VISCOUS COSMOLOGIES WITH EXTRA DIMENSIONS

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Abstract

We present an analysis of a n-dimensional vacuum Einstein field equations in which 4-dimensional space-time is described by a Friedmann Robertson-Walker (FRW) metric and that of the extra dimensions by a Kasner type Euclidean metric. The field equations are interpreted as four dimensional Einstein equations with effective matter properties. The effective matter is then treated as viscous fluid. We consider the theories of imperfect fluid given by Eckart, truncated Israel-Stewart and full Israel-Stewart theories and obtain cosmological solutions for a flat model of the universe. The imperfect fluid assumption admits here power law inflation for the 3-physical space in some cases.

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Model building in higher dimensions was initiated by Kaluza and Klein\(^1\) who tried to unify gravity with electromagnetic interaction by introducing an extra dimension. Kaluza-Klein theory is basically an extension of Einstein general relativity in 5D which is of much interest in Particle Physics and Cosmology. In the last few decades, the study of a higher dimensional theory has been revived and considerably generalized after realizing that many interesting theories of particle interactions need more than four dimensions for their formulation. Attempts have been made to build cosmological models\(^2\) in higher dimensions which may undergo a spontaneous compactification leading to a product space \(M^4 \times M^d\), with \(M^d\) describing the compact inner space. In the usual approach one uses Einstein’s field equation

\[
G_{ab} = 8\pi G T_{ab}
\]  

which match the Einstein tensor and the energy momentum tensor. Recently another approach\(^3\) has been developed taking \(G_{ab} = 0\), the extra terms which appear in these equations due to extra dimensions are interpreted as induced or effective properties of matter in ordinary 4D space-time. Ibáñez and Verdaguer [henceforth IV]\(^4\) obtained a set of solutions of Einstein’s equation in an n-dimensional vacuum. They found homogeneous solutions with expanding three dimensional isotropic space. The solutions are then identified with the observable four dimensional subspace with perfect fluid. Later a class of solutions in flat universe model have been obtained by Gleiser and Diaz.\(^5\) Krori \textit{et al}.\(^6\) using a higher dimensional anisotropic cosmology (Bianchi-I) obtained 4D perfect fluid solutions. It was shown that the perfect fluid solutions obtained by them are compatible with contraction of all the extra dimensions. The solutions are identified with a 4 dimensional anisotropic cosmolog-
ical model. However, it is known that the matter distribution in the early universe may not be as simple as predicted by perfect fluid due to a number of dissipative processes. Hence it is essential to study cosmological solutions in the framework of imperfect fluid description. The simple theory is the Eckart theory, however, it suffers from serious drawbacks concerning causality and stability. These difficulties can, however, be removed by considering higher order theories i.e., extended irreversible thermodynamics (EIT). In this paper we intend to study a higher dimensional vacuum Einstein equations which leads to an observed 4 dimensional universe with imperfect fluid. We obtain exact solutions in the framework of the Eckart, truncated Israel-Stewart (TIS) and full Israel-Stewart (FIS) theories.

We now consider a higher dimensional metric in the form

\[ ds^2 = ds_{FW}^2 + \sum_{i=1}^{d} b_i^2(t) \, dx_i^2 \]  

where \( d \) is the number of extra dimensions \( (d = n - 4) \) and \( ds_{FW}^2 \) represents the line element of the FRW metric in four dimensions which is given by

\[ ds_{FW}^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - k r^2} + r^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \right] \]

where \( a(t) \) is the scale factor of the 4 dimensional spacetime and \( b_i \)'s are the scale factors for the extra dimensions and \( k = 0, +1, -1 \) represents flat, closed and open universe respectively.

The vacuum Einstein’s equation in \( n \)-dimensions using the metric (2) can be written as

\[ 3 \left( \frac{\dot{a}^2 + k}{a^2} \right) = \frac{\ddot{D}D}{D} - \frac{1}{8} \left( \sum_{i=1}^{d} \ddot{b}_i \right)^2 + \frac{1}{8} \sum_{i=1}^{d} \left( \ddot{b}_i \right)^2, \]

\[ 2 \frac{\dddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2} = \frac{\dot{a}\dot{D}}{aD} + \frac{1}{8} \sum_{i=1}^{d} \left( \dot{b}_i \right)^2 - \frac{1}{8} \sum_{i=1}^{d} \left( \dot{b}_i \right)^2, \]
\[ \frac{\ddot{b}_i}{b_i} + 3 \dot{a} \dot{b}_i + \frac{\dot{D} b_i}{D b_i} - \frac{b_i^2}{b_i^2} = 0 \]  \hspace{1cm} (6)

where we denote \( D^2 = \Pi_{i=1}^{d} b_i(t) \).

The field eqs. (4)-(6) can be rewritten as the Einstein equations with imperfect fluid in 4 dimensions. Using \( b_i = D^{2p_i} \) with \( \sum_{i=1}^{d} p_i = 1 \) the field equations reduce to

\[ 3 \left( \frac{\dot{a}^2 + k}{a^2} \right) = \rho \]  \hspace{1cm} (7)

\[ 2 \ddot{a} + \frac{\dot{a}^2 + k}{a^2} = -P_{eff} \]  \hspace{1cm} (8)

where \( \rho = \frac{\dot{D}}{D} + \alpha \left( \frac{\dot{D}}{D} \right)^2 \), \( P_{eff} = p + \Pi = -\frac{\dot{a} \dot{D}}{a D} + \alpha \left( \frac{\dot{D}}{D} \right)^2 \) and \( \alpha = \frac{1}{2} \left[ \sum_{i=1}^{d} p_i^2 - 1 \right] \).

In the above \( p \) represents the thermodynamic pressure and \( \Pi \) represents the bulk viscous stress. Using the relation \( p = (\gamma - 1) \rho \) \((1 \leq \gamma \leq 2)\) one obtains the bulk viscous stress which is given by

\[ \Pi = -\left( \gamma - \frac{4}{3} \right) \frac{\dot{D}}{D} - \alpha(\gamma - 2) \left( \frac{\dot{D}}{D} \right)^2. \]  \hspace{1cm} (9)

It is evident above that for \( \alpha = 0 \) and \( \gamma = \frac{4}{3} \) one recovers the cosmological solutions for a radiation dominated universe i.e. without viscosity. The solutions are obtained by IV in the context of perfect fluid in 4 dimensions. We study here cosmological solutions for \( \alpha \neq 0 \), which leads to interesting results. The evolution of the bulk viscous stress in \((3+1)\) dimensions is given by

\[ \Pi + \tau \dot{\Pi} = -3\zeta H - \frac{\epsilon}{2} \tau \Pi \left( 3H + \frac{\dot{\zeta}}{\zeta} - \frac{\dot{T}}{T} \right), \]  \hspace{1cm} (10)

where \( \zeta \) is the coefficient of bulk viscosity, \( \tau \) is the relaxation time and \( T \) is the temperature and \( H \left( = \frac{\dot{a}}{a} \right) \)

is the Hubble parameter. Here the parameter \( \epsilon \) can take the value either 0 or 1; \( \epsilon = 0 \) for TIS and \( \epsilon = 1 \) for FIS theory. One obtains Eckart theory for \( \tau = 0 \). The
system of eqs. (7)-(10) are not closed as number of unknowns are more than the number of equations. It is, therefore, necessary to assume the following adhoc but commonly chosen relations for the bulk viscosity co-efficient and relaxation time:

$$\zeta = \beta \rho^q \quad \text{and} \quad \tau = \beta \rho^{q-1}$$

(11)

where \( q \geq 0 \) and \( \beta \) is a constant parameter. The behavior of temperature \( T \) of a universe with imperfect fluid is obtained in FIS theory using eq.(10).

We now look for cosmological solutions. Let us consider a power law model of the expansion of the universe given by

$$a = a_o t^m$$

(12)

where \( a_o \) and \( m \) are constants. We consider a flat universe ( \( k = 0 \) ) and obtain solutions for Eckart, TIS and FIS theories for two cases \( \gamma = \frac{4}{3} \) and \( \gamma = 2 \) respectively.

**Case I : Eckart theory :**

Eckart theory corresponds to \( \tau = 0 \). The eq.(10) determines the viscous stress, using \( q = \frac{1}{2} \) i.e., \( \zeta = \beta \rho^{1/2} \) we get the following solutions:

(i) For \( \gamma = \frac{4}{3} \) one gets power law expansion with \( \beta = \frac{2(2m-1)}{\sqrt{3m}} \). The coefficient of bulk viscosity \( \beta \) is a positive quantity for \( m > \frac{1}{2} \). The model admits a large power law inflation \( (m \rightarrow \infty) \) when \( \beta \rightarrow \frac{4}{\sqrt{3}} \).

(ii) For \( \gamma = 2 \), we get \( \beta = \frac{2(3m-1)}{3\sqrt{3m}} \). In this regime a large power law inflation is obtained when \( \beta \rightarrow \frac{2}{\sqrt{3}} \).

**Case II : TIS theory ( \( \epsilon = 0 \) ) :**

(i) For \( \gamma = \frac{4}{3} \), eq.(9) determines the bulk viscous stress which is given by

$$\Pi = \frac{2\alpha(1 - 3m)^2}{3t^2}.$$  

(13)
The bulk viscous stress depends both on $\alpha$ and on the exponent $m$. For $m = \frac{1}{3}$, the co-efficient of bulk viscosity vanishes. Thus one requires $\alpha < 0$ and $m \neq \frac{1}{3}$ for an acceptable solution. The first constraint in the model is satisfied for $\sum_{i=1}^{d} p_i^2 < 1$. For simplicity we consider $q = \frac{1}{2}$ and obtain $\beta = \frac{2\sqrt{3}(2m-1)m}{9m^2+8m-4}$ from eq.(10). In this case a realistic scenario is obtained for $m > \frac{1}{2}$. One gets a large power law inflation when $\beta \to \frac{4\sqrt{3}}{9}$.

(ii) For $\gamma = 2$, the bulk viscous stress is given by

$$\Pi = -\frac{2m(3m-1)}{t^2},$$

which remains negative for $m > \frac{1}{3}$. The bulk viscous stress is independent of $\alpha$. For $q = \frac{1}{2}$, we get $\beta = \frac{2\sqrt{3}m(3m-1)}{9m^2+12m-4}$, which allows a realistic scenario for $m > \frac{1}{3}$. In this case a large power law expansion is derived when $\beta \to \frac{2\sqrt{3}}{3}$.

Case III : FIS theory ( $\epsilon = 1$ ) :

In this case temperature of the universe can be evaluated from the eq.(10).

(i) For radiation ( $\gamma = \frac{4}{3}$ ) and viscous fluid model, the bulk viscous stress is evaluated which is given by

$$\Pi = \frac{2\alpha (1-3m)^2}{3t^2}.$$  

For simplicity we take $q = \frac{1}{2}$ and obtain the temperature of the universe which is given by

$$T = T_0 t^{a_1}$$

where $a_1 = 3m - 4 + \frac{2\sqrt{3}m}{\beta} + \frac{27m^3}{\alpha(1-3m)^2}$ and $T_0$ is an integration constant. One gets a decreasing mode of temperature in this regime for $\beta > \frac{2\sqrt{3}m(2m-1)}{3m^2+11m-4}$.

which in turn determines the relaxation time and coefficient of bulk viscosity.

Thus a lower bound on viscosity is obtained depending on the exponent of the scale
factor. A realistic scenario is obtained for \( m \geq \frac{1}{2} \). However, for a large power law inflation one requires a large value of the bulk viscosity (as \( \beta > \frac{4\sqrt{3}}{3} \)).

(ii) For a stiff matter (\( \gamma = 2 \)) and viscous fluid model, the bulk viscous stress is

\[
\Pi = -\frac{2m(3m - 1)}{t^2},
\]

which is independent of \( \alpha \). Here \( \Pi < 0 \) is obtained for \( m > \frac{1}{3} \). Using \( q = \frac{1}{2} \) one gets temperature of the universe which varies as

\[
T = T_{02} t^{a_2}
\]

where \( a_2 = \frac{4 - 15m}{3m - 1} + \frac{2\sqrt{3}m}{\beta} \) and \( T_{02} \) is an integration constant. In this case a decreasing mode of temperature is found for \( \beta > \frac{2\sqrt{3}m(3m - 1)}{15m - 4} \). For a realistic scenario one requires \( m \geq \frac{1}{3} \) in this case. One gets a large power law expansion for \( \beta > \frac{2\sqrt{3}}{5} \).

Thus we obtain cosmological solutions of a higher dimensional universe considering matter sector of the theory which arises from the extra dimensions. The evolution of the extra dimensions scale factor is described by Kasner-type behavior. The solutions obtained here are identified with the \((3 + 1)\) dimensional flat \((k = 0)\) universe with imperfect fluid. The solutions are obtained in Eckart, TIS and FIS theories respectively. The causal fluid distribution considered here admit cosmological models with power law inflation as well as models with slow power law expansion of the early universe. The overall evolution of the internal space is contracting in nature (as \( \sum_{i=1}^{d} p_i^2 < 1 \) and \( \sum_{i=1}^{d} p_i = 1 \)) whereas the 3-physical space expands. We note the following:

- In the TIS or FIS theory one may get a universe with comparatively large power law expansion when \( \gamma = \frac{4}{3} \) than that when \( \gamma = 2 \) for a given bulk viscosity. However
for $\gamma = \frac{4}{3}$ with $m = \frac{1}{2}$ one gets back the solution corresponding to perfect fluid distribution. The usual radiation dominated universe ($a(t) \sim \sqrt{t}$) is recovered in the case.$^4$

- A large viscosity drives a power law inflation. In this model the viscosity parameter $\beta$ is least in TIS theory and greatest in FIS theory for a given expansion of the universe for $\gamma = \frac{4}{3}$ as well as for $\gamma = 2$. Thus in the TIS theory, one gets a large power law inflation for a comparatively lower bulk viscosity. However one gets a range of values of $\beta$ in FIS theories which requires a higher bulk viscosity to begin with for $\gamma = \frac{4}{3}$ compared to the other models. The viscosity of the universe decreases as the universe evolves.

- The temperature of the universe can be determined explicitly in the FIS theory which is found to have a decreasing mode determined by the viscous fluid distribution. The bulk viscosity of the universe decreases with time with the choice $q = \frac{1}{2}$ considered here in eq. (11). One interesting aspect of the solution is that for a given expansion of the universe $\beta$ is fixed in Eckart and TIS theories with values $\beta(\text{Eckart}) > \beta(\text{TIS})$. However one finds a lower bound on the value of viscosity in the FIS theory. In this case a hot universe cools rapidly when $\beta$ i.e., viscosity of the universe is large.

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