WELL-POSEDNESS FOR ONE-DIMENSIONAL DERIVATIVE NONLINEAR SCHröDINGER EQUATIONS

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Abstract. In this paper, we investigate the one-dimensional derivative nonlinear Schrödinger equations of the form

\[ iu_t - u_{xx} + i\lambda |u|^k u_x = 0 \]

with non-zero \( \lambda \in \mathbb{R} \) and any real number \( k \geq 5 \). We establish the local well-posedness of the Cauchy problem with any initial data in \( H^{1/2} \) by using the gauge transformation and the Littlewood-Paley decomposition.

1. Introduction. In the present paper, we consider the following Cauchy problem for the derivative nonlinear Schrödinger equation

\[ \begin{aligned}
    iu_t - u_{xx} + i\lambda |u|^k u_x &= 0, \\
    u(0, x) &= u_0(x),
\end{aligned} \tag{1.1, 1.2} \]

where \( u = u(t, x) : \mathbb{R}^2 \rightarrow \mathbb{C} \) is a complex-valued wave function, both \( \lambda \neq 0 \) and \( k \geq 5 \) are real numbers.

A great deal of interesting research has been devoted to the mathematical analysis for the derivative nonlinear Schrödinger equations \([3, 4, 6, 7, 8, 9, 10, 11, 13, 18, 21]\). In \([13]\), C. E. Kenig, G. Ponce and L. Vega studied the local existence theory for the Cauchy problem of the derivative nonlinear Schrödinger equations

\[ \begin{aligned}
    iu_t + u_{xx} + f(u, \bar{u}, u_x, \bar{u}_x) &= 0, \\
    u(0, x) &= u_0(x),
\end{aligned} \]

with small data \( u(0, x) = u_0(x) \) in \( H^{7/2} \) where \( f \) is a polynomial having no constant or linear terms with the lowest order term of degree being greater than or equal to 3. Subsequently, it was improved to \( H^3 \) by N. Hayashi and T. Ozawa \([11]\).

If the nonlinearity consists mostly of the conjugate wave \( \bar{u} \), then it can be done much better. In the case \( f = (\bar{u}_x)^k \), A. Grünrock, in \([8]\), obtained local well-posedness when \( s > s_c = 3/2 - 1/(k - 1) \), \( s \geq 1 \), and \( k \geq 2 \) was an integer. In particular, the global well-posedness in \( H^1 \) is obtained when \( f = i(\bar{u}_x)^2 \) with the help of the Bourgain spaces (cf. \([2, 23]\)).

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In [21], H. Takaoka discussed the derivative nonlinear Schrödinger equation of the form
\[ u_t - iu_{xx} + |u|^2u_x = 0, \quad (t, x) \in \mathbb{R}^2, \]
and obtained the local well-posedness in \( H^s \) for \( s \geq 1/2 \) by performing a fixed point argument in an adapted Bourgain space \( X_{s,b} \), which yields a \( C^\infty \)-solution map.

A very similar equation to (1.1) is the generalized Benjamin-Ono equation
\[ u_t + \mathcal{H}u_{xx} \pm u^k u_x = 0, \quad (t, x) \in \mathbb{R}^2, \quad (1.3) \]
where \( u \) is a real-valued function, \( \mathcal{H} \) is the Hilbert transform defined by
\[ \mathcal{H}f(x) = -i \int_{\mathbb{R}} e^{ix\xi} \text{sgn}(\xi) \hat{f}(\xi) d\xi, \]
and \( k \geq 2 \) is an integer, the symbol \( : \) (or \( \mathcal{F} \)) denotes the spatial Fourier transform. For this equation, L. Molinet and F. Ribaud [16, 17] obtained the local preservation of the Hamiltonian and the following compactness argument with a priori estimates with the help of the by combining the gauge transformation with a Littlewood-Paley decomposition and

For any \( C > 0 \), we have to reconstruct new and complicated estimates for the case \( k \gg 1 \) and obtained the local well-posedness in \( X \) later the function space \( N \) and obtained the local well-posedness in \( X \) later the function space \( N \).

Throughout this paper, we often use the Littlewood-Paley theorem (cf. [20, 23])
\[ \left\| \left( \sum_N |P_N \phi|^2 \right)^{1/2} \right\|_{L_p} \sim \| \phi \|_{L_p}, \]

In the present paper, we shall generalize the above results to the derivative nonlinear Schrödinger equation with \( k \geq 5 \) by using some ideas in [14]. However, we have to reconstruct new and complicated estimates for the case \( k \geq 5 \) which is quite different from the case \( k = 2 \).

We first state the main result of this paper as follows, though we shall define later the function space \( X_T \) at the end of this section.

**Theorem 1.1.** For any \( u_0 \in H^{1/2} \), there exist a \( T = T(\|u_0\|_{H^{1/2}}) \) and a unique solution \( u \) of (1.1)-(1.2) satisfying
\[ u \in C([-T, T]; H^{1/2}) \cap X_T. \]

For convenience, we now introduce some notations. For nonnegative real numbers \( A, B \), we use \( A \lesssim B \) to denote \( A \leq CB \) for some \( C > 0 \) which is independent of \( A \) and \( B \). \( A \sim B \) means \( A \lesssim B \lesssim A \), and \( A \ll B \) denotes \( A \ll CB \) for some small \( C > 0 \) which is also independent of \( A \) and \( B \).

To give the Littlewood-Paley decomposition, let \( \psi \) be a fixed even \( C^\infty \) function with a compact support, \( \text{supp} \psi \subset \{ |\xi| < 2 \} \), and \( \psi(\xi) = 1 \) for \( |\xi| \leq 1 \). Define \( \varphi(\xi) = \psi(\xi) - \psi(2\xi) \). Let \( N \) be a dyadic number of the form \( N = 2^j, j \in \mathbb{N} \cup \{ 0 \} \) or \( N = 0 \). Writing \( \varphi_N(\xi) = \varphi(\xi/N) \) for \( N \geq 1 \), we define the convolution operator \( P_N \) by \( P_N u = \varphi_N * u \), where the symbol \( : \) (or \( \mathcal{F}^{-1} \)) denotes the spatial Fourier inverse transform. We define the function \( \varphi_0 \) by \( \varphi_0(\xi) = 1 - \sum_N \varphi_N(\xi) \) and denote \( P_0 u = \varphi_0 * u \). Then we introduce a spatial Littlewood-Paley decomposition [20]
for $1 < p < \infty$. We also use more general operators $P_{\leq N}$ and $P_{\geq N}$ which are defined by

$$P_{\leq N} = \sum_{M \leq N} P_M, \quad P_{\geq N} = \sum_{M \geq N} P_M,$$

and $P_{\geq N}, P_{\leq N}$ and $P_{\sim N}$ which can be defined in a similar way. The Littlewood-Paley operators commute with derivative operators (including $|\nabla|^s$ and $i\partial_t - \partial_{xx}$), the propagator $S(t) = e^{-i\partial_x^2 t}$, and conjugation operations, are self-adjoint, and are bounded on every Lebesgue space $L^p$ and homogeneous Sobolev space $H^s$ if $1 \leq p \leq \infty$. Furthermore, they obey the following Sobolev and Bernstein estimates for $\mathbb{R}$ with $s \geq 0$ and $1 \leq p \leq \infty$ (which is similar to those of three dimensions [5]):

$$\|P_{\geq N} f\|_{L^p} \lesssim N^{-s} \|\nabla|^s P_{\geq N} f\|_{L^p},$$

$$\|P_{\leq N} \nabla|^s f\|_{L^p} \lesssim N^s \|P_{\leq N} f\|_{L^p},$$

$$\|P_N \nabla|^{\pm s} f\|_{L^p} \lesssim N^{\pm s} \|P_N f\|_{L^p},$$

which can be verified by combining the Bernstein multiplier theorem [1] and the interpolation theorem of Sobolev spaces.

We define the Lebesgue spaces $L^q_T L^p_x$ and $L^q_T L^p_T$ by the norms

$$\|f\|_{L^q_T L^p_x} = \left\|\left\|f\right\|_{L^p_x(\mathbb{R})}\right\|_{L^q_T(\mathbb{R})},\quad \|f\|_{L^q_T L^p_T} = \left\|\left\|f\right\|_{L^p_T(\mathbb{R})}\right\|_{L^q_T(\mathbb{R})}.$$ 

In particular, we abbreviate $L^q_T L^p_x$ or $L^q_T L^p_T$ as $L^p_{x,T}$ in the case $p = q$.

We also use the elementary inequality [5]

$$\left\|\left(\sum_N |f_N|^2\right)^{1/2}\right\|_{L^q_T L^p_x} \leq \left(\sum_N \|f_N\|_{L^q_T L^p_x}^2\right)^{1/2},$$

for all $2 \leq q, p \leq \infty$ and arbitrary functions $f_N$, and the dual version

$$\left(\sum_N \|N\|_{L^{p'}_{x,T}}^2\right)^{1/2} \leq \left\|\left\|f_N\right\|_{L^{p'}_{x,T}}^2\right\|_{L^q_T L^{p'}_x},$$

where $p'$ is the conjugate number of $p$ given by $1/p + 1/p' = 1$. It is easy to verify that they also hold if we replace the norm $L^q_T L^p_x$ by the norm $L^p_T L^q_x$ in both side of the above inequalities.

Let $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$. We use the fractional differential operators $D_x^s$ and $\langle D_x \rangle^s$ defined by

$$D_x^s f = \mathcal{F}^{-1}(\xi^s \mathcal{F} f), \quad \langle D_x \rangle^s f = \mathcal{F}^{-1}(\xi^s \mathcal{F} f).$$

Thus, we can introduce the resolution space. For $T > 0$, we define the function space $X_T$ in a similar way as in [13] by

$$X_T := \{u \in \mathcal{D}'((\mathbb{R} \times (-T,T)) \times \mathbb{R}) : \|u\|_{X_T} < \infty\},$$

where

$$\|u\|_{X_T} = \|u\|_{L_T^{p,H^{1/2}}} + \left(\sum_N \|\partial_x P_N u\|_{L_T^{p,H^1}}^2\right)^{1/2},$$

for $1 < p < \infty$. We also use more general operators $P_{\leq N}$ and $P_{\geq N}$ which are defined by

$$P_{\leq N} = \sum_{M \leq N} P_M, \quad P_{\geq N} = \sum_{M \geq N} P_M,$$
By computation, we have the following complex-valued function for a dyadic number $N$ given by

$$v_N(t, x) = e^{-\frac{i}{4} \int_{-\infty}^{\infty} |P_{\leq N} u(t, y)|^4 dy} P_N u.$$  \hfill (2.1)

By computation, we have

$$i \partial_t v_N - \partial_x^2 v_N = -i \lambda e^{-\frac{i}{4} \int_{-\infty}^{\infty} |P_{\leq N} u(t, y)|^4 dy} \left[ P_N (|u|^k u_x) - |P_{\leq N} u|^k P_N u_x \right]$$

$$- \frac{i \lambda}{2} e^{-\frac{i}{4} \int_{-\infty}^{\infty} |P_{\leq N} u|^4 dy} P_N u (i \partial_t - \partial_x^2) \int_{-\infty}^{\infty} |P_{\leq N} u(t, y)|^k dy$$

$$+ \frac{\lambda^2}{4} e^{-\frac{i}{4} \int_{-\infty}^{\infty} |P_{\leq N} u|^4 dy} |P_{\leq N} u|^{2k} P_N u.$$  \hfill (2.2)

For the second term, we integrate by parts and have

$$\int_{-\infty}^{\infty} |P_{\leq N} u(t, y)|^k dy$$

$$= \int_{-\infty}^{\infty} \frac{k}{2} |P_{\leq N} u|^k \left( \partial_t P_{\leq N} u \overline{P_{\leq N} u} + P_{\leq N} u \partial_t \overline{P_{\leq N} u} \right) dy$$

$$- \frac{k}{2} |P_{\leq N} u|^k \left( \partial_x P_{\leq N} u \overline{P_{\leq N} u} + P_{\leq N} u \partial_x \overline{P_{\leq N} u} \right)$$

$$= \int_{-\infty}^{\infty} \frac{k(k-2)}{4} |P_{\leq N} u|^k \left[ (|P_{\leq N} u|^2 - (P_{\leq N} u \overline{P_{\leq N} u})^2 \right] dy$$

$$- \int_{-\infty}^{\infty} \frac{i \lambda k}{2} |P_{\leq N} u|^k \left( P_{\leq N} u \partial_x \overline{P_{\leq N} u} + P_{\leq N} u \partial_x \overline{P_{\leq N} u} \right) dy$$

Thus, $v_N$ obeys the following differential-integral equation

$$i \partial_t v_N - \partial_x^2 v_N (t, x)$$

$$= -i \lambda e^{-\frac{i}{4} \int_{-\infty}^{\infty} |P_{\leq N} u|^4 dy} \left[ P_N (|u|^k u_x) - |P_{\leq N} u|^k P_N u_x \right]$$

$$- \frac{i \lambda k(k-2)}{8} e^{-\frac{i}{4} \int_{-\infty}^{\infty} |P_{\leq N} u|^4 dy} P_N u \int_{-\infty}^{\infty} |P_{\leq N} u|^{k-4}$$

$$\left[ (|P_{\leq N} u|^2 - (P_{\leq N} u \overline{P_{\leq N} u})^2 \right] dy$$

$$- \frac{\lambda^2 k}{4} e^{-\frac{i}{4} \int_{-\infty}^{\infty} |P_{\leq N} u|^4 dy} P_N u \int_{-\infty}^{\infty} |P_{\leq N} u|^{k-2} P_{\leq N} u \partial_x \overline{P_{\leq N} u}$$

$$+ \frac{i \lambda k}{2} e^{-\frac{i}{4} \int_{-\infty}^{\infty} |P_{\leq N} u|^4 dy} |P_{\leq N} u|^{k-2} P_{\leq N} u \overline{P_{\leq N} u}$$

$$+ \frac{\lambda^2}{4} e^{-\frac{i}{4} \int_{-\infty}^{\infty} |P_{\leq N} u|^4 dy} |P_{\leq N} u|^{2k} P_N u.$$
The equivalent integral equation reads
\[
\begin{align*}
\int_0^t S(t-\tau)|I_{N,1} + I_{N,2} + I_{N,3} + I_{N,4} + I_{N,5}|(\tau)d\tau.
\end{align*}
\] (2.4)

3. Preliminaries. In order to prove the a priori estimate for the equation of \(v_N\), we need the linear estimates associated with the one-dimensional Schrödinger equation. We first recall the Strichartz estimates, smoothing effects and maximal function estimates. For the proofs, one can see [13, 14].

Lemma 3.1. For all \( \phi \in \mathcal{S}(\mathbb{R}) \), \( \theta \in [0, 1] \) and \( T \in (0, 1) \),
\[
\begin{align*}
\|S(t)\phi\|_{L_T^\theta L_x^\infty} &\lesssim \|\phi\|_{L^\theta}, \\
\|S(t)P_N\phi\|_{L_T^{\theta/2} L_x^\infty} &\lesssim \langle N \rangle^{\frac{\theta}{2}} \|\phi\|_{L^\theta}, \\
\|S(t)\phi\|_{L_T^1 L_x^\infty} &\lesssim \|\phi\|_{H^\theta}.
\end{align*}
\] (3.1)

We also need the \( L_T^q L_x^p \) and \( L_T^p L_x^q \) estimates for the linear operator \( f \mapsto \int_0^t S(t-\tau)f(\tau)d\tau \). For the proofs, one can see [14].

Lemma 3.2. For \( f \in \mathcal{S}(\mathbb{R}^2) \), \( \theta \in [0, 1] \) and \( T \in (0, 1) \),
\[
\begin{align*}
\left\| \int_0^t S(t-\tau)f(\tau)d\tau \right\|_{L_T^\frac{2}{\theta} L_x^{2\theta}} &\lesssim \|f\|_{L_T^\frac{2}{\theta} L_x^{2\theta}}, \\
\left\| (D_x)^{\frac{\theta}{4}} \int_0^t S(t-\tau)f(\tau)d\tau \right\|_{L_T^\theta L_x^2} &\lesssim \|f\|_{L_T^\theta L_x^2}, \\
\left\| (D_x)^{\frac{\theta}{2}} \int_0^t S(t-\tau)P_Nf(\tau)d\tau \right\|_{L_T^\theta L_x^2} &\lesssim \langle N \rangle^{\frac{\theta}{2}} \|f\|_{L_T^\theta L_x^2}, \\
\left\| (D_x)^{\frac{\theta}{2} - \frac{1}{4}} \int_0^t S(t-\tau)P_Nf(\tau)d\tau \right\|_{L_T^\theta L_x^2} &\lesssim \|f\|_{L_T^\theta L_x^2}, \\
\left\| \int_0^t S(t-\tau)P_Nf(\tau)d\tau \right\|_{L_T^\theta L_x^2} &\lesssim \langle N \rangle^{\frac{\theta}{2} - \theta} \|f\|_{L_T^\theta L_x^2}, \\
\left\| \int_0^t S(t-\tau)f(\tau)d\tau \right\|_{L_T^\theta L_x^2} &\lesssim \|f\|_{L_T^\theta H_x^\theta}, \\
\left\| \int_0^t S(t-\tau)f(\tau)d\tau \right\|_{L_T^\theta L_x^2} &\lesssim \|f\|_{L_T^\theta L_x^2}, \\
\left\| \partial_x \int_0^t S(t-\tau)f(\tau)d\tau \right\|_{L_T^\frac{2}{\theta} L_x^{2\theta}} &\lesssim \|f\|_{L_T^\frac{2}{\theta} L_x^{2\theta}},
\end{align*}
\] (3.4)

where \( p' \) is the conjugate number of \( p \in [1, \infty] \), i.e. \( 1/p + 1/p' = 1 \), and
\[
\begin{align*}
\frac{1}{p(\theta)} &\equiv \frac{3 + \theta}{4}, \\
\frac{1}{q(\theta)} &\equiv \frac{3 - \theta}{4}.
\end{align*}
\] (3.10)

Next, we recall the Leibniz’ rule for a product of the form \( e^{itF}g \) where \( F \) is the spatial primitive of some function \( f \). For the proof, we refer to [13, 17].
Lemma 3.3 ([9] Lemma 3.5). Let $\alpha \in (0, 1)$, $p$, $p_1$, $p_2$, $q$, $q_1 \in (1, \infty)$, $q_2 \in (0, \infty]$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, and let $F(t, x) = \int_{-\infty}^{x} f(t, y)dy$, with real-valued function $f$. Then

$$
\|D_x^\alpha e^{iF} g\|_{L_T^p L_x^q} \lesssim \|F\|_{L_T^{p_1} L_x^{q_1}} \|g\|_{L_T^{p_2} L_x^{q_2}} + \|(D_x)^\alpha g\|_{L_T^p L_x^q}.
$$

4. Bilinear estimates. In this section, we prove the following space-time estimate which is crucial to the proof of the nonlinear estimates.

Proposition 4.1. Let $u \in H^\infty$ and $p \geq 4$ be a real number. Then we have

$$
\|u \tilde{u}_x\|_{L_T^p L_x^q} \lesssim T^{\frac{1}{2}} \|u\|_{X_T}^2 + (1 + T^{\frac{1}{2}} \|u\|_{X_T}) \|u\|_{X_T} \|P_{\gg 1} u\|_{X_T}.
$$

Proof. By the Littlewood-Paley decomposition, we can write

$$
\|u \tilde{u}_x\|_{L_T^p L_x^q} = \sum_{N_1 \sim N_2} \|P_{N_1} u P_{N_2} \tilde{u}_x\|_{L_T^p L_x^q}.
$$

Now, we derive the estimates for $I_1$, $I_2$ and $I_3$, respectively.

From the Hölder inequality, the Bernstein type inequalities and the real interpolation theorem, we have

$$
I_1 \lesssim \sum_{N_1 \sim N_2} \|P_{N_1} u\|_{L_T^{2p} L_x^4} \|P_{N_2} \tilde{u}_x\|_{L_T^{2p} L_x^4} \lesssim \sum_{N_1 \sim N_2} \|P_{N_1} u\|_{L_T^{2p} L_x^4} N_2 \|P_{N_2} \tilde{u}_x\|_{L_T^{2p} L_x^4}.
$$

Applying the Sobolev embedding theorem and the Hölder inequality to the first term, and Bernstein estimates to the second term, we can see that it is bounded by

$$
\lesssim T^{1/2} \|u\|_{X_T}^2 + \sum_{N} \|P_{N} P_{\gg 1} u\|_{L_T^{2p} L_x^4} \|\partial_x P_{N} P_{\gg 1} u\|_{L_T^{2p} L_x^4}.
$$
By the Cauchy-Schwartz inequality and the Sobolev embedding theorem (i.e. \( H^{1/4}_4 \subset H^{1/4-1/p}_4 \subset L^p \) for the real number \( p \geq 4 \)), we can bound it by

\[
\lesssim T^{1/2} \| u \|^2_{L^T} + \| P_{\gg 1} u \|^2_{L^T}.
\]

For \( I_2 \) or \( I_3 \), it is suffice to consider one of them, e.g. \( I_2 \), in view of symmetry. For \( N_1 \ll N_2 \), we have

\[
P_{N_1} u P_{N_2} \tilde{u}_x = \hat{P}_N (P_{N_1} u P_{N_2} \tilde{u}_x),
\]

where \( \hat{P}_N = \sum_{j=-2}^2 P_{2^j N} \). We split these into three cases, i.e. \( N_1 \ll 1 \ll N_2 \), \( N_1 \ll N_2 \ll 1 \) and \( 1 \ll N_1 \ll N_2 \). For the case \( N_1 \ll 1 \ll N_2 \), from the Hölder inequality and the Littlewood-Paley theorem, we can get

\[
\left\| \sum_{N_1 \ll 1 \ll N_2} P_{N_1} u P_{N_2} \tilde{u}_x \right\|_{L^p_x L^2_t} \lesssim \left\| P_{\lesssim 1} u P_{\gg 1} \tilde{u}_x \right\|_{L^p_x L^2_t} \lesssim \left\| P_{\leq 1} u \right\|_{L^p_x L^\infty_t} \left\| P_{\gg 1} u \right\|_{L^\infty_x L^2_t}
\]

\[
\lesssim \left( \sum_M \| P_M P_{\leq 1} u \|^2_{L^p_x L^\infty_t} \right)^{1/2} \left( \sum_M \| P_M P_{\gg 1} u \|^2_{L^\infty_x L^2_t} \right)^{1/2}
\]

\[
\lesssim \left( \sum_{M \leq 1} \| P_M u \|^2_{L^p_x L^\infty_t} \right)^{1/2} \left( \sum_{M \geq 1} \| P_M P_{\gg 1} u \|^2_{L^\infty_x L^2_t} \right)^{1/2} \| u \|_{L^p_x L^\infty_t}.
\]

For the case \( N_1 \ll N_2 \ll 1 \), we have, by the Hölder inequality and the Littlewood-Paley theorem, that

\[
\left\| \sum_{N_1 \ll N_2 \ll 1} P_{N_1} u P_{N_2} \tilde{u}_x \right\|_{L^p_x L^2_t} = \left\| \sum_{N_2 \ll 1} P_{\ll N_2} u P_{N_2} \tilde{u}_x \right\|_{L^p_x L^2_t}
\]

\[
\lesssim T^{1/2} \left( \sum_{N_2 \ll 1} \| P_{\ll N_2} u \|^2_{L^p_x L^\infty_t} \right)^{1/2} \| u \|_{L^p_x L^\infty_t}.
\]

For \( N_2 \ll 1 \), we have, by the Sobolev embedding theorem, that

\[
\| P_{\ll N_2} u \|_{L^p_x L^\infty_t} \sim \left( \sum_M \| P_M P_{\ll N_2} u \|^2_{L^p_x L^\infty_t} \right)^{1/2} \lesssim \left( \sum_M \| P_M u \|^2_{L^p_x L^\infty_t} \right)^{1/2}
\]

\[
\lesssim \left( \sum_{M \ll N_2} N_2^2 M^{-2\epsilon} \| D_x^\epsilon P_M u \|^2_{L^p_x L^\infty_t} \right)^{1/2} \lesssim \left( \sum_{M \ll N_2} N_2^{2\epsilon} \| D_x^\epsilon P_M u \|^2_{L^p_x L^\infty_t} \right)^{1/2}
\]
where \( \varepsilon = (p - 1)/2p \). Thus, (4.5) can be bounded by

\[
\lesssim T^{1/2} \left( \sum_{N_2 \leq 1} N_2^{2\varepsilon} \right)^{1/2} \| u \|_{X_T}^{1/2} \lesssim T^{1/2} \| u \|_{X_T}^{1/2}.
\]

Now, we turn to the case \( 1 \ll N_1 \ll N_2 \). From the Littlewood-Paley theorem, we have for \( p \geq 4 \)

\[
\left\| \sum_{1 < N_1 < N_2} P_{N_1} u P_{N_2} \bar{u} \right\|_{L_x^p L_t^q} \approx \left\| \sum_{N_1 \gg 1} P_{N_1} u P_{N_1} \bar{u} \right\|_{L_x^p L_t^q}
\]

\[
\lesssim \sum_{N_1 \gg 1} \| P_{N_1} u P_{N_1} \bar{u} \|_{L_x^p L_t^q} \lesssim \sum_{N_1 \gg 1} \| P_{N_1} u \|_{L_x^p L_t^q} \| P_{N_1} \bar{u} \|_{L_x^p L_t^q}.
\]  \quad (4.6)

Noticing that

\[
\| P_{N_1} \bar{u} \|_{L_x^1 L_t^{p^*}} \sim \left( \sum_M |P_M P_{N_1} \bar{u}|^2 \right)^{1/2} \lesssim \left( \sum_{M \gg N_1} \| P_M u \|_{L_x^p L_t^{p^*}}^2 \right)^{1/2}
\]

and for \( N_1 \gg 1 \), \( \varepsilon = 1/p \) and \( p \geq 4 \)

\[
\| P_{N_1} u \|_{L_x^p L_t^{p^*}} = \| P_{N_1} (P_{N_1} u + P_{N_1} \bar{u}) \|_{L_x^p L_t^{p^*}} = \| P_{N_1} P_{N_1} u \|_{L_x^p L_t^{p^*}} \sim N_1^{-\varepsilon} \| D_x^\varepsilon P_{N_1} P_{N_1} u \|_{L_x^p L_t^{p^*}}
\]

\[
\lesssim N_1^{-\varepsilon} \| D_x^\varepsilon P_{N_1} P_{N_1} u \|_{L_x^1 L_t^{p^*}},
\]  \quad (4.7)

we can bound (4.6) by

\[
\lesssim \| u \|_{X_T} \sum_{N_1 \gg 1} N_1^{-\varepsilon} \| D_x^\varepsilon P_{N_1} P_{N_1} u \|_{L_x^1 L_t^{p^*}}
\]

\[
\lesssim \| u \|_{X_T} \sum_{N_1 \gg 1} N_1^{-2\varepsilon} \left( \sum_{N_1 \gg 1} \| D_x^\varepsilon P_{N_1} P_{N_1} u \|_{L_x^1 L_t^{p^*}}^2 \right)^{1/2}
\]

\[
\lesssim \| u \|_{X_T} \int_{X_T} \| P_{N_1} u \|_{X_T}, \quad \forall p \geq 4.
\]  \quad (4.8)

in view of the Hölder inequality. Thus, we have obtained

\[
\| u \|_{Y_T} = \| u \|_{L_x^p H^{1/2}_T} + \| \partial_x u \|_{L_x^p L_t^2} + \| u \|_{L_x^p L_t^\infty} + \| (D_x)_{1/2} u \|_{L_x^p L_t^\infty}.
\]

Therefore, we have the desired result (4.1) for any real number \( p \geq 4 \). \( \square \)

5. Nonlinear estimates. To state the estimates for the nonlinearities \( I_{N,j} \), we define the function space \( Y_T \) equipped with the following norm:

\[
\| u \|_{Y_T} = \| u \|_{L_x^p H^{1/2}_T} + \| \partial_x u \|_{L_x^p L_t^2} + \| u \|_{L_x^p L_t^\infty} + \| (D_x)_{1/2} u \|_{L_x^p L_t^\infty}.
\]

We have the following proposition for the nonlinearities.
**Proposition 5.1.** Let \( u \) be a \( H^\infty \)-solution to (1.1)–(1.2). Then,

\[
\left( \sum_{N \geq 1} \left( \int_0^t S(t - \tau) \sum_{j=1}^5 I_{N,j}(\tau) d\tau \right)^2 \right)^{1/2}
\lesssim (1 + \|u\|_{L^\infty_x}^k) \left[ T^{\frac{1}{2}} \|u\|_{L^2_x}^{k+1} \|P_{>1}u\|_{L^2_x}^{k-k} + (1 + T^{\frac{1}{2}} \|u\|_{L^2_x}) \|u\|_{L^2_x}^k \|P_{>1}u\|_{L^2_x}^{k+1-k} \right]
\]

\[
+ T^{\frac{1}{2}} \left( \|u\|_{L^2_x}^{2k-1} + \|u\|_{L^2_x}^{(5k-2)/2} \right) \|P_{>1}u\|_{L^2_x}
\]

\[
+ T^{\frac{1}{2}} (1 + T^{\frac{1}{2}} \|u\|_{L^2_x}) \frac{1}{2} \|u\|_{L^2_x}^{2k-1} \|P_{>1}u\|_{L^2_x}^{\frac{3}{2}}
\]

\[
+ (1 + \|u\|_{L^2_x}^k) \left[ T^{\frac{1}{2}} \|u\|_{L^2_x}^k \|P_{>1}u\|_{L^2_x} + (1 + T^{\frac{1}{2}} \|u\|_{L^2_x})^2 \|u\|_{L^2_x}^{k-1} \|P_{>1}u\|_{L^2_x}^2 \right]
\]

\[
+ T^{\frac{1}{2}} \|u\|_{L^2_x}^{3k} \|P_{>1}u\|_{L^2_x},
\]

where \( \tilde{k} \) denotes the maximal integer that is less than \( k \) (i.e. \( \tilde{k} = [k] \) if \( k \) is not an integer and \( \tilde{k} = k - 1 \) if \( k \) is an integer where \([k]\) denotes the maximal integer that is less than or equal to \( k \)).

We consider each nonlinearity separately.

### 5.1. Nonlinear estimates of \( I_{N,1} \)

Noting that the term \( P_N(|P_{<N}u|^k u_x) \) has Fourier support in \( |\xi| \sim N \), we have

\[
P_N(|u|^k u_x) - |P_{<N}u|^k P_N u_x
\]

\[
= P_N((|u|^k - |P_{<N}u|^k) u_x) + P_N(|P_{<N}u|^k u_x) - |P_{<N}u|^k P_N u_x
\]

\[
= P_N((|u|^k - |P_{<N}u|^k) u_x) + P_N(|P_{<N}u|^k \tilde{P}_N u_x) - |P_{<N}u|^k P_N \tilde{P}_N u_x, \quad (5.1)
\]

where \( \tilde{P}_N = P_{N/2} + P_N + P_{2N} \).

For the second term in (5.1), we have the following estimate.

### Lemma 5.1

Let \( u \) be a solution of (1.1)–(1.2). Then, we have for any \( k \geq 4 \)

\[
\left( \sum_{N \geq 1} \left( \left\| P_N(|P_{<N}u|^k \tilde{P}_N u_x) - |P_{<N}u|^k P_N \tilde{P}_N u_x \right\|_{L^2_t L^2_x} \right)^2 \right)^{1/2}
\]

\[
\lesssim T^{\frac{1}{2}} \|u\|_{L^2_x}^k \|P_{>1}u\|_{L^2_x} + (1 + T^{\frac{1}{2}} \|u\|_{L^2_x}) \|u\|_{L^2_x}^{k-1} \|P_{>1}u\|_{L^2_x}^2.
\]

**Proof.** To shift a derivative from the high-frequency function \( P_N u_x \) to the low-frequency function \( |P_{<N}u|^k \), we require the following Leibniz rule for \( P_N \) from [11]:

\[
(P_N(f)g - f P_N g)(x) = \int_0^1 \left( \int \varphi_N(y) y f_x(x - \eta y) g(x - y) dy \right) d\eta.
\]

Thus, we have

\[
\left\| P_N(|P_{<N}u|^k \tilde{P}_N u_x) - |P_{<N}u|^k P_N \tilde{P}_N u_x \right\|_{L^1_t L^2_x}
\]

\[
\lesssim \left\| \varphi_N(y) \right\|_{L^\infty_y} \left\| (|P_{<N}u|^k)_x \right\|_{L^2_y L^\infty_t} \| \tilde{P}_N u_x \|_{L^2_x L^\infty_t}
\]

\[
\lesssim \left\| \varphi_1(y) \right\|_{L^\infty_y} \left\| (|P_{<N}u|^k)_x \right\|_{L^2_y L^\infty_t} \| \tilde{P}_N u_x \|_{L^2_x L^\infty_t}
\]

\[
\lesssim \left\| (|P_{<N}u|^k)_x \right\|_{L^2_y L^\infty_t} \| \tilde{P}_N u_x \|_{L^2_x L^\infty_t}
\]
Thus,

\[ \lesssim \| P_{\ll N} u \|^{k-2}_{L^2_x L^\infty_t} \| P_{\ll N} u \|_{L^2_x L^\infty_t} \| \tilde{P}_N u \|_{L^2_x L^\infty_t}. \]  \tag{5.4} \]

Decomposing \( P_{\ll N} u = P_{\ll 1} u + P_{1<\ll N} u \) for \( N \gg 1 \), we have

\[ P_{\ll N} u = P_{\ll 1} u + P_{1<\ll N} u \]

for \( N \gg 1 \), we have

\[ P_{\ll N} u = P_{\ll 1} u + P_{1<\ll N} u \]

By the Littlewood-Paley theorem, we can obtain

\[ \| P_{\ll N} u \|_{L^2_x L^\infty_t} \lesssim \left( \sum_{M} \| P_M P_{\ll N} u \|^{2}_{L^2_x L^\infty_t} \right)^{1/2} \lesssim \left( \sum_{M} \| P_M u \|^{2}_{L^2_x L^\infty_t} \right)^{1/2} \]

\[ \lesssim \left( \sum_{M>1} \| P_M u \|^{2}_{L^2_x L^\infty_t} \right)^{1/2} \lesssim \| P_{1<} u \|_{X_T}. \]  \tag{5.5} \]

For the first term in \( \text{(5.5)} \), we have

\[ \| P_{\ll 1} u \|_{L^2_x L^\infty_t} \lesssim \| u \|_{X_T}. \]  \tag{5.6} \]

In the similar way, we have

\[ \| P_{\ll N} u \|_{L^2_x L^\infty_t} \lesssim \| u \|_{X_T}. \]  \tag{5.7} \]

For the last two term in \( \text{(5.5)} \), in a similar way as in the proof of Proposition \ref{prop:control}, we can obtain the following bound:

\[ \| P_{1<} u \|_{L^2_x L^\infty_t} \lesssim T^{1/2} \| u \|_{X_T} \| P_{1>} u \|_{X_T}. \]  \tag{5.8} \]

\[ \| P_{1<} u \|_{L^2_x L^\infty_t} \lesssim T^{1/2} \| u \|_{X_T} + (1 + T^{1/2} \| u \|_{X_T}) \| P_{1>} u \|_{X_T}. \]  \tag{5.9} \]

From the Sobolev embedding theorem and \( \text{(5.7)} \), \( \text{(5.9)} \), we obtain that \( \text{(5.4)} \) can be bounded by

\[ \lesssim \left( T^{1/2} \| u \|_{X_T} + (1 + T^{1/2} \| u \|_{X_T}) \| u \|_{X_T} \| P_{\gg 1} u \|_{X_T} \right) \| \tilde{P}_N u \|_{L^2_x L^\infty_t}. \]

Thus, we can bound \( \text{(5.2)} \) by

\[ \lesssim \left( T^{1/2} \| u \|_{X_T} + (1 + T^{1/2} \| u \|_{X_T}) \| u \|_{X_T} \| P_{\gg 1} u \|_{X_T} \right) \left( \sum_{N \gg 1} \| P_N u \|^{2}_{L^2_x L^\infty_t} \right)^{1/2} \]

\[ \lesssim T^{1/2} \| u \|_{X_T} \| P_{\gg 1} u \|_{X_T} + (1 + T^{1/2} \| u \|_{X_T}) \| u \|_{X_T} \| P_{\gg 1} u \|_{X_T}^{2}, \]

which yields the desired result. \qed

For the first term in \( \text{(5.1)} \), we have the following estimate:
Lemma 5.2. Let \( u \) be a solution of (1.1)-(1.2). Then, we have for any \( k \geq 4 \)
\[
\left( \sum_{N \gg 1} \| P_N (|u|^k - |P_{\leq N} u|^k) u_x \|^2_{L^1_T L^2_x} \right)^{1/2} \lesssim T^\frac{1}{4} \| u \|^k_{X^k_T} \| P_{\gg 1} u \|_{X^k_T} + (1 + T^\frac{1}{4} \| u \|_{X^k_T}) \| u \|^k_{X^k_T} \| P_{\gg 1} u \|_{X_T^k}.
\]
(5.10)

Proof. We split (5.11) into several terms for \( N \gg 1 \) and \( k \geq 4 \)
\[
P_N (|u|^k - |P_{\leq N} u|^k) u_x
= P_N (|u|^{k-2} \bar{u} u x P_{\geq N} u)
+ P_N (|u|^k - |P_{\leq N} u|^k) \bar{u} u x P_{\leq N} u)
+ P_N (|P_{\leq N} u|^{k-2} u_x P_{\leq N} u P_{\geq N} \bar{u}).
\]
(5.12)
(5.13)
(5.14)

Notice that
\[
\| P_{\geq N} u \|_{L^1_T L^2_x} \lesssim \left( \sum_{M \geq N} \| P_M u \|^2_{L^1_T L^2_x} \right)^{1/2} \lesssim \left( \sum_{M \geq N} N^{-2\varepsilon_k} M^{2\varepsilon_k} \| P_M u \|^2_{L^1_T L^2_x} \right)^{1/2}
\lesssim \left( \sum_{M \geq N} N^{-2\varepsilon_k} \| D_x^{\varepsilon_k} P_M u \|^2_{L^1_T L^2_x} \right)^{1/2}
\lesssim N^{-\varepsilon_k} \left( \sum_{M \geq N} \| D_x \| \| P_M u \|^2_{L^1_T L^2_x} \right)^{1/2}
\lesssim N^{-\varepsilon_k} \| P_{\gg 1} u \|_{X_T^k}, \quad \forall k \geq 4,
\]
where \( \varepsilon_k > 0 \) is defined by \( \varepsilon_k = 1/k \).

Thus, for the first term (5.12), from the fact \( \| \tilde{\varphi}_N \|_{L^1_T} \lesssim 1 \) and Proposition 4.1 we have for \( k \geq 4 \)
\[
\| P_N (|u|^{k-2} \bar{u} u_x P_{\geq N} u) \|_{L^1_T L^2_x} \lesssim \| |u|^{k-2} \bar{u} u_x P_{\geq N} u \|_{L^1_T L^2_x}
\lesssim \| |u|^{k-2} \bar{u} u_x \|_{L^1_T L^2_x} \| P_{\geq N} u \|_{L^1_T L^2_x}
\lesssim N^{-\varepsilon_k} \left[ T^\frac{1}{4} \| u \|^k_{X^k_T} \| P_{\gg 1} u \|_{X^k_T} + (1 + T^\frac{1}{4} \| u \|_{X^k_T}) \| u \|^k_{X^k_T} \| P_{\gg 1} u \|^2_{X^k_T} \right].
\]

Therefore, we obtain, for any \( k \geq 4 \), that
\[
\left( \sum_{N \gg 1} \| P_N (|u|^{k-2} \bar{u} u_x P_{\geq N} u) \|^2_{L^1_T L^2_x} \right)^{1/2} \lesssim T^\frac{1}{4} \| u \|^k_{X^k_T} \| P_{\gg 1} u \|_{X^k_T} + (1 + T^\frac{1}{4} \| u \|_{X^k_T}) \| u \|^k_{X^k_T} \| P_{\gg 1} u \|^2_{X^k_T}.
\]

For (5.13), in the same way as the case (5.12), we have
\[
\left( \sum_{N \gg 1} \| P_N (|P_{\leq N} u|^{k-2} u_x P_{\leq N} u P_{\geq N} \bar{u}) \|^2_{L^1_T L^2_x} \right)^{1/2} \lesssim T^\frac{1}{4} \| u \|^k_{X^k_T} \| P_{\gg 1} u \|_{X^k_T} + (1 + T^\frac{1}{4} \| u \|_{X^k_T}) \| u \|^k_{X^k_T} \| P_{\gg 1} u \|^2_{X^k_T}.
\]

Now, we derive the estimate for (5.14) by using the induction argument in \( k \).
For $k = 4$, we have
\[ |u|^{-2} - |P_{<N}u|^{-2} \lesssim |P_{>N}u|^2 + |P_{<N}uP_{<N}u|. \]

From the Young inequality, the Hölder inequality, (5.6) and Proposition 4.1, we can get for $k = 4$
\[ \left\| P_N((|u|^{-2} - |P_{<N}u|^{-2})\bar{u}u_xP_{<N}u) \right\|_{L^1_t L^2_x} \]
\[ \lesssim \left\| |u|^{-2} - |P_{<N}u|^{-2} \right\|_{L^{5/4}(L^2_x)} \left\| \bar{u}u_x \right\|_{L^5_t L^2_x} \left\| P_{<N}u \right\|_{L^5_t L^5_x} \]
\[ \lesssim \left\| |u|^{-2} - |P_{<N}u|^{-2} \right\|_{L^{5/4}(L^2_x)} \left\| \bar{u}u_x \right\|_{L^5_t L^2_x} \left\| P_{<N}u \right\|_{L^5_t L^5_x} \]
\[ \lesssim N^{-1/8} \left\| P_{>1}u \right\|_{X^T}^2 \left\| \bar{u}u_x \right\|_{L^5_t L^2_x} \left\| P_{<N}u \right\|_{L^5_t L^5_x} \]
\[ \lesssim N^{-1/8} \left[ T_{\frac{k}{2}} \left\| \bar{u}u_x \right\|_{X^T}^2 \left\| P_{>1}u \right\|_{X^T} + (1 + T_{\frac{k}{2}} \left\| u \right\|_{X^T}) \left\| \bar{u}u_x \right\|_{X^T} \left\| P_{>1}u \right\|_{X^T} \right]. \]

From the triangle inequality for complex number, i.e. $|z_1| - |z_2| \lesssim |z_1 - z_2|$ for $z_1, z_2 \in \mathbb{C}$, we can get $|z_1| - |z_2|$ for any $\theta \in (0, 1)$.

For $k \in (4, 5)$, we have
\[ \left\| |u|^{-2} - |P_{<N}u|^{-2} \right\|_{L^1_t L^2_x} \]
\[ \lesssim \left\| |u|^{-2} - |P_{<N}u|^{-2} \right\|_{L^{5/4}(L^2_x)} \left\| \bar{u}u_x \right\|_{L^5_t L^2_x} \left\| P_{<N}u \right\|_{L^5_t L^5_x} \]
\[ \lesssim N^{-\varepsilon_k} \left[ T_{\frac{k}{2}} \left\| \bar{u}u_x \right\|_{X^T}^2 \left\| P_{>1}u \right\|_{X^T} + (1 + T_{\frac{k}{2}} \left\| u \right\|_{X^T}) \left\| \bar{u}u_x \right\|_{X^T} \left\| P_{>1}u \right\|_{X^T} \right]. \]

where $\varepsilon_k = (k - 4)/2k$ for $k \in (4, 5)$. By the same procedure, we can obtain for any $k \geq 4$
\[ \left\| P_N((|u|^{-2} - |P_{<N}u|^{-2})\bar{u}u_xP_{<N}u) \right\|_{L^1_t L^2_x} \]
\[ \lesssim N^{-\varepsilon_k} \left[ T_{\frac{k}{2}} \left\| \bar{u}u_x \right\|_{X^T}^2 \left\| P_{>1}u \right\|_{X^T} + (1 + T_{\frac{k}{2}} \left\| u \right\|_{X^T}) \left\| \bar{u}u_x \right\|_{X^T} \left\| P_{>1}u \right\|_{X^T} \right]. \]

where $\varepsilon_k = (k - \hat{k})/2k > 0$. Therefore, we have for any $k \geq 4$
\[ \left( \sum_{N \geq 1} \left\| P_N((|u|^{-2} - |P_{<N}u|^{-2})\bar{u}u_xP_{<N}u) \right\|_{L^1_t L^2_x} \right)^{1/2} \]
\[ \lesssim T_{\frac{k}{2}} \left\| \bar{u}u_x \right\|_{X^T}^2 \left\| P_{>1}u \right\|_{X^T} + (1 + T_{\frac{k}{2}} \left\| u \right\|_{X^T}) \left\| \bar{u}u_x \right\|_{X^T} \left\| P_{>1}u \right\|_{X^T}. \]

Thus, we have proved the desired result. $\square$

**Remark 5.1.** From the proof of Lemma 5.2, we can see that
\[ \left( \sum_{N \geq 1} \left\| P_N((|u|^{-2} - |P_{<N}u|^{-2})\bar{u}u_xP_{<N}u) \right\|_{L^1_t L^2_x} \right)^{1/2} \]
holds for any \( \varepsilon \in (0, 1) \) in view of Proposition 4.1.

We turn to the proof of Proposition 5.4 for the nonlinearity \( I_{N, 1} \). We also consider the decomposition in (5.1). For convenience, we denote \( B_N = P_N((P_{\ll N} u)^k \tilde{P}_N u_x) - |P_{\ll N} u|^k P_N \tilde{P}_N u_x). \) From (3.3), (3.11), (3.6) and (3.7), we have

\[
\left( \sum_{N \gg 1} \left\| \int_0^t S(t - \tau) e^{-\frac{\partial}{\partial \tau}} f_{-\infty}^\tau |P_{\ll N} u|^k dy B_N d\tau \right\|_{Y_T}^2 \right)^{1/2} \\
\lesssim \left( \sum_{N \gg 1} \| B_N \|_{L_t^1 L_x^\infty}^2 + \left( \sum_{N \gg 1} \left( \sum_M \| P_M(e^{-\frac{\partial}{\partial x}} f_{-\infty}^\tau |P_{\ll N} u|^k dy B_N) \right)_{L_t^1 L_x^\infty}^2 \right)^{1/2} \right). 
\]

(5.17)

By Lemma 5.1, the first term can be bounded by

\[
\lesssim T^{\frac{1}{2}} \| u \|_{X_T}^k \| P_{\gg 1} u \|_{X_T} + (1 + T^{\frac{1}{2}} \| u \|_{X_T}) \| u \|_{X_T}^{k-1} \| P_{\gg 1} u \|_{X_T}^2. 
\]

For the second term, we split the sum \( \sum_M \) into three parts \( \sum_{M \sim N} + \sum_{M \ll N} + \sum_{M \gg N} \) as in (14). For the part of \( M \sim N \), it is the same as Lemma 5.1 by summing in \( M \) such that \( M \sim N \). For the part \( M \ll N \), we can add the projection operator \( P_{\ll N} \) to \( e^{-\frac{\partial}{\partial x}} f_{-\infty}^\tau |P_{\ll N} u|^k dy \) since \( B_N \) has Fourier support in \( |\xi| \sim N \). Thus, by the Hölder inequality, we have

\[
\left( \sum_{N \gg 1} \left( \sum_{M \ll N} \| P_M(e^{-\frac{\partial}{\partial x}} f_{-\infty}^\tau |P_{\ll N} u|^k dy B_N) \right)_{L_t^1 L_x^\infty}^2 \right)^{1/2} \\
\lesssim \left( \sum_{N \gg 1} \left( \sum_{M \ll N} \| P_{\ll N} e^{-\frac{\partial}{\partial x}} f_{-\infty}^\tau |P_{\ll N} u|^k dy B_N \right)_{L_t^1 L_x^\infty}^2 \right)^{1/2} \\
\lesssim \left( \sum_{N \gg 1} (\ln N)^2 \| P_{\ll N} e^{-\frac{\partial}{\partial x}} f_{-\infty}^\tau |P_{\ll N} u|^k dy \|_{L_t^{1/(1-\varepsilon)} L_x^\infty}^2 \| B_N \|_{L_t^{1/(1-\varepsilon)} L_x^\infty}^2 \right)^{1/2}, 
\]

(5.18)

where \( \varepsilon \in (0, 1/k) \).

By the Bernstein inequality, we have

\[
N \| P_{\ll N} e^{-\frac{\partial}{\partial x}} f_{-\infty}^\tau |P_{\ll N} u|^k dy \|_{L_t^{1/k} L_x^\infty} \lesssim \| \partial_x P_{\ll N} e^{-\frac{\partial}{\partial x}} f_{-\infty}^\tau |P_{\ll N} u|^k dy \|_{L_t^{1/k} L_x^\infty} \\
\lesssim \| P_{\ll N} u \|_{L_t^{1/k} L_x^\infty}^k \lesssim \| u \|_{X_T}^k,
\]

and from (5.3) and the Hölder inequality, we can get, as a similar way as in (5.4), that

\[
\| B_N \|_{L_t^{1/(1-\varepsilon)} L_x^\infty} = \|(P_{\ll N} u)^k \|_{L_t^{2/(1-2\varepsilon)} L_x^\infty} \| \tilde{P}_N u \|_{L_t^2 L_x^\infty} \\
\lesssim \| P_{\ll N} u \|_{L_t^{2/(1-2\varepsilon)} L_x^\infty} \| P_{\ll N} u_x \|_{L_t^1 L_x^\infty} \| \tilde{P}_N u \|_{L_t^1 L_x^\infty} \\
\lesssim \left( T^{\frac{1}{2}} \| u \|_{X_T}^k + (1 + T^{\frac{1}{2}} \| u \|_{X_T}) \| u \|_{X_T}^{k-1} \| P_{\gg 1} u \|_{X_T} \| P_N u \|_{L_t^2 L_x^\infty}. 
\]

Similarly, from (3.5), (3.11), (3.6) and (3.7), we can get

\[ A \]

For the part in (5.1), we denote it by

\[ B \]

Thus, (5.18) can be bounded by

\[ C. C. HAO \]

\[ M \]

In a similar way with the part

\[ F \]

Therefore, we have obtained

\[ G \]

Noticing that (5.16), and in the same way as in dealing with the second term of

\[ H \]

For the part

\[ I \]

For the part

\[ J \]

In a similar way with the part

\[ K \]

Therefore, we have obtained

\[ L \]

From Lemma [5.2], the first term is bounded by

\[ M \]

Noticing that (5.19), and in the same way as in dealing with the second term of

\[ N \]

Therefore, we have obtained

\[ O \]
5.2. Nonlinear estimates of $I_{N,2}$. From (3.4), (3.8), (3.9) and (3.10), we have

$$
\left( \sum_{N \geq 1} \left\| \int_0^t S(t-\tau) I_{N,2}(\tau) d\tau \right\|_{L_T^2} \right)^{1/2}
$$

$$
\lesssim \left( \sum_{N \geq 1} \left\| e^{-i \frac{\partial}{\partial x} \int_{-\infty}^{x} |P_{\leq N} u|^4 dy} P_{\leq N} u B_{N,2} \right\|_{L_T^2 H_x^{1/2}} \right)^{1/2} 
$$

$$
+ \left( \sum_{N \geq 1} \left( \sum_M \left\| P_M(e^{-i \frac{\partial}{\partial x} \int_{-\infty}^{x} |P_{\leq N} u|^4 dy} P_{\leq N} u B_{N,2}) \right\|_{L_T^2 H_x^{1/2}} \right)^2 \right)^{1/2},
$$

(5.20)

where $B_{N,2} = \int_{-\infty}^{x} |P_{\leq N} u|^{k-4} \left[ (\overline{P_{\leq N} u_x} P_{\leq N} u)^2 - (P_{\leq N} u_x \overline{P_{\leq N} u})^2 \right] dy$. For the first term (5.20), from Lemma 5.3 and the Hölder inequality, it can be bounded by

$$
\lesssim \sum_{N \geq 1} \left\| P_{\leq N} u B_{N,2} \right\|_{L_T^2 L_x^2}^2
$$

$$
+ \left\| P_{\leq N} u B_{N,2} \right\|_{L_T^2 L_x^2} \left\| \partial_x(e^{-i \frac{\partial}{\partial x} \int_{-\infty}^{x} |P_{\leq N} u|^4 dy} P_{\leq N} u B_{N,2}) \right\|_{L_T^2 L_x^2}^{1/2}
$$

$$
\lesssim \left( \sum_{N \geq 1} \left\| P_{\leq N} u \right\|_{L_T^\infty L_x^2}^2 \| B_{N,2} \|_{L_T^2 L_x^\infty}^2 + \left\| P_{\leq N} u \right\|_{L_T^2 H_x^{1/2}} \| B_{N,2} \|_{L_T^2 L_x^\infty}^2
$$

$$
+ \left\| P_{\leq N} u \right\|_{L_T^\infty L_x^2} \| B_{N,2} \|_{L_T^2 L_x^\infty} \left\| P_{\leq N} u \right\|_{L_T^2 L_x^\infty}^{k-4} \left\| P_{\leq N} u \right\|_{L_T^2 L_x^\infty}^{2(k-1)} \left\| P_{\leq N} u \right\|_{L_T^2 L_x^\infty}^{2(k-1)}
$$

$$
+ \left\| P_{\leq N} u \right\|_{L_T^\infty L_x^2} \| B_{N,2} \|_{L_T^2 L_x^\infty} \left\| P_{\leq N} u \right\|_{L_T^2 L_x^\infty} \left\| \partial_x B_{N,2} \right\|_{L_T^2 L_x^\infty} \right)^{1/2}.
$$

(5.22)

By the Hölder inequality, we have for $k \geq 5$

$$
\| B_{N,2} \|_{L_T^\infty L_x^\infty} \lesssim \left\| (\overline{P_{\leq N} u_x} P_{\leq N} u)^2 - (P_{\leq N} u_x \overline{P_{\leq N} u})^2 \right\|_{L_T^1} \lesssim \left\| P_{\leq N} u \right\|_{L_T^\infty L_x^2}^{k-4} \left\| P_{\leq N} u \right\|_{L_T^\infty L_x^\infty}^{2(k-1)}
$$

$$
\lesssim \left( T \left\| u \right\|_{X_T}^k \right) \left( 1 + T^{\frac{k}{4}} \left\| u \right\|_{X_T} \right)^2 \left\| u \right\|_{X_T}^{k-2} \left\| P_{\geq 1} u \right\|_{X_T}^2,
$$

(5.23)

and from Proposition 4.1 and the proof of Lemma 5.1

$$
\left\| P_{\leq N} u \right\|_{L_T^\infty L_x^2} \left\| \partial_x B_{N,2} \right\|_{L_T^2 L_x^\infty}
$$

$$
= \left\| P_{\leq N} u \right\|_{L_T^\infty L_x^2} \left\| (\overline{P_{\leq N} u_x} P_{\leq N} u)^2 - (P_{\leq N} u_x \overline{P_{\leq N} u})^2 \right\|_{L_T^2 L_x^\infty}
$$

$$
\lesssim \left\| P_{\leq N} u_x \right\|_{L_T^\infty L_x^2} \left\| P_{\leq N} u_x \right\|_{L_T^2 L_x^\infty} \left\| P_{\leq N} u \right\|_{L_T^2 L_x^\infty}^{k-2} \left\| P_{\leq N} u \right\|_{L_T^2 L_x^\infty}^{2(k-1)}
$$

$$
\lesssim \left\| P_{\leq N} u_x \right\|_{L_T^\infty L_x^2} \left[ T^{\frac{k}{4}} \left\| u \right\|_{X_T}^k \right] \left( 1 + T^{\frac{k}{4}} \left\| u \right\|_{X_T} \right) \left\| u \right\|_{X_T}^{k-1} \left\| P_{\geq 1} u \right\|_{X_T}.
$$

Thus, we can bound (5.20) by

$$
\lesssim (1 + \left\| u \right\|_{X_T}^k) \left( T^{1/2} \left\| u \right\|_{X_T} \left\| P_{\geq 1} u \right\|_{X_T} \right) + \left( 1 + T^{\frac{k}{4}} \left\| u \right\|_{X_T} \right)^2 \left\| u \right\|_{X_T}^{k-2} \left\| P_{\geq 1} u \right\|_{X_T}.
$$
For \((5.21)\), we split the sum \(\sum_{M} \) into two parts \(\sum_{M \leq N} + \sum_{M \gg N}\); which gives the bound by

\[
\lesssim \left( \sum_{N \gg 1} \left( \sum_{M \leq N} \langle M \rangle^{k} \|P_{N}uB_{N,2}\|_{L_{t}^{1}L_{x}^{2}} \right)^{2} \right)^{1/2} \tag{5.24}
\]

\[
+ \left( \sum_{N \gg 1} \left( \sum_{M \gg N} \|P_{M}(D_{x})^{1}e^{-\frac{i}{4} \int_{-\infty}^{x} P_{\leq N}|u|^k dy}P_{N}uB_{N,2}\|_{L_{t}^{1}L_{x}^{2}} \right)^{2} \right)^{1/2} \tag{5.25}
\]

For the first term \((5.24)\), noticing that \(\sum_{M \leq N} \langle M \rangle^{k} \lesssim N^{1/2}\) and \((5.23)\), we can bound it by

\[
\lesssim \left( \sum_{N \gg 1} \left( \left\|D_{x}^{1}P_{N}u\right\|_{L_{t}^{\infty}L_{x}^{\infty}} \left\|B_{N,2}\|_{L_{t}^{1}L_{x}^{2}} \right) \right)^{2} \right)^{1/2}
\]

\[
\lesssim \|u\|_{X_{T}}^{k} \|P_{\gg 1}u\|_{X_{T}}. \]

For the second term \((5.25)\), in a similar way with \((5.22)\), we bound it by

\[
\lesssim \left( \sum_{N \gg 1} \left( \sum_{M \gg N} M^{-\frac{k}{2}} \|P_{M}(D_{x})e^{-\frac{i}{4} \int_{-\infty}^{x} P_{\leq N}|u|^k dy}P_{N}uB_{N,2}\|_{L_{t}^{1}L_{x}^{2}} \right)^{2} \right)^{1/2}
\]

\[
\lesssim \left( \sum_{N \gg 1} \|P_{N}uB_{N,2}\|_{L_{t}^{1}L_{x}^{2}}^{2} + \left\|\partial_{x}e^{-\frac{i}{4} \int_{-\infty}^{x} P_{\leq N}|u|^k dy}P_{N}uB_{N,2}\|_{L_{t}^{1}L_{x}^{2}} \right)^{2} \right)^{1/2}
\]

\[
\lesssim (1 + \|u\|_{X_{T}}^{k}) \left( T^{1/2} \|u\|_{X_{T}}^{k} \|P_{\gg 1}u\|_{X_{T}} + (1 + T^{\frac{k}{4}} \|u\|_{X_{T}} \right)^{2} \|u\|_{X_{T}}^{k-2} \|P_{\gg 1}u\|_{X_{T}}^{3}. \]

Therefore, we obtain

\[
\left( \sum_{N \gg 1} \int_{0}^{T} S(t-\tau)I_{N,2}(\tau) d\tau \right)^{2} \lesssim (1 + \|u\|_{X_{T}}^{k}) \left( T^{1/2} \|u\|_{X_{T}}^{k} \|P_{\gg 1}u\|_{X_{T}} + (1 + T^{\frac{k}{4}} \|u\|_{X_{T}} \right)^{2} \|u\|_{X_{T}}^{k-2} \|P_{\gg 1}u\|_{X_{T}}^{3}. \]

5.3. Nonlinear estimates of \(I_{N,3}\). From \((3.21)\), \((8.8)\), \((3.9)\) and \((3.10)\), we have

\[
\left( \sum_{N \gg 1} \int_{0}^{T} S(t-\tau)I_{N,3}(\tau) d\tau \right)^{2} \lesssim \left( \sum_{N \gg 1} \left\|e^{-\frac{i}{4} \int_{-\infty}^{x} P_{\leq N}|u|^k dy}P_{N}uB_{N,3}\|_{L_{t}^{1}H_{x}^{1/2}} \right)^{2} \right)^{1/2} \tag{5.26}
\]

\[
\lesssim \left( \sum_{N \gg 1} \left\|P_{N}(e^{-\frac{i}{4} \int_{-\infty}^{x} P_{\leq N}|u|^k dy}P_{N}uB_{N,3})\|_{L_{t}^{1}H_{x}^{1/2}} \right)^{2} \right)^{1/2}, \tag{5.27}
\]

where \(B_{N,3} = \int_{-\infty}^{x} P_{\leq N}|u|^{k-2} P_{\leq N}|u|^{k}(u_{x} + \bar{u}) dy\). By Hölder inequality, we get

\[
\|B_{N,3}\|_{L_{t}^{1}L_{x}^{2}} \lesssim \|P_{\leq N}|u|^{k-2} P_{\leq N}|u|^{k}(u_{x} + \bar{u})\|_{L_{t}^{1}L_{x}^{2}}.
\]
\[ \lesssim T^{\frac{1}{2}} \| P_{\leq N} u \|_{L_x^{k-2} L_t^{\infty}}^{k-2} \| P_{\leq N} u | u|^k (u_x + \bar{u}_x) \|_{L_x^{(k-2)/k} L_t^1}. \]

\[ \lesssim T^{\frac{1}{2}} \| P_{\leq N} u \|_{L_x^{k-2} L_t^{\infty}}^{k-2} \| u \|_{L_x^{(k-2)/k} L_t^\infty} \| u_x \|_{L_x^\infty L_t^1}. \]

\[ \lesssim T^{\frac{1}{2}} \| u \|_{L_x^{2k-1} L_t^\infty}. \]

From the Hölder inequality and Proposition \[\text{[4.4]}\], we have
\[ \| \partial_x B_{N, 3} \|_{L_x^2 L_t^2} = \| P_{\leq N} u \|_{L_x^{k-2} L_t^{\infty}}^{k-2} \| P_{\leq N} u | u|^k (u_x + \bar{u}_x) \|_{L_x^2 L_t^2}. \]

\[ \lesssim \| P_{\leq N} u \|_{L_x^{k-2} L_t^{\infty}}^{k-2} \| u \|_{L_x^{(k-2)/k} L_t^\infty} \| \bar{u}_x \|_{L_x^{(k-2)/k} L_t^\infty} \]

\[ \lesssim T^{\frac{1}{2}} \| u \|_{L_x^{2k-1} L_t^\infty}^{2k-1} + (1 + T^{\frac{1}{2}} \| u \|_{L_t^\infty}) \| P_{\geq 1} u \|_{L_t^\infty}^{2}. \]

In addition, for \( N \gg 1 \), we have \( \| P_N u \|_{L_x^2 L_t^2} \lesssim \| P_N u_x \|_{L_x^2 L_t^2} \). Thus, in the same way as in the case \( I_{N, 2} \), we can bound \[\text{(5.20)}\] by
\[ \lesssim T^{\frac{1}{2}} \| u \|_{L_x^{2k-1} L_t^\infty}^{2k-1} \| P_{\geq 1} u \|_{L_t^\infty}^{2}. \]

5.4. Nonlinear estimates of \( I_{N, 4} \). From \[\text{(3.5)}, \text{(8.11)}, \text{(5.6)} \text{ and } \text{(3.7)}, \] we have
\[ \left( \sum_{N \gg 1} \left\| \int_0^T S(t - \tau) e^{-it \Delta} f_x \infty | P_{\leq N} u |^4 \, dy B_{N, 4} d\tau \right\|_{L_t^{\infty}} \right)^{1/2} \]
\[ \leq \left( \sum_{N \gg 1} \| B_{N, 4} \|_{L_x^2 L_t^2} \right)^{1/2}
\]
\[ + \left( \sum_{N \gg 1} \left( \sum_M \| P_M (e^{-it \Delta} f_x \infty | P_{\leq N} u |^4 \, dy B_{N, 4}) \|_{L_t^{\infty}} \right)^{1/2} \right)^{1/2}, \quad (5.29) \]

where \( B_{N, 4} = | P_{\leq N} u |^{k-2} P_N u P_{\leq N} u \mathcal{F} \). By the Hölder inequality, we have
\[ \| B_{N, 4} \|_{L_x^2 L_t^2} \lesssim \| P_{\leq N} u \|_{L_x^{k-2} L_t^{\infty}}^{k-2} \| P_N u \|_{L_x^k L_t^\infty} \| P_{\leq N} u \mathcal{F} \|_{L_x^2 L_t^\infty}. \]

\[ \lesssim \left[ T^{\frac{1}{2}} \| u \|_{L_x^k} + (1 + T^{\frac{1}{2}} \| u \|_{L_t^\infty}) \| u \|_{L_x^k} \right] \| P_{\geq 1} u \|_{L_t^\infty}. \]

Thus, the first term in \[\text{(5.20)}\] can be bounded by
\[ \lesssim T^{\frac{1}{2}} \| u \|_{L_x^k} \| P_{\geq 1} u \|_{L_t^\infty} + (1 + T^{\frac{1}{2}} \| u \|_{L_t^\infty}) \| u \|_{L_x^k} \| P_{\geq 1} u \|_{L_t^\infty}. \]

By the Hölder inequality, we get
\[ \| B_{N, 4} \|_{L_x^2 L_t^2} \lesssim \| P_{\leq N} u \|_{L_x^{k-2} L_t^\infty} \| P_N u \|_{L_x^k L_t^\infty} \| P_{\leq N} u \mathcal{F} \|_{L_x^2 L_t^\infty}. \]

\[ \lesssim \left[ T^{\frac{1}{2}} \| u \|_{L_x^k} + (1 + T^{\frac{1}{2}} \| u \|_{L_t^\infty}) \| u \|_{L_x^k} \right] \| P_{\geq 1} u \|_{L_t^\infty}. \]

Noticing that \( B_{N, 4} \) has Fourier support in \(|\xi| \sim N\), we can repeat the procedure which we use to deal with the second term in \[\text{(5.17)}, \] and obtain that the second term in \[\text{(5.20)}\] can be bounded by
\[ \lesssim T^{\frac{1}{2}} \| u \|_{L_x^2} \| P_{\geq 1} u \|_{L_t^\infty} + (1 + T^{\frac{1}{2}} \| u \|_{L_t^\infty}) \| u \|_{L_x^2} \| P_{\geq 1} u \|_{L_t^\infty}. \]
Therefore, we obtain
\[
\left( \sum_{N \gg 1} \left\| \int_0^t S(t-\tau)I_{N,4}(\tau) d\tau \right\|^2_{Y_T} \right)^{1/2} \lesssim (1 + \|u\|_{X_T}^k) \left[ T^k \|u\|_{X_T}^k \|P_{\geq 1}u\|_{X_T} + (1 + T^k \|u\|_{X_T}) \|u\|_{X_T}^{k-1} \|P_{\geq 1}u\|_{X_T}^2 \right].
\]

5.5. **Nonlinear estimates of** $I_{N,5}$. From (3.5), (3.11), (3.6) and (3.7), we have
\[
\left( \sum_{N \gg 1} \left\| \int_0^t S(t-\tau)e^{-\frac{\tau}{T}} \int_0^\infty |P_{\leq N}u|^4 dy B_{N,5} d\tau \right\|^2_{Y_T} \right)^{1/2} \lesssim \left( \sum_{N \gg 1} \|B_{N,5}\|_{L^1_tL^2_x}^2 \right)^{1/2} + \left( \sum_{N \gg 1} \left( \sum_M \|P_M(e^{-\frac{T}{N}} \int_0^\infty |P_{\leq N}u|^4 dy B_{N,5})\|_{L^1_tL^2_x} \right)^2 \right)^{1/2},
\]
where $B_{N,5} = |P_{\leq N}u|^{2k} P_Nu$. By the Hölder inequality, we have
\[
\|B_{N,5}\|_{L^1_tL^2_x} \lesssim T^{k} \|P_{\leq N}u\|_{L^4_tL^\infty_x}^{2k} \|P_Nu\|_{L^2_tL^\infty_x}
\lesssim T^{k} \|u\|_{L^2_tL^\infty_x}^{2k} \|P_Nu\|_{L^2_tL^\infty_x},
\]
and
\[
\|B_{N,5}\|_{L^1_tL^2_x} \lesssim T^{k} \|P_{\leq N}u\|_{L^4_tL^\infty_x}^{2k} \|P_Nu\|_{L^2_tL^\infty_x}
\lesssim T^{k} \|u\|_{L^2_tL^\infty_x}^{2k} \|P_Nu\|_{L^2_tL^\infty_x}.
\]
Thus, in a similar way as dealing with $I_{N,1}$ and $I_{N,4}$, and noticing that $B_{N,5}$ has Fourier support in $|\xi| \sim N$, we can bound (5.30) by
\[
\lesssim T^{k} \|u\|_{X_T}^{3k} \|P_{\geq 1}u\|_{X_T}.
\]

6. **A priori estimates for solutions.** By the scaling argument
\[
u(t, x) \mapsto u_\gamma(t, x) = \frac{1}{\gamma^{1/k}} u \left( \frac{t}{\gamma}, \frac{x}{\gamma} \right),
\]
we have
\[
\|u_{0,\gamma}\|_{L^2} = \gamma^{\frac{1}{2} - \frac{1}{k}} \|u_0\|_{L^2},
\]
\[
\|u_{0,\gamma}\|_{H^{\frac{1}{2}}} = \frac{1}{\gamma^{1/k}} \|u_0\|_{H^{\frac{1}{2}}}.
\]
Thus, we may rescale
\[
\|P_{\leq 1}u_{0,\gamma}\|_{L^2} \lesssim \gamma^{\frac{1}{2} - \frac{1}{k}} \|u_0\|_{L^2} = \mathcal{C}_{low},
\]
\[
\|P_{\geq 1}u_{0,\gamma}\|_{H^{\frac{1}{2}}} \lesssim \frac{1}{\gamma^{1/k}} \|u_0\|_{H^{\frac{1}{2}}} < \mathcal{C}_{high} \ll 1,
\]
where we choose $\gamma = \gamma(\|u_0\|_{H^{1/2}}) \gg 1$, and take the time interval $T$ depending on $\gamma$ later. We now drop the writing of the scaling parameter $\gamma$ and assume
\[
\|P_{\leq 1}u_0\|_{L^2} \leq \mathcal{C}_{low}.
\]
We now apply this to the norms $X_T$ and $H^{1/2}$, and define new version of the norms of $X_T$ and $H^{1/2}$, given by with the decomposition $I = P_{\leq 1} + P_{\gg 1}$,

$$||u||_{X_T} = \frac{1}{C_{low}} ||P_{\leq 1}u||_{X_T} + \frac{1}{C_{high}} ||P_{\gg 1}u||_{X_T},$$

and

$$||\phi||_{H^{1/2}} = \frac{1}{C_{low}} ||P_{\leq 1}\phi||_{L^2} + \frac{1}{C_{high}} ||P_{\gg 1}\phi||_{H^{1/2}},$$

which implies that $||u_0||_{H^{1/2}} \leq 2$.

For the low frequency part, we have the following estimates.

**Lemma 6.1.** Let $u$ be a solution of (1.1)–(1.2). Then

$$||P_{\leq 1}u||_{X_T} \lesssim C_{low} + T^{1/2} ||u||^{k+1}_{X_T}.$$

**Proof.** Using the integral equation of (1.1)

$$u(t) = S(t)u_0 - \chi \int_0^t S(t - \tau)|u(\tau)|^k u_x(\tau)d\tau,$$

and by (3.1), (3.2), (3.3), (3.8), (3.9), (3.10) and the Hölder inequality, we have

$$||P_{\leq 1}u||_{X_T} \lesssim ||S(t)P_{\leq 1}u_0||_{X_T} + \int_0^t S(t - \tau)P_{\leq 1}(|u|^k u_x)(\tau)d\tau_{X_T}$$

$$\lesssim ||P_{\leq 1}u_0||_{L^2} + ||P_{\leq 1}(|u|^k u_x)||_{L^1_T H^{1/2}} \lesssim C_{low} + ||u||^k ||u_x||_{L^2_T H^{1/2}}$$

$$\lesssim C_{low} + T^{1/2} ||u||^{k} ||u_x||_{L^2_T H^{1/2}}$$

$$\lesssim C_{low} + T^{1/2} ||u||^{k+1}_{X_T},$$

which is the desired result.  

For the high frequency part, we have

**Lemma 6.2.** Let $u$ and $v_N$ be given in (2.1). Then

$$||P_{\gg 1}u||_{X_T} \lesssim (1 + ||u||^{2k}_{L_T^2 H^{1/2}}) \left( \sum_{N \gg 1} ||v_N||^2_{Y_T} \right)^{1/2}.$$

**Proof.** By (2.1), we have

$$P_N u = e^{i t \phi} \int_0^t \sum_{|\xi| \leq N} |p_{\xi} u|^{k} d\eta_{\xi} v_N.$$ 

For $L_T^2 H^{1/2}_x$-norm, by the interpolation theorem, we obtain for $N \gg 1$,

$$||P_N u||_{H^{1/2}} \lesssim ||P_N u||^{1/2}_{L^2_T} ||P_N u||^{1/2}_{H^1} \lesssim ||v_N||^{1/2}_{L^2_T} (||P_N u||_{L^2_T} + ||\partial_x P_N u||_{L^2_T})^{1/2}$$

$$\lesssim ||v_N||^{1/2}_{L^2_T} \left(||v_N||_{L^2_T} + ||P_{\leq N} u_k v_N||_{L^2_T} + ||\partial_x v_N||_{L^2_T} \right)^{1/2}$$

$$\lesssim ||v_N||^{1/2}_{L^2_T} \left(||P_{\leq N} u||_{L^2_T} ||v_N||_{H^1_T} + ||v_N||_{H^1_T} \right)^{1/2}$$

$$\lesssim \left(1 + ||P_{\leq N} u||_{H^{1/2}} \right)^{1/2} ||v_N||_{H^{1/2}} \lesssim \left(1 + ||P_{\leq N} u||_{H^{1/2}} \right) ||v_N||_{H^{1/2}},$$

which yields the desired estimate by summing on $l_N^2$. 

For the $L^\infty_x L^2_T$-norm, noticing that

$$\partial_x P_N u = e^{\frac{i}{2} \int_{-\infty}^{\infty} [P_{\ll N} u]^k dy} (\partial_x v_N + \frac{i\lambda}{2} [P_{\ll N} u]^k v_N),$$

we have

$$\|\partial_x P_N u\|_{L^\infty_x L^2_T} \lesssim \|\partial_x v_N\|_{L^\infty_x L^2_T}$$

$$+ \left\| P^1_N \left( \sum_{N_1} P_{N_1} \left( e^{\frac{i}{2} \int_{-\infty}^{\infty} [P_{\ll N} u]^k dy} [P_{\ll N} u]^k \sum_{N_2} P_{N_2} v_N \right) \right) \right\|_{L^\infty_x L^2_T}. \quad (6.1)$$

To estimate the second term $\|(6.1)\|_\infty$, we split the sum $\sum_{N_2} = \sum_{N_2 \sim N} + \sum_{N_2 \gg N}$. For $N_2 \sim N$, from the Bernstein inequality, we bound $\|(6.1)\|_\infty$ by

$$\lesssim \left\| \sum_{N_2 \sim N} [P_{\ll N} u]^k \sum_{N_2} P_{N_2} v_N \right\|_{L^\infty_x L^2_T} \lesssim \left\| P_{\ll N} u \right\|_{L^\infty_x L^2_T} \sum_{N_2 \sim N} \|P_{N_2} v_N\|_{L^\infty_x L^2_T}$$

$$\lesssim N \left\| D_x^{-1/k} P_{\ll N} u \right\|_{L^\infty_x L^2_T} \sum_{N_2 \sim N} \|P_{N_2} v_N\|_{L^\infty_x L^2_T}$$

$$\lesssim \left\| P_{\ll N} u \right\|_{L^\infty_x H^{1/2}_T} \sum_{N_2 \sim N} \|P_{N_2} \partial_x v_N\|_{L^\infty_x L^2_T}$$

$$\lesssim \|u\|_{L^\infty_x H^{1/2}_T} \sum_{N_2 \sim N} \|P_{N_2} \partial_x v_N\|_{L^\infty_x L^2_T}.$$
Therefore, summing on \( l \), we complete the proof for the \( L^\infty L^2_T \)-norm.

For the \( L^2_T L^\infty_x \)-norm, it is easy to obtain the desired result since \(|P_N u| = |v_N|\).

We turn to estimate the \( L^4_T L^\infty_x \)-norm. It is similar with the proof for the \( L^\infty_T L^2_x \)-norm, since \( \| (D_x)^{1/4} P_N u \|_{L^1_T L^4_x} \sim N^{1/4} \| P_N u \|_{L^4_T L^2_x} \) for \( N \gg 1 \). In fact, we have

\[
\left\| (D_x)^{1/4} P_N u \right\|_{L^1_T L^4_x} \sim N^{1/4} \left\| \sum_{N_1} P_{N_1} \left( e^{\frac{i\lambda}{2} \int_{-\infty}^x \langle |P_{\leq N} u| \rangle^4 dy} \right) \sum_{N_2} P_{N_2} v_N \right\|.
\]  

We also split \( \sum_{N_2} = \sum_{N_2 \sim N} + \sum_{N_2 \gg N} \). For \( N_2 \sim N \), we bound (6.3) by

\[
\lesssim N^{1/4} \sum_{N_2 \sim N} \| P_{N_2} v_N \|_{L^1_T L^4_x} \lesssim \left\| (D_x)^{1/4} v_N \right\|_{L^1_T L^4_x}.
\]

For the part \( N_2 \sim N \), we split it as \( \sum_{N_2 \sim N} = \sum_{N_2 \lesssim N} + \sum_{N_2 \gg N} \). Noticing that for \( N_2 \ll N \),

\[
\tilde{P}_N \left( \sum_{N_1} P_{N_1} \left( e^{\frac{i\lambda}{2} \int_{-\infty}^x \langle |P_{\leq N} u| \rangle^4 dy} \right) \sum_{N_2} P_{N_2} v_N \right)
\]

and for \( N_2 \gg N \),

\[
= \tilde{P}_N \left( \sum_{N_1 \sim N_2 \gg N} P_{N_1} \left( e^{\frac{i\lambda}{2} \int_{-\infty}^x \langle |P_{\leq N} u| \rangle^4 dy} \right) P_{N_2} v_N \right),
\]

we can bound (6.3), in view of the Bernstein inequality and the Hölder inequality, by

\[
\lesssim \left\| (D_x)^{1/4} v_N \right\|_{L^1_T L^4_x} + N^{1/4} \left\| P_{\leq N} \left( e^{\frac{i\lambda}{2} \int_{-\infty}^x \langle |P_{\leq N} u| \rangle^4 dy} \right) P_{\leq N} v_N \right\|_{L^\infty_T L^4_x} + \sum_{N_1 \sim N_2 \gg N} \left\| P_{N_1} \left( e^{\frac{i\lambda}{2} \int_{-\infty}^x \langle |P_{\leq N} u| \rangle^4 dy} \right) P_{N_2} v_N \right\|_{L^1_T L^4_x}
\]

\[
\lesssim \left\| (D_x)^{1/4} v_N \right\|_{L^1_T L^4_x} + N^{-3/4} \left\| P_{\leq N} u \right\|_{L^\infty_T L^4_x} \left\| (D_x)^{1/4} P_{\leq N} v_N \right\|_{L^4_T L^\infty_x} + \sum_{N_1 \sim N_2 \gg N} N_1^{-1/3} N_2^{-1/3} N^{-1/3} \left\| P_{\leq N} u \right\|_{L^\infty_T L^4_x} \left\| (D_x)^{1/4} P_{N_2} v_N \right\|_{L^4_T L^\infty_x}
\]

\[
\lesssim (1 + \| u \|_{L^\infty_T H^{1/2}_x}) \left\| (D_x)^{1/4} v_N \right\|_{L^4_T L^\infty_x},
\]

which yields the desired estimate by applying \( I_N^2 \)-sum.

Thus, we complete the proof of this Lemma.

Of course, we need the following estimate of the data.
Lemma 6.3. For any $u_0 \in H^{1/2}$, we have

$$\left( \sum_{N \geq 1} \left\| \left( e^{-\frac{i}{N} \int_{-\infty}^{x} |P_{\leq N} u_0|^4 dy} P_N u_0 \right)^2 \|_{L^2_{x, t}} \right\|^{1/2}_{L^2_{x, t}} \right)^2 \lesssim (1 + \|u_0\|_{H^{1/2}}^2) \|P_{\geq 1} u_0\|_{H^{1/2}}. \quad (6.5)$$

Proof. From (3.1), (3.2) and (3.3), we bound the left hand side of (6.5) by

$$\left( \sum_{N \geq 1} \left\| e^{-\frac{i}{N} \int_{-\infty}^{x} |P_{\leq N} u_0|^4 dy} P_N u_0 \right\|_{H^{1/2}}^2 \right)^{1/2} \quad (6.6)$$

and

$$+ \left( \sum_{N \geq 1} \left( \sum_{M} \left\| P_M \left( e^{-\frac{i}{N} \int_{-\infty}^{x} |P_{\leq N} u_0|^4 dy} P_N u_0 \right) \right\|_{H^{1/2}}^2 \right)^{1/2} \right. \quad (6.7)$$

From Lemma 3.3, we have

$$\lesssim \left( \sum_{N \geq 1} \left\| P_{\leq N} u_0 \right\|_{L^2_{x, t}}^{2} \left\| P_N u_0 \right\|_{L^4_{x}}^2 + \left\| P_N u_0 \right\|_{H^{1/2}}^2 \right)^{1/2} \lesssim (1 + \|u_0\|_{H^{1/2}}^2) \|P_{\geq 1} u_0\|_{H^{1/2}}. \quad (6.8)$$

For the second term (6.7), it is similar with (6.4). We split the sum $\sum_{M} = \sum_{M \leq N} + \sum_{M \geq N}$. By the Bernstein inequality, the Hölder inequality and the Sobolev embedding theorem, we bound (6.7) by

$$\lesssim \left( \sum_{N \geq 1} \left( \sum_{M \leq N} \left\| \sum_{M \leq N} \frac{1}{M} \left\| P_N u_0 \|_{L^2_{x, t}}^2 \right\|^{1/2} \right) \right)^2$$

$$+ \left( \sum_{N \geq 1} \left( \sum_{M \geq N} \left\| \sum_{M \geq N} \frac{1}{M} \left\| P_N u_0 \|_{L^2_{x, t}}^2 \right\|^{1/2} \right) \right)^2$$

$$\lesssim \left( \sum_{N \geq 1} \left( N^{1/2} \left\| P_N u_0 \|_{L^2_{x, t}}^2 \right\|^{1/2} \right) \right)^2$$

$$+ \left( \sum_{N \geq 1} \left( \sum_{M \geq N} \left\| P_{M} \left( e^{-\frac{i}{N} \int_{-\infty}^{x} |P_{\leq N} u_0|^4 dy} P_N u_0 \right) \right\|_{L^2_{x, t}}^2 \right)^{1/2} \right)$$

$$\lesssim \left( \sum_{M \geq 1} \left( \sum_{M \geq N} \left\| \sum_{M \geq N} \frac{1}{M} \left\| P_N u_0 \|_{L^2_{x, t}}^2 \right\|^{1/2} \right) \right)^2$$

$$+ \left( \sum_{N \geq 1} \left( \sum_{M \geq N} \left\| P_{M} \left( e^{-\frac{i}{N} \int_{-\infty}^{x} |P_{\leq N} u_0|^4 dy} P_N u_0 \right) \right\|_{L^2_{x, t}}^2 \right)^{1/2} \right)$$

$$\lesssim \|P_{\geq 1} u_0\|_{H^{1/2}}$$

$$+ \left( \sum_{N \geq 1} \left( \sum_{M \geq N} \left\| \sum_{M \geq N} \frac{1}{M} \left\| P_{M} \partial_x e^{-\frac{i}{N} \int_{-\infty}^{x} |P_{\leq N} u_0|^4 dy} P_N u_0 \right\|_{L^4_{x}} \right\|_{L^4_{x}}^{2} \right)^{1/2} \right)$$

$$\lesssim \|P_{\geq 1} u_0\|_{H^{1/2}}$$

$$+ \left( \sum_{N \geq 1} \left( \sum_{M \geq N} \left\| \sum_{M \geq N} \frac{1}{M} \left\| P_{M} \partial_x e^{-\frac{i}{N} \int_{-\infty}^{x} |P_{\leq N} u_0|^4 dy} P_N u_0 \right\|_{L^4_{x}} \right\|_{L^4_{x}}^{2} \right)^{1/2} \right)$$

$$\lesssim \|P_{\geq 1} u_0\|_{H^{1/2}}.$$
\[ \lesssim \|P_{\geq 1} u_0\|_{H^{1/2}} + \left( \sum_{N \geq 1} \left( \|P_{\leq N} u_0\|_{L^k}^k \|P_N u_0\|_{L^c}^c \right)^2 \right)^{1/2} \]

which yields the desired result. \[ \square \]

With the help of the above lemmas, we can prove the following proposition which yields the a priori estimate.

**Proposition 6.1.** Let \( u \) be a smooth solution to (1.1)-(1.2) and \( 0 < T \leq C_{\text{high}}^4 \). Then we have

\[ \|u\|_{\bar{X}_T} \lesssim C(C_{\text{low}} + C(C_{\text{low}} + \|u\|_{\bar{X}_T})^{3k}(T^{1/4} + C_{\text{high}}) \|u\|_{\bar{X}_T}. \]

**Proof.** Noticing that

\[ \|P_{\leq 1} u\|_{X_T} \lesssim C_{\text{low}} \|u\|_{\bar{X}_T}, \quad \|P_{> 1} u\|_{X_T} \lesssim C_{\text{high}} \|u\|_{\bar{X}_T}, \]

and from Lemmas 6.1, 6.2, 6.3 and Proposition 5.1 we obtain through a complicated computation

\[ \|u\|_{\bar{X}_T} = \frac{1}{C_{\text{low}}} \|P_{\leq 1} u\|_{X_T} + \frac{1}{C_{\text{high}}} \|P_{> 1} u\|_{X_T} \]

\[ \lesssim 1 + \frac{1}{C_{\text{low}}} T^{\frac{1}{2}} \|u\|_{X_T}^{k+1} + \frac{1}{C_{\text{high}}} (1 + \|u\|_{L^k T^{1/2}}^{2k} \left( \sum_{N \geq 1} \|u_N\|_{Y_T}^2 \right)^{1/2} \]

\[ \lesssim 1 + \frac{1}{C_{\text{low}}} T^{\frac{1}{2}} \|u\|_{X_T}^{k+1} + \frac{1}{C_{\text{high}}} (1 + \|u\|_{L^k T^{1/2}}^{2k} \|P_{> 1} u\|_{X_T} (1 + \|u\|_{X_T}^{k}) \|P_{> 1} u\|_{X_T}^{k+1}) \]

\[ + (1 + \|u\|_{X_T}^k) \left( T^{\frac{1}{2}} \|u\|_{X_T}^{k+1} \|P_{> 1} u\|_{X_T} + (1 + T^{\frac{1}{2}} \|u\|_{X_T}) \|u\|_{X_T} \right) \]

\[ + T^{\frac{1}{2}} (1 + T^{\frac{1}{2}} \|u\|_{X_T}) \|u\|_{X_T}^{2k-1} \|P_{> 1} u\|_{X_T} \]

\[ + (1 + \|u\|_{X_T}^k) \left( T^{\frac{1}{2}} \|u\|_{X_T} \|P_{> 1} u\|_{X_T} + (1 + T^{\frac{1}{2}} \|u\|_{X_T})^2 \|u\|_{X_T} \right) \]

\[ \lesssim C + C(C_{\text{low}}) T^{\frac{1}{2}} \|u\|_{X_T}^{k+1} + (1 + \|u\|_{L^k T^{1/2}}^{2k} \|P_{> 1} u\|_{X_T}^{k+1}) \]

\[ + (1 + \|u\|_{X_T}^k) \left( T^{\frac{1}{2}} \|u\|_{X_T} \|P_{> 1} u\|_{X_T} + (1 + T^{\frac{1}{2}} \|u\|_{X_T})^2 \|u\|_{X_T} \right) \]

Notice that

\[ \|u(t)\|_{H^{1/2}} \lesssim \|P_{\leq 1} u(t)\|_{L^2} + C_{\text{high}} \|P_{> 1} u\|_{H^{1/2}}. \]

The high frequency part \( C_{\text{high}} \|P_{> 1} u\|_{H^{1/2}} \) can be absorbed into the \( \bar{X}_T \)-norm. Then substituting Lemma 6.1 again in estimating the low frequency part of the norm \( \|P_{\leq 1} u\|_{L^\infty T^{1/2}} \), we complete the proof of Proposition 6.1. \[ \square \]
From Proposition 6.1, we have the following a priori estimate for the solution of (1.1)-(1.2) if we take \( T \) and \( C_{\text{high}} \) small enough.

**Corollary 6.1.** Let \( u \) be a smooth solution to (1.1)-(1.2), we have

\[
\|u\|_{\bar{X}_T} \lesssim C_{\text{low}} + C_{\text{high}},
\]

for \( T \) and \( C_{\text{high}} \) small enough.

For the proof of Theorem 1.1, we can follow the compactness argument with the a priori estimate. Since the proof is standard, we omit the details and refer to the papers [14, 15, 16, 17, 19, 22].

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**REFERENCES**

[1] J. Bergh, J. Löfström, Interpolation Spaces, An Introduction, Springer-Verlag, Berlin Heidelberg (1976).

[2] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations, Parts I, II*, Geom. Funct. Anal. 3 (1993), 107–156, 209–262.

[3] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, *Global well-posedness for the Schrödinger equations with derivative*, SIAM J. Math. Anal. 33 (2001), 649–669.

[4] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, *A refined global well-posedness for the Schrödinger equations with derivative*, SIAM J. Math. Anal. 34 (2002), 64–86.

[5] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, *Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in \( \mathbb{R}^3 \)*, to appear in Annals Math., arXiv:math.AP/0402129.

[6] I. Fukuda, Y. Tsutsumi, *On solutions of the derivative nonlinear Schrödinger equation: existence and uniqueness theorem*, Funkcial. Ekvac. 23 (1980), 259–277.

[7] I. Fukuda, Y. Tsutsumi, *On solutions of the derivative nonlinear Schrödinger equation II*, Funkcial. Ekvac. 234 (1981), 85–94.

[8] A. Grünrock, *On the Cauchy and periodic boundary value problem for a certain class of derivative nonlinear Schrödinger equations*, preprint.

[9] N. Hayashi, *The initial value problem for the derivative nonlinear Schrödinger equation in the energy space*, Nonlinear Anal. 20 (1993), 823–833.

[10] N. Hayashi, T. Ozawa, *On the derivative nonlinear Schrödinger equation*, Phys D 55 (1992), 14–36.

[11] N. Hayashi, T. Ozawa, *Remarks on nonlinear Schrödinger equations in one space dimension*, Diff. Integ. Eq. 7 (1994), 453–461.

[12] C. Kenig, G. Ponce, L. Vega, *Oscillatory integrals and regularity of dispersive equations*, Indiana Univ. Math. J. 40 (1991), 33–69.

[13] C. E. Kenig, G. Ponce, L. Vega, *Small solutions to nonlinear Schrödinger equations*, Ann. Inst. H. Poincaré Anal. Non Lineaire, 10 (1993), 255–288.

[14] C. E. Kenig, H. Takaoka, *Global wellposedness of the modified Benjamin-Ono equation with initial data in \( H^{1/2} \)*, Int. Math. Res. Not., (2006), Art. ID 95702, 44 pages.

[15] H. Koch, N. Tzvetkov, *On the local well-posedness of the Benjamin-Ono equation in \( H^s(\mathbb{R}) \)*, Int. Math. Res. Not., 26 (2003), 1449–1464.

[16] L. Molinet, F. Ribaud, *Well-posedness results for the generalized Benjamin-Ono equation with small initial data*, J. Math. Pures Appl., 83 (2004), 277–311.

[17] L. Molinet, F. Ribaud, *Well-posedness results for the generalized Benjamin-Ono equation with arbitrary large initial data*, Int. Math. Res. Not., 70 (2004), 3757–3795.

[18] T. Ozawa, *On the nonlinear Schrödinger equations of derivative type*, Indiana Univ. Math. J. 45 (1996), 137–163.

[19] G. Ponce, *On the global well-posedness of the Benjamin-Ono equation*, Diff. Int. Equas., 4 (1991), 527–542.
[20] E. M. Stein, Harmonic analysis, Real-variable methods, orthogonality, and oscillatory integrals, Princeton University Press, Princeton (1993).
[21] H. Takaoka, Well-posedness for the one-dimensional nonlinear Schrödinger equation with the derivative nonlinearity, Adv. Diff. Eqns., 4 (1999), 561-580.
[22] T. Tao, Global well-posedness of the Benjamin-Ono equation in $H^1(\mathbb{R})$, J. Hyperbolic Diff. Eqns., 1 (2004), 27–49.
[23] T. Tao, Nonlinear Dispersive Equations: Local and Global Analysis, CBMS Regional Conference Series in Mathematics, 106, American Mathematical Society, Providence, RI, (2006).

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