bures measure ($V_B$) and the measure induced by partial trace on composite systems. The Hilbert–Schmidt measure is induced by the Hilbert–Schmidt metric which induces the flat and Euclidean geometry into $D$. Hence, a lot of known techniques, such as techniques from geometric functional analysis and convex geometry, can be used to estimate the Hilbert–Schmidt volume of (convex) subsets of $D$, e.g., $S$ and $\mathcal{PPT}$ in [12, 13]. These two papers provided answers to questions (1) and (3) for large $N$ in the sense of Hilbert–Schmidt volume. In [14], the author obtained similar results in the sense of Bures volume, by comparing the Bures volume with the Hilbert–Schmidt volume.

The present paper strives to answer questions (1) and (3) for large $N$ in the sense of the $\alpha$-volume (see section 2 for its definition). The $\alpha$-volume is a natural generalization of the Hilbert–Schmidt volume and the volume induced by partial trace on composite systems. The latter one has appeared in many places and attracted a lot of attention [15–21]. It is inspired by the important procedure of purification and the (up to a multiplicative constant) unique measure on the space of pure states of the (complex) Hilbert space $\mathcal{H}$. Any quantum state $\rho$ on $\mathcal{H}$ may be obtained by partial tracing of some pure quantum state $\langle \phi | \phi \rangle$ over the auxiliary subsystem $\mathcal{H}'$ (with dimension $K \geq N$), where $\langle \phi |$ is a unit (column) vector in the composite Hilbert space $\mathcal{H} \otimes \mathcal{H}'$. The space of pure states of $\mathcal{H}$ is isomorphic with the complex projective space $\mathbb{C}P^{N-1}$, and hence the unique measure on it, $P_N(\cdot)$, is the one induced by the Haar measure on the unitary group $U(N)$. Therefore, to obtain $V_{N,K}$, the volume induced by partial trace on the composite Hilbert space $\mathcal{H} \otimes \mathcal{H}'$, one chooses the natural measure $P_{N,K}(\cdot)$ on the space of pure states of $\mathcal{H} \otimes \mathcal{H}'$, and considers its push-forward induced by the operation of partial trace [21]. When $K = N$, $V_{N,K}$ coincides with the Hilbert–Schmidt measure. Thanks to the work of Sommers and Życzkowski [20, 21], one knows the precise mathematical formula of $V_{N,K}$ and some statistical properties of $V_{N,K}$. In particular, they calculated the exact value of $V_{N,K}(D)$ for all $N$ and $K$. However, the calculation of the exact values of $V_{N,K}(S)$ and $V_{N,K}(\mathcal{PPT})$ seems to be difficult because the geometry of these sets is not very well understood. These quantities can be used to measure the probabilities of separability and of PPT within $D$. In the present paper, we provide estimates of $V_{N,K}(S)$ and $V_{N,K}(\mathcal{PPT})$, as well as the $\alpha$-volumes of $S$ and $\mathcal{PPT}$. Our results show that the probability of finding separable states in $D$ is extremely small and so the Peres–Horodecki PPT criterion is not precise for (even moderate) large $N$, in the sense of $\alpha$-volume.

The present paper is organized as follows. In section 2, we introduce some necessary mathematical background and notations; all the results cited there are known or straightforward. Our main results are presented in section 3, that is, we compare the $\alpha$-volume with the Hilbert–Schmidt volume and the Bures volume. Section 4 contains estimates of the $\alpha$-volume of $S$ and $\mathcal{PPT}$, and these new estimates supply solutions for questions (1) and (3) for large $N$. Moreover, we provide some numerical examples to show the effectiveness of our new estimates. Conclusions, comments and final remarks are given in section 5.

2. Mathematical background and notations

We work on the (complex) Hilbert space $\mathcal{H} = \mathbb{C}^{D_1} \otimes \mathbb{C}^{D_2} \otimes \cdots \otimes \mathbb{C}^{D_n}$ with $n \geq 2$ and $D_i \geq 2$ for all $i = 1, 2, \ldots, n$. The (complex) dimension of $\mathcal{H}$ equals $N = D_1D_2 \cdots D_n$. 

Any quantum state on \( \mathcal{H} \) can be represented as a density matrix, i.e. an \( N \times N \) positive (semi-) definite matrix with trace 1. Here the trace of a matrix is the sum of its diagonal elements. We use \( \mathcal{D} = \mathcal{D}(\mathcal{H}) \) to denote the set of all quantum states on \( \mathcal{H} \). The set of separable quantum states on \( \mathcal{H} \) [22] is denoted by \( S \), that is,
\[
S = S(\mathcal{H}) := \text{conv}\{\rho_1 \otimes \cdots \otimes \rho_n, \rho_i \in \mathcal{D}(\mathbb{C}^{D_i})\}.
\]
The complement of \( S \) within \( \mathcal{D} \) is the set of entangled quantum states, i.e. \( \mathcal{E} := \mathcal{D} \setminus S \). Note that the entangled quantum states play crucial roles in quantum information and quantum computations. Both \( \mathcal{D} \) and \( S \) are convex subsets with (real) dimension \( d = N^2 - 1 \).

For any quantum state \( \rho \in \mathcal{D} \), there are some unitary matrices \( U \in \mathcal{U}(N) \) and some diagonal matrices \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_N) \) with \( (\lambda_1, \ldots, \lambda_N) \in \Delta \), such that, \( \rho = U \Lambda U^\dagger \). Hereafter, \( A^\dagger \) is the complex conjugate of the matrix \( A \), and \( \mathcal{U}(N) \) denotes the \( N \)th unitary group, that is, the set of all \( N \times N \) unitary matrices. \( \Delta \) refers to the regular simplex in \( \mathbb{R}^N \), i.e. \( \Delta = \{ (\lambda_1, \ldots, \lambda_N) \in \mathbb{R}^N : \lambda_i \geq 0, \sum_{i=1}^N \lambda_i = 1 \} \). Obviously, the eigenvalue decomposition \( \rho = U \Lambda U^\dagger \) is not unique. One can change the order in which the eigenvalues of \( \rho \) occur by using unitary permutation matrices. Without loss of generality, one can choose \( (\lambda_1, \ldots, \lambda_N) \in \Delta_1 \), the subset of \( \Delta \) with the order \( \lambda_1 \geq \cdots \geq \lambda_N \geq 0 \). This corresponds to divide \( \Delta \) into \( N! \) parts and to pick just one of them. The reason for establishing an order in the eigenvalues of \( \rho \) is to avoid duplication in calculating the volume of subsets of \( \mathcal{D} \). In the case of a non-degenerate spectrum, i.e. \( \lambda_1 > \lambda_2 > \cdots > \lambda_N \), it is easy to verify that
\[
\rho = U \Lambda U^\dagger = U B \Lambda B^\dagger U^\dagger,
\]
where \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_N) \) and \( B = \text{diag}(z_1, \ldots, z_N) \) with \( |z_j| = 1 \) for \( j = 1, \ldots, N \). That is, the matrix \( U \) is determined up to the \( N \) arbitrary phases entering \( B \), and the orbit will be the coset space \( \mathcal{F}^N = \mathcal{U}(N)/[\mathcal{U}(1)]^N \). Again to avoid duplication in calculating the volume of subsets of \( \mathcal{D} \), we will use the measure on \( \mathcal{F}^N \) instead of the measure on \( \mathcal{U}(N) \). Note that if degeneracies occur in the spectrum of \( \rho \), the matrix \( B \) need not be diagonal in order to commute with \( \Lambda \); however, the degenerate spectrum can be ignored because its total measure is 0.

Measures on \( \mathcal{D} \), such as the Hilbert–Schmidt measure, the Bures measure and the measure induced by partial trace, are invariant under conjugation by a unitary matrix. Moreover, they use definite matrix with trace 1. Here the trace of a matrix is the sum of its diagonal elements. We will use the measure on \( \mathcal{F}^N \) instead of the measure on \( \mathcal{U}(N) \). Note that if degeneracies occur in the spectrum of \( \rho \), the matrix \( B \) need not to be diagonal in order to commute with \( \Lambda \); however, the degenerate spectrum can be ignored because its total measure is 0.

Measures on \( \mathcal{D} \), such as the Hilbert–Schmidt measure, the Bures measure and the measure induced by partial trace, are invariant under conjugation by a unitary matrix. Moreover, they all have the product form: \( d\nu \times dy \), where \( \nu \) are some measures on the simplex \( \Delta \) and \( y \) is the invariant measure on \( \mathcal{F}^N \). The \( y \) measure may be written as
\[
dy = \prod_{1 \leq i < j \leq N} 2 \text{Re}(U^{-1}dU)_{ij}\text{Im}(U^{-1}dU)_{ij},
\]
where \( d\nu \) is the variation of \( U \) such that \( U, U + dU \in \mathcal{U}(N) \). Note that the \( y \) measure is the unique measure (up to a multiplicative constant) induced by the Haar measure on the unitary group \( \mathcal{U}(N) \). It is known that \( y(\mathcal{F}^N) \), the total \( y \) measure of \( \mathcal{F}^N \) (see [23]), is equal to
\[
Z_N = \frac{(2\pi)^{N(N-1)/2}}{E(N)} = \prod_{j=1}^N \Gamma(j),
\]
where \( \Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\,dt \) is the Gamma function. We point out that
\[
\frac{1}{\vartheta x} \leq \Gamma(x) \leq \frac{1}{x}, \quad \text{or} \quad \frac{1}{\vartheta} \leq \Gamma(1+x) \leq 1 \quad \text{for all} \quad x \in (0, 1),
\]
where \( \vartheta \approx 1.12917 \) [24].

For all \( \alpha > 0 \), we define the \( \alpha \)-volume, \( V_\alpha \), as
\[
dV_\alpha = \prod_{i=1}^N \lambda_i^{\alpha-1} \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j)^2 \, d\Delta \, dy.
\]
Here, for simplicity, we let
\[ d\Lambda = \delta_0 \left( \sum_{i=1}^{N} \lambda_i - 1 \right) \prod_{i=1}^{N} d\lambda_i, \]
where \( \delta_0 \) is the Dirac measure at 0. The \( \alpha \)-volume of \( \mathcal{D} \) can be calculated as follows (see [21, 23, 25, 26]):
\[
V_{\alpha}(\mathcal{D}) = \int_{\mathcal{D}} \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j|^2 \prod_{i=1}^{N} \lambda_i^{\alpha-1} d\Lambda = \frac{(2\pi)^{N(N-1)/2}}{\Gamma(\alpha N + N(N - 1))} \prod_{j=1}^{N} \Gamma(j + \alpha - 1).
\] (4)

The Hilbert–Schmidt distance between any two states \( \rho, \sigma \in \mathcal{D} \) is defined as
\[
D_{\text{HS}}(\rho, \sigma) = \|\rho - \sigma\|_{\text{HS}} = \sqrt{\text{tr}((\rho - \sigma)^2)}.
\]
This (natural) metric equips \( \mathcal{D} \) with the flat, Euclidean geometry on \( \mathcal{D} \), but it is not monotone [27]. It induces the Hilbert–Schmidt measure \( V_{\text{HS}} \), which equals to \( \sqrt{N} V_{1} \) [21, 23], namely
\[
dV_{\text{HS}} = \sqrt{N} \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j)^2 d\Lambda d\gamma.
\]
Note that the Hilbert–Schmidt measure is same as the usual (translation invariant) Lebesgue measure on \( \mathcal{D} \). Moreover, the Hilbert–Schmidt measure does not have singularities. By formula (4), the precise value of the Hilbert–Schmidt volume of \( \mathcal{D} \) [23] equals
\[
V_{\text{HS}}(\mathcal{D}) = (2\pi)^{N(N-1)/2} \frac{\sqrt{N} E(N)}{\Gamma(N^2)}.
\] (5)

In later sections, we are interested in the ratio \( \frac{V_{\text{HS}}(\mathcal{K})}{V_{\text{HS}}(\mathcal{D})} \) for \( \mathcal{K} \subset \mathcal{D} \), which is equal to the ratio \( \frac{V_{1}(\mathcal{K})}{V_{1}(\mathcal{D})} \). Hence, it is convenient to ignore the constant \( \sqrt{N} \) for our analysis, and we keep using \( V_{\text{HS}} \) instead of \( V_{1} \) to represent the 1-measure, i.e.
\[
dV_{\text{HS}} = \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j)^2 d\Lambda d\gamma.
\] (6)

Another special case of the \( \alpha \)-volume is the volume induced by partial tracing of the composite quantum system \( \mathcal{H} \otimes \mathcal{H}' \). Here \( \mathcal{H} \) and \( \mathcal{H}' \) are \( N \)- and \( K \)-dimensional Hilbert spaces respectively. Without loss of generality, we assume \( K \geq N \). The partial tracing over \( \mathcal{H}' \) gives a reduced density matrix of size \( N \times N \). Any state \( \rho \) on \( \mathcal{H} \otimes \mathcal{H}' \) can be expressed uniquely as
\[
\rho = \sum_{i,j} \rho_{i\alpha,j\beta} |e_i \otimes f_\alpha\rangle \langle e_j \otimes f_\beta|,
\]
where \( \{e_i\}_{i=1}^{N} \) and \( \{f_\alpha\}_{\alpha=1}^{K} \) are the canonical bases of \( \mathcal{H} \) and \( \mathcal{H}' \) respectively. Define the partial trace over \( \mathcal{H}' \) as
\[
\rho^A = \text{Tr}_B(\rho), \quad \text{where} \quad \rho_{i\beta}^A = \sum_{\beta=1}^{K} \rho_{i\beta,j\beta} \quad \text{for} \quad i, j = 1, \ldots, N.
\]

The measure induced by partial trace is an alternative way to derive measures on \( \mathcal{D} \). In fact, the partial trace process allows us to view states on \( \mathcal{H} \) as a (pure) state on (much) higher-dimensional space \( \mathcal{H} \otimes \mathcal{H}' \). Then the measures induced by partial trace may be considered as a projection of the \( (NK - 1) \)-dimensional simplex of eigenvalues into simplex of \( (N - 1) \)-dimension \( \Delta \) [21]. It takes the form
\[
dV_{N,K} = \prod_{i=1}^{N} \lambda_i^{K-N} \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j)^2 d\Lambda d\gamma,
\]
where \((\lambda_1, \ldots, \lambda_N) \in \Delta_1\). This is the \(\alpha\)-volume with \(\alpha = K - N + 1\), a natural number bigger than or equal to \(1\), and thus, (integer) \(\alpha\)-measure is an induced measure on \(D\). In particular, \(V_{N,N}\) is just the Hilbert–Schmidt measure. In other words, the Hilbert–Schmidt measure can be viewed as an induced measure on \(D\).

The Bures measure \(V_B(\cdot)\) [16, 25, 28] can be formulated as follows:

\[
dV_B = \frac{2^{-N/2}}{\lambda_1 \cdots \lambda_N} \prod_{1 \leq i < j \leq N} \frac{(\lambda_i - \lambda_j)^2}{\lambda_i + \lambda_j} \, d\Lambda d\gamma,
\]

where \((\lambda_1, \ldots, \lambda_N) \in \Delta_1\). It is induced by the Bures metric \(D_B(\cdot, \cdot)\) [29, 30], which takes the following form: for any two states \(\rho, \sigma \in D\), \(D_B(\rho, \sigma) = \sqrt{2 - 2 \tr \sqrt{\rho \sigma} \sqrt{\rho}}\). The Bures metric is proven to be equivalent to [31]

\[
D_B(\rho, \sigma) = \sup_{\{P_i\}} \left( \sum_{i=1}^{N} \left( \sqrt{\tr(P_i \rho)} - \sqrt{\tr(P_i \sigma)} \right)^2 \right)^{1/2},
\]

where the supremum runs over all the projection measurements \(\{P_i\}_{i=1}^{N}\), i.e. all \(P_i\) are rank 1 projections satisfying \(\sum_{i=1}^{N} P_i = I_{DN}\). Employing the projection measurement \(\{P_i\}_{i=1}^{N}\) to a state \(\rho\) results in the \(i\)th outcome with probability \(\tr(P_i \rho)\), \(i = 1, \ldots, N\). Note that \(\left( \sum_{i=1}^{N} \left( \sqrt{\tr(P_i \rho)} - \sqrt{\tr(P_i \sigma)} \right)^2 \right)^{1/2}\) is the Hellinger distance, which measures the statistical distinguishability between the discrete probability distributions \(\{\tr(P_i \rho)\}_{i=1}^{N}\) and \(\{\tr(P_i \sigma)\}_{i=1}^{N}\). Hence, the Bures distance provides the measurement of statistical distinguishability between \(\rho\) and \(\sigma\). We point out that the Bures metric is Riemannian but not flat. Moreover, it is monotone, that is, it does not increase under the quantum channels (completely positive, trace preserving maps) [32]. The precise value of the Bures volume of \(D\) was obtained in [28]:

\[
V_B(D) = \int_{\Delta_1 \times \mathbb{R}^N} \frac{2^{-N/2}}{\lambda_1 \cdots \lambda_N} \prod_{1 \leq i < j \leq N} \frac{(\lambda_i - \lambda_j)^2}{\lambda_i + \lambda_j} \, d\Lambda d\gamma = 2^{1-N^2} \frac{\pi^{N^2/2}}{\Gamma(N^2/2)},
\]

which is the \(d\)-dimensional volume of the \(d\)-dimensional hemisphere with radius \(\frac{1}{2}\). (There is no satisfactory explanation for this mysterious fact.) Note that the Bures volume is not translation invariant and has singularities on the boundary of \(D\), where at least one of the \(\lambda_i\)'s are 0. (If two or more of them are 0, then some denominators in (7) vanish.) As in the case of the Hilbert–Schmidt volume, it is convenient to ignore the constant term \(2^{1-N^2} \pi^{N^2/2}/\Gamma(N^2/2)\) in \(dV_B\). Hereafter, we keep using \(dV_B\) to represent the measure

\[
dV_B = \frac{1}{\lambda_1 \cdots \lambda_N} \prod_{1 \leq i < j \leq N} \frac{(\lambda_i - \lambda_j)^2}{\lambda_i + \lambda_j} \, d\Lambda d\gamma. \tag{7}
\]

The primary goal in [14] is to compare \(V_{B}(\mathcal{K}, D)\) with \(V_{HS}(\mathcal{K}, D)\) for any subset \(\mathcal{K}\) in \(D\). Here the Bures volume radii ratio, \(VR_B(\mathcal{K}, \mathcal{L})\), for two subsets \(\mathcal{K}, \mathcal{L} \subset D\), was defined as [14]

\[
VR_B(\mathcal{K}, \mathcal{L}) = \left( \frac{V_B(\mathcal{K})}{V_B(\mathcal{L})} \right)^{1/d},
\]

where \(d = N^2 - 1\) and the Hilbert–Schmidt volume radii ratio was defined as

\[
VR_{HS}(\mathcal{K}, \mathcal{L}) = \left( \frac{V_{HS}(\mathcal{K})}{V_{HS}(\mathcal{L})} \right)^{1/d}.
\]
These quantities aim to compare the volume ratio of $K$ to $L$. Similarly, we define here the $\alpha$-volume radii ratio, $\text{VR}_\alpha(K, L)$, as

$$\text{VR}_\alpha(K, L) = \left( \frac{V_\alpha(K)}{V_\alpha(L)} \right)^{1/d}.$$ 

This can be used as a measure of the relative size of $K$ to $L$ in the $\alpha$-volume sense and does have geometric meanings. This paper strives to estimate $\text{VR}_\alpha(K, D)$ from both upper and below in terms of $\text{VR}_{\text{HS}}(K, D)$. Our proofs rely on the Stirling approximation, which can be written as

$$\Gamma(z) = \sqrt{2\pi} \left( \frac{z}{e} \right)^z \left( 1 + O\left( \frac{1}{z} \right) \right). \quad (8)$$

We refer the readers to references [14, 16, 21, 23, 25, 28, 33] for more detailed background and for motivation.

3. $\alpha$-volume versus Hilbert–Schmidt and Bures volumes

This section aims to compare the $\alpha$-volume of subsets of quantum states in terms of their Hilbert–Schmidt volume and Bures volume. Hereafter, we let $K$ be any (Borel) measurable subset of $D(\mathcal{H})$, the set of quantum states on (complex) Hilbert space $\mathcal{H}$. The following lemma, which is independent of possible tensor product structures of $\mathcal{H}$, is our main tool to estimate the $\alpha$-volume in terms of the Hilbert–Schmidt volume. The same approach will give corresponding comparison results for real Hilbert space $\mathcal{H}$.

**Lemma 1.** Let $\alpha > 0$ be a (fixed) constant independent of the dimension of $\mathcal{H}$. Let $\mathcal{H}$ be any (complex) Hilbert space with (complex) dimension $N$ and $K$ be any measurable subset in $D(\mathcal{H})$.

(i) For all $p > \alpha > 1$,

$$\text{V}_{\text{HS}}(K)^p (\text{V}_{\frac{\alpha-1}{1-p}}(D))^{1-p} \leq V_\alpha(K) \leq N^{(1-\alpha)N} \text{V}_{\text{HS}}(K). \quad (9)$$

(ii) For all $0 < p < \alpha < 1$,

$$N^{(1-\alpha)N} \text{V}_{\text{HS}}(K) \leq V_\alpha(K) \leq \text{V}_{\text{HS}}(K)^p (\text{V}_{\frac{\alpha-1}{1-p}}(D))^{1-p}. \quad (10)$$

**Remark.** From formula (4) for $\frac{\alpha-1}{1-p} > 0$, one has

$$V_{\frac{\alpha-1}{1-p}}(D) = \frac{(2\pi)^{N(N-1)/2}}{\Gamma\left(\left(\frac{\alpha-1}{1-p}\right)N + N(N-1)\right)} \prod_{j=1}^N \Gamma\left( j + \left( \frac{\alpha - 1}{1 - p} \right) \right).$$

The above formula, as a function of $p$, can be extended to $\frac{\alpha-1}{1-p} < 0$; however, there are singularities whose exact locations depend on $N$ and $\alpha$.

**Proof.** We only prove the case of $\alpha > 1$. Similar approach will lead to the argument for the case of $\alpha < 1$.

To that end, let $h : \Delta \to \mathbb{R}$ be $h(\lambda_1, \ldots, \lambda_N) = \prod_{i=1}^N \lambda_i^{\alpha-1}$. Note that $\partial \Delta$, the boundary of the simplex $\Delta$, consists of sequences for which one or more of the $\lambda_i$’s equal to 0. Therefore, $h(\lambda_1, \ldots, \lambda_N)$ is always 0 on $\partial \Delta$ if $\alpha > 1$ and is strictly positive in the interior of the simplex $\Delta$. On the other hand, $h(\lambda_1, \ldots, \lambda_N)$ has a unique critical point $(1/N, \ldots, 1/N)$ in the interior of simplex $\Delta$ by the Lagrange multiplier method. The compactness of $h(\lambda_1, \ldots, \lambda_N)$ implies that $h(\lambda_1, \ldots, \lambda_N)$ must have a maximum inside the interior of the simplex $\Delta$ for
\(\alpha > 1\), and hence the (unique) critical point \((1/N, \ldots, 1/N)\) must be the (only) maximizer of \(h(\lambda_1, \ldots, \lambda_N)\) on \(\Delta\). Therefore,

\[
h(\lambda_1, \ldots, \lambda_N) = \prod_{i=1}^{N} \lambda_i^{\alpha - 1} \leq N^{(1-\alpha)N} \quad \text{for} \quad \alpha > 1.
\]

(11)

By formula (3), the \(\alpha\)-volume of \(K\) can be calculated as follows:

\[
V_{\alpha}(K) = \int_{K} \prod_{i=1}^{N} \lambda_i^{\alpha - 1} \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j)^2 \, d\mu \, dy.
\]

(12)

Therefore, formulas (6) and (11) imply

\[
V_{\alpha}(K) \leq N^{(1-\alpha)N} V_{\text{HS}}(K) \quad \text{for} \quad \alpha > 1.
\]

Now let us prove the lower bound for \(\alpha > 1\). By (12)

\[
V_{\alpha}(K) = \int_{K} \prod_{i=1}^{N} \lambda_i^{\alpha - 1} \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j)^2 \, d\mu \, dy = \int_{K} f \, d\mu \, dy,
\]

(13)

where, to reduce the clutter, we denoted

\[
g(\lambda_1, \ldots, \lambda_N) = \prod_{i=1}^{N} \lambda_i^{\alpha - 1} \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j|^{2p},
\]

\[
f(\lambda_1, \ldots, \lambda_N) = \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j|^{2p}.
\]

For any \(p > 1\), we employ the Hölder inequality (see [34]) to (13) and get

\[
V_{\alpha}(K) \geq \left( \int_{K} f^\frac{p}{\alpha} \, d\mu \, dy \right)^\alpha \left( \int_{K} g^\frac{p}{\alpha} \, d\mu \, dy \right)^{1-\alpha}.
\]

(14)

By formula (6), and substituting \(f\) into the first integral of (14), one gets

\[
\left( \int_{K} \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j)^2 \, d\mu \, dy \right)^\alpha = V_{\text{HS}}(K)^p.
\]

(15)

By substituting \(g\) into the second integral of (14), one has, for \(p > \alpha > 1\),

\[
\left( \int_{K} \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j|^{\frac{p}{\alpha}} \, d\mu \, dy \right)^{1-\alpha} \geq \left( \int_{D} \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j|^2 \lambda_i^{\frac{p}{\alpha} - 1} \, d\mu \, dy \right)^{1-\alpha} = \left( V_{\frac{p}{\alpha}}(D) \right)^{1-\alpha}.
\]

(16)

where the inequality follows \(K \subset D\) and \(p > \alpha > 1\). The lower bound is then an immediate consequence of inequalities (15) and (16).

\[\square\]

Remark. More generally, one can compare \(V_{\alpha}(K)\) in terms of \(V_\beta(K)\) for all \(\alpha, \beta > 0\). The above approach gives the following corollary.

Corollary 1. Let \(\alpha, \beta > 0\) be (fixed) constants independent of the dimension of \(\mathcal{H}\). Let \(\mathcal{H}\) be any (complex) Hilbert space with (complex) dimension \(N\) and \(K\) be any measurable subset in \(D(\mathcal{H})\).
(i) For all $p > \frac{\alpha}{\beta} > 1$,\[ V_\beta(K)^p \left( \frac{V_\beta(D)}{V_\alpha(D)} \right)^{1-p} \leq V_\alpha(K) \leq N^{(\beta - \alpha)N} V_\beta(K). \]

(ii) For all $0 < p < \frac{\alpha}{\beta} < 1$,
\[ N^{(\beta - \alpha)N} V_\beta(K) \leq V_\alpha(K) \leq \frac{V_\beta(K)^p}{(\frac{V_\beta(D)}{V_\alpha(D)})^{1-p}}. \]

The following lemma is simple but important for the proof of our main results.

**Lemma 2.** Let $\alpha > 0$ be a constant independent of $N$. Then
\[ \lim_{N \to \infty} \left( \frac{V_{\text{HS}}(D)}{V_\alpha(D)} \right)^{1/(N^2 - 1)} = 1. \]

**Remark.** A direct consequence of this lemma is as follows: let $\alpha, \beta > 0$ be two constants independent of $N$; then
\[ \lim_{N \to \infty} \left( \frac{V_\beta(D)}{V_\alpha(D)} \right)^{1/(N^2 - 1)} = 1. \]

**Proof.** Now by formula (4), one gets
\begin{equation}
V_{\text{HS}}(D) = \frac{\Gamma(N^2 - N + \alpha N)}{\Gamma(N^2)} \prod_{j=1}^{N} \left( \frac{\Gamma(j)}{\Gamma(j + \alpha - 1)} \right). \tag{17}
\end{equation}

For each $j = 1, \ldots, N$, and by $\alpha > 1$, one has
\[ \frac{\Gamma(j)}{\Gamma(\alpha + j - 1)} = \frac{1 \times 2 \times \cdots (j-1)}{\Gamma(\alpha) \times \alpha \times (\alpha + 1) \times \cdots (\alpha + j - 2)} \leq \frac{1}{\Gamma(\alpha)}. \]

Combining with the Stirling approximation formula (8), one has for $\alpha > 1$
\[ \lim_{N \to \infty} \left( \frac{V_{\text{HS}}(D)}{V_\alpha(D)} \right)^{1/(N^2 - 1)} \leq 1. \]

Now we bound $\left( \frac{V_{\text{HS}}(D)}{V_\alpha(D)} \right)^{1/d}$ from below. By induction, for all $j = 1, \ldots, N$ and by $\alpha > 1$, one has
\[ \frac{\Gamma(j)}{\Gamma(j + \alpha - 1)} \geq \frac{\Gamma(N)}{\Gamma(N + \alpha - 1)}, \]

and hence
\begin{equation}
\prod_{j=1}^{N} \left( \frac{\Gamma(j)}{\Gamma(j + \alpha - 1)} \right) \geq \left( \frac{\Gamma(N)}{\Gamma(N + \alpha - 1)} \right)^N. \tag{18}
\end{equation}

Together with the Stirling approximation formula (8) and formula (17), one has
\[ \lim_{N \to \infty} \left( \frac{V_{\text{HS}}(D)}{V_\alpha(D)} \right)^{1/(N^2 - 1)} \geq 1, \]
which completes the proof. \hfill \Box

**Theorem 1.** Let $\alpha > 0$ be a (fixed) constant independent of the dimension of $\mathcal{H}$. There are universal constants (independent of dimension $N$, but depending on $\alpha$) $C_1, c'_1 > 0$, such that for any Hilbert space $\mathcal{H}$ and any subset $K \subset D$,
(i) for $\alpha > 1$, $\text{VR}_\alpha(K, D) \leq C_1 \text{VR}_{\text{HS}}(K, D)$,
(ii) for $0 < \alpha < 1$, $\text{VR}_\alpha(K, D) \geq c_1 \text{VR}_{\text{HS}}(K, D)$.

Moreover, the bounds are optimal in general.

**Remark.** In fact, as $N \to \infty$, $C_1(N, \alpha) \to 1$ (and $c_1(N, \alpha) \to 1$). For small dimension $N$, one can precisely calculate $C_1(N, \alpha)$ (and $c_1(N, \alpha)$) by formula (19), if $\alpha$ is given. In fact, one can let

$$C_1(N, \alpha) = \left( \frac{N^{(1-\alpha)N} \text{VR}_{\text{HS}}(D)}{V_\alpha(D)} \right)^{1/\left(\alpha^2 - 1\right)}.$$  

Moreover, by lemma 1, $C_1(N, \alpha) \geq 1$ (and $c_1(N, \alpha) \leq 1$) for all $\alpha > 0$ and all $N$. Let us consider $\mathcal{H} = (\mathbb{C}^2)^{\otimes n}$ with $N = 2^n$. Then, if $\alpha = 2$, $C_1(4, 2) \approx 1.19861$, $C_1(8, 2) \approx 1.10785$, $C_1(16, 2) \approx 1.05706$, $C_1(32, 2) \approx 1.02958$ which seems to decrease to 1 as $N$ increases, while if $\alpha = 1/2$, $c_1(4, 1/2) \approx 0.873608$, $c_1(8, 1/2) \approx 0.936271$, $c_1(16, 1/2) \approx 0.968159$, $c_1(32, 1/2) \approx 0.984148$ which seems to increase to 1 if $N$ increases.

Similar phenomenon happens to other fixed $\alpha$, and this strongly suggests that $C_1(4, \alpha)$ for $\alpha > 1$ or $c_1(4, \alpha)$ for $\alpha < 1$ should work for all $2^n \geq 4$ (and, in fact, all other integers $N \geq 4$).

**Proof.** We only prove the case of $\alpha > 1$. Now dividing $V_\alpha(D)$ from both sides of the upper bound of inequality (9),

$$\frac{V_\alpha(K)}{V_\alpha(D)} \leq \frac{N^{\left(1-\alpha\right)N} \text{VR}_{\text{HS}}(K)}{V_\alpha(D)} = \frac{\text{VR}_{\text{HS}}(K)}{V_\alpha(D)} \left( \frac{N^{\left(1-\alpha\right)N} \text{VR}_{\text{HS}}(D)}{V_\alpha(D)} \right)^{1/\left(\alpha^2 - 1\right)}.$$  

Taking $d$th root from both sides,

$$\text{VR}_\alpha(K, D) \leq \text{VR}_{\text{HS}}(K, D) \left( \frac{N^{\left(1-\alpha\right)N} \text{VR}_{\text{HS}}(D)}{V_\alpha(D)} \right)^{1/\left(\alpha^2 - 1\right)}. \quad (19)$$

By lemma 2, for $\alpha > 1$,

$$\lim_{N \to \infty} \left( \frac{N^{\left(1-\alpha\right)N} \text{VR}_{\text{HS}}(D)}{V_\alpha(D)} \right)^{1/\left(\alpha^2 - 1\right)} = 1,$$

and hence, there is a universal constant $C_1 > 0$ such that $\text{VR}_\alpha(K, D) \leq C_1 \text{VR}_{\text{HS}}(K, D)$.

To see the optimality of the upper bound, we let $\mathcal{K}_t = tD + (1-t)\rho_{\text{max}}$ for $0 < t < 1$, i.e.

$$\mathcal{K}_t = \left\{ UXU^\dagger : X = \text{diag} \left( \frac{1-t}{N}, \frac{1-t}{N}, \ldots, \frac{1-t}{N}, \frac{1-t}{N} + t\lambda_N \right), \lambda_1, \ldots, \lambda_N \in \Delta \text{ and } U \in \mathcal{U}(N) \right\}.$$  

Let $Z_N$ be as in (1).

Clearly $\text{VR}_{\text{HS}}(\mathcal{K}_t, D) = t$ by homogeneity of the Hilbert–Schmidt volume. Now we estimate $V_\alpha(\mathcal{K}_t)$ from below. In fact

$$V_\alpha(\mathcal{K}_t) = Z_N t^{N^2-1} \int_{\Delta_t} \prod_{j=1}^N \left( \frac{1-t}{N} + t\lambda_j \right)^{\alpha-1} \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j)^2 \prod_{j=1}^{N-1} d\lambda_j.$$  

As $\frac{1-t}{N} + t\lambda_j \geq t\lambda_j$ for all $j = 1, \ldots, N$, and $\alpha > 1$, one has

$$V_\alpha(\mathcal{K}_t) \geq t^{N^2-1+N(\alpha-1)} Z_N \int_{\Delta_t} \prod_{j=1}^N \lambda_j^{\alpha-1} \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j)^2 \prod_{j=1}^{N-1} d\lambda_j = t^{N^2-1+N(\alpha-1)} V_\alpha(D).$$
Equivalently,
\[
VR_\alpha(K, D) \geq t \cdot \frac{\log t}{\log \alpha}.
\]
Hence, if \( t \geq \exp(-c^2(t)(N^2 - 1)/N) \) for some constant \( c^2(t) > 0 \), then
\[
VR_\alpha(K, D) \geq \exp(-c^2(t)(\alpha - 1))t = \exp(-c^2(t)(\alpha - 1))VR_{\alpha}(K, D).
\]
Part (i) guarantees that \( VR_\alpha(K, D) \leq C_1 VR_{\alpha}(K, D) \). So the upper bound can be obtained for \( \alpha > 1 \), and it is optimal in general. \( \square \)

**Theorem 2.** Let \( \alpha > 0 \) be a (fixed) constant independent of the dimension of \( \mathcal{H} \). There are universal constants (independent of \( N \), but depending on \( \alpha \)) \( c_1, C_1 > 0 \), such that for any Hilbert space \( \mathcal{H} \) and any subset \( K \subseteq D \) (let \( \xi = VR_{\mathcal{H}}(K, D) \)),

(i) for \( \alpha > 1 \), \( VR_\alpha(K, D) \geq c_1 VR_{\alpha}(K, D) \) \( \exp \left( \frac{(1-\alpha)\ln N \ln(\xi)}{N^{\alpha}} \right) \),

(ii) for \( 0 < \alpha < 1 \), \( VR_\alpha(K, D) \leq C_1 VR_{\alpha}(K, D) \) \( \exp \left( \frac{(1-\alpha)\ln N \ln(\xi)}{N^{\alpha}} \right) \).

**Proof.** We only prove (i). By the lower bound of (9), for all \( p > \alpha > 1 \),
\[
V_{HS}(K)^p \left( V_{\Gamma^p}(D) \right)^{1-p} \leq V_\alpha(K).
\]
Dividing \( V_\alpha(D) \) from both sides, one has for all \( p > \alpha > 1 \),
\[
\frac{V_\alpha(K)}{V_\alpha(D)} \geq \left( \frac{V_{HS}(K)}{V_{HS}(D)} \right)^p \left( \frac{V_{HS}(D)}{V_\alpha(D)} \right)^p \left( \frac{V_{\Gamma^p}(D)}{V_\alpha(D)} \right)^{1-p}.
\]
Equivalently, by taking the \( d \)th root from both sides,
\[
VR_\alpha(K, D) \geq \left( VR_{\alpha}(K, D) \right)^p \left( \frac{V_{HS}(D)}{V_\alpha(D)} \right)^{\frac{1}{p}} \left( \frac{V_{\Gamma^p}(D)}{V_\alpha(D)} \right)^{\frac{1}{p}}. \tag{20}
\]
We first bound \( \left( \frac{V_{\Gamma^p}(D)}{V_\alpha(D)} \right)^{\frac{1}{p}} \) from below. To that end, we let \( \tau = \tau(N, \xi) = \frac{p-\alpha}{p-1} = \frac{1}{N \ln(\xi)} \) be a function depending on \( N \) and \( \xi = VR_{\mathcal{H}}(K, D) \). This implies that \( p(N, \xi) = \frac{p-\alpha}{p-1} \), and \( p(N, \xi) \to \alpha \) as \( N \to \infty \). Moreover, \( 0 < \tau N = \frac{1}{\ln(\xi)} < 1 \), and hence \( \tau j < 1 \) for all \( j = 1, \ldots, N-1 \).

By formula (4), one has
\[
\frac{V_\tau(D)}{V_\alpha(D)} = \frac{\Gamma(N^2 - N + \alpha N)}{\Gamma(N^2 - N + \tau N)} \prod_{j=1}^{N} \frac{\Gamma(j + \tau - 1)}{\Gamma(j + \alpha - 1)}.
\]
Formula (2) and induction show that for all \( j = 2, \ldots, N \),
\[
\frac{\Gamma(j + \tau - 1)}{\Gamma(j + \alpha - 1)} \leq \frac{\Gamma(\tau + 1)}{\Gamma(\alpha + 1)} \leq \frac{1}{\Gamma(\alpha + 1)},
\]
because \( 0 < \tau < 1 \). Thus, again by (2)
\[
\prod_{j=1}^{N} \frac{\Gamma(j + \tau - 1)}{\Gamma(j + \alpha - 1)} \leq \left( \frac{1}{\Gamma(\alpha + 1)} \right)^{N-1} \frac{\Gamma(\tau)}{\Gamma(\alpha)} \leq \frac{\alpha}{\tau \Gamma(1 + \alpha) N}.
\]
This further implies, by \( p = \frac{q - t}{t - 1} > 1 \) and \( \tau = \frac{1}{N \ln(c/\xi)} \),

\[
\lim_{N \to \infty} \left( \prod_{j=1}^{N} \frac{\Gamma(j + \tau - 1)}{\Gamma(j + \alpha - 1)} \right)^{\frac{1 - p}{\tau}} \geq \lim_{N \to \infty} \left( \frac{\alpha}{\tau \Gamma(1 + \alpha)} \right)^{\frac{1 - p}{\tau}} \left( N \right) \geq \lim_{N \to \infty} \left( \frac{\alpha}{\Gamma(1 + \alpha) N} \right)^{\frac{1 - p}{\tau}} \exp \left( \frac{1}{N^2 - 1} \right) \times \exp \left( \frac{(1 - \alpha) \ln(e/\xi)}{N^2 - 1} \right) = \lim_{N \to \infty} \exp \left( \frac{(1 - \alpha) \ln(e/\xi)}{N^2 - 1} \right). \tag{21}
\]

The Stirling approximation formula \((8)\) implies that

\[
\lim_{N \to \infty} \left( \frac{\Gamma(N^2 - N + \alpha N)}{\Gamma(N^2 - N + \tau N)} \right)^{\frac{1 - p}{\tau}} = 1.
\]

Together with inequality \((21)\), one has

\[
\lim_{N \to \infty} \left( \frac{V_H \left( \mathcal{D} \right) \xi}{V_H \left( \mathcal{D} \right)} \right)^{\frac{1 - p}{\tau}} \geq \lim_{N \to \infty} \exp \left( \frac{(1 - \alpha) \ln(e/\xi)}{N^2 - 1} \right). \tag{22}
\]

Inequality \((9)\) implies that

\[
\lim_{N \to \infty} \left( \frac{V_H \left( \mathcal{D} \right) \xi}{V_H \left( \mathcal{D} \right)} \right)^{\frac{1 - p}{\tau}} \geq \lim_{N \to \infty} N^\frac{\alpha - 1}{N^2 - 1} \geq \lim_{N \to \infty} \exp \left( \frac{\alpha(\alpha - 1) \ln N}{N} \right) = 1. \tag{23}
\]

Since \( \tau \ln \zeta = \frac{\ln \zeta}{\frac{N^2 - N}{1 + \ln(1/\xi)}} \geq \frac{1}{N} \) and \( 0 < \tau < 1 \), one has

\[
\lim_{N \to \infty} \text{VR}_{\mathcal{H}}(\mathcal{K}, \mathcal{D})^{\alpha - \alpha} = \lim_{N \to \infty} \exp \left( \frac{\alpha - 1}{1 - \tau} \ln \zeta \right) \geq \lim_{N \to \infty} \exp \left( \frac{(1 - \alpha)}{N} \right) = 1. \tag{24}
\]

Combining \((20), (22), (23)\) and \((24)\), one gets \( \text{VR}_{\mathcal{D}}(\mathcal{K}, \mathcal{D}) \geq c_1 \text{VR}_{\mathcal{H}}(\mathcal{K}, \mathcal{D}) \exp \left( \frac{(1 - \alpha) \ln(e/\xi)}{N^2 - 1} \right) \quad \left( 1 + O \left( \frac{\ln N}{N} \right) \right) \)

**Remark.** A slightly more precise calculation shows that

\[
\text{VR}_{\mathcal{D}}(\mathcal{K}, \mathcal{D}) \geq c_1 \text{VR}_{\mathcal{H}}(\mathcal{K}, \mathcal{D}) \exp \left( \frac{(1 - \alpha) \ln(e/\xi)}{N^2 - 1} \right) \left( 1 + O \left( \frac{\ln N}{N} \right) \right).
\]

Moreover, from the above proof, one gets for any fixed \( \alpha > 1 \), the constant \( c_1(N, \alpha) \to 1 \) as \( N \to \infty \). Numerical results strongly suggest that, for any fixed \( \alpha > 1 \), \( c_1(N, \alpha) \) increases to 1 as \( N \) increases, and \( c_1(4, \alpha) \) should work for all \( N \geq 4 \). For instance, if we consider \( \mathcal{H} = (\mathbb{C}^2)^{2n} \) with \( N = 2^n \), then our proof yields that \( c_1(4, 2) \approx 0.219056, c_1(8, 2) \approx 0.374655, c_1(16, 2) \approx 0.539382, c_1(32, 2) \approx 0.686509, c_1(64, 2) \approx 0.800189, c_1(128, 2) \approx 0.878831, c_1(256, 2) \approx 0.929124 \) and \( c_1(512, 2) \approx 0.959595 \). A similar phenomenon happens in the case of \( 0 < \alpha < 1 \), namely \( c_1'(N, \alpha) \) decrease to 1 as \( N \) increases. For example, if \( \alpha = 0.5 \), then \( c_1'(4, 0.5) \approx 1.436603, c_1'(8, 0.5) \approx 1.24359, c_1'(16, 0.5) \approx 1.13648, c_1'(32, 0.5) \approx 1.07664, c_1'(64, 0.5) \approx 1.04292 \) and hence, \( c_1'(4, \alpha) \) should work for all \( N \geq 4 \).

As a consequence of corollary \(1\), one can prove the following result, whose proof is similar to theorems \(1 \) and \(2\).
Corollary 2. Let $\alpha, \beta > 0$ be (fixed) constants independent of the dimension of $\mathcal{H}$. Let $\mathcal{H}$ be any (complex) Hilbert space with (complex) dimension $N$ and $\mathcal{K}$ be any measurable subset in $\mathcal{D}(\mathcal{H})$. Let $\xi = \text{VR}_B(\mathcal{K}, \mathcal{D})$, $\alpha_{\max} = \max\{1, \frac{\alpha}{\beta}\}$ and $\alpha_{\min} = \min\{1, \frac{\alpha}{\beta}\}$. There exist universal constants (independent of $N$, but depending on $\alpha$ and $\beta$) $c_2, C_2 > 0$, such that

$$c_2 \xi^{\alpha_{\max}} \exp\left(\frac{1 - \alpha_{\max}}{N^2 - 1}\right) \leq \text{VR}_B(\mathcal{K}, \mathcal{D}) \leq C_2 \xi^{\alpha_{\min}} \exp\left(\frac{1 - \alpha_{\min}}{N^2 - 1}\right).$$

Remark. As in theorem 1, $K_j$ attains the upper bound if $\alpha > \beta$ and attains the lower bound if $\alpha < \beta$. Therefore, for $\alpha > \beta$ the upper bound is optimal in general, and for $\alpha < \beta$ the lower bound is optimal in general. Theorems 1 and 2 are special cases of this corollary with $\beta = 1$.

Theorems 1 and 2 in [14] compare the Bures volume of $\mathcal{K}$ in terms of the Hilbert–Schmidt volume. The following theorem compares the Bures volume with the $\alpha$-volume.

Theorem 3. Let $\alpha > 0$ be a (fixed) constant independent of the dimension of $\mathcal{H}$. Let $\mathcal{H}$ be any (complex) Hilbert space with (complex) dimensions $N$ and $\mathcal{K}$ be any measurable subset in $\mathcal{D}(\mathcal{H})$. There exist universal constants (independent of $N$, but depending on $\alpha$) $c_3, C_3 > 0$, such that

$$c_3 \xi^{\max\{1, \frac{\alpha}{\beta}\}} \exp\left(\frac{1 - \max\{1, \frac{\alpha}{\beta}\}}{N^2 - 1}\right) \leq \text{VR}_B(\mathcal{K}, \mathcal{D}) \leq C_3 \xi^{\min\{1, \frac{\alpha}{\beta}\}} \exp\left(\frac{\ln\left(e/\xi\right)}{2N}\right),$$

where $\xi = \text{VR}_B(\mathcal{K}, \mathcal{D})$.

Proof. Similar to the proof of lemma 1, $\prod_{1 \leq i < j \leq N} \frac{1}{\lambda_i + \lambda_j}$ attains the minimum at $\lambda_i = 1/N$ for all $i = 1, \ldots, N$. Hence,

$$V_B(\mathcal{K}) = \int_\mathcal{K} \frac{1}{\sqrt{\lambda_1 \cdots \lambda_N}} \prod_{1 \leq i < j \leq N} \frac{(\lambda_i - \lambda_j)^2}{\lambda_i + \lambda_j} \, d\Lambda \, d\gamma \geq \left(\frac{N}{2}\right)^{\frac{N^2 - N}{2}} \int_\mathcal{K} \frac{1}{\sqrt{\lambda_1 \cdots \lambda_N}} \prod_{1 \leq i < j \leq N} \frac{(\lambda_i - \lambda_j)^2}{\lambda_i + \lambda_j} \, d\Lambda \, d\gamma = \left(\frac{N}{2}\right)^{\frac{N^2 - N}{2}} V_1(\mathcal{K}).$$

Recall $d = N^2 - 1$. Dividing $V_B(\mathcal{D})$ from both sides and taking the $d$th root, one gets that for some universal constant $e'' > 0$,

$$\text{VR}_B(\mathcal{K}, \mathcal{D}) \geq \left(\frac{N}{2}\right)^{\frac{N^2 - N}{2d}} \text{VR}_1(\mathcal{K}, \mathcal{D}) \left(\frac{V_1(\mathcal{K}, \mathcal{D})}{V_1(\mathcal{D})}\right)^{1/d} \geq e'' \text{VR}_1(\mathcal{K}, \mathcal{D}),$$

where the second inequality follows lemma 2 and the limit [14]

$$\lim_{N \to \infty} \frac{V_H(\mathcal{D})}{V_B(\mathcal{D})} \left(\frac{V_H(\mathcal{D})}{V_B(\mathcal{D})}\right)^{1/d} = 2 e^{-1/4}.$$

The lower bound is an immediate consequence of corollary 2 and inequality (25). The upper bound is an immediate consequence of theorems 1, 2 and theorem 2 in [14]. \[\square\]
Remark. If $\alpha > \frac{1}{2}$, the lower bound is optimal in general, and if $\alpha \leq 1$, the upper bound is optimal in general. If $\alpha \in [1/2, 1]$, then there exist universal constants $c'_1, C'_1 > 0$, such that
\[
c'_1 VR_\alpha(\mathcal{K}, \mathcal{D}) \leq VR_\beta(\mathcal{K}, \mathcal{D}) \leq C'_1[VR_\alpha(\mathcal{K}, \mathcal{D})]^{1/2} \exp \left( \frac{\ln(e/\zeta)}{2N} \right),
\]
and the bounds are optimal in general. Moreover, there exist universal constants $c''_2, C''_2 > 0$, such that
\[
c''_2 VR_\alpha(\mathcal{K}, \mathcal{D}) \leq VR_\beta(\mathcal{K}, \mathcal{D}) \leq C''_2[VR_{\text{HS}}(\mathcal{K}, \mathcal{D})]^{1/2} \exp \left( \frac{\ln(e/\zeta)}{2N} \right),
\]
with $\zeta = VR_{\text{HS}}(\mathcal{K}, \mathcal{D})$ and the bounds are optimal in general. This inequality improves results in [14].

In fact, it was proved that $\tilde{c} t \leq VR_\beta(\mathcal{K}_t, \mathcal{D}) \leq \tilde{C} t$ if $t \leq \frac{3}{2}$ for some universal constants $\tilde{c}, \tilde{C} > 0$ [14]. Recall $\mathcal{K}_t = tD + (1-t)\rho_{\text{max}}$. Therefore, $\mathcal{K}_t$ attains the lower bound for $\alpha \geq 1/2$ by a similar calculation in the proof of theorem 1.

To see the optimality of the upper bound for $\alpha \leq 1$, we let $0 < t < 1$ and $\mathcal{K}_t = \{UXU^*: X = \text{diag}(1-t+t\lambda_1, t\lambda_2, \ldots, t\lambda_N), (\lambda_1, \ldots, \lambda_N) \in \Delta_1 \text{ and } U \in \mathcal{U}(N)\}$.

Recall that $\Delta_1$ is the chamber of $\Delta$ with order $\lambda_1 \geq \cdots \geq \lambda_N$. For $e^{(t-1-N)} < t < \frac{4}{5}$, one has $VR_\beta(\mathcal{K}_t, \mathcal{D}) \geq \tilde{c}_t VR_{\text{HS}}(\mathcal{K}_t, \mathcal{D})$ where $\tilde{c}_t, \tilde{C}_t > 0$ are two constants [14]. To prove the optimality of the upper bound for $\alpha \leq 1$, it is enough to prove that $VR_{\text{HS}}(\mathcal{K}_t, \mathcal{D}) \leq \tilde{c}_t VR_\alpha(\mathcal{K}_t, \mathcal{D})$ for some constant $\tilde{c}_t > 0$.

To that end, we first bound $V_\alpha(\mathcal{K}_t)$ from above. By formula (3), one has
\[
V_\alpha(\mathcal{K}_t) = Z_N t^{(N-1)^2+(a-1)(N-1)} \int_{\Delta_1} \prod_{j=2}^{N} \lambda_j^{\alpha-1} \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j)^2 \times (1 + t\lambda_1 - t)^{\alpha-1} \prod_{j=2}^{N} \prod_{1 \leq i < j \leq N} (1 - t + t\lambda_1 - t\lambda_j)^2 \prod_{j=1}^{N-1} d\lambda_j.
\]
As $1 - t \leq 1 + t\lambda_1 - t \leq 1, 0 \leq 1 - t + t\lambda_1 - t\lambda_j \leq 1$ and $\alpha \leq 1$, one has
\[
V_\alpha(\mathcal{K}_t) \leq Z_N t^{(N-1)^2+(a-1)(N-1)(1-t)^{\alpha-1}} \int_{\Delta_1} \prod_{j=2}^{N} \lambda_j^{\alpha-1} \prod_{2 \leq i < j \leq N} (\lambda_i - \lambda_j)^2 \prod_{j=1}^{N-1} d\lambda_j
\]
\[
= \frac{Z_N}{Z_{N-1}} t^{(N-1)^2+(a-1)(N-1)(1-t)^{\alpha-1}} V_\alpha(\mathcal{D}_{N-1}) \int_{0}^{\lambda_1} (1 - \lambda_1)^{N^2 - 2N + (N-1)(a-1)} d\lambda_1
\]
\[
\leq \frac{Z_N}{Z_{N-1}} t^{(N-1)^2+(a-1)(N-1)(1-t)^{\alpha-1}} V_\alpha(\mathcal{D}_{N-1}).
\]

The Stirling approximation formula (8) implies that for $e^{(t-1-N)} < t < \frac{4}{5}$,
\[
VR_\alpha(\mathcal{K}_t, \mathcal{D}) \leq \tilde{C} t
\]
for some constant $\tilde{C}_t > 0$.

Now we bound $VR_{\text{HS}}(\mathcal{K}_t)$ from below. That is,
\[
VR_{\text{HS}}(\mathcal{K}_t) = Z_N t^{(N-1)^2} \int_{\Delta_1} \prod_{2 \leq i < j \leq N} (\lambda_i - \lambda_j)^2 \prod_{j=2}^{N} (1 - t + t\lambda_1 - t\lambda_j)^2 \prod_{j=1}^{N-1} d\lambda_j
\]
\[
\geq Z_N t^{N^2-1} VR_{\text{HS}}(\mathcal{D}),
\]
where the inequality follows $1 - t + t\lambda_1 - t\lambda_j \geq t\lambda_1 - t\lambda_j$ for all $j = 2, \ldots, N$. Equivalently, $\text{VR}_{\text{HS}}(K', D) \geq t$. Together with inequality (26), one has

$$\text{VR}_{\text{HS}}(K', D) \geq \varepsilon \text{VR}_{a}(K', D)$$

for $\varepsilon = \frac{1}{e} > 0$. This concludes the optimality of the upper bound for $\alpha \leq 1$.

### 4. The $\alpha$-volumes of $\mathcal{S}$ and $\mathbb{P}^{\text{PPT}}$

In this section, we provide answers to questions (1) and (3) for large $N$ in the sense of $\alpha$-volume. These two questions are related to the ones listed on page 9 in [1], that is,

- Does the volume of the set of separable states go really to zero as the dimension of the composite system $N$ grows, and how fast?
- Has the set of separable states really a volume strictly smaller than the volume of the set of states with a positive partial transpose?

Here, we supply solutions to those two questions in the sense of $\alpha$-volume for large $N$.

The geometry of $D$ is reasonably well understood, and hence the volumes (e.g. Hilbert–Schmidt, Bures, $\alpha$) of $D$ can be calculated precisely [21, 23, 25, 28]. However, the geometry of $\mathcal{S}$ and $\mathbb{P}^{\text{PPT}}$ are more complicated, and less information for them is available. This makes the calculation of volumes (Hilbert–Schmidt, Bures, $\alpha$) of $\mathcal{S}$ and $\mathbb{P}^{\text{PPT}}$ quite intractable. Thanks to the remarkable works of [12, 13], one knows that $\text{VR}_{\text{HS}}(K, D)$ decays to 0 very quickly for large $N$ and $\text{VR}_{\text{HS}}(\mathbb{P}^{\text{PPT}}, D)$ is essentially a constant. In [14], the author obtained similar results in the sense of Bures volume. Therefore, the probability of finding separable quantum states within quantum states is extremely small and the Peres–Horodecki PPT criterion as tools to detect separability is imprecise for large $N$, in the sense of both Hilbert–Schmidt and Bures volumes.

The following two corollaries give the estimates of the $\alpha$-volume of $\mathcal{S}$. They are direct consequences of theorems 1 and 2, and the estimates for $\text{VR}_{\text{HS}}(\mathcal{S}, D)$ implicit in [12].

**Corollary 3 (Large number of small subsystems).** Let $\alpha > 0$ be a (fixed) constant independent of the dimension of $\mathcal{H}$. For the system $\mathcal{H} = (\mathbb{C}^D)^{\otimes n}$, there exist universal constants $c_4, C_4 > 0$, such that for all $D, n \geq 2$,

$$\left(\frac{c_4}{N^{1/2\alpha_D}}\right)^{\max\{1, \alpha\}} \leq \text{VR}_{a}(\mathcal{S}, D) \leq C_4 \left(\frac{(Dn \ln n)^{1/2}}{N^{1/2\alpha_D}}\right)^{\min\{1, \alpha\}},$$

where $\alpha_D = \frac{1}{2} \log_D (1 + \frac{1}{D}) - \frac{1}{2\pi} \log_D (D + 1)$.

**Remark.** Recall that for $\mathcal{H} = (\mathbb{C}^D)^{\otimes n}, N = D^n$, Corollary 3 means that, for fixed (small) $D$ and fixed $\alpha > 0$, $\text{VR}_{a}(\mathcal{S}, D)$ is bounded above by (up to a multiplicative universal constant) $\left(\frac{(Dn \ln n)^{1/2}}{N^{1/2\alpha_D}}\right)^{\min\{1, \alpha\}}$, and hence $\text{VR}_{a}(\mathcal{S}, D)$ goes to zero exponentially as $n \to \infty$. This implies that $\mathbb{P}_{a}(\mathcal{S}, N, D) := \frac{\text{VR}_{a}(\mathcal{S}, D)}{\text{VR}_{a}(\mathcal{S}, D)^{\otimes n}}$, the $\alpha$-probability of finding separable states within $\mathcal{D}$ (on $\mathcal{H} = (\mathbb{C}^D)^{\otimes n}$), goes to zero super-double-exponentially, since $\text{VR}_{a}(\mathcal{S}, D)$ is already the $(N^2 - 1)$st root of $\mathbb{P}_{a}(\mathcal{S}, N, D)$. Numerical results show that, for even (moderately) large $N$, $\mathbb{P}_{a}(\mathcal{S}, N, D)$ is very small. For instance, $\mathbb{P}_{2}(\mathcal{S}, 256, 2) \leq 2.1 \times 10^{-1595}$ with $n = 8, \mathbb{P}_{2}(\mathcal{S}, 512, 2) \leq 3.1 \times 10^{-43631}$ with $n = 9, \mathbb{P}_{3}(\mathcal{S}, 512, 2) \leq 1.1 \times 10^{-43412}$ with $n = 9$ and $\mathbb{P}_{3}(\mathcal{S}, 243, 3) \leq 1.52 \times 10^{-5301}$ with $n = 5$. Similarly, we list some probability for $0 < \alpha < 1, \mathbb{P}_{0.5}(\mathcal{S}, 256, 2) \leq 8.8 \times 10^{-479}$ with $n = 8, \mathbb{P}_{0.5}(\mathcal{S}, 512, 2) \leq 3.36 \times 10^{-21102}$ with $n = 9, \mathbb{P}_{0.2}(\mathcal{S}, 512, 2) \leq 6.62 \times 10^{-7905}$ with $n = 9$ and $\mathbb{P}_{0.5}(\mathcal{S}, 243, 3) \leq 9.5 \times 10^{-2351}$ with $n = 5$. 


Corollary 4 (Small number of large subsystems). Let $\alpha > 0$ be a (fixed) constant independent of the dimension of $\mathcal{H}$. For the system $\mathcal{H} = (\mathbb{C}^D)^{\otimes n}$, there exist universal computable constants $c_5, C_5 > 0$, such that for all $D, n \geq 2$,

\[
\left( \frac{c_5^n}{N^{1/2 - 1/(2n)}} \right)^{\max\{1,\alpha\}} \leq \nu_{\alpha}(\mathcal{S}, D) \leq C_5 \left( \frac{(t \ln n)^{1/2}}{N^{1/2 - 1/(2\alpha n)}} \right)^{\min\{1,\alpha\}}.
\]

Remark. Corollary 4 shows that, for fixed (small) $n$, ‘the order of decay’ of $\nu_{\alpha}(\mathcal{S}, D)$ is between $D^{(1/2 - 1/2)\max\{1,\alpha\}}$ and $D^{(1/2 - 1/2)\min\{1,\alpha\}}$ as $D \to \infty$. Therefore, $\mathbb{P}_a(\mathcal{S}, N, D)$ goes to zero with order between $D^{(1/2 - n/2)(\alpha^2 - 1)\max\{1,\alpha\}}$ and $D^{(1/2 - n/2)(\alpha^2 - 1)\min\{1,\alpha\}}$ as $D \to \infty$. Numerical results show that $\mathbb{P}_a(\mathcal{S}, N, D)$ is very small for (even moderately) large $N$, and its ‘order of decay’ relies not only on $D$ but also (heavily) on $n$. For instance, if $\alpha = 2$, then $\mathbb{P}_a(\mathcal{S}, 784, 28) \leq 1.35 \times 10^{-5230}$ with $n = 2$, $\mathbb{P}_a(\mathcal{S}, 900, 30) \leq 8.5 \times 10^{-19028}$ with $n = 2$, $\mathbb{P}_a(\mathcal{S}, 729, 9) \leq 4.9 \times 10^{-27218}$ with $n = 3$ and $\mathbb{P}_a(\mathcal{S}, 1000, 10) \leq 1.74 \times 10^{-97132}$ with $n = 3$. Similarly, if $\alpha = 1/2$, then $\mathbb{P}_{0.5}(\mathcal{S}, 900, 30) \leq 1.12 \times 10^{-8235}$ with $n = 2$ and $\mathbb{P}_{0.5}(\mathcal{S}, 729, 9) \leq 6.8 \times 10^{-12538}$ with $n = 3$.

For a bipartite system $\mathcal{H} = \mathbb{C}^{D_1} \otimes \mathbb{C}^{D_2}$, any state $\rho$ on $\mathcal{H}$ can be expressed uniquely as

$$\rho = \sum_{i,j} \sum_{\alpha, \beta} \rho_{\alpha, j\beta} |e_i \otimes f_{\alpha\beta}| \langle e_j \otimes f_{\alpha\beta}|,$$

where $\{e_i\}_{i=1}^{D_1}$ and $\{f_{\alpha\beta}\}_{\alpha=1}^{D_2}$ are the canonical bases of $\mathbb{C}^{D_1}$ and $\mathbb{C}^{D_2}$, respectively. Define the partial transpose $T(\rho)$ with respect to the first subsystem as

$$T(\rho) = \sum_{i,j} \sum_{\alpha, \beta} \rho_{\alpha, j\beta} |e_i \otimes f_{\alpha\beta}| \langle e_j \otimes f_{\alpha\beta}|.$$

We use $\mathbb{P}_{\text{PPT}} (\subset D)$ to denote the set of quantum states with PPT, i.e. $\rho \in \mathbb{P}_{\text{PPT}}$ if and only if $T(\rho) \geq 0$. Thus the Peres–Horodecki PPT criterion is equivalent to $\mathcal{S} \subset \mathbb{P}_{\text{PPT}}$. For systems $\mathbb{C}^2 \otimes \mathbb{C}^2$, $\mathbb{C}^2 \otimes \mathbb{C}^3$ and $\mathbb{C}^3 \otimes \mathbb{C}^2$, one has $\mathcal{S} = \mathbb{P}_{\text{PPT}}$. That is, the Peres–Horodecki PPT criterion is a necessary and sufficient condition for $\rho$ being a separable quantum state if dimension of $\mathcal{H}$ is less than or equal to 6.

The following corollary is to estimate the $\alpha$-volume of $\mathbb{P}_{\text{PPT}}$. It is a direct consequence of theorems 1 and 2, and theorem 1 in [12]. Note the upper bound is trivial because $\mathbb{P}_{\text{PPT}} \subset D$.

Corollary 5 ($\alpha$-Volume of $\mathbb{P}_{\text{PPT}}$). Let $\alpha > 0$ be a (fixed) constant independent of the dimension of $\mathcal{H}$. There exists an absolute computable constant (depending on $\alpha$) $c_0 > 0$, such that for any bipartite system $\mathcal{H} = \mathbb{C}^D \otimes \mathbb{C}^D$, $\alpha > 0$, $\nu_{\alpha}(\mathcal{P}_{\text{PPT}}, D) \leq 1$.

Remark. An immediate consequence of corollaries 4 and 5 is that for $\mathcal{H} = \mathbb{C}^D \otimes \mathbb{C}^D$ and large $D$, ‘the order of decay’ of $\nu_{\alpha}(\mathcal{S}, \mathbb{P}_{\text{PPT}})$ is between $D^{1/2 - 1/2\max\{1,\alpha\}}$ and $D^{1/2 - 1/2\min\{1,\alpha\}}$ as $D \to \infty$. The upper bound decreases to 0 as $D \to \infty$, and hence $\mathcal{S}$ really has a volume strictly small than $\mathbb{P}_{\text{PPT}}$. Moreover, the conditional $\alpha$-probability of separability given positive partial transpose condition is exceedingly small. In other words, the Peres–Horodecki PPT criterion is imprecise as tools to detect separability for large $N$. We point out that our proof yields $c_0(N, \alpha) \to (e^{-1/4}/4)^{\max\{1,\alpha\}} \approx (0.195)^{\max\{1,\alpha\}}$ as $N \to \infty$ (see also [12]). Any improvements of this constant are very much appreciated. Let us denote the conditional $\alpha$-probability of $\mathcal{S}$ given $\mathbb{P}_{\text{PPT}}$ as $\mathbb{P}_a(\mathcal{S}|\mathbb{P}_{\text{PPT}}, N, D) = \frac{\nu_{\alpha}(\mathcal{S}|\mathbb{P}_{\text{PPT}})}{\nu_{\alpha}(\mathbb{P}_{\text{PPT}})}$, for fixed $\alpha > 0$ and $N = D^n$. Below, we provide some numerical examples for $\mathbb{P}_a(\mathcal{S}|\mathbb{P}_{\text{PPT}}, N, D)$ to see the effectiveness of our estimates. For instance, $\mathbb{P}_{1.1}(\mathcal{S}|\mathbb{P}_{\text{PPT}}, 4096, 2) \leq 2.5 \times 10^{-272}$ with $n = 12$, $\mathbb{P}_{1.1}(\mathcal{S}|\mathbb{P}_{\text{PPT}}, 6561, 3) \leq 1.82 \times 10^{-122}$ with $n = 8$ and $\mathbb{P}_{0.9}(\mathcal{S}|\mathbb{P}_{\text{PPT}}, 6561, 3) \leq 3.74 \times 10^{-73561}$ with $n = 8$. 

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5. Conclusion and comments

In the present paper, we compare several important measures (Hilbert–Schmidt, Bures, α-measure) on the set of states \( D \) on \( \mathcal{H} \). Roughly speaking, for any fixed \( \alpha > 0 \) and any measurable subset \( \mathcal{K} \subseteq D \), the α-volume ratio \( \text{VR}_\alpha(\mathcal{K}, D) \) is bounded from below by \( \text{VR}_{\text{HS}}(\mathcal{K}, D)^{\max[1,\alpha]} \) and from above by \( \text{VR}_{\text{HS}}(\mathcal{K}, D)^{\min[1,\alpha]} \) (up to some universal constants). We also compare the α-volume radii ratio \( \text{VR}_\alpha(\mathcal{K}, D) \) with the Bures volume radii ratio \( \text{VR}_\beta(\mathcal{K}, D) \) for fixed \( \beta > 0 \). In particular, theorem 3 improves the estimates on \( \text{VR}_\beta(\mathcal{K}, D) \) in [14].

Employing these estimates to the set of separable states \( S \) and to \( \mathcal{P} \mathcal{P} \mathcal{T} \) (the set of states with positive partial transpose), we obtain both upper and lower bounds for \( \text{VR}_\alpha(S, D) \) and \( \text{VR}_\alpha(\mathcal{P} \mathcal{P} \mathcal{T}, D) \) for fixed \( \alpha > 0 \). It is shown that \( \text{VR}_\alpha(\mathcal{P} \mathcal{P} \mathcal{T}, D) \) is essentially a constant, while \( \text{VR}_\alpha(S, D) \) is of order (at most) \( N^{-k} \) for some \( k > 0 \) (and so is \( \text{VR}_\alpha(S, \mathcal{P} \mathcal{P} \mathcal{T}) \)). This means that a typical (PPT) quantum state is entangled; namely, randomly choosing a (PPT) quantum state in \( D \) on \( \mathcal{H} \) with (even moderately) large dimension, this state is entangled with probability (very) close to 1. This may be of importance in the analysis of quantum algorithms or quantum protocols that rely on entanglement. This also shows that, for not-too-small dimensions, the Peres–Horodecki PPT criterion is not a precise tool to establish separability. Consequently, more precise necessary and/or sufficient conditions for separability are in great demand.

In applications, one frequently considers \( \mathcal{H} \) as \( \mathcal{H} = (\mathbb{C}^D)^{\otimes n} \), often with \( D = 2 \) or \( D = 3 \). The estimates we obtain show that the α-probability of \( S \) in \( D \) or of \( S \) in \( \mathcal{P} \mathcal{P} \mathcal{T} \) is very small even for small values of \( n \). For example, the ratio of 2-volumes \( \frac{V_2(S)}{V_2(D)} \) is less than \( 2.1 \times 10^{-1595} \) if \( D = 2 \) and \( n = 8 \), and similarly, \( \frac{V_3(S)}{V_3(D)} \leq 1.52 \times 10^{-530} \) if \( D = 3 \) and \( n = 5 \). Likewise, the ratio of (1.1)-volumes \( \frac{V_{1.1}(S)}{V_{1.1}(\mathcal{P} \mathcal{P} \mathcal{T})} \) is less than \( 2.5 \times 10^{-272} \) if \( D = 2 \) and \( n = 12 \), and similarly, \( \frac{V_{1.1}(S)}{V_{1.1}(\mathcal{P} \mathcal{P} \mathcal{T})} \leq 1.8 \times 10^{-122} \) if \( D = 3 \) and \( n = 8 \).

Concerning question (4), the situation is much less clear. In fact, to answer this question, one needs to prove that, for any bipartite system \( \mathcal{H} = \mathbb{C}^D \otimes \mathbb{C}^D \), there exists a constant \( C_0 < 1 \), such that \( \text{VR}_\alpha(\mathcal{P} \mathcal{P} \mathcal{T}, D) \leq C_0 \). Similar questions can be asked by replacing the α-volume with the Hilbert–Schmidt volume and the Bures volume.

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References

[1] Žyczkowski K, Horodecki P, Sanpera A and Lewenstein M 1998 Phys. Rev. A 58 883
[2] Horodecki R, Horodecki P, Horodecki M and Horodecki K 2009 Rev. Mod. Phys. 81 865
[3] Nielsen M A and Chuang I L 2000 Quantum Computation and Quantum Information (Cambridge: Cambridge University Press)
[4] Einstein A, Podolsky B and Rosen N 1935 Phys. Rev. 47 777
[5] Schrödinger E 1935 Naturwissenschaften 23 807
[6] Gurvits L 2004 J. Comput. Syst. Sci. 69 448
[7] Peres A 1996 Phys. Rev. Lett. 77 1413
[8] Horodecki M, Horodecki P and Horodecki R 1996 Phys. Lett. A 223 1
[9] Størmer E 1963 Acta Math. 110 233
[10] Woronowicz S L 1976 Rep. Math. Phys. 10 165
[11] Horodecki P 1997 Phys. Lett. A 232 333
[12] Aubrun G and Szarek S J 2006 Phys. Rev. A 73 022109
[13] Szarek S J 2005 Phys. Rev. A 72 032304
[14] Ye D 2009 J. Math. Phys. 50 083502
[15] Braunstein S L 1996 Phys. Lett. A 219 169
[16] Hall M J W 1998 Phys. Lett. A 242 123
[17] Lloyd S and Pagels H 1988 Ann. Phys. (NY) 188 186
[18] Lubkin E 1978 J. Math. Phys. 19 1028
[19] Page D 1993 Phys. Rev. Lett. 71 1291
[20] Sommers H J and Życzkowski K 2004 J. Phys. A: Math. Gen. 37 8457
[21] Życzkowski K and Sommers H J 2001 J. Phys. A: Math. Gen. 34 7111
[22] Werner R F 1989 Phys. Rev. A 40 4277
[23] Życzkowski K and Sommers H J 2003 J. Phys. A: Math. Gen. 36 10115
[24] Deming W E and Colcord C G 1935 Nature (London) 135 917
[25] Bengtsson I and Życzkowski K 2006 Geometry of Quantum States (Cambridge: Cambridge University Press)
[26] Mehta M L 1990 Random Matrices 2nd edn (New York: Academic)
[27] Ozawa M 2000 Phys. Lett. A 268 158
[28] Sommers H J and Życzkowski K 2003 J. Phys. A: Math. Gen. 36 10083
[29] Bures D J C 1969 Trans. Am. Math. Soc. 135 199
[30] Uhlmann A 1976 Rep. Math. Phys. 9 273
[31] Barnum H, Caves C, Fuchs C, Jozsa R and Schumacher B 1996 Phys. Rev. Lett. 73 2818
[32] Petz D 1996 Linear Algebra Appl. 244 81
[33] Hiai F and Petz D 2000 The Semicircle Law, Free Random Variables and Entropy (Mathematical Surveys and Monographs vol 77) (Providence, RI: American Mathematical Society)
[34] Hardy G H, Littlewood J E and Pólya G 1952 Inequalities 2nd edn (Cambridge: Cambridge University Press)