PRIMES IN ARITHMETIC PROGRESSIONS TO
SPACED MODULI. III

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Abstract. Let
\[ E(x, q) = \max_{(a, q) = 1} \left| \sum_{n \leq x, n \equiv a \pmod{q}} \Lambda(n) - \frac{x}{\phi(q)} \right|, \]
where \( \Lambda \) is the von Mangoldt function. Let \( S_f = \{ f(k) : k \in \mathbb{N} \} \), where \( f \) is a polynomial of degree \( d \geq 2 \) with integer coefficients and positive leading coefficient. In analogy with the Bombieri-Vinogradov

1. Introduction

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theorem, we would like to show that

\[ \sum_{q \in S_f, Q < q \leq 2Q} E(x, q) \ll_{A, \varepsilon} x Q^{1/d-1} (\log x)^{-A} \]

for \( \varepsilon > 0, A > 0 \) and \( Q \leq x^{1/2-\varepsilon} \). In the general case, (1.1) is known only for \( Q \leq x^{9/20-\varepsilon} \), and in the special case \( f(X) = X^2 \), for \( Q \leq x^{43/90-\varepsilon} \) [2].

Here we refine the approach in [2] for \( f(X) = X^2 \).

**Theorem 1.** Let \( f(X) = X^2 \). Then (1.1) holds for \( Q \leq x^{1/2-\varepsilon} \).

To prove Theorem 1, we sharpen the auxiliary results on pp. 147–150 of [2]. With a little modification, we are then able to complete the proof of Theorem 1 by arguing as in [2]. The key new result is Lemma 2 below, which strengthens Lemma 11 of [2]. Thanks are due to James Maynard for suggesting in conversation the line of argument used to prove Lemma 2.

**Notation.** We write

\[ \|\theta\| = \min_{n \in \mathbb{Z}} |\theta - n| \]

and, for complex numbers \( c_1, \ldots, c_N \),

\[ \|c\|_2 = \left( \sum_{n=1}^{N} |c_n|^2 \right)^{1/2}. \]

The \( k \)-th Riesz mean is defined by

\[ A_k(x, q, a, d) = \frac{1}{k!} \sum_{\ell \leq x, \ell \equiv a (\text{mod} \ q), \ell \equiv 0 (\text{mod} \ d)} \left( \log \frac{x}{\ell} \right)^k \quad (k = 0, 1, \ldots) \]

and we write

\[ r_k(x, q, a, d) = A_k(x, q, a, d) - \frac{x}{qd}. \]

It is convenient to write \( a^{(q)} \) for an arbitrary integer with \( (a^{(q)}, q) = 1 \). We suppose, as we may, that \( x \) is large and \( \varepsilon \) is sufficiently small, and write \( \delta = \varepsilon^2 \). Except in Lemma 3, implied constants depend at most on \( \varepsilon \) or, when \( A \) appears in the result, on \( \varepsilon \) and \( A \).

The conductor of a primitive Dirichlet character \( \chi \) is denoted by \( C(\chi) \).
2. The large sieve for square moduli

Lemma 1. Let $\Delta > 0$ and $Q \geq 1$. For $\beta$ real, let $\mathcal{N}(\beta)$ denote the number of relatively prime pairs $a, q$, $1 \leq a \leq q^2$, $q \leq Q$, with

$$\left\| \frac{a}{q^2} - \beta \right\| \leq \Delta.$$

Then

$$\mathcal{N}(\beta) \ll (Q\Delta^{-1})^\varepsilon (Q^3 \Delta + Q^{1/2}).$$

Proof. This is due to Baier and Zhao [1, Section 11].

Lemma 2. Let $Q \geq 1$. Let $a_1, \ldots, a_N$ be complex numbers,

$$T(\alpha) = \sum_{n=1}^{N} a_n e(n\alpha).$$

Let $g \in \mathbb{N}$. Then

$$\sum_{q \leq Q} \sum_{a=1}^{gq^2} \left| T\left(\frac{q}{gq^2}\right) \right|^2 \ll (QN)^\varepsilon \left(1 + \frac{g}{N}\right) (gQ^3 + Q^{1/2}N) \|a\|_2^2.$$ 

Proof. We first show that, for real $\alpha$ and $\Delta > 0$, the number $\mathcal{M}(\alpha)$ of solutions of

$$\left\| \frac{a}{gq^2} - \alpha \right\| \leq \Delta, 0 \leq a \leq gq^2 - 1, (a, gq^2) = 1, 1 \leq q \leq Q,$$

satisfies

$$\mathcal{M}(\alpha) \ll (1 + g\Delta)(Q^3(g\Delta) + Q^{1/2})(Q\Delta^{-1})^\varepsilon.$$ 

To see this, write $a = b + q^2n$, $0 \leq n < q$, $0 \leq b < q^2$. Then (2.2) implies

$$\left\| \frac{b}{q^2} - g\alpha \right\| = \left\| \frac{b + q^2n}{q^2} - g\alpha \right\| \leq g\Delta.$$ 

The number of possible $b$ is

$$\ll (Q^3(g\Delta) + Q^{1/2})(Q\Delta^{-1})^\varepsilon$$

by Lemma 1. Once $b$ is fixed, (2.2) implies

$$\left\| \frac{b}{gq^2} + \frac{n}{q} - \alpha \right\| \leq \Delta.$$ 

There are at most $2g\Delta + 1$ possible $n$, and the bound (2.3) follows.

By [4, Theorem 2.1], the left-hand side of (2.1) is bounded by

$$\ll (N + \Delta^{-1}) \left(\max_{\alpha \in \mathbb{R}} \mathcal{M}(\alpha)\right) \|a\|_2^2,$$
for any $\Delta > 0$. We take $\Delta = N^{-1}$ and apply (2.3) to obtain the lemma.

Lemma 3. Let $Q = x^\theta$ and $0 < \lambda \leq \theta$, $x \geq M \gg x^{\theta - \lambda}$. Let $c_1, \ldots, c_M$ be complex numbers. Let

$$T(\lambda) = \sum_{Q < q^2 \leq 2Q} \sum_{\chi \pmod{q^2}} \left| \sum_{m=1}^{M} c_m \chi(m) \right|^2 .$$

Then

$$T(\lambda) \ll x^* (Q^{1/2} x^\lambda + Q^{3/4} M x^{-\lambda/2}) \|c\|_2^2 .$$

Proof. For a character $\chi \pmod{q^2}$ counted in $T(\lambda)$, induced by a primitive character $\chi' \pmod{C(\lambda)}$, we have

(2.5) $C(\chi) = g k^2 \in (x^\lambda, 2 x^\lambda]$ with $g$ squarefree, $k \in \mathbb{N}$; and

$$\chi(m) = \begin{cases} 
\chi'(m) & \text{if } (m, q) = 1 \\
0 & \text{if } (m, q) > 1 .
\end{cases}$$

Since $C(\chi) \mid q^2$, we have

$$v g k^2 = q^2 \in (x^\theta, 2x^\theta]$$

for a natural number $v$. Obviously $v = g t^2$, $t \in \mathbb{N}$,

(2.6) $x^{\theta / 2} < q = g t k \leq (2 x^\theta)^{1/2} .$

It follows that

(2.7)

$$T(\lambda) \leq \sum_{\frac{1}{2} x^{\theta - \lambda} < g t^2 \leq 2 x^{\theta - \lambda}} \sum_{\frac{g t^2}{2} < k \leq 2 \frac{g t^2}{\phi(g k^2)}} \chi' \pmod{g k^2} \left| \sum_{m \leq M \atop (m, t) = 1} c_m \chi'(m) \right|^2 .$$

Here $\sum^*$ denotes a sum restricted to primitive characters. By a standard inequality [3 Chapter 27, (10)],

$$\sum^* \chi' \pmod{g k^2} \left| \sum_{m \leq M \atop (m, t) = 1} c_m \chi'(m) \right|^2 \leq \frac{\phi(g k^2)}{g k^2} \sum_{a=1 \atop (a, g k^2) = 1}^{g k^2} \left| \sum_{m=1 \atop (m, t) = 1} c_m e \left( \frac{am}{g k^2} \right) \right|^2$$
Using Lemma 2 for fixed $g$ and $t$, the sum over $k$ on the right-hand side of (2.7) is
\[
\ll (QM)^{\varepsilon/3} \left( g \left( \frac{Q^{1/2}}{gt} \right)^3 + \frac{Q^{1/4}}{(gt)^{1/2}} M \right) \|c\|_2^2.
\]
(Note that $g \ll x^{\theta - \lambda} \ll M$ here.) For some $G \geq 1, T \geq 1$ with $GT^2 \asymp x^{\theta - \lambda}$, we have
\[
T(\lambda) \ll (\log x)^2 (QM)^{\varepsilon/3} \sum_{G \leq g < 2G} \sum_{T \leq t < 2T} \left\{ g \left( \frac{Q^{1/2}}{gt} \right)^3 + \frac{Q^{1/4}}{(gt)^{1/2}} M \right\} \|c\|_2^2
\]
\[
\ll x^{\varepsilon} (Q^{3/2} G^{-1} T^{-2} + Q^{1/4} MG^{1/2} T^{1/2}) \|c\|_2^2
\]
\[
\ll x^{\varepsilon} (Q^{1/2} x^\lambda + Q^{3/4} M x^{-\lambda/2}) \|c\|_2^2.
\]
This completes the proof of Lemma 3. \qed

**Lemma 4.** Let $1 \leq x^\lambda \leq Q \ll x^{1/2-\varepsilon}, x^\lambda \geq Q^{1/2} x^{\varepsilon/6}$. Let $H$ and $K$ satisfy
\[
Q x^{-\lambda} \ll K \ll H \ll x^{3/5}, \quad HK \ll x.
\]
Let $a_n (K < n \leq 2K)$ and $b_m (H < m \leq 2H)$ be complex numbers, $a_n \ll x^\delta, b_m \ll x^\delta$. Let
\[
K(s, \chi) = \sum_{K < n \leq 2K} a_n \chi(n) n^{-s},
\]
\[
H(s, \chi) = \sum_{H < m \leq 2H} b_m \chi(m) m^{-s},
\]
\[
S = \sum_{Q < q^2 \leq 2Q} \sum_{x^\lambda < C(\chi) \leq 2x^\lambda} \left| H \left( \frac{1}{2} + it, \chi \right) K \left( \frac{1}{2} + it, \chi \right) \right|.
\]
Then
\[
S \ll x^{1/2-\varepsilon/20} Q^{1/2}.
\]

**Proof.** We apply the Cauchy-Schwarz inequality to $S$, followed by applications of Lemma 3 to each of the two sums over $q, \chi$. The conditions
\[
H \gg x^{\theta - \lambda}, \quad K \gg x^{\theta - \lambda}
\]
are fulfilled since
\[
H \geq K \gg Q x^{-\lambda}.
\]
Lemma 6. A lemma is a variant of [2, Proposition 1].

Proof. This is a special case of [2, Lemma 6].

Let \( \beta \) be such that for each \( m \) we have
\[ \sum_m b_m m^{-\frac{1}{2}-it}\ll x^{3\delta} \]
and similarly for \( \sum_n a_n n^{-\frac{1}{2}-it}\ll x^{2\delta} \), we have
\[
S \ll x^{3\delta} (Q^{1/4} x^{\lambda/2} + Q^{3/8} K^{1/2} x^{-\lambda/2}) (Q^{1/4} x^{\lambda/2} + Q^{3/8} H^{1/2} x^{-\lambda/2}) \\
\ll x^{3\delta} (Q^{1/2} x^{\lambda} + Q^{3/4} x^{1/2-\lambda/2} + Q^{5/8} x^{\lambda/4} H^{1/2}) \\
\ll x^{3\delta} (Q^{3/2} + Q^{3/4} x^{1/2-\lambda/2} + Q^{7/8} x^{3/10}).
\]

Each of these three terms is \( x^{1/2-\varepsilon/20} Q^{1/2} \):
\[
Q^{3/2} x^{3\delta} (x^{1/2-\varepsilon/20} Q^{1/2})^{-1} \ll Q x^{\varepsilon/20+3\delta-1/2} \ll 1; \\
Q^{3/4} x^{1/2-\lambda/2+3\delta} (x^{1/2-\varepsilon/4} Q^{1/2})^{-1} \ll Q^{1/4} x^{-\lambda/2+3\delta+\varepsilon/20} \ll 1; \\
Q^{7/8} x^{3/10+3\delta} (Q^{1/2} x^{1/2-\varepsilon/20})^{-1} \ll Q^{3/8} x^{-1/5+\varepsilon/20+3\delta} \ll 1.
\]

This completes the proof of Lemma 4. \( \square \)

3. Proof of Theorem 1

It is convenient to write \( S(Q) = \{ q^2 : Q < q^2 \leq 2Q \} \).

Lemma 5. Let \( 0 < \gamma < 1 \). There is a subset \( F(Q) \) of \( S(Q) \) with
\[
\# F(Q) \ll Q^{1/2-\beta},
\]
such that for \( q^2 \in S(Q) \setminus F(Q) \), \( \chi \) a nonprincipal character \((\mod q^2)\) and \( \Re s = 1/2 \), we have
\[
\sum_{n \leq N} \chi(n) n^{-s} \ll |s| N^{\frac{1}{2}-\beta} \quad (N \geq q^\gamma).
\]

Here \( \beta = \beta(\gamma) > 0 \). The implied constants depend on \( \gamma \).

Proof. This is a special case of [2, Lemma 6]. \( \square \)

We shall refer to \( F(Q) \) in the remaining lemmas. The following lemma is a variant of [2, Proposition 1].

Lemma 6. Let \( M_1, \ldots, M_{15} \) be numbers with \( M_1 \geq \cdots \geq M_{15} \geq 1 \), and suppose that \( \{1, \ldots, 15\} \) has a partition into subsets \( A, B \) such that
\[
\prod_{i \in A} M_i \ll x^{1/2-3\varepsilon/4}, \quad \prod_{i \in B} M_i \ll x^{1/2-3\varepsilon/4}.
\]

Let \( a_i(m) \) \((M_i/2 < m \leq M_i, 1 \leq i \leq 15)\) be complex sequences with
\[
|a_i(m)| \leq \log m \quad (1 \leq i \leq 15, M_i/2 < m \leq M_i).
\]
Suppose that, whenever $M_i > x^{1/8}$, $a_i(m)$ is 1 ($M_i/2 < m \leq M_i$) or $\log m$ ($M_i/2 < m \leq M_i$). Let

$$M_i(s, \chi) = \sum_{M_i/2 < m \leq M_i} a_i(m)\chi(m)m^{-s},$$

$$L = x/(M_1 \ldots M_{15}), B_1(s, \chi) = \sum_{Lx^{-\varepsilon} < n \leq L} \chi(n)n^{-s}.$$

Then for $\text{Re } s = 1/2$ and $Q \ll x^{1/2-\varepsilon}$,

$$S := \sum_{q \in S(Q) \setminus F(Q)} \sum_{\chi \mod q} |B_1(s, \chi)M_1(s, \chi) \ldots M_{15}(s, \chi)| \ll |s|^{3}Q^{1/2}x^{1/2-3\delta}.$$

**Proof.** It suffices to show for $0 \leq \lambda \leq \theta$ that

$$(3.1) S(\lambda) \ll |s|^{3}Q^{1/2}x^{1/2-4\delta},$$

where $S(\lambda)$ is the subsum of $S$ defined by the additional condition $x^\lambda < C(\lambda) \leq 2x^\lambda$.

Arguing exactly as in the proof of [2, Lemma 10], (3.1) holds unless (writing as usual $Q = x^\theta$) we have

$$(3.2) \lambda > (5\theta + \varepsilon)/6,$$

We now suppose that (3.2) holds. We decompose $B_1(s, \chi)$ into $O(\log x)$ subsums $M_{16}(x, \chi)$ defined by a condition

$$M_{16}/2 < n \leq M_{16},$$

where $Lx^{-\varepsilon} \leq M_{16} < L$. It suffices to prove the analogue of (3.1) with $B(s, \chi)$ replaced by $M_i(s, \chi)$ and $6\delta$ in place of $4\delta$.

Rearranging $M_1, \ldots, M_{16}$ as $N_1 \geq \cdots \geq N_{16}$, write $N_i(s, \chi)$ for the corresponding Dirichlet polynomials and

$$N_i = x^{\beta_i}.$$

Then $\beta_1 \geq \cdots \geq \beta_{16} \geq 0, 1 - \varepsilon \leq \beta_1 + \cdots + \beta_{16} \leq 1$.

We can use the argument in the proof of [2, Lemma 15] to complete the present proof whenever $\beta_1 + \beta_2 > 3/5$. Suppose now that

$$\beta_1 + \beta_2 < 3/5.$$

As shown in the proof of [2, Lemma 15], there is a subset $W$ of \{1, \ldots, 16\} such that

$$x^{1/2} \ll H := \prod_{j \in W} 2M_j \ll x^{3/5}.$$
Let $K := \prod_{j \leq 16, j \notin W} 2M_j$. We see that
\[ x^{2/5-\varepsilon} \ll K \ll H, \quad HK \ll x. \]
Let
\[ H(s, \chi) = \prod_{j \in W} M_j(s, \chi), \quad K(s, \chi) = \prod_{1 \leq j \leq 16, j \notin W} M_j(s, \chi). \]
We note that
\[ K \gg x^{\theta-\lambda}, \quad \text{since} \quad \theta - \lambda < 1/12. \]
Hence we may apply Lemma 3 to obtain the desired bound in the form
\[ \sum_{q \in S(Q)} \sum_{\chi \pmod{q}} \left| H(s, \chi)K(s, \chi) \right| \ll x^{1/2-6\delta}Q^{1/2}. \]

Our final lemma is a variant of [2, Lemma 18].

**Lemma 7.** Let $a_i(m)$ ($1 \leq i \leq 15$) be nonnegative sequences satisfying the hypotheses of Lemma 6. Let
\[ u_d = \sum_{\substack{d = m_1 \ldots m_{15} \leq M, \forall_i \leq 2^{16}D \leq M_i}} a_1(m_1) \ldots a_{15}(m_{15}) \]
for $D_1 < d \leq D$, with $D = M_1 \ldots M_{15}$, $D_1 = 2^{-15}D$. Let $Q \ll x^{1/2-\varepsilon}$. Then for every $A > 0$,
\[ \sum_{q \in S(Q) \setminus F(Q)} \left| \sum_{D_1 < d \leq D} u_d r_0(x, q, a^{(q)}, d) \right| \ll \frac{x}{Q^{1/2}(\log x)^A}. \]

**Proof.** Just as in the proof of [2, Lemma 18], it suffices to show that
\[ \sum_{q \in S(Q) \setminus F(Q)} \left| \sum_{D_1 < d \leq D} u_d r_4(x, q, a^{(q)}, d) \right| \ll x^{1-\delta}Q^{-1/2} \]
The condition from small $\ell$ in (1.2), (1.3) to $r_4$ is negligible:
\[ \sum_{D_1 < d \leq D} \left| u_d \right| r_4(x^{1-\varepsilon}, q, a^{(q)}, d) \ll x^{\varepsilon/3} \left\{ \sum_{\substack{md \leq x^{1-\varepsilon} \leq \deg a \pmod{q} \leq x^{1-2\varepsilon/3}}} + \sum_{d \leq x^{1-\varepsilon}/q} \frac{x}{qd} \right\} \ll x^{1-\delta}Q^{-1} \]
thus it suffices to show that
\[ (3.3) \sum_{q \in S(Q) \setminus F(Q)} \sum_{D_1 < d \leq D} u_d (r_4(x, q, a^{(q)}, d) - r_4(x^{1-\varepsilon}, q, a^{(q)}, d)) \ll x^{1-\delta}Q^{-1/2}. \]
We now follow the argument in the proof of [2, Lemma 18] to show that (3.3) follows from
\begin{align*}
\int_{\text{Re } s=1/2} \sum_{q \in S(Q) \setminus F(Q)} \sum_{\chi \pmod{q}} \left| \sum_{d \leq D_1, d \chi \neq \chi_0} u_d \chi(d) d^{-s} \right| |B_1(s, \chi)| \left| \frac{ds}{s^5} \right| \ll x^{1/2-\delta} Q^{-1/2}.
\end{align*}
Here $B_1(s, \chi)$ is the Dirichlet polynomial in Lemma 6. At this point we see that (3.4) follows from Lemma 6. \hfill \Box

Proof of Theorem 4. Just as in [2], we reduce this to showing that
\begin{align*}
\sum_{q \in S(Q) \setminus F(Q)} \left| \sum_{m, n \leq Q^{x/4}} \Lambda(m) \mu(n) r_0(x, q, a(q), mn) \right| 
\ll xQ^{-1/2} (\log x)^{-A}.
\end{align*}
for every $A > 0$. We use Heath-Brown’s decomposition of $\Lambda(m)$, and a slight variant of this decomposition for $\mu(n)$, to show that (3.5) follows from Lemma 7; full details are given on page 158 of [2]. This completes the proof of Theorem 4. \hfill \Box

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