Magnetic fields in noncommutative quantum mechanics

Dedicated to the memory of Julius Wess

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Abstract

We discuss various descriptions of a quantum particle on noncommutative space in a (possibly non-constant) magnetic field. We have tried to present the basic facts in a unified and synthetic manner, and to clarify the relationship between various approaches and results that are scattered in the literature.

1We dedicate these notes to the memory of Julius Wess whose scientific work was largely devoted to the study of gauge fields and whose latest interests concerned physical theories on noncommutative space. The fundamental and inspiring contributions of Julius to Theoretical Physics will always bear with us, but his great kindness, his clear and enthusiastic presentations, and his precious advice will be missed by all those who had the chance to meet him.

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Contents

1 Introduction and overview

2 Reminder on a particle in a magnetic field in QM
   2.1 General magnetic fields
   2.1.1 Hamiltonian involving a vector potential
   2.1.2 Commutation relations involving the magnetic field
   2.2 Constant magnetic field (Landau system)

3 Noncommutative classical mechanics
   3.1 Poisson algebra and its representations
       3.1.1 Poisson brackets
       3.1.2 Transformation to canonical coordinates
       3.1.3 Case of non-constant magnetic field
   3.2 Dynamics
       3.2.1 Poisson brackets involving the magnetic field
       3.2.2 Hamiltonian involving a vector potential

4 NCQM: representations of the algebra
   4.1 Representations of the algebra
   4.2 About the relation between the parameters $\theta$ and $B$

5 NCQM systems: generalities
   5.1 About Hamiltonians
   5.2 Remarks on the ordering problem
   5.3 Properties of energy spectra

6 NCQM systems: operatorial approach
   6.1 Commutation relations involving a constant magnetic field
   6.2 Minimal coupling to a (possibly non-constant) magnetic field
   6.3 “Exotic” approach

7 NCQM systems: star product approaches
   7.1 Scalar potentials
   7.2 Magnetic fields
       7.2.1 Gauge potentials
       7.2.2 Gauge potentials and Seiberg-Witten map

8 Summary and concluding remarks

A Noncommuting coordinates in the truncated Landau problem (Peierls’ substitution)
1 Introduction and overview

**What noncommutativity is about:** Let us assume that the components \( \hat{X}_1, \ldots, \hat{X}_d \) of the quantum mechanical position operator in \( d \)-dimensional space do not commute with each other, but rather satisfy the commutation relations \( [\hat{X}_i, \hat{X}_j] = i\theta_{ij} \mathbb{1} \) where \( \theta_{ij} = -\theta_{ji} \) are real numbers (at least one of which is non-zero). The Cauchy-Schwarz inequality then implies the uncertainty relations \( (\Delta_\psi \hat{X}_i)(\Delta_\psi \hat{X}_j) \geq \frac{1}{2} |\theta_{ij}| \), i.e. the particle described by the wave function \( \psi \) cannot be localized in a precise way. This uncertainty relation for the position implies a certain fuzziness of points in space: one says that the space is *fuzzy, pointless* or that it has a *lattice, quantum or noncommutative structure* \([1]\). Obviously, the noncommutativity can only manifest itself if the configuration space is at least of dimension two. Thus, it represents a deformation of the classical theory which is quite different from the deformation by quantization which concerns all dimensions and which amounts to introducing a cellular structure (parametrized by \( \hbar \)) in phase space.

**History of the subject:** The idea of a fuzzy configuration space looks quite interesting for quantum field theories since it may help to avoid, or at least ameliorate, the problem of short-distance singularities, i.e. ultraviolet divergences. Indeed\(^3\), this argument has already been put forward in 1930 by Heisenberg \([4]\) and the message was successively passed on to Peierls, Pauli \([5]\), Oppenheimer and Snyder (who was a student of the latter). In 1933, Peierls worked out a quantum mechanical application concerning a particle in a constant magnetic field (see appendix) \([6]\) and, in 1947, Snyder considered noncommuting coordinates on space-time in order to discard the ultraviolet divergences in quantum field theory without destroying the Lorentz covariance \([7]\). In the sequel, the latter work was not further exploited due to the remarkable success of the renormalization scheme in quantum electrodynamics. Recently, noncommuting coordinates have regained interest, in particular in the framework of superstring theories and of quantum gravity. Roughly speaking, the intrinsic length scale of the strings induces a noncommutative structure on space-time at low scales. (For a review of these topics, and in particular of quantum field theory on noncommutative space-time and its renormalization, see for instance references \([8]\)-\([11]\).) Since quantum mechanics may be regarded as a field theory in zero spatial dimensions or as a non-relativistic description of the one-particle sector of field theory, it is well suited as a toy model for the introduction of noncommuting coordinates.

**Definition of NCQM:** Quantum mechanics on noncommutative configuration space is generally referred to as noncommutative quantum mechanics (NCQM). Somewhat more generally, one can consider the situation where the commutators for coordinates and momenta are non-canonical. After some precursory work dealing with noncommutative phase space variables in quantum mechanics \([12]\)-\([15]\), NCQM has been defined in a simple and direct manner using different approaches \([16]\)-\([20]\), and some basic physical models have been studied \([16]\)-\([25]\). In the sequel, various aspects of the subject have been further elaborated upon and the literature has grown to a considerable size over the last years\(^4\). The

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\(^3\)The history of the subject has been traced back by J. Wess and is reported upon in references \([2, 3]\).

\(^4\)We apologize to the authors whose work is not explicitly cited here and we refer to \([26]\) for a more
study of exactly solvable models of NCQM should hopefully lead to a better understanding of some issues in noncommutative field theory. There are some potential physical applications which include condensed matter physics (quantum Hall effect, superfluidity, ...) and spinning particles. By considering classical Poisson brackets (instead of commutators) and Hamilton’s equations of motion, one can also discuss noncommutative classical mechanics, as well as applications thereof like magneto-hydrodynamics [27] [2].

**About the physical interpretation and applications:** Since a cellular structure in configuration space is not observed at macroscopic scales, the noncommutativity parameters $\theta_{ij}$ should only manifest themselves at a length scale which is quite small compared to some basic length scale like the Planck length $\sqrt{\hbar G/c^3}$ [1]. (In the context of field theory, one usually writes $\theta_{ij} = \Lambda_{NC}^2 \tilde{\theta}_{ij}$, where the $\tilde{\theta}_{ij}$ are dimensionless and of order 1, so that $\Lambda_{NC}$ represents a characteristic energy scale for the noncommutative theory which is necessarily quite large.) Thus, noncommutativity of space may be related to gravity at very short distances and NCQM may be regarded as a deformation of classical mechanics that is independent of the deformation by quantization. From this point of view, a fully fledged theory of quantum gravity should provide a fuller understanding of the noncommutativity of space.

In the framework of noncommutative classical or quantum mechanics, the parameters $\theta_{ij}$ admit many close analogies with a constant magnetic field both from the algebraic and dynamical points of view [24]. Such an effective magnetic field, which is necessarily quite small, may play a role as primordial magnetic field in cosmological dynamics [28].

Though various physical applications of NCQM have been advocated (in particular for quantum mechanical systems coupled to a constant magnetic field), the only experimental signature of noncommuting spatial coordinates which is currently available appears to be the approximate noncommutativity appearing in the Landau problem for the limiting case of a very strong magnetic field – see appendix. (This is a realization of noncommutativity which is analogous to the one in string theory).

**Scope of the present paper:** The essential features of NCQM can be exhibited by considering the following three instances to which we limit ourselves in the present paper.

1. We focus on flat 2-dimensional space so that the antisymmetric matrix $(\theta_{ij})$ is given by $\theta_{ij} = \theta \varepsilon_{ij}$ where $\theta$ is a real number and $\varepsilon_{ij}$ are the components of the antisymmetric tensor normalized by $\varepsilon_{12} = 1$.

2. We assume that the fundamental commutator algebra has the following form (which is not the most general form that one can consider for NCQM [21]):

   $$[\hat{X}_1, \hat{X}_2] = i\theta \mathbb{1}, \quad [\hat{P}_1, \hat{P}_2] = iB \mathbb{1}, \quad [\hat{X}_i, \hat{P}_j] = i\delta_{ij} \mathbb{1}. \quad (1)$$

   Here, the real constant $B$ (which measures the noncommutativity of momenta) describes a constant magnetic field that is perpendicular to the $\hat{X}_1 \hat{X}_2$-plane. In equations $\mathbb{1}$ and in the following, the variables $\hat{X}_i, \hat{P}_j$ (which are to be viewed as self-adjoint Hilbert space operators) are denoted by a hat so as to distinguish them from complete list of references.
the basic operators $X_i, P_j$ of standard quantum mechanics which satisfy the Heisenberg algebra, i.e. the canonical commutation relations (CCR’s)

$$[X_1, X_2] = 0 = [P_1, P_2], \quad [X_i, P_j] = i\hbar \delta_{ij}. \quad (2)$$

As we will see, one can express the operators $\hat{X}_1, \hat{X}_2, \hat{P}_1, \hat{P}_2$ as linear combinations of $X_1, X_2, P_1, P_2$ with coefficients which depend on the noncommutativity parameters $\theta$ and $B$ (that describe the deviation from the canonical commutators). We note that the parameters $\theta$ and $B$ in equations (1) are not necessarily independent of each other and we will see that their interplay yields some particularly interesting results [20, 22, 23, 24].

3. We are interested in the case of a possibly non-constant magnetic field, i.e. $B$ possibly depending on the spatial coordinates.

**Short preview of results:** The main results concerning the representations of the algebra (1) and the spectra of Hamiltonian operators $H(\hat{X}, \hat{P})$ may be summarized as follows.

- For $B \neq 1/\theta$ (e.g. for $B = 0$), there exists a linear invertible transformation relating the operators $\hat{X}_i, \hat{P}_j$ to operators $X_i, P_j$ satisfying canonical commutation relations. By contrast, if $B$ and $\theta$ are related by $B = 1/\theta$, the transformation from $\hat{X}_i, \hat{P}_j$ to $X_i, P_j$ is no longer invertible. The singular behavior is also reflected by the fact that the so-called canonical limit $(\theta, B) \to (0, 0)$ does not exist if $B = 1/\theta$. Moreover, in this case the four-dimensional phase space degenerates to a two-dimensional one in the sense that for any irreducible representation of the commutator algebra, the representation of the $\hat{X}_i$ alone becomes irreducible.

- While the Jacobi identities for the algebra (1) involving the constant magnetic field $B$ are trivially satisfied, they are violated for a non-constant field $B$ if $\theta \neq 0$. Thus, contrary to standard quantum mechanics, such a magnetic field forces us to introduce the interaction into the Hamiltonian by means of a vector potential and a minimal coupling. The noncommutativity of configuration space then leads to a non-Abelian gauge structure and thus to self-interactions of the gauge potential which do not exist in standard quantum mechanics.

- The spectrum of a typical Hamiltonian operator like $H(\hat{X}, \hat{P}) = \frac{1}{2m} \hat{P}^2 + V(\hat{X}_1, \hat{X}_2)$ depends on the ordering of the noncommuting operators $\hat{X}_1, \hat{X}_2$. Apart from some special cases, the energy spectrum for $\theta \neq 0$ differs notably from the one for $\theta = 0$. Non-polynomial potentials $V(\hat{X}_1, \hat{X}_2)$ become non-local when expressed in terms of canonical coordinates $X_1, X_2$. Depending on the chosen parametrization of the phase space variables $X_i, P_j$ in terms of canonical variables $\hat{X}_i, \hat{P}_j$, the Hamiltonian may take different disguises. However, in many instances, the parameter $\theta$ resembles a constant magnetic field.
Outline of the presentation: To start with, we recall the description of a particle in a magnetic field within standard quantum mechanics since it represents a useful guideline for noncommutativity in quantum mechanics and a starting point for generalizations. Section 3 deals with noncommutative classical mechanics. In section 4, we discuss the representations of the commutator algebra of NCQM. The remainder of the text is devoted to the study of physical models of NCQM: we discuss some general properties of Hamiltonian operators and of their spectra, and then treat in detail the problem of a (possibly non-constant) magnetic field. The text concludes with a summary and some remarks.

2 Reminder on a particle in a magnetic field in QM

In non-relativistic quantum mechanics in \( \mathbb{R}^d \), the coordinates \( X_i \) and momenta \( P_j \) satisfy the CCR’s. It is worthwhile recalling that the Schrödinger representation on \( L^2(\mathbb{R}^d, d^d x) \) (i.e. \( P_j = \frac{\hbar}{i} \frac{\partial}{\partial x_j} \) and \( X_i = \text{operator of multiplication by the real variable } x_i \)) is the only possible irreducible realization of the CCR’s up to unitary equivalence \([29, 30]\).

2.1 General magnetic fields

Concerning the algebra of NCQM and its representations, it is useful to have in mind the standard quantum mechanical treatment \([31]\) of a charged particle in three dimensions which is coupled to a magnetic field \( \vec{B}(\vec{x}) \) deriving from a vector potential \( \vec{A}(\vec{x}) \). In the next two paragraphs, we recall the two different, though equivalent descriptions of the problem under consideration.

2.1.1 Hamiltonian involving a vector potential

We start from the coordinates \( X_i \) and the components \( P_j \) of the canonical momentum \( \vec{P} \) satisfying the CCR’s \([2]\). If the spin of the particle is not taken into account, the interaction with the magnetic field \( \vec{B} = \vec{\text{rot}} \vec{A} \) is simply described by means of the so-called minimal coupling, i.e. by the Hamiltonian

\[
H = \frac{1}{2m} (\vec{P} - \frac{e}{c}\vec{A})^2.
\]  

2.1.2 Commutation relations involving the magnetic field

The Hamiltonian \([3]\) is quadratic in the components of the kinematical (gauge invariant) \([32]\) momentum \( \vec{\Pi} \equiv \vec{P} - \frac{e}{c}\vec{A} \):

\[
H = \frac{1}{2m} \vec{\Pi}^2.
\]

\( ^5 \)In the relation \([X_k, P_k] = i\hbar \) at least one of the self-adjoint operators \( X_k, P_k \) has to be unbounded, henceforth it is not defined on all of Hilbert space, but only on some subspace. For this reason, one usually exponentiates all of these operators so as to obtain unitary (i.e. bounded) operators satisfying Weyl’s form of the CCR’s: the cited classification theorem only makes sense for this form of the CCR’s.
Thus, it has the same form as the free particle Hamiltonian $H_0 \equiv \frac{1}{2m} \vec{P}^2$ with $\vec{P}$ replaced by $\vec{\Pi}$. The momentum variables $\Pi_i$ do not commute with each other, rather they satisfy the non-canonical commutation relations

$$[\Pi_i, \Pi_j] = i\hbar \frac{e}{c} B_{ij}, \quad \text{where } B_{ij} \equiv \partial_i A_j - \partial_j A_i = \varepsilon_{ijk} B_k,$$

$$[X_i, X_j] = 0, \quad [X_i, \Pi_j] = i\hbar \delta_{ij} \mathbb{1} \quad \text{for } i, j \in \{1, 2, 3\}.$$

In other words, when starting from the description of a free particle, the coupling of the particle to a magnetic field can simply be described by replacing the canonical momentum $\vec{P}$ in the free Hamiltonian by the kinematical momentum $\vec{\Pi}$ whose components have a non-vanishing commutator: it is the magnetic field which measures this noncommutativity.

We note that the Jacobi identities for the quantum algebra (5) are satisfied for any field strength $\vec{B}(\vec{x})$ and, in particular, one has $[X_i, [\Pi_j, \Pi_k]] + \text{circular permutations} = 0$. For $\vec{B} \neq \vec{0}$, the transformation $(\vec{X}, \vec{P}) \mapsto (\vec{X}, \vec{\Pi})$ is not unitary since it modifies the commutators (2).

### 2.2 Constant magnetic field (Landau system)

In the particular case where the magnetic field is constant, the coordinate axes can be oriented such that $\vec{B} = B\vec{e}_3$. The three-dimensional problem then reduces to a two-dimensional one, namely the problem of a particle in the $x_1x_2$-plane subject to a vector potential $\vec{A} = (A_1, A_2)$ such that $B = \partial_1 A_2 - \partial_2 A_1$: the Hamiltonian then reads as

$$H = \frac{1}{2m} (\Pi_1^2 + \Pi_2^2),$$

if one discards the term $\frac{1}{2m} P_3^2$ describing a free particle motion along the $x_3$-direction. The present physical system (i.e. a charged particle in a plane subject to a constant magnetic field that is perpendicular to this plane) is often referred to as the Landau system. For this problem, convenient choices of the vector potential are the symmetric gauge $(A_1, A_2) = (-\frac{B}{2} x_2, \frac{B}{2} x_1)$ and the Landau gauge, i.e. $(A_1, A_2) = (0, B x_1)$ or $(A_1, A_2) = (-B x_2, 0)$.

The energy spectrum of the Landau system can be obtained by a simple argument. In fact, the Hamiltonian only depends on $\vec{\Pi} = (\Pi_1, \Pi_2)$ and the latter variables satisfy the commutation relation $[\Pi_1, \Pi_2] = i\hbar \frac{e}{c} B \mathbb{1}$, or equivalently

$$[Q, \Pi_2] = i\hbar \mathbb{1} \quad \text{with } Q \equiv \frac{e}{cB} \Pi_1.$$ 

Thus, the operators $Q$ and $\Pi_2$ can be viewed as canonically conjugate variables in terms of which the Hamiltonian reads as

$$H = \frac{1}{2m} \Pi_2^2 + \frac{1}{2} m \omega_B^2 Q^2 \quad \text{with } \omega_B \equiv \frac{|eB|}{mc}.$$ 

Accordingly, the Landau system is equivalent to a linear harmonic oscillator and its eigenvalues (the so-called Landau levels) are given by

$$E_n = \hbar \frac{|eB|}{mc} \left(n + \frac{1}{2}\right) \quad \text{with } n = 0, 1, 2, \ldots$$
Each of these energy levels is infinitely degenerate (with respect to the momentum eigenvalues \( p_1 \in \mathbb{R} \)) which reflects the fact that the Landau system is actually two-dimensional.

### 3 Noncommutative classical mechanics

Instead of an algebra of commutators, one can consider its classical analogon \([33, 15]\) involving Poisson brackets \( \{ \cdot, \cdot \}_\text{PB} \) of functions depending on the real variables \( x_1, x_2, p_1, p_2 \).

#### 3.1 Poisson algebra and its representations

##### 3.1.1 Poisson brackets

The classical algebra associated to (1) reads as \([33, 28]\)

\[
\{x_1, x_2\}_\text{PB} = \theta, \quad \{p_1, p_2\}_\text{PB} = B, \quad \{x_i, p_j\}_\text{PB} = \delta_{ij}.
\]

(10)

The so-called “exotic” algebra \([15, 23, 25]\) amounts to modifying these brackets by a factor \(1/\kappa\) where

\[
\kappa \equiv 1 - B\theta,
\]

(11)

i.e.

\[
\{x_1, x_2\}_\text{PB} = \frac{\theta}{\kappa}, \quad \{p_1, p_2\}_\text{PB} = \frac{B}{\kappa}, \quad \{x_i, p_j\}_\text{PB} = \frac{\delta_{ij}}{\kappa}.
\]

(12)

The expression \(\kappa\) is a characteristic parameter for the algebraic system under study and will frequently reappear in the sequel.

If we denote the phase space coordinates collectively by \(\xi^I \equiv (x_1, x_2, p_1, p_2)\), the Poisson bracket of two functions \(F\) and \(G\) on phase space reads as \(\{F, G\}_\text{PB} \equiv \sum_{I, J=1}^4 \Omega^{IJ} \frac{\partial F}{\partial \xi^I} \frac{\partial G}{\partial \xi^J}\), so that \(\{\xi^I, \xi^J\}_\text{PB} = \Omega^{IJ}\), and the symplectic 2-form is given by

\[
\omega = \frac{1}{2} \sum_{I, J=1}^4 \omega_{IJ} d\xi^I \wedge d\xi^J,
\]

(13)

with \(\omega_{IJ} \equiv (\Omega^{-1})_{IJ}\). The symplectic matrix \(\Omega \equiv (\Omega^{IJ})\) for the standard and exotic algebras \([10]\) and \([12]\) is respectively given by

\[
\Omega_{\text{standard}} = \begin{bmatrix}
0 & \theta & 1 & 0 \\
-\theta & 0 & 0 & 1 \\
-1 & 0 & 0 & B \\
0 & -1 & -B & 0
\end{bmatrix}, \quad \Omega_{\text{exotic}} = \frac{1}{\kappa} \Omega_{\text{standard}}.
\]

(14)

Note that \(\det \Omega_{\text{standard}} = \kappa^2\), hence \(\det \Omega_{\text{exotic}} = \kappa^{-2}\). For \(\theta = 0\) or \(B = 0\), the matrices \(\Omega_{\text{standard}}\) and \(\Omega_{\text{exotic}}\) coincide with each other. The brackets \([12]\) defining the exotic algebra diverge for \(\kappa \to 0\), but not so the associated symplectic form. For the “standard” algebra \([10]\), one has the reversed situation, i.e. one does not have a well defined symplectic form.
3.1.2 Transformation to canonical coordinates

By definition, the phase space \( \mathbb{R}^4 \) equipped with a non-degenerate, antisymmetric bilinear form \((13)\) is a symplectic vector space \([34, 35]\). (The condition of non-degeneracy is equivalent to \( \det (\omega_{IJ}) \neq 0 \).) For such a space, one can find a change of basis (i.e. an invertible linear transformation of coordinates, \((\xi^I) \mapsto (\xi'^I)\)) such that the matrix \((\omega_{IJ})\) goes over into the canonical (normal) form \( J = \begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix} \). This result \([34, 35]\), which is also referred to as the linear Darboux theorem, represents a special case of the Darboux theorem which applies to general phase space manifolds. The explicit form of the linear transformation \((\xi^I) \mapsto (\xi'^I)\) for the algebra \((1)\) will be discussed in section 4 in the context of quantum mechanics. Here, we only emphasize that this transformation is not canonical (in the sense of Hamiltonian mechanics) since it modifies the Poisson brackets.

3.1.3 Case of non-constant magnetic field

Let us consider the Poisson algebra \((10)\) with constant parameter \( \theta \neq 0 \) and with a parameter \( B \) which depends on \( \vec{x} \) (but not on \( \vec{p} \)). Then,

\[
\{x_i, \{p_1, p_2\}\}_{PB} + \text{circular permutations} = \theta \varepsilon_{ij} \frac{\partial B}{\partial x_j}, \quad \text{for } i = 1, 2,
\]

i.e. the Jacobi identities are only satisfied if \( B \) is a constant magnetic field. By contrast, the exotic algebra \((12)\) with constant parameter \( \theta \neq 0 \) allows for a \( \vec{x} \)-dependent \( B \)-field since such a field is compatible with the corresponding Jacobi identities \([25]\). Thus, in the noncommutative plane, the coupling of a particle to a non-constant magnetic field can be achieved by the exotic approach, but not by the standard one. For \( \theta = 0 \), the two approaches coincide with each other. We will come back to the exotic algebra in section 6.3.

3.2 Dynamics

The time evolution of a function \( F \) on phase space is given by Hamilton’s equation of motion \( \dot{F} = \{F, H\}_{PB} \) where \( H \) is the Hamiltonian function. We will only consider the case of a constant magnetic field.

3.2.1 Poisson brackets involving the magnetic field

The dynamics based on the Poisson algebra \((10)\) has been studied in references \([33, 28, 36]\). For the Hamiltonian \( H = \frac{1}{2m} \vec{p}^2 + V(x_1, x_2) \), one finds the equation of motion

\[
m\ddot{x}_i = -\kappa \frac{\partial V}{\partial x_i} + B \varepsilon_{ij} \dot{x}_j + m \theta \varepsilon_{ij} \frac{d}{dt} \left( \frac{\partial V}{\partial x_j} \right) \quad \text{for } i \in \{1, 2\},
\]

(15)

where \( \kappa \equiv 1 - B\theta \). If the singular limit \( \kappa \to 0 \) is taken, the first (Newton-like) force on the right-hand-side vanishes and the number of dynamical degrees of freedom is reduced by half. (This result is related to the Peierls substitution discussed in the appendix.)
Some specific potentials, e.g. the one of the anisotropic harmonic oscillator, are elaborated upon in reference [28]. Here, we will only discuss the case $V = 0$, i.e. the dynamics of a charged particle (of unit charge) in the noncommutative plane subject to a constant magnetic field which is perpendicular to this plane. The equation of motion (15) then reduces to

$$m\ddot{x}_i = B\varepsilon_{ij}\dot{x}_j \quad \text{for } i \in \{1, 2\}. \quad (16)$$

Thus, we have exactly the same equation of motion as for $\theta = 0$. Indeed, the fact that $H$ does not depend on the coordinates $x_i$ implies that the noncommutativity of these coordinates does not manifest itself in the classical trajectories, though it affects some physical quantities (like the density of states per unit area – see section 4 below for the quantum theory).

### 3.2.2 Hamiltonian involving a vector potential

Let us now consider the Poisson algebra (10) with $B = 0$ and introduce a constant magnetic field $B$ deriving from a vector potential $\vec{A} = (A_1, A_2)$, i.e. $\partial_1 A_2 - \partial_2 A_1 = B$. The equations of motion following from the minimally coupled Hamiltonian $H = \frac{1}{2m} (\vec{p} - \vec{A})^2$ then read as

$$m\dot{x}_i = (p_k - A_k) (\delta_{ik} - \theta \varepsilon_{ij} \partial_j A_k), \quad m\dot{p}_i = (p_k - A_k) \partial_i A_k. \quad (17)$$

In the symmetric gauge $A_i = -\frac{B}{2} \varepsilon_{ij} x_j$, these equations yield

$$m\ddot{x}_i = \varepsilon_{ij} \dot{x}_j B (1 + \frac{1}{4} \theta B) \quad (18)$$

and in the Landau gauge $(A_1, A_2) = (0, B x_1)$ we get

$$m\ddot{x}_1 = \dot{x}_2 \frac{B}{1 + \theta B}, \quad m\ddot{x}_2 = -\dot{x}_1 B (1 + \theta B). \quad (19)$$

Thus, it seems that for $\theta \neq 0$ the equations of motion (resulting from the minimally coupled Hamiltonian $H = \frac{1}{2m} (\vec{p} - \vec{A})^2$) depend on the chosen gauge, and that they differ from those obtained in equation (16) (by introducing the magnetic field into the Poisson brackets). This point can be elucidated by evaluating the Poisson bracket between the components of the kinematical momentum $\vec{p} - \vec{A}$:

$$\{p_1 - A_1, p_2 - A_2\}_\text{PB} = \partial_1 A_2 - \partial_2 A_1 + \{A_1, A_2\}_\text{PB} \equiv \mathcal{F}_{12}. \quad (20)$$

The resulting expression for the field strength $\mathcal{F}_{12}$ involves a quadratic term $\{A_1, A_2\}_\text{PB}$ which is characteristic for non-Abelian gauge field theories. The field $\mathcal{F}_{12}$ appears in the equation of motion for the velocity $v_i \equiv \dot{x}_i$: indeed, by differentiating equations (18) and (19), we obtain the equation of motion $\ddot{v}_i + \omega^2 v_i = 0$ where the frequency $\omega$ is defined by

$$\omega \equiv \frac{|\mathcal{F}_{12}|}{m}, \quad \text{with } \mathcal{F}_{12} = \begin{cases} \frac{B}{1 + \frac{1}{4} \theta B} & \text{for the symmetric gauge} \\ B & \text{for the Landau gauge}. \end{cases}$$
Thus, the gauge potentials $A_i = -\frac{B}{2} \varepsilon_{ij} x_j$ and $(A_1, A_2) = (0, B x_1)$ do not define the same noncommutative field strength $\mathcal{F}_{12}$. We will encounter a completely analogous situation in NCQM (sections 6.2 and 7) and we will discuss in that context the non-Abelian gauge transformations, the introduction of a constant $\theta$-independent field strength and the comparison between the different approaches to such a magnetic field.

4 NCQM: representations of the algebra

In the sequel of the text, we generally consider a charge $e = 1$ and a system of units such that $\hbar \equiv 1 \equiv c$. We always assume that the canonical operators $X_i$ and $P_j$ operate on $L^2(\mathbb{R}^2, dx_1 dx_2)$ as in the standard Schrödinger representation, i.e., $X_i$ acts as multiplication by $x_i$ and $P_j = \hbar^{-1} \partial_j$.

4.1 Representations of the algebra

If the canonical limit (or so-called commutative limit) $(\theta, B) \to 0$ exists and is considered, the quantum algebra (1) reduces to the Heisenberg algebra (2) so that the operators $\hat{X}_i, \hat{P}_j$ should reduce to the operators $X_i, P_j$ satisfying (2). Thus, it is natural to look for representations of the algebra (1) for which the operators $\hat{X}_i, \hat{P}_j$ are operator-valued functions of the real parameters $\theta, B$ and of the operators $X_i, P_j$.

To determine such representations, one generally considers a linear transformation $(\vec{X}, \vec{P}) \mapsto (\hat{X}, \hat{P})$ whose transformation matrix depends on the noncommutativity parameters, e.g., see references [24, 37]. In the following, we discuss different representations of (1) and in section 4.2 we will come back to the existence of the canonical limit.

(i) Case $\theta = 0$ : A simple particular case of (1) is given by the Landau system discussed in section 2, then, we have the algebra

$$[\hat{X}_1, \hat{X}_2] = 0, \quad [\hat{P}_1, \hat{P}_2] = iB \mathbb{I}, \quad [\hat{X}_i, \hat{P}_j] = i\delta_{ij} \mathbb{I}$$

and different gauge choices $(A_1(\vec{x}), A_2(\vec{x}))$ with $\partial_1 A_2 - \partial_2 A_1 = B$ provide different representations of the algebra (21), all of which have the form

$$\hat{X}_i = X_i, \quad \hat{P}_j = P_j \pm A_j(\vec{X}) \quad \text{with} \quad \partial_1 A_2 - \partial_2 A_1 = B.$$  

Vector potentials $\vec{A}, \vec{A}'$ giving rise to the same magnetic field $B$ are related by a gauge transformation $\vec{A} \mapsto \vec{A}' = \vec{A} + \nabla_{\alpha}(\vec{X})$ which goes along with a unitary transformation of operators (and states): $\hat{X}' = U \hat{X} U^{-1}$ and $\hat{P}' = U \hat{P} U^{-1}$ with $U = e^{i\alpha}$. In fact [38], all representations of (21) are unitarily equivalent to the representation (22).

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6For the reasons indicated in the footnote to section 2, one should exponentiate the unbounded self-adjoint operators $\hat{X}_i, \hat{P}_j$ so as to obtain Weyl’s form of commutation relations: the classification theorem for the representations actually refers to this form of the commutation relations.
(ii) Case $B = 0$: A completely analogous situation is the case where the spatial coordinates do not commute whereas the momenta commute:

$$[\hat{X}_1, \hat{X}_2] = i\theta I, \quad [\hat{P}_1, \hat{P}_2] = 0, \quad [\hat{X}_i, \hat{P}_j] = i\delta_{ij} I$$

(23)

Then,

$$\hat{P}_i = P_i, \quad \hat{X}_j = X_j - \hat{A}_j(\vec{P}) \quad \text{with} \quad \frac{\partial \hat{A}_1}{\partial P_2} - \frac{\partial \hat{A}_2}{\partial P_1} = \theta$$

(24)

provides a representation of the algebra (23). In fact, in the present context it is appropriate to consider Schrödinger’s momentum space representation for the canonical operators $P_i$ and $X_j$: the operators $\hat{P}_i$ and $\hat{X}_j$ then act on wave functions $\psi \in L^2(\mathbb{R}^2, dp_1 dp_2)$ according to

$$\hat{P}_i \psi = p_i \psi, \quad \hat{X}_j \psi = i \frac{\partial \psi}{\partial p_j} - \hat{A}_j \psi.$$  

(25)

The parameter $\theta$ then measures the noncommutativity of the $\vec{p}$-space covariant derivatives. Obviously, the vector field $(\hat{A}_1, \hat{A}_2)$ in the $\vec{p}$-plane is analogous to a vector potential in the $\vec{x}$-plane, and $\theta$ to a magnetic field. For a given $\theta$, different choices of the vector field $(\hat{A}_1, \hat{A}_2)$ may be viewed as different “gauges” for $\hat{X}$ (just as different choices of the vector potential $(A_1, A_2)$ define different gauges for the covariant derivatives $\hat{P}_j$ in equation (22)). A change of gauge, $\hat{A}_j \mapsto \hat{A}_j' = \hat{A}_j + \frac{\partial \tilde{\alpha}}{\partial p_j}(\vec{p})$ induces a unitary transformation of $\hat{X}_i$ and $\hat{P}_j$ by means of the operator $V = e^{i\tilde{\alpha}(\vec{P})}$. Analogously to case (i), all representations of (23) are unitarily equivalent to the representation (25).

(iii) General case: If we combine the noncommutativities (21) and (23), we obtain the algebra

$$[\hat{X}_1, \hat{X}_2] = i\theta I, \quad [\hat{P}_1, \hat{P}_2] = iB I, \quad [\hat{X}_i, \hat{P}_j] = i\delta_{ij} I,$$

(26)

which can be represented by combining particular realizations of (21) and (23), e.g. (20)

$$\hat{X}_1 = X_1 \quad \hat{X}_2 = X_2 + \theta P_1 \quad \hat{P}_1 = P_1 + B X_2 \quad \hat{P}_2 = P_2.$$  

(27)

While the ‘minimal’ representation (27) amounts to considering the “Landau gauge” for $\hat{X}$, we can also choose the “symmetric gauge” for $\hat{X}$ if $\theta \neq 0$ (24):

$$\hat{X}_1 = aX_1 - \frac{\theta}{2a} P_2 \quad \hat{P}_1 = cP_1 + dX_2 \quad \hat{X}_2 = aX_2 + \frac{\theta}{2a} P_1 \quad \hat{P}_2 = cP_2 - dX_1,$$

(28)

where

$$a \in \mathbb{R}^*, \quad c = \frac{1}{2a} (1 \pm \sqrt{\kappa}), \quad d = \frac{a}{\theta} (1 \mp \sqrt{\kappa}).$$  

(29)
with $\kappa$ as defined in (11): $\kappa = 1 - B\theta$. The expressions (27) and (28) specifying $(\hat{X}, \hat{P})$ in terms of $(X, P)$ are invertible if and only if the functional determinant
\[
\det \left[ \frac{\partial (\hat{X}, \hat{P})}{\partial (X, P)} \right] = \kappa
\] does not vanish, i.e. if and only if $B \neq 1/\theta$. As we noticed in section 3.1.2, this result reflects the linear Darboux theorem which holds in classical mechanics if the symplectic matrix defining the Poisson brackets of coordinates and momenta is invertible.

For $B = 1/\theta$, the canonical limit $(\theta, B) \to 0$ does not exist for the algebra (26) nor for the given representations: we will come back to this case in section 4.2. At this point, we only note that the representation (27) is reducible for $B = 1/\theta$ since the operators (27) then leave invariant the linear space of functions of the form $\psi(x_1, x_2) = f(x_1) \exp[i(\lambda - \frac{2i}{\theta})x_2]$.

If the coordinates and momenta are gathered into a phase space vector
\[
\hat{u} = (\hat{u}_I)^t_{I=1,...,4} = (X_1, P_1, X_2, P_2)^t,
\] then the commutation relations (26) take the form $[\hat{u}_I, \hat{u}_J] = i\hat{M}_{IJ}1$ where $\hat{M}$ is a constant antisymmetric $4 \times 4$ matrix. As discussed in references [21, 39], the matrix $\hat{M}$ can be block-diagonalized by means of an $O(4)$ matrix $R$ (i.e. $R^t R = 1$). For $B \neq 1/\theta$, the components of the transformed phase space vector $\hat{u} = R^t \hat{u}$ can be rescaled so as to obtain new phase space variables $X_1, P_1, X_2, P_2$ which satisfy CCR’s. Thus, for variables $(\hat{X}, \hat{P})$ satisfying the algebra (26) with $B \neq 1/\theta$, there exists an invertible linear transformation to canonical variables $(\hat{X}, \hat{P})$. Since all representations of the CCR’s are unitarily equivalent, this also holds for the representations of the algebra (26) if $B \neq 1/\theta$.

To conclude, we note that the algebra (26) can be decoupled by virtue of appropriate linear combinations of generators [20, 22]: for $\theta \neq 0$, we may consider the change of generators $(\hat{X}, \hat{P}) \to (\hat{X}, \hat{K})$ given by
\[
\hat{K}_j = \hat{P}_j - \frac{1}{\theta} \varepsilon_{jk} \hat{X}_k \quad \text{for} \quad j = 1, 2,
\] which yields a direct sum of two algebras of the same form:
\[
[\hat{X}_1, \hat{X}_2] = i\theta \mathbb{I}, \quad [\hat{K}_1, \hat{K}_2] = -i\frac{\kappa}{\theta} \mathbb{I}, \quad [\hat{X}_i, \hat{K}_j] = 0.
\] If $B = 1/\theta$ (i.e. $\kappa = 0$), one concludes from (32) that $\hat{K}_1$ and $\hat{K}_2$ commute with all generators of the algebra (i.e. they belong to the center of the algebra and represent Casimir operators). By Schur’s lemma, they are c-number operators for any irreducible representation, i.e. $\hat{K}_i = \lambda_i \mathbb{I}$ with $\lambda_i \in \mathbb{R}$. In this case, the only nontrivial commutator is the one involving $\hat{X}_1$ and $\hat{X}_2$ which may be rewritten as
\[
[\hat{X}_1, \hat{\Pi}] = i\mathbb{I} \quad \text{with} \quad \hat{\Pi} = \frac{1}{\theta} \hat{X}_2,
\] i.e. we have canonically conjugate variables in one dimension.
4.2 About the relation between the parameters $\theta$ and $B$

Let us start from noncommuting coordinates in the plane and assume as before that the commutator between $\hat{X}$ and $\hat{P}$ is canonical [17]:

\[
[\hat{X}_1, \hat{X}_2] = i\theta \mathbb{1}, \quad [\hat{X}_i, \hat{P}_j] = i\delta_{ij} \mathbb{1}, \quad [\hat{P}_1, \hat{P}_2] \propto i \mathbb{1}.
\] (34)

Furthermore, let us assume that the considered representation of this algebra is irreducible.

For the linear combination $\hat{K}_j = \hat{P}_j - \frac{i}{\theta} \varepsilon_{jk} \hat{X}_k$ introduced in equation (31), we have

\[
[\hat{K}_i, \hat{X}_j] = 0, \quad [\hat{K}_i, \hat{P}_j] = 0 = [\hat{K}_2, \hat{P}_2],
\]

\[
[\hat{K}_1, \hat{P}_2] = [\hat{P}_1, \hat{P}_2] - i \frac{1}{\theta} \mathbb{1} = -[\hat{K}_2, \hat{P}_1]
\] (35)

and $[\hat{K}_1, \hat{K}_2] = [\hat{P}_1, \hat{P}_2] - i \frac{1}{\theta} \mathbb{1}$. Thus, there are two particular cases:

\[
[\hat{P}_1, \hat{P}_2] = i \frac{1}{\theta} \mathbb{1} \quad \text{or} \quad [\hat{P}_1, \hat{P}_2] = i B \mathbb{1} \quad \text{with } B \neq \frac{1}{\theta}.
\]

For the first case, it follows from equations (35) that the operators $\hat{K}_1$ and $\hat{K}_2$ commute with $\hat{X}$ and $\hat{P}$. By Schur’s lemma they are constant multiples of the identity operator so that $\hat{P}$ is a function of $\hat{X}$, e.g. for vanishing constants: $\hat{P}_j = \frac{1}{\theta} \varepsilon_{jk} \hat{X}_k$. (For instance, for the irreducible representation (28) we have $\hat{K}_1 = 0 = \hat{K}_2$ if $B = 1/\theta$.) Henceforth, the only independent commutator is then given by $[\hat{X}_1, \hat{X}_2] = i\theta \mathbb{1}$ and describes a 1-dimensional canonical system: we have a degeneracy of the representation of the quantum algebra in the sense that the representation of the algebra of the $\hat{X}_i$ alone becomes irreducible [22, 25].

In view of this result, the phase space with $B = 1/\theta$ might be referred to as a “degenerate” or “singular” noncommutative space. For this space, the canonical limit $\theta \to 0$ does not exist and one cannot express the variables $\hat{X}$ and $\hat{P}$ in terms of canonical variables $\hat{X}$ and $\hat{P}$ by virtue of an invertible transformation – see equation (30).

By contrast, in the second case the canonical limit $(\theta, B) \to 0$ can be taken and there exists an invertible transformation relating the variables $(\hat{X}, \hat{P})$ and $(\hat{X}, \hat{P})$, e.g. the transformation (27). Particular instances of this case are given by $\theta = 0$, $B \neq 0$ and $\theta \neq 0$, $B = 0$.

5 NCQM systems: generalities

5.1 About Hamiltonians

A given Hamiltonian $H(\hat{X}, \hat{P})$ of standard quantum mechanics in $\mathbb{R}^d$ can be generalized to a Hamiltonian $H(\hat{X}, \hat{P})$ of NCQM by replacing the canonical coordinates and momenta by non-canonical ones. (The ordering problem for the variables $\hat{X}_1, \ldots, \hat{X}_d$ is to be discussed in
the next subsection.) The so-obtained Hamiltonian is still assumed to act on standard wave functions \( \psi \in L^2(\mathbb{R}^d, d^d) \). This action is clearly defined once an explicit representation of \( \hat{\mathbf{X}} \) and \( \hat{\mathbf{P}} \) in terms of \( \mathbf{X} \) and \( \mathbf{P} \) is given. The independence of the spectral properties of \( H(\hat{\mathbf{X}}, \hat{\mathbf{P}}) \) from the representation chosen for \( \hat{\mathbf{X}}, \hat{\mathbf{P}} \) will be addressed in section 5.3 below.

The spectrum of \( H(\hat{\mathbf{X}}, \hat{\mathbf{P}}) \) can be determined perturbatively by treating the noncommutativity parameters \( \theta_{ij} \) as small perturbative parameters. In some particular cases, the spectrum of \( H(\hat{\mathbf{X}}, \hat{\mathbf{P}}) \) can be determined exactly by algebraic methods without referring to any explicit representation of \( (\hat{\mathbf{X}}, \hat{\mathbf{P}}) \) in terms of \( (\mathbf{X}, \mathbf{P}) \).

### 5.2 Remarks on the ordering problem

In two-dimensional NCQM with position operators satisfying \([\hat{X}_1, \hat{X}_2] = i\theta \mathbb{1}\), one can consider different ordering prescriptions to define an operator \( V(\hat{X}_1, \hat{X}_2) \) associated to a classical function \( V(x_1, x_2) \), the most popular one being the Weyl ordering, i.e., the symmetrization prescription. After introducing the operator \( \hat{a} = \frac{1}{\sqrt{2\theta}} (\hat{X}_1 + i\hat{X}_2) \) and its adjoint which operators satisfy \( [\hat{a}, \hat{a}^\dagger] = \mathbb{1} \), one can also define the normal ordering (all operators \( \hat{a} \) to the right) or the anti-normal ordering (all operators \( \hat{a}^\dagger \) to the right) \([12, 25]\).

For instance, for a central potential given by a power law, one has at the classical level

\[
V(\hat{r}^2) \equiv \hat{r}^{2N} = (x_1^2 + x_2^2)^N \sim (2\theta)^N (\hat{a}^\dagger \hat{a})^N \quad (N = 1, 2, \ldots),
\]

where the tilde symbol indicates that some ordering prescription needs to be specified for the last expression at the quantum level. E.g. for the so-called holomorphic polarization à la Bargmann-Fock \([12, 25]\), one chooses the anti-normal order:

\[
(\hat{a}^\dagger \hat{a})^N_{\text{anti-normal order}} \equiv (\hat{a}^\dagger \hat{a})^N_{\text{anti-normal order}} = (\hat{a}^\dagger \hat{a} + \mathbb{1}) (\hat{a}^\dagger \hat{a} + 2\mathbb{1}) \cdots (\hat{a}^\dagger \hat{a} + N\mathbb{1}).
\]

Alternatively, the operator-valued function \( V(\hat{r}^2) \) (with \( \hat{r}^2 \equiv \hat{X}^2 \)), which is radially symmetric, may simply be assumed to act on Hilbert space as multiplication by the classical function \( V(\hat{r}^2) \).

For observables which may be probed experimentally, one can (and one should) resort to the experimental results for choosing the “good” quantization scheme \([40, 41]\). However in the case where no precise experimental data are available (as it appears to be the case for the models of NCQM), it is in general impossible to find a preferred quantization scheme (although one may invoke some extra criteria favoring certain choices \([42]\)).

### 5.3 Properties of energy spectra

Before looking at some specific examples, it is worthwhile to determine some general properties of the spectrum of a given Hamiltonian \( H(\hat{X}, \hat{P}) \).

As emphasized earlier, the reparametrization of \( (\hat{X}, \hat{P}) \) in terms of canonical variables \( (\mathbf{X}, \mathbf{P}) \) does not represent a unitary transformation since it modifies the commutators.
Henceforth the Hamiltonian \( H(\hat{\mathbf{X}}, \hat{\mathbf{P}}) \) is not unitarily equivalent to its commutative counterpart \( H(\hat{\mathbf{X}}, \hat{\mathbf{P}}) \). By way of consequence, their spectra are in general different.

As argued in section 4.1, all representations of the quantum algebra \( (26) \) are unitarily equivalent for \( B \neq 1/\theta \), i.e. for any two representations \((\hat{\mathbf{X}}, \hat{\mathbf{P}})\) and \((\hat{\mathbf{X}}', \hat{\mathbf{P}}')\), we have \( \hat{X}_i' = U \hat{X}_i U^{-1}, \hat{P}_j' = U \hat{P}_j U^{-1} \) with \( U \) unitary. Thus, the corresponding Hamiltonians are also unitarily equivalent,

\[
H(\hat{\mathbf{X}}', \hat{\mathbf{P}}') = U H(\hat{\mathbf{X}}, \hat{\mathbf{P}}) U^{-1},
\]

which ensures that the energy spectrum does not depend on the representation that has been chosen for the quantum algebra.

By way of example, we briefly elaborate on the particular case where \( B = 0 \), i.e. on the quantum algebra \( (23) \). This algebra is realized by the operators \( (24) \) which act on \( L^2(\mathbb{R}^2, dp_1 dp_2) \) as in equation \( (25) \). Different representations of the quantum algebra, i.e. different gauges for \( \hat{\mathbf{X}} \) are related by a gauge transformation described by a function \( \hat{\alpha}(\vec{p}) \):

\[
\hat{A}_j \mapsto \hat{A}_j' = \hat{A}_j + \frac{\partial \hat{\alpha}}{\partial p_j}.
\]

The induced transformation of \( \hat{X}_i \) and \( \hat{P}_j \) is a unitary transformation:

\[
\hat{X}_i' = U \hat{X}_i U^{-1} \quad \text{and} \quad \hat{P}_j' = U \hat{P}_j U^{-1} \quad \text{with} \quad U = e^{i \hat{\alpha}(\vec{p})}.
\]

The wave function \( \hat{\psi}(p_1, p_2) \) transforms according to \( \hat{\psi} \mapsto \hat{\psi}' = U \hat{\psi} = e^{i \hat{\alpha}(\vec{p})} \hat{\psi} \) and the eigenvalue equation for \( H(\hat{\mathbf{X}}, \hat{\mathbf{P}}) \), i.e. \( H \hat{\psi} = E \hat{\psi} \), becomes \( H' \hat{\psi}' = E \hat{\psi}' \). Thus, the spectrum of the Hamiltonian \( H(\hat{\mathbf{X}}, \hat{\mathbf{P}}) \) does not depend on the gauge chosen for \( \hat{\mathbf{X}} \), i.e. on the representation which is considered for the quantum algebra.

### 6 NCQM systems: operatorial approach

In this section, we discuss the standard operatorial approach to NCQM. The star product approach (“deformation quantization”) will be treated in section 7 while the path integral approach is to be commented upon elsewhere [20].

The noncommutative Landau problem (i.e. a particle in the noncommutative plane coupled to a constant magnetic field which is perpendicular to this plane [20]) can be treated along the lines of its commutative counterpart. Thus, two equivalent approaches can be considered: either the Hamiltonian is expressed in terms of the canonical momentum and a vector potential describing the magnetic field, or the Hamiltonian has the form of a free particle Hamiltonian, but involving a momentum whose components do not mutually commute, their commutator being given by the magnetic field strength. The latter approach will be described in the first subsection. We note that this formulation (which does not involve a gauge potential) cannot be generalized to non-constant magnetic fields since the Jacobi identities for the commutator algebra are violated for \( \theta \neq 0 \) and \( B \) depending on \( \hat{\mathbf{X}} \) (see section 3.1.3).

In subsection 6.2 we will discuss the approach relying on a vector potential and we will see that the noncommutativity of the configuration space variables then leads to expressions
which are characteristic for non-Abelian Yang-Mills (YM) theories. A comparison between the different approaches to magnetic fields in NCQM will be made in section 8.

As before and unless otherwise stated, the canonical operators $\hat{X}$ and $\hat{P}$ are assumed to act on $L^2(\mathbb{R}^2, dx_1 dx_2)$ in the standard manner.

### 6.1 Commutation relations involving a constant magnetic field

For a particle of unit mass and charge, we consider the Hamiltonian $H = \frac{1}{2} \hat{\mathbf{P}}^2$ which depends on momentum variables $\hat{P}_j$ which do not commute with each other, but rather satisfy the algebra (26):

$$[\hat{X}_1, \hat{X}_2] = i\theta \mathbb{I}, \quad [\hat{P}_1, \hat{P}_2] = iB \mathbb{I}, \quad [\hat{X}_i, \hat{P}_j] = i\delta_{ij} \mathbb{I}.\quad (38)$$

The spectrum of $H$ may be determined by mimicking the reasoning presented in equations (7)-(9): the calculation and the final result

$$E_n = |B| \left(n + \frac{1}{2}\right) \quad \text{with} \quad n \in \{0, 1, 2, \ldots\} \quad (37)$$

do not involve the parameter $\theta$ which means that the energy levels are identical in standard quantum mechanics ($\theta = 0$) and in NCQM ($\theta \neq 0$). This result reflects the fact that the Hamiltonian does not depend on the coordinates $\hat{X}_1, \hat{X}_2$ so that its spectrum is insensitive to the value of the commutator $[\hat{X}_1, \hat{X}_2]$. However, there are observable quantities which do not coincide in standard quantum mechanics and in NCQM, e.g. the density of states per unit area $\rho = \frac{1}{2\pi} \left| \frac{B}{1 - B\theta} \right|$. The latter diverges at the critical point $B = 1/\theta$.

A detailed study of the physical observables in the noncommutative Landau system is presented in reference [43].

### 6.2 Minimal coupling to a (possibly non-constant) magnetic field

We start from the algebra describing the noncommutativity of configuration space,

$$[\hat{X}_1, \hat{X}_2] = i\theta \mathbb{I}, \quad [\hat{P}_1, \hat{P}_2] = 0, \quad [\hat{X}_i, \hat{P}_j] = i\delta_{ij} \mathbb{I} \quad (38)$$

and from the Hamiltonian $H = \frac{1}{2m}(\hat{P} - e\hat{A})^2$. Here, $\hat{A}$ is a vector field depending on the non-canonical operators $\hat{X}_1, \hat{X}_2$, i.e. $\hat{A} \equiv \hat{A}(\hat{X})$, with some ordering prescription for the variables $\hat{X}_1$ and $\hat{X}_2$. The operator $\hat{X}_i$ is to be thought of as a function of the canonical operators $X_j$ and $P_k$ (see equation (24) for a general expression), while $\hat{P} = \hat{\mathbf{P}} = -i\nabla$. From relations (38), it follows that the components of the kinematical momentum $\hat{\Pi} = \hat{\mathbf{P}} - e\hat{A}$ satisfy

$$[\hat{\Pi}_1, \hat{\Pi}_2] = ie\hat{F}_{12} \mathbb{I}, \quad \text{with} \quad \hat{F}_{12} \equiv \partial_1 \hat{A}_2 - \partial_2 \hat{A}_1 - ie [\hat{A}_1, \hat{A}_2] \quad (39)$$

7 Concerning this result, it is worthwhile recalling from section 3.2.1 that the classical equations of motion for $\theta = 0$ and $\theta \neq 0$ are also identical.
where $\partial_j \equiv \partial / \partial x_j$. We note that $\hat{F}_{12}$ has the same expression as the field strength in non-Abelian YM theories. Since $\hat{X}_i$ depends on $\hat{P}$ (according to $(24)$), one has $[\hat{X}_i, \hat{P}_j] = i \delta_{ij} \mathbb{1}$ plus additional terms (specified in equation $(60)$ below within the star product approach).

Since the vector potential depends on the noncommuting variables $\hat{X}_i$, it is natural to assume that the gauge transformations also depend on $\hat{X}$. Thus, the wave function $\psi(\hat{x})$ transforms as $\psi \mapsto \psi' = \hat{U} \psi$ with $\hat{U} = e^{i \lambda(\hat{X})}$. By requiring the covariant derivative $\hat{\Pi}_i \psi$ to transform in the same manner as $\psi$, one finds

$$\hat{A}'_i = \hat{U} \hat{A}_i \hat{U}^{-1} + \frac{i}{e} \hat{U} \partial_i \hat{U}^{-1}$$

$$\hat{F}'_{ij} = \hat{U} \hat{F}_{ij} \hat{U}^{-1}.$$  

(40)

Henceforth, the consideration of noncommuting coordinates in configuration space yields transformation laws for the gauge fields that are characteristic for a non-Abelian gauge theory. If the noncommutative field strength $\hat{F}_{12}$ is constant (i.e. independent of $\hat{X}$), it is gauge invariant by virtue of equation (40).

More specifically, let us now consider the symmetric gauge for $\hat{A}$,

$$\hat{A} = \left( -\frac{\bar{B}}{2} \hat{X}_2, \frac{\bar{B}}{2} \hat{X}_1 \right),$$  

(41)

where $\bar{B}$ is a constant (which might depend on $\theta$). Substitution of (41) into (39) yields

$$\hat{F}_{12} = \bar{\Lambda} \bar{B}, \quad \text{with} \quad \bar{\Lambda} = 1 + \frac{e}{4} \theta \bar{B}. $$  

(42)

If $\bar{B}$ depends on $\theta$ and on a $\theta$-independent constant $B$ according to

$$\bar{B} = B(\theta; B ; \theta) \equiv \frac{2}{e \theta} \left( \sqrt{1 + e \theta B} - 1 \right)$$

$$= B \left( 1 + \frac{e}{4} \theta B \right) + O(\theta^2),$$  

(43)

then relation (42) implies that the noncommutative field strength $\hat{F}_{12}$ is a $\theta$-independent constant: $\hat{F}_{12} = B$. In this case, we have the same algebraic setting as in subsection 6.1 the Hamiltonian reads as $H = \frac{1}{2m} \hat{\Pi}^2$ and depends on the non-canonical momentum components $\hat{\Pi}_j$ which satisfy $[\hat{\Pi}_1, \hat{\Pi}_2] = ieB \mathbb{1}$ (where $B$ is a $\theta$-independent constant). Thus, the spectrum of $H$ is the same as in ordinary quantum mechanics: $E_n = \frac{|eB|}{m} (n + \frac{1}{2})$ with \( n = 0, 1, \ldots \).

The same result can be obtained by considering the Landau gauge $\hat{A} = \left( 0, B \hat{X}_1 \right)$ which yields the same constant field strength: $\hat{F}_{12} = B$. By contrast to the star product formalism described in section 7.2 below, the approach described in this subsection does not fix afore-hand a specific gauge for $\hat{X}$ (i.e. a specific representation of $(\hat{X}, \hat{P})$ in terms of $(\hat{X}, \hat{P})$) and it works in a simple way for any choice of gauge potential $\hat{A}$ describing a constant magnetic field strength $\hat{F}_{12}$. The commutation relations for $\hat{X}$ and $\hat{\Pi}$ have the form $(60)$ and they allow for a treatment of variable magnetic fields.
6.3 “Exotic” approach

We note that there exists another description of the NC Landau system, namely the so-called “exotic model” [25, 15, 23]. This approach is based on the Poisson algebra (12), i.e. it amounts to modifying the commutators (26) by a factor $1/\kappa$. Applications of the present model to the fractional quantum Hall effect and to the vortex dynamics in a thin superfluid $^4$He film are discussed in reference [25].

7 NCQM systems: star product approaches

Consider the CCR’s (2) of standard quantum mechanics in two dimensions:

$$[X_1, X_2] = 0 = [P_1, P_2], \quad [X_i, P_j] = i\delta_{ij} = 1 \text{ for } i, j \in \{1, 2\}.$$ (44)

In the operatorial approach, one works with functions of the noncommuting operators $X_i, P_j$ and with the ordinary product of such functions. Alternatively, one can consider functions $f(\vec{x}, \vec{p})$ depending on the ordinary commuting coordinates $x_i, p_j$ and multiply these functions by a noncommutative product, namely the so-called star product. This formalism, initiated by Weyl and Wigner [45], developed by Groenewold and Moyal [46] and further generalized in the sequel [47, 48, 49] allows for a phase space description of quantum mechanics. This autonomous approach to quantum mechanics is also referred to as deformation quantization - see [49, 50, 51] for a nice summary and overview. Here, we only note that the original formalism of Groenewold and Moyal assumes Weyl ordering of the noncommuting variables.

The procedure of deformation quantization can be generalized in different manners to NCQM as described (in its simplest form) by the algebra

$$[\hat{X}_1, \hat{X}_2] = i\theta \mathbb{1}, \quad [\hat{P}_1, \hat{P}_2] = 0, \quad [\hat{X}_i, \hat{P}_j] = i\delta_{ij} \mathbb{1} \text{ for } i, j \in \{1, 2\}. \quad (44')$$

The obvious generalization consists of defining a star product for phase space functions $f(\vec{x}, \vec{p})$ which implements all of the commutation relations (44). In this formulation, a quantum state is also a function on phase space, namely the so-called Wigner function. Such a phase space formulation of NCQM has been considered by some authors, e.g. [21, 52].

Another approach [16], to be treated in detail in this section, proceeds along the lines of the star product formulation of field theory on noncommutative space [53]: here one only introduces a star product in configuration space in order to implement the noncommutativity of the $x$-coordinates, i.e. the first of relations (44). By definition, the Groenewold-Moyal star product of two smooth functions $V$ and $\psi$ depending on $\vec{x}$ is a series in $\theta$ given by

$$V \star \psi = V\psi + \frac{1}{1!} \frac{i}{2} \theta^{ij} \partial_i V \partial_j \psi + \frac{1}{2!} \left( \frac{i}{2} \right)^2 \theta^{ij} \theta^{kl} \partial_i \partial_k V \partial_j \partial_l \psi + \ldots, \quad (45)$$

which implies that

$$[x_1 \star x_2] = x_1 \star x_2 - x_2 \star x_1 = i\theta.$$
For this formulation of NCQM, quantum states are described as usual by wave functions \( \psi(\vec{x}) \) on configuration space and the momentum variables are also assumed to act as usual on these functions, i.e. \( P_j = -i \partial / \partial x_j \) (as in the representation (24) of the commutation relations (44)).

Before considering a particle subject to a magnetic field, we investigate the simpler case of a particle in a scalar potential.

### 7.1 Scalar potentials

In the standard formulation of ordinary quantum mechanics, the potential energy \( V \) acts on the wave function \( \psi \in L^2(\mathbb{R}^2, dx_1 dx_2) \) as an operator of multiplication: \( \psi \mapsto V \cdot \psi \).

Accordingly, in the star product formulation of NCQM, the potential energy \( V \) acts on the wave function \( \psi \) by the star product (45). With the help of the Fourier transform, one can check [16, 14] that the expression (45) can be rewritten as

\[
(V \star \psi)(\vec{x}) = V(\hat{X}) \psi(\vec{x}) \quad \text{with} \quad \hat{X}_i \equiv X_i - \frac{1}{2} \theta^{ij} P_j .
\] (46)

Here, the quantity \( V(\hat{X}) \) is to be considered as an operator acting on the wave function \( \psi \in L^2(\mathbb{R}^2, dx_1 dx_2) \). In particular, we have

\[
(x_i \star \psi)(\vec{x}) = \hat{X}_i \psi(\vec{x}) .
\] (47)

Thus, we recover the representation

\[
\hat{X}_i = X_i - \frac{1}{2} \theta^{ij} P_j , \quad \hat{P}_j = P_j ,
\]

of the algebra (44), i.e. the “symmetric gauge” for \( \hat{X} \).

Obviously, equation (16) corresponds to a particular ordering prescription: the right-hand-side is defined in terms of the left-hand-side which is unambiguous. One can verify that this prescription is indeed the one of Weyl: by spelling out \( (x_i x_j) \star \psi \) with the help of (45), one finds

\[
((x_i x_j) \star \psi)(\vec{x}) = \frac{1}{2} (\hat{X}_i \hat{X}_j + \hat{X}_j \hat{X}_i) \psi(\vec{x}) .
\] (48)

*In summary:* For NCQM based on the algebra (44), there are two equivalent approaches (at least for sufficiently regular potential functions): first, the operatorial approach in which the Hamiltonian is expressed in terms of the operators \( P_j \equiv P_j \equiv -i \partial / \partial x_j \) and in terms of the non-canonical operators \( \hat{X}_i \) and, second, the star product approach in which the Hamiltonian is written in terms of \( P_j \equiv -i \partial / \partial x_j \) and real coordinates \( x_i \) acting on the wave function by the Groenewold-Moyal star product (17). The latter formulation relies on the Weyl ordering of the variables \( \hat{X}_i \) and corresponds to the choice of the symmetric gauge for \( \hat{X} \), see equations (16)-(18). (For a different ordering prescription, namely normal ordering (hence a different star product), we refer to [74].)
For the Hamiltonian $H = \frac{1}{2m} \vec{p}^2 + V(\vec{x})$ acting on wave functions by the star product, the probabilistic interpretation proceeds along the usual lines, all products being replaced by star products. Indeed, the time derivative of the probability density $\rho \equiv \bar{\psi} \star \psi$ can be determined by using the Schrödinger equation

$$H \star \psi = i\hbar \partial_t \psi, \quad \text{with} \quad H = \frac{1}{2m} \vec{p}^2 + V(\vec{x})$$

and its complex conjugate (taking into account the fact that $\bar{V} \star \psi = \bar{\psi} \star V$ for the real-valued potential $V$). Thus, one obtains the continuity equation

$$\partial_t \rho + \text{div} \vec{j} = 0, \quad \text{with} \quad \begin{cases} \rho = \bar{\psi} \star \psi \\ \vec{j} = \frac{\hbar}{2m} [\bar{\psi} \star \vec{\nabla} \psi - (\vec{\nabla} \bar{\psi}) \star \psi] \end{cases}$$

which implies that the total probability $\int_{\mathbb{R}^2} d^2x \bar{\psi} \star \psi$ is a conserved quantity.

### 7.2 Magnetic fields

Let us now consider a particle moving in the noncommutative plane subject to a (possibly non-constant) magnetic field which is perpendicular to this plane. The present section describes two approaches to this problem which are both based on the star product in configuration space and on the introduction of a vector potential (i.e. a $U(1)$ gauge field). The first approach solely relies on the star product \cite{16, 55} while the second one supplements the star product with the so-called Seiberg-Witten map \cite{56, 57, 58, 59}. The results obtained for the particular case of a constant magnetic field (i.e. the Landau system) will be compared with those obtained for this system in section 6.

#### 7.2.1 Gauge potentials

We use the following notation. The fields of NCQM (which depend on the coordinates $\vec{x} = (x_1, x_2)$ and which are multiplied by each other by means of the star product) are denoted by a ‘check’ e.g. $\check{\psi}, \check{A}_i, \check{\lambda}, \ldots$.

A local $U(1)$ gauge transformation of the wave function $\check{\psi}$ (describing a particle of charge $e$) is given by

$$\check{\psi} \mapsto \check{\psi}' = \check{U}_\lambda \star \check{\psi}, \quad \text{with} \quad \check{U}_\lambda = e^{i e \lambda},$$

where all products of $\check{\lambda}$ in the exponential $\check{U}_\lambda$ are star products. The inverse of $\check{U}_\lambda$ is determined by the relation $\check{U}_\lambda \star \check{U}^{-1}_\lambda = 1$. The covariant derivative of $\check{\psi}$, as defined by $\check{D}_i \check{\psi} = \partial_i \check{\psi} - ie \check{A}_i \star \check{\psi}$, transforms in the same way as $\check{\psi}$ if the vector potential $\check{A}_i$ transforms according to

$$\check{A}_i \mapsto \check{A}'_i = \check{U}_\lambda \star \check{A}_i \star \check{U}^{-1}_\lambda + \frac{i}{e} \check{U}_\lambda \star (\partial_i \check{U}^{-1}_\lambda).$$

---

8 We do not use the standard notation \cite{60}, i.e. a ‘hat’, in order to avoid any confusion with the hatted operators $\hat{X}_i$ and $\hat{P}_j$ considered in the earlier sections.
The commutator of the covariant derivatives determines the magnetic field strength:

\[
[D_i, D_j] \psi = -ie F_{ij} \star \psi,
\]

with \( F_{ij} = \partial_i \bar{A}_j - \partial_j \bar{A}_i - ie [\bar{A}_i, \bar{A}_j] \).

The latter transforms according to

\[
\bar{F}_{ij} \mapsto - \bar{U}_\lambda \star \bar{F}_{ij} \star \bar{U}_\lambda^{-1}.
\]

Thus, for the Abelian gauge group \( U(1) \), the noncommutativity of the star product induces expressions that are characteristic for non-Abelian Yang-Mills theories \( [16, 8, 61] \). For the two-dimensional configuration space that we consider here for simplicity, the only non-vanishing component of the tensor \( \bar{F}_{ij} \) is \( \bar{F}_{12} \equiv B^{\ast} \).

By contrast to the field strength \( F_{ij} \) of an ordinary \( U(1) \) gauge theory, the field strength \( \bar{F}_{ij} \) is not gauge invariant, but transforms covariantly as in equation (52). Instead of the fundamental representation for the matter field (that we consider here and for which the covariant derivative reads as \( D_i \psi = \partial_i \psi - ie \bar{A}_i \star \psi \)), one can also consider the anti-fundamental or the adjoint representation (for which we have, respectively, \( \bar{D}_i \psi \equiv \partial_i \psi + ie \psi \star \bar{A}_i \) and \( \bar{D}_i \psi \equiv \partial_i \psi - ie [\bar{A}_i, \psi] \)).

The Schrödinger equation \( \bar{H} \star \psi = i\partial_t \psi \) with

\[
\bar{H} = \frac{1}{2m} \left( \vec{P} - e\bar{A} \right)^2 \equiv \frac{1}{2m} \sum_i (P_i - e\bar{A}_i) \star (P_i - e\bar{A}_i)
\]

is invariant under the gauge transformations (49), (50) with a gauge parameter \( \lambda \) which does not depend on time. Let us now consider the case where the field strength \( \bar{F}_{ij} \) is constant. This case is quite particular since it follows from (52) that such a field \( \bar{F}_{ij} \) is gauge invariant.

Let us choose \( [16, 55] \) the symmetric gauge

\[
\bar{A}(\vec{x}) = \left( -\frac{\bar{B}}{2} x_2, \frac{\bar{B}}{2} x_1 \right),
\]

where the real constant \( \bar{B} \) parametrizes the magnetic field. From (51), we infer that the associated field strength is given by

\[
\bar{F}_{12} = \bar{A} B^{\ast}, \quad \text{with} \quad \bar{A} = 1 + \frac{e}{4} \theta \bar{B}.
\]

Henceforth, we presently have expressions of the same form as in the operatorial approach – compare (51) with (41) and (55) with (42).

The star products can be disentangled by virtue of relation (47):

\[
(P_i - e\bar{A}_i) \star \psi = \bar{A} \left( P_i - \frac{e}{\bar{A}} \bar{A}_i(\vec{x}) \right) \psi.
\]

By applying this relation once more, one finds \( [16] \) that \( \bar{H} \star \psi \) (with \( \bar{A} \) given by (54)) reads as

\[
\bar{H} \star \psi = H \psi,
\]

for which we consider here an \( U(1) \) gauge theory, the field strength \( \bar{F}_{ij} \) is not gauge invariant, but transforms covariantly as in equation (52). Instead of the fundamental representation for the matter field (that we consider here and for which the covariant derivative reads as \( D_i \psi = \partial_i \psi - ie \bar{A}_i \star \psi \)), one can also consider the anti-fundamental or the adjoint representation (for which we have, respectively, \( \bar{D}_i \psi \equiv \partial_i \psi + ie \psi \star \bar{A}_i \) and \( \bar{D}_i \psi \equiv \partial_i \psi - ie [\bar{A}_i, \psi] \)).

The Schrödinger equation \( \bar{H} \star \psi = i\partial_t \psi \) with

\[
\bar{H} = \frac{1}{2m} \left( \vec{P} - e\bar{A} \right)^2 \equiv \frac{1}{2m} \sum_i (P_i - e\bar{A}_i) \star (P_i - e\bar{A}_i)
\]

is invariant under the gauge transformations (49), (50) with a gauge parameter \( \lambda \) which does not depend on time. Let us now consider the case where the field strength \( \bar{F}_{ij} \) is constant. This case is quite particular since it follows from (52) that such a field \( \bar{F}_{ij} \) is gauge invariant.

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\]

By applying this relation once more, one finds \( [16] \) that \( \bar{H} \star \psi \) (with \( \bar{A} \) given by (54)) reads as

\[
\bar{H} \star \psi = H \psi,
\]
with
\[ H_* = \frac{1}{2m_*} \left( \vec{P} - e_* \vec{A}(\vec{x}) \right)^2, \quad \text{where} \quad m_* = \bar{\Lambda}^{-2} m, \quad e_* = \bar{\Lambda}^{-1} e. \]

Here, we put a lower index * on the quantities which have been modified due to the consideration of star products (i.e. the quantities \( H_*, m_* \) and \( e_* \)).

The Hamiltonian \( H_* \) has the same form as the corresponding Hamiltonian in ordinary quantum mechanics (the only difference being a redefined mass and charge which depend on \( \theta \)). As a consequence of this fact and of the relation \( \partial_1 \vec{A}_2 - \partial_2 \vec{A}_1 = \vec{B} \), the spectrum of \( H_* \) is given, in analogy to expression (9), by
\[ E_n = \hbar \frac{|e_* B|}{m_* c} \left( n + \frac{1}{2} \right) \]
\[ = \hbar \frac{|\vec{F}_{12}|}{mc} \left( n + \frac{1}{2} \right) \quad \text{with} \quad n \in \{0, 1, \ldots \}. \] (58)

These energy levels are the same as in ordinary quantum mechanics, but with the magnetic field \( \vec{B} = \partial_1 \vec{A}_2 - \partial_2 \vec{A}_1 \) replaced by the noncommutative field strength \( \vec{F}_{12} \). The latter field depends on \( \theta \) according to (55) except for the case where \( \vec{B} \) (and thereby \( \vec{A} \)) depends on \( \theta \) in such a way that \( \vec{F}_{12} \) is a \( \theta \)-independent constant \( B \): if relation (13) holds, i.e. if
\[ \vec{B} = \vec{B}(B; \theta) \equiv \frac{2}{eB} \left( \sqrt{1 + e\theta B} - 1 \right) \]
\[ = B \left( 1 + \frac{e}{4} \theta B \right) + O(\theta^2), \] (59)
then equation (51) yields \( \vec{F}_{12} = B \). The symmetric gauge (54) with \( \vec{B} \) given by (59) is gauge equivalent to the Landau gauge \( \vec{A} = (0, Bx_1) \).

In summary: There are two different viewpoints for the noncommutative Landau system described in terms of a vector potential and the star product. The first viewpoint is that the variable \( \vec{B} \) in the vector potential (54) is a \( \theta \)-independent constant: this leads to a \( \theta \)-dependent energy spectrum \( (E_n \propto |\bar{\Lambda} \vec{B}|) \) which differs from the one that we obtained by the approach of section 6.1 \( (E_n \propto |\vec{B}|) \) where we considered a free Hamiltonian and a commutator of momenta determined by the constant magnetic field: \( [\hat{P}_1, \hat{P}_2] \equiv i\vec{B}\mathbb{I} \). The second viewpoint consists of regarding the noncommutative field strength \( \vec{F}_{12} \) as the central variable, thereby considering this field as a \( \theta \)-independent constant: \( \vec{F}_{12} = B \). In this case, the vector potential (54) depends on \( \theta \) according to (59). This implies that the energy spectra obtained, respectively, by the approach of this section and the one of section 6.1 (with \( [\hat{P}_1, \hat{P}_2] \equiv iB\mathbb{I} \)) coincide with each other: \( E_n \propto |B| \). This spectrum also coincides with the one of ordinary quantum mechanics.

Remarks:
(i) As was already pointed out in reference (13), the argumentation (56),(57) relies on the symmetric gauge (54) for \( \vec{A} \) (e.g. equation (55) does not hold in a Landau-type gauge). This result is related to the fact that the star product amounts to choosing the symmetric
gauge for $\hat{X}$ – see equation (17). Henceforth, the spectrum for the minimally coupled Hamiltonian can only be determined by simple algebraic arguments if the symmetric gauge is chosen as well for the vector potential $\vec{A}$. Other gauge choices require more complicated computations.

(ii) The gauge invariance of the $\star$-eigenvalue equation $\hat{H} \star \hat{\psi} = E_n \hat{\psi}$ is ensured by construction. The ordinary eigenvalue equation $\hat{H} \hat{\psi} = E_n \hat{\psi}$ is invariant under ordinary gauge transformations of $\hat{A}$ and $\hat{\psi}$, the relationship between the different gauge transformations being the subject of the Seiberg-Witten map discussed in the next subsection.

(iii) If a scalar potential $\hat{V}$ is added to the Hamiltonian (53), the function $\hat{V} \star \hat{\psi}$ transforms in the same manner as $\hat{\psi}$ under a local $U(1)$ gauge transformation $\hat{U}_\lambda$, if the potential $\hat{V}$ transforms according to $\hat{V} \rightarrow \hat{V}' = \hat{U}_\lambda \star \hat{V} \star \hat{U}_\lambda^{-1}$. The example of a constant electric field $\vec{E} = -\vec{\nabla} V$ is treated in reference [55] and applied to the Hall effect and the Aharonov-Bohm effect.

(iv) If we introduce the components of the kinematical momentum, $\hat{\Pi}_k \equiv -i \hat{D}_k = -i \partial_k - e \hat{A}_k$, we obtain the $\star$-commutation relations

\begin{align*}
[x_1 \star x_2] &= i \theta \\
[\hat{\Pi}_1 \star \hat{\Pi}_2] &= ie \hat{F}_{12}, \quad \text{with } \hat{F}_{12} = \partial_1 \hat{A}_2 - \partial_2 \hat{A}_1 - ie [\hat{A}_1 \star \hat{A}_2] \\
[x_i \star \hat{\Pi}_j] &= i \delta_{ij} - e [x_i \star \hat{A}_j].
\end{align*}

(60)

By contrast to the commutation relations (26) involving a constant magnetic field $B$, i.e.

\begin{align*}
[\hat{X}_1, \hat{X}_2] &= i \theta \mathbb{1} \\
[\hat{P}_1, \hat{P}_2] &= i B \mathbb{1}, \quad [\hat{X}_i, \hat{P}_j] = i \delta_{ij} \mathbb{1},
\end{align*}

(61)

we may presently have a non-constant magnetic field, but the algebra (60) contains some extra terms on the right-hand side which are due to the fact that $x_1$ and $x_2$ do not $\star$-commute. These additional terms ensure the validity of the Jacobi identities for the algebra (60) for any vector potential $\vec{A}$. In particular, the Jacobi identities hold for a vector potential describing a variable magnetic field by contrast to the algebra (61) for which these identities do not hold if $B$ depends on $\vec{X}$. Yet, if the field strength $\hat{F}_{12}$ is not constant, it is not gauge invariant which renders its physical interpretation as a magnetic field somewhat unclear. As we will see in the next subsection, this issue can be settled by considering the Seiberg-Witten map which relates the noncommutative gauge fields and parameters to their commutative counterparts in a specific way.

7.2.2 Gauge potentials and Seiberg-Witten map

In the previous subsection, we saw that the consideration of star products in the Schrödinger equation coupled to a $U(1)$ gauge field leads to the emergence of a non-Abelian gauge structure in NCQM which is not present in ordinary quantum mechanics. The situation is completely analogous in relativistic field theory when passing from an ordinary $U(1)$ gauge theory to a noncommutative $U(1)$ gauge theory. In the latter context, Seiberg and
Witten [60] have introduced a map between both theories which relates noncommutative gauge fields and parameters to their commutative counterparts in such a way that gauge equivalent field configurations are mapped into each other. By virtue of this map, both theories can be considered as equivalent and the action functional or Hamiltonian of the noncommutative $U(1)$ theory can be expressed in terms of ordinary $U(1)$ gauge fields and the noncommutativity parameters $\theta_{ij}$ (which may be viewed as a constant “background field”).

In this section, we will discuss this so-called Seiberg-Witten map within the context of quantum mechanics [56, 57, 58, 59]. We use the same notation as in the previous subsection, i.e. the fields of NCQM (which depend on the coordinates $\vec{x} = (x_1, x_2)$ and which are multiplied with each other by means of the star product) are denoted by a ‘check’. We also spell out the coupling constant $e$ which is usually absorbed into the fields.

By definition, the Seiberg-Witten map is a mapping $A_i \mapsto \tilde{A}_i(A)$ and $\lambda \mapsto \tilde{\lambda}(\lambda, A)$ (the ‘checked’ fields being power series in the noncommutativity parameters $\theta_{ij} = \theta e^{i\epsilon_{ij}}$) which satisfies

$$\tilde{A}_i(A) + \delta_\lambda \tilde{A}_i(A) = \tilde{A}_i(A + \delta_\lambda A).$$

(62)

Here, $\delta_\lambda \tilde{A}_i = \tilde{D}_i \tilde{\lambda} \equiv \partial_i \tilde{\lambda} - ie [\tilde{A}_i, \tilde{\lambda}]$ and $\delta_\lambda A_i = \partial_i \lambda$ describe infinitesimal $U(1)$ gauge transformations in NCQM and in ordinary quantum mechanics, respectively. Equation (62) is solved [60] by

$$A_i \mapsto \tilde{A}_i(A; \theta) = A_i - \frac{e}{2} \theta^{kl} A_k (\partial_i A_l + F_{li}) + \mathcal{O}(\theta^2)$$

$$\lambda \mapsto \tilde{\lambda}(\lambda, A; \theta) = \lambda - \frac{e}{2} \theta^{ij} A_i \partial_j \lambda + \mathcal{O}(\theta^2),$$

and we have analogous maps [62] for the matter field $\psi$ and the field strength of $A$:

$$\psi \mapsto \tilde{\psi}(\psi, A; \theta) = \psi - \frac{e}{2} \theta^{ij} A_i \partial_j \psi + \mathcal{O}(\theta^2)$$

$$F_{ij} \mapsto \tilde{F}_{ij}(A; \theta) = F_{ij} + e \theta^{kl} F_{ik} F_{jl} + \mathcal{O}(\theta^2).$$

(64)

Here, the expression for $\tilde{F}_{ij}$ follows from the defining relation (51). The fact that $\tilde{\lambda}$ not only depends on $\lambda$, but also on $A$ implies there is no well-defined mapping between the gauge group $\{ U_\lambda = e^{i\epsilon \lambda} \}$ of ordinary gauge theory and the gauge group $\{ \tilde{U}_\lambda = e^{i\epsilon \lambda} \}$ of noncommutative gauge theory [60].

For a constant magnetic field $B \equiv F_{12}$, one has the following exact (i.e. valid to all orders in $\theta$) result for the noncommutative field strength $\tilde{F}_{12} \equiv \tilde{B}$ which is determined by the Seiberg-Witten map [60]:

$$\tilde{B} = \frac{B}{1 - e\theta B}.$$ 

(65)

Just as the ordinary magnetic field $B$ can be described by a symmetric gauge field configuration, $A_i = -\frac{1}{2} B \epsilon_{ij} x_j$ (with $i \in \{1, 2\}$), the Seiberg-Witten field strength $\tilde{B} \equiv \tilde{F}_{12}$

$$23$$
can be described by a symmetric field configuration [59]:

\[ \tilde{A}_i(A; \theta) = -\frac{1}{2} \tilde{B}(B; \theta) \varepsilon_{ij} x_j, \quad \text{with} \quad \tilde{B}(B; \theta) = \frac{2}{e \theta} \left( \frac{1}{\sqrt{1 - e \theta B}} - 1 \right) \]

\[ = B(1 + \frac{3}{4} e \theta B) + O(\theta^2). \]

Indeed, substitution of this expression into \( \tilde{F}_{ij} \), as defined by relation (51), yields the result (65). The relationship with the expressions in the previous subsection can easily be worked out.

To discuss the Landau problem, we again proceed as in the previous subsection, see equations (53)-(58). Thus, we consider the Hamiltonian

\[ \hat{H} = \frac{1}{2 \hat{m}} (\hat{P} - e \hat{A})^2 \equiv \frac{1}{2 \hat{m}} \sum_i (P_i - e \hat{A}_i) * (P_i - e \hat{A}_i), \]

acting on the Seiberg-Witten field \( \hat{\psi} \). In expression (67), the Seiberg-Witten gauge field \( \hat{A}_i \) describing a constant magnetic field is assumed to be given by the symmetric gauge (66).

The mass parameter has been denoted by \( \hat{m} \). Very much like the Seiberg-Witten fields, it may be viewed as a function of \( \theta \) satisfying \( \hat{m}(\theta = 0) = m \).

By using relation (17) to disentangle the star products, we get

\[ \hat{H} * \hat{\psi} = H_{SW} \hat{\psi}, \]

with

\[ H_{SW} = \frac{1}{2m_{SW}} \left( \hat{P} - e_{SW} \tilde{A}(\hat{x}) \right)^2, \quad \text{where} \quad m_{SW} = \tilde{\Lambda}^{-2} \hat{m}, \quad e_{SW} = \tilde{\Lambda}^{-1} e. \]

The Hamiltonian \( H_{SW} \) (which depends on the constant “background field” \( \theta \)) has the same form as the corresponding Hamiltonian in ordinary quantum mechanics. As a consequence of this fact and of the relation \( \partial_1 \tilde{A}_2 - \partial_2 \tilde{A}_1 = \tilde{B} \), the spectrum of \( H_{SW} \) is given by

\[ E_n = \hbar \left| \frac{e_{SW} \tilde{B}}{m_{SW} c} \right| (n + \frac{1}{2}) = \hbar \left| \frac{e \tilde{B}}{mc} \right| (n + \frac{1}{2}) \quad \text{with} \quad n \in \{0, 1, \ldots \}. \]

If we assume that

\[ \hat{m} = \frac{m}{1 - e \theta B}, \]

then it follows from (65) that the energy levels read as

\[ E_n = \hbar \left| \frac{e \tilde{B}}{mc} \right| (n + \frac{1}{2}) \quad \text{with} \quad n \in \{0, 1, \ldots \}. \]

\[ \text{In fact, in the present section we wrote} \quad \tilde{B} \equiv F_{12} \quad \text{rather than} \quad B \equiv F_{12} \quad \text{since the variable} \quad B \quad \text{was used in section 7.2.1 with a different meaning. As in subsection 7.2.1, we presently have} \quad \tilde{F}_{12} = \tilde{\Lambda} \tilde{B} \quad \text{with} \quad \tilde{\Lambda} = 1 + \frac{3}{4} \theta B. \quad \text{If} \quad B = B^{-1}, \quad \text{we recover the expressions of subsection 7.2.1:} \quad \tilde{B} \quad \text{as a function of} \quad B \quad \text{is then given by equation (59) and} \quad \tilde{F}_{12} = B. \]
This result coincides with the one of ordinary quantum mechanics (for a particle of mass $m$ and charge $e$ coupled to the constant magnetic field $B = \partial_1 A_2 - \partial_2 A_1$). In this sense, both theories are physically equivalent. Note that the singularity $B = (e\theta)^{-1}$ of the parameters $\bar{B}$ and $\bar{m}$ does not manifest itself in the spectrum (70).

A somewhat different treatment of the Landau problem within the Seiberg-Witten framework (leading to the same result for the spectrum) is presented in reference [59]. In the latter treatment, relation (69) follows from the requirement that the Hamiltonians $H_{\text{SW}}(\theta)$ and $H_{\text{SW}}(0)$ are related by a unitary transformation which ensures that the physics remains invariant under a change in $\theta$. The approach discussed in this subsection also allows to incorporate scalar potentials (see reference [57] for applications to the Hall effect).

We note that substitution of the symmetric gauge $A_i = -\frac{1}{2} B \varepsilon_{ij} x_j$ into $\bar{\psi}$, as given by equation (64), leads to

$$\bar{\psi}(\psi, A; \theta) = \psi + \frac{e\theta B}{4} \bar{\vec{x}} \cdot \bar{\nabla} \psi + O(\theta^2).$$

Here, the first order term in $\theta$ amounts to a $\theta$-dependent scale transformation of $\psi$, which shows that the Seiberg-Witten map (67) is not unitary [59]. This map may eventually be unitarized, see references [57, 59]. Different aspects of the Seiberg-Witten map (like singularities or ambiguities in the parametrization) are discussed in the work [63].

In summary: The approach of Seiberg-Witten to NCQM amounts to considering star products, and to assuming that the fields of NCQM (i.e. $\bar{\psi}, \bar{A}_i, \bar{\lambda}, \ldots$) and the coupling constants of NCQM (i.e. $e_{\text{SW}}$ and $m_{\text{SW}}$) are specific functions of those occurring in ordinary quantum mechanics. The star product is tantamount to the symmetric gauge for $\bar{\vec{X}}$ and thereby this approach to a constant magnetic field only works in a simple way if the symmetric gauge is also chosen for $\bar{A}$.

8 Summary and concluding remarks

Magnetic fields in NCQM – Summary: For $[\hat{X}_1, \hat{X}_2] = i\theta \mathbb{1}$, the coupling to a constant magnetic field can either be described by considering a free Hamiltonian $H = \frac{1}{2m} \vec{P}^2$ and a non-trivial commutator $[\hat{P}_1, \hat{P}_2] = ieB \mathbb{1}$, or by considering a trivial commutator $[\hat{P}_1, \hat{P}_2] = 0$ and a Hamiltonian involving a vector potential. For the latter case, we presented three approaches (subsections 6.2, 7.2.1 and 7.2.2) all of which lead to a noncommutative field strength. The vector potential or field strength appearing in these approaches can be expressed in various ways in terms of the constant $\theta$-independent field $B$. By choosing this dependence on $B$ in an appropriate way, one recovers the spectral results of the former formalism (where $[\hat{P}_1, \hat{P}_2] = ieB \mathbb{1}$). The issue of gauge invariance of the energy spectrum is unproblematic in the operatorial formulation (section 6.2) and somewhat subtle in the other approaches.

The three approaches based on vector potentials lead to a non-canonical commutator between the coordinates and the kinematical momentum – see equation (60). Thereby,
they allow for a treatment of a non-constant magnetic field to be identified with the noncommutative field strength $F_{12}$. However, the physical interpretation is somewhat subtle due to the fact that the field $F_{12}$ is not gauge invariant in this case.

**Concluding remarks:** In these notes, we discussed some general properties of NCQM while limiting ourselves to two dimensions and to the case of a canonical commutator between $\hat{X}_i$ and $\hat{P}_j$: $[\hat{X}_i, \hat{P}_j] = i\delta_{ij}\mathbb{1}$. The case of higher dimensions and of a more general algebra are to be addressed elsewhere [26] along with other physical systems involving both scalar and vector potentials. In that context, we will also review other methods of quantization like the path integral approach to NCQM.

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A Noncommuting coordinates in the truncatedLandau problem (Peierls’ substitution)

In the following, we will show that noncommuting coordinates naturally appear in theLandau problem in the limit of a very strong magnetic field \( B \). Since the separation ofLandau levels \( \ell \) is of order \( B/m \), a large magnetic field \( B \) is equivalent to a small mass\( m \) and the limit \( m \to 0 \) amounts to a restriction to the lowest Landau level (LLL). Beforepresenting this limiting procedure in a rigorous manner, we provide a simple heuristicargument \[12, 2\]. The classical dynamics of the Landau problem (supplemented by a weakscalar potential \( V(x_1, x_2) \) describing impurities in the plane) is described by the Lagrangian

\[
L_m = \frac{1}{2} m \dot{x}^2 + \frac{e}{c} \mathbf{A} \cdot \dot{x} - V(x)
\]

(71)

where we have chosen the Landau gauge \( (A_x, A_y) = (0, Bx_1) \). In the limit \( m \to 0 \), we areleft with

\[
L_0 = \frac{eB}{c} x_1 \dot{x}_2 - V(x_1, x_2),
\]
i.e. an expression of the form \( p \dot{q} - H_0(p, q) \) with a pair of canonically conjugate variables\( (p, q) = (\frac{eB}{c} x_1, x_2) \) and a Hamiltonian \( H_0 = V \). Upon quantization, the classical variablesbecome operators \( \hat{X}_1 \) and \( \hat{X}_2 \) satisfying CCR’s, \( [\hat{X}_1, \hat{X}_2] = -i\hbar \mathbb{I} \), i.e.

\[
[\hat{X}_1, \hat{X}_2] = -i \frac{\hbar \mathbb{I}}{eB}.
\]

(72)

Thus, the restriction of the particle dynamics to the LLL amounts to introducing noncom-muting coordinates in the plane which satisfy the algebra \( (72) \), the reduced Hamiltonian\( H_0 = V(\hat{X}_1, \hat{X}_2) \) describing this dynamics being given by the potential depending on thenoncommuting coordinates. This so-called Peierls’ substitution means that the effect ofan impurity (as described by the potential \( V \) in the Lagrangian \( (71) \)) on the eigenstate of theLLL can be evaluated to lowest order by computing the eigenvalues \( \epsilon_n \) of the effectiveHamiltonian \( H_0 = V(\hat{X}_1, \hat{X}_2) \), where the commutator of \( \hat{X}_1 \) and \( \hat{X}_2 \) is proportional to \( 1/B \).More explicitly, if \( V(\hat{X}_1, \hat{X}_2)|n\rangle = \epsilon_n |n\rangle \), then the energy levels for the quantum systemassociated to \( (71) \) are given by \( E_n = \frac{1}{2} \hbar \omega_B + \epsilon_n \) (with \( \omega_B \equiv \frac{|eB|}{mc} \)) in the approximationof strong \( B \) and weak \( V \) \[6, 12, 2\]. (Incidentally, the result \( (72) \) coincides – up to a sign – with the commutator \( [\hat{X}_1, \hat{X}_2] \) that appears in the algebra \( (26) \) at the singular point \( \kappa \equiv 1 - \frac{\epsilon}{\hbar c} B \theta = 0 \).)

To make the previous argument more precise \[63\] (see also \[2, 54, 65\]), we consider the Hamiltonian operator \( H = \frac{1}{2m} \sum_{j=1}^{2} (P_j - \frac{e}{c} A_j)^2 \) as written in the symmetric gauge. Recall that the eigenstate \( |n, k\rangle \) of \( H \) associated to the \( n \)-th Landau level \( E_n = \hbar \omega_B (n + \frac{1}{2}) \) is infinitely degenerate: \( k \in \mathbb{R} \). The projection operator \( P_n = \int \mathbb{R} dk |n, k\rangle \langle n, k| \) onto
the corresponding eigenspace can be expressed as follows in terms of the Hamiltonian \( H = \sum_{n=0}^{\infty} E_n P_n \): \[ P_n = \frac{4}{\pi(2n+1)} \sin \left[ \left( n + \frac{1}{2} \right) \pi \left( H_n - 1 \right) \right] \frac{\sin \left( \left( n + \frac{1}{2} \right) \pi \left( H_n + 1 \right) \right)}{(H_n - 1)(H_n + 1)}, \quad \text{with} \quad H_n \equiv \frac{1}{E_n} H. \]

Rather than truncating the theory to the LLL, we may consider more generally the theory obtained by cutting off at an energy \( E \) with \( E_N \leq E < E_{N+1} \), where \( N \in \{0, 1, 2, \ldots\} \) is fixed \([54, 65, 64]\). The operator \( \Pi_N = \sum_{n=0}^{N} P_n \) then projects onto the corresponding subspace of Hilbert space \( \mathcal{H} \). In particular, for any operator \( B \) on \( \mathcal{H} \), the expression \[ \hat{B} \equiv \Pi_N B \Pi_N \]

represents the truncation of \( B \). The authors of reference \([64]\) determined the commutators of the truncated canonical operators:

\[ [\hat{X}_1, \hat{X}_2] = -i \frac{\hbar c}{eB} (N + 1) \mathcal{P}_N \tag{73} \]
\[ [\hat{P}_1, \hat{P}_2] = -i \hbar \frac{eB}{4c} (N + 1) \mathcal{P}_N \]
\[ [\hat{X}_i, \hat{P}_j] = i \hbar \delta_{ij} \left[ 1 - \frac{1}{2} (N + 1) \right] \mathcal{P}_N. \]

In particular, the first equation implies that the LLL matrix elements of \([\hat{X}_1, \hat{X}_2]\) are given by

\[ \langle 0, k | [\hat{X}_1, \hat{X}_2] | 0, k' \rangle = -i \frac{\hbar c}{eB} \langle 0, k | 0, k' \rangle \]

in agreement with the result (72) suggested by an heuristic argument. Thus, the set of equations (73) gives a precise mathematical meaning to the procedure of ‘projection to LLL’. For \( N \rightarrow \infty \), one recovers the canonical commutators \([65]\).

The modifications to (73) brought about an interaction potential \( V_\lambda(x_1^2 + x_2^2) \) are discussed in reference \([66]\). In general, the commutator \([\hat{X}_1, \hat{X}_2]\) then depends on the parameter \( \lambda \) characterizing the potential \( V_\lambda \).
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