METHA-RAMANATHAN FOR $\varepsilon$ AND $k$-SEMISTABLE DECORATED SHEAVES

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ABSTRACT. This paper is devoted to generalizing the Mehta-Ramanathan restriction theorem to the case of $\varepsilon$-semistable and $k$-semistable decorated sheaves. After recalling the definition of decorated sheaves and their usual semistability we define the $\varepsilon$ and $k$-(semi)stability. We first prove the existence of a (unique) $\varepsilon$-maximal destabilizing subsheaf for decorated sheaves (Section 3.1). After some others preliminar results (such as the openness condition for families of $\varepsilon$-semistable decorated sheaves) we finally prove, in Section 3.7, a restriction theorem for slope $\varepsilon$-semistable decorated sheaves. In Section 4 we reach the same results in the $k$-semistability case that we did in the $\varepsilon$-semistability, but only for rank $\leq 3$ decorated sheaves.

1. Introduction

In the framework of bundles with a decoration we recall two types of objects which incorporate all others: decorated sheaves and the so-called tensors. The former were introduced by Schmitt while the latter by Gomez and Sols. We recall briefly the definitions of such objects. A decorated sheaf of type $(a,b,c,N)$ over $X$ is a pair $(\mathcal{E}, \varphi)$ where $\mathcal{E}$ is a torsion free sheaf over $X$ and

$$\varphi : (\mathcal{E}^\otimes a)^{\oplus b} \otimes (\det \mathcal{E}^\vee)^{\otimes c} \to N,$$

while a tensor of type $(a,b,c,D_u)$ is a pair $(\mathcal{E}, \varphi)$ where $\mathcal{E}$ is a torsion free sheaf and

$$\varphi : (\mathcal{E}^\otimes a)^{\oplus b} \otimes (\det \mathcal{E}^\otimes c) \otimes D_u,$$

where $D_u$ is a locally free sheaf belonging to a fixed family $\{D_u\}_{u \in R}$ parametrized by a scheme $R$. As one can easily see, these two objects are quite similar; they both incorporate many types of bundles with a morphism, such as framed bundles, Higgs bundles, quadratic, orthogonal and symplectic bundles, and many others. The problem with classifying decorated sheaves up to equivalence is therefore related to many classification problems in algebraic geometry. In order to solve this problem by establishing the existence of a coarse moduli space one needs to introduce a notion of semistability. The notion of semistability for decorated sheaves and tensors is the same. A semistability condition for these objects was introduced by Bradlow, Garcia-Prada, Gothen and Mundet i Riera in [1] and [2] and then studied by Schmitt in the case of decorated bundle (see for example [11]) and by Gomez-Sols in the case of tensors (see for example [3]).

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In both cases one tests the (semi)stability of an object \((E, \varphi)\) against saturated weighted filtrations of \(E\), namely against pairs \((E^\bullet, \alpha)\) consisting of a filtration
\[
E^\bullet : \quad 0 \subset E_1 \subset \cdots \subset E_s \subset E_{s+1} = E
\]
of saturated sheaves of \(E\), i.e., such that \(E/E_i\) is torsion free, and a tuple
\[
\alpha = (\alpha_1, \ldots, \alpha_s)
\]
of positive rational numbers. Then one says that a decorated sheaf or tensor \((E, \varphi)\) is (semi)stable with respect to \(\delta\) if and only if for any weighted filtration
\[
P(E^\bullet, \alpha) + \delta \mu(E^\bullet, \alpha; \varphi) \succ 0,
\]
where \(P(E^\bullet, \alpha)\) is a polynomial depending on the Hilbert polynomials of the sheaves of the weighted filtration, \(\delta\) is a fixed polynomial and \(\mu(E^\bullet, \alpha; \varphi)\) a number depending on the weighted filtration and on \(\varphi\) (see Definition 3).

As one can easily observe when looking at the definition of \(\mu(E^\bullet, \alpha; \varphi)\), the semistability for decorated sheaves is not easy to handle and is quite complicated to calculate in general. This fact affects the possibility of generalizing many basic tools that instead exist for vector bundles. For example, until now, there is no notion of a maximal destabilizing object for decorated sheaves nor is there a notion of Jordan-Hölder or Harder-Narasimhan filtration although some results in this direction can be found in [3] for similar objects. This paper is devoted to the study of the semistability condition of decorated bundles in order to better understand and simplify it in the hope that this will be useful in the study of decorated sheaves. In [8] we tried, and succeeded for the case of \(a = 2\), to bound the length of weighted filtrations on which one checks the semistability condition, while in this paper we approach the problem in a different way: we “enclose” the above semistability condition between a stronger semistability condition and a weaker one. To be more precise: we say that a decorated sheaf \((E, \varphi)\) of type \((a, b, c, N)\) is \(\varepsilon\)-(semi)stable with respect to a fixed polynomial \(\delta\) of degree \(\dim X - 1\) if for any subsheaf \(F \subset E\)
\[
\text{rk}(E) (P_F - a \delta \varepsilon(\varphi|_F)) \prec \text{rk}(F) (P_E - a \delta \varepsilon(\varphi)),
\]
where \(P_F\) denotes the Hilbert polynomial of a sheaf \(F\) and \(\varepsilon(\varphi) = 1\) if \(\varphi \neq 0\) and zero otherwise. Similarly, given a sheaf \(E\) and a subsheaf \(F \subset E\), we define a function \(k_{F, \varepsilon}\) with values in the set \(\{0, 1, \ldots, a\}\) and depending on the behaviour of \(\varphi\) on \(F\) as subsheaf of \(E\) (see Equation (16)). Then we say that a decorated sheaf \((E, \varphi)\) is \(k\)-(semi)stable if for any subsheaf \(F \subset E\) one has that
\[
\text{rk}(E) (P_F - \delta k_{F, \varepsilon}) \prec \text{rk}(F) (P_E - \delta k_{\varepsilon, \varepsilon}).
\]
What happens is that
\[
\varepsilon\text{-}(semi)\text{stable} \Rightarrow (semi)\text{stable} \Rightarrow k\text{-}(semi)\text{stable}
\]
and, if \(\text{rk}(E) = 2\), (semi)stability is equivalent to \(k\text{-}(semi)\text{stability}\). In this respect, we generalize some known results (in the case of vector bundles) to the case of \(\varepsilon\)-semistable decorated sheaf and, to a lesser extent, to the \(k\text{-}(semi)\text{stable}\) case. In fact, using \(\varepsilon\)-semistability, we find the \(\varepsilon\)-maximal destabilizing subsheaf, prove a Mehta-Ramanathan’s like theorem about the behavior of slope \(\varepsilon\)-semistability under restriction to curves. Since \(k\)-semistability is a little bit more complicated to
handle, we managed to find a $k$-maximal destabilizing subsheaf and prove a Mehta-Ramanathan theorem only for rank $\leq 3$.

2. Definition and first properties

Notation. Let $(X, \mathcal{O}_X(1))$ be a polarized projective smooth variety of dimension $n$, $\delta = \delta(x) = \delta_{n-1}x^{n-1} + \cdots + \delta_1 x + \delta_0 \in \mathbb{Q}[x]$ be a fixed polynomial with positive leading coefficient and let $\overline{\delta} = \delta_{n-1}$. Recall that given two polynomials $p(m)$ and $q(m)$ then $p \leq q$ if and only if there exists $m_0 \in \mathbb{N}$ such that $p(m) \leq q(m)$ for any $m \geq m_0$.

Definition 1. Let $N$ be a line bundle over $X$ and let $a, b, c$ be nonnegative integers. A decorated vector bundle of type $t = (a, b, c, N)$ over $X$ is the datum of a vector bundle $E$ over $X$ and a morphism $\varphi : E_{a, b, c} \cong (E^\otimes a) \otimes b \otimes (\text{det } E)^{\otimes -c} \to N$.

A decorated sheaf of type $t$ is instead a pair $(E, \varphi)$ such that $E$ is a torsion free sheaf and $\varphi$ is a morphism as in (1). Sometimes we will call these objects simply decorated sheaves (resp. bundles) instead of decorated sheaves (resp. bundles) of type $t = (a, b, c, N)$ if the input data are understood.

Remark 2. Note that, although $E$ is torsion free, the sheaf $E_{a, b, c}$ may have torsion.

Now we define morphisms between such objects. Let $(E, \varphi)$ and $(E', \varphi')$ be decorated sheaves (resp. bundles) of the same type $t$. A morphism of sheaves (resp. bundles) $f : E \to E'$ is a morphism of decorated sheaves (resp. bundles) if exists a scalar morphism $\lambda : N \to N$ making the following diagram commute:

\[ \begin{array}{ccc} E_{a, b, c} & \xrightarrow{f_{a, b, c}} & E'_{a, b, c} \\ \varphi \downarrow & & \varphi' \downarrow \\ N & \xrightarrow{\lambda} & N. \end{array} \]

We will say that a morphism of decorated sheaves (bundles) $f : (E, \varphi) \to (E', \varphi')$ is injective if exists an injective morphism of sheaves (bundles) $f : E \to E'$ and a non-zero scalar morphism $\lambda$ such that the above diagram commutes. Analogously we will say that a morphism of decorated sheaves (bundles) $f : (E, \varphi) \to (E', \varphi')$ is surjective if exists a surjective morphism of sheaves (bundles) $f : E \to E'$ and a scalar morphism $\lambda$ making diagram commute. Finally we will say that $(E, \varphi)$ and $(E', \varphi')$ are equivalent if exists an injective and surjective morphism of decorated sheaves (bundles) between them.

2.1. Semistability conditions. Let $(E, \varphi)$ be a decorated sheaf of type $t = (a, b, c, N)$ and let $r = \text{rk}(E)$. We want to recall the notion of semistability for these objects. To this end let

\[ 0 \subsetneq E_{i_1} \subsetneq \cdots \subsetneq E_{i_s} \subsetneq E_r = E \]

be a filtration of saturated subsheaves of $E$ such that $\text{rk}(E_{i_j}) = i_j$ for any $j = 1, \ldots, s$, let $\alpha = (\alpha_{i_1}, \ldots, \alpha_{i_s})$ be a vector of positive rational numbers and finally let $I = \{i_1, \ldots, i_s\}$ be the set of indexes appearing in the filtration. We will refer to the pair $(E^\bullet, \alpha | I)$ as weighted filtration of $E$ indexed by $I$ or simply weighted
filtration if the set of indexes is clear from the context, moreover we will denote by \(|I|\) the cardinality of the set \(I\). Such a weighted filtration defines the polynomial

\[
P_I(\mathcal{E}^\bullet, \omega) \doteq \sum_{i \in I} \alpha_i \left( P_{\phi_i} \cdot \text{rk}(\mathcal{E}_i) - \text{rk}(\mathcal{E}) \cdot P_{\phi_i} \right),
\]

and the rational number

\[
L_I(\mathcal{E}^\bullet, \omega) \doteq \sum_{i \in I} \alpha_i \left( \text{deg} \mathcal{E} \cdot \text{rk}(\mathcal{E}_i) - \text{rk}(\mathcal{E}) \cdot \text{deg} \mathcal{E}_i \right).
\]

Moreover we associate with \((\mathcal{E}^\bullet, \omega)\) the following rational number depending also on \(\varphi\):

\[
\mu_I(\mathcal{E}^\bullet, \omega; \varphi) \doteq - \min_{i_1, \ldots, i_a \in \overline{I}} \left\{ \gamma_1^{(i_1)} + \cdots + \gamma_I^{(i_a)} \mid \varphi_{(\mathcal{E}_{i_1} \otimes \cdots \otimes \mathcal{E}_{i_a}) \otimes b} \neq 0 \right\}
\]

where \(\overline{I} \doteq I \cup \{ r \}\), \(P_{\varphi_i}\) (respectively \(P_{\varphi}\)) is the Hilbert polynomial of \(\mathcal{E}\) (resp. \(\mathcal{E}_i\)) and

\[
\gamma_I = (\gamma_1^{(1)}, \ldots, \gamma_I^{(r)})
\]

\[
\doteq \sum_{i \in I} \alpha_i \left( \text{rk}(\mathcal{E}_i) - r, \ldots, \text{rk}(\mathcal{E}_i) - r, \text{rk}(\mathcal{E}_1), \ldots, \text{rk}(\mathcal{E}_j) \right) \text{\(r\)-times} \rightarrow \text{\(r\)-times} \}
\]

**Definition 3 (Semistability).** A decorated sheaf \((\mathcal{E}, \varphi)\) of type \((a, b, c, \mathbb{N})\) is \(\delta\)-(semi)stable if for any weighted filtration \((\mathcal{E}^\bullet, \omega)\) the following inequality holds:

\[
P_I(\mathcal{E}^\bullet, \omega) + \delta \mu_I(\mathcal{E}^\bullet, \omega; \varphi) \geq 0.
\]

It is slope \(\delta\)-(semi)stable if

\[
L_I(\mathcal{E}^\bullet, \omega) + \delta \mu_I(\mathcal{E}^\bullet, \omega; \varphi) > 0.
\]

**Notation.** The notation \(\succ\) (\(\geq\)) means that \(\succ\) (\(\geq\)) has to be used in the definition of stable and \(\succ\) (\(\geq\)) in the definition of semistable.

**Remark 4.** (1) The morphism \(\varphi : \mathcal{E}_{a,b,c} \rightarrow \mathbb{N}\) induces a morphism \(\mathcal{E}_{a,b} \rightarrow (\det \mathcal{E})^{\otimes c} \otimes \mathbb{N}\). With abuse of notation, we still refer to the former by \(\varphi\). In this context it is easy to see that a decorated sheaf of type \((a, b, c, \mathbb{N})\) corresponds (uniquely up to isomorphism) to a decorated sheaf of type \((a, b, 0, (\det \mathcal{E})^{\otimes c} \otimes \mathbb{N})\). Therefore the category of decorated sheaves (with fixed determinant \(= \mathcal{L}\)) of type \((a, b, c, \mathbb{N})\) is equivalent to the category of decorated sheaves (with fixed determinant \(= \mathcal{L}\)) of type \((a, b, 0, \mathcal{L}^{\otimes c} \otimes \mathbb{N})\). For this reason if \(P\) is any property which does not involve families of decorated sheaves but just a fixed decorated sheaf then \(P\) holds true for decorated sheaves of type \((a, b, c, \mathbb{N})\) if and only if it holds true for decorated sheaves of type \((a, b, \mathbb{N})\).

(2) Let \((\mathcal{E}^\bullet, \omega)\) be a weighted filtration and suppose that \(\mu_I = - (\gamma_1^{(i_1)} + \cdots + \gamma_I^{(i_a)})\), then there exists at least one permutation \(\sigma : \{i_1, \ldots, i_a\} \rightarrow \{i_1, \ldots, i_a\}\) such that \(\varphi_{(\mathcal{E}_{i_1} \otimes \cdots \otimes \mathcal{E}_{i_a}) \otimes b} \neq 0\). We can say that, although the morphism \(\varphi\) is not symmetric, the semistability condition has a certain symmetric behavior.

(3) From now on we will write

\[
\varphi_{(\mathcal{E}_{i_1} \otimes \cdots \otimes \mathcal{E}_{i_a}) \otimes b} \neq 0
\]
if there exists at least one permutation \( \sigma : \{ i_1, \ldots, i_a \} \to \{ i_1, \ldots, i_a \} \) such that \( \varphi_{| (E_{\sigma(i_1)} \otimes \cdots \otimes E_{\sigma(i_a)}) \oplus b} \neq 0 \).

Fix now a weighted filtration \((E^\bullet, \alpha)\) indexed by \( I \), define \( r_s \equiv r_{E_s} \) and suppose that the minimum of \( \mu_I(E^\bullet, \alpha; \varphi) \) is attained in \( i_1, \ldots, i_a \). Then
\[
\mu_I(E^\bullet, \alpha; \varphi) = - \left( \sum_{s \in I} \alpha_s r_s + \sum_{s \geq i_1} \alpha_s r + \cdots + \sum_{s \geq i_a} \alpha_s r \right)
\]

Then define
\[
R(I) \equiv \sum_{s \geq l, a \in I} \alpha_s r \text{ for } l \in I \text{ and } R_I(r) \equiv 0,
\]
(10)

and finally fix, for any \( i \in I \), the following quantities
\[
C_i \equiv r_i P_E - r P_{E_i} - ar_i, \quad c_i \equiv r_i \deg E - r \deg E_i - ar_i.
\]
(11)
(12)

Therefore the semistability condition \( [5] \) is equivalent to the following
\[
\sum_{i \in I} \alpha_i C_i + r \delta R_1 \geq 0,
\]
(13)

while the slope semistability condition \( [9] \) is equivalent to the following
\[
\sum_{i \in I} \alpha_i c_i + r \delta R_1 \geq 0.
\]
(14)

Sometimes, for convenience’s sake, we will write \( R_I(i_1, \ldots, i_a) \) instead of \( R_I(i_1) + \cdots + R_I(i_a) \).

2.2. Others notions of semistabilities. Decorated sheaves provide a useful machinery to study principal bundles or more generally vector bundles with additional structures. However in general it is quite hard to check semistability for decorated sheaves because one has to verify inequality \( [5] \) and therefore calculate \( \mu_I \) for any filtration. For this reason we introduce two notions of semistability for decorated sheaves which are more computable: \( \varepsilon \)-semistability and \( k \)-semistability. \( \varepsilon \)-semistability is quite similar to the semistability condition for framed sheaves given by Huybrechts and Lehn in \( [5] \) while \( k \)-semistability is the usual semistability restricted to subsheaves instead of filtrations. We will prove that \( \varepsilon \)-semistability is stronger than the usual one while \( k \)-semistability is weaker (Proposition 6).

Let \((E, \varphi)\) be as before and let \( F \) be a subsheaf of \( E \) then define
\[
\varepsilon_F = \varepsilon(F, \varphi) \equiv \begin{cases} 1 & \text{if } \varphi|_{F, b} \neq 0 \\ 0 & \text{otherwise,} \end{cases}
\]
(15)
and

\[ k_{F,E} = k(F,E,\varphi) \triangleq \begin{cases} a \text{ if } \varphi|_{I_{a,b}} \neq 0 \\ a - s \text{ if } \varphi|_{F^{(a-s)}(E)} \neq 0 \text{ and } \varphi|_{F^{(a-s-1)}(E)} \equiv 0 \\ 0 \text{ otherwise,} \end{cases} \]

Here with the notation $F^{(a-s)} \circ E^s$ we mean any tensor product between $E$ and $F$ where $E$ appears exactly $s$-times while $F$ appears $a$-times, and when we write $\varphi|_{F^{(a-s)} \circ E^s} \neq 0$ we mean that there exists at least one tensor product between $F$ and $E$ over which $\varphi$ is not identically zero (see Remark 4 point (3)).

**Definition 5** ($\varepsilon$-semistability, $k$-semistability). Let $(E, \varphi)$ be a decorated sheaf; we will say that $(E, \varphi)$ is $\varepsilon$-(semi)stable, slope $\varepsilon$-(semi)stable, $k$-(semi)stable or slope $k$-(semi)stable if and only if for any subsheaf $F$ the following inequalities hold:

\[
\begin{align*}
(17) & \quad \varepsilon\text{-}(semi)stable & & p_F - \frac{a\delta \varepsilon_F}{rk(F)} & \prec & & p_{E} - \frac{a\delta}{rk(E)} \\
(18) & \quad \text{slope } \varepsilon\text{-}(semi)stable & & \mu(F) - \frac{a\delta \varepsilon_F}{rk(F)} & \prec & & \mu(E) - \frac{a\delta}{rk(E)} \\
(19) & \quad k\text{-}(semi)stable & & p_F - \frac{\delta k_{F,E}}{rk(F)} & \prec & & p_{E} - \frac{a\delta}{rk(E)} \\
(20) & \quad \text{slope } k\text{-}(semi)stable & & \mu(F) - \frac{\delta k_{F,E}}{rk(F)} & \prec & & \mu(E) - \frac{a\delta}{rk(E)}
\end{align*}
\]

where $p_F \triangleq \frac{p_F}{rk(F)}$ is the reduced Hilbert polynomial.

The above conditions are equivalent to the following:

\[
\begin{align*}
(17) & \quad \iff & & P_F rk(F) - rk(E)P_{E} + a\delta(rk(E)\varepsilon_F - rk(F)) \succ 0 \\
(18) & \quad \iff & & \deg E rk(F) - rk(E) \deg F + a\delta(rk(E)\varepsilon_F - rk(F)) \succ 0 \\
(19) & \quad \iff & & P_F rk(F) - rk(E)P_{E} + \delta(rk(E)k_{F,E} - \text{ark}(F)) \succ 0 \\
(20) & \quad \iff & & \deg E rk(F) - rk(E) \deg F + \delta(rk(E)k_{F,E} - \text{ark}(F)) \succ 0.
\end{align*}
\]

Moreover note that $k$-(semi)stability is equivalent to the usual (semi)stability applied to filtrations of length one. In fact let $F$ be a subsheaf of $E$ and consider the filtration $0 \subset F \subset E$ with weight vector $\alpha = 1$. An easy calculation shows that

\[
P(0 \subset F \subset E, 1; \varepsilon) + \delta \mu(0 \subset F \subset E, 1; \varphi) = P_F rk(F) - rk(E)P_{E} + \delta(rk(E)k_{F,E} - \text{ark}(F)).
\]

More precisely, these three notions of semistability are related in the following way:

**Proposition 6.** $\varepsilon$-(semi)stable $\Rightarrow$ (semi)stable $\Rightarrow$ $k$-(semi)stable.

**Proof.** Let $(E, \varphi)$ be a $\varepsilon$-(semi)stable decorated sheaf of rank $rk(E) = r$, let $(E^*(\alpha))$ be a weighted filtration indexed by $I$ and let $r_i$ be the rank of $E_i$. For any $i \in I$,

\[
P_{E} r_i - rP_{E_{\alpha}} + a\delta(r\varepsilon_{E_{\alpha}} - r_i) \succ 0,
\]

therefore

\[
\sum_{i \in I} \alpha_i (P_{E} r_i - rP_{E_{\alpha}} + a\delta(r\varepsilon_{E_{\alpha}} - r_i)) = P_1 + a\delta \left( \sum_{i \in I} \alpha_i \varepsilon_{E_{\alpha}} - \sum_{i \in I} \alpha_i r_i \right) \succ 0.
\]
We want to show that $P_1 + \delta \mu_1 \succ 0$. Denote by $\varepsilon_i \div \varepsilon_{i_k}$ and let $j_0 = \min\{k \in I \mid \varepsilon_k \neq 0\}$. Therefore

$$
\mu_1 = -\min\{\gamma^{(i_k)}_I + \cdots + \gamma^{(i_k)}_I \mid \varphi_{|E_{i_k} \otimes \cdots \otimes E_{i_k}} \neq 0\}
\geq -a\gamma^{(j_0)}_I
= a \left( \sum_{i \geq j_0, i \in I} \alpha_i r - \sum_{i \in I} \alpha_i r_i \right)
= a \left( \sum_{i \in I} \alpha_i \varepsilon_i r - \sum_{i \in I} \alpha_i r_i \right)
= a \left( \sum_{i \in I} \alpha_i (\varepsilon_i r - r) \right).
$$

So

$$
P_1 + \delta \mu_1 \succ P_1 + a\delta \left( \sum_{i \in I} \alpha_i (\varepsilon_i r - r_i) \right)
= \sum_{i \in I} \alpha_i \left( P_{E_i r} - r P_{E_i} + a\delta (r \varepsilon_{E_i} - r_i) \right) \succ 0,
$$

and we are done.

Finally, given a (semi)stable decorated sheaf, we want to show that is $k$-(semi)stable. Let $\mathcal{F}$ be a subsheaf of $\mathcal{E}$ of rank $r_F$; if we consider the filtration $0 \subset \mathcal{F} \subset \mathcal{E}$ with weights identically 1, after small calculation ones get that

$$
0 \prec P(0 \subset \mathcal{F} \subset \mathcal{E}, \frac{1}{E_i}) + \delta \mu(0 \subset \mathcal{F} \subset \mathcal{E}, \frac{1}{E_i} \varphi) = P_{E_F r} - r P_{E_F} + \delta (r k_{F, E} - a r_F)
$$

and we have done.

Note that $\mu(\mathcal{E}^+; \alpha; \varphi)$ is not additive for all filtrations, i.e., it is not always true that

$$
(21) \quad \mu(\mathcal{E}^+, \alpha; \varphi) = \sum_{i \in I} \mu(0 \subset E_i \subset \mathcal{E}, \alpha_i; \varphi).
$$

We will call non-critical a filtration for which \ref{21} holds and critical otherwise. Finally we will say that $\varphi$ is additive if equality \ref{21} holds for any weighted filtration, i.e., there are no critical filtrations.

**Remark 7.**

(1) It easy to see that for any filtration (indexed by $I$)

$$
\mu_1(\mathcal{E}^+, \alpha; \varphi) \leq \sum_{i \in I} \mu(0 \subset E_i \subset \mathcal{E}, \alpha_i; \varphi).
$$

Therefore any subfiltration of a non-critical one is still non-critical. Indeed suppose that $\mathcal{E}^+$ is a non critical filtration indexed by $I$ and $J \subset I$ indexes a critical subfiltration of $\mathcal{E}^+$. Then $\mu_1(\mathcal{E}^+, \alpha; \varphi) = \sum_{i \in I} \mu(0 \subset E_i \subset \mathcal{E}, \alpha_i; \varphi)$ (since the whole filtration is non critical) and $\mu_1(\mathcal{E}^+, \alpha; \varphi) < \sum_{i \in J} \mu(0 \subset E_i \subset \mathcal{E}, \alpha_i; \varphi)$. Therefore $\mu_{1 \prec J}(\mathcal{E}^+, \alpha; \varphi) > \sum_{i \in I \setminus J} \mu(0 \subset E_i \subset \mathcal{E}, \alpha_i; \varphi)$ which is absurd.
(2) If $\varphi$ is additive $k$-(semi)stable implies (semi)stable and therefore the two conditions are equivalent.

(3) Checking semistability conditions over non-critical filtrations is the same to check them over subbundles.

(4) Thanks to the previous considerations, the following conditions are equivalent:
   
   - (a) $(E, \varphi)$ is $\delta$-(semi)stable;
   - (b) For any subsheaf $F$ and for any critical filtration $(E^\bullet, \alpha)$ the following inequalities hold
     \[0 \leq (\text{rk} F)P_{\delta} - rP_{\varphi} - \delta \text{rk}_{F, \epsilon} - \text{ark}(F),\]
     \[0 \leq P(E^\bullet, \alpha) + \delta \mu(E^\bullet, \alpha; \varphi).\]
   
   Observe that the first part of condition (2) is just requiring that $(E, \varphi)$ is $k$-(semi)stable.

(5) Note that Proposition 6 and points 2, 3 and 4 above hold also for slope semistability.

2.3. Decorated coherent sheaves. With the expression “decorated coherent sheaf” we mean a decorated sheaf $(A, \varphi)$ such that $A$ is just a coherent sheaf (and not necessarily torsion free).

Before proceeding we recall what a decorated coherent subsheaf is. If $i: (F, \psi) \rightarrow (A, \varphi)$ is an injective morphism of decorated sheaves we get immediately from condition (2) that $\lambda \cdot \psi = i^* \varphi$. From now on we will say that the triple $((F, \psi), i)$ is a decorated subsheaf of $(A, \varphi)$ and we will denote it just by $(F, \varphi, i)$. Note moreover that, if $F$ is a subsheaf of $A$ and $i: F \rightarrow A$ is the inclusion, then it defines a decorated subsheaf; in fact defining $\psi = \varphi|_{a(b,F_a,b)}$ the triple $((F, \psi), i)$ is a decorated subsheaf of $(A, \varphi)$.

For an arbitrary decorated coherent sheaf $(A, \varphi)$ define the $\varepsilon$-decorated degree

\[\deg(A, \varphi) \triangleq \deg(A) - a\delta \varepsilon(A, \varphi),\]

where $\deg A \triangleq c_1(A) \cdot \mathcal{O}_X(1)^{n-1}$, and the $\varepsilon$-decorated Hilbert polynomial

\[P_{(A, \varphi)}(m) \triangleq P_A(m) - a\delta (m) \varepsilon(A, \varphi)\]

If moreover $\text{rk}(A) > 0$ we define the $\varepsilon$-decorated slope and, respectively, the reduced $\varepsilon$-decorated Hilbert polynomial as:

\[\mu(A, \varphi) \triangleq \frac{\deg(A, \varphi)}{\text{rk}(A)}, \quad p_{(A, \varphi)}(m) \triangleq \frac{P_{(A, \varphi)}(m)}{\text{rk}(A)}\]

Sometimes, if the morphism $\varphi$ is clear from the context, we will write $\deg_{\varepsilon}(A)$ instead of $\deg(A, \varphi)$, $\mu_{\varepsilon}(A)$ instead of $\mu(A, \varphi)$ and $P^\varepsilon_A$ (respectively $p_A^\varepsilon$) instead of $P_{(A, \varphi)}$ (resp. $p_{(A, \varphi)}$).

**Definition 8.** Let $(A, \varphi)$ be a decorated coherent sheaf of positive rank, than we will say that $(A, \varphi)$ is $\varepsilon$-(semi)stable or, respectively, slope $\varepsilon$-(semi)stable with respect to $\delta$ (resp. $\theta$) if and only if for any proper non-trivial subsheaf $F \subset A$ the following inequality holds:

\[P^{\varepsilon}_{(F, \varphi, i)} \text{rk}(A) \leq p_{(A, \varphi)} \text{rk}(F).\]
or, respectively,

$$\deg(F, \varphi) \cdot \text{rk}(A) \preceq \text{rk}(F) \cdot \deg(A, \varphi)$$

If $\text{rk}(A) = 0$ we say that $(A, \varphi)$ is semistable (resp. slope semistable) or stable (resp. slope stable) if moreover $P_A = \delta$ (resp. $\deg A = \delta$).

In particular this $\varepsilon$-(semi)stability extends $\varepsilon$-semistability, defined in Section 2.2, to decorated coherent sheaves.

**Remark 9.** Note that $\text{slope } \varepsilon\text{-stable } \Rightarrow \varepsilon\text{-stable } \Rightarrow \varepsilon\text{-semistable } \Rightarrow \text{slope } \varepsilon\text{-semistable};$

and recall that $\varepsilon$-semistability (slope $\varepsilon$-semistability) is strictly stronger than the usual semistability (resp. slope semistability) introduced in Section 2.1.

The kernel of $\varphi$ lies in $A_{a,b}$, so for our purpose we need to define a subsheaf of $A$ that plays a similar role to the kernel of $\varphi$. Therefore we let

$$K \doteq \max \{ 0 \subseteq F \subseteq A \mid F_{a,b} \subseteq \ker \varphi \}$$

where the maximum is taken with respect to the partial ordering given by the inclusion of sheaves. Note that $K$ is unique, indeed if $K'$ is another maximal element, then $K \cup K'$ is a subsheaf of $A$ such that $(K \cup K')_{a,b} \subset \ker \varphi$ and this is absurd.

**Remark 10.**

(1) Let $T(A)$ be the torsion part of $A$. The torsion part $T(A_{a,b})$ lies in $\ker \varphi$, otherwise there would be a non zero morphism between a sheaf of pure torsion and the torsion free sheaf $N$ and this is impossible. Therefore also the twisted torsion part $T(A)_{a,b} \subset T(A_{a,b})$ lies in the kernel of $\varphi$.

(2) $T(A) \subset \ker \varphi$ (point (1)) therefore $T(A) \subset K$;

(3) $A$ is torsion free if and only if $K$ is torsion free. Indeed, suppose that $K$ is torsion free and that $T(A) \neq \emptyset$, then $K \subset K \cup T(A)$ which is absurd for maximality of $K$. The converse is obvious.

(4) $A_{a,b}$ is torsion free if and only if $\ker \varphi$ is torsion free.

(5) If $(A, \varphi)$ is semistable and $\text{rk}(A) > 0$ then $K$ is a torsion free sheaf. Indeed if $T(K)$ is the torsion part of $K$, $\text{rk}(T(K)) = 0$ and then, for the semistability condition, we get that:

$$\text{rk}(A)P_{(T(K); a,b \cap \varphi|T(K)_{a,b})} \preceq 0.$$ 

Therefore $T(K)$ is zero and $K$ is torsion free.

(6) If $(A, \varphi)$ is semistable and $\text{rk}(A) > 0$ then $A$ is pure of dimension $\dim X$ and therefore torsion free. Indeed let $F$ a subsheaf of $A$ of pure torsion, then by the semistability condition we get that

$$\text{rk}(A)\{P_x - a\delta \varepsilon(\varphi|_x)\} \preceq \text{rk}(F)\{P_A - a\delta\} = 0.$$ 

Moreover, for point (1), $F_{a,b} \subset \ker \varphi$ and so $P_x \preceq 0$, this immediately implies $F = 0$.

**Remark 11.** Note that Remark 10 holds also in the slope $\varepsilon$-semistable case.

**Proposition 12.** Let $(A, \varphi)$ be a decorated coherent sheaf of positive rank and let $T \doteq T(A)$ be the torsion of $A$. Then the following statements hold.

(1) $(A, \varphi)$ is $\varepsilon$-semistable with respect to $\delta$ $\Rightarrow$ $A/T$ is $\varepsilon$-semistable with respect to $\delta$. 

(2) \((A/T, \varphi)\) is \(\varepsilon\)-semistable with respect to \(\delta\) \(\Rightarrow\) \((A, \varphi)\) is \(\varepsilon\)-semistable with respect to \(\delta\) or \(T\) is the maximal destabilizing subsheaf of \(A\) in the sense of Remark \(\ref{rem:maxdest}
\).

**Proof.** First of all note that, since \(T \subset \ker \varphi\), the pair \((A/T, \varphi)\) is a well-defined decorated (torsion free) sheaf of the same type of \((A, \varphi)\).

(1) If \((A, \varphi)\) \(\varepsilon\)-semistable with respect to \(\delta\), then as in Remark \(\ref{rem:semistable}
\) one can prove that \(T = 0\) and so obviously \((A/T, \varphi)\) is semistable.

(2) Suppose that \((A/T, \varphi)\) \(\varepsilon\)-semistable with respect to \(\delta\). If \(T\) does not destabilize, then \(P_{\varepsilon}^T = P_T \preceq 0\) and so \(T = 0\) and \((A, \varphi)\) is \(\varepsilon\)-semistable. Otherwise \(P_{\varepsilon}^T = P_T \succeq 0\) and Remark \(\ref{rem:maxdest}
\) shows that is the maximal destabilizing subsheaf.

\(\blacksquare\)

3. Mehta-Ramanathan for slope \(\varepsilon\)-semistable decorated sheaves

In this section we want to prove a Mehta-Ramanathan theorem for slope \(\varepsilon\)-semistable decorated sheaves. Before we proceed we need some notation and preliminary results.

**Notation.** Let \(k\) be an algebraic closed field of characteristic 0, \(S\) an integral \(k\)-scheme of finite type. \(X\) will be a smooth projective variety over \(k\), \(\mathcal{O}_X(1)\) an ample line bundle on \(X\) and \(f : X \to S\) a projective flat morphism. Note that \(\mathcal{O}_X(1)\) is also \(f\)-ample. In this section, unless otherwise stated, we will suppose that any decorated sheaf is of type \((a, b, N)\). If \((W, \varphi)\) is a decorated coherent sheaf over \(X\) we denote by \(W_s\) the restriction \(W|_{X_s}\), where \(X_s = f^{-1}(s)\), and \(\varphi_s\) the restriction \(\varphi|_{W_s}\). Finally, if \(F\) is a sheaf, we will denote by \(r_F\) the quantity \(\text{rk}(F)\).

3.1. Maximal destabilizing subsheaf.

**Proposition 13.** Let \((E, \varphi)\) be a decorated sheaf over a nonsingular projective smooth variety \(X\). If \((E, \varphi)\) is not \(\varepsilon\)-semistable there is a unique, \(\varepsilon\)-semistable, proper subsheaf \(F\) of \(E\) such that:

1. \(p_F^\varepsilon \succeq p_W^\varepsilon\) for all subsheaf \(W\) of \(E\).
2. If \(p_F^\varepsilon = p_W^\varepsilon\) then \(W \subset F\).

The subsheaf \(F\), with the induced morphism \(\varphi|_F\), is called maximal destabilizing subsheaf of \((E, \varphi)\).

**Proof.** First we recall that by definition \(E\) is torsion free and therefore of positive rank.

We define a partial ordering on the set of decorated subsheaves of a given decorated sheaf \((E, \varphi)\). Let \(F_1, F_2\) two subsheaves of \(E\), then

\(\begin{align*}
F_1 \preceq F_2 & \iff F_1 \subseteq F_2 \land P_{(\varphi_1, \varphi_2)} \text{rk}(F_2) \preceq P_{(\varphi_1, \varphi_2)} \text{rk}(F_1) \\
\end{align*}\)

where \(\varphi_i = \varphi|_{F_i}\). Note that the set of the subsheaves of a sheaf \(E\) with this order relation \(\preceq\) satisfies the hypothesis of Zorn’s Lemma, so there exists a maximal
element (non unique in general). Let
\begin{equation}
\mathcal{F} \doteq \min \{ \mathcal{G} \subset \mathcal{E} \mid \mathcal{G} \text{ is } \preceq \text{-maximal} \}
\end{equation}
i.e., \( \mathcal{F} \) is a \( \preceq \)-maximal subsheaf with minimal rank among all \( \preceq \)-maximal subsheaves. Then we claim that \( (\mathcal{F}, \varphi_{|\mathcal{F}}) \) has the asserted properties.

Suppose that exists \( \mathcal{G} \subset \mathcal{E} \) such that
\begin{equation}
p^\mathcal{G}_\varepsilon \geq p^\mathcal{F}_\varepsilon
\end{equation}
First we show that we can assume \( \mathcal{G} \subset \mathcal{F} \) by replacing \( \mathcal{G} \) by \( \mathcal{F} \cap \mathcal{G} \). Indeed if \( \mathcal{G} \not\subset \mathcal{F} \), \( \mathcal{F} \) is a proper subsheaf of \( \mathcal{F} + \mathcal{G} \) in fact \( \mathcal{F} \not\subset \mathcal{G} \) (otherwise \( \mathcal{F} \preceq \mathcal{G} \) which is absurd for maximality of \( \mathcal{F} \)). By maximality one has that
\begin{equation}
p^\mathcal{G}_\varepsilon \succ p^\mathcal{F}_\varepsilon + p^\mathcal{G}_\varepsilon.
\end{equation}
Using the exact sequence
\[ 0 \rightarrow \mathcal{F} \cap \mathcal{G} \rightarrow \mathcal{F} \oplus \mathcal{G} \rightarrow \mathcal{F} + \mathcal{G} \rightarrow 0 \]
one finds \( \text{rk}(\mathcal{F} \cap \mathcal{G}) = \text{rk}(\mathcal{F} \oplus \mathcal{G}) = \text{rk}(\mathcal{F} \cap \mathcal{G}) + \text{rk}(\mathcal{F} + \mathcal{G}) \). Hence
\begin{equation}
r_{\mathcal{F} \cap \mathcal{G}}(p^\mathcal{G}_\varepsilon - p^\mathcal{G}_\varepsilon) = r_{\mathcal{F} + \mathcal{G}}(p^\mathcal{G}_\varepsilon - p^\mathcal{G}_\varepsilon) + (r^\mathcal{G}_\varepsilon - r_{\mathcal{F} \cap \mathcal{G}})(p^\mathcal{G}_\varepsilon - p^\mathcal{G}_\varepsilon).
\end{equation}
where we denote by \( r_{\mathcal{F}}, r_{\mathcal{G}}, r_{\mathcal{F} + \mathcal{G}} \) and \( r_{\mathcal{F} \cap \mathcal{G}} \) the rank of \( \text{rk}(\mathcal{F}), \text{rk}(\mathcal{G}), \text{rk}(\mathcal{F} + \mathcal{G}) \) and \( \text{rk}(\mathcal{F} \cap \mathcal{G}) \) respectively.

If the morphism \( \varphi \) is zero \( \varepsilon \)-semistability coincides with the usual semistability for torsion-free sheaves and the existence of the maximal destabilizing subsheaf is a well-known fact that one can find, for example, in \[6\] Lemma 1.3.6. So we suppose that \( \varepsilon(\varphi) = 1 \). From the above inequalities between reduced decorated Hilbert polynomial of \( \mathcal{F}, \mathcal{G} \) and \( \mathcal{F} + \mathcal{G} \) one can easily obtain:
\[ p_{\mathcal{F} + \mathcal{G}} - p_{\mathcal{F}} \leq a\delta \left( \frac{\varepsilon_{\mathcal{F} + \mathcal{G}}}{r_{\mathcal{F} + \mathcal{G}}} - \frac{\varepsilon_{\mathcal{F}}}{r_{\mathcal{F}}} \right) \]
\[ p_{\mathcal{F}} - p_{\mathcal{G}} \leq a\delta \left( \frac{\varepsilon_{\mathcal{F}}}{r_{\mathcal{F}}} - \frac{\varepsilon_{\mathcal{G}}}{r_{\mathcal{G}}} \right), \]
therefore, using equation (28) and after some easy computations, one gets
\[ r_{\mathcal{F} \cap \mathcal{G}}(p^\mathcal{G}_\varepsilon - p^\mathcal{F}_\varepsilon) = r_{\mathcal{F} + \mathcal{G}}(p^\mathcal{G}_\varepsilon - p^\mathcal{G}_\varepsilon) + (r^\mathcal{G}_\varepsilon - r_{\mathcal{F} \cap \mathcal{G}})(p^\mathcal{G}_\varepsilon - p^\mathcal{G}_\varepsilon) - a\delta r_{\mathcal{F} \cap \mathcal{G}} \left( \frac{\varepsilon_{\mathcal{F} + \mathcal{G}}}{r_{\mathcal{F} + \mathcal{G}}} - \frac{\varepsilon_{\mathcal{F}}}{r_{\mathcal{F}}} \right) + a\delta r_{\mathcal{F} \cap \mathcal{G}} \left( \frac{\varepsilon_{\mathcal{F}}}{r_{\mathcal{F}}} - \frac{\varepsilon_{\mathcal{G}}}{r_{\mathcal{G}}} \right) - a\delta \varepsilon_{\mathcal{F} \cap \mathcal{G}} \varepsilon_{\mathcal{G}} + a\delta \varepsilon_{\mathcal{F} \cap \mathcal{G}} \varepsilon_{\mathcal{G}} \]
\[ \leq 0. \]
Therefore we can suppose both \( \mathcal{G} \subset \mathcal{F} \) and \( p^\mathcal{G}_\varepsilon \geq p^\mathcal{F}_\varepsilon \), and, up to replacing \( \mathcal{G} \), we can suppose that \( \mathcal{G} \) is maximal in \( \mathcal{F} \) with respect to \( \preceq \). Let \( \mathcal{G}' \) be a \( \preceq \)-maximal in \( \mathcal{E} \) among all subsheaves (of \( \mathcal{E} \)) containing \( \mathcal{G} \). Then
\[ p^\mathcal{F}_\varepsilon \leq p^\mathcal{G}_\varepsilon \leq p^\mathcal{G}'_\varepsilon. \]
Note that neither \( G' \) is contained in \( F \), because \( F \) has minimal rank between all \( \leq \)-maximal subsheaves of \( E \), nor \( F \) is contained in \( G' \), for maximality of \( F \); therefore \( F \) is a proper subsheaf of \( F + G' \) and, for maximality, \( p^\varepsilon_F \geq p^\varepsilon_{F + G'} \). As before one gets
\[
p^\varepsilon_{F + G'} > p^\varepsilon_F \succeq p^\varepsilon_G,
\]
but \( G \subseteq F \cap G' \subseteq F \) and this contradicts the assumptions on \( G \). Therefore \( F \) satisfies the required properties. The uniqueness and the \( \varepsilon \)-semistability of \( F \) easily follow from properties (1) and (2).

Lemma 14. Let \((E, \varphi)\) be as before. If it is not slope \( \varepsilon \)-semistable there is a unique proper subsheaf \( F \) of \( E \) such that:
\[
\begin{align*}
(1) & \quad \mu_\varepsilon(F) \geq \mu_\varepsilon(W) \text{ for all subsheaves } W \text{ of } E. \\
(2) & \quad \text{If } \mu_\varepsilon(F) = \mu_\varepsilon(W) \text{ then } W \subset F.
\end{align*}
\]

Proof. The proof is the same of Proposition 13, it is sufficient to replace \( p^\varepsilon \) with \( \mu_\varepsilon \), \( P^\varepsilon \) with \( \deg_\varepsilon \), and \( \delta \) with \( \delta \).

Remark 15. Note that, if \((E, \varphi)\) is \( \varepsilon \)-semistable, or, respectively, slope \( \varepsilon \)-semistable, then the maximal decorated destabilizing (resp. slope destabilizing) subsheaf coincides with \( E \).

Proposition 16. Let \((A, \varphi)\) be a decorated coherent sheaf of positive rank, then Proposition 13 and Lemma 14 hold true, in the sense that if \((A, \varphi)\) is not \( \varepsilon \)-semistable (slope \( \varepsilon \)-semistable respectively) there is a unique, \( \varepsilon \)-semistable, proper subsheaf \( F \) of \( E \) such that:
\[
\begin{align*}
(1) & \quad P^\varepsilon_F \rk(E) \geq P^\varepsilon_W \rk(F) \text{ for all subsheaves } W \text{ of } E. \\
(2) & \quad \text{If } P^\varepsilon_F \rk(W) = P^\varepsilon_W \rk(F) \text{ then } W \subset F.
\end{align*}
\]
or, respectively
\[
\begin{align*}
1'. & \quad \deg_\varepsilon(F) \rk(W) \geq \deg_\varepsilon(W) \rk(F) \text{ for all subsheaves } W \text{ of } E. \\
2'. & \quad \text{If } \deg_\varepsilon(F) \rk(W) = \deg_\varepsilon(W) \rk(F) \text{ then } W \subset F.
\end{align*}
\]

Proof. Indeed, let \( F \) be a minimal rank sheaf between all \( \leq \)-maximal sheaves as in the proof of Proposition 13. Suppose that \( \rk(F) = 0 \), then \( F \subset K = T(A) \), and so, by maximality \( F = T(A) \). If \( G \) is such that \( \rk(G) > 0 \) and \( T(A) \leq G \) then \( T(A) \) stabilize and so \( T(A) \) is clearly unique and semistable. Otherwise, if \( \rk(F) > 0 \), then \( A \) has no nontrivial rank zero subsheaves, in particular is torsion free. Indeed if exists a subsheaf \( G \subset A \) with \( r_G = 0 \) then, by the above considerations exists \( G' \) with \( \rk(G') = 0 \), \( G \subset G' \) and \( G' \leq \maximal \) which is absurd by the assumptions on \( F \). Then the proof continues as the proof of Proposition 13.

The proof in the case of slope \( \varepsilon \)-semistability is the same.

Remark 17. Proposition 13 and Lemma 14 hold true also for decorated sheaves of type \((a, b, c, \mathbb{N})\) as pointed out in (1) Remark 4.

3.2. Families of decorated sheaves. Let \( f : Y \rightarrow S \) be a morphism of finite type of Noetherian schemes. Recall that a flat family of coherent sheaves on the fibre of the morphism \( f \) is a coherent sheaf \( A \) over \( Y \), which is flat over \( S \), i.e., for any \( y \in Y \) \( A_y \) is flat over the local ring \( \mathcal{O}_{S, (y)} \). If \( A \) is flat over the fibre of \( f \) the Hilbert polynomial \( P_A \) is locally constant as a function of \( s \). The converse is not true in general, but, if \( S \) is reduced, then the two assertions are equivalent.
Definition 18. Let \((\mathcal{E}, \varphi)\) be a decorated sheaf over \(Y\) of type \((a, b, N)\) and \(f : Y \to S\) be a morphism of finite type between Noetherian schemes. Then \((\mathcal{E}, \varphi)\) is a flat family over the fibre of \(f\) if and only if

- \(\mathcal{E}\) and \(N\) are flat families of coherent sheaves over the fibre of \(f : Y \to S\);
- \(\mathcal{E}_s = \mathcal{E}_{f^{-1}(s)}\) is torsion free for all \(s \in S\);
- \(N_s = N_{f^{-1}(s)}\) is locally free for all \(s \in S\);
- \(\varepsilon(\varphi_s) = \varepsilon(\varphi_{|\mathcal{E}_s})\) is locally constant as a function of \(s\).

Note that the above conditions imply that the \(\varepsilon\)-Hilbert polynomials \(P^\varepsilon_{\mathcal{E}_s}\) are locally constant for \(s \in S\).

Definition 19. Let \((A, \varphi)\) be a decorated coherent sheaf of positive rank. Then \((A, \varphi)\) is a flat family over the fibre of \(f\) if and only if

- \(A\) and \(N\) are flat families of coherent sheaves over the fibre of \(f : Y \to S\);
- \(N_s\) is locally free for all \(s \in S\);
- \(\varepsilon(\varphi_s)\) is locally constant as a function of \(s\);
- \(\text{rk}(A_s) > 0\) for any \(s \in S\).

As in Definition 18, the above conditions imply that the \(\varepsilon\)-Hilbert polynomials \(P^\varepsilon_{A_s}\) are locally constant for \(s \in S\).

3.3. Families of quotients. Let \((A, \varphi)\) be a decorated coherent sheaf over \(X\) and let \(q : A \to Q\) be surjective morphism of sheaves. Let \(F\) be the subsheaf of \(A\) defined by \(\ker q\), so the following succession of sheaves is exact:

\[
0 \to F \xrightarrow{i} A \to Q \to 0.
\]

Note that \(F\) is uniquely determined by \(Q\) and therefore also \((F, \varphi_{|F})\) is uniquely (up to isomorphism of decorated sheaves) determined by \(Q\). Indeed, let \((F, \psi)\) be another decorated subsheaf of \((A, \varphi)\), then, by definition of decorated subsheaf, there exists a non-zero scalar morphism \(\lambda : N \to N\) such that \(\lambda \circ \psi = \varphi\), then

\[
\begin{array}{ccc}
F_{a,b} & \xrightarrow{i_{a,b}} & A_{a,b} \\
\psi & \downarrow & \varphi \\
N & \xrightarrow{\lambda} & N \\
\end{array}
\]

Since the big square and the upper triangle commute, the entire diagram commutes and so it easy to see that \((F, \psi)\) and \((F, \varphi_{|F})\) are isomorphic as decorated sheaves.

Suppose now that \((F, \varphi_{|F})\) de-semistabilizes a decorated sheaf \((\mathcal{E}, \varphi)\) (with respect to the slope \(\varepsilon\)-semistability), then \(\mu^\varepsilon(F) > \mu^\varepsilon(\mathcal{E})\) and so

\[
\mu(F) - \frac{a \delta_{\mathcal{E}}}{r_{\mathcal{E}}} > \mu(\mathcal{E}) - \frac{a \delta_{\mathcal{E}}}{r_{\mathcal{E}}},
\]

\[
\deg(F) > r_{\mathcal{E}} \left[ \mu(\mathcal{E}) + a \delta \left( \frac{\varepsilon_{\mathcal{E}}}{r_{\mathcal{E}}} - \frac{\varepsilon_{F}}{r_{F}} \right) \right]
\]
Recalling that $\deg(\mathcal{E}) = \deg(\mathcal{F}) + \deg(\mathcal{Q})$

$$\deg(\mathcal{Q}) < \deg(\mathcal{E}) - r_F \left[ \mu(\mathcal{E}) + a\delta \left( \frac{\varepsilon_F}{r_F} - \frac{\varepsilon_E}{r_E} \right) \right]$$

$$= \mu(\mathcal{E})r_Q - a\delta \left( \varepsilon_F - \varepsilon_E \frac{r_F}{r_E} \right)$$

and therefore, if $\varepsilon_F = 1$,

$$\mu(Q) < \mu(E) - a\delta \left( \frac{r_F}{r_Q} - \frac{r_F}{r_Q r_E} \right) = \mu(E) + a\delta \cdot \begin{cases} \frac{r_F}{r_Q r_E} \div C_0 & \text{if } \varepsilon_F = 0 \\ -\frac{1}{r_E} \div -C_1 & \text{if } \varepsilon_F = 1, \end{cases}$$

otherwise, if $\varepsilon_F = 0$ then also $\varepsilon_E = 0$ and so we get that $\mu(Q) < \mu(E)$.

**Remark 20.** Defining $\deg_\varepsilon(Q) = \deg_\varepsilon(\mathcal{E}) - \deg_\varepsilon(\mathcal{F})$ and $\mu_\varepsilon(Q) = \deg_\varepsilon(Q)/r_Q$ one easily gets that $\mu_\varepsilon(\mathcal{F}) > \mu_\varepsilon(\mathcal{E})$ if and only if $\mu_\varepsilon(Q) \leq \mu_\varepsilon(\mathcal{E})$. Note that in general it is not possible to define a morphism $\psi$ over $\mathcal{Q}$ such that $(\mathcal{Q}, \psi)$ is a decorated sheaf and $\varepsilon(\mathcal{Q}, \psi) + \varepsilon(\mathcal{F}, \mathcal{F}) = \varepsilon(\mathcal{E}, \varphi)$. In fact it is possible to define a morphism to the quotient satisfying such properties if and only if $k_{\varepsilon, \varepsilon} = 0$ or $k_{\varepsilon, \varepsilon} = a$. This is because only in these two cases the morphism $\varphi$:

$$\mathcal{F}_{a,b} \xrightarrow{\varphi} \mathcal{E}_{a,b} \xrightarrow{\psi} \mathcal{N}$$

is well defined and so it is possible to give a well-defined structure of decorated sheaf to $(\mathcal{E}/\mathcal{F})$.

Analogously, if $(\mathcal{F}, \varphi|_x)$ de-semistabilizes $(\mathcal{E}, \varphi)$ with respect to the $\varepsilon$-semistability, i.e., if

$$p_F - a\delta \frac{\varepsilon_F}{r_F} \succ p_E - \frac{a\delta}{r_E},$$

then similar calculations show that

$$p_Q \prec p_E + a\delta \cdot \begin{cases} C_0 & \text{if } \varepsilon_F = 0 \\ -C_1 & \text{if } \varepsilon_F = 1, \end{cases}$$

Note that condition (29) implies condition (30), conversely, if $p_Q \prec p_E + \delta C$ then $\mu(Q) \leq \mu(\mathcal{E}) + \delta C$.

Let $(\mathcal{E}, \varphi)$ be a flat family of decorated sheaves over the fibre of a projective morphism $f : X \to S$. Let $P = P_{\varepsilon,s}$ and $p = p_{\varepsilon,s}$ the Hilbert polynomial and, respectively, the reduced Hilbert polynomial of $\mathcal{E}$ (which are constant because the family is flat over $S$). Define:

1. $\mathcal{S}$ as the family (over the fibre of $f$) of saturated subsheaves $\mathcal{F} \hookrightarrow \mathcal{E}_s$ such that the induced torsion free quotient $\mathcal{E}_s \to \mathcal{Q}$ satisfy $\mu(\mathcal{Q}) \leq \mu(\mathcal{E}_s) + a\delta C_0$;
2. $\mathcal{S}_o$ as the family of decorated subsheaves $(\mathcal{F}, \varphi|_{\mathcal{F}}) \to (\mathcal{E}_s, \varphi|_{\mathcal{E}_s})$ such that:
   - $\varepsilon(\mathcal{F}, \mathcal{F}) = 0$;
   - $p_F^s = p_F \succ p^s_E$ (i.e., $p_Q \prec p + a\delta C_0$ with $Q = \text{coker}(\mathcal{F} \hookrightarrow \mathcal{E}_s)$);
   - $\mathcal{F}$ is a saturated subsheaf of $\mathcal{E}_s$;
subsheaves of a flat family \( E \) of quotients sheaves \( Q \) of \( \mathcal{O}_Y \).

Lemma 26. Let \( S \) be a Noetherian scheme and denote by \( \mathcal{O}_Y \) a coherent sheaf relative to \( S \). Let \( \mathcal{O}_Y \) be a saturated subsheaf of \( \mathcal{E}_s \);

We want to prove that the set of Hilbert polynomials of destabilizing decorated subsheaves of a flat family \( (\mathcal{E}, \varphi) \) of decorated sheaves over the fibre of a projective morphism \( f : X \to S \) is a finite set. From this we conclude that the semistability condition is an open condition, i.e., the set \( \{ s \in S \mid (\mathcal{E}_s, \varphi_s) \text{ is slope } \varepsilon \text{-semistable} \} \) is open in \( S \). In order to prove this result we first need to recall some facts.

Definition 21. A family of isomorphism classes of coherent sheaves on a projective scheme \( Y \) over \( k \) is bounded if there is a \( k \)-scheme \( S \) of finite type and a coherent \( \mathcal{O}_{S \times Y} \)-sheaf \( \mathcal{G} \) such that the given family is contained in the following set: \( \{ \mathcal{G}_{|\text{spec}(k(x)) \times Y} \mid s \text{ is a closed point in } S \} \).

Definition 22. A sheaf \( \mathcal{A} \) over \( Y \) is said \( m \)-regular if

\[ H^i(Y, \mathcal{A}(m - i)) = 0 \text{ for all } i > 0. \]

Then the following statements hold:

Lemma 23 (Lemma 1.7.2 of [4]). If \( \mathcal{A} \) is \( m \)-regular, then

i) \( \mathcal{A} \) is \( m' \)-regular for all integers \( m' \geq m \).

ii) \( \mathcal{A}(m) = \mathcal{A} \otimes \mathcal{O}_X(m) \) is globally generated.

iii) For all \( n \geq 0 \) the natural homomorphisms

\[ H^0(X, \mathcal{A}(m) \otimes H^0(X, \mathcal{O}_X(n)) \to H^0(X, \mathcal{A}(m + n)) \]

are surjective.

Lemma 24 (Lemma 1.7.6 of [5]). The following properties of families of sheaves \( \{ \mathcal{A}_i \}_{i \in I} \) are equivalent:

i) the family is bounded;

ii) the set of Hilbert polynomials \( \{ P_{\mathcal{A}_i} \}_{i \in I} \) is finite and there is a uniform bound for \( \text{reg}(\mathcal{A}_i) \leq C \) for all \( i \in I \);

iii) the set of Hilbert polynomials \( \{ P_{\mathcal{A}_i} \}_{i \in I} \) is finite and there is a coherent sheaf \( \mathcal{A} \) such that all \( \mathcal{A}_i \) admit surjective morphisms \( \mathcal{A} \to \mathcal{A}_i \).

Definition 25. Let \( \mathcal{A} \) be a coherent sheaf. We call hat-slope the rational number

\[ \hat{\mu}(\mathcal{A}) = \frac{\beta_{\dim \mathcal{A} - 1}(\mathcal{A})}{\beta_{\dim \mathcal{A}}(\mathcal{A})} , \]

where \( \beta_i(\mathcal{A}) \) is defined as the coefficient of \( x^i \) of the Hilbert polynomial of \( \mathcal{A} \) multiplied by \( i! \), i.e., if \( P_x(x) = \sum_{i=0}^{\dim \mathcal{A}} \beta_i(\mathcal{A}) x^i \), then \( \beta_i(\mathcal{A}) = \beta_i \).

Lemma 26 (Lemma 2.5 in [4]). Let \( f : Y \to S \) be a projective morphism of Noetherian schemes and denote by \( \mathcal{O}_Y(1) \) a line bundle on \( Y \), which is very ample relative to \( S \). Let \( \mathcal{O}_Y \) be a coherent sheaf on \( Y \) and \( Q \) the set of isomorphism classes of quotients sheaves \( Q \) of \( \mathcal{A}_s \) for \( s \) running over the points of \( S \). Suppose that the dimension of \( Y_s \) is \( \leq r \) for all \( s \). Then the coefficient \( \beta_r(Q) \) is bounded from above and below, and \( \beta_{r-1}(Q) \) is bounded from below. If \( \beta_{r-1}(Q) \) is bounded from above, then the family of sheaves \( Q/T(Q) \) is bounded.
Proposition 27. Let \( \mathcal{A} \) be a flat family of coherent sheaves on the fibres of a projective morphism \( f : Y \to S \) of Noetherian schemes. Then the family of torsion free quotient \( \mathcal{Q} \) of \( \mathcal{A}_s \) for \( s \in S \) with hat slope bounded from above is a bounded family.

Proof. It is an easy corollary of Lemma 2.5 in [1].

Thanks to Proposition 27 the family \( \mathfrak{F} \) is bounded. Due the previous considerations both families \( \mathfrak{F}_\circ \) and \( \mathfrak{F}_1 \) can be regarded as subfamilies of \( \mathfrak{F} \) and therefore \( \mathfrak{F}_\circ \) and \( \mathfrak{F}_1 \) are bounded families as well. Thanks to Proposition 24 the sets \( \{ P_x | F \in \mathfrak{F}_\circ \} \) and \( \{ P_x | F \in \mathfrak{F}_1 \} \) are finite.

3.4. Quot schemes. Let \( \mathcal{A} \) be a coherent sheaf over \( X \) flat over the fibres of \( f : X \to S \). Let \( P \in \mathbb{Q}[x] \) be a polynomial. Define a functor

\[
\mathcal{Q} \coloneqq \mathcal{Q}_{\text{quot}}_{X/S}(\mathcal{A}, P) : (\text{Sch}/S) \to (\text{Sets})
\]

as follows: if \( T \to S \) is scheme over \( S \) let \( \mathcal{Q}(T) \) be the set of all \( T \)-flat coherent quotient sheaves \( \mathcal{A}_T \to \mathcal{Q} \) with Hilbert polynomial \( P \), where \( \mathcal{A}_T \) denotes the sheaf over \( X_T = X \times_S T \) induced by \( \mathcal{A} \). If \( g : T' \to T \) is an \( S \)-morphism, let \( \mathcal{Q}(g) : \mathcal{Q}(T) \to \mathcal{Q}(T') \) be the map that sends \( \mathcal{A}_T \to \mathcal{Q} \) to \( \mathcal{A}_T' \to g_X^* \mathcal{Q} \), where \( g_X : X_T' \to X_T \) is the map induced by \( g \).

Theorem 28 (Theorem 2.2.4 in [6]). The functor \( \mathcal{Q}_{\text{quot}}_{X/S}(\mathcal{A}, P) \) is represented by a projective \( S \)-scheme \( \pi : \mathcal{Q}_{\text{quot}}_{X/S}(\mathcal{A}, P) \to S \).

Consider now a decorated coherent sheaf \( (\mathcal{A}, \varphi) \) over \( X \), flat over the fibre of \( f : X \to S \) and let \( P \in \mathbb{Q}[x] \) be a polynomial. Define the functor

\[
\mathcal{Q}^\circ \coloneqq \mathcal{Q}^\circ_{\text{quot}}_{X/S}(\mathcal{A}, \varphi, P) : (\text{Sch}/S) \to (\text{Sets})
\]

as follows: if \( T \to S \) is scheme over \( S \) let \( \mathcal{Q}^\circ(T) \) be the set of all \( T \)-flat coherent quotient sheaves \( \mathcal{A}_T \to \mathcal{Q} \) with Hilbert polynomial \( P \) such that \( \varepsilon(\varphi_{T|\ker(\mathcal{A}_T \to \mathcal{Q})}) = 0 \), where \( \mathcal{A}_T \) denotes the sheaf over \( X_T = X \times_S T \) induced by \( \mathcal{A} \) and \( \varphi_T : (\mathcal{A}_T)_{a,b} \to \mathcal{A}_{a,b} \) is the morphism induced by \( \varphi \). If \( g : T' \to T \) is an \( S \)-morphism, let \( \mathcal{Q}^\circ(g) : \mathcal{Q}^\circ(T) \to \mathcal{Q}^\circ(T') \) be the map that sends \( \mathcal{A}_T \to \mathcal{Q} \) to \( \mathcal{A}_T' \to g_X^* \mathcal{Q} \), note that \( g_X^* \varphi_T \) is zero if restricted on \( \ker(\mathcal{A}_T \to \mathcal{Q}) \).

Theorem 29. The functor \( \mathcal{Q}^\circ_{\text{quot}}_{X/S}(\mathcal{A}, \varphi, P) \) is represented by a projective \( S \)-scheme \( \pi^\circ : \mathcal{Q}^\circ_{\text{quot}}_{X/S}(\mathcal{A}, \varphi, P) \to S \) that is a closed subscheme of \( \mathcal{Q}_{\text{quot}}_{X/S}(\mathcal{A}, P) \).

Proof. The additional property is closed and therefore, using the same arguments of the proof of Theorem 1.6 in [12], one can prove that \( \mathcal{Q}^\circ_{\text{quot}}_{X/S}(\mathcal{A}, \varphi, P) = \{ q \in \mathcal{Q}_{\text{quot}}_{X/S}(\mathcal{A}, P) | \varepsilon(\varphi_{\ker(q)}) = 0 \} \) is a closed projective subscheme of \( \mathcal{Q}_{\text{quot}}_{X/S}(\mathcal{A}, P) \).

3.5. Openness of semistability condition.

Proposition 30. Let \( f : X \to S \) be a projective morphism of Noetherian schemes and let \( (\mathcal{E}, \varphi) \) be a flat family of decorated sheaves over the fibre of \( f \). The set of points \( s \in S \) such that \( (\mathcal{E}_s, \varphi_s) \) is \( \varepsilon \)-(semi)stable with respect to \( \delta \) is open in \( S \).

Proof. Let \( P = P_\varepsilon \) and \( p = p_\varepsilon \) the Hilbert polynomial and, respectively, the reduced Hilbert polynomial of \( \mathcal{E} \). We first consider the semistable case. Let

\[
A \coloneqq \{ P'' \in \mathbb{Q}[x] | \exists s \in S, \exists q : \mathcal{E}_s \to \mathcal{Q} \text{ such that } P = P'' \text{ and } \ker(q) \in \mathfrak{F} \}
\]
and, for \( i = 0, 1 \), let

\[ A_i \doteq \{ P'' \in \mathbb{Q}[x] \mid \exists s \in S, \exists q : E_s \to Q \text{ such that } P_q = P'' \text{ and } \ker(q) \in \mathfrak{F}_i \} \]

The sets \( A_0, A_1 \) and \( A_2 \) are finite because the families \( \mathfrak{G}, \mathfrak{G}_0 \) and \( \mathfrak{G}_1 \) are bounded as proved in Section 3.3. For any \( P'' \in A_1 \) consider the Quot scheme \( \pi : \mathbb{Q}_X/S(E, P'') \to S \), while for \( P'' \in A_0 \) consider the Quot scheme \( \pi : \mathbb{Q}_X/S(E, \varphi, P'') \to S \). Both images \( S(P'') \) of \( \pi \) (for \( P'' \in A_1 \)) and \( S^\circ (P'') \) of \( \pi^\circ \) (for \( P'' \in A_0 \)) are closed sets of \( S \). Therefore the union

\[
\left( \bigcup_{P'' \in A_0} S^\circ (P'') \right) \cup \left( \bigcup_{P'' \in A_1} S(P'') \right)
\]

is a closed subset of \( S \), in fact it is finite union of closed sets. Finally it is easy to see that \((E_s, \varphi_s)\) is semistable if and only if \( s \) is not in the above union.

The proof of the stable case is similar to the semistable case, it is indeed sufficient to consider, for \( i = 0, 1 \), the sets

\[ A_i^* \doteq \{ P'' \in A \mid \text{with } p_q \leq p + (1 - i)(-a\delta C_0) + i(a\delta C_1) \} \]

and continue as in the semistable case.

### 3.6. Relative maximal destabilizing subsheaf.

**Theorem 31.** Let \((X, \mathcal{O}_X(1))\), \( S, f : X \to S \) and \((E, \varphi)\) as before. Then there is an integral \( k \)-scheme \( T \) of finite type, a projective birational morphism \( g : T \to S \), a dense open subset \( U \subset T \) and a flat quotient \( Q \) of \( \mathcal{E}_T \) such that for all points \( t \in U \), \( \mathcal{F}_t \doteq \ker(\mathcal{E}_t \to Q_t) \) with the induced morphism \( \varphi_t|_{\mathcal{F}_t} \) is the maximal destabilizing subsheaf of \((\mathcal{E}_t, \varphi_t)\) or \( Q_t = \mathcal{E}_t \).

Moreover the pair \((g, Q)\) is universal in the sense that if \( g' : T' \to S \) is any dominant morphism of \( k \)-integral schemes and \( Q' \) is a flat quotient of \( \mathcal{E}_{T'} \), satisfying the same property of \( Q \), there is an \( S \)-morphism \( h : T' \to T \) such that \( h^*(Q) = Q' \).

**Proof.** In the proof we apply the same arguments as in [10]. Define \( B_1 = A_1 \) and \( B_0 = A'_0 \), i.e.,

\[ B_0 = \{ P'' \in A \mid p_q \leq p - a\delta C_0 \} \]

\[ B_1 = \{ P'' \in A \mid p_q < p + a\delta C_1 \} \]

Then define

\[ \tilde{B}_0 \doteq \{ P'' \in B_0 \mid \pi^\circ(\mathbb{Q}_X/S(E, \varphi, P'')) = S \} \]

\[ \tilde{B}_1 \doteq \{ P'' \in B_1 \mid \pi(\mathbb{Q}_X/S(E, P'')) = S \text{ and } \forall s \in S \text{ s.t. } \pi^{-1}(s) \not\subset Q_0 \} \]

Note that \( B_0 \cup B_1 \) and \( \tilde{B}_0 \cup \tilde{B}_1 \) are nonempty. We want to define an order relation on \( B_0, \tilde{B}_0, B_1 \) and \( \tilde{B}_1 \) but first we need the following construction: let \( P''_1, P''_2 \) be polynomials in \( B_0, \tilde{B}_0, B_1 \) or \( \tilde{B}_1 \); then there exist surjective morphisms \( q_i : E_s \to Q_i \) (\( i = 1, 2 \)) such that \( P''_i = P_q \), Define, for \( i = 1, 2 \), \( P_i \doteq P_{q_i} \), \( r_i \doteq \text{rk}(\ker(q_i)) \) and \( p_i = P_i/r_i \). We will say that the polynomials \( P_i \) are associated with the polynomials \( P''_i \).

If \( P''_2 \in B_0 \) or \( \tilde{B}_0 \) define the following order relation:

\[ P''_1 \prec P''_2 \iff p_1 > p_2 \quad \text{or} \quad p_1 = p_2 \text{ and } r_1 > r_2, \]
otherwise, if \( P'' \in B_1 \) or \( \tilde{B}_1 \), define:

\[
P'' \prec P'^{a} \iff p_1 - \frac{a \delta}{r_1} > p_2 - \frac{a \delta}{r_2} \quad \text{or} \quad p_1 - \frac{a \delta}{r_1} = p_2 - \frac{a \delta}{r_2} \quad \text{and} \quad r_1 > r_2
\]

Let \( P''_i \), for \( i = 0, 1 \), be a \( \prec \)-minimal polynomial among all polynomials in \( \tilde{B}_i \) and \( P_i^\prime \) the associated polynomials. Then consider the following cases:

Case 1: \( p_0^0 > p^\prime_1 - \frac{a \delta}{r_1} \);
Case 2: \( p_0^0 < p^\prime_1 - \frac{a \delta}{r_1} \);
Case 3: \( p_0^0 = p^\prime_1 - \frac{a \delta}{r_1} \) and \( r_0^0 > r^\prime_1 \);
Case 4: \( p_0^0 = p^\prime_1 - \frac{a \delta}{r_1} \) and \( r_0^0 < r^\prime_1 \);

In the first and third case define \( P'' = P''_0 \), in the second and fourth case put \( P'' = P''_1 \). Note that the set

\[
U' = \left( \bigcup_{P'' \in B_0, P'' < P''_0} \pi^0(Q_{\mathcal{X}/S}(\mathcal{E}, \varphi, P'')) \right) \cup \left( \bigcup_{P'' \in B_1, P'' < P''_1} \pi(Q_{\mathcal{X}/S}(\mathcal{E}, P'')) \right)
\]

is a proper closed subscheme of \( S \). In fact it is proper and closed because it is a finite union of closed proper subschemes of \( S \). Call \( U' \) its complement in \( S \).

Suppose that \( P'' \in B_0 \). By definition the projective morphism

\[
\pi^0(Q_{\mathcal{X}/S}(\mathcal{E}, \varphi, P'')) \to S
\]

is surjective and for any point \( s \in S \) the fibre of \( \pi^0 \) at \( s \) parametrizes possible quotients with Hilbert polynomial \( P'' \). The associated subsheaf of any such quotient is, by construction, the maximal decorated destabilizing subsheaf. The case that \( P'' \in \tilde{B}_1 \) is similar. Finally by re-adapting the techniques used in the proof of the corresponding result in [10], one concludes.

3.7. Restriction theorem. Let \( X \) be a smooth projective variety and \( \mathcal{O}_X(1) \) be a fixed ample line bundle. Let \( (\mathcal{E}, \varphi) \) be a decorated sheaf of type \( (a, b, N) \) over \( X \) with non-zero decoration morphism. For a fixed positive integer \( a \in \mathbb{N}^+ \), we define:

- \( \Pi_a \triangleq |\mathcal{O}_X(a)| \) the complete linear system of degree \( a \) in \( X \);
- \( Z_a \triangleq \{(D, x) \in \Pi_a \times X \mid x \in D\} \) the incidence variety with projections

\[
\begin{array}{ccc}
\Pi_a \times X & \overset{\varphi_a}{\longrightarrow} & Z_a \\
\downarrow & & \downarrow \\
\Pi_a & \overset{p_a}{\longrightarrow} & X
\end{array}
\]

One can prove (see Section 2 of [3]) that:

\[
(32) \quad \text{Pic}(Z_a) = q^*_a \text{Pic}(X) \oplus p^*_a \text{Pic}(\Pi_a).
\]

For any sheaf \( \mathcal{G} \) over \( X \) one has \( P_{\mathcal{G}|\Pi_a}(n) = P_{\mathcal{G}}(n) - P_{\mathcal{G}}(n - a) \), therefore, given a decorated sheaf \( (\mathcal{E}, \varphi) \) over \( X \) with decoration of type \( \frac{a}{b} = (a, b, N) \), for all \( D \in \Pi_a \) the restrictions \( \mathcal{E}_{\mid D} \) and \( N_{\mid D} \) have constant Hilbert polynomials. Since \( \Pi_a \) is reduced, as remarked at the beginning of Section 3.2, it follows that \( q^*_a \mathcal{N} \) and \( q^*_a \mathcal{E} \) are flat families of sheaves on the fibre of \( p_a : Z_a \to \Pi_a \).
Remark 32. If for any $D \in \Pi_\alpha$ \( \varphi_{a,|q^*_D E|_{\mu^{-1}(D)}} = \varphi_{(\epsilon, D)} \neq 0 \), the family of decorated sheaves \((q^*_D E, q^*_D \varphi)\) is flat. Otherwise, since to be nonzero is open condition, there exists a dense open subset of $\Pi_\alpha$ over which \((q^*_D E, q^*_D \varphi)\) is flat.

Thanks to this remark and Theorem 41, there exist a dense open subset $V_\alpha$ of $\Pi_\alpha$ and a torsion-free sheaf $Q_\alpha$ over $Z_{V_\alpha}$ such that:
- \((E_\alpha, \varphi_\alpha) \approx (q^*_\alpha E, q^*_\alpha \varphi)\) is flat over $V_\alpha$;
- $Q_\alpha$ is flat over $V_\alpha$;
- $F_\alpha \approx \ker(E_\alpha \to Q_\alpha)$, with the induced morphism $\varphi_{a,|Q_\alpha}$, is the relative maximal decorated destabilizing subsheaf of $(E_\alpha, \varphi_\alpha)$; i.e., for any $D \in V_\alpha$ $F_{a,|Q_\alpha^{-1}(D)}$ (with the induced morphism) de-semistabilizes $(E_\alpha, \varphi_\alpha)|_{Q_\alpha^{-1}(D)}$.

Recall that:
- by construction of the relative maximal decorated destabilizing subsheaf, the quantity
\[
\varepsilon \left( \frac{F_{a,|Q_\alpha^{-1}(D)}; \varphi_{a,|Q_\alpha^{-1}(D)}}{E_{a,|Q_\alpha^{-1}(D)}} \right)
\]
depends only on $a$ and not on $D \in V_\alpha$ and for this reason from now on we will denote by $\varepsilon(a)$;
- $(E_\alpha, \varphi_\alpha)$, $(F_\alpha, \varphi_\alpha|_{Q_\alpha})$ and $Q_\alpha$ are flat families of decorated sheaves (resp. sheaves) over $V_\alpha$.

Let $G_\alpha$ be a line bundle which extends $\det(Q_\alpha)$ to all $Z_{a}$; in view of Theorem 92, the line bundle $G_\alpha$ can be uniquely decomposed as $G_\alpha = q^*_\alpha L_a \otimes p^*_\alpha M_a$ with $L_a \in \text{Pic}(X)$ and $M_a \in \text{Pic}(\Pi_\alpha)$. Note that $\deg(Q_{a,|Q_\alpha^{-1}(D)}) = \deg(L_a)$.

For a general divisor $D \in \Pi_\alpha = |\mathcal{O}_X(a)|$, let $\deg(a)$, $\text{rk}(a)$ and $\mu(a)$ denote the degree, rank and slope of the maximal decorated destabilizing subsheaf $(F_{a,|Q_\alpha^{-1}(D)}; \varphi_{a,|Q_\alpha^{-1}(D)})$ of $(E_{a,|Q_\alpha^{-1}(D)}; \varphi_{a,|Q_\alpha^{-1}(D)})$. Let $\mu_\varepsilon(a) = \mu(a) - \frac{\varepsilon(a)}{\text{rk}(a)} \deg^q(a) = \deg^q(E_{a,|Q_\alpha^{-1}(D)}) - \deg(a)$, $\text{rk}^\varepsilon(a) = \text{rk}(E_{a,|Q_\alpha^{-1}(D)}) - \text{rk}(a)$ and $\varepsilon^q(a) = \varepsilon(E_{a,|Q_\alpha^{-1}(D)}) - \varepsilon(a)$. Finally $\mu_\varepsilon^q(a) = \frac{\deg^q(a)}{\text{rk}^\varepsilon(a)}$, $\deg^\varepsilon(a) = \deg(a) - a\delta^\varepsilon(a)$ and $\mu_\varepsilon^\varepsilon(a) = \mu^\varepsilon(a) - a\delta^\varepsilon(a) = \frac{\deg^\varepsilon(a)}{\text{rk}^\varepsilon(a)}$.

Let $U_\alpha \subset V_\alpha$ denote the dense open set of points $D \in V_\alpha$ such that $D$ is smooth.

Lemma 33 (Lemma 7.2.3 in [1]). Let $a_1, \ldots, a_l$ be positive integers, $a = \sum_i a_i$ and $D_i \in U_\alpha$, divisors such that $D = \sum_i D_i$ is a divisor with normal crossing. Then there is a smooth locally closed curve $C \subset \Pi_\alpha$ containing the point $D$ such that $C \cap \{D\} \subset U_\alpha$ and $Z_C = C \times_{\Pi_\alpha} Z_\alpha$ is smooth in codimension 2.

Lemma 34. Let $a_1, \ldots, a_l$ be positive integers and $a = \sum_i a_i$. Then
- $\mu(a) \leq \sum_i \mu(a_i)$,
- $\mu^\varepsilon(a) \geq \sum_i \mu^\varepsilon(a_i)$,
- $\mu^\varepsilon(a) \geq \sum_i \mu^\varepsilon(a_i)$
and in case of equality $\text{rk}^\varepsilon(a) \leq \min_i \text{rk}^\varepsilon(a_i)$, or equivalently $\text{rk}(a) \geq \max_i \text{rk}(a_i)$.
Proof. Let $D_i \in U_{a_i}$, for $i = 1, \ldots, l$, be divisors satisfying the requirements of Lemma 33 be $D = \sum D_i$ and let $C$ be a curve with the properties of Lemma 33. There exists over $V_a$ a maximal decorated destabilizing subsheaf $F$ with the associated torsion free quotient $E|_{V_a} \to Q_a$. Recall that both sheaves are flat over $V_a$. Its restriction to $V_a \cap C$ can uniquely be extended to a $C$ flat quotient $E_C \to Q_C$ and let $F_C = \ker(E_C|_C \to Q_C)$, then also $F_C$ extends $F$ to all $C$. Note that also $F_C$ is flat over $C$ and so $F_{|C|_D} = P_{F_{|C|_D}}$ for any $c \in C$. Therefore $\mu(F_{|C|_D}) = \mu(a)$, $\text{rk}(F_{|C|_D}) = \text{rk}(a)$ and $\varepsilon(F_{|C|_D}) = \varepsilon(a)$. Let

- $\mathfrak{T}_D \equiv Q_{C|D}/T(Q_{C|D})$ and $\mathfrak{F}_D \equiv \ker(E_{|D} \to \mathfrak{T}_D)$, i.e., they fit in the exact sequence

$$0 \to \mathfrak{F}_D \to E_{|D} \to \frac{Q_{C|D}}{T(Q_{C|D})} \to 0;$$

- $Q_i \equiv \mathfrak{T}_{D|D_i}/T(\mathfrak{T}_{D|D_i})$ and $\mathfrak{F}_i \equiv \ker(E_{|D_i} \to Q_i)$, i.e., they fit in the exact sequence

$$0 \to \mathfrak{F}_i \to E_{|D_i} \to \frac{Q_{D|D_i}}{T(\mathfrak{T}_{D|D_i})} \to 0;$$

Then one gets

- $\text{rk}(a) = \text{rk}(F_{|C|_D}) = \text{rk}(F_D) = \text{rk}(\mathfrak{F}_D) = \text{rk}(\mathfrak{F}_i)$ and $\text{rk}^\theta(a) = \text{rk}(Q_{C|D}) = \text{rk}(\mathfrak{T}_D) = \text{rk}(\mathfrak{T}_{D|D_i}) = \text{rk}(Q_i);$  
- $\mu^\theta(a) = \mu(Q_{C|D}) \geq \mu(\mathfrak{T}_D)$ and $\mu(a) = \mu(F_{|C|_D}) \leq \mu(F_D);$  
- $\mu(\mathfrak{T}_{D|D_i}) \geq \mu(Q_i)$ and $\mu(\mathfrak{F}_{|D|D_i}) \leq \mu(\mathfrak{F}_i).$

Since $E_{|D}$ and $\mathfrak{T}_D$ are pure, and the sequences

$$0 \to \mathfrak{T}_D \to \bigoplus_{i,j} (\mathfrak{T}_D)|_{D_i \cap D_j} \to 0$$

$$0 \to E_{|D} \to \bigoplus_{i,j} (E_{|D})|_{D_i \cap D_j} \to 0$$

are exact modulo sheaves of dimension $n - 3$, following the same calculations of Lemma 7.2.5 in [4], one gets that

$$\mu(\mathfrak{T}_D) = \sum_i \left( \mu((\mathfrak{T}_D)|_{D_i}) - \frac{1}{2} \sum_{j \neq i} \left( \frac{\text{rk}(\mathfrak{T}_D)|_{D_i \cap D_j}}{\text{rk}^\theta(a)} - 1 \right) a_i a_j \right);$$

$$\mu(E_{|D}) = \sum_i \left( \mu((E_{|D})|_{D_i}) - \frac{1}{2} \sum_{j \neq i} \left( \frac{\text{rk}(E_{|D})|_{D_i \cap D_j}}{\text{rk}(E_{|D})} - 1 \right) a_i a_j \right),$$

and

$$\mu(Q_i) \leq \mu((\mathfrak{T}_D)|_{D_i}) - \frac{1}{2} \sum_{j \neq i} \left( \frac{\text{rk}(\mathfrak{T}_D)|_{D_i \cap D_j}}{\text{rk}(\mathfrak{T}_D)} - 1 \right) a_i a_j$$

Therefore $\mu(\mathfrak{T}_D) \geq \sum_i \mu(Q_i)$, $\deg(E_{|D}) \leq \sum_i \deg((E_{|D})|_{D_i})$ and so easy calculations show that $\mu(F_D) \leq \sum_i \mu(F_i)$.
Note that, since $T_{F} \cong T_{F}/(\mathcal{F}_{C})$ is pure torsion, $\varphi_{|F}$ is pure torsion, $\varphi_{|F}$ = 0 (see Remark 10) and so $\varepsilon(\varphi_{|F}/\mathcal{F}_{C}) = \varepsilon(\varphi_{|F})$. For the same reason $\varepsilon(\varphi_{|T_{F}}) = \varepsilon(\varphi_{|T_{F}})$. Moreover, if $\varepsilon(a) = \varepsilon(\mathcal{F}_{C}|_{D}) = 0$ then obviously also $\varepsilon_{i} = 0$ for all $i$; conversely if $\varepsilon(a) = 1$ then there exists at least one $i$ such that $\varepsilon_{i} = 1$. Therefore $\sum i \varepsilon_{i} \geq \varepsilon(a) \geq \varepsilon_{i}$.

Therefore, defining $\varepsilon_{0} = (1 - \varepsilon_{i})$ and $\varepsilon_{D} = (1 - \varepsilon_{i})$ as in Remark 10, thanks to the previous inequalities and considerations, one gets $\sum \varepsilon_{0} \geq \varepsilon_{D} \geq \varepsilon_{0}$, and so

$$\varepsilon_{0} = \mu_{c}(Q_{i}) \geq \sum_{i} \mu_{e}(Q_{i}) \geq \sum_{i} \varepsilon_{0}(a_{i})$$

If $\varepsilon_{0}(a) = \sum_{i} \varepsilon_{0}(a_{i})$ it follows that $\varepsilon_{0}(Q_{i}) = \varepsilon_{0}(a_{i})$. Since $\varepsilon_{0}(a)$ is the decorated slope of the minimal destabilizing quotient (i.e., its kernel is the maximal decorated destabilizing subsheaf), we have $rk^{0}(a) = rk(Q_{i}) \geq rk^{0}(a_{i})$ for all $i$.

**Corollary 35.** $rk^{0}(a)$, $\varepsilon^{0}(a)$, $\mu_{c}(a)$, $\mu_{e}(a)$, $\varepsilon^{0}(a)$, $\mu_{e}(a)$, $rk(a)$ and $\varepsilon(a)$ are constant for $a \gg 0$.

**Proof.** The quantities $rk^{0}(a)$ and $\varepsilon^{0}(a)$ are constant as proved in Corollary 7.6. The same arguments show that $\varepsilon^{0}(a)$ is constant as well. Therefore $\varepsilon^{0}(a)$ has to be constant too and easy calculations show that also $\varepsilon(a)$, $\mu_{c}(a)$ and $\mu_{e}(a)$ are constant.

**Corollary 36.** For $a \gg 0$ or $\varepsilon^{0}(a) = 0$ and $\varepsilon(a) = 1$ or $\varepsilon(\mathcal{E}_{a}, \varphi_{a}) = 0$.

**Proof.** Since $\varepsilon^{0}(a)$ is definitively constant, $\varepsilon^{0}(a) = 0$ for $a \gg 0$. Since $\varepsilon^{0}(a) = \varepsilon(\mathcal{E}_{a}, \varphi_{a}) - \varepsilon(a)$ or they are (definitely) both zero or both one.

**Lemma 37** (Lemma 7.2.7). There exist $a_{0} \in \mathbb{N}$ and a line bundle $L \in \text{Pic}(X)$ such that $L_{a} \cong L$ for any $a > a_{0}$.

In this way we have proved that for $a \gg 0$ an extension of $det(Q_{a})$ is of the form $L \boxtimes M_{a}$ with $L \in \text{Pic}(X)$ and $deg(Q_{a}|_{D}) = a \text{deg}(L)$ for any $D \in V_{a}$. Now we can state and prove the main theorem of this section:

**Theorem 38.** Let $X$ be a smooth projective surface and $\mathcal{O}_{X}(1)$ be a very ample line bundle. Let $(\mathcal{E}, \varphi)$ be a slope $\varepsilon$-semistable decorated sheaf. Then there is an integer $a_{0}$ such that for all $a \geq a_{0}$ there is a dense open subset $U_{a} \subset \mathcal{O}_{X}(a)$ such that for all $D \in U_{a}$ the divisor $D$ is smooth and $(\mathcal{E}, \varphi)|_{D}$ is slope $\varepsilon$-semistable.

**Proof.** We proof the theorem by reduction to absurd. Suppose the theorem is false: thanks to the previous constructions there exists a line bundle $L_{a}$ such that

$$\frac{\text{deg}(L_{a}) - a\varepsilon^{0}(a)}{\text{rk}^{0}(a)} < \mu_{e}(\mathcal{E})$$

and $1 \leq \text{rk}^{0}(a) \leq \text{rk}(\mathcal{E})$. We recall that $\text{rk}^{0}(a)$ and $L_{a}$ are constant for $a$ greater than a certain constant $a_{0}$, so from now on we suppose that $a$ is so and we call $L_{a} = L$ and $\text{rk}^{0}(a) = r^{0}$. We want to construct a rank $r^{0}$ quotient $Q$ of $\mathcal{E}$ such that $det(Q) = L$.

Let $a$ be a sufficiently large integer, $D \in U_{a}$ and let $(\mathcal{F}_{D}, \varphi_{|D})$ be the maximal decorated destabilizing subsheaf of $(\mathcal{E}, \varphi)|_{D}$ and $Q_{D} \cong \text{co ker}(\mathcal{F}_{D} \rightarrow \mathcal{E}|_{D})$ the associated minimal decorated destabilizing quotient. Put $L_{D} \cong \text{det}(Q_{D})$ and note
that $L_D = L_{|D}$ (by uniqueness of the maximal destabilizing subsheaf and so of the minimal destabilizing quotient). The surjective morphism $E_{|D} \to Q_D$ induces a surjective homomorphism $\sigma_D : \Lambda^r E_{|D} \to L_D$ and morphisms

$$i_{|D} : D \to \text{Grass}(E_{|D}, r^q) \to \mathbb{P}(\Lambda^r E_{|D}).$$

Consider the exact sequence

$$\text{Hom}(\Lambda^r E, L(-a)) \to \text{Hom}(\Lambda^r E, L) \to \text{Hom}(\Lambda^r E_{|D}, L_{|D}) \to \text{Ext}^1(\Lambda^r E, L(-a)).$$

By Serre’s theorem and Serre duality one has that for $i = 0, 1$ and $a \gg 0$

$$\text{Ext}^i(\Lambda^r E, L(-a)) = H^{n-i}(X, \Lambda^r E \otimes L^\vee \otimes \omega_X(a)) = 0.$$

Hence if $a$ is big enough $f$ is bijective and $\sigma_D$ extends uniquely to a homomorphism $\sigma \in \text{Hom}(\Lambda^r E, L)$. Using the same arguments of the final part of the proof of Theorem 7.2.1 in [6], $\sigma$ induces a morphism $i : X \to \mathbb{P}(\Lambda^r E)$ that factorize thorough Grass($E, r^q$) and so we obtain a quotient $q : E \to \mathcal{Q}$. Since $\det \mathcal{Q}_{|D} \equiv L_D = L_{|D}$ for all $D \in U_a$, by Lemma 7.2.2 [6], we get $L = \det \mathcal{Q}$. Define $\mathcal{F} \equiv \ker(E \to \mathcal{Q})$ and note that $\mathcal{F}_{|D} = \mathcal{F}_D$. Finally, thanks to Fujita’s vanishing theorem ([7] pg 66),

$$H^i(X, \mathcal{F}_{a,b} \otimes N^\vee \otimes \omega_X(a)) = 0$$

for $i > 0$ and $a$ big enough. Therefore

$$\text{Ext}^j(\mathcal{F}_{a,b}, N(-a)) = H^{n-j}(X, \mathcal{F}_{a,b} \otimes N^\vee \otimes \omega_X(a)) = 0$$

for $j = 0, 1$. The same holds also for $E$ and so the following diagram is commutative:

\[
\begin{array}{ccc}
\text{Hom}(\mathcal{F}_{a,b}, N) & \longrightarrow & \text{Hom}(\mathcal{F}_{a,b}, N_{|D}) \\
\downarrow & & \downarrow \\
\text{Hom}(E_{a,b}, N) & \longrightarrow & \text{Hom}(E_{a,b}, N_{|D})
\end{array}
\]

which proves that we can extend to all $\mathcal{F}_{a,b}$ the morphism we have over $\mathcal{F}_{a,b}|_D$ in such a way that $\varepsilon(\varphi_{|_D}) = \varepsilon(a)$. By construction $(\mathcal{F}, \varphi)_{|_D}$ destabilizes, with respect to the slope $\varepsilon$-semistability, the decorated sheaf $(\mathcal{E}, \varphi)$ and this contradicts the hypothesis. ♦

4. Mehta–Ramanathan theorem for slope $k$-semistable decorated sheaves of rank 2 and 3

**Notation.** Let $X$ be a smooth projective variety, $\mathcal{O}_X(1)$ a fixed ample line bundle over $X$, $k$ an algebraic closed field of characteristic 0, $S$ an integral $k$-scheme of finite type and $f : X \to S$ a projective flat morphism. Note that $\mathcal{O}_X(1)$ is also $f$-ample.

**Proposition 39 (Properties of $k_{\mathcal{F}, \mathcal{E}}$).** Let $(\mathcal{E}, \varphi)$ be a decorated sheaf of type $(a, b, N)$ and rank $r$. Let $\mathcal{G}, \mathcal{F}$ be subsheaves of $\mathcal{E}$. Then the following statements hold:

1. There exist an open subset $U \subseteq X$ and complex vector spaces $V'$ and $V$ of dimension $rk(\mathcal{F})$ and $r$ (respectively) such that $\mathcal{F}|_U \simeq V' \otimes \mathcal{O}_U$, $\mathcal{E}|_U \simeq V \otimes \mathcal{O}_U$ and $k_{\mathcal{F}, \mathcal{E}} = k_{\mathcal{E}|_U, \mathcal{F}|_U}$.

2. If there exists an open subset $U$ of $X$ such that $\mathcal{F}|_U$ is isomorphic to $\mathcal{G}|_U$, then $k_{\mathcal{F}, \mathcal{E}} = k_{\mathcal{G}, \mathcal{E}}$. 
(3) \( k_{F + G, \varepsilon} \geq \max\{k_{F, \varepsilon}, k_{G, \varepsilon}\} \).
(4) \( k_{F \cap G, \varepsilon} \leq \min\{k_{F, \varepsilon}, k_{G, \varepsilon}\} \).
(5) \( k_{F, \varepsilon} + k_{G, \varepsilon} \geq k_{F + G, \varepsilon} \).
(6) If \( k_{F \cap G, \varepsilon} = k_{F \cap G, \varepsilon'} \) then
\[
\begin{equation}
\tag{33}
k_{F + G, \varepsilon} + k_{F \cap G, \varepsilon} \leq k_{F, \varepsilon} + k_{G, \varepsilon}
\end{equation}
\]
in particular
\[
k_{F + G, \varepsilon} + k_{F \cap G, \varepsilon} \leq k_{F, \varepsilon} + k_{G, \varepsilon}.
\]

Proof. (1) Let \( U_F \) be a maximal open subset where \( F \) is locally free and admits a trivialization. Suppose that \( k_{F, \varepsilon} = k \), then there exist an open subset \( U' \subseteq X \), \( k \) local sections \( f_1, \ldots, f_k \in H^0(U', F|_{U'}) \) and \( a - k \) local sections \( c_1, \ldots, c_{a-k} \in H^0(U', E|_{U'}) \) such that
\[
\varphi((f_1, \ldots, f_k, c_1, \ldots, c_{a-k})^{\oplus b}) \neq 0.
\]
Let \( U = U' \cap U_F \), then \( E|_U \simeq V \otimes \mathcal{O}_U \) and \( k_{F, \varepsilon} = k_{E|_U, \varepsilon|_U} \).
(2) The statement follows directly from (1).
(3) Follows from the fact that \( F, G \subseteq F + G \).
(4) Since \( F \cap G \subseteq F, G \), it is easy to see that the statement holds.
(5) Let \( t = k_{F + G, \varepsilon} \) and \( k = k_{F, \varepsilon} \). Thanks to the first point, we can suppose that \( F \cap G, F, G \) and \( F + G \) are trivial sheaves. Let \( f_1 + g_1, \ldots, f_t + g_t \) and \( e_1, \ldots, e_{a-t} \) be sections of \( F + G \) and \( E \) respectively such that \( f_i \) are sections of \( F, g_i \) are sections of \( G \) and \( \varphi((f_1 + g_1, \ldots, f_t + g_t, e_1, \ldots, e_{a-t})^{\oplus b}) \neq 0 \). Let \( f = \sum f_i, g = \sum g_i \) and \( e = \sum e_i \), then also \( \varphi(((f + g)^{\otimes t} \otimes e^{\otimes a-t})^{\oplus b}) \neq 0 \).
But
\[
(f + g)^{\otimes t} \otimes e^{\otimes a-t} = \left( \sum_{i=0}^{t} \binom{t}{i} f^{\otimes t-i} \otimes g^{\otimes i} \right) \otimes e^{\otimes a-t}.
\]
Since \( k \leq t \) there exists \( i_0 \geq 0 \) such that \( t - i_0 = k \). Then for any \( 0 \leq i < i_0 \) one has \( \varphi((f^{\otimes t-i} \otimes g^{\otimes i} \otimes e^{\otimes a-t})^{\oplus b}) = 0 \), since \( k_{F, \varepsilon} = k \) and \( t - i > k \). Therefore
\[
\varphi \left( \left( \sum_{i=0}^{t} \binom{t}{i} f^{\otimes t-i} \otimes g^{\otimes i} \otimes e^{\otimes a-t} \right)^{\oplus b} \right) \neq 0
\]
and so \( k_{G, \varepsilon} \geq i_0 = t - k = k_{F + G, \varepsilon} - k_{F, \varepsilon} \).
(6) Let \( s = k_{F \cap G, \varepsilon} = k_{F \cap G, \varepsilon'} \). If \( s = 0 \) there is nothing to prove. Otherwise, similarly to the proof of the previous point, we can choose a section \( h \) of \( F \cap G \) and sections \( f \) and \( g \) of \( F \) and \( G \) respectively such that \( f + g \) is a section of \( F + G \) and \( \varphi((h^{\otimes s} \oplus (f + g)^{\otimes a-s})^{\oplus b}) \neq 0 \). In particular note that \( k_{F + G, \varepsilon} = k_{F + G, F \cap G} = a \). Then it is easy to see that \( k_{G, \varepsilon} \geq a - k_{F, \varepsilon} + s \).

Let \((E, \varphi)\) be a decorated sheaf and \( F \) a subsheaf of \( E \). As usual denote
\[
\begin{align*}
P_F^k & \doteq P_F - \delta k_{F, \varepsilon}, \\
p_F^k & \doteq P_F^k \div \text{rk}(F), \\
\deg(F)^k & \doteq \deg(F) - \delta k_{F, \varepsilon}, \\
\mu^k(F) & \doteq \deg(F)^k \div \text{rk}(F).
\end{align*}
\]
We recall that \((\mathcal{E}, \varphi)\) is \((\text{s}\text{-})\text{stable}\), respectively \((\text{s}\text{-})\text{\textit{semi}}\text{\text{-}}\text{stable}\), if and only if for any \(\mathcal{F} \subset \mathcal{E}\)
\[ p^k_{\mathcal{F}} \prec p^k_{\mathcal{E}}, \]
or
\[ \mu^k(\mathcal{F}) \preceq \mu^k(\mathcal{E}), \]
respectively.

4.1. Maximal destabilizing subsheaf.

**Notation.** In this section, unless otherwise stated, any decorated sheaf will have rank \(r \leq 3\).

**Proposition 40.** Let \((\mathcal{E}, \varphi)\) be a decorated sheaf of type \((a, b, N)\) and rank \(r = 2\) or \(r = 3\). If \((\mathcal{E}, \varphi)\) is not slope \(k\)-semistable then there exists a unique, \(k\)-slope-semistable subsheaf \(\mathcal{F}\) of \(\mathcal{E}\) such that:

1. \(\mu^k(\mathcal{F}) \geq \mu^k(\mathcal{W})\) for any \(\mathcal{W} \subset \mathcal{E}\).
2. If \(\mu^k(\mathcal{F}) = \mu^k(\mathcal{W})\) then \(\mathcal{W} \subseteq \mathcal{F}\).

The subsheaf \(\mathcal{F}\), with the induced morphism \(\varphi|_{\mathcal{F}}\), is called the **maximal slope \(k\)-destabilizing subsheaf**.

**Proof.** Define the following partial ordering on the set of decorated subsheaves of \(\mathcal{E}\); let \(\mathcal{F}_1, \mathcal{F}_2\) be two subsheaves of \(\mathcal{E}\); then
\[ \mathcal{F}_1 \preceq k \mathcal{F}_2 \iff \mathcal{F}_1 \subseteq \mathcal{F}_2 \text{ and } \mu^k(\mathcal{F}) \leq \mu^k(\mathcal{E}). \]

The set of subsheaves of \(\mathcal{E}\) with this ordering relation satisfies the hypotheses of Zorn’s Lemma, so there exists a maximal element (not unique in general). Choose an element \(\mathcal{F}\) in the following set:

\[ \min \{ \mathcal{G} \subset \mathcal{E} \mid \mathcal{G} \text{ is } \preceq k \text{-maximal} \}. \]

Then we claim that \((\mathcal{F}, \varphi|_{\mathcal{F}})\) has the asserted properties.

By contradiction, suppose that there exists \(\mathcal{G} \subset \mathcal{E}\) such that \(\mu^k(\mathcal{G}) \geq \mu^k(\mathcal{F})\), i.e.,
\[ \mu(\mathcal{G}) - \frac{3k_{\mathcal{G}, \mathcal{E}}}{r_{\mathcal{G}}} \geq \mu(\mathcal{F}) - \frac{3k_{\mathcal{F}, \mathcal{E}}}{r_{\mathcal{F}}}. \]

**Claim.** We can assume \(\mathcal{G} \subseteq \mathcal{F}\) by replacing \(\mathcal{G}\) by \(\mathcal{G} \cap \mathcal{F}\).

Indeed, if \(\mathcal{G} \not\subseteq \mathcal{F}\), \(\mathcal{F}\) is a proper subsheaf of \(\mathcal{F} + \mathcal{G}\) since (by the assumptions we made on the \(k\)-slope of \(\mathcal{G}\) and by maximality of \(\mathcal{F}\)) \(\mathcal{F} \not\subseteq \mathcal{G}\). By maximality
\[ \mu^k(\mathcal{F}) > \mu^k(\mathcal{F} + \mathcal{G}). \]

Using the exact sequence
\[ 0 \longrightarrow \mathcal{F} \cap \mathcal{G} \longrightarrow \mathcal{F} \oplus \mathcal{G} \longrightarrow \mathcal{F} + \mathcal{G} \longrightarrow 0 \]
on one finds, following calculations we made in the proof of Proposition 40 that
\[ r_{\mathcal{F} \cap \mathcal{G}}(\mu^k(\mathcal{G}) - \mu^k(\mathcal{F} \cap \mathcal{G})) < \delta(k_{\mathcal{F} + \mathcal{G}, \mathcal{E}} + k_{\mathcal{F} \cap \mathcal{G}, \mathcal{E}} - k_{\mathcal{F}, \mathcal{E}} - k_{\mathcal{G}, \mathcal{E}}). \]

Therefore if \(k_{\mathcal{F} + \mathcal{G}, \mathcal{E}} + k_{\mathcal{F} \cap \mathcal{G}, \mathcal{E}} - k_{\mathcal{F}, \mathcal{E}} - k_{\mathcal{G}, \mathcal{E}} \leq 0\) then \(\mu^k(\mathcal{F} \cap \mathcal{G}) \geq \mu^k(\mathcal{G})\) and the claim holds true.
First suppose that \( r = 2 \).
Consider \( \mathcal{F} \cap \mathcal{G} \). If \( \text{rk}(\mathcal{F} \cap \mathcal{G}) = 0 \) then \( k_{\mathcal{F} \cap \mathcal{G}} = 0 \) and, thanks to point (5) of Proposition III, the right part of equation (\ref{eq:33}) is less or equal to zero, and the claim holds true. If \( \text{rk}(\mathcal{F} \cap \mathcal{G}) = 2 \) then \( \text{rk} \mathcal{F} = \text{rk} \mathcal{G} = \text{rk} \mathcal{F} \cap \mathcal{G} = 2 \). \( \mathcal{F} \) coincides, up to a rank zero sheaf \( \mathcal{T} = \mathcal{E}/\mathcal{F} \), with \( \mathcal{E} \) and \( k_{\mathcal{F} \cap \mathcal{G}} = k_{\mathcal{F}} = k_{\mathcal{G}} = k_{\mathcal{F} \cap \mathcal{G}} = k_{\mathcal{E}} = a \). Since \( \deg(\mathcal{E}) = \deg(\mathcal{F}) + \deg(\mathcal{T}) \) and \( k_{\mathcal{E}} = a \), one gets that \( \mu^k(\mathcal{F}) \leq \mu^k(\mathcal{E}) \) and \( \mathcal{F} \) is not \( \leq^k \)-maximal, which is absurd. Therefore \( \text{rk}(\mathcal{F} \cap \mathcal{G}) = 1 \). If \( \text{rk} \mathcal{F} \) or \( \text{rk} \mathcal{G} \) are equal to 2 then, as before, one easily gets that \( \mathcal{F} \) is not maximal, against the assumptions.

The only chance is that \( \text{rk} \mathcal{F} = \text{rk} \mathcal{G} = \text{rk} \mathcal{F} \cap \mathcal{G} = 1 \) and all these sheaves coincide with each other up to rank zero sheaves. Thus \( k_{\mathcal{F} \cap \mathcal{G}} = k_{\mathcal{F}}, k_{\mathcal{G}}, k_{\mathcal{F} \cap \mathcal{G}} = k_{\mathcal{F} \cap \mathcal{G}} \). Therefore the inequality \( \ref{eq:33} \) holds true and \( \mu^k(\mathcal{F} \cap \mathcal{G}) \geq \mu^k(\mathcal{G}) \geq \mu^k(\mathcal{F}) \).

Now suppose that \( r = 3 \).
If \( \text{rk}(\mathcal{F} \cap \mathcal{G}) = 0 \) then \( k_{\mathcal{G}} = 0 \) and the right part of equation (\ref{eq:33}) is less or equal to zero and the claim holds true. If \( \text{rk}(\mathcal{F} \cap \mathcal{G}) = 3 \) as before we easily fall in contradiction. If \( \text{rk}(\mathcal{F}) = 3 \) then \( k_{\mathcal{G}} = a \) and \( \mathcal{F} \) coincides, up to a rank zero sheaf, with \( \mathcal{E} \); so \( \mu^k(\mathcal{F}) \leq \mu^k(\mathcal{E}) \) and \( \mathcal{F} \) is not maximal, that is absurd. Similarly if \( \text{rk}(\mathcal{G}) = 3 \), then \( \mu^k(\mathcal{F}) \leq \mu^k(\mathcal{G}) \leq \mu^k(\mathcal{E}) \), that is again absurd. Therefore the possible cases are the following:

| \( \text{rk}(\mathcal{F} \cap \mathcal{G}) \) | \( \text{rk}(\mathcal{F}) \) | \( \text{rk}(\mathcal{G}) \) | \( \text{rk}(\mathcal{F} + \mathcal{G}) \) | \text{implies} |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | \( k_{\mathcal{F} \cap \mathcal{G}} = k_{\mathcal{F}} = k_{\mathcal{G}} = k_{\mathcal{F} \cap \mathcal{G}} \) |
| 1 | 1 | 2 | 2 | \( k_{\mathcal{F} \cap \mathcal{G}} = k_{\mathcal{F}} \) and \( k_{\mathcal{G}} = k_{\mathcal{F} \cap \mathcal{G}} \) |
| 1 | 2 | 1 | 2 | \( k_{\mathcal{F} \cap \mathcal{G}} = k_{\mathcal{G}} \) and \( k_{\mathcal{F}} = k_{\mathcal{F} \cap \mathcal{G}} \) |
| 1 | 2 | 2 | 3 | \( k_{\mathcal{F} \cap \mathcal{G}} = k_{\mathcal{G}} = a \) |
| 2 | 2 | 2 | 2 | \( k_{\mathcal{F} \cap \mathcal{G}} = k_{\mathcal{G}} = a \) |

The non-trivial cases are the following: \( \text{rk}(\mathcal{F} \cap \mathcal{G}) = 1 \) and \( \text{rk}(\mathcal{F}) = \text{rk}(\mathcal{G}) = 2 \) or \( \text{rk}(\mathcal{F} \cap \mathcal{G}) = \text{rk}(\mathcal{F}) = \text{rk}(\mathcal{G}) = 2 \). In the first case \( \text{rk}(\mathcal{F} + \mathcal{G}) = 3 \) and so \( k_{\mathcal{F} \cap \mathcal{G}} = a \) and \( k_{\mathcal{F} \cap \mathcal{G}} = k_{\mathcal{F} \cap \mathcal{G}} \), in the second case \( \text{rk}(\mathcal{F} + \mathcal{G}) = 2 \) and so \( k_{\mathcal{F} \cap \mathcal{G}} = k_{\mathcal{F} \cap \mathcal{G}} = k_{\mathcal{F} \cap \mathcal{G}} = k_{\mathcal{F} \cap \mathcal{G}} \). Therefore in both cases equation (\ref{eq:33}) holds true and equation (\ref{eq:34}) holds with the less or equal than zero. Then \( \mu^k(\mathcal{F} \cap \mathcal{G}) \geq \mu^k(\mathcal{G}) \geq \mu^k(\mathcal{F}) \) and the claim holds true.

Since we have proved the claim, the proof may continue as the proof of Proposition III.

\[\begin{align*}
\text{Proposition 41.} \quad & \text{Let } (\mathcal{E}, \varphi) \text{ be a decorated sheaf of type } (a, b, N) \text{ and rank } r = 2 \text{ or } r = 3. \text{ If } (\mathcal{E}, \varphi) \text{ is not } k\text{-semistable then exists a unique, } k\text{-semistable subsheaf } \mathcal{F} \text{ of } \mathcal{E} \text{ such that:} \\
& (1) \; \mu^k(\mathcal{F}) \leq \mu^k(\varphi) \text{ for any } \mathcal{W} \subset \mathcal{E}. \\
& (2) \; \text{If } \mu^k(\varphi) = \mu^k(\mathcal{W}) \text{ then } \mathcal{W} \subset \mathcal{F}.
\end{align*}\]

The subsheaf \( \mathcal{F} \), with the induced morphism \( \varphi|_{\mathcal{F}} \), is called the \textit{k-maximal destabilizing subsheaf}.

\[\text{Proof.} \quad \text{The proof is similar to the proof of Proposition III.}\]
Remark 42. As in the \( \varepsilon \text{-semistable} \) case, if \((E, \varphi)\) is \( k \text{-semistable} \) (resp. slope \( k \text{-semistable} \)) the maximal \( k \text{-destabilizing} \) (resp. slope \( k \text{-destabilizing} \)) subsheaf coincide with \( E \).

4.2. Restriction theorem. In the previous section we proved that, given a decorated sheaf \((E, \varphi)\) of rank less or equal to 3, there exists a unique maximal \( k \text{-destabilizing} \) subsheaf \((F, \varphi|_F)\). Since, as we noticed in Section 3.3 there is a one-to-one correspondence between decorated subsheaves of \((E, \varphi)\) and quotients of \( E \), we will call minimal \( k \text{-destabilizing quotient} \) the (unique) sheaf \( Q \cong \text{coker}(F \hookrightarrow E) \) such that \( F \) is the maximal \( k \text{-destabilizing} \) subsheaf.

In analogy with Section 3 we will say that a decorated sheaf \((E, \varphi)\) over a Noetherian scheme \( Y \) is flat over the fibre of a morphism \( f : Y \to S \) of finite type between Noetherian schemes if and only if

- \( E \) and \( N \) are flat families of sheaves over the fibre of \( f : Y \to S \);
- \( k_{\varphi_s, \varphi_s} \) is locally constant as a function of \( s \), where \( E_s \cong E_{f^{-1}(s)} \).

Note that the above conditions imply that the \( k \text{-Hilbert polynomials} \) \( P^k_s \) are locally constant for \( s \in S \). The converse holds only if \( S \) is irreducible: i.e., asking that \( P^k_{\varphi_s} \) is locally constant as function of \( s \) is equivalent to ask that \( P^k_{\varphi_s} \) and \( k_{\varphi_s, \varphi_s} \) are locally constant as functions of \( s \).

If \((F, \varphi|_F)\) slope de-semistabalizes (resp. de-semistabalizes) \((E, \varphi)\) then \( \mu^k(F) > \mu^k(E) \) (resp \( P^k_{\varphi} > P^k_{\varphi} \)). Let \( Q \cong \text{coker}(F \hookrightarrow E) \); then

\[
\mu(Q) < \mu(E) - \delta \left( \frac{k_{\varphi_s, \varphi_s}}{r_Q} - \frac{ar_{\varphi_s}}{r_{\varphi_s}r_Q} \right),
\]

or

\[
P_{\varphi_s} < P_{\varphi_s} - \delta \left( \frac{k_{\varphi_s, \varphi_s}}{r_Q} - \frac{ar_{\varphi_s}}{r_{\varphi_s}r_Q} \right),
\]

respectively.

Define \( C_{k,i} = \left( \frac{1}{r_Q} - \frac{ar_{\varphi_s}}{r_{\varphi_s}r_Q} \right) \) for \( i = 0, \ldots, a \). Let \((E, \varphi)\) be a flat family of decorated sheaves over the fibre of a projective morphism \( f : X \to S \). Let \( P = P^k_{\varphi_s} \) and \( p = P^k_{\varphi_s} \), the Hilbert polynomial and, respectively, the reduced Hilbert polynomial of \( E_s \) (which are constant because the family is flat over \( S \)). Define:

1. \( \tilde{E}_k \) as the family on \( X \) parameterized by \( S \) of saturated subsheaves \( F \hookrightarrow E_s \) such that the induced torsion free quotient \( E_s \twoheadrightarrow Q \) satisfy \( \mu(Q) \leq \mu(E_s) + \delta C_{k,0} \);
2. \( \tilde{E}_{k,i} \) as the family of decorated subsheaves \((F, \varphi|_F) \hookrightarrow (E_s, \varphi|_{E_s}) \) such that:
   - \( k_{\varphi_s, \varphi_s} = l \);
   - \( p^k_{\varphi_s} > p^k_{\varphi_s} \) (i.e., \( p_s < p + \delta C_{k,1} \) with \( Q \cong \text{coker}(F \hookrightarrow E_s) \));
   - \( F \) is a saturated subsheaf of \( E_s \),
for \( i = 0, \ldots, a \).

It is easy to see that these families are bounded and therefore, using the same techniques used in Section 5.5 one can prove that \( k \)-semistability is open. More precisely

**Proposition 43.** Let \( f : X \to S \) be a projective morphism of Noetherian schemes and let \((E, \varphi)\) be a flat family of decorated sheaves over the fibre of \( f \). The set of points \( s \in S \) such that \((E_s, \varphi|_{E_s}) \) is \( k \)-semistable with respect to \( \delta \) is open in \( S \).
**Proof.** Let

\[ A \doteq \{ P'' \in \mathbb{Q}[x] \mid \exists s \in S, \exists q : E_s \rightarrow Q \text{ such that } P_q = P'' \text{ and } \ker(q) \in \mathfrak{F}_k \} \]

and, for \( i = 0, \ldots, a \),

\[ A_{k,i} \doteq \{ P'' \in \mathbb{Q}[x] \mid \exists s \in S, \exists q : E_s \rightarrow Q \text{ such that } P_q = P'' \text{ and } \ker(q) \in \mathfrak{F}_{k,i} \}. \]

Then, using the same techniques used in the proof of Proposition 30, one concludes the proof.

Thanks to the previous results and using the same arguments as in the proof of Theorem 31, it is easy to see that the following theorem holds true:

**Theorem 44 (Relative maximal k-destabilizing subsheaf).** Let \( X, \mathcal{O}_X(1), S \) and \( f : X \rightarrow S \) as before. Let \((E, \varphi)\) be a decorated sheaf of rank \( r \leq 3 \). Then there is an integral k-scheme \( T \) of finite type, a projective birational morphism \( g : T \rightarrow S \), a dense open subset \( U \subset T \) and a flat quotient \( Q \) of \( E_T \) such that for all points \( t \in U \), \( F_t \doteq \ker(E_t \rightarrow Q_t) \) with the induced morphism \( \varphi_t|_{F_t} \) is the maximal k-destabilizing subsheaf of \((E_t, \varphi_t)\) or \( Q_t = E_t \).

Moreover the pair \((g, Q)\) is universal in the sense that if \( g' : T' \rightarrow S \) is any dominant morphism of k-integral schemes and \( Q' \) is a flat quotient of \( E'_{T'} \), satisfying the same property of \( Q \), there is an \( S \)-morphism \( h : T' \rightarrow T \) such that \( h_X^* (Q) = Q' \).

Finally, following the constructions made in Section 3.7 and replacing \( k \) with \( \varepsilon \), one can prove the following

**Theorem 45 (Mehta-Ramanathan for slope k-semistable decorated sheaves).** Let \( X \) be a smooth projective surface and \( \mathcal{O}_X(1) \) be, as usual, a very ample line bundle. Let \((E, \varphi)\) be a slope k-semistable decorated sheaf of rank \( r \leq 3 \). Then there is an integer \( a_0 \) such that for all \( a \geq a_0 \) there is a dense open subset \( U_a \subset |\mathcal{O}_X(a)| \) such that for all \( D \in U_a \) the divisor \( D \) is smooth and \((E, \varphi)|_D\) is slope k-semistable.

4.3. Decorated sheaves of rank 2.

**Lemma 46.** Let \((E, \varphi)\) be a decorated sheaf of rank \( r = 2 \). Then the following conditions are equivalent:

- \((E, \varphi)\) is (semi)stable (in the sense of Definition 5);
- \((E, \varphi)\) is k-(semi)stable.

**Proof.** Since the rank of \( E \) is equal to 2 all filtrations of \( E \) are non-critical and of length one. Then the statement follows from the fourth point of Remark 7.

Thanks to the previous Lemma all results in the previous section holds true for rank 2 semistable decorated sheaves of type \((a, b, N)\). In particular, for such objects

- we have found the maximal destabilizing subsheaf and the relative maximal destabilizing subsheaf;
- we have provided the Harder-Narasimhan filtration and the relative Harder-Narasimhan filtration;
- we have proved that the semistability condition is open;
- we have proved a Mehta-Ramanathan’s like theorem.
5. Further remarks

(1) In Section 3 we never used that \( N \) is of rank 1 nor that it is a vector bundle. We only used that it is a pure dimensional torsion free sheaf (of positive rank). Therefore all results in this chapter can be easily generalized for pairs \((E, \varphi)\) of type \((a, b, c, N)\) where \( E \) and \( N \) are torsion free sheaves over \( X \), \( a, b, c \) are positive integers and

\[
\varphi: E_{a,b} \longrightarrow \det(E)^{\otimes c} \otimes N.
\]

(2) Let \((A, \varphi)\) be a decorated sheaf of rank \( r > 0 \). Define \( \tilde{A} \equiv A_{a,b} \); then the pair \((\tilde{A}, \varphi)\) can be regarded as a framed sheaf. Recall that a framed sheaf \((A, \alpha)\) of positive rank and with nonzero morphism \( \alpha \) is slope semistable with respect to \( \tilde{\delta} \) if and only if for any \( F \subset A \)

\[
\mu(F) - \frac{\tilde{\delta}}{r} \leq \mu(A) - \frac{\tilde{\delta}}{r}.
\]

Suppose now that \((\tilde{A}, \varphi)\) is frame semistable with respect to \( \tilde{\delta} \), then \((A, \varphi)\) is slope \( \varepsilon \)-semistable with respect to \( \hat{\delta} \equiv \tilde{\delta} \). In fact if

\[
\text{rk}(A_{a,b}) = b r^a \quad \text{deg}(A_{a,b}) = a b r^{(a-1)} \deg(A)
\]

and so if \( F \) is a subsheaf of \( A \) then

\[
\mu(F) - \frac{\tilde{\delta}}{r_{A_{a,b}}} \leq \mu(\tilde{A}) - \frac{\tilde{\delta}}{r_{\tilde{A}}},
\]

which implies that

\[
a \mu(F) - \frac{\tilde{\delta}}{br^a} \leq a \mu(A) - \frac{\tilde{\delta}}{br^a},
\]

and so

\[
\mu_{\varepsilon}(F) = \mu(F) - a \frac{\tilde{\delta}}{a^2 b (r_{A_{a,b}})^{a-1}} \leq \mu(A) - a \frac{\tilde{\delta}}{a^2 b r^{a-1}} = \mu_{\varepsilon}(E).
\]

Since the subsheaves of \( A \) correspond to subsheaves of \( A_{a,b} \) but this correspondence is not surjective, the converse does not hold in general but only if \( a = 1 \) (\( b \) and \( c \) generic). Thanks to the previous calculations and to Proposition 6, one has that

\[
(A_{a,b}, \varphi) \tilde{\delta} \text{ frame slope (semi)stable} \Rightarrow (A, \varphi) \hat{\delta} \text{ slope } \varepsilon-(\text{semi)stable}
\]

\[
\Rightarrow \hat{\delta} \text{ slope (semi)stable}
\]

\[
\Rightarrow \hat{\delta} \text{ k-(semi)stable}.
\]

Replacing \( \deg_{\varepsilon} \) by \( P^\varepsilon \) and \( \mu_{\varepsilon} \) by \( p^\varepsilon \), similar calculations show that the same result holds also for semistability and not only for slope semistability.

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