Multiplicative Limit Order Markets with Transient Impact and Zero Spread

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January 9, 2015

We study a multiplicative limit order book model for an illiquid market, where price impact by large orders is multiplicative in relation to the current price, transient over time, and non-linear in volume (market) impact. Order book shapes are specified by general density functions with respect to relative price perturbations. Market impact is mean reverting with possibly non-linear resilience. We derive optimal execution strategies that maximize expected discounted proceeds for a large trader over an infinite horizon in one- and also in two-sided order book models, where buying as well as selling is admitted at zero bid-ask spread. Such markets are shown to be free of arbitrage. Market impact as well as liquidation proceeds are stable under continuous Wong-Zakai-type approximations of strategies.

Keywords: Optimal trade execution, illiquidity, singular control, finite-fuel problem, stability, no-arbitrage, limit order book
MSC2010 subject classifications: 93E20, 91G80, 49L20, 60H30

1. Introduction

We consider the optimal execution problem for a large trader in a financial market with finite liquidity, who aims to sell (or buy) a given amount of a risky asset. Since orders of the large trader have an adverse impact on the prices at which they are executed, she needs to balance the incurred liquidity costs against her preference to complete the trade early. Since seminal work by [AC01, BL98], optimal execution problems have been a

*We like to thank Peter Bank for fruitful discussions on early version of the control problem.
†Support by German Science foundation DFG via Berlin Mathematical School BMS and research training group RTG1845 StoA is gratefully acknowledged.
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subject of extensive research. We mention [Alm03, OW13, ASI10, AFS10, PSS11, ASS12, LS13, BF14, HN14] and refer for a detailed discussion with further references to the recent survey by [GS13] and the introduction in [Løk14].

In this paper we develop a multiplicative limit order book model. It is close in spirit to the additive order book models of [OW13, AFS10, PSS11, Løk14], the key difference being that the price impact of orders is multiplicative instead of additive with respect to the (unperturbed) risky asset price. In absence of large trader activity, the risky asset price follows some unperturbed price evolution $S = (S_t) > 0$ of Black-Scholes type, i.e. geometric Brownian motion, as proposed by Paul Samuelson for non-negative asset prices. Because of illiquidity effects, the large trader’s actions cause the risky asset price to evolve as $S_t = S_t f(Y_t)$, $t \geq 0$, for a process $Y$ describing the level of market impact and defined by a mean-reverting stochastic differential equation $dY_t = -h(Y_t) dt + d\theta_t$ driven by the trading strategy $(\theta_t)$ of the large trader, with $\theta_t$ denoting the number of risky assets she holds at time $t$. Subject to suitable properties for the functions $f, h$ (see Assumption 3.2), the sales (buys) of the large trader then depress (increase) the level of market impact $Y_t$ and thereby the actual price $S_t$ in a transient way. Taking $f$ to be strictly positive, multiplicative price impact ensures that risky asset prices $S_t$ are strictly positive, like in the continuous-time variant [GS13, Sect. 3.2] of the Bertsimas-Lo model [BL98]. By contrast, negative prices can occur in additive impact models. We admit for general non-linear price impact functions $f$, corresponding to general density shapes of the multiplicative limit order book (see Sect. 2.1), where such shapes are specified with respect to relative price perturbations $S/S$ instead of absolute ones. Depth of the order book may be finite or infinite. The market impact process $Y$ can be interpreted as a volume effect process, as in [AFS10, PSS11], that reflects the volume displacement in the limit order book due to the large trader’s activity. It is tending back towards its neutral level zero over time thanks to the (finite) resilience of the market. We admit for general mean-reverting specifications of the resilience function $h$, as in [PSS11]; in particular, the resilience rate $h(Y_t)$ is not required to be linear, so resilience of the market impact $Y_t$ does not need to be exponential. The interesting recent article [GZ] also considers an optimal execution problem in a model with multiplicative instead of additive price impact. Important differences are that [GZ] considers permanent price impact, non-zero bid-ask spread (proportional transaction costs) and a specific exponential parametrization of price impact from block trades, whereas we consider transient price (or volume) impact, zero spread and general impact functions $f$.

Our exposition concentrates on the optimal liquidation problem, how to sell a given position of risky assets optimally in an illiquid market. The problem of how to optimally acquire an asset position can be treated analogously, see Remark 4.6. We formulate this problem in continuous time as a singular control problem of finite fuel type, cf. [KS86]. The strategy of the large investor is determined by the initial position and by the cumulative dynamic amount of risky assets that she sells. She could sell continuously at some rate but block sales are admitted as well. We consider the optimization objective to maximize the expected discounted proceeds over an infinite time horizon for which we are able to obtain explicit analytic solutions. The finite horizon problem within our liquidity model would require a further dimension for time in the dynamical programming
approach, which is left to future research. For the open-horizon problem to be well-posed, the rate of discounting has to be large enough such that discounted (unperturbed) risky asset prices are strict supermartingales. We obtain explicit solutions for two variants of the optimal liquidation problem, in a one-sided and in a two-sided order book setting. In the first variant (I) of the problem, solved in Section 4, the large investor is only admitted to sell but not to buy, whereas in the second variant (II), intermediate buying is admitted even though the investor ultimately wants to liquidate her position. Variant I is of interest, if for instance a bank selling a large position on behalf of a client is required by regulation to execute only sell orders, while the second variant may be appropriate otherwise. Moreover, the latter is essential to investigate whether the multiplicative limit order book model is free of arbitrage for large traders, or whether it admits for profitable round trips or transaction triggered price manipulations, as investigated e.g. in [AS10, ASS12] for additive models.

It turns out that optimal liquidation strategies are determined from the solution to a free boundary problem. They could involve an initial block sale or (in variant I) an initial waiting period respectively (in variant II) an initial block buy. Afterwards, it is optimal to sell continuously at rates \( \frac{dY_t}{dt} \) as to keep the level of market impact \( Y_t \) on a non-constant free boundary, separating the sell region from the wait region (in variant I), respectively from the buy regions (in variant II), cf. Fig. 2. Notably, the involved free boundary problem has the same solution for both variants. The time to complete liquidation is finite, varies continuously with the chosen level of discounting, i.e. the investor’s impatience, and tends to zero for increasing impatience in an order book of infinite depth with suitable parametrizations, cf. Example 4.5 and Fig. 3b. For problem variant II, it turns out that an initial buy order is optimal if the initial market impact \( Y_0 \) and thereby the asset price \( S_0 \) is at a very depressed level and thus \( S \) has strong drift upwards, whereas the optimal strategy does not involve intermediate buy orders if \( Y_0 \) is not too much below the neutral state 0. For the second variant (II) we consider a model with vanishing bid-ask spread, i.e. the best bid and the best ask price are taken to be identical. This can be seen as an idealization of the predominant one-tick-spread that have been observed in [CDL13] for common assets that are relatively liquid. The volume effect process \( Y \) can still be interpreted as the volume displacement within the multiplicative limit order book caused by the large trader. But large orders do not cause a gap in the order book, i.e. bid-ask spread, that decays only slowly over time (at rates). Instead, the order book is essentially instantly re-filled, for instance by market makers or high-frequency traders; afterwards the price \( S \) reverts slowly over time towards its unperturbed fundamental value \( \bar{S} \) by the finite resilience of the market. We note that this is different to a possible modeling of separate bid and ask prices in limit order book models, as mentioned in [AFS10, PSS11, Løk14]. Yet, results for the model with zero spread have implications for extensions with non-vanishing spread, see Remarks 5.2 and 7.3. Despite zero bid-ask spread, the model does not offer arbitrage opportunities to the large trader within a class of bounded semimartingale strategies.

As a further contribution, in addition to solving the optimal execution problems and showing viability in terms of no arbitrage, we prove that the liquidation proceeds as well as the market impact \( Y \) depend in a stable way on the liquidation strategies. This
builds on connections to the Marcus stochastic integral and an extension of results in [KPP95]. Indeed, if a general monotone liquidation strategy is approximated continuously by Wong-Zakai-type strategies, then the respective liquidation proceeds, being increasing processes that correspond to measures on the time axis, converge in the Prokhorov topology in probability. In particular, the proceeds of a block sale are approximated by selling at a high rate \( n \) over a short time period of length \( 1/n \).

The paper is organized as follows. In Section 2 we introduce the model and explain how the price impact function \( f \) and the impact level \( Y \) are related to a multiplicative order book setting. The solution to the optimal liquidation problem for the variant (I) where only selling is admitted is provided in Sections 3, 4 while Section 5 solves the variant (II) where intermediate buying is admitted. In Section 6 we show stability of liquidation proceeds with respect to continuous Wong-Zakai-type approximations of liquidation strategies. Section 7 shows that the multiplicative limit order book model without bid-ask spread does not permit arbitrage by bounded semimartingale strategies.

2. The model

We consider a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) with a one-dimensional \((\mathcal{F}_t)\)-Brownian motion \( W \). The filtration \((\mathcal{F}_t)_{t \geq 0}\) is assumed to satisfy the usual conditions of right-continuity and completeness, and that \( \mathcal{F}_0 \) is the trivial \( \sigma \)-field. Let also \( \mathcal{F}_{0-} \) denote the trivial \( \sigma \)-field. We consider a market with a single risky asset. Without activity of large traders, the unaffected discounted price process of the risky asset would evolve according to the stochastic differential equation

\[
    dS_t = S_t(\mu \, dt + \sigma \, dW_t), \quad S_0 > 0,
\]

with \( \mu \in \mathbb{R}, \sigma \in (0, \infty) \). The (discounted) price of the riskless asset is constant at 1.

To model the impact that trading strategies by a single large trader have on the risky asset price, let us denote by \((\theta_t)_{t \geq 0}\) the risky asset holdings of the large trader throughout time. We write this process in the form

\[
    \theta_t = \theta_{0-} - A_t,
\]

where \( \theta_{0-} \) is the number of shares the large trader holds initially and \((A_t)_{t \geq 0}\) is a predictable process with càdlàg paths and \( A_{0-} = 0 \). At first we do require \( A \) also to be increasing, but this will be generalized later in Sections 5 and 7 to finite variation strategies and bounded semimartingale strategies, respectively. We will analytically solve an optimal trade execution problem about how the large trader can optimally sell a given position of risky assets. To this end, \( A_t \) represents the cumulative number of assets sold up to time \( t \). The process \( A \) is the control strategy of the large investor who executes \( dA_t \) market orders in the limit order book at time \( t \).

The large trader is faced with illiquidity costs because her trading has an adverse impact at the prices at which her orders are executed as follows. A process \( Y \), that
we call market impact process, captures the impact on the price of the risky asset from implementing a strategy $A$, and is defined as the unique càdlàg adapted solution to
\begin{equation}
    dY_t = -h(Y_t) \, dt + d\theta_t \tag{2.3}
\end{equation}
for some initial condition $Y_{0-} \in \mathbb{R}$. For now, we only assume that $h : \mathbb{R} \to \mathbb{R}$ is a strictly increasing continuous function with $h(0) = 0$, but we will make suitable assumptions later on (see Assumption 3.2). Thus, the market impact recovers towards 0 whenever the large trader is not active. The function $h$ gives the speed of resilience at any level of $Y_t$ and we will refer to it as the resilience function. For example, when $h(y) = \beta y$ for some constant $\beta > 0$, the market recovers at exponential rate (as in [OW13] and [Løk14]).

The price $S$ at which trading occurs is affected by the actions of the large trader through the market impact process $Y$ and is modeled by
\begin{equation}
    S_t := f(Y_t) \tilde{S}_t, \tag{2.4}
\end{equation}
where the function $f$ is of the form
\begin{equation}
    f(y) = \exp \left( \int_0^y \lambda(x) \, dx \right), \quad y \in \mathbb{R}, \tag{2.5}
\end{equation}
for some locally integrable function $\lambda : \mathbb{R} \to (0, \infty)$. For continuous trading strategies $\theta$, the process $(S_t)_{t \geq 0}$ can be seen as the evolution of prices at which strategy $\theta$ is executed dynamically. That means, if the large trader’s strategy is given by a continuous process $A^c$, then her proceeds from trading on $[0, T]$ will be $\int_0^T S_u \, dA^c_u$. To allow for discontinuous trading, we take the proceeds from a block market sell order of size $\Delta A_t$, which is to be executed immediately at time $t$, to be given by
\begin{equation}
    \tilde{S}_t \int_0^{\Delta A_t} f(Y_t - x) \, dx. \tag{2.6}
\end{equation}
The expression in (2.6) is sensible in the context of a limit order book interpretation, see Section 2.1. Moreover, Section 6 justifies the definition (2.6) by stability considerations.

### 2.1. Limit order book with multiplicative market impact

Let $s = \rho \tilde{S}_t$ be some price near the current unaffected price $\tilde{S}_t$ and let $q(\rho) \, d\rho$ denote the infinitesimal number of (bid or ask) offers at that price $s$, i.e. at the relative price perturbation $\rho$. In terms of infinitesimal price changes $d\rho$, this leads to a measure with cumulative distribution function $Q(\rho) := \int_{\rho}^\infty q(x) \, dx$, $\rho \in (0, \infty)$. For Borel measurable $M \subset (0, \infty)$, the total number of offers for prices in $M \cdot \tilde{S}_t = \{ \rho \tilde{S}_t \mid \rho \in M \}$ is $\int_M q(x) \, dx$.

Now, selling $\Delta A_t$ shares at time $t$ shifts the price from $\rho \tilde{S}_t$ to $\rho \tilde{S}_t$, so that the volume change is $Q(\rho \tilde{S}_t) - Q(\rho_t) = \Delta A_t$. The proceeds of this sale are $\tilde{S}_t \int_{\rho_t}^{\rho \tilde{S}_t} \rho \, dQ(\rho)$. By change of variables, setting $Y_t := Q(\rho_t)$ and $f := Q^{-1}$, these proceeds can be expressed as in equation (2.6). In this sense, the process $Y$ from (2.3) can be understood as the volume effect process as in [PSS11] Section 2], illustrated in Fig. 1.
Example 2.1. Let the (one- or two-sided) limit order book density be given by $q(x) = c/x^r$ on $x \in (0, \infty)$ for constants $c, r > 0$. Parameters $c$ and $r$ determine the market depth. If $r < 1$, a trader can sell only finitely many but buy infinitely many assets at any point in time. In contrast, for $r > 1$ one could sell infinitely many but buy only finitely many assets. The case $r = 1$ describes infinite market depth in both directions. The antiderivative $Q$ and its inverse $f$ are determined for $x > 0$ and $(r - 1)y \neq c$ as

$$Q(x) = \begin{cases} c \log x, & \text{for } r = 1, \\ \frac{c}{1-r}(x^{1-r} - 1), & \text{otherwise}, \end{cases} \quad f(y) = \begin{cases} e^{y/c}, & \text{for } r = 1, \\ \left(1 + \frac{1-r}{c}y\right)^{1/(1-r)}, & \text{otherwise}. \end{cases}$$

For the parameter function $\lambda$ this yields $\lambda(y) = f'(y)/f(y) = (c + (1 - r)y)^{-1}$.

3. Optimal liquidation problem I: Without intermediate buying

This section describes the optimal trade execution problem central for our paper. The large investor is facing the task to execute a large trade of $\theta_0$ risky assets but has the possibility to split it into smaller orders to improve according to some performance criterion. Up to Section 5 we will admit only control strategies that do not allow for intermediate buying. The reason is twofold: first, banks acting on behalf of clients may be obliged by regulations to do one-sided trades (no intermediate buy orders); moreover, our analysis for this more restricted variant of control policies will be shown later to carry over to an alternative variant with a wider set of controls (being of finite variation, admitting also intermediate buy orders), see Section 5.

For an initial position of $\theta_0$ shares, let the set of admissible trading strategies for the
large trader be given by
\[ A_{\text{mon}}(\theta_{0-}) := \{ A \mid A \text{ is monotone increasing, càdlàg, previsible,} \]
with \( 0 =: A_{0-} \leq A_t \leq \theta_{0-} \}. \tag{3.1} \]

Here, the quantity \( A_t \) represents the number of shares sold up to time \( t \). Any admissible strategy \( A \in A_{\text{mon}}(\theta_{0-}) \) decomposes into a continuous and a discontinuous part
\[ A_t = A_t^c + \sum_{0 \leq s \leq t} \Delta A_s, \tag{3.2} \]
where \( A_t^c \) is continuous (and increasing) and \( \Delta A_s := A_s - A_{s-} \geq 0 \). Aiming for an explicit analytic solution, we consider trading on the infinite time horizon \([0, \infty)\) with discounting. The \( \gamma \)-discounted proceeds from implementing an admissible strategy \( A \) on the time interval \([t, T]\) for \( 0 \leq t \leq T \) take the form
\[ L_{t,T}(y; A) := \int_t^T e^{-\gamma s} f(Y_s) S_s \, dA_s^c + \sum_{t \leq s \leq T} \frac{e^{-\gamma s} S_s}{\Delta A_s \neq 0} \int_0^{\Delta A_s} f(Y_{s-} - x) \, dx, \tag{3.3} \]

where \( y = Y_{0-} \) indicates the initial state of the impact process \( Y \). Clearly, \( Y_{0-} \) and \( A \) determine \( Y \) from (2.3).

Remark 3.1. The (possibly) infinite sum in (3.3) has finite expectation. Indeed, for any \( A \in A_{\text{mon}}(\theta_{0-}) \) we have that \( \sup_{s \leq T} |Y_s| \leq C_1 \) for some constant \( C_1 > 0 \). Hence, the mean value theorem and the continuity of \( f \) give that
\[ 0 \leq \int_0^{\Delta A_s} f(Y_{s-} - x) \, dx \leq \Delta A_s \cdot \sup_{x \in (0, \Delta A_s)} f(Y_{s-} - x) \leq C_2 \Delta A_s, \]
for a constant \( C_2 > 0 \). Now, since \( E[\sup_{u \in [t,T]} S_u] < \infty \) and \( \sum_{s \in [t,T]} \Delta A_s \leq \theta_{0-} \) we conclude that the sum is bounded from above by \( C_2 \theta_{0-} E[\sup_{u \in [t,T]} S_u] < \infty \) in expectation.

The goal of the trader is to maximize expected proceeds by implementing an admissible strategy. Note that \( L_{0,T}(y; A) \) is increasing in \( T \), so \( L_{0,\infty}(y; A) := \lim_{T \to \infty} L_{0,T}(y; A) \) always exists. We consider the optimal liquidation problem
\[ \max_{A \in A_{\text{mon}}(\theta_{0-})} J(y; A) \tag{3.4} \]
for the gain functional \( J(y; A) := E[L_{0,\infty}(y; A)] \) being the expected proceeds from implementing the strategy \( A \) on \([0, \infty)\) with the value function being
\[ v(y, \theta) := \sup_{A \in A_{\text{mon}}(\theta)} J(y; A). \tag{3.5} \]
Since two parameters affect the time-value of assets in our model, one being the discounting rate \( \gamma \) which penalizes keeping shares, and the other being the drift rate \( \mu \) of the unaffected price process \( S \), we will need \( \delta := \gamma - \mu \). To solve the optimal liquidation problem explicitly, the following assumption will be assumed to hold throughout Sections 3 to 5.
We then have \( \rho_1 + \psi \). Moreover, if there exists Assumption 3.2.

Remark 3.3 (On the interpretation of \((h\lambda)'>0\)). Let the large trader be inactive in some time interval \((t - \varepsilon, t + \varepsilon)\), i.e., \( \theta \) be constant there. During that period, we have \( dY_t = -h(Y_t) dt \) and, by Section 2.1, it follows \( dY_t = q(\rho_t) d\rho_t \). Now, using \( \lambda(Y_t) = (Q^{-1})(Q(\rho_t)) = q(\rho_t) \rho_t^{-1} \), we find \( (h\lambda)(Y_t) = -(\log \rho)'(t) \). Now let \( Y_t < 0 \), i.e. \( \rho_t < 1 \). There \( Y_t \) increases since \( h(Y_t) < 0 \), so \( (\log \rho)'(t) < 0 \). This means, the multiplicative price impact \( \rho_t \) is logarithmically strict concave and increasing when \( \rho_t < 1 \). Analogously, for \( \rho_t > 1 \), we find that \( \rho_t \) is logarithmically strict convex and decreasing.

This is compatible with exponential decay of price impact in additive models, cf. [GSS11 OW13]. To compare multiplicative and additive price impact models, fix \( S_t \equiv 1 \). We then have \( \rho_t = f(Y_t) = 1 + \psi(Y_t) \). If \( \psi(Y_t) = Ce^{-ct} \) for \( c > 0 \) and \( C \in \mathbb{R} \) such that \( 1 + \psi(Y_t) > 0 \), our \( \rho_t \) is logarithmically strict convex for \( \psi(Y_t) > 0 \) and logarithmically strict concave for \( \psi(Y_t) < 0 \).

Remark 3.4. [PSS11] consider a similar optimal execution problem, with an additive price impact \( S_t = \overline{S}_t + \psi(Y_t) \) with volume effect process \( Y_t \) as in (2.3). Because they study execution on a fixed finite time horizon, they have no need for discounting. The execution costs, which they seek to minimize in expectation, are equal to the negative liquidation proceeds \(-L_{0,T}\) in our model (for \( \gamma, \mu = 0 \)) with fixed \( Y_{0-} := 0 \).

We will use the martingale optimality principle to solve the optimization problem (3.4) for each possible initial state of the impact process \( Y_{0-} = y \in \mathbb{R} \). In contrast, in the related additive model in [PSS11] the optimal buying strategy for finite time horizon and impact process starting at zero was characterized using convexity arguments. Cf. also [BPT14] for an application of convexity arguments in optimal execution.

Proposition 3.5. Let \( V : \mathbb{R} \times [0, \infty) \to [0, \infty) \) be a continuous function such that \( G_t(y; A) := L_{0,t}(y; A) + e^{-rt} \overline{S}_t \cdot V(Y_t, \theta_t) \) with \( Y_{0-} = y \) is a supermartingale for each \( A \in A_{\text{mon}}(\theta_{0-}) \) and additionally \( G_0(y; A) \leq G_{0-}(y; A) := \overline{S}_0 \cdot V(Y_{0-}, \theta_{0-}) \). Then

\[
\overline{S}_0 \cdot V(y, \theta) \geq v(y, \theta).
\]

Moreover, if there exists \( A^* \in A_{\text{mon}}(\theta_{0-}) \) such that \( G(y; A^*) \) is a martingale and it holds \( G_0(y; A^*) = G_{0-}(y; A^*) \), then \( \overline{S}_0 \cdot V(y, \theta) = v(y, \theta) \) and \( v(y, \theta) = J(y; A^*) \).

Remark 3.6. The processes \( Y \) and \( \theta \) are determined by \( A \), \( y \) and \( \theta_{0-} \). The additional condition on \( G_0 \) and \( G_{0-} \) can be regarded as extending the (super-)martingale property from time intervals \([0, T]\) to time “0−”. 

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Proof. Note that \( E[G_0(y; A)] = G_0(y; A) = S_0 \cdot V(y, \theta_0) \) and
\[
E[G_t(y; A)] = E[L_{0,t}(y; A)] + E[e^{-\gamma t} \overline{S}_t \cdot V(Y^A_t, \theta_t^A)]
\]
for each \( t \geq 0 \). Also, \( V(Y^A_t, \theta_t^A) \) is bounded uniformly on \( t \geq 0 \) and \( A \in \mathcal{A}_{mon}(\theta_0) \) by a finite constant \( C > 0 \), because \( V \) is assumed to be continuous (and hence bounded on compacts) and the state process \( (Y^A_t, \theta_t^A) \) takes values in the interval
\[
[\min(-y - \theta_0, y - \theta_0), \max(-y + \theta_0, y + \theta_0)] \times [0, \theta_0].
\]
Hence, \( E[e^{-\gamma t} \overline{S}_t \cdot V(Y^A_t, \theta_t^A)] \leq C e^{-\delta t} E[\overline{S}_t] = C e^{-\delta t} S_0 \) tends to 0 for \( t \to \infty \), since \( \delta > 0 \). Since \( E[L_{0,t}(y; A)] \to J(y; A) \) as \( t \to \infty \) by means of monotone convergence theorem, we conclude that \( S_0 \cdot V(y, \theta_0) \geq G_0(y; A) \geq E[G_t(y; A)] \to J(y; A) \). This implies the first part of the claim. The second part follows analogously.

In order to make use of Proposition 3.5, one applies Itô’s formula to \( G \), assuming that \( V \) is smooth enough,
\[
dG_t = e^{-\gamma t} \overline{S}_t \left( \sigma V(Y^A_t, \theta_t^A) dW_t 
+ (-\delta V - hV_y)(Y^A_t, \theta_t^A) \, dt 
+ (f - V_y - V_\theta)(Y^A_t, \theta_t^A) \, dA_t 
+ \int_0^{\Delta A_t} (f - V_y - V_\theta)(Y^A_t - x, \theta_t^A - x) \, dx \right) 
\]
with the abbreviating conventions \((-\delta V - hV_y)(a, b) := -\delta V(a, b) - h(a)V_y(a, b)\) and \((f - V_y - V_\theta)(a, b) := f(a) - V_y(a, b) - V_\theta(a, b)\). The classical martingale optimality principle now provides equations for regions where the optimal strategy should sell or wait, in that the \( dA \)-integrands need to be zero when there is selling and the \( dt \)-integrand must vanish when only time passes (waiting).

The task is now to find a \( C^{1,1} \) function \( V : \mathbb{R} \times [0, \infty) \to \mathbb{R} \) and a strictly decreasing \( C^2 \) function \( y(\cdot) : [0, \infty) \to \mathbb{R} \), such that
\[
\begin{align*}
-\delta V - h(y)V_y &= 0 & \text{in } \overline{W} \\
-\delta V - h(y)V_y &< 0 & \text{in } S \\
V_y + V_\theta &= f(y) & \text{in } S \\
V_y + V_\theta &> f(y) & \text{in } W \\
V(y, 0) &= 0 & \forall y \in \mathbb{R}
\end{align*}
\]
for wait region \( W \) and sell region \( S \) (cf. Fig. 2) defined as
\[
\begin{align*}
W &:= \{(y, \theta) \in \mathbb{R} \times [0, \infty) \mid y < y(\theta)\}, \\
S &:= \{(y, \theta) \in \mathbb{R} \times [0, \infty) \mid y > y(\theta)\}.
\end{align*}
\]
The optimal trade execution studied here is an example of finite-fuel stochastic control problems, which often lead to free boundary problems similar to the one derived above. See [KSS86] for an explicit solution of the so-called finite-fuel monotone follower problem, and [JJZ08] for further examples and an extensive list of references. In the next section, we present two ways to find the boundary \( y(\theta) \), then construct the value function and finally prove optimality of the constructed strategy.

**Remark 3.7.** (On the notation) We have three a priori independent dimensions at hand – time \( t \), the investor’s holdings \( \theta \) and her market impact \( y \). For the sake of intuition, we will write \( y(\theta) \) or \( y(t) \) for the \( y \)-coordinate as a function of holdings or of time along the boundary between \( S \) and \( W \), instead of introducing various function symbols for the relation between these coordinates. Accordingly, the inverse function of \( y(\theta) \) is \( \theta(y) \).

The advantage of this notation is that we can identify the meaning of individual terms at a glance, without having to look up the meaning of further symbols. Of course, these are very different functions, which is to be kept in mind, e.g. when differentiating.

### 4. Free boundary problem

In the next two subsections, we construct an explicit solution to our free boundary problem of finding \( \overline{W} \cap \overline{S} = \{(y(\theta), \theta) \mid \theta \geq 0\} = \{(y, \theta(y)) \mid \ldots\} \). We will find that under Assumption 3.2 the optimal strategy is described by the free boundary with

\[
\theta'(y) = 1 + \frac{h(y)\lambda(y)}{\delta} - \frac{h(y)h''(y)}{\delta h'(y)} + \frac{h(y)(h\lambda + h' + \delta)'(y)}{\delta(h\lambda + h' + \delta)}(y)
\]  

for \( y \) in some appropriate interval \( (y_\infty, y_0) \) and \( \theta(y_0) = 0 \), see Fig. 2 for a graphical visualization. In Section 4.3 we verify that (4.1) defines a monotone boundary with a vertical asymptote, and in Section 4.4 we construct \( V \) solving the free boundary problem (3.7) – (3.11), completing the verification of the optimal liquidation problem.

#### 4.1. Smooth-pasting approach

Motivated by the approach in the literature on finite-fuel stochastic control problems, e.g. [KSS86, Section 6], we apply in this section the principle of smooth fit to derive a candidate boundary given by (4.1) dividing the sell region and the wait region. To this end, we shall at first assume that a solution \((V, y(\cdot))\) is already constructed and is sufficiently smooth along the free boundary. This smoothness along the boundary will allow us to recover the free boundary together with the function \( V \) by applying a simple algebraic trick. We verify in Section 4.4 that this approach indeed leads to a construction of a classical solution to the free boundary problem.

The first guess we make is that the wait region \( \overline{W} \) is contained in \( \{(y, \theta) : y < c\} \) for some \( c < 0 \). In this case, the solution to (3.7) in the wait region would be of the form

\[
V(y, \theta) = C(\theta) \exp \left( \int_c^y \frac{-\delta}{h(x)} \, dx \right), \quad (y, \theta) \in \overline{W},
\]  

10
where $C : [0, \infty) \to [0, \infty)$. To shorten the expressions that will follow, set
\[
\phi(y) := \exp\left(\int_c^y \frac{-\delta}{h(x)} \, dx\right), \quad y \leq c.
\]
Suppose that $C$ is continuously differentiable. Calculating the directional derivative $V_y + V_\theta$ and the expression $V_{yy} + V_{y\theta}$ in the wait region (they would exist if $C$ is differentiable), we obtain for $(y, \theta) \in \mathcal{W}$:
\[
\begin{align*}
V_y(y, \theta) + V_\theta(y, \theta) &= -\delta C(\theta)\phi(y)/h(y) + C'(\theta)\phi(y), \\
V_{yy}(y, \theta) + V_{y\theta}(y, \theta) &= -\delta C'(\theta)\phi(y)/h(y) + \delta C(\theta)\phi(y)h^{-2}(y)(\delta + h'(y)).
\end{align*}
\]  
(4.3)  
(4.4)
On the other hand, the same expressions computed in the sell-region yield (for $(y, \theta) \in \mathcal{S}$)
\[
\begin{align*}
V_y(y, \theta) + V_\theta(y, \theta) &= f(y), \\
V_{yy}(y, \theta) + V_{y\theta}(y, \theta) &= f'(y).
\end{align*}
\]
Now, suppose that $V$ is a $C^2$-function. In particular, we must have for $y = y(\theta)$:
\[
\begin{align*}
f(y) &= -\delta C(\theta)\phi(y)/h(y) + C'(\theta)\phi(y), \\
f'(y) &= -\delta C'(\theta)\phi(y)/h(y) + \delta C(\theta)\phi(y)h^{-2}(y)(\delta + h'(y)).
\end{align*}
\]  
(4.5)
Solving (4.5) as a linear system for $C(\theta)$ and $C'(\theta)$, we get for $y = y(\theta)$:
\[
\begin{align*}
C(\theta) &= f(y) \cdot \frac{1}{\phi(y)} \cdot \frac{h(y)}{\delta h'(y)} \left(\delta + h(y)\lambda(y)\right) =: M_1(y), \\
C'(\theta) &= f'(y) \cdot \frac{1}{\phi(y)} \cdot \frac{\delta + h(y)\lambda(y) + h'(y)}{h'(y)} =: M_2(y).
\end{align*}
\]  
(4.6)
Now, (4.6) reads that we should have $C(\theta(y)) = M_1(y)$ and $C'(\theta(y)) = M_2(y)$, where $\theta(\cdot)$ is the inverse function of $y(\cdot)$ (in domains of definition to be specified later). Thus, by the chain rule it follows that $M_1'(y) = C''(\theta(y)) \cdot \theta'(y)$, and therefore
\[
\theta'(y) = \frac{M_1'(y)}{M_2(y)} = \frac{(\delta + 2h\lambda)(h')^2 + (\delta^2 + 2h\lambda + h^2\lambda^2 + h^2\lambda')h' - h(\delta + h\lambda)h''}{\delta h'(\delta + h\lambda + h')} (y) \quad (4.7)
\]
whenever $\theta(\cdot)$ is defined. Note that the right-hand sides of (4.1) and (4.7) are equal.
To derive the domain of definition of $\theta(\cdot)$, we use the boundary condition (3.11) together with (4.1) and (4.6) to get that $y_0 := y(0) = \theta^{-1}(0)$ solves $\delta + h(y_0)\lambda(y_0) = 0$. The denominator in (4.7) suggests that $y_\infty$ solving $\delta + h(y_\infty)\lambda(y_\infty) + h'(y_\infty) = 0$ is a vertical asymptote of the boundary. Note that Assumption 3.2 implies that $y_\infty < y_0 < 0$ and in particular we can choose $c \in (y_0, 0)$ in the beginning of this section. The discussion so far suggests to define a candidate boundary as follows: for $y \in (y_\infty, y_0)$ set
\[
\theta(y) := -\int_{y_0}^{y} \frac{(\delta + 2h\lambda)(h')^2 + (\delta^2 + 2h\lambda + h^2\lambda^2 + h^2\lambda')h' - h(\delta + h\lambda)h''}{\delta h'(\delta + h\lambda + h')}(x) \, dx. \quad (4.8)
\]
We verify in Lemma 4.1 that (4.8) defines a decreasing boundary with $\lim_{y \downarrow y_\infty} \theta(y) = +\infty$ and $\theta(y_0) = 0$. Having a candidate boundary, we can construct $V$ in the wait region $\mathcal{W}$ in the form (4.2), using (4.6), and in the sell region $\mathcal{S}$ using the directional derivative (3.9). In Section 4.4 we verify that this construction gives a solution to the free boundary problem (3.7) - (3.11), and consequently to the optimal liquidation problem.
4.2. Calculus of variation approach

To describe the task of finding the optimal boundary as a classical isoperimetric problem from calculus of variations, we postulate that the optimal strategy is deterministic (so by Fubini we can assume w.l.o.g. \( S = 1 \)) and that it will liquidate all \( \theta_0 \) risky assets in finite time \( T := \inf \{ t \geq 0 \mid \theta_t = 0 \} < \infty \). To simplify the analysis we observe time \( t \) backwards as time to liquidation (TTL) \( \tau = T - t \) and search for a strategy \( A_t = \theta_0 - \bar{\theta}(\tau) \) along the boundary \( (\bar{g}(\tau), \bar{\theta}(\tau)) \in \mathcal{W} \cap \mathcal{S} \), assuming \( C^1 \)-smoothness of that boundary. By the dynamics (2.3) of \( Y_t \), it follows

\[
\bar{\theta}'(\tau) = y'(\tau) - h(y(\tau)) \quad \text{(4.9)}
\]

for the function \( y(\tau) = y(t) = Y_t \). So the optimization problem (3.4) translates to finding \( y : [0, \infty) \to \mathbb{R} \) which maximizes

\[
J(y) := \int_0^T f(y(\tau)) e^{-\delta(T-\tau)}(y'(\tau) - h(y(\tau))) \, d\tau =: \int_0^T \mathcal{F}(\tau, \bar{g}(\tau), y'(\tau)) \, d\tau \quad \text{(4.10)}
\]

with subsidiary condition

\[
\theta = K(y) := \int_0^T (y'(\tau) - h(y(\tau))) \, d\tau =: \int_0^T \mathcal{G}(\tau, \bar{g}(\tau), y'(\tau)) \, d\tau \quad \text{(4.11)}
\]

for fixed position \( \theta := \theta_0 \). The Euler equation of this isoperimetric problem is

\[
\bar{F}_{y'} - \frac{d}{d\tau} \bar{F}_y + \bar{\lambda} \left( \bar{G}_y - \frac{d}{dt} \bar{G}_{y'} \right) = 0 \quad \text{(4.12)}
\]

with Lagrange multiplier \( \bar{\lambda} = \bar{\lambda}(T) \). However, terminal time \( T = T(\theta) \), final state \( \bar{g}(0) \) and initial state \( \bar{g}(T) \) are unknown. A priori, the final state \( \bar{g}(0) \) is free, which leads to the natural boundary condition

\[
\bar{F}_{y'} + \bar{\lambda} \bar{G}_{y'} \bigg|_{\tau=0} = 0. \quad \text{(4.13)}
\]

With \( y_0 := \bar{g}(0) \) and \( y := \bar{g}(\tau) \), equation (4.13) simplifies to \( \bar{\lambda} = -f(y_0) e^{-\delta T} \), and

\[
0 = f(y_0) h'(y) - f(y) e^{\delta \tau} \left( h(y) \lambda(y) + h'(y) + \delta \right) \quad \text{(4.14)}
\]

follows from equation (4.12). Solutions \( \bar{g}_1 \) and \( \bar{g}_2 \) for time horizons (TTL) \( T_1 < T_2 \) should coincide for \( \tau \in [0, T_1] \), because the optimal strategy depends only on the current position \( \theta = \bar{g}(T) \) and current market impact \( \bar{g}(T) \), but not on the past (Markov model). In particular, \( y_0 \) is independent of \( T \). So for \( \tau = 0 \) we get \( h(y_0) \lambda(y_0) + \delta = 0 \). Existence

\[\text{[1] See [GF00] chapter 2, section 12.1.}\]
and uniqueness of such \( y_0 \) is guaranteed by Assumption 3.2. It must hold that \( y_0 < 0 \), because \( \lambda > 0 \) and \( h(y) < 0 \iff y < 0 \). Rearranging (4.14) gives an explicit description for the time to liquidation along the boundary:

\[
e^{-\delta \tau} = \frac{f(y) h(y) \lambda(y) + h'(y) + \delta}{f(y_0) h'(y_0)}.
\] (4.15)

This defines \( \tau \mapsto \bar{g}(\tau) \) implicitly. Together with \( \bar{\theta}(\tau) = \int_0^\tau \left( \bar{g}'(\tau) - h(\bar{g}(\tau)) \right) d\tau \), this function describes the free boundary as a parametric curve. Differentiating equation (4.14) with respect to \( \tau \), we get

\[0 = f(y_0) h''(\bar{g}(\tau)) \bar{g}'(\tau) - f'(y_0) h'(\bar{g}(\tau)) \bar{g}'(\tau) e^{\delta \tau} (h\lambda + h' + \delta)(\bar{g}(\tau)) - \delta f'(\bar{g}(\tau)) e^{\delta \tau} (h\lambda + h' + \delta)(\bar{g}(\tau)) - f(y_0) h'(\bar{g}(\tau)) e^{\delta \tau} (h\lambda + h' + \delta)(\bar{g}(\tau)) \bar{g}'(\tau).
\]

Thus, for \( y = \bar{g}(\tau) \) we obtain

\[
\bar{g}'(\tau) = \frac{\delta f(y) (h\lambda + h' + \delta)(y)}{f(y_0) h''(y_0) e^{-\delta \tau} - f'(y_0) (h\lambda + h' + \delta)(y)} = \frac{\delta f(y) h'(y) + \delta}{y} \left( h\lambda + h' + \delta \right)(y) - \delta f(y) h'(y).
\] (4.16)

if the denominator is nonzero. Also note that

\[
\bar{g}'(0) = \frac{-\delta h'(y_0)}{h'(y_0) \lambda(y_0) + (h\lambda)'(y_0)} < 0
\] (4.17)

by Assumption 3.2 as \( h' > 0 \), \( (h\lambda)' > 0 \) and \( \lambda > 0 \). Hence, there exists a maximal \( T_\infty \in (0, \infty) \) such that \( \bar{g}(\tau) < 0 \) for \( \tau \in [0, T_\infty) \), so \( \bar{g} \) is bijective there. Call \( \tau := \bar{g}^{-1}(y) \) its inverse and let \( y_\infty := \lim_{\tau \nearrow T_\infty} \bar{g}(\tau) < y_0 \). By (4.15), equation (4.16) simplifies to

\[
\bar{g}'(\tau) = \frac{\delta (h\lambda + h' + \delta)(y) h'(y)}{(h'' - h'\lambda)(y) h'(y) - (h\lambda + h' + \delta)(y) h'(y)}
\] (4.18)

for \( y = \bar{g}(\tau) \). By definition of \( T_\infty \) and \( y_\infty \), we see that \( \bar{g}'(\tau) \) is negative on \([0, T_\infty)\) and 0 at \( y = y_\infty \), hence \( h(y_\infty) \lambda(y_\infty) + h'(y_\infty) + \delta = 0 \). By Assumption 3.2 such a unique solution \( y_\infty < y_0 \) exists. An ODE for \( \theta(y) \) on \( y \in [y_\infty, y_0] \) is obtained from (4.9) via

\[
\theta'(y) = \frac{d}{dy} \bar{g}(\tau(y)) = \bar{g}'(\tau(y)) \tau'(y)
\]

\[= \left( \bar{g}'(\tau(y)) - h(y) \right) \frac{1}{\bar{g}'(\tau(y))} = 1 - \frac{h(y)}{\bar{g}'(\tau(y))}
\]

\[= 1 - \frac{h(y)}{\delta h'(y)} (h'' - \lambda h')(y) + \frac{h(y)}{\delta (h\lambda + h' + \delta)(y)}
\]

(4.19)

with \( \theta(y_0) = 0 \). We also note that (4.19) equals (4.1).
4.3. Properties of the candidate for the free boundary

In order to justify the analysis above, we need to check the presumed properties of the candidate boundary, especially bijectivity of \( \theta : (y_\infty, y_0] \to [0, \infty) \).

**Lemma 4.1.** The function \( \theta : (y_\infty, y_0] \to \mathbb{R} \) defined in (4.8) is a strictly decreasing \( C^1 \) function that maps bijectively \( (y_\infty, y_0] \) to \( [0, \infty) \) with \( \theta(y_0) = 0 \) and \( \lim_{y \downarrow y_\infty} \theta(y) = +\infty \).

**Proof.** By Assumption 3.2 we have that \( h' > 0 \) and \( y \mapsto (\delta + h\lambda + h')(y) \) is strictly increasing, giving that the denominator in (4.7) is strictly positive when \( y > y_\infty \). Thus, to verify that \( \theta \) is decreasing it suffices to check that the numerator in (4.7) is negative. For this, we write the numerator as

\[
(\delta + h\lambda)(h')^2 + (\delta + h\lambda)^2h' + hh'(h\lambda)' - h(\delta + h\lambda)h''.
\]

Note that \( h(\delta + h\lambda)h'' \geq 0 \) for \( y \leq y_0 \) because of Assumption 3.2 and \( y_0 < 0 \). Similarly, we have that \( hh'(h\lambda)' < 0 \). Hence, \( \theta'(y) < 0 \) follows by

\[
(\delta + h\lambda)(h')^2 + (\delta + h\lambda)^2h' = h'(\delta + h\lambda)(\delta + h' + h\lambda) < 0.
\]

It is clear that \( \theta \) defined in (4.8) is \( C^1 \). So it remains to verify \( \lim_{y \downarrow y_\infty} \theta(y) = +\infty \). Note that the arguments above actually show that the numerator of the integrand is bounded from above by a constant \( c < 0 \) when \( x \in [y_\infty, y_0] \). Also, since the derivative of the denominator is bounded on \( [y_\infty, y_0] \), we have by the mean value theorem

\[
0 \leq \delta(h'(\delta + h\lambda h'))(x) \leq C(x - y_\infty), \quad x \in (y_\infty, y_0],
\]

for a finite constant \( C > 0 \). Thus, we can estimate

\[
\theta(y) \geq \int_y^{y_0} \frac{-c}{C(x - y_\infty)} \, dx = \frac{-c}{C} \left( \log(y_0 - y_\infty) - \log(y - y_\infty) \right) \quad \forall y \in (y_\infty, y_0],
\]

which converges to \( +\infty \) as \( y \downarrow y_\infty \). This finishes the proof. 

---

Figure 2: The division of the state space, for \( \delta = 0.5, h(y) = y \) and \( \lambda(y) \equiv 1 \).
4.4. Construction of the value function $V$ and the optimal strategy

The smooth pasting approach directly gives the value function $V$ along the boundary as

$$V(y, \theta) = V_{\text{bdry}}(\theta) := f(y)h(y) \frac{\delta + h(y)\lambda(y)}{\delta h'(y)} \Bigg|_{y=y(\theta)}$$  \hspace{1cm} (4.20)

via equations (4.2) and (4.6). In the calculus of variations approach, we get that equation as the solution to (4.10) after inserting equation (4.15), doing a change of variables with (4.9) and applying Lemma A.2. By equation (3.7), we can extend $V$ into the wait region:

$$V(y, \theta) = V^W(y, \theta) := V_{\text{bdry}}(\theta) \exp \left( \int_{y(\theta)}^{y} \frac{-\delta}{h'(x)} \, dx \right)$$

$$= \left( \frac{fh(\delta + h\lambda)}{\delta h'} \right)(y(\theta)) \exp \left( \int_{y}^{y(\theta)} \frac{\delta}{h'(x)} \, dx \right)$$  \hspace{1cm} (4.21)

for $(y, \theta) \in \overline{W}$. Using equation (3.9) we get $V$ inside $S_1 := S \cap \{(y, \theta) \mid y < y_0 + \theta\}$ as follows. For $(y, \theta) \in S_1$ let $\Delta := \Delta(y, \theta)$ be the $||\cdot||_1$-distance of $(y, \theta)$ to the boundary in direction $(-1, -1)$, i.e.

$$\theta = \theta_1 + \Delta, \quad y = y(\theta_1) + \Delta, \quad \Delta \geq 0.$$  \hspace{1cm} (4.22)

We then have for $y(\theta) \leq y \leq y_0 + \theta$, that

$$V(y, \theta) = V^{S_1}(y, \theta) := V_{\text{bdry}}(\theta_1) + \int_{0}^{\Delta} f(y_1 + x) \, dx$$

$$= \left( \frac{fh(\delta + h\lambda)}{\delta h'} \right)(y - \Delta) + \int_{y-\Delta}^{y} f(x) \, dx.$$  \hspace{1cm} (4.23)

Similarly, with equation (3.11) we get $V$ in $S_2 := S \setminus S_1$, i.e. for $y \geq y_0 + \theta$:

$$V(y, \theta) = V^{S_2}(y, \theta) := \int_{y-\theta}^{y} f(x) \, dx.$$  \hspace{1cm} (4.25)

Since $V_{\text{bdry}}(0) = 0$, we can combine $V^{S_1}$ and $V^{S_2}$ by extending $\Delta(y, \theta) := \theta$ inside $S_2$. So $\Delta := \Delta(y, \theta)$ is the $||\cdot||_1$-distance in direction $(-1, -1)$ of the point $(y, \theta) \in S$ to $\partial S$ and

$$V(y, \theta) = V^{S}(y, \theta) := V_{\text{bdry}}(\theta - \Delta) + \int_{y-\Delta}^{y} f(x) \, dx$$  \hspace{1cm} (4.26)

for all $(y, \theta) \in S$. But note that $y(\theta - \Delta) = y - \Delta$ only holds in $\overline{S}_1$, not $S_2$. After resuming the properties of $V$ in the next lemma (proved in Appendix A.1), we can state our main result.

**Lemma 4.2.** The function $V : R \times [0, \infty) \rightarrow R$ with

$$V(y, \theta) = \begin{cases} V_{\text{bdry}}(\theta - \Delta) + \int_{y-\Delta}^{y} f(x) \, dx, & \text{for } y \geq y(\theta), \\ V_{\text{bdry}}(\theta) \cdot \exp \left( \int_{y(\theta)}^{y} \frac{-\delta}{h'(x)} \, dx \right), & \text{for } y \leq y(\theta), \end{cases}$$

as defined by equations (4.20), (4.21) and (4.26) is in $C^1(R \times [0, \infty))$ and solves the free boundary problem (3.7) - (3.11).
Theorem 4.3. Let the model parameters $h$, $\lambda$, $\delta$ satisfy Assumption 3.2 and $\theta_0 \geq 0$ be given. Define $y_{\infty} < y_0 < 0$ as the unique solutions of $h(y_{\infty}) \lambda(y_{\infty}) + h'(y_{\infty}) + \delta = 0$ and $h(y_0) \lambda(y_0) + \delta = 0$, respectively, and let

$$
\tau(y) := \frac{1}{\delta} \log \left( \frac{f(y) h(y) \lambda(y) + h'(y) + \delta}{f(y_0) h'(y)} \right),
$$

(4.27)

for $y \in (y_{\infty}, y_0]$ with inverse function $\tau \mapsto \bar{y}(\tau) : [0, \infty) \rightarrow (y_{\infty}, y_0]$. Moreover, let $\theta(y)$, $y \in (y_{\infty}, y_0]$, be the strictly decreasing solution to the differential equation

$$
\begin{cases}
\theta'(y) = 1 + \frac{h(y) \lambda(y)}{\delta} - \frac{h(y) h'(y) + h(y) (h \lambda + h' + \delta)'}{\delta(h \lambda + h' + \delta)}(y), & y \in (y_{\infty}, y_0]
\end{cases}
$$

(4.28)

and call its inverse $\theta \mapsto y(\theta)$, $\theta \geq 0$. For given $\theta_{0-}$ and $Y_{0-}$, define the sell strategy $A = A_{opt}$ with $A_{0-} := 0$ as follows

1. If $Y_{0-} \geq y_0 + \theta_{0-}$, sell all assets at once: $A_0 = \theta_{0-}$.

2. If $y(\theta_{0-}) < Y_{0-} < y_0 + \theta_{0-}$, then sell a block of size $\Delta A_0 \equiv A_0 - A_{0-} = A_0$ such that $\theta_0 \equiv \theta_{0-} - \Delta A_0 > 0$ and $Y_0 \equiv Y_{0-} - \Delta A_0 = y(\theta_0)$.

3. If $Y_{0-} < y(\theta_{0-})$, wait until time $s = \inf \{t > 0 \mid y_w(t) = y(\theta_{0-})\} < \infty$, where $y_w$ solves $y_w'(t) = -h(y_w(t))$ with initial condition $y_w(0) = Y_{0-}$. That is, set $A_t = 0$ for $0 \leq t < s$. This leads to $Y_t = y_w(t)$ for $0 \leq t < s$.

4. As soon as step 2 or 3 lead to the state $Y_s = y(\theta_s)$ for some time $s \geq 0$, sell continuously: $A_t = \theta_{0-} - \theta(\bar{y}(T - t))$, $s \leq t \leq T$ until time $T = s + \tau(y(\theta_s))$.

5. Stop as soon as all assets are sold at some time $T < \infty$: $A_t = \theta_{0-}$, $t \in [T, \infty)$.

Then the strategy $A_{opt}$ is the unique maximizer to the problem (3.4) of optimal liquidation $\max_{A \in A_{mon}(\theta_{0-})} \mathbb{E}[L_{0, \infty}(y; A)]$ for $\theta_{0-}$-assets with initial market impact being $Y_{0-} = y$.

Proof. On admissibility of $A_{opt}$: Previsibility of $A_{opt}$ is obvious by continuity of $y(\theta)$. In fact, $A_{opt}$ is deterministic because $Y_t$ is so. As noted in the proof of Lemma A.5, the function $y \mapsto (f \cdot (h \lambda + h' + \delta))'(y)$ is increasing in $(y_{\infty}, y_0]$, so $\tau(y)$ and its inverse $\bar{y}(\tau)$ are decreasing, as is $\theta(y)$ by Lemma 4.1. This implies monotonic increase of $A_{opt}$. Right continuity follows from the description of the 5 steps above. So $A_{opt} \in A_{mon}(\theta_{0-})$.

On finite time to liquidation: By $h \lambda + h' + \delta > 0$ in $(y_{\infty}, y_0]$ and equation (4.15), it takes $\tau(Y_s) < \infty$ time to liquidation if $Y_s = y(\theta_s)$, i.e. along the boundary. This time only increases by some waiting time $s > 0$ in case $Y_{0-} < y(\theta_{0-})$ (step 3). But since $h(y_0) < 0$, we have $s < \infty$.

On optimality: Note that $(Y_t, \theta_t) \in [\min\{y - \theta_{0-}, 0\}, \max\{0, y\}] \times [0, \theta_{0-}]$ for $y = Y_{0-}$ because $h(0) = 0$ and $h' > 0$. So $V(Y_t, \theta_t)$ is bounded by continuity of $V$ (Lemma 4.2 above). So the local martingale part of $G$ in equation (3.6) is a true martingale. By construction of $V$ and Lemmas 4.2 and 4.3 to A.5, $G$ is a supermartingale with $G_0 - G_{0-} = \int_0^{\infty} \mathbb{E}_0^{A_{opt}} (f - V_y - V_0) (Y_{0-} - x, \theta_{0-} - x) \, dx \leq 0$ for every strategy and a true martingale with $G_{0-} = G_0$ for $A_{opt}$. So Proposition 3.5 applies. \qed
Remark 4.4. The result in Theorem 4.3 would still hold for more general unperturbed price process of the form $S_t = e^{\mu t} M_t$ with a positive martingale $M$ such that $(M_t)_{t \in [0,T]} \in \mathcal{H}^1$ for any $T \in (0, \infty)$ in a filtration satisfying the usual conditions, as can be seen from the proof of Proposition 3.5 above. The same remark applies for Theorem 5.1 below.

Example 4.5. Recall the limit order book with density $q(x) = c/x^r$ from Section 2.1. One might be skeptical about general exponents $r \neq 1$, because $\lambda(y) = (c + (1-r)y)^{-1}$ does not satisfy Assumption 3.2 everywhere; there is a pole at $c/(r - 1)$. However, as can be seen from the proofs, $\lambda$ is only needed at possible values of $Y_t$ and an exponent $r \neq 1$ effectively restricts our state space $\mathcal{W} \cup \bar{S}$ to

$$y > \frac{c}{r - 1} \quad \text{if } 0 \leq r < 1, \quad \text{and} \quad y < \frac{c}{r - 1} \quad \text{if } r > 1.$$ 

We only have to check that $Y_{0-}$, $y_0$ and $y_{\infty}$ satisfy these conditions. Note that the special case of no initial impact, $Y_{0-} = 0$, already satisfies them. Consider linear resilience speed $h(y) = \beta y$ for some $\beta > 0$. We have $(h \lambda)' > 0$ everywhere and get

$$y_0 = -\frac{c \delta}{\beta + (1-r) \delta} \quad \text{and} \quad y_{\infty} = -\frac{c(\beta + \delta)}{\beta + (1-r)(\beta + \delta)}.$$ 

It follows that $y_{\infty}$ and $y_0$ lie in the required range and satisfy $y_{\infty} < y_0 < 0$ if

$$0 \leq r < 1 + \frac{\beta}{\beta + \delta}.$$ 

Note that for $r = 1$ and increasing impatience $\delta \to \infty$, we have $y_0 \to -\infty$, so the time to liquidation for the optimal strategy tends to and ultimately equals 0, an initial block sale $A_0 = \theta_{0-}$ being optimal for sufficiently large $\delta$. For $r \neq 1$, $\delta \to \infty$, the boundary approaches a vertical line at $y_{\infty} = y_0 = c/(r - 1) < 0$. Figure 3 shows the time needed to liquidate $\theta_{0-} = 20$ assets with initial volume impact $Y_{0-}$ for different parameters $r$, $\beta$ and $\delta$. Note that time to liquidation can be written as a function of $y/c$ without further dependence on $c$. When the market can absorb more assets (larger $r > 1$), recovers faster (larger $\delta$) or price trend is downwards (larger $\delta$), the time to liquidation decreases.

Using equation (4.27) and (4.28), or directly by equations (4.9) and (4.18), we get the rate of continuous liquidation $dA_t/dt = \bar{\theta}'(T-t)$. As shown in Fig. 4, the rate becomes asymptotically constant for increasing times to liquidation $\tau = T-t \to \infty$ and decreases for $\tau \to 0$, the respective limits being

$$\lim_{\tau \to \infty} \bar{\theta}'(\tau) = -h(y_{\infty}) = \frac{\beta c(\beta + \delta)}{\beta + (1-r)(\beta + \delta)} \quad \text{and} \quad \bar{\theta}'(0) = \frac{\delta \beta c}{2 \beta + (1-r) \delta}.$$ 

Remark 4.6. How to purchase a position minimizing the expected costs is the natural counterpart problem of the one considered so far. In this case, if we represent the admissible strategies by increasing càdlàg processes $\theta$ starting at 0 (describing the cumulative number of shares purchased in time), then the cost of implementing an admissible (purchase) strategy $\theta$ takes the form

$$\int_0^\infty e^{\eta s} f(Y_{s-}) \mathcal{E}(\sigma W)_s \, d\theta_s + \sum_{s \geq 0, \Delta \theta_s \neq 0} e^{\eta s} \mathcal{E}(\sigma W)_s \int_0^{\Delta \theta_s} f(Y_{s-} + x) \, dx,$$  \hspace{1cm} (4.29)

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Figure 3: Dependence of the time to liquidation (TTL) on $Y_0$ for $\theta_0 = 20$, linear resilience speed $h(y) = \beta y$ and order book density $q(x) = 1/x^\gamma$. A red point marks $(y(\theta_0), \tau(\theta_0))$, where continuous trading begins.

Figure 4: Rate of liquidation (after the initial block trade) against time to liquidation. The lines end when $\theta(\tau)$ reaches 100.

where we write the discounted unaffected price process as $e^{-\gamma t} \mathcal{S}_t = e^{\eta t} \mathcal{E}(\sigma W)_t$ with $\eta := \mu - \gamma = -\delta$. To have a well-posed minimization problem, we need to assume that the discounted price process increases in expectation in time, i.e. $\eta > 0$.

In this case, the value function of the optimization problem will be described by the
variational inequality
\[ \min \{ f + V_y - V_\theta , \eta V - hV_y \} = 0. \]

The approach to the optimal liquidation problem taken so far permits to construct the classical solution to this free-boundary problem explicitly. Thereby, the state space is divided into a wait region and a buy region by the free boundary, that is described by
\[ \theta'(y) = -1 + \frac{h(y)\lambda(y)}{\eta} - \frac{h(y)h''(y)}{\eta h'(y)} + \frac{h(y)(h\lambda + h' - \eta)'(y)}{\eta(h\lambda + h' - \eta)(y)}, \quad y \geq y_0, \quad (4.30) \]
with initial condition \( \theta(y_0) = 0 \), where \( y_0 \) is the unique root of \( h(y)\lambda(y) = \eta \) (similar to (4.1) from the optimal liquidation problem). It might be interesting to point out that (4.30) defines an increasing (in \( y \)) boundary that does not necessarily have an asymptote; for example when \( h(y) = \beta y \), because the expression for the boundary becomes
\[ \theta(y) = \int_{y_0}^{y} \frac{u^2\lambda'(u) + u^2\lambda^2(u) - 2(\alpha - 1)u\lambda(u) + \alpha(\alpha - 1)}{\alpha(u\lambda(u) + 1 - \alpha)} \, du, \quad y \geq y_0, \]
with \( \alpha := \eta/\beta \) and integrand being at least \((h(y)\lambda(y) - \eta)/\eta\) that is bounded away from 0 for \( y \geq y_0 + \varepsilon \) with \( \varepsilon > 0 \).

5. Optimal liquidation problem II: With intermediate buying

In this section, we solve the optimal liquidation problem when the trading strategies allow for intermediate buy orders, under Assumption 3.2. We address this problem in a zero-spread model and, although favorable for the large trader, we show that the optimal trading strategy is monotone when \( Y_0^- \) is not too small (see Remark 5.2). More precisely, the state space decomposes into a buy region and a sell region with a common boundary that turns out to be the free boundary constructed in Section 4.

Until now, we were only considering selling in the market and specified our model for such tradings, or in the sense of Section 2.1 we specified only the bid side of the LOB. At this point we extend our model to accommodate for buying. In addition to a sell strategy \( A^+ \), suppose that the large trader has a buy strategy given by the increasing càdlàg process \( A^- \) with \( A^- = 0 \). In this case, the evolution of the risky asset holdings of the large trader is described by the process \( \theta = \theta_{0-} - A^+ + A^- \). We assume that the price impact process \( Y = Y^\theta \) is given by (2.3) with \( \theta = \theta_{0-} - A^+ + A^- \). In addition, the best bid price and the best ask price evolve according to the same process \( S = f(Y^\theta)S \), i.e. the bid-ask spread is zero. Similarly to the case of market sell orders, the proceeds from executing a market buy order at time \( t \) of size \( \Delta A^-_t > 0 \) are given by (2.6) with \( \Delta A_t = -\Delta A^-_t \); with negative proceeds we denote the cost the large trader pays in order to execute the trade. Thus, the \( \gamma \)-discounted proceeds from implementing a (finite variation)
trading strategy \((A^+, A^-)\) on \([0, T]\) are

\[
L_T := L_{0,T} = -\int_0^T e^{-\gamma t} f(Y_t) S_t \, d\theta_t - \sum_{\Delta \theta_t \neq 0} e^{-\gamma t} S_t \int_0^{\Delta \theta_t} f(Y_{t-} + x) \, dx.
\]  

Note that for strategies with \(\theta\) having paths of finite total variation, i.e. \(\theta^+_{\infty} + \theta^-_{\infty} < \infty\), as it will hold true for elements of \(A_{fv}(\theta_{0-})\) below, the infinite sum in (5.1) converges absolutely (see Remark 3.1), and so is well-defined.

For the optimization problem, the set of admissible trading strategies is

\[
A_{fv}(\theta_{0-}) := \{ A = A^+ - A^- \mid A^\pm \text{ are increasing, càdlàg, previsible, of bounded total variation on } [0, \infty), \text{ with } A^\pm_{0-} = 0 \text{ and } A_t \leq \theta_{0-} \text{ for } t \geq 0 \},
\]  

where \(A = A^+ - A^-\) denotes the minimal decomposition for a process \(A\) of finite (here even bounded) variation; \(A_t^+\) describes the cumulative number of assets sold while \(A_t^-\) describes the cumulative amount of assets bought up to time \(t\). The condition that an admissible strategy is always less than \(\theta_{0-}\) means that no short-selling is allowed.

For an admissible strategy \(A \in A_{fv}(\theta_{0-})\), \(L_{0,T}(y; A)\) as defined in (3.3) but for general finite variation process \(A\) describes the proceeds from implementing \(A\) on time interval \([t, T]\). It can be checked as in Section 3 that these proceeds have finite expectation and \(L_{0,\infty}(y; A) := \lim_{T \to \infty} L_{0,T}(y; A)\) is well-defined. Moreover, \(L_{0,\infty}(y; A)\) has finite expectation because \(A^+ + A^- \leq C\) for a constant \(C > 0\) and \(L_{0,\infty}(y; A^+), L_{0,\infty}(y; A^-)\) are integrable (essentially by the analysis in the optimal liquidation problem without intermediate buy orders). Thus \(J(y; A) := E[L_{0,\infty}(y; A)]\) is well-defined. We can now formulate the optimal liquidation problem with possible intermediate buy orders as

\[
\max_{A \in A_{fv}(\theta_{0-})} J(y; A).
\]

An analysis as in Section 3 (cf. Proposition 3.5 and equation (3.6)) shows that in this case it suffices to find a classical solution to the following problem

\[
\begin{align*}
V_y + V_\delta = f & \quad \text{on } \mathbb{R} \times [0, \infty) \\
-\delta V - h(y)V_y & \leq 0 \quad \text{on } \mathbb{R} \times [0, \infty)
\end{align*}
\]  

with appropriate boundary conditions. The boundary conditions should make sure that a classical solution exists and also that the (super-)martingale in Proposition 3.5 extends to such on the time horizons \([0-, T]\).

The optimal liquidation strategy in this case can be described by a sell region and a buy region, divided by a common boundary. The sell region turns out to be the same as for the problem with no intermediate buy orders in Section 3, i.e. the region \(S\), while the wait region \(\mathcal{W}\) there now becomes a buy region:

\[
\mathcal{B} := \mathbb{R} \times [0, \infty) \setminus \overline{S}.
\]
In view of Section 4.4, we extend the definition of $\Delta(y, \theta)$ to $B$. For $(y, \theta) \in \mathbb{R} \times [0, \infty)$, let $\Delta(y, \theta)$ be the signed $\|\cdot\|_1$ distance in direction $(-1, -1)$ of the point $(y, \theta)$ to the boundary $\partial S = \{(y, \theta) \mid \theta \geq 0\} \cup \{(y, 0) \mid y \geq y_0\}$, i.e. $(y - \Delta, \theta - \Delta) \in \partial S$. Recall the definition of $V^S$ in (4.26) and let

$$V^B(y, \theta) := V_{bdry}(\theta - \Delta(y, \theta)) - \int_y^{y - \Delta(y, \theta)} f(x) \, dx, \quad \text{for } (y, \theta) \in B.$$ 

The discussion so far suggests that the following function is a classical solution to the problem (5.4) – (5.5) describing the value function of the optimization problem (5.3):

$$V^{B,S}(y, \theta) := \begin{cases} V^S(y, \theta), & \text{if } (y, \theta) \in \overline{S}, \\ V^B(y, \theta), & \text{if } (y, \theta) \in B, \end{cases}$$

(5.6)

up to the multiplicative constant $S_0$. Note that both cases in (5.6) can be combined to

$$V^{B,S}(y, \theta) = V_{bdry}(\theta + \Delta(y, \theta)) + \int_y^{y + \Delta(y, \theta)} f(x) \, dx, \quad \text{for all } (y, \theta).$$

The next theorem proves the conjectures already stated in this section for solving the optimal liquidation problem with possible intermediate buy orders.

**Theorem 5.1.** Let the model parameters $h, \lambda, \delta$ satisfy Assumption 3.2. The function $V^{B,S}$ is in $C^1(\mathbb{R} \times [0, \infty))$ and solves (5.4) and (5.5). The value function of the optimization problem (5.3) is given by $S_0 \cdot V^{B,S}$. Moreover, for given number of shares $\theta_0 - \geq 0$ to liquidate and initial state of the market impact process $Y_0 = y$, the unique optimal strategy $A^{opt}$ is given by $A_0^{opt} = 0$ and:

1. If $(y, \theta_0 -) \in \overline{S}$, $A^{opt}$ is the liquidation strategy for $\theta_0 -$ shares and impact process starting at $y$ as described in Theorem 4.3.
2. If $(y, \theta_0 -) \in B$, $A^{opt}$ consists of an initial buy order of $\left|\Delta(y, \theta_0 -)\right|$ shares (so that the state process $(Y, \theta)$ jumps to the boundary between $B$ and $S$) and then trading according to the liquidation strategy for $\theta_0 - + \left|\Delta(y, \theta_0 -)\right|$ shares and impact process starting at $y + \left|\Delta(y, \theta_0 -)\right|$ as described in Theorem 4.3.

The proof of Theorem 5.1 is given in Appendix A.1.

**Remark 5.2.** The results in this section show that if the market is too depressed, i.e. if the market impact is sufficiently small ($Y_0 < y_0$), then the optimal liquidation strategy may comprise an initial block buy, followed by continuous selling of the risky asset position. However, this does not constitute an arbitrage opportunity (in the classical sense), because the proceeds from such trading strategies are positive only in expectation, but not almost surely. Indeed, the absence of arbitrage in our model (see Theorem 7.1) implies that such proceeds will be negative with positive probability.

On the other hand, if the level of market impact is not overly depressed, i.e. $Y_0 \geq y_0$, then an optimal liquidation strategy will never involve intermediate buy orders.
includes in particular the case of a neutral initial market impact (i.e. \(Y_0 = 0\), like it is assumed in [PSS11]), or of an only slightly depressed initial impact level (\(Y_0 \in [y_0, \infty)\)), as it would occur after a sufficiently long time of inactivity by the large trader due to resilience of the market. The monotonicity of the optimal liquidation strategy will carry over to the case of non-zero bid-ask spread, as explained in Remark 7.3.

6. Stability

For the remaining Sections 6 and 7, we do no longer impose Assumption 3.2, but will only require that \(\delta \in \mathbb{R}\) and functions \(f, h(\cdot)\) are in \(C^1\) with \(f > 0\) and \(h(y) \text{sgn}(y) \geq 0\) for all \(y \in \mathbb{R}\) (a rather weak notion of “mean reversion of \(Y\”)).

We will show stability of our model in the following sense: Let \(A_t = \theta_0 - \theta_t\) be some monotone admissible strategy with liquidation proceeds \(L_t = L_{0,t}(A)\). Note that in the following we speak of \(\theta\) as a strategy instead of referring to \(A\). Then, Wong-Zakai-type approximation \(\theta^h\) of \(\theta\), given by (6.5), have proceeds \(L^h_t\) which are near in probability in the Prokhorov metric: Each path of \(L\) (respectively \(L^h\)) defines (\(\omega\)-wise) a measure on the time axis \(([0, \infty), \mathcal{B}([0, \infty]))\) and those measures converge weakly \(L^h \Rightarrow L\) for \(h \to 0\), in probability. To obtain this result, we express the proceeds in terms of a solution to a SDE with Marcus integral as defined in [KPP95] and extend the results therein for our setup. In the more general class of bounded semimartingale strategies \(\theta\), we will identify the proceeds process \(L^\theta\) as a limit (in the ucp topology) of the proceeds \(L^h\) of a Wong-Zakai-type approximation sequence of absolutely continuous processes \(\theta^h\), in which case \(L^h\) is already defined as in (5.1).

**Definition (Marcus integral).** Let \(F : \mathbb{R}^d \to \mathbb{R}^{d \times k}\) be continuously differentiable and \(Z\) be a \(k\)-dimensional semimartingale. Then the notation

\[
X_t = X_0 + \int_0^t F(X_s) \circ dZ_s
\]

means that \(X\) satisfies the stochastic integral equation

\[
X_t = X_0 + \int_0^t F(X_{s-}) dZ_s
+ \sum_{0 \leq s \leq t, \Delta Z_s \neq 0} \left(\varphi(F(\cdot)\Delta Z_s, X_{s-}) - X_{s-} - F(X_{s-})\Delta Z_s\right),
\]

where \(F_{-j}\) is the \(j\)th column of \(F\), \(Z^j\) is the \(j\)th entry of \(Z\) and \(\varphi(g, x)\) denotes the value \(y(1)\) of the solution to

\[
y'(u) = g(y(u)) \quad \text{with} \quad y(0) = x.
\]

The quadratic (co-)variation process is denoted by \([\cdot] = [\cdot]^c + [\cdot]^d\), it decomposes into a continuous part (appearing in (6.2)) and a discontinuous part. The next lemma gives a representation of the different processes defining our model, namely time, impact,
proceeds and unperturbed price, in terms of a Marcus SDE. To this end, let the function \( F : \mathbb{R}^4 \to \mathbb{R}^{4 \times 3} \) for \( X = (X^1, X^2, X^3, X^4)^{tr} \in \mathbb{R}^4 \) be given by

\[
F(X) := \begin{pmatrix}
1 & 0 & 0 \\
-h(X^2) & 1 & 0 \\
0 & -e^{-\gamma X^1 \cdot X^4 \cdot f(X^2)} & 0 \\
(\mu - \frac{\sigma^2}{2})X^4 & 0 & \sigma X^4
\end{pmatrix}.
\] (6.4)

**Lemma 6.1.** Let \( \theta \) be a càdlàg process with paths of finite total variation, and \( L \) be defined by \([5.1]\) be the process describing the evolution of proceeds generated by \( \theta \). Set \( X_t := (t, Y_t, L_t, \bar{S}_t)^{tr} \), so \( X_{0-} = (0, Y_{0-}, 0, \bar{S}_0)^{tr} \), and \( Z_t := (t, \theta_t, W_t, \sigma_t)^{tr} \). Then the process \( X \) is the solution to the Marcus-SDE

\[
X_t = X_{0-} + \int_0^t F(X_s) \circ dZ_s.
\]

For the proof see Appendix A.2. Following the analysis in [KPP95 Section 6], we now derive a Wong-Zakai-type approximation result in our setup. This type of result will justify from analytical point of view the use of the integral terms in \([3.3] \) representing the proceeds from block trades that could be seen as a limit of continuous trades with increasing speed of trading. More precisely, for a bounded semimartingale process \( \theta \) and \( h > 0 \) consider the approximating absolutely continuous processes defined by

\[
\Theta_t^h := \frac{1}{h} \int_{t-h}^t \theta_s \, ds, \quad t \geq 0,
\] (6.5)

with the convention that \( \theta_t = \theta_{0-} \) for \( t < 0 \). Let \( Z_t^h := (t, \Theta_t^h, W_t)^{tr} \) and \( X^h \) be a solution to the following SDE in the Itô sense

\[
dX_t^h = F(X_t^h) \, dZ_t^h, \quad X_0^h = X_{0-}.
\] (6.6)

The following result is about the convergence of \( X^h \). For the proof, see Appendix A.2

**Theorem 6.2.** Let \( (\Theta_t)_{t \geq 0} \) be a bounded semimartingale. For \( h > 0 \), let \( \Theta^h \) be the Wong-Zakai-type approximations from \( [6.5] \). Let \( X^h \) be defined by \( [6.6] \) for \( Z_t^h := (t, \Theta_t^h, W_t)^{tr} \) and \( F \) be defined as in \( [6.4] \). For the time-changes \( \gamma_t(h) := \frac{1}{h} \int_{t-h}^t (\theta^d_s + s) \, ds \) consider the processes \( (Y_t^{h})_{t \geq 0} \) defined by \( Y_t^h := X^h_{\gamma_t^{-1}(t)} \). Then the processes \( Y_t^h \) converge, as \( h \) goes to \( 0 \), in probability in the compact uniform topology to a process \( (Y_t^0)_{t \geq 0} \), such that \( X_t = (X^1_t, X^2_t, X^3_t, X^4_t)^{tr} := Y_{0(t)}^0 \) is a solution to

\[
X_t = X_{0-} + \int_0^t F(X_s) \circ dZ_s - \left(0, 0, \frac{1}{2} \int_0^t \sigma e^{-\gamma X^1_s X^4_s f(X^2_s)} \, d[W, \theta_s, 0]^{tr} \right)^{tr}
\] (6.7)

where \( X_{0-} = (0, Y_{0-}, 0, \bar{S}_0)^{tr} \) and \( \gamma_0(t) := [\theta^d_t + t] \).
Remark 6.3. a) Note that boundedness of \( \theta \) implies that \( X^2 \) is bounded. Hence, \( f \) is globally Lipschitz continuous on the range of \( X^2 \). This implies absolute convergence of the infinite sum in (6.2), see [KPP95] p. 365. In particular, this ensures that the proceeds in (7.1) are well-defined.

b) The additional covariation term in the limiting equation (6.7) arises since only the strategies \( \theta \) are approximated in a Wong-Zakai sense, but not also time \( t \) and the Brownian motion \( W \). For strategies \( \theta \) being of finite variation (as it would be natural under proportional transaction costs), this additional covariation term clearly vanishes.

An important consequence of the previous result is a stability property for our model. It essentially implies that we can approximate each strategy by a sequence of absolutely continuous strategies, corresponding to small intertemporal shifts of reassigned trades, whose proceeds will approximate the proceeds of the original strategy. More precisely, if we restrict our attention to the class of monotone strategies, then we can restate this stability in terms of the Prokhorov metric on the pathwise proceeds (which are monotone and hence define measures on the time axis). This result on stability of proceeds with respect to small intertemporal Wong-Zakai-type re-allocation of orders may be compared to seminal work by [HHK92] on a different but related problem, who required that for economic reason the utility should be a continuous functional of cumulative consumption with respect to small intertemporal shifts of reassigned trades, if we restrict our attention to the class of monotone strategies, then we can restate this result on stability of proceeds with respect to small intertemporal Wong-Zakai-type re-allocation of orders may be compared to seminal work by [HHK92] on a different but related problem, who required that for economic reason the utility should be a continuous functional of cumulative consumption with respect to the Prokhorov metric, in order to satisfy the sensible property of intertemporal substitution for consumption.

Corollary 6.4. Let \( \theta \) be a bounded finite variation trading strategy and \( \theta^h \) be the continuous approximations defined by (6.5), with \( L, L^h \) denoting the proceeds processes from the respective strategies. Then \( L^h_t \to L_t \) at all continuity points \( t \) of \( L \) as \( h \to 0 \), in probability. In particular, for any bounded monotone strategy \( \theta \) with \( \theta = \theta^\tau := \theta_{\cdot \wedge \tau} \) for a finite stopping time \( \tau < \infty \), the Borel measures \( L^h( dt; \omega) \) and \( L( dt; \omega) \) on \([0, \infty)\) are finite (a.s.) and converge in the Lévy-Prokhorov metric \( \rho(L^h(\omega), L(\omega)) \) in probability, i.e. for any \( \varepsilon > 0 \),

\[
\mathbb{P} \left[ \rho(L^h(\omega), L(\omega)) > \varepsilon \right] \to 0 \quad \text{as } h \to 0.
\]

Proof. Theorem 6.2 gives for the third components \( L^h = X^{h,3} \), \( L = X^{0,3} \) that for any \( \varepsilon > 0 \) and any time horizon \( T \in [0, \infty) \) it holds

\[
\mathbb{P} \left[ \sup_{t \leq T} |L^h_{\gamma^{-1}_h(\gamma_0(t))} - L_t| \leq \varepsilon \right] \to 1 \quad \text{as } h \to 0.
\]

Since \( \gamma^{-1}_h(\gamma_0(t)) \to t \) at continuity points of \( \gamma_0 \) (which are the continuity points of \( \theta \) and thus of \( L \)) it follows that \( \mathbb{P} [\Omega^*_h] \to 1 \) as \( h \to 0 \) with

\[
\Omega^*_h := \{ \omega \mid \forall t \text{ with } \Delta L_t(\omega) = 0 : |L^h_t(\omega) - L_t(\omega)| \leq \varepsilon \}.
\]

This is the first statement of the corollary. For stopped strategies \( \theta = \theta^\tau \), the approximations \( \theta^h \) are stopped at \( \tau + h \), so the proceeds \( L \) and \( L^h \) are constant in \([\tau(\omega) + h, \infty)\) and hence they define finite measures on \([0, \infty)\). One can check that \( L^h_{t+\varepsilon}(\omega) - \varepsilon \leq L_t(\omega) \leq L^h_{t+\varepsilon}(\omega) + \varepsilon \) for all \( t \geq 0 \) and \( \omega \in \Omega^*_h \), that is \( \rho(L^h(\omega), L(\omega)) \leq \varepsilon \) on \( \Omega^*_h \) by the definition of \( \rho \) as in e.g. [HHK92] p. 407. \qed
Note that when \( \delta = \gamma - \mu > 0 \), the convergence result in Corollary 6.4 in terms of the Prokhorov metric still holds for general monotone liquidation strategies without the need of stopping at finite times. Indeed, in this case the proceeds on the full time horizon \([0, \infty)\) will be a.s. finite as a consequence of the optimality result in Theorem 4.3.

For continuous semimartingales \( \theta \), the time change \( \gamma_0 \) in Theorem 6.2 is the identity, and one can state the convergence result without considering time-changes.

**Corollary 6.5.** For a continuous bounded semimartingale \( \theta \), let \( \theta^h \) be the Wong-Zakai-type approximations with corresponding impact and proceeds processes \((Y^h, L^h)\), as defined by (2.3) and (5.1) respectively. Then \((Y^h, L^h)\) converges for \( h \to 0 \) in the ucp topology to the process \((Y, L)\) with market impact \( Y = Y^\theta \) as in (2.3) and

\[
L_t := - \int_0^t e^{-\gamma u} \mathbb{S}_u f(Y_{u-}) \circ d\theta_u - \frac{1}{2} \int_0^t \sigma e^{-\gamma u} \mathbb{S}_u f(Y_u) \, d[\theta, W]_u, \quad t \geq 0.
\]

**Proof.** In the case \([\theta]^d = 0\), the argument proving Theorem 6.2 goes through without a need for time-changes. \( \square \)

### 7. Absence of arbitrage

We now address the question of existence of arbitrage opportunities in our model. To this end, we extend our set of trading strategies to include (bounded) semimartingale processes. Since the following builds on the results of the previous section, we still rely on our relaxed assumptions on \( f, h \) from Section 6. To get stability in the sense of Theorem 6.2 (see also Corollary 6.5) with \( \gamma := 0 \), we set the

**Definition.** For a bounded semimartingale strategy \( \theta \), the proceeds \( L = L(\theta) \) of this strategy up to \( T \) \( \infty \) are given by

\[
L_T := - \int_0^T f(Y^\theta_{t-}) \mathbb{S}_t \circ d\theta_t - \frac{1}{2} \int_0^T \sigma f(Y^\theta_t) \, d[\theta, W]_t
- \sum_{\Delta \theta_t \neq 0} \mathbb{S}_t \left( \int_0^{\Delta \theta_t} f(Y^\theta_{t-} + x) \, dx - f(Y^\theta_{t-}) \Delta \theta_t \right), \tag{7.1}
\]

where the stochastic integral is understood in Itô’s sense and \( Y^\theta \) is given as in (2.3).

Note that (7.1) coincides with (5.1) if in addition \( \theta \) has paths of finite total variation. Now, consider a self-financing portfolio \((\beta_t, \theta_t)\) of the large investor, where \( \beta_t \) represents the bank account at time \( t \) with interest rate set to zero and \( \theta_t \) denotes the holdings of risky asset at time \( t \). We will consider bounded (càdlàg) semimartingale strategies \( \theta \) on the full time horizon \([0, \infty)\). The self-financing property in our model reads that the bank account evolves according to the dynamics

\[
\beta_t := \beta_0 + L_t \quad \forall t \geq 0, \tag{7.2}
\]
with L as in (7.1). In order to define the wealth dynamics induced by the large trader’s strategy, we have to specify the dynamics of the value of the risky asset position in the portfolio. If the large trader was forced to liquidate her stock position immediately by a single block trade, the contribution in the bank account would be given by a term of the form (2.6). In this sense, the instantaneous liquidation value \( V_t^\theta \) of her position is

\[
V_t^\theta := \beta_t + \mathcal{S}_t \int_0^{\theta_t} f(Y_t^\theta - x) \, dx. \tag{7.3}
\]

The dynamics of this liquidation value is mathematically tractable and will be useful for our notion of arbitrage opportunities. For instance, \( V_t^\theta \) is a continuous process, see (7.4) below. We will prove a no-arbitrage theorem for the set of admissible trading strategies

\[\mathcal{A}_{\text{semi}} := \{ (\theta_t)_{t \geq 0} \mid \text{bounded (cadlag) semimartingale, with } V_0 \text{ bounded from below, } \theta_{0-} = 0, \text{ and such that } \theta_t = 0 \text{ for } t \in [T, \infty) \text{ for some } T < \infty \}\}.
\]

Note that for such a strategy \( \theta \) clearly holds \( V^\theta = \beta \) on \([T, \infty)\), i.e. beyond some bounded horizon \( T < \infty \) the liquidation value coincides with the cash holdings \( \beta_T \). Boundness from below for \( V^\theta \) has an clear economical meaning, while the boundedness of \( \theta \) may be viewed as a more technical requirement. It ensures, besides proceeds in (7.1) being bounded (cadlag) semimartingale, with \( V_0 \) bounded from below, that \( \theta_t = 0 \) for \( t \in [T, \infty) \) for some \( T < \infty \).

**Theorem 7.1.** The market is free of arbitrage up to any finite time horizon \( T \in [0, \infty) \) in the sense that there exists no \( \theta \in \mathcal{A}_{\text{semi}} \) with \( \theta_t = 0 \) on \( t \in [T, \infty) \) such that for the self-financing strategy \((\beta, \theta)\) with \( \theta_{0-} = 0 \) we have \( \mathbb{P}(V_0^\theta \geq 0) = 1 \) and \( \mathbb{P}(V_T^\theta > 0) > 0 \).

**Proof.** Consider the dynamics of the instantaneous liquidation value process \( V_t^\theta \). With \( F(y) := \int_0^y f(x) \, dx \), one can write \( \mathcal{S}_t f_0^\theta f(Y_t^\theta - x) \, dx = \mathcal{S}_t (F(Y_t^\theta) - F(Y_t^\theta - \theta_t)) \). Hence,

\[
dV_t^\theta = d(\mathcal{S}_t(F(Y_t^\theta) - F(Y_t^\theta - \theta_t))) + d\beta_t
\]

\[
= (F(Y_t^\theta) - F(Y_t^\theta - \theta_t)) \, d\mathcal{S}_t + \mathcal{S}_t \, d(F(Y_t^\theta) - F(Y_t^\theta - \theta_t)) + d[\mathcal{S}, F(Y^\theta)]_t + d\beta_t
\]

\[
= (F(Y_t^\theta) - F(Y_t^\theta - \theta_t)) \, d\mathcal{S}_t + \sigma \mathcal{S}_t f(Y_t^\theta) \, d[\theta, W]_t
\]

\[
+ \mathcal{S}_t (F'(Y_t^\theta) \, dY_t^\theta + 1/2 F''(Y_t^\theta) \, d[\theta]_t - F'(Y_t^\theta - \theta_t) \, d(Y_t^\theta - \theta_t) + \text{j-term}) + d\beta_t,
\]

where the j-term equals \( F(Y_t^\theta) - F(Y_t^\theta - \theta_t) - F'(Y_t^\theta - \theta_t) \Delta Y_t^\theta \). Using (7.1) with (7.3) yields

\[
dV_t^\theta = (F(Y_t^\theta) - F(Y_t^\theta - \theta_t)) \, d\mathcal{S}_t - \mathcal{S}_t (F'(Y_t^\theta) - F'(Y_t^\theta - \theta_t)) \, h(Y_t^\theta) \, dt. \tag{7.4}
\]

Now, \( dQ := \mathcal{E}(-\mu/\sigma) \cdot W^\theta_T \, d\mathbb{P} \) defines a probability (martingale) measure \( Q \), equivalent on \( \mathcal{F}_T \) to \( \mathbb{P} \). Then, for the \( \mathcal{Q}\)-Brownian motion \( W^\alpha := (\mu/\sigma) \, t + W \) we have

\[
dV_t^\theta = (F(Y_t^\theta) - F(Y_t^\theta - \theta_t)) \, \mathcal{S}_t \sigma (dW_t^\alpha + \xi_t \, dt),
\]

for \( t \in [0, T] \) with

\[
\xi_t := -\frac{h(Y_t^\theta)}{\sigma} \cdot \frac{F'(Y_t^\theta) - F'(Y_t^\theta - \theta_t)}{F(Y_t^\theta) - F(Y_t^\theta - \theta_t)} 1_{\{\theta_t \neq 0\}}.
\]

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For each \( \theta \in A_{\text{semi}} \) we have that \((\theta, Y^\theta)\) is bounded. Thus, \( \xi \) is bounded as well because, if \( \theta_{t^-} \neq 0 \), by the mean value theorem we have

\[
\frac{F'(Y^\theta_{t^-}) - F'(Y^\theta_{t^-} - \theta_{t^-})}{F(Y^\theta_{t^-}) - F(Y^\theta_{t^-} - \theta_{t^-})} = \frac{f'(z_1)}{f(z_2)} \quad \text{for some } z_{1,2} \text{ between } Y^\theta_{t^-} \text{ and } Y^\theta_{t^-} - \theta_{t^-},
\]

and this is bounded from above because \( f, f' \) are continuous and \( f > 0 \) (so it is bounded away from zero on any compact set). Hence \( \mathbb{E}^Q[\exp\{1/2 \int_0^T \xi_t^2 \, dt\}] < \infty \), ensuring the existence of a probability measure \( Q^\theta \approx Q \) with \( dQ^\theta := \mathcal{E}(-\int_0^T \xi_t \, dW_t^\theta) \, dQ \) that is equivalent to \( \mathbb{P} \) on \( \mathcal{F}_T \). Under \( Q^\theta \), clearly \( (V^\theta_t)_{t \in [0,T]} \) is a local martingale bounded from below, and hence a supermartingale. This rules out an arbitrage opportunity under \( \mathbb{P} \approx Q \) as described in the theorem.

\( \square \)

Remark 7.2. The proof of Theorem 7.1 applies also for stochastic, time-inhomogeneous \( \sigma = \sigma(t,\omega) \) that is uniformly bounded from above. Note that Remark 4.4 applies if \( \sigma \) is also uniformly bounded from above.

Remark 7.3 (On general bid-ask spread). Absence of arbitrage in the two-sided limit order book model with zero bid-ask spread naturally implies no arbitrage for model extensions with spread, at least when the admissible trading strategies have paths of finite variation. To make this precise, let us model different impact processes \( Y^{A^+} \) and \( Y^{A^-} \) from selling and buying, respectively, according to (2.3), and best bid and ask price processes \( (S^b, S^a) := (f(Y^{A^+})S^b, f(Y^{A^-})S^a) \) with \( S^b \leq S^a \). Then, the proceeds from implementing \( (A^+, A^-) \) on \([0, T]\) would be

\[
\int_0^T S^b_t \, dA^+_t - \int_0^T S^a_t \, dA^-_t + \sum_{0 \leq t \leq T, \Delta A^+_t > 0} S^b_t \int_0^{\Delta A^+_t} f(Y^{A^+} - x) \, dx - \sum_{0 \leq t \leq T, \Delta A^-_t > 0} S^a_t \int_0^{\Delta A^-_t} f(Y^{A^-} + x) \, dx.
\]

Now, the initial relation \( Y^{A^+}_0 \leq Y^{\theta}_0 \leq Y^{A^-}_0 \) would imply \( Y^{A^+} \leq Y^{\theta} \leq Y^{A^-} \), hence \( S^b \leq S \leq S^a \) and the proceeds above for the model with non-vanishing spread would be dominated (a.s.) by those that we get in (5.1), i.e. in the model without bid-ask spread. In an alternative but different variant, one could extend the zero bid-ask spread model to a one-tick-spread model, motivated by insights in [CDL13], by letting \( (S^b, S^a) := (S, S + \delta) \) for some \( \delta > 0 \). Again, proceeds in this model would be dominated by those in the zero-spread model. In either variant, absence of arbitrage strategies in the zero bid-ask spread model rules out arbitrage opportunities in the extended model with spread.

Moreover, the results in Section 5 will be applicable for non-zero bid-ask spread models. Indeed, if the initial market impact is not too small \( (Y_{t^-} \geq y_0) \), the optimal liquidation strategy in a model with non-zero bid-ask spread would still be monotone (hence requires modeling only the bid side of the LOB) and would be described by Theorem 5.1 since

\[
\sup_{A \in A_{\text{mon}}(\theta_{t^-})} J(Y_{t^-}; A) = \sup_{A \in A_{\text{vo}}(\theta_{t^-})} J(Y_{t^-}; A) \geq \sup_{A \in A_{\text{vo}}(\theta_{t^-})} J^{\text{spr}}(Y_{t^-}; A),
\]

where \( J^{\text{spr}}(Y_{t^-}; A) \) would be the corresponding cost functional in a non-zero spread model, and \( J(Y_{t^-}, \cdot) \) and \( J^{\text{spr}}(Y_{t^-}, \cdot) \) coincide on \( A_{\text{mon}}(\theta_{t^-}) \).
A. Appendix

A.1. Proofs about the value function

Note that Assumption 3.2 is in force throughout Appendix A.1. To prove the inequalities for verification, it is useful to represent $V_{bdry}$ as in Lemma A.1.

For all $\theta \geq 0$ we have

$$V_{bdry}(\theta) = \int_{0}^{\theta} f(y(x)) \exp \left( \int_{0}^{\theta} \frac{\delta}{h(y(z))} \, dz \right) \exp \left( \int_{y(\theta)}^{y(x)} \frac{\delta}{h(y)} \, dy \right) \, dx.$$ 

**Proof.** Using equation \((4.1)\), consider:

$$\int_{0}^{\theta} \frac{\delta}{h(y(z))} \, dz = \int_{y_{0}}^{y(\theta)} \frac{\delta}{h(y)} \, \theta'(y) \, dy$$

$$= \int_{y_{0}}^{y(\theta)} \frac{\delta}{h(y)} \left( 1 + \frac{h(y)\lambda(y)}{\delta} - \frac{h(y)}{\delta h'(y)} \right) \, dy$$

$$= \int_{y_{0}}^{y(\theta)} \frac{\delta}{h(y)} \, dy + \int_{y_{0}}^{y(\theta)} \frac{\theta'(y)}{f(y)} \, dy - \int_{y_{0}}^{y(\theta)} \frac{\lambda(y)}{h'(y)} \, dy + \int_{y_{0}}^{y(\theta)} \frac{h'(y)}{(h\lambda + h' + \delta)'(y)} \, dy$$

$$= \int_{y_{0}}^{y(\theta)} \frac{\delta}{h(y)} \, dy + \left[ \log f(y) \right]_{y_{0}}^{y(\theta)} - \left[ \log |h'(y)| \right]_{y_{0}}^{y(\theta)} + \left[ \log \left( h\lambda + h' + \delta \right)(y) \right]_{y_{0}}^{y(\theta)}.$$

Thus it follows

$$\exp \left( \int_{0}^{\theta} \frac{\delta}{h(y(z))} \, dz \right) = \frac{1}{f(y_{0})} \exp \left( \int_{y_{0}}^{y(\theta)} \frac{\delta}{h(y)} \, dy \right) \left( \frac{f(h\lambda + h' + \delta)}{h'} \right)(y(\theta)),$$

which implies

$$\exp \left( \int_{0}^{\theta} \frac{\delta}{h(y(z))} \, dz + \int_{y(\theta)}^{y(x)} \frac{\delta}{h(y)} \, dy \right) = \left( \frac{f(h\lambda + h' + \delta)}{h'} \right)(y(\theta)) \left( \frac{h'}{f(h\lambda + h' + \delta)} \right)(y(x)).$$

Integration using Lemma A.2 after multiplication with $f(y(x))$ yields the claim. 

**Lemma A.2.** Let $\theta \geq 0$. Then

$$\int_{0}^{\theta} \left( \frac{h'}{h\lambda + h' + \delta} \right)(y(x)) \, dx = \left( \frac{h(h\lambda + \delta)}{\delta(h\lambda + h' + \delta)} \right)(y(\theta)).$$

**Proof.** At $\theta = 0$, both sides equal zero, so it suffices to show equality of their derivatives.
By equation (4.1), we have as functions of \( y = y(\theta) \):

\[
\frac{h'}{h \lambda + h' + \delta} \theta' = \frac{h'}{h \lambda + h' + \delta} \left( 1 + \frac{h \lambda}{\delta} - \frac{hh''}{h' + \delta} + \frac{h(h \lambda + h' + \delta)'}{\delta (h \lambda + h' + \delta)} \right) = \frac{h'(\delta + h \lambda)(h \lambda + h' + \delta) - hh''(h \lambda + h' + \delta) + hh'(h \lambda + h' + \delta)'}{\delta (h \lambda + h' + \delta)^2} = \frac{h'}{\delta} - \left( \frac{hh'}{\delta (h \lambda + h' + \delta)} \right)' = \left( \frac{h(h \lambda + \delta)}{\delta (h \lambda + h' + \delta)} \right)'.
\]

**Lemma A.3.** It holds (3.10): \( V^W_y + V^W_\theta > f \) in \( \mathcal{W} \).

**Proof.** Using notation from Section 4.1, we have for \( y < y(\theta) \):

\[
V^W_y(y, \theta) + V^W_\theta(y, \theta) = C(\theta) \phi'(y) + C'(\theta) \phi(y) = -\frac{fh(\lambda + \delta)}{\phi h'} \left. \frac{\phi(y)}{h(y)} + \frac{f(\lambda + \delta)}{\phi h'} \right|_{y(\theta)} \cdot \phi(y) = \phi(y) \frac{f(\lambda + \delta)}{h'} \left|_{y(\theta)} \right. \cdot \left( 1 - \frac{h(y(\theta))}{h(y)} \right) + f(y(\theta)) \right) \geq f(y),
\]

noting \( \frac{\delta}{\phi(y)} \in (0, \lambda(x)] \) for \( x \leq y(\theta) \). Similar calculation at \( y = y(\theta) \) yields equality. \( \square \)

Recall from Section 4.1 the regions \( S_1 \) and \( S_2 \):

\[
S_1 := \{(y, \theta) \in \mathbb{R} \times [0, \infty) \mid y(\theta) < y < y_0 + \theta \},
S_2 := \{(y, \theta) \in \mathbb{R} \times [0, \infty) \mid y_0 + \theta < y \}.
\]

**Lemma A.4.** It holds (3.8): \( -\delta V^{S_2} - h(y)V^{S_2}_y < 0 \) in \( S_2 \).

**Proof.** Fix \( y > y_0 \). We will see, that \( g(\theta) := \delta V^{S_2}(y, \theta) + h(y)V^{S_2}_y(y, \theta) \) increases for \( \theta \in (0, y - y_0) \). By equation (4.25) holds

\[
g'(\theta) = \frac{d}{d\theta} \left( \delta \int_{y-\theta}^{y} f(x) \, dx + h(y) \left( f(y) - f(y - \theta) \right) \right) = \delta f(y - \theta) + h(y) f'(y - \theta) = f(y - \theta) \left( \delta + h(y) \lambda(y - \theta) \right) \geq f(y - \theta) \left( \delta + h(y_0) \lambda(y_0) \right) = 0,
\]

by monotonicity of \( h \) and \( h \lambda \). Noting \( g(0) = 0 \), the claimed inequality follows. \( \square \)
Lemma A.5. It holds \([3.8]\): \[-δV^{S_1} - h(y)V_y^{S_1} < 0\] in \(S_1\). Moreover, we have

\[
V_{\text{bdry}}'(\theta) = f(y(\theta)) + \frac{δ}{h(y(\theta))} (1 - y'(\theta)) V_{\text{bdry}}(\theta) \tag{A.2}
\]

and

\[
V_y^{S_1}(y, \theta) = f(y) - f(y - Δ) - \frac{δ}{h(y - Δ)} V_{\text{bdry}}(\theta - Δ). \tag{A.3}
\]

Proof. Let \((y, \theta) ∈ \mathcal{F}_1\). By \((4.22)\), we have \(θ - Δ = θ(y - Δ),\) implying

\[
Δ_y = \frac{θ'(y - Δ)}{θ'(y - Δ) - 1} = \frac{1}{1 - y'(θ - Δ)}.
\]

Using Lemma A.1, we get \(V_{\text{bdry}}'(\theta) = f(y(θ)) + \frac{δ}{h(y(θ))} (1 - y'(θ)) V_{\text{bdry}}(θ)\) and thereby

\[
V_{\text{bdry}}'(θ - Δ) = f(y - Δ) + \frac{δ}{h(y - Δ)} (1 - y'(θ - Δ)) V_{\text{bdry}}(θ - Δ).
\]

With equation \((4.23)\) it follows that

\[
V_y^{S_1}(y, \theta) = V_{\text{bdry}}'(θ - Δ) · (-Δ_y) + f(y) - f(y - Δ)(1 - Δ_y)
\]

\[
= -f(y - Δ)Δ_y + \frac{δ}{h(y - Δ)} V_{\text{bdry}}(θ - Δ) + f(y) + f(y - Δ)(Δ_y - 1)
\]

\[
= f(y) - f(y - Δ) - \frac{δ}{h(y - Δ)} V_{\text{bdry}}(θ - Δ).
\]

Now we fix \((y_1, \theta_1) := (y - Δ, θ - Δ)\) on the boundary and vary \(Δ ≥ 0\) to show monotonicity of \(g(Δ) := \delta V^{S_1}(y_1 + Δ, \theta_1 + Δ) + h(y_1 + Δ)V_y^{S_1}(y_1 + Δ, \theta_1 + Δ),\) which equals

\[
δV_{\text{bdry}}(\theta_1)\left(1 - \frac{h(y_1 + Δ)}{h(y_1)}\right) + δ \int_{y_1}^{y_1 + Δ} f(x) \, dx + h(y_1 + Δ)(f(y_1 + Δ) - f(y_1))
\]

and gives \(g(0) = 0\). Therefore, one obtains

\[
g'(Δ) = δV_{\text{bdry}}(\theta_1)\left(-\frac{h'(y_1 + Δ)}{h(y_1)}\right) + δf(y_1 + Δ)
\]

\[
+ h'(y_1 + Δ)(f(y_1 + Δ) - f(y_1)) + h(y_1 + Δ)f'(y_1 + Δ)
\]

\[
= -h'(y_1 + Δ)\left(\frac{δ}{h(y_1)} V_{\text{bdry}}(\theta_1) + f(y_1)\right) + f(y_1 + Δ) (hλ + h' + δ)(y_1 + Δ)
\]

\[
= -h'(y_1 + Δ) f(y_1) \frac{(hλ + h' + δ)(y_1)}{h'(y_1)} + f(y_1 + Δ) (hλ + h' + δ)(y_1 + Δ).
\]

Note, that, since \((hλ + δ)(y) ≤ 0,\) for \(y < y_0,\) the function \(y ↦ \frac{hλ + h' + δ}{h'}(y)\) is increasing in the interval \((−∞, y_0)\). So \(y ↦ \left(f : \frac{hλ + h' + δ}{h'}\right)(y)\) is increasing in \((y∞, y_0),\) which implies
We have existence of the directed derivative \( V_{-\infty} \) which is continuous in \( \theta \) due to \( \Delta(\cdot) \). Finally, let \( (y,\theta) \) be such that \( y > y_0 \). For all \( \theta > 0 \) where \( y_0 \) is already done in the proof of Lemma A.5, we have \( V_{\text{bdry}}^{y'}(y,\theta) = f(y(\theta))\exp\left(\int_{y(\theta)}^{y} \frac{-\delta}{h(x)} \, dx\right) + \frac{\delta}{h(y(\theta))} V_{\text{bdry}}^{y'}(y,\theta) \). Let \( y < y_0 \). As shown in Lemma A.5 we have

\[
V_{y}^{S_1}(y_0 + \theta, \theta) = f(y_0 + \delta) - f(y_0) = V_{S_2}^{y}(y_0 + \theta, \theta).
\]

Finally, let \( (y, \theta) \in \mathbb{R} \times \{0\} \). Since \( h(y_0) \lambda(y_0) + \delta = 0 \), it follows \( V(\cdot, 0) = 0 \) directly. We only need to show existence and continuity of \( V_y(0, 0) = \lim_{\theta \downarrow 0} \frac{1}{\theta} V(y, \theta) \). Let \( y < y_0 \). As shown in Lemma A.5, \( V_{\text{bdry}}^{y}(y, \theta) = f(y(\theta)) + \delta(1 - y'(\theta)) V_{\text{bdry}}^{y}(\theta) / h(y(\theta)) \), which leads to

\[
V^{y'}(y, \theta) = f(y(\theta))\exp\left(\int_{y(\theta)}^{y} \frac{-\delta}{h(x)} \, dx\right) + \frac{\delta}{h(y(\theta))} V^{y'}(y, \theta)
\]

by definition of \( V^{y'} \). With l'Hôpital’s rule we get

\[
\lim_{\theta \downarrow 0} \frac{1}{\theta} V^{y'}(y, \theta) = \lim_{\theta \downarrow 0} V^{y'}(y, \theta) = f(y_0)\exp\left(\int_{y_0}^{y} \frac{-\delta}{h(x)} \, dx\right),
\]

which is continuous in \((-\infty, y_0)\) and which equals \( f(y_0) \) at \( y = y_0 \). For \( y > y_0 \) we get \( V(y, \theta) = V^{S_2}(y, \theta) \), if \( \theta \geq 0 \) is small enough. Again by l'Hôpital,

\[
\lim_{\theta \downarrow 0} \frac{1}{\theta} V^{S_2}(y, \theta) = \lim_{\theta \downarrow 0} V^{S_2}(y, \theta) = f(y).
\]

Now let \( y = y_0 \). For all \( \theta > 0 \) we have \( V(y_0, \theta) = V^{S_1}(y_0, \theta) \). By construction it is \( V^{S_1}(y_0, \theta) = f(y_0) - V^{S_2}(y_0, \theta) \). So by equation (A.3) it holds

\[
\lim_{\theta \downarrow 0} \frac{1}{\theta} V^{S_1}(y_0, \theta) = f(y_0) - \lim_{\theta \downarrow 0} V^{S_2}(y_0, \theta)
\]

\[
= f(y_0) - \lim_{\theta \downarrow 0} \left( f(y_0) - f(y_0 - \Delta) - \frac{\delta}{h(y_0 - \Delta)} V_{\text{bdry}}(\theta - \Delta) \right) = f(y_0),
\]

due to \( \Delta(y_0, \theta) \to 0 \) for \( \theta \to 0 \) and thanks to \( h(y_0) \neq 0 \).
Proof of Theorem 5.1. That $V^{B,S} \in C^1(\mathbb{R} \times [0, \infty))$ essentially follows from Lemma 4.2. We show that $V^{B,S}$ satisfies (5.4) – (5.5). It is clear by construction that (5.4) holds true, so it remains to show (5.5). For $(y, \theta) \in \bar{S}$ the inequality follows from Lemma A.4 and Lemma A.5; note that we have equality only when $(y, \theta)$ is on the boundary between $S$ and $B$, or $\theta = 0$. Now suppose that $(y, \theta) \in B$. For simplicity of the exposition let $\tilde{\Delta}(y, \theta) = -\Delta(y, \theta) \geq 0$ be the distance from $(y, \theta)$ to the boundary in direction $(1, 1)$. We shall omit the arguments of $\tilde{\Delta}$ to ease notation. Set $(y_b, \theta_b) := (y + \tilde{\Delta}, \theta + \tilde{\Delta})$. Then

$$V^{B,S}(y, \theta) = V^{B,S}(y_b, \theta_b) - \int_y^{y_b} f(x) \, dx$$

and moreover

$$V^y_{\theta}(y, \theta) = \frac{d}{dy}(V^{B,S}(y + \tilde{\Delta}, \theta + \tilde{\Delta}) - \int_y^{y + \tilde{\Delta}} f(x) \, dx)$$

$$= (\tilde{\Delta}_y + 1)V_y + \tilde{\Delta}_y V_\theta - (1 + \tilde{\Delta}_y)(f(y + \tilde{\Delta}) - f(y)) = f(y) - V^y_{\theta}(y_b, \theta_b),$$

where the last equality uses $f = V_y + V_\theta$. We set

$$g(\tilde{\Delta}) := -h(y_b - \tilde{\Delta})(f(y_b - \tilde{\Delta}) - V^y_{\theta}(y_b, \theta_b)) - \delta(V^{B,S}(y_b, \theta_b) - \int_{y_b - \tilde{\Delta}}^{y_b} f(x) \, dx).$$

Note that $g(0) = 0$ by construction of the boundary between $S$ and $B$ in Section 4.1. Thus, it suffices to verify that $g' \leq 0$. Recalling $y = y_b - \tilde{\Delta}$, we have

$$\frac{d}{d\tilde{\Delta}} g(\tilde{\Delta}) = f(y)[h(y)\lambda(y) + h'(y) + \delta] - h'(y)V^y_{\theta}(y_b, \theta_b).$$

Recall the following form for the function $V^{B,S}$ on the boundary (see (4.6)):

$$V^y_{\theta}(y_b, \theta_b) = f(y_b) \frac{h(y_b)\lambda(y_b) + h'(y_b) + \delta}{h'(y_b)}.$$  \hspace{1cm} \text{(A.4)}$$

Thus, checking that $g'(\tilde{\Delta}) \leq 0$ is equivalent to verifying

$$h(y)\lambda(y) + h'(y) + \delta - \frac{h'(y)}{h'(y_b)} \cdot \frac{f(y_b)}{f(y)} \cdot (h(y_b)\lambda(y_b) + h'(y_b) + \delta) \leq 0.$$  \hspace{1cm} \text{(A.5)}$$

Since $y \leq y_b$ we have that $f(y) \leq f(y_b)$. Hence it suffices to check the last inequality when $f(y_b)/f(y)$ is replaced by 1. This is equivalent to verifying that $(h(y)\lambda(y) + \delta)/h'(y)$ is at most $(h(y_b)\lambda(y_b) + \delta)/h'(y_b)$, which clearly holds true as $x \mapsto (h(x)\lambda(x) + \delta)/h'(x)$ is strictly increasing for $x \leq y_b$.

Note that the analysis above actually shows that equality in (5.5) holds if and only if $(y, \theta)$ is on the boundary between $S$ and $B$. This ensures uniqueness of the optimal strategy. The rest of the proof follows the same lines as in the one for Theorem 4.3. \hfill \square
A.2. Proofs about stability

Proof of Lemma 6.1. It is $d[Z^j, Z^m]_t^c = dt$ for $j = m = 3$, and 0 otherwise. So the \( \partial F_{-j}/\partial x_t \) terms in equation (6.2) simplify to

$$
\frac{1}{2} \sum_{j=m=1}^{3} \sum_{\ell=1}^{4} \int_0^t \partial F_{-j}(X_{s-})F_{\ell,m}(X_{s-}) \, d[Z^j, Z^m]_s^c = \left(0, 0, 0, \frac{1}{2} \int_0^t \sigma^2 S_s \, ds\right)^{tr}.
$$

(A.6)

Jumps of \( Z \) are of the form \( \Delta Z_s = (0, \Delta \theta_s, 0)^{tr} \), so

$$
g(X) := F(X) \Delta Z_s = \left(0, \Delta \theta_s, -e^{-\gamma X^1} X^4 \cdot f(X^2) \Delta \theta_s, 0\right)^{tr},
$$

which yields the solution \( y(u) = V_u = (V_1^u, V_2^u, V_3^u, V_4^u)^T \in \mathbb{R}^4 \) to (6.3) with \( V_0 = X_{s-} \),

\[
\begin{align*}
V_1^u &= s - s = s, \\
V_2^u &= Y_{s-} + \int_0^u \Delta \theta_s \, dx = Y_{s-} + u \Delta \theta_s, \\
V_3^u &= \mathbb{S}_{s-} = \mathbb{S}_s, \\
V_4^u &= L_{s-} - \int_0^u e^{-\gamma s} \mathbb{S}_s f(Y_{s-} + x \Delta \theta_s) \Delta \theta_s \, dx = L_{s-} - e^{-\gamma s} \mathbb{S}_s \int_0^u \Delta \theta_s \, f(Y_{s-} + x) \, dx.
\end{align*}
\]

Thus the jump terms in (6.2) become

$$
\varphi(F(\cdot) \Delta Z_s, X_{s-}) - X_{s-} - F(X_{s-}) \Delta Z_s
$$

\[
= \left(0, 0, -e^{-\gamma s} \mathbb{S}_s \int_0^{u \Delta \theta_s} f(Y_{s-} + x) \, dx + e^{-\gamma s} \mathbb{S}_s \int_0^{u \Delta \theta_s} f(Y_{s-}) \Delta \theta_s, 0\right)^{tr}.
\]

(A.7)

Furthermore, the Itô integral in (6.2) reads

$$
\int_0^t F(X_{s-}) \, dZ_s = \begin{pmatrix}
- \int_0^t h(Y_s) \, ds + \theta_t - \theta_0 - \\
- \int_0^t e^{-\gamma s} \mathbb{S}_s f(Y_{s-}) \, d\theta_s \\
\int_0^t (\mu - \frac{\sigma^2}{2}) \mathbb{S}_s \, ds + \int_0^t \sigma \mathbb{S}_s \, dW_s
\end{pmatrix}.
$$

(A.8)

Summing up \( X_{0-} \) and equations (A.6) to (A.8) yields the first, second and fourth components \( t, \ Y_0 - \int_0^t h(Y_s) \, ds + \theta_t - \theta_0 = Y_t \) and \( \mathbb{S}_0 + \int_0^t \mu \mathbb{S}_s \, ds + \int_0^t \sigma \mathbb{S}_s \, dW_s = \mathbb{S}_t \), respectively. To complete the proof, we note that for the third component we get

\[
L_{0-} - \int_0^t e^{-\gamma s} \mathbb{S}_s f(Y_{s-}) \, d\theta_s + \sum_{0 \leq s \leq t} \left(-e^{-\gamma s} \mathbb{S}_s \int_0^{\Delta \theta_s} f(Y_{s-} + x) \, dx + e^{-\gamma s} \mathbb{S}_s f(Y_{s-}) \Delta \theta_s\right)
\]

\[
= - \int_0^t e^{-\gamma s} \mathbb{S}_s f(Y_{s-}) \, d\theta_s^c - \sum_{0 \leq s \leq t} e^{-\gamma s} \mathbb{S}_s \int_0^{\Delta \theta_s} f(Y_{s-} + x) \, dx = L_t.
\]

\[33\]
Proof of Theorem 6.2. The proof follows the ideas in [KPP95, Section 6] where the statement is proved for one-dimensional SDEs, whereas here we need a multidimensional version. For convenience of the reader, we will indicate the changes in the arguments to accommodate our setup. Localizing along the Brownian motion part (the variable $X^i$), we can assume that $F, F', F''$ are globally Lipschitz continuous and bounded. The localized solutions can be easily pasted together because of the global existence and strong uniqueness of a solution to the Marcus SDE $dX_t = F(X_t) \circ dZ_t$ in our case; note also that the localizing sequence will not affect the time-changes $\gamma_h$ which additionally simplifies the argument. Now let $V^h_t := Z^h_{\gamma^{-1}_h(t)}$. Then $Y^h$ is the unique solution of

$$Y^h_t = X_0^- + \int_0^t F(Y^h_s) \, dV^h_s. \tag{A.9}$$

Note that $V^h_t - Z_{\gamma^{-1}_h(t)}(t) = (0, (\theta^h - \theta)_{\gamma^{-1}_h(t)}(t), 0)^{tr} =: (0, U^h_t, 0)^{tr}$. For the limit, we have $\lim_{h \to 0} U^h_t = \bar{\theta}_t - \theta_{\gamma^{-1}_0(t)} =: U_t$, where $\bar{\theta}_t$ is the limit (in the compact uniform topology) of $\theta^h_{\gamma^{-1}_h}$ given by

$$\bar{\theta}_t = \begin{cases} \theta_{\gamma^{-1}_0(t)} & \text{if } \eta_1(t) = \eta_2(t), \\ \theta_{\gamma^{-1}_0(t)} - \frac{t - \eta_1(t)}{\eta_2(t) - \eta_1(t)} \eta_2(t) - \frac{\eta_2(t) - t}{\eta_2(t) - \eta_1(t)} & \text{if } \eta_1(t) \neq \eta_2(t), \end{cases}$$

where $\eta_1(t) := \sup\{s \mid \gamma^{-1}_0(s) < \gamma^{-1}_0(t)\}$ and $\eta_2(t) := \inf\{s \mid \gamma^{-1}_0(s) > \gamma^{-1}_0(t)\}$; as in [KPP95, Lemma 6.2]. The last term in (A.9) is $(0, U^h_t, \int_0^t -e^{-\gamma^{-1}_h(s)} S_{\gamma^{-1}_h(s)} f(Y^{h,2}_s) \, dU^h_s, 0)^{tr}$. To identify the limit of the third component, we integrate by parts to obtain (with $\hat{S}_t := -e^{-\gamma t} S_t$)

$$\int_0^t \hat{S}_{\gamma^{-1}_h(s)} f(Y^{h,2}_s) \, dU^h_s$$

$$= \hat{S}_{\gamma^{-1}_h(t)} f(Y^{h,2}_t) U^h_t - \int_0^t U^h_s \, d(\hat{S}_{\gamma^{-1}_h(s)} f(Y^{h,2}_s)) - \left[ \hat{S}_{\gamma^{-1}_h(s)} f(Y^{h,2}_s), U^h_s \right]_t$$

$$= \hat{S}_{\gamma^{-1}_h(t)} f(Y^{h,2}_t) U^h_t - \int_0^t U^h_s f(Y^{h,2}_s) \, d\hat{S}_{\gamma^{-1}_h(s)}$$

$$+ \int_0^t f'(Y^{h,2}_s) h(Y^{h,2}_s) U^h_s \hat{S}_{\gamma^{-1}_h(s)} d\gamma^{-1}_h(s)$$

$$- \int_0^t \hat{S}_{\gamma^{-1}_h(s)} f(Y^{h,2}_s) U^h_s d\gamma^{-1}_h(s) - \left[ \hat{S}_{\gamma^{-1}_h(s)} f(Y^{h,2}_s), U^h_s \right]_t \tag{A.10}$$

Note that $\int_0^t U^h_s \, d\hat{S}_{\gamma^{-1}_h(s)} \Rightarrow \int_0^t U_s \, d\hat{S}_{\gamma^{-1}_h(s)} \equiv 0$ and $\int_0^t U^h_s \, d\gamma^{-1}_h(s) \Rightarrow \int_0^t U_s \, d\gamma^{-1}_h(s) \equiv 0$, where “$\Rightarrow$” denotes weak convergence of the processes (in the Skorokhod topology $J_1$ on
the path space). For the fourth term in (A.10), note that [KPP95] Lemma 6.3 gives
\[
\int_0^t \hat{S}_{\gamma h^{-1}(t)} dY^h \to \frac{1}{2} \int_0^t \hat{S}_{\gamma h^{-1}(t)} d((U_x)^2 - [\theta]_{\gamma h^{-1}(t)}).
\]
The quadratic covariation process in the last term of (A.10) can be written as
\[
-[\hat{S}_{\gamma h^{-1}(t)} f(Y^{h,2}) dU^h] = -[\hat{S}_{\gamma h^{-1}(t)} f(Y^{h,2})]_t = -[\hat{S}_{\gamma h^{-1}(t)} f(X^{h,2})]_t = [\hat{S} f(X^{h,2})]_{\gamma h^{-1}(t)} = -[\hat{S} f(X^{h,2})]_{\gamma h^{-1}(t)}
\]
where the time-changed equalities can be justified by [RY99 Proposition V.1.4]. The process \(Y^{h,2}\) satisfies the ODE \(dY^{h,2} = -h(Y^{h,2}) d\gamma h^{-1}(t) + d\theta h^{-1}(t) + dU^h\). Since \(\gamma h^{-1}\) is good and \((\gamma h^{-1}, \theta h^{-1}, U) \Rightarrow (\gamma_0^{-1}, \theta_0^{-1}, U)\) holds, it is \(Y^{h,2} \Rightarrow Y^{0,2}\) with
\[
dY^{0,2} = -h(Y^{0,2}) d\gamma_0^{-1}(t) + d\theta_0^{-1}(t) + dU_t.
\]

As in [KPP95] Section 6], we conclude that the right-hand side of (A.10) converges in distribution to
\[
\hat{S}_{\gamma h^{-1}(t)} f(Y^{0,2}) dU_t - \frac{1}{2} \int_0^t \hat{S}_{\gamma h^{-1}(s)} f'(Y^{0,2}) d((U_x)^2 - [\theta]_{\gamma h^{-1}(s)})
\]
\[
+ \frac{1}{2} \int_0^t \hat{S}_{\gamma h^{-1}(s)} f'(Y^{0,2}) d[\theta]_{\gamma h^{-1}(s)} + \int_0^t \sigma \hat{S}_{\gamma h^{-1}(s)} f(Y^{0,2}) d[\theta, W]_{\gamma h^{-1}(s)}.
\]

Let \(\{\tau_i\}\) be the jump times of \(\theta\). Note that \([\theta] d\) only changes at times \(\tau_i\) and \(U_t = 0\) if \(t \notin [\gamma_0(\tau_i^-), \gamma_0(\tau_i)]\) for any \(\tau_i\). Thus, the first line in (A.11) only changes when \(t \in [\gamma_0(\tau_i^-), \gamma_0(\tau_i)]\) for some \(\tau_i\). Now, for \(t \in [\gamma_0(\tau_i^-), \gamma_0(\tau_i)]\) we have that
\[
U_t = -\Delta \theta_{\tau_i} (\eta_2(t) - t)/(\eta_2(t) - \eta_1(t)),
\]
and so \(-\frac{1}{2} \int_0^t d((U_t)^2 - [\theta]_{\gamma h^{-1}(t)}) = |\Delta \theta_{\tau_i}|^2 (\gamma_0(t) - t) dt\) and \(Y^{0,2} = Y^{0,2}_{\gamma h^{-1}(\tau_i^-)} + \Delta \theta_{\tau_i} (\eta_2(t) - t)/(\eta_2(t) - \eta_1(t))\). Thus, using \(\eta_2(t) - \eta_1(t) = |\Delta \theta_{\tau_i}|^2\) and integrating by parts we get that the contribution from the first line in (A.11) over the full time interval \([\gamma_0(\tau_i^-), \gamma_0(\tau_i)]\) equals to
\[
\hat{S}_{\tau_i} \int_0^{\Delta \theta_{\tau_i}} f(Y^{0,2}_{\gamma h^{-1}(\tau_i^-)} + x) dx - \hat{S}_{\tau_i} f(Y^{0,2}_{\gamma h^{-1}(\tau_i^-)}) \Delta \theta_{\tau_i}.
\]

Note that this is exactly the jump term in the definition of the Marcus integral. So, collecting all the intermediate results so far we conclude that \(Y^h\) converges in distribution to a process \(Y^0\) such that \(X_t = Y^0_{\gamma h(t)}\). Now, the convergence in the compact uniform topology follows from the argument in the proof of [KPP95] Theorem 6.5].
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