THE AFFINE TRANSFORM FORMULA FOR AFFINE
JUMP-DIFFUSIONS WITH A GENERAL CLOSED
CONVEX STATE SPACE

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We establish existence of exponential moments and the validity of the affine transform formula for affine jump-diffusions with a general closed convex state space. This extends known results for affine jump-diffusions with a canonical state space. The key step is to prove the martingale property of an exponential local martingale, using the well-posedness of the associated martingale problem. By analytic extension we obtain the affine transform formula for complex exponentials, in particular for the characteristic function. Our results apply to a wide class of affine processes, including those with a matrix-valued state space, which have recently gained interest in the literature.

1. Introduction. Affine jump-diffusions, as introduced in [9, 10], are widely used in finance, due to their flexibility and mathematical tractability. Their main attraction lies in the so-called affine transform formula

\[ E_x \exp(u^\top X_t) = \exp(\psi_0(t, u) + \psi(t, u)^\top x), \quad u \in \mathbb{C}^p, X_0 = x, \]  

which relates exponential moments of the affine jump-diffusion \( X \) to solutions \((\psi_0, \psi)\) to certain ordinary differential equations, called generalized Riccati equations. The importance of this formula is particularly elucidated in option and bond pricing. For example, the affine transform formula yields a closed form expression for the zero-coupon bond price in an affine term structure model, see [9, 10]. Moreover, taking \( u \) purely imaginary in (1.1) gives the characteristic function of \( X_t \), which is of vital importance for calculating more general prices by using Fourier methods, e.g. those of [2].

The validity of the affine transform formula is not straightforward in general. In the literature most results in this respect are proved for affine jump-diffusions living on the state space \( \mathbb{R}_+^m \times \mathbb{R}^{p-m} \), see [8, 12, 13, 19, 22] amongst others. This state space, often called the canonical state space, was introduced in [6] and has traditionally been the standard choice in financial applications. Currently though, there is a growing number of papers
devoted to matrix-valued affine processes living on $S^p_{+}$, the cone of positive semi-definite matrices, or on variations of it, like $S^p_{+} \times \mathbb{R}$, see for instance [4, 5, 14, 15, 23]. Moreover, in an accompanying paper [24] we provide further examples of affine diffusions with a “non-canonical” state space, e.g. those with a quadratic state space, indicating that this class is rather rich. This feeds the demand to obtain results for the validity of (1.1) for more general state spaces than $\mathbb{R}^m_+ \times \mathbb{R}^{p-m}$, which is the scope of the present paper.

We highlight that one of our aims is to establish for arbitrary state spaces the affine transform formula for the characteristic function, a crucial feature for the application of affine processes in mathematical finance as pointed out in the first paragraph. To our knowledge, this important property has only been derived for affine processes living on a canonical state space, see [8, 12]. The complicated factor is that the so-called admissibility conditions that are required for stochastic invariance and for existence and uniqueness of the affine process, are much more involved for a non-canonical than for a canonical state space, due to the curvedness of the boundary. As a consequence, it is much harder for general state spaces to control the solutions of the Riccati equations by means of these admissibility conditions. We circumvent this difficulty by relying on probabilistic methods instead.

The contents and set-up of the paper are as follows. First we derive a general result in Section 2 on the martingale property of a stochastic exponential, building on results in [3]. Next we apply this in Section 3 to the stochastic exponential of affine jump-diffusions in order to obtain sufficient conditions on $\psi$ such that (1.1) holds, irrespective of the underlying state space. This is our first main result and extends the result in [19], which is limited to the canonical state space.

Our second main result concerns the full range of validity of (1.1) for affine jump-diffusions with an arbitrary closed convex state space, under some moment conditions on the jump-measure. We show existence of solutions to the Riccati equations under finiteness of exponential moments and establish the affine transform formula (1.1) whenever either side of (1.1) is well-defined, both for real and complex $u$. This generalizes a recent result by [12], which concerns affine diffusions on the canonical state space $\mathbb{R}^m_+ \times \mathbb{R}^{p-m}$ under absence of jumps.

The proof of the second main result is distributed over two sections. In Section 4 we establish the full range of validity for real-valued exponentials, while in Section 5 we extend this to complex ones. For the latter we use the analyticity of both the characteristic function and the solutions to the Riccati equations. A complicating matter is that an affine jump-diffusion with a general state space is in general not infinite divisible, as opposed to
those with a canonical state space. Hence, a priori it is not excluded that
the left-hand side of (1.1) vanishes for certain complex $u$, which would yield
an explosion of $\psi$. We tackle this problem by using properties of analytic
functions.

In Section 6 we relax the moment conditions on the jump-measure and
establish the validity of (a slight variation of) (1.1) in the case the left-
hand side is uniformly bounded in $x$ and $t$, which includes the characteristic
function. This yields our third main result and it enables us to obtain suffi-
cient conditions for infinite divisibility in Subsection 6.1 as well as proving
additional results for the case that the state space is a self-dual cone in
Subsection 6.2.

Finally, some technical results used throughout the text are put in the
appendix, in order to keep a fluid presentation.

2. Preliminary result on exponential martingales. In this section
we obtain sufficient conditions for the martingale property of a stochastic
exponential. This is the key-ingredient in obtaining our results concerning
the affine transform formula for affine jump-diffusions in the next sections.
We use the framework of [3] with some slight modifications and derive a
corollary of its main result, [3, Theorem 2.4], in Theorem 2.6.

Let $E \subset \mathbb{R}^p$ be a closed set and $E_\Delta = E \cup \{\Delta\}$ the one-point compacti-
fication of $E$. Every measurable function $f$ on $E$ is extended to $E_\Delta$ by setting $f(\Delta) = 0$. Throughout this section, $\Omega$ denotes a subset of $D_{E_\Delta}[0, \infty)$,
the space of càdlàg functions $\omega : [0, \infty) \to E_\Delta$. Unless mentioned other-
wise, $\Omega$ is equipped with the $\sigma$-algebra $\mathcal{F}^X = \sigma(X_s : s \geq 0)$ and filtration
$\mathcal{F}_t^X := \sigma(X_s : 0 \leq s \leq t)$, generated by the coordinate process $X$ given by
$X_t(\omega) = \omega(t)$.

Let us be given measurable functions $b : E \to \mathbb{R}^p$, $c : E \to S^p_+$ (space of
positive semi-definite $(p \times p)$-matrices) and a transition kernel $K$ from $E$ to
$F \subset \mathbb{R}^p \setminus \{0\}$ such that $E + F \subset E$. Assume that

\begin{equation}
(2.1) \quad b(\cdot), c(\cdot) \text{ and } \int (|z|^2 \wedge |z|)dK(\cdot, dz) \text{ are bounded on compacta of } E,
\end{equation}

and

\begin{equation}
(2.2) \quad \int_{\{|z| > 1\}} |z|^q K(x, dz) \leq C(1 + |x|^q), \quad \text{for some } C, q > 0, \text{ all } x \in E.
\end{equation}

Write $\nabla f$ for the gradient of $f$ (as a row vector) and $\nabla^2 f$ for the Hessian.
Then
\[ Af(x) = \nabla f(x)b(x) + \frac{1}{2} \text{tr} (\nabla^2 f(x)c(x)) \]
\[ + \int (f(x + z) - f(x) - \nabla f(x)z)K(x, dz) \]
\[ = \xi_A(f, x) \]
(2.3)
defines a linear operator \( A : C_c^\infty(E) \to B(E) \), see Lemma A.1 in the appendix. Here, \( C_c^\infty(E) \) denotes the space of \( C^\infty \)-functions on \( E \) with compact support and \( B(E) \) the space of bounded measurable functions on \( E \).

**Definition 2.1.** A probability measure \( \mathbb{P} \) on \( (\Omega, \mathcal{F}_X) \) is called a solution of the martingale problem for \( A \) if
\[ M_t^f = f(X_t) - f(X_0) - \int_0^t Af(X_s)ds \]
is a \( \mathbb{P} \)-martingale with respect to \( (\mathcal{F}_t^X) \) for all \( f \in C_c^\infty(E) \). If in addition \( \lambda \) is a probability measure on \( E \) such that \( \mathbb{P} \circ X_0^{-1} = \lambda \), then we say \( \mathbb{P} \) is a solution of the martingale problem for \((A, \lambda)\) and we often write \( \mathbb{P} = \mathbb{P}_\lambda \). If \( \lambda = \delta_x \), the Dirac-measure at \( x \) for some \( x \in E \), then we write \( \mathbb{P}_x \) instead.

**Remark 2.2.** 1. In case \( \Omega = \mathcal{D}_E[0, \infty) \), then it holds that \( \mathbb{P} \) is a solution of the martingale problem for \( A \) on \( (\Omega, \mathcal{F}_X^X) \) if and only if \( X \) is a special jump-diffusion on \( (\Omega, \mathcal{F}_X^X, (\mathcal{F}_t^X), \mathbb{P}) \) with differential characteristics \((b(X), c(X), K(X, dz))\), by [16, Theorem II.2.42] and a modification of [3, Proposition 3.2]. In that case, \( X \) can be decomposed according to its characteristics by
\[ X = X_0 + B + X^c + z * (\mu^X - \nu^X), \]
where \( B_t = \int_0^t b(X_s)ds \), \( \mu^X \) is the random measure associated to the jumps of \( X \), \( \nu^X(dt, dz) = K(X_t, dz)dt \) its compensator and \( X^c \) is the continuous local martingale part of \( X \) with quadratic variation \( (X^c)_t = \int_0^t c(X_s)ds \).

2. If the martingale problem for \( A \) is well-posed, then \( (\mathbb{P}_x)_{x \in E} \) is a transition kernel and for all probability measures \( \lambda \) on \( E \) it holds that \( \mathbb{P}_\lambda = \int \mathbb{P}_x \lambda(dx) \) is the unique solution of the martingale problem for \((A, \lambda)\). In addition, the strong Markov property holds, i.e.
\[ E_\lambda(f(X_{t+\tau})|\mathcal{F}_t^X) = E_X,f(X_t), \quad \mathbb{P}_\lambda \text{-a.s.} \]
for all integrable \( f, t \geq 0 \) and a.s. finite \( (\mathcal{F}_t^X) \)-stopping times \( \tau \). See the appendix for the proof of this assertion.

3. If for some \( x_0 \in E \), \( \mathbb{P} \) is a solution of the martingale problem for \( (\mathcal{A}, \delta_{x_0}) \), then \( \mathcal{A} f(x_0) = \lim_{t \downarrow 0} (E f(X_t) - f(x_0))/t \), for \( f \in C_c^\infty(E) \). This follows by taking expectations in (2.4) and applying Fubini, which is justified since \( \mathcal{A} f \) is bounded.

In addition to \( b, c \) and \( K \), let us be given a measurable function \( \tilde{b} : E \to \mathbb{R}^p \) and a transition kernel \( \tilde{K} \) from \( E \) to \( F \). Assume that

\[
\tilde{b}(\cdot), c(\cdot) \text{ are continuous, } (|z|^2 \wedge |z|)\tilde{K}(\cdot, dz) \text{ is weakly continuous,}
\]
and

\[
\int_{\{|z|>1\}} |z|^q \log |z| \tilde{K}(x, dz) \leq C(1 + |x|^q), \text{ some } C, q > 0, \text{ all } x \in E.
\]

Then

\[
\tilde{\mathcal{A}} f(x) = \nabla f(x)\tilde{b}(x) + \frac{1}{2} \text{tr}(\nabla^2 f(x)c(x)) + \int (f(x + z) - f(x) - \nabla f(x)z)\tilde{K}(x, dz)
\]

defines a linear operator \( \tilde{\mathcal{A}} : C_c^\infty(E) \to C^0(E) \), where \( C^0(E) \) denotes the space of continuous functions on \( E \) vanishing at infinity, see Lemma A.1. Here, weak continuity means that \( x \mapsto \int f(z)(|z|^2 \wedge |z|)\tilde{K}(x, dz) \) is continuous for all \( f \in C_0(F) \), the space of bounded continuous functions on \( F \). As in [3], we assume there exist measurable mappings \( h : E \to \mathbb{R}^p \), \( w : E \times F \to (-1, \infty) \) such that \( \tilde{b} \) and \( \tilde{K} \) are related to \( b \) and \( K \) by

\[
\tilde{b}(x) = b(x) + c(x)h(x) + \int zw(x, z)K(x, dz)
\]

\[
\tilde{K}(x, dz) = (w(x, z) + 1)K(x, dz).
\]

Our aim is to show the martingale property of a stochastic exponential with the aid of [3, Theorem 2.4], under the assumption that the martingale problem for \( \mathcal{A} \) is well-posed. This requires the existence of a solution of the martingale problem for \( \mathcal{A} \) on \( (D_E[0, \infty), \mathcal{F}^X) \), which is part of the assumptions in [3, Theorem 2.4]. In our case though, we are able to derive the existence by invoking [11, Theorem 4.5.4], as the range of \( \mathcal{A} \) is contained in \( C^0(E) \), due to the additional continuity conditions (2.6). Note that these conditions are similar as those in [25, Theorem 2.2], where existence is derived for the case \( E = \mathbb{R}^p \).
The next lemma will be used to obtain the maximum principle for \( \hat{A} \) in the ensuing proposition, where we establish the existence of a solution of the martingale problem for \( \hat{A} \).

**Lemma 2.3.** Let \( x_0 \in E \) and suppose the martingale problem for \((A, \delta_{x_0})\) has a solution \( \mathbb{P} \) on \((\Omega, \mathcal{F}_t^X)\) with \( \Omega = \mathcal{D}_E[0, \infty) \). Suppose \( f \in C_c^\infty(E) \) attains its maximum at \( x_0 \). Then it holds that

1. \( \nabla f(x_0)c(x_0) = 0 \),
2. \( \int \nabla f(x_0)zK(x_0, dz) \) is well-defined and finite,
3. \( \nabla f(x_0)b(x_0) - \int \nabla f(x_0)zK(x_0, dz) + \frac{1}{2} \text{tr} (\nabla^2 f(x_0)c(x_0)) \leq 0 \).

**Proof.** By Remark 2.2 part 1, \( X \) is a jump-diffusion on \((\Omega, \mathcal{F}_t^X, \mathcal{F}_{t+}^X, \mathbb{P})\) with differential characteristics \((b(X), c(X), K(X, dz))\). Let \( \lambda \in \mathbb{R}^p \) and \( \varepsilon > 0 \) be arbitrary, define \( h(x) = \lambda 1_{\{x = x_0\}} \) and \( w(x, z) = (\varepsilon - 1)1_{\{x = x_0\}\cap\{|z| > \varepsilon\}} \) and write \( H_t = h(X_t), W(t, z) = w(X_t, z) \) and

\[
Z = H \cdot X + W \ast (\mu^X - \nu^X).
\]

For \( T > 0 \) it holds that \( E(Z)^T = E(Z^T) \) is a uniformly integrable martingale by [21, Theorem IV.3], since

\[
\frac{1}{2}\langle Z^c \rangle_T + (W + 1) \log(W + 1) - W \ast \nu_T^X
= \int_0^T \left( \frac{1}{2}\lambda^T c(x_0)\lambda + \int_{\{|z| > \varepsilon\}} (\varepsilon \log\varepsilon - \varepsilon + 1)K(x_0, dz) \right) 1_{\{X_s = x_0\}} ds
\]

has finite expectation as it is bounded. By Girsanov’s Theorem [18, Proposition 4], \( \mathbb{Q} = \mathcal{E}(Z)_T \cdot \mathbb{P} \) is a probability measure on \( \mathcal{F}_t^X \) equivalent to \( \mathbb{P} \) and \( X \) is a special jump-diffusion on \([0, T]\) with differential characteristics \((\tilde{b}(X_t), c(X_t), \tilde{K}(X_t, dz))\) under \( \mathbb{Q} \) given by

\[
\tilde{b}(x) = b(x) + c(x)h(x) + \int z w(x, z)K(x, dz)
\]

\[
\tilde{K}(x, dz) = (w(x, z) + 1)K(x, dz).
\]

Therefore, [16, Theorem II.2.42] yields that \( \mathbb{Q} \) is a solution of the martingale problem for \((\hat{A}, \delta_{x_0})\) on \((\Omega, \mathcal{F}_t^X)\) with time restricted to \([0, T]\), with the linear operator \( \hat{A} : C_c^\infty(E) \to B(E) \) defined by

\[
\hat{A}f(x) = \nabla f(x)\tilde{b}(x) + \frac{1}{2} \text{tr} (\nabla^2 f(x)c(x)) + \int (f(x + z) - f(x) - \nabla f(x)z)\tilde{K}(x, dz) = A f(x) + \nabla f(x)c(x)h(x) + \int (f(x + z) - f(x))w(x, z)K(x, dz).
\]
Hence \( \hat{A}f(x_0) \) equals

\[
\hat{A}f(x_0) = Af(x_0) + \nabla f(x_0)c(x_0)\lambda + \int_{\{|z|>\varepsilon\}} (f(x_0 + z) - f(x_0))(\varepsilon - 1)K(x_0, dz).
\]

Since \( f \) attains its maximum at \( x_0 \), Remark 2.2 part 3 yields that \( \hat{A}f(x_0) \leq 0 \). Therefore, (2.10) is non-positive for all \( \lambda \in \mathbb{R}^p \) and \( \varepsilon > 0 \). This yields that \( \nabla f(x_0)c(x_0) = 0 \), which is the first assertion. It follows that

\[
Af(x_0) + \int_{\{|z|>\varepsilon\}} (f(x_0 + z) - f(x_0))(\varepsilon - 1)K(x_0, dz) \leq 0,
\]

for all \( \varepsilon > 0 \). Letting \( \varepsilon \downarrow 0 \) in (2.11) and applying the Monotone Convergence Theorem gives

\[
Af(x_0) - \int (f(x_0 + z) - f(x_0))K(x_0, dz) \leq 0.
\]

The left-hand side equals

\[
\nabla f(x_0)b(x_0) - \int \nabla f(x_0)zK(x_0, dz) + \frac{1}{2}\text{tr}(\nabla^2 f(x_0)c(x_0)),
\]

which yields the second and third assertion. \( \square \)

**Proposition 2.4.** Suppose for all \( x \in E \) there exists a solution of the martingale problem for \( (A, \delta_x) \) on \((D_E[0, \infty), \mathcal{F}_X)\). Then for all \( x \in E \) there exists a solution of the martingale problem for \( (\hat{A}, \delta_x) \) on \( \Omega \) given by

\[
\Omega = \{\omega \in D_{E\Delta}[0, \infty) : if \omega(t-) = \Delta or \omega(t) = \Delta then \omega(s) = \Delta for s \geq t\}.
\]

**Proof.** We check the conditions of [11, Theorem 4.5.4]. Let \( f \in C^\infty_c(E) \) attain its maximum at some point \( x_0 \in E \). By Lemma 2.3, we can write \( \hat{A}f(x_0) \) as the sum of two non-positive terms, namely

\[
\nabla f(x_0)b(x_0) - \int \nabla f(x_0)zK(x_0, dz) + \frac{1}{2}\text{tr}(\nabla^2 f(x_0)c(x_0))
\]

and

\[
\int (f(x_0 + z) - f(x_0))(w(x_0, z) + 1)K(x_0, dz).
\]

Hence \( \hat{A}f(x_0) \leq 0 \). This yields that \( \hat{A} \) satisfies the (positive) maximum principle. Since \( \hat{A} : C^\infty_c(E) \to C_0(E) \) and \( C^\infty_c(E) \) is dense in \( C_0(E) \), [11,
Theorem 4.5.4] yields for all \( x \in E \) the existence of a solution \( \mathbb{P}_x \) of the martingale problem for \((\mathcal{A}, \delta_x)\) on \((D_{E\Delta}[0,\infty), \mathcal{F}^{X})\). In order to obtain a solution on \( \Omega \), we define the stopping time

\[
T_\Delta = \inf\{t \geq 0 : X_{t-} = \Delta \text{ or } X_t = \Delta\},
\]

and write \( X' = X^{T_\Delta} \). Then \( X'(\omega) \in \Omega \) for all \( \omega \in D_{E\Delta}[0,\infty) \) and for all \( f \in C^\infty_c(E) \) it holds that (recall \( \tilde{A}f(\Delta) = 0 \))

\[
f(X'_t) - f(X'_0) - \int_0^t \tilde{A}f(X'_s)ds = f(X^{T_\Delta}_t) - f(X^{T_\Delta}_0) - \int_0^{t \wedge T_\Delta} \tilde{A}f(X_s)ds
\]

\[
= (M^f)_{T_\Delta}^t,
\]

where \( M^f \) is given by (2.4) with \( \mathcal{A} \) replaced by \( \tilde{A} \). Since \( M^f \) is a right-continuous \( \mathbb{P}_x \)-martingale on \((\mathcal{F}^{X}_t)\) for \( f \in C^\infty_c(E) \), \( (M^f)_{T_\Delta}^t \) is a martingale on \((\mathcal{F}^{X}_t)\). Hence \( \mathbb{P}_x \circ (X')^{-1} \) is a solution of the martingale problem for \((\tilde{A}, \delta_x)\) on \((\Omega, \mathcal{F}^{X})\) for all \( x \in E \), as we needed to show.

**Proposition 2.5.** Let \( x_0 \in E \) and suppose there exists a solution \( \mathbb{P} \) of the martingale problem for \((\tilde{A}, \delta_{x_0})\) on \((\Omega, \mathcal{F}^{X})\) with \( \Omega \) given by (2.12). Assume the growth condition

\[
|\tilde{b}(x)|^2 + |c(x)| + \int |z|^2 \tilde{K}(x, dz) \leq C(1 + |x|^2), \text{ some } C > 0, \text{ all } x \in \mathbb{R}^p.
\]

Then it holds that \( \mathbb{P}(X \in D_E[0,\infty)) = 1 \).

**Proof.** By the remark preceding [3, Proposition 3.2], a transition to \( \Delta \) can only occur by explosion. Define stopping times

\[
T_n = \inf\{t \geq 0 : |X_{t-}| \geq n \text{ or } |X_t| \geq n\} \land n.
\]

By [3, Proposition 3.2], \( X^{T_n} \) is a special semimartingale with differential characteristics \((\bar{b}(X^{T_n})1_{[0,T_n]}, \bar{c}(X^{T_n})1_{[0,T_n]}, \bar{K}(X^{T_n}, dz)1_{[0,T_n]})\). Lemma A.2 yields

\[
\mathbb{E} \sup_{t \leq T \wedge T_n} |X_t| \leq C(T) < \infty,
\]

for all \( T > 0 \), with \( C(T) \) a positive constant that does not depend on \( n \). Letting \( n \to \infty \) we get

\[
\mathbb{E} \sup_{t \leq T \wedge T_\Delta} |X_t| < \infty,
\]

for all \( T > 0 \), where \( T_\Delta \) is given by (2.13). Hence \( T_\Delta > T \) almost surely for all \( T \). This proves the assertion. \( \square \)
Having derived the existence of a solution of the martingale problem for $\tilde{A}$ from the existence of a solution for $A$, we are now ready to prove the martingale property of a stochastic exponential by the use of [3, Theorem 2.4].

**Theorem 2.6.** Suppose (2.14) holds and

$$x \mapsto h(x)^T c(x) h(x) \text{ and } x \mapsto \int (w(x, z) - \log(w(x, z) + 1))K(x, dz)$$

are bounded on compacta.

Let $\Omega = D_E[0, \infty)$, write $H_t = h(X_t)$, $W(t, z) = w(X_t, z)$ and suppose $\mathbb{P}$ is a solution of the martingale problem for $A$ on $(\Omega, \mathcal{F}^X)$, which yields the decomposition (2.5) for $X$. If the martingale problem for $A$ is well-posed, then

$$L = \mathcal{E}(H \cdot X^c + W \ast (\mu^X - \nu^X))$$

is an $(\mathcal{F}^X_t, \mathbb{P})$-martingale and the martingale problem for $\tilde{A}$ is well-posed.

**Proof.** First assume $\mathbb{P} = \mathbb{P}_x$ is a solution of the martingale problem for $(A, \delta_x)$ for some $x \in E$. By Proposition 2.4 and 2.5, there exists a solution $Q_x$ of the martingale problem for $(\tilde{A}, \delta_x)$ on $(\Omega, \mathcal{F}^X)$. We can apply [3, Theorem 2.4] with the roles of $(A, \mathbb{P})$ and $(\tilde{A}, Q)$ reversed. Indeed, in the notation of [3] we have $\phi_1 = -h$, $\phi_2 = 0$, $\phi_3 = 1/(w + 1)$ and these functions satisfy the criterion mentioned in [3, Remark 2.5] by the assumptions. This yields $\mathbb{P}_x|_{\mathcal{F}^X_t} \sim Q_x|_{\mathcal{F}^X_t}$ for all $t > 0$ and the existence of a positive $Q_x$-martingale $D$ such that

$$\mathbb{P}_x|_{\mathcal{F}^X_t} = D_t \cdot Q_x|_{\mathcal{F}^X_t} \text{ for all } t \geq 0.$$  

By Remark 2.2 part 1, $X$ is a special semimartingale on $(\Omega, \mathcal{F}^X, (\mathcal{F}^X_t), Q_x)$ with decomposition

$$X = X_0 + \tilde{B} + \tilde{X}^c + z \ast (\mu^X - \tilde{\nu}^X),$$

where $\tilde{B}_t = \int_0^t \tilde{b}(X_s)ds$, $\tilde{\nu}^X(dt, dz) = \tilde{K}(X_t, dz)dt$ and $\tilde{X}^c$ the continuous local martingale part with quadratic variation $\langle \tilde{X}^c \rangle_t = \int_0^t c(X_s)ds$. A close inspection of the proof of [3, Theorem 2.4] reveals that

$$D = \mathcal{E}(\phi_1(X) \cdot \tilde{X}^c + (\phi_3(X, z) - 1) \ast (\mu^X - \tilde{\nu}^X)).$$

Applying the product rule for stochastic exponentials one verifies that

$$D^{-1} = \mathcal{E}(-\phi_1(X) \cdot X^c + (1 - \phi_3(X, z)) \phi_3(X, z) \ast (\mu^X - \nu^X)).$$
so that \( D^{-1} = \mathcal{E}(H \cdot X^c + W * (\mu^X - \nu^X)) = L \). Since

\[
\mathbb{Q}_x|_{\mathcal{F}_t} = D_t^{-1} \cdot \mathbb{P}_x|_{\mathcal{F}_t}, \quad \text{for all } t > 0,
\]

it follows that \( L \) is a \( \mathbb{P}_x \)-martingale as well as the martingale problem for \( \tilde{A} \) on \( \Omega \) is well-posed.

Now assume \( \mathbb{P} = \mathbb{P}_\eta \) is a solution of the martingale problem for \( (A, \eta) \) with \( \eta \) an arbitrary probability measure on \( E \). By Remark 2.2 part 2, \( \mathbb{Q}_\lambda = \int \mathbb{Q}_x \lambda(dx) \) is the (unique) solution of the martingale problem for \( (\tilde{A}, \eta) \) on \( \Omega \). Hence we can repeat the above argument with \( \mathbb{P}_x \) and \( \mathbb{Q}_x \) replaced by \( \mathbb{P}_\eta \) and \( \mathbb{Q}_\eta \) to see that \( L \) is a \( \mathbb{P}_\eta \)-martingale.

\[\square\]

3. Affine jump-diffusions and affine processes.

3.1. Definitions. We start with the definition of affine jump-diffusions and affine processes. The former are defined from the point of view of semimartingale theory as being jump-diffusions with affine differential characteristics. The latter are characterized from the point of view of Markov process theory as having an exponentially affine expression for their characteristic functions. As in the previous section we restrict ourselves to special semimartingales.

**Definition 3.1.** The martingale problem for \( A \) given by (2.3) is called an affine martingale problem if \( b, c \) and \( K \) are affine in the sense that

\[
\begin{align*}
    b(x) &= a^0 + \sum_{i=1}^{p} a^i x_i, \\
    c(x) &= A^0 + \sum_{i=1}^{p} A^i x_i, \\
    K(x, dz) &= K^0(dz) + \sum_{i=1}^{p} K^i(dz)x_i,
\end{align*}
\]

for some column vectors \( a^i \in \mathbb{R}^p \), symmetric matrices \( A^i \in \mathbb{R}^{p \times p} \) and (signed) measures \( K^i \) on \( F \) satisfying \( \int ((z)^2 \wedge |z|)|K^i|(dz) < \infty \). If the affine martingale problem is well-posed and \( \mathbb{P} \) is a solution, then the coordinate process \( X \) is called an affine jump-diffusion on \( (\Omega, \mathcal{F}^X, (\mathcal{F}^X_t), \mathbb{P}) \) with state space \( E \).

**Definition 3.2.** If the coordinate process \( X \) on \( \Omega = D_E[0, \infty) \) is a Markov process with state space \( E \) and transition kernel \( (\mathbb{P}_x)_{x \in E} \) such that
for all $u \in i\mathbb{R}^p$, $t \geq 0$ we have
\begin{equation}
E_x \exp(u^\top X_t) = \exp(\psi_0(t,u) + \psi(t,u)^\top x), \quad \text{for all } x \in E,
\end{equation}
for some $\psi_0 : [0,\infty) \times i\mathbb{R}^p \to \mathbb{C}$ and $\psi : [0,\infty) \times i\mathbb{R}^p \to \mathbb{C}^p$, then $(X,(P_x)_{x \in E})$ is called an affine process. Note that $\psi_0(t,u)$ may be altered by multiples of $2\pi i$. If in addition $\psi_0$ and $\psi$ are continuously differentiable in their first argument, it is called a regular affine process. In that case we put $\psi_0(0,u) = 0$, so that $\psi_0$ and $\psi$ are uniquely determined by (3.2).

For existence of an affine jump-diffusion, restrictions need to be imposed on the state space $E$ and the parameters $(a_i, A_i, K_i)_{0 \leq i \leq p}$ in order that $c(x)$ is a positive semi-definite matrix and $K(x, dz)$ is a non-negative measure for $x \in E$, while in addition $E$ is stochastic invariant for $X$ (that is, $X$ does not leave the set $E$). These parameter conditions are called admissibility conditions and the corresponding parameter set $(a_i, A_i, K_i)_{0 \leq i \leq p}$ is called admissible.

Possible state spaces amongst others are the canonical state space $\mathbb{R}_m^m \times \mathbb{R}^{p-m}$, the cone of positive semi-definite matrices $S_p^+$ and quadratic state spaces including the parabolic state space $\{x \in \mathbb{R}^p : x_1 \geq \sum_{i=2}^p x_i^2\}$ and the Lorentz cone $\{x \in \mathbb{R}^p : x_1 \geq 0, x_1^2 \geq \sum_{i=2}^p x_i^2\}$, see respectively [4, 8, 24] for the existence and uniqueness of the associated affine jump-diffusion. We note that the matrix-valued affine jump-diffusions are contained in the framework of Definition 3.1 as we can identify symmetric matrices with vectors using the half-vectorization operator $\text{vech} : S_p \to \mathbb{R}^{p(p+1)/2}$ (the linear operator that stacks the elements from the upper triangle of a symmetric matrix into a vector).

Equivalence of affine jump-diffusions and affine processes has only been proved for the canonical state space $\mathbb{R}_m^m \times \mathbb{R}^{p-m}$ in [8] with the use of the admissibility conditions. For other state spaces this appears much harder as the admissibility conditions become more involved, while for arbitrary state space one has no access at all to these conditions. One of the aims in this paper is to establish the equivalence between affine jump-diffusions and (regular) affine processes with an arbitrary state space under well-posedness of the martingale problem for $\mathcal{A}$. One direction is relatively easy and has been proved for the diffusion case in [12, Theorem 2.2]. The next proposition also incorporates jumps. The converse direction is much harder to establish and will be proved with the least restrictions in Section 6 in Theorem 6.2.

**Proposition 3.3.** Let $E \subset \mathbb{R}^p$ be closed with non-empty interior, $E = \overline{E^0}$ and suppose the martingale problem for $\mathcal{A}$ is well-posed. Let $\mathbb{P}$ be a solution of the martingale problem for $\mathcal{A}$ on $\Omega$ and $\mathbb{P}_x$ for $(\mathcal{A}, \delta_x)$, $x \in E$. If
(X, (P_x)_{x \in E}) is a regular affine process, then X is an affine jump-diffusion on \((\Omega, \mathcal{F}^X, (\mathcal{F}^X_t), \mathbb{P})\) with state space E, say with differential characteristics \((b(X), c(X), K(X, dz))\) given by (3.1). Moreover, for all \(u \in \mathbb{R}^p\) it holds that 
\((\psi_0(\cdot, u), \psi(\cdot, u))\) characterized by (3.2) and \(\psi_0(0, u) = 0\), solves the system of generalized Riccati equations

\[
\dot{\psi}_t = R_i(\psi), \quad \psi_t(0) = u_i, \quad i = 0, \ldots, p,
\]

with

\[
R_i(y) = y^\top a_i + \frac{1}{2} y^\top A_i y + \int (e^{y^\top z} - 1 - y^\top z) K_i(dz),
\]

where we write \(\psi\) instead of \(\psi(T - t, u)\) and \(\dot{\psi}(T - t, u)\). By Remark 2.2 part 1, X is a special jump-diffusion and admits the decomposition (2.5). Itô’s formula gives

\[
\frac{d f(t, x)}{f(t, x)} = (-\dot{\psi}_0 - \psi^\top X_t)dt + \psi^\top dX_t + \frac{1}{2} \psi^\top c(X_t) \psi dt
\]

\[
+ \int_{z \in F} (e^{\psi^\top z} - 1 - \psi^\top z) \mu^X(dt, dz)
\]

\[
= \psi^\top dX_t^c + \int_{z \in F} (e^{\psi^\top z} - 1)(\mu^X - \nu^X)(dt, dz) + I(t, x)dt,
\]

with

\[
I(t, x) = -\dot{\psi}_0 - \psi^\top x + \psi^\top b(x) + \frac{1}{2} \psi^\top c(x) \psi + \int (e^{\psi^\top z} - 1 - \psi^\top z) K(x, dz),
\]

and all expression are well-defined as \(f\) is bounded, see [16, Theorem II.2.42].

Since \(f(t, X_t)\) is a \(\mathbb{P}\)-martingale, it follows that \(\int_0^T I(s, X_s)ds = 0\), \(\mathbb{P}\)-a.s. Right-continuity of \(I(t, X_t)\) yields that \(I(t, X_t) = 0\) for all \(t \geq 0\), \(\mathbb{P}\)-a.s. In particular \(I(0, X_0) = 0\), \(\mathbb{P}\)-a.s. Choosing \(\mathbb{P} = P_x\) for \(x \in E\), we obtain

\[
\dot{\psi}_0(T, u) + \psi(T, u)^\top x = \psi(T, u)^\top b(x) + \frac{1}{2} \psi(T, u)^\top c(x) \psi(T, u)
\]

\[
+ \int (e^{\psi(T, u)^\top z} - 1 - \psi(T, u)^\top z) K(x, dz).
\]
This holds for all $T \geq 0$, $u \in \mathbb{R}^p$. In particular it holds for $T = 0$. We have
\[ \psi(0, u) = u \] for $u \in \mathbb{R}^p$. Write $u = iy$ for $y \in \mathbb{R}^p$, then we get
\[ \psi_0(0, iy) + \psi(0, iy) = iy^\top x = iy^\top c(x)y + \int (e^{iy^\top z} - 1 - iy^\top z) K(x, dz) \]
for all $y \in \mathbb{R}^p$. Differentiating the left- and right-hand side with respect to $y_i$ in $y_i = 0$ and putting $y_k = 0$ for $k \neq i$ gives that $b_i(x)$ is affine for all $i \leq p$. Dividing the left- and right-hand side by $y_i y_j$ for $i, j \leq p$, putting $y_k = 0$ for $k \neq i, j$ and letting $y_i \to \infty$, $y_j \to \infty$, we deduce that $c_{ij}(x)$ is affine. Hence $c(x)$ is affine and also $\int (e^{iy^\top z} - 1 - iy^\top z) K(x, dz)$ is affine in $x$ for all $y \in \mathbb{R}^p$. To show that $K(x, dz)$ is affine in $x$, we fix $k \in E^o$ arbitrary and take $\varepsilon > 0$ such that
\[ \{ x \in \mathbb{R}^p : k_i \leq x_i \leq k_i + \varepsilon \text{ for all } i \} \subset E. \]
Define
\[ K^0(dz) = K(k, dz) - \sum_{i=1}^p (K(k + \varepsilon e_i, dz) - K(k, dz)) k_i / \varepsilon \]
\[ K^i(dz) = (K(k + \varepsilon e_i, dz) - K(k, dz)) / \varepsilon, \quad \text{for } i = 1, \ldots, p. \]
Then it follows that
\[ \int (e^{u^\top z} - 1 - u^\top z) K(x, dz) = \int (e^{u^\top z} - 1 - u^\top z)(K^0(dz) + \sum_{i=1}^p K^i(dz)x_i), \]
for all $u \in \mathbb{R}^p$, $x \in E$, since the left-hand side is affine and is uniquely determined by the values at $x = k$ and $x = k + \varepsilon e_i$, $i = 1, \ldots, p$. Equality of the left- and right-hand side at these points follows from the identity
\[ K^0(dz) + \sum_{i=1}^p K^i(dz)x_i = K(k, dz)(1 + \sum_{i=1}^p (k_i - x_i) / \varepsilon) \]
\[ + \sum_{i=1}^p K(k + \varepsilon e_i, dz)(x_i - k_i) / \varepsilon. \]
Note that the right-hand side is a non-negative measure for $x \in B_k$, where $B_k$ is given by
\[ B_k := \{ x \in \mathbb{R}^p : k_i \leq x_i \leq k_i + \varepsilon / p \text{ for all } i \}. \]
By uniqueness of the Lévy triplet (see [16, Lemma II.2.44]), this yields that
\[ K(x, dz) = K^0(dz) + \sum_{i=1}^p K^i(dz)x_i \] for $x \in B_k$. Since $k \in E^o$ is chosen
arbitrarily, we have an affine expression for $K(x, dz)$ on a neighborhood of each $x \in E^c$. From this it follows that $K(x, dz)$ is affine on the whole of $E = E^c$. Hence $X$ is an affine jump-diffusion. Let the differential characteristics $(b(X), c(X), K(X, dz))$ be given by (3.1). Plugging these into (3.6) and separating first order terms in $x$ gives (3.3).

3.2. The affine transform formula. The expression (3.2) where $(\psi_0, \psi)$ solve the system of Riccati equations (3.3), is called the affine transform formula. In the previous subsection we obtained this formula for the characteristic function of an affine process, with a general state space. This subsection is devoted to the validity of the affine transform formula for affine jump-diffusions with a general state space, for arbitrary parameters $u \in C^p$. The key step is the following proposition which is a direct application of Theorem 2.6.

**Proposition 3.4.** Suppose the affine martingale problem for $A$ given by (2.3) and (3.1) is well-posed. Let $h : E \to \mathbb{R}^p$ and $w : E \times F \to (-1, \infty)$ be measurable, write $H_t = h(X_t)$, $W(t, z) = w(X_t, z)$ and let $\mathbb{P}$ be a solution of the martingale problem for $A$ on $\Omega$, which yields the decomposition (2.5) for $X$. Then

$$L = \mathcal{E}(H \cdot X^c + W * (\mu^X - \nu^X))$$

is an $(\mathcal{F}_{t+}^X, \mathbb{P})$-martingale under the additional assumptions

1. $h$ is bounded and continuous,
2. $x \mapsto \int |z| w(x, z) |K^i| (dz)$ is continuous and finite
3. $x \mapsto \int (|z|^2 \wedge |z|) (w(x, z) + 1) |K^i| (dz)$ is continuous and finite,
4. $\int |z|^2 (w(x, z) + 1) |K^i| (dz) |x_i| \leq C (1 + |x|^2)$, for some $C > 0$, all $x \in E$,
5. $x \mapsto \int (w(x, z) - \log (w(x, z) + 1)) |K^i| (dz)$ is bounded on compacta,
6. $\int |z|^q \log |z| (w(x, z) + 1) |K^i| (dz) |x_i| \leq C (1 + |x|^q)$, for some $C > 0$, $q > 0$, all $x \in E$,

for all $i = 0, \ldots, p$, where we write $x_0 := 1$. Furthermore, the martingale problem for $\tilde{A}$ given by (2.8) and (2.9) is well-posed.

**Proof.** This is a reformulation of Theorem 2.6 for the affine martingale problem. One has to check conditions (2.1), (2.2), (2.6), (2.7), (2.14) and (2.15), which is left to the reader. □

Using the above proposition we validate the affine transform formula under existence of the solutions to the Riccati equations in the following theorem, which is the first main result of the paper. The imposed assumptions are in the same spirit as [19, Theorem 5.1].
THEOREM 3.5. Let $X$ be an affine jump-diffusion with differential characteristics $(b(X), c(X), K(X, d z))$ given by (3.1) on $(D_E(0, \infty], \mathcal{F}^X, (\mathcal{F}^X_t), \mathbb{P})$. Let $u \in \mathbb{R}^p$, $T > 0$ and suppose $\psi_0 \in C^1([0, T], \mathbb{R})$ and $\psi \in C^1([0, T], \mathbb{R}^p)$ solve the system of generalized Riccati equations given by (3.3) (with $u_0 := 0$). Under the assumptions

1. $\sup_{t \in \mathbb{R}^+} \int |z|^2 e^{\psi(t)^T z} |K^i|(dz) < \infty$, for $i = 0, \ldots, p$,
2. $t \mapsto \int_{\{|z| > 1\}} |z| e^{\psi(t)^T z} |K^i|(dz)$ is continuous for all $i = 0, \ldots, p$,
3. $E \exp(\psi(T)^T X_0) < \infty$,

it holds that

$$E(\exp(u^T X_T)|\mathcal{F}^X_t) = \exp(\psi_0(T - t) + \psi(T - t)^T X_t), \text{ for all } t \leq T.$$  

PROOF. To prove Theorem 3.5 it suffices to show that $f(t, X_t)$ given by $f(t, X_t) = \exp(\psi_0(T - t) + \psi(T - t)^T X_t)$ is an $((\mathcal{F}^X_t), \mathbb{P})$-martingale on $[0, T]$, since $f(T, X_T) = \exp(u^T X_T)$ in view of the initial condition of $(\psi_0, \psi)$. We restrict time to $[0, T]$. We have (3.5) with $I(t, X_t) = 0$, since $(\psi_0, \psi)$ satisfy (3.3). Hence $M_t := f(t, X_t)$ satisfies

$$M = M_0 E(\psi(T - t) \cdot X^c + (e^{\psi(T - t)^T z} - 1) \ast (\mu^X - \nu^X)).$$

Write $Y = (X_t, t)$ and note $Y$ is an affine jump-diffusion with state space $E \times [0, T]$. We define $h(x, t) = \psi(T - t)$ and $w(x, t, z) = e^{\psi(T - t)^T z} - 1$. Write $H = h(Y)$, $W(t, z) = w(Y_t, z)$, then we deduce that

$$L := E(H \cdot Y^c + W \ast (\mu^Y - \nu^Y))$$

is an $((\mathcal{F}^X_t), \mathbb{P})$-martingale by applying Proposition 3.4 to the affine jump-diffusion $Y$. One easily verifies that the assumptions in that proposition are met. Since $M_t = M_0 L_t$ and $E M_0 < \infty$, it follows that $M$ is an $((\mathcal{F}^X_t), \mathbb{P})$-martingale on $[0, T]$, as we needed to show. \qed

Theorem 3.7 below is our second main result. We establish the full-range of validity of the affine transform formula under all finite exponential moments for the tails of the jump-measures $K^i$, for affine jump-diffusion with a general closed convex state space, extending [12, Theorem 3.3]. The proof is divided over the next two sections. We use the results and notation from [12, Lemma 2.3 and Lemma A.2], which we state as a proposition for ease of reference.
Proposition 3.6. Suppose

\begin{equation}
\int_{\{|z|>1\}} e^{k^T z} |K^z| (dz) < \infty, \text{ for all } k \in \mathbb{R}^p, i = 0, \ldots, p.
\end{equation}

Let $\mathbb{K}$ be a placeholder for either $\mathbb{R}$ or $\mathbb{C}$. It holds that

(i) For all $u \in \mathbb{K}^p$ there exists an “explosion-time” $t_\infty(u) > 0$ such that there exists a unique solution $(\psi_0(\cdot, u), \psi(\cdot, u)) : [0, t_\infty(u)) \to \mathbb{K} \times \mathbb{K}^p$ to the system of Riccati equations (3.3), where either $t_\infty(u) = \infty$ or $\lim_{t \uparrow t_\infty(u)} \|\psi(t, u)\| = \infty$. In particular $t_\infty(0) = \infty$.

(ii) The set

$D_\mathbb{K} := \{(t, u) \in [0, \infty) \times \mathbb{K}^p : t < t_\infty(u)\},$

is open in $[0, \infty) \times \mathbb{K}^p$ and the $\psi_i$ are analytic on $D_\mathbb{K}$. In addition, for all $t \geq 0$

$D_\mathbb{K}(t) := \{u \in \mathbb{C}^p : (t, u) \in D_\mathbb{K}\}$

is an open neighborhood of $0$ and $D_\mathbb{K}(t_2) \subset D_\mathbb{K}(t_1)$ for $0 \leq t_1 \leq t_2$.

(iii) If $O \subset \mathbb{R}^p$ is an open set and $\nu$ is a bounded measure such that we have $\int \exp(u^T x) d\nu(x) < \infty$ for all $u \in O$, then $u \mapsto \int \exp(u^T x) d\nu(x)$ is analytic on the open strip

$S(O) := \{z \in \mathbb{C}^p : \Re z \in O\}$.

Theorem 3.7. Suppose $E \subset \mathbb{R}^p$ is closed convex with non-empty interior and let $X$ be an affine jump-diffusion on $(\mathcal{D}_E(0, \infty], \mathcal{F}_X, (\mathcal{F}_t^X), \mathbb{P})$ with differential characteristics $(b(X, c(X), K(X, dz))$ given by (3.1). Assume (3.7) and let the notation of Proposition 3.6 be in force. Then for $t > 0$ it holds that

(i) $D_\mathbb{R}(t) = M(t)$, where

$M(t) = \{u \in \mathbb{R}^p : E_x(\exp(u^T X_t)) < \infty \text{ for all } x \in E\}$.

(ii) $S(D_\mathbb{R}(t)) \subset D_C(t)$.

(iii) The affine transform formula (3.2) holds for all $u \in S(D_\mathbb{R}(t))$.

(iv) $D_\mathbb{R}(t)$ and $D_\mathbb{R}$ are convex sets.

(v) $M(t) \subset M(s)$ for $0 \leq s \leq t$.

Proof. Theorem 3.5 yields $D_\mathbb{R}(t) \subset M(t)$. The proof of $D_\mathbb{R}(t) \supset M(t)$ is the content of Section 4, while Section 5 is devoted to the proof of (ii) and (iii). Assertions (iv) and (v) follow from (i).
4. Full range of validity for real exponentials. Let $T > 0$. In order to prove $M(T) \subset D_R(T)$, we show that $\psi(T, u)$ explodes when $u \in D_R(T)$ approaches the boundary $\partial(D_R(T))$. This is not immediate as the following example demonstrates.

Example 4.1. Consider the Riccati equation $\dot{x} = x^2$. Its solution $x$ with initial condition $u \in \mathbb{C}$ is given by $x(t, u) = u/(1 - ut)$ and we have $t_\infty(u) = u^{-1}$ for $u \in \mathbb{R}_{>0}$ and $t_\infty(u) = \infty$ otherwise. Hence $D_C(T) = \{ u \in \mathbb{C} : u \not\in [T^{-1}, \infty) \}$ and $\partial D_C(T) = [T^{-1}, \infty)$. Obviously $x(T, u)$ does not explode if $u \in D_C(T)$ tends to $u_0 \in (T^{-1}, \infty)$. If we take real and imaginary part, then we obtain a 2-dimensional system of Riccati equations given by
\[
\begin{align*}
\dot{x}_1 &= x_2^2 - x_2^2 \\
\dot{x}_2 &= 2x_1x_2.
\end{align*}
\]
In this case $D_R(T) = \{ u \in \mathbb{R}^2 : u \not\in [T^{-1}, \infty) \}$ and again $x(t, u)$ does not explode if $u \in D_R(T)$ tends to $u_0 \in (T^{-1}, \infty)$. If we take real and imaginary part, then we obtain a 2-dimensional system of Riccati equations given by
\[
\begin{align*}
\dot{x}_1 &= x_2^2 - x_2^2 \\
\dot{x}_2 &= 2x_1x_2.
\end{align*}
\]
In this case $D_R(T) = \{ u \in \mathbb{R}^2 : u \not\in [T^{-1}, \infty) \}$ and again $x(t, u)$ does not explode if $u \in D_R(T)$ tends to $u_0 \in (T^{-1}, \infty)$. Note that the Riccati equations are of the form (3.3) (excluding the equation for $\psi_0$) with
\[
A^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad a = 0.
\]
However, they are not related to an affine diffusion where the state space has non-empty interior. Indeed, the corresponding diffusion matrix would be
\[
c(x) = \begin{pmatrix} x_1 & x_2 \\ x_2 & -x_1 \end{pmatrix},
\]
which is positive semi-definite if and only if $x = 0$.

In Lemma 4.2 below we derive a formula that relates solutions to Riccati equations to the expectation of the corresponding affine diffusion. This will turn out to be most useful in Proposition 4.4 to derive that $M(T) \subset D_R(T)$, which proves Theorem 3.7 (i).

Proposition 4.2. Consider the situation of Theorem 3.7. Define the non-negative function $k : E \times \mathbb{R}^p \to \mathbb{R}$ by
\[
k(x, y) = \frac{1}{2} y^\top c(x) y + \int (e^{y^\top z} - 1 - y^\top z) K(x, dz),
\]
for $x \in E$ and $y \in \mathbb{R}^p$. Then for all $x \in E$, $u \in \mathbb{R}^p$, $t < t_\infty(u)$ it holds that
\[
\psi_0(t, u) + \psi(t, u)^\top x = u^\top E_x X_t + \int_0^t k(E_x X_{t-s}, \psi(s, u)) ds,
\]
and $E_x X_t$ solves the linear ODE

\begin{equation}
\dot{x} = b(x), \quad x(0) = x.
\end{equation}

**Proof.** Fix $u \in \mathbb{R}^p$ and write $\psi(\cdot)$ instead of $\psi(\cdot, u)$. We can write the ODE for $(\psi_0, \psi)$ as an inhomogeneous linear ODE, namely

\[
\begin{pmatrix}
\dot{\psi}_0 \\
\dot{\psi}
\end{pmatrix} = A \begin{pmatrix}
\psi_0 \\
\psi
\end{pmatrix} + g,
\]

where we write $a$ for the $(p \times p)$-matrix with columns $a_i$, $i = 1, \ldots, p$ and $g = (g_0, g_1, \ldots, g_p)$ is the function given by

\[
g_i = \frac{1}{2} \psi^\top A_i \psi + \int (e^{\psi^\top z} - 1 - \psi^\top z) K^i(dz), \quad i = 0, \ldots, p.
\]

By an application of a variation of constants, the solution can be written as

\[
\begin{pmatrix}
\psi_0(t) \\
\psi(t)
\end{pmatrix} = e^{At} \begin{pmatrix}
0 \\
u
\end{pmatrix} + \int_0^t e^{A(t-s)} g(s) ds,
\]

which yields

\begin{equation}
\psi_0(t) + \psi(t)^\top x = \begin{pmatrix}
\psi_0(t) \\
\psi(t)
\end{pmatrix}^\top \begin{pmatrix}
1 \\
x
\end{pmatrix}
= \begin{pmatrix}
0 & u^\top
\end{pmatrix} e^{A^\top t} \begin{pmatrix}
1 \\
x
\end{pmatrix} + \int_0^t g(s) e^{A^\top(t-s)} \begin{pmatrix}
1 \\
x
\end{pmatrix} ds.
\end{equation}

Write $f(t, x)$ for the solution to the linear ODE (4.3) with $f(0, x) = x$. Then we have

\[
\begin{pmatrix}
y(t) \\
z(t)
\end{pmatrix} := e^{A^\top t} \begin{pmatrix}
1 \\
x
\end{pmatrix} = \begin{pmatrix}
1 \\
f(t, x)
\end{pmatrix}.
\]

Indeed, since

\[
\begin{pmatrix}
y \\
z
\end{pmatrix} = A^\top \begin{pmatrix}
y \\
z
\end{pmatrix} = \begin{pmatrix}
0 \\
a^0 y + az
\end{pmatrix},
\]

it holds that $y = 1$ and $\dot{z} = az + a^0 = b(z)$ with $z(0) = x$, whence $z(t) = f(t, x)$. Noting that

\[
g^\top \begin{pmatrix}
1 \\
x
\end{pmatrix} = \frac{1}{2} \psi^\top x c(x) \psi + \int (e^{\psi^\top z} - 1 - \psi^\top z) K(x, dz), \quad \text{for all } x \in E,
\]
and $E_x X_t \in E$ for all $x \in E$, $t \geq 0$, by convexity of $E$, we obtain (4.2) from (4.4) after we have shown that $E_x X_t = f(t, x)$. The latter follows from Lemma A.2, as it yields

$$E_x X_t = x + E_x \int_0^t (a^0 + a X_s) ds = \int_0^t (a^0 + a E_x X_s) ds.$$  

In the following we make use of the fact that for $c_n \in \mathbb{R}^p$ it holds that

$$(4.5) \quad \lim_{n \to \infty} \|c_n\| = \infty \Rightarrow \exists x \in \{-1, 1\}^p, \varepsilon > 0 : \limsup_{n \to \infty} \inf_{y \in B(x, \varepsilon)} c_n^T y = \infty.$$  

Indeed, if $\lim_{n \to \infty} \|c_n\| = \infty$, then there exists a subsequence $c_{n_k}$ such that all components $c_{n_k, i}$ are convergent in $[-\infty, \infty]$. In addition, one of them converges to either $+\infty$ or $-\infty$. Define $x \in \mathbb{R}^p$ by taking $x_i = -1$ if $c_{n_k, i} \to -\infty$ and $x_i = 1$ otherwise. Then obviously for $y \in B(x, \varepsilon)$ with $0 < \varepsilon < 1$ we have

$$\inf_{y \in B(x, \varepsilon)} c_{n_k}^T y \to \infty, \text{ as } k \to \infty.$$  

**Lemma 4.3.** Consider the situation of Theorem 3.7. Let $u \in \mathbb{R}^p$ and suppose $T := t_\infty(u) < \infty$. Then there exists $x \in E$ such that $E_x \exp(u^T X_T) = \infty$.

**Proof.** Without loss of generality we may assume that $\{-1, 1\}^p \subset E^\circ$ (thus by convexity also $0 \in E^\circ$). Since $\|\psi(t, u)\| \to \infty$ for $t \uparrow T$ and in view of (4.5), there exists a ball $B := B(x_0, \varepsilon) \subset E$ (with $x_0 \in \{-1, 1\}^p \subset E^\circ$, $\varepsilon > 0$) and a sequence $t_n \uparrow T$ such that

$$\inf_{y \in B} \psi(t_n, u)^T y \to \infty \text{ as } n \to \infty.$$  

Moreover, it holds that $\psi_0(t, u) \geq u^T E_0(X_t)$ for $t < T$ by Proposition 4.2. In particular we have $\liminf_{t \uparrow T} \psi_0(t, u) > -\infty$. Hence

$$\lim_{n \to \infty} \inf_{y \in B} (\psi_0(t_n, u) + \psi(t_n, u)^T y) = \infty.$$  

By right-continuity of $X$, it follows that

$$\lim_{n \to \infty} (\psi_0(t_n, u) + \psi(t_n, u)^T X_{T-t_n}) = \infty, \text{ } \mathbb{P}_{x_0}-\text{a.s.}$$
The Markov property and Theorem 3.5 give
\[ \text{E}_x \exp(u^\top X_T) = \text{E}_x \left( \text{E}_{X_{T-t}} \exp(u^\top X_t) \right) = \text{E}_x \exp(\psi_0(t, u) + \psi(t, u)^\top X_{T-t}), \]
for \( 0 \leq t < T, \ x \in E \). Applying the previous together with Fatou’s Lemma we get
\[
\text{E}_{x_0} \exp(u^\top X_T) = \liminf_{n \to \infty} \text{E}_{x_0} \exp(u^\top X_T)
\]
\[
= \liminf_{n \to \infty} \text{E}_{x_0} \exp(\psi_0(t_n, u) + \psi(t_n, u)^\top X_{T-t_n})
\]
\[
\geq \text{E}_{x_0} \liminf_{n \to \infty} \exp(\psi_0(t_n, u) + \psi(t_n, u)^\top X_{T-t_n}) = \infty.
\]

\[ \square \]

Proposition 4.4. Consider the situation of Theorem 3.7. Let \( T \geq 0 \). Then \( M(T) = D_R(T) \) and (3.2) holds for \( u \in M(T), \ t \leq T \).

Proof. In view of Theorem 3.5 it is sufficient to prove \( M(T) \subset D_R(T) \). Without loss of generality we may assume that \( \{-1, 1\}^p \subset \mathcal{E} \). Let \( u \in \mathbb{R}^p \) and suppose \( t_\infty(u) < \infty \). We need to show that for all \( T \geq t_\infty(u) \) there exists \( x \in E \) such that \( \text{E}_x \exp(u^\top X_T) = \infty \). Lemma 4.3 gives the result for \( T = t_\infty(u) \). Therefore, let \( T > t_\infty(u) \). Arguing by contradiction, assume \( \text{E}_x \exp(u^\top X_T) < \infty \) for all \( x \in E \). Then by Jensen’s inequality we have
\[
\text{E}_x \exp(\lambda u^\top X_T) \leq (\text{E}_x \exp(u^\top X_T))^\lambda \leq 1 + \text{E}_x \exp(u^\top X_T) < \infty,
\]
for all \( 0 \leq \lambda \leq 1, \ x \in E \). Let \( \lambda^* = \inf\{\lambda \geq 0 : \lambda u \not\in D_C(T)\} \). Note that \( 0 < \lambda^* \leq 1 \) and \( \lambda^* u \not\in D_C(T) \), since \( u \not\in D_C(T) \) and \( D_C(T) \) is an open neighborhood of 0. Considering \( \lambda^* u \) instead of \( u \), we may assume without loss of generality that \( \lambda^* = 1 \). In the following, we let \( u_n = \lambda_n u \), for arbitrary \( \lambda_n \in \{0, 1\} \) such that \( \lambda_n \uparrow 1 \) as \( n \to \infty \), so that \( u_n \in D_C(T) \) and \( u_n \to u \). We divide the proof into a couple of steps.

Step 1. If for some \( t \leq T \) and \( x \in E \) we have
\[
\lim_{n \to \infty} (\psi_0(t, u_n) + \psi(t, u_n)^\top x) = \infty,
\]
then \( \limsup_{n \to \infty} \|\psi(t, u_n)\| = \infty \). To prove this, suppose (4.7) holds for some \( t \leq T \), but \( \limsup_{n \to \infty} \|\psi(t, u_n)\| < \infty \). Then \( \lim_{n \to \infty} \psi_0(t, u_n) = \infty \) and (4.7) holds for all \( x \). Since \( u_n \in D_C(T) \subset D_C(t) \), the Markov property and Theorem 3.5 give
\[
\text{E}_x \exp(u_n^\top X_T) = \text{E}_x \left( \text{E}_{X_{T-t}} \exp(u_n^\top X_t) \right)
\]
\[
= \text{E}_x \exp(\psi_0(t, u_n) + \psi(t, u_n)^\top X_{T-t}).
\]
Fatou’s Lemma yields
\[
\infty = E_x \liminf_{n \to \infty} \exp(\psi_0(t, u_n) + \psi(t, u_n)^T X_{T-t})
\leq \liminf_{n \to \infty} E_x \exp(\psi_0(t, u_n) + \psi(t, u_n)^T X_{T-t}) = \liminf_{n \to \infty} E_x \exp(u_n^T X_T),
\]
which contradicts (4.6) as \( u_n = \lambda_n u \) with \( 0 \leq \lambda_n < 1 \).

Step 2. It holds that
\[
\limsup_{n \to \infty} \|\psi(t_\infty(u), u_n)\| = \infty.
\]
Indeed, since \( u_n \in D_\mathbb{C}(T) \subset D_\mathbb{C}(t_\infty(u)) \), Fatou’s Lemma together with Theorem 3.5 gives
\[
E_x \exp(u^T X_{t_\infty(u)}) \leq \liminf_{n \to \infty} E_x \exp(u_n^T X_{t_\infty(u)})
= \liminf_{n \to \infty} \exp(\psi_0(t_\infty(u), u_n) + \psi(t_\infty(u), u_n)^T x),
\]
for all \( x \in E \). In view of Lemma 4.3 there exists an \( x_0 \in E \) such that we have \( E_{x_0} \exp(u^T X_{t_\infty(u)}) = \infty \), whence
\[
\psi_0(t_\infty(u), u_n) + \psi(t_\infty(u), u_n)^T x_0 \to \infty, \quad \text{as} \quad n \to \infty.
\]
Step 1 yields (4.8).

Step 3. It holds that \( \limsup_{n \to \infty} \|\psi(T, u_n)\| = \infty \). To prove this, we show that there exists \( \varepsilon > 0 \) such that if \( \limsup_{n \to \infty} \|\psi(t_0, u_n)\| = \infty \) for some \( t_0 \in [t_\infty(u), T] \), then \( \limsup_{n \to \infty} \|\psi(t_1, u_n)\| = \infty \) for \( t_1 = T \wedge (t_0 + \varepsilon) \). By Step 2 and an iteration of the above implication, it follows that \( \limsup_{n \to \infty} \|\psi(T, u_n)\| = \infty \).

Write \( f(t, x) \) for the solution to the linear ODE (4.3) with \( f(0, x) = x \). By continuity of \( f \) and the assumption \( \{-1, 1\}^p \subset E_0 \), there exists \( \varepsilon > 0 \) such that \( f(-t, x) \in E \) for all \( x \in \{-1, 1\}^p, \ 0 \leq t \leq \varepsilon \). Let \( t_0 \in [t_\infty(u), T] \) and \( t_1 = T \wedge (t_0 + \varepsilon) \). Suppose \( \limsup_{n \to \infty} \|\psi(t_0, u_n)\| = \infty \). Then in view of (4.5), there exist \( x \in \{-1, 1\}^p \) and a subsequence of \( u_n \) (also denoted by \( u_n \)) such that
\[
\lim_{n \to \infty} \psi(t_0, u_n)^T x = \infty.
\]
As in the proof of Lemma 4.3 we have \( \liminf_{n \to \infty} \psi_0(t_0, u_n) > -\infty \). Hence
\[
\lim_{n \to \infty} (\psi_0(t_0, u_n) + \psi(t_0, u_n)^T x) = \infty.
\]
Since \( t_0 - t_1 \geq -\varepsilon \), we have \( y := f(t_0 - t_1, x) \in E \) and by the semi-group property of the flow it holds that
\[
E_y X_{t_1 - s} = f(t_1 - s, f(t_0 - t_1, x)) = f(t_0 - s, x) = E_x X_{t_0 - s}, \quad \text{for} \quad s \leq t_0.
\]
Let $k$ be the non-negative function given by (4.1). It follows from Proposition 4.2 that

$$
\psi_0(t_1, u_n) + \psi(t_1, u_n) \top y = u_n \top E_y X_{t_1} + \int_{t_1}^{t_0} k(E_y X_{t_1-s}, \psi(s, u_n)) \, ds
$$

\[\geq u_n \top E_y X_{t_1} + \int_{t_0}^{t} k(E_y X_{t_1-s}, \psi(s, u_n)) \, ds\]

\[= u_n \top E_y X_{t_1} + \int_{t_0}^{t} k(E_x X_{t_0-s}, \psi(s, u_n)) \, ds\]

\[= u_n \top (E_y X_{t_1} - E_x X_{t_0}) + \psi_0(t_0, u_n) + \psi(t_0, u_n) \top x ,\]

which tends to infinity as $n \to \infty$. Step 1 yields $\limsup_{n \to \infty} \|\psi(t_1, u_n)\| = \infty$.

**Step 4.** We are now able to conclude the proof. By Step 3 and (4.9) with $t_0 = T$, there is an $x \in \{-1, 1\}^p$ and a subsequence of $u_n$ (also denoted by $u_n$) such that

$$\lim_{n \to \infty} (\psi_0(T, u_n) + \psi(T, u_n) \top x) = \infty.$$

From (4.6) and Theorem 3.5 we obtain

$$1 + E_x \exp(u \top X_T) \geq E_x \exp(u_n \top X_T) = \exp(\psi_0(T, u_n) + \psi(T, u_n) \top x),$$

for all $n$. The right-hand side tends to infinity, whence $E_x \exp(u \top X_T) = \infty$, contrary to the assumption. \(\square\)

5. **Extending the validity to complex exponentials.** To show that $S(M(T)) \subset D_C(T)$ we need continuity of $x \mapsto E_x \exp(u \top X_T)$. We prove this first in the next lemma, together with some additional results needed in Section 6.

**Lemma 5.1.** Let $X$ be an affine jump-diffusion on $(D_E(0, \infty], F^X, (F^X_t, \mathbb{P})$ with differential characteristics $(b(X), c(X), K(X, dz))$ given by (3.1). Assume

(5.1) \[\int |z|^2 |K^i|(dz) < \infty, \text{ for all } i = 0, \ldots, p,\]

and let $u \in \mathbb{C}^p$ be such that $\sup_{x \in E} \Re u \top x < \infty$. Suppose there exists functions $\Psi_0 : [0, T) \mapsto \mathbb{C}$, $\psi : [0, T) \mapsto \mathbb{C}^p$ such that $\Psi_0(t) \neq 0$ for $t < T$ and

$$E_x \exp(u \top X_t) = \Psi_0(t) \exp(\psi(t) \top x) \text{ for all } x \in E, t < T.$$

Then there exists a function $\psi_0$ such that $\Psi_0(t) = \exp(\psi_0(t))$ and $(\psi_0, \psi)$ solve the system of generalized Riccati equations (3.3) on $[0, T)$. Moreover, $x \mapsto E_x \exp(u \top X_T)$ is continuous on $E$. 
**Proof.** Recall that $K$ is a transition kernel from $E$ to $F$ satisfying $F + E \subset E$. Iterating this relation yields $nF + E \subset E$ for all $n \in \mathbb{N}$. Since $\sup_{x \in E} \Re u^\top x < \infty$, it follows that $\Re u^\top z \leq 0$ for $z \in F$. Hence $f$ given by

$$f(x) = u^\top b(x) + \frac{1}{2} u^\top c(x) + \int (e^{u^\top z} - 1 - u^\top z) K(x, dz)$$

is well-defined and by Itô’s formula

$$\exp(u^\top X_t) - \int_0^t \exp(u^\top X_s)f(X_s)ds$$

is a local martingale. Therefore, there exists a sequence of stopping times $T_n \uparrow \infty$ such that

$$E_x \exp(u^\top X_{t\wedge T_n}) = \exp(u^\top x) + E_x \int_0^{t\wedge T_n} \exp(u^\top X_s)f(X_s)ds.$$ 

Note that $|\exp(u^\top X_s)f(X_s)| \leq C(1 + \sup_{s \leq t} |X_s|)$ for some $C > 0$. In view of Lemma A.2, we can apply the Dominated Convergence Theorem as well as Fubini’s Theorem to derive that

$$(5.2) \quad E_x \exp(u^\top X_t) = \exp(u^\top x) + \int_0^t E_x(\exp(u^\top X_s)f(X_s))ds.$$ 

By the same lemma together with the Dominated Convergence Theorem we get that $s \mapsto E_x(\exp(u^\top X_s)f(X_s))$ is continuous, as $X_s$ is right-continuous and quasi left-continuous. The Fundamental Theorem of Calculus yields that $t \mapsto E_x \exp(u^\top X_t)$ is continuously differentiable for all $x \in E$, which implies that $\Psi_0$ and $\psi_i$ are continuously differentiable in $t$. Define $\psi_0$ by

$$\psi_0(t) = \int_0^t \frac{\Psi_0'(s)}{\Psi_0(s)} ds, \quad t < T.$$ 

Then $\psi_0$ is also continuously differentiable and $\Psi_0(t) = \exp(\psi_0(t))$ (indeed, the quotient of the left- and right-hand side has derivative 0 and equality holds for $t = 0$, whence it holds for all $t$). Necessarily $(\psi_0, \psi)$ has to satisfy the generalized Riccati equations (3.3), in view of (3.6).

To show the second assertion we note that by (5.2) and the previous we have

$$E_x(\exp(u^\top X_t)f(X_t)) = \left. \frac{\partial}{\partial t} E_x \exp(u^\top X_t) \right|_{t=0} = (\dot{\psi}_0(t, u) + \dot{\psi}(t, u^\top x) \exp(\psi_0(t, u) + \psi(t, u^\top x)).$$
So \( x \mapsto E_x(\exp(u^\top X_t)f(X_t)) \) is continuous for \( t < T \). By Lemma A.2
and the Dominated Convergence Theorem we see that
\[
x \mapsto \int_0^T E_x(\exp(u^\top X_s)f(X_s))ds \text{ is continuous},
\]
whence \( x \mapsto E_x \exp(u^\top X_T) \) is continuous.

To extend the validity of the affine transform formula from real to complex
exponentials, we use the analyticity of the characteristic function and the
solutions to the Riccati equations. This is demonstrated in the next lemma,
which we apply in Proposition 5.3 below to derive the desired assertion.

**Lemma 5.2.** Consider the situation of Theorem 3.7. For \( t \geq 0 \), if \( U \subset S(M(t)) \cap D_C(t) \) is
connected and \( 0 \in U \), then (3.2) holds for all \( u \in U \).

**Proof.** By Proposition 4.4 equality (3.2) holds for \( u \in M(t) \). The left-
hand side of (3.2) as a function of \( u \) is analytic on \( S(M(t)) \) and the right-hand
side is analytic on \( D_C(t) \), see Proposition 3.6 (ii) and (iii). By assumption
and the fact that \( S(M(t)) \cap D_C(t) \) is an open neighborhood of \( 0 \) (since
\( M(t) = D_R(t) \) by Proposition 4.4 and \( D_C(t) \) is an open neighborhood of \( 0 \)
by Proposition 3.6 (ii)), there exists an open domain \( B \subset S(M(t)) \cap D_C(t) \)
containing the connected set \( U \cup M(t) \) (as \( M(t) \) is convex). It holds that
\( M(t) \), being an open set in \( \mathbb{R}^p \), is a set of uniqueness for \( B \), whence we can
extend the equality in (3.2) to \( u \in B \), in particular to \( u \in U \).

**Proposition 5.3.** Consider the situation of Theorem 3.7 and let \( T_0 > 0 \)
be arbitrary. Then \( S(D_R(T_0)) \subset D_C(T_0) \) and the affine transform formula
(3.2) holds for all \( u \in S(D_R(T_0)) \), \( t = T_0 \).

**Proof.** In view of Lemma 5.2 it suffices to show \( S(M(T_0)) \subset D_C(T_0) \).
We argue by contradiction. Suppose there exists \( u^* \in S(M(T_0)) \cap D_C(T_0)^c \).
We divide the proof into a couple of steps. In the following we write \([0,u]\)
for the line segment in \( \mathbb{C}^p \) with endpoints 0 and \( u \). For a function \( f \) we write
\( f([0,t]) \) for the path \( s \mapsto f(s), s \in [0,t] \). Furthermore, throughout we use
that (3.2) holds for \( u \in M(t) \), which follows from Proposition 4.4.

**Step 1.** There exists \( u_0 \in [0,u^*] \) such that
\[
\begin{align*}
[0,u_0] & \subset S(M(t)) \cap D_C(t), & \text{ for } t < T := t_\infty(u_0), \\
[0,u_0] & \subset S(M(T)) \cap D_C(T).
\end{align*}
\]
We prove this as follows. Since \( S(M(T_0)) \) is convex, the line from \([0, u^*]\) is contained in \( S(M(T_0)) \). Define
\[
\lambda_0 = \inf \{ \lambda \geq 0 : \lambda u^* \notin D_C(T_0) \}.
\]
Then \( 0 < \lambda_0 \leq 1 \), since \( D_C(T_0) \) is an open neighborhood of 0, by Proposition 3.6 (ii). Moreover, \( \lambda u^* \in D_C(T_0) \) for \( \lambda < \lambda_0 \) and \( \lambda_0 u^* \notin D_C(T_0) \). Take \( u_0 = \lambda_0 u^* \), \( T = t_{\infty}(u_0) \). Note that \( T \leq T_0 \), so \( D_C(T_0) \subset D_C(T) \). Then by the previous we have \( (0, u_0) \subset D_C(T_0) \subset D_C(T) \subset D_C(t) \), for \( t < T \). Moreover, \( u_0 \in D_C(t) \) for \( t < T = t_{\infty}(u_0) \). This yields the assertion.

**Step 2.** For all open \( B \subset E^o \) there exists \( x \in B \) such that \( E_x \exp(u_0^\top X_T) = 0 \). To see this, first note that \( \Re u_0 \in M(T) \subset M(t) \) for \( t \leq T \) and that (3.2) holds for \( u = \Re u_0 \). Therefore,
\[
E_x \exp(\Re u_0 X_t) = \exp(\psi_0(t, \Re u_0) + \psi(t, \Re u_0)^\top x) \\
\rightarrow \exp(\psi_0(T, \Re u_0) + \psi(T, \Re u_0)^\top x) = E_x \exp(\Re u_0 X_T) < \infty,
\]
for \( t \uparrow T \), \( x \in E \). By quasi-left continuity we have \( X_t \rightarrow X_T \), \( \mathbb{P}_x \)-a.s. Since \( |\exp(u_0^\top X_t)| \) is bounded by \( \exp(\Re u_0 X_t) \) (indeed it is equal), an extended version of the Dominated Convergence Theorem [17, Theorem 1.21] yields
\[
\lim_{t \uparrow T} E_x \exp(u_0^\top X_t) = E_x \exp(u_0^\top X_T),
\]
for all \( x \in E \). In particular
\[
\lim_{t \uparrow T} \exp(\psi_0(t, u_0) + \psi(t, u_0)^\top x) \text{ exists and is finite, for all } x \in E.
\]
Since \( T = t_{\infty}(u_0) \), we have \( \lim_{t \uparrow T} |\psi(t, u_0)| = \infty \), by Proposition 3.6 (i). It follows that for all open balls \( B \subset E^o \) there exists \( x \in B \) such that
\[
E_x \exp(u_0^\top X_T) = 0,
\]
as otherwise \( \lim_{t \uparrow T} (\psi_0(t, u_0) + \psi(t, u_0)^\top x) \) would be finite on some ball \( B \), which would give a finite limit for \( \psi(t, u_0) \), a contradiction.

**Step 3.** Fix \( 0 < \varepsilon < T \). There exists \( 0 < \delta < T - \varepsilon \) such that
\[
\psi([0, \varepsilon + \delta], u_0) \subset S(M(T - \varepsilon)).
\]

The proof is as follows. Step 1 together with Lemma 5.2 implies that (3.2) holds for \( u = u_0 \) and \( t < T \). Hence by Jensen’s inequality and the Markov
property we have for \( t < \varepsilon, x \in E \) that
\[
E_x \exp(\Re \psi_0(t, u_0) + \Re \psi(t, u_0)X_{T-\varepsilon}) = E_x |\exp(\psi_0(t, u_0) + \psi(t, u_0)\top X_{T-\varepsilon})| \\
= E_x |E_{X_{T-\varepsilon}} \exp(u_0\top X_t)| \\
\leq E_x E_{X_{T-\varepsilon}} \exp(\Re u_0\top X_t) \\
= E_x \exp(\Re u_0\top X_{T-\varepsilon+t}).
\]

Since \( \Re u_0 \in M(T) \subset M(T-\varepsilon) \) it follows that for \( t < \varepsilon, x \in E \) we have
\[
E_x \exp(\Re \psi(t, u_0)X_{T-\varepsilon}) \leq \exp(-\Re \psi_0(t, u_0))E_x \exp(\Re u_0\top X_{T-\varepsilon+t}) \\
= \exp(-\Re \psi_0(t, u_0) + \psi_0(T-\varepsilon + t, \Re u_0) \\
\quad + \psi(T-\varepsilon + t, \Re u_0)\top x).
\]

Fatou’s Lemma yields
\[
E_x \exp(\Re \psi(t, u_0)X_{T-\varepsilon}) \leq \liminf_{t \uparrow \varepsilon} E_x \exp(\Re \psi(t, u_0)X_{T-\varepsilon}) \\
\leq \exp(-\Re \psi_0(t, u_0) + \psi_0(T, \Re u_0) + \psi(T, \Re u_0)\top x) \\
< \infty,
\]
for all \( x \in E \). Hence \( \psi([0, \varepsilon], u_0) \subset S(M(T-\varepsilon)) \). Since \( S(M(T-\varepsilon)) \) is open and \( t \mapsto \psi(t, u_0) \) is continuous on \([0, T]\), the result follows.

**Step 4.** It holds that \( x \mapsto E_x \exp(u_0\top X_T) \) is not continuous. To show this, we argue by contradiction and assume it is continuous. Then we have \( E_x \exp(u_0\top X_T) = 0 \) for all \( x \in E \), by Step 2 and the fact that \( E = \overline{E^\circ} \). The Markov property gives
\[
0 = E_x \exp(u_0\top X_T) \\
= E_x E_{X_{T-\varepsilon}} \exp(u_0\top X_t) \\
= E_x \exp(\psi_0(t, u_0) + \psi(t, u_0)\top X_{T-\varepsilon}), \text{ for all } 0 \leq t < T, x \in E,
\]
so \( E_x \exp(\psi(t, u_0)\top X_{T-\varepsilon}) = 0 \) for all \( 0 \leq t < T, x \in E \). Fix \( 0 < \varepsilon < T \) and write \( v = \psi(\varepsilon, u_0) \) and \( s = T-\varepsilon \). By the semi-group property of the flow we have \( \psi(t, v) = \psi(t+\varepsilon, u_0) \) for \( t < s \), whence the previous yields
\[
E_x \exp(\psi(t, v)\top X_{s-t}) = 0, \text{ for all } 0 \leq t < s, x \in E.
\]
Let \( \delta \) be as in Step 3. Then \( E_x \exp(\psi(t, v)\top X_s) \) is well-defined for \( t \leq \delta, x \in E \). Applying the Markov property yields
\[
E_x \exp(\psi(t, v)\top X_s) = E_x E_{X_t} \exp(\psi(t, v)\top X_{s-t}) = 0, \text{ all } 0 \leq t \leq \delta, x \in E,
\]
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Plugging back $v = \psi(\varepsilon, u_0)$ and $s = T - \varepsilon$ and using the semi-group property of the flow, we see that

\[(5.5)\quad \mathbb{E}_x \exp(\psi(t + \varepsilon, u_0) \top X_{T - \varepsilon}) = 0, \quad \text{for all } 0 \leq t \leq \delta, \quad x \in E.\]

Now fix $x \in E$. It holds that $u \mapsto \mathbb{E}_x \exp(u \top X_{T - \varepsilon})$ and $t \mapsto \psi(t, u_0)$ are analytic on $S(M(T - \varepsilon))$ respectively $[0, T)$, see Proposition 3.6 (ii) and (iii). Step 3 yields (5.3). Therefore, there exists an open domain $B \subset \mathbb{C}^p$ with $[0, \varepsilon + \delta] \subset B$ such that $\psi(z, u_0) \in S(M(T - \varepsilon))$ for $z \in B$. The composition of analytic functions is analytic, whence

$$z \mapsto \mathbb{E}_x \exp(\psi(z, u_0) \top X_{T - \varepsilon})$$

is analytic on $B$. Equation (5.5) yields it is zero on $[\varepsilon, \varepsilon + \delta]$, whence it is zero on the whole of $B$, as $[\varepsilon, \varepsilon + \delta]$ is a set of uniqueness for $B$. In particular it is zero for $z = 0$, i.e. $\mathbb{E}_x \exp(u_0 \top X_{T - \varepsilon}) = 0$. However, by Step 1 and Lemma 5.2 we have

$$\mathbb{E}_x \exp(u_0 \top X_{T - \varepsilon}) = \exp(\psi_0(T - \varepsilon, u_0) + \psi(T - \varepsilon, u_0) \top x) \neq 0,$$

a contradiction.

**Step 5.** It holds that $i\mathbb{R}^p \subset D_C(T)$ and the affine transform formula (3.2) holds for $u \in i\mathbb{R}^p$, $t = T$. Indeed, if $u^* \in i\mathbb{R}^p$, then also $u_0 \in i\mathbb{R}^p$, as $u_0 \in [0, u]$. Step 1 together with Lemma 5.2 yields (3.2) for $u = u_0$, $t < T$. However, Lemma 5.1 then gives that $x \mapsto \mathbb{E}_x(\exp(u_0 \top X_T))$ is continuous, which contradicts Step 2. Hence $i\mathbb{R}^p \subset D_C(T)$. By Lemma 5.2 again we get validity of (3.2) for $u \in i\mathbb{R}^p$, $t = T$.

**Step 6.** We conclude the proof by showing that $x \mapsto \mathbb{E}_x \exp(u \top X_T)$ is continuous for all $u \in S(M(T))$, which contradicts Step 2. Let $x_n \to x$, some $x_n, x \in E$. By Step 5 we have for all $u \in i\mathbb{R}^p$ that

$$\mathbb{E}_{x_n} \exp(uX_T) = \exp(\psi_0(T, u) + \psi(T, u) \top x_n)$$

$$\to \exp(\psi_0(T, u) + \psi(T, u) \top x) = \mathbb{E}_x \exp(uX_T),$$

as $n \to \infty$. Hence $\mathbb{P}_{x_n} \circ X_T^{-1} \to \mathbb{P}_x \circ X_T^{-1}$ weakly. By Skorohod’s Representation Theorem [17, Theorem 4.30] there exist random variables $Y_n, Y$ defined on a common probability space $(\Omega, \mathcal{F}, P)$ such that $P \circ Y_n^{-1} = \mathbb{P}_{x_n} \circ X_T^{-1}$, $P \circ Y^{-1} = \mathbb{P}_x \circ X_T^{-1}$ and $Y_n \to Y$, $P$-a.s. Now let $u \in S(M(T))$ be arbitrary. It holds that $|\exp(u \top Y_n)| = \exp(\Re u \top Y_n)$ and

$$\int \exp(\Re u \top Y_n) dP = \exp(\psi_0(T, \Re u) + \psi(T, \Re u) \top x_n)$$

$$\to \exp(\psi_0(T, \Re u) + \psi(T, \Re u) \top x) = \int \exp(\Re u \top Y) dP,$$
for \( n \to \infty \), since \( \Re u \in M(T) \). An extended version of the Dominated Convergence Theorem [17, Theorem 1.21] yields
\[
E_x \exp(u^\top X_T) = \int \exp(u^\top Y) dP \\
\to \int \exp(u^\top Y) dP = E_x \exp(u^\top X_T),
\]
for \( n \to \infty \), whence \( x \mapsto E_x \exp(u^\top X_T) \) is continuous. \( \square \)

6. Additional results for bounded exponentials. In this section we relax condition (3.7) of Theorem 3.7 on the exponential moments of the \( K^i \) and consider the validity of the affine transform formula when the left-hand side of (3.2) is uniformly bounded in \( t \) and \( x \) (which includes the characteristic function). The following theorem is the third main result of this paper.

**Theorem 6.1.** Suppose \( E \subset \mathbb{R}^p \) is closed convex with non-empty interior and let \( X \) be an affine jump-diffusion on \((D_E(0, \infty], \mathcal{F}^X, (\mathcal{F}^X_t), \mathbb{P})\) with differential characteristics \((b(X), c(X), K(X, dz))\) given by (3.1). Assume (5.1) and write \( U = \{ u \in \mathbb{C}^p : \sup_{x \in E} \Re u^\top x < \infty \} \). Then for all \( u \in U \) there exists a \( t_\infty(u) \in (0, \infty] \) and a solution \((\psi_0(\cdot, u), \psi(\cdot, u)) : [0, t_\infty(u)) \to \mathbb{C} \times \mathbb{C}^p\) to the system of generalized Riccati equations given by (3.3) and for all \( x \in E \) it holds that
\[
E_x \exp(u^\top X_T) = \begin{cases} 
\exp(\psi_0(t, u) + \psi(t, u)^\top x), & t \in [0, t_\infty(u)) \\
0, & t \in [t_\infty(u), \infty) 
\end{cases}
\]

**Proof.** For \( n \in \mathbb{N} \) define
\[
b^n(x) = b(x) + \int z(e^{-\frac{1}{2}z^2} - 1)K(x, dz), \quad K^n(x, dz) = e^{-\frac{1}{2}|z|^2}K(x, dz),
\]
and the operator \( A^n : C_c^\infty(E) \to C_0(E) \) by
\[
A^n f(x) = \nabla f(x) b^n(x) + \frac{1}{2} \text{tr} \left( \nabla^2 f(x) c(x) \right) \\
+ \int (f(x + z) - f(x) - \nabla f(x) z) K^n(x, dz).
\]
Then the affine martingale problem for \( A^n \) is well-posed by Proposition 3.4. Let \( Q^n_x \) be the solution for \((A^n, \delta_x)\) and write \( E^{n}_x \) for the expectation with respect to \( Q^n_x \). Since \( K^n \) satisfies (3.7), Theorem 3.7 yields
\[
E^{n}_x \exp(u^\top X_T) = \exp(\psi^n_0(t, u) + \psi^n(t, u)^\top x),
\]
for all $u \in U$, $x \in E$, $t \geq 0$, where $(\psi_0^n, \psi^n)$ satisfies (3.3) with $b$ and $K$ replaced by $b^n$ and $K^n$. Fix $x \in E$ arbitrarily and let (2.5) be the decomposition of $X$ under $\mathbb{P}_x$. By Proposition 3.4 it holds that $Q^n|x_{F_x} = L^n_t \cdot \mathbb{P}_x|x_{F_x}$ for all $t \geq 0$, where

$$L^n = \mathcal{E}((e^{-\frac{1}{3}|z|^2} - 1) * (\mu^X - \nu^X)) = \exp((1 - e^{-\frac{1}{3}|z|^2}) * \nu^X - \frac{1}{3}|z|^2 * \mu^X).$$

For all $u \in U$ there is a constant $C > 0$ such that $|\exp(u^\top X_t)L^n_t| \leq CL^n_t$. Since $E_xL^n_t = 1$ for all $n$ and $\lim_{n \to \infty}L^n_t = 1$, an extended version of the Dominated Convergence Theorem [17, Theorem 1.21] yields

$$\lim_{n \to \infty} E^n_x \exp(u^\top X_t) = \lim_{n \to \infty} E_x \exp(u^\top X_t)L^n_t = E_x \exp(u^\top X_t),$$

for all $t \geq 0, u \in U$. Since $x \in E$ was taken arbitrarily, this yields

$$E_x \exp(u^\top X_t) = \lim_{n \to \infty} \exp(\psi_0^n(t,u) + \psi^n(t,u)^\top x),$$

for all $u \in U$, $x \in E$, $t \geq 0$. If $E_x \exp(u^\top X_t) \neq 0$ for all $u \in U, x \in E, t \geq 0$, then $\lim_{n \to \infty} \psi^n_0(t,u)$ and $\lim_{n \to \infty} \psi^n(t,u)$ exist and are finite for all $t \geq 0, u \in U$, and the result follows from Lemma 5.1.

Suppose $E_{x_0} \exp(u^\top X_T) = 0$ for some $u \in U, T > 0, x_0 \in E$. We first show that then $E_x \exp(u^\top X_T) = 0$ for all $x \in E^\circ$. If $\limsup_{n \to \infty} |\Re\psi^n(T,u)| < \infty$, then necessarily $\limsup_{n \to \infty} \Re\psi^n_0(T,u) = -\infty$ and the assertion follows immediately. Otherwise, there exists a subsequence of $\psi^n$ (also denoted by $\psi^n$) and an $i \in \{1, \ldots, p\}$ such that

$$\lim_{n \to \infty} \Re\psi^n_0(T,u) = \pm \infty.$$

Then if there exists $x \in E^\circ$ such that

$$\liminf_{n \to \infty}(\Re\psi^n_0(T,u) + \Re\psi^n(T,u)^\top x) > -\infty,$$

then $y$ with $y_j = x_j$ for $j \neq i$ and $y_i = x_i \pm \varepsilon$ for some small $\varepsilon > 0$ satisfies

$$\liminf_{n \to \infty}(\Re\psi^n_0(T,u) + \Re\psi^n(T,u)^\top y) = \infty.$$

This is impossible, since $E_y \exp(u^\top X_T)$ is finite.

Thus $E_x \exp(u^\top X_T) = 0$ for all $x \in E^\circ$. Let $t_\infty(u)$ be given by

$$t_\infty(u) = \inf\{t \geq 0 : E_x \exp(u^\top X_t) = 0 \text{ for some } x \in E\}.$$

Then $t_\infty(u) > 0$. Indeed, otherwise for all $t > 0$ there exists $x \in E$ and $s < t$ such that $E_x \exp(u^\top X_s) = 0$. But then for all $t > 0$ there exists
s < t such that \( E_x \exp(u^\top X_s) = 0 \) for all \( x \in E^o \), in view of the previous. Right-continuity of \( t \mapsto X_t \) in 0 yields \( \exp(u^\top x) = 0 \) for all \( x \in E^o \), which is absurd.

Note that \( E_x \exp(u^\top X_{t_{\infty}(u)}) = 0 \) for all \( x \in E^o \), as \( X \) is right-continuous. For \( t < t_{\infty}(u) \) we have existence of finite limits for \( \psi_0^i(t, u) \) and \( \psi^s(t, u) \).

Lemma 5.1 yields (3.2) where \( (\psi_0, \psi) \) are solutions to the generalized Riccati equations for \( t < t_{\infty}(u) \). In addition it implies that \( x \mapsto E_x \exp(u^\top X_{t_{\infty}(u)}) \) is continuous, whence we have \( E_x \exp(u^\top X_{t_{\infty}(u)}) = 0 \) for all \( x \in E \). Applying the Markov property we see that for \( t \geq t_{\infty}(u) \) it holds that

\[
E_x \exp(u^\top X_t) = E_x E_{X_{t_{\infty}(u)}} \exp(u^\top X_{t_{\infty}(u)}) = 0,
\]

which concludes the proof. \( \square \)

Under analyticity of the Riccati functions \( R_i \), we can sharpen the assertion in Theorem 6.1.

**Theorem 6.2.** Consider the situation of Theorem 6.1. Assume there exists an open domain \( B \supset U \) such that

\[
\int_{\{|z| > 1\}} e^{\sum_{i=0}^p K_i^\top |z|^i} \, dz < \infty, \text{ for all } k \in B \cap \mathbb{R}^p, i = 0, \ldots, p.
\]

Then \( t_{\infty}(u) = \infty \) and (3.2) holds for all \( u \in U, t \geq 0 \). In particular, \( X \) is a regular affine process.

**Proof.** We argue as in Proposition 5.3, Step 4. Let \( u_0 \in U \) and suppose \( T := t_{\infty}(u_0) < \infty \). For \( t < T \), \( u = u_0 \) we have (3.2), which implies that \( \psi(t, u_0) \in U \), as \( E_x \exp(u_0^\top X_t) \) is bounded in \( x \). Similar as in (5.4) we deduce that \( E_x \exp(\psi(t, u_0)^\top X_{T-t}) = 0 \) for all \( 0 \leq t < T \), \( x \in E \). Fix \( 0 < \varepsilon < T \) and write \( v = \psi(\varepsilon, u_0) \) and \( s = T - \varepsilon \). We have \( \psi(t, v) \in U \), so \( E_x \exp(\psi(t, v)X_s) \) is well-defined for \( t < s, x \in E \). By the same argument as in Step 4 of Proposition 5.3, we get (5.5), with \( \delta < T - \varepsilon \). Since \( \psi(t, u_0) \in U \subset B \) for all \( t < T \) and \( R_i \) given by (3.4) is analytic on \( B \), it follows by standard ODE results (e.g. [7, Theorem 10.4.5]) that \( t \mapsto \psi(t, u_0) \) is analytic on \( [0, T) \). Moreover, for all \( u \in B \) there exists a solution \( (\psi_0, \psi) \) to (3.3) on a non-empty interval \([0, t_{\infty}(u)]\) with

\[
t_{\infty}(u) = \lim_{n \to \infty} \inf\{t \geq 0 : \psi(t, u) \in \partial B \text{ or } |\psi(t, u)| \geq n\},
\]

and

\[
D(t) = \{u \in B : t < t_{\infty}(u)\}.
\]
is an open set containing $U$, for all $t \geq 0$, see [1, Theorems 7.6 and 8.3]. Theorem 3.5 implies that (3.2) holds for $u \in D(t) \cap \mathbb{R}^p$ for all $t \geq 0$. By Proposition 3.6 (iii) we obtain that $u \mapsto \mathbb{E}_x \exp(u^T X_t)$ is analytic on $U$ for $x \in E$, for all $t \geq 0$. It follows that $\mathbb{E}_x \exp(\psi(t,u_0)^T X_{T-t})$ is analytic in $t$. Since it is zero on $[\varepsilon, \varepsilon + \delta]$, it is zero everywhere, in particular it is zero at $t = 0$. This contradicts the fact that $T - \varepsilon < t_\infty(u_0)$.

6.1. Infinite divisibility. As a corollary of Theorem 6.1 we obtain a sufficient criterium for infinite divisibility of an affine jump-diffusion with a general closed convex state space.

**Theorem 6.3.** Consider the situation of Theorem 6.1. Suppose for all $n \in \mathbb{N}$ it holds that

\[(a^i, nA^i, \frac{1}{n}K^i(\frac{1}{n} \, \mathrm{d}z))_{0 \leq i \leq p}\]

is an admissable parameter set. Then $\mathbb{P}_x \circ X_t^{-1}$ is infinitely divisible for all $t \geq 0$, $x \in E$. Consequently, $t_\infty(u) = \infty$ and (3.2) holds for all $u \in U$, $t \geq 0$.

**Proof.** Let $(\psi_0, \psi)$ be the solution to the Riccati equations as given in Theorem 6.1. Define $\psi^n_i = \frac{1}{n} \psi_i$, for $i = 0, \ldots, p$. Then $(\psi^n_0, \psi^n)$ solve the system of Riccati equations corresponding to an affine jump-diffusion with parameter set (6.1). Let $\mathbb{P}_x^n$ be the solution of the associated affine martingale problem with initial condition $\delta_x$ and write $\mathbb{E}_x^n$ for the expectation with respect to this probability measure. From Theorem 6.1 it follows that

\[\left(\mathbb{E}_x \exp(uX_t)\right)^{1/n} = \mathbb{E}_x^n \exp(uX_t),\]

for all $x \in E$, $u \in U$. In particular it holds for $u \in i \mathbb{R}^p$, which yields the result. $\square$

6.2. Self-dual cone. We can strengthen the conditions of Theorem 6.2 in case $E$ is a self-dual cone. Recall that $E$ is a self-dual cone with respect to an inner product $\langle \cdot, \cdot \rangle$ if

\[E = \{ x \in \mathbb{R}^p : \langle x, y \rangle \geq 0 \text{ for all } y \in E \}.\]

In that case we also have

\[E^o = \{ x \in \mathbb{R}^p : \langle x, y \rangle > 0 \text{ for all } y \in E \setminus \{0\} \}.\]

For $x, y \in \mathbb{R}^p$ we write $x \preceq y$ if $y - x \in E$ and $x < y$ if $y - x \in E^o$. An inner product on $\mathbb{R}^p$ can always be written as $\langle x, y \rangle = x^T M y$ for some positive
definite matrix $M$. By applying the linear transformation $x \mapsto M^{1/2}x$ on the state space $E$, we may assume without loss of generality that the underlying inner product is the usual Euclidean inner product and we write $x^\top y$ instead of $\langle x, y \rangle$.

Part of the following proposition extends [20, Proposition 3.4] and [4, Lemma 3.3] from the state spaces $\mathbb{R}^p_+$ and $S^p_+$ to general self-dual cones. We adapt their proofs slightly by using the analyticity of $t \mapsto \psi_i(t, u)$ in a neighborhood of 0 for $u \in -E^o$, which is a consequence of Theorem 6.1.

**Proposition 6.4.** Consider the situation of Theorem 6.1. Assume the state space $E$ is a self-dual cone and in addition assume $E^o \subset \{x \in \mathbb{R}^p : \Phi(x) > 0\}$ and $\partial E \subset \{x \in \mathbb{R}^p : \Phi(x) = 0\}$, for some analytic function $\Phi : \mathbb{R}^p \to \mathbb{R}$. Then for $U = -E + i\mathbb{R}^p$ it holds that

1. $\psi_0(t, u) \leq \psi_0(t, v)$ and $\psi(t, u) \leq \psi(t, v)$ for $u \leq v$ with $u, v \in RU$, $t \geq 0$;
2. $\psi(t, u) \in RU^o$ for all $u \in RU^o$, $t \geq 0$;
3. $t_\infty(u) = \infty$ for $u \in U^o$ and $\psi(t, u) \in U^o$ for all $u \in U^o$, $t \geq 0$;

**Remark 6.5.** Examples of such state spaces are $\mathbb{R}^p_+$, $\vech(S^p_+)$ (with inner product $\langle x, y \rangle = \text{tr} (\vech^{-1}(x)\vech^{-1}(y))$) and the Lorentz cone $\{x \in \mathbb{R}^p : x_1 \geq (\sum_{i=2}^p x_i^2)^{1/2}\}$. The analytic function $\Phi(x)$ can be chosen to be respectively $\Phi(x) = \prod_{i=1}^p x_i$, $\Phi(x) = \det(\vech^{-1}(x))$ and $\Phi(x) = x_1^2 - \sum_{i=1}^p x_i^2$.

**Proof.** If $u \leq v$ with $u, v \in RU$, then $u^\top x \leq v^\top x$ for all $x \in E$. Hence

$$E_x \exp(u^\top X_t) \leq E_x \exp(v^\top X_t), \text{ for all } x \in E, t \geq 0.$$ 

Since the affine transform formula is valid for $u \in RU$ by Theorem 6.1, it follows that

$$\psi_0(t, u) + \psi(t, u)^\top x \leq \psi_0(t, v) + \psi(t, v)^\top x, \text{ for all } x \in E, t \geq 0.$$ 

Taking $x = nx_0$ with $x_0 \in E \setminus \{0\}$, $n \in \mathbb{N}$ and letting $n$ tend to infinity, we obtain the first assertion.

We prove the second assertion from an argument by contradiction. Suppose $u_0 \in RU^o$ and suppose

$$t := \inf\{s > 0 : \psi(s, u_0) \not\in RU^o\} < \infty.$$ 

Then $\psi(t, u_0) \in \partial E$, so $\psi(t, u_0)^\top x_0 = 0$ for some $x_0 \in E$. If $u_0 \leq v$, then $\psi(t, u_0) \leq \psi(t, v)$ by the first assertion and since $\psi(t, v) \in -E$ we have
ψ(t, u₀)ᵀ x ≤ ψ(t, v)ᵀ x ≤ 0 for all x ∈ E. Hence ψ(t, v)ᵀ x₀ = 0 and ψ(t, v) ∈ \partial E. Thus we have

Φ(ψ(t, v)) = 0, for all v ≥ u₀.

It holds that \{v ∈ RU₀ : v ≥ u₀\} is a set of uniqueness. Moreover, u ↦ Eₓ exp(uᵀ Xₜ) is analytic on \(U^₀\) for all x ∈ E, by Proposition 3.6 (iii). This implies that u ↦ ψ(t, u) is analytic on \(RU^₀\). It follows that

Φ(ψ(t, u)) = 0, for all u ∈ RU.

In particular (take u = ψ(s, u₀)) we have

Φ(ψ(t + s, u₀)) = Φ(ψ(t, ψ(s, u₀))) = 0, for all s > 0.

Let ε > 0 be such that ψ(s, u) ∈ RU₀ for −ε < s < ε. Then s ↦ ψ(s, u) is analytic on \((-ε, ε)\) in view of (3.3) and the analyticity of (3.4). Hence s ↦ Φ(ψ(t + s, u)) is analytic on \((-ε, ε)\) and it follows that it is zero on this interval, as it is zero on \([0, ε)\). This contradicts ψ(s, u₀) ∈ RU₀ for s < t.

For the third assertion, let u ∈ U₀. Then

\[
\exp(\Reψ₀(t, u) + \Reψ(t, u)ᵀ x) = |Eₓ(\exp(uᵀ Xₜ))| \\
≤ Eₓ(\exp(\Re uᵀ Xₜ) \\
= \exp(ψ₀(t, \Re u) + ψ(t, \Re u)ᵀ x),
\]

for all x ∈ E, t < t∞(u). Take x₀ ∈ E\{0\} and x = nx₀ for n ∈ N and let n tend to infinity. Then the right-hand side of the above display tends to zero, which implies \(\Reψ(t, u)ᵀ x < 0\) for all x ∈ E\{0\}, i.e. \(ψ(t, u) ∈ U^₀\). The proof of t∞(u) = ∞ goes along the same lines as the proof of Theorem 6.2.

**Corollary 6.6.** Consider the situation of Proposition 6.4. Write K for the vector of signed measures Kᵢ, i = 1, \ldots, p, let L_u = \{z ∈ E : uᵀ z ≠ 2kπ, for all k ∈ Z\} and assume K(L_u) > 0 for all u ∈ \(\Re^p\). Then t∞(iu) = ∞ for all u ∈ \(\Re^p\), whence X is a regular affine process in the sense of Definition 3.1.

**Proof.** For u = 0 there is nothing to prove. Let u ∈ \(\Re^p\)\{0\} be arbitrary. It suffices to prove ψ(0, iu) ∈ U₀. Indeed, by continuity we then have ψ(t, iu) ∈ U₀ for t > 0 small enough. Hence ψ(t, iu) = iu + \(\int_0^t ψ(s, iu)ds ∈ U^₀\) for t > 0 small enough. The result then follows from Proposition 6.4.

We first show that c(x) − A₀ is positive semi-definite and K(dz)ᵀ x is a positive measure, for all x ∈ E. Since E is a cone we have nx ∈ E, for all n ∈ N, x ∈ E. We can write

\[
c(nx) = A₀ + n(c(x) − A₀), \quad K(nx, dz) = K₀(dz) + nK(dz)ᵀ x.
\]
Since \( c(x) \) is positive semi-definite and \( K(x, dz) \) is a positive measure for all \( x \in E \), we have the same properties for \( c(x) - A^0 \) respectively \( K(dz)^\top x \), in view of the above display.

Next we note that \( \int (\cos(u^\top z) - 1)K(dz) < 0 \). Indeed, by the assumption \( K(L_u) > 0 \) and the fact that \( f(z) := \cos(u^\top z) - 1 < 0 \) for \( z \in L_u \) we have

\[
\int f(z)K(dz)^\top x = \int_{E\setminus L_u} f(z)K(dz)^\top x + \int_{L_u} f(z)K(dz)^\top x < 0,
\]

for \( x \in E\setminus\{0\} \).

Now let \( x \in E\setminus\{0\} \) be arbitrary. Then the previous together with (3.4) yields

\[
\Re \dot{\psi}(0, iu)^\top x = -\frac{1}{2} u^\top (c(x) - A^0)u + \int (\cos(u^\top z) - 1)K(dz)^\top x < 0,
\]

whence \( \Re \dot{\psi}(0, iu)^\top < 0 \), as we needed to show.

**APPENDIX A**

**Proof of Remark 2.2 part 2.** Let \( f_k \) be a sequence in \( C_\infty^c \) with \( 0 \leq f_k \leq 1 \) and \( f_k = 1 \) on the ball with center 0 and radius \( k \). We define

\[
\mathcal{C} = \{ f \in C_\infty^c(E) : f(x) = \cos(u^\top x)f_k(x) \text{ or } f(x) = \sin(u^\top x)f_k(x), \text{ for some } u \in \mathbb{Q}, k \in \mathbb{N} \}.
\]

Then \( P \) is a solution of the martingale problem for \( A \) on \( \Omega \) if and only if

\[
f(X_t) - f(X_0) - \int_0^t Af(X_s)ds
\]

is an \((\mathcal{F}_t^X, \mathbb{P})\)-martingale for all \( f \in \mathcal{C} \). Indeed, suppose the latter holds, then following the proof of [3, Proposition 3.2] we deduce that

\[
f(X_t) - f(X_0) - \int_0^t \nabla f(X_s)b(X_s) + \frac{1}{2} \text{tr} (\nabla^2 f(X_s)c(X_s))
\]

\[
+ \int (f(X_s + z) - f(X_s) - \nabla f(X_s)z)K(X_s, dz)ds
\]

is an \((\mathcal{F}_t^X, \mathbb{P})\)-local martingale for \( f(x) = e^{iu^\top x} \), for all \( u \in \mathbb{Q}^p \), whence for all \( u \in \mathbb{R}^p \) by dominated convergence. [16, Theorem II.2.42] yields that \( \mathbb{P} \) is a solution of the martingale problem for \( A \) on \( \Omega \).

Applying [11, Theorem 4.4.6] to the operator \( A|_\mathcal{C} \) gives that \( x \mapsto \mathbb{P}_x(B) \) is measurable for all Borel sets \( B \), i.e. \((\mathbb{P}_x)_{x \in E}\) is a transition kernel. We
note that although we don’t have well-posedness for all initial values in the sense of [11, Theorem 4.4.6], the assertion in that theorem still holds under the weaker assumption of well-posedness for degenerate initial distributions. This is a consequence of the fact that the set \( \{ P \in \mathcal{P}(E) : P \text{ is degenerate} \} \) is measurable with respect to the Borel \( \sigma \)-algebra induced by the Prohorov metric (in fact, it is even a closed set).

Following the last part of the proof of [17, Theorem 21.10] we see that \( \mathbb{P}_\lambda := \int \mathbb{P}_x \lambda(dx) \) is the unique solution for \((A, \lambda)\). The strong Markov property is a consequence of [11, Theorem 4.4.2(c)].

**Lemma A.1.** Let \( A \) and \( \tilde{A} \) be given by (2.3) and (2.8) and assume (2.1), (2.2), (2.6) and (2.7). Then for all \( f \in C_c^\infty(E) \) it holds that \( A f \in B(E) \) and \( \tilde{A} f \in C_0(E) \).

**Proof.** Take \( f \in C_c^\infty(E) \) with \( f(x) = 0 \) for \( |x| > M \), some \( M > 0 \). Then for \( |x| > M + 1 \) it holds that

\[
|Af(x)| = \left| \int f(x + z)K(x, dz) \right|
\leq \|f\|_\infty \int_{\{|z|>|x|-M\}} |z|^q/(|x|-M)^q K(x, dz)
\leq \|f\|_\infty C(1 + |x|)^q/(|x|-M)^q,
\]

which is bounded for \( x \geq M + 1 \). Hence \( Af \in B(E) \) and likewise one can show that \( \tilde{A} f \to 0 \) if \( |x| \to \infty \). It remains to show that \( \tilde{A} f \) is continuous. Write \( g(x) = f(x + z) - f(x) - \nabla f(x)z \) and

\[
\int g(x)\tilde{K}(x, dz) - \int g(y)\tilde{K}(y, dz) = \int (g(x) - g(y))\tilde{K}(x, dz)
+ \int g(y)(\tilde{K}(x, dz) - \tilde{K}(y, dz)).
\]

The integrand of the first term on the right-hand side equals (where \( f_{ijk} \) is short-hand notation for \( \partial_i \partial_j \partial_k f \))

\[
\sum_{i,j,k} \int_0^1 \int_0^1 \int_0^1 f_{ijk}((1-u)y + ux + sz)(x_i - y_i)s t z_j z_k \, du \, ds \, dt \, 1_{\{|z| \leq 1\}}
+ \sum_{i,j} \int_0^1 \int_0^1 (f_{ij}((1-t)y + tx + sz)
- f_{ij}((1-t)y + tx))(x_i - y_i)st z_j \, ds \, dt \, 1_{\{|z| > 1\}},
\]
whence its integral tends to zero for \( x \to y \) since \( \int (|z|^2 \wedge |z|) \tilde{K}(. , dz) \) is bounded on compacta. The integrand in the second term on the right-hand side can be bounded by a constant times \( |z|^2 \wedge |z| \), whence the integral tends to zero by weak continuity of \( x \mapsto (|z|^2 \wedge |z|) \tilde{K}(x,dz) \). It now easily follows that \( \tilde{A}f \) is continuous.

**Lemma A.2.** Let \( \Omega = D_E[0, \infty) \) and suppose \( X \) is a special jump-diffusion on \( (\Omega,F^X,\{F^X_t\},\mathbb{P}) \) with decomposition (2.5) and differential characteristics \( (b(X)1_{[0,\tau]}, c(X)1_{[0,\tau]}, K(X, dz)1_{[0,\tau]}) \) for some \( (F^X_t) \)-stopping time \( \tau \). Assume \( \mathbb{E}|X_0|^2 < \infty \) and

\[
|b(x)|^2 + |c(x)| + \int |z|^2 K(x, dz) \leq C(1 + |x|^2), \text{ for some } C > 0, \text{ all } x \in E.
\]

Then for all \( T \geq 0 \) it holds that

\[
\mathbb{E}\sup_{t \leq T} |X_t|^2 \leq (4\mathbb{E}|X_0|^2 + C(T))e^{C(T)T},
\]

with \( C(T) \) a constant depending on \( C \) and \( T \). In addition, \( X^c \) and \( z*(\mu^X - \nu^X) \) are proper martingales.

**Proof.** Define stopping times \( T_n = \inf\{t \geq 0 : |X_t| \geq n \text{ or } |X_{t-}| \geq n\} \).

It holds that

\[
\langle X^c \rangle^T_{T_n} = \int_0^{t \wedge T_n} c(X_s)1_{[0,\tau]}(s)ds
\]

and

\[
\langle z*(\mu^X - \nu^X) \rangle^T_{T_n} = \int_0^{t \wedge T_n} \int |z|^2 K(X_s, dz)1_{[0,\tau]}(s)ds
\]

have finite expectation, as they are bounded. This yields that both \( (X^c)^T_{T_n} \) and \( z*(\mu^X - \nu^X)^T_{T_n} \) are martingales, by [16, Proposition I.4.50]. For \( t \geq 0 \) write \( ||X||_t = \sup_{s \leq t} |X_s| \) and let \( T > 0 \) be fixed. Then it holds that

\[
\frac{1}{T}||X^\tau ||_T^2 \leq |X_0|^2 + \sup_{t \leq T} \left| \int_0^t b(X_s)1_{[0,\tau \wedge T_n]}(s)ds \right|^2 + \sup_{t \leq T} |X^c_{t \wedge T_n}|^2
\]

Cauchy-Schwarz gives

\[
\sup_{t \leq T} \left| \int_0^t b(X_s)1_{[0,\tau \wedge T_n]}(s)ds \right|^2 \leq T \int_0^T |b(X_s)|^2 1_{[0,\tau \wedge T_n]}(s)ds
\]

\[
\leq CT \int_0^T (1 + ||X^\tau ||_T^2)ds.
\]
Doob's inequality gives
\[
E \sup_{t \leq T} |X^c_{t \wedge T_n}|^2 \leq 4E(X^c_{t \wedge T_n})^2 \leq 4E \int_0^T c(X_s) 1_{[0,\tau \wedge T_n]} ds
\]
\[
\leq 4C \int_0^T (1 + E\|X^{T_n}\|_2^2) ds
\]
and
\[
E \sup_{t \leq T} |1_{[0,T_n]} \ast (\mu^X - \nu^X)_t|^2 \leq 4E \int_0^{T \wedge \tau \wedge T_n} \int |z|^2 \nu^X(ds,dz)
\]
\[
\leq 4C \int_0^T (1 + E\|X^{T_n}\|_2^2) ds,
\]
It follows that for \( t \leq T \) we have
\[
E\|X^{T_n}\|_2^2 \leq 4|X_0|^2 + C'(T)(1 + \int_0^t E\|X^{T_n}\|_2^2 ds),
\]
with \( C'(T) \) a constant depending on \( C \) and \( T \). Since
\[
E\|X^{T_n}\|_2^2 \leq E|X_0|^2 + n^2 + E|\Delta X_{T \wedge T_n}|^2
\]
\[
\leq E|X_0|^2 + n^2 + E \int_0^{T \wedge T_n} \int |z|^2 \mu^X(dt,dz)
\]
\[
= E|X_0|^2 + n^2 + E \int_0^{T \wedge T_n} \int |z|^2 \nu^X(dt,dz)
\]
\[
\leq E|X_0|^2 + n^2 + C E \int_0^{T \wedge T_n} (1 + |X_s|^2) ds < \infty,
\]
Grownwall’s lemma yields
\[
E\|X^{T_n}\|_2^2 \leq (4E|X_0|^2 + C(T)) e^{C(T)T}.
\]
for some constant \( C(T) \) depending on \( C \) and \( T \). Let \( n \to \infty \), then the left-hand side converges by the Monotone Convergence Theorem to \( E\|X\|_2^2 \), which is bounded by the right-hand side. This yields the first assertion of the lemma. The second assertion is an immediate consequence in view of [16, Proposition I.4.50], since
\[
\langle X^c \rangle_t = \int_0^{t \wedge \tau} c(X_s) ds \quad \text{and} \quad \langle z \ast (\mu^X - \nu^X) \rangle_t = \int_0^{t \wedge \tau} \int |z|^2 K(X_s, dz) ds
\]
have finite expectation due to the growth-condition (A.2) and the derived moment inequality for \( |X_t|^2 \). □
REFERENCES

[1] Amann, H. (1990). *Ordinary differential equations. de Gruyter Studies in Mathematics 13*. Walter de Gruyter & Co., Berlin. An introduction to nonlinear analysis, Translated from the German by Gerhard Metzen. MR1071170 (91e:34001)

[2] Carr, P. and Madan, D. B. (1999). Option valuation using the fast fourier transform. *Journal of Computational Finance* 2.

[3] Cheridito, P., Filipović, D. and Yor, M. (2005). Equivalent and absolutely continuous measure changes for jump-diffusion processes. *Ann. Appl. Probab.* 15 1713–1732. MR2152242 (2006c:60099)

[4] Cuchiero, C., Filipović, D., Mayerhofer, E. and Teichmann, J. (2009). Affine processes on positive semidefinite matrices. *Ann. Appl. Probab.* To appear.

[5] Da Fonseca, J., Grasselli, M. and Tebaldi, C. (2008). A multifactor volatility Heston model. *Quant. Finance* 8 591–604. MR2457710 (2009j:91090)

[6] Dai, Q. and Singleton, K. (2000). Specification analysis of affine term structure models. *Journal of Finance* 55 1943–1978.

[7] Dieudonné, J. (1969). *Foundations of modern analysis*. Academic Press, New York. Enlarged and corrected printing, Pure and Applied Mathematics, Vol. 10-I. MR0349288 (50 #1782)

[8] Duffie, D., Filipović, D. and Schachermayer, W. (2003). Affine processes and applications in finance. *Ann. Appl. Probab.* 13 984–1053. MR1994043 (2004g:60107)

[9] Duffie, D. and Kan, R. (1996). A yield-factor model of interest rates. *Math. Finance* 6 379–406.

[10] Duffie, D., Pan, J. and Singleton, K. (2000). Transform analysis and asset pricing for affine jump-diffusions. *Econometrica* 68 1343–1376. MR1793362 (2001m:91081)

[11] Ethier, S. N. and Kurtz, T. G. (1986). *Markov processes*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Inc., New York. Characterization and convergence. MR838085 (88a:60130)

[12] Filipović, D. and Mayerhofer, E. (2009). Affine diffusion processes: theory and applications. In *Advanced financial modelling*. Radon Ser. Comput. Appl. Math. 8 125–164. Walter de Gruyter, Berlin. MR2648460

[13] Glasserman, P. and Kim, K.-K. (2010). Moment explosions and stationary distributions in affine diffusion models. *Math. Finance* 20 1–33. MR2599675

[14] Gourieroux, C., Jasiak, J. and Sufana, R. (2009). The Wishart autoregressive process of multivariate stochastic volatility. *J. Econometrics* 150 167–181. MR2535514

[15] Grasselli, M. and Tebaldi, C. (2008). Solvable affine term structure models. *Math. Finance* 18 135–153. MR2380943 (2008k:91082)

[16] Jacod, J. and Shiryayev, A. (2003). *Limit theorems for stochastic processes*, Second ed. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences] 288. Springer-Verlag, Berlin.

[17] Kalnberg, O. (2002). *Foundations of modern probability*, Second ed. Probability and its Applications (New York). Springer–Verlag, New York.

[18] Kallsen, J. (2006). A didactic note on affine stochastic volatility models. In *From stochastic calculus to mathematical finance* 343–368. Springer, Berlin. MR2233549

[19] Kallsen, J. and Muhle-Karbe, J. (2010). Exponentially affine martingales, affine measure changes and exponential moments of affine processes. *Stochastic Process. Appl.* 120 163–181. MR2576885

[20] Keller-Ressel, M., Schachermayer, W. and Teichmann, J. (2010). Affine processes are regular. *Probab. Theory Relat. Fields*. 
THE AFFINE TRANSFORM FORMULA

[21] Lépingle, D. and Mémin, J. (1978). Sur l’intégrabilité uniforme des martingales exponentielles. Z. Wahrsch. Verw. Gebiete 42 175–203. MR489492

[22] Levendorski˘ı, S. (2004). Consistency conditions for affine term structure models. Stochastic Process. Appl. 109 225–261. MR2031769 (2004k:91111)

[23] Muhle-Karbe, J., Pfaffel, O. and Stelzer, R. (2010). Option pricing in multivariate stochastic volatility models of OU type. Preprint arXiv:math.PR/1001.3223.

[24] Spreij, P. and Veerman, E. (2010). Affine diffusions with non-canonical state space. Preprint arXiv:math.PR/1004.0429.

[25] Stroock, D. W. (1975). Diffusion processes associated with Lévy generators. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 32 209–244. MR0433614 (55 #6587)

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