Freidlin-Wentzell exit problems for stochastic equations in Banach spaces

Michael Salins

Department of Mathematics
University of Maryland
College Park
Maryland, USA

Abstract

Freidlin and Wentzell characterized the logarithmic asymptotics of the exit time from a basin of attraction for a finite dimensional diffusion with small noise. After that, several authors studied the same properties for exit problems associated to specific infinite dimensional system. In this paper, we present a general method, based on the control theoretic approach, to establish exit time and exit place results for a large class of stochastic equations in Banach spaces.

1 Introduction

In [10], Freidlin and Wentzell study finite dimensional stochastic differential equations of the form

\[
\begin{cases}
  dX^\epsilon_x(t) = f(X^\epsilon_x(t))dt + \sqrt{\epsilon}\sigma(X^\epsilon_x(t))dW(t), \\
  X^\epsilon_x(0) = x \in \mathbb{R}^d.
\end{cases}
\]

They investigate the asymptotics of the exit time,

\[\tau_x^\epsilon = \inf\{t > 0 : X^\epsilon_x(t) \notin G\},\]

where \(G \subset \mathbb{R}^d\) is an open set such that the unperturbed system, \(X^0_x\), is uniformly attracted to one asymptotically stable equilibrium point \(a \in G\), without leaving \(G\). Because the unperturbed system never leaves \(G\), \(\tau_x^\epsilon\) will diverge as \(\epsilon \to 0\). More specifically, using the theory of large deviations, they show that this divergence is of exponential type and that for any \(x \in G\),

\[
\lim_{\epsilon \to 0} \epsilon \log \mathbb{E}\tau_x^\epsilon = \inf_{y \in \partial G} V(y)
\]

where \(V(y)\) is a non-negative function called the quasipotential. Additionally, if there is a unique \(y_0 \in \partial G\) such that \(V(y_0) = \inf_{y \in \partial G} V(y)\), then

\[
\lim_{\epsilon \to 0} X^\epsilon_x(\tau_x^\epsilon) = y_0 \text{ in probability.}
\]
In other words, $X_\varepsilon^x$ exits $G$ near $y_0$ with overwhelming probability.

In this paper, we study the following class of stochastic equations in a Banach space $E$,

\[
\begin{aligned}
&dX_\varepsilon^x(t) = (AX_\varepsilon^x(t) + F(X_\varepsilon^x(t)))dt + \sqrt{\varepsilon}B(X_\varepsilon^x(t))dw(t) \\
&X_\varepsilon^x(0) = x \in E.
\end{aligned}
\]  

(1.2)

and we establish analogous exit time and exit place results. In the above equation, $A$ is the generator of a $C_0$ semigroup, and both $F$ and $B$ can be unbounded nonlinear operators. The inspiration for this problem comes from the theory of partial differential equations, where $E$ would be a function space, for example, an $L^p$ space, the space of continuous functions, or a space of Hölder-continuous functions, and $A$ would be some linear differential operator. The exit time asymptotics have previously been characterized for a variety of infinite dimensional equations including stochastic reaction diffusion equations [4, 3, 6, 11], stochastic semilinear wave equations [5], and stochastic Navier-Stokes equations [1]. In the current paper, we introduce a unified approach to study the exit problem for solutions of abstract stochastic evolution equations in Banach spaces that can apply to a wide variety of problems. We identify six sufficient conditions that imply our results and we state these as hypotheses. We also give examples to show that each assumption is both reasonable and quite general.

For an open set $G \subset E$ that contains 0, we study the exit times

\[\tau_\varepsilon^x = \inf\{t > 0 : X_\varepsilon^x(t) \notin G\}.\]

We assume that $\{X_\varepsilon^x\}$ satisfies a large deviations principle on the space $C([0,T];E)$ with respect to the rate functions $I_{0,T}$. The functional $I_{-\infty,0}$ is defined as an extension of these rate functions to the space $C((\infty,0);E)$. This extension is possible because (1.2) is time homogeneous. We then define the quasipotential as

\[V(y) = \inf\{I_{-\infty,0}(\varphi) : \lim_{t \to -\infty} |\varphi(t)|_E = 0, \varphi(0) = y\}.\]

Under certain assumptions, we show that $\tau_\varepsilon^x$ diverges exponentially as $\varepsilon \to 0$. Specifically, we prove that

\[\lim_{\varepsilon \to 0} \varepsilon \log \mathbb{E} \tau_\varepsilon^x = \inf_{x \in \partial G} V(x),\]  

(1.3)

and

\[\lim_{\varepsilon \to 0} \varepsilon \log \tau_\varepsilon^x = \inf_{x \in \partial G} V(x) \text{ in probability}.\]  

(1.4)

We also prove that $X_\varepsilon^x$ is likely to exit $G$ near the points that minimize $V$ on the boundary of $G$. This means that if $N \subset \partial G$ is closed and

\[\inf_{y \in N} V(y) > \inf_{y \in \partial G} V(y),\]

then

\[\lim_{\varepsilon \to 0} \mathbb{P}(X_\varepsilon^x(\tau_\varepsilon^x) \in N) = 0.\]

In particular, if there exists a unique $y^* \in \partial G$ such that

\[V(y^*) = \inf_{x \in \partial G} V(x),\]

then

\[\lim_{\varepsilon \to 0} \mathbb{P}(X_\varepsilon^x(\tau_\varepsilon^x) = y^*) = 1.\]
then by setting \( N = \{ y \in \partial G : |y - y^*|_E > \delta \} \), for any \( \delta \), we can show that

\[
\lim_{\epsilon \to 0} X_x^\epsilon (\tau_x^\epsilon) = y^* \text{ in probability.}
\]

Our proofs of the exit time and exit place results are largely based on the proofs of [8, Chapter 5], but important and nontrivial modifications have to be introduced to allow us to deal with the infinite dimensionality of the problem. Notice that because \( G \) is open and infinite dimensional, it is never compact. We also do not assume that \( G \) is bounded because when there are second order derivatives in time such as in [5] and [9], we often want \( G \) to be an unbounded set in the phase space. Our results are generalizations, therefore, of the results of Chenal and Millet [6], where they studied the exit time from compact subsets of \( E = C([0, T]) \).

The famous proofs of the exit time results for finite dimensional systems ([8, 10]) similarly took advantage of the fact that bounded sets of \( \mathbb{R}^d \) are compact. In the infinite dimensional setting, however, it is very natural to study the exit from a noncompact subset \( E \) because the open balls in these spaces are noncompact. In the current paper, we show that the compactness of \( G \) is unnecessary, and instead, we require that the level sets of \( I_{-\infty,0} \) are compact. This assumption, which we call Hypothesis \( \text{H4} \) is both very powerful and surprising general. We prove, for example, that this hypothesis is satisfied if \( E \) is a Hilbert space, \( (1.2) \) is a linear equation, and the semigroup generated by \( A \) is of negative type. This condition on the level sets of \( I_{-\infty,0} \) also explains why our results are tighter than those of Zabczyk [7, 13]. There, the author proves that

\[
\limsup_{\epsilon \to 0} \epsilon \log \mathbb{E} \tau_x^\epsilon \leq \inf_{y \in \partial G} V(y)
\]

and

\[
\liminf_{\epsilon \to 0} \epsilon \log \mathbb{E} \tau_x^\epsilon \geq \xi
\]

for some quantity, \( \xi \leq \inf_{y \in \partial G} V(y) \). The requirement that the level sets of \( I_{-\infty,0} \) are compact guarantees that \( \xi = \inf_{y \in \partial G} V(y) \) which then implies \( (1.3) \).

In the next section, we introduce our notations and we identify our six hypotheses. After each hypothesis, we include a discussion or examples to show that these hypotheses are reasonable and general. In section 3 we state and prove the main results.

2 Assumptions and Preliminaries

Let \((E, |\cdot|_E)\) be a Banach space. For any set \( G \subset E \), we use the following notations. The complement of \( G \) is denoted as \( G^c = E \setminus G \). The closure of \( G \) is denoted as \( G \). We study the following equation

\[
\begin{aligned}
\begin{cases}
    dX_x^\epsilon(t) = (AX_x^\epsilon(t) + F(X_x^\epsilon(t)))dt + \sqrt{\epsilon}B(X_x^\epsilon(t))dw(t), \\
    X_x^\epsilon(0) = x,
\end{cases}
\end{aligned}
\]

(2.1)

where \( A : D(A) \subseteq E \to E \) is the generator of a \( C_0 \) semigroup, \( S(t) \), and \( F \) and \( B \) are possibly unbounded, nonlinear mappings. In the above equation, \( w(t) \) is a cylindrical Wiener process
on some Hilbert space, \( H \). This means that formally

\[
w(t) = \sum_{k=1}^{\infty} \beta_k(t)e_k,
\]

where \( \{\beta_k\} \) is a family of independent one-dimensional Brownian motions on some stochastic basis \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\), and \( \{e_k\} \) is a complete orthonormal system of \( H \).

**Definition 2.1** (Mild solution). An adapted process \( X^\varepsilon_x \) is called a mild solution for \((2.1)\) if for \( t \geq 0 \),

\[
X^\varepsilon_x(t) = S(t)x + \int_0^t S(t-s)F(X^\varepsilon_x(s))ds + \sqrt{\varepsilon} \int_0^t S(t-s)B(X^\varepsilon_x(s))dw(s).
\]

**Hypothesis 1.** For any \( T > 0 \), the mild solution of \( X^\varepsilon_x \) exists and is unique in the space \( L^2(\Omega; C([0,T]; E)) \).

For sufficient conditions that imply existence and uniqueness see [7, Chapter 7].

We also want to study the deterministic control systems associated with this equation. Namely for \( \psi \in L^2([0,T]; H) \) we study the problem

\[
\begin{cases}
    \frac{dX^\psi_x}{dt}(t) = F(X^\psi_x(t)) + B(X^\psi_x(t))\psi(t), \\
    X^\psi_x(0) = x,
\end{cases}
\tag{2.2}
\]

which can be rewritten in mild form as

\[
X^\psi_x(t) = S(t)x + \int_0^t S(t-s)F(X^\psi_x(s))ds + \int_0^t S(t-s)B(X^\psi_x(s))\psi(s)ds.
\tag{2.3}
\]

**Hypothesis 2.** There exists an increasing positive function \( \kappa_2 : \mathbb{R}_+ \to \mathbb{R}_+ \) such that for any \( t > 0, x_1, x_2 \in E, \) and \( \psi \in L^2_{loc}((0, +\infty); H) \),

\[
|X^\psi_{x_1}(t) - X^\psi_{x_2}(t)|_E \leq \kappa_2 \left( \frac{1}{2} |\psi|_{L^2((0,t]; E)}^2 \right) |x_1 - x_2|_E.
\tag{2.4}
\]

Additionally, for any \( R > 0, T > 0 \),

\[
\lim_{r \to 0} \sup_{\|x\|_E \leq R} \sup_{t \leq T} \left\{ |X^\psi_x - X^0_x|_{C([0,t]; E)} : |\psi|_{L^1((0,t]; H)} \leq r \right\} = 0.
\tag{2.5}
\]

In other words, \((2.4)\) says that the map \( x \in E \mapsto X^\psi_x \in C([0,t]; E) \) is continuous uniformly for \( |\psi|_{L^2((0,t]; H)} \leq R \) and \( t \leq T \), and \((2.5)\) says that the map \( \psi \in L^1((0,t]; H) \mapsto X^\psi_x \in C([0,t]; E) \) is continuous at \( \psi = 0 \) uniformly in \( |x|_E \leq R \) and \( t \leq T \).

Hypothesis \( 2 \) is satisfied in many general situations.
Then Hypothesis 2 is satisfied.

Now let
\[ \phi \]
such that
\[ \lambda > \epsilon \]
d\leq
\[ \left\langle A(X_{x_1}(t) - X_{x_2}(t)) + F(X_{x_1}(t)) - F(X_{x_2}(t)), x^*(t) \right\rangle \leq 0. \]

Therefore, (2.4) follows because
\[ |X_{x_1}(t) - X_{x_2}(t)|_E \leq |x_1 - x_2|_E. \]

By a similar argument, we see that
\[ \frac{d^-}{dt} |X_{x_1}(t) - X_{x_2}(t)|_E \leq |Q \psi(t)|_E. \]

Thereby, (2.5) follows because
\[ |X_{x_1}(t) - X_{x_2}(t)|_E \leq \|Q\|_{L(H,E)}\|\psi\|_{L^1([0,T];H)}. \]

Example 2.3 (Multiplicative noise, strongly dissipative nonlinearities). Assume that there exists \( \lambda > 0 \) such that for any \( x, y \in D(F) \cap D(A) \), there exists \( x^* \in \partial|x - y|_E \) such that
\[ \langle A(x - y) + F(x) - F(y), x^* \rangle \leq -\lambda|x - y|_E \]

and that \( B \) is Lipschitz continuous with linear growth in the sense that there exists \( \kappa_1 \) such that for any \( x, y \in E \),
\[ \|B(x) - B(y)\|_{L(H,E)} \leq \kappa_1 |x - y|_E \]

and \( \|B(x)\|_{L(H,E)} \leq \kappa_1 (1 + |x|) \).

Now let \( \varphi(t) = X_{x_1}^\psi(t) - X_{x_2}^\psi(t) \). For any \( t \), there exists \( x^*(t) \) such that
\[ \frac{d^-}{dt} |\varphi(t)|_E \leq \left\langle \frac{d^-}{dt} \varphi(t), x^*(t) \right\rangle \]
\[ \leq \left\langle A \varphi(t) + F(X_{x_1}^\psi(t)) - F(X_{x_2}^\psi(t)), x^*(t) \right\rangle + |B(X_{x_1}^\psi(t))\psi(t) - B(X_{x_2}^\psi(t))\psi(t)|_E \]
\[ \leq -\lambda|\varphi(t)|_E + \kappa_1 |\varphi(t)|_E |\psi(t)|_H. \]

By a comparison argument, we see that
\[ |X_{x_1}^\psi(t) - X_{x_2}^\psi(t)|_E \leq \exp \left( -\lambda t + \kappa_1 \int_0^t |\psi(s)|_H ds \right) |x_1 - x_2|_E \]
\[ \leq \exp \left( -\lambda t + \sqrt{\kappa_1} |\psi|_{L^2([0,T];H)} \right) |x_1 - x_2|_E \]
\[ \leq \exp \left( \frac{\kappa_1^2 |\psi|_{L^2([0,T];H)}^2}{2\lambda} \right) |x_1 - x_2|_E. \]
Further, Hypothesis (2.5) holds with \( \kappa_2(r) = \exp\left( \frac{r^2}{\lambda} \right) \).

By similar calculations we see that
\[
\frac{d}{dt} |X_x^\psi(t) - X_x^0(t)|_E \leq -\lambda |X_x^\psi(t) - X_x^0(t)|_E + |B(X_x^\psi(t))\| \psi(t)|_E \\
\leq -\lambda |X_x^\psi(t) - X_x^0(t)|_E + |(B(X_x^\psi(t)) - B(X_x^0(t))\| \psi(t)|_E + |B(X_x^0(t))\| \psi(t)|_E \\
\leq -\lambda |X_x^\psi(t) - X_x^0(t)|_E + \kappa_1 |X_x^\psi(t) - X_x^0(t)|_E \| \psi(t)|_H + \kappa_1(1 + |X_x^0(t)|_E) \| \psi(t)|_H.
\]

By a comparison argument with the same calculations that we used above,
\[
|X_x^\psi(t) - X_x^0(t)|_E \leq \exp \left( \kappa_1 \left( 1 + \sup_{s\geq 0} |X_x^0(s)|_E \right) \right) \| \psi\|_{L^1([0,t];H)}.
\]

Because of our dissipativity assumptions, the unperturbed system has the property that
\[
\frac{d}{dt} |X_x^0(t)|_E \leq -\lambda |X_x^0(t)|_E
\]
and therefore,
\[
\sup_{s\geq 0} |X_x^0(s)|_E \leq |x|_E.
\]

These calculations show that
\[
|X_x^\psi - X_x^0|_{C([0,T];E)} \leq \kappa_1 (1 + |x|_E) \exp(|\psi|_{L^1([0,T];H)}) \| \psi\|_{L^1([0,T];H)},
\]
and therefore, (2.5) holds.

Now we establish some large deviations results for the trajectories of \( X_x^\eps \). For any \( T > 0 \), define the rate functions
\[
I_{0,T}(\phi) = \inf \left\{ \frac{1}{2} |\phi|_{L^2([0,T];H)}^2 : \phi(t) = X_x^\psi(t), \ 0 \leq t \leq T \right\}
\]
with the standard convention that \( \inf \emptyset = +\infty \). We also define the corresponding level sets, \( K_{0,T}(r) \), by
\[
K_{0,T}(r) = \{ \phi \in C([0,T];E) : I_{0,T}(\phi) \leq r \}.
\]

In this paper, we state the large deviations principle as an assumption.

**Hypothesis 3** (Large deviations principle). For any \( x \in E \) and any \( T > 0 \),
\[
\limsup_{\eps \to 0} \sup_{x \in E} |X_x^\eps - X_x^0|_{C([0,T];E)} = 0 \text{ in probability.}
\]

Further, for any \( \phi \in L^2([0,T];H) \) and \( \delta > 0 \),
\[
\liminf_{\eps \to 0} \eps \log \left( \inf_{x \in E} \mathbb{P} \left( |X_x^\eps - X_x^0|_{C([0,T];E)} < \delta \right) \right) \geq - \frac{1}{2} |\phi|_{L^2([0,T];H)}^2, \tag{2.8}
\]
and any \( r > 0 \) and \( \delta > 0 \),
\[
\limsup_{\eps \to 0} \eps \log \left( \sup_{x \in E} \mathbb{P} \left( \text{dist}_{C([0,T];E)}(X_x^\eps, K_{0,T}(r)) > \delta \right) \right) \leq -r. \tag{2.9}
\]
Remark 2.4. For sufficient conditions implying Hypothesis 3, see [2].

Now, we want to extend the domain of the rate functions to include trajectories on the negative half-line. Because (2.1) is time homogeneous, we use translation to define, for any \( t < T \in \mathbb{R} \),

\[
I_{t,T}(\varphi) = I_{0,T-t}(\varphi(\cdot + t)).
\]  

We then define

\[
I_{-\infty,T}(\varphi) = \sup_{t < T} I_{t,T}(\varphi).
\]  

Now, as mentioned in the introduction, we define the quasipotential to be

\[
V(x) = \inf \left\{ I_{-\infty,0}(\varphi) : \lim_{t \to -\infty} |\varphi(t)|_E = 0, \varphi(0) = x \right\}
\]  

where once again we use the standard convention that \( \inf \emptyset = +\infty \). One can think of this as the minimal amount of energy required to reach the point \( x \) starting from the point \( 0 \) in an infinite amount of time. We also use the notation that for any \( N \subseteq E \),

\[
V(N) = \inf_{x \in N} V(x),
\]

as this notation causes no confusion.

**Hypothesis 4 (Compact level sets).** For any \( r \in \mathbb{R} \), the set

\[
K_{-\infty}(r) = \{ \varphi \in C((-\infty,0)) : \lim_{t \to -\infty} |\varphi(t)|_E = 0, I_{-\infty,0}(\varphi) \leq r \}
\]

is compact in the topology of uniform convergence on bounded intervals.

Remark 2.5. Hypothesis 4 is a technical assumption that provides the main power for the proof of Theorem 3.2. In the case that \( E \) is finite dimensional (see [8] and [10]), it is not difficult to show that the quasipotential is Lipschitz continuous in the sense that there exists \( \kappa > 0 \) such that

\[
|V(x) - V(y)| \leq \kappa|x - y|_E.
\]

This is not true when \( E \) infinite dimensional. In fact, in many situations one can show that \( V(x) = +\infty \) except on a dense subset. The compactness of level sets allows us to overcome the fact that \( V \) is not continuous. Hypothesis 4 is true in many general situations. Proposition 2.6, below, gives a sufficient condition for Hypothesis 4 to hold when (2.1) is linear. Then in Example 2.7, we show that Hypothesis 4 holds whenever \( E \) is a Hilbert space, \( S(t) \) is of negative type, and (2.1) is linear. In example 2.8, we give an Example in the case where \( E = C([0,1]) \).

**Proposition 2.6.** Assume that that (2.1) is linear, that is, \( F(x) \equiv 0 \), \( B(x) \equiv Q \). Assume that there exists \( M > 1 \) such that for all \( t > 0 \), \( \|S(t)\|_L(E) \leq M \), and that the linear operator \( L : L^2((0, +\infty); H) \to E \) given by

\[
L\psi = \int_0^{\infty} S(s)Q\psi(s)ds
\]

is a compact operator. Then Hypothesis 4 holds.
Proof. By (2.13), if \( \varphi \in K_{-\infty}(r) \) then \( \varphi \) is a trajectory on the negative half line for which

\[
I_{-\infty,0}(\varphi) \leq r \text{ and } \lim_{t \to -\infty} |\varphi(t)|_E = 0.
\]

By (2.11), \( \varphi \) must be a weak solution of

\[
\frac{d}{dt} \varphi(t) = A\varphi(t) + Q\psi(t)
\]

for some \( \psi \) with

\[
\frac{1}{2} |\psi|_{L^2((-\infty,0);H)}^2 \leq r.
\]

Because \( A \) is the generator of the semigroup, \( S(t) \), we have that for any \( T > 0 \),

\[
\varphi(t) = S(t + T)\varphi(-T) + \int_{-T}^{t} S(t - s)Q\psi(s)ds.
\]

By letting \( T \to +\infty \), we see that

\[
\varphi(t) = \int_{-\infty}^{t} S(t - s)\psi(s)ds.
\]

By a time change, therefore,

\[
\varphi(t) = \int_{0}^{\infty} S(s)\psi(t - s)ds = L\psi(t - \cdot),
\]

where \( L \) is defined in (2.14). By these arguments, given any sequence \( \{\varphi_n\} \subset K_{-\infty}(r) \), there exists a sequence \( \{\psi_n\} \subset L^2((-\infty,0);H) \) with the property that

\[
\varphi_n(t) = L\psi_n(t - \cdot) \text{ and } \frac{1}{2} |\psi_n|_{L^2((-\infty,0);H)}^2 \leq r.
\]

By Alaoglu’s theorem, there exists a subsequence which we relabel \( \psi_n \) and a limit \( \psi \) such that

\[
\psi_n \rightharpoonup \psi \text{ weakly in } L^2((-\infty,0);H).
\]

It is not difficult to see that for any fixed \( t > 0 \)

\[
\psi_n(t - \cdot) \rightharpoonup \psi(t - \cdot) \text{ weakly in } L^2((0, +\infty);H).
\]

We define

\[
\varphi(t) = L\psi(t - \cdot).
\]

Because \( L \) is a compact operator from the norm topology on \( L^2((0, +\infty);H) \) to the norm topology on \( E \), it is a continuous operator from the weak topology on \( L^2((0, +\infty);H) \) to the norm topology on \( E \) (see for example [12, Chapter 21]). It follows that for any fixed \( t < 0 \)

\[
|\varphi_n(t) - \varphi(t)|_E = |L\psi_n(t - \cdot) - L\psi(t - \cdot)|_E \to 0.
\]
To finish the proof, we must show that (possibly for a further subsequence) this convergence is uniform on bounded intervals of \( t \). By the Arzela-Ascoli theorem, it only remains to prove that \( \{ \varphi_n \} \) is a equicontinuous set. For any \( s < t \leq 0 \),

\[
\varphi_n(t) - \varphi_n(s) = L\psi_n(t - \cdot) - L\psi_n(s - \cdot) \\
= \int_0^{t-s} S(r)Q\psi_n(t - r)dr - \int_0^{\infty} S(r)Q\psi_n(s - r)dr \\
= \int_0^{t-s} S(r)Q\psi_n(t - r)dr + (S(t-s)-I)\int_0^{\infty} S(r)Q\psi_n(s - r)dr \\
= L(\psi_n(t - \cdot)\mathbb{1}_{[0,t-s]}(\cdot)) + (S(t-s)-I)L\psi_n.
\]

Because \( L\psi_n \to L\psi \),

\[
\lim_{|t-s|\to 0} \sup_n |(S(t-s)-I)L\psi_n|_E = 0.
\]

Next, we observe that as \( |t-s| \to 0 \)

\[
\psi_n(t - \cdot)\mathbb{1}_{[0,t-s]}(\cdot) \to 0 \text{ weakly in } L^2((0, +\infty); H), \text{ uniformly in } n
\]

To see this, take any test function \( \phi \in L^2((0, +\infty); H) \). By the Cauchy-Schwarz inequality

\[
\sup_n \left| \int_0^{t-s} \langle \psi_n(r), \phi(r) \rangle_H \right| \leq \sup_n |\psi_n|_{L^2((0, +\infty); H)} |\phi|_{L^2((0, t-s); H)} \to 0.
\]

Therefore, because \( L \) is a continuous operator from the weak topology on \( L^2((0, +\infty); H) \) to \( E \), it follows that

\[
\lim_{|t-s|\to 0} \sup_n |L(\psi_n(t - \cdot)\mathbb{1}_{[0,t-s]}(\cdot))|_E = 0.
\]

Therefore, the family \( \{ \varphi_n \} \) is equicontinuous, and for each \( t < 0 \) \( \varphi_n(t) \to \varphi(t) \). By the Arzela-Ascoli theorem, it follows that there exists a subsequence, also relabeled as \( \varphi_n \), such that for any \( T > 0 \)

\[
\lim_{n \to +\infty} |\varphi_n - \varphi|_{C([-T,0]; E)} = 0.
\]

Finally we can conclude that because \( \varphi(t) = L\psi(t - \cdot) \) and \( L \) is a bounded operator

\[
\lim_{t \to -\infty} |\varphi(t)|_E = \lim_{t \to -\infty} |L\psi(t - \cdot)|_E \leq \lim_{t \to -\infty} \|L\|_{L(E; L^2((0, +\infty); H), E)} |\psi|_{L^2((0, t-s); H)} = 0.
\]

Therefore, \( \varphi \in K_{-\infty}(\varphi) \).

\[\square\]

**Example 2.7** (In a Hilbert space). If \( E = U \) is a Hilbert space, \( (2.1) \) is linear, and there exists \( M \geq 0 \) and \( \omega > 0 \) such that \( \|S(t)\|_{L(E)} \leq M e^{-\omega t} \), then Hypothesis 3 is satisfied.

The mild solution to \( (2.1) \) in this case is

\[
X^\varepsilon_x(t) = S(t)x + \sqrt{\varepsilon} \int_0^t S(t-s)Qdw(s).
\]

By Itô formula, we can calculate that for any fixed time \( T > 0 \),

\[
E \left[ \int_0^T S(T-s)Qdw(s) \right]^2_U = \int_0^T \text{Tr} S(T-s)QQ^*S^*(T-s)ds = \int_0^T \text{Tr} S(s)QQ^*S^*(s)ds.
\]
Therefore, a necessary condition for Hypothesis 1 to hold is that for any $T > 0$,

$$
\int_0^T \text{Tr} S(s)QQ^* S^*(s)ds < +\infty.
$$

Actually, it follows that

$$
\int_0^\infty \text{Tr} S(s)QQ^* S^*(s)ds < +\infty.
$$

(2.15)

To see this, notice that because $S$ is a semigroup with $\|S(t)\|_{\mathcal{L}(U)} \leq Me^{-\omega t}$, for any $t,s>0$,

$$
\text{Tr} S(t+s)QQ^* S^*(t+s) = \sum_{k=1}^\infty |S(t)S(s)Qe_k|^2_U
\leq M^2e^{-2\omega t} \sum_{k=1}^\infty |S(s)Qe_k|^2_U = M^2e^{-2\omega t} \text{Tr} S(s)QQ^* S^*(s).
$$

From these observations we can show that

$$
\int_0^\infty \text{Tr} S(s)QQ^* S^*(s)ds = \sum_{n=0}^\infty \int_0^1 \text{Tr} S(n+s)QQ^* S^*(n+s)ds
\leq \sum_{n=0}^\infty M^2e^{-2\omega n} \int_0^1 \text{Tr} S(s)QQ^* S^*(s)ds \leq c \int_0^1 \text{Tr} S(s)QQ^* S^*(s)ds < +\infty.
$$

The operator

$$
L\psi = \int_0^\infty S(s)Q\psi(s)ds
$$

has the property that

$$
\text{Tr} LL^* = \int_0^\infty \text{Tr} S(t)QQ^* S^*(t)dt < +\infty.
$$

Therefore, $L$ is a compact operator and it follows by Proposition 2.6 that Hypothesis 4 holds.

In the next example, we show that the situation is a little bit more complicated when $E$ is not a Hilbert space, but that Hypothesis 4 can be proven.

**Example 2.8** (Stochastic heat equation). Consider the stochastic heat equation on $E = C([0,1])$,

$$
\begin{align*}
\frac{\partial X}{\partial t}(t,\xi) &= \frac{\partial^2 X}{\partial \xi^2}(t,\xi) + \frac{\partial w}{\partial t}(t,\xi) \\
X(t,0) &= X(t,1) = 0, \quad X(0,\xi) = X_0(\xi).
\end{align*}
$$

(2.16)

In the above equation, $\partial w/\partial t$ is a space-time white noise. It is well known that the operator $A = \frac{\partial^2}{\partial \xi^2}$ with Dirichlet boundary conditions has a sequence of eigenfunctions

$$
e_k(\xi) = 2\sin(k\pi\xi)
$$
which form a complete orthonormal basis of $H := L^2([0,1])$. Furthermore, the space-time white noise can be represented as the formal sum
\[
\frac{\partial w}{\partial t}(t, \xi) = \sum_{k=1}^{\infty} e_k(\xi) d\beta_k(t)
\]
where $\{\beta_k(t)\}$ is a sequence of i.i.d. one-dimensional Brownian motions. The operator, $A$, generates a $C_0$ semigroup, $S(t)$, on $E := C([0,1])$, and this semigroup has the property that
\[
S(t)e_k(\xi) = e^{-k^2\pi^2 t}e_k(\xi).
\]
By Proposition 2.6, Hypothesis 4 will hold if we can prove that the operator
\[
L\psi = \int_0^\infty S(s)\psi(s)ds
\]
is a compact operator from $L^2((0, +\infty); H) \to E$. We show that the operator is compact by the Arzela-Ascoli theorem. First, by writing
\[
\psi(s) = \sum_{k=1}^{\infty} \langle \psi(s), e_k \rangle_H e_k
\]
we estimate that
\[
|L\psi|_E = \left| \sum_{k=1}^{\infty} \int_0^\infty S(s) e_k \langle \psi(s), e_k \rangle_H ds \right|_E
\]
\[
= \left| \sum_{k=1}^{\infty} \int_0^\infty e^{-k^2\pi^2 s} e_k \langle \psi(s), e_k \rangle_H ds \right|_E.
\]
By the Cauchy-Schwarz inequality and the fact that $|e_k|_E \leq 2,$
\[
|L\psi|_E \leq \sqrt{2} \left( \sum_{k=1}^{\infty} \frac{1}{k^2 \pi^2} \right)^{1/2} \|\psi\|_{L^2((0, +\infty); H)} = c \|\psi\|_{L^2((0, +\infty); H)}.
\]
This shows that the $L\psi$ are equibounded. Now we need to show that they are equicontinuous. We begin by observing that by the mean-value theorem, for any $\xi, \eta \in [0,1]$
\[
|e_k(\xi) - e_k(\eta)| \leq 2k\pi|\xi - \eta|.
\]
In addition, we have the trivial bound
\[
|e_k(\xi) - e_k(\eta)| \leq |e_k(\xi)| + |e_k(\eta)| \leq 4.
\]
Therefore, for any $\gamma \in [0,1],$
\[
|e_k(\xi) - e_k(\eta)| \leq |e_k(\xi) - e_k(\eta)|^\gamma |e_k(\xi) - e_k(\eta)|^{1-\gamma} \leq c_\gamma k^\gamma |\xi - \eta|^\gamma.
\]
Then, we calculate that
\[
|L\psi(\xi) - L\psi(\eta)| \leq \left| \sum_{k=1}^{\infty} \int_0^\infty e^{-k^2\pi^2 s} (e_k(\xi) - e_k(\eta)) \langle \psi(s), e_k \rangle_H ds \right|.
\]
By the Cauchy-Schwarz inequality, as long as $0 < \gamma < 1$

$$|L\psi(\xi) - L\psi(\eta)| \leq c_\gamma |\xi - \eta|^\gamma \left( \sum_{k=1}^{\infty} \frac{1}{k^{2(1-\gamma)}} \right)^{\frac{1}{2}} |\psi|_{L^2((0,\infty);H)} \leq c_\gamma |\xi - \eta|^\gamma |\psi|_{L^2((0,\infty);H)}.$$ 

Therefore, by the Arzela-Ascoli theorem, $L$ is a compact operator, and by Proposition 2.6 Hypothesis 7 holds.

Let $G \subset E$ be open, connected, and let $0 \in G$. The goal of the next chapter is to characterize the exit time and exit place of the process $X_\varepsilon^t$ from the domain $G$. We now state our hypotheses on the set $G$. For $r > 0$ let $G_r$ be the enlargement of $G$ given by $G_r = \{x \in E : \text{dist}_E(x,G) < r\}$.

We also let $\Lambda_r$ be the set of initial conditions for which the unperturbed process leaves $G_r$. That is,

$$\Lambda_r = \{x \in G : \exists t, X^0_\varepsilon(t) \notin G_r\}.$$ 

**Hypothesis 5 (Attraction to stable equilibrium).**

i. There exists $M > 0$ such that for all $t \geq 0$,

$$|X^0_\varepsilon(t)|_E \leq M|x|_E \quad (2.17)$$

ii. There exists $r > 0$, $T > 0$ such that for any $x \in \Lambda_r$ there exists $t(x) \leq T$ such that

$$X^0_\varepsilon(t(x)) \notin G_r. \quad (2.18)$$

iii. $X^0_\varepsilon(t) \to 0$ uniformly for $x \in G \setminus \Lambda_r$, that is

$$\lim_{t \to +\infty} \sup_{x \in G \setminus \Lambda_r} |X^0_\varepsilon(t)|_E = 0. \quad (2.19)$$

This means that if $x \in G$, either $X^0_\varepsilon$ leaves $G_r$ in a finite amount of time, or $X^0_\varepsilon$ converges to 0.

**Remark 2.9.** In some situations we will be able to prove that

$$\lim_{t \to +\infty} \sup_{x \in G} |X^0_\varepsilon(t)|_E = 0$$

and that $X^0_\varepsilon$ never leaves $G$ if $x$ is in $G$. In this case, $\Lambda_r$ is empty. Our hypothesis is more general than this because in many equations with second-order derivatives in time, such as [5] and [9], some unperturbed trajectories will leave $G$, but we still can prove exit time and exit place results.

**Hypothesis 6 (Boundary Regularity).** Assume that

$$V(\partial G) = V(\overline{G}^c).$$
Remark 2.10. First, we observe that it always holds that

\[ V(\partial G) \leq V(\bar{G}^c)). \]

If \( \tilde{y} \in \bar{G}^c \) and \( V(\tilde{y}) < +\infty \), then by the definition of \( V \), and the compactness of the level sets of \( I_{-\infty,0} \) (Hypothesis[4]), there exists a trajectory \( \varphi \in C((-\infty,0); E) \) such that

\[ \lim_{t \to -\infty} \varphi(t) = 0 \text{ and } \varphi(0) = \tilde{y}. \]

with

\[ I_{-\infty,0}(\varphi) = V(\tilde{y}) < +\infty. \]

Because this trajectory is continuous and we assumed that \( 0 \in G \), and \( \tilde{y} \in \bar{G}^c \), there must be some \( t < 0 \) for which \( \varphi(t) \in \partial G \). But then, by the definition of \( V \),

\[ V(\partial G) \leq V(\varphi(t)) \leq I_{-\infty,t}(\varphi) \leq I_{-\infty,0}(\varphi) = V(\bar{G}^c). \]

To see why such a boundary regularity assumption is important, consider a punctured ball. Let \( a \in E \), be such that \( 0 < |a|_E < 1 \) and \( V(a) < \inf_{|x| = 1} V(x) \). Let \( G = \{ x \in E : |x| < 1, x \neq a \} \). We should not expect exit place results to be based on the values of \( V \) on \( \partial G \), because \( a \in \partial G \) but \( a \) is far from \( \bar{G}^c \).

3 Exit time and exit place

In this section we state and prove Theorem 3.2, the main theorem of our paper. Define the stopping times

\[ \tau^\epsilon_x = \inf \{ t > 0 : X^\epsilon_x(t) \notin G \}. \]  

(3.1)

First we state and prove a somewhat straightforward special case.

Proposition 3.1. Suppose that \( x \) is such that the unperturbed trajectory, \( X^0_x \), has the property that the non-random time, \( \tau^0_x \), is finite and that there exists some \( \delta_0 > 0 \) such that for any \( \delta \in (0,\delta_0) \), \( X^0_x(\tau^0_x + \delta) \in \bar{G}^c \). In other words, we assume that \( X^0_x \) exits \( G \) in a finite amount of time and then immediately leaves \( \bar{G} \) too. Then, it follows that

\[ \tau^\epsilon_x \to \tau^0_x < +\infty \text{ and } X^\epsilon_x(\tau^\epsilon_x) \to X^0_x(\tau^0_x). \]

Proof. This a consequence of the fact that

\[ |X^\epsilon_x - X^0_x|_{C([0,\tau^0_x+\delta];E)} \to 0. \]

Proposition 3.1 tell us that there is no need to use the large deviations principle to study the exit time and exit place problems for such initial conditions. In the next theorem, we show what happens in \( X^0_x(t) \in G \) for all \( t > 0 \).

Theorem 3.2. For any \( x \) such that \( X^0_x(t) \in G \) for all \( t > 0 \),
Proof. First, by Hypothesis 6, there exists

\[ \lim_{\epsilon \to 0} \epsilon \log \mathbb{E} \tau_x^\epsilon = V(\partial G). \quad (3.2) \]

ii. For any \( \eta > 0 \),

\[ \lim_{\epsilon \to 0} \mathbb{P} \left( \epsilon \tau_x^\epsilon \leq \tau_x^\epsilon \leq \epsilon e^{\frac{1}{2}V(\partial G) + \eta} \right) = 1. \quad (3.3) \]

iii. For any closed \( N \subset \partial G \) with \( V(N) > V(\partial G) \),

\[ \lim_{\epsilon \to 0} \mathbb{P} (X_x^\epsilon \tau_x^\epsilon \in N) = 0. \quad (3.4) \]

Remark 3.3. If \( x \) is such that \( X_x^0 \) reaches \( \partial G \) but does not immediately leave \( G \), then the exact exit time asymptotics and exit place characterizations are unknown. Fortunately, from the proof of Theorem 3.2 which is below, we can conclude that for any \( x \in G \) the upper bounds of (3.2) and (3.3) hold. That is,

\[ \limsup_{\epsilon \to 0} \epsilon \log \mathbb{E} \log \tau_x^\epsilon \leq V(\partial G) \]

and

\[ \lim_{\epsilon \to 0} \mathbb{P} \left( \tau_x^\epsilon \leq e^{\frac{1}{2}V(\partial G) + \eta} \right) = 1. \]

Also, in [13], the author proves the same upper bound and he also proves a lower bound for the exit time which may be strictly less than the upper bound.

Before proving Theorem 3.2 we state and prove the following lemmas.

Lemma 3.4. For any \( \eta > 0 \), there exist \( \rho > 0 \), \( T_1 > 0 \), \( \delta > 0 \) and \( \psi \in L^2([0,T];H) \) such that

\[ \frac{1}{2} |\psi|^2_{L^2([0,T];H)} \leq V(\partial G) + \eta. \]

and for all \( |x| \leq \rho \),

\[ \text{dist}_E(X_x^\psi(T_1), G) > \delta. \]

Proof. First, by Hypothesis 6 there exists \( y \in \bar{G}^c \) such that \( V(y) < V(\partial G) + \frac{\eta}{3} \). Then by (2.11), there must exist a trajectory \( \varphi \in C((-\infty, 0); E) \) with

\[ \lim_{t \to -\infty} |\varphi(t)|_E = 0, \quad \varphi(0) = y, \quad \text{and} \quad I_{-\infty,0}(\varphi) < V(\partial G) + \frac{2\eta}{3}. \]

Because \( y \in \bar{G}^c \), the distance, \( d := \text{dist}_E(y, G) \), is strictly positive. Because \( \lim_{t \to -\infty} \varphi(t) = 0 \), we can choose \( T_1 > 0 \) to be large enough so that \( |\varphi(-T_1)|_E < \frac{d}{3\kappa_2(r)} \), where \( \kappa_2 \) is from Hypothesis 2 and \( r = V(\partial G) + \eta \). Set \( x_1 = \varphi(-T_1) \). By (2.10), it is clear that \( \varphi(t - T_1) = X_x^\psi(t) \) for some \( \psi \) with

\[ \frac{1}{2} |\psi|^2_{L^2([0,T_1];H)} < V(\partial G) + \frac{2\eta}{3}. \]

Then, by (2.4), if \( |x|_E < \frac{d}{3\kappa_2(r)} \),

\[ |y - X_x^\psi(T_1)|_E = |X_{x_1}^\psi(T_1) - X_x^\psi(T_1)|_E \leq \kappa_2(r)|x_1 - x| \leq \frac{2d}{3}, \]
In particular, this means that for all $|x|_E < \frac{d}{3\eta_2(r)}$,

$$\text{dist}_E(X^\psi_x(T_1), G) \geq \text{dist}_E(y, G) - |y - X^\psi_x(T_1)|_E > \frac{d}{3}.$$ 

Our result follows with $\rho = \frac{d}{3\eta_2(r)}$ and $\delta = \frac{d}{3}$. \hfill \Box

**Lemma 3.5.** For any $\eta > 0$, we can find $T > 0$ such that

$$\liminf_{\epsilon \to 0} \epsilon \log \left( \inf_{x \in G} \mathbb{P}(\tau^\epsilon_x < T) \right) > -(V(\partial G) + \eta).$$ (3.5)

*Proof.* Let $\psi, \rho, \delta,$ and $T_1$ satisfy Lemma 3.4. Then, because $\text{dist}_E(X^\psi_x(T_1), G) > \delta$ for $|x|_E < \rho$, the events

$$\{\tau^\epsilon_x \leq T_1\} \subset \{|X^\epsilon_x - X^\psi_x|_{C([0,T_1];H)} < \delta\}.$$ 

By (2.8),

$$\lim_{\epsilon \to 0} \epsilon \log \left( \inf_{|x|_E \leq \rho} \mathbb{P}(\tau^\epsilon_x \leq T_1) \right) \geq \lim_{\epsilon \to 0} \log \left( \inf_{|x|_E \leq \rho} \mathbb{P} \left( |X^\epsilon_x - X^\psi_x|_{C([0,T_1];H)} < \delta \right) \right) \geq -(V(\partial G) + \eta).$$ (3.6)

Let $r > 0$ satisfy Hypothesis 3. For $|x|_E > \rho$, there are two cases. Either $x \in \Lambda_r$ or $x \in G \setminus \Lambda_r$. By (2.18), there exists $T_2 > 0$ such for all $x \in \Lambda_r$ there exists $t(x) \leq T_2$ such that $\text{dist}_E(X^0_x(t(x)), G) > r$. This means that

$$\liminf_{\epsilon \to 0} \inf_{x \in \Lambda_r} \mathbb{P}(\tau^\epsilon_x \leq T_2) \geq \liminf_{\epsilon \to 0} \inf_{x \in \Lambda_r} \mathbb{P} \left( |X^\epsilon_x - X^0_x|_{C([0,T_2];E)} < r \right) = 1.$$ (3.7)

By (2.19), there exists $T_3$ such that $\sup_{x \in G \setminus \Lambda_r} |X^0_x(T_3)|_E < \frac{d}{7}$. By the Markov property,

$$\inf_{x \in G \setminus \Lambda_r} \mathbb{P}(\tau^\epsilon_x \leq T_1 + T_3) \geq \inf_{x \in G \setminus \Lambda_r} \mathbb{P} \left( |X^\epsilon_x(T_3)| < \rho \text{ and } \tau^\epsilon_x(T_3) \leq T_1 \right) \geq \left( \inf_{x \in G \setminus \Lambda_r} \mathbb{P} \left( |X^\epsilon_x - X^0_x|_{C([0,T_3])} < \frac{d}{7} \right) \right) \left( \inf_{|x|_E < \rho} \mathbb{P}(\tau^\epsilon_x \leq T_1) \right)$$

which means that, by (3.6),

$$\liminf_{\epsilon \to 0} \epsilon \log \left( \inf_{x \in G \setminus \Lambda_r} \mathbb{P}(\tau^\epsilon_x \leq T_1 + T_3) \right) \geq -(V(\partial G) + \eta).$$

Therefore, by the above equation and (3.7),

$$\lim_{\epsilon \to 0} \epsilon \log \left( \inf_{x \in G} \mathbb{P}(\tau^\epsilon_x \leq T_1 + T_2 + T_3) \right) \geq -(V(\partial G) + \eta).$$
Proof. Suppose by contradiction there exist sequences \( \{T_n\} \subset \mathbb{R} \), \( \{x_n\} \subset E \), \( \{\psi_n\} \subset L^2([0,T_n];H) \), such that
\[
\lim_{n \to 0} |x_n|_E = 0, \quad \text{dist}_E(X_{x_n}^\psi(T_n), N) < |x_n|, \quad \text{and} \quad I_{0,T_n}(X_{x_n}^\psi) \leq \frac{1}{2} |\psi_n|^2_{L^2([0,T];H)} \leq \nu.
\]
Then we define the processes on \((-\infty, 0)\)
\[
\varphi_n(t) = \begin{cases} 
0 & \text{if } t \leq -T_n \\
X_{x_n}^\psi(t + T_n) & \text{if } -T_n < t \leq 0
\end{cases}
\]
Notice that
\[
I_{-\infty,0}(\varphi_n) = I_{0,T}(X_0^\psi) \leq \nu
\]
By Hypothesis 5, \( I_{-\infty,0} \) has compact level sets. Therefore, because \( I_{-\infty,0}(\varphi_n) \leq \nu \) for all \( n \), a subsequence of \( \varphi_n \) (which we relabel as \( \varphi_n \)) converges to a limit \( \varphi \) with \( I_{-\infty,0}(\varphi) \leq \nu \).

We also notice that by Hypothesis 2,
\[
\text{dist}_E(\varphi_n(0), N) \leq |X_{x_n}^\psi(T_n) - X_{x_0}^\psi(T_n)|_E + \text{dist}_E(X_{x_n}^\psi(T_n), N) \leq (\kappa_2(\nu) + 1)|x_n|_E
\]
Therefore, because \( N \) is closed and \( x_n \to 0 \), it follows that \( \lim_{n \to \infty} \varphi_n(0) = \varphi(0) \in N \). This is a contradiction because \( V(N) \leq I_{-\infty,0}(\varphi) \leq \nu < V(N) \). \( \square \)

Let \( \Gamma_\rho = \{ x \in E : |x|_E = 2M\rho \} \), where \( M \) is the constant from (2.17), and let \( \gamma_\rho = \{ x \in E : |x|_E \leq \rho \} \). Let \( \tau_{1,x}^\epsilon = \inf\{ t > 0 : X_x^\psi(t) \in \gamma_\rho \cup \partial G \} \).

Lemma 3.7. For any \( \rho > 0 \), such that \( \gamma_\rho \subset G \),
\[
\limsup_{t \to +\infty} \limsup_{\epsilon \to 0} \log \left( \sup_{x \in G} \mathbb{P} \left( \tau_{1,x}^\epsilon \geq t \right) \right) = -\infty. \tag{3.8}
\]

Proof. By Hypothesis 6, there exists \( T > 0 \) such that for any \( x \in G \), \( X_0^\psi \) either leaves \( G_r \) or enters \( \gamma_{\rho/4} \) before time \( T \). Without loss of generality, assume that \( \frac{\rho}{2} < r \). Then this means that if \( X_x^\psi \) is a controlled trajectory with the property that \( X_x^\psi(t) \in G_{r/2} \setminus \gamma_{\rho/2} \) for all \( t \in [0,T] \), then,
\[
\frac{\rho}{4} \leq \left| X_x^\psi - X_0^\psi \right|_{C([0,T];E)}.
\]
By (2.3) and the Cauchy-Schwartz inequality, there must exist some \( c := c \left( \sup_{x \in G} |x|_E, T, \frac{\rho}{4} \right) > 0 \) such that
\[
\frac{\rho}{4} \leq \left| X_x^\psi - X_0^\psi \right|_{C([0,T];E)} \implies c \leq |\psi|_{L^1([0,T];H)} \leq \sqrt{T} |\psi|^2_{L^2([0,T];H)}.
\]

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From these observations, it follows that if $X^\psi_x$ is a control system such that $X^\psi_x(t) \in G_{r/2} \setminus \gamma_{\rho/2}$ for all $t \in [0, T]$, then

$$I_{0,T}(X^\psi_x) = \frac{1}{2} |\psi|_{L^2([0,T];H)}^2 \geq \frac{\epsilon^2}{2T} := a > 0.$$ 

Another way to say this is

$$K_{0,T}(a) \subseteq \{ \varphi \in C([0,T];E) : \varphi(t) \notin G_{r/2} \setminus \gamma_{\rho/2} \text{ for some } t \in [0, T] \}$$

where $K_{0,T}$ is the level set defined by $\text{(2.7)}$. Then, if $\varphi$ is a trajectory such that $\varphi(t) \in G \setminus \gamma_{\rho}$ for all $t \in [0, T]$, 

$$\text{dist}_{C([0,T];E)}(\varphi, K_{0,T}(a)) > \frac{\rho}{2}.$$ 

Because the event

$$\{ \tau_{1,x} \geq T \} \subseteq \{ X^\epsilon_x(t) \in G \setminus \gamma_{\rho}, \text{ for all } t \in [0, T] \} \subseteq \{ \text{dist}_{C([0,T];E)}(X^\epsilon_x, K_{0,T}(a)) > \frac{\rho}{4} \},$$

by the large deviations principle,

$$\limsup_{\epsilon \to 0} \epsilon \log \left( \sup_{x \in G} \mathbb{P} \left( \tau_{1,x}^\epsilon \geq T \right) \right) \leq \limsup_{\epsilon \to 0} \epsilon \log \left( \sup_{x \in G} \mathbb{P} \left( \text{dist}_{C([0,T];E)}(X^\epsilon_x, K_{0,T}(a)) > \frac{\rho}{4} \right) \right) \leq -a.$$

By the Markov property, for any $k \in \mathbb{N}$,

$$\sup_{x \in G} \mathbb{P} \left( \tau_{1,x}^\epsilon \geq kT \right) \leq \left( \sup_{x \in G} \mathbb{P} \left( \tau_{1,x}^\epsilon \geq T \right) \right)^k$$

and therefore,

$$\limsup_{\epsilon \to 0} \epsilon \log \left( \sup_{x \in G} \mathbb{P} \left( \tau_{1,x}^\epsilon \geq kT \right) \right) \leq -ka.$$ 

Our result follows because we can choose $k$ to be arbitrarily large.

**Lemma 3.8.** Let $N \subseteq \partial G$ be closed. Then,

$$\limsup_{\rho \to 0} \limsup_{\epsilon \to 0} \epsilon \log \left( \sup_{x \in \Gamma_{\rho}} \mathbb{P} \left( X^\epsilon_x(\tau_{1,x}^\epsilon) \in N \right) \right) \leq -V(N). \quad (3.9)$$

This lemma remains true if $V(N) = +\infty$.

**Proof.** Let $\nu_0 < V(N)$. Let $\rho_0$ be the number from Lemma 3.6 for $\nu_0$, and choose $\rho < \frac{\rho_0}{2M}$. Then, if $\varphi \in C([0,T];E)$ with $\varphi(0) \in \Gamma_{\rho}$ and $I_{0,T}(\varphi) \leq \nu_0$, it follows from Lemma 3.6 that

$$\inf_{t \in [0,T]} \text{dist}_E(\varphi(t), N) > |\varphi(0)| = 2M \rho.$$

Therefore, for any $T > 0$, the event

$$\{ X^\epsilon_x(\tau_{1,x}^\epsilon) \in N, \tau_{1,x}^\epsilon \leq T \} \subset \{ X^\epsilon_x(t) \in N \text{ for some } t \leq T \} \subset \{ \text{dist}_{C([0,T];E)}(X^\epsilon_x, K_{0,T}(\nu_0)) > 2M \rho \}.$$ 

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By the large deviations principle, for any $T > 0$,
\[
\limsup_{\epsilon \to 0} \epsilon \log \left( \sup_{x \in \Gamma_\rho} \mathbb{P} \left( X^\epsilon_x(\tau_{1,x}^\rho) \in N, \tau_{1,x}^\rho \leq T \right) \right) \\
\leq \limsup_{\epsilon \to 0} \epsilon \log \left( \sup_{x \in \Gamma_\rho} \mathbb{P} \left( \text{dist}_{C([0,T];E)}(X^\epsilon_x, K_{0,T}(\nu_0)) > 2M\rho \right) \right) \\
\leq -\nu_0.
\]

Furthermore, by Lemma 3.7, we can find $T$ large enough so that
\[
\limsup_{\epsilon \to 0} \epsilon \log \left( \sup_{x \in \Gamma_\rho} \mathbb{P} \left( \tau_{1,x}^\rho \geq T \right) \right) \leq -V(N).
\]

Then we observe that
\[
\mathbb{P} \left( X^\epsilon_x(\tau_{1,x}^\rho) \in N \right) \leq \mathbb{P} \left( X^\epsilon_x(\tau_{1,x}^\rho) \in N, \tau_{1,x}^\rho \leq T \right) + \mathbb{P} \left( \tau_{1,x}^\rho \geq T \right)
\]
and therefore,
\[
\limsup_{\epsilon \to 0} \epsilon \log \left( \sup_{x \in \Gamma_\rho} \mathbb{P} \left( X^\epsilon_x(\tau_{1,x}^\rho) \in N \right) \right) \leq -\nu_0.
\]

The result follows because $\nu_0 < V(N)$ was arbitrary. \qed

**Lemma 3.9.** For any $\rho > 0$ such that $\gamma_\rho \subset G$, and any $x \in G$ such that $X^0_x(t) \to 0$ without leaving $G$,
\[
\lim_{\epsilon \to 0} \mathbb{P} \left( X^\epsilon_x(\tau_{1,x}^\rho) \in \gamma_\rho \right) = 1. \tag{3.10}
\]

**Proof.** Let $\rho_0 < \min \{ \frac{\rho}{2}, \inf_{t > 0} \text{dist}_E(X^\epsilon_x(t), \bar{G}^\rho) \}$. Let $X^0_x$ be the unperturbed trajectory starting at $x \in G$. Because $X^0_x \to 0$, we can find $T > 0$ such that $|X^0_x(T)|_E < \frac{\rho}{2}$. Then, because $X^\epsilon_x \to X^0_x$ uniformly on finite time intervals,
\[
\mathbb{P} \left( X^\epsilon_x(\tau_{1,x}^\rho) \in \gamma_\rho \right) \geq \mathbb{P} \left( |X^\epsilon_x - X^0_x|_{C([0,T];E)} < \rho_0 \right) \to 1.
\]
\qed

**Lemma 3.10.** For fixed $\rho > 0$,
\[
\lim_{T \to 0} \limsup_{\epsilon \to 0} \epsilon \log \left( \sup_{x \in \Gamma_\rho} \mathbb{P} \left( \text{There exists } t \in [0,T], X^\epsilon_x(t) \in \Gamma_\rho \right) \right) = -\infty.
\]

**Proof.** By (2.17), the unperturbed system has the property that
\[
\sup_{x \in \Gamma_\rho} |X^0_x(t)|_E \leq M\rho.
\]

If we set
\[
\Phi_T = \left\{ \varphi \in C([0,T];E) : \varphi(0) \in \gamma_\rho, \text{ and there exists } t \in [0,T] \text{ such that } |\varphi(t)|_E = \frac{3M\rho}{2} \right\}
\]

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and let

\[ a(T) := \inf_{\varphi \in \Phi_T} I_{0,T}(\varphi) \]

then because \( \Gamma_\rho = \{ x \in E : |x|_E = 2M\rho \} \), it follows that

\[
\begin{align*}
\limsup_{\epsilon \to 0} \epsilon \log \left( \sup_{x \in \gamma_\rho} \mathbb{P} \left( \text{there exists } t_0 \in [0,t] \text{ such that } X^\epsilon_x(t_0) \in \Gamma_\rho \right) \right)
&\leq \limsup_{\epsilon \to 0} \epsilon \log \left( \sup_{x \in \gamma_\rho} \mathbb{P} \left( \text{dist}_{C([0,T];E)}(X^\epsilon_x,K_0,T(a(T))) > \frac{M\rho}{2} \right) \right) \\
&\leq -a(T).
\end{align*}
\]

Therefore, it remains to show that

\[
\lim_{T \to 0} a(T) = +\infty.
\]

If \( X^\psi_x \in \Phi_T \), then,

\[
\frac{3M\rho}{2} \leq |X^\psi_x|_{C([0,T];E)} \leq |X^\psi_x - X^0_x|_{C([0,T];E)} + |X^0_x|_{C([0,T];E)} \leq |X^\psi_x - X^0_x|_{C([0,T];E)} + M\rho.
\]

Therefore,

\[
\frac{M\rho}{2} \leq |X^\psi_x - X^0_x|_{C([0,T];E)}.
\]

By (2.5), there must exist some \( c := c(|x|_E,M\rho) > 0 \) such that for \( T \leq 1 \),

\[
\frac{M\rho}{2} \leq |X^\psi_x - X^0_x|_{C([0,T];E)} \implies c \leq |\psi|_{L^1([0,T];H)}.
\]

Then, because \( |\psi|_{L^1([0,T];H)} \leq \sqrt{T}|\psi|_{L^2([0,T];H)} \), we see that if \( X^\psi_x \in \Phi_T \), then

\[
\frac{1}{2} |\psi|_{L^2([0,T];H)}^2 \geq \frac{c^2}{2T}.
\]

This implies that

\[
\lim_{T \to 0} a(T) = +\infty
\]

which is what we were trying to show.

\[ \square \]

**Proof of Theorem 3.2.** The following proofs are based on the proofs in [8]. For completeness, they are included below.

**Upper Bound**

By the Markov property, for fixed \( \epsilon > 0 \) and \( T > 0 \),

\[
\sup_{x \in G} \mathbb{P}(\tau^\epsilon_x \geq kT) \leq \left( \sup_{x \in G} \mathbb{P}(\tau^\epsilon_x \geq T) \right)^k.
\]

It follows that

\[
\begin{align*}
E(\tau^\epsilon_x) &\leq T \sum_{k=1}^\infty \mathbb{P}(\tau^\epsilon_x \geq kT) \\
&\leq T \sum_{k=1}^\infty \left( \sup_{x \in G} \mathbb{P}(\tau^\epsilon_x \geq T) \right)^k \\
&\leq \frac{1}{1 - \left( \sup_{x \in G} \mathbb{P}(\tau^\epsilon_x \geq T) \right)} \leq \inf_{x \in G} \mathbb{P}(\tau^\epsilon_x < T).
\end{align*}
\]

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Fix $\eta > 0$. By Lemma 3.3, we can find a $T > 0$ such that

$$\liminf_{\epsilon \to 0} \epsilon \log \left( \inf_{x \in G} \mathbb{P}(\tau_x^\epsilon < T) \right) > -\left( V(\partial G) + \frac{\eta}{2} \right).$$

Therefore,

$$\limsup_{\epsilon \to 0} \epsilon \log \mathbb{E} \tau_x^\epsilon \leq \limsup_{\epsilon \to 0} \epsilon \log \left( \frac{1}{\inf_{x \in G} \mathbb{P}(\tau_x^\epsilon < T)} \right) \leq (V(\partial G) + \eta)$$

and because $\eta > 0$ was arbitrary, the upper bound of (3.2) follows. The upper bound of (3.3) follows by a straightforward application of Chebyshev’s inequality.

**Lower Bound**

Let $\gamma_\rho = \{ x \in E : |x|_E \leq \rho \}$ and $\Gamma_\rho = \{ x \in E : |x|_E = 2M\rho \}$. Define the stopping times

$$\tau_{1,x}^\epsilon = \inf\{ t > 0 : X_x^\epsilon(t) \in \gamma_\rho \cup \partial G \},$$

$$\tau_{n+1,x}^\epsilon = \inf\{ t > \tau_{n,x}^\epsilon : X_x^\epsilon(t) \in \Gamma_\rho \},$$

$$\tau_{n+1,x}^\epsilon = \inf\{ t > \tau_{n,x}^\epsilon : X_x^\epsilon(t) \in \gamma_\rho \cup \partial G \}.$$ (3.11)

Let $\eta > 0$. Using Lemma 3.3, find $\rho > 0$ small enough so that

$$\limsup_{\epsilon \to 0} \epsilon \log \left( \sup_{x \in \Gamma_\rho} \mathbb{P}(X_x^\epsilon(\tau_{1,x}^\epsilon) \in \partial G) \right) < -V(\partial G) + \frac{\eta}{2}.$$

Notice that for $m \geq 2$, by the Markov property,

$$\sup_{x \in G} \mathbb{P}(\tau_x^\epsilon = \tau_{m,x}^\epsilon) \leq \left( \sup_{x \in G} \mathbb{P}(\tau_{\neq}^\epsilon \in [1, m-1]) \right) \left( \sup_{x \in \Gamma_\rho} \mathbb{P} \left( \tau_{1,x}^\epsilon \in \partial G \right) \right) \leq \sup_{x \in \Gamma_\rho} \mathbb{P}(\tau_{1,x}^\epsilon \in \partial G).$$

Next, using Lemma 3.10, we find $T_0 > 0$ small enough so that

$$\limsup_{\epsilon \to 0} \epsilon \log \left( \sup_{x \in \gamma_\rho} \mathbb{P}(\text{There exists } t \in [0, T_0], X_x^\epsilon(t) \in \Gamma_\rho) \right) \leq -V(\partial G).$$

A consequence of this is that for any $n \in \mathbb{N}$

$$\limsup_{\epsilon \to 0} \epsilon \log \mathbb{P}(\tau_{n+1,x}^\epsilon - \tau_{n,x}^\epsilon \leq T_0) \leq -V(\partial G).$$

In [8], they observe that the event $\{ \tau_x^\epsilon \leq kT_0 \}$ implies either that $\{ \tau_x^\epsilon = \tau_{m,x}^\epsilon \}$ for some $m \leq k + 1$ or that at least one of the excursion times, $\tau_{m+1,x}^\epsilon - \tau_{m,x}^\epsilon \leq T_0$. Therefore, for any $k \in \mathbb{N}$, $x \in G$, and small enough $\epsilon$,

$$\mathbb{P}(\tau_x^\epsilon \leq kT_0) \leq \sum_{m=1}^{k+1} \left( \mathbb{P}(\tau_x^\epsilon = \tau_{m,x}^\epsilon) + \mathbb{P}(\tau_{m+1,x}^\epsilon - \tau_{m,x}^\epsilon \leq T_0) \right) \leq \mathbb{P}(\tau_x^\epsilon = \tau_{1,x}^\epsilon) + 2ke^{-\frac{1}{4}(V(\partial G) - \eta)}$$

If we set $k = \left[ e^{\frac{1}{4}(V(\partial G) - \eta)} T_0 \right] + 1$ then we see that for small enough $\epsilon > 0$,

$$\mathbb{P}(\tau_x^\epsilon \leq e^{\frac{1}{4}(V(\partial G) - \eta)}) \leq \mathbb{P}(\tau_{1,x}^\epsilon \in \partial G) + \frac{4}{T_0} e^{-\frac{\eta}{4x}}.$$
We apply Lemma 3.9 to see that the above quantity converges to 0 as $\epsilon \to 0$. We have proven the lower bound for (3.3). The lower bound for (3.2) follows by a straightforward application of Chebyshev inequality.

**Exit Place**

The proof of the exit place is very similar to the proof of the lower bound of the exit time. Let $N \subset \partial G$ be closed and have the property that $V(N) > V(\partial G)$. Let $0 < \eta < \frac{1}{3}(V(N) - V(\partial G))$. Using Lemma 3.8, we find $\rho$ small enough so that

$$\limsup_{\epsilon \to 0} \epsilon \log \left( \sup_{x \in \Gamma, \rho} \mathbb{P}(X^\epsilon_x(\tau^\epsilon_{1,x}) \in N) \right) \leq - \left( V(N) - \frac{\eta}{2} \right).$$

Next, using Lemma 3.10 we choose $T_0$ small enough so that

$$\limsup_{\epsilon \to 0} \epsilon \log \left( \sup_{x \in \partial G} \mathbb{P}(\text{There exists } t \in [0, T_0], X^\epsilon_x(t) \in \Gamma, \rho) \right) \leq - \left( V(N) - \frac{\eta}{2} \right).$$

Using the same stopping times defined in (3.11), we observe that for $x \in G$ with the property that the unperturbed system starting at $x$, $X^0_x(t)$, never leaves $G$, for $k$ to be chosen later, and for small enough $\epsilon$,

$$\mathbb{P}(X^\epsilon_x(\tau^\epsilon_x) \in N) \leq \mathbb{P}(\tau^\epsilon_x > \tau^\epsilon_{k,x}) + \sum_{m=1}^{k} \mathbb{P}(\tau^\epsilon_x \geq \tau^\epsilon_{m,x}) \mathbb{P}(X^\epsilon_x(\tau^\epsilon_{m,x}) \in N | \tau^\epsilon_x \geq \tau^\epsilon_{m,x})$$

$$\leq \mathbb{P}(\tau^\epsilon_x > (k-1)T_0) + \mathbb{P}(\tau^\epsilon_{k,x} \leq (k-1)T_0) + \mathbb{P}(X^\epsilon_x(\tau^\epsilon_{1,x}) \in N) + \sum_{m=2}^{k} \mathbb{P}(X^\epsilon_{m,x}(\tau^\epsilon_{1,x}) \in N)$$

$$\leq \mathbb{P}(\tau^\epsilon_x > (k-1)T_0) + 2\mathbb{P}(X^\epsilon_x(\tau^\epsilon_{1,x}) \in N) + 2ke^{-\frac{1}{2}(V(N)-\eta)}.$$

In the last line, we used the fact that

$$\mathbb{P}(\tau^\epsilon_{k,x} \leq (k-1)T_0) \leq \sum_{m=2}^{k} \mathbb{P}(\text{There exists } t \in [0, T_0], X^\epsilon_x(t) \in \Gamma, \rho).$$

This is because if $\tau^\epsilon_{k,x} \leq kT_0$, then at least one of $\tau^\epsilon_{m,x} - \tau^\epsilon_{m-1,x}$, $m = 2..k$ must be less than $T_0$. By (3.2) and Chebyshev inequality, for small enough $\epsilon$,

$$\mathbb{P}(\tau^\epsilon_x > (k-1)T_0) \leq \frac{e^{\frac{1}{2}(V(\partial G)+\eta)}}{(k-1)T_0}.$$

Then, we choose $k = \lceil e^{\frac{1}{2}(V(\partial G)+2\eta)} \rceil$. Because $V(N) - V(\partial G) > 3\eta$,

$$2ke^{-\frac{1}{2}(V(N)-\eta)} \leq 2e^{\frac{1}{2}(V(\partial G)-V(N)+3\eta)} \to 0$$

and we are left with

$$\limsup_{\epsilon \to 0} \mathbb{P}(X^\epsilon_x(\tau^\epsilon_x) \in N) \leq 2 \limsup_{\epsilon \to 0} \mathbb{P}(X^\epsilon_x(\tau^\epsilon_{1,x}) \in N).$$

Finally, by Lemma 3.9

$$\lim_{\epsilon \to 0} \mathbb{P}(X^\epsilon_x(\tau^\epsilon_{1,x}) \in N) = 0$$

and our result follows. \qed
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