SELFSIMILAR EQUIVALENCE OF POROUS MEDIUM AND $p$-LAPLACIAN FLOWS

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Abstract. We demonstrate the equivalence between the two popular models of nonlinear diffusion, the porous medium equation and the $p$-Laplacian equation. The equivalence is shown at the level of selfsimilar solutions.

1. Introduction

The theory of nonlinear evolution equations of degenerate parabolic type has been intensively studied in the last decades both for its importance in a number of applications in mechanics and heat propagation and also because it combines the theory of nonlinear evolution PDEs with geometry in the form of free boundaries. Two of the most studied models have been the porous medium equation, shortly PME, and the $p$-Laplacian equation, shortly PLE.

We write the PME as

\[ u_t = \Delta (u^m/m), \quad m > 1. \]

It generalizes the heat equation, which is the linear case $m = 1$, and we also include the range of parameters $m < 1$ where it is known as the fast diffusion equation, $\text{[10]}$ ($m \leq 0$ is also allowed in the general theory). On the other hand, the standard PLE is

\[ u_t = \Delta_p u := \nabla \cdot (|\nabla u|^{p-2} \nabla u), \]

where the parameter ranges in the interval $1 < p < \infty$. The heat equation is the case $p = 2$. There has been strong interest in recent times in the case $p = 1$ as a geometrical flow, $\text{[1]}$.

It was soon remarked that the theory of both equations offers striking parallels. To mention just one, the PME has the property of finite speed of propagation for $m > 1$, while the PLE has the same property for $p > 2$. The consequence in both cases is that solutions with compactly supported initial data $u(x,0) \geq 0$ preserve the property of compact support w.r.t the space variable for all $t > 0$, and this implies the existence of a clear-cut interface or free boundary separating the non-empty regions where $u > 0$ and where $u = 0$. Many other properties are known that reinforce that parallelism, cf. $\text{[6]}, \text{[9]}$.

In this note we show that the very strong connection between the two equations can be improved into complete equivalence when we consider special classes of solutions. Such solutions are the self-similar solutions, i.e., solutions of one of the two forms,

\[ u(x,t) = t^{-\alpha} f(x t^{-\beta}) = t^{-\alpha} f(\eta), \quad u(x,t) = (T-t)^{\alpha} f(x (T-t)^{\beta}) = (T-t)^{\alpha} f(\eta). \]

We will call the solutions of the form $\text{[3]}$-left selfsimilar solutions of Type I, those of the form $\text{[3]}$-right are called Type II. The practical importance of such solutions lies in the fact that they usually describe the asymptotic behavior of solutions of Cauchy problem associated to parabolic equations, as the extensive literature shows, cf. $\text{[3], [10]}$ and their references. In view of the definite progress obtained recently in the study of the PME, as shown in the latter reference, and the key role actually played by suitable selfsimilar solutions in the classification of the basic qualitative and asymptotic properties, the equivalence is of clear significance to establish a similar theory for the PLE. We remark that for our equations there is still a possible selfsimilarity of Type III, with solutions of the form $u(x,t) = e^{\alpha t} f(x e^{\beta t})$, but it plays a minor role in the theory. Since the details are almost the same we will make only side mention to it.

A popular technique used for the description of these solutions is phase-plane analysis, cf. $\text{[3], [5], [10]}$. Our main contribution in this Note consists in introducing a new set of variables in the phase-plane description of the self-similar solutions of the $p$-Laplacian equation. They are not straightforward at all, but they have the property that the resulting phase plane can be exactly identified with a known version of the phase-plane for the self-similar
solutions of the PME, the only apparent difference amounting to different formulas for the values of the parameters. Here is our main result.

**Theorem 1.1.** The analysis of radial selfsimilar solutions for both the PME (when \( m \neq 1 \)) and the PLE (when \( p \neq 2 \)) can be reduced to a particular case of the autonomous ODE system

\[
\begin{align*}
\dot{\Phi} &= \Psi \Phi \\
\dot{\Psi} &= c_1 \Phi^2 - c_2 \Psi \Phi - c_3 \Phi \pm \Psi + \text{sgn}(b)
\end{align*}
\]

where the parameters \( c_1, c_2, c_3 \) and \( b \) are explicit functions of \( n, m \) and \( \beta \) in the PME case; of \( n, p \) and \( \beta \) in the PLE case. The variables \( \Phi \) and \( \Psi \) are given by algebraic expressions in terms of \( \eta, f, f' \), which are different for both equations. It follows that a translation rule can be set up from the PME into the PLE and vice versa, which involves changing the space dimension in the process. Besides, there are in general two options for the equivalence map in either direction.

Obviously, in the exceptional case \( m = 1, p = 2 \) the two equations coincide and the identity transformation solves the equivalence problem. We recall that there exist a large number of selfsimilar analyses of the PME and the PLE but the equivalence has not been remarked.

In the sequel we give whole details of these assertions: a careful selection of the correspondence between the two sets of parameters allows for a complete identification of the phase planes, which implies a transformation rule for the set of selfsimilar solutions from one equation to the other. We then derive some interesting applications. We will also provide self-maps of the solutions of the PME (resp. the PLE) into themselves based on a repeated use of the transformations, using a different option at each turn. We point out that when performing the selfsimilar analysis with radial profiles by means of ordinary differential equations we may assume that the space dimension is any real positive number.

## 2. Phase plane analysis for the PME

We want to find solutions in the self-similar forms (3) for equation (1) posed for \( x \in \mathbb{R}^n \) with \( m \neq 1 \). This is well-known, and is described in detail in [9] for instance. First, the relation between the similarity exponents \( \alpha \) and \( \beta \) under the usual assumption of radial symmetry the profile \( f \) solves the equivalence problem. We recall that there exist a large number of selfsimilar analyses of the PME and it is any real positive number.

We want to find solutions in the self-similar forms (3) for equation (1) posed for \( x \in \mathbb{R}^n \) with \( m \neq 1 \). This is well-known, and is described in detail in [9] for instance. First, the relation between the similarity exponents \( \alpha \) and \( \beta \) reads \( (m - 1)\alpha + 2\beta = 1 \) for Type I selfsimilarity, and \( (m - 1)\alpha + 2\beta = -1 \) if it belongs to Type II. Then, under the usual assumption of radial symmetry the profile \( f \) must satisfy the ODE

\[
n^{1-n}(n-1)f^{m-1}f' + \alpha f + \beta f' = 0,
\]

where \( \eta > 0 \). From now on we assume that \( n \) is any positive real number. We first introduce the variables:

\[
X = \eta f'/f, \quad Y = \eta^2 f^{1-m}.
\]

We also replace the \( \eta \) variable by \( r = \log \eta \). The functions \( X(r) \) and \( Y(r) \) satisfy the autonomous ODE system:

\[
\dot{X} = (2 - n)X - mX^2 - (\alpha + \beta X)Y, \quad \dot{Y} = (2 + (1 - m)X)Y.
\]

where the over-dot indicates differentiation with respect to \( r \). Now, we introduce a new pair of variables

\[
\Phi = (2 + (1 - m)X)/\sqrt{|b|}, \quad \Psi = Y/|b|,
\]

where \( b = 2n(m - m_c)/(m - 1) \) and \( m_c = (n - 2)/n \). This assumes that \( m \neq m_c \) so that \( b \neq 0 \). Replacing then the \( r \) variable by \( r_1 = \sqrt{|b|}r \), so that over-dot indicates differentiation with respect to \( r_1 \), the system takes the desired quadratic form (4), with precise values for the constants given by

\[
c_1 = \frac{m}{m - 1}, \quad c_2 = \beta \sqrt{|b|}, \quad c_3 = \frac{(n + 2)(m - m_c)}{(m - 1)\sqrt{|b|}}, \quad m_\ast = \frac{n - 2}{n + 2},
\]

and the \( \pm \) of the equation is + for the Type I and it is − for the Type II. With these values System (4) has free parameters \( m, n \) and \( \beta \), since \( \alpha \) can be calculated from them. The case \( m = m_c \) will be discussed below.
3. Phase plane analysis for the PLE

In this case, the relation between the similarity exponents $\alpha$ and $\beta$ is $(p-2)\alpha + p\beta = 1$ for Type I selfsimilarity, and $(p-2)\alpha + p\beta = -1$ if it is of Type II. The radially symmetric profile $f$ satisfies the ODE
\begin{equation}
\eta^{1-n}(\eta^{-1}|f'|^{p-2}f')' + \alpha f + \beta f' = 0,
\end{equation}
where prime denotes differentiation with respect to $\eta > 0$. From now on we assume that $n$ is any positive real number. In a similar way as before, for $p \neq 2$ we introduce phase plane variables, a bit different from the ones in the PME case:
\begin{equation}
X = -\eta^2|f'|^{1-p}f', \quad Z = \eta^\gamma f, \quad \gamma = \frac{p}{2-p}
\end{equation}
We thus get the autonomous ODE system
\begin{equation}
\frac{p-1}{2-p} \dot{X} = -(n-\gamma)X + \alpha Z|X|^{\frac{1}{p-2}} - \beta|X|X, \quad \dot{Z} = \gamma Z - |X|^\frac{p-1}{2-p} X,
\end{equation}
where we have replaced the $\eta$ variable by $r = \log \eta$, so that over-dot indicates differentiation with respect to $r$. This system is not quadratic so that we perform further change with this objective in mind. We introduce $Y = |X|^\frac{p-1}{2-p} X = -\eta|f'|^{1-p}f'$ and the flow equations become
\begin{equation}
\dot{X} = \frac{2-p}{p-1} X (\gamma - n + \alpha Y - \beta|X|), \quad \dot{Y} = -\alpha Y^2 + nY + \beta Y|X| - |X|,
\end{equation}
which is a quadratic system if $X$ has a sign. For the next step we assume that $X > 0$ and we set
\begin{equation}
\Psi = aX, \quad \Phi = -\frac{p-2}{(p-1)\sqrt{|b|}} (\gamma - n + \alpha Y - \beta X),
\end{equation}
where
\begin{align*}
a &= \frac{1}{|b|(p-1)}, \quad b = \frac{p(n+1)(p-p_c)}{(p-2)(p-1)}, \quad p_c = \frac{2n}{n+1}.
\end{align*}
In order to proceed further we assume that $p \neq p_c$. After all these transformations, the flow equations become exactly the desired [4]. The $\pm$ of the equation is $+$ for the Type I and it is $-$ for the Type II, and the constants are now given by
\begin{align*}
c_1 &= \frac{p-1}{p-2}, \quad c_2 = \beta \sqrt{|b|}, \quad c_3 = \frac{(n+2)(p-p_s)}{(p-2)\sqrt{|b|}}, \quad p_s = \frac{2n}{n+2}.
\end{align*}
This system has free parameters $p, n$ and $\beta$, and $p = 2$ is excluded. We have replaced the $r$ variable by $r_1 = \sqrt{|b|r}$, so that over-dot indicates differentiation with respect to $r_1$.

The critical cases, $m_c$ and $p_c$. Some changes have to be made in the special cases $m = m_c$ for the PME or $p = p_c$ for the PLE, since our definitions imply that $b = 0$. We change it into $\sqrt{b} = (n-2)$ for PME and $\sqrt{b} = n$ for PLE so that $c_3 = -1$ and the independent term $sgn(b)$ disappears from the second equation of system [4], which becomes
\begin{equation}
\dot{\Psi} = \Psi \Phi, \quad \dot{\Phi} = c_1 \Phi^2 - c_2 \Psi \Phi + \Phi \pm \Psi.
\end{equation}
with $c_1$ and $c_2$ as before.

Type III selfsimilarity. It offers few novelties. The exponent relationships are $\alpha(1-m) = 2\beta$ and $\alpha(2-p) = p\beta$ respectively. We obtain a similar system, except for the fact that the term $\pm \Psi$ disappears. Summing up, the coefficient of $\Psi$ is $+$ for Type I, $-$ for Type II, and 0 for Type III.

4. Equivalence

Identifying the two systems implies taking parameters $(m, n, \beta)$ in the first system and $(p', n', \beta')$ in the second, so that the expressions for the free constants of System [4] obtain the same values. (i) Identification of $c_1$ gives the relation $p = m + 1$, which is well-known from different dimensional considerations. (ii) Identifying $c_3$ gives
\begin{equation}
\frac{(n+2)(m-m_s)}{(m-1)\sqrt{|b|}} = \frac{(n'+2)(p-p_s)}{(p-2)\sqrt{|b'|}}.
\end{equation}
Note that \( b' = p(n' + 1)(p - c)/(p - 2)(p - 1) \) with \( p_c = 2n'/(n' + 1) \). This leads to
\[
2m(n(m - 1) + 2)(n'(m - 1) + 2(m + 1))^2 = (m + 1)(n'(m - 1) + m + 1)(n(m - 1) + 2(m + 1))^2,
\]
which can be written as a quadratic condition: \( A(n')^2 + Bn' + C = 0 \), with
\[
A = 2m(n - 1)^2[n(m - 1) + 2], \quad B = -(m + 1)(m - 1)^3(n - 2)^2, \quad C = -(m - 1)(m + 1)^2(n - 2)^2.
\]
This gives the values for \( n' \) as a double function of \( n \), representing the change of dimension from PME to PLE
\[
n_1' = \frac{(n - 2)(m + 1)}{2m}, \quad n_2' = \frac{(n - 2)(m + 1)}{n - 2 - nm} = \frac{m_c(m + 1)}{m_c - m}.
\]
The double solution exists for \( m \neq 0, m_c \) or 1. The two values of \( n' \) coincide for \( m = m_s \), and then \( n' = n \) and \( p = p_s \). Clearly, we have
\[
\frac{1}{n_1'} + \frac{1}{n_2'} = -\frac{B}{C} = \frac{1 - m}{m + 1} = \frac{2 - p}{p},
\]
which expresses a possible self-map of the PLE (for fixed \( p \)) by change of dimension. The situation is similar in the other sense: if \( p \neq p_c(n') = 2n'/(n' + 1) \), there are two possible values of \( n \) for every given \( n' \) and we have the relationship
\[
\frac{1}{n_1 - 2} + \frac{1}{n_2 - 2} = \frac{1 - m}{2m},
\]
which expresses a possible self-map of the PME. A precedent of self-relations can be found in King’s [8].

(iii) Identifying \( c_2 \) implies that \( \beta\sqrt{|b|} = \beta'\sqrt{|b'|} \), so that
\[
\frac{\beta^2}{\beta'} = \frac{(m - 1)p(n' + 1)(p - c)}{2n(m - m_c)(p - 2)(p - 1)},
\]
which gives the corresponding two values for \( \beta' \) in terms of \( \beta \)
\[
\beta_1' = \frac{2m}{m + 1} = \frac{n - 2}{n_1'}, \quad \beta_2' = \frac{n(m - 1) + 2}{m + 1} = \frac{2 - n}{n_2'}.
\]
Note that in both cases \( \beta^2(n - 2)^2 = (\beta' n')^2 \).

(iv) We still have to check that \( sgn(b) = sgn(b') \). We have
\[
b'/b = \frac{p(n' + 1)(p - c)(m - 1)}{2n(p - 2)(p - 1)(m - m_c)} = \frac{(m - 1)n' + m + 1(m + 1)}{2m(n(m - 1) + 2)} = \frac{(n')^2}{(n - 2)^2}
\]
We have used that \( p = m + 1, p_c = 2n'/(n' + 1) \), and \( (m - 1)n' + m + 1 = 2m(n(m - 1) + 2)(n')^2/(m + 1)(n - 2)^2 \)
given by the quadratic condition for \( n' \).

Some general conclusions. (1) The transformation is not applicable for \( n = 2 \) since it implies \( n' = 0 \) for both branches. Besides, the study is different for the cases \( n < 2 \) and \( n > 2 \).

(2) In the case \( n = 1 \) we have the same dimension \( n_1' = 1 \) for all \( m \) and then \( \beta' = \beta \). This is well known since formally differentiating the one-dimensional PLE gives the PME. For the other branch corresponding to \( n = 1 \) we have
\[
n_1' = -\frac{m + 1}{2m}, \quad \beta' = -\frac{\beta}{n_1'},
\]
so that \( n_1' > 0 \) if \(-1 \leq m < 0 \), a case of very fast diffusion studied in [4], see also [3]. Also, \( n_1' \geq 1 \) if \(-1/3 \leq m < 0 \).

(3) Cases \( n \geq 3 \). (i) We have \( n' = n \) if and only if \( m = m_s \) in both branches. This is the Yamabe case and it allows us to get an explicit equivalent of the Loewner-Nirenberg solution of the fast Diffusion Equation for the \( p \)-Laplacian equation.

(ii) In the case of the first branch, \( n' > 0 \) if \( m > 0 \) or \( m < -1 \) (a case we may disregard since then \( p < 0 \)). In the first branch \( n' \) decreases with \( m \) from \( \infty \) as \( m \to 0 \) to \((n - 2)/2\) as \( m \to \infty \).

(iii) In the second branch \( n' \) is positive for \(-1 < m < m_c = (n - 2)/n \) and increases from \( 0 \) to \( \infty \) in that interval; more precisely, \( n_1' = 1 \) for \( m = 0 \).

(4) Identification of the critical cases \( m = m_c \) in the PME and \( p = p_c \) in the PLE takes place for \( n' = n - 1 \) and \( \beta' = \beta(n - 2)/n' = \beta(n - 2)/(n - 1) \).

(5) In the limit \( m \to 1, p \to 2 \), both equations reduce to the linear heat equation (there is also the possibility of converging to the eikonal equation \( u_t = |\nabla u|^2 \) with a different limit process, cf. [2]). Now, extrapolating our transformation to the linear case, \( m = 1, p = 2 \), we have \( n_1' = n - 2 \) with \( \beta' = \beta \), and \( n_2' = 2 - n \) with \( \beta' = \beta \).
5. Application. Some explicit solutions

Let us now analyze some self-similar solutions given by the plane of phases for the equation of porous medium, and compare these solutions with the corresponding solution of the $p$-Laplacian equation. We will be mainly interested in dimensions $n \geq 3$ but lower dimensions appear in the final results.

**Barenblatt solution and straight lines.** Consider the equation of the trajectories with $\text{sgn}(b) = 1$ and selfsimilarity of Type I:

$$
\frac{d\Phi}{d\Psi} = \frac{c_1 \Phi^2 - c_2 \Psi \Phi - c_3 \Phi + \Psi + 1}{\Psi \Phi}.
$$

We want to find the linear solutions to (14). In order to do that we assume that $\Phi = a_1 \Psi + a_2$ and insert this form into (14), obtaining the following conditions: $a_1 = a_2 = c_2/(c_1 - 1)$, and $c_2$ must be a solution of the equation

$$
c_1 c_2^2 - (c_1 - 1) c_3 c_2 + (c_1 - 1)^2 = 0.
$$

Since $c_2 = \beta \sqrt{|b|}$, we get two values for $\beta$ and $\beta'$ which are

$$
\beta_1 = \frac{1}{n(m - 1) + 2} \quad \text{or} \quad \beta_2 = \frac{1}{2m} \quad \text{(porous medium)}
$$

and

$$
\beta'_1 = \frac{1}{n'(p - 2) + p} \quad \text{or} \quad \beta'_2 = \frac{1}{p} \quad \text{(p-Laplacian)}.
$$

The profiles corresponding to the trajectories for $\beta_1$ and $\beta'_1$ produce the so-called Barenblatt solution, both for the PME and the PLE. The other case is as follows: $\beta_2$ corresponds to the dipole solution of the porous medium,

$$
f = \eta^{-\frac{n-2}{2}} \left( K - \frac{1}{b} \eta^{(m-n+2)/m} \right)^{\frac{1}{m-1}} +
$$

cf. [10], Section 4.6, and $\beta'_2$ corresponds $\alpha'_2 = 0$, and a profile whose derivative is

$$
f'(\eta) = \eta^{-(n-1)/(p-1)} \left( c - \frac{1}{(p-1)b} \eta^{(p-2)b/p} \right)^{1/(p-2)} +
$$

**Yamabe case.** It corresponds to $p_s = m_s + 1 = 2n/(n+2)$ (cf. [9], Section 7.5 and Appendix AIII). Choosing Type II and putting $\beta = 0$, so that the self-similar solution is a separate variables solution, we have $c_1 = -(n-2)/4$, $c_2 = c_3 = 0$ and $\text{sgn}(b) = 1$. The equation of the trajectories (14) turns out to be

$$
\frac{d\Phi}{d\Psi} = \frac{c_1 \Phi^2 - \Psi + 1}{\Psi \Phi}.
$$

Solving the above Bernoulli’s equation, we obtain that the curve $\Psi = n(1/(n - 2) - 1/4\Phi^2)$ is one of its solutions. The explicit profiles that we get after undoing all the changes of variables are:

$$
f(\eta) = (k_1 + \frac{\eta^2}{4nk_1})^{-\frac{n+2}{2}} \quad \text{(Loewner-Nirenberg profile for the PME with } m = m_s)$$

and

$$
f(\eta) = C(1 + k_2 \eta^{\frac{2}{n+2}})^{-\frac{n}{n+2}}, \quad C = \frac{4n}{n+2} \left( \frac{4n^3 k_2}{n^2 - 4} \right)^{(n-2)/4} \quad \text{(new p-Laplacian profile for } p = p_s)$$

where $k_1, k_2 > 0$ are arbitrary.

Note. A detailed account of the results and some applications will appear as a separate publication [7].

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