SUPPORTS OF MEASURES IN A FREE ADDITIVE
CONVOLUTION SEMIGROUP

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Abstract. In this paper, we study the supports of measures in the free additive
convolution semigroup $\{\mu \boxplus t : t > 1\}$, where $\mu$ is a Borel probability measure on $\mathbb{R}$. We give a formula for the density of the absolutely continuous part of $\mu^\boxplus t$ and use this formula to obtain certain regularizing properties of $\mu^\boxplus t$. We show that the number $n(t)$ of the components in the support of $\mu^\boxplus t$ is a decreasing function of $t$ and give equivalent conditions so that $n(t) = 1$ for sufficiently large $t$. Moreover, a measure $\mu$ so that $\mu^\boxplus t$ has infinitely many components in the support for all $t > 1$ is given.

1. Introduction

Given two Borel probability measures $\mu$ and $\nu$ on $\mathbb{R}$, denote by $\mu \boxplus \nu$ their free additive convolution, which is the probability distribution of $X + Y$, where $X$ and $Y$ are free selfadjoint random variables with distributions $\mu$ and $\nu$, respectively. Free convolution is a binary operation on the set of Borel probability measures on $\mathbb{R}$, which is analogue of classical convolutions. We refer to [19] for a systematic exposition of this subject and [5] for analytic methods for the computations of free convolutions.

One of the important properties for free convolution is subordination. That is, the Cauchy transform (see section 2 for definition) $G_{\mu \boxplus \nu}$ is subordinated to $G_{\mu}$, in the sense that $G_{\mu \boxplus \nu} = G_{\mu} \circ \omega$ for some analytic self-mapping $\omega$ of the complex upper half-plane. We refer the reader to [7,9, and 17] for details. One of the applications of subordination functions is to study the regularity for free convolution. Free convolution is a highly nonlinear operation, however, it has a stronger regularizing effect than classical convolution. Systematic study of regularity for free convolution by means of subordination functions can be found in [6,7,8, and 17]. In particular, the atoms of free convolutions are determined. Moreover, if $\gamma$ is the semicircular distribution then the density of $\mu \boxplus \gamma$ is continuous on $\mathbb{R}$ and analytic wherever it is positive.

For $n \in \mathbb{N}$, the $n$-fold free convolution $\mu \boxplus \cdots \boxplus \mu$ is denoted by $\mu^{\boxplus n}$. It is known that the discrete semigroup $\{\mu^{\boxplus n} : n \in \mathbb{N}\}$ can be embedded in a continuous family $\{\mu^{\boxplus t} : t \geq 1\}$ such that $\mu^{\boxplus t_1} \boxplus \mu^{\boxplus t_2} = \mu^{\boxplus (t_1 + t_2)}$ for all $t_1, t_2 \geq 1$. This was first proved by Bercovici and Voiculescu in [6] for measures $\mu$ with compact support and for large values of $t$. Later, in [14] Nica and Speicher generalized this result to compactly

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1. **2000 Mathematics Subject Classification:** Primary 46L54, Secondary 30A99.
2. **Key words and phrases.** Free Convolution, Cauchy transform, regularity.
supported measures and $t > 1$ by exhibiting explicit random variables with the corresponding distributions whose calculations of the distribution are based on moments and combinatorics. The existence of subordination functions $\omega_t$ in this context and related consequences for the regularity for the measures $\mu^{\Xi t}$ were thoroughly studied by Belinschi and Bercovici in [1] and [2]. More precisely, for $t > 1$ let $\mu^{\Xi t}$ be the measure satisfying the requirement $F_{\mu^{\Xi t}} = F_{\mu} \circ \omega_t$, where $F_{\mu^{\Xi t}} = 1/G_{\mu^{\Xi t}}$ and $F_{\mu} = 1/G_{\mu}$ are reciprocal Cauchy transforms. This gives an alternate proof of the existence of $\mu^{\Xi t}$.

With the help of $\omega_t$, it was shown that the free convolution power $\mu^{\Xi t}$ of $\mu$ has no singular continuous part and its density in the absolutely continuous part is locally analytic.

In this present paper, we use different points of views to obtain some properties of $F_\mu$ and analyze the relation between the supports $\text{supp}(\mu)$ and $\text{supp}(\mu^{\Xi t})$ of $\mu$ and $\mu^{\Xi t}$, $t > 1$, respectively. By means of the methods developed in this paper, we give a complete description about how $\text{supp}(\mu^{\Xi t})$ changes when $t$ increases. Motivated by the free central limit theorem (see [6,10,11,12,15, and 20] for details), one of the purposes of this paper is to study the number $n(t)$ of components in the support of $\mu^{\Xi t}$. We show that $n(t)$ is a decreasing function of $t$ for $t > 1$ and that $n(t) = 1$ for sufficiently large $t$ if and only if $n(t')$ is finite for some $t' > 1$. We also construct a measure $\mu$ so that $n(t)$ is infinite for all $t > 1$.

The paper is organized as follows. Section 2 contains some definitions and basic propositions from free probability theory. Section 3 provides some properties of $F_\mu$ which are related to the subordination function $\omega_t$, and an explicit formula for the density of the absolutely continuous part of $\mu^{\Xi t}$. Section 4 contains complete investigations on the supports of the measures $\mu^{\Xi t}$.

2. Preliminary

For any complex $z$ in $\mathbb{C}$, let $\Re z$ and $\Im z$ be the real and imaginary parts of $z$, respectively, and denote by $\mathbb{C}^+ = \{z \in \mathbb{C} : \Re z > 0\}$ the complex upper half-plane. For any Borel probability measure $\mu$ on $\mathbb{R}$, define its Cauchy transform as

$$G_\mu(z) = \int_{\mathbb{R}} \frac{d\mu(s)}{z - s}, \quad z \in \mathbb{C}^+.\$$

As shown in [5], the composition inverse $G_\mu^{-1}$ of $G_\mu$ is defined in an appropriate Stolz angle

$$D_{\eta,\epsilon} = \{x + iy : |x| < -\eta y, \ 0 < -y < \epsilon\}$$

of zero in the lower half-plane. The Voiculescu transform $\mathcal{R}_\mu$ of the measure $\mu$ defined as

$$\mathcal{R}_\mu(z) = G_\mu^{-1}(z) - z^{-1}, \quad z \in D_{\eta,\epsilon},$$

is the linearizing transform for the operation $\boxplus$. That is, the identity

$$\mathcal{R}_{\mu^{\Xi t}} = \mathcal{R}_\mu + \mathcal{R}_\nu$$
holds on a domain where these three functions are defined. The measure \( \mu \) can be recovered from the Cauchy transform \( G_\mu \) as the weak*\-limit of the measures

\[
d\mu_\epsilon(s) = -\frac{1}{\pi} \Im G_\mu(s + i\epsilon) \, ds
\]

as \( \epsilon \to 0^+ \), which we refer to as the Stieltjes inversion formula. Particularly, if the function \( \Im G_\mu(x) \) is continuous at the point \( x \in \mathbb{R} \) then in the Lebesgue decomposition the density of the absolutely continuous part of \( \mu \) at \( x \) is given by \( -\Im G_\mu(x)/\pi \).

Therefore, this inversion formula gives a way to extract the density function from the Cauchy transform.

The reciprocal Cauchy transform \( F_\mu = 1/G_\mu \) is an analytic self-mapping of \( \mathbb{C}^+ \). By Nevanlinna representation, it can be expressed as

\[
F_\mu(z) = a + z + \int_\mathbb{R} \frac{1 + sz}{s - z} \, d\rho(s),
\]

where \( a = \Re F_\mu(i) \) and \( \rho \) is some finite positive Borel measure on \( \mathbb{R} \) uniquely determined by \( F_\mu \). Indeed, \( \rho \) is the weak*\-limit as \( \epsilon \to 0^+ \) of the measures

\[
d\rho_\epsilon(s) = \frac{\Im F_\mu(s + i\epsilon)}{\pi(s^2 + 1)} \, ds.
\]

It was shown in [13] that \( F_\mu \) can be expressed in terms of a Cauchy transform of some measure if \( \mu \) has finite mean and unit variance. More precisely, a measure \( \mu \) satisfies

\[
\int_\mathbb{R} s \, d\mu(s) = 0 \quad \text{and} \quad \int_\mathbb{R} s^2 \, d\mu(s) = 1
\]

if and only if

\[
F_\mu(z) = z - G_\nu(z), \quad z \in \mathbb{C}^+,
\]

where \( \nu \) is some Borel probability measure on \( \mathbb{R} \).

Denote by \( z \to \alpha \) if \( z \to \alpha \) nontangentially to \( \mathbb{R} \), i.e., \((\Re z - \alpha)/\Im z\) stays bounded.

A useful criterion for locating an atom \( \alpha \) of \( \mu \) is the limit

\[
\lim_{z \to \alpha} (z - \alpha) G_\mu(z) = \mu(\{\alpha\}).
\]

An equivalent statement of the above limit is that a point \( \alpha \in \mathbb{R} \) is an atom of \( \mu \) if and only if \( F_\mu(\alpha) = 0 \) and the Julia-Carathéodoty derivative \( F_\mu'(\alpha) \) (which is the limit of

\[
\frac{F_\mu(z) - F_\mu(\alpha)}{z - \alpha}
\]

as \( z \to \alpha \) nontangentially and \( z \in \mathbb{C}^+ \)) is finite, in which case

\[
F_\mu'(\alpha) = \frac{1}{\mu(\{\alpha\})}.
\]

Similarly, there is an effective way to identify the atoms of the measure \( \rho \) in the Nevanlinna representation (2.1) for \( F_\mu \). The following lemma is well known, however, we have no recollection of seeing it stated explicitly. We provide a full proof without claiming any paternity.
Lemma 2.1. If $\alpha \in \mathbb{R}$ then
\[
\lim_{z \to \alpha} (z - \alpha)F_\mu(z) = -(\alpha^2 + 1)\rho(\{\alpha\}).
\]

Proof. By considering the positive measure $\rho - \rho(\{\alpha\})\delta_\alpha$, we may assume that $\alpha$ is not an atom of $\rho$, in which case we must show that $(z - \alpha)F_\mu(z) \to 0$ as $z \to \alpha$ nontangentially. Write $z = x + iy$ and suppose
\[
|x - \alpha| \leq M \quad \text{and} \quad |z - \alpha| < 1,
\]
where $0 < M < \infty$. Let $N = 2(|\alpha| + 1)$. Note that if $s \geq N$ then
\[
\left| \frac{s}{s - z} \right| = \left| \frac{s - x - iy}{s - x} \right| \leq \frac{s}{s - x} \leq \frac{s}{s - (\alpha + 1)} \leq 2,
\]
whence
\[
|z - \alpha| \left| \frac{sz}{s - z} \right| \leq 2(\alpha + 1).
\]
Similarly, the above inequality also holds for $s \leq -N$. For $s \in (-N, N)$, we have
\[
|z - \alpha| \left| \frac{sz}{s - z} \right| \leq \frac{|z - \alpha|}{y}|sz| \leq N(|\alpha| + 1)\sqrt{M^2 + 1}.
\]
Moreover, for any $s \in \mathbb{R}$ we have
\[
\left| \frac{z - \alpha}{s - z} \right| = \left| \frac{x - \alpha + iy}{s - x} \right| \leq \frac{x - \alpha + iy}{\sqrt{x^2 + y^2}} \leq \sqrt{M^2 + 1}.
\]
Since
\[
\left| \frac{(z - \alpha)\int_{\mathbb{R}} \frac{1+sz}{s-z} d\rho(s)}{\int_{\mathbb{R}} \frac{1+sz}{s-z} d\rho(s)} \right| \leq \int_{\mathbb{R}} \frac{|z - \alpha| |sz| |z - \alpha|}{|s - z|} d\rho(s),
\]
and as $z \to \alpha$ nontangentially the integrand converges to zero $\rho$-a.e. and stays bounded by the above discussions, it is easy to see the desired result holds. \hfill \square

As in the introduction, the collection of the $t$-th convolution power $\mu^{\mathbb{T}t}$ of $\mu$ forms a semigroup $\{\mu^{\mathbb{T}t} : t > 1\}$ which interpolates the discrete semigroup $\{\mu^{\mathbb{N}n} : n \in \mathbb{N}\}$. The construction of $\mu^{\mathbb{T}t}$ for arbitrary $\mu$ by using subordination functions and analytic methods is introduced in following two paragraphs.

For any $t > 1$, consider the function
\[
H_t(z) = tz - (t - 1)F_\mu(z), \quad z \in \mathbb{C}^+.
\]
Then we have $\Im H_t(z) \leq \Im z$ for $z \in \mathbb{C}^+$ and
\[
\lim_{y \to \infty} \frac{H_t(iy)}{iy} = 1.
\]
Moreover, there exists a continuous function $\omega_t : \mathbb{C}^+ \to \mathbb{C}^+$ such that $\omega_t(\mathbb{C}^+) \subset \mathbb{C}^+$, $\omega_t|\mathbb{C}^+$ is analytic, and $H_t(\omega_t(z)) = z$ for $z \in \mathbb{C}^+$. If the set $\{z \in \mathbb{C}^+ : \Im H_t(z) > 0\}$ is denoted by $\Omega_t$ then
\[
\Omega_t = \omega_t(\mathbb{C}^+)
\]
is a simply connected set whose boundary \( \omega_t(\mathbb{R}) \) is a simple curve. Then the equation 
\[ \omega_t(H_t(\omega_t(z))) = \omega_t(z), \quad z \in \mathbb{C}^+, \]
shows that the equation \( \omega_t(H_t(z)) = z \) holds for \( z \in \Omega_t \), and hence the inverse of \( \omega_t \) is \( H_t |_{\Omega_t} \). Moreover, if \( x \in \mathbb{R} \) and \( \Im \omega_t(x) > 0 \) then \( \omega_t \) can be continued analytically to a neighborhood of \( x \).

Let \( \mu^{\mathbb{E}t} \) be the measure defined by the requirement

\[
F_{\mu^{\mathbb{E}t}}(z) = F_\mu(\omega_t(z)), \quad z \in \mathbb{C}^+.
\]

It turns out that the measure \( \mu^{\mathbb{E}t} \) can be characterized in terms of Voiculescu transform, i.e., the measure \( \mu^{\mathbb{E}t} \) obtained in this way is the unique measure satisfying

\[
\mathcal{R}_{\mu^{\mathbb{E}t}}(z) = t \mathcal{R}_\mu(z),
\]
where \( z \) is in the common domain of these two functions. By using the equation \( H_t(\omega_t(z)) = z \) we can rewrite (2.5) as

\[
F_{\mu^{\mathbb{E}t}}(z) = \frac{t\omega_t(z) - z}{t - 1}, \quad z \in \mathbb{C}^+,
\]
which shows, in particular, that \( F_{\mu^{\mathbb{E}t}} \) extends continuously to \( \mathbb{C}^+ \). Also, by the equation \( \omega_t(H_t(z)) = z, z \in \Omega_t \), we have

\[ F_{\mu^{\mathbb{E}t}}(H_t(z)) = F_\mu(z), \quad z \in \Omega_t. \]

Regularity properties of \( \mu^{\mathbb{E}t} \) have been studied thoroughly in [1] and [2]. We now review some of these results which are relevant to our investigation on the number of the components in the support of \( \mu^{\mathbb{E}t} \). For any \( t > 1 \), the singular part of the measure \( \mu^{\mathbb{E}t} \) with respect to Lebesgue measure is purely atomic and \( \mu^{\mathbb{E}t} \) has an atom \( \alpha \) if and only if \( \mu(\{t\alpha\}) > 1 - t^{-1} \) in which case \( \mu^{\mathbb{E}t}(\{\alpha\}) = t\mu(\{\alpha\}) - (t - 1) \). A weaker statement, i.e., a point \( \alpha \) satisfying \( F_{\mu^{\mathbb{E}t}}(t\alpha) = 0 \) if and only if \( \mu(\{\alpha\}) \geq 1 - t^{-1} \), also holds. Hence for any measure \( \mu \) which is not a point mass, the atoms of \( \mu^{\mathbb{E}t} \) disappear for sufficiently large \( t \), and hence \( \mu^{\mathbb{E}t} \) is absolutely continuous for such \( t \).

3. Decreasing number of components in the support of \( \mu^{\mathbb{E}t} \)

Throughout this section, \( t \) is a parameter with \( t > 1 \) and for any given and fixed Borel probability measure \( \mu \) on \( \mathbb{R} \), \( \rho \) is the unique finite positive Borel measure in the Nevanlinna representation (2.1) of \( F_\mu \). Our analysis of the support of \( \mu^{\mathbb{E}t} \) will be based on the following function \( g : \mathbb{R} \to \mathbb{R} + \{\infty\} \) defined as

\[
g(x) = \int_{\mathbb{R}} \frac{s^2 + 1}{(s - x)^2} \, d\rho(s), \quad x \in \mathbb{R}.
\]
Associated with the function \( g \) and the parameter \( t \) are the following sets

\[
V_t^+ = \left\{ x \in \mathbb{R} : g(x) > \frac{1}{t - 1} \right\},
\]
\[
V_t = \left\{ x \in \mathbb{R} : g(x) = \frac{1}{t - 1} \right\},
\]
\[
V_t^- = \left\{ x \in \mathbb{R} : g(x) < \frac{1}{t - 1} \right\}.
\]
and the function
\[ f_t(x) = \inf \left\{ y \geq 0 : \int_{\mathbb{R}} \frac{s^2 + 1}{(x-s)^2 + y^2} \, d\rho(s) \leq \frac{1}{t-1} \right\} \]
which will play important roles in the analysis.

**Lemma 3.1.** If \( V_t \cup V_t^- \) contains an open interval \( I \) then \( \rho(I) = 0 \) and \( g \) is strictly convex on \( I \).

**Proof.** Let \((a,b)\) be any interval contained in \( I \). Then for any \( x \in (a,b) \) we have
\[
\frac{1}{t-1} \geq \int_a^b \frac{s^2 + 1}{(s-x)^2} \, d\rho(s) \geq \int_a^b \frac{1}{(b-a)^2} \, d\rho(s) = \frac{\rho((a,b))}{(b-a)^2},
\]
from which we obtain that \( \rho((a,b)) = 0 \). Since the second order derivative of \( g \) is positive on \((a,b)\), the second assertion follows. This completes the proof. \( \square \)

By the definition of \( H_t \) in (2.4), we have
\[
(3.1) \quad \Im H_t(z) = (\Im z) \left( 1 - (t-1) \int_{\mathbb{R}} \frac{s^2 + 1}{|z-s|^2} \, d\rho(s) \right), \quad z \in \mathbb{C}^+.
\]

**Proposition 3.2.** The function \( H_t \) satisfies \( \Im H_t(z) > 0 \) for all \( z \in \mathbb{C}^+ \) if and only if \( \mu \) is a Dirac measure in which case \( \mu^\natural t \) is a Dirac measure as well.

**Proof.** It follows from (3.1) that \( \Im H_t(z) > 0 \) holds for all \( z \in \mathbb{C}^+ \) if and only if
\[
\int_{\mathbb{R}} \frac{s^2 + 1}{|z-s|^2} \, d\rho(s) < \frac{1}{t-1}, \quad z \in \mathbb{C}^+,
\]
which happens if and only if \( g(x) \leq (t-1)^{-1} \) holds for all \( x \in \mathbb{R} \) by monotone convergence theorem. Then Lemma 3.1 shows that \( \Im H_t(z) > 0, \ z \in \mathbb{C}^+ \), if and only if \( \rho \) is a zero measure or, equivalently, \( F_{\mu} = a + z \), i.e., \( \mu \) is the point mass \( \delta_{a+z} \). It is easy to see from (2.4) and (2.5) that \( F_{\mu^\natural t} = ta + z \), whence \( \mu^\natural t = \delta_{-ta} \). \( \square \)

From the proof of the preceding proposition, \( \mu \) is a point mass if and only if \( V_t^+ \) is an empty set for all \( t > 1 \). For the rest of the paper, we confine our attention to the case of \( \mu \) which is not a Dirac measure.

The following lemmas are essentially similar to the results in [8] and [11]. For the completeness, we provide the proofs here.

**Lemma 3.3.** The set \( V_t^+ \) coincides with \( \{ x \in \mathbb{R} : f_t(x) > 0 \} \) and \( \{ x \in \mathbb{R} : f_t(x) = 0 \} = V_t \cup V_t^- \). Moreover, for any \( x \in \mathbb{R} \) we have
\[
(3.2) \quad \int_{\mathbb{R}} \frac{s^2 + 1}{(x-s)^2 + f_t^2(x)} \, d\rho(s) \leq \frac{1}{t-1},
\]
where the equality holds for \( x \in V_t^+ \).

**Proof.** First note that if \( z = x + iy \in \mathbb{C}^+ \), i.e., \( y > 0 \), then
\[
\Im F_{\mu}(z) = y \left( 1 + \int_{\mathbb{R}} \frac{s^2 + 1}{(x-s)^2 + y^2} \, d\rho(s) \right),
\]

which gives, particularly,
\[ \int_{\mathbb{R}} \frac{s^2 + 1}{(x-s)^2 + y^2} \, d\rho(s) < \infty. \]

It follows that for any fixed \( x \in \mathbb{R} \), the mapping
\[ y \mapsto \int_{\mathbb{R}} \frac{s^2 + 1}{(x-s)^2 + y^2} \, d\rho(s) \]
is decreasing and continuous on \((0, \infty)\) by Fatou’s lemma. Indeed, for any sequence \( \{y_n\} \) converging to \( y \in (0, \infty) \),
\[ \int_{\mathbb{R}} \frac{s^2 + 1}{(x-s)^2 + y_n^2} \, d\rho(s) \leq \liminf_{n \to \infty} \int_{\mathbb{R}} \frac{s^2 + 1}{(x-t)^2 + y_n^2} \, d\rho(s) \leq \limsup_{n \to \infty} \int_{\mathbb{R}} \frac{s^2 + 1}{(x-t)^2 + y_n^2} \, d\rho(s) \leq \int_{\mathbb{R}} \frac{s^2 + 1}{(x-t)^2 + y^2} \, d\rho(s) \]
which shows the continuity of the mapping. If \( f_t(x) > 0 \) then the definition of \( f_t \) shows that for small \( \epsilon > 0 \) we have
\[ \int_{\mathbb{R}} \frac{s^2 + 1}{(x-s)^2 + (1 + \epsilon)f_t^2(x)} \, d\rho(s) \leq \frac{1}{t-1} < \int_{\mathbb{R}} \frac{s^2 + 1}{(x-s)^2 + (1 - \epsilon)f_t^2(x)} \, d\rho(s), \]
which gives the equality in (3.2) by letting \( \epsilon \to 0 \). If \( f_t(x) = 0 \) then it is easy to see from Fatou’s lemma that \( x \in V_t \cup V_t^- = \mathbb{R} \setminus V_t^+ \), whence \( V_t^+ = \{ x \in \mathbb{R} : f_t(x) > 0 \} \) and \( V_t \cup V_t^- = \{ x \in \mathbb{R} : f_t(x) = 0 \} \). This completes the proof. \( \square \)

With the help of preceding lemma, we are able to provide another description of the set \( \Omega_t = \{ z \in \mathbb{C}^+ : \Im H_t(z) > 0 \} \).

**Lemma 3.4.** The set \( \{ x + iy \in \mathbb{C}^+ : y > f_t(x) \} \) coincides with the set \( \Omega_t \) and
\[ \int_{\mathbb{R}} \frac{s^2 + 1}{|z-s|^2} \, d\rho(s) < \frac{1}{t-1}, \quad z \in \Omega_t. \]
The function \( f_t \) is continuous on \( \mathbb{R} \) and the set \( V_t^+ \) is open. Moreover, if \( 1 < t_1 < t_2 \) then \( \Omega_{t_2} \subset \Omega_{t_1} \) and \( f_{t_1}(x) \leq f_{t_2}(x) \) for all \( x \in \mathbb{R} \).

**Proof.** The equation (3.1) shows that \( z \in \Omega_t \) if and only if (3.3) holds. By Lemma 3.3 and the definition of \( f_t \), we have \( \Omega_t = \{ x + iy \in \mathbb{C}^+ : y > f_t(x) \} \), and hence the boundary \( \partial \Omega_t \) is the graph of \( y = f_t(x) \). Since \( \partial \Omega_t \) is simply connected and \( V_t = \{ x \in \mathbb{R} : f_t(x) > 0 \} \), the second assertion holds. The last assertion follows from (3.3). \( \square \)

It was shown in [8] that the Cauchy transform \( G_\mu \) is Lipschitz continuous on some subset of \( \mathbb{C}^+ \). The following proposition gives similar results for the reciprocal Cauchy transform \( F_\mu \) on \( \overline{\Omega_t} \).

**Proposition 3.5.** The reciprocal Cauchy transform \( F_\mu \) extends continuously to \( \overline{\Omega_t} \) and satisfies
\[ |F_\mu(z_1) - F_\mu(z_2)| \leq \frac{t}{t-1}|z_1 - z_2|, \quad z_1, z_2 \in \overline{\Omega_t}. \]
Moreover, this continuous extension can be represented as

\[ F_\mu(z) = a + z + \int_{\mathbb{R}} \frac{1 + sz}{s - z} \, d\rho(s), \quad z \in \overline{\Omega_t}. \]

If \( t > 2 \) then

\[ \frac{t - 2}{t - 1} |z_1 - z_2| \leq |F_\mu(z_1) - F_\mu(z_2)|, \quad z_1, z_2 \in \overline{\Omega_t}, \]

and, consequently, \( F_\mu \) is one-to-one.

**Proof.** Using (3.3) and monotone convergence theorem gives the inequality

\[ \int_{\mathbb{R}} \frac{s^2 + 1}{|s - z|^2} \, d\rho(s) \leq \frac{1}{t - 1}, \quad z \in \overline{\Omega_t}, \]

whence the integral

\[ \int_{\mathbb{R}} \frac{s}{|s - z|^2} \, d\rho(s) \]

converges for all \( z \in \overline{\Omega_t} \) by Hölder inequality. This implies that

\[ \int_{\mathbb{R}} \frac{1 + sz}{s - z} \, d\rho(s) = \int_{\mathbb{R}} \frac{(1 - |z|^2)s + x(s^2 - 1)}{|s - z|^2} \, d\rho(s) + iy \int_{\mathbb{R}} \frac{s^2 + 1}{|s - z|^2} \, d\rho(s) \]

converges as well for \( z = x + iy \in \overline{\Omega_t} \). This implies that (3.5) holds, and moreover for \( z_1, z_2 \in \overline{\Omega_t} \) we have

\[
\left| \frac{F_\mu(z_1) - F_\mu(z_2)}{z_1 - z_2} \right| \leq 1 + \left| \int_{\mathbb{R}} \frac{s^2 + 1}{(s - z_1)(s - z_2)} \, d\rho(s) \right| \\
\leq 1 + \left( \int_{\mathbb{R}} \frac{s^2 + 1}{|s - z_1|^2} \, d\rho(s) \right)^{1/2} \left( \int_{\mathbb{R}} \frac{s^2 + 1}{|s - z_2|^2} \, d\rho(s) \right)^{1/2} \\
\leq 1 + \frac{1}{t - 1} = \frac{t}{t - 1},
\]

where the Hölder inequality is used in the second inequality. Similarly, if \( t > 2 \) then

\[
\left| \frac{F_\mu(z_1) - F_\mu(z_2)}{z_1 - z_2} \right| \geq 1 - \frac{1}{t - 1},
\]

which implies the desired inequality. This completes the proof. \( \square \)

For \( t > 1 \), it could happen that the boundary \( \partial \Omega_t \) contains an interval \( I \subset \mathbb{R} \), i.e., \( f_t \) vanishes on \( I \). The following corollary characterizes such an interval.

**Corollary 3.6.** Let \( I \) be an open interval in \( \mathbb{R} \). Then \( \rho(I) = 0 \) if and only if for any closed interval \( J \subset I \) there exists some \( t > 1 \) such that \( f_t(x) = 0 \) for \( x \in J \). Moreover, if \( \rho(I) = 0 \) then the expression (3.5) for \( F_\mu \) holds for \( x \in I \) and \( F_\mu \) is strictly increasing on \( I \).

**Proof.** First, assume that \( I \) is bounded, \( \rho(I) = 0 \), and \( J \) is any closed interval contained in \( I \). Since \( g \) is continuous on \( I \), there exists a number \( t > 1 \) such that \( g(x) \leq (t - 1)^{-1} \) or, equivalently, \( f_t(x) = 0 \) for all \( x \in J \). If \( I = (a, \infty) \) (resp. \( I = (-\infty, a) \)) for some finite \( a \) and \( \rho(I) = 0 \) then \( g \) is decreasing (resp. increasing) on \( I \), whence it is easy to see that the necessity follows. The sufficiency follows from Lemma 3.1 and 3.3.
By Proposition 3.5 and the first assertion of the corollary, it is easy to see that (3.5) holds on any interval \( I \) having \( \rho \)-measure zero. For such an interval \( I \), taking the derivative gives that
\[
F'_\mu(x) = 1 + \int \frac{s^2 + 1}{(s - x)^2} d\rho(s), \quad x \in I,
\]
and therefore \( F_\mu \) is strictly increasing on \( I \).

From the proof of Proposition 3.5, the function \( H_t \) also satisfies similar estimates. That is,
\[
|H_t(z_1) - H_t(z_2)| \leq 2|z_1 - z_2|, \quad z_1, z_2 \in \Omega_t,
\]
from which we deduce that \( H_t \) is also Lipschitz continuous and has a continuous extension to \( \overline{\Omega_t} \). If this continuous extension is still denoted by \( H_t \) then the equation
\[
H_t(\omega_t(z)) = z, \quad z, \omega_t(z) \in \mathbb{C}^+.
\]
The results in the following corollary are direct consequences of (3.3) and preceding discussions. We refer the reader to [2] for related results of the function \( H_t \) and the subordination function \( \omega_t \).

**Corollary 3.7.** The function \( H_t \) is a conformal mapping from \( \Omega_t \) onto \( \mathbb{C}^+ \) and is a homeomorphism from \( \overline{\Omega_t} \) to \( \mathbb{C}^+ \).

Note that the continuity of \( f_t \) on \( \mathbb{R} \) ensures that the map \( x \mapsto x + if_t(x) \) is a homeomorphism of \( \mathbb{R} \) onto \( \partial \Omega_t \). Define the map \( \psi_t : \mathbb{R} \to \mathbb{R} \) as
\[
\psi_t(x) = H_t(x + if_t(x)).
\]
Then \( \psi_t \) is a homeomorphism as well by Corollary 3.7 and the equation
\[
\omega_t(\psi_t(x)) = \omega_t(H_t(x + if_t(x))) = x + if_t(x) \quad \text{holds for} \quad x \in \mathbb{R}.
\]

It is apparent that we have
\[
F_{\mu \oplus t}(\psi_t(x)) = \frac{tx - \psi_t(x) + itf_t(x)}{t - 1}, \quad x \in \mathbb{R}.
\]
Since \( \omega_t \) can be continued analytically to a neighborhood of \( \psi_t(x) \) if \( f_t(x) > 0 \), it follows that the function \( f_t \) is analytic on \( V_t^+ \).

Now we are in a position to state the main theorem of this section. For any measure \( \nu \) on \( \mathbb{R} \), let \( \text{supp}(\nu) \) be the support of \( \nu \) and denote by \( \nu^{ac} \) the absolutely continuous part of \( \nu \) with respect to Lebesgue measure.

**Theorem 3.8.** Suppose that \( \mu \) is a Borel probability measure (not a point mass) on \( \mathbb{R} \) and \( t > 1 \). Then the following statements hold.
\( (i) \) The absolutely continuous part of \( \mu \oplus t \) is concentrated on the closure of \( \psi_t(V_t^+) \).
\( (ii) \) The density of \( (\mu \oplus t)^{ac} \) on the set \( \psi_t(V_t^+) \) is given by
\[
\frac{d(\mu \oplus t)^{ac}}{dx}(\psi_t(x)) = \frac{t(t - 1)f_t(x)}{\pi[tx - \psi_t(x) + itf_t(x)]^2}, \quad x \in V_t^+.
\]
(iii) The density of \((\mu^\otimes t)^{ac}\) is analytic on the set \(\psi_t(V_t^+)\) and at the point \(\psi_t(x)\), \(x \in V_t^+\), it is bounded by \((t - 1)/(\pi t f_t(x))\).

(iv) The number of the components in \(\text{supp}(\mu^\otimes t)^{ac}\) is a decreasing function of \(t\).

Proof. Since \(F_{\mu^\otimes t}\) is continuous on \(\mathbb{C}^+\), the density of the nonatomic part of \(\mu^\otimes t\) is continuous except at the points \(x\) at which \(F_{\mu^\otimes t}(x) = 0\). In view of (3.6) and the Stieltjes inversion formula, the density of \(\mu^\otimes t\) at \(\psi_t(x)\) where \(x \in V_t^+\) is given by

\[
\frac{d\mu^\otimes t}{dx}(\psi_t(x)) = -\frac{1}{\pi} \Im G_{\mu^\otimes t}(\psi_t(x)) = \frac{t(t - 1)f_t(x)}{\pi |tx - \psi_t(x) + itf_t(x)|^2}.
\]

This shows that the support of \((\mu^\otimes t)^{ac}\) is the closure of \(\psi_t(V_t^+)\) and they have the same number of components since \(\psi_t\) is a homeomorphism. The assertion (iii) follows from part (ii) and the fact that \(f_t\) is analytic on \(V_t^+\). To verify the statement (iv), we need to show the the number of the components in \(V_t^+\) is nonincreasing as \(t\) increases. This will hold if we show that \(g\) never has a local maximum in any open interval \((a, b)\) in \(\mathbb{R}\setminus V_t^+\). Indeed, \(g\) is strictly convex on such an interval by Lemma 3.1, whence (iv) holds.

Before we state the next result, observe that the number of atoms of \(\mu^\otimes t\) decreases as a function of \(t\). Indeed, this is a direct consequence of the fact that a point \(\alpha\) is an atom of \(\mu^\otimes t\) if and only \(\mu(\{\alpha/t\}) > 1 - t^{-1}\). Since the open set \(V_t^+\) can be written as a countable union of open intervals, combining the result for atoms and Theorem 3.8 gives the following conclusion.

Corollary 3.9. With the same assumption in Theorem 3.8, the measure \(\mu^\otimes t\) has at most countably many components in the support, which consists of finitely many points (atoms) and countably many intervals. Moreover, the number of the components in \(\text{supp}(\mu^\otimes t)\) is a decreasing function of \(t\).

4. Support and Regularity for \(\mu^\otimes t\)

In this section, we will investigate the support and regularity property for the free convolution power \(\mu^\otimes t\) of \(\mu\) where \(t > 1\) and \(\mu\) is a Borel probability measure (not a point mass so that \(V_t^+\) is nonempty) on \(\mathbb{R}\). By analyzing the formula

\[
F_{\mu^\otimes t}(\psi_t(x)) = \frac{tx - \psi_t(x) + itf_t(x)}{t - 1}, \quad x \in \mathbb{R},
\]

established in the previous section, we are able to understand the relation between supports and regularities for the measures \(\mu\) and \(\mu^\otimes t\). If \(f_t(x) = 0\) for some \(x \in \mathbb{R}\) then \(H_t(x) = \psi_t(x)\) and (4.1) can be rewritten as

\[
F_{\mu}(x) = F_{\mu^\otimes t}(\psi_t(x)) = \frac{tx - \psi_t(x)}{t - 1}.
\]

The following result gives another basic property of the homeomorphism \(\psi_t\).
Lemma 4.1. The derivative $\psi_{\mu}'(x) > 0$ for $x \in V_i^+$ and $\psi_{\mu}$ is a strictly increasing function on $\mathbb{R}$.

Proof. Let $z = x + if_i(x)$ be any point with $x \in V_i^+$. Since functions $H_i$ and $f_i$ are analytic at points $z$ and $x$, respectively, it follows that

$$
\psi_{\mu}'(x) = \left( \frac{dH_i}{dz}(x + if_i(x)) \right) (1 + if_i'(x))
= \Re \left( \frac{dH_i}{dz}(x + if_i(x)) \right) - \Im \left( \frac{dH_i}{dz}(x + if_i(x)) \right) f_i'(x).
$$

Observe that

$$
\frac{dH_i}{dz}(x + if_i(x)) = 1 - (t - 1) \int_{\mathbb{R}} \frac{s^2 + 1}{(s - z)^2} \, d\rho(s)
= 1 - (t - 1) \int_{\mathbb{R}} \frac{(s^2 + 1)(s - x)^2 - f_i^2(x) + 2i(s - x)f_i(x)}{|s - z|^4} \, d\rho(s)
= 1 - (t - 1) \int_{\mathbb{R}} \frac{(s^2 + 1)|s - z|^2 - 2f_i^2(x) + 2i(s - x)f_i(x)}{|s - z|^4} \, d\rho(s)
= 2(t - 1) \left[ \int_{\mathbb{R}} \frac{(s^2 + 1)f_i^2(x)}{|s - z|^4} \, d\rho(s) - i \int_{\mathbb{R}} \frac{(s^2 + 1)(s - x)f_i(x)}{|s - z|^4} \, d\rho(s) \right]
$$

where the identity

$$
(4.3) \quad \int_{\mathbb{R}} \frac{s^2 + 1}{|s - z|^2} \, d\rho(s) = \frac{1}{t - 1}
$$

is used in the forth equality. On the other hand, differentiating (4.3) with respect to $x$ gives

$$
\int_{\mathbb{R}} \frac{(s^2 + 1)(s - x)}{|s - z|^4} \, d\rho(s) = f_i(x)f_{\mu}'(x) \int_{\mathbb{R}} \frac{s^2 + 1}{|s - z|^4} \, d\rho(s),
$$

whence

$$
\frac{dH_i}{dz}(x + if_i(x)) = 2(t - 1)f_i^2(x) \left[ \int_{\mathbb{R}} \frac{s^2 + 1}{|s - z|^4} \, d\rho(s) - if_{\mu}'(x) \int_{\mathbb{R}} \frac{s^2 + 1}{|s - z|^4} \, d\rho(s) \right].
$$

Combining these results gives that

$$
\psi_{\mu}'(x) = 2(t - 1)f_i^2(x) \left[ \int_{\mathbb{R}} \frac{s^2 + 1}{|s - z|^4} \, d\rho(s) + f_i^2(x) \int_{\mathbb{R}} \frac{s^2 + 1}{|s - z|^4} \, d\rho(s) \right]
= 2(t - 1)f_i^2(x)(f_i^2(x) + 1) \int_{\mathbb{R}} \frac{s^2 + 1}{|s - z|^4} \, d\rho(s)
> 0,
$$
as desired. Since $\psi_{\mu}$ is one-to-one and continuous on $\mathbb{R}$ with nonnegative derivative on a nonempty subset of $\mathbb{R}$, it must be strictly increasing. \qed

Note that if $x \in V_i$ then $f_i(x) = 0$, and Proposition 3.5 shows that the limit $F_{\mu}(x) = \lim_{\epsilon \to 0} F_{\mu}(x + i\epsilon)$ exists and is a finite number. The next result characterizes the atoms of $\mu$ in terms of $V_i$ and $F_{\mu}$.
Lemma 4.2. A point $\alpha \in \mathbb{R}$ is an atom of $\mu$ with mass $\mu(\{\alpha\}) = 1 - t^{-1}$ if and only if $\alpha \in V_t$ and $F_\mu(\alpha) = 0$ in which case the Julia-Carathéodory derivative of $F_\mu$ at $\alpha$ is

$$F'_\mu(\alpha) = \frac{1}{\mu(\{\alpha\})} = 1 + \int_{\mathbb{R}} \frac{s^2 + 1}{(s - \alpha)^2} \, d\rho(s).$$

Proof. First assume that $\alpha$ is an atom of $\mu$, i.e., the limit in (2.3) is $1 - t^{-1} > 0$. This implies that

$$\lim_{\epsilon \to 0} \frac{\Re F_\mu(\alpha + i\epsilon)}{i\epsilon} = \frac{1}{2} \lim_{\epsilon \to 0} \left( \frac{F_\mu(\alpha + i\epsilon)}{i\epsilon} + \frac{F_\mu(\alpha - i\epsilon)}{i\epsilon} \right)$$

$$= \frac{1}{2} \lim_{\epsilon \to 0} \left( \frac{1}{i\epsilon G_\mu(\alpha + i\epsilon)} - \frac{1}{i\epsilon G_\mu(\alpha - i\epsilon)} \right) = 0.$$

Using the limit shown above gives

$$\frac{1}{\mu(\{\alpha\})} = \lim_{\epsilon \to 0} \frac{F_\mu(\alpha + i\epsilon)}{i\epsilon} = \lim_{\epsilon \to 0} \frac{\Re F_\mu(\alpha + i\epsilon)}{\epsilon}$$

$$= \lim_{\epsilon \to 0} \frac{\Re F_\mu(\alpha + i\epsilon)}{\epsilon} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\mathbb{R}} \frac{s^2 + 1}{(s - \alpha)^2 + \epsilon^2} \, d\rho(s)$$

$$= 1 + \int_{\mathbb{R}} \frac{s^2 + 1}{(s - \alpha)^2} \, d\rho(s),$$

where in the last equality monotone convergence theorem is used. Hence $\alpha$ belongs to the set $V_t$, and $F_\mu(\alpha)$ is defined and equals zero.

To show the sufficiency, by the arguments shown above it suffices to show that

$$\lim_{\epsilon \to 0} \frac{\Re F_\mu(\alpha + i\epsilon)}{i\epsilon} = 0$$

or, equivalently,

$$\lim_{\epsilon \to 0} \frac{\Re[F_\mu(\alpha + i\epsilon) - F_\mu(\alpha)]}{\epsilon} = 0. \tag{4.4}$$

First, in view of (3.5) we obtain

$$\frac{F_\mu(\alpha + i\epsilon) - F_\mu(\alpha)}{\epsilon} = i \left( 1 + \int_{\mathbb{R}} \frac{s^2 + 1}{(s - \alpha - i\epsilon)(s - \alpha)} \, d\rho(s) \right).$$

Since

$$\int_{\mathbb{R}} \left| \frac{s^2 + 1}{(s - \alpha - i\epsilon)(s - \alpha)} \right| \, d\rho(s) \leq \int_{\mathbb{R}} \frac{s^2 + 1}{(s - \alpha)^2} \, d\rho(s) = \frac{1}{t - 1},$$

it follows that

$$\lim_{\epsilon \to 0} \frac{F_\mu(\alpha + i\epsilon) - F_\mu(\alpha)}{\epsilon} = i \frac{t}{t - 1}$$

by dominated convergence theorem which implies (4.4). The expression of the Julia-Carathéodory derivative of $F_\mu$ is clear. \qed
It was proved in [2] that if a point \( x \in \mathbb{R} \) belongs to \( \omega_t(\mathbb{R}) \), i.e., \( f_t(x) = 0 \) then the Julia-Carathéodory derivative \( H'_t(x) \geq 0 \), where the existence of

\[
H'_t(x) = \lim_{\epsilon \to 0} \frac{H(x + i\epsilon) - H(x)}{i\epsilon}
\]

is guaranteed by the existence of \( H_t(x) \in \mathbb{R} \). Indeed, by a similar proof in Lemma 4.2 and Lemma 3.3 at such a point \( x \) the Julia-Carathéodory derivative of \( F_\mu \) is given by

\[
(4.5) \quad F'_\mu(x) = 1 + \int_\mathbb{R} \frac{s^2 + 1}{(s - x)^2} \, d\rho(s) \leq t/(t - 1),
\]

from which we deduce that

\[
(4.6) \quad H'_t(x) = 1 - (t - 1) \int_\mathbb{R} \frac{s^2 + 1}{(s - x)^2} \, d\rho(s) \geq 0.
\]

Note that (4.6) shows that if \( f_t(x) = 0 \) then \( H'_t(x) > 0 \) if and only if \( x \in V_t^- \). In the following proposition, by means of functions \( f_t \) and \( \psi_t \) we are able to obtain certain regularity results which were proved in [2].

**Proposition 4.3.** Let \( \alpha \) be a point in \( \mathbb{R} \) and \( t > 1 \). Then the following conditions are equivalent:

(i) \( f_t(\alpha) = 0 \) and \( F_\mu(\alpha) = 0 \);

(ii) \( F'_\mu(t\alpha) = 0 \);

(iii) \( \mu(\{\alpha\}) \geq 1 - t^{-1} \).

**Proof.** The equivalence of (i) and (ii) follows from the bijectivity of \( \psi_t \) and (4.1). Indeed, letting \( \psi_t(\alpha') = t\alpha \) for some \( \alpha' \in \mathbb{R} \) implies that \( F'_\mu(t\alpha) = 0 \) if and only if \( t\alpha' = \psi_t(\alpha) \) and \( f_t(\alpha') = 0 \) which translate to the conditions in (i). By Lemma 4.2 and (4.5), it is easy to see that (i) implies (iii). On the other hand, by Lemma 4.2 we see that if \( \mu(\{\alpha\}) = 1 - t_0^{-1} \geq 1 - t^{-1} \) then \( F_\mu(\alpha) = 0 \) and \( f_{t_0}(\alpha) = 0 \). Since \( t \leq t_0 \), \( f_t(\alpha) = 0 \) by Lemma 3.4, whence (iii) implies (i). This completes the proof. \( \square \)

Next, we investigate those intervals on which \( f_t \) vanishes.

**Proposition 4.4.** If \( f_t \) vanishes on some interval \( I \) then the following statements are equivalent:

(i) \( \mu(I) = 0 \);

(ii) \( F_\mu(x) \neq 0 \) for any \( x \) in \( I \).

If (i) or (ii) is satisfied then \( \mu_{\mathbb{R}t}(\psi_t(I)) = 0 \).

**Proof.** The implication that (i) implies (ii) follows from Proposition 4.3. Indeed, if \( \mathbb{R} F_\mu(x) = 0 \) for some \( x \in I \) then \( x \) is an atom of \( \mu \), and hence \( \mu(I) > 0 \). Next, we prove that (ii) implies (i). First we consider the case that \( I \) is a compact set \([a, b]\). Since \( F_\mu \) is continuous and nonzero on \([a, b]\), \( G_\mu(x) \) is a continuous and real-valued function on \([a, b]\), and consequently \( \mu([a, b]) = 0 \) by the Stieltjes inversion formula. Next, suppose that \( I \) is bounded but not closed. If \( I = (a, b) \) then we have \( \mu([a + \epsilon, b]) = 0 \) for any small \( \epsilon > 0 \) by the result established above, and therefore letting \( \epsilon \to 0 \) gives \( \mu((a, b)) = 0 \). Similarly, the assertion (i) holds if \( I = [a, b] \) or \( I = (a, b) \). Finally, if \( I \) is unbounded
(closed or not closed) then by the results shown above and the countable additivity of μ, it is easy to see that μ(I) = 0. This completes the proof that (ii) implies (i).

By similar arguments and (4.2), it is easy to see that the last statement follows from (i) or (ii).

\[ \square \]

**Corollary 4.5.** Assume that \( I \subset \mathbb{R} \) is an interval on which \( f_1 \) vanishes and \( \mu(I) > 0 \). Then the interval \( I \) contains one and only one atom \( \alpha \) of \( \mu \) and this only atom has mass \( \mu(\{\alpha\}) \geq 1 - t^{-1} \). Moreover, \( \mu(I \setminus \{\alpha\}) = 0 \) and \( \mu^\mathbb{R}(\psi_t(I)) = t\mu(\{\alpha\}) - (t - 1) \).

**Proof.** First observe that \( F_\mu \) has zeros in \( I \). Since \( F_\mu \) is strictly increasing on \( I \) by Corollary 3.6, it only has one zero \( \alpha \), i.e., \( \alpha \) is the only atom of \( \mu \) in \( I \) by Lemma 4.2. The atom \( \alpha \) with the desired mass is a direct consequence of Proposition 4.3. It is clear that the set \( I \setminus \{\alpha\} \) must have \( \mu \)-measure zero. Since \( \bigcap \{\alpha\} \subset V_I \cup V_I^- \) and \( \mu^\mathbb{R}(\{\alpha\}) = t\mu(\{\alpha\}) - (t - 1) \), \( \psi_t(I) \) has the desired \( \mu^\mathbb{R} \)-measure. \[ \square \]

**Theorem 4.6.** Let \( I \) be any component in \( \text{supp}(\mu) \) and \( t > 1 \). If \( I \) is an interval then it does not contain a closed interval on which \( f_1 \) vanishes and the interval \( \psi_t(I) \) is contained in some component of \( \text{supp}(\mu^\mathbb{R}) \).

**Proof.** Let \( \alpha_1, \cdots, \alpha_n \) be all the atoms of \( \mu \) in \( I \) with \( \mu(\{\alpha_j\}) \geq 1 - t^{-1}, j = 1, \cdots, n. \) Then it suffices to show that there does not exist an open interval \( J \subset I \setminus \{\alpha_1, \cdots, \alpha_n\} \) such that \( f_1 \) vanishes on \( J \). If such an interval \( J \) exists then the fact \( \mu(J) > 0 \) gives the existence of an atom \( \alpha \) of \( \mu \) in \( J \) with mass \( \mu(\{\alpha\}) \geq 1 - t^{-1} \) by Corollary 4.5, a contradiction. Hence the first desired result follows. This result also reveals that \( g > (t - 1)^{-1} \) on the interval \( I \) except at points \( x \) such that \( f_1(x) = 0 \). Since \( \mu^\mathbb{R} \) does not contain the singular continuous part, the last assertion follows from Theorem 3.8(i). \[ \square \]

It is known that the measure \( \mu^\mathbb{R} \) has fewer atoms when \( t \) increases. The next result explains where these disappearing atoms go.

**Proposition 4.7.** Assume that \( \alpha \) is an atom of \( \mu \) and \( \mu(\{\alpha\}) = 1 - t_0^{-1} \). Then \( \mu^\mathbb{R}(\{\psi_t(\alpha)\}) > 0 \) for \( 1 < t < t_0 \) and \( \psi_t(\alpha) \in \text{supp}(\mu^\mathbb{R}) \) for \( t \geq t_0 \).

**Proof.** For \( 1 < t < t_0 \), we have \( \mu(\{\alpha\}) > 1 - t^{-1} \), which implies that \( \psi_t(\alpha) = t\alpha \) is an atom of \( \mu^\mathbb{R} \) by (4.1) and Proposition 4.3. Next note that there does not exist an open interval \( I \) containing \( \alpha \) such that \( f_1(x) = 0 \) for all \( x \in I \). Indeed, if such an interval \( I \) exists then the facts that \( g(\alpha) = (t_0 - 1)^{-1} \) and \( g \) is strictly convex on \( I \) will lead to a contradiction. Hence \( \psi_t(\alpha) \) must be in the closure of \( \psi_t(t_0^+) \). Since \( \alpha \in V_t^+ \) for \( t > t_0 \), the desired result follows from Theorem 3.8(i). \[ \square \]

Theorem 4.6 and Proposition 4.7 give the inclusion \( \psi_t(\text{supp}(\mu)) \subset \text{supp}(\mu^\mathbb{R}) \) or, equivalently, the complement \( \mathbb{R} \setminus \text{supp}(\mu^\mathbb{R}) \) is contained in \( \psi_t(\mathbb{R} \setminus \text{supp}(\mu)) \) for all \( t > 1 \). Since \( \mathbb{R} \setminus \text{supp}(\mu) \) is a countable union of open intervals, the preceding observation leads us to investigate open intervals \( I \) which have \( \mu \)-measure zero. First note that the Cauchy transform \( G_\mu \) extends analytically through the interval \( I \) and takes real values.
Lemma 4.8. Suppose that $(\mathbb{R}, ..., G)$ and $\mu$ with $\rho(\{x_0\}) = - \frac{1}{(x_0^2 + 1)G'_{\rho}(x_0)}$ and $\rho(I \setminus \{x_0\}) = 0$. (i) If $G_{\mu}$ has a zero $x_0$ in $I$ then $x_0$ is an atom of $\rho$ with mass $\rho(\{x_0\}) = - \frac{1}{(x_0^2 + 1)G'_{\rho}(x_0)}$ and $\rho(I \setminus \{x_0\}) = 0$. (ii) At any point $x \in I$, the function $g$ can be expressed as $g(x) = \frac{-G'_{\rho}(x)}{G_{\rho}(x)} - 1$, where the equality is interpreted as $+\infty$ on both sides at the unique zero $x_0$, if it exists, of $G_{\mu}$ on $I$. In addition, if $I = (a, b)$ is bounded then $\inf_{x \in I} g(x) \geq \frac{\rho(\mathbb{R})}{1 + \max\{a^2, b^2\}}$. 

Proof. First observe that $F_{\mu}$ is continuous and takes real values on $I \setminus \{x_0\}$, whence the weak* limit of the measures $d\rho_{\epsilon}(s)$ in (2.2) as $\epsilon \downarrow 0$ is zero which gives $\rho(I \setminus \{x_0\}) = 0$. By Lemma 2.1, we have $\lim_{\epsilon \downarrow 0} \epsilon F_{\mu}(x_0 + \epsilon) = -(x_0^2 + 1)\rho(\{x_0\})$. On the other hand, 

$$\lim_{\epsilon \downarrow 0} \frac{G_{\mu}(x_0 + \epsilon)}{\epsilon} = \lim_{\epsilon \downarrow 0} \frac{G_{\mu}(x_0 + \epsilon) - G_{\mu}(x_0)}{\epsilon} = G'_{\rho}(x_0),$$ 

which gives the desired mass for $\rho(\{x_0\})$ in (i). To verify the statements in (ii), by (i) we may assume that $G_{\mu}$ is nonzero on $I$. Then observe that for any $x \in I$ and $\epsilon > 0$ we have 

$$\Im F_{\mu}(x + \epsilon) = \Im \frac{G_{\mu}(x + \epsilon)}{|G_{\mu}(x + \epsilon)|^2} = \frac{-\Im G_{\mu}(x + \epsilon)}{|G_{\mu}(x + \epsilon)|^2} = \epsilon \int_{-\epsilon}^{\epsilon} \frac{d\mu(s)}{|s + \epsilon|^2}. $$

On the other hand, the Nevanlinna representation for $F_{\mu}$ gives 

$$\Im F_{\mu}(x + \epsilon) = \epsilon \left( 1 + \int_{-\epsilon}^{\epsilon} \frac{s^2 + 1}{(s - x)^2 + \epsilon^2} \ d\rho(s) \right),$$ 

from which we obtain 

$$(4.7) \int_{-\epsilon}^{\epsilon} \frac{s^2 + 1}{(s - x)^2 + \epsilon^2} \ d\rho(s) = \frac{\epsilon}{|s + \epsilon|^2} - 1.$$
Since \( \mu(I) = 0 \) and \( G_\mu \) is nonzero on \( I \), it follows that
\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\mathbb{R}} \frac{d\mu(s)}{(x - s)^2 + \epsilon} = \int_{\mathbb{R}} \frac{d\mu(s)}{(x - s)^2} < \infty
\]
and
\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\mathbb{R}} \frac{d\mu(s)}{(x - s) + i\epsilon} = \int_{\mathbb{R}} \frac{d\mu(s)}{x - s} \neq 0
\]
for all \( x \in I \). Therefore, by the equation (4.7) we have
\[
g(x) = \left( \int_{\mathbb{R}} \frac{s^2 + 1}{(s - x)^2} \, d\rho(s) \right) - 1
\]
Finally, for any \( x \) in the bounded interval \((a, b)\) and \( s \in \mathbb{R}\setminus(a, b) \) we have
\[
\frac{s^2 + 1}{(s - x)^2} \geq \frac{1}{1 + x^2} \geq \frac{1}{1 + \max\{a^2, b^2\}},
\]
from which we deduce that
\[
g(x) = \int_{-\infty}^{x} \frac{s^2 + 1}{(s - x)^2} \, d\rho(s) + \int_{x}^{\infty} \frac{s^2 + 1}{(s - x)^2} \, d\rho(s)
\]
\[
\geq \frac{\rho(\mathbb{R})}{1 + \max\{a^2, b^2\}}.
\]
as desired. This finishes the proof. \( \square \)

**Proposition 4.9.** If \( I \) is a bounded component in \( \mathbb{R}\setminus\text{supp}(\mu) \) then for each \( t > 1 \) the interval \( \psi_t(I) \) contains at most two components of \( \mathbb{R}\setminus\text{supp}(\mu^{\otimes t}) \).

**Proof.** If \( G_\mu \) has no zero in \( I \) then the strict convexity of \( g \) on \( I \) shows that there exists at most one subinterval \( J \) of \( I \) such that \( f_t \) vanishes on \( J \) when \( t \) varies from one to infinity. If \( G_\mu \) has a zero \( x_0 \) in \( I = (a, b) \) then applying the preceding argument to the intervals \((a, x_0)\) and \((x_0, b)\) gives that two subintervals at most. Hence the desired conclusion follows from Theorem 3.8(i). \( \square \)

For any \( t > 1 \), denote by \( n(t) \) the number of components in \( \text{supp}(\mu^{\otimes t}) \).

**Theorem 4.10.** If \( \mu \) is any Borel probability measure on \( \mathbb{R} \) then the following statements are equivalent:

(i) \( n(t) = 1 \) for sufficiently large \( t \);
(ii) \( n(t) < \infty \) for some \( t > 1 \);
(iii) the infimum \( m \) of \( g \) on the set of all bounded components of \( \mathbb{R} \setminus \text{supp}(\mu) \) is nonzero.
Moreover, if \( m > 0 \) then \( n(t) = 1 \) for \( t > t_0 \), where
\[
t_0 = \max \left( 1 + \frac{1}{m}, \frac{1}{1 - \mu(\{\alpha\})} \right)
\]
and \( \alpha \) is one of the atoms of \( \mu \) with the largest mass.

Proof. It is clear that (i) implies (ii). Next, suppose that \( n(t_1) < \infty \) for some \( t_1 > 1 \) and \( \mu(\{\alpha\}) < 1 - t_1^{-1} \) for any atom \( \alpha \) of \( \mu \). Let \( S_{t_1} \) be the set of all components which are bounded intervals in \( \{ x \in \mathbb{R} : f_{t_1}(x) = 0 \} \). Then \( S_{t_1} \) is a finite set and by Corollary 4.5 the set \( S_{t_1} \) is contained in the union of some components \( I_1, \ldots, I_n \) of \( \mathbb{R} \setminus \text{supp}(\mu) \). Note that each \( I_k \) is bounded. Indeed, if \( \mu((M, \infty)) = 0 \) for some finite number \( M \) then \( G_{\mu} > 0 \) on \( (M, \infty) \), whence \( \rho((M, \infty)) = 0 \) and \( g \) is strictly decreasing on \( (M, \infty) \). This implies that \( g(x) \leq (t_1 - 1)^{-1} \) for sufficiently large \( x \), and so \( f_{t_1} \) vanishes on \((c, \infty)\) for some finite number \( c \). Similarly, no \( I_k \) is of the form \((-\infty, M)\). Let \([a, b]\) be the smallest closed interval containing these \( I_k \)'s. Then
\[
\inf \{ g(x) : x \in (\mathbb{R} \setminus \text{supp}(\mu)) \cap [a, b] \} \geq \frac{\rho(\mathbb{R})}{1 + \max \{ a^2, b^2 \}}
\]
by Lemma 4.8(ii). If \( I \) is the intersection of any bounded component in \( \mathbb{R} \setminus \text{supp}(\mu) \) with \((b, \infty)\) and \( I \neq \emptyset \) then \( g(x) \geq (t_1 - 1)^{-1} \) on \( I \). Indeed, if \( g(x) < (t_1 - 1)^{-1} \) for some \( x \in I \) then there exists some interval \( J \subset I \) such that \( g < (t_1 - 1)^{-1} \) on \( J \), i.e., \( f_{t_1} = 0 \) on \( J \), which violates the definition of \( S_{t_1} \). Similarly, \( g \geq (t_1 - 1)^{-1} \) on any bounded component in \( (\mathbb{R} \setminus \text{supp}(\mu)) \cap (-\infty, a) \), and hence (ii) implies (iii). Finally, assume that the number \( m \) in the statement (iii) is nonzero and let \( t > t_0 \), where \( t_0 \) is defined as in theorem. Then with the help of Theorem 4.6, we see that \( \{ x \in \mathbb{R} : f_t(x) = 0 \} \) contains at most two components which are intervals, in which case these two intervals must be unbounded. In other words, the closure of \( V_t^+ \) is an interval, whence \( n(t) = 1 \) and (iii) implies (i).

In the proof of the preceding theorem, the set \( \{ x \in \mathbb{R} : f_t(x) = 0 \} \) might contain two unbounded components for any \( t > 1 \). Actually, this happens when \( \mu \) is compactly supported which was proved in [6] and [14]. In the following, we provide an analytic way to prove this fact.

Corollary 4.11. Let \( t > 1 \). Then the following statements hold.
(i) The function \( f_t \) vanishes on \((a, \infty)\) (resp. \((-\infty, a)\)) for some finite number \( a \) if and only if \( \mu((M, \infty)) = 0 \) (resp. \((-\infty, M)) \) for some finite number \( M \).
(ii) The measure \( \mu \) has compact support if and only if so does \( \mu^{\#} \).
(iii) If \( \mu \) is compactly supported then \( n(t) = 1 \) for large \( t \).

Proof. If \( f_t(x) = 0 \) for \( x \in (a, \infty) \) then by Corollary 4.5 there exists a point \( M \geq a \) such that \( \mu((M, \infty)) = 0 \). Similarly, if \( f_t \) vanishes on \((-\infty, a) \) then \( \mu((\infty, M)) = 0 \) for some finite number \( M \leq a \). Conversely, if \( \mu((b, \infty)) = 0 \) for some \( b \) then as shown in the preceding theorem we have \( f_t(x) = 0 \) for sufficiently large \( x \). Similarly,
if \( \mu((\infty, b]) = 0 \) then \( f_t(x) = 0, \ x \leq a \), for some finite number \( a \), and so (i) follows. The assertion (ii) follows from (i), while (iii) is a direct consequence of Lemma 4.8(ii) and Theorem 4.10.

Now we are able to give a complete description of \( \text{supp}(\mu \otimes t) \) in terms of \( \text{supp}(\mu) \).

(i) The set \( \mathbb{R} \setminus \text{supp}(\mu) \) contains a component \((b, c)\) in which case \( b \) is an atom.

(ii) There exists a point \( c > b \) such that \((b, c)\) is contained in a component of \( \text{supp}(\mu) \).

(iii) There exists a sequence of components \((a_n, b_n)\) in \( \mathbb{R} \setminus \text{supp}(\mu) \) such that \( b_{n+1} \leq a_n \) for all \( n \) and \( a_n \downarrow b \) as \( n \to \infty \).

Cases (i) and (ii) are analyzed in Proposition 4.3, Theorem 4.6, and Proposition 4.9.

Applying Corollary 4.5 to case (iii), it is easy to see that for each \( t > 1 \) there exists a point \( \mu \otimes t((t_\alpha, t\beta)) > 0 \) if \( t_\alpha, t\beta \) are atoms of \( \mu \otimes t \). Using functions \( f_t \) and \( \psi_t \), we can prove a somewhat stronger result.

**Proposition 4.12.** If \( \mu \) has atoms \( \alpha < \beta \) such that \( \mu(\{\alpha\}), \mu(\{\beta\}) \geq 1 - t^{-1} \) then \( \mu \otimes t((t_\alpha, t\beta)) > 0 \).

**Proof.** First observe that \( f_t(\alpha) = f_t(\beta) = F_\mu(\alpha) = F_\mu(\beta) = 0 \) and \( \psi_t(\alpha) = t\alpha, \psi_t(\beta) = t\beta \). Without loss of generality, we assume that \( F_\mu(\psi_t(x)) \neq 0 \) for all \( x \in (t\alpha, t\beta) \) or, equivalently, \( F_\mu(\psi_t(x)) \neq 0 \) for all \( x \in (\alpha, \beta) \). By Theorem 3.8(i), it suffices to show that \( f_t \) is positive at some point in \([\alpha, \beta]\). If \( f_t \) vanishes on \([\alpha, \beta]\) then \( F_\mu \) is continuous and strictly increasing on \([\alpha, \beta]\) by Proposition 3.5 and Corollary 3.6, a contradiction. This completes the proof. □

By the work of Maassen [13], for any probability measure \( \nu \) on \( \mathbb{R} \) there exists a unique probability measure \( \mu \) on \( \mathbb{R} \) with mean zero and unit variance such that

\[
F_\mu(z) = z - G_\nu(z), \quad z \in \mathbb{C}^+.
\]

If \( \rho \) denotes the finite positive Borel measure in the Nevanlinna representation of \( F_\mu \) as before then

\[
(s^2 + 1) \, dp(s) = dv(s)
\]

by the Stieltjes inversion formula and (2.2). In this case,

\[
H_t(z) = z + (t - 1)G_\nu(z), \quad z \in \mathbb{C}^+,
\]

and hence \( z \in \Omega_t \) if and only if

\[
\int_{\mathbb{R}} \frac{dv(s)}{|z - s|^2} < \frac{1}{t - 1}.
\]

Moreover, the function \( g \) defined in section 3 is given by the formula

\[
g(x) = \int_{\mathbb{R}} \frac{dv(s)}{(s - x)^2}, \quad x \in \mathbb{R}.
\]
Proposition 4.13. There exists a Borel probability measure $\mu$ on $\mathbb{R}$ such that the number of components in $\text{supp}(\mu^{\oplus t})$ is infinite for any $t > 1$.

Proof. Define the probability measure $\nu$ on $\mathbb{R}$ as
\[ \nu = \sum_{n=1}^{\infty} 2^{-n} \delta_{2^n}, \]
and let $\mu$ be the unique measure satisfying the requirement (4.8). Then (4.9) shows that the function $g$ is continuous and strictly convex on the intervals $I_n = (2^n, 2^{n+1})$, $n \geq 1$, and
\[ \lim_{x \uparrow 2^{n+1}} g(x) = \lim_{x \downarrow 2^n} g(x) = \infty, \]
from which we see that $g$ reaches its minimum $m_n$ on each $I_n$. If $x_n$ is the middle point of $I_n$ then for each $s \in \mathbb{R} \setminus I_n$,
\[ (s - x_n)^2 \geq \left( \frac{2^{n+1} - 2^n}{2} \right)^2 = 2^{2n-2}, \]
from which we deduce that $m_n \leq g(x_n) = \int_{\mathbb{R} \setminus I_n} \frac{d\nu(s)}{(s - x_n)^2} \leq 2^{-2n+2}$.

This shows that for any $t > 1$, there are infinitely many intervals $\{I_{nk}\}_{k=1}^{\infty}$ each of which contains a closed subinterval $J_{nk}$ such that $f_t$ vanishes on $J_{nk}$. Note that (4.10) shows that these intervals $J_{nk}$ are separated by intervals on which $f_t$ is positive. Hence we conclude that the closure of the set $V_t^+ = \{ x \in \mathbb{R} : g(x) > (t - 1)^{-1} \}$ contains infinitely many components, whence the desired result follows. $\square$

We conclude this section with an example. For any $0 < \epsilon < 1$, consider the measure
\[ \mu_{\epsilon} = \frac{\epsilon}{2} (\delta_{-1} + \delta_1) + (1 - \epsilon) \delta_0. \]
Since
\[ G_{\mu_{\epsilon}}(z) = \frac{z^2 - 1 + \epsilon}{z(z^2 - 1)} \]
has zeros at $\pm \sqrt{1 - \epsilon}$, by Lemma 4.8 and some simple manipulations we obtain
\[ \rho_{\epsilon} = \frac{\epsilon}{2(2 - \epsilon)} (\delta_{-\sqrt{1-\epsilon}} + \delta_{\sqrt{1-\epsilon}}) \]
and
\[ g_{\epsilon}(x) = \int_{\mathbb{R}} \frac{s^2 + 1}{(s - x)^2} \, d\rho_{\epsilon}(s) = \frac{\epsilon x^2 + \epsilon(1 - \epsilon)}{(x^2 - 1 + \epsilon)^2}. \]
Note that $g_{\epsilon}(x) = \epsilon(1 - \epsilon)^{-1}$ at points $x = 0, \pm \sqrt{3(1 - \epsilon)}$ which implies that for $1 < t < \epsilon^{-1}$ the set $V_t^+ = \{ x : g_{\epsilon}(x) > (t - 1)^{-1} \}$ consists of two components which are contained in intervals $(-\sqrt{3(1 - \epsilon)}, 0)$ and $(0, \sqrt{3(1 - \epsilon)})$, respectively. Moreover, when $t = \epsilon^{-1}$ the closure of the union of these two components is the interval $[-\sqrt{3(1 - \epsilon)}, \sqrt{3(1 - \epsilon)}]$. Hence the support $\text{supp}(\mu_{\epsilon}^{\oplus t})$ consists of two components for $1 < t < \epsilon^{-1}$ and these two components merge into one piece when $t \geq \epsilon^{-1}$. On the
other hand, if $\epsilon < 2/3$ then the measure $\mu^{|t_\epsilon|}$ has three atoms $0, \pm t$ for $1 < t < 2(2 - \epsilon)^{-1}$, has one atom $0$ for $2(2 - \epsilon)^{-1} \leq t < \epsilon^{-1}$, and has no atom for $t \geq \epsilon^{-1}$. If $\epsilon = 2/3$ then the measure $\mu^{|t_\epsilon|}$ has three atoms $0, \pm t$ for $1 < t < 3/2$ and has no atom for $t \geq 3/2$. If $2/3 < \epsilon$ then the measure $\mu^{|t_\epsilon|}$ has three atoms $0, \pm t$ for $1 < t < \epsilon^{-1}$, has two atoms $\pm t$ for $\epsilon^{-1} \leq t < 2(2 - \epsilon)^{-1}$, and has no atom for $t \geq 2(2 - \epsilon)^{-1}$. By Proposition 4.7 we see that

$$0 \in \text{supp}((\mu^{|t_\epsilon|})^{ac}) \quad \text{for} \quad t \geq \epsilon^{-1}$$

and

$$\pm t \in \text{supp}((\mu^{|t_\epsilon|})^{ac}) \quad \text{for} \quad t \geq 2(2 - \epsilon)^{-1}.$$ 

From the above discussions, we conclude that the numbers of components in $\text{supp}(\mu^{|t_\epsilon|})$ as follows.

If $0 < \epsilon < 2/3$ then

$$n(t) = \begin{cases} 
5, & \text{for } t \in (1, 2(2 - \epsilon)^{-1}) \\
3, & \text{for } t \in [2(2 - \epsilon)^{-1}, \epsilon^{-1}) \\
1, & \text{for } t \in [\epsilon^{-1}, \infty).
\end{cases}$$

If $\epsilon = 2/3$ then

$$n(t) = \begin{cases} 
5, & \text{for } t \in (1, 3/2) \\
1, & \text{for } t \in [3/2, \infty).
\end{cases}$$

Similarly, if $2/3 < \epsilon < 1$ then $n(t)$ begins with 5, reduces to 3, and then becomes 1 for large $t$.

**Acknowledgments** The author wishes to thank his advisor, Professor Hari Bercovici, for his generosity, encouragement, and invaluable discussion during the course of the investigation.

**References**

[1] S.T. Belinschi, H. Bercovici, Atoms and regularity for measures in a partially defined free convolution semigroup, *Math. Z.* **248** (4) 665-674 (2004).

[2] S.T. Belinschi, H. Bercovici, Partially defined semigroups relative to multiplicative free convolution, *Int. Math. Res. Not.* **2** 65-101 (2005).

[3] S.T. Belinschi, The Lebesgue decomposition of the free additive convolution of two probability distributions, *Probab. Theory Relat. Fields* **142** 125-150 (2008).

[4] H. Bercovici, D. Voiculescu, Lévy-Hinčin type theorems for multiplicative and additive free convolution *Pacific Journal of Mathematics* **153** No.2 217-248 (1992).

[5] H. Bercovici, D. Voiculescu, Free Convolutions of measures with unbounded support, *Indiana Univ. Math. J.* **42** (3) 733-773 (1993).

[6] H. Bercovici, D. Voiculescu, Superconvergence to the central limit and failure of the Cramér theorem for free random variables, *Probab. Theory Relat. Fields* **103** 215-222 (1995).

[7] H. Bercovici, D. Voiculescu, Regularity questions for free convolution, in: Nonselfadjoint Operator Algebras, Operator Theory, and Related Topics, in: Oper. Theory Adv. Appl., vol. 104, Birkhauser, Basel, 1998, pp. 37-47.

[8] P. Biane, On the free convolution with a semi-circular distribution, *Indiana Univ. Math. J.* **46** (3) 705-718 (1997).

[9] P. Biane, Processes with free increments, *Math. Z.* **227** (1) 143-174 (1998).

[10] G. P. Chistyakov, F. Götze, Limit theorems in free probability theory. I, *Ann. Probab.* **36** No.1 54-90 (2008).

[11] G. P. Chistyakov, F. Götzte, Asymptotic Expansions in the CLT in Free Probability. ArXiv: 1109.4844.

[12] G. P. Chistyakov, F. Götzte, Rate of Convergence in the entropic free CLT. ArXiv: 1112.5087.
[13] H. Maassen, Addition of freely independent random variables, *J. Funct. Anal.* **106** 409-438 (1992).
[14] A. Nica, R. Speicher, On the multiplication of free N-tuples of noncommutative random variables. *Amer. J. Math.* **118**(4), 799-837 (1996).
[15] V. Pata, The central limit theorem for free additive convolution, *J. Funct. Anal.* **140** 359-380(1996).
[16] D.V. Voiculescu, Addition of certain non-commuting random variables, *J. Funct. Anal.* **66** 323-346(1986).
[17] D.V. Voiculescu, The analogues of entropy and of Fisher’s information measure in free probability theory I, *Comm. Math. Phys.* **155** (1) 411-440 (1993).
[18] D.V. Voiculescu, The coalgebra of the free difference quotient and free probability, *Internat. Math. Res. Notices* **2** 79-106 (2000).
[19] D.V. Voiculescu, K.J. Dykema, A. Nica, Free Random Variables. CRM Monograph Series, Vol. 1 Am. Math. Soc. Providence, RI, (1992).
[20] J.-C. Wang, Local Limit Theorem in Free Probability Theory, *Ann. Probab.* **38** No.4 1492-1506 (2010).

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