Universal Approximation with Neural Intensity Point Processes

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Abstract

We propose a class of neural network models that universally approximate any point process intensity function. Our model can be easily applied to a wide variety of applications where the distribution of event times is of interest, such as, earthquake aftershocks, social media events, and financial transactions. Point processes have long been used to model these events, but more recently, neural network point process models have been developed to provide further flexibility. However, the theoretical foundations of these neural point processes are not well understood. We propose a neural network point process model which uses the summation of basis functions and the function composition of a transfer function to define point process intensity functions. In contrast to prior work, we prove that our model has universal approximation properties in the limit of infinite basis functions. We demonstrate how to use positive monotonic Lipschitz continuous transfer functions to shift universal approximation from the class of real valued continuous functions to the class of point process intensity functions. To this end, the Stone-Weierstrass Theorem is used to provide sufficient conditions for the sum of basis functions to achieve point process universal approximation. We further extend the notion of universal approximation mentioned in prior work for neural point processes to account for the approximation of sequences, instead of just single events. Using these insights, we design and implement a novel neural point process model that achieves strong empirical results on synthetic and real world datasets; outperforming state-of-the-art neural point process on all but one real world dataset.

Keywords: universal approximation, point process, sequence modelling, neural networks

1. Introduction

The temporal point process abstraction Daley and Vere-Jones (2007) have been successful at characterising and predicting sequences of events in many application domains, including, earthquake modelling Hawkes (1971a,b); Hawkes and Oakes (1974); Ogata (1988), social media Zipkin et al. (2016); Rizoiu et al. (2017a,b), and finance Embrechts et al. (2011); Bacry et al. (2015). With the many application domains point processes have been used in, many different variants have been developed for specific scenarios Kingman (2005); Daley and Vere-Jones (2007); Bacry et al. (2015); Laub et al. (2015); Isham and Westcott (1979). Traditionally, point processes were defined by simple intensity functions such as the Hawkes model with exponential triggering kernel Bacry et al. (2015); Laub et al. (2015); however, the form of this intensity function must be carefully chosen to match the underlying generating process, or else poor performance will likely result Bacry et al. (2015);
Figure 1: Overview of our method of universally approximating point processes. A RNN is used to parameterise a set of basis functions at each interarrival time $\tau_i$. Then, the sum of basis functions is used to approximate a continuous function, which is composed with a transfer function $f_\varepsilon$ to universally approximate the class of valid intensity functions.

Mishra et al. (2016). It is therefore desirable to choose a class of intensity functions that can approximate any valid intensity function to arbitrary precision. This type of property is referred to as universal approximation Cybenko (1989); Hornik et al. (1989); Hornik (1991); Debao (1993).

A known method for approximating a wide variety of functions is the use of neural networks Hornik et al. (1989); Hornik (1991). Recent work has utilised recurrent neural networks to define point process intensity functions Du et al. (2016); Mei and Eisner (2017); Omi et al. (2019); Shchur et al. (2020). A central concept amongst these models is the usage of a recurrent neural network to encode the past events in a sequence, initially proposed by the Recurrent Marked Temporal Point Process model Du et al. (2016). Early neural point process models presented strong empirical results when compared to their tradition counterparts Du et al. (2016); Mei and Eisner (2017), but it remains unknown whether they are flexible enough to represent all intensity functions. Newer neural point processes have provided some discussions about approximation guarantees for single events Omi et al. (2019); Shchur et al. (2020), but the discussion in these cases were not the main focus of the studies and it is not clear if the universal approximation property extends to entire sequence.

We provide theoretical foundations to point process approximation, which guides the design of a novel neural point process model, summarised in Figure 1. First, we provide theoretical results that establish universal approximation of point process intensity functions within specific time intervals. To achieve this we use the sum of flexible basis functions, which can achieve arbitrary approximation error for a single event. Next, we extend the notion of universal approximation to full event sequences by utilising the universal approximation properties of dynamical system with recurrent neural networks Schäfer and Zimmermann (2007). We crucially identify that to extend approximation from a single event to full event sequences, the functional form of the intensity function approximator must have bounded perturbation under parameterisation error, dictated by a surjective function. Finally, we implement and evaluate UNIPoint, a new neural point process model that is a universal approximator in the limit of infinite basis functions. In our experiments, we find that UNI-
Point has consistent performance across multiple datasets, and outperforms our traditional point process and state-of-the-art neural point process baselines in all but one dataset.

Our primary contributions are:

(C1) A foundational result for constructing universal approximators of point process intensity functions from universal approximators of continuous functions (Theorem 3);

(C2) Sufficient conditions for generalising universal approximation of intensity functions between events to event sequences. In particular, we consider the effect of intensity function parameterisation errors on the final error of the intensity function (Theorem 8);

(C3) UNIPoint: A novel neural network point process architecture with proven universal approximation guarantees that demonstrates strong empirical results on both synthetic and real world datasets. (Sections 4.3, 5, & 6).

2. Related Work & Notation

This study lies in the intersection of the point process, neural network, and universal approximation literature. Thus, we present relevant studies from the point process, neural network point process, and universal approximation literature. We further introduce the shared notation used throughout this paper.

2.1 Types of Point Processes

There are many different types of point processes, and a variety of applications in which they are employed, for example analysing social networks Mishra et al. (2016); Wilhelm et al. (2018). The homogeneous Poisson process is often considered the simplest type of point process, where the number of events over a finite time interval is a random variable with a Poisson distribution Kingman (2005); Daley and Vere-Jones (2007). For the homogeneous Poisson process, the distribution of points is independent of the history and the distribution does not change over time. The inhomogeneous Poisson process extends the homogeneous version by relaxing the condition that the distribution is fixed over time. A particular type of inhomogenous Poisson process, the Hawkes process, aims to model self-excitation and so has a distribution of points that depends on the history Bacry et al. (2015); Laub et al. (2015). Note that the Hawkes process is not the only type of point process that models self-interaction; the self-correcting point process is another example Isham and Westcott (1979). Concrete definitions of the aforementioned point processes are introduced in Section 5 since they are used as baselines in our study. A major downside of the aforementioned traditional point process models is that they assume a particular form for the underlying process and thus are prone to model misspecification and under-fitting. The model we propose is related to the Hawkes process but is much more flexible and thus does not suffer as greatly from these issues.

2.2 Neural Point Processes

A number of recent studies have proposed neural network point processes which marry modern neural network architectures with a point processes framework. A major motiva-
tion for neural point processes is to increase the expressive power of point processes by leveraging the flexibility of neural networks. Recurrent Marked Temporal Point Process (RMTPP) Du et al. (2016) was among the first models to employ an recurrent neural network (RNN) to encode the event history and generate parameters defining the intensity function. RMTPP uses an exponential intensity function, a choice which has also been adopted by (Upadhyay et al., 2018). Other methods have employed piecewise constant functions Li et al. (2018); Huang et al. (2019).

More complex intensity functions are used in Neural Hawkes Mei and Eisner (2017) and a fully neural network based model Omi et al. (2019). These models utilise multilayer perceptrons to define their intensity functions. Neural Hawkes introduces a flexible intensity function at the cost of requiring numeric integration to evaluate its loss function. The fully neural method of (Omi et al., 2019) avoids this issue by directly modelling the integral of the intensity function as a neural network and thus implicitly defines the intensity function as the derivative of this network. Normalising flows have also been used to define point process intensity functions Shchur et al. (2020).

A number of the aforementioned neural point processes are applied in reinforcement learning settings Upadhyay et al. (2018); Li et al. (2018). Other approaches do not use the traditional point process log-likelihood function as the loss function Xiao et al. (2017b,a); Huang et al. (2019). In this work, we do not consider these two settings, instead we opt to focus on the non-reinforcement learning setting and neural processes which closely resemble traditional point processes with loss functions defined as the log-likelihood of point processes.

Mixtures of kernel functions have previously been used for point process modelling Taddy et al. (2012); Tabibian et al. (2017); Okawa et al. (2019). This work is similar to our own since the sum of basis functions we use could be considered as a mixture. However, these studies are substantially different to our own as they do not use a RNN to encode the event history.

2.3 Universal Approximation Theorems

Universal approximation theorems are important theoretical results for understanding the approximation power of neural networks. It has been proven that a one layer neural network with infinite width can approximate any arbitrary continuous function on compact support Cybenko (1989); Debo (1993); Hornik et al. (1989); Hornik (1991). The neural networks considered in these studies where those with sigmoid activation functions. Later studies generalised these results to other types of activation function, such as the rectified linear unit Sonoda and Murata (2017). Some studies have also considered neural network variants for universally approximating monotone functions Kay and Ungar (2000); Daniels and Velikova (2010), and explored how neural networks with infinite depth achieve universal approximation Lu et al. (2017); Hanin and Sellke (2017).

The approximation capabilities of more complex neural network architectures have their own corresponding universal approximation theorems. Recurrent neural networks have been shown to universally approximate dynamic systems Schäfer and Zimmermann (2007). While a different study showed transformer networks are universal approximators of sequence-to-sequence functions Yun et al. (2020).
Some neural point processes have been shown to be universal approximators between events. For example, the fully neural method Omi et al. (2019) claims universal approximation using the result that RNNs can universally approximate dynamic systems Schäfer and Zimmermann (2007) along with the result that positive weighted neural networks are universal approximators for monotone functions Kay and Ungar (2000); Daniels and Velikova (2010). Taking a different approach (Shchur et al., 2020) uses neural flows which have universal approximation properties for probability densities Huang et al. (2018); Jaini et al. (2019). However, both studies use completely different classes of model to ours and their claims of universal approximation cannot be extended to our own. Moreover, neither study considers how to propagate approximation errors from the recurrent neural network to the the point process intensity function over an entire sequence.

2.4 Notation

\( C(X,Y) \) denotes the class of continuous functions mapping from domain \( X \) to range \( Y \). Denote \( \mathbb{R} \) as the set of real numbers, \( \mathbb{R}_+ \) as the positive reals, and \( \mathbb{R}_{++} \) as the strictly positive reals. For a function \( f \) and a class of functions \( \mathcal{F} \), define the composition \( f \circ \mathcal{F} = \{ f \circ g : g \in \mathcal{F} \} \). The sigmoid function \( [1 + \exp(x)]^{-1} \) is denoted \( \sigma(x) \).

3. Temporal Point Process

A temporal point process is an ordered set of time points \( \{t_i\}_{i=1}^N \). We typically describe a point process by its conditional intensity function \( \lambda(t \mid \mathcal{H}_{t-}) \) which can be interpreted as the instantaneous probability of an event occurring at time \( t \) given history \( \mathcal{H}_{t-} \), where the history consists of the set of all events before time \( t \). This can be written as Daley and Vere-Jones (2007); Bacry et al. (2015); Laub et al. (2015):

\[
\lambda(t \mid \mathcal{H}_{t-}) = \lim_{h \downarrow 0^+} \frac{P(N[t, t+h) > 0 \mid \mathcal{H}_{t-})}{h},
\]

where \( N[t_1, t_2) \) is the number of events occurring between two arbitrary times \( t_1 < t_2 \).

Note that given a history, the conditional intensity is a deterministic function of time \( t \) only. Following standard conventions we will refer to the conditional intensity function as simply the intensity function, and abbreviate \( \lambda(t \mid \mathcal{H}_{t-}) \) to \( \lambda^*(t) \).

Point processes can be specified by choosing a functional form for the intensity function. For example, the Hawkes process, which can be thought of as the simplest interacting point process Bacry et al. (2015); Laub et al. (2015), can be defined as follows:

\[
\lambda^*(t) = \mu + \sum_{t_i < t} \varphi(t - t_i),
\]

where \( \mu \) specifies the background intensity and \( \varphi(t - t_i) \) is the triggering kernel which characterises the self-exciting effects of prior events \( t_i \).

Many point processes, including the Hawkes process, have intensity functions that are discontinuous at each event \( t_i \), but otherwise are continuous between events \( t \in (t_{i-1}, t_i] \). Thus, we reparametrise the intensity function with the interarrival time \( \tau = t - t_{i-1} \) — thereby allowing us to assume continuity between events. We only consider events up to
some final time $T > 0$ as this assumption frequently holds in practice. Thus valid intensity
functions $\mathcal{F}_{\text{INT}}$ are restricted to:

1. The set of strictly positive continuous functions;
2. With interarrival compact domain $[0, T]$ for some $T > 0$.

This can also be written as $\mathcal{F}_{\text{INT}} = C([0, T], \mathbb{R}_{++})$. We note that typically intensity func-
tions can be zero, however, our definition still allows for arbitrarily low intensity.

The likelihood of a point process is Daley and Vere-Jones (2007); Bacry et al. (2015);
Laub et al. (2015):

$$L = \left[ \prod_{i=1}^{N} \lambda^*(t_i) \right] \exp \left( \int_{0}^{T} \lambda^*(s) \, ds \right).$$

(3)

4. Point Process Universal Approximation

We establish universal approximation of point processes by first considering the approxi-
mation between two consecutive events; and then the approximation for an event sequence.
These theoretical results are used to design and implement a practical system that is a
universal approximator of intensity functions.

4.1 Approximation Between Two Events

Although we wish to approximate all valid intensity functions $\mathcal{F}_{\text{INT}}$ with respect to event
sequences, we will start with approximating intensity functions between two consecutive
events.

First we must define what it means for a function to be close to another function. For
this we use the uniform metric in Equation 4, which quantifies the maximum error between
an intensity function and an approximation. This metric has previously been used for
proving universal approximation properties of neural networks Hornik et al. (1989); Debao
(1993) and recurrent neural networks Schäfer and Zimmermann (2007). Given two functions
$f, g : \mathbb{R} \to \mathbb{R}$ the uniform metric is:

$$d(f, g) = \sup_{x \in \mathbb{R}} |f(x) - g(x)|.$$  \hspace{1cm} (4)

For the remainder of this paper we will consider universal approximation with respect
to the uniform metric.

To approximate an intensity function, we aim to leverage the extensive literature on
universal approximations to the class of continuous functions in $C(K, \mathbb{R})$ — where $K$ is a
compact subset of $\mathbb{R}$. In order to do this we employ a class of transfer functions
which shift approximations of functions in $C(K, \mathbb{R})$ to functions in $\mathcal{F}_{\text{INT}}$. These transfer function
are defined using the definition of Lipschitz continuous functions which is stated formally
below:

\textbf{Definition 1 (From Royden and Fitzpatrick (1988))} A function $f$ is a $M$-Lipschitz
continuous if

$$|f(x) - f(y)| \leq M|x - y|, \text{ for all } x, y \in \mathbb{R}. \hspace{1cm} (5)$$


We define the \emph{M-transfer functions} which project functions from the class of continuous functions $C(K, \mathbb{R})$ to the class of positive continuous functions $C(K, \mathbb{R}^+)$, with $M$ corresponding to the Lipschitz constant in Equation 5.

\textbf{Definition 2} A function $f_+ : \mathbb{R} \to \mathbb{R}^+$ is a M-transfer function if the function satisfies the following:

1. $f_+$ is $M$-Lipschitz continuous;
2. $\mathbb{R}^+ \subseteq f_+[\mathbb{R}]$;
3. And $f_+$ is strictly increasing on $f_+^{-1}[\mathbb{R}^+]$.

The definition of the transfer function preserves the universal approximation property under composition — as we formalise in Theorem 3.

\textbf{Theorem 3} Given a class of functions $\mathcal{F}$ which is uniformly dense in $C(K, \mathbb{R})$ and a $M$-transfer function $f_+$, the composed class of functions $f_+ \circ \mathcal{F}$ is uniformly dense in $C(K, \mathbb{R}^+)$ for any compact subset $K \subset \mathbb{R}$.

\textbf{Proof} Let $f \in C(K, \mathbb{R}^+)$ and $\varepsilon > 0$ be arbitrary. Since $f_+$ is strictly increasing and continuous on the preimage of $\mathbb{R}^+$ then $f_+^{-1}$ exists, is continuous, and restricted to subdomain $\mathbb{R}^+$. Thus, there exists some $g \in C(K, \mathbb{R})$ such that $f = f_+ \circ g$.

As $\mathcal{F}$ is dense with respect to the uniform metric, for $\varepsilon/M$ there exists some $h \in \mathcal{F}$ such that $d(h, g) < \varepsilon/M$. Thus for any $x \in K$,

$$
|(f_+ \circ h)(x) - f(x)| = |(f_+ \circ h)(x) - (f_+ \circ g)(x)|
\leq M|h(x) - g(x)| < \varepsilon.
$$

We have $d(f_+ \circ h, f) < \varepsilon.$

Note that Theorem 3 restricts the set of approximators to those defined over compact subdomain $K$ because valid intensity functions $\mathcal{F}_{\text{INT}}$ are defined over $[0, T]$. Further, although Theorem 3 holds for any $M$-transfer function, in practice it is often convenient to use the softplus function $f_{\text{SP}}(x) = \log(1 + \exp(x))$ which is a 1-transfer function.

Corollary 1 (below) which corresponds to Theorem 1 provides sufficient conditions on the candidate function family to ensure universal approximation of $C(K, \mathbb{R})$. Specifically, we consider the function family consisting of the sum of basis functions $\phi(x; p_j)$, where $p_j \in \mathcal{P}$ denotes the parameterisation of the basis function $\phi$. Consider the following definition of the sum of basis functions:

\textbf{Definition 4} Denote $\Sigma(\phi)$ as the class of functions corresponding to the sum of basis functions $\phi : \mathbb{R} \times \mathcal{P} \to \mathbb{R}$, with parameter space $\mathcal{P}$, as follows:

$$
\left\{ f : \mathbb{R} \to \mathbb{R} \mid f(x) = \sum_{j=1}^{J} \phi(x; p_j), \ p_j \in \mathcal{P}, \ J \in \mathbb{N} \right\}.
$$
The parameter space $\mathcal{P}$ of $\phi(x; p_j)$ determines the chosen basis function from a family. For example, the class composed of exponential basis functions could be defined with parameter space $\mathcal{P} = \mathbb{R}^2$ such that $\Phi_{\text{EXP}} = \{ \phi : \mathbb{R} \to \mathbb{R} | \alpha \exp(\beta x), \alpha, \beta \in \mathbb{R} \}$. Definition 4 encompasses a wide range of function classes, including neural networks with sigmoid Cybenko (1989); Hornik et al. (1989); Debao (1993) or rectified linear unit activations Sonoda and Murata (2017).

We can now use the Stone-Weierstrass theorem to find basis functions for universal approximation. The Stone-Weierstrass theorem can be defined as the following:

**Theorem 5 (Stone-Weierstrass Theorem Rudin et al. (1964); Royden and Fitzpatrick (1988))**

Suppose an subalgebra $\mathcal{A}$ of $C(X, \mathbb{R})$ satisfies the following conditions:

1. For all $x, y \in X$, there exists some $f \in \mathcal{A}$ such that $f(x) \neq f(y)$;
2. For all $x_0 \in X$, there exists $f \in \mathcal{A}$ such that $f(x_0) \neq 0$.

Then $\mathcal{A}$ is uniformly dense in $C(X, \mathbb{R})$.

Note the Stone-Weierstrass theorem provides sufficient conditions to approximate the continuous functions — for instance, polynomial approximations satisfy the conditions of the theorem Rudin et al. (1964); Royden and Fitzpatrick (1988). Thus, by using Theorem 3 and the Stone-Weierstrass theorem, Theorem 5, we arrive at Corollary 6, which gives sufficient conditions for basis functions $\phi$ to ensure that $f_+ \circ \sum(\phi)$ is a universal approximator for $C(K, \mathbb{R}^+)$. 

**Corollary 6** For any compact subset $K \subset \mathbb{R}$ and for any $M$-transfer function $f_+$, if a basis function $\phi(\cdot; p)$ parametrised by $p \in \mathcal{P}$ satisfies the following conditions:

1. $\sum(\phi)$ is closed under product;
2. For any distinct points $x, y \in K$, there exists some $p \in \mathcal{P}$ such that $\phi(x; p) \neq \phi(y; p)$;
3. For all $x_0 \in K$, there exists some $p \in \mathcal{P}$ such that $\phi(x_0; p) \neq 0$.

Then $f_+ \circ \sum(\phi)$ is uniformly dense in $C(K, \mathbb{R}^+)$. 

Given the conditions of Corollary 6, some interesting choices for valid basis functions $\phi(x; p)$ are the exponential basis function $\phi_{\text{EXP}}(x) = \alpha \exp(\beta x), \ (\alpha, \beta) \in \mathbb{R}^2$ and the power law basis function $\phi_{\text{PL}}(x) = \alpha(1+x)^{-\beta}, \ (\alpha, \beta) \in \mathbb{R} \times \mathbb{R}_+$. These basis functions are inspired by the exponential and power law Hawkes triggering kernels, which have seen far-reaching success in many domains Ogata (1988); Bacry et al. (2015); Laub et al. (2015); Rizoiu et al. (2017b).

Corollary 6 only provides sufficient conditions. For example, the sigmoid basis function $\phi_{\text{SIG}}(x) = \alpha \sigma(\beta x + \delta), \ (\alpha, \beta, \delta) \in \mathbb{R}^3$ does not allow $\sum(\phi_{\text{SIG}})$ to be closed under product and thus does not satisfy the conditions of Corollary 6. However, as the sum of sigmoid basis functions is equivalent to the class of single hidden layer neural networks, it does have the universal approximation property through Theorem 3 Hornik et al. (1989); Debao (1993). As Corollary 6 does not provide a complete selection of possible basis functions, we present
Table 1: Basis function universal approximators for intensity functions between two consecutive events. † indicates which functions that satisfy Corollary 6. The corresponding citation for the sigmoidal basis function can be found in Cybenko (1989); Hornik et al. (1989); Debao (1993) and the ReLU basis function can be found in Sonoda and Murata (2017).

| Basis Function | Functional Form $\phi$ | Parameter Space $\mathcal{P}$ |
|----------------|------------------------|--------------------------------|
| $\phi_{\text{EXP}}^\dagger$ | $\alpha \exp(\beta x)$ | $(\alpha, \beta) \in \mathbb{R}^2$ |
| $\phi_{\text{PL}}^\dagger$ | $\alpha (1 + x)^{-\beta}$ | $(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}_+$ |
| $\phi_{\text{COS}}^\dagger$ | $\alpha \cos(\beta x + \delta)$ | $(\alpha, \beta, \delta) \in \mathbb{R}^3$ |
| $\phi_{\text{SIG}}$ | $\alpha \sigma(\beta x + \delta)$ | $(\alpha, \beta, \delta) \in \mathbb{R}^3$ |
| $\phi_{\text{ReLU}}$ | $\max(0, \alpha x + \beta)$ | $(\alpha, \beta) \in \mathbb{R}^2$ |

Table 1 which details a number of interesting basis function to universally approximate $C(K, \mathbb{R}_{++})$ and by extension $\mathcal{F}_{\text{INT}}$ between events. Another direct result of Corollary 6 is that the finite mixture of $\sum(\phi_i)$ classes which satisfies the preconditions of Corollary 6 will also result in universal approximation, i.e., each element is the sum of the many $\phi_i$ basis functions. We explore this in practice in Section 5 and 6.

4.2 Approximation for Event Sequences

Having established universal approximation using sums of a range of basis functions in Section 4.1, we now bridge the gap between approximating continuous intensities between events, to approximating piece-wise continuous intensity functions over the whole even history in $[0, T]$.

To this end, we rely on the universal approximation property of RNNs for dynamic systems Schäfer and Zimmermann (2007). Specifically, we consider the parameterisation of the intensity function as a dynamic system:

$$s_{i+1} = g(s_i, t_{i+1})$$

$$p_i = \nu(s_i),$$

where $s_{i+1}$ is the internal state of the dynamic system, $p_i \in \mathcal{P}$ are the parameters of the intensity function after event $t_i$, $g$ updates the internal state at each step, and $\nu$ maps from the internal state to the desired intensity function parameters.

A key concern when using this setup is the potential compounding of approximation errors, since the RNN generates an approximate set of parameters for the intensity function approximation. Thus, we first define the effects of perturbation in the parameter space of a class of functions.

**Definition 7** The $\delta > 0$ perturbation of a class of functions $\mathcal{F} = \{f(\cdot; p) : p \in \mathcal{P}\}$ is defined as:

$$\Delta \mathcal{F}(\delta) = \sup \left\{ d(f(\cdot; p), f(\cdot; p + x)) : x \in [-\delta, \delta]^{\lvert \mathcal{P} \rvert} \right\}.$$  

where $(p + x)$ is restricted to parameter space $\mathcal{P}$. 

9
We now formally state our theorem for the universal approximation of point processes over an event sequence.

**Theorem 8** Let $\mathcal{F} = \{f(\cdot; p) : p \in \mathcal{P}\}$ universally approximate continuous functions $C([0,T], \mathbb{R})$ with respect to the uniform metric and have bounded perturbation $\Delta \mathcal{F}(\delta) = O(U(\delta))$, where $U : \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}$ is a surjective function restricted to subdomain $\mathbb{R}_{++}$.

Given a $M$-transfer function $f_+$, then for any sequence of events $\{t_i\}_{i=1}^N$, with $t_i \in [0,T]$, and for any valid intensity function $\lambda^*(t)$, there exists a RNN (Equations 9 and 10) such that for all $i$ and for any valid intensity function $C$ appropriate weights $[W, v, b, A]$,

\begin{align*}
  h_i &= \sigma(W h_{i-1} + v t_i + b) \quad (9) \\
  \hat{p}_i &= A h_i \quad (10) \\
  \hat{\lambda}_i(t; \hat{p}_i) &= (f_+ \circ f)(t - t_{i-1}; \hat{p}_i), \quad (11)
\end{align*}

such that for all $i$, $\hat{\lambda}_i(t; \hat{p}_i)$ approximates $\lambda^*(t)$ with arbitrary precision on $(t_{i-1}, t_i]$.

**Proof** Let $\varepsilon > 0$ be arbitrary. For each time interval $t \in (t_{i-1}, t_i]$ (where $t_0 = 0$) define $\lambda_i(t) = \lambda^*(t)$, with fixed history $H_{t_i}$. Thus, from Theorem 3 there exists $p_i \in \mathcal{P}$ such that $|\lambda_i(t) - (f_+ \circ f)(t - t_{i-1}; p_i)| < \varepsilon / 2$ for each $i \in \{1, \ldots, N\}$.

As $\Delta \mathcal{F}(\delta) = O(U(\delta))$, there exists some constant $c > 0$ such that $\Delta \mathcal{F}(\delta) < cU(\delta)$.

Additionally, from the surjective property of $U(\delta)$ and the universal approximation property of RNNs Schäfer and Zimmermann (2007), there exists a RNN (Equations 9 and 10) which can approximate the following dynamic system with $U(\epsilon_{\text{RNN}}) < \varepsilon / 2Mc$ precision:

\begin{align*}
  s_{i+1} &= \hat{g}(s_i, t_{i+1}) \\
  \hat{p}_i &= \hat{v}(s_i),
\end{align*}

where $\hat{g}(s_i, t_{i+1})$ is a measurable function defined with respect to the required event sequence and $\hat{v}(s_i)$ is a continuous function defined to achieve correct $\hat{p}_i$.

Thus, for any $i \in \{1, \ldots, N\}$ and any $t \in (t_{i-1}, t_i]$,

\begin{align*}
  |(f_+ \circ f)(t - t_{i-1}; p_i) - (f_+ \circ f)(t - t_{i-1}; \hat{p}_i)| &
  \leq M|f(t - t_{i-1}; p_i) - f(t - t_{i-1}; \hat{p}_i)| \\
  &< McU(\varepsilon_{\text{RNN}}) < \varepsilon / 2.
\end{align*}

By the triangle inequality, for any $i \in \{1, \ldots, N\}$ we have that $d(\lambda_i(\cdot), (f_+ \circ f)(\cdot; \hat{p}_i)) < \varepsilon$. ■

Theorem 8 provides an argument for universal approximation of event sequences through the use of a RNN and a suitable family of functions $\mathcal{F}$. The surjective perturbation condition in the theorem ensures that any nonzero error in approximation of the parameter space can be compensated by the universal approximation property of the RNN.

For example, consider the family of rectified linear unit basis (ReLU) functions $\Phi_{\text{ReLU}}$. The perturbation of this family of functions can be upper bounded by a linear shift $\Delta \Phi_{\text{ReLU}}(\delta) = O(\delta)$. Thus, Theorem 8 shows that the class of ReLU basis function sums can be used to universally approximate intensity functions with respect to event sequences. Similar to
Figure 2: *(Left)* Shows how UNIPoint generates the point process intensity for the entire sequence. For each interarrival time, the RNN (blue circle) feeds into a multi-layer perceptron (green square) to produce a point process intensity. *(Right)* A more detailed view, showing how UNIPoint adds multiple basis functions to create the intensity function. $p_0, p_1, p_2, p_3$ are the basis functions parameters generated by the neural network using the hidden state $h_1$ and interarrival time $\tau_2$.

classical universal approximation theorems, no guarantees on learnability are given for the perturbation condition. In section 6, we include a discussion on the experimental performance of basis functions which satisfy the perturbation condition.

In prior work Omi et al. (2019); Shchur et al. (2020) the triangle inequality is invoked to propagate the RNN approximation error to the intensity function approximation error; however, we note that the functional form of the intensity function is important for approximation guarantees — not all approximation methods give universal approximation for the context of an event sequence.

### 4.3 Approximation with Neural Networks

We propose the *UNIPoint* neural network architecture which is a universal approximator by Theorem 8 and Corollary 6. Given a dataset of event sequences $\{t_i\}_{i=1}^{N}$ and corresponding interarrival times $\tau_i = t_i - t_{i-1}$, *UNIPoint* produces an intensity function $\hat{\lambda}$. To construct this intensity function *UNIPoint* combines a RNN with a transfer function and a sum of basis functions. We use $J$ many $\phi$ basis function defined over parameter space $\mathcal{P}$, where the RNN with fully connected output layer generates points in this parameter space. We set $\tau_1 = 0$ as a starting token for the RNN.

The UNIPoint model is depicted in Figure 2. The left part of Figure 2 shows how the RNN is used to form a point process intensity function over multiple time-steps. This RNN setup is similar to previous work on neural point process architectures (Du et al., 2016; Mei and Eisner, 2017; Omi et al., 2019). The right part of Figure 2 shows how basis functions are added together to form the intensity function between events – this is unique to our model and relies on Theorem 8 with specific basis functions.
Recurrent Neural Network. As per Theorem 8, we use a vanilla RNN (though other popular variants would also work, e.g., LSTM, or GRU) to produce hidden states \( h_i \in \mathbb{R}^M \) which are mapped into the basis function parameter space for each normalised interarrival time \( \tau_i \). The RNN is defined as:

\[
h_i = f(W h_{i-1} + v \tau_i + b),
\]

where \( W, v, b \) and \( h_0 \) are learnable parameters. \( f \) is any activation function such that the RNN is a universal approximator; for example the sigmoid function Schäfer and Zimmermann (2007).

Parameter Generation. To generate parameters for our basis functions, we define a linear transformation from the RNN hidden state vector into parameter matrix:

\[
P = A h_i + B, \quad t_i < t \leq t_{i+1},
\]

where \( P \in \mathbb{R}^{J \times |P|}, A \in \mathbb{R}^{J \times |P| \times M}, \) and \( B \in \mathbb{R}^{J \times |P|} \). We further define \( p_j = P_{(j, \cdot)} \in \mathbb{R}^{|P|}, \) for \( j \in \{1, \ldots, J\} \), which corresponds to the parameter of the \( j \)th basis function at some time \( t_i < t \leq t_{i+1} \).

Intensity Function. Using the parameters \( p_1, \ldots, p_J \) the intensity function with respect to interarrival time \( \tau = t - t_i \) is:

\[
\hat{\lambda}(\tau) = f_{SP} \left[ \sum_{j=1}^{J} \phi(\tau; p_j) \right], \quad t_i < t \leq t_{i+1},
\]

where \( f_{SP}(x) = \log(1 + \exp(x)) \) is the softplus function.

In practice, we normalise the interarrival times by their standard deviation to avoid numerical issues.

A variation one can use is a combination of different basis functions. As long as Equation 14 is composed of a finite combination of different basis functions types, which satisfy the preconditions for Corollary 6, universal approximation will hold in the limit of the number of basis functions. We consider combination of basis functions in our evaluation section below, where we use 4 ReLU basis functions and 4 power law basis functions.

Loss Function. We use the point process negative log-likelihood, as seen in Equation 3. In most cases the integral cannot be calculated analytically so instead we calculate it numerically using Monte-Carlo integration Press et al. (2007).

5. Evaluation

To test our UNIPoint model we consider two experiment settings: (1) we compare the performance of UNIPoint models to the baseline models described above; and (2) we compare the performance of the UNIPoint model over different numbers of basis functions. To evaluate the performance of our neural network point process model we use four synthetic datasets and two real world datasets. We also compare our model to traditional point process models and state-of-the-art neural network point processes.
5.1 Synthetic Datasets

We synthesise datasets from traditional point process models. For each model, we generate 2048 event sequences each containing 128 events. These point processes are chosen because they cover three of the four possible combinations of monotonic and non-monotonic intensity functions between events, and self-exciting and non-self-exciting point processes – with only the simplest combination of monotonic and non-self-exciting being omitted as we found no substantial findings. For each synthetic dataset we use a 60 : 20 : 20 training-validation-test split. The following traditional point processes are used for our synthetic datasets:

**Sine Inhomogeneous Poisson (InHom).** The intensity function is sinusoidal, given by

\[ \lambda^*(t) = \mu + \alpha \sin(2\pi t/\beta), \]

where \( \mu = 1.5, \alpha = 1.0, \) and \( \beta = 50. \)

**Self Correcting Process (SC).** The intensity function is

\[ \lambda^*(t) = \exp\left(\nu t - \sum_{t_i < t} \gamma\right), \]

where \( \nu = 1 \) and \( \gamma = 1. \)

**Exponential Hawkes Process (ExpHP).** The intensity function is a Hawkes process with exponential decaying triggering kernel, given by

\[ \lambda^*(t) = \mu + \alpha \beta \sum_{t_i < t} \exp(-\beta(t - t_i)), \]

where \( \mu = 0.5, \alpha = 0.8, \) and \( \beta = 1. \)

**Decaying Sine Hawkes Process (DSHP).** The intensity function is a Hawkes process with a sinusoidal triggering kernel product with an exponential decaying triggering kernel, given by

\[ \lambda^*(t) = \mu + \gamma \sum_{t_i < t} (1 + \sin(\alpha(t - t_i)) \exp(-\beta(t - t_i))), \]

where \( \mu = 0.5, \alpha = 5\pi, \beta = 2, \) and \( \gamma = 1. \)

5.2 Real World Dataset

We further test the practicality of our model with two real world datasets. Similarly to the synthetic datasets, we use a 60 : 20 : 20 training-validation-test split for each of the real world datasets. We use the following real world datasets for evaluation:

**MOOC\(^1\).** A dataset of student interaction in online courses Kumar et al. (2019), previously used for evaluating neural point processes Shchur et al. (2020). Events correspond to different types of interaction, e.g., watching videos.

**Reddit\(^1\).** A dataset of user posts on a social media platform Kumar et al. (2019), previously used for evaluating neural point processes Shchur et al. (2020). Each event sequence corresponds to a user’s post behaviour.

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1. [https://github.com/srijankr/jodie/](https://github.com/srijankr/jodie/)
5.3 Baselines

The following traditional and neural network based point process models are used as baseline comparison for our models:

**Exponential Hawkes Process (ExpHP).** The point process likelihood is optimised to determine parameter $\mu$, $\alpha$, and $\beta$ in intensity function

$$\lambda^\star(t) = \mu + \alpha \beta \sum_{t_i < t} \exp(-\beta(t - t_i)).$$

**Power Law Hawkes Process (PLHP).** The point process likelihood is only optimised to determine parameter $\mu$, $\alpha$, and $\beta$ in intensity function

$$\lambda^\star(t) = \mu + \alpha \sum_{t_i < t} (t - t_i + \delta)^{-(1+\beta)}.$$  

The optimisation choice is to compensate for the difficulty of the power law Hawkes process fitting Bacry et al. (2015).

**RMTPP Du et al. (2016).** We implement the RMTPP neural network architecture as one of the neural point process baselines. The intensity function of RMTPP

$$\lambda^\star(t) = \exp(v^T h_i + w(t - t_{i-1}) + b)$$  \hspace{1cm} (15)

is defined with respect to the RNN hidden state $h_i$. We use an RNN size of 32 for testing.

**Fully Neural (FN) Omi et al. (2019).** Another neural architecture we implement is the fully neural network point process. The integral of the intensity function (compensator) is defined as a neural network with RNN hidden state and event time input. We use a RNN size of 32 and fully connected layer of size 32 to produce the compensator.

5.4 Basis Functions and Experiments

Different configurations of our UNIPoint model are used in our two experiments. For the first experiment, we fit a UNIPoint model for each of the basis function types described in Table 1. We use the softplus transfer functions and 8 basis functions. Additionally, we consider the mixed combination of basis functions (ReLU+PL), where we consider a UNIPoint model with 4 powerlaw basis functions and 4 ReLU basis functions. We fit models for all synthetic and real world datasets.

For the second experiment, for each real world dataset we fit 5 sigmoid basis function UNIPoint models over with vary number of basis functions. We test $\{2, 4, 8, 16, 32\}$ basis functions.

To numerically calculate the corresponding loss function for all of our model variants, we use Monte-Carlo integration with 200 per event interval Press et al. (2007). All UNIPoint models tested uses a configuration where the RNN has a hidden layer size of 48, a batch size of 64, is trained with Adam Kingma and Ba (2014), and a $L^2$ weight decay set to $10^{-5}$. The RNN hidden layer size was chosen such that the number of learnable parameters for the UNIPoint model are comparable to the FullyNeural model ($\approx 3000$). The $L^2$ weight decay hyperparameter was selected through additional tests of additional synthetic data, independent of that used in experiments. Furthermore, the validation set described in the dataset section is used for an early stopping criteria, where training halts if the validation loss does not improve above $10^{-4}$ for 50 successive epochs.
5.5 Evaluation Metrics

We use two evaluation metrics: (1) holdout log-likelihood; and (2) total variation.

**Holdout Log-likelihood.** As each of the baseline and UNIPoint models have a corresponding intensity function, we are able to calculate the log-likelihood using Equation 3. We numerically calculate the integral term with Monte-Carlo integration Press et al. (2007) if it cannot be calculated analytically. Additionally, we calculate the per event difference of log-likelihood score for each dataset, as per prior work Shchur et al. (2020).

**Total Variation.** We use total variation since it is similar to the uniform metric in that they both depend on the difference of true and approximate intensity functions. It is defined as:

\[
TV(f, g) = \int |f(s) - g(s)|^2 ds.
\]  

Total variation can only be used on synthetic datasets where the true intensity function is known. We calculate the total variation via Monte-Carlo integration Press et al. (2007).

6. Results

We present the experimental results used to measure the performance of our UNIPoint point process. Specifically we consider the two sets of experiments: (1) a comparison between the baseline models and basis-sum models; and (2) a comparison of basis-sum models over different number of basis functions. The log-likelihood scores and total variation scores are used to evaluate the synthetic datasets. Whilst only log-likelihood scores are used for evaluating the real world datasets since calculating the total variation requires a ground truth intensity function.

6.1 Comparison to Baselines

UNIPoint performs significantly better than the traditional Hawkes process models on both synthetic and real world datasets. This can be seen in Table 2 and Figure 3. In Table 2 all UNIPoint models perform better than the exponential Hawkes process and the powerlaw Hawkes process, which is expected as they all share the universal approximation property in the limit of basis functions. In Figure 3, which shows total variation, UNIPoint also outperforms all traditional Hawkes process models except on the ExpHP dataset where the exponential Hawkes process performs best since it has a functional form that exactly matches the underlying process. Over the real world datasets the UNIPoint model shows even greater gains in performance over traditional baseline models. For example, on the MOOC dataset the best performing traditional Hawkes process model scores 0.532 in average log-likelihood while the best UNIPoint model scores 4.855 in average log-likelihood. This highlights the importance of model flexibility when modelling complex real world phenomena.

The neural point process baselines produce log-likelihood and total variation scores that are much closer to the UNIPoint model when compared to that of the traditional Hawkes processes on both synthetic and real world datasets. Still, UNIPoint outperforms RMTPP as shown in Table 2 regardless of which basis function is used for UNIPoint. UNIPoint has similar performance to the state-of-the-art neural network point process model Fully Neural...
In the synthetic datasets, if the form of the point process used to generate the data matches that of the fitted model, then the model tends to perform well – even better than the neural models. For example, see the ExpHP dataset and exponential Hawkes baseline in Figure 3. Intuitively, the exponential basis function seems a good choice for UNIPoint in the ExpHP dataset, but we find that the exponential basis function model does not significantly
outperform the other basis functions. This leaves open the question of how to pick a good basis function even in very simple cases.

The choice of basis functions does affect the performance of the UNIPoint model on the real world datasets. For example, on the MOOC dataset the power law basis function scores 4.805 in log-likelihood while the sigmoid basis function scores 4.471. There does not appear to be any single best basis function since the power law performs the best on MOOC while the sigmoid performs the best on Reddit. Further, a combination of basis functions can help increase the expressiveness of a UNIPoint model, where the mixed basis UNIPoint model performs best, with a log-likelihood score of 4.855, which consists of a both ReLU and power law basis functions. This mixture was chosen by mixing the best two individual basis functions. It is therefore still important to choose a good basis function for the particular dataset; however, the choice of basis function is much less significant than for traditional Hawkes models.

The ReLU basis function satisfies the perturbation condition in Theorem 8 and performs well in all evaluations; however, other basis satisfying this criteria did not perform so well. The cosine and sigmoid basis functions both satisfy the perturbation condition (as they are bounded functions). However, the cosine basis function performs poorly on both real world datasets and the sigmoid basis function performs poorly on the MOOC dataset in comparison to other basis functions. We note that the good performance of the ReLU basis functions could also be attributed to the primarily linear form of the ReLU being particularly conducive to optimisation Goodfellow et al. (2016). However, with the mixture of the power law basis function, which does not satisfy the perturbation condition, the mixed (ReLU+PL) UNIPoint model does not have consistent performance over all datasets. Although it performs very well for MOOC, it performs worse than the pure ReLU model, and even worse than the power law model.

6.2 Number of Basis Functions

Evaluations of the UNIPoint model with different numbers of basis functions shows the performance is not particularly sensitive to the number of basis functions for the synthetic datasets. For example, when using sigmoid basis functions on the self-correcting dataset, the log-likelihood was $-0.777 \pm 0.003, -0.779 \pm 0.003, -0.775 \pm 0.003, -0.776 \pm 0.003$, and
Figure 4: The log-likelihood (higher is better) of sigmoid basis function UNIPoint model over various numbers of basis functions. The plots are over the MOOC dataset (left) and Reddit dataset (right). Each point corresponds to the average log-likelihood and the error bar corresponds to the 95% confidence interval.

0.775 ± 0.003 for 2, 4, 8, 16, and 32 basis functions respectively. For brevity, results for all synthetic datasets are omitted as they show a similar pattern.

In real world datasets, we find that performance increases with the number of basis functions but reaches a plateau, after which increasing the number of basis functions does not lead to improvement. Figure 4 shows the average log-likelihood over the MOOC and Reddit dataset when using sigmoid basis functions. For both of the datasets we see that 2 and 4 basis functions perform worse than the 8 basis function configuration used in the first experiment. Interestingly, the UNIPoint model still out-performs the traditional point process baselines and the RMTPP baselines even with only 2 basis functions. For 16 and 32 basis functions we see little improvement even measuring a slight decrease in performance. The cases in which performance decreases is could due to the UNIPoint model becoming more difficult to fit. Although Theorem 8 states that we should be able to achieve arbitrary accuracy using the UNIPoint model as basis functions increase, it does not account for the learnability of the model. We leave this problem for future work. Alternatively, the performance decrease could be due to the larger number of basis functions over-fitting the training data.

6.3 Number of Integration Steps

Changing the number of numerical integration steps per time interval in log-likelihood evaluation did not significantly change the loss function calculation for our UNIPoint model. For example, the fitted UNIPoint with 8 sigmoid basis functions evaluated on the self-correcting test set gives a log-likelihood of $-0.7739 \pm 0.0031$ for 200 steps per interval, a log-likelihood of $-0.7736\pm 0.0030$ for 100 steps per interval, and a log-likelihood of $-0.7739\pm 0.0030$ for 50 steps per interval. Thus UNIPoint is not particularly sensitive to the number of integration steps.

7. Conclusion

Focusing on the theory universal approximation, we develop a new method for approximating temporal point processes. We present a method for turning universal approximators
for continuous functions into universal approximators for point process intensity functions. Furthermore, we provide the distinction between universal approximation for a single event versus a sequence of events, bridging this gap in prior works. With these theoretical considerations, we develop UNIPoint: a new neural network point process model which performs well on synthetic and real world benchmarks. Although UNIPoint does not outperform the fully neural network model Omi et al. (2019) on all datasets, investigating the learnability of UNIPoint in relation to different basis functions could lead a method with stronger performance. Moreover, UNIPoint could be adapted to directly approximate different aspects of a point process such as the point process compensator Omi et al. (2019) or the event distribution Shchur et al. (2020), which could also lead to broader theoretical foundation and better performance.

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