A Lagrange subspace approach to dissipation inequalities

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Abstract—The standard dissipation inequality for passivity is extended by replacing storage functions by general Lagrange subspaces. This leads to some interesting consequences, including loss of controllability. A classical fundamental factorization result for passive systems is extended to this generalized case, making use of the newly defined concept of the Hamiltonian lift of a differential-algebraic equation (DAE) system.

I. INTRODUCTION

Lyapunov and dissipation inequalities are at the core of systems and control theory. Within the context of Riccati equations it is well-known that the symmetric solutions of them can be obtained by computing invariant Lagrange subspaces of a corresponding Hamiltonian matrix. Apart from the standard Lyapunov inequality $A^TQ+QA \preceq 0$ in the symmetric matrix $Q$, also the dual Lyapunov inequality $AX+XA^T \preceq 0$ arises at many places in the theory; notably in controllability and (stochastic) filtering. This already suggests the consideration of Lagrange subspaces as a means for their unification. In this paper we take a close look at Lyapunov and dissipation inequalities by starting from a general Lagrange subspace point of view, and exploring its consequences. At least for dissipation inequalities this seems to be relatively new. After treating some generalities with regard to Lagrange subspaces and their invariance (Section II, see also the Appendix), we show in Section III how the standard dissipation inequality for passivity can be naturally extended from storage functions (and their gradient vectors) to general Lagrange subspaces. Importantly, we note that this implies an ordinary dissipation inequality (with storage function determined by the Lagrange subspace) for an associated differential-algebraic equation (DAE) system. Then, in Section IV, we analyze the consequences of the satisfaction of this generalized dissipation inequality for the structure of the system. In particular we show that whenever the Lagrange subspace does not correspond to a storage function defined on the whole state space this necessarily implies loss of controllability. The rest of the paper is devoted to the factorization of systems satisfying the generalized dissipation inequality. First in Section V it is shown, via direct transfer matrix computations, how the satisfaction of the generalized dissipation inequality leads to the same factorization result as in the ‘classical’ dissipation case, but now with respect to the transfer matrix of the DAE system alluded to above. In Section VI the same result is proved in state space terms, using a natural extension of the DAE system to a Hamiltonian input-output system, called its Hamiltonian lift. Finally Section VII contains conclusions and outlook for further work.

II. LAGRANGE SUBSPACES AND LYAPUNOV INEQUALITIES

Consider an $n$-dimensional linear state space $\mathcal{X}$, with dual space $\mathcal{X}^*$. A subspace $\mathcal{L} \subset \mathcal{X} \times \mathcal{X}^*$ is called a Lagrange subspace if the canonical symplectic form $\mathcal{J}$ on $\mathcal{X} \times \mathcal{X}^*$, in matrix representation given as

$$\mathcal{J} = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix},$$

is zero restricted to $\mathcal{L}$, and furthermore $\mathcal{L}$ is maximal with respect to this property. Equivalently, $\mathcal{L} = \mathcal{L}^\perp$, where $\perp$ denotes the orthogonal companion with respect to $\mathcal{J}$. It follows that $\dim \mathcal{L} = n$ for any Lagrange subspace.

It is easily seen (see Proposition 8.1 in the Appendix) that any Lagrange subspace $\mathcal{L}$ admits an image representation

$$\mathcal{L} = \text{im} \begin{bmatrix} P \\ S \end{bmatrix}$$

(with im denoting ‘image’), for certain square matrices $P, S$ satisfying

$$S^TP = P^TS, \quad \text{rank} \begin{bmatrix} P \\ S \end{bmatrix} = n.$$  

Conversely, any subspace $\mathcal{L}$ as in (2) with $P$ and $S$ satisfying (3) is a Lagrange subspace. Denoting elements of $\mathcal{X}$ by $x$, and of its dual space $\mathcal{X}^*$ by $p$, it follows that all elements $(x, p) \in \mathcal{L}$ can be expressed as

$$x = Pz, \quad p = Sz, \quad z \in \mathcal{Z},$$

where $\mathcal{Z}$ is an $n$-dimensional parametrization space. The ‘canonical’ choice of $\mathcal{Z}$ is $\mathcal{L}$, considered as a linear space in its own right. In this case, $P$ is given as the projection of $\mathcal{L} \subset \mathcal{X} \times \mathcal{X}^*$ to $\mathcal{X}$, and $S$ is the projection of $\mathcal{L} \subset \mathcal{X} \times \mathcal{X}^*$ to $\mathcal{X}^*$.

We note that any Lagrange subspace is endowed with an intrinsic bilinear form, which is given as

$$\langle (x_1, p_1), (x_2, p_2) \rangle = p_1^T x_2 - p_2^T x_1, \quad (x_i, p_i) \in \mathcal{L}, i = 1, 2.$$

For $\mathcal{L}$ given by (2), (3), (4) this bilinear form is simply given as

$$\langle (x_1, p_1), (x_2, p_2) \rangle = z_1^T S^TP z_2.$$  

Obvious examples of a Lagrange subspace are (2) with $P = I, S = Q$, where $Q = Q^T$ is a symmetric matrix, or $P = I, S = Q^T$ with $Q$ being a non-symmetric matrix.

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\(X, S = I\), where \(X = X^\top\) is a symmetric matrix. However, there are many more cases ‘in between’ these two subclasses; see also Proposition 4.3.

Given any Hamiltonian function \(H : \mathcal{X} \times \mathcal{X}^* \to \mathbb{R}\) one defines the standard symplectic Hamiltonian dynamics on \(\mathcal{X} \times \mathcal{X}^*\) as

\[
\begin{bmatrix}
0 & -I \\
I & 0
\end{bmatrix}
\begin{bmatrix}
\dot{x} \\
\dot{p}
\end{bmatrix}
= \begin{bmatrix}
\frac{\partial H}{\partial x} \\
\frac{\partial H}{\partial p}
\end{bmatrix} 
\tag{7}
\]

A Lagrange subspace \(\mathcal{L}\) is invariant for (7) if and only if (see e.g. [2])

\[H(x, p) = c, \quad \text{for all } (x, p) \in \mathcal{L}\]  
(8)

for some constant \(c\). In the case of a quadratic Hamiltonian (as in all of this paper) the constant \(c\) is actually zero.

With respect to the Hamiltonian \(H(x, p) = p^\top Ax\), with \(A\) some real \(n \times n\) matrix, the Hamiltonian dynamics (7) is given as

\[
\begin{bmatrix}
\dot{x} \\
\dot{p}
\end{bmatrix}
= \begin{bmatrix}
Ax \\
-A^\top p
\end{bmatrix},
\tag{9}
\]

and thus \(\mathcal{L}\) is invariant for this Hamiltonian dynamics if and only if \(p^\top Ax = 0\) for all \((x, p) \in \mathcal{L}\). For any Lagrange subspace as given by (2), (3), (4) this holds if and only if

\[z^\top S^\top APz = 0\]  
(10)

for all \(z\), or equivalently

\[S^\top AP + P^\top A^\top S = 0, \quad S^\top P = P^\top S, \quad \rank(P) = n.\]  
(11)

This overarches the two traditional Lyapunov equations

\[A^\top Q + QA = 0, \quad Q = Q^\top, \quad AX + XA^\top = 0, \quad X = X^\top,\]  
(12)

corresponding to taking \(P = I, \ S = Q\), respectively \(P = X, \ S = I\).

Equation (11) therefore will be called the generalized Lyapunov equation; see [9] for related developments. Of course, in the case that \(Q \) or \(X \) are invertible, the two Lyapunov equations (12) are equivalent in the sense that by pre- and post-multiplication with the inverse of \(Q \) or \(X \) they can be transformed into each other. However, they are not equivalent if this is not the case.

Furthermore, the generalized Lyapunov inequality corresponds to \(H(x, p) \leq 0\) for all \((x, p) \in \mathcal{L}\). This means \(z^\top S^\top APz \leq 0\) for all \(z\), or equivalently

\[S^\top AP + P^\top A^\top S \leq 0.\]  
(13)

Classical examples are

\[A^\top Q + QA = -C^\top C,\]  
(14)

with \(Q\) the observability Gramian of the system \(\dot{x} = Ax, \ y = Cx\), and

\[AX + XA^\top = -BB^\top,\]  
(15)

with \(X\) the controllability Gramian of the system \(\dot{x} = Ax + Bu\). (The same equation appears in the covariance equation for linear stochastic systems.) The intrinsic difference between the two Lyapunov inequalities (14) and (15) is especially clear in case the system is unobservable, corresponding to a singular \(Q\), or uncontrollable, corresponding to a singular \(X\).

### III. The Generalized Dissipation Inequality

The ideas from the last section can be used for generalizing the dissipation inequality corresponding to passivity. In the case of a standard input-state-output system

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{align*}
\]  
(16)

this takes the following form. Consider the Hamiltonian

\[H(x, p, u) := p^\top (Ax + Bu) - u^\top (Cx + Du).\]  
(17)

Then the traditional dissipation inequality corresponding to passivity [17] is obtained by substituting \(p = Qx, \ Q = Q^\top\), in \(H(x, p, u) \leq 0\), so as to obtain

\[x^\top Q(Ax + Bu) - u^\top (Cx + Du) \leq 0, \quad \text{for all } x, u.\]  
(18)

This leads to the well-known linear matrix inequality, see e.g. [17],

\[
\begin{bmatrix}
A^\top Q + QA & Q^\top B - C^\top \\
B^\top Q - C & -D - D^\top
\end{bmatrix}
\leq 0.
\]  
(19)

This inequality and its ramifications extends the famous Kalman-Yakubovich-Popov lemma; see e.g. [17] and the references quoted therein.

By replacing the storage function \(\frac{1}{2}x^\top Qx\) (leading to the gradient vector \(p = Qx\)) by a Lagrangian subspace \(\mathcal{L} = \im (P)\), this extends to the generalized dissipation inequality

\[
\begin{bmatrix}
S^\top AP + P^\top A^\top S & \quad S^\top B - P^\top C^\top \\
B^\top S - CP & \quad -D - D^\top
\end{bmatrix}
\leq 0,\]  
(20)

obtained by substituting \(x = Pz, \ p = Sz\) in \(H(x, p, u) \leq 0\). If \(P\) is singular, then (20) can not be rewritten as a traditional dissipation inequality (19) corresponding to a storage function expressed in the state \(x\). On the other hand, as we have seen above, there is a bilinear form, and thus a quadratic function, associated to the Lagrange subspace given by \(P, S\), namely \(V(z) := \frac{1}{2}z^\top S^\top Pz\); see [4]. Then, using \(Pz = \dot{x} = APz + Bu\), the generalized dissipation inequality (20) is seen to be equivalent to

\[
\frac{d}{dt} V(z) = z^\top S^\top P \dot{z} \leq u^\top (CPz + Du).
\]  
(21)

Thus, although (20) cannot be interpreted as a dissipation inequality for the original system (16) in case \(P\) is singular, it still can be interpreted as a dissipation inequality for the DAE system in the state variables \(z\)

\[
\begin{align*}
P \dot{z} &= APz + Bu, \quad z \in \mathbb{Z} \\
y &= CPz + Du
\end{align*}
\]  
(22)
with singular pencil $s P - A P$, and storage function $V(z) = \frac{1}{2} z^\top S^\top P z$. Note that the equation space for the DAE system (22), i.e., the co-domain of the mappings $P$ and $A P$, is given by $X$.

**Remark 3.1:** Throughout this paper we do not impose any nonnegativity assumptions on $Q$ in (19), or on $P$, $S$ in (20). Thus, strictly speaking we are dealing with cyclo-passivity instead of passivity; cf. [14]. In particular, we do not assume nonnegativity of $S^\top P$. Geometrically the nonnegativity condition $S^\top P \geq 0$ is formalized by requiring that the symmetric canonical form on $X \times X^*$, in matrix representation given as

$$
\begin{bmatrix}
0 & I_n \\
I_n & 0
\end{bmatrix},
$$

is nonnegative on $L = \text{im} \begin{bmatrix} P \\ S \end{bmatrix}$.

**Remark 3.2:** The consideration of the generalized dissipation inequality (20) suggests as a special case the “dual” dissipation inequality, resulting from taking $S = I, P = P^\top =: X$ in (20),

$$
\begin{bmatrix}
A X + X A^\top & B - X C^\top \\
B^\top - C X & -D - D^\top
\end{bmatrix} \leq 0. \tag{24}
$$

**IV. COORDINATE EXPRESSIONS**

Consider a general Lagrange subspace $L = \text{im} \begin{bmatrix} P \\ S \end{bmatrix}$. By allowing for coordinate transformations on $X$ and $Z$ we can always transform $P$ to the form

$$
P = \begin{bmatrix} I_k & 0 \\
0 & 0
\end{bmatrix}, \tag{25}
$$

where $k \leq n$ is the rank of $P$. Using $S^\top P = P^\top S$ it follows that $S$ needs to be of the corresponding form

$$
S = \begin{bmatrix}
S_{11} & 0 \\
S_{12} & S_{22}
\end{bmatrix}, \quad S_{11} = S_{11}^\top. \tag{26}
$$

Furthermore, since rank $\begin{bmatrix} P \\ S \end{bmatrix} = n$, $S_{22}$ is invertible. Hence by post-multiplication with the additional coordinate transformation (which does not affect $P$)

$$
\begin{bmatrix}
I \\
-S_{22}^{-1} S_{21}
\end{bmatrix},
$$

the matrix $S$ transforms into

$$
S = \begin{bmatrix}
S_{11} & 0 \\
0 & I
\end{bmatrix}, \quad S_{11} = S_{11}^\top. \tag{28}
$$

(See [11] for much more refined results.) In such coordinates, with partitioning $A, B, C$ accordingly, the generalized Lyapunov inequality (13) can be seen to take the form

$$
\begin{bmatrix}
A_{11}^\top S_{11} + S_{11} A_{11} & A_{21}^\top S_{11} B_1 - C_1^\top \\
A_{21} & 0 & B_2 \\
B_1^\top S_{11} - C_1 & B_2^\top & -D - D^\top
\end{bmatrix} \leq 0. \tag{29}
$$

Because of the 0-block this implies that $A_{21} = 0$ and $B_2 = 0$. Hence the system (16) takes the block-triangular form

$$
\begin{align*}
\dot{x}_1 &= A_{11} x_1 + A_{12} x_2 + B_1 u, \\
\dot{x}_2 &= A_{22} x_2, \\
y &= C_1 x_1 + C_2 x_2 + D u,
\end{align*} \tag{30}
$$

satisfying the reduced dissipation inequality

$$
\begin{bmatrix}
A_{11}^\top S_{11} + S_{11} A_{11} & S_{11} B_1 - C_1^\top \\
B_1^\top S_{11} - C_1 & -D - D^\top
\end{bmatrix} \leq 0. \tag{31}
$$

In particular, it follows that for any initial condition $x(0)$ with $x_2(0) = 0$, the system (30) leads to the reduced system

$$
\begin{align*}
\dot{x}_1 &= A_{11} x_1 + B_1 u, \\
y &= C_1 x_1 + D u,
\end{align*} \tag{32}
$$

that is satisfying the dissipation inequality with storage function $\frac{1}{2} x_1^\top S_{11} x_1$. Furthermore, it follows that any system (16) satisfying the generalized dissipation inequality (20) with $P$ singular is necessarily uncontrollable (since the dynamics of $x_2$ is independent of $u$ and $x_1$). In fact, it follows from (30) that the controllable part of the system (46) is contained in $\text{im} P$.

**Remark 4.1:** This is in line with the satisfaction of the dissipation inequality for the DAE system (22), with storage function $\frac{1}{2} x_1^\top S^\top P z$; see also [4]. In fact, using $A_{21} = 0$, $B_2 = 0$ it follows that the DAE system (22) reduces to the ordinary differential equation system

$$
\begin{align*}
\dot{z}_1 &= A_{11} z_1 + B_1 u, \\
y &= C_1 z_1 + D u,
\end{align*} \tag{33}
$$

solely in $z_1 = x_1$; i.e., the part of (30) corresponding to any initial condition $x(0)$ with $x_2(0) = 0$.

**Remark 4.2:** When applied to the system $\dot{x} = A x$ without inputs and outputs, (30) specializes to the triangular form

$$
\begin{align*}
\dot{x}_1 &= A_{11} x_1 + A_{12} x_2, \\
\dot{x}_2 &= A_{22} x_2,
\end{align*} \tag{34}
$$

satisfying the reduced Lyapunov inequality

$$
A_{11}^\top S_{11} + S_{11} A_{11} \leq 0. \tag{35}
$$

Without using general coordinate transformations on $X$ and $Z$ as above one can still use the following general representation of Lagrange subspaces.

**Proposition 4.3:** Consider a Lagrange subspace $L \subset X \times X^*$ given as $L = \text{im} \begin{bmatrix} P \\ S \end{bmatrix}$ for $n \times n$ matrices $P, S$ satisfying (3). Suppose $\text{rank} P = m \leq n = \dim X$. Then, possibly after permutation of the elements of $x$ and correspondingly $p$, there exists a splitting $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, p = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$ with $x_1, p_1$ both $m$-dimensional, and $x_2, p_2$ both $(n-m)$-dimensional, such that $L$ is represented as

$$
L = \{ (x, p) \in X \times X^* \mid \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = W \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \} \tag{36}
$$

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with \( W = W^\top \).

This proposition is well-known in symplectic geometry [3], [16] (and in fact extends to Lagrange submanifolds). A direct linear-algebraic proof of Proposition 4.3 can be found in [15]. The quadratic function (in the subvectors \( x_1, p_2 \)) defined by the symmetric matrix \( W \) is commonly called a generating function of the Lagrange subspace.

As a consequence of Proposition 4.3, by defining the following parametrization vector \( z \) and partitioning \( W \) accordingly, i.e.,
\[
  z = \begin{bmatrix} x_1 \\ p_2 \end{bmatrix}, \quad W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix},
\]
we obtain
\[
P = \begin{bmatrix} I & 0 \\ -W_{21} & -W_{22} \end{bmatrix}, \quad S = \begin{bmatrix} W_{11} & W_{12} \\ 0 & I \end{bmatrix}.
\]
In particular, we see that
\[
S^\top P = \begin{bmatrix} W_{11} & 0 \\ 0 & -W_{22} \end{bmatrix}.
\]

V. TRANSFER MATRIX FACTORIZATION

Consider the DAE system (22), with \( \mathcal{L} = \text{im} \begin{bmatrix} P \\ S \end{bmatrix} \) satisfying the generalized dissipation inequality (20). It follows that there exist matrices \( M, N \) such that
\[
\begin{bmatrix} S^\top AP + P^\top A^\top S & S^\top B - P^\top C^\top \\ B^\top S - CP & -D - D^\top \end{bmatrix} = - \begin{bmatrix} M^\top \\ N^\top \end{bmatrix} \begin{bmatrix} M & N \end{bmatrix}.
\]

(See [12] for related developments.)

We have the following result, directly generalizing a corresponding result for the classical dissipation inequality (19); cf. [1], [17]. Since \( Ps - AP \) is singular, we cannot define the transfer matrix of the DAE system (22) in the standard way using the inverse of \( Ps - AP \). However in the present case the transfer matrix of (22) still can be uniquely defined in a generalized form as
\[
G(s) := CP(Ps - AP)^{-1}B + D,
\]
where for a matrix function \( T, T^{-1} \) denotes an appropriate generalized inverse. This is justified because 1) (as we have seen in the previous section) the generalized dissipation inequality implies that \( \text{im} B \) is contained in \( \text{im} P \) (see (30), 2) the generalized inverse of \( Ps - AP \) is premultiplied by \( P \). Thus the expression \( P(Ps - AP)^{-1}B \) is indeed well defined. We obtain the following result.

**Proposition 5.1:** Consider the generalized transfer matrix of the DAE system (22) given by (41). Then (40) corresponds to the following factorization of transfer matrices
\[
G(s) + G^\top (-s) = K^\top (-s)K(s),
\]
where
\[
K(s) := M(Ps - AP)^{-1}B + N.
\]

(Note that \( \text{ker} P \subset \text{ker} M \) so \( K(s) \) is again well defined.)

**Proof:** For simplicity assume \( D = 0 \), in which case \( N = 0 \) in (40), and thus (40) amounts to
\[
S^\top AP + P^\top A^\top S = -M^\top M, \quad CP = B^\top S.
\]
Then, using similar arguments as in [1], and additionally using \( S^\top P = P^\top S \)
\[
G(s) + G^\top (-s) = CP(Ps - AP)^{-1}B + B^\top(Ps - AP)^{-1}P^\top C^\top = B^\top S(Ps - AP)^{-1}B + B^\top(Ps - AP)^{-1}S^\top B = \left[ B^\top S + B^\top(Ps - P^\top A^\top)^{-1} S^\top (Ps - AP) \right].
\]
\[
(Ps - AP)^{-1}B = B^\top(Ps - P^\top A^\top)^{-1}.
\]
\[
[(-P^\top A^\top)S - S^\top S(Ps - AP)].
\]
\[
(Ps - AP)^{-1}B = B^\top(Ps - P^\top A^\top)^{-1}.
\]
\[
[(-P^\top A^\top)S - S^\top AP] \cdot (Ps - AP)^{-1}B = B^\top(Ps - P^\top A^\top)^{-1}M^\top P(Ps - AP)^{-1}B = K^\top (-s)K(s).
\]
The proof for the case \( D \neq 0 \) can be performed in a similar way (see [1] for details in a similar context), or by first removing the feedthrough term by an extension as discussed in [10].

VI. THE HAMILTONIAN LIFT OF DAE SYSTEMS AND FACTORIZATION

In this section we will show how the factorization, performed in the previous section using direct transfer matrix computations, can be also obtained from a pure state space point of view; thereby providing additional insight and possible generalizations to other settings.

First recall, see [7], that any ordinary input-state-output system
\[
\dot{x} = Ax + Bu,
\]
\[
y = Cx + Du,
\]
on \( \mathcal{X} \) can be lifted to a Hamiltonian input-output system on \( \mathcal{X} \times \mathcal{X}^* \). (See [6], [13], [7], [14] for the definition of a Hamiltonian input-output system.) Indeed, one considers the canonical symplectic form \( \mathcal{J} \) on \( \mathcal{X} \times \mathcal{X}^* \) as in (1), together with the Hamiltonian as considered before
\[
H(x, p, u) = p^\top(Ax + Bu) - u^\top(Cx + Du).
\]
This leads to the Hamiltonian input-output system
\[
\mathcal{J} \begin{bmatrix} \dot{z} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial p} \end{bmatrix} = \begin{bmatrix} A^\top p - C^\top u \\ Ax + Bu \end{bmatrix},
\]
\[
y = -\frac{\partial H}{\partial u} = -B^\top p + Cx + (D + D^\top)u,
\]
also called the Hamiltonian lift of the original system (46). (The definition immediately extends to the nonlinear case as
well; see [7] for details.) It is immediately verified that (48) equals the parallel interconnection of (46) with its adjoint system, and that the transfer matrix of (48) is given as
\[ G'(s) + G'^T(-s), \]
where \( G'(s) = C(I s - A)^{-1} B + D \) is the transfer matrix of (46). This lifting is instrumental for various purposes; see e.g. [14] for a detailed discussion and applications.

The described lifting construction can be extended to DAE systems as follows. Consider first a general linear DAE system
\[
\begin{align*}
E \dot{x} &= Ax + Bu, \\
y &= Cx + Du,
\end{align*}
\]
where \( E \) and \( A \) are mappings \( E : \mathcal{X} \to \mathcal{R}, A : \mathcal{X} \to \mathcal{R} \), for some linear equation space \( \mathcal{R} \), together with \( B : \mathcal{U} \to \mathcal{R} \), with \( \mathcal{U} = \mathbb{R}^m \) the input space. Then define the skew-symmetric form \( J_E \) on \( \mathcal{X} \times \mathcal{R}^* \) as
\[
J_E := \begin{bmatrix} 0 & -E^T \\ E & 0 \end{bmatrix}.
\]

Since in general \( E \) is not invertible, the skew-symmetric form \( J_E \) is degenerate, and defines a pre-symplectic form [2], [3]. Similarly to (17), define the Hamiltonian \( H' : \mathcal{X} \times \mathcal{R}^* \times \mathcal{U} \to \mathbb{R} \) as
\[
H'(x, v, u) := v^T(Ax + Bu) - u^T(Cx + Du).
\]

Then define the Hamiltonian lift of (50) as the input-output Hamiltonian DAE system
\[
\begin{align*}
\dot{x} &= A^T v - C^T u \\
y &= Ax + Bu,
\end{align*}
\]
\[
\begin{align*}
y_H &= -B^T v + Cx + (D + D^T) u.
\end{align*}
\]

Applied to the specific DAE system (22) with the Hamiltonian
\[ H(z, v, u) := v^T(APz + Bu) - u^T(CPz + Du), \]
this yields the following Hamiltonian lift of (22)
\[
\begin{bmatrix} 0 & -P^T \\ P & 0 \end{bmatrix} \begin{bmatrix} \dot{z} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} P^T A^T v - P^T C^T u \\ APz + Bu \end{bmatrix},
\]
\[
\begin{align*}
\dot{y}_h &= -B^T v + CPz + (D + D^T) u,
\end{align*}
\]
living on \( \mathcal{Z} \times \mathcal{X}^* \). See [8] for a related Hamiltonian in the context of optimal control problems with DAE constraints.

The associated generalized transfer matrix of the Hamiltonian lift (55) is given as
\[
\begin{align*}
&\begin{bmatrix} CP & -B^T \\ sP - AP & 0 \end{bmatrix} \begin{bmatrix} -P^T C^T \\ B \end{bmatrix} \\
&\text{+ (D + D^T) = } CP(sP - AP) - B + D \\
&\text{+ } B^T(-sP^T - P^T A^T) - P^T C^T + D^T.
\end{align*}
\]

Thus, similar to the Hamiltonian lift of the standard input-state-output system (46), the generalized transfer matrix of the Hamiltonian lift of (22) is equal to \( G(s) + G'^T(-s) \), with \( G(s) = CP(sP - AP) - B + D \) the generalized transfer matrix of (22). Furthermore, (55) can be seen to be equal to the parallel interconnection of (22) with its adjoint system
\[
\begin{align*}
-P^T \dot{v} &= P^T A^T v - P^T C^T u_a \\
y_a &= -B^T v + D^T u_a,
\end{align*}
\]
with \( u_a = u \) and \( y_h = y + y_a \).

We will now show how the factorization result \( G(s) + G'^T(-s) = K^T(-s) K(s) \) as obtained in the previous section by generalized transfer matrix computations can also be obtained by state space methods applied to the Hamiltonian lift (55). This provides extra insight, and is expected to facilitate the generalization of the nonlinear case; cf. [14] for the standard nonlinear input-state-output case.

Indeed, let \( L = \text{im} \begin{bmatrix} P \\ S \end{bmatrix} \) be a solution to (20). Then consider the state space transformation
\[
z = \tilde{z}, v = \tilde{v} + S z
\]
for (55). This transforms the Hamiltonian \( \mathcal{H}(z, v, u) \) defined in (54) into
\[
\mathcal{H}(\tilde{z}, \tilde{v}, u) = \tilde{v}^T(AP\tilde{z} + Bu) - u^T(CP\tilde{z} + Du), \\
+ \tilde{z}^T S^T AP\tilde{z} + \tilde{z}^T S^T Bu - u^T CP\tilde{z} - u^T Du.
\]

Making use of (40) this yields
\[
\mathcal{H}(\tilde{z}, \tilde{v}, u) = \tilde{v}^T(AP\tilde{z} + Bu) - u^T(CP\tilde{z} + Du) - \frac{1}{2} w^T w,
\]
where \( w := M\tilde{z} + Nu \). Furthermore, the state space transformation (58) leaves the pre-symplectic form \( \begin{bmatrix} 0 & -P^T \\ P & 0 \end{bmatrix} \) invariant, since
\[
\begin{bmatrix} I & S^T \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & -P^T \\ P & 0 \end{bmatrix} \begin{bmatrix} I \\ S \end{bmatrix} = \begin{bmatrix} 0 & -P^T \\ P & 0 \end{bmatrix},
\]
in view of \( S^T P = P^T S \). Hence the state space transformation (58) transforms the Hamiltonian lift (55) into another representation of the same input-output Hamiltonian system, still defined with respect to the same presymplectic form, but now with Hamiltonian \( \mathcal{H} \). This new representation (in the new coordinates \( \tilde{z}, \tilde{v} \)) is thus given as
\[
\begin{bmatrix} 0 & -P^T \\ P & 0 \end{bmatrix} \begin{bmatrix} \tilde{z} \\ \tilde{v} \end{bmatrix} = \begin{bmatrix} P^T A^T \tilde{v} - M^T M\tilde{z} + (M^T N - P^T C^T u) \\ AP\tilde{z} + Bu \end{bmatrix},
\]
\[
\dot{y}_h = -B^T \tilde{v} + CP\tilde{z} + N^T M\tilde{z} + (D + D^T + N^T N) u.
\]
It can be directly verified that the generalized transfer matrix of (62) is given by $K^{\top}(-s)K(s)$, with $K(s)$ defined by (43). Furthermore, from a state space point of view, (62) is immediately seen to be the series interconnection of the system with modified output equation

\[ P\ddot{z} = AP\dot{z} + Bu \]
\[ \ddot{y} = M\dot{z} + Nu, \]

with its adjoint system

\[ -P^{\top}\ddot{v} = P^{\top}A^{\top}\dot{v} - M^{\top}u_a \]
\[ \dot{y}_a = -B^{\top}\dot{v} + D^{\top}u_a, \]

via the series interconnection constraints $u_a = \ddot{y}$, $\dot{y}_h = \ddot{y}_a$.

VII. CONCLUSIONS AND OUTLOOK

It has been shown how the Lyapunov and dissipation inequality can be naturally generalized from quadratic Lyapunov and storage functions to general Lagrange subspaces. This gives rise to the consideration of an associated DAE system. In fact it has been shown that this DAE system satisfies the classical dissipation inequality with respect to a storage function on any parametrization space determined by the Lagrange subspace. Furthermore, it has been shown how the same factorization result as in the classical case can be obtained in case of the generalized dissipation inequality. Especially the state space treatment of this result in Section VII, making use of the Hamiltonian lift of a DAE system, looks promising for generalizations to the nonlinear case. Another venue for future research concerns the question how to extend the Lagrange subspace methodology from standard ODE systems (as in this paper) to DAE systems; see [8] for a similar analysis in the context of optimal control with DAE constraints. As a final remark, it has been noted how the satisfaction of the generalized dissipation inequality often implies lack of controllability. This is another indication that the traditional emphasis in systems and control theory on minimal systems may not be so relevant anymore; also in the light of classical electrical circuit theory and modern learning networks.

VIII. APPENDIX

Proposition 8.1: A subspace $\mathcal{L} \subset \mathcal{X} \times \mathcal{X}^*$ with $\dim \mathcal{X} = n$ is a Lagrange subspace if and only if there exist $n \times n$ matrices $P, S$ satisfying

\[ S^{\top}P = P^{\top}S, \quad \text{rank} \left[ S^{\top} \quad P^{\top} \right] = n \]

such that (see (2))

\[ \mathcal{L} = \{(x, p) \in \mathcal{X} \times \mathcal{X}^* \mid \exists z \in \mathbb{R}^n \text{ s.t. } \begin{bmatrix} x \\ p \end{bmatrix} = \begin{bmatrix} P \\ S \end{bmatrix} \begin{bmatrix} z \end{bmatrix} \}. \]

Proof: The ‘if’ direction follows by checking that

\[ \begin{bmatrix} x^1 \\ p_1^1 \\ 0 \\ -I \end{bmatrix} \begin{bmatrix} 0 \\ I \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} x_2 \\ p_2 \end{bmatrix} = 0 \text{ for any two pairs } (x_i, p_i) \text{ with } x_i = Pz_i, p_i = Sz_i, i = 1, 2, \text{ and } P, S \text{ satisfying (65).} \]

For the ‘only if’ direction we note that any $n \times n$ matrices $P, S$ satisfying rank $[P^{\top} \quad S^{\top}] = n$. Then take any two pairs $(x_1, p_1) \in \mathcal{L}$ with $x_1 = Pz_1, p_1 = Sz_1, i = 1, 2$. Since $\mathcal{L}$ is a Lagrange subspace it follows that

\[ 0 = \begin{bmatrix} x_1^{\top} \\ p_1^{\top} \end{bmatrix} \begin{bmatrix} 0 \\ -I \end{bmatrix} \begin{bmatrix} x_2 \\ p_2 \end{bmatrix} = z_2^{\top}SPz_1 - z_1^{\top}S^TPz_2 \]

\[ = -z_1^{\top}(S^{\top}P - P^T)z_2 \]

for all $z_1, z_2$, implying that $S^{\top}P = P^T S$.

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