A continuum of incomplete intermediate logics*
(corrected version)

Tadeusz Litak
Department of Logic, Jagiellonian University
Grodzka 52, 31-044 Cracow
lt@konto.pl

Note 2018: This paper was originally published in Reports on Mathematical Logic 36, 2002, pp. 131–141. I have recently noted that the proof of one of theorems in it was incorrect; it was also independently discovered by Guillaume Massas (UC Irvine). This does not concern the main result claimed in the title (Theorem 11), which seems unassailable, but rather my attempt to present the proof of Theorem 5 essentially due to Shehtman, without, as I say below, “a superfluous use of transfinite induction” (i.e., differing with the original paper [Sh77] and my own Master’s Thesis). My version of proof is fixable; I would like to thank Guillaume for coming up with the idea. Hopefully, some of his work towards generalizing such results will be published soon. Let us also note that Valentin Shehtman himself points out that the proof in the 1980 paper [Sh80] or in his more recent Habilitation Thesis has already been simplified compared with the one in the original reference [Sh77]. Apart from this crucial fix (and adjusting one reference), I left this earliest paper of mine unchanged on principle, even though I was tempted to polish up—at the very least—its style, narration and English.

Abstract

Although in 1977 V.B. Shehtman constructed the first Kripke incomplete intermediate logic, no-one in the known literature has completed his work by constructing a continuum of such logics. After a substantial reminder on how an incomplete logic can be obtained, I will construct a sequence of frames similar to those used by Jankov and Fine. None of these frames can be reduced by a p-morphism to another; at the same time, there are no p-morphisms from generated subframes of the Fine frame onto any frame from the considered sequence. All of the frames satisfy all of Shehtman’s axioms. Therefore, by using the characteristic formulas of the frames from the sequence it is possible to obtain the desired conclusion.

*This article is based on a paper delivered at the 4th International Tbilisi Symposium on Language, Logic and Computation (September 2001).
In the 1970’s, a number of important, deep and technically complicated results concerning relational semantics for modal logics was obtained by such authors as S. Thomason, K. Fine, M.S. Gerson, R.I. Goldblatt, J. F. A. K. van Benthem and W. Blok; it was the Golden Age of the subject, see [Bu82], [Bu83] and [ChZ97] for references and summaries of the most important works. The main goal of my paper is to draw attention to the fact that many important results lack superintuitionistic analogues, although the task of transferring them is highly nontrivial.

This gap may be partially due to the fact that Kripke semantics never became as popular in the realm of intermediate logics as they are in the realm of modal logics, which are more suitable and flexible tools to deal with frames. There were fewer experts working on relational semantics for intuitionistic logics. In 1977, one of the most distinguished persons in the field, V. B. Shehtman, constructed the first Kripke incomplete intermediate propositional logic. His construction was based mainly on a frame from [Fi74b], but he very ingeniously used a formula introduced in [GdJ74]. Nevertheless, he did not follow Fine’s suggestion that it seems to be possible to construct a continuum of incomplete logics. Such a continuum of $S4$ logics was presented in [Ry77] in the same year as Shehtman’s construction; it is known, however, that the incompleteness of a modal logic does not imply the incompleteness of its intuitionistic equivalent. In [On72] one may find the claim that there exists a continuum of incomplete predicate superintuitionistic logics. Unfortunately, this claim is given without proof; besides, it is far easier to construct an incomplete predicate superintuitionistic logic than to construct an incomplete propositional superintuitionistic logic. It is truly surprising but up to this day no-one has presented a proof that there exists a continuum of such logics. I shall attempt to fill in this gap.

In this paper I shall try to conform to the standard definitions and symbols which may be found, for example, in a monograph by Chagrov & Zakharyaschev [ChZ97]. Nevertheless, for the sake of convenience, let me remind the most standard ones. Unless otherwise stated, by a logic I shall mean a superintuitionistic (intermediate) logic.

**Definition 1** A (Kripke) structure/frame consists of a set and a relation of partial order $F = \langle W, \leq \rangle$.

**Definition 2** A substructure/subframe of a structure $F = \langle W, \leq \rangle$ is a frame $G = \langle V, \leq_1 \rangle$ where $V \subseteq W$ and $\leq_1 = V^2 \cap \leq$.

**Definition 3** A (Kripke) model is an ordered pair $M = (F, \mathcal{B})$ consisting of a frame $F = \langle W, \leq \rangle$ and a function $\mathcal{B}$ from the set of propositional
variables to the set of upward closed subsets of $W$. Valuation is extended to all formulas in the usual way.

I would like now to introduce two technical notions, weaker than finite approximability (finite model property) and stronger than completeness.

**Definition 4** A logic is fa-approximable iff the set of its theorems coincides with the set of all formulas which are true in some class of rooted frames with no infinite antichains.

**Definition 5** A logic is ac-approximable iff the set of its theorems coincides with the set of all formulas true in some class of frames with no infinite ascending chains — Chagrov & Zakharayashev call such orders Noetherian.

Professor A. Wroński has suggested that fa-approximability implies ac-approximability. This would give rise to the following picture:

\[
\text{finite approximability } \Rightarrow \text{fa-approximability } \Rightarrow \text{ac-approximability } \Rightarrow \text{completeness.}
\]

In my paper, I shall prove that there exists a continuum of propositional logics even outside the broadest class, i.e. the class of all complete logics. Nevertheless, first let me describe how an incomplete logic can be obtained — it is an easy generalization of Shehtman’s method [Sh77].

**Theorem 1** A logic $L$ lacks ac-approximability iff its modal companion above $\text{Grz} \tau L$ is incomplete.

**Proof.** It is enough to recall that $\text{Grz}$ is complete with respect to all partial orders without infinite ascending chains. ⊣

**Theorem 2** If there exists a rule of the form

\[
(\psi \lor (\psi \rightarrow e(\chi))) \rightarrow \chi
\]

($e$ is any uniform substitution) which is not admissible in some intermediate logic, then this logic lacks ac-approximability and thus lacks the finite model property.
Figure 1: A model refuting $\chi$ but verifying $\psi \lor (\psi \rightarrow e(\chi)) \rightarrow \chi$

**Proof.** (sketch) In any family of frames adequate for the logic (if there exists such) there must be a frame validating

$$(\psi \lor (\psi \rightarrow e(\chi))) \rightarrow \chi$$

with all substitutions (because the formula belongs to the logic) and refuting $\chi$ under some valuation. It can be easily seen that such a frame must contain an infinite ascending chain — see figure 1.

**Corollary 3** If an intermediate logic satisfies the assumptions of theorem 2, then its companion above $\text{Grz}$ is incomplete.

**Proof.** A consequence of theorems 1 and 2.

In fact far more can be proved about such a logic — see my forthcoming paper [Li02].

**Theorem 4** If there exists a rule of the form

$$(\psi \lor (\psi \rightarrow e(\chi))) \rightarrow \chi$$

which is not admissible in a logic $L$, then in any class of frames adequate for $L$ (if there exists any) there must be a structure containing an infinite comb or a willow (see fig. 2) as a substructure; thus, $L$ must lack both $ac$-approximability and $fa$-approximability.
Proof. Similar to the proof of theorem 2 — see fig. 4.

Let me recall the celebrated Gabbay-de Jongh axioms \cite{GdJ74}

\[ bb_n := \bigwedge_{i=0}^{n} ((p_i \rightarrow \bigvee_{j \neq i} p_j) \rightarrow \bigvee_{j \neq i} p_j) \rightarrow \bigvee_{i=0}^{n} p_i \ (n \geq 1) \]

which are complete with respect to the class of all finite frames of branching \( n \). It is well known that they can be refuted in the infinite comb. Nevertheless, not every frame containing the infinite comb as a substructure refutes these axioms — see figure 3. Therefore the following theorem is nontrivial:

**Theorem 5** If there exists a rule of the form

\[
\begin{align*}
\psi & \rightarrow e(\chi) \\
\psi & \leftrightarrow \varsigma \\
\varsigma & \rightarrow \tau \\
\chi & \leftrightarrow \psi \lor e(\tau)
\end{align*}
\]

which is not admissible in some intermediate logic \( L \), then in any class of frames adequate for \( L \) (if there exists any) there must exist a structure refuting \( bb_n(n \geq 2) \). Thus, if \( L \) contains any of Gabbay-de Jongh axioms, it must be incomplete.

Proof. It may be carried out in a manner similar to that of Shehtman \cite{Sh77}, but it is needlessly complicated, e.g. with a superfluous use of transfinite induction. Therefore I would like to sketch a more elegant and intuitive proof. Assume then that there is a frame \( F \) for \( L \), a valuation \( V \) and a point \( x \) in \( F \) such that \( x \not\models_V \chi \). It is easy to check that \( x \) must be

![Figure 2: An infinite comb](image-url)
the root of the submodel of \( \langle \mathcal{F}, \mathcal{V} \rangle \) depicted by picture \( \text{图} \). Now let me define a new valuation \( \mathcal{B} \) based on \( \mathcal{V} \) and inspired by figure \( \text{图} \). TML2018: Here is where the original 2002 text is edited.

\[
\mathcal{B}(p_i) := \bigcap_{\forall m \in \omega \ n \neq 3n+i} \mathcal{V}(e^n(\psi)).
\]

Axioms of \( L \) and Figure \( \text{图} \) ensure that sets \( \mathcal{B}(p_0), \mathcal{B}(p_1) \) and \( \mathcal{B}(p_2) \) are distinct and non-empty. It is easily seen that the consequent of \( \text{bb}_2 \) is refuted at \( x \) under the valuation \( \mathcal{B} \). Now suppose that there is some \( y \geq x \) such that some conjunct of the premise of \( \text{bb}_2 \) is classically refuted at \( y \), e.g.,

\[
y \models_{\mathcal{B}} p_0 \implies (p_1 \lor p_2)
\]

and

\[
y \not\models_{\mathcal{B}} p_0 \lor p_1 \lor p_2.
\]

(2) and (3) taken together imply

\[
y \not\models_{\mathcal{B}} p_0 \lor p_1 \lor p_2.
\]

(4)

We claim that

\[
\exists n \in \omega \ y \not\models_{\mathcal{V}} e^n(\chi).
\]

(5)

To see this, assume (5) does not hold, that is, \( e^n(\psi) \lor e^{n+1}(\tau) \) is \( \mathcal{V} \)-satisfied at \( y \) for every \( n \). Pick the smallest \( m \) s.t. \( y \not\models_{\mathcal{V}} e^m(\psi) \); it exists by (3). This means \( e^{m+1}(\tau) \) must be satisfied, thus yielding \( y \models_{\mathcal{V}} e^{m'}(\psi) \) for every \( m' > m \). As by the assumption on \( m \) we have the same for every \( m' < m \) as well, we thus contradict (4).

Hence, we can pick the smallest \( m \) s.t. \( y \not\models_{\mathcal{V}} e^m(\chi) \). Note that for any \( m' \leq m \), \( y \not\models_{\mathcal{V}} e^{m'}(\tau) \), and hence our assumption on \( m \) holds only if for any \( m' < m \), \( y \models_{\mathcal{V}} e^m(\psi) \). We can find an infinite comb similar to

Figure 3: A structure containing an infinite comb as a substructure where Gabbay-de Jongh axiom \text{bb}_2 \) is true
Figure 4: A submodel of \( \langle F, \mathcal{V} \rangle \) whose root refutes \( \chi \).

the one in Figure 4, but whose root this time is \( y \) and whose labelling is obtained by replacing each formula in Figure 4 by its suitably iterated \( e \)-substitution; think of the subframe generated by the \( m \)-th point up the trunk. It is consequently possible to find some (in fact, infinitely many) points from this comb classically refuting \( p_0 \rightarrow (p_1 \lor p_2) \), contradicting (2).

TML2018: The rest of the paper is left in the form it was written in 2002.

It may be worth mentioning that rule \( \text{I} \) is as a matter of fact inspired by the form of axioms in Shehtman’s later paper [Sh80]. In his paper from 1977 [Sh77] the axioms were more complicated and to make Shehtman’s 1977 theorem a consequence of theorem \( \text{E} \) — as I am going to do — rule \( \text{I} \) should be replaced by the following one:

\[
\begin{align*}
\psi & \leftrightarrow \varsigma \rightarrow \tau \\
\tau & \rightarrow e(\tau) \\
\chi & \leftrightarrow \psi \lor e(\psi) \\
e(\psi) & \rightarrow \psi \lor e(\tau)
\end{align*}
\]

(6)

Now let me consider a family of formulas introduced by Shehtman:

\[
\begin{align*}
\beta_{-1} & := p, & \gamma_{-1} & := q, \\
\beta_0 & := q \rightarrow p, & \gamma_0 & := p \rightarrow q, \\
\beta_{n+1} & := \gamma_n \rightarrow \beta_n \lor \gamma_{n-1}, & \\
\gamma_{n+1} & := \beta_n \rightarrow \gamma_n \lor \beta_{n-1}, & \\
\alpha_n & := \beta_{n+2} \land \gamma_{n+2} \rightarrow \beta_{n+1} \lor \gamma_{n+1} & (n \in \omega), \\
\eta & := \alpha_0 \rightarrow \alpha_1 \lor \alpha_2, & \epsilon & := \alpha_0 \lor \alpha_1, \\
\delta & := \eta \rightarrow \epsilon, & \kappa & := \alpha_1 \rightarrow \alpha_0 \lor \beta_2.
\end{align*}
\]
If $\varsigma$ stands for $\beta_2 \land \gamma_2$, $\tau$ stands for $\beta_1 \lor \gamma_1$ and $e$ is defined as follows:

$$e(p) := q \lor (q \rightarrow p), \quad e(q) := p \lor (p \rightarrow q),$$

then the following observation allows me to use a variant of theorem 5 concerning rule 6.

- $\alpha_0$ is of the form $\psi$, i.e. $\varsigma \rightarrow \tau$;
- $\epsilon$ is of the form $\chi$, i.e. $\psi \lor e(\psi)$;
- $\delta$ is equivalent to $(\psi \lor (\psi \rightarrow e(\chi))) \rightarrow \chi$;
- $\kappa$ intuitionistically implies $e(\psi) \rightarrow \psi \lor e(\tau)$;
- $\tau \rightarrow e(\tau)$ is an Int-tautology.

Of course, it would also be possible to use theorem 5 without any modification. In this case one should define $\epsilon$ as $\alpha_0 \lor \beta_2 \lor \gamma_2$ or even $\alpha_0 \lor \beta_2$, $\delta$ as $(\alpha_0 \rightarrow \alpha_1 \lor \beta_3) \rightarrow \alpha_0 \lor \beta_2$ and no $\kappa$ is needed at all. Nevertheless, I am going to stick to the first paper of Shehtman to make references easier; the paper from 1980 [Sh80] is less known.

**Lemma 6** Axioms $\delta$ and $\kappa$ are true in a structure known as the Fine frame (see figure 5). Axiom $bb_2$ is true in a general frame based on the Fine frame and generated by the two upward closed singletons. The same general frame refutes axiom $\epsilon$.

**Proof.** It is quite easy and may be found, for example, in [Sh77]. ⊣

**Corollary 7 (Shehtman)** An intermediate logic $L$ determined by axioms $\delta$, $\kappa$, and $bb_2$ is incomplete.

**Proof.** A consequence of theorem 5 and lemma 6. ⊣

Now I may construct a continuum of incomplete logics inspired by ideas from Kit Fine’s classical papers [Fi74a], [Fi74b]. I will construct a sequence of frames $F_n$ (see fig. 6) very similar to the sequence from [Fi74a].

**Lemma 8** For any $n \in \omega$, $F_n \models \delta \land \kappa \land bb_2$. Besides, $F_n \not\models \epsilon$.

**Proof.** The fact that the Gabbay-de Jongh axioms are true in all of those frames is obvious. It is impossible to simultaneously refute $\alpha_0$ and $\alpha_1$ in any of the frames, which implies that $F_n \models \delta \land \epsilon$. The validity of $\kappa$ may be shown in the same way as in case of the Fine frame. ⊣
Figure 5: The Fine frame

Figure 6: Frames $\mathcal{F}_0$, $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$
Lemma 9  For any $n \in \omega$, there exists no $p$-morphism from any generated subframe of $F_n$ onto $F_m$ ($m \neq n$). In other words,

$$F_n \models \beta^#(F_m, \bot)(m \neq n),$$

where $\beta^#(F_m, \bot)$ is a Jankov formula for $F_m$.

PROOF. It is similar to the one in [Fi74a] (by induction). ⊢

Lemma 10  For any $n \in \omega$, there exists no $p$-morphism from any generated subframe of the Fine frame onto $F_n$. In other words, Jankov formulas for the entire sequence are satisfied in the Fine frame.

PROOF. As above. ⊢

Theorem 11  Distinct subsets of natural numbers generate distinct intermediate logics whose axioms are $\delta$, $\kappa$, $\mathbb{B}_2$, and the Jankov formulas of those frames from the sequence whose indices belong to a given subset of $\omega$. All of these logics are incomplete.

PROOF. The fact that these logics are all distinct is a consequence of lemmas 8 and 9. The fact that these logics are incomplete follows from theorems 5 and lemmas 9 and 10 — a suitable inference rule is not admissible in any of the logics. ⊢

I would like to thank Professor A. Wróński, the supervisor of my master’s thesis, for his constant help and advice.

References

[Bu82] R.A. BULL. Review. Journal of Symbolic Logic, 47:440-445, 1982

[Bu83] R.A. BULL. Review. Journal of Symbolic Logic, 48:488-495, 1983

[ChZ97] A.V. CHAGROV and M.V. ZAKHARYASCHEV. Modal Logic. Clarendon Press, Oxford 1997.

[Fi74a] K. FINE. An Ascending Chain of $S^4$ Logics. Theoria, 40:110-116, 1974.

[Fi74b] K. FINE. An Incomplete Logic Containing $S^4$. Theoria, 40:23-29, 1974.
[GdJ74] D. M. GABBAY AND D. H. J. DE JONGH. A Sequence of Decidable Finitely Axiomatizable Intermediate Logics with the Disjunction Property. *Journal of Symbolic Logic*, 39:67-78, 1974.

[Ja68] V. A. JANKOV. Constructing a Sequence of Strongly Independent Superintuitionistic Propositional Calculi. *Soviet Mathematics Doklady*, 9:806-807, 1968.

[Li02] T. LITAK. Modal incompleteness revisited. To appear in *Studia Logica*, 2003.

[On72] H. ONO. A Study of Intermediate Predicate Logics. *Publ. RIMS, Kyoto University*, 8:619-649, 1972/73.

[Ry77] V. V. RYBAKOV. Noncompact Extensions of the Logic $S_4$. *Algebra and logic*, 16:321-334, 1977.

[Sh77] V. B. SHEHTMAN. On Incomplete Propositional Logics. *Soviet Mathematics Doklady*, 18:985-989, 1977.

[Sh80] V. B. SHEHTMAN. Topological models of propositional logics. *Semiotics and Information Science*, 15:74–98, 1980. (Russian)