MULTISYMPLECTIC FORMULATION OF LAGRANGIAN MODELS IN GRAVITATION

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Abstract

We apply the multisymplectic formulation of classical field theories \[11, 12, 14\] to describe the Einstein-Hilbert and the Einstein-Palatini (or metric-affine) Lagrangian models of General Relativity.

Key words: Classical field theories, jet bundles, multisymplectic forms, Hilbert-Einstein action, Einstein-Palatini action, constraints.

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1 Introduction

The geometrization of the theory of gravity (General Relativity (GR)) and, in particular, the multisymplectic framework, allows us to understand several inherent characteristics of it. It is studied by different authors, such as \[1, 2, 3, 7, 8, 9, 10, 13, 15\].

We present the main Lagrangian models for GR using the multisymplectic framework: first the Einstein-Hilbert model which is described by a 2nd-order singular Lagrangian (and so GR is formulated as a higher-order premultisymplectic field theory with constraints), and second the Einstein-Palatini (metric-affine) model described by a 1st-order singular Lagrangian (and so GR is formulated as a 1st-order premultisymplectic field theory with constraints).

2 Geometric structures: Jet bundles and multivector fields

First we introduce some fundamental geometrical tools which are used in the exposition.

Let \( \pi: E \rightarrow M \) be a fiber bundle (with adapted coordinates \((x^\mu, y^i)\)). A section of \( \pi \) is a map \( \phi: U \subset M \rightarrow E \) such that \( \pi \circ \phi = Id_M \). The set of sections is denoted \( \Gamma(\pi) \). Two sections \( \phi_1, \phi_2 \in \Gamma(\pi) \) are k-equivalent at \( x \in M \) if \( \phi_1(x) = \phi_2(x) \) and their partial derivatives until order k at \( x \) are equal. This is an equivalence relation in \( \Gamma_x(\pi) \) and each equivalence class is a jet field at \( x \); denoted \( j^k\phi_x \). The kth-order jet bundle of \( \pi \) is the set \( J^k\pi := \{ j^k_x\phi \mid x \in M, \phi \in \Gamma_x(\pi) \} \). Natural projections are:

\[ \pi^k_r: J^k\pi \rightarrow J^r\pi \quad (r < k) \quad , \quad \pi^k: J^k\pi \rightarrow E \quad , \quad \pi^k: J^k\pi \rightarrow M \quad . \]
Definition 1: The kth-prolongation of a section $\phi \in \Gamma(\pi)$ to $J^k\pi$ is the section $J^k\phi \in \Gamma(\pi^k)$ defined as $J^k\phi(x) := J^k_x\phi$; $x \in M$. A section $\psi \in \Gamma(\pi^k)$ in $J^k\pi$ is holonomic if $\psi = J^k\phi$; that is, $\psi$ is the kth prolongation of a section $\phi = \pi^k \circ \psi \in \Gamma(\pi)$.

If $\phi = (x, y^i(x))$, then $\psi = J^k\phi = \left(x, y^i(x), \frac{\partial y^i}{\partial x^\mu}(x), \frac{\partial^2 y^i}{\partial x^\mu \partial x^\nu}(x), \ldots\right)$.

Definition 2: An m-multivector field in $J^k\pi$ is a skew-symmetric contravariant tensor of order $m$ in $J^k\pi$. The set of m-multivector fields in $J^k\pi$ is denoted $\mathfrak{X}^m(J^k\pi)$. A multivector field $X \in \mathfrak{X}(J^k\pi)$ is said to be locally decomposable if, for every $p \in J^k\pi$, there is an open neighbourhood $U_p \subset J^k\pi$ and $X_1, \ldots, X_m \in \mathfrak{X}(U_p)$ such that $X|_{U_p} = X_1 \wedge \ldots \wedge X_m$. Locally decomposable m-multivector fields $X \in \mathfrak{X}^m(J^k\pi)$ are locally associated with m-dimensional distributions $D \subset TJ^k\pi$. Then, $X$ is integrable if its associated distribution is integrable. In particular, $X$ is holonomic if it is integrable and its integral sections are holonomic sections of $\pi^k$.

If $\Omega \in \Omega^r(J^k\pi)$ is a differential r-form in $J^k\pi$ and $X \in \mathfrak{X}(J^k\pi)$ is locally decomposable, the contraction between $X$ and $\Omega$ is $i(X)\Omega \mid_U := i(X_1)\ldots i(X_m)\Omega$.

3 Einstein-Hilbert model (without sources)

The configuration bundle for the Einstein-Hilbert model is $\pi: E \to M$, where $M$ is an oriented, connected 4-dimensional manifold representing space-time, with volume form $\omega \in \Omega^4(M)$, and $E$ is the manifold of Lorentzian metrics on $M$. Thus $\dim E = 14$. Adapted fiber coordinates in $E$ are $(x^\alpha, g_{\alpha\beta})$, (with $0 \leq \alpha \leq \beta \leq 3$), where $g_{\alpha\beta}$ are the components of the metric, and such that $\omega = dx^\alpha \wedge dx^\beta \wedge dx^\gamma \wedge dx^\delta \equiv d^4x$. (We also use the notation $d^3x_\mu \equiv i\left(\frac{\partial}{\partial x^\mu}\right)d^4x$.

The Lagrangian formalism is developed in $J^3\pi$, with the induced coordinates denoted as $(x^\mu, g_{\alpha\beta}, g_{\alpha\beta,\mu}, g_{\alpha\beta,\mu\nu}, g_{\alpha\beta,\mu\nu\lambda})$, $(0 \leq \alpha \leq \beta \leq 3; 0 \leq \mu \leq \nu \leq \lambda \leq 3)$. The bundle $J^3\pi$ has some canonical structures; in particular, the total derivatives are

$$D_\tau = \frac{\partial}{\partial x^\tau} + g_{\alpha\beta,\tau} \frac{\partial}{\partial g_{\alpha\beta}} + g_{\alpha\beta,\mu\nu} \frac{\partial}{\partial g_{\alpha\beta,\mu}} + g_{\alpha\beta,\mu\nu\lambda} \frac{\partial}{\partial g_{\alpha\beta,\mu\nu}}$$

The Hilbert-Einstein Lagrangian function (without energy-matter) is

$$L_{EH} = \sqrt{|\det(g_{\alpha\beta})|} R \equiv g_{\alpha\beta} R g^{\alpha\beta} \in C^\infty(J^2\pi);$$

where $R_{\alpha\beta}$ are the Ricci tensor components, $\Gamma^\rho_{\mu\nu}$ are the Christoffel symbols of the Levi-Civita connection of $g$, and $R$ is the scalar curvature (which contains 2nd-order derivatives of $g_{\mu\nu}$). The Hilbert-Einstein Lagrangian density is $L = L d^4x \in \Omega^4(J^3\pi)$, where $L = (\pi^3)^* L_{EH} \in C^\infty(J^3\pi)$. We denote

$$L^{\alpha\beta,\mu\nu} = \frac{1}{n(\mu\nu)} \frac{\partial L}{\partial g_{\alpha\beta,\mu\nu}} = \frac{n(\alpha\beta)}{2} \rho(g^{\alpha\mu} g^{\beta\nu} + g^{\alpha\nu} g^{\beta\mu} - 2g^{\alpha\beta} g^{\mu\nu})$$

$$L^{\alpha\beta,\mu} = \frac{\partial L}{\partial g_{\alpha\beta,\mu}} - \sum_{\nu=0}^3 \frac{1}{n(\mu\nu)} D_\nu \left(\frac{\partial L}{\partial g_{\alpha\beta,\mu\nu}}\right) - \sum_{\nu=0}^3 D_\nu L^{\alpha\beta,\mu\nu}$$

$$H = \sum_{\alpha\beta,\mu\nu,\rho} L^{\alpha\rho,\mu\nu} g_{\alpha\rho,\mu} + \sum_{\alpha\beta,\mu} L^{\alpha\beta,\mu} g_{\alpha\beta,\mu} - L = \rho g_{\alpha\beta,\mu} g_{\lambda\kappa,\nu} H^{\alpha\beta\kappa\lambda\mu\nu}$$

$$H^{\alpha\beta\kappa\lambda\mu\nu} = \frac{1}{4} \rho g^{\alpha\beta} g^{\kappa\lambda} g^{\mu\nu} - \frac{1}{4} g^{\alpha\kappa} g^{\beta\lambda} g^{\mu\nu} + \frac{1}{2} g^{\alpha\kappa} g^{\beta\lambda} g^{\mu\nu} - \frac{1}{2} g^{\alpha\beta} g^{\kappa\lambda} g^{\mu\nu},$$
They define the Lagrangian final constraint submanifold \( S \) particular, J. Gaset, N. Román-Roy: Multisymplectic Lagrangian models in gravitation.  

As \( \psi \) sections are the solutions and it is a premultisymplectic form because \( L \) is a singular Lagrangian.  

The problem stated by the Hamilton variational principle for the system \( (J^3\pi, \Omega_L) \) consists in finding holonomic sections \( \psi_L = j^3\overline{\phi} \in \Gamma(\overline{\pi}^3) \) satisfying any of the following equivalent conditions:  

(a) \( \psi_L \) is a solution to the equation \( \psi_L^* i(X)\Omega_L = 0 \), for every \( X \in \mathcal{X}(J^3\pi) \).  

(b) \( \psi_L \) is an integral section of a holonomic multivector field \( X_L \in \mathcal{X}^4(J^3\pi) \) satisfying the equation \( i(X_L)\Omega_L = 0 \).  

As \( \Omega_L \) is a premultisymplectic form, these field equations have no solution everywhere in \( J^3\pi \). Applying the premultisymplectic constraint algorithm we obtain the following constraints (see [5]):  

\[
L^{\alpha\beta} := -g(n(\alpha\beta))(R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R) = 0 . \tag{1}
\]

\[
D_\tau L^{\alpha\beta} = D_\tau(-g(n(\alpha\beta))(R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R)) = 0 . \tag{2}
\]

They define the Lagrangian final constraint submanifold \( S_f \hookrightarrow J^3\pi \) where solutions exist and, in particular,

\[
X_L = \bigwedge_3 \sum_{\tau=0}^3 \sum_{\alpha \leq \beta \leq \mu \leq \nu \leq \lambda} \left( \frac{\partial}{\partial x^\tau} + g_{\alpha\beta,\tau} \frac{\partial}{\partial g_{\alpha\beta}} + g_{\alpha\beta,\mu\tau} \frac{\partial}{\partial g_{\alpha\beta,\mu}} + g_{\alpha\beta,\mu\nu\tau} \frac{\partial}{\partial g_{\alpha\beta,\mu\nu}} + D_\tau D_\lambda(g_{\lambda\sigma} \frac{\partial}{\partial g_{\alpha\beta,\mu\sigma}} + \tau(\sigma,\mu,\lambda)) \right) \]

is a holonomic multivector field solution to the equation in (b), tangent to \( S_f \). Their integral sections are the solutions \( \psi_L(x) = (x^\mu, g_{\alpha\beta}(x), g_{\alpha\beta,\mu}(x), g_{\alpha\beta,\mu\nu}(x), g_{\alpha\beta,\mu\nu\lambda}(x)) \) to the equation in (a), which gives

\[
g_{\alpha\beta,\mu} - \frac{\partial g_{\alpha\beta}}{\partial x^\mu} = 0 , \tag{3}
\]

\[
g_{\alpha\beta,\mu\nu} - \frac{1}{n(\mu\nu)} \left( \frac{\partial g_{\alpha\beta,\mu}}{\partial x^\nu} + \frac{\partial g_{\alpha\beta,\nu}}{\partial x^\mu} \right) = 0 , \tag{4}
\]

\[
g(n(\alpha\beta))(R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R) = 0 . \tag{5}
\]

In this set of equations, (3) and (4) are (part of the) holonomy conditions; meanwhile (5) are the physical relevant equations, which are the constraints (1) evaluated on the image of sections, \( L^{\alpha\beta}|_{\psi_L} = 0 \), and constitute the Euler-Lagrange equations of the theory; that is, the Einstein equations.

As a consequence of the singularity of \( L \), the form \( \Theta_L \) is \( \pi_1^3 \)-projectable onto a form in \( J^1\pi \) (but it is not the Poincaré-Cartan form of any 1st-order Lagrangian). Then, Einstein equations are 2nd-order PDE’s, instead of 4th-order as it correspond to a 2nd-order Lagrangian. So they are defined as a submanifold of \( J^3\pi \) (and appear as constraints).

The constraints (2) are of geometrical nature and arise because we are using a manifold prepared for a theory of a 2nd-order Lagrangian that, really, is physically equivalent to a 1st-order Lagrangian. These constraints hold automatically when they are evaluated on the image of the sections \( \psi_L \) which are solutions to the Einstein equations. Furthermore, the Einstein-Hilbert model is a gauge theory (because \( L_{EH} \) is singular). Then, the constraints (1) and (2) fix partially the gauge. To remove the remaining gauge degrees of freedom leads to a submanifold of \( S_f \) diffeomorphic to \( J^1\pi \).
4 Einstein-Palatini (metric-affine) model (without sources)

The configuration bundle of the Einstein-Palatini (metric-affine) model is $\Pi: \mathcal{E} \rightarrow M$, where $\mathcal{E} = E \times_M C(LM)$, where $E$ is the manifold of Lorentzian metrics on $M$ and $C(LM)$ is the manifold of linear connections in $TM$. Adapted fiber coordinates in $E$ are $(x^\mu, g_{\alpha\beta}, \Gamma^\gamma_{\alpha\beta})$, $(0 \leq \alpha \leq \beta \leq 3)$, and the induced coordinates in $J^1\Pi$ are $(x^\mu, g_{\alpha\beta}, \Gamma^\gamma_{\alpha\beta}, g_{\alpha\beta\mu}, \Gamma^\lambda_{\alpha\beta\mu})$. Thus $\dim E = 78$ and $\dim J^1\Pi = 374$.

The Einstein-Palatini Lagrangian (without energy-matter) is a singular 1st-order Lagrangian depending on the components of the metric $g$ and of a connection $\Gamma$,

$$L_{EP} = g^{\alpha\beta} R_{\alpha\beta} = g^{\alpha\beta} (\Gamma^\gamma_{\beta\alpha \gamma} - \Gamma^\gamma_{\alpha\beta \gamma} + \Gamma^\gamma_{\gamma\alpha\beta} - \Gamma^\gamma_{\beta\gamma\alpha}) \in C^\infty (J^1\Pi).$$

The Lagrangian density is $\mathcal{L} = L_{EP} \, dx$ in $\Omega^4(\mathcal{J}^1\Pi)$, and its Poincaré-Cartan 5-form is

$$\Omega_\mathcal{L} = d\left( \frac{\partial L_{EP}}{\partial \Gamma^\alpha_{\beta\gamma\mu}} \Gamma^\alpha_{\beta\gamma\mu} - L_{EP} \right) \wedge dx - d\frac{\partial L_{EP}}{\partial \Gamma^\alpha_{\beta\gamma\mu}} \wedge d\Gamma^\alpha_{\beta\gamma} \wedge d^3x \in \Omega^5(\mathcal{J}^1\Pi),$$

which is a premultisymplectic form since $L_{EP}$ is also a singular Lagrangian.

The Lagrangian problem for the system $(J^1\Pi, \Omega_\mathcal{L})$ consists in finding holonomic sections $\psi_\mathcal{L} = j^1 \phi \in \Gamma(\mathcal{P}^1)$ ($\phi \in \Gamma(\Pi)$) satisfying any of the following equivalent conditions:

(a) $\psi_\mathcal{L}$ is a solution to the equation $\psi_\mathcal{L}^* i(X) \Omega_\mathcal{L} = 0$, for every $X \in \mathcal{X}(J^1\Pi)$.

(b) $\psi_\mathcal{L}$ is an integral section of a holonomic multivector field $X_\mathcal{L} \in \mathcal{X}^4(J^1\Pi)$ satisfying the equation $i(X_\mathcal{L}) \Omega_\mathcal{L} = 0$.

Now, the premultisymplectic constraint algorithm leads to the constraints (see [6]):

$$0 = \frac{\partial H}{\partial g_{\mu\nu}} - \frac{\partial L_{EP}^\beta\gamma\sigma}{\partial g_{\mu\nu}} \Gamma^\alpha_{\beta\gamma\sigma},$$

$$0 = g_{\rho\sigma, \mu} - g_{\sigma\lambda} \Gamma^\lambda_{\mu\rho} - g_{\rho\lambda} \Gamma^\lambda_{\gamma\mu} - \frac{2}{3} g_{\rho\sigma} T^\lambda_{\gamma\mu},$$

$$0 = T^\alpha_{\beta\gamma} - \frac{1}{3} \delta^\alpha_{\beta} T^\mu_{\mu\gamma} + \frac{1}{3} \delta^\alpha_{\gamma} T^\mu_{\mu\beta},$$

$$0 = T^\alpha_{\beta\gamma, \nu} - \frac{1}{3} \delta^\alpha_{\beta} T^\mu_{\mu\gamma, \nu} + \frac{1}{3} \delta^\alpha_{\gamma} T^\mu_{\mu\beta, \nu},$$

$$0 = g_{\rho\sigma} \Gamma^\gamma_{[\nu\lambda]} \Gamma^\lambda_{\mu\sigma} + g_{\sigma\gamma} \Gamma^\gamma_{[\nu\lambda]} \Gamma^\lambda_{\mu\sigma} + g_{\rho\lambda} \Gamma^\lambda_{[\nu\sigma, \beta]} + g_{\sigma\lambda} \Gamma^\lambda_{[\mu, \nu, \beta]} + \frac{2}{3} g_{\rho\sigma} T^\lambda_{[\mu, \nu, \beta]}.$$

where $T^\alpha_{\beta\gamma} \equiv \Gamma^\alpha_{\beta\gamma} - \Gamma^\alpha_{\gamma\beta}$. They define the submanifold $S_f \rightarrow J^1\Pi$, where there are holonomic multivector fields solution to the equations in (b), tangent to $S_f$.

A consequence of the singularity of $\mathcal{L}$ is that $\Omega_\mathcal{L}$ is $\Pi^1$-projectable onto a form in $\mathcal{E}$ and then, the Euler-Lagrange equations (Einstein’s eqs.) are 1st-order PDE’s, instead of 2nd-order. So they are defined as a submanifold of $J^1\pi$, and appear as constraints [6]. On the other hand, the equalities (7) are related to the metricity condition for the Levi-Civita connection and they are called premetricity constraints. Furthermore, there are the torsion constraints that impose conditions on the torsion of the connection (3) and on their derivatives (9). Finally, the additional integrability constraints (10) appear as a consequence of demanding the integrability of the multivector fields which are solutions to the equations in (b).

The Einstein-Palatini model is a gauge theory (as $\mathcal{L}$ is singular) with higher gauge freedom than in the Einstein-Hilbert model. The above constraints fix partially the gauge. To remove the remaining gauge degrees of freedom leads to a submanifold of $S_f$ diffeomorphic to $J^1\pi$ in the Einstein-Hilbert model. The conditions of the connection to be torsionless and metric (which allows us to recover the Einstein-Hilbert model from the Einstein-Palatini model) are a consequence of the constraints and a partial fixing of this gauge freedom [4].
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