Black Rings and the Physical Process Version of the First Law of Thermodynamics

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We consider the problem of physical process version of the first law black ring thermodynamics in n-dimensional Einstein gravity with additional (p+1)-form field strength and dilaton fields. The first order variations of mass, angular momentum and local charge for black ring were derived. By means of them we prove physical process version of the first law of thermodynamic for stationary black rings.

I. INTRODUCTION

During the past decade there has been growing interest in higher dimensional black holes motivated by various attempts of building unified theories. It has been recently shown that for static n-dimensional black holes the uniqueness theorem is valid [1], while for the stationary axisymmetric ones even in five-dimensional spacetime there is a counterexample, black ring [2]. The black ring solution with the topology horizon of $S^2 \times S^1$ has the same mass and angular momentum as spherical black hole. Nevertheless, when one assumes the spherical topology of the horizon $S^3$ one can prove the uniqueness of vacuum five-dimensional stationary axisymmetric solution [3] as well as stationary axisymmetric self-gravitating $\sigma$-models [4].

The black ring solutions can be extended to possess the electric as well as magnetic dipole charge [5,6]. The static black ring solution in five-dimensional Einstein-Maxwell-dilaton gravity was presented in Ref. [7]. In Ref. [8] the solution characterized by three conserved charges, three dipole charges, two equal angular momentum and a parameter that measured the deviation from the supersymmetry configuration was presented. It also turned out that supersymmetric black rings exist [9].

The first connection between black holes and thermodynamics was presented in the seminal paper of Bardeen, Carter and Hawking [10]. They considered linear perturbations of a stationary electrovac black hole to another stationary black hole. Sudarsky and Wald [11] derived the first law of black hole thermodynamics valid for arbitrary asymptotically flat perturbations of a stationary black hole. There have been several derivations of the first law of black hole thermodynamics valid for an arbitrary diffeomorphism invariant Lagrangians with metric and matter fields possessing stationary and axisymmetric black hole solutions [12]. The higher curvature terms and higher derivative terms in the metric were considered in Ref. [13], as well as the Lagrangian being arbitrary function of metric, Ricci tensor and a scalar field [14] were taken into account. The case of a charged and rotating black hole where fields were not smooth through the event horizon was treated in Ref. [15]. On the other hand, the physical process version of the first law of black hole thermodynamics obtained by changing a stationary black hole by some infinitesimal physical
process, e.g., when matter was thrown into black hole was considered. Assuming that the black hole eventually settle down to a stationary state and calculating the changes of black hole’s parameters one can find this law. If the resulting relation fails comparing to the known version of the first law of black hole thermodynamics it provides inconsistency with the assumption that the black hole settles down to a final stationary state. This fact will give a strong evidence against cosmic censorship. The physical process version of the first law of black hole thermodynamics in Einstein theory was proved in Ref. [16]. Then, it was generalized for Einstein-Maxwell (EM) black holes in Ref. [17] and for Einstein-Maxwell axion-dilaton (EMAD) gravity black holes in [18].

The first law of black hole thermodynamics was also intensively studied in the realm of n-dimensional black holes. The equilibrium state version was elaborated in Ref. [19] under the assumption of spherical topology of black holes. Some of the works assume that four-dimensional black hole uniqueness theorem extends to higher dimensional case [20]. The physical process of the first law of black hole thermodynamics in n-dimension was treated in Ref. [21].

One hopes that the analysis of the physical processes in the spacetime of black rings will deepen our understanding of these objects. Recently, several works were devoted to this problem. Namely, the process of Penrose extraction was analyzed [22] and ultrarelativistic boost was taken into account [23]. The scalar perturbations in the background of both nonsupersymmetric and supersymmetric black rings was considered in Ref. [24]. The general form of the first law of mechanics for black rings, taking into account dipole charges was established in Ref. [25].

In our paper we shall investigate the problem of physical process version of the first law of thermodynamics in the higher dimensional gravity containing (p + 1)-form field strength and dilaton fields. This theory constitutes the simplest generalization of five-dimensional one, which in turn contains stationary black ring solution with dipole charge [8].

II. PHYSICAL PROCESS VERSION OF THE FIRST LAW OF BLACK HOLE MECHANICS

We begin with the Lagrangian of higher dimensional generalization of the five-dimensional theory with three form field strength and dilaton fields which contains stationary black ring solution. It is subject to the relation as follows:

$$L = \epsilon R - \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi - \frac{1}{2(p+1)!} e^{-\alpha \phi} H_{\mu_1...\mu_{p+1}} H^{\mu_1...\mu_{p+1}}$$

where by $\epsilon$ we denote the volume element, $\phi$ is the dilaton field while $H_{\mu_1...\mu_{p+1}} = (p+1)! \nabla_{[\mu_1} B_{\mu_2...\mu_{p+1}]}$ is (p+1)-form field strength. The equations of motion for the underlying theory imply

$$G_{\mu\nu} - T_{\mu\nu}(B, \phi) = 0,$$

$$\nabla_j \left( e^{-\alpha \phi} H^{j...p+1} \right) = 0,$$

$$\nabla_\mu \nabla^\mu \phi + \frac{\alpha}{2(p+1)!} e^{-\alpha \phi} H_{\mu_1...\mu_{p+1}} H^{\mu_1...\mu_{p+1}} = 0,$$

while the energy momentum tensor for (p + 1)-form field strength and dilatons has the form as

$$T_{\mu\nu}(B, \phi) = \frac{1}{2} \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{4} g_{\mu\nu} \nabla_\alpha \phi \nabla^\alpha \phi + \frac{1}{2(p+1)!} e^{-\alpha \phi} \left[ (p+1) H_{\mu_2...\nu_{p+1}} H^{\nu_2...\nu_{p+1}} - \frac{1}{2} g_{\mu\nu} H_{\mu_1...\mu_{p+1}} H^{\mu_1...\mu_{p+1}} \right].$$
In order to find the physical version of the first law of black rings thermodynamics we shall try to find the explicit expressions for the variation of mass and angular momentum. To begin with we perform variation of the Lagrangian (1). On evaluating the variations of the adequate fields, we find that one finally left with

\[ \delta L = \epsilon \left( G_{\mu \nu} - T_{\mu \nu}(B, \phi) \right) \delta g^{\mu \nu} - \epsilon \nabla_j \left( e^{-\alpha \phi} H^{j1...j_{p+1}} \right) \delta B_{j2...j_{p+1}} \]

\[ + \epsilon \left( \nabla_\mu \nabla^\mu \phi + \frac{\alpha}{2(p+1)} e^{-\alpha \phi} H_{\mu_1...\mu_{p+1}} H^{\mu_1...\mu_{p+1}} \right) \delta \phi + d\Theta. \]

For brevity, in what follows, we shall denote fields in the underlying theory by \( \psi_\alpha \), while their variations by \( \delta \psi_\alpha \). By virtue of relation (6) we get the symplectic \((n-1)\)-form \( \Theta_{j1...j_{n-1}}[\psi_\alpha, \delta \psi_\alpha] \), which yields

\[ \Theta_{j1...j_{n-1}}[\psi_\alpha, \delta \psi_\alpha] = \epsilon_{\mu j1...j_{n-1}} \left[ \omega^\mu - e^{-\alpha \phi} H_{m_2...m_{p+1}} \delta B_{m_2...m_{p+1}} - \nabla^m \phi \delta \phi \right], \]

where \( \omega_\mu = \nabla^\alpha g_{\alpha \mu} - \nabla_\mu g_\alpha^\beta \).

From Eq.(6) one can see that equation of motion can be read off. As in Ref. [17], we identify variations of fields with a general coordinate transformations induced by an arbitrary Killing vector field \( \xi_\alpha \). In the next step, we calculate the Noether \((n-1)\)-form with respect to this above mentioned Killing vector, i.e., \( J_{j1...j_{n-1}} = \epsilon_{\mu j1...j_{n-1}} J^m[\psi_\alpha, \xi_\alpha]. \)

The result of doing that is

\[ J_{j1...j_{n-1}} = d \left( Q^{GR} + Q^B \right)_{j1...j_{n-1}} + 2 \epsilon_{\mu j1...j_{n-1}} \left( G^\delta_{\eta} - T^\delta_{\eta}(B, \phi) \right) \xi^\eta \]

\[ + p \epsilon_{\mu j1...j_{n-1}} \xi^d B_{d_2...d_{p+1}} \nabla_{a_2} \left( e^{-\alpha \phi} H^{a_2...a_{p+1}} \right), \]

where \( Q^{GR}_{j1...j_{n-2}} \) yields

\[ Q^{GR}_{j1...j_{n-2}} = -\epsilon_{j1...j_{n-2}ab} \nabla^a \xi^b, \]

while \( Q^B_{j1...j_{n-2}} \) has the following form:

\[ Q^B_{j1...j_{n-2}} = \frac{p}{(p+1)} \epsilon_{\alpha j1...j_{n-1}} \xi^d B_{d_3...d_{p+1}} e^{-\alpha \phi} H^{a_3...a_{p+1}}. \]

As in Ref. [17], one has in mind that \( J[\xi] = dQ[\xi] + \xi^\alpha C_\alpha \), where \( C_\alpha \) is an \((n-1)\)-form constructed from dynamical fields, i.e., from \( g_{\mu \nu}, (p+1)\)-form field \( H^{j1...j_{p+1}} \) and dilaton fields. Consequently, one may also identify \( Q_{j1...j_{n-1}} = (Q^{GR} + Q^B)_{j1...j_{n-1}} \) with the Noether charge for the considered theory. Thus, \( C_\alpha \) reduces to

\[ C_{d1...j_{n-1}} = 2 \epsilon_{mj1...j_{n-1}} \left[ G^m - T^m_{d_1}(B, \phi) \right] + p \epsilon_{mj1...j_{n-1}} \nabla_{a_2} \left( e^{-\alpha \phi} H^{a_2...a_{p+1}} \right) B_{d_3...d_{p+1}}. \]

The case when \( C_\alpha = 0 \) is responsible for the source-free Eqs. of motion. On the other hand, when this is not the case, it follows directly that we have the following:

\[ G_{\mu \nu} - T_{\mu \nu}(B, \phi) = T_{\mu \nu}(\text{matter}), \]  

\[ \nabla_\mu \left( e^{-\alpha \phi} H^{\mu_1...\mu_{p+1}} \right) = j^{\mu_2...\mu_{p+1}}(\text{matter}). \]
\[ \delta C_{\alpha_1...\alpha_{n-1}} = 2 \varepsilon_{\alpha_1...\alpha_{n-1}} \left[ \delta T_a^m (\text{matter}) + p B_{\alpha_3...\alpha_{p+1}} \delta j^{\alpha_3...\alpha_{p+1}} (\text{matter}) \right]. \] (14)

For since the Killing vector field \( \xi_\alpha \) describes also a symmetry of the background matter field, one gets the formula for a conserved quantity connected with \( \xi_\alpha \), namely

\[ \delta H_\xi = -2 \int_\Sigma \varepsilon_{\alpha_1...\alpha_{n-1}} \left[ \delta T_a^m (\text{matter}) \xi_\alpha + p B_{\alpha_3...\alpha_{p+1}} \delta j^{\alpha_3...\alpha_{p+1}} (\text{matter}) \right] + \int_{\partial \Sigma} \left[ \delta Q(\xi) - \xi \cdot \Theta \right]. \] (15)

Let us choose \( \xi_\alpha \) to be an asymptotic time translation \( t^\alpha \), then one can conclude that \( M = H_t \) and finally obtain the variation of the ADM mass

\[ \alpha \delta M = -2 \int_\Sigma \varepsilon_{\alpha_1...\alpha_{n-1}} \left[ \delta T_a^m (\text{matter}) t^\alpha + p \ t^\alpha B_{\alpha_3...\alpha_{p+1}} \delta j^{\alpha_3...\alpha_{p+1}} (\text{matter}) \right] + \int_{\partial \Sigma} \left[ \delta Q(t) - t \cdot \Theta \right], \] (16)

where \( \alpha = \frac{n-3}{n-2} \). Next, if we take the Killing vector fields \( \phi_{(i)} \) which are responsible for the rotation in the adequate directions, we arrive at the relations for angular momenta

\[ \delta J_{(i)} = 2 \int_\Sigma \varepsilon_{\alpha_1...\alpha_{n-1}} \left[ \delta T_a^m (\text{matter}) \phi^a_{(i)} + p \ \phi^a_{(i)} B_{\alpha_3...\alpha_{p+1}} \delta j^{\alpha_3...\alpha_{p+1}} (\text{matter}) \right] - \int_{\partial \Sigma} \left[ \delta Q(\phi_{(i)}) - \phi_{(i)} \cdot \Theta \right]. \] (17)

Consider now stationary black ring solution to the Eqs. of motion (2)-(4). Let us perturb the black ring by dropping in some matter and assume that in the process of this action black ring will be not destroyed and settle down to a stationary final state. Just one can find the changes of the black ring parameters.

In order to study the physical process version of the first law of black ring thermodynamics let us assume that \( (g_{\mu\nu}, B_{\alpha_1...\alpha_p}, \phi) \) are solutions to the source free Einstein equations with \( (p+1) \) form fields and scalar dilaton fields. Suppose, moreover that the event horizon the Killing vector field \( \xi_\mu \) is of the form as

\[ \xi_\mu = t^\mu + \sum_i \Omega_{(i)} \phi^{\mu^{(i)}}. \]

In our considerations we shall assume that \( \Sigma_0 \) is an asymptotically flat hypersurface which terminates on the event horizon. Then, we take into account the initial data on \( \Sigma_0 \) for a linearized perturbations \( \delta g_{\mu\nu}, \delta B_{\alpha_1...\alpha_p}, \delta \phi \) with \( \delta T_{\mu\nu} (\text{matter}) \) and \( \delta j^{\alpha_1...\alpha_{p+1}} (\text{matter}) \). We require that \( \delta T_{\mu\nu} (\text{matter}) \) and \( \delta j^{\alpha_1...\alpha_{p+1}} (\text{matter}) \) disappear at infinity and the initial data for \( \delta g_{\mu\nu}, \delta B_{\alpha_1...\alpha_p}, \delta \phi \) vanish in the vicinity of the black ring horizon \( \mathcal{H} \) on the hypersurface \( \Sigma_0 \). It envisages the fact that for the initial time \( \Sigma_0 \), the considered black hole is unperturbed. Because of the fact that perturbations vanish near the internal boundary \( \partial \Sigma_0 \) it follows from relations (16) and (17) that the following is fulfilled:

\[ \alpha \delta M - \sum_i \Omega_{(i)} \delta J_{(i)} = \]

\[ -2 \int_{\Sigma_0} \varepsilon_{\alpha_1...\alpha_{n-1}} \left[ \delta T_a^m (\text{matter}) \phi^a_{(i)} + p \ \phi^a_{(i)} B_{\alpha_3...\alpha_{p+1}} \delta j^{\alpha_3...\alpha_{p+1}} (\text{matter}) \right] + \int_{\mathcal{H}} \gamma_\alpha k_\alpha \bar{\varepsilon}_{\alpha_1...\alpha_{n-1}}, \] (19)
where \( \bar{\epsilon}_{j_1...j_{n-1}} = n^a \epsilon_{\delta j_1...j_{n-1}} \) and \( n^a \) is the future directed unit normal to the hypersurface \( \Sigma_0 \), \( k_\alpha \) is tangent vector to the affinely parametrized null geodesics generators of the event horizon. Due to the fact of the conservation of current \( \gamma^\alpha \) and the assumption that all of the matter falls into the considered black ring we replace in Eq.(19) \( n^a \) by the vector \( k^a \). In order to find the integral over the event horizon we take into account the following relation:

\[
\rho L \xi [B_{a_2...a_{p+1}} \delta j_2^{a_2...a_{p+1}} - \xi^d H_{d a_2...a_{p+1}} \delta j_2^{a_2...a_{p+1}} = \rho \rho |\nabla_{a_2} (\xi^d B_{d a_2...a_{p+1}}) \delta j_2^{a_2...a_{p+1}}.
\]

The first term of the left-hand side of Eq.(20) is equal to zero because \( \xi_\alpha \) is symmetry of the background solution. Furthermore, let us consider \( n \)-dimensional Raychauduri equation of the form

\[
d\theta = -\frac{\theta^2}{(n-2)} - \sigma_{ij} \sigma^{ij} - R_{\mu \nu} \xi^\mu \xi^\nu,
\]

where \( \lambda \) denotes the affine parameter corresponding to vector \( k_\alpha \), \( \theta \) is the expansion and \( \sigma_{ij} \) is shear. They both vanish in the stationary background. An inspection of Eq.(21) reveals the fact that \( R_{\alpha \beta} k^\alpha k^\beta |_H = 0 \) which in turn implies the following:

\[
\frac{1}{2} k^d \nabla_\mu \phi k^\nu \nabla_\nu \phi + \frac{1}{2} p e^{-\alpha \phi} H_{\mu \nu 2... \mu_{p+1}} H_{\nu \mu 2... \mu_{p+1}} k^\mu k^\nu |_H = 0.
\]

Using the fact that \( L_k \phi = 0 \), it is easily seen that, \( H_{\mu \nu 2... \mu_{p+1}} k^\mu = 0 \). Since \( H_{\mu \nu 2... \mu_{p+1}} k^\mu k^\nu = 0 \), then by asymmetry of \( H_{\mu_1...\mu_{p+1}} \) it follows that \( H_{\mu \nu 2... \mu_{p+1}} k^\mu \sim k_{\mu_2} ... k_{\mu_{p+1}} \). The pull-back of \( H_{\mu \nu 2... \mu_{p+1}} k^\mu \) to the event horizon is equal to zero. Thus, \( \xi^d H_{d a_2...a_{p+1}} \) is a closed \( p \)-form on the horizon. Due to the Hodge theorem (see e.g., [26]) it may be rewritten as a sum of an exact and harmonic form. An exact one does not contribute to the above expression because of the field Eqs. are fulfilled. The only contribution stems from the harmonic part of \( \xi^d H_{d a_2...a_{p+1}} \). Having in mind the duality between homology and cohomology, one can conclude that there is a harmonic form \( \eta \) dual to \( n-p-1 \) cycle \( S \) in the sense of the equality of the adequate surface integrals. Just, it follows that the surface term will have the form of \( \Phi_l \delta q_l \), where \( \Phi_l \) is the constant relating to the harmonic part of \( \xi^d H_{d a_2...a_{p+1}} \) and \( \delta q_l \) is the variation of a local charge [25]. These allow one to write down the following:

\[
\alpha \delta M - \sum_l \Omega_{(i)} \delta f^{(i)} + \Phi_l \delta q_l = 2 \int_H \delta T_{\mu \nu} \xi^\mu k^\nu.
\]

In order to calculate the right-hand side of Eq.(23), one can use the same procedure as described in Refs. [17,18,21]. Namely, considering \( n \)-dimensional Raychauduri Eq. and using the fact that the null generators of the event horizon of the perturbed black ring coincide with the null generators of the unperturbed stationary black ring, lead to the conclusion that

\[
\kappa \delta A = \int_H \delta T_{\mu \nu} \xi^\mu k^\nu,
\]

where \( \kappa \) is the surface gravity.

In the light of what has been shown we obtained the physical process version of the first law of black ring mechanics in Einstein gravity with additional \((p+1)\)-form field strength and dilaton fields. It is of the same form as known from Ref. [25], namely
\[ \alpha \delta M - \sum_i \Omega^{(i)} \delta J^{(i)} + \Phi_i \delta q_i = \kappa \delta A. \] (25)

We finally remark that a proof of physical process version of the first law of thermodynamics for \(n\)-dimensional black rings also provides support for cosmic censorship.

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