An efficient technique to solve coupled–time fractional Boussinesq–Burger equation using fractional decomposition method

Mahmoud S Alrawashdeh and Shifaa Bani-Issa

Abstract
For this work, a novel numerical approach is proposed to obtain solution for the class of coupled time-fractional Boussinesq–Burger equations which is a nonlinear system. This system under consideration is endowed with Caputo time-fractional derivative. By means of the natural decomposition approach, approximate solutions of the proposed nonlinear fractional system are obtained where the exact solutions are presented in the classical case of fractional order at $\gamma = 1$. Some numerical examples are given to support the theoretical framework and to point out the role and the effectiveness of the intended method. Our results clearly show the approximate analytical solutions eventually will converge quickly to the already known exact solutions.

Keywords
Decomposition method, fractional derivatives, Caputo derivative, Boussinesq–Burger equations

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Introduction
It was only the starts when Leibniz sent an amazing notice to l’Hospital in 1695 which led to the theory of fractional derivatives and integrals of arbitrary order such as one-half. Since then, scholars started developing theories of fractional derivative as theoretical field of pure Mathematics for three centuries. For the past three decades, researchers started bringing the attention about non-integer fractional integrals order along with its non-integer derivatives order. Moreover, they clarify the fact that they are very useful and more adequate than classical integer order in description the properties of real materials such as polymers, modelling mechanical and electrical properties of real materials, description of archaeological properties of rocks, theory of fractals.1

Surveys of the history of fractional derivatives can be found in Miller and Ross,2 Podlubny,3 Al-Smadi et al.,4 and Eid et al.5 We can summarize all the above describing the relation between these two concepts in one sentence: fractional derivative is the generalized form of classical integer derivative. In addition, the subject of fractional derivatives can be widely applied in many real life applications, such as; engineering, mathematical biology, quantum physics, fluid mechanics fields.6–11 Due to the fast development of software programs such as Mat Lab, Mathematica and Maple, many new powerful analytical techniques have been
proposed to find new and approximate solutions for fractional linear and nonlinear differential equations such as; the sub-equation method,\(^\text{12}\) Exponential function method,\(^\text{13}\) first integral method,\(^\text{14}\) the expansion method,\(^\text{15}\) fractional reproducing kernel method,\(^\text{16–18}\) fractional Adomian decomposition method,\(^\text{19}\) fractional homotopy analysis,\(^\text{21}\) fractional homotopy perturbation,\(^\text{20}\) fractional residual power series,\(^\text{22–25}\) fractional Caputo derivative will be used through out the paper. The methodology of FDM is the Heaviside function. Consider the non-linear fractional Boussinesq–Burger equation which is a non-is to look for exact solution to coupled time-fractional differential equations. Section 6 is for discussion and conclusion of this paper.

**Background of fractional calculus**

In recent years, scientists showed some interest in non-local field theories and their interest really became more consistence. This development became more clear due to the expectation and the needs to use these theories so that they can treat problems in a more elegant and effective fashion. A particular group of non-local field theories plays an outstanding role and may be described with operators of fractional nature and is specified within the framework of fractional calculus.\(^\text{4–9}\)

**Definition 2.1.** If \(m \in \mathbb{N}, x > 0\) with \(g \in C^m_{-1}\) and \(m - 1 < \gamma \leq m\). Then, the fractional Caputo derivative of \(g\) is defined as: \(D^\gamma g(x) = J^{m-\gamma} D^mg(x) = \frac{1}{\Gamma(m-\gamma)} \int_0^x (x - y)^{m-\gamma-1} g^{(m)}(y)dy\).

Note that, since the I.C and B.C will be included in the formulation of our applications, then the Caputo fractional derivative will be used through out the paper.

**The natural transform method**

Here, we refer the reader to Belgacem and Silambarasan\(^\text{38}\) to view some of the background and history of the (NTM).

**Definition 3.1.** Suppose \(H(t)\) is the Heaviside function. Consider a real-valued function, where the natural transform \(\mathbb{N}t\) is well-defined on the half plane \(s > bu\) for some \(b > 0\). Let \(g(t)\) be continuous on \(\mathbb{R}\). Let \(K, b > 0\), then we define

\[
B = \{g(t) : |g(t)| < Ke^{bt}H(t) + Ke^{-bt}H(-t)\}.
\]

**Definition 3.2.** The natural transform of the function \(g(t)\) for \(t \in \mathbb{R}\) is given by\(^\text{38}\):

\[
\mathbb{N}[g(t)] = R(s,u) = \int_{-\infty}^{\infty} e^{-st} g(ut)dt;\ s, u \in \mathbb{R}, \quad (3)
\]

where \(\mathbb{N}[g(t)]\) is the natural transform of \(g(t)\).

Note that equation (3) can be written in the form\(^\text{38}\)

\[
\mathbb{N}[g(t)] = R^+(s,u) + R^-(s,u).
\]

**Definition 3.3.** Suppose that \(g(t)H(t)\) is defined on \(\mathbb{R}^+\), and \(s, u \in \mathbb{R}^+\), where \(g(t) \in B\). Then, we define the N-transform as

\[
\mathbb{N}[g(t)H(t)] = \mathbb{N}^+[g(t)] = R^+(s,u) = \int_{0}^{\infty} e^{-st} g(ut)dt. \quad (4)
\]

Consequently, if \(u = 1\) we obtain the Laplace transform and if \(s = 1\), we get the Sumudu transform. Also, one can find most of the properties of the N-transforms in Rawashdeh\(^\text{30}\) and Belgacem and Silambarasan.\(^\text{38}\) For example, \(\mathbb{N}^+[1] = \frac{1}{s}\) and \(\mathbb{N}^+\{\gamma t\} = \frac{\gamma^{\gamma u}}{\Gamma(\gamma)}\), where \(\gamma > -1\).

**The methodology of FDM**

In this section, we give the methodology of the FDM which also can be found in Rawashdeh\(^\text{30}\) and Rawashdeh and Darweesh.\(^\text{35}\)

**Theorem 4.1.** If \(m \in \mathbb{Z}^+\), where \(m - 1 \leq \gamma < m\) and \(\mathbb{N}^+[g(t)] = R^+(s,u)\), then the natural transform of \(g(t)\) is given by
\[ N^+ \left[ D^\gamma g(t) \right] = \frac{s^\gamma}{\mu^\gamma} R(s, u) - \sum_{k=0}^{n-1} \frac{s^{\gamma-(k+1)}}{\mu^{\gamma-k}} \left[ D^k g(t) \right]_{t=0^-}. \]

For the sake of explanation of the method algorithm, let us consider a nonlinear fractional system in the general form:

\[
D^\gamma_0 v(x, t) + L_1 v(x, t) + N_1(w, v) = g(x, t),
\[
D^\beta_0 w(x, t) + L_2 v(x, t) + N_2(w, v) = h(x, t),
\tag{5}
\]

along with the following initial conditions

\[
v(x, 0) = g_1(x),
\]
\[
w(x, 0) = g_2(x),
\tag{6}
\]

where \(D^\gamma_0 w(x, t)\) and \(D^\beta_0 v(x, t)\) is the Caputo fractional derivative of the function \(w(x, t)\) and \(v(x, t)\) respectively, \(L_1\) and \(L_2\) are the linear differential operators, \(g(x, t)\) and \(h(x, t)\) are the non-homogeneous terms and \(N_1(w, v), N_2(w, v)\) represent the nonlinear differential operators.

Now, by applying the \(N^+\) to equation (5) and theorem (4.1), we have

\[
V(x, s, u) = \frac{u^\gamma}{s^\gamma} \sum_{k=0}^{n-1} \frac{s^{\gamma-(k+1)}}{\mu^{\gamma-k}} \left[ D^k v \right]_{t=0^-} + \frac{u^\gamma}{s^\gamma} N^+ \left[ h(x, t) \right] - \frac{u^\gamma}{s^\gamma} N^+ \left[ L_1 v(x, t) \right] - \frac{u^\gamma}{s^\gamma} N^+ \left[ N_1(w, v) \right],
\]
\[
W(x, s, u) = \frac{u^\beta}{s^\beta} \sum_{k=0}^{n-1} \frac{s^{\beta-(k+1)}}{\mu^{\beta-k}} \left[ D^k w \right]_{t=0^-} + \frac{u^\beta}{s^\beta} N^+ \left[ g(x, t) \right] - \frac{u^\beta}{s^\beta} N^+ \left[ L_2 w(x, t) \right] - \frac{u^\beta}{s^\beta} N^+ \left[ N_2(w, v) \right].
\tag{7}
\]

Thus, we apply \(N^{-1}\) to the above equation and we get

\[
v(x, t) = K(x, t) - N^{-1} \left[ \frac{u^\gamma}{s^\gamma} N^+ \left[ L_1 v(x, t) \right] \right]
\]
\[
- N^{-1} \left[ \frac{u^\gamma}{s^\gamma} N^+ \left[ N_1(w, v) \right] \right],
\tag{8}
\]
\[
w(x, t) = M(x, t) - N^{-1} \left[ \frac{u^\beta}{s^\beta} N^+ \left[ L_2 w(x, t) \right] \right]
\]
\[
- N^{-1} \left[ \frac{u^\beta}{s^\beta} N^+ \left[ N_2(w, v) \right] \right].
\]

From the above equation, then equation (8) becomes:

\[
\sum_{n=0}^{\infty} v_n(x, t) = K(x, t)
\]
\[
- N^{-1} \left[ \frac{u^\gamma}{s^\gamma} N^+ \left[ L_1 \sum_{n=0}^{\infty} w_n(x, t) \right] + \sum_{n=0}^{\infty} A_n \right],
\tag{10}
\]
\[
\sum_{n=0}^{\infty} w_n(x, t) = M(x, t)
\]
\[
- N^{-1} \left[ \frac{u^\beta}{s^\beta} N^+ \left[ L_2 \sum_{n=0}^{\infty} v_n(x, t) \right] + \sum_{n=0}^{\infty} B_n \right].
\]

Looking at both sides of equation (10), one can obtain

\[
v_0(x, t) = K(x, t), w_0(x, t) = M(x, t),
\]
\[
v_1(x, t) = - N^{-1} \left[ \frac{u^\gamma}{s^\gamma} N^+ \left[ L_1 v_0(x, t) + A_0 \right] \right],
\]
\[
w_1(x, t) = - N^{-1} \left[ \frac{u^\beta}{s^\beta} N^+ \left[ L_2 w_0(x, t) + B_0 \right] \right].
\]
\[
v_2(x, t) = - N^{-1} \left[ \frac{u^\gamma}{s^\gamma} N^+ \left[ L_1 v_1(x, t) + A_1 \right] \right],
\]
\[
w_2(x, t) = - N^{-1} \left[ \frac{u^\beta}{s^\beta} N^+ \left[ L_2 w_1(x, t) + B_1 \right] \right].
\]

If we proceed as before one can obtain this recursive relation

\[
v_{n+1}(x, t) = - N^{-1} \left[ \frac{u^\gamma}{s^\gamma} N^+ \left[ L_1 v_n(x, t) + A_n \right] \right], n \geq 0,
\]
\[
w_{n+1}(x, t) = - N^{-1} \left[ \frac{u^\beta}{s^\beta} N^+ \left[ L_2 w_n(x, t) + B_n \right] \right], n \geq 0.
\tag{11}
\]

Hence, our intended approximate solutions are as follows:

\[
v(x, t) = \sum_{n=0}^{\infty} v_n(x, t); \quad w(x, t) = \sum_{n=0}^{\infty} w_n(x, t).
\]

**Numerical examples**

It has been demonstrated that the FDM deals efficiently with the fractional nonlinear system of differential equations when compared with the other methods that exist in literature. However, this section provides some applications of nonlinear coupled time-fractional PDEs using the FDM, including coupled Boussinesq–Burger equations which is an application of dynamical system.

**Example 1.** Consider the coupled time-fractional Boussinesq–Burger equation:
\[ D_y^2 w(x, t) + v_y(x, t) + w(x, t)w_x(x, t) = 0, \quad (0 < \gamma \leq 1), \]
\[ D_y^2 v(x, t) + (w(x, t)v_y(x, t))_x + w_{xxx}(x, t) = 0, \quad (0 < \gamma \leq 1), \]
\[ \sum_{n=0}^{\infty} w_n(x, t) = 1 + \tanh\left(\frac{X}{2}\right) - N^{-1}\left[\frac{\mu^y}{\gamma} N^+ \left[ \sum_{n=0}^{\infty} A_n \right]\right], \]
\[ N^+ \left[ D_y^2 w(x, t) \right] + N^+ \left[ v_y(x, t) \right] + N^+ \left[ w(x, t)w_x(x, t) \right] = 0, \]
\[ N^+ \left[ D_y^2 v(x, t) \right] + N^+ \left[ (w(x, t)v_y(x, t))_x \right] + N^+ \left[ w_{xxx}(x, t) \right] = 0. \]

So equation (14) becomes
\[ \frac{s^y}{\gamma} N^+ [w] = \sum_{k=0}^{\infty} \frac{s^y}{\gamma^k} \left[ D_y^k w \right]_{x=0} + N^+ [v_y] + N^+ [w_{xx}], \]
\[ \frac{s^y}{\gamma} N^+ [v] = \sum_{k=0}^{\infty} \frac{s^y}{\gamma^k} \left[ D_y^k v \right]_{x=0} + N^+ [(wv)_x] + N^+ [w_{xxx}], \]

Plug equation (13) into equation (14) to conclude
\[ N^+ [w(x, t)] = \frac{1}{s} + \frac{1}{2s} \tanh\left(\frac{X}{2}\right) - \frac{\mu^y}{\gamma} N^+ [v_x] - \frac{\mu^y}{\gamma} N^+ [w_{xx}], \]
\[ N^+ [v(x, t)] = \frac{1}{2s} - \frac{1}{2s} \tanh^2\left(\frac{X}{2}\right) - \frac{\mu^y}{\gamma} N^+ [w_{xx}], \]
\[ - \frac{\mu^y}{\gamma} N^+ [w_{xxx}]. \]

Now, taking the N^{-1} of the above equation, we get
\[ w(x, t) = 1 + \tanh\left(\frac{X}{2}\right) - N^{-1}\left[\frac{\mu^y}{\gamma} N^+ [v_x]\right] - N^{-1}\left[\frac{\mu^y}{\gamma} N^+ [w_{xx}]\right], \]
\[ v(x, t) = \frac{1}{2} - \frac{1}{2} \tanh^2\left(\frac{X}{2}\right) - N^{-1}\left[\frac{\mu^y}{\gamma} N^+ [(wv)_x]\right] - N^{-1}\left[\frac{\mu^y}{\gamma} N^+ [w_{xxx}]\right]. \]

Consequently, let us assume the infinite series solutions of the unknown functions \( w(x, t), v(x, t) \) and the non-linear terms \( w_{xx}, wv \) can be expanded in the forms
\[ w(x, t) = \sum_{n=0}^{\infty} w_n(x, t), \quad v(x, t) = \sum_{n=0}^{\infty} v_n(x, t), \]
\[ w_{xx} = \sum_{n=0}^{\infty} A_n(x, t), \quad wv = \sum_{n=0}^{\infty} B_n(x, t). \]

From equations (17) and (18), we conclude that
\[ v_2(x, t) = \frac{\cosh(x) - 2}{4\Gamma(2\gamma + 1)} \sinh^4\left(\frac{\gamma}{2}\right). \]
By continuing in the same way, we conclude:

\[
w(x, t) = \sum_{n=0}^{\infty} w_n(x, t) \]
\[
= w_0(x, t) + w_1(x, t) + w_2(x, t) + \ldots
\]
\[
= \frac{\text{sech}^2 \left( \frac{x}{2} \right) t^\gamma}{2 \Gamma(\gamma + 1)} + \frac{8 \text{csch}^3(x) \sinh^4 \left( \frac{x}{2} \right) t^{2\gamma}}{\Gamma(2\gamma + 1)} + \ldots
\]
\[= 1 + \tanh \left( \frac{x - \frac{\gamma}{2}}{2} \right), \]

and

\[
v(x, t) = \sum_{n=0}^{\infty} v_n(x, t) \]
\[
= v_0(x, t) + v_1(x, t) + v_2(x, t) + \ldots
\]
\[
= \frac{4 \text{csch}^3(x) \sinh^4 \left( \frac{x}{2} \right) t^\gamma}{\Gamma(\gamma + 1)}
+ \frac{(\cosh(x) - 2) \sinh^4 \left( \frac{x}{2} \right) t^{2\gamma}}{4 \Gamma(2\gamma + 1)} + \ldots
\]
\[= \frac{1}{2} - \frac{1}{2} \tanh^2 \left( \frac{x - \frac{\gamma}{2}}{2} \right). \]

With the help of Taylor expansion, the intended approximate solutions for \(\gamma = 1\) are given by

\[
w(x, t) = 1 + \tanh \left( \frac{x - \frac{\gamma}{2}}{2} \right), \quad v(x, t) = \frac{1}{2} - \frac{1}{2} \tanh^2 \left( \frac{x - \frac{\gamma}{2}}{2} \right),
\]

which coincides with the exact solutions of coupled time-fractional Boussinesq–Burger equations (12). In order for us, to see our proposed method (FDM) is reliable and efficient, the fractional behaviour of the approximate solutions \(\{w_n(x, t), v_n(x, t)\}\) of Example 5.1 above is discussed when \(n = 5\) by utilizing the 2D and 3D graphs as follows: The 3D plots of \(w(x, t)\) and approximate solutions \(w_n(x, t)\) together with absolute error are presented in Figure 1. While the behaviour of approximate solution \(w_n(x, t)\) are plotted in Figure 2 for different values of fractional order \(\gamma\) such that \(\gamma \in \{0.6, 0.75, 0.9\}\). The 2D graphs of fractional levels of approximate solution \(w_n(x, y)\) are presented in Figure 3. Similar graphical representations of approximate solution \(v_n(x, t)\) are provided in Figures 4 to 6. From these graphs, it can be seen that the FDM approximations are in closed agreement with each other for various values of \(\gamma\) and with exact solution.

Further, we can illustrate the validity of the FDM by looking at both, the exact and approximate solutions for different values of fractional order. In Tables 1 and 2, some numerical results of the approximate solutions \(w(x, t)\) and \(v(x, t)\) are considered at various values of \(x\) and \(t\) such that \(x = \{-5, 0, 5\}\) and \(t = \{0.01, 0.03, 0.05\}\).

**Example 2.** Consider the fractional system below:
\[ D^\alpha x(t) = 2y^2(t), \quad 0 < \alpha \leq 1, \]
\[ D^\beta y(t) = tx(t), \quad 0 < \beta \leq 1, \]
\[ D^\gamma z(t) = y(t)z(t), \quad 0 < \gamma \leq 1, \]

along with the I.C’s
\[ x(0) = 0; \quad y(0) = 1; \quad z(0) = 1. \]  

Employ the fractional N-transform algorithm on equation (20), one can conclude:
\[ N^+ [D^\alpha x(t)] = 2N^+ [y^2(t)], \]
\[ N^+ [D^\beta y(t)] = N^+ [tx(t)], \]
\[ N^+ [D^\gamma z(t)] = N^+ [y(t)z(t)]. \]
so equation (22) becomes

\[
\frac{s^a}{\mu^a} \mathbb{N}^+ [x(t)] - \sum_{k=0}^{n-1} \frac{s^{a-k}}{\mu^{a-k}} [D^k x(t)]_{t=0} = 2N^+ [y^2(t)],
\]

\[
\frac{s^b}{\mu^b} \mathbb{N}^+ [y(t)] - \sum_{k=0}^{n-1} \frac{s^{b-k}}{\mu^{b-k}} [D^k y(t)]_{t=0} = N^+ [tx(t)],
\]

\[
\frac{s^c}{\mu^c} \mathbb{N}^+ [z(t)] - \sum_{k=0}^{n-1} \frac{s^{c-k}}{\mu^{c-k}} [D^k z(t)]_{t=0} = N^+ [y(t)z(t)].
\]

(23)

Combine equations (23) and (21), to accomplish

\[
\mathbb{N}^+ [x(t)] = \frac{2\mu^a}{s^a} \mathbb{N}^+ [y^2(t)],
\]

\[
\mathbb{N}^+ [y(t)] = \frac{1}{s} + \frac{\mu^b}{s^b} \mathbb{N}^+ [tx(t)],
\]

\[
\mathbb{N}^+ [z(t)] = \frac{1}{s} + \frac{\mu^c}{s^c} \mathbb{N}^+ [y(t)z(t)].
\]

(24)

Now, we take the \( N^{-1} \) of the above equation to obtain

\[
x(t) = 2N^{-1} \left[ \frac{\mu^a}{s^a} \mathbb{N}^+ [y^2(t)] \right],
\]

\[
y(t) = 1 + N^{-1} \left[ \frac{\mu^b}{s^b} \mathbb{N}^+ [tx(t)] \right],
\]

\[
z(t) = 1 + N^{-1} \left[ \frac{\mu^c}{s^c} \mathbb{N}^+ [y(t)z(t)] \right].
\]

(25)

Consequently, let us assume the infinite series solutions for \( x(t), y(t) \) and \( z(t) \) which can be written in the forms

\[
x(t) = \sum_{n=0}^{\infty} x_n(t), \quad y(t) = \sum_{n=0}^{\infty} y_n(t), \quad z(t) = \sum_{n=0}^{\infty} z_n(t),
\]

(26)

and the nonlinear terms \( y^2(t) \) and \( y(t)z(t) \) as

\[
y^2(t) = \sum_{n=0}^{\infty} A_n(t), \quad y(t)z(t) = \sum_{n=0}^{\infty} B_n(t).
\]

(27)

If we combine equations (26) and (27), then equation (25) implies

\[
\sum_{n=0}^{\infty} x_n(t) = 2N^{-1} \left[ \frac{\mu^a}{s^a} \mathbb{N}^+ \left[ \sum_{n=0}^{\infty} A_n(t) \right] \right], \quad n \geq 0,
\]

(28)

\[
\sum_{n=0}^{\infty} y_n(t) = 1 + N^{-1} \left[ \frac{\mu^b}{s^b} \mathbb{N}^+ \left[ \sum_{n=0}^{\infty} t x_n(t) \right] \right], \quad n \geq 0,
\]

\[
\sum_{n=0}^{\infty} z_n(t) = 1 + N^{-1} \left[ \frac{\mu^c}{s^c} \mathbb{N}^+ \left[ \sum_{n=0}^{\infty} B_n(t) \right] \right], \quad n \geq 0.
\]

Now looking at equation (28), one can come up with the recursive relation

\[
x_0(0) = 0, \quad y_0(0) = 1, \quad z_0(0) = 1.
\]

Thus,
\[ x_1(t) = 2N^{-1} \left[ \frac{\alpha^\gamma}{\nu^\gamma N^+} A_0(t) \right] = 2N^{-1} \left[ \frac{\nu^\gamma}{\alpha^\gamma N^+} \left( y_0 \right)^2 \right] = \frac{2t^\alpha}{\Gamma(\alpha + 1)}, \]

\[ y_1(t) = N^{-1} \left[ \frac{\beta^\gamma}{\nu^\gamma N^+} [tx_0(t)] \right] = 0, \]

\[ z_1(t) = N^{-1} \left[ \frac{\nu^\gamma}{\beta^\gamma N^+} [B_0(t)] \right] = N^{-1} \left[ \frac{\nu^\gamma}{\beta^\gamma N^+} [y_0 z_0] \right] = \frac{t^\gamma}{\Gamma(\gamma + 1)}, \]

and

\[ x_2(t) = 2N^{-1} \left[ \frac{\alpha^\gamma}{\nu^\gamma N^+} [A_1(t)] \right] = 2N^{-1} \left[ \frac{\nu^\gamma}{\alpha^\gamma N^+} [2y_0 y_1] \right] = 0, \]

\[ y_2(t) = N^{-1} \left[ \frac{\beta^\gamma}{\nu^\gamma N^+} [x_1(t)] \right] = N^{-1} \left[ \frac{\nu^\gamma}{\beta^\gamma N^+} \left( \frac{2t^\alpha + 1}{\Gamma(\alpha + 1)} \right) \right] = \frac{2(\alpha + 1)t^\alpha + \beta + 1}{\Gamma(\alpha + \beta + 2)}, \]

\[ z_2(t) = N^{-1} \left[ \frac{\nu^\gamma}{\beta^\gamma N^+} [B_1(t)] \right] = N^{-1} \left[ \frac{\nu^\gamma}{\beta^\gamma N^+} [y_1 z_0 + y_0 z_1] \right] = \frac{t^\gamma}{\Gamma(\gamma + 1)} = \frac{t^\gamma}{\Gamma(2\gamma + 1)}. \]

Subsequently, the 3rd approximate solutions can be obtained as

\[ x_3(t) = \frac{8(\alpha + 1)t^2\alpha + \beta + 1}{\Gamma(2\alpha + \beta + 2)}, \]

\[ y_3(t) = 0, \]

\[ z_3(t) = \frac{2(\alpha + 1)t^\alpha + \beta + \gamma + 1}{\Gamma(\alpha + \beta + \gamma + 2)} + \frac{t^\gamma}{\Gamma(3\gamma + 1)}. \]

Proceeding in this way, one comes up with these approximate solutions

\[ x(t) = \frac{2t^\alpha}{\Gamma(\alpha + 1)} + \frac{8(\alpha + 1)t^2\alpha + \beta + 1}{\Gamma(2\alpha + \beta + 2)} + \ldots, \]

\[ y(t) = 1 + \frac{2(\alpha + 1)t^\alpha + \beta + 1}{\Gamma(\alpha + \beta + 2)} + \frac{8(\alpha + 1)(2\alpha + \beta + 2)t^2\alpha + 2\beta + 2}{\Gamma(2\alpha + 2\beta + 3)} + \ldots, \]

\[ z(t) = 1 + \frac{t^\gamma}{\Gamma(\gamma + 1)} + \frac{t^\gamma}{\Gamma(2\gamma + 1)} + \frac{2(\alpha + 1)t^\alpha + \beta + \gamma + 1}{\Gamma(\alpha + \beta + \gamma + 2)} + \frac{t^\gamma}{\Gamma(3\gamma + 1)} + \ldots. \]

For graphical representation, 2D plots of approximate solutions \( x(t), y(t) \) and \( z(t) \) are respectively presented in

Figures 7 to 9 for \( 0 \leq t \leq 1.5 \) and different fractional order \( \alpha, \beta, \gamma = \{0.6, 0.75, 0.9, 1\} \).

Clearly, from Figures 7 to 9, the FDM exact and approximation solutions are in close agreement for various values of fractional order. Hence, the approximate solution is convergent fast enough to the exact solution.

**Conclusion**

We successfully employed the The fractional decomposition method and we obtained analytical approximate and exact solutions for two time-fractional order nonlinear systems. We were being able to find exact solutions to nonlinear system of coupled time-fractional Boussinesq–Burger equation. To the best of our knowledge, we are the first to find such exact solutions for
the proposed systems. Since the exact solutions of most FDE’s cannot be found easily, then analytical and numerical methods like (FDM) can be used more often. In all cases, the (FDM) provided us with exact solutions in the case when \( \gamma = 1 \). The results showed that (FDM) is simple and easy mathematical technique to accomplish exact and numerical solutions of nonlinear time-fractional equations. Finally, one can conclude the FDM can be employ to investigate and study numerous applications of fractional differential equations which usually shows up in many areas of Physics and engineering.

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**ORCID iD**

Mahmoud S Alrawashdeh  https://orcid.org/0000-0003-1543-7443

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