WILD CHARACTER VARIETIES, POINTS ON THE RIEMANN SPHERE AND CALABI’S EXAMPLES

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1. Introduction

Wild character varieties are moduli spaces of monodromy data of connections on bundles on smooth algebraic curves. They were shown to admit holomorphic symplectic structures in [B99, B01a], to admit (complete) hyperkähler metrics in [BB04], and to arise as finite dimensional quasi-Hamiltonian quotients in [B02b, B07, B14a]. A simple example was shown to underlie the Drinfeld–Jimbo quantum group in [B01b] (as conjectured in [B99, B01a]) and further it was shown in [B02a] that Lusztig’s symmetries (a.k.a. the quantum Weyl group generators) are the quantization of a simple example of a wild mapping class group action on a wild character variety.

The wild character varieties generalize the tame character varieties, which are moduli spaces of monodromy data of regular singular connections, i.e. spaces of representations of the fundamental group. The extra monodromy data, enriching the fundamental group representation, needed to classify irregular connections is known as “Stokes data”. There are at least two approaches to Stokes data. One approach “Stokes structures” (due to Deligne [Del78, Ber80, Mal83, Mal91, DMR07], building on work of Malgrange, Sibuya and others) involves adding flags on sectors at each pole, measuring the possible exponential growth rates of solutions. In general it is complicated to classify such flags. A different, but algebraically equivalent, approach was developed by Balser–Jurkat–Lutz [Jur78, BJL79], Martinet–Ramis [MR91], Loday–Richaud [LR94] and others. It involves canonical Stokes matrices and leads to the notion of “Stokes local system”. This approach was extended to arbitrary reductive groups $G$ in [B02a] and used in the description of the wild character varieties as multiplicative symplectic quotients.

The aim of this article is to describe a simple class of examples of wild character varieties (studied in depth by Sibuya [Sib75]) from both points of view, to illustrate this dichotomy. In these examples it is not so difficult to directly bridge the gap between the two viewpoints. A key point is that for connections on rank two bundles the flags amount to points of the Riemann sphere, and so in simple cases the wild character varieties are specific moduli spaces of points on $\mathbb{P}^1$, studied by Sibuya when he considered the distinguished “subdominant” solutions in sectors at the poles.

For example we will explain the following theorem. Let

$$\mathcal{M}_{2k}^{\text{Sibuya}} = \{p_1, \ldots, p_{2k} \in \mathbb{P}^1(\mathbb{C}) \mid p_1 \neq p_2 \neq \cdots \neq p_{2k} \neq p_1\}/\text{PSL}_2(\mathbb{C})$$
be the moduli space of $2k$-tuples of points of the Riemann sphere such that cyclically-consecutive points are distinct. The prescription

$$\varphi(p_1, \ldots, p_{2k}) = (-1)^k \frac{(p_1 - p_2)(p_3 - p_4) \cdots (p_{2k-1} - p_{2k})}{(p_2 - p_3)(p_4 - p_5) \cdots (p_{2k} - p_1)}$$

gives a well-defined map $\varphi : \mathcal{M}_{2k}^{\text{Sibuya}} \to \mathbb{C}^*$, generalizing the cross-ratio, and the subvarieties $\mathcal{M}_{2k}^{\text{Sibuya}}(q) := \varphi^{-1}(q)$ have dimension $2k - 4$ for any $q \in \mathbb{C}^*$.

**Theorem 1.** For any $q \in \mathbb{C}^*$ the space $\mathcal{M}_{2k}^{\text{Sibuya}}(q)$ is a wild character variety (by [Sib75]), and it is complex symplectic (by [B01a]). If $q \neq 1$ it is smooth, and it admits a complete hyperkähler metric (by [BB04]).

If $k = 3$ (i.e. 6-tuples of points) then each space $\mathcal{M}_{2k}^{\text{Sibuya}}(q)$ has real dimension four. Physicists refer to such complete hyperkähler manifolds as “gravitational instantons”. However these examples were not known to physicists. On the other hand the underlying complex algebraic surfaces have appeared frequently, in relation to the second Painlevé equation, as will be explained.

The layout of this article is as follows. Section 2 gives the direct “canonical Stokes matrices” approach to a simple class of wild character varieties, and explains how they arise as finite dimensional multiplicative symplectic quotients. Section 3 then recalls the direct approach to the spaces of points on $\mathbb{P}^1$ considered by Sibuya and then gives the direct proof of the isomorphism between the two approaches. Next Section 4 describes the quiver approach and shows that these wild character varieties are multiplicative analogues of a family of hyperkähler manifolds introduced by Calabi. Finally Section 5 relates these examples to a 1764 paper of Euler, and shows that Euler’s continuant polynomials are group valued moment maps.

**2. Abelian Fission Spaces and Wild Character Varieties**

The quasi-Hamiltonian approach involves constructing the wild character varieties as finite dimensional *multiplicative* symplectic quotients. The symplectic/Poisson structure on the wild character variety then arises algebraically from the quasi-Hamiltonian two-form upstairs. For the present examples only some simple quasi-Hamiltonian spaces will be needed.

Let $G = \text{GL}_2(\mathbb{C})$ and consider the following subgroups:

$$U_+ = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \subset B_+ = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \subset G \supset B_- = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \supset U_- = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix},$$

and $T = B_- \cap B_+$ the diagonal subgroup. The wild character varieties we will consider are as follows. Choose an integer $k \geq 1$ and consider the variety

(1) $$\mathcal{M}_B = \{ S \in (U_+ \times U_-)^k \mid S_{2k} \cdots S_2 S_1 \in T \} / T$$

where $T$ acts by diagonal conjugation, and we take the affine geometric invariant theory quotient (the affine variety associated to the ring of $T$-invariant functions).
Here $S = (S_1, \ldots, S_{2k})$ with $S_{\text{even}} \in U_-$ and $S_{\text{odd}} \in U_+$. Further, for any fixed $t \in T$ of determinant one, consider the subvariety
\begin{equation}
\mathcal{M}_B(t) = \{ S \in (U_+ \times U_-)^k \mid S_{2k} \cdots S_2 S_1 = t \}/T \subset \mathcal{M}_B
\end{equation}
which has dimension $2k - 4$. It is a hypersurface in $\mathcal{M}_B$.

To obtain these wild character varieties as multiplicative symplectic quotients consider the smooth affine variety
\begin{equation}
\mathcal{A} = G\mathcal{A}_T^k := G \times (U_+ \times U_-)^k \times T.
\end{equation}

In this section we will explain and illustrate the following result.

**Theorem 2** ([B02b]). The abelian fission space $\mathcal{A} = G\mathcal{A}_T^k$ is an algebraic quasi-Hamiltonian $G \times T$-space.

This result means there is an action of $G \times T$ on $\mathcal{A}$, an invariant algebraic two-form $\omega$ on $\mathcal{A}$, and a group-valued moment map
\begin{equation}
\mu = (\mu_G, \mu_T) : \mathcal{A} \to G \times T;
\end{equation}
\begin{equation}
\mu_G(C, S, h) = C^{-1} h S_{2k} \cdots S_2 S_1 C \in G,
\end{equation}
\begin{equation}
\mu_T(C, S, h) = h^{-1} \in T
\end{equation}
satisfying various axioms, which are multiplicative analogues of the usual axioms for a Hamiltonian $G \times T$-space. The formula for $\omega$ is deferred to Remark 6 below. In particular we will consider the reduction by $G$
\begin{equation}
\mathcal{B} = \mathcal{B}^k := G\mathcal{A}_T^k \!\!/G = \mu_G^{-1}(1)/G,
\end{equation}
which is a smooth affine variety of dimension $2k - 2$, with a residual action of $T$. Some immediate consequences of Theorem 2, and the general quasi-Hamiltonian/quasi-Poisson yoga [AMM98, AKSM02], are as follows.

**Corollary 3.** 1) The quotient
\begin{equation}
\mathcal{A}/G \cong (U_+ \times U_-)^k \times T
\end{equation}
is an algebraic Poisson manifold with symplectic leaves
\begin{equation}
\mathcal{A} \!\!/ G = \mu_G^{-1}(C)/G
\end{equation}
for conjugacy classes $C \subset G$,

2) $\mathcal{B} = \mathcal{A} \!\!/ G$ is an algebraic symplectic manifold,

3) The quotient by $T$
\begin{equation}
\mu_G^{-1}(C)/(G \times T)
\end{equation}
of any symplectic leaf from 1) is a Poisson variety with symplectic leaves $\mu^{-1}(C \times \{t\})/(G \times T)$, for elements $t \in T$.

\[1\text{In fact [B02b] proves this for arbitrary complex reductive groups } G \text{ (with } B_{\pm} \text{ opposite Borels). The spaces } \mathcal{A} \text{ are denoted } C/L \text{ in [B02b] Rmk 4 p.6. The “non-abelian” extension appears in [B14a] (with } B_{\pm} \text{ replaced by arbitrary opposite parabolics, and } T \text{ by their common Levi subgroup).} \]
4) For any $t \in T$ the reduction $\mathcal{B} \sslash_t T \cong \mu^{-1}([1] \times \{t\})/G \times T$ is a symplectic variety.

Proof. 1) is a general result about quasi-Poisson manifolds; the quotient is a quasi-Poisson $T$-space, which means it is Poisson since $T$ is abelian. The leaves are the quasi-Hamiltonian reductions by $G$ (and are a priori quasi-Hamiltonian $T$-spaces, but this implies they are symplectic since $T$ is abelian). 2) is a special case of 1), taking $\mathcal{C} = \{1\} \subset G$. 3) follows the same pattern as in 1), considering the full action of $G \times T$. 4) is a special case of 3). □

Now it is easy to obtain the wild character varieties from the fission spaces. The action of $G \times T$ on $\mathcal{A}$ is given by

$$(g, t)(C, S, h) = (tCg^{-1}, tSt^{-1}, h)$$

where $(g, t) \in G \times T$ and $tSt^{-1} = (tS_1t^{-1}, \ldots, tS_kt^{-1})$. Thus

$$\mathcal{B} = \mathcal{A} \sslash G = \mu_G^{-1}(1)/G$$

(3)

$$= \{(S, h) \in (U_+ \times U_-)^k \times T \mid hS_2 \cdots S_2S_1 = 1\}$$

$$\cong \{S \in (U_+ \times U_-)^k \mid S_2 \cdots S_2S_1 \in T\}$$

which is a quasi-Hamiltonian $T$-space with moment map $h^{-1}$, and in particular a symplectic manifold (as $T$ is abelian). In turn $\mathcal{M}_\mathcal{B} = \mathcal{B} / T$ is thus a Poisson variety, with symplectic leaves

$$\mathcal{M}_\mathcal{B}(t) = \mathcal{B} \sslash_t T = \{(S, h) \in \mathcal{B} \mid h^{-1} = t\}/T.$$ (4)

Equivalently if $\mathcal{C} \subset G \times T$ is the conjugacy class containing $1 \times t \in G \times T$ then

$$\mathcal{M}_\mathcal{B}(t) = \mathcal{A} \sslash C \cong \mu^{-1}(\mathcal{C})/G \times T.$$ (5)

As mentioned above, the general quasi-Hamiltonian yoga implies that $\mathcal{M}_\mathcal{B}(t)$ is a symplectic variety: the restriction of $\omega$ to $\mu^{-1}(\mathcal{C}) \subset \mathcal{A}$ descends to give the symplectic form on $\mathcal{M}_\mathcal{B}(t)$.

The standard examples [AMM98] of quasi-Hamiltonian spaces are the conjugacy classes $\mathcal{C} \subset G$ (with the inclusion being the moment map), and the double $\mathcal{D} \cong G \times G$, which are multiplicative analogues of the coadjoint orbits $\mathcal{O} \subset \mathfrak{g}^*$ and the cotangent bundle $T^*G$ in the usual Hamiltonian world. Using the fusion and reduction processes these examples give a clean algebraic construction of the Atiyah–Bott symplectic/Poisson structure on spaces of fundamental group representations of Riemann surfaces [AMM98]. The fission spaces give an algebraic construction of the more general irregular Atiyah–Bott symplectic/Poisson structure of [B99, B01a] on spaces of monodromy/Stokes data. The holomorphic symplectic manifolds in 3),4) above are examples of this general construction. In these examples the symplectic form was then computed explicitly by Woodhouse [Woo01] and this led to the above quasi-Hamiltonian approach (which works for arbitrary numbers of poles on arbitrary genus Riemann surfaces).
In turn the irregular Atiyah–Bott complex symplectic quotients were upgraded into hyperkähler quotients in [BB04]. In the present context this implies:

**Theorem 4** ([BB04]). If \( t \neq 1 \) then \( \mathcal{M}_B(t) \) is a complete hyperkähler manifold of real dimension \( 4k - 8 \).

The condition \( t \neq 1 \) implies that all the points are stable, and so the spaces are smooth (cf. [BB04] §8). In the set-up of [B01a, BB04] this class of examples appears by considering meromorphic connections on rank two bundles on the Riemann sphere, having just one pole of order \( k + 1 \). We then use the irregular Riemann–Hilbert correspondence to pass to the wild character variety. (In general the hyperkähler metric will depend on the choice of irregular type of the connection at the pole—cf. also [B12]). Note that in the context of compact kähler manifolds, it is known that any holomorphic symplectic manifold admits a hyperkähler metric, but the situation is more subtle in the noncompact case. In the case \( k = 3 \), \( \mathcal{M}_B(t) \) is a complete hyperkähler manifold of real dimension four, i.e. a “gravitational instanton” in the physics terminology. We will see below, when discussing quivers, that they may be viewed as multiplicative analogues of the Eguchi–Hanson spaces (the \( A_1 \) ALE spaces).

**Remark 5.** In the case \( k = 3 \) it is easy to describe the complex surface \( \mathcal{M}_B(t) \) explicitly. It is isomorphic to the affine cubic surface

\[
x y z + x + y + z = b - b^{-1}
\]

where \( b \) is a non-zero constant such that \( t = \text{diag}(q^{-1}, q) \) and \( q = -b^2 \). These cubic surfaces appear in [FN80] (3.24). In more detail let \( s_i \) be the nontrivial off-diagonal matrix entry of \( S_i \). Then (cf. (2)) the equation \( S_n \cdots S_1 = t \) is equivalent to the three equations: \( s_1 = -q(s_3 s_4 + s_3 + s_5) \), \( s_6 = -q(s_3 s_4 + s_2 + s_4) \) (allowing to eliminate \( s_1 \) and \( s_6 \)), and \( s_2 s_3 s_4 + s_2 s_3 + s_2 s_5 + s_4 s_5 + 1 = 1/g \). To quotient by \( T \) we pass to invariants \( s_{23}, s_{25}, s_{34}, s_{45} \) where \( s_{ij} = s_i s_j \), and thus find \( s_{25} = 1/q - (1 + s_{45} + s_{23} + s_{23} s_{45}) \). Substituting this in the relation \( s_{23} s_{45} = s_{34} s_{25} \) yields the Flaschka–Newell surface (5) after relabelling \( s_{45} = x/b - 1, s_{23} = y/b - 1 \) and \( s_{34} = -1 - b z \). Note that Flaschka–Newell find these cubic surfaces as wild character varieties in a slightly different context (using a non-standard Lax pair for the Painlevé 2 equation), but their approach is known to be equivalent to the usual approach (cf. [JKT09]), explaining why the cubic surfaces are the same. See §5 for higher \( k \).

**Remark 6.** An explicit formula for the algebraic two-form \( \omega \) on \( \mathcal{A} \) is as follows. Define maps \( C_i : \mathcal{A} \to G \) by

\[
C_i = S_i \cdots S_2 S_1 C
\]

so that \( C = C_0 \), and let \( b = h S_{2k} \cdots S_2 S_1 \). Then

\[
2 \omega = (\overline{\tau}, \text{Ad}_b \tau) + (\overline{\tau}, \overline{\beta}) + (\overline{\tau}_{2k}, \eta) - \sum_{i=1}^{2k} (\gamma_i, \gamma_{i-1})
\]

where the brackets \((, )\) denote a fixed bilinear form on \( \mathfrak{g} \) and the Greek letters denote the following Lie algebra valued one-forms:

\[
\gamma_i = C_i^*(\theta), \quad \overline{\tau}_i = C_i^*(\overline{\theta}), \quad \eta = h^*(\theta_T), \quad \overline{\beta} = b^*(\overline{\theta})
\]
where $\theta = g^{-1}dg, \bar{\theta} = dg^{-1}$ are the left and right invariant Maurer–Cartan forms respectively. This expression equals that in [B02b].

3. Sibuya spaces

Sibuya [Sib75] studied the Stokes data at $\infty$ of differential equations of the form
\[
\frac{d^2y}{dz^2} = p(z)y
\]
for complex monic polynomials $p(z)$ of degree $m$. In brief there are $m+2$ distinguished directions at $\infty$ ("Stokes directions") and a preferred solution $v_i$ (the subdominant solution) on each sector between two consecutive Stokes directions. More precisely only the ray $\langle v_i \rangle$ spanned by $v_i$ is canonically determined. Since the (rank two) local system of solutions on the plane is trivial, this determines $m+2$ rays in $\mathbb{C}^2$, i.e. $m+2$ points of $\mathbb{P}^1$. Sibuya thus considered the following spaces and related them to Stokes matrices.

Let $n = m+2$ and let
\[
X_n^{\text{Sibuya}} = \{p_1, \ldots, p_n \in \mathbb{P}^1(\mathbb{C}) \mid p_1 \neq p_2 \neq \cdots \neq p_n \neq p_1\}
\]
be the configuration space of $n$-tuples of cyclically ordered points of $\mathbb{P}^1$ such that consecutive points are distinct. Let
\[
\mathcal{M}_n^{\text{Sibuya}} = X_n^{\text{Sibuya}} / \text{PSL}_2(\mathbb{C})
\]
be the geometric invariant theory quotient, which has dimension $n-3$. Note that there is an inclusion
\[
\mathcal{M}_{0,n} \subset \mathcal{M}_n^{\text{Sibuya}}
\]
where $\mathcal{M}_{0,n}$ is the moduli space of genus zero curves with $n$ distinct marked points.

Suppose now that $n$ is even and set $n = 2k$. Consider the function
\[
\varphi : X_{2k}^{\text{Sibuya}} \to \mathbb{C}^*
\]
defined by
\[
\varphi(p_1, \ldots, p_{2k}) = (-1)^k \frac{(p_1 - p_2)(p_3 - p_4) \cdots (p_{2k-1} - p_{2k})}{(p_2 - p_3)(p_4 - p_5) \cdots (p_{2k} - p_1)}.
\]
Up to a sign this is the "multiratio" of the $2k$ points (see e.g. [KS03]). It is invariant under diagonal Möbius transformations.

Choose an element $q \in \mathbb{C}^*$ and let
\[
\mathcal{M}_{2k}^{\text{Sibuya}}(q) = \varphi^{-1}(q) / \text{PSL}_2(\mathbb{C}) \subset \mathcal{M}_{2k}^{\text{Sibuya}}
\]
be the subvariety of points with fixed multiratio. It has dimension $2k-4$.

The stability condition for tuples of points on $\mathbb{P}^1$ is well-known ([Mum62]) and leads to the following
Lemma 7. If \( q \neq 1 \) then all the points are stable, and so \( \mathcal{M}^{\text{Sibuya}}_{2k}(q) \) is the set of PSL\(_2(\mathbb{C})\)-orbits in \( \varphi^{-1}(q) \subset X^{\text{Sibuya}}_{2k} \).

**Proof.** A \( 2k \) tuple \( p \) is stable if no point has multiplicity \( k \) or more. In the current set-up (with consecutive points distinct) the multiplicity is at most \( k \). Clearly if \( k \) of the points are equal (e.g. \( p_{\text{odd}} = 0 \) then \( \varphi(p) = 1 \). □

**Remark 8.** Note that if \( k = 3 \) then the condition \( \varphi(p) = 1 \) means that the multiratio of the 6 points is \(-1\), which is a condition much-studied, even classically (see [KS03] and references therein).

Our aim now is to explain the following (which follows easily from [Sib75]).

**Proposition 9.** The Sibuya moduli space \( \mathcal{M}^{\text{Sibuya}}_{2k} \) is algebraically isomorphic to the Poisson wild character variety \( \mathcal{M}_B \) in (1), and the multi-ratio function cuts out the symplectic leaves: \( \mathcal{M}^{\text{Sibuya}}_{2k}(q) \) is algebraically isomorphic to the wild character variety \( \mathcal{M}_B(t) \), where \( t = \text{diag}(q^{-1}, q) \).

As discussed in the previous section, standard results on wild character varieties then imply that \( \mathcal{M}^{\text{Sibuya}}_{2k}(q) \) is a symplectic variety, and even a complete hyperkähler manifold whenever \( q \neq 1 \).

**Proof.** Write \( n = 2k \) and consider the affine variety

\[
V = V_n := \{ v_1, \ldots, v_n \in \mathbb{C}^2 \setminus \{0\} \mid v_1 \parallel v_2 \parallel \cdots \parallel v_n \parallel v_1 \}
\]

where the symbol \( \parallel \) means “is not parallel to”. The torus \((\mathbb{C}^*)^n\) acts on \( V \) by scaling the vectors, and \( G := \text{GL}_2(\mathbb{C}) \) acts diagonally. The quotient is \( \mathcal{M}^{\text{Sibuya}}_n \). Viewing the \( v_i \) as column vectors define \( 2 \times 2 \) matrices

\[
\Psi_i = (v_i v_{i+1})
\]

(where the indices are taken modulo \( n \)). By assumption these are all invertible matrices. Thus following Sibuya we can define some matrices (“Stokes matrices”):

\[
B_i := \Psi_i^{-1} \Psi_{i-1}
\]

so that \( (v_{i-1} v_i) = (v_i v_{i+1}) B_i \) (cf. [Sib75] 21.32 p.86). Clearly, by construction,

\[
B_n \cdots B_2 B_1 = 1
\]

(cf. [Sib75] 21.31) and moreover

\[
B_i \in W := \left\{ \begin{pmatrix} * & 1 \\ * & 0 \end{pmatrix} \right\} \subset G
\]

(as in [Sib75] 21.30). This leads to the space

\[
W := \{ B_1, \ldots, B_n \in W \mid B_n \cdots B_2 B_1 = 1 \}
\]

and the procedure above defines a map \( \pi : V \to W \).
Lemma 10. The map $\pi : V \to W$ is a trivial principal $G$-bundle. In other words the action of $G$ on $V$ is free, has quotient $W$ and admits a global slice; $V \cong G \times W$.

Proof. Define an extended map $\tilde{\pi} : V \to G \times W$ by setting $C = (v_1, v_2) = \Psi_1 \in G$. The formula $\Psi_i = \Psi_{i+1}B_{i+1}$ shows that each $\Psi_i$ is determined by $C = \Psi_1$ and the Stokes matrices, and in turn the $\Psi_i$ determine the $v_i$, so $\tilde{\pi}$ is an isomorphism. Specifically $\Psi_i = CB_1B_nB_{n-1}\cdots B_{i+1}$. The condition (7) ensures this holds for all $i$ modulo $n$. \hfill \Box

This lemma holds even if $n$ is odd. To get to $\mathcal{M}_B$ we adjust the ordering of some of the basis vectors to put the Stokes data in alternating Borels. Suppose we swap the order of the columns of $\Psi_{2i}$, i.e. we redefine $\Psi_{2i}$ as $\Psi_{2i} = (v_{2i+1}, v_{2i})$ (and leave $\Psi_{\text{odd}}$ unchanged). Then if we set $G_i = \Psi_i^{-1}\Psi_{i-1}$ we have $G_{2i} \in B_-, G_{2i+1} \in B_+$. More precisely

$$G_{2i} = PB_{2i} \in W_- := \left\{ \begin{pmatrix} * & 0 \\ * & 1 \end{pmatrix} \right\},$$

$$G_{2i+1} = B_{2i+1}P \in W_+ := \left\{ \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \right\}$$

where $P = (0 \ 1 \ 1 \ 0)$. This allows to rewrite Lemma 10, showing that

$$V/G = W \cong W' := \{ G_1, \ldots, G_{2k} \in (W_+ \times W_-)^k \mid G_{2k} \cdots G_2G_1 = 1 \}.$$ 

Now it is easy to compute the formal monodromy, and discover the multi-ratio.

Lemma 11. The formal monodromy is $\text{diag}(q, q^{-1}) \in T$ where $q = \varphi(p_1, \ldots, p_{2k})$ is $(-1)^k$ times the multi-ratio of the points $p_i = \langle v_i \rangle \in \mathbb{P}^1$.

Proof. The formal monodromy is the monodromy of the associated graded local system (as in [Del78]), which here means we ignore the off-diagonal entries of the $G_i$ and only consider their diagonal entries. If we set $d(i, j) = \det(v_i v_j)$ then these nontrivial diagonal entries are

$$\det(G_i) = \frac{d(i-1, i)}{d(i+1, i)}$$

so that the formal monodromy is $\text{diag}(q, q^{-1})$ where

$$q = \det(G_{2k}) \det(G_{2k-2}) \cdots \det(G_2) = \frac{d(2k-1, 2k) \cdots d(3, 4) d(1, 2)}{d(1, 2k) \cdots d(5, 4) d(3, 2)}$$

$$= (-1)^k \frac{(p_1 - p_2)(p_3 - p_4) \cdots (p_{2k-1} - p_{2k})}{(p_2 - p_3)(p_4 - p_5) \cdots (p_{2k} - p_1)} = \varphi(p_1, \ldots, p_{2k}).$$ \hfill \Box

Finally we need to consider the induced action of the torus $\tilde{T} := (\mathbb{C}^*)^{2k}$ on $W'$ and show it reduces to an action of $T \cong (\mathbb{C}^*)^2$ as in the definition of $\mathcal{M}_B$. 
Write an element of $\tilde{T}$ as $c = (c_1, \ldots, c_{2k})$. This acts by scaling the $v_i$, and the induced action on the $G_i$ is:
\[
c \cdot G_{2i} = \text{diag}(c_{2i+1}, c_{2i}) G_{2i} \text{diag}(c_{2i-1}, c_{2i})^{-1},
\]
\[
c \cdot G_{2i+1} = \text{diag}(c_{2i+1}, c_{2i+2}) G_{2i+1} \text{diag}(c_{2i+1}, c_{2i})^{-1}.
\]
Thus in all cases the action on the nontrivial diagonal entry of $G_i$ is by multiplication by $c_{i+1}/c_{i-1}$. We can thus use up most of this torus action by setting most of the $G_i$ to be unipotent: Namely we can restrict to the subtorus $T \cong (\mathbb{C}^*)^2 \subset \tilde{T}$ where
\[
c_1 = c_3 = \cdots = c_{2k-1} \quad \text{and} \quad c_2 = c_4 = \cdots = c_{2k}
\]
and restrict to the subset $\mathcal{W}'' \subset \mathcal{W}$ where $G_2, G_3, \ldots, G_{2k-1}$ are unipotent (i.e. $\det(G_i) = 1$ if $i \neq 1, 2k$). Then the quotient $\mathcal{W}'/\tilde{T}$ is identified with the quotient $\mathcal{W}''/T$. If we write
\[
G_1 = S_1 \text{diag}(1, q^{-1})
\]
with $S_1 \in U_+$, and
\[
G_{2k} = \text{diag}(q, 1) S_{2k}
\]
with $S_{2k} \in U_-$ and set $S_i = G_i \in U_\pm$ otherwise, then the relation $G_{2k} \cdots G_1 = 1$ becomes
\[
h S_{2k} \cdots S_2 S_1 = 1
\]
where $h = \text{diag}(q, q^{-1})$ is the formal monodromy, so that
\[
\mathcal{W}'' \cong \{ S \in (U_+ \times U_-)^k \mid S_{2k} \cdots S_1 \in T \} = B.
\]
Moreover the $T$-action on $\mathcal{W}''$ matches the $T$-action in the definition (1) of $\mathcal{M}_B$, and so the proof is complete. \hfill $\Box$

Remark 12. This algebraic isomorphism is an example of the equivalence of categories between Stokes structures and Stokes local systems (both of which are equivalent to a category of connections, cf. [Del78], [B14a] Thm A.3). It is a consequence of the simple analytic fact that the columns of the canonical fundamental solutions (used to define the canonical Stokes matrices, as in [BJL79] Thm A) are consecutive subdominant solutions (when suitably scaled and ordered). Loday-Richaud’s algorithm [LR94] gives an algebraic procedure for translating between Stokes structures and Stokes local systems in general.

Remark 13. Note that Sibuya related the spaces $\mathcal{M}_r^{\text{Sibuya}}$ to Nevanlinna’s theory of Riemann surfaces (see [Sib75] p.ix).

4. Multiplicative quiver varieties

There is a standard way to associate hyperkähler manifolds to graphs [Kro89, Nak94], known as additive/Nakajima quiver varieties. Here we will briefly recall the algebraic approach to the underlying holomorphic symplectic varieties (cf. also [CS98, CB01]) and then discuss the multiplicative version relevant to the present
examples. Let $\Gamma$ be a graph with nodes $I$. Suppose we are given a vector space $V_i$ for each node $i \in I$. Then we can consider the vector space

$$\text{Rep}(\Gamma, V) = \bigoplus_{a \in \Gamma} \text{Hom}(V_{t(a)}, V_{h(a)})$$

of maps along each edge of the graph in both directions. We will call this the space of representations of the graph on the $I$-graded vector space $V = \bigoplus V_i$. Here $\Gamma$ is the set of oriented edges of $\Gamma$, i.e. the set of pairs $(e, o)$ where $e$ is an edge of $\Gamma$ and $o$ is one of the two possible orientations of $e$. Given an oriented edge $a \in \Gamma$, the head $h(a) \in I$ and tail $t(a) \in I$ are well-defined.

The group $H := \prod \text{GL}(V_i)$ acts on $\text{Rep}(\Gamma, V)$ via its natural action on $V$ preserving the grading. Further a choice of orientation of the graph $\Gamma$ determines a holomorphic symplectic structure on $\text{Rep}(\Gamma, V)$, and then the action of $H$ is Hamiltonian with a moment map

$$\tilde{\mu} : \text{Rep}(\Gamma, V) \to \mathfrak{h}^* = \text{Lie}(H)^* \cong \prod_{i \in I} \text{End}(V_i).$$

The Nakajima quiver varieties are defined by choosing a central value $\lambda \in \mathbb{C}^I$ of the moment map and taking the symplectic quotient:

$$\mathcal{N}(\Gamma, d, \lambda) = \text{Rep}(\Gamma, V) \sslash H = \{ \rho \in \text{Rep}(\Gamma, V) \mid \tilde{\mu}(\rho) = \lambda \}/H$$

where $\lambda$ is identified with the central element $\sum \lambda_i \text{Id}_{V_i}$ of $\text{Lie}(H)^*$. Here $d \in \mathbb{Z}^I$ denotes the dimension vector, with components $d_i = \dim(V_i)$, and the spaces are empty unless $\sum \lambda_i d_i = 0$. The quotient is the affine quotient, taking the variety associated to the ring of $H$ invariant functions (although often one adds an extra parameter choosing a nontrivial linearization as well—for simplicity here we won’t do this).

$$n = k - 1$$

**Figure 1.** The graph $\Gamma$.

If $\Gamma$ is an affine Dynkin graph and $d$ is the minimal imaginary root, then as discovered in [Kro89], $\mathcal{N}(\Gamma, d, \lambda)$ has complex dimension two, and is a deformation of the Kleinian singularity $\mathbb{C}^2/G(\Gamma)$, where $G(\Gamma) \subset \text{SL}_2(\mathbb{C})$ is the McKay group of $\Gamma$. For type $A$, these hyperkähler four-manifolds were known before: for $A_1$ they are the Eguchi-Hanson spaces [EH78] which in one complex structure are generic coadjoint orbits of $\text{SL}_2(\mathbb{C})$ (and in another they are $T^*\mathbb{P}^1$). Just after this example was found Calabi [Cal79] found examples in all dimensions: In one complex structure Calabi’s examples are minimal semisimple coadjoint orbits of $\text{SL}_n(\mathbb{C})$ (and in another they are
$T^*\mathbb{P}^{n-1}$). As quiver varieties, Calabi’s examples arise by considering the simple graph in Figure 1, with two nodes and $n$ edges (with dimension vector $d = (1, 1)$).

In this example $\text{Rep}(\Gamma, V)$ has dimension $2n$ and $\mathcal{N}$ is the symplectic reduction by a torus $H = (\mathbb{C}^*)^2$, and has complex dimension $2n - 2$ (the diagonal subgroup of $H$ acts trivially). The aim of the rest of this section is to explain the following statement:

The wild character varieties $\mathcal{M}_B(t)$ in (2) are multiplicative analogues of Calabi’s examples.

First of all note that if we set $k = n + 1$ then $\dim(\mathcal{M}_B(t)) = \dim(\mathcal{N}(\Gamma, d, \lambda)) = 2n - 2 = 2k - 4$.

Secondly note that $\dim(\mathcal{B}) = \dim(\text{Rep}(\Gamma, V)) = 2n$, and moreover:

**Lemma 14** (cf. [B13] Rmk 5.4). The space $\mathcal{B} = \mathcal{B}^k$ may be identified with a Zariski open affine subset of $\text{Rep}(\Gamma, V)$.

**Proof.** In brief $V = V_1 \oplus V_2$ with both $V_1, V_2$ one dimensional complex vector spaces, and $\rho \in \text{Rep}(\Gamma, V)$ consists of $n$ maps $V_1 \rightarrow V_2$ and $n$ maps $V_2 \rightarrow V_1$. Suppose we label these maps

$$s_2, s_4, \ldots s_{2n} : V_1 \rightarrow V_2, \quad s_1, s_3, \ldots s_{2n-1} : V_2 \rightarrow V_1.$$ 

Then we may identify $\text{Rep}(\Gamma, V)$ with $(U_- \times U_+)^n \subset \text{GL}(V)^{2n}$ by setting

$$S_{2i} = \begin{pmatrix} 1 & 0 \\ s_{2i} & 1 \end{pmatrix} \in U_- \quad \text{and} \quad S_{2i+1} = \begin{pmatrix} 1 & s_{2i+1} \\ 0 & 1 \end{pmatrix} \in U_+.$$ 

Now recall from (3) that, with $k = n + 1$

$$\mathcal{B} = \{ S \in (U_+ \times U_-)^k \mid (S_{2n+2}S_{2n+1})S_{2n} \cdots S_2S_1 \in T \}$$

(8)

$$\cong \{ S \in (U_+ \times U_-)^n \mid S_{2n} \cdots S_2S_1 \in G^o \}$$

where $G^o = U_+TU_- = U_+U_-T \subset G$ is the opposite Gauss cell. Thus (8) and the Gauss decomposition says that

$$\mathcal{B} \cong \{ \rho = (s_1, \ldots s_{2n}) \in \text{Rep}(\Gamma, V) \mid (S_{2n} \cdots S_2S_1)_{22} \neq 0 \}$$

so that $\mathcal{B}$ is indeed a Zariski open affine subset of $\text{Rep}(\Gamma, V)$.

Now define the invertible representations of the graph $\Gamma$ to be this open subset:

$$\text{Rep}^*(\Gamma, V) = \{ \rho \in \text{Rep}(\Gamma, V) \mid S_{2n} \cdots S_2S_1 \in G^o \} \subset \text{Rep}(\Gamma, V).$$

Thus, noting that $H = T$ is the maximal torus of $G = \text{GL}(V)$, and that $\mathcal{B}$ is a quasi-Hamiltonian $T$-space (cf. (3)), there is a group-valued moment map

(9)

$$\mu : \text{Rep}^*(\Gamma, V) \rightarrow H$$

defined by taking the $T$ component of $S_{2n} \cdots S_2S_1 \in G^o$. Beware $\mu$ is not the restriction of the usual moment map $\tilde{\mu} : \text{Rep}(\Gamma, V) \rightarrow \text{Lie}(H)$. 

 españoles
In turn it is natural to define the multiplicative quiver varieties of $\Gamma$ to be the multiplicative symplectic reductions of $\text{Rep}^* (\Gamma, V)$ at central values of the moment map. Namely we choose $q \in (\mathbb{C}^*)^I$ and define

$$\mathcal{M}(\Gamma, d, q) = \text{Rep}^* (\Gamma, V) \sslash H = \mu^{-1}(q)/H.$$  

These will be empty unless $q^d := \prod q_i^{d_i} = 1$, which in the current setup means $q_1q_2 = 1$. Of course, this is just a rephrasing of the construction of the wild character variety $\mathcal{M}_B(t)$ in (4), with $t = \text{diag}(q_1, q_2)$, so that

$$\mathcal{M}(\Gamma, d, q) \cong \mathcal{M}_B(t)$$

but now we see the link to graphs, and thus to Kac–Moody root systems. (Many other examples appear in [B13].) In the simplest nontrivial example $k = 3$ the graph $\Gamma$ is the affine $A_1$ Dynkin graph (with two edges) and so the corresponding wild character varieties are multiplicative analogues of the Eguchi–Hanson spaces. Note that Okamoto [Oka92] already related the second Painlevé equation to the affine $A_1$ Weyl group (cf. also [B09] Exercise 3 and [B08] Appendix C).

If we are prepared to work analytically then a more direct link between the additive and multiplicative quiver varieties is available. Namely the Riemann–Hilbert–Birkhoff map plays a role analogous to the exponential map:

**Theorem 15.** The Riemann–Hilbert–Birkhoff map, taking a connection to its monodromy data, gives a holomorphic map from the additive quiver varieties for the graph $\Gamma$ to the corresponding multiplicative quiver varieties. It relates the holomorphic symplectic structures (but not the hyperkähler metrics).

**Proof.** This follows from [B01a] Thm 6.1 modulo relating both sides to quivers: on the multiplicative side this follows from the discussion above plus the quasi-Hamiltonian approach in [B02b] (see especially Cor. p.3). The general dictionary relating the additive side to Nakajima quiver varieties was written down (and established in some cases) in [B08] Appendix C and justified in general in [HY13].

In more detail recall the diagram of moduli spaces (from [B01a] (29) p.181):

$$\begin{align*}
\tilde{\mathcal{M}}_{\text{DR}} & \xrightarrow{\cong} \tilde{\mathcal{A}}_B/G_1 \\
\bigcup \tilde{O}_1 \times \cdots \times \tilde{O}_m \sslash G & \cong \tilde{\mathcal{M}}^*_{\text{DR}} \xrightarrow{\nu} \tilde{\mathcal{M}}_B.
\end{align*}$$

Here $\tilde{\mathcal{M}}^*_{\text{DR}}$ is a moduli space of framed meromorphic connections on the trivial bundle on the Riemann sphere with $m$ poles, and $\tilde{\mathcal{M}}_B$ is the corresponding space of monodromy/Stokes data. (See [B01a] for the full definitions.) Thm 6.1 of [B01a] says that the map $\nu$ taking monodromy/Stokes data is symplectic. In turn in [B02b] the space $\tilde{\mathcal{M}}_B$ is identified (as a symplectic manifold) with a fusion product

$$\tilde{\mathcal{M}}_B \cong \tilde{C}_1 \otimes \cdots \otimes \tilde{C}_m \sslash G$$
for certain quasi-Hamiltonian $G \times T$ spaces $\tilde{C}_i$. Thus the (framed) Riemann–Hilbert–Birkhoff map is a symplectic map

$$\tilde{O}_1 \times \cdots \times \tilde{O}_m \rightarrow \tilde{C}_1 \otimes \cdots \otimes \tilde{C}_m / G.$$ 

It is injective and intertwines the $T$-actions changing the framings. (This injectivity is due to [JMU81]—it was extended to a bijective correspondence $\tilde{M}_{\text{DR}} \cong \tilde{M}_{\text{B}}$ in [B01a] Cor. 4.9.) Now if we specialise to the case $m = 1$ with just one pole and use [B01a] Lem. 2.4 to “decouple”

$$\tilde{O}_1 \cong O_B \times T^* G$$

the space $\tilde{O}_1$, then we obtain a $T$-equivariant symplectic map

$$O_B \rightarrow \tilde{C}_1 / G.$$ 

Here $O_B$ is a coadjoint orbit of the (unipotent) group of based jets of bundle automorphisms, and thus has global Darboux coordinates (as noted in [B01a] p.190). The conjecture of [B08] Appendix C proved in [HY13] identifies $O_B$ (as a Hamiltonian $T$-space) with a space of graph representations, and thus the reduction by $T$ with a Nakajima quiver variety. If the group $G$ is $\text{GL}_2$ then the graph is the graph $\Gamma$ of Figure 1 if the connections have poles of order $k + 1$. On the other side we now just need to identify the $T$ reductions of $\tilde{C}_1 / G$ with the $T$ reductions of $\mathcal{B}$, which is immediate from the definitions. \qed

5. Eulerian hypersurfaces

On p.55 of Euler’s 1764 article “Specimen Algorithmi Singularis” [Eul64] the reader will find the list of polynomials:

$$(a) = a$$

$$(a, b) = ab + 1$$

$$(a, b, c) = abc + c + a$$

$$(a, b, c, d) = abcd + cd + ad + ab + 1$$

$$(a, b, c, d, e) = abcde + cde + ade + abc + e + c + a$$

etc.

They appear when computing continued fractions and nowadays they are known as “Euler’s continuants”. The latter monomials in the $n$th continuant arise by forgetting all possible pairs of consecutive letters from the first monomial. Recalling Remark 5 we thus see that, for $k = 3$, the space $\mathcal{M}_B(t)$ is the quotient of the “Eulerian hypersurface”

$$(s_2, s_3, s_4, s_5) = 1/q$$
by the action of $\mathbb{C}^*$, and the expressions for $s_1, s_6$ involve continuants of degree one less. Moreover the quasi-Hamiltonian $T$-space $B$ is isomorphic to the open subset

$$(s_2, s_3, s_4, s_5) \neq 0$$

of $\mathbb{C}^4$ and the continuant $(s_2, s_3, s_4, s_5)$ is a group-valued moment map for the $\mathbb{C}^*$ action on this subset. The aim of this section is to point out that this all holds for any $k$:

**Proposition 16.** For any $k$ the quasi-Hamiltonian $T$-space $B = B^k$ is isomorphic to the open subset

$$(s_2, s_3, \ldots, s_{2k-1}) \neq 0$$

of $\mathbb{C}^{2k-2}$, the continuant $(s_2, s_3, \ldots, s_{2k-1})$ is a group-valued moment map for the $\mathbb{C}^*$ action on this subset and the wild character variety $\mathcal{M}_B(t)$ is the quotient of the Eulerian hypersurface

$$(s_2, s_3, \ldots, s_{2k-1}) = 1/q$$

by the action of $\mathbb{C}^*$.

**Proof.** The continuants can be defined (see e.g. [Knu98]) by the recurrence

$$(x_1, x_2, \ldots, x_n) = x_1(x_2, x_3, \ldots, x_n) + (x_3, x_4, \ldots, x_n)$$

and using this one can easily show

$$(11) \quad \begin{pmatrix} x_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} x_n & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} (x_1, \ldots, x_n) & (x_1, \ldots, x_{n-1}) \\ (x_2, \ldots, x_n) & (x_2, \ldots, x_{n-1}) \end{pmatrix}.$$ 

Now using the Gauss decomposition $B$ is isomorphic to

$$\{(S_2, \ldots, S_{2k-1}) \in (U_- \times U_+)^{k-1} \mid (S_{2k-1} \cdots S_2)_{11} \neq 0\}.$$ 

Thus the key point is to show that $(S_{2k-1} \cdots S_2)_{11} = (s_2, s_3, \ldots, s_{2k-1})$ where $s_i$ is the active off-diagonal entry of $S_i$. But this is a direct consequence of (11), upon noting that $PS_2 = (s_2, 1)_{11}$ and $S_3P = (s_3, 1)_{11}$ etc., where $P = (0, 1, 0)$.

**Remark 17.** Although it is beyond the scope of this article let us mention that this gives a presentation of the fission algebra $\mathcal{F}^q(\Gamma)$ of the graph $\Gamma$: it is isomorphic to the quotient of the path algebra $P(\overline{\Gamma})$ of the quiver $\overline{\Gamma}$ by the relations

$$(a_1, b_1, a_2, b_2, \ldots, a_n, b_n)e_1 = q_1e_1$$

$$(b_n, a_n, \ldots, b_2, a_2, b_1, a_1)e_2 = q_2^{-1}e_2$$

where $q_i \in \mathbb{C}^*$, the $a$’s are the arrows to the left, the $b$’s are the arrows to the right and $e_1/e_2$ is the idempotent for the left/right node respectively. Here the ordering of the symbols in the continuants is as written by Euler. For $n = 1$ one gets the multiplicative preprojective algebra $\Lambda^q$ of [CBS06] (which in this case is isomorphic to a deformed preprojective algebra), but in general $\mathcal{F}^q \neq \Lambda^q$ (cf. [B13] Rmk 6.10).
Conclusion

We have described four ways to think about a simple class of wild character varieties. In these examples the structure group was $\text{GL}_2(\mathbb{C})$ and the underlying curve was the complex plane—the connections just had one singularity, at $\infty$, and we assumed there was an even number of Stokes directions at the singularity. Clearly these spaces admit lots of generalizations, and we hope they serve as a helpful introduction to the more general moduli spaces constructed in [B01a, B02b, BB04, B14a], complementing the survey [B14b]. As mentioned above, Loday-Richaud's algorithm [LR94] gives an algebraic procedure for translating between Stokes structures and Stokes local systems in general. Note that under this dictionary the spaces constructed in [B01a, B02b] involve full flags, whereas the spaces constructed in [BB04, B14a] involve arbitrary partial flags. In a subsequent article we will describe in detail the “odd” case (also considered by Sibuya)—this is in some sense simpler since the spaces $\mathcal{M}^{\text{Sibuya}}_{2k+1}$ are smooth symplectic varieties directly, with the same expression for the two-form $\omega$. Moreover the analysis of [BB04] extends directly (as pointed out in [Sab02, Wit08]) to show they are complete hyperkähler manifolds. In another direction a nice class of examples generalizing the present ones come from the following graphs (bearing in mind the dictionary in [B08] Apx C, [B13] §3.3). In this case the additive quiver varieties are arbitrary coadjoint orbits of $\text{GL}_n(\mathbb{C})$ (and in special complex structures they are cotangent bundles of flag varieties via [Nak94] §7, deframed in the sense of [CB01] p.261).

\begin{center}
\begin{tikzpicture}
\node (1) at (0,0) {$1$};
\node (2) at (1,0) {$d_1$};
\node (3) at (2,0) {$d_2$};
\node (4) at (3,0) {$d_{r-1}$};
\node (5) at (4,0) {$d_r$};
\node (n) at (1.5,1) {$n$};
\draw (1) -- (2);
\draw (2) -- (3);
\draw (3) -- (4);
\draw (4) -- (5);
\end{tikzpicture}
\end{center}

Acknowledgments. Thanks are due to Alistair Scott MacLeod for providing a copy of [Knu98] at an opportune moment.

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