GL(3, ℂ) Invariance of Type B 3-fold Supersymmetric Systems

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Abstract

Type B 3-fold supersymmetry is a necessary and sufficient condition for a quantum Hamiltonian to admit three linearly independent local solutions in closed form. We show that any such a system is invariant under GL(3, ℂ) homogeneous linear transformations. In particular, we prove explicitly that the parameter space is transformed as an adjoint representation of it and that every coefficient of the characteristic polynomial appeared in 3-fold superalgebra is algebraic invariants. In the type A case, it includes as a subgroup the GL(2, ℂ) linear fractional transformation studied in the literature. We argue that any N-fold supersymmetric system has a GL(N, ℂ) invariance for an arbitrary integral N.

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I. INTRODUCTION

One of the most important roles of mathematical science is to unveil universal mathematical structure which is hidden in various models. For this aim, the concept of symmetry has played a central and crucial role, see, for instance, a classical reference [1]. In many cases, one would first consider a symmetry, regardless of it is exact or only approximate, which characterize a system under consideration, and then build a corresponding model such that the symmetry is realized manifestly or may be broken dynamically in it. In other cases, however, a symmetry is discovered a posteriori or even accidentally in an existing model. Recently developed $\mathcal{N}$-fold supersymmetry (SUSY) is such an example, which has been recognized that it exists in any (quasi-)solvable quantum one-body Hamiltonian since its equivalence to weak quasi-solvability was proved in [2]. It means in particular that any second-order linear ordinary differential equation has this symmetry provided that it admits a certain number of local solutions in closed form, since such an operator can be always transformed into a Schrödinger one. For a review of $\mathcal{N}$-fold SUSY, see Ref. [3].

In our previous paper [4], we showed that a particular type of $\mathcal{N}$-fold SUSY, called type B, for $\mathcal{N}_3$ is a necessary and sufficient condition for a quantum Hamiltonian to admit three linearly independent local solutions in closed form, and obtained its most general analytical form. Contrary to the virtually every exactly solvable model whose analytical solutions are expressible in terms of a polynomial system in a certain variable, type B systems are peculiar in the sense that their solutions are essentially of non-polynomial character. In addition, they are not Lie-algebraic, in contrast with many quasi-solvable systems like the famous $\mathfrak{sl}(2)$ ones constructed in [5] and others [6, 7]. Hence, most of their aspects have still remained unknown and uncovered.

One of issues to be interested in next is, what kind of potentials is realizable in type B 3-fold SUSY. In the case of type A $\mathcal{N}$-fold SUSY, the problem was entirely solved by considering the $GL(2, \mathbb{C})$ symmetry in [8]. The use of the latter transformation group is originated from the investigation into the normalizability of $\mathfrak{sl}(2)$ Lie-algebraic quasi-solvable models in [9]. The essential point is that one can classify the models by considering the inequivalent classes with respect to the symmetry transformation. Later in [8], it was recognized that the most general type A systems were invariant under the same $GL(2, \mathbb{C})$ transformation and thus that they were classified in the same way as done in [9]. Hence, it is reasonable to expect that discovery of symmetry in the most general type B 3-fold SUSY models would provide one a useful and powerful tool to examine the classification problem.

In this paper, we explore in detail effect of a general variable transformation on a type B 3-fold SUSY system and its invariance under a $GL(3, \mathbb{C})$ homogeneous linear transformation. For the purpose, we first introduce a change of variable in a three-dimensional linear function space in a gauged space which entirely characterize the system under consideration. We then examine the resultant transformations of all the constituent functions and operators involved in the system. Based on these arrangements, we proceed to a study of a $GL(3, \mathbb{C})$ transformation which preserves the linear space. Due to the latter property, every component quantities of the system has invariance under the transformation. After completing the full analysis, we consider a type A 3-fold SUSY system as a particular case of the type B. In particular, we reconsider the $GL(2, \mathbb{C})$ invariance of any type A model in [8] and show that it is a subgroup of the $GL(3, \mathbb{C})$ in this work. Hence, we reproduce all the results derived in the latter reference. Finally, we generalize the argument to assert that any $\mathcal{N}$-fold SUSY system has a $GL(\mathcal{N}, \mathbb{C})$ invariance.
We organize the paper as follows. In Section II, we briefly summarize the ingredients of type B 3-fold SUSY which constitute a basis for the present purpose. In Section III, we develop effects of a general variable transformation in type B 3-fold SUSY. We present transformation formulas for various quantities there. In Section IV, we proceed to a homogeneous linear transformation $GL(3, \mathbb{C})$ as a particular case of the general one in Section III. We show explicitly that any type B 3-fold SUSY system is invariant under the $GL(3, \mathbb{C})$ transformation and that there exist three invariants of the transformation, in terms of which both the Hamiltonians and 3-fold supercharge components are entirely expressible. We prove that the parameter space, which is dual to the linear space of quasi-solvable operators, is transformed as an adjoint representation of $GL(3, \mathbb{C})$. In Section V, we present a product of the type B 3-fold supercharges which constitutes a part of type B 3-fold superalgebra and provides a characteristic polynomial for the spectrum. We show that its coefficients are all invariants of the $GL(3, \mathbb{C})$ transformation. In Section VI, we consider type A 3-fold SUSY as a special case of the most general type B. In particular, we show that the $GL(2, \mathbb{C})$ linear fractional transformation in [8] is a subgroup of the $GL(3, \mathbb{C})$. In Section VII, we argue by following the same reasoning as in Section IV that any $\mathcal{N}$-fold SUSY system has a $GL(\mathcal{N}, \mathbb{C})$ invariance for an arbitrary $\mathcal{N} \in \mathbb{N}$. In Section VIII, we summarize the results and discuss prospects in the future development.

II. INGREDIENTS OF TYPE B 3-FOLD SUSY

Type B $\mathcal{N}$-fold SUSY was first discovered in Ref. [10] by a simple deformation of type A $\mathcal{N}$-fold supercharge. The component of type B 3-fold supercharge is given by

$$P_3^- = \left( \frac{d}{dq} + W(q) - E(q) - F(q) \right) \left( \frac{d}{dq} + W(q) + E(q) \right), \quad (2.1)$$

where the three functions $W(q)$, $E(q)$, and $F(q)$ are at present arbitrary. A pair of Hamiltonians

$$H^\pm = -\frac{1}{2} \frac{d^2}{dq^2} + V^\pm(q), \quad (2.2)$$

is said to have type B 3-fold SUSY if it is intertwined by $P_3^-$ in (2.1) as

$$P_3^+ H^\mp = H^\pm P_3^+, \quad (2.3)$$

where $P_3^+$ is the transposition of $P_3^-$ in the $q$-space, $P_3^+ = (P_3^-)^T$. It was shown that the intertwining relation (2.3) holds if and only if the potential terms in (2.2) have the following form

$$V^\pm = \frac{1}{2} W^2 - \frac{1}{3} (2E' - E^2) - \frac{1}{6} (2F' + 2WF - 2EF - F^2) \pm \frac{1}{2} (3W' - F'), \quad (2.4)$$

and simultaneously the three functions $W(q)$, $E(q)$, and $F(q)$ satisfy

$$\left( \frac{d}{dq} - E \right) F'_1 - \frac{F}{2} \left( F'_1 - \frac{F''}{6} \right) = 0, \quad (2.5)$$

$$\left( \frac{d}{dq} - 2E - \frac{3}{2} F \right) \left( \frac{d}{dq} - E \right) F'_2 + \frac{3}{2} (2F' - 2EF - F^2) \left( F'_1 - \frac{F''}{6} \right) = 0, \quad (2.6)$$
where \( F_1(q) \) and \( F_2(q) \) are given by
\[
F_1 = W' + EW - \frac{1}{4} (F' - 2WF + 2EF + F^2), \\
F_2 = E' + E^2 + \frac{1}{2} (F' - 2WF + 2EF + F^2).
\]
(2.7)

They can be easily integrated analytically, but the problem gets simpler if we make a change of variable \( z = z(q) \) and introduce three functions \( f(z) \), \( A(z) \), and \( Q(z) \) defined by
\[
2A(z(q)) = z'(q)^2, \quad E(q) = \frac{z''(q)}{z'(q)}, \\
F(q) = \frac{f'''(z(q))}{f''(z(q))} z'(q), \quad W(q) = -\frac{Q(z(q))}{z'(q)}.
\]
(2.8)

Then, the pair of type B 3-fold SUSY Hamiltonians \( H^\pm \) having the potential terms (2.4) can be written as
\[
H^\pm = e^{-W_3^+(q)} \tilde{H}[z] e^{W_3^+(q)} \big|_{z=z(q)},
\]
(2.9)

where the gauged Hamiltonian \( \tilde{H}^- \) is a second-order linear differential operator
\[
\tilde{H}^-[z] = -A(z) \frac{d^2}{dz^2} - B(z) \frac{d}{dz} - C(z), \quad B(z) = Q(z) - \frac{A'(z)}{2},
\]
(2.10)

and the coefficients \( A(z) \), \( B(z) \), and \( C(z) \) are functions of \( z \) given by
\[
A(z) f''(z) = \left[ (c_2 z - b_2) f(z) + c_1 z^2 + (c_0 - b_1) z - b_0 \right] f'(z) \\
- \left[ c_2 f(z) + c_1 z + c_0 - a_2 \right] f(z) + a_1 z + a_0, \\
B(z) = - (c_2 z - b_2) f(z) - c_1 z^2 - (c_0 - b_1) z + b_0, \\
C(z) = c_2 f(z) + c_1 z + c_0.
\]
(2.11)–(2.13)

In the gauged \( z \)-space, the component of type B 3-fold supercharge reads as
\[
\tilde{P}_3^-[z] = e^{W_3^-} P_3^- e^{-W_3^-} = z'(q)^3 \left( \frac{d}{dz} - \frac{f'''(z)}{f''(z)} \right) \frac{d^2}{dz^2},
\]
(2.14)

and it annihilates all the elements of a three-dimensional linear function space \( \tilde{V}_3^- \):
\[
\tilde{V}_3^-[z] = \ker \tilde{P}_3^-[z] = \langle 1, z, f(z) \rangle.
\]
(2.15)

On the other hand, it follows from the gauge-transformed version of the intertwining relation (2.3) with upper signs, namely, \( \tilde{P}_3^+ \tilde{H}^- = \tilde{H}^+ \tilde{P}_3^- \), that \( \tilde{H}^- \ker \tilde{P}_3^- \subset \ker \tilde{P}_3^- \). Hence, any gauged type B 3-fold SUSY Hamiltonian \( \tilde{H}^- \) in (2.10) with (2.11)–(2.13) preserves the linear space (2.15), which is thus called a solvable sector of \( \tilde{H}^- \).

A solvable sector \( \tilde{V}_3^+ \) preserved by \( \tilde{H}^+ \) is characterized by the gauge-transformed operator of the transposed component \( P_3^+ \) of type B 3-fold supercharge
\[
\tilde{P}_3^+[z] = e^{W_3^+} P_3^+ e^{-W_3^+} = -z'(q)^3 \frac{d^2}{dz^2} \left( \frac{d}{dz} + \frac{f'''(z)}{f''(z)} \right).
\]
(2.16)

In fact, the gauge transformation with \( W_3^+ \) of the intertwining relation (2.3) with lower signs, namely, \( \tilde{P}_3^+ \tilde{H}^+ = \tilde{H}^- \tilde{P}_3^+ \), immediately leads to \( \tilde{H}^+ \ker \tilde{P}_3^+ \subset \ker \tilde{P}_3^+ \). Hence, we obtain from (2.16)
\[
\tilde{V}_3^+[z] = \ker \tilde{P}_3^+[z] = \frac{1}{f''(z)} \langle 1, f'(z), zf'(z) - f(z) \rangle.
\]
(2.17)
III. GENERAL TRANSFORMATION IN TYPE B 3-FOLD SUSY

Let us first review and then develop further the effect of a general transformation on the three-dimensional type B space studied in Ref. [4], Section 5:

\[ \mathcal{V}_3^{-}[z] = \langle 1, z, f(z) \rangle \rightarrow \hat{\mathcal{V}}_3^{-}[w] = \langle \tilde{\varphi}_1(w), \tilde{\varphi}_2(w), \tilde{\varphi}_3(w) \rangle. \] (3.1)

Since the latter space is expressed as

\[ \hat{\mathcal{V}}_3^{-}[w] = \tilde{\varphi}_1(w) \mathcal{V}_3^{-}[z] \bigg|_{z = \tilde{\varphi}_2 / \tilde{\varphi}_1, f(z) = \tilde{\varphi}_3 / \tilde{\varphi}_1}, \] (3.2)

an operator of a certain property acting on it is transformed as

\[ \hat{O}[z] \rightarrow \hat{O}[w] = \tilde{\varphi}_1(w) O[z] \hat{\varphi}_1(w)^{-1} \bigg|_{z = \tilde{\varphi}_2 / \tilde{\varphi}_1, f(z) = \tilde{\varphi}_3 / \tilde{\varphi}_1}. \] (3.3)

Several useful formulas needed for performing a transformation are summarized in Appendix A. For instance, the gauged type B 3-fold supercharge component \( \hat{P}_3^+ \) is transformed as

\[ \hat{P}_3^+[w] = w'(q)^3 \left( \frac{d}{dw} \frac{\tilde{\varphi}'_1(w)}{\tilde{\varphi}_1(w)} + \frac{W'_{2,1}(w)}{W_{2,1}(w)} - \frac{W'_{3,1,21}(w)}{W_{3,1,21}(w)} \right) \times \left( \frac{d}{dw} \frac{\tilde{\varphi}'_1(w)}{\tilde{\varphi}_1(w)} - \frac{W'_{2,1}(w)}{W_{2,1}(w)} \right) \left( \frac{d}{dw} \frac{\tilde{\varphi}'_1(w)}{\tilde{\varphi}_1(w)} \right), \] (3.4)

where the Wronskians \( W_{i,j}(w) \) and \( W_{ij,kl}(w) \) are defined in (A3) and (A5). The plus component of the gauged type B 3-fold supercharge \( \hat{P}_3^+[w] \) and the linear space \( \hat{\mathcal{V}}_3^+[w] \) annihilated by it in the \( w \)-space are related to the corresponding quantities in the \( z \)-space, \( \hat{P}_3^+[z] \) in (2.16) and \( \hat{\mathcal{V}}_3^+[z] \) in (2.17), respectively, as

\[ \hat{P}_3^+[w] = \tilde{\varphi}_1(w)^3 W_{2,1}(w)^{-2} \hat{P}_3^+[z] W_{2,1}(w) \hat{\varphi}_1(w)^{-3} \bigg|_{z = \tilde{\varphi}_2 / \tilde{\varphi}_1, f(z) = \tilde{\varphi}_3 / \tilde{\varphi}_1}, \]
\[ \hat{\mathcal{V}}_3^+[w] = \tilde{\varphi}_1(w)^3 W_{2,1}(w)^{-2} \hat{\mathcal{V}}_3^+[z] \bigg|_{z = \tilde{\varphi}_2 / \tilde{\varphi}_1, f(z) = \tilde{\varphi}_3 / \tilde{\varphi}_1}. \] (3.5)

With these formulas, we obtain

\[ \hat{P}_3^+[w] = -w'(q)^3 \left( \frac{d}{dw} \frac{\tilde{\varphi}'_1(w)}{\tilde{\varphi}_1(w)} \right) \left( \frac{d}{dw} \frac{\tilde{\varphi}'_1(w)}{\tilde{\varphi}_1(w)} + \frac{W'_{2,1}(w)}{W_{2,1}(w)} \right) \times \left( \frac{d}{dw} \frac{\tilde{\varphi}'_1(w)}{\tilde{\varphi}_1(w)} - \frac{W'_{2,1}(w)}{W_{2,1}(w)} \right) \left( \frac{d}{dw} \frac{\tilde{\varphi}'_1(w)}{\tilde{\varphi}_1(w)} \right), \] (3.6)

and

\[ \hat{\mathcal{V}}_3^+[w] = \frac{\tilde{\varphi}_1(w)}{W_{31,21}(w)} \langle W_{2,1}(w), W_{3,1}(w), W_{3,2}(w) \rangle. \] (3.7)

We shall next consider the effect of the above transformation in the original \( q \)-space. For this purpose, we must know the transformations of the functions \( E(q), F(q), \) and \( W(q) \).
Comparing the (3.11) and (3.12), and using (A2), we obtain the transformation rule as

\[
\tilde{E}(q) = \frac{w''(q)}{w'(q)}, \quad \tilde{F}(q) = \frac{f''(w(q))}{f''(w(q))} w'(q), \quad \tilde{W}(q) = -\frac{\tilde{Q}(w(q))}{w'(q)}.
\] (3.8)

To begin with, we note that with the aid of the formulas (A6), we have

\[
E(q) = \frac{w''(q)}{w'(q)} + \left( \frac{W'_{2,1}(w(q))}{W_{2,1}(w(q))} - \frac{2\varphi'_{1}(w(q))}{\varphi_{1}(w(q))} \right) w'(q),
\]

\[
F(q) = \left( \frac{W'_{3,21}(w(q))}{W_{3,21}(w(q))} - \frac{3W'_{2,1}(w(q))}{W_{2,1}(w(q))} + \frac{2\varphi'_{1}(w(q))}{\varphi_{1}(w(q))} \right) w'(q).
\] (3.9)

Hence, the transformation of \( E(q) \) immediately reads as

\[
\tilde{E}(q) = E(q) - \left( \frac{W'_{2,1}(w(q))}{W_{2,1}(w(q))} - \frac{2\varphi'_{1}(w(q))}{\varphi_{1}(w(q))} \right) w'(q).
\] (3.10)

On the other hand, the transformation of \( F(q) \) cannot be determined unless the dependence of \( \tilde{\varphi}_{i}(w) \) (\( i = 1, 2, 3 \)) on the function \( f(w) \) is specified, which we shall later consider.

Next, to establish the transformation of the function \( W(q) \), we must first know the transformation of \( Q(z) \) introduced in (2.10). The transformed functions \( \tilde{A}(w) \) and \( \tilde{Q}(w) \) are thus defined as

\[
\tilde{H}^{-}[w] = -\tilde{A}(w) \frac{d^{2}}{dw^{2}} - \left( \tilde{Q}(w) - \frac{\tilde{A}'(w)}{2} \right) \frac{d}{dw} - \tilde{C}(w).
\] (3.11)

Applying the formula (3.3) to \( \tilde{H}^{-}[z] \), we have

\[
\tilde{H}^{-}[w] = -A(z) \left( \frac{dw}{dz} \right)^{2} \frac{d^{2}}{dw^{2}} + \left[ 2A(z) \left( \frac{dw}{dz} \right)^{2} \frac{\varphi'_{1}}{\varphi_{1}} - A(z) \frac{d^{2}w}{dz^{2}} 
- \left( Q(z) - \frac{A'(z)}{2} \right) \frac{dw}{dz} \right] \frac{d}{dw} + A(z) \left( \frac{dw}{dz} \right)^{2} \left( \frac{\varphi'_{1}}{\varphi_{1}} - \frac{2(\tilde{\varphi}_{1})^{2}}{(\tilde{\varphi}_{1})^{2}} \right)
+ \left[ A(z) \frac{d^{2}w}{dz^{2}} + \left( Q(z) - \frac{A'(z)}{2} \right) \frac{dw}{dz} \right] \frac{\varphi'_{1}}{\varphi_{1}} - C(z) \right]_{z=\tilde{\varphi}_{2}/\tilde{\varphi}_{1}, f(z)=\tilde{\varphi}_{3}/\tilde{\varphi}_{1} .
\] (3.12)

Comparing the (3.11) and (3.12), and using (A2), we obtain the transformation rule as

\[
\tilde{A}(w) = A(z) \frac{(\tilde{\varphi}_{1})^{4}}{(W_{2,1})^{2}}, \quad \tilde{B}(w) = B(z) \frac{(\tilde{\varphi}_{1})^{2}}{W_{2,1}} - A(z) \frac{W'_{2,1}(\tilde{\varphi}_{1})^{4}}{(W_{2,1})^{3}} ,
\]

\[
\tilde{C}(w) = C(z) - B(z) \tilde{\varphi}_{1} \tilde{\varphi}_{1}' \frac{W_{2,1}}{W_{2,1}} + A(z) \frac{W'_{2,1}(\tilde{\varphi}_{1})^{4}}{(W_{2,1})^{3}} ,
\] (3.13)

\[
\tilde{Q}(w) = Q(z) \frac{(\tilde{\varphi}_{1})^{2}}{W_{2,1}} - 2A(z) \left( \frac{W'_{2,1}}{W_{2,1}} - \frac{\tilde{\varphi}_{1}'}{\tilde{\varphi}_{1}} \right) (\tilde{\varphi}_{1})^{4} .
\]
The last formula determines the transformation of $W(q)$ as

$$\hat{W}(q) = W(q) + \left( \frac{W'_{2,1}(w(q))}{W_{2,1}(w(q))} - \frac{\hat{\varphi}'_1(w(q))}{\hat{\varphi}_1(w(q))} \right) w'(q).$$  \hspace{1cm} (3.14)

For a concrete calculation of the transformation in (3.13), it is convenient to rewrite the functions $A(z), B(z),$ and $C(z)$ given in (2.11)–(2.13) in matrix form. Let us arrange the set of parameters $\{a_i, b_i, c_i\}$ in a three-by-three matrix $\Omega$ and the bases of $\hat{V}_3^- [z]$ and $\hat{V}_3^- [w]$ in three-component column vectors $\varphi_0(z)$ and $\varphi(w)$, respectively, as

$$\Omega = \begin{pmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ c_0 & c_1 & c_2 \end{pmatrix}, \quad \varphi_0(z) = \begin{pmatrix} 1 \\ z \\ f(z) \end{pmatrix}, \quad \varphi(w) = \begin{pmatrix} \hat{\varphi}_1(w) \\ \hat{\varphi}_2(w) \\ \hat{\varphi}_3(w) \end{pmatrix}. \hspace{1cm} (3.15)$$

Then, the formulas (2.11)–(2.13) are rewritten as

$$A(z)f''(z) = \xi^T(z)\Omega\varphi_0(z),$$

$$B(z) = -\zeta_0^T(z)\Omega\varphi_0(z),$$

$$C(z) = \zeta_0^T(z)\Omega\varphi_0(z), \hspace{1cm} (3.16)$$

where the superscript $T$ denotes the transposition of a matrix, and the column vectors $\xi(z)$ and $\zeta_0(z)$ are defined by

$$\xi(z) = \begin{pmatrix} zf'(z) - f(z) \\ -f'(z) \\ 1 \end{pmatrix}, \quad \zeta_0(z) = \begin{pmatrix} z \\ -1 \\ 0 \end{pmatrix}. \hspace{1cm} (3.17)$$

Applying the formulas (A1) and (A2), they are expressed in terms of the new variable $w$ as

$$A(z)f''(z) = \frac{1}{W_{2,1}(w)}W^T(w)\Omega\varphi(w),$$

$$B(z) = -\frac{1}{\hat{\varphi}_1(w)}\zeta^T(w)\Omega\varphi(w),$$

$$C(z) = \frac{1}{\hat{\varphi}_1(w)}\zeta_0^T(w)\Omega\varphi(w), \hspace{1cm} (3.18)$$

where

$$W(w) = \begin{pmatrix} W_{3,2}(w) \\ -W_{3,1}(w) \\ W_{2,1}(w) \end{pmatrix}, \quad \zeta(w) = \begin{pmatrix} \hat{\varphi}_2(w) \\ -\hat{\varphi}_1(w) \\ 0 \end{pmatrix}. \hspace{1cm} (3.19)$$

Substituting them into (3.13), we obtain the transformed functions $\hat{A}(w), \hat{B}(w),$ and $\hat{C}(w)$ in matrix form as

$$\hat{A}(w)f''(w) = \frac{\hat{\varphi}_1(w)}{W_{31,21}(w)}W^T(w)\Omega\varphi(w),$$

$$\hat{B}(w) = -\frac{\hat{\varphi}_1(w)}{W_{31,21}(w)}W^T(w)\Omega\varphi(w),$$

$$\hat{C}(w) = \frac{\hat{\varphi}_1(w)}{W_{31,21}(w)}W^T(w)\Omega\varphi(w), \hspace{1cm} (3.20)$$
where the column vector $W'(w)$ is defined by

$$
W'(w) = \begin{pmatrix}
W_{3',2'}(w) \\
-W_{3',1'}(w) \\
W_{2',1'}(w)
\end{pmatrix}.
$$

(3.21)

The Wronskian $W_{i,j'}(w)$ appeared in the components is defined in (A4).

IV. $GL(3, \mathbb{C})$ INVARIANCE OF TYPE B 3-FOLD SUSY

We are now in a position to consider a homogeneous linear transformation $GL(3, \mathbb{C})$ of the basis:

$$
\varphi(w) = \Lambda \varphi_0(w), \quad \Lambda = \begin{pmatrix}
\lambda_{11} & \lambda_{12} & \lambda_{13} \\
\lambda_{21} & \lambda_{22} & \lambda_{23} \\
\lambda_{31} & \lambda_{32} & \lambda_{33}
\end{pmatrix} \in GL(3, \mathbb{C}),
$$

(4.1)

where $\lambda_{ij} \in \mathbb{C}$ ($i, j = 1, 2, 3$) are all constants satisfying $\det \Lambda \neq 0$. It is evident that the three-dimensional type B space is invariant under any of the $GL(3, \mathbb{C})$ transformation, that is,

$$
\hat{\mathcal{V}}_3^-[w] = \mathcal{V}_3^-[w].
$$

(4.2)

It means in particular that type B 3-fold SUSY Hamiltonians $H^\pm$ and supercharge component $P_3^-$ must have the same forms both in terms of the variable $z(q)$ and in terms of $w(q)$. To see it, we firstly need to know more explicit forms of the Wronskians in the $GL(3, \mathbb{C})$ case. We summarize them in (A8). We note that, applying them to (3.7) and comparing the result with (2.17), we can also check the invariance of the other subspace:

$$
\hat{\mathcal{V}}_3^+[w] = \mathcal{V}_3^+[w].
$$

(4.3)

Substituting the formula for $W_{31,21}(w)$ into (3.9), we eventually obtain the transformation of the function $F(q)$:

$$
\hat{F}(q) = F(q) + 3 \left( \frac{W_{21}'(w(q))}{W_{2,1}(w(q))} - \frac{\hat{\varphi}_1'(w(q))}{\hat{\varphi}_1(w(q))} \right) w'(q).
$$

(4.4)

From the transformation formulas (3.10), (3.14), and (4.4) together with (A8), we can show that there are three functions $I_i$ ($i = 1, 2, 3$) composed of $E(q)$, $W(q)$, and $F(q)$ which are invariant under the $GL(3, \mathbb{C})$ as the followings:

$$
I_1[E, W, F] = W - \frac{1}{3} F, \quad I_2[E, W, F] = 2 E' + F' - E^2 - EF - \frac{1}{3} F^2,
$$

$$
I_3[E, W, F] = F'' - E' F - 3 E F' + 2 F F' + 2 E^2 F + 2 E F^2 + \frac{4}{9} F^3.
$$

(4.5)

That is, all the above satisfy

$$
I_i[\hat{E}, \hat{W}, \hat{F}] = I_i[E, W, F], \quad (i = 1, 2, 3).
$$

(4.6)
The invariance of $I_1$ is trivial from (3.14) and (4.4). To show it for $I_2$ and $I_3$, we need the following identity which is easily shown with the formula (A8):

$$J(w) = (W_{2,1}'' \cdot \varphi_1 - W_{2,1}' \cdot \varphi_1' + W_{2,1}'' \cdot \varphi_1')f'' - W_{2,1}'' \cdot \varphi_1 f''' = 0.$$  (4.7)

Indeed, a direct calculation shows that

$$I_2[\hat{E}, \hat{W}, \hat{F}] = I_2[E, W, F] + \frac{3}{W_{2,1}'' \cdot \varphi_1} w'(q)^2,$$

$$I_3[\hat{E}, \hat{W}, \hat{F}] = I_3[E, W, F] - 6 \frac{\varphi_1 f'' - \varphi_1'' f'''}{\varphi_1 f''} w'(q)^3 + \left[3(W_{2,1}'' \cdot \varphi_1 - W_{2,1}' \cdot \varphi_1') f'' - 5 W_{2,1}'' \cdot \varphi_1 f''']J + 3 W_{2,1}'' \cdot \varphi_1 f'' f'J' \right] w'(q)^3,$$

and their invariance is now manifest since $\varphi_1'' f'' = \varphi_1'' f'''$ by (A8). Then, the type B 3-fold supercharge component $P_3^-$ in (2.1) and the pair of type B 3-fold SUSY potentials $V^\pm$ in (2.4) are all expressible solely in terms of these invariants as

$$P_3^- = \frac{d^3}{dq^3} + 3 I_1(q) \frac{d^2}{dq^2} + \left[3 I_1'(q) + 3 I_1(q)^2 + I_2(q)\right] \frac{d}{dq} I_1''(q) + 3 I_1(q)^3 + I_1(q)I_2(q) + \frac{1}{2} I_2'(q) - \frac{1}{6} I_3(q),$$

$$V^\pm = \frac{1}{2} I_1(q)^2 - \frac{1}{3} I_2(q) \pm \frac{3}{2} I_1(q)^2,$$  (4.8, 4.9)

and hence their invariance under the $GL(3, \mathbb{C})$ is clearly shown. However, we note that each of the factors $P_{3i}^-$ $(i = 1, 2, 3)$ in the type B 3-fold supercharge $P_3^- = P_{31}^- P_{32}^- P_{33}^-$, where

$$P_{31}^- = \frac{d}{dq} + W - E - F, \quad P_{32}^- = \frac{d}{dq} + W, \quad P_{33}^- = \frac{d}{dq} + W + E,$$  (4.10)

is not invariant; actually they transform as

$$P_{31}^-[\hat{E}, \hat{W}, \hat{F}] = P_{31}^-[E, W, F] - \frac{W_{2,1}'}{W_{2,1}} w'(q),$$

$$P_{32}^-[\hat{E}, \hat{W}, \hat{F}] = P_{32}^-[E, W, F] + \left(\frac{W_{2,1}'}{W_{2,1}} - \frac{\varphi_1'}{\varphi_1}\right) w'(q),$$

$$P_{33}^-[\hat{E}, \hat{W}, \hat{F}] = P_{33}^-[E, W, F] + \frac{\varphi_1'}{\varphi_1} w'(q),$$  (4.11)

which can be regarded as a generalization of (5.1) in Ref. [11] in the case of type A. It indicates in particular that there is an infinite number of different factorizations of $P_3^-$. The $GL(3, \mathbb{C})$ transformations of the functions $A(z)$, $B(z)$, and $C(z)$ are calculated as follows. Applying the formulas (A8) to the expression (3.20), we have

$$\hat{A}(w)f''(w) = \xi^T(w) \Lambda^{-1} \Omega \Lambda \varphi_0(w),$$

$$\hat{B}(w) = -\xi^T_0(w) \Lambda^{-1} \Omega \Lambda \varphi_0(w),$$

$$\hat{C}(w) = \xi^{\prime T}_0(w) \Lambda^{-1} \Omega \Lambda \varphi_0(w).$$  (4.12)
Comparing them with (3.16), we obtain the $GL(3, \mathbb{C})$ transformation of the set of parameters $\{a_i, b_i, c_i\}$ to $\{\hat{a}_i, \hat{b}_i, \hat{c}_i\}$ as
\[
\hat{\Omega} = \begin{pmatrix}
\hat{c}_0 & \hat{c}_1 & \hat{c}_2 \\
\hat{b}_0 & \hat{b}_1 & \hat{b}_2 \\
\hat{a}_0 & \hat{a}_1 & \hat{a}_2
\end{pmatrix} = \Lambda^{-1}\Omega\Lambda,
\] (4.13)
which means that $\Omega$ transforms as an adjoint representation.

V. TYPE B 3-FOLD SUPERALGEBRA

Another notable feature in $\mathcal{N}$-fold SUSY is that the anti-commutator of $\mathcal{N}$-fold supercharges is a polynomial of degree $\mathcal{N}$ in the corresponding superHamiltonian \cite{2,12} which constitutes a generalized superalgebra, called $\mathcal{N}$-fold superalgebra. The emerged $\mathcal{N}$th-degree polynomial determines the spectrum of $H^\pm$ in the respective solvable sectors $\mathcal{V}_N^\pm$ by its $\mathcal{N}$ roots. In addition, given that there is a symmetry in an $\mathcal{N}$-fold SUSY system, each coefficient of the polynomial is composed of an invariant quantity of the symmetry. For instance, type A $\mathcal{N}$-fold SUSY for any $\mathcal{N} \in \mathbb{N}$ has the $GL(2, \mathbb{C})$ symmetry composed of linear fractional transformations, and as a consequence every coefficients of the polynomial involved in type A $\mathcal{N}$-fold superalgebra are composed of algebraic invariants of the $GL(2, \mathbb{C})$ transformations \cite{8}, cf. Section VI.

In our present type B 3-fold case, the third-degree polynomial to be appeared is calculated directly via e.g., $P_3^+ P_3^-$, by using (2.1). In practice, it is easier to carry out the calculation in the gauged $z$-space by noting the fact that any algebraic relation among operators is preserved by a gauge (similarity) transformation. A direct calculation shows
\[
\hat{P}_3^+ \hat{P}_3^- = -(z')^3 \left( \frac{d}{dz} + \frac{2A' + B}{A} \right)^2 \left( \frac{d}{dz} + \frac{2A' + B}{A} \frac{f'''}{f''} \right) \left( \frac{d}{dz} - \frac{f'''}{f''} \right) \frac{d^2}{dz^2}
= 8 \left[ (\hat{H}^- + C_0(\Omega))^3 + C_1(\Omega)(\hat{H}^- + C_0(\Omega)) + C_2(\Omega) \right],
\] (5.1)
where the constants $C_i(\Omega)$ ($i = 0, 1, 2$) are expressed in terms of the set of parameters $\{a_i, b_i, c_i\}$ as
\[
3C_0(\Omega) = a_2 + b_1 + c_0,
\] (5.2)
\[
3C_1(\Omega) = -(a_2)^2 + a_2 b_1 - 3a_1 b_2 + a_2 c_0 - 3a_0 c_2 - (b_1)^2 + b_1 c_0 - 3b_0 c_1 - (c_0)^2,
\]
(5.3)
\[
27C_2(\Omega) = 2(a_2)^3 - 3(a_2)^2 b_1 - 3(a_2)^2 c_0 + 9a_1 a_2 b_2 + 9a_0 a_2 c_2 - 3a_2 (b_1)^2 + 12a_2 b_1 c_0 - 18a_2 b_0 c_1 - 3a_2 (c_0)^2 - 18a_1 b_2 c_0 + 9a_1 b_1 b_2 + 27a_1 b_0 c_2 + 27a_0 b_2 c_1 - 18a_0 b_1 c_2 + 9a_0 c_0 c_2 + 2(b_1)^3 - 3(b_1)^2 c_0 + 9b_0 b_1 c_1 - 3b_1 (c_0)^2 + 9b_0 c_0 c_1 + 2(c_0)^3.
\] (5.4)

Next, we shall consider the effect of the $GL(3, \mathbb{C})$ transformation on the operator identity (5.1). Given that $\hat{P}_3^\pm$ and $\hat{H}^-$ are all transformed according to the rule (3.3), it is evident that the algebraic relation among them is maintained under the transformation and thus the same relation among the transformed quantities $\hat{P}_3^\pm$ and $\hat{H}^-$ holds. On the other hand,
the parameter set \( \{a_i, b_i, c_i\} \) also transforms according to (4.13). Hence, we must have 
\[ C_i(\Omega) = C_i(\Omega^\prime) \quad (i = 0, 1, 2) \]
so that
\[ \hat{P}_3^+ \hat{P}_3^- = 8 \left[ (\hat{H}^- + C_0(\Omega))^3 + C_1(\Omega) (\hat{H}^- + C_0(\Omega)) + C_2(\Omega) \right], \tag{5.5} \]
holds. Actually, the latter fact can be immediately derived if we notice that the constants 
\[ C_i(\Omega) \quad (i = 0, 1, 2) \]
are all expressible in terms of invariants of a matrix under a similarity transformation, namely, traces and determinants, as
\[ 3C_0(\Omega) = \text{Tr} \Omega, \quad 3C_1(\Omega) = -(\text{Tr} \Omega)^2 + 3(\text{det} \Omega) \text{Tr} \Omega^{-1}, \]
\[ 27C_2(\Omega) = 2(\text{Tr} \Omega)^3 - 9(\text{det} \Omega)(\text{Tr} \Omega) \text{Tr} \Omega^{-1} + 27 \text{det} \Omega. \tag{5.6} \]
This result generalizes the fact in the type A case that every coefficient of the polynomials appeared in an \( \mathcal{N} \)-fold superalgebra consists of solely algebraic invariants.

VI. TYPE A LIMIT AND THE SUBGROUP \( GL(2, \mathbb{C}) \)

In the case of type A \( \mathcal{N} \)-fold SUSY, it is invariant under the \( GL(2, \mathbb{C}) \) linear fractional transformations for arbitrary \( \mathcal{N} \in \mathbb{N} \) [8]. On the other hand, type A 3-fold SUSY is a particular case of type B one, and hence it is evident that the former must admit the larger \( GL(3, \mathbb{C}) \) symmetry discussed in the preceding sections. In this section, we shall show that the \( GL(2, \mathbb{C}) \) symmetry in type A 3-fold SUSY is actually a subgroup of the \( GL(3, \mathbb{C}) \) and thus any result followed from the former can be reproduced by the present latter formulation.

In the gauged \( z \)-space, any type A \( \mathcal{N} \)-fold SUSY system is characterized by two polynomials \( A^{(A)}(z) \) and \( Q^{(A)}(z) \), one of which is of fourth degree and another of second:
\[ A^{(A)}(z) = a_4^{(A)} z^4 + a_3^{(A)} z^3 + a_2^{(A)} z^2 + a_1^{(A)} z + a_0^{(A)}, \tag{6.1} \]
\[ Q^{(A)}(z) = b_2^{(A)} z^2 + b_1^{(A)} z + b_0^{(A)}. \tag{6.2} \]
For \( \mathcal{N} = 3 \), the corresponding gauged Hamiltonian \( \hat{H}^- \) is expressed in terms of them as
\[ \hat{H}^-[z] = -A^{(A)}(z) \frac{d^2}{dz^2} - \left( Q^{(A)}(z) - \frac{A^{(A)\prime}(z)}{2} \right) \frac{d}{dz} \]
\[ -\frac{A^{(A)\prime\prime}(z)}{6} + Q^{(A)\prime\prime}(z) - R^{(A)}, \tag{6.3} \]
where \( R^{(A)} \) is a constant. Type A 3-fold SUSY is a special case of type B one realized by setting \( f(z) = z^2 \) [4]. By comparison between (2.10) with (2.11)–(2.13) where \( f(z) = z^2 \) is substituted and (6.3), each of the coefficients \( a_i^{(A)} \) \((i = 0, \ldots, 4)\), \( b_i^{(A)} \) \((i = 0, 1, 2)\), and \( R^{(A)} \) are related to the parameters in type B as
\[ a_2 = 2b_1^{(A)} + c_0, \quad a_1 = a_1^{(A)} + 2b_0^{(A)}, \quad a_0 = 2a_0^{(A)}, \]
\[ 2b_2 = -a_3^{(A)} - 2b_2^{(A)}, \quad b_1 = -a_2^{(A)} + b_1^{(A)} + c_0, \quad 2b_0 = -a_1^{(A)} + 2b_0^{(A)}, \]
\[ c_2 = 2a_4^{(A)}, \quad c_1 = a_3^{(A)} - 2b_2^{(A)}, \quad 3c_0 = a_2^{(A)} - 3b_1^{(A)} + 3R^{(A)}. \tag{6.4} \]
Substituting (6.4) into (5.2)–(5.4), we see that the three constants \( C_i(\Omega) \) \((i = 0, 1, 2)\) which characterize 3-fold superalgebra of type B reduce to the correct type A formulas in [8]:

\[
\begin{align*}
C_0(\Omega) &= R^{(A)}, \\
3C_1(\Omega) &= -i_2[A^{(A)}] + 3D_2[Q^{(A)}], \\
27C_2(\Omega) &= 2j_3[A^{(A)}] + 18I_{1,2}[A^{(A)}, Q^{(A)}],
\end{align*}
\]

(6.5)

where \( D_2[Q], i_2[A], j_3[A], \) and \( I_{1,2}[A, Q] \) are the algebraic invariants, called transvectants, composed of fourth- and second-degree polynomials \( A \) and \( Q \) (see, e.g., [13] and references in [9] for details), and are given in terms of their respective coefficients \( a_i \) \((i = 0, \ldots, 4)\) and \( b_i \) \((i = 0, 1, 2)\) by

\[
\begin{align*}
D_2[Q] &= 4b_0b_2 - (b_1)^2, \\
i_2[A] &= 12a_0a_4 - 3a_1a_3 + (a_2)^2, \\
2j_3[A] &= 72a_0a_2a_4 - 27(a_0a_3)^2 - 27(a_1)^2a_4 + 9a_1a_2a_3 - 2(a_2)^3, \\
I_{1,2}[A, Q] &= 6a_4(b_0)^2 - 3a_3b_0b_1 + 2a_2b_0b_2 + a_2(b_1)^2 - 3a_1b_1b_2 + 6a_0(b_0)^2,
\end{align*}
\]

(6.6)

where the superscript \((A)\) has been omitted for the simplicity.

In Ref. [11], we considered the \( GL(2, \mathbb{C}) \) linear fractional transformation

\[
z = \frac{\alpha w + \beta}{\gamma w + \delta}, \quad (\alpha, \beta, \gamma, \delta \in \mathbb{C}, \quad \Delta = \alpha\delta - \beta\gamma \neq 0).
\]

(6.7)

The three-dimensional type A monomial subspace is transformed as

\[
\tilde{V}_3^{(A)}[z] = \langle 1, z, z^2 \rangle \rightarrow \tilde{V}_3^{-}(w) = (\gamma w + \delta)^2\tilde{V}_3^{(A)}[z] \big|_{z=(\alpha w+\beta)/(\gamma w+\delta)} = \langle (\gamma w + \delta)^2, (\alpha w + \beta)(\gamma w + \delta), (\alpha w + \beta)^2 \rangle = \tilde{V}_3^{(A)}[w].
\]

(6.8)

Comparing it with (3.1) and (4.1), and noting \( f(w) = w^2 \) in the type A case, we recognize that the above \( GL(2, \mathbb{C}) \) transformation is in fact a subgroup of the \( GL(3, \mathbb{C}) \) transformation (4.1) characterized by the following specific form of \( \Lambda \):

\[
\Lambda = \begin{pmatrix}
\delta^2 & 2\gamma \delta & \gamma^2 \\
\beta \delta & \alpha \delta + \beta \gamma & \alpha \gamma \\
\beta^2 & 2\alpha \beta & \alpha^2
\end{pmatrix}.
\]

(6.9)

The Wronskian \( W_{2,1}(w) \) in this case is calculated as

\[
W_{2,1}(w) = \Delta(\gamma^2w^2 + 2\gamma\delta w + \delta^2) = \Delta \tilde{\varphi}_1(w).
\]

(6.10)

Hence, the \( GL(2, \mathbb{C}) \) transformations of \( E(q), W(q), \) and \( F(q) \) in (3.10), (3.14), and (4.4), respectively, as a subgroup of the \( GL(3, \mathbb{C}) \) one, namely, read as

\[
\begin{align*}
\tilde{E}(q) &= E(q) + \frac{2\gamma w'(q)}{\gamma w(q) + \delta} = E(q) - \frac{2\gamma z'(q)}{\gamma z(q) - \alpha}, \\
\tilde{W}(q) &= W(q), \\
\tilde{F}(q) &= F(q) = 0.
\end{align*}
\]

(6.11)
which coincides exactly with (2.12) in [11]. In particular, each factor of the factorized type A 3-fold supercharge \( P_3^{-} = P_{31}^{-}P_{32}^{-}P_{33}^{-} \) is transformed as

\[
P_{31}^{-}[\hat{W}, \hat{E}, 0] = P_{31}^{-}[W, E, 0] + \frac{2\gamma'(q)}{\gamma(q) - \alpha},
\]

\[
P_{32}^{-}[\hat{W}, \hat{E}, 0] = P_{32}^{-}[W, E, 0],
\]

\[
P_{33}^{-}[\hat{W}, \hat{E}, 0] = P_{33}^{-}[W, E, 0] - \frac{2\gamma'(q)}{\gamma(q) - \alpha},
\]

and thus (5.1) of [11] is reproduced.

As a final remark, we note that the three invariants in (4.5) reduce to the two ones of the \( GL(2, \mathbb{C}) \) in the case of type A 3-fold SUSY (cf., Eqs. (2.12) and (2.13) in Ref. [11]) in the type A limit \( F \to 0 \):

\[
I_1[E, W, 0] = W, \quad I_2[E, W, 0] = 2E' - E^2, \quad I_3[E, W, 0] = 0.
\]

The \( GL(2, \mathbb{C}) \) considered in the type A case is a subgroup of the present \( GL(3, \mathbb{C}) \), and thus a consistency has been checked.

VII. \( GL(N, \mathbb{C}) \) INVARIANCE OF \( N \)-FOLD SUSY

By generalizing the argument about the \( GL(3, \mathbb{C}) \) invariance of type B 3-fold SUSY in Section IV, we easily come to the conclusion that any \( N \)-fold SUSY system has \( GL(N, \mathbb{C}) \) invariance. As shown in [14], a specific \( N \)-dimensional linear function space

\[
\tilde{V}_N^{-}[z] = \langle \tilde{\varphi}_1(z), \ldots, \tilde{\varphi}_N(z) \rangle,
\]

(7.1)

uniquely determines an \( N \)-fold SUSY system and vice versa. On the other hand, under any \( GL(N, \mathbb{C}) \) transformation on \( \tilde{V}_N^{-}[z] \) defined by

\[
\tilde{\varphi}_i(w) = \sum_{j=1}^{N} \lambda_{ij} \varphi_j(w), \quad (i = 1, \ldots, N),
\]

(7.2)

it is evident that the vector space \( \tilde{V}_N^{-}[z] \) is invariant:

\[
\tilde{V}_N^{-}[w] = \langle \tilde{\varphi}_1(w), \ldots, \tilde{\varphi}_N(w) \rangle = \tilde{V}_N^{-}[w].
\]

(7.3)

Hence, the assertion must follow in the same sense as in Section IV.

The argument about the subgroup \( GL(2, \mathbb{C}) \) in the limiting case of type A in Section VI further suggests that the \( GL(2, \mathbb{C}) \) linear fractional transformation on any type A \( N \)-fold SUSY system for an arbitrary \( N \in \mathbb{N} \) considered in [8] is also a subgroup of the above symmetry group \( GL(N, \mathbb{C}) \). In fact, the \( GL(2, \mathbb{C}) \) transformation acts on the type A solvable sector \( \tilde{V}_N^{(A)}[z] = \langle 1, z, \ldots, z^{N-1} \rangle \) as

\[
\tilde{V}_N^{(A)}[z] \rightarrow \tilde{\tilde{V}}_N^{-}[w] = (\gamma w + \delta)^{N-1} \tilde{V}_3^{(A)}[z] \big|_{z=(\alpha w + \beta)/(\gamma w + \delta)} = \langle (\gamma w + \delta)^{N-1}, (\alpha w + \beta)(\gamma w + \delta)^{N-2}, \ldots, (\alpha w + \beta)^{N-1} \rangle = \tilde{V}_3^{(A)}[w],
\]

(7.4)
and thus it is a particular $GL(N, \mathbb{C})$ transformation:

$$\hat{\varphi}_i(w) = \sum_{j=1}^{N} \lambda_{ij} w^{j-1}, \quad (i = 1, \ldots, N),$$

(7.5)

where the coefficient $\lambda_{ij}$ $(i, j = 1, \ldots, N)$ is explicitly given by

$$\lambda_{ij} = \sum_{k=0}^{j-1} \binom{N - i}{j - k - 1} \alpha^k \beta^i \gamma^{-k} \delta^{N - i - j + k + 1}.$$  

(7.6)

In the above, any binomial coefficient $\binom{m}{n}$ is understood to be zero for $m < n$.

VIII. DISCUSSION AND SUMMARY

In this paper, we have developed the effects of a general variable transformation in type B 3-fold SUSY and then applied them to the $GL(3, \mathbb{C})$ homogeneous linear transformation. Then, we have shown that there are three invariant functions $I_i(q)$ $(i = 1, 2, 3)$ in (4.5) composed of the three functions $E(q)$, $W(q)$, and $F(q)$ which characterize a type B 3-fold SUSY system, and that such a system is completely expressible in terms of them, (4.8) and (4.9), and hence is invariant under the $GL(3, \mathbb{C})$. We have also shown that the parameter set $\{a_i, b_i, c_i\}$ transform as an adjoint representation (4.13). We have calculated the characteristic polynomial of third degree emerged from the product of type B 3-fold supercharges and then confirmed that its coefficients are all $GL(3, \mathbb{C})$ invariants (5.6). We have considered the type A limit and shown that the $GL(2, \mathbb{C})$ linear fractional transformation which leaves any type A system invariant is a subgroup of the $GL(3, \mathbb{C})$. In the last, we have argued that any $N$-fold SUSY system has invariance under a $GL(N, \mathbb{C})$ transformation for any $N \in \mathbb{N}$.

The present results have several applications in the future development. Noting first the fact that the $GL(2, \mathbb{C})$ invariance was so efficient in classifying the type A $N$-fold SUSY models in [8], we expect that the present $GL(3, \mathbb{C})$ will enable one to investigate systematically what kind of type B 3-fold SUSY potentials can be realized. In this respect, it would be also interesting to reconsider the classification of type A 3-fold SUSY in view of the more general $GL(3, \mathbb{C})$ than the previous $GL(2, \mathbb{C})$. The latter study would further give a suggestion on what can be anticipated if we review type A $N$-fold SUSY for any $N \in \mathbb{N}$ by utilizing a $GL(cN, \mathbb{C})$ transformation.

Another important aspect of the transformation properties is that each of factorized components of the type B 3-fold supercharge is not invariant under the $GL(3, \mathbb{C})$, cf. (4.11). In the type A 2- and 3-fold SUSY cases, it was shown in [11, 15] that the non-invariance resulted in the existence of different sets of intermediate Hamiltonians. In the study of the number and classification of such inequivalent sets, the $GL(2, \mathbb{C})$ transformation played a central and crucial role. Hence, we expect that the present $GL(3, \mathbb{C})$ transformation will also provide an indispensable tool for investigating both the existence and classification of different sets of intermediate Hamiltonians in the most general type B 3-fold SUSY system.

Regarding the subject of intermediate Hamiltonians in $N$-fold SUSY, we would like to recall its relevance on the concept of shape invariance [16] and its muti-step generalization [17], which have been practical methods in constructing solvable quantum Hamiltonians. Not only
Using them, we can calculate derivatives of $z$ and throughout the paper we have employed the following notation for the Wronskians:

\[ W_{i,j}(w) = \varphi_i'(w)\varphi_j(w) - \varphi_i(w)\varphi_j'(w), \]

\[ W_{i',j'}(w) = \varphi_i''(w)\varphi_j'(w) - \varphi_i'(w)\varphi_j''(w), \]

\[ W_{i,j;kl}(w) = W_{i,j}(w)W_{k,l}(w) - W_{i,j}(w)W_{k,l}(w). \]

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Appendix A: Formulas for the Transformation

The general transformation from the $z$-space to the $w$-space in Section III is carried out based on the formulas:

\[ z = \tilde{\varphi}_2(w)/\tilde{\varphi}_1(w), \quad f(z) = \tilde{\varphi}_3(w)/\tilde{\varphi}_1(w). \quad (A1) \]

Using them, we can calculate derivatives of $z$ and $f(z)$ with respect to $w$ as

\[
\begin{align*}
\frac{dz}{dw} &= \frac{W_{2,1}(w)}{\tilde{\varphi}_1(w)^2}, & \frac{d^2z}{dw^2} &= \frac{W_{2,1}'(w)}{\tilde{\varphi}_1(w)^2} - 2\frac{W_{2,1}(w)\tilde{\varphi}_1'(w)}{\tilde{\varphi}_1(w)^3}, \\
\frac{d^3z}{dw^3} &= \frac{W_{2,1}''(w)}{\tilde{\varphi}_1(w)^2} - 4\frac{W_{2,1}'(w)\tilde{\varphi}_1'(w) + 2W_{2,1}(w)\tilde{\varphi}_1''(w)}{\tilde{\varphi}_1(w)^3} + \frac{6W_{2,1}(w)\tilde{\varphi}_1'(w)^2}{\tilde{\varphi}_1(w)^4}, \\
\frac{df(z)}{dw} &= \frac{W_{3,1}(w)}{\tilde{\varphi}_1(w)^2}, & \frac{d^2f(z)}{dw^2} &= \frac{W_{3,1}'(w)}{\tilde{\varphi}_1(w)^2} - 2\frac{W_{3,1}(w)\tilde{\varphi}_1'(w)}{\tilde{\varphi}_1(w)^3}, \\
\frac{d^3f(z)}{dw^3} &= \frac{W_{3,1}''(q)}{\tilde{\varphi}_1(q)^2} - \frac{4W_{3,1}'(q)\tilde{\varphi}_1'(q) + 2W_{3,1}(q)\tilde{\varphi}_1''(q)}{\tilde{\varphi}_1(q)^3} + \frac{6W_{3,1}(q)\tilde{\varphi}_1'(q)^2}{\tilde{\varphi}_1(q)^4},
\end{align*}
\]

where and throughout the paper we have employed the following notation for the Wronskians:

\[ W_{i,j}(w) = \varphi_i'(w)\varphi_j(w) - \varphi_i(w)\varphi_j'(w), \quad (A3) \]

\[ W_{i',j'}(w) = \varphi_i''(w)\varphi_j'(w) - \varphi_i'(w)\varphi_j''(w), \quad (A4) \]

\[ W_{i,j;kl}(w) = W_{i,j}(w)W_{k,l}(w) - W_{i,j}(w)W_{k,l}(w). \quad (A5) \]
With these formulas, we obtain

\[ z'(q) = \frac{W_{2,1}(w)}{\tilde{\varphi}_1(w)^2} u'(q), \quad zf'(z) = \frac{W_{3,2}(w)}{W_{2,1}(w)}, \]

\[ z''(q) = \frac{W_{2,1}(w)}{\tilde{\varphi}_1(w)^2} u''(q) + \left( \frac{W'_{2,1}(w)}{\tilde{\varphi}_1(w)^2} - \frac{2W_{2,1}(w)\tilde{\varphi}'_1(w)}{\tilde{\varphi}_1(w)^3} \right) u'(q)^2, \]

\[ f'(z) = \frac{W_{3,1}(w)}{W_{2,1}(w)}, \quad f''(z) = \frac{W_{31,21}(w)\tilde{\varphi}_1(w)^2}{W_{2,1}(w)^3}, \quad f'''(z) = \frac{W_{31,21}(w)\tilde{\varphi}_1(w) + 2W_{31,21}(w)\tilde{\varphi}'_1(w)}{W_{2,1}(w)^4} \tilde{\varphi}_1(w)^3 \]

\[ \quad - \frac{3W_{31,21}(w)W_{2,1}(w)}{W_{2,1}(w)^5}. \]

For the $GL(3, \mathbb{C})$ transformation in Section IV, we need to calculate several Wronskians defined in (A3)–(A5). Each component of the transformation (4.1) is

\[ \tilde{\varphi}_i(w) = \lambda_{i1} + \lambda_{i2}w + \lambda_{i3}f(w). \]

Then, we have the following:

\[ W_{2,1}(w) = \tilde{\lambda}_{33} - \tilde{\lambda}_{32}f'(w) + \tilde{\lambda}_{31}(wf'(w) - f(w)), \]

\[ W_{3,1}(w) = -\tilde{\lambda}_{23} + \tilde{\lambda}_{22}f'(w) - \tilde{\lambda}_{21}(wf'(w) - f(w)), \]

\[ W_{3,2}(w) = \tilde{\lambda}_{13} - \tilde{\lambda}_{12}f'(w) + \tilde{\lambda}_{11}(wf'(w) - f(w)), \]

\[ W_{31,21}(w) = (\det \mathbf{A})\tilde{\varphi}_1(w)f''(w), \quad W_{2',1'}(w) = \tilde{\lambda}_{31}f'''(w), \]

\[ W_{3',1'}(w) = -\tilde{\lambda}_{21}f'''(w), \quad W_{3',2'}(w) = \tilde{\lambda}_{11}f'''(w), \]

where $\tilde{\lambda}_{ij}$ is the cofactor of the matrix element $\lambda_{ij}$ in $\mathbf{\Omega}$.

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