The generalized Chern character and Lefschetz numbers in W*-modules

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Abstract

We define $N$-theory being some analogue of $K$-theory on the category of von Neumann algebras such that $K_0(A) \subset N_0(A)$ for any von Neumann algebra $A$. Moreover, it turns out to be possible to construct the extension of the Chern character to some homomorphism from $N_0(A)$ to even Banach cyclic homology of $A$. Also, we define generalized Lefschetz numbers for an arbitrary unitary endomorphism $U$ of an $A$-elliptic complex. We study them in the situation when $U$ is an element of a representation of some compact Lie group.

Key words: $N$-theory, generalized Chern character, Banach cyclic homology

1991 mathematics subject classification: 46L80, 46L10

1 Introduction

For an arbitrary von Neumann algebra $A$ we introduce an abelian group $N_0(A)$ in the following way. It is possible to define some equivalence relation between normal elements of the inductive limit $M_\infty(A) = \lim \to M_n(A)$ such that for projections it coincide with the usual stable equivalence relation. Then the set of all equivalence classes of normal elements from $M_\infty(A)$ is an abelian semigroup (with respect to the direct sum operation) and $N_0(A)$ is its symmetrization. The first part of our paper is devoted to the consideration

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of some properties of $N$-groups. More detail on this subject can be found in [12].

Further, we introduce Banach cyclic homology of $A$ as some analogue of usual cyclic homology and construct the generalized Chern character as a map from $N_0(A)$ to even Banach cyclic homology. Furthermore, this map is an extension of the classic Chern character to the group $N_0(A) \supset K_0(A)$ in some natural sense.

In the final section we define generalized Lefschetz numbers for an arbitrary unitary endomorphism $U$ of an $A$-elliptic complex. Besides, in the case when $U$ is an element of a representation of some compact Lie group we describe the connection of the generalized Lefschetz numbers with the $W^*$-Lefschetz numbers of the first and of the second types introduced in [14, 15, 1].

Some results of the present paper were formulated in [11].

The paper is organized as follows:

1. Introduction
2. Some properties of $N$-groups
3. The group $N_0(A)_{fin}$
4. Banach cyclic homology
5. The generalized Chern character
6. Generalized Lefschetz numbers
   References

2 Some properties of $N$-groups

Suppose $A$ is a von Neumann algebra, $M_r(A)$ is the set of $r \times r$ matrices with entries in $A$, $M_\infty(A)$ is the inductive limit of the sequence $\{M_r(A)\}_{r=1}^\infty$, and $M_\infty(A)_\nu$ is the set of normal elements for $M_\infty(A)$. Denote by $\mathcal{B}(\mathbb{C})$ the family of all Borel subsets of the complex plane. If $a \in M_\infty(A)_\nu$ and $E \in \mathcal{B}(\mathbb{C})$, then by $P_a(E)$ we denote the spectral projection of $a$ corresponding to the set $E$. We remark that $P_a(E) \in M_\infty(A)_\nu$ since von Neumann algebras are closed with respect to the Borel calculus. We denote the stable equivalence relation (see [9]) of projections $p, q \in M_\infty(A)$ by $p \simeq q$. Finally, let $sp(a)$ denote the spectrum of an element $a$. 
A Borel set $E \subset \mathbb{C}$ is called admissible if zero does not belong to the closure of $E$. Denote by $\mathcal{B}_s(\mathbb{C})$ the family of all admissible Borel subsets of the complex plane.

**Definition 1.** Call elements $a, b \in M_\infty(A)$ equivalent (and denote by $a \simeq b$) if and only if $P_a(E) \simeq P_b(E)$ for all $E \in \mathcal{B}_s(\mathbb{C})$.

Note that this equivalence relation coincides with the usual stable equivalence relation whenever $a, b$ are projections. It is easy to see that $a \oplus 0_m \simeq a$, where $0_m$ is the zero $m \times m$ matrix and $a \in M_\infty(A)_\nu$. We put

$$\mathcal{N}(A) = M_\infty(A)_\nu/\simeq.$$  

For $a \in M_\infty(A)_\nu$ let us denote by $[a]$ the equivalence class of $a$ in $\mathcal{N}(A)$. Since

$$P_{a \oplus b}(E) = P_a(E) \oplus P_b(E)$$

for all $a, b \in M_\infty(A)_\nu, E \in \mathcal{B}(\mathbb{C})$, this implies that the set $\mathcal{N}(A)$ is an abelian semigroup with respect to the direct sum operation.

**Definition 2.** The symmetrization of $\mathcal{N}(A)$ is called the $N$-group of $A$ and is denoted by $N_0(A)$.

Under the previous considerations, the following result is clear.

**Proposition 1.** $K_0(A)$ is a subgroup of the group $N_0(A)$. □

**Proposition 2.** If the group $K_0(A)$ is trivial, then the group $N_0(A)$ is trivial too.

**Proof.** Suppose $[a] - [b] \in N_0(A)$. Since $K_0(A)$ is trivial, this implies that $P_a(E) \simeq 0 \simeq P_b(E)$ for all $E \in \mathcal{B}_s(\mathbb{C})$. Whence, $a \simeq b$. Thus the group $N_0(A)$ is trivial. □

Note that $\mathcal{N}(A)$ is a cancellation semigroup, i.e., the condition $[a] + [c] = [b] + [c]$ implies $[a] = [b]$ for any $[a], [b], [c] \in \mathcal{N}(A)$. In particular, the symmetrization homomorphism $s : \mathcal{N}(A) \longrightarrow N_0(A), s([a]) = [a] - [0]$ is injective.
Our next aim is to establish a functorial property for $N_0$. We recall that an arbitrary *-homomorphism of C*-algebras is a contraction, i.e., its norm does not exceed 1 [9, Theorem 2.1.7]. Besides, an arbitrary surjective *-homomorphism of von Neumann algebras is continuous with respect to the ultra-strong topology [1, Theorem 2.4.23].

Now let $A, B$ be von Neumann algebras and $\varphi : A \to B$ an ultra-strong continuous unital *-homomorphism. By definition, put $\varphi(a) = (\varphi(a_{ij}))$ and $\varphi([a]) = [\varphi(a)]$ for each matrix $a = (a_{ij}) \in M_\infty(A)_\nu$.

**Theorem 1.** The map $\varphi_* : N_0(A) \to N_0(B)$ is a well defined homomorphism of abelian groups.

The following lemma is the main ingredient of the proof of Theorem 1.

**Lemma 1.** $P_{\varphi(a)}(E) = \varphi(P_a(E))$ for each $a \in M_\infty(A)_\nu$ and for each Borel subset $E \subset sp(a)$.

**Proof.** We can assume that $a \in M_r(A)$ for some $r \geq 1$. It is clear that $sp(\varphi(a)) \subset sp(a)$. It can be directly verified that $\varphi(R(a)) = R(\varphi(a))$ for an arbitrary polynomial $R$. Further, for any function $f$, which is continuous on $sp(a)$, there exists a sequence of polynomials $\{R_n\}_{n=1}^\infty$ such that it converges uniformly to $f$. Then

$$\varphi(f(a)) = \varphi(\lim_n R_n(a)) = \lim_n \varphi(R_n(a)) = \lim_n R_n(\varphi(a)) = f(\varphi(a)).$$

Now let $\chi_E$ be the characteristic function of $E$. Then we can find a sequence $\{f_n\}_{n=1}^\infty$ of continuous functions on the compact space $sp(a)$ such that $\{f_n\}_{n=1}^\infty$ converges to $\chi_E$ pointwise, i.e., with respect to the strong topology. Moreover, we can assume that the family $\{f_n\}_{n=1}^\infty$ is norm-bounded. In this case the sequence $\{f_n(a)\}_{n=1}^\infty$ of elements of the von Neumann algebra $M_r(A)$ converges strongly to the element $\chi_E(a) = P_a(E)$ and the family $\{f_n(a)\}_{n=1}^\infty$ is norm-bounded. Since strong and ultra-strong topologies coincide on bounded sets, we have the convergence with respect to the ultra-strong topology: $P_a(E) = \sigma\text{-lim}_n f_n(a)$. Finally, using the ultra-strong continuity of the *-homomorphism $\varphi$, we obtain:

$$\varphi(P_a(E)) = \varphi(\sigma\text{-lim}_n f_n(a)) = \sigma\text{-lim}_n \varphi(f_n(a)) = \sigma\text{-lim}_n f_n(\varphi(a)) = \chi_E(\varphi(a)) = P_{\varphi(a)}(E). \square$$
Proof of Theorem \[\Box]. We have to establish that \(\varphi\) is well defined. Let elements \(a, b \in M_\infty(A)_\nu\) be equivalent. Then \(\varphi(P_a(E)) \simeq \varphi(P_b(E))\) for any \(E \in \mathcal{B}_s(C)\). Now it immediately follows from Lemma \[\Box\] that the elements \(\varphi(a)\) and \(\varphi(b)\) are equivalent too. \(\Box\)

3 The group \(N_0(A)_{fin}\)

Let \(M_\infty(A)_{fin} \subset M_\infty(A)_\nu\) be the subset of all elements \(a \in M_\infty(A)\) such that their spectrum is finite. Suppose, \(N_0(A)_{fin} = \{[a] : a \in M_\infty(A)_{fin}\}\). We remark that \(N_0(A)_{fin}\) is a subsemigroup of \(N_0(A)\). By \(N_0(A)_{fin}\) we denote the symmetrization of the abelian monoid \(N(A)_{fin}\). So \(N_0(A)_{fin}\) is a subgroup of the group \(N_0(A)\). Any element of the group \(N_0(A)_{fin}\) we can represent in the form \([\bigoplus_{i=1}^n \lambda_i p_i] - [\bigoplus_{i=1}^n \lambda_i q_i]\), where \(\lambda_i \in \mathbb{C}\) and \(p_i, q_i\) are some (possibly zero) projections in \(M_\infty(A)\).

For an arbitrary map \(f : \mathbb{C} \setminus \{0\} \rightarrow K_0(A)\) let us put \(\Lambda_f = \{\lambda \in \mathbb{C} \setminus \{0\} : f(\lambda) \neq 0\}\). Let us denote by

\[\mathcal{M}_{fin; A} = \mathcal{M}_{fin; C \setminus \{0\}, K_0(A)}\]

the set of all maps from \(\mathbb{C} \setminus \{0\}\) to \(K_0(A)\) such that \(\Lambda_f\) is finite (or empty). The set \(\mathcal{M}_{fin; A}\) is an abelian group with respect to the pointwise addition of maps. Let us consider the map

\[\phi : N_0(A)_{fin} \rightarrow \mathcal{M}_{fin; A}, \quad (\phi([a] - [b]))(\lambda) = [P_a(\{\lambda\})] - [P_b(\{\lambda\})].\]

Theorem 2. The map \(\phi\) is an isomorphism of groups.

Proof. It is clear that \(\phi\) is a well defined homomorphism. Let \([a] - [b] \in N_0(A)_{fin}\), and \(\phi([a] - [b]) = 0\). Therefore for each \(\lambda \in \mathbb{C} \setminus \{0\}\) the projections \(P_a(\{\lambda\})\) and \(P_b(\{\lambda\})\) are stably equivalent. Since the spectra of elements \(a, b\) are finite, we conclude that these elements are equivalent. Hence, \(\phi\) is injective.

Let us examine a map \(f \in \mathcal{M}_{fin; A}\) such that \(\Lambda_f = \{\lambda_i\}_{i=1}^n\) and \(f(\lambda_i) = [p_i] - [q_i] \quad (1 \leq i \leq n)\), where \(p_i, q_i\) are projections from \(M_\infty(A)\). Since
\( p_i \simeq 0_k \oplus p_i \), we can assume that the projections \( \{p_i\}_{i=1}^n \) (and \( \{q_i\}_{i=1}^n \)) are pairwise orthogonal. We put \( a = \bigoplus_{i=1}^n \lambda_i p_i \) and \( b = \bigoplus_{i=1}^n \lambda_i q_i \). Then the elements \( a, b \) belong to \( M_\infty(A)_{\text{fin}} \). Furthermore, \( \phi([a] - [b]) = f \). Hence, \( \phi \) is surjective. \( \square \)

**Corollary 1.** The groups \( N_0(M_r(C)) \) and \( M_{\text{fin}}(C \setminus \{0\}, \mathbb{Z}) \) are isomorphic.

**Proof.** The spectrum of any element from \( M_\infty(C) \) is finite. Therefore, \( N_0(M_r(C)) = N_{\text{fin}}(M_r(C)) \). To complete the proof, it remains to use Theorem 2. \( \square \)

Assume \( h_{\lambda, \mu} := [p(\lambda + \mu)] - [p\lambda \oplus p\mu] \) and \( g_{\lambda, p}^{(n)} := [\lambda p \oplus n] - [n\lambda p] \), where \( n \in \mathbb{N}, \lambda, \mu \in C \) and \( p \) is a projection in \( M_\infty(A) \). Then \( h_{\lambda, \mu} \) and \( g_{\lambda, p}^{(n)} \) are elements from \( N_0(A)_{\text{fin}} \).

Let us denote by \( H \) the subgroup of \( N_0(A)_{\text{fin}} \) with the following system of generators \( \{h_{\lambda, \mu} : \lambda, \mu \in C, p \) is a projection in \( M_\infty(A)\} \), and by \( G \) the subgroup of \( N_0(A)_{\text{fin}} \) with the following system of generators \( \{g_{\lambda, p}^{(n)} : n \in \mathbb{N}, \lambda \in C, p \) is a projection in \( M_\infty(A)\} \).

**Lemma 2.** The group \( G \) is a subgroup of \( H \).

**Proof.** Take \( \lambda \in C \) and a projection \( p \) from \( M_\infty(A) \). We have to demonstrate that \( g_{\lambda, p}^{(k)} \) belongs to \( H \) for all \( k \geq 1 \). Let us prove this statement by induction over \( k \). The case \( k = 1 \) is clear. Suppose \( g_{\lambda, p}^{(k)} \in H \) for all \( k \leq n - 1 \). In particular,

\[
g_{\lambda, p}^{(n-1)} = [\lambda p \oplus (n-1)] - [(n-1)\lambda p] = y
\]

for some \( y \in H \). Therefore,

\[
g_{\lambda, p}^{(n)} = [\lambda p \oplus (n-1)] + [\lambda p] - [(n-1)\lambda + \lambda)p
\]

\[
= [(n-1)\lambda p] + y + [\lambda p] - [(n-1)\lambda + \lambda)p
\]

\[
= y - h_{(n-1)\lambda, \lambda, p}.
\]

Thus \( g_{\lambda, p}^{(n)} \in H \) and by induction we obtain the desired statement. \( \square \)

Let us consider the map

\[
h : N_0(A)_{\text{fin}} \longrightarrow K_0(A) \otimes C
\]  

(1)
defined as follows
\[
h([a] - [b]) = \sum_{i=1}^{n} [P_a(\lambda_i)] \otimes \lambda_i - \sum_{j=1}^{m} [P_b(\mu_j)] \otimes \mu_j,
\]
where \(a, b \in M_{\infty}(A)_{\text{fin}}\) and \(sp(a) = \{\lambda_1, \ldots, \lambda_n\}\), \(sp(b) = \{\mu_1, \ldots, \mu_m\}\).

**Proposition 3.** The map \(h\) is a surjective homomorphism of groups. Besides, the kernel of \(h\) coincides with the group \(H\).

**Proof.** It is obvious that \(h\) is a well defined surjective homomorphism. Also, it is clear that \(H\) belongs to the kernel of \(h\). To complete the proof let us construct the inverse for \(h\) homomorphism \(t\):
\[
K_0(A) \otimes C \longrightarrow N_0(A)_{\text{fin}}/H.
\]
We put
\[
t(\sum_{i=1}^{n} ([p_i] - [q_i]) \otimes \lambda_i) = \sum_{i=1}^{n} [p_i \lambda_i] - \sum_{i=1}^{n} [q_i \lambda_i] + H.
\]
Let us demonstrate that the map \(t\) is well defined. In the other words, we have to verify that the homomorphism \(t\) is trivial on the elements \(c_{\lambda, \mu}^{(1)} = [p] \otimes (\lambda + \mu) - [p] \otimes \lambda - [p] \otimes \mu\), \(c_{\lambda, p}^{(2)} = [p] z \otimes \lambda - [p] \otimes z \lambda\), and \(c_{\lambda, q}^{(3)} = ([p] + [q]) \otimes \lambda - [p] \otimes \lambda - [q] \otimes \lambda\), where \(\lambda, \mu \in C\), \(z \in Z\) and \(p, q \in M_{\infty}(A)\) are projections. We derive \(t(c_{\lambda, \mu}^{(1)}) = h_{\lambda, \mu} + H = H\). Besides, \(c_{\lambda, -\lambda}^{(2)} = c_{\lambda, -\lambda}^{(1)}\).

Therefore it suffices to regard the elements \(c_{\lambda, z}^{(2)}\) provided \(z > 0\). In this case we conclude \(t(c_{\lambda, p}^{(2)}) = t([p] z \otimes \lambda - [p] \otimes z \lambda) = g_{\lambda, p}^{(2)} + H = H\), where we have used Lemma 2. Finally, it can be directly verified that \(t(c_{\lambda, q}^{(3)}) = 0 + H = H\). To complete the proof, it remains to note that \(t = h^{-1}\). \(\square\)

## 4 Banach cyclic homology

As above, let \(A\) be a von Neumann algebra. First let us recall some concepts from noncommutative geometry (see, for example, [4, 6, 13]). Consider the complex vector space \(C_n(A) = A \otimes (n+1)\), where \(A \otimes (n+1) = A \otimes A \otimes \ldots \otimes A\).
The cyclic operator $\tau_n : C_n(A) \rightarrow C_n(A)$ is defined on generators by the formula

$$\tau_n(a_0 \otimes \ldots \otimes a_n) = (-1)^n a_n \otimes a_0 \otimes \ldots \otimes a_{n-1}. $$

The cokernel of the endomorphism $1 - \tau_n : C_n(A) \rightarrow C_n(A)$ we denote by

$$CC_n(A) = A^{\otimes (n+1)}/\text{Im}(1 - \tau_n).$$

Further, we define the face operator $b_n : C_n(A) \rightarrow C_{n-1}(A)$ by the formula

$$b_n(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n +$$

$$( -1)^n a_n a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1}.$$ 

It is clear that

$$b = \sum_{i=0}^{n} (-1)^i d_i, \quad (2)$$

where the linear maps $d_i : A^{\otimes (n+1)} \rightarrow A^{\otimes n}$ are defined as follows

$$d_i(a_0 \otimes \ldots \otimes a_n) = a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n, \quad 0 \leq i \leq n-1,$$

$$d_n(a_0 \otimes \ldots \otimes a_n) = a_n a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1}. $$

It can be verified by the direct calculation that the family of linear spaces $CC_*(A) = \{CC_n(A), b_n\}$ is a chain complex. Homology of this complex is called cyclic homology of $A$ and is denoted by $HC_n(A) = H_*(CC_*(A))$, $n \geq 0$.

The trace map $Tr : M_r(A)^{\otimes (n+1)} \rightarrow A^{\otimes (n+1)}$ is defined by the formula

$$Tr(\xi^{(0)} \otimes \xi^{(1)} \otimes \ldots \otimes \xi^{(n)}) = \sum_{i_0, \ldots, i_n = 1}^{r} \xi^{(0)}_{i_0 i_1} \otimes \xi^{(1)}_{i_1 i_2} \otimes \ldots \otimes \xi^{(n)}_{i_n i_0},$$

where $\xi^{(k)} = (\xi^{(k)}_{i,j})_{i,j=1}^{r}$ $\in M_r(A)$. It can be directly verified that the trace map $Tr : CC_*(M_r(A)) \rightarrow CC_*(A)$ is a morphism of chain complexes. Furthermore, the induced map

$$Tr_* : HC_*(M_r(A)) \rightarrow HC_*(A)$$

is an isomorphism.
Now let $X, Y$ be normed spaces, and $x \in X, y \in Y$. A representative of the equivalence class $x \otimes y \in X \otimes Y$ we shall denote by $x \square y$. Assume

$$\| \sum_i x_i \square y_i \| := \sum_i \| x_i \| \| y_i \| \quad (x_i \in X, y_i \in Y).$$

Then the projective norm of an equivalence class $\xi \in X \otimes Y$ is defined as follows

$$\| \xi \| = \inf \{ \| \sum_i x_i \square y_i \| : x_i \in X, y_i \in Y \text{ and } \sum_i x_i \square y_i \in \xi \}.$$ 

Below we assume that all tensor products of normed spaces are equipped with the projective norm.

Under the previous conventions, let us set

$$CC_n(A) = A^{\otimes (n+1)} / \overline{\text{Im}(1 - \tau_n)}.$$ 

Note that $CC_n(A)$ is a Banach space. For $\xi \in A^{\otimes (n+1)}$ we denote by $[\xi]_{CC_n(A)}$ the quotient class of $\xi$ in $CC_n(A)$, and by $[\xi]_{CC_n(A)}$ the quotient class of $\xi$ in $CC_n(A)$.

**Lemma 3.** The face operator $b_n : A^{\otimes (n+1)} \to A^{\otimes n}$ is a continuous map.

**Proof.** Under equality (2), it is sufficiently to prove that all maps $d_i$ ($0 \leq i \leq n$) are continuous. Given any $\varepsilon > 0$. For each $\xi \in A^{\otimes (n+1)}$ we can find an element $\sum_k a_0^{(k)} \square \ldots \square a_n^{(k)} \in \xi$ such that

$$\| \sum_k a_0^{(k)} \square \ldots \square a_n^{(k)} \| - \| \xi \| < \varepsilon.$$ 

Therefore,

$$\|d_i(\xi)\| = \| \sum_k a_0^{(k)} \otimes \ldots \otimes a_n^{(k)} \|$$

$$= \| \sum_k a_0^{(k)} \otimes \ldots \otimes a_i^{(k)} a_{i+1}^{(k)} \otimes \ldots \otimes a_n^{(k)} \|$$

$$\leq \| \sum_k a_0^{(k)} \square a_i^{(k)} a_{i+1}^{(k)} \square \ldots \square a_n^{(k)} \|$$

$$= \sum_k \| a_0^{(k)} \| \ldots \| a_i^{(k)} a_{i+1}^{(k)} \ldots \| a_n^{(k)} \|$$

$$\leq \sum_k \| a_0^{(k)} \| \ldots \| a_i^{(k)} \| \| a_{i+1}^{(k)} \ldots \| a_n^{(k)} \|$$

$$= \| \sum_k a_0^{(k)} \square \ldots \square a_n^{(k)} \| < \| \xi \| + \varepsilon$$

Therefore,
for all $\varepsilon > 0$. Hence, $\|d_i(\xi)\| \leq \|\xi\|$. □

Let us define the map $\beta_n : \mathbb{CC}_n(A) \rightarrow \mathbb{CC}_{n-1}(A)$ by the formula

$$\beta_n([x]_{\mathbb{CC}_n(A)}) = [b_n(x)]_{\mathbb{CC}_{n-1}(A)}.$$ 

From Lemma 3 we conclude that the family $\mathbb{CC}_*(A) = \{\mathbb{CC}_n(A), \beta_n\}$ is a well defined chain complex. Let us put

$$\mathcal{H}C_n(A) = \text{Ker } \beta_n / \text{Im } \beta_{n+1}.$$ 

The quotient space $\mathcal{H}C_*(A)$ we shall call Banach (cyclic) homology of $A$ (cf. [4, 5]). Note that $\mathcal{H}C_n(A)$ is a Banach space (for each $n \geq 0$).

For $\xi \in M_r(A)^{(n+1)}$ let us denote by $\langle \xi \rangle_{\mathcal{H}C}(\mathcal{H}C_*(A))$ the cyclic homology class of $\xi$ and by $\langle \xi \rangle_{\mathcal{H}C}(\mathcal{H}C_*(M_r(A)))$ the Banach cyclic homology class of $\xi$.

Let $p \in M_r(A)$ be projection. Then

$$b_{2l}(p^{\otimes(2l+1)}) = \sum_{k=0}^{2l} d_k(p^{\otimes(2l+1)}) = \sum_{k=0}^{2l} (-1)^k p^{\otimes 2l} = p^{\otimes 2l}.$$ 

On the other hand, we have $[p^{\otimes 2l}]_{\mathbb{CC}_r(M_r(A))} = (-1)^{(2l-1)}[p^{\otimes 2l}]_{\mathbb{CC}_r(M_r(A))}$. So $[p^{\otimes 2l}]_{\mathbb{CC}_r(M_r(A))} = 0$. Therefore $p^{\otimes(2l+1)}$ is a cycle. Now let us define the Chern character

$$Ch^0_{2l} : K_0(A) \rightarrow HC_{2l}(A)$$

by $Ch^0_{2l}([p]) = Tr_* ((-1)^l p^{\otimes(2l+1)}|_{HC})$. The Chern character is a well defined homomorphism of groups [3, Theorem 8.3.2].

Let us study the linear epimorphism $\pi_n : CC_n(A) \rightarrow CC_n(A)$, where

$$\pi_n([\xi]_{CC_n(A)}) = [\xi]_{CC_n(A)}, \xi \in A^{\otimes(n+1)}.$$ 

It is obvious that the family of the maps $\{\pi_n\} : CC_*(A) \rightarrow CC_*(A)$ is a chain homomorphism, i.e., $\pi_{n-1}b_n = \beta_n \pi_n$ for all $n \geq 1$. So $\pi_n(\text{Ker } b_n) \subset \text{Ker } \beta_n$ and $\pi_n(\text{Im } b_{n+1}) \subset \overline{\text{Im } \beta_{n+1}}$. Therefore,

$$\pi_* : \mathcal{H}C_*(A) \rightarrow \mathcal{H}C_*(A)$$

is a well defined map of homology spaces.

Below we shall need the following result.
Lemma 4. The trace $T r : M_r(A)^{\otimes(n+1)} \longrightarrow A^{\otimes(n+1)}$ is a continuous map.

Proof. Given any $\varepsilon > 0$. For each quotient class $\xi \in M_r(A)^{\otimes(n+1)}$ of the tensor product we can find a representative $\sum_{k=1}^{N} \xi^{(0),k} \otimes \xi^{(1),k} \otimes \ldots \otimes \xi^{(n),k}$ of $\xi$ such that

$$\| \sum_{k=1}^{N} \xi^{(0),k} \otimes \xi^{(1),k} \otimes \ldots \otimes \xi^{(n),k} - \| \xi \| < \varepsilon.$$ 

Therefore,

$$\| Tr(\xi) \| = \| \sum_{k=1}^{N} Tr(\xi^{(0),k} \otimes \xi^{(1),k} \otimes \ldots \otimes \xi^{(n),k}) \|$$

$$\leq \| \sum_{k=1}^{N} \sum_{i_0, \ldots, i_n = 1}^{r} \xi^{(0),k}_{i_0 i_1} \otimes \xi^{(1),k}_{i_1 i_2} \otimes \ldots \otimes \xi^{(n),k}_{i_n i_0} \|$$

$$\leq \sum_{k=1}^{N} \sum_{i_0, \ldots, i_n = 1}^{r} \| \xi^{(0),k} \| \| \xi^{(1),k} \| \ldots \| \xi^{(n),k} \|$$

$$\leq r^{n+1} \sum_{k=1}^{N} \| \sum_{i_0, \ldots, i_n = 1}^{r} \xi^{(0),k} \otimes \xi^{(1),k} \otimes \ldots \otimes \xi^{(n),k} \| < r^{n+1}(\| \xi \| + \varepsilon)$$

for all $\varepsilon > 0$. So, $\| Tr(\xi) \| \leq r^{n+1}\| \xi \|$. □

From Lemma 4 we conclude that the map

$$Tr : CC_*(M_r(A)) \longrightarrow CC_*(A), \ [\xi]_{CC_*(M_r(A))} \longmapsto [Tr(\xi)]_{CC_*(A)} \quad (4)$$

is well defined. Furthermore, it can be directly checked up that trace map (4) is a morphism of chain complexes. By Lemma 4, this implies that the induced homomorphism

$$Tr_* : HC_*(M_r(A)) \longrightarrow HC_*(A)$$

of Banach homology is well defined.

5 The generalized Chern character

Given $l \geq 0$. We want to define a map

$$T : M_\infty(A)_\nu \longrightarrow HC_{2l}(A)$$

to the even Banach homology by the following construction. Let an element $a$ belong to $M_\infty(A)_\nu$. Then we can suppose that $a \in M_r(A)$ for some $r \geq 1$.

For each natural number $n$ let us consider a cover $E^{(n)} = \{ E^{(n)}_k \}_{k=1}^{k_n}$ of the
spectrum of \( a \) by disjoint Borel sets such that the diameter of each of these sets does not exceed \( 1/n \). Moreover, we can assume that \( \{E_k^{(m)}\}_{k=1}^{k_n} \) is a subdivision of the cover \( \{E_k^{(n)}\}_{k=1}^{k_n} \) when \( m \geq n \). Therefore we can write

\[
E_k^{(n)} = \bigcup_{j=1}^{j(k)} E_{k,j}^{(m)},
\]

where \( E_{k,j}^{(m)} \) are some elements of the cover \( \mathcal{E}^{(m)} \). Thus, \( \mathcal{E}^{(m)} = \{E_{k,j}^{(m)}\}_{k=1;j=1}^{k_n;j(k)} \) and \( \sum_{k=1}^{k_n} j(k) = k_m \).

Also, for any \( \lambda_k^{(m)} \in E_k^{(n)} \) let us consider a sequence \( \{a_n\}_{n=1}^{\infty} \) from \( M_r(A) \), where

\[
a_n = \sum_{k=1}^{k_n} P_a(E_k^{(n)})\lambda_k^{(n)}.
\]

It follows from the spectral theorem that \( \{a_n\}_{n=1}^{\infty} \) converges uniformly to \( a \). Further, for each \( a_n \) let us examine

\[
\tilde{a}_n = \sum_{k=1}^{k_n} P_a(E_k^{(n)}) \otimes (2l+1) \lambda_k^{(n)} \in M_r(A) \otimes (2l+1).
\]

Given natural numbers \( n, m \ (m \geq n) \). Under the previous notation, we see that

\[
a_n = \sum_{k=1}^{k_n} \sum_{j=1}^{j(k)} P_a(\bigcup_{j=1}^{j(k)} E_{k,j}^{(m)}) \lambda_k^{(n)} = \sum_{k=1}^{k_n} \sum_{j=1}^{j(k)} P_a(E_{k,j}^{(m)}) \lambda_k^{(n)}
\]

and \( a_m = \sum_{k=1}^{k_n} \sum_{j=1}^{j(k)} P_a(E_{k,j}^{(m)}) \lambda_k^{(m)} \), where \( \lambda_k^{(m)} \in E_{k,j}^{(m)} \). This yields that

\[
a_n - a_m = \sum_{k=1}^{k_n} \sum_{j=1}^{j(k)} P_a(E_{k,j}^{(m)}) (\lambda_k^{(n)} - \lambda_k^{(m)}).
\] (5)

Furthermore,

\[
\begin{align*}
\tilde{a}_n &= \sum_{k=1}^{k_n} P_a(E_k^{(n)}) \otimes (2l+1) \lambda_k^{(n)} = \sum_{k=1}^{k_n} \sum_{j=1}^{j(k)} P_a(\bigcup_{j=1}^{j(k)} E_{k,j}^{(m)}) \otimes (2l+1) \lambda_k^{(n)} \\
&= \sum_{k=1}^{k_n} \sum_{j_0=1}^{j(k)} \cdots \sum_{j_{2l-1}=1}^{j(k)} P_a(E_{k,j_0}^{(m)}) \otimes \cdots \otimes P_a(E_{k,j_{2l}}^{(m)}) \lambda_k^{(n)} \\
&= \sum_{k=1}^{k_n} \sum_{j_0=1}^{j(k)} \cdots \sum_{j_{2l-1}=1}^{j(k)} P_a(E_{k,j_0}^{(m)}) \otimes \cdots \otimes P_a(E_{k,j_{2l}}^{(m)}) \lambda_k^{(n)}
\end{align*}
\]
and \( \tilde{a}_m = \sum_{k=1}^{k_n} \sum_{j=1}^{j(k)} P_a(E_{k,j}^{(m)})^{\otimes(2l+1)} \lambda_{k,j}^{(m)}. \)

For brevity we shall use the following notation
\[
\tilde{\sum}_{j_0,\ldots,j_{2l}=1} := \sum_{\{1 \leq j_0 \leq \ldots \leq j_{2l} \leq j(k) : \exists j_p \neq j_q\}}
\]
for the sum over \( j_0, \ldots, j_{2l} \) from one to \( j(k) \), where not all indices coincide.

So we obtain
\[
\tilde{a}_n - \tilde{a}_m = \sum_{k=1}^{k_n} \sum_{j=1}^{j(k)} P_a(E_{k,j}^{(m)})^{\otimes(2l+1)} (\lambda_k^{(n)} - \lambda_{k,j}^{(m)})
\]
\[
+ \sum_{k=1}^{k_n} \tilde{\sum}_{j_0,\ldots,j_{2l}=1} P_a(E_{k,j_0}^{(m)}) \otimes \ldots \otimes P_a(E_{k,j_{2l}}^{(m)}) \lambda_k^{(n)}
\]
\[= \alpha_{n,m} + \gamma_{n,m}, \tag{6}\]

where by \( \alpha_{n,m} \) (\( \gamma_{n,m} \)) we denote the first (the second) summand in expression (5).

The following result is the main ingredient of our definition of the map \( T \).

**Theorem 3.** Let \( \{p_i\}_{i=1}^N \in M_r(A) \) be a family of pairwise orthogonal projections and \( \eta = \tilde{\sum}_{j_0,\ldots,j_{2l}=1} p_{j_0} \otimes \ldots \otimes p_{j_{2l}}. \) Then the element \( \eta \) belongs to the kernel of the face operator \( \beta_{2l} \). Besides, \( \langle \eta \rangle_{HC_{2l}(M_r(A))} = 0. \)

**Proof.** We have
\[
\eta = (\sum_{j=1}^N p_j)^{\otimes(2l+1)} - \sum_{j=1}^N p_j^{\otimes(2l+1)}.
\]
So \( \eta \) belongs to the kernel of the face operator \( \beta_{2l} \) as a difference of elements from \( \text{Ker} \beta_{2l} \).

Now let us examine the following element
\[
\alpha := Tr_*(\langle \sum_{j=1}^N p_j \rangle^{\otimes(2l+1)})_{HC}
\]
\[
= \sum_{j=1}^N (-1)^i Ch_2^0(\sum_{j=1}^N [p_j]) = \sum_{j=1}^N (-1)^i Ch_2^0([p_j])
\]
\[
= \sum_{j=1}^N Tr_*(p_j^{\otimes(2l+1)})_{HC} = Tr_*(\sum_{j=1}^N p_j^{\otimes(2l+1)})_{HC}.
\]
On the other hand,
\[
\alpha = Tr_*(\sum_{j_0,\ldots,j_{2l}=1} p_{j_0} \otimes \ldots \otimes p_{j_{2l}})_{HC}.
\]
Thus,

\[
0 = Tr_*\langle \sum_{j_0,\ldots,j_2l=1}^N p_{j_0} \otimes \cdots \otimes p_{j_2l} \rangle_{HC} - Tr_*\langle \sum_{j=1}^N p_j^{(2l+1)} \rangle_{HC}
\]

\[
= Tr_*\langle \sum_{j_0,\ldots,j_2l=1}^N p_{j_0} \otimes \cdots \otimes p_{j_2l} - \sum_{j=1}^N p_j^{(2l+1)} \rangle_{HC}
\]

\[
= Tr_*\langle \sum_{j_0,\ldots,j_2l=1}^N p_{j_0} \otimes \cdots \otimes p_{j_2l} \rangle_{HC}.
\]

The trace map is an isomorphism. Therefore we conclude

\[
\langle \eta \rangle_{HC} = \langle \sum_{j_0,\ldots,j_2l=1}^N p_{j_0} \otimes \cdots \otimes p_{j_2l} \rangle_{HC} = 0.
\]

Whence, \( \langle \eta \rangle_{HC} = \pi_*\langle \langle \eta \rangle_{HC} \rangle = 0 \), where \( \pi_* \) is map (4). \( \square \)

Now let us return to expressions \( (6), (7) \). By the previous theorem, we see that \( \langle \gamma_{n,m} \rangle_{HC} = 0 \) so \( \langle \tilde{a}_n - \tilde{a}_m \rangle_{HC} = \langle \alpha_{n,m} \rangle_{HC} \).

We claim that \( \|\alpha_{n,m}\| = \|a_n - a_m\| \). Indeed, it is clear that elements \( a_n - a_m \) and \( \alpha_{n,m} \) are normal. Besides, \( sp(a_n - a_m) \setminus \{0\} \) and \( sp(\alpha_{n,m}) \setminus \{0\} \) coincide with the set \( \{\lambda_k^{(n)} - \lambda_{k,j}^{(m)} \}_{k=1; j=1}^{n,m} \). This implies that spectral radii of these elements coincide too. Thus we obtain the desired statement.

So we can write

\[
\|\langle \tilde{a}_n - \tilde{a}_m \rangle_{HC}\| = \|\langle \alpha_{n,m} \rangle_{HC}\| \leq \|\alpha_{n,m}\| = \|a_n - a_m\|. \tag{7}
\]

By Lemma \( [4] \) and inequality \( (7) \), we conclude that \( \{Tr_*\langle \langle \tilde{a}_n \rangle_{HC} \rangle\}_{n=1}^\infty \) is a Cauchy sequence. Therefore it converges to some element \( T(a; \{a_n\}) \in \mathcal{HC}_{2l}(A) \).

It remains to verify that the limit \( T(a; \{a_n\}) \) does not depend on \( \{a_n\} \).

Let us regard covers \( \mathcal{E}^{(n)} = \{E_k^{(n)}\}_{k=1}^{n} \) and \( \mathcal{F}^{(n)} = \{F_j^{(n)}\}_{j=1}^{n} \) of the spectrum of \( a \) by disjoint Borel sets. Besides, we shall assume that the diameter of each of these sets does not exceed \( 1/n \). Also, for any \( \mu_j^{(n)} \in F_j^{(n)} \) let us examine a sequence \( \{c_n\}_{n=1}^\infty \), where \( c_n = \sum_{j=1}^{j_n} P_a(F_j^{(n)})\mu_j^{(n)} \).

**Theorem 4.** The elements \( T(a; \{a_n\}) \) and \( T(a; \{c_n\}) \) coincide. Thus the map

\[
T : M_{\infty}(A)_\nu \to \mathcal{HC}_{2l}(A), \quad a \mapsto T(a; \{a_n\}) = T(a)
\]

is well defined.
Thus we obtain $X^{(n)}_{k,j} = E^{(n)}_k \cap F^{(n)}_j$. Then

$$a_n = \sum_{k=1}^{k_n} P_a(E^{(n)}_k) \lambda^{(n)}_k = \sum_{k=1}^{k_n} P_a(\bigcup_{j=1}^{j_n} E^{(n)}_k \cap F^{(n)}_j) \lambda^{(n)}_k = \sum_{k=1}^{k_n} \sum_{j=1}^{j_n} P_a(X^{(n)}_{k,j}) \lambda^{(n)}_k$$

and by the same reason $c_n = \sum_{k=1}^{k_n} \sum_{j=1}^{j_n} P_a(X^{(n)}_{k,j}) \mu^{(n)}_j$. Hence,

$$a_n - c_n = \sum_{k=1}^{k_n} \sum_{j=1}^{j_n} P_a(X^{(n)}_{k,j}) (\lambda^{(n)}_k - \mu^{(n)}_j). \quad (8)$$

If $X^{(n)}_{k,j} = \emptyset$, then $P_a(X^{(n)}_{k,j}) = 0$. Therefore we can assume that $X^{(n)}_{k,j} \neq \emptyset$ in expression $(8)$. In this case let us consider $z \in X^{(n)}_{k,j}$. We deduce that $|\lambda^{(n)}_k - \mu^{(n)}_j| \leq |\lambda^{(n)}_k - z| + |z - \mu^{(n)}_j| \leq 1/n + 1/n = 2/n$ for all $1 \leq k \leq k_n$, $1 \leq j \leq j_n$. Note that $a_n - c_n$ is a normal element. Therefore,

$$\|a_n - c_n\| = \sup \{|\lambda^{(n)}_k - \mu^{(n)}_j| : X^{(n)}_{k,j} \neq \emptyset, 1 \leq k \leq k_n, 1 \leq j \leq j_n\} \leq 2/n. \quad (9)$$

On the other hand, we have

$$\tilde{a}_n = \sum_{k=1}^{k_n} P_a(E^{(n)}_k) \otimes (2l+1) \lambda^{(n)}_k = \sum_{k=1}^{k_n} \left( P_a(\bigcup_{j=1}^{j_n} X^{(n)}_{k,j}) \right) \otimes (2l+1) \lambda^{(n)}_k = \sum_{k=1}^{k_n} \sum_{j_0,\ldots,j_{2l+1}=1}^{j_n,\ldots,j_n} P_a(X^{(n)}_{k,j_0}) \otimes \ldots \otimes P_a(X^{(n)}_{k,j_{2l+1}}) \lambda^{(n)}_k$$

and by the same reason

$$\tilde{c}_n = \sum_{j=1}^{j_n} \sum_{k_0,\ldots,k_{2l+1}=1}^{k_n} P_a(X^{(n)}_{j,k_0}) \otimes \ldots \otimes P_a(X^{(n)}_{j,k_{2l+1}}) \mu^{(n)}_j. \quad (8)$$

Thus we obtain

$$\tilde{a}_n - \tilde{c}_n = \sum_{k=1}^{k_n} \sum_{j=1}^{j_n} P_a(X^{(n)}_{k,j}) \otimes (2l+1) (\lambda^{(n)}_k - \mu^{(n)}_j) + \sum_{k=1}^{k_n} \sum_{j_0,\ldots,j_{2l+1}=1}^{j_n,\ldots,j_n} P_a(X^{(n)}_{k,j_0}) \otimes \ldots \otimes P_a(X^{(n)}_{k,j_{2l+1}}) \lambda^{(n)}_k - \sum_{j=1}^{j_n} \sum_{k_0,\ldots,k_{2l+1}=1}^{k_n} P_a(X^{(n)}_{j,k_0}) \otimes \ldots \otimes P_a(X^{(n)}_{j,k_{2l+1}}) \mu^{(n)}_j = \gamma^{(1)}_n + \gamma^{(2)}_n - \gamma^{(3)}_n.$$

From Theorem $3$ we conclude that $\langle \gamma^{(2)}_n \rangle_{HC} = \langle \gamma^{(3)}_n \rangle_{HC} = 0$. The elements $a_n - c_n$ and $\gamma^{(1)}_n$ are normal. Furtermore, the sets $sp(a_n - c_n) \setminus \{0\}$ and
$sp(\gamma_n^{(1)}) \setminus \{0\}$ coincide. Therefore, $\|a_n - c_n\| = \|\gamma_n^{(1)}\|$. Using inequality (3), we obtain

$$\|\langle \tilde{a}_n \rangle_{HC} - \langle \tilde{c}_n \rangle_{HC}\| = \|\langle \gamma_n^{(1)} \rangle_{HC}\| \leq \|\gamma_n^{(1)}\| = \|a_n - c_n\| \leq 2/n.$$ 

Therefore, $\|Tr_\ast(\langle \tilde{a}_n \rangle_{HC}) - Tr_\ast(\langle \tilde{c}_n \rangle_{HC})\| \leq 2\|Tr_\ast\|/n$. This estimate implies that

$$T(a, \{a_n\}) = \lim_n Tr_\ast(\langle \tilde{a}_n \rangle_{HC}) = \lim_n Tr_\ast(\langle \tilde{c}_n \rangle_{HC}) = T(a, \{c_n\}).$$

The proof is complete. $\Box$

**Proposition 4.** Suppose $a, b \in M_\infty(A)_\nu$ are equivalent in the sense of Definition [4]. Then $T(a) = T(b)$.

**Proof.** For each $n \in \mathbb{N}$ let us cover the space $sp(a) \cap sp(b)$ by disjoint Borel sets and enlarge this system of sets to a disjoint cover $\{E_k^{(n)}\}_{k=0}^{k_n}$ of the space $sp(a) \cup sp(b)$. As above, we suppose that $\text{diam}(E_k^{(n)}) \leq 1/n$ for all $0 \leq k \leq k_n$. Also, let us consider $\gamma_k^{(n)} \in E_k^{(n)}$. If $0 \in sp(a) \cup sp(b)$, then we shall assume that $0 \in E_0^{(n)}$ and $\gamma_0^{(n)} = 0$. In the opposite case, we put $E_0^{(n)} = \emptyset$. Furthermore, for any $1 \leq k \leq k_n$ we can assume that $E_k^{(n)}$ is an admissible Borel set.

Projections $P_a(E_k^{(n)})$ and $P_b(E_k^{(n)})$ are stably equivalent for all $k, n \geq 1$. Therefore, $Ch_{2l}[P_a(E_k^{(n)})] = Ch_{2l}[P_b(E_k^{(n)})]$. Since the trace map is an isomorphism, we conclude that $\langle P_a(E_k^{(n)})^{\otimes(2l+1)} \rangle_{HC} = \langle P_b(E_k^{(n)})^{\otimes(2l+1)} \rangle_{HC}$. Thus,

$$\langle P_a(E_k^{(n)})^{\otimes(2l+1)} \rangle_{HC} = \pi_\ast(\langle P_a(E_k^{(n)})^{\otimes(2l+1)} \rangle_{HC}) = \pi_\ast(\langle P_b(E_k^{(n)})^{\otimes(2l+1)} \rangle_{HC}) = \langle P_b(E_k^{(n)})^{\otimes(2l+1)} \rangle_{HC},$$

where $\pi_\ast$ is map (3). So we obtain that

$$\langle \tilde{a}_n \rangle_{HC} = \sum_{k=0}^{k_n} \langle P_a(E_k^{(n)})^{\otimes(2l+1)} \rangle_{HC} \gamma_k^{(n)} = \sum_{k=0}^{k_n} \langle P_b(E_k^{(n)})^{\otimes(2l+1)} \rangle_{HC} \gamma_k^{(n)} = \langle \tilde{b}_n \rangle_{HC}$$

for all $n \geq 1$ so $T(a) = T(b)$. $\Box$
**Definition 3.** We define the *generalized Chern character* as the map
\[ Ch_0^{2l} : N_0(A) \longrightarrow HC_{2l}(A), \quad [a] - [b] \longmapsto (-1)^l(T(a) - T(b)). \]

It follows from Proposition 4 that the generalized Chern character is well defined. An immediate verification gives us

**Proposition 5.** The generalized Chern character is a homomorphism of groups. \(\square\)

**Theorem 5.** For any \(l \geq 0\) there is a commutative diagram

\[
\begin{array}{ccc}
K_0(A) & \xrightarrow{\pi_*} & N_0(A) \\
\downarrow Ch_0^{2l} & & \downarrow Ch_0^{2l} \\
HC_{2l}(A) & \xrightarrow{\pi_*} & HC_{2l}(A),
\end{array}
\]

where \(\pi_*\) is map \((3)\).

**Proof.** Under the notation of the beginning of this section let us argue as follows. Let \(p \in M_\infty(A)\) be a projection. In this case the cover \(E^{(n)}\) of the spectrum of \(p\) coincide with the set \(\{\{1\}, \{0\}\}\) for all \(n \geq 1\). Therefore for all \(n \geq 1\) we have \(\tilde{p}_n = p^{\otimes(2l+1)}\). Finally, we obtain

\[
Ch_0^{2l}([p]) = \lim_n Tr_*((-1)^l \langle \tilde{p}_n \rangle_{HC_{2l}}) = \lim_n Tr_*(\langle (-1)^l p^{\otimes(2l+1)} \rangle_{HC_{2l}}) = Tr_*(\langle (-1)^l p^{\otimes(2l+1)} \rangle_{HC_{2l}}) = \langle (-1)^l Tr(p^{\otimes(2l+1)}) \rangle_{HC_{2l}} = \pi_* Ch_0^{2l}([p]).
\]

The proof is complete. \(\square\)

**Theorem 6.** For any \(l \geq 0\) there is a commutative diagram

\[
\begin{array}{ccc}
N_0(A)_{fin} & \xrightarrow{h} & K_0(A) \otimes \mathbb{C} \\
\downarrow Ch_0^{2l} & & \downarrow \overline{Ch}_0^{2l} \\
HC_{2l}(A) & \xleftarrow{\pi_*} & HC_{2l}(A),
\end{array}
\]

where \(h\) is map \((4)\) and \(\overline{Ch}_0^{2l}([p] \otimes \lambda) = Ch_0^{2l}([p]) \lambda.\)
Proof. Let \( a \) be an element of \( M_\infty(A)_{\text{fin}} \) such that \( sp(a) = \{ \lambda_1, \ldots, \lambda_n \} \). Assume \( p_i := P_a(\{ \lambda_i \}) \). Then we have
\[
\pi_* \widetilde{Ch}_0^{2l} h([a]) = \pi_* \widetilde{Ch}_0^{2l} \left( \sum_{i=1}^n [p_i] \otimes \lambda_i \right)
= \pi_* \left( \sum_{i=1}^n Ch_0^{2l}(p_i) \lambda_i \right)
= \sum_{i=1}^n Tr_* \left( \langle (-1)^l p_i \otimes (2l+1) \rangle_{HC} \right) \lambda_i.
\]

On the other hand, we can suppose that \( \mathcal{E}^{(k)} = \{ \{ \lambda_1 \}, \ldots, \{ \lambda_n \} \} \) and \( \tilde{a}_k = \sum_{i=1}^n p_i \otimes (2l+1) \lambda_i \) for all \( k \geq 1 \). Whence,
\[
Ch_0^{2l}([a]) = (-1)^l T(a) = (-1)^l \lim_k Tr_* \langle \tilde{a}_k \rangle_{HC}
= (-1)^l \sum_{i=1}^n Tr_* \langle (p_i \otimes (2l+1))_{HC} \rangle \lambda_i. \]
\]

In particular, Theorem 3 implies that one can extend the generalized Chern character to the map from the quotient group \( N_0(A)/\text{Ker} h \) to the even Banach homology.

6 Generalized Lefshetz numbers

Suppose \( A \), as above, is a von Neumann algebra, \( G \) is a compact Lie group, and \( X \) is a compact \( G \)-manifold. Let us denote by \( \mathcal{P}(A) \) the category of finitely generated projective modules over \( A \).

Let us recall some notation from [13]. The set of all \( G-A \)-bundles over \( X \) is an abelian semigroup with respect to the direct sum operation. The symmetrization of this semigroup is denoted by \( K_G(X; A) \). Assume \( K^G(A) := K_G(pt; A) \). In this situation there is an isomorphism
\[
K^G(A) \cong K_0(A) \otimes R(G),
\]
where \( R(G) \) is the ring of representations for \( G \).

Let us consider a sequence \( \{ E^i \} \) of \( G-A \)-bundles over \( X \) together with equivariant pseudo-differential operators \( \{ d_i : \Gamma(E^i) \longrightarrow \Gamma(E^{i-1}) \} \), where by \( \Gamma(E^i) \) we denote the Banach (with respect to the uniform topology) \( A \)-module of continuous sections of \( E^i \). Besides, let us denote by \( \sigma_i \) the symbol of \( d_i \). Then this sequence of bundles and operators is called a \( G-A \)-elliptic complex (and is denoted by \( (E, d) \)) if it satisfies the following conditions:
(i) $d_i d_{i+1} = 0$,

(ii) the sequence of symbols

$$0 \longrightarrow \pi^* E^n \xrightarrow{\sigma_n} \pi^* E^{n-1} \longrightarrow \ldots \xrightarrow{\sigma_1} \pi^* E^0 \longrightarrow 0$$

is exact out of some compact neighbourhood of the zero section $X \subset T^* X$. Here $\pi : T^* X \to X$ is the natural projection.

The index of the elliptic operator $F = d + d^* : \Gamma(E_{ev}) \to \Gamma(E_{od})$ is an element of the group $K^G(A)$. Furthermore, for any $g \in G$ by computation of the character we can define the map $g : R(G) \to \mathbb{C}$. Whence, using isomorphism (10), we obtain the map

$$g : K^G(A) \to K_0(A) \otimes \mathbb{C}.$$  

Then the Lefschetz number of the first type is defined as follows

$$L_1(E, g) = g(\text{index}(F)) \in K_0(A) \otimes \mathbb{C}.$$ 

Note that there exists a connection between these Lefschetz numbers and fixed points of $g$ (see [14, 17]).

Now let us consider an $A$-elliptic complex $(E, d)$ and its unitary endomorphism $U$. Furthermore, we shall assume that $U = U_g$ for some representation $U_g$ of a compact Lie group $G$.

Let $\mathcal{M}$ be a Hilbert $A$-module (see, for example, [10]). We denote by $\text{End}_A(\mathcal{M})$ the Banach algebra of all bounded $A$-homomorphisms of $\mathcal{M}$. Now let us consider some strongly continuous representation $G \to \text{End}_A(\mathcal{M})$. Then this representation is called unitary, if $\langle gx, gy \rangle = \langle x, y \rangle$ for any $g \in G, x, y \in \mathcal{M}$. A Hilbert $A$-module together with a unitary representation of the group $G$ is called a Hilbert $G$-$A$-module. Besides, a set $\{ x \}_{\beta \in B} \subset \mathcal{M}$ is a system of generators for $\mathcal{M}$, if finite sums $\{ \sum_k x_k a_k : a_k \in A \}$ are dense in $\mathcal{M}$.

We need the following result of [7].

**Theorem 7.** Let $\mathcal{M}$ be a countably generated Hilbert $G$-$A$-module. Besides, let $\{ V_\pi \}$ be a full system of pairwise not isomorphic unitary finite-dimensional irreducible representations for $G$. Then there exists a $G$-$A$-isomorphism

$$\mathcal{M} \cong \bigoplus_\pi \text{Hom}_G(V_\pi, \mathcal{M}) \otimes_\mathbb{C} V_\pi.$$
Here the algebra $A$ (the group $G$) acts on the first (on the second) multiplier of the space $\text{Hom}_G(V_\pi, \mathcal{M}) \otimes C V_\pi$. □

Let the $G$-$A$-module $\mathcal{M}$ belong to the class $\mathcal{P}(A)$. Then it is clear that $\mathcal{M}_\pi := \text{Hom}_G(V_\pi, \mathcal{M}) \in \mathcal{P}(A)$. Furthermore, only finite number of terms in the sum $\bigoplus \mathcal{M}_\pi \otimes C V_\pi$ is not equal to zero (see [13, 1.3.49]).

In particular, we obtain for $\mathcal{M} = A^n$ the following formula

$$A^n \cong \bigoplus_{k=1}^{M} Q_k \otimes V_k,$$

where $V_k \cong C^{L_k}$ and $Q_k \in \mathcal{P}(A)$. Therefore,

$$U_g(\sum_{k=1}^{M} x_k \otimes v_k) = \sum_{k=1}^{M} x_k \otimes u^k_g v_k = \sum_{k=1}^{M} \sum_{s=1}^{L_k} x_k \otimes e^{ix_k} v_k^s f_s. \quad (11)$$

Here $f_1, \ldots, f_{L_k}$ is a basis for $V_k$ such that the operator $u^k_g$ is diagonal with respect to it; $v_k = \sum v_k^s f_s$. In this case let us define

$$\tau(U_g) = \sum_{k=1}^{M} Ch_0^{2l}(Q_k) \cdot \text{Trace}(u^k_g) \in HC_{2l}(A).$$

The following result was proved in [3].

**Lemma 5.** For the $A$-Fredholm operator $F = d + d^*: \Gamma(E_{ev}) \longrightarrow \Gamma(E_{od})$ there exists a decomposition

$$F: M_0 \oplus \tilde{N}_0 \longrightarrow M_1 \oplus \tilde{N}_1, \quad F: M_0 \cong M_1 \quad (12)$$

such that

$$\tilde{N}_0 = \bigoplus_{j=0}^{T} N_{2j}, \quad \tilde{N}_1 = \bigoplus_{j=1}^{T} N_{2j-1}, \quad N_m \subset \Gamma(E_m),$$

where $N_m$ are projective $U$-invariant Hilbert $A$-modules. □

Now the Lefschetz number of the second type is defined as follows

$$L_{2l}(E, U_g) = \sum_{j} (-1)^j \tau(U_g|N_j) \in HC_{2l}(A).$$

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This definition is well.

For more detail about $W^*$-Lefschetz numbers we refer to the works [3], [10], [17].

Now let us consider an $\mathcal{A}$-elliptic complex $(E, d)$ and an arbitrary unitary endomorphism $U$ of it ($U$ is not necessarily an element of some representation of $G$). In this situation let us formulate the following

**Definition 4.** We define the generalized Lefschetz number $L_1$ as follows

$$L_1(E, U) = \sum_j (-1)^j [U|N_j] \in N_0(A).$$

Note that generalized Lefschetz numbers are well defined. This follows by the same reason that for the Lefschetz numbers of the second type (see [13], 5.2.21).

**Theorem 8.** Let $U$ be a unitary endomorphism of an $\mathcal{A}$-elliptic complex $(E, d)$. Besides, suppose that $U = U_g$ for some representation $U_g$ of a compact Lie group $G$. Then $L_1(E, U_g)$ belongs to the group $N_0(A)_{\text{fin}}$ and $h(L_1(E, U_g)) = L_1(E, U_g)$, where $h$ is map (1).

**Proof.** Let us examine decomposition (12) for $F$. We have shown above that there exist isomorphisms

$$N_{2j} \cong \bigoplus_{k_j=1}^{K_j} P^{(j)}_{k_j} \otimes V^{(j)}_{k_j}, \quad N_{2j-1} \cong \bigoplus_{l_j=1}^{L_j} Q^{(j)}_{l_j} \otimes W^{(j)}_{l_j},$$

where $V^{(j)}_{k_j}$ and $W^{(j)}_{l_j}$ are complex vector spaces of irreducible unitary representations of $G$, $P^{(j)}_{k_j}$ and $Q^{(j)}_{l_j}$ are $G$-trivial modules from $\mathcal{P}(A)$. Thus we get

$$\text{index}(F) = \sum_{j=0}^{T} \sum_{k_j=1}^{K_j} [P^{(j)}_{k_j}] \otimes \chi(V^{(j)}_{k_j}) - \sum_{j=1}^{T} \sum_{l_j=1}^{L_j} [Q^{(j)}_{l_j}] \otimes \chi(W^{(j)}_{l_j})$$

and

$$L_1(E, g) = \sum_{j=0}^{T} \sum_{k_j=1}^{K_j} [P^{(j)}_{k_j}] \otimes \text{Trace}(g|V^{(j)}_{k_j}) - \sum_{j=1}^{T} \sum_{l_j=1}^{L_j} [Q^{(j)}_{l_j}] \otimes \text{Trace}(g|W^{(j)}_{l_j}).$$

Here we have denoted by $\chi$ the character of the representation of $G$. 

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On the other hand, using expression (11), we conclude that
\[ \text{sp}(U_g|P_{k_j}^{(j)} \otimes V_{k_j}^{(j)}) = \text{sp}(u^{(j),k_j}_g|V_{k_j}^{(j)}). \]
This implies that \( \mathcal{L}_1(E,U_g) \) belongs to the group \( N_0(A)_{\text{fin}} \). Furthermore, for any \( e^{i\varphi} \) from \( \text{sp}(u^{(j),k_j}_g|V_{k_j}^{(j)}) \) the spectral projection of \( U_g \) corresponding to this point is equal to \( P_{k_j}^{(j)} \). Thus we obtain
\[
\begin{align*}
  h(\mathcal{L}_1(E,U_g)) &= h(\sum_{j=0}^{2T} (-1)^j [U_g|N_j]) \\
  &= h(\sum_{j=0}^{2T} \sum_{k_j=1}^{K_j} [Id_{p_{k_j}^{(j)}} \otimes u^{(j),k_j}_g|V_{k_j}^{(j)}]) \\
  &\quad - \sum_{j=1}^{T} \sum_{l_j=1}^{L_j} [Id_{Q_{l_j}^{(j)}} \otimes u^{(j),l_j}_g|W_{l_j}^{(j)}]) \\
  &= h(\sum_{j=0}^{2T} \sum_{k_j=1}^{K_j} [P_{k_j}^{(j)} \text{Trace}(u^{(j),k_j}_g|V_{k_j}^{(j)})] \\
  &\quad - \sum_{j=1}^{T} \sum_{l_j=1}^{L_j} [Q_{l_j}^{(j)} \text{Trace}(u^{(j),l_j}_g|W_{l_j}^{(j)})]) \\
  &= \sum_{j=0}^{2T} \sum_{k_j=1}^{K_j} [P_{k_j}^{(j)} \otimes \text{Trace}(u^{(j),k_j}_g|V_{k_j}^{(j)})] \\
  &\quad - \sum_{j=1}^{T} \sum_{l_j=1}^{L_j} [Q_{l_j}^{(j)} \otimes \text{Trace}(u^{(j),l_j}_g|W_{l_j}^{(j)})].
\end{align*}
\]
Thus we establish the required statement.

**Theorem 9.** Under the assumptions of the previous theorem, we have
\[ \pi_*(L_{2l}(E,U_g)) = \text{Ch}_{2l}^0(\mathcal{L}_1(E,U_g)). \]
Here \( \pi_* \) is map (3).

**Proof.** From Theorems 6, 8 we deduce that
\[ \text{Ch}_{2l}^0(\mathcal{L}_1(E,U_g)) = \pi_* \text{Ch}_{2l}^0(h(\mathcal{L}_1(E,U_g))) = \pi_* \text{Ch}_{2l}^0(\mathcal{L}_1(E,g)). \]
Furthermore, it follows from [13, Theorem 5.2.22] that \( \text{Ch}_{2l}^0(L_1(E,g)) = L_{2l}(E,U_g) \). The proof is complete.

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References

[1] Bratteli O., Robinson D. Operator algebras and quantum statistical mechanics. — New York – Heidelberg – Berlin, 1979.

[2] Connes A. Non-commutative differential geometry. \(\text{Publ. Math. I.H.E.S. — 1985, V. 62, 41-144.}\)

[3] Frank M., Troitsky E.V. Lefschetz numbers and geometry of operators in \(W^*-\)modules. \(\text{Func. Anal. i Pril. — 1996, V.30, N 4, 45-57 (in Russian); English transl. in Funct. Anal. Appl. — 1996, V.30, 257-266.}\)

[4] Helemskii A.Ya. Banach cyclic (co)homology and the Connes-Tzygan exact sequence. \(\text{London Math. Soc. — 1992, V. 46, N 2, 449-462.}\)

[5] Helemskii A.Ya. Banach cyclic (co)homology in terms of Banach derived functors. \(\text{St. Petersburg Math. J. — 1992, V.3, N 5, 1149-1164.}\)

[6] Loday J.-L. Cyclic homology. — Springer-Verlag, 1992.

[7] Mishchenko A.S. Representations of compact groups on Hilbert modules over \(C^*\)-algebras. \(\text{Trudy Math. Inst. im. V.A. Steklova. — 1984, V.166, 161-176 (in Russian); English transl. in Proc. Steklov Inst. Math. — 1986, V.166, 179-195.}\)

[8] Mishchenko A.S., Fomenko A.T. The index of elliptic operators over \(C^*\)-algebras. \(\text{Izv. Akad. Nauk USSR, Ser. Math. — 1979, V. 43, N 4, 831-859 (in Russian); English transl. in Math. USSR-Izv. — 1980, V. 15, 87-112.}\)

[9] Murphy G.J. \(C^*\)-algebras and operator theory. — Academic Press, 1990.

[10] Paschke W.L. Inner product modules over \(B^*\)-algebras \(\text{Trans. Amer. Math. Soc. — 1973, V. 182, 443-468.}\)

[11] Pavlov A.A. Generalized Lefschetz numbers of unitary endomorphisms of \(W^*\)-elliptic complexes. \(\text{International Conference Dedicated to the 80th Anniversary of V.A. Rokhlin, August 19-25, 1999, St. Petersburg. Abstracts, 56-58.}\)
[12] *Pavlov A.A.* The functor $N_0$ over the category of von Neumann algebras and its relation to the operator K-theory. \( \text{Vestnik Mosc. Univ. Ser. 1. Mat. Meh.} \) — 2000, N 4, to appear (in Russian); English transl. in Moscow Univ. Math. Bull.

[13] *Solovyov Yu.P., Troitsky E.V.* C*-algebras and elliptic operators in differential topology. — Moscow: Factorial Publish., 1996 (in Russian); English transl. in Amer. Math. Soc., to appear.

[14] *Troitsky E.V.* The equivariant index of elliptic operators over C*-algebras. \( \text{Izv. Akad. Nauk USSR, Ser. Math.} \) — 1986, V.50, N 4, 849-865 (in Russian); English transl. in Math. USSR-Izv. — 1987, V.29, 207-224.

[15] *Troitsky E.V.* Lefschetz numbers of C*-complexes. \( \text{Springer Lecture Notes in Math.} \) — 1991, V.1474, 193-206.

[16] *Troitsky E.V.* Traces, Lefschetz numbers of C*-elliptic complexes and the even cyclic homology. \( \text{Vestnik Mosc. Univ. Ser. 1. Mat. Meh.} \) — 1993, N 5, 36-39 (in Russian); English transl. in Moscow Univ. Math. Bull. — 1993, V.48, N 5.

[17] *Troitsky E.V.* Orthogonal complements and endomorphisms of Hilbert modules and C*-elliptic complexes. \( \text{Novikov Conjectures, Index Theorems and Rigidity, V.2} \) (London Math. Soc. Lect. Notes Series V.227), 1995, 309-331.

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