A Laughlin-like approach to some symmetry protected topological phases

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The Laughlin wavefunctions offer an elegant and intuitive picture of the quantum Hall effect. Besides being extremely good approximations, they provide an algebraic playground to study various quantum Hall phases, their structure embodies the concepts of flux attachment and hidden order, and furthermore they provide the basis for efficient numerical methods. Here we construct analogous tensor product states (TPSs), describing the ground states of various symmetry protected topological phases (SPTs) in one and two dimensions. Their analogy with the Laughlin wavefunction follows from replacing the $U(1)$ electromagnetic symmetry obeyed by these functions, with the symmetry preserving the SPT. In particular, both wavefunctions are written in the charge basis of their symmetry and both are a product of a broken symmetry state times a phase factor controlling the flux attachment. For the 1D case, we validate our approach by making explicit relations with the group cohomology formalism and matrix product state classification. For the 2D Ising SPT, analytical and numerical evidence is given to show that the TPS indeed describes the topological phase.

I. INTRODUCTION

In the past three decades, topological phases have attracted a large amount of interest due to their tendency to exhibit highly robust quantum phenomena which has various applications in quantum engineering and metrology. One of the current frontiers in the field aims at understanding the variety of novel topological phases which arise when some extra symmetries are not allowed to be broken. For example, considering the case of time reversal symmetry, one may ask whether new topological phases exist in spin systems, provided that no magnetic fields or magnetic impurities are allowed. For this case, it was shown that there are two topologically distinct Ising paramagnets in both one and two dimensions (1D, 2D). Both phases are “integer” (or short range entangled) paramagnetic phases; however they are distinguished by the absence or presence of protected boundary excitations. By now a variety of integer and fractional (long range entangled) SPTs are known to be theoretically possible along with some experimental realizations. The latter include 2D and 3D topological insulators, the Haldane chain, and recently an experiment realizing a 2D bosonic SPT has been proposed.

At least in 2D, the quantum Hall effect (QHE) is the basic building block of most topological phases. For example, upon replacing the layer index by a spin index, a quantum Hall bilayer in which each layer experiences an opposite magnetic field (a $(1,1,0)$ state) describes a topological insulator. Similarly a $(0,0,1)$ quantum Hall bilayer state can be thought of as a bosonic SPT, protected by the $U(1)$ symmetry associated with the charge difference between the layers. Furthermore, by condensing charges such that this symmetry is broken down to a discrete subgroup $G$, one can obtain a large variety of abelian 2D SPT phases. Also in two recent works, effects of flux and charge binding, analogous to those occurring in the QHE, have been established for several types of SPTs.

A powerful tool in analyzing the quantum Hall effect are Laughlin’s wavefunctions. Besides being extremely good approximations, they offer a simple picture of the groundstate and excitations of these phases, they embody the idea of flux attachment which is the basis of composite particle theory. They allow the construction of parent Hamiltonians for which they are the exact ground state, and they are amendable to numerical explorations. Furthermore, they reveal an intriguing connection between topological phases and broken symmetry phases via the concept of hidden long range order. Indeed, with complex phases removed, Laughlin’s wavefunctions appear as states in which the $U(1)$ charge symmetry is spontaneously broken. Curiously, a similar type of hidden long range order exists in other 1D SPTs.

Here we generalize the Laughlin wavefunctions to the SPT context by exchanging the $U(1)$ symmetry of these functions, by the symmetry, $G$, protecting the SPT. This results in Laughlin-like wavefunctions describing some bosonic SPTs in one and two dimensions. More specifically, we write an ansatz for the groundstate of SPTs on a lattice using a specific type of tensor product state (TPS). The TPS is written in the basis where $G$ acts diagonally (the symmetry-charge basis) and appears as a broken symmetry state dressed by a phase factor which attaches fluxes to charges of the symmetry $G$. For each $Z_N \times Z_N$ symmetry in 1D, we use our ansatz to constructively obtain the groundstates of the $N$ possible SPTs in this class. We validate our construction either by presenting the states as a matrix product states (MPSs) and appealing to the known classification or by making a local unitary transformation which maps our states onto the ones obtained using group cohomology. In 2D we study the Ising ($G = Z_2$) SPT, we again obtain a candidate state for the groundstate and give analytical and numerical evidence that our state indeed describes this topological phase.
II. THE TPS ANSATZ

We motivate our TPS ansatz by first rewriting the Laughlin wavefunction as a TPS. Of course a finite rank TPS representation of a Laughlin state, which holds in the thermodynamic limit, is believed to be impossible [24]. Accordingly, our TPS has both a continuous internal index (infinite rank) and it indeed fails in the thermodynamic limit. Both these pathologies will be removed once we replace the $U(1)$ symmetry by the discrete symmetry protecting the SPT.

Consider the Laughlin wavefunction at $1/m$ filling with $n$ particles

$$
\Psi_m(z_1, ..., z_n) = \Pi_{i<j}(z_i - z_j)^m e^{-\sum_i \frac{1}{2} |z_i|^2}. \tag{1}
$$

We introduce a $U(1)$ scalar field $\phi \in [0,1]$ and use it to decouple the particle-particle terms

$$
\Psi_m(z_1, ..., z_n) = \int D\phi e^{-\int d^2 r \frac{1}{2} (\nabla \phi)^2 + i \pi m \phi(\rho(z))} \Pi_i A_i \tag{2}
$$

where $\rho(z) = \sum_i \delta(z - z_i)$, $O_{zi}$ is some circle around $z_i$ with an $\epsilon$ radius, $\beta$ is to be determined below, and we only allow vorticity around $z_i$’s and nowhere else. To verify the correctness of the above equation, note that the delta function constraints can be removed by the following transformation $\phi \rightarrow \phi + \sum_i \frac{1}{2\pi} \Im \log(z - z_i)$ (where $\Im$ denotes taking the imaginary part). As described in detail in App. (A), two additional particle terms are generated by this: The first is $\frac{i\pi m}{2} \int d\phi \rho(z) \sum_i \Im \log(z - z_i) = \frac{i\pi m}{2} \sum_{i,j} \Im \log(z_j - z_i)$, coming from $i\pi m \phi(z)$. The second is $\frac{1}{2\pi} \sum_{i,j} \Im \log(z_j - z_i)$ coming from the extra field energy due to vorticity. Completing the square on preexisting $\int i\pi m \phi(z) = i\pi \sum_i \phi(z_i)$ term, simply results in an addition to the second term with all particle terms now decoupled from particle terms. The $\phi$ field can then be integrated out leading to some normalization constant. Choosing $\beta = \pi m$, the particle terms sum to $\frac{1}{2\pi} \sum_{i,j} \log(z_j - z_i)$, and give the desired $\Pi_{i<j}(z_j - z_i)^m$ factor [25]. Notably the above $\beta$, for $m > 4$, is in the ordered side of the Kosterlitz-Thouless transition ($\beta > \beta_c \approx 8\pi$ [26]).

Next we restrict position to a dense lattice $(z_{ab})$, where $a$ and $b$ are the $x$ and $y$ coordinates, with spacing much smaller than 1 (the magnetic length), and discretize $\rho(z)$ and $\phi(z)$ to $\rho_{ab} \in \mathbb{Z}_+$ and $\phi_{ab} \in [0,1]$. This yields

$$
\langle \{\rho_{ab}\}|\Psi_m\rangle = \int \Pi_{ab}[d\phi_{ab}] A_\phi(\rho_{ab}) \tag{3}
$$

where $\langle \{\rho_{ab}\}\rangle$ is a state in the occupation basis and derivatives should be understood as their lattice versions involving nearest neighboring lattice sites. The above expression now appears as a TPS with $\phi \in U(1)$ serving as the tensor index. Notably, beyond a certain system size, this construction will fail due to screening of vortices by circle discontinuities (see App. (A)).

To carry Eq. (3) into the SPT context we note that the charges $\rho_{ab} \in \mathbb{Z}_+$ label positive representations of $\phi \in U(1)$. We thus replace the $U(1)$ symmetry by an abelian discrete symmetry $G$, and replace charges by representations of $G$. Since $G$ representations have a cyclic group structure, there is no sense of restricting to positive representations. Accordingly, we allow all representations and remove the term, $e^{-\sum_i \frac{1}{2} |z_i|^2}$, which was introduced to neutralize the overall charge [14]. For simplicity, we also remove the $\phi$ field strength term $(\nabla \phi)^2$.

Last, we fix our definition of a lattice derivative by demanding it to fulfill a consistency condition, given below, which is inspired by the notion of flux attachment. Interestingly, this appears to work well both for 1D and 2D SPT phases.

We turn to write the TPS explicitly. We consider $Z_N$ groups and represent the group elements $(\phi)$ as the numbers $0, 1, ..., N - 1$ and group action as addition modulo $N$. A product $\phi_a \phi_b$ means adding $\phi_b$’s $\phi_a$-times. Representations are labeled by the symmetry-charges $\alpha \in Z_N$ and for the character tables we use $\chi_{\phi}(\alpha) = e^{\frac{2\pi i}{N} \alpha \phi}$. Our Hilbert space is either a 1D lattice or a 2D triangular lattice where each site is occupied by a symmetry-charge degree of freedom, again thought of as an integer in $0, 1, ..., N - 1$.

Unlike some other approaches [1], we work in the on-site symmetry-charge basis $|\{\alpha\}\rangle = |\alpha_1\rangle|\alpha_2\rangle...$. On this basis the action of $g \in G$, on a site $i$ is diagonal and given by

$$
\langle \{\alpha\}|\chi_g(\alpha_1)\rangle = \langle \{\alpha\}|\chi_g(\alpha_2)\rangle = \cdots \tag{4}
$$

Tensor indices $(\phi_i)$ are elements in $Z_N$ and also sit on sites rather than bonds. Our TPS ansatz for an SPT wavefunction in a phase $m$ $|\psi_m\rangle$ is

$$
|\{\alpha\}|\psi_m\rangle = \sum_{\{\phi\} \in G} \Pi_i A_{\phi_{i,j}}(\alpha_j) \tag{5}
$$

where $\phi_{i,j}$ denotes the nearest neighbors of $\phi_i$. The lattice derivative operator for dimension $d$ $d_{\phi_{i,j}}(\{\phi\})$, which is some function of $\phi_{i,j}$, and the local phase factors $(e^{i\theta_m(\phi_{i,j})})$, are determined next.

We demand the following generalized flux attachment condition on the overall phase factor of the TPS $(\Theta_m = \sum_j \theta_m(\phi_{i,j}))$

$$
\Delta_j e^{i\theta_m(\phi)} = \frac{e^{i\theta_m(\phi_i) + 1_j}}{e^{i\theta_m(\phi_i)}} = \chi_m(d_{\phi_{i,j}}(\phi)), \tag{6}
$$

where $\{\phi\} + 1_j$ means adding 1 to $\phi_j$ within the set $\{\phi\}$. Note that the same condition is obeyed by Eq. (3) in
the bosonic case. The above equation implies the following consistence condition on $d_1$, derived by requiring $\chi_m(d_{[\{\phi\}]})$ to be an exact “differential” for $e^{i\Theta_m(\{\phi\})}$

$$\frac{\chi_m(d_{[\{\phi\}] + 1_{j,j^\prime}})\chi_m(d_{[\{\phi\}]_{j,j^\prime}})}{\chi_m(d_{[\{\phi\} + 1_{j,j^\prime}})\chi_m(d_{[\{\phi\}]_{j,j^\prime}})} = 1,$$

(7)

where $j$ and $j^\prime$ are two neighboring sites. One can also write down an onsite consistence condition, requiring $N$ consecutive applications of Eq. (6) on the same site to yield an overall factor of 1. Provided that $d_{[\{\phi\}]}$ is not a function of site $j$ itself, this will be automatically obeyed since $(\chi_m(\alpha))^N = \chi_m(\alpha) = \chi_0(\alpha) = 1$.

### III. ONE DIMENSION

We proceed by solving the above consistence condition for a 1D lattice. As one can verify, a generic solution is a staggered derivative

$$d_1([\{\phi\}]_{j}) = \phi_{2j+1} - \phi_{2j-1},$$

(8)

Notably, this equation along with Eq. (6) determine $e^{i\Theta_m}$ up to a global phase. Explicitly we find that

$$\theta_m(\phi_{-j}) = 2\pi m(-1)^j \phi_j \phi_{j+1},$$

(9)

plus some constant.

Next we establish the topological properties of our state. To this end, consider reversing the role of $\alpha$ and $\phi$ in the above TPS, such that $\phi$’s become the physical degrees of freedom and $\alpha$’s are traced over. By grouping elements in pairs, the resulting state $|\tilde{\psi}_m\rangle$ can be written as

$$\langle \{\phi\}|\tilde{\psi}_m\rangle = \Pi_k \nu_m(1, \{\phi_{2k-1}, \phi_{2k}\}, \{\phi_{2k+1}, \phi_{2k+2}\})$$

(10)

$$\nu_m(1, \{\phi_{2k-1}, \phi_{2k}\}, \{\phi_{2k+1}, \phi_{2k+2}\}) = e^{2\pi i \phi_{2k+1} - \phi_{2k-1}}$$

Consider $[\sigma, \tau]$ as a group element in $Z_N \times Z_N$ and extend $\nu_m$ to be a function of all three arguments via symmetry $\nu_m(\{\sigma_0, \tau_0\}, \{\sigma_1, \tau_1\}, \{\sigma_2, \tau_2\}) = \nu_m(1, \{\sigma_1 - \sigma_0, \tau_1 - \tau_0\}, \{\sigma_2 - \sigma_0, \tau_2 - \tau_0\})$. One then finds that $\nu_m(a, b, c) \cdot \nu_m(a, c, d) \cdot \nu_m(b, c, d) = 1$ for any $a, b, c, d \in Z_N \times Z_N$, and so $\nu_m$ are cocycles in the second cohomology group $H^2(Z_N \times Z_N, U(1))$. Furthermore in App. (B) we show that varying $m$, all cocycles of $H^2(Z_N \times Z_N, U(1))$ are obtained. Thus $|\tilde{\psi}_m\rangle$ are just the ground states obtained in Ref. (11) for 1D SPTs with $G = Z_N \times Z_N$. Moreover they are related to our TPS via

$$\hat{d}_1|\tilde{\psi}_m\rangle = |\tilde{\psi}_m\rangle,$$

$$\hat{d}_1 = \sum_{\{\phi\}} |d_{[\{\phi\}]}\rangle \langle \{\phi\}|.$$

(11)

Notably the coboundary arbitrariness which exists in the cohomology formalism has been fixed quite naturally by our choice of $d_1$. Thus our state can be viewed as standard SPT phase, presented in the dual basis. This is short of implying topological equivalence between $|\tilde{\psi}_m\rangle$ and $|\tilde{\psi}_m\rangle$, since $d_1$ cannot be written as a product of local unitary transformations (even for fixed boundary conditions). To see this, assume that it can be written in such a form. If so its inverse, $d_1^{-1}$, must also have a local unitary form whereas actually it is a highly non-local string-like transformation.

Next we regroup the tensor and indices such that they appear as MPSs. This will allow us to use the known classification of MPSs [3]. Accordingly, the $\phi$’s and $\alpha$’s are paired into $[\alpha_k, \tau_k] = [\phi_{2k-1}, \phi_{2k}]$, and $[\beta_k, \gamma_k] = [\alpha_{2k}, \alpha_{2k+1}]$. The tensors are paired as

$$B_m([\beta_k, \gamma_k]|_{\alpha_k, \alpha_{k+1}, \tau_k, \tau_{k+1}}) =$$

$$\delta_{\beta_k, \alpha_{k+1}} \delta_{\gamma_k, \tau_{k+1}} \nu_{m}(1, \{\alpha_k, \tau_k\}, \{\alpha_{k+1}, \tau_{k+1}\}) =$$

(12)

$$\delta_{\beta_k, \alpha_{k+1}} \delta_{\gamma_k, \tau_{k+1}} \nu_{m}(1, \{\alpha_k, \tau_k\}, \{\alpha_{k+1}, \tau_{k+1}\}),$$

where in the first line we used the definition of $\nu_m$ appearing in Eq. (10), and in the second line we used the fact that flipping just the $\tau$’s is equivalent to inverting $m$ (i.e. adding $N$ to it). As a result, our state appears as an MPS with site dimension $(d = N^2)$ and bond dimension $(D = N^2)$

$$\langle \{[\beta, \gamma]\}|\psi_m\rangle = Tr[\Pi_k B_m([\beta_k, \gamma_k])].$$

(13)

Next we discuss the symmetries of our state. As one can verify (see also App. (B)), $B_m([\beta, \gamma])$’s furnish a projective unitary representation of $G = Z_N \times Z_N$

$$B_m(g)B_m(g') = B_m(gg')\omega_{-m}(g, g')$$

$$\omega_{m}(g, g') = \nu_{m}(1, g, g'),$$

(14)

with $g, g' \in G$. Let us associate with each element $g \in G$, the matrix $B_m(g)$. Consider conjugating each matrix in the MPS by $B_m(g)$ such that the trace in Eq. (13) remains invariant. Note that $B_m(g)B_m([\beta, \gamma])B_m(g)^\dagger = \omega_{m}(g, [\beta, \gamma])B_m([\beta, \gamma])$. Interestingly the right hand side is equal to the character of $-mg$ in the representation labelled by $[\beta, \gamma] \in G$ [27]. In our notations, this character is $\chi_{-mg}([\beta, \gamma], g \equiv [g_l, g_r])$. Thus this conjugation is equivalent to acting on the state with the following unitary operators, representing $G$

$$\hat{G}(g) = \sum_{\{[\beta, \gamma]\}} |\{[\beta, \gamma]\} \Pi_k \chi_{-mg}([\beta_k, \gamma_k])\langle [\beta, \gamma]| \rangle.$$

(15)

Using Eq. (4), the above operators can be interpreted as separate symmetry operations on the even and odd sublattices. Since they are equivalent to conjugation, we find that our state obeys the above symmetry.

As shown in Ref. [28], the topological phase of a MPS with a symmetry $G$, can be determined by the projective representation of the matrices which implement the symmetry using conjugation. Following the
previous discussion, these matrices here are simply the $B_n([\beta,\gamma])$’s themselves. Since the set $\{\nu_m\}_{m=1}^N$ spans $H^2(Z_N \times Z_N, U(1))$, the different $B_m$’s span all projective representation and hence all SPTs in this symmetry class [28].

In App. (C) we also show that an onsite unitary transformation ($\hat{F}$) which rotates between the symmetry-charge basis and the regular basis of Ref. [1], gives a mapping between the two sets of states

$$\hat{F}|\tilde{\psi}_m\rangle = \hat{d}_1|\tilde{\psi}_m\rangle = |\psi_m\rangle.$$  \hspace{1cm} (16)

Since the symmetry $G$ acts regularly on $|\psi_m\rangle$ then conjugated by $\hat{F}$, it should indeed act diagonally on our TPS (see Eq. [1]). The above relation also implies a non-local symmetry of our states given by bond-site duality $(\hat{d}_1)$ times a local change of basis $(\hat{F})$. For a specific choice of coboundary this is also a symmetry of the states obtained in Ref. [1].

The Laughlin-like form of $|\psi_m\rangle$ further includes the hidden long range order structure. Note that $|\psi_m\rangle$ contains all possible charge configuration and thus one can use it to construct a unitary transformation given by $U = \sum_{\{\alpha\}} |\{\alpha\}\rangle \langle \psi_m|\{\alpha\}\rangle |\{\alpha\}\rangle$. Being diagonal in the symmetry-charge basis, this transformation commutes with the symmetry operation and furthermore it removes all phases from $|\psi_m\rangle$. Ignoring boundary conditions, the disentangled wavefunction $U|\psi_m\rangle$ is a product-state consisting of $\sum_{\alpha} |\alpha\rangle$ on each site. This state is not an eigenstate of $G$ and thus breaks the symmetry. Different boundary conditions constrain $\sum \alpha_i$ thus allowing various directions of symmetry breaking. We comment that $U$ also preserves the locality of symmetry respecting operators, and in particular maps the Hamiltonian of the SPT to local Hamiltonian of a ferromagnet. The ground state degeneracy of the ferromagnet reflects the degeneracy caused by the boundary states in the SPT. Furthermore $U$ coincides with the disentanglers obtained in Ref. [24].

IV. TWO DIMENSIONS

We turn to discuss the two dimensional case. We again solve Eq. [4], only this time on a triangular lattice with both $\alpha_i$ and $\phi_i$ sitting on sites (vertices). We focus on a $Z_2$ symmetry, where the solution is quite simple and given by

$$d_2[\phi] = \frac{\sum_{(i,j)} (\phi_i - \phi_{i-1})Z_N}{N} Z_N^2 + a_0.$$  \hspace{1cm} (17)

where $\phi_i$ are arranged in clockwise order, $\langle...\rangle Z_N$ means taking modulo $N$, $a_0 \in [0..N-1]$ is an arbitrary number which we shall later fix, and we take $N = 2$. Considering a generic $N$, one is forced to tripartite the triangular lattice and, as shown in App. [D], the above operator maintains a similar form only on one of the three sublattices.

Despite appearances Eq. [17] is a natural algebraic definition of vorticity (see Fig. [1]). To show this consider the limit of large $N$. By presenting $Z_N$ as points on the complex circle, we asymptotically obtain the $U(1)$ group. A $U(1)$ single valued function defined on a disk with a hole, may show any integer vorticity. A way of locally measuring this vorticity is the following: (a) take a differential of the $U(1)$ phase $\partial \phi = i/e$ lift it to be a variable in $R$ within $[-\epsilon,\epsilon]$, (b) divide the result by the periodicity of $U(1)$ to obtain a number in $Z = e^{i2\pi}$ equal to the vorticity. This procedure is well-defined because $U(1) = R/Z$. In the above equations we do the same only interchange $R$ by $Z_N$, integration by a discrete sum and $\epsilon$ by $1/N$. This procedure is well defined since $Z_N = Z_{N^2}/Z_N$. We note that our discrete vorticity is closely related to the Bockstein homomorphism [29] [30] in singular cohomology (or lattice gauge theory) where it is used for lifting a 1-cocycle (gauge field) to 2-cocycle (curvature).

We proceed by analyzing the $N = 2$ case and leave a more generic study for future work. Given $d_2$ with $a_0 = 1$, and Eq. [6], the overall phase factor is determined and given by

$$e^{i\Theta_m(\{\phi\})} = (-1)^{\#dw(\{\phi\})}$$  \hspace{1cm} (18)

where $\#dw(\{\phi\})$ denotes the number of domain walls in the $\{\phi\}$ configuration. To verify the above note that in the case where all surrounding $\phi$’s around a site $j_0$ are equal, flipping $\phi_{j_0}$ (i.e. adding 1 modulo 2) either creates or destroys a domain wall in the $\phi$ variables and so changes the domain wall number parity and the above right hand side receives a minus sign. Consistently with Eq. [6], the discrete vorticity in this case is $a_0 = 1$ and $\chi_{m=1}(a_0) = -1$. Similarly one can check all other $\phi$ combinations around the hexagon surrounding $j_0$. Interestingly, this overall phase factor can be divided into product of local factors thereby allowing a local TPS rep-
the above relation is again short of implying that both wavefunctions are in the same phase. There is a further qualitative difference compared to 1D, as the transfer operator associated with the TPS and its transpose, on a cylindrical geometry of circumferences 4, 5, and 6 sites. Using brute-force numerics, we find that it is 0.82(6), 0.12(4) and 0.08(1) respectively, while it is strictly one for \(|\psi_{Z_2}⟩\). Last note we that in the bulk, \(\phi\) flips can only create/annihilate \(\alpha = 1\) sites in pairs. Thus in direct analogy with regular vorticity, in periodic boundary conditions there cannot be an overall odd vorticity (charge). Since the global Ising symmetry acts here as \((-1)\) to the number of nonzero charges, \(|\psi_{Z_2}⟩\) obeys the global Ising symmetry. In particular this implies that any product of an odd number of \(\sigma_z\) or \(\sigma_y\) averages to zero.

To proceed, we use a variation of the PEPS algorithm\(^{31}\) suitable for our TPS (see App. \(^{32}\) for details). We obtain the spectrum and eigenvalues of the transfer operator associated with the TPS and its transpose, on a cylindrical geometry of circumferences 4, 5, and 6 sites. Using this data we bound the correlation length and obtain the entanglement spectrum a half infinite cylinder\(^{33}\). The results for the TPS are shown in the upper panel of Fig. \(^{34}\). The correlation length is close to one and decreases. This justifies our finite size method and strongly suggests that the TPS describes a non-critical state (note that unlike MPSs, finite bond di-
mension TPSs may also describe critical states\cite{32}. The entanglement spectrum, showing the six smallest eigenvalues, suggests a non-gapped spectrum.

![Correlation length and sixth lowest entanglement spectrum eigenvalues for the TPS in Eq.~(19) (upper panel) and the TPS without complex phase factors (lower panel) on an infinite cylinder of circumference 4,5 and 6 lattice sites.](image)

The TPS shows a small and decreasing correlation length as well as a non-gapped entanglement spectrum, perhaps gapless. Without the phase factor, the correlation length is comparable to the circumference and grows with it. The entanglement spectrum shows two low lying states and no gapless tendencies. These features suggest a critical broken symmetry state.

To study the hidden long range order structure, we disentangle the TPS by removing the phase factor of \((-1)^{\#\text{core}(\{\phi\})}\). The state obtained is simply a superposition of all the \(\{\alpha\}\) configurations, weighted by a non-negative integer equal to the number of source \(\phi\) configurations \((\{\alpha\} = d_2[\{\phi\}]\). Under the assumption that each \(\{\alpha\}\), with an even total charge, has exactly two source \(\{\phi\}\)'s, the disentangled TPS would be a simple sum of two product states— one with \(\sigma_z = 1\) on each site and its Ising conjugated state. Namely, the caricature broken Ising symmetry state. The fact that the entropy of \(\{\alpha\}\) is slightly smaller than that of \(\{\phi\}\) means that the above assumption is however only roughly true. To establish what actually occurs we turn to numerics.

As in the previous case we use PEPS numerics to obtain the correlation length and entanglement spectrum of this state (see Fig.~(3), lower panel). Interestingly the correlation length is comparable to the circumference and grows with it, suggesting a critical state. The entanglement spectrum shows a degeneracy which can be understood as a signature of broken symmetry. Indeed we have checked that upon making linear combinations of the two largest transfer matrix eigenvalues, the average of \(\sigma_z\) is non-zero and typically of the order of one. Notably a form of hidden critically, somewhat analogous to ours, has been discussed in Ref.~\cite{33}.

\section{Summary}

Motivated by the Laughlin wavefunction, we have written down an ansatz for a TPS describing abelian, bosonic SPTs in 1D and 2D. We tested our ansatz and established its validity for 1D SPTs with \(G = \mathbb{Z}_N \times \mathbb{Z}_N\). This was carried out either by rewriting it as an MPS and using the known classification \cite{5} or by directly mapping it, through a local basis change, to the wavefunctions obtained using the group cohomology approach \cite{1}. Furthermore, a unitary transformation which maps these SPT phases onto broken symmetry states (a disentangler \cite{23}) was read-off the phase factor of these states.

In 2D we used the same ansatz to derive a TPS which we conjecture to be in the Ising SPT phase. As evidence, we found that it is an Ising symmetry respecting state with strictly short range correlations. It has strongly fluctuating symmetry charges \(\sigma_x\) with zero average and thus it is markedly different from the representative wavefunction of the trivial phase—a simple product state of \(\sigma_z = 1\) on every site. Furthermore based on the conjecture that the entanglement spectrum and edge spectrum shared the same topological features \cite{34,35}, we may relate its non-gapped entanglement spectrum to a non-gapped edge spectrum, as one expects from an SPT phase. Last when removing the phase factors from the TPS, the resulting state appeared as a critical broken symmetry state, implying a hidden long range order structure which is a feature of several other topological phases \cite{21,23}.

Our approach, which is a hybrid of Laughlin’s approach and TPSs, has several advantages. It is a constructive microscopic approach, derived directly from the physical concept of composite particles and flux attachment. This concept has proven extremely useful in past studies of bosonic, fermionic, integer, and fractional quantum Hall states. One may therefore hope that its adaptation to SPTs will provide for similar generalizations and help explore these phases away from the well mapped realm of integer (short ranged entangled)
bosonic SPTs. Furthermore having a TPS representation for the groundstate allows the use of efficient numerical techniques \[28\] and also derivation of parent Hamiltonians \[32\]. Lastly, this approach touches upon the intriguing correspondence between topological phases and broken symmetry phases \[21\] [23].

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Appendix A: Detailed derivation of a finite-size Laughlin TPS

Here we derive a finite-size TPS representation of the Laughlin state. Various subtleties may arise when one attempts to take the thermodynamic limit of our TPS, or discretize the indices to a dense lattice. As it is not essential for the main part of the work, we leave these issues aside for now, and focus below on the limit of an infinitely dense lattice ($\epsilon \to 0$) while keeping the system size fix.

Consider a bosonic Laughlin wavefunction at $1/m$ filling with $n$ particles in a droplet of radius of approximately $R = \sqrt{2(n-1)m}$, in units of the magnetic length which we set to unity

$$\Psi_m(z_1, ..., z_n) = \Pi_{i<j}(z_i - z_j)^m e^{-\sum_i \frac{1}{2}|z_i|^2}, \quad (A1)$$

also consider its discrete version $(\tilde{\Psi}_m)$ where $z_i$ take values only on a lattice of resolution $\epsilon \to 0$. Note that there is no obstruction to discretizing space under a Laughlin state. In fact there are lattice models for which the discrete Laughlin wave function is an exact, topologically distinct, ground state \[34\]. Here we show that one can find a tensor product state ($TPS_m$) with continuous indices such that the overlap $(\tilde{\Psi}_m|TPS_m) > 1 - f(\epsilon/r_0)$. Where $r_0$ is the radius on which the tensors acts, and $f$ is some smooth function such that $f(0) = 0$.

To this end we introduce a $U(1)$ scalar field $\phi \in [0,1]$ which lives on the lattice and use it to decouple the flux from the particles as follows

$$\tilde{\Psi}_m(z_1, ..., z_n) = C_0 \int \text{D}\phi e^{-\frac{1}{\epsilon} \int \delta(z_i - z_j) e^{-\frac{|z_i - z_j|^2}{\epsilon}}, \quad (A2)$$

$$A_i = \delta \left(\int_{O_i} d\lambda \nabla \phi - 1\right) e^{-\frac{|z_i|^2}{\epsilon}},$$

where $C_0$ is a normalization constant, $\rho(z) = \sum_j \delta(z-z_j)$ and $O_i$ is an $r_0$-circle around $z_i$, $\beta$ is to be determined below, and derivatives and spatial delta functions should be thought of as their lattice counterparts. The boundary conditions for $\phi(r >> R)$ are $(2\pi)^{-1}n3 \log(z)$. Notably when $\beta > \beta_c \approx 8\pi \cdot 1.12$, the scalar field theory is on the ordered side of the Kosterlitz-Thouless transition \[26\].

Let us be concrete about the vorticity operator $(\int_{O_i} d\lambda \nabla \phi)$. As a first attempt at defining it, one may consider the $r_0$ radius of integration, perform the discrete derivative, lift it from $U(1)$ to $R$, and sum in $R$. Whenever the field is smooth the resulting derivative is $\epsilon$-close to unity and so there is no ambiguity in lifting it to $R$. When the field is not smooth, the discrete derivative may be of the order of 1 and ambiguities arise. Although this requires more rigorous study, the impact of such singularities should be suppressed by increasing $(r_0/\epsilon)$. The reason is that, as shown later, $\beta > \beta_c$ for $m > 4$. The field-theory on the scale of $r_0$, can then be made arbitrarily close to the smooth gaussian fixed-point \[37\] by increasing $r_0$. Consequently, if we measure the vorticity on a scale $r_0$, we expect the effect of singularities to decrease with $r_0/\epsilon$. To define the vorticity operator on the scale $r_0$, we smooth it out by averaging the former definition of vorticity with some smooth weight function which decays quickly outside $r_0$.

Let us discuss the validity of Eq. (A2). Note that the delta function constraint can be removed by transforming $\phi \rightarrow \phi + \phi_0$ where $\phi_0 = \sum_i \frac{1}{2} i \pi m 3 \log(z - z_i)$. Consequently the exponent of the $\phi$ field $(-S[\phi])$ changes to $S[\phi] + S_0[\phi_0]$ where

$$S_0[\phi_0] = \int d^2r \frac{\beta}{2}(\nabla \phi_0)^2 + \beta \nabla \phi \nabla \phi_0 + i \pi m \phi_0 \rho(z). \quad (A3)$$

Note that boundary conditions of $\phi$ are now constant, and therefore allow no net vorticity. The last term on the above right hand side is given by

$$\int d^2r i \pi m \phi_0 \rho(z) = \sum_{i,j} \frac{m}{2} i \pi m 3 \log(z_i - z_j) \quad (A4)$$

$$= \sum_{i,j} im \left[3 \log(z_i - z_j) + \frac{1}{2} 3 \log(-1)\right] + \sum_i \frac{im}{2} 3 \log(0).$$

Out of the resulting three terms, the second is just $i \pi m N(N-1)/4$ and can be absorbed into $C_0$. To define the third term we require a lattice definition of the $\phi_0$ shift. We choose it such that $\log(0) = 0$.

Next we discuss the second term on the right hand side of Eq. (A3). We use integration by parts to obtain $\phi \nabla^2 \phi_0$ along with a vanishing boundary term. Away from the singularities of $\phi_0$ we may use the continuum limit where $\nabla^2 \phi_0 = 0$. In a few $\epsilon$ radius around the singularities of $\phi_0$, a more delicate analysis is required. Let us taylor expand $\phi$ around some particle $(z_j)$ as $\phi(z + z_j) = \phi(z_j) + z \partial_\phi \phi(z_j) + ...$. As one can check the $\phi(z_j)\nabla^2 \phi_0$ term vanishes exactly even on the lattice. Thinking of the underlying lattice as rectangular, $\pi$ rotation symmetry also removes the next order term. Following this, contributions from the vortex cores will be smaller than $\sum_j (\epsilon^2)(\phi(z_j))$ and therefore negligible.

We turn our attention to $\frac{\partial}{\partial t}(\nabla \phi_0)^2$. Away from the singularities we may again use the continuum approximation with minor $\epsilon$ corrections of the type we had
previously. By a \( \pi/2 \) rotation of \( \nabla \phi_0 \), we find that it describes the electric field \( (E) \) of a set of 2D coulomb charges at positions \( z_i \) (or equivalently the electric field of a set of 1D wires parallel to a fictitious \( z \) axis). The action is therefore \( \int d^2r E^2 \) and proportional to the overall electrostatic energy of this field configuration. Rewriting \( E = \nabla \varphi \), where \( \varphi = \sum_i (2\pi)^{-1} \log (z - z_i) \), we use integration by parts to get Coloumb’s law for the charges \( \left( \sum_{i,j} \frac{\beta}{4\pi} \log (z_i - z_j) \right) \) along with a surface term which gives the infrared divergence proportional to \( \eta^2 \log(R) \). These infrared divergence can again be absorbed into the normalization, provided \( \epsilon \) is small enough. Focusing on the Coulomb part, terms in this sum with equal \( i \) and \( j \), give rise to a some core energy term \( E_{\text{core}} \propto 1/\epsilon^{-1} \). Since the vortex number is fixed here, their contribution is constant can be absorbed into \( C_0 \). Note that when \( z_i \) and \( z_j \) come \( \epsilon \)-close, the continuum description breaks down. Also before this, there may be running coupling effect in \( \beta \). The length scale on which these effects start interfering is again controlled by \( \epsilon \) and therefore we allow ourselves to ignore this.

The last term which couples the \( U(1) \) field to the particles, is the preexisting \( \int i\pi m\rho_i \) term. Completing the square, this results in an additional vortex repulsion term \( \sum_{i,j} \frac{\pi m^2}{4\epsilon} \log (z_i - z_j) \). The \( \phi \) field then becomes decoupled and can be integrated out leading to yet another change in \( C_0 \).

Finally we choose \( \beta \), such that \( \frac{\pi m^2}{4\epsilon^2} + \frac{\beta}{2\pi} = m \) so that the coefficients of the real and imaginary part of the log are both equal to \( m \). This yields a single solution which is \( (\beta = \pi m) \) such that the desired \( \sum_i \frac{\pi m^2}{4}\log (z_i - z_j) \) term appears in the exponent. Notably, \( \beta > \beta_c \) for \( m > 4 \).

Last we recast Eq. (A2) into tensor product form. Instead of describing the position of each particle we work with a discrete density field, \( \rho_{ab} \), and describe the particle number at each position. The resulting expression is

\[
\langle \{\rho_{ab}\} \rangle_{TPS_m} = \int \Pi_{ab}[d\phi_{ab}] A_\phi(\rho_{ab})
\]

\[
A_\phi(\rho_{ab}) = \delta \left( \int_{O_{ab}} \delta \nabla \phi - \rho_{ab} \right) e^{i\pi m \phi_{ab} \rho_{ab}}
\]

\[
\times e^{-\frac{\pi m}{\epsilon} (\nabla \phi_{ab})^2 - \frac{\epsilon}{4} \rho_{ab} |\rho_{ab}|^2}
\]

where \( \{\rho_{ab}\} \) is a state in the occupation basis. The above expression now appears as a tensor product state with \( \phi \) serving as the tensor indices and taking values in the group \( U(1) \). Notably, as well as forcing vorticity at particle positions, the above form also forces zero vorticity everywhere else. As long as \( \beta > \beta_c \), this should not matter much as net vorticity, on the scale \( r_0 \), is already exponentially suppressed. When \( \beta < \beta_c \), a naive guess will be that these constraints suppress the Kosterlitz-Thouless transition.

Appendix B: \( \nu_m \) covers \( H^2(Z_N \times Z_N, U(1)) \)

Here we show that the cocycles used in Eq. (13), fully cover the second cohomology group \( H^2(Z_N \times Z_N, U(1)) \) in the sense that they give a representative cocycle in each equivalence class. From any such cocycle one can construct a projective representation of \( Z_N \times Z_N \) as follows. Consider a vector space \( \{[\sigma, \tau]\} \), labeled by group elements and have the matrices \( D([\sigma_0, \tau_0]) \) as \( D([\sigma_0, \tau_0]|\{[\sigma, \tau]\}) = |[\sigma + \sigma_0, \tau + \tau_0]| \). This is gives the regular (and non-projective) representation of \( Z_N \times Z_N \). Next we use \( D \) and the cocycle to define a projective representation of \( Z_N \times Z_N \) by

\[
[R([\sigma_0, \tau_0])|([\sigma, \tau], [\sigma', \tau']) = |D([\sigma_0, \tau_0])|([\sigma, \tau], [\sigma', \tau']) (B1)
\]

\[
\times \nu_m(1, [\sigma, \tau], [\sigma', \tau'])
\]

using the cocycle condition \( (\nu_m(a, b, c) \cdot \nu_m(a, c, d) \cdot \nu_m^{-1}(a, b, d) \cdot \nu_m^{-1}(a, b, d) = 1) \), one can verify that this is a projective representation and that

\[
R([\sigma_0, \tau_0])R([\sigma_1, \tau_1]) = R([\sigma_0 + \sigma_1, \tau_0 + \tau_1])
\]

\[
\times \omega_m([\sigma_0, \tau_0], [\sigma_1, \tau_1])
\]

\[
\omega_m([\sigma, \tau], [\sigma', \tau']) \equiv \nu_m(1, [\sigma, \tau], [\sigma' - \sigma, \tau' - \tau]) (B3)
\]

\[
= e^{-2\pi i m \sigma'\tau'}.
\]

For any \( m \) different than zero, the representation obtained is non-abelian and must therefore be projective. As a result, we find that all the cocycles are non-trivial.

We further note that \( H^2(Z_N \times Z_N, U(1)) = Z_N \) has a group structure under the action of multiplying cocycles. Since \( \nu_m^{-1} \nu_m = \nu_{m'} \) results in a regular representation only when \( m = m' \), we find that all cocycles are distinct. Having \( N \) of them means they cover the entire cohomology group.

Appendix C: Self duality in 1D SPTs

Here we construct a local, symmetry respecting, unitary transformation which maps between \( |\psi_m\rangle \) and \( |\psi_m\rangle \). To this end we limit ourselves to prime cyclic groups. This way multiplying \( \phi \) by \( m \) is an automorphism from the group to itself. Consider the local unitary Fourier transform, \( |E_m\rangle_{a,\phi} = \chi_m(\phi) \), which changes basis from the \( \phi \) (symmetry-phase) basis and the \( \alpha \) (symmetry-charge) basis. We argue that when acting on \( |\psi_m\rangle \), this transform is equivalent to \( d_1 \) up to complex conjugation. A useful tool in proving this is the following parent Hamiltonian for \( |\psi_m\rangle \)

\[
H_{sd} = -\sum_i \sum_{p=0}^{N-1} e^{2\pi i m \alpha (\tau_i - \tau_{i-1})} \sigma_i^{+p} + e^{2\pi i m \alpha (\tau_i - \tau_{i+1})} \sigma_i^{+p},
\]

where \( [\tau_i, \tau_j] = [\phi_{2j-1}, \phi_{2j}] \), \( \sigma_i^{+p} \) is a raise-\( \alpha \)-by-\( p \)-and-take-modulo-N operator (for example in the \( N = 2 \) case,
\( \sigma^{+0} = I_{2 \times 2}; \sigma^{+1} = \sigma_x \). Note that this Hamiltonian obeys two separate \( Z_N \) symmetries, acting on the \( \sigma \) and \( \tau \) degrees of freedom with the regular action.

Next we show that \( H_{sd} \) obeys a self-duality symmetry and that its groundstate is unique. First recall the action of \( \hat{d}_1 \) and \( \hat{F}_m \) on the double site basis, spanned by all configurations of the form \( |\ldots, [\sigma_{i-1}, \tau_{i-1}], [\sigma_i, \tau_i] \ldots \rangle \)

\[
\hat{d}_1 |\ldots, [\sigma_{i-1}, \tau_{i-1}], [\sigma_i, \tau_i], [\sigma_{i+1}, \tau_{i+1}] \ldots \rangle = |\ldots, [\sigma_{i-1}, \tau_{i-1}], [\sigma_i, \tau_i-1], [\sigma_{i+1}, \tau_{i+1}] \ldots \rangle.
\]

\[ \hat{F}_m |\ldots, [\sigma_i, \tau_i], [\sigma_i, \tau_i] \ldots \rangle = \frac{1}{\sqrt{N}} e^{\frac{2\pi i m (\sigma_i + \tau_i)}{N}}. \]

For interpreting \( \hat{d}_1 \) as a symmetry, we need its inverse which, strictly speaking, does not exist. This is because any two symmetry-phase configurations which differ by a global rotation are mapped to the same state. Focusing on bulk effects, one can remedy this problem by fixing the leftmost two sites to be \([0, 0]\).

Conjugating \( H_{sd} \) by \( \hat{d}_1 \) we find

\[
\hat{d}_1 H_{sd} \hat{d}_1^{-1} = -\sum_i \sum_{p=0}^{N-1} e^{\frac{2\pi i m (\sigma_i + \tau_i)}{N}} \tau_i + p \tau_i - p + \frac{e^{\frac{2\pi i m (\tau_i - \tau_i)}{N}}}{N}.
\]

Next using the fact that

\[
\hat{F}_m e^{\frac{2\pi i m \tau}{N}} \hat{F}_m = \tau + p,
\]

\[
\hat{F}_m \tau + p \hat{F}_m = e^{-\frac{2\pi i m \tau}{N}},
\]

\[
\hat{F}_m e^{\frac{2\pi i m \sigma}{N}} \hat{F}_m = \sigma + p,
\]

\[
\hat{F}_m \sigma + p \hat{F}_m = e^{-\frac{2\pi i m \sigma}{N}},
\]

we find that

\[
\hat{F}_m \hat{d}_1 H_{sd} \hat{d}_1^{-1} \hat{F}_m = \sum_i \sum_{p=0}^{N-1} e^{-\frac{2\pi i m (\tau_i - \tau_i)}{N}} \sigma_i + p + e^{-\frac{2\pi i m (\sigma_i - \sigma_{i+1})}{N}} \tau_i + p
\]

\[ = H_{sd}^*. \]

Next we wish to show that the ground state of this Hamiltonian is unique on closed boundary conditions. To this end we conjugate the Hamiltonian with the following unitary transformation

\[ U = \ldots, [\sigma_i, \tau_i], [\sigma_{i+1}, \tau_{i+1}] \ldots \}

\[ \nu_m(1, \phi_a, \phi_b, \phi_c) = e^{-\frac{2\pi i m (\phi_a - \phi_b)}{N} + (\phi_b - \phi_c) N - (\phi_c - \phi_a) N} \]

where \( \langle \ldots \rangle_N \) means taking modulo \( N \).

Note that the above cocycle already possess a flux attachment structure: Let us label the bonds of the lattice by the difference \((\phi_b - \phi_a) N\), when \( a \) points to \( b \) in the branching structure. Following this the term in the square brackets can be viewed as discrete line integration over all the bond values in the triangle, weighted by the branching orientation. Borrowing one \( N \) from the \( N^2 \) denominator and using it to divide the result of this discrete integration, one obtains the discrete vorticity for this triangle. The entire cocycle then appears as a phase variable \( a \) times a flux, or as a discrete Chern-Simons factor on a simplex.

We next use Eq. \( (6) \) to obtain \( d_2 \). Clearly this way the consistency condition in Eq. \( (7) \) is automatically satisfied. We begin by varying \( \phi \) on an \( A \) site \( (\phi_a) \). Taking into account the branching structure \( \Pi \), the phase factor obtained is

\[ \Delta_a e^{i \Theta_m(\{\phi\})} = \prod_{j=0}^{5} \nu_j(1, \phi_a, 1, \phi_j, \phi_{j+1}) \]

where \( \phi_{0, 5} \) are the phases on the six sites around \( \phi_a \) (where the index 0 is at 2-O’clock and goes clockwise, the

\[ \text{Appendix D: } d_2 \text{ for } Z_N \]

Here we define \( d_2 \) for a generic \( Z_N \) group. To this end, we will reverse our line of argument. We will start from \( e^{i \Theta_m} \) and use the flux attachment condition to obtain \( d_2 \). As shown in the main text for the \( 1D \) case, \( e^{i \Theta_m} \) for a particular \( \phi \) configuration comes out to be the amplitude of the wavefunction of Ref. \( \Pi \) for this configuration. Following this, we take \( e^{i \Theta_m} \) to be the amplitude of the wavefunctions of Ref. \( \Pi \). To define this amplitude two things are needed: a cocycle in \( H^2(\mathbb{Z}_N; U(1)) \), and a branching structure. For the branching structure, we imagine dividing our triangular lattice into \( A, B \) and \( C \) sublattices. For each hexagon centered around an \( A \) site, we take all inner bonds pointing outward and the outer bonds to have staggered orientation. This gives a consistent branching structure so that there are no completely clockwise or completely anti-clockwise oriented triangles. For the cocycle we take \( \nu_m \)
branch direction between $\phi_0$ and $\phi_1$ is also clockwise) and inversion factor $(-1)^j$ is $-1$ (1) if the triangle is mainly clockwise (anti-clockwise) oriented. Due to this factor, all inner bonds are canceled in both the numerator and denominator. Since the outer bonds do not depend on $\phi_0$ we obtain

$$\Delta_a e^{i\Theta_m}((\phi)) = e^{2\pi i m \sum_{x}(-1)^{j+1}((-1)^j(\phi_{j+1} - \phi_j))}$$

and one can perform the same procedure on the $B$ sublattice (with all bonds oriented in the incoming direction) and the remaining $C$ sublattice. The final result is

$$d_a^B[\{\phi\}]_b = \frac{\langle \sum (-1)^{j+1}((-1)^j(\phi_{j+1} - \phi_j)) \rangle}{N}$$

$$d_a^B[\{\phi\}]_c = \sum_{k=0}^2 \tilde{\phi}_{2k+1} - \phi_{2k+2}$$

$$d_a^C[\{\phi\}]_c = \sum_{k=0}^2 \tilde{\phi}_{2k} - \phi_{2k+1},$$

where $1_x$ is 1 (0) if $1 + x \geq N$ ($g + x < N$) and $\tilde{1}_x$ is 1 (0) if $x - 1 < 0$ ($x - 1 \geq 0$). Notably the above equations can be used to determine $e^{i\Theta_m}$ up to a global phase factor. The first of the above factors is very similar to the discrete vorticity defined in the main text, containing however extra $(-1)^j$ factors which keep track of orientation. The terms on the other two sublattices are, admittedly, somewhat convoluted. This suggests, on aesthetical grounds, that we may have missed something in this generalization. Nonetheless, as pointed earlier, the flux attachment form of the basic cocycle suggests that the general approach should be applicable.

**Appendix E: Numerics**

Here we study the correlation length and entanglement spectrum of the TPS in Eq. (19) both with and without the $(-1)^{d\text{vec}[\{\phi\}]}$ factor. We work in an infinite cylindrical geometry aligned along the $x$-direction. The underlying triangle lattice is skewed into a square lattice containing an extra diagonal bond on each square going in the right-up direction. To obtain the correlation length and entanglement spectrum, we use the transfer operator approach as described in Ref. [28]. Since our TPS differs slightly from standard PEPS, we re-derive some of their formulation.

Consider two local operators placed at $x_1$ and $x_2$ given by

$$O_1 = \sum_{\alpha_1,\alpha_1'} c_{\alpha_1,\alpha_1'} |\alpha_1\rangle \langle \alpha_1'| \otimes I$$

$$O_2 = \sum_{\alpha_2,\alpha_2'} c_{\alpha_2,\alpha_2'} |\alpha_2\rangle \langle \alpha_2'| \otimes I$$

where we make a slight abuse of notation and use the dummy indices to denote the position on the lattice. Calculating their joint expectation value involves tracing two conjugated TPSs

$$\langle O_1 O_2 \rangle = \frac{\sum_{\phi,\phi',\alpha,\alpha'} \Pi_1 A_{\phi_1,0}(\alpha_1) A_{\phi_1,0}^+(\alpha_1') \langle \{\alpha\}|O_1 O_2|\{\alpha'\} \rangle}{\sum_{\phi,\phi',\alpha,\alpha'} \Pi_1 A_{\phi_1,0}(\alpha_1) A_{\phi_1,0}^+(\alpha_1') \langle \{\alpha\}|\{\alpha\} \rangle},$$

where $A$ tensors are used in the main text however due to technical reasons, the cocycle differs by an insignificant boundary term. More specifically we take $\nu'|(1, a, b, c)$ which is 1 (i) whenever the majority of $a, b$ and $c$ are zero (one) and add an additional $(-1)^\phi$ term. One can check that such factor also solves equation Eq. (7) with the previously used $d_2$. For example if all $\phi$’s surrounding a site $j$ are zero, $d_2[\{\phi\}]_j = a_0 = 1$ and consequently a minus sign is expected. Accordingly, there will be no change in $\nu'$ in the six triangles around $j$ however due to the $(-1)^\phi$ factor there will be an overall minus sign. Since both this choice and the one used in the main text agree on periodic boundary conditions, they may only differ by a boundary term. Focusing on bulk properties one may ignore this subtlety. Indeed the boundary term can be absorbed into a unitary transformation of the transfer matrix defined below.

Let us first discuss how to use the transfer matrix method to efficiently calculate the above denominator. Consider dividing the cylinder into thin ring regions ($R$) whose width is two sites. The transfer operator of such a ring is defined as

$$[T]_{\phi_1,\bar{\phi}_1,\phi_0,\bar{\phi}_0} = \sum_{\{\alpha\}_R} \Pi_{1,R,L} A_{\phi_1,L,\phi_0,R}(\alpha_1) A_{\phi_1,L,\bar{\phi}_0,R}^+(\alpha_1),$$

where $\phi_{1,L}$ and $\phi_{1,R}$ are the left and right subsets of the set $\{\phi_{1,L}, \phi_0\}$, since these two subsets overlap, a delta function forcing them to be equal is implicit in the above notation. The set $R_T$ denotes the indices on the right column of the region $R$. Using $T$, one can express $\langle \psi_{Z_1}|\psi_{Z_2}\rangle$ for a cylinder of length $N$ rings, as

$$\langle \psi_{Z_1}|\psi_{Z_2}\rangle = \langle \chi_L|T^{N}|\chi_R\rangle,$$

where $\langle \chi_L|$ and $|\chi_R\rangle$ are determined by the boundary conditions. Applying similar ideas we can express the average as

$$\langle O_1 O_2 \rangle = \frac{\langle \chi_L|T^{m} M_1 T^{m} M_2 T^{N-m-n-2}|\chi_R\rangle}{\langle \chi_L|T^{N}|\chi_R\rangle},$$

with $M_k$ being

$$[M_k]_{\phi_0,\bar{\phi}_0,\phi_0,\bar{\phi}_0} = \sum_{\{\alpha,\alpha'\}_R} \Pi_{1,R} A_{\phi_1,L,\phi_0,R}(\alpha_1) A_{\phi_1,L,\bar{\phi}_0,R}^+(\alpha_1) [O_k]_{\{\alpha\}_R,\{\alpha'\}_R}.$$
Next we discuss an important symmetry of $T$ which stems from the fact that flipping all the indices does not affect the charge configuration. As an initial guess for what the symmetry associated with this redundancy may be, consider flipping either the $\phi$ or $\bar{\phi}$ indices of $T$. As far as $\alpha$’s are concerned, this is clearly a symmetry. However the phase factor changes. Since $R$ contains two columns, the $R_i \in \mathbb{R}$, is always symmetric under such a flip. However the $\nu$ factors change such that $1$ becomes $i$ and $i$ becomes $1$ (recall that $\nu(1, \phi_1, \phi_2, \phi_3) = 1$ if the majority of $\phi$’s are $0$ ($1$)). Thus each triangle in $R$ gets multiplied by either $i$ or $-i$ depending on its majority spins. Since the number of triangles is even, this always results in a $\pm 1$ relative phase factor. For even circumference, this factor is simply $(-1)$ to the number of majority $1$ triangles. Conveniently, we find that this factor, which is naively a function of both the left and right indices of $T$, can be written as a product of two functions acting on the left and right indices of $T$ separately. As a result we obtain the following Ising-advanced ($I_A$) and Ising-retarded ($I_R$) symmetries of $T$

\[ [I_A]_{\{\phi, \bar{\phi}\}, \{\phi', \bar{\phi}'\}} = \delta_{\{\phi\} = \{-\phi'\}} \delta_{\{\bar{\phi}\} = \{\bar{\phi}'\}} (-1)^{[11-\text{bonds}](\{\phi'\})} \]

(E7)

where $[11 - \text{bonds}](\{\phi'\})$ denotes the number of adjacent sites whose $\phi$’s are both equal to $1$ on the right column of $\{\phi'\}$. The Ising-retarded symmetry is defined in exactly the same way with $\phi$ and $\bar{\phi}$ exchanged.

Notably, the above two Ising symmetries are also symmetries of $M_1$ and $M_2$ and more generally of any such matrix representing an operator acting on the physical space. Indeed, physical operators such as the above $O_1$ and $O_2$, when presented in transfer matrix form, are mapped to different pairing of $A(\alpha_j)$ between the retarded and advance sector (see Eq. (E6)). To maintain the above lifted-Ising symmetries, one just requires the $A$ tensors within each sector to be function of $\phi$ fluxes and not $\phi$ themselves. Consequently these symmetries are automatically obeyed by any transfer matrix involved in calculating physical observables. It reflects a redundancy in our description rather than an actual physical symmetry.

To obtain the decay of correlations, we numerically obtain the six maximal eigenvalues of $T$ for cylinders of circumference 4, 5 and 6. The results are

\[ \lambda_4 = [17.60(2), 16.67(7), 8.75(3), 8.40(0), 8.34(5), -7.85(7)] \]
\[ \lambda_5 = [34.47(8), 34.31(4), 16.42(2), 4.40(5) + 13.55(7)i, 14.30(0), 4.40(5) - 13.55(7)i] \]
\[ \lambda_6 = [70.04(8), 69.33(6), -14.72(5) + 25.50(5)i, -14.72(5) + 25.50(5)i, -28.71(9), 29.45(1)] \]

(E8)

(E9)

In both cases, the two maximal eigenvalues show a different $I_A$ and $I_R$ eigenvalues. The (marginally) larger of the two has a $+1$ eigenvalue for both $I_A$ and $I_R$ and the smaller one has a $-1$ eigenvalue for both symmetries. Consequently, no operator can mix the two dominant eigenvalues. Taking this symmetry constraint into account, as well as Eq. (E5), the correlation length $\chi_n$ is given by $e^{-\lambda_n} = \lambda_n[2]/\lambda_n[0]$ (as these are the two closest eigenvalues within the same sector), and we find $\chi_4 = 1.55(1), \chi_5 = 1.35(7), \chi_6 = 1.15(4)$. This short scale, suggests that the above results are already close to the thermodynamic limit.

Here the definition of the $I_A$ and $I_R$ symmetries are simply spin flips, as there is no phase factor. Checking the $I_A$ and $I_R$ eigenvalues we find that the largest eigenvector has both positive ($+, +$), unlike before the two smaller ones are a mixture of ($+, +$) and ($-, -$). Consequently the symmetry does not protect mixing by physical operators. Indeed taking mixed boundary conditions containing the above three top eigenvectors, we find various values for the magnetization, typically of the order of $1/2$. We also obtain large correlation lengths $\chi_4 = 6.84(2), \chi_5 = 7.03(5), \chi_6 = 8.06(7)$ suggesting a critical phase.

To obtain the entanglement spectrum we again follow Ref. [28]. For simplicity, we normalize $T$ such that its largest eigenvalue is exactly $1$. Consider the reduced density matrix of a very long cylinder cut in the middle. One can define the following two quantities capturing the “quantum state” of the unphysical indices at the cut

\[ [\sigma_R]_{\{\phi\}, \{\bar{\phi}\}} = \langle\{\phi\}, \{\bar{\phi}\}|T^{N/2}|\chi_R \rangle \]
\[ [\sigma_L]_{\{\phi\}, \{\bar{\phi}\}} = \langle\chi_L|T^{N/2}|\{\phi\}, \{\bar{\phi}\} \rangle \]

(E10)

(E11)

Clearly for $N \rightarrow \infty$ only the largest eigenvalue dominates and so $\sigma_{L/R}$ are simply a repackaging of this maximal eigenvalue into a matrix in retarded-advance space. In our case, there are several dominant eigenvalues which seem to become degenerate in the long circumference limit. To be concrete we will assume $N$ goes to infinity last and keep only the largest of them. Conveniently, this way our results are independent of the boundary conditions. Following Ref. [28], the entanglement spectrum is given by the spectrum of $\sqrt{\sigma_L^{*} \sigma_R} \sqrt{\sigma_L}$. The results obtained are shown in Fig. 4.
