Low-temperature universal dynamics of the bidimensional Potts model in the large $q$ limit

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Abstract. We study the low temperature quench dynamics of the two-dimensional Potts model in the limit of a large number of states, $q \gg 1$. We identify a $q$-independent crossover temperature (the pseudo spinodal) below which no high-temperature metastability stops the curvature driven coarsening process. At short length scales, the latter is decorated by freezing for some lattice geometries, notably the square one. With simple analytic arguments, we evaluate the relevant time-scale in the coarsening regime, which turns out to be of Arrhenius form and independent of $q$ for large $q$. Once taken into account, dynamic scaling is universal.

Keywords: coarsening processes, metastable states, classical Monte Carlo simulations, kinetic Ising models

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1. Introduction

The Potts model [1–3] is one of the best known models of statistical mechanics. It is an extension of the Ising model in which the variables are upgraded to take \( q \) integer values. The model appears in many areas of physics, as well as at interfacing with other branches of science. In the physical context, the large \( q \) limit of the ferromagnetic model is used to describe grain growth, soap froth evolution, re-crystallization and late stage sintering [4–6] (see also [7] and references therein). Mappings to other celebrated models of statistical mechanics, such as loop and spin ice models [8], opened the way to a myriad of studies in mathematical physics. The analysis of its critical properties helped developing the conformal field theory apparatus [9] and, very recently, the bootstrap approach has also been applied to this problem [10–12]. The anti-ferromagnetic Potts model represents the colouring problem of computer science [13, 14]. Other applications in this realm are community detection in complex networks [15–17] or the inverse problem in biophysics [18]. Concomitantly, the Potts model has also been used to mimic a variety of biophysical problems, see e.g. [19, 20]. The way in which quench randomness affects the order and universality of phase transitions was addressed using mostly the
weakly disordered Potts ferromagnets as a paradigm [21–23]. Mean-field Potts models with strong disorder realise the random first-order phase transition scenario for the glassy arrest [24, 25]. Last but not least, atomic physics realisations of the Potts model have been recently proposed [26] and particle physics applications of the same model have also appeared in the literature [27].

One of the interests of the ferromagnetic Potts model is that beyond a critical value of the number of single spin states, \( q_c \), the transition is of first order. Thus, the quench dynamics across this phase transition not only involves the more familiar coarsening phenomena described by the dynamic scaling hypothesis [28–32] but also the peculiarities of metastability and nucleation [33–36]. It is, therefore, much richer and, quite surprisingly, still far from being fully understood.

In this paper, and in an accompanying manuscript [37], we build upon the analysis of metastability in the large \( q \) bidimensional ferromagnetic Potts model presented in [38]. In short, we study the dynamical behaviour after a rapid quench for sufficiently large \( q \) so that the transition is of first order. While curvature driven coarsening should be the leading mechanism for ordering [39–44], the Potts model dynamics also present low temperature freezing [45–50] and metastability close to the critical temperature [38, 52–56] that may conspire against the system reaching equilibrium after the quench.

In both works we aim to improve our understanding of the interplay between the coarsening process, the low temperature freezing and the metastability close to criticality. In particular, we study the influence of the final reduced temperature \( T/T_c(q) \) and the number of states \( q \) on the dynamic properties. In this work, we identify the crossover temperature below which high-temperature metastability in sub-critical quenches is no longer important (the pseudo spinodal [54]) for various lattice geometries, and we focus on the low temperature freezing, escape from it, and scaling in the subsequent coarsening regime. In [37], instead, the multi-nucleation mechanism at higher temperatures is investigated.

The layout of the paper is as follows. In section 2 we recall the definition of the Potts model and the parameter dependence of the critical temperature on different bidimensional lattices. We also present in this section a short summary of our results. After discussing some general arguments for metastability (section 3) and freezing (section 4) in the \( q \to \infty \) limit, we show the outcome of the numerical simulations for various \( q \) and reduced temperatures \( T/T_c(q) \), using different lattices (all with periodic boundary conditions) in section 5. Details on the heat-bath algorithm that we use, and its analysis in the \( q \gg 1 \) or \( T/T_c(q) \ll 1 \) limits, are given in section 4.3. In the last section, we draw our conclusions.

2. The model

The Potts model [1] is a generalisation of the Ising model in which the spin variables take \( q \) integer values (often associated to colours) and are locally coupled ferromagnetically; that is to say, nearest neighbour exchanges favour equal values of the spins (colours).
The Hamiltonian is
\[ H_{J\{s_i\}} = -\frac{J}{2} \sum_{\langle ij \rangle} \delta_{s_i s_j} \]  
(1)
with \( J > 0 \), \( s_i = 1, \ldots, q \), and the sum runs over the nearest-neighbours on the lattice (each bond contributing twice to the sum). The model undergoes an equilibrium phase transition at a critical temperature that can be of first or second order depending on \( q \) and the dimension of space, \( d \). In two dimensions, \( d = 2 \), the transition is of second-order for \( 2 \leq q \leq 4 \), while it is of first-order for \( q > 4 \). The critical temperature depends on the coupling strength, \( J \), the dimension, \( d \), and the coordination of the lattice, \( z \). On the square lattice (SL), \( z = 4 \) and \([1-3]\)
\[ T_{\text{square}}^c \approx \frac{2J}{\ln q} \text{ for } q \gg 1 \]  
(2)
\((k_B = 1 \text{ henceforth})\). On the triangular lattice (TL) and honeycomb lattice (HL), the critical temperatures are given by implicit expressions \([2]\)
\[ 0 = x^3 - 3x^2 - 3(q-1)x + 3q - 1 - q^2 \quad \text{honeycomb } z = 3, \]  
(3)
with \( x = e^{\beta c J} \) and \( \beta = 1/T \). In the large \( q \) limit \( \beta_c J \gg 1 \), implying \( x^3 \gg x^2 \gg x \), and
\[ T_{\text{triang}}^c \approx \frac{3J}{\ln q}, \quad T_{\text{honey}}^c \approx \frac{3J}{2 \ln q}. \]  
(4)
Therefore, in the three cases
\[ T_c \approx \frac{zJ}{2 \ln q} \quad \text{for } q \gg 1 \]  
(5)
and \( T_c \) diminishes logarithmically with \( q \).

Before entering into the details of our study, let us here give a concise description of our results, found with a combination of analytic arguments and numerical simulations. We use systems with \( N = L^2 \) and \( L = 10^4 \) sites, and \( q \) ranging from \( 10^2 \) to \( \infty \). The various regimes in the \((T,q)\) plane are sketched in figure 1. We consider either a subcritical quench starting from a completely disorder configuration or an inverse quench above the critical temperature, starting from a completely ordered configuration, that is, a zero temperature ground state. Exploiting the ideas developed in \([38]\), we identify a finite temperature interval around the critical one in which the \( q \to \infty \) model remains metastable (blocked in the initial state forever) after both lower and upper critical quenches. The lower limit of this interval is at \( T = T_c(q) / 2 \) with \( T_c(q) \) being the critical temperature of the corresponding lattice. For finite \( q \) the lifetime of the metastable states can be extremely long and go beyond any reachable time-span, even for not-so-large values of \( q \). This occurs in the region labelled ‘metastability’ for the sub-critical quench and is painted in pink in figure 1. Below the lower limit of this metastability region, two kinds of mechanisms can lead to equilibration: either multi-nucleation \([37]\) followed by coarsening or just curvature driven coarsening. We identify the spinodal
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Figure 1. A sketch of the phase diagram of the 2d Potts SL model. The \( (T/T_c(q) \leq 1, q \gg 4) \) plane with the crossover lines between different types of dynamic behaviour is displayed. The black dots sitting on the limit between the (pink) metastability and (white) multi-nucleation regions were obtained in [38]. This paper focuses on the dynamics in the (light green) regime \( T < T_c(q)/2 \), while [37] concentrates on the multi-nucleation and further coarsening arising in the white region.

Cross-over between the metastable and unstable regions at the temperature \( T = T_c(q)/2 \), the vertical line separating the green and white sectors in figure 1. Finally, at sufficiently low temperatures, the systems quickly get trapped in a partially ordered state with lifetime diverging in the zero temperature and infinite \( q \) limits. Only at finite temperature and finite \( q \), after leaving these blocked states at a parameter dependent time-scale that we determine here, do the dynamics enter the proper curvature driven asymptotic regime. In the latter, the typical length-scale grows algebraically with the expected universal power 1/2 and the prefactor is simply given by a \( q \)-independent Arrhenius factor.

The main focus of this paper is the study of the coarsening evolution of the large \( q \) Potts model quenched from high temperature and, in particular, the analysis of the cross-over from the temporarily blocked states and the coarsening regime, and its scaling properties. In a companion paper [37], we studied the metastability and escape from it arising closer to the critical temperature (see the white sector in figure 1), and the peculiar finite size effects in the dynamic evolution.

Concretely, we measure the time evolution and parameter (\( q \) and \( T/T_c \)) dependence of the growing length, \( R(t; q, T/T_c) \), which quantifies the typical linear extent of the ordered patches in the low temperature dynamics (geometric spin clusters). A way to measure \( R \) is to monitor the energy per spin at time \( t \), \( e(t; q, T/T_c) \), which is associated with the total length of the interfaces. Accordingly,

\[
R(t; q, T/T_c) = \frac{e(t \to \infty; q, T/T_c)}{e(t \to \infty; q, T/T_c) - e(t; q, T/T_c)}.
\]
For a random initial condition, $e(t = 0) \approx 0$ for all $q$ and $T/T_c$. At very low temperatures $e(t \to \infty) \neq 0$ (e.g. for the SL, $e(t \to \infty) \simeq -2J$) for all $q \geq 2$ and $R(t = 0) \simeq 1$.

The dynamic scaling hypothesis [28–32] states that the time-dependence of $R$ should be universal in the coarsening regime, and is expected to apply for $\delta \ll R \ll L$ with $\delta$ the lattice spacing and $L$ the linear system size. This means that if, as in curvature driven coarsening with non-conserved order parameter, kinetic ordering is ruled by the algebraic law $t^{1/2}$, the exponent should not depend on the parameters, and all of their dependencies must appear in a pre-factor, in such a way that

$$R(t; q, T/T_c) \simeq [\lambda_q(T/T_c)t]^{1/2}. \quad (7)$$

We will examine this guess and confirm that it holds in the asymptotic coarsening regime, after leaving any frozen state, in the restricted temperature interval

$$T/T_c \leq 1/2 \quad \text{in the large } q \text{ limit.} \quad (8)$$

In the $q \to \infty$ model, at higher though still subcritical temperatures, the system stays blocked in the disordered initial state. For finite $q$ and $T$ beyond the limit in equation (8), the system performs multi-nucleation before reaching a coarsening regime [37]. We will also show that for sufficiently large $q$ or low reduced temperature $T/T_c$ the parameter dependence of the time-scale $\lambda_q^{-1}(T/T_c)$ is

$$t_S(q, T/T_c) = \lambda_q^{-1}(T/T_c) \simeq a e^{J/T} \simeq a q^{2T_c/(zT)} \quad (9)$$

with $a$ being a constant. Finally, we will prove that, on the SL and HL, the growing length (7) with the time-scale (9) establishes when the length $R(t; q, T/T_c)$ detaches from a long-lived plateau

$$R(t; q, T/T_c) \simeq R_p(q, T/T_c) \quad \text{at} \quad t \simeq t_S(q, T/T_c) \simeq e^{J/T}. \quad (10)$$

$R_p$ is the typical linear extent of ordered patches in the zero temperature blocked states. It is of the order of a few $\delta$ (concretely, 3.63 $\delta$ for $q \to \infty$ on the SL) and it very weakly depends on $q$ and $T/T_c$. The TL model does not show this kind of blocking (see also [45, 50]).

Therefore, we conclude that, in quenches to $T/T_c \leq 1/2$, after a short transient, the large $q$ Potts model reaches a state of the kind of the blocked configurations at zero temperature on the SL and HL, while it progressively orders with no arrest on the TL. On the first two lattices, the blocked state survives until a time-scale $t_S$ of the order of $e^{J/T}$ when the dynamics cross over from $R \simeq R_p$ to the conventional curvature driven coarsening, $R \simeq R_p(t/t_S)^{1/2}$. Consistently, $t_S$ diverges for $T/J \to 0$. This study is complemented by the one that two of us and collaborators present in [37], where we study the dynamics of quenches to $T/T_c > 1/2$ for various (not that large) values of $q$.

3. Metastability in the $q \to \infty$ limit

As there exist an ordered and a disordered phase separated by a phase transition at $T_c$, one could naively expect that, starting from an initial configuration typical of the
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ordered phase and suddenly changing the temperature beyond the critical one, after some (short) transient the system should disorder. Conversely, starting from an initial state in the disordered phase, and changing the temperature below the critical one, the expectation would be that the system orders, at least over long length scales, after some time. However, some simple considerations allow us to show that this does not happen everywhere in the high temperature phase for the upper critical quench, nor everywhere in the low temperature phase for the lower critical one, in the infinite \( q \) limit.

In this limit, the critical temperatures of the SL, TL and HL Potts models are given by equation (5), which can also be expressed as

\[
ed^\beta J \simeq q^{2/z}. \tag{11}\]

The Boltzmann equilibrium weight of one (out of \( q \)) fully ordered configurations, one of the ground states, is \( P_{\text{ground}} = e^{\beta(zJ/2)N}/Z \), with \( Z \) being the partition function and \( N \) being the number of spins. A first excited state is obtained from this ground state by changing a single spin to any of the other \( q - 1 \) orientations. The energy gap is \( E_{\text{exc}} - E_{\text{ground}} = zJ \) and the ratio of the two probabilities tends to \( P_{\text{exc}}/P_{\text{ground}} \simeq q e^{-\beta zJ} \) in the large \( q \) limit. Accordingly, the change of a spin can occur in the large \( q \) limit, if and only if \( e^{\beta zJ} < q \). On the SL, TL and HL, we can now use equation (11) to prove that the disordered dynamics can be active only for

\[
\beta < \beta_c/2 \quad \text{or} \quad T > 2T_c. \tag{12}\]

Otherwise, if \( T < 2T_c \), no spin can flip, \( P_{\text{exc}}/P_{\text{ground}} \to 0 \) in the large \( q \) limit, and the system remains blocked in the ground state. Thus, starting from a completely ordered state, the system either (i) remains blocked in this ground state at any temperature \( T < 2T_c \) or (ii) disorders completely at \( T > 2T_c \).

Next, we can consider the case in which the initial state is disordered. Because of the \( q \to \infty \) limit, typically, each spin takes a different value. The energy of such a fully disordered configuration vanishes, \( E_{\text{dis}} = 0 \), and its probability weight is \( P_{\text{dis}} = q^N/Z \). After a quench to a sub-critical temperature, one of the \( N \) available spins will try to align with one of its neighbours. The energy of a state with a ‘bond’ is then \( E_{\text{bond}} = -J \) and its probability \( P_{\text{bond}} = (e^{\beta J}/q) P_{\text{dis}} \). The condition to start ordering is then

\[
P_{\text{bond}}/P_{\text{dis}} \geq 1 \Rightarrow e^{\beta J} \geq q \Rightarrow e^{\beta zJ/2} \Rightarrow \beta \geq \frac{z\beta_c}{2} \quad \text{or} \quad T \leq \frac{2T_c}{z}. \tag{13}\]

However, in practice, this does not mean that after a quench to \( T \leq 2T_c/z \) the state will order completely. For concreteness, let us focus on the SL problem. Indeed, each spin will try to align with one of its neighbours. Thus, after a full update of the lattice, all of the spins will have created a ‘satisfied’ bond with a random neighbour. At the next update, any of these bonds can break if one of the spins in the pair changes to take the value of another neighbour. It is easy to observe that if a spin has the same value as two of its neighbours, \( E_{\text{2bonds}} = -2J \), it will then be much more stable than if it aligns with only one of its neighbours, \( E_{\text{bond}} = -J \). If the two bonds form a corner, as shown in the left part of the following sketch, then another spin which closes a square will have

\[\text{We use heat-bath dynamics. More details are given in section 4.3.}\]
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Figure 2. Asymptotic configurations of the Potts model in the infinite $q$ limit after a quench to $T < T_c/2$. The lattice is a squared one with periodic boundary conditions and linear sizes $L = 10$ (a) and $L = 10^2$ (b). The dynamic updates follow the heat-bath rules given in equation (15).

a large probability to take the same value. Thus, the grey spin will flip to a blue spin, adding two more bonds.

![Diagram showing spin flips](https://doi.org/10.1088/1742-5468/ac0f67)

It is then easy to see that squares and rectangles form the more stable small structures. Consequently, after a few iterations, the configurations are filled with small squares and rectangles. In particular, one observes the existences of so called T-junctions that were identified as the main reason for which blocked states occur in finite $q$ Potts models at zero temperature [6, 39, 48]:

![Diagram showing T-junctions](https://doi.org/10.1088/1742-5468/ac0f67)

Typical snapshots displaying this fact for $L = 10$ and $L = 10^2$ are shown in figure 2, at a time $t = 10^3$ after a quench of the SL Potts model towards $T < T_c/2$. In the infinite $q$ limit, the dynamics at $T < T_c/2$ are at effectively vanishing temperature, and these configurations are stable. Thus, this run has only partially ordered on the SL. (It is worth noticing that a similar phenomenon is found in the Ising model on the HL, where it is possible to form hexagons of any size that will be stable under zero-temperature single spin-flip dynamics [51]. We will show below that such blocked states also exist for $q > 2$ on this lattice.)

From these simple arguments we conclude that the non-trivial ordering process in the infinite $q$ limit is restricted to a quench from a disordered state to $T \leq 2T_c/z$. For $2T_c/z < T$ the dynamics are blocked and the systems remain frozen in the disordered state.
initial state. Depending on the lattice, the phase ordering kinetics at $T < 2T_c/z$ can, however, go through temporarily blocked states with interesting patterns, as in the ones illustrated in figure 2 for the square geometry. These are typical blocked states at zero temperature, as the ones studied in [44, 48–50]. After a $T/T_c$ dependent time-scale is needed to escape such states, the system enters the proper dynamic scaling regime that takes it towards equilibrium.

4. Early approach to temporarily blocked states

In this section, we discuss the similarity of the $q \to \infty$ model quenched to $T < 2T_c/z < T_c$ and the finite $q$ model quenched to zero temperature. We first illustrate this feature with some numerical data and we then explain the origin of the equivalence by studying in detail the two limits of the heat bath transition rates.

4.1. Numerical method

We focus on sub-critical quenches. In this section, we consider as a starting condition a completely disordered configuration, and we then quench the system to a temperature, $T < 2T_c/z < T_c$, at the initial time $t = 0$. Next, we start updating with the new temperature. In Monte Carlo simulations, one chooses one site at random and changes the value of the spin according to a microscopic stochastic rule. For a system with $N$ spins, the $N$ update attempts correspond to a single time-step. We find it convenient to use heat-bath dynamics, since each move is actually an update in this case. Moreover, a continuous time version [57] of the algorithm can be implemented very easily. Details on the transition probabilities and their large $q$ and low $T$ limits are given in section 4.3.

4.2. The typical linear size in the blocked states

The similarity between the growth laws in the $q \to \infty$ model quenched to $T/T_c < 2/z$ and the finite $q$ one quenched to $T = 0$ is demonstrated by the curves in figure 3, where we show the growing length $R$ as a function of time in various cases of interest\(^4\). In the SL model with $q \to \infty$ quenched to $T < T_c/2$ the dynamics eventually block, in the way illustrated in figure 2, with $R$ approaching, very quickly at $t \simeq 10$, a plateau at $R_p \simeq 3.63$ (we use lattice spacing units such that $\delta = 1$). Besides, the zero temperature quench of a $q = 10^3$ model shows a very similar behaviour, with asymptotically blocked states of the same kind, and roughly the same value of $R_p$. We checked that the similarity persists for all $q \geq 10^2$ quenched to $T = 0$.

Exceptions to the equivalence are found for sufficiently small $q$. For example, for $q = 10$, $R_p$ fluctuates from sample to sample. We did not make a detailed check of all cases between $q = 10$ and $q = 10^2$; we will simply focus on large enough $q$. Moreover, for large lattices, such as $L = 10^3$ and $q = 10$, even at zero temperature, some samples can escape the temporarily blocked configurations and reach a nearly equilibrated state.

\(^4\)Here and in what follows, we do not write explicitly the $q$ and $T/T_c$ dependence of $R$.
Figure 3. The growing length $R$ vs $t$ of the Potts model on the SL, HL and TL with $L = 10^3$. Different curves correspond to values of $q$ and $T$ given in the key. Note the absence of freezing in the TL case.

The same equivalence is observed on the HL, with coordination number $z = 3$, see the other pair of curves in figure 3, which approach $R_p \approx 4$. The curves show data for the infinite $q$ limit with the condition $T < 2T_c/3$, and the zero temperature quench of the Ising model, $q = 2$. The latter case was already considered in [58], where it was observed that $R$ saturates to a value $R_p \approx 4$ after a rather short time, $t \approx 10$. This is due to the existence of frozen configurations on the odd-coordinated HL, see section 6.1 in [58]. Then, for this lattice, the behaviour at $T < 2T_c/3$ and infinite $q$ is similar to the one at $T = 0$ for any $q \geq 2$.

Last, we also show data for the TL for the infinite $q$ limit with the condition $T < T_c/3$ and at $T = 0$ and $q = 10^3$. For this lattice, the dynamics do not block. For the largest times, the growing length $R(t) \approx t^{1/2}$ as expected for standard curvature driven coarsening. The absence of blocking states at zero temperature for the TL has been known for a very long time [45, 46] and was discussed in a recent work for small values of $q$ [49].

4.3. Large $q$ or $T \to 0$ limits of the heat-bath rules

For concreteness, we focus on the SL case, and we explain, from the behaviour of the microscopic updates, the origin of the plateau at $R_p \approx 3.63$ in the growing length curves shown in figure 3.

Let us recall the general rules for the heat-bath dynamics on the SL, valid for any $q \geq 5$. This is done by considering all possible local configurations and their central spin flip evolution.

We follow the scheme introduced in [38]. Starting from one spin $S(i)$, we first count the number of neighbouring spins, $S(j)$, taking the same value, $S(j) = S(i)$, and we call this number $n_1$. Next, we count the number of neighbours taking other spin values and we organise them in decreasing order, $n_2, n_3$, etc, according to their frequency of appearance. We denote each possible configuration by $[n_1, n_2, \text{etc}]$ where only the values

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$n_i \neq 0$ are included. On a SL there are 12 possible local configurations that we also label with an integer $k = 0, \ldots, 11$ and we write this label in parenthesis.

Now, all possible single spin flip transitions are

$$(0) : [4] \rightarrow (0), (7) \quad (1) : [3, 1] \rightarrow (1), (4), (8)$$

$$(2) : [2, 2] \rightarrow (2), (2), (9) \quad (3) : [2, 1, 1] \rightarrow (3), (5), (10)$$

$$(4) : [1, 3] \rightarrow (4), (1), (8) \quad (5) : [1, 1, 1] \rightarrow (5), (3), (10)$$

$$(6) : [1, 1, 1, 1] \rightarrow (6), (11) \quad (7) : [0, 4] \rightarrow (7), (0)$$

$$(8) : [0, 3, 1] \rightarrow (8), (1), (4) \quad (9) : [0, 2, 2] \rightarrow (9), (2)$$

$$(10) : [0, 2, 1, 1] \rightarrow (10), (3), (5) \quad (11) : [0, 1, 1, 1, 1] \rightarrow (11), (6)$$

and the $(0)–(11)$ states were represented in a figure in section 3.2 in [38].

The reasoning behind these formulae is the following. The first configuration, named $(0)$, corresponds to one spin surrounded by four neighbours with the same colour. The central spin can either keep the same value, thus the $(0)$ on the right of the arrow, or flip to another value, thus the new configuration $(7) : [0, 4]$. The local configuration $(0)$ will remain the same with the probability $\simeq e^{4J}$ and change with the probability $e^0 = 1$ for each other possible value of the flipped spin. There are $q-1$ such values. Then, normalising the transition probabilities and writing them, for simplicity, for $J = 1$, we have

$$P_{0 \rightarrow 0} = \frac{e^{4J}}{e^{4J} + q - 1}, \quad P_{0 \rightarrow 7} = \frac{q - 1}{e^{4J} + q - 1}. \tag{14}$$

Following the same kind of analysis, we find that all other transition probabilities are

$$P_{1 \rightarrow 1} = \frac{e^{3J}}{e^{3J} + e^J + q - 2}, \quad P_{1 \rightarrow 11} = \frac{e^J}{e^{3J} + e^J + q - 2}, \quad P_{1 \rightarrow 8} = \frac{q - 2}{e^{3J} + e^J + q - 2}$$

$$P_{2 \rightarrow 2} = \frac{2e^{3J}}{e^{3J} + 2e^J + q - 2}, \quad P_{2 \rightarrow 9} = \frac{q - 2}{2e^{3J} + q - 2}, \quad P_{2 \rightarrow 10} = \frac{q - 3}{e^{3J} + 2e^J + q - 3}$$

$$P_{3 \rightarrow 3} = \frac{e^{3J}}{e^{3J} + 2e^J + q - 3}, \quad P_{3 \rightarrow 5} = \frac{2e^J}{e^{3J} + 2e^J + q - 3}, \quad P_{3 \rightarrow 10} = \frac{q - 3}{2e^{3J} + e^J + q - 3}$$

$$P_{4 \rightarrow 1} = \frac{e^J}{e^J + e^{3J} + q - 2}, \quad P_{4 \rightarrow 8} = \frac{e^J}{e^J + e^J + q - 2}, \quad P_{4 \rightarrow 11} = \frac{q - 4}{4e^J + q - 4}$$

$$P_{5 \rightarrow 5} = \frac{2e^J}{2e^J + e^{2J} + q - 3}, \quad P_{5 \rightarrow 3} = \frac{e^{2J}}{2e^J + e^{2J} + q - 3}, \quad P_{5 \rightarrow 10} = \frac{q - 3}{2e^J + e^{2J} + q - 3}$$

$$P_{6 \rightarrow 6} = \frac{4e^J}{4e^J + q - 4}, \quad P_{6 \rightarrow 11} = \frac{4e^J}{4e^J + q - 4}, \quad P_{6 \rightarrow 10} = \frac{e^{4J}}{e^{4J} + q - 1}$$

$$P_{7 \rightarrow 7} = \frac{q - 1}{e^{4J} + q - 1}, \quad P_{7 \rightarrow 0} = \frac{e^{4J}}{e^{4J} + q - 1}, \quad P_{7 \rightarrow 1} = \frac{e^{3J}}{e^{3J} + e^J + q - 2}$$

$$P_{8 \rightarrow 8} = \frac{q - 2}{e^{3J} + e^J + q - 2}, \quad P_{8 \rightarrow 1} = \frac{e^{3J}}{e^{3J} + e^J + q - 2}, \quad P_{8 \rightarrow 4} = \frac{e^J}{e^{3J} + e^J + q - 2}$$

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In the large $q$ limit, the transition probabilities simplify considerably. Since $e^\beta = e^{\beta_c T_c/T} = (1 + \sqrt{q})^{T_c/T} \approx q^{T_c/T}$, we deduce that, in this limit, $P_{11 \rightarrow 11}$ goes to (i) 0 for $T < T_c/2$, and (ii) 1 for $T_c/2 < T < T_c$. Similar simplifications apply to the other probabilities with (6) and (11) states, while the other probabilities take values 0 or 1 for $T < T_c$. In summary, the large $q$ limits of the transition probabilities are

\[
\begin{align*}
P_{0 \rightarrow 0} &= 1 & P_{0 \rightarrow 7} &= 0 \\
P_{1 \rightarrow 1} &= 1 & P_{1 \rightarrow 4} &= 0 & P_{1 \rightarrow 8} &= 0 \\
P_{2 \rightarrow 2} &= 1 & P_{2 \rightarrow 9} &= 0 \\
P_{3 \rightarrow 3} &= 1 & P_{3 \rightarrow 5} &= 0 & P_{3 \rightarrow 10} &= 0 \\
P_{4 \rightarrow 4} &= 0 & P_{4 \rightarrow 1} &= 0 & P_{4 \rightarrow 8} &= 0 \\
P_{5 \rightarrow 5} &= 0 & P_{5 \rightarrow 3} &= 0 & P_{5 \rightarrow 10} &= 0 \\
P_{6 \rightarrow 6} &= 1 / 0 & P_{6 \rightarrow 11} &= 0 / 1 \\
P_{7 \rightarrow 7} &= 0 & P_{7 \rightarrow 10} &= 0 \\
P_{8 \rightarrow 8} &= 0 & P_{8 \rightarrow 1} &= 0 & P_{8 \rightarrow 4} &= 0 \\
P_{9 \rightarrow 9} &= 0 & P_{9 \rightarrow 2} &= 0 \\
P_{10 \rightarrow 10} &= 0 & P_{10 \rightarrow 3} &= 0 & P_{10 \rightarrow 5} &= 0 \\
P_{11 \rightarrow 11} &= 0 / 1 & P_{11 \rightarrow 6} &= 1 / 0
\end{align*}
\]

The two values of the transition probabilities involving (6) and (11) states are written in order for $T < T_c/2$ and $T_c/2 < T < T_c$.

Next, we notice that the rules for $T < T_c/2$ in the large $q$ limit are the same as the ones for finite $q$ in the limit $T \rightarrow 0$, thus finding the explanation of the similarity between the two dynamics shown in section 4.2 (see figure 3). Let us now give more details on how the evolution takes place in the two cases.

In the $q \rightarrow \infty$ limit, the initial configuration contains only (11) states. If $T > T_c/2$, the state (11) is stable and the system remains disordered forever. If $T < T_c/2$, the (11) states change into (6) states. In some cases, a spin connecting to another spin is already in a (6) state, so this will form a (3) state (figure 4).

After some iterations, all of the states will become (0), (1), (2) or (3) states. The latter can still evolve if a neighbour is flipped, but otherwise they are stable.

For finite $q$ and zero temperature, the dynamics are also very similar. The main difference is that the initial configuration does not only contain (11) states. A simple
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Figure 4. Examples of transitions which, starting from a fully disordered configuration (a), partially order the system. (b) The blue circled spin has changed its value to form a bond with the neighbour on its right. (c) The orange circled spin has evolved in order to have the same colour as the spin to its right. We can see the appearance of two (6) states in (b) and a (3) state in (c). The grey spins have the same colour, while the blue bonds indicate a non-trivial interaction between spins, as used in [38].

check shows that the value of the plateau has a weak dependence on $q$. Using a lattice with $L = 10^3$, we measure $R_p \simeq 3.898(1)$ for $q = 10^2$, $R_p = 3.635(1)$ for $q = 10^3$, and $R_p = 3.632(1)$ for $q = 10^5$, this last value being compatible with the infinite $q$ one.

A similar analysis can be attempted for the microscopic updating heat-bath rules on the honeycomb and triangular cases. For the HL, there exist six states in the heat-bath formulation and such an analysis is even easier than for the SL, with the 12 states discussed above [38]. For the TL, there are 30 states and it is much more tedious to write a heat-bath algorithm. Still, in the zero temperature limit, the transition probabilities simplify and it is manageable. We will show below some results at finite temperature using the three lattice geometries. In the TL case, we employed conventional Metropolis updates.

4.4. Blocked states on the SL

Consider again the blocked configuration obtained after a quench from a disordered initial state to $T < T_c/2$ in the $q$ infinite limit of the Potts model on a $L = 10$ SL. Such a configuration is shown in figure 5(a), where we highlighted the state in which each boundary spin is, using the notation introduced in section 4.3. For this particular configuration, a direct inspection shows that only four types of states exist, (0), (1), (2) and (3). The (3) states, shown as green squares ■, lie on the corners of the interfaces. They are spins with two neighbours taking the same value and two neighbours taking different values from the central one and also being different from each other. The (2) states, shown with blue crosses (X), are blinking states: the central spin has two neighbours with the same value and two neighbours with an identical value which is, however, different from the central one. For the configuration shown in figure 5(a), there is only one blinking state (close to the upper right corner). The (1) states, shown as

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Figure 5. Snapshots of the Potts model in the large $q$ limit at low temperature. The lattice is a squared one with periodic boundary conditions and $L = 10$. The domains within which the spins take the same value are painted with the same colour. See the text for details on the convention used to identify the states of the spins on the domain walls, shown with small data-points with different form and colour. The neighbouring configurations differ by a single spin flip.

red crosses ($+$), are spins at a flat interface with three neighbours being identical to the central one and one neighbour taking a different value. The (0) states, corresponding to spins in the bulk of the domains, are not shown.

Following the transition rules in equation (15), the states (0), (1) and (3) are stable and the corresponding spins cannot flip. Only the (2) state can change, producing the configuration shown in figure 5(b). Note that under this update, the central spin in the (2) state changes but the state (2) is not modified. As a consequence of the flip of the central spin, the four neighbouring spins will also change state but again towards stable states: two (1) states are changed in (3) states, one (3) state in a (1) state and a (1) state in a (0) state. Thus, in this new configuration, only the same spin in the (2) state can change, which will again produce the configuration of figure 5(a). In the infinite $q$ limit, the only evolution corresponds to blinking between the configurations shown in figures 5(a) and (b).

Next, we consider the case of a large but finite $q$ with the condition $e^{\beta} > q$ corresponding to a low temperature $T < T_c/2$. Let us take the configuration in figure 5(a). According to the general rules for heat-bath dynamics, a (0) state changes to a (7) state.
with probability $\simeq q e^{-4\beta}$, a (1) state changes to a (4) state with probability $\simeq e^{-2\beta}$ and to a (8) state with probability $\simeq q e^{-3\beta}$, a (2) state changes to another (2) state with probability $\simeq 1/2$ and to a (9) state with probability $\simeq q e^{-2\beta}$, and a (3) state changes to a (5) state with probability $\simeq e^{-\beta}$ and to a (10) state with probability $\simeq q e^{-2\beta}$.

In the large $q$ limit with the condition $T < T_c/2$, $e^{-\beta}$ is very small and $q e^{-\beta} < 1$. Then, the dominant changes are the flips of spins in the (2) state towards another (2) state (thus, a blinking) and the next leading changes are the spins in the (3) state changed in a (5) state. The flip of a spin in the (2) state will again produce the configuration shown in figure 5(b), which will almost surely flip back to the configuration in (a), etc. Thus, this leads to the same blinking behaviour as in the infinite $q$ limit. But for a large and finite $q$, after a time $\simeq e^\beta$, a spin in a (3) state can also flip. Such an example is shown in figure 5(c). The spin that changed value (from red to yellow) is now in a (5) state, shown as a purple triangle (▲). We also observe the appearance of another (2) state just below. Of course, the spin in the (5) state is short lived (the probability that it flips back to its previous value is $\simeq 1 - 2e^{-\beta}$). But there exists a finite probability $\simeq 1/2$ that the blinking state below is updated first, producing the configuration shown in figure 5(d). We then see that the (5) state is changed in a (more stable) (3) state, while the blinking state moves down. Next, there is a finite probability that the new (2) state changes colour to produce the configuration seen in figure 5. This configuration again contains a short-lived (5) state. Therefore, from this state, there is a large probability to end in the configuration shown in figure 5(f).

The conclusion is that starting from configuration (a), one will observe blinking between (a) and (b) configurations until, after a time $\simeq e^\beta$, a transition to a (c) configuration occurs. Next, with a finite probability, the state evolves to the (d) configuration and, again with a finite probability, it goes to the (e) configuration from which it evolves to the (f) one. This shows that the time scale for quitting the low temperature and large $q$ blocked state is given by
\[ t_s \simeq e^{\beta J} = e^{J/T}, \] (16)
where we restored the coupling constant $J$. The same time-scale was found with different arguments by Spirin et al [47] and Ferrero and Cannas [43].

Finally, note that this simple scenario is valid for all values of $T$ and $q$ as far as the conditions $e^{-2\beta} \ll e^{-\beta}$ and $q e^{-4\beta} \ll e^{-\beta}$ are satisfied. This can be rephrased by saying that we expect universal behaviour at low temperature and for large $q$. In the next sections, we will test these predictions with numerical simulations. We stress here that a very similar phenomenology is found on the HL and a slightly different one, with no blocked states and no plateau in $R$, on the triangular one, also to be discussed below.

5. The growing length

In this section, we study the parameter $(q$ and $T/T_c)$ dependence of the growing length $R$ with the aim of proving the hypothesis (7) and finding the explicit form of the pre-factor $\lambda_q(T/T_c)$ or, equivalently, the time-scale $T_s$. In most of this section, we work with...
Figure 6. The growing length $R$ vs $t$ for $q = 10^2$ (a) and (b), and $10^6$ (c) and (d), in a SL system with $L = 10^3$ and various values of $T/T_c$ written in the keys. (b), (d) A rescaled version in which time is divided by $t_S = e^{J/T}$. The horizontal dotted lines are at the plateau $R_p \simeq 3.63$ and the inclined dashed lines are the expected asymptotic $t^{1/2}$ law.

the SL. By the end of it, we present data for honeycomb geometry, which confirms the same kind of universality found on the square one. We have considered the TL too but, in this case, we observe a different behaviour, which will also be discussed.

5.1. Parameter dependence

In the left panels of figure 6, we show $R$ vs $t$ for $q = 10^2$ and $10^6$ (from top to bottom) and the reduced temperatures $T/T_c$ given in the keys. In the right panels the time is rescaled by $t_S = e^{J/T}$, the time-scale that we identified in equation (16).

First of all, for both $q$, the curves show the predicted crossover at $T/T_c = 1/2$. At very short times, the data for $T/T_c > 1/2$ demonstrate the early establishment of the (high-temperature) metastable state with $R \simeq R_m \simeq 1$ and later evolution via the multi-nucleation process [37], while the early evolution of the curves at $T/T_c < 1/2$
is temperature independent and rapidly approaches $R_p \simeq 3.7$, to only later enter the coarsening regime (figure 7).

Let us now first discuss the case $q = 10^2$ in more detail. At relatively low temperatures, up to $T/T_c \simeq 0.30$, $R$ makes a small initial jump from $R(0) \simeq 1$ to a finite value $R_p \simeq 3.63$ and keeps this value during a time, which we call $t_p$, that increases as we decrease the temperature. Next, at later times, there is a crossover towards a regime with the $R(t) \simeq t^{1/2}$ characteristic of the standard curvature driven non-conserved order parameter coarsening [28–32]. At higher temperatures, there is a similar initial rapid increase to $R_p \simeq 3.63$, next an inflection point, and then the coarsening regime. This can be seen up to $T/T_c \simeq 0.5$. For even higher temperatures, $R$ keeps a small value close to one, corresponding to the disordered metastable state, and this for longer and longer times $t_m$ as we increase temperature. For example, for $T/T_c = 0.95$, the metastable state survives up to $t_m \simeq 10^5$. Next, $R$ increases very rapidly up to a large value $R_l$. After this very rapid variation, $R$ increases very slowly first and next faster towards the conventional coarsening regime with $R(t) \simeq t^{1/2}$. Note that $R_l$ increases as we increase $T/T_c$, and $t_m$ practically diverges at $T/T_c \simeq 0.99$. More details on the multi-nucleation processes taking place above $T_c/2$ are given in [37].

In summary, for $q = 10^2$ we observe:

- At $T/T_c \leq 0.5$, there is an initial jump of $R$ to $R_p \simeq 3.63$, a value that remains constant up to a time $t_p$, which increases as $T/T_c$ decreases. Afterwards, the dynamics reach the coarsening regime with $R$ growing as $t^{1/2}$.

- At $T/T_c > 0.5$, the system remains in the metastable high-temperature state with a very small value $R_m$ up to a time $t_m$, which increases with $T/T_c$. At later times, $t > t_m$, we observe a jump towards a finite value $R_l$, which increases in a way similar to the increase of $t_m$. Next, the typical length grows very slowly, and finally reaches the curvature driven $t^{1/2}$ law.
The situation is similar for other values of $q$ (see the other left panel in figure 6), a system with $q = 10^6$. The main difference is that for $T/T_c > 0.5$, after the jump towards $R_t$, we observe that the slow growth is replaced by a long-lasting plateau as we increase $q$. This is particularly clear for large $q$ (see the third left panel in figure 6).

This change of behaviour at $T/T_c \simeq 0.5 = 2/5$ is even better seen if we rescale time by the time-scale $t_s(q, T/T_c) = e^{1/T}$, determined in the previous section. In the right panels of figure 6, we show $R$ as a function of $t/t_s(q, T/T_c)$. We observe that, for each $q$, $R$ first goes to a plateau at temperatures $T/T_c \leq 0.5$ up to a rescaled time $t/t_s \simeq 10^{-2}$, which does not depend on $q$ (confirmed by other values of $q$ not shown). For longer times, the plateau will be escaped in a universal way and for longer (rescaled) times, the coarsening regime will be reached with $R(t) \simeq t^{1/2}$. Note that for $q = 10^4$ and $q = 10^6$, the curves are identical for $T/T_c \leq 0.5$. For $q = 10^2$, the scaling is not as good for $T/T_c$ close to 0.5. This is in agreement with the previous observation that the zero temperature behaviour becomes universal for large $q$ and with deviations up to $q \simeq 10^2$.

Then, for $T/T_c < 0.5$, we claim that the behaviour of $R$ is universal if we introduce a rescaling of time by $e^{1/T}$ such that

$$R(t; T/T_c, q) \simeq f(e^{-1/T}t) \quad \text{with} \quad f(x) \simeq \begin{cases} 3.63 & \text{for } x \ll 1, \\ x^{1/2} & \text{for } x \gg 1. \end{cases} \quad (17)$$

As usual with scaling laws, this behaviour is restricted to $R \gg \delta$, with $\delta$ being a length scale of the order of the lattice spacing, and $R \ll L$ with $L$ being the system size where equilibration of at least some samples comes into play. Our main finding is that for large $q$ ($q \geq 10^3$) and small $T$ ($T/T_c \leq 0.5$), after a short transient the system reaches a state equivalent to the blocked state at zero temperature, with $R_p$ determined by the lattice geometry and microscopic dynamics, and that this blocked state survives up to $t_p \simeq R_p^2 e^{1/T} = R_p^2 t_s$ when the dynamics cross over to the conventional coarsening one. The behaviour on the TL is different and we discuss it below. A much more detailed analysis of the nucleation process and further phase ordering kinetics in the cases $T/T_c(q) > 1/2$ is reserved to [37].

5.2. Snapshots

In the previous subsection, we argued that there is universal behaviour as a function of temperature after a proper rescaling of time. This was shown by considering the behaviour of the growing length $R(t; q, T/T_c)$. We want to confirm this result by showing some snapshots of a system with $L = 10^2$ and $q = 10^2$ (see figure 8). We present the instantaneous configurations at the times $R = 5, 10, 20, 40$ and 80 from left to right, reaching the relative temperatures $T/T_c = 0.2, 0.3, 0.4, 0.5$ from top to bottom. The main observation is that the snapshots look very much the same for a fixed value of $R(t; q, T/T_c)$. We have also checked similar snapshots for other values of the number of colours, $q = 10^3$ up to $10^6$, and they also look the same for the same $R$ and relative temperatures $T/T_c$.

5 In [41, 42] a rather weak dependence of $\lambda_0(T = T_c/2)$ on $q$ was claimed. Differently from here, in those papers only small values of $q$, $q = 2, 3, 8$, and the special temperature $T = T_c/2$ were considered.

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Figure 8. Snapshots at different times such that $R = 5, 10, 20, 40$ and $80$ (left to right). $L = 10^2$, $q = 10^2$ and from top to bottom $T/T_c = 0.2, 0.3, 0.4$ and $0.5$.

5.3. Honeycomb lattice

To check the universality of our results over the lattice kind, we now show some measurements using a HL. We recall that the critical temperature is given by $e^{2J/q} \approx q^{2/3}$ in the large $q$ limit. In figure 9, we study the time and reduced temperature dependence of the growing length $R$ in a system with $L = 10^3$ and $q = 10^4$. In this case $T_c \approx 0.162$. The time-dependence of $R$ is similar to the one found on the SL with the same $q$ (see figure 6). Again, at low temperatures, i.e. $T < 2T_c/3$, $R$ first goes to a plateau at $R_p \approx 4$, the same value obtained in the infinite $q$ limit after a quench towards $T < 2T_c/3$ (see figure 3). In the right part of figure 9, one can see that this plateau exists during a time $\approx e^{J/T}$ (as was the case on the SL for $T < T_c/2$). At reduced temperatures above $2T_c/z$, the system remains in the high temperature metastable state with $R \approx 1$ until a sudden jump in $R$ towards $R_l$ takes it out of it. $R_l$ increases with $T$.

We build the HL starting from a SL of linear size $L \times L$. At each site, the spin is connected to the left and right, and alternating, to the upper or lower row. For a more detailed description, see [58].
Figure 9. The time and temperature dependence of the growing length $R$ in the Potts model with $q = 10^4$ on the HL with periodic boundary conditions and $L = 10^3$. (a) Bare data for many $T/T_c(q)$ are given in the key. (b) $R$ against $t$ rescaled by $t_S = e^{J/T}$ for four temperatures $T < 2T_c/z$ and also one temperature $T > 2T_c/z$, which approaches the asymptotic $t^{1/2}$ without scaling at short times. The dashed inclined line is the curvature driven law $t^{1/2}$.

Figure 10. The growing length $R$ vs $t$ for the TL model with $q = 10^2$ and $L = 10^3$ at different values of $T/T_c$ given in the key. For comparison, we also show results for the SL and the HL at $T/T_c = 0.90$. The dashed inclined line is the curvature driven law $t^{1/2}$.

5.4. Triangular lattice

We now consider the case of the TL with coordination number $z = 6$. In the large $q$ limit, $e^{\Delta_J} \approx q^{1/3}$ and the interesting regime is the one of quenches below $T_c/3$. Different from what observed on the SL and HL, in such quenches there is no plateau in $R$ and the growing length does not slow down strongly. This was already observed in [7] for the $q = 10^2$ model at $T = 0.1$, while $T_c = 0.635$ for this $q$. We show our results, also for
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$q = 10^2$, in figure 10 for various values of $T/T_c$. We observe that $R$ does not depend on $T$ for small $T/T_c$ and we do not observe a plateau. For the smallest value $T/T_c = 0.15$, $R$ takes the same value as for the zero temperature shown in figure 3 (either at $q = 10^3$ or in the infinite $q$ limit). Only for $T/T_c = 0.4 > T/T_c = 1/3$ do we see a small deviation. For large values of $T/T_c$, the behaviour is similar to the one of the other lattices. We have also included, in figure 10, the time dependence of $R$ at $T/T_c = 0.90$ for the SL and HL models.

Note that the measurements using the TL at a finite temperature have been done with a Metropolis algorithm since the heat bath one is difficult to implement. We then had to rescale the time by a factor $\simeq q$ to compare the two. We found that with a factor 50, the behaviour of $R$ is similar for the three lattices at $T/T_c = 0.90$.

6. Conclusions

Exploiting the $q \to \infty$ limit of the heat-bath Monte Carlo algorithm [38], we identified the temperature interval $[2T_c/z, 2T_c]$ in which high or low temperature initial conditions are metastable after sudden sub-critical or upper-critical quenches, respectively, with $z$ the coordination of the lattice. In other words, we located the spinodals. In the $q \to \infty$ limit, the metastability is everlasting while, for finite $q$, the initial states will eventually die out. Once this done, we focused on sub-critical quenches for temperatures below the lowest temperatures at which nucleation is observed, that is $T < 2T_c/z$. For these processes, we showed that on the SL and HL, after a rapid evolution, the systems temporarily block in configurations that are typical of the asymptotic states of the zero temperature dynamics [48, 49]. At non-vanishing temperatures, these states are not fully blocking and the systems escape them in a time-scale $t_s \simeq e^{J/T}$ independently of $q$ for large $q$ (see also [43, 47]). The proper curvature driven coarsening then takes over with the universal algebraic growing length $R \simeq (t/t_s)^{1/2}$. On the TL we see no freezing, similarly to what was found in [45, 46, 50], and $R$ is independent of temperature, for $T/T_c < 2/z = 1/3$, within our numerical accuracy.

In a companion paper [37], the multi-nucleation process in the SL model at $T > 2T_c/z = T_c/2$ was thoroughly studied. The analysis proves that the finite size of the system plays a determinant role in deciding how many phases nucleate, with a $\ln L$ growth of this number with the linear system’s size. The ordered phases nucleate in a background that is strongly reminiscent of, and even quantitatively similar to, the critical state.

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