Generalized oscillator representations for Calogero Hamiltonians

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Abstract
This paper is a natural continuation of the previous paper (Gitman et al 2011 J. Phys. A: Math. Theor. 44 425204), where oscillator representations for nonnegative Calogero Hamiltonians with coupling constant \( \alpha \geq -1/4 \) were constructed. In this paper, we present generalized oscillator representations for all Calogero Hamiltonians with \( \alpha \geq -1/4 \). These representations are generally highly nonunique, but there exists an optimum representation for each Hamiltonian.

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1. Introduction
This paper is a natural continuation of the previous papers [6] and [2]. In [6] (see also section 7.2 in [5] to which we mainly refer in what follows), all one-particle Calogero Hamiltonians \( \hat{H}_c \) associated with the self-adjoint (referred to as s.a. in what follows) Calogero differential operation

\[
\hat{H} = -d^2_x + \alpha x^2, \quad x \in \mathbb{R}_+, \quad \alpha \in \mathbb{R},
\]

where \( \alpha \) is a dimensionless coupling constant, were constructed as s.a. operators in \( L^2(\mathbb{R}_+) \) and their spectra and (generalized) eigenfunctions were evaluated including inversion formulae. In [2], the so-called oscillator representations for nonnegative Calogero Hamiltonians \( \hat{H}_c \), \( \hat{H}_c \geq 0 \), with \( \alpha \geq -1/4 \) were constructed. An oscillator representation is a representation of the form

\[
\hat{H}_c = \hat{c}^+ \hat{c},
\]

where \( \hat{c} \) and \( \hat{c}^+ \) are a pair of closed mutually adjoint first-order differential operators, \( \hat{c}^+ = (\hat{c})^*, \hat{c} = (\hat{c}^+)^* \). Such a representation makes it evident that the Hamiltonian \( \hat{H}_c \) is nonnegative. Here, the results in [2] are generalized to all Calogero Hamiltonians with \( \alpha \geq -1/4 \), generally not nonnegative, in the form of generalized oscillator representations.

The initial basic ideas of the generalization are as follows.

As is known, all Calogero Hamiltonians with \( \alpha \geq -1/4 \) are bounded from below [5]. From the general standpoint, \( \hat{c} \) and \( \hat{c}^+ \) are of the previous meaning; \( \hat{c} \) is an oscillator representation. Such representations are one more aspect of the general Calogero problem. They can be useful for many reasons, including the spectral analysis of the Hamiltonians. In particular, representation (3) for a Hamiltonian \( \hat{H}_c \) makes it evident that \( \hat{H}_c \) is bounded from below, and its spectrum is bounded from below by \( -\langle sk_0 \rangle^2 \), which is the lower boundary of the spectrum if the kernel of the operator \( \hat{c} \) is nontrivial, \( \ker \hat{c} \neq \{0\} \); then \( \ker \hat{c} \) is the groundstate (ground state) of the Hamiltonian and \( E_0 = -\langle sk_0 \rangle^2 \) is its ground-state energy.

A starting point for constructing oscillator representations (2) was the oscillator representation for this is a consequence of the fact that the initial symmetric operator \( \hat{H} \) associated with \( \hat{H}_c \) with \( \alpha \geq -1/4 \) and defined on the subspace \( D(\mathbb{R}_+) \) of smooth compactly supported functions is nonnegative and therefore all its s.a. extensions \( \hat{H}_c \), which are just the Calogero Hamiltonians, are bounded from below [1, 7]. This implies that for each Calogero Hamiltonian \( \hat{H}_c \) with \( \alpha \geq -1/4 \), there exists a nonnegative constant \( u \) such that the operator \( \hat{H}_c + u^2 \hat{I} \), where \( \hat{I} \) is the identity operator, is nonnegative and may allow an oscillator representation of the form (1). The operator \( \hat{H}_c + u^2 \hat{I} \) is evidently associated with the differential operation \( \hat{H} + u^2 \). The parameter \( u \) is dimension of inverse length, and it is convenient to represent it as \( u = sk_0 \), where \( s \geq 0 \) is a dimensionless parameter and \( k_0 > 0 \) is a fixed parameter of the dimension of inverse length. By a generalized oscillator representation for the Calogero Hamiltonian \( \hat{H}_c \), we mean a representation of the form

\[
\hat{H}_c = \hat{c}^+ \hat{c} - (sk_0)^2 \hat{I}, \quad s \geq 0,
\]
the respective differential operation $\hat{H}$ that is a representation of the form

$$\hat{H} = \hat{b} \hat{a},$$

$$\hat{b} = \hat{a}^* = (\hat{b}^*)^*, \quad \hat{a} = (\hat{a}^*)^*,$$

where $\hat{a}$ is a first-order differential operation and $\hat{a}^*$ is its adjoint by Lagrange, see [1]. Accordingly, a starting point for constructing generalized oscillator representations (3) for Calogero Hamiltonians with the coupling constant $\alpha \geq -1/4$ should be a generalized oscillator representation

$$\hat{H} = \hat{b} \hat{a} - (sk_0)^2, \quad s \geq 0,$$  \hspace{1cm} (4)

for the respective $\hat{H}$ with certain $\hat{a}$ and $\hat{b}$ of the previous meaning.

Let differential operation $\hat{H}$ (1) allow generalized oscillator representation (4). If we introduce the pair of initial differential operators $\hat{a}$ and $\hat{b}$ in $L^2(\mathbb{R}_\alpha)$ defined on $\mathcal{D}(\mathbb{R}_\alpha)$ and associated with the pair of the respective differential operations $\hat{a}$ and $\hat{b}$, then the initial symmetric operator $\hat{H}$ is evidently represented as

$$\hat{H} = \hat{b} \hat{a} - (sk_0)^2 \hat{1}.$$  \hspace{1cm} (5)

Let $\hat{c}$ and $\hat{c}^*$ be a pair of closed mutually adjoint operators that are closed extensions of the respective initial operators $\hat{a}$ and $\hat{b}$, $\hat{a} \subset \hat{c}, \hat{b} \subset \hat{c}^*$. Then the operator

$$\hat{H}_{\text{ext}} = \hat{c}^* \hat{c} - (sk_0)^2 \hat{1}$$

is an evident extension of $\hat{H}$, $\hat{H} \subset \hat{H}_{\text{ext}}$. By the von Neumann theorem [4] (for a proof, see also [1]), the operator $\hat{N} = \hat{c}^* \hat{c}$, where $\hat{c}$ is closed, $\hat{c}^* = \hat{c}^*$ is s.a. and nonnegative, $\hat{N} = \hat{N}^* \geq 0$; in addition, if $\ker \hat{c} \neq \{0\}$, then $\ker \hat{c}$ is an eigenspace with the minimum eigenvalue 0 which is the lower boundary of the spectrum of $\hat{N}$. Therefore, the operator $\hat{H}_{\text{ext}}$ (5) is s.a., which means that $\hat{H}_{\text{ext}}$ is a certain Calogero Hamiltonian $\hat{H}_c$ represented in the generalized oscillator form (3) providing its above-mentioned properties.

We note that the generalized oscillator representation (3) for a Calogero Hamiltonian is equivalent to the representation

$$\hat{H}_c = \hat{d} \hat{a}^* - (sk_0)^2 \hat{1},$$

where $\hat{d}$ and $\hat{a}^*$ are a pair of closed mutually adjoint operators that are extensions of the respective initial operators $\hat{b}$ and $\hat{a}$: it is sufficient to make the identifications $\hat{c} = \hat{a}^*, \hat{c}^* = \hat{d}$. Constructing a pair $\hat{c} \supset \hat{a}, \hat{c}^* \supset \hat{b}$ or a pair $\hat{d} \supset \hat{b}, \hat{d}^* \supset \hat{a}$ is a matter of convenience: we can start with extending $\hat{a}$ to its closure or with extending $\hat{b}$ to its closure.

Varying the parameter $s$ in (4) and involving all possible mutually adjoint extensions of the initial $\hat{a}$ and $\hat{b}$ with given $s$, we can hope to construct generalized oscillator representations (3) for all Calogero Hamiltonians with $\alpha \geq -1/4$. We show below that these expectations are realized. An identification of Hamiltonians $\hat{H}_c$ (3) with the known Calogero Hamiltonians in [5] is straightforward for $\alpha \geq 3/4$ because the Hamiltonian with given $\alpha \geq 3/4$ is unique, while for $-1/4 \leq \alpha < 3/4$, an identification is achieved by evaluating the asymptotic behavior of functions belonging to the domain of $\hat{H}_c$ (3) at the origin and comparing it with the asymptotic boundary conditions specifying different Calogero Hamiltonians with given $\alpha \in [-1/4, 3/4]$ [5].

We say in advance that the generalized oscillator representation (3) for a given Calogero Hamiltonian is generally highly nonunique; in fact, there exists a one- or even two-parameter family of generalized oscillator representations for each Hamiltonian, among which there exists an optimum representation.

As to generalized oscillator representations for Calogero Hamiltonians with the coupling constants $\alpha < -1/4$, there are no such representations and there cannot be because these Hamiltonians are not bounded from below [5].

2. General

We begin by looking into the possibility of generalized oscillator representation (4) for the Calogero differential operation $\hat{H}$ (1), i.e. representing $\hat{H}$ as a product of two finite-order differential operations mutually adjoint by Lagrange minus a nonnegative constant:

$$\hat{H} = \hat{b} \hat{a} - (sk_0)^2, \quad s \geq 0,$$

$$\hat{b} = \hat{a}^* = (\hat{b}^*)^*, \quad \hat{a} = (\hat{a}^*)^*,$$  \hspace{1cm} (6)

the fixed parameter $k_0$ is of the dimension of inverse length. In fact, we deal with a family $[\hat{a}(s), \hat{b}(s)]$ of mutually adjoint by Lagrange differential operations. We often omit the argument $s$ in the symbols $\hat{a}$ and $\hat{b}$ for brevity and write it when needed. The representation (6) with $s = 0$ is an oscillator representation for $\hat{H}$; it was considered in [2]. It is desirable that $\hat{a}(s)$ and $\hat{b}(s)$ be continuous in $s$; in particular, it is desirable to obtain the known oscillator representations for $\hat{H}$ from generalized oscillator representations (6) for $\hat{H}$ in the limit $s \to 0$.

It is easy to see that $\hat{a}$ and $\hat{b}$ have to be first-order differential operations,

$$\hat{a} = e^{i\theta(x)}[d_x - h(x)], \quad \hat{b} = -[d_x + \overline{h(x)}]e^{-i\theta(x)}.$$  \hspace{1cm} (7)

The substitution of (7) into (6) results in the following necessary and sufficient conditions on the function $h(x)$ for representation (6) to hold:

$$\text{Im}\ h(x) = 0, \quad h'(x) + h^2(x) = ax^{-2} + s^2k_0^2,$$

which is the Ricatti equation. We additionally require that the functions $\theta(x)$ and $h(x)$ be nonsingular in $(0, \infty)$. The arbitrary phase factors $e^{i\theta(x)}$ in (7) are irrelevant because they trivially cancel in the product $\hat{b} \hat{a}$; their fixing is a matter of convenience. We set $\theta(x) = 0$, so that in what follows, $\hat{a}$ and $\hat{b}$ in representation (6) are given by

$$\hat{a} = d_x - h(x), \quad \hat{b} = -d_x - h(x),$$  \hspace{1cm} (8)

$$\text{Im}\ h(x) = 0, \quad h'(x) + h^2(x) = ax^{-2} + s^2k_0^2.$$  \hspace{1cm} (9)

We recall that in fact, we have a family $[\hat{a}(s) = d_x - h(x); s], \hat{b}(s) = -d_x - h(x); s]$ of mutually adjoint first-order differential operations and the corresponding family $[h(s); s]$ of functions $h$. 

2. General
The representation \((6), (8)\) and \((9)\) with given \(s\), if it exists, is generally nonunique: if nonlinear equation \((9)\) with given \(s\) has a family of different admissible solutions \(h\), there is a family of the respective different pairs \(\tilde{a}, \tilde{b}\) \((8)\) providing the desired representation, so that apart from \(s\), the symbols \(\tilde{a}\) and \(\tilde{b}\) can contain a certain additional argument, let it be \(\mu\), parametrizing the family of admissible \(h\) with given \(s\), and we actually have a two-parameter family \(\{h(s, \mu, x)\}\) and the respective family \(\{\tilde{a}(s, \mu, \cdot), \tilde{b}(s, \mu, \cdot)\}\). Where possible, this parametrization must be such that it might provide a smooth transition to the limit \(s \to 0\) that reproduces the known oscillator representations for \(\tilde{H}\). We write the arguments \(s\) and \(\mu\) where needed and omit them for brevity if this does not lead to misunderstanding.

It is easy to prove that differential operation \(\tilde{H}\) \((1)\) allows the generalized oscillator representation \((6), (8)\) and \((9)\) iff the homogeneous differential equation
\[
-\phi''(x) + \frac{\alpha}{x^2} \phi(x) + (sk_0)^2 \phi(x) = 0
\]
(10)
or the eigenvalue problem
\[
\tilde{H} \phi = -\phi''(x) + \frac{\alpha}{x^2} \phi(x) = -(sk_0)^2 \phi(x)
\]
(which can be considered as a stationary Schrödinger equation with ‘energy’ \(-(sk_0)^2\)) has a real-valued positive solution (with no zeros in \((0, \infty)\)),
\[
\text{Im} \phi(x) = 0, \quad \phi(x) > 0, \quad x > 0,
\]
and in this case
\[
h(x) = \phi'(x)/\phi(x) = -\phi(x) \left( \frac{1}{\phi(x)} \right),
\]
so that \(\tilde{a}\) and \(\tilde{b}\) allow the representations
\[
\tilde{a} = \phi(x) d_x \left( \frac{1}{\phi(x)} \right), \quad \tilde{b} = -\frac{1}{\phi(x)} d_x \phi(x).
\]
(11)
It is evident that the function \(\phi\) is defined up to a positive constant factor.

We also note that by \(\phi(x)\) is actually meant a family \(\{\phi(s; x)\}\); we write the argument \(s\) when needed.

\textbf{Necessity.} Let \(\tilde{H}\) allow the representation \((6), (8)\) and \((9)\) with a function \(h(x)\) absolutely continuous in \((0, \infty)\). We introduce the real-valued positive function \(\phi(x)\) defined up to a positive constant factor by
\[
\phi(x) = \exp \int_1^x d\xi \ h(\xi),
\]
so that the function \(h(x)\) can be represented as
\[
h(x) = \phi'(x)/\phi(x) = -\phi(x) \left( \frac{1}{\phi(x)} \right).
\]
It is easy to verify that equation \((9)\) for \(h(x)\) implies equation \((10)\) for \(\phi(x)\).

\textbf{Sufficiency.} Let \(\phi(x)\) be a real-valued positive solution of equation \((10)\). It is easy to verify that the function \(h(x) = \phi'(x)/\phi(x)\) is absolutely continuous in \((0, \infty)\) and satisfies equation \((9)\), thus providing the representation \((6), (8)\) and \((9)\) for \(\tilde{H}\).

We thus obtain that existence and a possible nonuniqueness of the representation \((6)\) are formulated in terms of equation \((10)\) as follows. If equation \((10)\) has no real-valued positive solution, there exists no representation \((6)\). If equation \((10)\) has a unique, up to a positive constant factor, real-valued positive solution \(\phi\), there exists a unique representation \((6), (11)\). If equation \((10)\) has two linearly independent real-valued positive solutions \(\phi_1\) and \(\phi_2\), there exists a one-parameter family \(\{\tilde{a}, \tilde{b}\}\) of different admissible pairs \(\tilde{a}, \tilde{b}\) providing the desired representation. This family is in one-to-one correspondence given by \((11)\) with a family \(\{A \phi_1 + B \phi_2, A, B : = -A \phi_1(x) + B \phi_2(x) > 0, x > 0\}\) of pairwise linearly independent real-valued positive solutions of equation \((10)\) defined modulo a positive constant factor: the constant coefficients \(A\) and \(B\) that differ by a positive constant factor yield the same pair \(\tilde{a}, \tilde{b}\). The range of admissible coefficients \(A\) and \(B\) has to be established. Where possible, a single parameter, let it be \(\mu\), parametrizing these coefficients, \(A = A(\mu)\) and \(B = B(\mu)\), must be introduced in such a way as to provide a proper oscillator representation for \(\tilde{H}\) in the limit \(s \to 0\).

The general solution of equation \((10)\) with \(s > 0\) is given by
\[
\phi(s; x) = A \sqrt{x} I_{\sigma}(sk_0 x) + B \sqrt{x} K_{\sigma}(sk_0 x),
\]
\[
\sigma = \begin{cases} \sqrt{\alpha + 1/4} \geq 0, & \alpha \geq -1/4, \\ i \sigma, & \sigma = \sqrt{|\alpha| - 1/4} > 0, \quad \alpha < -1/4 \end{cases}
\]
(12)
or by
\[
\phi(s; x) = A \sqrt{x} I_{\sigma}(sk_0 x) + B \sqrt{x} I_{-\sigma}(sk_0 x)
\]
(13)
if \(\sigma \notin \mathbb{Z}\).

Here, \(I_{\pm\sigma}\) are the modified first-order Bessel functions and \(K_{\sigma}\) is the McDonald function (another name for these functions is the Bessel functions of imaginary argument), and \(A\) and \(B\) are arbitrary complex constants. Whether the right-hand sides in \((12)\) or in \((13)\) can be real-valued and positive under an appropriate choice of the coefficients \(A\) and \(B\) crucially depends on the value of the coupling constant \(\alpha\). Two regions of the coupling constant differ drastically, \(\alpha < -1/4\) and \(\alpha \geq -1/4\), which we consider separately. In the second region, the point \(\alpha = -1/4\) is naturally distinguished.

We first consider the region \(\alpha < -1/4\) where a situation with generalized oscillator representations is simplest.

\textbf{3. Region} \(\alpha < -1/4\) \((\kappa = i \sigma)\)

In this region of the coupling constant, we use form \((13)\) with \(\kappa = i \sigma\) of the general solution of equation \((10)\) with \(s > 0\),
\[
\phi(s; x) = A \sqrt{x} I_{\sigma}(sk_0 x) + B \sqrt{x} I_{-\sigma}(sk_0 x),
\]
\[
\sigma = \sqrt{|\alpha| - 1/4} > 0.
\]
(14)
Because the functions \(I_{\pm\sigma}\) of the real argument are linearly independent and complex conjugate, \(I_{-\sigma}(sk_0 x) = \overline{I_{\sigma}(sk_0 x)}\), the condition \(\text{Im} \phi(sk_0; x) = 0\) requires that
In this region of the coupling constant, we use form (10) of the form

\[ \phi(s; x) = 2\Re[A \sqrt{1} I_\nu(s k_0 x)]. \]

The asymptotic behavior of such a function as \( x \to 0 \) is

\[ \phi(s; x) = 2\Re \left\{ \rho \phi(s; x) \right\} = 2\sqrt{\pi} \left[ 1 + O(x^2) \right] \cos(\sigma \ln(0) + \varphi + O(x^2)), \]

\[ \rho \phi(s; x) = \frac{A(1/2)^{\nu}}{\Gamma(1 + i \sigma)}, \]

which demonstrates that any real-valued solution of equation (10) with \( \alpha < -1/4 \) and \( s > 0 \) has an infinite number of zeros in any neighborhood of the origin. This means that the Calogero differential operation \( \bar{H} \) of equation (1) with \( \alpha < -1/4 \) does not allow generalized oscillator representation (6) with \( s > 0 \).

We recall that the same holds for an oscillator representation for \( \bar{H} \), which corresponds to the case \( s = 0 \), see [2]. We note that to have the general solution of equation \( \bar{H} \phi = 0 \) from the general solution (14) of equation (10) with \( s \neq 0 \) as a proper limit \( s \to 0 \), it is sufficient to make the substitutions \( A \to \bar{A} \omega, B \to \bar{B} \omega \).

4. Region \( \alpha > -1/4 \) (\( \kappa > 0 \))

4.1. Generalized oscillator representations for \( \bar{H} \), differential operations \( \bar{a} \) and \( \bar{b} \)

In this region of the coupling constant, we use form (12) of the general solution of equation (10) with \( s > 0 \). To include the point \( s = 0 \) in the range of admissible \( s \), \( s > 0 \), smoothly, it is appropriate to make the substitutions

\[ A \to A \Gamma(1 + \kappa) \left( \frac{s}{2} \right)^{-\frac{\alpha}{\kappa}} \sqrt{k_0}, \quad B \to B \frac{2}{\Gamma(1 + \kappa)} \left( \frac{s}{2} \right)^{\frac{\alpha}{\kappa}} \sqrt{k_0}. \]

Under these substitutions, the general solution of equation \( \bar{H} \phi = 0 \) is properly obtained from the general solution (12) of equation (10) with \( s \neq 0 \) in the limit \( s \to 0 \), which allows us to obtain the known oscillator representations for \( \bar{H} \) (1), see [2], from generalized oscillator representations (6), (11) with \( s > 0 \) in the limit \( s \to 0 \).

The general solution of equation (10) with \( \alpha > -1/4 \) and \( s > 0 \) is then given by

\[ \phi(s; x) = A \Gamma(1 + \kappa) \left( \frac{s}{2} \right)^{-\frac{\alpha}{\kappa}} \sqrt{k_0} I_\nu(s k_0 x) \]

\[ + B \frac{2}{\Gamma(1 + \kappa)} \left( \frac{s}{2} \right)^{\frac{\alpha}{\kappa}} \sqrt{k_0} K_\nu(s k_0 x), \quad x = \frac{\alpha}{\kappa} + 1/4 > 0. \]

(15)

Because \( \sqrt{k_0} I_\nu(s k_0 x) \) and \( \sqrt{k_0} K_\nu(s k_0 x) \) are real-valued and linearly independent solutions, a condition \( \Im \phi(s; x) = 0 \) requires that \( \Im A = \Im \bar{B} = 0 \). The function \( I_\nu(s k_0 x) \) is positive in \( (0, \infty) \) and monotonically increases from zero at \( x = 0 \) to infinity as \( x \to \infty \) (see [3, 8.432.1, 8.486.16, 8.486.11, 8.451.6, 8.485, 8.466]), whereas the function \( K_\nu(s k_0 x) \) is positive in \( (0, \infty) \) and monotonically decreases from infinity at \( x = 0 \) to zero as \( x \to \infty \) (see [3, 8.432.1, 8.486.16, 8.486.11, 8.451.6, 8.485, 8.446]).

It follows that \( \phi(s; x) \) (15) is real-valued and positive in \( (0, \infty) \) if \( A > 0, B > 0, A + B > 0 \). As noted above, a common constant positive factor in \( \phi(s; x) \) is irrelevant because it does not enter the generalized oscillator representation (6), (11) for \( \bar{H} \).

To make this evident, we can set \( A = \sin \mu, B = \cos \mu, \mu \in [0, \pi/2] \), in (15) without loss of generality.

Thus, for each \( \alpha > -1/4, (\kappa > 0) \), we have a one-parameter family \( \{\phi(s, \mu; x)\} \) of pairwise linearly independent real-valued positive solutions \( \phi(s, \mu; x) \) of equation (10) with \( s > 0 \) and the corresponding two-parameter family \( \{\bar{a}(s, \mu, \nu), \bar{b}(s, \mu)\} \) of different pairs of mutually adjoint first-order differential operations \( \bar{a}(s, \mu, \nu) \) and \( \bar{b}(s, \mu) \) given by (11) with \( \phi = \phi(s, \mu, \nu) \),

\[ \bar{a}(s, \mu, \nu) = \bar{b}^*(s, \mu, \nu) = \frac{d_x}{\nu} - h(\mu, s; x), \]

\[ = \mu \Gamma(1 + \kappa) \left( \frac{s}{2} \right)^{-\frac{\alpha}{\kappa}} \sqrt{k_0} I_\nu(s k_0 x) \]

\[ + \cos \mu \left( \frac{s}{2} \right)^{\frac{\alpha}{\kappa}} K_\nu(s k_0 x), \]

\[ h(\mu, s; x) = \frac{\phi'(s, \mu; x)}{\phi(s, \mu; x)}, \]

\[ \phi(\mu, s; x) = \sin \mu \Gamma(1 + \kappa) \left( \frac{s}{2} \right)^{-\frac{\alpha}{\kappa}} \sqrt{k_0} I_\nu(s k_0 x) \]

\[ + \cos \mu \left( \frac{s}{2} \right)^{\frac{\alpha}{\kappa}} K_\nu(s k_0 x), \]

\[ \mu \in \left[ 0, \frac{\pi}{2} \right], \quad s \in [0, \infty), \quad \kappa > 0, \]

(16)

providing a two-parameter family of different generalized oscillator representations (6) for \( \bar{H} \) (1) with given \( \alpha > -1/4, \kappa > 0 \).

As a rule, we indicate the ranges of parameters \( \mu \) and \( s \) in formulae to follow only if they differ from the whole ranges; these are \( [0, \pi/2] \) for \( \mu \), and \( [0, \infty) \) for \( s \); the range of \( \kappa \) is clear from the title of the section, subsection or subsubsection.

As to the main resulting formulae, we indicate the ranges of all the parameters including \( \kappa \).

For the asymptotic behavior of the functions \( \phi(s, \mu; x) \) and the functions \( 1/\phi(s, \mu, \nu; x) \) at the origin, which we need below, we have

\[ \phi(s, \mu, \nu; x) = \left\{ \begin{array}{ll}
\cos \mu \left( \frac{s}{2} \right)^{\frac{\alpha}{\kappa}} \left[ 1 + O(x^{2\kappa}) \right], & \mu \in [0, \pi/2),
\cos \mu \left( \frac{s}{2} \right)^{\frac{\alpha}{\kappa}} + O(x^{3/2} \ln x), & \kappa > 1,
\cos \mu \left( \frac{s}{2} \right)^{\frac{\alpha}{\kappa}} + O(x^{3/2} \ln x), & \mu \in [0, \pi/2),
\cos \mu \left( \frac{s}{2} \right)^{\frac{\alpha}{\kappa}} + O(x^{3/2} \ln x), & \kappa = 1, x \to 0,
\cos \mu \left( \frac{s}{2} \right)^{\frac{\alpha}{\kappa}} + O(x^{3/2} \ln x), & \mu \in [0, \pi/2), 0 < \kappa < 1,
\end{array} \right. \]

\[ \bar{A}(s, \mu, \nu) = \sin \mu \Gamma(1 - \kappa) \left( \frac{s}{2} \right)^{\frac{\alpha}{\kappa}} \cos \mu \]

(18)
and
\[
\frac{1}{\phi(\mu, s; x)} = \begin{cases}
\frac{1}{\cos^\mu (k_0 x)^{1/2}} \left[ 1 + O(x^2) \right], & \mu \in [0, \pi/2), \\
\frac{1}{\cos^\mu (k_0 x)^{1/2} + O(x^{5/2} \ln 1/x)}, & \mu \in [0, \pi/2), \\
\frac{1}{\cos^\mu (k_0 x)^{1/2} + O(x^{3/2})}, & \mu \in [0, \pi/2), \quad x > 1,
\end{cases}
\]
\[
\frac{1}{\cos^\mu (k_0 x)^{1/2}} \left[ 1 + O(x^2) \right], & \mu = \pi/2,
\]
\[
\frac{1}{\cos^\mu (k_0 x)^{1/2} + O(x^{5/2} \ln 1/x)}, & \mu = \pi/2,
\]
\[
\frac{1}{\cos^\mu (k_0 x)^{1/2} + O(x^{3/2})}, & \mu = \pi/2, \quad x = 1, \quad x \to 0.
\]

For the respective functions \(h(\mu, s; x) = \phi'(\mu, s; x)/\phi(\mu, s; x)\), we have
\[
h(\mu, s; x) = \frac{1}{2x} + sk_0 \left[ \sin \mu \Gamma(1 + x) \left( \frac{1}{2} \right)^{1/2} \right]^N I_n'(sk_0 x) + \cos \mu \Gamma(1 + x) \left( \frac{1}{2} \right)^{1/2} K_n'(sk_0 x),
\]
\[
h(\mu, s; x) = \left\{ \begin{array}{ll}
sk_0 + O(x^{-1}), & \mu \in (0, \pi/2), \\
-\sk_0 + O(x^{-1}), & \mu = 0,
\end{array} \right. \quad x \to \infty,
\]
\[
(\psi, \hat{a}\xi) = (\hat{b}\psi, \xi), \quad \forall \psi, \xi \in D(\mathbb{R}^+),
\]
which is easily verified by integration by parts.

It is evident from (17) that the initial symmetric operator \(\hat{H}\) with \(\alpha > -1/4\) associated with \(\hat{H}\) and defined on \(D(\mathbb{R}^+)\) can be represented as
\[
\hat{H} = \hat{b}(\mu, s) \hat{a}(\mu, s) - (sk_0)^2 \hat{I}.
\]

These representations provide a basis for constructing generalized oscillator representations for all s.a. Calogero Hamiltonians \(\hat{H}_e\) with \(\alpha > -1/4\) in accordance with the program formulated in section 1. Namely, we should construct all possible extensions of any pair \(\hat{a}(\mu, s), \hat{b}(\mu, s)\) of initial operators with given \(\mu\) and \(s\) to a pair of closed mutually adjoint operators \(\hat{c}(\mu, s), \hat{c}^\ast(\mu, s), \hat{a}(\mu, s) \subset \hat{c}(\mu, s), \hat{b}(\mu, s) \subset \hat{c}^\ast(\mu, s)\), or equivalently, to a pair of closed mutually adjoint operators \(\hat{a}(\mu, s), \hat{d}(\mu, s), \hat{d}^\ast(\mu, s) \subset \hat{a}(\mu, s), \hat{b}(\mu, s) \subset \hat{d}^\ast(\mu, s)\). These extensions produce the respective operators
\[
\hat{H}_{e(\mu, s)} = \hat{c}^\ast(\mu, s) \hat{c}(\mu, s) - (sk_0)^2 \hat{I}
\]
and
\[
\hat{H}_{e, \hat{d}(\mu, s)} = \hat{d}(\mu, s) \hat{d}^\ast(\mu, s) - (sk_0)^2 \hat{I}.
\]

The arguments \(a\) and \(b\) are written in accordance with the natural conventions denoted by \(D^\ast\) for special reasons, see [1, 5, 7].
\( \hat{f} \psi(x) \) has a sense\(^3\) and is also square integrable, \( D^n_f = \{ \psi(x) : \hat{\psi} \in L^2(R_n) \} \). The natural domain is the domain of the maximum operator associated with a given differential operation.

We outline the results of the evaluation. The operators \( \hat{a}^* \) and \( \tilde{b} \) are associated with the differential operation \( \hat{b} = \tilde{a}^* \), whereas the operators \( \tilde{b}^* \) and \( \tilde{a} \) are associated with the differential operation \( \tilde{a} = \hat{b}^* \). We therefore dwell on the domains of the operators involved, which either coincide with or belong to the natural domains for the respective differential operations.

1. The domain of the operator \( \hat{a}^* \) is the natural domain for\(^4\) \( \tilde{b} \):

\[
D_{\hat{a}^*} = D_{\tilde{b}}^* = \{ \psi(x) : \psi \text{ is a.c. in } R_+ \},
\]

\[
\tilde{b} \psi(x) = - \frac{1}{\phi(x)} \frac{d}{dx} (\phi(x) \psi(x)) = - \psi' + h \psi = \eta \in L^2(R_+). \tag{26}
\]

Because the functions \( h \) are bounded at infinity, see (20), it is immediately established in a standard way that the functions \( \psi \) belonging to \( D_{\tilde{b}}^* \) vanish at infinity,

\[
\psi(x) \to 0, \quad x \to \infty, \quad \forall \psi \in D_{\tilde{b}}^*. \tag{27}
\]

A generic function \( \psi \) belonging to \( D_{\tilde{b}}^* \) can be considered as the general solution of the inhomogeneous differential equation

\[
\tilde{b} \psi(x) = \eta(x) \tag{28}
\]

under the additional conditions that both the \( \psi(x) \) and \( \eta(x) \) are square integrable on \( R_+ \). Therefore, taking (18) and (19) into account, \( \psi \) belonging to \( D_{\tilde{b}}^* \) allows the representation

\[
\psi(x) = \frac{1}{\phi(\mu, s; x)} \left[ C - \int_{x_0}^x dy \phi(\mu, s; y) \eta(y) \right],
\]

\[
\eta(x) = \tilde{b} \psi(x) \in L^2(R_+),
\]

\( x_0 = 0 \) for \( 0 < \kappa < 1 \) and for \( \mu = \pi/2, \kappa \geq 1 \); \( x_0 > 0 \)

for \( \mu \in [0, \pi/2), \kappa \geq 1 \),

\( C \) is an arbitrary constant for \( \mu \in [0, \pi/2); \)

\( C = 0 \) for \( \mu = \pi/2 \). \tag{29}

Estimating the integral term in (29) with the Cauchy–Bunyakovskii inequality, we obtain that the asymptotic behavior of functions \( \psi \in D_{\tilde{b}}^* \) at the origin is
given by

\[
\psi(x) = \begin{cases} 
O(x^{1/2}), & \kappa > 1, \\
O \left( {x^{1/2} \ln^{1/2} x} \right), & \mu \in [0, \pi/2), \kappa = 1, \\
O(x^{1/2}), & \mu = \pi/2, \\
O \left( \frac{\zeta(x^2)}{\cos \mu} \right) + O(x^{1/2}), & x \to 0,
\end{cases}
\]

where \( \zeta(x) \) is a fixed smooth function equal to 1 in the neighborhood of the origin and equal to 0 for \( x \geq x_\infty > 0 \), and \( D_{\tilde{b}}^* \) is the subspace of functions belonging to \( D_{\tilde{b}}^* \) and vanishing at the origin:

\[
\tilde{D}_{\tilde{b}}^* = \left\{ \psi(x) \in D_{\tilde{b}}^* : \psi(x) = O(x^{1/2}), \quad x \to 0 \right\},
\]

\( \psi \in [0, \pi/2), \quad 0 < \kappa < 1 \). \tag{31}

The final result is given by

\[
D_{\hat{a}^*} = \left\{ D_{\tilde{b}}^* : \mu \in [0, \pi/2), \kappa \geq 1 \right\}
\]

\[
\tilde{D}_{\hat{b}} = \left\{ \chi(x) : \chi \text{ is a.c. in } R_+ ; \chi, \tilde{a} \chi = \phi d_s \left( \frac{1}{\phi} \chi \right) = \chi' - h \chi = \eta \in L^2(R_+) \right\}. \tag{33}
\]

By the same reasoning as for the case of functions \( \psi \) belonging to \( D_{\tilde{b}}^* \), the functions \( \chi \) belonging to \( D_{\tilde{b}}^* \) vanish at infinity,

\[
\chi(x) \to 0, \quad x \to \infty, \quad \forall \chi \in D_{\tilde{b}}^*. \tag{34}
\]
Using arguments similar to those for a generic function \( \psi \) belonging to \( D^n_{a(\mu,s)} \), with the natural interchange \( \phi \leftrightarrow 1/\phi \), we establish that a generic function \( \chi \) belonging to \( D^n_{a(\mu,s)} \) allows the representation

\[
\chi(x) = \phi(\mu, s; x) \left[ D + \int_{x_0}^{\infty} dy \frac{1}{\phi(\mu, s; y)} \eta(y) \right],
\]

\[
\eta(x) = \tilde{a} \chi(x) \in L^2(\mathbb{R}^+_s),
\]

\( x_0 = 0 \) for \( \mu \in [0, \pi/2) \); \( x_0 > 0 \) for \( \mu = \pi/2 \),

\( D \) is an arbitrary constant for \( \mu = \pi/2, \kappa \geq 1 \)

and for \( 0 < \kappa < 1; D = 0 \) for \( \mu \in [0, \pi/2), \kappa \geq 1 \).

The asymptotic behavior of functions \( \chi \in D^n_{a(\mu,s)} \) at the origin is given by

\[
\chi(x) = \begin{cases} 
O(x^{1/2}), & \kappa \geq 1, \\
D \cos(\mu k_0 x)^{1/2-x} + O(x^{1/2}), & \mu \in [0, \pi/2), \quad 0 < \kappa < 1, \quad x \to 0, \\
O(x^{1/2}), & \mu = \pi/2,
\end{cases}
\]

(35)

We note that for \( 0 < \kappa < 1 \), the natural domain \( D^n_{a(\mu,s)} \) for \( \tilde{a}(\mu, s) \) with \( \mu \in [0, \pi/2) \) can be represented as a direct sum of the form

\[
D^n_{a(\mu,s)} = \{ D\varphi_0(\mu, s) \} + \tilde{D}^n_{a(\mu,s)},
\]

\( \mu \in [0, \pi/2), \quad 0 < \kappa < 1, \quad (36) \)

where the function \( \varphi_0(\mu, s; x) \) belonging to \( D^n_{a(\mu,s)} \) is given by

\[
\varphi_0(\mu, s; x) = \phi(\mu, s; x) \zeta(x), \quad \text{so that}
\]

\[
\tilde{a}(\mu, s) \varphi_0(\mu, s; x) = \phi(\mu, s; x) \zeta'(x),
\]

\( \zeta(x) \) is a fixed smooth function equal to 1 in the neighborhood of the origin and equal to 0 for \( x \geq x_\infty > 0 \), and \( \tilde{D}^n_{a} \) is the subspace of functions belonging to \( D^n_{a} \) and vanishing at the origin:

\[
\tilde{D}^n_{a(\mu,s)} = \{ \chi(x) \in D^n_{a(\mu,s)} : \chi(x) = O(x^{1/2}), \quad x \to 0 \},
\]

\( \mu \in [0, \pi/2), \quad 0 < \kappa < 1. \)

(37)

The final result is given by

\[
D^{n}_{a(\mu,s)} = \begin{cases} 
D^n_{a(\mu,s)}, & \mu \in [0, \pi/2) \quad \kappa \geq 1 \quad \text{and} \\
D^n_{a(\mu,s)} & \mu = \pi/2, \quad \kappa > 0, \quad \text{and} \\
D^n_{a(\mu,s)} = \{ D\varphi_0(\mu, s) \} + \tilde{D}^n_{a(\mu,s)}, & \mu \in [0, \pi/2), \quad 0 < \kappa < 1,
\end{cases}
\]

(38)

with \( D^n_{a(\mu,s)} \) given by (33) and with estimates (34) at infinity and (36) at the origin.

3. The operator \( \tilde{a} \) is evaluated in accordance with (25), \( \bar{a} = (\bar{a}^*)^* \subset \bar{b}^* \): as a restriction of \( \bar{b}^* \), this operator is associated with \( \tilde{a} \) and its domain belongs to or coincides with \( D^n_{\bar{a}} \), while the defining equation for \( \bar{a} \) as \( (\bar{a}^*)^* \),

\[
(\chi, \bar{a}^* \psi - (\bar{a} \chi, \psi) = 0, \quad \chi \in D_{\bar{a}}, \quad \forall \psi \in D^n_{\bar{b}},
\]

is reduced to the equation for \( D_{\bar{a}} \), i.e. for the functions \( \chi \in D_{\bar{a}} \subset D^n_{\bar{a}} \), of the form

\[
(\chi, \bar{b}^* \psi - (\bar{a} \chi, \psi) = 0, \quad \chi \in D_{\bar{a}} \subset D^n_{\bar{a}}, \quad \forall \psi \in D^n_{\bar{b}}.
\]

(40)

Integrating by parts in \( (\bar{a} \chi, \psi) \) and taking asymptotic estimates (27), (34) and (36) into account, we establish that for \( \mu \in [0, \pi/2), \kappa \geq 1 \) and for \( \mu = \pi/2, \kappa > 0 \), equation (40) holds identically for all \( \chi \in D^n_{a(\mu,s)} \), while for \( \mu \in [0, \pi/2), \quad 0 < \kappa < 1 \), equation (40) is reduced to

\[
\bar{D}_{\bar{a}} C = 0, \quad \forall C,
\]

which requires that \( D = 0 \).

We finally obtain that

\[
\tilde{a}(\mu, s) = \bar{b}^*(\mu, s), \quad \mu \in [0, \pi/2), \quad \kappa \geq 1 \quad \text{and} \quad \mu = \pi/2, \quad \kappa > 0,
\]

in particular, \( D_{\bar{a}(\mu,s)} = \tilde{D}^n_{a(\mu,s)} \)

(41)

and

\[
\tilde{a}(\mu, s) \subset \bar{b}^*(\mu, s), \quad D_{\bar{a}(\mu,s)} = \tilde{D}^n_{a(\mu,s)} \quad \text{for} \quad 0 < \kappa < 1.
\]

(42)

4. Quite similarly, we find that

\[
\tilde{b}(\mu, s) = (\bar{b}^*)^*(\mu, s), \quad \mu \in [0, \pi/2), \quad \kappa \geq 1 \quad \text{and} \quad \mu = \pi/2, \quad \kappa > 0,
\]

in particular, \( D_{\bar{a}(\mu,s)} = \tilde{D}^n_{a(\mu,s)} \)

(43)

and

\[
\tilde{b}(\mu, s) \subset (\bar{b}^*)^*(\mu, s), \quad \tilde{D}^n_{a(\mu,s)} = \tilde{D}^n_{b(\mu,s)} \quad \text{for} \quad 0 < \kappa < 1.
\]

(44)

We note that equality (43) and inclusion (44) directly follow from the respective previous equality (41) and inclusion (42) by taking the adjoints, and only the domain \( D_{\bar{a}(\mu,s)} \) in the last case has to be evaluated.

We thus show that each pair \( \tilde{a}(\mu, s), \tilde{b}(\mu, s) \) of mutually adjoint by Lagrange differential operations (16) providing generalized oscillator representation (17) for \( \hat{H} \) (1) with \( \alpha > -1/4 \) (\( \kappa > 0 \)) generates a unique pair \( \bar{a}(\mu, s) = \tilde{b}^*(\mu, s), \quad \bar{a}^*(\mu, s) = \tilde{b}(\mu, s) \) of closed mutually adjoint operators for \( \mu \in [0, \pi/2), s \in [0, \infty), \quad \kappa \geq 1 \) and for \( \mu = \pi/2, s \in [0, \infty), \quad \kappa > 0 \), while for \( \mu \in [0, \pi/2), s \in [0, \infty), \quad 0 < \kappa < 1 \), each
pair $\tilde{a}(\mu, s), \tilde{b}(\mu, s)$ generates two different pairs $\tilde{a}(\mu, s), \tilde{a}^*(\mu, s)$ and $\tilde{b}^*(\mu, s), \tilde{b}(\mu, s)$ of closed mutually adjoint operators such that $\tilde{a}(\mu, s) \subseteq \tilde{b}^*(\mu, s)$ and $\tilde{b}(\mu, s) \subseteq \tilde{a}^*(\mu, s)$. The operators $\tilde{a}(\mu, s)$ and $\tilde{b}^*(\mu, s)$ are extensions of the initial operator $\tilde{a}(\mu, s)$, they are associated with $\tilde{a}$, and their domains are given by the respective (41), (42) and (39), (33). The operators $\tilde{b}(\mu, s)$ and $\tilde{a}^*(\mu, s)$ are extensions of the initial operator $\tilde{b}(\mu, s)$, they are associated with $\tilde{b}$, and their domains are given by the respective (43), (44) and (32), (26).

It is easy to prove that there are no other pairs of closed mutually adjoint operators that are extensions of each pair $\tilde{a}(\mu, s), \tilde{b}(\mu, s)$. Indeed, let $\tilde{g}, \tilde{g}^*$ be such a pair; then because $\tilde{a}(\mu, s)$ and $\tilde{b}(\mu, s)$ are minimum closed extensions of the respective $\tilde{a}(\mu, s)$ and $\tilde{b}(\mu, s)$, we have

$$\tilde{a}(\mu, s) \subseteq \tilde{a}(\mu, s) \subseteq \tilde{g} = (\tilde{g}^*)^*, \quad \tilde{b}(\mu, s) \subseteq \tilde{b}(\mu, s) \subseteq \tilde{g}^*.$$  

It follows by taking the adjoints of these inclusions that

$$\tilde{g}^* \subseteq \tilde{a}^*(\mu, s), \quad \tilde{g} \subseteq \tilde{b}^*(\mu, s),$$

so that we finally have

$$\tilde{a}(\mu, s) \subseteq \tilde{g} \subseteq \tilde{b}^*(\mu, s),$$

in particular,

$$\mathcal{D}_{\tilde{a}(\mu, s)} \subseteq \mathcal{D}_g \subseteq \mathcal{D}_{\tilde{b}^*(\mu, s)}.$$  

It then directly follows from (41) that $\tilde{g} = \tilde{a}(\mu, s) = \tilde{b}^*(\mu, s)$ and therefore $\tilde{g}^* = \tilde{b}(\mu, s) = \tilde{a}^*(\mu, s)$ for $\mu \in [0, \pi/2), s \in [0, \infty), x > 1$ and for $\mu = \pi/2, s \in [0, \infty), x > 0$, while for $\mu \in [0, \pi/2), s \in [0, \infty), 0 < x < 1$, it follows from (42), (37) that the domains $\mathcal{D}_{\tilde{a}(\mu, s)} = \mathcal{D}_{\tilde{g}(\mu, s)}$ and $\mathcal{D}_{\tilde{a}(\mu, s)} = \mathcal{D}_{\tilde{g}(\mu, s)}$ differ by a one-dimensional subspace, so that either $\tilde{g} = \tilde{a}(\mu, s)$, and therefore $\tilde{g}^* = \tilde{a}^*(\mu, s)$, or $\tilde{g} = \tilde{b}^*(\mu, s)$, and therefore $\tilde{g}^* = \tilde{b}(\mu, s)$.

5. The point $\alpha = -1/4 (x = 0)$

A consideration in this section is completely similar to that in the previous section for the case of $\alpha > -1/4 (x > 0)$. A difference with the previous case is that for $\alpha = -1/4$, the inclusion of the point $s = 0$ in the range of admissible $s$ and getting the known oscillator representation for $\tilde{H}$, see [2], from generalized oscillator representations (6) with $s > 0$ in the limit $s \to 0$ calls for a special investigation.

We therefore distinguish the case of $s > 0$ and the case of $s = 0$.

5.1. The region $s > 0$

5.1.1. Generalized oscillator representations for $\tilde{H}$ and differential operations $\tilde{a}$ and $\tilde{b}$. For the coupling constant $\alpha = -1/4$, we use form (12) with $x = 0$ of the general solution of equation (10) with substitutions $A \to A\sqrt{K_0}, B \to B\sqrt{K_0}$,

$$\phi(s; x) = A\sqrt{K_0}I_0(sK_0x) + B\sqrt{K_0}K_0(sK_0x).$$  

Because $\sqrt{K_0}I_0(sK_0x)$ and $\sqrt{K_0}K_0(sK_0x)$ are real-valued linearly independent solutions, a condition $\text{Im } \phi(x) = 0$ requires that $\text{Im } A = \text{Im } B = 0$. The function $I_0(sK_0x)$ monotonically increases from 1 at $x = 0$ to infinity as $x \to \infty$, whereas the function $K_0(sK_0x)$ monotonically decreases from infinity at $x = 0$ to 0 as $x \to \infty$. It follows that $\phi(s; x)$ (45) is real-valued and positive in $(0, \infty)$ iff $A \geq 0, B \geq 0, A + B > 0$. Once again, a common constant positive factor in coefficients $A$ and $B$ is irrelevant from the standpoint of the generalized oscillator representation (6), (11) for $\tilde{H}$, so that we can set $A = \sin \mu, B = \cos \mu, \mu \in [0, \pi/2]$ without loss of generality.

As a result, we have a two-parameter family $\{\tilde{a}(\mu, s), \tilde{b}(\mu, s)\}$ of different pairs of mutually adjoint first-order differential operations,

$$\tilde{a}(\mu, s) = \tilde{b}^*(\mu, s) = d_x - h(\mu, s; x)$$

$$= \phi(\mu, s; x)d_x, \quad = \phi(\mu, s; x),$$

$$\tilde{b}(\mu, s) = \tilde{a}^*(\mu, s) = -d_x - h(\mu, s; x)$$

$$= -\frac{1}{\phi(\mu, s; x)}d_x, \quad = \phi(\mu, s; x),$$

$$h(\mu, s; x) = \phi(\mu, s; x), \quad \phi(\mu, s; x) = \sin \mu \sqrt{K_0}I_0(sK_0x) + \cos \mu \sqrt{K_0}K_0(sK_0x),$$

$$\mu \in \left[0, \frac{\pi}{2}\right], \quad s > 0,$$  

providing a two-parameter family of different generalized oscillator representations (6) for $\tilde{H}$ (1) with $\alpha = -1/4$,

$$\tilde{H} = \tilde{b}(\mu, s)\alpha(\mu, s) - (sK_0)^2.$$  

As before, we indicate the ranges of parameters $\mu$ and $s$ in the formulae to follow only if they differ from the whole ranges; these are $[0, \pi/2]$ for $\mu$ and $(0, \infty)$ for $s$. In the main resulting formulae, we indicate the ranges of the both parameters.

For the asymptotic behavior of the functions $\phi(\mu, s; x)$ and the functions $1/\phi(\mu, s; x)$ at the origin, which we need below, we have

$$\tilde{A}(\mu, s) = \tilde{A}(\mu, s) - \cos \mu \sqrt{K_0} \ln(sK_0x) + \cos \mu \sqrt{K_0} \ln(sK_0x),$$

$$\phi(\mu, s; x) = \begin{cases} \tilde{A}(\mu, s)\sqrt{K_0} \ln(sK_0x) - \cos \mu \sqrt{K_0} \ln(sK_0x) + O(x^{5/2} \ln x), \mu \in [0, \pi/2), x \to 0, \\ \sqrt{K_0} + O(x^{3/2}), \mu = \pi/2, \end{cases}$$

$$\tilde{A}(\mu, s) = \sin \mu + \cos \mu \psi(1) - \ln(sK_0x),$$  

and

$$\frac{1}{\phi(\mu, s; x)} = \begin{cases} -\frac{1}{\sin \mu \sqrt{K_0} \ln(sK_0x)} - \frac{1}{\cos \mu \sqrt{K_0} \ln(sK_0x)}, \mu \in [0, \pi/2), x \to 0, \\ +O(x^{3/2} \ln x), \mu = \pi/2, \end{cases}$$

$$\frac{1}{\sqrt{K_0}} + O(x^{3/2}), \mu = \pi/2,$$  

and
For the respective functions
\[ h(\mu, s; x) = \phi'(\mu, s; x)/\phi(\mu, s; x), \]
we have
\[ h(\mu, s; x) = \frac{1}{2s} + \frac{\sin \mu I_1(s_k)x - \cos \mu K_1(s_k)x}{\sin \mu I_0(s_k)x + \cos \mu K_0(s_k)x} = \begin{cases} s_k + O(x^{-1}), & \mu \in [0, \pi/2], \\ -s_k + O(x^{-1}), & \mu = 0, \end{cases}, \quad x \to \infty. \]

### 5.1.3 Adjoint operators

We introduce the initial differential operators \( \hat{a}(\mu, s) \) and \( \hat{b}(\mu, s) \) associated with each pair of the respective differential operators \( \hat{a}(\mu, s) \) and \( \hat{b}(\mu, s) \) (46) and defined on \( D(\mathbb{R}_+) \). These operators and the initial symmetric operator \( \hat{H} \) with \( \alpha = -1/4 \) have the properties that are copies of (21) and (22), with the change \( s \to s, \in [0, \infty) \) to \( s \in (0, \infty) \), which provides a basis for constructing generalized oscillator representations similar to (23), (24) for all s.a. Calogero Hamiltonians \( \hat{H}_s \) with \( \alpha = -1/4 \) by constructing all possible extensions of any pair \( \hat{a}(\mu, s), \hat{b}(\mu, s) \) of initial operators to a pair of closed mutually adjoint operators.

A procedure for extending presented below is completely similar to that in the previous section for the case of \( \alpha > -1/4 \). Once again, we omit the arguments \( \mu \) and \( s \) of operators \( \hat{a} \) and \( \hat{b} \) for brevity, writing them when needed.

### 5.1.2 Initial operators \( \hat{a} \) and \( \hat{b} \)

Using arguments similar to those in section 4.3, we prove that the operators \( \hat{a} \) and \( \hat{b} \) have adjoints, the respective \( \hat{a}^* \), \( \hat{b}^* \), and closures, the respective \( \hat{a}^* \) and \( \hat{b}^* \), which form the chains of inclusions similar to (25),

\[ \hat{a} \subset \hat{a} = (\hat{a}^*)^* \leq \hat{b}^* \leq \hat{b} = (\hat{b}^*)^* \leq \hat{a}^*. \] (49)

An evaluation of the operators \( \hat{a}^*, \hat{b}^* \), \( \hat{a} \) and \( \hat{b} \) is completely similar to that in section 4.4 for the case of \( \alpha > -1/4 \). The operators \( \hat{b} \) and \( \hat{a}^* \) are associated with the differential operator \( \hat{b} = \hat{a}^* \), while the operators \( \hat{a} \) and \( \hat{b}^* \) are associated with the differential operator \( \hat{a} = \hat{b}^* \).

1. The domain of the operator \( \hat{a}^* \) is the natural domain for \( \hat{b}, D_{\hat{a}^*} = D^a_\infty \), given by a copy of (26). A generic function \( \psi \) belonging to \( D^a_\infty \) allows the representation
\[
\psi(x) = \frac{1}{\phi(\mu, s; x)} \left[ \left. C - \int_0^x dy \frac{1}{\phi(\mu, s; y)} \eta(y) \right] \right.,
\]
\[ \eta(x) = \hat{b}\psi(x) \in L^2(\mathbb{R}_+), \]
\[ C \text{ is an arbitrary constant for } \mu \in [0, \pi/2), \]
\[ C = 0 \text{ for } \mu = \pi/2. \]

The asymptotic behavior of functions \( \psi \in D^a_\infty \) at infinity and at the origin is given by
\[
\psi(x) \to 0, \quad x \to \infty,
\]
\[
\psi(x) = \begin{cases} 
\frac{1}{x^{\cos \mu}} \left[ 1 + O(\frac{1}{|\ln x|}) \right], & \mu \in [0, \pi/2), \\
O(x^{1/2}), & \mu = \pi/2,
\end{cases}
\quad x \to 0.
\]

The natural domain \( D^a_{\hat{a}(\mu, s)} \) for \( \hat{a}(\mu, s) \) with \( \mu \in [0, \pi/2) \) can be represented as a direct sum of the form
\[
D^a_{\hat{a}(\mu, s)} = \{ C\psi(\mu, s) + \hat{D}^a_{\hat{a}(\mu, s)} \}, \quad \mu \in [0, \pi/2),
\]
where the function \( \psi(\mu, s; x) \) belonging to \( \hat{D}^a_{\hat{a}(\mu, s)} \) is given by
\[
\psi(\mu, s; x) = \frac{1}{\phi(\mu, s; x)} \xi(x), \quad \text{so that}
\]
\[ \hat{b}(\mu, s)\psi(\mu, s; x) = -\frac{1}{\phi(\mu, s; x)} \xi'(x), \]
\[ \xi(x) \text{ is a fixed smooth function equal to } 1 \text{ in the neighborhood of the origin and equal to } 0 \text{ for } x \geq x_\infty > 0,
\]
and \( \hat{D}^a_{\hat{a}(\mu, s)} \) is the subspace of functions belonging to \( D^a_{\hat{a}(\mu, s)} \) and vanishing at the origin:
\[
\hat{D}^a_{\hat{a}(\mu, s)} = \{ \psi(x) \in D^a_{\hat{a}(\mu, s)} : \psi(x) = O(x^{1/2}), \quad x \to 0 \},
\]
\[ \mu \in [0, \pi/2). \] (51)

The final result is given by
\[
D^a_{\hat{a}(\mu, s)} = \left\{ \begin{array}{l}
\{ C\psi(\mu, s) + \hat{D}^a_{\hat{a}(\mu, s)} \}, \\
\mu \in [0, \pi/2),
\end{array} \right.
\]
\[ \hat{D}^a_{\hat{a}(\mu, s)} \}, \quad \mu = \pi/2, \] (52)
with \( \hat{D}^a_{\hat{a}(\mu, s)} \) given by a copy of (26) and with estimates (50) at infinity and at the origin.

2. The domain of the operator \( \hat{b}^* \) is the natural domain for \( \hat{a}, D_{\hat{b}^*} = D^b_\infty \), given by a copy of (33). A generic function \( \chi \) belonging to \( D^b_\infty \) allows the representation
\[
\chi(x) = \phi(\mu, s; x) \left[ D + \int_{x_0}^x dy \frac{1}{\phi(\mu, s; y)} \eta(y) \right],
\]
\[ \eta(x) = \hat{a}\chi(x) \in L^2(\mathbb{R}_+), \\
x_0 = 0 \quad \text{for } \mu \in [0, \pi/2), \quad x_0 > 0 \quad \text{for } \mu = \pi/2, \\
D \text{ is an arbitrary constant.} \] (53)

The asymptotic behavior of functions \( \chi \in D^b_\infty \) at infinity and at the origin is given by
\[
\chi(x) \to 0, \quad x \to \infty,
\]
\[
\chi(x) = \left\{ \begin{array}{l}
- D \cos \mu \sqrt{k_0} \ln(k_0x) + O(x^{1/2} \ln^{1/2}(\frac{1}{x})), \\
\mu \in [0, \pi/2),
\end{array} \right.
\]
\[ O(x^{1/2} \ln^{1/2}(\frac{1}{x})), \quad \mu = \pi/2, \] (54)
The natural domain \( D^b_{\hat{b}(\mu, s)} \) for \( \hat{b}(\mu, s) \) with \( \mu \in [0, \pi/2) \) can be represented as a direct sum of the form
\[
D^b_{\hat{b}(\mu, s)} = \{ D\chi(\mu, s) + \hat{D}^b_{\hat{b}(\mu, s)} \}, \quad \mu \in [0, \pi/2),
\]
where the function $\chi_0(\mu, s; x)$ belonging to $D^\mu_{\hat{a}(\mu, s)}$ is given by

$$\chi_0(\mu, s; x) = \phi(\mu, x, \zeta(x), x) = \phi(\mu, s; x) \zeta'(x),$$

$\zeta(x)$ is a fixed smooth function equal to 1 in the neighborhood of the origin and equal to 0 for $x \geq x_{\infty} > 0$, and $\hat{D}^\mu_{\hat{a}(\mu, s)}$ is the subspace of functions belonging to $D^\mu_{\hat{a}(\mu, s)}$ and vanishing at the origin:

$$\hat{D}^\mu_{\hat{a}(\mu, s)} = \left\{ \psi(x) \in D^\mu_{\hat{a}(\mu, s)} : \chi(x) = O \left( x^{1/2} \ln^{1/2} \left( \frac{1}{x} \right) \right), x \to 0 \right\}, \quad \mu \in [0, \pi/2).$$

The final result is given by

$$D^\mu_{\hat{b}(\mu, s)} = \left\{ \begin{array}{ll} D^\mu_{\hat{a}(\mu, s)} = \{ D\chi_0(\mu, s, x) \} + \hat{D}^\mu_{\hat{a}(\mu, s)}, \\
\hat{D}^\mu_{\hat{a}(\mu, s)}, \quad \mu \in [0, \pi/2), \\
D^\mu_{\hat{a}(\mu, s)}, \quad \mu = \pi/2, \end{array} \right.$$  

with $D^\mu_{\hat{a}(\mu, s)}$ given by a copy of (33) and with estimates (54) at infinity and at the origin.

3. The operator $\hat{a}$ is evaluated in accordance with (49), $\hat{a} = (\hat{a}^*)^*$, as a restriction of $\hat{b}^*$, this operator is associated with $\hat{a}$ and its domain belongs to or coincides with $D^\mu_{\hat{a}}$, while the defining equation for $\hat{a}$ as $(\hat{a}^*)^*$ is reduced to the equation for its domain $D_a \subseteq D^\mu_{\hat{a}}$ of the form

$$(\chi, \hat{b}\psi) - (\hat{a}\chi, \psi) = 0, \quad \chi \in D_a \subseteq D^\mu_{\hat{a}}, \forall \psi \in D^\mu_{\hat{a}}.$$  

Integrating by parts in $(\hat{a}\chi, \psi)$ and taking asymptotic estimates (50) and (54) into account, we establish that for $\mu = \pi/2$, equation (57) holds identically for all $\chi \in D^\mu_{\hat{a}(\pi/2, s)}$, while for $\mu \in [0, \pi/2)$, equation (57) is reduced to

$$\mathcal{T}C = 0, \quad \forall C,$$

which requires that $D = 0$.

We finally obtain that

$$\overline{\hat{a}}(\pi/2, s) = \hat{b}^*(\pi/2, s),$$

in particular, $D_{\hat{a}(\pi/2, s)} = D^\mu_{\hat{a}(\pi/2, s)}$  

and

$$\overline{\hat{a}}(\mu, s) \subset \hat{b}^*(\mu, s), \quad D_{\hat{a}(\mu, s)} = \hat{D}^\mu_{\hat{a}(\mu, s)}, \quad \mu \in [0, \pi/2).$$

4. Quite similarly, we find that

$$\overline{\hat{b}}(\pi/2, s) = \hat{a}^*(\pi/2, s),$$

in particular, $D_{\hat{b}(\pi/2, s)} = D^\mu_{\hat{b}(\pi/2, s)}$  

and

$$\overline{\hat{b}}(\mu, s) \subset \hat{a}^*(\mu, s), \quad D_{\hat{b}(\mu, s)} = \hat{D}^\mu_{\hat{b}(\mu, s)}, \quad \mu \in [0, \pi/2).$$

We note that equality (60) and inclusion (61) directly follow from the respective previous equality (58) and inclusion (59) by taking the adjoints, and only the domain $D_{\hat{a}(\mu, s)}$ in the last case has to be evaluated.

We thus show that each pair $\hat{a}(\mu, s), \hat{b}(\mu, s)$ of mutually adjoint by Lagrange differential operations (46) providing generalized oscillator representations (47) for $\hat{H} (1)$ with $\alpha = -1/4 (x = 0)$ generates a unique pair $\overline{a}(\pi/2, s) = \hat{b}^*(\pi/2, s), \overline{a}^*(\pi/2, s) = \overline{b}(\pi/2, s)$ of closed mutually adjoint operators for $\mu = \pi/2, s \in (0, \infty)$, while for $\mu \in [0, \pi/2), s \in (0, \infty)$, each pair $\hat{a}(\mu, s), \hat{b}(\mu, s)$ generates two different pairs $\overline{a}(\mu, s), \overline{a}^*(\mu, s)$ and $\overline{b}(\mu, s), \overline{b}^*(\mu, s)$ of closed mutually adjoint operators such that $\overline{a}(\mu, s) \subset \overline{b}^*(\mu, s)$ and $\overline{b}(\mu, s) \subset \overline{a}^*(\mu, s)$. The operators $\overline{a}(\mu, s)$ and $\overline{b}^*(\mu, s)$ are extensions of the initial operator $\hat{a}(\mu, s)$, they are associated with $\hat{a}$, and their domains are given by the respective (58), (59) and (56), the operators $\overline{b}(\mu, s)$ and $\overline{a}^*(\mu, s)$ are extensions of the initial operator $\hat{b}(\mu, s)$, they are associated with $\hat{b}$, and their domains are given by the respective (60), (61) and (52).

Using arguments similar to those in section 4.4, it is easy to prove that there are no other pairs of closed mutually adjoint operators that are extensions of each pair $\hat{a}(\mu, s), \hat{b}(\mu, s)$.

5.2. The point $s = 0$

In this case, the general real-valued solution of equation (10) is given by

$$\phi(x) = A\sqrt{k_0x} + B\sqrt{k_0x} \ln(k_0x)$$

$$= \left\{ \begin{array}{ll} B\sqrt{k_0x} \ln(k_0x), & B \neq 0, \\
A\sqrt{k_0x}, & B = 0, \end{array} \right.$$  

$$\sigma = e^{A/B}, \quad \text{Im}A = \text{Im}B = 0.$$  

It is evident that the real-valued $\phi(x)$ is positive in $(0, \infty)$ if $A > 0, B = 0$. Because a constant positive factor is irrelevant from the standpoint of generalized oscillator representation (6), (11) for $\hat{H}$, we can set $A = 1$ without loss of generality.

As a result, we have a unique pair $\hat{a}, \hat{b}$ of mutually adjoint first-order differential operations,

$$\hat{a} = \hat{b}^* = d_i - \frac{1}{2x} = \phi(x) d_i \frac{1}{\phi(x)}, \quad \hat{b} = \hat{a}^* = -d_i - \frac{1}{2x} = -\frac{1}{\phi(x)} d_i \phi(x),$$

$$h(\mu, s; x) = \frac{\phi'(x)}{\phi(x)} = \frac{1}{2x}, \quad \phi(x) = \sqrt{k_0x}.$$  

providing a unique oscillator representation (6) with $s = 0$ for $\hat{H}$ with $\alpha = -1/4$.

$$\hat{H} = \hat{h}\hat{a}.$$  

We then introduce the initial operators $\hat{a}$ and $\hat{b}$ associated with the respective $\hat{a}$ and $\hat{b}$ and construct the pairs of operators $\overline{\hat{a}}, \overline{\hat{a}}^*, \overline{\hat{b}}, \overline{\hat{b}}^*$ as all possible extensions of the pair $\hat{a}, \hat{b}$.
to a pair of mutually adjoint closed operators. The procedure follows the standard way adopted in the previous subsection. The result can be formulated as follows.

It is easy to see that formulae (63) and (64) can be obtained from formulae (46) and (47) by setting $\mu = \pi/2$ and taking the limit $s \to 0$. Moreover, we can verify that all the results concerning the properties of operators $\hat{a}^*$, $\hat{b}^*$, $\overline{a}$, $\overline{b}$, including their domains and the equalities $\overline{a} = \hat{b}^*$, $\overline{b} = \hat{a}^*$, can be obtained from the corresponding results for the operators $\hat{a}^*(\mu, s)$, $\hat{b}^*(\mu, s)$, $\overline{a}(\mu, s)$ and $\overline{b}(\mu, s)$ in the preceding subsection by setting $\mu = \pi/2$ and taking the limit $s \to 0$, so that we can set $\hat{a}^* = \hat{a}^*(\pi/2, 0)$, $\hat{b}^* = \hat{b}^*(\pi/2, 0)$, $\overline{a} = \overline{a}(\pi/2, 0)$, $\overline{b} = \overline{b}(\pi/2, 0)$.

The final conclusion of this section in the case of the coupling constant $\alpha = -1/4$ is that the results of the previous subsection for the operators $\hat{a}^*(\mu, s)$, $\hat{b}^*(\mu, s)$, $\overline{a}(\mu, s)$ and $\overline{b}(\mu, s)$, where $\mu \in [0, \pi/2]$, $s \in (0, \infty)$, actually hold for $\mu \in [0, \pi/2]$, $s \in (0, \infty)$ and for $\mu = \pi/2$, $s \in [0, \infty)$.

6. Oscillator representations

Now we are in a position to answer the question on generalized oscillator representations

$$\hat{H}_e = \hat{c}^*\hat{c} - (sk_0)^2\hat{I}, \quad s \geq 0, \quad (65)$$

or equivalently

$$\hat{H}_e = \hat{a}^*\hat{a} - (sk_0)^2\hat{I}, \quad s \geq 0, \quad (66)$$

where $\hat{c}$ and $\hat{a}$ are densely defined closed first-order differential operators and $\hat{c}^*$ and $\hat{a}^*$ are their respective adjoints, for all Calogero Hamiltonians $\hat{H}_e$ with any coupling constant $\alpha \in (-\infty, \infty)$. We recall that any $\hat{H}_e$ is an s.a. operator associated with Calogero differential operation $\hat{H}$ (1) with the same $\alpha$. The answer to the question is essentially different for the region $\alpha < -1/4$ and for the region $\alpha \geq -1/4$.

We can say immediately that any Calogero Hamiltonian $\hat{H}_e$ with $\alpha < -1/4$ does not allow generalized oscillator representations (65) or (66) because such a representation would imply that $\hat{H}_e$ is bounded from below, whereas any Calogero Hamiltonian with $\alpha < -1/4$ is not bounded from below [5]. This conclusion is in complete agreement with that according to section 3, there is no generalized oscillator representation (6), (11) for Calogero differential operation $\hat{H}$ (1) with $\alpha < -1/4$.

As to the second region of the coupling constant $\alpha \geq -1/4$, we show in what follows that any Calogero Hamiltonian with $\alpha \geq -1/4$ does allow generalized oscillator representations (65) or (66); in fact, a family of such representations, one- or two-parameter. In accordance with the program formulated in section 1 and according to the above results, we have two families of Calogero Hamiltonians in a generalized oscillator form,

$$\hat{H}_{ea(\mu, s)} = \hat{a}^*(\mu, s)\overline{a}(\mu, s) - (sk_0)^2\hat{I}, \quad \mu \in [0, \pi/2], \quad s \in [0, \infty), \quad (67)$$

and

$$\hat{H}_{eb(\mu, s)} = \overline{b}(\mu, s)\hat{b}^*(\mu, s) - (sk_0)^2\hat{I}, \quad \mu \in [0, \pi/2], \quad s \in [0, \infty), \quad (68)$$

for any $\alpha \geq -1/4$. It turns out that these families cover all of the set of known Calogero Hamiltonians with given $\alpha \geq -1/4$. Namely, each Calogero Hamiltonian with given $\alpha \geq -1/4$ can be identified with one or more members of family (67) or family (68). This identification is trivial in the case of $\alpha \geq 3/4$ where there is a unique Calogero Hamiltonian with given $\alpha$. In the case of $\alpha$ such that $-1/4 \leq \alpha < 3/4$, the procedure of identification is more complicated. We recall that for each $\alpha \in [-1/4, 3/4)$, there exists a one-parameter family $\{\hat{H}_{e(a, s)} = \hat{a}^*(\mu, s)\overline{a}(\mu, s) - (sk_0)^2\hat{I}, \forall \mu \in [0, \pi/2], s \in (0, \infty), \forall \mu \in [0, \pi/2], s \in (0, \infty)\}$

6.1. Hamiltonians $\hat{H}_{ea}$

6.1.1. The region $\alpha \geq 3/4$ ($\alpha \geq 1$). For each $\alpha$ in this region, there exists only one s.a. Calogero Hamiltonian $\hat{H}_1$ defined on the natural domain $D_{\hat{H}_1}$ see [5]. That is why we can immediately conclude that

$$\hat{H}_1 = \hat{a}^*(\mu, s)\overline{a}(\mu, s) - (sk_0)^2\hat{I}, \quad \forall \mu \in [0, \pi/2], \quad \forall s \in [0, \infty), \quad \alpha \geq 1, \quad (69)$$

which represents a two-parameter family of oscillator representations for a unique Calogero Hamiltonian $\hat{H}_1$ with a given coupling constant $\alpha \geq 3/4$.

Taking $s = 0$ in the rhs of (69), we obtain the known one-parameter family of oscillator representations for the nonnegative $\hat{H}_1$ [2] which are the optimum representations from the standpoint of an optimum estimate on its spectrum from below. According to [5], the spectrum of $\hat{H}_1$ is given by $\text{spec} \hat{H}_1 = [0, \infty)$ and is continuous, which agrees with that $\ker\overline{a}(\mu, s) = \{0\}, \forall \mu, \forall s$.

6.1.2. The region $-1/4 < \alpha < 3/4$ ($0 < \alpha < 1$). By definition of the operator $\hat{H}_{ea(\mu, s)}$, its domain belongs to or coincides with the domain of the operator $\overline{a}(\mu, s)$, $D_{\hat{H}_{ea(\mu, s)}} \subseteq D_{\overline{a}(\mu, s)}$ given by (41) for $\mu = \pi/2$ and (42) for $\mu \in [0, \pi/2)$. According to (36) and (38), the asymptotic behavior of functions $\chi$ belonging to $D_{\overline{a}(\mu, s)}$ is estimated for the case of $0 < \chi < 1$ by $\chi = O(x^{1/2})$ as $x \to 0$. It follows that the functions belonging to $D_{\hat{H}_{ea(\mu, s)}}$ tend to zero not weaker than $x^{1/2}$ as $x \to 0$.

According to [5], there exists only one s.a. Calogero Hamiltonian with such an asymptotic behavior of functions
6.2. Hamiltonians

6.2.1. The region $\alpha \geq 3/4$ ($\kappa \geq 1$). For each $\alpha$ in this region, we have the identities $b(\mu, s) = \tilde{a}(\mu, s)$ and $\tilde{b}(\mu, s) = \tilde{a}(\mu, s)$, see (43) and (41), so that with taking into account section 6.1.1, formula (69), we find that

$$
\tilde{H}_1 = \tilde{b}(\mu, s)\tilde{b}(\mu, s) - (s_{0k})^2 \tilde{I}, \quad \forall \mu, s \in [0, \pi/2],
$$

which is another representation of the known two-parameter family of oscillator representations (69) for $\tilde{H}_1$ followed by an appropriate comment.

6.2.2. The region $-1/4 < \alpha < 3/4$ ($0 < \kappa < 1$), $\mu = \pi/2$. A reasoning concerning $\tilde{H}_{eb}$ in this case is completely similar to that in the previous subsection for the case of $\alpha \geq 3/4$. According to (43) and (41), we have the identities $\tilde{b}(\pi/2, s) = \tilde{a}(\pi/2, s)$ and $\tilde{b}(\pi/2, s) = \tilde{a}(\pi/2, s)$, so that with taking into account section 6.1.2, formula (70), we find that

$$
\tilde{H}_{2,\pi/2} = \tilde{b}(\pi/2, s)\tilde{b}(\pi/2, s) - (s_{0k})^2 \tilde{I}, \quad \forall s \geq 0, 0 < \kappa < 1,
$$

which is another representation of the one-parameter family of oscillator representations for $\tilde{H}_{2,\pi/2}$ that is a restriction of the known two-parameter family of oscillator representations (70) to $\mu = \pi/2$. Of course, the comment following (70) holds.

6.2.3. $-1/4 < \alpha < 3/4$ ($0 < \kappa < 1$), $0 \leq \mu < \pi/2$. We have to find the asymptotic behavior of functions belonging to the domain $D_{H_{eb,\mu}}$, of the operator $\hat{H}_{eb}(\mu, s)$, $\mu \in [0, \pi/2]$, $s \in [0, \infty]$, $\kappa \in [0, 1)$, at the origin.

We begin with representing the asymptotic behavior of functions $\phi(\mu, s, x)$ (16), $\mu \in [0, \pi/2]$, $s \in [0, \infty]$, $\kappa \in (0, 1)$, at the origin given in (18) in a new form:

$$
\phi(\mu, s, x) = c[(k_0x)^{1/2} \sin \theta(\mu, s) + (k_0x)^{1/2-x} \times \cos \theta(\mu, s)] + O(x^{5/2-x}), \quad x \to 0,
$$

$$
\tan \theta(\mu, s) = \tan \mu - \frac{\Gamma(1-x)}{\Gamma(1+x)} \left(\frac{s}{2}\right)^{2x}, \quad c = \frac{\cos \mu}{\cos \theta(\mu, s)},
$$

$$
\mu = 0, \quad s \in [0, \infty], \quad \kappa \in (0, 1), \quad \theta = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).
$$

By definition of the operator $\hat{H}_{eb}(\mu, s)$, its domain $D_{H_{eb,\mu}}$ consists of functions $\chi \in D_{D_{eb,\mu}}$ such that $\tilde{b}(\mu, s)\chi = \eta \tilde{a}(\mu, s)\chi \in D_{D_{eb,\mu}} \subset L^2(\mathbb{R}_x)$. The first condition implies that $\chi$ allows representation (35) with $x_0 = 0$ and, in general, $D \neq 0$, while the second condition implies that $\eta(\chi) = O(x^{1/2}), \chi \to 0$, see (44) and (31). Estimating the integral term in (35) with such $\eta$, we obtain that the asymptotic behavior of functions $\chi \in D_{H_{eb,\mu}}$, $\mu \in [0, \pi/2]$, $s \in [0, \infty]$, $\kappa \in (0, 1)$, at the origin is given by

$$
\chi(x) = C[(k_0x)^{1/2} \sin \theta(\mu, s) + (k_0x)^{1/2-x} \times \cos \theta(\mu, s)] + O(x^{5/2-x}), \quad x \to 0.
$$

According to [5], for each $\alpha \in (-1/4, 3/4)$ ($\kappa \in (0, 1)$), there is a one-parameter family of s.a. Calogero Hamiltonians with such an asymptotic behavior of functions belonging to their domains, namely the family $\{H_{2,v}: v \in (-\frac{\pi}{2}, \frac{\pi}{2})\}$. The parameter $v$ is naturally identified with the angle $\theta(\mu, s)$ in (74), and we establish that

$$
\tilde{H}_{2,v} = \tilde{b}(\mu, s)\tilde{b}(\mu, s) - (s_{0k})^2, \quad v = \theta(\mu, s),
$$

$$
v \in (-\pi/2, \pi/2), \quad \mu \in [0, \pi/2], \quad s \in [0, \infty], \quad \kappa \in (0, 1).
$$
which represents a one-parameter family of generalized oscillator representations for $H_{2,\nu}$ with given $\alpha \in (-1/4, 3/4)$, $0 < \nu < 1$, and $\nu \in (-\pi/2, \pi/2)$. It is convenient to take $\mu$ as the independent parameter, then $s$ is easily determined from the relation $\tan \theta(\mu, s) = \tan \nu$ to yield

$$s = s(\mu, \nu) = 2 \left( \frac{\tan \nu - \tan \nu}{\Gamma(1 + \nu) \Gamma(1 - \nu)} \right)^{1/2},$$

$$\nu \in (-\pi/2, \pi/2), \; \mu \in [0, \pi/2), \; \kappa \in (0, 1).$$

(76)

with due regard to the condition $s \geq 0$. For fixed $\nu$, the function $s(\mu, \nu)$ (76) is monotonically increasing from $s_{\min}(\nu)$ to $\infty$ as $\mu$ ranges admissible values from $\mu_{\min}(\nu)$ to $\pi/2 - 0$, where

$$\mu_{\min}(\nu) = \nu, \; s_{\min}(\nu) = 0, \; 0 \leq \nu < \pi/2,$$

$$\mu_{\min}(\nu) = 0, \; s_{\min}(\nu) = 2 \left( \frac{\tan \nu}{\Gamma(1 + \nu) \Gamma(1 - \nu)} \right)^{1/2},$$

$$\nu - \pi/2 < \nu < 0.$$

It is evident that the spectrum of $H_{2,\nu}$ is bounded from below by $-(s_{\min}(\nu)k_0)^2$.

If $0 \leq \nu < \pi/2$, this boundary is zero, and according to [5], this is an exact lower bound of the spectrum, $\text{spec} \hat{H}_{2,\nu} = [0, \infty)$, and the spectrum is continuous, which agrees with that ker $\hat{b}^*(\mu, s(\nu, \mu)) = \{0\}$, for all $\nu \in [\nu, \pi/2]$.

Taking $\mu = \mu_{\min}(\nu) = \nu$ and $s = s(\mu_{\min}(\nu), \nu) = s_{\min}(\nu) = 0$, we obtain the known oscillator representation for the nonnegative $H_{2,\nu}$, $0 \leq \nu < \pi/2$, [2], which is the optimum representation.

If $-\pi/2 < \nu < 0$, we have ker $\hat{b}^*(0, s_{\min}(\nu)) = \{c\phi(0, s_{\min}(\nu); x) \neq 0 \}$ because $\phi(0, s_{\min}(\nu); x) \sim x^{1/2} K_{\nu}(s_{\min}(\nu)k_0x)$ is square integrable on the whole semi-axis $\mathbb{R}_+$, whereas ker $\hat{b}^*(\mu, s(\nu, \mu)) = \{0\}$, $0 < \mu < \pi/2$. This implies that

$$-(s_{\min}(\nu)k_0)^2 = -4k_0^2 \left( \frac{\tan \nu}{\Gamma(1 + \nu) \Gamma(1 - \nu)} \right)^{1/2} = E_2(v)$$

is an exact lower bound of the spectrum of $\hat{H}_{2,\nu}$, $E_2(\nu)$ is an eigenvalue of $\hat{H}_{2,\nu}$, the energy of its negative ground level, and the normalized eigenfunction of the ground state is given by

$$U_2(\nu, x) = \frac{\sqrt{2\sin(\pi\nu)E_2(\nu)}}{\pi \kappa} x^{1/2} K_{\nu}(E_2(\nu)|x|).$$

According to [5], the spectrum of $\hat{H}_{2,\nu}$ is given by $\text{spec} \hat{H}_{2,\nu} = \{E_2(\nu)\} \cup [0, \infty)$, and the semi-axis $[0, \infty)$ is a continuous part of the spectrum.

Setting $\mu = 0$, $s = s_{\min}(\nu)$ in (75), we obtain the optimum generalized oscillator representation for $H_{2,\nu}$, $-\pi/2 < \nu < 0$, bounded from below, but not nonnegative.

$6.2.4. \; \alpha = \frac{1}{2} (\nu = 0), \; \mu = \pi/2$. The reasoning in this subsection is completely similar to those in sections 6.2.1 and 6.2.2.

According to (58) and (60), we have the identities $\tilde{b}(\pi/2, s) = \tilde{a}^*(\pi/2, s)$ and $\tilde{b}^*(\pi/2, s) = \tilde{a}(\pi/2, s)$, the point $s = 0$ included, see section 6.2.3, formula (71), we find that

$$\hat{H}_{3,\nu} = \hat{b}(\nu/2, s) \tilde{b}^*(\pi/2, x) - (sk_0)^2 I, \quad 0 < \kappa = 0,$$

which is another form of the one-parameter family of oscillator representations for $\hat{H}_{3,\nu}$, that is a restriction of the known two-parameter family of oscillator representations (71) to $\mu = \pi/2$. Of course, the comment following (71) holds.

$6.2.5. \; \alpha = \frac{1}{2} (\nu = 0), 0 \leq \mu < \pi/2$. The reasoning in this section is completely similar to that in section 6.2.3 for the case of $0 < \kappa < 1$.

We begin with representing the asymptotic behavior of functions $\phi(\mu, \nu; x)$ (46), $\mu \in [0, \pi/2), x \in (0, \infty)$, $\kappa = 0$, at the origin given by (48) in a new form:

$$\phi(\mu, \nu; x) = c(\sqrt{k_0x \sin \theta(\mu, \nu) + \sqrt{k_0x} \ln(k_0x) \times \cos \theta(\mu, \nu))} + O(x^{5/2} \ln x),$$

$$\tan \theta(\mu, s) = \ln(s/2) - \ln \mu - \psi(1), \; c = -\frac{\cos \mu}{\cos \theta(\mu, s)}, \; \mu \in [0, \pi/2), \; s \in (0, \infty), \; \kappa = 0, \; \theta(\mu, s) \in \left(\frac{\pi}{2}, \frac{\pi}{2}\right).$$

We now determine the asymptotic behavior of functions belonging to the domain $D_{\hat{H}_{3,\nu}(\mu, s)}$ of the operator $\hat{H}_{3,\nu}(\mu, s)$, $\mu \in [0, \pi/2), \; s \in (0, \infty)$, $\kappa = 0$, at the origin.

By definition of the operator $\hat{H}_{3,\nu}(\mu, s)$, its domain $D_{\hat{H}_{3,\nu}(\mu, s)}$ consists of functions $\chi \in D_{\hat{H}_{3,\nu}(\mu, s)}$ such that

$$\hat{b}(\mu, \nu)\chi = \eta \in D_{\hat{H}_{3,\nu}(\mu, s)} \subset L^2(\mathbb{R}_+).$$

The first condition implies that $\chi$ allows representation (53) with $x_0 = 0$ and, in general, $D \neq 0$, whereas the second condition implies that $\eta(x) = O(x^{1/2}), x \to 0$, see (61) and (51).

Estimating the integral term in (53) with such $\eta$, we obtain that the asymptotic behavior of functions $\chi \in D_{\hat{H}_{3,\nu}(\mu, s)}$, $\mu \in [0, \pi/2), \; s \in (0, \infty)$, $\kappa = 0$, at the origin is given by

$$\chi(x) = C(k_0x)^{1/2} \sin \theta(\mu, s) + (k_0x)^{1/2} \ln(k_0x) \times \cos \theta(\mu, s) + O(x^{3/2}), \; x \to 0.$$ (77)

According to [5], for $\alpha = -1/4 (\kappa = 0)$, there is a one-parameter family of $s$-a. Calogero Hamiltonians with such an asymptotic behavior of functions belonging to their domains, namely the family $\{\hat{H}_{3,\nu}, \nu \in (-\pi/2, \pi/2)\}$.

The parameter $\nu$ is naturally identified with the angle $\theta(\mu, s)$ in (77), and we establish that

$$\hat{H}_{3,\nu} = \tilde{b}(\nu/2, s) \tilde{a}^*(\pi/2, s) - (sk_0)^2 \tilde{b}(\nu/2, s) - (sk_0)^2 \tilde{b}^*(\pi/2, s), \; \nu = \theta(\mu, s), \; \mu \in [0, \pi/2), \; s \in (0, \infty), \; \kappa = 0,$$

(78)

which represents a one-parameter family of generalized oscillator representations for $\hat{H}_{3,\nu}$ with $\alpha = -1/4 (\kappa = 0)$ and $\nu \in (-\pi/2, \pi/2)$.

It is convenient to take $\mu$ as the independent parameter, then $s$ is easily determined from the relation $\tan \theta(\mu, s) = \tan \nu$ to yield

$$s = s(\mu, \nu) = 2\tan \nu + \tan \mu + \psi(1).$$ (79)
For fixed $\nu$, the function $s(\mu, \nu)$ (79) is monotonically increasing from $s_{\min}(\nu)$ to $\infty$ as $\mu$ ranges from $0$ to $\pi/2 - 0$, where

$$s_{\min}(\nu) = s(0, \nu) = 2\sqrt{e\tan\nu+\psi(1)}.$$ 

It is evident that the spectrum of $\hat{H}_{3,\nu}$ is bounded from below by

$$-(s_{\min}(\nu)k_0)^2 = -4k_0^2e^{2(\tan\nu+\psi(1))} = E_{\nu}(\nu).$$

Because $\ker \hat{b}^+(0, s_{\min}(\nu)) = [ce^{1/2}K_0(s_{\min}(\nu)k_0x)] \neq \{0\}$, whereas $\ker \hat{b}^+(\mu, s(\mu, \nu)) = \{0\}$, $0 < \mu < \pi/2$, this boundary is an exact lower boundary of the spectrum of $\hat{H}_{3,\nu}$, $E_{\nu}(\nu)$ is an eigenvalue of $\hat{H}_{3,\nu}$, the energy of its negative ground level, and the normalized eigenfunction of the ground state is given by

$$U_{\nu}(\nu, x) = \sqrt{2/E_{\nu}(\nu)}x^{1/2}K_0(|E_{\nu}(\nu)|^{1/2}x).$$

According to [5], the spectrum of $\hat{H}_{3,\nu}$ is given by $\text{spec} \hat{H}_{3,\nu} = \{E_{\nu}(\nu)\} \cup [0, \infty)$, and the semiaxis $[0, \infty)$ is a continuous part of the spectrum.

Setting $\mu = 0$, $s = s_{\min}(\nu)$, in (78), we obtain the optimum generalized oscillator representation for $\hat{H}_{3,\nu}$, $\nu \in (-\pi/2, \pi/2)$, bounded from below, but not nonnegative.

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