On synchronized non-sofic subshifts

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Abstract. We show that a synchronized coded system $X$ is intrinsically ergodic of full support if and only if $h(Y)$, the topological entropy of $Y$, is less than $h(X)$ whenever $Y$ is a proper subsystem of $X$. We also show that alike systems with specification property, SVGL’s the non-mixing version of systems with specification property, are intrinsically ergodic of full support. Moreover, we compute the entropy of the underlying graph of the Fischer cover of a synchronized system.

Keywords: shift of finite type, sofic, entropy, zeta function, intrinsically ergodic, specification.

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1. Introduction

Coded systems were first coined by Blanchard and Hansel [2] as a generalization of irreducible sofic shifts in topological dynamical systems. A well-known subclass containing shifts with the specification property is the family of synchronized systems. Also, a non-mixing version of shifts with specification property, that is, the family of shifts with variable gap length (SVGL) are synchronized. All these families are fairly easy to describe and show very rich dynamical properties.

The investigation for the existence and uniqueness of a measure of maximal entropy has a long history and those systems with this unique invariant measure are called intrinsically ergodic. This measure, if exists, is the most natural measure on subshifts and is the main tool for studying their statistical properties. One goal of this note is to look for intrinsically ergodicity in systems and especially among the synchronized systems.

Parry [14] and Weiss [19, 20] established intrinsically ergodic for topologically transitive shifts of finite type and all their subshift factors (sofic shifts) and Bowen [5] proved for shifts with specification property. We will show that
for a subshift \((X, \sigma)\) with positive entropy, intrinsically ergodicity of full support implies that \(h(Y) < h(X)\) whenever \(Y\) is a proper subsystem of \(X\) where \(h(X)\) is the topological entropy of \(X\). The converse is only true when \(X\) is synchronized [Theorem 3.2]. In particular, we show that a SVGL is intrinsically ergodic with full support [Theorem 3.8].

In section 4, we show that a synchronized system is mixing if and only if it is totally irreducible [Theorem 4.2] and also, the zeta function of a synchronized system \(X\) whose depth is \(n < \infty\) is either rational or transcendental [Theorem 4.3]. In Subsection 4.1, for a synchronized system \(X\) with the underlying graph \(G\) for its Fischer cover, we show that \(h(G) = h_{syn}(X)\) where \(h(G)\) and \(h_{syn}(X)\) are the entropy of \(G\) and the synchronized entropy of \(X\) respectively. [Theorem 4.4].

2. Background and Notations

The notations has been borrowed from [12] and the proofs of the claims in this section can be found there. Let \(\mathcal{A}\) be an alphabet, that is a nonempty finite set. The full \(\mathcal{A}\)-shift denoted by \(\mathcal{A}^\mathbb{Z}\) and equipped with the product topology on the discret space \(\mathcal{A}\), is the collection of all bi-infinite sequences of symbols from \(\mathcal{A}\). A block (or word) over \(\mathcal{A}\) is a finite sequence of symbols from \(\mathcal{A}\). The shift function \(\sigma\) on the full shift \(\mathcal{A}^\mathbb{Z}\) maps a point \(x = \{x_i\}\) to the point \(y = \{y_i\} = \sigma(x)\) whose \(i\)th coordinate is \(y_i = x_{i+1}\).

Let \(B_n(X)\) denote the set of all admissible \(n\) blocks. The Language of \(X\) is the collection \(B(X) = \bigcup_{n=0}^{\infty} B_n(X)\) of all finite words. For \(u \in B(X)\), let \([u]\) denote the closed and open set \(\{x \in X : x_{[|u|+1]} = u\}\), which is called a cylinder.

A point \(x\) in a shift space \(X\) is doubly transitive if every word in \(X\) appears in \(x\) infinitely many often to the left and to the right. A word \(v \in B(X)\) is synchronizing if whenever \(uv\) and \(vw\) are in \(B(X)\), we have \(uvw \in B(X)\).

A shift space \(X\) is irreducible if for every ordered pair of blocks \(u, v \in B(X)\) there is a word \(w \in B(X)\) so that \(uvw \in B(X)\). It is called weak mixing if for every ordered pair \(u, v \in B(X)\), there is a thick set (a subset of integers containing arbitrarily long blocks of consecutive integers) \(P\) such that for every \(n \in P\), there is a word \(w \in B_n(X)\) such that \(uvw \in B(X)\). It is mixing if for every ordered pair \(u, v \in B(X)\), there is an \(N\) such that for each \(n \geq N\) there is a word \(w \in B_n(X)\) such that \(uvw \in B(X)\) and it is totally irreducible if for every ordered pair of blocks \(u, v \in B(X)\) and for \(n \in \mathbb{N}\) there is a word \(w \in B(X)\) so that \(uvw \in B(X)\) and \(|w| = kn\) for some \(k \in \mathbb{N}\).

An edge shift, denoted by \(X_G\), is a shift space which consists of all bi-infinite walks in a directed graph \(G = (\mathcal{V}(G), \mathcal{E}(G))\) where \(\mathcal{V}(G)\) and \(\mathcal{E}(G)\) are the set of vertices and edges respectively.

A labeled graph \(G\) is a pair \((G, \mathcal{L})\) where \(G\) is a graph with edge set \(\mathcal{E}\), and the labeling \(\mathcal{L} : \mathcal{E} \rightarrow \mathcal{A}\). A soft shift \(X_G\) is the set of sequences obtained by
On synchronized non-sofic subshifts

We say $G$ is a presentation of $X_G$. Every SFT is sofic, but the converse is not true.

A labeled graph $G = (G, L)$ is right-resolving if for each vertex $I$ of $G$ the edges starting at $I$ carry different labels. A minimal right-resolving presentation of a sofic shift $X$ is a right-resolving presentation of $X$ having the fewest vertices among all right-resolving presentations of $X$. Any two minimal right-resolving presentations of an irreducible sofic shift must be isomorphic [12, Theorem 3.3.18]. So we can speak of “the” minimal right-resolving presentation of an irreducible sofic shift $X$, which is called the Fischer cover of $X$. Every irreducible sofic shift has a synchronizing word.

Now we review the concept of Fischer cover for a not necessarily sofic system [8]. Let $x \in B(X)$. Then $x_+ = (x_i)_{i \in \mathbb{Z}^+}$ (resp. $x_- = (x_i)_{i < 0}$) is called right (resp. left) infinite $X$-ray. For a left infinite $X$-ray, say $x_-$, its follower set is $\omega_+(x_-) = \{ x_+ \in X^+ : x_-x_+ \text{ is a point in } X \}$. Consider the collection of all follower sets $\omega_+(x_-)$ as the set of vertices of a graph $X^+$. There is an edge from $I_1$ to $I_2$ labeled $a$ if and only if there is an $X$-ray $x_-$ such that $x_-a$ is an $X$-ray and $I_1 = \omega_+(x_-)$, $I_2 = \omega_+(x_-a)$. This labeled graph is called the Krieger graph for $X$. If $X$ is a synchronized system with synchronizing word $\alpha$, the irreducible component of the Krieger graph containing the vertex $\omega_+(\alpha)$ is called the (right) Fischer cover of $X$.

The entropy of a shift space $X$ is defined by $h(X) = \lim_{n \to \infty} (1/n) \log |B_n(X)|$. There are some other entropies which will be used in this note. One is the entropy related to graphs given by Gurevich. Let $G$ be a connected oriented graph. Then for any vertices $I, J$

$$h(G) = \lim_{n \to \infty} \frac{1}{n} \log B_{I,J}(n)$$

where $B_{I,J}(n)$ is the number of paths of length $n$ which start at $I$ and end at $J$ [15, Proposition 1.2].

Let $(X, M, f)$ be a measurable dynamical system. A measure $\mu$ on $(X, M)$ is said to be invariant under $f$ if for every measurable set $A \in M$, $\mu(f^{-1}(A)) = \mu(A)$. The measure $\mu$ is called of full support, if $\mu$ is positive on open sets.

3. Systems with entropies larger than proper subsystems

A sliding block code $\varphi : X \to Y$ is finite-to-one if there is an integer $M$ such that $\varphi^{-1}(y)$ contains at most $M$ points for every $y \in Y$. If $X$ is an irreducible sofic shift, then by [12, Corollary 4.4.9],

$$h(Y) < h(X), \ Y \text{ is a proper subsystem of } X.$$
This condition is sufficient to have the double transitivity as a totally invariant property for the finite-to-one factor codes; a fact that has been already established for sofics. That is,

**Theorem 3.1.** Suppose $X$ is compact and satisfies (3.1) and $\varphi : X \to Y$ is a finite-to-one factor code. Then $x \in X$ is doubly transitive if and only if $\varphi(x)$ is.

**Proof.** A similar result holds for irreducible sofic shifts [12, Lemma 9.1.13]. The main ingredients for the proof of that result is to have $X$ compact and the fact that (3.1) holds for irreducible sofics [12, Corollary 4.4.9]. Both of them are provided here. □

Recall that dynamical systems with a unique invariant measure of maximal entropy are called *intrinsically ergodic*. The following is a natural extension of [12, Corollary 4.4.9].

**Theorem 3.2.** Let $X$ be a compact shift space with positive topological entropy. If $X$ is intrinsically ergodic of full support, then $h(Y) < h(X)$ whenever $Y$ is a proper subsystem of $X$. The converse is true if $X$ is synchronized.

**Proof.** First let $\mu_X$ be the unique invariant measure on $X$ with maximal entropy of full support; so $h(X) = h_{\mu_X}$. Suppose $Y$ is a proper subsystem of $X$ and $h(Y) = h(X)$. By the variational principle we have

$$h(Y) = \sup \{ h_\nu : \nu \in \mathcal{M}(Y, \sigma) \}$$

where $\mathcal{M}(Y, \sigma)$ is the set of all invariant measures. Since the shift map $\sigma$ is expansive, by [18, Theorems 8.2 and 8.7], there is a measure $\nu_Y$ with $h(Y) = h_{\nu_Y}$. Now define $\nu_X(A) = \nu_Y(A \cap Y)$ for $A \in \mathcal{M}(X)$. Then $\nu_X$ is an invariant measure on $X$ vanishing at the open set $X \setminus Y$. Therefore, $\nu_X$ is different from $\mu_X$. But by a direct verification, $h_{\nu_X} = h_{\nu_Y} = h(Y) = h(X)$ which is absurd by the uniqueness of $\mu_X$.

Now let $X$ be synchronized. For the converse recall that since the shift map $\sigma$ is expansive, by [18, Theorems 8.2 and 8.7], there is a measure on $X$, say $\mu$, with maximal entropy. Since a maximal measure of full support for synchronized systems is unique [16], it is sufficient to show that any such $\mu$ in our situation is of full support. Assume the contrary. Then there has to be a cylinder $[u]$ in $X$ with $\mu([u]) = 0$. Now $Y = X \setminus \bigcup_{i=-\infty}^{\infty} \sigma^{-i}([u])$ is a closed invariant subset of $X$ and in fact a proper subsystem of $X$ with $\mu(Y) = \mu(X)$. Restrict $\mu$ to $Y$ and call it $\mu_Y$. Then $h_{\mu_Y}(Y) = h_{\mu}(X) = h(X)$ and this in turn by applying (3.2) implies that $h(Y) = h(X)$ which violates our hypothesis. □

**Example 3.3.** We will show that the converse of Theorem 3.2 is not necessarily true in non-synchronized systems. We give our example for a half-synchronized system.
Let $X$ be the Dyck shift on 2 pairs \cite{11}. Its alphabet $A$ consists of four delimiters, say \{\ell_1 := (, \ell_2 := [, r_1 :=), r_2 :=]\}. A point $(x_i)_{i \in \mathbb{Z}} \in A^\mathbb{Z}$ is a point of $X$ if and only if any finite subblock $x_j \cdots x_k$, $j \leq k$, obeys the standard bracket rules. This system is not synchronized but a well-known half-synchronized system and it is not intrinsically ergodic of full support \cite{11}. To prove our assertion, we show that $h(Y) < h(X)$ whenever $Y$ is a proper subsystem of $X$.

Recall that $h(X) = \log 3$ \cite{11}. Let $X_F$ denote a subsystem of $X$ where $w \in B(X_F)$ if and only if no subwords of $w$ are in $F \subseteq B(X)$. If $F' \subseteq F$, then $X_F$ is a subsystem of $X_{F'}$. So we may assume that $F$ consists of just one word $f$. Also, if $F = \{f\}$ and $F = \{ufu\}$, then $h(X_F) \leq h(X_{F'})$. Therefore, we assume that $f$ is a balanced word by adding the required delimiters to make $f$ a balanced word. That is, any left (resp. right) delimiter in $w$ has its matching right (resp. left) delimiter. For our last reduction, we may assume $f = (\cdots)$ consists of $q \ell_1$ and $q r_1$.

Our computation of $h(X_F)$ is inspired by computing $h(X)$ in \cite{13}. First let $f = ()$.

An admissible word has the general form

$$w = b_0 r_1 \ell_1 b_1 r_2 \ell_2 \cdots b_k r_k \ell_k b_k \ell_{k+1} b_{k+1} \cdots \ell_{m} b_{k+m}$$

where each $b_a = 0, \ldots, k + m$, is a (possibly empty) balanced subword, and the $k \geq 0$ right delimiters which are not matched in $w$ all occur to the left of the $m \geq 0$ unmatched left delimiters in $w$. This leads to a natural decomposition of any admissible word as a concatenation of three (possibly empty) subwords $w = ABC$ where $B = b_k$ is balanced, while $A = b_0 r_1 \ell_1 b_1 r_2 \cdots b_{k-1} r_{k-1}$ (resp. $C = \ell_{j_1} b_{k+1} \cdots \ell_{j_m} b_{k+m}$) ends (resp. starts) with an unmatched right (resp. left) delimiter. When $A$ has length $n$, we denote it by $A_n$ and $B_n$, $C_n$ are similarly defined.

We estimate the cardinality of the sets $A_n$, $B_n$, $C_n$ and we will begin with the set $C_n$. Note that the initial subword of length $k \leq n$ of any $c \in C_n$ belongs to $C_k$. Given $c \in C_n$, we have $c \ell_i \in C_{n+1}$ for $i = 1, 2$. Also, if $c = c' \ell_2 b$ where $b$ (possibly empty) is a balanced word, then $c r_2 \in C_{n+1}$. But if $c = c' \ell_1 b$, then $cr_1$ is a forbidden word. In average at most one half of words in $C_n$ are from the latter case, so this gives us

$$|C_{n+1}| \leq (2 + \frac{1}{2})|C_n|$$

and since $|C_1| = 2$, $|C_n| \leq \left(\frac{5}{2}\right)^n$. A similar estimate can be obtained for $|A_n|$, either by repeating the above argument or by noting the bijection between $A_n$ and $C_n$ obtained by reversing letter orders and interchanging $\ell$ with $r$ (keeping indices).
When \( n \) is odd \( B_n = \emptyset \). So let \( n \) be even and \( b \in B_n \). then by definition \( \ell_i b \in C_{n+1} \). Therefore, \(|B_n| \leq |C_{n+1}| \leq (2 + \frac{1}{2})|C_n| \) and this rule is valid for \( f = \((\cdots)\) \) of length \( 2q \).

Finally, to estimate \(|B_n|\) we consider, for each ordered triple \((i, j, k)\) of nonnegative integers summing up to \( n \), the set of words of the form \( w = ABC \) with \(|A| = i, |B| = j, \) and \(|C| = k\). Since an arbitrary factoring is possible, the number of such words is

\[
|A_i| \cdot |B_j| \cdot |C_k| \leq \left( \frac{5}{2} \right)^{i+j+k+1} = \left( \frac{5}{2} \right)^{n+1}
\]

But the number of possible triples \((i, j, k)\) summing up to \( n \) is less than \((n+1)^3\), so \(|B_n| \leq (n + 1)^3\left( \frac{5}{2} \right)^{n+1} \) which means that \( h(X_F) \leq \log \frac{5}{2} \).

Now let \( f = (()) \). Again if \( c \in C_n, c \ell_i \in C_{n+1} \) for \( i = 1, 2 \). But \( c \ell_i \not\in C_{n+1} \) if \( i = 1 \) and \( c = c'(()) \). Since we have four symbols, \(|B_3(X_F)| = 64 \) and since the probability of having \((())\) in the end of \( c \) in \( C_n \) is \( \frac{1}{64} \),

\[
|C_{n+1}| \leq (2 + \frac{63}{64})|C_n|.
\]

So \(|C_n| \leq (2 + \frac{63}{64})^n \). Now a similar argument as above will show that \( h(X_F) \leq \log(2 + \frac{63}{64}) \).

The same routines works for when \(|f| = 2q \). In fact, we will have \( h(X_F) \leq \log(2 + \frac{4q^2-3-1}{4q^2-4}) < \log 3 = h(X) \).

If \( X \) is a synchronized system with Fischer cover \( G \), \( X \) has a maximal measure of full support if and only if \( G \) is positive recurrent and \( h(G) = h(X) \) where \( h(G) \) is defined as (2.1) [16, Corollary 6.8]. So by Theorem 3.2, we have the following.

**Corollary 3.4.** Let \( X \) be a synchronized system with Fischer cover \( G \). Then the following are equivalent.

1. \( X \) is intrinsically ergodic of full support.
2. \( G \) is positive recurrent and \( h(G) = h(X) \).
3. \( h(Y) < h(X) \) whenever \( Y \) is a proper subsystem of \( X \).

Specification property was first defined by Bowen [4]. In subshifts, a shift space \( X \) has specification property if there exists \( N \in \mathbb{N} \) such that for all \( u, v \in B(X) \), there exists \( w \in B_N(X) \) with \( uwv \in B(X) \).

**Definition 3.5.** [10] A shift space \( X \) has specification with variable gap length (SVGL) if there exists \( N \in \mathbb{N} \) such that for all \( u, v \in B(X) \), there exists \( w \in B(X) \) with \( uwv \in B(X) \) and \(|w| \leq N \). We call \( N \) a transition length for \( X \).

A mixing SVGL has specification property.
Theorem 3.6. [17] Let \( X \) be a synchronized system. Then there exist a unique \( p \in \mathbb{N} \) and closed sets \( D_i \subseteq X \), \( i = 0, 1, \ldots, p - 1 \), called the cyclic cover for \( X \), such that

1. \( X = \bigcup_{i=0}^{p-1} D_i \),
2. \( \sigma(D_i) = D_{(i+1) \mod p} \),
3. \( \sigma^p|_{D_i} \) is mixing for all \( i = 0, 1, \cdots, p - 1 \) and
4. \( D_i \cap D_j \) has empty interior when \( i \neq j \).

Lemma 3.7. [10] Let \( X \) be a SVGL shift. If \( \{D_0, \cdots, D_{p-1}\} \) is the cyclic cover for \( X \), then \( \sigma^p|_{D_i} \) has the specification property.

Theorem 3.8. If \((X, \sigma)\) is SVGL, then it is intrinsically ergodic of full support.

Proof. By Lemma 3.7, \( \sigma^p : D_i \to D_i \) has specification property and by [5], it is intrinsically ergodic. Call that unique ergodic measure of full support \( \mu_i \). For \( 0 \leq i \leq p-1 \), the homeomorphism \( \sigma^i : D_0 \to D_i \) defines a topological conjugacy between \((D_0, \sigma^p)\) and \((D_i, \sigma^p)\) and if \( \mu_i^* = \mu_0 \circ \sigma^{-i} \) is the projection of \( \mu_0 \) to \( D_i \), then clearly \( \mu_i = \mu_i^* \). This implies that \( \sigma^i \) defines an isomorphism between \((D_0, \mathcal{M}_0, \mu_0, \sigma^p)\) and \((D_i, \mathcal{M}_i, \mu_i, \sigma^p)\) where \( \mathcal{M}_i \) is the Borel \( \sigma \)-algebra on \( D_i \). Observe that \( D = \bigcap_{i=0}^{p-1} D_i \) is a closed invariant subset of \( D_i \) under \( \sigma^p \) and since \( \mu_i(\sigma^p) \) is ergodic, \( \mu_i(\sigma^p) \) is either 1 or 0. But \( D \) is a closed set with empty interior, so \( \mu_i(D) = 0 \). Now for any \( A \in \mathcal{M} \), the Borel \( \sigma \)-algebra of \( X \), set \( A_i := A \cap D_i \) and define \( \mu \) on \( X \) as \( \mu(A) = \mu_0(\bigcup_{i=0}^{p-1} \sigma^{-i} A_i) = \mu_j(\bigcup_{i=0}^{p-1} \sigma^{(j-i) \mod p} A_i) \). Another equivalent formula for \( \mu \) is

\[
\mu(A) = \mu_0(A_0) + \mu_0(\sigma^{-1}(A_1) \setminus A_0) + \cdots + \mu_0(\sigma^{-p+1}(A_{p-1}) \setminus \bigcup_{i=0}^{p-2} \sigma^{-i} A_i).
\]

This shows \( \mu(\sigma^{-1} A) = \mu_0(\sigma^{-1}(\bigcup_{i=0}^{p-1} \sigma^{-i} A_i)) = \mu(A) \). Moreover, if \( \sigma^{-1}(A) = A \), then \( \sigma^{-p}(\bigcup_{i=0}^{p-1} \sigma^{-i} A_i) = \bigcup_{i=0}^{p-1} \sigma^{-i} A_i \) for \( \sigma^p : D_0 \to D_0 \). But then ergodicity of \( \sigma^p \) implies the ergodicity of \( \sigma \). Conjugacy among \((D_i, \mathcal{B}_i, \mu_i, \sigma^p)\)’s show that

\[
h_{\mu}(\sigma, X) = \frac{1}{p} h_{\mu_i}(\sigma^p, D_i) = \frac{1}{p} h(\sigma^p, D_i) = h(\sigma, X),
\]

where the last equality is because entropy equals the maximal entropy on subsystems. That means \( \mu \) is a maximal measure. On the other side, each \( \mu_i \) is unique. So \( \mu \) is a unique maximal measure.

Specification guarantees a unique measure of maximal entropy of full support [3]. So Lemma 3.7 extends this result to SVGL’s.

In particular, by this result, (2) and (3) of Corollary 3.4 are satisfied for any SVGL shift.
4. On the mixing properties, zeta function and synchronized entropy of a synchronized system

In a synchronized system $X$, for any synchronized word $\alpha = \alpha_1 \cdots \alpha_p$, $X$ is generated by

$$(4.1) \quad \mathcal{S}_\alpha = \{ v\alpha \in B(X) : \alpha v\alpha \in B(X), \alpha \not\subseteq v \} \cup \{\varepsilon\},$$

where $\varepsilon$ represents the empty word.

**Theorem 4.1.** A synchronized system $X$ with the set of generators $\mathcal{S}_\alpha$ is mixing if and only if $\gcd\{|v\alpha| : v\alpha \in \mathcal{S}_\alpha\} = 1$.

**Proof.** If $X$ is mixing, then there exists $N > 0$ such that for all $n \geq N$, there exists $w \in B_n(X)$ with $\alpha w\alpha \in B(X)$. Thus there are words of length $N+1$ and $N+2$ of the form $\alpha v_1 \alpha \cdots v_{m-1} \alpha v_m$ with $v_i \alpha \in \mathcal{S}_\alpha$ implying that $\gcd\{|v\alpha| : v\alpha \in \mathcal{S}_\alpha\} = 1$.

For the converse suppose that $\gcd\{|v\alpha| : v\alpha \in \mathcal{S}_\alpha\} = 1$. Then for all sufficiently large $n$, there is a word $w_n$ of length $n$ of the form $\alpha v_1 \alpha \cdots v_{m-1} \alpha v_m \alpha$, $v_i \alpha \in \mathcal{S}_\alpha$. Assume $u$ and $v$ are arbitrary words. By irreducibility, there are $u_0$ and $v_0$ such that $u_0 \alpha \omega v_0 \alpha$ and $\alpha v_0 \alpha$ are words in $X$. Then $u_0 \alpha \omega v_0 \alpha$ is also a word and we are done. \qed

**Theorem 4.2.** Let $X$ be a synchronized system. Then the following are equivalent.

1. $X$ is mixing.
2. $X$ is weak mixing.
3. $X$ is totally irreducible.

**Proof.** Clearly every mixing shift space is weak mixing. Furthermore, any weak mixing shift space is totally irreducible [9].

(3) $\Rightarrow$ (2). This is because if $X$ is totally irreducible and has a dense set of periodic points, it is weak mixing [1].

(2) $\Rightarrow$ (1). Let $\alpha$ be a synchronizing word and let $X$ be generated by $\mathcal{S}_\alpha$ as in (4.1). By Theorem 4.1, it suffices to show that $\gcd\{|v\alpha| : v\alpha \in \mathcal{S}_\alpha\} = 1$. Since $X$ is weak mixing, there is a thick set $P$ such that for every $n \in P$ there exists a word $w \in B_n(X)$ with $\alpha w\alpha \in B(X)$. Thus there are words of length $m, m+1 \in P$ of the form $\alpha u$, implying that $\gcd\{|v\alpha| : v\alpha \in \mathcal{S}_\alpha\} = 1$. \qed

Let $\text{Per}X$ denote the set of periodic points of the shift space $X$ and let $R(X) = \text{Per}(X)$ be its closure. The *derived shift space* of $X$

$$\partial X = \{ x \in R(X) : x \text{ contains no words that are synchronizing for } R(X) \}$$

was first introduced in [17]. The derived shift space is a subshift of $X$ and the construction can be iterated:

$$\partial^j X = \partial(\partial^{j-1} X), \quad j \geq 2.$$
Thus we have a (possibly trivial) sequence
\[ \partial^0 X = X \supseteq \partial^1 X \supseteq \partial^2 X \supseteq \ldots \]
of subshifts inside \( X \), called the derived shift spaces of \( X \). If there is \( n \in \mathbb{N} \) such that \( \partial^n X \neq \emptyset \); while \( \partial^{n+1} X = \emptyset \), we call \( n \) the depth of \( X \).

For a dynamical system \((X, T)\), let \( p_n(X) \) be the number of periodic points in \( X \) having period \( n \). When \( p_n(X) < \infty \), the zeta function \( \zeta_X(t) \) is defined as the formal series

\[
\zeta_X(t) = \exp \left( \sum_{n=1}^{\infty} \frac{p_n(X)}{n} t^n \right).
\]

A loop graph \( G_\ell \) consists of disjoint loops of lengths \( l_1, l_2, \ldots \) with \( l_i \leq l_{i+1} \) (Figure 1). Let \( f_n \) be the number of first-return loops of length \( n \) from \( v \) to \( v \) and define \( f(t) = \sum_{n=1}^{\infty} f_n t^n \). By [6], the zeta function of \( X_\ell \) is

\[
\zeta_{X_\ell}(t) = \frac{1}{1 - f(t)}.
\]

**Theorem 4.3.** Let \( X \) be a synchronized system whose depth is \( n < \infty \). Then the zeta function of \( X \) is either rational or transcendental.

**Proof.** Let \( \alpha \) be a synchronizing word. By (4.2),

\[
\zeta_X(t) = \exp \left( \sum_{n=1}^{\infty} \frac{p_n(X \setminus \partial X) + p_n(\partial X)}{n} t^n \right) = \exp \left( \sum_{n=1}^{\infty} \frac{p_n(X \setminus \partial X)}{n} t^n \right) \zeta_{\partial X}(t).
\]

Construct a loop graph \( G \) representing \( X \setminus \partial X \). A periodic point of \( X \setminus \partial X \) is of the form \((v_1 \alpha \cdots v_n \alpha)^\infty\) where \( v_i \alpha \in \mathbb{S}_\alpha = \{v \alpha \in \mathcal{B}(X) : \alpha v \alpha \in \mathcal{B}(X), \alpha \nsubseteq v\} \} \cup \{\varepsilon\} \). So choose a base vertex \( b \) such that for any \( v \alpha \in \mathbb{S}_\alpha \), there is a cycle
of length $|v\alpha|$ from $b$ to $b$. By (4.3),

$$\zeta_{X_G}(t) = \frac{1}{1 - f(t)}.$$  

On the other hand,

$$\zeta_{X_G}(t) = \exp \left( \sum_{n=1}^{\infty} \frac{p_n(X \setminus \partial X)}{n} t^n \right).$$

Therefore, by (4.4), (4.5) and (4.6),

$$\zeta_X(t) = \frac{1}{1 - f(t)} \zeta_{\partial X}(t).$$

Set $f^1 := f$ and continue the process on the derived sets to have

$$\zeta_X(t) = \prod_{i=1}^{n} \frac{1}{1 - f^i(t)}.$$

Now the proof is complete by using the celebrated result of Fatou [7] which states that a function whose power series expansion has integer coefficients and positive radius of convergence is either rational or transcendental. □

### 4.1. Synchronized entropy.

First let $X$ be synchronized and fix a synchronizing word $\alpha \in \mathcal{B}(X)$. Let $C_n(X)$ be the set of words $v \in \mathcal{B}_n(X)$ such that $\alpha v \alpha \in \mathcal{B}(X)$. Then the synchronized entropy $h_{\text{syn}}(X)$ is defined by

$$h_{\text{syn}}(X) = \limsup_{n \to \infty} \frac{1}{n} \log |C_n(X)|.$$  

This value is independent of $\alpha$ and $h(X) \geq h_{\text{syn}}(X)$. In general, $h(X) \neq h_{\text{syn}}(X)$; however, Thomsen [17] showed that for irreducible sofic shifts $h(X) = h_{\text{syn}}(X)$. Later Jung [10] extended this result to SVGL shifts.

The following can be deduced from [17, Theorem 3.2]; however, it was not stated directly. We found it more convenient to give a new proof which essentially uses the main ingredients of the proof of [17, Lemma 3.1].

**Theorem 4.4.** Let $X$ be a synchronized system with Fischer cover $G$. Then $h(G) = h_{\text{syn}}(X)$. In particular, when any (hence all) of the conditions of Corollary 3.4 holds, $h(X) = h_{\text{syn}}(X)$.

**Proof.** First we show that $h(G) \leq h_{\text{syn}}(X)$. Let $\alpha \in \mathcal{B}(X)$ be a synchronizing word. Then all elements of $\mathcal{L}^{-1}(\alpha)$ have the same terminal vertex, say $I$. Set $v = \mathcal{L}^{-1}(\alpha)$. Since $G$ is irreducible, there is a path $u$ in $G$ with initial vertex $I$ such that $uv \in \mathcal{B}(X_G)$. Then $m' = \mathcal{L}(uv)$ is synchronizing for $X$. Let $L_n$ denote the set of cycles in $X_G$ of length $n$ starting and terminating at $I$. Since $\mathcal{L}_\infty$ is right-resolving,

$$\mathcal{L} : L_n \rightarrow \{ w \in \mathcal{B}_n(X) : m'wm' \in \mathcal{B}(X) \}.$$
is injective. So,
\[ \limsup_{n \to \infty} \frac{1}{n} \log L_n \leq \limsup_{n \to \infty} \frac{1}{n} \log \# \{ w \in B_n(X) : m'wm' \in B(X) \} = h_{\text{syn}}(X). \]

By \((2.1)\), \( \limsup_{n \to \infty} \frac{1}{n} \log L_n = h(G) \). So \( h(G) \leq h_{\text{syn}}(X) \).

The converse is quite similar! Let \( v \in \{ w \in B_n(X) : \alpha w \alpha \in B(X) \} \) with \( v = L^{-1}(\pi) \). Thus \( \alpha v \alpha \in B(X) \). Since all elements of \( L^{-1}(\alpha) \) have the same terminal vertex \( I \), then for some \( \pi' \in L^{-1}(\alpha) \), \( \pi \pi' \) is a cycle in \( G \) starting and terminating at \( I \). So
\[ h_{\text{syn}}(X) = \limsup_{n \to \infty} \frac{1}{n} \log \{ w \in B_n(X) : \alpha w \alpha \in B(X) \} \leq \limsup_{n \to \infty} \frac{1}{n} \log L_n. \]

\[ \square \]

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