The strict and relaxed stochastic maximum principle for optimal control problem of backward systems

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Abstract

We consider a stochastic control problem where the set of controls is not necessarily convex and the system is governed by a nonlinear backward stochastic differential equation. We establish necessary as well as sufficient conditions of optimality for two models. The first concerns the strict (classical) controls. The second is an extension of the first to relaxed controls, who are a measure valued processes.

Keywords. Backward stochastic differential equation, strict control, relaxed control, maximum principle, adjoint equation, variational inequality, variational principle.

AMS Subject Classification. 93 Exx

1 Introduction

In this paper we study a stochastic control problem where the system is governed by a nonlinear backward stochastic differential equation (BSDE for short) of the type

\[
\begin{align*}
    dy_t^v &= b(t, y_t^v, z_t^v, v_t) \, dt + z_t^v \, dW_t, \\
    y_T^v &= \xi,
\end{align*}
\]

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where \( b \) is given function, \( \xi \) is the terminal data and \( W = \{ W_t \}_{t \geq 0} \) is a standard \( d \)-dimensional Brownian motion, defined on a filtered probability space \( (\Omega, \mathcal{F}, \{ \mathcal{F}_t \}_{t \geq 0}, \mathcal{P}) \) satisfying the usual conditions. The control variable \( v = (v_t) \), called strict (classical) control, is an \( \mathcal{F}_t \)-adapted process with values in some set \( U \) of \( \mathbb{R}^k \). We denote by \( \mathcal{U} \) the class of all strict controls.

The criteria to be minimized, over the set \( \mathcal{U} \), has the form

\[
J(v) = \mathbb{E} \left[ g(y^v_0) + \int_0^T h(t, y^v_t, z^v_t, v_t) \, dt \right],
\]

where \( g \) and \( h \) are given maps, and \( (y^v_t, z^v_t) \) is the trajectory of the system controlled by \( v \).

A control \( u \in \mathcal{U} \) is called optimal if it satisfies

\[
J(u) = \inf_{v \in \mathcal{U}} J(v).
\]

Stochastic control problems for the backward and forward-backward systems have been studied by many authors. The first contribution of control problems of forward-backward systems is made by Peng [30], he obtained the maximum principle with the control domain being convex. Xu [34] established the maximum principle for this kind of problem in the case where the control domain is not necessary convex, with uncontrolled diffusion coefficient and a restricted functional cost. The work of Peng [30] (convex control domain) is generalized by Wu [33], where the system is governed by a fully coupled forward-backward stochastic differential equation. Shi and Wu [32] extend the result of Xu [34] to the fully coupled forward-backward systems, with convex control domain and uncontrolled diffusion coefficient. Ji and Zhou [22] use the Ekeland variational principle and establish a maximum principle of controlled forward-backward systems, while the forward state is constrained in a convex set at the terminal time, and apply the result to state constrained stochastic linear-quadratic control models and a recursive utility optimization problem are investigated. All the cited previous works on stochastic control of forward-backward systems are obtained by introducing two adjoint equations. In the recent works on the subject, Bahlali and Labed [3] and Bahlali [6] introduce three adjoint equations to establish necessary as well as sufficient optimality conditions. In [3] the authors establish the results in the case where the control domain being nonconvex and uncontrolled diffusion coefficient. The results of [6], are obtained while the control domain
is convex and with controlled diffusion coefficient, moreover the author apply
his theory to solve the financial model of cash flow valuation.

On the other hand, stochastic maximum principle of backward systems
was studied by El-Karoui et al [14], where the linear case is solved and some
applications in finance are treated. Dokuchaev and Zhou [9] established
necessary as well as sufficient optimality conditions, where the control domain
is not convex.

Our objective in this paper is to establish necessary as well as sufficient
optimality conditions, of the Pontryagin maximum principle type, for two
models.

Firstly, we derive necessary as well as sufficient optimality conditions for
strict controls. Since the set of strict controls is nonconvex, the classical way
to use, is the spike variation method. More precisely, if $u$ is an optimal strict
control and $v$ is arbitrary, then with a sufficiently small $\theta > 0$, we define a
perturbed control as follows

$$u^\theta_t = \begin{cases} v & \text{if } t \in [\tau, \tau + \theta], \\ u_t & \text{otherwise}. \end{cases}$$

We then derive the variational equation from the state equation, and the
variational inequality from the fact that

$$0 \leq J(u^\theta) - J(u).$$

The major difficulty in doing this is that the state of a backward system
and the functional cost depends on two variables $y_t$ and $z_t$. Then, we can’t
derive directly the variational inequality, because $z_t$ is hard to handle, there
is no convenient pointwise (in $t$) estimation for it, as opposed to the first vari-
able $y_t$. To overcome this difficulty, we introduce a new method which consist
to transform the initial control problem to a restricted problem without in-
tegral cost, by adding an unidimensional BSDE. We establish then necessary
optimality conditions for the restricted control problem and by an adequate
transformation on the adjoint process and the adjoint equation associated
with the restricted problem, we reformulate necessary optimality conditions
for the initial control problem.

To achieve this part of the paper, we study when these necessary optimi-
mality conditions becomes sufficient.

The second main result in this paper concerns necessary as well as suf-
fficient optimality conditions for relaxed controls. In the relaxed model, the
controller chooses at time $t$ a probability measure $q_t(da)$ on the control set $U$, rather than an element $v_t$ of $U$. The system is then governed by the BSDE

$$
\begin{aligned}
&dy^q_t = \int_U b(t, y^q_t, z^q_t, a) q_t(da) \ dt + z^q_t dW_t, \\
&y^q_T = \xi.
\end{aligned}
$$

The criteria to be minimized, over the set $\mathcal{R}$ of relaxed controls, has the form

$$
\mathcal{J}(q) = \mathbb{E} \left[ g(y^q_0) + \int_0^T \int_U h(t, y^q_t, z^q_t, a) q_t(da) \ dt \right].
$$

A control $\mu \in \mathcal{R}$ is called optimal if it satisfies

$$
\mathcal{J}(\mu) = \inf_{q \in \mathcal{R}} \mathcal{J}(q).
$$

The relaxed control problem is an extension of the previous model of strict controls. Indeed, if $q_t(da) = \delta_{v_t}(da)$ is a Dirac measure concentrated at a single point $v_t$, then we get a strict control problem as a particular case of the relaxed one.

By using the Ekeland’s variational principle, we are able to establish necessary optimality conditions for near optimal strict controls converging in some sense to the relaxed optimal control, by the so called chattering lemma. The relaxed necessary optimality conditions are then derived by using some stability properties of the trajectories and the adjoint process with respect to the control variable.

We note that necessary optimality conditions for relaxed controls, where the systems are governed by a stochastic differential equation, were studied by Mezerdi and Bahlali [27], Bahlali, Dječić and Mezerdi [4].

The paper is organized as follows. In Section 2, we formulate the problem and give the various assumptions used throughout the paper. Section 3 is devoted to restrict the initial control problem to a problem without integral cost and we derive a restricted necessary optimality conditions. In Section 4, we give our first main result, the necessary optimality conditions for the initial control problem and under additional hypothesis, we prove that these conditions becomes sufficient. Finally, in the last Section, we give necessary optimality conditions for near optimal controls and from this we derive our second main result in this paper, necessary as well as sufficient optimality conditions for relaxed controls.
Along this paper, we denote by $C$ some positive constant, $\mathcal{M}_{n \times d}(\mathbb{R})$ the space of $n \times d$ real matrix and $\mathcal{M}^d_{n \times n}(\mathbb{R})$ the linear space of vectors $M = (M_1, ..., M_d)$ where $M_i \in \mathcal{M}_{n \times n}(\mathbb{R})$. We use the standard calculus of inner and matrix product.

2 Formulation of the problem

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathcal{P})$ be a probability space equipped with a filtration satisfying the usual conditions, on which a $d$-dimensional Brownian motion $W = (W_t)_{t \geq 0}$ is defined. We assume that $(\mathcal{F}_t)$ is the $\mathcal{P}$-augmentation of the natural filtration of $(W_t)_{t \geq 0}$.

Let $T$ be a strictly positive real number and $U$ a non empty subset of $\mathbb{R}^k$.

**Definition 1** An admissible control is an $\mathcal{F}_t$-adapted process with values in $U$ such that

$$\mathbb{E} \left[ \sup_{t \in [0,T]} |v_t|^2 \right] < \infty.$$  

We denote by $\mathcal{U}$ the set of all admissible controls.

For any $v \in \mathcal{U}$, we consider the following BSDE

$$\begin{cases} dy_t^v = b(t, y_t^v, z_t^v, v_t) \, dt + z_t^v \, dW_t, \\ y_T^v = \xi, \end{cases}$$

(1)

where

$$b : [0,T] \times \mathbb{R}^n \times \mathcal{M}_{n \times d}(\mathbb{R}) \times U \longrightarrow \mathbb{R}^n,$$

and $\xi$ is an $n$-dimensional $\mathcal{F}_T$-measurable random variable such that

$$\mathbb{E} |\xi|^2 < \infty.$$

The expected cost is defined from $\mathcal{U}$ into $\mathbb{R}$ by

$$J(v) = \mathbb{E} \left[ g(y_0^v) + \int_0^T h(t, y_t^v, z_t^v, v_t) \, dt \right],$$

(2)

where

$$g : \mathbb{R}^n \longrightarrow \mathbb{R},$$

$$h : [0,T] \times \mathbb{R}^n \times \mathcal{M}_{n \times d}(\mathbb{R}) \times U \longrightarrow \mathbb{R}.$$
A control \( u \in \mathcal{U} \) is called optimal, if that solves
\[
J(u) = \inf_{v \in \mathcal{U}} J(v).
\]  
(3)

Our goal is to establish necessary as well as sufficient optimality conditions for controls in the form of stochastic maximum principle.

The following assumptions will be in force throughout this paper

The functions \( b, g \) and \( h \) are continuous in \( (y, z, v) \), they are differentiable with respect to \( (y, z) \), and they derivatives \( b_y, b_z, g_y, h_y \) and \( h_z \) are continuous in \( (y, z, v) \) and uniformly bounded.

\( b \) and \( h \) are bounded by \( C (1 + |y| + |v|) \) and bounded in \( z \).

Under the above hypothesis, for every \( v \in \mathcal{U} \), equation (1) has a unique strong \( (\mathcal{F}_t)_t \)-adapted solution and the functional cost \( J \) is well defined from \( \mathcal{U} \) into \( \mathbb{R} \).

### 3 Problem with restricted cost

Since the function \( h \) of the cost depend explicitly on \( z_t \), we can’t treat our problem directly. Thus, let us in this section restrict the initial control problem \{ (1), (2), (3) \} to a problem without integral cost. For this end, consider the following unidimensional BSDE

\[
\begin{aligned}
\left\{
\begin{array}{l}
\frac{d x^v_t}{dt} = h(t, y^v_t, z^v_t, v_t) dt + k^v_t dW_t, \\
x^v_T = \eta,
\end{array}
\right.
\end{aligned}
\]

where \( k^v \) is an \((1 \times d)\) matrix, \((y^v_t, z^v_t)\) is the solution of equation (1) and \( \eta \) is an one-dimensional \( \mathcal{F}_T \)-measurable random variable such that

\[
\mathbb{E} |\eta|^2 < \infty.
\]

The above equation admits a unique strong \( (\mathcal{F}_t)_t \)-adapted solution.

We put

\[
\tilde{y}_t = \begin{pmatrix} y^v_t \\ x^v_t \end{pmatrix},
\]
and consider now the following \((n + 1)\)-dimensional BSDE

\[
\begin{aligned}
&\left\{ \begin{array}{l}
d\tilde{y}_t = \tilde{b}_t(t, \tilde{y}_t, \tilde{z}_t, v_t) \, dt + \tilde{z}_t dW_t, \\
\tilde{y}_T = \left( \begin{array}{c}
\xi \\
\eta
\end{array} \right),
\end{array} \right. \\
\end{aligned}
\]  

(5)

where the functions \(\tilde{b}\) is defined from \([0, T] \times \mathbb{R}^{n+1} \times \mathcal{M}_{(n+1) \times d} (\mathbb{R}) \times U \) into \(\mathbb{R}^{n+1}\) by

\[
\tilde{b}_t(t, \tilde{y}_t, \tilde{z}_t, v_t) = \left( \begin{array}{l}
b(t, y^v_t, z^v_t, v_t) \\
h(t, y^v_t, z^v_t, v_t)
\end{array} \right),
\]

and \(\tilde{z}_t\) is a \((n + 1) \times d\) real matrix given by

\[
\tilde{z}_t = \left( \begin{array}{l}
z^v_t \\
\v^v_t \\
k^v_t
\end{array} \right) = \left( \begin{array}{l}
z^v_{11} & z^v_{12} & \cdots & z^v_{1d} \\
z^v_{21} & z^v_{22} & \cdots & z^v_{2d} \\
\vdots & \vdots & \ddots & \vdots \\
z^v_{n1} & z^v_{n2} & \cdots & z^v_{nd} \\
k^v_1 & k^v_2 & \cdots & k^v_d
\end{array} \right).
\]

From (4), \(\tilde{b}\) is uniformly Lipschitz in \((\tilde{y}_t, \tilde{z}_t)\), then equation (1) admits a unique strong solution \((\tilde{y}_t, \tilde{z}_t)\) adapted to the filtration \((\mathcal{F}_t)_{t \geq 0}\).

Define now the function \(\tilde{g}\) from \(\mathbb{R}^{n+1}\) into \(\mathbb{R}\) by

\[
\tilde{g}(\tilde{y}_t) = g(y^v_t) - x^v_t,
\]

and the new functional cost from \(\mathcal{U}\) into \(\mathbb{R}\) by

\[
\tilde{J}(v) = \mathbb{E}\left[ \tilde{g}(\tilde{y}_0) \right] + \mathbb{E}[\eta].
\]

(6)

It’s easy to see that

\[
\tilde{J}(v) = J(v).
\]

Consequently, it’s sufficient to minimize the restricted cost \(\tilde{J}\) over \(\mathcal{U}\). If \(u \in \mathcal{U}\) is an optimal solution, that is

\[
\tilde{J}(u) = \inf_{v \in \mathcal{U}} \tilde{J}(v).
\]

(7)

From this transformation, we have reduce our initial problem \{1, 2, 3\} to a new problem without integral cost. We can now study the restricted
problem \{(5), (6), (7)\} by using a classical way of spike variation method. We establish necessary optimality conditions for a restricted problem and by an adequate transformation on the adjoint process and the adjoint equation associated with the restricted problem, we reformulate necessary optimality conditions for the initial control problem \{(1), (2), (3)\}.

### 3.1 Preliminary results

Suppose that \(u \in U\) is an optimal control and denote by \((\tilde{y}_t, \tilde{z}_t)\) the solution of (5) corresponding to \(u\). Introduce the following perturbation (spike variation) of the optimal control \(u\)

\[
u_t^\theta = \begin{cases} v & \text{if } t \in [\tau, \tau + \theta], \\ u_t & \text{otherwise}, \end{cases}
\]

where \(0 \leq \tau \leq T\) is fixed, \(\theta > 0\) is sufficiently small and \(v\) is an arbitrary \(\mathcal{F}_t\)-measurable random variable with values in \(U\) such that \(\mathbb{E}[|v|^2] < \infty\).

The control \(u^\theta\) is admissible and let \((\tilde{y}_t^\theta, \tilde{z}_t^\theta)\) be the solution of (5) associated with \(u_t^\theta\).

Since \(u\) is optimal, the variational inequality will be derived from the fact that

\[
0 \leq \tilde{J}(u^\theta) - \tilde{J}(u).
\]

For this end, we need the following lemmas.

**Lemma 2** Under assumptions (4), we have

\[
\mathbb{E}\left[\sup_{t \in [0, T]} |\tilde{y}_t^\theta - \tilde{y}_t|^2\right] \leq C\theta^2,
\]

\[
\mathbb{E}\int_0^T |\tilde{z}_t^\theta - \tilde{z}_t|^2 dt \leq C\theta^2.
\]

**Proof.** We have

\[
\begin{align*}
d(\tilde{y}_t^\theta - \tilde{y}_t) &= \left[\tilde{b}(t, \tilde{y}_t^\theta, \tilde{z}_t^\theta, u_t^\theta) - \tilde{b}(t, \tilde{y}_t, \tilde{z}_t, u_t)\right] dt \\
&\quad + \left(\tilde{z}_t^\theta - \tilde{z}_t\right) dW_t,
\end{align*}
\]

\[
(\tilde{y}_T^\theta - \tilde{y}_T) = 0.
\]
Put

\[ Y_t^\theta = \tilde{y}_t^\theta - \tilde{y}_t, \]
\[ Z_t^\theta = \tilde{z}_t^\theta - \tilde{z}_t, \]

and

\[
\varphi^\theta (t, Y_t^\theta , Z_t^\theta ) = \int_0^1 b_y (t, \tilde{y}_t + \lambda (\tilde{y}_t^\theta - \tilde{y}_t) , \tilde{z}_t + \lambda (\tilde{z}_t^\theta - \tilde{z}_t) , u_t^\theta ) Y_t^\theta d\lambda \quad (12)
\]
\[ + \int_0^1 b_z (t, \tilde{y}_t + \lambda (\tilde{y}_t^\theta - \tilde{y}_t) , \tilde{z}_t + \lambda (\tilde{z}_t^\theta - \tilde{z}_t) , u_t^\theta ) Z_t^\theta d\lambda 
\]
\[ + \tilde{b} (t, \tilde{y}_t, \tilde{z}_t, u_t^\theta ) - \tilde{b} (t, \tilde{y}_t, \tilde{z}_t, u_t). \]

Then

\[
\begin{cases}
  dY_t^\theta = \varphi^\theta (t, Y_t^\theta , Z_t^\theta ) dt + Z_t^\theta dW_t, \\
  Y_T^\theta = 0.
\end{cases}
\]

The above equation is a linear BSDE with bounded coefficients and with terminal condition \( Y_T^\theta = 0 \). Then by applying a priori estimates (see Briand et al [8, Proposition 3.2, Page 7]), we get

\[
E \left[ \sup_{t \in [0, T]} |Y_t|^2 + \int_0^T |Z_t|^2 dt \right] \leq C E \left[ \int_0^T \left| \varphi^\theta (t, 0, 0) \right| dt \right]^2.
\]

From (12), we get

\[
E \left[ \sup_{t \in [0, T]} |Y_t|^2 + \int_0^T |Z_t|^2 dt \right] \leq C E \left[ \int_0^T \left| b (t, \tilde{y}_t, \tilde{z}_t, u_t^\theta ) - \tilde{b} (t, \tilde{y}_t, \tilde{z}_t, u_t) \right| dt \right]^2.
\]

By the definition of \( u^\theta \), we have

\[
E \left[ \sup_{t \in [0, T]} |Y_t|^2 + \int_0^T |Z_t|^2 dt \right] \leq C E \left[ \int_\tau^{\tau+\theta} \left| \tilde{b} (t, \tilde{y}_t, \tilde{z}_t, v) - \tilde{b} (t, \tilde{y}_t, \tilde{z}_t, u_t) \right| dt \right]^2
\]
\[ \leq C E \left[ \sup_{t \in [0, T]} \left| \tilde{b} (t, \tilde{y}_t, \tilde{z}_t, v) - \tilde{b} (t, \tilde{y}_t, \tilde{z}_t, u_t) \right| \int_\tau^{\tau+\theta} dt \right]^2.
\]

By (4), \( b \) is with linear growth with respect to \((y, v)\) and bounded in \( z \), then \( \tilde{b} \) satisfy the same properties, and we get

\[
E \left[ \sup_{t \in [0, T]} |Y_t|^2 + \int_0^T |Z_t|^2 dt \right] \leq C \theta^2.
\]

The lemma is proved. \( \blacksquare \)
3.2 Necessary optimality conditions for restricted problem

We can now state necessary optimality conditions for a restricted control problem \{(5), (6), (7)\}.

**Theorem 3** (necessary optimality conditions for restricted problem) Let \((u, \tilde{y}, \tilde{z})\) be an optimal solution of the restricted control problem \{(5), (6), (7)\}. Then there exists a unique adapted process 
\[
\tilde{p} \in L^2 ([0, T] ; \mathbb{R}^{n+1}),
\]
which is solution of the following forward stochastic differential equation
\[
\begin{cases}
-d\tilde{p}_t = \tilde{H}_y (t, \tilde{y}_t, \tilde{z}_t, \tilde{p}_t, u_t) \, dt + \tilde{H}_z (t, \tilde{y}_t, \tilde{z}_t, \tilde{p}_t, u_t) \, dW_t, \\
\tilde{p}_0 = \tilde{g}_y (\tilde{y}_0),
\end{cases}
\]

such that
\[
\tilde{H} (t, \tilde{y}_t, \tilde{z}_t, \tilde{p}_t, u_t) = \max_{v \in U} \tilde{H} (t, \tilde{y}_t, \tilde{z}_t, \tilde{p}_t, v) ; \text{ a.e., a.s.},
\]

where the Hamiltonian \(\tilde{H}\) is defined from \([0, T] \times \mathbb{R}^{n+1} \times M_{(n+1) \times d} (\mathbb{R}) \times \mathbb{R}^{n+1} \times U\) into \(\mathbb{R}\) by
\[
\tilde{H} (t, \tilde{y}_t, \tilde{z}_t, \tilde{p}_t, u_t) = \tilde{b} (t, \tilde{y}_t, \tilde{z}_t, u_t) \tilde{p}_t.
\]

**Proof.** For simplicit, we put
\[
\Lambda^\theta_t = (t, \tilde{y}_t + \lambda \left( \tilde{y}^\theta_0 - \tilde{y}_0 \right), \tilde{z}_t + \lambda \left( \tilde{z}^\theta_0 - \tilde{z}_0 \right), u^\theta_t).
\]

Since \(u\) minimizes the cost \(\tilde{J}\) over \(\mathcal{U}\), then
\[
0 \leq \tilde{J} (u^\theta) - \tilde{J} (u) \leq \mathbb{E} \left[ \tilde{g} (\tilde{y}^\theta_0) - \tilde{g} (\tilde{y}_0) \right]
\leq \mathbb{E} \int_0^1 \tilde{g}_y \left[ \tilde{y}_0 + \lambda \left( \tilde{y}^\theta_0 - \tilde{y}_0 \right) \right] \left( \tilde{y}^\theta_0 - \tilde{y}_0 \right) \, d\lambda
\leq \mathbb{E} \left[ \tilde{g}_y (\tilde{y}_0) (\tilde{y}^\theta_0 - \tilde{y}_0) \right] + \mathbb{E} \int_0^1 \left[ \tilde{g}_y \left( \tilde{y}_0 + \lambda \left( \tilde{y}^\theta_0 - \tilde{y}_0 \right) \right) - \tilde{g}_y (\tilde{y}_0) \right] \left( \tilde{y}^\theta_0 - \tilde{y}_0 \right) \, d\lambda.
\]
We remark from (14) that
\[ \tilde{p}_0 = \tilde{g}_y(\tilde{y}_0). \]

Then
\[ 0 \leq \mathbb{E} \left[ \tilde{p}_0 (\tilde{y}_0^\theta - \tilde{y}_0) \right] + \mathbb{E} \int_0^1 \left[ \tilde{g}_y (\tilde{y}_0 + \lambda (\tilde{y}_0^\theta - \tilde{y}_0)) - \tilde{g}_y (\tilde{y}_0) \right] (\tilde{y}_0^\theta - \tilde{y}_0) \, d\lambda. \]

By applying Itô’s formula to \( \tilde{p}_t (\tilde{y}_t^\theta - \tilde{y}_t) \), we get
\[
\mathbb{E} \left[ \tilde{p}_0 (\tilde{y}_0^\theta - \tilde{y}_0) \right] = \mathbb{E} \int_0^T \int_0^1 \left[ \tilde{b}_y (\Lambda_t^\theta) - \tilde{b}_y (t, \tilde{y}_t, \tilde{z}_t, u_t) \right] (\tilde{y}_t - \tilde{y}_t^\theta) \, \tilde{p}_t \, d\lambda \, dt
+ \mathbb{E} \int_0^T \int_0^1 \left[ \tilde{b}_z (\Lambda_t^\theta) - \tilde{b}_z (t, \tilde{y}_t, \tilde{z}_t, u_t) \right] (\tilde{z}_t - \tilde{z}_t^\theta) \, \tilde{p}_t \, d\lambda \, dt
+ \mathbb{E} \int_0^T \left[ \tilde{b} (t, \tilde{y}_t, \tilde{z}_t, u_t) - \tilde{b} (t, \tilde{y}_t, \tilde{z}_t, u_t^\theta) \right] \, \tilde{p}_t \, dt.
\]

Then (16) becomes
\[
0 \leq \mathbb{E} \int_0^T \left[ \tilde{H} (t, \tilde{y}_t, \tilde{z}_t, \tilde{p}_t, u_t) - \tilde{H} (t, \tilde{y}_t, \tilde{z}_t, \tilde{p}_t, u_t^\theta) \right] \, dt
- \mathbb{E} \int_0^1 \left[ \tilde{g}_y (\tilde{y}_0 + \lambda (\tilde{y}_0^\theta - \tilde{y}_0)) - \tilde{g}_y (\tilde{y}_0) \right] (\tilde{y}_0 - \tilde{y}_0^\theta) \, d\lambda
+ \mathbb{E} \int_0^T \int_0^1 \left[ \tilde{b}_y (\Lambda_t^\theta) - \tilde{b}_y (t, \tilde{y}_t, \tilde{z}_t, u_t) \right] (\tilde{y}_t - \tilde{y}_t^\theta) \, \tilde{p}_t \, d\lambda \, dt
+ \mathbb{E} \int_0^T \int_0^1 \left[ \tilde{b}_z (\Lambda_t^\theta) - \tilde{b}_z (t, \tilde{y}_t, \tilde{z}_t, u_t) \right] (\tilde{z}_t - \tilde{z}_t^\theta) \, \tilde{p}_t \, d\lambda \, dt
\]

Let us show that
\[
\mathbb{E} \int_0^T \int_0^1 \left[ \tilde{b}_y (\Lambda_t^\theta) - \tilde{b}_y (t, \tilde{y}_t, \tilde{z}_t, u_t) \right] (\tilde{y}_t - \tilde{y}_t^\theta) \, \tilde{p}_t \, d\lambda \, dt \leq C\theta^{3/2},
\]
and
\[
\mathbb{E} \int_0^T \int_0^1 \left[ \tilde{b}_z (\Lambda_t^\theta) - \tilde{b}_z (t, \tilde{y}_t, \tilde{z}_t, u_t) \right] (\tilde{z}_t - \tilde{z}_t^\theta) \, \tilde{p}_t \, d\lambda \, dt \leq C\theta^{3/2}.
\]
Indeed, by using the Cauchy-Schwartz inequality to term in the left hand side of (19), we get

$$\mathbb{E} \int_0^T \int_0^1 \left[ \tilde{b}_z (\Lambda_t^\theta) - \tilde{b}_z (t, \tilde{y}_t, \tilde{z}_t, u_t) \right] (\tilde{z}_t - \tilde{z}_t^\theta) \tilde{p}_t \, d\lambda \, dt$$

$$\leq \left( \mathbb{E} \int_0^T \int_0^1 \left| \tilde{b}_z (\Lambda_t^\theta) - \tilde{b}_z (t, \tilde{y}_t, \tilde{z}_t, u_t) \right| \tilde{p}_t \, d\lambda \, dt \right)^{1/2} \left( \mathbb{E} \int_0^T |\tilde{z}_t - \tilde{z}_t^\theta|^2 \, dt \right)^{1/2}.$$

By (11), we obtain

$$\mathbb{E} \int_0^T \int_0^1 \left[ \tilde{b}_z (\Lambda_t^\theta) - \tilde{b}_z (t, \tilde{y}_t, \tilde{z}_t, u_t) \right] (\tilde{z}_t - \tilde{z}_t^\theta) \tilde{p}_t \, d\lambda \, dt$$

$$\leq C\theta \left( \mathbb{E} \int_0^T \int_0^1 \left| \tilde{b}_z (\Lambda_t^\theta) - \tilde{b}_z (t, \tilde{y}_t, \tilde{z}_t, u_t) \right| \tilde{p}_t \, d\lambda \, dt \right)^{1/2}.$$

By the definition of $u^\theta$, we have

$$\mathbb{E} \int_0^T \int_0^1 \left[ \tilde{b}_z (\Lambda_t^\theta) - \tilde{b}_z (t, \tilde{y}_t, \tilde{z}_t, u_t) \right] (\tilde{z}_t - \tilde{z}_t^\theta) \tilde{p}_t \, d\lambda \, dt$$

$$\leq C\theta \left( \mathbb{E} \int_\tau^{\tau+\theta} \int_0^1 \left| \tilde{b}_z (\Lambda_t^\theta) - \tilde{b}_z (t, \tilde{y}_t, \tilde{z}_t, u_t) \right| \tilde{p}_t \, d\lambda \, dt \right)^{1/2}.$$

Since $\tilde{b}_y$ is bounded, we get

$$\mathbb{E} \int_0^T \int_0^1 \left[ \tilde{b}_z (\Lambda_t^\theta) - \tilde{b}_z (t, \tilde{y}_t, \tilde{z}_t, u_t) \right] (\tilde{z}_t^\theta - \tilde{z}_t) \tilde{p}_t \, d\lambda \, dt$$

$$\leq C\theta \left( \int_\tau^{\tau+\theta} \mathbb{E} |\tilde{p}|^2 \, dt \right)^{1/2}.$$

Since $\tilde{p} \in \mathcal{L}^2 ([\tau, T]; \mathbb{R}^{n+1})$, we obtain

$$\mathbb{E} \int_0^T \int_0^1 \left[ \tilde{b}_z (\Lambda_t^\theta) - \tilde{b}_z (t, \tilde{y}_t, \tilde{z}_t, u_t) \right] (\tilde{z}_t^\theta - \tilde{z}_t) \tilde{p}_t \, d\lambda \, dt$$

$$\leq \left( C\int_\tau^{\tau+\theta} dt \right)^{1/2} C\theta = C\theta^{3/2}.$$

Relation (19) is proved.
(18) is proved by the same method and by using (10) and the fact that \(b_y\) is bounded.

Now, by (17), (18) and (19) we get

\[
0 \leq E \int_0^T \left[ \tilde{H}(t, \tilde{y}_t, \tilde{z}_t, \tilde{p}_t, u_t) - \tilde{H}(t, \tilde{y}_t, \tilde{z}_t, \tilde{p}_t, u^0_t) \right] dt
+ E \int_0^1 \left[ \bar{g}_y \left( \tilde{y}_0 + \lambda \left( \tilde{y}_0^0 - \bar{y}_0 \right) \right) - \bar{g}_y (\bar{y}_0) \right] (\tilde{y}_0^0 - \bar{y}_0) d\lambda
+ C\theta^{3/2}.
\]

By applying the Cauchy-Schwartz inequality to the second term in the right hand side of the above inequality, we get

\[
0 \leq E \int_0^T \left[ \tilde{H}(t, \tilde{y}_t, \tilde{z}_t, \tilde{p}_t, u_t) - \tilde{H}(t, \tilde{y}_t, \tilde{z}_t, \tilde{p}_t, u^0_t) \right] dt
+ \left( \int_0^1 E \left| \bar{g}_y \left( \tilde{y}_0 + \lambda \left( \tilde{y}_0^0 - \bar{y}_0 \right) \right) - \bar{g}_y (\bar{y}_0) \right|^2 d\lambda \right)^{1/2}
\left( E |\tilde{y}_0^0 - \bar{y}_0|^2 \right)^{1/2}
+ C\theta^{3/2}.
\]

By (10), we deduce

\[
0 \leq E \int_0^T \left[ \tilde{H}(t, \tilde{y}_t, \tilde{z}_t, \tilde{p}_t, u_t) - \tilde{H}(t, \tilde{y}_t, \tilde{z}_t, \tilde{p}_t, u^0_t) \right] dt
+ C\theta \left( \int_0^1 E \left| \bar{g}_y \left( \tilde{y}_0 + \lambda \left( \tilde{y}_0^0 - \bar{y}_0 \right) \right) - \bar{g}_y (\bar{y}_0) \right|^2 d\lambda \right)^{1/2}
+ C\theta^{3/2}.
\]

From the definition of \(u^0_t\), we have

\[
0 \leq E \int_{\tau}^{\tau+\theta} \left[ \tilde{H}(t, \tilde{y}_t, \tilde{z}_t, \tilde{p}_t, u_t) - \tilde{H}(t, \tilde{y}_t, \tilde{z}_t, \tilde{p}_t, v) \right] dt
+ C\theta \left( \int_0^1 E \left| \bar{g}_y \left( \tilde{y}_0 + \lambda \left( \tilde{y}_0^0 - \bar{y}_0 \right) \right) - \bar{g}_y (\bar{y}_0) \right|^2 d\lambda \right)^{1/2}
+ C\theta^{3/2}.
\]
Dividing by \( \theta \), we get
\[
0 \leq \frac{1}{\theta} \mathbb{E} \int_{\tau}^{\tau + \theta} \left[ \tilde{H} (t, \tilde{y}_t, \tilde{z}_t, \tilde{p}_t, u_t) - \tilde{H} (t, \tilde{y}_t, \tilde{z}_t, \tilde{p}_t, v) \right] dt \tag{20}
\]
\[
+ C \left( \int_0^1 \mathbb{E} \left| \tilde{g}_y (\tilde{y}_0 + \lambda (\tilde{y}_0^g - \tilde{y}_0)) - \tilde{g}_y (\tilde{y}_0) \right|^2 d\lambda \right)^{1/2}
\]
\[
+ C \theta^{1/2}.
\]

Since \( \tilde{g}_y \) is continuous and bounded, then by (10) and the dominated convergence theorem, we have
\[
\lim_{\theta \to 0} C \int_0^1 \left( \mathbb{E} \left| \tilde{g}_y (\tilde{y}_0 + \lambda (\tilde{y}_0^g - \tilde{y}_0)) - \tilde{g}_y (\tilde{y}_0) \right|^2 \right)^{1/2} d\lambda = 0.
\]

Then, by taking the limit as \( \theta \to 0 \) in (20), we obtain
\[
0 \leq \lim_{\theta \to 0} \frac{1}{\theta} \mathbb{E} \int_{\tau}^{\tau + \theta} \left[ \tilde{H} (t, \tilde{y}_t, \tilde{z}_t, \tilde{p}_t, u_t) - \tilde{H} (t, \tilde{y}_t, \tilde{z}_t, \tilde{p}_t, v) \right] dt.
\]

This implies that
\[
0 \leq \mathbb{E} \left[ \tilde{H} (\tau, \tilde{y}_\tau, \tilde{z}_\tau, \tilde{p}_\tau, u_\tau) - \tilde{H} (\tau, \tilde{y}_\tau, \tilde{z}_\tau, \tilde{p}_\tau, a) \right], \ d\tau - a.e.
\]

Now, let \( a \in U \) be a deterministic element and \( F \) be an arbitrary element of the \( \sigma \)-algebra \( \mathcal{F}_t \), and set
\[
w_t = a1_F + u_t1_{\Omega - F}.
\]

It is obvious that \( w \) is an admissible control.

Since \( 0 \leq \tau \leq T \), then for every bounded \( U \)-valued, \( \mathcal{F}_t \)-measurable random variable \( v \) such that \( \mathbb{E}|v|^2 < +\infty \), we get
\[
0 \leq \mathbb{E} \left[ \tilde{H} (t, \tilde{y}_t, \tilde{z}_t, \tilde{p}_t, u_t) - \tilde{H} (t, \tilde{y}_t, \tilde{z}_t, \tilde{p}_t, v) \right], \ dt - a.e,
\]

Applying the above inequality with \( w \), we get
\[
0 \leq \mathbb{E}[1_F(\tilde{H} (t, \tilde{y}_t, \tilde{z}_t, \tilde{p}_t, u_t) - \tilde{H} (t, \tilde{y}_t, \tilde{z}_t, \tilde{p}_t, a))], \ \forall F \in \mathcal{F}_t,
\]

which implies that
\[
0 \leq \mathbb{E}[\tilde{H} (t, \tilde{y}_t, \tilde{z}_t, \tilde{p}_t, u_t) - \tilde{H} (t, \tilde{y}_t, \tilde{z}_t, \tilde{p}_t, a) / \mathcal{F}_t].
\]

The quantity inside the conditional expectation is \( \mathcal{F}_t \)-measurable, and thus the result follows immediately. This prove theorem 3.
4 Necessary and sufficient optimality conditions for strict controls

Starting from the results of the last section, we can now reformulate the restricted necessary optimality conditions given by theorem 3, and state necessary as well as sufficient optimality conditions for the initial control problem \{(1), (2), (3)\}.

4.1 Necessary optimality conditions

**Theorem 4** (necessary optimality conditions for strict controls) Let \((u, y^u, z^u)\) be an optimal solution of the initial control problem \{(1), (2), (3)\}. Then there exists a unique adapted processes

\[ p^u \in L^2 ([0, T]; \mathbb{R}^n) , \]

which are solution of the following forward stochastic differential equation

\[
\begin{cases}
-dp^u_t = H_y (t, y^u_t, z^u_t, p^u_t, u_t) \, dt + H_z (t, y^u_t, z^u_t, p^u_t, u_t) \, dW_t, \\
p^u_0 = g_y (y^u_0),
\end{cases}
\]

such that

\[ H (t, y^u_t, z^u_t, p^u_t, u_t) = \max_{v \in U} H (t, y^u_t, z^u_t, p^u_t, v) ; \ a.e \ , \ a.s, \ (22) \]

where the Hamiltonian \(H\) is defined from \([0, T] \times \mathbb{R}^n \times \mathcal{M}_{n \times d} (\mathbb{R}) \times \mathbb{R}^n \times U\) into \(\mathbb{R}\) by

\[ H (t, y, z, p, v) = pb (t, y, z, v) - h (t, y, z, v) . \]

**Proof.** We put

\[ \tilde{p}_t = \begin{pmatrix} p^u_t \\ -1 \end{pmatrix} . \]

From the definition of \(\tilde{H}, \tilde{p}, \tilde{b}\) and \(\tilde{z}\), we have

\[ \tilde{H} (t, \tilde{y}_t, \tilde{z}_t, \tilde{p}_t, u_t) = H (t, y^u_t, z^u_t, p^u_t, u_t) , \]

and from the adjoint equation (14), we can easily deduce (21). Finally (22) is derived immediately from (23) and (15).
4.2 Sufficient optimality conditions

Theorem 5 (Sufficient optimality conditions for strict controls). If we assume that, \( U \) is convex and for every \( v \in U \) and for all \( t \in [0, T] \), the function \( g \) is convex and \((y_t, z_t, v_t) \rightarrow H(t, y_t, z_t, p_t, v_t)\) is concave. Then \( u \) is an optimal control of the problem \{\( (1), (2), (3) \)\} if it satisfies (22).

Proof. Let \( u \) be an arbitrary admissible control (candidate to be optimal) and \((y^u_t, z^u_t)\) the solution of (1) associated with \( u \). For any admissible control \( v \), with associated trajectory \((y^v_t, z^v_t)\), we have

\[
J(v) - J(u) = \mathbb{E} \left[ g(y^v_0) - g(y^u_0) \right] + \mathbb{E} \int_0^T \left[ h(t, y^v_t, z^v_t, v_t) - h(t, y^u_t, z^u_t, u_t) \right] dt.
\]

Since \( g \) is convex, then

\[
g(y^v_t) - g(y^u_t) \geq g_y(y^u_0) (y^v_0 - y^u_0).
\]

Then

\[
J(v) - J(u) \geq \mathbb{E} \left[ g_y(y^u_0) (y^v_0 - y^u_0) \right] + \mathbb{E} \int_0^T \left[ h(t, y^v_t, z^v_t, v_t) - h(t, y^u_t, z^u_t, u_t) \right] dt.
\]

We remark from (21) that

\[
p^u_0 = g_y(y^u_0).
\]

Then, we have

\[
J(v) - J(u) \geq \mathbb{E} \left[ p^u_0 (y^v_0 - y^u_0) \right] + \mathbb{E} \int_0^T \left[ h(t, y^v_t, z^v_t, v_t) - h(t, y^u_t, z^u_t, u_t) \right] dt.
\]

By applying Itô’s formula to \( p^u_t (y^v_t - y^u_t) \), we obtain

\[
J(v) - J(u) \geq \mathbb{E} \int_0^T \left[ H_y(t, y^u_t, z^u_t, p^u_t, u_t) (y^v_t - y^u_t) + H_z(t, y^u_t, z^u_t, p^u_t, u_t) (z^v_t - z^u_t) \right] dt
\]

\[
+ \mathbb{E} \int_0^T \left[ H(t, y^v_t, z^v_t, p^v_t, v_t) - H(t, y^v_t, z^v_t, p^u_t, v_t) \right] dt.
\]
Since $H$ is concave in $(y, z, u)$, then
\[
H(t, y_t^v, z_t^v, p_t^u, v_t) - H(t, y_t^u, z_t^u, p_t^u, u_t) \\
\leq H_y(t, y_t^u, z_t^u, p_t^u, u_t) (y_t^v - y_t^u) \\
+ H_z(t, y_t^u, z_t^u, p_t^u, u_t) (z_t^v - z_t^u) + H_v(t, y_t^u, z_t^u, p_t^u, u_t) (v_t - u_t). 
\]

Or equivalently
\[
H_v(t, y_t^u, z_t^u, p_t^u, u_t) (u_t - v_t) \\
\leq H(t, y_t^u, z_t^u, p_t^u, u_t) - H(t, y_t^v, z_t^v, p_t^u, v_t) \\
+ H_y(t, y_t^u, z_t^u, p_t^u, u_t) (y_t^v - y_t^u) + H_z(t, y_t^u, z_t^u, p_t^u, u_t) (z_t^v - z_t^u). 
\]

Then, we get
\[
J(v) - J(u) \geq \mathbb{E} \int_0^T H_v(t, y_t^u, z_t^u, p_t^u, u_t) (u_t - v_t) dt. \tag{24}
\]

We know that $H(t, y_t^u, z_t^u, p_t^u, .)$ is concave, then $-H(t, y_t^u, z_t^u, p_t^u, .)$ is convex from $U$ into $\mathbb{R}$. Furthermore $U$ is convex and $-H(t, y_t^u, z_t^u, p_t^u, .)$ is continuous, Gâteaux-differentiable, with differential continuous, then from the convex optimization principle (see Ekeland-Temam [11, prop 2.1, page 35]), we have
\[
-H(t, y_t^u, z_t^u, p_t^u, v_t) = \inf_{v_t \in U} -H(t, y_t^u, z_t^u, p_t^u, v_t) \iff -H_v(t, y_t^u, z_t^u, p_t^u, u_t) (v_t - u_t) \geq 0. 
\]

Or equivalently
\[
H(t, y_t^u, z_t^u, p_t^u, u_t) = \max_{v_t \in U} H(t, y_t^u, z_t^u, p_t^u, v_t) \iff H_v(t, y_t^u, z_t^u, p_t^u, u_t) (u_t - v_t) \geq 0. 
\]

Then from the necessary condition of optimality (22), we deduce that
\[
H_v(t, y_t^u, z_t^u, p_t^u, u_t) (u_t - v_t) \geq 0. 
\]

And from (24), we have
\[
J(v) - J(u) \geq 0. 
\]

The theorem is proved.
5 The relaxed model

In this section, we generalize the results of the above section to a relaxed control problem. The idea for relaxed the strict control problem defined above is to embed the set $U$ of strict controls into a wider class which gives a more suitable topological structure. In the relaxed model, the $U$-valued process $v$ is replaced by a $\mathbb{P}(U)$-valued process $q$, where $\mathbb{P}(U)$ denotes the space of probability measure on $U$ equipped with the topology of stable convergence.

Let $V$ the set of positive random measures on $[0, T] \times U$ whose projection on $[0, T]$ coincide with the Lebesgue measure $dt$. Equipped with the topology of stable convergence of measures, $V$ is a compact metrizable space. The stable convergence is required for bounded measurable functions $f(t, a)$ such that for each fixed $t \in [0, T]$, $h(t, .)$ is continuous. The space $V$ is equipped with its Borel $\sigma$-field, which is the smallest $\sigma$-field such that the mapping $q \mapsto \int f(s, a) q(ds, da)$ are measurable for any bounded measurable function $f$, continuous with respect to $a$ (Instead of functions bounded and continuous with respect to the pair $(t, a)$ for the weak topology).

For more details, see Jacod-Memin [18, page 629-630] and El Karoui et al [9, Page 4-5].

**Definition 6** A relaxed control $(q_t)_t$ is a $\mathbb{P}(U)$-valued process, progressively measurable with respect to $(\mathcal{F}_t)_t$ and such that for each $t$, $1_{[0,t]} q$ is $\mathcal{F}_t$-measurable.

We denote by $\mathcal{R}$ the set of all relaxed controls.

Every relaxed control $q$ may be desintegrated as $q(dt, da) = q_t(da) dt = q_t(da) dt$, where $q_t(da)$ is a progressively measurable process with value in the set of probability measures $\mathbb{P}(U)$.

The set $U$ is embedded into the set $\mathcal{R}$ of relaxed process by the mapping

$$f : v \in U \mapsto f_v(dt, da) = \delta_{v_t}(da) dt \in \mathcal{R}$$

where $\delta_v$ is the atomic measure concentrated at a single point $v$.

For more details on relaxed controls, see [2], [4], [5], [12], [16], [26], [27].
For any $q \in \mathbb{R}$, we consider the following relaxed BSDE

$$
\begin{align*}
\tau \frac{dy_t^q}{dt} &= \int_U b(t, y_t^q, z_t^q, a) q_t(da) dt + z_t^q dW_t, \\
y_T^q &= \xi.
\end{align*}
$$

(25)

The expected cost associated to a relaxed control $q$ is defined as follows

$$
J(q) = \mathbb{E} \left[ g(y_0^q) + \int_0^T \int_U h(t, y_t^q, z_t^q, a) q_t(da) dt \right].
$$

(26)

Our objective is to minimize the functional $J$ over $\mathcal{R}$. If $\mu \in \mathcal{R}$ is an optimal relaxed control, that is

$$
J(\mu) = \inf_{q \in \mathcal{R}} J(q).
$$

(27)

Throughout this section we suppose moreover that

- $U$ is compact,
- $b$ and $h$ are bounded,
- $b_y, h_y, b_z$ and $h_z$ are Lipschitz continuous in $z$.

Remark 7 If we put

$$
\begin{align*}
\overline{b}(t, y_t^q, z_t^q, q_t) &= \int_U b(t, y_t^q, z_t^q, a) q_t(da), \\
\overline{h}(t, y_t^q, z_t^q, q_t) &= \int_U h(t, y_t^q, z_t^q, a) q_t(da),
\end{align*}
$$

then equation (25) becomes

$$
\begin{align*}
\tau \frac{dy_t^q}{dt} &= \overline{b}(t, y_t^q, z_t^q, q_t) dt + z_t^q dW_t, \\
y^q(T) &= \xi,
\end{align*}
$$

with a functional cost given by

$$
J(q) = \mathbb{E} \left[ g(y_0^q) + \int_0^T \overline{h}(t, y_t^q, z_t^q, q_t) dt \right].
$$

Hence by introducing relaxed controls, we have replaced $U$ by a larger space $\mathbb{P}(U)$. We have gained the advantage that $\mathbb{P}(U)$ is both compact and convex, the new drift and the integral coefficient of $J$ are linear in $q$. 

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On the other hand, the coefficients $\overline{b}$ (defined above) check the same assumptions as $b$. Then, under assumptions (4), $\overline{b}$ is uniformly Lipschitz and with linear growth. Then, by classical results on BSDEs (The Pardoux-Peng theorem, see: Pardoux-Peng [28]), for every $q \in \mathcal{R}$, equation (25) has a unique solution.

Moreover, it is easy to see that $\overline{h}$ checks the same assumptions as $h$. Then, the functional cost $J$ is well defined from $\mathcal{R}$ into $\mathbb{R}$.

Remark 8 If $q_t = \delta_{v_t}$ is an atomic measure concentrated at a single point $v_t$, then for each $t \in [0,T]$ we have

$$\int_U b(t, y_t^q, z_t^q, a) q_t(da) = \int_U b(t, y_t^q, z_t^q, a) \delta_{v_t}(da) = b(t, y_t^q, z_t^q, v_t),$$

$$\int_U h(t, y_t^q, z_t^q, a) q_t(da) = \int_U h(t, y_t^q, z_t^q, a) \delta_{v_t}(da) = h(t, y_t^q, z_t^q, v_t).$$

In this case $(y^q, z^q) = (y^v, z^v)$, $J(v) = J(q)$ and we get an ordinary admissible control problem. So the problem of strict controls defined in the section 2 is a particular case of the problem of relaxed one.

5.1 Approximation of trajectories

The next lemma, known as the Chattering Lemma, tells us that any relaxed control is a stable limit of a sequence of strict controls. This lemma was first proved for deterministic measures and then extended to random measures in [12] and [16].

Lemma 9 (Chattering Lemma). Let $q_t$ be a predictable process with values in the space of probability measures on $U$. Then there exists a sequence of predictable processes $(u^n)_n$ with values in $U$ such that

$$dtq^n_t(da) = dt\delta_{u^n_t}(da) \longrightarrow dtq_t(da) \text{ stably, } \mathcal{P} - a.s. \quad (29)$$

Proof. See El Karoui et al [12].

Lemma 10 Let $q$ be a relaxed control and $(u^n)_n$ be a sequence of strict controls such that (29) holds. Then for any bounded function $f : [0,T] \times U \to \mathbb{R}$, measurable in $t$ and continuous in $a$, we have

$$\int_U f(t, a) \delta_{u^n_t}(da) \longrightarrow \int_U f(t, a) q_t(da). \quad (30)$$
Proof. By the Chattering lemma and the definition of the stable convergence (see Jacod-Memin [21, definition 1.1, page 529], we have

\[
\int_0^T \int_U f(t, a) \delta_{u^n_t}(da) \, dt \xrightarrow{n \to \infty} \int_0^T \int_U f(t, a) q_t(da) \, dt.
\]

Put \(g(s, a) = 1_{[0,t]}(s) f(s, a)\).

It's clear that

\[
\int_0^T \int_U g(s, a) \delta_{u^n_s}(da) \, ds \xrightarrow{n \to \infty} \int_0^T \int_U g(s, a) q_s(da) \, ds.
\]

Then

\[
\int_U \int_0^t f(s, a) \delta_{u^n_s}(da) \, ds \xrightarrow{n \to \infty} \int_U \int_0^t f(s, a) q_s(da) \, ds.
\]

The set \(\{(s, t) \, ; \, 0 \leq s \leq t \leq T\}\) generate \(\mathcal{B}_{[0,T]}\). Then \(\forall B \in \mathcal{B}_{[0,T]}\), we have

\[
\int_B \int_U f(s, a) \delta_{u^n_s}(da) \, ds \xrightarrow{n \to \infty} \int_B \int_U f(s, a) q_s(da) \, ds.
\]

This implies that

\[
\int_U f(s, a) \delta_{u^n_s}(da) \xrightarrow{n \to \infty} \int_U f(s, a) q_s(da), \quad dt - a.e.
\]

The lemma is proved. \(\blacksquare\)

The next lemma gives the stability of the controlled stochastic differential equation with respect to the control variable.

Lemma 11 Let \(q_t \in \mathcal{R}\) be a relaxed control and \((y^n, z^n)\) the corresponding trajectory. Then there exists a sequence \((u^n)_n \subset \mathcal{U}\) such that

\[
\lim_{n \to \infty} \mathbb{E} \left[ \sup_{t \in [0,T]} |y^n_t - y^q_t|^2 \right] = 0, \quad (31)
\]

\[
\lim_{n \to \infty} \mathbb{E} \int_0^T |z^n_t - z^q_t|^2 \, dt = 0, \quad (32)
\]

\[
\lim_{n \to \infty} J(u^n) = J(q), \quad (33)
\]

where \((y^n, z^n)\) denotes the solution of equation (1) associated with \(u^n\).
Proof. We have
\[ d (y^n_t - y^q_t) = \left[ b \left( t, y^n_t, z^n_t, u^n_t \right) - b \left( t, y^q_t, z^q_t, u^q_t \right) \right] dt \]
\[ + \left[ b \left( t, y^n_t, z^n_t, u^n_t \right) - \int b \left( t, y^n_t, z^n_t, a \right) q_t \left( da \right) \right] dt \]
\[ + (z^n_t - z^q_t) dW_t \]

Put
\[ Y^n_t = y^n_t - y^q_t, \]
\[ Z^n_t = z^n_t - z^q_t, \]
and
\[ \varphi^n (t, Y^n_t, Z^n_t) = b \left( t, y^n_t, z^n_t, u^n_t \right) - \int b \left( t, y^n_t, z^n_t, a \right) q_t \left( da \right) \]
\[ + \int_0^1 b_y (t, y^n_t + \lambda (y^n_t - y^q_t), z^n_t + \lambda (z^n_t - z^q_t), u^n_t) Y^n_t d\lambda \]
\[ + \int_0^1 b_z (t, y^n_t + \lambda (y^n_t - y^q_t), z^n_t + \lambda (z^n_t - z^q_t), u^n_t) Z^n_t d\lambda. \]

Then
\[ \begin{cases} dY^n_t = \varphi^n (t, Y^n_t, Z^n_t) dt + Z^n_t dW_t, \\
Y^n_T = 0. \end{cases} \]

The above equation is a linear BSDE with bounded coefficients and with terminal condition \( Y^n_T = 0 \), then by applying a priori estimates (see Briand et al [8]), we get
\[ \mathbb{E} \left[ \sup_{t \in [0,T]} \left| Y^n_t \right|^2 + \int_0^T \left| Z^n_t \right|^2 dt \right] \leq C \mathbb{E} \left[ \int_0^T |\varphi^n (t, 0, 0)| dt \right]^2. \]

From (34), we get
\[ \mathbb{E} \left[ \sup_{t \in [0,T]} \left| Y^n_t \right|^2 + \int_0^T \left| Z^n_t \right|^2 dt \right] \leq C \mathbb{E} \left[ \int_0^T \left| \varphi^n (t, 0, 0) \right| dt \right]^2. \]
By (30) and the dominated convergence theorem, the term in the right hand side of the above inequality tends to zero as \( n \) tends to infinity. This prove (31) and (32).

Let us prove (33)

Since \( g \) and \( h \) are Lipshitz continuous in \((y, z)\), then by using the Cauchy-Schwartz inequality, we have

\[
|J(u^n) - J(q)|
\leq C \left( \mathbb{E} |y_0^n - y_0^q|^2 \right)^{1/2} + C \left( \int_0^T \mathbb{E} |y_t^n - y_t^q|^2 \, ds \right)^{1/2} + C \left( \int_0^T |z_t^n - z_t^q|^2 \, dt \right)^{1/2}
+ C \left( \mathbb{E} \int_0^T \left| \int_U b(t, y_t^n, z_t^n, a) \delta_{u^n_t} (da) - \int_U h(t, y_t^q, z_t^q, a) q_t (da) \right|^2 \, dt \right)^{1/2}.
\]

From (31) and (32) the first, the second and the third terms in the right hand side converge to zero, and by (30) and the dominated convergence theorem, the fourth term in the right hand side tends to zero. □

Remark 12 As a consequence, it is easy to see that the strict and relaxed optimal control problems have the same value function.

5.2 necessary optimality conditions for near controls

In this section we derive necessary optimality conditions for near optimal controls. This result is based on Ekeland’s variational principle which is given by the following.

Lemma 13 (Ekeland’s variational principle). Let \((E, d)\) be a complete metric space and \( f : E \rightarrow \mathbb{R} \) be lower-semicontinuous and bounded from below. Given \( \varepsilon > 0 \), suppose \( u^\varepsilon \in E \) satisfies \( f(u^\varepsilon) \leq \inf (f) + \varepsilon \). Then for any \( \lambda > 0 \), there exists \( v \in E \) such that

1. \( f(v) \leq f(u^\varepsilon) \).
2. \( d(u^\varepsilon, v) \leq \lambda \).
3. \( f(v) < f(w) + \frac{\varepsilon}{\lambda} d(v, w) \), \( \forall w \neq v \).
Proof. See Ekeland [10].

To apply Ekeland’s variational principle, we have to endow the set \( \mathcal{U} \) of strict controls with an appropriate metric. For any \( u, v \in \mathcal{U} \), we set

\[
d(u, v) = \mathcal{P} \otimes dt \left\{ (\omega, t) \in \Omega \times [0, T], \ u(t, \omega) \neq v(t, \omega) \right\},
\]

where \( \mathcal{P} \otimes dt \) is the product measure of \( \mathcal{P} \) with the Lebesgue measure \( dt \).

Let us summarize some of the properties satisfied by \( d \).

Lemma 14

1. \( (\mathcal{U}, d) \) is a complete metric space.

2. The cost functional \( J \) is continuous from \( \mathcal{U} \) into \( \mathbb{R} \).

Proof. See Mezerdi [25].

Now let \( \mu \in \mathcal{R} \) be an optimal relaxed control and denote by \( (y^n_\mu, z^n_\mu) \) the trajectory of the system controlled by \( \mu \). From lemmas 9, 10 and 11, there exists a sequence \( (u^n) \) of strict controls such that

\[
dt \mu^n_t (da) = dt \delta_{u^n_t} (da) \xrightarrow{n \to \infty} dt \mu_t (da) \quad \text{Stably,} \quad \mathcal{P}\text{-a.s},
\]

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |y^n_t - y^n_\mu|^2 \right] \xrightarrow{n \to \infty} 0,
\]

\[
\mathbb{E} \int_0^T |z^n_t - z^n_\mu|^2 dt \xrightarrow{n \to \infty} 0.
\]

where \( (y^n_t, z^n_t) \) is the solution of equation (25) controlled by \( \mu^n \).

According to the optimality of \( \mu \) and (29), there exists a sequence \( (\varepsilon_n) \) of positive real numbers with \( \lim_{n \to \infty} \varepsilon_n = 0 \) such that

\[
J(u^n) = J(\mu^n) \leq J(\mu) + \varepsilon_n.
\]

A suitable version of lemma 13 implies that, given any \( \varepsilon_n > 0 \), there exists \( (u^n) \in \mathcal{U} \) such that

\[
J(u^n) \leq \inf_{u \in \mathcal{U}} J(u) + \varepsilon_n,
\]

\[
J(u^n) \leq J(u) + \varepsilon_n d(u^n, u) \quad \forall u \in \mathcal{U}.
\]

(37)
Let us define the perturbation
\[ u_{t}^{n,\theta} = \begin{cases} 
  v & \text{if } t \in [\tau, \tau + \theta], \\
  u_{t}^{n} & \text{Otherwise.}
\end{cases} \tag{38} \]

From (37) we have
\[ 0 \leq J\left(u_{t}^{n,\theta}\right) - J(u^{n}) + \varepsilon_{n} d\left(u^{n,\theta}, u_{t}^{n}\right). \tag{39} \]

From the definition of the metric \( d \), we obtain
\[ 0 \leq J\left(u_{t}^{n,\theta}\right) - J(u^{n}) + \varepsilon_{n} C\theta. \tag{39} \]

From these above inequalities, we shall establish necessary optimality conditions for near optimal controls.

**Theorem 15** (Necessary optimality conditions for near controls). For each \( \varepsilon_{n} > 0 \), there exists \( (u^{n})_{n} \in U \) such that there exists a unique adapted process
\[ p^{n} \in L^{2}\left([0, T]; \mathbb{R}^{n}\right), \]
solution of the following forward stochastic differential equation
\[ \begin{cases} 
  -dp_{t}^{n} = H_{y}(t, y_{t}^{n}, z_{t}^{n}, p_{t}^{n}, u_{t}^{n}) dt + H_{z}(t, y_{t}^{n}, z_{t}^{n}, p_{t}^{n}, u_{t}^{n}) dW_{t}, \\
  p_{0}^{n} = g_{y}(y_{0}^{n}),
\end{cases} \tag{40} \]
such that for all \( v \in U \),
\[ 0 \leq [H(t, y_{t}^{n}, z_{t}^{n}, p_{t}^{n}, u_{t}^{n}) - H(t, y_{t}^{n}, z_{t}^{n}, p_{t}^{n}, v)] + C\varepsilon_{n}. \tag{41} \]

**Proof.** From inequality (39), we use the same method as in the last sections with index \( n \). \( \blacksquare \)

### 5.3 Necessary and sufficient optimality conditions for relaxed controls

In this subsection, we will state and prove necessary as well as sufficient optimality conditions for relaxed controls. For this end, let us summarize and prove some of lemmas that we will use thereafter.
Introduce the following adjoint equation in the relaxed form
\[
\begin{aligned}
- dp_t^\mu &= H_y^\mu (t, y_t^\mu, z_t^\mu, p_t^\mu, \mu_t) \, dt + H_z^\mu (t, y_t^\mu, z_t^\mu, p_t^\mu, \mu_t) \, dW_t, \\
p_0^\mu &= g_y (y_0^\mu),
\end{aligned}
\]
where the Hamiltonian $H^\mu$ in the relaxed form is defined from $[0, T] \times \mathbb{R}^n \times \mathcal{M}_{n \times d} (\mathbb{R}) \times \mathbb{R}^n \times \mathbb{P} (U)$ into $\mathbb{R}$ by
\[
H^\mu (t, y_t^\mu, z_t^\mu, p_t^\mu, \mu_t) = p_t^\mu \int_U b (t, y_t^\mu, z_t^\mu, a) \mu_t (a) - \int_U h (t, y_t^\mu, z_t^\mu, a) \mu_t (a).
\]
For simplicity of notation, we denote
\[
\begin{aligned}
f_t^n (t) &= f (t, y_t^n, z_t^n, a_t^n), \\
f_t^\mu (t) &= \int_U (t, y_t^\mu, z_t^\mu, a) \mu_t (a),
\end{aligned}
\]
where $f$ stands for one of the functions $b_y, b_z, h_y, h_z$.

**Lemma 16** The following estimations hold
\[
\begin{aligned}
\lim_{n \to \infty} \mathbb{E} \int_0^t |b_y^n (s) - b_y^\mu (s)|^2 \, ds &= 0, \\
\lim_{n \to \infty} \mathbb{E} \int_0^t |b_z^n (s) - b_z^\mu (s)|^2 \, ds &= 0, \\
\lim_{n \to \infty} \mathbb{E} \int_0^t |h_y^n (s) - h_y^\mu (s)|^2 \, ds &= 0, \\
\lim_{n \to \infty} \mathbb{E} \int_0^t |h_z^n (s) - h_z^\mu (s)|^2 \, ds &= 0.
\end{aligned}
\]

**Proof.** We have
\[
\begin{aligned}
\mathbb{E} \int_0^t |b_y^n (s) - b_y^\mu (s)|^2 \, ds &= \mathbb{E} \int_0^t \left| b_y (s, y_s^n, z_s^n, u_s^n) - \int_U b_y (s, y_s^\mu, z_s^\mu, a) \mu_s (a) \right|^2 \, ds \\
&\leq \mathbb{E} \int_0^t \left| b_y (s, y_s^n, z_s^n, u_s^n) - b_y (s, y_s^\mu, z_s^\mu, u_s^n) \right|^2 \, ds \\
&\quad + \mathbb{E} \int_0^t \left| b_y (s, y_s^\mu, z_s^\mu, u_s^n) - b_y (s, y_s^\mu, z_s^\mu, u_s^\mu) \right|^2 \, ds \\
&\quad + \mathbb{E} \int_0^t \left| b_y (s, y_s^\mu, z_s^\mu, u_s^\mu) - \int_U b_y (s, y_s^\mu, z_s^\mu, a) \mu_s (a) \right|^2 \, ds.
\end{aligned}
\]
Since $b_y$ is Lipschitz continuous in $z$, then
\[
\mathbb{E} \int_0^t |b^n_y(s) - b^\mu_y(s)|^2 \, ds \leq \mathbb{E} \int_0^t |b_y(s, y^n_s, z^n_s, u^n_s) - b_y(s, y^\mu_s, z^n_s, u^n_s)|^2 \, ds \\
+ C \mathbb{E} \int_0^t |z^n_s - z^\mu_s|^2 \, ds \\
+ \mathbb{E} \int_0^t |b_y(s, y^n_s, z^n_s, u^n_s) - \int_U b_y(s, y^\mu_s, z^\mu_s, a) \mu_s(a)|^2 \, ds.
\]

From (32), we have
\[
\lim_{n \to \infty} \mathbb{E} \int_0^t |z^n_s - z^\mu_s|^2 \, ds = 0.
\]

Since $b_y$ is bounded and continuous, then by (31) and the dominate convergence theorem, we have
\[
\lim_{n \to \infty} \mathbb{E} \int_0^t |b_y(s, y^n_s, z^n_s, u^n_s) - b_y(s, y^\mu_s, z^n_s, u^n_s)|^2 \, ds = 0.
\]

On the other hand, by the chattering lemma and the dominate convergence theorem, we have
\[
\lim_{n \to \infty} \mathbb{E} \int_0^t \left| \int_U b_y(s, y^\mu_s, z^\mu_s, a) \delta_{w^n_s}(da) - \int_U b_y(s, y^\mu_s, z^\mu_s, a) \mu_s(a) \right|^2 \, ds = 0.
\]

By (47) and these above three limits, we deduce (43). Using the same method and arguments, we prove (44), (45) and (46).

**Lemma 17** Let $p^n$ and $p^\mu$ respectively the solutions of (40) and (42), then we have
\[
\lim_{n \to \infty} \mathbb{E} \left[ \sup_{t \in [0, T]} |p^n_t - p^\mu_t|^2 \right] = 0.
\]

**Proof.** From (40) and (42), we have
\[
p^n_t = g_y(y^n_0) - \int_0^t H^n_y(s) \, ds - \int_0^t H^n_z(s) \, dW_s,
\]
\[
p^\mu_t = g_y(y^\mu_0) - \int_0^t H^\mu_y(s) \, ds - \int_0^t H^\mu_z(s) \, dW_s,
\]
where

\[ H^n_y (t) = H_y (t, y^n_t, z^n_t, p^n_t, u^n_t) ; \quad H^n_y (t) = \int_U H_y (t, y^n_t, z^n_t, p^n_t, a) \mu_t (a) , \]

\[ H^n_z (t) = H_z (t, y^n_t, z^n_t, p^n_t, u^n_t) ; \quad H^n_z (t) = \int_U H_z (t, y^n_t, z^n_t, p^n_t, a) \mu_t (a) . \]

Then

\[
\mathbb{E} |p^n_t - p^\mu_t|^2 \leq C \mathbb{E} |g_y (y^n_0) - g_y (y^\mu_0)|^2 + C \mathbb{E} \int_0^t \left| H^n_y (s) - H^\mu_y (s) \right|^2 ds
\]

\[
\quad + C \mathbb{E} \int_0^t \left| H^n_z (s) - H^\mu_z (s) \right|^2 ds
\]

\[
\quad \leq C \mathbb{E} \int_0^t \left| b^n_y (s) (p^n_s - p^\mu_s) \right|^2 ds + C \mathbb{E} \int_0^t \left| b^n_z (s) (p^n_s - p^\mu_s) \right|^2 ds + C\alpha^n_t ,
\]

where

\[
\alpha^n_t = \mathbb{E} |g_y (y^n_0) - g_y (y^\mu_0)|^2 + \mathbb{E} \int_0^t \left| h^n_y (s) - h^\mu_y (s) \right|^2 ds
\]

\[
\quad + \mathbb{E} \int_0^t \left| (b^n_y (s) - b^\mu_y (s)) p^n_s \right|^2 ds + \mathbb{E} \int_0^t \left| h^n_z (s) - h^\mu_z (s) \right|^2 ds
\]

\[
\quad + \mathbb{E} \int_0^t \left| (b^n_z (s) - b^\mu_z (s)) p^n_s \right|^2 ds .
\]

Since \( b_y \) and \( b_z \) are bounded then

\[
\mathbb{E} |p^n_t - p^\mu_t|^2 \leq 2C \mathbb{E} \int_0^t \left| p^n_s - p^\mu_s \right|^2 ds + C\alpha^n_t .
\]

Let us prove that \( \lim_{n \to \infty} \alpha^n_t = 0 \)

Since \( g_y \) is bounded and continuous, then by (31) and the dominated convergence theorem, we have

\[
\lim_{n \to \infty} \mathbb{E} |g_y (y^n_0) - g_y (y^\mu_0)|^2 = 0 .
\]

On the other hand, since \( b_y \) is bounded, then

\[
\left| (b^n_y (s) - b^\mu_y (s)) p^n_s \right| \leq 2C \left| p^n_s \right| .
\]
Hence by the Cauchy-Schwartz inequality we get,
\[
E \int_0^t \left| b^n_y(s) - b^\mu_y(s) \right| p^\mu_s ds \leq \left( E \int_0^t \left| b^n_y(s) - b^\mu_y(s) \right|^2 ds \right)^{1/2} \left( E \int_0^t |p^\mu_s|^2 ds \right)^{1/2}.
\]

Since \( p^\mu \in \mathcal{L}^2([0, T]; \mathbb{R}^n) \), then
\[
E \int_0^t \left| b^n_y(s) - b^\mu_y(s) \right| p^\mu_s ds \leq C \left( E \int_0^t \left| b^n_y(s) - b^\mu_y(s) \right|^2 ds \right)^{1/2}.
\]

By (43), we have
\[
\lim_{n \to \infty} E \int_0^t \left| b^n_y(s) - b^\mu_y(s) \right|^2 ds = 0.
\]

Then, we deduce that
\[
\lim_{n \to \infty} E \int_0^t \left| b^n_y(s) - b^\mu_y(s) \right| p^\mu_s ds = 0. \tag{53}
\]

By using the dominated convergence theorem we obtain
\[
\lim_{n \to \infty} E \int_0^t \left| b^n_y(s) - b^\mu_y(s) \right| p^\mu_s|^2 ds = 0. \tag{54}
\]

Similarly, using (44), the boundeness of \( b_z \) and the dominated convergence theorem, it follows that
\[
\lim_{n \to \infty} E \int_0^t \left| b^n_z(s) - b^\mu_z(s) \right| p^\mu_s|^2 ds = 0. \tag{55}
\]

From (45), (46), (51), (54) and (55), it is easy to see that
\[
\lim_{n \to \infty} \alpha^n_t = 0. \tag{56}
\]

Finally from (50), (56), Gronwall’s lemma and Bukholder-Davis-Gundy inequality, we have the desired result. \( \blacksquare \)
Theorem 18 \textbf{(Necessary optimality conditions for relaxed controls).} Let $\mu$ be an optimal relaxed control minimizing the cost $J$ over $\mathcal{R}$ and $(y^\mu_t, z^\mu_t)$ the corresponding optimal trajectory. Then there exists a unique adapted processes $p^\mu \in L^2([0, T]; \mathbb{R}^n)$, solution of the stochastic forward differential equation (42), such that for all $q \in \mathcal{R}$, we have

$$H^\mu (t, y^\mu_t, z^\mu_t, p^\mu_t, \mu_t) = \max_{q \in \mathcal{P}(U)} H^\mu (t, y^\mu_t, z^\mu_t, p^\mu_t, q).$$

(57)

\textbf{Proof.} Let $\mu$ be an optimal relaxed control. By the necessary condition for near controls (Theorem 15), there exists a sequence $(u^n)_n \subset \mathcal{U}$ such that for all $v \in \mathcal{U}$

$$0 \leq [H(t, y^n_t, z^n_t, p^n_t, u^n_t) - H(t, y^n_t, z^n_t, p^n_t, v)] + C \varepsilon_n,$$

where $\lim_{n \to \infty} \varepsilon_n = 0$.

According to (29), (31), (31) and (48), the result follows immediately by letting $n$ going to infinity in the last inequality. \hfill \blacksquare

\textbf{Remark 19} If $\mu_t (da) = \delta_{u(t)} (da)$, we recover the strict necessary optimality conditions (Theorem 4).

Theorem 20 \textbf{(Sufficient optimality conditions for relaxed controls).} We know that the set $\mathcal{R}$ of relaxed controls is convex and the function $H^q (t, y^q_t, z^q_t, p^q_t, q_t)$ is linear in $q_t$. If we assume that for every $q \in \mathcal{R}$ and for all $t \in [0, T]$, the functions $q$ is convex and $(y^q_t, z^q_t) \rightarrow H^q (t, y^q_t, z^q_t, p^q_t, q_t)$ is concave, then $\mu$ is an optimal relaxed control if it satisfies (57).

\textbf{Proof.} The proof is the same that in theorem 5. \hfill \blacksquare

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