The field of invariants for the adjoint action of the Borel group in the nilradical of a parabolic subalgebra

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Abstract. In this paper the field of invariants for the adjoint action of the Borel group in the nilradical of a parabolic subalgebra is studied. We construct the set of $B$-invariant rational functions generating the field of invariants.

§1. Introduction

Let $G$ be the general linear group $\text{GL}(n, K)$ over an algebraically closed field $K$ of characteristic zero. Let $B$ ($N$, respectively) be its Borel (maximal unipotent, respectively) subgroup, which consists of upper triangular matrices with nonzero (unit, respectively) elements on the diagonal. We fix a parabolic subgroup $P \supset B$. Let $\mathfrak{p}$, $\mathfrak{b}$ and $\mathfrak{n}$ be the Lie subalgebras in $\mathfrak{gl}(n, K)$ corresponding to $P$, $B$ and $N$, respectively. We represent $\mathfrak{p} = \mathfrak{r} \oplus \mathfrak{m}$ as the direct sum of the nilradical $\mathfrak{m}$ and a block diagonal subalgebra $\mathfrak{r}$ with sizes of blocks $(r_1, r_2, \ldots, r_u)$. The subalgebra $\mathfrak{m}$ is invariant relative to the adjoint action of the group $P$:

for any $g \in P$ we have $x \in \mathfrak{m} \mapsto \text{Ad}_g x = gxg^{-1}$.

Therefore $\mathfrak{m}$ is invariant relative to the adjoint action of the subgroups $B$ and $N$. We extend this action to the representation in the algebra $\mathbb{K}[\mathfrak{m}]$ and in the field $\mathbb{K}(\mathfrak{m})$:

for any $g \in P$ we have $f(x) \in \mathbb{K}[\mathfrak{m}] \mapsto f(\text{Ad}_g^{-1}x)$.

The complete description of the field of invariants $\mathbb{K}(\mathfrak{m})^N$ for any parabolic subalgebra is a result of [S1]. In this paper a notion of an extended base is

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introduced. The extended base is a subset of the set of positive roots such that
elements of the extended base correspond to a set of algebraically independent
$N$-invariants. These invariants generate the field of invariants $K(m)^N$.

The aim of this paper is to study the field of invariants $K(m)^B$. It
continues the series of works [PS], [S1], [S2], [S3]. In the paper we introduce
an analog of extended base for $B$-action. We determine a subset $\Psi$ of the
extended base. Every root of $\Psi$ corresponds to a rational function. We show
that these rational functions are $B$-invariant (Lemmas 3.2 and 3.6) and
algebraically independent (Proposition 3.7) and generate the field of invariants
$K(m)^B$ (Corollary 4.4). We also construct a representative of any $B$-orbit in
general position (Theorem 4.1).

Further in the papers [S2], [S3] the structure of the algebra of invariants
$K[m]^N$ is considered. If the sizes of diagonal blocks are $(2, k; 2)$, $k > 2$, or $(1, 2, 2, 1)$,
then the invariants constructed on the extended base do not
generate the algebra of invariants and the algebra of invariants is not free.
Besides, the additional invariants in both cases are constructed, which together
with the main list of invariants constructed on the extended base generate the
algebra of invariants $K[m]^N$. Also, the relations between these invariants are
provided. In [S3] it was shown that the algebra of invariants $K[m]^N$ is finitely
generated. Since $B = T \ltimes N$, where $T$ is the reductive group of nondegenerate
diagonal matrices, the algebra of invariants $K[m]^B = K[K[m]^N]^T$ is finitely
generated too. The structures of $K[m]^N$ and $K[m]^B$ seem to be very mysterious
and are a considerable challenge.

§2. Main statements and definitions

We begin with definitions. Let $b = n \oplus h$ be a triangular decomposition.
Let $\Delta$ be the root system relative to $h$ and let $\Delta^+$ be the set of positive roots.
Let $\{\varepsilon_i\}_{i=1}^n$ be the standard basis of $C^n$. Every positive root $\gamma$ in $\mathfrak{gl}(n, K)$ can
be represented as $\gamma = \varepsilon_i - \varepsilon_j$, $1 \leq i < j \leq n$ (see [GG]). We identify a root $\gamma$
with the pair $(i, j)$ and the set of the positive roots $\Delta^+$ with the set of pairs
$(i, j)$, $i < j$. The system of positive roots $\Delta^+_r$ of the reductive subalgebra $r$ is
a subsystem in $\Delta^+$. Let 

\[ \{E_{i,j} : i < j\} \]

be the standard basis in $n$. Let $E_\gamma$ denote the basis element $E_{i,j}$, where $\gamma = (i, j)$.

Let $M$ be a subset of $\Delta^+_r$ corresponding to $m$ that is

\[ m = \bigoplus_{\gamma \in M} E_\gamma. \]

We identify the algebra $K[m]$ with the polynomial algebra in variables $x_{i,j}$,
$(i, j) \in M$. 

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We define a relation in $\Delta^+$ setting $\gamma' > \gamma$ whenever $\gamma' - \gamma \in \Delta^+$. Note that the relation $>$ is not an order relation.

The roots $\gamma$ and $\gamma'$ are called comparable, if either $\gamma' > \gamma$ or $\gamma > \gamma'$.

We will introduce a subset $S$ in the set of positive roots such that every root from this subset corresponds to some $N$-invariant.

**Definition 2.1.** A subset $S$ in $M$ is called a base if the elements in $S$ are not pairwise comparable and for any $\gamma \in M \setminus S$ there exists $\xi \in S$ such that $\gamma > \xi$.

Let us show that the base exists. We need the following

**Definition 2.2.** Let $A$ be a subset in $M$. We say that $\gamma$ is a minimal element in $A$ if there is no $\xi \in A$ such that $\xi < \gamma$.

For a given parabolic subgroup we will construct a diagram in the form of a square array. The cell of the diagram corresponding to a root of $S$ is labeled by the symbol $\otimes$. Symbols $\times$ will be explained below.

**Example 2.3.** Diagram 1 represents the parabolic subalgebra with sizes of its diagonal blocks $(2, 1, 3, 2)$. In this case minimal elements in $M$ are $(2, 3)$, $(3, 4)$ and $(6, 7)$.

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\otimes & & & & & & & 1 \\
1 & \otimes & & & & & & 2 \\
1 & \otimes & & & & & & 3 \\
& & & \otimes & & & & 4 \\
& & \times & \times & & & & 5 \\
& & & & & \times & \otimes & 6 \\
& & & & & & 1 & 7 \\
& & & & & & 1 & 8 \\
\end{array}
\]

Diagram 1

We construct the base $S$ by the following algorithm.

**Step 1.** Put $M_0 = M$ and $i = 1$. Let $S_1$ be the set of minimal elements in $M_0$.

**Step 2.** Put $M_i = M_{i-1} \setminus \{S_i \cup \{\gamma \in M_{i-1} : \exists \xi \in S_i, \xi < \gamma\}\}$. Let $S_i$ be the set of minimal elements of $M_{i-1}$. Increase $i$ by 1 and repeat Step 2 until $M_i$ is empty.

Denote $S = S_1 \cup S_2 \cup \ldots$ The base $S$ is unique.

We have $S_1 = \{(2, 3), (3, 4), (6, 7)\}$ and $S_2 = \{(1, 5), (5, 8)\}$ in Example 2.3.
Let \((r_1, r_2, \ldots, r_s)\) be the sizes of the diagonal blocks in \(\mathfrak{r}\). Put

\[ R_k = \sum_{i=1}^{k} r_i. \]

Let us present \(N\)-invariant corresponding to a root of the base. Consider the formal matrix \(X\) of variables

\[(X)_{i,j} = \begin{cases} x_{i,j} & \text{if } (i, j) \in M; \\ 0 & \text{otherwise}. \end{cases}\]

The matrix \(X\) can be represented as a block matrix

\[ X = \begin{pmatrix} 0 & X_{1,2} & X_{1,3} & \cdots & X_{1,s} \\ 0 & 0 & X_{2,3} & \cdots & X_{1,s} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & X_{s-1,s} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \]

where the size of \(X_{i,j}\) is \(r_i \times r_j\),

\[ X_{i,j} = \begin{pmatrix} x_{R_i-R_i+1,R_j-1+1} & x_{R_i-R_i+1,R_j-1+2} & \cdots & x_{R_i-R_i+1,R_j} \\ x_{R_i-R_i+1,R_j-1+2} & x_{R_i-R_i+2,R_j-1+2} & \cdots & x_{R_i-R_i+2,R_j} \\ \vdots & \vdots & \ddots & \vdots \\ x_{R_i,R_j-1+1} & x_{R_i,R_j-1+2} & \cdots & x_{R_i,R_j} \end{pmatrix}. \] (1)

**Lemma 2.4.** The roots corresponding to the antidiagonal elements in \(X_{i,i+1}\) (from the lower left element towards right upper direction) are in the base.

Thus the roots of the base in the blocks \(X_{i,i+1}\) are as follows.

\[ \begin{array}{cccc} \otimes & \otimes & \otimes & \otimes \\ \otimes & \otimes & \otimes & \otimes \\ \otimes & \otimes & \otimes & \otimes \\ \end{array} \] or

\[ \begin{array}{cccc} \otimes & \otimes & \otimes & \otimes \\ \otimes & \otimes & \otimes & \otimes \\ \otimes & \otimes & \otimes & \otimes \\ \end{array} \]

**Proof.** By definition \[2.2\] for any \(i\) the root \((R_i, R_i + 1)\) is minimal. Therefore \(M \setminus M_1\) contains roots corresponding to all cells in the row \(R_i\) and the column \(R_i + 1\). Hence \((R_i - 1, R_i + 2) \in S_2\) if \(r_i, r_{i+1} > 1\) and all roots of \(M\) in the rows \(R_i, R_i - 1\) and in the columns \(R_i + 1, R_i + 2\) belong to \(M \setminus M_2\). Hence \((R_i - 2, R_i + 3) \in S_3\) if \(r_i, r_{i+1} > 2\) etc. \(\square\)
There are roots in $S$ such that these roots do not correspond to elements of the secondary diagonal in $X_{i,i+1}$, for example $(1,5)$ in Example 2.3.

**Lemma 2.5.** Let the maximal size of blocks in $r$ between the $i$th and $j$th blocks be $k$ and $k < r_i$ and $k < r_j$, then the roots of the base in the block $X_{i,j}$ are following

\[
\begin{array}{c}
\begin{array}{c}
\vdots \\
\otimes \\
\vdots
\end{array}
\end{array}
\]

or

\[
\begin{array}{c}
\begin{array}{c}
\vdots \\
\otimes \\
\vdots
\end{array}
\end{array}
\]

where there are $k$ empty cells on the antidiagonal (from the lower left element towards right upper direction). There is a root of $S$ under every cell of these $k$ empty cells and there is a root of $S$ to the left of every cell of these $k$ empty cells. Besides, for the first diagram there is not a root of $S$ to the left of every first empty rows and for the second diagram there is not a root of $S$ under every last empty columns.

**Example 2.6.** In Example 2.3 the block $X_{1,3}$ corresponds to the following diagram.

\[
\begin{array}{c}
\begin{array}{c}
\otimes \\
\end{array}
\end{array}
\]

**Proof.** We show the statement by induction on $k$. To prove the base of induction we consider $k = 0$ and $j = i + 1$, the case follows from Lemma 2.4. Suppose that the statement is true for any maximal size of blocks between blocks $i$ and $j$ less that $k$. Let us prove for $k$. Let $k$ is the size of $m$th block, $k = r_m$. Consider the block $X_{i,m}$. The sizes of blocks between $i$th and $m$th blocks are less than $k$. By assumption of induction, the structure of $X_{i,m}$ is as in the first diagram in Lemma 2.5. Therefore there is a root of $S$ to the left of every last $k$ rows and there is not a root of $S$ to the left of the other rows of $X_{i,m}$. Hence the same is true for $X_{i,j}$.

Similarly, there is a root of $S$ under every first $k$ columns and there is not a root of $S$ under the other columns of $X_{i,j}$. Hence we have the statement of the lemma. $\square$

For any root $\gamma = (a,b) \in M$ let $S_{\gamma} = \{(i,j) \in S : i > a, j < b\}$. Let $S_{\gamma} = \{(i_1,j_1), \ldots, (i_k,j_k)\}$. Note that if $\gamma$ is minimal in $M$, then $S_{\gamma} = \emptyset$. 5
Denote by $M_\gamma$ a minor $X^I_J$ of the matrix $X$ with ordered systems of rows $I$ and columns $J$, where

$$I = \text{ord}\{a, i_1, \ldots, i_k\}, \quad J = \text{ord}\{j_1, \ldots, j_k, b\}.$$ 

**Example 2.7.** Let us continue Example 2.3. For the root $(1, 6)$ we have $S_{(1,6)} = \{(2,3), (3,4)\}$, $I = \{1, 2, 3\}$, $J = \{3, 4, 6\}$, and

$$M_{(1,6)} = \begin{vmatrix} x_{1,3} & x_{1,4} & x_{1,6} \\ x_{2,3} & x_{2,4} & x_{2,6} \\ 0 & x_{3,4} & x_{3,6} \end{vmatrix}.$$ 

All minors $M_\xi$ for $\xi \in S$ are following

$$M_{(2,3)} = x_{2,3}, \quad M_{(3,4)} = x_{3,4}, \quad M_{(6,7)} = x_{6,7},$$

$$M_{(5,8)} = \begin{vmatrix} x_{5,7} & x_{5,8} \\ x_{6,7} & x_{6,8} \end{vmatrix}, \quad M_{(1,5)} = \begin{vmatrix} x_{1,3} & x_{1,4} & x_{1,5} \\ x_{2,3} & x_{2,4} & x_{2,5} \\ 0 & x_{3,4} & x_{3,5} \end{vmatrix}.$$ 

**Lemma 2.8.** For any $\xi \in S$ the minor $M_\xi$ is $N$-invariant.

**Notation 2.9.** The group $N$ is generated by the one-parameter subgroups $g_{i,j}(t) = I + tE_{i,j}$, where $1 \leq i < j \leq n$ and $I$ is the identity matrix. The adjoint action of any $g_{i,j}(t)$ makes the following transformations of a matrix:

1) the $j$th row multiplied by $t$ is added to the $i$th row,

2) the $i$th column multiplied by $-t$ is added to the $j$th column, i.e. for a variable $x_{a,b}$ we have

$$\text{Ad}_{g^{-1}_{i,j}(t)}x_{a,b} = \begin{cases} x_{a,b} + tx_{j,b} & \text{if } a = i; \\ x_{a,b} - tx_{a,i} & \text{if } b = j; \\ x_{a,b} & \text{otherwise}. \end{cases}$$

**Proof.** By the notation it is sufficient to prove that for any $\xi = (k, m) \in S$ the minor $M_\xi$ is invariant under the adjoint action of $g_{i,j}(t)$ for any $i < j$. If $i < k$, then the $i$th row does not belong to the minor $M_\xi$ and the adding of the $j$th row to the $i$th row leaves $M_\xi$ unchanged. Let $M_\xi = X^I_J$ for some collections of rows $I$ and columns $J$. If $i \geq k$, then since the numbers in $I$ are consecutive, the number of any nonzero row $j$ at the intersection with columns $J$ belongs to $I$. Then the adding of the $j$th row to the $i$th row leaves
$M_\xi$ unchanged again. Using the similar reasoning for columns, we get that $M_\xi$ is $N$-invariant. □

The set $\{M_\xi, \xi \in S\}$ does not generate all the $N$-invariants. There is the other series of $N$-invariants. To present it we need

**Definition 2.10.** An ordered set of positive roots

$$\{\varepsilon_{i_1} - \varepsilon_{j_1}, \varepsilon_{i_2} - \varepsilon_{j_2}, \ldots, \varepsilon_{i_s} - \varepsilon_{j_s}\}$$

is called a *chain* if $j_1 = i_2, j_2 = i_3, \ldots, j_{s-1} = i_s$.

**Definition 2.11.** We say that two roots $\xi, \xi' \in S$ form an *admissible pair* $q = (\xi, \xi')$ if there exists $\alpha_q$ in the set $\Delta^+_i$ corresponding to the reductive part $r$ such that the ordered set of roots $\{\xi, \alpha_q, \xi'\}$ is a chain. In other words, roots $\xi = \varepsilon_i - \varepsilon_j$ and $\xi' = \varepsilon_k - \varepsilon_l$ are an admissible pair if $\alpha_q = \varepsilon_j - \varepsilon_k \in \Delta^+_i$. Note that the root $\alpha_q$ is uniquely determined by $q$.

**Example 2.12.** In the case of Diagram 1 we have three admissible pairs $q_1 = (\xi_1, \xi_3), q_2 = (\xi_2, \xi_3), q_3 = (\xi_1, \xi_4), \text{ where } \xi_1 = (2,3), \xi_2 = (1,5), \xi_3 = (6,7), \text{ and } \xi_4 = (5,8)$.

Let the set $Q := Q(p)$ consist of admissible pairs. For every admissible pair $q = (\xi, \xi')$ we construct a positive root $\varphi_q = \alpha_q + \xi'$, where $\{\xi, \alpha_q, \xi'\}$ is a chain. Consider the subset $\Phi = \{\varphi_q : q \in Q\}$ in the set of positive roots.

The cell of the diagram corresponding to a root of $\Phi$ is labeled by $\times$.

**Example 2.13.** The roots of $\Phi$ for the admissible pairs in Example 2.12 are $\varphi_{q_1} = (4,7), \varphi_{q_2} = (5,7), \varphi_{q_3} = (4,8)$.

Now we are ready to present the $N$-invariant corresponding to a root $\varphi \in \Phi$.

Let admissible pair $q = (\xi, \xi')$ correspond to $\varphi_q \in \Phi$. We construct the polynomial

$$L_{\varphi_q} = \sum_{\substack{\alpha_1, \alpha_2 \in \Delta^+_i \cup \{0\} \\ \alpha_1 + \alpha_2 = \alpha_q}} M_{\xi + \alpha_1} M_{\alpha_2 + \xi'}.$$  \hspace{1cm} (2)

**Example 2.14.** Continuing the previous example, we have

$$L_{(4,7)} = x_{3,4}x_{4,7} + x_{3,5}x_{5,7} + x_{3,6}x_{6,7},$$

$$L_{(4,8)} = x_{3,4} \begin{vmatrix} x_{4,7} & x_{4,8} \\ x_{6,7} & x_{6,8} \end{vmatrix} + x_{3,5} \begin{vmatrix} x_{5,7} & x_{5,8} \\ x_{6,7} & x_{6,8} \end{vmatrix},$$

$$L_{(5,7)} = \begin{vmatrix} x_{1,3} & x_{1,4} & x_{1,5} \\ x_{2,3} & x_{2,4} & x_{2,5} \\ 0 & x_{3,4} & x_{3,5} \end{vmatrix} + \begin{vmatrix} x_{1,3} & x_{1,4} & x_{1,6} \\ x_{2,3} & x_{2,4} & x_{2,6} \\ 0 & x_{3,4} & x_{3,6} \end{vmatrix} x_{5,7}.\hspace{1cm} x_{6,7}.$$
Lemma 2.15. The polynomial $L_{\varphi}$ is $N$-invariant for any $\varphi = \varphi_q \in \Phi$, $q = (\xi, \xi')$.

Proof. By Notation 2.9 it is sufficient to prove for the action of $g_{i,j}(t)$. Let $\xi = (a,b)$, $\xi' = (a',b')$. Using the definition of admissible pair, we have $a < b < a' < b'$, $\alpha_q = (b,a') \in \Delta^+_r$, and $\varphi = (b,b')$. If $i < b$ or $j > a'$, then using the same arguments as in the proof of the invariance of $M_\xi$ for $\xi \in S$, we have that the minors of the right part of (2) are $g_{i,j}(t)$-invariant.

Let $b \leq i < j \leq a'$. Denote $\gamma_1 = (b,i)$, $\gamma_2 = (j,a')$, $\beta = (i,j)$; then $\alpha_q = \gamma_1 + \beta + \gamma_2$ and $\gamma_1 + \beta, \beta + \gamma_2 \in \Delta^+_r \cup \{0\}$. We have

$$
\left\{ \begin{array}{l}
T_{g_{i,j}(t)} M_{\xi+\gamma_1+\beta} = M_{\xi+\gamma_1+\beta} + t M_{\xi+\gamma_1}, \\
T_{g_{i,j}(t)} M_{\beta+\gamma_2+\xi'} = T_{\beta+\gamma_2+\xi'} - t M_{\gamma_2+\xi'}.
\end{array} \right.
$$

The other minors of (2) are invariant under the action of $g_{i,j}(t)$. Combining (2) and (3), we get

$$(T_{g_{i,j}(t)} L_{\varphi}) - L_{\varphi} = M_{\xi+\gamma_1} (M_{\beta+\gamma_2+\xi'} - t M_{\gamma_2+\xi'}) + (M_{\xi+\gamma_1+\beta} + t M_{\xi+\gamma_1}) M_{\gamma_2+\xi'} - M_{\xi+\gamma_1} M_{\beta+\gamma_2+\xi} - M_{\xi+\gamma_1+\beta} M_{\gamma_2+\xi'} = 0, \quad \square$$

Thus we proved the first part of

Theorem 2.16. For an arbitrary parabolic subalgebra, the system of polynomials

$$
\{ M_\xi, \xi \in S, L_{\varphi}, \varphi \in \Phi, \}
$$

is contained in $K[\mathfrak{m}]^N$ and is algebraically independent over $K$.

To show the algebraic independence, consider the restriction homomorphism $f \mapsto f|_\mathcal{Y}$, where

$$
\mathcal{Y} = \left\{ \sum_{\xi \in S \cup \Phi} c_\xi E_\xi : c_\xi \neq 0 \ \forall \xi \in S \cup \Phi \right\},
$$

from $K[\mathfrak{m}]$ to the polynomial algebra $K[\mathcal{Y}]$ of $x_\xi, \xi \in S$, and of $x_\varphi, \varphi \in \Phi$. Direct calculations show that the system of the images

$$
\{ M_\xi|_\mathcal{Y}, \xi \in S, L_{\varphi}|_\mathcal{Y}, \varphi \in \Phi \}
$$

is algebraically independent over $K$. Therefore, the system (4) is algebraically independent over $K$ (see details in [PS]).

Definition 2.17. The set $S \cup \Phi$ is called an extended base.

Definition 2.18. The matrices of $\mathcal{Y}$ are called canonical.
By [S1] one has the following theorems.

**Theorem 2.19.** There exists a nonempty Zariski-open subset $W \subset \mathfrak{m}$ such that the $N$-orbit of any $x \in W$ intersects $Y$ at a unique point.

**Theorem 2.20.** The field of invariants $K(\mathfrak{m})^N$ is the field of rational functions of $M_\xi$, $\xi \in S$, and $L_\varphi$, $\varphi \in \Phi$.

§3. Invariants of the Borel group in the nilradical of a parabolic subalgebra

In this paragraph we will introduce some subset $\Psi \subset S \cup \Phi$ and construct a corresponding invariant in $K[\mathfrak{m}]^B$ to every root $\xi \in \Psi$.

We describe two series of invariants in $K[\mathfrak{m}]^B$. The first (second, respectively) series corresponds to a subset $\Psi_1 \subset \Psi$ ($\Psi_2 \subset \Psi$, respectively) and $\Psi = \Psi_1 \cup \Psi_2$. The first series is following. Let $\psi = (i, b)$ be a root in $\Phi$ such that there exist three roots $\xi_1 = (i, a)$, $\xi_2 = (j, a)$, $\xi_3 = (j, b)$ in $S \cup \Phi$, where $i < j$ and $a < b$. Since there is no more than one root from $S$ in a row and in a column, the roots $\xi_1, \xi_2$ belong to the set $\Phi$. In other words, these four roots correspond to one of two following diagrams

\[
\begin{array}{cc|c}
 a & b & i \\
 \times & \times & j \\
 b & a & j \\
 \end{array}
\quad \text{or} \quad
\begin{array}{cc|c}
 a & b & i \\
 \times & \times & j \\
 b & a & j \\
 \end{array}
\]

If there are such three roots $\xi_1, \xi_2, \xi_3$ with the above described properties, then we say that $\psi$ belongs to the set $\Psi_1$.

Since $\xi_2 \in \Phi$, it corresponds to an admissible pair $(\gamma, \gamma')$. Denote

\[
A_\psi = \begin{cases} 
\frac{L_\psi L_{\xi_2}}{L_{\xi_1} L_{\xi_3}} & \text{if } \xi_3 \in \Phi, \\
\frac{L_\psi L_{\xi_2}}{L_{\xi_1} M_\gamma M_{\xi_3}} & \text{if } \xi_3 \in S.
\end{cases}
\]

Evidently, the rational function $A_\psi$ is $N$-invariant and depends on a choice of $\xi_1, \xi_2$, and $\xi_3$. We will show below that $A_\psi$ is $B$-invariant.

**Example 3.1.** Consider the case when the diagonal blocks of $r$ are $(2, 3, 2)$. 

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The unique root $\psi \in \Psi_1$ is $(3, 7)$. We will denote roots of $\Psi$ by the symbol $\boxtimes$. In this case we have $\xi_1 = (3, 6)$, $\xi_2 = (4, 6)$, and $\xi_3 = (4, 7) \in S$. The root $\xi_3$ corresponds to the admissible pair $((1, 4), (5, 6))$, then

$$A_{\xi_3} = \frac{L_{(3, 7)} L_{(4, 6)}}{L_{(3, 6)} M_{(1, 4)} M_{(4, 7)}},$$

where

$$L_{(3, 7)} = x_{2, 3} \begin{vmatrix} x_{3, 6} & x_{3, 7} \\ x_{5, 6} & x_{5, 7} \end{vmatrix} + x_{2, 4} \begin{vmatrix} x_{4, 6} & x_{4, 7} \\ x_{5, 6} & x_{5, 7} \end{vmatrix},$$

$$L_{(4, 6)} = x_{1, 3} \begin{vmatrix} x_{1, 4} \\ x_{2, 4} \end{vmatrix} x_{4, 6} + x_{1, 5} \begin{vmatrix} x_{1, 3} \\ x_{2, 3} \end{vmatrix} x_{2, 5} x_{5, 6},$$

$$L_{(3, 6)} = x_{2, 3} x_{3, 6} + x_{2, 4} x_{4, 6} + x_{2, 5} x_{5, 6},$$

$$M_{(1, 4)} = \begin{vmatrix} x_{1, 3} & x_{1, 4} \\ x_{2, 3} & x_{2, 4} \end{vmatrix}, \quad M_{(4, 7)} = \begin{vmatrix} x_{4, 6} & x_{4, 7} \\ x_{5, 6} & x_{5, 7} \end{vmatrix}.$$

If $\xi = (i, j)$, then denote $\text{row}(\xi) = i$ and $\text{col}(\xi) = j$.

**Lemma 3.2.** Rational functions $A_\psi$, $\psi \in \Psi_1$, are invariant under the adjoint action of $B$.

**Proof.** To prove the statement note that $A_\psi$ is $N$-invariant. Since $B = T \rtimes N$, where $T$ is the set of nondegenerate diagonal matrices, then it is sufficient to prove that $A_\psi$ is $T$-invariant.

Let $X^J_I$ be a square minor at the intersection of rows $I = \{i_1, \ldots, i_k\}$ and columns $J = \{j_1, \ldots, j_k\}$. The adjoint action of the diagonal element

$$t = \begin{pmatrix} t_1 & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \ldots & t_n \end{pmatrix} = \text{diag}(t_1, \ldots, t_n)$$

on $X^J_I$ is follows

$$\text{Ad}_t X^J_I = \frac{t_J}{t_I} X^J_I,$$
where \( t_I = t_{i_1} \ldots t_{i_k} \) and \( t_J = t_{j_1} \ldots t_{j_k} \).

Take any \( \varphi \in \Phi \). Let \((\mu_1, \mu_2)\) be an admissible pair for \( \varphi \). If \( M_{\mu_1} = X^J I \) and \( M_{\mu_2} = X^I J \) for some collections of rows \( I \) and \( I_1 \) and columns \( J \) and \( J_1 \), then

\[
D^{I,J}_{I',J'} = \begin{vmatrix}
X^J_{I'} & (X^I_{J'}) \\
0 & X^I_{I'}
\end{vmatrix}
\]

is called a combined minor, where \( I' = I_2 \setminus \{\text{row}(\mu_2)\} \) and \( J' = J_1 \setminus \{\text{col}(\mu_1)\} \).

By Proposition 2.5 from [S2] the combined minor (5) is equal to the \( N \)-invariant \( L_{\varphi} \).

**Example 3.3.** Let us show combined minors for invariants \( L_{\varphi} \) in Example 3.1.

We have

\[
L_{(3,6)} = D^{(6),\emptyset}_{(2),\emptyset} = (X^2)^{(6)}_{(2)},
\]

\[
L_{(3,7)} = D^{(6,7),\emptyset}_{(2),\{5\}} = \begin{vmatrix}
(X^2)^{(6,7)}_{(2)} \\
X^6_{(7)}
\end{vmatrix},
\]

\[
L_{(4,6)} = D^{(6),\{3\}}_{\{1,2\},\emptyset} = X^3_{1,2} \begin{vmatrix}
(X^2)^{(6)}_{\{1,2\}}
\end{vmatrix}.
\]

Note that

\[
D^{(6,7),\{3\}}_{\{1,2\},\emptyset} = \begin{vmatrix}
X^3_{\{1,2\}} & (X^2)^{(6,7)}_{\{1,2\}} \\
0 & X^6_{\{7\}}
\end{vmatrix} = M_{(2,3)} M_{(4,7)}.
\]

Since \( \psi \in \Psi_1 \), there are roots \( \xi_1, \xi_2, \xi_3 \in S \cup \Phi \) such that \( \psi = (i, b) \), \( \xi_1 = (i, a) \), \( \xi_2 = (j, a) \), \( \xi_3 = (j, b) \) for some \( i < j \) and \( a < b \). Using the submission (5) we have

\[
L_{\psi} = D^{I_1,J'_n}_{I_1,J'}, \quad L_{\xi_1} = D^{I_2,J''}_{I_2,J'}, \quad L_{\xi_2} = D^{I_2,J''}_{I_2,J''}
\]

for some ordered sets of rows \( I_1 \subset I_2 \), \( I' \supset I'' \) and columns \( J_1 \supset J_2 \), \( I_1 \subset I_2 \).

If \( \xi_3 \in \Phi \), then \( L_{\xi_3} = D^{I_2,J'}_{I_2,J'} \). Let \( \xi_3 \in S \); since there exists an admissible pair \( (\gamma, \gamma') \) for \( \xi_3 \), then there is the root \( \gamma \in S \) in the column \( j \). We have \( M_{\xi_3} M_{\gamma} = D^{I_2,J''}_{I_2,J''} \). In any case we get

\[
A_{\psi} = \frac{D^{I_1,J'}_{I_1,J'} \cdot D^{I_2,J''}_{I_2,J''}}{D^{I_1,J'}_{I_1,J'} \cdot D^{I_2,J''}_{I_2,J''}}
\]

Then

\[
\text{Ad}_t A_{\psi} = \begin{vmatrix}
t_{I_1,I'} & t_{I_1,J'} \\
t_{J_1,J'} & t_{J_1,I'}
\end{vmatrix} \begin{vmatrix}
t_{I_1,J'} & t_{I_1,J''} \\
t_{J_1,J'} & t_{J_1,J''}
\end{vmatrix} = A_{\psi},
\]

where \( t_I = t_{i_1} \ldots t_{i_k} \) and \( t_J = t_{j_1} \ldots t_{j_k} \).
i.e. a function $A_{\psi}$ is $T$-invariant. □

Now we construct the second series of $B$-invariants. Suppose for some $t$ we have $\psi = (R_{t-1} + k, R_t + 1) \in \Phi$, where $k < r_t$, i.e. $x_\psi$ is in the first column and in the $k$th row of the block $X_{t-1,t}$. We say that the root $\psi$ belongs to the set $\Psi_2$ of the second series if there exist $k$ roots

$$(R_{s-1} + 1, j_1), (R_{s-1} + 1, j_2), \ldots, (R_{s-1} + 1, j_k) \in \Phi$$

for some $j_1 < j_2 < \ldots < j_k$ and $s < t$. The roots (6) correspond to variables in the first row of the blocks $X_{s-1,s}, X_{s-1,s+1}, \ldots$. Suppose that the number $s$ is maximal. We can assume without loss of generality that there are no roots of $\Phi$ in the row $R_{s-1} + 1$ between the roots (6), i.e. for any $a$ there is not a column $j$, $j_a < j < j_{a+1}$, such that $(R_{s-1} + 1, j) \in \Phi$. Denote by $\xi_1$ the last root $(R_{s-1} + 1, j_k)$ in (6). Let $\gamma_1 = (R_{s-1}, R_{s-1} + 1)$, evidently $\gamma_1 \in S$. Since $\psi \in \Phi$, then there is a corresponding admissible pair $(\gamma_2, \gamma_3)$ for $\psi$, where $\gamma_2$ is in the $(R_{t-1} + k)$th column and $\gamma_3 = (R_t, R_t + 1)$. See the diagram below.

![Diagram 3](image)

Let us show that the sizes of blocks between the $s$th and $t$th blocks in $r$ are less or equal to $k$. Suppose $r_p > k$, $s < p < t$. Therefore by Lemmas 2.4 and 2.5 there are $r_p$ roots of $S$ in rows $R_{p-1} + 1, \ldots, R_p$, i.e. there is a root in every row to the right of $p$th block. Every of these roots of $S$ in rows $R_{p-1}, \ldots, R_p$ and a root $(R_{p-1}, R_{p-1} + 1)$ make an admissible pair. Hence there exist $r_p - 1$ roots of $\Phi$ in the row $R_{p-1} + 1$. Since $r_p - 1 \geq k$, it is a contradiction with the maximality of $s$.

From Lemmas 2.4 and 2.5 it follows that there exists a root $\gamma_4 \in S$ in the column $R_{t-1} + k + 1$. Since the sizes of blocks between the $s$th and $t$th blocks in $r$ are less or equal to $k$, $\gamma_4$ is to the right of the $s$th block in $r$. 12
There exist three cases $r_s = r_t$, $r_s < r_t$, and $r_s > r_t$. We illustrate all three cases and point all roots $\psi, \xi_1, \gamma_i$ on Diagrams 3, 4, and 5. Diagram 3 (4 and 5, respectively) corresponds to the case $r_s = r_t$ ($r_s < r_t$ and $r_s > r_t$, respectively).

Let us analyze these three cases.

1. If $r_s = r_t$, then denote $\gamma_5 = (R_{s-1} + 1, R_t)$. Obviously $\gamma_5 \in S$. 

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2. If \( r_s < r_t \), then there exists a root in \( S \) in the row \( R_{s-1} + 1 \). Denote this root by \( \gamma_5 \). In this case the pair \( (\gamma_5, \gamma_3) \) is admissible, therefore the pair corresponds to some \( \xi_2 \in \Phi \). The root \( \xi_2 \) is in the column \( R_t + 1 \).

3. In the case \( r_s > r_t \) there exists a root in \( S \) in the column \( R_t \), we denote it by \( \gamma_5 \). Then \( (\gamma_1, \gamma_5) \) is admissible and we denote a corresponding root \( (R_{s-1} + 1, R_t) \in \Phi \) by \( \xi_3 \).

Denote

\[
D = \begin{cases} 
M_{\gamma_1}M_{\gamma_3}M_{\gamma_5} & \text{if } r_s = r_t; \\
M_{\gamma_1}L_{\xi_2} & \text{if } r_s < r_t; \\
M_{\gamma_3}L_{\xi_3} & \text{if } r_s > r_t.
\end{cases}
\]

Now we are ready to show the second series of \( B \)-invariants. Firstly, consider the simplest case, when \( t = s + 1 \) and \( r_s = 2 \) or \( r_t = 2 \) see the diagrams below.

In these cases denote

\[
B_\psi = \frac{L_\psi L_{\xi_1}}{D}.
\]

**Example 3.4.** The simplest case \((1, 2, 2, 1)\) is following.

We have a unique root \( \psi = (4, 6) \in \Psi_2 \) in \( \Psi \) and the unique \( B \)-invariant.

\[
B_\psi = \frac{L_{(2,4)}L_{(4,6)}}{M_{(1,2)}M_{(5,6)}M_{(2,5)}} = \frac{(x_{1,2}x_{2,4} + x_{1,3}x_{3,4})(x_{3,4}x_{4,6} + x_{3,5}x_{5,6})}{x_{1,2}x_{5,6}}.
\]
Let us construct invariants for more difficult case. We define a relation in \( \Delta^+ \) such that for roots \( \gamma = (a_1, b_1) \) and \( \gamma' = (a_2, b_2) \) we have \( \gamma \succ \gamma' \) whenever \( a_1 < a_2 \) and \( b_1 > b_2 \).

If \( r_s > 2 \) and \( r_t > 2 \) or \( s \neq t + 1 \) and \( r_s, r_t \geq 2 \), then a \( B \)-invariant of the second series is defined as follows:

\[
B_\psi = \frac{L_{\xi_1} L_{\psi} \cdot \prod_{\mu \prec \gamma_5} M_\mu}{D \cdot \prod_{\mu \prec \gamma_4} M_\mu \cdot \prod_{\mu' \prec \mu \prec \gamma_4} M_{\mu'}}. 
\]

The product in the numerator (the first product in the denominator resp.) are taken on all roots \( \mu \prec \gamma_5 \) (\( \mu \prec \gamma_4 \) resp.) such that \( \mu \in S \) and \( \mu \) is maximal in the sense of the relation \( \prec \). For example, \( \prod_{\mu \prec (1,5)} M_\mu = M_{(2,3)} M_{(3,4)} \) for Diagram 1. The second product in the denominator is taken on all roots \( \mu' \prec \mu \), where \( \mu \prec \gamma_4 \), such that \( \mu, \mu' \in S \) and \( \mu, \mu' \) are maximal in the sense of the relation \( \prec \).

**Example 3.5.** Consider more difficult example than Example 3.4. Let \((2, 2, 3, 3, 2)\) be the sizes of diagonal blocks in \( \tau \).

We have \( \Psi_2 = \{(5, 8), (8, 11), (9, 11)\} \). For the root \((5, 8)\) we have the simple case of the second series

\[
B_{(5,8)} = \frac{L_{(5,8)} L_{(3,5)}}{L_{(6,8)} M_{(2,3)}}.
\]
For roots (8, 11) and (9, 11) one has $\gamma_1 = (4, 5), \gamma_3 = (10, 11), \gamma_5 = (5, 10),$ and $\prod_{\mu \prec \gamma_5} M_{\mu} = M_{(6,9)}$. For the root (8, 11) we have $\gamma_4 = (6, 9), \prod_{\mu \prec \gamma_4} M_{\mu} = M_{(7,8)},$ and there is not a root $\mu'$ such that $\mu' < \mu < \gamma_4$, then

$$B_{(8,11)} = \frac{L_{(8,11)}L_{(5,8)}M_{(6,9)}}{M_{(4,5)}M_{(5,10)}M_{(10,11)}M_{(7,8)}}.$$  

For (9, 11) we get $\gamma_4 = (5, 10), \prod_{\mu \prec \gamma_4} M_{\mu} = M_{(6,9)}, \prod_{\mu' \prec \mu \prec \gamma_4} M_{\mu'} = M_{(7,8)},$ and

$$B_{(9,11)} = \frac{L_{(9,11)}L_{(5,9)}}{M_{(4,5)}M_{(5,10)}M_{(10,11)}M_{(7,8)}}.$$  

Let us also write $B$-invariants of the first series, $\Psi_1 = \{(5, 9), (8, 12)\},$

$$A_{(5,9)} = \frac{L_{(5,9)}L_{(6,8)}}{L_{(5,8)}M_{(3,6)}M_{(6,9)}}, \quad A_{(8,12)} = \frac{L_{(8,12)}L_{(9,11)}}{L_{(8,11)}M_{(5,9)}M_{(9,12)}}.$$  

**Lemma 3.6.** Rational functions $B_\psi, \psi \in \Psi_2,$ are invariant under the adjoint action of $B$.

**Proof.** Similarly to the case $A_\psi,$ it is sufficient to prove that $B_\psi$ are $T$-invariant, where $T$ is the set of nondegenerate diagonal matrices. We prove the statement for the case $r_s = r_t$. The others cases are similar.

If $\psi \in \Psi_2$ and $r_s = r_t$, then there exist roots $\xi_1 \in \Phi$ and $\gamma_1, \ldots, \gamma_5 \in S$ describing before. We have

$$L_\psi = D_{[R_s+1]}^{J_1}, \quad L_{\xi_1} = D^{J_1}_{\{R_{s-1}\}}, \quad M_{\gamma_1} = X^{\{R_{s-1}+1\}}_{\{R_{s-1}\}}, \quad M_{\gamma_3} = X^{\{R_t+1\}}_{\{R_t\}}, \quad M_{\gamma_5} = X^{\{R_{s-1}+R_s+R_t+1\}}_{\{R_{s-1}, R_s, R_t\}}, \quad \prod_{\mu \prec \gamma_5} M_{\mu} = X_1^{J_1'}, \quad \prod_{\mu' \prec \mu \prec \gamma_4} M_{\mu} = X_1^{J_1 \cup \{\text{row}(\gamma_4)\}}, \quad \prod_{\mu \prec \gamma_1} M_{\mu} = X_1^{J_1'}, \quad \prod_{\mu \prec \gamma_3} M_{\mu} = X_1^{J_1 \cup \{\text{col}(\gamma_4)\}}.$$  

where

$I_1 = \{\text{row}(\gamma_4)+1, \text{row}(\gamma_4)+2, \ldots, R_{t-1}\}, \quad J_1 = \{R_s+1, R_s+2, \ldots, \text{col}(\gamma_4)-1\},$  

$I' = I_1 \setminus \{\text{row}(\gamma_4)+1\}, \quad J' = J_1 \setminus \{\text{col}(\gamma_4)-1\}.$  

Note that the number of elements in the sets $I_1$ and $J_1$ are the same because $X_1^{J_1 \cup \{\text{col}(\gamma_4)\}} = M_{\gamma_4}$ is a square minor.

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Then
\[ \text{Ad}_t B_\psi = \text{Ad}_t \frac{L_{\xi_1} L_\psi \cdot \prod_{\mu < \gamma_5} M_\mu}{D \cdot \prod_{\mu < \gamma_4} M_\mu \cdot \prod_{\mu' < \mu < \gamma_4} M_{\mu'}} = \frac{t_{\gamma_1} t_p L_{\xi_1} \cdot t_{J_1}}{t_{\gamma_1} t_{R_{\gamma_1} + 1} t_{J'}} \times \frac{t_{R_{\gamma_1} + 1} M_{\gamma_1} \cdot t_{R_{\gamma_1} + 1} M_{\gamma_3}}{t_{R_{\gamma_1} + 2} t_{R_{\gamma_1} + 3} \cdots t_{R_{\gamma_1} - 1} \prod_{\mu < \gamma_5} M_\mu} \times \frac{t_{R_{\gamma_1} + 2} \cdots t_{R_{\gamma_1} - 1} \prod_{\mu < \gamma_5} M_\mu}{t_{R_{\gamma_1} + 2} \cdots t_{R_{\gamma_1}} M_{\gamma_5} \cdot \prod_{\mu < \gamma_4} M_{\mu}} = B_\psi. \]

So \( B_\psi \) is \( T \)-invariant. \( \square \)

Denote
\[ X = \sum_{\psi \in \Psi} c_\psi E_\psi + \sum_{\xi \in (S \cup \Phi) \setminus \Psi} E_\xi. \]

We will show below that a \( B \)-orbit in general position has a representative in \( X \).

Let \( \pi : K(m)^B \to K(X) \) be the restriction map
\[ \pi(x_\xi) = x_\xi|_X = \begin{cases} c_\xi & \text{if } \xi \in \Psi, \\ 1 & \text{if } \xi \in (S \cup \Phi) \setminus \Psi, \\ 0 & \text{if } \xi \notin S \cup \Phi, \end{cases} \quad (7) \]

where the field \( K(X) \) is a field of fractions for the polynomial algebra \( K[X] \) of variables \( c_\psi, \psi \in \Psi \).

**Proposition 3.7.** The system of rational functions
\[ \{ A_{\psi_1}, B_{\psi_2} : \psi_1 \in \Psi_1, \psi_2 \in \Psi_2 \} \]

is algebraically independent.

**Proof.** It is sufficient to prove that the system
\[ \{ A_{\psi_1}|_X, B_{\psi_2}|_X, \psi_1 \in \Psi_1, \psi_2 \in \Psi_2 \} \]

is algebraically independent.

Let us number roots \( \psi \in \Psi \) from the bottom up in columns from left to right.

**Example 3.8.** Roots \( \psi \in \Psi \) are numbered, roots from \( S \) and from \( \Phi \setminus \Psi \) are labeled by the symbols \( \otimes \) and \( \times \) as before.
We prove the lemma by induction on number of roots in $\Psi$. Since for the first root $\psi \in \Psi$ we have $A_\psi = c_\psi$ if $\psi \in \Psi_1$ and $B_\psi = c_\psi$ if $\psi \in \Psi_2$, the system consisting of one $B$-invariant is algebraically independent. Therefore the base of induction is obvious. Suppose that the system $\{A_\psi, B_\psi\}$, where $\psi'$ and $\psi''$ are all roots in $\Psi$ such that numbers of $\psi'$ and $\psi''$ are less than $k$, is algebraically independent. Let $\psi \in \Psi$ be a root with the number $k$. For any $\xi \in S$ and $\varphi \in \Phi$ we have

$$M_\xi|_X = \begin{cases} c_\varphi & \text{if } \varphi \in \Psi, \\ 1 & \text{if } \varphi \in \Phi \setminus \Psi. \end{cases}$$

Denote

$$\tilde{c}_\varphi = \begin{cases} c_\varphi & \text{if } \varphi \in \Psi, \\ 1 & \text{if } \varphi \in \Phi \setminus \Psi. \end{cases}$$

for any $\varphi \in \Phi$. Then

$$A_\psi|_X = \begin{cases} \frac{c_\varphi \tilde{c}_{\xi_2}}{\tilde{c}_{\xi_1}} \tilde{c}_{\xi_3} & \text{if } \xi_3 \in \Phi, \\ \frac{c_\varphi \tilde{c}_{\xi_2}}{\tilde{c}_{\xi_1}} & \text{if } \xi_3 \in S; \end{cases} \quad B_\psi|_X = \begin{cases} \frac{c_\varphi \tilde{c}_{\xi_2}}{\tilde{c}_{\xi_1}} & \text{if } r_s = r_t, \\ \frac{c_\varphi \tilde{c}_{\xi_1}}{\tilde{c}_{\xi_3}} & \text{if } r_s < r_t, \\ \frac{c_\varphi \tilde{c}_{\xi_1}}{\tilde{c}_{\xi_3}} & \text{if } r_s > r_t, \end{cases}$$

where $\xi_1, \xi_2, \xi_3$ are the mentioned above roots corresponding to $\psi$. Note that the roots $\xi_1, \xi_2, \xi_3$ are below of the root $\psi$ or the numbers of columns for $\xi_1, \xi_2, \xi_3$ are less than $\text{col}(\psi)$ for the both cases $\psi \in \Psi_1$ and $\psi \in \Psi_2$. Therefore
the numbers of $\xi_1, \xi_2, \xi_3$ are less than $k$. Hence there is no the variable $c_\psi$ in any function of the system $\{A_\psi, B_\psi\}$, where numbers of $\psi'$ and $\psi''$ are less than $k$. It means that the image of the $B$-invariant corresponding to $\psi$ does not depend on $\{A_\psi', B_\psi''\}$, where numbers of $\psi'$ and $\psi''$ are less than $k$. So we get the statement of the proposition. \Square

§4. The canonical representative for $B$-orbits in general position

In this section we show that a $B$-orbit in general position has a unique representative in the set

$$X = \sum_{\psi \in \Psi} c_\psi E_\psi + \sum_{\xi \in (S \cup \Phi) \setminus \Psi} E_\xi, \quad c_\psi \neq 0.$$  

**Theorem 4.1.** There exists an nonempty open set $U \subset \mathfrak{m}$ such that for any $x \in U$ there is $g \in B$ satisfying $\text{Ad}_g x \in X$.

**Proof.** Let $T$ be the set of nondegenerate diagonal matrices. By Theorem 2.19 and using $B = T \ltimes N$ it is sufficient to show that $W = U$ and there is $t \in T$ such that for any $x \in \mathcal{Y} \cap W$ the $T$-orbit of $x$ intersects $X$ at a unique point, where $W$ is nonempty open from Theorem 2.19.

We describe an algorithm of bringing $x \in \mathcal{Y} \cap W$ to the canonical form. Denote

$$h_i(b) = \text{diag}(\underbrace{1, \ldots, 1}_i, b, \underbrace{1, \ldots, 1}_{n-i}).$$

Then the adjoin action of $h_i(b)$ on $y \in \mathfrak{m}$ consists of two transformations:

1) the row $i$ is multiplied by $b$,

2) the column $i$ is multiplied by $b^{-1}$.

Let $a_1 = 1$ and $a_1 < a_2 < \ldots < a_p$ are numbers of blocks in $r$ such that $1 < r_{a_2} < \ldots < r_{a_p}$ and $p$ is maximal.

Let the map $\omega_{i,j} : \mathfrak{m} \to K$ take each $y \in \mathfrak{m}$ to the element in the intersection of the $i$th row and the $j$th column of $y$. Denote $\omega_\xi = \omega_{i,j}$ if $\xi = (i, j)$.

We need making units in positions of the matrix $x$ corresponding to $\xi$ for $\xi \in (S \cup \Phi) \setminus \Psi$. The algorithm consists of 5 steps.

**Step 1.** Consider the first column of the block $X_{a_2,a_2+1}$, it is the column $R_{a_2} + 1$. By the definition of $a_1, a_2, \ldots, a_p$ it follows that $r_2 = r_3 = \ldots =$
\( r_{a_2 - 1} = 1 \). Therefore there are not roots of \( \Phi \) in the rows 1, 2, \ldots, \( R_{a_2 - 1} \). The pair
\[
q = \left( \left( R_{a_2 - 1}, R_{a_2 - 1} + 1 \right), \left( R_{a_2}, R_{a_2} + 1 \right) \right)
\]
is admissible. Hence the root \( \xi = (R_{a_2 - 1} + 1, R_{a_2} + 1) \in \Phi \) correspond to \( q \) and the row \( R_{a_2 - 1} + 1 \) is the first row with a root of \( \Phi \). So \( \xi \not\in \Psi \). Similarly, there are not roots of \( \Psi \) in the column \( R_{a_2} + 1 \).

Note that since \( x \in \mathcal{Y} \), we have \( \omega_\xi(x) \neq 0 \) for any \( \xi \) of the extended base. Therefore for any \( b \in T \) and \( \xi \in S \cup \Phi \) we have \( \omega_\xi(\text{Ad}_b x) \neq 0 \).

The adjoint action of the element
\[
g_1 = \prod_{i=R_{a_2 - 1}+1}^{R_{a_2}} h_i(\omega_{i,R_{a_2}+1}(x)^{-1})
\]
on \( x \) makes units in the positions \( (R_{a_2 - 1} + i, R_{a_2} + 1) \) for \( i = 1, 2, \ldots, r_{a_2} \), i.e. the action of \( g_1 \) makes units in the first column of \( X_{a_2,a_2+1} \). The numbers 1 label the cells in the diagram Example 4.2, where we make units by \( g_1 \).

**Example 4.2.** Let the diagonal sizes are \((2, 1, 3, 1, 4, 2)\), then \( p = 3 \) and \( a_1 = 1, a_2 = 3, a_3 = 5 \). Note that if \( \xi \in S \cup \Phi \) belongs to \( \Psi \), then \( \xi \) is labeled by the symbol \( \otimes \). If \( \xi \in (S \cup \Phi) \setminus \Psi \), then \( \xi \) is labeled by numbers of steps.

![Diagram 12](image)

**Step 2.** Any row \( 1, 2, \ldots, R_{a_2 - 1} \) of a diagram contains a single symbol \( \otimes \). The adjoint action of
\[
g_2 = \prod_{i=1}^{R_{a_2-1}} h_i(b_i)
\]
with a suitable choice of \( b_i \) on \( \text{Ad}_{g_1} x \) makes units in the positions corresponding to roots of \( S \) in rows \( 1, 2, \ldots, R_{a_2 - 1} \) and \( g_2 \) leaves the rest part of \( \text{Ad}_{g_1} x \) unchanged. We have \( b_i = \omega_{\xi_i}(\text{Ad}_{g_1} x)^{-1} \) for the root \( \xi_i \) in the \( i \)th row. The step 2 makes units in the positions that are label by 2 in Diagram 12. Now we have that the first \( R_2 + 1 \) columns have the canonical form.

Take \( k := 3 \).

**STEP 3** provides the canonical form in the columns \( R_{a_k - 1} + 2, \ldots, R_a \).

Let \( j \) be any column of \( R_{a_k - 1} + 2, \ldots, R_a \). Let us show that if there exists \( \psi \in \Phi \) in this column, then \( \psi \in \Psi \). There are two cases, the first case is \( j = R_t + 1 \) for some \( t \), where \( a_{k-1} + 1 < t \leq a_k \), and the second one is \( j \neq R_t + 1 \) for any \( t \).

For the first case \( j = R_t + 1 \) by the definition of \( a_1, \ldots, a_p \) we have \( r_t \leq r_{a_k - 1} \). By Lemmas 2.4 and 2.5 there exists a root of the base in every row \( R_{a_k - 1} + 1, R_{a_k - 1} + 2, \ldots, R_a \). Therefore the root \( (R_{a_k - 1}, R_{a_k - 1} + 1) \) and every of these roots except the root in the row \( R_{a_k - 1} + 1 \) make an admissible pair. Hence there are \( r_{a_k - 1} - 1 \) roots of \( \Phi \) in the row \( R_{a_k - 1} + 1 \). Since \( r_t \leq r_{a_k - 1} \) there are no more that \( r_t - 1 \) roots of \( \Phi \) in the column \( j = R_t + 1 \). Therefore by the definition of \( \Psi \) every root of \( \Phi \) in the column \( j \) belongs to \( \Psi \).

Consider the second case. If \( \psi = (i, j) \) is in \( \Phi \) for some \( i \) and \( R_t + 1 < j \leq R_{t+1} \) for some \( t \), then let us prove that \( \psi \in \Psi \). It is sufficient to prove that \( \xi_1 = (i, R_t + 1) \in \Phi, \xi_2 = (i + 1, R_t + 1) \in \Phi, \) and \( \xi_3 = (i + 1, j) \in S \cup \Phi \). Since \( (i, j) \in \Phi \), then there is the first root \( \gamma_1 \in S \) of the admissible pair for \( \psi \) in the column \( i \). Then \( (\gamma_1, (R_t, R_t + 1)) \) is an admissible pair and the corresponding root \( (i, R_t + 1) \) is in \( \Phi \). Since \( r_t \leq r_{k-1} \), by Lemmas 2.4 and 2.5 we have that there is a root of \( S \) in every column \( R_{t-1} + 1, R_{t-1} + 2, \ldots, R_t \), i.e. in every column above the block \( r_t \) in \( r \). Therefore there exists \( \gamma_2 \in S \) in the column \( i + 1 \), then \( (\gamma_2, (R_t, R_t + 1)) \) is admissible and \( \xi_2 \in \Phi \). Similar reasoning shows \( \xi_3 \in S \cup \Phi \). Thus we have \( \psi \in \Psi \).

So we need constructing units in columns \( R_{a_k - 1} + 2, \ldots, R_a \) only in the positions \( \varphi \), where \( \varphi \in S \). The action of

\[
g_{2k-3} = \prod_{j=R_{a_k-1}+2}^{R_a} h_j(b_j), \quad b_j = \omega_{\varphi_j}(\text{Ad}_{g_{2k-4 \ldots 1} x})
\]

on \( \text{Ad}_{g_{2k-4 \ldots 1} x} \), where \( \varphi_j \in S \) is the root in the \( j \)th column, constructs units in positions corresponding to roots in \( S \) in columns \( j \), \( R_{a_k - 1} + 1 < j \leq R_a \).

If there is no a root of \( S \) in the \( j \)th column, then \( h_j \) is equal to the identity matrix. The action of \( g_{2k-3}g_{2k-4} \ldots g_1 \) multiplies the rows \( R_{a_k - 1} + 2, \ldots, R_a \) and the same columns on \( b_i \) and therefore it makes the canonical view in the columns \( 1, 2, \ldots, R_a \).
In Example 4.2 Step 3 constructs units in positions that was labeled by 3.

STEP 4 is performed if $k \neq p$ or $R_{ap} < n$.

Consider all roots of $(S \cup \Phi) \setminus \Psi$ in the column $R_k + 1$ to the right of the $r_k$th block in $\tau$. We show in the previous step that for any $m, 1 < m < p$, there are exactly $a_m - 1$ roots of $\Phi$ in the row $R_{a_m - 1} + 1$. Therefore the maximal number of roots of $\Phi$ in a row $i$, where $i \leq R_{a_i - 1}$, is $r_{a_i - 1} + 1$. Hence if $1 \leq i < r_{a_k - 1}$, then $\xi = (R_{a_k - 1} + i, R_{a_k} + 1) \in \Psi$ and if $r_{a_k - 1} \leq i \leq r_{a_k}$, then $\xi = (R_{a_k - 1} + i, R_{a_k} + 1) \in \Phi \setminus \Psi$. By Lemmas 2.4 and 2.5 there exist $r_{a_k - 1}$ roots of the base in the first $r_{a_k - 1}$ columns above the $r_k$th block in $\tau$. Besides, the root of $S$ in the last column $R_{a_k - 1} + r_{a_k - 1}$ of these columns is in the row $R_{a_k - 1} + 1$. Therefore there is not a root of $\Phi$ in the column $R_{a_k - 1} + r_{a_k - 1}$.

Besides there are not roots of the extended base in columns $R_{a_k - 1} + r_{a_k - 1} + 1, \ldots, R_{a_k}$.

The action of element

$$g_{2k-2} = \left( \prod_{i=R_{a_k - 1} + r_{a_k - 1} + 1}^{R_a} h_i(b_i) \right) \cdot h_{R_{a_k} + 1}(b_{R_{a_k} + 1})$$

for $b_{R_{a_k} + 1} = \omega_{R_{a_k - 1} + r_{a_k - 1}, R_{a_k} + 1}(\text{Ad}_{g_{2k-1} \cdots g_1}x)$

and $b_i = \omega_{i, R_{a_k} + 1}(\text{Ad}_{g_{2k-1} \cdots g_1}x)^{-1}$

makes units in the positions $(R_{a_k - 1} + i, R_{a_k} + 1)$, where $r_{a_k - 1} \leq i \leq R_{a_k}$ and does not change the rows $1, 2, \ldots, R_{a_k - 1}$ of the matrix $\text{Ad}_{g_{2k-1} \cdots g_1}x$.

We increase the variable $k$ by 1 and go to the Step 3 until $k = p$.

STEP 5 is performed if $R_{ap} + 1 < n$. Using the similar reasoning, one has that any root of $\Phi$ in columns $j, j > R_{ap} + 1$, belongs to $\Psi$.

We make units in positions corresponding to roots of $S$ in the columns $R_{ap} + 2, \ldots, n$ as in Step 3. □

**Theorem 4.3.** The restriction map (7) is a bijection from $K(m)^B$ to $K(\mathcal{X})$.

**Proof.** Let us show that $\pi$ is an injection. Suppose $\pi(f) = 0$ for some $f \in K(m)^B$. It means $f(\mathcal{X}) = 0$. By Theorem 4.1 we have $\text{Ad}_B \mathcal{X} = m$. Therefore since $f$ is a $B$-invariant, we obtain

$$f(\text{Ad}_B \mathcal{X}) = f(\mathcal{X}) = 0.$$ 

Hence $f \equiv 0$ and $\text{Ker}(\pi) = \{0\}$.

Let us check that any variable $c_\psi, \psi \in \Psi$, has a preimage in $K(m)^B$ under the action of $\pi$. We use the numbering $\psi_1, \psi_2, \ldots, \psi_k, k = |\Psi|$, for the roots

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of $\Psi$ from the proof of Theorem 3.7. Let us prove the theorem by induction on the number of roots from $\Psi$.

The base of induction is showed as follows. If $\psi_1 \in \Psi_1$, then by definition of $A_{\psi_1}$ there exist roots $\xi_1, \xi_2, \xi_3 \in S \cup \Phi$. Since the number of $\psi_1$ is 1 and $\xi_1, \xi_2, \xi_3$ are below or to the left of $\psi$, then $\xi_1, \xi_2, \xi_3$ do not belong to $\Psi$. Therefore, $c_{\psi_1} = \pi(A_{\psi_1})$. Similarly, if $\psi_1 \in \Psi_2$, then $c_{\psi_1} = \pi(B_{\psi_1})$.

Suppose that the statement is true for any number less than $m$. If $\psi_m \in \Psi_1$, then

$$A_{\psi_m}|\chi = \frac{c_{\psi_m}c_{\xi_2}}{c_{\xi_1}c_{\xi_3}} \text{ if } \xi_3 \in \Phi \text{ or } A_{\psi_m}|\chi = \frac{c_{\psi_m}c_{\xi_2}}{c_{\xi_1}} \text{ if } \xi_3 \in S,$$

for the roots $\xi_1, \xi_2, \xi_3$ defined for $\psi_m$. Since $\xi_1, \xi_2, \xi_3$ are under and to the left of $\psi_m$, then the numbers of $\xi_1, \xi_2, \xi_3$ are less than the number of $\psi_m$. Therefore by the inductive assumption there exist $B$-invariant rational functions $f, g, h$ such that

$$c_{\xi_1} = \pi(f(A_{\psi}, B_{\phi})_{\phi, \psi \in \Psi}),$$
$$c_{\xi_2} = \pi(g(A_{\psi}, B_{\phi})_{\phi, \psi \in \Psi}),$$
$$c_{\xi_3} = \pi(h(A_{\psi}, B_{\phi})_{\phi, \psi \in \Psi}).$$

Then in the case $\xi_3 \in \Phi$ we have

$$c_{\psi_m} = \pi\left(\frac{A_{\psi_m}f(A_{\phi}, B_{\phi})h(A_{\phi}, B_{\phi})}{g(A_{\phi}, B_{\phi})}\right).$$

If $\xi_3 \in S$, then

$$c_{\psi_m} = \pi\left(\frac{A_{\psi_m}f(A_{\phi}, B_{\phi})}{g(A_{\phi}, B_{\phi})}\right).$$

So $c_{\psi_m}$ has a preimage.

The case $\psi_m \in \Psi_2$ is proved similarly. □

**Corollary 4.4.** The field of invariants $K(m)^B$ is generated by rational functions $A_{\psi_1}$ and $B_{\psi_2}$, $\psi_1 \in \Psi_1$, $\psi_2 \in \Psi_2$.

Using Corollary of Theorem 2.3 of [PV] one has

**Corollary 4.5.** The dimension of a $B$-orbit in general position equals

$$\dim m - |\Psi|.$$

The rational functions $A_{\psi_1}$ and $B_{\psi_2}$, $\psi_1 \in \Psi_1$, $\psi_2 \in \Psi_2$, separate $B$-orbits in general position.

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