SPECIAL POLYNOMIALS RELATED TO THE SUPERSYMMETRIC EIGHT-VERTEX MODEL. II. SCHRÖDINGER EQUATION.

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Abstract. We show that symmetric polynomials previously introduced by the author satisfy a certain differential equation. After a change of variables, it can be written as a non-stationary Schrödinger equation with elliptic potential, which is closely related to the Knizhnik–Zamolodchikov–Bernard equation and to the canonical quantization of the Painlevé VI equation. In a subsequent paper, this will be used to construct a four-dimensional lattice of tau functions for Painlevé VI.

1. Introduction

The present work is the second part of a series, devoted to the study of certain symmetric polynomials related to the eight-vertex model and other elliptic solvable lattice models of statistical mechanics. In the first part [R3], we introduced these polynomials and studied their behaviour at special parameter values corresponding to cusps of the relevant modular group \( \Gamma_0(12) \). In the present work, we continue this study by proving that our polynomials solve a non-stationary Schrödinger equation with elliptic potential.

To be more precise, let \( m \) be a non-negative integer and \( \mathbf{k} \in \mathbb{Z}^4 \) be such that \(|\mathbf{k}| + m = 2n\) is even (throughout, \(|\mathbf{k}| = \sum_j k_j\)). In [R3], we introduced a certain space \( \Theta^k_n \) of quasi-periodic meromorphic functions; see §2. We proved that \( \dim \Theta^k_n = m \), and constructed explicit symmetric rational functions \( T_n^{(k)} \) of \( m \) variables such that, up to an elementary factor and a change of variables, \( T_n^{(k)} \) spans the one-dimensional space \( (\Theta^k_n)^\wedge m \). Since the denominator in \( T_n^{(k)} \) is elementary, they are essentially symmetric polynomials.

The functions \( T_n^{(k)} \) include as special cases various polynomials related (sometimes conjecturally) to elliptic lattice models of statistical mechanics, at the parameter values \( \Delta = \pm 1/2 \). Indeed, they appear as the ground state eigenvalue for the \( Q \)-operator of the eight-vertex model [BM1, BM2], in expressions for the domain wall partition function of the eight-vertex-solid-on-solid and three-colour models [R1, R2] and in expressions for ground state eigenvectors of the XYZ spin chain [MB, RS2, Z] and related chains [BH, FH, H].

In the present paper, we show that the elements in the space \( (\Theta^k_n)^\wedge m \) satisfy a non-stationary Schrödinger equation with elliptic potential, see Theorem 3.1.
When $m = 1$, this equation takes the form
\[ \psi_t = \frac{1}{2} \psi_{xx} - V \psi, \]  
(1.1)

where $V$ is the Darboux potential
\[ V(x, t) = \sum_{j=0}^{3} \frac{k_j(k_j + 1)}{2} \wp(x - \gamma_j|1, 2\pi it), \]
with $\gamma_j$ the four half-periods of the $\wp$-function. The $m$-variable case is simply the equation for $m$ non-interacting particles with the same potential. The case $m = 1$, $k = (0, n, n, -1)$ corresponds to the non-stationary Lamé equation in [BM1].

The equation (1.1) has appeared in the literature in several contexts. It is the canonical quantization of Painlevé VI, and has been studied from this viewpoint by Nagoya [N1, N2], Suleimanov [S1, S2] and Zabrodin and Zotov [ZZ], see [No, ZS] for related work. To explain this, recall the elliptic form of Painlevé VI,
\[ \frac{d^2 q}{dt^2} = \sum_{j=0}^{3} \nu_j \wp'(q - \gamma_j|1, 2\pi it). \]

It is equivalent to the Hamiltonian system
\[ \frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}, \]
where
\[ H = \frac{p^2}{2} - \sum_{j=0}^{3} \nu_j \wp(q - \gamma_j|1, 2\pi it). \]

In imaginary time, the canonical quantization of this system is the quantum Painlevé VI equation
\[ \hbar \psi_t = \frac{1}{2} \psi_{xx} - \sum_{j=0}^{3} \nu_j \wp(x - \gamma_j|1, 2\pi it) \psi, \]  
(1.2)

which for $\hbar = 1$ reduces to (1.1), with $\nu_j = k_j(k_j + 1)/2$.

The equation (1.2) also appears in conformal field theory and the representation theory of affine Lie algebras. At least under some extra condition on the parameters, it is the one-dimensional case of the Knizhnik–Zamolodchikov–Bernard heat equation satisfied by conformal blocks of Wess–Zumino–Witten theory on a torus [B, EK]. The general case also appears in conformal field theory [E]. Recently, Kolb [K] identified the corresponding Schrödinger operator with the radial part of the Casimir operator for $\hat{\mathfrak{sl}}(2)$ with respect to zonal spherical functions. Interestingly, the condition $\hbar = 1$ corresponds to central charge $c = 1$, a case known to have close connections to Painlevé VI, see e.g. [ER, GL]. Finally, we mention the recent paper [LT], where a more general equation, representing interacting particles, is used to study the Inozemtsev model.
By a change of variables, the Schrödinger equation can be transformed to an algebraic differential equation for the functions $T_n^{(k)}$, see Theorem 3.3. Special cases of this equation have been obtained by Bazhanov and Mangazeev [BM1, MB] (without complete proof) and Zinn-Justin [Z].

An important application of the Schrödinger equation is that, when combined with minor relations for the determinants defining $T_n^{(k)}$, it can be used to derive bilinear relations for the polynomials. Although many such relations exist, in the present paper we just give two examples, see Theorem 4.1.

In the next paper of this series [R4], Theorem 4.1 will be used to identify the case $m = 0$ of $T_n^{(k)}$ with tau functions of Painlevé VI, obtained from one of Picard’s solutions by acting with the full four-dimensional lattice of Bäcklund transformations. These tau functions can be obtained from $m = 1$ instances of $T_n^{(k)}$, that is, from solutions to (1.1), by specializing the variable to a half period. (More precisely, the solutions to (1.1) that we construct satisfy $\psi(x) = O((x-\gamma_j)^{k_j+1})$; we claim that there is a natural rescaling of these solutions so that their leading behaviour at the points $x = \gamma_j$ is given by Painlevé tau functions.) A similar observation was made in [N2] for another class of solutions. Presumably, this phenomenon is linked to the close relation between (1.2) with $\hbar = 1$ and the Lax representation of Painlevé VI described in [S1, ZZ].

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2. Preliminaries

We recall some relevant facts from [R3]. For $\tau$ fixed in the upper half-plane, we write $p = e^{\pi i \tau}$. We will also write $\omega = e^{2\pi i / 3}$. We will use the notation

$$
(x; p)_\infty = \prod_{j=0}^{\infty} (1 - xp^j),
$$

$$
\theta(x; p) = (x; p)_\infty (p/x; p)_\infty.
$$

Repeated variables are used as a short-hand for products; for instance,

$$
\theta(a, \pm b; p) = \theta(a; p) \theta(b; p) \theta(-b; p).
$$

The function

$$
\psi(z) = \psi(z, \tau) = p^{\frac{1}{12}} (p^2; p^2)_\infty e^{-\pi iz} \theta(e^{2\pi iz}, \pm pe^{2\pi iz}; p^2)
$$

satisfies

$$
\psi(z + 1) = \psi(-z) = -\psi(z), \quad \psi(z + \tau) = e^{-3\pi i (\tau + 2z)} \psi(z),
$$

the heat equation

$$
12\pi^2 \frac{\partial \psi}{\partial \tau} = \frac{\partial^2 \psi}{\partial z^2}
$$
and
\[ \psi(z) = \psi \left( z + \frac{1}{3} \right) + \psi \left( z - \frac{1}{3} \right). \] (2.4)

We will write
\[ x(z) = x(z, \tau) = \frac{\theta(-p; p^2)\theta(\omega e^{\pm 2\pi i z}; p^2)}{\theta(-\omega; p^2)\theta(p\omega e^{\pm 2\pi i z}; p^2)}, \]
\[ \zeta = \zeta(\tau) = \frac{\omega^2\theta(-1, -p\omega; p^2)}{\theta(-p, -\omega; p^2)}. \]

The function \( x \) generates the field of even elliptic functions with periods \( (1, \tau) \). Moreover, \( \tau \mapsto \zeta(2\tau) \) generates the field of modular functions for the group \( \Gamma_0(12) \).

We will need the identity
\[ x(z) - x(w) = -\frac{\omega \theta(p, p\omega; p^2)\theta(-p\omega; p^2)^2}{e^{2\pi i z}\theta(-\omega; p^2)^2} \frac{\theta(e^{2\pi i(z+w)}; p^2)}{\theta(p\omega e^{\pm 2\pi i z}; p^2)} \] (2.5)
and its limit case
\[ x'(z) = \frac{2\pi i \omega(p^2; p^2)\omega \theta(p, p\omega; p^2)\theta(-p\omega; p^2)^2}{\theta(-\omega; p^2)^2} e^{-2\pi i z} \frac{\theta(e^{4\pi i z}; p^2)}{\theta(p\omega e^{\pm 2\pi i z}; p^2)}. \] (2.6)

Denoting the half-periods in \( \mathbb{Z} + \tau \mathbb{Z} \) by
\[ \gamma_0 = 0, \quad \gamma_1 = \frac{\tau}{2}, \quad \gamma_2 = \frac{\tau}{2} + \frac{1}{2}, \quad \gamma_3 = \frac{1}{2}, \]
the values \( \xi_j = x(\gamma_j) \) and \( \eta_j = x(\gamma_j + 1/3) \) are given by
\[ \xi_0 = 2\zeta + 1, \quad \xi_1 = \frac{\zeta}{\zeta + 2}, \quad \xi_2 = \frac{\zeta(2\zeta + 1)}{\zeta + 2}, \quad \xi_3 = 1, \] (2.7a)
\[ \eta_0 = 0, \quad \eta_1 = \infty, \quad \eta_2 = \frac{2\zeta + 1}{\zeta + 2}, \quad \eta_3 = \zeta. \] (2.7b)

Moreover \([R2, Lemma 9.1]\),
\[ \zeta + 1 = -\frac{\theta(p, -p\omega; p^2)}{\theta(-p, -p\omega; p^2)}, \] (2.8a)
\[ \zeta - 1 = -\frac{\theta(p, p\omega; p^2)\theta(\omega; p^2)^2}{\theta(-p, -p\omega; p^2)\theta(-\omega; p^2)^2}, \] (2.8b)
\[ \zeta + 2 = p \frac{\theta(-1, -\omega; p^2)\theta(\omega; p^2)^2}{\theta(-p, -p\omega; p^2)\theta(p\omega; p^2)^2}, \] (2.8c)
\[ 2\zeta + 1 = \frac{\theta(-p\omega, \omega; p^2)^2}{\theta(-\omega, p\omega; p^2)^2}. \] (2.8d)

For \( n \in \mathbb{Z} \) and \( k = (k_0, k_1, k_2, k_3) \in \mathbb{Z}^4 \), such that \( 2n \geq |k| = \sum_j k_j \), we define a function space \( \Theta_n^k \) as follows \([R3, Lemma 2.3]\). The elements in the space are
meromorphic functions, which are analytic outside the lattice \( \frac{1}{6}\mathbb{Z} + \frac{5}{2}\mathbb{Z} \), satisfy
\[
    f(z + 1) = f(z), \quad f(z + \tau) = e^{-6\pi i n(\tau + 2z)} f(z), \quad f(-z) = -f(z),
\]
\[
    (2.9a)
\]
\[
    f(z) + f \left( z + \frac{1}{3} \right) + f \left( z - \frac{1}{3} \right) = 0
\]
\[
    (2.9b)
\]
and, for \( j = 0, 1, 2, 3 \),
\[
    \lim_{z \to \gamma_j} (z - \gamma_j)^{1-2k_j} f(z) = \lim_{z \to \gamma_j} (z - \gamma_j)^2 \left( f \left( z + \frac{1}{3} \right) - f \left( z - \frac{1}{3} \right) \right) = 0.
\]
\[
    (2.10)
\]
Writing \( m = 2n - |k| \), we have proved that \( \dim \Theta_n^k = m \) \([3, \text{Thm. 2.4}]\). Moreover, realizing the maximal exterior power \( (\Theta_n^k)^\wedge m \) as a space of anti-symmetric functions in \( z_1, \ldots, z_m \), it is spanned by
\[
    \prod_{j=1}^m \left( M_n(z_j) \prod_{l=0}^3 (x_j - \xi_l)^{k_l} \right) \Delta(x_1, \ldots, x_m) T_n(x_1, \ldots, x_m),
\]
\[
    (2.11)
\]
where
\[
    M_n(z) = e^{-2\pi i \theta (e^{4\pi i z}; p^2) \theta (\omega p e^{2\pi i z}; p^2)^3 n^{-2}},
\]
\[
    x_j = x(z_j), \quad \Delta(x) = \prod_{i<j} (x_j - x_i) \quad \text{and} \quad T_n(x_1, \ldots, x_m)
\]
\[
    \text{is a certain symmetric rational function, depending also rationally on the parameter } \zeta.
\]
To describe the construction of \( T_n^k \), we start with the case
\[
    T_n^{(0,0,0,0)}(x_1, \ldots, x_{2n})
\]
\[
    = \frac{\prod_{i,j=1}^n G(x_i, x_{n+j})}{\Delta(x_1, \ldots, x_n) \Delta(x_{n+1}, \ldots, x_{2n})} \det_{1 \leq i,j \leq n} \left( \frac{1}{G(x_i, x_{n+j})} \right),
\]
\[
    (2.13)
\]
where
\[
    G(x, y) = (\zeta + 2)xy(x + y) - \zeta(x^2 + y^2) - 2(\zeta^2 + 3\zeta + 1)xy + \zeta(2\zeta + 1)(x + y).
\]
Then, \( T_n^{(0,0,0,0)} \) is a symmetric polynomial in all its variables, depending also as a polynomial on the parameter \( \zeta \). Up to a change of variables, \( T_n^{(0,0,0,0)} \) coincides with the polynomial \( H_{2n} \) of \([3]\). For \( k \in \mathbb{Z}_{\geq 0} \), \( T_n^k \) is obtained from \( T_n^{(0,0,0,0)} \) by specializing \( k_j \) of the variables to \( \xi_j \), for \( 0 \leq j \leq 3 \). If \( k_j < 0 \) for some \( j \), the definition is more complicated.

To explain the general definition of \( T_n^k \), let
\[
    (\sigma f)(z) = f(z + 1/3) - f(z - 1/3)
\]
and
\[
    a(x) = (x - (2\zeta + 1))(x - 1)(((\zeta + 2)x - \zeta)((\zeta + 2)x - (2\zeta + 1)\zeta).
\]
\[
    (2.14)
\]
\[
    = (\zeta + 2)^2 \prod_{l=0}^3 (x - \xi_l).
\]
Then, \( a(x(z)) \) has a meromorphic square root that we will denote \( \sqrt{a} \). (In \([R3]\), \( \sigma \) and \( \sqrt{a} \) are denoted \( \sqrt{3}\sigma/i \) and \( i\phi/\sqrt{3} \).) We can then write
\[
\sigma\bigg|_{\Theta^{(0,0,0)}} = \frac{M_n}{\sqrt{a}} \hat{\sigma}_n \frac{1}{M_n},
\]
where \( \hat{\sigma}_n \) is an operator acting between appropriate spaces of polynomials in \( x = x(z) \). It is determined by
\[
\hat{\sigma}_n \left( (x-a) \prod_{j=1}^{n-1} (x-b_j) G(x, b_j) \right) = \prod_{j=1}^{n-1} (x-b_j) G(x, b_j)
\times \left( x(x-2\zeta-1)((\zeta+2)x-3\zeta) - a((\zeta+2)x-\zeta)(2\zeta+1-3x) \right),
\]
where \( a \) and \( b_j \) are arbitrary.

We may now define, for \( 0 \leq k \leq 2n \),
\[
T(x_1, \ldots, x_k; x_{k+1}, \ldots, x_{2n}) = \frac{(id^\otimes k \otimes \hat{\sigma}_n^\otimes (2n-k)) \Delta(x_1, \ldots, x_{2n}) T_n^{(0,0,0)}(x_1, \ldots, x_{2n})}{\Delta(x_1, \ldots, x_k) \Delta(x_{k+1}, \ldots, x_{2n})}.
\]

The identity (2.16) can be applied termwise to (2.13) to give an explicit formula for (2.17) as a block determinant, see \([R3]\) Eq. (2.39).

The general definition of \( T_n^{(k)} \) can now be stated as
\[
T_n^{(k)}(x_1, \ldots, x_m) = \frac{(-1)^{\binom{k}{2}} T(x_1, \ldots, x_m; \xi^k; \xi^{-k})}{2^{k+1} \prod_{i,j=0}^3 G(\xi_i; \xi_j)^{k_i^+ k_j^-} \prod_{i=0}^m \prod_{j=1}^3 G(x_j, \xi_i)^{k_i}},
\]
where \( k_j^\pm = \max(\pm k_j, 0) \) and
\[
\xi^k = (\xi_0, \ldots, \xi_0, \underbrace{\xi_3, \ldots, \xi_3}_{k_3}).
\]

Then,
\[
T_n^{(k+1)}(x_1, \ldots, x_m) = T_n^{(k)}(x_1, \ldots, x_m, \xi^1), \quad 1 \in \mathbb{Z}_{\geq 0}.
\]

In §3.3 we will also need dual functions defined by
\[
U_n^{(k)}(x_1, \ldots, x_m) = \frac{(-1)^{\binom{k+1}{2}} T(\xi^{k^+}; x_1, \ldots, x_m; \xi^{k^-})}{2^{k+1} \prod_{i,j=0}^3 G(\xi_i; \xi_j)^{k_i^+ k_j^-} \prod_{i=0}^m \prod_{j=1}^3 G(x_j, \xi_i)^{k_i}}.
\]

They satisfy
\[
U_n^{(k-1)}(x_1, \ldots, x_m) = U_n^{(k)}(x_1, \ldots, x_m; \xi^1), \quad 1 \in \mathbb{Z}_{\geq 0}.
\]
which can be proved similarly as \((2.18)\). It follows easily from \((2.17)\) that
\[
\hat{\sigma}^\otimes_m \prod_{j=1}^m \prod_{i=0}^3 (x_j - \xi_i)^{k_i} G(x_j, \xi_i)^{k_i^-} (\Delta T^{(k)}_n)(x_1, \ldots, x_m)
\]
\[
= (-1)^{(k_2)} 2^{k_2} \prod_{j=1}^m (x_j - \xi_i)^{k^-} G(x_j, \xi_i)^{k^-} (\Delta U^{(k)}_n)(x_1, \ldots, x_m). \tag{2.21}
\]
By \([R3\text{ Prop. 2.20}]\), we have up to an explicit factor independent of the variables \(x_j\),
\[
U^{(k_0, k_1, k_2, k_3)}_n(x_1, \ldots, x_m) \sim \prod_{j=1}^m x_j \left( x_j - \frac{2\zeta + 1}{\zeta + 2} \right) (x_j - \zeta)
\]
\[
\times T^{(-k_0-1, -k_1-1, -k_2-1, -k_3-1)}_{m-2-n}(x_1, \ldots, x_m);
\]
however, we will not need this fact.

3. Schrödinger equation

3.1. Schrödinger equation with elliptic potential. In this Section, we show that the elements in the one-dimensional space \((\Theta^{(k)}_n)^\wedge m\) satisfy a Schrödinger equation with elliptic potential. We let \(\wp = \wp(z|1/3, \tau)\) denote Weierstrass’s \(\wp\)-function as defined in \([WW]\). It is an even elliptic function with periods \(1/3\) and \(\tau\), with no singularities except double poles at the lattice points, such that
\[
\lim_{z \to 0} z^2 \wp(z) = 1. \tag{3.1}
\]
These properties determine \(\wp\) uniquely up to an additive constant, whose value is irrelevant for our purposes.

**Theorem 3.1.** Let \(\Psi(z_1, \ldots, z_m, \tau)\) be a meromorphic function, which for fixed \(\tau\) belongs to \((\Theta^{(k)}_n)^\wedge m\), and let
\[
\Phi = \prod_{j=1}^m \left( e^{-3\pi i z_j} \theta(e^{6\pi i z_j}; p^6)^{k_0} \theta(p^3 e^{6\pi i z_j}; p^6)^{k_1} \right.
\]
\[
\times \theta(-p^3 e^{6\pi i z_j}; p^6)^{k_2} \left( e^{-3\pi i z_j} \theta(-e^{6\pi i z_j}; p^6) \right)^{k_3}).
\]
Then, \(\Phi^{-1} \Psi\) satisfies the Schrödinger equation
\[
\mathcal{H} \Phi^{-1} \Psi = C \Phi^{-1} \Psi, \tag{3.2}
\]
where
\[
\mathcal{H} = -12\pi i m \frac{\partial}{\partial \tau} + \sum_{j=1}^m \left( \frac{\partial^2}{\partial z_j^2} - V(z_j) \right), \tag{3.3}
\]
C is independent of the variables $z_j$ and

$$V(z) = \sum_{j=0}^{3} k_j (k_j + 1) \varphi(z - \gamma_j).$$

Note that $\Psi$ is only determined up to a factor depending on $\tau$; the factor $C$ depends on this choice of normalization. If we choose $C = 0$ and use that $\varphi(z|1/3, \tau) = 9 \varphi(3z|1, 3\tau)$, we find that the case $m = 1$ of (3.2) reduces to (1.1), with $z = x/3$, $\tau = 2\pi iT/3$.

For the proof of Theorem 3.1, we first state the following elementary consequence of the chain rule.

**Lemma 3.2.** If $f(z, \tau)$ is a meromorphic function in two variables satisfying

$$f(z + \tau, \tau) = \varepsilon e^{-\lambda(\tau + 2z)} f(z, \tau),$$

where $\varepsilon$ and $\lambda$ are arbitrary constants, then the same identity holds with $f$ replaced by

$$\frac{\partial^2 f}{\partial z^2} - 4\lambda \frac{\partial f}{\partial \tau}.$$

Let us express the potential in terms of the function

$$\phi(z) = \frac{i}{3\pi} \frac{(-p^6; p^6)_\infty^2}{(p^6; p^6)_\infty^2} \frac{\theta(e^{6\pi i z}; p^6)}{\theta(-e^{6\pi i z}; p^6)}.$$

(3.4)

It is easy to see that [WW, §20.53, Example 1]

$$\varphi(z) - \varphi(1/6) = \frac{1}{\phi(z)^2}.$$

Thus, up to a change of the constant $C$, we may as well prove that (3.2) holds with the modified potential

$$V(z) = \sum_{j=0}^{3} \frac{k_j (k_j + 1)}{\phi(z - \gamma_j)^2}.$$  

(3.5)

Note that (3.1) translates to

$$\phi'(0)^2 = 1$$  

(3.6)

(indeed, one may check directly from (3.4) that $\phi'(0) = 1$).

Since, by [R3, Thm. 2.4], $\dim(\Theta_n^k)^\wedge m = 1$, it is enough to show that $\Xi = \Phi H \Phi^{-1} \Psi \in (\Theta_n^k)^\wedge m$. As a function of each $z_j$, $\Phi$ satisfies

$$\Phi(z + 1/3) = (-1)^{k_0 + k_3} \Phi(z),$$

$$\Phi(-z) = (-1)^{k_0} \Phi(z),$$

$$\Phi(z + \tau) = (-1)^{k_0 + k_1} e^{-3\pi i (k_0 + k_1 + k_2 + k_3)(\tau + 2z)} \Phi(z).$$
Thus, (3.8) follows from
\[ (\Phi^{-1}\Psi)(z + 1) = (-1)^{k_0+k_3}(\Phi^{-1}\Psi)(z), \]
\[ (\Phi^{-1}\Psi)(-z) = (-1)^{k_0+1}(\Phi^{-1}\Psi)(z), \]
\[ (\Phi^{-1}\Psi)(z + \tau) = (-1)^{k_0+k_1}e^{-3\pi im(\tau+2z)}(\Phi^{-1}\Psi)(z), \]
\[ (\Phi^{-1}\Psi)(z) + (-1)^{k_0+k_3}((\Phi^{-1}\Psi)(z + 1/3) + (\Phi^{-1}\Psi)(z - 1/3)) = 0. \]

We must show that these relations are preserved by \( \mathcal{H} \). Since \( V \) is an even elliptic function with periods 1/3 and \( \tau \), this is clear except for the third relation, which is covered by Lemma 3.2. Thus, \( \Xi \) satisfies (2.9) as a function of each \( z_j \). It is also obviously antisymmetric.

It remains to show that
\[ \lim_{z_1 \to \gamma_j} \phi(z_1 - \gamma_j)^{1-2k_j}\Xi(z_1) = 0, \]  
(3.8)
\[ \lim_{z_1 \to \gamma_j} \phi(z_1 - \gamma_j)^2 \left( \Xi \left( z_1 + \frac{1}{3} \right) - \Xi \left( z_1 - \frac{1}{3} \right) \right) = 0. \]  
(3.9)

To prove (3.8), we write
\[ \Phi^{-1}\Psi = \phi(z_1 - \gamma_j)^{k_j+1} \cdot \Phi^{-1}\Psi, \]  
where, by (2.10) for \( f = \Psi(z_1) \), the second factor is regular at \( z_1 = \gamma_j \). Acting with \( \mathcal{H} \), only the term \( \partial^2/\partial z_1^2 - V(z_1) \) contributes to (3.8). Moreover, both derivatives must hit the first factor in (3.10), which is then reduced to
\[ k_j(k_j + 1)\phi'(z_1 - \gamma_j)^2\phi(z_1 - \gamma_j)^{k_j-1}. \]

Thus, (3.8) follows from
\[ \lim_{z \to \gamma_j} \left( k_j(k_j + 1)\phi'(z - \gamma_j)^2 - V(z)\phi(z - \gamma_j)^2 \right) = 0, \]  
(3.11)
which is true in view of (3.6).

The proof of (3.9) is similar. We start from the factorization
\[ \Phi(z) = \phi(z_1 - \gamma_j)^{-k_j} \cdot \frac{\Phi(z_1 + 1/3) - \Psi(z_1 - 1/3)}{\phi(z_1 - \gamma_j)^{-k_j}} \cdot \frac{\Psi(z_1 + 1/3) - \Psi(z_1 - 1/3)}{\Phi(z)\phi(z_1 - \gamma_j)^{-k_j}}. \]

In view of (3.7) and the fact that \( V \) is 1/3-periodic, the operator \( \Phi\mathcal{H}\Phi^{-1} \) commutes with translations by 1/3. Using this fact, (3.9) can be reduced to (3.11).

3.2. Uniformized Schrödinger equation. The following result is a uniformized version of Theorem 3.1. The special case \( m = 1, k = (0, n, n, -1) \), is equivalent to [BM1, Eq. (27)] (given there without a complete proof, since it was not known at the time that \( \dim \Theta_n^{(0,n,n,-1)} = 1 \)). The case \( m = 1, k = (n, n, 0, -1) \) was conjectured in [MB]; it is in fact equivalent to the case \( k = (0, n, n, -1) \) by the symmetries [R3, Cor. 2.19]. Moreover, the case \( m = 2n, k = (0, 0, 0, 0) \) is equivalent to [Z, Eq. (50)].
Theorem 3.3. The function $T_n^{(k)}$ satisfies the differential equation

$$\left( \sum_{j=1}^{m} \left( a(x_j) \frac{\partial^2}{\partial x_j^2} + b(x_j) \frac{\partial}{\partial x_j} + c(x_j) \right) + m d \frac{\partial}{\partial \zeta} \right) \left( \prod_{j=1}^{m} F(x_j) \Delta(x_1, \ldots, x_m) T_n^{(k)}(x_1, \ldots, x_m) \right) = 0,$$

where

$$F(x) = \prod_{j=0}^{3} \left( \frac{x - \xi_j}{G(x, \xi_j)} \right)^{k_j},$$

$$f = \left( \zeta + 1 \right)^{\frac{1}{2}} k_1(k_1 - 3) + \frac{1}{2} k_2(k_2 - 3) - k_0 k_3 \left( \zeta - 1 \right)^{\frac{1}{2}} k_2(k_2 + k_3 - 2) (2\zeta + 1)^{\frac{1}{2}} + \frac{1}{2} k_2 k_3,$$

$$a(x) \text{ is given by } x^{2m}, \quad b(x) \text{ is a polynomial in } (x, \zeta) \text{ of bidegree } (3, 3), \quad \text{which we give in terms of the partial fraction decomposition}$$

$$\frac{b(x)}{a(x)} = \frac{3(\zeta + 1) + m(\zeta - 1)(\zeta + 2)}{2(\zeta + 1)(x - 2\zeta - 1)} + \frac{(\zeta + 2)(3\zeta(\zeta + 1) - m(2\zeta + 1)(\zeta - 1))}{2\zeta((\zeta + 2)x - \zeta)}$$

$$+ \frac{(\zeta + 2)(3\zeta - m(\zeta^2 + 4\zeta + 1))}{2\zeta((\zeta + 2)x - \zeta(2\zeta + 1))} + \frac{3}{2(x - 1)}, \quad (3.12)$$

c(x) = c_0(x) + W(x), \quad \text{with}

$$c_0(x) = \frac{3(m - 2)(3m - 4)}{4} (\zeta + 2)^2 x^2$$

$$- \frac{3m - 4}{2} (\zeta + 2)(2m - 3)(\zeta^2 + 1) + (7m - 12)\zeta x$$

$$- \frac{2(2m^2 - 5)}{3} \zeta^4 - \frac{7m^2 + 66m - 112}{6} \zeta^3 + \frac{7(m - 2)(7m - 8)}{4} \zeta^2$$

$$+ \frac{(5m - 8)(19m - 14)}{6} \zeta + \frac{11m^2 - 24m + 10}{3}, \quad (3.13)$$

$$W(x) = -k_0(k_0 + 1)(2\zeta + 1)^3 \frac{(x - 1)(x - \zeta)^2}{x^2(x - (2\zeta + 1))}$$

$$- k_1(k_1 + 1) \frac{((\zeta + 2)x - \zeta(2\zeta + 1))(\zeta + 2)x - (2\zeta + 1)^2}{(\zeta + 2)x - \zeta}$$

$$+ k_2(k_2 + 1) \frac{(\zeta + 1)(\zeta - 1)^2(2\zeta + 1)^3((\zeta + 2)x - \zeta)}{((\zeta + 2)x - (2\zeta + 1))(\zeta + 2)x - (2\zeta + 1)^2}.$$
\[ k_3 (k_3 + 1) (\zeta + 1) (\zeta - 1)^3 \frac{x^2 (x - (2\zeta + 1))}{(x - 1) (x - \zeta)^2} \]  

and

\[ d = 2\zeta (\zeta - 1) (\zeta + 1) (\zeta + 2) (2\zeta + 1). \]

The factors \( f \) and \( F(x_j) \) have been introduced in order to simplify the expressions for the coefficients \( b \) and \( c \). For later use, we note that commuting them across the differential operator leads to

\[
\left( \sum_{j=1}^{m} \left( a(x_j) \frac{\partial^2}{\partial x_j^2} + b_F(x_j) \frac{\partial}{\partial x_j} + c_F(x_j) \right) + md \frac{\partial}{\partial \zeta} + e \right) \Delta T_n^{(k)} = 0, \tag{3.15}
\]

where

\[
\begin{align*}
b_F &= 2d \frac{\partial F}{\partial x} + b, \\
c_F &= a \frac{\partial^2 F}{\partial x^2} + b \frac{\partial F}{\partial x} + c + md \frac{\partial F}{\partial \zeta}, \\
e &= md \frac{f'}{f}.
\end{align*}
\tag{3.16}
\]

To prove Theorem 3.3, we first note that, up to a factor independent of the variables \( z_j \), (2.11) is proportional to

\[
\Phi \prod_{j=1}^{m} \psi(z_j)^m E(x_j) F(x_j) \Delta(x_1, \ldots, x_m) T_n^{(k)}(x_1, \ldots, x_m), \tag{3.17}
\]

where

\[
E(x) = (x - \xi_0)^{-\frac{1-m}{2}} (x - \xi_1)^{-\frac{1-m}{2}} (x - \xi_2)^{-\frac{1-m}{2}} (x - \xi_3)^{-\frac{1}{2}}.
\]

This can be seen either from the fact that the quotient of (2.11) and (3.17) is periodic without zeroes or poles, or using (2.5). We choose the function \( \Psi \) in Theorem 3.1 as (3.17).

We first study the action of the Schrödinger operator \( \mathcal{H} \) on \( \psi(z_j)^m \). We have

\[
\prod_{j=1}^{m} \psi(z_j)^{-m} \mathcal{H} \prod_{j=1}^{m} \psi(z_j)^m = \sum_{j=1}^{m} \left( m \frac{\psi''(z_j)}{\psi(z_j)} + m(m - 1) \frac{\psi'(z_j)^2}{\psi(z_j)^2} - 12 \pi i m^2 \frac{\psi(z_j)}{\psi(z_j)} \right)
\]

\[ = \sum_{j=1}^{m} m(m - 1) \left( \frac{\psi'(z_j)^2}{\psi(z_j)^2} - \frac{\psi''(z_j)}{\psi(z_j)} \right) = - \sum_{j=1}^{m} m(m - 1) (\log \psi(z_j))'',
\]
where $\psi' = \partial \psi / \partial z$, $\dot{\psi} = \partial \psi / \partial \tau$ and we used (2.3) in the second step. It follows that, for any function $X = X(z_1, \ldots, z_m, \tau)$,

$$
\prod_{j=1}^{m} \psi'(z_j)^{-m} \prod_{j=1}^{m} \psi(z_j)^{m} X = -12\pi i m \dot{X}
$$

$$
+ \sum_{j=1}^{m} \left( \frac{\partial^2 X}{\partial z_j^2} + 2m (\log \psi(z_j))' \frac{\partial X}{\partial z_j} - (m(m-1)(\log \psi(z_j))'' + V(z_j)) X \right).
$$

Thus, if $X$ can be expressed in terms of the variables $x_j = x(z_j, \tau)$ and $\zeta = \zeta(\tau)$,

$$
\prod_{j=1}^{m} \psi(z_j)^{-m} \prod_{j=1}^{m} \psi(z_j)^{m} X
$$

$$
= -12\pi i m \dot{X} \frac{\partial X}{\partial \zeta} + \sum_{j=1}^{m} \left( (x_j')^2 \frac{\partial^2 X}{\partial x_j^2} + (x_j'' + 2m (\log \psi(z_j))' x_j' - 12\pi i m \dot{x}_j) \frac{\partial X}{\partial x_j} - (m(m-1)(\log \psi(z_j))'' + V(z_j)) X \right).
$$

(3.18)

We must express the coefficients in (3.18) in terms of the variables $x_j$ and $\zeta$. We formulate the relevant elliptic function identities as a series of lemmas. It is convenient to introduce the parameter

$$
\chi = \chi(\tau) = 4\pi^2 \rho(p^2; p^2)^4 \theta(-1; p^2) \theta(-\omega; p^2)^3.
$$

Lemma 3.4. We have

$$
(x')^2 = -\frac{\chi}{2\zeta(\zeta + 1)(\zeta + 2)} a,
$$

(3.19)

$$
x'' = -\frac{\chi}{4\zeta(\zeta + 1)(\zeta + 2)} \frac{\partial a}{\partial x},
$$

(3.20)

where $a$ is as in (2.14).

Proof. Using (2.5) and (2.6), we can write

$$
x'(z) = \frac{4\pi^2 \omega^2 (p^2; p^2)^4 \theta(p, p\omega; p^2)^2 \theta(-p\omega; p^2)^4}{\theta(-\omega; p^2)^4}
$$

$$
\times \frac{\theta(e^{\pm 2\pi i z}; -e^{\pm 2\pi i z}, pe^{\pm 2\pi i z}, -pe^{\pm 2\pi i z}; p^2)}{\theta(p\omega e^{\pm 2\pi i z}; p^2)^4}
$$

$$
= -\frac{4\pi^2 \omega^2 (p^2; p^2)^4 \omega^2 \theta(-\omega; p^2)^6 \theta(\omega; p^2)^2}{\theta(p; p^2)^2 \theta(-p\omega; p^2)^2 (\zeta + 2)^2} a(x).
$$

By (2.8), this can be written in the form (3.19), which is then differentiated to yield (3.20). □
It follows from Lemma 3.4 that
\[ \frac{\partial^2}{\partial z^2} = -\frac{\chi}{4\zeta(\zeta + 1)(\zeta + 2)} \left( 2a \frac{\partial^2}{\partial x^2} + \frac{\partial a}{\partial x} \frac{\partial}{\partial x} \right). \] (3.21)

**Lemma 3.5.** The function \( \psi \) satisfies
\[ \frac{\psi'(1/3)}{\psi(1/3)} = -2\pi i \left( \frac{p^2; p^2}_\infty \theta(-\omega; p^2) \right), \] (3.22)
\[ \frac{\psi'(1/3 + \tau/2)}{\psi(1/3 + \tau/2)} + 3\pi i = -2\pi i \omega^2 \theta(-p\omega; p^2) \theta(-p; p^2). \] (3.23)

**Proof.** Since \( \psi \) is odd, differentiating (2.4) gives
\[ \psi'(1/3) = \frac{\psi'(0)}{2}. \]
On the other hand, by (2.1) we can write
\[ \psi(z) = \frac{1}{1 - e^{4\pi iz}} \frac{p^{\frac{1}{12}} \theta(-e^{2\pi iz}; p^2)_\infty}{\theta(-x^2; p^2)_\infty}, \]
which for \( z \to 0 \) reduces to
\[ \psi'(0) = -4\pi i \frac{p^{\frac{1}{12}} (p^2; p^2)_\infty^3}{\theta(-1; p^2)}. \] (3.24)

After simplification, this yields (3.22). Similarly, it follows from (2.2) and (2.4) that
\[ \psi(z) = \psi\left(z + \frac{1}{3}\right) - e^{3\pi i(\tau - 2z)} \psi\left(-z + \tau + \frac{1}{3}\right). \]
Differentiating this identity and letting \( z = \tau/2 \) gives
\[ \frac{\psi'(1/3 + \tau/2)}{\psi(1/3 + \tau/2)} + 3\pi i = \frac{\psi'(1/3 + \tau/2)}{2\psi(1/3 + \tau/2)}. \] (3.25)

We now let \( z \to \tau/2 \) in the identity
\[ \frac{\psi(z)}{1 - p^2 e^{4\pi iz}} = \frac{p^{\frac{1}{12}} e^{-\pi iz} (p^2, e^{4\pi iz}, p^4 e^{-4\pi iz}, p^2)_\infty}{\theta(-e^{2\pi iz}; p^2)_\infty}, \]
to obtain
\[ \psi'\left(\frac{\tau}{2}\right) = 4\pi i \frac{p^{-\frac{1}{12}} (p^2; p^2)_\infty^3}{\theta(-p; p^2)}. \]
Combining this with (3.25) gives (3.23). \( \square \)

**Lemma 3.6.** In the notation above,
\[ 2 \frac{\psi'}{\psi} x' - 12\pi i x = -\frac{\chi}{\zeta + 2} B, \] (3.26)
where
\[ B(x, \zeta) = (x - 1)((\zeta + 2)x + 2\zeta + 1). \]
Proof. Let \( q \) denote the left-hand side of (3.26). It is easy to check that \( q \) is an even elliptic function with periods 1, \( \tau \). Thus, as a function of \( z \), it is a rational function of \( x(z) \). By (2.6), \( x' \) vanishes at all zeroes of \( \psi \), so \( q \) can have poles only where \( x \) has poles. This means that \( q \) is a polynomial in \( x \). Moreover, since \( x \) has only single poles, \( q \) has at most double poles, which means that \( q \) is a polynomial of degree at most 2, say \( q(x) = \alpha x^2 + \beta x + \gamma \).

Since \( x(1/3) = 0 \), it is clear that \( \dot{x}(1/3) = 0 \), and thus

\[
\gamma = q(0) = q(x(1/3)) = 2\frac{\psi'}{\psi}(1/3)x'(1/3).
\]

Using (3.22) and (2.6), one readily computes

\[
\gamma = 4\pi^2(p^2; p^2)^4\theta(-p; p^2)\theta(-p\omega; p^2)^3.
\]

Next, differentiating the identity \( x^{-1}(1/3 + \tau/2) = 0 \) gives

\[
\left. \frac{x' + 2\dot{x}}{x^2} \right|_{z = 1/3 + \tau/2} = 0.
\]

It follows that

\[
\alpha = \lim_{z \to 1/3 + \tau/2} \frac{1}{x^2} \left( 2\frac{\psi'}{\psi}x' - 12\pi i \dot{x} \right) = \lim_{z \to 1/3 + \tau/2} 2\frac{\psi'}{x^2} \left( \psi' + 3\pi i \right).
\]

Using (3.23) and (2.6), this can be simplified to \( \alpha = -\chi \). By (2.8), it follows that

\[
\frac{\gamma}{\alpha} = -\frac{\pi}{\theta(-1; \omega)}\frac{\theta(-p; p^2)\theta(-p\omega; p^2)^3}{\theta(-\omega; p^2)^2} = -\frac{2\zeta + 1}{\zeta + 2}.
\]

Finally, we let \( z = 1/2 \). Since \( x(1/2) = 1, \dot{x}(1/2) = 0 \). Moreover, it is clear from (2.6) that \( x'(1/2) = 0 \), while \( \psi(1/2) \neq 0 \). This shows that \( q(x(1/2)) = q(1) = 0 \).

We conclude that indeed

\[
q = \alpha(x - 1) \left( x - \frac{\gamma}{\alpha} \right) = -\chi(x - 1) \left( x + \frac{2\zeta + 1}{\zeta + 2} \right).
\]

\[\Box\]

Lemma 3.7. The function \( \dot{\zeta} = \partial \zeta / \partial \tau \) can be expressed as

\[
12\pi i \dot{\zeta} = \chi(\zeta - 1)(2\zeta + 1).
\]

Proof. Let \( z = 0 \) in (3.26). By (2.6), (2.8) and (3.24),

\[
\frac{\psi'x'}{\psi}(0) = \frac{8\pi^2 \omega(p^2; p^2)^4\theta(p; p^2)\theta(-p\omega; p^2)^2}{\theta(-\omega; p^2)^2\theta(p\omega; p^2)^3} = -\frac{2\chi(\zeta + 1)(2\zeta + 1)}{\zeta + 2}.
\]

Since \( x(0) = 2\zeta + 1 \), we have \( \dot{x}(0) = 2\dot{\zeta} \) and

\[
B(x(0), \zeta) = 2\zeta(2\zeta + 1)(\zeta + 3).
\]
Combining these facts, we find that
\[12\pi i \hat{\zeta} = \chi \left( \frac{\zeta(2\zeta + 1)(\zeta + 3)}{\zeta + 2} - \frac{2(\zeta + 1)(2\zeta + 1)}{\zeta + 2} \right),\]
which simplifies to the desired result. \(\Box\)

**Lemma 3.8.** One may write
\[
(\log \psi)'' = C(\tau) + \frac{\chi}{\zeta + 2} D,
\]
where \(C\) is independent of \(z\) and
\[
D(x, \zeta) = \frac{(x - \zeta)(x(\zeta + 2) + \zeta(2\zeta + 1))}{(x - (2\zeta + 1))((\zeta + 2)x - \zeta)((\zeta + 2)x - \zeta(2\zeta + 1))}.
\]

**Proof.** It is easy to see that both \((\log \psi)''\) and \(D\) are even elliptic function with periods 1 and \(\tau\), the only singularities being double poles at the zeroes of \(\psi\). If we can show that
\[
\lim_{\psi(z) \to 0} \psi^2 ((\log \psi)'' - \chi D/(\zeta + 2)) = 0,
\]
then the conclusion follows from Liouville’s theorem.

Clearly,
\[
\lim_{\psi(z) \to 0} \psi^2 (\log \psi)'' = - \lim_{\psi(z) \to 0} (\psi')^2.
\]
If we let \(P\) and \(Q\) denote the numerator and denominator of \(D\), respectively, then l’Hôpital’s rule and (3.21) give
\[
\lim_{\psi(z) \to 0} \frac{\psi^2 \chi D}{\zeta + 2} = \lim_{\psi(z) \to 0} \frac{2(\psi')^2 \chi P}{(\zeta + 2)Q''} = - \lim_{\psi(z) \to 0} \frac{8\zeta(\zeta + 1)(\psi')^2 P}{2a \frac{\partial^2 Q}{\partial x^2} + \frac{\partial a}{\partial x} \frac{\partial Q}{\partial x}}
\]
\[
= - \lim_{\psi(z) \to 0} \frac{8\zeta(\zeta + 1)(\psi')^2 P}{\frac{\partial a}{\partial x} \frac{\partial Q}{\partial x}}.
\]
We are now reduced to verifying the polynomial identity
\[
8\zeta(\zeta + 1)P = \frac{\partial a}{\partial x} \frac{\partial Q}{\partial x}
\]
at the three points \(x = 2\zeta + 1, x = \zeta/(\zeta + 2)\) and \(x = \zeta(2\zeta + 1)/(\zeta + 2)\), corresponding to the three zeroes modulo \(\mathbb{Z} + \tau \mathbb{Z}\) of \(\psi\). \(\Box\)

**Lemma 3.9.** The modified potential (3.5) can be expressed as
\[
V(z) = \frac{\chi}{2\zeta(\zeta + 1)(\zeta + 2)} W(x),
\]
where \(W\) is as in (3.14).

**Proof.** Although it is straightforward to check this from (2.5) and (2.7), we will use a different method. By Liouville’s theorem, the first term in (3.5) can be written
\[
\frac{k_0(k_0 + 1)}{\phi(z)^2} = C \frac{(x - 1)(x - \zeta)^2}{x^2(x - (2\zeta + 1))},
\]
with $C$ independent of $z$. We rewrite this as

$$C \phi(z)^2 = \frac{k_0(k_0 + 1)x^2(x - (2\zeta + 1))}{(x - 1)(x - \zeta)^2},$$

and apply $\partial^2/\partial z^2$ at the point $z = 1/3$ to both sides. Using (3.19) and the fact that $\phi(1/3) = 0$ and $\phi'(1/3) = 1$, we obtain

$$2C = 2k_0(k_0 + 1)x'(1/3)^2 \frac{(x - (2\zeta + 1))}{(x - 1)(x - \zeta)^2} \bigg|_{x=0} = -k_0(k_0 + 1) \frac{\chi(2\zeta + 1)^3}{\zeta(\zeta + 1)(\zeta + 2)}.$$ This gives the first term in the expression for $W$. The other terms can be treated similarly, or be derived from the first term using [R3, Lemma 2.7].

We can now write (3.2) in algebraic form. We choose the function $\Psi$ in Theorem 3.1 as in (3.17). We express the left-hand side of (3.2) using (3.18), and then apply Lemmas 3.4, 3.6, 3.7, 3.8 and 3.9. The term involving the constant $C(\tau)$ from Lemma 3.8 is moved to the right-hand side. Finally, we multiply the resulting equation through with $-2\zeta(\zeta + 1)(\zeta + 2)/\chi$. We find that, up to a factor independent of the variables $x_j$,

$$\left(\sum_{j=1}^{m} \left( a(x_j) \frac{\partial^2}{\partial x_j^2} + b(x_j) \frac{\partial}{\partial x_j} + C(x_j) \right) + md \frac{\partial}{\partial \zeta}\right) \prod_{j=1}^{m} F(x_j) \Delta T_n^{(k)}$$

$$\sim \prod_{j=1}^{m} F(x_j) \Delta T_n^{(k)}, \quad (3.27)$$

where $a$ and $d$ are as in Theorem 3.3,

$$b = \frac{1}{2} \frac{\partial a}{\partial x} + 2m\zeta(\zeta + 1)B + 2a \frac{\partial E/\partial x}{E},$$

which agrees with (3.12) and $C = W + C_0$, with

$$C_0 = 2m(m - 1)\zeta(\zeta + 1)D + a \frac{\partial^2 E/\partial x^2}{E}$$

$$+ \left( \frac{1}{2} \frac{\partial a}{\partial x} + 2m\zeta(\zeta + 1)B \right) \frac{\partial E/\partial x}{E} + md \frac{\partial E/\partial \zeta}{E}.$$ One may check that, with $c_0$ as in (3.13), $C_0 - c_0$ is independent of $x$. Thus, (3.27) can be written

$$\Omega \prod_{j=1}^{m} F(x_j) \Delta(x_1, \ldots, x_m) T_n^{(k)}(x_1, \ldots, x_m) = 0, \quad (3.28)$$

where

$$\Omega = \sum_{j=1}^{m} \left( a(x_j) \frac{\partial^2}{\partial x_j^2} + b(x_j) \frac{\partial}{\partial x_j} + c(x_j) \right) + ma \frac{\partial}{\partial \zeta} + e, \quad (3.29)$$
for some yet unknown function \( e \) of \( \zeta \). To prove Theorem 3.3 it remains to show that \( e \) is given by \( (3.16) \).

3.3. **The constant term.** To compute the constant term \( e \) in \( (3.29) \), we will first prove \( (3.16) \) for \( k = (0,0,0,0) \), and then proceed by induction on \( \sum |k_j| \). For both the base case and the induction step, our method is based on investigating limits of \( (3.28) \) when all the variables coincide.

For the proof of the next lemma, we will need the elementary identity

\[
\sum_{j=1}^{m} x_j^k \frac{\partial^k}{\partial x_j^k} \Delta(x) = k! \binom{m}{k+1} \Delta(x). \tag{3.30}
\]

To see this, note that the left-hand side is an anti-symmetric homogeneous polynomial of the same degree as \( \Delta(x) \), and thus proportional to \( \Delta(x) \). The value of the constant follows since the coefficient of \( x_2x_3^2 \cdots x_m^{m-1} \) on the left-hand side is

\[
\sum_{j=1}^{m} (j-1)(j-2) \cdots (j-k) = k! \binom{m}{k+1}.
\]

**Lemma 3.10.** Suppose that \( P \) is a symmetric formal power series in \( m \) variables, whose Taylor expansion at \( 0 \) is given by

\[
\alpha + \beta \sum_{j=1}^{m} x_j + \gamma \sum_{j=1}^{m} x_j^2 + \delta \sum_{1 \leq j < k \leq m} x_j x_k + \text{higher order terms}
\]

and let \( f \) be a formal power series

\[
f(x) = a + bx + cx^2 + \text{higher order terms}.
\]

Then,

\[
\text{C.T.} \frac{1}{\Delta(x)} \sum_{j=1}^{m} f(x_j) \frac{\partial^2}{\partial x_j^2} \Delta(x) P(x) = 2 \binom{m}{3} c \alpha + 2 \binom{m}{2} b \beta + 2m^2 \alpha \gamma - 2 \binom{m}{2} a \delta, \tag{3.31}
\]

\[
\text{C.T.} \frac{1}{\Delta(x)} \sum_{j=1}^{m} f(x_j) \frac{\partial}{\partial x_j} \Delta(x) P(x) = \binom{m}{2} b \alpha + m a \beta, \tag{3.32}
\]

where C.T. stands for the constant term.

**Proof.** We split the left-hand side of \( (3.31) \) as

\[
\text{C.T.} \frac{1}{\Delta} \sum_{j=1}^{m} f(x_j) \left( \frac{\partial^2 \Delta}{\partial x_j^2} P(x) + 2 \frac{\partial \Delta}{\partial x_j} \frac{\partial P}{\partial x_j} + \Delta \frac{\partial^2 P}{\partial x_j^2} \right) = S_1 + S_2 + S_3.
\]

By homogeneity, only the quadratic term in \( f \) contributes to \( S_1 \). It then follows from \( (3.30) \) that \( S_1 = 2 \binom{m}{3} c \alpha \). In \( S_2 \), we get contributions from the linear and
constant terms in $f$. Again by (3.30), the linear term contributes $2^{(m)} b \beta$, whereas the constant term contributes

$$C. T. \frac{2a}{\Delta} \sum_{j=1}^{m} \frac{\partial \Delta}{\partial x_j} \frac{\partial}{\partial x_j} \left( \gamma \sum_{k=1}^{m} x_k^2 + \delta \sum_{1 \leq k < l \leq m} x_k x_l \right)$$

$$= C. T. \frac{2a}{\Delta} \sum_{j=1}^{m} \frac{\partial \Delta}{\partial x_j} \left( (2\gamma - \delta) x_j + \delta \sum_{k=1}^{m} x_k \right) = 2 \left( \frac{m}{2} \right) a (2\gamma - \delta),$$

where we used that $\sum_{j=1}^{m} \partial \Delta / \partial x_j = 0$ (as it is an anti-symmetric homogeneous polynomial of lower degree than $\Delta$). Finally, $S_3 = 2ma\gamma$, which completes the proof of (3.31). The proof of (3.32) is similar. \hfill \Box

We want to apply Lemma 3.10 to the case $k = (0, 0, 0)$ of (3.28). For this purpose, we need the lowest terms in the Taylor expansion of $T_n^{(0,0,0,0)}$.

**Lemma 3.11.** For $m = 2n \geq 1$, the Taylor expansion of $T_n^{(0,0,0,0)}(x_1, \ldots, x_m)$ around $x_1 = \cdots = x_m = 0$ has the form

$$\alpha_n + \beta_n \sum_{j=1}^{m} x_j + \gamma_n \sum_{j=1}^{m} x_j^2 + \delta_n \sum_{1 \leq j < k \leq m} x_j x_k + \text{higher order terms},$$

where

$$\alpha_n = \zeta^{n(n-1)}(2\zeta + 1)^{n(n-1)}$$

$$\beta_n = -(n-1)\zeta^{n(n-1)}(2\zeta + 1)^{n(n-1)} - 1,$$

$$\gamma_n = \frac{(n-1)(n-2)}{2} \zeta^{n(n-1)}(2\zeta + 1)^{(n+1)(n-2)},$$

$$\delta_n = (n-1)^2 \zeta^{n(n-1)}(2\zeta + 1)^{(n+1)(n-2)}.$$

**Proof.** Expand (2.13) along the first row and then let $x_1 = x_{n+1} = 0$. Then, only the first term gives a non-zero contribution. Rewriting the complementary minor in terms of $T_n^{(0,0,0,0)}$ gives after relabelling the parameters

$$T_n^{(0,0,0,0)}(x_1, \ldots, x_{2n-2}, 0, 0) = \zeta^{2n-2} \prod_{j=1}^{2n-2} (2\zeta + 1 - x_j) T_{n-1}^{(0,0,0,0)}(x_1, \ldots, x_{2n-2}).$$

This leads to the system of recursions

$$\alpha_n = \zeta^{2n-2}(2\zeta + 1)^{2n-2} \alpha_{n-1},$$

$$\beta_n = \zeta^{2n-2}(2\zeta + 1)^{2n-3} ((2\zeta + 1) \beta_{n-1} - \alpha_{n-1}),$$

$$\gamma_n = \zeta^{2n-2}(2\zeta + 1)^{2n-3} ((2\zeta + 1) \gamma_{n-1} - \beta_{n-1}),$$

$$\delta_n = \zeta^{2n-2}(2\zeta + 1)^{2n-4} ((2\zeta + 1)^2 \delta_{n-1} - 2(2\zeta + 1) \beta_{n-1} + \alpha_{n-1}),$$

which is easily solved from the initial value $\alpha_1 = 1, \beta_1 = \gamma_1 = \delta_1 = 0$. \hfill \Box
We can now prove \((3.28)\) for \(k = (0, 0, 0, 0)\). This result has been obtained by Zinn-Justin \([Z, \S 4.2.2]\), see \([R3, \S 5.3]\) for the precise relation to the notation used there.

**Lemma 3.12 (Zinn-Justin).** Theorem 3.3 holds for \(k = (0, 0, 0, 0)\).

**Proof.** Applying Lemma 3.10 to the case \(k = (0, 0, 0, 0)\) of \((3.28)\) gives
\[
\left. \left( \left( \frac{m}{3} \right) \frac{\partial^2 a}{\partial x^2} + \left( \frac{m}{2} \right) \frac{\partial b}{\partial x} + m c_0 + e \right) \right|_{x=0} \alpha_n + \left. \left( \frac{2}{2} \frac{\partial a}{\partial x} + m b \right) \right|_{x=0} \beta_n \\
+ a(0) \left( 2 m^2 \gamma_n - 2 \right) \delta_n \right) + m d \frac{\partial \alpha_n}{\partial \zeta} = 0,
\]
where \(m = 2n\). Inserting the expressions given in Theorem 3.3 and Lemma 3.11 yields \(e = 0\), in agreement with \((3.16)\).

It seems difficult to extend this proof to general \(k\). To proceed, we write \((3.28)\) as in \((3.15)\) (with \(e\) still unknown), and then let all variables \(x_j\) tend to \(\xi_l\). Although, in general, \(b_F, c_F\) and \(T_n^{(k)}\) have poles, they are regular at the point \(\xi_l\). Thus, we may apply Lemma 3.10, with \(x_j\) replaced by \(x_j - \xi_l\). Since \(a(\xi_l) = 0\), the result simplifies to
\[
\left. \left( \left( \frac{m+1}{3} \right) \frac{\partial^2 a}{\partial x^2} + \left( \frac{m+1}{2} \right) \frac{\partial b_F}{\partial x} + (m+1) c_F + e \right) \right|_{x=\xi_l} \alpha + \left. \left( \frac{2}{2} \frac{\partial a}{\partial x} + m b_F \right) \right|_{x=\xi_l} \beta \\
+ (m+1) d \frac{\partial \alpha_n}{\partial \zeta} = 0, \tag{3.33a}
\]
where
\[
\alpha = T_n^{(k)}(\xi_l^{(m)}), \quad \beta = \frac{\partial T_n^{(k)}}{\partial x_1}(\xi_l^{(m)}), \quad \varepsilon = \frac{\partial T_n^{(k)}}{\partial \zeta}(\xi_l^{(m)}).
\]

Consider now \((3.33a)\), with \(k\) replaced by \(k - e_l\) (with \(e_l\) a unit vector) and \(m\) by \(m+1\). By \(2.18\),
\[
T_n^{(k)}(x_1, \ldots, x_m) = T_n^{(k-e_l)}(x_1, \ldots, x_m, \xi_l).
\]
It follows that, as the indices change, \(\alpha\) and \(\beta\) remain the same whereas \(\varepsilon\) is replaced by \(\varepsilon - \beta \partial \xi_l / \partial \zeta\). Thus,
\[
\left. \left( \left( \frac{m+1}{3} \right) \frac{\partial^2 a}{\partial x^2} + \left( \frac{m+1}{2} \right) \frac{\partial b_F}{\partial x} + (m+1) c_F + \tilde{e} \right) \right|_{x=\xi_l} \alpha \\
+ \left. \left( \frac{2}{2} \frac{\partial a}{\partial x} + (m+1) \tilde{b}_F - (m+1) d \frac{\partial \xi_l}{\partial \zeta} \right) \right|_{x=\xi_l} \beta + (m+1) d \varepsilon = 0, \tag{3.33b}
\]
where \(\sim\) signifies the change \((k, m) \mapsto (k - e_l, m+1)\) in the coefficients depending on these indices.
Eliminating $\varepsilon$ from the equations (3.33), the resulting coefficient of $\beta$ is

$$
(m + 1) \left( 2 \frac{m}{2} \frac{\partial a}{\partial x} + mb_F \right) - m \left( 2 \frac{m + 1}{2} \frac{\partial a}{\partial x} + (m + 1)b_F - (m + 1)d \frac{\partial \xi}{\partial \zeta} \right) \bigg|_{x=\xi_l} = 0,
$$

by a direct computation. Since we know from [R3, Cor. 3.9] that $\alpha = T_n^{(k+\mu)}$ does not vanish identically, it follows that

$$
(m + 1) \left( \frac{m}{3} \frac{\partial^2 a}{\partial x^2} + \frac{m}{2} \frac{\partial b_F}{\partial x} + mc_F + e \right) - m \left( \frac{m + 1}{3} \frac{\partial^2 a}{\partial x^2} + \frac{m + 1}{2} \frac{\partial b_F}{\partial x} + (m + 1)\bar{c}_F + \bar{e} \right) \bigg|_{x=\xi_l} = 0. \tag{3.34}
$$

We view this as a recursion for obtaining the unknown coefficient $e$ from $\bar{e}$. By another direct computation, it is consistent with the explicit expression (3.16). This proves the following induction step.

**Lemma 3.13.** If Theorem 3.3 holds for fixed $k$ and $m \geq 1$, then it also holds when $k$ is replaced by $k + e$ and $m$ by $m - 1$.

We will need another recursion, which follows from the following differential equation for the polynomials $U_n^{(k)}$ defined in (2.19).

**Proposition 3.14.** The polynomials $U_n^{(k)}$ satisfy the differential equation

$$
\Omega \prod_{j=1}^{m} K(x_j) \Delta(x_1, \ldots, x_m) U_n^{(k)}(x_1, \ldots, x_m) = 0, \tag{3.35}
$$

where $\Omega$ is as in (3.29), and $K = 1/F \sqrt{a}$.

The main point of Proposition 3.14 is that (3.35) holds with the same $e$ as in (3.28). It is easier to see that it holds for some $e$ or, equivalently, for some function $K \sim 1/F \sqrt{a}$ up to a $\zeta$-dependent factor.

**Proof.** The operator $\Omega$ has been constructed so that

$$
H^{(k)} M_n \Omega (H^{(k)} M_n)^{-1} = \Phi^{(k)} H \Phi^{(-k)} + C,
$$

where

$$
H^{(k)} = \prod_{j=1}^{m} \prod_{l=0}^{3} (x_j - \xi_l) G(x_j, \xi_l)^{\frac{k}{2}},
$$

$M_n$ is given in (2.12), $H$ in (3.3) and $C$ is independent of the variables $z_j$. We conjugate this identity with $\Phi^{(2k^-)}$. Using that

$$
\Phi^{(2k^-)} H^{(k)} M_n \sim H^{(k^+ + k^-)} M_{n+k^-}
$$
up to a factor independent of the variables $z_j$, we find that (with a change of $C$)
\[ H^{(k^+ + k^-)} M_{n+|k^-|} \Omega (H^{(k^+ + k^-)} M_{n+|k^-|})^{-1} = \Phi^{(k^+ + k^-)} H \Phi^{(-k^+ - k^-)} + C. \]  
(3.36)

It follows from (3.7) that $\sigma \otimes m$ commutes with the right-hand side of (3.36). If $L$ denote the left-hand side of (3.36), we apply (2.15), with $n$ replaced by $n + |k^+|$, to the left-hand side of $L \sigma \otimes m = \sigma \otimes m L$. This gives
\[ H^{(k^+ + k^-)} M_{n+|k^-|} \Omega H^{(-k^+ - k^-)}(\sqrt{a^{-1}}) \otimes \hat{\sigma} \otimes m \]
\[ n+|k^-| = \sigma \otimes m H^{(k^+ + k^-)} M_{n+|k^-|} \Omega H^{(-k^+ - k^-)}, \]  
(3.37)

which holds on the domain of $\hat{\sigma} \otimes m$. If we act with (3.37) on $H^{(k^+ + k^-)} F^{\otimes m} \Delta T^{(k)}_n$, the right-hand side vanishes by (3.28). We can then deduce (3.35) from (2.21). □

We can now repeat the analysis leading to (3.34), using (2.20) and (3.35) rather than (2.18) and (3.28). We find that
\[ (m+1) \left( \begin{array}{c} m+1 \frac{\partial^2 a}{\partial x^2} + \frac{m+1}{2} \frac{\partial b_K}{\partial x} + mc_K + e \\ \frac{3}{2} \frac{\partial^2 a}{\partial x^2} + \frac{m+1}{2} \frac{\partial b_K}{\partial x} + (m+1)\tilde{c}_K + \tilde{e} \end{array} \right) \bigg|_{x=\xi_i} = 0. \]

where $\sim$ now denotes the change of indices $(k, m) \mapsto (k+e_l, m+1)$. Again, this is consistent with (3.16), which proves the following lemma.

**Lemma 3.15.** If Theorem 3.3 holds for fixed $k$ and $m \geq 1$, it also holds when $k$ is replaced by $k - e_l$ and $m$ by $m - 1$.

It is clear that Lemmas 3.12, 3.13 and 3.15 together imply Theorem 3.3 by induction on $\sum_j |k_j|$.

4. Bilinear identities

The case $m = 2n - |k| = 0$, when $T^{(k)}_n$ depends only on $\zeta$, is of particular interest. We will write $t^{(k)} = T^{(k)}_{|k|/2}$. In the subsequent paper [R4], we will show that $t^{(k)}$ can be identified with tau functions of Painlevé VI. We will now explain how bilinear identities for tau functions arise from our construction. Rather than giving a complete list, we will just give two examples of such relations, which will in fact be used in [R4] to obtain the identification with tau functions. Once this idenfication has been established, one can obtain further bilinear identities from Painlevé theory; some examples are discussed in [R4].

The point of the following result is to characterize $t^{(k)}$ by a very short list of properties. Recall from [R3, Cor. 2.19 and Cor. 2.21] that the lattice of functions $t^{(k)}$ is symmetric under the group $G = S_4 \times S_2$ in the following sense. If $S_4$ acts by
permuting \((k_0, k_1, k_2, k_3)\) and \(S_2\) by the reflection \((k_0, k_1, k_2, k_3) \mapsto (-k_0 - 1, -k_1 - 1, -k_2 - 1, -k_3 - 1)\), then for any \(\sigma \in G\) there holds an identity

\[
t^{(k)}(\zeta) = \phi(\zeta)t^{(\sigma k)}(\psi(\zeta)),
\]

with \(\phi = \phi_{k,\sigma}\) and \(\psi = \psi_{\sigma}\) rational functions that can be given explicitly.

**Theorem 4.1.** The functions \(t^{(k)}\) satisfy the two identities

\[
t^{(k-2e_0)}t^{(k+e_0+e_1)} = \zeta^2(\zeta + 1)(\zeta - 1)(2\zeta + 1)^2
\times \left( \frac{1}{2k_0 - 1} t^{(k)} \frac{dt^{(k-e_0+e_1)}}{d\zeta} - \frac{1}{2k_0 + 1} \frac{dt^{(k-e_0)}}{d\zeta} \right) + \frac{\zeta(2\zeta + 1)}{2(2k_0 - 1)(2k_0 + 1)(\zeta + 2)} A^{(k)}t^{(k-e_0+e_1)},
\]

\[
t^{(k-2e_0)}t^{(k+e_0-e_1)} = \frac{(\zeta + 1)(\zeta - 1)(2\zeta + 1)^2(\zeta + 2)^2}{\zeta^2}
\times \left( \frac{1}{2k_0 - 1} t^{(k)} \frac{dt^{(k-e_0-e_1)}}{d\zeta} - \frac{1}{2k_0 + 1} \frac{dt^{(k-e_0)}}{d\zeta} \right) + \frac{(2\zeta + 1)(\zeta + 2)}{2(2k_0 - 1)(2k_0 + 1)\zeta^3} B^{(k)}t^{(k-e_0-e_1)},
\]

where

\[
A^{(k)} = (2\zeta^4 - 23\zeta^3 - 36\zeta^2 - 5\zeta + 8)k_0^2
- \zeta(2\zeta + 1)(3\zeta^2 + 10\zeta + 5)k_1(2k_0 + k_1)
- \zeta(6\zeta^3 + 19\zeta^2 + 4\zeta - 11)k_2^2
- \zeta(2\zeta + 1)(3\zeta^2 + 2\zeta + 1)k_3^2
- 2\zeta(\zeta - 1)(2\zeta + 1)(\zeta + 3)(k_0 + k_1)k_2
- 2(\zeta - 1)(2\zeta + 1)(3\zeta^2 + 9\zeta + 4)(k_0 + k_1)k_3
- 2(2\zeta + 1)(\zeta^3 + 6\zeta^2 + 3\zeta - 4)k_2k_3
- 4(2\zeta + 1)(\zeta^2 + 5\zeta + 3)(k_0 + k_1)
+ 4(2\zeta + 1)(2\zeta^3 + 5\zeta^2 - \zeta - 3)k_2
+ 4(2\zeta + 1)(\zeta^2 + \zeta + 1)k_3
- 4(\zeta + 1)^2(2\zeta^2 - \zeta + 2),
\]

\[
B^{(k)} = (10\zeta^4 + 13\zeta^3 - 28\zeta^2 - 41\zeta - 8)k_0^2
- \zeta(2\zeta + 1)(3\zeta^2 + 10\zeta + 5)k_1(k_1 - 2k_0)
- \zeta(6\zeta^3 + 19\zeta^2 + 4\zeta - 11)k_2^2
- \zeta(2\zeta + 1)(3\zeta^2 + 2\zeta + 1)k_3^2
\]
Moreover, the lattice of functions $t^{(k)}$, where $k = (k_0, k_1, k_2, k_3) \in \mathbb{Z}^4$ with $\sum_j k_j$ even, is uniquely determined by (4.11), (4.12) and the three values

$$
\begin{align*}
t^{(0,0,0,0)} &= t^{(1,1,0,0)} = 1, \\
t^{(0,-1,-1,0)} &= -\frac{2\zeta^2(\zeta - 1)(\zeta + 1)^2(2\zeta + 1)}{(\zeta + 2)^2}.
\end{align*}
$$

Proof. We start from the Jacobi–Desnanot identity in the form [R3, Eq. (2.41a)]

$$(a - b)(c - d)T(x; y)T(a, b, c, d; x; y) = G(a, d)G(b, c)T(a, c, x; y)T(b, d; x; y)
- G(a, c)G(b, d)T(a, d; x; y)T(b, c; x; y).$$

When $b = c = \xi_0$, $d = \xi_1$, $x = \xi^{(k^+)}$ and $y = \xi^{(k^-)}$, this can be written

$$(a - \xi_0)(\xi_0 - \xi_1)t^{(k)}T_{n+2}^{(k+2\xi_0+\xi_1)}(a)
= G(a, \xi_1)G(\xi_0, \xi_0)T_{n+1}^{(k+\xi_0)}(a)t^{(k+\xi_0+\xi_1)} - G(a, \xi_0)G(\xi_0, \xi_1)t^{(k+2\xi_0)}T_{n+1}^{(k+\xi_1)}(a),$$

where $|k| = 2n$. Differentiating with respect to $a$ and letting $a = \xi_0$ gives

$$(\xi_0 - \xi_1)t^{(k)}t^{(k+3\xi_0+\xi_1)}
= \left( G(\xi_0, \xi_0)\frac{\partial G}{\partial x}(\xi_0, \xi_1) - G(\xi_0, \xi_0)G(\xi_0, \xi_1) \right) t^{(k+2\xi_0)}t^{(k+\xi_0+\xi_1)}
+ G(\xi_0, \xi_0)G(\xi_0, \xi_1) \left( \frac{\partial T^{(k+\xi_0)}_{n+1}}{\partial x}(\xi_0)t^{(k+\xi_0+\xi_1)} - t^{(k+2\xi_0)}\frac{\partial T^{(k+\xi_1)}_{n+1}}{\partial x}(\xi_0) \right).$$

(4.3)

The main point is now that the specialized derivatives of $T$-functions in (4.13) can be expressed in terms of $t$-functions using the Schrödinger equation. The relevant identity is a special case of (3.33b), but for clarity we repeat the argument. If we let $x_1 \to \xi_0$ in the case $m = 1$ of (3.15), we get

$$
b_F(\xi_0)\frac{\partial T^{(k)}_n}{\partial x}(\xi_0) + (c_F(\xi_0) + c)T^{(k)}_n(\xi_0) + d\frac{\partial T^{(k)}_n}{\partial \xi}(\xi_0) = 0.$$
On the other hand, differentiating the equality \( T_n^{(k)}(\xi_0) = t^{(k+e_0)} \) gives
\[
2 \frac{\partial T_n^{(k)}}{\partial x}(\xi_0) + \frac{\partial T_n^{(k)}}{\partial \zeta}(\xi_0) = \frac{dt^{(k+e_0)}}{d\zeta}.
\]

Eliminating \( \partial T_n^{(k)}/\partial \zeta \) from these two equations, we find that
\[
\frac{\partial T_n^{(k)}}{\partial x}(\xi_0) = \frac{1}{2d - b_F(\xi_0)} \left( (c_F(\xi_0) + e)t^{(k+e_0)} + d \frac{dt^{(k+e_0)}}{d\zeta} \right).
\]
(4.4)

Using (4.4) on the right-hand side of (4.3) gives, after replacing \( k_0 \) by \( k_0 - 2 \), (4.2a).

The identity (4.2b) is proved similarly, starting instead from [R3, Eq. (2.41b)]
\[
(a - b)T(x; y)T(a, b, c, x; d, y) = (a - d)G(a, d)G(b, c)T(a, c, x; y)T(b, x; d, y)
- (b - d)G(a, c)G(b, d)T(a, x; d, y)T(b, c, x; y).
\]

To show that \( t^{(k)} \) can be constructed from the given data, we apply induction on \( N(k) = \sum_{j=0}^3 |k_j + 1/2| \). Thus, fixing \( k \), suppose that \( t^{(l)} \) is known for all \( l \) with \( N(l) < N(k) \). By the symmetries (4.14) and the fact that \( N(k) \) is invariant under the group action, we may replace \( k \) by any element in the same orbit. We choose this element so that \( k_0 + 1/2 \geq \max_{1 \leq j \leq 3} |k_j + 1/2| \). In particular, \( k_0 \geq 0 \). If \( k_1 \geq 0 \), we use (4.2a), with \( k \) replaced by \( k - e_0 - e_1 \), to define \( t^{(k)} \). If \( k_1 \leq -1 \), we use instead (4.2b), with \( k \) replaced by \( k - e_0 + e_1 \). This does not lead to division by zero, since \( t^{(k)} \) never vanishes identically [R3, Cor. 3.9].

For the construction just described to work, the remaining functions \( t^{(l)} \) appearing in (4.2) must satisfy \( N(l) < N(k) \). Thus, in the case \( k_1 \geq 0 \) we must have
\[
\begin{align*}
\left| k_0 - \frac{5}{2} \right| + \left| k_1 - \frac{1}{2} \right| < k_0 + k_1 + 1, \\
\left| k_0 - \frac{3}{2} \right| < k_0 + \frac{1}{2}, \\
\left| k_0 - \frac{1}{2} \right| + \left| k_1 - \frac{1}{2} \right| < k_0 + k_1 + 1,
\end{align*}
\]

It is an elementary exercise to check that this is true except in the cases \((k_0, k_1) = (0, 0)\) and \((k_0, k_1) = (1, 0)\). Similarly, when \( k_1 \leq 0 \) the construction works except if \((k_0, k_1) = (0, -1)\) or \((k_0, k_1) = (1, -1)\). In conclusion, the induction step works unless \( k_0 \in \{0, 1\}, k_1 \in \{0, -1\} \). Repeating the same construction with \( k_1 \) replaced by \( k_2 \) or \( k_3 \) we are left with the exceptional cases \( k_0 \in \{0, 1\}, k_1, k_2, k_3 \in \{0, -1\} \).

Since \(|k|\) is even there are eight such cases, which split into four \( G \)-orbits represented by \( k = (0, 0, 0, 0), (0, -1, -1, 0), (1, -1, 0, 0), (1, -1, -1, -1) \). We have chosen the first three as initial values. The case \( k = (0, 0, 0, 0) \) of (4.2b) expresses a point in the fourth orbit in terms of the other three. \( \square \)
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