SUPER REVIVALS AND SUB-PLANCK SCALE STRUCTURES
OF A SLIGHTLY RELATIVISTIC PARTICLE IN A BOX

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1. Introduction

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detail in the context of the so-called quantum carpets, i.e. highly regular space-time structures in the probability density, as well as in phase space by means of the Wigner distribution.

There are few examples, however, where quantum revivals have been investigated in the relativistic regime. In this context, an analysis has been made for a slightly relativistic particle caught in an infinitely deep square well. This treatment takes into account the relativistic effects as perturbations, that produce a quartic correction to the non-relativistic energy spectrum. For very small values of the relativistic parameter, the relativistic corrections determine a shift in the characteristic revival time of the system, whereas larger values completely wash out the highly regular spatio-temporal patterns in the probability density. Recently, attention has been paid to the fully relativistic case, by solving Dirac equation for a particle confined to a circumference of radius $R$. This analysis has confirmed that generally revivals do not occur within a relativistic regime. Only carefully designed wavepackets, with specific mathematical properties, can display quantum revivals and weave quantum carpets within a fully relativistic theory.

Here, we investigate more carefully the role and the effects of the relativistic corrections for a slightly relativistic particle in a box. We find out that the modifications to the quadratic spectrum of the system result in a new revival time scale, giving rise to super revivals. In particular, we focus on the dependence of the energy eigenvalues on the fourth power of the quantum number. This feature suggests the possible existence of a different and new time scale, governing the system dynamics. Surprisingly, even in the slightly relativistic regime, this modest modification can produce large nonlinear effects on a sufficiently long time scale. Our theoretical analysis clearly demonstrates that a perfect revival of the original wave packet takes place at a definite time $T^{(4)}_{sr}$, depending on the relativistic effects. This revival manifests itself both in the space-time representation of the probability density and in phase space. Here, we can appreciate even the smallest features of the Wigner function, the so-called sub-Planck scale structures.

These highly nonclassical features are extremely sensitive to perturbations and decoherence, as pointed out in a variety of systems. Therefore, they seem to be an ideal tool to monitor and characterize the Wigner function at or nearby a fractional revival time. To provide quantitative evidence to our predictions, we calculate the relevant sub-Planck dimension for the Wigner function at typical fractional revival times. The sensitivity of these interference structures allows us to compare even the finest details of the wave packet dynamics for different values of the relativistic corrections. This quantitative measure provides a clear proof that the system evolution exhibits full and fractional revivals on a new time scale $T^{(4)}_{sr}$, set by the relativistic effects. These revivals reproduce all the features of the original wave packet, exactly as it happens in the unperturbed non-relativistic case.

The paper is organized as follows. In Sec. 2 we briefly review the basics of a slightly relativistic particle in a box. The rich structure of quantum carpets and
the phase space Wigner distribution are presented and discussed in Sec. 3, where we analyse the wave packet dynamics. Both approaches clearly show the existence of a new revival time for the system, due to the relativistic corrections. To quantify differences and similarities between the Wigner functions for different values of the relativistic corrections, we consider the variation of sub-Planck dimension for these phase space structures. A sensitivity analysis of these structures is discussed in Sec. 4. Finally, we conclude by summarizing our main results in Sec. 5.

2. Slightly relativistic particle in an infinite square well

In ordinary laboratory conditions, the electronic energy is of the order of few eV. Hence, one would expect that corrections, much smaller than this order of magnitude, should be negligible. However, such a little modification can produce a large nonlinearity and introduce a new time scale in the dynamics of a slightly relativistic particle.

The relativistic Hamiltonian of a particle of mass $m$ is given by

$$H = \sqrt{m^2c^4 + p^2c^2} - mc^2.$$  \hspace{1cm} (1)

To our purposes it suffices to consider an approximate version of this Hamiltonian, by making an expansion of Eq. (1)

$$H \simeq \frac{p^2}{2m} - \frac{1}{2mc^2} \left(\frac{p^2}{2m}\right)^2.$$  \hspace{1cm} (2)

We note that Hamiltonian, Eq. (2), includes quartic corrections to the non-relativistic Hamiltonian of a particle in an infinitely deep square well. The corresponding eigenvectors and eigenvalues satisfy the following time independent Schrödinger equation for a slightly relativistic particle

$$H|u_n\rangle = E_n|u_n\rangle.$$  \hspace{1cm} (3)

Enforcing the boundary conditions at the walls of the box located at $x = 0$ and $x = L$

$$u_n(x = 0) = u_n(x = L) = 0$$  \hspace{1cm} (4)

and trying the following ansatz

$$u_n(x) = \sin(k_n x),$$  \hspace{1cm} (5)

the Schrödinger equation \hspace{1cm} (3) reads

$$\left[\frac{\hbar^2 k_n^2}{2m} - \frac{1}{2mc^2} \left(\frac{\hbar^2 k_n^2}{2m}\right)^2 - E_n\right] \sin(k_n x) = 0,$$  \hspace{1cm} (6)

where $k_n = n\pi/L$. It leads to the dispersion relation

$$E_n = \frac{\hbar^2 k_n^2}{2m} - \frac{1}{2mc^2} \left(\frac{\hbar^2 k_n^2}{2m}\right)^2,$$  \hspace{1cm} (7)
with the normalized energy eigen functions of the slightly relativistic particle being
\[ u_n(x) = \sqrt{\frac{2}{L}} \sin \left( n\pi \frac{x}{L} \right). \tag{8} \]
In the non-relativistic case, the spectrum is quadratic in the quantum number \( n \) and the energy levels are all integer multiples of the lowest eigenvalue. Therefore, the system dynamics is periodic with a characteristic revival time defined as
\[ T_{\text{rev}} = \frac{4mL^2}{\pi\hbar}. \tag{9} \]
If we insert the revival time, Eq. (9), in the expression, Eq. (7), for the energy eigenvalues, we obtain
\[ E_n = (n^2 - q^2 n^4) \frac{2\pi}{T_{\text{rev}}}, \tag{10} \]
where we introduced the relativistic parameter
\[ q = \frac{1}{4L} \left( \frac{2\pi\hbar}{mc} \right) = \frac{\lambda_c}{4L}, \tag{11} \]
as the ratio between the Compton wavelength \( \lambda_c \) of the particle and the box width \( L \). The parameter \( q \) controls the size of the relativistic effects. For an electron, the Compton wavelength is approximately equal to \( 2.43 \times 10^{-12} \) m. Therefore, even in nanostructures, like quantum dots, realistic values of the parameter \( q \) are of the order of \( 10^{-3} \).

Despite the smallness of the relativistic parameter \( q \), the quartic term in the expression, Eq. (10), for the energy eigenvalues leads to a new revival time, given by
\[ T^{(4)}_{\text{sr}} = \frac{4mL^2}{q^2\pi\hbar} = \frac{T_{\text{rev}}}{q^2}. \tag{12} \]
This second revival time is well separated from the non-relativistic one. Indeed, the ratio between the two revival times \( T_{\text{rev}}/T^{(4)}_{\text{sr}} \) is equal to the relativistic parameter \( q^2 \), which is typically quite small. Hence, the two time scales are utterly different.

### 3. Revival dynamics of a slightly relativistic particle

**3.1. Gaussian wave packet and quantum carpets**

A particle in an infinitely deep square box can be well approximated by a Gaussian wave packet
\[ \psi(x) = \frac{1}{(\sqrt{\pi}\Delta x)^{1/2}} \exp \left[ -\frac{1}{2} \left( \frac{x - \bar{x}}{\Delta x} \right)^2 + i \tilde{p} \left( x - \bar{x} \right) \frac{\hbar}{\Delta x} \right], \tag{13} \]
where \( \bar{x} \) and \( \tilde{p} \) represent, respectively, the initial position and the average momentum. To study the time evolution of the particle, it is convenient to expand the
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Fig. 1. (Color online) Population distribution $|a_n|^2$ as a function of the quantum number $n$ for an initial wave packet with width $\Delta x = L/10$ and average momentum $\bar{p} = 50\hbar/L$. The distribution is peaked around $\bar{n} = 16$ with a spreading $\Delta n \simeq 6$.

The wave packet in terms of the energy eigenstates $|u_n\rangle$

$$\psi(x, t) = \sum_{n=1}^{\infty} a_n u_n(x) e^{-iE_n t/\hbar}, \quad (14)$$

where the expansion coefficients are given by

$$a_n = \frac{1}{2i} \sqrt{\frac{4\Delta x \pi}{L \sqrt{\pi}}} \left[ e^{in\pi \bar{x}/L} e^{-\Delta x^2 (\bar{p} + n\pi \hbar /L)^2 / 2\hbar^2} - e^{-in\pi \bar{x}/L} e^{-\Delta x^2 (\bar{p} - n\pi \hbar /L)^2 / 2\hbar^2} \right] \quad (15)$$

and satisfy the normalization condition

$$\sum_{n=1}^{\infty} |a_n|^2 = \int_0^L |\psi(x, t)|^2 dx = 1. \quad (16)$$

From the expression, Eq. (15), for the expansion coefficients $a_n$, we see that the wave packet, describing the particle in the box, is peaked around the energy eigenstate with average quantum number $\bar{n} = pL/(\hbar \pi)$. Other energy eigenstates with a non-negligible overlap with the initial wave packet are characterized by quantum numbers in the range $\bar{n} \pm \Delta n$. As an example, we plot in Fig. 1 the population distribution $|a_n|^2$ for an initial wave packet with $\bar{p} = 50\hbar/L$, that is peaked around the quantum number $\bar{n} = 16$. In this case, the spreading $\Delta n \simeq 6$, since it is enough to include the energy eigenstates ranging from $n = 10$ to 22 to satisfy the normalization condition up to more than 99%.

Equation (14) is the starting point to numerically investigate the probability density to find the particle in the box at a given time and position. Density plots of the probability density of the evolved wave packet are shown in Fig. 2. Here, we have chosen the wave packet initially centered at $\bar{x} = L/2$ and with average...
Fig. 2. (Color online) Plot of the probability density to find the particle at time $t$ and position $x$ in an infinitely deep square well. The initial Gaussian wave packet is centered at $\bar{x} = L/2$ and has a width $\Delta x = L/10$. The average momentum is $\bar{p} = 50\hbar/L$ for all cases. (a) It shows the space-time structures of a non-relativistic particle, whereas (b) and (c) include small relativistic corrections with $q^2 = 10^{-5}$ and $q^2 = 5 \times 10^{-4}$, respectively. Position is expressed in units of the box length $L$, while time is scaled by the revival time $T_{rev}$.

The probability density, expressed in units of $L$, shows the particle's position at different times. The momentum $\bar{p} = 50\hbar/L$. The three panels present different cases, ranging from the non-relativistic one ($q = 0$), to others with small relativistic corrections ($q^2 = 10^{-5}$ and $5 \times 10^{-4}$). During the time evolution, the wave packet breaks into small replicas of itself, thus giving rise to the phenomenon of fractional revivals. Finally, it regains its original shape at the time $T_{rev}/2$. For this reason, here we consider the time evolution only up to time $T_{rev}/2$. Fractional revivals take place at times $t = rT_{rev}/s$, with $r$ and $s$ mutually prime integers. The scenario of full and fractional revivals characterizes case (a), which depicts a non-relativistic particle. Also in the presence of very small relativistic corrections, plot (b), the main features are preserved. However, already for a slightly larger value of the relativistic parameter $q^2 = 5 \times 10^{-4}$ (c), the overall quantum carpet disappears, being replaced by a more classical trajectory, which reminds of a ball bouncing between the two walls. The relativistic corrections completely change the space-time structures in the probability density and reveal a more classical nature marked by periodic oscillations. This classical behaviour, which governs the system dynamics at earlier times, is characterized by
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The oscillation period

\[ T_{cl} = \frac{2L}{v_{cl}} = \frac{T_{rev}}{2\bar{n}} \]  

(17)

where the average velocity of the particle is \( v_{cl} = \bar{p}/m \). Thus, this classical time scale \( T_{cl} \) depends on the average value of the distribution, i.e., \( \bar{n} \). In our analysis, we have chosen \( \bar{p} = 50\hbar/L \) and the corresponding average quantum number is \( \bar{n} = 16 \), as shown in Fig. 1. Interestingly, for relativistic corrections of the order \( q^2 = 5 \times 10^{-4} \), one can observe almost \( \bar{n} \) oscillations of the particle trajectory up to the time \( T_{rev}/2 \) [see Fig. 2(c)].

Alternatively, whenever we deal with a wave packet, characterized by a well defined average quantum number \( \bar{n} \) and encompassing a few other energy eigen states, we can resort to a Taylor expansion of the energy eigenvalues around \( \bar{n} \)

\[ E_n \approx E_{\bar{n}} + E_{\bar{n}}'(n - \bar{n}) + \frac{1}{2!}E_{\bar{n}}''(n - \bar{n})^2 + \frac{1}{3!}E_{\bar{n}}'''(n - \bar{n})^3 + \ldots, \]  

(18)

where primes denote derivatives with respect to \( n \). The coefficients of the Taylor series are then directly related to the relevant time scales of the wave packet evolution. This method, when applied to the energy eigenvalues, Eq. (10), for a slightly relativistic particle, not only amends the time periods characterizing the dynamics of the non-relativistic case, but also introduces new time scales. In particular, the classical time scale \( T_{cl} \), which controls the initial behavior of the wave packet, becomes

\[ \bar{T}_{cl} = \frac{2\pi \hbar}{|E_{\bar{n}}|} = \frac{T_{rev}}{2\bar{n} - 4q^2\bar{n}^3} = \frac{T_{cl}}{1 - 2q^2\bar{n}^2}. \]  

(19)

This expression reveals that \( \bar{T}_{cl} \) is slightly larger than \( T_{cl} \), as \( q^2 \) is a very small quantity. Similarly, the quadratic term in Taylor series expansion yields a new revival time

\[ \bar{T}_{rev} = \frac{2\pi \hbar}{1/2|E_{\bar{n}}'|} = \frac{T_{rev}}{1 - 6q^2\bar{n}^2}. \]  

(20)

Again the relativistic corrections lead to an increase of the revival time in comparison to the non-relativistic case. This small shift is visible in Fig. 2(b), where the wave packet is not yet back at the initial position \( \bar{x} = L/2 \) at exactly \( T_{rev}/2 \). A small extra time is required to observe the full and complete revival.

Moreover, super revival times appear in correspondence with the cubic and quartic terms of Taylor series

\[ T_{sr}^{(3)} = \frac{2\pi \hbar}{4|E_{\bar{n}}''|} = \frac{T_{rev}}{4\bar{n}q^2} \text{ and } T_{sr}^{(4)} = \frac{2\pi \hbar}{24|E_{\bar{n}}'''|} = \frac{T_{rev}}{q^2}. \]  

(21)

These two time scales \( T_{sr}^{(3)} \) and \( T_{sr}^{(4)} \) both depend on the relativistic parameter \( q^2 \). When \( q^2 \) is very small, these two times are much greater than the revival time \( \bar{T}_{rev} \). In this case, the system time evolution is still controlled by \( \bar{T}_{rev} \), while \( T_{sr}^{(3)} \) and \( T_{sr}^{(4)} \) are not yet playing a significant role. On the contrary, for a non-negligible
relativistic parameter as in Fig. 2(c), a drastic change takes place in the system behavior. Classical oscillations persist for a comparatively longer time, at least of the order of \( T_{\text{rev}}/2 \). Under these conditions, \( T_{\text{sr}}^{(3)} \) and \( T_{\text{sr}}^{(4)} \) rule the system dynamics and govern the appearance of the usual fractional and full revivals. Hence, the existence of the super revival time scales appears only for relativistic corrections. We will investigate further this behaviour in our phase space analysis.

3.2. Phase space Wigner distribution

An alternative tool to investigate full and fractional revivals is represented by the Wigner function, which provides a complete description of these phenomena in phase space. Moreover, the negativity of the Wigner function is a hallmark of the non-classicality of quantum states. The Wigner function is defined as

\[
W(x, p, t) = \frac{1}{\pi \hbar} \int_{-\infty}^{+\infty} \psi^*(x - x', t) \psi(x + x', t) e^{-2ipx'/\hbar} dx'.
\] (22)

With our choice of an initial Gaussian wave packet, Eq. (13), the corresponding Wigner function at time \( t = 0 \) is

\[
W(x, p) = \frac{1}{\pi \hbar} \frac{1}{\sqrt{2\pi}} \exp \left[ -\left( \frac{x - \bar{x}}{\Delta x} \right)^2 \right] \exp \left[ -\left( \frac{p - \bar{p}}{\hbar/\Delta x} \right)^2 \right].
\] (23)

It is a Gaussian distribution, centered at \( \bar{x} \) and \( \bar{p} \) in phase space. During its time evolution, the wave packet undergoes a spreading, before reviving at characteristic times. For instance, the well known Schrödinger cat state will appear at \( t = T_{\text{rev}}/4 \), when the wave packet consists of two components with opposite momenta. This feature is apparent in the corresponding Wigner function, as depicted in Fig. 3(a). The wave function is the superposition of two Gaussian wave packets with the same average position but different average momenta \( \bar{p} \) and \( -\bar{p} \), respectively. The density plot of the Wigner function clearly shows the two-way breakup in phase space and the interference ripples appearing in the middle. This scenario is modified when dealing with a slightly relativistic particle. A small relativistic correction \( q^2 = 10^{-5} \), same as that chosen in Fig. 2(b), is considered in the case of Fig. 3(b). It depicts a small deviation from the expected fractional revival, which is not clearly visible in the carpet structure of Fig. 2(b). Figure 3(c) shows the density plot of the Wigner function again at time \( t = T_{\text{rev}}/4 \) for a comparatively larger relativistic correction, \( q^2 = 5 \times 10^{-4} \), same as that chosen for the probability density plot of Fig. 2(c). This small correction completely destroys the two-way breakup structure.

The reason, behind this phenomenon, is that the revival time, \( T_{\text{rev}} \), comes from the quadratic term in the expression, see Eq. (10), for the energy eigenvalues. However, the relativistic corrections can produce large non-linearities of higher order, which introduce new time scales, Eq. (21). So the wave packet motion for a slightly relativistic particle is governed by the super revival times \( T_{\text{sr}}^{(3)} \) and \( T_{\text{sr}}^{(4)} \). The system will revive at times, which are multiple of both \( T_{\text{sr}}^{(3)} \) and \( T_{\text{sr}}^{(4)} \).
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Fig. 3. (Color online) Snapshots of the Wigner distribution describing a particle in an infinitely deep square well. The initial wave function is a Gaussian wave packet centered in \( \bar{x} = \frac{L}{2} \) with average momentum \( \bar{p} = 50\hbar/L \). (a) Non-relativistic particle at time \( t = T_{\text{rev}}/4 \); (b) same time but for small relativistic corrections with \( q^2 = 10^{-5} \) and (c) comparatively larger relativistic corrections with \( q^2 = 5 \times 10^{-4} \). A slightly relativistic particle \((q^2 = 5 \times 10^{-4})\) shows again a perfect two-way break up but on a longer time scale at time \( t = T_{sr}^{(4)}/4 \), as found in (d).

Therefore, a fractional revival will take place at time \( t_{\text{frac}} \) provided that

\[
t_{\text{frac}} = \frac{r_1}{s_1} T_{sr}^{(3)} = \frac{r_2}{s_2} T_{sr}^{(4)},
\]

where \( r_1, s_1, \) and \( r_2, s_2 \) are relative prime integers. For very small relativistic corrections, such as \( q^2 = 10^{-5} \), just a deviation from the non-relativistic behaviour is observed on the shorter time scale, at times \( t \) of the order of the first revival time \( T_{\text{rev}} \). So, in this range of relativistic corrections, the first revival time is still playing a role in the overall system dynamics. Moreover, the super revival time \( T_{sr}^{(4)} = 4n T_{sr}^{(3)} \) defines the full and perfect revival of the system state. For instance, our numerical investigations predict a perfect two-way break up of the Wigner function at time \( T_{sr}^{(4)}/4 \), also when the relativistic parameter is just \( q^2 = 10^{-5} \). If we further increase the relativistic parameter \( q^2 \), as shown in Fig. 3(c), on the short time scale, around \( T_{\text{rev}} \), the system dynamics is completely different from the non-relativistic case. Apparently the ordered space-time structures of quantum carpets are lost [see Fig. 2(c)]. To recover the periodic time behaviour we should wait for a longer time. In this case, the system dynamics is dominated by the super revival time. Fractional and full revivals take place only on the longer time scale, set by \( T_{sr}^{(4)} \), as demonstrated in Fig 3(d).

The explanation behind this effect is the competition between the two time scales. For non-negligible relativistic corrections, the relevant time scale is set by \( T_{sr}^{(4)} \). From Eq. (12) it is evident that whenever the system satisfies the revival condition for \( T_{sr}^{(4)} \), automatically also the other condition, involving \( T_{\text{rev}} \) is satisfied. The vice versa is not true and fractional revivals on the shorter time scale disappears. For a non-relativistic particle the unique revival time is \( T_{\text{rev}} \). At intermediate regimes, there is a coexistence of both the revival times \( T_{\text{rev}} \) and \( T_{sr}^{(4)} \), whereas for slightly larger relativistic effects the dynamics is dominated by \( T_{sr}^{(4)} \) only.
4. Sensitivity of sub-Planck structures in phase space

Phase space interference structures of dimension smaller than $\hbar$, called sub-Planck scale structures, are well studied in the context of their sensitivity to decoherence and perturbations. Their dimension is defined as $a \approx \hbar^2/A$, where $A$ represents the classical action. Interestingly, these structures give an indication of the system sensitivity against perturbations. For this reason they appear to be an appropriate tool to quantify the modifications, due to the relativistic effects, of phase space structures in the Wigner function of a particle in the box. For instance, they can provide a quantitative measure of the differences between the Wigner functions of Figs. 3(a) and (b), pertaining, respectively, to the non-relativistic case ($q^2 = 0$) and to a slightly relativistic one ($q^2 = 10^{-5}$).

To this end, we focus on the fractional revival appearing at time $T_{\text{rev}}/4$ and characterize the interference fringes through their sub-Planck dimension. Our numerical analysis yields the classical action for the non-relativistic particle being $A \approx \Delta x \Delta p = 3.57 \hbar$ and the corresponding sub-Planck structure size is $a = 0.28 \hbar$. As shown in Fig. 3(b), at the time $T_{\text{rev}}/4$, the two-way break up of the Wigner function is also present when the relativistic parameter is just $q^2 = 10^{-5}$. This small relativistic correction causes a perturbation in the system, which results in a change of sub-Planck dimension $a_q = 0.17 \hbar$. Figure 4 shows the variation of sub-Planck dimension, at $t = T_{\text{rev}}/4$ (solid line), as a function of the relativistic parameter $q^2$. In particular, we introduce the quantity $\delta \equiv a_q/a$, which provides the ratio of sub-Planck dimension of the slightly relativistic case to the non-relativistic one. The plot takes into account values of the relativistic parameter up to $q^2 = 10^{-5}$. It clearly depicts the fall of sub-Planck dimension of the interference fringes with the increase of relativistic effects. Thus, even the smallest relativistic corrections are
captured by the size of sub-Planck structures.

It is then interesting to compare the fractional revival [see Fig. 3(d)], taking place at $T_{sr}^{(4)}/4$ in the slightly relativistic case, to the one at $T_{rev}/4$ for the non-relativistic particle [see Fig. 2(a)]. Hence, we have calculated the ratio between the sub-Planck dimensions of the interference fringes of the Wigner function at these two fractional revival times. The results are presented in Fig. 4 (dashed line): the sub-Planck dimension remains constant for all the values of the relativistic parameter $q^2$. This is a quantitative evidence that, in the slightly relativistic case, the wave packet dynamics not only revives on the time scale set by $T_{sr}^{(4)}$, but maintains all the characteristics of fractional and full revivals typical of the non-relativistic case.

5. Conclusions

Relativistic corrections to the energy spectrum of a particle in an infinitely deep square box not only amend the typical revival time but, most notably, introduce a super revival on a much longer time scale. Hence, quartic terms in the energy spectrum of a slightly relativistic particle are not simply a higher-order correction, but play a definite role in the system dynamics. For very small values of the characteristic relativistic parameter, the system time evolution still exhibits full and fractional revivals at the expected times. However, one can perceive some distortions both in the regular pattern of space-time probability density, the so-called quantum carpets, and in the phase space representation of Wigner function. These deviations are better captured by the sensitivity of sub-Planck structures.

For slightly larger values of the relativistic parameter, there is a dramatic change in the overall picture. The particle probability density does not spread but gives rise to a more or less localized trajectory, reminiscent of a classical ball bouncing between the two walls. Interference effects seem to disappear even at moderate relativistic regimes. This behaviour obscures the expected full and fractional revivals on a time scale of the order $T_{rev}$. However, the system dynamics is now governed by a new time scale, $T_{sr}^{(4)}$, which depends on the quartic terms in the energy spectrum and is only due to relativistic effects. The familiar full and fractional revivals, that mark the time evolution of the non-relativistic particle, manifest themselves on this new time scale $T_{sr}^{(4)}$. Again the analysis of sub-Planck dimension of the relevant interference fringes in the Wigner function allows for a comparison between the fractional revivals for different values of the relativistic parameter. The sub-Planck scale structures provide a quantitative evidence for the perfect reshaping of the original wave packet around $T_{sr}^{(4)}$.

In conclusion, the slightly relativistic particle in a box exhibits super revivals at a time $T_{sr}^{(4)}$, which is set by the relativistic effects and has no counterpart in the non-relativistic case. Remarkably, the observation of these super revivals does not require a specially tailored or designed wave packet, as it happens, instead, in the fully relativistic case considered in Ref. [26].

These effects may be observed in a cyclotron maser or gyrotron based on slightly
relativistic mass effect of the free electron in vacuum. Also in narrow-gap semiconductors, conduction electrons exhibit a pseudo-relativistic behavior. Indeed, the dispersion relation between the conduction-band energy and the momentum closely resembles the expression for the energy of a relativistic particle, in which the speed of light is replaced by a characteristic velocity, determined by the energy gap and the effective mass of the conduction electron. Therefore, our considerations apply to these systems as well.

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