BARYON DECUPLLET MASSES FROM THE VIEWPOINT OF $q$-EQUIDISTANCE

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Summary

Masses of baryons $\frac{3}{2}^+$ are calculated on the base of representations of dynamical "pseudounitary" $q$-deformed algebra $u(4,1)_q$ which provides necessary breaking of the 4-flavor symmetry realized by the assumed $q$-algebra $su(4)_q$. It is demonstrated that, contrary to the case of $su(3)_q$-octet baryons $\frac{1}{2}^+$, one and the same $q$-analog of mass relation for baryons $\frac{3}{2}^+$ from decuplet embedded into 20-plet of $su(4)_q$ follows from evaluations within all the different admissible "dynamical" representations.

1. Quantum algebras (the $q$-algebra $su(2)_q$ as most popular among them) introduced in the eighties in the context of integrable systems, now occupy a very important place both in pure mathematical disciplines (e.g., such as theory of knots and links) and in mathematical as well as theoretical physics. A couple of years ago $su(2)_q$ has appeared in the framework of some interesting phenomenological models aimed to describe rotational spectra of (super)deformed heavy nuclei and diatomic molecules [1]. Concerning miscellaneous applications of quantum groups/$q$-algebras in a more wide context let us quote refs. [2-4] and references given therein.

In attempts to find phenomenological application of higher rank quantum algebras, one may encounter some new features absent in $su(2)_q$ case. Among such features we mention the necessity to deal with nonsimple-root elements (hence, with $q$-Serre relations) of those algebras, nontriviality of the concepts of (and formulas for) Casimir operators, Clebsch-Gordan coefficients (CGC’s), etc.

Recently, the use of higher rank quantum algebras $su(n)_q$ (or $u(n)_q$) in order to replace conventional unitary groups $SU(n)$ and their irreducible representations (irreps) in describing flavor symmetries of hadrons (vector mesons $1^-$ and baryons $\frac{1}{2}^+$) has been proposed [5-7]. With the help of the corresponding algebras $u(n+1)_q$ or $u(n,1)_q$ of "dynamical" symmetry, one can realize necessary breaking of $n$-flavor symmetries up to
exact (for strong interactions alone) isospin symmetry $su_q(2)_I$ and obtain some $q$-analogs of mass relations (MR’s). Such an application of the $q$-algebras $u(n)_q$ for obtaining $q$-analogs of hadron MR’s uses generators that correspond to both simple-root elements and nonsimple-root ones (thus, the $q$-Serre relations play definite role [6]); from another side, it exploits rather simple model which allows one to circumvent difficulties related with $q$-CGC’s and $q$-Casimirs.

As implied by the results of [5-7], replacement of the conventional unitary groups of hadronic flavor symmetries by their quantum counterparts may also lead to further interesting consequences. Let us sketch some of them briefly.

In the case of the ordinary (non-deformed) $SU(n)$ symmetry of hadrons with $n$ quark flavors, the approach based on dynamical unitary groups allows one to obtain [8] besides the well-known octet mass sum rule (MSR) $3m_{\omega_8} + m_\rho = 4m_{K^*}$, also the higher-flavor set of MSR’s for vector mesons $1^-$. A comparison of the octet MSR with existing data is impossible without introducing certain mixing between isosinglet $\omega_8$ and $SU(3)$ singlet (i.e. $\omega_8$ must be considered as a superposition of realistic mesons $\phi$ and $\omega$) with some mixing angle to be determined from empirical data. Likewise, in cases of more flavors ($n > 3$) one needs $n - 2$ mixing angles. But, extending the approach of [8] to $q$-algebras $su(n)_q$, one can derive for vector mesons the $q$-deformed MR’s which admit (at $|q| = 1$, $q \neq \pm 1$) some principally different (from the $q = 1$ case) treatment which allows to avoid [5,6] manifest introducing of singlet mixing.

Next, it turns out that all the $q$-dependence in vector meson masses and in coefficients of their MR’s is expressible in terms of certain Lorant-type polynomials (of variable $q + q^{-1}$) which were noticed to be related [6] with some knot invariants (Alexander polynomials $P_n(q)$ ). In a sense, the polynomial $P_n(q)$ through its root $q(n)$ enables one to determine the strength of deformation at every fixed $n$, and due this property may be called a defining polynomial for the corresponding vector meson MSR. This way is principally different from the choice of $q$ by fitting procedure [1,4]. Further, by utilizing the quantum groups/$q$-algebras instead of conventional unitary groups of flavor symmetries, together with ‘dynamical’ quantum algebras, we get as a result that the collection of torus knots is put into correspondence [6] with heavy vector quarkonia. In other words, application of the embeddings $u(n)_q \subset u(n + 1)_q$ to analysis of (vector) meson masses opens an appealing possibility of definite topological characterization of heavy flavors, since the
number $n$ just corresponds to $2n-1$ overcrossings of 2-strand braids whose closures give those $(2n-1)$-torus knots.

The approach of [5,6] was recently extended in order to treat the case of baryons $\frac{1}{2}^+$ (including charmed ones), by adopting again the algebra $u(4)_q$ for the 4-flavor symmetry [7]. Unlike the case of vector mesons where 'compact' $q$-algebras $u(n+1)_q$ were used for dynamical symmetry, and in some analogy to the case of baryon MR’s obtained with non-deformed dynamical $u(4,1)$-symmetry [8], it was more convenient there to exploit representations of the 'noncompact' dynamical symmetry, realized by the $q$-algebra $u(4,1)_q$ in order to effect necessary symmetry breakings.

On the base of evaluations within certain concrete irrep it was demonstrated [7] that the resulting $q$-analog of baryon octet MR yields either the usual Gell-Mann–Okubo (GMO) mass sum rule [9] $m_N + m_\Xi = \frac{3}{2} m_\Lambda + \frac{1}{2} m_\Sigma$ or a very successful novel MSR if one fixes respectively $q = 1$ or $q = e^{i\frac{\pi}{6}}$. These values are the roots of one and the same defining polynomial $A_q$ appearing in the $q$-analog. Another $q$-analogs (with different defining $q$-polynomials) of octet MR can be obtained by performing calculations within another specific representations of $u(4,1)_q$. Therefore, one can say that the use of quantum algebras in place of non-deformed ones, in a sense, "removes degeneracy".

The purpose of the present letter is to apply the scheme developed in [7] to the case of baryons $\frac{3}{2}^+$ which constitute the decuplet irrep of $su(3)_q$. According to conventional descriptions, symmetry breaking results in equidistant differences between masses of isopleft members of the decuplet [9]. However, the empirical data show that actually there is a deviation from exact equidistance. Use of the $q$-algebras $su(n)_q$ instead of ordinary $SU(n)$ seems to provide natural and rather simple accounting of such a deviation. After presenting the results of evaluation of decuplet baryon masses in two specific irreps of the dynamical $u(4,1)_q$, an assertion is proved which states that certain $q$-analog of MR, see eq. (5) below, holds in all the admissible dynamical representations. Also, it is argued that the deformation parameter $q$ must be pure phase.

2. As already mentioned, among possible approaches to $SU(n)_q$-symmetry breaking necessary in order to obtain mass relations for hadrons of $n$ quark flavors, we prefer the scheme which was used in ref.[7] for the case of baryons $\frac{1}{2}^+$ and which is nothing but a straightforward extension to quantum groups of the approach based on noncompact (pseudo-unitary) dynamical groups, see [8] and references given therein.
In what follows, we use the 20-dimensional irrep $T_{20}$ of $u(4)_q$ (instead of $su(4)_q$ ) which is characterized by the 4-tuple \{p + 2, p - 2, p - 3, p - 4\} with some $p \in \mathbb{Z}$ (explicit value of $p$ will not enter final expressions).

To form state vectors for baryons $3^+_2$ from decuplet, we use the (orthonormalized) Gel’fand-Zetlin basis elements constructed in accordance with the chain $u(4)_q \supset u(4)_q \supset u(3)_q \supset u(2)_q$, namely,

$$ |B_i \rangle \leftrightarrow |\chi; m_4, m_3, m_{2(i)}, Q_i \rangle, \quad i = 1, \ldots, 4, $$

where $\chi \equiv \{l_1, l_2, l_3; c_1, c_2\}$ labels irreps of $u(4)_q$; $m_4 \equiv \{p + 2, p - 2, p - 3, p - 4\}$ and $m_3 \equiv \{p + 2, p - 2, p - 3\}$ label respectively 20-plet of $u(4)_q$ and 10-plet of its subalgebra $u(3)_q$; $m_{2(i)}$, with $i = 1, \ldots, 4$, characterizes isoplets (of dimensions 1, \ldots, 4,) from the 10-plet (see ref. [7] for more details concerning the approach and irreps of $u(4)_q$).

Another ingredient of this scheme, the mass operator, is composed of "noncompact" generators of the $q$-algebra $u(4)_q$ and is taken in the form [7]

$$ \tilde{M}_4 = M_o + aI_{45}I_{54} + bI_{54}I_{45} $$

$$ + \alpha I_{35} \tilde{I}_{53} + \beta \tilde{I}_{53}I_{35} + \tilde{\alpha}I_{35}I_{53} + \tilde{\beta}I_{53} \tilde{I}_{35}. $$

(2)

Since $I_{51}$, $I_{55}$ with $i = 1, 2$ and their dual (tilded) counterparts are absent in eq.(2), this operator commutes with "isospin" $su_q(2)_I$. To reduce the number of independent parameters we impose $\alpha_i = \tilde{\alpha}_i$, $\beta_i = \tilde{\beta}_i$.

The representation $T_\chi$ of 'dynamical’ algebra is called $T_{20}$-compatible if it contains irrep $T_{20}$ of its subalgebra $u(4)_q$. From definitions of integer-type infinitesimally unitary irreps of $u(4)_q$ which include the corresponding branching rules under restriction $u_q(4,1)|_{u_q(4)}$, it is an easy task to check that the following assertion is valid.

**Proposition 1.** The integer type irreps of $u(4,1)_q$ which are $T_{20}$-compatible are contained in the following list (here $s \in \mathbb{Z}_+$, and either $t = 0$ or $t = 1$):

(i) $D_{12}^1(p-1, p-3, p-4; p+1+s, p-2)$ and $D_{12}^1(p+t, p-3, p-4; p+1+s, p-1+t)$;

(ii) $\tilde{D}_{24}^1(p, p-3, p-4; p-3, p-5)$ and $\tilde{D}_{24}^1(p-1, p-3, p-4; p-3, p-5)$;

(iii) $\tilde{D}_{34}^1(p-1, p-3, p-4; p-4, p-5)$ and $\tilde{D}_{34}^1(p, p-3, p-4; p-4, p-5)$;

(iv) $\tilde{D}_{41}^1(p-1, p-3, p-4; p-1, p)$;

(v) $D_{4}^4(p, p-3, p-4; p-4-s, p-4-s)$ and $D_{4}^4(p-1, p-3, p-4; p-4-s, p-4-s)$;
(vi) $D^1_-(p, p-3, p-4; p+1+s, p+1+s)$ and $D^1_-(p-1, p-3, p-4; p+1+s, p+1+s)$.

3. We are now in a position to obtain the ($q$-dependent) expressions for masses of the hadrons under consideration. Using first the irrep $D^{12}_{12}(p-1, p-3, p-4; p+2, p-2)$ we calculate $\langle B_i | \hat{M}_4 | B_i \rangle$, $i = 1, \ldots, 4$, and obtain the following results for baryon mass from $10$-plet of $su(3)_q$ (here $[k] \equiv [k]_q = \frac{q^k - q^{-k}}{q - q^{-1}}$):

\[
\begin{align*}
    m_{\Delta_4} &= m_{10} + a[4] + \alpha[2][4], \\
    m_{\Sigma_3} &= m_{10} + a[4] + \alpha\{[2]^2[4] + \frac{[2]}{[6]}([2] + [4][5]) - 4[4]\}, \\
    m_{\Xi_2} &= m_{10} + a[4] + \alpha\{[2][3][4] + \frac{[2]^2}{[6]}([2] + [4][5]) - 4[2][4]\}, \\
    m_{\Omega_1} &= m_{10} + a[4] + \alpha\{[2][4]^2 + \frac{[2][3]}{[6]}([2] + [4][5]) - 4[3][4]\}. \\
\end{align*}
\]

We observe that the parameters $b$ and $\beta$ (see (2)) do not enter the above expressions, and this fact is a characteristic feature of the representation just exploited. Indeed, in this specific irrep there is no freedom to raise (or to lower) the $l$-coordinates $l_{14}, l_{24}, l_{34}$ in the highest weight $m_4$ of its $u(4)_q$-subrepresentation since that is forbidden by $u_q(4,1)|u_q(4)$-branching rules (the corresponding matrix elements of representation operators $I_{45}, I_{54}$ do vanish). Only $l_{44}$ can be lowered.

In terms of the denotation

\[
m = m_{10} + a[4], \quad \alpha' = \alpha[2][4], \quad \alpha'' = \alpha\{\frac{[2]}{[6]}([2] + [4][5]) - 4[4]\}
\]

the expressions (3) take essentially more transparent form, namely

\[
\begin{align*}
    m_{\Delta_4} &= m + \alpha', \\
    m_{\Sigma_3} &= m + \alpha'[2] + \alpha'', \\
    m_{\Xi_2} &= m + \alpha'[3] + \alpha''[2], \\
    m_{\Omega_1} &= m + \alpha'[4] + \alpha''[3].
\end{align*}
\]

The $q$-equidistance (it generalizes usual equidistance of the case of $SU(n)$-symmetries) is manifest here. In the 'classical' (non-deformed) limit $q \to 1$, the conventional equidistance (with the 'step' or distance equal to $\left(\alpha' + \alpha''\right)|_{q=1} = -\frac{2}{3}\alpha$), is recovered:

\[
m_{\Omega_1} - m_{\Xi_2} = m_{\Xi_2} - m_{\Sigma_3} = m_{\Sigma_3} - m_{\Delta_4} = -\frac{2}{3}\alpha
\]
(recall that it was the equidistance relation (4) that led to prediction and discovery of
the famous $\Omega^-$-particle [9]).

4. To make some contact with empirical situation in the $q$-deformed case, it is
important to fix the parameter $q$ appropriately. Unfortunately, at the moment we have
not a conclusive idea of how to do this (contrary to the cases of vector mesons [5-6] and
octet baryons $\frac{1}{2}^+$, see [7]). Nevertheless, we can decide which of the two options,

$(a) \quad q \in \mathbb{R}$ \quad or \quad $(b) \quad q = e^{ih}, \quad h \in \mathbb{R},$

is realized. To this end, we deduce from (3) or (3') the relation for masses which does
not contain undetermined parameters. From the differences $m_{\Sigma_3} - m_{\Delta_4}, \quad m_{\Xi_2} - m_{\Sigma_3}, \quad$ and $m_{\Omega_1} - m_{\Xi_2}$ we form a combination that results in the desired mass relation (it is of
the ”$q$-average” type):

$$\frac{m_{\Sigma_3} - m_{\Delta_4} + m_{\Omega_1} - m_{\Xi_2}}{[2]_q} = m_{\Xi_2} - m_{\Sigma_3}. \quad (5)$$

The dependence on deformation parameter here is the simplest possible one. Now recall
that empirical values for the participating baryon masses averaged over isoplets are [10]
$m_{\Delta_4} = 1232$ Mev, $m_{\Sigma_3} = 1384.6$ Mev, $m_{\Xi_2} = 1533.4$ Mev, and $m_{\Omega_1} = 1672.4$ Mev.
Substitution into eq.(5) of the empirical data shows that we must have $[2]_q \approx 1.95$. Since
at real $q$ other than 1 necessarily $[2]_q \equiv q + q^{-1} > 2$, we conclude definitively in favor of
the option (b).

5. Now let us go over to exploiting another 'dynamical' representation, namely,
$D_{12}^{12}(p,p-3,p-4;p+2,p-1)$. Performing necessary evaluations within it, this time we
obtain the following expressions for isoplet masses:

$$m_{\Delta_4} = m + \alpha \frac{[2]_q[4][5][3]}{3} + \beta \frac{[2]_q^2}{3},$$

$$m_{\Sigma_3} = m + \alpha \left\{ \frac{[2]}{3} \left( \frac{[2]_q[4][5]^2}{6} + [4]^2 + [4][6] \right) - 4 \frac{[4][5]}{3} \right\} + \beta \frac{[2]_q^3}{3^2},$$

$$m_{\Xi_2} = m + \alpha \left\{ \frac{[2]}{3} \left( \frac{[2]_q^2 + [2][4][5]^2}{6} + [2][4][6] \right) + [2][4] - 4 \frac{[2][4][5]}{3} \right\} + \beta \frac{[2]_q^2}{3^2}, \quad (6)$$

$$m_{\Omega_1} = m + \alpha \left\{ \frac{[2]}{3} \left( \frac{[2]_q[4][5]^2}{6} + [4][6] + \frac{[2][4]}{3} \right) - 4[4][5] \right\}$$

where $m = m_{10} + \frac{[4][5]}{3}a + \frac{[2]}{3}b$. Classically, again we are led to the equidistance (the
‘step’ now equals to $-2\alpha - \frac{4}{9}\beta$).
At $q \neq 1$ the expressions (6) apparently differ from eqs.(3). Indeed, now masses depend both on the parameter $\alpha$ and on the parameter $\beta$. Furthermore, introducing the denotion $A = \alpha [\frac{2[4]}{[3][6]}], \ A' = \alpha \{[2](\frac{[2]+[4][5]}{[3][6]} + \frac{[4][6]}{[3][5]}) - 4 \frac{[4][5]}{[3][6]}\},$ and $B = \beta [\frac{2}{[3][7]}]$, we rewrited eq.(6) in the form

$$
m_{\Delta_4} = m + [5]A + [3]B,$$
$$m_{\Sigma_3} = m + [4]A + A' + [2]B,$$
$$m_{\Xi_2} = m + [3]A + [2]A' + B,$$
$$m_{\Omega_1} = m + [2]A + [3]A',$$

and observe here both decreasing and growing sequences of $q$-numbers simultaneously in the coefficients when reading from the top down (generalized $q$-equidistance).

Like before, we are interested in obtaining the mass relation which is independent of undetermined parameters $m$, $\alpha$ and $\beta$. Forming the differences $m_{\Sigma_3} - m_{\Delta_4}$, $m_{\Xi_2} - m_{\Sigma_3}$, and $m_{\Omega_1} - m_{\Xi_2}$ we arrive at the relation which coincides with eq.(5).

Is such a coincidence merely an accidental fact, or may be the $q$-relation (5) is in a sense universal (i.e. holds independently of the choice of dynamical representation)? The following assertion gives the answer.

**Proposition 2.** The $q$-average mass relation (5) for isoplet masses from the 10-plet of $su(3)_q$ is valid in any of the $T_{20}$-compatible irreps $T_\chi$ from the list given above (see proposition 1).

To prove this, we have to calculate the matrix elements $\langle i | \hat{M}_4 | i \rangle$ for every isoplet $B_i$ ( $i$ runs over $\Delta_4$, $\Sigma_3$, $\Xi_2$, and $\Omega_1$) within the 10-plet of $su(3)_q$ embedded into (an arbitrary) $T_{20}$-compatible irrep $T_\chi$.

From the explicit action formulas for the representation operators $T_\chi(I_{n,n+1})$ and $T_\chi(I_{n+1,n})$ we infer two facts. First, in arbitrary representation $T_\chi$, matrix elements of the invariant operator $\hat{M}_0$ and of the terms $I_{54}I_{45}$, $I_{45}I_{54}$ (see eq. (2) ) are the same for all the isoplets $B_i$. Denote these as $\langle i | (\hat{M}_0 + aI_{45}I_{54} + bI_{54}I_{45}) | i \rangle \equiv F_\chi$. Second, the dependence of $\langle i | \hat{M}_4 | i \rangle$ on the signature of specific isoplet $B_i$ has its origin only in the action of the operators $I_{34}$ and $I_{43}$ which compose the operators $I_{35}$, $\bar{I}_{35}$, $I_{53}$, $\bar{I}_{53}$. Moreover, denoting $\langle A_{53} \rangle_i \equiv \langle i | (I_{53}\bar{I}_{35} + \bar{I}_{53}I_{35}) | i \rangle$ and $\langle A_{35} \rangle_i \equiv \langle i | (I_{35}\bar{I}_{53} + \bar{I}_{35}I_{53}) | i \rangle$, it
is not difficult to show that
\[
\langle A_{53} \rangle_i = A_1[|l_{13} - p + 2 - i|] + A_2[|l_{33} - p + 2 - i|] = A_1[k] + A_2[u],
\]
\[
\langle A_{35} \rangle_i = C_1[|l_{13} + 1 - p + 2 - i|] + C_2[|l_{23} + 1 - p + 2 - i|] = C_1[k + 1] + C_2[u - 2],
\]
where \( k = 4 - i, \ u = 1 + i, \ (i = 1, ..., 4) \), and the coefficients \( A_1, A_2, C_1, \) and \( C_2 \) depend on \( \chi \), on the signature \( m_4 \) of 20-plet, on the signature \( m_3 \) of the 10-plet, but do not depend on particular isoplet \( B_i \).

From the calculated expressions for isoplet masses we find the differences:
\[
m_{\Sigma_3} - m_{\Delta_4} = A_1([k + 1] - [k]) + A_2([u - 1] - [u])
+ C_1([k + 2] - [k + 1]) + C_2([u - 3] - [u - 2]),
\]
\[
m_{\Xi_2} - m_{\Sigma_3} = A_1([k + 2] - [k + 1]) + A_2([u - 2] - [u - 1])
+ C_1([k + 3] - [k + 2]) + C_2([u - 4] - [u - 3]),
\]
\[
m_{\Omega_1} - m_{\Xi_2} = A_1([k + 3] - [k + 2]) + A_2([u - 3] - [u - 2])
+ C_1([k + 4] - [k + 3]) + C_2([u - 5] - [u - 4]).
\]

Finally, with the q-number identity \([n + 2] = [2][n + 1] - [n]\) taken into account, the relation (5) follows and this completes the proof.

6. Summarizing, we have shown that, unlike the case of \( SU_q(3) \)-octet baryons \( \frac{1}{2}^+ \) (see [7]), in the present case of decuplet baryons \( \frac{3}{2}^+ \) the obtained q-average mass relation (5) holds for any of \( T_{20} \)-compatible irreps \( T_\chi \) of ”dynamical” q-algebra \( u(4,1)_q \).

Taking into account the empirical values of baryon masses we have made a conclusion that the deformation parameter \( q \) must be a pure phase (see the option (b) in sec.4) in order that the relation (5) would give realistic mass sum rule. However, the problem of more exact fixation of the deformation parameter remains open since it depends drastically on an improvement of the accuracy of data for empirical masses (remark that it is the isoquartet masses the values of which are most ambiguous).

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