BPS and non–BPS Domain Walls in Supersymmetric QCD with SU(3) Gauge Group

A.V. Smilga and A.I. Veselov

ITEP, B. Cheremushkinskaya 25, Moscow 117218, Russia

Abstract

We study the spectrum of the domain walls interpolating between different chirally asymmetric vacua in supersymmetric QCD with the SU(3) gauge group and including 2 pairs of chiral matter multiplets in fundamental and anti-fundamental representations. For small enough masses $m < m_\ast \approx .286 \Lambda_{SQCD}$, there are two different domain wall solutions which are BPS–saturated and two types of “wallsome sphalerons”. At $m = m_\ast$, two BPS branches join together and, in the interval $m_\ast < m < m_{\ast\ast} \approx 3.704 \Lambda_{SQCD}$, BPS equations have no solutions but there are solutions to the equations of motion describing a non–BPS domain wall and a sphaleron. For $m > m_{\ast\ast}$, there are no solutions whatsoever.
1 Introduction

Supersymmetric QCD is the theory involving a gauge vector supermultiplet $V$ and some number of chiral matter supermultiplets. The models of this class attracted attention of theorists since the beginning of the eighties and many interesting and non-trivial results concerning their non-perturbative dynamics have been obtained [1]. The dynamics depends in an essential way on the gauge group, the matter content, the masses of the matter fields and their Yukawa couplings.

The most simple in some sense variant of the model is based on the $SU(N)$ gauge group and involves $N-1$ pairs of chiral matter supermultiplets $S_i \alpha, S'_i \alpha$ in the fundamental and anti-fundamental representations of the gauge group with a common mass $m$. The lagrangian of the model reads

\[ \mathcal{L} = \left( \frac{1}{4g^2} \text{Tr} \int d^2 \theta \left( W^2 + \text{H.c.} \right) \right) + \sum_{i=1}^{N-1} \left[ \frac{1}{4} \int d^2 \theta d^2 \bar{\theta} \ S_i e^{V} S_i \right. \\
+ \left. \frac{1}{4} \int d^2 \theta d^2 \bar{\theta} \ S'_i e^{-V} S'_i - \frac{m}{2} \left( \int d^2 \theta \ S'_i S_i + \text{H.c.} \right) \right], \tag{1} \]

color and Lorentz indices are suppressed. In this case, the gauge symmetry is broken completely and the theory involves a discrete set of vacuum states. The presence of $N$ chirally asymmetric states has been known for a long time. They are best seen in the weak coupling limit $m \ll \Lambda_{SQCD}$ where the chirally asymmetric states involve large vacuum expectation values of squark fields $\langle s_i \rangle \gg \Lambda_{SQCD}$ and the low energy dynamics of the model is described in terms of the colorless composite fields $M_{ij} = 2S'_i S_j$. The effective lagrangian presents a Wess–Zumino model with the superpotential

\[ W = -\frac{2 \Lambda_{SQCD}^{2N+1}}{3 \det M} - \frac{m}{2} \text{Tr} \ M \tag{2} \]

The second term in Eq.(2) comes directly from the lagrangian (1) and the first term is generated dynamically by instantons. Assuming $M_{ij} = X^2 \delta_{ij}$ and solving the equation $\partial W/\partial \chi = 0$ ($\chi$ is the scalar component of the superfield $X$), we find $N$ asymmetric vacua

\[ \langle \chi \rangle_k = \left( \frac{4 \Lambda_{SQCD}^{2N+1}}{3 \ m} \right)^{1/2N} e^{\pi i k/N} \tag{3} \]

(the vacua “k” and “k + N” have the same value of the moduli $\langle \chi^2 \rangle_k$ and are physically equivalent). These vacua are characterized by a finite gluino condensate

\[ \langle \text{Tr} \lambda^2 \rangle_k = 8\pi^2 m \langle \chi^2 \rangle_k \tag{4} \]

It was noted recently [2] that on top of (3) also a chirally symmetric vacuum with the zero value of the condensate exists. It cannot be detected in the framework of
Eq. (2) which was derived assuming that the scalar v.e.v. and the gluino condensate are nonzero and large, but is clearly seen if writing down the effective lagrangian due to Taylor, Veneziano, and Yankielowicz (TVY) involving also the composite field

\[ \Phi^3 = \frac{3}{32\pi^2} \text{Tr} \ W^2 \]  

The corresponding superpotential reads

\[ W = \frac{2}{3} \Phi^3 \left[ \ln \left( \frac{\Phi^3 \det M}{\Lambda_{SQCD}^{2N+1}} \right) - 1 \right] - \frac{m^2}{2} \text{Tr} \ M \]  

The presence of different degenerate physical vacua in the theory implies the existence of domain walls — static field configurations depending only on one spatial coordinate \( z \) which interpolate between one of the vacua at \( z = -\infty \) and another one at \( z = \infty \) and minimizing the energy functional. As was shown in, in many cases the energy density of these walls can be found exactly due to the fact that the walls present the BPS–saturated states.

The energy density of a BPS–saturated wall in SQCD with \( SU(N) \) gauge group satisfies a relation

\[ \epsilon = \frac{N}{8\pi^2} \left( \langle \text{Tr} \lambda^2 \rangle_\infty - \langle \text{Tr} \lambda^2 \rangle_{-\infty} \right) \]  

where the subscript \( \pm \infty \) marks the values of the gluino condensate at spatial infinities. Bearing Eqs. (7, 4) in mind, the energy densities of the BPS walls are

\[ \epsilon_r = N \left( \frac{4m^{N-1}}{3} \right)^{1/N} \]  

for the real walls and

\[ \epsilon_c = 2\epsilon_r \sin \frac{\pi}{N} \]  

for the complex walls. The RHS of Eqs. (7, 4) presents an absolute lower bound for the energy of any field configuration interpolating between different vacua.

The relation (7) is valid assuming that the wall exists and is BPS–saturated. However, whether such a BPS–saturated domain wall exists or not is a non–trivial dynamic question which can be answered only in a specific study of a particular theory in interest. This question has already been studied in our previous works \[ \textbf{[6, 7, 8, 9]} \]. In particular, in \[ \textbf{[5, 7, 8]} \] the simplest model of the class \( \textbf{[1]} \) with \( N_c = 2, \ N_f = 1 \) was analyzed. The results are the following:

\[ \text{The factor } 2/3 \text{ in Eq.} \textbf{[1]} \text{ and the corresponding factor in Eq.} \textbf{[2]} \text{ match the chosen (by historical reasons) normalization factor in the definition} \textbf{[3]} \text{ of } \Phi. \]

\[ \text{A relation of this kind can be derived also for other variants of the theory involving exotic groups and more complicated matter content, but in general case the energy of a BPS wall and the gluino condensate are not related so directly} \textbf{[3]} \]
1. For any value of the mass of the matter fields \( m \), there are domain walls interpolating between a chirally asymmetric and the chirally symmetric vacua (we call them real walls). They are BPS–saturated.

2. There are also complex BPS solutions interpolating between different chirally asymmetric vacua. But they exist only if the mass is small enough \( m \leq m^* = 4.67059 \ldots \Lambda_{\text{SQCD}} \). When \( m > m^* \), BPS walls are absent.

3. In a narrow range of masses \( m^* < m \leq m^*\approx 4.83 \Lambda_{\text{SQCD}} \), complex domain walls still exist, but they are not BPS saturated anymore. At \( m > m^*\approx 4.83 \Lambda_{\text{SQCD}} \), there are no such walls whatsoever.

In Ref.[9], we studied the problem of existence of BPS–saturated domain walls in the model (1) with \( N \geq 3 \). The results are basically the same as for \( N = 2 \): the real walls exist for any \( m \) and are BPS–saturated, and there are two complex BPS branches which exist in a limited range \( m < m^* \). The value of \( m^* \) goes down with \( N \): \( m^* = .28604 \ldots \Lambda_{\text{SQCD}} \) for \( SU(3) \), \( m^* = .07539 \ldots \Lambda_{\text{SQCD}} \) for \( SU(4) \), etc. For large \( N \), \( m^*(N) \propto N^{-3} \).

The results concerning the BPS walls were obtained by solving numerically the first order BPS equations

\[
\partial_z \phi = e^{i\delta} \partial \bar{W} / \partial \bar{\phi}, \quad \partial_z \chi = e^{i\delta} \partial \bar{W} / \partial \bar{\chi}
\]

associated with the TVY lagrangian. The phase \( \delta \) depends on particular vacua between which the wall interpolates (see Refs.[5, 10, 9] for details).

To study the spectrum of the domain walls which are not BPS–saturated, one has to solve the equations of motion which are of the second order, and, technically, the problem is a little bit more involved. We did it earlier for \( N = 2 \) [8]. This paper is devoted to the numerical solution of the equations of motion for the \( SU(3) \) gauge group.

2 Solving equations of motion

The scalar potential corresponding to the superpotential (3) is

\[
U(\phi, \chi) = \left| \frac{\partial W}{\partial \phi} \right|^2 + \left| \frac{\partial W}{\partial \chi} \right|^2 = 4 \left| \phi^2 \ln(\phi^3 \chi^{2(N-1)}) \right|^2 + (N-1)^2 \left| m \chi - \frac{4\phi^3}{3\chi} \right|^2
\]

(from now on we set \( \Lambda_{\text{SQCD}} \equiv 1 \)). The potential (11) has \( N + 1 \) degenerate minima. One of them is chirally symmetric: \( \phi = \chi = 0 \). There are also \( N \) chirally asymmetric vacua with \( \langle \chi \rangle_k \) given in Eq.(3) and

\[
\langle \phi \rangle_k = \left( \frac{3m}{4} \right)^{(N-1)/(3N)} e^{-\frac{2(N-1)\phi k}{3N}}
\]
To study the domain wall configurations, we should add to the potential (11) the kinetic term which we choose in the simplest possible form

\[ \mathcal{L}_{\text{kin}} = |\partial \phi|^2 + |\partial \chi|^2 \]  

(13)

and solve the equations of motion with boundary conditions

\[ \phi(-\infty) = \langle \phi \rangle_0 \equiv R_*, \quad \phi(\infty) = R_* e^{-2\pi i (N-1)/3N} \]

\[ \chi(-\infty) = \langle \chi \rangle_0 \equiv \rho_*, \quad \chi(\infty) = \rho_* e^{\pi i / N} \]  

(14)

Thereby we are studying the walls interpolating between “adjacent” complex vacua. Actual calculations will be performed for \( N = 3 \) where all the vacua are adjacent. In principle, one could also study numerically the walls interpolating between the vacua \( k = 0 \) and \( k = 2 \) for, say, \( SU(5) \) gauge group, etc. We expect the physical results for all such walls to be qualitatively the same.

It is convenient to introduce the polar variables \( \chi = \rho e^{i\alpha}, \phi = R e^{i\beta} \) after which the equations of motion acquire the form

\[ R'' - R\beta'^2 = 8R^3[L(L + 3/2) + \beta^2_+] + (N - 1)^2 \left[ 16R^5 \frac{3}{3\rho^2} - 4mR^2 \cos(\beta_-) \right] \]

\[ R\beta'' + 2R'\beta' = 12R^3\beta_+ + 4(N - 1)^2mR^2 \sin(\beta_-) \]

\[ \rho'' - \rho\alpha'^2 = (N - 1)^2 \frac{8R^4}{\rho} L + (N - 1)^2 \left( m^2 \rho - \frac{16R^6}{9\rho^3} \right) \]

\[ \rho\alpha'' + 2\rho'\alpha' = (N - 1)^2 \frac{8R^4}{\rho} \beta_+ - (N - 1)^2 \frac{8mR^3}{3\rho} \sin(\beta_-) \]  

(15)

where \( L = \ln[R^2\rho^{2(N-1)}] \), \( \beta_+ = 3\beta + 2(N - 1)\alpha \), \( \beta_- = 3\beta - 2\alpha \), with the boundary conditions

\[ \rho(-\infty) = \rho(\infty) = \rho_*; \quad R(-\infty) = R(\infty) = R_*; \]

\[ \alpha(-\infty) = \beta(-\infty) = 0; \quad \alpha(\infty) = \pi / N; \quad \beta(\infty) = -\frac{2(N - 1)\pi}{3N} \]  

(16)

When \( N = 2 \), the system (15) is reduced to that studied in Ref. [8]. The system (15) involves one integral of motion

\[ T - U = R'^2 + \rho'^2 + R^2\beta'^2 + \rho^2\alpha'^2 - 4R^4(L^2 + \beta^2_+) - (N - 1)^2 \left[ m^2 \rho^2 + \frac{16R^6}{9\rho^2} - \frac{8mR^3}{3} \cos(\beta_-) \right] = \text{const} \]  

(17)

In our case, \( \text{const} = 0 \) due to boundary conditions (16). The phase space of the system (15) is 8–dimensional and a general Cauchy problem involves 8 initial conditions. The problem is simplified, however, when noting that the wall solution should
Figure 1: The ratio $\eta = R(0)/R_0(0)$ for the solutions of the equations of motion as a function of mass for the $SU(3)$ theory. The solid lines describe the BPS solutions and the dashed lines describe the non–BPS wall and the sphalerons.

Indeed, one can be easily convinced that the Ansatz (18) goes through the equations (15). It is convenient to solve the equations (15) numerically on the half–interval from $z = 0$ to $z = \infty$. The symmetry (18) dictates $\rho'(0) = R'(0) = 0$, $\alpha(0) = \pi/(2N)$, $\beta(0) = -(N-1)\pi/(3N)$ which fixes 4 initial conditions. Four others satisfy the relation (17). Thus, we are left with 3 free parameters, say, $\rho(0)$, $R(0)$, and $\beta'(0)$, which should be fitted so that the solution approach the complex vacuum in Eq.(14) at $z \to \infty$.

All the solutions obtained with such a procedure are presented in Fig. 1 where the parameter $R(0)$, one of the fitted initial conditions, is plotted as a function of mass (we normalized $R(0)$ at its value $R_0(0) = .918 \ldots R_*$ for the upper BPS branch in the limit $m \to 0$; see Ref. [9] for details). For small masses, there are several solutions. We obtain first of all the solutions studied in Ref. [9] and describing the BPS–saturated domain walls. (solid lines in Fig. 1). We find also two new solution branches drawn with the dashed lines in Fig. 1. We see that, similarly to the BPS branches, two new dashed branches fuse together at some $m = m_{**} \approx 3.704$. No solution for the system (15) exist at $m > m_{**}$.

The picture is rather analogous to what we had for $N = 2$ [8] and the physical
interpretation is similar. Let us assume first that \( m < m_\ast \) and draw the energy functional \( E \) for field configurations with wall boundary conditions minimized over all parameters except the value of \( R(0) \) which is kept fixed (see Fig. 2a). For very small \( R(0) \), our configuration nearly passes the chirally symmetric minimum and the minimum of the energy corresponds to two widely separated real walls. Thus \( E[R(0) = 0] = 2\epsilon_r \) with \( \epsilon_r \) given in Eq.(8). Two minima in Fig. 2a correspond to BPS solutions with the energy \( \epsilon_c = \sqrt{3}\epsilon_r \). They are separated by an energy barrier. The top of this barrier (actually, this is a saddle point with only one unstable mode corresponding to \( R(0) \), in other words — a sphaleron) is a solution described by the upper dashed line in Fig. 1. The lower dashed line corresponds to the local maximum on the energy barrier separating the lower BPS branch and the configuration of two distant real walls at \( R(0) = 0 \).

At \( m = m_\ast \), two BPS minima fuse together and the energy barrier separating them disappears. The upper sphaleron branch coincides with the BPS solution at this point. When \( m \) is increased above \( m_\ast \), the former BPS minimum is still a minimum of the energy functional, but its energy is now above the BPS bound (see Fig. 2b). The corresponding solution is described by the analytic continuation of the upper sphaleron branch. The lower dashed branch in the region \( m < m_\ast < m < m_\ast^\ast \) is

\[ E[R(0) = 0] = 2\epsilon_r \]

In contrast to the case \( N = 2 \), the existence of such a barrier could not be established from the BPS spectrum alone. The matter is that, while \( 2\epsilon_r = \epsilon_c \) for \( N = 2 \) and the presence of the maximum is guaranteed by the Roll theorem, \( 2\epsilon_r > \epsilon_c \) in our case and one could in principle imagine a situation where \( E \) falls down monotonically when \( R(0) \) is increased from zero up to its value at the lower BPS branch. As will be discussed later, our numerical results are not good enough to make a rigid statement on the form of the function \( E[R(0)] \) at small \( R(0) \) in the region \( m < m_\ast \), but they strongly suggest that the energy barrier (though a tiny one) is present.
still a sphaleron. At the second critical point \( m = m_{**} \), the picture is changed again (see Fig.2c). The local maximum and the local minimum fuse together and the only one remaining stationary point does not correspond to an extremum of the energy functional anymore. At larger masses, no non-trivial stationary points are left.

Our findings are illustrated in Figs. 3, 4 where the energies of the non-BPS wall and of the lower sphaleron branch are plotted as a function of \( m \). In Fig. 3, the ratios of the energies of both branches to the BPS bound \((9)\) are plotted. The lower line in Fig. 3 corresponds to the stable wall solution and the upper line to the sphaleron branch. For almost all \( m < m_{**} \), the wall solution is globally stable. When \( m_{**} - m \) becomes very small, it is stable only locally: we see that, at \( m = m_{**} \) where two branches are fused together, their energy exceeds slightly the energy of two real walls \( 2\epsilon_r \).

In Fig. 4 the sphaleron energy is redrawn in logarithmic scale in the units of \( 2\epsilon_r \). Unfortunately, we do not have good numerical data at \( m \lesssim 0.7 \) — our relative uncertainty becomes large. We see, however, that the logarithmic plot in Fig. 4 is pretty much linear, the best fit is

\[
\kappa'(m) = 5.49 \cdot 10^{-7} \exp\{2.11m\} \tag{19}
\]

The fit \((19)\) cannot be valid for very small masses: we expect \( \kappa'(0) = 0 \) which means that the straight line in Fig. 4 should bend down at small enough masses due to a preexponential factor \( \sim m^\alpha \) which we cannot determine from our data. Anyway, it is seen from Fig. 4 that \( \ln \kappa' \) does not “want” to hit infinity at a finite mass, and we assume that it does not (though we cannot exclude it as a logical possibility).
Figure 4: Logarithmic plot for the ratio $\kappa'(m) = \frac{E_{\text{lower sphaleron}}}{2\epsilon_r} - 1$ vs. mass.

In terms of Fig. 2, that means that the energy barrier on the left (for illustrative purposes, it is very much exaggerated) is present for all masses.

Finally, we present for illustration the profile $R(z)$ for the lower sphaleron branch at $m = .7$. As was expected, it resembles very much a combination of two separate real domain walls.

3 Discussion.

Our main conclusion is that, besides the critical mass $m_\star$ beyond which BPS solutions disappear, also a second critical mass $m_{\star\star}$ exists beyond which no complex wall solution can be found whatsoever. This was the case for $SU(2)$ \[8\] and, as we see now, this is also the case for $SU(3)$. Seemingly, the same situation holds for any $N$. That means in particular that no domain walls connecting different chirally asymmetric vacua are left in the pure supersymmetric Yang–Mills theory corresponding to the limit $m \to \infty$, and only the real domain walls connecting the chirally symmetric and a chirally asymmetric vacuum states survive in this limit. That contradicts an assumption of Ref.\[11\] that it is complex rather than real domain walls which are present in the pure SYM theory (Witten discussed them in the context of brane dynamics).

One has to make a reservation here: our result was obtained in the framework of the TVY effective lagrangian \[3\] whose status [in contrast to that of the lagrangian \[3\]] is not absolutely clear: the field $\Phi$ describes heavy degrees of freedom (viz. a scalar glueball and its superpartner) which are not nicely separated from all the rest. However, the form of the superpotential \[3\] and hence the form of the lagrangian
for static field configurations is rigidly dictated by symmetry considerations; the uncertainty involves only kinetic terms. It is reasonable to assume that, as far as the vacuum structure of the theory is concerned (but not e.g. the excitation spectrum — see Ref. for detailed discussion), the effective TVY potential can be trusted. A recent argument against using Eq. that the chirally symmetric phase whose existence follows from the TVY lagrangian does not fulfill certain discrete anomaly matching conditions is probably not sufficient. First, it assumes that the excitation spectrum in the symmetric phase is the same as it appears in the TVY lagrangian which is not justified. Second, it was argued recently that the TVY lagrangian describes actually all the relevant symmetries of the underlying theory and the absence of the anomaly matching is in a sense an “optical illusion”.

The main distinction of the $SU(3)$ case considered here compared to the $SU(2)$ theory is that the values of two critical values are rather different (in $SU(2)$ case they were pretty close: $m_\ast = 4.67059 \ldots$ and $m_{\ast\ast} \approx 4.83$). This is due to the fact that the energy of the complex BPS wall $\epsilon_c$ is less in this case than the energy of two real walls $2\epsilon_r$. When we increase the mass and go above $m_\ast$, the energy of the wall first has to rise from $\epsilon_c$ to $2\epsilon_r$. Only then the complex domain wall “bound state” can break apart into its “constituents”, the real walls.

One can expect that $m_\ast$ and $m_{\ast\ast}$ differ more and more as $N$ grows. A tentative guess is that $m_{\ast\ast}$ is roughly $N$–independent (to be compared with $m_\ast(N) \propto N^{-3}$). Of course, that can be confirmed or disproved by only actual numerical study. Note, however, that numerical calculations become more and more difficult as $N$ grows.

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4 Actually, as we have seen, the energy barrier in Fig. does not allow the complex wall to break apart until its energy goes a little bit above the limit $2\epsilon_r$. 
the instabilities characterized by eigenvalues of the Jacobi matrix of the system (14) near the minima (3, 12) grow as $N^2$.

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