Braided tensor products and the covariance of quantum noncommutative free fields

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Abstract

We introduce the free quantum noncommutative fields as described by braided tensor products. The multiplication of such fields is decomposed into three operations, describing the multiplication in the algebra $\mathcal{M}$ of functions on noncommutative spacetime, the product in the algebra $\mathcal{H}$ of deformed field oscillators, and the braiding by factor $\Psi_{\mathcal{M},\mathcal{H}}$ between algebras $\mathcal{M}$ and $\mathcal{H}$. For noncommutativity of quantum spacetime generated by the twist factor, we shall employ the $\ast$-product realizations of the algebra $\mathcal{M}$ in terms of functions on the standard Minkowski space. The covariance of single noncommutative quantum fields under deformed Poincare symmetries is described by the algebraic covariance conditions which are equivalent to the deformation of generalized Heisenberg equations on a Poincare group manifold. We shall calculate the braided field commutator covariant under deformed Poincare symmetries, which for free quantum noncommutative fields provides the field quantization condition and is given by the standard Pauli–Jordan function. For the illustration of our new scheme, we present explicit calculations for the well-known case in the literature of canonically deformed free quantum fields.

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1. Introduction

It is at present a common view that the most important problem in the theory of fundamental interactions is to quantize in a consistent way the dynamical theory of spacetime, i.e. gravity, and incorporate it into the description of all other interactions. The consistent quantization of gravitation theory is sought in different ways: by employing new (nonlocal, discrete) descriptions of Einstein gravity (loop quantum gravity [1], spin foam models [2]) or by embedding standard gravity in a larger dynamical framework ((super)-string [3], brane world models [4]). In most of these approaches the classical commutative spacetime geometry is modified because the quantization of gravitational dynamics leads to a new algebraic structure of relativistic phase space with the noncommutative quantum spacetime sector. In particular...
if we consider the quantum fields used e.g. in the formulation of the standard model in the presence of a quantum gravity background, the appearance of quantum spacetime implies that the standard framework for classical and quantum fields should be suitably deformed.

In the last 15 years, various ways were proposed to introduce the deformation of quantum free fields, which due to the quantization procedure of gravity (see e.g. [5]) or quantized D = 10 string effects in a tensorial gauge fields background (see e.g. [6]), were defined on noncommutative spacetime [7–16]. It was proposed that the fields defined on relativistic spacetime (Minkowski space) should be replaced by new fields which algebraically take into account the quantum gravity effects and are defined on quantum spacetime. In these new field-theoretic models, one replaces the commutative spacetime coordinates \( x_\mu \) by the noncommutative ones

\[
[x_\mu , x_\nu ] = 0 \quad \Rightarrow \quad [\hat{x}_\mu , \hat{x}_\nu ] = \frac{i}{\kappa^2} \Theta_{\mu\nu}(\kappa \hat{x}),
\]

where (\( \kappa \)-masslike deformation parameter) \( \Theta_{\mu\nu}(\kappa \hat{x}) = \Theta_{\mu\nu}^{(0)} + \kappa \Theta_{\mu\nu}^{(1)} \hat{x}_\rho + \cdots \). The choice \( \Theta_{\mu\nu}(\kappa \hat{x}) = \Theta_{\mu\nu}^{(0)} \) corresponds to the so-called canonical deformation [5, 6] and \( \Theta_{\mu\nu}(\kappa \hat{x}) = \Theta_{\mu\nu}^{(1)} \hat{x}_\rho \) provides the Lie-algebraic noncommutativity of spacetime with the most known special case describing the \( \kappa \)-deformation of Minkowski space [17–19]. It has also been shown in the examples of deformed quantum free fields that the noncommutativity of spacetime algebra is linked in various ways with the deformation of field oscillators algebra (see e.g. [20–28]).

One introduces the symmetries in theories with noncommutativity (1) in two distinct ways:

1. One keeps the classical Poincare symmetries and the noncommutativity (1) is treated as introducing the terms which break the classical relativistic symmetries [5, 29–31]. In such a way, the choice of function \( \Theta_{\mu\nu}(\kappa \hat{x}) \) can be arbitrary (modulo Jacobi identities) and the set of constant tensors \( \Theta_{\mu\nu\rho_1...\rho_n}(\rho = 0, 1, 2, \ldots) \) describe the classical Poincare symmetry breaking parameters.

2. One can introduce the modified quantum Poincare–Hopf symmetries in a way which leads to the covariance of the relations (1) (see e.g. [12–16]). In such an approach quantum covariance means that the relation (1) has the same algebraic form in all deformed Poincare frames and \( \Theta_{\mu\nu}^{(n)}(\kappa \hat{x}) \) can be treated as a set of constant parameters, which also enter into the deformed Poincare–Hopf algebra. It appears that such a method is more restrictive; it selects only the particular classes of functions \( \Theta_{\mu\nu}(\kappa \hat{x}) \) occurring in (1).

In this paper we shall follow the second approach.

The basic aim of this paper is to study a new framework for the description of covariant twist-deformed free quantum fields, with the covariant deformation of field oscillator algebra. It is known already from earlier studies of Faddeev–Zamolodchikov and \( q \)-deformed Heisenberg algebras (see e.g. [32–36]) that the deformed commutators of field oscillators which are covariant under the quasitriangular quantum symmetry takes a braided form, with the covariant braid factor described by the corresponding universal \( R \)-matrix (see e.g. [37], section 7.2). Such an \( R \)-matrix-dependent braid factor will also characterize the \( c \)-number commutator of deformed free quantum fields as well as the braided products of these fields.

In section 2 we describe the properties of a general twisting procedure of Hopf algebras and specify the description for the canonical twist. In section 3 we consider the deformed noncommutative free quantum fields \( \phi(\hat{x}) \) as described by binary braided tensor products \( \mathcal{M} \otimes \mathcal{H} \), where \( \mathcal{M} \) is the algebra of functions on noncommutative spacetime and \( \mathcal{H} \) denotes the algebra of deformed field oscillators (see also [38–40], where however the unbraided tensor product is used). We point out that the nontrivial braiding between quantum symmetry
modules \( \mathcal{M} \) and \( \mathcal{H} \) will be only revealed in the multiplication procedure of quantum fields. In section 4 we describe two covariant actions of the deformed Poincare generators on a deformed quantum field. In the definition of a single noncommutative quantum field after introduction in \( \mathcal{M} \) of the Weyl map, the tensor structure \( \mathcal{M} \otimes \mathcal{H} \) collapses into the field oscillator algebra module \( \mathcal{H} \), which is also required by the correct no-deformation limit of the deformed quantum fields. Algebraically, the covariance condition will be expressed on deformed fields as generalized Heisenberg field equations. Furthermore in section 5, we shall consider the products of deformed free fields with the braided multiplication \( m_\star \). Such a new multiplication rule introduces the braiding factor \( \Psi_{\mathcal{M},\mathcal{H}} \) between the algebra of spacetime functions and deformed field oscillators (see also [15, 28, 36]). If the deformation of Poincare symmetries is described by the universal \( \mathcal{R} \)-matrix (we denote \( \mathcal{R} = \mathcal{R}_{(1)} \otimes \mathcal{R}_{(2)} \) and \( \mathcal{R}_{21} = \mathcal{R}_{(2)} \otimes \mathcal{R}_{(1)} \)), the braiding factor \( \Psi_{\mathcal{M},\mathcal{H}} \) will be defined by \( \mathcal{R}_{21} \) and the quantization of a free quantum field \( \phi \in \mathcal{M} \otimes \mathcal{H} \) will be described by the covariant braided commutator\(^2\)

\[
[\phi(\hat{x}), \phi(\hat{y})]^{BR}_\star \equiv \phi(\hat{x}) \cdot \phi(\hat{y}) - \mathcal{R}_{21} \triangleright (\phi(\hat{y}) \star \phi(\hat{x})),
\]

where we shall argue in section 4 that the action \( g \triangleright \phi \) of deformed Poincare algebra generators occurring in \( \mathcal{R}_{21} \) is fully described in covariant theory by the operation \( \triangleright \) of quantum adjoint in the algebra \( \mathcal{H} \) of deformed field oscillators. Our basic point here (see also [42]) is that the deformed Poincare covariance of the free quantum field commutator requires the braided form (2).

In this paper we restrict our explicit considerations to the triangular deformations described by a Drinfeld twist [43]. Our main aim is to show that using a suitable twist-form (2), the deformed Poincare covariance of the free quantum field commutator requires the braided in the algebra \( \mathcal{H} \).

In order to illustrate our approach, we shall consider in an explicit way the simple example of canonically deformed QFT \( (\Theta_{\mu\nu}(x) = \Theta_{\mu\nu}^{(0)} \equiv \Theta_{\mu\nu}) \) in (1)), with the Weyl map described by the Moyal–Weyl \( \star \)-product. It can be also shown (see e.g. [45]) that an important class of noncanonically deformed relations (1) with the rhs containing the linear and quadratic terms can be made covariant under twisted Poincare symmetries. On such a class of noncommutative spacetimes, one can define as well the corresponding deformed free quantum fields with covariant braided field commutator (2). We conclude that our scheme provides the description of deformed free QFT which is covariant under twisted Hopf–Poincare algebra but in principle can be extended to the deformations described by any quasitriangular universal \( \mathcal{R} \)-matrix.

2. The quantum Poincare–Hopf algebras and the deformation by a canonical twist

The classical relativistic symmetries are generated by a Poincare Lie algebra \( \mathcal{P} \), which can be endowed with a trivial (primitive) coalgebraic structure by the coproducts

\[
\Delta_0(g) = g \otimes 1 + 1 \otimes g,
\]

\( g = (M_{\mu\nu}, P_\mu) \)

(3)

describing the homomorphic map \( \Delta_0 : \mathcal{U}_0(\mathcal{P}) \rightarrow \mathcal{U}_0(\mathcal{P}) \otimes \mathcal{U}_0(\mathcal{P}) \) of the enveloping classical Poincare algebra \( \mathcal{U}_0(\mathcal{P}) \). For the description of quantum relativistic symmetries, one uses the quantum Poincare–Hopf algebras \( \mathcal{H}(\mathcal{U}(\mathcal{P}), m, \eta, \Delta, \epsilon, S) \), where \( \mathcal{U}(\mathcal{P}) \) describes the

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1 A similar braiding factor has been considered also in [41] but with different aims.

2 We introduced the shorthand notation. The operator \( \mathcal{R}_{21} \) acts in definition (2) of braided commutator as follows:

\[
\mathcal{R}_{21} \triangleright (\phi(\hat{x}) \star \phi(\hat{y})) = m_\star \circ \mathcal{R}_{21} \triangleright (\phi(\hat{y}) \otimes \phi(\hat{x})) = (\mathcal{R}_{21} \triangleright \phi(\hat{x})) \star (\mathcal{R}_{11} \triangleright \phi(\hat{y})).
\]
enveloping deformed Poincare algebra with a multiplication map \( m : \mathcal{U}(\mathcal{P}) \otimes \mathcal{U}(\mathcal{P}) \to \mathcal{U}(\mathcal{P}) \); \( \Delta : \mathcal{U}(\mathcal{P}) \to \mathcal{U}(\mathcal{P}) \otimes \mathcal{U}(\mathcal{P}) \) denotes the nonprimitive (noncommutative) coproduct map and \( S : \mathcal{U}(\mathcal{P}) \to \mathcal{U}(\mathcal{P}) \) defines the antipode or coinverse, which for classical Lie algebras is given by \( S_0(g) = -g \). For a complete definition of a Hopf algebra (see e.g. [50]), one also introduces the unit \( \eta \) \( (m(\eta \otimes a) = a) \) and counit \( \epsilon \), playing the role of unit for coalgebra maps.

A simple way of passing from the classical to quantum Poincare–Hopf algebra is achieved by introducing the twist \( \mathcal{F} \). During the twisting procedure of the undeformed Poincare–Hopf algebra, the algebraic sector remains unchanged; therefore, the classical Poincare algebra remains valid and the formulae for Casimirs are not changed. The twist factor \( \mathcal{F} \in \mathcal{U}(\mathcal{P}) \otimes \mathcal{U}(\mathcal{P}) \) deforms only the co-structure of Hopf algebra by modification of the coproduct and antipode by the following formulae:

\[
\Delta_{\mathcal{F}} (g) = \mathcal{F}\Delta_0(g)\mathcal{F}^{-1},
\]

\[
S_{\mathcal{F}} = vS_0(g)v^{-1}, \quad v = f_1S(f_2),
\]

where we denote \( \mathcal{F} = f_1 \otimes f_2 \). When the coproduct is changed, the action of the algebra \( \mathcal{U}(\mathcal{P}) \) on the tensor product representation is modified [50] because the coproduct (4) enters into the covariant action as follows \( \Delta_{\mathcal{F}}(g) = g_{(1)} \otimes g_{(2)} \) :

\[
g \triangleright (vw) = m[\Delta_{\mathcal{F}}(g) \triangleright (v \otimes w)] = \sum (g_{(1)} \triangleright v)(g_{(2)} \triangleright w), \quad g \triangleright 1 = \epsilon(g) \triangleright 1.
\]

If twist \( \mathcal{F} \) does not generate a nontrivial coassociator, it should satisfy the 2-cocycle condition

\[
\mathcal{F}_{12}(\Delta \otimes id)\mathcal{F} = \mathcal{F}_{23}(id \otimes \Delta)\mathcal{F}.
\]

In such a case, the leading term describing the twist factor \( \mathcal{F} \) is described by the classical \( r \)-matrix (by \( \xi \), we denote the deformation parameter)

\[
\mathcal{F} = 1 \otimes 1 + \xi r + \mathcal{O}(\xi^2),
\]

where \( r \in g \otimes g \) satisfies the classical Yang–Baxter equation; one obtains for any generator \( g \) that

\[
\Delta(g) = \Delta^{(0)}(g) + \xi [r, \Delta^{(0)}(g)] + \mathcal{O}(\xi^2).
\]

From Lie algebra relations for \( g \) it follows that the first-order correction in (9) belongs to the tensor product \( g \otimes g \).

The algebra of the canonically deformed spacetime

\[
[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}
\]

can be described as covariant under deformed (twisted) Poincare symmetries if one introduces the canonical twist factor [8, 12, 13]

\[
\mathcal{F} = e^{i\theta_{\mu\nu}p_\mu \otimes p_\nu}.
\]

For canonical deformation, from twist factor (11), we obtain only the modification of coproduct for Lorentz generators \( M_{\mu\nu} \)

\[
\Delta_{\mathcal{F}}(P_\mu) = \Delta_0(P_\mu),
\]

\[
\Delta_{\mathcal{F}}(M_{\mu\nu}) = \Delta_0(M_{\mu\nu}) - \theta^{\rho\sigma}[(\eta_{\rho\mu}P_\rho - \eta_{\rho\nu}P_\rho) \otimes P_\sigma + P_\rho \otimes (\eta_{\sigma\mu}P_\rho - \eta_{\sigma\nu}P_\rho)].
\]

The coinverse as well as counit remains undeformed

\[
S(g) = -g, \quad \epsilon(g) = 0.
\]
For twisted canonical spacetime generated by twist \((11)\), the multiplication \(\star_{\mathcal{M}}\) \((21)\) in the algebra \(\mathcal{M}\) is defined by the twist factor \(\mathcal{F}\) as follows:
\[
e^{ipx} \star_{\mathcal{M}} e^{iqy} = m \circ \mathcal{F}^{-1} \triangleright [e^{ipx} \otimes e^{iqy}]
= m \circ (\mathcal{F}^{-1}(1) \triangleright e^{ipx} \otimes \mathcal{F}^{-1}(2) \triangleright e^{iqy}) = \mathcal{F}^{-1}(p, q) e^{ipx} e^{iqy},\]
where in the general case we denote \(\mathcal{F}^{-1} = \mathcal{F}^{-1}(1) \otimes \mathcal{F}^{-1}(2)\) and in the canonical case \(\mathcal{F}^{-1}(p, q) = e^{-i p q},\ p q \equiv p^\mu \theta_{\mu \nu} q^\nu\).

The algebra \((10)\) can be extended consistently to the collection of noncommutative plane waves \(e^{i p_\mu x_\mu} \phi(h)\), \(\phi\in\Phi_1\) in the following way \([15, 40]\) \((i, j = 1, 2, \ldots, n)\)
\[
\left[\tilde{x}_\mu^{(i)}, \tilde{x}_\nu^{(j)}\right] = i \theta_{\mu \nu}.
\]
If we introduce the corresponding Weyl map for the bilocal product \(e^{i p_\mu x_\mu} e^{i q_\nu y_\nu}\) \((\tilde{x}_\mu \equiv \tilde{x}_\mu^{(1)}, \tilde{y}_\mu \equiv \tilde{x}_\mu^{(2)})\), it can be shown that formula \((15)\) can be extended as follows:
\[
e^{ipx} \star_{\mathcal{M}} e^{iqy} = m \circ \mathcal{F}^{-1} \triangleright [e^{ipx} \otimes e^{iqy}] = \mathcal{F}^{-1}(p, q) e^{ipx} e^{iqy}.
\]

The twisted canonical Poincare algebra is the triangular Hopf algebra, with a universal \(R\)-matrix satisfying the relation
\[
\Delta_{21} = R \Delta R^{-1},
\]
and given by the formula
\[
R = \mathcal{F}_{21} \mathcal{F}^{-1} = \mathcal{F}^{-2}, \quad \mathcal{F}_{21} = \mathcal{F}^{-1} = \mathcal{F}^2,
\]
where in \((19)\) we used the relation \(\mathcal{F}_{21} = \mathcal{F}^{-1}\). Formulae \((19)\) describe the universal \(R\)-matrix for triangular Hopf algebras.

### 3. Deformed quantum fields and the braided tensor product

If we construct deformed QFT, we deal with the following three algebras: the algebra \(\mathcal{M}\) of functions \(f \in \mathcal{M}\) on noncommutative spacetime, the algebra \(\mathcal{H}\) of deformed field oscillators \(h \in \mathcal{H}\) and the algebra \(\Phi\) of deformed fields \(\phi \in \Phi\). The deformed relativistic free quantum fields \(\phi\) can be described as the infinite sum (in fact continuous integral) of the braided tensor products of the noncommutative plane waves \(e^{ipx}\) from \(\mathcal{M}\) and the elements \(A(p)\) from the algebra of field oscillators \(\mathcal{H}\)
\[
\phi \in \mathcal{M} \otimes \mathcal{H},
\]
where underlining of the tensor product describes its braided nature.

The algebra \(\mathcal{M}(e^{ipx}, \cdot)\) of basic functions on noncommutative spacetime \(\tilde{x} = (\tilde{x}_i, \tilde{x}_0)\) shall be further represented isomorphically by the star product \(\star_{\mathcal{M}}\) of commutative functions \(e^{ipx}\) provided by means of the Weyl map \(\mathcal{M}(e^{ipx}, \cdot)\)
\[
e^{ipx} e^{iqy} \mapsto m(e^{ipx} \otimes e^{iqy}) \xrightarrow{\text{Weyl map}} m_\mathcal{M}(e^{ipx} \otimes e^{iqy}) = e^{ipx} \star_{\mathcal{M}} e^{iqy},
\]
where \(x = (x_i, x_0)\) describes the standard Minkowski spacetime coordinates. We add that for the consideration of bilocal products \(\phi(\tilde{x})\phi(\tilde{y})\), we should introduce as well the Weyl map \(e^{ipx} e^{iqy} \mapsto e^{ipx} \star_{\mathcal{M}} e^{iqy}\) generalizing formula \((21)\) (see also \((15)\) and \((17)\)). Effectively in formula \((20)\), we describe \(\mathcal{M}\) by the commutative functions on the classical Minkowski space with the modified star-multiplication law \(m \mapsto m_{\mathcal{M}}\).

The Poincare algebra as well as its Casimirs are not modified by the twist deformation, i.e. if we consider only twisted Poincare symmetries, they do not change the mass shell which
is provided by the mass Casimir, or equivalently the relativistic energy–momentum dispersion relation \( \omega(\vec{p}) = \sqrt{\vec{p}^2 + m^2} \) remains valid after twisting. In accordance with (20), in twisted field theory one can use the standard Fourier decomposition of free scalar quantum fields with noncommutative Fourier exponentials

\[
\phi(\vec{x}) = \frac{1}{(2\pi)^4} \int d^4 p \delta(p^2 - m^2) e^{i\vec{p}\cdot\vec{x}} A(p) \\
= \frac{1}{(2\pi)^3} \int \frac{d^3 \vec{p}}{2\omega(\vec{p})} \left( e^{i\vec{p}\cdot\vec{x}} a(\vec{p}) + e^{-i\vec{p}\cdot\vec{x}} a(\vec{p}) \right)_{\text{p\text{m}\text{p}}} ,
\]

where \( a(\vec{p}) = A(\vec{p}, \omega(\vec{p})) \), \( (a(\vec{p}) = A(-\vec{p}, -\omega(\vec{p})) \) are the deformed annihilation (creation) field oscillators.

The algebra of deformed field oscillators \( \mathcal{H}(A(p); \cdot) \) with standard multiplication in \( \mathcal{H} \) \( (m(A(p) \otimes A(q)) = A(p)A(q)) \) in the noncommutative case will be described in accordance with the quantum-deformed covariance condition by the braided commutation relations of the oscillator algebra with suitable multiplication in \( \mathcal{H} \).

If we use the Weyl map (21), the elements of algebra \( \mathcal{M} \) are replaced by classical functions (e.g. \( e^{i\vec{p}\cdot\vec{x}} \rightarrow e^{ip\cdot x} \)). Further using the isomorphism \( \mathcal{C} \otimes \mathcal{H} \simeq \mathcal{H} \), one can introduce the field \( \phi(\vec{x}) \sim \varphi(x) \) (\( W \) denotes the Weyl map) having a well-known standard form

\[
\varphi(x) = \frac{1}{(2\pi)^4} \int d^4 p \delta(p^2 - m^2) e^{i\vec{p}\cdot\vec{x}} A(p) \\
= \frac{1}{(2\pi)^3} \int \frac{d^3 \vec{p}}{2\omega(\vec{p})} \left( e^{ip\cdot x} a(\vec{p}) + e^{-ip\cdot x} a(\vec{p}) \right)_{\text{p\text{m}\text{p}}} ,
\]

where \( \varphi(x) \in \mathcal{H} \), and we point out that oscillators \( A(p) \) will satisfy the deformed binary relations.

The tensor structure of the deformed quantum field (see (20) and (22)) should be consistent with the actions of the spacetime symmetry generator \( g \). On the deformed symmetry algebra modules \( \mathcal{M} \) and \( \mathcal{H} \), the actions of \( g \) are defined by the coproduct \( \Delta(g) = g(1) \otimes g(2) \) as follows:

\[
g \cdot a = ad_{g}a = \sum g_{(1)}ag_{(2)}. \quad a \in \mathcal{H},
\]

\[
g \triangleright f = ad_{g}f = \hat{D}(g)f. \quad f \in \mathcal{M},
\]

where \( \hat{D}(g) \) describes the differential realization of \( g \) on noncommutative functions \( f \in \mathcal{M} \).

If we wish to define the actions \( \triangleright \) and \( \cdot \) on the tensor product \( \mathcal{M} \otimes \mathcal{H} \), we postulate in consistency with the symmetry properties of the no-deformation classical limit the trivial actions on the ‘wrong’ part of the tensor product (20)

\[
g \cdot f = \epsilon(g)f .
\]

\[
g \triangleright a = \epsilon(g)a,
\]

where \( \epsilon(1) = 1 \) and \( \epsilon(g) = 0 \) for any Poincare generators. We see that the module \( \mathcal{M} \) under action \( \triangleright \) and the module \( \mathcal{H} \) under action \( \cdot \) behave as a number (scalar spectator). In particular, e.g. if we choose the product of basic modules \( e^{ip\cdot x} \otimes a(\vec{p}) \), one obtains for example

\[
g \cdot (e^{ip\cdot x} \otimes a(\vec{p})) = (ad \otimes ad) \Delta^{\mathcal{F}}(g)(e^{ip\cdot x} \otimes a(\vec{p}))
\]

\[
= adg_{(1)}e^{ip\cdot x}adg_{(2)}a(\vec{p}) = (g_{(1)} \cdot e^{ip\cdot x} \otimes (g_{(2)} \triangleright a(\vec{p}))).
\]

Because the general coproduct can be written as \( (\Delta^{(0)}(g) = g \otimes 1 + 1 \otimes g; \text{see also (9)}) \)

\[
\Delta^{\mathcal{F}}(g) = \Delta^{(0)}(g) + \text{terms not containing components 1} \otimes g \text{ and } g \otimes 1,
\]
the only term contributing to \( g(1) \triangleright e^{ip\hat{x}} \) comes from \( 1 \otimes g \), and one obtains
\[
g \triangleright (e^{ip\hat{x}} \otimes a(\hat{p})) = (1 \triangleright e^{ip\hat{x}}) \otimes (g \triangleright a(\hat{p})) = e^{ip\hat{x}} \otimes (g \triangleright a(\hat{p})).
\] (30)

The relation (30) has the correct classical limit when the factor \( e^{ip\hat{x}} \) becomes a classical function \( e^{px} \). Similarly
\[
g \triangleright (e^{ip\hat{x}} \otimes a(\hat{p})) = (g \triangleright e^{ip\hat{x}}) \otimes a(\hat{p}).
\] (31)

In the consideration of deformed symmetry properties, we shall need only the actions (30), (31) of the generators \( g \).

In this paper we consider the extension of the usual framework by introducing the algebra of deformed quantum fields \( \Phi_1(\phi, \psi) \) with the braided multiplication rule. We should multiply the noncommutative fields (20) in a way that takes into account the braid factor \( \Psi_1 \) introduced as follows:
\[
\phi(\hat{x}) \cdot \phi(\hat{y}) = m_{\mathcal{MH}} \phi(\hat{x}) \otimes \phi(\hat{y})
\] (32)
\[
= (m \otimes m) \circ (id \otimes \Psi_{\mathcal{MH}} \otimes id) \phi(\hat{x}) \otimes \phi(\hat{y}).
\]

The braiding factor \( \Psi_{\mathcal{MH}} \) describes effectively the noncommutativity of factors \( A(p) \) and \( e^{ip\hat{x}} \) in the product of field operators in accordance with the general formula for covariant braiding acting on two modules of the quasitriangular Hopf algebra. It is defined by a universal matrix \( R = R_1 \otimes R_2 \) \((f \in \mathcal{M}, h \in \mathcal{H})\); see e.g. [37]
\[
\Psi_{\mathcal{MH}}(h \otimes f) = (R_2 \triangleright f) \otimes (R_1 \triangleright h).
\] (33)

We see that if we specify in \( \mathcal{M} \otimes \mathcal{H} \) the braid factor \( R_{21} \), the generators \( g \) (we recall that \( R \) is defined in terms of Poincare algebra generators \( g \)) act nontrivially on both legs of the tensor product. In fact before applying the Weyl map to the product (32), we should first use the braid factor \( \Psi_{\mathcal{MH}} \) in order to introduce ordered elements of \( \mathcal{M} \) and \( \mathcal{H} \), with field oscillators on the right and noncommutative Fourier exponentials on the left.

The covariant braid factor (33) will be necessary in order to introduce the multiplication in the covariant algebra of deformed quantum fields \( \Phi(\phi, m_{\mathcal{MH}}) \). The commutative (undeformed) limit of the relation (32) is obtained by putting \( \Psi_{\mathcal{MH}} = \tau \), where \( \tau \) is the flip operator describing the no-deformation limit of \( R_{21} \). In the standard Poincare symmetry case, one assumes that
\[
[e^{px}, A(q)] = (1 - \tau)e^{px}A(q) = 0.
\] (34)

That is, the standard free quantum fields can be described by unbraided tensor products \((m_{\mathcal{MH}} \rightarrow m_{\mathcal{MH}}))\), which provides the ordering
\[
m_{\mathcal{MH}}[(e^{px} \otimes A(p)) \otimes (e^{py} \otimes A(q))]= e^{i(p_x+q_y)}A(p)A(q) \simeq e^{i(p_x+q_y)}A(p)A(q).
\] (35)

In this paper we describe the algebra of deformed quantum free fields with noncommutative spacetime arguments endowed with nontrivial braided multiplication (32) consistent with twisted Poincare covariance. The deformed covariant field quantization will be described by the \( c \)-number braided field commutator (see section 5). We shall perform explicit calculations for the case of quantum deformation described by the canonical twist (11). In the following section, we first describe the covariance conditions satisfied by twist-deformed quantum fields.

4. The twisted covariance of a single quantum field

Let us consider first the Poincare covariance of the standard (undeformed) free quantum field \( \psi(x) \), given by (23) with algebra \( \mathcal{H} \) describing standard field oscillator algebra. The classical Poincare covariance is given by the known formula
\[
U(\Lambda, a) \psi(x) U^{-1}(\Lambda, a) = \psi(\Lambda x + a),
\] (36)
where $U(\Lambda, a) = \exp(i a^\mu P_\mu + i a_{\mu\nu} M_{\mu\nu})$ describes the unitary representation of the Poincare group in the Hilbert–Fock space, generated from the Poincare-invariant vacuum by the products of field oscillators. For infinitesimal $a_\mu$ and $a_{\mu\nu}$ ($\Lambda^\mu_\nu = \delta^\mu_\nu + a^\mu_\nu$), it follows from relation (36) that there are two ways of expressing the action of Poincare generators on a quantum free field.

We obtain therefore in the deformed case the following deformed generalized Heisenberg equations (37) and (38) describe in an infinitesimal form the standard global Poincare covariance of quantum free fields, given by relation (36).

In the undeformed quantum free field case, the generators $P_\mu$, $M_{\mu\nu}$ are the bilinear functionals of the oscillators $a(p)$, $a^\dagger(p)$ and on the rhs of (37) and (38) we have the known relativistic differential realizations $D^{(0)}(\xi_A)$ ($\xi_A = (P_\mu, M_{\mu\nu})$) of the standard Poincare–Hopf algebra on the scalar fields with Minkowski spacetime arguments

$$D^{(0)}(P_\mu) = -i\partial_\mu, \quad D^{(0)}(M_{\mu\nu}) = -i(x_\nu \partial_\mu - x_\mu \partial_\nu).$$

The generalized Heisenberg equations (37) and (38) can be written as

$$M_{\mu\nu} \triangleright \varphi = -g^{(0)} \triangleright \varphi, \quad \varphi = [P_\mu, \varphi] = -D^{(0)}(P_\mu)\varphi = -P_\mu \triangleright \varphi, \quad \varphi = [M_{\mu\nu}, \varphi] = -D^{(0)}(M_{\mu\nu})\varphi = -M_{\mu\nu} \triangleright \varphi.$$

We see that relations (37)–(38) can be written as

$$g^{(0)} \triangleright \varphi = -g^{(0)} \triangleright \varphi,$$

where $g^{(0)}$ denotes the undeformed Poincare algebra generators providing the generalized Heisenberg equation on the classical Poincare group manifold.

In a noncommutative framework the description should be modified in accordance with formula (20) and corresponding Hopf-algebraic language. The deformed counterpart of equations (37) and (38) should be derived with the use of formulae (28)–(30). If we observe that the twisted coproduct is given by formula (9), due to the vanishing values of counit $\epsilon(g)$, only $\Delta^{(0)}(g)$ in (9) will contribute to the adjoint action $g \triangleright \phi$ in accordance with (29).

We obtain therefore in the deformed case the following deformed generalized Heisenberg equations:

$$g \triangleright \phi = S(g) \triangleright \phi, \quad g = (P_\mu, M_{\mu\nu})$$

or more explicitly

$$P_\mu \triangleright \phi = (ad_{P_\mu}(\otimes) \otimes ad_{P_\mu(\otimes)}) \phi = (1 \otimes ad_{P_\mu}) \phi = (D[S(P_\mu)] \otimes 1) \phi = S(P_\mu) \triangleright \phi,$$

$$M_{\mu\nu} \triangleright \phi = (ad_{P_\mu}(\otimes) \otimes ad_{P_\mu(\otimes)}) \phi = (1 \otimes ad_{M_{\mu\nu}}) \phi = (D[S(M_{\mu\nu})] \otimes 1) \phi = S(M_{\mu\nu}) \triangleright \phi,$$

where the adjoint action of deformed symmetry generators $g$ in $H$ is given by (24) and $\hat{D}(g)$ is the suitably deformed differential realization on $M$ of standard Poincare algebra, consistent with the deformed coalgebra structure. In particular after the Weyl map $f(\hat{x}) \xrightarrow{W} f(x)$, $g(\hat{x}) \xrightarrow{W} g(x)$, $\hat{D}(g) \xrightarrow{W} D(g)$, we should have the following form of the modified Leibnitz rule

$$D(g)(f(x) \star_M g(x)) = m_M \circ [(D \otimes D)\Delta^T(g) \circ f(x) \otimes g(x)].$$

In the case of canonical deformation, as follows from (12), we obtain the undeformed covariance relation (37) for the momentum generators, because from (24) it follows that

$$(1 \otimes ad_{P_\mu}) \phi = (1 \otimes ad^{(0)}_{P_\mu}) \phi,$$

and $D(P_\mu) = D^{(0)}(P_\mu)$. For the Lorentz algebra generators $M_{\mu\nu}$, it follows from (13), (14) and (24) that

$$(1 \otimes ad_{M_{\mu\nu}}) \phi = [1 \otimes M_{\mu\nu}, \phi] + \theta_{\mu\nu} \cdot (1 \otimes P_\mu)(1 \otimes P_{\mu\nu}) + \theta_{\mu\nu} \cdot (1 \otimes P_\mu)(1 \otimes P_{\mu\nu}).$$
In order to obtain in relation (43) the suitably deformed differential realization of Lorentz algebra generators $D(M_{\mu\nu})$, one can use either a differential realization on the noncommutative Minkowski space or, after performing the Weyl map, we can consider the deformation $D(M_{\mu\nu})$ of $D^{(0)}(M_{\mu\nu})$ (see (40)) on the standard Minkowski space extended to relativistic phase space. We shall use below the second possibility. Let us observe that the noncommutative spacetime coordinates (10) can be described by the relativistic nondeformed quantum phase space variables $(x_\mu, p_\mu)$ as follows:

$$\tilde{x}_\mu = x_\mu + \theta_{\mu\nu} p^\nu,$$

(47)

where $[x_\mu, p_\nu] = i \eta_{\mu\nu} (\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1))$, $[x_\mu, x_\nu] = [p_\mu, p_\nu] = 0$. Using relation (47) and the standard Lorentz transformations of the variables $x_\mu, p_\nu$, the modified Lorentz transformation of canonically deformed spacetime variables $\tilde{x}_\mu$ can be expressed as

$$\tilde{x}'_\mu = \Lambda^\mu_\rho x_\rho + \theta_{\mu\nu} \Lambda^\nu_\rho p^\rho = \Lambda^\mu_\nu \tilde{x}_\nu + \tilde{\zeta}_\mu,$$

(48)

where $\tilde{\zeta}_\mu = (\theta_{\mu\rho} \Lambda^\rho_\nu - \Lambda^\mu_\rho \theta_{\rho\nu}) p^\nu$ is the momentum-dependent translation. Using the infinitesimal Lorentz transformation ($\Lambda = \delta + \omega$), we obtain for the noncommutative spacetime coordinates the following infinitesimal Lorentz transformations

$$\tilde{x}'_\mu = \tilde{x}_\mu + \omega_{\mu\nu} \tilde{x}_\nu + \tilde{\zeta}_\mu,$$

(49)

where $\tilde{\zeta}_\mu = (\omega_{\mu\rho} x^\rho_\nu - \omega_{\mu\rho} x^\rho_\nu) p^\rho$. If we perform the Weyl map $\tilde{x}_\mu \rightarrow x_\mu$ and observe from (47) that $[\tilde{x}_\mu, p_\nu] = i \eta_{\mu\nu} \rightarrow [x_\mu, p_\nu] = i \eta_{\mu\nu}$, the canonically conjugated momenta $p_\mu$ can be represented by the derivatives ($p_\mu \rightarrow -i \partial_\mu$) and formula (22) is replaced by (23). Then the infinitesimal deformed Lorentz transformation, as follows from (49), implies the following change of the field operator (23):

$$i \omega^{\mu\nu} D(M_{\mu\nu}) \varphi = [\omega^{\alpha\beta} (\partial_\alpha x^\mu_\beta - \partial_\beta x^\mu_\alpha) + \tilde{\zeta}^\mu] \partial_\mu \varphi,$$

(50)

and we obtain that

$$D(M_{\mu\nu}) = -i (x_\mu \partial_\nu - x_\nu \partial_\mu) - [\theta^{\rho\sigma} \partial_\rho \partial_\sigma + \theta^{\rho\sigma} \partial_\sigma \partial_\rho].$$

(51)

We recall here that formula (51) is known [15, 46]. Expressing the relation (51) for a deformed differential realization as the relation between classical Poincare algebra generators, we obtain the deformed Lorentz generators as given by nondeformed Poincare generators as follows (we recall that $P_\mu = P^{(0)}_\mu$):

$$M_{\mu\nu} = M_{\mu\nu}^{(0)} + \theta^{\rho\sigma} P_\sigma P_\rho + \theta^{\rho\sigma} P_\rho P_\sigma.$$

(52)

The covariance relations (42) describe the equality of the action on the field (22) of the Poincare algebra symmetry generators $g$ in classical (differential) spacetime and quantum-mechanical (adjoint) field oscillators realizations. For an arbitrary twist deformation, the quantum adjoint action (24) and differential field realizations are determined by the coalgebra relations of quantum Poincare symmetry (see (4)). Several explicit examples of differential realizations $D(g)$ satisfying (42) and (44) have been presented in the literature, but the formulae providing $D(g)$ for an arbitrary twist were not given.

5. Braided multiplication of fields and the covariant field commutator

5.1. The covariance of a binary product of noncommutative free quantum fields

The bilocal $\star_M$-multiplication for a canonically deformed Poincare symmetry is given by formula (17). In the algebra of oscillators $\mathcal{H}(A(p), \cdot)$, we use the standard multiplication rule.

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4 See however [47] where the generalization of formula (47) for an arbitrary twist is proposed.
We shall introduce the braid factor $\Psi_{M,H}$ in the physical basis of $M$ and $H$ in accordance with (33)
\[ \Psi_{M,H}(A(p) \otimes e^{i\hat{q}^2}) = R_{(2)} \triangleright e^{i\hat{q}^2} \otimes R_{(1)} \triangleright A(p). \] (53)

For twist-deformed theory the multiplication prescription (32) is determined if we know the twist factor $\mathcal{F}$ and the braid $\Psi_{M,H}$ given by (53). The explicit form of the product (32) of fields on canonical noncommutative spacetime has therefore the form
\[ \phi(\hat{x}) \otimes \psi(\hat{y}) = m_{M \otimes H}[\phi(x) \otimes \psi(y)] \]
\[ = \frac{1}{(2\pi)^8} \int d^4p \int d^4q \delta(p^2 - m^2) \delta(p^2 - m^2) \]
\[ e^{ip\hat{x}}(R_{(2)} \triangleright e^{i\hat{q}^2}) \otimes (R_{(1)} \triangleright A(p)A(q)), \] (54)
\[ \simeq \frac{1}{(2\pi)^8} \int d^4p \int d^4q \delta(p^2 - m^2) \delta(p^2 - m^2) \]
\[ (\hat{f}_{(1)} \triangleright e^{i\hat{q}^2})(\hat{f}_{(2)}R_{(2)} \triangleright e^{i\hat{q}^2})(R_{(1)} \triangleright A(p)A(q)), \]
where we recall that $\mathcal{F}^{-1} = \hat{f}_{(1)} \otimes \hat{f}_{(2)}$ and by the notation $\simeq$ we denote the Weyl homomorphism in $M$ with the $\star$-product of classical Fourier exponentials representing the product $e^{ip\hat{x}}e^{i\hat{q}^2}$. In the actions on the classical plane waves which are present in last formula (54), one uses the differential realization $D(g_A)$ (for deformation by canonical twist, see (51)), and on the field oscillators, the Poincare generators act by the quantum adjoint action (see (24), (45) and (46)).

Before explicit calculation of the braided commutator (2), we shall show that it is twist covariant. Using formula (6) for the algebra $\Phi(\phi, m_{M \otimes H})$ of deformed free quantum fields, we choose
\[ g \triangleright (\phi(\hat{x}) \otimes \psi(\hat{y})) = m_{M \otimes H}[\Delta(g) \triangleright \phi(\hat{x}) \otimes \psi(\hat{y})] \]
\[ = (g_{(1)} \triangleright \phi(\hat{x})) \otimes (g_{(2)} \triangleright \psi(\hat{y})), \] (55)
where the quantum action (24) on the algebra $H$ is used. Furthermore ($R_{21} = R_{(2)} \otimes R_{(1)}$)
\[ g \triangleright (R_{21} \triangleright [\phi(\hat{x}) \otimes \psi(\hat{y})]) = g \triangleright ((R_{(2)} \triangleright \phi(\hat{x})) \otimes (R_{(1)} \triangleright \psi(\hat{y}))) \]
\[ = m_{M \otimes H}[(\Delta(g)R_{21} \triangleright \phi(\hat{x}) \otimes \psi(\hat{y}))]. \] (56)

The twisted covariance of the braided commutator (2) means that
\[ g \triangleright [\phi(\hat{x}) \otimes \psi(\hat{y})]^{BR} = (g_{(1)} \triangleright \phi(\hat{x})) \otimes (g_{(2)} \triangleright \psi(\hat{y})) \]
\[ - R_{21} \triangleright [(g_{(2)} \triangleright \phi(\hat{x})) \otimes (g_{(1)} \triangleright \psi(\hat{y}))], \] (57)
which requires the choice (53) of the braid factor defining $\star$-multiplication and the validity of the relation (see also (18))
\[ \Delta(g)R_{21} - R_{21}\Delta_{21}(g) = 0. \] (58)

Indeed, if we use formulae (4) and (19) one shows easily that in twist-deformed theory, the relation (58) is valid
\[ \Delta(g)R_{21} - R_{21}\Delta_{21}(g) = \mathcal{F}[\Delta_0(g), \tau]\mathcal{F}^{-1} = 0. \] (59)
5.2. Calculation of the covariant braided field commutator: braided field oscillators algebra and braided locality

By using (54) and (2) we shall calculate explicitly the braided commutator. We obtain

\[ [\phi(x), \phi(y)]^{BR}_\star \cong \frac{1}{(2\pi)^8} \int d^4 p \int d^4 q \delta(p^2 - m^2)\delta(q^2 - m^2) \]

\[ \left[ (\mathcal{R}_{(1)}^{\star} \triangleright e^{ip_1}) (\mathcal{R}_{(2)}^{\star} \triangleright e^{ip_2}) (\mathcal{R}_{(1)}^{\star} \triangleright A(p) A(q) - (\mathcal{R}_{(2)}^{\star} \triangleright A(q)) (\mathcal{R}_{(1)}^{\star} \triangleright A(p)). \right. \]

(60)

Let us consider further the canonical deformation described by the twist (11). As follows from (19), \( \mathcal{R}_{21} \) depends only on the four-momentum generators and their actions follow from the formulæ

\[ P_\mu \triangleright e^{ip} = p_\mu e^{ip}, \quad P_\mu \triangleright A(p) = -p_\mu A(p). \]

(61)

In the canonically deformed case, the universal \( \mathcal{R} \)-matrix (19) acts as follows:

\[ \mathcal{R}_{21} \triangleright [e^{ip_1} \otimes e^{ip_2}] = e^{i\theta_{\mu\nu} p_\mu q_\nu} e^{ip_1} \otimes e^{ip_2}, \]

\[ \mathcal{R}_{21} \triangleright [A(p) \otimes A(q)] = e^{i\theta_{\mu\nu} p_\mu q_\nu} A(p) \otimes A(q), \]

(62)

and the braid \( \Psi_{\mathcal{M}, \mathcal{H}} \) has the explicit form

\[ \Psi_{\mathcal{M}, \mathcal{H}}^{\otimes A(q)} = \mathcal{R}_{(2)}^{\star} \triangleright e^{ip_1} \otimes \mathcal{R}_{(1)}^{\star} \triangleright A(q) = e^{-i\theta_{\mu\nu} p_\mu q_\nu} e^{ip_1} \otimes A(q). \]

(63)

In order to obtain \( c \)-number braided field commutator, one should be able to factor out in (60) the binary relations satisfied by the field oscillators (the field oscillators algebra). If we use formula (62) the required factorization in formula (60) is achieved (we use in the last term of (64) the shorthand notation described in footnote 2)

\[ [\phi(x), \phi(y)]^{BR}_\star \cong \frac{1}{(2\pi)^8} \int d^4 p \int d^4 q \delta(p^2 - m^2)\delta(q^2 - m^2) e^{ipx} e^{ipy} \]

\[ [A(p) \star_{\mathcal{H}} A(q) - \mathcal{R}_{21} \triangleright (A(q) \star_{\mathcal{H}} A(p))]. \]

(64)

provided that we introduce the following multiplication for the description of binary products

\[ A(p) \star_{\mathcal{H}} A(q) = m \circ \mathcal{F} \triangleright [A(p) \otimes A(q)] = e^{i\theta_{\mu\nu} p_\mu q_\nu} A(p) A(q). \]

(65)

We add that the nonstandard multiplication \( \star_{\mathcal{H}} \) is different from \( \star_{\mathcal{M}} (\mathcal{F}^{-1}) \) in (15) is replaced by \( \mathcal{F} \) but it is known from the literature and was used e.g. in [15].

We recall that in undeformed theory the covariant formulation of the field oscillator algebra is provided by the relation (see e.g. [44])

\[ \delta(p^2 - m^2)\delta(q^2 - m^2)[A(p), A(q)] = \epsilon(p_0)\delta(p^2 - m^2)\delta^{(4)}(p + q). \]

(66)

The following modification of relation (66) describes the binary relation for deformed field oscillators which due to the presence of braid factor \( \mathcal{R}_{21} \) is covariant under quantum symmetries and leads to \( c \)-number value of the braided commutator (64)

\[ \delta(p^2 - m^2)\delta(q^2 - m^2)[A(p) \star_{\mathcal{H}} A(q) - \mathcal{R}_{21} \triangleright (A(q) \star_{\mathcal{H}} A(p))] \]

\[ = \epsilon(p_0)\delta(p^2 - m^2)\delta^{(4)}(p + q). \]

(67)

If we substitute (67) into (64), we obtain that

\[ [\phi(x), \phi(y)]^{BR}_\star \cong \Delta(x - y; m^2), \]

(68)
with the braided commutator for canonically deformed free quantum fields given by the known classical Pauli–Jordan function
\[
\Delta(x − y; m^2) = \frac{-i}{(2\pi)^3} \int \frac{d^3 p}{\omega(\vec{p})} \sin[\omega(\vec{p})(x_0 − y_0)] e^{i\vec{p}(\vec{x} − \vec{y})}. \tag{69}
\]

It should be noted that the choice of \( \star_{M} \) (see (15) for the canonical case) and of the covariant braid \( \Psi_{1}^{M} \) (see (33) and (53)) is necessary for getting the twist-covariant algebra of deformed field operators. The braid factor \( R_{21} \) which is an intertwiner in the quantum quasitriangular Poincare–Hopf algebra appears in our framework on three levels:

1. In the Weyl realization of the algebra \( M \) as expressing the ‘braided commutativity’ of the \( \star_{M} \)-multiplication (see e.g. [42], \( f, h \in M \))
   \[
   [f, h]^{BR}_{\star_{M}} := f \star_{M} h = (R_{(2)} \triangleright h) \star_{M} (R_{(1)} \triangleright f) = 0. \tag{70}
   \]
   In particular, putting \( f = x_{\mu}, h = x_{\nu} \) in (70) we reproduce the spacetime noncommutativity corresponding to a given \( R \). We add that the relation (70) is covariant for any choice of \( f \) and \( h \) under the action of deformed Poincare generators \( g = (P_{\mu}, M_{\mu\nu}) \)
   \[
   g \triangleright [f, h]^{BR}_{\star_{M}} = 0. \tag{71}
   \]
2. In the algebra (67) of quantized field oscillators which is covariant under the action \( \triangleright \) of the symmetry generators \( g \).
3. In the algebra of the deformed free quantum field \( \phi \) (see (2)) and the multiplication \( m_{M\otimes H} \), which leads to the covariance of the braided field commutator (68).

We point out that the nonstandard multiplication (65) is selected by the validity of braided ⊗-locality described by formula (68). If we look at formula (67) from the point of view of its deformed Poincare covariance, one can show that only the braid factor \( R_{21} \) is required, but the choice of multiplication in \( H \) is not determined.

6. Final remarks

The aim of this paper was to study quantum Poincare covariance in the framework of the braided formulation of the theory of noncommutative quantum free fields. We restricted our considerations to binary products of such fields, but for twist-deformed noncommutative fields the extension of our formalism to \( n \)-ary associative products is straightforward. It can be shown (see e.g. [50]) that the associativity of braided products will follow from the hexagon relation satisfied by braid \( R_{21} \).

We stress that we employ the triple-twisted covariance requirement, obtain the braided form (2) of the deformed field commutator, the braided form (67) of the deformed oscillator algebra and braided product (32) in the algebra of deformed quantum free fields. We show that the derivation of a covariant braided c-number field commutator describing braided locality requires the validity of the relation (67) for deformed oscillators. It should be stressed that separate elements of our construction were present in previous papers (e.g. the Weyl map (21) is used in all papers with the realization of noncommutative fields in the standard Minkowski space; see also [11, 28]) but all elements of our construction occur together only in this paper.

Our explicit calculations have been given for the simplest case of canonical twist deformation, with the additional numerical phase space factors in momentum space characterizing the canonical deformation. If however the twist factor depends as well on the Lorentz generators \( M_{\mu\nu} \) (see e.g. [45]), the formulae describing deformed fields are more complicated. In such a case, due to the realization (40) of Lorentz generators after the Weyl map (21), the bidifferential operator describing star product \( \star_{M} \) depends as well on the spacetime
coordinate $x_\mu$. Analogously, the adjoint action of the braid factor $R_{21}$ on oscillators in relations (67) is described effectively by the bidifferential operators in four-momentum space acting on the product of two field oscillators which are the operator-valued functions depending on the four-momenta $p_\mu, q_\mu$.\(^7\)

An important question which should be considered in the future is the application of braided-deformed free quantum fields for the description of deformed interacting QFT. For that purpose one should use the deformed version of the formulae expressing interacting quantum fields in terms of free fields, within the perturbative framework. There are two ways of approaching this problem:

1. One can deform the perturbative rules for Feynman diagrams, with braid-deformed free Feynman propagators and suitably modified vertices. Such a perturbative description in the case of canonical deformation however without the use of braiding and quantum group symmetries was first proposed by Filk [29], and leads to a nonlocal generalization of Dyson $S$-matrix formula (the nonlocality is obtained if we replace after the Weyl map the standard point-wise multiplication of fields by nonlocal $\ast$-multiplication). At present it is a challenge to derive a braid-deformed counterpart of the Dyson formula describing deformed perturbative $S$-matrix expansion (for a clue in this direction, see e.g. [51, 52]).

2. Another way of defining deformed interacting QFT is to modify the formulae expressing the interacting fields in terms of free asymptotic fields (they define the so-called Haag expansion [53]). The basic dynamical tool for such an approach is provided by perturbative solution of the deformed Yang–Feldmann equation.

Finally let us observe that, the presented formalism with deformed braided fields can in principle be applied to general quasitriangular quantum deformation of free quantum fields. Such general deformations can be described by the use of the twist technique only in the Drinfeld category of quasi-Hopf algebras, with a nontrivial coassociator introducing a nonassociative product of three field operators [48–50, 54].

$$\phi(\hat{x}) \ast (\phi(\hat{y}) \ast \phi(\hat{z})) \equiv \Psi_{1(2(3}}(\phi(\hat{x}) \ast \phi(\hat{y})) \ast \phi(\hat{z}).$$  \((72)\)

In particular, because the $\kappa$-deformation of Poincare symmetries is not described by the triangular Poincare–Hopf algebra, the description of $\kappa$-deformed free quantum fields covariant under $\kappa$-deformed Poincare symmetries can be described by twist only with a nontrivial coassociator and requires the application of a non-coassociative framework of quasi-Hopf algebras [54, 55]. We should mention that effective application of the approach presented here to the $\kappa$-deformed field-theoretic framework is under consideration.

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\(^7\) See e.g. the derivation of the nonlocal product of $\kappa$-deformed field oscillators given in [27].
