REMARKS ON THE INTERSECTION OF FILTERS

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Abstract. We will show that the existence of an uncountable family of nonmeager filters whose intersection is meager is consistent with MA(Suslin).

1. Introduction

Suppose that $F$ is a filter on $\omega$. We identify elements of $F$ with their characteristic functions and think of $F$ as a subset of $2^\omega$. It is well known that if $F$ is a nonprincipal filter (which we assume to be always the case) then $F$ is meager or $F$ does not have the property of Baire. Similarly, $F$ has measure zero or is nonmeasurable. Since an intersection of filters is again a filter we want to know how many nonmeager filters one needs to intersect to produce a meager filter. Define

\[ f_M = \min \{ |H| : \forall F \in H \text{ $F$ is a filter without the Baire property and } \bigcap H \text{ has the Baire property} \} , \]

\[ f_N = \min \{ |H| : \forall F \in H \text{ $F$ is a nonmeasurable filter and } \bigcap H \text{ has measure zero} \} . \]

**Theorem 1.1** (<4>). $t \leq f_M$. In particular, MA implies that the intersection of less than $2^{\aleph_0}$ filters without the Baire property does not have the Baire property.

In [3] Plewik found sharper estimates. In particular, he showed that $h \leq f_M \leq d$. Repicky improved the lower bound and showed that $g \leq f_M$, where $g = \min \{ |H| : \forall H \in H \text{ $H$ is groupwise dense and } \bigcap H = \emptyset \}$. Recall that a family $H \subseteq [\omega]^\omega$ is groupwise dense if

1. $\forall x \in [\omega]^\omega \exists y \in [\omega]^\omega \text{ such that } x \neq y \in H$,
2. if $x \subseteq y$ and $y \in H$, then $x \in H$, and
3. for every partition of $\omega$ into finite sets, $\{ I_n : n \in \omega \}$, there is $a \in [\omega]^\omega$ such that $\bigcup_{n \in a} I_n \in H$.

For measure the situation is quite different.

**Theorem 1.2** (Fremlin, [1]). Assume MA. Then there exists a family $\{ F_\xi : \xi < 2^{\aleph_0} \}$ of nonmeasurable filters such that $\bigcap_{\xi \in X} F_\xi$ is a measurable filter for every uncountable set $X \subseteq 2^{\aleph_0}$. In particular, MA implies that there exists a family of $\aleph_1$ nonmeasurable filters with measurable intersection.

The goal of this note is to show that the equality $f_M = \aleph_1$ is consistent with a relatively strong version of Martin’s Axiom.

Recall that a forcing notion $(P, \leq)$ is Suslin if

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1. \( \mathcal{P} \) is ccc,
2. \( \mathcal{P} \) is \( \Sigma_1 \),
3. relations \( \leq, \perp \) are \( \Sigma_1 \).

Let \( \text{MA}(\text{Suslin}) \) denote the Martin’s Axiom for Suslin partial orders. It is well known that \( \text{MA}(\text{Suslin}) \) implies that additivity of measure, b, etc. are all equal to \( 2^{\aleph_0} \).

2. Consistency result

The goal of this section is to show \( \text{MA}(\text{Suslin}) \) is consistent with \( f_M = \aleph_1 \).

**Theorem 2.1.** There exists a model of \( V' \models ZFC \) and a family of filters \( \{ \mathcal{F}_\alpha : \alpha < 2^{\aleph_0} \} \in V' \) such that:

1. \( \mathcal{F}_\alpha \) is not meager for each \( \alpha \),
2. \( \bigcap_{\alpha \in X} \mathcal{F}_\alpha \) is meager for every uncountable set \( X \),
3. \( V' \models \text{MA}(\text{Suslin}) + 2^{\aleph_0} = \aleph_2 \).

**Proof** Let \( (\mathcal{P}_\alpha, \mathcal{Q}_\alpha : \alpha < \omega_2) \) be a finite support iteration such that

1. \( \forces_{\mathcal{P}_\alpha} \mathcal{Q}_\alpha \) is Suslin,
2. if \( \alpha \) is a successor ordinal then \( \mathcal{Q}_\alpha \) adds a Cohen real.

The second requirement is purely technical, its purpose is to simplify notation later on.

By careful bookkeeping we can ensure that \( V^{\mathcal{P}_{\omega_2}} \models \text{MA}(\text{Suslin}) + 2^{\aleph_0} = \aleph_2 \).

Let \( Z = \{ X_\alpha : \alpha < \omega_2 \} \) be the sequence of Cohen reals added by \( \mathcal{Q}_\alpha \)'s (represented as elements of \( [\omega]^{\omega_1} \)).

**Lemma 2.2.** \( Z \) is a generalized Luzin set. In particular, every subset of \( Z \) of size \( \aleph_2 \) in nonmeager.

**Proof** Suppose that \( A \subseteq [\omega]^{\omega_1} \) is a Borel meager set. Let \( \gamma \) be such that \( A \in V[\mathcal{P}_\gamma \cap G] \). Then \( X_\alpha \not\in A \) for \( \alpha > \gamma \). \( \square \)

**Lemma 2.3.** If \( \alpha_1 < \alpha_2 < \cdots < \alpha_n < \omega_2 \) then \( X_{\alpha_1} \cap X_{\alpha_2} \cap \cdots \cap X_{\alpha_n} \) is a Cohen real over \( V[G \cap \mathcal{P}_{\alpha_1-1}] \).

**Proof** Let \( \varphi : [\omega]^{\omega_1} \to [\omega]^{\omega_1} \) be defined as \( \varphi(X_1, \ldots, X_n) = X_1 \cap \cdots \cap X_n \). Suppose that \( A \subseteq [\omega]^\omega \) is a meager Borel set in \( V[G \cap \mathcal{P}_{\alpha_1-1}] \). Then \( B_0 = \varphi^{-1}(A) \) is a meager set. Let

\[
C_0 = \{ X : \langle X_2, \ldots, X_n \rangle : \langle X, X_2, \ldots, X_n \rangle \in B_0 \} \text{ is meager} \}.
\]

\( C_0 \) is a comeager set so \( X_{\alpha_1} \in C_0 \).

Let

\[
B_1 = \{ \langle X_2, \ldots, X_n \rangle : \langle X_{\alpha_1}, X_2, \ldots, X_n \rangle \in B_0 \}.
\]

\( B_1 \) is a meager set coded in \( V[G \cap \mathcal{P}_{\alpha_1}] \subseteq V[G \cap \mathcal{P}_{\alpha_2-1}] \). We apply the construction above to get the set

\[
C_1 = \{ X : \langle X_3, \ldots, X_n \rangle : \langle X, X_3, \ldots, X_n \rangle \in B_1 \} \text{ is meager} \},
\]

and continue in this fashion.

It follows that \( \langle X_{\alpha_1}, X_{\alpha_2}, \ldots, X_{\alpha_n} \rangle \not\in \varphi^{-1}(A) \) which finishes the proof. \( \square \)

Let \( \{ Z_\alpha : \alpha < \omega_2 \} \) be a partition of the set \( \{ \alpha : \alpha = \beta + 1, \beta < \omega_2 \} \) into disjoint cofinal sets.
Let $F_\beta$ be a filter generated by sets $\{X_\alpha : \alpha \in Z_\beta\}$. Since $X_\alpha$'s are Cohen reals each $F_\beta$ is indeed a filter.

Note that $F_\beta \supseteq \{X_\alpha : \alpha \in Z_\beta\}$. Thus it follows from 2.2 that $F_\beta$ is not meager for every $\beta < \omega_2$.

**Theorem 2.4.** $\bigcap_{\alpha \in X} F_\alpha$ is the filter of cofinite sets for every uncountable set $X \subseteq \omega_2$.

**Proof** For simplicity assume that $X = \omega_1$. The proof of the general case is the same.

Suppose that $X \in \bigcap_{\xi < \omega_1} F_\xi$ and $|\omega \setminus X| = \aleph_0$.

For each $\xi < \omega_1$ we can find $\alpha_1^\xi < \alpha_2^\xi < \cdots < \alpha_n^\xi \in Z_\xi$ such that

$$X_{\alpha_1^\xi} \cap X_{\alpha_2^\xi} \cap \cdots \cap X_{\alpha_n^\xi} \subseteq^* X.$$  

By passing to a subsequence we can assume that $n^\xi = n$ for all $\xi < \omega_1$. Moreover, we can assume that $n$ is minimal, that is,

$$\forall \xi < \omega_1 \quad X_{\alpha_1^\xi} \cap X_{\alpha_2^\xi} \cap \cdots \cap X_{\alpha_n^\xi} \subseteq^* X.$$  

Let $\gamma$ be the least ordinal such that $\{\xi < \omega_1 : \alpha_n^\xi < \gamma\}$ is uncountable. By the minimality of $n$, $\gamma$ is a limit ordinal and $\text{cf}(\gamma) = \aleph_1$. By passing to a subsequence again we can assume that

1. $\alpha_n^\xi < \gamma$ for all $\xi < \omega_1$,
2. $\forall \delta < \gamma \exists \alpha \forall \xi > \alpha \alpha_1^\xi > \delta$ (since $Z_\alpha$'s are disjoint).

We will show that $X \notin V[P_\delta \cap G]$ for all $\delta < \omega_2$ and this contradiction will finish the proof.

For $\xi < \omega_1$ let $X_\xi = X_{\alpha_1^\xi} \cap X_{\alpha_2^\xi} \cap \cdots \cap X_{\alpha_n^\xi}$. By the assumption $X_\xi \subseteq^* X$ for all $\xi < \omega_1$.

**Lemma 2.5.** $X \notin V[P_\gamma \cap G]$.

**Proof** Arguing as in 2.3 and using 2.3 and condition (2) above, we show that $\{X_\xi : \xi < \omega_1\}$ is a Luzin set in $V[P_\gamma \cap G]$. Since the set $\{Z \in [\omega]^{\omega} : Z \subseteq^* X\}$ is meager it follows that $\{\xi : X_\xi \subseteq^* X\}$ is countable. □

**Lemma 2.6.** $X \notin V[P_\delta \cap G]$ for $\gamma < \delta < \omega_2$.

**Proof** We will work in a model $V' = V[P_\delta \cap G]$. Suppose that the lemma is false. Let $\dot{X}$ be a $P_{\gamma, \omega_2}$-name for $X$. Let $M$ be a countable elementary submodel of $H(\chi)$ containing $\dot{X}$ and $P_{\omega_2}$. Define a finite support iteration $(P_\alpha(M), \dot{Q}_\alpha(M) : \alpha < \omega_2)$ as follows:

$$\vdash \alpha \dot{Q}_\alpha = \begin{cases} \dot{Q}_\alpha & \text{if } \alpha \in M \\ \emptyset & \text{if } \alpha \notin M \end{cases} \quad \text{for } \alpha < \omega_2.$$  

Let $P = \lim P_\alpha(M)$.

$P$ is isomorphic to a countable iteration of Suslin forcings. It may not be Suslin itself but it has enough absoluteness properties to carry out the rest of the proof (see 2 or 3 lemma 9.7.4). In particular, $P$ has a definition that can be coded as a real number.

From Suslinness it follows that $P \leq P_{\omega_2}$ and that $\dot{X}$ is a $P$-name.
Let $N \prec \mathbf{H}(\chi)$ be a countable model containing $M, \hat{X}$ and $P$. Since $\{X_\xi : \xi < \omega_1\}$ is a Luzin set in $\mathbf{V}'$ we can find $\xi$ such that $Y = X_\xi$ is a Cohen real over $N$. By the assumption $\Vdash Y \subseteq^* \hat{X}$.

By absoluteness, $N[Y][G \cap N[Y]] \models Y \subseteq^* \hat{X}[G \cap N[Y]]$ and therefore

\[ N[Y] \models "\Vdash Y \subseteq^* \hat{X}". \]

Represent Cohen algebra as $C = [\omega]^{< \omega}$ and let $\dot{Y}$ be the canonical name for a Cohen real. There is a condition $p \in C$ such that

\[ N \models p \Vdash C \models "\Vdash \dot{Y} \subseteq^* \dot{X}". \]

Let $Y' = p \cup ((\omega \setminus Y) \setminus \text{max}(p))$. $Y'$ is also a a Cohen real over $N$ and since $p \subseteq Y'$ we get that $N[Y'] \models "\Vdash Y' \subseteq^* \dot{X}"$. It follows that

\[ N[Y'][G \cap N[Y']] \models Y' \subseteq^* \dot{X}[G \cap N[Y']]. \]

Note that $\dot{X}[G] = \dot{X}[G \cap N[Y']] = \dot{X}[G \cap N[Y]]$. Thus $\mathbf{V}[G] \models Y \cup Y' \subseteq^* \dot{X}[G]$ which means that $\dot{X}[G]$ is cofinite. Contradiction. \qed

References

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