Convexity of reachable sets of nonlinear ordinary differential equations

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Abstract

We present a necessary and sufficient condition for the reachable set, i.e., the set of states reachable from a ball of initial states at some time, of an ordinary differential equation to be convex. In particular, convexity is guaranteed if the ball of initial states is sufficiently small, and we provide an upper bound on the radius of that ball, which can be directly obtained from the right hand side of the differential equation. In finite dimensions, our results cover the case of ellipsoids of initial states. A potential application of our results is inner and outer polyhedral approximation of reachable sets, which becomes extremely simple and almost universally applicable if these sets are known to be convex. We demonstrate by means of an example that the balls of initial states for which the latter property follows from our results are large enough to be used in actual computations.

1 Introduction

Reachability problems play a central part in a wide range of control related problems, including safety and liveness verification, diagnosis, controller synthesis, optimization and others [1, 2, 3, 4, 5, 6, 7]. The vast majority of methods developed in that context compute approximations of reachable sets in an intermediate step [8, 3, 5], which may simplify considerably if the reachable set is known to be convex. Consider, for example, an autonomous ordinary differential equation $\dot{x} = f(x)$ with smooth flow $\varphi: U \subseteq \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, a subset $\Omega \subseteq \mathbb{R}^n$ of initial states and some $t_1 \in \mathbb{R}$ with $\{t_1\} \times \Omega \subseteq U$, and assume $\Omega$ is closed with nonempty interior and smooth boundary $\partial \Omega$. Inner and outer polyhedral approximations to the reachable set $\varphi(t_1, \Omega)$ from $\Omega$ at time $t_1$
are then computed easily. In particular, if \( v \) is an outside normal to \( \partial \Omega \) at \( x \in \partial \Omega \), an outside normal to the boundary \( \partial \varphi(t_1, \Omega) \) at \( \varphi(t_1, x) \in \partial \varphi(t_1, \Omega) \) can be obtained from the solution of the adjoint to the variational equation along \( \varphi(\cdot, x) \) with initial value \( v \) [9]. Thus, a convex reachable set may be efficiently approximated by inner and outer polyhedra up to arbitrary precision, see Fig. 1. Similar ideas apply to systems with inputs, e.g. [10, 8, 11]. Thus, the question arises under what conditions reachable sets are convex.

The more general problem of whether the image of a nonlinear map is convex appears in the context of optimization and optimal control [12, 13, 14] and is related to some geometric problems with a long history [15, 16, 17]. Recently, Zampieri and Gorni [18] have obtained a criterion for a local homeomorphism between open subsets of real finite dimensional spaces to be one-to-one and to have a convex image. They have also shown the image is convex provided that a certain matrix is positive semi-definite everywhere and the local homeomorphism actually is a global \( C^2 \)-diffeomorphism. Polyak [19] has presented a sufficient condition for the image of a ball under a local \( C^{1,1} \)-submersion (\( C^1 \) with Lipschitz-continuous derivative) between real Hilbert spaces to be convex, from which a duality result and an efficient algorithm for nonconvex optimization problems restricted to a sufficiently small ball follow. Further sufficient conditions for the convexity of the image of convex compact subsets of real finite dimensional spaces under homeomorphisms and \( C^\infty \)-subimmersions, respectively, have been presented by Bobylev, Emel’yanov and Korovin [20] and Vakhrameev [21].

Consider now the control system

\[
\dot{x} = f(t, x, u(t)),
\]

(1)
where $f : U \subseteq \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $u$ is from a set $U$ of admissible controls, and denote the reachable set of (1) from $\Omega \subseteq \mathbb{R}^n$ at time $t_1$ by $\mathcal{R}(t_1, \Omega)$,

$$\mathcal{R}(t_1, \Omega) = \{ \varphi(t_1) \mid \varphi(0) \in \Omega \text{ and } \varphi : [0, t_1] \to \mathbb{R}^n \text{ is a solution of (1) for some } u \in U \}.$$ 

A result of Pliš implies that, under suitable assumptions, which include convexity conditions on $\Omega$ and on images of $f(t, x, \cdot)$, the reachable set $\mathcal{R}(t_1, \Omega)$ is convex for sufficiently small $t_1 > 0$ [22]. Lojasiewicz improved upon that result by giving an explicit upper bound on $t_1$ [23]. For prescribed $t_1 > 0$ and $\Omega$ a singleton, Polyak has shown under different hypotheses that $\mathcal{R}(t_1, \Omega)$ is convex if $U$ is a ball of sufficiently small radius in the space of square integrable functions $[0, t_1] \to \mathbb{R}^m$ [24]. Recently, Azhmyakov, Flockerzi and Raisch [25] have presented a related result for a closed-loop variant of (1) to which we give a counterexample in section 4. Further sufficient conditions for convexity of the reachable set of (1) are known for rather special classes of right hand sides of (1), e.g. [26, 27, 28].

When applied to the problem described at the beginning of this section, the results from [22, 18, 19] cited above ensure that the reachable set $\varphi(t_1, \Omega)$ is convex if $\Omega$ is a Euclidean ball of radius $r$, and $t_1$ [22, 23] or $r$ [23, 18, 19] does not exceed some bound. However, that bound could be extremely small and of no practical value, as is the case with the reachability problem studied by Polyak [24, p. 262].

In this paper, we present a necessary and sufficient condition for the convexity of the reachable set of the ordinary differential equation (ODE)

$$\dot{x} = f(t, x)$$ (2)

from a ball $\Omega$ of initial values. (Note that the uncertainty comes from a set of initial values only. In contrast to the control system (1) investigated in [22, 23, 24], there are no inputs to (2).) In particular, convexity is guaranteed if $\Omega$ is sufficiently small, and we provide an upper bound on the radius of $\Omega$, which can be directly obtained from the right hand side $f$ of (2). We also demonstrate by means of an example that the balls of initial states for which our results imply the convexity of the reachable set are large enough to be used in actual computations, such as in local programming techniques [19] and polyhedral approximation of reachable sets discussed at the beginning of this section. Our results extend those in [29, 30, 31].

The remaining of this paper is structured as follows. After having introduced basic terminology in section 2, we establish a criterion for the convexity
of a sublevel set $\Omega$,

$$\Omega = \{ x \in U \mid g(x) \leq 0 \},$$  \hspace{1cm} (3)

in terms of generalized second-order directional derivatives of $g$ in section 3, where $g: U \subseteq X \to \mathbb{R}$ is of class $C^{1,1}$ and $X$ is a real Banach space. We also present a criterion, rather than a sufficient condition, for the image $F(\Omega)$ of $\Omega$ under a $C^{1,1}$-diffeomorphism $F$ to be convex. In section 4 we investigate reachable sets from a ball $\Omega$ of initial states through solutions of the ordinary differential equation (ODE) (2), where $f: U \subseteq \mathbb{R} \times X \to X$ is continuous and $X$ is a real Hilbert space. We establish a sharp upper bound on the radius of $\Omega$ that ensures convexity of the reachable set under the assumption that $f$ is of class $C^{1,1}$ with respect to its second argument and also present a necessary and sufficient condition for convexity under the assumption that $f$ is of class $C^2$ with respect to its second argument. In section 5 we apply our results to the equations of the damped mathematical pendulum.

The reader will notice that stronger smoothness assumptions than those adopted in this paper would have simplified both notation and arguments considerably. However, such simplification would have come at the expense of narrowing applicability of our results since many commonly used models of physical systems involve $C^{1,1}$-functions that are not of class $C^2$, e.g. [32, Sec. 9.1]. On the other hand, if we had weakened smoothness requirements further, beyond $C^{1,1}$, certain geometric properties of the boundary of $\Omega$ and $F(\Omega)$, respectively, that are related to curvature, would have become lost. See also the short discussion at the end of section 3. To conclude, we believe that for the problems investigated in this paper, $C^{1,1}$-smoothness of both maps and sets is a rather natural assumption.

2 Preliminaries

Throughout this paper, “iff” abbreviates “if and only if”, and $X$ and $Y$ denote real Banach spaces with norm $\| \cdot \|$ unless specified otherwise. $B(x, r)$ and $\bar{B}(x, r)$ denote the open and closed, respectively, ball of radius $r$ centered at $x$, and the space of continuous linear operators $X \to Y$ is denoted by $\mathcal{L}(X, Y)$.

$\mathbb{R}$ and $\mathbb{R}_+$ denote the field of real numbers and its subset of nonnegative real numbers, respectively, and $[a, b]$, $]a, b[$, $[a, b[$, and $]a, b]$ denote the closed, open and halfopen, respectively, intervals with end points $a$ and $b$, $a < b$. 


sign denotes the signum function. We write \( y \geq x \) and \( x \leq y \) for \( x, y \in \mathbb{R}^n \) if \( y - x \in \mathbb{R}^n_+ \).

The domain of a map \( f \) is denoted by \( \text{dom } f \), \( f \circ g \) denotes the composition of \( f \) and \( g \), \( (f \circ g)(x) = f(g(x)) \), \( \text{id} \) denotes the identity map, \( f^{-1} \) is used for the inverse of \( f \) as well as for preimages, and \( \ker f \) denotes the nullspace of \( f \) if \( f \) is linear. If \( L \) is \( k \)-linear, we set \( Lh^k := L(h, \ldots, h) \).

Arithmetic operations involving subsets of a linear space are defined pointwise, e.g. \( \alpha M := \{ \alpha y \mid y \in M \} \), \( M + N := \{ y + z \mid y \in M, z \in N \} \) if \( \alpha \in \mathbb{R} \) and \( M, N \subseteq X \). \( \partial M \) denotes the boundary of \( M \subseteq X \), and \( \dim M \) denotes the dimension of a linear subspace \( M \subseteq X \).

\( D^jf \) denotes the derivative of order \( j \) of \( f \), and \( D^j_i f \), the partial derivative of order \( j \) with respect to the \( i \)th argument of \( f \), and \( D_i f := D^1_i f \), \( f' := Df := D^1 f \), and \( f'' := D^2 f \). \( C^k \) denotes the class of \( k \) times continuously differentiable maps, and \( C^{k,1} \), the class of maps in \( C^k \) with (locally) Lipschitz-continuous \( k \)th derivative. Let \( U \subseteq X \) be open. \( f : U \to \mathbb{R} \) is a submersion at \( x \in U \) if \( f \) is of class \( C^1 \) on a neighborhood of \( x \) and \( f'(x) \) is surjective. \( f \) is a submersion on \( V \subseteq U \) if \( f \) is a submersion at each point \( x \in V \). A \( K \)-submersion is a submersion of class \( K \) whenever \( K \) is one of the classes of maps defined above.

For \( f : U \subseteq X \to \mathbb{R} \) of class \( C^1 \) with \( U \) open, we define the four generalized second-order directional derivatives \( D^2_{\pm} f \) and \( \overline{D}^2_{\pm} f \),

\[
D^2_{\pm} f(x, h, k) = \liminf_{\pm t \to 0} \frac{f'(x + th)k - f'(x)k}{t},
\]

\[
\overline{D}^2_{\pm} f(x, h, k) = \limsup_{\pm t \to 0} \frac{f'(x + th)k - f'(x)k}{t},
\]

for all \( x \in U \) and all \( h, k \in X \). If \( D^2 \) is any of the operators defined above, we note that \( D^2 f \) is positively homogeneous in both its second and its third argument and define \( D^2 f(x, h^2) := D^2 f(x, h, h) \). Furthermore, it is easily verified that

\[
D^2_{\pm} f(x, -h, -k) = D^2_{\pm} f(x, h, k),
\]

\[
\overline{D}^2_{\pm} f(x, -h, -k) = \overline{D}^2_{\pm} f(x, h, k).
\]

2.1 Proposition. Let \( U \subseteq X \) be open and convex, \( f : U \to \mathbb{R} \) be of class \( C^1 \), and let \( D^2 \) be one of the four operators defined in (4)-(5). Then \( f \) is convex iff \( D^2 f(x, h^2) \geq 0 \) for all \( x \in U \) and all \( h \in X \).
For these and related concepts and results, see [33, 34] and the references given there.

Let \( I \subseteq \mathbb{R} \) be an interval, \( U \subseteq I \times X \) be relatively open in \( I \times X \), the map \( f : U \to X \) be (locally) Lipschitz-continuous with respect to its second argument and continuous, and \( V \subseteq \{ (\tau, t, x) \in \mathbb{R} \times \mathbb{R} \times X \mid (t, x) \in U \} \). \( \varphi : V \to X \) is called the general solution of (2) if for all \( (t_0, x_0) \in U, \varphi(\cdot, t_0, x_0) \) is the maximal solution of the initial value problem composed of (2) and the initial condition \( x(t_0) = x_0 \) [9]. The map \( (t, x) \mapsto \varphi(t, 0, x) \) is called the flow of (2) if \( \varphi \) is the general solution of (2) and (2) is autonomous.

Let now \( X \) be a Hilbert space with inner product \( \langle \cdot | \cdot \rangle \). Two vectors \( x \) and \( y \) are perpendicular, \( x \perp y \), if \( \langle x | y \rangle = 0 \). \( \| x \| \) and \( L^* \) denote the norm of \( x \) and the adjoint of the linear map \( L \), respectively, with respect to \( \langle \cdot | \cdot \rangle \). We define continuous maps \( \mu_{\pm} : \mathcal{L}(X, X) \to \mathbb{R} \) by
\[
\mu_-(A) = \inf \{ \langle Ax | x \rangle \mid \| x \| = 1 \}, \\
\mu_+(A) = \sup \{ \langle Ax | x \rangle \mid \| x \| = 1 \}.
\]

The following result is sometimes referred to as Ważewski’s inequality. Its proof given in [35] carries over to the Hilbert space setting.

**2.2 Proposition.** Let \( I \subseteq \mathbb{R} \) be an interval, \( A : I \to \mathcal{L}(X, X) \) be continuous, and \( x : I \to X \) be a solution of \( \dot{x} = A(t)x \). Then
\[
\| x(t_0) \| e^{\int_{t_0}^t \mu_-(A(\tau))d\tau} \leq \| x(t) \| \leq \| x(t_0) \| e^{\int_{t_0}^t \mu_+(A(\tau))d\tau}
\]
for all \( t, t_0 \in I \) with \( t \geq t_0 \).

### 3 Convexity of images of sublevel sets under diffeomorphisms

In this section, we present a necessary and sufficient condition for the convexity of the image \( F(\Omega) \) of a sublevel set \( \Omega \) from (3) under a diffeomorphism \( F \). We assume that both the map \( g \) from (3) and the diffeomorphism \( F \) are of class \( C^{1,1} \); see our remarks at the end of section 1. We therefore use the generalized derivatives defined in (4)-(5); the corresponding differentiation operators are denoted \( \mathcal{D} \) throughout this section.

We first present a characterization of the convexity of sublevel sets on which our subsequent results are based. Note that the requirement that \( \Omega \) be closed is automatically met if \( U = X \).
3.1 Theorem. Let $U \subseteq X$ be open, $g: U \to \mathbb{R}$ be continuous and a $C^{1,1}$-submersion on its zero set, and let $\Omega$ defined by (3) be closed and connected. Let further $\mathcal{D}^2$ be one of the four operators defined in (4)-(5). Then $\Omega$ is convex iff \( \mathcal{D}^2 g(x, h^2) \geq 0 \) for all $x \in \partial \Omega$ and all $h \in \ker g'(x)$.

Proof. Assume $\partial \Omega \neq \emptyset$ without loss and observe that our hypotheses imply $g^{-1}(0) = \partial \Omega$. Let us call $\mu: W \to \mathbb{R}$ a representation of $\partial \Omega$ about $x$ in $Z$ with respect to the direction $v$ if $x \in \partial \Omega$, $v \in X$, $W \subseteq \ker g'(x)$ is a convex open neighborhood of the origin, and there is an open interval $V$ containing the origin such that $Z = x + W + V v$ and $\mu(h) \in V$ for all $h \in W$, and

$$Z \cap \Omega = \{x + h + \lambda v \mid h \in W, \lambda \in V, \lambda \geq \mu(h)\}.$$  \hfill (6)

An application of the implicit function theorem to the equation $g(x + h + \lambda v) = 0$ for $h \in \ker g'(x)$ and $\lambda \in \mathbb{R}$ shows that for all $x \in \partial \Omega$ and all $v \in X$ with $g'(x)v < 0$ there is a representation $\mu$ of $\partial \Omega$ about $x$ with respect to $v$ that is of class $C^1$ and fulfills

$$\mu'(h)\xi = - (g'(p(h))v)^{-1} g'(p(h))\xi$$  \hfill (7)

for all $h \in \text{dom} \mu$ and all $\xi \in \ker g'(x)$, where $\text{dom} \mu$ denotes the domain of $\mu$ and $p(h) = x + h + v \mu(h)$.

Let $\mu$ be such a representation. It follows from (6) that $Z \cap \Omega$ is convex iff $\mu$ is; see [36] for a proof of an analogous result on epigraphs. Further, (6) yields $g'(x)v < 0$, and (7) implies that $\mu$ is actually of class $C^{1,1}$. We prove

$$\mathcal{D}^2 \mu(h, \xi^2) = - (g'(p(h))v)^{-1} \mathcal{D}^2 g(p(h), (g'(h)\xi)^2)$$  \hfill (8)

for all $h \in \text{dom} \mu$ and all $\xi \in \ker g'(x)$, which is the key relation.

Let $\mathcal{D}^2 = \mathcal{D}^2_+$. For $h = 0$, (8) reduces to

$$\liminf_{t \to 0} \frac{g'(x + t \xi + v \mu(t \xi))\xi}{t} = \liminf_{t \to 0} \frac{g'(x + t \xi)\xi}{t}$$  \hfill (9)

since $\mu$ and $g'$ are continuous, $g'(x)v < 0$, and $g'(x)\xi = 0$. As $g'$ is uniformly Lipschitz-continuous in a neighborhood of $x$ and $\mu'(0) = 0$ we obtain

$$\lim_{t \to 0} \frac{g'(x + t \xi + v \mu(t \xi))\xi - g'(x + t \xi)\xi}{t} = 0.$$
which implies (9). Therefore, for $D = D^2$,

$$D^2 \kappa(0, \zeta^2) = -(g'(y)v)^{-1} D^2 g(y, \zeta^2)$$  \hspace{1cm} (10)

for all representations $\kappa$ of $\partial \Omega$ about $y$ with respect to the direction $v$ and all $\zeta \in \ker g'(y)$. For the other three operators defined in (4)-(5), (10) is obtained in exactly the same way.

Let now $h \in \text{dom} \mu$ be arbitrary, let $P$ be the projection operator along $v$ onto $\ker g'(x)$, let $y = x + h + v\mu(h)$, and define $\kappa$ on a neighborhood of the origin in $\ker g'(y)$ by $v\kappa(s) = v\mu(h + Ps) - v\mu(h) - (\text{id} - P)s$, which implies

$$D^2 \kappa(0, \zeta^2) = D^2 \mu(h, (P\zeta)^2)$$  \hspace{1cm} (11)

for all $\zeta \in \ker g'(y)$. Choose convex open neighborhoods of the origin $W' \subseteq \ker g'(y)$ and $V' \subseteq \mathbb{R}$ such that $Z' := y + W' + V'v \subseteq Z$ and $\kappa(s) \in V'$ whenever $s \in W'$. It is easily verified that the restriction of $\kappa$ to $Z'$ is a representation of $\partial \Omega$ about $y$ in $Z'$ with respect to $v$. Hence, (11) and (10) for $\zeta = p'(h)\xi$ give (8).

With (8) at our disposal, we are now in a position to prove the theorem. It follows from (8) that $D^2 g(x, h^2) \geq 0$ for all $x \in \partial \Omega$ and all $h \in \ker g'(x)$ iff for all $x \in \partial \Omega$ there is a representation $\mu$ of $\partial \Omega$ about $x$ with $D^2 \mu(h, \xi^2) \geq 0$ for all $h \in \text{dom} \mu$ and all $\xi \in \ker g'(x)$. By Prop. 2.1, the latter condition is equivalent to the convexity of $\mu$, which in turn is equivalent to the convexity of $Z \cap \Omega$ for some neighborhood $Z \subseteq X$ of $x$. As $\Omega$ is closed and connected, application of a (generalization of a) theorem of Tietze-Nakajima [37] completes the proof.

The criterion for the convexity of the set $\Omega$ presented in Theorem 3.1 takes the form of a condition on the map $g$ that defines $\Omega$. In finite dimensions, i.e., if $X = \mathbb{R}^n$, the oriented distance function of $\Omega$ is a $C^{1,1}$-submersion on its zero set under our assumptions [38], and hence, is a natural choice for the map $g$ in Theorem 3.1. However, the condition presented in Theorem 3.1 actually describes metric properties of the boundary of $\Omega$ rather than properties of maps defining $\Omega$. In particular, if $X = \mathbb{R}^n$ and the map $g$ is of class $C^2$, then $D^2 g(x, h^2) = g''(x)h^2$, and the restriction of $g''(x)$ to $\ker g'(x)$ coincides with the second fundamental form [39] of $\partial \Omega$ at $x$ up to a positive scalar factor. Hence, Theorem 3.1 implies the following well-known result: A closed, connected set of class $C^2$ is convex iff the second fundamental form of its boundary is positive semi-definite everywhere.
Next we present a criterion for the convexity of the image of a sublevel set \( \Omega \) under a diffeomorphism. Note that the requirement that \( F(\Omega) \) be closed is automatically met if \( \Omega \) is compact or \( V = Y \). We do not assume that \( \Omega \) itself is convex.

\[ 3.2 \text{ Theorem. Let } U \subseteq X \text{ and } V \subseteq Y \text{ be open, } g: U \to \mathbb{R} \text{ be continuous and a } C^{1,1}\text{-submersion on its zero set, let } \Omega \text{ defined by (3) be closed and connected, } F: U \to V \text{ be a } C^{1,1}\text{-diffeomorphism, and } F(\Omega) \text{ be closed. Let further } \mathcal{D}^2 \text{ be one of the four operators defined in (4)-(5). Then } F(\Omega) \text{ is convex iff }
\]

\[
\mathcal{D}^2 \left( g'(x)F'(x)^{-1}F(\cdot) \right)(x, h^2) \leq \mathcal{D}^2 g(x, h^2)
\]  

for all \( x \in \partial \Omega \) and all \( h \in \ker g'(x) \).

**Proof.** Under our hypotheses, \( F(\Omega) \) is closed and connected, \( F(\partial \Omega) = \partial F(\Omega) \), and

\[
F(\Omega) = \{ y \in V \mid f(y) \leq 0 \}
\]

for \( f := g \circ F^{-1}: V \to \mathbb{R} \). In addition, \( F'(x)h \in \ker f'(F(x)) \) iff \( h \in \ker g'(x) \), for all \( x \in \partial \Omega \). Therefore, by Theorem 3.1, \( F(\Omega) \) is convex iff

\[
d^2 f(F(x), (F'(x)h)^2) \geq 0
\]  

for all \( x \in \partial \Omega \) and all \( h \in \ker g'(x) \), whenever \( d^2 \) is one of the four operators defined in (4)-(5). We first establish the relations

\[
\mathcal{D}^2_\pm f(F(x), (F'(x)h)^2) \leq \mathcal{D}^2_\pm g(x, h^2) - \mathcal{D}^2_\pm (g'(x)F'(x)^{-1}F(\cdot))(x, h^2),
\]

\[
\overline{\mathcal{D}}^2_\pm f(F(x), (F'(x)h)^2) \geq \overline{\mathcal{D}}^2_\pm g(x, h^2) - \overline{\mathcal{D}}^2_\pm (g'(x)F'(x)^{-1}F(\cdot))(x, h^2),
\]

\[
\underline{\mathcal{D}}^2_\pm f(F(x), (F'(x)h)^2) \leq \underline{\mathcal{D}}^2_\pm g(x, h^2) - \underline{\mathcal{D}}^2_\pm (g'(x)F'(x)^{-1}F(\cdot))(x, h^2),
\]

\[
\overline{\mathcal{D}}^2_\pm f(F(x), (F'(x)h)^2) \geq \overline{\mathcal{D}}^2_\pm g(x, h^2) - \overline{\mathcal{D}}^2_\pm (g'(x)F'(x)^{-1}F(\cdot))(x, h^2).
\]

Let \( x \in \partial \Omega \) and \( h \in \ker g'(x) \). We assume \( X = Y \), \( x = F(x) = 0 \), and \( F'(0) = \text{id} \) without loss of generality to obtain

\[
\mathcal{D}^2_\pm f(0, h^2) = \liminf_{\pm t \to 0} \frac{f'(th)h}{t},
\]

\[
\overline{\mathcal{D}}^2_\pm f(0, h^2) = \limsup_{\pm t \to 0} \frac{f'(th)h}{t},
\]

\[
\underline{\mathcal{D}}^2_\pm (g'(0)F(\cdot))(0, h^2) = \liminf_{\pm t \to 0} \frac{f'(0)F'(th)h}{t}.
\]
Continuity of \( f' \) and Lipschitz-continuity of \( F' \) imply \( \lim_{t \to 0} (f'(th) - f'(0))(F'(th)h - h)/t = 0 \), hence

\[
D_2^2 g(0, h^2) = \liminf_{\pm t \downarrow 0} \frac{f'(F(th))F'(th)h}{t} = \liminf_{\pm t \downarrow 0} \left( \frac{f'(th)h}{t} + \frac{f'(0)F'(th)h}{t} \right). \tag{21}
\]

(14) and (15) follow from (18), (19), (20), and (21). (16) and (17) are shown by analogous arguments.

Assume now \( F(\Omega) \) is convex. Then (13), (14) and (16) imply (12). Conversely, (12), (15) and (17) imply (13) for at least one of the operators defined in (4)-(5), and hence, \( F(\Omega) \) is convex.

\[ \square \]

3.3 Corollary. Let \( X, Y \) be real Hilbert spaces, \( U \subseteq X \) and \( V \subseteq Y \) be open, \( F: U \to V \) be a \( C^{1,1} \)-diffeomorphism, \( \Omega \subseteq U \) be a closed ball centered at \( x_0 \), \( F(\Omega) \) be closed, and \( D^2 \) be one of the four operators defined in (4)-(5). Then \( F(\Omega) \) is convex iff

\[
D^2 \langle x - x_0 | F'(x)^{-1} F'(\cdot) \rangle (x, h^2) \leq 1 \tag{22}
\]

for all \( x \in \partial \Omega \) and all \( h \perp (x - x_0) \) with \( \|h\| = 1 \).

Proof. Set \( g(x) := \|x - x_0\|^2 - r^2 \), where \( r \) is the radius of \( \Omega \), and apply Theorem 3.2. \[ \square \]

In contrast to related results in [18, 40, 20], the conditions in Theorem 3.2 and in Corollary 3.3 are to be checked on the boundary of \( \Omega \) and for tangent vectors only. Further, as with Theorem 3.1, the criteria in Theorem 3.2 and Corollary 3.3 take particularly simple forms if the maps \( F \) and \( g \) are smooth. If \( F \) is of class \( C^2 \), the left hand side of (12) and (22) equals \( g'(x)F'(x)^{-1} F''(x)h^2 \) and \( \langle x - x_0 | F'(x)^{-1} F''(x)h^2 \rangle \), respectively, and if \( g \) is of class \( C^2 \), then \( D^2 g(x, h^2) = g''(x)h^2 \) in Theorem 3.2.

The following is an immediate consequence of Corollary 3.3: The image of a ball centered at the origin under a \( C^{1,1} \)-diffeomorphism defined on a neighborhood of the origin in a Hilbert space is convex provided the radius of the ball is sufficiently small [18, 40]. The assumption that the diffeomorphism be of class \( C^{1,1} \) rather than merely \( C^1 \) is essential [18]. For \( C^2 \)-diffeomorphisms the result is obvious [18] and can also be concluded from a well-known result on the existence of geodetically convex neighborhoods in Riemannian manifolds; see [41] and also [18, Sec. 6]. Polyak has raised the question of whether the result extends to uniformly convex Banach spaces [42]. It does not, as the following example shows.
3.4 Example. Endow \( \mathbb{R}^2 \) with the norm \( \| \cdot \| \) defined by \( \| x \| = (|x_1|^p + |x_2|^p)^{1/p} \) for some real \( p > 2 \), which makes \( \mathbb{R}^2 \) a uniformly convex space. Then \( \bar{B}(0,r) = \{ x \in \mathbb{R}^2 \mid g(x) \leq 0 \} \) for arbitrary \( r > 0 \), where we have set \( g(x) = \| x \|^p - r^p \). The map \( g \) is of class \( C^2 \). For \( x = (r,0) \in \partial B(0,r) \), \( h = (0,1) \in \ker g'(x) \), \( \varepsilon > 0 \), and \( F: x \mapsto (x_1 + \varepsilon x_2^2, x_2) \) we obtain \( g''(x)h^2 = 0 \) and \( g'(x)F'(x)^{-1}F''(x)h^2 = 2\varepsilon pr^{p-1} \). By Theorem 3.2, \( F(\bar{B}(0,r)) \) is not convex, no matter how small \( r \) and \( \varepsilon \) are.

We finally remark that the assumption in Theorem 3.2 and Corollary 3.3 that \( F \) be a global diffeomorphism on a neighborhood of \( \Omega \) could easily be relaxed.

4 Convexity of reachable sets

In this section, we investigate reachable sets from a set \( \Omega \) of initial states through solutions of the ordinary differential equation (ODE) (2), where the right hand side \( f: U \subseteq \mathbb{R} \times X \rightarrow X \) of (2) is of class \( C^{1,1} \) or \( C^2 \) with respect to its second argument and continuous. We also give a counterexample to a related result for control systems recently presented by Azhmyakov, Flockerzi and Raisch [25]. For the sake of simplicity, we restrict ourselves to the case where \( \Omega \) is a ball and \( X \) is a real Hilbert space, so that Corollary 3.3 applies.

The general solution of (2) is denoted \( \varphi \) throughout this section. In order to avoid confusion of ideas, we would like to remind the reader that according to the notation adopted in section 2, operators \( D_2 \) and \( D_3 \) refer to the partial derivative of a map with respect to its second and third, respectively, argument, whereas \( D^2 \) refers to the second order derivative, and \( D_2^2 \) and \( D_3^2 \), to the second order partial derivative with respect to the second and third, respectively, argument.

Our first result is a sufficient condition for the set of states reachable from a ball of initial values to be convex. That condition takes the form of an upper bound on the radius of the ball of initial values, which will be shown to be sharp later in Corollary 4.4.

4.1 Theorem. Let \( I \subseteq \mathbb{R} \) be an interval, \( U \subseteq I \times X \) be relatively open in \( I \times X \), and the right hand side \( f: U \rightarrow X \) of (2) be of class \( C^{1,1} \) with respect to its second argument and continuous. Let further \( x_0 \in X \), \( r > 0 \) and \( t_0, t_1 \in I \) be such that \( \{t_1\} \times \{t_0\} \times \bar{B}(x_0,r) \subseteq \text{dom} \varphi \). Finally, assume
there are $M_1, M_2 \in \mathbb{R}$ that

\[
M_1 \geq \begin{cases} 
2 \mu_+ (D_2 f(\tau, x)) - \mu_- (D_2 f(\tau, x)), & \text{if } t_1 \geq t_0, \\
\mu_+ (D_2 f(\tau, x)) - 2 \mu_- (D_2 f(\tau, x)), & \text{otherwise},
\end{cases}
\]

\[
M_2 \geq \limsup_{h \to 0} \frac{\|D_2 f(\tau, x + h) - D_2 f(\tau, x)\|}{\|h\|}
\]

holds for all $(\tau, x) \in U$, and define $K$ by

\[
K(\alpha) = \begin{cases} 
|t_1 - t_0|, & \text{if } \alpha = 0, \\
(\exp(\alpha |t_1 - t_0|) - 1) / \alpha, & \text{otherwise}.
\end{cases}
\]

Then the reachable set $\varphi(t_1, t_0, B(x_0, r))$ is convex if

\[
r M_2 K(M_1) \leq 1.
\]

**Proof.** Pick arbitrary $x \in \partial B(x_0, r)$ and $h \perp (x - x_0)$ with $\|h\| = 1$. We show that (22) holds for $F := \varphi(t_1, t_0, \cdot)$, which implies $\varphi(t_1, t_0, B(x_0, r))$ is convex by Corollary 3.3. To this end, we assume without loss of generality $t_0 = 0$, $t_1 \neq 0$, $x = 0$, and $I = [0, t_1]$ if $t_1 > 0$ and $I = [t_1, 0]$, otherwise.

As $I$ is compact, $\varphi$ is continuous, and $U$ is relatively open in $I \times X$, we may choose $\delta > 0$ such that $(\tau, \varphi(\tau, 0) + h) \in U$ for all $\tau \in I$ and all $h \in B(0, \delta)$. Then (24) implies

\[
\|D_2 f(\tau, \varphi(\tau, 0) + h) - D_2 f(\tau, \varphi(\tau, 0))\| \leq M_2 \|h\|
\]

for all $\tau \in I$ and all $h \in B(0, \delta)$.

Pick arbitrary $\varepsilon > 0$. By the relative openness of the domain of $\varphi$ in $I \times I \times X$, the continuity of $\varphi$ and $D_3 \varphi$, and the compactness of $I$, there is some neighborhood $W \subseteq B(0, \delta)$ of the origin in $X$ such that for all $w \in W$ and all $\tau \in I$ we have $I \subseteq \text{dom} \varphi(\cdot, 0, w)$ as well as the following estimates:

\[
\|D_3 \varphi(\tau, 0, w)\| \leq (1 + \varepsilon)\|D_3 \varphi(\tau, 0, 0)\|,
\]

\[
\|\varphi(\tau, 0, w) - \varphi(\tau, 0, 0)\| < \delta.
\]

Define maps $Z$ and $A$ on $I \times W$ by

\[
Z(\tau, w) = D_3 \varphi(\tau, 0, w) - D_3 \varphi(\tau, 0, 0),
\]

\[
A(\tau, w) = (D_2 f(\tau, \varphi(\tau, 0, w)) - D_2 f(\tau, \varphi(\tau, 0, 0))) D_3 \varphi(\tau, 0, w)
\]

12
and use the variational equation of (2) along \( \varphi(\cdot, 0, w) \) and \( \varphi(\cdot, 0, 0) \), respectively, to obtain

\[
D_1 Z(\tau, w) = D_2 f(\tau, \varphi(\tau, 0, 0)) Z(\tau, w) + A(\tau, w) \tag{30}
\]

for all \((\tau, w) \in I \times W\). (30) is a linear differential equation in \( Z(\cdot, w) \), and \( Z(0, \cdot) = 0 \). Hence \( Z(t_1, w) = D_3 \varphi(t_1, 0, 0) \int_0^{t_1} D_3 \varphi(\tau, 0, 0)^{-1} A(\tau, w) d\tau \), which implies

\[
\|D_3 \varphi(t_1, 0, 0)^{-1} Z(t_1, w)\| \leq \left| \int_0^{t_1} \|D_3 \varphi(\tau, 0, 0)^{-1}\| \cdot \|A(\tau, w)\| d\tau \right| \tag{31}
\]

for all \( w \in W \). Use the mean value theorem to estimate \( \|\varphi(\tau, 0, w) - \varphi(\tau, 0, 0)\| \) and then apply (27), (28), and (29) to obtain

\[
\|A(\tau, w)\| \leq (1 + \varepsilon)^2 M_2 \|D_3 \varphi(\tau, 0, 0)\|^2 \|w\|. \tag{32}
\]

From the variational equation of (2) along \( \varphi(\cdot, 0, 0) \), its adjoint, (23), Ważewski’s inequality (Prop. 2.2), and (25) we obtain \( \left| \int_0^{t_1} \|D_3 \varphi(\tau, 0, 0)^{-1}\| \cdot \|D_3 \varphi(\tau, 0, 0)\|^2 d\tau \right| \leq K(M_1) \), regardless of the sign of \( t_1 \), so that \( \|F'(0)^{-1}(F'(w) - F'(0))\| \leq (1 + \varepsilon)^2 M_2 K(M_1) \|w\| \) for all \( w \in W \) by (31) and (32). Now let \( \varepsilon \) tend to 0 to obtain (22) from \( r M_2 K(M_1) \leq 1 \).

The following is an immediate consequence of Theorem 4.1.

4.2 Corollary. Let \( U, f, x_0, r, t_0, t_1, M_2 \) and \( K \) as in Theorem 4.1, assume there are \( \lambda_-, \lambda_+ \in \mathbb{R} \) that

\[
\lambda_- \leq \mu_-(D_2 f(\tau, x)) \leq \mu_+(D_2 f(\tau, x)) \leq \lambda_+ \tag{33}
\]

holds for all \((\tau, x) \in U\), and define \( M_1 \) by

\[
M_1 = \begin{cases} 2\lambda_+ - \lambda_- & \text{if } t_1 \geq t_0, \\ \lambda_+ - 2\lambda_- & \text{otherwise.} \end{cases} \tag{34}
\]

Then the reachable set \( \varphi(t_1, t_0, \bar{B}(x_0, r)) \) is convex if (26) holds.

The main advantage of Theorem 4.1 over the results from section 3 is that the bound on the radius can be determined directly from properties of the right hand side of (2). Note that Theorem 4.1 cannot be obtained from applying any of the estimates from the literature [18, 40, 20] to the map...
\(F := \varphi(t_1, t_0, \cdot)\). In fact, separately estimating \(|F'(x)|^{-1}\) and \(|F''(x)|\) gives a larger bound in general.

We would like to comment on our hypotheses. First note that \(\mu_-(A)\) and \(\mu_+(A)\) equal the minimum and maximum, respectively, eigenvalues of the self-adjoint part

\[
\frac{1}{2}(A + A^*)
\]

of \(A\) if \(X = \mathbb{R}^n\). Hence, (23) and (33) reduce to bounds on eigenvalues of (35). If the ball \(\bar{B}(x_0, r)\) of initial values in Theorem 4.1 and Corollary 4.2 is an Euclidean ball, or equivalently, if the inner product \(\langle \cdot | \cdot \rangle\) equals the Euclidean inner product \((\cdot | \cdot)\) given by

\[
(x|y) = \sum_{i=1}^{n} x_i y_i,
\]

then \(A^*\) is just the transpose \(A^T\) of \(A\). If the ball \(\bar{B}(x_0, r)\) is an ellipsoid rather than Euclidean, then there is a symmetric positive definite matrix \(Q\) such that \(\langle x | y \rangle = (x | Qy)\), from which \(A^* = Q^{-1}A^TQ\) follows. This shows conditions (23) and (33) can be readily verified.

Second, note that in the Euclidean case, the result of Lojasiewicz [23] for the control system (1) applied to the special case investigated in this section would result in a similar bound on radii. In fact, apart from some technical hypotheses, the sufficient condition (26) for the convexity of the reachable set is obtained from [23], with \(M_1\) defined by \(M_1 \geq 3\|D_2f(\tau, x)\|\) rather than by (23) or (34). This gives a bound on the radius \(r\) that is never larger and in general smaller than the bounds presented in Theorem 4.1 and Corollary 4.2 since \(2 \mu_+(A) - \mu_-(A) \leq \frac{3}{2} \|A + A^T\| \leq 3 \|A\|\).

Third, note that if \(U = I \times \mathbb{R}^n\) in Theorem 4.1 or in Corollary 4.2, then the condition \(\{t_1\} \times \{t_0\} \times \bar{B}(x_0, r) \subseteq \text{dom} \varphi\) is automatically fulfilled. Indeed, it is straightforward to obtain \(\text{dom} \varphi = I \times I \times \mathbb{R}^n\) from condition (23) in this case.

Finally, note that condition (24) is just a bound on \(|D_2^2f(\tau, x)|\) in case \(f\) is of class \(C^2\) with respect to its second argument. For this case the proof of Theorem 4.1 can be specialized to obtain a necessary and sufficient condition:

**4.3 Theorem.** Let \(U, f, x_0, r, t_0\) and \(t_1\) as in Theorem 4.1 and assume in addition that \(f\) is of class \(C^2\) with respect to its second argument. Then
\[ \varphi(t_1, t_0, \bar{B}(x_0, r)) \text{ is convex iff} \]
\[
\int_{t_0}^{t_1} \langle x - x_0 | D_3 \varphi(\tau, t_0, x)^{-1} D_2^2 f(\tau, \varphi(\tau, t_0, x))(D_3 \varphi(\tau, t_0, x)h)^2 \rangle \, d\tau \leq 1
\]

for all \( x \in \partial B(x_0, r) \) and all \( h \perp (x - x_0) \) with \( \|h\| = 1 \).

**Proof.** Pick arbitrary \( x \in \partial B(x_0, r) \) and \( h \perp (x - x_0) \) with \( \|h\| = 1 \). We show that the left hand sides of (37) and (22) coincide if \( F = \varphi(t_1, t_0, \cdot) \).

As \( F \) is of class \( C^2 \), we obtain

\[ F'(x)^{-1} F''(x) h^2 = D_3 \varphi(t_1, t_0, x)^{-1} D_3^2 \varphi(t_1, t_0, x) h^2. \]

Since \( D_3 \varphi(\cdot, t_0, x) \) is a solution of the variational equation, \( D_3 \varphi(\cdot, t_0, x)h \) solves the initial value problem

\[
\dot{z}(\tau) = D_2^2 f(\tau, \varphi(\tau, t_0, x)) z(\tau),
\]
\[ z(t_0) = h, \]

which implies that \( D_3^2 \varphi(\cdot, t_0, x)h^2 \) solves

\[
\dot{z}(\tau) = D_2^2 f(\tau, \varphi(\tau, t_0, x)) z(\tau) + D_2^2 f(\tau, \varphi(\tau, t_0, x)) (D_3 \varphi(\tau, t_0, x)h)^2,
\]
\[ z(t_0) = 0. \]

Hence, \( F'(x)^{-1} F''(x) h^2 = \int_{t_0}^{t_1} D_3 \varphi(\tau, t_0, x)^{-1} D_2^2 f(\tau, \varphi(\tau, t_0, x)) (D_3 \varphi(\tau, t_0, x)h)^2 \, d\tau. \)

\[ \square \]

In contrast to the hypotheses of Theorem 4.1 and Corollary 4.2, which may be verified by direct inspection of the right hand side \( f \) of (2), the necessary and sufficient condition for convexity of the image \( \varphi(t_1, t_0, \bar{B}(x_0, r)) \) of the diffeomorphism \( \varphi(t_1, t_0, \cdot) \) that is established in Theorem 4.3 contains the diffeomorphism itself. The advantage of the latter result over a direct application of (the \( C^2 \)-version of) Corollary 3.3 is that the second derivative of that diffeomorphism does not appear in condition (37). Hence, in order to estimate the left hand side of (37) for a particular example of (2), one has to study the variational equation of (2) and its adjoint only. That way one may obtain bounds on the radius strictly greater than the one given in Theorem 4.1. This is demonstrated in section 5.

With the criterion from Theorem 4.3 at our disposal, we are now able to prove that the bound on the radius given in Corollary 4.2 is sharp, from which the sharpness of the bound in Theorem 4.1 easily follows.
4.4 Corollary. Let there be given \( t_1, t_0, \lambda_-, \lambda_+, M_2, r \in \mathbb{R} \), where \( \lambda_- < \lambda_+ \), \( M_2 > 0 \), let \( M_1 \) and \( K \) be defined by (34) and (25), and assume \( \dim X \geq 2 \) and \( r M_2 K(M_1) > 1 \).

Then there is an autonomous ODE (2) with analytic right hand side \( f \) defined on \( \mathbb{R} \times X \) such that (33) and (24) hold for all \((\tau, x) \in \mathbb{R} \times X\), yet \( \varphi(t_1, t_0, \bar{B}(0, r)) \) is not convex.

Proof. Assume \( X = \mathbb{R}^2 \) and \( t_0 = 0 \) without loss of generality. Since \( K \) is continuous and \( \lambda_- < \lambda_+ \) we may pick \( \varepsilon \in ]0, \lambda_+ - \lambda_-[ \) such that \( r M_2 K(M_1 - \varepsilon) > 1 \). Set \( \nu_\pm = \lambda_\pm \mp \varepsilon/3 \) and define \( g \) and \( f \) by

\[
g(s) = \alpha^2 (2M_2)^{-1} \left( \sin \left( \frac{M_2 s}{\alpha} \right) \right)^2 ,
\]

\[
f(\tau, x) = (\nu_- x_1 + g(x_2), \nu_+ x_2)
\]

to obtain \( \|D^2 f(\tau, x)\| \leq M_2 \) for all \( x \) and any \( \alpha > 0 \). By our choice of \( \nu_\pm \) and the continuity of eigenvalues, and since \( |g'(s)| \leq \alpha \) for all \( s \in \mathbb{R} \), we may choose \( \alpha > 0 \) such that \( \mu_-(D_2 f(\tau, x)), \mu_+(D_2 f(\tau, x)) \in [\lambda_-, \lambda_+] \) for all \( x \in X \). Thus, (33) and (24) hold.

Direct calculation shows that for \( x = 0, h = (0, 1) \), \( x_0 = (-r, 0) \), and \( t_1 > 0 \), the left hand side of (37) equals \( r M_2 K(M_1 - \varepsilon) \), and thus, application of Theorem 4.3 shows \( \varphi(t_1, 0, \bar{B}(x_0, r)) \) is not convex. If \( t_1 < 0 \), set \( \nu_\pm = \lambda_\pm \mp \varepsilon/3 \) and \( x_0 = (r, 0) \) instead to proceed in exactly the same way. As \( K(M_1) \neq 0 \) implies \( t_1 \neq t_0 \), the claim is proved after applying the change of coordinates \( x \mapsto x - x_0 \).

Recently, Azhmyakov, Flockerzi and Raisch have investigated the closed-loop variant

\[
\begin{align*}
\dot{x} &= f(x, u(x)) \quad \text{(38a)} \\
x(0) &= x_0, u(x) \in U & \text{(38b)}
\end{align*}
\]

of the control system (1), where the control set \( U \subseteq \mathbb{R}^m \) is compact and convex, solutions to (38) are assumed to exist on \([0, t_1]\) for any \( U \)-valued measurable feedback \( u \) and to be uniformly bounded, the right hand side \( f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \) is merely Lipschitz-continuous, and admissible controls are \( U \)-valued and Lipschitz-continuous [25]. It has been claimed that the reachable set

\[
\mathcal{R}(t_1, x_0) = \{ x(t_1) \mid x \text{ is a solution of (38) for some admissible feedback } u \}
\]

16
is convex provided \( t_1 \) is small enough [25]. It follows from geometric ideas employed in [22, 23, 24] and the present paper, in particular from Example 3.4 and Corollary 4.4, that such a claim must be wrong. We provide a simple counterexample below.

4.5 Example. Consider the special case

\[
\begin{align*}
\dot{x}_1 &= u(x), \\
\dot{x}_2 &= u(x)^2, \\
x(0) &= (0, 0), u(x) \in [0, 1]
\end{align*}
\]

of (38) on some time interval \([0, t_1]\). As the constant feedbacks 0 and 1 are admissible, we have \((0, 0), (t_1, t_1) \in \mathcal{R}(t_1, (0, 0))\) for any \( t_1 > 0 \). If \( \mathcal{R}(t_1, (0, 0)) \) were convex, then \((t_1/2, t_1/2) \in \mathcal{R}(t_1, (0, 0))\), which implies \( \int_0^{t_1} u(x(\tau)) d\tau = t_1/2 = \int_0^{t_1} u(x(\tau))^2 d\tau \) for some admissible control \( u \) and corresponding solution \( x \). Hence, the integral \( \int_0^{t_1} u(x(\tau)) - u(x(\tau))^2 d\tau \) vanishes, and so does its continuous, nonnegative integrand. It follows that either \( u \circ x = 0 \) or \( u \circ x = 1 \), which is a contradiction. So, \( \mathcal{R}(t_1, (0, 0)) \) is not convex, which proves [25, Theorems 2 and 3] wrong. It is easily seen that the same system is also a counterexample to [25, Theorem 1].

5 Application

In this section, we demonstrate the application of the results from section 4 to the equations of the damped mathematical pendulum,

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -\omega^2 \sin(x_1) - 2\gamma x_2,
\end{align*}
\]  

where \( \omega > 0 \) and \( \gamma \geq 0 \) [35]. Note that the investigation of the convexity of reachable sets of more general systems, such as a cart-pole system with a piecewise constant control, can be reduced to the autonomous system (39) [30].

The results of sections 3 and 4 cover the case of images of ellipsoids as we have allowed for arbitrary inner products. In this section, we restrict ourselves to the case of images of Euclidean balls for the sake of simplicity.

The above ODE is of the form

\[
\dot{x} = f(x),
\]  

17
where \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) is given by

\[
f(x) = \left( -\omega^2 \sin(x_1) - 2\gamma x_2 \right).
\]

We first demonstrate a straightforward application of Corollary 4.2:

**5.1 Theorem.** Assume \( \omega > 0, \gamma \geq 0 \) and \( t_1 \neq 0 \), let \( M_1 \) and \( R \) be given by

\[
M_1 = -\text{sign}(t_1)\gamma + 3\sqrt{\gamma^2 + (1 + \omega^2)^2/4}, \tag{41}
\]

\[
R = \frac{M_1}{\omega^2(\exp(M_1|t_1|) - 1)}, \tag{42}
\]

and let \( \varphi \) be the flow of the pendulum equation (39). Then the image of any ball with radius not exceeding \( R \) under the map \( \varphi(t_1, \cdot) \) is convex.

**Proof.** From

\[
f'(x) = \begin{pmatrix} 0 & 1 \\ -\omega^2 \cos(x_1) & -2\gamma \end{pmatrix}
\]

(43)

we obtain the minimum and maximum eigenvalues \( \lambda_{\pm} = -\gamma \pm \sqrt{\gamma^2 + (1 + \omega^2)^2/4} \) of the symmetric part \( \frac{1}{2}(f'(x) + f'(x)^*) \) of \( f'(x) \). As

\[
f''(x)h^2 = \begin{pmatrix} 0 \\ \omega^2 h_1^2 \sin(x_1) \end{pmatrix}, \tag{44}
\]

Corollary 4.2 is applicable with \( M_1 \neq 0 \) and \( M_2 = \omega^2 \) if the ball is closed. The theorem is proved since the image of an open ball under the map \( \varphi(t_1, \cdot) \) is the interior of the image of the closure of that ball. \( \square \)

The following larger bound on the radius is obtained from Theorem 4.3.

**5.2 Theorem.** Let \( \omega, \gamma, \kappa_{\pm}, t_1, R \in \mathbb{R} \) with \( 0 \leq \gamma \leq \omega, 1 \leq \omega, \kappa_{\pm} = \sqrt{\omega^2 \pm \gamma^2}, 2\kappa_{-}t_1 \leq \pi, 0 \leq t_1 \),

\[
R = \frac{6\omega \kappa_{+}}{(1 + (\omega + \gamma)^2)^{3/2} \sinh(\kappa_{+}t_1)(\cosh(2\kappa_{+}t_1) + 5 - 10 \exp(-\omega))}, \tag{45}
\]

and let \( \varphi \) be the flow of the pendulum equation (39). Then the image of any ball with radius not exceeding \( R \) under the map \( \varphi(t_1, \cdot) \) is convex.
Proof. According to Theorem 4.3 it suffices to prove
\[
\int_0^{t_1} \langle (-h_2, h_1) | D_2 \varphi(\tau, x_0)^{-1} f''(\varphi(\tau, x_0)) (D_2 \varphi(\tau, x_0) h)^2 \rangle d\tau \leq 1/R
\] (46)
for all \(x_0, h \in \mathbb{R}^2\) with \(\|h\| = 1\). \(D_2 \varphi(\cdot, x_0)\) is an operator solution of the variational equation \(\dot{x} = f'(\varphi(t, x_0)) x\) of (39) along \(\varphi(\cdot, x_0)\), and \(D_2 \varphi(0, x_0) = \text{id}\). Hence, by (43), (44), Cramer’s rule, and the formula of Abel–Liouville, (46) reduces to
\[
\omega^2 \int_0^{t_1} e^{2\gamma \tau} \sin (\varphi(\tau, x_0)_1) (D_2 \varphi(\tau, x_0) h)_1^3 d\tau \leq 1/R.
\] (47)
We shall establish (47) under the assumption \(\gamma < \omega\). The result for \(\gamma = \omega\) then follows from a continuity argument.

1.) In order to estimate the integrand in (47) we investigate initial value problems
\[
\begin{align*}
\dot{x} &= A_\rho x + \begin{pmatrix} 0 \\ u(t) \end{pmatrix}, \\
x(0) &= h
\end{align*}
\] (48a)
for continuous \(u\) and \(h \geq 0, h \neq 0\), where
\[
A_\rho = \begin{pmatrix} 0 & 1 \\ \rho & -2\gamma \end{pmatrix}
\]
for any \(\rho \in \mathbb{R}\). For \(\lambda(t) := \exp(A_\rho t) h\) we obtain
\[
\lambda_1(t) = \frac{1}{2\kappa_+} e^{-\gamma t} \left( h_1 (\kappa_+ + \gamma) + h_2 \right) e^{\kappa_+ t} + (h_1 (\kappa_- - \gamma) - h_2) e^{-\kappa_- t}
\] (49)
if \(\rho = \omega^2\), and \(\lambda_1(t) = e^{-\gamma t} \left( h_1 \cos(\kappa_- t) + \kappa_-^{-1} (h_1 \gamma + h_2) \sin(\kappa_- t) \right)\), if \(\rho = -\omega^2\). In particular, \(\lambda_1\) is positive on \([0, t_1]\) in the latter case. Moreover, for \(\rho = \omega^2\), (48a) is cooperative as \(A_{\omega^2}\) is essentially nonnegative [43]. Hence, \(x(t) \leq \lambda(t)\) for all \(t \in [0, t_1]\) if \(u\) is nonpositive on \([0, t_1]\).

2.) We claim \(0 \leq (D_2 \varphi(t, x_0) h)_1 \leq (\exp (A_{\omega^2} t) h)_1\) for all \(t \in [0, t_1]\) and all \(x_0, h \in \mathbb{R}^2\) with \(h \geq 0\). First note that \(x := D_2 \varphi(\cdot, x_0) h\) is a solution of the initial value problem (48) with \(\rho = -\omega^2\) and \(u(t) = \omega^2 (1 - \cos (\varphi(t, x_0)_1)) x_1(t)\). This implies
\[
x_1(t) - (\exp (A_{-\omega^2} t) h)_1 = \int_0^t (\exp (A_{-\omega^2} (t - \tau)))_{1,2} u(\tau) d\tau
\] (50)
for all $t \in \mathbb{R}$. Let $h_1 > 0$, assume $x_1$ has a zero in $[0, t_1]$, and let $s$ be the smallest such zero. Then, by step 1, the left hand side of (50) is negative at $t = s$. On the other hand, $u$ is nonnegative on $[0, s]$, which by step 1 implies the integrand in (50) is nonnegative if $t = s$, and hence, the right hand side of (50) is nonnegative at $t = s$. This is a contradiction, so $x_1$ is positive on $]0, t_1[.

Observe now that $x$ is a solution of the initial value problem (48), this time with $\rho = \omega^2$ and $u(t) = -\omega^2 (1 + \cos(\phi(t, x_0_1))) x_1(t)$. As $u$ is nonpositive on $[0, t_1]$, application of step 1 and a continuity argument (in $h_1$) completes the proof of the claim.

3.) From step 2, (49), $1 \leq \omega$, and $| (D_2 \varphi(\tau, x_0) h_1) | \leq \| D_2 \varphi(\tau, x_0_1). \|$, we obtain the bound

$$e^{-\gamma \tau} (1 + (\kappa_+ + \gamma)^2)^{3/2} \kappa_+^{-3} \left( \cosh(\kappa_+ \tau)^2 - \frac{1}{\omega^2 + 1} \right)^{3/2}$$

(51)

for the modulus of the integrand in (47). We next show

$$g(x) := \left( 1 - \frac{5}{3} x \exp(-\alpha) \right) - (1 - x/(\alpha^2 + 1))^{3/2} \geq 0$$

(52)

for all $\alpha \geq 1$ and all $x \in [0, 1]$. As $g'$ is monotone decreasing on $[0, 1]$, the map $g$ is concave. Therefore, since $g(0) = 0$, it suffices to show that $g(1) \geq 0$, i.e., that

$$z(\alpha) := \frac{3}{5} \exp(\alpha) \left( 1 - \frac{\alpha^3}{(\alpha^2 + 1)^{3/2}} \right) \geq 1$$

(53)

for all $\alpha \geq 1$. It is easy to see that $z'(\alpha)$ is a positive multiple of $(\alpha^2 + 1)^{5/2} - \alpha^5 - \alpha^3 - 3\alpha^2$ and that $(\alpha^2 + 1)^{5} - (\alpha^5 + \alpha^3 + 3\alpha^2)^2$ is a polynomial in $s$ with nonnegative coefficients if $1 + s$ is substituted for $\alpha$. Hence $z'(\alpha) \geq 0$ for all $\alpha \geq 1$. As $z(1) > 1$ is easily verified, (53), and hence (52), have been established for all $\alpha \geq 1$ and all $x \in [0, 1]$. From (51), (52) and the fact that $(1 + (\kappa_+ + \gamma)^2)^{3/2} \kappa_+^{-3} \leq (1 + (\omega + \gamma)^2)^{3/2} \omega^{-3}$ we obtain the bound

$$(1 + (\omega + \gamma)^2)^{3/2} \omega^{-3} \left( \cosh(\kappa_+ \tau)^3 - \frac{5}{3} \exp(-\omega) \cosh(\kappa_+ \tau) \right)$$

for the modulus of the integrand in (47), from which (47) follows by integration. □
Figure 2: Bound $R$ on the radius of $\Omega$ that ensures convexity of the reachable set $\varphi(t_1, \Omega)$ over $t_1$, where $\lg R$ denotes the logarithm to base 10 of $R$. (a) $\omega = 1, \gamma = 0$. (b) $\omega = 6.1, \gamma = 0.2$. (●●: numerical bound, — —: bound from Theorem 5.1, —: bound from Theorem 5.2, - -: bound from [23].)

We would like to emphasize that the results we have established in this section are of a global type, i.e., the bounds $R$ on the radius of a ball in Theorems 5.1 and 5.2 do not depend on the location of the ball in phase space. Instead, convexity of reachable sets from arbitrary balls of initial states is guaranteed, provided their radii do not exceed $R$.

As we have already remarked in section 4, the bound (42) on radii is never smaller and in general larger than the bound from [23]. For the pendulum equations (39) one can show that these bounds coincide iff $\omega = 1$ and $\gamma = 0$. In particular, in the undamped case, $\gamma = 0$, (41) reads $M_1 = \frac{2}{3}(1 + \omega^2)$, whereas [23] would give the bound (42) with $M_1 = 3 \max\{1, \omega^2\}$. Fig. 2(a) and Fig. 2(b) show the bounds obtained from Theorems 5.1 and 5.2 and from [23] in comparison to a bound obtained numerically for two sets of parameters, $\omega = 1$ and $\gamma = 0$, and $\omega = 6.1$ and $\gamma = 0.2$. The latter parameters are obtained from a model of an experimental cart-pole system when time is measured in seconds [44]. The results show that in both cases, the balls of initial states for which Theorem 5.2 proves the reachable sets are convex are large enough to be used in actual computations, such as in local programming techniques [19] and polyhedral approximation of reachable sets discussed in section 1. For the second set of parameters, the bounds obtained from Theorem 5.1 and [23] seem to be less useful.
6 Conclusions

We have presented a novel necessary and sufficient condition for the image of a sublevel set under a diffeomorphism to be convex. That result has been applied to reachable sets from a ball of initial states through solutions of an ordinary differential equation, which has resulted in a necessary and sufficient condition for the convexity of those reachable sets. We have also established an upper bound on the radius of the ball of initial states that ensures the reachable set is convex. That bound is sharp and can be directly obtained from the right hand side of the differential equation. In finite dimensions, our results cover the case of ellipsoids of initial states. We have also demonstrated by means of an example that the balls of initial states that result in convex reachable sets are large enough to be used in actual computations, for example in local programming techniques [19] and polyhedral approximation of reachable sets discussed in section 1.

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