Functional Linear Regression of CDFs

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Abstract

The estimation of cumulative distribution functions (CDF) is an important learning task with a great variety of downstream applications, e.g., risk assessments in predictions and decision making. We study functional regression of contextual CDFs where each data point is sampled from a linear combination of context dependent CDF bases. We propose estimation methods that estimate CDFs accurately everywhere. In particular, given $n$ samples with $d$ bases, we show estimation error upper bounds of $O(d/n)$ for fixed design, random design, and adversarial context cases. We also derive matching information theoretic lower bounds, establishing minimax optimality for CDF functional regression. To complete our study, we consider agnostic settings where there is a mismatch in the data generation process. We characterize the error of the proposed estimator in terms of the mismatched error, and show that the estimator is well-behaved under model mismatch.

1 Introduction

Estimating cumulative distribution functions (CDF) of random variables is a salient theoretical problem that underlies the study of many real-world phenomena. For example, Huang et al. [2021] recently showed that estimating CDFs is sufficient for risk assessment, thereby making CDF estimation a key building block for such decision-making problems. In a similar vein, it is known that CDFs can also be used to directly compute distorted risk functions in [Wirch and Hardy, 2001], coherent risks in [Artzner et al., 1999], spectral risks in [Acerbi, 2002], value-at-risk, conditional value-at-risk, and mean-variance in [Cassel et al., Sani et al., 2013, Vakili and Zhao, 2015, Zimin et al., 2014], and cumulative prospect theory risks in [Prashanth et al., 2016]. Furthermore, CDFs are also useful in calculating various risk functionals appearing in insurance premium design, portfolio design, behavioral economics, behavioral finance, and healthcare applications, cf. [Sharpe, 1966, Rockafellar et al., 2000, Krokhmal, 2007, Shapiro et al., 2014, Acerbi, 2002, Prashanth et al., 2016, Jie et al., 2018]. Given the broad utility of estimating CDFs, there is a vast (and fairly classical) literature that tries to understand this problem.

In particular, the renowned Glivenko–Cantelli theorem [Cantelli, 1933, Glivenko, 1933] (also
known as the fundamental theorem of mathematical statistics [Devroye et al., 2013]) states that given some independent samples of a random variable, one can construct a consistent estimator for its CDF. Tight non-asymptotic sample complexity rates for such estimation using the Kolmogorov-Smirnov (KS) distance as the loss have also been established in the literature [Cantelli, 1933, Glivenko, 1933, Dvoretzky et al., 1956, Massart, 1990]. However, these results are all limited to the setting of a single random variable. In contrast, many modern learning problems, such as doubly-robust estimators in contextual bandits, treatment effects, and Markov decision processes [Huang et al., 2021, Kallus et al., 2019, Huang et al., 2022], require us to simultaneously learn the CDFs of potentially infinitely many random variables from limited data. Hence, the classical results on CDF estimation do not address the needs of such emerging learning applications.

**Contributions** In this work, as a first step towards developing general CDF estimation methods that fulfill the needs of the aforementioned learning problems, we study functional linear regression of CDFs, where samples are generated from CDFs that are linear (or convex) combinations of context-dependent CDF bases. As our main contribution, we define both least-squares regression and ridge regression estimators for the unknown linear weight parameter, and establish corresponding estimation error bounds for the fixed design, random design, adversarial, and self-normalized settings. In particular, given \( n \) samples with \( d \) CDF bases, we prove estimation error upper bounds that scale like \( \tilde{O}(\sqrt{d/n}) \) (neglecting sub-dominant factors). Our results achieve the same problem-dependent scaling as in canonical finite dimensional linear regression [Abbasi-Yadkori et al., 2011b,a, Peña et al., 2008, Hsu et al., 2012b]. On the other hand, our results specialize the functional regression setting of [Benatia et al., 2017, Wang et al., 2020] to CDF estimation, where minimal assumptions are made on the data generation process. Moreover, we also derive \( \Omega(\sqrt{d/n}) \) information theoretic lower bounds for functional linear regression of CDFs. This establishes minimax estimation rates of \( \tilde{O}(\sqrt{d/n}) \) for the CDF functional regression problem. We later show that this result directly implies the concentration of CDFs in KS distance. To complete our study, we consider agnostic settings where there is a mismatch between our linear model and the actual data generation process. We characterize the estimation error of the proposed estimator in terms of the mismatch error, and demonstrate that the estimator is well-behaved under model mismatch. Finally, we present numerical simulation results for a few synthetic and controlled experiments to illustrate the performance of our estimation methods.

It is worth mentioning that a complementary approach to the proposed CDF regression framework is quantile regression [Koenker and Bassett Jr, 1978]. Although quantile regression may appear to be closely related to CDF regression at first glance, the two problems have very different flavors. Indeed, unlike CDFs, quantiles are not sufficient for law invariant risk assessment. Furthermore, due to their infinite range, quantile estimation is quite challenging, resulting in analyses that only consider pointwise estimation [Takeuchi et al., 2006]. Perhaps more importantly, quantile regression can be ill-posed in many machine learning settings. For example, quantiles are not estimatable in decision-making problems and games with mixed random variables (which take both discrete and continuous values). For these reasons, our focus in this paper will be on CDF regression.
2 Preliminaries

Notation  Let $\mathbb{N}$ denote the set of positive integers. For any $n \in \mathbb{N}$, let $[n]$ denote the set \{1, \ldots, n\}. For any probability space $(\Omega, \mathcal{F}, \mathbb{P})$, define the Hilbert space $L^2(\Omega, \mathbb{P}) := \{f : \Omega \to \mathbb{R} \mid \int_\Omega |f|^2 \mathbb{P} < \infty\}$ with $L^2$-norm $\|f\|_{L^2(\Omega, \mathbb{P})} := \sqrt{\int_\Omega |f|^2 \mathbb{P}}$ for $f \in L^2(\Omega, \mathbb{P})$. For any positive definite matrix $A \in \mathbb{R}^{d \times d}$, define $\|\cdot\|_A$ to be the weighted $\ell^2$-norm in $\mathbb{R}^d$ induced by $A$, i.e., $\|x\|_A = \sqrt{x^\top A x}$ for $x \in \mathbb{R}^d$. For the standard Euclidean (or $\ell^2$-) norm $\|\cdot\|_{\ell_2}$, we omit the subscript $I_d$, where $I_d$ denotes the $d \times d$ identity matrix. Let $\mu_{\min}(A)$ denote the smallest eigenvalue of $A$. Let $\text{KS}(F_1, F_2) := \sup_{x \in \mathbb{R}} |F_1(x) - F_2(x)|$ denote the KS distance for two CDFs $F_1$ and $F_2$. Finally, let $1\{\cdot\}$ denote the indicator function.

Problem setup  In this paper, we consider the problem of functional linear regression of CDFs. To define this problem, let $\mathcal{X}$ denote the context space, and let $F(x, \cdot) : \mathbb{R} \to [0, 1]$ be the CDF of some $\mathbb{R}$-valued random variable for any $x \in \mathcal{X}$. We assume that $\mathcal{X}$ is a Polish space throughout the paper. For a context $x \in \mathcal{X}$, we observe a sample $y$ from its corresponding CDF $F(x, \cdot)$. We next summarize two schemes to generate $(x, y)$ samples:

- **Scheme I (Adversarial).** For each $j \in \mathbb{N}$, an adversary picks $x^{(j)} \in \mathcal{X}$ (either deterministically or randomly) in an adaptive way given knowledge of the previous $y^{(i)}$’s for $i < j$, and then $y^{(j)} \in \mathbb{R}$ is sampled from $F(x^{(j)}, \cdot)$. This includes the canonical fixed design setting as a special case, where all $x^{(j)}$’s are fixed a priori without knowledge of $y^{(j)}$’s.

- **Scheme II (Random).** For each $j \in \mathbb{N}$, $x^{(j)} \in \mathcal{X}$ is sampled from some probability distribution $P_x^{(j)}$ on $\mathcal{X}$ independently, and then $y^{(j)} \in \mathbb{R}$ is sampled from $F(x^{(j)}, \cdot)$ independently. This is commonly known as the random design setting in the regression context.

Note that although the random design setting in Scheme II is a special case of Scheme I, we emphasize it because it has specific properties that deserve a separate treatment. The adversarial setting in Scheme I is more general than what is typically considered for regression, and our corresponding self-normalized analysis has several potential future applications in risk assessment for reinforcement learning, e.g., in contextual bandits [Abbasi-Yadkori et al., 2011a].

The task of CDF regression is to recover $F$ from a sample $\{(x^{(j)}, y^{(j)})\}_{j \in [n]}$ of size $n$. To do this, we consider a linear model for $F$. Let $d$ be a fixed positive integer. For each $i \in [d]$ and $x \in \mathcal{X}$, let $\phi_i(x, \cdot) : \mathbb{R} \to [0, 1]$ be a feature function that is a CDF of a $\mathbb{R}$-valued random variable with range contained in some Borel set $S \subseteq \mathbb{R}$, and assume that $\phi_i$ is measurable. Then, we define the vector-valued function $\Phi : \mathcal{X} \times \mathbb{R} \to [0, 1]^d$, $\Phi(x, t) = [\phi_1(x, t), \ldots, \phi_d(x, t)]^\top$. We assume that there exists some unknown $\theta^* \in \Delta^{d-1}$, where $\Delta^{d-1} := \{(\theta_1, \ldots, \theta_d) \in \mathbb{R}^d : \sum_{i=1}^d \theta_i = 1, \theta_i \geq 0 \text{ for } 1 \leq i \leq d\}$ denotes the probability...
simplex in $\mathbb{R}^d$, such that
\[
F(x,t) = \theta_\ast^\top \Phi(x,t), \quad \forall \ x \in \mathcal{X}, t \in \mathbb{R}.
\]
(1)

Thus, we can view $\Phi$ as a “basis” for contextual CDF learning. (Note that due to (1), for every $x \in \mathcal{X}$, the random variable with CDF $F(x, \cdot)$ also takes values in $S$.)

As explained in the sampling schemes above, given $x^{(j)}$ at the $j$th sample, the observation $y^{(j)}$ is generated according to the CDF $F(x^{(j)}, \cdot) = \theta_\ast^\top \Phi(x^{(j)}, \cdot)$. For notational convenience, we will often refer to the vector-valued function $\Phi(x^{(j)}, \cdot)$ as $\Phi_j(\cdot)$ for all $j \in [n]$, so that $F(x^{(j)}, \cdot) = \theta_\ast^\top \Phi_j(\cdot)$. Under the linear model in (1), our goal is to estimate the unknown parameter $\theta_\ast$ from the sample $\{(x^{(j)}, y^{(j)})\}_{j \in [n]}$ in a (regularized) least-squares error sense. This in turn recovers the contextual CDF function $F$.

### 3 Results on CDF functional linear regression

In this section, we propose an estimation paradigm for the a priori unknown parameter $\theta_\ast$, derive both upper and lower bounds on the associated estimation error, and then establish bounds in the mismatched model setting. We begin by formally stating our least-squares functional regression optimization problem to learn $\theta_\ast$.

Given a probability measure $m$ on $S$, the sample $\{(x^{(j)}, y^{(j)})\}_{j \in [n]}$, and the set of basis functions $\{\Phi_j\}_{j \in [n]}$, we propose to estimate $\theta_\ast$ by minimizing the (ridge or) $\ell^2$-regularized squared $L^2(S, m)$-distance between the estimated and empirical CDFs as follows:

\[
\hat{\theta}_\lambda := \arg \min_{\theta \in \mathbb{R}^d} \sum_{j=1}^{n} \|I_{y^{(j)}} - \theta^\top \Phi_j\|_{L^2(S, m)}^2 + \lambda \|\theta\|^2,
\]

(2)

where $\lambda \geq 0$ is the hyper-parameter that determines the level of regularization, and the function observation $I_{y^{(j)}}(t) := 1\{y^{(j)} \leq t\}$ is an empirical CDF of $y^{(j)}$ that forms an unbiased estimator for $F(x^{(j)}, \cdot)$ conditioned on past contexts and observations. Hence, in Scheme I, we only require that $I_{y^{(j)}} - \theta^\top \Phi_j$ is a zero-mean function given past contexts and observations, making our analysis suitable for online learning problems where the later contexts can depend on the past contexts and observations. Notice further that $\hat{\theta}_\lambda$ in (2) is an improper estimator since it may not lie in $\Delta^{d-1}$. However, since $\Delta^{d-1}$ is compact in $\mathbb{R}^d$, $\theta_\ast := \arg \min_{\theta \in \Delta^{d-1}} \|\theta - \hat{\theta}_\lambda\|_A$ exists for any positive definite $A \in \mathbb{R}^{d \times d}$. Moreover, since $\Delta^{d-1}$ is also convex, we have $\|\hat{\theta}_\lambda - \theta\|_A \leq \|\hat{\theta}_\lambda - \theta_\ast\|_A$ [Beck, 2014, Theorem 9.9]. This means that an upper bound on $\|\hat{\theta}_\lambda - \theta_\ast\|_A$ is also an upper bound on $\|\hat{\theta}_\lambda - \theta\|_A$. Therefore, we focus our analysis on the improper estimator $\hat{\theta}_\lambda$, and note that its projection onto $\Delta^{d-1}$ yields an estimator $\hat{\theta}_\lambda$ for which the same upper bounds hold.

When $\lambda > 0$, the objective function in (2) is a $(2\lambda)$-strongly convex function of $\theta \in \mathbb{R}^d$ (see, e.g., Bertsekas et al. [2003] for the definition), and is uniquely minimized at

\[
\hat{\theta}_\lambda = \left( \sum_{j=1}^{n} \int_S \Phi_j \Phi_j^\top dm + \lambda I_d \right)^{-1} \left( \sum_{j=1}^{n} \int_S I_{y^{(j)}} \Phi_j dm \right).
\]

(3)
For the unregularized case where \( \lambda = 0 \), we omit the subscript \( \lambda \) and write \( \hat{\theta} \) to denote a corresponding estimator in (2). Note that when \( \lambda = 0 \), if \( \mu_{\min}(\sum_{j=1}^{n} \Phi_j^T \Phi_j \, dm(d\mu)) > 0 \), the objective function in (2) is still strongly convex, and is uniquely minimized at \( \hat{\theta} \) given in (3) with \( \lambda = 0 \). In practice, one can deploy standard numerical methods to compute the integral in (3), and the computational complexity of the matrix inversion is cubic in the dimension \( d \). However, iterative methods can be used for better dimension dependence in the running time.

### 3.1 Upper bounds on estimation error

When the sample is generated according to Scheme I, we prove self-normalized upper bounds on the error term \( \hat{\theta}_\lambda - \theta_* \). For any probability measure \( m \) on \( S \), define \( U_n := \sum_{j=1}^{n} \Phi_j^T \Phi_j \, dm(d\mu) \) and \( U_n(\lambda) = U_n + \lambda I_d \) for \( n \in \mathbb{N} \) and \( \lambda \geq 0 \). Moreover, for \( n, N, d \in \mathbb{N} \), \( \delta \in (0, 1) \), \( \lambda, \tau \in (0, \infty) \), \( \theta \in \mathbb{R}^d \), and \( A \in \mathbb{R}^{d \times d} \) positive definite, define

\[
\varepsilon_\lambda(n, d, \delta) := \sqrt{d \log \left( 1 + \frac{n}{\lambda} \right) + 2 \log \frac{1}{\delta}} + \sqrt{\lambda \|\theta_*\|}
\]

and

\[
\varepsilon(n, d, \delta, \tau) := \frac{\sqrt{d + \sqrt{8d \log \frac{1}{\delta} + \frac{4}{3} \sqrt{d/n} \log \frac{1}{\tau}}}}{\sqrt{\tau}}
\]

Using these definitions, the next theorem states our self-normalized upper bounds on the estimation error.

**Theorem 1** (Self-normalized bound in adversarial setting). Assume \( m \) is a probability measure on \( S \) and \( \{(x^{(j)}, y^{(j)})\}_{j \in \mathbb{N}} \) is sampled according to Scheme I with \( F \) defined in (1). For any \( \lambda > 0 \) and \( \delta \in (0, 1) \), with probability at least \( 1 - \delta \), the estimator defined in (2) satisfies

\[
\|\hat{\theta}_\lambda - \theta_*\|_{U_n(\lambda)} \leq \varepsilon_\lambda(n, d, \delta)
\]

for all \( n \in \mathbb{N} \).

Moreover, for unregularized estimator, we have the following result.

**Proposition 2** (Self-normalized bound in adversarial setting for unregularized estimator). Under the same assumption as Theorem 1, if \( U_N \) is positive definite for some fixed \( N \in \mathbb{N} \), then for any \( \delta \in (0, 1) \) and \( n \geq N \), with probability at least \( 1 - \delta \), the estimator defined in (2) with \( \lambda = 0 \) satisfies

\[
\|\hat{\theta} - \theta_*\|_{U_n} \leq \varepsilon(n, d, \delta, \mu_{\min}(U_n)/n).
\]

The proofs of Theorem 1 and Proposition 2 are provided in Appendix A. Informally, Theorem 1 and Proposition 2 convey that with high probability, the self-normalized errors \( \|\hat{\theta}_\lambda - \theta_*\|_{U_n(\lambda)} \) and \( \|\hat{\theta} - \theta_*\|_{U_n} \) scale as \( O(\sqrt{d}) \) in the \( \ell^2 \)-regularized and unregularized cases, where \( O(\cdot) \) ignores logarithmic and other sub-dominant factors. We note that Theorem 1 and Proposition 2 also
imply upper bounds on the (un-normalized) error $\| \hat{\theta}_\lambda - \theta_* \|$. Indeed, for any positive definite matrix $A \in \mathbb{R}^{d \times d}$ and vector $a \in \mathbb{R}^d$, we have $\| a \| \leq \mu_{\min}(A)^{-1/2} \| a \|_A$. Thus, for example, (6) in Theorem 1 implies that

$$\| \hat{\theta}_\lambda - \theta_* \| \leq \mu_{\min}(U_n(\lambda))^{-1/2}\varepsilon_\lambda(n, d, \delta) = \tilde{O}\left(\sqrt{d/n}\right)$$

with high probability when $\mu_{\min}(U_n)$ grows linearly with $n$.

The key idea in the proof of Theorem 1 is to first notice that $\hat{\theta}_\lambda = U_n(\lambda)^{-1}W_n - U_n(\lambda)^{-1}(\lambda \theta_*)$, where $W_n := \sum_{j=1}^n \int_S (I_{y(j)} \Phi_j - \theta_*^T \Phi_j) \, dm$. We next show that $\{ \overline{M}_n \}_{n \geq 0}$ where

$$\overline{M}_n := \frac{\lambda^{d/2}}{\det(U_n(\lambda))^{1/2}} \exp\left( \frac{1}{2} \frac{\| W_n \|^2_{U_n(\lambda)^{-1}}} \right)$$

is a super-martingale. Doob’s maximal inequality for super-martingales can then be used in conjunction with extra developed algebra steps to establish (6).

To prove Proposition 2, we use a vector Bernstein inequality for bounded martingale difference sequence [Hsu et al., 2012a, Proposition 1.2] to show a high probability upper bound for $\| W_n \|$. Note that $U_N$ is positive definite implies that $U_n$ is also positive definite for any $n \geq N$. Since $\| \hat{\theta} - \theta_* \|_{U_n} = \| W_n \|_{U_n^{-1}} \leq \| W_n \| / \sqrt{\mu_{\min}(U_n)}$, we are able to establish (7).

Since the fixed design setting is a special case of the adversarial setting, Theorem 1 and Proposition 2 immediately imply the same $\tilde{O}(\sqrt{d})$-style upper bounds as a corollary in the fixed design setting.

**Corollary 3 (Self-normalized bound in fixed design setting).** For an arbitrary probability measure $m$ on $S$ and an arbitrary sequence $\{ x^{(j)} \}_{j \in \mathbb{N}} \in X^\mathbb{N}$, assume that $y^{(j)}$ is sampled from $F(x^{(j)}, \cdot)$ independently for each $j \in \mathbb{N}$ with $F$ defined in (1). For any $\lambda > 0$ and $\delta \in (0, 1)$, with probability at least $1 - \delta$, the estimator defined in (2) satisfies (6) for all $n \in \mathbb{N}$.

If $U_N$ is positive definite for some fixed $N \in \mathbb{N}$, then for any $\delta \in (0, 1)$ and $n \geq N$, with probability at least $1 - \delta$, the estimator defined in (2) with $\lambda = 0$ satisfies (7).

The proof is the same as those of Theorem 1 and Proposition 2 and is therefore omitted.

Furthermore, based on Theorem 1 and Proposition 2, we prove self-normalized upper bounds on the estimation error when the sample is generated under Scheme II, which corresponds to the random design setting in linear regression. For convenience, for any probability measure $m$ on $S \subseteq \mathbb{R}$, define $\Sigma^{(j)} := \mathbb{E}_{x^{(j)} \in P^{(j)}} \left[ \sum_j \Phi_j \Phi_j^T \right] \, dm$ and $\Sigma_n := \sum_{j=1}^n \Sigma^{(j)}$ for $j, n \in \mathbb{N}$.

**Theorem 4 (Self-normalized bound in random setting).** Assume $m$ is a probability measure on $S$, $\{(x^{(j)}, y^{(j)})\}_{j \in \mathbb{N}}$ is sampled according to Scheme II with $F$ defined in (1), and $\mu_{\min}(\Sigma^{(j)}) \geq \sigma_{\min}$ for some $\sigma_{\min} > 0$ and all $j \in \mathbb{N}$. For any $\delta \in (0, 1/2)$ and $n \geq \frac{32d^2}{\sigma_{\min}^2} \log \frac{d}{\delta}$, with probability at least $1 - 2\delta$, the estimator defined in (2) with $\lambda = 0$ satisfies

$$\| \hat{\theta} - \theta_* \|_{\Sigma_n} \leq 2\varepsilon(n, d, \delta, \sigma_{\min}).$$

Moreover, for regularized estimator, we have the following result.
Proposition 5 (Self-normalized bound in random setting for regularized estimator). Under the same assumptions as Theorem 4, for any \( \lambda > 0, \delta \in (0, 1/2) \), and \( n \geq \frac{32d^2 \log(\frac{2}{\delta})}{\delta^2} \), with probability at least \( 1 - 2\delta \), the estimator defined in (2) satisfies

\[
\left\| \hat{\theta}_\lambda - \theta_* \right\|_{\Sigma_n} \leq \sqrt{2} \varepsilon_\lambda(n, d, \delta).
\] (9)

The proofs of Theorem 4 and Proposition 5 are given in Appendix B. As before, they convey that in the random design setting, the self-normalized errors \( \left\| \hat{\theta}_\lambda - \theta_* \right\|_{\Sigma_n} \) and \( \left\| \hat{\theta} - \theta_* \right\|_{\Sigma_n} \) scale as \( \tilde{O}(\sqrt{d/n}) \) with high probability in the \( \ell^2 \)-regularized and unregularized cases. Moreover, we once again note that Theorem 4 and Proposition 5 imply upper bounds on the (un-normalized) error \( \left\| \hat{\theta}_\lambda - \theta_* \right\| \). For example, (8) implies that \( \left\| \hat{\theta} - \theta_* \right\| \leq 2\mu_{\min}(\Sigma_n)^{-1/2} \varepsilon(n, d, \delta, \sigma_{\min}) = \tilde{O}(\sqrt{d/n}) \) with high probability since \( \mu_{\min}(\Sigma_n) \rangle n\sigma_{\min} \) by Weyl’s inequality [Weyl, 1912].

The main idea in the proofs of Theorem 4 and Proposition 5 is to establish a high probability lower bound on \( \mu_{\min}(\Delta_n) \), where \( \Delta_n := \Sigma_n^{-1/2} (U_n - \Sigma_n) \Sigma_n^{-1/2} \). This can be achieved using the matrix Hoeffding’s inequality [Tropp, 2012, Theorem 1.3]. Then, we show that for any \( \lambda \geq 0 \),

\[
\left\| \hat{\theta}_\lambda - \theta_* \right\|_{\Sigma_n} \leq (1 + \mu_{\min}(\Delta_n))^{-1/2} \left\| \hat{\theta}_\lambda - \theta_* \right\|_{I_n(\lambda)}.
\]

For Theorem 4, we prove that \( \mu_{\min}(U_n) \geq \mu_{\min}(\Sigma_n)(\mu_{\min}(\Delta_n) + 1) \). Then, we can lower bound \( \mu_{\min}(U_n) \) in (7) by some factor times \( \mu_{\min}(\Sigma_n) \) with high probability. Thus, (8) follows from (7) and the high probability lower bound on \( \mu_{\min}(\Delta_n) \). For Proposition 5, (9) immediately follows from (6) and the high probability lower bound on \( \mu_{\min}(\Delta_n) \).

At this stage, it is worth remarking upon the choice of measure \( m \) used in the results in this section. In order for the estimator in (2) to be well-defined, since \( I_y(t), \theta^T \Phi(x, t) \in [0, 1] \) for any \( t, y \in \mathbb{R} \) and \( x \in X \), it suffices to ensure that \( m(S) < \infty \) (i.e., \( m \) is a finite measure). So, we choose to normalize the measure \( m \) and set \( m(S) = 1 \). This is the reason why we restrict \( m \) to be a probability measure on \( S \). Furthermore, the probability measure \( m \) can in general be chosen to adapt to specific problem settings. For example, the uniform measure \( m_U \) on \( S \) is often easy to compute for some choices of \( S \). Specifically, if \( 0 < \text{Leb}(S) < \infty \), where \( \text{Leb} \) denotes the Lebesgue measure, \( m_U \) is defined by \( \frac{dm_U}{d\text{Leb}} = \frac{1}{\text{Leb}(S)} \), where \( \frac{dm_U}{d\text{Leb}} \) is the Radon-Nikodym derivative. If \( S \) is a finite set with cardinality \( \#S \), \( m_U = \frac{1}{\#S} \sum_{s \in S} \delta_s \), where \( \delta_s \) denotes the Dirac measure at \( s \). On the other hand, when \( S = \mathbb{R} \), \( m \) can be set to the Gaussian measure \( \gamma_{c, \sigma^2} \) defined by \( \gamma_{c, \sigma^2}(dx) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-c)^2/2\sigma^2} dx \) with \( c \in \mathbb{R} \) and \( \sigma^2 > 0 \).

We briefly compare our results in this section with related results in the literature. In the (canonical, finite dimensional) adversarial linear regression setting, Abbasi-Yadkori et al. [2011b] show an \( \tilde{O}(\sqrt{d}) \) upper bound on the self-normalized error of the (ridge or) \( \ell^2 \)-regularized least-squares estimator. Our functional regression upper bound in (6) in Theorem 1 matches this scaling as shown earlier (neglecting sub-dominant factors). In the (canonical, finite dimensional) random design linear regression setting, Hsu et al. [2012b] show an \( \tilde{O}(\sqrt{d}) \) upper bound on the self-normalized error of the unregularized least-squares estimator under some conditions on the distribution of the covariates. Our functional regression upper bound in (8) in Theorem 4 for the unregularized case also matches the scaling in [Hsu et al., 2012b] (neglecting...
Theorem 6

where the infimum is over all (possibly randomized) estimators

To show that our estimator (2) is minimax optimal, we prove information theoretic lower

where we use the Cauchy-Schwarz inequality and the fact that

Finally, we note that an upper bound on \( \| \hat{\theta}_\lambda - \theta_* \| \) immediately implies an upper bound on the KS distance between our estimated CDF and the true one. Let \( \hat{F}_\lambda(x, \cdot) := \hat{\theta}_\lambda^\top \Phi(x, \cdot) \) denote the estimated CDF for any \( x \in X \). Then, under the linear model (1), we have

\[
\text{KS}(\hat{F}_\lambda(x, \cdot), F(x, \cdot)) = \sup_{t \in S} |(\hat{\theta}_\lambda - \theta_*)^\top \Phi(x, t)| \leq \| \hat{\theta}_\lambda - \theta_* \| \sup_{t \in S} \| \Phi(x, t) \| \leq \sqrt{d} \| \hat{\theta}_\lambda - \theta_* \| = \tilde{O}(d/\sqrt{n})
\]

where we use the Cauchy-Schwarz inequality and the fact that \( \| \hat{\theta}_\lambda - \theta_* \| = \tilde{O}(\sqrt{d/n}) \) (see discussion below Theorems 1 and 4).

3.2 Minimax lower bounds

To show that our estimator (2) is minimax optimal, we prove information theoretic lower bounds on the Euclidean norm of the estimation error for any estimator. Recall that for any distribution family \( \mathcal{Q} \) and (parameter) function \( \xi : \mathcal{Q} \to \mathbb{R}^d \), the minimax \( \ell^2 \)-risk is defined as

\[
\mathfrak{R}(\xi(\mathcal{Q})) := \inf_{\xi} \sup_{Q \in \mathcal{Q}} \mathbb{E}_{z \sim Q}[\| \hat{\xi}(z) - \xi(Q) \|], \quad (10)
\]

where the infimum is over all (possibly randomized) estimators \( \hat{\xi} \) of \( \xi \) based on a sample \( z \), and the supremum is over all distributions in the family \( \mathcal{Q} \). To specialize this definition for our problem, for any \( x \in X \) and \( \theta \in \mathbb{R}^d \), let \( P_{Y|x,\theta} \) denote the probability measure defined by the CDF \( \theta^\top \Phi(x, \cdot) \). Moreover, for any sequence \( x^{1:n} := (x^{(1)}, \ldots, x^{(n)}) \in X^n \), define the collection of product measures \( \mathcal{P}_{x^{1:n}} := \{ \otimes_{j=1}^n P_{Y|x^{(j)}, \theta} : \theta \in \Delta^{d-1} \} \), and for any distribution \( P \in \mathcal{P}_{x^{1:n}} \), let \( \theta(P) \) be a parameter in \( \Delta^{d-1} \) such that \( P = \otimes_{j=1}^n P_{Y|x^{(j)}, \theta} \). Then, we have the following theorem in the adversarial setting.

**Theorem 6** (Information theoretic lower bound in adversarial setting). For any sequence \( x^{1:n} = (x^{(1)}, \ldots, x^{(n)}) \in X^n \), we have

\[
\mathfrak{R}(\theta(\mathcal{P}_{x^{1:n}})) = \Omega \left( \sqrt{d/n} \right) \quad (11)
\]

The proof uses Fano’s method [Fano, 1961] and is given in Appendix C. Note that strictly speaking, the above theorem is written for the fixed design setting. However, a lower bound in the fixed design setting also implies the very same \( \Omega(\sqrt{d/n}) \) lower bound in adversarial setting. Furthermore, by our discussion below Theorem 1, (6) implies that in the adversarial setting, \( \mathbb{P}[\| \hat{\theta}_\lambda - \theta_* \|^2 \geq C_1 \frac{d}{n} \log(n) + C_2 \frac{1}{n} + C_3 \frac{r}{n}] \leq e^{-r} \) for \( r > 0 \) and some constants \( C_1, C_2, \ldots \),
and $C_3$, which immediately implies that $\mathbb{E}[\|\hat{\theta}_x - \theta_*\|] = \tilde{O}(\sqrt{d/n})$. Thus, our estimator (2) is minimax optimal with rate $\tilde{\Theta}(\sqrt{d/n})$ in the adversarial setting.

In the proof of Theorem 6, we construct a family of $\Omega(a/\sqrt{d})$-packing subsets of $\Delta^{d-1}$ for $a \in (0, 1)$ under $\ell^2$-distance. We then show that when $\phi_1, \ldots, \phi_d$ are the CDFs of $d$ Bernoulli distributions, for any $\theta^{(1)} \neq \theta^{(2)}$ in such a packing subset, the Kullback-Leibler (KL) divergence (see definition in Appendix C) satisfies $D(P_{Y|\{x,j\},\theta^{(1)}} \| P_{Y|\{x,j\},\theta^{(2)}}) = O(a^2/d)$ for any $j \in [n]$. Since the above family of Bernoulli distributions is a subset of $\mathcal{P}_{2^{1:n}}$, we are able to show that $\mathcal{R}(\theta(P_{2^{1:n}})) = \Omega(\sqrt{d/n})$ using Fano’s method and the aforementioned bound on KL divergence.

Next, to analyze minimax $\ell^2$-risk under the random setting, let $\mathcal{D}_X$ denote the set of all probability distributions on $\mathcal{X}$. For any $P_X \in \mathcal{D}_X$, let $P_XP_{Y|X,\theta}$ denote the joint distribution of $(X, Y)$ such that the marginal distribution of $X$ is $P_X$ and the conditional distribution of $Y$ given $X = x$ is $P_{Y|X,\theta}$. Define the distribution family $\mathcal{P} := \{ \otimes^n_{j=1} P^{(j)}_{X,Y|X,\theta} : \theta \in \mathbb{R}^d, P^{(j)}_{X} \in \mathcal{D}_X \}$, and for any $P \in \mathcal{P}$, let $\theta(P)$ denote the parameter in $\Delta^{d-1}$ such that $P = \otimes^n_{j=1} P^{(j)}_{X,Y|X,\theta}$. Clearly, for any $x^{1:n} \in \mathcal{X}^n$, we have $\{ \otimes^n_{j=1} \delta_{x^{(j)}} P_{Y|X,\theta} : \theta \in \Delta^{d-1} \} \subseteq \mathcal{P}$. Thus, each $P_{2^{1:n}}$ is a collection of marginal distributions of elements belonging to such subsets of $\mathcal{P}$. Then, by the definition of minimax $\ell^2$-risk, Theorem 6 immediately implies the following corollary.

**Corollary 7** (Information theoretic lower bound in random setting).

$$\mathcal{R}(\theta(P)) = \Omega\left(\sqrt{d/n}\right),$$

(12)

The detailed proof is given in Appendix D. The lower bound on the Euclidean norm of the estimation error is also $\Omega(\sqrt{d/n})$ in random setting. Again, by our discussion below Theorem 4, (8) implies that in random setting, $\mathbb{P}[\|\hat{\theta} - \theta_*\| \geq C_1 \sqrt{d/n} + C_2 \sqrt{r d/n} + C_3 r \sqrt{d/n}] \leq e^{-r}$ for $r > 0$ and some constants $C_1$, $C_2$, and $C_3$, which immediately implies that $\mathbb{E}[\|\hat{\theta} - \theta_*\|] = \tilde{O}(\sqrt{d/n})$. Thus, our estimator (2) is minimax optimal with rate $\tilde{\Theta}(\sqrt{d/n})$ in random setting.

### 3.3 Mismatched model

In general, a mismatch may exist between the true target function and our linear model (1) with basis $\Phi$. So, in analogy with canonical linear regression where additive Gaussian random variables are used to model the error term [Montgomery et al., 2021], we consider the following **mismatched model**:

$$F(x,t) = \theta_*^T \Phi(x,t) + e(x,t), \quad \forall \; x \in \mathcal{X}, \; t \in \mathbb{R},$$

(13)

where an additive error function depending on the context is included to model the mismatch in (1). Note that in (13), each $F(x, \cdot)$ is a CDF and $e : \mathcal{X} \times S \rightarrow [-1, 1]$ is a measurable function. One equivalent interpretation of (13) is as follows. Suppose that their exists another contextual CDF function $\phi_e$ such that $F(x, \cdot)$ is a mixture of the linear model $\theta_*^T \Phi(x, \cdot)$ and the new feature function $\phi_e(x, \cdot)$, i.e., for some $q \in [0, 1]$,

$$F(x,t) = (1-q)\theta_*^T \Phi(x,t) + q \phi_e(x,t) = \theta_*^T \Phi(x,t) + q \left( \phi_e(x,t) - \theta_*^T \Phi(x,t) \right), \quad \forall \; x \in \mathcal{X}, \; t \in \mathbb{R}.$$
Then, we naturally obtain an additive error function \( e(x, t) = q \left( \phi_c(x, t) - \theta^* \Phi(x, t) \right) \).

Given a sample \( \{(x^{(j)}, y^{(j)})\}_{j \in [n]} \) generated using the mismatched model (13), let \( e_j(t) \) denote \( e(x^{(j)}, t) \) for \( j \in [n] \). Moreover, define \( E_n := \sum_{j=1}^{n} \int_S e_j \Phi_j \, dm \) and \( B_n := \mathbb{E}[E_n] = \sum_{j=1}^{n} \mathbb{E} \left[ \int_S e_j \Phi_j \, dm \right] \). Then, we have the following theoretical guarantees for the task of estimating \( \theta^* \) using the estimator in (2) in the adversarial and random settings.

**Theorem 8** (Self-normalized bound in mismatched adversarial setting). Assume \( m \) is a probability measure on \( S \) and \( \{(x^{(j)}, y^{(j)})\}_{j \in \mathbb{N}} \) is sampled according to Scheme I with \( F \) defined in (13). For any \( \lambda > 0 \) and \( \delta \in (0, 1) \), with probability at least \( 1 - \delta \), the estimator defined in (2) satisfies

\[
\| \hat{\theta}_\lambda - \theta^* \|_{U_n(\lambda)} \leq \varepsilon_\lambda(n, d, \delta) + \frac{1}{\sqrt{\lambda}} \| E_n \| 
\]  

for all \( n \in \mathbb{N} \).

The proof of Theorem 8 follows the same approach as the proof of Theorem 1, and it is provided in Appendix E. Furthermore, Theorem 8 implies the following corollary.

**Corollary 9** (Self-normalized bound in mismatched random setting). Assume \( m \) is a probability measure on \( S \), \( \{(x^{(j)}, y^{(j)})\}_{j \in \mathbb{N}} \) is sampled according to Scheme II with \( F \) defined in (13), and \( \mu_{\min}(\Sigma^{(j)}) \geq \sigma_{\min} \) for some \( \sigma_{\min} > 0 \) and all \( j \in \mathbb{N} \). For any \( \lambda > 0 \), \( \delta \in (0, 1/2) \), and \( n \geq \frac{32d^2}{\sigma_{\min}^2} \log \left( \frac{d}{\delta} \right) \), with probability at least \( 1 - 2\delta \), the estimator defined in (2) satisfies

\[
\| \hat{\theta}_\lambda - \theta^* \|_{\Sigma_n} \leq \sqrt{2} \varepsilon_\lambda(n, d, \delta) + \sqrt{\frac{2}{\lambda}} \| B_n \|. 
\]  

The proof of Corollary 9 is given in Appendix F. It follows from the proofs of Theorem 8 and Proposition 5.

In the adversarial setting, comparing (14) in Theorem 8 with (6) in Theorem 1, we see that the effect of the additive error in the mismatched model is captured by the additional \( \| E_n \|/\sqrt{\lambda} \) term in our self-normalized error upper bound. Similarly, in the random setting, comparing (15) in Corollary 9 with (9) in Proposition 5, we again see that the effect of the additive error is captured by the additional \( \sqrt{2/\lambda} \| B_n \| \) term in the self-normalized error upper bound.

### 4 Numerical simulations

#### 4.1 Simulation studies

We first evaluate the performance of our estimator (2) on simulated samples. We consider the following feature functions:

\[
\phi_i(x, t) = \begin{cases} 
0, & \text{if } t < 0, \\
(xt)^{r(i)}, & \text{if } 0 \leq t \leq \frac{1}{2}, \\
1, & \text{otherwise}, 
\end{cases} 
\]  

(16)
The KS distance

(c) Real data experiment

3000

13

2000

KS distance

4

4

1000

3

(b) Simulation: dimension

˜

4000

11

error

log of error

log of error

ℓ2 error KS distance

Figure 1: Means and confidence intervals of estimation errors in simulation and real data experiments. These plots demonstrate that the estimation error of our proposed estimator scales desirably with the sample size and dimension.

where \(r(i) = i\) if \(1 \leq i \leq \frac{d+1}{2}\) and \(r(i) = \frac{2}{2i-d+1}\) if \(\frac{d+1}{2} < i \leq d\). To simulate samples, we first choose a true parameter \(\theta_\ast\). For each \(j \in [n]\), \(x_j\) is sampled independently from the uniform distribution on \([0.5, 2]\). Then, we sample \(y_j\) independently from the CDF \(\theta_\ast^\top \Phi(x_j, \cdot)\) using the inverse CDF method for \(j \in [n]\). Given the simulated sample, we calculate \(\hat{\theta}_\lambda\) using (3) with \(S = [0, 2]\), \(\lambda = 0.001\), and \(m\) chosen as the uniformly distribution \(\mu_d\) on \(S\). We measure the performance by evaluating two errors: the \(\ell^2\) estimation error \(\|\hat{\theta}_\lambda - \theta_\ast\|\) and the KS distance \(\text{KS}(\hat{F}_\lambda(x, \cdot), F(x, \cdot))\). Moreover, to obtain stable results, we repeat the simulation independently 100 times to calculate 90\% confidence intervals and means of the errors.

In Figures 1a and 1b, we plot the curves of confidence intervals and means for \(\ell^2\) estimation errors and KS distances of our estimator (2) against the sample size \(n\), which ranges from \(10^4\) to \(10^6\) (Figure 1a), and the dimension \(d\), which ranges from 10 to 100 (Figure 1b), in logarithmic scale. In Figure 1a, we fix \(d = 5\) and in Figure 1b, we fix \(n = 10^5\). According to our discussion below Theorem 1 and at the end of Section 3.1, the \(\ell^2\) error and KS distance are \(\tilde{O}(\sqrt{d/n})\) and \(\tilde{O}(d/\sqrt{n})\), respectively, when \(\mu_{\min}(U_n)\) grows linearly with \(n\). We have verified numerically that for this simulated data, \(\mu_{\min}(U_n)\) does grow linearly with \(n\). We note that in Figure 1, the slopes of the curves in Figure 1a are around \(-0.5\) and the slopes of the curves in Figure 1b are at most 0.5, which indicate that the scalings of the \(\ell^2\) error and the KS distance with respect to sample size \(n\) and dimension \(d\) are indeed upper bounded by \(\tilde{O}(\sqrt{d/n})\) and \(\tilde{O}(d/\sqrt{n})\), respectively, in this example.

4.2 Real data experiments

We also evaluate the performance of our estimator (2) on a dataset on financial wealth and 401(k) plan participation of size 9,915 from the R package ‘hdm’ [Chernozhukov et al., 2016] collected during wave 4 of the 1990 Survey of Income and Program Participation (SIPP).

Similar to [Chernozhukov and Hansen, 2004, Kallus et al., 2019], we use participation in 401(k), age, income, family size, education, marital status, two-earner status, defined benefit (DB) pension status, IRA participation status, and homeownership status as contexts \(x\) (\(d = 10\),

\[a\]The official website is https://www.census.gov/programs-surveys/sipp/data/datasets/1990-panel/wave-4.html.
and net financial assets as samples $y$ from the target CDF function. We split the whole dataset into two parts. To construct the basis $\Phi$, we use $1/3$ of the dataset to fit a Gaussian linear model for each context $x_i$ individually, and obtain a coefficient $\beta_i^{(1)}$, intercept $\beta_i^{(0)}$, and variance of residuals $\sigma_i^2$ for $i \in [d]$. Then, we define $\phi_i(x, t)$ to be the CDF of the Gaussian distribution $N(\beta_i^{(1)} x_i + \beta_i^{(0)}, \sigma_i^2)$. To evaluate the performance, we calculate $\hat{\theta}_\lambda$ using (3) with $m = 70100$ and $\lambda = 10$ on a subset of size $n$ of the remaining 6,610 data points, and denote the estimated parameter by $\hat{\theta}_\lambda^{(n)}$. We use $\hat{\theta}_\lambda^{(n)}$ to denote the $\ell^2$ projection of $\hat{\theta}_\lambda^{(n)}$ onto the probability simplex. Then, we calculate the $\ell^2$ errors $\|\hat{\theta}_\lambda^{(n)} - \hat{\theta}_\lambda^{(N)}\|$ and $\|\hat{\theta}_\lambda^{(n)} - \hat{\theta}_\lambda^{(N)}\|$ fixing $N = 6,610$. This time, to get stable results, we permute the dataset uniformly at random independently, and repeat the above procedure 100 times to obtain the means and 90% confidence intervals of the $\ell^2$ errors that are plotted in Figure 1c against sample size $n$ ranging from 1 to 4,000. As shown in Figure 1c, our estimator (2) generalizes quite well on real data, and the projected estimator $\hat{\theta}_\lambda$ has smaller error than (2) (as expected).\footnote{The repository for the implementation of the numerical experiments is provided at https://github.com/QianZhang20/Functional-Linear-Regression-of-CDFs.}

## 5 Conclusion

In this paper, we propose a linear model for contextual CDFs and an estimator for the coefficient parameter in this model. We prove $O(\sqrt{d/n})$ upper bounds on the estimation error of our estimator under the adversarial and random settings, and show that the upper bounds are tight up to logarithmic factors by proving $\Omega(\sqrt{d/n})$ information theoretic lower bounds. Furthermore, when a mismatch exists in the linear model, we prove that the estimation error of our estimator only increases by an amount commensurate with the mismatch error. Our current work has the limitation that the number of bases is finite and the bases are completely known. So, a fruitful future research direction would be to focus on the basis selection problem for CDF regression with possibly infinitely many base functions. Finally, since this work is mainly theoretical, it has no immediate societal consequences.
Appendices

We first briefly expand on the notation for the proofs of our theoretical results. For any topological space $A$, let $\mathcal{B}(A)$ denote the Borel $\sigma$-algebra of $A$. For any two spaces with $\sigma$-algebras, $(A_1, \mathcal{A}_1)$ and $(A_2, \mathcal{A}_2)$, a function $f : A_1 \to A_2$ is $A_1/A_2$-measurable if for any $E \in \mathcal{A}_2$, we have $f^{-1}(E) \in \mathcal{A}_1$. When $A_2$ is the Borel $\sigma$-algebra on $A_2$, we sometimes write $f$ is $A_1$-measurable. Then, according to [Bogachev, 2007, Proposition 10.7.6], for each distribution on $\mathcal{A}$, then, according to Caratheodory’s extension theorem. Then, $(A_1 \times A_2, \mathcal{A}_1 \otimes \mathcal{A}_2)$ is the product $\sigma$-algebra of $(A_1, \mathcal{A}_1, \nu_1)$ and $(A_2, \mathcal{A}_2, \nu_2)$. When $A_1 = A_2$ and $A_1 = A_1$, we will write $A_1^2$ to represent $A_1 \otimes A_2$. When $A_1 = A_2, A_1 = A_2$, and $\nu_1 = \nu_2$, we will write $\nu_1^2$ to represent $\nu_1 \otimes \nu_1$.

Note that according to our assumptions, $\mathcal{X}$ is a Polish space equipped with the Borel $\sigma$-algebra $\mathcal{B}(\mathcal{X})$, $\phi_i : \mathcal{X} \times S \to [0, 1]$ is $(\mathcal{B}(\mathcal{X}) \otimes \mathcal{B}(S))/\mathcal{B}([0, 1])$-measurable for each $i \in [d]$, and $e : \mathcal{X} \times S \to [-1, 1]$ is $(\mathcal{B}(\mathcal{X}) \otimes \mathcal{B}(S))/\mathcal{B}([-1, 1])$-measurable.

In the proof of Theorem 1 (Appendix A.1), Proposition 2 (Appendix A.2), Theorem 4 (Appendix B.1), Proposition 5 (Appendix B.2), Theorem 8 (Appendix E), and Corollary 9 (Appendix F), we consider an arbitrary probability measure $m$ on $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$. Since there is no ambiguity, for brevity, we omit “$d\mathcal{m}$” in the notation of integral. Notice that some quantities defined below depend on the chosen probability measure $m$.

A Proofs of Theorem 1 and Proposition 2

In the proofs of Theorem 1 (Appendix A.1) and Proposition 2 (Appendix A.2), we make the following measure-theoretic treatment of the probability spaces. Notation we use can be found at the beginning of the appendix. The underlying probability space for the sample $\{(x^{(j)}, y^{(j)})\}_{j \in \mathbb{N}}$ is $([0, 1]^N, \mathcal{B}([0, 1]^N), \mathbb{P})$ where $[0, 1]^N = \{ (\xi^{(1)}, \xi^{(2)}, \ldots) : \xi^{(j)} \in [0, 1] \}$, $\mathcal{B}([0, 1]^N) = \sigma(\{B_1 \times \cdots \times B_n : B_1, \ldots, B_n \in \mathcal{B}([0, 1]), n \in \mathbb{N})$ is the $\sigma$-algebra generated by all the finite product of Borel sets on $[0, 1]$, and $\mathbb{P}_{|[0,1]^n} = \mathcal{L} \otimes_{j=1}^n \mathcal{L}$ with $\mathcal{L}$ being the Lebesgue measure on $([0, 1], \mathcal{B}([0, 1]))$. The existence of the above probability space is guaranteed by Kolmogorov’s extension theorem. Define the random vector $\Xi = (\Xi^{(j)})_{j \in \mathbb{N}}$ on $([0, 1]^N, \mathcal{B}([0, 1]^N))$ to be the identity mapping, i.e., $\Xi : [0, 1]^N \to [0, 1]^N$, $(\xi^{(j)})_{j \in \mathbb{N}} \to (\xi^{(j)})_{j \in \mathbb{N}}$. Then, $\mathbb{P}$ is also the probability measure on $([0, 1]^N, \mathcal{B}([0, 1]^N))$ induced by $\Xi$ and $\Xi$ follows the uniform distribution on $[0, 1]^N$. Suppose $\{(x^{(j)}, y^{(j)})\}_{j \in \mathbb{N}}$ is sampled according to Scheme I with $F$ defined in (1). Then, according to [Bogachev, 2007, Proposition 10.7.6], for each $j \in \mathbb{N}$, there exist some $(\mathcal{B}(\mathcal{X}) \otimes \mathcal{B}(S))^{-1} \otimes \mathcal{B}([0, 1])/\mathcal{B}(\mathcal{X})$-measurable function $h^{(j)}_X : (\mathcal{X} \times S)^{j-1} \times [0, 1] \to \mathcal{X}$ and $(\mathcal{B}(\mathcal{X}) \otimes \mathcal{B}(S))^{-1} \otimes \mathcal{B}([0, 1])/\mathcal{B}(S)$-measurable function
\[
h_Y^{(j)} : (X \times S)^{j-1} \times X \times [0, 1] \to S \text{ such that } x^{(j)} = h_X^{(j)}(x(1), y(1), \ldots, x^{(j-1)}, y^{(j-1)}, \Xi^{(2j-1)}), \\
y^{(j)} = h_Y^{(j)}(x(1), y(1), \ldots, x^{(j-1)}, y^{(j-1)}, x(j), \Xi^{(2j)})\text{, and}
\]

\[
E \left[ \int \{ h_Y^{(j)}(x(1), y(1), \ldots, x^{(j-1)}, y^{(j-1)}, x(j), \Xi^{(2j)}) \leq t \} | F_{j-1} \right] = \theta^*_s \Phi(x^{(j)}, t) \tag{17}
\]

for any \( t \in S \) and \( j \in \mathbb{N} \), where \( F_j := \sigma \left( \{ \Xi(k) : k \in [2j+1] \} \right) \) is the sub-\( \sigma \)-algebra of \( B([0, 1])^N \) generated by the random variables \( \Xi^{(1)}, \ldots, \Xi^{(2j+1)} \). By definition, we have that \( y^{(j)} \) is \( F_j \cap B(S) \)-measurable for each \( j \in \mathbb{N} \) and \( \{ F_j \}_{j=0}^\infty \) forms a filtration of \( ([0, 1]^N, B([0, 1])^N, \mathbb{P}) \). Therefore, \( \{ y^{(j)} \}_{j=0}^{\infty} \) is \( \{ F_j \}_{j=0}^{\infty} \)-adapted.

By the above construction, for each \( j \in \mathbb{N} \), \( x^{(j)} : [0, 1]^N \to X, \xi \mapsto x^{(j)}(\xi) \) is a \( F_{j-1} \cap B(X) \)-measurable function. Thus, for each \( j \in \mathbb{N} \), the function \( \tilde{h}_X : [0, 1]^N \times S \to X \times S, (\xi, t) \mapsto (x^{(j)}(\xi), t) \) is \( (F_{j-1} \otimes B(S))/(B(X) \otimes B(S)) \)-measurable. Since \( \phi_j : X \times S \to [0, 1] \), \( (x, t) \mapsto \phi_j(x, t) \) is \( B(X) \otimes B(S)/B([0, 1]) \)-measurable, we know that \( \tilde{\phi}^{(j)}_i : [0, 1]^N \times S \to [0, 1] \), \( \xi \mapsto \tilde{\phi}^{(j)}_i(\xi, t) \) is \( (F_{j-1} \otimes B(S))/(B([0, 1])) \)-measurable. Therefore, the vector-valued function \( \Phi_j : [0, 1]^N \times S \to [0, 1]^d, (\xi, t) \mapsto \{ \phi_1(x^{(j)}(\xi), t), \ldots, \phi_d(x^{(j)}(\xi), t) \} = [\tilde{\phi}^{(j)}_1(\xi, t), \ldots, \tilde{\phi}^{(j)}_d(\xi, t)] \) is \( \tilde{F}_{j-1} \otimes B(S)/B([0, 1]^d) \)-measurable for each \( j \in \mathbb{N} \).

### A.1 Proof of Theorem 1

Define \( V_j := \int_S I_{y^{(j)}} \Phi_j - \int_S \theta^*_s \Phi_j \Phi_j \). Since we have proved above that for each \( j \in \mathbb{N} \), \( y^{(j)} \) is \( F_j \)-measurable and the function \( S \times S \to [0, 1], (y, t) \mapsto 1 \{ y \leq t \} \) is \( B(S)^2 \)-measurable, we have that \( I_{y^{(j)}} : [0, 1]^N \times S \to [0, 1], (\xi, t) \mapsto 1 \{ y^{(j)}(\xi) \leq t \} \) is \( F_j \otimes B(S) \)-measurable. Since we have also proved above that for each \( j \in \mathbb{N} \), \( \Phi_j \) is \( \tilde{F}_{j-1} \otimes B(S) \)-measurable, by Fubini’s theorem and (17), we have that \( \int_S \theta^*_s \Phi_j \Phi_j \) is \( \tilde{F}_{j-1} \)-measurable, \( V_j \) is \( F_j \)-measurable, and

\[
E[V_j | F_{j-1}] = E \left[ \int_S I_{y^{(j)}} \Phi_j | F_{j-1} \right] - \int_S \theta^*_s \Phi_j \Phi_j \\
= \int_S E \left[ I_{y^{(j)}} | F_j \right] \Phi_j - \int_S \theta^*_s \Phi_j \Phi_j \\
= \int_S \theta^*_s \Phi_j \Phi_j - \int_S \theta^*_s \Phi_j \Phi_j \\
= 0. \tag{18}
\]

For any \( \alpha \in \mathbb{R}^d \), define \( M_0(\alpha) = 1 \). Then, \( M_0(\alpha) \) is \( F_0 \)-measurable for any \( \alpha \in \mathbb{R}^d \). If \( n \in \mathbb{N} \), define \( M_n(\alpha) := \exp \{ \alpha^T W_n - \frac{1}{2} \| \alpha \|^2_{U_n} \} \) with \( W_n := \sum_{j=1}^n V_j \) and \( U_n = \sum_{j=1}^n \int_S \Phi_j \Phi_j \). Since \( \Phi_j \) is \( \tilde{F}_{j-1} \otimes B(S) \)-measurable and \( V_j \) is \( F_j \)-measurable, by Fubini’s theorem, \( U_n \) is \( F_{n-1} \)-measurable and \( W_n \) is \( F_n \)-measurable for each \( n \in \mathbb{N} \). Thus, \( M_n(\alpha) \) is also \( F_n \)-measurable for any \( \alpha \in \mathbb{R}^d \) and \( n \in \mathbb{N} \). Moreover, note that the function \( \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{d \times d} \to (0, \infty), (\alpha, W, U) \mapsto \exp \{ \alpha^T W - \frac{1}{2} \| \alpha \|^2_{U} \} \) is measurable. Hence, \( M_n : [0, 1]^N \times \mathbb{R}^d \to (0, \infty), (\xi, \alpha) \mapsto \exp \{ \alpha^T W_n(\xi) - \frac{1}{2} \| \alpha \|^2_{U_n(\xi)} \} \) is \( \tilde{F}_n \otimes B(\mathbb{R}^d) \)-measurable. Thus, for any \( \alpha \in \mathbb{R}^d \), \( \{ M_n(\alpha) \}_{n \geq 0} \) is
$(\mathcal{F}_n)_{n \geq 0}$-adapted. Besides, for any $\alpha \in \mathbb{R}^d$ and $n \in \mathbb{N}$, we have

$$
\mathbb{E}[M_n(\alpha) | \mathcal{F}_{n-1}] = M_{n-1}(\alpha) \mathbb{E} \left[ \exp \left\{ \alpha^T \mathbb{V}_n - \frac{1}{2} \alpha^T \left( \int_S \Phi_n \Phi_n^T \right) \alpha \right\} | \mathcal{F}_{n-1} \right]
$$

$$
= M_{n-1}(\alpha) \frac{\mathbb{E} \left[ \exp \left\{ \alpha^T \mathbb{V}_n \right\} | \mathcal{F}_{n-1} \right]}{\exp \left\{ \frac{1}{2} \int_S (\alpha^T \Phi_n)^2 \right\}}.
$$

(19)

Since $-\int_S |\alpha^T \Phi_n| \leq \alpha^T \mathbb{V}_n \leq \int_S |\alpha^T \Phi_n|$ almost surely (a.s.), we have

$$
\mathbb{E} \left[ \exp \left\{ \alpha^T \mathbb{V}_n \right\} | \mathcal{F}_{n-1} \right] \leq \exp \left\{ \frac{4}{8} \left( \int_S |\alpha^T \Phi_n| \right)^2 \right\}
$$

(20)

$$
\leq \exp \left\{ \frac{1}{2} \int_S (\alpha^T \Phi_n)^2 \right\}
$$

(21)

$$
\leq \exp \left\{ \frac{1}{2} \int_S (\alpha^T \Phi_n)^2 \right\}
$$

(22)

where (20) follows from Hoeffding’s lemma [Hoeffding, 1963], (21) follows from Cauchy-Schwarz inequality and the fact that $\int_S 1 = m(S) = 1$. Then, by (19) and (22), we have

$$
\mathbb{E}[M_n(\alpha) | \mathcal{F}_{n-1}] \leq M_{n-1}(\alpha) \frac{\exp \left\{ \frac{1}{2} \int_S (\alpha^T \Phi_n)^2 \right\}}{\exp \left\{ \frac{1}{2} \int_S (\alpha^T \Phi_n)^2 \right\}} = M_{n-1}(\alpha).
$$

(23)

Since $M_0(\alpha) = 1$ and $M_n(\alpha) \geq 0$, for any $\alpha \in \mathbb{R}^d$, $(M_n(\alpha))_{n \geq 0}$ is a super-martingale.

Now, for any $n \geq 0$, define $\overline{M}_n := \int_{\mathbb{R}^d} M_n(\alpha) h(\alpha) d\alpha$ with $d\alpha$ denoting $\text{Leb}(d\alpha)$ where the Lebesgue measure is on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ and

$$
h(\alpha) = \left( \frac{\lambda}{2\pi} \right)^{\frac{d}{2}} \exp \left\{ -\frac{\lambda}{2} \alpha^T \alpha \right\} = \left( \frac{\lambda}{2\pi} \right)^{\frac{d}{2}} \exp \left\{ -\frac{1}{2} \|\alpha\|^2_{\lambda I_d} \right\}.
$$

(24)

Recall that $U_n(\lambda) = \mathbb{V}_n + \lambda I_d$. Then, for $n \geq 1$, we have

$$
\overline{M}_n = \left( \frac{\lambda}{2\pi} \right)^{\frac{d}{2}} \int_{\mathbb{R}^d} \exp \left\{ \alpha^T \mathbb{V}_n - \frac{1}{2} \|\alpha\|^2_{\mathbb{V}_n} - \frac{1}{2} \|\alpha\|^2_{\lambda I_d} \right\} d\alpha
$$

$$
= \left( \frac{\lambda}{2\pi} \right)^{\frac{d}{2}} \int_{\mathbb{R}^d} \exp \left\{ \frac{1}{2} \|\mathbb{V}_n\|^2_{U_n(\lambda)^{-1}} - \frac{1}{2} \|\alpha - U_n(\lambda)^{-1} \mathbb{V}_n\|^2_{U_n(\lambda)} \right\} d\alpha
$$

(25)

$$
= \frac{\lambda^{\frac{d}{2}}}{\det(U_n(\lambda))^{\frac{1}{2}}} \exp \left\{ \frac{1}{2} \|\mathbb{V}_n\|^2_{U_n(\lambda)^{-1}} \right\}.
$$

(26)
Thus, by the triangle inequality,
\[
\|W_n\|_{U_n(\lambda)^{-1}}^2 - \|\alpha - U_n(\lambda)^{-1} W_n\|_{U_n(\lambda)}^2 \\
= \|W_n\|_{U_n(\lambda)^{-1}}^2 - (\alpha^T - W_n^T U_n(\lambda)^{-1}) U_n(\lambda) (\alpha - U_n(\lambda)^{-1} W_n) \\
= \|W_n\|_{U_n(\lambda)^{-1}}^2 - \|\alpha\|_{U_n(\lambda)}^2 - \|W_n\|_{U_n(\lambda)^{-1}}^2 + 2\alpha^T W_n \\
= 2\alpha^T W_n - \|\alpha\|_{U_n(\lambda)}.
\]

For \(n = 0\), \(M_0 = \int_{\mathbb{R}^d} M_0(\alpha) h(\alpha) d\alpha = \int_{\mathbb{R}^d} h(\alpha) d\alpha = 1\).

Moreover, since we have shown that \(M_n\) is \(\mathcal{F}_n \otimes \mathcal{B}(\mathbb{R}^d)\)-measurable, by Fubini’s theorem and (23), \(\bar{M}_n\) is \(\mathcal{F}_n\)-measurable for any \(n \geq 0\) and for any \(n \in \mathbb{N}\),
\[
\mathbb{E}[\bar{M}_n|\mathcal{F}_{n-1}] = \mathbb{E}\left[\int_{\mathbb{R}^d} M_n(\alpha) h(\alpha) d\alpha | \mathcal{F}_{n-1}\right] \\
= \int_{\mathbb{R}^d} \mathbb{E}[M_n(\alpha)|\mathcal{F}_{n-1}] h(\alpha) d\alpha \\
\leq \int_{\mathbb{R}^d} M_{n-1}(\alpha) h(\alpha) d\alpha \\
= \bar{M}_{n-1}.
\]

Thus, \(\{\bar{M}_n\}_{n \geq 0}\) is also a super-martingale. By Doob’s maximal inequality for super-martingales,
\[
\mathbb{P}\left[\sup_{n \in \mathbb{N}} \bar{M}_n \geq \delta\right] \leq \frac{\mathbb{E}[\bar{M}_0]}{\delta} = \frac{1}{\delta}
\]
which, together with (26), implies that,
\[
\mathbb{P}\left[\exists n \in \mathbb{N} \text{ s.t. } \|W_n\|_{U_n(\lambda)^{-1}} \geq \sqrt{\log \frac{\det(U_n(\lambda))}{\lambda^d} + 2 \log \frac{1}{\delta}}\right] \leq \delta.
\]

Since
\[
\theta_\ast = \left(\sum_{j=1}^n \int_{S} \Phi_j \Phi_j^T + \lambda I_d\right)^{-1} \left(\sum_{j=1}^n \int_{S} \Phi_j \Phi_j^T \theta_\ast + \lambda \theta_\ast\right)
\]
by (3), we have
\[
\hat{\theta}_\lambda - \theta_\ast = U_n(\lambda)^{-1} \left(\sum_{j=1}^n V_j - \lambda \theta_\ast\right) = U_n(\lambda)^{-1} W_n - U_n(\lambda)^{-1} (\lambda \theta_\ast).
\]

Thus, by the triangle inequality,
\[
\|\hat{\theta}_\lambda - \theta_\ast\|_{U_n(\lambda)} \leq \|U_n(\lambda)^{-1} W_n\|_{U_n(\lambda)} + \lambda \|U_n(\lambda)^{-1} \theta_\ast\|_{U_n(\lambda)} \\
= \|W_n\|_{U_n(\lambda)^{-1}} + \lambda \|\theta_\ast\|_{U_n(\lambda)^{-1}} \\
\leq \|W_n\|_{U_n(\lambda)^{-1}} + \sqrt{\lambda} \|\theta_\ast\|
\]
where the last inequality follows from the facts that \( U_n(\lambda)^{-1} = \frac{1}{\lambda} \left( I - U_n(\lambda)^{-1} U_n \right) \) and \( \| I - U_n(\lambda)^{-1} U_n \|_2 \leq 1 \).

By (28) and (30), with probability at least \( 1 - \delta \), for all \( n \in \mathbb{N} \), we have
\[
\| \hat{\theta}_\lambda - \theta_* \|_{U_n(\lambda)} \leq \sqrt{\log \frac{\det(U_n(\lambda))}{\lambda^d}} + 2 \log \frac{1}{\delta} + \sqrt{\lambda} \| \theta_* \|. \tag{31}
\]

By the arithmetic mean-geometric mean (AM–GM) inequality, we have
\[
\log \det(U_n(\lambda)) \leq d \log \left( \frac{\text{trace}(U_n(\lambda))}{d} \right) = d \log \left( \frac{1}{d} \text{trace} \left( \sum_{j=1}^{n} \int_S \Phi_j \Phi_j^\top + \lambda I_d \right) \right).
\]

Since
\[
\text{trace} \left( \sum_{j=1}^{n} \int_S \Phi_j \Phi_j^\top + \lambda I_d \right) = d\lambda + \sum_{j=1}^{n} \int_S \text{trace} \left( \Phi_j \Phi_j^\top \right)
= d\lambda + \sum_{j=1}^{n} \int_S \| \Phi_j \|_2^2
\leq d\lambda + nd,
\]
we have
\[
\log \det(U_n(\lambda)) \leq d \log \left( \frac{1}{d} (d\lambda + nd) \right) = d \log (\lambda + n). \tag{32}
\]

By (31) and (32), for any \( \lambda > 0 \), \( \delta \in (0, 1) \), with probability at least \( 1 - \delta \), for all \( n \in \mathbb{N} \), we have
\[
\| \hat{\theta}_\lambda - \theta_* \|_{U_n(\lambda)} \leq \sqrt{d \log \left( 1 + \frac{n}{\lambda} \right)} + 2 \log \frac{1}{\delta} + \sqrt{\lambda} \| \theta_* \|. \tag{33}
\]
Thus, Theorem 1 is proved for any probability measure \( \mathfrak{m} \) on \( (S, \mathcal{B}(S)) \).

A.2 Proof of Proposition 2

When \( U_N \) is non-singular for some fixed \( N \in \mathbb{N} \), since \( \int_S \Phi_j \Phi_j^\top \) is positive semi-definite for any \( j \in \mathbb{N} \), it immediately follows that \( U_n \) are non-singular for any \( n \geq N \). Then, \( \hat{\theta} \) is unique and is given by (3) with \( \lambda = 0 \) for any \( n \geq N \), i.e.,
\[
\hat{\theta} = \left( \sum_{j=1}^{n} \int_S \Phi_j \Phi_j^\top \right)^{-1} \left( \sum_{j=1}^{n} \int_S I_{b(j)} \Phi_j \right).
\]
for any \( n \geq N \). Since
\[
\theta_* = \left( \sum_{j=1}^{n} \int_S \Phi_j \Phi_j^\top \right)^{-1} \left( \sum_{j=1}^{n} \int_S \Phi_j \Phi_j^\top \theta_* \right), \tag{34}
\]
we have
\[ \hat{\theta} - \theta_* = U_n^{-1} W_n. \]  
(35)

By definition and the triangle inequality for integrals, we have
\[ \|V_j\| \leq \int_S |y(\omega) - \hat{\theta}_*^\top \Phi_j| \|\Phi_j\| \leq \int_S \sqrt{d} = \sqrt{d} \]  
(36)

which also implies that
\[ \sum_{j=1}^n \mathbb{E}[\|V_j\|^2 | F_{j-1}] \leq \sum_{j=1}^n d = nd \]  
(37)

Since \( W_n = \sum_{j=1}^n V_j \), by (18), (36), (37), and Proposition 1.2 in [Hsu et al., 2012a], we have
\[ \mathbb{P}[\|W_n\| \geq \sqrt{nd} + \sqrt{8nd} \alpha + (4/3)\sqrt{d}\alpha] \leq e^{-a} \]  
for any \( a > 0 \). Thus, for any \( \delta \in (0,1) \) and \( n \in \mathbb{N} \), with probability at least \( 1 - \delta \), we have
\[ \|W_n\| \leq \sqrt{nd} + \sqrt{8nd} \log \frac{1}{\delta} + \frac{4}{3}\sqrt{d}\log \frac{1}{\delta}. \]  
(38)

Since \( U_n \) is positive definite, by (38), we have
\[ \|W_n\|_{U_n^{-1}} = \sqrt{W_n^\top U_n^{-1} W_n} \leq \frac{\|W_n\|}{\sqrt{\mu_{\min}(U_n)}} \leq \frac{\sqrt{nd} + \sqrt{8nd} \log \frac{1}{\delta} + \frac{4}{3}\sqrt{d}\log \frac{1}{\delta}}{\sqrt{\mu_{\min}(U_n)}} \]  
(39)

with probability at least \( 1 - \delta \). Hence, by (35), and (39), we have that for any \( n \geq N \),
\[ \|\hat{\theta} - \theta_*\|_{U_n} = \|U_n^{-1} W_n\|_{U_n} = \|W_n\|_{U_n^{-1}} \leq \frac{\sqrt{nd} + \sqrt{8nd} \log \frac{1}{\delta} + \frac{4}{3}\sqrt{d}\log \frac{1}{\delta}}{\sqrt{\mu_{\min}(U_n)}} \]
with probability at least \( 1 - \delta \). In conclusion, Proposition 2 is proved for any probability measure \( \mathbb{m} \) on \((S, \mathcal{B}(S))\).

B  Proofs of Theorem 4 and Proposition 5

In this section, we follow the same construction of the probability space as in Appendix A. In particular, noting Scheme II is a special case of Scheme I, we consider the underlying probability space for the sample \( \{(x^{(j)}, y^{(j)})\}_{j \in \mathbb{N}} \) to be \(([0,1]^N, \mathcal{B}([0,1]^N), \mathbb{P})\). Define the random vector \( \Xi \) to be the identity mapping from \([0,1]^N \) onto itself as in Appendix A. Then, \( \Xi \) follows the uniform distribution on \([0,1]^N \). Suppose \( \{(x^{(j)}, y^{(j)})\}_{j \in \mathbb{N}} \) is sampled according to Scheme II with \( F \) defined in (1). Then, according to [Bogachev, 2007, Proposition 10.7.6], for each \( j \in \mathbb{N} \), there exist some \( \mathcal{B}([0,1])/\mathcal{B}(X) \)-measurable function \( h_X^{(j)} : [0,1] \rightarrow X \) and
We also need to introduce the following notation. For any square matrix $B$, consider the following random matrix for any $\Phi$, $\Phi(x^{(j)}, \Xi^{(2j)})$, and

$$E \left[ 1 \left\{ h^{(j)}_Y(x^{(j)}, \Xi^{(2j)}) \leq t \right\} \mid F_{j-1} \right] = \theta^{(j)}_Y \Phi(x^{(j)}, t)$$

for any $t \in S$ and $j \in \mathbb{N}$, where $F_j := \sigma \left( \{ \Xi(k) : k \in [2j + 1] \} \right)$ is the sub-$\sigma$-algebra of $\mathcal{B}([0,1])^N$ generated by the random variables $\Xi^{(1)}, \ldots, \Xi^{(2j+1)}$. With the same proof provided at the beginning of Appendix A, $\{y^{(j)}\}_{j \in \mathbb{N}}$ is independent, which implies that $\{\Phi_j(t)\}_{j \in \mathbb{N}}$ is independent for each $j \in \mathbb{N}$. Moreover, $\{x^{(j)}\}_{j \in \mathbb{N}}$ is independent, which implies that $\{\Phi_j(t)\}_{j \in \mathbb{N}}$ is independent for any $t \in S$ and $\{y^{(j)}\}_{j \in \mathbb{N}}$ is independent.

We also need to introduce the following notation. For any square matrix $A$, let $\mu_{\max}(A)$ denote the largest eigenvalue of $A$ and let $\|A\|_2$ denote the spectral norm of the matrix $A$, i.e., $\|A\|_2 = \sqrt{\mu_{\max}(A^TA)}$.

### B.1 Proof of Theorem 4

By definition and Fubini’s theorem, we have $\Sigma^{(j)} = E \left[ \int_S \Phi_j \Phi_j^\top \right] = \int_S E \left[ \Phi_j \Phi_j^\top \right]$ for each $j \in [n]$, $\Sigma_n = \sum_{j=1}^n \Sigma^{(j)} = E \left[ U_n \right]$. For the proof, we need to define $\Delta_n := \Sigma_n^{-\frac{1}{2}} (U_n - \Sigma_n) \Sigma_n^{-\frac{1}{2}}$, $\Sigma_{n}^{(j)} := \Sigma_n^{-\frac{1}{2}} \Sigma^{(j)} \Sigma_n^{-\frac{1}{2}}$, and $\tilde{\Phi}_j(t) := \Sigma_n^{-\frac{1}{2}} \Phi_j(t)$ for any $t \in \mathbb{R}$ and $j \in [n]$. For any $j \in \mathbb{N}$, we have

$$\|\Sigma^{(j)}\|_2 = \mu_{\max} \left( \Sigma^{(j)} \right) = \mu_{\max} \left( E \left[ \int_S \Phi_j \Phi_j^\top \right] \right) \leq E \left[ \int_S \|\Phi_j\|_2^2 \right] \leq d. \quad (40)$$

By the assumption that $\mu_{\min}(\Sigma^{(j)}) \geq \sigma_{\min}$ for all $j \in \mathbb{N}$ and Weyl’s inequality [Weyl, 1912], we have

$$\mu_{\min}(\Sigma_n) \geq n\sigma_{\min}. \quad (41)$$

By (40) and (41), for each $j \in [n]$, we have

$$\mu_{\max} \left( \tilde{\Sigma}_n^{(j)}(\lambda) \right) \leq \mu_{\max} \left( \Sigma^{(j)} \right) \mu_{\min} \left( \Sigma_n(\lambda) \right) \leq \frac{d}{n\sigma_{\min}}. \quad (42)$$

Consider the following random matrix for $j \in [n]$.

$$Z_j := \int_S \tilde{\Phi}_j \tilde{\Phi}_j^\top - \tilde{\Sigma}_n^{(j)} = \Sigma_n^{-\frac{1}{2}} \left( \int_S \Phi_j \Phi_j^\top - \Sigma^{(j)} \right) \Sigma_n^{-\frac{1}{2}}.$$ 

We have

$$\Delta_n = \sum_{j=1}^n Z_j, \quad (43)$$

$$E[Z_j] = 0, \quad (44)$$

$$\|Z_j\|_2 = \max\{\lambda_{\max}(Z_j), -\lambda_{\min}(Z_j)\} \leq \max \left\{ \mu_{\max} \left( \int_S \tilde{\Phi}_j \tilde{\Phi}_j^\top \right), \mu_{\max} \left( \tilde{\Sigma}_n^{(j)}(\lambda) \right) \right\} \leq \frac{d}{n\sigma_{\min}}. \quad (45)$$
where (45) follows from (42) and
\[ \mu_{\text{max}} \left( \int_S \Phi_j^2 \Phi_j^\top \right) \leq \int_S \|\Phi_j\|^2 \leq \frac{1}{\mu_{\text{min}}(\Sigma_n(\lambda))} \int_S \|\Phi_j\|^2 \leq \frac{d}{n\sigma_{\text{min}}} . \]

By (43), (44), (45), and [Tropp, 2012, Theorem 1.3], we have
\[ \mathbb{P} \left[ \mu_{\text{min}}(\Delta_n) \leq -a \right] \leq d \exp \left( -\frac{n\sigma_{\text{min}}^2 a^2}{8d^2} \right) \]
for any \( a \geq 0 \). Thus, with probability at least \( 1 - \delta \),
\[ \mu_{\text{min}}(\Delta_n) \geq -\frac{d}{\sigma_{\text{min}}} \sqrt{\frac{8}{n} \log \left( \frac{d}{\delta} \right)} . \]

Since \( \Delta_n = \Sigma_n^{-\frac{1}{2}} U_n \Sigma_n^{-\frac{1}{2}} - I_d \), we have \( \mu_{\text{min}}(\Sigma_n^{-\frac{1}{2}} U_n \Sigma_n^{-\frac{1}{2}}) = \mu_{\text{min}}(\Delta_n) + 1 \) which together with the fact that \( U_n = \Sigma_n^{-\frac{1}{2}} \Sigma_n^{-\frac{1}{2}} U_n \Sigma_n^{-\frac{1}{2}} \Sigma_n^{-\frac{1}{2}} \) implies that
\[ \mu_{\text{min}}(U_n) \geq \mu_{\text{min}}(\Sigma_n) \mu_{\text{min}}(\Sigma_n^{-\frac{1}{2}} U_n \Sigma_n^{-\frac{1}{2}}) = \mu_{\text{min}}(\Sigma_n)(\mu_{\text{min}}(\Delta_n) + 1) \]

By (48), we have
\[ \mu_{\text{min}}(\Delta_n) \geq -\frac{1}{2} \implies \mu_{\text{min}}(U_n) \geq \frac{1}{2} \mu_{\text{min}}(\Sigma_n) \geq \frac{n}{2} \sigma_{\text{min}} > 0 . \]

Note that when \( U_n \) is positive definite, we have
\[ \Sigma_n^{-\frac{1}{2}} U_n^{-\frac{1}{2}} \Sigma_n^{-\frac{1}{2}} = \Sigma_n^{-\frac{1}{2}} U_n^{-\frac{1}{2}} \left( \Sigma_n^{-\frac{1}{2}} U_n^{-\frac{1}{2}} \right)^\top , \]
\[ U_n^{-\frac{1}{2}} \Sigma_n U_n^{-\frac{1}{2}} = \left( \Sigma_n^{-\frac{1}{2}} U_n^{-\frac{1}{2}} \right)^\top \Sigma_n^{-\frac{1}{2}} U_n^{-\frac{1}{2}} . \]

Thus,
\[ \| U_n^{-\frac{1}{2}} \Sigma_n U_n^{-\frac{1}{2}} \|_2 = \| \Sigma_n^{-\frac{1}{2}} U_n^{-\frac{1}{2}} \|_2 = \| \left( \Sigma_n^{-\frac{1}{2}} U_n^{-\frac{1}{2}} \right)^{-1} \|_2 = \| (I_d + \Delta_n)^{-1} \|_2 = \frac{1}{\mu_{\text{min}}(I_d + \Delta_n)} = \frac{1}{1 + \mu_{\text{min}}(\Delta_n)} . \]

By (49) and (50), we have
\[ \mu_{\text{min}}(\Delta_n) \geq -\frac{1}{2} \implies \mu_{\text{min}}(U_n) \geq \frac{n}{2} \sigma_{\text{min}} \text{ and } \| U_n^{-\frac{1}{2}} \Sigma_n U_n^{-\frac{1}{2}} \|_2 \leq 2 . \]
By (47), for any \( \delta \in (0, 1) \), if \( n \geq \frac{32d^2}{\delta_{\text{min}}} \log(d/\delta) \), we have \( \mu_{\text{min}}(\Delta_n) \geq -\frac{1}{2} \) with probability at least \( 1 - \delta \). Then, by (51), we have

\[
\|U_n^{-\frac{1}{2}} \Sigma_n U_n^{-\frac{1}{2}}\|_2^2 \leq 2 \text{ and } \mu_{\text{min}}(U_n) \geq \frac{n}{2} \sigma_{\text{min}}.
\]

with probability at least \( 1 - \delta \).

Still define \( W_n := \sum_{i=1}^n (\mathbb{I}_S \mathbb{I}_{y(i)} \Phi_j - \mathbb{I}_S \theta_* \Phi_j \Phi_j) \). By (35), we have \( \hat{\theta} - \theta_* = U_n^{-1} W_n \) and \( \|\hat{\theta} - \theta_*\|_U \leq \|W_n\|_{U_n^{-1}} \).

By (7), (52), and union bound, for any \( \delta_1 \in (0, 1) \) and \( \delta_2 \in (0, 1 - \delta_1) \), if \( n \geq \frac{32d^2}{\delta_{\text{min}}^2} \log \frac{d}{\delta_1} \), we have

\[
\|\hat{\theta} - \theta_*\|_{\Sigma_n} = \sqrt{\text{tr}(W_n^T U_n^{-\frac{1}{2}} W_n U_n^{-\frac{1}{2}} \Sigma_n U_n^{-\frac{1}{2}})}
\]

\[
\leq \sqrt{\|U_n^{-\frac{1}{2}} \Sigma_n U_n^{-\frac{1}{2}}\|_2 \|W_n\|_{U_n^{-1}}^2}
\]

\[
= \sqrt{\|U_n^{-\frac{1}{2}} \Sigma_n U_n^{-\frac{1}{2}}\|_2 \|\hat{\theta} - \theta_*\|_{U_n}^2}
\]

\[
\leq \sqrt{2} \|\hat{\theta} - \theta_*\|_{U_n}
\]

\[
\leq \sqrt{2nd} + 4 \sqrt{nd \log \frac{1}{\delta_1} + \frac{4}{3} \sqrt{2d \log \frac{1}{\delta_2}}}
\]

\[
\leq \frac{\sqrt{\mu_{\text{min}}(U_n)}}{\sigma_{\text{min}}}
\]

\[
\leq \frac{2\sqrt{nd} + 4 \sqrt{2nd \log \frac{1}{\delta_2} + \frac{8}{3} \sqrt{d \log \frac{1}{\delta_2}}}}{\sqrt{\sigma_{\text{min}}}}
\]

\[
= \frac{2\sqrt{d} + 4 \sqrt{2d \log \frac{1}{\delta_1} + \frac{8}{3} \sqrt{d/n \log \frac{1}{\delta_1}}}}{\sqrt{\sigma_{\text{min}}}}
\]

with probability at least \( 1 - \delta_1 - \delta_2 \).

By letting \( \delta_1 = \delta_2 = \delta \), (8) is proved. In conclusion, Theorem 4 is proved for any probability measure \( \mathbb{m} \) on \( (S, \mathcal{B}(S)) \).

**B.2 Proof of Proposition 5**

Since

\[
\Sigma_n^{-\frac{1}{2}} U_n(\lambda) \Sigma_n^{-\frac{1}{2}} = \Sigma_n^{-\frac{1}{2}} (\Sigma_n + \lambda I_d + U_n - \Sigma_n) \Sigma_n^{-\frac{1}{2}} = I_d + \lambda \Sigma_n^{-1} + \Delta_n
\]
and \( \lambda \Sigma_n^{-1} \) is positive semi-definite for any \( \lambda \geq 0 \), we have
\[
\| \Sigma_n^{-\frac{1}{2}} U_n(\lambda)^{-1} \Sigma_n^{-\frac{1}{2}} \|_2 = \left\| \left( \Sigma_n^{-\frac{1}{2}} U_n(\lambda) \Sigma_n^{-\frac{1}{2}} \right)^{-1} \right\|_2 = \left\| (I_d + \lambda \Sigma_n^{-1} + \Delta_n)^{-1} \right\|_2 = \frac{1}{\mu_{\min}(I_d + \lambda \Sigma_n^{-1} + \Delta_n)} 
\leq \frac{1}{1 + \mu_{\min}(\Delta_n)}. \tag{53}
\]

Since
\[
\Sigma_n^{\frac{1}{2}} U_n(\lambda)^{-1} \Sigma_n^{\frac{1}{2}} = \left( \Sigma_n^{\frac{1}{2}} U_n(\lambda)^{-\frac{1}{2}} \right)^T, \\
U_n(\lambda)^{-\frac{1}{2}} \Sigma_n U_n(\lambda)^{-\frac{1}{2}} = \left( \Sigma_n^{\frac{1}{2}} U_n(\lambda)^{-\frac{1}{2}} \right)^T \Sigma_n^{\frac{1}{2}} U_n(\lambda)^{-\frac{1}{2}},
\]
by (53), we have
\[
\| U_n(\lambda)^{-\frac{1}{2}} \Sigma_n U_n(\lambda)^{-\frac{1}{2}} \|_2 = \left\| \Sigma_n^{\frac{1}{2}} U_n(\lambda)^{-1} \Sigma_n^{\frac{1}{2}} \right\|_2 \leq \frac{1}{1 + \mu_{\min}(\Delta_n)}.
\]
Define \( R_n = \sum_{i=1}^n V_j - \lambda \theta_* \) where \( V_j := \int_S I_{y(j)} \Phi_j - \int_S \theta_*^T \Phi_j \Phi_j \). Then, by (3) and (29), we have
\[
\hat{\theta}_\lambda - \theta_* = U_n(\lambda)^{-1} R_n.
\]
Thus,
\[
\| \hat{\theta}_\lambda - \theta_* \|_{\Sigma_n} \leq \frac{\| \hat{\theta}_\lambda - \theta_* \|_{U_n(\lambda)}}{\sqrt{1 + \mu_{\min}(\Delta_n)}} \leq \frac{\sqrt{d \log \frac{1}{\delta_1}} + 2 \log \frac{1}{\delta_2} + \sqrt{\lambda} \| \theta_* \|}{\sqrt{1 - \frac{d}{\sigma_{\min}} \sqrt{\frac{8}{n} \log \left( \frac{d}{\delta} \right)}}}
\]
with probability at least \( 1 - \delta_1 - \delta_2 \). Then, when \( n \geq \frac{32d^2}{\sigma_{\min}} \log(d/\delta_1) \), by the above inequality, we have
\[
\| \hat{\theta}_\lambda - \theta_* \|_{\Sigma_n} \leq \sqrt{2 \left( d \log \left( \frac{1}{\delta_1} \right) + 2 \log \frac{1}{\delta_2} \right) + \sqrt{2 \lambda} \| \theta_* \|} \tag{54}
\]
with probability at least \( 1 - \delta_1 - \delta_2 \). Thus, (9) is obtained from (54) by setting \( \delta_1 = \delta_2 = \delta \in (0, 1/2) \). Proposition 5 is proved for any probability measure \( m \) on \( (S, \mathcal{B}(S)) \).
C Proof of Theorem 6

Let $d_{\ell^2}$ denote the $\ell^2$ distance. For $\delta \in (0, 1)$, let $P(\Delta^{d-1}, d_{\ell^2}, \delta)$ denote the $\delta$-packing number of the set $\Delta^{d-1}$. Then, by [Vershynin, 2018, Proposition 4.2.12], we have

$$P(\Delta^{d-1}, d_{\ell^2}, \delta) \geq \frac{\text{Vol}(\Delta^{d-1})}{\text{Vol}(B_\delta^{d-1}(0))}$$

(55)

where $B_\delta^{d-1}(0) := \{x \in \mathbb{R}^{d-1} : \|x\|_2 \leq \delta\}$ and for any $E \subseteq \mathbb{R}^{d-1}$, Vol($E$) is the volume of $E$ under the Lebesgue measure in $\mathbb{R}^{d-1}$. According to [Stein, 1966] and [DLMF], we have

$$\text{Vol}(\Delta^{d-1}) = \frac{\sqrt{d}}{(d-1)!},$$

(56)

$$\text{Vol}(B_\delta^{d-1}(0)) = \frac{(\sqrt{\pi} \delta)^{d-1}}{\Gamma(d^{1/2})}.$$

(57)

Thus,

$$P(\Delta^{d-1}, d_{\ell^2}, \delta) \geq \frac{\Gamma(d^{1/2}) \sqrt{d}}{(d-1)! (\sqrt{\pi} \delta)^{d-1}} \frac{\Gamma(d^{1/2}) \sqrt{d}}{\Gamma(d)(\sqrt{\pi} \delta)^{d-1}}.$$

(58)

When $d \geq 3$, we have $\frac{d^{1/2}}{2} \geq 2$ and $d \geq 2$. According to [Batir, 2008, Theorem 1.5], we have $2((x-1/2)/e)^{x-1/2} < \Gamma(x) < 3((x-1/2)/e)^{x-1/2}$ for any $x \geq 2$. Thus, for $d \geq 3$, we have

$$\frac{\Gamma(d^{1/2})}{\Gamma(d)} > \frac{2}{3} \left(\frac{d}{2e}\right)^{d/2}. $$

We verify that the above inequality also holds when $d = 2$. Therefore, for any $d \geq 2$, we have

$$\frac{\Gamma(d^{1/2})}{\Gamma(d)} > \frac{2}{3} \left(\frac{d}{2e}\right)^{d/2} $$

which implies that

$$P(\Delta^{d-1}, d_{\ell^2}, \delta) \geq \frac{2\sqrt{d}}{3(\sqrt{\pi} \delta)^{d-1}} \left(\frac{d}{2e}\right)^{d/2} \left(\frac{d}{2e}\right)^{d-1/2}$$

$$\geq \frac{2\sqrt{d}}{3(\sqrt{\pi} \delta)^{d-1}} \left(\frac{d}{2e}\right)^{d/2}$$

$$= \frac{2\sqrt{d}}{3(\sqrt{\pi} \delta)^{d-1}} \frac{1}{2^{d/2} e^{d/2}} d^{-d/2}.$$
Let \( \delta = \frac{\sqrt{e}}{2 \sqrt{\pi d}} \left( \frac{\sqrt{d}}{3} \right)^{\frac{1}{d-1}} \left( \frac{1}{\sqrt{2}} \right)^{\frac{1}{d-1}} \). Then, by (59), we have
\[
P(\Delta^{d-1}, d'2, \delta) > 2^d,
\]
which implies that there exits a \( \delta \)-separated subset \( \mathcal{V}_1 \) of \( \Delta^{d-1} \) of size \( |\mathcal{V}_1| \geq 2^d \). For \( d \geq 2 \), we have \( \delta \geq \frac{\sqrt{e}}{4 \sqrt{\pi d}} \left( \frac{\sqrt{d}}{3} \right)^{\frac{1}{d-1}} \). Consider the function \( f(x) = \frac{1}{x-1} \log \left( \frac{\sqrt{x}}{3} \right) \) with \( x \geq 2 \). We have that
\[
f'(x) = \frac{1 - \frac{1}{x} - \log x + 2 \log 3}{2(x-1)^2}.
\]
Since the function \( g: x \mapsto -\frac{1}{x} - \log x \) is a decreasing function when \( x \geq 2 \) and \( f'(2) > 0 \), \( f'(e^5) < 0 \), we have that \( f \) first increases and then decreases when \( x \) increases from 2 to infinity. Since \( \lim_{x \to \infty} f(x) = 0 \), \( f(2) = \log(\sqrt{2}/3) \), we have that \( f(x) \geq f(2) = \log(\sqrt{2}/3) \). Therefore, for any \( d \geq 2 \), we have \( \left( \frac{\sqrt{d}}{3} \right)^{\frac{1}{d-1}} \geq \sqrt{2}/3 \) and
\[
\delta \geq \frac{\sqrt{2e}}{12 \sqrt{\pi d}} \quad (61)
\]
Define \( \mathcal{V}_a := \{ l_a(\theta) : \theta \in \mathcal{V}_1 \} \) where \( l_a(\theta) = \left[ a\theta_1, \ldots, a\theta_{d-1}, 1 - a \sum_{i=1}^{d-1} \theta_i \right]^\top \) for \( 0 \leq a < \frac{1}{\sup_{\theta \in \mathcal{V}_1} \sum_{i=1}^{d-1} \theta_i} \). Then, for any \( \theta^{(1)}, \theta^{(2)} \in \mathcal{V}_1 \) and any \( j \in [n] \), we have
\[
\| l_a(\theta^{(1)}) - l_a(\theta^{(2)}) \| = \sqrt{a^2 \sum_{i=1}^{d} \left( \theta_i^{(1)} - \theta_i^{(2)} \right)^2} = a \| \theta^{(1)} - \theta^{(2)} \|
\]
Thus, we have
\[
\| l_a(\theta^{(1)}) - l_a(\theta^{(2)}) \| \leq a \sup_{x, y \in \Delta^{d-1}} \| x - y \| = \sqrt{2}a \quad (62)
\]
and
\[
\| l_a(\theta^{(1)}) - l_a(\theta^{(2)}) \| \geq a \delta
\]
which implies that \( \mathcal{V}_a \) is a \((a \delta)\)-separated subset of \( \Delta^{d-1} \) of size \( |\mathcal{V}_a| \geq 2^d \).

Let \( D(Q_1||Q_2) \) and \( \chi^2(Q_1||Q_2) \) denote the Kullback-Leibler (KL) divergence and \( \chi^2 \)-divergence between two probability measures \( Q_1 \) and \( Q_2 \) on \( \mathbb{R} \), respectively, where \( Q_1 \) is absolutely continuous w.r.t. \( Q_2 \). Their definitions are given below:
\[
D(Q_1||Q_2) := \int_{\mathbb{R}} \log \left( \frac{dQ_1}{dQ_2} \right) dQ_1, \\
\chi^2(Q_1||Q_2) := \int_{\mathbb{R}} \left( \frac{dQ_1}{dQ_2} - 1 \right)^2 dQ_2,
\]
where \( \frac{dQ_1}{dQ_2} \) denotes the Radon–Nikodym derivative of \( Q_1 \) w.r.t. \( Q_2 \).
For any $\theta^{(1)}, \theta^{(2)} \in \mathcal{V}_a$, we have

$$D \left( P_{Y|X^{(j)}, \theta^{(1)}} \| P_{Y|X^{(j)}, \theta^{(2)}} \right) \leq \chi^2 \left( P_{Y|X^{(j)}, \theta^{(1)}} \| P_{Y|X^{(j)}, \theta^{(2)}} \right)$$ (63)

where (63) follows from the bound on KL divergence w.r.t. $\chi^2$-divergence [Su, 1995] (also see [Makur, 2019, Lemma 2.3] or [Makur and Zheng, 2020, Lemma 3] and the references therein).

By the tensorization of KL divergence, we have

$$D \left( \otimes_{j=1}^n P_{Y|X^{(j)}, \theta^{(1)}} \| \otimes_{j=1}^n P_{Y|X^{(j)}, \theta^{(2)}} \right) = \sum_{j=1}^n D \left( P_{Y|X^{(j)}, \theta^{(1)}} \| P_{Y|X^{(j)}, \theta^{(2)}} \right).$$ (64)

Now suppose $\Phi_{ji}(t) = \mathbb{P}[Z_i \leq t]$ with $Z_i \sim \text{Bernoulli}(p_i)$ for any $1 \leq i \leq d, 1 \leq j \leq n$. Then, for any $\theta \in \Delta^{d-1}$, we have that $\sum_{i=1}^d \theta_i \Phi_{ji}(t) = \mathbb{P}[Z_\theta \leq t]$ with $Z_\theta \sim \text{Bernoulli}(p^T \theta)$ where $p = [p_1, \ldots, p_d]^T$. Let $P_{p^ \theta}$ be the probability measure induced by the Bernoulli distribution with parameter $p \in \mathbb{R}$. By definition, the $\chi^2$-divergence between two different Bernoulli distributions with parameters $p^T \theta^{(1)}$ and $p^T \theta^{(2)}$ is

$$\chi^2 \left( P_{Y|X^{(j)}, \theta^{(1)}} \| P_{Y|X^{(j)}, \theta^{(2)}} \right) = \chi^2 \left( P_{p^ \theta^{(1)}} \| P_{p^ \theta^{(2)}} \right) = \frac{(p^T (\theta^{(1)} - \theta^{(2)}))^2}{p^T \theta^{(2)}}$$

$$= \frac{(p^T (\theta^{(1)} - \theta^{(2)}))^2}{1 - p^T \theta^{(2)}}$$

$$\leq \frac{2a^2 \sum_{i=1}^d p_i^2}{1 - p^T \theta^{(2)}}$$ (65)

where (65) is by Cauchy-Schwarz inequality and (62). Suppose $d \geq 2$ and $\frac{1}{2d^2} \leq p_i \leq \frac{1}{d^2}$. Then, we have

$$\sum_{i=1}^d p_i^2 \leq \frac{1}{d^3}$$

and

$$\frac{1}{2d^2} \leq p^T \theta^{(2)} \leq \frac{1}{d^2}.$$ (66)

Thus, for any $d \geq 2$, we have

$$\frac{2a^2 \sum_{i=1}^d p_i^2}{(1 - p^T \theta^{(2)}) p^T \theta^{(2)}} \leq \frac{2a^2}{d^3} \cdot \frac{2d^2}{1 - \frac{1}{2d^2}} \leq \frac{32a^2}{7d}$$

which, together with (63) and (65), implies that

$$D \left( P_{Y|X^{(j)}, \theta^{(1)}} \| P_{Y|X^{(j)}, \theta^{(2)}} \right) \leq \frac{32a^2}{7d}. \quad (67)$$
By (61), (64), (67) and Fano's method [Fano, 1961], we have

\[
\mathcal{R}(\theta(P_{x^{1:n}})) \geq \mathcal{R}(\theta(P_B))
\]

\[
= \inf_{\hat{\theta}} \sup_{P \in \mathcal{P}_B} \mathbb{E}_P[|\hat{\theta}(y^{(1)}, \ldots, y^{(n)}) - \theta(P)|]
\]

\[
\geq a \delta \left( 1 - \frac{\sup_{\theta(1, \theta(2)) \in \mathcal{V}_a} D\left( \sum_{j=1}^{n} F_{\theta(1)}(x^{(j)}, .) \bigg\| \sum_{j=1}^{n} F_{\theta(2)}(x^{(j)}, .) \right) + \log 2}{\log |\mathcal{V}_a|} \right)
\]

\[
\geq a \delta \left( 1 - \frac{32n^2 + \log 2}{6d \log 2} \right)
\]

\[
\geq \frac{a \sqrt{2e}}{12 \sqrt{\pi d}} \left( 1 - \frac{32n^2 + 7d \log 2}{7d^2 \log(2)} \right)
\]

(68)

where (68) follows from the fact that \( \mathcal{P}_B \subseteq \mathcal{P}_{x^{1:n}} \).

Choosing \( a = \Theta\left(\frac{4}{\sqrt{n}}\right) \), by (69), we have \( \mathcal{R}(\theta(P_{x^{1:n}})) = \Omega\left(\frac{\sqrt{2}}{n}\right) \).

D Proof of Corollary 7

Assume that \( X^{(1)}, \ldots, X^{(n)} \) are independent random variables in \( \mathcal{X} \). For any fixed sequence \( x^{1:n} = (x^{(1)}, \ldots, x^{(n)}) \in \mathcal{X}^n \), denote by \( \mathcal{P}_{X,Y^{1:1:n}} \subseteq \mathcal{P} \) the family of the joint distributions of \( (Y^{(1)}, X^{(2)}, \ldots, Y^{(n)}, X^{(n)}) \) whose marginal distribution on \( (X^{(1)}, \ldots, X^{(n)}) \) is \( 1_{x^{1:n}} \), i.e., the delta mass on \( x^{1:n} \). Then, we have

\[
\mathcal{R}(\theta(\mathcal{P})) = \inf_{\hat{\theta}} \sup_{P \in \mathcal{P}} \mathbb{E}_P[|\hat{\theta}(y^{(1)}, \ldots, y^{(n)}) - \theta(P)|]
\]

\[
= \inf_{\hat{\theta}} \sup_{P \in \mathcal{P}_{X,Y^{1:1:n}}} \mathbb{E}_P\left[ |\hat{\theta}(y^{(1)}, \ldots, y^{(n)}) - \theta(P)||X^{(1)}, \ldots, X^{(n)}| \right]
\]

\[
\geq \inf_{\hat{\theta}} \sup_{P \in \mathcal{P}_{X,Y^{1:1:n}}} \mathbb{E}_P\left[ |\hat{\theta}(y^{(1)}, \ldots, y^{(n)}) - \theta(P)||X^{(1)}, \ldots, x^{(n)}| \right]
\]

\[
\geq \frac{a \sqrt{2e}}{12 \sqrt{\pi d}} \left( 1 - \frac{32n^2 + 7d \log 2}{7d^2 \log(2)} \right)
\]

(70)

where (70) follows from (69).

Choosing \( a = \Theta\left(\frac{4}{\sqrt{n}}\right) \), we have that \( \mathcal{R}(\theta(\mathcal{P})) = \Omega\left(\frac{\sqrt{2}}{n}\right) \).

E Proof of Theorem 8

In the setting of Theorem 8, the sample \( \{(x^{(j)}, y^{(j)})\}_{j \in \mathbb{N}} \) is generated according to Scheme I, and similar to setting of Appendix A, we consider the underlying probability space for the sample to be \([0, 1]^N, \mathcal{B}([0, 1]^N), \mathbb{P}\) which is already defined at the beginning of Appendix A. Define the random vector \( \Xi \) to be the identity mapping from \([0, 1]^N\) onto itself as in Appendix A. Then, \( \Xi \) follows the uniform distribution on \([0, 1]^N\). Suppose \( \{(x^{(j)}, y^{(j)})\}_{j \in \mathbb{N}} \) is sampled according to Scheme I with \( F \) defined in (13). Then, according to [Bogachev, 2007, Proposition 10.7.6],
for each \( j \in \mathbb{N} \), there exist some \((\mathcal{B}(\mathcal{X}) \otimes \mathcal{B}(S))^{j-1} \otimes \mathcal{B}([0, 1])/\mathcal{B}(\mathcal{X})\)-measurable function \( h_{X}^{(j)} : (\mathcal{X} \times S)^{j-1} \times [0, 1] \to \mathcal{X} \) and \((\mathcal{B}(\mathcal{X}) \otimes \mathcal{B}(S))^{j-1} \otimes \mathcal{B}(\mathcal{X}) \otimes \mathcal{B}([0, 1])/\mathcal{B}(S)\)-measurable function \( h_{Y}^{(j)} : (\mathcal{X} \times S)^{j-1} \times \mathcal{X} \times [0, 1] \to S \) such that \( x^{(j)} = h_{X}^{(j)}(x^{(1)}, y^{(1)}, \ldots, x^{(j-1)}, y^{(j-1)}, \Xi^{(j-1)}) \), \( y^{(j)} = h_{Y}^{(j)}(x^{(1)}, y^{(1)}, \ldots, x^{(j-1)}, y^{(j-1)}, x^{(j)}, \Xi^{(j)}) \), and

\[
\mathbb{E}\left[\mathbb{1}\left\{ h_{Y}^{(j)}(x^{(1)}, y^{(1)}, \ldots, x^{(j-1)}, y^{(j-1)}, x^{(j)}, \Xi^{(j)}) \right\}\mid \mathcal{F}_{j-1}\right] = \theta_{*}^{\top}\Phi(x^{(j)}, t) + e(x^{(j)}, t) \tag{71}
\]

for any \( t \in S \) and \( j \in \mathbb{N} \), where \( \mathcal{F}_{j} := \sigma\left(\{\Xi^{(k)} : k \in [2j + 1]\}\right) \). With the same proof provided at the beginning of Appendix A, \( \{y^{(j)}\}_{j \in \mathbb{N}} \) is \( \{\mathcal{F}_{j}\}_{j \in \mathbb{N}} \)-adapted, \( x^{(j)} \) is \( \mathcal{F}_{j-1}/\mathcal{B}(\mathcal{X}) \)-measurable, and \( \Phi_{j} \) is \( (\mathcal{F}_{j-1} \otimes \mathcal{B}(S))/\mathcal{B}([0, 1]^{d}) \)-measurable for each \( j \in \mathbb{N} \). Since \( e : \mathcal{X} \times S \to [-1, 1] \), \( (x, t) \mapsto e(x, t) \) is \( \mathcal{B}(\mathcal{X}) \otimes \mathcal{B}(S) \)-measurable and \( x^{(j)} \) is \( \mathcal{F}_{j-1}/\mathcal{B}(\mathcal{X}) \)-measurable, we have that \( e_{j} : [0, 1]^{\mathbb{N}} \times S \to [-1, 1] \), \( (\xi, t) \mapsto e(x^{(j)}(\xi), t) \) is \( \mathcal{F}_{j-1} \otimes \mathcal{B}(S) \)-measurable.

Define \( V_{j} := \int_{S} I_{y^{(j)}} \Phi_{j} - \int_{S} (\theta_{*}^{\top}\Phi_{j} + e_{j}) \Phi_{j} \). Since \( I_{y^{(j)}} \) is \( \mathcal{F}_{j} \otimes \mathcal{B}(S) \)-measurable and \( e_{j} \) and \( \Phi_{j} \) are \( \mathcal{F}_{j-1} \otimes \mathcal{B}(S) \)-measurable, by Fubini’s theorem and (71), We have

\[
\mathbb{E}[V_{j} \mid \mathcal{F}_{j-1}] = \mathbb{E}\left[\int_{S} I_{y^{(j)}} \Phi_{j} \mid \mathcal{F}_{j-1}\right] - \int_{S} (\theta_{*}^{\top}\Phi_{j} + e_{j}) \Phi_{j}
\]

\[
= \int_{S} \mathbb{E}\left[I_{y^{(j)}} \mid \mathcal{F}_{j-1}\right] \Phi_{j} - \int_{S} (\theta_{*}^{\top}\Phi_{j} + e_{j}) \Phi_{j}
\]

\[
= \int_{S} (\theta_{*}^{\top}\Phi_{j} + e_{j}) \Phi_{j} - \int_{S} (\theta_{*}^{\top}\Phi_{j} + e_{j}) \Phi_{j}
\]

\[
= 0.
\]

For any \( \alpha \in \mathbb{R}^{d} \), if \( n = 0 \), define \( M_{n}(\alpha) = 1 \). If \( n \geq 1 \), define \( M_{n}(\alpha) := \exp\left\{ \alpha^{\top}W_{n} - \frac{1}{2}\|\alpha\|_{U_{n}}^{2}\right\} \) for \( W_{n} := \sum_{j=1}^{n} V_{j} \) and \( U_{n} = \sum_{j=1}^{n} \int_{S} \Phi_{j} \Phi_{j}^{\top} \). Then, with the similar proof as in Appendix A.1, we can show that \( W_{n} \) is \( \mathcal{F}_{n} \)-measurable, \( U_{n} \) is \( \mathcal{F}_{n-1} \)-measurable, and \( M_{n} \) is \( \mathcal{F}_{n} \otimes \mathcal{B}(\mathbb{R}^{d}) \)-measurable for any \( n \in \mathbb{N} \). Thus, for any \( \alpha \in \mathbb{R}^{d} \), \( \{M_{n}(\alpha)\}_{n \geq 0} \) is \( \{\mathcal{F}_{n}\}_{n \geq 0} \)-adapted. Moreover, for any \( \alpha \in \mathbb{R}^{d} \) and \( n \in \mathbb{N} \), we have

\[
\mathbb{E}[M_{n}(\alpha) \mid \mathcal{F}_{n-1}] = M_{n-1}(\alpha) \mathbb{E}\left[\exp\left\{ \alpha^{\top}V_{n} - \frac{1}{2}\alpha^{\top}\left(\int_{S} \Phi_{n} \Phi_{n}^{\top}\right) \alpha\right\} \mid \mathcal{F}_{n-1}\right]
\]

\[
= M_{n-1}(\alpha) \mathbb{E}\left[\frac{\exp\left\{ \alpha^{\top}V_{n}\right\}}{\exp\left\{ \frac{1}{2}\int_{S} (\alpha^{\top}\Phi_{n})^{2}\right\}} \right]. \tag{72}
\]

Since \( -\int_{S} |\alpha^{\top}\Phi_{n}| \leq \alpha^{\top}V_{n} \leq \int_{S} |\alpha^{\top}\Phi_{n}| \) a.s., we have

\[
\mathbb{E}\left[\exp\left\{ \alpha^{\top}V_{n}\right\} \mid \mathcal{F}_{n-1}\right] \leq \exp\left\{ \frac{4}{8}\left(\int_{S} |\alpha^{\top}\Phi_{n}|\right)^{2}\right\}
\]

\[
\leq \exp\left\{ \frac{1}{2}\int_{S} (\alpha^{\top}\Phi_{n})^{2}\right\} \tag{73}
\]

\[
\leq \exp\left\{ \frac{1}{2}\int_{S} (\alpha^{\top}\Phi_{n})^{2}\right\} \tag{74}
\]
where (73) is by Cauchy-Schwarz inequality and \( \int_S 1 = m(S) = 1 \). Then, by (72) and (74), we have

\[
\mathbb{E}[M_n(\alpha) | \mathcal{F}_{n-1}] \leq M_{n-1}(\alpha) \frac{\exp\left\{ \frac{1}{2} \int_S (\alpha^T \Phi_n)^2 \right\}}{\exp\left\{ \frac{1}{2} \int_S (\alpha^T \Phi_n)^2 \right\}} = M_{n-1}(\alpha).
\]

Thus, for any \( \alpha \in \mathbb{R}^d \), \( \{M_n(\alpha)\}_{n \geq 0} \) is a super-martingale.

Now define \( \bar{M}_n := \int_{\mathbb{R}^d} M_n(\alpha) h(\alpha) d\alpha \) for

\[
h(\alpha) = \left( \frac{\lambda}{2\pi} \right)^{\frac{d}{2}} \exp\left\{ -\frac{\lambda}{2} \alpha^T \alpha \right\} = \left( \frac{\lambda}{2\pi} \right)^{\frac{d}{2}} \exp\left\{ -\frac{1}{2} \|\alpha\|^2 \right\}. \]

Then, with the same calculation as (26) in Appendix A.1, we have \( \bar{M}_n = \frac{\lambda^{d/2}}{\det(U_n(\lambda))^{1/2}} \exp\left( \frac{1}{2} \|W_n\|^2 \|U_n(\lambda)^{-1}\right) \). By Fubini’s theorem, \( M_n \) is \( \mathcal{F}_n \otimes \mathcal{B}(\mathbb{R}^d) \)-measurable implies that \( \bar{M}_n \) is \( \mathcal{F}_n \)-measurable for any \( n \geq 0 \). With the same analysis as (27), \( \{\bar{M}_n\}_{n \geq 0} \) is a super-martingale.

By Doob’s maximal inequality for super-martingales, we have that

\[
P\left[ \sup_{n \in \mathbb{N}} \bar{M}_n \geq \delta \right] \leq \frac{\mathbb{E}[\bar{M}_0]}{\delta} = \frac{1}{\delta}
\]

which implies that for any \( N \in \mathbb{N} \),

\[
P \left[ \exists n \in [N] \text{ s.t. } \|W_n\|_{U_n(\lambda)^{-1}} \geq \sqrt{\log \frac{\det(U_n(\lambda))}{\lambda^d} + 2 \log \frac{1}{\delta}} \right] \leq \delta.
\]

According to (32), we have

\[
\|W_n\|_{U_n(\lambda)^{-1}} \leq \sqrt{d \log \left( 1 + \frac{n}{\lambda} \right) + 2 \log \frac{1}{\delta}} \tag{75}
\]

for all \( n \in \mathbb{N} \) with probability at least \( 1 - \delta \).

By (3), (29), and the definition of \( V_j \), we have

\[
\hat{\theta}_\lambda - \theta_* = U_n(\lambda)^{-1}\left( \sum_{j=1}^n V_j + E_n - \lambda \theta_* \right) = U_n(\lambda)^{-1}W_n + U_n(\lambda)^{-1}(E_n - \lambda \theta_*). \tag{76}
\]

where \( E_n = \sum_{j=1}^n \int_S e_j \Phi_j \) by definition. Thus,

\[
\|\hat{\theta}_\lambda - \theta_*\|_{U_n(\lambda)} \leq \|U_n(\lambda)^{-1}W_n\|_{U_n(\lambda)} + \|U_n(\lambda)^{-1}(E_n - \lambda \theta_*\|_{U_n(\lambda)} \leq \|W_n\|_{U_n(\lambda)^{-1}} + \|\lambda \theta_*\|_{U_n(\lambda)^{-1}} + \|E_n\|_{U_n(\lambda)^{-1}} \leq \|W_n\|_{U_n(\lambda)^{-1}} + \sqrt{\lambda}\|\theta_*\| + \|E_n\|_{U_n(\lambda)^{-1}} \tag{77}
\]

where (77) is because of \( U_n(\lambda)^{-1} = \frac{1}{\lambda} (I - U_n(\lambda)^{-1}U_n) \) and \( \|I - U_n(\lambda)^{-1}U_n\|_2 \leq 1 \).

By (75) and (77), for any \( \delta \in (0, 1) \), with probability at least \( 1 - \delta \), we have

\[
\|\hat{\theta}_\lambda - \theta_*\|_{U_n(\lambda)} \leq \sqrt{d \log \left( 1 + \frac{n}{\lambda} \right) + 2 \log \frac{1}{\delta} + \sqrt{\lambda}\|\theta_*\| + \|E_n\|_{U_n(\lambda)^{-1}}. \tag{78}
\]
for all \( n \in \mathbb{N} \). Since \( U_n(\lambda) - \lambda I_d \) is positive semi-definite, (78) immediately implies that

\[
\|\hat{\theta}_\lambda - \theta\|_{U_n(\lambda)} \leq \sqrt{d \log \left(1 + \frac{n}{\lambda}\right) + 2 \log \frac{1}{\delta} + \sqrt{\lambda}\|\theta_\ast\| + \frac{1}{\sqrt{\lambda}}\|E_n\|}
\]  

(79)

which is exactly (14).

F Proof of Corollary 9

In the setting of Corollary 9, the sample \( \{(x^{(j)}, y^{(j)})\}_{j \in \mathbb{N}} \) is generated according to Scheme II. In the following proof, we consider the underlying probability space for the sample \( \{(x^{(j)}, y^{(j)})\}_{j \in \mathbb{N}} \) to be \( ([0,1]^N, \mathcal{B}([0,1]^N), \mathbb{P}) \) which has already been defined at the beginning of Appendix A. Define the random vector \( \Xi \) to be the identity mapping from \([0,1]^N\) onto itself as in Appendix A. Then, \( \Xi \) follows the uniform distribution on \([0,1]^N\). Suppose \( \{(x^{(j)}, y^{(j)})\}_{j \in \mathbb{N}} \) is sampled according to Scheme II with \( F \) defined in (13). Then, according to [Bogachev, 2007, Proposition 10.7.6], for each \( j \in \mathbb{N} \), there exist some \( \mathcal{B}([0,1])/\mathcal{B}(\mathcal{X}) \)-measurable function \( h_X^{(j)} : [0,1] \to \mathcal{X} \) and \( \mathcal{B}(\mathcal{X}) \otimes \mathcal{B}([0,1])/\mathcal{B}(S) \)-measurable function \( h_Y^{(j)} : \mathcal{X} \times [0,1] \to S \) such that \( x^{(j)} = h_X^{(j)}(\Xi^{(2j-1)}) \), \( y^{(j)} = h_Y^{(j)}(x^{(j)}, \Xi^{(2j)}) \), and

\[
\mathbb{E}\left[ \mathbb{I}\left\{ h_Y^{(j)}(x^{(j)}, \Xi^{(2j)}) \leq t \right\} \mid F_{j-1} \right] = \theta_\ast^T \Phi(x^{(j)}, t) + e(x^{(j)}, t)
\]

(80)

for any \( t \in S \) and \( j \in \mathbb{N} \), where \( F_j := \sigma\left( \{ \Xi^{(k)} : k \in [2j+1] \} \right) \). With the same proof provided at the beginning of Appendix A, \( \{y^{(j)}\}_{j \in \mathbb{N}} \) is \( (F_j)_{j \in \mathbb{N}} \)-adapted and \( \Phi_j \) is \( (F_{j-1} \otimes \mathcal{B}(S)) / \mathcal{B}([0,1]^d) \)-measurable for each \( j \in \mathbb{N} \). Moreover, \( \{x^{(j)}\}_{j \in \mathbb{N}} \) is independent, which implies that \( \{\Phi_j(t)\}_{j \in \mathbb{N}} \) is independent for any \( t \in S \), \( \{e_j(t)\}_{j \in \mathbb{N}} \) is independent for any \( t \in S \), and \( \{y^{(j)}\}_{j \in \mathbb{N}} \) is independent.

Let \( b_j(t) := \mathbb{E}[e_j(t) \Phi_j(t)] \) for \( t \in S \) and \( j \in [n] \). Then, by Fubini’s theorem, \( b_j \) is measurable with \( b_j(t) \in [-1, 1] \) for \( t \in S \), \( j \in \mathbb{N} \) and \( i \in [d] \). By definition and Fubini’s theorem, we have \( B_n = \sum_{j=1}^n \int_S b_j \).

Define \( V_j := \int_S I_{y^{(j)}} \Phi_j - \int_S \theta_\ast^T \Phi_j \Phi_j - \int_S b_j \). By Fubini’s theorem and (80), we have

\[
\mathbb{E}[V_j] = \mathbb{E}\left[ \int_S (I_{y^{(j)}} - \theta_\ast^T \Phi_j) \Phi_j \right] - \int_S b_j
\]

\[
= \int_S \mathbb{E}\left[ (I_{y^{(j)}} - \theta_\ast^T \Phi_j) \Phi_j \right] - \int_S b_j
\]

\[
= \int_S \mathbb{E}\left[ \mathbb{E}[I_{y^{(j)}} - \theta_\ast^T \Phi_j] | F_{j-1} \right] \Phi_j - \int_S b_j
\]

\[
= \int_S \mathbb{E}[e_j \Phi_j] - \int_S b_j
\]

\[
= \int_S b_j - \int_S b_j
\]

\[
= 0.
\]
For any $\alpha \in \mathbb{R}^d$, if $n = 0$, define $M_n(\alpha) = 1$. If $n \geq 1$, define $M_n(\alpha) := \exp \{ \alpha^\top W_n - \frac{1}{2} \| \alpha \|^2_{U_n} \}$ for $W_n := \sum_{j=1}^n V_j$ and $U_n = \sum_{j=1}^n \int_S \Phi_j \Phi_j^\top$. Similar to Appendix E, we can show that $M_n$ is $F_n \otimes \mathcal{B}(\mathbb{R}^d)$-measurable for any $n \geq 0$. Moreover, for any $n \in \mathbb{N}$,

$$
\mathbb{E}[M_n(\alpha) | F_{n-1}] = M_{n-1}(\alpha) \mathbb{E} \left[ \exp \left\{ \alpha^\top V_n - \frac{1}{2} \alpha^\top \left( \int_S \Phi_n \Phi_n^\top \right) \alpha \right\} | F_{n-1} \right] = M_{n-1}(\alpha) \frac{\mathbb{E} \left\{ \exp \left\{ \alpha^\top V_n \right\} | F_{n-1} \right\}}{\exp \left\{ \frac{1}{2} \int_S (\alpha^\top \Phi_n)^2 \right\}} \tag{81}
$$

with $-\int_S |\alpha^\top \Phi_n| - \int_S \alpha^\top b_n \leq \alpha^\top V_n \leq \int_S |\alpha^\top \Phi_n| - \int_S \alpha^\top b_n$ a.s.. Thus,

$$
\mathbb{E} \left[ \exp \left\{ \alpha^\top V_n \right\} | F_{n-1} \right] \leq \exp \left\{ \frac{1}{2} \int_S (\alpha^\top \Phi_n)^2 \right\} \leq \exp \left\{ \frac{1}{2} \int_S (\alpha^\top \Phi_n)^2 \right\} \tag{82}
$$

Then, by (81) and (82), we have

$$
\mathbb{E}[M_n(\alpha) | F_{n-1}] \leq M_{n-1}(\alpha) \frac{\exp \left\{ \frac{1}{2} \int_S (\alpha^\top \Phi_n)^2 \right\}}{\exp \left\{ \frac{1}{2} \int_S (\alpha^\top \Phi_n)^2 \right\}} = M_{n-1}(\alpha).
$$

Thus, for any $\alpha \in \mathbb{R}^d$, $\{M_n(\alpha)\}_{n \geq 0}$ is a super-martingale. With the same approach as in Appendix E, for any $\lambda \in (0, \infty)$, we can show that

$$
\| \hat{\theta}_n - \theta_*\|_{U_n(\lambda)} \leq \sqrt{d \log \left( 1 + \frac{n}{\lambda} \right) + 2 \log \frac{1}{\delta} + \sqrt{\lambda} \| \theta_* \| + \frac{1}{\sqrt{\lambda}} \| B_n \|} \tag{83}
$$

for all $n \in \mathbb{N}$ with probability at least $1 - \delta$. Then, using the same analysis as in Appendix B.2, we can show that for any $\delta_1 \in (0, 1)$, $\delta_2 \in (0, 1 - \delta_1)$, and $n \geq \frac{2dE}{\delta_2} \log(d/\delta_1)$, we have

$$
\| \hat{\theta}_n - \theta_*\|_{\Sigma_n} \leq \sqrt{2 \left( d \log \left( 1 + \frac{1}{\lambda} \right) + 2 \log \frac{1}{\delta_2} \right) + \sqrt{2} \| \theta_* \| + \frac{2}{\lambda} \| B_n \|}
$$

with probability at least $1 - \delta_1 - \delta_2$. Then, (15) is obtained by setting $\delta_1 = \delta_2 = \delta \in (0, 1/2)$. 

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