THE MITTAG-LEFFLER CONDITION DESCENTS VIA PURE MONOMORPHISMS

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Abstract. This note aims to clarify the proof given by Raynaud and Grus on [6] that the Mittag-Leffler property descents via pure rings monomorphism of commutative rings. A consequence of that is that projectivity descents via such ring homomorphisms (cf. [5]), a revision of the proof also allows to prove that the property of being pure-projective also descents via pure monomorphisms between commutative rings [4, Section 8].

In their fundamental paper [6], Raynaud and Gruson introduced the class of Mittag-Leffler modules. They proved how useful such notion was, showing an important number of striking results. One of them was the descent of projectivity via pure ring monomorphisms [6, Théorème II.3.1.3] (or universally injective maps, as they are named in [5]) of commutative rings.

It seems there has been some misunderstanding in the literature because, as noted by Gruson in the paper [3], statement [6, Proposition II.2.5.2] is wrong. The descent of projectivity via pure monomorphisms is stated in [5, Examples II.3.1.4] that are presented as a consequence of the wrong statement, and no correction for that is given in [3]. However, to conclude the descent of projectivity via pure monomorphisms only [6, Proposition II.2.5.1, Théorème II.3.1.3] are needed and these results are perfectly correct in the original paper.

The descent of projectivity means that if \( R \rightarrow T \) is a pure ring monomorphism of commutative rings, and \( M \) is a flat \( R \)-module then \( M_R \) is projective if and only if \( M \otimes_R T_T \) is a projective \( T \)-module. This result was reproved in [5] in the case the ring homomorphism \( R \rightarrow T \) is faithfully flat. In [1], if was reproved for the case of pure-monomorphisms. In both papers, it was also observed that results of Brewer and Rutter [2, Theorem 2] allow to state the result in the following way:

Let \( R \rightarrow T \) be a pure ring monomorphism of commutative rings, and let \( M \) be an \( R \)-module. Then \( M_R \) is projective if and only if \( M \otimes_R T_T \) is a projective \( T \)-module.

In a recent preprint, Herbera, Prihoda and Wiegand have observed that suitable modifications of the original arguments due to Raynaud and Gruson allow also to show that pure projectivity descents via pure monomorphisms of commutative rings [4]. The proof by Raynaud and Gruson is based on [5, Proposition II.2.5.1] which shows that the Mittag-Leffler property descents via pure monomorphism of commutative rings. This result is reproduced in Proposition 2.1.

To make clear that the result is [4] is correct, we have written this short note. In Section 1, we introduce the characterization of Mittag-Leffler modules that allows to prove [5, Proposition II.2.5.1]. The proof of the latter result is then included in Proposition 2.3. Finally, in the third section we include the detailed proof of the descent of pure projectivity and, as a consequence, the descent of projectivity.

We stress the fact that we are just reproducing arguments that are already in [6].
1. Characterizations of Mittag-Leffler modules

Definition 1.1. [6] Let $M$ be a right module over a ring $R$. Then $M$ is a Mittag-Leffler module if the canonical map

$$\rho: M \bigotimes_R \prod_{i \in I} Q_i \to \prod_{i \in I} (M \bigotimes_R Q_i)$$

is injective for any family $\{Q_i\}_{i \in I}$ of left $R$-modules.

The following characterization of Mittag-Leffler modules is also due to Raynaud and Gruson. It is also reproved in [5].

Proposition 1.2. The following are equivalent conditions for a right $R$-module $M$.

(i) $M$ is a Mittag-Leffler module.
(ii) Let $M = \varinjlim F_\alpha$ where $(F_\alpha, u_{\beta\alpha}: F_\alpha \to F_\beta)_{\alpha \leq \beta \in \Lambda}$ is a directed system of finitely presented modules. Then for any $\alpha \in \Lambda$ there exists $\beta \geq \alpha$ such that, for any left $R$-module $Q$, $\ker (u_{\beta\alpha} \otimes_R Q) = \ker (u_{\gamma\alpha} \otimes_R Q)$ for any $\gamma \geq \beta \in \Lambda$.
(iii) There exists a directed system of finitely presented modules $(F_\alpha, u_{\beta\alpha}: F_\alpha \to F_\beta)_{\alpha \leq \beta \in \Lambda}$ such that $M = \varinjlim F_\alpha$, and satisfying that for any $\alpha \in \Lambda$ there exists $\beta \geq \alpha$ such that, for any left $R$-module $Q$, $\ker (u_{\beta\alpha} \otimes_R Q) = \ker (u_{\gamma\alpha} \otimes_R Q)$ for any $\gamma \geq \beta \in \Lambda$.

The key to prove Proposition 1.2 is the following Lemma.

Lemma 1.3. Let $u: M \to N$ and $v: M \to M'$ be homomorphisms of right $R$-modules. Consider the push-out diagram

\[
\begin{array}{ccc}
M & \xrightarrow{u} & N \\
\downarrow v & & \downarrow w \\
M' & \xrightarrow{u'} & B
\end{array}
\]

Then, there are exact sequences

$$0 \to \ker u \cap \ker v \xrightarrow{u} \ker u \to \ker u' \to 0$$

$$0 \to \ker u \cap \ker v \xrightarrow{v} \ker u' \to \ker w \to 0$$

In particular, $\ker u = \ker v$ if and only if $u'$ and $w$ are monomorphisms.

Proof. The proof is easily done using the push-out property combined with element-chasing.

Using Lemma 1.3, the characterization of Proposition 1.2 can be rewritten in the following fancy way:

Proposition 1.4. The following are equivalent conditions for a right $R$-module $M$.

(i) $M$ is a Mittag-Leffler module.
(ii) Let $M = \varinjlim F_\alpha$ where $(F_\alpha, u_{\beta\alpha}: F_\alpha \to F_\beta)_{\alpha \leq \beta \in \Lambda}$ is a directed system of finitely presented modules. Then for any $\alpha \in \Lambda$ there exists $\beta \geq \alpha$ such that, for any $\gamma \geq \beta \in \Lambda$, the homomorphisms $w_{\gamma\alpha}$ and $u_{\beta\alpha}'$ in the push-out diagram

\[
\begin{array}{ccc}
F_\alpha & \xrightarrow{u_{\beta\alpha}} & F_\beta \\
\downarrow u_{\gamma\alpha} & & \downarrow w_{\gamma\alpha} \\
F_\gamma & \xrightarrow{u_{\beta\alpha}'} & N_{\gamma\beta}
\end{array}
\]
are pure monomorphisms.

(iii) There exists a directed system of finitely presented modules \((F_\alpha, u_{\beta\alpha}: F_\alpha \to F_\beta)_{\alpha \leq \beta \in \Lambda}\) such that \(M = \varinjlim F_\alpha\), and satisfying that for any \(\alpha \in \Lambda\) there exists \(\beta \geq \alpha\) such that, for any \(\gamma \geq \beta \in \Lambda\) the homomorphisms \(w_{\gamma\alpha}\) and \(u_{\beta\alpha}'\) in the push-out diagram

\[
\begin{array}{ccc}
F_\alpha & \xrightarrow{u_{\beta\alpha}} & F_\beta \\
\downarrow{w_{\gamma\alpha}} & & \downarrow{w_{\gamma\alpha}} \\
F_\gamma & \xrightarrow{u_{\beta\alpha}'} & N_{\gamma\beta}
\end{array}
\]

are pure monomorphisms.

2. Descent of the Mittag-Leffler condition via pure monomorphisms

**Proposition 2.1.** Let \(\varphi: R \to T\) be a pure ring monomorphism of commutative rings. Let \(M, N\) be \(R\)-modules and let \(u: M \to N\) an \(R\)-module homomorphism. Then \(u\) is a pure monomorphism of \(R\)-modules if and only if \(u \otimes_R T: M \otimes_R T \to N \otimes_R T\) is a pure monomorphism of \(T\)-modules.

**Proof.** It is clear that if \(u\) is a pure monomorphism then so is \(u \otimes_R T\).

To prove the converse, assume that \(u \otimes_R T\) is a pure monomorphism of \(T\)-modules. We claim it is also a pure monomorphism of \(R\)-modules. Indeed, for any \(R\)-module \(Q\) there is a commutative diagram

\[
\begin{array}{ccc}
M \otimes_R T \otimes_R Q & \xrightarrow{u \otimes_R \varphi \otimes_R Q} & N \otimes_R T \otimes_R Q \\
\downarrow{\cong} & & \downarrow{\cong} \\
M \otimes_R T \otimes_T (T \otimes_R Q) & \xrightarrow{(u \otimes_R T) \otimes_T (T \otimes_R Q)} & N \otimes_R T \otimes_T (T \otimes_R Q)
\end{array}
\]

in which the lower row is a monomorphism, then so is the upper row.

Note also that for any \(R\)-module \(X\), the embedding \(X \otimes_R \varphi: X \to X \otimes_R T\) is a pure monomorphism of \(R\)-modules.

Finally, the commutativity of the diagram

\[
\begin{array}{ccc}
M & \xrightarrow{M \otimes_R \varphi} & M \otimes_R T \\
\downarrow{u} & & \downarrow{u \otimes_R T} \\
N & \xrightarrow{N \otimes_R \varphi} & N \otimes_R T
\end{array}
\]

in which \((u \otimes_R T) \circ (M \otimes_R \varphi)\) is a pure monomorphism of \(R\)-modules, implies that \(u\) is also a pure monomorphism of \(R\)-modules, as we wanted to prove. ■

**Corollary 2.2.** Let \(\varphi: R \to T\) be a pure ring monomorphism of commutative rings, and let \(M_R\) be an \(R\) module. Then \(M_R\) is flat if and only if the \(T\)-module \(M \otimes_R T\) is flat.

**Proof.** Since a module if flat if and only if it fits in a short exact sequence

\[0 \to K \xrightarrow{f} F \to M \to 0\]

such that \(F\) is free and \(f\) is a pure monomorphisms, the statement is a direct consequence of Proposition [2.1]. ■

Now we are ready to prove the crucial [6, Proposition II.2.5.1].
Proposition 2.3. ([6 Proposition II.2.5.1]) Let $\varphi: R \to T$ be a pure ring monomorphism of commutative rings, and let $M$ be an $R$-module. Then $M$ is a Mittag-Leffler $R$-module if and only if $M \otimes R T$ is a Mittag-Leffler $T$-module.

Proof. Assume that $M_R$ is a Mittag-Leffler $R$-module. For any family of $T$-modules $\{Q_i\}_{i \in I}$ the canonical map $T \otimes_T \prod_{i \in I} Q_i \to \prod_{i \in I} T \otimes_T Q_i$ is an isomorphism. Hence, the composition of maps
\[
(M \otimes R T) \otimes_T \prod_{i \in I} Q_i \to M \otimes_R \left( \prod_{i \in I} T \otimes_T Q_i \right) \to \prod_{i \in I} M \otimes_R T \otimes_T Q_i
\]
is injective and, therefore, $M \otimes R T$ is Mittag-Leffler as $T$-module.

Now assume that $M \otimes R T$ is a Mittag-Leffler $T$-module. Let $(F_\alpha, u_{\beta \alpha}: F_\alpha \to F_\beta)_{\alpha \leq \beta \in \Lambda}$ be a directed system of finitely presented $R$-modules such that $M = \lim_{\to} F_\alpha$. Then $(F_\alpha \otimes_R T, u_{\beta \alpha} \otimes_R T: F_\alpha \otimes_R T \to F_\beta \otimes_R T)_{\alpha \leq \beta \in \Lambda}$ be a directed system of finitely presented $T$-modules such that $M \otimes_R T = \lim_{\to} F_\alpha \otimes_R T$.

By Proposition 1.2, for any $\alpha \in \Lambda$ there exists $\beta \geq \alpha$ such that, for any left $T$-module $Q$, $\ker (u_{\beta \alpha} \otimes_R T \otimes_T Q) = \ker (u_{\gamma \alpha} \otimes_R T \otimes_T Q)$ for any $\gamma \geq \beta \in \Lambda$. In view of Proposition 1.4, and since tensor products preserves push-out diagrams this is equivalent to say that for any $\alpha$ there exists $\beta \geq \alpha$ such that, for any $\gamma$, in the push out diagram
\[
\begin{array}{ccc}
F_\alpha & \xrightarrow{u_{\beta \alpha}} & F_\beta \\
\downarrow u_{\gamma \alpha} & & \downarrow w_{\gamma \alpha} \\
F_\gamma & \xrightarrow{u'_{\beta \alpha}} & N_{\gamma \beta}
\end{array}
\]
$w_{\gamma \alpha} \otimes R T$ and $u'_{\beta \alpha} \otimes R T$ are pure monomorphisms of $T$-modules. By Proposition 2.1 we deduce that $w_{\gamma \alpha}$ and $u'_{\beta \alpha}$ are pure monomorphisms of $R$-modules. By Proposition 1.5 we deduce that $M$ is a Mittag-Leffler $R$-module.

3. Descent of pure projectivity

A module $M$ is said to be pure projective if the functor $\text{Hom}_R(M, -)$ is exact with pure short exact sequences. Equivalently, $M$ is pure projective if it is a direct summand of a direct sum of finitely presented modules.

Pure-projective modules always decompose into a direct sum of countably presented pure-projective submodules.

Now we include the proof that pure projectivity descents via pure monomorphisms. We reproduce this result from [1, §8].

First we recall the results that relate pure-projective modules and Mittag-Leffler modules.

Remark 3.1. The map $\rho$ in the definition of Mittag-Leffler module is obviously bijective if $M$ is a finitely generated free module. An easy diagram chase shows that it is also bijective if $M$ is finitely presented. Thus finitely presented modules are Mittag-Leffler. Since the class of Mittag-Leffler modules is closed under direct summands and arbitrary direct sums, all pure-projective modules are Mittag-Leffler modules.

Lemma 3.2. ([6 Théorème 2.2.1 p. 73]) Any countably generated submodule $X$ of a Mittag-Leffler module $Y$ is contained in a pure-projective countably generated pure submodule $Y'$ of $Y$.

The following Proposition is a variation of [6 Théorème 3.1.3 p. 78] adapted to the pure-projective situation.
Proposition 3.3. [4] Let $R \subseteq T$ be a pure extension of commutative rings. Let $M$ be an $R$-module. Then $M_R$ is pure projective if and only if $M \otimes_R T$ is pure projective as a $T$-module.

Proof. If $M_R$ is pure projective then, clearly, $M \otimes_R T$ is pure projective as a $T$-module. For the converse, assume that $M \otimes_R T$ is a pure projective $T$-module. By Remark 3.1 $M \otimes_R T$ is a Mittag-Leffler $T$-module, and then Proposition 2.3 implies that $M$ is a Mittag-Leffler $R$-module. We need to prove that, in addition, $M_R$ is pure-projective.

Write $M \otimes_R T = \oplus_{i \in I} Q_i$, where each $Q_i$ is a countably generated $T$-module. Let $F = \{Q_i\}_{i \in I}$. A submodule $X$ of $M$ is said to be adapted (to $F$) if it is pure and the canonical image of $X \otimes_R T$ in $M \otimes_R T$ is a direct sum of modules in $F$. If $X$ is an adapted submodule of $M$, then the sequence

$$0 \to X \otimes_R T \to M \otimes_R T \to (M/X) \otimes_R T \to 0$$

is split exact. Therefore $X \otimes_R T$ and $(M/X) \otimes_R T$, being isomorphic to direct summands of $M \otimes_R T$, are pure-projective as $T$-modules.

Step 1. Every countably generated pure submodule of $M$ is contained in a countably generated adapted submodule of $M$.

Let $X$ be a countably generated pure submodule of $M$. As $M$ is Mittag-Leffler and the modules $Q_i$ are countably generated, we can construct a sequence $\{X_n, I_n\}_{n \in \mathbb{N}}$ such that

1. $X_0 = X$ and, for every $n \geq 0$, $X_n$ is a countably generated pure submodule of $M$ and $X_n \subseteq X_{n+1}$;
2. for any $n \geq 0$, $I_n$ is a countable subset of $I$ and it consists of the elements $i \in I$ such that the canonical image of $X_n \otimes_R T$ in $Q_i$ is different from zero;
3. for any $n \geq 0$, the image of $X_{n+1} \otimes_R T$ contains $\oplus_{i \in I_n} Q_i$.

To be more specific, suppose $X_n, I_n$ were defined. Since each $Q_i$ is countably generated, there exists a countable set $G_n \subseteq M$ such that the canonical image of $G_n R \otimes_R T$ in $M \otimes_R T$ contains $\oplus_{i \in I_n} Q_i$. By Lemma 3.2 there exists a countably generated $X_{n+1}$ which is a pure submodule of $M$ containing $X_n + G_n R$. Then $I_{n+1}$ is chosen as described in (2).

Set $Y = \bigcup_{n \in \mathbb{N}} X_n$. By construction, $Y$ is an adapted submodule of $M$.

Step 2. Let $X$ be an arbitrary adapted submodule of $M$ such that $X \neq M$. Then there exists an adapted submodule $X'$ of $M$ such that $X \subseteq X'$ and $X'/X$ is a countably generated adapted submodule of $M/X$. Hence $X'/X$ is pure-projective; therefore the pure exact sequence

$$0 \to X \to X' \to X'/X \to 0$$

splits.

By definition, if $X$ is an adapted submodule of $M$ then $(M/X) \otimes_R T \cong \oplus_{i \in I'} Q_i$ for a certain $I' \subseteq I$. Hence, it makes sense to talk about adapted submodules of $M/X$ with respect to the decomposition induced by $F' = \{Q_i\}_{i \in I'}$. By Step 1, there exists a submodule $X'$ of $M$ containing $X$ and such that $X'/X$ is a countably generated adapted submodule of $M/X$. Therefore $X'$ is also an adapted submodule of $M$. Since $X$ is an adapted submodule of $X'$ the rest of the statement is clear.

Finally, combining the first and the second steps, we deduce that there exist an ordinal $\kappa$ and a continuous chain $\{X_\alpha\}_{\alpha < \kappa}$ of adapted submodules of $M$ such that

(i) $X_0 = 0$, and
(ii) for any $\alpha + 1 < \kappa$, $X_{\alpha+1}/X_\alpha$ is pure projective and a direct summand of $X_{\alpha+1}$.
In this situation $M \cong \oplus (X_{\alpha+1}/X_\alpha)$ (cf. [6, Lemme 3.1.2, p. 81]). Therefore $M$ is pure projective.

**Corollary 3.4.** Let $R \to T$ be a pure ring monomorphism of commutative rings, and let $M$ be an $R$-module. Then $M_R$ is projective if and only if $M \otimes_R T$ is a projective $T$-module.

**Proof.** Since projective modules are exactly the pure-projective modules that, in addition, are flat, the statement follows from Proposition[5.3] and Corollary[2.2].

**References**

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