Particle Acceleration in Underdense Plasmas

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Abstract
An effective theory of laser-plasma based particle acceleration is presented. Here we treated the plasma as a continuous medium with an index of refraction $n_m$ in which a single electron propagates. Because of the simplicity of this model, we did not need to perform PIC simulations in order to study the properties of the electron acceleration. We studied the properties of the electron motion due to the Lorentz force and the relativistic equations of motion were numerically solved and analysed. We found that if the plasma density is much below the critical density then the acceleration of the electrons in plasma ($n_m < 1$) can be quite well approximated with the vacuum model ($n_m = 1$).

1 Introduction

In an early work [Tajima and Dawson (1979)] showed that an intense laser pulse is capable to create large amplitude waves in plasmas due to the ponderomotive force. Their studies were based both on analytic calculations and computer simulations. These wakes are responsible for the high energy gain of the accelerated electrons. This basic idea opens a new horizon of building compact particle accelerators. Tajima and Dawson suggested two methods for electron acceleration in plasmas: the first one is based on beating two monochromatic laser pulses and the second one needs the usage of ultra-intense monochromatic pulses. The names of these methods are Plasma Beat Wave Acceleration (PBWA) and Plasma Wakefield Acceleration (PWFA), respectively. The key point is the resonant excitation of the plasma in order to create large amplitude plasma waves. In PBWA the frequency of the beat wave, which is the difference of the laser frequencies $\omega_1$ and $\omega_2$ has to match the plasma frequency: $\omega_1 - \omega_2 = \omega_p$, while in PWFA the $\omega_L$ frequency of the monochromatic laser beam has to meet the plasma frequency: $\omega_L = \omega_p$. In the early '80s only the
PBWA method was accessible in the experiments. However in the middle of the '80s a remarkable improvement has been made: Strickland and Mourou (1985) invented the chirped pulse amplification (CPA) method which made possible to reach higher and higher laser intensities without damaging the lasing medium. In the early '90s ultra-high intensities ($\geq 10^{18}$ W/cm$^2$) began to be accessible in the experimental routine.

The invention of the CPA method induced a considerable advance of the plasma based particle accelerators. Numerous reviews and experimental studies have been published in the last two decades. Some important ones have been written by Esarey et al. (2009); Geddes et al. (2005); Gonsalves et al. (2011); Malka et al. (2002); Nakajima et al. (1995). In theoretical plasma physics the popular Particle-in-Cell (PIC) simulation approach became predominant in the last decade. In the field of modelling plasma based particle accelerators PIC is an essential tool. We should also mention the important works by Pukhov and Meyer-ter Vehn (2002); Vieira and Mendonça (2014). A robust, versatile state-of-the-art PIC code, called for OSIRIS, has been also developed by Fonseca et al. (2002).

In fact, there are alternative ways for modelling electron acceleration in strong electromagnetic fields. One way is the direct integration of the relativistic equations of motion, as it has been done by Wang et al. (1999, 2000), for instance. For some special cases the equations can be integrated analytically, however, in general the equations can be solved only by numerical means. A very important case is—both from theoretical and experimental side—the purely laser based particle acceleration. In this scheme the electrons are accelerated by the strong electromagnetic field of Gaussian laser pulses, as presented by Sohbatzadeh and Aku (2011); Sohbatzadeh et al. (2006). Varró (2014) has shown that there exist closed analytic solutions of the relativistic wave equations (Dirac and Klein-Gordon equations) if one takes into account the effect of the plasma through a phenomenological index of refraction ($n_{\text{in}}$). Needless to say, such an approach may also be meaningful in the classical (non-quantum) regime, where the index of refraction is contained in the Lorentz force.

The present work together with a former one, written by Aladi et al. (2014) is part of the Hungarian contribution to the planned CERN AWAKE experiment, see AWAKE Design Report (2013).

In the second section we give a brief overview of the applied mathematical formalism. Our results are presented and interpreted in the third section. Our paper ends with a short summary. For a better transparency $\hbar = c = k_B = 1$ units are used throughout the paper if not stated otherwise.

2 Theory

In the presence of an electromagnetic field the Lorentz-force acts on the electron:

$$\mathbf{F} = e (\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

with $e$ the electron charge, $\mathbf{E}$ the electric field, $\mathbf{B}$ the magnetic field, $\mathbf{v}$ the velocity of the electron and $\mathbf{F}$ the Lorentz-force. The equations of motions for
a relativistic electron are the following:

\[
\frac{d \mathbf{p}}{dt} = e \left( \mathbf{E} + \frac{\mathbf{p}}{m \gamma} \times \mathbf{B} \right), \quad (2a)
\]

\[
\frac{d \gamma}{dt} = \frac{1}{mc^2} \mathbf{F} \cdot \mathbf{v}. \quad (2b)
\]

It is important to note that one cannot write any arbitrary function in place of \( \mathbf{E} \) and \( \mathbf{B} \) since the electromagnetic field has to satisfy the electromagnetic wave equation. Hence, the most general form of \( \mathbf{E} \) and \( \mathbf{B} \) is as follows:

\[
\mathbf{E}(t, \mathbf{r}) = \varepsilon \varepsilon \varepsilon E_0 f[\omega \Theta(t, \mathbf{r})], \quad \text{with} \quad \varepsilon \varepsilon \varepsilon \text{the polarization vector,} \quad E_0, \quad \omega \quad \text{and} \quad \mathbf{n} \quad \text{the amplitude, angular frequency and} \quad \text{the unit vector of the propagation of the electromagnetic field and} \quad f \quad \text{an arbitrary, smooth function, respectively.} \quad \text{For a better transparency, we introduced the following notation:}
\]

\[
\Theta(t, \mathbf{r}) := t - \mathbf{n} \cdot \mathbf{r}. \quad (5)
\]

Note that the components of the electromagnetic field (\( \mathbf{E} \) and \( \mathbf{B} \)) depend only on \( \Theta \).

If we also want to take into account the presence of a medium with an index of refraction \( n_m < 1 \), we need to generalize the definition of \( \Theta(t, \mathbf{r}) \) in the following way:

\[
\Theta(t, \mathbf{r}, n_m) := t - n_m \mathbf{n} \cdot \mathbf{r}. \quad (6)
\]

Since the index of refraction of plasmas is less than unity the generalized definition of \( \Theta \) naturally describes the situation in which an electron propagates in a plasma. The refraction index of the plasma depends only on the plasma frequency:

\[
n_m = \sqrt{1 - \frac{\omega_p^2}{\omega_L^2}} \quad (7)
\]

with

\[
\omega_p^2 = \frac{n_e e^2}{\varepsilon_0 m_e} \quad (8)
\]

with \( n_e \) the electron density in the plasma, \( \varepsilon_0 \) the permittivity of vacuum and \( m_e \) the electron mass.

Now we shortly summarize the effective theory of electron acceleration in plasmas. Depending on the plasma density we determine \( \omega_p \) and \( n_m \). We treat the electromagnetic field, defined with equations (3) and (4) as a function of \( \Theta(t, \mathbf{r}, n_m) \) and solve the relativistic equations of motion (2) numerically. The special case \( n_m = 1 \) describes the electron acceleration with electromagnetic waves—with lasers, for instance. Worthy of note that Várro and Kocsis (1992) showed that the relativistic equations of motion can be integrated exactly with \( n_m = 1 \) for a circularly polarized, linearly chirped plane wave. Sohbatzadeh et al. (2006) did the same work for linear polarization. In both cases the solutions can be expressed with the Fresnel integrals; see Abramowitz and Stegun (1972).

In the present work we investigate the acceleration of a single electron with a chirped electromagnetic plane wave with linear polarization, sine-square shaped...
temporal envelope and in plasmas. We also study the properties of the pure laser acceleration, more accurately, the interaction of a single electron with a chirped Gaussian laser pulse. For a chirped sinusoidal plane wave pulse \( (n_m = 1) \)

\[
\begin{align*}
f(\Theta(t, r, n_m)) & = \begin{cases} 
\sin^2 \left[ \frac{\pi \Theta(t, r, n_m)}{T} \right] \times \sin \left[ \omega \Theta(t, r, n_m) + \sigma \Theta^2(t, r, n_m) + \varphi \right] & \text{if } t \in [0, T] \\
0 & \text{otherwise}
\end{cases}
\end{align*}
\]

(9)

with \( T \) the pulse duration, \( \sigma \) the chirp parameter and \( \varphi \) the carrier–envelope phase. The plane wave itself is polarized in the \( x \) direction and propagates in the \( z \) direction, that is, \( \mathbf{e} = e_x, \mathbf{n} = e_y \). A series of plane wave pulses is given with a sum of single plane waves:

\[
\sum_{l=0}^{N} f(\Theta(t, r, n_m) - lT),
\]

(10)

which has to be substituted into (9). As we mentioned, in case of plasma based acceleration \( n_m < 1 \) has to be applied.

For Gaussian pulses the formulation for the most general form of the electro-magnetic field \( E \) cannot be applied conveniently. Hence we give the \( x, y \) and \( z \) components explicitly for an \( x \)-polarized Gaussian pulse that propagates in the \( z \) direction. According to the works by Sohbatzadeh and Aku (2011); Sohbatzadeh et al. (2006) the components of such an electric field are:

\[
\begin{align*}
E_x & = E_0 \frac{W_0}{W(z)} \exp \left[ -\frac{r^2}{W^2(z)} \right] \exp \left[ -\frac{\Theta^2(t, r, n_m)}{T^2} \right] \times \cos \left[ 2kr^2 \frac{2R(z)}{W(z)} - \Phi(z) + \omega \Theta(t, r, n_m) + \sigma \Theta^2(t, r, n_m) + \varphi \right] \\
E_y & = 0 \tag{11a}
\end{align*}
\]

\[
\begin{align*}
E_z & = -\frac{x}{R(z)} E_x + E_0 \frac{2x}{kW^2(z)} \frac{W_0}{W(z)} \exp \left[ -\frac{r^2}{W^2(z)} \right] \exp \left[ -\frac{\Theta^2(t, r, n_m)}{T^2} \right] \times \sin \left[ 2kr^2 \frac{2R(z)}{W(z)} - \Phi(z) + \omega \Theta(t, r, n_m) + \sigma \Theta^2(t, r, n_m) + \varphi \right] \tag{11c}
\end{align*}
\]

and the magnetic field is given by

\[
\begin{align*}
B_x & = 0 \tag{12a}
\end{align*}
\]

\[
\begin{align*}
B_y & = E_x \tag{12b}
\end{align*}
\]

\[
\begin{align*}
B_z & = \frac{y}{R(z)} E_x + E_0 \frac{2y}{kW^2(z)} \frac{W_0}{W(z)} \exp \left[ -\frac{r^2}{W^2(z)} \right] \exp \left[ -\frac{\Theta^2(t)}{T^2} \right] \times \sin \left[ 2kr^2 \frac{2R(z)}{W(z)} - \Phi(z) + \omega \Theta(t) + \sigma \Theta^2(t) + \varphi \right] \tag{12c}
\end{align*}
\]

with \( W_z = \left[ 1 + (z/z_R)^2 \right]^{1/2} \) the beam waist, \( R(z) = z \left[ 1 + (z_R/z)^2 \right] \) the radius of curvature, \( \Phi(z) = \tan^{-1}(z/z_R) \) the Guoy phase, \( W_0 = \sqrt{\lambda z_R / \pi} \) the half of the focused spot size, \( z_R \) the Rayleigh length, \( \varphi \) the carrier–envelope phase, \( \lambda \) the wavelength of the laser and \( T \) the pulse duration.
3 Results

First we introduce the results obtained from the electron acceleration in a chirped electromagnetic plane wave. In this case the electromagnetic field is given by equations (3), (4) and (9). We investigated the dependence of the electron energy gain on the parameters, i.e. the initial momentum of the electron, the chirp parameter, the carrier–envelope phase and the field strength of the electromagnetic fields. After that we optimized the parameters in order to get the maximal gain. The numerical solution of the equations (2) have been obtained by using Wolfram Mathematica\textsuperscript{©} [Copyright 1988–2012 Wolfram Research, Inc.]. The tolerance was set to $10^{-5}$ both for the relative and absolute errors. Complete technical details can be found in the Master’s Thesis written by Pocsai (2014).

The energy gain is defined as follows:

$$\Delta E := m_e \left[ \gamma(t = T) - \gamma(t = 0) \right].$$

which is the difference of the final and the initial kinetic energy. Without chirp we did not experience any energy gain. The energy of the electron varied periodically in time with $T$ period. By positive chirp the energy gain was negligibly small, hence we only took negative values for the chirp parameter. The significant gain via negative chirp can be explained graphically in two different ways: the down-chirp can be interpreted as a down-conversion. Due to the conservation of energy, the decrement of the laser frequency results in an increment in the electron energy. From an other point of view, it can be seen that any kind of chirp causes a significant effect if the chirped pulse contains only a few optical cycles. It can also be seen that the oscillations of the field strength at the front of the pulse are approximately the same of the order of magnitude, but this difference increases in time. At the end of the pulse, the amplitude of the last oscillation is significantly smaller than that of the last but one. That is, the sharp, rising edge in the field strength cannot be compensated. Cheng and Xu (1999) discussed this phenomenon in details.

It is important to note that for acceleration in plane wave the energy gain of the electron does not depend on the initial position, it depends only on the initial momentum and the parameters. Without loss of generality the choice $p_z \equiv 0$ can be made. Because of simplicity we chose the electron to be started from the origin.

The following three figures show the optimisation of the initial momentum and the plane wave parameters. Their initial values were $p_0 = (0, 0, 0)$, $\lambda = 800 \text{ nm}$, $T = 35 \text{ fs}$, $\sigma = -0.03886 \text{ fs}^{-2}$ and $I = 10^{17} \text{ W} \cdot \text{cm}^{-2}$. Fig. 1 shows that for maximal acceleration the electron must not propagate in parallel with the plane wave. $\Delta E$ has got a maximum of approximately 550 keV by $p_0 = (-1570 \text{ keV/c}, 450 \text{ keV/c}, 0)$. Applying this initial momentum on the electron we sought the optimal values for the carrier–envelope phase and the pulse duration. The optimal values for these parameters are $\varphi = 4.21 \text{ rad}$ and $T = 75 \text{ fs}$, respectively. This further optimisation resulted in a growth by an order of magnitude in the energy gain. The maximal gain by these parameters is $\Delta E = 5600 \text{ keV}$. This is shown on Fig. 2. Applying these optimal parameters we optimised the chirp parameter and the intensity of the plane wave. The maximum of the energy gain grew again by an order of magnitude: its maximal value is approximately $\Delta E = 58 \text{ MeV}$ by $\sigma = -0.03698 \text{ fs}^{-2}$ and $I = 10^{21} \text{ W} \cdot \text{cm}^{-2}$.
and if the other parameters take their optimal values as given above. As a short summary we can say that by properly chosen parameters a single electron is able to gain as much as 58 MeV energy from a single plane wave pulse.

We also investigated the effective theory of the electron acceleration in underdense plasmas. As mentioned earlier, we needed to perform the same analysis by $n_m < 1$. We took $n_p = 10^{15}$ cm$^{-3}$ which is a typical value for the plasma density in the AWAKE experiment as indicated by Xia et al. (2011). The index of refraction by this density is $n_m = 0.9999997$ which means that the $n_m < 1$ case can be quite well approximated with the $n_m = 1$ case. Our calculations showed that the energy gain values for these two cases differ less than a per mill from each other.

Finally, we analysed the interaction of a single electron with a short, chirped Gaussian laser pulse. This latter model provides a more realistic description of the acceleration process. Lax et al. (1975) and Davis (1979) proved that Gaussian pulses are solutions of Maxwell’s equations. Wang et al. (1999) also provided a proof and investigated the dynamics of the acceleration process. From a practical point of view, it is known that the state-of-the-art laser systems emit a series of Gaussian laser pulses.

In the effect of the chirp parameter there is a small difference from the case of a plane wave: by a Gaussian pulse we also experienced some gain by zero chirp. However, this gain was negligibly small. This statement is also valid for positive chirp values, hence here we also took only negative values for $\sigma$. There is a major difference in the role of the initial position: since a Gaussian pulse has got also a spatial envelope that causes the laser intensity to decrease rapidly radially we had to place the electron on-axis, far enough for the electron to not to feel the electric field of the laser pulse. We also had to set the initial momentum parallel...
Figure 2: The dependence of the energy gain on the carrier–envelope phase and the pulse duration. The parameters are $p_0 = (-1570 \text{ keV/c, } 450 \text{ keV/c, } 0)$, $\lambda = 800 \text{ nm}$, $I = 10^{17} \text{ W} \cdot \text{cm}^{-2}$, $\sigma = -0.03886 \text{ fs}^{-2}$. The optimal values of the carrier–envelope phase and the pulse duration are $\phi = 4.21 \text{ rad}$ and $T = 75 \text{ fs}$, respectively.

Figure 3: The dependence of the energy gain on the chirp parameter and the laser intensity. The parameters are $p_0 = (-1570 \text{ keV, } 450 \text{ keV, } 0)$, $\lambda = 800 \text{ nm}$, $T = 75 \text{ fs}$, $\phi = 4.21$. The optimal values of the chirp parameter and the laser intensity are $\sigma = -0.03698 \text{ fs}^{-2}$ and $I = 10^{21} \text{ W} \cdot \text{cm}^{-2}$, respectively.
with the direction of the propagation of the laser pulse so that the electron is able to interact with the pulse as for much time as possible. This means that the initial momentum is parametrized as $p_0 = p_0 e_z$ with $p_0 \equiv |p_0|$.

We analysed the gain as a function of the initial momentum, the beam waist and the pulse duration. Initially we took the following parameters: $\lambda = 800 \text{nm}$, $T = 35 \text{fs}$, $I = 10^{21} \text{W cm}^{-2}$ and $W_0$ as some integer multiple of $\lambda$. These are typical values for Ti:Sapphire lasers. Mostly we chose $W_0 := 100\lambda$ because larger focused spot size means less spatial—and temporal—beam divergence.

First we investigated the energy gain as a function of the initial momentum. Our calculations showed that by sufficiently wide focused spot size an electron is able to gain as much as 200 MeV energy from a simple pulse. This is shown on Figs. 4 and 5. The energy gain can be increased by shortening the pulse. The dependence of the gain on the pulse duration is presented on Fig. 6. The maximal value of the energy gain from a single pulse by the optimal parameters $p_0 = 1533 \text{keV/c}$, $W_0 = 100\lambda$, $T = 31 \text{fs}$ and $I = 10^{21} \text{W cm}^{-2}$ is approximately $\Delta E = 275 \text{MeV}$. The result presented in this paper agree quite well with other theoretical calculations. Without completeness, see (Sohbatzadeh and Aku, 2011; Sohbatzadeh et al., 2006).
Figure 6: The energy gain as a function of the pulse duration. $p_0 = 1533 \text{ keV/c}$, $W_0 = 100 \lambda$, $I = 10^{21} \text{ W cm}^{-2}$, $\sigma = -0.0194 \text{ fs}^{-2}$.

4 Summary

In the present paper we studied the electron acceleration with electromagnetic plane waves and Gaussian laser pulses. An effective theory has also been presented for describing the electron acceleration in underdense plasmas. The key idea was the introduction of an effective refraction index for the plasma which can be incorporated into the phase of the laser pulse. Our results have been obtained by numerically solving the relativistic equations of motion. Our calculations showed that in this simple model the acceleration in plasmas can be quite well approximated with the vacuum model, that is, with the $n_m = 1$ case. We showed that by properly chosen parameters a single electron can gain as much as 58 MeV from a single plane wave pulse. This value is even higher, $\Delta E = 275 \text{ MeV}$, for a short Gaussian pulse.

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