Forces between Kinks and Antikinks with Long-range Tails

N. S. Manton

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, U.K.

Abstract

In a scalar field theory with a symmetric octic potential having a quartic minimum and two quadratic minima, kink solutions have long-range tails. We calculate the force between two kinks and between a kink and an antikink when their long-range tails overlap. This is a nonlinear problem, solved using an adiabatic ansatz for the accelerating kinks that leads to a modified, first-order Bogomolny equation. We find that the kink-kink force is repulsive and decays with the fourth power of the kink separation. The kink-antikink force is attractive and decays similarly. Remarkably, the kink-kink repulsion has four times the strength of the kink-antikink attraction.

Keywords: Kinks, Long-range tail, Scalar field, Force

email: N.S.Manton@damtp.cam.ac.uk
1 Introduction

Consider a Lorentz-invariant scalar field theory in 1 + 1 dimensions with one real field $\phi(x, t)$ and Lagrangian density

$$
\mathcal{L} = \frac{1}{2} \left( \frac{\partial \phi}{\partial t} \right)^2 - \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 - V(\phi) \, .
$$

(1)

Its field equation is

$$
\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + \frac{dV}{d\phi} = 0 \, .
$$

(2)

Assume that the potential $V(\phi)$ is smooth and non-negative, and attains its global minimum value $V = 0$ for one or more values of $\phi$. These field values are vacua of the theory, and if $V$ has more than one vacuum, then it has static kink solutions interpolating between them as $x$ increases from $-\infty$ to $\infty$. Much is known about kinks for generic $V$ of this type, and also in particular examples [1, 2, 3]. The classic kink occurs for the quartic potential $V(\phi) = \frac{1}{2}(1 - \phi^2)^2$. The solution $\phi(x) = \tanh(x)$ connects the vacua $\phi = -1$ and $\phi = 1$, and there is an antikink going the other way. If the theory is extended to higher spatial dimensions, then the kink becomes a domain wall, but we will not consider this generalisation further.

If $V$ has a finite or infinite sequence of vacua $\phi_n$ (in increasing order) then there is a kink connecting $\phi_n$ to $\phi_{n+1}$ and an antikink connecting $\phi_{n+1}$ to $\phi_n$ for each $n$, but there are no kinks connecting vacua whose indices differ by more than one [1]. Field configurations connecting non-adjacent vacua can be interpreted as nonlinear superpositions of more than one kink, but these configurations are never static. Physically, the kinks repel each other and separate. Field configurations connecting $\phi_n$ to close to $\phi_{n+1}$ and then back to $\phi_n$ can be regarded as kink-antikink pairs. These are not static either, as a kink and antikink attract each other. Note that the assumption that the kinks and antikinks are interpolating between vacua is important. It is possible for a kink and antikink to be in static equilibrium at a finite separation if the asymptotic field value $\phi$ is metastable, i.e. a local but not global minimum of $V$ [4].

The field equation for a static kink is a second-order ODE, but it can be reduced to a first-order equation. This is a special case of the Bogomolny trick [5] (although known much earlier in this context). We shall refer to the first-order equation as the Bogomolny equation for the kink, and use it repeatedly, even in the context of kink dynamics.

The minima of a generic potential are all quadratic, with $V$ having positive second derivative. In this case the tails of kinks are short-ranged, i.e.
the field $\phi$ approaches the vacuum values, between which it is interpolating, exponentially fast as $x \to \pm \infty$. However, Lohe [6], and more recently Khare et al. [7] and Bazeia et al. [8], have drawn attention to several examples where at least one minimum is not quadratic. This is not generic, but easily occurs as parameters in $V$ are varied. If one of the global minima of $V$ is quartic (the next simplest case) then one tail of the kink that approaches it is long-ranged; $\phi$ approaches the quartic minimum with a $1/x$ behaviour.

When the tails are short-ranged, then it is possible to calculate the force between two well-separated kinks [2]. This is for the situation where the kink on the left (smaller $x$) interpolates between $\phi_{n-1}$ and $\phi_n$, and the kink on the right interpolates between $\phi_n$ and $\phi_{n+1}$. The calculation relies on a linear superposition of the exponentially small tails in the region between the kinks. The result is a force that decays exponentially fast with the kink separation. An example is discussed in Appendix B.

There are a number of ways of approaching the force calculation. The force on one of the kinks can be found using a version of Noether’s theorem to determine the rate of change of its momentum [9]. This is equivalent to using the energy-momentum tensor to find the stress exerted on the half-line containing the kink. An alternative is to attempt to approximately solve the full, time-dependent field equation. One can make an ansatz describing an accelerating kink, and then match the tail of this to the tail of the other kink [10]. This determines the acceleration. A cruder approach is simply to set up a static field configuration that incorporates both kinks, satisfying the appropriate boundary conditions, and calculate the energy as a function of the separation. The negative of the derivative of this energy with respect to the separation is an estimate of the kink-kink force. Such a static field configuration does not of course satisfy the full, time-dependent field equation (2).

None of these methods is completely straightforward to implement, as each depends on an ansatz for an interpolating field. The methods might not agree. One also needs to know where the centres of the kinks are, to have a precise notion of separation. The centres have to be carefully defined, especially for kinks with an asymmetric profile, and we shall clarify in Appendix A what a good definition is. Because of the extended character of kinks, any formula for the force usually makes sense only to leading order in the separation, even when the separation is large, and subleading terms are meaningless.

The force between two kinks with long-range, $1/x$ tails has apparently not been accurately determined. Certainly, the calculation in [2] breaks down in this case. The force between a kink and antikink with these tails was estimated to decay with the fourth power of the separation by González and
The main purpose of this paper is to establish a similar result for the kink-kink case, and to find the numerical coefficients. We shall apply the different methods outlined above, and show to what extent they give consistent results. Our results show unambiguously that the force decays with the fourth power of the kink separation.

Using the calculated force, we can find an effective equation of motion for the positions of the kinks. Two kinks repel, so they can approach slowly from infinity, instantaneously stop at a large separation, and move out again to infinity. The separation remains large throughout, so the effective equation of motion should be reliable. The dynamics of a kink and antikink is more complicated; they attract and can then annihilate, resonate as an oscillating pair, or scatter.

To test these pictures, it is helpful to perform numerical simulations of the kink-kink and kink-antikink dynamics. Recently, Belendryasova and Gani studied kink-antikink dynamics numerically in the same model as ours, but their initial configuration was simply the sum of the kink and antikink fields. In this configuration, the long-range tail of the kink extends right across the antikink, producing an asymptotic field whose difference from the local vacuum value has the wrong sign. This introduces some unwanted initial energy that converts quickly into radiation, and substantially influences the magnitude and even the sign of the force, as has been clarified by Christov et al. Further refinement of the numerical algorithms is therefore desirable.

## 2 A Model for a Kink with a Long-range Tail

There are an unlimited number of scalar field theories with kinks having long-range tails. Several of them, arising from a variety of polynomial potentials $V$, are discussed in refs. and elsewhere. However, in many cases, the kink solution is not explicit, so the algebra needed to investigate the kink’s properties is complicated. We shall therefore focus on the simplest symmetric potential that admits two related kinks with long-range tails, and their antikinks, and calculate the forces between these. Our results should generalise.

The potential we consider is the octic polynomial

$$V(\phi) = \frac{1}{2} (1 - \phi^2)^2 \phi^4.$$  \hspace{1cm} (3)

This has quadratic minima at $\phi = \pm 1$, and a quartic minimum at $\phi = 0$. The kink interpolates between $\phi = 0$ and $\phi = 1$, and there is a mirror kink that
interpolates between $\phi = -1$ and $\phi = 0$, with the same energy. The antikink interpolates between $\phi = 1$ and $\phi = 0$, and there is an antimirror-kink too.

The potential $V$ can be expressed in terms of a superpotential $W$ as

$$V = \frac{1}{2} \left( \frac{dW}{d\phi} \right)^2 \quad (4)$$

where

$$\frac{dW}{d\phi} = (1 - \phi^2)\phi^2, \quad (5)$$

so

$$W(\phi) = \frac{1}{3}\phi^3 - \frac{1}{5}\phi^5 + \text{const}. \quad (6)$$

The constant has no significance, but it will be convenient to set it to $-\frac{2}{15}$, so that $W(1) = 0$ and $W(0) = -\frac{2}{15}$.

Starting with the Lagrangian density (1), and writing $V$ in terms of $W$, we obtain the energy expression for a static field

$$E = \int_{-\infty}^{\infty} \left( \frac{1}{2} \left( \frac{d\phi}{dx} \right)^2 + \frac{1}{2} \left( \frac{dW}{d\phi} \right)^2 \right) dx. \quad (7)$$

Using the Bogomolny rearrangement \cite{5} (completing the square and integrating the cross term), the energy can be reexpressed as

$$E = \int_{-\infty}^{\infty} \frac{1}{2} \left( \frac{d\phi}{dx} - \frac{dW}{d\phi} \right)^2 dx + W(1) - W(0), \quad (8)$$

where the field is assumed to interpolate between $\phi = 0$ as $x \to -\infty$ and $\phi = 1$ as $x \to \infty$. The kink minimises the energy (for the given boundary conditions) by satisfying the Bogomolny equation

$$\frac{d\phi}{dx} = \frac{dW}{d\phi} = (1 - \phi^2)\phi^2, \quad (9)$$

and its energy is

$$E = W(1) - W(0) = \frac{2}{15}. \quad (10)$$

The Bogomolny equation can be rewritten as

$$\left( \frac{1}{2(1 - \phi)} + \frac{1}{2(1 + \phi)} + \frac{1}{\phi^2} \right) d\phi = dx, \quad (11)$$

whose implicit solution is \cite{6}

$$\frac{1}{2} \log \frac{1 + \phi}{1 - \phi} - \frac{1}{\phi} = x - A, \quad (12)$$
where $A$ is a constant parameter. Inverting, we obtain the kink solution $\phi(x - A)$, which is shown in Fig.1 for $A = 0$. We call $A$ the location of the kink. Another interesting position in the kink is where the potential $V$ has its maximum value. This is where $\phi = \frac{1}{\sqrt{2}}$, because $\frac{d^2W}{d\phi^2}$ is zero here. We call this the centre of the kink. In the present example it occurs at $x_{\text{centre}} = A + \log(1 + \sqrt{2}) - \sqrt{2} \approx A - 0.533$. This is rather awkward to deal with algebraically, so we usually work with the location $A$. The notion of kink centre is further clarified in Appendix A. Nothing physical depends on the distinction between location and centre.

![Figure 1: Kink located at $A = 0$.](image)

We need to know the tail behaviours of the kink. On the left, where $\phi$ is near zero, the Bogomolny equation simplifies to $\frac{d\phi}{dx} = \phi^2$, with tail solution

$$\phi = \frac{1}{A - x}. \quad (13)$$

$A$ is the same constant as before. The next term in the expansion of $\phi$ is cubic in $\frac{1}{A - x}$, which makes the constant $A$ in the leading term unambiguous. (A small quadratic term $\frac{\epsilon}{(A - x)^2}$ could be interpreted as a shift of $A$ by $-\epsilon$ in the
leading term.) $A$ is the location where the extrapolated tail field diverges. On the right, where $\phi = 1 - \eta$ with $\eta$ small and positive, the Bogomolny equation linearises to $\frac{d\eta}{dx} = -2\eta$, with tail solution $\eta = \exp(-2(x - b))$. The constant $b$ equals $A - 1 + \frac{1}{2}\log 2 \simeq A - 0.653$. Our main interest will be in the long-range tail on the left.

The transition between the tail on the left and the asymptotic field value on the right is rather fast, so a crude approximation to the kink with location $A = 0$ is $\phi = -\frac{x}{2}$ for $x \leq -1$ and $\phi = 1$ for $x \geq -1$.

In the Lorentz invariant theory we are considering, the energy $E$ of a static kink is the same as its rest mass $M$. $M$ is of course the conversion factor between force and acceleration for kinks moving non-relativistically. We will also need to consider the energy (mass) of a kink with its long-range tail truncated. Consider therefore the kink with location $A$ on the half-line $X \leq x < \infty$, with $X \ll A$. Using the Bogomolny rearrangement again, we estimate that the tail truncation reduces the energy by

$$E_{\text{tail}} = W(\phi(X)) - W(0) = \frac{1}{3(A - X)^3}, \quad (14)$$

where we have just retained the leading terms in the kink solution, and in $W$.

When $\phi(x - A)$ is the kink solution located at $A$, $-\phi(-x - A)$ is the mirror kink located at $-A$. Note that the mirror kink obeys the same Bogomolny equation (9) as the kink. This is rather unusual, and is a consequence of $W$ having a cubic stationary point at $\phi = 0$. The equation still has no static solution with both a mirror kink and kink, interpolating between $\phi = -1$ and $\phi = 1$. This is because the Bogomolny equation is a gradient flow equation for the superpotential $W$, and solutions cannot pass through stationary points of $W$. The antikink, and the antimirror-kink, obey the Bogomolny equation with reversed sign, $\frac{d\phi}{dx} = -\frac{dW}{d\phi} = -(1 - \phi^2)\phi^2$. Since kink and antikink obey Bogomolny equations with opposite signs, there are no static kink-antikink solutions.

We conclude this section with some remarks about the well-known mechanical reinterpretation of a kink solution. The equation for a static field, obtained from the Lagrangian density (1), is

$$\frac{d^2\phi}{dx^2} = \frac{dV}{d\phi}. \quad (15)$$

This can be interpreted as the Newtonian equation of motion for a unit mass particle with “position” $\phi$ moving in “time” $x$ in the inverted “potential” $-V$. For our kink, the particle falls off the “potential” maximum at
\( \phi = 0 \) and eventually stops at the “potential” maximum at \( \phi = 1 \). Because the total “energy” is zero, the motion also obeys the first-order Bogomolny equation. The motion away from the maximum at \( \phi = 0 \) is particularly slow, because that maximum is quartic, leading to the long-range kink tail, but the approach to the maximum at \( \phi = 1 \) occurs more rapidly. Below, we will consider an accelerating kink (in the true time \( t \)) and will find that it approximately satisfies a modified static equation. In the mechanical reinterpretation, this is a Newtonian equation of motion with friction, so to get a solution for which the particle eventually stops at \( \phi = 1 \), the particle needs to leave \( \phi = 0 \) with a positive “velocity” at a finite “time”. Note that the friction is operative mainly during the long, slow “descent” of the particle away from \( \phi = 0 \), and is negligible during the rapid “ascent” to \( \phi = 1 \). The frictional “force” is comparable during these stages, since the “velocities” are similar, but the “times” over which it acts are very different.

3 Estimating Forces using Static Interpolating Fields

3.1 The Force between Kink and Mirror Kink

We expect a kink and mirror kink to repel each other, and shall estimate their accelerations when the kink is located at \( A \) and the mirror kink is at \(-A\), for \( A \gg 0 \). The field may be assumed to be antisymmetric in \( x \) at all times.

We need to produce a sensible interpolating field between \( \phi = -1 \) and \( \phi = 1 \). Simply adding the kink and mirror kink fields is not a good idea, as already mentioned. Instead we split up the spatial line at points \(-X\) and \( X\), with \( 0 \ll X \ll A \), so that for \( x \leq -X \) we have an exact mirror kink field, and for \( x \geq X \) an exact kink field. In between, for \( -X \leq x \leq X \), we assume the interpolating field has the linear behaviour \( \phi(x) = \mu x \). This is justified by the linearised, static field equation for small \( \phi \), which is simply \( \frac{d^2\phi}{dx^2} = 0 \). We require the field and its first spatial derivative to be continuous at \( X \) (and also \(-X\)). This leads to the conditions

\[
X = \frac{1}{2} A \quad \text{and} \quad \mu = \frac{4}{A^2},
\]

(16)

where we have assumed the tail formula for the kink field, which is approximately valid at \( X \). The interpolating field for \( A = 10 \) is shown in Fig.2.
Figure 2: Interpolating field for kink at $A = 10$ and mirror kink at $A = -10$, with linear interpolation between $x = -5$ and $x = 5$.

Note that if we had solved the static field equation exactly between the mirror kink and kink, the appropriate continuity conditions would have been impossible to solve, as they would have given us a global smooth, static solution, which doesn’t exist. This impasse is resolved by using an approximate solution, as we have done.

Let us now calculate the total energy of this interpolating field, to leading order in $\frac{1}{A}$. This is the sum of the energies of the kink and mirror kink, with their tails truncated, and the energy of the linear interpolating field. We calculated the tail energy in eq.(14); here it is $\frac{8}{3A^3}$. For the linear part of the field, the energy is

$$E_{\text{lin.}} = \int_{-X}^{X} \left( \frac{1}{2} \left( \frac{d\phi}{dx} \right)^2 + \frac{1}{2} \left( \frac{dW}{d\phi} \right)^2 \right) dx. \quad (17)$$

We set $\phi = \mu x$ and make the approximation $\frac{dW}{d\phi} = \phi^2 = \mu^2 x^2$. The integral gives $\mu^2 X + \frac{1}{5} \mu^4 X^5$. Now we use the values (16) and find that $\mu^2 X$ and $\mu^4 X^5$
are both of the same order in $\frac{1}{A}$ as the tail energy, and the total energy is

$$E = \frac{4}{15} - \frac{16}{3A^3} + \frac{8}{A^3} + \frac{8}{5A^3} = \frac{4}{15} + \frac{64}{15A^3}. \quad (18)$$

Using this as the potential energy, and dropping the constant, we obtain an effective Lagrangian for the motion of the two kinks at large separation

$$L_{\text{kinks}} = \frac{2}{15} \dot{A}^2 - \frac{64}{15A^3}, \quad (19)$$

where the kinetic energy is twice that of a single, freely moving kink. The equation of motion is

$$\ddot{A} = \frac{48}{A^4}, \quad (20)$$

showing that the kink has acceleration $\frac{48}{A^4}$ to the right. The mirror kink has opposite acceleration, so the kink and mirror kink repel each other. The force acting is the mass $M = \frac{2}{15}$ times the acceleration, and is

$$F = \frac{32}{5A^4}. \quad (21)$$

Alternatively, $F$ is the negative of the derivative of $E$ with respect to the separation of the kink and mirror kink. The separation is $2A$ with an ambiguity of order 1 because of the asymmetry of the kinks and the finite widths of their cores, and because the notion of kink location or centre is to some extent a matter of definition, but this ambiguity does not affect a force falling off as a power of $\frac{1}{A}$, nor its coefficient, to leading order.

This calculation gives the expected dependence of $F$ on $A$, but the coefficient is not correct. This is because the coefficient is sensitive to the interpolation used. Also, the force is not acting uniformly throughout each kink, but only in the region of the linear interpolating field. One knows this because in the regions of the exact, static kink and mirror kink solutions, no force is acting.

### 3.2 The Force between Kink and Antikink

A well-separated kink at $A$ and an antikink at $-A$ are expected to attract, and the field can be assumed to be symmetric in $x$ at all times. Again, we split up the spatial line at $-\frac{1}{2}A$ and $\frac{1}{2}A$, so that for $x \leq -\frac{1}{2}A$ we have an exact antikink field, and for $x \geq \frac{1}{2}A$ an exact kink field. In between, for $-\frac{1}{2}A \leq x \leq \frac{1}{2}A$, we assume the interpolating field has quadratic behaviour.
\( \phi(x) = \alpha + \beta x^2 \), and we fix the values of \( \alpha \) and \( \beta \) so that \( \phi \) is continuous and has continuous first derivative at \( \pm \frac{1}{2}A \). These continuity conditions are satisfied if

\[
\alpha = \frac{1}{A} \quad \text{and} \quad \beta = \frac{4}{A^3},
\]

where we have assumed the tail formula for the kink field, which is approximately valid at \( \frac{1}{2}A \). The splitting points are the same as those used in the kink-kink case. There could be a better choice, \(-X \) and \( X \) with \( 0 \ll X \ll A \) and \( X \neq \frac{1}{2}A \), but the algebra gets more complicated. The interpolating field is shown in Fig.3.

![Interpolating field for kink at A = 10 and antikink at A = -10, with quadratic interpolation between x = -5 and x = 5.](image)

Figure 3: Interpolating field for kink at \( A = 10 \) and antikink at \( A = -10 \), with quadratic interpolation between \( x = -5 \) and \( x = 5 \).

The total energy of this interpolating field, to leading order in \( \frac{1}{A} \), is the sum of the energies of the truncated kink and antikink, with their tail energies \( \frac{8}{3A^2} \) removed, and the energy of the quadratic interpolating field

\[
E_{\text{quad.}} = \int_{-\frac{1}{2}A}^{\frac{1}{2}A} \left( \frac{1}{2} \left( \frac{d\phi}{dx} \right)^2 + \frac{1}{2} \phi^4 \right) dx,
\]

(23)
where \( \phi = \frac{1}{A} + \frac{4}{A^3} x^2 \) and we have again made the approximation \( \frac{dW}{d\phi} = \phi^2 \) as \( \phi \) is small. The integral is elementary, but a bit more elaborate than for the linear interpolation in the kink-kink case. We find \( E_{\text{quad.}} = \frac{1504}{315A^3} \), so the total energy is

\[
E = \frac{4}{15} - \frac{16}{3A^3} + \frac{1504}{315A^3} = \frac{4}{15} - \frac{176}{315A^3}.
\]  

(24)

The estimated force between the kink and antikink is minus the derivative of \( E \) with respect to the separation \( 2A \),

\[
F = -\frac{88}{105A^4}.
\]  

(25)

It decays with the fourth power of the separation and is attractive. As the kink mass is \( M = \frac{2}{15} \), the acceleration of the kink is

\[
\ddot{A} = -\frac{44}{7A^4}.
\]  

(26)

The strength of the kink-kink repulsive force (using the same approximate approach) was \( \frac{32}{5A^3} \), so the strength of the attraction in the kink-antikink case appears to be about one eighth of this. However, these calculations are sensitive to the choice of splitting points and interpolating fields, so the numerical coefficients are not very reliable. In the next section we will use a different approach, where the kinks are accelerating and slightly deformed, and will derive more reliable forces.

4 Accelerating Kinks and Antikinks

4.1 Kink and Mirror Kink

In this approach to the force, we model the field \( \phi(x,t) \), assuming that the kink and mirror kink are located at well-separated, time-dependent positions \( A(t) \) and \( -A(t) \). The idea is to find an implicit profile for the kink with acceleration \( a = \ddot{A} \) that describes the field for \( x > 0 \), and at least approximately solves the field equation. The mirror kink has the reflected field (sign-reversed \( x, \phi \) and \( a \)) for \( x < 0 \). The accelerating kink has a distorted profile, whose tail is no longer \( \frac{1}{A^2} \). Instead, \( \phi \) is zero at some finite distance to the left of the kink, the distance depending on the acceleration. The kink profile is linear near here, so it can be continuously glued to the profile of
the accelerating mirror kink. Gluing at \( x = 0 \) creates a field antisymmetric in \( x \), with a continuous first derivative.

This interpolation does not give an exact solution, because the acceleration jumps discontinuously from \( a \) to \( -a \) at \( x = 0 \), and because we need to make various approximations that are explained in more detail below. Tail gluing nevertheless creates a convenient interpolation.

We introduce here a result related to Noether’s theorem for conserved momentum. By a standard argument \( [2] \), the momentum density in the field theory we are considering is \(-\frac{\partial \phi}{\partial t}\frac{\partial \phi}{\partial x}\), and its integral over the whole spatial line is conserved provided \( \phi \) satisfies its field equation. The time derivative of the momentum integral \( P \) over a finite interval is a sum of endpoint (surface) terms. In particular, using the field equation, one finds that for a field on the half-line \( x \geq X \), obeying the kink boundary condition \( \phi \to 1 \) as \( x \to \infty \),

\[
\frac{dP}{dt} = -\frac{d}{dt} \int_{X}^{\infty} \frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial x} \, dx = \left. \left( \frac{1}{2} \left( \frac{\partial \phi}{\partial t} \right)^2 + \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 - V(\phi(x)) \right) \right|_{x=X}.
\]

(27)

For the fields we are interested in, the term \( \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 \) is negligible. One may interpret \( \frac{dP}{dt} \) as the force \( F \) acting on the entire field to the right of \( X \), and it appears to be exerted at the point \( X \), although we shall refine this interpretation later. For the field of the accelerating kink and mirror kink we can choose \( X = 0 \), and there \( \phi = 0 \) and \( V = 0 \). The force on the kink becomes

\[
F = \left. \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 \right|_{x=0}.
\]

(28)

The last formula gives a rough estimate of the force, even when one does not attempt to solve the field equation. For example, if we simply add the static tail fields of the kink and mirror kink, then near \( x = 0 \),

\[
\phi = \frac{1}{A - x} - \frac{1}{A + x},
\]

(29)

so \( \frac{d\phi}{dx} = \frac{2}{x^2} \) at \( x = 0 \), giving the force estimate \( F = \frac{2}{x^2} \), and the kink acceleration estimate \( \ddot{A} = \frac{15}{4A^4} \). This has the right quartic dependence on \( \frac{1}{A} \), but the coefficient does not agree with what we found earlier, nor with what we claim is the more accurate value determined below.

Now let’s return to finding an approximate solution of the field equation representing the accelerating kink. We do not need to know that the acceleration is produced by the mirror kink to the left. We model the kink by a field of the form

\[
\phi(x, t) = \chi(x - A(t)).
\]

(30)
The acceleration is \( a = \ddot{A} \), which we assume is small. We also assume the squared velocity \( A^2 \) remains much less than 1, so the motion is non-relativistic, and Lorentz contraction of the kink and radiation can be ignored. Throughout, we work to linear order in \( a \). It is convenient to denote the argument of \( \chi \) by \( y \) and denote a derivative of \( \chi \) with respect to \( y \) by \( \prime \).

The first time derivative of \( \phi \) is \( -\dot{A}\chi' \), and we approximate the second time derivative by \( -\ddot{A}\chi' = -a\chi' \), dropping the term proportional to \( \dot{A}^2 \). (The term \( \frac{1}{2}(\frac{\partial \phi}{\partial t})^2 \) on the right hand side of eq. (27) is similarly proportional to \( A^2 \), which justifies its neglect.) Substituting the accelerating field (30) into the field equation (2) then gives

\[
\chi'' + a\chi' - \frac{dV(\chi)}{d\chi} = 0.
\] (31)

The profile \( \chi \) satisfies this static equation with \( a \) as parameter, so it evolves adiabatically with time as \( a \) varies. \( \phi \) evolves both because of this evolution of \( \chi \), and because \( A(t) \) occurs in the argument of \( \chi \).

Using the mechanical analogy discussed in Section 2, eq. (31) is interpreted as the motion of a “particle” with “position” \( \chi \) in the inverted potential \(-V\), now subject to friction with a friction coefficient \( a \). The kink boundary condition is \( \chi \to 1 \) as \( y \to \infty \). The potential \(-V\) is zero at both \( \chi = 0 \) and \( \chi = 1 \), so for the “particle” to reach \( \chi = 1 \) eventually and stop there, it must leave \( \chi = 0 \) with a finite “velocity” at a finite, but arbitrary time. This “velocity” is determined by \( a \).

Reverting back to field language, we anticipate a solution of eq. (31) where \( \chi = 0 \) and \( \chi' \) is positive at some finite \( y \), and \( \chi \to 1 \) as \( y \to \infty \). The equation for \( \chi \) is translation invariant, so any solution can be shifted to the left or right. The solution we require is the one where \( \chi(x - A) \) agrees as closely as possible with the static kink solution \( \phi(x - A) \) in the core region of the kink.

This requirement is not so easy to implement. The short-range tails to the right have slightly different exponential forms, so their coefficients cannot really be matched. Instead, one could require, for example, that \( \phi \) and \( \chi \) take identical values \( \frac{1}{\sqrt{2}} \) at the same position. This could be implemented numerically, but it cannot be done analytically because we do not have even an implicit solution for \( \chi \).

The approach we have adopted to this problem is to match the long-range tails of \( \phi \) and \( \chi \) close to the core regions, and in particular to arrange that the positions where the extrapolated long-range tails diverge are the same. This assumes that the main effect of the friction term is on these tails, and its effect in the kink core and further to the right is negligible, as we argued earlier.
For the static kink, the extrapolated tail $\frac{1}{A-x}$ diverges at $x = A$. For $\chi$ we calculate as follows. In the small term $a\chi'$ in eq. (31) we can assume that $\chi$ is the undeformed kink, for which $\chi' = \frac{dW}{d\chi}$. We can therefore trade the friction term for a modified potential $\tilde{V} = V - aW$ and obtain a first integral of eq. (31) of the form

$$\chi'^2 = 2\tilde{V} + \text{const.} \quad (32)$$

The constant is zero, because $\chi'$, $V$ and $W$ are all zero as $y \to \infty$. In the long-range tail region we can now make the approximations $V(\chi) = \frac{1}{2}\chi^4$ and $aW = -\frac{2a}{15}$, obtaining from eq. (32) the simplified equation

$$\frac{d\chi}{dy} = \sqrt{\frac{4a}{15} + \chi^4}. \quad (33)$$

As a consistency check, note that the linear behaviour of $\chi$ near its zero has slope $\mu = \sqrt{\frac{4a}{15}}$, so the force on the kink, according to eq. (27), is $\frac{2a}{15}$. This is the product of the kink’s mass and acceleration, as it should be.

The solution of eq. (33) still involves an elliptic integral of the first kind, but fortunately we just need the definite integral. The solution $\chi(y)$ should have the properties $\chi(-A) = 0$ and $\chi(0)$ diverges. Then $\chi(x - A)$ will be zero at $x = 0$ and $\chi(x - A)$ will diverge at $x = A$, as we require for the extrapolated tail field. Therefore

$$\int_0^\infty \frac{d\chi}{\sqrt{\frac{4a}{15} + \chi^4}} = A. \quad (34)$$

After changing variable to $\chi = (\frac{4a}{15})^{\frac{1}{4}} \lambda$, this relation simplifies to

$$I \equiv \int_0^\infty \frac{d\lambda}{\sqrt{1 + \lambda^4}} = \left(\frac{4a}{15}\right)^{\frac{1}{4}} A. \quad (35)$$

$I$ is a complete elliptic integral, and the further change of variable $\lambda^2 = \sinh \nu$ gives the equivalent relation

$$I = \frac{1}{2} \int_0^\infty (\sinh \nu)^{-\frac{1}{2}} d\nu = \left(\frac{4a}{15}\right)^{\frac{1}{4}} A. \quad (36)$$

Using either form of the integral $I$ [15, 16] we find that

$$a = \frac{\dot{A}}{A} = 15 \left(\frac{\Gamma \left(\frac{1}{4}\right)^2}{4\sqrt{\pi}}\right)^4 \frac{1}{A^4} \simeq 15 \left(\frac{1.854}{4}ight)^4 \frac{1}{A^4} = \frac{44.3}{A^4}. \quad (37)$$
This is our result for the acceleration. The coefficient 44.3 is different from the previous estimates of 48 and 15. The force on the kink is

\[ F = \frac{2}{15} \dot{A} \simeq \frac{1}{2} (1.854)^4 \frac{1}{A^4} = \frac{5.91}{A^4}. \]  

(38)

We can go back through the calculation, and verify that terms that were dropped, e.g. \( \frac{1}{3} a \chi^3 \) in \( aW \), are small compared to those retained.

The equation of motion for the kink, \( \ddot{A} = \frac{44.3}{A^3} \), implies the energy conservation equation

\[ \frac{1}{2} \dot{A}^2 + \frac{1}{3} \frac{44.3}{A^3} = \text{const.} \]  

(39)

and from this we deduce that if the kink and mirror kink approach from infinity with a small speed \( v \), then at closest approach

\[ A = \left( \frac{29.5}{v^2} \right)^{\frac{1}{3}}. \]  

(40)

This calculation, involving an accelerating kink, seems to give a more reliable result than the earlier, static approach because the force is properly distributed along the kink. To see this, we multiply eq.\( (31) \) by \( \chi' \) and integrate from \( X \) to \( \infty \), for \( X \geq 0 \), obtaining

\[ a \int_X^\infty \chi'^2 \, dy = \left( \frac{1}{2} \chi'^2 - V(\chi(y)) \right) \bigg|_{y=X}. \]  

(41)

By analogy with eq.\( (27) \), the right hand side is the force acting on the half-line to the right of \( X \). Since \( a \) is small, on the left hand side we can replace \( \chi' \) by the derivative of the static kink. The Bogomolny equation then implies that \( \chi'^2 \) is the energy density of the static kink, to the accuracy we need. (Of course, we can’t use the Bogomolny equation on the right hand side, because that would give zero.) Therefore the left hand side is the mass on the half-line to the right of \( X \), times the acceleration \( a \). As \( X \) is not fixed, we deduce that the force is everywhere of the correct strength.

We have apparently lost track of the mirror kink, but this occupies the half-line \( x \leq 0 \), and its field is \(-\phi(-x,t) = -\chi(-x-A(t))\). The acceleration of the mirror kink is \(-a\). The kink and mirror kink fields meet at \( x = 0 \) and have a continuous derivative. The mirror kink satisfies an equation like \( (31) \) but with the sign of \( a \) reversed. The second spatial derivative of the field \( \phi \) at \( x = 0 \) therefore has a discontinuity \( 2a \chi' \), which we see from eq.\( (33) \) is of order \( a^3 \). This discontinuity is presumably negligible, but is a consequence of the several approximations we have made. An even better approximation would involve a smoother interpolation of the acceleration through \( x = 0 \).
4.2 Kink and Antikink

Consider next a kink and antikink located at $A(t)$ and $-A(t)$, with $A \gg 0$. The profile of the accelerating kink now has a tail whose spatial derivative is expected to be zero at some finite position to the left of the kink. We arrange that this position is at $x = 0$, which gives a relation between the acceleration and kink location $A$. The antikink has the reflected field (sign-reversed $x$) for $x \leq 0$. At $x = 0$ we can glue the kink and antikink profiles together to create a field symmetric in $x$, which is continuous and has continuous first and second spatial derivatives. For similar reasons as in the kink-kink case, this approach does not give an exact solution.

Recall the Noether formula (27) for the rate of change of field momentum $P$ on the half-line $x \geq X$ (with $X \geq 0$), interpreted as the force $F$ acting on the field to the right of $X$. While the kink and antikink are slowly moving, the squared time-derivative term can again be neglected. We choose $X = 0$, and as $\frac{\partial \phi}{\partial x} = 0$ at $x = 0$ the force on the kink becomes

$$F = -V(\phi(0)).$$

This estimates the force even in the absence of a proper solution of the field equation. For example, $\phi(0) = \frac{1}{A}$ for the static interpolating field we constructed in Section 3, and if we use the approximation $V(\phi) = \frac{1}{2}\phi^4$ then $F = -\frac{1}{2A^4}$, so $\dddot{A} = -\frac{15}{4A^4}$. This has the expected sign and quartic dependence on $\frac{1}{A}$. The coefficient is similar to what we found earlier, using the energy of the static field.

As in the kink-kink case, we model the accelerating kink by a field $\phi(x, t) = \chi(x - A(t))$, and let $y$ denote the argument of $\chi$. Because of the expected attraction to the antikink, we assume $\dddot{A}$ to be negative, and introduce $a = -\dddot{A}$, with $a$ positive and small. As before, $\dddot{A}^2$ is supposed small enough to be neglected. The first time derivative of $\phi$ is $-A\chi'$, and we approximate the second time derivative by $-A\chi'' = a\chi'$, dropping the $\dddot{A}^2$ term. Substituting the accelerating field into the field equation (2) then gives

$$\chi'' - a\chi' - \frac{dV(\chi)}{d\chi} = 0,$$

the same equation as (31) but with reversed sign of $a$.

In the mechanical analogy, eq. (43) describes the motion of a “particle” subject to negative friction with coefficient $a$. For the “particle” to reach $\chi = 1$ eventually and stop there, it now needs to depart at zero “velocity” at some finite “time” from a positive “position” $\chi$ where $-V$ is negative. The initial “energy” is then negative, but increases with the “time” $y$.  

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In field language, we anticipate a solution of eq. (43) where $\chi$ is positive and $\chi' = 0$ at some finite $y$, and which satisfies the kink boundary condition $\chi \to 1$ as $y \to \infty$. The required solution, with $\chi(x - A)$ matching the static kink $\phi(x - A)$ in the kink core, is again found by arranging that the extrapolated long-range tails diverge at the same location $A$.

The calculation of the tail of $\chi$ differs now that the term proportional to $a$ in eq. (43) has the opposite sign. Replacing $\chi'$ by $dW/d\chi$ in this term leads to the modified potential $\tilde{V} = V + aW$ and the first integral

$$\chi'^2 = 2\tilde{V}. \quad (44)$$

In the long-range tail region we make the approximations $V(\chi) = \frac{1}{2} \chi^4$ and $aW = -\frac{2a}{15}$, obtaining from eq. (44)

$$\frac{d\chi}{dy} = \sqrt{\chi^4 - \frac{4a}{15}}. \quad (45)$$

The range of $\chi$, allowing for the divergence of the extrapolated tail, is from $(\frac{4a}{15})^{\frac{1}{4}}$ to $\infty$. At the lower end of this range, $\frac{d\chi}{dy} = 0$ and $V(\chi) = \frac{2a}{15}$, which implies that the force (42) is $-\frac{2a}{15}$, the product of the kink’s mass and acceleration. We can therefore consistently require that $\chi = (\frac{4a}{15})^{\frac{1}{4}}$ at $x = 0$.

As before, we just need the definite integral of eq. (45). The solution $\chi(y)$ should have the properties $\chi(-A) = (\frac{4a}{15})^{\frac{1}{4}}$ and $\chi(0)$ diverges. Then $\chi(x - A) = (\frac{4a}{15})^{\frac{1}{4}}$ at $x = 0$ and $\chi(x - A)$ diverges at $x = A$, as required. Therefore

$$\int_{(\frac{4a}{15})^{\frac{1}{4}}}^{\infty} \frac{d\chi}{\sqrt{\chi^4 - \frac{4a}{15}}} = A. \quad (46)$$

After rescaling, this simplifies to

$$J = \int_{1}^{\infty} \frac{d\lambda}{\sqrt{\lambda^4 - 1}} = \left(\frac{4a}{15}\right)^{\frac{1}{4}} A. \quad (47)$$

$J$, like the integral $I$, is a complete elliptic integral, and the further change of variable $\lambda^2 = \cosh \nu$ gives

$$J = \frac{1}{2} \int_{0}^{\infty} (\cosh \nu)^{-\frac{1}{2}} d\nu = \left(\frac{4a}{15}\right)^{\frac{1}{4}} A. \quad (48)$$

Knowing the integral $J$ [15, 16], we find

$$\left(\frac{4a}{15}\right)^{\frac{1}{4}} A = \frac{1}{\sqrt{2}} \frac{\Gamma(\frac{3}{4})^2}{4\sqrt{\pi}}. \quad (49)$$
The acceleration of the kink is therefore
\[ \dot{A} = -a = -\frac{15}{16} \left( \Gamma \left( \frac{1}{4} \right)^2 \right)^4 \frac{1}{A^4} \approx -11.1 \frac{1}{A^4}. \] (50)

The coefficient 11.1 differs from the previous estimates of \( \frac{44}{7} \) and \( \frac{15}{4} \) for the kink-antikink case. The force on the kink is
\[ F = \frac{2}{15} \dot{A} \approx -1.48 \frac{1}{A^4}, \] (51)
confirming that the kink and antikink attract.

The antikink occupying the half-line \( x \leq 0 \) has the field \( \phi(-x, t) = \chi(-x - A(t)) \), and the antikink’s acceleration is \( a \). The kink and antikink fields meet at \( x = 0 \) with zero spatial derivative. The antikink satisfies an equation like (31) but with the sign of \( a \) reversed. Therefore, the second spatial derivative of the field \( \phi \) at \( x = 0 \) is continuous, because the first derivative is zero there, but curiously, the third derivative has a small discontinuity.

The most interesting aspect of the kink-antikink force is the extra factor of \( \frac{1}{4} \) (and the reversed sign) compared with the force between a kink and mirror kink. This arises from the extra factor of \( \frac{1}{\sqrt{2}} \) in the integral \( J \), compared to \( I \). One can understand this factor using a contour integral argument, without evaluating the integrals.\(^2\)

Recall that
\[ I = \int_0^\infty \frac{d\lambda}{\sqrt{1 + \lambda^4}}, \quad J = \int_1^\infty \frac{d\lambda}{\sqrt{\lambda^4 - 1}}, \] (52)
and note that by a change of variable
\[ J = \int_0^1 \frac{d\lambda}{\sqrt{1 - \lambda^4}}. \] (53)

By considering the integral of \( \frac{1}{\sqrt{1+z^4}} \) along the contour from 0 to \( \infty \) to \( \infty e^{\pi i/4} \) and back to 0 via the branch point at \( z = e^{\pi i/4} \), we find that
\[ I + e^{\pi i/4} J - e^{\pi i/4} J = 0, \] (54)
so \( J = \frac{1}{\sqrt{2}} I \).

\(^{2}\)I am grateful to Joe Davighi and Alex Abbott for this insight.
5 Conclusions

We have investigated a simple example of a kink with a long-range tail in scalar field theory. The tail has $\frac{1}{x^2}$ behaviour, because the potential $V$ in the field theory has a quartic minimum. We have derived the repulsive force between a kink and a mirror kink, and the attractive force between a kink and an antikink, when their long-range tails overlap. The forces are proportional to the inverse fourth power of the separation when the separation is large. The numerical coefficients have been calculated by allowing for the kink accelerations and then solving modified Bogomolny equations. We have argued that this approach is likely to give the most reliable coefficients multiplying the power of the separation. The coefficients are transcendental.

Interestingly, the strength of the repulsion between kink and mirror kink is four times the attraction between kink and antikink, contrasting with the equal strengths of the kink-kink repulsion and kink-antikink attraction for kinks with short-range tails. This seems to be a consequence of the long-range tails being solutions of a nonlinear equation of type $\frac{d^2\phi}{dx^2} = 2\phi^3$ when $V$ has a quartic minimum, so the tails cannot simply be superposed, but it would be valuable to find a robust proof. It should not be difficult to generalise our discussion to variant kinks with long-range tails, and investigate this phenomenon. It would also be interesting to solve the equation of motion for incoming kinks with small velocities, using the forces we have obtained, and compare with a numerical simulation of the dynamics using the full field equation.

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\[ 3 \text{Since completion of this work, I.C. Christov et al. have calculated the forces between kinks and antikinks with long-range tails in the family of scalar field theories with potentials } V(\phi) = \frac{1}{2}(1 - \phi^2)^2\phi^{2n}, \text{ for } n = 2, 3, 4, \ldots. \text{ The forces decay with an inverse power of the separation depending on } n, \text{ and the ratio of the kink-kink repulsion and kink-antikink attraction also depends on } n. \text{ Numerical simulations of the full field equations confirm that the approach to the force calculation allowing for kink and antikink accelerations, as described in Section 4, gives reliable coefficients multiplying the inverse power of the separation.} \]
producing the figures. This work has been partially supported by STFC consolidated grant ST/P000681/1.

Appendix A: Kink Centre

Let $\phi_n$ and $\phi_{n+1}$ be adjacent zeros of $\frac{dW}{d\phi}$, and hence of the potential $V = \frac{1}{2} \left( \frac{dW}{d\phi} \right)^2$. We assume $\frac{dW}{d\phi}$ is positive between these zeros. Then the Bogomolny equation $\frac{d\phi}{dx} = \frac{dW}{d\phi}$ has the correct sign for a kink interpolating between $\phi_n$ and $\phi_{n+1}$, and the kink solution $\phi(x)$ increases monotonically with $x$.

$V$ could have several local maxima and minima between $\phi_n$ and $\phi_{n+1}$, but let us assume that it has just one maximum, denoted by $\phi_{n+\frac{1}{2}}$. For the following reasons, the position $x = x_{\text{centre}}$ where $\phi = \phi_{n+\frac{1}{2}}$ can be regarded as the centre of the kink. At $x_{\text{centre}}$,

$$\frac{dV}{d\phi} = \frac{dW}{d\phi} \frac{d^2W}{d\phi^2} = 0,$$

so $\frac{d^2W}{d\phi^2} = 0$, as $\frac{dW}{d\phi}$ is non-zero. Differentiating the Bogomolny equation gives, at $x_{\text{centre}}$,

$$\frac{d^2\phi}{dx^2} = \frac{d^2W}{d\phi^2} \frac{d\phi}{dx} = 0,$$

so $x_{\text{centre}}$ is the point of inflection in the kink profile, the position where $\phi$ is increasing with $x$ most rapidly. The energy density of a kink satisfying the Bogomolny equation is $2V$, so this is also maximal at $x_{\text{centre}}$. Let $\phi(x)$ be a centred kink, for which $x_{\text{centre}} = 0$. The general kink is then $\phi(x-c)$ with centre $c$.

For the familiar $\phi^4$ and sine-Gordon kinks, the centres are the obvious points about which the kink is antisymmetric. The centre of the kink with long-range tail, considered in the main part of this paper, was clarified in Section 2. Another non-trivial example is the $\phi^6$ kink [6]. Here,

$$V(\phi) = \frac{1}{2} (1 - \phi^2)^2 \phi^2,$$

with quadratic minima at $\pm 1$ and 0. The quadratic behaviours are different at 0 and 1, so the interpolating kink solution is not reflection (anti)symmetric. As $\frac{dW}{d\phi} = (1 - \phi^2)\phi$,

$$W(\phi) = \frac{1}{2} \phi^2 - \frac{1}{4} \phi^4 + \text{const}.$$
and the Bogomolny equation is
\[ \frac{d\phi}{dx} = (1 - \phi^2)\phi. \quad (59) \]

Using partial fractions, this can be integrated to give
\[ \phi(x) = \left(1 + 2e^{-2(x-c)}\right)^{-\frac{1}{2}}. \quad (60) \]

The kink centre is where \( \frac{d^2W}{dx^2} = 1 - 3\phi^2 = 0 \), i.e. where \( \phi = \frac{1}{\sqrt{3}} \). The expression \((60)\) has been carefully normalised so that \( x_{\text{centre}} = c \). The kink energy is \( E = W(1) - W(0) = \frac{1}{4} \).

**Appendix B: Force between \( \phi^6 \) Kinks**

Here, for completeness, we illustrate the simple method that gives the force between two kinks having short-range tails. The following example was not explicitly considered in \[9, 2\].

The \( \phi^6 \) theory has the kink solution \((60)\) interpolating between \( \phi = 0 \) and \( \phi = 1 \), and a mirror kink interpolating between \( \phi = -1 \) and \( \phi = 0 \) obtained by reversing the signs of \( x \) and \( \phi \). A field with a kink centred at \( c \) and mirror kink centred at \(-c\), with \( c \gg 0 \), is well described by the linear superposition
\[ \phi(x) = \left(1 + 2e^{-2(x-c)}\right)^{-\frac{1}{2}} - \left(1 + 2e^{2(x+c)}\right)^{-\frac{1}{2}}. \quad (61) \]

The kinks tails are short-ranged here, i.e. exponentially decaying, so the linear superposition is better justified than for the kinks with long-range tails we discussed earlier. The kink obeys the Bogomolny equation and the mirror kink the equation with reversed sign. So the linear superposition does not obey either Bogomolny equation, and is therefore not an exact static solution. However, both kink tails obey the linearised second-order static field equation in the region between the kinks, so the sum of the tails does too. This explains why the linear superposition is a good interpolating field.

To find the force exerted by the mirror kink on the kink, we use the Noether formula for the rate of change of momentum \([27]\). At a point \( X \) between the kinks, with \(-c \ll X \ll c\), the force acting on the kink to the right is
\[ F = \left. \left( \frac{1}{2} \left( \frac{d\phi}{dx} \right)^2 - \frac{1}{2} \phi^2 \right) \right|_{x=X}, \quad (62) \]
where we have made the approximation $V(\phi) = \frac{1}{2} \phi^2$, as $\phi$ is near zero. The superposed tail field, obtained from the leading exponentially small terms in (61), is

$$\phi(x) = \frac{1}{\sqrt{2}} e^{x-c} - \frac{1}{\sqrt{2}} e^{-x-c}.$$  \hspace{1cm} (63)

Each term separately would give no force, so it is the cross terms that produce a non-zero result. Substituting into (62), we find the repulsive force

$$F = e^{-2c},$$  \hspace{1cm} (64)

independent of $X$. As the mass of the kink is $\frac{1}{4}$, its acceleration is $4e^{-2c}$, and the mirror kink has the opposite acceleration. Because the kink tail has exponentially small energy, the kink mass to the right of $X$ is effectively constant even as $X$ varies, and therefore an $X$-independent force is what’s needed to produce a definite acceleration.

The effective equation of motion for the kink is

$$\frac{1}{4} \ddot{c} = e^{-2c},$$  \hspace{1cm} (65)

implying conservation of energy

$$\frac{1}{4} \dot{c}^2 + e^{-2c} = \text{const.}$$  \hspace{1cm} (66)

From this one determines that the closest approach is $c = \log 2 - \log v$ if the kink and mirror kink approach from infinity with small speed $v$.

Note that if we had decided on a different notion of the kink location, say $A = c + \delta$ with $\delta$ of order 1, then we would have derived the equation of motion

$$\frac{1}{4} \ddot{A} = e^{2\delta} e^{-2A}.$$  \hspace{1cm} (67)

This shows that the coefficient in front of an exponentially small force is only meaningful if one is careful to specify where the kink is located. Equations (67) and (65) are completely equivalent, although they seem to predict a closest approach for the kinks differing by $\delta$.

For a kink and antikink, one simply reverses the sign of the second term on the right hand side of (63) and finds an attractive force of the same strength.

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