SURFACES GIVEN WITH THE MONGE PATCH IN \( \mathbb{E}^4 \)

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Abstract

A depth surface of \( \mathbb{E}^3 \) is a range image observed from a single view can be represented by a digital graph (Monge patch) surface. That is, a depth or range value at a point \((u,v)\) is given by a single valued function \(z = f(u,v)\).

In the present study we consider the surfaces in Euclidean 4-space \( \mathbb{E}^4 \) given with a Monge patch \( z = f(u,v), w = g(u,v) \). We investigated the curvature properties of these surfaces. We also give some special examples of these surfaces which are first defined by Yu. Aminov. Finally, we proved that every Aminov surface is a non-trivial Chen surface.

1 Introduction

In recent years there has been a tremendous increase in computer vision research using range images (or depth maps) as sensor input data \[1\]. The most attractive feature of range images is the explicitness of the surface information. Many industrial and navigational robotic tasks will be more easily accomplished if such explicit depth information can be efficiently obtained and interpreted. Classical differential geometry provides a complete local description of smooth surfaces \[4\], \[13\]. The first and second fundamental forms of surfaces provide a set of differential-geometric shape descriptors that capture domain-independent surface information. Gaussian curvature is an intrinsic surface property which refers to an isometric invariant of a surface \[4\]. Both Gaussian and mean curvatures have the attractive characteristics of translational and rotational invariance. A depth surface is a range image observed from a single view can be represented by a digital graph (Monge patch) surface. That is, a depth or range value at a point \((u,v)\) is given by a single valued function \(z(u,v)\).

One interesting class of surfaces in \( \mathbb{E}^3 \) is that of translation surfaces, which can be parameterized, locally, as \( z(u,v) = f(u) + g(v) \), where \( f \) and \( g \) are smooth functions. From the definition, it is clear that translation surfaces are double curved surfaces. Therefore, translation surfaces are made up of quadrilaterals, that is, four sided, facets. Because of this property, translation surfaces are used in architecture to design and construct free-form glass roofing structures.

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see [9]. Scherk’s surface, obtained by H. Scherk in 1835, is the only non flat minimal surface, that can be represented as a translation surface [10]. Translation surfaces have been investigated from the various viewpoints by many differential geometers. L. Verstraelen, J. Walrave and S. Yaprak have investigated minimal translation surfaces in n-dimensional Euclidean spaces [17].

In [5] B.Y. Chen defined the allied vector field \( a(v) \) of a normal vector field \( v \). In particular, the allied mean curvature vector field is orthogonal to \( H \). Further, B.Y. Chen defined the \( \mathcal{A} \)-surface to be the surfaces for which \( a(H) \) vanishes identically. Such surfaces are also called Chen surfaces [10]. The class of Chen surfaces contains all minimal and pseudo-umbilical surfaces, and also all surfaces for which \( \text{dim}N_1 \leq 1 \), in particular all hypersurfaces. These Chen surfaces are said to be Trivial \( \mathcal{A} \)-surfaces [11]. In [15], B. Rouxel considered ruled Chen surfaces in \( \mathbb{E}^n \). For more details, see also, [6] and [12].

This paper is organized as follows: Section 2 gives some basic concepts of the surfaces in \( \mathbb{E}^4 \). Section 3 tells about the surfaces given with a Monge patch in \( \mathbb{E}^4 \). Further this section provides some basic properties of surfaces in \( \mathbb{E}^4 \) and the structure of their curvatures. In the third section we consider Aminov surfaces given with the Monge patch in \( \mathbb{E}^4 \). We also present some examples of these surfaces. We obtain few new interesting results. Namely, we obtain some equations on \( r(u) \), when on \( M \) the equation \( K + K_N = 0 \) has place. Then we obtain the condition for the case \( M \) is a Wintgen ideal surface. We remark that on the Wintgen ideal surfaces the equation \( K + K_N = \|H\|^2 \) has place. In the final section we obtain an important equation on the coefficients of the second quadratic form for Chen surfaces, when it is given at arbitrary parametrization. We also proved that every Aminov surfaces in \( \mathbb{E}^4 \) are non-trivial Chen surfaces.

## 2 Basic Concepts

Let \( M \) be a smooth surface in \( \mathbb{E}^4 \) given with the patch \( X(u,v) : (u,v) \in D \subset \mathbb{E}^2 \). The tangent space to \( M \) at an arbitrary point \( p = X(u,v) \) of \( M \) span \( \{X_u, X_v\} \). In the chart \( (u,v) \) the coefficients of the first fundamental form of \( M \) are given by

\[
E = \langle X_u, X_u \rangle, \quad F = \langle X_u, X_v \rangle, \quad G = \langle X_v, X_v \rangle, \quad (1)
\]

where \( \langle , \rangle \) is the Euclidean inner product. We assume that \( W^2 = EG - F^2 \neq 0 \), i.e. the surface patch \( X(u,v) \) is regular. For each \( p \in M \), consider the decomposition \( T_p \mathbb{E}^4 = T_p M \oplus T_p \perp M \) where \( T_p \perp M \) is the orthogonal component of \( T_p M \) in \( \mathbb{E}^4 \).

Let \( \nabla \) be the Riemannian connection of \( \mathbb{E}^4 \). Given any local vector fields \( X_i, X_j \) tangent to \( M \).

Let \( \chi(M) \) and \( \chi^\perp(M) \) be the space of the smooth vector fields tangent to \( M \) and the space of the smooth vector fields normal to \( M \), respectively. Consider the second fundamental map: \( h : \chi(M) \times \chi(M) \to \chi^\perp(M) \);

\[
h(X_i, X_j) = \tilde{\nabla}_X_i X_j - \nabla_X_i X_j, \quad 1 \leq i, j \leq 2. \quad (2)
\]

where \( \tilde{\nabla} \) is the induced. This map is well-defined, symmetric and bilinear.

For any arbitrary orthonormal normal frame field \( \{N_1, N_2\} \) of \( M \), recall the shape operator \( A : \chi^\perp(M) \times \chi(M) \to \chi(M) \);

\[
A_{N_i} X_i = - (\nabla_{N_i} N_i)^T, \quad X_i \in \chi(M). \quad (3)
\]
This operator is bilinear, self-adjoint and satisfies the following equation:

\[ \langle A_{N_k}X_i, X_j \rangle = \langle h(X_i, X_j), N_k \rangle = c_{ij}^k, \quad 1 \leq i, j, k \leq 2. \quad (4) \]

The equation (2) is called Gaussian formula, and

\[ h(X_i, X_j) = \sum_{k=1}^{2} c_{ij}^k N_k, \quad 1 \leq i, j \leq 2 \quad (5) \]

where \( c_{ij}^k \) are the coefficients of the second fundamental form.

Further, the Gaussian curvature and Gaussian torsion of a regular patch \( X(u, v) \) are given by

\[ K = \frac{1}{W^2} \sum_{k=1}^{2} (c_{11}^k c_{22}^k - (c_{12}^k)^2), \quad (6) \]

and

\[ K_N = \frac{1}{W^2} \left( E \left( c_{12}^1 c_{22}^2 - c_{12}^2 c_{22}^1 \right) - F \left( c_{11} c_{22}^1 - c_{11}^1 c_{22}^2 \right) + G \left( c_{11}^1 c_{12} - c_{11} c_{12}^1 \right) \right), \quad (7) \]

respectively.

Further, the mean curvature vector of a regular patch \( X(u, v) \) is defined by

\[ \overrightarrow{H} = \frac{1}{2W^2} \sum_{k=1}^{2} (c_{11}^k G + c_{22}^k E - 2c_{12}^k F) N_k. \quad (8) \]

Recall that a surface \( M \) is said to be \textit{minimal} if its mean curvature vector vanishes identically [5].

The surface patch \( X(u, v) \) is called \textit{pseudo-umbilical} if the shape operator with respect to \( H \) is proportional to the identity (see, [5]).

### 3 Surfaces Given with a Monge Patch in \( \mathbb{E}^4 \)

2-dimensional surfaces in \( \mathbb{E}^4 \) are interesting object for investigation of geometers. Here we have some difficult problems which wait its solutions. For example, it is unknown does there exist an isometric regular immersion of the whole Lobachevsky plane into \( \mathbb{E}^4 \). Hence the investigation of various classes of surfaces in \( \mathbb{E}^4 \) with point of view of influence of the principal invariants - Gauss curvature \( K \), Gauss torsion \( K_N \) and the vector of mean curvature \( \overrightarrow{H} \) on the behavior of surfaces is an actual problem.

In the considering work we use the representation of surfaces in the explicit form

\[ r(u, v) = (u, v, f(u, v), g(u, v)), \quad (9) \]

where \( f \) and \( g \) are some smooth functions. The parametrization (9) is called \textit{Monge patch in} \( \mathbb{E}^4 \).

First we obtain the following result.
Theorem 3.1. Let $M$ be a smooth surface given with the Monge patch \[[9]. Then the mean curvature vector of $M$ becomes

$$\overrightarrow{H} = \frac{1}{2\sqrt{AW^2}}(Gf_{uu} - 2Ff_{uv} + Ef_{vv})N_1$$

$$+ \frac{1}{2\sqrt{AW^3}}(G(-Bf_{uu} + Ag_{uu}) - 2F(-Bf_{uv} + Ag_{uv}) + E(-Bf_{vv} + Ag_{vv}))N_2$$

where

\[
A = 1 + (f_u)^2 + (f_v)^2,
B = f_u g_u + f_v g_v,
C = 1 + (g_u)^2 + (g_v)^2
\]

such that $EG - F^2 = AC - B^2$.

Proof. The tangent space of $M$ is spanned by the vector fields

$$\frac{\partial X}{\partial u} = (1, 0, f_u, g_u),$$

$$\frac{\partial X}{\partial v} = (0, 1, f_v, g_v).$$

Hence, the coefficients of the first fundamental form of the surface are

$$E = \langle X_u(u,v), X_u(u,v) \rangle = 1 + (f_u)^2 + (g_u)^2,$$

$$F = \langle X_u(u,v), X_v(u,v) \rangle = f_u f_v + g_u g_v,$$

$$G = \langle X_v(u,v), X_v(u,v) \rangle = 1 + (f_v)^2 + (g_v)^2,$$

where $\langle , \rangle$ is the standard scalar product in $\mathbb{R}^4$.

The second partial derivatives of $X(u,v)$ are expressed as follows

$$X_{uu}(u,v) = (0, 0, f_{uu}, g_{uu}),$$

$$X_{uv}(u,v) = (0, 0, f_{uv}, g_{uv}),$$

$$X_{vv}(u,v) = (0, 0, f_{vv}, g_{vv}).$$

Further, the normal space of $M$ is spanned by the vector fields

$$N_1 = \frac{1}{\sqrt{A}}(-f_u, -f_v, 1, 0)$$

$$N_2 = \frac{1}{\sqrt{A}}(Bf_u - Ag_u, Bf_v - Ag_v, -B, A).$$

Using \[[11], \[[13] and \[[14] we can calculate the coefficients of the second fun-
damental form $h$ are as follows:

$$
c^1_{11} = \langle X_{uu}(u,v), N_1 \rangle = \frac{f_{uu}}{\sqrt{A}},
$$

$$
c^1_{12} = \langle X_{uv}(u,v), N_1 \rangle = \frac{f_{uv}}{\sqrt{A}},
$$

$$
c^1_{22} = \langle X_{vv}(u,v), N_1 \rangle = \frac{f_{vv}}{\sqrt{A}}, \tag{15}
$$

$$
c^2_{11} = \langle X_{uu}(u,v), N_2 \rangle = \frac{-Bf_{uu} + Ag_{uv}}{W\sqrt{A}},
$$

$$
c^2_{12} = \langle X_{uv}(u,v), N_2 \rangle = \frac{-Bf_{uv} + Ag_{uv}}{W\sqrt{A}},
$$

$$
c^2_{22} = \langle X_{vv}(u,v), N_2 \rangle = \frac{-Bf_{vv} + Ag_{vv}}{W\sqrt{A}}. \tag{16}
$$

Further, substituting (12) and (15) into (8) we get (10). This completes the proof of the theorem. \qed

In [2] Yu. Aminov proved the following result.

**Theorem 3.2.** [2] Let $M$ be a smooth surface given with the Monge patch (9). Then the Gaussian curvature $K$ and Gaussian torsion $K_N$ of $M$ become

$$
K = \frac{C(f_{uw}f_{vw} - f_{uv}^2) - B(f_{uw}g_{uv} + g_{uu}f_{uw} - 2f_{uw}g_{uv}) + A(g_{uu}g_{uv} - g_{uv}^2)}{W^4} \tag{16}
$$

and

$$
K_N = \frac{E(f_{uw}g_{uv} - g_{uw}f_{uv}) - F(f_{uw}g_{uv} - g_{uu}f_{uw}) + G(f_{uw}g_{uv} - g_{uv}f_{uw})}{W^4} \tag{17}
$$

respectively.

**Proposition 3.1.** Let $M$ be a smooth surface given with the Monge patch of the form

$$
\begin{align*}
f(u,v) &= \phi_u(u,v), \\
g(u,v) &= \phi_v(u,v). \tag{18}
\end{align*}
$$

Then the Gaussian curvature $K$ coincides with the Gaussian torsion $K_N$ of $M$.

**Proof.** Suppose $M$ is a smooth surface given with the Monge patch (9). Then by the use of (11) with (12) we get

$$
\begin{align*}
E &= A = 1 + (\phi_{uu})^2 + (\phi_{uv})^2, \\
F &= B = \phi_{uu}\phi_{uv} + \phi_{uv}\phi_{vv}, \\
G &= C = 1 + (\phi_{uv})^2 + (\phi_{vv})^2. \tag{19}
\end{align*}
$$

Furthermore, substituting (18) into (16)-(17) and using partial derivatives of the functions given in the equation (18) we obtain $K = K_N$. \qed
Example 3.1. For the surface $M$ given with the Monge patch
\[
  f(u, v) = \phi_u(u, v) = e^u \cos v, \\
  g(u, v) = \phi_v(u, v) = -e^u \sin v
\]
the Gaussian curvature $K$ coincides with the Gaussian torsion $K_N$ of $M$ [1].

Definition 3.1. The surface given with the parametrization [7] by the parametrization
\[
f(u, v) = f_3(u) + g_3(v), \quad g(u, v) = f_4(u) + g_4(v)
\]
is called translation surface in Euclidean 4-space $E^4$ [7].

In the case [20] we obtain simple expressions for $K, K_N$ and $\overrightarrow{H}$. As a consequence of Theorem 1 and Theorem 2 we get the following results.

Corollary 3.1. Let $M$ be a translation surface given with the Monge patch [20]. Then the Gaussian curvature $K$ and Gaussian torsion $K_N$ of $M$ becomes
\[
  K = \frac{f''_3(u)g''_3(v)C - (f''_3(u)g''_4(v) + f''_4(u)g''_3(v))B + f''_3(u)g''_4(v)A}{W^4},
\]
and
\[
  K_N = \frac{F(f''_3(u)g''_3(v) - f''_3(u)g''_4(v))}{W^4},
\]
respectively, where
\[
  E = 1 + (f'_3(u))^2 + (f'_4(u))^2,
  F = f'_3(u)g'_3(v) + f'_4(u)g'_4(v),
  G = 1 + (g'_3(v))^2 + (g'_4(v))^2,
\]
and
\[
  A = 1 + (f'_3(u))^2 + (g'_3(v))^2,
  B = f'_3(u)f'_4(u) + g'_3(v)g'_4(v),
  C = 1 + (f'_4(u))^2 + (g'_4(v))^2.
\]

Corollary 3.2. Let $M$ be a translation surface given with the Monge patch [20]. Then the mean curvature vector of $M$ becomes
\[
  \overrightarrow{H} = \frac{f''_3(u)G + g''_3(v)E}{2\sqrt{AW^2}}N_1 + \frac{G(f''_4(u)A - f''_3(u)B) + E(g''_4(v)A - g''_3(v)B)}{2\sqrt{AW^3}}N_2.
\]

Example 3.2. The translation surface given with the surface patch of
\[
  X(u, v) = (u, v, u^2 + v^2, u^2 - v^2)
\]
has vanishing Gaussian curvature and Gaussian torsion [2].

Theorem 3.3. [7] Let $M$ be a translation surface in $E^4$. Then $M$ is minimal if and only if each $M$ is a plane or
\[
  f_k(u) = \frac{c_k}{c^2_3 + c^2_4} (\log |\cos(\sqrt{3u})| + cu) + e_k u,
  g_k(v) = \frac{c_k}{c^2_3 + c^2_4} (\log |\cos(\sqrt{4v})| + dv) + p_k v, \quad k = 3, 4,
\]
where $c_k, e_k, p_k, a > 0, b > 0, c, d$ are real constants.
4 Aminov Surfaces in $\mathbb{E}^4$

In the present section we consider the surfaces $M$ with

$$f(u, v) = r(u) \cos v, \ g(u, v) = r(u) \sin v. \quad (21)$$

which earlier were been considering in the work [1]. We call such surfaces Aminov surfaces in Euclidean 4-space $\mathbb{E}^4$.

As a consequence of Theorem 2 we get the following result.

**Corollary 4.1.** Let $M$ be an Aminov surface given with the Monge patch (21). Then the Gaussian curvature $K$ and Gaussian torsion $K_N$ of $M$ becomes

$$K = -\frac{r(u)r''(u)(1 + r^2(u)) + (r'(u))^2(1 + (r'(u))^2)}{(1 + r^2(u))^2(1 + (r'(u))^2)^2}, \
(22)$$

and

$$K_N = \frac{r'(u)r''(u)(1 + r^2(u)) + r(u)r'(u)(1 + (r'(u))^2)}{(1 + r^2(u))^2(1 + (r'(u))^2)^2}, \
(23)$$

respectively.

**Proposition 4.1.** Let $M$ be an Aminov surface given with the Monge patch (21). If $K + K_N = 0$, then the equality

$$(r(u) - r'(u)) ((r'(u)(1 + (r'(u))^2) - r''(u)(1 + r^2(u))) = 0 \quad (24)$$

holds.

**Proof.** Using (22) and (23) we get the result. \hfill $\Box$

As a consequence of Proposition 8 we can give the following example.

**Example 4.1.** The Aminov surface given with the surface patch of

$$X(u, v) = (u, v, \lambda e^u \cos v, \lambda e^u \sin v). \quad (25)$$

satisfies the relation $K + K_N = 0$.

As a consequence of Theorem 1 we get the following results.

**Proposition 4.2.** Let $M$ be an Aminov surface given with the Monge patch (21). Then the mean curvature vector of $M$ becomes

$$\overrightarrow{H} = \frac{(Gr''(u) - Er(u))}{2W^2\sqrt{A}} \left\{ \cos v N_1 + \left( \frac{A \sin v - B \cos v}{W} \right) N_2 \right\}. \quad (26)$$

where

$$A = 1 + (r'(u))^2 \cos^2 v + r^2(u) \sin^2 v,$$

$$B = ((r'(u))^2 - r^2(u)) \cos v \sin v,$$

$$C = 1 + (r'(u))^2 \sin^2 v + r^2(u) \cos^2 v.$$

and

$$E = 1 + (r'(u))^2,$$

$$F = 0,$$

$$G = 1 + r^2(u).$$

such that $EG - F^2 = AC - B^2$. 

Corollary 4.2. Let $M$ be an Aminov surface given with the Monge patch (21). Then the mean curvature of $M$ becomes

$$H = \frac{r''(u)(1 + r^2(u)) - r(u)(1 + (r'(u))^2)}{2(1 + r^2(u))(1 + (r'(u))^2)^{3/2}}$$

(27)

Corollary 4.3. Let $M$ be an Aminov surface given with the Monge patch (21). If $M$ is minimal then

$$r(u) = \frac{1}{2a} \left( a^2 e^{\frac{2(u+b)}{a}} + a^2 - 1 \right) e^{\frac{(u+b)}{a}},$$

(28)

where, $a$ and $b$ are real constants.

Proof. Suppose that $M$ is minimal then using the equality (27) we get

$$r''(u)(1 + r^2(u)) - r(u)(1 + (r'(u))^2) = 0.$$  

(29)

Further, by the use of Maple and easy calculation shows that (28) is a non-trivial solution of (29). \qed

Definition 4.1. A surface $M$ is said to be Wintgen ideal surface in $E^4$ if the equality

$$K + |K_N| = \left\| \overrightarrow{H} \right\|^2$$

(30)

holds \cite{18}.

We obtain the following result.

Theorem 4.1. Let $M$ be an Aminov surface given with the Monge patch (21). If $M$ is Wintgen ideal surface then the equality

$$2r''(1 + r^2)(1 + (r')^2)(2r' - r) + (1 + (r')^2)^2(4rr' - 4(r')^2 - r^2) - (r'')^2(1 + r^2)^2 = 0$$

holds.

Proof. Substituting (22), (23), (27) and (30) into we get (31). \qed

5 Chen Surfaces in $E^4$

Let $M$ be a smooth surface in $E^4$ given with the patch $X(u, v) : (u, v) \in D \subset \mathbb{E}^2$. If we chose an orthonormal tangent frame field $\{X, Y\}$

$$X = \frac{X_u}{\sqrt{E}},$$

$$Y = \frac{\sqrt{E}}{W} \left( X_v - \frac{FX_u}{E} \right),$$

then the coefficients of the second fundamental form are given by

$$h_{11}^\alpha = \langle h(X, X), N_\alpha \rangle = \frac{c_{11}^\alpha}{E}, \ 1 \leq \alpha \leq 2,$$

$$h_{12}^\alpha = \langle h(X, Y), N_\alpha \rangle = \frac{1}{W} \left( c_{12}^\alpha - \frac{F}{E} c_{11}^\alpha \right),$$

$$h_{22}^\alpha = \langle h(Y, Y), N_\alpha \rangle = \frac{1}{W^2} \left( E c_{22}^\alpha - 2Fc_{12}^\alpha + \frac{F^2}{E} c_{11}^\alpha \right).$$

(33)
Further, the shape operator matrix of the surface $M \subset \mathbb{E}^4$ becomes

$$A_{N_\alpha} = \begin{pmatrix} h_{11}^\alpha & h_{12}^\alpha \\ h_{12}^\alpha & h_{22}^\alpha \end{pmatrix}.$$ 

Hence, the mean curvature vector of a regular patch $X(u,v)$ is defined by

$$\vec{H} = \frac{1}{2}(tr(A_{N_1}) + tr(A_{N_2})) = H_1N_1 + H_2N_2,$$

(34)

where the functions

$$H_1 = \frac{1}{2}(h_{11}^1 + h_{22}^1), \quad H_2 = \frac{1}{2}(h_{11}^2 + h_{22}^2)$$

(35)

are called the first and second harmonic curvatures of $M$ respectively.

For any arbitrary orthonormal normal frame field $N_1, N_2$ of $M$ such that the vector field $N_1$ is parallel to mean curvature vector $\vec{H}$. In [5] B-Y. Chen defined the allied vector field $a(\vec{H})$ of the mean curvature vector field $\vec{H}$ by the formula

$$a(\vec{H}) = \frac{\|\vec{H}\|}{2} \left\{ tr(A_{N_1}A_{N_2}) \right\} N_2.$$ 

(36)

In particular, the allied mean curvature vector field of the mean curvature vector $\vec{H}$ is a well-defined normal vector field orthogonal to $\vec{H}$. If the allied mean vector $a(\vec{H})$ vanishes identically, then the surface $M$ is called $A$-surface of $\mathbb{E}^4$. Furthermore, $A$-surfaces are also called Chen surfaces [10]. The class of Chen surfaces contains all minimal and pseudo-umbilical surfaces, and also all surfaces for which $\dim N_1 \leq 1$, in particular all hypersurfaces. These Chen surfaces are said to be trivial $A$-surfaces [11].

**Theorem 5.1.** Let $M$ be a smooth surface in $\mathbb{E}^4$ given with the patch $X(u,v) : (u,v) \in D \subset \mathbb{E}^2$. Then $M$ is a non-trivial Chen surfaces if and only if

$$((h_{11}^1)^2 - (h_{12}^1)^2 + (h_{22}^1)^2 - (h_{22}^2)^2 + 2(h_{12}^1)^2 - 2(h_{12}^2)^2) H_1H_2 + (h_{11}^1h_{11}^2 + h_{12}^1h_{22}^2 + 2h_{12}^1h_{12}^2)(H_2^2 - H_1^2) = 0$$

(37)

holds, where $H_1$ and $H_2$ are the first and second harmonic curvatures of $M$ as defined before.

**Proof.** Suppose $M$ is a non-minimal surface in $\mathbb{E}^4$. Then we can construct another orthonormal normal frame field

$$\tilde{N}_1 = \frac{H_1N_1 + H_2N_2}{\sqrt{H_1^2 + H_2^2}}, \quad \tilde{N}_2 = \frac{H_2N_1 - H_1N_2}{\sqrt{H_1^2 + H_2^2}},$$

(38)

such that $\tilde{N}_1$ is parallel to $\vec{H}$. 

Furthermore, with respect to this frame we can obtain

\[
\begin{align*}
\tilde{h}_{11} &= \left\langle h(X, X), \tilde{N}_1 \right\rangle = \frac{H_1 h_{11}^1 + H_2 h_{11}^2}{\sqrt{H_1^2 + H_2^2}}, \\
\tilde{h}_{12} &= \left\langle h(X, Y), \tilde{N}_1 \right\rangle = \frac{H_1 h_{12}^1 + H_2 h_{12}^2}{\sqrt{H_1^2 + H_2^2}}, \\
\tilde{h}_{22} &= \left\langle h(Y, Y), \tilde{N}_1 \right\rangle = \frac{H_2 h_{22}^1 + H_2 h_{22}^2}{\sqrt{H_1^2 + H_2^2}}, \\
\tilde{h}_{11} &= \left\langle h(X, X), \tilde{N}_2 \right\rangle = \frac{H_2 h_{11}^1 - H_1 h_{11}^2}{\sqrt{H_1^2 + H_2^2}}, \\
\tilde{h}_{12} &= \left\langle h(X, Y), \tilde{N}_2 \right\rangle = \frac{H_2 h_{12}^1 - H_1 h_{12}^2}{\sqrt{H_1^2 + H_2^2}}, \\
\tilde{h}_{22} &= \left\langle h(Y, Y), \tilde{N}_2 \right\rangle = \frac{H_2 h_{22}^1 - H_1 h_{22}^2}{\sqrt{H_1^2 + H_2^2}},
\end{align*}
\]

(39)

So, the shape operator matrices of \( M \) with respect to \( \tilde{N}_1 \) and \( \tilde{N}_2 \) become

\[
A_{\tilde{N}_1} = \begin{pmatrix}
\frac{H_1 h_{11}^1 + H_2 h_{11}^2}{\sqrt{H_1^2 + H_2^2}} & \frac{H_1 h_{12}^1 + H_2 h_{12}^2}{\sqrt{H_1^2 + H_2^2}} \\
\frac{H_1 h_{22}^1 + H_2 h_{22}^2}{\sqrt{H_1^2 + H_2^2}} & \frac{H_2 h_{12}^1 - H_1 h_{12}^2}{\sqrt{H_1^2 + H_2^2}}
\end{pmatrix},
\quad
A_{\tilde{N}_2} = \begin{pmatrix}
\frac{H_2 h_{11}^1 - H_1 h_{11}^2}{\sqrt{H_1^2 + H_2^2}} & \frac{H_2 h_{12}^1 - H_1 h_{12}^2}{\sqrt{H_1^2 + H_2^2}} \\
\frac{H_2 h_{22}^1 - H_1 h_{22}^2}{\sqrt{H_1^2 + H_2^2}} & \frac{H_1 h_{22}^1 - H_2 h_{22}^2}{\sqrt{H_1^2 + H_2^2}}
\end{pmatrix},
\]

(40)

respectively.

Suppose \( M \) is a non-trivial Chen surface then by definition \( tr(A_{\tilde{N}_1} A_{\tilde{N}_2}) = 0 \). So by the use of (40) we get the result.

Conversely, if the equality (37) holds then \( tr(A_{\tilde{N}_1} A_{\tilde{N}_2}) = 0 \). So, \( M \) is a non-trivial Chen surface.

We obtain the following result.

**Theorem 5.2.** Let \( M \) be an Aminov surface in \( \mathbb{E}^4 \) given with the Monge patch (21). Then \( M \) is a non-trivial Chen surface.

**Proof.** Suppose \( M \) is an Aminov surface in \( \mathbb{E}^4 \) given with the parametrization (21). By the use of (15) with (33) a simple calculation gives

\[
\begin{align*}
h_{11} &= \frac{r''(u) \cos v}{\varphi \psi^3}, & h_{12} &= \frac{-r'(u) \sin v}{\varphi \psi^3}, \\
h_{22} &= \frac{-r(u) \cos v}{\varphi \omega^2}, & h_{11} &= \frac{-r'(u) \sin v}{\varphi \omega^3}, \\
h_{12} &= \frac{r'(u) \cos v}{\varphi \omega^2}, & h_{22} &= \frac{-r(u) \sin v}{\varphi \psi \omega},
\end{align*}
\]

(41)

where \( \varphi, \psi \) and \( \omega \) are differentiable functions defined by

\[
\begin{align*}
\varphi &= \sqrt{1 + (r'(u))^2 \cos^2 v + (r(u))^2 \sin^2 v}, \\
\psi &= \sqrt{1 + (r'(u))^2}, \\
\omega &= \sqrt{1 + (r(u))^2}.
\end{align*}
\]

Substituting (41) into (37) we get the result. \( \square \)
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