The parallel composition of processes

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Abstract. We suggest that the canonical parallel operation of processes is composition in a well-supported compact closed category of spans of reflexive graphs. We present the parallel operations of classical process algebras as derived operations arising from monoid objects in such a category, representing the fact that they are protocols based on an underlying broadcast communication.

1 Introduction

The algebraic structure of sequential operations on processes has been studied since the beginning of computer science, with recent important contributions being [3, 6]. Parallel operations have been studied with less success, with a variety of different process algebras arising and no real consensus on the basic operations. In this note we would like to argue that the well-supported compact closed category (wscc) of spans of reflexive graphs introduced in [10] is in fact a canonical algebra for parallel composition. We will present the particular algebras introduced by Milner [16], Hoare [7] and others as derived operations of the wscc structure and additional commutative monoid objects in Span(RGraph) (generalizing Winskell’s synchronization algebras [23]). One reason previous authors have considered these derived operations is their desire for a single point of observation of a process, which has been confused with the quite different idea of interleaving semantics. Another reason is that conventional process algebras assume a form of broadcast communication between processes, exactly achieved by the operations of the monoid object.

Our suggestion is coherent with remarks made by Abramsky in [1] which we quote here. He makes these criticisms, namely that in traditional process algebras

(i) “interaction becomes extrinsic: we must add some additional structure, typically a ‘synchronization algebra’ on the labels, which implicitly refers to some external agency for matching up labels and generating communication events, rather than finding the meaning of interaction in the structure we already have.”

(extra monoid object)

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(ii) “interaction becomes ad hoc: because it is an ‘invented’ additional structure, many possibilities arise, and it is hard to identify any as canonical” (non-canoncity)

(iii) “interaction becomes global: using names to match up communications implies some large space in which potential communications ‘swim’” (broadcast).

We agree with these three points, while firmly disagreeing with Abramsky’s acceptance in that paper that (“extensional”) behaviours should be described rather than (“intensional”) systems. It is our contention that there should be a common algebra for systems and behaviours, compositionality being the existence of a morphism (actual behaviour) between the two. Unfortunately this is lacking even in classical treatments of sequential processes - Kleene expressions describe behaviour but not automata. In [19] the algebra of this paper applied to cospans rather than spans to provide an algebra of automata and behaviours for which so that the Kleene theorem is a statement of compositionality. In the case of process algebras there is no notion which corresponds to the systems, only the behaviour - but the behaviour of what? It will be clear from this paper that we are firmly in favour of an automata semantics of process algebras, rather than the often incompatible, but commonly accepted, structural operational semantics. Of course, from any algebra of automata one may produce a rewrite system, as we have done for our algebra in [14].

The abstract algebra described in this paper was introduced in [4, 21] and developed in the context of computer science in [8–13]. The algebra has also found application in quantum field theory [15] and quantum experiments [2].

We describe the contents of the paper in more detail. In section 2 we give the abstract result that a pair of objects \( X, Y \) in a symmetric monoidal category, \( X \) with a comonoid structure and \( Y \) with a monoid structure induce a monoid structure on \( \text{Hom}(X, Y) \).

In section 3 we introduce a simpler variant \( 2LTS \) of the category \( \text{Span}(\mathbf{RGraph}) \), whose arrows are two-sided labelled transition systems, by which we mean spans of reflexive graphs which are jointly monic on arcs - there is at most one arc between two states with a given (double) labelling. This has the same algebraic structure as \( \text{Span}(\mathbf{RGraph}) \), but is more convenient in making comparison with classical labelled transition systems. We describe briefly the algebraic structure.

In section 4 we note that the synchronization algebras of Winskel [23] are particular commutative semigroups in \( 2LTS \), and we show that the parallel composition of transition systems in [23] is exactly our construction of section 2. Notice that familiar examples of synchronization algebras are merely semigroups in the setting of [23], but considered in \( 2LTS \) they are actually monoids with the results that processes under the associated parallel operations are monoids, not merely semigroups. Notice that the usual interpretation that processes have many channels is seen in our analysis to be misleading - in fact processes communicate on a single channel or bus, the mediation with this bus is provided by the monoid object. This is made particularly clear by the geometry corresponding to the algebra. Naturally, on the bus there is interleaving but by no means are the
various processes acting in interleaving internally. A further technicality which
has lead to the confusion between the fact that processes interact through a bus
and interleaving semantics is the lack of appreciation of the reflexive graphs. In
[23] it is prohibited that the synchronization of two actions results in the null
action $\varepsilon$. This means that internal actions are always mirrored on the bus.

Further comments on process algebras by the second and third authors may
be found at [22].

2 Monoid objects

A commutative monoid object in a symmetric strict monoidal category $\mathbf{M}$ (with
tensor $\otimes$, identity of tensor $I$, and symmetry $tw$) consists of an object $M$ with
arrows

$$e : I \to M, \quad m : M \otimes M \to M$$

such that the following diagrams commute:

$$M \otimes M \otimes M \xrightarrow{1 \otimes m} M \otimes M$$

$$M \otimes M \xrightarrow{m} M$$

$$I \otimes M \xrightarrow{e \otimes 1} M \otimes M$$

$$M \otimes M \xrightarrow{m} M$$

$$M \otimes M \xrightarrow{1 \otimes e} M \otimes I$$

$$M \otimes M \xrightarrow{tw} M \otimes M$$

There is a geometrical representation of expressions and equations in a symmetric
monoidal category (see, for example, [9]). The arrows $m$ and $e$ are represented
respectively as:

$$\begin{array}{c}
M \xrightarrow{m} M \\
M \end{array} \quad \begin{array}{c}
\quad e \\
M
\end{array}$$

Then axiom (1) becomes

$$\begin{array}{c}
m \quad m \\
\quad = \\
\quad m \quad m
\end{array}$$
If the arrow $e$ is omitted in the above definition we get the notion of \textit{commutative semigroup object} in $\mathbf{M}$. Dually, a \textit{cocommutative comonoid object} in a monoidal category $\mathbf{M}$ is a monoid $M$ in the dual category $\mathbf{M}^{\text{op}}$. That is, an object $M$ with two arrows

$$c : M \to M \otimes M, \quad e' : M \to I$$

satisfying the axioms dual to the monoid axioms. There is similarly an obvious notion of cocommutative cosemigroup.

\textbf{Proposition 1.} If $X$ and $Y$ are respectively a cocommutative comonoid and a commutative monoid object of a symmetric monoidal category $\mathbf{M}$, then $\text{Hom}(X,Y)$ has an induced commutative monoid structure with multiplication being

$$\bullet : \text{Hom}(X,Y) \times \text{Hom}(X,Y) \to \text{Hom}(X,Y)$$

$$(A,B) \mapsto f \bullet g := m(A \otimes B)c$$

and identity

$$e = X \xrightarrow{e'} I \xrightarrow{e} Y.$$  

An analogous result holds for $X$ a cosemigroup, and $Y$ a semigroup.

\textbf{Proof.} By the associativity of $m$ and $c$, $\bullet$ is associative:

$$A \bullet (B \bullet C) = m\left( A \otimes (m(B \otimes C)c) \right) = m(1 \otimes m)(A \otimes B \otimes C)(1 \otimes c)c =$$

$$= m(m \otimes 1)(A \otimes B \otimes C)(c \otimes 1)c = m\left( (m(A \otimes B)c) \otimes C \right)c = (A \bullet B) \bullet C.$$

The identity axiom follows since

$$A \bullet e = m\left( A \otimes (e e') \right)c = m(1 \otimes e)(A \otimes 1)(1 \otimes e')c = 1 (A \otimes 1) 1 = A$$

$$e \bullet A = m\left( (ee') \otimes A \right)c = m(e \otimes 1)(1 \otimes A)(e' \otimes 1)c = 1 (1 \otimes A) 1 = A.$$

The commutative law follows since

$$A \bullet B = m\left( A \otimes B \right)c = m \cdot \text{tw} \cdot (A \otimes B) \cdot c = m \cdot (B \otimes A) \cdot \text{tw} \cdot c = m(B \otimes A)c = B \bullet A.$$

It is useful to visualize the operation:

\[
\begin{array}{ccc}
X & \xrightarrow{A \bullet B} & Y \\
& & \xrightarrow{c} \\
& & \xrightarrow{A} \\
& & \xrightarrow{B} \\
& & \xrightarrow{m} \\
\end{array}
\]
3 2LTS and its algebraic structure

3.1 Reflexive graphs

A graph $X$ consists of a set $X_0$ of vertices of $X$, a set $X_1$ of edges and two functions $d_0, d_1 : X_1 \rightarrow X_0$ (domain and codomain functions). A reflexive graph $X$ is a graph with a function $\varepsilon : X_0 \rightarrow X_1$ such that $d_0 \varepsilon = d_1 \varepsilon$. For every $x \in X_0$, $\varepsilon_x$ is the reflexive edge of $x$.

Let $X$ and $Y$ be two reflexive graphs. A morphism of reflexive graphs $\phi : X \rightarrow Y$ is a graph morphism such that $\phi(\varepsilon_x) = \varepsilon_{\phi(x)}$.

The product $X \times Y$ is the reflexive graph such that

$(X \times Y)_0 := X_0 \times Y_0$, \quad $(X \times Y)_1 := X_1 \times Y_1$, \quad $\varepsilon_{X \times Y} := \varepsilon_X \times \varepsilon_Y$.

We denote by $RGrph$ the category of reflexive graphs and morphisms between them.

3.2 The category of 2LTS

The name 2LTS comes from the fact that the arrows of 2LTS are “two-sided labelled transition systems”.

**Definition 1.** Given two sets $X, Y$ both containing the symbol $\varepsilon$, a two-sided transition system $A$ labelled in $X$ and $Y$ consists of a set $A_0$ (of states), and a subset $A_1$ of $A_0 \times X \times Y \times A_0$ (of transitions) containing $(a, \varepsilon, \varepsilon, a)$ for each $a \in A_0$.

It is convenient sometimes to write the transition $(a, x, y, a')$ as $(a, x/y, a')$ or even $a \xrightarrow{z/y} a'$.

Strictly speaking we regard two transition systems $A, B$ with labels in $X$ and $Y$ as the same if $A_0$ is bijective with $B_0$ and the bijection respects edges and their labelling.

We now define the category 2LTS.

**Definition 2.** The objects of 2LTS are sets containing the symbol $\varepsilon$, which we may think of as alphabets. Given objects $X, Y$ an arrow $A$ from $X$ to $Y$ is a two-sided transition system labelled in $X$ and $Y$. The composition $A \cdot B$ of $A : X \rightarrow Y$ and $B : Y \rightarrow Z$ is defined to be

$(A \cdot B)_0 = A_0 \times B_0,$

$(A \cdot B)_1 = \{(a, b, x/z, a', b') : \exists y \in Y \text{ such that } (a, x/y, a') \in A_0, (b, y/z, b') \in B_0\}.$

The identity arrow of $X$ has one state $*$ and transitions $\{(*, x/x, *) : x \in X\}.$
The category $2LTS$ bears a straightforward relationship with $\text{Span}(RGraph)$ - it is in fact a full subcategory of a quotient of $\text{Span}(RGraph)$, analogous to the fact that the category of relations is a quotient of $\text{Span}(Sets)$. However we have preferred here to define $2LTS$ explicitly. To see how an arrow of $2LTS$ may be considered a span of reflexive graphs one must first regard the objects as one vertex graphs, the alphabet being the set of edges, including $\varepsilon$ as the reflexive edge. Further given an arrow $A : X \to Y$ in $2LTS$ the two sets $A_1$ and $A_0$ form the arrows and edges of a graph; the two functions $d_0, d_1$ are defined by $d_0(a, x/y, b) = a$, $d_1(a, x/y, b) = b$. Finally the arrow $A : X \to Y$ of $2LTS$ yields a span of reflexive graphs

$$X \xleftarrow{\delta_0^A} A \xrightarrow{\delta_1^A} Y$$

(4)

defined by $\delta_0(a, x/y, b) = x$, $\delta_1(a, x/y, b) = y$. Composition in $2LTS$ is composition in $\text{Span}(RGraph)$, followed by the reflection of general spans into spans jointly monic on arcs.

**Examples** To see examples of two-sided labelled transition systems and their use in modelling concurrent systems, we refer to papers of the authors, beginning with [9].

### 3.3 Relations

We will now see that the category $\text{Rel}_*$ of relations between pointed sets is a subcategory of $2LTS$. Given a relation $\rho$ between two pointed sets $X$ and $Y$ (the points both denoted $\varepsilon$) with the property that $\varepsilon \rho \varepsilon$, we obtain an arrow $\bar{\rho} : X \to Y$ of $2LTS$ as follows: $\bar{\rho}_0 = \{\ast\}$, $\bar{\rho}_1 = \{(\ast, x/y, \ast); x \rho y\}$. It is immediately clear that composition of relations in $2LTS$ agrees with the usual composition of relations.

### 3.4 The well-supported compact closed structure of $2LTS$

Since $RGrph$ has finite limits, $RGrph$ is a monoidal category. The tensor product $\otimes$ of reflexive graphs is their product. Each object $X$ of $RGrph$ has a structure of commutative monoid in $\text{Span}(RGrph)$. In fact the spans

$$\nabla := (\Delta_X, 1_X) : X \otimes X \to X \quad e := (!, 1) : I \to X$$

satisfy the axioms of the definition of monoid object, and the multiplication $\nabla$ is compatible with the twist map.

Each reflexive graph $X$ has also a structure of commutative comonoid. The comultiplication is

$$\Delta := (1, \Delta_X) : X \to X \otimes X$$

There is a symmetric monoidal structure on spans. Given two spans $A : X \to Y$, $B : Z \to W$ the tensor of $A$ and $B$ is defined as

$$A \otimes B := (\delta_0^A \times \delta_0^B, \delta_1^A \times \delta_1^B) : X \otimes Z \to Y \otimes W$$
Given two objects $X, Y$ there is a twist span

$$X \times Y \xrightarrow{1} X \times Y \xrightarrow{tw} Y \times X$$

where $tw$ is the twist map in $RGrph$.

**Definition 3.** A wscc category is a symmetric monoidal category with for each object $X$ the structure of commutative monoid and comonoid satisfy the following axioms:

1) **Frobenius axiom:** $(\nabla \otimes 1)(1 \otimes \Delta) = \Delta \nabla$

2) **Separable axiom:** $\nabla \Delta = 1$

Derived operations are:

- the projection $X \otimes Y \xrightarrow{X \otimes 1} X \otimes I \xrightarrow{1} X$,
- the opposite projection $I \xrightarrow{1} X \otimes I \xrightarrow{X \otimes 1} X \otimes Y$,
- the unit $\eta_X : I \xrightarrow{1} X \xrightarrow{\Delta} X \otimes X$,
- the counit $\epsilon_X : X \otimes X \xrightarrow{\nabla} X \xrightarrow{1} I$.

### 3.5 Monoids in $Rel$.

What is a commutative monoid in $2LTS$ in which the structure arrows of the monoid are relations? It is easy to verify the following proposition:

**Proposition 2.** A commutative monoid structure in $2LTS$ on object $X$, for which all the structure maps are pointed relations, amounts to (i) a subset $e$ of $X$ containing $\varepsilon$, (ii) a function $m : X \times X \rightarrow \wp(X)$ satisfying, for all $x, y, z$ in $X$, $m(e, x) = \{x\}$, $m(x, e) = \{x\}$, $m(x, y) = m(y, x) = m(x, y, z) = m(m(x, m(y, z)))$. Notice that the conditions involve extending the definition of $m$ in the obvious way to subsets of $X$. 

4 Parallel composition in Process Algebras based on broadcast

It is undoubtedly true that one of the most common ways of connecting components is by broadcast; that is, each component may communicate which any other directly. The geometry is not

![Diagram showing two components connected by a line](image)

but something like

![Diagram showing three components connected](image)

We might call the bottom line here a bus. The components can talk directly to each other through this medium, though naturally in interleaving. We maintain however that the first geometry above is the canonical one, while the second is a special derived operation, namely, in terms of the wscc operations of 2LTS,

\[
\nabla (C \otimes 1) \nabla (B \otimes 1) A = \nabla (C \otimes (\nabla (B \otimes A)))
\]

or geometrically

![Diagram showing three components connected](image)

This expression certainly acts like pure broadcast: in a transition of the whole systems the transitions of each component must have the same label on the bus. Note that we could have as easily, and perhaps more naturally, used $\Delta$ rather than $\nabla$; however the comparison with synchronization algebras is simpler using $\nabla$. 
What we will describe next is a modification of pure broadcast in which there is a protocol between the processes and the bus.

4.1 Classical labelled transition systems and synchronization algebras

Let \( L \) be an alphabet which does not include the symbols 0 and \( \varepsilon \). Define

\[
L_\varepsilon := L \cup \{\varepsilon\} \quad L_{\varepsilon,0} := L \cup \{\varepsilon, 0\}
\]

**Definition 4.** [23] A synchronization algebra on \( L \) is a binary, commutative and associative operation \( \diamond \) on \( L_{\varepsilon,0} \) such that for all \( \alpha, \beta \in L_{\varepsilon,0} \)

(i) \( \alpha \diamond 0 = 0 \),

(ii) \( \alpha \diamond \beta = \varepsilon \) if and only if \( \alpha = \beta = \varepsilon \).

The idea is that the element 0 denotes those synchronizations which are not allowed, \( \varepsilon \) allows asynchrony, and \( \alpha \diamond \beta \) is the resulting signal on the bus when messages \( \alpha \) and \( \beta \) are passed from components.

**Remark**

If we regard \( \alpha \diamond \beta = 0 \) as meaning that \( \alpha \diamond \beta \) is undefined, then we may think of the operation of a synchronization algebra as being a partial function. Property (i) assures us that no information is lost. Then clearly by Proposition 2 a synchronization algebra is a commutative semigroup object in \( \text{Rel}_* \), and hence in \( 2\text{LTS} \). In fact a synchronization algebra on alphabet \( L \) is the same thing as a commutative semigroup object in \( \text{Rel}_* \) on object \( L_\varepsilon \) whose multiplication is a partial function, and which satisfies the additional property that \( \alpha \diamond \beta = \varepsilon \) implies \( \alpha = \beta = \varepsilon \).

As usual, a transition system \( A \) labelled in \( L \) consists of a set \( S \) of states, and a transition relation \( T \subseteq S \times L \times S \).

**Definition 5.** [23] Let \( A = (S_A, L, T_A), B = (S_B, L, T_B) \) be transition systems on the same alphabet \( L \). Given a synchronization algebra \( \diamond \) on \( L \), the parallel composition of \( A \) and \( B \) is the transition system \( A \parallel B := (S_{\parallel}, L, T_{\parallel}) \) where:

\[
S_{\parallel} := S_A \times S_B, \\
T_{\parallel} := \{(a,b),(\alpha',\beta') | \lambda = \alpha \diamond \beta \neq 0, (x \xrightarrow{\alpha} x', y \xrightarrow{\beta} y') \in T_x\},
\]

where \( T_x = T_A \times T_B + T_A \times \{b \xrightarrow{\varepsilon} b \mid b \in S_B\} + \{a \xrightarrow{\varepsilon} a \mid a \in S_A\} \times T_B \).

It is straightforward to check the following proposition:
Proposition 3. Consider a synchronization algebra $\diamond$ on the alphabet $\mathcal{L}$, considered now as a commutative semigroup object in $2LTS$. Let $\bullet$ be the commutative semigroup operation induced on $\text{Hom}(I, \mathcal{L}_\varepsilon)$, as in Proposition 1 ($I$ has a trivial cocommutative comonoid structure). Then

$$A \parallel B = A \bullet B.$$ 

In the geometry of expressions in the wscc category $2LTS$, $A \parallel B$ is:

\[
\begin{array}{c}
A \\
\downarrow \\
B \\
\searrow ^m \\
\end{array}
\]

It is also straightforward to see that $A \parallel B \parallel C$ has the following geometry, analogous to the example of pure broadcast above:

\[
\begin{array}{c}
A \\
\searrow ^m \\
B \\
\searrow ^m \\
C \\
\end{array}
\]

The expression here is $m(C \otimes 1)m(B \otimes 1)A$.

Notice that $A \parallel B \parallel C=(C \parallel A) \parallel B$ and hence $C$ may communicate directly with $A$, and also that the order on the bus is irrelevant.

4.2 Examples

Pure Broadcast We have already discussed this case which arises from the comonoid structure of objects which is part of the wscc structure of $2LTS$. The comultiplication is the arrow

$$\nabla : \mathcal{L}_\varepsilon \otimes \mathcal{L}_\varepsilon \rightarrow \mathcal{L}_\varepsilon$$

which is actually the partial function $\mathcal{L}_\varepsilon \times \mathcal{L}_\varepsilon \xleftarrow{\Delta} \mathcal{L}_\varepsilon \xrightarrow{1} \mathcal{L}_\varepsilon$. Notice however that the unit of the monoid structure is not a partial function, which means that the synchronization algebra is only a semigroup. It is our view that the extension of the notion of synchronization algebra to monoids in $\text{Rel}_\varepsilon$ is important.
The alphabet $\mathcal{L}$ contains a special letter $\tau$ and to each other letter $\alpha \in \mathcal{L}$, $\alpha \neq \tau$, there exists the complementary label $\bar{\alpha} \in \mathcal{L}$. The multiplication $\diamond$ on $\mathcal{L}_\varepsilon$ is the partial map defined as follows:

(i) $\alpha \diamond \varepsilon = \varepsilon \diamond \alpha = \alpha$ for all $\alpha$ (including $\tau$),
(ii) $\alpha \neq \tau$, implies that $\alpha \diamond \bar{\alpha} = \tau = \bar{\alpha} \diamond \alpha$,
(iii) on all other pairs $\diamond$ is undefined.

This multiplication does have an identity element, namely the element $\varepsilon$.

**Non-reflexive graphs and synchronization** An important special case of broadcast is the clock signal in synchronous machines. The clock has one vertex and one non-reflexive edge, the clock signal.

![Diagram of non-reflexive graph with clock signal](image)

If each non-reflexive edge of $A, B, C, \cdots$ is labelled by the clock signal, then this expression evaluates to the product, in non-reflexive Graphs, of the graphs consisting of the non-reflexive edges of $A, B, C, \cdots$.

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