TWO WAVELET MULTIPLIERS AND LANDAU-POLLAK-SLEPIAN OPERATORS ON LOCALLY COMPACT ABELIAN GROUPS ASSOCIATED TO RIGHT-\(H\)-TRANSLATION-INVARIANT FUNCTIONS

APARAJITA DASGUPTA, SWARAJ PAUL, AND SANTOSH KUMAR NAYAK

Abstract. By using a coset of closed subgroup, we define a Fourier like transform for locally compact abelian (LCA) topological groups. We studied two wavelet multipliers and Landau-Pollak-Slepian operators on locally compact abelian topological groups associated to the transform and show that the transforms are \(L^p\)bounded linear operators, and are in Schatten p-class for \(1 \leq p \leq \infty\). Finally, we determine their trace class and also obtain a connection with the generalized Landau-Pollak-Slepian operators.

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Date: March 11, 2021.

2020 Mathematics Subject Classification. Primary 47G10, 47G30, Secondary 42C40.

Key words and phrases. Short time Fourier transform, Directional short time Fourier transform, Generalized multiplier, Generalized two wavelet multipliers.

The research of A. Dasgupta and S. Paul was supported by Core Research Grant(RP03890G), Science and Engineering Research Board (SERB), DST, India. S. Paul acknowledge Kinara Roy for her support during the preparation of the work.
1. Introduction

The short time Fourier transform (STFT) has many applications specially in signal and image processing for providing good time-frequency localization. Singularities of discontinuity across a curve such as edges in an image in multidimensional often hold the key information. To tackle the directional singularity Candès [4] first introduced ridgelet transform, which is the wavelet transform in Radon domain. Ridgelets are constant along the ridge or hyperplane. Now a days Curvelets and shearlets play important role to represent directional selectivity as they give optimal sparse approximations for a class of bivariate functions exhibiting anisotropic features [5, 6, 8, 9]. The relation between wavelet transform and the above directional representation is the above transform projects a hyperplane singularity into a point singularity, then takes one-dimensional wavelet transform. We recall that the short time Fourier transform (STFT) [10] of a function \( f \in L^2(\mathbb{R}^n) \) w.r.t. a window \( g \in L^2(\mathbb{R}^n) \), is the function

\[
V_g f(x, \omega) = \int_{\mathbb{R}^n} f(t) g(t-x) e^{-2\pi i \omega t} dt,
\]

i.e.,

\[
(1)
V_g f(x, \omega) = \hat{f}.T_x \hat{g}(\omega).
\]

Since the STFT enjoy the orthogonality relation, it gives a full reconstruction of a function/distribution \( f \) for \( f, g, h \in L^2(\mathbb{R}^n) \) and \( \hat{f}, \hat{G} \in L^1(\mathbb{R}^n) \) as

\[
(2)
f(u) = \frac{1}{\langle h, g \rangle} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} V_g f(x, \omega) M_\omega T_x h(u) d\omega dx, \text{ for all } u \in \mathbb{R}^n,
\]

where \( M_\omega \) and \( T_x \) are the modulation and translation operator respectively. But the STFT does not give the directional information about the function or distribution. For directional sensitivity of time-frequency decomposition Grafakos and Sansing [11] introduced a variant of STFT of function \( f \in L^2(\mathbb{R}) \) using Gabor ridge function \( e^{2\pi i m(s-t)}g(s-t) \) on \( S^{n-1} \times \mathbb{R} \times \mathbb{R} \) as

\[
(3) (\xi, x, \omega) \rightarrow \int_{\mathbb{R}^n} f(t) g(\xi, t-x) e^{-2\pi i \omega(\xi, t)} dt.
\]

But this transform does not provide a full reconstruction of a function \( f \) and for this the authors in [11] use the derivative of Gabor ridge function, i.e. “weighted Gabor ridge functions” for the analysis and synthesis of the functions \( f \). Moreover, this transform lost the Fourier transform representation [11] of STFT, since \( x \in \mathbb{R}^n \) and \( \omega \in \mathbb{R} \). Modifying the idea, Giv [3] introduced directional short-time Fourier transform and studied some useful properties like orthogonality, full reconstruction formula. Recently Mejjaoli and Omri [1] introduced a generalized two-wavelet multiplier using directional STFT and studied the \( L^p \) boundedness and compactness of the two-wavelet multiplier.

Our aim in this paper is to give a generalization of two-wavelet multiplier on locally compact
abelian topological groups $G$ associated to right $H$-translation invariant functions. For this as a generalization of directional STFT, we will define a transform $D_H^0 f(\omega, zH)$ of $f \in L^2(G)$ using a character $\omega \in \hat{G}$ and a window $g \in L^2(G/H)$, where $H$ is a closed subgroup of $G$. In Section 2 we first define the STFT, $D_H^0 f, f \in L^1(G)$ and $g \in L^\infty(G/H)$ on $G/H$ which we have shown satisfies certain orthogonality relations. In this section we have also defined the generalized multiplier and generalized two-wavelet multiplier. In Section 3, 4 and 5 we have shown that the generalized multiplier is bounded, Schatten class and $L^p$-bounded, $1 \leq p \leq \infty$ for all symbol $\sigma$ in $L^p(\hat{G} \times G/H), 1 \leq p \leq \infty$. Finally, in Section 6 we have shown that the generalized wavelet multipliers are compact operators for all symbol $\sigma$ in $L^1(\hat{G} \times G/H)$. In Section 7 we have defined the generalized Landau-Pollak-Slepian operator and shown that its is actually unitarily equivalent to scaler multiples of the generalized two-wavelet multiplier.

2. Fourier like transform on locally compact abelian topological groups associated to the coset of closed subgroup

$G$ denotes a locally compact abelian topological group with the Haar measure $dm_G$ and $\hat{G}$ is the dual group of $G$ with the Haar measure $dm_{\hat{G}}$ such that $dm_{\hat{G}}$ is the dual measure of $dm_G$. Let $H$ be a closed subgroup of an LCA group $G$ with the Haar measure $dm_H$. The annihilator of $H$ is the set $H^\perp \subset \hat{G}$ given by $H^\perp = \{ \eta \in \hat{G} : \langle y, \eta \rangle = 1 \text{ for all } y \in H \}$. Moreover, $H^\perp$ is a closed subgroup of $\hat{G}$, is topologically isomorphic with the character group of $G/H$ and we have the followings:

$$ (H^\perp)^\perp = H \quad \text{and} \quad \hat{H} = \hat{G}/H^\perp. $$

Let $f$ be any function in $L^1(G)$ then according to Theorem 28.54 in [2], the function $x \to \int_H f(xy)dm_H(y)$ depends only on the [left] coset of $H$ containing $x$, is a function on the quotient group $G/H$. Moreover, the function $R_H$ is Haar measurable on $G/H$ and belongs to $L^1(G/H)$, where $R_H$ is the function $xH \to \int_H f(xy)dm_H(y)$. We normalize the Haar measure so that

$$ (4) \quad \int_{G/H} R_H(xH)dm_{G/H}(xH) = \int_{G/H} \int_H f(xy)dm_H(y)dm_{G/H}(xH) = \int_G f(x)dm_G(x). $$

For $R_H \in L^1(G/H)$, the Fourier transform of $R_H$ is defined by

$$ \widehat{R_H}(\chi^+) = \int_{G/H} R_H(xH)\overline{\chi^+(xH)}dm_{G/H}(xH). $$

where the character $\chi^+(\in H^\perp)$ of the group $G/H$ is defined by $\chi^+(xH) = \chi(x)$ for $\chi \in \hat{G}$.

The relation between the Fourier transform of the function on group $G/H$ and function on $G$ is the
Theorem 2.1. If \( g \in L^\infty(G/H) \) and \( f \in L^1(G) \), then for every \((\omega, zH) \in \hat{G} \times G/H\)

\[
\mathcal{D}_H^g f(\omega, zH) = (f(x)g(z^{-1}xH))^\wedge(\omega) = (R_H(M_{-\omega}f), T_{zH}g)_{G/H} = (R_H(M_{-\omega}f) * g)(zH).
\]

Proof. The result can be easily followed in view of Weil’s formula (1) and the following:

\[
\begin{align*}
\mathcal{D}_H^g f(\omega, zH) &= \int_G f(x)w(x)g(z^{-1}xH)dx \\
&= \int_{G/H} \int_H f(xy)w(xy)g(z^{-1}xyH)dm_H(y)dm_{G/H}(xH) \\
&= \int_{G/H} R_H(M_{-\omega}f)(xH)g(z^{-1}xH)dm_{G/H}(xH) \\
&= (R_H(M_{-\omega}f) * g)(zH),
\end{align*}
\]
where for the functions \( f_1, f_2 \in L^1(G/H) \) the convolution in the quotient space is defined by
\[
(f_1 \ast f_2)(xH) = \int_{G/H} f_1(yH)f_2(y^{-1}xH)dm_{G/H}(yH).
\]

Theorem 2.2. If \( f \in L^1(G) \), \( \hat{f} \in L^1(\hat{G}) \) and \( g \in L^1(G/H) \cap L^\infty(G/H) \), then the transform \( D_H^g f(\omega, zH) \) can be regarded as a STFT in the quotient group \( G/H \) through the following theorem:

\[
\text{Theorem 2.2. If } f \in L^1(G), \ \hat{f} \in L^1(\hat{G}) \text{ and } g \in L^1(G/H) \cap L^\infty(G/H), \text{ then the transform } D_H^g f(\omega, zH) \text{ is the STFT of the function } T_{-\omega} \hat{f} \text{ with respect to the window } \hat{G}, \text{ evaluated at } (0, -zH).
\]

**Proof.** From the previous theorem we have \( D_H^g f(\omega, zH) = \langle R_H(M_{-\omega}f), T_{zH}g \rangle_{G/H} \). Since \( f \in L^1(G) \) and \( \hat{f} \in L^1(\hat{G}) \) then \( R_H(M_{-\omega}f) \in L^1(G/H) \) and \( R_H(M_{-\omega}f) \in L^1(G/H) \). So \( R_H(M_{-\omega}f) \in L^2(G/H) \). Also \( g \in L^2(G/H) \). Hence using Plancherel’s theorem and (5) we can write \( D_H^g f(\omega, zH) \) the followings for \( \eta \in H^\perp:\n\]
\[
D_H^g f(\omega, zH) = \langle R_H(M_{-\omega}f)(\eta), \hat{T}_{zH}\hat{g}(\eta) \rangle_{G/H} = \langle M_{-\omega}\hat{f}(\eta), M_{-zH}\hat{g}(\eta) \rangle_{H^\perp} = \langle T_{-\omega}\hat{f}(\eta), M_{-zH}\hat{g}(\eta) \rangle_{H^\perp}.
\]

Hence proved. \( \square \)

The transform \( D_H^g f(\omega, zH) \) satisfy the following orthogonality relations:

**Theorem 2.3.**

(i) For every directional window \( g \in L^\infty(G/H) \), the operator \( D_g \) is bounded from \( L^1(G) \) into \( L^\infty(\hat{G} \times G/H) \) and the operator norm satisfies

\[
\|D_H^g\| \leq \|g\|_{L^\infty(G/H)}.
\]

(ii) Suppose \( g_1, g_2 \in L^\infty(G/H) \) and \( f_1, f_2 \in L^1(G) \cap L^2(G) \). If at least one of the \( g_i \)’s is in \( L^1(G/H) \), then \( D_H^g f(\omega, zH) \) satisfies

\[
\int_{\hat{G} \times G/H} D_H^{g_1} f_1(\omega, zH)D_H^{g_2} f_2(\omega, zH)dm_{\hat{G}}(w)dm_{G/H}(zH) = \langle f_1, f_2 \rangle_{L^2(G)} \langle g_2, g_1 \rangle_{L^2(G/H)}.
\]

Moreover, if \( g \in L^1(G/H) \cap L^\infty(G/H) \) and \( f \in L^1(G) \cap L^2(G) \), then \( D_H^g f(\omega, zH) \in L^2(G/H \times \hat{G}) \) and

\[
\|D_H^g f\|_{L^2(G/H \times \hat{G})} = \|g\|_{L^2(G/H)} \|f\|_{L^2(G)}.
\]
Proof. The proof of first part follows from Equation (8).

Since \(g_i \in L^\infty(G/H)\) and \(f_i \in L^1(G) \cap L^2(G)\), \(f_i(x)w(x)g_i(z^{-1}xH) \in L^1(G) \cap L^2(G)\) for \(i = 1, 2\).

Using Plancherel’s theorem we can write the followings:

\[
\int_{G \times G/H} D^q_H f_1(\omega, zH) D^q_H f_2(\omega, zH) dm_G(w) dm_{G/H}(zH)
\]

\[
= \int_{G \times G/H} (f_1(x)\overline{g_1(z^{-1}xH)})^\omega (f_2(x)\overline{g_2(z^{-1}xH)})^\omega dm_G(w) dm_{G/H}(zH)
\]

\[
= \int_{G \times G/H} f_1(x) g_1(z^{-1}xH) f_2(x) g_2(z^{-1}xH) dm_G(w) dm_{G/H}(zH)
\]

\[
= \int_{G} f_1(x) f_2(x) \left( \int_{G/H} g_1(z^{-1}xH) g_2(z^{-1}xH)(w) dm_{G/H}(zH) \right) dm_G(x)
\]

Hence the proof of (11) follows by noting

\[
\langle g_2, g_1 \rangle_{G/H} = \int_{G/H} g_1(z^{-1}xH) g_2(z^{-1}xH)(w) dm_{G/H}(zH),
\]

since \(G/H\) is unimodular group. The equation (12) follows as a consequence of (11). \(\square\)

Corollary 2.4. Suppose \(g_1, g_2 \in L^\infty(G/H)\) and \(f \in L^1(G) \cap L^2(G)\). If at least one of the \(g_i\)’s is in \(L^1(G/H)\) and \(\langle g_2, g_1 \rangle \neq 0\), then

\[
f(x) = \frac{1}{\langle g_2, g_1 \rangle} \int_{G \times G/H} D^q_H f(\omega, zH) \omega(x) g_2(z^{-1}xH) dm_G(w) dm_{G/H}(zH).
\]

Moreover, for non-zero function \(g \in L^1(G/H) \cap L^\infty(G/H)\)

\[
f(x) = \frac{1}{\|g\|^2_{L^p(G/H)}} \int_{G \times G/H} D^q_H f(\omega, zH) \omega(x) g(z^{-1}xH) dm_G(w) dm_{G/H}(zH)
\]

Proof. The proof follows by using Theorem 2.3. \(\square\)

Remark 1. The weak version of the above Corollary 2.4 means that for every function \(u\) in \(L^1(G) \cap L^2(G)\) there exist unique \(f \in L^2(G)\), s.t.

\[
\langle f, u \rangle = \frac{1}{\langle g_2, g_1 \rangle} \int_{G \times G/H} D^q_H f_1(\omega, zH) \overline{D^q_H u(\omega, zH)} dm_G(w) dm_{G/H}(zH)
\]

which is nothing but the equation (11).

Proposition 2.5. We assume that \(g \in L^1(G/H) \cap L^\infty(G/H)\), \(f \in L^1(G) \cap L^2(G)\) and \(p\) belongs in \([2, \infty]\). We have

\[
\|D_g(f)\|_{L^p(G \times G/H)} \leq \|g\|_{L^p(G/H)} \|f\|_{L^{p'}(G)}.
\]
Example 2.1. Here we give one example where \( G = \mathbb{R}^n, H = \{ x \in G : x \cdot \theta = 1, \ \theta \in S^{n-1} \} \). For any element in \( G/H \) we can write \( xH = (x \cdot \theta)\theta H \). Then the quotient group \( G/H = \{ t\theta H, t \in \mathbb{R} \} \). Hence the right-\( H \)-translation-invariant functions can be written as

\[
R_{\theta}f(xH) = R_{\theta}f(t\theta H) = \int_H f(t\theta h) dm_H(h) = \int_{z.\theta = t} f(z) dz = R_{\theta}f(t),
\]

where \( R_{\theta}f(t) \) is the function on \( S^{n-1} \times \mathbb{R} \), which is the Radon transform of \( f \). In this case the transform \( D^\theta_H f(\omega, zH) \) is represented using Theorem 2.1 as

\[
D^\theta_H f(\omega, zH) = \int_{G/H} R_{\theta}(M_{-\omega}f)(xH) g(z^{-1}xH) dm_{G/H}(xH).
\]

The quotient group is defined by \( G/H = \{ t\theta H, t \in \mathbb{R} \} \). Also, the characterization of the quotient group is the subgroup \( H \) of \( G \) defines an equivalence relation in \( G \) by \( g_1 = g_2 \iff g_2^{-1} g_1 \in H \), i.e., for \( g_1, g_2 \in \mathbb{R}^n \) and \( \theta \in S^{n-1} \), \( g_1 \cdot \theta = g_2 \cdot \theta \). So the norms are equal and hence the quotient group \( G/H \) is characterized by \( \mathbb{R} \). Hence for \( z \in \mathbb{R} \), \( D^\theta_H f(\omega, zH) \) is defined on \( S^{n-1} \times \mathbb{R} \times \mathbb{R}^n \) as

\[
D^\theta_H f(\omega, zH) = \int_{t \in \mathbb{R}} R_{\theta}(M_{-\omega}f)(t) g(t - z) dt = \int_{t \in \mathbb{R}} \int_{x, \theta = t} f(x) w(x) dxg(t - z) dt
\]

\[
= \int_{t \in \mathbb{R}} \int_{x, \theta = t} f(x) w(x) g(x, \theta - z) dx dt
\]

\[
= \int_{\mathbb{R}^n} f(x) w(x) g(x, \theta - z) dx dt,
\]

which is the directional short-time Fourier transform of the function \( f \) with respect to window \( g \) \([3,4]\).

We define the generalized multiplier and generalized two-wavelet multiplier in the following.

Definition 2.6. Let \( \sigma \in L^\infty(\hat{G} \times G/H) \), we define the linear operator \( M_{\sigma,g} : L^2(G) \rightarrow L^2(G) \) by

\[
M_{\sigma,g}(f) = (D^\theta_H)^{-1}(\sigma D^\theta_H f).
\]

This operator is called the generalized multiplier, where \( 0 \neq g \in L^1(G/H) \cap L^\infty(G/H) \).

Definition 2.7. Let \( u, v \) be a measurable functions on \( G \) and \( \sigma \) be measurable function on \( \hat{G} \times G/H \), we define the generalized two wavelet multiplier operator denoted by \( P_{u,v,g}(\sigma) \) on \( L^p(G), 1 \leq p \leq \infty \) defined by, for \( \neq 0, g \in L^1(G/H) \cap L^\infty(G/H) \),

\[
P_{u,v,g}(\sigma)(f)(t) = \int_{\hat{G} \times G/H} \sigma(w, zH) D^\theta_H(u \sigma)(w, zH) g_{w, zH}(t) v(t) dm_{\hat{G}}(\omega) dm_{G/H}(zH),
\]
where \( g_{w,zH}(x) = g(z^{-1}xH)w(x) \).

\( P_{u,v,g}(\sigma) \) in a weak sense, for \( f \in L^p(G), 1 \leq p \leq \infty \) and \( h \in L^{p'}(G) \),

\[
\langle P_{u,v,g}(\sigma)(f), h \rangle = \int_{\hat{G}} \int_{G/H} \sigma(w, zH)D^D_{H}(uf)(w, zH)\overline{D^D_{H}(vh)(w, zH)} dm_{\hat{G}}(\omega)dm_{G/H}(zH).
\]

**Proposition 2.8.** Let \( f \) be in \( L^p(G) \) and \( h \) in \( L^{p'} \), where \( 1 \leq p < \infty \), \( \frac{1}{p} + \frac{1}{p'} = 1 \). Then

\[
P_{u,v,g}(\sigma) = P_{v,u,g}(\sigma).
\]

**Proof.** For \( f \) be in \( L^p(G) \) and \( h \) in \( L^{p'} \),

\[
\langle P_{u,v,g}(\sigma)(f), h \rangle_{L^2(G)} = \int_{\hat{G}} \int_{G/H} \sigma(w, zH)D^D_{H}(uf)(w, zH)\overline{D^D_{H}(vh)(w, zH)} dm_{\hat{G}}(\omega)dm_{G/H}(zH)
\]

\[
= \langle P_{v,u,g}(\sigma)(f), h \rangle_{L^2(G)}
\]

\[
= \langle f, P_{v,u,g}(\sigma)(h) \rangle_{L^2(G)}.
\]

Hence we get,

\[
P_{u,v,g}(\sigma)(h) = P_{v,u,g}(\sigma)(h).
\]

This complete the proof. \( \square \)

**Proposition 2.9.** Let \( \sigma \in L^1(\hat{G} \times G/H) \cup L^\infty(\hat{G} \times G/H) \) and \( u, v \in L^2(G) \cap L^\infty(G) \). Then

\[
\langle P_{u,v,g}(\sigma)(f), h \rangle_{L^2(G)} = \|g\|_{L^2(G/H)}^2 \langle \overline{\nabla M_{\sigma,g}(uf)}, h \rangle_{L^2(G)}.
\]

**Proof.** In view of Definition 2.6 and Theorem 2.3 we conclude

\[
\langle P_{u,v,g}(\sigma)(f), h \rangle = \int_{\hat{G}} \int_{G/H} \sigma(w, zH)D^D_{H}(uf)(w, zH)\overline{D^D_{H}(vh)(w, zH)} dm_{\hat{G}}(\omega)dm_{G/H}(zH)
\]

\[
= \int_{\hat{G}} \int_{G/H} \overline{D^D_{H}(M_{\sigma,g}(uf))}(w, zH)\overline{D^D_{H}(vh)(w, zH)} dm_{\hat{G}}(\omega)dm_{G/H}(zH)
\]

\[
\langle \nabla M_{\sigma,g}(uf), h \rangle_{L^2(G)}
\]

\[
(14)
\]

\( \square \)
3. $L^2$-Boundedness of Generalized Two Wavelet Multiplier

In this section we will show the operators

$$P_{u,v,g} : L^2(G) \to L^2(G)$$

are bounded linear operators for all symbol $\sigma$ in $L^p(\hat{G} \times G/H)$, $1 \leq p \leq \infty$.

Let us assume $u, v$ be in $L^2(G) \cap L^\infty(G)$ such that

$$\|u\|_{L^2(G)} = \|v\|_{L^2(G)} = 1.$$  

**Proposition 3.1.** Let $\sigma$ be in $L^1(\hat{G} \times G/H)$, then the generalized two wavelet multiplier $P_{u,v,g}(\sigma)$ is in $S_\infty$.

**Proof.** For every functions $f$ and $h$ in $L^2(G)$, it follows from Definition 2.7, Equation (8) and Cauchy-Schwarz inequality,

$$\langle (P_{u,v,g}(\sigma))(f), h \rangle_{L^2(G)}$$

$$\leq \int_{\hat{G} \times G/H} |\sigma(w, zH) D_H^q(u)(w) D_H^q(vh)(w, zH)| dm_G(\omega) dm_{G/H}(zH)$$

$$\leq \|D^q_H(u)\|_{L^\infty(\hat{G} \times G/H)} \|D^q_H(vh)\|_{L^\infty(\hat{G} \times G/H)} \int_{\hat{G} \times G/H} |\sigma(w, zH)| dm_G(\omega) dm_{G/H}(zH)$$

$$\leq \|u\|_{L^2(G)} \|f\|_{L^2(G)} \|g\|_{L^\infty(G/H)} \|v\|_{L^2(G)} \|h\|_{L^2(G)} \|\sigma\|_{L^1(\hat{G} \times G/H)}.$$  

Hence

$$\|P_{u,v,g}(\sigma)\|_{S_\infty} \leq \|\sigma\|_{L^1(\hat{G} \times G/H)} \|g\|_{L^\infty(G/H)}^2.$$  

\[ \square \]

**Proposition 3.2.** Let $\sigma$ be in $L^\infty(\hat{G} \times G/H)$, then the generalized two wavelet multiplier operator $P_{u,v,g}(\sigma)$ is in $S_\infty$.

**Proof.** Here $P_{u,v,g}(\sigma) : L^2(G) \to L^2(G)$. For every functions $f$ and $h$ in $L^2(G)$ we have the followings from Definition 2.7, Cauchy-Schwarz's inequality and Plancherel formula 12:

$$\langle (P_{u,v,g}(\sigma))(f), h \rangle_{L^2(G)}$$

$$\leq \int_{\hat{G} \times G/H} |\sigma(\omega, zH) D_H^q(u)(\omega) D_H^q(vh)(\omega, zH)| dm_G(\omega) dm_{G/H}(zH)$$

$$\leq \|D^q_H(u)\|_{L^2(\hat{G} \times G/H)} \|D^q_H(vh)\|_{L^2(\hat{G} \times G/H)} \|\sigma\|_{L^\infty(\hat{G} \times G/H)}$$

$$\leq \|u\|_{L^\infty(G)} \|f\|_{L^2(G)} \|g\|_{L^2(G/H)}^2 \|v\|_{L^\infty(G)} \|h\|_{L^2(G)} \|\sigma\|_{L^\infty(\hat{G} \times G/H)}.$$  

\[ \square \]
Now we can proceed to interpret the generalized two wavelet multiplier

\[ P_{u,v,g}(\sigma) : L^2(G) \to L^2(G) \]

to all symbol \( \sigma \) in \( L^p(\widehat{G} \times G/H) \), \( 1 \leq p \leq \infty \) being in \( S_\infty \). It is given in the following theorem.

**Theorem 3.3.** Let \( \sigma \in L^p(\widehat{G} \times G/H), 1 \leq p \leq \infty \). Then there exists a unique bounded linear operator \( P_{u,v,g}(\sigma) : L^2(G) \to L^2(G) \) s.t.

\[
\| P_{u,v,g}(\sigma) \|_\infty \leq \| g \|_{L^\infty(G/H)}^{\frac{2(p-1)}{p}} \| g \|_{L^2(G/H)} \left( \| u \|_{L^\infty(G)} \| v \|_{L^\infty(G)} \right)^{\frac{p-1}{p}} \| \sigma \|_{L^p(\widehat{G} \times G/H)}.
\]

**Proof.** Let \( f \in L^2(G) \) and \( T : L^1(\widehat{G} \times G/H) \cap L^\infty(\widehat{G} \times G/H) \to L^2(G) \) given by

\[ T(\sigma) := P_{u,v,g}(\sigma)f. \]

We have from Proposition 3.1

\[ \| P_{u,v,g}(\sigma)f \|_{L^2(G)} \leq \| f \|_{L^2(G)} \| g \|_{L^2(G/H)} \| \sigma \|_{L^1(\widehat{G} \times G/H)} \]

and from Proposition 3.2

\[ \| P_{u,v,g}(\sigma)f \|_{L^2(G)} \leq \| f \|_{L^2(G)} \| g \|_{L^2(G/H)} \| \sigma \|_{L^1(\widehat{G} \times G/H)} \| u \|_{L^\infty(G)} \| v \|_{L^\infty(G)}. \]

By Riesz-Thorin interpolation theorem, \( T \) may uniquely extended to \( L^p(\widehat{G} \times G/H), 1 \leq p \leq \infty \) and

\[ \| T(\sigma) \|_{L^2(G)} = \| P_{u,v,g}(\sigma)f \|_{L^2(G)}. \]

So

\[
\| P_{u,v,g}(\sigma)f \|_{L^2(G)} \leq \| g \|_{L^\infty(G/H)}^{\frac{2(p-1)}{p}} \| g \|_{L^2(G/H)} \left( \| u \|_{L^\infty(G)} \| v \|_{L^\infty(G)} \right)^{\frac{p-1}{p}} \| f \|_{L^2(G)} \| \sigma \|_{L^p(\widehat{G} \times G/H)}.
\]

\[ \square \]

4. **Schatten class boundedness of generalized two wavelet multiplier**

Our aim is to show the linear operators

\[ P_{u,v,g} : L^2(G) \to L^2(G) \]

are in Schatten class \( S_p \) for all symbol \( \sigma \) in \( L^p(\widehat{G} \times G/H) \), \( 1 \leq p \leq \infty \).

**Proposition 4.1.** Let \( \sigma \) be in \( L^1(\widehat{G} \times G/H) \), then the generalized two wavelet multiplier \( P_{u,v,g}(\sigma) \) is in \( S_2 \) and we have

\[ \| P_{u,v,g}(\sigma) \|_{S_2} \leq \| g \|_{L^\infty(G/H)} \| \sigma \|_{L^1(\widehat{G} \times G/H)}. \]
Proof. Let \( \{ \phi_j : j = 1, 2, \cdots \} \) be an orthonormal basis for \( L^2(G) \).

\[
\sum_{j=1}^{\infty} \| P_{u,v,g}(\sigma)(\phi_j) \|_{L^2(G)}^2 = \sum_{j=1}^{\infty} \langle P_{u,v,g}(\sigma)\phi_j, P_{u,v,g}(\sigma)\phi_j \rangle.
\]

Now using Definition \ref{def:Fubini} Fubini’s theorem, Parseval’s identity and Proposition \ref{prop:Parseval} we have

\[
\langle P_{u,v,g}(\sigma)\phi_j, P_{u,v,g}(\sigma)\phi_j \rangle
= \int_{\hat{G} \times G/H} \sigma(w, zH) D_H^0(u\phi_j)(w, zH) D_H^0(vP_{u,v,g}(\sigma)\phi_j)(w, zH) dm_G(\omega) dm_{G/H}(zH)
= \int_{\hat{G} \times G/H} \sigma(w, zH) \langle u\phi_j, g w, z H \rangle_{L^2(G)} \langle vP_{u,v,g}(\sigma)\phi_j, g w, z H \rangle_{L^2(G)} dm_G(\omega) dm_{G/H}(zH)
= \int_{\hat{G} \times G/H} \sigma(w, zH) \langle \phi_j, g w, z H \rangle_{L^2(G)} \langle P_{u,v,g}(\sigma)\phi_j, g w, z H \rangle_{L^2(G)} dm_G(\omega) dm_{G/H}(zH).
\]

So

\[
\sum_{j=1}^{\infty} \| P_{u,v,g}(\sigma)(\phi_j) \|_{L^2(G)}^2
= \sum_{j=1}^{\infty} \int_{\hat{G} \times G/H} \sigma(\omega, zH) \langle \phi_j, g w, z H \rangle_{L^2(G)} \langle P_{u,v,g}(\sigma)(\phi_j), g w, z H \rangle_{L^2(G)} dm_G(\omega) dm_{G/H}(zH)
= \int_{\hat{G} \times G/H} \sigma(\omega, zH) \sum_{j=1}^{\infty} \langle \phi_j, g w, z H \rangle_{L^2(G)} \langle P_{u,v,g}(\sigma)(\phi_j), g w, z H \rangle_{L^2(G)} dm_G(\omega) dm_{G/H}(zH)
= \int_{\hat{G} \times G/H} \sigma(\omega, zH) \langle P_{u,v,g}(\sigma)(\phi_j), g w, z H \rangle_{L^2(G)} dm_G(\omega) dm_{G/H}(zH).
\]

Thus

\[
\sum_{j=1}^{\infty} \| P_{u,v,g}(\sigma)(\phi_j) \|_{L^2(G)}^2
\leq \| g \|_{L^2(G/H)}^2 \int_{\hat{G} \times G/H} \| \sigma(\omega, zH) \| \| P_{u,v,g}(\sigma) \|_{s_\infty} dm_G(\omega) dm_{G/H}(zH)
\leq \| g \|_{L^2(G/H)}^2 \| \sigma \|_{L^1(\hat{G} \times G/H)}^2.
\]

Proposition 4.2. Let \( \sigma \) be a symbol in \( L^p(\hat{G} \times G/H), 1 \leq p < \infty \). Then the generalized two-wavelet multiplier \( P_{u,v,g}(\sigma) \) is compact.

Proof. We know \( L^1(\hat{G} \times G/H) \cap L^\infty(\hat{G} \times G/H) \) is dense in \( L^p(\hat{G} \times G/H), (1 \leq p < \infty) \). Let \( \sigma \in L^p(\hat{G} \times G/H) \), then there exist \( \{ \sigma_n \}_{n \in \mathbb{N}} \subset L^1(\hat{G} \times G/H) \cap L^\infty(\hat{G} \times G/H) \) s.t. \( \sigma_n \to \sigma \) in
Now by Bessel’s inequality, Cauchy-Schwarz’s inequality, Fubini’s theorem and the fact

Then

\[ P_{u,v,g}(\sigma) = \sum_{j=1}^{\infty} s_j \langle f, \phi_j \rangle_{L^2(G)} \psi_j, \]

where \( s_j, j = 1, 2, 3, \ldots \) are positive singular values of \( P_{u,v,g}(\sigma) \) corresponding to \( \phi_j \).

Then

\[ \| P_{u,v,g}(\sigma) \|_{S_1} = \sum_{j=1}^{\infty} s_j = \sum_{j=1}^{\infty} \langle P_{u,v,g}(\sigma)(\phi_j), \psi_j \rangle_{L^2(G)}. \]

Now by Bessel’s inequality, Cauchy-Schwarz’s inequality, Fubini’s theorem and the fact \( \| u \|_{L^2(G)} = \| u \|_{L^2(G)} = 1 \), we get

\[
\sum_{j=1}^{\infty} \langle P_{u,v,g}(\sigma)(\phi_j), \psi_j \rangle_{L^2(G)}
= \sum_{j=1}^{\infty} \int_{\hat{G} \times G/H} \sigma(\omega, zH) D^2_H(u(\phi_j)(\omega, zH)D^2_H(v\psi_j))(\omega, zH) \, dm_{G}(\omega) \, dm_{G/H}(zH)
= \int_{\hat{G} \times G/H} \sigma(\omega, zH) \sum_{j=1}^{\infty} \langle \phi_j, u\xi_{\omega,zH} \rangle_{L^2(G)} \langle u\xi_{\omega,zH}, \psi_j \rangle_{L^2(G)} \, dm_{G}(\omega) \, dm_{G/H}(zH)
\leq \int_{\hat{G} \times G/H} |\sigma(\omega, zH)| \left( \sum_{j=1}^{\infty} |\langle \phi_j, u\xi_{\omega,zH} \rangle_{L^2(G)}|^2 \right)^{1/2} \left( \sum_{j=1}^{\infty} |\langle u\xi_{\omega,zH}, \psi_j \rangle_{L^2(G)}|^2 \right)^{1/2} dm_{G}(\omega) \, dm_{G/H}(zH)
\leq \int_{\hat{G} \times G/H} |\sigma(\omega, zH)| \| u\xi_{\omega,zH} \|_{L^2(G)} \| u\xi_{\omega,zH} \|_{L^2(G)} dm_{G}(\omega) \, dm_{G/H}(zH)
\leq \| \sigma \|_{L^1(\hat{G} \times G/H)} \| g \|_{L^2(G/H)^2}^2.
\]
Corollary 4.4. For $\sigma \in L^1(\hat{G} \times G/H)$, we have the following trace formula;

$$\text{tr}(P_{u,v,g}(\sigma)) = \int_{\hat{G} \times G/H} \sigma(\omega, zH)\langle ug_{\omega, zH}, ug_{\omega, zH}\rangle_{L^2(G)} dm\hat{G}(\omega)dm_{G/H}(zH).$$

Proof. Let $\{\phi_k : k = 1, 2, \cdots \}$ be an orthonormal basis for $L^2(G)$. By Theorem 4.3, the generalized two wavelet multiplier $P_{u,v,g}(\sigma)$ is in $S_1$.

Now by definition of trace formula, Fubini’s theorem and Parseval’s identity, we have

$$\text{tr}(P_{u,v,g}(\sigma)) = \sum_{k=1}^{\infty} \langle P_{u,v,g}(\sigma)(\phi_k), \phi_k \rangle_{L^2(G)}$$

$$= \sum_{k=1}^{\infty} \int_{\hat{G} \times G/H} \sigma(\omega, zH)\langle \phi_k, ug_{\omega, zH}\rangle_{L^2(G)}\langle \phi_k, vg_{\omega, zH}\rangle_{L^2(G)} dm\hat{G}(\omega)dm_{G/H}(zH)$$

$$= \int_{\hat{G} \times G/H} \sigma(\omega, zH)\sum_{k=1}^{\infty} \langle \phi_k, vg_{\omega, zH}\rangle_{L^2(G)}\langle \phi_k, ug_{\omega, zH}\rangle_{L^2(G)} dm\hat{G}(\omega)dm_{G/H}(zH)$$

$$= \int_{\hat{G} \times G/H} \sigma(\omega, zH)\langle vg_{\omega, zH}, ug_{\omega, zH}\rangle_{L^2(G)} dm\hat{G}(\omega)dm_{G/H}(zH).$$

\[ \square \]

We give a result

$$P_{u,v,g}(\sigma) : L^2(G) \rightarrow L^2(G)$$

will be in $S_p$ for $\sigma \in L^p(G)$, $1 \leq p \leq \infty$.

Corollary 4.5. Let $\sigma \in L^p(\hat{G} \times G/H), 1 \leq p \leq \infty$. Then the generalized two wavelet multiplier $P_{u,v,g}(\sigma) : L^2(G) \rightarrow L^2(G)$ is in $S_p$ and we have

$$\|P_{u,v,g}\|_p s_p \leq \|g\|_{L^p(G/H)}^{2(p-1)} \|g\|_{L^2(G/H)}^{2(p-1)} \left(\|u\|_{L^\infty(G)}\|v\|_{L^\infty(G)}\right)^{\frac{p-1}{p}} \|\sigma\|_{L^p(\hat{G} \times G/H)}.$$

Proof. The proof follows from above Theorem 4.3 and Proposition 3.2 and interpolation theorem in [12]. \[ \square \]

5. $L^p$-boundedness of generalized two wavelet multiplier for $1 \leq p \leq \infty$

Our aim is to show the linear operators

$$P_{u,v,g} : L^p(G) \rightarrow L^p(G)$$

are bounded operators for all symbol $\sigma$ in $L^r(\hat{G} \times G/H)$, $1 < r \leq \infty$ for all $p \in \left[\frac{2r}{r+1}, \frac{2r}{r-1}\right]$ and for $r = 1, p \in [1, \infty]$.

Let us assume $0 \neq g \in L^\infty(G/H)$ for this section.
Proposition 5.1. Let $\sigma$ be in $L^1(\hat{G} \times G/H)$, $u \in L^\infty(G)$ and $v \in L^1(G)$, then the generalized two wavelet multiplier

$$P_{u,v,g}(\sigma) : L^1(G) \to L^1(G)$$

is a bounded operator and we have

$$\|P_{u,v,g}(\sigma)\|_{B(L^1(G))} \leq \|u\|_{L^\infty(G)} \|v\|_{L^1(G)} \|g\|_{L^2(G/H)}^2 \|\sigma\|_{L^1(\hat{G} \times G/H)}.$$\[\square\]

Proof. We know that

$$P_{u,v,g}(\sigma)(f)(t) = \int_{\hat{G} \times G/H} \sigma(\omega, zH)D_H^\varphi(uf)(\omega, zH)g_{\omega,zH}(t)v(t)dm_\varphi(\omega)dm_{G/H}(zH).$$

Now by Equation 8 and Equation 10

$$\|P_{u,v,g}(\sigma)(f)\|_{L^1(G)} \leq \int_{\hat{G} \times G/H} \int_G |\sigma(\omega, zH)|D_H^\varphi(uf)(\omega, zH)|g_{\omega,zH}(t)v(t)|dm_\varphi(\omega)dm_{G/H}(zH)dt$$

$$\leq \int_{\hat{G} \times G/H} \int_G |\sigma(\omega, zH)||uf||g_{\omega,zH}|_{L^2(G)}|v(t)|dm_\varphi(\omega)dm_{G/H}(zH)dt$$

$$\leq \|g\|_{L^\infty(G/H)}\|f\|_{L^1(G)}\|u\|_{L^\infty(G)} \int_{\hat{G} \times G/H} |\sigma(\omega, zH)|dm_\varphi(\omega)dm_{G/H}(zH) \int_G |v(t)|dt$$

$$= \|g\|_{L^\infty(G/H)}\|f\|_{L^1(G)}\|u\|_{L^\infty(G)}\|\sigma\|_{L^1(\hat{G} \times G/H)} \|v\|_{L^1(G)}.$$\[\square\]

Proposition 5.2. Let $\sigma$ be in $L^1(\hat{G} \times G/H)$, $u \in L^1(G)$ and $v \in L^\infty(G)$, then the generalized two wavelet multiplier

$$P_{u,v,g}(\sigma) : L^\infty(G) \to L^\infty(G)$$

is a bounded operator and we have

$$\|P_{u,v,g}(\sigma)\|_{B(L^\infty(G))} \leq \|u\|_{L^1(G)} \|v\|_{L^\infty(G)} \|g\|_{L^2(G/H)}^2 \|\sigma\|_{L^1(\hat{G} \times G/H)}.$$\[\square\]

Proof. Let $f$ be in $L^\infty(G)$, then by Equation 10 and Equation 8 we have

$$|P_{u,v,g}(\sigma)(f)(t)|$$

$$\leq \int_{\hat{G} \times G/H} |\sigma(\omega, zH)|D_H^\varphi(uf)(\omega, zH)|g_{\omega,zH}(t)v(t)|dm_\varphi(\omega)dm_{G/H}(zH)$$

$$\leq \|u\|_{L^1(G)}\|v\|_{L^\infty(G)}\|g\|_{L^2(G/H)}^2\|\sigma\|_{L^1(\hat{G} \times G/H)}\|f\|_{L^\infty(G)}^2.$$
Theorem 5.3. Let \( u, v \in L^1(G) \cap L^\infty(G) \). Then for all \( \sigma \in L^1(\hat{G} \times G/H) \), there exist a unique bounded linear operator

\[
P_{u,v,g}(\sigma) : L^p(G) \to L^p(G), \quad 1 \leq p \leq \infty,
\]
s.t.

\[
\|P_{u,v,g}(\sigma)\|_{B(L^p(G))} \leq \|u\|_{L^1(G)}^{1/p'} \|v\|_{L^\infty(G)}^{1/p} \|u\|_{L^\infty(G)}^{1/p'} \|v\|_{L^1(G)}^{1/p} \|g\|_{L^\infty(G/H)} \|
\]

Proof. We know

\[
P_{u,v,g}(\sigma) : L^1(G) \to L^1(G)
\]
is the adjoint of

\[
P_{v,u,g}(\sigma) : L^\infty(G) \to L^\infty(G).
\]

Now by Proposition 5.1 and Proposition 5.2 and interpolation theorem, for \( 1 \leq p \leq \infty \),

\[
\|P_{u,v,g}(\sigma)\|_{B(L^p(G))} \leq \|u\|_{L^1(G)}^{1/p'} \|v\|_{L^\infty(G)}^{1/p} \|u\|_{L^\infty(G)}^{1/p'} \|v\|_{L^1(G)}^{1/p} \|g\|_{L^\infty(G/H)} \|\sigma\|_{L^1(\hat{G} \times G/H)}.
\]

We give another version of \( L^p \)-boundedness of Theorem 5.3.

Theorem 5.4. Let \( \sigma \) be in \( L^1(\hat{G} \times G/H) \), \( u \) and \( v \) in \( L^1(G) \cap L^\infty(G) \). Then there exists a unique bounded linear operator \( P_{u,v,g}(\sigma) : L^p(G) \to L^p(G) \) \( 1 \leq p \leq \infty \) s.t.

\[
\|P_{u,v,g}(\sigma)\|_{B(L^p(G))} \leq \max\left(\|u\|_{L^1(G)} \|v\|_{L^\infty(G)}, \|u\|_{L^\infty(G)} \|v\|_{L^1(G)}\right) \|g\|_{L^\infty(G/H)}^2 \|\sigma\|_{L^1(\hat{G} \times G/H)}.
\]

Proof. From Definition 2.7

\[
P_{u,v,g}(\sigma)(f)(t)
\]

\[
= \int_{\hat{G} \times G/H} \sigma(\omega, zH) D_H^n(u \circ f)(\omega, zH) g_{\omega, zH}(t) v(t) \overline{v(t)} d\hat{G}(\omega) dm_{G/H}(zH)
\]

\[
= \int_{\hat{G} \times G/H} \sigma(\omega, zH) \int_G u(s) f(s) g_{\omega, zH}(s) ds g_{\omega, zH}(t) v(t) \overline{v(t)} d\hat{G}(\omega) dm_{G/H}(zH)
\]

So the integral operator is

\[
P_{u,v,g}(\sigma)(f)(t) = \int_G N(t; s) f(s) ds,
\]

with the kernel

\[
N(t; s) = \int_{\hat{G} \times G/H} \sigma(\omega, zH) u(s) g_{\omega, zH}(s) g_{\omega, zH}(t) v(t) \overline{v(t)} d\hat{G}(\omega) dm_{G/H}(zH).
\]
Now
\begin{equation}
\int_G |N(t; s)|dt \leq \int_{\mathcal{G} \times G/H} |\sigma(\omega, zH)||u(s)||g_{\omega, zH}(s)||g_{\omega, zH}(t)||v(t)|d\tilde{m}(\omega)d\tilde{m}_{G/H}(zH)dt
\end{equation}
\leq \|u\|_{L^\infty(G)}\|g\|_{L^\infty(G/H)}\|g\|_{L^\infty(G/H)}\|v\|_{L^1(G)}\|\sigma\|_{L^1(\tilde{G} \times G/H)}.

Similarly
\begin{equation}
\int_G |N(t; s)|ds \leq \|u\|_{L^1(G)}\|g\|_{L^\infty(G/H)}\|g\|_{L^\infty(G/H)}\|v\|_{L^\infty(G)}\|\sigma\|_{L^1(\tilde{G} \times G/H)}.
\end{equation}

Thus by Schur’s test [7], we can conclude $P_{u,v,g}(\sigma): L^p(G) \to L^p(G)$ bounded and
\begin{equation}
\|P_{u,v,g}(\sigma)\|_{B(L^2(G))} \leq \max\left(\|u\|_{L^1(G)}\|v\|_{L^\infty(G)}, \|u\|_{L^\infty(G)}\|v\|_{L^1(G)}\right)\|g\|_{L^\infty(G)}^2\|\sigma\|_{L^1(\tilde{G} \times G/H)}.
\end{equation}

\begin{proposition}
Let $\sigma$ be in $L^1(\tilde{G} \times G/H)$, $v$ in $L^p(G)$ and $u$ in $L^p'(G)$ for $1 < p \leq \infty$ then the generalized two-wavelet multiplier $P_{u,v,g}(\sigma): L^p(G) \to L^p(G)$ is a bounded linear operator and we have
\begin{equation}
\|P_{u,v,g}(\sigma)\|_{B(L^2(G))} \leq \|u\|_{L^p'(G)}\|v\|_{L^p(G)}\|g\|_{L^\infty(G)}^2\|\sigma\|_{L^1(\tilde{G} \times G/H)}.
\end{equation}
\end{proposition}

\begin{proof}
For any $f \in L^p(G)$, consider the linear functional $I_f: L^p'(G) \to \mathbb{C}$ defined by
\begin{equation}
I_f(h) = \langle h, P_{u,v,g}(\sigma)(f) \rangle_{L^2(G)}.
\end{equation}

Now from Definition 2.7, Equation 8 and Holder’s inequality, we have
\begin{equation}
\|P_{u,v,g}(\sigma)(f), h\|_{L^2(G)}
\end{equation}
\begin{equation}
= \left| \int_{\tilde{G} \times G/H} |\sigma(\omega, zH)||u(\omega)||v(\omega, zH)||g_{\omega, zH}(s)||g_{\omega, zH}(t)||D_H^g(u, v, g, h)ds\tilde{d}m_{\tilde{G}}(\omega)d\tilde{m}_{G/H}(zH) \right|
\end{equation}
\begin{equation}
\leq \|\sigma\|_{L^1(\tilde{G} \times G/H)}\|u\|_{L^p'(G)}\|v\|_{L^p(G)}\|g\|_{L^\infty(G/H)}^2\|f\|_{L^p(G)}\|h\|_{L^p'(G)}d\tilde{m}_{\tilde{G}}(\omega)d\tilde{m}_{G/H}(zH)
\end{equation}
\begin{equation}
\leq \|\sigma\|_{L^1(\tilde{G} \times G/H)}\|u\|_{L^p'(G)}\|v\|_{L^p(G)}\|g\|_{L^\infty(G/H)}^2\|f\|_{L^p(G)}\|h\|_{L^p'(G)}.
\end{equation}

By Riesz representation theorem
\begin{equation}
\|P_{u,v,g}(\sigma)(f)\| = \|I_f\|_{B(L^p'(G))}.
\end{equation}
By multilinear interpolation theory, we get a unique bounded operator
\[ |I_f(h)| \leq \|\sigma\|_{L^1(\hat{G} \times G/H)} \|u\|_{L^p(G)} \|v\|_{L^p(G)} \|g\|_{L^\infty(G/H)} \|f\|_{L^p(G)} \|h\|_{L^{p'}(G)}. \]

Then
\[ \|I_f\|_{B(L^{p'}(G))} \leq \|\sigma\|_{L^1(\hat{G} \times G/H)} \|u\|_{L^{p'}(G)} \|v\|_{L^p(G)} \|g\|_{L^\infty(G/H)} \|f\|_{L^p(G)}, \]

and hence
\[ \|P_{u,v,g}(\sigma)\|_{B(L^p(G))} \leq \|\sigma\|_{L^1(\hat{G} \times G/H)} \|u\|_{L^{p'}(G)} \|v\|_{L^p(G)} \|g\|_{L^\infty(G/H)}^2. \]

\[ \square \]

We give the result for \( p = 1 \) of Proposition 5.5

**Theorem 5.6.** Let \( \sigma \) be in \( L^1(\hat{G} \times G/H) \), \( v \) in \( L^p(G) \) and \( u \) in \( L^{p'}(G) \) for \( 1 \leq p \leq \infty \) then the generalized two-wavelet multiplier \( P_{u,v,g}(\sigma) : L^p(G) \to L^p(G) \) is a bounded linear operator and we have
\[ \|P_{u,v,g}(\sigma)\|_{B(L^p(G))} \leq \|u\|_{L^{p'}(G)} \|v\|_{L^p(G)} \|g\|_{L^\infty(G/H)} \|\sigma\|_{L^1(\hat{G} \times G/H)}. \]

**Proof.** We can show using Proposition 5.5 and Proposition 5.1. \( \square \)

Let us consider \( 0 \neq g \in L^1(G/H) \cap L^\infty(G/H) \subset L^q(G/H), \ (1 < q < \infty) \) for rest of this section.

**Theorem 5.7.** Let \( \sigma \) be in \( L^r(\hat{G} \times G/H) \), \( r \in [1,2] \) and \( u, v \in L^1(G) \cap L^\infty(G) \). Then there exists a unique bounded linear operator \( P_{u,v,g}(\sigma) : L^p(G) \to L^p(G) \) for all \( p \in [r, r'] \) and we have
\[ \|P_{u,v,g}(\sigma)\|_{B(L^p(G))} \leq c_1 c_2^{1-t} \|\sigma\|_{L^r(\hat{G} \times G/H)}, \quad \frac{t}{r} + \frac{1-t}{r'} = \frac{1}{p}. \]

**Proof.** Consider the linear functional
\[ I : (L^1(\hat{G} \times G/H) \cap L^2(\hat{G} \times G/H)) \times (L^1(G) \cap L^2(G)) \to L^1(G) \cap L^2(G), \]
\[ (\sigma, f) \mapsto P_{u,v,g}(\sigma)(f). \]

By Proposition 5.1 for \( \sigma \in L^1(\hat{G} \times G/H), f \in L^1(G), \)
\[ \|I(\sigma,f)\|_{L^1(G)} = \|P_{u,v,g}(\sigma)(f)\|_{L^1(G)} \leq \|f\|_{L^1(G)} \|u\|_{L^\infty(G)} \|v\|_{L^1(G)} \|g\|_{L^\infty(G/H)} \|\sigma\|. \]

By Theorem 3.2 for \( \sigma \in L^2(\hat{G} \times G/H), f \in L^2(G), \)
\[ \|I(\sigma,f)\|_{L^2(G)} = \|P_{u,v,g}(\sigma)(f)\|_{L^2(G)} \leq \|f\|_{L^2(G)} \left(\|u\|_{L^\infty(G)} \|v\|_{L^\infty(G)} \right)^{1/2} \]
\[ \|\sigma\|_{L^2(\hat{G} \times G/H)} \|g\|_{L^\infty(G/H)} \|L^2(G/H). \]

By multilinear interpolation theory, we get a unique bounded operator
\[ I : L^r(\hat{G} \times G/H) \times L^r(\hat{G} \times G/H) \to L^r(G) \]
such that
\[ \|I(\sigma, f)\|_{L'(G)} \leq c_1 \|f\|_{L'(G)} \|\sigma\|_{L'(\hat{G} \times G/H)}, \]
where
\[ c_1 = \left( \|g\|_{L^2(G/H)}^2 \|u\|_{L^\infty(G)} v_{L^1(G)} \right)^{\frac{1}{2}} \left( \|g\|_{L^2(G/H)} \|g\|_{L^\infty(G/H)} \|u\|_{L^\infty(G)} v_{L^\infty(G)} \right)^{1 - \frac{1}{2}}, \]
and \( \frac{\theta}{1} + \frac{1 - \theta}{2} = \frac{1}{2} \).

By definition of \( I \),
\[ \|P_{u,v,g}(\sigma)\|_{B(L'(G))} \leq c_1 \|\sigma\|_{L'(\hat{G} \times G/H)}. \]

Also \( P_{u,v,g}(\sigma) \) is the adjoint of \( P_{u,v,g}(\sigma) \), so \( P_{u,v,g}(\sigma) \) is a bounded linear operator on \( L'(G) \) with the operator norm
\[ \|P_{u,v,g}(\sigma)\|_{B(L'(G))} = \|P_{u,v,g}(\sigma)\|_{B(L'(G))} \leq c_2 \|\sigma\|_{L'(\hat{G} \times G/H)}, \]
where
\[ c_2 = \left( \|g\|_{L^2(G/H)} \|u\|_{L^1(G)} v_{L^\infty(G)} \right)^{\frac{1}{2}} \left( \|g\|_{L^2(G/H)} \|g\|_{L^\infty(G/H)} \|u\|_{L^\infty(G)} v_{L^\infty(G)} \right)^{1 - \frac{1}{2}}. \]

By interpolation theorem, for \( p \in [r, r'] \),
\[ \|P_{u,v,g}(\sigma)\|_{B(L^p(G))} \leq c_1 c_2^{1 - t} \|\sigma\|_{L'(\hat{G} \times G/H)}. \]

\[ \Box \]

**Theorem 5.8.** Let \( \sigma \) be in \( L'(\hat{G} \times G/H) \), \( r \in [1, 2] \) and \( u, v \in L'(G) \cap L^\infty(G) \). Then there exists a linear bounded operator \( P_{u,v,g}(\sigma) : L^p(G) \rightarrow L^p(G) \) for all \( p \in [r, r'] \) and we have
\[ \|P_{u,v,g}(\sigma)\|_{B(L^p(G))} \]
\[ \leq \|g\|_{L^\infty(G/H)} \|g\|_{L^r(G/H)} \left( \|u\|_{L^r(G)} \|v\|_{L^\infty(G)} \right)^{\frac{p}{r}} \left( \|v\|_{L^r(G)} \|u\|_{L^\infty(G)} \right)^{1 - \frac{p}{r}} \|\sigma\|_{L'(\hat{G} \times G/H)}, \]
where \( t = \frac{r - p}{p(r - 2)} \).

**Proof.** For any \( f \in L'(G) \) and \( h \in L'(G) \)
\[ \|(P_{u,v,g}(\sigma)(f), h)_{L^2(G)}\| \]
\[ \leq \int_{\hat{G} \times G/H} |\Delta(\omega, zH)||D^q_H(uf)(\omega, zH)||D^q_H(vh)(\omega, zH)| dm(\omega) dm_{G/H}(zH) \]
\[ \leq \|D^q_H(uf)\|_{L^\infty(\hat{G} \times G/H)} \|D^q_H(vh)\|_{L'(\hat{G} \times G/H)} \|\sigma\|_{L'(\hat{G} \times G/H)}. \]

By Equation (5)
\[ \|D^q_H(uf)\|_{L^\infty(\hat{G} \times G/H)} \leq \|g\|_{L^\infty(G/H)} \|u\|_{L^r(G)} \|f\|_{L^r(G)}. \]
By Equation (13)
\[ \| D^g_H(vh) \|_{L^r(\hat{G} \times G/H)} \leq \| g \|_{L^r(G/H)} \| v \|_{L^\infty(G)} \| h \|_{L^r(G)}. \]

So
\[ \| P_{u,v,g}(\sigma)(f), h \|_{L^2(G)} \]
\[ \leq \sigma \| L^r(\hat{G} \times G/H) \| g \|_{L^\infty(G/H)} \| u \|_{L^r(G)} \| f \|_{L^r(G)} \| g \|_{L^r'(G/H)} \| v \|_{L^\infty(G)} \| h \|_{L^r(G)}. \]

Thus
\[ \| P_{u,v,g}(\sigma) \|_{B(L^r'(G))} \leq \sigma \| L^r(\hat{G} \times G/H) \| g \|_{L^\infty(G/H)} \| u \|_{L^r(G)} \| g \|_{L^r'(G/H)} \| v \|_{L^\infty(G)}. \]

\[ \square \]

6. Compactness of generalized two wavelet multipliers

Our aim is to show the linear operators

\[ P_{u,v,g}(\sigma) : L^p(G) \rightarrow L^p(G) \]

are compact operators for all symbol \( \sigma \) in \( L^1(\hat{G} \times G/H) \).

**Proposition 6.1.** Under the same hypothesis of Theorem 5.3, the generalized two wavelet multiplier \( P_{u,v,g}(\sigma) : L^1(G) \rightarrow L^1(G) \) is compact.

**Proof.** Let \( f_n \in L^1(G) \) s.t. \( f_n \rightarrow 0 \) weakly in \( L^1(G) \). We need to show, \( P_{u,v,g}(\sigma)(f_n) \rightarrow 0 \) in \( L^1(G) \).

\[ \| P_{u,v,g}(\sigma)(f_n) \|_{L^1(G)} \leq \int_{\hat{G} \times G/H} \int_G |\sigma(\omega, zH)||f_n, g, zH u|_{L^2(G)}|g, zH(t)v(t)|dm_G(\omega)dm_{G/H}(zH). \]

Now
\[ \sigma(\omega, zH)||f_n, g, zH u|_{L^2(G)}|g, zH(t)v(t)| \leq C\| g \|_{L^\infty(G/H)}^2 \sigma(\omega, zH)\| u \|_{L^\infty(G)}|v(t)|. \]

By Dominated convergence theorem we conclude
\[ \lim_{n \rightarrow \infty} \| P_{u,v,g}(\sigma)f_n \|_{L^1(G)} = 0 \]

\[ \square \]

**Theorem 6.2.** Under the hypothesis of Theorem 5.3, the bounded operator

\[ P_{u,v,g}(\sigma) : L^p(G) \rightarrow L^p(G) \]

is compact for \( 1 \leq p \leq \infty \).
Proof. We know $P_{u,v,g}(\sigma) : L^\infty(G) \to L^\infty(G)$ is the adjoint operator of $P_{u,v,g}(\sigma) : L^1(G) \to L^1(G)$, which is compact by Proposition 6.1. Hence by interpolation operator theorem for compactness on $L^1(G)$ and on $L^\infty(G)$ can be extended to compactness of $P_{u,v,g}(\sigma) : L^p(G) \to L^p(G)$ for $1 < p < \infty$. \hfill \Box

7. Landau-Pollak-Slepian operator

Suppose $C_i, D$ and $\Omega$ are a compact neighbourhood of identity elements of $G$, $G/H$ and $\hat{G}$ respectively for $i = 1, 2$. Define the operators $Q_R$ and $P_{R_i}$ as

$$Q_R : L^2(G/H \times \hat{G}) \to L^2(G/H \times \hat{G}) \quad \text{and} \quad P_{R_i} : L^2(G/H \times \hat{G}) \to L^2(G/H \times \hat{G})$$

by

$$Q_R g(xH, \omega) = \chi_{D \times \Omega}(xH, \Omega) g(xH, \Omega)$$

and

$$P_{R_i} g(xH, \omega) = \mathcal{D}_H^g \left( \chi_{C_i}(x)(\mathcal{D}_H^g)^{-1} g(xH, \omega) \right).$$

Now we present some properties of the above operator in the next proposition.

**Proposition 7.1.** The operators $Q_R$ and $P_{R_i}$ are self-adjoint projections.

Proof. For $h, \phi \in G/H \times \hat{G}$, we have

$$\langle P_{R_i} h, \phi \rangle = \langle \mathcal{D}_H^g \left( \chi_{C_i}(x)(\mathcal{D}_H^g)^{-1} h \right), \phi \rangle$$

Now we can write the following in view of Theorem 2.3

$$\langle P_{R_i} h, \phi \rangle = \langle \chi_{C_i}(x)(\mathcal{D}_H^g)^{-1} h, (\mathcal{D}_H^g)^{-1} \phi \rangle \| g \|^2_{L^2(G/H)}$$

$$= \langle (\mathcal{D}_H^g)^{-1} h, \chi_{C_i}(x)(\mathcal{D}_H^g)^{-1} \phi \rangle \| g \|^2_{L^2(G/H)}$$

$$= \langle h, (\mathcal{D}_H^g)^{-1} \phi \rangle \chi_{C_i}(x)(\mathcal{D}_H^g)^{-1} \phi \rangle$$

$$= \langle h, P_{R_i} \phi \rangle.$$

Hence the operator $P_{R_i}$ is the self-adjoint operator. Also $P_{R_i}$ is the projection operator by noting

$$\langle P_{R_i}^2 h, \phi \rangle = \langle P_{R_i} h, P_{R_i} \phi \rangle$$

$$= \langle (\mathcal{D}_H^g \chi_{C_i}(x)(\mathcal{D}_H^g)^{-1} h), (\mathcal{D}_H^g \chi_{C_i}(x)(\mathcal{D}_H^g)^{-1} \phi) \rangle$$

$$= \| g \|^2_{L^2(G/H)} \langle \chi_{C_i}(x)(\mathcal{D}_H^g)^{-1} h, \chi_{C_i}(x)(\mathcal{D}_H^g)^{-1} \phi \rangle$$

$$= \| g \|^2_{L^2(G/H)} \langle \chi_{C_i}(x)(\mathcal{D}_H^g)^{-1} h, (\mathcal{D}_H^g)^{-1} \phi \rangle.$$
\[ = \langle D_H^g (\chi_{C_1}(x) (D_H^g)^{-1} h) , \phi \rangle = \langle P_R h, \phi \rangle. \]

Similarly, \( Q_R \) is also self-adjoint projection operator.

The linear operator \( P_R Q_R P_R \) : \( L^2(G/H \times \hat{G}) \to L^2(G/H \times \hat{G}) \) is called the generalized Landau-Pollak-Slepian operator. Now we will obtain the relation between the generalized Landau-Pollak-Slepian operator and the generalized two-wavelet multiplier.

**Theorem 7.2.** Let \( u \) and \( v \) are the functions of \( G \) defined by

\[ u = \frac{1}{\sqrt{|C_1|}} \chi_{C_1}(x) \quad \text{and} \quad v = \frac{1}{\sqrt{|C_2|}} \chi_{C_2}(x) \]

then the generalized Landau-Pollak-Slepian operator

\[ P_R Q_R P_R : L^2(G/H \times \hat{G}) \to L^2(G/H \times \hat{G}) \]

is unitary equivalent to a scalar multiple of the generalized tw-o-wavelet multiplier

\[ P_{u,v,g} (\chi_{D_2} \chi_{\Omega}) : L^2(G) \to L^2(G). \]

**Proof.** Clearly, \( \|u\|_{L^2(G)} = 1 = \|v\|_{L^2(G)} \). We have

\[ \langle P_{u,v,g} (\chi_{D_2} \chi_{\Omega}) f_1, f_2 \rangle_{L^2(G)} = \int_{\chi_{D_2} \chi_{\Omega}} D_H^g (u f_1)(\omega, zH) \overline{D_H^g (v f_2)(\omega, zH)} dm_{\hat{G}}(\omega) dm_{G/H}(zH), \]

where

\[ D_H^g (u f_1) = \int_G u(x) f_1(x) \overline{w(x) g(z^{-1} x H)} dm_G(x) \]

\[ = \frac{1}{\sqrt{|C_1|}} \int_{C_1} f_1(x) \overline{w(x) g(z^{-1} x H)} dm_G(x) \]

\[ = \frac{1}{\sqrt{|C_1|}} P_{R_1} (D_H^g (f_1)). \]

Hence we can write

\[ \langle P_{u,v,g} (\chi_{D_2} \chi_{\Omega}) f_1, f_2 \rangle_{L^2(G)} = \frac{1}{\sqrt{|C_1| C_2}} \int_{\chi_{D_2} \chi_{\Omega}} P_{R_1} (D_H^g (f_1)) \overline{P_{R_2} (D_H^g (f_2))} dm_{\hat{G}}(\omega) dm_{G/H}(zH) \]
\[ = \frac{1}{\sqrt{|C_1C_2|}} \int_{G/H \times \hat{G}} Q_R P_R_1 \left( D_H^g (f_1) \right) \overline{P_R_2 \left( D_H^g (f_2) \right)} \, dm (\omega) \, dm_{G/H} (zH) \]
\[ = \frac{1}{\sqrt{|C_1C_2|}} \langle Q_R P_R_1 \left( D_H^g (f_1) \right), P_R_2 \left( D_H^g (f_2) \right) \rangle_{L^2(G/H \times \hat{G})} \]
\[ = \frac{1}{\sqrt{|C_1C_2|}} \langle P_R_2 Q_R P_R_1 \left( D_H^g (f_1) \right), D_H^g (f_2) \rangle_{L^2(G/H \times \hat{G})} \]
\[ = \frac{\|g\|_{L^2(G/H)}^2}{\sqrt{|C_1C_2|}} \langle (D_H^g)^{-1} P_R_2 Q_R P_R_1 \left( D_H^g (f_1) \right), f_2 \rangle_{L^2(G/H \times \hat{G})} \]

\[ \square \]

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Department of Mathematics  
Indian Institute of Technology, Delhi, Hauz Khas  
New Delhi-110016, India  

Email address: swaraj.lie@gmail.com

Santosh Kumar Nayak:  
Department of Mathematics  
Indian Institute of Technology, Delhi, Hauz Khas  
New Delhi-110016  
India  

Email address: nayaksantosh212@gmail.com