GROUND STATE ENERGY
OF CURRENT CARRIERS IN GRAPHENE

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Abstract. The ground state energy of current carriers in graphene considered as a zero-gap semiconductor was calculated in the two-band approximation. The condition of the electronic (hole) system stability in graphene was obtained. The possibility of the zero-gap semiconductor–semimetal transition was discussed.

It is known that thin graphite films exhibit semimetallic properties [1]; however, a single-atomic layer of carbon atoms forming a regular hexagonal lattice (graphene) has such a band structure that the energy gap is zero at six K points of the Brillouin zone. Therefore, graphene can be considered as a two-dimensional zero-gap semiconductor or a semimetal with zero conduction and valence band overlap [2]. The former approach makes it possible to describe current carriers in graphene within the two-band Dirac model [7, 8]

\[ u\alpha \cdot \hat{p}\Psi = \varepsilon_p \Psi, \]

where \( \alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} \) are Dirac \( \alpha \)-matrices, \( \hat{p} = -i\hbar \nabla \) is the two-dimensional momentum operator (hereafter \( \hbar = 1 \)), \( u = \frac{3}{2}\gamma a_0 = 9.84 \times 10^7 \text{ cm/s} \) is the quantity similar to the Kane matrix element of the interband transition rate, \( \gamma \simeq 3 \text{ eV} \) is the band parameter, and \( a_0 = 1.44 \text{ Å} \) is the interatomic distance in the graphene lattice [9]. In the vicinity of K points of the Brillouin zone, the dispersion relation of current carriers is linear, \( \varepsilon_p = \pm up \) (± signs correspond to electrons and holes, respectively).

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\[ \text{Dirac equation (1) is equivalent with accuracy of the unitary transformation of the Hamiltonian and the wave function of a pair of Weyl equations (see book [3], p. 79). As is known, the Weyl equation describes the two-component neutrino in quantum electrodynamics (QED) (see, e.g., book [4]). The use of the Dirac equation as a } 4 \times 4 \text{ matrix equation in the two-dimensional system is possible since the } 4 \times 4 \text{ matrix representation in case of two spatial dimensions can be used equivalently with } 2 \times 2 \text{ matrix representation (see book [5], chap. XIV). This fact allows us to extend the formalism of the QED diagram technique to the case of the two-dimensional system of Dirac fermions (graphene). The Weyl equation was first applied to the problem of describing current carriers in a zero-gap semiconductor in [6].} \]
For a two-dimensional electron (hole) gas arising during electron doping of a zero-gap semiconductor [2], the ground state energy per one particle is the sum of three terms

\[ E_{gs} = E_{\text{kin}} + E_{\text{exch}} + E_{\text{corr}}, \tag{2} \]

where \( E_{\text{kin}} = \frac{3}{2} u p_F \) is the average kinetic energy, \( p_F = \sqrt{\frac{2e n_{2D}}{\nu}} \) is the Fermi momentum, \( n_{2D} \) is the two dimensional particle concentration, \( \nu \) is the degeneration multiplicity. If the Fermi level \( E_F \) lies above \( E = 0 \), the system contains only electrons as current carriers in the conduction band with the number of valleys \( \nu_e = 2 \); if \( E_F < 0 \), the system contains only holes as current carriers with \( \nu_h = 2 \). The Fermi level position can be varied by applying an electric field [2]. We can see that both cases in the Dirac model are equivalent. In follows, for definiteness, we shall consider the case of electrons.

The exchange energy is given by the first-order exchange diagram (Fig. 1)

\[ E_{\text{exch}} = -\frac{\nu}{2n_{2D}} \int \frac{d^2 p d\varepsilon}{(2\pi)^3} \frac{d^2 k d\omega}{(2\pi)^3} Sp \left\{ \gamma^\mu G(p, \varepsilon) \gamma^\nu G(k, \omega) \right\} D_{\mu\nu}(p - k, \varepsilon - \omega), \tag{3} \]

where the photon propagator \( D_{\mu\nu}(p - k, \varepsilon - \omega) \approx V(p - k) \delta_{\mu4} \delta_{\nu4} \) (we neglect the photon pole at \( \omega = \pm c |p - k| \), whose contribution to the integral in frequencies \( \varepsilon \) and \( \omega \) is of the order of \( (u/c)^2 \sim 10^{-5} \) in comparison with the contribution of Green’s function poles), \( V(q) = \frac{2\pi e^2}{\kappa_{eff}|q|} \) is the Coulomb law in the two-dimensional case, and \( \kappa_{eff} \) is the effective permittivity of graphene.

The free-electron Green’s function at \( \Delta = 0 \) is [10]

\[ G(p, \varepsilon) = -\frac{u \hat{p}}{u^2 p^2 - \varepsilon^2 - i0} + 2\pi i \delta(u^2 p^2 - \varepsilon^2) N_p u \hat{p}, \tag{4} \]

where \( \hat{p} = p_\beta \gamma^\beta (\beta = 0, 1, 2) \) is the convolution with Dirac matrices \( \gamma^k = -i \gamma^0 \alpha^k \) for \( k = 1, 2 \) and \( \gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \), \( I \) is the \( 2 \times 2 \) unit matrix, \( N_p = \theta(|p| - p_F)\theta(\varepsilon); \theta(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases} \)

Expression (3) is transformed to the form (see the Appendix)

\[ E_{\text{exch}} = -\frac{\alpha^* I_1}{\sqrt{2\pi}} \left( \frac{n_{2D}}{\nu} \right)^{1/2}, \tag{5} \]

Fig. 1. First-order exchange diagram.

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\(^3\)In the general case for two spin components, the degeneration multiplicity is \( \nu = \nu_{e,h} \). It will be shown below that the spin-unpolarized state is more energetically favorable than spin-polarized one, for which \( \nu \rightarrow \nu_{e,h}/2 \).
where \( I_1 = \int_0^1 dx \int_0^1 dy \int_0^{2\pi} d\chi \frac{(1+\cos \chi)xy}{\sqrt{x^2+y^2-2xy\cos \chi}} = \frac{8}{3} \left( G + \frac{1}{2} \right) \), \( G = 0.915965 \ldots \) is the Catalan constant [11], \( \alpha^* = \frac{e^2}{\hbar c m^*} \) is the analogue of the fine structure constant.

The correlation energy is given by the formula [12]

\[
E_{\text{corr}} = \frac{1}{2n_{2D}} \int \frac{d^2k}{(2\pi)^3} \int \frac{d\lambda}{\lambda} \left[ \frac{-\lambda \nu V(k) \Pi_{44}(k, i\omega)}{1 - \lambda \nu V(k) \Pi_{44}(k, i\omega)} + \lambda \nu V(k) \Pi_{44}^{(0)}(k, i\omega) \right].
\]

(6)

The total polarization operator is written as

\[
\Pi_{44}(k, i\omega) = \Pi_{44}^{(0)}(k, i\omega) + \Pi_{44}^{(1)}(k, i\omega) + \ldots,
\]

(7)

which corresponds to the sum of the diagrams

\[\gamma^\nu \gamma^\mu = \gamma^\nu + \gamma^\nu + \ldots\]

The polarization operator \( \Pi_{44}^{(0)}(k, i\omega) \) in the lowest order in the interaction for the two-dimensional case is given by [10]

\[
\Pi_{44}^{(0)}(k, i\omega) = 16 \int \frac{d^2p}{(2\pi)^2} \frac{\theta (|p| - p_F)}{2\epsilon_p} \frac{(k \cdot p)^2 - \frac{|k|^2 \epsilon_p}{u^2}}{(|k|^2 + \frac{\omega^2}{u^2})^2 - 4 \left( k \cdot p - \frac{i\epsilon_p \omega}{u^2} \right)^2}.
\]

(8)

We calculate the correlation energy using a method similar in many respects to the known Nozieres–Pines method [13, 14] which is applied to calculate the electron gas and electron-hole liquid using asymptotics of the dimensionless polarization operator [15]

\[
\tilde{\Pi}_{44}^{(0)}(r, \theta) = \begin{cases} \\
-\frac{\sin \theta}{24\pi r^2}, & r \gg 1, \\
-\frac{1-|\cos \theta|}{16\pi}, & r \ll 1,
\end{cases}
\]

(9)

where dimensionless variables \( r = \sqrt{\omega^2 + u^2k^2}/u_F \) and \( \sin \theta = u k/\sqrt{\omega^2 + u^2k^2} \) are introduced.

To determine the smallness parameter of expansion (7), we estimate the first-order correction \( \Pi_{44}^{(1)} \) in the interaction to the polarization operator (8)

\[
\Pi_{44}^{(1)}(k, i\omega) = -\int \frac{d^2p \, d\Omega}{(2\pi)^3} \frac{d^2q}{(2\pi)^3} \frac{S(p + q + \Omega) G(p, \epsilon + \Omega) G(q, \epsilon)}{G(p, \epsilon)}.
\]

\[
\gamma^\nu G(p - k, \epsilon - \omega) \gamma^\beta G(p + q - k, \epsilon + \Omega - \omega) \times V(q) \delta_{\alpha \beta} \approx -\frac{\alpha^*}{16\pi u_F} \frac{p_F}{(2\pi)^2} \frac{2\pi \alpha^*}{|q|} \theta(p_F - |p|) \theta(p_F - |q|) = -\frac{\alpha^*}{8\pi^2} \frac{p_F}{u^2}.
\]

(10)
Taking into account that the main contribution to $\Pi_{44}^{(1)}(k, i\omega)$ is made by small transferred momenta due to $V(q)$ [15], (10) should be compared with the asymptotics $\Pi_{44}^{(0)}(k, i\omega)$ at small $|k|$, from which we obtain that (10) is small in the parameter $\frac{\omega_{\pi}}{4\pi} \ll 1$, which is simultaneously the condition of the applicability of the random-phase approximation.

Substitution of the polarization operator asymptotics at $r \gg 1$ into (6) yields the contribution of large momenta

$$E^\infty_{\text{corr}} = -\frac{u_0^2 \nu}{3(2\pi)^2 n_{2D}} [(1 + g_1) \cdot \ln(1 + g_1) - g_1],$$

where $g_1 = \frac{\alpha^* \nu}{12} \ll 1$ at $\nu = 2$ and $\alpha^* \ll 1$; expanding (11), we obtain

$$E^\infty_{\text{corr}} = -\frac{\alpha^* \nu^{1/2}}{864\sqrt{2\pi}} u n_{2D}^{1/2},$$

which coincides with the contribution of the second-order ring diagram (Fig. 2)

$$E^{(2)}_{\text{corr}} = -\frac{i \nu}{4n_{2D}} \int \frac{d^2p d^2q d^2k \epsilon d\Omega d\omega}{(2\pi)^9} \text{Sp} \left\{ \gamma^\alpha G(p, \epsilon) \gamma^\beta G(p - k, \epsilon - \omega) \right\} \times$$

$$\text{Sp} \left\{ \gamma^\mu G(q, \Omega) \gamma^\nu G(q + k, \Omega + \omega) \right\} D_{\alpha\mu}(k, \omega) D_{\beta\nu}(k, \omega) \approx (13)$$

Thus, according to the Nozieres–Pines method [13, 14], when calculating the correlation energy at large transferred momenta by formula (6), with an accuracy of the terms of the order of $g^2_1$, the analysis can be restricted to the second-order of the perturbation theory. Apart from the second-order ring diagram (Fig. 2), let us also consider the second-order exchange diagram (Fig. 3)

$$\tilde{E}^{(2)}_{\text{corr}} = -\frac{i \nu}{4n_{2D}} \int \frac{d^2p d^2q d^2k d\epsilon d\Omega d\omega}{(2\pi)^9} \text{Sp} \left\{ \gamma^\alpha G(p, \epsilon) \gamma^\beta G(p - k, \epsilon - \omega) \right\} \times$$

$$\gamma^\mu \cdot G(p - q - k, \epsilon - \Omega - \omega) \gamma^\nu G(p - q, \epsilon - \Omega) D_{\alpha\mu}(q, \Omega) D_{\beta\nu}(k, \omega).$$

Evaluation of integral (14) is very laborious; however, estimations show that, as in the nonrelativistic case, its contribution is positive and is smaller in magnitude by a factor of $\frac{1}{2\nu}$ than (13) [16]. Finally, for the contribution of large transferred momenta to the correlation energy, we obtain

$$E^\infty_{\text{corr}} = -\frac{\alpha^* \nu^{1/2}}{864\sqrt{2\pi}} u n_{2D}^{1/2},$$

Substitution of the polarization operator asymptotics at $r \ll 1$ from (9) into (6) yields the contribution of small transferred momenta
The following notations are introduced:
\[ p_1 = p - k, \quad \varepsilon_1 = \varepsilon - \omega; \]
\[ p_2 = p - q, \quad \varepsilon_2 = \varepsilon - \Omega. \]

\[
E_{corr}^0 = -\frac{u p_l^3 g_2^2}{3(2\pi)^2 n_{2D}} \int_0^{\pi/2} d\theta (1 - \cos \theta)^2 \left[ 1 - g_2 \frac{1 - \cos \theta}{\sin \theta} \cdot \ln \left( 1 + \frac{\sin \theta}{g_2(1 - \cos \theta)} \right) \right. + \\
\left. \sin \theta \frac{1 - \cos \theta}{g_2(1 - \cos \theta)^2} \cdot \ln \left( 1 + g_2 \frac{1 - \cos \theta}{\sin \theta} \right) \right], \tag{16}
\]

where \( g_2 = \frac{\alpha^* \nu}{8} \ll 1 \), and expression (16) can be simplified,
\[
E_{corr}^0 = -\frac{\alpha^* \nu^{1/2}}{128 \sqrt{2\pi} u n_{2D}^{1/2}}. \tag{17}
\]

We can see from (15) and (17) that the order of smallness of the correlation energy and exchange energy (5) is \( \alpha^* \) and \( \alpha^* \), respectively; thus, we obtain the ground state energy of the electron gas as a power series of \( \alpha^* \), which we cut off after the terms of the order \( \alpha^* \),
\[
E_{gs} = \frac{2\sqrt{2\pi}}{3} u \left( \frac{n_{2D}}{\nu} \right)^{1/2} - \alpha^* \frac{I_1}{\sqrt{2\pi}} u \left( \frac{n_{2D}}{\nu} \right)^{1/2} - \frac{\alpha^* \nu}{64 \sqrt{2\pi}} \left( \frac{3\pi}{8} - \frac{25}{27} \right) u \left( \nu n_{2D} \right)^{1/2}. \tag{18}
\]

It can be seen from (18) that the spin-unpolarized state with the degeneracy multiplicity \( \nu = \nu_{e,h} \) is more energetically favorable than the spin-polarized state with the degeneracy multiplicity \( \nu = \nu_{e,h}/2 \). The main contribution is given by the kinetic energy; therefore, \( E_{gs} > 0 \). The condition \( E_{gs} > 0 \) in the case of \( \Delta = 0 \) means that graphene as a zero-gap semiconductor is stable (at \( E_{gs} < 0 \), it would be favorable to generate electron–hole pairs). The transition to the semimetallic state, i.e., spontaneous electron–hole pair generation, occurs at certain \( \alpha_0^* \) such that \( E_{gs} < 0 \) at \( \alpha^* > \alpha_0^* \). Equation \( E_{gs} = 0 \) upon the substitution of \( \nu = 2 \) yields
\[
\alpha_0^* = \sqrt{A^2 + B - A} = 1.1044, \tag{19}
\]

where \( A = \frac{64I_1}{3\pi - 202/27} \) and \( B = \frac{512\pi}{9\pi - 202/9} \).
Let us now consider the effect of some parameters on the band structure of graphene in more detail.

**Effective multivalley structure.** A structure (superlattice) containing $N$ graphene layers, in the absence of transitions between layers, effectively contains $\tilde{\nu} = \nu \cdot N$ electron (hole) valleys with the number of electrons (holes) $N_{e(h)} = N_{e(h)} \cdot N$, where $N_{e(h)}$ is the number of electrons (holes) in each graphene layer. Let graphene layers be interfaced by a wide-gap semiconductor (insulator).

In the general case the Coulomb law for periodic structures is given by the expression \[ V(q, w) = \frac{2\pi e^2}{\kappa |q|} \sinh \left( \frac{|q| d}{\cosh \left( \frac{|q| d}{\cos w} \right) - \cos w} \right), \] where $0 \leq w \leq 2\pi$; however, in the case of large transferred momenta such that $|q| d \gg 1$, the second fraction in (20) tends to unity. Formula (11) for $E_{\infty}^{corr}$ remains valid with an accuracy of the substitution $g_1 \rightarrow \tilde{g}_1 = \alpha^* \nu^{12} \gg 1$ and $n_{2D} \rightarrow \tilde{n}_{2D} = \frac{N_{e(h)}}{S} = N \cdot n_{2D}$ ($S$ is the area of layers), \[ E_{\infty}^{corr} = -\frac{\alpha^*}{36\sqrt{2\pi}} u \left( \frac{n_{2D}}{\nu} \right)^{1/2} \cdot \ln \left( \frac{\alpha^* \nu^{12}}{12 N} \right). \] At small transferred momenta, $|q| d \ll 1$, the Coulomb law (20) is not singular at $|q| = 0$. In this limit, $E_{0}^{corr}$ is small in comparison with $E_{\infty}^{corr}$ due to appearance of $\ln N$; therefore, it can be omitted. The ground state energy is given by \[ E_{gs} = \left[ \frac{2\sqrt{2\pi}}{3} - \frac{\alpha^*}{\sqrt{2\pi}} \left( I_1 + \frac{1}{36} \ln \left( \frac{\alpha^* \nu^{12}}{12 N} \right) \right) \right] \cdot u \left( \frac{n_{2D}}{\nu} \right)^{1/2}. \] **Electric field effect.** As shown above, application of an electric field can produce a nonzero electron (or holes when the electric field direction is changed) density in graphene on substrate [2]. In this case, the two-dimensional current-currier concentration is proportional to the gate voltage $V_g$ [2] \[ n_{2D} = \frac{\epsilon}{4\pi e l} V_g, \] where $\epsilon$ is the substrate permittivity.

According to the calculation [17], in graphene containing several layers, the energy gap opens at K points of the Brillouin zone under a sufficiently strong electric field. This means that the system transits from a semimetallic state (studied in [1]) to the semiconductor state. Let us find out whether or not a similar phenomenon exists in single-layer graphene. Assume that application of a rather weak electric field results in opening the gap, $\Delta = 0$, small in comparison with the Fermi energy $E_F = u_F$: $\Delta \ll E_F = \frac{\nu}{2\sqrt{\pi} V_g}$; in this case, the correction to the kinetic energy is $\delta E_{kin} = \Delta^2 / E_F$. According to (A1), the correction to the exchange energy is given by \[ \delta E_{exch} = -\frac{\alpha^* I_2}{2\pi} \frac{\Delta^2}{E_F}, \] (24)
The correction to the ground state energy contains the additional small factor $r, \theta$
Changing to dimensionless variables $E_{up}$ the transition). We can estimate $p$ is

$$
\kappa = \frac{l}{\cos \chi}
$$
due to image forces, i.e., the variation in the effective permittivity $\kappa$ relative to the transition to nonzero-gap semiconductor phase in an external electric field. moreover, this correction is positive, which suggests that the zero-gap semiconductor phase is therefore, even in the presence of a gap (small in comparison with $E_F$), expression (18) is correct; moreover, this correction is positive, which suggests that the zero-gap semiconductor phase is stable relative to the transition to nonzero-gap semiconductor phase in an external electric field.

**Transition to the semimetallic phase.** For graphene, the parameter $\alpha^*$ can efficiently vary due to image forces, i.e., the variation in the effective permittivity $\kappa_{eff}$ of graphene depending on its environment (insulator or vacuum); at the substrate thickness $l \gg n_{2D}^{-1/2}$; it is given by $\kappa_{eff} = \epsilon + \epsilon'$, where $\epsilon'$ is the permittivity of a medium above graphene [18]. For a SiO$_2$ substrate, $\kappa_{eff} = 5$ and $\alpha^* \approx 0.44$; for a SiC substrate, $\kappa_{eff} = 3$ and $\alpha^* \approx 0.73$ [19]. Let us estimate the valence and conduction band overlap $\delta E$ in the semimetallic state (Fig. 4). Let initially $E_F > 0$ and $\Delta = 0$; and after the transition, the conduction band was lowered with respect to the level $E = 0$ by $\delta E/2$ and the valence band rose by $\delta E/2$; then the number $\Delta N_e$ of electrons transferred to the valence band is $\frac{\nu}{4\pi} p^2 F S$; the new Fermi momentum is $p_2 : N'_e = \frac{\nu}{4\pi} p^2 F S = N_e + \Delta N_e$, where $N_e = \frac{\nu}{4\pi} p^2 F S$ ($p F$ is the Fermi momentum before the transition). We can estimate $ap_2 \approx \delta E/2$. The average kinetic energy of electrons is $E'_{kin} = \frac{\nu}{3} up_2$; for holes, $E''_{kin} = \frac{\nu}{3} up_1$ (the number of holes is equal to the number of transferred electrons, $N_h = \Delta N_e$; therefore, their Fermi momentum is equal to $p_1$). The ground state energy in the semimetallic phase is given by

$$
E'_{gs} = E'_{kin} + E''_{kin} + E_{exch} + E_{corr},
$$

$$
\delta \Pi^{(0)}_{44} (k, i \omega) = \frac{3}{4 \pi \omega^3} \frac{u^2 |k|^4 \Delta^2}{(u^2 |k|^2 + \omega^2)^2} \arctan \left( \frac{2up_F \omega}{u^2 |k|^2 + \omega^2} \right), \quad (25)
$$

$$
\delta E_{corr} = -\frac{\nu^2}{2n_{2D}} \int \frac{d^2k d\omega}{(2\pi)^3} \frac{V^2 (k) \Pi^{(0)}_{44} (k, i \omega)}{1 - \nu V (k) \Pi^{(0)}_{44} (k, i \omega)} \delta \Pi^{(0)}_{44} (k, i \omega). \quad (26)
$$

Changing to dimensionless variables $r, \theta$ and substituting the asymptotics $\tilde{\Pi}^{(0)}_{44}(r, \theta)$ at $r \gg 1$ and $r \ll 1$, noticing that the main contribution to $\delta E_{corr}$ given by small transferred momenta due to $V^2(k)$ in (26), we obtain

$$
\delta E_{corr} = \frac{9\pi \alpha^2 \nu}{256} \ln \left( 1 + \frac{8}{\alpha^* \nu} \right) \frac{\Delta^2}{E_F}. \quad (27)
$$

The correction to the ground state energy contains the additional small factor

$$
\delta E_{gs} = \delta E_{kin} + \delta E_{exch} + \delta E_{corr} = \left[ 1 - \frac{8G\alpha^*}{\pi} + \frac{9\pi \alpha^2 \nu}{256} \ln \left( 1 + \frac{8}{\alpha^* \nu} \right) \right] \frac{\Delta^2}{E_F} \approx 0.1202 \frac{\Delta^2}{E_F}; \quad (28)
$$

therefore, even in the presence of a gap (small in comparison with $E_F$), expression (18) is correct; moreover, this correction is positive, which suggests that the zero-gap semiconductor phase is stable relative to the transition to nonzero-gap semiconductor phase in an external electric field.
Fig. 4. Transition to the semimetallic state of graphene depending on a substrate material: the conduction band is overlapped with the valence band at all three pairs of neighboring points K and K' of the Brillouin zone.

where $E_{\text{corr}}$ is calculated by the polarization operator $\Pi_{44}^{(0)} = \Pi_{44}^{(0)e} + \Pi_{44}^{(0)h}$; $E_{\text{exch}}$ is set equal to (5) if $N_h \simeq N_e$ so that $E_{\text{exch}}^h \simeq E_{\text{exch}}^e = E_{\text{exch}}$. Neglecting $E_{\text{corr}}$, on the assumption that $E_{gs}' = 0$, we obtain

$$
\delta E \simeq \left( b - \frac{1}{b} \right) u p_F,
$$

where $b = \left| \frac{E_{\text{exch}}}{E_{\text{kin}}} \right| = \frac{2a^*}{\pi} \left( G + \frac{1}{2} \right)$. The condition $b > 1$ means that the zero-gap semiconductor phase is unstable ($E_{gs} < 0$); in this case, $\delta E > 0$, i.e., the transition to the semimetallic state, the transition from the spin-unpolarized to the spin-polarized occurs.

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**Appendix**

Using the Green’s function from [10], we obtain the expression for the exchange energy at arbitrary energy gap $\Delta$

$$
E_{\text{exch}} = -\frac{\nu}{2n_{2D}} \int \frac{d^2p d^2k}{(2\pi)^4} \frac{\Delta^2 + u^2 p \cdot k + \varepsilon_p \varepsilon_k}{\varepsilon_p \varepsilon_k} \frac{2\pi e^2}{\kappa_{\text{eff}}} |p - k| \theta (p_F - |p|) \theta (p_F - |k|). \quad (A1)
$$

From (A1), in the nonrelativistic limit, $\Delta \gg u p_F, \varepsilon_p \varepsilon_k \approx \Delta^2 \gg u^2 |p \cdot k|$, we obtain

$$
E_{\text{exch}}^{\text{nonrel}} = -\frac{\nu}{n_{2D}} \int \frac{d^2p d^2k}{(2\pi)^4} \frac{2\pi e^2}{\kappa_{\text{eff}}} |p - k| \theta (p_F - |p|) \theta (p_F - |k|), \quad (A2)
$$

which coincides with the known expression (see, e.g., [16]). In the ultrarelativistic limit, $\Delta \ll u p_F, \varepsilon_p \varepsilon_k \approx u^2 |p \cdot k| \gg \Delta^2$, we obtain

$$
E_{\text{exch}}^{\text{ultrarel}} = -\frac{\nu}{2n_{2D}} \int \frac{d^2p d^2k}{(2\pi)^4} \left( 1 + \frac{p \cdot k}{|p| |k|} \right) \frac{2\pi e^2}{\kappa_{\text{eff}}} |p - k| \theta (p_F - |p|) \theta (p_F - |k|). \quad (A3)
$$

Dedimensionalizing the integrand in (A3) and integrating over momenta, we obtain the answer in the form of (5). Expression (A3) is equivalent to formula (7) in [20].
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