Rim Hook Tableaux and Kostant’s $\eta$-Function Coefficients

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Abstract

Using a 0/1 encoding of Young diagrams and its consequences for rim hook tableaux, we prove a reduction formula of Littlewood for arbitrary characters of the symmetric group, evaluated at elements with all cycle lengths divisible by a given integer. As an application, we find explicitly the coefficients in a formula of Kostant for certain powers of the Dedekind $\eta$-function, avoiding most of the original machinery.

1 Introduction

A well known classical result due to Kostant [6], based on previous work of Macdonald [8], expresses certain powers of the Dedekind $\eta$-function as power series, summing over the irreducible representations of suitable Lie groups. An important factor in these expressions was shown to obtain only the values 0, 1, and $-1$ when the group is simply laced. The proof used the representation theory of Lie groups. Stated explicitly (for explanation of notations see Subsection 5.1 below), the result is

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Theorem 1.1 [6, Theorem 1] For any simple, simply connected and simply laced compact Lie group $K$,
\[ \phi(x)^{\dim K} = \sum_{\lambda \in D} \epsilon(\lambda) \cdot \dim V_{\lambda} \cdot x^{c(\lambda)} \]
and also \[ \epsilon(\bar{\lambda}) \in \{0, 1, -1\} \quad (\forall \bar{\lambda} \in D). \]

In this paper we give an explicit computation of $\epsilon(\bar{\lambda})$ for $K = SU(k)$ (i.e., Lie type A), leading to a very clear description of those $\bar{\lambda} \in D$ for which $\epsilon(\bar{\lambda}) \neq 0$. Explicitly, our main result is

Theorem 1.2 (see Corollary 5.9) Let $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_{k-1})$ be a dominant $SU(k)$-weight, and define $\bar{\lambda}_k := 0$. If the numbers $(\bar{\lambda}_i + k - i)_{i=1}^k$ have $k$ distinct residues (mod $k$) then $\epsilon(\bar{\lambda})$ is the sign of the permutation needed to transform the sequence of residues (mod $k$) of $(\bar{\lambda}_i - \bar{p} + k - i)_{i=1}^k$ into the sequence $(k-i)_{i=1}^k$, where $\bar{p} := (\bar{\lambda}_1 + \ldots + \bar{\lambda}_k)/k$. In all other cases, $\epsilon(\bar{\lambda}) = 0$.

The proof uses combinatorial tools related to the symmetric group. The connection is made via the Schur-Weyl double commutant theorem. Our arguments use the 0/1 encoding of partitions, a corresponding interpretation of rim hook tableaux, and a result (known to Littlewood) concerning the computation of $S_n$-character values on permutations with all cycle lengths divisible by a given integer.

The structure of the paper is as follows. Section 2 contains some preliminaries. In Section 3 we describe the 0/1 encoding of partitions and the concepts of $k$-quotient and $k$-core. Section 4 deals with rim hook tableaux; the zero permutation and the $k$-quotient of a rim hook tableau are defined and related to each other. Finally, in Section 5 we compute Kostant’s coefficients for type A.

2 Preliminaries

In this section we collect some definitions, notation and results that will be used in the rest of this paper.
2.1 Partitions and Characters

A partition is a weakly decreasing infinite sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of non-negative integers $\lambda_1 \geq \lambda_2 \geq \ldots \geq 0$ in which only finitely many terms are nonzero. Its size is

$$|\lambda| := \sum_i \lambda_i$$

and its length is

$$\ell(\lambda) := \max \{i \geq 1 \mid \lambda_i > 0\}.$$ 

By definition, the empty partition $\emptyset = (0, 0, \ldots)$ has length (and size) zero.

If $|\lambda| = n$ then we say that $\lambda$ is a partition of $n$, and write $\lambda \vdash n$.

A partition $\mu$ with $m_1$ “1”s, $m_2$ “2”s, etc., can also be written in multiset notation: $\mu = (\ldots 2^m 1^m \ldots)$. For such a partition, let

$$z_\mu := m_1! 1^{m_1} \cdot m_2! 2^{m_2} \cdot \ldots.$$ 

Naturally, $z_\mu := 1$ for the empty partition $\mu = \emptyset$.

The cycle type of a permutation $\pi \in S_n$ is the multiset of sizes of cycles in $\pi$. We can (and will) order this multiset into a partition of $n$.

**Proposition 2.1** If $\mu$ is a partition of $n$ then the number of permutations in $S_n$ having cycle-type $\mu$ is $n!/z_\mu$.

Let $\chi^\lambda$ be the irreducible character of $S_n$ indexed by $\lambda \vdash n$. For $\mu \vdash n$, let $\chi^\lambda_\mu$ be the value of that character on an element of $S_n$ having cycle type $\mu$. The orthogonality property of characters, together with Proposition 2.1, yield

**Proposition 2.2** For any partition $\lambda$ of $n$ ($n \geq 0$),

$$\sum_{\mu \vdash n} \frac{1}{z_\mu} \chi^\lambda_\mu = \frac{1}{n!} \sum_{\sigma \in S_n} \chi^\lambda(\sigma) = \delta_{\lambda, (n)},$$

where $\delta$ is the Kronecker delta and $(n)$ is the partition of $n$ with (at most) one part.
2.2 Weak Compositions and Rim Hook Tableaux

The definitions below follow [12, pp. 345–347].

A weak composition of a nonnegative integer \( n \) is an infinite sequence \( \mu \), not necessarily decreasing, of nonnegative integers adding up to \( n \). We write \( |\mu| = n \). If \( \mu = (\mu_1, \mu_2, \ldots) \) is a weak composition and \( k \) is a positive integer, let \( k\mu := (k\mu_1, k\mu_2, \ldots) \).

A rim hook (or border strip, or ribbon) is a connected skew shape containing no \( 2 \times 2 \) square. The length of a rim hook is the number of boxes in it, and its height is one less than its number of rows. By convention, the height of an empty rim hook is zero.

Let \( \lambda \) be a partition and \( \mu = (\mu_1, \mu_2, \ldots) \) be a weak composition of the same integer \( n \). A rim hook tableau of shape \( \lambda \) and type \( \mu \) is an assignment of positive integers to the boxes of the Young diagram of \( \lambda \) such that

1. every row and column is weakly increasing;
2. each positive integer \( i \) appears \( \mu_i \) times; and
3. the set of boxes occupied by \( i \) forms a (possibly empty) rim hook.

Equivalently, we may think of a rim hook tableau of shape \( \lambda \) as a sequence \( \emptyset = \lambda^0 \subseteq \lambda^1 \subseteq \cdots \subseteq \lambda^r = \lambda \) of partitions such that each skew shape \( \lambda^i/\lambda^{i-1} \) is a rim hook of length \( \mu_i \) (including the empty rim hook \( \emptyset \), when \( \mu_i = 0 \)). The height \( \text{ht}(T) \) of a rim hook tableau \( T \) is the sum of heights of all (nonempty) rim hooks in it.

Let \( \text{RHT}^\lambda_\mu \) be the set of all rim hook tableaux of shape \( \lambda \) and type \( \mu \). For example,

\[
T = \begin{bmatrix}
1 & 1 & 4 \\
3 & 4 & 4 \\
3
\end{bmatrix} \in \text{RHT}^\lambda_\mu
\]

where \( \lambda = (3, 3, 1) \) and \( \mu = (2, 0, 2, 3) \).

Proposition 2.3 (Murnaghan-Nakayama Formula) [5, p. 60] For any partition \( \lambda \) and weak composition \( \tilde{\mu} \) of the same integer,

\[
\sum_{T \in \text{RHT}^\lambda_{\tilde{\mu}}} (-1)^{\text{ht}(T)} = \lambda^\lambda_\mu
\]

where \( \mu \) is the unique partition obtainable from \( \tilde{\mu} \) by reordering the parts.
It follows that the sum on the left-hand side is invariant under reordering
the weak composition $\tilde{\mu}$. The size of the set $RHT^\lambda_\tilde{\mu}$ itself is, however, not
invariant; this set may even be empty for some orderings and nonempty for
others.

3 The 0/1 Encoding of Partitions

In this section we shall describe an encoding of partitions (or Young dia-
grams) in terms of 0/1 sequences, as well as the concepts of $k$-quotient and
$k$-core of a partition. This will lead, in Section 4, to a very transparent
description of rim hook tableaux. This description, in turn, will be used in
Section 5 to give a reduction of a character sum over a coset in a large group
to a similar sum in a smaller group, via the Murnaghan-Nakayama character
formula. For historical and other remarks regarding the 0/1 encoding, see
Subsection 3.5 below.

3.1 The 0/1 Sequence of a Partition

Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a partition of length $n$, so that $\lambda_1 \geq \ldots \geq \lambda_n >
0 = \lambda_{n+1} = \ldots (n \geq 0)$. This is usually denoted graphically by a Young

diagram, which has $n$ left-justified rows of boxes, with $\lambda_i$ boxes in row $i$
$(1 \leq i \leq n)$. Extend the South-Eastern borderline of the diagram by a
vertical ray extending down (South) from the South-Western corner, and a
horizontal ray extending to the right (East) from the North-Eastern corner.
Denote the South-Eastern borderline, together with these two rays, by $\partial \lambda$,
and call it the extended borderline of $\lambda$.

Now trace the edges of $\partial \lambda$, starting from the vertical ray and ending at
the horizontal ray. Each step is either up (North) or to the right (East).
Encode this walk by a 0/1 sequence $s(\lambda)$, where each “up” step is encoded
by “0” and each “right” step is encoded by “1”. The resulting doubly-

infinite sequence is initially all “0”s (corresponding to the vertical ray) and
eventually all “1”s (corresponding to the horizontal ray).

Each digit (“0” or “1”) in the sequence corresponds to an edge of the
extended borderline $\partial \lambda$, and each “space” between digits corresponds to a
vertex of $\partial \lambda$. A special “space” in the sequence corresponds to the unique
vertex of $\partial \lambda$ which is on the same diagonal as the North-Western corner of
$\lambda$. Call this space the median of $s(\lambda)$ and denote it by “|”; it is the unique
Many natural parameters of a partition \( \lambda \) can be easily described using \( s(\lambda) \). The “0”s of \( s(\lambda) \) correspond (in reverse order) to the rows of the diagram of \( \lambda \). The length \( \lambda_i \) of the \( i \)th row is the number of “1”s to the left of the \( i \)th (from the right) “0” in \( s(\lambda) \). In particular, the empty rows correspond to the infinite sequence of leading “0”s in \( s(\lambda) \). Similarly, the length \( \lambda'_j \) of the \( j \)th column is the number of “0”s to the right of the \( j \)th (from the left) “1” in \( s(\lambda) \). The length \( \ell(\lambda) = \lambda'_1 \) of \( \lambda \) is the number of “0”s to the right of the first “1” in \( s(\lambda) \). The size of \( \lambda \) is the number of inversions in \( s(\lambda) \):

\[
|\lambda| = \#\{(i, j) \mid i < j, s_i = 1, s_j = 0\}.
\]

The side-length \( r \) of the Durfee square of \( \lambda \) (the largest square contained in the diagram of \( \lambda \)) is the value of either side of equation (1). The Frobenius notation \( \lambda = (\alpha_1, \ldots, \alpha_r \mid \beta_1, \ldots, \beta_r) \) is obtained by letting \( \alpha_i \) (respectively, \( \beta_i \)) be the distance from the median to the \( i \)th from the right (left) “0” (“1”); only “0”s to the right (left) of the median are taken into account here. The empty partition corresponds, of course, to the sequence \( \ldots 00|11 \ldots \).

### 3.2 The \( k \)-Quotient

For an integer \( k \geq 2 \), define the \( k \)-quotient of a partition \( \lambda \) by

\[
\text{quot}_k(\lambda) := (\lambda_1, \ldots, \lambda_k),
\]

where \( \lambda_i \) (\( 1 \leq i \leq k \)) is the partition whose 0/1 sequence \( s(\lambda_i) \) is the subsequence containing every \( k \)th digit of \( s(\lambda) \), including the \( i \)th digit after the median of \( s(\lambda) \).

**Example 3.1** Let \( \lambda = (4, 4, 3, 2) \) and \( k = 3 \). Then \( \text{quot}_3(\lambda) = (\lambda_1, \lambda_2, \lambda_3) \) where \( \lambda_1 = (1, 1) \), \( \lambda_2 = \emptyset \) and \( \lambda_3 = (2) \), since

\[
s(\lambda) = \ldots 001101|010011 \ldots \rightarrow \ldots 00|11 \ldots = s(\lambda_1)
\]

\[
s(\lambda) = \ldots 001101|010011 \ldots \rightarrow \ldots 00|11 \ldots = s(\lambda_2)
\]

\[
s(\lambda) = \ldots 11|01 \ldots = s(\lambda_3)
\]

Note that the medians shown in the example are the ones “inherited” from \( s(\lambda) \), and do not necessarily coincide with the intrinsically defined
medians of the various $s(\lambda_i)$. Actually, in order to reconstruct $\lambda$ from its $k$-quotient we need to know the relative positions of the medians of the various subsequences $s(\lambda_i)$. This information is provided by the $k$-core, to be defined in the sequel.

### 3.3 Removing a Rim Hook

Removing a rim hook of length $k$ from a Young diagram $\lambda$ is equivalent to finding a “1” and a “0” at distance $k$ apart in $s(\lambda)$, with the “1” preceding the “0”, and interchanging these two digits. The height $h$ of the rim hook is equal to the number of “0”s strictly between the two digits to be interchanged.

**Example 3.2** Denoting by “$\hat{1}$” and “$\hat{0}$” the two digits to be interchanged, here is an example of removing a rim hook of length 4 and height 1:

\[
\ldots 00\hat{1}101|010011\ldots \rightarrow \ldots 00\hat{0}101|\hat{1}10011\ldots
\]

It follows that removing from $\lambda$ a rim hook of length divisible by $k$ affects only one of the subsequences $s(\lambda_1), \ldots, s(\lambda_k)$, where $(\lambda_1, \ldots, \lambda_k) = \text{quot}_k(\lambda)$. In particular, removing a rim hook of length exactly $k$ is equivalent to a transposition $10 \rightarrow 01$ of consecutive digits in one of these subsequences. Note also that, by condition (1), this operation changes neither the position of the median of $s(\lambda)$ nor the intrinsic median of any subsequence $s(\lambda_i)$.

### 3.4 The $k$-Core

The $k$-core of a partition $\lambda$, $\text{core}_k(\lambda)$, is the unique partition obtained from $\lambda$ by removing as many rim hooks of length $k$ as possible. By the remarks in the previous subsection, this corresponds to “squeezing” all the “0”s to the left (and all the “1”s to the right) in each of the subsequences $s(\lambda_1), \ldots, s(\lambda_k)$ separately. This interpretation shows that the $k$-core is well-defined, i.e., does not depend on the choice of rim hooks to be removed. It also follows that the $k$-core has trivial $k$-quotient, i.e.,

\[
\text{quot}_k(\text{core}_k(\lambda)) = (\emptyset, \ldots, \emptyset) \quad (\forall \lambda).
\]

Thus, the $k$-core encodes only the relative positions of the intrinsic medians of the various subsequences (mod $k$) of $s(\lambda)$; their “average” is the median of $s(\lambda)$. By the comment at the end of Subsection 3.2 we now get the following result.
Lemma 3.3 [5, Theorem 2.7.30] For any $k \geq 2$, a partition is uniquely determined by its $k$-core and $k$-quotient.

The following consequence of the remarks above is noted here for future reference.

Lemma 3.4 (Empty $k$-Core Criterion)
A partition $\lambda$ has an empty $k$-core if and only if the median of each subsequence (mod $k$) of $s(\lambda)$ coincides with the median of $s(\lambda)$.

Example 3.5 ($k = 2$)
Each of the following two examples shows the maximal rim hook removal from a 0/1 sequence and the corresponding operation on its two subsequences (mod 2).

1. Empty 2-core:

...1011|0001... → ...0000|1111...

...1 1 |0 0 ... → ...0 0 |1 1 ...

...0 1| 0 1... → ...0 0| 1 1...

2. Nonempty 2-core:

...1011|0010... → ...0010|1011...

...1 1 |0 1 ... → ...0 1 |1 1 ...

...0 1| 0 0... → ...0 0| 0 1...

We shall also need the following quantitative relation.

Lemma 3.6 If

$$\text{quot}_k(\lambda) = (\lambda_1, \ldots, \lambda_k)$$

and

$$\text{core}_k(\lambda) = \lambda_0$$

then

$$|\lambda| = |\lambda_0| + k \cdot (|\lambda_1| + \ldots + |\lambda_k|).$$
Proof. Recall expression (2) for the size of a partition in terms of the number of inversions in its 0/1 sequence. Removal of a rim hook of length \(k\) from \(\lambda\) corresponds to an adjacent transposition \(10 \rightarrow 01\) in one of the subsequences \(s(\lambda_i)\), thus reducing by 1 its inversion number (and the size of \(\lambda_i\)), not affecting any of the other subsequences. This removal clearly reduces by \(k\) the size of \(\lambda\), as can also be seen using \(s(\lambda)\).

\[\square\]

Note the following special case, which actually follows directly from the definition of \(k\)-core.

**Corollary 3.7** If \(\lambda\) is a partition with \(\text{core}_k(\lambda) = \emptyset\) then \(k\) divides \(|\lambda|\).

### 3.5 Historical Remarks

The 0/1 encoding is a succinct and useful way to represent a Young diagram. It was rediscovered many times; e.g., by Shahar Mozes and the present authors in the 1980’s (unpublished). See [12, pp. 467, 517] for references, dating from 1959 on. An essentially equivalent technique, using bead configurations and abaci, appears in [5, Ch. 2.7]. Fairy sequences (or, equivalently, content vectors), used for example in [2], are essentially partial sums of 0/1 sequences. The introduction of the median and the statement of Lemma 3.4 (the empty \(k\)-core criterion) seem to be new.

### 4 The Decomposition Theorem

#### 4.1 The \(k\)-Quotient of a Rim Hook Tableau

Using the 0/1 encoding of partitions, a rim hook tableau \(T \in RHT^\lambda\) corresponds to a finite sequence of operations of either the form \(1 \ldots 0 \rightarrow 0 \ldots 1\) or the form “do nothing”, transforming \(s(\lambda)\) into \(s(\emptyset) = \ldots 00|11\ldots\), and whose sequence of lengths is the weak composition \(\mu\) in reverse order. Note that we view \(T\) here as “peeling” rather than “constructing” the diagram \(\lambda\), and that empty rim hooks are allowed. If \(\mu\) is a partition (i.e., weakly decreasing) then rim hooks are removed in order of weakly increasing length.

Let \(\lambda\) be a partition with empty \(k\)-core, and let \(\text{quot}_k(\lambda) = (\lambda_1, \ldots, \lambda_k)\) be its \(k\)-quotient. If \(|\lambda_i| = p_i\) \((1 \leq i \leq k)\) then, by Lemma 3.6, \(p_1 + \ldots + p_k = p\) where \(|\lambda| = kp\). Let \(T\) be a rim hook tableau of shape \(\lambda\), with all rim hook lengths divisible by \(k\). Its type has the form \(k\mu\), with \(\mu\) a weak composition
of $p$. In the sequel we shall usually assume, for simplicity, that $\mu$ is actually a partition. Clearly, for any $t \geq 0$, removing a rim hook of length $kt$ from $\lambda$ is equivalent to removing a rim hook of length $t$ from a suitable $\lambda_i$. The peeling of $T$ thus corresponds to peeling $k$ rim hook tableaux $T_1, \ldots, T_k$. If $\mu$ is a partition then the type of each $T_i$ is also a partition. We can then define a mapping

$$\omega_{k\mu}^\lambda : RHT_{k\mu}^\lambda \rightarrow \bigcup_{(\mu_1, \ldots, \mu_k) \in P^\lambda_\mu} \left( RHT_{\mu_1}^{\lambda_1} \times \cdots \times RHT_{\mu_k}^{\lambda_k} \right)$$

where $P^\lambda_\mu$ is the set of all $k$-tuples $(\mu_1, \ldots, \mu_k)$ of partitions satisfying

$$|\mu_i| = |\lambda_i| \quad (1 \leq i \leq k)$$

and

$$\mu_1 \oplus \cdots \oplus \mu_k = \mu.$$

Here the direct sum of $\mu_1, \ldots, \mu_k$ is the partition obtained by reordering (in weakly decreasing order) all the parts of all the $\mu_i$.

Recall the definition of $z_\mu$ from Subsection 2.1.

**Lemma 4.1** If $\lambda$ and $\mu$ are partitions satisfying $\text{core}_k(\lambda) = \emptyset$ and $|\lambda| = k|\mu|$ then the map $\omega_{k\mu}^\lambda$ defined above is surjective, and each $k$-tuple

$$(T_1, \ldots, T_k) \in RHT_{\mu_1}^{\lambda_1} \times \cdots \times RHT_{\mu_k}^{\lambda_k}$$

in its range is obtained from exactly $z_\mu / (z_{\mu_1} \cdots z_{\mu_k})$ elements (rim hook tableaux) in its domain.

**Proof.** Let $\mu = (\ldots 2^{m_2} 1^{m_1})$. Given $(T_1, \ldots, T_k)$ we can reconstruct $T$ by successively adding the rim hooks of all the $T_i$. The only ambiguity is in deciding, for each $j \geq 1$, which of the $m_j$ rim hooks of length $j$ in $T$ should come from each $T_i$ ($1 \leq i \leq k$). The number of possibilities is a product (over $j$) of multinomial coefficients, and boils down to the claimed formula. $\Box$
4.2 The Zero Permutation

Let $\lambda$ be a partition. Label the zeros in $s(\lambda)$ by distinct labels, say by the positive integers. It will actually suffice to label only the finitely many zeros corresponding to non-empty rows of $\lambda$, i.e., the zeros succeeding the first “1” in $s(\lambda)$. Let $\ell(\lambda)$ be the length of $\lambda$, and let $T$ be a rim hook tableau of shape $\lambda$ and arbitrary type $\mu$. The zero permutation of $T$ is the permutation $\pi_T \in S_{\ell(\lambda)}$ of the labeled zeros in the sequence $s(\emptyset)$ obtained from $s(\lambda)$ by successively removing rim hooks according to $T$, namely in order of decreasing entries.

**Example 4.2** Let

$$T = \begin{array}{ccc}
1 & 1 & 4 \\
3 & 4 & 4 \\
3 & & \\
\end{array} \in RHT^\lambda_{\mu}$$

where $\lambda = (3, 3, 1)$ and $\mu = (2, 0, 2, 3)$. Label the zeros in $s(\lambda)$, and remove rim hooks from $\lambda$ according to $T$ (starting with the rim hook with entries equal to 4):

$$\begin{array}{c}
\ldots 10_1 \hat{1} | 10_2 0_3 \ldots \xrightarrow{4} \ldots \hat{1} 0_1 \hat{0}_3 | 10_2 1 \ldots \xrightarrow{3} \ldots 0_3 0_1 1 | 10_2 1 \\
\xrightarrow{2} \ldots 0_3 0_1 \hat{1} | \hat{1}_0 2 1 \ldots \xrightarrow{1} \ldots 0_3 0_1 0_2 | 11 1 \ldots ;
\end{array}$$

hence $\pi_T = 312 \in S_3$.

Recall the definition of $ht(T)$ from Subsection 2.2.

**Lemma 4.3** The sign of the zero permutation

$$\text{sign}(\pi_T) = (-1)^{ht(T)}.$$  

**Proof.** Recall that

$$\text{sign}(\pi_T) = (-1)^{\text{inv}(\pi_T)},$$

where the inversion number of a permutation $\pi$ is

$$\text{inv}(\pi) := \#\{(i, j) \mid i < j, \pi(i) > \pi(j)\}.$$ 

Both $\text{inv}(\pi_T)$ and $ht(T)$ are equal to zero for the empty tableau $T = \emptyset$. They both change by $h(\text{mod} \ 2)$ when we remove from a tableau $T$ a nonempty rim hook of length $k$ and height $0 \leq h \leq k - 1$ (since $\pi_T$ records
only the zeros of $s(\lambda)$, and during the rim hook removal one of these zeros "skips" exactly $h$ others). Therefore

$$\text{inv}(\pi_T) \equiv \text{ht}(T) \pmod{2}$$

and the proof is complete.

\[\square\]

**Corollary 4.4** If $T$ is a rim hook tableau of type $(1^n)$ (i.e., a standard Young tableau) then $\text{sign}(\pi_T) = 1$.

**Proof.** The height of a rim hook of length 1 is 0, so that $\text{ht}(T) = 0$ here. Indeed, the permutation $\pi_T$ is the identity permutation, since removing a rim hook of length 1 is an adjacent transposition $10 \rightarrow 01$ in $s(\lambda)$ (where $\lambda$ is the shape of $T$), so does not affect the order of zeros.

\[\square\]

### 4.3 The Sign Decomposition Theorem

This subsection is devoted to the proof of Theorem 4.6 below. An essentially equivalent formulation of this theorem, in terms of character values, was given by Littlewood [7, pp. 143–146].

We first state an important special case as a lemma.

**Lemma 4.5** Let $\lambda$ be a partition with empty $k$-core, and let $p := |\lambda|/k$. Then, for any $T \in \text{RHT}^\lambda_{(kp)}$, the zero permutation $\pi_T$ depends only on $\lambda$ (and not on the choice of $T$), and will be denoted $\pi_\lambda$. In particular,

$$\chi^\lambda_{(kp)} = \text{sign}(\pi_\lambda) \cdot |\text{RHT}^\lambda_{(kp)}|.$$

**Proof.** $T$ is of shape $\lambda$ and type $(kp)$. Removing a rim hook of length $k$ from $s(\lambda)$ corresponds to an adjacent transposition $10 \rightarrow 01$ in a suitable subsequence $s(\lambda_i)$, and does not change the zero permutation in this subsequence. Thus, the effect of removing all the rim hooks in $T$ is merely to shift, in each subsequence $s(\lambda_i)$, all the zeros as much to the left as possible, without changing their order. The resulting zero permutation $\pi_T$, therefore, depends only on the partition $\lambda$ and not on the specific choice of $T$. Denote this permutation by $\pi_\lambda$. The claimed formula follows from Proposition 2.3 (with $\mu = \tilde{\mu} = (kp)$), Lemma 4.3 and the definition of $\pi_\lambda$.

\[\square\]
Theorem 4.6 (Sign Decomposition Theorem)
Let $\lambda$ be a partition with empty $k$-core, $\mu$ a partition of $p := |\lambda|/k$, and $T \in RHT_{k\mu}^\lambda$. If $\omega_{k\mu}^\lambda(T) = (T_1, \ldots, T_k)$ then

$$
\text{sign}(\pi_T) = \text{sign}(\pi_\lambda) \cdot \text{sign}(\pi_{T_1}) \cdot \cdots \cdot \text{sign}(\pi_{T_k}),
$$

where $\pi_\lambda$ is as defined in Lemma 4.5.

In our proof of this theorem we shall use the following result of Garsia and Stanley [11]. We note that this result may also be proved using 0/1 sequences.

Lemma 4.7 [11, Lemma 7.3] If $H$ is a skew shape which is a rim hook of length $kt$, then there exists a skew rim hook tableau of the shape of $H$ consisting of $t$ rim hooks of length $k$ each.

Corollary 4.8 If $RHT_{k\mu}^\lambda \neq \emptyset$ for some weak composition $\mu$ of size $p := |\lambda|/k$, then this also holds for the partition $\mu = (1^p)$.

Proof of Theorem 4.6. $T$ is a rim hook tableau of shape $\lambda$, with all rim hook lengths divisible by $k$. Consider a rim hook in $T$, of length $kt \ (t \geq 1)$. By Lemma 4.7, we can replace this rim hook by a suitable sequence of $t$ rim hooks of length $k$ each. The resulting set of zero positions is the same, whether we remove the original long rim hook or the $t$ short ones; the two zero permutations, though, differ by a cycle of length $h + 1$, where $h$ is the height of the rim hook of length $t$ corresponding, in one of the tableaux $T_i$, to the original rim hook of length $kt$.

Example 4.9 $k = 2$, $t = 3$, $h = 1$.
Removing one rim hook of length $kt = 6$ from $s(\lambda)$:

$$
\ldots \hat{1}0\hat{1}0_2|03041\ldots \rightarrow \ldots 0_410_2|0311\ldots
$$

Removing the corresponding rim hook of length $t = 3$ and height $h = 1$ from $s(\lambda_1)$:

$$
\ldots \hat{1}0_1|\hat{1}0_4\ldots \rightarrow \ldots 0_40_1|11\ldots
$$

Removing $t = 3$ rim hooks of length $k = 2$ each from $s(\lambda)$:

$$
\ldots \hat{1}\hat{1}0_10_2|03041\ldots \rightarrow \ldots 0_11\hat{1}0_2|03041\ldots \rightarrow \ldots 0_11\hat{1}0_2|040311\ldots
\rightarrow \ldots 0_110_40_2|0311\ldots
$$

The two resulting zero permutations, 4123 and 1423, differ by a cycle of length $h + 1 = 2$ (i.e., a transposition).
It follows that this replacement multiplies both $\text{sign}(\pi_T)$ and the corresponding $\text{sign}(\pi_{T_i})$ by $(-1)^h$, without changing $\text{sign}(\pi_\lambda)$. Iterating this process, we eventually get to the case of $T$ with all rim hooks of the same length $k$, and Lemma 4.5 completes the proof of our claim.

\[\square\]

5 Computation of Kostant’s Coefficients

5.1 Kostant’s Formula

Let us recall Kostant’s formula for powers of the $\phi$-function

\[ \phi(x) := \prod_{n=1}^\infty (1 - x^n), \]

which is a slight modification of the Dedekind $\eta$-function

\[ \eta(x) := x^{1/24} \phi(x). \]

There are nice power-series expansions of $\phi(x)^{-1}$ (the generating function for partition numbers), $\phi(x)$ (due to Euler), $\phi(x)^3$ (due to Jacobi), $\phi(x)^{24}$ (whose coefficients are the values of the Ramanujan $\tau$-function), and others. Note that 3 and 24 are the dimensions of the simple Lie algebras of types $A_1$ and $A_4$ (or the simple compact Lie groups $SU(2)$ and $SU(5)$).

Kostant [6], building on previous work of Macdonald [8], gave an algebraic interpretation to the power-series expansion of certain powers of $\phi(x)$, including the ones mentioned above. We now state his result.

Let $K$ be a compact, simple, simply connected, simply laced Lie group, and let $D$ be its set of dominant weights. For $\tilde{\lambda} \in D$, let $V_{\tilde{\lambda}}$ be an irreducible $K$-module corresponding to $\tilde{\lambda}$; and let

\[ c(\tilde{\lambda}) := (\tilde{\lambda} + \rho, \tilde{\lambda} + \rho) - (\rho, \rho), \]

where $\rho$ is half the sum of all the positive roots. Let $V_{\tilde{\lambda}}^T$ be the zero-weight subspace of $V_{\tilde{\lambda}}$, i.e., the subspace of all vectors which are pointwise invariant under a fixed maximal torus $T$ in $K$. Let $W$ be the Weyl group of $K$, and let

\[ \theta_{\tilde{\lambda}} : W \to \text{Aut} V_{\tilde{\lambda}}^T \]

be the representation of $W$ on $V_{\tilde{\lambda}}^T$. Denote

\[ \epsilon(\tilde{\lambda}) := \text{tr} \theta_{\tilde{\lambda}}(\tau), \]

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where \( \tau \) is any Coxeter element in \( W \). Using these notations, here is the remarkable result of Kostant:

**Theorem 5.1** [6, Theorem 1] For any (simply laced) \( K \) as above,

\[
\phi(x)^{\text{dim } K} = \sum_{\bar{\lambda} \in D} \epsilon(\bar{\lambda}) \cdot \dim V_{\bar{\lambda}} \cdot x^{c(\bar{\lambda})}
\]

and also

\[
\epsilon(\bar{\lambda}) \in \{0, 1, -1\} \quad (\forall \bar{\lambda} \in D).
\]

We shall discuss in this paper the case of type \( A \), i.e., \( K = SU(k) \) (for which the Weyl group \( W = S_k \), the symmetric group), and compute explicitly the value of \( \epsilon(\bar{\lambda}) \) by “elementary” means.

### 5.2 From \( SU(k) \) to \( S_m \)

In this section we express \( \epsilon(\bar{\lambda}) \) as a sum over the symmetric group \( S_m \), for suitable values of \( m \).

We can replace the compact Lie group \( SU(k) \) by the algebraic group \( SL_k \), which has the same representations. In fact, it will be even more convenient to use the (nonsimple) group \( GL_k \). The irreducible representations of \( GL_k \) are indexed by their maximal weights, which may be represented by partitions \( \lambda \) (of an arbitrary nonnegative integer) with \( \ell(\lambda) \leq k \). The corresponding irreducible representation of \( SL_k \) (or \( SU(k) \)) is actually determined by the dominant weight \( \bar{\lambda} = (\lambda_1 - \lambda_k, \lambda_2 - \lambda_k, \ldots, \lambda_k - 1 - \lambda_k) \). The correspondence between irreducible representations of \( GL_k \) and \( SL_k \) is not one-to-one.

Let \( U \) be a \( k \)-dimensional vector space over \( \mathbb{C} \), and consider its \( m \)th tensor power \( U \otimes^m \). This tensor power carries a \( GL_k \)-action, via the natural action of \( GL_k \) on \( U \) (after choosing a basis for \( U \)), and also an \( S_m \)-action \( \rho \) by permuting the factors:

\[
\rho(\sigma)(v_1 \otimes \cdots \otimes v_m) := v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(m)} \quad (\forall \sigma \in S_m, v_1, \ldots, v_m \in U).
\]

The following classical result expresses Schur-Weyl duality between \( S_m \) and \( GL_k \).

**Proposition 5.2** (Double Commutant Theorem) [4, p. 374]

\[
U \otimes^m \cong_{S_m \times GL_k} \bigoplus_{\lambda \in \text{Par}_k(m)} (S^\lambda \otimes V_{\bar{\lambda}})
\]

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is an isomorphism of $(S_m \times \text{GL}_k)$-modules, where $\lambda$ runs through all partitions of $m$ with at most $k$ parts, $S^\lambda$ is the irreducible $S_m$-module corresponding to $\lambda$, and $V^\lambda$ is the corresponding irreducible $\text{GL}_k$-module.

Now let $T$ be a maximal torus in $\text{SL}_k$, and $W = N(T)/T$ the corresponding Weyl group; thus $W \cong S_k$. Let $(U^\otimes m)^T$ be the zero-weight subspace of $U^\otimes m$, consisting of its $T$-invariant vectors. By definition, it carries an action of $W = N(T)/T$.

The following statement is clear.

**Lemma 5.3**

(a) 

$$(U^\otimes m)^T \neq \{0\} \iff k \mid m.$$ 

(b) If $m = kp$ and $\{e_1, \ldots, e_k\}$ is a basis of $U$ consisting of (simultaneous) eigenvectors for $T$, then 

$$B := \{e_{i_1} \otimes \cdots \otimes e_{i_m} \mid \text{as a multiset, } \{i_1, \ldots, i_m\} = \{1^p 2^p \ldots k^p\}\}$$ 

is a basis of $(U^\otimes m)^T$.

Now take 

$$\hat{\tau} = \begin{pmatrix} 0 & \cdots & 0 & (-1)^{k-1} \\ 1 & 0 & 0 \\ \ddots & \ddots & \ddots \\ 1 & 0 \end{pmatrix} \in \text{SL}_k.$$ 

Actually, $\hat{\tau} \in N(T) \subseteq \text{SL}_k$ where $T$ is the maximal torus consisting of diagonal matrices in $\text{SL}_k$. The corresponding vectors $e_1, \ldots, e_k$ form the standard basis of $U \cong \mathbb{C}^k$. The representative of $\hat{\tau}$ in the Weyl group $W = N(T)/T$ corresponds to the Coxeter element $\tau = (1, 2, \ldots, k) \in S_k$ under the natural isomorphism between the group of $k \times k$ permutation matrices and the symmetric group $S_k$. Recall the definitions of 

$$\theta^\lambda : W \longrightarrow \text{Aut} V^T_\lambda$$

and of $\epsilon(\bar{\lambda})$ from Subsection 5.1. Our goal is to prove the claim $\epsilon(\bar{\lambda}) \in \{0, 1, -1\}$ of Theorem 5.1 by “elementary” means, i.e., by using the representation theory of the symmetric group instead of highest weight theory.
For $\lambda \in \text{Par}_k(m)$, let $\chi^\lambda$ be the corresponding irreducible $S_m$-character. The element

$$e_\lambda = \frac{\chi^\lambda(id)}{m!} \sum_{\sigma \in S_m} \chi^\lambda(\sigma)\sigma \in \mathbb{C}S_m$$

is a central idempotent in the group algebra [9, p. 50]. It defines, via the representation $\rho$, a linear operator $\rho(e_\lambda) \in \text{End}_\mathbb{C}(U^{\otimes m})$ which is a projection from $U^{\otimes m}$ onto its $\chi^\lambda$-isotypic component [9, p. 21], isomorphic (by Proposition 5.2) to a direct sum of $\dim V_\lambda$ copies of $S^\lambda$. Note that complex conjugation of character values may (and will) be suppressed in this paper, since all the characters of $S_m$ are real (actually, integer) valued. Also, $\chi^\lambda(id) = \dim S^\lambda$ is the multiplicity of the representation $V_\lambda$ in $U^{\otimes m}$, viewed as a $GL_k$-module.

Let $\bar{\lambda}$ be the dominant weight corresponding to the partition $\lambda$. Since $\dim S^\lambda = \chi^\lambda(id)$,

$$\epsilon(\bar{\lambda}) = \text{tr} \theta_\lambda(\tau) = \text{tr} \left( \hat{\tau}|_{V^T_\lambda} \right) = \frac{1}{\chi^\lambda(id)} \text{tr} \left( \hat{\tau}|_{S^\lambda \otimes V^T_\lambda} \right).$$

The projection $\rho(e_\lambda)$ acts like the identity map on $S^\lambda \otimes V^T_\lambda$, and like the zero map on a complementary subspace of $(U^{\otimes m})^T$. It follows that

$$\epsilon(\bar{\lambda}) = \frac{1}{\chi^\lambda(id)} \text{tr} \left( \hat{\tau}\rho(e_\lambda)|_{(U^{\otimes m})^T} \right).$$

Recalling Lemma 5.3 we may assume that $k \mid m$, say $m = kp$. Choose one of the elements of the basis $B$ for $(U^{\otimes m})^T$, say

$$v_0 := (e_1 \otimes \ldots \otimes e_1) \otimes (e_2 \otimes \ldots \otimes e_2) \otimes \ldots \otimes (e_k \otimes \ldots \otimes e_k),$$

where each of the vectors $e_1, \ldots, e_k$ appears in $p$ consecutive factors. Every other basis element in $B$ has the form $\rho(\sigma)(v_0)$ for some $\sigma \in S_m$. Let the subgroup $H$ of $S_m$ be the stabilizer of $v_0$. Clearly $H \cong S_p \times \ldots \times S_p$ ($k$ factors).

Let $\langle \cdot, \cdot \rangle$ be the inner product on $(U^{\otimes m})^T$ for which the basis $B$ is orthonormal.

**Lemma 5.4** For $\bar{\lambda}$ corresponding to $\lambda \in \text{Par}_k(m)$ as above,

$$\epsilon(\bar{\lambda}) = \frac{1}{|H|} \sum_{\sigma \in S_m} \chi^\lambda(\sigma) \langle \hat{\tau}\rho(\sigma)(v_0), v_0 \rangle.$$
Proof. Let $B$ be the above basis of $(U^\otimes m)^T$. By (4) and (3)
\[
\epsilon(\overline{\lambda}) = \frac{1}{\chi^\lambda(id)\tr\left(\hat{\tau}\rho(e_\lambda)|_{(U^\otimes m)^T}\right)} = \\
= \frac{1}{m!} \sum_{\sigma \in S_m} \chi^\lambda(\sigma) \tr\left(\hat{\tau}\rho(\sigma)|_{(U^\otimes m)^T}\right) = \\
= \frac{1}{m!} \sum_{\sigma \in S_m} \chi^\lambda(\sigma) \sum_{v \in B} \langle \hat{\tau}\rho(\sigma)(v), v \rangle.
\]

$S_m$ acts transitively on the orthonormal basis $B$, with $H$ as the stabilizer of $v_0$; and $\rho(\sigma)$ is a unitary operator, for each $\sigma \in S_m$. Thus
\[
\epsilon(\overline{\lambda}) = \frac{1}{m!} \sum_{\sigma \in S_m} \chi^\lambda(\sigma) \frac{1}{|H|} \sum_{\tilde{\sigma} \in S_m} \langle \hat{\tau}\rho(\sigma)(v_0), \rho(\tilde{\sigma})(v_0) \rangle = \\
= \frac{1}{m!|H|} \sum_{\sigma \in S_m} \sum_{\tilde{\sigma} \in S_m} \chi^\lambda(\sigma) \langle \rho(\tilde{\sigma}^{-1})\hat{\tau}\rho(\sigma\tilde{\sigma})(v_0), v_0 \rangle.
\]
The actions of $\hat{\tau} \in GL_k$ and $\tilde{\sigma} \in S_m$ commute; therefore
\[
\epsilon(\overline{\lambda}) = \frac{1}{m!|H|} \sum_{\sigma \in S_m} \sum_{\tilde{\sigma} \in S_m} \chi^\lambda(\sigma) \langle \hat{\tau}\rho(\tilde{\sigma}^{-1}\sigma\tilde{\sigma})(v_0), v_0 \rangle.
\]
Denoting $\sigma' := \tilde{\sigma}^{-1}\sigma\tilde{\sigma}$, we now get (summing over $\sigma'$ and $\tilde{\sigma}$ instead of $\sigma$ and $\tilde{\sigma}$):
\[
\epsilon(\overline{\lambda}) = \frac{1}{m!|H|} \sum_{\sigma' \in S_m} \sum_{\tilde{\sigma} \in S_m} \chi^\lambda(\tilde{\sigma}\sigma'\tilde{\sigma}^{-1}) \langle \hat{\tau}\rho(\sigma')(v_0), v_0 \rangle = \\
= \frac{1}{|H|} \sum_{\sigma' \in S_m} \chi^\lambda(\sigma') \langle \hat{\tau}\rho(\sigma')(v_0), v_0 \rangle,
\]
as claimed.

\[\Box\]

5.3 From $S_{kp}$ to $S_p^{\times k}$

We shall now show that the summation in Lemma 5.4 is actually not on all of $S_m = S_{kp}$, but rather on a certain coset of the subgroup defined above,
\[H \cong S_p^{\times k} := S_p \times \ldots \times S_p \quad (k \text{ factors}).\]
Lemma 5.5 There exists a permutation $\sigma_0 \in S_{kp}$ such that

$$\epsilon(\bar{\lambda}) = \frac{(-1)^{(k-1)p}}{|H|} \sum_{\sigma \in \sigma_0 H} \chi^\lambda(\sigma),$$

Proof. Write

$$\{1, \ldots, kp\} = C_1 \cup \ldots \cup C_k,$$

where

$$C_i := \{(i-1)p+1, (i-1)p+2, \ldots, (i-1)p+p\} \quad (1 \leq i \leq k).$$

This decomposition has the property that $j \in C_i$ if and only if the $j$th factor in the tensor product (5) is $e_i$.

Now clearly, for any $\sigma \in S_{kp}$

$$\langle \hat{\tau} \rho(\sigma)(v_0), v_0 \rangle \in \{0, (-1)^{(k-1)p}\},$$

where the sign comes from the action of $\hat{\tau}$ on $p$ copies of $e_k$. Indeed, by our choice of $\hat{\tau}$,

$$\hat{\tau}(e_i) = \pm e_{i+1} \quad (1 \leq i \leq k)$$

where indices (here and in the sequel) are computed modulo $k$. Thus $\langle \hat{\tau} \rho(\sigma)(v_0), v_0 \rangle \neq 0$ if and only if $\hat{\tau} \rho(\sigma)(v_0) = \pm v_0$, that is

$$\sigma(C_i) = C_{i+1} \quad (1 \leq i \leq k).$$

This happens if and only if there are $k$ permutations $\sigma_1, \ldots, \sigma_k \in S_p$ such that

$$\sigma((i-1)p+j) = ip + \sigma_i(j) \quad (1 \leq i \leq k, 1 \leq j \leq p).$$

Let $\sigma_0 \in S_m$ correspond to $\sigma_1 = \ldots = \sigma_k = id \in S_p$, so that

$$\sigma_0((i-1)p+j) = ip + j \quad (1 \leq i \leq k, 1 \leq j \leq p).$$

Then

$$\langle \hat{\tau} \rho(\sigma)(v_0), v_0 \rangle \neq 0 \iff \sigma_0^{-1}\sigma(C_i) = C_i \quad (\forall i) \iff \sigma_0^{-1}\sigma \in H,$$

and we conclude from Lemma 5.4 that

$$\epsilon(\bar{\lambda}) = \frac{(-1)^{(k-1)p}}{|H|} \sum_{\sigma \in \sigma_0 H} \chi^\lambda(\sigma),$$

as claimed. \qed
5.4 From $S_p^{\times k}$ to $S_p$

Denote now
\[
\epsilon_1(\lambda) := \frac{1}{|H|} \sum_{\sigma \in \sigma_0 H} \chi^\lambda(\sigma),
\]
so that
\[
\epsilon(\bar{\lambda}) = (-1)^{(k-1)p} \epsilon_1(\lambda).
\]

Recall that each dominant weight $\bar{\lambda}$ may be obtained from many partitions $\lambda \in \text{Par}_k(m)$. There is a minimal such $\lambda = (\lambda_1, \ldots, \lambda_k)$, with $\lambda_k = 0$. All the other partitions are obtained from it by adding the same integer $c$ to each of $\lambda_1, \ldots, \lambda_k$, thus adding $c$ to $p = (\lambda_1 + \ldots + \lambda_k)/k$.

**Lemma 5.6** If $\lambda \vdash kp$ then
\[
\epsilon_1(\lambda) = \sum_{\mu \vdash p} \frac{1}{z_\mu} \chi^{\lambda}_{k\mu}.
\]

In particular, if $\text{core}_k(\lambda) \neq \emptyset$ then $\epsilon_1(\lambda) = 0$.

**Proof.** Recall the definition of $\sigma_0 \in S_{kp}$ from the proof of Lemma 5.5. For each $\sigma \in \sigma_0 H$, $\sigma(C_i) = C_{i+1}$ ($1 \leq i \leq k$) so that all the cycle lengths of $\sigma$ are divisible by $k$. As in the proof of Lemma 5.5, there exist $k$ permutations $\sigma_1, \ldots, \sigma_k \in S_p$ such that
\[
\sigma((i-1)p+j) = ip + \sigma_i(j) \quad (1 \leq i \leq k, 1 \leq j \leq p).
\]

The character value $\chi^\lambda(\sigma)$ depends only on the cycle lengths of $\sigma$, which in turn depend only on the product $\bar{\sigma} := \sigma_k \cdots \sigma_1 \in S_p$. To each cycle of length $t$ in $\bar{\sigma}$ there corresponds a cycle of length $kt$ in $\sigma$.

Define a map $\phi : S_p^{\times k} \to S_p$ by
\[
\phi(\sigma_1, \ldots, \sigma_k) := \sigma_k \cdots \sigma_1 \quad (\forall \sigma_1, \ldots, \sigma_k \in S_p).
\]

Clearly, the size of the inverse image
\[
|\phi^{-1}(\bar{\sigma})| = (p!)^{k-1} = \frac{|H|}{|S_p|} \quad (\forall \bar{\sigma} \in S_p).
\]

If $\bar{\sigma} \in S_p$ has cycle type $\mu$, then each $\sigma \in \sigma_0 H$ corresponding to a member of $\phi^{-1}(\bar{\sigma})$ has cycle type $k\mu$, so that $\chi^\lambda(\sigma) = \chi^{\lambda}_{k\mu}$. By Proposition 2.1, there are $|S_p|/z_\mu$ permutations in $S_p$ of any given cycle type $\mu$, so that
\[
\epsilon_1(\lambda) := \frac{1}{|H|} \cdot \frac{|H|}{|S_p|} \sum_{\mu \vdash p} \frac{|S_p|}{z_\mu} \chi^{\lambda}_{k\mu} = \sum_{\mu \vdash p} \frac{1}{z_\mu} \chi^{\lambda}_{k\mu},
\]

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as claimed.  

\[ \text{Theorem 5.7} \]

(a) If \( \lambda \vdash kp \) has core \( k(\lambda) = \emptyset \) and quot \( k(\lambda) = (\lambda_1, \ldots, \lambda_k) \), where each partition \( \lambda_i \) is either empty or has one part, then
\[
\epsilon_1(\lambda) = \text{sign}(\pi_\lambda)
\]
and
\[
\epsilon(\bar{\lambda}) = (-1)^{(k-1)p}\text{sign}(\pi_\lambda).
\]
Here \( \pi_\lambda \) is the zero permutation of \( \lambda \), as in Lemma 4.5.

(b) For any other partition \( \lambda \),
\[
\epsilon(\bar{\lambda}) = \epsilon_1(\lambda) = 0.
\]

Proof. Assume that \( \lambda \vdash kp \) and core \( k(\lambda) = \emptyset \); otherwise we are in case (b) of our theorem, and \( \epsilon_1(\lambda) = 0 \) by Lemma 5.3(a), equation (4), and Lemma 5.6. Now, by Lemma 5.6, Proposition 2.3 (the Murnaghan-Nakayama formula), and Lemma 4.3:

\[
\epsilon_1(\lambda) = \sum_{\mu \vdash p} \frac{1}{z_{\mu}} \chi_{k\mu}^\lambda =
\]

\[
= \sum_{\mu \vdash p} \frac{1}{z_{\mu}} \sum_{T \in \text{RHT}_{k\mu}} (-1)^{ht(T)} =
\]

\[
= \sum_{\mu \vdash p} \frac{1}{z_{\mu}} \sum_{T \in \text{RHT}_{k\mu}} \text{sign}(\pi_T).
\]

By Lemma 3.6, quot \( k(\lambda) = (\lambda_1, \ldots, \lambda_k) \) satisfies
\[
|\lambda_1| + \ldots + |\lambda_k| = p \quad (= |\lambda|/k).
\]

Denote now \( p_i := |\lambda_i| \) \((1 \leq i \leq k)\). Recalling the definition of \( \omega_{k\mu}^\lambda \) from Subsection 4.1, let \( \omega_{k\mu}^\lambda(T) = (T_1, \ldots, T_k) \). By Lemma 4.1 and Theorem 4.6:

\[
\epsilon_1(\lambda) = \sum_{\mu \vdash p} \frac{1}{z_{\mu}} \sum_{\mu_1, \ldots, \mu_k} \frac{z_{\mu_1} \cdots z_{\mu_k}}{z_{T_1} \cdots z_{T_k}} \sum_{T_1, \ldots, T_k} \text{sign}(\pi_{T_1}) \prod_{i=1}^{k} \text{sign}(\pi_{T_i}),
\]

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where the second summation is over all $\mu_i \vdash p_i$ such that $\mu_1 \oplus \ldots \oplus \mu_k = \mu$ and the third summation is over all $T_i \in RHT_{\mu_i}$. Equivalently,

$$
\epsilon_1(\lambda) = \text{sign}(\pi_\lambda) \cdot \sum_{\mu_1, \ldots, \mu_k} \prod_{i=1}^k \frac{1}{z_{\mu_i}} \sum_{T_i \in RHT_{\mu_i}} \text{sign}(\pi_{T_i}),
$$

where the first summation is over all $\mu_i \vdash p_i$. Thus, by Lemma 4.3 and Propositions 2.3 and 2.2:

$$
\epsilon_1(\lambda) = \text{sign}(\pi_\lambda) \cdot \prod_{i=1}^k \frac{1}{z_{\mu_i}} \lambda_{\mu_i} = \text{sign}(\pi_\lambda) \cdot \prod_{i=1}^k \delta_{\lambda_i, (p_i)}.
$$

This completes the proof. \qed

This result can be given a very explicit reformulation.

**Lemma 5.8** For a partition $\lambda \vdash kp$, $\epsilon_1(\lambda) \neq 0$ if and only if $\lambda$ has at most $k$ nonzero parts: $\lambda_1 \geq \ldots \geq \lambda_k \geq 0$, and the numbers $(\lambda_i + k - i)^k_{i=1}$ have $k$ distinct residues (mod $k$). In this case, $\epsilon_1(\lambda)$ is the sign of the permutation needed to transform the sequence of residues (mod $k$) of $(\lambda_i + k - i)^k_{i=1}$ into the sequence $(k - i)^k_{i=1}$.

**Proof.** By Subsection 3.1, a partition $\lambda$ has at most one part if and only if, in its 0/1 sequence $s(\lambda)$, all the digits before the median, except possibly the last one, are zeros. Thus, by Lemma 3.4, $\lambda$ satisfies the assumptions of Theorem 5.7(a) if and only if: $s(\lambda)$ has only zeros until the $k$th “space” before the median; has exactly $k$ zeros after this “space $-k$”; and the indices in $s(\lambda)$ of these $k$ zeros have distinct residues (mod $k$). In particular, $\lambda$ has at most $k$ parts. The index in $s(\lambda)$ of the zero corresponding to the $i$th part of $\lambda$ $(1 \leq i \leq k)$ is $\lambda_i + k - i$, where index 0 corresponds to the digit immediately following “space $-k$”; in $s(\emptyset)$, the index is $k - i$. Thus the zero permutation $\pi_\lambda$ is exactly the permutation described in the statement of the lemma. \qed

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Finally, recall that $\epsilon(\bar{\lambda}) = (-1)^{(k-1)p}\epsilon_1(\lambda)$. Adding 1(mod $k$) to each of $k$ distinct residues is equivalent to permuting them by a cycle of length $k$, whose sign is $(-1)^{k-1}$. Thus we get the following result.

Corollary 5.9 Let $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_{k-1})$ be a dominant SU($k$)-weight, and define $\bar{\lambda}_k := 0$. If the numbers $(\bar{\lambda}_i + k - i)_{i=1}^k$ have $k$ distinct residues (mod $k$) then $\epsilon(\bar{\lambda})$ is the sign of the permutation needed to transform the sequence of residues (mod $k$) of $(\bar{\lambda}_i - \bar{p} + k - i)_{i=1}^k$ into the sequence $(k - i)_{i=1}^k$, where $\bar{p} := (\bar{\lambda}_1 + \ldots + \bar{\lambda}_k)/k$. In all other cases, $\epsilon(\bar{\lambda}) = 0$.

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