The Full Kostant–Toda Hierarchy on the Positive Flag Variety

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Abstract: We study some geometric and combinatorial aspects of the solution to the full Kostant–Toda (f-KT) hierarchy, when the initial data is given by an arbitrary point on the totally non-negative (tnn) flag variety of $\text{SL}_n(\mathbb{R})$. The f-KT flows on the tnn flag variety are complete, and we show that their asymptotics are completely determined by the cell decomposition of the tnn flag variety given by Rietsch (Total positivity and real flag varieties. Ph.D. Thesis, Massachusetts Institute of Technology, Cambridge, 1998). Our results represent the first results on the asymptotics of the f-KT hierarchy (and even the f-KT lattice); moreover, our results are not confined to the generic flow, but cover non-generic flows as well. We define the f-KT flow on the weight space via the moment map, and show that the closure of each f-KT flow forms an interesting convex polytope which we call a Bruhat interval polytope. In particular, the Bruhat interval polytope for the generic flow is the permutahedron of the symmetric group $\mathfrak{S}_n$. We also prove analogous results for the full symmetric Toda hierarchy, by mapping our f-KT solutions to those of the full symmetric Toda hierarchy. In the appendix we show that Bruhat interval polytopes are generalized permutohedra, in the sense of Postnikov (Int. Math. Res. Not. IMRN (6):1026–1106, 2009).

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1. Introduction

The Toda lattice, introduced by Toda in 1967 (see [Tod89] for a comprehensive treatment), is an integrable Hamiltonian system representing the dynamics of $n$ particles of unit mass, moving on a line under the influence of exponential repulsive forces. The dynamics can be encoded by a matrix equation called the Lax equation

$$\frac{dL}{dt} = [\pi_{so}(L), L],$$

(1.1)

where $L$ is a tridiagonal symmetric matrix and $\pi_{so}(L)$ is the skew-symmetric projection of $L$. More specifically,

$$L = L(t) = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & 0 \\ a_1 & b_2 & a_2 & \cdots & 0 \\ 0 & a_2 & b_3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & a_{n-1} & b_n \end{pmatrix} \quad \text{and} \quad \pi_{so}(L) := (L)_{>0} - (L)_{<0},$$

where $(L)_{>0}$ (respectively $(L)_{<0}$) is the strictly upper (resp. lower) triangular part of $L$, so that $\pi_{so}(L)$ represents a skew-symmetrization of the matrix $L$. The entries $a_i$ and $b_j$ of $L$ are functions of $t$.

The Toda lattice gives an iso-spectral deformation of the eigenvalue problem of $L$, that is, the eigenvalues of $L(t)$ are independent of $t$. It is an immediate consequence of the Lax equation that for any positive integer $k$, the trace $\text{tr}(L^k)$ of $L^k$ is a constant of motion (it is invariant under the Toda flow). These invariants are the power sum symmetric functions of the eigenvalues, and are sometimes referred to as Chevalley invariants. Note that assuming $\text{tr}(L) = 0$ (i.e., $L \in \mathfrak{sl}_n(\mathbb{R})$), we have $n-1$ independent Chevalley invariants, $H_k := \text{tr}(L^{k+1})$ for $k = 1, \ldots, n-1$.

One remarkable property of the Toda lattice is that for generic initial data, i.e., $L$ has distinct eigenvalues

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n$$

and $a_k(0) \neq 0$ for all $k$, the asymptotic form of the Lax matrix is given by

$$L(t) \rightarrow \begin{cases} \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) & \text{as } t \rightarrow -\infty \\ \text{diag}(\lambda_n, \lambda_{n-1}, \ldots, \lambda_1) & \text{as } t \rightarrow \infty. \end{cases}$$

(1.2)

In other words, all off-diagonal elements $a_i(t)$ approach 0, and the time evolution of the Toda lattice sorts the eigenvalues of $L$ (see [DNT83,Sym82]). This property is referred to as the sorting property, which has important applications to matrix eigenvalue algorithms. It is also known that if we let $L$ range over all tridiagonal matrices with fixed eigenvalues $\lambda_1 < \cdots < \lambda_n$, the set of fixed points of the Toda lattice—i.e., those points $L$ such that $dL/dt = 0$—are precisely the diagonal matrices. Therefore, there are $|\mathfrak{S}_n| = n!$ fixed points of the Toda lattice, where $\mathfrak{S}_n$ is the symmetric group on $n$ letters.

The full symmetric Toda lattice is a generalization of the Toda lattice: it is defined using the Lax equation (1.1), but now $L$ can be any symmetric matrix. For generic $L$, the...