STACK-SORTING, SET PARTITIONS, AND LASSALLE’S SEQUENCE

COLIN DEFANT¹, MICHAEL ENGEN², AND JORDAN A. MILLER³

Abstract. We exhibit a bijection between recently-introduced combinatorial objects known as valid hook configurations and certain weighted set partitions. When restricting our attention to set partitions that are matchings, we obtain three new combinatorial interpretations of Lassalle’s sequence. One of these interpretations involves permutations that have exactly one preimage under the (West) stack-sorting map. We prove that the sequences obtained by counting these permutations according to their first entries are symmetric, and we conjecture that they are log-concave. We also obtain new recurrence relations involving Lassalle’s sequence and the sequence that enumerates valid hook configurations. We end with several suggestions for future work.

1. Introduction

In 2012, Lassalle [12] introduced a sequence \((A_m)_{m \geq 1}\) defined by the recurrence relation
\[
A_m = (-1)^{m-1}C_m + \sum_{j=1}^{m-1} (-1)^{j-1} \left( \frac{2m-1}{2m-2j-1} \right) A_{m-j} C_j
\]
and subject to the initial condition \(A_1 = 1\). Here, \(C_n = \frac{1}{n+1} \binom{2n}{n}\) is the \(n\)th Catalan number. The first few terms of this sequence, which has now come to be known as Lassalle’s sequence, are

1, 1, 5, 56, 1092, 32670, 138715, 7938930, 5882844968, 548129834616.

It is not at all obvious from the definition that the terms of Lassalle’s sequence should be positive. Indeed, Lassalle’s primary focus was to prove that the terms are positive and increasing, settling a conjecture of Zeilberger. This was reproven in [1], and the sequence was studied further in [6,10,15,16]. In particular, Josuat-Vergès found a combinatorial interpretation of \(A_m\) that relates these numbers to set partitions and cumulants in free probability theory; we briefly discuss this in Section 2.

One of the primary purposes of this article is to provide three new combinatorial interpretations of the numbers \(A_m\). The equivalence of these three interpretations follows from known results, but it is useful to have a variety of perspectives. The first interpretation answers a very natural question concerning the (West) stack-sorting map, whose background we now review.

Throughout this article, the word permutation refers to a permutation of a finite set of positive integers. We write permutations as words. Let \(S_n\) denote the set of permutations of \(\{1, 2, \ldots, n\}\). A permutation is called normalized if it is an element of \(S_n\) for some positive integer \(n\).

¹Princeton University
²University of Florida
³Washington State University

E-mail addresses: cdefant@princeton.edu, engenmt@ufl.edu, jordan.a.miller@wsu.edu.
In his seminal monograph *The Art of Computer Programming*, Knuth introduced an algorithm that “sorts” permutations through the use of a vertical “stack” [11]. West later studied a slight variant of this algorithm in his 1990 Ph.D. thesis [17]. Specifically, West studied the function $s$, known as the stack-sorting map, defined by the following procedure. Suppose we are given an input permutation $\pi = \pi_1\pi_2\cdots\pi_n$. At any point in time during the procedure, if the next entry in the input permutation is larger than the entry at the top of the stack or if the stack is empty, the next entry in the input permutation is placed at the top of the stack. Otherwise, the entry at the top of the stack is annexed to the end of the growing output permutation. This algorithm terminates when the output permutation has length $n$, and $s(\pi)$ is defined to be this output permutation. The following figure illustrates this procedure and shows that $s(3142) = 1324$.

We will not attempt to discuss all of the literature concerning the stack-sorting map. Instead, we state only the background information that we need for our purposes and refer the interested reader to [2,3,8].

West called $|s^{-1}(\pi)|$ the fertility of the permutation $\pi$ and computed the fertilities of a few very special types of permutations [17]. Bousquet-Mélou later studied so-called sorted permutations, which are permutations that have positive fertilities [4]. We say a permutation is uniquely sorted if its fertility is 1. A descent of a permutation $\pi = \pi_1\cdots\pi_n$ is an index $i \in \{1, \ldots, n-1\}$ such that $\pi_i > \pi_{i+1}$. Suppose $\pi \in S_n$ has exactly $k$ descents. We show in Section 3 that $\pi$ is uniquely sorted if and only if it is sorted and $n = 2k+1$. In particular, there are no uniquely sorted permutations of even length. When considering uniquely sorted permutations of odd length, we are led to our first combinatorial interpretation of Lassalle’s sequence. Specifically, we will show that $A_{k+1}$ is precisely the number of uniquely sorted permutations in $S_{2k+1}$.

One of the central notions concerning the stack-sorting map is that of a $t$-stack-sortable permutation. This is simply a permutation $\pi \in S_n$ such that $s^t(\pi) = 123\cdots n$, where $s^t$ denotes the $t$th iterate of $s$. Let $W_t(n)$ denote the number of $t$-stack-sortable permutations of length $n$. It follows from Knuth’s work in [11] that $W_1(n) = C_n$. West conjectured [17], and Zeilberger later proved [18], that

$$W_2(n) = \frac{2}{n+1} \binom{3n}{n}.$$ 

It follows from a theorem of West and Stankova [3, Theorem 3.4] that

$$W_t(n) \leq (t+1)^{2n}.$$ 

For several years, this was the best known upper bound for $W_t(n)$ when $t \geq 3$; it is still the best known upper bound when $t \geq 5$. The first author [8] improved these bounds when $t = 3$ and $t = 4$, showing that

$$W_3(n) < n^2 \cdot 12.53296^n \quad \text{and} \quad W_4(n) < n^5 \cdot 21.97225^n.$$ 

However, Bóna [2] has conjectured the much stronger estimate $W_t(n) \leq \frac{(t+1)n}{n}$.

In her study of sorted permutations, Bousquet-Mélou mentioned that it would be interesting to obtain a method for computing the fertility of any given permutation. This was achieved in much greater generality in [7] using new combinatorial objects called “valid hook configurations.” Roughly speaking, a valid hook configuration of a permutation $\pi$ is a configuration of L-shaped “hooks” that connect points in the plot of $\pi$ subject to certain restrictions. When we speak of a valid
hook configuration on \( n \) points, we simply mean a valid hook configuration of some permutation of length \( n \). The theory of valid hook configurations was the key ingredient used in [8] in order to obtain the above-mentioned upper bounds for \( W_3(n) \) and \( W_4(n) \).

We lack a thorough understanding of valid hook configurations; this is why an explicit formula for \( W_3(n) \) remains out of reach. Therefore, one of the other main purposes of this paper is to study these new objects. In Section 3, we review the major definitions and results concerning valid hook configurations. Our presentation differs slightly from that given in [7] and [8]. Because those two papers focus on using valid hook configurations to prove other results, they define these structures in fairly technical terms. In contrast, our approach in the current paper is meant to elucidate the constructions and aid comprehension. We also discuss valid hook configurations of uniquely sorted permutations. This allows us to obtain our second interpretation of Lassalle’s sequence. Namely, \( A_{k+1} \) is the number of normalized valid hook configurations on \( 2k + 1 \) points that use exactly \( k \) “hooks.”

Our final interpretation of Lassalle’s sequence, given in Section 4, states that \( A_{k+1} \) counts the number of decreasing binary plane trees with some specific properties. The advantage of viewing Lassalle’s sequence in terms of trees is that we will be able to easily detect a simple recursive combinatorial construction that builds objects counted by Lassalle’s sequence from smaller such objects.

In order to prove that these objects are counted by Lassalle’s sequence, we actually establish a bijection from the set of normalized valid hook configurations on \( n \) points to the set \( \overline{\mathcal{P}}^c(n + 1) \) defined in Section 2. This is done in Section 5. The bijection provides an interesting new way of viewing valid hook configurations. We will show that when \( n = 2k + 1 \), the preimage of the set \( \mathcal{M}^c(2k + 2) \) (also defined in Section 2) under this map is the set of normalized valid hook configurations on \( 2k + 1 \) points that use exactly \( k \) hooks, proving our second new combinatorial interpretation of Lassalle’s sequence. The first and third interpretations then follow from the second interpretation and known results.

We also show that the sequences \( (A_{k+1}(m))^{2k+1}_{m=1} \) are symmetric, where \( A_{k+1}(m) \) denotes the number of uniquely sorted permutations in \( S_{2k+1} \) with first entry is \( m \). This is somewhat surprising since we expect the stack-sorting map to output permutations that are in some sense “close” to the identity permutation. In other words, one might naturally suspect that the sequences \( (A_{k+1}(m))^{2k+1}_{m=1} \) should be decreasing. To conclude Section 4, we show that \( A_{k+1}(m) \) also counts uniquely sorted permutations according to another statistic that we call the hotspot of the permutation.

In Section 5, we exploit the structures of valid hook configurations in order to obtain a recurrence relation that generates the numbers \(-k_n(-1) = |\mathcal{P}^c(n)|\) (defined in Section 2). It turns out that the same recurrence with different initial conditions generates the Lassalle numbers \( A_n \). Finally, we end with several open problems and suggestions for future work.

2. Lassalle’s Sequence and Free Probability

In this section, we review the combinatorial interpretation of Lassalle’s sequence that Josuat-Vergès found. Let \( \mathcal{P}(n) \) denote the collection of partitions of the set \( \{0, 1, \ldots, n-1\} \). If \( \rho \in \mathcal{P}(n) \), we say two distinct blocks \( B \) and \( B' \) of \( \rho \) form a crossing if there exist \( i, k \in B \) and \( j, \ell \in B' \) such that either \( i < j < k < \ell \) or \( i > j > k > \ell \). The crossing graph \( G(\rho) \) is the graph whose vertices are the blocks of \( \rho \) in which two blocks are adjacent if and only if they form a crossing. We say a
A partition $\rho \in \mathcal{P}(n)$ is connected if $G(\rho)$ is a connected graph. Let $\mathcal{P}^c(n)$ denote the set of connected partitions in $\mathcal{P}(n)$. A matching is a set partition in which every block has exactly 2 elements. Let $\mathcal{M}(n)$ denote the set of matchings in $\mathcal{P}(n)$, and put $\mathcal{M}^c(n) = \mathcal{M}(n) \cap \mathcal{P}^c(n)$.

In free probability theory, the free counterpart of the classical Poisson law is known as the free Poisson law. It is characterized by the fact that all of the free cumulants are equal to a single parameter $\lambda > 0$, known as the rate (we will not actually make use of any free probability theory, but the interested reader can consult [13]). The free Poisson law also appears in random matrix theory in relation to Wishart matrices [13]. The $n^{th}$ moment is given by

$$m_n(\lambda) = \sum_{k=1}^{n} \lambda^k N(n,k),$$

where $N(n,k) = \frac{1}{n} \binom{n}{k} \binom{n}{k-1}$ is a Narayana number. Define the cumulants $k_n(\lambda)$ of the free Poisson law by

$$\sum_{n \geq 1} k_n(\lambda) \frac{z^n}{n!} = \log \left( 1 + \sum_{n \geq 1} m_n(\lambda) \frac{z^n}{n!} \right).$$

Josuat-Vergès [10] has proven that

$$k_n(\lambda) = - \sum_{\rho \in \mathcal{P}^c(n)} (-\lambda)^{\#\rho} T_{G(\rho)}(1,0),$$

where $\#\rho$ is the number of blocks of $\rho$ and $T_{G}(x,y)$ denotes the Tutte polynomial of the graph $G$ (see [10] for the definition of the Tutte polynomial of a graph). A source in a directed graph is a vertex with in-degree 0. We will make use of the following theorem due to Greene and Zaslavsky.

**Theorem 2.1** [9]. Fix a vertex $v$ in a simple graph $G$. The number of acyclic orientations of $G$ in which $v$ is the unique source is $T_G(1,0)$.

We can apply this theorem to the crossing graph of a partition $\rho \in \mathcal{P}^c(n)$ to see that $T_{G(\rho)}(1,0)$ is the number of acyclic orientations of $G(\rho)$ such that the block of $\rho$ containing the element 0 is the only source. Let $\mathcal{P}_c(n)$ be the set of ordered pairs $(\rho, r)$, where $\rho \in \mathcal{P}^c(n)$ and $r$ is an acyclic orientation of $G(\rho)$ whose only source is the block containing 0. According to (1), we have

$$- k_n(-1) = |\mathcal{P}_c(n)|.$$

By studying the cumulants of the $q$-semicircular law, Josuat-Vergès also proved that

$$A_m = |\mathcal{M}^c(2m)|,$$

where $\mathcal{M}^c(2m)$ is the collection of ordered pairs $(\rho, r) \in \mathcal{P}_c^c(2m)$ such that $\rho$ is a matching.

3. Valid Hook Configurations

Valid hook configurations were introduced in [7] as a tool for constructing all of the decreasing plane trees of certain types that have a given permutation as their postorder readings. In that paper, $\mathcal{H}(\pi)$ denotes the set of valid hook configurations of a permutation $\pi$, while $\mathcal{H}_{(0,2)}(\pi)$ denotes a specific subset of $\mathcal{H}(\pi)$ that is used to find the preimages of $\pi$ under the stack-sorting map. In this

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1. Although the free Poisson law is usually defined for $\lambda > 0$, Josuat-Vergès’ proof does not rely on the positivity of $\lambda$. 

section, we define only the valid hook configurations in the subset \( \mathcal{H}_{\{0,2\}}(\pi) \) because these are the only types of valid hook configurations we will need. The reader wishing to compare our treatment with that given in either \([7]\) or \([8]\) should be aware that the definition given here is, strictly speaking, different from that given in those two papers. Specifically, the valid hook configurations in \( \mathcal{H}_{\{0,2\}}(\pi) \) were originally defined so that some hooks have horizontal length 1. We have ignored these “small” hooks in our definition since they do not give any additional information relevant for our purposes. The reader who is seeing valid hook configurations here for the first time can ignore these remarks.

Let us begin the definition by choosing a permutation \( \pi = \pi_1 \pi_2 \cdots \pi_n \) with descent set \( \{d_1, \ldots, d_k\} \), where \( d_1 < \cdots < d_k \) (we do not require \( \pi \) to be normalized). Our running example will be the permutation \( 2 7 3 5 9 10 11 4 8 1 6 12 13 14 15 16 \).

The plot of a permutation is the graph displaying the points \((i, \pi_i)\) for \(1 \leq i \leq n\). Figure 1 portrays the plot of our example permutation.

A hook of \( \pi \) is drawn by starting at a point \((i, \pi_i)\) in the plot of \( \pi \), moving vertically upward, and then moving to the right until reaching another point \((j, \pi_j)\). We must necessarily have \( i < j \) and \( \pi_i < \pi_j \). The point \((i, \pi_i)\) is called the southwest endpoint of the hook while \((j, \pi_j)\) is called the northeast endpoint. Figure 2 shows our example permutation with a hook that has southwest endpoint \((2, 7)\) and northeast endpoint \((7, 11)\).

Definition 3.1. A valid hook configuration of a permutation \( \pi \) is a configuration of hooks drawn on the plot of \( \pi \) subject to the following restrictions:

1. The southwest endpoints of the hooks are precisely the descent tops of the permutation.
2. A point in the plot cannot lie directly above a hook.
3. Hooks cannot intersect each other except in the case that the northeast endpoint of one hook is the southwest endpoint of the other.

Figure 3 shows four placements of hooks that are forbidden by conditions 2 and 3 in Definition 3.1. Figure 4 shows a valid hook configuration of our example permutation. Note that the total number of hooks in a valid hook configuration of \( \pi \) is exactly \( k \), the number of descents of \( \pi \). Because the southwest endpoints of the hooks are the points \((d_i, \pi_{d_i})\), we have a natural ordering of
Figure 2. The plot of 27359101148161213141516 along with a single hook that has southwest endpoint (2, 7) and southwest endpoint (7, 11).

Figure 3. Four placements of hooks that are forbidden in a valid hook configuration.

Figure 4. A valid hook configuration of 27359101148161213141516.

the hooks. Namely, the $i$th hook is the hook whose southwest endpoint is $(d_i, \pi_{d_i})$. We say a valid hook configuration is normalized if it is a valid hook configuration of a normalized permutation.

Each valid hook configuration of $\pi$ induces a coloring of the points in the plot of $\pi$. To begin the process of coloring the plot, we first draw a “sky” over the entire diagram. Of course, we color the sky blue. Next, assign distinct colors other than blue to the $k$ hooks in the valid hook configuration.

There are $k$ northeast endpoints of hooks, and these points remain uncolored. However, all of the other $n - k$ points will be colored. In order to decide how to color a point $(i, \pi_i)$ that is not a northeast endpoint, imagine that this point simply looks directly upward. If this point sees a hook when looking upward, it receives the same color as the hook that it sees. If the point does not see a hook, it must see the sky, so it receives the color blue. There is one caveat here: if $(i, \pi_i)$ is the
southwest endpoint of a hook, then it looks around (on the left side of) the vertical part of that hook. Figure 5 shows the coloring of the plot of our example permutation induced from the valid hook configuration from Figure 4. Observe that the point $(2, 7)$ is colored blue because this point looks around the first (green) hook and sees the sky. Similarly, $(9, 8)$ is red because this point looks around the third (brown) hook and sees the second (red) hook.

To summarize, we started with a permutation $\pi$ with exactly $k$ descents. We chose a valid hook configuration of $\pi$ by drawing $k$ hooks according to the rules 1, 2, and 3 in Definition 3.1. This valid hook configuration then induced a coloring of the plot of $\pi$. Specifically, $n - k$ points were colored, and $k + 1$ colors were used (one for each hook and one for the sky). Let $q_i$ be the number of points colored the same color as the $i^{th}$ hook, and let $q_0$ be the number of points colored blue (sky color). Then $(q_0, q_1, \ldots, q_k)$ is a composition of $n - k$ into $k + 1$ parts. We call a composition obtained in this way a valid composition of $\pi$. Let $H_{\{0,2\}}(\pi)$ be the set of valid hook configurations of $\pi$ (as mentioned above, the notation is an artifact of the original definition of a valid hook configuration given in [7]). Let $V(\pi)$ be the set of valid compositions of $\pi$.

Although we will not use this fact, it is useful to be aware of Lemma 3.1 from [8], which states that the map $H_{\{0,2\}}(\pi) \rightarrow V(\pi)$ obtained by sending a valid hook configuration to the induced valid composition is a bijection. The motivation for studying valid hook configurations comes from the following theorem concerning the fertility of a permutation.

**Theorem 3.1 ([7]).** Let $\pi$ be a permutation with exactly $k$ descents. The fertility of $\pi$ is given by the formula

$$|s^{-1}(\pi)| = \sum_{(q_0, \ldots, q_k) \in V(\pi)} \prod_{t=0}^{k} C_{q_t},$$

where $C_j = \frac{1}{j+1} \binom{2j}{j}$ is the $j^{th}$ Catalan number.

Using valid hook configurations, one can also count preimages of a permutation $\pi$ under the map $s$ according to certain statistics. For example, Corollary 5.1 in [7] provides a formula for the number of preimages of $\pi$ with a given number of valleys. Theorem 5.2 in the same paper gives a formula for the number of preimages with a prescribed number of descents.

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2The problem of finding the fertility of a permutation is a special case of a more general problem involving postorder readings of decreasing plane trees, and this more general problem provides a motivation for studying the more general valid hook configurations defined in [7].
One immediate application of Theorem 3.1 comes from Exercise 18 in Chapter 8 of Bóna’s _Combinatorics of Permutations_ [2], which asks for the maximum number of descents that a sorted permutation of length $n$ can have. Recall that a permutation is called sorted if its fertility is positive. Suppose $\pi = \pi_1 \cdots \pi_n$ is a sorted permutation with $k$ descents. It follows from Theorem 3.1 that $V(\pi)$ is nonempty. Since the elements of $V(\pi)$ are compositions of $n - k$ into $k + 1$ parts, we must have $k + 1 \leq n - k$. Thus, $k \leq \left\lfloor \frac{n-1}{2} \right\rfloor$. Using valid hook configurations, it is not difficult to construct sorted permutations of length $n$ with $\left\lfloor \frac{n-1}{2} \right\rfloor$ descents. Indeed, the only valid composition of such a permutation is $(1,1,\ldots,1)$, so the fertility is $\prod_{r=0}^{k} C_1 = 1$.

On the other hand, suppose $\pi$ is a uniquely sorted permutation of length $n$ with $k$ descents. According to Theorem 3.1, we must have $V(\pi) = \{(1,1,\ldots,1)\}$, where $(1,1,\ldots,1)$ is a composition of $n - k$ into $k + 1$ parts that are all equal to 1. This proves the following proposition.

**Proposition 3.1.** Let $\pi$ be a permutation of length $n$ with $k$ descents. The permutation $\pi$ is uniquely sorted if and only if it is sorted and $n = 2k + 1$.

As mentioned in the introduction, this proves that there are no uniquely sorted permutations of even length.

**Corollary 3.1.** Uniquely sorted permutations in $S_{2k+1}$ are in bijection with normalized valid hook configurations on $2k + 1$ points with $k$ hooks.

**Proof.** A uniquely sorted permutation in $S_{2k+1}$ has a unique valid hook configuration, which must have $k$ hooks. On the other hand, if we are given a normalized valid hook configuration on $2k + 1$ points with $k$ hooks, then the underlying permutation must be a sorted permutation in $S_{2k+1}$ with $k$ descents. By Proposition 3.1, this permutation is uniquely sorted. \qed

Corollary 3.1 establishes the equivalence of our first two combinatorial interpretations of Lassalle’s sequence. We can describe uniquely sorted permutations (or equivalently, their valid hook configurations) via the following recursive combinatorial construction.

Begin by choosing two uniquely sorted permutations $\tau$ and $\mu$ such that $\tau \mu \in S_{2k}$ for some $k$. Make sure that the largest entry of $\tau$ is greater than the first entry of $\mu$. Now form the permutation $\tau \mu(2k+1)$. It might be easier to visualize this procedure by picturing valid hook configurations. Figure 6 shows a uniquely sorted permutation with its valid hook configuration; the two permutations from which this larger permutation was formed are shaded separately. In general, if we are given a uniquely sorted permutation $\pi = \pi_1 \cdots \pi_{2k+1} \in S_{2k+1}$, it is easy to reobtain the two uniquely sorted permutations from which it was built. We simply draw the unique valid hook configuration of the given permutation. The point $(2k+1,2k+1)$ must be a point in the plot, and it must be a northeast endpoint of a hook. The southwest endpoint of that hook is of the form $(d_r, \pi_d_r)$ for some descent $d_r$. The two permutations from which $\pi$ was built are $\tau = \pi_{d_r+1} \cdots \pi_{2k}$ and $\mu = \pi_1 \cdots \pi_{d_r}$.

This recursive construction suggests a link with trees, which leads us to our third combinatorial interpretation of Lassalle’s sequence.
Figure 6. The uniquely sorted permutation 547621389 is built from the smaller uniquely sorted permutations 547 and 62138.

4. Decreasing Plane Trees

A decreasing plane tree is a rooted plane tree whose nodes are labeled with distinct positive integers such that every non-root node has a label that is smaller than the label of its parent. A plane tree is called binary if each vertex has at most two children. If a vertex has exactly one child, we make a distinction between whether or not this child is a left or right child. A labeled tree is called normalized if its set of labels is of the form \( \{1, \ldots, n\} \) for some positive integer \( n \). See Figure 7 for an example of a decreasing binary plane tree.

To read a decreasing binary plane tree in in-order (sometimes called symmetric order), we first read the left subtree of the root, then the root, and finally the right subtree of the root. Each subtree is itself read in in-order. The symmetric order reading of the tree in Figure 7 is 2635741. Let \( S(\tau) \) denote the in-order reading of the decreasing binary plane tree \( \tau \). The map \( S \) is a bijection from the set of normalized decreasing binary plane trees on \( n \) vertices to the set \( S_n \). [2,14]

\[
\begin{align*}
7 & \quad 6 \quad 4 \\
2 & \quad 5 \quad 1 \\
3 &
\end{align*}
\]

Figure 7. A (normalized) decreasing binary plane tree.

To read a decreasing binary plane tree in postorder, we first read the left subtree of the root, then the right subtree of the root, and finally the root. Each subtree is itself read in postorder. The postorder reading of the tree in Figure 7 is 2356147. Let \( P(\tau) \) denote the postorder reading of a decreasing binary plane tree \( \tau \). It turns out that the stack-sorting map can be described in terms of symmetric order and postorder readings [2]. Specifically,

\[
s = P \circ S^{-1}.
\]

For example, \( s(2635741) = 2356147 = P(S^{-1}(2635741)) \), where \( S^{-1}(2635741) \) is the tree in Figure 7.

The first author has shown [7] how to construct the decreasing binary plane trees whose postorders are equal to a given permutation \( \pi \). We will not review this construction here. Instead, we simply discuss the mechanics of this construction in the (much simpler) specific case in which \( \pi \) is uniquely sorted. Refer to Figure 8 for an illustration.
Figure 8. The left image shows a valid hook configuration of $\pi = 547621389$ along with some additional dotted lines. The hooks and dotted lines transform into the edges of the tree on the left, which is the unique decreasing binary plane tree with postorder $\pi$.

Suppose $\pi = \pi_1 \cdots \pi_{2k+1}$ is uniquely sorted, and draw its valid hook configuration. For each northeast endpoint $(j, \pi_j)$, consider the point $(j - 1, \pi_{j-1})$. Draw a dotted line between these two points. Now replace each point $(i, \pi_i)$ with a vertex with label $\pi_i$. “Unbend” each hook to transform it into a left edge. Similarly, transform each dotted line into a right edge. This produces the unique decreasing binary plane tree $\tau$ with postorder $\pi$. Note also that $S(\tau)$ is the unique permutation in $s^{-1}(\pi)$.

For convenience, we say a decreasing binary plane tree $\tau$ is lonely if no other decreasing binary plane tree has the same postorder as $\tau$. The normalized lonely trees on $n$ vertices are precisely the decreasing binary plane trees whose postorders are uniquely sorted permutations in $S_n$. This provides our third combinatorial interpretation of Lassalle’s sequence. More precisely, there are no normalized lonely trees with an even number of vertices, while there are precisely $A_{k+1}$ normalized lonely trees on $2k + 1$ vertices.

It is possible to describe lonely trees without referring to permutations or the stack-sorting map. The description is recursive and is essentially equivalent to the recursive construction of uniquely sorted permutations discussed at the end of the previous section. The proof that the construction has the desired properties amounts to combining the recursive construction of uniquely sorted permutations with the above bijection between uniquely sorted permutations and normalized lonely trees.

Given a decreasing binary plane tree $\tau$, we call the vertex that is read first in $S(\tau)$ the leftmost vertex of $\tau$. Suppose $a$ is a vertex in $\tau$ with two children. Let $b$ be the left child of $a$. By the leftmost nephew of $b$, we mean the leftmost vertex in the right subtree of $a$. This is also the vertex that is read immediately after $a$ in $S(\tau)$. Alternatively, it is the vertex read immediately after $b$ in $P(\tau)$. The leftmost vertex of the tree in Figure 7 has label 2. In that tree, the leftmost nephew of the vertex labeled 2 is the vertex labeled 3. We say a decreasing binary plane tree is full if every vertex has either 0 or 2 children.

Our alternative descriptions of lonely trees are as follows. Of course, a single vertex with a positive integer label is a lonely tree. A lonely tree with more than one vertex is a decreasing binary plane tree that consists of a root whose left and right subtrees are themselves lonely and that has the additional property that the left child of the root has a label that is larger than the label of its leftmost nephew. Alternatively, a lonely tree is simply a full decreasing binary plane tree in which every vertex that is a left child has a label that is larger than the label of its leftmost nephew.
5. The Main Bijection

Now that we have described our three combinatorial interpretations of Lassalle’s sequence and shown that they are in bijection with each other, we can move on to actually proving that these objects are counted by Lassalle’s sequence. This will follow as a consequence of the following more general theorem. First, we need a short lemma and some observations about valid hook configurations.

Lemma 5.1. Let $\alpha$ be a valid hook configuration of a permutation $\pi$. Consider the coloring of the plot of $\pi$ induced by $\alpha$. If $i_1 < \cdots < i_r$ are indices such that $(i_1, \pi_{i_1}), \ldots, (i_r, \pi_{i_r})$ are all given the same color, then $\pi_{i_1} < \cdots < \pi_{i_r}$.

Proof. It suffices to prove the lemma in the case $r = 2$. Assume instead that $\pi_{i_1} > \pi_{i_2}$. There must be a descent $d$ of $\pi$ such that $i_1 \leq d < i_2$ and $\pi_d > \pi_{i_2}$. Assume that $d$ is chosen maximally subject to these conditions. There must be a hook whose southwest endpoint is $(d, \pi_d)$. The point $(i_2, \pi_{i_2})$ lies below this hook while $(i_1, \pi_{i_1})$ does not. This means that $(i_1, \pi_{i_1})$ and $(i_2, \pi_{i_2})$ cannot have the same color, which contradicts our hypothesis. \qed

Suppose $\alpha$ is a valid hook configuration of a permutation $\pi$. There is a canonical decomposition of $\alpha$ that makes use of what we call the top hook. This is simply the hook whose northeast endpoint is farthest to the north. For example, the top hook in Figure 4 is the hook with southwest endpoint $(7, 11)$ and northeast endpoint $(15, 15)$. The top hook separates $\alpha$ into two smaller valid hook configurations. We call these the sheltered and unsheltered pieces of $\alpha$. Specifically, the sheltered piece consists of all of the points and hooks that lie strictly underneath the top hook. The unsheltered piece consists of all of the other points and hooks except for the northeast endpoint of the top hook and the top hook itself. Let $\alpha_S$ denote the set of entries $\pi_i$ such that $(i, \pi_i)$ is in the sheltered piece. Define $\alpha_U$ similarly for the unsheltered piece. In the example depicted in Figure 4 we have

$$\alpha_S = \{1, 4, 6, 8, 12, 13, 14\} \quad \text{and} \quad \alpha_U = \{2, 3, 5, 7, 9, 10, 11, 16\}.$$ 

This decomposition of valid hook configurations will be crucial in our proof of Theorem 5.1 below.

We are now ready to define our main bijection. Recall from Section 2 that $\tilde{P}(n)$ is the set of ordered pairs $(\rho, r)$, where $\rho$ is a set partition of $\{0, \ldots, n - 1\}$ whose crossing graph $G(\rho)$ is connected and $r$ is an acyclic orientation of $G(\rho)$ whose only source is the block containing 0. Let

$$\mathcal{H}_{\{0,2\}}(S_n) = \bigcup_{\pi \in S_n} \mathcal{H}_{\{0,2\}}(\pi)$$

denote the set of all normalized valid hook configurations on $n$ points.

We define a map

$$\Phi : \mathcal{H}_{\{0,2\}}(S_n) \rightarrow \tilde{P}(n + 1)$$

as follows. Let $\alpha$ be a valid hook configuration of a permutation $\pi = \pi_1 \cdots \pi_n \in S_n$. Suppose $\alpha$ has $k$ hooks (equivalently, $\pi$ has $k$ descents). As discussed in Section 3, $\alpha$ induces a coloring of the plot of $\pi$. The $k$ northeast endpoints of hooks in $\alpha$ remain uncolored in this coloring. However, for the purpose of this proof, let us actually color the northeast endpoints as well. We do this by giving the northeast endpoint of a hook the same color as that hook. We now obtain a coloring of the elements of $\{1, \ldots, n\}$ by giving $\pi_i$ the same color as the point $(i, \pi_i)$ for each $i$. For example, $\pi_1$ must be blue (sky-colored) because the point $(1, \pi_1)$ must be blue. Let us also color the number 0
blue. This yields a partition $\rho$ of $\{0, 1, \ldots, n\}$ into color classes. For each block $B$ of this partition, let $\hat{B} = \{i : \pi_i \in B, 1 \leq i \leq n\}$.

We need to choose an acyclic orientation $r$ of $G(\rho)$. To do this, suppose we have an edge of $G(\rho)$ with endpoints $B$ and $B'$. In other words, $B$ and $B'$ are blocks of $\rho$ that form a crossing. If $\min \hat{B} < \min \hat{B}'$, orient this edge from $B$ to $B'$. If $\min \hat{B}' < \min \hat{B}$, orient the edge from $B'$ to $B$. This defines the acyclic orientation $r$, so put $\Phi(\alpha) = (\rho, r)$.

**Example 5.1.** Let $\alpha$ be the valid hook configuration in Figure 4. To obtain the pair $\Phi(\alpha) = (\rho, r)$, begin by coloring the diagram as in Figure 5. For each hook $H$, color the northeast endpoint of $H$ the same color as $H$. This yields the diagram shown on the left in Figure 9. The blocks in $\rho$ are the heights of the points in each color class, where we put 0 in the blue block. Specifically, the blocks of $\rho$ are

$$B_{\text{blue}} = \{0, 2, 7, 16\}, \quad B_{\text{green}} = \{3, 5, 9, 10, 11\}, \quad B_{\text{red}} = \{4, 8, 14, 15\}, \quad B_{\text{brown}} = \{1, 6, 12, 13\}.$$  

We also have

$$\hat{B}_{\text{blue}} = \{1, 2, 16\}, \quad \hat{B}_{\text{green}} = \{3, 4, 5, 6, 7\}, \quad \hat{B}_{\text{red}} = \{8, 9, 14, 15\}, \quad \hat{B}_{\text{brown}} = \{10, 11, 12, 13\}.$$  

In this example, every pair of blocks in $\rho$ forms a crossing, so $G(\rho)$ is a complete graph on 4 vertices. In the acyclic orientation $r$, depicted in the right image of Figure 9, we orient the edge connecting $B_{\text{blue}}$ and $B_{\text{green}}$ away from $B_{\text{blue}}$ since $\min \hat{B}_{\text{blue}} = 1 < 3 = \min \hat{B}_{\text{green}}$. We orient an edge from $B_{\text{red}}$ to $B_{\text{brown}}$ since $\min \hat{B}_{\text{red}} = 8 < 10 = \min \hat{B}_{\text{brown}}$. The other edges are oriented similarly.

**Theorem 5.1.** The map $\Phi : \mathcal{H}_{\{0,2\}}(S_n) \to \mathcal{P}^c(n+1)$ defined above is a bijection.

**Proof.** We first check that the pair $(\rho, r)$ is in fact an element of $\mathcal{P}^c(n+1)$. It is clear that the orientation $r$ is acyclic. Let $B_{\text{blue}}$ denote the blue block of $\rho$. We know that 0 $\in B_{\text{blue}}$ and $\min \hat{B}_{\text{blue}} = 1$. This means that the block containing 0 is a source; we must show that it is the only source. This will also imply that $G(\rho)$ is connected since each connected component contains a source for the acyclic orientation (this is a standard fact about acyclic orientations of graphs).
For each color $c$ that we use, let $B_c$ be the block of $\rho$ with the color $c$. If $c$ is not blue, let $H_c$ be the hook with the color $c$. Choose a non-blue color $c_1$. Our goal is to find a block $B^*$ of $\rho$ such that $B_{c_1}$ and $B^*$ form a crossing and $\min \hat{B}^* < \min \hat{B}_{c_1}$. Let $(u_2, \pi_{u_2})$ be the southwest endpoint of $H_{c_1}$, and let $c_2$ be the color of $(u_2, \pi_{u_2})$. If $c_2$ is not blue, then let $(u_3, \pi_{u_3})$ be the southwest endpoint of $H_2$, and let $c_3$ be the color of $(u_3, \pi_{u_3})$. Continue in this fashion until eventually defining a point $(u_m, \pi_{u_m})$ whose color $c_m$ is blue. It follows from the properties of valid hook configurations that $\min \hat{B}_{c_m} < \min \hat{B}_{c_{m-1}} < \cdots < \min \hat{B}_{c_1}$. Therefore, it suffices to show that there is some $i \in \{2, \ldots, m\}$ such that $B_{c_1}$ and $B_{c_i}$ form a crossing (we can then put $B^* = B_{c_i}$).

For $i \in \{2, \ldots, m\}$, the point $(u_i + 1, \pi_{u_i+1})$ is the color $c_{i-1}$. This means that $\pi_{u_i+1} \in B_{c_{i-1}}$, so $\min B_{c_{i-1}} \leq \pi_{u_i+1}$. Furthermore, $\max B_{c_{i-1}}$ is the height of the northeast endpoint of $H_{c_{i-1}}$. Because $(u_i, \pi_{u_i})$ is the southwest endpoint of $H_{c_{i-1}}$, we have $\pi_{u_i} < \max B_{c_{i-1}}$. We also know that $(u_i, \pi_{u_i})$ is a descent top of $\pi$, so $\pi_{u_i+1} < \pi_{u_i}$. Combining these inequalities yields $\min B_{c_{i-1}} < \pi_{u_i} < \max B_{c_{i-1}}$. This is important because $\pi_{u_i} \in B_{c_i}$. Suppose by way of contradiction that none of the blocks $B_{c_2}, \ldots, B_{c_m}$ form a crossing with $B_{c_1}$. Because $B_{c_2}$ does not form a crossing with $B_{c_1}$, we must have $\min B_{c_1} < \min B_{c_2} < \pi_{u_3} < \max B_{c_1}$. Because $B_{c_3}$ does not form a crossing with $B_{c_1}$, we must have $\min B_{c_1} < \min B_{c_3} < \pi_{u_4} < \max B_{c_3} < \max B_{c_1}$. Continuing in this manner, we eventually find that $\min B_{c_1} < \pi_{u_m} < \max B_{c_1}$. However, $B_{c_m}$ is the blue block, so $0, \pi_{u_m} \in B_{c_m}$. We have $0 < \min B_{c_1} < \pi_{u_m} < \max B_{c_1}$, which means that $B_{c_m}$ does form a crossing with $B_{c_1}$ after all, a contradiction.

It remains to show that $\Phi$ is a bijection. To do so, we exhibit its inverse. Suppose we are given a pair $(\rho, r) \in \overline{P}(n+1)$. We want to reobtain the valid hook configuration $\alpha$ with $\Phi(\alpha) = (\rho, r)$. We can assume that $\rho$ has more than one block; otherwise, $\alpha$ is the valid hook configuration consisting of the identity permutation and no hooks. Here is where we make use of the “top hook decomposition” discussed before the definition of the map $\Phi$. If we can determine the southwest and northeast endpoints of the top hook of $\alpha$ along with the sets $\alpha_S$ and $\alpha_U$, then we can proceed inductively to reconstruct all of $\alpha$. We will see that these endpoints and sets are completely determined by $(\rho, r)$, from which it will follow that there is a unique $\alpha \in \mathcal{H}_{|\{0,2\}|}(S_n)$ with $\Phi(\alpha) = (\rho, r)$.

Begin by coloring the elements of $\{0, 1, \ldots, n\}$ so that two elements have the same color if and only if they are in the same block of $\rho$. Make sure to use the color blue to color the elements of the block containing 0. Let $c$ be the largest element of $\{0, 1, \ldots, n\}$ that is not blue. Because $c+1, \ldots, n$ are all blue, we need the points $(c+1, \pi_{c+1}), \ldots, (n, \pi_n)$ to see the sky when they look up. This forces us to put $\pi_j = j$ for all $j \in \{c, \ldots, n\}$ (otherwise, there would be a hook preventing one of these points from seeing the sky). The northeast endpoint of the top hook of $\alpha$ must be $(c, c)$.

Now, the acyclic orientation $r$ defines a partial order $\preceq$ on the blocks of $\rho$, where we declare that $B \preceq B'$ if and only if there is a directed path from $B$ to $B'$ in $G(\rho)$ or $B = B'$. Let $C$ be the block of $\rho$ containing $c$. One can show that $\alpha_S$ must be the union of all of the blocks $D$ satisfying $C \preceq D$. We then know that $\alpha_U$ must be $\{1, \ldots, n\} \setminus (\alpha_S \cup \{c\})$. Observe that the numbers $c+1, \ldots, n$ are elements of $\alpha_U$; the next-largest entry of $\alpha_U$ must be the height of the southwest endpoint of the top hook of $\alpha$. More precisely, this southwest endpoint is $(a, b)$, where $a = |\alpha_U| - (n - c)$ and $b = \max(\alpha_U \setminus \{c+1, \ldots, n\})$.

Using the notation of Section 2, we now deduce from (2) that $-k_{n+1}(-1)$ is the total number of normalized valid hook configurations on $n$ points. In the previous two sections, we found bijective correspondences among uniquely sorted permutations in $S_{2k+1}$, normalized valid hook configurations on $2k+1$ points with $k$ hooks, and normalized lonely trees on $2k+1$ vertices. We can now
finally show that these objects are counted by Lassalle’s sequence. Let $H^k_{(0,2)}(S_n)$ be the set of normalized valid hook configurations on $n$ points with $h$ hooks.

**Corollary 5.1.** When $n = 2k + 1$, the map $\Phi$ from Theorem 5.1 restricts to a bijection 
\[ \Phi' : H^k_{(0,2)}(S_{2k+1}) \rightarrow \tilde{M}^c(2k + 2). \]
In particular, 
\[ |H^k_{(0,2)}(S_{2k+1})| = A_{k+1}. \]

**Proof.** Let $\alpha$ be a normalized valid hook configuration on $2k + 1$ points with $h$ hooks, and put $\Phi(\alpha) = (\rho, r)$. The valid composition $(q_0, \ldots, q_h)$ induced from $\alpha$ is a composition of $2k + 1 - h$ into $h + 1$ parts. It follows from the definition of $\Phi$ that $\rho$ has $h + 1$ blocks, where the blocks are of sizes $q_0 + 1, \ldots, q_h + 1$ in some order. We find that $\Phi(\alpha) \in \tilde{M}^c(2k + 2)$ (that is, $\rho$ is a matching) if and only if $q_i = 1$ for all $i$. This occurs if and only if $h = k$. This proves the first statement of the corollary. The second statement follows from the first and from (3). \(\square\)

In the following additional corollary to Theorem 5.1, we adopt a notational convention from [10]. Given a sequence $(u_n)_{n \geq 1}$ and a set partition $\rho$, write 
\[ u_\rho = \prod_{B \in \rho} u_{|B|}. \]
For example, if $\rho = \{(0, 3), \{1, 6, 7, 8\}, \{2, 4, 5\}\}$, then $C_{\rho - 1} = C_{2-1}C_{4-1}C_{3-1} = 1 \cdot 5 \cdot 2 = 10$. Recall the notation from (1).

**Corollary 5.2.** We have 
\[ \sum_{\rho \in \mathcal{P}^c(n+1)} C_{\rho - 1}T_{G(\rho)}(1, 0) = n!. \]

**Proof.** Let $\Phi$ be the bijection from Theorem 5.1. Given a normalized valid hook configuration $\alpha$ on $n$ points, let $\Psi(\alpha)$ be the set partition which is the first coordinate of $\Phi(\alpha)$. In other words, if $\Phi(\alpha) = (\rho, r)$, then $\Psi(\alpha) = \rho$. We know from Theorem 3.1 and the definition of $\Phi$ that a valid hook configuration $\alpha$ of a permutation $\pi \in S_n$ gives rise to $C_{\Psi(\alpha) - 1}$ preimages of $\pi$ under the stack-sorting map $s$. Invoking Theorem 5.1 we find that 
\[ n! = \sum_{\pi \in S_n} |s^{-1}(\pi)| = \sum_{\alpha \in H_{(0,2)}(S_n)} C_{\Psi(\alpha) - 1} = \sum_{(\rho, r) \in \mathcal{P}^c(n+1)} C_{\rho - 1} = \sum_{\rho \in \mathcal{P}^c(n+1)} C_{\rho - 1}T_{G(\rho)}(1, 0). \]
We have used the fact, which we mentioned in the paragraph following Theorem 2.1, that $T_{G(\rho)}(1, 0)$ is the number of acyclic orientations $r$ such that $(\rho, r) \in \mathcal{P}^c(n+1)$. \(\square\)

Let $A_{k+1}(\ell)$ be the number of uniquely sorted permutations in $S_{2k+1}$ whose first entry is $\ell$. Corollary 5.1 provides us with a means for proving the following somewhat surprising result concerning this refinement of the Lassalle numbers.

**Theorem 5.2.** For each nonnegative integer $k$, the sequence $(A_{k+1}(\ell))_{\ell=1}^{2k+1}$ is symmetric.

**Proof.** Let $Q_{k+1}(\ell)$ be the set of ordered pairs $(\rho, r) \in \tilde{M}^c(2k + 2)$ in which 0 and $\ell$ are in the same block of $\rho$. Note that $|Q_{k+1}(\ell)| = A_{k+1}(\ell)$. This follows from Corollary 5.1 and the correspondence between uniquely sorted permutations in $S_{2k+1}$ and normalized valid hook configurations on $2k + 1$ points with $k$ hooks. Now, suppose we are given $(\rho, r) \in Q_{k+1}(\ell)$. We can picture the matching
ρ as a collection of arches as in the left part of Figure 10. Reflect all of the points other than 0 through a vertical line passing through k + 1. When performing this reflection, do not break any of the arches. This process, which is illustrated in Figure 10, produces a new matching ρ’.

The crossing graphs G(ρ) and G(ρ’) are naturally isomorphic. Indeed, each block (vertex) {a, b} in G(ρ) that does not contain 0 corresponds to the block {2k + 2 − a, 2k + 2 − b} in G(ρ’). The block {0, ℓ} corresponds to {0, 2k + 2 − ℓ}. Using this isomorphism, we transfer the acyclic orientation r on G(ρ) to an acyclic orientation r’ on G(ρ’) in the obvious way. This correspondence between pairs (ρ, r) ∈ Qk+1(ℓ) and (ρ’, r’) ∈ Qk+1(2k + 2 − ℓ) proves the desired symmetry. □

Numerical evidence suggests that (Ak+1(ℓ))2k+1 is log-concave (and therefore unimodal) for each nonnegative integer k. We state this as a conjecture in Section 6, where we collect several other suggestions for future work.

Each uniquely sorted permutation π ∈ S2k+1 has a unique valid hook configuration α, and α has a top hook (assuming k ≥ 1). As discussed above, this hook separates α into a sheltered piece and an unsheltered piece. Suppose (j, πj) is the leftmost point in the sheltered piece. We call πj the hotspot of π. If we let Φ′(α) = ρ, where Φ’ is the map from Corollary 5.1, then the hotspot of π is also the entry in the same block as 2k + 1 in ρ. We saw in Section 3 that π corresponds to a normalized lonely tree T. Let a be the left child of the root of T. In the notation introduced at the end of Section 3, the hotspot of π is the label of the leftmost neighbor of a. The following theorem shows an interesting relationship between the first entry and the hotspot of a uniquely sorted permutation and also provides an alternative method for studying the numbers Ak+1(ℓ) from Theorem 5.2.

**Theorem 5.3.** If k ≥ 1, then there are exactly Ak+1(ℓ) uniquely sorted permutations in S2k+1 with hotspot ℓ − 1.

**Proof.** Given a matching ρ of the set {0, 1, ..., 2k + 1}, let parρ(i) denote the partner of i in ρ (that is, {i, parρ(i)} is a block of ρ). In addition, let ˜ρ be the matching obtained by reflecting ρ through a vertical line passing through the point k + 1. More precisely, if the blocks of ρ are {a1, b1}, ..., {ak+1, bk+1}, then the blocks of ˜ρ are {2k + 1 − a1, 2k + 1 − b1}, ..., {2k + 1 − ak+1, 2k + 1 − bk+1}. It is easy to see that G(ρ) and G(˜ρ) are isomorphic, so T̃G(ρ)(1, 0) = T̃G(˜ρ)(1, 0).

Furthermore, parρ(0) = 2k + 1 − parρ(2k + 1).

We know that there is a bijective correspondence between uniquely sorted permutations π ∈ S2k+1 and ordered pairs (ρ, r) ∈ M̃c(2k + 2). Under this correspondence, the first entry of π is parρ(0) while the hotspot of π is parρ(2k + 1). Therefore, our goal is to show that

\[
A_{k+1}(\ell) = \left| \left\{ (\rho, r) \in M\overline{c}(2k + 2) : \text{par}_\rho(2k + 1) = \ell - 1 \right\} \right|.
\]

It follows from Theorem 2.1 and Theorem 5.2 that

\[
A_{k+1}(\ell) = A_{k+1}(2k + 2 - \ell) = \left| \left\{ (\rho, r) \in M\overline{c}(2k + 2) : \text{par}_\rho(0) = 2k + 2 - \ell \right\} \right|.
\]
\[
T_{G(\rho)}(1, 0) = \sum_{\rho \in M_c(2k+2)} \text{par}_\rho(0) = 2k + 2 - \ell T_G(\rho)(1, 0),
\]

and this is precisely the right-hand side of (4). \hfill \square

6. New Recurrences

We know from (2) and Theorem 5.1 that the total number of normalized valid hook configurations on \(n\) points is \(-k_{n+1}(-1)\). We know from (3) and Corollary 5.1 that the total number of normalized valid hook configurations on \(2k+1\) points with \(k\) hooks is \(A_{k+1}\). In this section, we study the combinatorial properties of valid hook configurations in order to derive new recurrence relations for these numbers. These recurrences keep track of a permutation statistic that we call the tail length.

In what follows, the normalization of a permutation \(\pi = \pi_1 \cdots \pi_n \in S_n\) is the unique permutation in \(S_n\) that is order isomorphic to \(\pi\). For example, the normalization of 26589 is 13245.

**Definition 6.1.** The tail length of a permutation \(\pi = \pi_1 \cdots \pi_n \in S_n\), denoted \(tl(\pi)\), is the smallest nonnegative integer \(i\) such that \(\pi_n - i \neq n - i\). The tail length of an arbitrary permutation is the tail length of its normalization.

For example, the permutation 31524678 has tail length 3 while the permutation 26589 has tail length 2. For an indication of the relevance of this statistic for our purposes, observe that a sorted permutation (equivalently, a permutation that has a valid hook configuration) must have a positive tail length. Heuristically, we should expect the fertility of a permutation in \(S_n\) with a large tail length to be larger than the fertility of a permutation in \(S_n\) with a small tail length.

Let
\[
D_m(n) = \sum_{\pi \in S_n\atop tl(\pi) = m} |H_{\{0,2\}}(\pi)|
\]
be the total number of valid hook configurations whose underlying permutations are elements of \(S_n\) with tail length \(m\). Let
\[
D_{\geq m}(n) = \sum_{\ell = m}^{n} D_{\ell}(n).
\]

In particular, \(D_{\geq 0}(n) = -k_{n+1}(-1)\) is the total number of normalized valid hook configurations on \(n\) points.

**Theorem 6.1.** The numbers \(D_m(n)\) and \(D_{\geq m}(n)\) defined above satisfy the recurrence
\[
D_m(n) = \sum_{j=1}^{m} \sum_{i=1}^{n-m-1} \binom{n-m-1}{i-1} D_{\geq j}(i + j - 1) D_{\geq m-j}(n - j - i)
\]
for \(0 \leq m < n\). The initial conditions are given by \(D_0(n) = 1\).

**Proof.** The initial condition \(D_0(n) = 1\) is the statement that the identity permutation has a unique valid hook configuration (the one with no hooks). The recurrence is obvious when \(0 = m < n\) since a permutation in \(S_n\) with tail length 0 has no valid hook configurations.

Now suppose \(0 < m < n\). To produce a valid hook configuration of a permutation \(\pi = \pi_1 \cdots \pi_n \in S_n\) with \(tl(\pi) = m\), begin by choosing the index \(i \in \{1, \ldots, n - m - 1\}\) such that \(\pi_i = n - m\).
Figure 11. Two small valid hook configurations combine into a larger one as described in the proof of Theorem 6.1.

Note that $i$ must be a descent of this permutation. This implies that there must be a hook $H$ with southwest endpoint $(i, n - m)$. The northeast endpoint of this hook is of the form $(n - j, n - j)$ for some $j \in \{0, \ldots, m - 1\}$. There are $\binom{n - m - 1}{i - 1}$ choices for the entries in the set $\{\pi_1, \ldots, \pi_{i-1}\}$. Note that $\pi_{i+1} \cdots \pi_{n-j}$ will be a (not necessarily normalized) permutation of length $n - j - i$ with tail length at least $m - j$. Choosing the part of the valid hook configuration that lies below $H$ amounts to choosing $\pi_{i+1} \cdots \pi_{n-j}$ and choosing a valid hook configuration on this permutation. There are $D_{\geq m-j}(n - j - i)$ ways to do this. Similarly, there are $D_{\geq j}(i + j - 1)$ ways to choose the hooks on the points that are not $(n - j, n - j)$ and do not lie below $H$.

Example 6.1. Figure 11 shows the construction of a valid hook configuration of a permutation $\pi \in S_{12}$ with tail length 4. Here, we have chosen $i = 5$, $j = 3$, and $\{\pi_1, \pi_2, \pi_3, \pi_4\} = \{1, 2, 4, 6\}$. We have also chosen two small valid hook configurations, which combine to form the large valid hook configuration on the right. The points coming from the first small valid hook configuration are represented as squares while the points coming from the second one are represented as triangles. Note that the point $(10, 10)$ is represented by a disc because it does not come from either of these small valid hook configurations (this is because it is the northeast endpoint of the hook $H$).

Let $E_m(n)$ be the number of uniquely sorted permutations in $S_n$ with tail length $m$. Of course, $E_m(n) = 0$ when $n$ is even since there are no uniquely sorted permutations of even length. Let

$$E_{\geq m}(n) = \sum_{\ell=m}^{n} E_{\ell}(n).$$

In particular, $E_{\geq 0}(2k + 1) = A_{k+1}$ is the total number of uniquely sorted permutations in $S_{2k+1}$.

Theorem 6.2. The numbers $E_m(n)$ and $E_{\geq m}(n)$ defined above satisfy the recurrence

$$E_m(n) = \sum_{j=1}^{m} \sum_{i=1}^{n-m-1} \binom{n-m-1}{i-1} E_{\geq j}(i + j - 1) E_{\geq m-j}(n - j - i)$$

for $0 \leq m < n$. The initial conditions are given by $E_n(n) = \delta_{n1}$, where $\delta$ denotes the Kronecker delta.

Proof. Suppose $\alpha$ is a valid hook configuration of a permutation $\pi$. It follows from the discussion preceding Proposition 3.1 that $\pi$ is uniquely sorted if and only if the coloring of the plot of $\pi$ induced by $\alpha$ does not give any two distinct points the same color. The proof of Theorem 6.2 is now virtually identical to that of Theorem 6.1. Specifically, we start to construct the valid hook
configuration of a uniquely sorted permutation $\pi \in S_n$ with $tl(\pi) = m$ by first choosing the index $i$ such that $\pi_i = n - m$. As before, $(i, n-m)$ must be the southwest endpoint of a hook $H$. We choose $j$ such that $(n - j, n - j)$ is the northeast endpoint of $H$. We then choose the set $\{\pi_1, \ldots, \pi_{i-1}\}$ in $\binom{n-m-1}{i-1}$ ways. Finally, we choose the part of the valid hook configuration lying below $H$ in $E_{\geq m-j}(n-j-i)$ ways and choose the part not lying below $H$ in $E_{\geq j}(i+j-1)$ ways. 

7. Future Work

Through different ways of phrasing the main results of our paper, we obtain several possible avenues for potential generalizations. For example, it would be interesting to enumerate doubly sorted permutations, which are simply permutations with fertility 2. Arguing as in Section 2, one can show that there are no doubly sorted permutations of odd length. Letting $B_m$ denote the number of doubly sorted permutations of length $2m$, we have $B_1 = 1$, $B_2 = 3$, $B_3 = 31$, and $B_4 = 1186$. This sequence appears to be new.

A permutation in $S_n$ is uniquely sorted if and only if it is sorted and has $\frac{n-1}{2}$ descents. From this point of view, it would be interesting to count sorted permutations in $S_n$ with exactly $k$ descents, where $k$ could be a function of $n$. For example, we could ask how many sorted permutations in $S_n$ have exactly $\frac{n-2}{2}$ descents.

Of course, uniquely sorted permutations in $S_n$ are in bijection with permutations $\pi \in S_n$ such that $s(\pi)$ is uniquely sorted. As mentioned in the previous paragraph, $s(\pi)$ is uniquely sorted if and only if $s(\pi)$ has exactly $\frac{n-1}{2}$ descents. This leads us to ask for the total number of permutations $\pi \in S_n$ such that $s(\pi)$ has exactly $k$ descents. Again, $k$ could be a function of $n$ here.

We have seen that every uniquely sorted permutation has exactly one valid hook configuration. It could be interesting to count the total number of permutations in $S_n$ that have exactly one valid hook configuration.

Combining (2) and Theorem 5.1 allows us to see that $|H_{\{0,2\}}(S_n)| = -k_{n+1}(-1)$. Suppose we choose some interesting set of permutations $\mathcal{U}$. For example, $\mathcal{U}$ might be the set of all permutations that avoid the pattern 321. Let $\mathcal{U}_n = \mathcal{U} \cap S_n$. It would be worthwhile to see if there are nice expressions for $|H_{\{0,2\}}(\mathcal{U}_n)|$, the total number of valid hook configurations of permutations in $\mathcal{U}_n$.

In Theorem 5.2, we saw that the sequence $(A_{k+1}(\ell))_{\ell=1}^{2k+1}$ is symmetric for each nonnegative integer $k$. Recall that a sequence $a_1, \ldots, a_m$ is called unimodal if there exists $j \in \{1, \ldots, m\}$ such that $a_1 \leq \cdots \leq a_{j-1} \leq a_j \geq a_{j+1} \geq \cdots \geq a_m$ and is called log-concave if $a_j^2 \geq a_{j-1}a_{j+1}$ for all $j \in \{2, \ldots, m-1\}$. It is well-known that a log-concave sequence of nonnegative real numbers is unimodal.

**Conjecture 7.1.** For each nonnegative integer $k$, the sequence $(A_{k+1}(\ell))_{\ell=1}^{2k+1}$ is log-concave.

Even if Conjecture 7.1 is too difficult to prove, it would still be of great interest to prove the weaker claim that these sequences are unimodal. We have verified Conjecture 7.1 for $0 \leq k \leq 4$.

When defining the bijection $\Phi$ that appears in Theorem 5.1, we described how to obtain a partition $\rho = \{B_1, \ldots, B_\ell\}$ of $\{0, 1, \ldots, n\}$ from a given valid hook configuration $\alpha$ of a permutation $\pi = \pi_1 \cdots \pi_n \in S_n$. This is done by first coloring the points in the plot of $\pi$ and then coloring the number $\pi_i$ the same color as the point $(i, \pi_i)$. After coloring 0 blue, we obtain a partition of $\{0, 1, \ldots, n\}$ into color classes. In the proof of Theorem 5.1, we showed that this partition is
connected (that is, its crossing graph is connected). We can obtain another set partition \( \hat{\rho} = \{ \hat{B}_1, \ldots, \hat{B}_\ell \} \) from \( \alpha \). To do this, color the points \((i, \pi_i)\) as before, but this time, color a number \(i\) the same color as \((i, \pi_i)\). This will partition \( \{1, \ldots, n\} \) into color classes. It follows from the rules defining valid hook configurations that \( \hat{\rho} \) is a noncrossing partition (that is, its crossing graph has no edges). It would be interesting to investigate possible connections between the partitions \( \rho \) and \( \hat{\rho} \) that are obtained from the same valid hook configuration \( \alpha \). We could also study the noncrossing partitions arising in this way in their own right.

As mentioned in the introduction and in Section 3, the valid hook configurations considered in this paper are actually special types of more general valid hook configurations studied in [7]. It is possible that some of the results from this paper generalize to other classes of valid hook configurations. In particular, it would be interesting to find analogues of the map \( \Phi \) from Theorem 5.1 in the context of other types of valid hook configurations.

Finally, we mention the possibility of defining valid hook configurations on words. It could certainly be possible to generalize the results in this paper and in [7] and [8] to the setting of words. In particular, valid hook configurations on words might provide a means by which to study a suitably-defined generalization of the stack-sorting algorithm.

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