ON SOME SUBCLASSES OF M-FOLD SYMMETRIC BI-UNIVALENT FUNCTIONS

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Abstract. In this work, we introduce two new subclasses $S_m(\alpha, \lambda)$ and $S_{\Sigma_m}(\beta, \lambda)$ of $\Sigma_m$ consisting of analytic and $m$-fold symmetric bi-univalent functions in the open unit disc $U$. Furthermore, for functions in each of the subclasses introduced in this paper, we obtain the coefficient bounds for $|a_{m+1}|$ and $|a_{2m+1}|$.

1. Introduction

Let $A$ denote the class of functions $f$ which are analytic in the open unit disc $U = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \}$, with in the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1.1)$$

Let $S$ be the subclass of $A$ consisting of the form (1.1) which are also univalent in $U$. It is well known that every function $f \in S$ has an inverse $f^{-1}$, satisfying $f^{-1}(f(z)) = z$, $(z \in U)$ and $f(f^{-1}(w)) = w$, $(|w| < r_0(f), r_0(f) \geq \frac{1}{4})$, where

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots \quad (1.2)$$

A function $f \in A$ is said to be bi-univalent in $U$ if both $f$ and $f^{-1}$ are univalent in $U$. Let $\Sigma$ denote the class of bi-univalent functions defined in the unit disc $U$. For a brief history and interesting examples in the class $\Sigma$, see [11], (see also [1], [3], [8], [9], [12], [15], [16], [20], [21]).

For each function $f \in S$, the function

$$h(z) = \sqrt[m]{f(z^m)} \quad (z \in U, \ m \in \mathbb{N}) \quad (1.3)$$

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is univalent and maps the unit disc $U$ into a region with $m$-fold symmetry. A function is said to be $m$-fold symmetric (see [7], [10]) if it has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \quad (z \in U, \ m \in \mathbb{N}).$$  \hspace{1cm} (1.4)

We denote by $S_m$ the class of $m$-fold symmetric univalent functions in $U$, which are normalized by the series expansion (1.4). In fact, the functions in the class $S$ are one-fold symmetric.

Analogous to the concept of $m$-fold symmetric univalent functions, we here introduced the concept of $m$-fold symmetric bi-univalent functions. Each function $f \in \Sigma$ generates an $m$-fold symmetric bi-univalent function for each integer $m \in \mathbb{N}$. The normalized form of $f$ is given as in (1.4) and the series expansion for $f^{-1}$, which has been recently proven by Srivastava et al. [13], is given as follows:

$$g(w) = w - a_{m+1} w^{m+1} + \left[ (m+1) a_{m+1}^2 - a_{2m+1} \right] w^{2m+1}$$

$$- \left[ \frac{1}{2} (m+1)(3m+2) a_{m+1}^3 - (3m+2) a_{m+1} a_{2m+1} + a_{3m+1} \right] w^{3m+1}$$

$$+ \cdots$$  \hspace{1cm} (1.5)

where $f^{-1} = g$. We denote by $\Sigma_m$ the class of $m$-fold symmetric bi-univalent functions in $U$. For $m = 1$, the formula (1.5) coincides with the formula (1.2) of the class $\Sigma$. Some examples of $m$-fold symmetric bi-univalent functions are given as follows:

$$\left( \frac{z^m}{1 - z^m} \right)^{1 \over m}, \ [-\log(1 - z^m)]^{1 \over m}, \ \left[ \frac{1}{2} \log \left( \frac{1 + z^m}{1 - z^m} \right) \right].$$

Thus, following Altinkaya and Yalçın [3] constructed the subclasses $S_{\Sigma}(\lambda, \alpha)$ and $S_{\Sigma}(\lambda, \beta)$ of bi-univalent functions and obtained estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses. Furthermore, in [4], Altinkaya and Yalçın obtained the second Hankel determinant, for the class $S_{\Sigma}(\lambda, \beta)$.

Recently, certain subclasses of $m$-fold bi-univalent functions class $\Sigma_m$ similar to subclasses of introduced and investigated by Altinkaya and Yalçın [2], (see also [13], [14], [17], [18], [19]).

The aim of the this paper is to introduce two new subclasses of the function class $\Sigma_m$ and derive estimates on the initial coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in these new subclasses of the function class $\Sigma$ employing the techniques used earlier by Srivastava et al. [11] (see also [6]).

Let $P$ denote the class of functions consisting of $p$, such that

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots = 1 + \sum_{n=1}^{\infty} p_n z^n,$$
which are regular in the open unit disc $U$ and satisfy $\Re(p(z)) > 0$ for any $z \in U$.

Here, $p(z)$ is called Caratheodory function [5].

We have to remember the following lemma so as to derive our basic results:

**Lemma 1.** (see [10]) If $p \in P$, then

$$|p_n| \leq 2 \quad (n \in \mathbb{N} = \{1, 2, \ldots\}).$$

2. **Coefficient bounds for the function class $S_{\Sigma_m}(\alpha, \lambda)$**

**Definition 1.** A function $f \in \Sigma_m$ is said to be in the class $S_{\Sigma_m}(\alpha, \lambda)$ if the following conditions are satisfied:

$$\left|\arg \left[ \frac{1}{2} \left( \frac{zf'(z)}{f(z)} + \left( \frac{zf'(z)}{f(z)} \right)^{\frac{1}{2}} \right) \right]\right| < \frac{\alpha \pi}{2} \quad (0 < \alpha \leq 1, \ 0 < \lambda \leq 1, \ z \in U)$$

and

$$\left|\arg \left[ \frac{1}{2} \left( \frac{wg'(w)}{g(w)} + \left( \frac{wg'(w)}{g(w)} \right)^{\frac{1}{2}} \right) \right]\right| < \frac{\alpha \pi}{2} \quad (0 < \alpha \leq 1, \ 0 < \lambda \leq 1, \ w \in U)$$

where the function $g = f^{-1}$.

**Theorem 1.** Let $f$ given by (1.4) be in the class $S_{\Sigma_m}(\alpha, \lambda)$, $0 < \alpha \leq 1$. Then

$$|a_{m+1}| \leq \frac{4\lambda \alpha}{m\sqrt{(1+\lambda)[4\lambda \alpha + (1+\lambda)(1-\alpha)]} + 2\alpha(1-\lambda)}$$

and

$$|a_{2m+1}| \leq \frac{2\lambda \alpha}{m(1+\lambda)} + \frac{8(m+1)\lambda^2 \alpha^2}{m^2(1+\lambda)^2}.$$ 

**Proof.** Let $f \in S_{\Sigma_m}(\alpha, \lambda)$. Then

$$\frac{1}{2} \left( \frac{zf'(z)}{f(z)} + \left( \frac{zf'(z)}{f(z)} \right)^{\frac{1}{2}} \right) = [p(z)]^\alpha \quad (2.1)$$

$$\frac{1}{2} \left( \frac{wg'(w)}{g(w)} + \left( \frac{wg'(w)}{g(w)} \right)^{\frac{1}{2}} \right) = [q(w)]^\alpha \quad (2.2)$$

where $g = f^{-1}$, $p, q$ in $P$ and have the forms

$$p(z) = 1 + p_m z^m + p_{2m} z^{2m} + \cdots$$

and

$$q(w) = 1 + q_m w^m + q_{2m} w^{2m} + \cdots.$$ 

Now, equating the coefficients in (2.1) and (2.2), we get

$$m(1+\lambda) = \frac{2\lambda}{2\lambda} a_{m+1} = \alpha p_m, \quad (2.3)$$

$$m(1+\lambda) \left( 2a_{2m+1} - a_{2m+1}^2 \right) + \frac{m^2(1-\lambda)}{4\lambda^2} a_{m+1}^2 = \alpha p_{2m} + \frac{\alpha(\alpha-1)}{2} p_m. \quad (2.4)$$
and

\[- \frac{m(1 + \lambda)}{2\lambda} a_{m+1} = \alpha q_m, \quad (2.5)\]

\[
m(1 + \lambda) \left( (2m + 1) a_{m+1}^2 - 2a_{2m+1} \right) + \frac{m^2(1 - \lambda)}{4\lambda^2} a_{m+1}^2 = \alpha q_{2m} + \frac{\alpha(a - 1)}{2} q_m^2. \quad (2.6)
\]

Making use of (2.3) and (2.5), we obtain

\[p_m = -q_m. \quad (2.7)\]

and

\[
\frac{m^2(1 + \lambda)^2}{2\lambda^2} a_{m+1}^2 = \alpha^2 (p_m^2 + q_m^2). \quad (2.8)
\]

Also from (2.4), (2.6) and (2.8) we have

\[
\left[ \frac{m^2(1 + \lambda)}{\lambda} + \frac{m^2(1 - \lambda)}{2\lambda^2} \right] a_{m+1}^2 = \alpha \left( p_{2m} + q_{2m} \right) + \frac{\alpha(a - 1)}{2} \left( p_m^2 + q_m^2 \right).
\]

\[= \alpha \left( p_{2m} + q_{2m} \right) + \frac{\alpha(a - 1)}{2} \frac{m^2(1 + \lambda)^2}{2\lambda^2} a_{m+1}^2.
\]

Therefore, we have

\[
a_{m+1}^2 = \frac{4\lambda^2 \alpha^2 (p_{2m} + q_{2m})}{m^2 \left\{ (1 + \lambda) [4\lambda \alpha + (1 + \lambda)(1 - \alpha)] + 2\alpha (1 - \lambda) \right\}}. \quad (2.9)
\]

Applying Lemma 1 for the coefficients \(p_{2m}\) and \(q_{2m}\), we obtain

\[|a_{m+1}| \leq \frac{4\lambda \alpha}{m \sqrt{(1 + \lambda) [4\lambda \alpha + (1 + \lambda)(1 - \alpha)] + 2\alpha (1 - \lambda)}}.
\]

Next, in order to find the bound on \(|a_{2m+1}|\), by subtracting (2.6) from (2.4), we get

\[
\frac{2m(1 + \lambda)}{\lambda} a_{2m+1} - \frac{m(m + 1)(1 + \lambda)}{\lambda} a_{m+1}^2 = \alpha \left( p_{2m} - q_{2m} \right) + \frac{\alpha(a - 1)}{2} \left( p_m^2 - q_m^2 \right).
\]

Then, in view of (2.7) and (2.8), and applying Lemma 1 for the coefficients \(p_{2m}, p_m\) and \(q_{2m}, q_m\), we have

\[|a_{2m+1}| \leq \frac{2\lambda \alpha}{m (1 + \lambda)} + \frac{8(m + 1) \lambda^2 \alpha^2}{m^2 (1 + \lambda)^2}.
\]

which completes the proof of Theorem 1. \(\square\)

3. Coefficient bounds for the function class \(S_{\Sigma_m}(\beta, \lambda)\)

**Definition 2.** A function \(f \in \Sigma_m\) given by (1.4) is said to be in the class \(S_{\Sigma_m}(\beta, \lambda)\) if the following conditions are satisfied:

\[
\Re \left\{ \frac{1}{2} \left( \frac{zf'(z)}{f(z)} + \left( \frac{zf'(z)}{f(z)} \right)^2 \right) \right\} > \beta, \quad (0 \leq \beta < 1, \ 0 < \lambda \leq 1, \ z \in U) \quad (3.1)
\]
Theorem 2. Let \( f \) given by (1.4) be in the class \( S_{\Sigma m}(\beta, \lambda) \), \( 0 \leq \beta < 1 \). Then

\[
|a_{m+1}| \leq \frac{2\lambda}{m} \sqrt{\frac{2(1-\beta)}{2\lambda^2 + \lambda + 1}}
\]

and

\[
|a_{2m+1}| \leq \frac{8(m+1)\lambda^2 (1-\beta)^2}{m^2(1+\lambda)^2} + \frac{2\lambda (1-\beta)}{m (1+\lambda)}.
\]

Proof. Let \( f \in S_{\Sigma m}(\beta, \lambda) \). Then

\[
\frac{1}{2} \left( \frac{zf'(z)}{f(z)} + \left( \frac{zf'(z)}{f(z)} \right)^{\frac{1}{2}} \right) = \beta + (1-\beta)p(z)
\]

and

\[
\frac{1}{2} \left( \frac{wg'(w)}{g(w)} + \left( \frac{wg'(w)}{g(w)} \right)^{\frac{1}{2}} \right) = \beta + (1-\beta)q(w)
\]

where \( p, q \in P \) and \( g = f^{-1} \).

It follows from (3.3) and (3.4) that

\[
\frac{m(1+\lambda)}{2\lambda} a_{m+1} = (1-\beta)p_m,
\]

\[
\frac{m(1+\lambda)}{2\lambda} (2a_{2m+1} - a_{m+1}^2) + \frac{m^2(1-\lambda)}{4\lambda^2} a_{m+1}^2 = (1-\beta)p_{2m},
\]

and

\[
\frac{m(1+\lambda)}{2\lambda} a_{m+1} = (1-\beta)q_m,
\]

\[
\frac{m(1+\lambda)}{2\lambda} [(2m+1)a_{m+1}^2 - 2a_{2m+1}] + \frac{m^2(1-\lambda)}{4\lambda^2} a_{m+1}^2 = (1-\beta)q_{2m}.
\]

Then, by making use of (3.5) and (3.7), we get

\[
p_m = -q_m.
\]

and

\[
\frac{m^2(1+\lambda)^2}{2\lambda^2} a_{m+1}^2 = (1-\beta)^2(p_m^2 + q_m^2).
\]

Adding (3.6) and (3.8), we have

\[
\left[ \frac{m^2(1+\lambda)}{\lambda} + \frac{m^2(1-\lambda)}{2\lambda^2} \right] a_{m+1}^2 = (1-\beta)(p_{2m} + q_{2m}).
\]
Therefore, we obtain
\[ a_{m+1}^2 = \frac{2\lambda^2(1-\beta)(p_{2m} + q_{2m})}{m^2(2\lambda^2 + \lambda + 1)}. \]

Applying Lemma 1 for the coefficients \( p_{2m} \) and \( q_{2m} \), we obtain
\[ |a_{m+1}| \leq \frac{2\lambda}{m} \sqrt{\frac{2(1-\beta)}{2\lambda^2 + \lambda + 1}}. \]

Next, in order to find the bound on \( |a_{2m+1}| \), by subtracting (3.8) from (3.6), we obtain
\[ \frac{2m(1+\lambda)}{\lambda} a_{2m+1} - \frac{m(m+1)(1+\lambda)}{\lambda} a_{m+1}^2 = (1-\beta)(p_{2m} - q_{2m}). \]

Then, in view of (3.9) and (3.10), applying Lemma 1 for the coefficients \( p_{2m}, p_m \) and \( q_{2m}, q_m \), we have
\[ |a_{2m+1}| \leq \frac{8(m+1)\lambda^2(1-\beta)^2}{m^2(1+\lambda)^2} + \frac{2\lambda(1-\beta)}{m(1+\lambda)}, \]
which completes the proof of Theorem 2.

If we set \( \lambda = 1 \) in Theorems 1 and 2, then the classes \( S_{\mathcal{C}}(\alpha, \lambda) \) and \( S_{\mathcal{C}}(\beta, \lambda) \) reduce to the classes \( S_{\mathcal{C}}^\alpha \) and \( S_{\mathcal{C}}^\beta \) and thus, we obtain the following corollaries:

**Corollary 1.** (see [2]) Let \( f \) given by (1.4) be in the class \( S_{\mathcal{C}}^\alpha \) \( (0 < \alpha \leq 1) \). Then
\[ |a_{m+1}| \leq \frac{2\alpha}{m\sqrt{\alpha+1}} \]
and
\[ |a_{2m+1}| \leq \frac{\alpha}{m} + \frac{2(m+1)\alpha^2}{m^2}. \]

**Corollary 2.** (see [2]) Let \( f \) given by (1.4) be in the class \( S_{\mathcal{C}}^\beta \) \( (0 \leq \beta < 1) \). Then
\[ |a_{m+1}| \leq \frac{\sqrt{2(1-\beta)}}{m} \]
and
\[ |a_{2m+1}| \leq \frac{2(m+1)(1-\beta)^2}{m^2} + \frac{1-\beta}{m}. \]

**Remark 1.** For one-fold symmetric bi-univalent functions, if we put \( \lambda = 1 \) in our Theorems, then we obtain the Corollary 1 and Corollary 2 which were proven earlier by Murugunsundaramoorthy et al. [9].
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