Mode I Elastic Solution of Cracked Composite Plate

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Abstract. The basic equation of composite mechanics has been established according to the orthotropic material properties. The analytic function of the complex variable including the material parameter is analyzed fully. Boundary problems of composite plate with Mode I crack are solved by using the complex variable method. The formulae for stress fields and strain fields are derived. The derivation of elastic displacement fields is carried out, and the whole displacement expression is determined by the complex variable function.

1. Introduction

Fracture mechanics of non-homogeneous materials or anisotropic materials has more applications in new engineering structure materials. Prediction of crack initiation and propagation must be based on the fracture mechanics. Fiber-reinforced polymer matrix materials are the most typical composites and usually modeled as anisotropic materials at the macroscopic level [1]. The orthotropic plate may have been the base of composites in common use. So this paper focuses on the fracture of orthotropic materials. Mode I stress fields and displacement fields in the cracked plate are mainly analyzed.

Linear elastic fracture mechanics (LEFM) has been found to be a very useful tool for design purposes and investigated in great detail for many engineering materials, whether isotropic or anisotropic [2]. The analysis of stresses near the crack tip holds an essential part of LEFM. The methods of elasticity are used to obtain stresses and displacements in cracked bodies [3]. The only viable method to solve stress-field problems in anisotropic composites is using complex analytic function theory, and the results have been reported [4–6]. But the general solution may have some weakness. So it is necessary to make up a new solution, and this is the paper purpose.

2. Basic Equations and Stress Function

The plane stress state of composite sheets is common and very importance for the application. It is the key point to solve stress-field problems in orthotropic materials. In plane elasticity, the equilibrium equations of three stress components are (body forces are absent):

\[ \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \]

For two-dimensional problems, three strain components can be expressed by the partial derivatives of two displacement components \( u \) and \( v \), namely:

\[ \varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \]
Suppose the principal elastic directions of the plate coincide with the coordinate directions, and let the directions 1, 2 parallel to the axes x, y, respectively. Now consider the linear elastic strain-stress relations, the constitutive equations for the orthotropic materials are given as (plane stress state):

\[ \varepsilon_x = \frac{\sigma_x}{E_1} - \frac{v_{12}\sigma_y}{E_1} , \quad \varepsilon_y = \frac{\sigma_y}{E_2} - \frac{v_{12}\sigma_x}{E_2} , \quad \gamma_{xy} = \frac{\tau_{xy}}{G_{12}} \]  

(3)

It is well known that the stress components can be expressed by a real stress function \( F = F(x, y) \), which is defined as follows:

\[ \sigma_x = \frac{\partial^2 F}{\partial y^2} , \quad \sigma_y = \frac{\partial^2 F}{\partial x^2} , \quad \tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y} \]  

(4)

The equilibrium equations in Eqs (1) can be satisfied naturally. For the plane stress problems, the stress function \( F \) must be selected reasonably in terms of the boundary conditions.

3. General Complex Function

In the consideration of cracked orthotropic materials, and also in the construction of suitable stress functions, it is very advantageous to use complex variables. It is well known that two real numbers \( x \) and \( y \) form the complex number \( z = x + iy \) conventionally \( (i = \sqrt{-1}) \). And the conjugate complex number is expressed by \( \bar{z} = x - iy \). For the convenience of general investigation, now we introduce another complex variable \( (w) \), that is:

\[ w = x + ihy , \quad \bar{w} = x - ihy \]  

(5)

Where \( h \) is a real constant. And we suppose the constant \( h \) to be positive, \( h > 0 \), also it can be called tensile or compressive ratio for the coordinate system. The derivative relation must be given as:

\[ \frac{\partial w}{\partial x} = \frac{\partial \bar{w}}{\partial x} = 1 , \quad \frac{\partial w}{\partial y} = -\frac{\partial \bar{w}}{\partial y} = ih \]  

(6)

For the crack problem, rectangular and polar coordinates are shown in Figure 1. The polar coordinate system centered at the crack tip may be adequate for local stress analysis. In terms of the polar coordinates, the complex variables can be written as:

\[ w = x + iy = a + r \cos \theta + ihr \sin \theta \]

\[ \bar{w} = x - iy = a + r \cos \theta - ihr \sin \theta \]  

(7)
Consider an analytic function of complex variable $\Psi = \Psi(w)$ and its conjugate, $\Psi = \Psi(\overline{w})$. And they can be expressed as:

$$\Psi(w) = \text{Re}\, \Psi + i\, \text{Im}\, \Psi, \quad \Psi(\overline{w}) = \text{Re}\, \Psi + i\, \text{Im}\, \Psi$$

Furthermore, another analytic function $\Phi$ and its conjugate $\Phi$ are introduced as below:

$$\Phi = \Phi(w) = \frac{d\Psi}{dw} = \Psi', \quad \Phi = \Phi(\overline{w}) = \frac{d\Psi}{d\overline{w}} = \Psi'$$

The partial derivatives of $\Psi(w)$ with respect to $x$ and $y$ should be given by:

$$\frac{\partial \Psi}{\partial x} = \frac{d\Psi}{dw} \frac{\partial w}{\partial x} = \Phi = \text{Re}\, \Phi + i\, \text{Im}\, \Phi = \frac{\partial \text{Re}\, \Psi}{\partial x} + i\, \frac{\partial \text{Im}\, \Psi}{\partial x}$$

$$\frac{1}{h} \frac{\partial \Psi}{\partial y} = \frac{1}{h} \frac{d\Psi}{dw} \frac{\partial w}{\partial y} = i\, \text{Im}\, \Phi = i\, \text{Im}\, \Phi = \frac{1}{h} \left( \frac{\partial \text{Re}\, \Psi}{\partial y} + i\, \frac{\partial \text{Im}\, \Psi}{\partial y} \right)$$

Obviously, it is easy to derive the following differential equation:

$$\text{Re}\, \Phi = \frac{\partial \text{Re}\, \Psi}{\partial x} = \frac{1}{h} \frac{\partial \text{Im}\, \Psi}{\partial y}, \quad \text{Im}\, \Phi = \frac{\partial \text{Im}\, \Psi}{\partial x} = -\frac{1}{h} \frac{\partial \text{Re}\, \Psi}{\partial y}$$

Once more, they can be written as follows:

$$\frac{\partial^2 \text{Re}\, \Psi}{\partial x^2} + \frac{1}{h^2} \frac{\partial^2 \text{Re}\, \Psi}{\partial y^2} = 0, \quad \frac{\partial^2 \text{Im}\, \Psi}{\partial x^2} + \frac{1}{h^2} \frac{\partial^2 \text{Im}\, \Psi}{\partial y^2} = 0$$

Similarly, the partial derivatives of $\Phi(w)$ ought to be given by:

$$\text{Re}\, \Phi' = \frac{\partial \text{Re}\, \Phi}{\partial x} = \frac{1}{h} \frac{\partial \text{Im}\, \Phi}{\partial y}, \quad \text{Im}\, \Phi' = \frac{\partial \text{Im}\, \Phi}{\partial x} = -\frac{1}{h} \frac{\partial \text{Re}\, \Phi}{\partial y}$$

$$\frac{\partial^2 \text{Re}\, \Phi}{\partial x^2} + \frac{1}{h^2} \frac{\partial^2 \text{Re}\, \Phi}{\partial y^2} = 0, \quad \frac{\partial^2 \text{Im}\, \Phi}{\partial x^2} + \frac{1}{h^2} \frac{\partial^2 \text{Im}\, \Phi}{\partial y^2} = 0$$
4. Stress Function and Solution

The Airy stress function \( U \) can be expressed by the complex variable functions. Therefore, we consider an infinite plane with the central crack of 2a-length along the x-axis shown in Figure 1, and consider the problem of Mode I crack (Opening Mode) in tension loading along the y-direction. The derivation of elastic stress field equations will be limited to Mode I load because it is the predominant mode in many practical cases. Once the derivation is completed, it is possible to obtain a number of useful expressions for stresses and displacements in the crack tip region. In terms of the stress intensity factor and the \( T \)-stress in references [7–9], and by experience, the stress function can be determined by the below form:

\[
F = A \text{Re} \Psi + B y \text{Im} \Phi + \frac{T}{2} y^2
\]  

To take advantage of the preceding equations, the partial derivatives of \( F(x, y) \) with respect to \( x \) or \( y \) can be given as follows:

\[
\frac{\partial F}{\partial x} = A \partial \text{Re} \Phi + B y \partial \text{Im} \Phi
\]

\[
\frac{\partial F}{\partial y} = (B - hA) \text{Im} \Phi + B y \frac{\partial \text{Im} \Phi}{\partial y} + Ty
\]

\[
\frac{\partial^2 F}{\partial x^2} = A \frac{\partial^2 \text{Re} \Phi}{\partial x^2} + B y \frac{\partial^2 \text{Im} \Phi}{\partial x^2}
\]

\[
\frac{\partial^2 F}{\partial y^2} = (2B - hA) \frac{\partial \text{Im} \Phi}{\partial y} + B y \frac{\partial^2 \text{Im} \Phi}{\partial y^2} + T
\]

\[
\frac{\partial^2 F}{\partial x \partial y} = (B - hA) \frac{\partial \text{Im} \Phi}{\partial x} + B y \frac{\partial^2 \text{Im} \Phi}{\partial x \partial y}
\]

By substituting above formulae into the expression (4), the stress components can be expressed by

\[
\sigma_x = (2B - hA) h \text{Re} \Phi' - h^2 B y \text{Im} \Phi' + T
\]

\[
\sigma_y = A \text{Re} \Phi' + B y \text{Im} \Phi'
\]

\[
\tau_{xy} = -(B - hA) \text{Im} \Phi' - hB y \text{Re} \Phi'
\]

Evidently, the analytic function \( \Phi \) can be as a new usual stress function. For Mode I crack problem, the plane with a line crack is subjected to symmetric loading \( \sigma \) at infinity along y direction, and the stress boundary conditions are:

\[
\sigma_y = \tau_{xy} = 0 \quad \text{at} \quad |x| < a \quad \text{and} \quad y = 0 \quad \text{(free crack surfaces)}
\]

\[
\sigma_y = \sigma_x = \tau_{xy} = 0 \quad \text{at} \quad |x| \rightarrow \infty \quad \text{and} \quad y = 0
\]

In order for the stress function to meet the preceding boundary value problem, the complex function can be selected as:

\[
\Phi(w) = w \sqrt{1 - \frac{a^2}{w^2}}
\]  

Then the derivatives of \( \Phi(w) \) must be given by:
\[ \Phi' = \sqrt{\frac{w^2}{w^2 - a^2}} , \quad \Phi'' = -\frac{a^2 w}{(w^2 - a^2)^2} \sqrt{1 - \frac{a^2}{w^2}} = \frac{a^2}{w^2} \left( \frac{w^2}{w^2 - a^2} \right)^{3/2} \]

Substituting the expressions into Equation (17), and to make the stress components to meet the stress boundary conditions, then the coefficients can be determined as:

\[ A = \sigma , \quad B = hA = h\sigma , \quad T = -\sigma h^2 \]

Therefore, the stress components can be expressed as follows:

\[ \sigma_x = h^2 \text{Re} \left[ \frac{w^2}{w^2 - a^2} + h^3 \text{Im} \left[ \frac{a^2 y}{w^3} \sqrt{\frac{w^2}{w^2 - a^2}} \right] - h^2 \right] \]

\[ \sigma_y = \text{Re} \left[ \frac{w^2}{w^2 - a^2} - h \text{Im} \left[ \frac{a^2 y}{w^3} \sqrt{\frac{w^2}{w^2 - a^2}} \right] \right] \]

\[ \tau_{xy} = h^2 \text{Re} \left[ \frac{a^2 y}{w^3} \sqrt{\left( \frac{w^2}{w^2 - a^2} \right)^3} \right] \] (21)

5. Deformation Analysis

For discussing the deformation of above elastic body as shown in Figure 1, let \( u \) and \( v \) express the displacements of any point in the \( x \) and \( y \) directions. Now suppose the displacement components can be given by the complex function as follows:

\[
\begin{align*}
\begin{cases}
    u = A_1 \text{Re} \Phi + A_2 y \text{Im} \Phi' + A_3 x \\
v = B_1 \text{Im} \Phi + B_2 y \text{Re} \Phi' + B_3 y
\end{cases}
\end{align*}
\] (22)

Substituting the expressions into equation (2), the strain components can be determined as:

\[
\begin{align*}
\begin{cases}
    \epsilon_x = A_1 \text{Re} \Phi' + A_2 y \text{Im} \Phi'' + A_3 \\
    \epsilon_y = (B_2 + hB_1) \text{Re} \Phi' - B_2 h y \text{Im} \Phi'' + B_3 \\
    \gamma_{xy} = (B_1 - A_1 h + A_2) \text{Im} \Phi' + (B_2 + A_2 h) y \text{Re} \Phi''
\end{cases}
\end{align*}
\] (23)

And again, in terms of the constitutive equation (3) and the stress equation (21), the identical relation of the strain components can be built. On the basis of the strain equivalence, the constants can be solved and given as follows:

\[
\begin{align*}
A_1 &= \frac{\sigma}{E_1} (h^2 - \nu_{12}) , \quad A_2 = -\frac{\sigma h}{E_1} (h^2 + \nu_{12}) , \quad A_3 = -\frac{\sigma}{E_1} h^2 \\
B_1 &= \frac{2\sigma}{hE_2} , \quad B_2 = -\frac{\sigma}{E_1} \left( \frac{E_1}{E_2} + \nu_{12} h^2 \right) , \quad B_3 = \frac{\sigma}{E_1} \nu_{12} h^2 \\
h^2 &= \frac{E_1}{E_2} - \nu_{12} = \sqrt{\frac{E_1}{E_2}}
\end{align*}
\] (24)

It is evident that the constant \( h \) is relative to the elasticity of the orthotropic material. Up to now, all the constants have been determined. Therefore, the displacement components can be given by:
\[
\begin{align*}
\left\{ 
\begin{array}{l}
u = \frac{2\sigma}{hE_2} \text{Im}[w \cdot \sqrt{1 - \frac{a^2}{w^2}} - \frac{\sigma_h}{E_1} (h^2 + \nu_{12}) y \text{Im} \sqrt{\frac{w^2}{w^2 - a^2} - \frac{\sigma}{E_1} h^2 x} \\
u = \frac{2\sigma}{hE_2} \text{Im}[w \cdot \sqrt{1 - \frac{a^2}{w^2}} - \frac{\sigma}{E_1} (E_1 + \nu_{12} h^2) y \text{Re} \sqrt{\frac{w^2}{w^2 - a^2} + \frac{\sigma}{E_1} \nu_{12} h^2 y} \end{array}
\end{align*}
\] (25)

Obviously, the displacement fields for the orthotropic material are quite complex. For the isotropic materials \((h = 1)\), \(E_1 = E_2 = E = 2G(1 + \nu)\), \(\nu_{12} = \nu\), \(G_{12} = G\) and \(w = z = x + iy\). And so the expressions of the displacements can be reduced to:

\[
\begin{align*}
\{u &= \frac{\sigma}{E} (1 - \nu) \text{Re}[z \cdot \sqrt{1 - \frac{a^2}{z^2}} - \frac{\sigma}{E} (1 + \nu) y \text{Im} \sqrt{\frac{z^2}{z^2 - a^2} - \frac{\sigma}{E} x} \\
\nu &= \frac{2\sigma}{E} \text{Im}[z \cdot \sqrt{1 - \frac{a^2}{z^2}} - \frac{\sigma}{E} (1 + \nu) y \text{Re} \sqrt{\frac{z^2}{z^2 - a^2} + \frac{\sigma}{E} y} \}
\end{align*}
\] (26)

Obviously, the expressions are conventional displacement fields of isotropic materials.

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7. References
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