Generating branes via sigma-models

D.V. Gal’tsov\textsuperscript{1} and O.A. Rytchkov\textsuperscript{2}
Department of Theoretical Physics, Moscow State University,
Moscow 119899, Russia

Abstract

Starting with the $D$-dimensional Einstein-dilaton-antisymmetric form equations and assuming a block-diagonal form of a metric we derive a $(D - d)$-dimensional $\sigma$-model with the target space $SL(d, R)/SO(d) \times SL(2, R)/SO(2) \times R$ or its non-compact form. Various solution-generating techniques are developed and applied to construct some known and some new $p$-brane solutions. It is shown that the Harrison transformation belonging to the $SL(2, R)$ subgroup generates black $p$-branes from the seed Schwarzschild solution. A fluxbrane generalizing the Bonnor-Melvin-Gibbons-Maeda solution is constructed as well as a non-linear superposition of the fluxbrane and a spherical black hole. A new simple way to endow branes with additional internal structures such as plane waves is suggested. Applying the harmonic maps technique we generate new solutions with a non-trivial shell structure in the transverse space (‘matrioshka’ $p$-branes). Similar $\sigma$-model is constructed for the intersecting branes. It is shown that the intersection rules have a simple geometric interpretation as conditions ensuring the symmetric space property of the target space. The null-geodesic method is used to find intersecting ‘matrioshka’ $p$-branes in Type IIA supergravity. Finally, a Bonnor-type symmetry relating the four-dimensional vacuum $SL(2, R)$ with the corresponding sector of the above global symmetry group is used to construct a new magnetic 6-brane with a dipole moment in the ten-dimensional IIA theory.

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\textsuperscript{1}email: galtsov@grg.phys.msu.su
\textsuperscript{2}email: rytchkov@grg1.phys.msu.su
1 Introduction

Investigation of classical $p$-brane solutions to supergravities in various dimensions has led to considerable progress in understanding interlinks between different string models. Of particular interest are IIA and IIB supergravities in ten dimensions which are the low-energy limits of the corresponding superstring theories, and the eleven-dimensional supergravity, which is supposed to be the low-energy limit of M-theory. Recent progress in the string theory is also connected with the discovery of non-perturbative objects called $D$-branes. In the low-energy approximation they correspond to certain solutions of appropriate classical equations. There are several types of $p$-branes which attracted attention. The most important type includes those purely bosonic solutions which preserve a part of the initial supersymmetry, so called BPS states (for a review see [1, 2]). Another family consists of not saturating Bogomol’nyi bound black $p$-branes possessing a regular event horizon. Both families may form intersecting multiple-brane structures. The solutions may be also endowed with additional structures such as traveling waves.

Solving highly non-linear bosonic equations in the multidimensional supergravities constitutes a formidable technical task. In many cases $p$-brane solutions were obtained using some special ansätze for the metric and matter fields [3, 4, 5, 6]. In the BPS-saturated cases one can also use the first order Bogomol’nyi type equations instead of direct solving the equations of motion [7, 8]. However this method is efficient mostly in eleven and ten-dimensional supergravities where the Killing spinor equations are relatively simple. Once such solutions are obtained, certain lower-dimensional solutions may be found via appropriate compactification schemes. Some ad hoc prescriptions are also known which allow to perform ‘blackening’ deformations of $p$-branes from extremal configurations [9, 10, 11]. All these techniques are applicable only for rather restricted classes of solutions.

An approach which opens a way to explore more general solution classes consists in the use of ‘hidden’ symmetries (dualities) arising in dimensionally reduced theories. This method allows to generate new non-trivial solutions from known ones, as well as to suggest some new integration algorithms. This approach appeared in the four-dimensional General Relativity, where it has achieved a high level of sophistication. For a class of vacuum solutions effectively depending on three, rather than four coordinates, the hidden symmetry group is $SL(2,\mathbb{R})$ which acts non-linearly on two moduli. One of the symmetry transformations (Ehlers transformation) is non-trivial and may be used as generating symmetry. $N = 2$ supergravity in four dimensions being restricted to the class of solutions possessing a non-null Killing vector field leads to the famous Kramer-Neugebauer-Kinnersley group $SU(1, 2)$ [12, 13], while the (truncated to one vector) $N = 4$ supergravity generates the $Sp(4,\mathbb{R})$ symmetry [14]. The crucial role of three dimensions is due to the fact that the vector fields can be traded there for scalars thus leading to a non-linear $\sigma$-model description of the system.

The same approach can be used in higher dimensional supergravities to construct multidimensional solutions. Here also the $\sigma$-model description of the full space of solutions with a sufficient number of commuting Killing vectors can be achieved only in three dimensions. In higher dimensions one has to deal with a great number of residual forms of various ranks, originating from the initial forms and produced by the Kaluza-Klein reduction. The symmetries of such reduced theories are called dualities ($U$-dualities). It
is well-known, for example, that the IIB supergravity, compactified to nine dimensions, exhibits the $SL(2, R)$ symmetry (S-duality) mixing the NS and RR fields. Also there is a correspondence between the IIA and IIB supergravities reduced to nine dimensions, which is called $T$-duality [14]. Using these dualities, accompanied by appropriate boosts (or by the dimensional reduction and uplifting), it is possible to construct a variety of new solutions from the known ones [14, 17, 18]. To fully exploit global symmetries arising due to dimensional reduction in the $p$-brane context one has to construct explicit non-linear realisations of the $U$-duality groups on the space of physical variables, what is generally a highly non-trivial problem. The first step towards this goal is to consider a truncated theory, in which only scalar fields are exited (what leads to certain restrictions on the metric and the initial forms). In this case we obtain a rather simple non-linear sigma-model, which can be exhaustively analysed and fruitfully exploited.

Starting with the $D$-dimensional Einstein-dilaton-antisymmetric form equations and assuming a block-diagonal form of a metric we construct a $\sigma$-model on the transverse space (of any dimension) with the isometry (duality) group $SL(d, R) \times SL(2, R) \times R$. Applying non-trivial transformations of this group one can generate charged $p$–brane solutions from the seed (multidimensional) Schwarzschild metric, to find a $p$-brane generalization of the Melvin solution (a fluxbrane), to generate intersecting branes and to put plane waves on branes. Apart from a direct application of the target space isometries, one can use $\sigma$-model approach to develop alternative integration schemes, such as harmonic maps onto geodesic subspaces etc. Ultimately one can find a completely integrable system assuming dependence of solutions on only two variable.

The paper is organized as follows. In Section 2 we consider the simple containing $p$-branes bosonic theory, which describes the gravity coupled $d$-form field and the dilaton. Using a block-diagonal ansatz for the metric we derive the corresponding $\sigma$-model action and examine its symmetries. Section 3 is devoted to generation of the general black $p$-brane solution by applying $\sigma$-model transformations. We argue that the prescription of ‘blackening’ the extremal $p$-branes is a manifestation of the target space isometries. In Section 4 using the $\sigma$-model transformations we generate a fluxbrane, which is a multidimensional analog of the Bonnor–Melvin universe. We also find the non–linear superposition of the fluxbrane and a black hole. In Section 5 we apply the technique of harmonic maps to obtain new solutions of the $p$-brane type and study their properties. In Section 6 we discuss the intersecting $p$-brane type and study their properties. It is shown that the well-known intersection rules restricting dimensionalities and the coupling constants for known classes of composite $p$-branes are equivalent to the symmetric space condition for the target space. In this case the coset models may be formulated which open a way to construct more general classes of intersecting branes, an example is given for the case of the IIA supergravity. In Section 7 we use the null geodesic method to generate the Brinkmann wave and demonstrate its independence on the $p$-brane structure. In Section 8 we discuss a Bonnor-type map relating four-dimensional solutions of the vacuum Einstein equations to multidimensional $p$-brane type solutions and derive an apparently new $p$-brane solution to the IIA supergravity in ten dimensions endowed with a dipole moment. We conclude with some remarks on further perspectives of the suggested approach.
2 Sigma-model representation

Except for some particular applications to ten-dimensional IIA supergravity in Sections 5 and 7 we will consider the model theory with the following action in the $D$-dimensional spacetime

$$S = \frac{1}{2\kappa^2} \int d^Dx \sqrt{-G} \left( R - \frac{1}{2} (\nabla \phi)^2 - \frac{e^{-\alpha\phi}}{2(d+1)!} F_{d+1}^2 \right),$$

where $F_{d+1}$ is a $(d+1)$-differential form, $F_{d+1} = dA$, $\phi$ is a dilaton. The corresponding equations of motion are

$$R_{MN} - \frac{1}{2} G_{MN} R = e^{-\alpha\phi} T^{(F)}_{MN} + T^{(\phi)}_{MN},$$

$$\partial_M \left( e^{-\alpha\phi} \sqrt{-G} F_{d+1}^{M_1...M_d} \right) = 0,$$

$$\partial_M \left( \sqrt{-G} G_{MN} \partial_N \phi \right) + \frac{\alpha}{2(d+1)!} e^{-\alpha\phi} F_{d+1}^2 = 0.$$  

The energy-momentum tensors for the matter fields have the form

$$T^{(F)}_{MN} = \frac{1}{2d!} \left( F_{M_1...M_d} F_{N}^{M_1...M_d} - \frac{G_{MN}}{2(d+1)!} F_{d+1}^2 \right),$$

$$T^{(\phi)}_{MN} = \frac{1}{2} \left( \partial_M \phi \partial_N \phi - \frac{1}{2} g_{MN} (\nabla \phi)^2 \right).$$

Let us suppose that the space-time has $k$ commuting Killing vectors orthogonal to hypersurfaces, one of them being time-like (the case with only space-like Killing vectors will be discussed below). Then we can use the following ansatz for the metric

$$ds^2 = g_{\mu\nu}(x) dy^\mu dy^\nu + (\sqrt{-g})^{-\frac{d}{2}} h_{\alpha\beta}(x) dx^\alpha dx^\beta,$$

where $g_{\mu\nu}$ and $h_{\alpha\beta}$ are arbitrary $d$- and $s+2$-dimensional metrics, with $\mu, \nu$ running from 0 to $d-1$, and $\alpha, \beta$ running from 1 to $s+2$, $s \geq 1$, $D = d + s + 2$, $g = \det(g_{\mu\nu})$. The factor $(\sqrt{-g})^{-\frac{d}{2}}$ is introduced for future convenience. Both metric tensors depend only on (transverse) coordinates $x^\alpha$.

For the antisymmetric form we assume either electric or magnetic ansätze. In the electric case the $d$-form has only one non-trivial component

$$A_{01...d-1} = v(x).$$

With this choice of the metric and the $d$-form one obtains a reduced theory in $s+2$-dimensional space. Clearly this reduction is not the most general one, namely we have tacitly assumed that all Kaluza-Klein vectors as well as the lower-dimensional antisymmetric forms arising in full dimensional reduction are not excited. However this truncated theory is still reach enough to be explored in details.

In terms of the functions $g_{\mu\nu}$, $h_{\alpha\beta}$, $\phi$ and $v$ the equations of motion read

$$\frac{1}{\sqrt{h}} \partial_\alpha (\sqrt{h} h^{\alpha\beta} \partial_\beta g_{\mu\lambda} g^{\lambda\sigma}) g_{\sigma\nu} = \frac{s}{d + s} e^{-\psi} g_{\mu\nu} h^{\alpha\beta} \partial_\alpha v \partial_\beta v,$$
\[ \partial_\alpha (\sqrt{|h|} h^{\alpha\beta} e^{-\psi} \partial_\beta v) = 0, \]  
(10)

\[ \partial_\alpha (\sqrt{|h|} h^{\alpha\beta} \partial_\beta v) = \frac{\alpha}{2} e^{-\psi} h^{\alpha\beta} \partial_\alpha v \partial_\beta v, \]  
(11)

\[ R^{(h)}_{\alpha\beta} = \frac{1}{2} \partial_\alpha \phi \partial_\beta \phi + \frac{1}{s} \partial_\alpha (\ln \sqrt{-g}) \partial_\beta (\ln \sqrt{-g}) \]

\[ + \frac{1}{4} g^{\mu\lambda} \partial_\alpha g_{\lambda\nu} g^{\nu\sigma} \partial_\beta g_{\sigma\mu} - \frac{1}{2} e^{-\psi} h^{\alpha\beta} \partial_\alpha v \partial_\beta v, \]  
(12)

where

\[ \psi = \alpha \phi + 2 \ln \sqrt{-g}. \]  
(13)

It is straightforward to check that the field equations (9), (10), (11) and (12) can be obtained from a new action of the \( \sigma \)-model type

\[ S = \frac{1}{2 \kappa^2} \int d^{s+2}x \sqrt{|h|} \left\{ R^{(h)} - h^{\alpha\beta} \left( \frac{1}{2} \partial_\alpha \phi \partial_\beta \phi + \frac{1}{s} \partial_\alpha (\ln \sqrt{-g}) \partial_\beta (\ln \sqrt{-g}) \right) \right. \]

\[ \left. + \frac{1}{4} g^{\mu\lambda} \partial_\alpha g_{\lambda\nu} g^{\nu\sigma} \partial_\beta g_{\sigma\mu} - \frac{1}{2} e^{-\psi} h^{\alpha\beta} \partial_\alpha v \partial_\beta v \right\}. \]  
(14)

The similar action can be obtained assuming purely magnetic ansatz for the \( d \)-form

\[ F^{\alpha_1...\alpha_{s+1}} = \frac{1}{\sqrt{-G}} \epsilon^{\alpha_1...\alpha_{s+1}\beta} \partial_\beta u(x), \]  
(15)

in this case one has to set in the metric \( s = d \). The Maxwell equations (3) are trivially satisfied, while the equation for \( u \) follows from the Bianchi identity. In this case the \( \sigma \)-model action still has the form (17) with the replacement of \( v \) on \( u \) and reversing the sign of \( \alpha \). This fact is a manifestation of the electric-magnetic duality. In what follows we consider explicitly an electric case, the corresponding magnetic solutions can be obtained by the above dualization procedure.

For subsequent analysis of the action (17) it is convenient to renormalize the world-volume metric \( g_{\mu\nu} \) introducing the matrix

\[ \tilde{g}_{\mu\nu} = (\sqrt{-g})^{-\frac{2}{d}} g_{\mu\nu}, \]  
(16)

such that \( \det(\tilde{g}_{\mu\nu}) = -1 \). Then the action will read

\[ S = \frac{1}{2 \kappa^2} \int d^{s+2}x \sqrt{|h|} \left\{ R^{(h)} - h^{\alpha\beta} \left( \frac{1}{2} \partial_\alpha \phi \partial_\beta \phi + \frac{s + d}{sd} \partial_\alpha (\ln \sqrt{-g}) \partial_\beta (\ln \sqrt{-g}) \right) \right. \]

\[ \left. + \frac{1}{4} g^{\mu\lambda} \partial_\alpha \tilde{g}_{\lambda\nu} g^{\nu\sigma} \partial_\beta \tilde{g}_{\sigma\mu} - \frac{1}{2} e^{-\psi} \partial_\alpha v \partial_\beta v \right\}. \]  
(17)

Note, that the matrix \( \tilde{g}_{\mu\nu} \) now is decoupled from the rest of the \( \sigma \)-model variables, interacting with them only through the gravitational field \( h_{\alpha\beta} \). Since \( \tilde{g}_{\mu\nu} \) is a symmetric matrix with (minus) unit determinant (the sign of the determinant is in fact irrelevant since the action remains unchanged under a multiplication of \( \tilde{g}_{\mu\nu} \) on a constant matrix with the determinant minus one), this matrix parametrizes a coset \( SL(d, R)/SO(1, d-1) \).
Therefore the metric on the world-volume of the $p$-brane is to high extent independent of the other $\sigma$-model variables, which only influence its determinant.

To simplify the rest of the action we introduce together with (13) another variable

$$\xi = sd\phi - \alpha(s + d) \ln \sqrt{-g},$$

so that the inverse transformations read

$$\phi = \frac{1}{\Delta} \left( \alpha \psi + \frac{2\xi}{(s + d)} \right),$$

$$\ln \sqrt{-g} = \frac{1}{\Delta(s + d)} (sd\psi - \alpha \xi),$$

where $\Delta = \alpha^2 + 2sd/(s + d)$.

In the new variables the part of the action not including the matrix $\tilde{g}_{\mu\nu}$ will read

$$S = \frac{1}{2\kappa^2} \int d^{s+2}x \sqrt{h} \left\{ R^{(h)} - h^{\alpha\beta} \left( A \partial_\alpha \xi \partial_\beta \xi + B \partial_\alpha \psi \partial_\beta \psi - \frac{1}{2} e^{-\psi} \partial_\alpha v \partial_\beta v \right) \right\},$$

where

$$A = \frac{1}{\alpha^2 sd(s + d) + 2s^2 d^2}, \quad B = \frac{s + d}{2\alpha^2(s + d) + 4sd},$$

Now the $\xi$-part is also decoupled. The remaining fields $\psi$ and $v$ parametrize a coset $SL(2,R)/SO(1,1)$. Therefore the action (17) corresponds to a non-linear $\sigma$-model with the target space $SL(d,R)/SO(1,d - 1) \times SL(2,R)/SO(1,1) \times R$.

Note that the possibility of description in the $\sigma$-model terms on coset target spaces is typical for many dimensionally reduced gravitational theories. The four-dimensional Einstein–Maxwell theory in the presence of one non-null Killing vector field is equivalent to the $SU(2,1)/SU(2) \times U(1)$ $\sigma$-model [12, 13]. More complicated example is given by the dilaton–axion coupled Einstein–Maxwell theory, in which case one has the coset target space $Sp(4,R)/U(2)$ [14]. Several $\sigma$-models were derived in multidimensional supergravities [1, 14, 20], but the geometric structure of the target spaces was not studied.

Let us discuss our $\sigma$-model (17) in details. Since the potential space is the direct product of three independent cosets, one can analyse each of them separately. The transverse $SL(2,R)/SO(1,1)$ part can be conveniently described by an analog of the Ernst potentials [21]

$$\Phi = \frac{v}{2\sqrt{2B}}, \quad \mathcal{E} = e^\psi - \frac{v^2}{8B},$$

using which the target space metric can be rewritten in a familiar form [13]

$$dl^2 = \frac{1}{2} F^{-2} (d\mathcal{E} + 2\Phi d\Phi)^2 - 2F^{-1} d\Phi d\Phi,$$

$$F = \mathcal{E} + \Phi^2.$$
The action of $SL(2, R)$ on the potentials is realized non-linearly. It can be presented in terms of the following three one-parametric subgroups

I. $E = a^2 E_0, \Phi = a\Phi_0, \quad (26)$

II. $E = E_0 - 2b\Phi_0 - b^2, \Phi = \Phi_0 + b, \quad (27)$

III. $E' = \frac{E}{1 - 2c\Phi - c^2E'}, \quad \Phi' = \frac{\Phi + cE}{1 - 2c\Phi - c^2E'}, \quad (28)$

where $a, b$ and $c$ are parameters. Transformations I and II are pure gauge ones, while the third (Harrison transformation) acts on the space-time variables and matter fields non-trivially.

Similarly one can consider the symmetry transformations realized on the variables $\xi$ and $\tilde{g}$. Subgroup $R$ acts only on $\xi$:

$$\xi \rightarrow \xi + a.$$ 

In terms of initial fields it corresponds to the shift of the dilaton on a constant accompanied by the rescaling of the metric. The matrix $\tilde{g}$ parametrizes the coset $SL(d, R)/SO(1, d-1)$, the representation of the group $SL(d, R)$ is realized in a natural way

$$\tilde{g} \rightarrow U^{-1}\tilde{g}U,$$

where $U$ is a constant element of $SL(d, R)$.

So far we have considered the equations of motion (2), (3), (4) assuming that the space-time metric admits $d$ commuting Killing vectors one of which is time-like. One can also investigate the case when all Killing vectors are space-like. In this case the $\sigma$-model action will read

$$S = \frac{1}{2\kappa^2} \int d^{d+2}x \sqrt{-h} \left\{ R^{(h)} - h^{\alpha\beta} \left( A \partial_\alpha \xi \partial_\beta \xi + \frac{1}{4} \tilde{g}^{\mu\lambda} \partial_\alpha \tilde{g}_{\lambda\nu} \tilde{g}^{\nu\sigma} \partial_\beta \tilde{g}_{\sigma\mu} ight) ight.$$ 

$$+ B \partial_\alpha \psi \partial_\beta \psi + \frac{1}{2} e^{-\psi} \partial_\alpha \psi \partial_\beta \psi \right\}, \quad (29)$$

where $A$ and $B$ are the same (22). This action differs from the previous one by the sign of the last term. As a result the metric on the target space is positively definite so we deal with the coset $SL(d, R)/SO(d) \times SL(2, R)/SO(2) \times R$. Introducing modified Ernst potentials for the $SL(2, R)/SO(2)$ sector

$$\Phi = \frac{v}{2\sqrt{2B}}, \quad E = -e^{\psi} - \frac{v^2}{8B}, \quad (30)$$

we come back to the Eq. (24). Hence the Harrison transformation again is given by (28).

3 Generation of black $p$-branes

As the first application of the above method we will consider the generation of black $p$-branes. Black $p$-brane is a multidimensional generalization of the Reissner-Nordström
black hole. The simplest case $p = 1$ corresponds to the black string [22]. Another important example is the black membrane of $D = 11$ supergravity [23]. Black $p$-branes for general dimensions were constructed in [9, 24]. Also there is a great number of papers where the intersections of black $p$-branes are considered (see for example [10, 11, 25] and references therein).

Black $p$-branes are usually treated as a special deformation of the corresponding extremal $p$-branes, specified by the one-center harmonic functions. The process of deformation is called ‘blackening’, the relevant prescription was given in [9, 10]. Also it is known that black $p$-brane solutions can be obtained from Schwarzschild solution by sequences of boosts and dualities [26]. We demonstrate that the existence of such prescriptions is a manifestation of the hidden symmetry contained in the model. This symmetry is nothing but the $SL(2, R)$, which was considered in the previous section. In this Section we use this symmetry for an explicit generation of a single black $p$-brane, while in Section 6 we will explain the generation of intersecting black $p$-branes.

Let us start with the Schwarzschild solution in the $D$-dimensional spacetime corresponding to a ‘neutral’ ($d - 1$)-brane

$$ds^2 = -\left(1 - \frac{2M}{r^s}\right)dt^2 + dy_1^2 + \ldots + dy_{d-1}^2 + \left(1 - \frac{2M}{r^s}\right)^{-1} dr^2 + r^2 d\Omega_{s+1}. \quad (31)$$

Using the equations (13) and (18) we obtain

$$\psi_0 = \ln \left(1 - \frac{2M}{r^s}\right), \quad \xi_0 = -\frac{1}{2} \alpha (s + d) \ln \left(1 - \frac{2M}{r^s}\right), \quad v_0 = 0, \quad (32)$$

what corresponds to the following seed Ernst potentials

$$\Phi_0 = 0, \quad \mathcal{E}_0 = 1 - \frac{2M}{r^s}. \quad (33)$$

The Harrison transformations (28) and the rescaling of potentials yield the new functions $\psi$ and $v$

$$\psi = \ln \left(1 - \frac{2M}{r^s}\right) + \ln \left(1 + \frac{2Q}{r^s}\right)^{-2}, \quad v = 2c\sqrt{2B} \left(1 - \frac{2M}{r^s}\right) \left(1 + \frac{2Q}{r^s}\right)^{-1}, \quad (34)$$

where

$$Q = \frac{Mc^2}{1 - c^2}. \quad (35)$$

The function $\xi$ remains the same. The resulting metric is

$$ds^2 = \left(1 + \frac{2Q}{r^s}\right)^{-\nu} \left\{ -\left(1 - \frac{2M}{r^s}\right) dt^2 + dy_1^2 + \ldots + dy_{d-1}^2 \right\}$$

$$+ \left(1 + \frac{2Q}{r^s}\right)^{\nu d} \left\{ \left(1 - \frac{2M}{r^s}\right)^{-1} dr^2 + r^2 d\Omega_{s+1} \right\}, \quad (36)$$

where $\nu = 4\Delta^{-1}(s + d)^{-1}$. It coincides with the metric of the black $p$-brane solution [9].

The corresponding dilaton field is given by

$$e^{-\alpha \phi} = \left(1 + \frac{2Q}{r^s}\right)^{2\alpha^2/\Delta}. \quad (37)$$

Note that the extremal limit of this solution is $M \rightarrow 0, c \rightarrow 1$ so that $Q$ is finite.
4 Generation of the fluxbrane

Assuming that all Killing vectors are space-like we can apply the same technique to obtain the solution which is called a fluxbrane. The fluxbrane is a multidimensional generalization of the Bonnor–Melvin solution [27], which is well-known in the usual Einstein-Maxwell gravity. The Bonnor–Melvin solution with a dilaton was constructed by Gibbons and Maeda [28]. We give its generalization for the case of the arbitrary rank $d$-form of either electric, or magnetic type.

Our starting point is a flat $D$-dimensional space-time presented in the multicylindrical coordinates

$$ds^2 = -dt^2 + (\rho_1^2 d\varphi_1^2 + \ldots + \rho_d^2 d\varphi_d^2) + d\rho_1^2 + \ldots + d\rho_d^2 + dx_\alpha dx^\alpha,$$  \hspace{1cm} (38)

where $\alpha = 1, \ldots, s + 1 - d$. This yields

$$\psi_0 = 2 \ln \rho_1 \ldots \rho_d, \quad \xi_0 = -\alpha(s + d) \ln \rho_1 \ldots \rho_d, \quad v_0 = 0,$$  \hspace{1cm} (39)

so that the corresponding Ernst potentials are

$$\mathcal{E}_0 = -\rho_1^2 \ldots \rho_d^2, \quad \Phi_0 = 0.$$  \hspace{1cm} (40)

Applying the electric Harrison transformation (28) we obtain

$$\psi = 2 \ln \rho_1 \ldots \rho_d - 2 \ln(1 + c^2 \rho_1^2 \ldots \rho_d^2), \quad v = -\frac{2c\sqrt{2B\rho_1^2 \ldots \rho_d^2}}{1 + c^2 \rho_1^2 \ldots \rho_d^2},$$  \hspace{1cm} (41)

with $\xi$ remaining the same. As a result we get the following metric

$$ds^2 = (1 + c^2 \rho_1^2 \ldots \rho_d^2)^{-\nu_s} \left(\rho_1^2 d\varphi_1^2 + \ldots + \rho_d^2 d\varphi_d^2\right)$$

$$+ \left(1 + c^2 \rho_1^2 \ldots \rho_d^2\right)^{\nu_d} \left(-dt^2 + d\rho_1^2 + \ldots + d\rho_d^2 + dx_\alpha dx^\alpha\right).$$  \hspace{1cm} (42)

The corresponding dilaton field is

$$e^{-\alpha \phi} = \left(1 + c^2 \rho_1^2 \ldots \rho_d^2\right)^{2\alpha / \Delta},$$  \hspace{1cm} (43)

while the $d$-form potential has the non-vanishing component

$$A_{\varphi_1 \ldots \varphi_d} = -\frac{2c\sqrt{2B\rho_1^2 \ldots \rho_d^2}}{1 + c^2 \rho_1^2 \ldots \rho_d^2},$$  \hspace{1cm} (44)

where the coefficient $B$ is given by (22).

Applying similar technique one can easily construct more complicated solutions. As an example let us derive the metric describing a six-dimensional dilatonic black hole in the magnetic field of the 1-fluxbrane. Now we start not with the flat space-time, but with the six-dimensional Schwarzschild solution writing the metric on the 4-sphere in the form [29]:

$$ds^2 = -\left(1 - \frac{2M}{r^3}\right) dt^2 + \left(1 - \frac{2M}{r^3}\right)^{-1} dr^2$$

\hspace{1cm} (9)
\[ r^2(d\theta^2 + \cos^2 \theta d\psi^2 + \sin^2 \theta d\varphi_1^2 + \cos^2 \theta \sin^2 \psi d\varphi_2^2). \]  

According to (13) and (18),

\[ \psi_0 = 2 \ln \left[ \left( 1 - \frac{2M}{r^3} \right)^{-1} r^2 \sin \theta \cos \sin \psi \right], \]

\[ \xi_0 = -4\alpha \ln \left[ \left( 1 - \frac{2M}{r^3} \right)^{-1} r^2 \sin \theta \cos \sin \psi \right], \]

thus the seed Ernst potentials have the form

\[ \mathcal{E}_0 = - \left( 1 - \frac{2M}{r^3} \right)^{-2} r^4 \sin^2 \theta \cos^2 \psi, \quad \Phi_0 = 0. \]

Using Harrison transformations (28) we obtain the new solution with the metric

\[ ds^2 = \left\{ 1 + c^2 \left( 1 - \frac{2M}{r^3} \right)^{-2} r^4 \sin^2 \theta \cos^2 \psi \right\}^{\frac{2}{\alpha^2 + 2}} \left\{ - \left( 1 - \frac{2M}{r^3} \right) dt^2 \right. \]

\[ + \left. \left( 1 - \frac{2M}{r^3} \right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \cos^2 \theta d\psi^2 \right\} \]

\[ + \left\{ 1 + c^2 \left( 1 - \frac{2M}{r^3} \right)^{-2} r^4 \sin^2 \theta \cos^2 \psi \right\}^{-\frac{2}{\alpha^2 + 2}} r^2 (\sin^2 \theta d\varphi_1^2 + \cos^2 \theta \sin^2 \psi d\varphi_2^2). \]

The corresponding dilaton field is

\[ e^{-\alpha \phi} = \left\{ 1 + c^2 \left( 1 - \frac{2M}{r^3} \right)^{-2} r^4 \sin^2 \theta \cos^2 \psi \right\}^{\frac{2c^2}{\alpha^2 + 2}}, \]

while the non-vanishing component of the 2-form potential is given by

\[ A_{\varphi_1 \varphi_2} = - \frac{2c}{\alpha^2 + 2} \left\{ 1 + c^2 \left( 1 - \frac{2M}{r^3} \right)^{-2} r^4 \sin^2 \theta \cos^2 \psi \right\}. \]

It is easy to see that the obtained solution is indeed a non-linear superposition of the black hole and the fluxbrane. The limit \( c \to 0 \) returns us back to the Schwarzshild solution, while putting \( M = 0 \) we recover the fluxbrane.

### 5 Harmonic maps

For further analysis we rewrite the \( \sigma \)-model action (17) in the following matrix form

\[ S = \frac{1}{2\kappa^2} \int d^{s+2}x \sqrt{h} \left\{ R^{(h)} + h^{\alpha\beta} \left( 2B Tr \partial_\alpha M \partial_\beta M^{-1} + \frac{1}{4} Tr \partial_\alpha g \partial_\beta g^{-1} \right) \right\}. \]
where the matrix $M$ is built from the fields $\Psi$, $v$ and $\xi$ as follows

$$M = \exp(-\frac{\psi}{2}) \begin{pmatrix} 2 & \frac{v}{2\sqrt{2B}} & 0 \\ \frac{v}{2\sqrt{2B}} & -\frac{1}{2}(\exp \psi - \frac{v^2}{8B}) & 0 \\ 0 & 0 & \exp \frac{\psi}{2} + \frac{\xi}{\sqrt{sd(s+d)}} \end{pmatrix}$$

This representation is a convenient starting point for an application of the harmonic maps technique. In particular we will be interested here in constructing solutions corresponding to the null geodesics of the target space $[30, 31]$. 

Consider the transverse part of the action (47)

$$S = \frac{1}{2\kappa^2} \int d^{s+2}x \sqrt{h} \left( R^{(h)} + 2B h^{\alpha\beta} \text{Tr} \left( \partial_\alpha M \partial_\beta M^{-1} \right) \right),$$

where $M$ is an element of the appropriate coset space $G/H$. The equations of motion read

$$\frac{1}{\sqrt{h}} \partial_\alpha (\sqrt{h} h^{\alpha\beta} M^{-1} \partial_\beta M) = 0,$$

$$R^{(h)}_{\alpha\beta} = -2B \text{Tr} (\partial_\alpha M \partial_\beta M^{-1}).$$

It was noticed [12], that if the matrix $M$ depends on $x$-coordinates through a single function, $M = M(\sigma(x))$, then $\sigma(x)$ can be chosen to be a harmonic function on the curved space with the metric $h$, i.e.

$$\frac{1}{\sqrt{h}} \partial_\alpha (\sqrt{h} h^{\alpha\beta} \partial_\beta \sigma) = 0.$$

The equation (50) then reduces to the matrix equation

$$\frac{d}{d\sigma} \left( M^{-1} \frac{dM}{d\sigma} \right) = 0,$$

whose solution can be expressed in the exponential form

$$M = M_0 e^{K\sigma},$$

where $K$ belongs to the Lie algebra of the group $G$, and $M_0 \in G/H$. Substituting this into the Einstein equations (51) one gets

$$R^{(h)}_{\alpha\beta} = 2B \text{Tr} (K^2) \partial_\alpha \sigma \partial_\beta \sigma.$$

It is clear that in the particular case $\text{Tr}(K^2)=0$ the metric $h$ is Ricci-flat (and hence can be chosen flat). This is a constructive way to build null–geodesic solutions to an arbitrary $\sigma$-model. Let us apply it to our model with the target space $SL(2, R)/SO(1, 1) \times R$. Here we are interested in the asymptotically flat solutions, so we choose the harmonic function $\sigma$ such that $\sigma(\infty) = 0$. According to the above general scheme, we present $M$ in the form (54), where $M_0$ is an element of the coset $SL(2, R)/SO(1, 1) \times R$ and the generator $K$.
belongs to the algebra \( sl(2, R) \times R \). \( M_0 \) has to be taken corresponding to an assumed asymptotic behaviour, the Eq.\((48)\) gives
\[
M_0 = \text{diag}(2, -\frac{1}{2}, 1).
\]
A convenient parametrization of the matrix \( K \) is
\[
K = \begin{pmatrix}
a & c & 0 \\
d & -a & 0 \\
0 & 0 & b
\end{pmatrix}.
\]
One has to distinguish two different cases: \( \det K \neq 0 \) and \( \det K = 0 \).

1) Degenerate case: \( \det K = 0 \)

The general constraint \( \text{Tr} K^2 = 0 \) gives \( 2(a^2 + cd) + b^2 = 0 \). Together with the restriction \( \det K = 0 \) this means \( b = 0, \ a^2 + cd = 0 \). In terms of the matrix \( K \) this leads to \( K^2 = 0 \), so the exponentiation is essentially different from that in the non-degenerate case:
\[
e^{K\sigma} = I + K\sigma.
\]
Therefore for the matrix \( M \) one obtains
\[
M = \begin{pmatrix}
2 + 2a\alpha & 2c\alpha & 0 \\
-\frac{1}{2}d\sigma & -\frac{1}{2}(1 - a\sigma) & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]
This matrix should be symmetric what gives an additional constraint on the parameters, so the resulting matrix depends on a single parameter \( a \)
\[
M = \begin{pmatrix}
2 + 2a\alpha & a\alpha & 0 \\
a\alpha & -\frac{1}{2}(1 - a\sigma) & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]
Comparing this with the Eqs. \((48)\) and \((13)\) we get
\[
\psi = -2\ln(1 + a\sigma), \quad \xi = 0, \quad \nu = 2\sqrt{2B} \left(1 - \frac{1}{1 + a\sigma}\right).
\]
Since \( \sigma \) is an arbitrary harmonic function, it is defined up to a scale parameter. Thus without loss of generality one can put \( a = 1 \). The resulting metric is
\[
ds^2 = (1 + \sigma)^{-\nu_s} \left(-dt^2 + dy_1^2 + \ldots + dy_{d-1}^2\right) + (1 + \sigma)^{\nu_d} \left(dx_1^2 + \ldots + dx_{s+2}^2\right).
\]
This metric is nothing but the usual \( p \)-brane solution with the harmonic function \( H = 1 + \sigma \) \([32]\). The corresponding dilaton field is given by
\[
e^{-\alpha\phi} = (1 + \sigma)^{2\alpha^2/\Delta}.
\]
This solution saturates the Bogomol'nyi bound
\[
M = \frac{\Omega_{s+1}}{2\kappa^2} 8sBQ.
\]
2) Non-degenerate case: \( \det K \neq 0 \)

Once again we have a constraint \( \text{Tr} K^2 = 0 \), what implies \( 2(a^2 + cd) + b^2 = 0 \). Performing a direct exponentiation one obtains

\[
M = \begin{pmatrix}
2 \cos \frac{b r}{\sqrt{2}} + \frac{2 \sqrt{2}}{b} \sin \frac{b r}{\sqrt{2}} & \frac{2 \sqrt{2}}{b} \sin \frac{b r}{\sqrt{2}} & 0 \\
-\frac{2 \sqrt{2}}{b} \sin \frac{b r}{\sqrt{2}} & -\frac{1}{2} \cos \frac{b r}{\sqrt{2}} + \frac{a \sqrt{2}}{2 b} \sin \frac{b r}{\sqrt{2}} & 0 \\
0 & 0 & \sqrt{2} e^{b r} \end{pmatrix}.
\]

(65)

This matrix should be symmetric, so taking into account the constraints on the coefficients we obtain

\[
M = \begin{pmatrix}
\frac{\sin(\sigma + \varphi)}{\sin \sigma} & \frac{\sin \sigma}{\sin(\sigma - \varphi)} & 0 \\
\frac{\sin \sigma}{\sin \varphi} & \frac{2 \sin \varphi}{\sin(\sigma - \varphi)} & 0 \\
0 & 0 & e^{\sqrt{2} \sigma} \end{pmatrix},
\]

(66)

where we put \( b = \sqrt{2} \) because of the scaling freedom for the harmonic function and denoted

\[
\sin \varphi = \frac{1}{\sqrt{a^2 + 1}}.
\]

(67)

The Eqs. (48) and (13) yield

\[
\xi = \sigma \sqrt{2sd(s + d)}, \quad \psi = -2 \ln \left[ \frac{\sin(\sigma + \varphi)}{\sin \varphi} \right], \quad v = \frac{2\sqrt{2B} \sin \sigma}{\sin(\sigma + \varphi)},
\]

(68)

Now it is easy to construct the whole metric

\[
ds^2 = \left[ \frac{\sin(\sigma + \varphi)}{\sin \varphi} \right]^{-\nu s} e^{-\sqrt{\frac{2(s + d)}{2d}} \nu_{\alpha \sigma}} \left( -dt^2 + dy_1^2 + \ldots + dy_{d-1}^2 \right) \\
+ \left[ \frac{\sin(\sigma + \varphi)}{\sin \varphi} \right]^{-\nu d} e^{-\sqrt{\frac{2(s + d)}{2s}} \nu_{\alpha \sigma}} \left( dx_1^2 + \ldots + dx_{s+2}^2 \right),
\]

(69)

and the dilaton field

\[
e^{-\alpha \phi} = \left[ \frac{\sin(\sigma + \varphi)}{\sin \varphi} \right]^{\frac{2a^2}{\nu}} e^{-\sqrt{\frac{2(s + d)}{2s}} \nu_{\alpha \sigma}}.
\]

(70)

The structure of this solution is similar to that of the usual \( p \)-brane, but the metric functions are essentially different (for the 0-brane case see [31]). The full \( r \)-range solution contains a sequence of compact singular transverse hypersurfaces lying between the subsequent roots \( r_k \) of the equation

\[
\sigma(r_k) + \varphi = \pi k, \quad k = 1, 2, \ldots
\]

(71)

and forming ‘matrioshka’-type structure in the transverse space. Curvature invariants diverge at \( r_k \). The outer solution is asymptotically flat, and for it one can calculate the ADM mass and the Page charge. It easy to check that the Bogomol’nyi bound is saturated
indeed (as could be expected since the solution corresponds to a null geodesic in the target space) if the parameter $\varphi$ satisfies the constraints

$$\sin(\varphi + \chi) = \sqrt{\frac{sd}{2(s + d)}}, \quad \cos \chi = \alpha \sqrt{2B}. \tag{72}$$

As the realistic example let us take the IIA supergravity, whose bosonic action in the Einstein frame is given by

$$S = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-G} \left( R - \frac{1}{2}(\nabla \phi)^2 - \frac{e^{-\phi}}{2 \cdot 3!} F_3^2 - \frac{e^{\frac{\phi}{4}}}{2 \cdot 2!} F_4^2 \right) - \frac{1}{4\kappa^2} \int F_4 \wedge F_4 \wedge A_2, \tag{73}$$

where

$$F_4' = dA_3 + A_1 \wedge F_3. \tag{74}$$

We will consider the NS part of the action consisting of the metric, dilaton and 2-form. The usual extremal 1-brane solution corresponds to the elementary NS-string and has the form

$$ds^2 = H^{-\frac{1}{4}}(-dt^2 + dy^2) + H^{\frac{1}{4}}(dx_1^2 + \ldots + dx_8^2), \tag{75}$$

The ‘matrioshka’ 1-brane line element reads

$$ds^2 = \left[ \frac{\sin(\sigma + \varphi)}{\sin \varphi} \right]^{-\frac{1}{4}} e^{-\frac{\sqrt{3}}{4} \sigma} (-dt^2 + dy^2)$$

$$+ \left[ \frac{\sin(\sigma + \varphi)}{\sin \varphi} \right]^{\frac{1}{4}} e^{\frac{\sqrt{3}}{4} \sigma} (dx_1^2 + \ldots + dx_8^2), \tag{76}$$

while the dilaton is

$$e^{-\phi} = \left[ \frac{\sin(\sigma + \varphi)}{\sin \varphi} \right]^{\frac{1}{4}} e^{-\frac{\sqrt{3}}{4} \sigma}. \tag{77}$$

6 Intersecting $p$-branes

In order to describe within the same approach the intersecting $p$-branes we have to change our basic ansatz (7), (8). Now assume that the $d$-form has two nontrivial components

$$A_{01...q-1q...q+r-1} = v_1(x), \quad A_{01...q-1q+r...q+2r-1} = v_2(x), \tag{78}$$

where $d = r + q$. A suitable parametrization for the metric is

$$ds^2 = g_{\mu\nu}^{(0)}(x)dy^\mu dy^\nu + g_{ij}^{(1)}(x)dz_i^1 dz_i^1 + g_{ij}^{(2)}(x)dz_i^2 dz_i^2$$

$$+ (\sqrt{-g^{(0)}} \sqrt{g^{(1)}} \sqrt{g^{(2)}})^{-\frac{3}{2}} h_{\alpha \beta}(x) dx^\alpha dx^\beta, \tag{79}$$
where \( g_{\mu\nu}^{(i)} \) and \( h_{\alpha\beta} \) are arbitrary symmetric tensors, \( i,j = 1 \ldots r \). As in the Section 2 we substitute this ansatz into the equations of motion and obtain the corresponding \( \sigma \)-model. Introducing as in (16) the “internal” metrics \( \tilde{g}_{\mu\nu}^{(i)} \)

\[
g_{\mu\nu}^{(0)} = (\sqrt{-g^{(0)}})^2 \tilde{g}_{\mu\nu}^{(0)}, \quad g_{ij}^{(1)} = (\sqrt{g^{(1)}})^2 \tilde{g}_{ij}^{(1)}, \quad g_{ij}^{(2)} = (\sqrt{g^{(2)}})^2 \tilde{g}_{ij}^{(2)}
\]

it is easy to obtain the following. These renormalized metric tensors are decoupled from the rest of the \( \sigma \)-model and we find them only in the sector \( \frac{1}{4} \text{Tr} \partial_\alpha \tilde{g}_{\mu}^{(i)} \partial_\beta \tilde{g}_{\nu}^{(i)^{-1}} \). The rest of the \( \sigma \)-model action reads

\[
S = \frac{1}{2\kappa^2} \int d^{s+2}x \sqrt{h} \left\{ R^{(h)} - h^{\alpha\beta} \left( \frac{1}{2} \partial_\alpha \phi \partial_\beta \phi + \frac{s + q}{sq} \partial_\alpha (\ln \sqrt{-g^{(0)}}) \partial_\beta (\ln \sqrt{-g^{(0)}}) \right. \\
+ \frac{s + r}{sr} \partial_\alpha (\ln \sqrt{g^{(1)}}) \partial_\beta (\ln \sqrt{g^{(1)}}) + \frac{s + r}{sr} \partial_\alpha (\ln \sqrt{g^{(2)}}) \partial_\beta (\ln \sqrt{g^{(2)}}) \\
+ \frac{2}{s} \partial_\alpha (\ln \sqrt{g^{(1)}}) \partial_\beta (\ln \sqrt{g^{(1)}}) + \frac{2}{s} \partial_\alpha (\ln \sqrt{g^{(2)}}) \partial_\beta (\ln \sqrt{g^{(2)}}) \\
\left. + \frac{2}{s} \partial_\alpha (\ln \sqrt{g^{(1)}}) \partial_\beta (\ln \sqrt{g^{(2)}}) \right) - \frac{1}{2} e^{-\alpha\phi - 2 \ln \sqrt{-g^{(0)}} - 2 \ln \sqrt{g^{(1)}}} \partial_{\alpha} v_{1} \partial_{\beta} v_{1} \\
- \frac{1}{2} e^{-\alpha\phi - 2 \ln \sqrt{-g^{(0)}} - 2 \ln \sqrt{g^{(2)}}} \partial_{\alpha} v_{2} \partial_{\beta} v_{2} \right\}
\]

Now the target space is six-dimensional, it is parametrized by \( \phi, \ln \sqrt{-g^{(0)}}, \ln \sqrt{g^{(1)}}, \ln \sqrt{g^{(2)}}, v_{1} \) and \( v_{2} \).

The explicit solutions known for the intersecting branes were found only assuming certain conditions on the parameters (intersection rules). It turns out that these conditions correspond to the symmetric space property of the sigma-model target space. Let us remind that the metric space is called symmetric, if the Riemann tensor is covariantly constant, i.e.

\[
\nabla_{a} R_{bcde} = 0.
\]

Straightforward calculations yield the following. All non-zero components of the five-index tensor \( \nabla_{a} R_{bcde} \) are proportional to

\[
\exp(-\alpha\phi - 2 \ln \sqrt{-g^{(0)}} - 2 \ln \sqrt{g^{(1)}}) \exp(-\alpha\phi - 2 \ln \sqrt{-g^{(0)}} - 2 \ln \sqrt{g^{(2)}}) \left( \frac{\alpha^2}{2} + \frac{qs - r^2}{q + 2r + s} \right).
\]

This means that the target space is symmetric when the parameters \( s, q, r \) and \( \alpha \) satisfy the following condition

\[
\frac{\alpha^2}{2} + \frac{qs - r^2}{q + 2r + s} = 0.
\]

This condition can be rewritten as

\[
\frac{\alpha^2}{2} + q - \frac{d}{D - 2} = 0,
\]
showing that we deal with the usual $p$-brane intersection rule \( [3] \). Thus, the $\sigma$-model approach gives a simple geometrical interpretation of the intersection rule \( [3] \): only when \( (84) \) is satisfied, the target space is a symmetric (pseudo)Riemannian space.

If the parameters of our configuration satisfy \( (84) \), the $\sigma$-model action \( (81) \) could be diagonalized and reduced to the simple form similar to \( (21) \):

\[
S = \frac{1}{2\kappa^2} \int d^{s+2}x \sqrt{h} \left\{ R^{(h)} - h^{\alpha\beta} \left( A_1 \partial_\alpha \xi_1 \partial_\beta \xi_1 + A_2 \partial_\alpha \xi_2 \partial_\beta \xi_2 + B_1 \partial_\alpha \psi_1 \partial_\beta \psi_1 - \frac{1}{2} e^{-\psi_1} \partial_\beta \psi_1 v_1 + B_2 \partial_\alpha \psi_2 \partial_\beta \psi_2 - \frac{1}{2} e^{-\psi_2} \partial_\beta \psi_2 v_2 \right) \right\},
\]

where

\[
\psi_1 = \alpha \varphi + 2 \ln \sqrt{-g^{(0)}} + 2 \ln \sqrt{g^{(1)}},
\]

\[
\psi_2 = \alpha \varphi + 2 \ln \sqrt{-g^{(0)}} + 2 \ln \sqrt{g^{(2)}},
\]

\[
\xi_1 = -\alpha s \varphi - 2(s - r) \ln \sqrt{-g^{(0)}} + 2r \ln \sqrt{g^{(1)}} + 2r \ln \sqrt{g^{(2)}},
\]

\[
\xi_2 = q(r + s) \varphi - \alpha(q + 2r + s) \ln \sqrt{-g^{(0)}},
\]

and the constants are

\[
A_1 = \frac{1}{4sr^2}, \quad A_2 = \frac{1}{2(q + 2r + s)r^2d}, \quad B_1 = \frac{1}{4r}, \quad B_2 = \frac{1}{4r}.
\]

Thus we have obtained the $\sigma$-model with the $SL(2, R)/SO(1, 1) \times SL(2, R)/SO(1, 1) \times R \times R$ target space. This structure means that two $p$-branes can be generated separately. As an example one can construct two intersecting black $p$-branes. The procedure is similar to that discussed in Section 3, but now one has to apply Harrison transformations with different parameters to each $SL(2, R)/SO(1, 1)$ component. Thus one obtains two intersecting non-extremal $p$-branes with different charges \([10, 11]\). This derivation demonstrates that the existence of such configurations is a consequence of the $\sigma$-model target space symmetries.

In the Sec. 5 we have constructed some new solutions using the null geodesic method applied to the $\sigma$-model \( (21) \). The same strategy applied to the $\sigma$-model \( (85) \) leads to extremal intersecting $p$-branes with two charges (in the case of the degenerate matrix $K$) and to the intersecting ‘matrioshka’-type $p$-branes (in the case of the non-degenerate matrix $K$). Thus we can speculate that ‘matrioshka’ $p$-branes are subject of the usual intersection rule. As an example we exhibit the metric of two intersecting ‘matrioshka’-type 1-branes in the Type IIA supergravity:

\[
ds^2 = \left[ \frac{\sin (\varphi_1 + \varphi)}{\sin \varphi_1} \right]^\frac{3}{2} \left[ \frac{\sin (\varphi_2 + \varphi)}{\sin \varphi_2} \right]^\frac{3}{2} e^{-\frac{\varphi}{2}\varphi_1 - \frac{\varphi}{2}\varphi_2} \left( -\left[ \frac{\sin (\varphi_1 + \varphi)}{\sin \varphi_1} \right]^{-1} \left[ \frac{\sin (\varphi_2 + \varphi)}{\sin \varphi_2} \right]^{-1} e^{-\frac{\varphi}{2}\varphi_1 + \frac{\varphi}{2}\varphi_2} (-dt^2 + dy^2) \right.
\]

\[
+ \left. \left[ \frac{\sin (\varphi_1 + \varphi)}{\sin \varphi_1} \right]^{-1} e^{\frac{\varphi}{2}\varphi_1} \left( dz_1^2 + dz_2^2 \right) + \left[ \frac{\sin (\varphi_2 + \varphi)}{\sin \varphi_2} \right]^{-1} e^{\frac{\varphi}{2}\varphi_2} \left( dz_3^2 + dz_4^2 \right) + dx_\alpha dx^\alpha \right).
\]

(91)
7 Brinkmann waves on branes

The null geodesic method can also be applied to the $SL(d,R)/SO(1, d-1)$ part of the initial $\sigma$-model [17]. The matrix $\tilde{g}$ should be taken in the form, similar to (54)

$$\tilde{g} = \tilde{g}_0 e^{K\sigma},$$

(92)

where $\text{Tr} K^2 = 0$, $K$ belongs to an algebra $sl(d,R)$. Asymptotic flatness conditions imply

$$\tilde{g}_0 = \text{diag}(-1, 1, \ldots).$$

(93)

In the simplest case $d = 2$ the condition $\text{Tr} K^2 = 0$ leads to $\det K = 0$, i.e. the resulting matrix $\tilde{g}$ has the form

$$\tilde{g} = \begin{pmatrix} -(1 + a\sigma) & -c\sigma \\ d\sigma & 1 - a\sigma \end{pmatrix}, \quad a^2 + dc = 0.$$

(94)

The matrix $\tilde{g}$ should be symmetric, so $c = -d = \pm a$, and after rescaling of the harmonic function $\sigma$

$$\tilde{g} = \begin{pmatrix} -(1 + \sigma) & \pm\sigma \\ \pm\sigma & 1 - \sigma \end{pmatrix}.$$ 

(95)

This metric can be rewritten in the light-cone coordinates as

$$ds^2 = -du dv - \sigma du^2.$$ 

(96)

It corresponds to the well-known Brinkmann wave [34] and the decoupling of $\tilde{g}$ from the action (17) reflects the possibility of a superposition of $p$-branes and waves [33].

8 Bonnor-type map

One can obtain new non-trivial solutions from the old one using a map between similar cosets describing physically different theories. This idea traces back to the Bonnor construction of the metric of a magnetic dipole in General Relativity using a correspondence of two $SL(2,R)/SO(1,1)$ describing stationary vacuum gravity and static electrovacuum. Since we have the same subspace in the $p$-brane case (17), one can use the same correspondence to generate new $p$-brane solutions.

For the vacuum Einstein theory in four dimensions the target space describing stationary solutions has the form

$$ds^2 = \frac{1}{2f^2}(df^2 + d\chi^2),$$

(97)

where $f = g_{tt}$, and $\chi$ is the twist-potential. One can check that the correspondence between two $\sigma$-models can be achieved only if $B = 1/8$. Note that the appropriate map is complex

$$\Psi = 2 \ln f, \quad v = i\chi,$$

(98)
so, in order to obtain real solutions in the Minkowskian space, we should take the complexified seed solutions.

As an example let us consider the complexified Kerr-NUT solution of the Einstein theory taking pure imaginary rotation and NUT parameters $\tilde{a} = ia$, $\tilde{N} = iN$

$$ds^2 = -\frac{\Delta + \tilde{a}^2 \sin^2 \theta}{\Sigma} (dt - \omega d\varphi^2) + \Sigma \left( \frac{dr^2}{\Delta} + \frac{\Delta \sin^2 \theta}{\Delta + \tilde{a}^2 \sin^2 \theta} d\varphi^2 \right),$$

(99)

where

$$\Delta = r^2 - 2Mr - \tilde{a}^2 + \tilde{N}^2,$$

(100)

$$\Sigma = r^2 + \delta^2, \quad \delta = -i(\tilde{a} \cos \theta + \tilde{N}),$$

(101)

$$\omega = \frac{2i}{\Delta + \tilde{a}^2 \sin^2 \theta} (\tilde{N} \Delta \cos \theta + \tilde{a} \sin^2 \theta (Mr - \tilde{N}^2)).$$

(102)

The potentials $f$ and $\chi$ will be given by

$$f = \frac{\Delta + \tilde{a}^2 \sin^2 \theta}{\Sigma}, \quad \chi = -\frac{2i(M\tilde{a} \cos \theta + M\tilde{N} - \tilde{N}r)}{\Sigma}. \quad (103)$$

Since the potential $\chi$ is pure imaginary, the complex transformation (98) will give a real solution.

Consider the Type IIA supergravity (73). It contains a 1-form which can be connected with electric black hole or with the magnetic D6-brane. We will construct the metric of the D6-brane. It is easy to check that in this case $B$ is equal to $1/8$, so we can use the above technique. The map (98) applied to the potentials (103) and the Eqs. (19), (20) lead the following metric

$$ds^2 = f^\frac{1}{8} (-dt^2 + dy_1^2 + \ldots + dy_6^2) + f^\frac{1}{8} \Sigma dr^2 + f^\frac{1}{8} \Sigma d\theta^2 + f^{-\frac{5}{8}} \Delta \sin^2 \theta d\varphi^2,$$

(104)

where $f$, $\Delta$ and $\Sigma$ are given by (103), (100) and (101). This metric describes the magnetic D6-brane with the dipole moment which is generated by the parameter $\tilde{a}$. The non-trivial components of the 1-form field strength are

$$F^{\rho \varphi} = -\frac{f^{-5/4}}{\Sigma \sin \theta} \partial_\theta u, \quad F^{\theta \varphi} = \frac{f^{-5/4}}{\Sigma \sin \theta} \partial_\varphi u,$$

(105)

where $u$ is given by

$$u = 2(M\tilde{a} \cos \theta + M\tilde{N} - \tilde{N}r)/\Sigma. \quad (106)$$

The corresponding dilaton field can be expressed through the function $f$ as follows

$$e^{\frac{1}{4} \phi} = f^{\frac{1}{8}}. \quad (107)$$

It is easy to check that the Bogomol’nyi bound is saturated if $M = N$. In the limit $\tilde{a} = 0$, $M = N$ the configuration obtained reduces to the usual extremal magnetic D6-brane.
9 Concluding remarks

In this paper, we have focused on the technical aspects of getting $p$-brane solutions via the $\sigma$-model formulation of the simplest brane-containing theories. Although the idea of using dualities of dimensionally reduced theories is not new, we have shown that knowing an explicit non-linear realization of dualities in terms of the target space variables one can exploit ‘hidden symmetries’ to higher extent. Apart from a direct use of transformations to get new solutions from old ones, one can also apply various integration methods developed earlier in the context of General Relativity. In particular, using a technique of harmonic maps we have found new classes of $p$-branes with a nontrivial ‘matrioshka’-type structure of the transverse space. We have shown that some $p$-brane ‘rules’, such as intersection rules for composite branes, or ‘blackening’ prescriptions, have a rather natural geometric interpretation in the $\sigma$-model terms. Since the main subgroup involved is $SL(2, R)$, one can effectively use solutions to other theories sharing the same group structure to get new $p$-brane solutions, this Bonnor-type correspondence is somewhat similar to duality between different theories which was widely discussed recently in the context of the superstrings.

Our formulation also opens a way to apply techniques of integrable systems assuming that the target space variables depend only on two of the transverse coordinates. In the four-dimensional theories the full space-time metric can be recovered once the solution of the corresponding integrable system is found. In the multidimensional cases additional assumptions are needed about the structure of the transverse space to ensure complete solvability. More general lagrangians including several antisymmetric forms and dilatons may be also investigated under the assumption of the block-diagonal metrics. However in the non–diagonal cases one encounters serious technical complications while attempting to find an explicit non–linear realization of ‘hidden’ symmetries.

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