Sparse event-triggered control of linear systems

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Abstract
In event-triggered control, a situation where the control input must be sparse often arises. Therefore, in this study, we propose sparse event-triggered control, meaning that the control input is sparse and updated in an event-triggered manner. First, we present a model-based method for sparse event-triggered control of linear systems, where the event condition is defined by a Lyapunov function. The resulting control input is proven to be sparse and the control system is confirmed to be asymptotically stable. Second, we extend it to a data-driven version, where the event condition is adaptively updated from online data on the state trajectory. Finally, we discuss the possibility of extending our framework to two cases of disturbance and nonlinear dynamics.

1 INTRODUCTION
There is growing interest in networked control, and event-triggered control is known to be a promising method of reducing the amount of computation and transmission in such applications. Therefore, many relevant studies have been conducted to date.1–13 Moreover, in recent years, data-driven methods have been developed for cases in which a plant model is unavailable.9–12,14

On the other hand, a situation often arises in which the control input must be sparse, in the sense that the value becomes zero for several time intervals. A typical case involves a system featuring the so-called “sleep mode,” where the control input is zero. Furthermore, in recent years, the concept of an energy Internet has been proposed15–20 as an analogy of TCP/IP, where an on-demand energy supply is assumed (see, e.g., references for a power pocket network17–20). If a control system is connected to an energy Internet, the actuator is driven in an intermittent manner. In such a case, the control input must be sparse, as shown in Figure 1.

Nevertheless, the existing results for event-triggered control cannot be used in the above situations. In fact, the resulting control inputs are not always sparse in usual event-triggered control.1–14 In contrast, an event-based control framework with a sparse input has been presented in the work,21 however, the period when the control input is applied (i.e., the period $\epsilon$ in Figure 1) cannot be specified.

In this study, we propose sparse event-triggered control, meaning that the control input is sparse and updated in an event-triggered manner. In this framework, state feedback is applied to the plant according to time intervals that start when an event occurs and end after a specified period, as illustrated in Figure 1. First, we present a model-based method for sparse event-triggered control, where the event condition is defined by a Lyapunov function. The resulting control input is proven to be sparse and the control system is confirmed to be asymptotically stable. Second, we extend it to a...
data-driven version, where the event condition is adaptively updated from online data on the state trajectory. Finally, we discuss the possibility of extending our framework to two cases of disturbance and nonlinear dynamics. 

Although the event conditions proposed in this article are similar to those of the existing papers,\cite{7,8,22} our result is distinguished from them. The results in References 7 and 8 are not for sparse control and cannot be straightforwardly extended to our sparse case. In fact, in the sparse control case, it is essential to prove the existence of a lower bound of the periods of zero input because the sparsity of control input might disappear without it. However, no existence result for sparse cases has been given in the existing paper. The result in Reference 22 has addressed a similar sparse control problem with piecewise constant input. However, the periods of nonzero input cannot be arbitrarily selected, which will be a restriction in practice. By considering the above circumstances, this article establishes a complete framework for sparse event-triggered control for linear systems.

This article is organized as follows. The problems to be studied are formulated in Section 2. Section 3 presents a model-based sparse event-triggered control. Based on this result, a data-driven version is proposed in Section 4. The extension to sparse event-triggered control is discussed in Section 5. Finally, the article is concluded in Section 6.

This article is based on our preliminary versions\cite{23,24} published in conference proceedings. However, this journal article contains (a) complete proofs omitted in the preliminary versions, (b) the existence result of a lower bound of the periods of zero input, (c) characterization of the data that achieve data-driven control, and (d) discussion on the extensions to two cases of disturbance and nonlinear dynamics.

Notation.

(i) Sets: Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Q}_+$, and $\mathbb{Z}_+$ denote the sets of real numbers, positive numbers, non-negative numbers, and non-negative integers, respectively. The set $B_r \subset \mathbb{R}^n$, represents the closed ball in $\mathbb{R}^n$ with a radius $r \in \mathbb{R}_+$, that is, $B_r = \{x \in \mathbb{R}^n \mid ||x|| \leq r\}$, where $||x||$ is the Euclidean norm of $x$. We denote by $S_n \subset \mathbb{R}^{n \times n}$ the set of $n \times n$ symmetric matrices. For $\mathcal{S} \subset S_n$, $\text{span}(\mathcal{S})$ represents the space resulted by the linear combinations of symmetric matrices in $\mathcal{S}$. Moreover, $\Lambda(A)$ is the set of all eigenvalues of $A \in \mathbb{R}^{n \times n}$.

(ii) Scalars: For $a, b \in \mathbb{R}$, $\min(a, b)$ and $\max(a, b)$ are the minimum and maximum element of $\{a, b\} \subset \mathbb{R}$, respectively.

(iii) Vectors: The functions $rvec(x)$ and $lvec(P)$ are used to represent the quadratic form $x^T Px$: For $P \in S_n$, with $(i, j)$th element $p_{ij}$ (where $p_{ij} = p_{ji}$), $rvec(P) \in \mathbb{R}^{1/2mn(n+1)}$ denotes the column vector satisfying $[p_{11} p_{12} p_{22}] \cdots [p_{1n} p_{2n} \cdots p_{nn}]^T_1$ and $lvec(x)$ \(\in \mathbb{R}^{1(1/2n(n+1))}\) is the row vector satisfying $x^T Px = lvec(x)\overrightarrow{rvec(P)}$. For example, $lvec(x) = [x_1^2 2x_1 x_2 x_2^2]$ and $rvec(P) = [p_{11} p_{12} p_{22}]^T$ for

$$
\begin{align*}
    x &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2, \\
    P &= \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \in S_2.
\end{align*}
$$

Furthermore, for the matrix $[a_1 a_2 \cdots a_m] \in \mathbb{R}^{m \times m}$, $\overrightarrow{\text{vec}(A)}$ represents $[a_1^T a_2^T \cdots a_m^T]^T \in \mathbb{R}^{nm}$.

(iv) Matrices: The identity matrix of order $n$ is denoted by $I_n \in S_n$. For $P \in S_n$ (whose eigenvalues are real numbers), we use $\lambda_{\min}(P) \in \mathbb{R}$ and $\lambda_{\max}(P) \in \mathbb{R}$ to represent the minimum and maximum eigenvalues, respectively. The Kronecker product of the matrices $A$ and $B$ is denoted by $A \otimes B$.

Fundamental facts: In this article, we use the following mathematical facts.
(I) For $P \in S_n$ and $x \in \mathbb{R}^n$,
$$\lambda_{\min}(P)\|x\|^2 \leq x^TPx \leq \lambda_{\max}(P)\|x\|^2.$$  

(II) (Comparison theorem\textsuperscript{25}) Consider a differentiable function $f : \mathbb{R}_{0+} \rightarrow \mathbb{R}$ satisfying
$$\frac{df}{dt}(t) \leq Kf(t), \ t \in [a, b]$$
for some constant $K \in \mathbb{R}$. Then,
$$f(t) \leq f(a)e^{K(t-a)}$$
for every $t \in [a, b]$.

(III) For matrices $A \in \mathbb{R}^{n \times m}, X \in \mathbb{R}^{m \times p}$, and $B \in \mathbb{R}^{p \times q}$,
$$\text{vec}(AXB) = (B^T \otimes A) \text{vec}(X).$$

(IV) For matrices $A, B \in \mathbb{R}^{n \times n}$,
$$\Lambda(A \otimes B) = \bigcup_{\mu \in \Lambda(B)} \bigcup_{\lambda \in \Lambda(A)} \{\lambda \mu\}.$$

2 PROBLEM FORMULATION

Let us consider the control system $\Sigma$, as shown in Figure 2, which is composed of a plant $P$ and a controller $K$.

The plant $P$ is given by
$$P : \dot{x}(t) = Ax(t) + Bu(t), \quad (1)$$
where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, and $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ are constant matrices, for which the pair $(A, B)$ is stabilizable.

The controller $K$ is given by
$$K : u(t) = \begin{cases} Fx(t) & \text{if } t \in [t_k, t_k + \epsilon), \\ 0 & \text{if } t \in [t_k + \epsilon, t_{k+1}), \end{cases} \quad (2)$$
where $F \in \mathbb{R}^{m \times n}$ is a feedback gain such that $A + BF$ is Hurwitz, $t_k \in \mathbb{R}_{0+}$ ($k = 0, 1, \ldots$) is the sequence of time instants such that $t_0 = 0$ and $t_0 < t_1 < \cdots$, and $\epsilon \in \mathbb{R}_+$ is a constant number. This controller intermittently applies the feedback control $u(t) = Fx(t)$ to $P$. The time instants $t_k$ are called the start time instants and $\epsilon$ is called the control period. The resulting control input of this controller is illustrated in Figure 1.

In (2), the start time instants $t_k$ ($k = 0, 1, \ldots$) are determined in an event-triggered manner. For a function $s : \mathbb{R}_{0+} \times \mathbb{R}^n \times \mathbb{R}_{0+} \times \mathbb{R}^n \rightarrow \mathbb{R}$, the start time instant $t_{k+1} \in [t_k + \epsilon, \infty)$ is defined as the minimum time instant when
$$s(t, x(t), t_k, x(t_k)) < 0 \quad (3)$$

\[\text{FIGURE 2} \quad \text{Control system } \Sigma\]
is violated, that is, \(s(t, x(t), t_k, x(t_k)) = 0\). The function \(s\) is called the start function.

Then, our sparse event-triggered control problem is formulated as follows.

**Problem 1.** Consider the control system \(\Sigma\). Suppose that a control period \(\epsilon \in \mathbb{R}_+\) is given. Then, find a start function \(s\) such that

(i) the control system \(\Sigma\) is globally asymptotically stable,

(ii) there exists a \(\tau_{\text{min}} \in \mathbb{R}_+\) satisfying

\[
t_{k+1} - t_k > \epsilon + \tau_{\text{min}}
\]

for every \(k \in \mathbb{Z}_{0+}\).

Four remarks are given for Problem 1.

First, a similar problem has been addressed in Reference 22, where the controller is a piecewise constant version of (2), that is,

\[
u(t) = \begin{cases} 
Fx(t_k) & \text{if } t \in [t_k, t_k + \epsilon), \\
0 & \text{if } t \in [t_k + \epsilon, t_{k+1}).
\end{cases}
\]

However, it has been clarified for the piecewise constant controller that the control period \(\epsilon\) is limited to be within a certain range determined by the dynamics of the plant. To overcome this limitation, we employ the controller in (2) generating piecewise continuous signals.

Second, (ii) is concerned with the vanishing sparsity. In our setting, the length of \(k\)th time slot of zero input is expressed as \(d_k := t_{k+1} - t_k - \epsilon\). It might be possible that \(d_k \to 0\) as \(k \to \infty\), which implies that the sparsity vanishes. To avoid this phenomenon, (ii) is imposed on the problem.

Third, (ii) is similar to the requirement that a Zeno behavior does not exist for event-triggered control, that is,

(ii') there exists a \(\tau_{\text{min}} \in \mathbb{R}_+\) satisfying

\[
t_{k+1} - t_k > \tau_{\text{min}}
\]

for every \(k \in \mathbb{Z}_{0+}\).

However, (ii) is a strictly stronger condition than (ii') since the control period \(\epsilon\) is a given positive constant.

Fourth, the control period \(\epsilon\) is a given parameter, which corresponds to the period of time during which the actuator can operate continuously. For example, if the actuator gains energy via a power pocket network, \(^{17-20}\) which is a kind of time division multiplexing of power supply as illustrated in Section 1, the control period \(\epsilon\) is set to the time length of a power pocket.

Next, let us formulate the problem for the case where the mathematical model of the plant is not available.

For \(r_1 \in \mathbb{R}_{0+}\) and \(x_1 \in \mathbb{R}^n\), let \(x(t, r_1, x_1)\) be the solution of \(\Sigma\) for \(x(r_1) = x_1\). Moreover, for \(r_2 \in (r_1, \infty)\), let

\[
X([r_1, r_2], x_1) := \bigcup_{r \in [r_1, r_2]} \{(t, x(t, r_1, x_1))\}.
\]

This represents the segment of the state trajectory \(x(t, r_1, x_1)\) on a time interval \([r_1, r_2]\).

Then, the data-driven version of Problem 1 is given as follows.

**Problem 2.** Consider the control system \(\Sigma\) with unknown \(A\) and \(B\). Suppose that a control period \(\epsilon \in \mathbb{R}_+\) and a stabilizing feedback gain \(F \in \mathbb{R}^{m \times n}\) are given, and assume that the state trajectory data \(X([0, \epsilon], x(0))\) are available at each \(t\) in \(\Sigma\). Then, find a start function \(s\) satisfying (i) and (ii) in Problem 1.

In the above setting, it is assumed that a stabilizable feedback gain \(F\) is available although the matrices \(A\) and \(B\) are unknown. This assumption is reasonable in some cases, for example, when \(F\) is a PID controller designed by the so-called...
PID tuning without a mathematical model or when \(A\) and \(B\) are varied by deterioration caused by aging but \(F\) is still stabilizing.

3 | MODEL-BASED SPARSE EVENT-TRIGGERED CONTROL

3.1 | Solution to Problem 1

In intermittent control, it is reasonable to set a nonzero value to the control input when the plant is in a “bad” state. This motivates us to design the start function \(s\) so as to quantify the badness.

To quantify the badness of the plant, we employ the quadratic function

\[
V(x) = x^T P^* x
\]

for a unique solution \(P^*\) to the Lyapunov equation

\[
(A + BF)^TP + P(A + BF) = -Q.
\]

Moreover, we exploit \(V(x(t_k))e^{-\lambda(t-t_k)}\) (which is equal to the function \(V(x(0))e^{-\lambda t}\)) as a threshold of the badness. This implies that, if \(V(x(t)) \leq V(x(t_k))e^{-\lambda(t-t_k)}\) holds for each \(k \in \mathbb{Z}_+\), \(V(x(t))\) is forced to be bounded by \(V(x(0))e^{-\lambda t}\). Motivated by this fact, we propose the following start function \(s\):

\[
s(t, x(t), t_k, x(t_k)) = V(x(t)) - V(x(t_k))e^{-\lambda(t-t_k)}.
\]

For this start function, the following result is obtained.

**Theorem 1.** Consider Problem 1. Suppose that a positive-definite matrix \(Q \in \mathbb{S}_n\) is given. Let \(\lambda \in (0, \lambda^*)\) be a constant number for the positive number

\[
\lambda^* := \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P^*)},
\]

where \(P^*\) is a unique solution to (6). Then, (7) is a solution to Problem 1.

**Proof.** First, we prove (i). For each \(k \in \mathbb{Z}_+\), we consider \(\Sigma\) on the time interval \([t_k, t_k + \epsilon]\), during which the dynamics of \(\Sigma\) is given by

\[
\dot{x}(t) = (A + BF)x(t).
\]

From \(\dot{V}(x(t)) = -x^T(t)Qx(t)\) (which is given by (5), (6), and (9)) and Fact (I) in the end of Section 1, we have

\[
V(x(t)) \leq \lambda_{\max}(P^*)\|x(t)\|^2,
\]

\[
\dot{V}(x(t)) \leq -\lambda_{\min}(Q)\|x(t)\|^2.
\]

This imply

\[
\dot{V}(x(t)) \leq -\lambda^* V(x(t)).
\]

By applying Fact (II) into (11), it follows that

\[
V(x(t)) - V(x(t_k))e^{-\lambda^*(t-t_k)} \leq 0
\]
on \([t_k, t_k + \epsilon)\). Furthermore, this inequality and the continuity of \(x(t)\) imply that

\[
V(x(t)) - V(x(t_k))e^{-\lambda(t-t_k)} < 0
\]

(13)

holds on \((t_k, t_k + \epsilon)\) for each \(\lambda \in (0, \lambda^*).\) In particular, \(t_{k+1}\) is the time instant when (3) is violated, which gives

\[
V(x(t)) - V(x(t_k))e^{-\lambda(t-t_k)} \leq 0
\]

(14)

on \([t_k, t_{k+1}].\) Moreover, from the definition of \(t_{k+1},\) it follows that

\[
V(x(t_{k+1})) = V(x(t_k))e^{-\lambda(t_{k+1}-t_k)},
\]

(15)

which gives

\[
V(x(t_k)) = V(x(t_0))e^{-\lambda(t_k-t_0)}
\]

(16)

by considering (15) as a difference equation with respect to the variable \(t_k.\) Therefore, from (16) and the fact that (14) holds on \([t_k, t_{k+1}],\) we obtain

\[
V(x(t)) \leq V(x(0))e^{-\lambda t}
\]

(17)

for every \(t \in [t_0, \infty).\) In (17), \(0 \leq V(x(t)), \lim_{t \to \infty} V(x(0))e^{-\lambda t} = 0,\) and \(V(x(0))e^{-\lambda t} < \infty\) hold for every \(t \in \mathbb{R}_{0+}.\) Therefore, (17) implies \(\lim_{t \to \infty} V(x(t)) = 0,\) that is, \(\lim_{t \to \infty} x(t) = 0.\) This proves (i).

Next, we prove (ii) for

\[
\tau_{\text{min}} = \frac{\lambda^* - \lambda}{2(\mu + \lambda)}\epsilon,
\]

(18)

where \(\mu \in \mathbb{R}_{0+}\) is given by

\[
\mu := \frac{|\lambda_{\text{max}}(A^TP^* + P^*A)|}{\lambda_{\text{min}}(P^*)}.
\]

(19)

Note here that \(P^* > 0.\)

Let us consider the time interval \([t_k + \epsilon, t_{k+1}).\) Then, the dynamics of \(\Sigma\) is written by

\[
\dot{x}(t) = Ax(t)
\]

(20)

on \([t_k + \epsilon, t_{k+1}).\) Hence, the time derivative of \(V(x(t))\) along the state trajectory \(x(t)\) is given by

\[
\dot{V}(x(t)) = x^T(t) \left(A^TP^* + P^*A\right)x(t)
\]

(21)

from (5) and (20). In addition, we have

\[
\dot{V}(x(t)) \leq \lambda_{\text{max}}(A^TP^* + P^*A)||x(t)||^2,
\]

\[
\lambda_{\text{min}}(P^*)||x(t)||^2 \leq V(x(t))
\]

(22)

from (21), (5), and Fact (I). Thus, it follows that

\[
\dot{V}(x(t)) \leq |\lambda_{\text{max}}(A^TP^* + P^*A)||x(t)||^2 \leq \mu V(x(t))
\]

(23)

on the time interval \([t_k + \epsilon, t_{k+1}).\) Moreover, applying Fact (II) to (23) provides

\[
V(x(t)) \leq e^{\mu(t-t_k-\epsilon)}V(x(t_k + \epsilon))
\]

(24)
on \([t_k + \epsilon, t_{k+1}]\). On the other hand, since (12) holds for \(t = t_k + \epsilon\) because of the continuity of \(V(x(t))\), we have

\[
V(x(t_k + \epsilon)) - V(x(t_k)) e^{-\lambda^* \epsilon} \leq 0.
\]

This fact and (24) imply

\[
V(x(t)) \leq e^{\mu(t - t_k - \epsilon)} V(x(t_k)) e^{-\lambda^* \epsilon}.
\] (25)

Then, from (25) and (7), the start function \(s\) is bounded on \([t_k + \epsilon, t_{k+1}]\) as follows:

\[
s(t, x(t), t_k, x(t_k)) \leq e^{\mu(t - t_k - \epsilon)} V(x(t_k)) e^{-\lambda^* \epsilon} - V(x(t_k)) e^{-\mu(t - t_k - \epsilon)}.
\] (26)

In particular, under \(t \leq t_k + \epsilon + \tau_{\min}\), we have

\[
s(t, x(t), t_k, x(t_k)) \leq e^{\mu \tau_{\min}} V(x(t_k)) e^{-\lambda^* \epsilon} - V(x(t_k)) e^{-\mu(\epsilon + \tau_{\min})}
\]

\[
= (e^{\mu(\epsilon + \tau_{\min})} - 1) V(x(t_k)) e^{-\mu(\epsilon + \tau_{\min})} e^{-\lambda^* \epsilon} - V(x(t_k)) e^{-\mu(\epsilon + \tau_{\min})}.
\]

from (18). This fact indicates (4) because of the definition of \(t_k\). This completes the proof. \(\blacksquare\)

One may consider that (ii) can be obtained in the same manner as the proof of the nonexistence of Zeno behaviors in typical event-triggered control, for example, References 7 and 8; however, the actual situation is different. This is because the resulting dynamics is distinct from those of the typical framework. In fact, the closed-loop system of the standard event-triggered control is given by

\[
\dot{x}(t) = Ax(t) + BFx(t_k), \quad \text{if } t \in [t_k, t_{k+1}),
\]

while the closed-loop system in this article has a switching dynamics as in (9) and (20), that is,

\[
\dot{x}(t) = \begin{cases} 
(A + BF)x(t) & \text{if } t \in [t_k, t_k + \epsilon), \\
Ax(t) & \text{if } t \in [t_k + \epsilon, t_{k+1}).
\end{cases}
\]

**Example 1.** Consider the system \(\Sigma\), which is given by

\[
A = \begin{bmatrix} 1 & -5 \\ 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} -4 \\ 5 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & -1 \end{bmatrix}
\]

and the start function (7) for

\[
\epsilon = 0.1, \quad Q = \begin{bmatrix} 7 & 3 \\ 3 & 7 \end{bmatrix}, \quad \lambda = 1.3.
\]

Figure 3 shows the simulation results for \(x(0) = [-6 - 4]^T\). It is observed that the control input is sparse and the state approaches zero over time. In the bottom figure, we see that \(V(x(t))\) decreases when the control input is nonzero but it does not always increase when the control input is zero.

### 3.2 Application: Rotor angle regulation of DC motor via a power packet network

In this section, we apply our sparse event-triggered control to rotor angle regulation of a DC motor to which energy is supplied by a power packet network.17-20
Consider the electrical circuit illustrated in Figure 4. The dynamics of this system is given by (1) for $x(t) = [\theta(t) \dot{\theta}(t) i(t)]^T$, $u(t) = v(t)$, and

$$
A = \begin{bmatrix}
0 & 1 & 0 \\
0 & -\frac{K_b}{J} & \frac{K}{J} \\
0 & -\frac{K_e}{L} & -\frac{R}{L}
\end{bmatrix}, \quad B = \begin{bmatrix}
0 \\
0 \\
\frac{1}{L}
\end{bmatrix},
$$

(27)

where $\theta(t), i(t)$, and $v(t)$ are the rotor angle, armature current, and input voltage, respectively. On the other hand, $R, L, J, K, K_b$, and $K_e$ are positive constants that describe the characteristics of the motor, which are given as $(R, L, J, K, K_b, K_e) = (3, 0.5, 0.1, 0.5, 0.1, 0.2)$. 

**FIGURE 3** Simulation result for model-based sparse event-triggered control in Example 1

**FIGURE 4** The circuit of a DC motor driven by a power packet
Since the power to drive the motor is supplied by a power packet, the control input $u(t)$ has to be sparse. Thus, we regulate the state $x(t)$ of this system into the origin 0 by using the sparse event-triggered control in Theorem 1. For this system, we construct a sparse event-triggered controller by (2), (7), and

$$F = \begin{bmatrix} -7 & -7 & -1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \lambda = 0.7, \quad \varepsilon = 0.1. \quad (28)$$

The simulation result of the system for $x(0) = [2 \ -1 \ 0]^T$ is illustrated in Figure 5. This figure shows that the rotor angle $\theta(t)$ decreases with each intermittent control and converges to 0 with time. In this way, the proposed method is useful for the control under intermittent power supply.

4 | DATA-DRIVEN SPARSE EVENT-TRIGGERED CONTROL

Now, we address Problem 2, that is, the data-driven version of Problem 1.

4.1 | Data-driven solution to Lyapunov equations

In solving Problem 2, a data-driven method to construct a Lyapunov function plays an important role. Thus, we first present it based on the data on state trajectories.
Consider the linear system
\[ \dot{x}(t) = \tilde{A}x(t), \]  

(29)

where \( x(t) \in \mathbb{R}^n \) is the state and \( \tilde{A} \in \mathbb{R}^{n \times n} \) is a Hurwitz matrix that is assumed to be unknown.

We then consider the following problem, which is useful for constructing a quadratic Lyapunov function \( V : \mathbb{R}^n \rightarrow \mathbb{R}_0^+ \) in a data-driven manner.

**Problem 3.** Consider the system in (29). Suppose that a positive-definite matrix \( Q \in S_n \) and \( m \) state trajectory data \( X([t_{i1}, t_{i2}], x_{i1}) \) \((i = 1, 2, \ldots, m)\) are given for some \( x_{i1} \in \mathbb{R}^n \) and \((t_{i1}, t_{i2}) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+ \) satisfying \( t_{i1} < t_{i2} \). Then, find a solution \( P \in S_n \) to the Lyapunov equation

\[ \tilde{A}^T P + P\tilde{A} = -Q. \]  

(30)

Note that (30) has a unique positive-definite solution \( P \) because \( \tilde{A} \) is Hurwitz and \( Q \) is positive-definite.

Now, we derive a solution to Problem 3. If (30) holds, we obtain

\[ x^T(t_{i2}, t_{i1}, x_{i1})Px(t_{i2}, t_{i1}, x_{i1}) - x^T_{i1}Px_{i1} = -\int_{t_{i1}}^{t_{i2}} x^T(t, t_{i1}, x_{i1})Qx(t, t_{i1}, x_{i1}) \, dt \quad (i = 1, 2, \ldots, m) \]  

(31)

because

\[
x^T(t_{i2}, t_{i1}, x_{i1})Px(t_{i2}, t_{i1}, x_{i1}) - x^T_{i1}Px_{i1} = \int_{t_{i1}}^{t_{i2}} \frac{d}{dt} (x^T(t, t_{i1}, x_{i1})Px(t, t_{i1}, x_{i1})) \, dt \\
= \int_{t_{i1}}^{t_{i2}} x^T(t, t_{i1}, x_{i1}) \left( \tilde{A}^T P + P\tilde{A} \right) x(t, t_{i1}, x_{i1}) \, dt \\
= -\int_{t_{i1}}^{t_{i2}} x^T(t, t_{i1}, x_{i1})Qx(t, t_{i1}, x_{i1}) \, dt.
\]

In Problem 3, it is assumed that the positive-definite matrix \( Q \) and the state trajectory data \( x(t, t_{i1}, x_{i1}) \) on \([t_{i1}, t_{i2}]\) are given. This implies that, in (31), \( x(t_{i2}, t_{i1}, x_{i1}), x_{i1}, \) and \( \int_{t_{i1}}^{t_{i2}} x^T(t, t_{i1}, x_{i1})Qx(t, t_{i1}, x_{i1}) \, dt \) are exactly known. Thus, we can consider (31) as a linear equation with an unknown matrix \( P \in S_n \), and thus the solution to (31) is obtained by solving the linear equation.

This idea is formalized as follows. By using the functions rvec and lvec defined in the end of Section 1, (31) is transformed into

\[ \Delta \text{rvec}(P) = \Gamma, \]  

(32)

where

\[
\Delta := \begin{bmatrix}
\text{lvec}(x(t_{i2}, t_{i1}, x_{i1})) - \text{lvec}(x_{i1}) \\
\text{lvec}(x(t_{i2}, t_{i1}, x_{i1})) - \text{lvec}(x_{i2}) \\
\vdots \\
\text{lvec}(x(t_{i2}, t_{i1}, x_{i1})) - \text{lvec}(x_{m1})
\end{bmatrix} \in \mathbb{R}^{m \times l}, \quad \Gamma := \begin{bmatrix}
\int_{t_{i1}}^{t_{i2}} x^T(t, t_{i1}, x_{i1})Qx(t, t_{i1}, x_{i1}) \, dt \\
\int_{t_{i1}}^{t_{i2}} x^T(t, t_{i1}, x_{i1})Qx(t, t_{i1}, x_{i1}) \, dt \\
\vdots \\
\int_{t_{i1}}^{t_{i2}} x^T(t, t_{i1}, x_{i1})Qx(t, t_{i1}, x_{i1}) \, dt
\end{bmatrix} \in \mathbb{R}^l,
\]  

(33)

and \( l := \frac{1}{2}n(n + 1) \). Note that the equation in (31) is equivalent to the equation in (32). Moreover, (32) is a linear equation with \( l \) unknown parameters. Thus, the following result is obtained.

**Theorem 2.** If

\[ \text{rank}(\Delta) = l, \]  

(34)
then the following two statements hold:

1) There exists a unique symmetric solution \( P \in S_n \) to the equation in (31).
2) The solution to the Lyapunov equation in (30) is equal to the solution \( P \in S_n \) to the equation in (31).

Proof. The following three facts prove Theorem 2.

(a) If \( \bar{A} \) is Hurwitz, there exists a unique positive-definite solution \( P \) to (30).
(b) If (34) holds, (31) has a unique symmetric solution given by

\[
P^* = \int_0^\infty \left( e^{At} \right)^\top Q e^{At} \, dt,
\]

which is positive-definite.
(c) \( P^* \) in (35) is a solution to (30).

Fact (a) is a well-known result (see, e.g., Reference 27).

Next, we prove (b). The linear equation in (31) is equivalently transformed into the linear equation in (32). Moreover, (32) has a unique solution subject to (34). These facts indicate that (31) has a unique symmetric solution \( P \) under (34). Furthermore, (31) holds for \( P^* \) because we have

\[
x^\top(\tau, \tau_0, x_0)P^*x(\tau, \tau_0, x_0) = x^\top(\tau, \tau_0, x_0) \left( \int_0^\infty \left( e^{At} \right)^\top Q e^{At} \, dt \right) x(\tau, \tau_0, x_0)
\]

\[
= \int_0^\infty \left( e^{At}x(\tau, \tau_0, x_0) \right)^\top Q e^{At}x(\tau, \tau_0, x_0) \, dt
\]

\[
= \int_0^\infty x^\top(t + \tau, \tau_0, x_0)Qx(t + \tau, \tau_0, x_0) \, dt
\]

\[
= \int_\tau^\infty x^\top(t, \tau_0, x_0)Qx(t, \tau_0, x_0) \, dt
\]

for each \( \tau_0 \in \mathbb{R}_{n+} \), \( \tau \in [\tau_0, \infty) \), and \( x_0 \in \mathbb{R}^n \). In fact, (36) indicates

\[
x^\top(\tau_{i2}, \tau_{i1}, x_{i1})P^*x(\tau_{i2}, \tau_{i1}, x_{i1}) - x_{i1}^\top P^*x_{i1} = \int_{\tau_{i2}}^\infty x^\top(t, \tau_{i1}, x_{i1})Qx(t, \tau_{i1}, x_{i1}) \, dt - \int_{\tau_{i1}}^\infty x^\top(t, \tau_{i1}, x_{i1})Qx(t, \tau_{i1}, x_{i1}) \, dt
\]

\[
= -\int_{\tau_{i1}}^{\tau_{i2}} x^\top(t, \tau_{i1}, x_{i1})Qx(t, \tau_{i1}, x_{i1}) \, dt
\]

for \( i = 1, 2, \ldots, m \). This implies that \( P^* \) is a solution to (31). In addition, \( P^* \) is clearly positive-definite if \( Q \) is positive-definite. These facts prove (b).

Finally, we provide (c). Substituting \( P^* \) for \( P \) in the left-hand side of (30), we obtain

\[
\bar{A}^\top P^* + P^* \bar{A} = \bar{A}^\top \left( \int_0^\infty \left( e^{At} \right)^\top Q e^{At} \, dt \right) + \left( \int_0^\infty \left( e^{At} \right)^\top Q e^{At} \, dt \right) \bar{A}
\]

\[
= \int_0^\infty \left( \bar{A}^\top e^{At} Q e^{At} + e^{At} \bar{A}^\top Q e^{At} \right) \, dt
\]

\[
= \int_0^\infty \left( e^{At} \right)^\top Q e^{At} \, dt
\]

\[
= \left[ \left( e^{At} \right)^\top Q e^{At} \right]_0^\infty
\]

\[
= -Q.
\]

This completes the proof.
Lemma 1. Suppose that $\sigma \in \mathbb{R}_+$ is given. Assume that $m \geq l$ and $\tau_{i2} - \tau_{i1} = \sigma$ ($i = 1, 2, \ldots, m$). Then, $W(T_1, T_2)$ is a zero-measure set in $\mathbb{R}^{nxm}$.

Proof. Lemma 1 is proved by the following three facts.

(a) If $\tau_{i2} - \tau_{i1} = \sigma$ ($i = 1, 2, \ldots, m$) holds, $W(T_1, T_2)$ is equal to

$$W := \{X \in \mathbb{R}^{nxm} | p(X) = 0\}, \tag{37}$$

where

$$p(X) := \det \left( G(X)(G(X))^\top \right)$$

for $X = [x_1 \ x_2 \ \ldots \ x_m]$ and $G(X) := [rvec(x_1x_1^\top) \ rvec(x_2x_2^\top) \ \ldots \ rvec(x_mx_m^\top)] \in \mathbb{R}^{l \times m}$.

(b) If $m \geq l$ holds, then 1) $p$ is a polynomial function with $nm$ variables composed of the elements of $X$ and 2) it holds $p(X) \neq 0$ for some $X \in \mathbb{R}^{nxm}$.

(c) If 1) and 2) in (b) hold, then $W$ in (37) is zero-measure in $\mathbb{R}^{nxm}$.

Fact (c) is straightforwardly derived by the fact that \{ $x \in \mathbb{R}^n$ | $p(x) = 0$ \} is zero-measure in $\mathbb{R}^n$ if $p : \mathbb{R}^n \rightarrow \mathbb{R}$ is a polynomial function satisfying $p(x) \neq 0$ for some $x \in \mathbb{R}^n$. In the following, we prove (a) and (b).

First, we show (a). It holds if $\text{rank}(\Delta(X, T_1, T_2)) = \text{rank}(G(X))$ because of $\text{rank}(G(X)) < l \Leftrightarrow G(X)(G(X))^\top \in \mathbb{R}^{l \times l}$ is a singular matrix $\Rightarrow p(X) = 0$. Therefore, we prove $\text{rank}(\Delta(X, T_1, T_2)) = \text{rank}(G(X))$.

This fact is obtained by

$$\text{rank}(\Delta(X, T_1, T_2)) = \text{rank} \left( (\Delta(X, T_1, T_2))^\top \right)$$

$$= \text{rank} \left( \begin{bmatrix} \text{Crvec}(x_{11}x_{11}^\top) & \text{Crvec}(x_{21}x_{21}^\top) & \ldots & \text{Crvec}(x_{ml}x_{ml}^\top) \end{bmatrix} \right)$$

$$= \text{rank}(CG(X))$$

$$= \text{rank}(G(X)),$$

where $C \in \mathbb{R}^{k \times l}$ is a nonsingular matrix. The first, third, and fourth equalities are all trivial. To show the second equality, we prove that there exists a nonsingular $C \in \mathbb{R}^{k \times l}$ satisfying

$$(lvec(x_{i2}, \tau_{i1}, x_{i1})) - lvec(x_{i1}) \in \mathbb{R}^{k \times 1}$$

for all $i \in \{1, 2, \ldots, m\}$.

From the definitions of lvec, vec, and rvec, we have

$$(lvec(x))^\top = C_1 \text{vec}(xx^\top), \tag{38}$$

$$\text{vec}(xx^\top) = C_2 \text{rvec}(xx^\top), \tag{39}$$
where \( C_1 \in \mathbb{R}^{bn \times i} \) is a full column rank matrix and \( C_2 \in \mathbb{R}^{i \times l} \) is a full row rank matrix. On the other hand, from \( r_{12} - r_{11} = \sigma \) \((i = 1, 2, \ldots, m)\) and Fact (III) in the end of Section 1,
\[
\text{vec}(x(t_{12}, t_{11}, x_{11})x^T(t_{12}, t_{11}, x_{11})) = \text{vec}(e^{A_x}x_{11}x_{11}^T(e^{A_x})^T) = (e^{A_x} \otimes e^{A_x}) \text{ vec } (x_{11}x_{11}^T).
\]
(40)
holds. Thus, (38), (39), and (40) implies
\[
(l \text{vec}(x(t_{12}, t_{11}, x_{11}))) - l \text{vec}(x(t_1)) = C_1 (e^{A_x} \otimes e^{A_x} - I_n) \text{ vec } (x_{11}x_{11}^T) = C \text{ vec } (x_{11}x_{11}^T),
\]
where we use \( C := C_1(e^{A_x} \otimes e^{A_x} - I_n) \). Moreover, \( C \) is nonsingular because 1) \( C_1 \) is full column rank, 2) \((e^{A_x} \otimes e^{A_x} - I_n) \in \mathbb{R}^{bn \times n^2}\) is nonsingular (which is derived by Fact (IV) and the fact that \( e^{A_x} \) has no eigenvalue 1 if \( A \) is Hurwitz), and 3) \( C_2 \) is full row rank. These facts prove (a).

Next, we give the proof of (b). It is clear that \( p \) is a polynomial function from the definition of \( p \) and \( \text{vec} \). Therefore, we prove that there exists \( X \in \mathbb{R}^{nxm} \) satisfying \( p(X) \neq 0 \).

We consider \( m \geq l \) and \( X = [x_1 x_2 \cdots x_m] \in \mathbb{R}^{nxm} \) that holds
\[
\{ x_1, x_2, \ldots, x_l \} = \bigcup_{(i,j) \in \{1, 2, \ldots, n\}^2} \{ e_i + e_j \}
\]
and \( x_i = 0 \) \((i = l + 1, l + 2, \ldots, m)\), where \( e_i \in \mathbb{R}^n \) \((i = 1, 2, \ldots, n)\) are the vectors satisfying \([e_1 e_2 \cdots e_n] = I_n\). Then, we have
\[
\text{span } \{ \{ x_1 x_1^T, x_2 x_2^T, \ldots, x_m x_m^T \} \} = S_n.
\]
(41)
This is because 1) \((e_i e_i^T + e_j e_j^T) \in S_n \) \((i, j = 1, 2, \ldots, n)\) holds
\[
e_i e_i^T + e_j e_j^T = (e_i + e_j)(e_i + e_j)^T - \frac{1}{4}(e_i + e_i)(e_i + e_i)^T - \frac{1}{4}(e_j + e_j)(e_j + e_j)^T
\]
and 2) it spans \( S_n \). Moreover, from the definitions of \( \text{vec} \) and \( \text{rvec} \) and (41), we have
\[
\text{rank}(G(X)) = \text{rank } \left( \begin{array}{c|c|c|c} \text{vec}(x_1 x_1^T) & \text{vec}(x_2 x_2^T) & \cdots & \text{vec}(x_m x_m^T) \end{array} \right) = \text{dim } \left( \text{span } \{ \{ x_1 x_1^T, x_2 x_2^T, \ldots, x_m x_m^T \} \} \right) = \text{dim}(S_n) = l.
\]
Furthermore, \( \text{rank}(G(X)) = l \) is equivalent to \( p(X) \neq 0 \) as described in the proof of (a). Hence, there exists \( X \in \mathbb{R}^{nxm} \) satisfying \( p(X) \neq 0 \). This completes the proof of (b).

**Example 2.** Consider the system in (29) for
\[
\hat{A} = \begin{bmatrix} -1 & 3 \\ -1 & -1 \end{bmatrix}.
\]
(42)
We address Problem 3 for
\[
Q = \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix}
\]
and the data on the three state trajectories in Figure 6.

In this case, \( m = l = 3 \) and we obtain the linear equation
FIGURE 6 Segments of state trajectories for data-driven design of Lyapunov function. (A) State trajectory for initial state \(x(0) = [5 - 5]^\top\). Bi-directional arrows represent time intervals \([\tau_{11}, \tau_{12}]\) and \([\tau_{21}, \tau_{22}]\). (B) State trajectory for initial state \(x(0) = [2 6]^\top\). Bi-directional arrow represents a time interval \([\tau_{31}, \tau_{32}]\).

\[
\begin{bmatrix}
7.6855 & -8.2556 & -10.3133 \\
-2.4512 & 2.5013 & -0.1324 \\
-21.9440 & -1.9609 & 1.6359
\end{bmatrix}
\quad rvec(P) = \begin{bmatrix}
7.0810 \\
3.2855 \\
21.7835
\end{bmatrix},
\] (43)

which corresponds to (32). Since \(\text{rank}(\Delta) = 3\), (43) has a unique symmetric solution:

\[
rvec(P) = \begin{bmatrix}
1.125 \\
-0.125 \\
1.625
\end{bmatrix}.
\]

Thus, the solution to (31) is given by

\[
P = \begin{bmatrix}
1.125 & -0.125 \\
-0.125 & 1.625
\end{bmatrix}.
\] (44)

From Theorem 2, \(P\) corresponds to the solution to the Lyapunov equation in (30). In fact, by substituting \(P\) in (44) and \(\bar{A}\) in (42) into the left-hand side of (30), we find

\[
\begin{bmatrix}
-1 & 3 \\
-1 & -1
\end{bmatrix}^{\top} \begin{bmatrix}
1.125 & -0.125 \\
-0.125 & 1.625
\end{bmatrix} + \begin{bmatrix}
1.125 & -0.125 \\
-0.125 & 1.625
\end{bmatrix} \begin{bmatrix}
-1 & 3 \\
-1 & -1
\end{bmatrix} = \begin{bmatrix}
-2 & 2 \\
2 & -4
\end{bmatrix} = -Q.
\]

Therefore, we obtain a Lyapunov function of the linear system in (29) for (42) as

\[
V(x) = x^{\top} \begin{bmatrix}
1.125 & -0.125 \\
-0.125 & 1.625
\end{bmatrix} x.
\]

4.2 Solution to Problem 2

In this section, we provide a solution to Problem 2. As explained above, if \(A\) and \(B\) are unknown for the plant, we cannot directly obtain a Lyapunov function for the system given in (9). Then, a start function \(s\) cannot be constructed for sparse
event-triggered control. We therefore consider adaptive construction of the start function $s$ from the online state trajectory data. This is achieved by employing the method derived in the previous section.

The proposed method is based on the sparse event-triggered strategy in (2) and (3), and on an adaptive update rule for the start function $s$ from the online state trajectory data. The update rule is composed of the following three steps for each $k \in \mathbb{Z}_0+$:

(i) **Data collection:** In the time interval $[t_k, t_k + \epsilon]$, when the state feedback is applied to the plant, we collect the state trajectory data.

(ii) **Estimation of $P$:** Using the method in Section 4.1, we estimate the solution $P$ to the Lyapunov equation in (6) from the data collected until the time $t_k + \epsilon$. The estimation of the solution $P$ on the time interval $[0, t_k]$ is denoted by $P_k$.

(iii) **Update of $s$:** The start function $s$, which is used in the time interval $[t_k + \epsilon, t_{k+1}]$, is updated according to the estimation $P_k$ by (7) for $V(x(t)) = V_k(x(t))$, where

$$V_k(x) := x^TP_kx. \quad (45)$$

This idea is formulated as follows. Consider the control system $\Sigma$ with a start-time sequence $t_k$ ($k = 0, 1, \ldots$). Let $x(t, t_0, x_0)$ denote the state $x(t)$ for the initial state $x(t_0) = x_0$. We introduce

$$\Delta_k := \begin{bmatrix}
\text{lvec}(x(t_0 + \epsilon, t_0, x(t_0))) - \text{lvec}(x(t_0)) \\
\text{lvec}(x(t_1 + \epsilon, t_1, x(t_1))) - \text{lvec}(x(t_1)) \\
\vdots \\
\text{lvec}(x(t_k + \epsilon, t_k, x(t_k))) - \text{lvec}(x(t_k))
\end{bmatrix}, \quad 
\Gamma_k := - \begin{bmatrix}
\int_{t_0}^{t_0+\epsilon} x^T(t, t_0, x(t_0))Qx(t, t_0, x(t_0)) \, dt \\
\int_{t_1}^{t_1+\epsilon} x^T(t, t_1, x(t_1))Qx(t, t_1, x(t_1)) \, dt \\
\vdots \\
\int_{t_k}^{t_k+\epsilon} x^T(t, t_k, x(t_k))Qx(t, t_k, x(t_k)) \, dt
\end{bmatrix}.$$

By noting that $\Delta_k$ and $\Gamma_k$ are composed of the data collected until the time $t_k + \epsilon$, as stated in (ii), we obtain the following linear equation:

$$\Delta_k rvec(P_k) = \Gamma_k, \quad (46)$$

which corresponds to (32). Moreover, we use

$$\lambda_k := \begin{cases} 
\alpha \frac{\lambda_{\max}(Q)}{\lambda_{\max}(P_k)} & \text{if } \lambda_{\max}(P_k) > 0, \\
0 & \text{otherwise},
\end{cases}$$

where $\alpha \in (0, 1)$ is an arbitrary given number and $P_k$ is a solution to (46).

Then, the solution to Problem 2 is obtained as follows.

**Theorem 3.** Consider Problem 2. Suppose that a positive-definite matrix $Q \in \mathbb{S}_n$, $\alpha \in (0, 1)$ and $h \in \mathbb{R}_+$ are given. Let

$$s(t, x(t), t_k, x(t_k)) := \begin{cases} 
V_k(x(t)) - V_k(x(t_k))e^{-\lambda_k(t-t_k)} & \text{if } \text{rank}(\Delta_k) = l, \\
t - t_k - \epsilon - h & \text{otherwise.}
\end{cases} \quad (47)$$

If there exists a $k \in \mathbb{Z}_0+$ satisfying

$$\text{rank}(\Delta_k) = l, \quad (48)$$

then (47) is a solution to Problem 2.

**Proof.** See Section 4.3.

We make two remarks regarding Theorem 3.
First, the condition expressed in (48) tends to be satisfied as \( k \) grows (i.e., with time), because the matrix \( \Delta_k \) expands as new data are acquired and \( \text{rank}(\Delta_k) \) is nondecreasing with respect to \( k \).

Second, the start function \( s \) in (47) is a conditional version. The former is the same as in (7), while the latter is newly introduced due to the following reason. In our data-driven method, the estimated matrix \( P_k \) is different from \( P^* \) before (48) is satisfied. As the result, the existence of \( \tau_{\min} \), which is specified in Problem 1(ii) as the lower bound of the periods of zero input, cannot be guaranteed. On the other hand, the latter start function plays a role in ensuring a certain period (which is of length \( h \)) of zero input until the time when (48), that is, \( P_k = P^* \) for some \( k \). This guarantees the existence of \( \tau_{\min} \) as proven in Section 4.3.2.

**Example 3.** Consider the control system \( \Sigma \) in Example 1. For this system, we select \( h = 0.1 \) and \( \alpha = 0.6 \).

Then, the simulation result for \( x(0) := [-6 \ -4]^T \) is illustrated in Figure 7. It is clear that the resulting control input is sparse and the state approaches zero. Moreover, the solution to the Lyapunov equation given in (6) is obtained in this process. The time evolution of \( V_k(x(t)) \) and \( V_k(x(t_k))e^{-\lambda_k(t-t_k)} \) is shown in Figure 8, where we can find the followings: (a) On an early stage of the control, \( V_k(x(t)) \) and \( V_k(x(t_k))e^{-\lambda_k(t-t_k)} \) are discontinuous due to their update with insufficient data, and (b) the behavior of the system becomes similar to that of the model-based case in Section 3 in the latter half of the simulation.

**Example 4.** Consider the system in Section 3.2. We select \( h = 0.1 \) and \( \alpha = 0.9 \).

Figure 9 shows the simulation result for \( x(0) = [2 \ -1 \ 0] \), where \( \theta(t) \), \( \dot{\theta}(t) \), and \( i(t) \) converge to 0 by a sparse input \( u(t) \). Although nonzero inputs are frequently applied on the time interval \([0, 1.1)\) to collect the information of the system, only few inputs are applied on the time interval \([1.1, 10]\).

**FIGURE 7** Simulation result for data-driven sparse event-triggered control in Example 3. In the third figure, the solid circles at time \( t = 5 \) are the elements of the (true) solution to the Lyapunov equation given in (6).
4.3 Proof of Theorem 3

We prove that (i) and (ii) in Problem 2 hold for the resulting control system with (47).

4.3.1 Proof that (i) holds

The following three facts prove that (i) holds for (47).

(a) If there exists a $k \in \mathbb{Z}_0^+$ satisfying (48), then $\text{rank}(\Delta_k) = \text{rank}(\Delta_{k+1}) = \text{rank}(\Delta_{k+2}) = \cdots = l$. 
(b) If \( \text{rank}(\Delta_k) = \text{rank}(\Delta_{k+1}) = \cdots = l \) holds for some \( k \in \mathbb{Z}_{0+} \), then \( P_k, P_{k+1}, \ldots \) are equal to \( P^* \) in (35).

(c) If \( P_k, P_{k+1}, \ldots \) are equal to \( P^* \) for some \( k \in \mathbb{Z}_{0+} \), the control system \( \Sigma \) is globally asymptotically stable.

Fact (a) is proved by (48) and the fact that the rank of \( \Delta_k \) is nondecreasing with respect to \( k \). In fact,

\[
\text{rank}(\Delta_{k+1}) = \text{rank} \left( \begin{bmatrix} \Delta_k & \vec{l}(x(t_{k+1} + \xi, t_{k+1}, x(t_{k+1} + \xi))) - \vec{l}(x(t_k)) \end{bmatrix} \right) \geq \text{rank}(\Delta_k)
\]

holds for all \( k \in \mathbb{Z}_{0+} \). Moreover, (b) is straightforwardly derived from Theorem 2.

Next, we prove (c). Without loss of generality, we assume that \( k = 0 \), that is, \( P_0, P_1, \ldots \) are equal to \( P^* \). Since \( P^* \) is the solution to the Lyapunov equation in (6) and \( \lambda \) is a positive number satisfying (8) as defined in Theorem 1, we have \( P_k = P^* \) and \( \lambda_k = \lambda (k = 0, 1, \ldots) \). These relations imply that the start function \( s(t, x(t), t_k, x(t_k)) \) in (47) is equivalent to that in (7). Thus, (c) is directly derived from Theorem 1(ii).

4.3.2 Proof that (ii) holds

First, we consider \( k < K \), where \( K \) is the minimum \( k \) satisfying (48). Then, it follows from (47) that

\[
s(t, x(t), t_k, x(t_k)) = t - t_k - \varepsilon - h,
\]

which indicates \( t_{k+1} = t_k + \varepsilon + h \). Hence, we have \( t_{k+1} - t_k - \varepsilon > h/2 \).

Next, when \( k \geq K \), the start function is written by (7) because \( \Delta_k \) is nondecreasing. This implies that there exists a \( \tau'_{\min} \in \mathbb{R}_+ \) such that \( t_{k+1} - t_k - \varepsilon > \tau'_{\min} \), which is derived in the same manner as the proof of Theorem 1(ii).

Therefore, (4) holds for \( \tau_{\min} = \min(h/2, \tau'_{\min}) \).

5 EXTENSIONS

In this section, we extend sparse event-triggered control in Section 3 into two cases with disturbance and with nonlinear dynamics.

5.1 Case with disturbance

Consider the control system \( \Sigma \) in Figure 2. The plant \( P \) is given by

\[
P : \ \dot{x}(t) = Ax(t) + Bu(t) + Dw(t),
\]

where \( D \in \mathbb{R}^{nxr} \) is a constant matrix and \( w(t) \in \mathbb{R}^r \) is disturbance. We assume that \( w(t) \) is unknown but bounded by a known constant \( d \in \mathbb{R}_+ \), that is, \( ||w(t)|| < d \) holds for every \( t \in \mathbb{R}_{0+} \). The controller \( K \) is given by (2).

In this case, we have the following result for the corresponding sparse event-triggered control problem.

**Theorem 4.** Let \( \lambda \in (0, \lambda^*) \) be a constant number for the positive number \( \lambda^* \in \mathbb{R}_+ \) in (8). Suppose that a start function \( s \) is given by

\[
s(t, x(t), t_k, x(t_k)) = V(x(t)) - \max \left( V(x(t_k))e^{-\lambda(t-t_k)}, \lambda_{\max}(P^*)c^2d^2 \right),
\]

where the function \( V \) is given by (5) and \( c := 4||D||/(\lambda^* - \lambda) \). Then, the following two statements hold:

1) There exists a tuple \( (a, b) \in \mathbb{R}_+^2 \) satisfying \( ||x(t)|| \leq \max(ac^{-b}||x(0)||, \sqrt{\lambda_{\max}(P^*)/\lambda_{\min}(P^*)}cd) \) for every \( t \in \mathbb{R}_{0+} \).
2) Statement (ii) in Problem 1 holds.
Theorem 4 indicates that \( x(t) \) is steered into a neighborhood of origin by sparse control input.

### 5.2 Case with nonlinear dynamics

Let us consider the control system \( \Sigma \) in Figure 2, where the plant \( P \) and controller \( K \) are given by

\[
P : \quad \dot{x}(t) = f(x(t), u(t)),
\]

\[
K : \quad u(t) = \begin{cases} 
  g(x(t)) & \text{if } t \in [t_k, t_k + \varepsilon), \\
  0 & \text{if } t \in [t_k + \varepsilon, t_{k+1}).
\end{cases}
\]

We assume that (a) the functions \( f_0 \) and \( f_\text{cl} \), defined by \( f_0(x) := f(x, 0) \) and \( f_\text{cl}(x) := f(x, g(x)) \) for all \( x \in \mathbb{R}^n \), are globally Lipschitz continuous, (b) the origin of the system \( \dot{x}(t) = f_\text{cl}(x(t)) \) is locally exponentially stable. Note that (a) guarantees an existence and uniqueness of the state trajectory of \( \Sigma \).

Then, there exist an \( \alpha_i \in \mathbb{R}_+ (i = 1, 2, 3, 4) \), \( r \in \mathbb{R}_+ \), and \( V : \mathbb{B}_r \to \mathbb{R}_0^+ \) be positive numbers and a continuous function satisfying

\[
\alpha_1 \|x\|^2 \leq V(x) \leq \alpha_2 \|x\|^2,
\]

\[
\frac{\partial V(x)}{\partial x} f_\text{cl}(x) \leq -\alpha_3 \|x\|^2,
\]

\[
\left\| \frac{\partial V(x)}{\partial x} \right\| \leq \alpha_4 \|x\|
\]

for all \( x \in \mathbb{B}_r \).

For the \( r \) and \( V \), let \( \Omega \subseteq \mathbb{R}^n \) be given by

\[
\Omega := \{x \in \mathbb{B}_r \mid V(x) \leq \alpha_1 r^2\},
\]

which is the level set of a function \( V \). Note that \( \Omega \) is an invariant set of the system in (53); however, it is not clear that the same statement holds for the system \( \Sigma \).

Then, the following theorem is obtained.

**Theorem 5.** Consider system \( \Sigma \) composed of (51) and (52). Let \( \lambda \in (0, \lambda^*) \) be a constant number for the positive number

\[
\lambda^* := \frac{\alpha_3}{\alpha_1}.
\]

Moreover, let the start function \( s \) be given by (7), where \( V \) is the function in (54)–(56). Then, the following statements hold:

1) There exists a tuple \( (a, b) \in \mathbb{R}_+^2 \) satisfying \( \|x(t)\| \leq ae^{-bt}\|x(0)\| \) for all \( x(0) \in \Omega \).

2) If \( x(0) \in \Omega \) holds, then there exists a \( \epsilon_{\text{min}} \in \mathbb{R}_+ \) satisfying (4) for all \( k \in \mathbb{Z}_0^+ \).

**Proof.** First, 1) is proved by the following two facts.

(a) If \( x(0) \in \Omega \) holds, then \( x(t) \in \Omega \) holds for all \( t \in \mathbb{R}_0^+ \).

(b) If \( x(t) \in \Omega \) holds for all \( t \in \mathbb{R}_0^+ \), then there exists a tuple \( (a, b) \in \mathbb{R}_+^2 \) satisfying \( \|x(t)\| \leq ae^{-bt}\|x(0)\| \).

First, in order to prove (a), we give the proof that

\[
x(t_k) \in \Omega \Rightarrow \forall t \in [t_k, t_{k+1}] x(t) \in \Omega
\]
holds for each \( k \in \mathbb{Z}_+ \), which is sufficient for (a).

We consider \( \Sigma \) on \([t_k, t_{k+1}]\). The dynamics of \( \Sigma \) on \([t_k, t_k + \epsilon] \) is given by (53). Since \( \Omega \) is an invariant set of (53) contained in \( \mathbb{B}_r \), \( x(t) \in \mathbb{B}_r \) holds on \([t_k, t_k + \epsilon] \) if \( x(t_k) \in \Omega \). In contrast, we have (11) for \( x^* \) defined by (57), which is straightforwardly derived by (54) and (55). From this, we obtain (14) on \([t_k, t_{k+1}]\) in the same manner as the proof of Theorem 1. Therefore, \( x(t) \in \Omega \) holds on \([t_k, t_{k+1}]\) under \( x(t_k) \in \Omega \) because of \( V(x(t)) \leq V(x(t_k)) \leq \alpha_1 r^2 \). This proves (a).

Next, let us prove (b). From (a), we obtain (14) on \([t_k, t_{k+1}]\) for all \( k \in \mathbb{Z}_+ \). Thus, (17) holds for every \( t \in \mathbb{R}_+ \). This implies that \( x(t) \) converges to 0 exponentially for all \( x(0) \in \Omega \). This completes the proof of (b).

Finally, we prove 2). The dynamics of \( \Sigma \) on \([t_k + \epsilon, t_{k+1}]\) is written by \( \dot{x}(t) = f_0(x(t)) \). Therefore, the time derivative of \( V(x(t)) \) along the state trajectory \( x(t) \) of \( \Sigma \) is bounded by

\[
\dot{V}(x(t)) = \frac{\partial V}{\partial x} f_0(x(t)) \leq \left\| \frac{\partial V}{\partial x} \right\| \| f_0(x(t)) \| \leq \alpha_4 L \| x(t) \|^2, \tag{58}
\]

where we use (56) and \( L \in \mathbb{R}_+ \) satisfying \( \| f_0(x) \| \leq L \| x \| \) for all \( x \in \mathbb{R}^n \) (i.e., \( L \) is a Lipschitz constant of \( f_0 \)). Moreover, from (58) and (54), we obtain

\[
\dot{V}(x(t)) \leq \mu V(x(t)), \tag{59}
\]

where \( \mu \in \mathbb{R}_+ \) is the positive number given by \( \mu := (\alpha_4/\alpha_1)L \). From (59), we have (24) on \([t_k + \epsilon, t_{k+1}]\). Hence, we eventually obtain (26); this implies that \( \tau_{\text{min}} \in \mathbb{R}_+ \) defined in (18) satisfies (4) for all \( x(0) \in \Omega \). This completes the proof of 2).

In Theorem 5, 1) indicates the exponential stability of the origin of \( \Sigma \). Moreover, 2) guarantees that the stabilization is always conducted by a sparse input.

6 | CONCLUSION

In this article, we established a framework of sparse event-triggered control. First, we presented a model-based method for sparse event-triggered control, where the event condition is defined by a Lyapunov function. The resulting control input is proven to be sparse and the control system is confirmed to be asymptotically stable. Second, we extended it to a data-driven version, where the event condition is adaptively updated from online data on the state trajectory. Finally, we discussed the possibility of extending our framework to the cases of disturbance and nonlinear dynamics.

Our proposed data-driven method was derived in the absence of noise. In future work, this approach must be extended to the case where the data are exposed to noise.

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CONFLICT OF INTEREST

The authors declare no conflicts of interest associated with this manuscript.

DATA AVAILABILITY STATEMENT

The data are available on request from the authors.

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APPENDIX A. PROOF OF THEOREM 4

A.1 Proof of 1)
Statement 1) is proved by the following four facts.

(a) Suppose that $\sigma \in (0, 1)$ is given. Let $c' := 2||D||/(\sigma(\lambda^* - \lambda))$. Then, $V(x(t))$ is bounded as
on the time interval $[t_k, t_k + \epsilon)$ for each $k \in \mathbb{Z}_0^+$.  

(b) If (a) holds, then $V(x(t))$ is bounded as

$$V(x(t)) \leq \max \left( V(x(t_k))e^{-((1-\sigma)\Lambda^* + \sigma \lambda)(t-t_k)}, \lambda_{\max}(P^*)c^2d^2 \right)$$

(A1)

for every $t \in \mathbb{R}_0^+$.  

(c) If $V(x(t)) \leq V(x(0))e^{-\Lambda^*t}$, there exists a tuple $(a, b) \in \mathbb{R}_+^2$ such that

$$||x(t)|| \leq ae^{-bt}||x(0)||.$$  

(A3)

(d) If $V(x(t)) \leq \lambda_{\max}(P^*)c^2d^2$, then

$$||x(t)|| \leq \sqrt{\lambda_{\max}(P^*)/\lambda_{\min}(P^*)cd}.$$  

(A4)

Items (c) and (d) are derived by Fact (I) and simple calculation. In fact, we have (22) and $V(x(0))e^{-\Lambda^*t} \leq \lambda_{\max}(P^*)||x(0)||e^{-\Lambda^*t}$ from Fact (I), which imply (A3) for $(a, b) = (\sqrt{\lambda_{\max}(P^*)/\lambda_{\min}(P^*)}, \Lambda^*/2)$. Moreover, (A4) is obtained by using (22). Therefore, we prove (a) and (b) in the following sections.

### A.1.1 Proof of (a)

Suppose that $k \in \mathbb{Z}_0^+$ is given. Then, either one of the following two statements holds: (a-1) $||x(t)|| > c'd$ for all $t \in [t_k, t_k + \epsilon)$ and (a-2) $||x(t)|| \leq c'd$ for some $t \in [t_k, t_k + \epsilon)$.

First, let us show (a) under (a-1). Consider $\Sigma$ on the time interval $[t_k, t_k + \epsilon)$, during which the dynamics of $\Sigma$ is given by

$$\dot{x}(t) = (A + BF)x(t) + Dw(t).$$  

(A5)

From (5), (6), and (A5), we have

$$\dot{V}(x(t)) = -x^T(t)Qx(t) + w^T(t)D^TP^*x(t) + x^T(t)P^*Dw(t) \leq -\lambda_{\min}(Q)||x(t)||^2 + 2\lambda_{\max}(P^*)d||D||||x(t)||$$

(A6)

on $[t_k, t_k + \epsilon)$. Therefore, if $||x(t)|| > c'd$,

$$\dot{V}(x(t)) < -\left( \lambda_{\min}(Q) - \frac{2\lambda_{\max}(P^*)d||D||}{c'd} \right)||x(t)||^2 = -\lambda_{\min}(Q)\left( \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P^*)} - \sigma(\lambda^* - \lambda) \right)||x(t)||^2$$

(A7)

and

$$\dot{V}(x(t)) = \lambda_{\max}(P^*)(1 - \sigma)\lambda^* + \sigma \lambda ||x(t)||^2.$$  

(A8)

This inequality, (10), and Fact (II) imply

$$V(x(t)) \leq V(x(t_k))e^{-(1-\sigma)\Lambda^* + \sigma \lambda(t-t_k)}$$

(A9)

on $[t_k, t_k + \epsilon)$. Thus, (A1) holds on $[t_k, t_k + \epsilon)$.

Next, we prove (a) under (a-2). Let $t' \in \mathbb{R}_+$ be the minimum $t \in [t_k, t_k + \epsilon)$ satisfying $||x(t)|| \leq c'd$. Then, since $||x(t)|| > c'd$ for all $t \in [t_k, t')$, we have (A9) on the time interval $[t_k, t')$ from a similar discussion to the case (a-1). Meanwhile, on $[t', t_k + \epsilon)$, $V(x(t))$ is bounded as

$$V(x(t)) \leq \lambda_{\max}(P^*)c^2d^2,$$

(A10)

because, if $||x(t)|| \leq c'd$, we have (A10) from Fact (I); otherwise (i.e., if $||x(t)|| > c'd$), $\dot{V}(x(t)) < 0$ holds because of (A8). These two facts imply (A10) on $[t', t_k + \epsilon)$ from the continuity of $V(x(t))$. Thus, (a) is proved.
A.1.2 Proof of (b)

We prove (b) by dividing into the following two cases: (b-1) \( V(x(0)) \leq \lambda_{\text{max}}(P^*)e^{cd} \) and (b-2) \( V(x(0)) > \lambda_{\text{max}}(P^*)e^{cd} \).

First, let us consider the case (b-1). From (A1) for \( \sigma = 1/2 \), we obtain

\[
V(x(t)) \leq \max \left( V(x(t_k))e^{-(1/2)\lambda^* t(t_k-t)} , \lambda_{\text{max}}(P^*)c^2 d^2 \right) \leq \max \left( V(x(t_k))e^{-(1-\lambda) t} , \lambda_{\text{max}}(P^*)c^2 d^2 \right)
\]

on \([t_k, t_k + \epsilon)\) for each \( k \). This inequality with \( k = 0 \) and \( V(x(t_0))e^{-(1-\lambda) t_0} \leq V(x(t_0)) \leq \lambda_{\text{max}}(P^*)c^2 d^2 \) indicate

\[
V(x(t)) \leq \lambda_{\text{max}}(P^*)c^2 d^2
\] (A11)

for every \( t \in [t_0, t_0 + \epsilon) \). Moreover, from the definition of \( t_1 \) and the continuity of \( V(x(t)) \), we have (A11) on \([t_0, t_1] \). Meanwhile, if (A11) holds at the time instant \( t_k \) for some \( k \in \mathbb{Z}_{0+} \), then (A11) also holds on \([t_k, t_{k+1}] \), which is derived by a similar discussion to the case for \( k = 0 \). These facts imply (A11) on the time interval \([t_k, t_{k+1}] \) for all \( k \in \mathbb{Z}_{0+} \), that is, for every \( t \in \mathbb{R}_{0+} \).

Next, we assume (b-2). Let \( p \in \mathbb{Z}_{0+} \) be the maximum \( k \in \mathbb{Z}_{0+} \) satisfying \( V(x(0))e^{-\lambda t_k} \geq \lambda_{\text{max}}(P^*)c^2 d^2 \). Then, we have (17) on the time interval \([0, t_p] \) and (A2) on \([t_p, t_{p+1}] \) in a similar manner to the proof of Theorem 1(ii). Furthermore, we obtain (A11) on \([t_{p+1}, \infty) \) from a similar discussion to the case (b-1). These facts derive (A2) for every \( t \in \mathbb{R}_{0+} \). Thus, (b) is proved.

A.2 Proof of 2)

Statement 2) is proved by the following two facts.

(a) The relation

\[
V(x(t)) \leq e^{(\mu + \lambda')(t - t_k - \epsilon)} \max \left( V(x(t_k))e^{-\lambda \epsilon} , \frac{1}{2} \lambda_{\text{max}}(P^*)c^2 d^2 \right)
\] (A12)

holds on \([t_k + \epsilon, t_{k+1}] \), where \( \lambda' := \left( 1 - \sqrt{2}/2 \right) \lambda^* + \left( \sqrt{2}/2 \right) \lambda \).

(b) Assume (a) and let

\[
r_\text{min} = \min \left( \frac{\lambda^* - \lambda}{2(\mu + \lambda^* + \lambda)} , \frac{\ln(3/2)}{\mu + \lambda^*} \right),
\]

where \( \mu \in \mathbb{R}_+ \) is the positive number in (19). Then (4) holds for every \( k \in \mathbb{Z}_{0+} \).

These facts are proved as follows.

A.2.1 Proof of (a)

From the continuity of \( x(t) \), either one of the following two statements holds:

(a-1) For every \( t \in [t_k + \epsilon, t_{k+1}] \),

\[
\|x(t)\| \geq \frac{1}{\sqrt{2}} cd.
\] (A13)

(a-2) For some \( t \in [t_k + \epsilon, t_{k+1}] \),

\[
\|x(t)\| < \frac{1}{\sqrt{2}} cd.
\] (A14)

Therefore, let us prove (a) under (a-1) and (a-2).

First, consider the case (a-1). From (5), Fact (I), (A13), and (10), we have
Equations (A19) and (A20) imply
\[ V(x(t)) \leq \lambda_{\max}(A^TP^* + P^*A)\|x(t)\|^2 + \frac{2\lambda_{\max}(P^*)d\|D\|}{(1/\sqrt{2})cd} \|x(t)\|^2 = \lambda_{\max}(A^TP^* + P^*A)\|x(t)\|^2 + \frac{\sqrt{2}}{2} \lambda_{\max}(P^*) (\lambda^* - \lambda)\|x(t)\|^2 \]
\[ \leq (\|\lambda_{\max}(A^TP^* + P^*A)\| + \lambda_{\max}(P^*)\lambda^*)\|x(t)\|^2 \leq (\mu + \lambda^*)V(x(t)) \]  
(A15)
on the time interval \([t_k + \epsilon, t_{k+1})\). This inequality and Fact (II) imply
\[ V(x(t)) \leq e^{(\mu + \lambda^*)(t-t_k-\epsilon)}V(x(t_k + \epsilon)) \]  
(A16)on \([t_k + \epsilon, t_{k+1})\). Moreover, (A16) holds at the time instant \(t_{k+1}\) from the continuity of \(V(x(t))\). On the other hand, \(V(x(t_k + \epsilon))\) in (A16) is bounded as
\[ V(x(t_k + \epsilon)) \leq \max \left( V(x(t_k))e^{-\lambda^*}, \frac{1}{2} \lambda_{\max}(P^*)c^2 \right)^2 \]  
(A17)which is derived as follows: From (A1) with \(\sigma = 1/\sqrt{2}\), we obtain
\[ V(x(t)) \leq \max \left( V(x(t_k))e^{-\lambda^*(t-t_k)}, \frac{1}{2} \lambda_{\max}(P^*)c^2 \right)^2 \]on \([t_k, t_k + \epsilon);\) this fact and the continuity of \(V(x(t))\) imply (A17). Thus, (A16) and (A17) imply (A12) on \([t_k + \epsilon, t_{k+1})\).

Next, we prove (a) under (a-2) by contradiction. Assume that (A12) is violated at some time instant \(\tilde{t} \in [t_k + \epsilon, t_{k+1})\), that is,
\[ V(x(\tilde{t})) > e^{(\mu + \lambda^*)(\tilde{t}-t_k-\epsilon)} \max \left( V(x(t_k))e^{-\lambda^*}, \frac{1}{2} \lambda_{\max}(P^*)c^2 \right)^2 \]  
(A18)Since \(V(x(\tilde{t})) > (1/2)\lambda_{\max}(P^*)c^2\), we have
\[ \|x(\tilde{t})\| > \frac{1}{\sqrt{2}}cd \]from Fact (I). Meanwhile, there exists a \(\tilde{t} \in [t_k + \epsilon, \tilde{t}]\) violating (A13) from the discussion in the case (b-1). Therefore, by letting \(\tilde{t} \in \mathbb{R}_{0+}\) be the maximum \(t \in [t_k + \epsilon, \tilde{t}]\) satisfying
\[ \|x(t)\| = \frac{1}{\sqrt{2}}cd, \]we have (A13) on the time interval \([\tilde{t}, \tilde{t}].\) This fact, (A15), and Fact (II) derive
\[ V(x(t)) \leq e^{(\mu + \lambda^*)(\tilde{t}-\tilde{t})}V(x(\tilde{t})) = \frac{1}{2} e^{(\mu + \lambda^*)(\tilde{t}-\tilde{t})} \lambda_{\max}(P^*)c^2 \]on \([\tilde{t}, \tilde{t})], where we use Fact (I) and the definition of \(\tilde{t}.\) In particular, \(V(x(\tilde{t}))\) is bounded as
\[ V(x(\tilde{t})) \leq \frac{1}{2} e^{(\mu + \lambda^*)(\tilde{t}-\tilde{t})} \lambda_{\max}(P^*)c^2. \]  
(A19)On the other hand, (A18) indicates
\[ V(x(\tilde{t})) > \frac{1}{2} e^{(\mu + \lambda^*)(\tilde{t}-\tilde{t}-\epsilon)} \lambda_{\max}(P^*)c^2. \]  
(A20)Equations (A19) and (A20) imply \(\tilde{t} < t_k + \epsilon;\) however, this contradicts the definition of \(\tilde{t}.\) Thus, (A12) holds on \([t_k + \epsilon, t_{k+1}].\)
A.2.2 Proof of (b)

Suppose that \(k \in \mathbb{Z}_{0+}\) is given. We show (3) on the time interval \([t_k + \varepsilon, t_k + \varepsilon + \tau_{\min}]\) by dividing into the following two cases: (b-1) \(V(x(t_k))e^{-\lambda^\varepsilon} \leq (1/2)\lambda_{\max}(P^\varepsilon)c^2d^2\) and (b-2) \(V(x(t_k))e^{-\lambda^\varepsilon} > (1/2)\lambda_{\max}(P^\varepsilon)c^2d^2\).

First, let us consider the case (b-1). From (50), (A12), and the relation \(\alpha \leq \max(\alpha, \beta)\) for all \((\alpha, \beta) \in \mathbb{R}^2\), we have

\[
s(t, x(t), t_k, x(t_k)) \leq e^{(\mu + \lambda^\varepsilon)(t - t_k - \varepsilon)}V(x(t_k))e^{-\lambda^\varepsilon} - \max \left( V(x(t_k))e^{-\lambda^\varepsilon}, \lambda_{\max}(P)c^2d^2 \right)
\]

\[
\leq e^{(\mu + \lambda^\varepsilon)(t - t_k - \varepsilon)}V(x(t_k))e^{-\lambda^\varepsilon} - V(x(t_k))e^{-\lambda^\varepsilon}
\]

on \([t_k + \varepsilon, t_k + \varepsilon + \tau_{\min}]\). Therefore, when \(t \leq t_k + \varepsilon + \tau_{\min}\), we can derive

\[
s(t, x(t), t_k, x(t_k)) \leq e^{(\mu + \lambda^\varepsilon)\tau_{\min}}V(x(t_k))e^{-\lambda^\varepsilon} - V(x(t_k))e^{-\lambda^\varepsilon}
\]

\[
= \left( e^{(\mu + \lambda^\varepsilon)\tau_{\min} - (\lambda^\varepsilon - \lambda)(\varepsilon - 1)} - 1 \right) V(x(t_k))e^{-\lambda^\varepsilon}
\]

\[
\leq \left( e^{-(1/2)(\lambda^\varepsilon - \lambda)} - 1 \right) V(x(t_k))e^{-\lambda^\varepsilon} < 0
\]

from \(\tau_{\min} \leq (\lambda^\varepsilon - \lambda)/(2(\mu + \lambda^\varepsilon + \lambda))\).

Next, we assume (b-2). From (50) and (A12), the following inequality holds on \([t_k + \varepsilon, t_k + 1]\).

\[
s(t, x(t), t_k, x(t_k)) < \frac{1}{2} e^{(\mu + \lambda^\varepsilon)(t - t_k - \varepsilon)}\lambda_{\max}(P)c^2d^2 - \max \left( V(x(t_k))e^{-\lambda^\varepsilon}, \lambda_{\max}(P)c^2d^2 \right)
\]

\[
\leq \frac{1}{2} e^{(\mu + \lambda^\varepsilon)(t - t_k - \varepsilon)}\lambda_{\max}(P)c^2d^2 - \lambda_{\max}(P)c^2d^2
\]

\[
= \left( \frac{1}{2} e^{(\mu + \lambda^\varepsilon)(t - t_k - \varepsilon) - 1} \right) \lambda_{\max}(P)c^2d^2.
\]

Thus, when \(t \leq t_k + \varepsilon + \tau_{\min}\), it follows that

\[
s(t, x(t), t_k, x(t_k)) < \left( \frac{1}{2} e^{(\mu + \lambda^\varepsilon)\tau_{\min} - 1} \right) \lambda_{\max}(P)c^2d^2 \leq -\frac{1}{4} \lambda_{\max}(P)c^2d^2 < 0
\]

because \(\tau_{\min} \leq \ln(3/2)/(\mu + \lambda^\varepsilon)\). Therefore, 2) is proved.