OPTIMAL CONVERGENCE RATES OF THE MAGNETOHYDRODYNAMIC MODEL FOR QUANTUM PLASMAS WITH POTENTIAL FORCE

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Abstract. In this paper, we consider the quantum magnetohydrodynamic model for quantum plasmas with potential force. We prove the optimal decay rates for the solution to the stationary state in the whole space in the $L^q - L^2$ norm with $1 \leq q \leq 2$. The proof is based on the optimal decay of the linearized equations, multi-frequency decompositions and nonlinear energy estimates.

1. Introduction. The viscous quantum magnetohydrodynamic (vQMHD) model for plasmas with potential force is governed by the following system of equations

$$
\begin{cases}
\rho_t + \nabla \cdot (\rho u) = 0, \\
(\rho u)_t + \nabla \cdot (\rho u \otimes u) + \nabla P - \nabla \cdot \left( \frac{\hbar^2}{2} \rho \nabla \left( \frac{\nabla \sqrt{\rho}}{\sqrt{\rho}} \right) \right) + \nabla \times (\nabla \times B) = (\nabla \times B) \times B + \mu \Delta u + (\mu + \lambda) \nabla \nabla \cdot u + \rho F, \\
B_t - \nabla \times (u \times B) = -\nabla \times (\nu \nabla \times B), \quad \nabla \cdot B = 0,
\end{cases}
$$

where $\rho \geq 0$, $u = (u^1, u^2, u^3)$, and $B = (B^1, B^2, B^3)$ are functions of time $t$ and spatial position $x \in \mathbb{R}^3$, denoting respectively the density, velocity and magnetic field. In addition, $P = P(\rho)$ is pressure and $F = F(x)$ is an external force. The constants $\mu > 0$ and $\lambda$ are referred to as the first and second coefficients of viscosity, satisfying the usual condition $3\lambda + 2\mu \geq 0$, $\hbar > 0$ is the Planck constant and $\nu > 0$ is the magnetic diffusivity acting as a magnetic diffusion coefficient of the magnetic field. The vQMHD system is supplemented with initial data

$$
(\rho, u, B)(x, 0) = (\rho_0, u_0, B_0)(x) \to (\rho_\infty, 0, 0), \quad \text{as } |x| \to \infty,
$$

where $\rho_\infty$ is a positive constant.

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Without the quantum effects, the system (1) is well known as the compressible magnetohydrodynamic equations. The interested reader may refer to [2, 9, 10, 25, 26, 28] and references therein. Umeda et al [28] studied the global existence to the linearized compressible MHD equations in three dimensional problem. In [9, 10], authors showed the global weak solutions to the nonlinear compressible MHD equations under the general initial data. Li and Yu [18] established the optimal decay rate of classical solutions, while Chen and Tan [2] showed the large time behavior of smooth solutions when the initial data moves towards a constant state. In this paper, we put emphasis on the quantum terms in (1). The quantum terms data back to Wigner [30] and one may would like to refer to Haas [7, 8] for quantum plasma. Furthermore, for the derivation of the quantum KdV equations from the quantum Euler-Poisson equations is studied [17].

Before we state the main results, let us review some of the historic results of closely related models. Indeed, there is a vast literature addressing the decay and other related mathematical problems of hydrodynamic fluid equations, among which one may refer to [1, 2, 3, 4, 6, 11, 12, 13, 14, 15, 16, 19, 20, 21, 22, 23, 29, 33, 34, 35, 37]. In particular, when there is no external force, Matsumura and Nishida [20] obtained the optimal $L^2$ decay rate

$$\| (\rho - \rho_\infty, u, \theta - \theta_\infty)(t) \|_{L^2} \leq C_0 (1 + t)^{-\frac 32},$$

for the compressible viscous and heat conductive fluid in $\mathbb{R}^3$ when the initial perturbation belongs to $H^3 \cap L^1$. For the small initial perturbation belonging to $H^m \cap W^{m,1}$ with $m \geq 4$, Ponce [22] showed the optimal $L^p$ decay rate

$$\| \nabla^k (\rho - \rho_\infty, u, \theta - \theta_\infty)(t) \|_{L^p} \leq C_0 (1 + t)^{-\frac 32 (1 - \frac 1p) - \frac k2}, \quad \text{for } 2 \leq p \leq \infty, \quad 0 \leq k \leq 2.$$

For the full MHD equations, Pu and Guo [23] established $(1 + t)^{-\frac 32 (\frac 12 - \frac 1p) - \frac 14}$ time decay rates with $q \in [1, \frac 65)$ in $L^2$-norm for the $k^{th}$ $(1 \leq k \leq 3)$ derivative of solutions. In [5], Gao et al provided faster time decay rates for the higher order spatial derivatives of classical solutions than the work of Pu and Guo [23]. When there is an external potential force, Duan et al [4] obtained the optimal decay rates for the solution to the stationary profile $(\rho_\star, 0, \theta_\infty)$ in the whole space

\[
\begin{align*}
\| (\rho - \rho_\star, u, \theta - \theta_\infty)(t) \|_{L^p} & \leq C_0 (1 + t)^{-\frac 32 (1 - \frac 1p)}, \quad \text{for } 2 \leq p \leq 6, \quad t \geq 0, \\
\| \nabla (\rho - \rho_\star, u, \theta - \theta_\infty)(t) \|_{H^2} & \leq C_0 (1 + t)^{-\frac 34}, \\
\| \partial_t (\rho, u, \theta)(t) \|_{L^2} & \leq C_0 (1 + t)^{-\frac 34},
\end{align*}
\]

if the small initial disturbance belongs to $H^3 \cap L^1$. For the compressible Navier-Stokes equations, when the small initial perturbation belongs to $H^3 \cap L^q$ with $q \in [1, \frac 65)$, Duan et al [3] obtained the convergence rates

\[
\begin{align*}
\| (\rho - \rho_\star)(t) \|_{L^p} & \leq C_0 (1 + t)^{-\frac 32 (\frac 1p - \frac 12)}, \quad \text{for } 2 \leq p \leq 6, \\
\| \nabla^k (\rho - \rho_\star, u)(t) \|_{L^2} & \leq C_0 (1 + t)^{-\frac 32 (\frac 12 - \frac k2)} - \frac k2, \quad \text{for } k = 1, 2, 3.
\end{align*}
\]

If the small initial perturbation belongs to $(H^4 \cap L^1) \times (H^3 \cap L^1)$, Li [16] obtained the optimal decay rates for the Navier-Stokes-Korteweg equations

\[
\begin{align*}
\| \nabla^k (\rho - \rho_\star)(t) \|_{L^2} & \leq C_0 (1 + t)^{-\frac 32}, \quad \text{for } k = 1, 2, 3, 4, \\
\| \nabla^k u(t) \|_{L^2} & \leq C_0 (1 + t)^{-\frac 32}, \quad \text{for } k = 1, 2, 3, \\
\| (\rho - \rho_\star, u)(t) \|_{L^q} & \leq C_0 (1 + t)^{-\frac 32 (1 - \frac 1q)}, \quad \text{for } 2 \leq q \leq 6.
\end{align*}
\]
Theorem 1.1. For the stationary state equations

This paper. First, we give the existence of stationary solutions of (1) in the following

\textup{L}^2 \textup{equations from the quantum Euler-Poisson equations [17]. However, the optimal}

\textup{compressible quantum Navier-Stokes equations [24] and derived the famous KdV}

Hu and his coauthors studied the global existence of smooth solutions for the full

\textup{provided the time decay rates for the high-order spatial derivatives. Moreover,}

the potential force is not known yet, which is the main topic in the present paper.

All the above results are obtained via utilizing the Fourier analysis of the linearized

system and the Green function. From another point of view, introducing the negative

Sobolev space estimates, Guo and Wang [6] obtained the decay rate for the compressible Navier-Stokes equations, whose method was later generalized to various models [26, 27, 31].

For the vQMHD system, the global existence of weak solutions for large initial
data was recently established by Yang and Ju [38], via Galerkin approximation. Recently, Pu and Xu [25] obtained the optimal decay rates for the solutions to the constant state in the \textup{L}^p \textup{norm with } 2 \leq p \leq 6 and its first derivatives in \textup{L}^2 \textup{norm under the initial perturbation belongs to } (H^5{\cap}L^1) \times (H^4{\cap}L^1) \times (H^4{\cap}L^1). Xi et al [36] provided the time decay rates for the high-order spatial derivatives. Moreover, Pu and his coauthors studied the global existence of smooth solutions for the full compressible quantum Navier-Stokes equations [24] and derived the famous KdV equations from the quantum Euler-Poisson equations [17]. However, the optimal \textup{L}^q \textup{L}^2 \textup{time decay for smooth solutions of the vQMHD system when there is the}

potential force is not known yet, which is the main topic in the present paper.

It will be assumed that \( P(\rho) \) is smooth in a neighborhood of \( \rho_\infty \) with \( P_\rho(\rho_\infty) > 0 \) throughout this paper. Besides, we only consider the potential force \( F = -\nabla \Phi \) in this paper. First, we give the existence of stationary solutions of (1) in the following

\textbf{Theorem 1.1.} For the stationary state equations

\begin{equation}
\begin{aligned}
\nabla \cdot (\rho_\ast u_\ast) &= 0, \\
\rho_\ast \nabla \cdot (u_\ast \cdot \nabla) u_\ast + \nabla P(\rho_\ast) - \frac{k^2}{2} \rho_\ast \nabla (\frac{\Delta \sqrt{\rho_\ast}}{\sqrt{\rho_\ast}}) &\quad= (\nabla \times B_\ast) \times B_\ast + \mu \Delta u_\ast + (\mu + \lambda) \nabla \nabla \cdot u_\ast - \rho_\ast \nabla \Phi, \\
-\nu \Delta B_\ast &= B_\ast \cdot \nabla u_\ast - B_\ast \nabla \cdot u_\ast - u_\ast \cdot \nabla B_\ast,
\end{aligned}
\end{equation}

with

\begin{equation}
\rho_\ast \rightarrow \rho_\infty, \quad u_\ast \rightarrow 0, \quad B_\ast \rightarrow 0,
\end{equation}

there exists \( \epsilon_1 > 0 \) such that if \( ||\Phi||_{H^5} \leq \epsilon_1 \), the problem (3)-(4) has a solution \((\rho_\ast, u_\ast, B_\ast)\) satisfying

\begin{equation}
\rho_\ast - \rho_\infty \in H^7, \quad u_\ast = 0, \quad B_\ast = 0.
\end{equation}

Next, the global existence and asymptotic behavior of smooth solutions for (1) are stated in the following theorem:

\textbf{Theorem 1.2.} Assume that \((\rho_0 - \rho_\infty, u_0, B_0) \in H^5 \times H^4 \times H^4\). There exists a constant \( \epsilon \) such that if

\begin{equation}
||\rho_0 - \rho_\infty||_{H^5} + ||u_0||_{H^4} + ||B_0||_{H^4} + ||\Phi||_{H^5} + \sum_{k=0}^{5} ||(1 + |x|)\nabla^k \Phi||_{L^2} \leq \epsilon,
\end{equation}
there is a unique global solution \((\rho, u, B)\) of the Cauchy problem (1)-(2) satisfying
\[
\rho - \rho_* \in C([0, \infty); H^5(\mathbb{R}^3)) \cap C^1([0, \infty); H^4(\mathbb{R}^3)),
\]
\[
u, B \in C([0, \infty); H^4(\mathbb{R}^3)) \cap C^1([0, \infty); H^3(\mathbb{R}^3)).
\]
Furthermore, if in addition \(\|(\rho_0 - \rho_\infty, u_0, B_0)(t)\|_{L^q} < \infty\) for any given \(1 \leq q \leq 2\),
then we have
\[
\|\nabla (\rho - \rho_*, h\nabla (\rho - \rho_*), u, B)(t)\|_{H^5} \leq C_0(1 + t)^{-\frac{3}{2}(\frac{1}{2} - \frac{1}{4}) - \frac{1}{2}}, \quad (6)
\]
\[
\|(\rho - \rho_*, u, B)(t)\|_{L^p} \leq C_0(1 + t)^{-\frac{3}{2}(\frac{1}{2} - \frac{1}{4})}, \quad \forall p \in [2, 6], \quad (7)
\]
and
\[
\|\partial_t (\rho - \rho_*, u, B)(t)\|_{L^2} \leq C_0(1 + t)^{-\frac{3}{2}(\frac{1}{2} - \frac{1}{4}) - \frac{1}{2}}, \quad (8)
\]
for all \(t \geq 0\) and some positive constant \(C_0\).

**Remark 1.** According to the standard iteration technique and fixed point principle,
the existence of smooth solutions stated in Theorem 1.1 for (3)-(4) can be obtained.
Moreover, similar to the proof of [16], in addition to assumptions in Theorem 1.1,
if \(\sum_{k=0}^5 \|(1 + |x|)\nabla^k \Phi\|_{L^2} \leq \epsilon_1\), we can obtain
\[
\frac{1}{3} \rho_\infty \leq \rho_* \leq \frac{3}{2} \rho_\infty, \quad (9)
\]
\[
\|\rho_* - \rho_\infty\|_{H^7} \leq C\epsilon_1, \quad (10)
\]
\[
\|(1 + |x|)(\rho_* - \rho_\infty)\|_{H^7} \leq C\epsilon_1. \quad (11)
\]
Furthermore, (10) implies the uniqueness of smooth solutions for (3)-(4).

**Remark 2.** In Theorem 1.2, using the Sobolev imbedding inequalities, (5) together with Remark 1.3 yields
\[
\|\rho_* - \rho_\infty\|_{H^7} + \sum_{k=1}^6 \|(1 + |x|)\nabla^k (\rho_* - \rho_\infty)\|_{L^2 \cap L^3} \leq C\epsilon. \quad (12)
\]

To establish our main results, we need to carefully analyze the quantum potential term,
which is strongly nonlinear degenerate and should be understood as a consequence from dispersive properties of the quantum fluid. When the potential force is small, there exists a unique stationary solution \((\rho_*(x), 0, 0)\) for the equations (3). Therefore, we need to consider the decay rates of smooth solutions toward the stationary solution \((\rho_*(x), 0, 0)\). Moreover, we need also to obtain the energy inequalities via multi-frequency decompositions. The basic plan is to establish some crucial energy estimates and then apply the Gronwall inequality to complete the proof. Finally, we obtain the optimal \(L^q - L^2\) with \(1 \leq q \leq 2\) time decay rates of the global solutions to the stationary profile for the vQMHD model in a low frequency and high frequency decompositions, by combining the linear optimal decay rate of spectral analysis and the energy method.

The rest of this paper is organized as follows. In Section 2, we reformulate the system (1) and make some crucial energy estimates for solutions and their higher-order derivatives. In Section 3, we use the energy estimates derived in Section 2 and multi-frequency decompositions to deduce the Gronwall energy inequalities, which together with the linear decay estimates at a low frequency imply Theorem 1.2.
2. Energy estimates. In this section we will drive some \textit{a priori} energy estimates for the solutions to the system (13).

Define
\[
\tilde{\rho}(x,t) = \rho(x,t) - \rho_*(x), \quad \tilde{u}(x,t) = u(x,t), \quad \tilde{B}(x,t) = B(x,t),
\]
and
\[
\bar{\rho}(x,t) = \rho_*(x) - \rho_\infty.
\]
Then the vQMHD system (1) are transformed as the following
\[
\begin{align*}
\tilde{\rho}_t + \rho_\infty \nabla \cdot u = \tilde{F}_1, \\
\tilde{u}_t + \frac{\mu}{\rho_\infty} \Delta \tilde{u} - \frac{\mu + \lambda}{\rho_\infty} \nabla \cdot \tilde{u} + \frac{P_\rho(\rho_\infty)}{\rho_\infty} \nabla \tilde{\rho} - \frac{\hbar^2}{4\rho_\infty} \nabla \Delta \tilde{\rho} = \tilde{F}_2, \\
\tilde{B}_t - \nu \Delta \tilde{B} = \tilde{F}_3,
\end{align*}
\]
and the initial condition (2) becomes
\[
(\tilde{\rho}, \tilde{u}, \tilde{B})|_{t=0} = (\rho_0 - \rho_*, u_0, B_0)(x) \to (0, 0, 0), \text{ as } |x| \to \infty, \tag{14}
\]
where
\[
\begin{align*}
\tilde{F}_1 &= -\nabla \cdot (\tilde{\rho} \tilde{u}) - \nabla \cdot (\tilde{\rho} \tilde{u}), \\
\tilde{F}_2 &= -\tilde{u} \cdot \nabla \tilde{u} + (\frac{\mu}{\rho_\infty} - \frac{\mu}{\rho_*}) \tilde{u} \cdot \nabla \tilde{u} + \frac{\mu + \lambda}{\rho_\infty} \nabla \tilde{u} \cdot \nabla \tilde{u} \\
&\quad - \frac{P_\rho(\tilde{\rho} + \rho_*)}{\rho_\infty} \nabla \tilde{\rho} - \frac{P_\rho(\tilde{\rho} + \rho_*)}{\rho_*} \nabla \tilde{\rho} \\
&\quad + \frac{\hbar^2}{4} \frac{|\nabla (\tilde{\rho} + \rho_*)|^2 \nabla (\tilde{\rho} + \rho_*)}{(\rho + \rho_*)^3} - \frac{|\nabla \rho_\infty|^2 |\nabla \rho_\infty|}{\rho_*^2} \\
&\quad - \frac{\hbar^2}{4} \frac{\nabla (\tilde{\rho} + \rho_*) \cdot \nabla (\tilde{\rho} + \rho_*)}{(\rho + \rho_*)^2} - \frac{\nabla \rho_\infty \cdot \nabla \rho_\infty}{\rho_*^2} \\
&\quad + \frac{\hbar^2}{4} \frac{1}{\rho_*} - \frac{1}{\rho_*} \nabla \Delta \tilde{\rho} + \frac{1}{\rho_*} (\tilde{B} \cdot \nabla \tilde{B} - \frac{1}{2} \nabla |\tilde{B}|^2), \\
\tilde{F}_3 &= \tilde{B} \cdot \nabla \tilde{u} - \nabla \cdot \tilde{B} = \tilde{u} \cdot \nabla \tilde{B}.
\end{align*}
\]
Set
\[
\varrho = \tilde{\rho}, \quad v = \frac{\rho_\infty}{\sqrt{P_\rho(\rho_\infty)}} \tilde{u}, \quad b = \tilde{B},
\]
and
\[
\mu_1 = \frac{\mu}{\rho_\infty}, \quad \mu_2 = \frac{\mu + \lambda}{\rho_\infty}, \quad \gamma = \sqrt{P_\rho(\rho_\infty)},
\]
then the system (13) and (14) is reformulated as
\[
\begin{align*}
\varrho_t + \gamma \nabla \cdot v = F_1, \\
v_t + \gamma \nabla \varrho - \mu_1 \Delta v - \mu_2 \nabla \nabla \cdot v = \frac{\hbar^2}{4\gamma^2 \rho + \rho_\infty} \nabla \Delta \varrho + F_2, \\
(\varrho, v, b)(x, t)|_{t=0} = (\varrho_0, v_0, b_0)(x),
\end{align*}
\]
where
\[
\begin{align*}
F_1 &= \bar{F}_1, \\
F_2 &= \frac{\rho_\infty}{\sqrt{P_\rho(\rho_\infty)}} \left( \bar{F}_2 - \frac{\hbar^2}{4} \left( \frac{1}{\rho_*} + \frac{1}{\rho_\infty} \right) \nabla \Delta \tilde{\rho} \right), \\
F_3 &= \bar{F}_3, \quad (\varrho_0, v_0, b_0)(x) = (\rho_0 - \rho_*, \rho_\infty \sqrt{P_\rho(\rho_\infty)} u_0, B_0)(x) \to (0, 0, 0), \text{ as } |x| \to \infty.
\end{align*}
\]
Moreover, the nonlinear terms $F_1$, $F_2$, and $F_3$ have the following equivalence properties:

$$
\begin{align*}
F_1 & = v \cdot \nabla \rho + g \nabla \cdot v + v \cdot \nabla \bar{\rho} + \rho \nabla \cdot v, \\
F_2 & = v \cdot \nabla v + g \Delta v + \bar{\rho} \Delta v + g \nabla \nabla \cdot v + \rho \nabla \nabla \cdot v + g \nabla \rho + \rho \nabla \rho + g \nabla \bar{\rho} \\
& \quad + (\rho + \bar{\rho} + 1)(b \cdot \nabla b + \frac{\gamma}{2} |b|^2) + h^2 \nabla \Delta \bar{\rho} + h^2 (|\nabla g|^2 \nabla \rho + \nabla \theta \cdot \nabla \bar{\rho} \nabla \rho) \\
& \quad + |\nabla g|^2 \nabla \bar{\rho} + |\nabla \bar{\rho}|^2 \nabla g + \nabla \theta \cdot \nabla \bar{\rho} \nabla \rho + |\nabla \rho|^2 \nabla \rho (g^2 + \rho_\infty^2 + \rho^2 + \bar{\rho} g^2) \\
& \quad + \rho^2 (\rho + \bar{\rho}) + h^2 (\nabla \theta \Delta \rho + \nabla \bar{\rho} \Delta \bar{\rho} + \nabla \rho \Delta \rho + \nabla \bar{\rho} \Delta \bar{\rho} (g^2 + \rho + \bar{\rho} g)) \\
& \quad + h^2 (\nabla g \nabla^2 \rho + \nabla \bar{\rho} \nabla^2 \rho + \nabla \rho \nabla^2 \bar{\rho} + \nabla \bar{\rho} \nabla^2 \bar{\rho} (g^2 + \rho + \bar{\rho} g)), \\
F_3 & = v \cdot \nabla b + b \cdot \nabla v + b \nabla \cdot v.
\end{align*}
$$

We assume a priori that for sufficiently small $\delta > 0(\epsilon \ll \delta)$,

$$
\|(\rho, v, b)(t)\|_{H^4}^2 + \|h \nabla \rho(t)\|_{H^4}^2 \leq \delta.
$$

By (9) and Sobolev’s inequality, we then obtain $1/2 \rho_\infty \leq \rho + \bar{\rho} \leq 2 \rho_\infty$.

First, we will obtain the dissipation estimate for $v$.

**Lemma 2.1.** Let $(\rho, v, b)$ be a smooth solution to (15) and satisfy (5) and (17), then

$$
\frac{1}{2} \frac{d}{dt} \int |\nabla \rho|^2 + |v|^2 + |b|^2 + \frac{h^2 \sqrt{\rho_\infty}}{8 \gamma^2 \rho_\infty} |\nabla g|^2 d\nu + C \|
abla v\|_{L^2}^2 + \nu \|\nabla b\|_{L^2}^2 \leq C \delta (\|\nabla \rho\|_{L^2}^2 + h^2 \|\nabla^2 \theta\|_{L^2}^2).
$$

**Proof.** For $k = 0$, applying to (15) and taking $L^2$-inner product with $(\rho, v, b)$, we have

$$
\frac{1}{2} \frac{d}{dt} \int \frac{\nabla^2 \rho}{\rho + \bar{\rho} + \rho_\infty} d\nu + \int F_1 d\nu + \int F_2 d\nu + \int F_3 d\nu = I_1 + I_2 + I_3 + I_4.
$$

We will estimate each term on the right hand side. First, for the term $I_1$, by the continuity equation, integration by parts twice, Hölder’s inequality, Young’s
inequality and Sobolev embedding, we have

\[ I_1 = \int \frac{h^2}{4\gamma^2} \frac{\sqrt{\rho_\infty}}{(\rho + \bar{\rho} + \rho_\infty)^2} \nabla q \cdot \nabla q \, dx + \int \frac{h^2}{4\gamma} \frac{\rho}{(\rho + \bar{\rho} + \rho_\infty)^2} (\nabla \rho \cdot v + \nabla \bar{\rho} \cdot v) \, dx - \int \nabla (\frac{1}{(\rho + \bar{\rho} + \rho_\infty)^2}) \Delta q \cdot v \, dx \]

\[ \leq -\frac{1}{2} \frac{d}{dt} \int \frac{h^2}{4\gamma^2} \frac{\sqrt{\rho_\infty}}{(\rho + \bar{\rho} + \rho_\infty)^2} |\nabla q|^2 \, dx \]

\[ + \int \frac{h^2}{4\gamma} \frac{\sqrt{\rho_\infty}}{(\rho + \bar{\rho} + \rho_\infty)^2} \cdot \nabla q \cdot v \, dx \]

\[ - C \int \frac{h^2}{4\gamma^2} \frac{\sqrt{\rho_\infty}}{(\rho + \bar{\rho} + \rho_\infty)^2} \cdot \nabla q(v \cdot \nabla v + \rho \nabla \cdot v + \nabla \bar{\rho}) \]

\[ + \bar{\rho} \nabla \cdot v) \, dx + \int \frac{h^2}{4\gamma} \frac{\Delta q}{(\rho + \bar{\rho} + \rho_\infty)^2} (\nabla q \cdot v + \nabla \bar{\rho} \cdot v) \, dx \]

\[ - \int \frac{h^2}{4\gamma} \frac{1}{(\rho + \bar{\rho} + \rho_\infty)^2} \Delta q \cdot v \, dx \]

\[ \leq -\frac{1}{2} \frac{d}{dt} \int \frac{h^2}{4\gamma^2} \frac{\sqrt{\rho_\infty}}{(\rho + \bar{\rho} + \rho_\infty)^2} |\nabla q|^2 \, dx + C h^2 \|\nabla (\rho, \bar{\rho})\|_{L^\infty} \|\nabla q\|_{L^2} \|\nabla \cdot v\|_{L^2} \]

\[ + C h^2 \|\nabla (\rho, \bar{\rho})\|_{L^\infty} \|\nabla q\|_{L^2} \left( \|v\|_{L^6} + \|\nabla \cdot v\|_{L^2} \right) \left( \|\nabla (\rho, \bar{\rho})\|_{L^3} + \| (\rho, \bar{\rho})\|_{L^\infty} \right) \]

\[ + \|\Delta q\|_{L^2} \|\nabla (\rho, \bar{\rho})\|_{L^3} \|v\|_{L^6} + C h^2 \|\nabla (\rho, \bar{\rho})\|_{L^1} \|\Delta q\|_{L^2} \|v\|_{L^6} \]

\[ \leq -\frac{1}{2} \frac{d}{dt} \int \frac{h^2}{4\gamma^2} \frac{\sqrt{\rho_\infty}}{(\rho + \bar{\rho} + \rho_\infty)^2} |\nabla q|^2 \, dx + C h^2 \delta \left( \|\nabla q\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 \right). \]

For the term \( I_2 \), using (12), Hölder’s inequality, Young’s inequality, Hardy’s inequality, and Sobolev embedding, we have

\[ I_2 \leq C \left( \|\nabla q\|_{L^2} \|v\|_{L^3} + \|\nabla \cdot v\|_{L^2} \| (\rho, \bar{\rho})\|_{L^3} \right) \|q\|_{L^6} \]

\[ + C \|1 + |x|\|\nabla \bar{\rho}\|_{L^1} \|v\|_{L^6} \left\| \frac{\rho}{1 + |x|} \right\|_{L^2} \]

\[ \leq C \delta \left( \|\nabla q\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 \right). \]

The term \( I_3 \) is much more complicated, which can be further decomposed into

\[ I_3 \leq C \int (v \cdot \nabla v + \rho \Delta v + \rho \nabla \cdot v + \rho \nabla \bar{q} + (\rho + 1)(b \cdot \nabla b + \frac{1}{2} \nabla |b|^2)) \cdot v \, dx \]

\[ + C \int (\bar{\rho} \Delta v + \bar{\rho} \nabla \cdot v + \bar{\rho} \nabla \bar{q} + \rho \nabla \rho + \bar{\rho} (b \cdot \nabla b + \frac{1}{2} \nabla |b|^2)) \cdot v \, dx \]

\[ + C h^2 \int (|\nabla q|^2 \nabla q + \nabla q \Delta q + \nabla q \nabla^2 q) \cdot v \, dx \]

\[ + C h^2 \int (\nabla q \cdot \nabla \bar{q} + \nabla q \nabla q + \nabla q \nabla \rho + \nabla q \nabla \rho + \nabla q \Delta \bar{q} \]

\[ + \nabla \rho \Delta q + \nabla q \nabla^2 \bar{q} + \nabla \rho \nabla^2 q) \cdot v \, dx + C h^2 \int (|\nabla \rho|^2 \nabla \bar{q} (\rho^3 + \rho^2) + \rho q \]

\[ + \rho^2 \rho + \rho \rho q + \nabla \rho \Delta \rho (\rho^2 + \rho + \rho \rho) + \nabla \rho \nabla^2 \rho (\rho^2 + \rho + \rho \rho) \) \cdot v \, dx \]

\[ = I_{31} + I_{32} + I_{33} + I_{34} + I_{35}. \]
For the term $I_{31}$, by Hölder’s inequality, Sobolev embedding and integration by parts, we have

$$I_{31} = C \int (v \cdot \nabla v + \rho \nabla \rho + (\rho + 1)(b \cdot \nabla b + \frac{1}{2} \nabla |b|^2)) \cdot vdx$$

$$- C \int \partial_i \rho \partial_i u_j \nabla \cdot \rho \nabla v^j + \partial_j \rho \partial_i v^j + \rho \partial_i v^j \partial_j v^i + \rho \partial_i v^j \partial_j v^i dx$$

$$\leq C(\|\nabla v\|_{L^2} + \|\nabla \rho\|_{L^2} \|\rho\|_{L^2} + (\|\rho\|_{L^2} + 1)\|\nabla b\|_{L^2} \|b\|_{L^6} \|v\|_{L^3}$$

$$+ C(\|\nabla v\|_{L^3} \|\rho\|_{L^5} + \|\nabla \cdot v\|_{L^3} \|\rho\|_{L^5})\|\nabla v\|_{L^2}$$

$$\leq C\delta(\|\nabla \rho\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\nabla b\|_{L^2}^2).$$

For the term $I_{32}$, by Hölder’s inequality, Young’s inequality, Hardy’s inequality, Sobolev embedding and integration by parts, we have

$$I_{32} = C \int (\rho \nabla \rho + \rho \nabla \rho + \rho \bar{b} \cdot \nabla b + \frac{1}{2} \nabla |b|^2)) \cdot vdx$$

$$- C \int \partial_i \rho \partial_i u_j \nabla \cdot \rho \nabla v^j + \partial_j \rho \partial_i v^j + \rho \partial_i v^j \partial_j v^i + \rho \partial_i v^j \partial_j v^i dx$$

$$\leq C(\|\nabla \rho\|_{L^2} \|\bar{b}\|_{L^2} + (1 + |x|)\|\nabla \rho\|_{L^2} \|\rho\|_{L^3}$$

$$+ \|\bar{b}\|_{L^2} \|\rho\|_{L^5} \|\nabla \rho\|_{L^2} + (\|\nabla \rho\|_{L^2} \|\rho\|_{L^5} + \|\bar{b}\|_{L^2} \|\nabla \rho\|_{L^2})\|\nabla v\|_{L^2}$$

$$\leq C\delta(\|\nabla \rho\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\nabla b\|_{L^2}^2).$$

For the term $I_{33}$, by Hölder’s inequality, Young’s inequality and Sobolev embedding, we have

$$I_{33} \leq C\delta(\|\nabla \rho\|_{L^2}^2 + \|\nabla v\|_{L^2}^2).$$

For the term $I_{34}$ and $I_{35}$, similar to $I_{32}$, we have

$$I_{34} \leq C\delta(\|\nabla \rho\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2 \|\rho\|_{L^2} + \|\nabla \rho\|_{L^2}^2 \|\nabla \rho\|_{L^3} + \|\nabla \rho\|_{L^2}^2 \|\nabla \rho\|_{L^2}$$

$$+ \|\nabla \rho\|_{L^2}^2 \|\Delta \rho\|_{L^2} + \|\nabla \rho\|_{L^2}^2 \|\Delta \rho\|_{L^2} + \|\nabla \rho\|_{L^2}^2 \|\nabla \rho\|_{L^2}$$

$$\leq C\delta(\|\nabla \rho\|_{H^1}^2 + \|\nabla v\|_{L^2}^2),$$

and

$$I_{35} \leq C\delta(\|\nabla \rho\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 \|\rho\|_{L^3} \|\nabla \rho\|_{L^2} \|\rho\|_{L^3}$$

$$+ \|\rho\|_{L^3} + 1 + \|\nabla \rho\|_{L^2} \|\nabla \rho\|_{L^3} \|\rho\|_{L^3}) \|\nabla \rho\|_{L^3} \|\rho\|_{L^3}$$

$$\times \|\nabla \rho\|_{L^3} \|\rho\|_{L^3} \|\rho\|_{L^3} + 1 + \|\nabla \rho\|_{L^3} \|\rho\|_{L^3}$$

$$\leq C\delta(\|\nabla \rho\|_{L^2}^2 + \|\nabla v\|_{L^2}^2).$$

Collecting these terms, we obtain

$$I_3 \leq C\delta(\|\nabla v\|_{L^2}^2 + \|\nabla v\|_{H^1}^2).$$

For the term $I_4$, we have

$$I_4 \leq C\delta(\|\nabla \rho\|_{L^2}^2 + \|\nabla v\|_{L^2}^2).$$

Summing up above terms, by the smallness of $\delta$, we conclude our lemma.

$\square$
In the next lemma, we derive the higher order dissipative estimate.

**Lemma 2.2.** Let \((\varrho, v, b)\) be a smooth solution to (15) and satisfy (5) and (17), then

\[
\frac{d}{dt} E_1(t) + \|\nabla^2 v\|_{L^2}^2 + \|\nabla^2 b\|_{L^2}^2 \leq C\delta \left( \|\nabla \varrho\|_{H^3}^2 + \hbar^2 \|\nabla^2 \varrho\|_{H^3}^2 + \|\nabla v\|_{H^2}^2 + \|\nabla b\|_{H^2}^2 \right),
\]

where the energy functional \(E_1(t)\) is equivalent to \(\|\nabla (\varrho, v, b)\|_{H^3}^2 + \hbar^2 \|\nabla^2 \varrho\|_{H^3}^2\).

**Proof.** For \(0 \leq k \leq 3\), applying \(\nabla^{k+1}\) to (15) and then taking \(L^2\)-inner product with \(\nabla^{k+1} \varrho, \nabla^{k+1} v, \nabla^{k+1} b\), we have

\[
\frac{1}{2} \frac{d}{dt} \int \left|\nabla^{k+1} \varrho\right|^2 + \left|\nabla^{k+1} v\right|^2 + \left|\nabla^{k+1} b\right|^2 dx
+ \int \mu_1 \left|\nabla^{k+2} v\right|^2 + \mu_2 \left|\nabla^{k+1} \nabla \cdot v\right|^2 + \nu \left|\nabla \nabla^{k+2} b\right|^2 dx \\
= \int \hbar^2 \nabla^{k+1} \left( \frac{\nabla \varrho}{\varrho + \bar{\rho} + \rho_{\infty}} \right) \cdot \nabla^{k+1} v dx + \int \nabla^{k+1} F_1 \nabla^{k+1} \varrho dx \\
+ \int \nabla^{k+1} F_2 \cdot \nabla^{k+1} v dx + \int \nabla^{k+1} F_3 \cdot \nabla^{k+1} b dx \\
= J_1 + J_2 + J_3 + J_4.
\]

We will estimate each term on the right hand side. First, we split \(J_1\) as

\[
J_1 = \int \frac{\hbar^2}{4\gamma} \left( \frac{1}{\varrho + \bar{\rho} + \rho_{\infty}} \right) \nabla^{k+1} \nabla \Delta \varrho \cdot \nabla^{k+1} v dx \\
+ \int \frac{\hbar^2}{4\gamma} \sum_{1 \leq l \leq k+1} C^l_{k+1} \nabla^l \left( \frac{1}{\varrho + \bar{\rho} + \rho_{\infty}} \right) \nabla^{k-l+1} \nabla \Delta \varrho \cdot \nabla^{k+1} v dx
\]

\[
(20)
\]

For the term \(J_{11}\), by integration by parts, we have

\[
J_{11} = - \int \frac{\hbar^2}{4\gamma} \nabla \left( \frac{1}{\varrho + \bar{\rho} + \rho_{\infty}} \right) \nabla^{k+1} \Delta \varrho \cdot \nabla^{k+1} v dx \\
- \int \frac{\hbar^2}{4\gamma} \frac{1}{\varrho + \bar{\rho} + \rho_{\infty}} \nabla^{k+1} \Delta \varrho \cdot \nabla^{k+1} \nabla \cdot v dx \\
= \int \frac{\hbar^2}{4\gamma} \nabla^{k+2} \varrho \nabla^2 \left( \frac{1}{\varrho + \bar{\rho} + \rho_{\infty}} \right) \cdot \nabla^{k+1} v dx \\
+ \int \frac{\hbar^2}{4\gamma} \nabla^{k+2} \varrho \nabla \left( \frac{1}{\varrho + \bar{\rho} + \rho_{\infty}} \right) \cdot \nabla^{k+2} v dx \\
- \int \frac{\hbar^2}{4\gamma} \frac{1}{\varrho + \bar{\rho} + \rho_{\infty}} \nabla^{k+1} \Delta \varrho \cdot \nabla^{k+1} \nabla \cdot v dx,
\]

where the first two terms can be easily bounded by

\[
C\delta \hbar^2 \left( \|\nabla^{k+2} \varrho\|_{L^2}^2 + \|\nabla^{k+2} v\|_{L^2}^2 \right).
\]
For the last term in $J_{11}$, by the continuity equation, we can be further decomposed into

$$
\int \frac{h^2}{4\gamma} \nabla^{k+2} \varrho \nabla \left( \frac{1}{\varrho + \bar{\rho} + \rho_\infty} \right) \cdot \nabla^{k+1} \nabla \cdot v + \frac{h^2}{4\gamma} \varrho \nabla^{k+2} \varrho \nabla^{k+2} \nabla \cdot v \, dx
$$



$$
= \int \frac{h^2}{4\gamma} \nabla^{k+2} \varrho \nabla \left( \frac{1}{\varrho + \bar{\rho} + \rho_\infty} \right) \cdot \nabla^{k+1} \nabla \cdot v \, dx
$$

- \int \frac{h^2}{4\gamma^2 (\varrho + \bar{\rho} + \rho_\infty)^2} \nabla^{k+2} \varrho \nabla^{k+2} \varrho \, dx

- \int \frac{h^2}{4\gamma^2 \varrho + \bar{\rho} + \rho_\infty} \sum_{0 \leq l \leq k+1} C_{k+2}^l \nabla^l \varrho \nabla^{k+2-l} \left( \frac{1}{\varrho + \bar{\rho} + \rho_\infty} \right) \nabla^{k+2} \varrho \, dx

- \int \frac{h^2}{4\gamma^2 \varrho + \bar{\rho} + \rho_\infty} \nabla^{k+2} \varrho \nabla^{k+2} \left( \frac{\nabla \cdot v}{\varrho + \bar{\rho} + \rho_\infty} \right) \, dx

- \int \frac{h^2}{4\gamma^2 \varrho + \bar{\rho} + \rho_\infty} \nabla^{k+2} \varrho \nabla^{k+2} \left( \frac{\nabla \bar{\rho} \cdot v}{\varrho + \bar{\rho} + \rho_\infty} \right) \, dx

= L_1 + L_2 + L_3 + L_4 + L_5.

The terms $L_1$ and $L_2$ can be easily bounded by

$$
L_1 + L_2 \leq - \frac{1}{2} \frac{d}{dt} \int \frac{h^2}{4\gamma^2 (\varrho + \bar{\rho} + \rho_\infty)^2} |\nabla^{k+2} \varrho|^2 \, dx + C\delta h^2 (||\nabla^{k+2} \varrho||_{L^2}^2 + ||\nabla^{k+2} v||_{L^2}^2).
$$

For the term $L_3$, using the continuity equation and Hölder’s inequality, we have

$$
L_3 \leq Ch^2 ||\nabla^{k+2} \varrho||_{L^2} \sum_{0 \leq l \leq k+1} ||\nabla^l \varrho \nabla^{k+2-l} \left( \frac{1}{\varrho + \bar{\rho} + \rho_\infty} \right)||_{L^2}
$$

$$
\leq Ch^2 ||\nabla^{k+2} \varrho||_{L^2} \sum_{0 \leq l \leq k+1} \left( ||\nabla^l \nabla \cdot v \nabla^{k+2-l} \left( \frac{1}{\varrho + \bar{\rho} + \rho_\infty} \right)||_{L^2}
$$

+ ||\nabla^l (\nabla \varrho \cdot v) \nabla^{k+2-l} \left( \frac{1}{\varrho + \bar{\rho} + \rho_\infty} \right)||_{L^2}

+ ||\nabla^l (\nabla \bar{\rho} \cdot v) \nabla^{k+2-l} \left( \frac{1}{\varrho + \bar{\rho} + \rho_\infty} \right)||_{L^2}

+ ||\nabla^l (\nabla \rho \cdot v) \nabla^{k+2-l} \left( \frac{1}{\varrho + \bar{\rho} + \rho_\infty} \right)||_{L^2} \right).
$$

(21)
For the second term of (21), separating the case of \( l = 0, 1 \) and \( k + 1 \) from the other cases, we bound the summation by

\[
\begin{align*}
&\quad C \delta^2 \| \nabla^{k+2} \varrho \|_{L^2} \left( \| \nabla \varrho \cdot v \nabla^{k+2} \left( \frac{1}{\varrho + \bar{\rho} + \varrho_\infty} \right) \|_{L^2} \\
+ &\quad \| \nabla (\nabla \varrho \cdot v) \nabla^{k+1} \left( \frac{1}{\varrho + \bar{\rho} + \varrho_\infty} \right) \|_{L^2} + \| \nabla^{k+1} (\nabla \varrho \cdot v) \nabla \left( \frac{1}{\varrho + \bar{\rho} + \varrho_\infty} \right) \|_{L^2} \\
+ &\quad \sum_{2 \leq l \leq k} \| \nabla^l (\nabla \varrho \cdot v) \nabla^{k+2-l} \left( \frac{1}{\varrho + \bar{\rho} + \varrho_\infty} \right) \|_{L^2} \right) \\
&\quad \leq C \delta^2 \| \nabla^{k+2} \varrho \|_{L^2} \left( C \delta \| \nabla^{k+2} \varrho \|_{L^2} + C \delta \| \nabla \varrho \|_{L^2} + C \delta \| \nabla^{k+1} \varrho \|_{L^2} + C \delta \| \nabla^2 \varrho \|_{L^2} \\
&\quad + C \delta \| \nabla^{k+1} (\nabla \varrho \cdot v) \|_{L^2} + \sum_{2 \leq l \leq k} \| \nabla^l (\nabla \varrho \cdot v) \|_{L^2} \| \nabla^{k+2-l} \left( \frac{1}{\varrho + \bar{\rho} + \varrho_\infty} \right) \|_{L^\infty} \right) \\
&\quad \leq C \delta \delta^2 \left( \| \nabla^{k+2} \varrho \|_{L^2} + \| \nabla \varrho \|_{H^{k+1}}^2 \right),
\end{align*}
\]

where

\[
\| \nabla^{k+1} (\nabla \varrho \cdot v) \|_{L^2} = \sum_{0 \leq l \leq k+1} \| \nabla^{l+1} \varrho \nabla^{k+1-l} \varrho \|_{L^2}
\]

\[
= \| \nabla \varrho \nabla^{k+1} v \|_{L^2} + \| \nabla^2 \varrho \nabla^k v \|_{L^2} + \sum_{2 \leq l \leq k+1} \| \nabla^{l+1} \varrho \nabla^{k+1-l} v \|_{L^2}
\]

\[
\leq C \left( \| \nabla \varrho \|_{L^\infty} \| \nabla^{k+1} v \|_{L^2} + \| \nabla^2 \varrho \|_{L^\infty} \| \nabla^k v \|_{L^2} \right.
\]

\[
+ \sum_{2 \leq l \leq k+1} \| \nabla^{l+1} \varrho \|_{L^2} \| \nabla^{k+1-l} v \|_{L^\infty} \right)
\]

\[
\leq C \delta \left( \| \nabla^{k+2} v \|_{L^2} + \| \nabla^2 \varrho \|_{H^{k+1}} \right),
\]

and

\[
\sum_{2 \leq l \leq k} \| \nabla^l (\nabla \varrho \cdot v) \|_{L^2} = \sum_{2 \leq l \leq k} \sum_{0 \leq m \leq l} C_l^m \| \nabla^{m+1} \varrho \nabla^{l-m} v \|_{L^2} \leq C \delta \| \nabla^2 \varrho \|_{H^k}.
\]

Similarly, we bound the remaining terms in (21) by

\[
C \delta (\| \nabla v \|_{H^k}^2 + \| \nabla \varrho \|_{H^{k+1}}^2).
\]

Collecting these terms, we obtain

\[
L_3 \leq C \delta \delta^2 \left( \| \nabla v \|_{H^k}^2 + \| \nabla^2 \varrho \|_{H^{k+1}}^2 \right).
\]

Similar to the term \( J_1 \), we split \( L_4 \) as

\[
L_4 = - \int \frac{h^2}{4\gamma} \nabla^{k+2} \varrho \nabla^{k+3} \varrho \cdot \frac{\sqrt{\varrho_\infty v}}{(\varrho + \bar{\rho} + \varrho_\infty)^2} dx
\]

\[
- \int \frac{h^2}{4\gamma} \nabla^{k+2} \varrho \cdot \sum_{0 \leq l \leq k+1} C_{k+2}^l \nabla^{l+1} \varrho \nabla^{k+2-l} \left( \frac{v}{\varrho + \bar{\rho} + \varrho_\infty} \right) dx
\]

\[
= L_{41} + L_{42}.
\]
For the term of $L_{41}$, we have by integration by parts that

$$L_{41} = -\frac{1}{2} \int \frac{h^2}{4\gamma} \nabla ((\nabla^k \varrho)^2) \cdot \frac{\sqrt{\rho} \varrho}{(\varrho + \rho_\infty)^2} \, dx$$

$$= \frac{1}{2} \int \frac{h^2}{4\gamma} \sqrt{\rho} \rho_\infty |\nabla^k \varrho|^2 \nabla \cdot \left( \frac{\varrho}{(\varrho + \rho_\infty)^2} \right) \, dx$$

$$\leq C\delta h^2 \|\nabla^{k+2} \varrho\|^2_{L^2}.$$

For the term of $L_{42}$, separating the case of $l = 0, k$ and $k + 1$ from the other cases, we bound the summation by

$$L_{42} = C h^2 \|\nabla^{k+2} \varrho\|_{L^2} \sum_{0 \leq l \leq k+1} \|\nabla^{l+1} \varrho \nabla^{k+2-l} \left( \frac{\varrho}{\varrho + \rho + \rho_\infty} \right) \|_{L^2}$$

$$= C h^2 \|\nabla^{k+2} \varrho\|_{L^2} \left( \|\nabla^{l+1} \varrho \nabla^{k+2-l} \left( \frac{\varrho}{\varrho + \rho + \rho_\infty} \right) \|_{L^2} + \|\nabla^{k+1} \varrho \nabla^{k+2-l} \left( \frac{\varrho}{\varrho + \rho + \rho_\infty} \right) \|_{L^2} \right)$$

$$+ \|\nabla^{k+2} \varrho \nabla \left( \frac{\varrho}{\varrho + \rho + \rho_\infty} \right) \|_{L^2} + \sum_{1 \leq l \leq k-1} \|\nabla^{l+1} \varrho \nabla^{k+2-l} \left( \frac{\varrho}{\varrho + \rho + \rho_\infty} \right) \|_{L^2}$$

$$\leq C h^2 \|\nabla^{k+2} \varrho\|_{L^2} \left( C \|\nabla^{k+2} \varrho\|_{L^2} + C \sum_{1 \leq l \leq k-1} \|\nabla^{k+2-l} \left( \frac{\varrho}{\varrho + \rho + \rho_\infty} \right) \|_{L^2} \right)$$

$$\leq C \delta h^2 \left( \|\nabla^2 \varrho\|_{H^k}^2 + \|\nabla^2 \varrho\|_{H^k}^2 \right).$$

where

$$\|\nabla^{k+2} \left( \frac{\varrho}{\varrho + \rho + \rho_\infty} \right) \|_{L^2} \leq C \sum_{0 \leq l \leq k+2} \|\nabla^{l+1} \varrho \nabla^{k+2-l} \left( \frac{1}{\varrho + \rho + \rho_\infty} \right) \|_{L^2}$$

$$= \| \varrho \nabla^{k+2} \left( \frac{1}{\varrho + \rho + \rho_\infty} \right) \|_{L^2} + \| \nabla^{k+2} \varrho \nabla \left( \frac{1}{\varrho + \rho + \rho_\infty} \right) \|_{L^2}$$

$$+ \sum_{1 \leq l \leq k} \|\nabla^{l+1} \varrho \nabla^{k+2-l} \left( \frac{1}{\varrho + \rho + \rho_\infty} \right) \|_{L^2}$$

$$\leq C \delta \left( \|\nabla^2 \varrho\|_{H^k}^2 + \|\nabla^2 \varrho\|_{H^k}^2 \right).$$

For the term of $L_5$, similar to $L_4$, we have

$$L_5 \leq C h^2 \|\nabla^{k+2} \varrho\|_{L^2} \sum_{0 \leq l \leq k+1} \|\nabla^{l+1} \rho \nabla^{k+2-l} \left( \frac{\varrho}{\varrho + \rho + \rho_\infty} \right) \|_{L^2}$$

$$= C h^2 \|\nabla^{k+2} \varrho\|_{L^2} \left( \|\nabla^{l+1} \rho \nabla^{k+2-l} \left( \frac{\varrho}{\varrho + \rho + \rho_\infty} \right) \|_{L^2} + \|\nabla^{k+1} \rho \nabla^{2} \left( \frac{\varrho}{\varrho + \rho + \rho_\infty} \right) \|_{L^2} \right)$$

$$+ \|\nabla^{k+2} \rho \nabla \left( \frac{\varrho}{\varrho + \rho + \rho_\infty} \right) \|_{L^2} + \|\nabla^{k+3} \varrho \|_{H^3} \|\varrho\|_{L^5}$$

$$+ \sum_{1 \leq l \leq k-1} \|\nabla^{l+1} \rho \nabla^{k+2-l} \left( \frac{\varrho}{\varrho + \rho + \rho_\infty} \right) \|_{L^2}$$

$$\leq C \delta h^2 \left( \|\nabla^2 \varrho\|_{H^{k+1}}^2 + \|\nabla^2 \varrho\|_{H^{k+1}}^2 \right).$$
Collecting these terms, we obtain

\[ J_{11} \leq -\frac{1}{2} \frac{d}{dt} \int \frac{\hbar^2}{4\gamma} \frac{\sqrt{\rho_{\infty}}}{(\varrho + \bar{\rho} + \rho_{\infty})^2} |\nabla^{k+2} \varrho|^2 \, dx + C \delta h^2 \left( \| \nabla \varrho \|_{H^{k+1}}^2 + \| \nabla v \|_{H^{k+1}}^2 \right). \]

For the second term in (20), by Hölder’s inequality and integration by parts, we have

\[ J_{12} = -\int \frac{\hbar^2}{4\gamma} C^2_{k+1} \varrho \nabla \left( \frac{1}{\rho + \bar{\rho} + \rho_{\infty}} \right) \cdot \nabla^{k+1} \varrho \, dx \]

\[ - \int \frac{\hbar^2}{4\gamma} \left( C^2_{k+1} \nabla \left( \frac{1}{\rho + \bar{\rho} + \rho_{\infty}} \right) \cdot \nabla^{k+1} \varrho \right) \, dx \]

\[ - C^2_{k+1} \nabla \left( \frac{1}{\rho + \bar{\rho} + \rho_{\infty}} \right) \nabla^{k+1} \varrho \cdot v \, dx \]

\[ - \int \frac{\hbar^2}{4\gamma} \sum_{3 \leq l \leq k+1} C^l_{k+1} \nabla^l \left( \frac{1}{\rho + \bar{\rho} + \rho_{\infty}} \right) \nabla^{k-l+1} \nabla \varrho \cdot \nabla^{k+1} \varrho \, dx \]

\[ \leq C h^2 \left( \| \nabla^2 \varrho \|_L^2 \| \nabla^{k+1} \varrho \|_L^6 \right) \]

\[ + \| \nabla \left( \frac{1}{\rho + \bar{\rho} + \rho_{\infty}} \right) \|_{L^2} \| \nabla^{k+2} \varrho \|_{L^2} \| \nabla^{k+2} \varrho \|_{L^2} \]

\[ + C h^2 \| \nabla^2 \|_L^2 \| \nabla^{k+2} \varrho \|_L^2 \| \nabla^{k+1} \varrho \|_L^6 \]

\[ + C h^2 \sum_{3 \leq l \leq k+1} \| \nabla^l \left( \frac{1}{\rho + \bar{\rho} + \rho_{\infty}} \right) \|_{L^2} \| \nabla^{k-l+2} \varrho \|_{L^2} \| \nabla^{k+1} \varrho \|_L^6 \]

\[ \leq C \delta h^2 \left( \| \nabla^2 \varrho \|_{H^k}^2 + \| \nabla^{k+2} \varrho \|_{L^2}^2 \right). \]

Summing up \( J_{11} \) and \( J_{12} \), we have

\[ J_1 \leq \frac{1}{2} \frac{d}{dt} \int \frac{\hbar^2}{4\gamma} \frac{\sqrt{\rho_{\infty}}}{(\varrho + \bar{\rho} + \rho_{\infty})^2} |\nabla^{k+2} \varrho|^2 \, dx + C \delta h^2 \left( \| \nabla \varrho \|_{H^{k+1}}^2 + \| \nabla v \|_{H^{k+1}}^2 \right). \]

For the term \( J_2 \), we split it as

\[ J_2 \leq C \int \nabla^{k+1} (v \cdot \nabla \varrho + \varrho \nabla \cdot v + v \cdot \nabla \varrho + \bar{\rho} \nabla \cdot v) \nabla^{k+1} \varrho \, dx \]

\[ = J_{21} + J_{22} + J_{23} + J_{24}. \]

The first term \( J_{21} \) can be bounded by

\[ J_{21} = C \int \nabla^{k+1} \varrho \cdot v \nabla^{k+1} \varrho \, dx + \int \sum_{0 \leq l \leq k} (C^l_{k+1} \nabla^{l+1} \varrho \nabla^{k+1-l} v) \nabla^{k+1} \varrho \, dx \]

\[ \leq C \| \nabla^{k+1} \varrho \|_{L^2}^2 \| \nabla \cdot v \|_{L^\infty} + C \| \nabla^{k+1} \varrho \|_{L^2} \left( \| \nabla \varrho \nabla^{k+1} \varrho \|_{L^2} + \| \nabla^2 \varrho \nabla^k \varrho \|_{L^2} \right) \]

\[ + \| \nabla^2 \varrho \nabla^k \varrho \|_{L^2} + \sum_{2 \leq l \leq k} \| \nabla^{l+1} \varrho \nabla^{k+1-l} v \|_{L^2} \]

\[ \leq C \delta \left( \| \nabla^{k+1} \varrho \|_{L^2}^2 + \| \nabla \varrho \|_{H^k}^2 \right). \]

For the remaining terms \( J_{22}, J_{23}, J_{24} \), we bound the summation by

\[ J_{22} + J_{23} + J_{24} \leq C \delta \left( \| \nabla \varrho \|_{H^k}^2 + \| \nabla \varrho \|_{H^k}^2 \right). \]

Summing up \( J_{21} \sim J_{24} \), we obtain

\[ J_2 \leq C \delta \left( \| \nabla \varrho \|_{H^k}^2 + \| \nabla \varrho \|_{H^{k+1}}^2 \right). \]
For the term in $J_3$, we have
\[
J_3 \leq C \int \nabla^{k+1} (v \cdot \nabla v + \rho \Delta v + \rho \nabla \cdot v + \rho \nabla \rho) 
+ (\rho + 1)(b \cdot \nabla b + \frac{1}{2} \nabla |b|^2)) \cdot \nabla^{k+1} v dx 
+ C \int \nabla^{k+1} (\rho \Delta v 
+ \rho \nabla \cdot v + \rho \nabla \rho + \rho (b \cdot \nabla b + \frac{1}{2} \nabla |b|^2)) \cdot \nabla^{k+1} v dx 
+ Ch^2 \int \nabla^{k+1} (|\nabla \rho|^2 \nabla \rho + \nabla \rho \Delta \rho + \nabla \rho \nabla^2 \rho) \cdot \nabla^{k+1} v dx 
+ Ch^2 \int \nabla^{k+1} (\nabla \rho \cdot \nabla \rho + |\nabla \rho|^2 \nabla \rho + \nabla \rho \cdot \nabla \rho \nabla \rho 
+ \nabla \rho \Delta \rho + \nabla \rho \Delta \rho + \nabla \rho \nabla^2 \rho + \nabla \rho \nabla^2 \rho) \cdot \nabla^{k+1} v dx 
+ Ch^2 \int \nabla^{k+1} (|\nabla \rho|^2 \nabla \rho (\rho^3 + \rho^2 + \rho + \rho^2 + \rho + \rho^2 + \rho)) 
+ \nabla \rho \Delta \rho (\rho^2 + \rho + \rho \rho) + \nabla \rho \nabla^2 \rho (\rho^2 + \rho + \rho \rho)) \cdot \nabla^{k+1} v dx.
\]
Separating the case of $k+1$ from the other cases and using Hölder’s inequality, Young’s inequality and integration by parts, we have
\[
Ch^2 \int \nabla^{k+1} (\nabla \rho \Delta \rho \rho) \cdot \nabla^{k+1} v 
= -Ch^2 \int \nabla^k (\nabla \rho \Delta \rho) \rho^2 \cdot \nabla^{k+2} v 
- Ch^2 \int \sum_{0 \leq l \leq k-1} C_l^k \nabla^l (\nabla \rho \Delta \rho) \nabla^{k-l} (\rho^2) \cdot \nabla^{k+2} v 
\leq C \delta h^2 (\|\nabla \rho\|_{H^{k+1}}^2 + \|\nabla \rho\|_{L^2}^2 + \|\nabla^2 \rho\|_{L^2}^2).
\]
Similarly, we bound the remaining terms in $J_3$ by
\[
C \delta h^2 (\|\nabla \rho\|_{H^{k+1}}^2 + \|\nabla \rho\|_{H^{k+1}}^2 + \|\nabla \rho\|_{H^{k+1}}^2).
\]
Summing up these terms, we have
\[
J_3 \leq C \delta h^2 (\|\nabla \rho\|_{H^{k+1}}^2 + \|\nabla \rho\|_{H^{k+1}}^2 + \|\nabla \rho\|_{H^{k+1}}^2).
\]
For the last term $J_4$, it is easy to show
\[
J_4 \leq C \delta (\|\nabla \rho\|_{H^{k+1}}^2 + \|\nabla \rho\|_{H^{k+1}}^2).
\]
Consequently, summing up $J_1 \sim J_4$, by the smallness of $\delta$, we have
\[
\frac{d}{dt} \int |\nabla^{k+1} \rho|^2 + |\nabla^{k+1} v|^2 + |\nabla^{k+1} b|^2 + \frac{h^2 \sqrt{\rho_\infty}}{4\gamma^2 (\rho + \rho_\infty)^2} |\nabla^{k+2} \rho|^2 dx 
+ C (\|\nabla^{k+2} \rho\|_{L^2}^2 + \|\nabla^{k+2} b\|_{L^2}^2) 
\leq C \delta (\|\nabla \rho\|_{H^{k+1}}^2 + h^2 \|\nabla^2 \rho\|_{H^{k+1}}^2 + \|\nabla \rho\|_{H^{k+1}}^2 + \|\nabla \rho\|_{H^{k+1}}^2).
\]
Summing up above estimates for from $k = 0$ to $k = 3$, by the smallness of $\delta$, we conclude our lemma.

Next, we derive the dissipation estimate for $\rho$. \hfill \Box
Lemma 2.3. Let \((\rho, v, b)\) be a smooth solution to (15) and satisfy (5) and (17), then

\[
\frac{d}{dt} \left( \sum_{k=0}^{3} \nabla^{k} v \cdot \nabla^{k+1} \rho dx + \| \nabla \rho \|_{H^3}^2 \right) + C(\| \nabla \rho \|_{H^3}^2 + h^2 \| \nabla^2 \rho \|_{H^3}^2) \\
\leq C\| \nabla v \|_{H^3}^2 + C\delta(\| \nabla v \|_{H^3}^2 + \| \nabla b \|_{H^3}^2).
\]  

(23)

Proof. For \(0 \leq k \leq 3\), applying the operator \(\nabla^k\) to the second equation in (15), and then multiplying by \(\nabla^{k+1} \rho\), we have

\[
\gamma \| \nabla^{k+1} \rho \|_{L^2}^2 + \frac{h^2}{4\gamma} \| \nabla^{k+2} \rho \|_{L^2}^2 \\
= -\int \nabla^k v \cdot \nabla^{k+1} \rho dx + \int \mu_1 \nabla^k \Delta v \cdot \nabla^{k+1} \rho dx + \int \mu_2 \nabla^{k+1} \nabla \cdot v \cdot \nabla^{k+1} \rho dx \\
- \int \frac{h^2}{4} \nabla^k \left( \frac{\rho + \bar{\rho}}{\rho + \bar{\rho} + \rho_{\infty}} \nabla \Delta \rho \right) \cdot \nabla^{k+1} \rho dx + \int \nabla^k F_2 \cdot \nabla^{k+1} \rho dx \\
= K_1 + K_2 + K_3 + K_4 + K_5.
\]

We will estimate each term on the right hand side. First, for the term \(K_1\), by integration by parts twice, the continuity equation, Hölder’s inequality, Young’s inequality and Sobolev embedding, we have

\[
K_1 = -\frac{d}{dt} \int \nabla^k v \cdot \nabla^{k+1} \rho dx + \int \gamma |\nabla^k \nabla \cdot v|^2 dx - \int \nabla^k \nabla \cdot v \nabla F_1 dx \\
\leq -\frac{d}{dt} \int \nabla^k v \cdot \nabla^{k+1} \rho dx + \gamma \| \nabla^{k+1} v \|_{L^2}^2 \\
+ C(\| \nabla^k (\nabla \rho \cdot \nabla \bar{\rho} \cdot v) \|_{L^2} + \| \nabla^k (\rho \nabla \cdot v, \rho \nabla \cdot v) \|_{L^2} \\
+ \| \nabla^k v \|_{L^3}^3 + \| \nabla^k v \|_{L^5} \| \nabla^k v \|_{L^5} + \sum_{1 \leq i \leq k} \| \nabla^i \nabla \cdot v \|_{L^3} \| \nabla^k \bar{\rho} \|_{L^3} \\
+ \| \nabla \cdot v \|_{L^3} \| \nabla \rho \|_{L^3} + \sum_{1 \leq i \leq k} \| \nabla^i \nabla \cdot v \|_{L^5} \| \nabla^k \bar{\rho} \|_{L^{5}}) \| \nabla^{k+1} v \|_{L^2} \\
\leq -\frac{d}{dt} \int \nabla^k v \cdot \nabla^{k+1} \rho dx + \gamma \| \nabla^{k+1} v \|_{L^2}^2 + C\delta(\| \nabla v \|_{H^{k+1}}^2 + \| \nabla \rho \|_{H^3}^2).
\]

For the terms \(K_2\) and \(K_3\), recalling from the estimate of \(J_1\), we have

\[
K_2 = \int \mu_1 \nabla^{k+1} \nabla \cdot v \nabla^{k+1} \rho \\
\leq \frac{1}{2} \frac{d}{dt} \int \frac{\mu_1 \sqrt{\rho_{\infty}}}{\rho + \bar{\rho} + \rho_{\infty}} |\nabla^{k+1} \rho|^2 dx + C\delta(\| \nabla^{k+1} v \|_{L^2}^2 + \| \nabla \rho \|_{H^3}^2),
\]

\[
K_3 \leq \frac{1}{2} \frac{d}{dt} \int \frac{\mu_2 \sqrt{\rho_{\infty}}}{\rho + \bar{\rho} + \rho_{\infty}} |\nabla^{k+1} \rho|^2 dx + C\delta(\| \nabla^{k+1} v \|_{L^2}^2 + \| \nabla \rho \|_{H^3}^2).
\]
For the term $K_4$, we obtain by Hölder’s inequality, Young’s inequality and integration by parts that

\[
K_4 = \int \frac{h^2}{4\gamma} \nabla \left( \frac{\varrho + \bar{\varrho}}{\varrho + \bar{\varrho} + \rho_{\infty}} \right) \nabla^{k+1} \varrho \cdot \nabla^{k+1} \varrho \, dx + \int \frac{h^2}{4\gamma} \frac{\varrho + \bar{\varrho}}{\varrho + \bar{\varrho} + \rho_{\infty}} |\nabla^{k+2} \varrho|^2 \, dx \\
- \int \frac{h^2}{4\gamma} \sum_{1 \leq l \leq k} C_k^l \nabla^l \left( \frac{\varrho + \bar{\varrho}}{\varrho + \bar{\varrho} + \rho_{\infty}} \right) \nabla^{k-l+1} \Delta \varrho \cdot \nabla^{k+1} \varrho \, dx \\
\leq C h^2 \left( \| \nabla \left( \frac{\varrho + \bar{\varrho}}{\varrho + \bar{\varrho} + \rho_{\infty}} \right) \|_{L^\infty} \| \nabla^{k+2} \varrho \|_{L^2} \| \nabla^{k+1} \varrho \|_{L^2} \right) \\
+ \| \frac{\varrho + \bar{\varrho}}{\varrho + \bar{\varrho} + \rho_{\infty}} \|_{L^\infty} \| \nabla^{k+2} \varrho \|_{L^2} \\
+ C h^2 \| \nabla \left( \frac{\varrho + \bar{\varrho}}{\varrho + \bar{\varrho} + \rho_{\infty}} \right) \|_{L^\infty} \| \nabla \Delta \varrho \|_{L^2} \| \nabla^{k+1} \varrho \|_{L^2} \\
+ C h^2 \sum_{2 \leq l \leq k} \| \nabla^l \left( \frac{\varrho + \bar{\varrho}}{\varrho + \bar{\varrho} + \rho_{\infty}} \right) \nabla^{k-l+1} \Delta \varrho \|_{L^2} \| \nabla^{k+1} \varrho \|_{L^2} \\
\leq C \delta h^2 \| \nabla \varrho \|_{H^{k+1}}^2.
\]

For the term $K_5$, we obtain by Hölder’s inequality and Young’s inequality that

\[
K_5 \leq C \int \nabla^k (v \cdot \nabla v + \varrho \Delta v + \varrho \nabla \cdot v + \varrho \nabla \varrho + (\varrho + 1)(b \cdot \nabla b) \\
+ \frac{1}{2} \nabla |b|^2) \cdot \nabla^{k+1} \varrho \, dx + C \int \nabla^k (\bar{\varrho} \Delta v + \bar{\varrho} \nabla \cdot v + \bar{\varrho} \nabla \varrho + \varrho \bar{\varrho} \\
+ \bar{\varrho} (b \cdot \nabla b + \frac{1}{2} \nabla |b|^2) \cdot \nabla^{k+1} \varrho \, dx + C h^2 \int \nabla^k (|\nabla \varrho|^2 \varrho \varrho + \varrho \nabla \Delta \varrho \\
+ \nabla \varrho \cdot \nabla \varrho \varrho + \nabla \varrho \nabla \varrho + \nabla \Delta \varrho + \nabla \varrho \varrho^2 + \nabla \varrho \varrho^2 \varrho) \cdot \nabla^{k+1} \varrho \, dx \\
+ C h^2 \int \nabla^k (|\nabla \bar{\varrho}|^2 \bar{\varrho} \bar{\varrho} + \bar{\varrho}^2 + \bar{\varrho}^2 \varrho + \bar{\varrho} \varrho^2 \\
+ \nabla \bar{\varrho} \Delta \varrho \varrho + \bar{\varrho} \nabla \bar{\varrho} \varrho + \bar{\varrho} \Delta \varrho \varrho + \bar{\varrho} \nabla \varrho \varrho + \bar{\varrho} \varrho^2 \varrho + \bar{\varrho} \varrho^2 \varrho) \cdot \nabla^{k+1} \varrho \, dx \\
\leq C \delta (\| \nabla v \|_{H^{k+1}}^2 + \| \varrho \varrho \|_{H^k}^2 + \| b \|_{H^k}^2) + C \delta h^2 \| \nabla \varrho \|_{H^{k+1}}^2.
\]

Summing up $K_1 \sim K_5$, we have

\[
\frac{d}{dt} \int \nabla^k v \cdot \nabla^{k+1} \varrho + \frac{(\mu_1 + \mu_2) \sqrt{\rho_{\infty}}}{2(\varrho + \bar{\varrho} + \rho_{\infty})} |\nabla^{k+1} \varrho|^2 \, dx + C (\| \nabla^{k+1} \varrho \|_{L^2}^2 + h^2 \| \nabla^{k+2} \varrho \|_{L^2}^2) \\
\leq \gamma \| \nabla^{k+1} v \|_{L^2}^2 + C \delta (\| \nabla v \|_{H^{k+1}}^2 + \| \nabla B \|_{H^{k+1}}^2 + \| \nabla \varrho \|_{H^k}^2 + h^2 \| \nabla^2 \varrho \|_{H^k}^2).
\]

Summing up above estimates for from $k = 0$ to $k = 3$, by the smallness of $\delta$, we conclude our lemma.

3. **Convergence rates.** In this section, we shall study the $L^q - L^p$ estimates of solutions to the initial value problem in $\mathbb{R}^3$. We consider the reformulated nonlinear
system
\[
\begin{align*}
\begin{cases}
\rho_t + \gamma \nabla \cdot v &= F_1, \\
v_t + \gamma \nabla \rho - \mu_1 \Delta v - \mu_2 \nabla \nabla \cdot v - \frac{\hbar^2}{4\gamma\rho_\infty} \nabla \Delta \rho &= \frac{\hbar^2}{4\gamma\rho_\infty} \left( \frac{1}{\rho + \rho_\infty} - \frac{1}{\rho_\infty} \right) \nabla \Delta \rho + F_2 := \tilde{F}_2, \\
\end{cases}
\end{align*}
\]
(24)

Let \( U = (\rho, v)^t \),
\[
A = \begin{pmatrix}
\gamma \nabla - \frac{\hbar^2}{4\gamma\rho_\infty} \nabla \Delta & -\mu_1 \Delta - \mu_2 \nabla \nabla \cdot \\
\end{pmatrix},
\]
the linearized system for the first two equations in (24) is written as
\[
\begin{align*}
\begin{cases}
U_t + AU = 0, \text{ for } t > 0, \\
U|_{t=0} = U(0).
\end{cases}
\end{align*}
(25)

Applying the Fourier transform to the system (25), we have
\[
\begin{align*}
\begin{cases}
\hat{\rho}_t &= -i\gamma \xi \cdot \hat{v}, \\
\hat{v}_t &= (-\mu_1 |\xi|^2 I - \mu_2 \xi^t \hat{\rho}) \hat{v} - i\gamma \xi \hat{\rho} - i\frac{\hbar^2}{4\gamma\rho_\infty} |\xi|^2 \xi \hat{\rho}.
\end{cases}
\end{align*}
(26)

From (26), we obtain the following initial problem for \( \hat{\phi}(\xi, t) \):
\[
\begin{align*}
\begin{cases}
\hat{\phi}_{tt} + (\mu_1 + \mu_2) |\xi|^2 \hat{\phi}_t + (\gamma^2 |\xi|^2 + \frac{\hbar^2}{4\rho_\infty} |\xi|^4) \hat{\phi} &= 0, \\
\hat{\phi}(\xi, 0) = \hat{\phi}_0(\xi), \quad \hat{\phi}_t(\xi, 0) = -i\gamma \xi \cdot \hat{\phi}_0(\xi).
\end{cases}
\end{align*}
(27)

Solving the ordinary differential equation with respect to \( t \), when \( \lambda_+ \neq \lambda_- \), we have
\[
\hat{\phi}(\xi, t) = \frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-} \hat{\phi}_0(\xi) - i\gamma \left( \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-} \right) \xi \cdot \hat{\phi}_0(\xi),
\]
(28)

where \( \lambda_+ \) and \( \lambda_- \) are the roots of the equation
\[
\lambda^2 + (\mu_1 + \mu_2) |\xi|^2 \lambda + (\gamma^2 |\xi|^2 + \frac{\hbar^2}{4\rho_\infty} |\xi|^4) = 0.
\]

Substituting (28) into (26), we obtain
\[
\begin{align*}
\hat{\phi}(\xi, t) = & -i(\gamma + \frac{\hbar^2}{4\gamma\rho_\infty} |\xi|^2) \left( \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-} \right) \xi \hat{\phi}_0(\xi) \\
& + \left( e^{-\mu_1 |\xi|^2 t} I + \frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-} - e^{-\mu_1 |\xi|^2 t} \frac{\xi \xi^t}{|\xi|^2} \right) \xi \hat{\phi}_0(\xi).
\end{align*}

Let \( E(t) \) be the semigroup generated by the linear operator \( A \), then we have \( E(t) = e^{-tA} (t \geq 0) \). We define \( E(t) f = \mathcal{F}^{-1}(e^{-tA} f(\xi, t)) \) with
\[
\hat{A}(\xi) = \begin{pmatrix}
0 & i\gamma \xi^t \\
\gamma \xi + i\frac{\hbar^2}{4\rho\gamma} |\xi|^2 \xi & \mu_1 |\xi|^2 + \mu_2 \xi^t
\end{pmatrix}, \quad \text{for } \xi = \begin{pmatrix}
\xi_1 \\
\xi_2 \\
\xi_3
\end{pmatrix}.
\]

We give the following properties on the eigenvalues \( \lambda_+ \) and \( \lambda_- \):

**Lemma 3.1.** Let \( \eta = \mu_1 + \mu_2, \kappa' = \frac{\hbar^2}{4\gamma\rho_\infty} \). When \( \eta^2 - 4\kappa' \gamma \leq 0 \) or \( \eta^2 - 4\kappa' \gamma \geq 0 \),
\[
|\xi| \leq \sqrt{\frac{4\kappa^2}{\eta^2 - 4\kappa' \gamma}},
\]
\[
\lambda_+(\xi) = \lambda_-(\xi) = \frac{-\eta^2}{2} |\xi|^2 + \frac{i}{2} \sqrt{4\gamma^2|\xi|^2 - (\eta^2 - 4\kappa' \gamma)}|\xi|^4,
\]
where \( \lambda_{+/-}(\xi) = \lambda_{+/-}(\xi) \).
and there exists a constant \( r_1 > 0 \) such that \( \lambda_\pm \) has a Taylor series expansion for \( |\xi| < r_1 \) as follows:

\[
\lambda_+(\xi) = \lambda_-(\xi) = -\frac{\eta^2}{2}|\xi|^2 + i\gamma|\xi| + iO(|\xi|^2).
\]

Similarly, when \( |\xi| \leq \sqrt{\frac{4\gamma^2}{\eta^2 - 4\kappa'\gamma}} \),

\[
\lambda_+(\xi) = \left(\frac{\eta^2}{2}|\xi|^2 + \frac{1}{2}\sqrt{(\eta^2 - 4\kappa'\gamma)|\xi|^4 - 4\gamma^2|\xi|^2}\right),
\]

\[
\lambda_-(\xi) = \left(-\frac{\eta^2}{2}|\xi|^2 - \frac{1}{2}\sqrt{(\eta^2 - 4\kappa'\gamma)|\xi|^4 - 4\gamma^2|\xi|^2}\right),
\]

and there exists a constant \( r_2 > 0 \) such that \( \lambda_\pm \) has a Laurent series expansion for \( |\xi| > r_2 \) as follows:

\[
\lambda_+(\xi) = \frac{1}{2}\left(\sqrt{\eta^2 - 4\kappa'\gamma} - \eta\right)|\xi|^2 - \frac{\gamma^2}{\sqrt{\eta^2 - 4\kappa'\gamma}} - \frac{\gamma^4}{(\eta^2 - 4\kappa'\gamma)^{3/2}|\xi|^2} + O\left(\frac{1}{|\xi|^4}\right),
\]

\[
\lambda_-(\xi) = \frac{1}{2}\left(\sqrt{\eta^2 - 4\kappa'\gamma} + \eta\right)|\xi|^2 + \frac{\gamma^2}{\sqrt{\eta^2 - 4\kappa'\gamma}} + \frac{\gamma^4}{(\eta^2 - 4\kappa'\gamma)^{3/2}|\xi|^2} + O\left(\frac{1}{|\xi|^4}\right).
\]

Let \( R = \sqrt{\frac{4\gamma^2}{\eta^2 - 4\kappa'\gamma}} \) and \( \varphi_0(\xi) \) be a function in \( C_0^\infty(\mathbb{R}^3) \) such that

\[
\varphi_0(\xi) = \begin{cases} 
1, & |\xi| \leq \frac{R}{2}, \\
0, & |\xi| \geq R,
\end{cases}
\]

(29)

Basing on the Fourier transform and (29), we may define the low and high frequency decompositions \( (f_L(x), f_H(x)) \) for \( f(x) \) as follows

\[
f_L = \mathcal{F}^{-1}(\varphi(\xi)\hat{f}(\xi)), \quad \text{and} \quad f_H = f - f_L.
\]

(30)

According to the Plancherel theorem, we can obtain the following estimates

\[
\|\nabla f\|_{L^2} \leq C\left(\|\nabla f_L\|_{L^2} + \|\nabla f_H\|_{L^2}\right),
\]

(31)

and

\[
C\|\nabla f_H\|_{L^2} \leq \|\nabla^k f_H\|_{L^2}, \quad C\|\nabla^k f_H\|_{L^2} \leq \|\nabla^k f\|_{L^2}, \quad \text{for } k \geq 1.
\]

(32)

Denote \( U_L := \mathcal{F}^{-1}(\varphi_0(\xi)\hat{U}(\xi, t)) \) and \( E_L(t)f := \mathcal{F}^{-1}(\varphi_0(\xi)\hat{e}^{-iA(\xi)}\hat{f}(\xi, t)) \) as the definition (30).

If we denote the nonlinear terms for the first two equations in (24) as \( N = (F_1, F_2) \), then (24) becomes

\[
U(t) = E(t)U_0 + \int_0^t E(t - \tau)N(U(\tau), b(\tau))d\tau,
\]

\[
b(t) = S(t)b_0 + \int_0^t S(t - \tau)F_3(U(\tau), b(\tau))d\tau,
\]

(33)

where \( S(t) \) is the semigroup generated by \( \nu \Delta \). Note that for \( S(t) \), we have

\[
\|\nabla^k S(t)b_0\|_{L^p} \leq C(1 + t)^{-\frac{1}{2}\left(\frac{d}{2} - \frac{1}{2}\right) - \frac{k}{2}}\|b_0\|_{L^q},
\]

with

\[
\gamma = \frac{\sqrt{(\eta^2 - 4\kappa'\gamma)^{3/2}|\xi|^2}}{|\xi|^2}.
\]
there exist constants $\beta$ such that
\[ ||\nabla^k b(t)||_{L^p} \leq C(1 + t)^{-\frac{3}{2} \left( \frac{1}{p} - \frac{1}{q} \right) - \frac{1}{2}} ||b_0||_{L^p} \]
\[ + C \int_0^t (1 + t - \tau)^{-\frac{3}{2} \left( \frac{1}{p} - \frac{1}{q} \right) - \frac{1}{2}} ||F_3(\tau)||_{L^q} \, d\tau, \]
for any $t \geq 0$ and $1 \leq p, q \leq \infty$.

Then from (33), we have
\[
U_L(t) = E_L(t)U_0 + \int_0^t E_L(t - \tau)N(U(\tau), b(\tau)) \, d\tau, \\
b_L(t) = S_L(t)b_0 + \int_0^t S_L(t - \tau)F_3(U(\tau), b(\tau)) \, d\tau.
\]

In order to estimate $U_L$, we need the $L^q - L^p$ type of the time decay estimates on the low frequency part of the semigroup $E(t)$. In the following, we give the following property on the decay in time for the semigroup $E_L(t)$.

**Lemma 3.2.** Let $k \geq 0$ be integers and $1 \leq q \leq 2 \leq p \leq \infty$, then for any $t > 0$, it holds
\[
||\nabla^k E_L(t)f||_{L^2} \leq C(k, q, p)(1 + t)^{-\frac{3}{2} \left( \frac{1}{q} - \frac{1}{p} \right) - \frac{1}{2}} ||f||_{L^q}.
\]

**Proof.** See [12, p. 629, Theorem 3.1]. It is noteworthy that, in view of Lemma 3.1, there exist constants $\beta_0, \beta_1$ and $C > 0$ such that $-\beta_0 |\xi|^2 \leq \Re \lambda_\pm(\xi) \leq -\beta_1 |\xi|^2$ and $|\xi^\alpha \partial_\xi^\ell \lambda_\pm(\xi)| \leq C(\alpha)|\xi|$. \(\square\)

We need the following elementary inequality [33]:

**Lemma 3.3.** Let $r_1, r_2 > 0$, then it holds that
\[
\int_0^t (1 + t - s)^{-r_1} (1 + s)^{-r_2} \leq C(r_1, r_2)(1 + t)^{-\min\{r_1, r_2, r_1 + r_2 - 1 - \eta\}},
\]
for an arbitrarily small $\eta > 0$.

Next, we will obtain the decay estimate of $(U_L, b_L)$ as the $L^q - L^p$ type estimate (36) in the following.

**Lemma 3.4.** Let $(U, b)$ be a smooth solution to (24) and satisfy (5) and (17). Assume that $\epsilon \ll \delta$, $k \geq 0$ is integers and $1 \leq q \leq 2 \leq p \leq \infty$, then
\[
||\nabla^k (U_L, b_L)(t)||_{L^2} \leq CE_0(1 + t)^{-\frac{3}{2} \left( \frac{1}{q} - \frac{1}{p} \right) - \frac{1}{2}}
\]
\[ + C\delta \int_0^t (1 + t - \tau)^{-\frac{3}{2} - \frac{1}{2}} ||\nabla(U, b)(\tau)||_{H^2} \, d\tau,
\]
where $E_0 = ||(U_0, b_0)||_{L^q}$ is finite.

**Proof.** From (35) and (36), we have
\[
||\nabla^k (U_L, b_L)(t)||_{L^2} \leq C(1 + t)^{-\frac{3}{2} \left( \frac{1}{q} - \frac{1}{p} \right) - \frac{1}{2}} ||(U_0, b_0)||_{L^q}
\]
\[ + C \int_0^t (1 + t - \tau)^{-\frac{3}{2} - \frac{1}{2}} ||(N(U), F_3)(\tau)||_{L^q} \, d\tau.
\]
We have by Gronwall’s inequality that
\[ \|F_1\|_{L^1} \leq C\|\nabla \rho\|_{L^2} \|v\|_{L^2} + C\|\langle \rho, \delta \rangle\|_{L^2} \|\nabla v\|_{L^2} + C\|1 + |x|\|\nabla \rho\|_{L^2} \|v\|_{L^2} \frac{v}{1 + |x|} \|L^2 \]
\[ \leq C\delta \|\langle \nabla \rho, \nabla v\rangle\|_{L^2} \]

The second term \(F_2\) is much more complicated, by using Hölder’s inequality, Sobolev embedding and Hardy’s inequality, we have
\[ \|F_2\|_{L^1} \leq C\|\nabla v\|_{L^2} \|v\|_{L^2} + \|\langle \rho, \delta \rangle\|_{L^2} \|\Delta v\|_{L^2} + \|\langle \rho, \delta \rangle\|_{L^2} \|\nabla \nabla \cdot v\|_{L^2} \]
\[ + \|\langle \rho, \delta \rangle\|_{L^2} \|\nabla \rho\|_{L^2} + \|\rho + \delta + 1\|_{L^\infty} \|b\|_{L^2} \|\nabla \rho\|_{L^2} \]
\[ + \|\langle 1 + |x|\|\nabla \rho\|_{L^2} \|\frac{\rho}{1 + |x|}\|_{L^2} \right) + C\delta h^2 \left(\|\rho + \delta\|_{L^2} \|\nabla \rho\|_{L^2} \right) \]
\[ + \|\langle 1 + |x|\|\Delta \rho\|_{L^2} \|\frac{\rho}{1 + |x|}\|_{L^2} + \left(\|\nabla^2 \rho, \Delta \rho\|_{L^2} + \|\nabla \rho\|_{L^2} \|\nabla v\|_{L^2} \right) \]
\[ + \|\nabla \rho\|_{L^2} \|\nabla \rho\|_{L^2} + \|\nabla \rho\|_{L^2} \|\nabla \rho\|_{L^2} \|\nabla \rho\|_{L^\infty} \|\nabla \rho\|_{L^2} \]
\[ + \|\nabla \rho\|_{L^2} \|\|\frac{\rho}{1 + |x|}\|_{L^2} \left(\|\langle \rho, \delta \rangle\|_{L^2} \|\nabla \rho\|_{L^2} \right) \]
\[ + C\delta \left(\|\nabla \rho\|_{H^1} + \|\nabla \rho\|_{L^2} + \|\nabla b\|_{L^2}\right) \]
\[ + C\delta h^2 \left(\|\nabla \rho\|_{L^2} + \|\nabla \rho\|_{H^1}\right) \]
\[ \leq C\delta \left(\|\nabla \rho\|_{L^1} + \|\nabla \rho\|_{L^2} + \|\nabla b\|_{L^2}\right) \]

In a similar way, we have
\[ \|F_3\|_{L^1} \leq C\delta \left(\|\nabla \rho\|_{L^2} + \|\nabla \rho\|_{H^1}\right) \]

Summing these terms, then we complete the proof of Lemma 3.4.

**Proof of Theorem 1.2.** Summing up (18) and (19), since \(\delta > 0\) is small, we obtain
\[ \frac{d}{dt}\left(\|\langle \rho, v, b\rangle\|_{H^3}^2 + \|\rho\|_{H^3}^2 + \|\nabla \rho\|_{H^3}^2\right) \]
\[ \leq C\delta \left(\|\nabla \rho\|_{H^3}^2 + \|\rho\|_{H^3}^2 + \|\nabla \rho\|_{H^3}^2 + \|\nabla b\|_{H^3}^2\right) \]
\[ \leq C\delta \left(\|\nabla \rho\|_{H^3}^2 + \|\rho\|_{H^3}^2 + \|\nabla b\|_{H^3}^2\right) \]
\[ \leq C\delta \left(\|\nabla \rho\|_{H^3}^2 + \|\rho\|_{H^3}^2 + \|\nabla b\|_{H^3}^2\right) \]

In view of Lemma 2.3, we have
\[ \frac{d}{dt}\left(\sum_{k=0}^3 \|\nabla^k \rho\|_{H^3}^2 + \|\nabla \rho\|_{H^3}^2\right) \]
\[ \leq C\delta \left(\|\nabla \rho\|_{H^3}^2 + \|\rho\|_{H^3}^2 + \|\nabla b\|_{H^3}^2\right) \]

Multiplying (40) by \(\frac{C_\delta}{C_\delta}\), adding it with (39), since \(\delta > 0\) is small, then we deduce
\[ \frac{d}{dt}\left(\sum_{k=0}^3 \|\nabla^k \rho\|_{H^3}^2 + \|\nabla \rho\|_{H^3}^2\right) + \int_0^t \|\nabla (v, b, h\rho)\|_{H^4}^2 \leq C\delta \left(\|\rho_0\|_{H^5}^2 + \|v_0\|_{H^5}^2 + \|b_0\|_{H^5}^2\right) \]

We have by Gronwall’s inequality that
\[ \|\langle \rho, v, b\rangle\|_{H^5}^2 + \|\nabla \rho\|_{H^5}^2 + \int_0^t \|\nabla (v, b, \rho\|_{H^5}^2 \leq C\left(\|\rho_0\|_{H^5}^2 + \|v_0\|_{H^5}^2 + \|b_0\|_{H^5}^2\right) \]

This closes the *a priori* estimates if we assume \(\|\rho_0 - \rho_0\|_H^5 + \|v_0\|_H^5 + \|B_0\|_H^5 \leq \delta\) is sufficiently small.
Now we turn to prove (6)~(8). Multiplying (23) by \( \frac{C_\delta}{C_9} \), adding it with (19), since \( \delta > 0 \) is small, we deduce
\[
\frac{d}{dt} M(t) + \| \nabla (\varrho, h\nabla^2 \varrho, \nabla^2 v, \nabla^2 b) \|^2_{H^3} \leq C_6 \| \nabla (v, b) \|^2_{H^3},
\]
(41)
where \( M(t) \) is equivalent to
\[
\| \nabla (\varrho, v, b) \|^2_{H^3} + \| h\nabla^2 \varrho \|^2_{H^3},
\]
that is, there exists a constant \( C_6 > 0 \) such that
\[
C_6^{-1} (\| \nabla (\varrho, v, b) \|^2_{H^3} + \| h\nabla^2 \varrho \|^2_{H^3}) \leq M(t) \leq C_6 (\| \nabla (\varrho, v, b) \|^2_{H^3} + \| h\nabla^2 \varrho \|^2_{H^3}).
\]
Using (31) and (32), from (41), there exists a constant \( C_7 > 0 \) such that
\[
\frac{d}{dt} M(t) + C_7 \| \nabla (\varrho, h\nabla \varrho, v, b) \|^2_{L^2} + \frac{1}{2} \| \nabla (\varrho, h\nabla^2 \varrho, \nabla^2 v, \nabla^2 b) \|^2_{H^3} \leq C_\delta (\| \nabla (\varrho, v, b) \|^2_{L^2} + \| \nabla (\varrho, h\nabla \varrho, v, b) \|^2_{H^3}).
\]
(42)
It follows from (42) and the smallness of \( \delta \) that
\[
\frac{d}{dt} M(t) + C_8 \| \nabla (\varrho, v, b) \|^2_{L^2} + \frac{1}{2} \| \nabla (\varrho, h\nabla \varrho, v, b) \|^2_{H^3} \leq C_\delta \| \nabla (\varrho, v, b) \|^2_{L^2},
\]
(43)
where \( C_8 > 0 \) is a constant. Adding \( C_8 \| \nabla (\varrho, h\nabla \varrho, v, b) \|^2_{L^2} \) to the both sides of (43), with the definition of \( M(t) \), there exists a constant \( D_1 > 0 \) such that
\[
\frac{d}{dt} M(t) + D_1 M(t) \leq C \| \nabla (U, b_L) \|^2_{H^1}.
\]
(44)
We define
\[
N(t) := \sup_{0 \leq \tau \leq t} (1 + \tau)^{3(\frac{1}{2} - \frac{1}{2}) + 1} M(\tau),
\]
(45)
then \( N(t) \) satisfies
\[
\| \nabla (\varrho, v, B) \|^2_{H^3} + \| h\nabla^2 \varrho \|^2_{H^3} \leq C \sqrt{M(\tau)} \leq C (1 + \tau)^{-\frac{3}{2}(\frac{1}{2} - \frac{1}{2}) + \frac{1}{2}} \sqrt{N(t)}, \quad 0 \leq \tau \leq t.
\]
From Lemma 3.3 and 3.4, we have
\[
\| \nabla (U, b_L) \|^2_{L^2} \leq C E_0 (1 + t)^{-\frac{3}{2}(\frac{1}{2} - \frac{1}{2}) - \frac{1}{2}}
\]
\[
+ C_\delta \int_0^t (1 + t - \tau)^{-\frac{3}{2}(\frac{1}{2} - \frac{1}{2}) - \frac{1}{2}} d\tau \sqrt{N(t)} \leq C (1 + t)^{-\frac{3}{2}(\frac{1}{2} - \frac{1}{2}) - \frac{1}{2}} (E_0 + \delta \sqrt{N(t)}),
\]
and
\[
\| \nabla^2 (U, b_L) \|^2_{L^2} \leq C E_0 (1 + t)^{-3(\frac{1}{2} - \frac{1}{2}) - 1}
\]
\[
+ C_\delta \int_0^t (1 + t - \tau)^{-\frac{3}{2}(\frac{1}{2} - \frac{1}{2}) - \frac{1}{2}} d\tau \sqrt{N(t)} \leq C (1 + t)^{-3(\frac{1}{2} - \frac{1}{2}) - \frac{1}{2}} (E_0 + \delta \sqrt{N(t)}).
\]
Then we obtain
\[
\| \nabla (U, b_L) \|^2_{H^1} \leq C E_0 (1 + t)^{-\frac{3}{2}(\frac{1}{2} - \frac{1}{2}) - \frac{1}{2}} (E_0 + \delta \sqrt{N(t)}).
\]
By Gronwall’s inequality, we have from (44) that

\[
M(t) \leq M(0)e^{-D_1t} + C \int_0^t e^{-D_1(t-\tau)} \|\nabla(U_L, b_L)(\tau)\|_{H^1}^2 d\tau \\
\leq M(0)e^{-D_1t} + C \int_0^t e^{-D_1(t-\tau)}(1 + \tau)^{-3(\frac{1}{4}-\frac{1}{2})-1} d\tau (E_0 + \delta \sqrt{N(t)})^2 \\
\leq C(1 + t)^{-3(\frac{1}{4}-\frac{1}{2})-1}(M(0) + E_0^2 + \delta^2 N(t)).
\]

Then, from (45), we have

\[
N(t) \leq C(M(0) + E_0^2 + \delta^2 N(t)),
\]

which implies that if \( \delta \) is small enough, then

\[
N(t) \leq C(M(0) + E_0^2).
\]

This in turn gives

\[
\|\nabla(\varrho, v, b)\|_{H^3}^3 + \|\mathcal{A}\nabla^2 \varrho\|_{H^3}^2 \leq C_0(1 + t)^{-\frac{3}{2}(\frac{1}{4}-\frac{1}{2})-\frac{1}{2}},
\]

which proves (6).

Define

\[
M_1(t) = \|(\varrho, v, b)\|_{H^4}^2 + \mathcal{A}\|\nabla \varrho\|_{H^4}^2 + \frac{C_2 \delta}{C_3} \sum_{k=0}^3 \int \nabla^k v \cdot \nabla^{k+1} \varrho dx,
\]

where \( M_1(t) \) is equivalent to \( \|(\varrho, v, b)\|_{H^4}^2 + \mathcal{A}\|\nabla \varrho\|_{H^4}^2 \), for the constant \( \delta > 0 \) can be sufficiently small. Then we have

\[
\frac{d}{dt}M_1(t) + \|\nabla(\varrho, \mathcal{A}\nabla \varrho, v, b)\|_{H^4}^2 \leq 0. \tag{46}
\]

Similar to (44), there exists a \( D_2 > 0 \) such that

\[
\frac{d}{dt}M_1(t) + D_2 M_1 \leq C\|(U_L, b_L)\|_{H^1}^2. \tag{47}
\]

From Lemma 3.3 and 3.4, we have

\[
\|(U_L, b_L)(t)\|_{L^2} \leq CE_0(1 + t)^{-\frac{3}{2}(\frac{1}{4}-\frac{1}{2})} + C\delta \int_0^t (1 + t - \tau)^{-\frac{3}{2}(\frac{1}{4}-\frac{1}{2})-\frac{1}{2}} d\tau \sqrt{N(t)} \\
\leq C(1 + t)^{-\frac{3}{2}(\frac{1}{4}-\frac{1}{2})} (E_0 + C),
\]

where take \( r_1 = \frac{3}{2}, \quad r_2 = \frac{3}{2}(\frac{1}{4}-\frac{1}{2}) + \frac{1}{2}, \quad \eta = \frac{1}{2} \). Then we obtain

\[
\|(U_L, b_L)(t)\|_{H^1} \leq CE_0(1 + t)^{-\frac{3}{2}(\frac{1}{4}-\frac{1}{2})} (E_0 + C).
\]

By Gronwall’s inequality, we have from (44) that

\[
M_1(t) \leq M_1(0)e^{-D_2t} + C \int_0^t e^{-D_2(t-\tau)} \|(U_L, b_L)(\tau)\|_{H^1}^2 d\tau \\
\leq M_1(0)e^{-D_2t} + C \int_0^t e^{-D_2(t-\tau)}(1 + \tau)^{-3(\frac{1}{4}-\frac{1}{2})-1} d\tau (E_0 + C)^2 \\
\leq C(1 + t)^{-3(\frac{1}{4}-\frac{1}{2})}(M_1(0) + E_0^2 + C).
\]

This in turn gives

\[
\|(\varrho, \mathcal{A}\nabla \varrho, v, b)\|_{H^4} \leq C_0(1 + t)^{-\frac{3}{2}(\frac{1}{4}-\frac{1}{2})}.
\]
Finally, by the interpolation, for any $2 \leq \varrho \leq 6$, we have
\[
\|(U, b)(t)\|_{L^p} \leq \|(U, b)(t)\|_{L^6}^{3(p-2)/2} \|(U, b)(t)\|_{L^2}^{1-3(p-2)/2p} \\
\leq C\|\nabla(U, b)(t)\|_{L^{2p}}^{3(p-2)/2p} \|(U, b)(t)\|_{L^2}^{1-3(p-2)/2p} \\
\leq C_0(1 + t)^{-\frac{3}{2}(\frac{1}{6} - \frac{1}{p}) - \frac{1}{2}},
\]
which prove (7).

Moreover, by using (15), (6), (12) and Sobolev embedding, we obtain
\[
\|\partial_t(\varrho, v, b)(t)\|_{L^2} \leq C\|\nabla \cdot v\|_{L^2} + \|F_1\|_{L^2} + C\|\nabla \varrho\|_{L^2} \\
+ C\|\Delta v\|_{L^2} + \|\nabla \nabla \cdot v\|_{L^2} + \|F_2\|_{L^2} + \|\Delta b\|_{L^2} + \|F_3\|_{L^2} \\
\leq C\left(\|\nabla \varrho\|_{H^1} + \|\Delta \varrho\|_{L^2} + \|\nabla v\|_{H^1} + \|\nabla b\|_{H^1}\right) \\
\leq C(1 + t)^{-\frac{3}{2}(\frac{1}{6} - \frac{1}{2}) - \frac{1}{2}},
\]
for any $0 \leq t \leq T$. The proof of Theorem 1.2 is complete. \hfill \Box

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