Influence of velocity curl on conservation laws

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The paper discusses impact of the velocity curl on some conservation laws in the gravitational field and electromagnetic field, by means of the characteristics of quaternions. When the velocity curl cannot be neglected, it will cause the predictions to departure slightly from the conservation laws, which include mass continuity equation, charge continuity equation, and conservation of spin, etc. And the scalar potential of gravitational field has an effect on the speed of light, the conservation of mass, and conservation of charge, etc. The results explain how the velocity curl influences some conservation laws in the gravitational field and electromagnetic field.

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I. INTRODUCTION

In the gravitational field, the rotating physical object can be considered as the particle with the velocity curl. But the existing theories can not deal with the theoretical model. The paper attempts to solve this problem, and reason out why the angular velocity has an influence on the conservation laws about the particle motion.

The quaternion can be used to describe the property of the gravitational field, including the scalar invariants and conservation laws. In the gravitational field, there exist the conservation of mass, the mass invariant, the energy invariant, and the conservation of energy. Making use of the scalar property of the quaternion, the potential and the strength of gravitational field both have the influence on these conservations. While the angular velocity has an impact on some physical quantities, such as the gyroscopic torque and Coriolis force.

With the features of the quaternions, we find the mass density, the spin density, and the energy density may be variable respectively under the quaternion coordinate transformation in the gravitational field and electromagnetic field. So are the mass continuity equation, the charge continuity equation, the spin continuity equation, and the energy continuity equation, and the continuity equation of spin magnetic moment, etc.

II. SCALAR INVARIANTS OF GRAVITATIONAL FIELD

Making use of the characteristics of the quaternions, we may obtain some kinds of scalar invariants in the case for coexistence of gravitational field and electromagnetic field with the velocity curl, under the octonion coordinate transformation. And each octonion definition of physical quantity possesses one invariant equation.

A. Quaternion transformation

In the quaternion space, the coordinates \( d_0, d_1, d_2, \) and \( d_3 \) are all real, the basis vector is \( \mathbf{E}_g = (1, \mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3) \). The quaternion \( \mathbb{D}(d_0, d_1, d_2, d_3) \) is defined as,

\[
\mathbb{D} = d_0 + d_1 \mathbf{i}_1 + d_2 \mathbf{i}_2 + d_3 \mathbf{i}_3
\]

When the coordinate system is transformed into the other, the quaternion \( \mathbb{D} \) will be become to the physical quantity \( \mathbb{D}'(d'_0, d'_1, d'_2, d'_3) \).

\[
\mathbb{D}' = \mathbb{K}^* \circ \mathbb{D} \circ \mathbb{K}
\]

where, the \( \mathbb{K} \) is the quaternion, and \( \mathbb{K}^* \circ \mathbb{K} = 1; \) the \( * \) denotes the quaternion multiplication; the \( \circ \) indicates the conjugate of quaternion.

Both sides’ scalar parts keep unchanged in the above, when the quaternion coordinate system is transforming. Therefore

\[
d_0 = d'_0
\]

Making use of the above equation, we can obtain some kinds of scalar invariant equations under the quaternion coordinate transformation in the gravitational field.

B. Radius vector

In the quaternion space, the coordinates are \( r_0, r_1, r_2, \) and \( r_3 \), while the radius vector is \( \mathbb{R}_g = \Sigma(r_i \mathbf{i}_i) \). It can be combined with the quaternion \( \mathbb{X}_g = \Sigma(x_i \mathbf{i}_i) \) to become the compounding radius vector

\[
\bar{\mathbb{R}}_g = \mathbb{R}_g + k_{rx} \mathbb{X}_g .
\]

where, \( r_0 = v_0 t; \) \( x_0 = a_0 t; \) \( v_0 \) is the speed of light beam; \( a_0 \) is the scalar potential of gravitational field; \( t \) denotes the time; \( k_{rx} = 1 \) and \( k_{cg} \) are the coefficients.

In other words, the \( \bar{\mathbb{R}}_g \) can be considered as the radius vector in the quaternion compounding space, with the basis vector \( (\mathbf{i}_0, \mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3) \), and \( \mathbf{i}_0 = 1 \).
TABLE I: The quaternion multiplication table.

| r   | i1  | i2  | i3  |
|-----|-----|-----|-----|
| 1   | 1   | i1  | i2  |
| i1  | i1  | -1  | -i2 |
| i2  | i2  | -i3 | i1  |
| i3  | i3  | i2  | -i1 |

When the coordinate system is transformed into the other, we have a radius vector \( \bar{r}_0 = (v'_0 + k_{rx}a_0)t' \) from Eq.(2).

From Eqs.(3) and (4), we obtain
\[
\bar{v}_0 = \bar{v}'_0
\]
where, \( \bar{v}_0 = \bar{v}'_0 ; \bar{v}_0 = v_0 + k_{rx}a_0 \).

In some special cases, we may substitute a quaternion quantity \( \bar{z}_g(\bar{z}_0, \bar{z}_1, \bar{z}_2, \bar{z}_3) \) for the quaternion radius vector \( \bar{R}(\bar{r}_0, \bar{r}_1, \bar{r}_2, \bar{r}_3) \). The former quantity is defined as
\[
\bar{z}_g = \bar{R}_g \circ \bar{R}_g
\]
where, \( \bar{z}_0 = (\bar{r}_0)^2 - (\bar{r}_1)^2 - (\bar{r}_2)^2 - (\bar{r}_3)^2 \).

When the quaternion coordinate system is rotated, the physical quantity \( \bar{z}_g'(\bar{z}'_0, \bar{z}'_1, \bar{z}'_2, \bar{z}'_3) \) from Eq.(2).

From Eqs.(3) and (6), we gain
\[
(\bar{r}_0)^2 - \Sigma(\bar{r}_j)^2 = (\bar{r}'_0)^2 - \Sigma(\bar{r}'_j)^2
\]
where, \( j = 1, 2, 3; i = 0, 1, 2, 3 \).

The above means that there may exist the special case of the \( x_i \neq 0 \) when the \( r_i = 0 \). We can obtain different invariants from the physical definitions, by means of the characteristics of quaternion coordinate transformation in the gravitational field.

C. Speed of light

In the quaternion space, the velocity \( \bar{v}_g(\bar{v}_0, \bar{v}_1, \bar{v}_2, \bar{v}_3) \) is combined with gravitational potential \( \bar{A}_g(a_0, a_1, a_2, a_3) \) to become the compounding velocity \( \bar{V}_g(\bar{v}_0, \bar{v}_1, \bar{v}_2, \bar{v}_3) \).
\[
\bar{v}_g = \bar{v}_g + k_{rx}\bar{A}_g
\]
where, the component \( a_j = 0 \) in Newtonian gravity.

When the quaternion coordinate system is transformed into the other, we have one velocity \( \bar{V}_g'(\bar{v}'_0, \bar{v}'_1, \bar{v}'_2, \bar{v}'_3) \) from Eq.(2) in the quaternion compounding space.

From Eq.(3) and the definition of velocity, we obtain the invariant equation about the speed of light.
\[
\bar{v}_0 = \bar{v}'_0
\]

Choosing the definition combination of the quaternion velocity and radius vector, we find the invariants, Eqs.(5) and (9), and the Galilean transformation.
\[
\bar{r}_0 = \bar{r}'_0, \quad \bar{v}_0 = \bar{v}'_0.
\]

In some cases, we obtain Lorentz transformation from the invariant equations, Eqs.(7) and (9).
\[
(\bar{r}_0)^2 - \Sigma(\bar{r}_j)^2 = (\bar{r}'_0)^2 - \Sigma(\bar{r}'_j)^2, \quad \bar{v}_0 = \bar{v}'_0.
\]

The above states the speed of light, \( \bar{v}_0 \), may be variable in some cases, due to the influence of scalar potential of gravitational field. And we can obtain some kinds of coordinate transformations from the different definition combinations in quaternion compounding space, such as the Galilean and Lorentz transformations, etc.

D. Potential and strength

In the quaternion compounding space, the potential \( \bar{A}_g(a_0, a_1, a_2, a_3) \) is defined from Eq.(8).
\[
\bar{A}_g = \bar{A}_g + K_{rx}\bar{V}_g
\]
where, \( K_{rx} = 1/k_{rx}; \bar{A}_g \) is the gravitational potential.

The strength \( \bar{B}_g(b_0, b_1, b_2, b_3) \) is defined from Eq.(12).
\[
\bar{B}_g = \bigcirc \bar{A}_g = \bar{B}_g + K_{rx}\bar{U}_g
\]
where, \( \bigcirc = (i\partial_i) \), with \( \partial_i = \partial/\partial r_i ; \nabla = \Sigma (i\partial_i) \); most cases, \( \bar{r}_i \approx r_i \), and then \( \partial/\partial \bar{r}_i \approx \partial/\partial r_i \); the velocity \( \bar{V}_g = v_0[\bigcirc \bar{B}_g - \nabla \{\Sigma (r_j \partial_j)\}] \); velocity curl \( \bar{U}_g = \bigcirc \bar{B}_g \); \( \bar{A}_g = v_0 \bigcirc \bar{X}_g \); \( \bar{B}_g = \bigcirc \bar{A}_g \); \( b_0 = \partial_0 a_0 - \Sigma (\partial_0 a_1) \).

In the planar polar coordinates, the velocity \( \bar{v} = \bar{v} \times \bar{a} \), with \( \bar{a} \) being the angular velocity. And then, we have the relation \( \nabla \times \bar{v} = 2\bar{a} \). Where, \( \bar{v} = \Sigma (v_j \bar{e}_j) \).

In the paper, we choose the gauge condition, \( \bar{b}_0 = 0 \), to simplify succeeding calculation. And the gravitational strength is written as \( \bar{B}_g = \bar{B} / \bar{v}_0 + \bar{B} \).
\[
\bar{B} / \bar{v}_0 = \partial_0 \bar{\pi} + \nabla a_0
\]
\[
\bar{B} = \nabla \times \bar{\pi}
\]
where, \( \bar{\pi} = \Sigma (a_j \bar{e}_j) ; \bar{B} = \Sigma (b_j \bar{e}_j) \).

When the quaternion coordinate system is transformed into the other, we have the potential \( \bar{A}_g'(a'_0, a'_1, a'_2, a'_3) \) and strength \( \bar{B}_g'(b'_0, b'_1, b'_2, b'_3) \) respectively from Eq.(2).

From Eqs.(3) and (12), we will obtain the invariant equation about the scalar potential,
\[
a_0 = a'_0
\]
and the strength invariant from Eqs.(3) and (13).
\[
\bar{b}_0 = \bar{b}'_0
\]

The above equations state the gravitational field in the quaternion compounding space possesses not only the potential and strength but also the velocity and velocity curl. While, the scalar potential \( a_0 \) and scalar strength \( b_0 \) of the gravitational field are variable, due to the influence of the velocity and the velocity curl.
E. Conservation of mass

The source density $S'$ is defined from the strength $B_g$ in the quaternion compounding space.

$$
\mu S' = -\left(\frac{\partial}{\partial t} \hat{\omega}_0 + \hat{\Omega}\right) \ast \circ B_g
= \mu^B_g \frac{\partial S}{\partial t} - \frac{\partial}{\partial t} \frac{\partial}{\partial \hat{\omega}_0} \frac{\partial}{\partial \hat{\Omega}}
$$

where, $\mu$ and $\mu^B_g$ are the constants; $S'$ covers the linear momentum density $S_g$ and an extra part $B_g \ast \circ (\hat{\omega}_0 \mu^B_g)$; and $B_g \circ \circ (2\mu^B_g)$ is the energy density of gravitational field, and is similar to that of electromagnetic field.

The linear momentum density $P_g(\hat{p}_0, \hat{p}_1, \hat{p}_2, \hat{p}_3)$ is the extension of $S_g = \hat{m} \hat{\Omega} / \hat{g}$.

$$
P_g = \mu \hat{S}' / \mu^B_g
$$

where, $\hat{p}_0 = \hat{m} \hat{\omega}_0$, $\hat{p}_j = \hat{m} \hat{\omega}_j \hat{t}_j$; $\hat{m} = m + \Delta m$; $m$ is the inertial mass density, and $\hat{m}$ is the gravitational mass density; $\Delta m = -B_g \ast \circ (\hat{\omega}_0 \mu^B_g)$.

When the quaternion coordinate system rotates, we have the linear momentum density $P_g(\hat{p}_0, \hat{p}_1, \hat{p}_2, \hat{p}_3)$, and the scalar invariant from Eqs.(3) and (19).

$$
\hat{m} \hat{\omega}_0 = \hat{m}' \hat{\omega}_0'
$$

By Eqs.(9) and (20), the gravitational mass density $\hat{m}$ remains unchanged in the gravitational field. And then we have the conservation of mass as follows.

$$
\hat{m} = \hat{m}'
$$

In the same way, the inertial mass density $m$ is the invariant from the definitions of $\hat{\Omega}_g$ and $S_g$.

$$
m = m'
$$

The above can also be reduced from Eq.(21), in the case of the $\hat{b}_j = 0$ and $\Delta m = 0$.

The above means that if we emphasize the definitions of the velocity and linear momentum, the gravitational mass density will remain the same under the coordinate transformation in Eq.(2) in the compounding space. The results are the same as that in the quaternion space.

F. Mass continuity equation

The applied force density $\hat{P}_g(\hat{f}_0, \hat{f}_1, \hat{f}_2, \hat{f}_3)$ is defined from the linear momentum density $P_g$.

$$
\hat{P}_g = \hat{v}_0(\hat{B}_g / \hat{v}_0 + \hat{\Omega}) \ast \circ \hat{P}_g
$$

where, the applied force density includes the inertial force density and the gravitational force density, etc. While, the scalar $\hat{f}_0 = \hat{v}_0 \hat{p}_0 / \hat{v}_0 + \hat{v}_0 \Sigma (\hat{p}_j / \hat{v}_j) + \Sigma (\hat{b}_j \hat{p}_j)$.

From the above, the cross product of velocity curl with linear momentum is the part of force, which is similar to the Coriolis force in Newtonian gravity.

When the quaternion coordinate system rotates, we have the force density $\hat{P}_g(\hat{f}_0, \hat{f}_1, \hat{f}_2, \hat{f}_3)$ from Eqs.(2), (3) and (22). And then, we have

$$
\hat{f}_0 = \hat{f}_0'
$$

In the equilibrium state, there is $\hat{f}_0 = 0$. And we have the mass continuity equation by Eqs.(9) and (23).

$$
\partial \hat{v}_0 / \partial t + \Sigma (\partial \hat{p}_j / \partial \hat{r}_j) + \Sigma (\hat{b}_j \hat{p}_j) / \hat{v}_0 = 0
$$

further, if $a_j = b_j = 0$, the above is reduced to

$$
\partial m / \partial t + \Sigma (\partial p_j / \partial r_j) = 0
$$

The above states the potential $\hat{A}_g$, strength $\hat{B}_g$, and curl $U_g$ have influences on the mass continuity equation, although the term $\Sigma (\hat{b}_j \hat{p}_j) / \hat{v}_0$ and extra mass density $\Delta m$ are very tiny. The mass continuity equation is the invariant under the quaternion transformation, if we choose the definition combination of the velocity and applied force in the gravitational field.

G. $f_0$ continuity equation

In a similar way, we obtain a new physical quantity density, $\hat{C}_g(\hat{c}_0, \hat{c}_1, \hat{c}_2, \hat{c}_3)$, which can be defined from the applied force density $\hat{P}_g$.

$$
\hat{C}_g = \hat{v}_0(\hat{B}_g / \hat{v}_0 + \hat{\Omega}) \ast \circ \hat{P}_g
$$

where, $\hat{c}_0 = \hat{v}_0 \hat{f}_0 / \hat{v}_0 + \hat{v}_0 \Sigma (\partial \hat{f}_j / \partial \hat{r}_j) + \Sigma (\hat{b}_j \hat{f}_j)$ is the scalar part of $\hat{C}_g$.

When the quaternion coordinate system rotates, we have the density $\hat{C}_g(\hat{c}_0, \hat{c}_1, \hat{c}_2, \hat{c}_3)$ from Eqs.(2), (3), and (26). And then, we have

$$
\hat{c}_0 = \hat{c}'_0
$$

In some cases, there is $\hat{C}_g = 0$. And then we have the continuity equation about $\hat{f}_0$ by Eqs.(9) and (27).

$$
\partial \hat{f}_0 / \partial t + \Sigma (\partial \hat{f}_j / \partial \hat{r}_j) + \Sigma (\hat{b}_j \hat{f}_j) / \hat{v}_0 = 0
$$

further, if $\hat{b}_j = 0$, the above is reduced to

$$
\partial \hat{f}_0 / \partial t + \Sigma (\partial \hat{f}_j / \partial \hat{r}_j) = 0
$$

The above states the strength $\hat{B}_g$ and velocity curl $U_g$ have an influence on the continuity equation about $\hat{f}_0$, although we do not know much about the physical quantity $\hat{C}_g$. This continuity equation is also the invariant under the quaternion transformation.

| Table II: The physical quantities of gravitational field in the quaternion compounding space. |
|----------------------------------|-----------------|-----------------|-----------------|
| space | field | mechanics | matter |
|--------|--------|-----------|--------|
| $\hat{B}_g$ | $\hat{X}_g$ | $\hat{P}_g$ | $\hat{L}_g$ |
| $\hat{\Omega}_g$ | $\hat{A}_g$ | $\hat{F}_g$ | $\hat{W}_g$ |
| $\hat{U}_g$ | $\hat{B}_g$ | $\hat{C}_g$ | $\hat{N}_g$ |
H. Spin density

The angular momentum density $\hat{L}_g(\vec{l}_0, \vec{l}_1, \vec{l}_2, \vec{l}_3)$ can be defined from the linear momentum density $\hat{p}_g$ and radius vector $\hat{r}_g$ in the quaternion compounding space.

$$\hat{L}_g = \hat{r}_g \circ \hat{p}_g$$  \hspace{1cm} (30)

where, the scalar part $\vec{l}_0 = \vec{r}_0 \vec{p}_0 - \Sigma(\vec{r}_j \vec{p}_j)$.

The scalar $l_0$ is regarded as the density of spin angular momentum in the quaternion compounding space. And the $l_0$ covers the spin angular momentum density, $\bar{l}_0$, in the gravitational field.

When the quaternion coordinate system rotates, we have the angular momentum density $\bar{L}_g(\vec{p}_0, \vec{p}_1, \vec{p}_2, \vec{p}_3)$ from Eq.(2). And then we have the conservation of spin from the above and Eq.(3).

$$\bar{l}_0 = \vec{p}_0$$ \hspace{1cm} (31)

The above means that the velocity and gravitational strength both have the influence on the orbital angular momentum and the spin angular momentum. And the spin angular momentum, $\bar{l}_0$, will be variable under the quaternion transformation in Eq.(2) in the quaternion compounding space.

I. Conservation of energy

The total energy density $\bar{W}_g(\bar{w}_0, \bar{w}_1, \bar{w}_2, \bar{w}_3)$ is defined from the angular momentum density $\bar{L}_g$ in the quaternion compounding space.

$$\bar{W}_g = \bar{v}_0(\bar{\mathbf{g}}/\bar{v}_0 + \odot) \circ \bar{L}_g$$ \hspace{1cm} (32)

where, the scalar part $\bar{w}_0 = \bar{g} \cdot \vec{1} + \bar{v}_0 \partial_0 \bar{v}_0 + \bar{v}_0 \nabla \cdot \vec{1}$ is the energy density; $\bar{w} = \Sigma(\bar{w}_j \vec{i}_j); \vec{1} = \Sigma(\bar{l}_j \vec{i}_j)$.

The total energy incorporates the potential energy, the kinetic energy, the torque, and the work, etc.

| definition | invariant | meaning |
|------------|-----------|---------|
| $\hat{r}_g$ | $\bar{r}_0 = \vec{r}_0$ | Galilean invariant |
| $\hat{g}_g \circ \hat{r}_g$ | $\bar{z}_0 = \vec{z}_0$ | Lorentz invariant |
| $\bar{v}_0$ | $\bar{v}_0 \circ \bar{v}_0$ | invariable speed of light |
| $\bar{a}_0$ | $\bar{a}_0 \circ \bar{a}_0$ | invariable scalar potential |
| $\hat{B}_g$ | $\bar{b}_0 = \vec{b}_0$ | invariable gauge |
| $\hat{p}_g$ | $\bar{p}_0 = \vec{p}_0$ | conservation of mass |
| $\hat{F}_g$ | $\bar{f}_0 = \vec{f}_0$ | mass continuity equation |
| $\hat{L}_g$ | $\bar{l}_0 = \vec{l}_0$ | invariable spin density |
| $\bar{w}_0$ | $\bar{w}_0 = \vec{w}_0$ | conservation of energy |
| $\bar{N}_g$ | $\bar{n}_0 = \vec{n}_0$ | energy continuity equation |

In the above, the cross product of angular momentum with velocity curl is the part torque, which is similar to the gyroscopic torque in Newtonian gravity. While the cross product of angular momentum with gravitational strength is the part torque also.

When the quaternion coordinate system rotates, we have the total energy density $\bar{W}_g(\bar{w}_0, \bar{w}_1, \bar{w}_2, \bar{w}_3)$ from Eq.(2). And then, we have the conservation of energy by Eq.(3) and the above.

$$\bar{w}_0 = \bar{w}_0'$$ \hspace{1cm} (33)

In some special cases, there exists $\bar{w}_0' = 0$, we obtain the spin continuity equation by Eqs.(9) and Eq.(33).

$$\partial \bar{v}_0/\partial r_0 + \nabla \cdot \vec{1} + \bar{g} \cdot \vec{1}/\bar{v}_0 = 0$$ \hspace{1cm} (34)

If the last term is neglected, the above is reduced to

$$\partial \bar{v}_0/\partial r_0 + \nabla \cdot \vec{1} = 0$$ \hspace{1cm} (35)

where, when the time $t$ is only the independent variable, the $\partial \bar{v}_0/\partial t$ will become the $d\bar{v}_0/\partial t$.

The above means that the energy density, $w_0$, may be variable under the quaternion transformation in Eq.(2) in the gravitational field, because of the gravitational strength and the velocity curl have the influence on the angular momentum density.

J. Energy continuity equation

The external power density $\bar{N}_g(\bar{n}_0, \bar{n}_1, \bar{n}_2, \bar{n}_3)$ is defined from the total energy density $\bar{W}_g$.

$$\bar{N}_g = \bar{v}_0(\bar{\mathbf{g}}_g/\bar{v}_0 + \odot) \circ \bar{W}_g$$ \hspace{1cm} (36)

where, the scalar $\bar{n}_0 = \bar{g} \cdot \bar{w} + \bar{v}_0 \partial_0 \bar{v}_0 + \bar{v}_0 \nabla^\star \cdot \bar{w}$ is the power density; the external power density $\bar{N}_g$ includes the power density $n_0$ etc. in the gravitational field.

From the above, the dot product of velocity curl with torque is the part of power in Newtonian gravity. In some cases, comparing Eq.(26) with Eq.(36), we find $C_g$ must equal to zero. Otherwise, $\bar{N}_g$ will change with the time.

When the quaternion coordinate system rotates, we have the external power density $\bar{N}_g(\bar{n}_0', \bar{n}_1', \bar{n}_2', \bar{n}_3')$ from Eq.(2). And then, we find that the power density will remain unchanged by Eq.(3).

$$\bar{n}_0 = \bar{n}_0'$$ \hspace{1cm} (37)

In some special cases, the $\bar{n}_0' = 0$, and then we obtain the energy continuity equation from Eqs.(9) and (37).

$$\partial \bar{v}_0/\partial r_0 + \nabla^\star \cdot \bar{w} + \bar{g} \cdot \bar{w}/\bar{v}_0 = 0$$ \hspace{1cm} (38)

If the last term is neglected, the above is reduced to

$$\partial \bar{v}_0/\partial r_0 + \nabla^\star \cdot \bar{w} = 0.$$ \hspace{1cm} (39)
where, when the time $t$ is only the independent variable, the $\partial \vec{w}_0/\partial t$ will become the $d\vec{w}_0/dt$.

The above means the strength $\vec{D}$ and torque density $\vec{\omega}$ have the influence on the energy continuity equation. And the power density, $n_0$, may be variable under the quaternion transformation in the gravitational field.

III. MECHANICAL INVARIANTS OF ELECTROMAGNETIC FIELD

The octonion $\mathbb{O}$ can be used to describe the property of the electromagnetic field $\vec{B}$ and gravitational field, including the scalar invariants in the case of existence of strength and velocity curl etc.

In the case for coexistence of the electromagnetic field and the gravitational field, there are the mechanical invariants, which include the conservation of mass, the conservation of spin, and the conservation of energy [11]. With the property of octonions, we find that the velocity curl and strength have the influence on conservation laws in the electromagnetic and gravitational fields.

From the above equations, we may obtain some kinds of mechanical invariants with the velocity curl under the coordinate transformation in the case for coexistence of gravitational field and electromagnetic field. And we find that the speed of light, mass density, spin density, and energy density are all variable in the case for coexistence of the electromagnetic field and gravitational field, under the octonion coordinate transformation.

A. Octonion transformation

In the octonion space, the basis vector $\vec{E}$ consists of the quaternion basis vectors $\vec{E}_g$ and $\vec{E}_e$. The basis vector $\vec{E}_g = (1, \vec{i}_1, \vec{i}_2, \vec{i}_3)$ is the basis vector of the quaternion space for the gravitational field, and $\vec{E}_e = (\vec{I}_0, \vec{I}_1, \vec{I}_2, \vec{I}_3)$ for the electromagnetic field. While the basis vector $\vec{E}_e$ is independent of the $\vec{E}_g$, with $\vec{E}_e \circ \vec{I}_0$.

\[ \vec{E} = (1, \vec{i}_1, \vec{i}_2, \vec{i}_3, \vec{I}_0, \vec{I}_1, \vec{I}_2, \vec{I}_3) \] (40)

The octonion quantity $\mathbb{D}(d_0, d_1, d_2, d_3, D_0, D_1, D_2, D_3)$ is defined as follows.

\[ \mathbb{D} = d_0 + \Sigma(d_i \vec{t}_j) + \Sigma(D_i \vec{I}_j) \] (41)

where, $d_i$ and $D_i$ are all real; $i = 0, 1, 2, 3; j = 1, 2, 3$.

When the coordinate system is transformed into the other, the physical quantity $\mathbb{D}$ will be transformed into the octonion $\mathbb{D}'(d'_0, d'_1, d'_2, d'_3, D'_0, D'_1, D'_2, D'_3)$.

\[ \mathbb{D}' = \mathbb{K}^* \circ \mathbb{D} \circ \mathbb{K} \] (42)

where, $\mathbb{K}$ is the octonion, and $\mathbb{K}^* \circ \mathbb{K} = 1$; $\ast$ denotes the conjugate of octonion; $\circ$ is the octonion multiplication.

The octonion $\mathbb{D}$ satisfies the following equations.

\[ d_0 = d'_0 \] (43)

\[ \mathbb{D}' \circ \mathbb{D} = (\mathbb{D}')^* \circ \mathbb{D}' \] (44)

In the above equation, the scalar part $d_0$ is preserved during the octonion coordinates are transforming. Some scalar invariants of electromagnetic field will be obtained from the characteristics of the octonion.

B. Radius vector

In the octonion space for the gravitational and electromagnetic fields, the radius vector $\vec{R} = \Sigma(r_i \vec{e}_i) + \Sigma(R_i \vec{I}_i)$. And it can be combined with the octonion $\vec{X} = \Sigma(x_i \vec{e}_i) + \Sigma(X_i \vec{I}_i)$ to become the compounding radius vector $\vec{R}$.

\[ \vec{R} = \vec{R} + k_{rx} \vec{X} \] (45)

where, $R_0 = V_0 T$; $V_0$ represents the speed of light-like, $T$ is a time-like quantity; $X_0 = A_0 T$; $A_0$ is the scalar potential of electromagnetic field.

In other words, the $\vec{R}$ can be considered as the radius vector in the octonion compounding space, with the basis vector $(1, \vec{i}_1, \vec{i}_2, \vec{i}_3, \vec{I}_0, \vec{I}_1, \vec{I}_2, \vec{I}_3)$.

When the octonion coordinate system is rotated, we obtain the radius vector $\vec{R}'(r'_0, r'_1, r'_2, r'_3, R'_0, R'_1, R'_2, R'_3)$ from Eqs.(42) and (45).

From Eqs.(43) and (45), we have

\[ r'_0 = r_0 \] (46)

where, $\vec{r}_i = r_i + k_{rx} x_i$ , $\vec{R}_i = R_i + k_{rx} X_i$.

Sometimes, the radius vector $\vec{R}$ can be replaced by the physical quantity $\vec{Z}(\vec{z}_0, \vec{z}_1, \vec{z}_2, \vec{z}_3, \vec{Z}_0, \vec{Z}_1, \vec{Z}_2, \vec{Z}_3)$, which is defined as

\[ \vec{Z} = \vec{R} \circ \vec{R} \] (47)

where, $\vec{z}_0 = (\vec{r}_0)^2 - \Sigma(\vec{r})^2 - \Sigma(\vec{R})^2$.

By Eqs.(43) and (47), we have

\[ \vec{z}_0 = \vec{r}_0 \] (48)

The above represents that the scalar invariant $\vec{r}_0$ and $\vec{z}_0$ remain unchanged when the coordinate system rotates in the octonion compounding space. And there may exist the special case of the $x_i \neq 0$ when $r_i = R_i = 0$. 

| Table IV: The octonion multiplication table. |
|---------------------------------------------|
| $d_0 = d'_0$ |
| $\mathbb{D}' \circ \mathbb{D} = (\mathbb{D}')^* \circ \mathbb{D}'$ |
C. Speed of light

The velocity \( \mathbf{V} = \Sigma(v_i \mathbf{i}_n) + \Sigma(V_i \mathbf{I}_n) \) can be combined with the potential \( \mathbf{A} = \Sigma(a_i \mathbf{i}_n) + \Sigma(A_i \mathbf{I}_n) \) to become the velocity \( \mathbf{V} \) in the octonion compounding space.

\[
\mathbf{\bar{V}} = \mathbf{V} + k_{rx} \mathbf{A}
\]

(49)

where, \( \bar{v}_i = v_i + k_{rx}a_i \); \( \bar{V}_i = V_i + k_{rx}A_i \).

In Eq.(49), the potential \( \mathbf{A} \) consists of the gravitational potential \( \mathbf{A}_g = \Sigma(a_i \mathbf{i}_n) \), and the electromagnetic potential \( \mathbf{A}_e = \Sigma(A_i \mathbf{I}_n) \).

\[
\mathbf{A} = \mathbf{A}_g + k_{eg} \mathbf{A}_e
\]

(50)

where, \( k_{eg} \) is the coefficient.

When the octonion coordinate system is rotated, we have the velocity \( \mathbf{\bar{V}}(\bar{v}_0, \bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{V}_1, \bar{V}_2, \bar{V}_3, \bar{V}_4) \) from Eqs.(42) and (49).

We have the scalar invariant about the speed of light in the octonion compounding space by Eqs.(43) and (49).

\[
\bar{v}_0 = v_0
\]

(51)

If we emphasize the definitions of radius vector Eq.(45) and velocity Eq.(49), we obtain Galilean transformation from Eqs.(46) and (51).

\[
\bar{v}_0 = v'_0, \quad \bar{v}_0 = v'_0
\]

(52)

In the same way, in case of we choose definitions of physical quantity Eq.(47) and velocity Eq.(49), we have Lorentz transformation \[12\] from Eqs.(48) and (51).

\[
\bar{z}_0 = z'_0, \quad \bar{v}_0 = v'_0
\]

(53)

In some special cases, the \( \Sigma(\mathbf{R}_i \mathbf{I}_n) \) do not partake the octonion coordinate transformation, we have the result \( \Sigma(\mathbf{R}_i)^2 = \Sigma(\mathbf{R}_i)^2 \) in Eq.(48).

The above means that the speed of light, \( v_0 \), will be variable, due to the existence of the scalar potential, \( a_0 \), of the gravitational field. But it is not associated with the scalar potential of electromagnetic field.

D. Potential and strength

In the octonion compounding space, the potential \( \mathbf{\bar{A}} = \Sigma(\mathbf{a}_i \mathbf{i}_n) + \Sigma(\mathbf{A}_i \mathbf{I}_n) \) is defined from the velocity \( \mathbf{\bar{V}} \).

\[
\mathbf{\bar{A}} = \mathbf{A} + K_{rx} \mathbf{V}
\]

(54)

where, \( a_i = a_i + K_{rx}v_i \); \( \mathbf{A} = \mathbf{A}_i + K_{rx} \mathbf{V}_i \).

When the coordinate system is rotated, we have the potential \( \mathbf{\bar{A}}'(\bar{a}_0', \bar{a}_1', \bar{a}_2', \bar{a}_3', \bar{A}_0', \bar{A}_1', \bar{A}_2', \bar{A}_3') \) from Eqs.(42) and (54). And we have the invariant about the scalar potential of gravitational field by Eqs.(43) and (54).

\[
\bar{a}_0 = \bar{a}_0'
\]

(55)

In the octonion compounding space, the strength \( \mathbf{\bar{B}} = \Sigma(\bar{b}_i \mathbf{i}_n) + \Sigma(\bar{B}_i \mathbf{I}_n) \) is defined from the potential \( \mathbf{\bar{A}} \).

\[
\mathbf{\bar{B}} = \mathbf{\bar{A}} \mathbf{\bar{A}} + K_{rx} \mathbf{U}
\]

(56)

where, \( \bar{b}_i = b_i + K_{rx}u_i \); \( \bar{B}_i = B_i + K_{rx}U_i \); the velocity \( \mathbf{\bar{V}} = v_0(\hat{\mathbf{r}} \mathbf{V} - \nabla \cdot \{r \mathbf{i}_j\}) \), the velocity curl \( \mathbf{\bar{U}} = \hat{\mathbf{r}} \mathbf{V} \); \( \mathbf{\bar{A}} = v_0(\hat{\mathbf{r}} \mathbf{V}) \); \( \mathbf{\bar{B}} = \hat{\mathbf{r}} \mathbf{V} \).

In Eq.(56), the strength \( \mathbf{\bar{B}} = \Sigma(\bar{b}_i \mathbf{i}_n) + \Sigma(\bar{B}_i \mathbf{I}_n) \) consists of the gravitational strength \( \mathbf{\bar{B}}_g \) and the electromagnetic strength \( \mathbf{\bar{B}}_e \).

\[
\mathbf{\bar{B}} = \hat{\mathbf{r}} \mathbf{V} = \bar{B}_g + \bar{B}_e
\]

(57)

In the above equation, we choose the following gauge conditions to simplify succeeding calculation.

\[
\bar{b}_0 = 0, \quad \bar{B}_0 = 0
\]

(58)

When the coordinate system is rotated, we have the strength \( \mathbf{\bar{B}}'(\bar{b}_0', \bar{b}_1', \bar{b}_2', \bar{b}_3', \bar{B}_0', \bar{B}_1', \bar{B}_2', \bar{B}_3') \) from Eqs.(42) and (56). And we have the invariant about the scalar strength of gravitational field by Eqs.(43) and (56).

\[
\bar{b}_0 = \bar{b}_0'
\]

(59)

The above means that the scalar potential, \( a_0 \), and the scalar strength, \( b_0 \), of gravitational field will be variable in the octonion compounding space, although the \( a_0 \) and the \( b_0 \) both are the scalar invariants.

E. Conservation of mass

The source density \( \mathbf{\bar{S}} \) is defined from the strength \( \mathbf{\bar{B}} \) in the octonion compounding space.

\[
\mu \mathbf{\bar{S}} = -(\mathbf{\bar{B}} / \bar{v}_0 + \hat{\mathbf{r}} \mathbf{V}) \mathbf{\bar{B}}
\]

\[
= \mu_0 \mathbf{\bar{S}} + k_{eg} \mu_0 \mathbf{S}_{ce} - \mathbf{B}^* \mathbf{\bar{B}} / \bar{v}_0
\]

(60)

where, \( k_{eg} = \mu_0 / \mu_e^2 \); \( q \) is the electric charge density; \( \mu_e^2 \) is the constant; the velocity \( \mathbf{V} = \Sigma(V_i \mathbf{I}_n) \); the electric current density \( \mathbf{S}_{ce} = q \mathbf{V} \) is the source of electromagnetic field.

In some cases, the electric charge is combined with the mass to become the electron or proton etc., we have the condition \( \mathbf{\bar{R}}_i \mathbf{I}_n = \bar{v}_i \mathbf{i}_n \mathbf{I}_n \) and \( \mathbf{\bar{V}}_i \mathbf{I}_n = \bar{v}_i \mathbf{i}_n \mathbf{I}_n \).

The \( \mathbf{\bar{B}}^* \mathbf{\bar{B}} / (2\mu_e^2) \) is the energy density, and includes that of the electromagnetic field.

\[
\mathbf{\bar{B}}^* \mathbf{\bar{B}} / (2\mu_e^2) = \mathbf{\bar{B}}^* \mathbf{\bar{B}}_g / \mu_e^2 + \mathbf{\bar{B}}^* \mathbf{\bar{B}}_e / \mu_e^2
\]

(61)

The linear momentum density \( \mathbf{\bar{P}} = \mu \mathbf{\bar{S}} / \mu_e^2 \) is written as

\[
\mathbf{\bar{P}} = \mathbf{\bar{m}} \bar{v}_0 + \Sigma(m \bar{v}_i \mathbf{i}_n) + \Sigma(M \mathbf{\bar{V}}_i \mathbf{i}_n)
\]

(62)

where, \( \mathbf{\bar{m}} = m - (\mathbf{\bar{B}}^* \mathbf{\bar{B}} / \mu_e^2) / \bar{v}_0^2 \); \( M = k_{eg}q^2 / \mu_e^2 \).

The above means that the gravitational mass density \( \mathbf{\bar{m}} \) is changed with either the strength or the velocity curl in the electromagnetic and gravitational fields.
From Eq.(42), we have the linear momentum density, \( \mathbf{F}'(m'v_0', m'v_1', m'v_2', M'v_0', M'v_1', M'v_2', M'v_3') \), when the coordinate system is rotated. And we obtain the invariant equation from Eqs.(43) and (62).

\[
\tilde{m}v_0 = m'v_0' 
\]  

(63)

Under Eqs.(51) and (63), we find that the gravitational mass density \( \tilde{m} \) remains unchanged. And then we have the conservation of mass.

\[
\tilde{m} = \tilde{m}' 
\]  

(64)

The above means that if we choose the definitions of velocity and linear momentum, the inertial mass density and the gravitational mass density will keep unchanged, under the octonion coordinate transformation in Eq.(42) in the octonion compounding space. The results are the same as that in the octonion space in the electromagnetic field and gravitational field.

**F. Mass continuity equation**

In the octonion compounding space, the applied force density \( \mathbf{F} = \Sigma(f_i \hat{\mathbf{e}}_i) + \Sigma(F_i I_i) \) is defined from the linear momentum density \( \mathbf{P} \) in Eq.(62).

\[
\mathbf{F} = \tilde{v}_0(\mathbf{\hat{B}}/\tilde{v}_0 + \mathcal{O})^* \circ \mathbf{P} 
\]  

(65)

where, \( \tilde{f}_0 = \tilde{v}_0 \partial \tilde{p}_0 / \partial \tilde{r}_0 + \tilde{v}_0 \Sigma(\partial \tilde{p}_j / \partial \tilde{r}_j) + \Sigma(\tilde{b}_j \tilde{p}_j + \tilde{B}_j \tilde{P}_j) \); \( \tilde{p}_0 = \tilde{m}v_0, \tilde{p}_j = m\tilde{v}_j; \tilde{P}_i = MV_i; \) and the applied force includes the gravity, the inertial force, the Lorentz force, and the interacting force between magnetic strength with magnetic moment, etc.

When the coordinate system rotates, we have the force density \( \mathbf{F}'(\tilde{f}_0', \tilde{f}_1', \tilde{f}_2', \tilde{f}_3', \tilde{F}_0', \tilde{F}_1', \tilde{F}_2', \tilde{F}_3') \).

By Eq.(43), we have

\[
\tilde{f}_0 = f_0'. 
\]  

(66)

When \( \tilde{f}_0' = 0 \) in the above, we have the mass continuity equation in the case for coexistence of the gravitational field and electromagnetic field.

\[
\partial \tilde{p}_0 / \partial \tilde{r}_0 + \Sigma(\partial \tilde{p}_j / \partial \tilde{r}_j) + \Sigma(\tilde{b}_j \tilde{p}_j + \tilde{B}_j \tilde{P}_j)/\tilde{v}_0 = 0 
\]  

(67)

If the \( a_i = A_i = 0 \) and \( \tilde{b}_i = B_i = 0 \), the above will be reduced to the following equation.

\[
\partial m / \partial t + \Sigma(\partial p_j / \partial r_j) = 0. 
\]  

(68)

The above states that the potential, the strength, and the velocity curl have the small influence on the mass continuity equation in the gravitational field and electromagnetic field, although the \( \Sigma(\tilde{b}_j \tilde{p}_j + \tilde{B}_j \tilde{P}_j)/\tilde{v}_0 \) and \( \Delta m \) both are usually very tiny when the fields are weak. In case of we choose the definitions of the applied force and velocity in the octonion compounding space, the mass continuity equation will be the invariant equation under the octonion transformation in Eq.(42).

**G. \( \tilde{f}_0 \) continuity equation**

In the octonion compounding space, the new density \( \tilde{C} = \Sigma(\tilde{c}_j \hat{\mathbf{e}}_j) + \Sigma(C_i I_i) \) is defined from the applied force density \( \mathbf{F} \) in Eq.(65).

\[
\tilde{C} = v_0(\mathbf{\hat{B}}/v_0 + \mathcal{O})^* \circ \mathbf{F} 
\]  

(69)

where, \( \tilde{c}_0 = \tilde{v}_0 \partial \tilde{f}_0 / \partial \tilde{r}_0 + \tilde{v}_0 \Sigma(\partial \tilde{f}_j / \partial \tilde{r}_j) + \Sigma(\tilde{b}_j \tilde{f}_j + \tilde{B}_j \tilde{F}_j) \).

When the octonion coordinate system rotates, we have the density \( \tilde{C}'(\tilde{c}_0', \tilde{c}_1', \tilde{c}_2', \tilde{c}_0', \tilde{C}_1', \tilde{C}_2', \tilde{C}_3') \).

By Eq.(43) and the above, we have

\[
\tilde{c}_0 = c_0'. 
\]  

(70)

If \( c_0' = 0 \) in the above, we have the continuity equation about \( \tilde{f}_0 \) in the case for coexistence of gravitational field and electromagnetic field.

\[
\partial \tilde{f}_0 / \partial \tilde{r}_0 + \Sigma(\partial \tilde{f}_j / \partial \tilde{r}_j) + \Sigma(\tilde{b}_j \tilde{f}_j + \tilde{B}_j \tilde{F}_j)/\tilde{v}_0 = 0 
\]  

(71)

When the \( a_i = A_i = 0 \) and \( \tilde{b}_i = B_i = 0 \), the above will be reduced to the following equation.

\[
\partial \tilde{f}_0 / \partial \tilde{r}_0 + \Sigma(\partial \tilde{f}_j / \partial \tilde{r}_j) = 0 
\]  

(72)

The above states that the potential, the strength, and the velocity curl have small influence on the continuity equation about \( \tilde{f}_0 \), although the \( \Sigma(\tilde{b}_j \tilde{f}_j + \tilde{B}_j \tilde{F}_j)/\tilde{v}_0 \) is usually very tiny when the fields are weak. When we choose definitions of \( \mathbf{V} \) and \( \mathbb{C} \), this continuity equation will be the invariant equation under the octonion coordinate transformation.

**H. Spin density**

The angular momentum density \( \tilde{L} = \Sigma(\tilde{\mathbf{e}}_j \hat{\mathbf{e}}_j) + \Sigma(\mathbf{L}_i I_i) \) is defined from the radius vector \( \tilde{R} \) and linear momentum density \( \mathbf{P} \) in the octonion compounding space.

\[
\tilde{L} = \tilde{R} \circ \mathbf{P} 
\]  

(73)

where, the scalar \( \tilde{L}_0 = \tilde{r}_0 \partial \tilde{R}_0 - \Sigma(\tilde{r}_j \partial \tilde{R}_j) - \Sigma(\tilde{R}_j \tilde{P}_j) \).

The \( \tilde{L}_0 \) is considered as the spin angular momentum density in the octonion compounding space. The angular momentum includes the orbital angular momentum and spin angular momentum in the gravitational field and the electromagnetic field.

When the octonion coordinate system is rotated, we have the angular momentum density \( \tilde{L}' = \Sigma(\tilde{L}_i I_i) \).

Under the octonion coordinate transformation, the spin density remains unchanged from Eq.(43). In other words, we have the conservation of spin as follows.

\[
\tilde{L}_0 = \tilde{L}_0'. 
\]  

(74)

The above means the velocity, velocity curl, potential, and strength have the influence on the orbital angular momentum and spin angular momentum. Meanwhile the spin angular momentum density \( \tilde{L}_0 \) will be variable in the gravitational field and electromagnetic field, although the \( \tilde{L}_0 \) is invariable under the octonion transformation.
I. Conservation of energy

The total energy density \( \bar{W} = \Sigma(\bar{w}_i \hat{i}) + \Sigma(\bar{W}_i \bar{I}_i) \) is defined from the angular momentum density \( \bar{L} \).

\[
\bar{W} = \bar{v}_0 (\bar{E}/\bar{v}_0 + \circ) \circ \bar{L}
\]  
(75)

where, the scalar \( \bar{w}_0 = \bar{v}_0 \partial_0 \bar{w}_0 + (\bar{v}_0 \nabla \cdot \bar{h}) \circ \bar{J} + \bar{h} \circ \bar{J} \); \( \bar{h} = \Sigma(b_j \hat{j}) \); \( \bar{H} = \Sigma(B_j \bar{L}_j) \); \( \bar{J} = \Sigma(J_j \bar{I}_j) \); the total energy includes the potential energy, the kinetic energy, the torque, and the work, etc. in the gravitational field and the electromagnetic field \[13\].

When the coordinate system is rotated, we have the energy density \( \bar{W}' = \Sigma(\bar{w}_i' \hat{i} + \bar{W}_i' \bar{I}_i) \). Under the octonion transformation, the scalar part of total energy density is the energy density and remains unchanged \[14\]. So we have the conservation of energy as follows.

\[
\bar{w}_0 = \bar{w}_0'
\]  
(76)

In some special cases, the right side is equal to zero. We obtain the spin continuity equation.

\[
\partial_0 \partial_0 \bar{r}_0 + \nabla \cdot \bar{J} + (\bar{h} \cdot \bar{J} + \bar{H} \cdot \bar{J}) / \bar{v}_0 = 0
\]  
(77)

If the last term is neglected, the above is reduced to

\[
\partial_0 \partial_0 \bar{r}_0 + \nabla \cdot \bar{J} = 0
\]  
(78)

further, if the last term is equal to zero, we have

\[
\partial_0 \partial_0 / \partial t = 0.
\]  
(79)

where, when the time \( t \) is only the independent variable, the \( \partial_0 \partial_0 / \partial t \) will become the \( d0/\partial t \).

The above means the energy density \( w_0 \) is variable in the case for coexistence of the gravitational field and the electromagnetic field, because the velocity, velocity curl, potential, and strength have the influence on the angular momentum density. While the scalar \( \bar{w}_0 \) is the invariant under the octonion coordinate transformation from Eqs.(43) and (76).

| definition | invariant | meaning |
|------------|-----------|---------|
| \( \bar{W} \) | \( \bar{N}_0 = \bar{N}_0' \) | energy continuity equation |

J. Energy continuity equation

In the octonion compounding space with the velocity curl, the external power density \( \bar{N} \) can be defined from the total energy density \( \bar{W} \) in Eq.(75).

\[
\bar{N} = \bar{v}_0 (\bar{E}/\bar{v}_0 + \circ) \circ \bar{W}
\]  
(80)

where, the external power density \( \bar{N} \) includes the power density in the gravitational and electromagnetic fields.

The external power density can be rewritten as follows.

\[
\bar{N} = \bar{n}_0 + \Sigma(\bar{n}_j \bar{t}_j) + \Sigma(\bar{N}_j \bar{I}_j)
\]  
(81)

where, the scalar \( \bar{n}_0 = \bar{v}_0 \partial_0 \bar{n}_0 + (\bar{v}_0 \nabla \cdot \bar{h}) = \bar{v}_0 \cdot \bar{Y} + \bar{h} \cdot \bar{Y} \); \( \bar{Y} = \Sigma(\bar{n}_j \bar{t}_j) \); \( \bar{Y} = \Sigma(\bar{N}_j \bar{I}_j) \).

When the coordinate system is rotated, we have the external power density \( \bar{W}' = \Sigma(\bar{n}_i' \hat{i} + \bar{N}_i' \bar{I}_i) \). Under the octonion coordinate transformation, the scalar part of external power density is the power density and remains unchanged by Eq.(43). That is the conservation of power.

\[
\bar{n}_0 = \bar{n}_0'
\]  
(82)

In the special cases, the right side is equal to zero. And then, we obtain the energy continuity equation.

\[
\partial_0 \bar{n}_0 + \nabla^* \cdot \bar{v} + (\bar{h} \cdot \bar{Y} + \bar{H} \cdot \bar{Y}) / \bar{v}_0 = 0
\]  
(83)

If the last term is neglected, the above is reduced to

\[
\partial_0 \bar{n}_0 + \nabla^* \cdot \bar{v} = 0
\]  
(84)

further, if the last term is equal to zero, we have

\[
\partial_0 \bar{n}_0 / \partial t = 0.
\]  
(85)

where, when the time \( t \) is only the independent variable, the \( \partial_0 \partial_0 / \partial t \) will become the \( d0/\partial t \).

The above means that the power density \( n_0 \) will be variable in the case for coexistence of the gravitational field and electromagnetic field, although the \( \bar{n}_0 \) is the scalar invariant under the octonion transformation. And the potential, strength, velocity curl, or torque density, etc. have the influence on the energy continuity equation in the gravitational field and the electromagnetic field.

IV. ELECTRIC INVARIANTS OF ELECTROMAGNETIC FIELD

In the octonion space for the electromagnetic field and gravitational field, there exist the electric invariants, which include the conservation of charge, conservation of spin magnetic moment, charge continuity equation, and conservation of energy-like, etc.

The electric invariants of electromagnetic field can be illustrated by octonions in the case for coexistence of the electromagnetic field and gravitational field.

With the property of the algebra of octonions, we find that the velocity curl and strength have the influence on the electric invariants in the electromagnetic field and gravitational field with the strength and velocity curl.
A. Radius vector

In the octonion compounding space, one new octonion quantity \( \mathbb{R}_q = \mathbb{R} \circ \mathbf{I}_0 \) can be defined from Eq.(45).

\[
\mathbb{R}_q = \Sigma(\mathbf{R}_i \mathbf{i}_i) \tag{86}
\]

When the octonion coordinate system is rotated, we obtain the radius vector \( \mathbb{R}'(\mathbf{r}_0, \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{R}_0, \mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3) \) from Eqs.(42) and (86).

From Eqs.(43) and (86), we have

\[
\mathbf{R}_0 = \mathbf{R}'_0 \tag{87}
\]

Further, the radius vector \( \mathbb{R} \) can be replaced by the physical quantity \( \mathbf{Z}(\mathbf{z}_0, \mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \mathbf{Z}_0, \mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) \), which is defined as

\[
\mathbf{Z} = \mathbb{R} \circ \mathbb{R} \tag{88}
\]

where, \( \mathbf{Z}_0 = 2\mathbf{r}_0 \mathbf{R}_0 \).

Similarly, the new octonion quantity \( \mathbb{Z}_q = \mathbf{Z} \circ \mathbf{I}_0 \) can be defined from the above.

\[
\mathbb{Z}_q = \Sigma(\mathbf{Z}_i \mathbf{i}_i) - \Sigma((\mathbf{z}_i \mathbf{i}_i) \tag{89}
\]

By Eqs.(43) and (89), we have

\[
\mathbf{Z}_0 = \mathbf{Z}'_0 \tag{90}
\]

The above represents that the scalar invariant \( \mathbf{R}_0 \) and \( \mathbf{Z}_0 \) remain unchanged when the coordinate system rotates in the octonion compounding space.

B. Speed of light-like

In the octonion compounding space, one new octonion quantity \( \mathbf{V}_q = \mathbf{V} \circ \mathbf{I}_0 \) can be defined from Eq.(49).

\[
\mathbf{V}_q = \Sigma(\mathbf{V}_i \mathbf{i}_i) - \Sigma((\mathbf{v}_i \mathbf{i}_i) \tag{91}
\]

When the octonion coordinate system is rotated, we have the velocity \( \mathbf{V}'(\mathbf{v}_0', \mathbf{v}_1', \mathbf{v}_2', \mathbf{v}_3', \mathbf{V}_0', \mathbf{V}_1', \mathbf{V}_2', \mathbf{V}_3') \) from Eqs.(42) and (49). And we have the scalar invariant about the speed of light in the octonion compounding space from the above.

\[
\mathbf{V}_0 = \mathbf{V}'_0 \tag{92}
\]

If we choose the definition of radius vector Eq.(45) and velocity Eq.(49), we obtain Galilean transformation from Eqs.(87) and (92).

\[
\mathbf{R}_0 = \mathbf{R}'_0 , \mathbf{V}_0 = \mathbf{V}'_0 \tag{93}
\]

Furthermore, if we choose the definitions of the radius vector Eq.(88) and velocity Eq.(49), we shall obtain one new transformation rather than Lorentz transformation from Eqs.(90) and (92).

\[
\mathbf{Z}_0 = \mathbf{Z}'_0 , \mathbf{V}_0 = \mathbf{V}'_0 \tag{94}
\]

The above means that the speed of light-like, \( V_0 \), will be variable, due to the existence of the scalar potential, \( A_0 \), of the electromagnetic field. In some special cases, if \( \mathbf{R}_0 = \mathbf{r}_0 \), Eq.(94) will be reduced to Eq.(93).

C. Potential and strength

In the octonion compounding space, one new octonion quantity \( \mathbb{A}_q = \mathbb{A} \circ \mathbf{I}_0 \) can be defined from Eq.(54).

\[
\mathbb{A}_q = \Sigma(\mathbf{A}_i \mathbf{i}_i) - \Sigma((\mathbf{a}_i \mathbf{i}_i) \tag{95}
\]

When the coordinate system is rotated, we obtain the potential \( \mathbb{A}'(\mathbf{a}_0', \mathbf{a}_1', \mathbf{a}_2', \mathbf{a}_3', \mathbf{A}_0', \mathbf{A}_1', \mathbf{A}_2', \mathbf{A}_3') \) from Eqs.(42) and (54). And we have the invariant about the scalar potential of electromagnetic field by Eqs.(43) and (95).

\[
\mathbf{A}_0 = \mathbf{A}'_0 \tag{96}
\]

In the octonion compounding space, one new octonion quantity \( \mathbb{B}_q = \mathbb{B} \circ \mathbf{I}_0 \) can be defined from Eq.(56).

\[
\mathbb{B}_q = \Sigma(\mathbf{B}_i \mathbf{i}_i) - \Sigma((\mathbf{b}_i \mathbf{i}_i) \tag{97}
\]

When the coordinate system is rotated, we have the strength \( \mathbb{B}'(\mathbf{b}_0', \mathbf{b}_1', \mathbf{b}_2', \mathbf{b}_3', \mathbf{B}_0', \mathbf{B}_1', \mathbf{B}_2', \mathbf{B}_3') \) from Eqs.(42) and (56). And we have the invariant about the scalar strength of electromagnetic field by Eqs.(43) and (97).

\[
\mathbf{B}_0 = \mathbf{B}'_0 \tag{98}
\]

The above means that the scalar potential, \( A_0 \), and the scalar strength, \( B_0 \), of electromagnetic field will be variable in the octonion compounding space, although the \( A_0 \) and the \( B_0 \) both are the scalar invariants.

D. Conservation of charge

In the octonion compounding space, one new octonion quantity \( \mathbb{P}_q = \mathbb{P} \circ \mathbf{I}_0 \) can be defined from Eq.(62).

\[
\mathbb{P}_q = \Sigma(\mathbf{P}_i \mathbf{i}_i) - \Sigma((\mathbf{p}_i \mathbf{i}_i) \tag{99}
\]

From Eq.(42), we have the linear momentum density, \( \mathbf{P}'(\mathbf{m}'\mathbf{v}_0', \mathbf{m}'\mathbf{v}_1', \mathbf{m}'\mathbf{v}_2', \mathbf{m}'\mathbf{v}_3', \mathbf{M}'\mathbf{V}_0', \mathbf{M}'\mathbf{V}_1', \mathbf{M}'\mathbf{V}_2', \mathbf{M}'\mathbf{V}_3') \), when the coordinate system is rotated. Thus we obtain the invariant equation from Eqs.(43) and (99). And the scalar \( M \) is about the electric charge density \( q \).

\[
\mathbf{M} \mathbf{V}_0 = \mathbf{M}' \mathbf{V}_0' \tag{100}
\]

Under Eqs.(92) and (100), we have the conservation of charge as follows.

\[
M = M' \tag{101}
\]

The above means that if we choose the definition of the velocity and linear momentum, the charge density will keep unchanged, under the coordinate transformation in Eq.(42) in the octonion compounding space.
E. Charge continuity equation

In the octonion compounding space, one new octonion quantity \( \bar{F}_q = \bar{F} \circ \bar{I}_0 \) can be defined from Eq.(65).

\[
\bar{F}_q = \Sigma(\bar{F}_i \bar{i}_i) - \Sigma(\bar{f}_i \bar{I}_i) \tag{102}
\]

When the coordinate system rotates, we have the force density \( \bar{F}' = (\bar{F}_1', \bar{F}_2', \bar{F}_3', \bar{F}_4', \bar{F}_5', \bar{F}_6', \bar{F}_7', \bar{F}_8') \).

By Eq.(43) and the above, we have

\[
\bar{F}_0 = \bar{F}'_0 \tag{103}
\]

When the right side is zero in the above, we have the charge continuity equation in the case for coexistence of the gravitational field and electromagnetic field.

\[
\partial \bar{F}_0/\partial \bar{r}_0 + \Sigma(\partial \bar{F}_j/\partial \bar{r}_j) + \Sigma(\bar{b}_j \bar{P}_j - \bar{B}_j \bar{p}_j)/\bar{v}_0 = 0 \tag{104}
\]

If the \( a_i = A_i = 0 \) and \( \bar{b}_i = \bar{B}_i = 0 \), the above will be reduced to the following equation.

\[
\partial M/\partial t + \Sigma(\partial P_j/\partial r_j) = 0 \tag{105}
\]

The above states that the potential, the strength, and the velocity curl have the small influence on the charge continuity equation in the gravitational field and electromagnetic field, although the \( \Sigma(\bar{b}_j \bar{P}_j - \bar{B}_j \bar{p}_j)/\bar{v}_0 \) and \( \Delta m \) both are usually very tiny when the fields are weak.

F. \( \bar{F}_0 \) continuity equation

In the octonion compounding space, one new octonion quantity \( \bar{C}_q = \bar{C} \circ \bar{I}_0 \) can be defined from Eq.(69).

\[
\bar{C}_q = \Sigma(\bar{C}_i \bar{i}_i) - \Sigma(\bar{c}_i \bar{I}_i) \tag{106}
\]

When the octonion coordinate system rotates, we have the density \( \bar{C}'(\bar{c}_1', \bar{c}_2', \bar{c}_3', \bar{c}_4', \bar{C}_1', \bar{C}_2', \bar{C}_3') \).

By Eq.(43) and the above, we have

\[
\bar{C}_0 = \bar{C}'_0. \tag{107}
\]

When the right side is zero in the above, we have the continuity equation about \( \bar{F}_0 \) in the case for coexistence of the gravitational field and electromagnetic field.

\[
\partial \bar{F}_0/\partial \bar{r}_0 + \Sigma(\partial \bar{F}_j/\partial \bar{r}_j) + \Sigma(\bar{b}_j \bar{F}_j - \bar{B}_j \bar{f}_j)/\bar{v}_0 = 0 \tag{108}
\]

If the \( a_i = A_i = 0 \) and \( \bar{b}_i = \bar{B}_i = 0 \), the above will be reduced to the following equation.

\[
\partial \bar{F}_0/\partial \bar{r}_0 + \Sigma(\partial \bar{F}_j/\partial \bar{r}_j) = 0 \tag{109}
\]

The above states that the potential, the strength, and the velocity curl have the small influence on the continuity equation about \( \bar{F}_0 \) in the gravitational field and electromagnetic field, although the \( \Sigma(\bar{b}_j \bar{F}_j - \bar{B}_j \bar{f}_j)/\bar{v}_0 \) is usually very tiny when the fields are weak.

G. Spin magnetic moment

In the octonion compounding space, one new octonion quantity \( \bar{L}_q = \bar{L} \circ \bar{I}_0 \) can be defined from Eq.(73).

\[
\bar{L}_q = \Sigma(\bar{L}_i \bar{i}_i) - \Sigma(\bar{I}_i \bar{I}_i) \tag{110}
\]

When the coordinate system rotates, we have the angular momentum density \( \bar{L}' = \Sigma(i'_i \bar{i}_i + \bar{I}_i \bar{I}_i) \). Under the coordinate transformation, the scalar part \( \bar{L}_0 \) about the spin magnetic moment density will remain unchanged from Eqs.(43) and (110). And we have the conservation of spin magnetic moment.

\[
\bar{L}_0 = \bar{L}'_0 \tag{111}
\]

The above means the velocity, velocity curl, potential, and strength have the influence on the angular momentum, including the spin magnetic moment density. While the scalar \( \bar{L}_0 \) about the spin magnetic moment density will be variable in the gravitational and electromagnetic fields, although the \( \bar{L}_0 \) is invariable under the octonion coordinate transformation.

H. Conservation of energy-like

In the octonion compounding space, one new octonion quantity \( \bar{W}_q = \bar{W} \circ \bar{I}_0 \) can be defined from Eq.(75).

\[
\bar{W}_q = \Sigma(\bar{W}_i \bar{i}_i) - \Sigma(\bar{w}_i \bar{I}_i) \tag{112}
\]

When the coordinate system is rotated, we have the energy density \( \bar{W}' = \Sigma(\bar{w}'_i \bar{i}_i + \bar{W}'_i \bar{I}_i) \). Under the octonion transformation, the scalar part \( \bar{W}_0 \) remains unchanged by Eqs.(43) and (75). And then we have the conservation of energy-like as follows.

\[
\bar{W}_0 = \bar{W}'_0 \tag{113}
\]

| Table VI: The definitions of electric invariants of gravitational field and electromagnetic field in the octonion compounding space. |
|---|---|---|
| definition | invariant | meaning |
| \( \bar{F}_q \) | \( \bar{F}_0 = \bar{F}'_0 \) | Galilean invariant |
| \( \bar{F}_q \circ \bar{I}_0 \) | \( \bar{Z}_0 = \bar{Z}'_0 \) | new invariant |
| \( \bar{V}_q \) | \( \bar{V}_0 = \bar{V}'_0 \) | speed of light – like |
| \( \bar{X}_q \) | \( \bar{X}_0 = \bar{X}'_0 \) | scalar potential |
| \( \bar{E}_q \) | \( \bar{E}_0 = \bar{E}'_0 \) | gauge equation |
| \( \bar{P}_q \) | \( \bar{P}_0 = \bar{P}'_0 \) | conservation of charge |
| \( \bar{F}_q \) | \( \bar{F}_0 = \bar{F}'_0 \) | charge continuity equation |
| \( \bar{L}_q \) | \( \bar{L}_0 = \bar{L}'_0 \) | spin magnetic moment density |
| \( \bar{W}_q \) | \( \bar{W}_0 = \bar{W}'_0 \) | conservation of energy – like |
| \( \bar{N}_q \) | \( \bar{N}_0 = \bar{N}'_0 \) | conservation of power – like |
In some special cases, if $\tilde{W}'_0 = 0$ in the above, we obtain the continuity equation of spin magnetic moment.

$$\partial L_0/\partial t + \sum (\partial L_j/\partial r_j) + \sum (\tilde{b}_j L_j - \tilde{B}_j \tilde{L}_j)/\tilde{v}_0 = 0$$

If the last term is neglected, the above is reduced to,

$$\partial L_0/\partial t + \sum (\partial L_j/\partial r_j) = 0$$

(114) further, if the last term is equal to zero, we have

$$\partial L_0/\partial t = 0.$$ 

(115)

where, when the time $t$ is only the independent variable, the $\partial L_0/\partial t$ will become the $dL_0/\partial t$.

The above means that the energy-like density $W_0$ is variable in the case for coexistence of the gravitational field and the electromagnetic field, because the velocity, velocity curl, potential, and strength have the influence on the spin magnetic moment density. While the scalar $W_0$ is the invariant under the octonion transformation from Eqs.(43) and (113).

### I. Energy-like continuity equation

In the octonion compounding space, one new octonion quantity $\tilde{N}_q = \tilde{N} \circ \tilde{I}_0$ can be defined from Eq.(80).

$$\tilde{N}_q = \Sigma(\tilde{N}_i \tilde{i}_i) - \Sigma(\tilde{n}_i \tilde{I}_i)$$

(116)

When the octonion coordinate system rotates, we have the external power density $\tilde{N}' = \Sigma(\tilde{n}'_i \tilde{i}'_i + \tilde{N}'_i \tilde{I}_i)$. Under the octonion coordinate transformation, the scalar part $\tilde{N}_0$ will remain unchanged by Eqs.(43) and (116). And then we have conservation of power-like.

$$\tilde{N}_0 = \tilde{N}'_0$$

(117)

In a special case, the right side is equal to zero. And then, we obtain the energy-like continuity equation.

$$\partial \tilde{W}_0/\partial r_0 + \sum (\partial \tilde{W}_j/\partial r_j) + \sum (\tilde{b}_j \tilde{W}_j - \tilde{B}_j \tilde{W}_j)/\tilde{v}_0 = 0$$

If the last term is neglected, the above is reduced to,

$$\partial \tilde{W}_0/\partial r_0 + \sum (\partial \tilde{W}_j/\partial r_j) = 0$$

(118) further, if the last term is equal to zero, we have

$$\partial \tilde{W}_0/\partial t = 0.$$ 

(119)

where, when the time $t$ is only the independent variable, the $\partial \tilde{W}_0/\partial t$ will become the $d\tilde{W}_0/\partial t$.

The above means that the power-like density $\tilde{N}_0$ will be variable in the case for coexistence of gravitational field and electromagnetic field, although the $\tilde{N}_0$ is the scalar invariant under the octonion transformation. And the potential, strength, velocity curl, or torque density, etc. have the influence on the energy-like continuity equation in the gravitational field and electromagnetic field.

### V. CONCLUSIONS

In the quaternion compounding space, the inferences about conservation laws depend on the combinations of physical definitions. By means of definition combination of radius vector and velocity, the gravitational potential is found to have the influence on the speed of light in the gravitational field. With the definition combination of the velocity and linear momentum, it is found that the mass density is variable, and the gravitational strength has the impact on the conservation of mass. Depending on the definition combination of the angular momentum and velocity, we obtain that the spin density and energy density both are variable, and the gravitational strength has the impact on the conservation of energy.

In the octonion compounding space, the results about the mechanics invariants of electromagnetic field depend on the definition combinations, in the case for coexistence of the gravitational field and electromagnetic field. With the definitions of angular momentum and velocity, the mass density will be variable and the conservation of mass will be changed with the strength and velocity curl. From the definitions of angular momentum and velocity, the spin density and the energy density will be variable for the influence of the potential and velocity curl etc., while the conservation of spin and the conservation of energy will be changed with the impact of the velocity curl and strength etc.

In the octonion compounding space, the results about the scalar invariants with the velocity curl depend on the definition combinations of physical quantities, in the case for coexistence of gravitational field and electromagnetic field. The mechanical invariants are different from the electric invariants. The former is related to the mass, and the latter is associated with the charge. With the definitions of linear momentum and velocity, the charge density will be variable and the conservation of charge will be changed with the strength and velocity curl. From the definitions of angular momentum and velocity, the spin magnetic moment density and the energy-like density both will be variable for the influence of the potential and velocity curl etc., while the conservation of spin magnetic moment and the conservation of energy-like will be changed with the impact of the velocity curl and strength etc. in the gravitational field and electromagnetic field.

It should be noted that the study for scalar invariants of the electromagnetic field and gravitational field examined only one simple case with very weak strength and low velocity curl in the gravitational field and electromagnetic field. Despite its preliminary character, this study can clearly indicate the strength and velocity curl in the gravitational field and electromagnetic field have the limited influence on the scalar invariants. For the future studies, the related investigation will concentrate on only the predictions of scalar invariants in the strong strength with high velocity curl in the gravitational field and electromagnetic field.
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[1] W. R. Hamilton, *Elements of Quaternions*, (Longmans, Green & Co., London, 1866).
[2] A. Lavoisier, *Elements of Chemistry*, trans. R. Kerr, (Dover Publications Inc., New York, 1965).
[3] T. Young, *Lectures on natural philosophy and the mechanical arts*, (Thoemmes Continuum Press, Bristol, 2002).
[4] G. E. Uhlenbeck and S. Goudsmit, Spinning Electrons and the Structure of Spectra, *Nature*, 17, 264, 1926.
[5] Z.-H. Weng, Field Equations of Electromagnetic and Gravitational Fields, arXiv:0709.2486
[6] L. V. Hau, S. E. Harris, Z. Dutton, and C. H. Behroozi, Light speed reduction to 17 metres per second in an ultracold atomic gas, *Nature*, 397, 594, 1999.
[7] G. G. Coriolis, Memoire sur les equations du mouvement relatif des systemes de corps, *Journal de l’ecole Polytechnique*, 15, 142-145, 1835.
[8] A. Cayley, *The Collected Mathematical Papers*, (Johnson Reprint Co., New York, 1963).
[9] J. C. Maxwell, *A Treatise on Electricity and Magnetism*, (Dover Publications, New York, 1954)
[10] I. Newton, *The Mathematical Principles of Natural Philosophy*, trans. A. Motte, (Dawsons of Pall Mall, London, 1968).
[11] J. P. Joule, *The Scientific Papers of James Prescott Joule*, Vol. I & II, (Dawsons of Pall Mall, London, first published 1887, reprinted 1963).
[12] H. A. Lorentz, *The Theory of Electrons*, (Dover Publications Inc., New York, 1952).
[13] O. Heaviside, A Gravitational and Electromagnetic Analogy, *The Electrician*, 31, 281-282, 359, 1893.
[14] A. Einstein, Does the Inertia of a Body Depend Upon Its Energy Content, *Annalen der Physik*, 18, 639-641, 1905.