Determinantal invariant gravity

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Abstract

Einstein-Hilbert action with a determinantal invariant has been considered. The obtained field equation contains the inverse Ricci tensor, $\mathcal{R}_{\alpha\beta}$. The linearized solution of invariant has been examined, and constant curvature space-time metric solution of the field equation gives different curvature constant for each values of $\sigma$. $\sigma = 0$ gives a trivial solution for constant curvature, $R_0$.

1 Introduction

Observations of the universe accumulate many investigation on Einstein theory of general relativity. One of them is the modification of Einstein-Hilbert (EH) action. There are numerus investigations on the modified EH action with different context [1]. Such modifications cast a vital role in the inflationary cosmological model of Starobinsky [2]. Modeling the exponentially expanding the early universe, i.e. the inflation, is the most capable theory to explain the natural structure of the present universe; such as horizon, flatness, isotropy, homogeneity etc. There are various inflationary models of the universe were introduced by different studies with different context [3, 4, 5, 6, 7]. For more information one can see the review [8] and references there in. According to the Planck observations [9], the most working model of inflation is that of Starobinsky. Our goal in this paper is to construct a determinantal invariant parameter which can be used in the EH action. The constant curvature solution of the invariant coincides with Starobinsky cosmological inflationary model. Also, constant curvature space-time solution of equation has been examined.

One can construct such a determinantal invariant with the same analogy in [10, 11]. The ratio of determinant of the Ricci tensor and metric tensor [12] is

$$r = \frac{\tilde{R}}{g}$$

(1)

Where $\tilde{R}$ is the determinant of Ricci tensor, $R_{\mu\nu}$, $g$ is that of metric tensor, $g_{\mu\nu}$. This parameter is our determinantal invariant. Accordingly a parameter function can be given as
\[ f(r) = \xi \frac{(r)^\sigma}{M^{4(2\sigma-1)}} \] (2)

Where \( \sigma, \xi \) are dimensionless constant numbers. \( \sigma \) is relating mass parameter, \( M \), and determinantal parameter, \( \tilde{R} \) to fix \( f \) as a dimensionfull parameter.

One can construct an action integral of EH with the determinantal invariant function, \( f(r) \), as follows

\[
S = \int d^4x \sqrt{-g} \left\{ \frac{M^2_{pl}}{2} R + \xi \frac{1}{M^{4(2\sigma-1)}} \left( \frac{\tilde{R}}{g} \right)^\sigma \right\} + S_{\text{matter}}. \] (3)

Where \( R \) is curvature scalar. Variation of equation (3) with respect to the metric tensor produces the field equation

\[
G_{\mu\nu} + \frac{1}{M^2_{pl}} g_{\mu\nu} (2\sigma - 1)f(r) + \frac{\sigma}{M^2_{pl}} \left\{ g_{\mu\nu} \nabla_\alpha \nabla_\beta [\tilde{R}^{\alpha\beta}] + \nabla_\alpha \nabla_\beta [\tilde{R}_{\mu\nu}] - 2 \nabla_\mu \nabla_\alpha [\tilde{R}^\alpha_\nu] \right\} = \frac{1}{M^2_{pl}} T_{\mu\nu}. \] (4)

Where \( \text{d} \) is derivative with respect to the determinantal invariant, \( r \), and \( G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \) is the Einstein tensor. The equation (4) has the novel structure, because it contains inverse Ricci tensor, \( \tilde{R}^{\alpha\beta} \). Comparing with the \( f(R) \) gravity theories [1], our field equation is very different, because of it has three extra terms with inverse Ricci tensor, \( \tilde{R}^{\alpha\beta} \). \( \sigma = 0 \) case, simplifies the field equation as follows

\[
G_{\mu\nu} - \xi \frac{M^4}{M^4_{pl}} g_{\mu\nu} = \frac{1}{M^2_{pl}} T_{\mu\nu}. \] (5)

The vacuum solution of this equation is the maximally symmetric solution of field equation (4).

2 Linearized solution

In the linearized approximation (up to the first order of \( h_{\mu\nu} \)) the metric tensor \( g_{\mu\nu} \), and its inverse \( g^{\mu\nu} \) become

\[
g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \] (6)

\[
g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + \frac{1}{2} h^{\alpha\mu} h_\alpha^\nu. \] (7)

Where \( \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1) \) is the flat Minkowski space-time metric. In the Minkowski background the linearized Ricci tensor, and curvature scalar, become

\[
R_{\mu\nu} = \frac{1}{2} (\partial_\alpha \partial_\mu h_\nu^\alpha + \partial_\alpha \partial_\nu h_\mu^\alpha - \partial_\mu \partial_\nu h - \partial^\alpha \partial_\alpha \partial_\mu h) \] (8)
\[ R = g^{\mu\nu} R_{\mu\nu} = \partial_\mu \partial_\nu h^{\mu\nu} - \partial^\alpha \partial_\alpha h \] 

respectively. In this section we consider the behavior of field equation (4) in the linearized approximation. The linearized form (expanding \( f \) up to the first order of \( h_{\mu\nu} \)) of determinantal invariant is

\[ f = -\frac{\tilde{R}_{\text{lin}}}{M^4(1 + h + \ldots)} \approx -M^{-4}\tilde{R}_{\text{lin}} = t_{\text{lin}} \] 

for \( \sigma = 1 \). Where \( \tilde{R}_{\text{lin}} \) is the determinant of the linearized Ricci tensor.

Using empty space condition for energy momentum tensor of matter, \( T_{\mu\nu} = 0 \), the linearized solution [11, 13] of equation (4) in the Minkowski background, expanding determinantal potential about \( r = 0 \), is obtained as

\[ G_{\mu\nu}^{\text{lin}} = \frac{1}{M_c^2} t_{\mu\nu} \] 

Where \( G_{\mu\nu}^{\text{lin}} \) is the linearized Einstein tensor, and \( t_{\mu\nu} \) is the 1st order perturbed (gravitational field) energy momentum tensor.

\[ G_{\mu\nu}^{\text{lin}} = \frac{1}{2}(\partial_\alpha \partial_\nu h^\alpha_\mu + \partial_\alpha \partial_\mu h^\alpha_\nu - \partial_\mu \partial_\nu h - \partial^\alpha \partial_\alpha h_{\mu\nu} - \eta_{\mu\nu} \partial_\alpha \partial_\beta h^{\alpha\beta} + \eta_{\mu\nu} \partial^\alpha \partial_\alpha h) \] 

The linearized Einstein field (11) equation takes the form of

\[ \partial_\alpha \partial_\nu h^\alpha_\mu + \partial_\alpha \partial_\mu h^\alpha_\nu - \partial_\mu \partial_\nu h - \partial^\alpha \partial_\alpha h_{\mu\nu} - \eta_{\mu\nu} \partial_\alpha \partial_\beta h^{\alpha\beta} + \eta_{\mu\nu} \partial^\alpha \partial_\alpha h = \frac{2}{M_c^2} t_{\mu\nu} \] 

3 Constant curvature space-time solution

The space-time metric with constant curvature, \( R_0 \), is characterized by the condition

\[ R_{\mu\nu\alpha\beta} = \frac{1}{12} R_0 (g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha}) \] 

on the Reimann tensor. So, the Ricci tensor satisfies

\[ R_{\mu\nu} = \frac{1}{4} R_0 g_{\mu\nu} \] 

from this, one can readily find the inverse Ricci tensor as follows

\[ \mathbb{R}^{\mu\nu} = \frac{4}{R_0} g^{\mu\nu}. \]

The maximally symmetric solution of equation (4) in vacuum is
\[ \frac{1}{4} R_0 g_{\mu\nu} - \frac{2}{M_{pl}^2} g_{\mu\nu} r f_r + \frac{1}{M_{pl}^2} g_{\mu\nu} f = 0. \] (17)

Where

\[ r = \frac{1}{256} R_0^4, \] (18)

\[ f(r) = \xi \left( \frac{R_0}{4M^2} \right)^{4\sigma} M^4, \] (19)

and

\[ df = f_r = \sigma \xi \left( \frac{R_0}{4M^2} \right)^{4\sigma - 1} \frac{1}{M^4}. \] (20)

Contracting the equation (17), gives us the algebraic

\[ R_0 - \xi \frac{4}{M_{pl}^2} \left( \frac{R_0}{4M^2} \right)^{4\sigma} M^4 (2\sigma - 1) = 0 \] (21)

equation. The solution for \( R_0 \) is not trivial for all values of \( f \neq 0 \). But, one can get the trivial value of \( R_0 \) for \( \sigma = 1/2 \). \( \sigma = 0 \) gives us the coupling constant \( \xi \) which linearly related to the \( R_0 \) as follows

\[ \xi = -\frac{M_{pl}^2}{4M^4} R_0. \] (22)

This coupling constant is positive just for negative constant curvature, \( R_0 \). Setting \( \sigma = 1 \), \( \xi \) becomes function of constant curvature, and Planck mass

\[ \xi = M_{pl}^2 M^4 \left( \frac{4}{R_0} \right)^3. \] (23)

This is positive just for positive values of \( R_0 \).

In the case of \( \sigma = 1/2 \), the determinantal invariant, \( f \), for 4-dimensional constant curvature space-time becomes

\[ f_c = \xi \frac{1}{16} R_0^2. \] (24)

This result is compatible with Starobinsky inflationary model [2], \( R^2 \). Then the constant curvature solution of equation (1) is

\[ - M_{pl}^2 g_{\mu\nu} R_0 - \xi \frac{1}{8} g_{\mu\nu} R_0^2 = 0. \] (25)

One can compare this result with the special case (constant curvature space-time) of Starobinsky inflationary parameter. The Starobinsky model of inflation can be written as follows

\[ f_s(R) = R - \frac{1}{6m^2} R^2. \] (26)
This is known as the chaotic inflationary model of Starobinsky, and it is perfectly well fitted with Planck data [9]. From the comparison of inflationary parameters of equation (25) with that of Starobinsky, the inflation mass can be given as

\[ m \simeq \frac{M_{pl}}{\sqrt{\xi}}. \] (27)

Inflaton mass [14] can be related to the reduced Planck mass with \( \xi \sim 1 \) limit in the early universe.

4 Conclusion

Determinantal invariant modification of EH action (1), does not affect the linearized solution, equation (13). However, constant curvature space-time solution of EH action with determinantal invariant, equation (24), mimics the Starobinsky inflationary parameter, \( R^2 \), equation (26). The maximally symmetric solution of action (3) gives us very different results for coupling constant, \( \xi \). Field equation (4) contains inverse Ricci tensor. Thus, the field equation (4) may produce novel results for physical or mathematical problems considered. As a result one can guess the mass of inflaton from equation (27).

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