A two-parametric deformation of $U[sl(2)]$, its representations and complex "spin"

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Abstract

A two-parametric deformation of $U[sl(2)]$ and its representations are considered. This newly introduced two-parametric quantum group denoted as $U_{p,q}[sl(2)]$ admits a class of infinite-dimensional representations which have no classical (non-deformed) and one-parametric deformation analogues, even at generic deformation parameters. Interestingly that finite-dimensional representations of $U_{p,q}[sl(2)]$ allow arbitrary complex "spins" (i.e., not necessary they to be integral or half-integral numbers), unlike those in the classical and one-parametric deformation cases.

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1. Introduction

Quantum groups are one of the most fascinating mathematical concepts with a physical origin [1] - [5]. Depending on points of view, quantum groups can be approached to in several ways. One of the approaches to quantum groups are the so-called Drinfel’d-Jimbo deformation of universal enveloping algebras [2] [4]. The quantum groups of this kind (also called quantum algebras) are non-commutative and non-cocommutative Hopf algebras [2]. By construction, such a quantum group depends on a, complex in general, parameter called a (quantum) deformation parameter. It is logical to ask if these one-parametric deformations can be extended to multi-parametric ones. This question was considered first by Manin [3] and later by a number of authors (see [6] - [16] and references therein). The structure of a multi-parametric deformation is usually richer than that of an one-parametric deformation. Unfortunately, in comparison with one-parametric deformations, multi-parametric deformations are less understood, in spite of some progress made in this direction. Therefore, the latter deserve more comprehensive investigations in both
the mathematical and physical aspects. Here we consider a relatively simple case, a two-parametric deformation of $U[sl(2)]$, which, however, has interesting structure and features.

The quantum group $U_q[sl(2)]$ as an one-parametric deformation of the universal enveloping algebra $U[sl(2)]$ is one of the best investigated quantum groups [17] - [21]. What about two-parametric deformations of $U[sl(2)]$, they have been considered in several versions and in different aspects (see, for example, [7] - [11] and references therein) though some of them are, in fact, equivalent to one-parametric deformations (upto some rescales). In this paper we introduce and consider one more two-parametric deformation of $U[sl(2)]$ denoted as $U_{p,q}[sl(2)]$. Following the method of highest-weight representations we can construct its representations. It turns out that this new quantum group admits a class of infinite-dimensional representations which have no analogue in the cases of the non-deformed $sl(2)$ and previously introduced deformations of $U[sl(2)]$. Moreover, the ”spin” (highest weight) corresponding to a finite-dimensional representation of $U_{p,q}[sl(2)]$ could be an arbitrary complex number, unlike the finite-dimensional representations of $sl(2)$ and its one-parametric deformations for which a ”spin” is an (half) integral number. We note that $U_{p,q}[sl(2)]$ is not equivalent to the one-parametric $U_q[sl(2)]$ unless at very special choices of $p$ and $q$ such as $p = q$.

In the next section, for compare, we briefly recall the non-deformed $sl(2)$ and its representations. The quantum group $U_{p,q}[sl(2)]$ is introduced and considered in Sect. 3. Some discussions and conclusions are made in the last section, Sect.4.

2. $sl(2)$ and representations

The algebra $sl(2)$ can be generated by three generators, say $E_+, E_-$ and $H$, subject to the commutation relations

$$[H, E_{\pm}] = \pm E_{\pm}, \quad [E_+, E_-] = 2H. \quad (1)$$

Demanding $(H)^\dagger = H$ and $(E_\pm)^\dagger = E_{\mp}$, a (unitary) representation induced from a (normalised) highest weight state $|j, j\rangle$ with a highest weight (”spin”) $j$,

$$H |j, j\rangle = j |j, j\rangle, \quad E_+ |j, j\rangle = 0, \quad (2)$$

has the matrix elements

$$H |j, m\rangle = m |j, m\rangle,$$

$$E_\pm |j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)} |j, m \pm 1\rangle, \quad (3)$$

where $|j, m\rangle$, $m \leq j$, is one of the orthonormalised states,

$$\langle j, m_1 | j, m_2 \rangle = \delta_{m_1 m_2}.$$
obtained from the highest weight state $|j, j\rangle$ by acting on the latter a monomial of the generator $E_-$ of an appropriate order, say $n$,

$$|j, m\rangle = A_n(E_-)^n |j, j\rangle, \quad n \in \mathbb{N}, \quad m = j - n,$$

(4) with $A_n$ a normalizing coefficient, which for a finite-dimensional representation (i.e., for a non-negative (half) integral $j$) equals

$$A_n = \sqrt{\frac{(2j)!}{(2j - n)!}}. \quad (5)$$

In this case, the representations constructed are simultaneously highest weight (above-bounded) and lowest weight (below-bounded), that is,

$$E_+ |j, j\rangle = 0, \quad E_- |j, -j\rangle = 0. \quad (6)$$

They are finite-dimensional (and also unitary and irreducible) representations of dimension $2j+1$. The situation is similar in the case of the one-parametric quantum group $U_q[sl(2)]$ at a generic deformation parameter $q$ (i.e., at $q$ not a root of unity) \[22\].

3. Two-parametric deformation $U_{p,q}[sl(2)]$

The two-parametric quantum group $U_{p,q}[sl(2)]$ as a two-parametric deformation of $U[sl(2)]$ is generated also by three generators $E_+, E_-$ and $H$ which now satisfy the deformed defining relations

$$[H, E_\pm] = \pm E_\pm, \quad [E_+, E_-] = [2H]_{p,q}, \quad (7)$$

where

$$[x]_{p,q} = \frac{q^x - p^{-x}}{q - p^{-1}}. \quad (8)$$

is a two-parametric deformation of $x$ (a number or an operator) with $p$ and $q$ being complex, in general, deformation parameters ($p^2 \neq q^2$). When $p = q$ this two-parametric deformation is reduced to the one-parametric deformation $U_q[sl(2)]$.

At generic $p$ and $q$, the center of $U_{p,q}[sl(2)]$ is spanned on the Casimir operator

$$C = \frac{1}{1 - q^{-2}}(k_q)^2 - \frac{1}{1 - p^2}(k_p^{-1})^2 + (q - p^{-1})E_-E_+, \quad (9)$$

where $k_q := q^H$, $k_p^{-1} := p^{-H}$. The latter can be used to quickly construct unitary representations of $U_{p,q}[sl(2)]$. We find the representations of $U_{p,q}[sl(2)]$ corresponding to those of $sl(2)$ in (3),

$$H |j, m\rangle = m |j, m\rangle,$$
When the unitary condition $(E_{\pm})^\dagger = E_{\mp}$ is imposed, the coefficient $A_n$ satisfies, instead of (4), the recurrent formula

$$\left| \frac{A_{n-1}}{A_n} \right|^2 = \frac{q^{2j-n+1}[n]_q - p^{-2j+n-1}[n]_p}{q - p^{-1}}, \quad n = j - m$$

if the latter is meaningful (i.e., if its r.h.s is a real positive number for some choice of $j$, $p$ and $q$). Generally speaking, however, the representations constructed are not unitary and we do not need this recurrent formula to obtain (10). These representations of $U_{p,q}[sl(2)]$, even at (half) integral $j$, are, in general, infinite-dimensional, unlike the constructed in a similar way representations of $sl(2)$ and other its deformations [22] which are finite-dimensional for non-negative (half) integral $j$. At arbitrary $p$ and $q$, the representations (10) of $U_{p,q}[sl(2)]$ are highest weight (by construction) but, as can be seen from (10), they may not be lowest weight (unbounded from below), i.e., not finite-dimensional any more, even for an integral $2j$. Therefore, it is a new class of infinite-dimensional representations of $U_{p,q}[sl(2)]$ not found before in the cases of $sl(2)$ and its previously considered deformations.

**Proposition 1:** Highest weight representations of the two-parametric quantum group $U_{p,q}[sl(2)]$ given in (10) are in general infinite-dimensional, even for non-negative (half) integral highest weights.

The next interesting phenomenon is related to the finite-dimensional representations. The representations (10) are finite-dimensional if the matrix element of $E_-$ vanishes for some value of $n \equiv j - m$ which is a non-negative integer ($n \in \mathbb{N}$).

**Proposition 2:** For a given, not necessary (half) integral, $j$, the representation (10) is reducible and contains a finite-dimensional subrepresentation iff the equation

$$f(x) \equiv q^{2j-x}[x + 1]_q - p^{-2j+x}[x + 1]_p = 0$$

has a non-negative integral solution.

The request for an integral solution makes the last equation resembling an equation of Diophantine type (the difference here is the coefficients in (11) are not necessarily integral). In general, to prove the Eq. (11) to have or not an integral solution (and when) is a hard mathematical problem which needs further investigations. Suppose $x = \mathcal{N}$ is the smallest non-negative integer solving Eq. (11), the dimension of the
corresponding finite-dimensional representation extracted from (10) is \( D = N + 1 \) which may not equal \( 2j + 1 \). Contrarily, we can choose \( j \) to get a representation of a given finite dimension.

**Proposition 3:** For a representation of a given finite dimension \( D \) the highest weight is given by

\[
2j = \log_{pq} \left( \frac{[D]_p}{[D]_q} \right) + D - 1 \quad \text{or} \quad 2j = \frac{\ln \left( \frac{[D]_p}{[D]_q} \right)}{\ln(pq)} + D - 1
\]

(12) if the logarithms here are well-defined.

When the logarithms in (12) are not well-defined we should keep the Eq. (11) and solve it for \( j \) with a given value of \( x = D - 1 \in \mathbb{N} \). The feature is \( 2j \), if found, may not be a non-negative integer at all, but an arbitrary complex number, \( 2j \in \mathbb{C} \).

The representations (10) have a relatively simple structure at generic \( p \) and \( q \). In the case of one or both parameters being roots of unity, the Casimir operator (9) may not exhaust an (extended) center of \( U_{p,q}[sl(2)] \) anymore and the representation structure becomes in general more complex (but sometimes the representation structure is similar to that at generic parameters). This interesting case deserves to be separately investigated in details.

Note that \( U_{p,q}[sl(2)] \) is by no means equivalent to the one-parametric \( U_q[sl(2)] \) (unless at very special choices of \( p \) and \( q \)) and other previously deformations of \( U[sl(2)] \).

4. Conclusion

The quantum group \( U_{p,q}[sl(2)] \) which is a two-parametric deformation of \( U[sl(2)] \) has been introduced and its representations have been investigated. This quantum group, even at generic deformation parameters, has interesting features. It is showed that \( U_{p,q}[sl(2)] \) admits a class of infinite-dimensional representations which have no analogues in the case of \( sl(2) \) and its previously considered deformations [22]. It is a new phenomenon of \( U_{p,q}[sl(2)] \). In general, the representations found are irreducible. They may become reducible under certain circumstances and then finite-dimensional irreducible subrepresentations can be extracted. Another feature is the highest weight (the "spin") of a finite-dimensional representation of \( U_{p,q}[sl(2)] \) is not necessarily a (half) integral number but a complex one, unlike that of \( sl(2) \) and its one-parametric deformation. This fact may have an interesting physical interpretation. Other classes of infinite - dimensional representations of \( U_{p,q}[sl(2)] \) and applications may be found by using the methods of [23, 24].

The next problem one could consider is \( U_{p,q}[sl(2)] \) at roots of unity. It is also
interesting to investigate this quantum group and its co-algebra structure in the light of the $R$-matrix formalism and to look for possible associated integrable models. Some of these problems turn out to be harder than expected at first sight. In this spirit, the results obtained in the present paper are far from being complete but we think they could be of independent interest.

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