Strong decays of heavy baryons in Bethe-Salpeter formalism

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Abstract

In this paper we study the properties of diquarks (composed of $u$ and/or $d$ quarks) in the Bethe-Salpeter formalism under the covariant instantaneous approximation. We calculate their BS wave functions and study their effective interaction with the pion. Using the effective coupling constant among the diquarks and the pion, in the heavy quark limit $m_Q \to \infty$, we calculate the decay widths of $\Sigma_Q^{(*)} (Q = c, b)$ in the BS formalism under the covariant instantaneous approximation and then give predictions of the decay widths $\Gamma(\Sigma_b^{(*)} \to \Lambda_b + \pi)$.

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1 Introduction

A light baryon composed of $u$, $d$, or $s$ quarks is a very complex three-body system in which all the three light quarks play important roles in the dynamics of the baryon. However, for a baryon containing a heavy quark, in the heavy quark limit, dynamics in the baryon is greatly simplified theoretically. In the heavy quark limit, the heavy quark effective theory (HQET) [1] shows that the light quarks move in an effective static color field (in the rest frame of the heavy baryon) and can not see the spin and flavor degrees of the heavy quark. The heavy baryon system has an extra $SU(2)_f \otimes SU(2)_s$ symmetry. Therefore, the spin and isospin of the light quarks and the heavy quark are conserved separately.

In recent years, more and more experimental results about heavy baryons have been reported by various experimental collaborations, e.g. the discovery and measurements of $\Xi^{++}_{cc}$ [2, 3], $\Sigma^{(*)}_b$ [4], and $\Xi^-_b$ [5], etc. in addition to much more results about baryons containing one $c$ or $b$ quark [2]. However, the structures and properties of these baryons are not very clear, hence more precise experimental measurements and detailed theoretical studies are urgent.

For a baryon with one heavy quark, the diquark picture has been taken in various references, see e.g. Refs. [6, 7]. Since the spin and the isospin of two light quarks within the heavy baryon are conserved, they can be regarded as a two-quark system, “diquark”, and the diquark then combines with the heavy quark to form the heavy baryon. In this diquark picture, the heavy baryon can be reduced to a two-body system. In this paper, we will focus on the study of heavy baryons containing one heavy quark and two light quarks ($u$ and/or $d$). We shall not consider the effects of the isospin symmetry violation.

The purpose of this paper is to calculate the decay widths of the strong decay processes $\Sigma^{(*)}_Q \rightarrow \Lambda_Q + \pi$ with $Q = c$ or $b$. These processes can be shown schematically in Fig. 1. From this figure, we can see that the light diquark from the mother baryon $\Sigma^{(*)}_Q$ combines with the spectator heavy quark to form the daughter baryon $\Lambda_Q$ after emitting a very soft pion.

From Fig. 1 we can see that, to calculate the amplitude $\langle \Lambda_Q(P_\Lambda)\pi(q)|\Sigma^{(*)}_Q(P_\Sigma)\rangle$, we need to know the wave functions of $\Sigma^{(*)}_Q$ and $\Lambda_Q$ and the effective interaction among the diquarks and the pion. In the Bethe-Salpeter (BS) equation approach, one can show that the baryonic wave functions appearing in these decay amplitudes are the
BS wave functions of $\Sigma_Q^{(*)}$ and $\Lambda_Q$ in the heavy-quark and light-diquark picture, which have been given in Refs. [6, 7]. On the other hand, since the pion emitted by the diquark is very soft, the interaction among the diquarks and the pion can be treated by chiral perturbation theory. In order to calculate the (lowest order) effective coupling constant among the diquarks and the pion, $G_{\pi\phi\phi}$, we will establish the BS equations for the scalar diquark and the axial-vector diquark, respectively. Then $G_{\pi\phi\phi}$ can be expressed as the overlap integral of the BS wave functions of the scalar diquark and the axial-vector diquark. To simplify the BS equations for the diquarks to tractable forms, we will impose the so-called covariant instantaneous approximation in the kernels of these BS equations. This approximation was also applied to the BS equations for $\Sigma_Q^{(*)}$ and $\Lambda_Q$ [6,7]. Furthermore, we will assume that the kernels contain a scalar confinement term and a one-gluon-exchange term. Throughout our calculations, we will take the heavy quark limit (that is to say, to neglect all the $1/m_Q$ corrections except for those coming from the kinematic factors).

Now let us give some more detailed explanation about the points mentioned above. First, the decay amplitude shown in Fig. 1 can be written out in terms of the following equation:

$$
\langle \Lambda_Q(P_\Lambda)\pi(q)\big|\Sigma_Q^{(*)}(P_\Sigma)\rangle = \int d^4(x_1x_2y_1y_2uv) \overline{\chi}_{P_\Lambda}(x_2,x_1)S_Q(x_1 - y_1)^{-1}\chi_{P_\Sigma,\lambda}(y_1, y_2)
\times \Delta^{-1}_\phi(x_2 - u)\Delta^{-1,\nu\lambda}_\phi(v - y_2) \sum_i \langle\pi(q)|T\phi^i(u)\overline{\psi}^i_\nu(v)|0\rangle,
$$

where $P_\Lambda$, $P_\Sigma$, and $q$ are the momenta of $\Lambda$, $\Sigma_Q^{(*)}$, and $\pi$, respectively, $\overline{\chi}_{P_\Lambda}$ and $\chi_{P_\Sigma,\lambda}$ are the Bethe-Salpeter (BS) wave functions of $\Lambda_Q$ and $\Sigma_Q^{(*)}$, respectively, $\Delta^{-1}_\phi$ and $\Delta^{-1,\nu\lambda}_\phi$ are
the propagators of the scalar diquark \( \phi \) and the axial-vector diquark \( \varphi \), respectively, \( S_Q \) is the propagator of the heavy quark \( Q \), and \( i \) is the color index.

Second, let us discuss the effective coupling among the diquarks and the pion. Since the pion emitted by the diquark is very soft, one can expand the low energy effective Lagrangian of the diquarks and the pion in terms of the pion’s momentum. To lowest order, we only need to consider the following interaction vertex:

\[
L_{\pi\phi\varphi} = G_{\pi\phi\varphi} \sum_{b,i} \phi^i \partial_\mu \pi^b \overline{\varphi}^{b,i}_\mu + \text{h.c.},
\]

where \( b \) is the isospin index. Then, the matrix element \( \langle \pi(q)|T\phi^i(u)\overline{\varphi}^{i}(v)|0 \rangle \) in Eq. (1) can be calculated out from this interaction vertex,

\[
\langle \pi(q)|T\phi^i(u)\overline{\varphi}^{i}(v)|0 \rangle \sim G_{\pi\phi\varphi} q^\mu \int d^4 z e^{iqz} \Delta_{\phi}(u-z) \Delta_{\varphi}(z-v)_{\mu\nu}.
\]

On the other hand, the effective coupling constant can be related to a transition amplitude (more details will be shown in Sect. 3),

\[
G_{\pi\phi\varphi} \sim \frac{1}{f_\pi} A(0),
\]

where \( A(q^2) \sim q^\mu (P_\phi, i|A^{-}_\mu(q)|P_-\varphi, r, i) \), which can be presented by the overlap integral of the BS wave functions of the diquarks \( \phi \) and \( \varphi \). To calculate this transition amplitude, we need the BS wave functions of \( \phi \) and \( \varphi \).

The diquark systems have been studied in Ref. [8] in which the authors first solve the BS equations in the rest frame of diquarks and then boost their solutions to a moving frame. Different from the approach of Ref. [8], we will solve the BS equations directly under the so-called covariant instantaneous approximation in a general coordinate system. Furthermore, the kernel in our paper is different from that used in Ref. [8]: there are only Coulomb part (arising from one-gluon-exchange) and the scalar linear part in our kernel, \( V \sim -\frac{2\alpha_s}{3r} + \kappa_r \).

One can see from Eq. (1) that, to calculate the decay amplitudes of heavy baryons, we also need the BS wave functions of heavy baryons. The BS wave functions of \( \Lambda_Q \) and \( \Sigma_Q^{(*)} \) have been studied in Refs. [6, 7]. We will take the results given there as the input to calculate the decay amplitudes of \( \Sigma_Q^{(*)} \) in this paper.

We will constrain the ranges of parameters in our model, \( m_\phi \) (or \( m_\varphi \), which is related to \( m_\phi \)) and \( \kappa_B \), by comparing the theoretical and experimental results about
the average momentum of the heavy quark in heavy baryons [10]. With these ranges of parameters, we calculate the decay widths of $\Sigma_Q^{(*)} \rightarrow \Lambda_Q + \pi$.

The remainder of this paper is organized as follows. In Sect. 2, we discuss the BS formalism under the covariant instantaneous approximation in some details. After a brief discussion about the BS equation of a general two-quark system, we derive the BS equations for the scalar diquark $\phi$ and the axial-vector diquark $\varphi$, respectively. We also give the normalization conditions of the BS wave functions in this section. In Sect. 3, we calculate the effective coupling constant among the diquarks and the pion. In Sect. 4, the calculation of the decay widths of $\Sigma_Q^{(*)} \rightarrow \Lambda_Q + \pi$ are carried out in the BS formalism under the covariant instantaneous approximation. In Sect. 5, we discuss how to constrain the parameters in our model and give numerical results for the decay widths of $\Sigma_Q^{(*)} \rightarrow \Lambda_Q + \pi$. Sect. 6 is reserved for our conclusions and some discussions. We also include three appendices (A, B, and C) in this paper. Appendix A contains some definitions. We review some results of Refs. [6, 7] in Appendix B. In appendix C, we discuss the normalization conditions of the BS wave functions of heavy baryons.

2 Bethe-Salpeter equations for diquarks

In this section, we will derive the BS equations for a scalar diquark $\phi$ and an axial-vector diquark $\varphi$. Before doing this, we will first discuss the general BS formalism for a two-quark system (other than the system composed of a quark and an anti-quark). The BS wave functions of the two-quark system are defined as the following:

$$\chi_p(x_1, x_2)_{\alpha\beta}^{ijk} = \langle 0|T\psi_\alpha^i(x_1)\psi_\beta^j(x_2)|P, k\rangle, \quad (5)$$

$$\chi_p(x_2, x_1)_{\beta\alpha}^{kji} = \langle P, k|T\psi_\beta^k(x_2)^*\psi_\alpha^i(x_1)^*|0\rangle, \quad (6)$$

where $i, j, k$ are the color indices, $\alpha$ and $\beta$ are spinor indices, and $P$ is the momentum of the diquark. Note that the flavors of the quark fields are not written out explicitly, so the flavors of the quarks could be the same or different. Since the diquark (like an anti-quark) must furnish the representation $\mathbf{3}$ of the color group $SU(3)_c$, we can define a colorless BS wave function by $\chi_p(x_1, x_2)_{\alpha\beta}^{ijk} = \frac{1}{3!}\varepsilon^{ijk}\chi_p(x_1, x_2)_{\alpha\beta}$,

$$\chi_p(x_1, x_2)_{\alpha\beta} = e^{ijk}\langle 0|T\psi_\alpha^i(x_1)\psi_\beta^j(x_2)|P, k\rangle = e^{-iPX}\int \frac{d^4p}{(2\pi)^4}\chi_p(p)_{\alpha\beta}e^{-ipx}, \quad (7)$$

$$\chi_p(x_2, x_1)_{\alpha\beta} = e^{ijk}\langle P, k|T\psi_\beta^k(x_2)^*\psi_\alpha^i(x_1)^*|0\rangle. \quad (8)$$
where \( X \) is the coordinate of the mass-center, \( p \) and \( x \) are the relative momentum and the relative coordinate, respectively. As usual, we start from a four point function,

\[
S(x_1,x_2;y_1,y_2)_{\alpha_1\alpha_2,\beta_2\beta_1} = \langle 0 | \psi_{\alpha_1}^i(x_1) \psi_{\alpha_2}^{j_2}(x_2) \psi_{\beta_2}^{j_2}(y_2) \psi_{\beta_1}^i(y_1) | 0 \rangle .
\]  

The Fourier transform of this four point function is defined by (with indices suppressed)

\[
S(x_1,x_2;y_1) = \int \frac{d^4P d^4P' d^4p d^4p'}{(2\pi)^{16}} e^{-ipx+ip'y} \tilde{S}(p,p',P,P')
\]

\[
= \int \frac{d^4P d^4p d^4p'}{(2\pi)^{12}} e^{-ip(x-y)+ip'y} \tilde{S}(p,p'),
\]

where, ignoring the effects of the isospin violation, \( p = (p_1-p_2)/2 \), \( p' = (p_1'-p_2')/2 \), \( P = p_1+p_2 \), \( P' = p_1'+p_2' \), and \( p_1, p_2, p'_1, p'_2 \) are the momenta of \( \psi_{\alpha_1}^i(x_1) \), \( \psi_{\alpha_2}^{j_2}(x_2) \), \( \psi_{\beta_1}^i(y_1) \), \( \psi_{\beta_2}^{j_2}(y_2) \), respectively. The relative coordinates and coordinates of the mass-center are defined by \( x = x_1 - x_2 \), \( y = y_1 - y_2 \), \( X = (x_1 + x_2)/2 \), and \( Y = (y_1 + y_2)/2 \). Then the BS wave function of a bound state composed of two quarks satisfies the following BS equation:

\[
\chi_\rho(p)^{i_1j_1}_{\alpha_1\alpha_2} = \int \frac{d^4k d^4k'}{(2\pi)^8} \tilde{S}_P(0,p,k)^{i_1j_1}_{\alpha_1\alpha_2,\alpha'_2\alpha'_1} \tilde{K}_P(k,k')^{i_1j_1}_{\alpha'_1\alpha'_2,\beta_2\beta_1} \chi_\rho(k')^{\beta_1\beta_2},
\]

where \( \tilde{K}_P(k,k') \) is the Fourier transform of the two-particle-irreducible kernel of the four point function (9). In terms of the colorless BS wave function \( \chi_\rho(x_1,x_2)_{\alpha\beta} \), this equation can be written as

\[
\chi_\rho(p)_{\alpha_1\alpha_2} = \frac{1}{6} \delta_{\alpha_1\alpha_2} \int \frac{d^4k d^4k'}{(2\pi)^8} \tilde{S}_P(0,p,k)^{i_1j_1}_{\alpha_1\alpha_2,\alpha'_2\alpha'_1} \tilde{K}_P(k,k')^{i_1j_1}_{\alpha'_1\alpha'_2,\beta_2\beta_1} \chi_\rho(k')^{\beta_1\beta_2},
\]

where \( \delta_{\alpha_1\alpha_2} \equiv \delta_{\alpha_1} \delta_{\alpha_2} - \delta_{\alpha_1} \delta_{\alpha_2} \).

To obtain the normalization conditions for the BS wave functions we also need an inhomogeneous equation for the four point function, \( \tilde{S}_P(p,p') \),

\[
\tilde{S}_P(p,p')^{i_1j_1}_{\alpha_1\alpha_2,\beta_2\beta_1} = \tilde{S}_P(0,p,p')^{i_1j_1}_{\alpha_1\alpha_2,\beta_2\beta_1}
\]

\[
+ \int \frac{d^4k d^4k'}{(2\pi)^8} \tilde{S}_P(0,p,k)^{i_1j_1}_{\alpha_1\alpha_2,\alpha'_2\alpha'_1} \tilde{K}_P(k,k')^{i_1j_1}_{\alpha'_1\alpha'_2,\beta_2\beta_1} \tilde{S}_P(k',p')^{\beta_1\beta_2}_{\beta_2\beta_1}.
\]

Near the diquark pole, we can isolate the contribution of the “bound state” (diquark) to the above four-point function,

\[
\tilde{S}_P(p,p')^{i_1j_1}_{\alpha_1\alpha_2,\beta_2\beta_1} = \sum_{i,j=1}^{N_c} \delta_{i_1} \sum_{p_0 - E_P - i\epsilon} \chi_\rho(p)_{\alpha_1\alpha_2} \chi_\rho(p')_{\beta_2\beta_1}
\]

\[
+ \text{terms regular at } P^0 = E_P,
\]
where $E_\mathbf{P} = \sqrt{\mathbf{P}^2 + m_\text{diquark}^2}$ is the ‘on-shell’ energy of the diquark. The inverse of $\tilde{S}^{(0)}_\mathbf{P}(p, p')$, $I_\mathbf{P}(p, p')$, is defined to satisfy the following equation:

$$
\int \frac{d^4 k}{(2\pi)^4} \left[ \tilde{S}^{(0)}_\mathbf{P}(p, k) I_\mathbf{P}(k, p') \right]_{\alpha_1 \alpha_2, \beta_1 \beta_2}^{i_1 i_2, j_1 j_2} = \frac{1}{2} (2\pi)^4 \left[ \delta^4(p - p') \delta_{\alpha_1 \beta_1} \delta_{\alpha_2 \beta_2} \delta^{i_1 j_1} \delta^{i_2 j_2} \pm \delta^4(p + p') \delta_{\alpha_1 \beta_2} \delta_{\alpha_2 \beta_1} \delta^{i_1 j_2} \delta^{i_2 j_1} \right],
$$

(15)

where ‘+’ is for the scalar diquark and ‘−’ is for the axial-vector diquark. We also need an auxiliary quantity,

$$
Q_\mathbf{P}(p, p') = \int \frac{d^4 k}{(2\pi)^4} \left( P_0^0 - E_\mathbf{P} \right) \tilde{S}_\mathbf{P}(p, k) \frac{\partial}{\partial P_0} \left\{ I_\mathbf{P}(k, p') - \frac{1}{2} \left[ \tilde{K}_\mathbf{P}(k, p') \pm \tilde{K}_\mathbf{P}(-k, p') \right] \right\}^{i_1 i_2, j_1 j_2} \chi_\mathbf{P}(p')
$$

Operating this quantity upon $\chi_\mathbf{P}$ gives the normalization condition for the BS wave function near the diquark pole $P_0^0 = E_\mathbf{P}$ (for more details see, e.g. Ref. [11]),

$$
\frac{i}{36} \delta^{i_1 i_2}_{j_1 j_2} \int \frac{d^4 p}{(2\pi)^8} \frac{d^4 p'}{(2\pi)^8} \chi_\mathbf{P}(p) \left( \frac{\partial}{\partial P_0^0} I_\mathbf{P}(p, p') \right)^{i_1 i_2, j_1 j_2} \chi_\mathbf{P}(p') = 1,
$$

(16)

where the spinor indices are suppressed. Since the kernel in our approximation is independent of $P_0^0$ (see the following subsections) the normalization condition is reduced to

$$
\frac{i}{36} \delta^{i_1 i_2}_{j_1 j_2} \int \frac{d^4 p}{(2\pi)^8} \frac{d^4 p'}{(2\pi)^8} \chi_\mathbf{P}(p) \left( \frac{\partial}{\partial P_0^0} I_\mathbf{P}(p, p') \right)^{i_1 i_2, j_1 j_2} \chi_\mathbf{P}(p') = 1, \quad P_0^0 = E_\mathbf{P}.
$$

Now, let us turn to the discussion about the kernel for the interaction between quarks. The success of the potential model for mesons tells us that the strong interaction between a quark and an anti-quark can be modeled by two important interactions, a one-gluon-exchange part (with an effective strong coupling) and a linear confinement

* In general, the inversion of $S^{(0)}_\mathbf{P}(p, p')$ will be more complicated. However, in our case (omitting the isospin violation), the following is enough.

† Eq. (17) is different from the expression in Lurie’s book [11]. The reason is that we have used a different convention for the normalization of one-particle state: $\langle \mathbf{p} | \mathbf{q} \rangle = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{q})$. In deriving this normalization condition, we have used the fact that $\chi_\mathbf{P}(-p)_{\beta \alpha} = -\chi_\mathbf{P}(p)_{\alpha \beta}$ for an iso-scalar diquark and $\chi_\mathbf{P}(-p)_{\beta \alpha} = \chi_\mathbf{P}(p)_{\alpha \beta}$ for an iso-vector diquark which will be shown explicitly in the following subsections.
part. We assume that this is also true for a two-quark system. Furthermore, the parameter in the kernel of the diquark system can be related to that of the meson system by the so-called one-half rule [12]: the kernel for the diquark is just one-half of that for the meson.

### 2.1 BS equation for the scalar diquark

In the following we will discuss the BS equation for a Lorentz-scalar and isospin-scalar diquark $\phi$ composed of $d$ and $u$ quarks. The definition of the BS wave function of the scalar diquark can be written as

$$
\chi_{\phi}(x_1, x_2)_{\alpha\beta} = e^{ijk} \langle 0 | T \{ d_i^\dagger(x_1) u_j^\dagger(x_2) - u_i^\dagger(x_1) d_j^\dagger(x_2) \} | \phi, k \rangle
$$

$$
= e^{-ip_x} \int \frac{d^4p}{(2\pi)^4} e^{-ip_x} \chi_{\phi}(p)_{\alpha\beta}.
$$

From this definition we can see that $\chi_{\phi}(-p)_{\beta\alpha} = -\chi_{\phi}(p)_{\alpha\beta}$. The free two-particle propagators reads

$$
\tilde{S}^{(0)}_{\phi}(p, p')_{i_1 j_1 i_2 j_2} = 2(2\pi)^4 \left\{ \delta^4(p - p') | S(p_1) \gamma_1^{0i_1 j_1} | S(p_2) \gamma_1^{0i_2 j_2} + ((\beta_1, j_1, p') \leftrightarrow (\beta_2, j_2, -p')) \right\},
$$

where $S(p_1)$ and $S(p_2)$ are quark propagators. The kernel arising from the one-gluon-exchange diagram is (we don’t consider the effects of the isospin violation throughout this paper)

$$
\tilde{K}^{(1g)}_{\phi}(p, p')_{i_1 j_1 i_2 j_2} = \frac{(ig_8)^2}{8} \left\{ \left( \gamma_1^{0\alpha\mu} T^\alpha \right)^{i_1 j_1} \left( \gamma_1^{0\gamma\nu} T^\gamma \right)^{i_2 j_2} \Delta_{\mu\nu}(p' - p) + ((\beta_1, j_1, p') \leftrightarrow (\beta_2, j_2, -p')) \right\},
$$

where $T^\alpha$ are generators of the fundamental representation of the color group $SU(3)_c$ and $\Delta_{\mu\nu}$ is the propagator of the gluon field in Feynman gauge. The confinement part of the kernel is assumed to be

$$
\tilde{K}^{(cf)}_{\phi}(p, p')_{i_1 j_1 i_2 j_2} = \frac{1}{8} \left\{ \left( \gamma_1^{0\alpha\mu} T^\alpha \right)^{i_1 j_1} \left( \gamma_1^{0\gamma\nu} T^\gamma \right)^{i_2 j_2} \tilde{K}^{(cf)}_{\phi}(p - p') + ((\beta_1, j_1, p') \leftrightarrow (\beta_2, j_2, -p')) \right\},
$$

where (after imposing the covariant instantaneous approximation [9], for some explanation about this approximation, see the following text)

$$
\tilde{K}^{(cf)}_{\phi}(p_1 - p'_1) = \frac{c}{\sqrt{[(p_1 - p'_1)^2 + \mu^2]}} - (2\pi)^3 \delta^3(p_1 - p'_1) \int \frac{d^3q_t}{(2\pi)^3} \frac{c}{\sqrt{[(p_1 - q_t)^2 + \mu^2]^2}},
$$

(22)
where the second term is introduced to remove the infra-red singularity of the confining kernel near the points \( p' = p \), and a small parameter \( \mu \) is introduced to avoid the divergence in numerical calculations. To determine the constant \( c \) in the above confinement kernel, one can compare the non-relativistic approximation of the BS equation with the Schrödinger equation where the effective potential is used in the potential model for mesons. In the non-relativistic limit one can show that the effective potential from the BS equation is

\[
V(r) = -\frac{2 \alpha_s}{3} r + \frac{-ic}{8\pi} r .
\]  

(23)

Comparing Eq. (23) with the effective potential in the meson case, \( V(r)_{\text{meson}} = -\frac{4 \alpha_s}{3} r + \kappa r \), we can see that setting \( c = 4i\pi\kappa \) is suitable from the viewpoint of the one-half rule [12].

With the interaction kernel in Eqs. (20) and (21), the BS equation (12) becomes (using the relation \( \chi_{P\phi}(p)_{\beta\alpha} = -\chi_{P\phi}(p)_{\alpha\beta} \))

\[
\chi_{P\phi}(p) = [S(p_1) \otimes S(p_2)] \int \frac{d^4p'}{(2\pi)^4} \cdot \left[ \gamma^\mu \otimes \gamma_\mu K^{(1g)}(p - p') + 1 \otimes 1 K^{(cf)}(p - p') \right] \cdot \chi_{P\phi}(p') ,
\]

(24)

where \( K^{(1g)} = \frac{2}{3} g_s^2 \frac{-i}{(p - p')^2} \). For convenience, we define a deformed BS wave function,

\[
\tilde{\chi}_{P\phi}(p)_{\alpha_1\alpha_2} = \chi_{P\phi}(p)\gamma_{\alpha_2} C^{-1}_{\gamma_\alpha_1} = (C\chi_{P\phi}(p))_{\alpha_1\alpha_2} ,
\]

(25)

where \( C \) is the charge conjugation matrix. With this deformed BS wave function, the BS equation becomes

\[
\tilde{\chi}_{P\phi}(p) = [CS(p_1)C^{-1} \otimes S(p_2)] \int \frac{d^4p'}{(2\pi)^4} \cdot \left[ C\gamma^\mu C^{-1} \otimes \gamma_\mu K^{(1g)}(p - p') + 1 \otimes 1 K^{(cf)}(p - p') \right] \cdot \tilde{\chi}_{P\phi}(p') .
\]

(26)

This equation can be written in a more usual matrix form,

\[
\tilde{\chi}_{P\phi}(p)^T = S(p_2) \int \frac{d^4p'}{(2\pi)^4} \left[ -\gamma^\mu \tilde{\chi}_{P\phi}(p')^T \gamma_\mu K^{(1g)} + \tilde{\chi}_{P\phi}(p')^T K^{(cf)} \right] S(-p_1) ,
\]

(27)

where the superscript ‘\( T \)’ represents the transpose of the spinor index. In deriving the normalization condition of the BS wave function, we also need its conjugation defined by \( \tilde{\chi}_{P\phi}(p) = C^{-1} \gamma^0 \tilde{\chi}_{P\phi}(p) \gamma^0 \), which satisfies the following BS equation:

\[
\tilde{\chi}_{P\phi}(p) = S(-p_2) \int \frac{d^4p'}{(2\pi)^4} \left[ -\gamma^\mu \tilde{\chi}_{P\phi}(p') \gamma_\mu K^{(1g)} + \tilde{\chi}_{P\phi}(p') K^{(cf)} \right] S(p_1) .
\]

(28)
From Eqs. (27) and (28), we can see that \( \tilde{\chi}_{P_{\phi}}(-p) \) and \( \tilde{\chi}_{P_{\phi}}(p)^{T} \) satisfy completely the same equation. Therefore, once we obtain the solution for \( \tilde{\chi}_{P_{\phi}}(p) \) we also obtain the solution for its conjugate by using \( \tilde{\chi}_{P_{\phi}}(p) = \tilde{\chi}_{-P_{\phi}}(-p)^{T} \).

Now, using the decomposition of a general matrix of Dirac fields
\[
\bar{\psi}^{c_{i}} \psi^{j}_{\alpha} = \frac{1}{4} \left[ (\bar{\psi}^{c_{i}} \psi^{j}) + \gamma^{\mu} (\bar{\psi}^{c_{i}} \gamma_{\mu} \psi^{j}) \right] + \frac{1}{2} \sigma^{\mu\nu} (\bar{\psi}^{c_{i}} \sigma_{\mu\nu} \psi^{j}) + \gamma_{5} (\bar{\psi}^{c_{i}} \gamma_{5} \psi^{j}) - \gamma_{5} \gamma^{\mu} (\bar{\psi}^{c_{i}} \gamma_{5} \gamma_{\mu} \psi^{j}) \right]_{\beta\alpha},
\]
we can parametrize the BS wave functions as (note the fact that the intrinsic parities of the quarks are the same)
\[
\tilde{\chi}_{P_{\phi}}(p)_{\alpha_{1} \alpha_{2}} = \left[ \gamma_{5} f_{1} + \gamma_{5} \gamma_{\mu} (P_{\phi}^{\mu} f_{2} + p_{l}^{\mu} f_{3})/m_{\phi} + (-i) \gamma_{5} \sigma_{\mu\nu} P_{\phi}^{\mu} p_{l}^{\nu} / m_{\phi}^{2} \right]_{\alpha_{2} \alpha_{1}} ,
\]
where \( p_{l} = p \cdot P_{\phi} / m_{\phi} \) is the longitudinal projection of \( p \) along the diquark momentum \( P_{\phi} \), \( p_{l}^{\mu} = p^{\mu} - p_{l} P_{\phi}^{\mu} / m_{\phi} \) is transverse to \( P_{\phi} \), \( f_{a} (a = 1, \ldots, 4) \) are Lorentz-scalar functions of \( p_{l}^{2} \) and \( p_{l} \), \( \sigma_{\mu\nu} = \frac{i}{2} [\gamma_{\mu}, \gamma_{\nu}] \).

To simplify the BS equation (26), we impose the so-called covariant instantaneous approximation in the kernel [9]: \( p_{l} = p_{l}' \). In this approximation the projection of the momentum of each constituent in the diquark along the total momentum \( P_{\phi} \) is not changed. For a diquark at rest, this requires the ‘energy’ \( (p_{l}^{0} or p_{l}^{0}) \) of the constituent particle not to be changed. This approximation is appropriate since the energy exchange between the constituents in the diquark is expected to be small when we use the constituent quark masses in the BS equation. Under this approximation, the kernel in the BS equation is reduced to \( \tilde{K}_{P_{\phi}}(p_{l} - p_{l}') \) which will be used in the following calculations.

Furthermore, we define a three-momentum BS wave function as \( \tilde{\chi}_{P_{\phi}}(p_{l}) = \int \frac{dp}{2\pi} \tilde{\chi}_{P_{\phi}}(p) \), the \( p_{l}' \)-integration in Eq. (26) can be carried out with the following redefinitions:
\[
\tilde{f}_{a}(p) \equiv \int \frac{dp}{2\pi} f_{a}(p), \quad a = 1, \ldots, 4.
\]
As usual, these four functions are not independent of each other. Their relations can be obtained by projecting the BS wave function with (see, e.g. Refs. [13,14])
\[
\Lambda_{1,2}^{\pm} = \frac{E_{1,2} \pm H_{1,2}}{2E_{1,2}}, \quad H_{1} = \frac{P_{\phi}}{m_{\phi}} (p_{l}^{1} + m_{1}), \quad H_{2} = \frac{P_{\phi}}{m_{\phi}} (-p_{l}^{1} + m_{2}),
\]
where the energy is defined by \( E_{1,2} = \sqrt{-p_{l}^{2} + m_{1,2}^{2}} \). The square of the covariant ‘Hamiltonian’ \( H_{1,2} \) is the square of the energy, \( H_{1,2}^{2} = E_{1,2}^{2} \), and the projection operators
satisfy $\Lambda^\pm_i \Lambda^\pm_j = \Lambda^\pm_i \Lambda^\pm_j = 0$, $f = 1, 2$. Furthermore, we will take the quark propagators to have the form of the free one which can be written as

\[
\frac{P_\phi}{m_\phi} S(p_1) = i \frac{m_\phi/2 + p_\ell + H_1}{(m_\phi/2 + p_\ell - E_1 + i\epsilon)(m_\phi/2 + p_\ell + E_1 - i\epsilon)},
\]

\[
\frac{P_\phi}{m_\phi} S(p_2) = -i \frac{-m_\phi/2 + p_\ell - H_2}{(-m_\phi/2 + p_\ell + E_2 - i\epsilon)(-m_\phi/2 + p_\ell - E_2 + i\epsilon)}.
\]

Operating the projection operators on both sides of Eqs. (33) and (34) leads to

\[
\Lambda^\pm_i \frac{P_\phi}{m_\phi} S(p_1) = i \left\{ \begin{array}{c} 1 \\ m_\phi/2 + p_\ell - E_1 + i\epsilon \\ m_\phi/2 + p_\ell + E_1 - i\epsilon \end{array} \right\} \Lambda^\pm_i,
\]

and

\[
\Lambda^\pm_i \frac{P_\phi}{m_\phi} S(p_2) = -i \left\{ \begin{array}{c} 1 \\ -m_\phi/2 + p_\ell + E_2 - i\epsilon \\ -m_\phi/2 + p_\ell - E_2 + i\epsilon \end{array} \right\} \Lambda^\pm_i.
\]

Now, by multiplying $\Lambda^\pm_i P_\phi$ and $\Lambda^\pm_j P_\phi$ on both sides of equation (26) and integrating out the longitudinal momentum $p_\ell$ along proper contour(s), we can obtain the following constraint equations:

\[
(L_1^+ P_\phi)\alpha'_1\alpha_1 (L_2^- P_\phi)\alpha'_2\alpha_2 \bar{\chi}_{P_\phi} (p_\ell)_{\gamma_\alpha_2} C_{\gamma_\alpha_1} = 0,
\]

\[
(L_1^- P_\phi)\alpha'_1\alpha_1 (L_2^+ P_\phi)\alpha'_2\alpha_2 \bar{\chi}_{P_\phi} (p_\ell)_{\gamma_\alpha_2} C_{\gamma_\alpha_1} = 0.
\]

Substituting the parametrization Eq. (30) into Eqs. (37) and (38) we obtain the following constraint relations:

\[
\bar{f}_3 = 0, \quad \bar{f}_4 = -\frac{m_\phi}{m} \bar{f}_2,
\]

where we have defined $m \equiv m_1 = m_2$ and consequently $\omega \equiv E_1 = E_2$.

In addition to the above constraint relations, we can also obtain other two equations by operating $\Lambda^\pm_i P_\phi$ and $\Lambda^\pm_j P_\phi$ upon both sides of equation (26):

\[
\left[ \Lambda^\pm_i \frac{P_\phi}{m_\phi} \otimes \Lambda^\pm_j \frac{P_\phi}{m_\phi} \right] \cdot \chi_{P_\phi} (p) = \left[ \Lambda^\pm_i \frac{P_\phi}{m_\phi} S(p_1) \otimes \Lambda^\pm_j \frac{P_\phi}{m_\phi} S(p_2) \right] \cdot \int \frac{d^4p}{(2\pi)^4} \left[ \gamma^\mu \otimes \gamma_\mu K^{(1g)}(p-p') + 1 \otimes 1 K^{(cf)}(p-p') \right] \cdot \chi_{P_\phi} (p'),
\]

\footnote{The above two equations can be written in matrix form: $L_2^- P_\phi \bar{\chi}_{P_\phi} (p)^T C (L_1^+ P_\phi)^T C^{-1} = 0$ and $L_2^+ P_\phi \bar{\chi}_{P_\phi} (p)^T C (L_1^- P_\phi)^T C^{-1} = 0$. These two equations are in fact linear combinations of independent matrices: $1$, $P_\phi$, $\not{p}$, and $P_\phi \not{p}$. From the equations of the coefficients of these matrices we can obtain the following consistent solutions.}
which, after taking the covariant instantaneous approximation and completing the integration $\int \frac{dp_\phi}{2\pi}$ on both sides, leads to (using $\chi = C^{-1}\tilde{\chi}$ and written in terms of the matrix form)

\[
-A_\pm^{\pm} = \frac{P_\phi}{m_\phi} \tilde{X}_{\phi}(p_t)^T \frac{P_\phi}{m_\phi} C(A_\pm^+) = \frac{i}{\pm m_\phi - 2\omega} \tag{41}
\]

\[
\times A_\pm^{\pm} \int \frac{d^4p_1}{(2\pi)^3} \left[ -\gamma^\mu \tilde{X}_{\phi}(p_t)^T \gamma_\mu K^{(1g)}(p_t - p'_t) + \tilde{X}_{\phi}(p'_t)^T K^{(1g)}(p_t - p'_t) \right] C(A_\pm^+) \tag{41}
\]

Multiplying both sides of Eq. (41) by $C^{-1}\gamma_5$ from the right and taking the trace over the spinor indices gives

\[
\tilde{f}_1(p_t) = \frac{1}{\omega(4\omega^2 - m_\phi^2)} \int \frac{d^3p'_t}{(2\pi)^3} \left\{ 2\omega^2(V^{(1g)} + 4V^{(1g)}) \tilde{f}_1(p'_t) - \frac{m_\phi}{m}(m^2 + p_t \cdot p'_t)V^{(1g)} - 2m^2V^{(1g)} \right\} \tilde{f}_2(p'_t),
\]

\[
\tilde{f}_2(p_t) = \frac{1}{\omega(4\omega^2 - m_\phi^2)} \int \frac{d^3p'_t}{(2\pi)^3} \left\{ -mm_\phi(V^{(1g)} + 4V^{(1g)}) \tilde{f}_1(p'_t) + 2\left[(m^2 + p_t \cdot p'_t)V^{(1g)} - 2m^2V^{(1g)} \right] \tilde{f}_2(p'_t) \right\}.
\]

where $V^{(1g)} = -iK^{(1g)}$ and $V^{(1g)} = -iK^{(1g)}$. At this point, let us define three-vectors $p'_t$ and $p_t$ which satisfy $p'^2_t = -P^2_t$, $p^2_t = -P^2_t$ and $p'_t \cdot p_t = -p_t \cdot p'_t$. Using these definitions, after completing the azimuthal integration, we can rewrite Eqs. (42) and (43) in the one-dimensional integration form,

\[
\tilde{f}_a(|p_t|) = \frac{1}{\omega(4\omega^2 - m_\phi^2)} \int d|p'_t| \sum_{b=1,2} \left[ A_{ab}(|p_t|, |p'_t|) \tilde{f}_b(|p'_t|) - D_{ab}(|p_t|, |p'_t|) \tilde{f}_b(|p'_t|) \right],
\]

where $a, b = 1, 2$, the ‘matrices’ are defined by

\[
A_{11} = 2\omega^2(L_0 + 4G_0), \quad A_{12} = -\frac{m_\phi}{m}(m^2L_0 + L_1 - 2m^2G_0),
\]

\[
A_{21} = -mm_\phi(L_0 + 4G_0), \quad A_{22} = 2(m^2L_0 + L_1 - 2m^2G_0),
\]

and the counter terms are

\[
D_{11} = 2\omega^2L_0, \quad D_{12} = -\frac{m_\phi}{m}(m^2 - |p_t|^2)L_0, \quad D_{21} = -mm_\phi L_0, \quad D_{22} = 2(m^2 - |p_t|^2)L_0,
\]

\[\text{§} \text{ Since } p'_t \text{ and } p_t \text{ are four-vectors perpendicular to the total momentum } P_\phi, \text{ which is a time-like four-vector, we can always accomplish this.}\]
where the definitions of \( L_0 \), \( L_1 \), and \( G_0 \) are given in Appendix A. The parameter \( \mu \) will be removed in the end of the calculation by letting \( \mu \to 0 \) (in fact, taking \( \mu \) to be sufficiently small is enough for practical calculations).

### 2.2 Normalization condition for the BS wave function of the scalar diquark

Now, we will discuss the normalization condition for the BS wave function of the scalar diquark \( \phi \). From Eqs. (15) and (19) we can define the inverse of \( \tilde{S}^{(0)} \) as

\[
I_{\phi}(p, p')_{ij} p_{ij} = \frac{1}{4} \delta_{i1}^{\alpha_1} \delta_{j1}^{\alpha_2} (2\pi)^4 \delta^{(4)}(p - p') [S(p_1)\gamma^0]_{\alpha_1 \beta_1} [S(p_2)\gamma^0]_{\alpha_2 \beta_2}^{-1}. \tag{45}
\]

From Eq. (17) we have the normalization condition for the BS wave function,

\[
\frac{i}{24} \int \frac{d^4p}{(2\pi)^4} \tilde{\chi}_{\phi}(p)_{j_2 \beta_2} \partial p_{\alpha_1} \left\{ \left[ S(p_1)\gamma^0]_{\alpha_1 \beta_1} [S(p_2)\gamma^0]_{\alpha_2 \beta_2}^{-1} \right] \chi_{\phi}(p)_{\alpha_1 \alpha_2} = 1. \tag{46}
\]

This equation can be recast in the matrix form,

\[
\frac{1}{48} \int \frac{d^4p}{(2\pi)^4} \text{Tr} \left\{ C\gamma^0 \tilde{\chi}_{\phi}(p)^T \gamma^0 \tilde{g} \tilde{\chi}_{\phi}(p)^T S(-p_1)^{-1} \right. \nonumber \n
- \left. C\gamma^0 \tilde{\chi}_{\phi}(p)^T \gamma^0 S(p_2)^{-1} \tilde{\chi}_{\phi}(p)^T \tilde{g} \right\} = 1, \tag{47}
\]

where \( \tilde{g} = (1, \tilde{0}) \). Furthermore, we can separate out the longitudinal momentum by using Eq. (27): \( \tilde{\chi}_{\phi}(p)^T = i S(p_2) \tilde{\chi}_{\phi}(p) S(-p_1) \) and define

\[
\tilde{\chi}_{\phi}(p) \equiv \gamma_5 \left\{ h_1(p) + \frac{P_\phi}{m_\phi} h_2(p) - i \sigma_{\mu\nu} \frac{P_\phi^\mu P_\phi^\nu}{m_\phi^2} h_4(p) \right\}, \tag{48}
\]

where

\[
h_1(p) = \int \frac{d^3p'}{(2\pi)^3} \left[ 4V^{(1g)}(p_\perp - p'_\perp) + V^{(cf)}(p_\perp - p'_\perp) \right] \tilde{f}_1(p'), \tag{49}
\]

\[
h_2(p) = \int \frac{d^3p'}{(2\pi)^3} \left[ -2V^{(1g)}(p_\perp - p'_\perp) + V^{(cf)}(p_\perp - p'_\perp) \right] \tilde{f}_2(p'), \tag{50}
\]

\[
h_4(p) = \int \frac{d^3p'}{(2\pi)^3} V^{(cf)}(p_\perp - p'_\perp) \frac{p_\perp \cdot p'_\perp}{p_\perp^2} \tilde{f}_4(p'). \tag{51}
\]

On the other hand, from Eq. (28) and the discussion following that equation, we have

\[
C\gamma^0 \tilde{\chi}_{\phi}(p)^T \gamma^0 = C \tilde{\chi}_{-\phi}(p)^T C^{-1} \equiv i S(-p_1) \tilde{\chi}_{\phi}(p)^{(c)} S(p_2), \tag{52}
\]

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where
\[
\tilde{\chi}_{p_\phi}(p_1)^{(c)} = \left\{ h_1(p_1) + \frac{P_\phi}{m_\phi} h_2(p_1) + i\sigma_{j\mu} \frac{P_\mu}{m_\phi^2} h_4(p_1) \right\} \gamma_5. \tag{53}
\]

With these definitions, we can write the normalization condition as
\[
-\frac{1}{48} \int \frac{d^4p}{(2\pi)^4} \text{Tr} \left\{ S(-p_1) \tilde{\chi}_{p_\phi}(p_1)^{(c)} S(p_2) \xi S(p_2) \tilde{\chi}_{p_\phi}(p_1) \right\} = 1. \tag{54}
\]

After integrating out the longitudinal momentum \( p_t \) and carrying out the trace calculation, we have the following one-dimensional integration equation:
\[
\frac{2E_\phi}{6m_\phi^2} \int \frac{|P_\perp|^2 d|P_\perp|}{2\pi^2 \omega(m_\phi^2 - 4\omega^2)^2} \left\{ 2|P_\perp|^4(h_4)^2 - |P_\perp|^2(m_\phi^2 + 4\omega^2)h_1 h_4 + 2m_\phi^2 m_\phi^2(h_2)^2 + 2m_\phi^2 \omega^2(h_1)^2 + 4mm_\phi|P_\perp|^2 h_2 h_4 - mm_\phi(m_\phi^2 + 4\omega^2)h_1 h_2 \right\} = 1, \tag{55}
\]
where \( E_\phi = P_\phi \cdot \xi = P_\phi^0 \) is the energy of the scalar diquark.

### 2.3 BS equation for the axial-vector diquark

Now we will derive the BS equation for the axial-vector diquark. Since we will not take the isospin violation into account, we can take the diquark composed of \( uu \) with \( I_3 = +1 \) as example in the following discussion. The BS wave function is defined by
\[
\chi_{p_\phi}^{(r)}(x_1, x_2)_{\alpha\beta} = \epsilon^{ijk} \langle 0 | T u_\alpha^i(x_1) u_\beta^j(x_2) | P_\phi, r, k \rangle = e^{-ip_\perp x} \int \frac{d^4p}{(2\pi)^4} \chi_{p_\phi}(p)_{\alpha\beta} e^{-ipx}, \tag{56}
\]
\[
\chi_{p_\phi}^{(r)}(x_2, x_1)_{\beta\alpha} = \epsilon^{ijk} \langle P_\phi, r, k | T u_\alpha^i(x_2) u_\beta^j(x_1) | 0 \rangle, \tag{57}
\]
where \( i, j, k \) are color indices, \( r \) is the index of the polarization vector of the axial-vector diquark. The free two-particle propagator reads
\[
\tilde{S}_{P_\phi}^{(0)}(p, p')_{\alpha_1 \alpha_2, \beta_1 \beta_2} = (2\pi)^4 \left\{ \delta^4(p - p') [S(p_1) \gamma^0 \gamma_{\alpha_1 \beta_1}^0] [S(p_2) \gamma^0 \gamma_{\alpha_2 \beta_2}^0] \right\}.
\]

The kernel arising from the one-gluon exchange diagram is
\[
\tilde{K}_{P_\phi}^{(1g)}(p, p')_{\alpha_1 \alpha_2, \beta_1 \beta_2} = \frac{(ig_s)^2}{4} \times \left\{ \gamma^\mu \gamma^\nu_{\alpha_1 \beta_1}^0 (\gamma^\nu \gamma_{\alpha_2 \beta_2}^0 \Delta_{\mu\nu}(p' - p) - (\beta_1 \leftrightarrow \beta_2, j_1 \leftrightarrow j_2, p' \leftrightarrow -p') \right\}. \tag{59}
\]
On the other hand, the confinement kernel is assumed to be

\[
\tilde{K}^{(cf)}_{P_{\varphi}}(p, p')^{i_1 i_2 j_1 j_2}_{\alpha_1 \alpha_2, \beta_2 \beta_1} = \left\{ \left( \gamma^0 \right)_{\alpha_1 \beta_1} \left( \gamma^0 \right)_{\alpha_2 \beta_2} V^{(cf)}(p - p') - (\beta_1 \leftrightarrow \beta_2, j_1 \leftrightarrow j_2, p' \leftrightarrow -p') \right\},
\]

(60)

From the above kernel and \( \tilde{S}^{(0)} \) we can obtain the BS equation for the axial-vector diquark,

\[
\tilde{\chi}^{(r)}_{P_{\varphi}}(p) = [CS(p_1)C^{-1} \otimes S(p_2)] \int \frac{d^4p'}{(2\pi)^4} \left[ C\gamma^\mu C^{-1} \otimes \gamma_\mu K^{(1g)}(p - p') + 1 \otimes 1 K^{(cf)}(p - p') \right] \cdot \tilde{\chi}^{(r)}_{P_{\varphi}}(p'),
\]

(61)

which has the same form as that for the scalar diquark, Eq. (26). Similar to the case of the scalar diquark, we can parametrize the BS wave function of the axial-vector diquark by several components \( g_f \) \((f = 1, \ldots, 8)\), which are functions of \( p_\rho \) and \( P_\varphi \):

\[
\tilde{\chi}^{(r)}_{P_{\varphi}}(p)_{\alpha_2 \alpha_1} = \varepsilon^{(r)}_{\rho} \left\{ p_\rho \frac{g_1}{m_\varphi} + \left[ p^\rho P_\mu \frac{g_2}{m_\varphi^2} + p^\rho p_\mu \frac{g_3}{m_\varphi^2} \right] \gamma_\mu + p_\mu P_\nu \gamma^{\mu\nu} \frac{g_5}{m_\varphi^2} + \gamma^\rho g_7 \\
- i \left[ p_\rho p_\mu P^\nu \frac{g_4}{m_\varphi^3} + g^{\mu\nu} P_\nu \frac{g_6}{m_\varphi} + g^{\mu\nu} p_\mu \frac{g_8}{m_\varphi} \right] \sigma^{\mu\nu} \right\}_{\alpha_1 \alpha_2},
\]

(62)

where \( \varepsilon^{(r)}_{\rho} \) is the \( r \)-th polarization state of the axial-vector diquark, which satisfies \( \varepsilon^{(r)}_{\rho} P_\varphi = 0 \). Performing the same procedure as that for the scalar diquark in the previous subsection, we have the following constraint relations for the coefficient functions:

\[
\tilde{g}_2 = \tilde{g}_8 \equiv 0, \quad \tilde{g}_1 = -\frac{p_\rho^2 \tilde{g}_3 + m_\varphi^2 \tilde{g}_7}{m_\varphi m}, \quad \tilde{g}_6 = \frac{m}{m_\varphi} \tilde{g}_5.
\]

(63)

With these constraint relations, the BS equation for the axial-vector diquark is composed of the following four integral equations:

\[
\tilde{g}_7 = \frac{1}{2\omega m_\varphi^2 p_\rho^2 (m_\varphi^2 - 4\omega^2)} \int \frac{d^3p'}{(2\pi)^3} \left\{ m m_\varphi \left[ p_\rho^2 p_\rho^2 - (p_\rho \cdot p_\rho')^2 \right] V^{(cf)} \tilde{g}_4 \right. \\
- 4m_\varphi p_\rho^2 \omega^2 (V^{(cf)} + 2V^{(1g)}) \tilde{g}_7 + 2m_\varphi^2 p_\rho^2 \left[ (m^2 + p_\rho \cdot p_\rho') V^{(cf)} - 2(p_\rho \cdot p_\rho') V^{(1g)} \right] \tilde{g}_5 \\
- 2\omega \left[ p_\rho^2 p_\rho^2 - (p_\rho \cdot p_\rho')^2 \right] (V^{(cf)} + 2V^{(1g)}) \tilde{g}_3 \right\},
\]

(64)
\[
\tilde{g}_3 = \frac{-1}{2\omega p_t^4(m_c^2 - 4\omega^2)} \int \frac{d^3p'}{(2\pi)^3} \left\{ mm_c (p_t^2 p'_t^2 + 3(p_t \cdot p'_t)^2) V^{(cf)} \tilde{g}_4 \right. \\
+ 4m^2 \varphi p_t^2 \left[ (p_t^2 + (p_t \cdot p'_t)) V^{(cf)} + 2(p_t^2 - 2(p_t \cdot p'_t)) V^{(1g)} \right] \tilde{g}_7 \\
+ 2m^2 \varphi p_t^2 (p_t \cdot p'_t) (V^{(cf)} - 2V^{(1g)}) \tilde{g}_5 \\
+ \left[ -2\omega^2 p_t^2 p'_t^2 + 4p_t^2 p'_t^2 (p_t \cdot p'_t) + 2(2m^2 + \omega^2) (p_t \cdot p'_t)^2 \right] V^{(cf)} \\
+ 4 \left[ -\omega^2 p_t^2 p'_t^2 - 2p_t^2 p'_t^2 (p_t \cdot p'_t) + (2m^2 + \omega^2) (p_t \cdot p'_t)^2 \right] V^{(1g)} \tilde{g}_3 \left\}, \quad (65) \right.
\]

\[
\tilde{g}_4 = \frac{1}{2m\omega p_t^4(m_c^2 - 4\omega^2)} \int \frac{d^3p'}{(2\pi)^3} \left\{ 2m \left[ m^2 p_t^2 p'_t^2 - (m^2 + 2\omega^2) (p_t \cdot p'_t)^2 \right] V^{(cf)} \tilde{g}_4 \\
+ 4m^2 \varphi p_t^2 \left[ (p_t^2 + (p_t \cdot p'_t)) V^{(cf)} - 2(p_t \cdot p'_t) V^{(1g)} \right] \tilde{g}_5 \\
+ 2m^2 \varphi p_t^2 (p_t \cdot p'_t) (V^{(cf)} - 4V^{(1g)}) \tilde{g}_7 \\
+ m^2 \varphi \left[ -m^2 p_t^2 p'_t^2 + 2p_t^2 p'_t^2 (p_t \cdot p'_t) + 3m^2 (p_t \cdot p'_t)^2 \right] V^{(cf)} \\
+ 2 \left[ -m^2 p_t^2 p'_t^2 - 2p_t^2 p'_t^2 (p_t \cdot p'_t) + 3m^2 (p_t \cdot p'_t)^2 \right] V^{(1g)} \tilde{g}_3 \left\}, \quad (66) \right.
\]

\[
\tilde{g}_5 = \frac{1}{2m\omega p_t^2(m_c^2 - 4\omega^2)} \int \frac{d^3p'}{(2\pi)^3} \left\{ 2m \left[ -p_t^2 p'_t^2 + (p_t \cdot p'_t)^2 \right] V^{(cf)} \tilde{g}_4 \\
+ 2m^3 \varphi p_t^2 (V^{(cf)} + 2V^{(1g)}) \tilde{g}_7 - 4m^2 \varphi p_t^2 \left[ (m^2 + p_t \cdot p'_t) V^{(cf)} - 2(p_t \cdot p'_t) V^{(1g)} \right] \tilde{g}_5 \\
+ m^2 \varphi \left[ p_t^2 p'_t^2 - (p_t \cdot p'_t)^2 \right] (V^{(cf)} + 2V^{(1g)}) \tilde{g}_3 \left\}, \quad (67) \right.
\]

where \( \omega^2 = m^2 - p_t^2 \equiv m^2 + |\mathbf{p}_t|^2 \). After carrying out the azimuthal integration of the three-momentum \( \mathbf{p}_t' \), we have

\[
\tilde{g}_a = \frac{-1}{2\omega |\mathbf{p}_t|^4(m_c^2 - 4\omega^2)} \sum_b \int d|\mathbf{p}_t'| \left[ B_{ab}(|\mathbf{p}_t|, |\mathbf{p}_t'|) \tilde{g}_b(|\mathbf{p}_t'|) - C_{ab}(|\mathbf{p}_t|, |\mathbf{p}_t'|) \tilde{g}_b(|\mathbf{p}_t|) \right], \quad (68) \right.
\]

where \( B_{ab}(|\mathbf{p}_t|, |\mathbf{p}_t'|) \) and \( C_{ab}(|\mathbf{p}_t|, |\mathbf{p}_t'|) \) \((a, b = 3, 4, 5, 7)\) are defined in Appendix A.

Using the Gaussian quadrature method to discretize the integral equations, we obtain
the following linear equations:

\[ \tilde{g}_7 = U_3 \tilde{g}_3 + U_4 \tilde{g}_4 + U_5 \tilde{g}_5 + U_7 \tilde{g}_7; \]
\[ 0 = R_3 \tilde{g}_3 + R_4 \tilde{g}_4 + R_5 \tilde{g}_5 + R_7 \tilde{g}_7; \]
\[ 0 = S_3 \tilde{g}_3 + S_4 \tilde{g}_4 + S_5 \tilde{g}_5 + S_7 \tilde{g}_7; \]
\[ 0 = T_3 \tilde{g}_3 + T_4 \tilde{g}_4 + T_5 \tilde{g}_5 + T_7 \tilde{g}_7, \] (69)

where \( U_i, R_i, S_i, T_i \) \((i = 1, \ldots, 4)\) can be read off from Eqs. (64), (65), (66), and (67), respectively. After some algebras we have the following eigenvalue equation for \( \tilde{g}_7 \):

\[ \tilde{g}_7 = U_7 \tilde{g}_7 - U_3 \left[ T_{45}^{-1} T_{43} - S_{45}^{-1} S_{43} \right]^{-1} \left[ T_{45}^{-1} T_{47} - S_{45}^{-1} S_{47} \right] \tilde{g}_7 
- U_4 \left[ T_{53}^{-1} T_{54} - S_{53}^{-1} S_{54} \right]^{-1} \left[ T_{53}^{-1} T_{57} - S_{53}^{-1} S_{57} \right] \tilde{g}_7 
- U_5 \left[ T_{34}^{-1} T_{35} - S_{34}^{-1} S_{35} \right]^{-1} \left[ T_{34}^{-1} T_{37} - S_{34}^{-1} S_{37} \right] \tilde{g}_7, \] (70)

where, for convenience, we have defined

\[ T_{ij} = T_i^{-1} T_j - R_i^{-1} R_j, \quad S_{ij} = S_i^{-1} S_j - R_i^{-1} R_j. \] (71)

After \( \tilde{g}_7 \) is solved out from Eq. (70), we can obtain \( \tilde{g}_{3,4,5} \) by the following equations

\[ \tilde{g}_3 = - \left[ T_{45}^{-1} T_{43} - S_{45}^{-1} S_{43} \right]^{-1} \left[ T_{45}^{-1} T_{47} - S_{45}^{-1} S_{47} \right] \tilde{g}_7, \] (72)
\[ \tilde{g}_4 = - \left[ T_{53}^{-1} T_{54} - S_{53}^{-1} S_{54} \right]^{-1} \left[ T_{53}^{-1} T_{57} - S_{53}^{-1} S_{57} \right] \tilde{g}_7, \] (73)
\[ \tilde{g}_5 = - \left[ T_{34}^{-1} T_{35} - S_{34}^{-1} S_{35} \right]^{-1} \left[ T_{34}^{-1} T_{37} - S_{34}^{-1} S_{37} \right] \tilde{g}_7. \] (74)

### 2.4 Normalization condition for the BS wave function of the axial-vector diquark

The inversion of the ‘free’ four-point function (58) can be defined to be

\[ I_{P_x}(p, p')^{\alpha_1 \alpha_2, \beta_2 \beta_1} = \frac{1}{2} \delta^{\alpha_1 \alpha_2} \delta^{\beta_2 \beta_1} (2\pi)^4 \delta^4(p - p') \langle S(p) \gamma_\alpha \rangle_{\alpha_1 \beta_1} \langle S(p') \gamma_\beta \rangle_{\alpha_2 \beta_2}, \] (75)

From this equation and using the fact \( \chi_{P_x}^{(r)}(p)_{\alpha \beta} = \chi_{P_x}^{(r)}(-p)_{\beta \alpha} \) we arrive at the normalization condition for the BS wave function (see Eq. (17)),

\[ \frac{1}{24} \int \frac{d^4p}{(2\pi)^4} \text{Tr} \left\{ C \gamma^0 \tilde{\chi}_{P_x}^{(r)}(p)T \gamma^0 \tilde{g} \tilde{\chi}_{P_x}^{(r)}(p)T S(-p_1)^{-1} 
- C \gamma^0 \tilde{\chi}_{P_x}^{(r)}(p)T \gamma^0 S(p_2)^{-1} \tilde{\chi}_{P_x}^{(r)}(p)T \tilde{g} \right\} = 1. \] (76)
Similar to the case for the scalar diquark, we define

$$\tilde{\chi}^{(r)}_{\rho}(p)^T \equiv i S(p_2) \tilde{\chi}^{(r)}_{\rho}(p_1) S(-p_1),$$

(77)

$$C_{\gamma^0} \tilde{\chi}^{(r)}_{\rho}(p)^T \gamma^0 \equiv i S(-p_1) \tilde{\chi}^{(r)}_{\rho}(p_1)^{\text{(c)}} S(p_2),$$

(78)

where

$$\tilde{\chi}^{(r)}_{\rho}(p_1) = \varepsilon^{(r)}_{\rho}(p_\varphi) \left\{ \begin{array}{'l'} p_t \tilde{g}_1' + p_{t\mu} P_{\varphi} \varepsilon^{\rho\mu\nu} \tilde{g}_5' M^2 \varepsilon^{(r)}_{\rho} + \gamma^\nu \tilde{g}_1' + p_t \tilde{g}_3' \\
- i g^{\mu\nu} P_{\varphi} \tilde{g}_6' M^2 \varepsilon^{(r)}_{\rho} \end{array} \right\},$$

(79)

$$\tilde{\chi}^{(r)}_{\rho}(p_1)^{\text{(c)}} = \varepsilon^{(r)}_{\rho}(p_\varphi) \left\{ - p_t \tilde{g}_1' + p_{t\mu} P_{\varphi} \varepsilon^{\rho\mu\nu} \tilde{g}_5' M^2 \varepsilon^{(r)}_{\rho} - \gamma^\nu \tilde{g}_1' - p_t \tilde{g}_3' \right\},$$

(80)

and $\tilde{g}_1', \ldots, \tilde{g}_7'$ are defined by the following integrations:

$$\tilde{g}_1'(p_1') = \int \frac{d^3p_1'}{(2\pi)^3} (-4V^{(1g)} + V^{(cf)}) \frac{p_1 \cdot p_1'}{p_1^2} \tilde{g}_1(p_1'),$$

(81)

$$\tilde{g}_5'(p_1) = \int \frac{d^3p_1'}{(2\pi)^3} (-2V^{(1g)} + V^{(cf)}) \frac{p_1 \cdot p_1'}{p_1^2} \tilde{g}_3(p_1'),$$

(82)

$$\tilde{g}_7'(p_1) = \int \frac{d^3p_1'}{(2\pi)^3} (2V^{(1g)} + V^{(cf)}) \left[ \tilde{g}_7(p_1') + \frac{p_1^2 p_1'^2 - (p_1 \cdot p_1')^2}{2m^2 p_1^2} \tilde{g}_3(p_1') \right],$$

(83)

$$\tilde{g}_6'(p_1) = \int \frac{d^3p_1'}{(2\pi)^3} V^{(cf)} \left[ \tilde{g}_6(p_1') + \frac{p_1^2 p_1'^2 - (p_1 \cdot p_1')^2}{2m^2 p_1^2} \tilde{g}_4(p_1') \right],$$

(84)

$$\tilde{g}_3'(p_1) = \frac{1}{2} \int \frac{d^3p_1'}{(2\pi)^3} \frac{3(p_1 \cdot p_1')^2 - p_1^2 p_1'^2}{p_1^4} (2V^{(1g)} + V^{(cf)}) \tilde{g}_3(p_1'),$$

(85)

$$\tilde{g}_4'(p_1) = \frac{1}{2} \int \frac{d^3p_1'}{(2\pi)^3} \frac{3(p_1 \cdot p_1')^2 - p_1^2 p_1'^2}{p_1^4} V^{(cf)} \tilde{g}_4(p_1').$$

(86)

Then the normalization condition can be written as

$$- \frac{1}{24} \int \frac{d^4p}{(2\pi)^4} \text{Tr} \left\{ S(-p_1) \tilde{\chi}^{(r)}_{\rho}(p_1)^{\text{(c)}} S(p_2) \tilde{\chi}^{(r)}_{\rho}(p_1) S(p_2) \tilde{\chi}^{(r)}_{\rho}(p_1) \right\} = 1,$$

(87)

which is similar to the case for the scalar diquark. After integrating out the longitudinal momentum $p_1$ and carrying out the trace calculation, we have the following...
one-dimensional integral equation:

\[
\frac{2E_\varphi}{3 m_\varphi^2} \int \frac{|\mathbf{P}_i|^2 d|\mathbf{P}_i|}{2\pi^2 \omega (m^2 - \omega^2)^2} \left\{ 2m_\varphi^3 |\mathbf{P}_i|^4 \tilde{g}_1'^2 + 4m_\varphi^3 |\mathbf{P}_i|^4 \tilde{g}_5'^2 + 2m_\varphi^2 |\mathbf{P}_i|^4 \tilde{g}_4'^2 \right. \\
+ 2m^2 m_\varphi |\mathbf{P}_i|^4 \tilde{g}_3'^2 + 2m^5 (2m^2 + \omega^2) \tilde{g}_6'^2 + 2m_\varphi^5 (m^2 + 2\omega^2) \tilde{g}_7'^2 \\
+ m_\varphi |\mathbf{P}_i|^4 (m^2 + 4\omega^2) \tilde{g}_1 \tilde{g}_4' - m |\mathbf{P}_i|^4 (m^2 + 4\omega^2) \tilde{g}_3' \tilde{g}_4' - 4m_\varphi^2 |\mathbf{P}_i|^4 \tilde{g}_1' \tilde{g}_3' \\
- 4m^2 m_\varphi^3 |\mathbf{P}_i|^2 \tilde{g}_1' \tilde{g}_3' + m^2 |\mathbf{P}_i|^2 (m^2 + 4\omega^2) \tilde{g}_7' \tilde{g}_4' + 2m_\varphi^3 |\mathbf{P}_i|^2 (m^2 + 4\omega^2) \tilde{g}_5' \tilde{g}_7' \\
+ 4m m_\varphi^4 |\mathbf{P}_i|^2 \tilde{g}_1' \tilde{g}_7' - 3m_\varphi^4 (m_\varphi^2 + 4\omega^2) \tilde{g}_6' \tilde{g}_7' - m_\varphi^3 |\mathbf{P}_i|^2 (m_\varphi^2 + 4\omega^2) \tilde{g}_6' \tilde{g}_7' \\
+ m m_\varphi^2 |\mathbf{P}_i|^2 (m_\varphi^2 + 4\omega^2) \tilde{g}_6' \tilde{g}_3' - 8m m_\varphi |\mathbf{P}_i|^2 \tilde{g}_6' \tilde{g}_5' - 4m_\varphi^2 \omega^2 |\mathbf{P}_i|^2 \tilde{g}_6' \tilde{g}_4' \right\} = 1 , \tag{88}
\]

where \( E_\varphi = P_\varphi \cdot \xi = P_\varphi^0 \) is the energy of the axial-vector diquark \( \varphi \).

### 3 The effective interaction of diquarks and the pion

As pointed out in the introduction, to study the decays of baryons in the diquark picture, we should first calculate the matrix element \( \langle \pi(q)|T \phi^i(x)\varphi^j(y)|0\rangle \). This can be determined by the effective low-energy interaction of diquarks and the pion. We will calculate the effective coupling constant in this section by presenting the decay amplitude of a process, where the axial-vector diquark decays into the scalar diquark and a very soft pion, in terms of BS wave functions of diquarks \( \phi \) and \( \varphi \).

Let us first introduce the effective interaction vertex of diquarks and the pseudo-Goldstone-boson pion:

\[
\mathcal{L}_{\pi\phi\phi} = G_{\pi\phi\phi} \sum_{b,i} \phi^i \partial^\mu \pi^b \overline{\varphi}^b_{\mu} + \text{h.c.} \, , \tag{89}
\]

where \( i \) is the index of the (anti-)fundamental representation of the color group \( SU(3)_c \), \( \pi^b(x) \) is \( \pi \) meson field with \( b (= \pm, 0) \) being the isospin index. Then the decay amplitude can be calculated out to the lowest order,

\[
\langle \phi^i(P_\phi)\pi^a(q)|\phi^a,i(P_\varphi), r \rangle = (2\pi)^4 \delta^4(q + P_\phi - P_\varphi) \frac{-3G_{\pi\phi\phi}}{\sqrt{2E_\pi} \sqrt{2E_\phi} \sqrt{2E_\varphi}} q^\mu \epsilon_\mu^{(r)}(P_\varphi) \, , \tag{90}
\]

where the summation over the repeated color index \( i \) is assumed and no summation is assumed for the repeated isospin index \( a \). On the other hand, we can present this decay amplitude in terms of the following transition amplitude with the aid of the partial
Then, the effective coupling constant can be calculated approximately by

\[
\langle \phi^i(P_\phi) \pi^a(q) | \phi^{a,i}(P_\varphi), r \rangle = \int d^4 x e^{i q x} i (\Box x + m_\pi^2) \langle \phi^i(P_\phi) | \pi^a(x)^\dagger | \phi^{a,i}(P_\varphi), r \rangle \]

\[
= - \frac{q^2 - m_\pi^2}{m_\pi^2 \sqrt{2E_\pi}} \frac{q^\mu}{\sqrt{2E_\pi}} \langle \phi^i(P_\phi) | A^a_\mu(q) | \phi^{a,i}(P_\varphi), r \rangle, \tag{91}
\]

where \( f_\pi \approx 93.3 \text{ MeV} \) is the pion decay constant, and the axial-vector current is \( A_\mu^-(z) = \overline{\psi}_d \gamma_5 \gamma_\mu \psi_u(z) \). Furthermore, momentum conservation and Lorentz invariance lead to

\[
q^\mu \langle \phi^i(P_\phi) | A^a_\mu(q) | \phi^{a,i}(P_\varphi), r \rangle \sqrt{2E_\phi 2E_\varphi} = P_\phi^\mu \cdot \varepsilon^{(r)} (2\pi)^4 \delta^4(q + P_\phi - P_\varphi) A(q^2), \tag{92}
\]

where \( q = P_\varphi - P_\phi \) and \( P_\phi \cdot \varepsilon^{(r)} = 0 \) have been used. The factors \( \sqrt{2E_\phi} \) and \( \sqrt{2E_\varphi} \) are introduced in Eq. (92) for latter convenience. This can be seen from the normalization equations (55) and (88): with these energy factors absorbed into the BS wave functions of the diquarks, we shall only normalize \( \chi_{P_\phi} \) and \( \chi_{P_\varphi}^{(r)} \), which are Lorentz covariant quantities, instead of \( \chi_{P_\phi} \) and \( \chi_{P_\varphi}^{(r)} \), respectively.

Combining Eqs. (90), (91), and (92) gives

\[
\frac{m_\pi^2}{q^2 - m_\pi^2} G_{\pi\phi\varphi} = \frac{-1}{3\sqrt{2f_\pi}} A(q^2). \tag{93}
\]

This equation shows that the amplitude \( A \) must develop a pole at \( q^2 = m_\pi^2 \). As usual, we can decompose this amplitude into two terms: one has the desired pole, the other is regular:

\[
A(q^2) = B_1(q^2) + \frac{q^2}{q^2 - m_\pi^2} B_2(q^2), \tag{94}
\]

where \( B_1 \) and \( B_2 \) are smooth functions without extra poles at \( q^2 = m_\pi^2 \), and their dependence on \( q^2 \) are expected to be very weak and hence are nearly constants when \( q^2 \) is small enough, e.g. \( q^2 \in (0, m_\pi^2) \). With these expectations in mind, we have the following Goldberger-Treiman-like relations:

\[
B_1(q^2) \approx 3\sqrt{2f_\pi} G_{\pi\phi\varphi}, \quad B_2(q^2) \approx -3\sqrt{2f_\pi} G_{\pi\phi\varphi}. \tag{95}
\]

Then, the effective coupling constant can be calculated approximately by

\[
G_{\pi\phi\varphi} \approx \frac{1}{3\sqrt{2f_\pi}} A(0). \tag{96}
\]
The transition amplitude \( A(q^2) \) at \( q^2 = 0 \) is regular and it is this value that will be calculated in the following.

To calculate \( A \), one can choose a specific polarization vector. Choosing \( \varepsilon^{(r)} = P^t_\phi \) and using Eq. (92), we have

\[
(2\pi)^4 \delta^4(q + P_\phi - P_\varphi) \, A(q^2) = \frac{q'' \langle \phi^i(P_\phi) | A^\mu_\mu(q) | \varphi^{+;i}(P_\varphi), r \rangle |_{\varepsilon^{(r)} = P^t_\phi}}{(P^t_\phi)^2} \sqrt{2E_\phi 2E_\varphi}. \tag{97}
\]

Let us now turn to the calculation of the matrix element \( \langle \phi^i(P_\phi) | A^\mu_\mu(q) | \varphi^{+;i}(P_\varphi), r \rangle \), which can be represented in terms of the BS wave functions of the diquarks \( \phi \) and \( \varphi \) (using Eqs. (52) and (77)),

\[
\langle \phi^i(P_\phi) | A^\mu_\mu(q) | \varphi^{+;i}(P_\varphi), r \rangle = \frac{1}{12} (2\pi)^4 \delta^4(q + P_\phi - P_\varphi) \int \frac{d^4p_1 d^4p'_1}{(2\pi)^8} (2\pi)^4 \delta^4(p_2 - p'_2) \times \text{Tr} \left\{ S(-p'_1) \tilde{\chi}_{P_\varphi} (p'_1)^{(c)} S(p_2) \tilde{\chi}_{P_\varphi} (p_1) S(-p_1) \gamma_{\mu} \gamma_5 \right\}
\]

\[
= \frac{1}{12} (2\pi)^4 \delta^4(q + P_\phi - P_\varphi) \int \frac{d^4p}{(2\pi)^4} \text{Tr} \left\{ S(-p'_1) \tilde{\chi}_{P_\varphi} (p'_1)^{(c)} S(p_2) \tilde{\chi}_{P_\varphi} (p_1) S(-p_1) \gamma_{\mu} \gamma_5 \right\}, \tag{98}
\]

where \( p_1 = P_\varphi/2 + p \), \( p_2 = P_\varphi/2 - p \), \( p'_1 = P_\phi/2 + p'_1 \), \( p'_2 = P_\phi/2 - p'_1 \), and the delta function \( \delta^4(p_2 - p'_2) \) imposes the following constraints upon the variables in the integral:

\[
p'_1 = \frac{-|P_\phi||P^t_\phi| \cos \theta + m^2_\phi - P_\phi \cdot P_\varphi + p_t P^t_\phi}{m_\phi}, \tag{99}
\]

\[
p'^2_1 = -|P_\phi|^2 - |P^t_\phi| \cos \theta - |P^t_\phi|^2/4 + (p_t + P^t_\phi/2 - m_\varphi/2)^2 - p'_t^2, \tag{100}
\]

\[
p'_1 \cdot P^t_\phi = -|P_\phi||P^t_\phi| \cos \theta - (1/2 - p'_1/m_\phi)|P^t_\phi|^2, \tag{101}
\]

\[
p'_1 \cdot p_t = -|P_\phi|^2 - (1/2 - p'_1/m_\phi)|P^t_\phi| \cos \theta, \tag{102}
\]

\[
p'_1 \cdot P_\varphi = p_t m_\varphi - m^2_\varphi/2 + (1/2 - p'_1/m_\phi)P_\phi \cdot P_\varphi, \tag{103}
\]

where \( p_t = p \cdot P_\varphi/m_\varphi \), \( p'_1 = p' \cdot P_\phi/m_\phi \), \( p_t = p - \frac{p_t}{m_\varphi} P_\varphi \), \( p'_1 = p' - \frac{p'_1}{m_\phi} P_\phi \), and \( \cos \theta \) is the azimuthal angle between \( p_t \) and \( P^t_\phi \). By momentum conservation, \( |P^t_\phi| \) and \( P_\phi \cdot P_\varphi \) can be written as:

\[
P_\phi \cdot P_\varphi = \frac{m^2_\phi + m^2_\varphi - q^2}{2} \equiv P^t_\phi m_\varphi, \quad |P^t_\phi|^2 = -m^2_\phi + (P^t_\phi)^2, \tag{104}
\]

where we have defined \( P^t_\phi = P_\phi - \frac{p_t^t}{m_\varphi} P_\varphi \).
To integrate out the longitudinal momentum \( p_1 \), we decompose the propagators \( S(-p'_1) \), \( S(p_2) \), and \( S(-p_1) \) into

\[
S(-p'_1) = \frac{(p'_1 - m)}{2\omega_q} \left[ \frac{-i}{p_1 + P_\phi^t - m_\phi/2 - \omega_q + i\epsilon} - \frac{-i}{p_1 + P_\phi^t - m_\phi/2 + \omega_q - i\epsilon} \right], \quad (105)
\]

\[
S(p_2) = \frac{-i}{p_t - m_\phi/2 + \omega_p - i\epsilon} \tilde{\Lambda}_2^+(p_t) + \frac{-i}{p_t - m_\phi/2 - \omega_p + i\epsilon} \tilde{\Lambda}_2^-(p_t), \quad (106)
\]

\[
S(-p_1) = \frac{-i}{p_t + m_\phi/2 - \omega_p + i\epsilon} \tilde{\Lambda}_1^+(p_t) + \frac{-i}{p_t + m_\phi/2 + \omega_p - i\epsilon} \tilde{\Lambda}_1^-(p_t), \quad (107)
\]

where \( p'_1 = p_t + \frac{p_\phi}{m_\phi} P_\phi + P_\phi^t + \left( \frac{p_\phi^t}{m_\phi} - \frac{1}{2} \right) P_\phi \), \( \omega_q = \sqrt{m^2 + |P_1|^2 + |P_\phi^t|^2 + 2|P_1||P_\phi^t| \cos \theta} \), \( \omega_p = \sqrt{m^2 + |P_1|^2} \), \( m \) is the mass of the constituent quark within the diquarks \( \phi \) and \( \phi' \).

The modified projection operators are defined by \( \tilde{\Lambda}_2^\pm = \frac{P_\phi}{m_\phi} \Lambda_2^\pm \) and \( \tilde{\Lambda}_1^\pm = C^{-1}(\Lambda_1^\pm) \), and can be written out explicitly as

\[
\tilde{\Lambda}_2^\pm(p_t) = \frac{P_\phi}{2m_\phi} \pm \frac{-p_t + m}{2\omega_p} \equiv \tilde{\Lambda}_1^\pm(p_t). \quad (108)
\]

Carrying out the integration over the longitudinal momentum \( p_1 \), the transition amplitude [MS] becomes

\[
\langle \phi^i(P_\phi) | A_\mu(q) | \phi'^{+i}(P_\phi'), r \rangle = \frac{1}{12} (2\pi)^4 \delta^4(q + P_\phi - P_\phi') \int \frac{d^3 p_1}{(2\pi)^3} \times \left\{ \frac{\text{Tr}(++) |_{p_1 = m_\phi/2 - \omega_p}}{(2\omega_p - m_\phi)(P_\phi^t - \omega_p - \omega_q)} + \frac{\text{Tr}(+-) |_{p_1 = m_\phi/2 + \omega_q - P_\phi^t}}{(\omega_p + \omega_q - P_\phi^t)(m_\phi + \omega_p + \omega_q - P_\phi^t)} \right.
\]

\[
- \left. \frac{\text{Tr}(--) |_{p_1 = -m_\phi/2 - \omega_p}}{(2\omega_p + m_\phi)(m_\phi + \omega_p - \omega_q - P_\phi^t)} + \frac{\text{Tr}(++) |_{p_1 = -m_\phi/2 + \omega_q}}{(2\omega_p - m_\phi)(m_\phi - \omega_p - \omega_q - P_\phi^t)} \right. \]

\[
- \left. \frac{\text{Tr}(-+) |_{p_1 = m_\phi/2 - \omega_q - P_\phi^t}}{(\omega_p + \omega_q + P_\phi^t)(m_\phi - \omega_p - \omega_q - P_\phi^t)} - \frac{\text{Tr}(+-) |_{p_1 = m_\phi/2 + \omega_p}}{(\omega_p + \omega_q + P_\phi^t)(m_\phi + 2\omega_p)} \right\}, \quad (109)
\]

where \( \text{Tr}(\pm\pm) \) are abbreviations for the following expressions:

\[
\text{Tr}(\pm\pm) = \text{Tr} \left\{ \frac{(p'_1 - m)}{2\omega_q} \tilde{\chi}_{P_\phi}^r(p'_1)^{(c)} \tilde{\Lambda}_2^\pm \tilde{\chi}_{P_\phi'}^{(r)}(p_t) \tilde{\Lambda}_1^\pm \gamma_\mu \gamma_5 \right\}. \quad (110)
\]

The traces are too lengthy to be expressed here (and so are the transition amplitudes). Therefore, we will give only the numerical results in the following.

**Numerical results**

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Motivated by the studies of mesons in the potential model [16] and in the BS formalism [9], we take the parameter $\kappa = 0.2 \text{ GeV}^2$ in the effective potential (23) and vary the effective coupling constant $\alpha_s$ to find the solutions of the BS equations. For each value of the diquark mass, $\alpha_s$ takes a certain value. On the other hand, from the analysis of the spectrum of the heavy baryons, Ref. [17] shows that the mass of the scalar diquark is related to that of the axial-vector diquark by $m_{\phi} - m_{\varphi} \approx 0.210 \text{ GeV}$. We find that for the scalar diquark, $(m_{\phi}(\text{GeV}), \alpha_s) = (0.70, 0.590), (0.75, 0.570), (0.80, 0.555)$, while for the axial-vector diquark, $(m_{\varphi}(\text{GeV}), \alpha_s) = (0.91, 0.490), (0.96, 0.307), (1.01, 0.093)$.

With the solutions of the BS equations as the input, the transition amplitude in Eq. (109) can be calculated out. Then, from Eqs. (96) and (97) we obtain the values of $A(0)$ and $G_{\pi\varphi \phi}$. The results are listed in Table 1. It should be noted that the sign of the coupling constant $G_{\pi\varphi \phi}$ is an artifact in numerical calculation (which is related to the arbitrariness of the sign for the BS wave functions). A physically observable quantity, e.g. the decay width discussed in the following, depends only on the norm of the effective coupling constant $|G_{\pi\varphi \phi}|$.

At this point, we want to point out that the numerical results show that the “++” component (the sum of the first and fourth terms in Eq. (109)) gives the most important contribution and the “+−” and “−+” components give less important contribution ($\leq 15\%$) to the total transition amplitude. The “−−” component (the sum of the third and sixth terms in Eq. (109)) gives very small contribution to the total diquark-transition amplitude.

Table 1: The transition amplitude $A(q^2)$ at $q^2 = 0$ and the effective coupling constant $G_{\pi\varphi \phi}$ corresponding to various values of the mass of the scalar diquark $\phi$ (the mass of the axial-vector diquark is related to that of the scalar diquark by $m_{\phi} - m_{\varphi} \approx 0.21 \text{ GeV}$).

| $m_{\phi}$ (GeV) | 0.70 | 0.75 | 0.80 |
|------------------|------|------|------|
| $A(0)$ (GeV)     | 1.32 | 1.43 | −1.53|
| $G_{\pi\varphi \phi}$ | 3.35 | 3.62 | −3.88|
4 Strong decays of heavy baryons

In this section, we will turn to the calculation of the decay widths of the processes \( \Sigma_Q^{(*)} \rightarrow \Lambda_Q + \pi \). As pointed out in the introduction, the heavy baryons \( \Lambda_Q \) and \( \Sigma_Q^{(*)} \) are regarded as bound states of the heavy quark \( Q \) and the diquarks, which are composed of two light quarks. In this picture, we can express the decay amplitudes in terms of the BS wave functions of \( \Lambda_Q \) and \( \Sigma_Q^{(*)} \) (taking \( \Sigma_Q^{++} \) as an example),

\[
\left\langle \Lambda_Q^+(P_\Lambda)\pi^+(q)|\Sigma_Q^{++}(P_\Sigma)\right\rangle \\
= \int d^4(x_1x_2y_1y_2uv)\bar{x}_{P_\Lambda}(x_2, x_1)S_Q(x_1 - y_1)^{-1}x_{P_\Sigma,\lambda}(y_1, y_2) \\
\times \Delta^{-1}_\phi(x_2 - u)\Delta^{-1,\mu\lambda}(v - y_2)\sum_i(\pi^+(q)|T\phi^i(u)\varphi^i_\mu(v)|0),
\]

where the superscript \( i \) is the color index, \( \phi^i \) is the field of the scalar diquark and \( \varphi^i_\mu \) is the field of the axial-vector diquark. \( \Delta^{-1}_\phi \) and \( \Delta^{-1,\mu\lambda}_\varphi \) are the inverse of the propagators of the scalar diquark and the axial-vector diquark, respectively. They are defined by \( \Delta^{-1}_\phi(x, y)\Delta_\phi(y, z) = \delta^4(x - z) \) and \( \Delta^{-1,\mu\lambda}_\varphi(x, y)\Delta_\varphi,\lambda(y, z) = \delta^\mu_\lambda\delta^4(x - z) \). \( S_Q \) \( (Q = c, b) \) is the quark propagator. The BS wave function of the heavy baryons are defined by

\[
\chi_{P_\Lambda}(x_1, x_2) = \langle 0|T\psi^i_Q(x_1)\phi^i(x_2)|P_\Lambda \rangle,
\]

\[
\chi_{P_\Sigma,\mu}(y_1, y_2) = \langle 0|T\psi^i_Q(y_1)\varphi^i_\mu(y_2)|P_\Sigma \rangle.
\]

If not explicitly pointed out, the summation is understood for repeated color index \( i \).

The complex conjugate of the BS wave function is defined by \( \bar{x}(x_2, x_1) = \chi(x_1, x_2)^*\gamma^0 \).

To simplify the analysis, we take, as usual, the propagators of the diquarks and the heavy quark to have the forms of the free ones. Then we have

\[
\Delta^{-1}_\phi(x, y) = i(\Box_x + m_\phi^2)\delta^4(x - y),
\]

\[
\Delta^{-1,\mu\lambda}_\varphi(x, y) = i\left[(\Box_x + m_\varphi^2)g^{\mu\lambda} + \partial_x^\mu\partial_x^\lambda\right]\delta^4(x - y),
\]

\[
S_Q(x, y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{\not{p} - m_Q + i\epsilon}e^{-ipx},
\]

where \( m_Q \) is the constituent mass of the heavy quark which can be determined by, e.g., fitting experimental data to the results of the potential model for mesons, \( m_\phi \) and \( m_\varphi \) are the masses of diquarks \( \phi \) and \( \varphi \), respectively.

The matrix element \( \langle \pi^+(q)|T\phi^i(u)\varphi^i_\mu(v)|0 \rangle \) can be calculated out to the lowest order with the effective interaction vertex defined in Eq. (89),

\[
\langle \pi^+(q)|T\phi^i(u)\varphi^i_\mu(v)|0 \rangle = -3G_{n,\phi,\varphi}q^\mu\sqrt{2E}\int d^4z e^{iqz}\Delta_\phi(u - z)\Delta_\varphi(z - v)\mu\nu,
\]
where the factor 3 comes from the summation over the color index. Then Eq. (111) becomes

$$
\langle \Lambda_Q^+(P_\Lambda)\pi^+(q)|\Sigma_Q^{++}(P_\Sigma)\rangle = \frac{-3G_F\delta(2\pi)^4\delta^4(q + P_\Lambda - P_\Sigma)}{\sqrt{2E_\pi}} q^\lambda
\times \int \frac{d^4p'd^4p}{(2\pi)^8}(2\pi)^4\delta^4(p'_1 - p_1)\bar{\chi}_{P_\Lambda}(p') S_Q(p_1)^{-1}\chi_{P_\Sigma\lambda}(p),
$$

(118)

where $p_1$ is the momentum of the heavy quark $Q$ within the baryon $\Sigma_Q^{(*)}$ and $p'_1$ is the momentum of the heavy quark $Q$ within the baryon $\Lambda_Q$, $p_2$ and $p'_2$ are the momenta of the diquarks $\varphi$ and $\phi$, respectively. These momenta are related to the total momenta of the bound states by the following equations:

$$
p'_1 = \lambda_1^1 P_\Lambda + p', \quad p'_2 = \lambda_1^2 P_\Lambda - p', \quad p_1 = \lambda_1 P_\Sigma + p, \quad p_2 = \lambda_2 P_\Sigma - p,
$$

(119)

where the parameters $\lambda_{1,2}$ and $\lambda_{1,2}'$ are defined by

$$
\lambda'_1 = \frac{m_Q}{m_Q + m_\varphi}, \quad \lambda'_2 = \frac{m_\phi}{m_Q + m_\phi}, \quad \lambda_1 = \frac{m_Q}{m_Q + m_\varphi}, \quad \lambda_2 = \frac{m_\varphi}{m_Q + m_\varphi}.
$$

(120)

FollowingRefs. [6] and [7], we can present the BS wave functions of the baryons as

$$
\bar{\chi}_{P_\Lambda}(p') = i u_\Lambda(v') N(p'_1, p'_2) \Delta_\psi(-p'_2) S_Q(p'_1),
$$

(121)

where $u_\Lambda$ is the Dirac spinor of $\Lambda_Q$. Since there is a delta-function $\delta(p'_1 - p_1)$ in Eq. (118), to the leading order in $1/m_Q$ expansion, we have

$$
\bar{\chi}_{P_\Lambda}(p') S_Q(p_1)^{-1} = \frac{i}{(p'_1 + m_\Sigma - P_\Lambda^2 - W^2_q + \varepsilon)} u_\Lambda(v') N(p'_1, p'_2).
$$

(122)

Similarly, to the leading order in $1/m_Q$ expansion, we can present the BS wave function in the baryon $\Sigma_Q^{(*)}$ as

$$
\chi_{P_\Sigma}^\lambda(p) = \frac{-i}{(p'_1 + E_0 + m_\varphi + \varepsilon)(p'_1^2 - W^2_q - \varepsilon)} M^{\lambda\mu}(p'_1, p') B^{(m)}(v),
$$

(123)

where $m = 1$ for $\Sigma_Q$, $m = 2$ for $\Sigma_Q^{*}$, $E_0$ is the binding energy of the baryon $\Sigma_Q^{(*)}$ which will be determined in Sect. [3]. For more details on discussions about the BS wave functions of heavy baryons, the readers are referred to the original references [6,7]. For convenience, we also give some results in Appendix [3]. In the above equations, we have defined the following quantities: $W_q = \sqrt{-(p'_1 - P_\Lambda^2)^2 + m_\varphi^2}$, $W_p = \sqrt{-p_1^2 + m_\varphi^2}$, $p'_1 = p'_1 - p' \cdot v' - \lambda_2 m_\Sigma$, $p'_1 = p' - (p' \cdot v')v'$ and $p_1 = p - (p \cdot v)v$. 

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are perpendicular to the “velocities” of the baryons, \( v' = P_\Lambda/m_\Lambda \) and \( v = P_\Sigma/m_\Sigma \), respectively. \( P_\Lambda^k = (m_\Lambda^2 + m_\Sigma^2 - q^2)/(2m_\Sigma) = m_\Lambda v \cdot v' \) is a constant, \( P_\Lambda = P_\Lambda - \frac{p_\Sigma}{m_\Sigma} P_\Sigma \) is perpendicular to \( P_\Sigma \). The polarization vectors of the baryons \( \Sigma_Q \) and \( \Sigma_Q^* \) are given by

\[
B^{(1)}_\mu(v) = \frac{1}{\sqrt{3}}(\gamma_\mu + v_\mu)\gamma_5 u_\Sigma(v), \quad B^{(2)}_\mu(v) = u_\mu(v),
\]

(124)

where \( u_\Sigma(v) \) is the Dirac spinor and \( u_\mu(v) \) is the Rarita-Schwinger vector spinor. \( B^{(m)}_\mu(v) \) satisfies the following conditions:

\[
f^\mu B^{(m)}_\mu(v) = B^{(m)}_\mu(v), \quad v^\mu B^{(m)}_\mu(v) = 0, \quad \gamma^\mu B^{(2)}_\mu(v) = 0.
\]

(125)

The above constraints for \( m = 1 \) can be seen from \( f u_\Sigma(v) = u_\Sigma(v) \) while for \( m = 2 \), they are the properties of the Rarita-Schwinger vector spinor with spin \( \frac{3}{2} \). The expressions of \( N(p'_i, p'_f) \) and \( M^{\lambda \mu}(p_i, p_l) \) are given in Appendix [13] (these expressions are extracted from Eq. (12) in Ref. [6] and Eq. (28) in Ref. [7], some notations in this paper are different from those in the original references). On the other hand, the delta-function \( \delta(p - p'_i) \) leads to the following relations:

\[
p_i \cdot P_\Lambda^k = -m_\Lambda |p_i| \sqrt{\Omega^2 - 1} \cos \theta, \quad p_i' = \Omega p_i - |p_i| \sqrt{\Omega^2 - 1} \cos \theta + \Omega m_\Sigma - m_\Lambda, \quad p_i'^2 = -|p_i|^2 \sin^2 \theta - \left[ \Omega |p_i| \cos \theta - (m_\Sigma + p_i) \sqrt{\Omega^2 - 1} \right]^2,
\]

(126)

(127)

(128)

where \( \Omega = v' \cdot v \).

The poles of \( p_i \) and \( p'_i \) are all from the prefactors in Eqs. (122) and (123) since \( N(p'_i, p'_f) \) depends linearly on \( p'_f \) and \( M^{\lambda \sigma}(p_i, p_l) \) contains \( p_l \) up to second order [6, 7]. After carrying out the integration over \( p_i \) in Eq. (118), we have

\[
\langle \Lambda_Q^0 (P_\Lambda) \pi^+(q) | \Sigma_Q^{*+} (P_\Sigma) \rangle = \frac{3iG_{\pi \phi \phi} \bar{u}_\Lambda(v') B^{(m)}_\rho(v)}{\sqrt{2E_\pi \sqrt{2E_\Lambda \sqrt{2E_\Sigma}}} C^\rho(q^2) \left( 2\pi \right)^4 \delta^4(q + P_\Lambda - P_\Sigma)},
\]

(129)

where

\[
C^\rho(q^2) = \sqrt{2E_\Lambda \sqrt{2E_\Sigma}} \int \frac{d^3 p_i}{(2\pi)^3} \left\{ \frac{q_\Lambda M^{\lambda \rho}(p) N(p')}{(2W_p W_\Sigma + E_\Sigma^2 + m_\varphi)\left( (W_p + m_\Sigma - P_\Lambda^k)^2 - W_q^2 \right)} \right. \\
+ \frac{q_\Lambda M^{\lambda \rho}(p) N(p')}{(2W_q (-m_\Sigma + P_\Lambda^k + W_q + E_\Sigma^2 + m_\varphi) \left( (m_\Sigma + P_\Lambda^k + W_q)^2 - W_p^2 \right)} \\
+ \frac{q_\Lambda M^{\lambda \rho}(p) N(p')}{(2W_q(-m_\varphi - E_0^\Sigma + m_\Sigma - P_\Lambda^k)^2 - W_q^2) \left( (m_\varphi + E_0^\Sigma)^2 - W_p^2 \right)} \right\}.
\]

(130)
The tensor function $M^\lambda^\rho$ is defined by

$$M^\lambda^\rho(p) = g^\lambda^\rho M_1(p) + \frac{v^\lambda^p}{m_\varphi^2} M_2(p) - \frac{p^\lambda^\rho}{m_\varphi^2} M_3(p),$$

(131)

where $g^{\mu\nu} = \text{diag}\{1, -1, -1, -1\}$ is the Lorentz metric tensor, $M_n \ (n = 1, 2, 3)$ are defined in Eqs. (B6)-(B8) in Appendix B. When calculating the contraction, we will use the orthogonal condition $v^\rho B^\rho_\nu(v) = 0$. Furthermore, the integrations of terms containing $p^\rho B^\rho_\nu(v)$ have the following form:

$$\int d^3p \, p^\rho B^\rho_\nu(v) h(p, P_\lambda^1) = P_\lambda^1 \cdot g(|P_\lambda^1|),$$

(132)

where $h$ is arbitrary smooth function and

$$g(|P_\lambda^1|) = \int d^3p \, \frac{p \cdot P_\lambda^1}{(P_\lambda^1)^2} h(p, P_\lambda^1).$$

(133)

Then only terms containing the factor $P_\lambda^1 \cdot B^\rho_\nu(v)$ can appear in the final expression. The calculation is straightforward and we have

$$C^\rho(q^2) = P_\lambda^1 \cdot D(q^2) = P_\lambda^1 \cdot \sqrt{2E_\Lambda \sqrt{2E_\Sigma}} \int_0^{\infty} \frac{|P_\lambda| \, |P| \, d\theta}{4\pi^2} \int_{-1}^{1} d\cos \theta \, \frac{1}{m_\varphi^2 |P_\lambda^1|}$$

$$\times \left\{ N_a \frac{(m_\Sigma - m_\Lambda \Omega) m_\varphi |p| \cos \theta M_{2a} - |P_\lambda^1|^2 (m_\varphi^2 M_{1a} + |p|^2 \cos^2 \theta M_{3a})}{2W_p (W_p + E_\Lambda^2 + m_\varphi) [(W_p + m_\Sigma - P_\lambda^1)^2 - W_q^2]}$$

$$+ N_b \frac{(m_\Sigma - m_\Lambda \Omega) m_\varphi |p| \cos \theta M_{2b} - |P_\lambda^1|^2 (m_\varphi^2 M_{1b} + |p|^2 \cos^2 \theta M_{3b})}{2W_q (-m_\Sigma + P_\lambda^1 \cdot W_q + E_\Lambda^2 + m_\varphi) [(m_\Sigma - W_q - P_\lambda^1)^2 - W_p^2]}$$

$$+ N_c \frac{(m_\Sigma - m_\Lambda \Omega) m_\varphi |p| \cos \theta M_{2c} - |P_\lambda^1|^2 (m_\varphi^2 M_{1c} + |p|^2 \cos^2 \theta M_{3c})}{(-m_\varphi - E_\Lambda^2 + m_\Sigma - P_\lambda^1)^2 - W_q^2} \right\},$$

(134)

where $N_a \equiv N(p)|_{p_\Lambda = W_p}$, $N_b \equiv N(p)|_{p_\Lambda = -m_\Sigma + W_q + P_\lambda^1}$, $N_c \equiv N(p)|_{p_\Lambda = -m_\varphi - E_\Sigma^2}$, $M_{na} \equiv M_n(p)|_{p_\Lambda = W_p}$, $M_{nb} \equiv M_n(p)|_{p_\Lambda = -m_\Sigma + W_q + P_\lambda^1}$, $M_{nc} \equiv M_n(p)|_{p_\Lambda = -m_\varphi - E_\Sigma^2} \ (n = 1, 2, 3)$.

Furthermore, $|P_\lambda^1| \equiv m_\Lambda \sqrt{\Omega^2 - 1}$ and $P_\lambda^1 \equiv m_\Lambda \Omega$ are constants. Note that the energy factors $\sqrt{2E_\Lambda}$ and $\sqrt{2E_\Sigma}$ will be absorbed into the corresponding BS wave functions.

This is convenient since they appear in the normalization conditions (C4) and (C9) given in Appendix C and make the normalization conditions be Lorentz invariant.

In the rest frame of $\Sigma(Q^\Sigma)$, the differential decay width of $\Sigma(Q^\Sigma) \rightarrow \Lambda_Q + \pi$ is

$$d\Gamma = \frac{1}{32\pi^2} \frac{|P_\lambda^1|}{m_\Sigma^2} |\mathcal{M}|^2 d\Omega,$$

(135)

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where $d\Omega$ is the solid angle of the particle in the final state, and $P_\Lambda$ is the three-momentum of the baryon $\Lambda_Q$. The Lorentz-invariant amplitude in our case can be written as

$$M = -3G_{\pi\phi\phi} \, D(q^2) \, \pi_\Lambda(v') B_\rho^{(m)}(v) P_\Lambda^\rho.$$  \hfill (136)

We will calculate the unpolarized decay width averaging over the spins of the initial states and summing over the spins of the final states. For $\Sigma_Q$, we have

$$\frac{1}{2} \sum_{s',s} \pi_\Lambda(v', s') B_\rho^{(1)}(v, s) P_\Lambda^\rho |^2 = \frac{m_\Lambda^2}{6} (\Omega - 1)(\Omega + 1)^2,$$  \hfill (137)

while for $\Sigma^*_Q$, we have

$$\frac{1}{4} \sum_{s',s} \pi_\Lambda(v', s') B_\rho^{(2)}(v, s) P_\Lambda^\rho |^2 = \frac{m_\Lambda^2}{6} (\Omega - 1)(\Omega + 1)^2.$$  \hfill (138)

While deriving the above two equations we have used the following formula for the spin sum of the Dirac spinor:

$$\sum_{s=1,2} u_B(k_B, s) \pi_B(k_B, s) = \frac{\gamma_B + 1}{2}, \quad B = \Lambda_Q \text{ or } \Sigma_Q,$$  \hfill (139)

with $v_{\Lambda_Q} = v'$, $v_{\Sigma_Q} = v$, and the formula for the spin sum of the Rarita-Schwinger spinor $[11, 18]$, \hfill (140)

$$\sum_{s=1}^4 u^\mu(k, s) \pi^\nu(k, s) = \frac{\gamma + 1}{2} \left\{ -g^{\mu\nu} + \frac{1}{3} \gamma^\mu \gamma^\nu + \frac{2}{3} v^\mu v^\nu + \frac{1}{3} (\gamma^\mu v^\nu - \gamma^\nu v^\mu) \right\},$$  \hfill (140)

where the fact $(\gamma + 1)\gamma = \gamma + 1$ has been used in the final step. The total unpolarized decay width in the final form then reads

$$\Gamma(\Sigma_Q^* \to \Lambda_Q + \pi) = \frac{3}{16\pi} |G_{\pi\phi\phi}|^2 |D(q^2)|^2 \frac{|P_\Lambda|}{m_\Sigma^*} \frac{m_\Lambda^2}{m_\Sigma^*} (\Omega - 1)(\Omega + 1)^2.$$  \hfill (141)

5 Numerical analysis

We will first discuss the parameters appearing in this paper. These parameters include $m_\phi$, $m_\varphi$, $\kappa_B$, $m_Q$, and $E_0$. The study of mesons in the BS equation approach $[9]$ shows that the values of the masses of the heavy quarks $m_c = 1.58 \text{ GeV}$ and $m_b = 5.02 \text{ GeV}$ lead to predictions in good agreement with experiment.
Now, we discuss the ranges of the masses of diquarks $\phi$ and $\varphi$. In contrast with the colorless hadronic states, diquarks (and other color states) are not free. Therefore, their (effective) masses cannot be measured in experiments. In our discussion, we will treat these masses as parameters varying in reasonable ranges which satisfy various constraints from physical considerations. From the analysis of the spectrum of the heavy baryons, Ref. [17] shows that the mass of the scalar diquark is related to that of the axial-vector diquark: $m_\varphi - m_\phi \approx 0.210 \text{ GeV}$. In our calculation, the mass of $\phi$ is taken to be in the range $m_\phi \in (0.70, 0.80) \text{ GeV}$ and the corresponding range of the mass of $\varphi$ is $m_\varphi \in (0.91, 1.01) \text{ GeV}$.

Now consider $\kappa_B$. It is argued in Refs. [6, 7] that the parameter $\kappa_B$ in the effective potential (See Eq. (B3)) can be ranged approximately from $0.02 \text{ GeV}^3$ to $0.1 \text{ GeV}^3$. Furthermore, by studying the average momentum of $b$-quark in $\Lambda_b$ and comparing with the value of this quantity derived from the experimental value of the average momentum of the $b$-quark in the $B$ meson with the aid of HQET, the authors in Ref. [10] show that $\kappa_B$ can be constrained to a narrower range: “When $m_\phi$ are 0.7 GeV and 0.8 GeV, $\kappa_B$ are roughly in the ranges ($0.02 - 0.06$) GeV$^3$ and ($0.02 - 0.04$) GeV$^3$, respectively.” Following this, in this paper, we will calculate the decay widths in the range $\kappa_B \in (0.02 - 0.06) \text{ GeV}^3$.

Now we will determine $m_\varphi + E_0$. In the heavy quark limit, the baryons $\Sigma_Q$ and $\Sigma^*_Q$ should be degenerate and the dynamics inside them are the same. In the heavy quark limit, we can write out the masses of the baryons as

$$m_{\Sigma_Q} = m_Q + m_\varphi + E_0 + \mathcal{O}\left(\frac{1}{m_Q}\right),$$

where $m_\varphi + E_0$ is the value to the leading order in $1/m_Q$ expansion and then is universal for all heavy baryons (with one heavy quark). Since $m_b \gg m_c$, the $1/m_Q$ corrections for $b$-baryons are much smaller than those for $c$-baryons, hence we shall take the data of $b$-baryons as the input to calculate this quantity. From the recent results of CDF Collaboration [4], we have

$$m_\varphi + E_0 \approx 0.81 \text{ GeV}$$

$^\parallel$ $m_\varphi - m_\phi = 0.211 \text{ GeV}$ for the $c$-baryons and $m_\varphi - m_\phi = 0.208 \text{ GeV}$ for the $b$-baryons (the masses of $b$-baryons are taken from Ref. [4]).

$^\star\star$ We calculate the following quantity by using the spin-averaged mass of $\Sigma_b^{(*)}$, $\overline{m}_{\Sigma_b} = (2m_{\Sigma^*_b} + 4m_{\Sigma_b^*})/6$, where $m_{\Sigma_b} = (m_{\Sigma^*_b} + m_{\Sigma_b^*})/2$ and $m_{\Sigma^*_b} = (m_{\Sigma_b^{*+}} + m_{\Sigma_b^{*-}})/2$. Notice that all effects of the isospin symmetry violation are omitted.
for $\Sigma^{(s)}_Q$ when the $1/m_Q$ corrections are omitted. For $\Lambda_Q$, we have

$$m_\phi + E_0 \approx 0.60 \text{ GeV}$$ (144)

when the $1/m_Q$ corrections are omitted.

With the parameters determined above, the decay widths of the processes $\Sigma^{(s)}_{c,b} \rightarrow \Lambda_{c,b} + \pi$ can be obtained. The results are shown in Table 2. From this table, one can

Table 2: The decay widths $\Gamma(\Sigma^{(s)}_{c,b} \rightarrow \Lambda_{c,b} + \pi)$ to the leading order in $1/m_Q$ expansion. The violation of $SU(2)$ symmetry is not taken into account. The unit of $m_\phi$ is GeV, the unit of $\Gamma$ is MeV, the unit of $\kappa_B$ is GeV$^3$. The mass of $\varphi$ is related to that of $\phi$ by $m_\varphi - m_\phi \approx 0.21 \text{ GeV}$.

| $m_\phi$ | 0.70 | 0.75 | 0.80 |
|----------|------|------|------|
| $\kappa_B$ | 0.02 0.04 0.06 | 0.02 0.04 0.06 | 0.02 0.04 0.06 |
| $\Gamma(\Sigma_c)$ | 6.61 4.83 3.75 | 4.91 3.85 3.15 | 3.95 3.26 2.77 |
| $\Gamma(\Sigma^{*}_c)$ | 18.88 14.83 12.11 | 17.54 14.26 12.01 | 15.99 13.49 11.69 |
| $\Gamma(\Sigma_b)$ | 13.45 10.20 8.10 | 11.16 8.88 7.34 | 9.47 7.88 6.73 |
| $\Gamma(\Sigma^{*}_b)$ | 17.74 13.76 11.09 | 15.76 12.68 10.57 | 13.92 11.65 10.00 |

see that the theoretical result for $\Gamma(\Sigma^{*}_c)$ is consistent with the experimental data [2], $\Gamma^{\text{exp}}(\Sigma^{*}_c) \approx (15 - 16) \text{ MeV}$. However, the theoretical value for $\Gamma(\Sigma_c)$ is bigger than the experimental data [2], $\Gamma^{\text{exp}}(\Sigma_c) \approx 2.2 \text{ MeV}$. We attribute this to the $1/m_Q$ corrections which are not taken into account in this paper. To look at this more transparently, one can estimate roughly the $1/m_Q$ corrections as follows. If the corrections to the magnitudes of the BS wave functions are $\Lambda_{QCD}/m_c = 0.16$ (for $\Lambda_{QCD} = 0.26 \text{ GeV}$), the corrections to the final results of the decay widths can be very large, $(\Gamma + \delta\Gamma)/\Gamma = 1.84$ to 0.49 for $c$-baryons. For $b$-baryons, $\Lambda_{QCD}/m_b = 0.05$, then we have $(\Gamma + \delta\Gamma)/\Gamma = 1.22$ to 0.81. Since the corrections for $b$-baryons are much smaller than those for $c$-baryons we expect that the predictions for the decay widths of $\Sigma^{(s)}_b$ are far more precise than those for $\Sigma^{(s)}_c$.  

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Discussions and conclusions

In this paper, we have first studied the properties of two kinds of diquarks, the scalar diquark $\phi$ and the axial-vector diquark $\varphi$, in the BS formalism. We have derived the BS equations for these two kinds of diquarks and studied all the BS equations under the covariant instantaneous approximation, which allows one to obtain the BS wave functions in a general coordinate system directly. With these BS wave functions of the diquarks, we have calculated the effective coupling constant among the diquarks and the pion, $G_{\pi\phi\varphi}$. We find that this effective coupling constant is $|G_{\pi\phi\varphi}| \in (3.35, 3.88)$.

With this effective coupling constant, we have calculated the decay widths of the heavy baryons $\Sigma_Q^{(*)} (Q = c, b)$ in the BS formalism in the heavy quark limit $m_Q \to \infty$.

There are two parameters $m_\phi$ (and $m_\varphi$) and $\kappa_B$ in our model which cannot be determined in this paper. Following the arguments in Ref. [10], we take $\kappa_B \in (0.02 - 0.06) \text{GeV}^3$. For the diquark-masses we take $m_\phi \in [0.70, 0.80] \text{GeV}$ and $m_\varphi \in [0.91, 1.01] \text{GeV}$. With these ranges of parameters, we give the predictions for the decay widths of $\Sigma_Q^{(*)} \to \Lambda_Q + \pi$ (the effects of the isospin violation are not taken into account):

\begin{align*}
\Gamma(\Sigma_c) &\approx (2.77 - 6.61) \text{MeV}, \quad \Gamma(\Sigma_c^*) \approx (11.69 - 18.88) \text{MeV}, \quad (145) \\
\Gamma(\Sigma_b) &\approx (6.73 - 13.45) \text{MeV}, \quad \Gamma(\Sigma_b^*) \approx (10.00 - 17.74) \text{MeV}. \quad (146)
\end{align*}

From Eq. (145), we can see that the calculated value for $\Gamma(\Sigma_c^*)$ is consistent with the experimental results [2], $\Gamma^\text{exp}(\Sigma_c^*) \approx (15 - 16) \text{MeV}$, but the calculated value for $\Gamma(\Sigma_c)$ deviates from the experimental results [2], $\Gamma^\text{exp}(\Sigma_c) \approx 2.2 \text{MeV}$. We attribute the deviation to the $1/m_c$ corrections which are not taken into account in this paper. However, we expect that the predictions for $b$-baryons are much more precise than those for $c$-baryons since $m_b \gg m_c$. Furthermore, the above results show that the decay widths of $\Sigma_b \to \Lambda_b + \pi$ will be much bigger than those of $\Sigma_c \to \Lambda_c + \pi$.

For comparison, let us quote the results in Ref. [19], where the decay widths of $\Sigma_b^{(*)}$ were calculated in the bag model (in units of MeV):

\begin{align*}
\Gamma(\Sigma_b^+) &= 4.35, \quad \Gamma(\Sigma_b^0) = 5.65, \quad \Gamma(\Sigma_b^-) = 5.77, \quad (147) \\
\Gamma(\Sigma_b^{*+}) &= 8.50, \quad \Gamma(\Sigma_b^{*0}) = 10.20, \quad \Gamma(\Sigma_b^{*-}) = 10.44, \quad (148)
\end{align*}

which are comparable with our results in Eq. (146).

In this paper, we omit all the $1/m_Q$ corrections in the calculations. If the $1/m_Q$ corrections are taken into account, we can take the value of $\Gamma(\Sigma_c)$, which has been
measured precisely in experiments, as the input to constrain the parameters $\kappa_B$, $m_\phi$, and $m_\varphi$ and make more precise predictions for $\Gamma(\Sigma^*_b)$ and $\Gamma(\Sigma^{*(b)})$. The full $1/m_Q$ corrections include the $1/m_Q$ corrections to the kernel, to the heavy quark propagators, and to the BS wave functions. The $1/m_Q$ corrections for $\Lambda_Q$ have been discussed in Ref. [20]. Since the study of these $1/m_Q$ corrections is very complicated, this is beyond the scope of the present paper and will be discussed elsewhere.

Finally, the effect of the isospin violation is not taken into account to make the presentation of the calculation more transparent. After taking into account the effect of the isospin symmetry violation one can obtain more information on the properties of heavy baryons (if including the strange quark $s$, one can also give predictions for $\Omega_Q$ and $\Xi_Q$). This will be the subjects in the future work.

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A Some definitions used in the previous sections

In this appendix, we give definitions of functions used in Subsections 2.1 and 2.3. The sixteen ‘matrices’ $B_{ij}$ in Subsection 2.3 can be written as

\begin{align}
B_{33} &= -2 \omega^2 |p|^2 |p'|^2 L_0 + 4 |p|^2 |p'|^2 L_1 + 2 (2m^2 + \omega^2) L_2 \\
&\quad - 4 \omega^2 |p|^2 |p'|^2 G_0 - 16 |p|^2 |p'|^2 G_1 + 4 (2m^2 + \omega^2) G_2, \\
B_{34} &= mm_\varphi (|p|^2 |p'|^2 L_0 - 3 L_2), \\
B_{35} &= -2m_\varphi^2 |p|^2 (L_1 - 2G_1), \\
B_{37} &= -4m_\varphi^2 |p|^2 (-|p|^2 L_0 + L_1 - 2|p|^2 G_0 - 4G_1),
\end{align}

\(^{11}\) In (and only in) this appendix, $p$ and $p'$ are in fact $p_t$ and $p'_t$, respectively, which appear in Sect. 
\(^{2}\) The change of the notation is to make the expressions more transparent.
\[
B_{43} = -\frac{m_\phi}{m} \left[ -m^2 p^2 + 2|p|^2 p' L_0 + 3m^2 L_2 - 2m^2 |p|^2 p' G_0 - 8|p|^2 |p'|^2 G_1 + 6m^2 G_2 \right], \quad (A5)
\]

\[
B_{44} = -2 \left[ m^2 |p|^2 p' L_0 - (m^2 + 2\omega^2) L_2 \right], \quad (A6)
\]

\[
B_{45} = 4mm_\phi |p|^2 \left[ -|p|^2 L_0 + L_1 - 2G_1 \right], \quad (A7)
\]

\[
B_{47} = \frac{2m^2 |p|^2}{m} (L_1 - 4G_1), \quad (A8)
\]

\[
B_{53} = |p|^2 \left[ |p|^2 p' L_0 - L_2 + 2|p|^2 |p'|^2 G_0 - 2G_2 \right], \quad (A9)
\]

\[
B_{54} = \frac{2m |p|^2}{m_\phi} \left[ -|p|^2 p' L_0 + L_2 \right], \quad (A10)
\]

\[
B_{55} = 4|p|^4 \left[ m^2 L_0 + L_1 - 2G_1 \right], \quad (A11)
\]

\[
B_{57} = -2m^2 |p|^4 (L_0 + 2G_0), \quad (A12)
\]

\[
B_{73} = -\frac{2\omega^2 |p|^2}{m_\phi} \left[ |p|^2 |p'|^2 L_0 - L_2 + 2|p|^2 |p'|^2 G_0 - 2G_2 \right], \quad (A13)
\]

\[
B_{74} = \frac{m |p|^2}{m_\phi} \left[ |p|^2 |p'| L_0 - L_2 \right], \quad (A14)
\]

\[
B_{75} = -2|p|^4 \left[ m^2 L_0 + L_1 - 2G_1 \right], \quad (A15)
\]

\[
B_{77} = 4\omega^2 |p|^4 (L_0 + 2G_0), \quad (A16)
\]

and the counter terms are
\[
C_{33} = 4|p|^4 (m^2 - |p|^2) L_0, \quad C_{34} = -2mm_\phi |p|^4 L_0,
\]
\[
C_{35} = 2m^2 |p|^4 L_0, \quad C_{37} = 8m^2 |p|^4 L_0, \quad (A17)
\]

\[
C_{43} = -\frac{2m_\phi |p|^4}{m} (m^2 - |p|^2) L_0, \quad C_{44} = 4\omega^2 |p|^4 L_0,
\]
\[
C_{45} = -8mm_\phi |p|^4 L_0, \quad C_{47} = -\frac{2m^2 |p|^4}{m} L_0, \quad (A18)
\]

\[
C_{53} = C_{54} = 0, \quad C_{55} = 4|p|^4 (m^2 - |p|^2) L_0, \quad C_{57} = -2m^2 |p|^4 L_0, \quad (A19)
\]

\[
C_{73} = C_{74} = 0, \quad C_{75} = -2|p|^4 (m^2 - |p|^2) L_0, \quad C_{77} = 4|p|^4 \omega^2 L_0, \quad (A20)
\]

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where the functions $L_i$ and $G_i$ $(i = 0, 1, 2)$ are defined by

$$
\int \frac{d^3p'}{(2\pi)^3} \left\{ \frac{4\pi \kappa}{[(p - p')^2 + \mu^2]^2} f(|p'|) \right\} = \int d|p'| L_0(|p|, |p'|) f(|p'|),
$$

$$
\int \frac{d^3p'}{(2\pi)^3} \left\{ \frac{4\pi \kappa(p \cdot p')}{[(p - p')^2 + \mu^2]^2} f(|p'|) \right\} = \int d|p'| L_1(|p|, |p'|) f(|p'|),
$$

$$
\int \frac{d^3p'}{(2\pi)^3} \left\{ \frac{4\pi \kappa(p \cdot p')^2}{[(p - p')^2 + \mu^2]^2} f(|p'|) \right\} = \int d|p'| L_2(|p|, |p'|) f(|p'|),
$$

$$
\int \frac{d^3p'}{(2\pi)^3} \left\{ \frac{2g_s^2}{3} \frac{1}{(p - p')^2 + \mu^2} f(|p'|) \right\} = \int d|p'| G_0(|p|, |p'|) f(|p'|),
$$

$$
\int \frac{d^3p'}{(2\pi)^3} \left\{ \frac{2g_s^2}{3} \frac{p \cdot p'}{(p - p')^2 + \mu^2} f(|p'|) \right\} = \int d|p'| G_1(|p|, |p'|) f(|p'|),
$$

$$
\int \frac{d^3p'}{(2\pi)^3} \left\{ \frac{2g_s^2}{3} \frac{(p \cdot p')^2}{(p - p')^2 + \mu^2} f(|p'|) \right\} = \int d|p'| G_2(|p|, |p'|) f(|p'|),
$$

with

$$
L_0 = \frac{2\kappa}{\pi} \frac{|p'|^2}{(|p|^2 + |p'|^2 + \mu^2)^2 - 4|p|^2|p'|^2},
$$

$$
L_1 = -\frac{\kappa}{4\pi} \frac{|p|}{|p'|} \left\{ \frac{4|p||p'|}{(|p|^2 + |p'|^2 + \mu^2)^2 - 4|p|^2|p'|^2} \log \frac{(|p| - |p'|)^2 + \mu^2}{(|p| + |p'|)^2 + \mu^2} \right\},
$$

$$
L_2 = \frac{\kappa}{4\pi} \frac{|p'|}{|p|} \left\{ \frac{4|p||p'|}{(|p|^2 + |p'|^2 + \mu^2)^2 - 4|p|^2|p'|^2} \log \frac{(|p| - |p'|)^2 + \mu^2}{(|p| + |p'|)^2 + \mu^2} \right\},
$$

$$
G_0 = -\frac{\alpha_s}{3\pi} \frac{|p'|}{|p|} \log \frac{(|p| - |p'|)^2 + \mu^2}{(|p| + |p'|)^2 + \mu^2},
$$

$$
G_1 = \frac{\alpha_s}{6\pi} \frac{|p'|}{|p|} \left\{ \frac{4|p||p'|}{(|p|^2 + |p'|^2 + \mu^2)} \log \frac{(|p| - |p'|)^2 + \mu^2}{(|p| + |p'|)^2 + \mu^2} \right\},
$$

$$
G_2 = -\frac{\alpha_s}{12\pi} \frac{(|p|^2 + |p'|^2 + \mu^2)}{|p|} \left\{ \frac{|p'|}{|p|} \frac{4|p||p'|}{(|p|^2 + |p'|^2 + \mu^2)} \log \frac{(|p| - |p'|)^2 + \mu^2}{(|p| + |p'|)^2 + \mu^2} \right\},
$$

where $\alpha_s = g_s^2/(4\pi)$, $p \cdot p' = -|p||p'| \cos \theta$. 

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B  Some results in Refs. [6, 7]

In the previous sections, we have used the BS wave functions of the heavy baryons \( \Lambda_Q \) and \( \Sigma_Q^{(*)} \) in Refs. [6, 7] as the input to calculate the decay amplitudes. In this appendix, for readers’ convenience, we write out some expressions (taking from Refs. [6, 7], but in a different notation) used in this paper explicitly. For more details, we refer the readers to the original references.

Following Ref. [6], one can write the BS wave function of \( \Lambda_Q \) to the leading order in \( 1/m_Q \) expansion in terms of the product of a scalar function \( N(p') \) and the spinor of \( \Lambda_Q \),

\[
\chi_{\Lambda}^i(p') = i\Delta (p') S(p') N(p') u_{\Lambda}(v'),
\]

where the scalar function \( N(p') \) is given by the following integration:

\[
N(p'_{i\ell}, p_{i\ell} |) = \int \frac{d^3k_i}{(2\pi)^3} (\tilde{V}_1 + 2p_i' \tilde{V}_2) \tilde{\phi}_\Lambda (k_i).
\]

\( \tilde{\phi}_\Lambda \) depends only on the norm of \( k_i \) and has been solved numerically in Ref. [6]. The kernel in the above equation is defined by

\[
\tilde{V}_1 = \frac{8\pi \kappa_B}{[(p_i' - k_i)^2 + \mu^2]^2} - (2\pi)^3 \delta^4(p_i' - k_i) \int \frac{d^3l}{(2\pi)^3} \frac{8\pi \kappa_B}{(l^2 + \mu^2)^2},
\]

\[
\tilde{V}_2 = -\frac{16\pi}{3} \frac{\alpha_s^2 B Q_0^2}{[(p_i' - k_i)^2 + \mu^2][(p_i' - k_i)^2 + Q_0^2]},
\]

where \( Q_0^2 = 3.2 \text{ GeV}^2 \), \( \mu \) is a parameter which is taken to be small enough so that the numerical result is insensitive to this parameter.

Following Ref. [7], one can write the BS wave functions of \( \Sigma_Q^{(*)} \) to the leading order in \( 1/m_Q \) expansion in terms of the product of a tensor function \( M^{\lambda\mu}(p) \) and the spinor of \( \Sigma_Q^{(*)} \), \( B_{\mu(m)}(m = 1, 2, \text{corresponding to } \Sigma_Q \text{ and } \Sigma_Q^{(*)}, \text{respectively}) \)

\[
\chi_{\Sigma_Q}(p) = \frac{-i}{(p_{\lambda} + E_0^{\Sigma} + m_\varphi + i\varepsilon)(p_{\lambda}^2 - W_p^2 + i\varepsilon)} M^{\lambda\mu}(p) B_{\mu}^{(m)}(v).
\]

The tensor function is given by

\[
M^{\lambda\mu}(p) = g^{\lambda\mu} M_1(p_{\lambda}, |p_{\lambda}|) + \frac{v^{\lambda} p^\mu_{\lambda}}{m_\varphi} M_2(p_{\lambda}, |p_{\lambda}|) - \frac{p^{\lambda}_{\lambda} p^\mu_{\lambda}}{m_\varphi^2} M_3(p_{\lambda}, |p_{\lambda}|),
\]

where

\[
M_1 = \int \frac{d^3k_i}{(2\pi)^3} \left\{ \tilde{A} + \tilde{D} \frac{(p_i \cdot k_i)^2 - p_i^2 k_i^2}{2p_i^2} \right\} (\tilde{V}_1 + 2p_i \tilde{V}_2) - \tilde{C} \frac{(p_i \cdot k_i)^2 - p_i^2 k_i^2}{2p_i^2} \tilde{V}_2,
\]

\[
M_2(p_{\lambda}, |p_{\lambda}|) = \int \frac{d^3k_i}{(2\pi)^3} \left\{ \tilde{D} \frac{(p_i \cdot k_i)^2 - p_i^2 k_i^2}{2p_i^2} \right\} (\tilde{V}_1 + 2p_i \tilde{V}_2),
\]

\[
M_3(p_{\lambda}, |p_{\lambda}|) = \int \frac{d^3k_i}{(2\pi)^3} \left\{ \tilde{C} \frac{(p_i \cdot k_i)^2 - p_i^2 k_i^2}{2p_i^2} \right\} (\tilde{V}_1 + 2p_i \tilde{V}_2).
\]
\[ M_2 = \frac{1}{m_\phi} \int \frac{d^3k_t}{(2\pi)^3} \left\{ -\tilde{A} + \tilde{D} \left( \frac{p_t \cdot k_t}{p_t^2} \right)^2 \right\} \left[ p_t \tilde{V}_1 + (p_t^2 + m_\phi^2) \tilde{V}_2 \right] \]

\[ -\tilde{C} \left[ (p_t^2 - m_\phi^2) \frac{p_t \cdot k_t}{p_t^2} \tilde{V}_1 + p_t \left( \frac{p_t \cdot k_t}{p_t^2} \right)^2 \tilde{V}_2 \right] \right\}, \tag{B7} \]

\[ M_3 = \int \frac{d^3k_t}{(2\pi)^3} \left\{ \tilde{A} (\tilde{V}_1 + p_t \tilde{V}_2) \right. \]

\[ + \tilde{C} \left[ p_t \frac{p_t \cdot k_t}{p_t^2} \tilde{V}_1 + \frac{m_\phi^2 (3(p_t \cdot k_t)^2 - p_t^2 k_t^2)}{2p_t^4} \tilde{V}_2 \right] \]

\[ + \tilde{D} \left[ -\frac{m_\phi^2 (3(p_t \cdot k_t)^2 - p_t^2 k_t^2)}{2p_t^4} (\tilde{V}_1 + 2p_t \tilde{V}_2) + p_t \left( \frac{p_t \cdot k_t}{p_t^2} \right)^2 \tilde{V}_2 \right] \} \right\}. \tag{B8} \]

The three wave functions, \( \tilde{A} \), \( \tilde{C} \), and \( \tilde{D} \) in Eqs. \( \text{(B6)} \)-\( \text{(B8)} \) are scalar functions which depend only on the norm of \( k_t \) and have been calculated numerically in Ref. [7].

It is worth pointing out that the following conventions have been used here, \( p_t^2 = |p_t|^2 \), \( p_t \cdot k_t = |p_t||k_t| \), etc., which are different from our conventions used to discuss the BS equations of diquarks.

### C Normalization of the BS wave functions of the heavy baryons

In this appendix, we will give the normalization conditions for the BS wave functions of \( \Lambda_Q \) and \( \Sigma_Q^{(*)} \). The normalization conditions of the BS wave functions of \( \Lambda_Q \) and \( \Sigma_Q^{(*)} \) have been discussed in Refs. [6, 7]. In order to calculate the decay amplitudes (other than the weak transition amplitude) of baryons in our case, it is necessary to obtain the BS wave function for each baryon separately. We start from a normalization equation which is similar to that of the diquark in Eq. \( \text{(17)} \). Furthermore, in the heavy quark limit \( m_Q \to \infty \), the normalization conditions in this section should take the same form as those obtained through the normalization of the Isgur-Wise functions at the zero recoil point, \( \xi(\Omega = 1) = 1 \) [6, 7]. This can be checked with explicit derivation.

*Normalization of \( \chi_{p_A} \)*
The normalization equation for the BS wave function of the baryon $\Lambda_Q$ is given by

$$i \int \frac{d^4 p \, d^4 p'}{(2\pi)^8} \chi_{p_A}^\dagger(p, s) \left\{ \frac{\partial}{\partial p'_A} I_{p_A}(p, p') \right\} \chi_{p_A}(p', s') = \delta_{s,s'},$$  \hspace{1cm} (C1)

where $s$ and $s'$ are indices of the spin of the baryon and

$$I_{p_A}(p, p') = \frac{1}{3} (2\pi)^4 \delta^4(p - p') S^{-1}_Q(p_1) \Delta^{-1}_\phi(p_2).$$  \hspace{1cm} (C2)

Then we have

$$i \int \frac{d^4 p}{(2\pi)^4} \chi_{p_A}^\dagger(p, s) \left\{ S^{-1}_Q(p_1) \Delta^{-1}_\phi(p_2) \right\} \chi_{p_A}(p, s') = \delta_{s,s'}.$$  \hspace{1cm} (C3)

After carrying out the integration over $p_1$, the normalization equation becomes (multiplying $\delta_{s,s'}$ on both sides and summing over the spin indices)

$$\frac{E_A}{6m_A} \int \frac{d^3 p}{(2\pi)^3} \frac{\tilde{\phi}_1 - 2W_p \tilde{\phi}_2}{W'_p(E_0 + m_\phi - W'_p)^2} \left\{ \tilde{\phi}_1 + 2 \left[ (1 - 2\lambda'_1)W_p - 2\lambda'_2(E_0 + m_\phi) \right] \tilde{\phi}_2 \right\} = 1,$$

which can be reduced further to a one-dimensional integral equation,

$$\frac{E_A}{6m_A} \int \frac{|p|}{2\pi} \frac{|p|}{2\pi} \frac{\tilde{\phi}_1 - 2W_p \tilde{\phi}_2}{W'_p(E_0 + m_\phi - W'_p)^2} \left\{ \tilde{\phi}_1 + 2 \left[ (1 - 2\lambda'_1)W_p - 2\lambda'_2(E_0 + m_\phi) \right] \tilde{\phi}_2 \right\} = 1,$$

(C4)

where $W_p = \sqrt{|p|^2 + m_\phi^2}$, $\tilde{\phi}_{1,2}$ are defined by

$$\tilde{\phi}_1(p) = \int \frac{d^3 k_t}{(2\pi)^3} \tilde{V}_1(p_t - k_t) \tilde{\phi}_{p_A}(k_t), \quad \tilde{\phi}_2(p) = \int \frac{d^3 k_t}{(2\pi)^3} \tilde{V}_2(p_t - k_t) \tilde{\phi}_{p_A}(k_t).$$  \hspace{1cm} (C5)

As pointed out in the beginning of this section, in the heavy quark limit, the normalization condition above should reduce to that obtained by the normalization of the Isgur-Wise function at the zero recoil point. This can be checked easily. In the heavy quark limit, $m_Q \to \infty$, we have $\lambda'_1 = 1$ and $\lambda'_2 = 0$, then Eq. (C4) becomes

$$\frac{E_A}{3m_A} \int \frac{|p|}{2\pi} \frac{|p|}{2\pi} \frac{\tilde{\phi}_1 - 2W_p \tilde{\phi}_2}{2W'_p(E_0 + m_\phi - W'_p)^2} = 1.$$  \hspace{1cm} (C6)

One can see that this is the same equation as that given in Ref. [6] at the zero recoil point $\Omega = 1$ (see Eq. (26) in Ref. [6], notice that $\Omega$ is written as $\omega$ there) \hspace{1cm} (C6)

In fact there is an extra factor $E_A/(3m_A)$ in this equation when compared with Eq. (26) of Ref. [6]. The energy factor $E_A/m_A$ is required to make the BS wave function $\tilde{\phi}$ be a Lorentz scalar in our convention of one-particle states, $\langle \mathbf{p} | \mathbf{p'} \rangle = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p'})$, which is different from that used in Ref. [6], $\langle \mathbf{p} | \mathbf{p'} \rangle = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p'})E_p/m$. Furthermore, the factor $1/3$, which comes from the summation of the color indices, does not appear in Eq. (26) of Ref. [6]. However, the ignorance of this color factor does not affect the results when one calculate quantities like those in Ref. [6].
Normalization of $\chi_{P_S}^\lambda$

For the baryon $\Sigma_Q^{(s)}$, the normalization equation of the BS wave function is

$$i \int \frac{d^4 p \, d^4 p'}{(2\pi)^8} \chi_{P_S}^\mu(p, s) \left\{ \frac{\partial}{\partial P_0} I_{P_S}(p, p')_{\mu\nu} \right\} \chi_{P_S}^{\nu'}(p', s') = \delta_{s, s'}, \quad (C7)$$

where

$$I_{P_S}(p, p')_{\mu\nu} = \frac{1}{3} (2\pi)^4 \delta^4(p - p') S_Q^{-1}(p_1) \Delta^{-1}_\varphi(p_2)_{\mu\nu}. \quad (C8)$$

Multiplying $\delta_{ss'}$ on both sides of Eq. $(C7)$ and summing over $s$ and $s'$, the normalization equation becomes (using the relations given in Appendix B)

$$1 = \int \frac{d^3 p}{(2\pi)^3} \frac{\lambda_3(v \cdot \eta)}{18m_p^4 \lambda(p - E_0 - m_\varphi)^2} \left\{ \frac{\lambda_1}{\lambda_2} W_p^2 \left[ -3m_\varphi M_1 M_1' + m_\varphi^2 |p_1|^2 (M_1 M_3' + M_2 M_2' - M_3 M_1') + |p_1|^4 M_3 M_3' \right] \right|_{p_x = -W_p}^{} + 3m_\varphi^2 W_p^2 (M_{11} - W_{p}M_{11}) [M_{11} + (W_p - 2\Delta_B) M_{11}]$$

$$+ m_\varphi^2 |p_1|^2 W_p^2 (M_{31} + m_\varphi M_{21}) [W_p (2\Delta_B - W_p) M_{31} - \Delta_B M_{31}]$$

$$+ W_p^2 (M_{10} - \Delta_B M_{31}) (2M_{30} + m_\varphi M_{21}) + m_\varphi M_{20} \left[ (\Delta_B - 2W_p) M_{10} + W_p^2 M_{11} \right]$$

$$+ |p_1|^4 W_p^2 (M_{31} + m_\varphi M_{22}) [W_p (2\Delta_B - W_p) M_{31} - \Delta_B M_{30}]$$

$$+ W_p^2 (M_{30} - \Delta_B M_{31}) (M_{30} + m_\varphi M_{21}) + m_\varphi M_{20} \left[ (\Delta_B - 2W_p) M_{30} + W_p^2 M_{31} \right], \quad (C9)$$

where $\Delta_B = E_0 + m_\varphi$, $W_p = \sqrt{|p_1|^2 + m_\varphi^2}$, $v \cdot \eta = E_{\Sigma}/m_{\Sigma}$, $M_n$ ($n=1,2,3$) have been defined in Eqs. $(B6)$-$(B8)$, and $M_{nm}(nm = 10, 11, 20, 21, 22, 30, 31)$ are defined by the expansion of $M_n$’s,

$$M_1 = M_{10} + p_{x} M_{11}, \quad M_2 = M_{20} + p_{x} M_{21} + p_{y}^2 M_{22}, \quad M_3 = M_{30} + p_{x} M_{31}. \quad (C10)$$

$M'_n$ ($n = 1, 2, 3$) in Eq. $(C9)$ are given by $M'_1 = M_1$, $M'_2 = M_2$, and $M'_3 = -(M_{30} + m_\varphi M_{21}) - p_{x} (M_{31} + m_\varphi M_{22})$. In the heavy quark limit, Eq. $(C9)$ reduces to

$$1 = \int \frac{d|p_1|}{2\pi^2} \frac{|p_1|^2 (v \cdot \eta)}{18m_p^4 \lambda(W_p - E_0 - m_\varphi)^2} \left\{ -3m_\varphi^2 M_1 M_1' + m_\varphi^2 |p_1|^2 (M_1 M_3' + M_2 M_2' - M_3 M_1') + |p_1|^4 M_3 M_3' \right\}_{p_x = -W_p}.$$
which gives the same result as that obtained through the normalization of Isgur-Wise function at the zero recoil point [7] (the discussion about the extra factor appearing in this equation is the same as that given in the previous footnote).
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