Generalized argument shift method and complete commutative subalgebras in polynomial Poisson algebras

Anton Izosimov*

Abstract

The Mischenko-Fomenko argument shift method allows to construct commutative subalgebras in the symmetric algebra \( S(g) \) of a finite-dimensional Lie algebra \( g \). For a wide class of Lie algebras, these commutative subalgebras appear to be complete, i.e. they have maximal transcendence degree. However, for many algebras, Mischenko-Fomenko subalgebras are incomplete or even empty. In this case, we suggest a natural way how to extend Mischenko-Fomenko subalgebras, and give a completeness criterion for these extended subalgebras.

1 Introduction

Let \( g \) be a finite-dimensional Lie algebra. For the sake of simplicity we shall assume that the ground field is \( \mathbb{C} \), however everything works over an arbitrary field of characteristic zero. The symmetric algebra \( S(g) \) can be naturally identified with the algebra of polynomials on the dual space \( g^\ast \) and carries a natural Poisson bracket (Lie-Poisson bracket) which is defined on linear functions by \( \{ \xi, \eta \} = [\xi, \eta] \) and is extended to all polynomials by the Leibnitz identity.

We will be interested in commutative subalgebras in \( S(g) \). Let \( C \subset S(g) \) be a commutative subalgebra. Then the maximal possible transcendence degree of \( C \) is

\[
b(g) = \frac{1}{2} (\dim g + \text{ind } g).
\]

If \( \text{tr.deg. } C = b(g) \), then \( C \) is called a complete commutative subalgebra. Each complete commutative subalgebra in \( S(g) \) can be interpreted as an integrable system on the Poisson manifold \( g^\ast \).

One of the most universal methods for constructing “large” commutative subalgebras in \( S(g) \) is the so-called argument shift method. This method was introduced by Mischenko and Fomenko \([1]\) as a generalization of the Manakov construction \([2]\) for the Lie algebra \( \mathfrak{so}(n) \).

The argument shift method can be described as follows. Let \( a \in g^\ast \) be an arbitrary regular element. Then there exist \( m = \text{ind } g \) analytic functionally-independent invariants of the coadjoint representation defined in a small neighborhood of \( a \). Denote these invariants by \( f_1, \ldots, f_m \). For each \( i = 1, \ldots, m \) expand the function \( f_i(a + \lambda x) \) in powers of \( \lambda \):

\[
f_i(a + \lambda x) = \sum_{j=0}^{\infty} f_{ij}(x)\lambda^j
\]

where all functions \( f_{ij}(x) \) are polynomials. Denote the algebra generated by all these polynomials by \( \mathcal{F}_a \). Then, as was proved by Mischenko and Fomenko, \( \mathcal{F}_a \) is a commutative subalgebra in \( S(g) \).

Moreover, if \( g \) is semisimple, then \( \mathcal{F}_a \) is complete.

Note that our description of the argument shift method is slightly different from the original description which required that the invariants \( f_1, \ldots, f_m \) are polynomial. The modification of the argument shift method presented here is due to Brailov (see Bolsinov \([3]\)).

The completeness criterion for subalgebras \( \mathcal{F}_a \) was found by Bolsinov \([4]\). Namely, he proved that \( \mathcal{F}_a \) is complete if and only if the set of singular elements \( \text{Sing} \subset g^\ast \) has codimension at least 2 (if the ground field \( K \) is not algebraically closed, one should consider the singular set in \( g^\ast \otimes K \) where \( K \) is the algebraic closure of \( K \); see Bolsinov and Zhang \([5]\) for details).

*Moscow State University and Higher School of Economics. E-mail: a.m.izosimov@gmail.com
Although the family $F_a$ is in general not complete, Mischenko and Fomenko stated the following conjecture: for each finite-dimensional Lie algebra $\mathfrak{g}$, there exists a complete commutative subalgebra $C \subseteq S(\mathfrak{g})$. This conjecture was proved by Sadetov [8], see also Bolsinov [7]. However, Sadetov’s construction is essentially different from the argument shift method. In particular, Sadetov’s subalgebras are commutative only with respect to the standard Poisson structure, while Mischenko-Fomenko subalgebras $F_a$ have the following remarkable property: they are commutative with respect to two Poisson structures, one of which is standard, and the second one (the so-called ‘frozen argument bracket’) is defined as follows. It is given on linear functions by $\{\xi, \eta\}_a = \langle a, [\xi, \eta] \rangle$, where $\langle , \rangle$ denotes the pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$, and is extended to all polynomials by the Leibnitz identity.

Our aim is to show that when the Mischenko-Fomenko subalgebra $F_a$ is incomplete, i.e. when the singular set $\text{Sing}$ has codimension one, then there is a natural extension $\tilde{F}_a \supset F_a$ which is also commutative with respect to both brackets $\{f, g\}$ and $\{f, g\}_a$, and to give a completeness criterion for $\tilde{F}_a$.

In their paper [8], Bolsinov and Zhang formulated the following “generalized argument shift” conjecture: for each Lie algebra $\mathfrak{g}$ there exists a subalgebra in $S(\mathfrak{g})$ which is commutative with respect to both brackets $\{f, g\}$ and $\{f, g\}_a$. In this way, our note is a step towards the proof of this conjecture.

2 Generalized argument shift method

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{C}$, and let $x \in \mathfrak{g}^*$. Let

$$\mathfrak{g}_x = \{\xi \in \mathfrak{g} | \text{ad}_x^*(\xi) = 0\}$$

be the stabilizer of $x$ w.r.t. the coadjoint representation. The set $\text{Sing}$ of singular elements in $\mathfrak{g}^*$ is

$$\text{Sing} = \{x \in \mathfrak{g}^* | \dim \mathfrak{g}_x > \text{ind} \mathfrak{g}\}$$

where

$$\text{ind} \mathfrak{g} = \min_{x \in \mathfrak{g}^*} \dim \mathfrak{g}_x$$

is the index of $\mathfrak{g}$.

We consider the case when $\text{Sing}$ has codimension one. Let $\text{Sing}_0$ be the union of all irreducible components of $\text{Sing}$ which have maximal dimension. The set $\text{Sing}_0$ is the zero set of a certain homogeneous polynomial $p_0(x)$. It is easy to see that $p_0$ is a semi-invariant of the coadjoint representation. Following Ooms and Van den Bergh [4], we call it the fundamental semi-invariant of $\mathfrak{g}$ (see also Joseph and Shafrir [10]).

More precisely, the fundamental semi-invariant is defined as follows. Let $t = \dim \mathfrak{g} - \text{ind} \mathfrak{g}$. Fix a basis in $\mathfrak{g}$, and let $c_{ij}^k$ be the structure constants in this basis. Then an element $x \in \mathfrak{g}^*$ is singular if and only if the rank of the matrix $A_x = c_{ij}^k x_k$ is less than $t$, i.e. if Pfaffians of all principal $t \times t$ minors of $A_x$ vanish. Define $p_0$ as the greatest common divisor of all these Pfaffians. Clearly, the zero set of $p_0$ coincides with $\text{Sing}_0$. However, $p_0$ is not necessarily the minimal polynomial which define $\text{Sing}_0$.

Now we use the fundamental semi-invariant $p_0$ to define the extended Mischenko-Fomenko subalgebra. Consider the polynomial $p_0(a + \lambda x)$ and expand it in powers of $\lambda$:

$$p_0(a + \lambda x) = \sum_{i=0}^n p_i(x) \lambda^i$$

where $n = \deg p_0$. Define the extended Mischenko-Fomenko subalgebra $\tilde{F}_a$ as a subalgebra in $S(\mathfrak{g})$ generated by all elements of the classical Mischenko-Fomenko subalgebra $F_a$ and the polynomials $p_1, \ldots, p_n$.

**Theorem 1.** For each regular $a \in \mathfrak{g}^*$, the extended Mischenko-Fomenko subalgebra $\tilde{F}_a$ is commutative with respect to both brackets $\{f, g\}$ and $\{f, g\}_a$.

**Remark 2.1.** This statement is, in fact, not new. Firstly, it was proved by Arhangel’skiĭ [12] that if two semi-invariants $f, g$ commute with respect to the Lie-Poisson bracket, then their shifts, i.e. functions of the form $f(x + a), g(x + ma)$ where $a \in \mathfrak{g}^*$ is fixed, also commute. This statement easily implies that the extended Mischenko-Fomenko subalgebra $\tilde{F}_a$ is commutative with respect to the Lie-Poisson bracket. Moreover, the proof of Arhangel’skiĭ can be easily modified to show that
\[ \tilde{F}_a \] is commutative with respect to the frozen argument bracket as well (see Section 3). We also note that the assumption that \( f \) and \( g \) commute is in fact satisfied for arbitrary semi-invariants (see Ooms and Van den Bergh [8] and Section 7).

Secondly, Theorem 1 can be viewed as a generalization of Proposition 7 of Bolsinov and Zhang [3] which asserts that the functions \( p_1, \ldots, p_n \) are in involution with respect to both Lie-Poisson bracket and frozen argument bracket.

Thirdly, Theorem 1 follows from the following general construction from the theory of compatible Poisson brackets. Let \( \mathcal{A} \) and \( \mathcal{B} \) be compatible Poisson brackets. Consider the family \( \mathcal{F} \) generated by Casimir functions of all generic linear combinations of \( \mathcal{A} \) and \( \mathcal{B} \). This family is commutative with respect to both \( \mathcal{A} \) and \( \mathcal{B} \) (see Reiman and Semenov-Tyan-Shanskii [12]), however it may be incomplete or even empty. In this case, \( \mathcal{F} \) can be extended by adding eigenvalues of the operator \( AB^{-1} \) (which is still well-defined on a certain quotient space even if \( \mathcal{B} \) is degenerate, see e.g. [13]). Denote the extended family by \( \tilde{\mathcal{F}} \).

**Proposition 2.1.** The family \( \tilde{\mathcal{F}} \) is commutative with respect to both brackets \( \mathcal{A} \) and \( \mathcal{B} \).

If we apply this construction to the Lie-Poisson bracket and the frozen argument bracket, we get Theorem 1.

Although we were not able to find the statement of Proposition 2.1 in the literature, we believe that it is well known to experts in the field. See [12, 14, 16] where different constructions of integrable systems related to compatible Poisson brackets are discussed. Also note that the relation between the argument shift method and compatible Poisson brackets was probably first mentioned by Meshcheryakov [17].

### 3 Completeness criterion

Let \( \text{Sing}_{sr} \subset \text{Sing}_0 \) be the subset of \( \text{Sing}_0 \) which consists of subregular elements, i.e.

\[
\text{Sing}_{sr} = \{ x \in \text{Sing}_0 \mid \dim g_x = \text{ind } g + 2 \}.
\]

It is clear that \( \text{Sing}_{sr} \) is open in \( \text{Sing}_0 \), however it is not necessarily dense and may be empty. Let also \( \text{Sing}_{sm} \subset \text{Sing}_0 \) be the set of points where \( \text{Sing}_0 \) is smooth. This set is open and dense in \( \text{Sing}_0 \).

Denote the two-dimensional non-Abelian Lie algebra by \( b_2 \), and the \( 2n+1 \)-dimensional Heisenberg algebra by \( h_{2n+1} \).

**Proposition 3.1.** Let \( x \in \text{Sing}_{sr} \cap \text{Sing}_{sm} \). Then \( g_x \) is isomorphic to one of the following Lie algebras:

1. \( b_2 \oplus \text{Abelian Lie algebra of dimension ind } g \);
2. \( b_3 \oplus \text{Abelian Lie algebra of dimension ind } g - 1 \);
3. \( \text{Abelian Lie algebra of dimension ind } g + 2 \).

**Proof.** Corollary 2.1 of [18] implies that the derived algebra of \( g_x \) is at most one-dimensional. It is easy to see that any Lie algebra with this property is either Abelian, or isomorphic to one of the following:

- \( b_2 \oplus \text{Abelian} \);
- \( h_{2n+1} \oplus \text{Abelian} \);

Now, using that \( \dim g_x = \text{ind } g + 2 \) and the inequality \( \text{ind } g_x \geq \text{ind } g \), we obtain the above list. \( \square \)

**Remark 3.1.** The inequality \( \text{ind } g_x \geq \text{ind } g \) is true for any Lie algebra \( g \) and any \( x \in g^\ast \). It is known as the Vinberg inequality. See Panyushev [15].

Consider the set

\[
\text{Sing}_g = \{ x \in \text{Sing}_{sr} \cap \text{Sing}_{sm} \mid g_x \cong b_2 \oplus \text{Abelian} \} \subset \text{Sing}_{sr} \cap \text{Sing}_{sm} \subset \text{Sing}_0.
\]

It is easy to see that \( \text{Sing}_g \) is open in \( \text{Sing}_{sr} \cap \text{Sing}_{sm} \) and thus in \( \text{Sing}_0 \).

**Theorem 2.** The extended Mischenko-Fomenko subalgebra \( \tilde{\mathcal{F}}_a \) is complete if and only if the set \( \text{Sing}_g \) is dense in \( \text{Sing}_0 \).

**Corollary 3.1.** Assume that for each irreducible component \( S_i \) of the variety \( \text{Sing}_0 \) there exists at least one \( x \in S_i \) such that \( g_x \cong b_2 \oplus C^{\text{ind } g} \). Then the extended Mischenko-Fomenko subalgebra \( \tilde{\mathcal{F}}_a \) is complete. In particular, if \( \text{Sing}_0 \) is irreducible, and there exists at least one \( x \in \text{Sing}_0 \) such that \( g_x \cong b_2 \oplus C^{\text{ind } g} \), then \( \tilde{\mathcal{F}}_a \) is complete.
Remark 3.2. Theorem 2 can also be formulated as follows. For each irreducible component $S_i$ of the variety $\text{Sing}_0$, there exists an open subset $\tilde{S}_i$ such that all elements of this subset have isomorphic stabilizers. The extended Mischenko-Fomenko subalgebra $\mathcal{F}_n$ is complete if and only if $g_x \simeq \mathfrak{b}_2 \oplus \mathfrak{c}^{\text{inf}}$ for each $x$ and each $x \in \tilde{S}_i$.

We also note that Theorem 2 can be generalized to the case of arbitrary compatible Poisson brackets.

4 Proof of Theorem 1

Though the statement of Theorem 1 follows from from considerations of Sections 4 and 5 we give an independent algebraic proof. As a matter of fact, we prove a stronger statement: if $\mathfrak{g}$ is a complex Lie algebra, and $a \in \mathfrak{g}^*$, then for any two semi-invariants $f, g \in S(\mathfrak{g})$ and any $\lambda, \mu \in \mathbb{C},$

$$\{f(a + \lambda x), g(a + \mu x)\}_a = 0.$$  

The proof given below follows the ideas of Arhangel'skii [11].

Recall that $f \in S(\mathfrak{g})$ is called a semi-invariant of $\mathfrak{g}$ if there exists a character $\chi_f : \mathfrak{g} \to \mathbb{C}$ such that

$$\{f, g\} = \chi_f(df)f$$

for any $g \in S(\mathfrak{g})$.

Proposition 4.1. Assume that $\{f, g\}$ is divisible by $f$ for any $g \in S(\mathfrak{g})$. Then $f$ is a semi-invariant.

Proof. Let $x_1, \ldots, x_n$ be a basis in $\mathfrak{g}$. Then

$$\{f, g\} = \sum_i \{f, x_i\} \frac{dg}{dx_i}.$$  \hspace{1cm} (1)

Since $\{f, x_i\}$ is divisible by $f$ and has the same degree as $f$, there exists $c_i \in \mathbb{C}$ such that $\{f, x_i\} = c_i f$. Define a linear function $\chi_f : \mathfrak{g} \to \mathbb{C}$ by setting $\chi_f(x_i) = c_i$. Then (1) can be rewritten as

$$\{f, g\} = \chi_f(df)f.$$

Further,

$$\{f, \{x_i, x_j\}\} = \{\{f, x_i\}, x_j\} + \{x_i, \{f, x_j\}\} = c_i \{f, x_j\} + c_j \{x_i, f\} = c_i c_j f - c_j c_i f = 0,$$

so

$$\chi_f(\{x_i, x_j\}) = \chi_f([x_i, x_j]) = 0,$$

i.e. $\chi_f$ is a character, q.e.d.

Proposition 4.2. Any semi-invariant $f$ is a product of irreducible semi-invariants.

Proof. Let $f = f_1^{k_1} \ldots f_m^{k_m}$ where $f_1, \ldots, f_m$ are irreducible and distinct. Then

$$\{f, g\} = \{f_1^{k_1} \ldots f_m^{k_m}, g\} = \sum_{i=1}^m k_i f_1^{k_1} \ldots f_i^{k_i - 1} \ldots f_m^{k_m} \{f_i, g\}.$$  \hspace{1cm} (2)

On the other hand,

$$\{f, g\} = \chi_f(df)f = \chi_f(df)f_1^{k_1} \ldots f_m^{k_m}.$$  \hspace{1cm} (3)

Comparing (2) and (3), we conclude that $\{f_i, g\}$ is divisible by $f_i$, hence $f_i$ is a semi-invariant.

Proposition 4.3. Let $f, g$ be two semi-invariants. Then $\{f, g\} = 0$.

Proof. Clearly, it suffices to prove the proposition for irreducible distinct $f$ and $g$. Let $h = \{f, g\}$. Then $h$ is divisible by both $f$ and $g$, and hence by $fg$. On the other hand,

$$\deg h \leq \deg f + \deg g - 1,$$

therefore $h = 0$.

Proposition 4.4. Let $f, g$ be two semi-invariants. Then $\{f, g(a + \lambda x)\} = 0$.

Proof. Since $\{f, g\} = 0$, we have $\chi_f(df(x)) = 0$ for any $x$. Let $h(x) = g(a + \lambda x)$. Then

$$dh(x) = \lambda df(a + \lambda x),$$

so $\chi_f(dh(x)) = 0$, and $\{f, h\} = \chi_f(df) = 0$. 

\hspace{1cm} \Box
Proposition 4.5. Let \( f, g \) be two semi-invariants. Then \( \{ f, g(a + \lambda x) \}_a = 0 \).

Proof. We shall use the following explicit formulas for the Lie-Poisson and frozen argument bracket:
\[
\{ f, g \}(x) = \{ x, [df(x), dg(x)] \}, \quad \{ f, g \}_a(x) = \{ a, [df(x), dg(x)] \}.
\]
Let \( h(x) = g(a + \lambda x) \). Then
\[
\{ f, h \}_a(x) = \{ a, [df(x), dh(x)] \} = \lambda \{ a, [df(x), dg(a + \lambda x)] \}.
\]
On the other hand,
\[
\lambda \{ x, [df(x), dg(a + \lambda x)] \} = \{ f, h \}(x) = 0,
\]
so
\[
\{ f, h \}_a(x) = \lambda \{ a + \lambda x, [df(x), dg(a + \lambda x)] \},
\]
and
\[
\{ f, h \}_a \left( \frac{x - a}{\lambda} \right) = \lambda \{ x, [df \left( \frac{x - a}{\lambda} \right), dg(x)] \} = \lambda^2 \{ f \left( \frac{x - a}{\lambda} \right), g \}(x).
\]
The latter Poisson bracket vanishes by Proposition [13] so
\[
\{ f, h \}_a \left( \frac{x - a}{\lambda} \right) = 0
\]
for any \( x \), and hence \( \{ f, h \}_a = 0 \). \hfill \Box

Proof of Theorem 1. Let \( f, g \) be two semi-invariants. We need to prove that
\[
\{ f(a + \lambda x), g(a + \mu x) \} = \{ f(a + \lambda x), g(a + \mu x) \}_a = 0.
\]
Let \( h(x) = f(a + \lambda x), k(x) = g(a + \mu x) \). Then
\[
\{ h, k \}(x) = \{ x, [dh(x), dk(x)] \} = \lambda \mu \{ x, [df(a + \lambda x), dg(a + \mu x)] \},
\]
so
\[
\{ h, k \} \left( \frac{x - a}{\lambda} \right) = \lambda \mu \{ x, [df(x), dg \left( a + \frac{\mu}{\lambda}(x - a) \right)] \} = \lambda \{ f, g \left( a + \frac{\mu}{\lambda}(x - a) \right) \}(x) - \lambda \{ f, g \left( a + \frac{\mu}{\lambda}(x - a) \right) \}_a(x) = 0,
\]
and hence \( \{ h, k \} = 0 \). Analogously,
\[
\{ h, k \}_a \left( \frac{x - a}{\lambda} \right) = \lambda \mu \{ a, [df(x), dg \left( a + \frac{\mu}{\lambda}(x - a) \right)] \} = \lambda^2 \{ f, g \left( a + \frac{\mu}{\lambda}(x - a) \right) \}_a(x) = 0,
\]
and hence \( \{ h, k \}_a = 0 \). \hfill \Box

5 Shifts of the fundamental semi-invariant

Recall that the shifts \( p_1, \ldots, p_n \) of the fundamental semi-invariant are defined by the formula:
\[
p_a(a + \lambda x) = \sum_{i=0}^n p_i(x) \lambda^i
\]
where \( n = \deg p_a \). Consider the factorization of \( p_a \) into irreducible factors:
\[
p_a = p^{k_1}_1 \cdots p^{k_m}_m,
\]
and let \( d = \sum \deg p_i \). Let \( x \in \mathfrak{g}^* \), and consider the equation
\[
p_a(x - \lambda a) = 0.
\]
Obviously, this equation has at most \( d \) distinct roots. Let us say that an element \( x \in \mathfrak{g}^* \) is nice if it has the following properties:
1. the equation \( p_a(x - \lambda a) = 0 \) has exactly \( d \) distinct roots \( \lambda_1, \ldots, \lambda_d \);
2. \( \lambda_1, \ldots, \lambda_d \) are locally analytic functions of \( x \);
3. for each \( i \), the dimension of the stabilizer of \( x - \lambda_i(x) a \) is locally constant;
4. the line \( x - \lambda a \) does not intersect the set \( \text{Sing} \setminus \text{Sing}_a \).

It is clear that the set \( \mathcal{N} \) of nice elements is Zariski dense in \( \mathfrak{g}^* \).
Proposition 5.1. Let \( x \in \mathcal{N} \). Then the space spanned by the differentials of \( \lambda_1, \ldots, \lambda_d \) coincides with the space spanned by the differentials of \( p_1, \ldots, p_n \).

Proof. Let \( s_i \) be the multiplicity of \( \lambda_i \). Then
\[
p_g(x - \lambda a) = p_g(a) \prod_{i=1}^{d} (\lambda_i(x) - \lambda)^{s_i},
\]
so
\[
p_g(a + \lambda x) = \lambda^n p_g(x + \frac{1}{\lambda} a) = p_g(a) \prod_{i=1}^{d} (1 + \lambda_i(x)\lambda)^{s_i},
\]
therefore the functions \( p_1, \ldots, p_n \) are, up to a constant factor, elementary symmetric polynomials of the functions \( \lambda_1, \ldots, \lambda_d \) taken with multiplicities, which easily implies the statement. \( \square \)

Let \( g_i(x) = g_{x - \lambda_i(x)a} \) be the stabilizer of \( x - \lambda_i(x)a \). The two following statements relate the differentials \( \lambda_1, \ldots, \lambda_d \) to the structure of stabilizers \( g_i(x), \ldots, g_d(x) \).

Proposition 5.2. Let \( x_0 \in \mathcal{N} \). Then \( d\lambda_i(x_0) \in g_i(x_0) \).

Proof. Consider the coadjoint orbit \( O \) passing through \( y_0 = x_0 - \lambda_i(x_0)a \), and let \( \xi \in T_{y_0}O \) be a vector tangent to the orbit at \( y_0 \). Let also \( y(t) \) be a curve such that \( y(t) \in O \), \( y(0) = y_0 \), and \( y(0) = \xi \). Let also \( x(t) = y(t) + \lambda_i(x_0)a \). Obviously, \( y(t) \in \text{Sing} \), and \( \lambda_i(x(t)) = \lambda_i(x_0) \). Differentiating this formula with respect to \( t \) at \( t = 0 \), we obtain
\[
\langle \dot{x}(0), d\lambda_i(x_0) \rangle = 0,
\]
and since \( \dot{x}(0) = \dot{y}(0) = \xi \), we have
\[
\langle \xi, d\lambda_i(x_0) \rangle = 0. \tag{4}
\]
Since (4) is true for any \( \xi \in T_{y_0}O \), we have
\[
\langle T_{y_0}O, d\lambda_i(x_0) \rangle = 0,
\]
which implies that \( d\lambda_i(x_0) \in g_{y_0} = g_i(x_0) \). \( \square \)

The following simple formula is of fundamental significance for the present paper.

Proposition 5.3. Let \( x_0 \in \mathcal{N} \), and let \( \xi, \eta \in g_i(x_0) \). Then
\[
\langle \xi, \eta \rangle = \langle a, [\xi, \eta] \rangle d\lambda_i(x_0). \tag{5}
\]

Proof. Choose a neighborhood \( U(x_0) \ni x_0 \) such that the dimension of \( g_i(x) \) is constant in \( U(x_0) \). Then it is possible to define smooth mappings \( \xi, \eta : U(x_0) \to g_i(x) \) such that \( \xi(x_0) = \xi, \eta(x_0) = \eta \), and \( \xi(x), \eta(x) \in g_i(x) \) for each \( x \in U(x_0) \). Differentiating the identity
\[
\langle x - \lambda_i(x)a, [\xi(x), \eta(x)] \rangle = 0.
\]
at \( x = x_0 \), we obtain (5). \( \square \)

6. Linear algebra related to a pair of skew-symmetric forms

Let \( P_0, P_\infty \) be two skew-symmetric forms on a vector space \( V \), and let
\[
P_\lambda = P_0 - \lambda P_\infty.
\]
Let also
\[
r = \min_{\lambda \in \mathbb{T}} \text{corank} P_\lambda, \quad \Lambda = \{ \lambda \in \mathbb{T} \mid \text{corank} P_\lambda > r \}, \quad L = \sum_{\lambda \in \mathbb{T} \setminus \Lambda} \text{Ker} P_\lambda.
\]

Proposition 6.1. The space \( L \) has the following properties:
1. it is isotropic with respect to any form \( P_\lambda \);
2. the skew-orthogonal complement to \( L \) given by \( L^+ = \{ \xi \in V \mid P_\lambda(\xi, L) = 0 \} \) does not depend on the choice of \( \lambda \in \mathbb{T} \);
3. if \( \lambda \notin \Lambda \), then \( P_\lambda \) is non-degenerate on \( L^+ / L \);
4. if \( \lambda \in \Lambda \), then \( \dim \text{Ker} P_\lambda \cap L = r \);
5. if \( \lambda \in \Lambda \), and \( \alpha \in \mathbb{T} \), then \( \text{Ker} (P_\alpha \mid_{\text{Ker} P_\lambda}) \supset \text{Ker} P_\lambda \cap L \).
Assume that $\infty \not\in \Lambda$, and define the recursion operator $R: L^\perp/L \to L^\perp/L$ by the formula

$$R = P_{\infty}^{-1} P_0.$$  

**Proposition 6.2.** The operator $R$ has the following properties:

1. the spectrum of $R$ coincides with the set $\Lambda$; the multiplicity of each eigenvalue is at least two;
2. the $\lambda$-eigenspace of $R$ coincides with the space $(\text{Ker } P_\lambda)/(\text{Ker } P_\lambda \cap L)$;
3. the eigenspaces of $R$ are pairwise orthogonal with respect to $P_\lambda$ for each $\lambda$;
4. the operator $R$ is diagonalizable if and only if for each $\lambda \in \Lambda$, the following identity holds

$$\dim \text{Ker } (P_{\infty} |_{\text{Ker } P_\lambda}) = r.$$  

For the proof of Propositions 6.1, 6.2 see [13]. They can also be easily deduced from the Jordan-Kronecker theorem [20–23].

**Proposition 6.3.** Let $\Lambda = \{\lambda_1, \ldots, \lambda_k\}$, and assume that $\xi_i \in \text{Ker } P_{\lambda_i}$. Let

$$U = L + \langle \xi_1, \ldots, \xi_k \rangle.$$  

Then

1. $U$ is isotropic with respect to $P_\lambda$ for any $\lambda \in \mathbb{C}$;
2. if $\lambda \not\in \Lambda$, then $U$ is maximal isotropic with respect to $P_\lambda$ if and only if all eigenvalues of $R$ have multiplicity two, and $\xi_i \not\in L$ for each $i$.

The proof easily follows from Propositions 6.1 and 6.2.

### 7 Proof of Theorem 2

Assume that $\text{Sing}_g$ is dense in $\text{Sing}_0$, and prove that the extended Mischenko-Fomenko subalgebra $\tilde{F}_a$ is complete. Let us take $x \in \mathcal{N}$ such that $x - \lambda_i(x)a \in \text{Sing}_g$ for each $i$, and prove that the space

$$d\tilde{F}(x) = \{df(x) | f \in \tilde{F}(x)\}$$

has dimension

$$b(g) = \frac{1}{2}(\dim g + \text{ind } g),$$

which immediately implies the completeness of $\tilde{F}_a$.

Consider skew-symmetric forms $P_0 = A_0$ and $P_{\infty} = A_{\infty}$ on the cotangent space $T_x^* g^* \simeq g$ which are given by

$$A_0(\xi, \eta) = (x, [\xi, \eta]), \quad A_{\infty}(\xi, \eta) = (a, [\xi, \eta]).$$

Let us apply the results of Section 6 to these two forms. We note that

$$r = \text{ind } g, \quad \text{Ker } P_\lambda = g_{x - \lambda a}, \quad \Lambda = \{\lambda_1(x), \ldots, \lambda_d(x)\}.$$  

The following lemma is due to Bolsinov [3].

**Lemma 7.1.** Let $\mathcal{F}$ be the classical Mischenko-Fomenko subalgebra, and let

$$d\mathcal{F}(x) = \{df(x) | f \in \mathcal{F}(x)\}.$$  

Then

$$d\mathcal{F}(x) = L.$$  

As follows from Proposition 5.1,

$$d\tilde{F}(x) = d\mathcal{F}(x) + \langle d\lambda_1(x), \ldots, d\lambda_d(x) \rangle.$$  

By Proposition 5.2 we have

$$d\lambda_i(x) \in g_i(x) = \text{Ker } P_{\lambda_i},$$

so we can use Proposition 6.3 to show that $d\tilde{F}(x)$ is maximal isotropic with respect to $A_0$ and hence is of dimension $b(g)$. In order to do this, we need to show that the eigenvalues of the recursion operator $R$ have multiplicity two, and that $d\lambda_i(x) \not\in L$. 

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Since $x - \lambda_i(x) a \in \text{Sing}_i$, we have $\dim g_i = r + 2$, and by item 4 of Proposition 6.1, $\dim (g_i \cap L) = r$, so
\[ \dim (g_i / (g_i \cap L)) = 2, \]
and all eigenspaces of $R$ are two-dimensional (see Proposition 6.2, item 2). Therefore, to prove that all eigenvalues of $R$ have multiplicity two, we need to show that $R$ has no Jordan blocks. By item 4 of Proposition 6.2, it suffices to prove that $\dim \ker (A_a | g_i) = r$. We have $\dim g_i = r + 2$, and $\dim g_i \cap L = r$. By item 5 of Proposition 6.1, $\ker (A_a | g_i) \supset g_i \cap L$, so $\dim \ker (A_a | g_i)$ can be either $r$ or $r + 2$. Assume that it is $r + 2$. Then $\ker (A_a | g_i) = g_i$, and $A_a | g_i = 0$. By Proposition 5.3 this implies that $g_i$ is Abelian, which is not the case.

Now, let us prove that $\partial \lambda_i(x) \not\in L$. Since $\ker (A_a | g_i) \supset g_i \cap L$, Proposition 5.3 implies that $g_i \cap L$ lies in the center $Z(g_i)$. So, if $\partial \lambda_i(x) \in L$, then
\[ \partial \lambda_i(x) \in Z(g_i). \]
On the other hand, since $g_i$ is not Abelian, Proposition 5.3 implies that
\[ \partial \lambda_i(x) \in [g_i, g_i], \quad \partial \lambda_i(x) \neq 0, \]
where $[g_i, g_i]$ is the derived subalgebra of $g_i$. But since $g_i \simeq b_2 \oplus \text{Abelian}$, we have $[g_i, g_i] \cap Z(g_i) = 0$, so $\partial \lambda_i(x) \not\in L$, which completes the proof of the if-part of the theorem. The proof of the only-if-part is analogous.

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