The partition function of a 3-dimensional topological scalar-vector model

Bogusław Broda* and Małgorzata Bakalarska
Department of Theoretical Physics, University of Łódź
Pomorska 149/153, 90-236 Łódź, Poland

Abstract

A study of the partition function of a 3-dimensional scalar-vector model formally related via duality to the Rozansky-Witten topological $\sigma$-model is presented. The partition function is shown to consist of such topological quantities of a 3-dimensional manifold $M$ like a lattice sum, the Reidemeister-Ray-Singer torsion $\tau_R(M)$ and the Massey product.

PACS: 02.40.-k, 11.10.Kk, 11.15.-q, 11.30.Pb.

Keywords: topological invariants of 3d manifolds, finite perturbative calculus.

1 Introduction

A twisted (topological) version of 3d $\mathcal{N} = 4$ SUSY $\sigma$-model with a hyper-Kähler manifold as a target space (RW model) has been analysed in detail by Rozansky and Witten in [1] (see, also [2] and [3]). In this letter, we will consider a scalar-vector (SV) $\sigma$-model with one variable dualized. More precisely, it consists of three scalar fields and one vector field which is dual to one out of four scalar fields appearing in RW $\sigma$-model. Our SV model can be interpreted as a variant related to low-energy version of 3d $\mathcal{N} = 4$ SUSY $SU(2)$ gauge model (Casson theory), or 3d $\mathcal{N} = 4$ SUSY abelian one with a matter hypermultiplet (3d Seiberg-Witten theory), or as a stand-alone model as well.

*e-mail: bobroda@krysia.uni.lodz.pl
Let us consider a compact four-manifold \( X^4 = S^1 \times X \) with a product metric
\[
\begin{pmatrix}
  g_{00} & 0 \\
  0 & g_{ij}(\varphi_1, \varphi_2, \varphi_3)
\end{pmatrix},
\]
as our target space. In this metric, we can perform a duality transformation for one scalar field, which replaces this field by a vector field in RW action. Since \( X^4 \) is not, in general, hyper-kählerian our SV model is not a priori topological.

We work on a 3-dimensional Euclidean manifold \( \mathcal{M} \) and denote local coordinates on \( \mathcal{M} \) as \( x^\mu, \mu = 1, 2, 3 \). \( \mathcal{M} \) is endowed with a metric tensor \( h_{\mu\nu} \). The bosonic scalar fields can be described as functions \( \varphi^i, i = 1, 2, 3 \) with a metric tensor \( g_{ij} \) on the target space \( X \). The fermions are a scalar \( \eta^I \) and a one-form \( \chi_I^\mu \), where \( I = 1, 2 \).

Classical action of our model assumes the following form
\[
S = \int_{\mathcal{M}} \sqrt{h}d^3x \left\{ \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} g_{ij} \partial_\mu \varphi^i \partial_\nu \varphi^j + \varepsilon_{IJ} \chi_I^\mu \nabla^\mu \eta^J + \frac{1}{2} \frac{1}{\sqrt{h}} \varepsilon^{\mu\nu\rho} \varepsilon_{IJ} \lambda^I_{\mu} \nabla_{\nu} \lambda^J_{\rho} + \frac{1}{6} \frac{1}{\sqrt{h}} \varepsilon^{\mu\nu\rho} \Omega_{IJKL} \chi_I^\mu \chi_J^\nu \chi_K^\rho \eta^L \right\},
\]
where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the usual \( U(1) \) gauge field strength. The "curvature" tensor \( \Omega_{IJKL} \) is a completely symmetric tensor field on \( X \). The covariant derivative of fermions, here denoted as \( \nabla_\mu \), is defined using the pullback of the Levi-Civita connection on \( X \),
\[
\nabla_\mu = \partial_\mu \delta_I^J + (\partial_\mu \varphi^i) \Gamma^I_{ij}.
\]
2 The classical contribution

Let us consider classical contribution to the partition function coming from $U(1)$ gauge field $A_\mu$, i.e. the contribution of classical saddle-points [5], [6]

$$Z_{cl} = \sum_{\text{saddle points}} e^{-S[A_{cl}]}.$$ (3)

When the three-manifold $\mathcal{M}$ has non-trivial homology two-cycles $\Sigma_I$, i.e. closed surfaces that are not boundaries, there exist field configurations with non-zero flux through these surfaces, that must obey generalized Dirac quantization condition

$$\int_{\Sigma_I} F = 2\pi m^I, \quad m^I \in \mathbb{Z},$$ (4)

where $I = 1, \ldots, b_2(\mathcal{M}) = \dim H_2(\mathcal{M})$, the second Betti number). This tells us that in the absence of sources $F$ can be written as

$$F = 2\pi \sum_I m^I \alpha_I,$$ (5)

where $\alpha_I$ is an integral basis of harmonic 2-forms, which by definition satisfy $d\alpha_I = d^*\alpha_I = 0$ and are normalized so that $\int_{\Sigma_I} \alpha_J = \delta^I_J$. The classical saddle-points are labelled by the integer magnetic fluxes $m^I$. The classical action for this field configuration is

$$S[m^I] = \pi^2 \sum_{I,J} m^I G_{IJ} m^J,$$ (6)

where

$$G_{IJ} = \int_{\mathcal{M}} \alpha_I \wedge \alpha_J,$$ (7)

represents the metric on the space of harmonic two-forms. Finally,

$$Z_{cl} = \sum_{m^I} e^{-S[m^I]}.$$ (8)
The zero modes of the vector field are tangent to the space of classical minima, which is a torus of dimension \( b_1(\mathcal{M}) \). The classical part should involve also integration over the \( b_1 \)-torus.

The minima of the action corresponding to the scalar field \( \varphi^i \) are the constant maps of \( \mathcal{M} \) to \( X \). So we will expand around those according to [1]. To take into account the bosonic zero modes one must introduce "collective coordinates" and integrate over the space of all constant maps of \( \mathcal{M} \) to \( X \). Thus, we split the bosonic field \( \varphi^i \) into a sum of a constant and fluctuating part,

\[
\varphi^i_0 + \varphi^i(x). \tag{9}
\]

We define a partition function \( Z_X(\mathcal{M}; \varphi^i_0) \) of fixed \( \varphi^i_0 \), and obtain the partition function \( Z_X(\mathcal{M}) \) as an integral over the three-dimensional target space \( X \)

\[
Z_X(\mathcal{M}) = \frac{1}{(2\pi\hbar)^{3/2}} \int_X Z(\mathcal{M}; \varphi^i_0) \sqrt{g} d^3 \varphi^i_0; \tag{10}
\]

where \( Z(\mathcal{M}; \varphi^i_0) \) is a product of two factors

\[
Z(\mathcal{M}; \varphi^i_0) = Z_0(\mathcal{M}; \varphi^i_0) Z_{\eta \chi \varphi}(\mathcal{M}, X; \varphi^i_0). \tag{11}
\]

Here \( Z_0(\mathcal{M}; \varphi^i_0) \) is the 1-loop contribution of non-zero modes of \( \varphi \) and \( A \), while \( Z_{\eta \chi \varphi}(\mathcal{M}, X; \varphi^i_0) \) is the exponential of the sum of all Feynman diagrams of two or more loops, in the background field of given \( \varphi^i_0 \).

### 3 The one-loop contribution

Let us first determine the one-loop contribution \( Z_0(\mathcal{M}; \varphi^i_0) \). We work with the part of the action which is quadratic in the vector field \( A_\mu(x) \), in fluctuating bosonic fields \( \varphi^i(x) \) and in fermionic fields \( \eta^I(x), \chi^I_\mu(x) \)

\[
S_0 = \int_\mathcal{M} d^3x \sqrt{h} \left\{ \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} g_{ij} \partial_\mu \varphi^i \partial_\mu \varphi^j + \varepsilon_{IJ} \chi^I_\mu \nabla_\nu \eta^J + \frac{1}{2} \frac{1}{\sqrt{h}} \varepsilon^{\mu\nu\rho} \varepsilon_{IJ} \chi^I_\mu \nabla_\nu \chi^J_\rho \right\}. \tag{12}
\]
The tensors $g_{ij}$, $\varepsilon_{IJ}$ and implicit Christoffel symbols $\Gamma^i_{jk}$ are taken at the point $\varphi^i_0$ of $X$. Gauge invariance of the action requires gauge fixing and introduction of the Faddeev-Popov ghost fields $c, \bar{c}$

$$S_{gauge} = \int_{\mathcal{M}} d^3x \sqrt{h} \left\{ \frac{1}{2} (\nabla^\mu A_\mu)^2 + \partial^\mu \bar{c} \partial_\mu c \right\}. \quad (13)$$

Supplementing the action (12) with (13), we obtain

$$S'_0 = \int_{\mathcal{M}} d^3x \sqrt{h} \left\{ \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} g_{ij} \partial_\mu \varphi^i \partial_\mu \varphi^j + \varepsilon_{IJ} \chi^I_\mu \nabla^\mu \eta^J + \frac{1}{2} \frac{1}{\sqrt{h}} \varepsilon^{\mu\nu\rho} \varepsilon_{IJ} \chi^I_\mu \nabla^\nu \chi^J_\mu + \frac{1}{2} (\nabla^\mu A_\mu)^2 + \partial^\mu \bar{c} \partial_\mu c \right\}. \quad (14)$$

It will appear that the partition function $Z_0$ corresponding to (14) essentially consists of the Reidemeister-Ray-Singer torsion of 3d manifold $\mathcal{M}$.

Namely,

$$Z_0(\mathcal{M}; \varphi^i_0) = \int [DY] \exp \left\{ -S'_0[Y] \right\}, \quad (15)$$

where the integration measure $[DY]$ is taken over all the fields: $A_\mu$, $\varphi^i$, $\eta^I$, $\chi^I_\mu$, $c, \bar{c}$. The path integral of the scalar bosonic and ghost fields gives a net contribution

$$\left( \det'(-\Delta_0) \right)^{-\frac{1}{2}}, \quad (16)$$

where $\Delta_i = \nabla^\mu \nabla_\mu$ ($i = 0, 1$) is a laplacian acting on $i$-forms on $\mathcal{M}$ and the prime means that we exclude zero modes. Now let us introduce an operator $L_-$ which acts on the direct sum of zero- and one-forms on $\mathcal{M}$

$$L_-(\eta, \chi_\mu) = \left( -\nabla^\mu \chi_\mu, \nabla_\mu \eta + h_{\mu\nu} \frac{1}{\sqrt{h}} \varepsilon^{\mu\nu\rho} \partial_\rho \chi_\lambda \right). \quad (17)$$

Then the fermionic part of the action (13) becomes a quadratic form

$$\frac{1}{2} \varepsilon_{IJ} \langle \eta^I, \chi^J_\mu \mid L_- \mid \eta^I, \chi^J_\mu \rangle. \quad (18)$$

The fermionic one-loop contribution with zero modes removed is

$$\det' L_- . \quad (19)$$
Finally, the integration over the gauge field $A_\mu$ yields

$$ (\det'(-\Delta_1))^{-\frac{1}{2}}, $$

so that the total one-loop contribution of non-zero modes is

$$ Z_0(\mathcal{M}; \varphi^i_0) = \frac{\det'L_-}{(\det'(-\Delta_0))^\frac{3}{2}(\det'(-\Delta_1))^\frac{1}{2}}. $$

The absolute value of the ratio of the determinants in (21) is related to the Reidemeister-Ray-Singer analytic torsion $\tau_R(\mathcal{M})$ [1], [8] and [9]

$$ \left| \frac{\det'L_-}{(\det'(-\Delta_0))^\frac{3}{2}(\det'(-\Delta_1))^\frac{1}{2}} \right| = \tau_R^{-2}(\mathcal{M}). $$

4 Zero modes and propagators

The partition function is plagued by zero modes, which we have temporarily removed by hand. There are the following four sorts of zero modes:

1. 3 scalar boson zero modes corresponding to $\varphi^i$;

2. $b_1$ vector boson zero modes of $A_\mu$;

3. 1 ghost zero mode for $c, \bar{c}$;

4. 2 scalar fermion zero modes of $\eta^I$ and 2 $b_1$ one-form fermion zero modes of $\chi^I_\mu$.

The ghost zero mode can be removed instantaneously without any consequences because it should not be present in the partition function from the very beginning at all, as the gauge transformation corresponding to the constant (zero) mode acts trivially on $A_\mu$. The rest of boson and fermion zero modes have been shifted from the one-loop calculation as they would produce trivial infinities and zeros in the partition function respectively. Actually, the boson
zero modes have been already dealt with—$\varphi_0$-integration for $\varphi$ and $b_1(M)$-torus integration for $A$. The fermion zero modes will saturate higher-order loops.

The propagators in our model are of the following form:

\[
\langle \varphi^i(x)\varphi^j(y) \rangle = -\hbar g^{ij} G^{(0)}(x, y),
\]

\[
\langle \chi^I_\mu(x)\eta^J(y) \rangle = \hbar \varepsilon^{IJ} \partial_\mu G^{(0)}(x, y),
\]

\[
\langle \chi^I_\mu(x)\chi^J_\nu(y) \rangle \equiv \hbar G^{(1)}_{\mu\nu}(x, y) = \frac{1}{2} \hbar \varepsilon^{IJ} h^{-1/2} h_{\mu\lambda} \varepsilon^{\lambda\kappa\rho} \partial_\rho G^{(1)}_{\kappa\nu}(x, y),
\]

where $G^{(1)}(x, y)$ is an inverted Laplacian for $i$-forms with zero modes removed.

5 Feynman diagrams

Let us analyse higher-order perturbative calculation, thus the diagrams that have a chance to absorb the fermionic zero modes. We may limit our attention to only those Feynman diagrams (analogously to [1]), whose contribution is of order $\hbar^s$, because the rest of diagrams is equal to zero.

Let us consider a diagram with $V$ vertices, $V = V_0 + V_1 + V_2 + V_3$, where $V_n$ ($n = 0, 1, 2, 3$) means the number of vertices with $n$ (fermionic) legs of type $\chi_\mu$. Let $L$ be the total number of legs. Each vertex in the diagram carries a factor of $h^{-1}$, so the vertices taken together bring a factor $h^{-V}$. Each propagator carries a factor of $\hbar$, and each fermionic zero mode carries a normalization factor of $h^{\frac{1}{2}}$. Therefore the total contribution of propagators and external legs is $h^{\frac{L}{2}}$. So we get

\[
\frac{L}{2} - V = s,
\]

7
where
\[ s = \frac{3}{2} + \frac{b_1}{2} - \frac{1}{2} = \frac{b_1}{2} + 1, \] (27)
which follows from the previous section. To narrow searching procedure it
is also necessary to take into account some inequalities among numbers of
vertices:
\[ L \geq 3V_0 + 4(V_1 + V_2 + V_3), \]
\[ V_1 + V_3 \geq 2, \] (28)
\[ V_1 + 2V_2 + 3V_3 \geq 2b_1, \]
which, in turn, imply the following conditions for the Feynman diagrams:
\[ 1 \geq V_0 + V_1 + V_2, \]
\[ V_3 \geq 2V_0 + V_1 + 2V_2, \]
\[ 4 \geq 2V_0 + 3V_1 + 2V_2 + V_3. \]

This set of inequalities yields exactly seven different solutions (types of
Feynman diagrams). Six of them vanish because of parity symmetry, or be-
cause of geometrical identities for (harmonic) one-forms: \( \omega \wedge \omega = 0, \) and
\( d\omega = 0. \) The only exception is the Feynman diagram (surviving for \( b_1(\mathcal{M}) = 2 \))
with \( V_3 = 2, \) where the vertices are connected by a single \( \langle \chi\chi \rangle \) propagator (all
remaining legs absorb the zero modes of \( \eta^I \) and \( \chi^I_{\mu} )\)

\[ \begin{array}{c}
\vdots \\
V_3 \\
\vdots \\
\langle \chi\chi \rangle \\
\vdots \\
V_3 \\
\vdots
\end{array} \]

**Fig. 1** The only non-vanishing higher-order Feynman diagram, giving rise
to the Massey product.
The integral corresponding to this diagram

\[ I(\mathcal{M}) = \int_{\mathcal{M}} \varepsilon^{\mu_1 \mu_2 \mu_3} \varepsilon_{\nu_1 \nu_2 \nu_3} \omega_{\mu_1}^{(1)}(x) \omega_{\nu_1}^{(1)}(y) \omega_{\mu_2}^{(2)}(x) \omega_{\nu_2}^{(2)}(y) G_{\mu_3 \nu_3}(x, y) d^3x d^3y, \]  

(29)
can be evaluated following [1], showing the appearance of the Massey product.

The weight function is equal to

\[ a(X) = \frac{1}{(2\pi)^{3/2}} \int x \sqrt{g} d^3 \varphi_0 \varepsilon^{J_1 J_2 J_3} \varepsilon^{I_1 I_2 I_3} \varepsilon^{I_4 I_5} \Omega_{I_1 I_2 I_3 I_4 I_5} \Omega_{J_1 J_2 J_3 J_4}. \]  

(30)

So the contribution of the above Feynman diagram to the partition function of our scalar-vector model is equal

\[ Z_{\eta \chi \varphi}(\mathcal{M}, X) = \frac{1}{2} a(X) I(\mathcal{M}). \]  

(31)

Finally, the partition function of the scalar-vector model has the following form

\[ Z_X(\mathcal{M}) = \frac{1}{2} Z_0(\mathcal{M}) a(X) I(\mathcal{M}), \]  

(32)

where \( Z_0(\mathcal{M}) \) is given by eq.(21).

6 Summary

For SV \( \sigma \)-model the partition function has been exactly calculated on the three-dimensional manifold \( \mathcal{M} \) giving the following topological quantities: a lattice sum containing the metric on the space of harmonic two-forms, the Reidemeister-Ray-Singer torsion \( \tau_R(\mathcal{M}) \) and the Massey product. The lattice sum is of classical origin, whereas the torsion is coming from the one-loop contribution (functional determinants without zero modes). Almost all “higher-order loops” are killed by fermion zero modes, and the only one Feynman diagram surviving (for \( b_1(\mathcal{M}) = 2 \)) yields the Massey product.

We should stress that SV \( \sigma \)-model discussed in the present paper is not equivalent to RW model. First of all, although SV action can be obtained
from RW action via duality, the duality performed, as a purely local operation transfers no global/topological information between the models. Besides, the hyper-kählerian condition is relaxed in our case, and a priori the model is not topological. Furthermore, a posteriori, the non-equivalence of both perturbative calculi is visible.

Acknowledgements

The paper has been supported by KBN grant 2P03B08415 and partially by UL grant 248. The authors would like to thank Guowu Meng for a critical remark.

References

[1] L. Rozansky and E. Witten, “Hyper-Kähler Geometry and Invariants of Three-Manifolds”, E-print hep-th/9612216.

[2] G. Thompson, “On the Generalized Casson Invariant”, E-print hep-th/9811199.

[3] M. Marino and G. Moore, “Three-manifold topology and the Donaldson-Witten partition function”, Nucl. Phys. B 547, 569 (1999).

[4] N. J. Hitchin, A. Karlhede, U. Lindström and M. Roček, Commun. Math. Phys. 108, 535 (1987).

[5] E. Verlinde, Nucl. Phys. B 455, 211 (1995).

[6] E. Witten, “On S-Duality In Abelian Gauge Theory”, E-print hep-th/9505180.

[7] E. Witten, Commun. Math. Phys. 121, 351 (1989).
[8] L. Rozansky, Commun. Math. Phys. 171, 279 (1995).

[9] L. Rozansky, “Witten’s invariant of 3-dimensional manifolds: loop expansion and surgery calculus”, Knots and Applications, ed. L. H. Kauffman, 271–299.