ON THE DEGREE OF FANO SCHEMES OF LINEAR SUBSPACES ON HYPERSURFACES

DANG TUAN HIEP

Abstract. This paper proves an explicit formula for computing the degree of Fano schemes of linear subspaces on general hypersurfaces. The method used here is based on the localization theorem and Bott’s residue formula in equivariant intersection theory.

1. Introduction

Let $X \subset \mathbb{P}^n$ be a general hypersurface of degree $d$. The Fano scheme $F_k(X)$ is defined to be the set of $k$-dimensional linear subspaces of $\mathbb{P}^n$ which are contained in $X$. This is a subscheme of the Grassmannian $G(k, n)$ of $k$-dimensional linear subspaces in $\mathbb{P}^n$. For convenience, we set

$$\delta = (k + 1)(n - k) - \binom{d + k}{d}.$$ 

Suppose that $d \neq 2$ (or $n \geq 2k + 1$) and $\delta \geq 0$. Langer [9] showed that $F_k(X)$ is smooth of expected dimension $\delta$. In this case, the degree of $F_k(X)$ is given by the following formula

$$\deg(F_k(X)) = \int_{G(k, n)} c_{\binom{d+k}{d}}(\text{Sym}^d S^\vee) \cdot c_1(Q)^\delta,$$

where $S$ and $Q$ are the tautological sub-bundle and quotient bundle on $G(k, n)$, $\text{Sym}^d S^\vee$ is the $d$-th symmetric power of the dual of $S$, and $c_i(E)$ is the $i$-th Chern class of the vector bundle $E$. Note that $\int_Y \alpha$ denotes the degree of the cycle class $\alpha$ on $Y$ defined in [7, Definition 1.4]. Formula (1) can be found, for example, in [7, Example 14.7.13] or [10, Section 3.5]. Using Schubert calculus, Debarre and Manivel [4, Theorem 4.3] showed that the degree of $F_k(X)$ is equal to a certain coefficient of an explicit polynomial, given as the product of linear

2010 Mathematics Subject Classification. 14C15; 14N15; 55N91.

Key words and phrases. Fano schemes, Schubert calculus, equivariant intersection theory, Bott’s residue formula.
forms. In this paper, we propose an explicit formula for computing the degree of $F_k(X)$ via equivariant intersection theory.

We denote by $\mathcal{I}$ the set of $(k+1)$-subsets of $\{0,1,\ldots,n\}$. Choose any set of integers $\lambda_0,\ldots,\lambda_n$ such that $\lambda_i \neq \lambda_j$ for $i \neq j$. For each $I \in \mathcal{I}$, we set

$$S_I = \prod_{v_i \in \mathbb{N},\sum_{i \in I} v_i = d} \left( \sum_{i \in I} v_i \lambda_i \right),$$

$$Q_I = \sum_{j \notin I} \lambda_j,$$

and

$$T_I = \prod_{i \in I} \prod_{j \notin I} (\lambda_i - \lambda_j).$$

Here is the main result of this paper:

**Theorem 1.1.** Let $k,d,n \in \mathbb{N}$ satisfy $d \neq 2$ (or $n \geq 2k+1$) and $\delta \geq 0$, and let $X \subset \mathbb{P}^n_C$ be a general hypersurface of degree $d$. Then the degree of the Fano scheme $F_k(X)$ of $k$-dimensional linear subspaces on $X$ is given by the following formula:

$$\deg(F_k(X)) = (-1)^\delta \sum_{I \in \mathcal{I}} \frac{S_I Q_I^\delta}{T_I}.$$

If $k,d,n \in \mathbb{N}$ satisfy $d \neq 2$ (or $n \geq 2k+1$) and $\delta = 0$, then the Fano scheme $F_k(X)$ is zero-dimensional. In this case, the degree of $F_k(X)$ is equal to the number of $k$-dimensional linear subspaces on $X$. As a special case, we have the following corollary.

**Corollary 1.2.** Let $k,d,n \in \mathbb{N}$ satisfy $d \neq 2$ (or $n \geq 2k+1$) and $\delta = 0$, and let $X \subset \mathbb{P}^n_C$ be a general hypersurface of degree $d$. Then the number of $k$-dimensional linear subspaces on $X$ is equal to $\sum_{I \in \mathcal{I}} S_I T_I$. In particular, the number of lines on a general hypersurface of degree $2n-3$ in $\mathbb{P}^n_C$ is equal to

$$\sum_{0 \leq i < j \leq n} \frac{\prod_{a=0}^{2n-3} (a\lambda_i + (2n-3-a)\lambda_j)}{\prod_{k \neq i,j} (\lambda_i - \lambda_k)(\lambda_j - \lambda_k)}.$$

For different formulas obtained by Schubert calculus, we refer to [1, 8, 10, 12]. The formulas in Theorem 1.1 and Corollary 1.2 can be easily implemented in computer algebra systems. See [3, Chapter 5] for the implementation in SAGE [11].
2. **Equivariant intersection theory**

Edidin and Graham \[5, 6\] gave an algebraic construction to equivariant intersection theory. In this section, we review the basic notions and results of this theory. Let \(G\) be a linear algebraic group and let \(X\) be a scheme of finite type over \(\mathbb{C}\) endowed with a \(G\)-action. For any non-negative integer \(i\), we can find a representation \(V\) of \(G\) together with a dense open subset \(U \subset V\) on which \(G\) acts freely and whose complement has codimension larger than \(\dim X - i\) such that the principal bundle quotient \(U \rightarrow U/G\) exists in the category of schemes (see \[5, Lemma 9\]). The diagonal action on \(X \times U\) is then also free, which implies that under mild assumption, a principal bundle quotient \(X \times U \rightarrow (X \times U)/G\) exists in the category of schemes (see \[5, Proposition 23\]). In what follows, we will tacitly assume that the scheme \((X \times U)/G\) exists and denote it by \(X_G\).

### 2.1. Equivariant Chow groups

We define the \(i\)-th \(G\)-equivariant Chow group of \(X\) to be

\[
A^G_i(X) := A_{i+\dim U-\dim G}(X_G),
\]

where \(A_*\) stands for the ordinary Chow group defined in \[7\]. By \[5, Proposition 1\], this is well-defined. The \(G\)-equivariant Chow group of \(X\) is defined to be

\[
A^G(X) = \bigoplus A^G_i(X).
\]

If the \(X_G\) are smooth, then \(A^G_i(X)\) inherits an intersection product from the ordinary Chow groups. This endows \(A^G_*\) with the structure of a graded ring, called the **\(G\)-equivariant Chow ring** of \(X\). For example, if \(G = T = (\mathbb{C}^*)^n\) is a split torus of dimension \(n\), then the \(T\)-equivariant Chow ring of a point is isomorphic to a polynomial ring in \(n\) variables (see \[5, Section 3.2\]). Throughout this paper, we denote this ring by \(R_T\).

### 2.2. Equivariant vector bundles and Chern classes

A \(G\)-equivariant vector bundle is a vector bundle \(E\) on \(X\) such that the action of \(G\) on \(X\) lifts to an action of \(G\) on \(E\) which is linear on fibers. By \[5, Lemma\], \(E_G\) is a vector bundle over \(X_G\). The **\(G\)-equivariant Chern classes** \(c^G_i(E)\) are defined to be the Chern classes \(c_i(E_G)\). If \(E\) has rank \(r\), then the
top Chern class $c_i^G(E)$ is called the \textit{G-equivariant Euler class} of $E$ and denoted $e^G(E)$.

Note that a $G$-equivariant vector bundle over a point is a representation of $G$ (see also \cite[Section 3.2]{example}). Our primary interest is when $G = T = \left(\mathbb{C}^*\right)^n$ is a split torus of dimension $n$ and $X = pt$ is a point. In this case, let $M(T)$ be the character group of the torus $T$. Suppose that $R_T = \mathbb{C}[h_1, \ldots, h_n]$. There is a group homomorphism $\psi : M(T) \to R_T$ given by $\rho_i \mapsto h_i$, where $\rho_i$ is the character of $T$ defined by $\rho_i(t_1, \ldots, t_n) = t_i$. This induces a ring isomorphism $\text{Sym}(M(T)) \simeq R_T$. We call $\psi(\rho)$ the \textit{weight} of $\rho$. In particular, $h_i$ is the weight of $\rho_i$.

\textbf{Example 2.1.} [\cite{example} Example 9.1.1.1] The diagonal action of $G = T = \left(\mathbb{C}^*\right)^n$ on $\mathbb{C}^n$ gives a $T$-equivariant vector bundle $E$ over a point. The corresponding representation of $T$ has characters $\rho_i$ for $i = 1, \ldots, n$, and their weights are the $h_i$. In this case, we have $E_T \cong \mathcal{O}(h_1) \oplus \cdots \oplus \mathcal{O}(h_n)$. This implies that $c_i^T(E) = c_i(E_T) = s_i(h_1, \ldots, h_n) \in R_T$, where $s_i$ is the $i$-th elementary symmetric function. In particular, the $T$-equivariant Euler class of $E$ is $e^T(E) = h_1 \cdots h_n \in R_T$.

\textbf{Example 2.2.} Consider the diagonal action of $T = \left(\mathbb{C}^*\right)^4$ on $\mathbb{C}^4$ given in coordinates by

\[
(t_1, t_2, t_3, t_4) \cdot (x_1, x_2, x_3, x_4) = (t_1x_1, t_2x_2, t_3x_3, t_4x_4).
\]

This induces an action of $T$ on the Grassmannian $G(2, 4)$ with the isolated fixed points $L_I$ corresponding to coordinate 2-planes in $\mathbb{C}^4$. Each $L_I$ is indexed by the 2-subset $I$ of the set $\{1, 2, 3, 4\}$ so that $L_I$ is defined by the equations $x_j = 0$ for $j \notin I$. Let $S$ be the tautological sub-bundle on $G(2, 4)$. At each $L_I$, the restriction of the action of $T$ on the fiber $S|_{L_I}$ gives a representation of $T$ with characters $\rho_i$ for $i \in I$. This representation gives a $T$-equivariant vector bundle of rank 2 over a point. We also denote it by $S|_{L_I}$. If $I = \{i_1, i_2\}$, then we have $c_i^T(S|_{L_I}) = h_{i_1} + h_{i_2}$ and $c_2^T(S|_{L_I}) = h_{i_1} \cdot h_{i_2}$, where $h_{i_1}$ and $h_{i_2}$ are the weights of $\rho_{i_1}$ and $\rho_{i_2}$ respectively.

\textbf{2.3. Localization and Bott’s residue formula.} Let $X$ be a scheme endowed with an action of $T = \left(\mathbb{C}^*\right)^n$. We denote the fixed point locus by $X^T$. The localization theorem states that up to $R_T$-torsion,
the $T$-equivariant Chow group of the fixed points locus $X^T$ is isomorphic to that of $X$. Moreover, the localization isomorphism is given by the equivariant push-forward induced by the inclusion of $X^T$ to $X$ (see [6, Theorem 1]). For smooth varieties, the inverse to the equivariant push-forward can be written explicitly (see [6, Theorem 2]). Using these results, Edidin and Graham gave an algebraic proof of Bott’s residue formula for Chern numbers of vector bundles on smooth complete varieties (see [6, Theorem 3]). Bott’s residue formula shows that we can compute the degree of a zero-dimensional cycle class on a smooth complete variety $X$ in terms of local contributions supported on the components of the fixed point locus of a torus action on $X$.

3. The proof of Theorem 1.1

Consider the diagonal action of $T = (\mathbb{C}^*)^{n+1}$ on $\mathbb{P}_\mathbb{C}^n$ given in coordinates by

$$(t_0, \ldots, t_n) \cdot (x_0 : \cdots : x_n) = (t_0x_0 : \cdots : t_nx_n).$$

This induces an action of $T$ on the Grassmannian $\mathbb{G}(k, n)$ with $\binom{n+1}{k+1}$ isolated fixed points $L_I$ corresponding to $\binom{n+1}{k+1}$ coordinate $k$-planes in $\mathbb{P}_\mathbb{C}^n$. Each fixed point $L_I$ is indexed by a $(k+1)$-subset $I$ of the set \{0, 1, \ldots, n\}. Let $S$ and $Q$ be the tautological sub-bundle and quotient bundle on $\mathbb{G}(k, n)$. The key idea is that, at each $L_I$, the torus action on the fibers $S|_{L_I}$ and $Q|_{L_I}$ have characters $\rho_i$ for $i \in I$ and $\rho_j$ for $j \notin I$ respectively. Since the tangent bundle on the Grassmannian is isomorphic to $S^\vee \otimes Q$, the characters of the torus action on the tangent space at $L_I$ are

$$\{\rho_j - \rho_i \mid i \in I, j \notin I\}.$$ 

The normal bundle $N_{L_I}$ of $L_I$ in $\mathbb{G}(k, n)$ is just the tangent space of $\mathbb{G}(k, n)$ at $L_I$. Hence

$$e^T(N_{L_I}) = \prod_{i \in I} \prod_{j \notin I} (h_j - h_i)$$

$$= (-1)^{(k+1)(n-k)} \prod_{i \in I} \prod_{j \notin I} (h_i - h_j),$$

where $h_i$ is the weight of $\rho_i$ defined above, and $R_T = \mathbb{C}[h_0, \ldots, h_n]$ is the $T$-equivariant Chow ring of a point.
At each $L_I$, we also need to compute $e^T(Sym^d S^\vee|_{L_I})$ and $c_1^T(Q|_{L_I})$. Since the characters of the torus action on $S^\vee|_{L_I}$ are $-\rho_i$ for $i \in I$, the torus action on $Sym^d S^\vee|_{L_I}$ has characters

\[
\left\{ \sum_{i \in I} v_i(-\rho_i) \mid v_i \in \mathbb{N}, \sum_{i \in I} v_i = d \right\}.
\]

Hence
\[
e^T(Sym^d S^\vee|_{L_I}) = \prod_{v_i \in \mathbb{N}, \sum_{i \in I} v_i = d} \left( \sum_{i \in I} v_i(-h_i) \right) = (-1)^{\binom{d+k}{d}} \prod_{v_i \in \mathbb{N}, \sum_{i \in I} v_i = d} \left( \sum_{i \in I} v_i h_i \right).
\]

Since the characters of the torus action on $Q|_{L_I}$ are $\rho_j$ for $j \notin I$, we have
\[
c_1^T(Q|_{L_I}) = \sum_{j \notin I} h_j.
\]

By (1) and Bott’s residue formula, we obtain
\[
\deg(F_k(X)) = \sum_{I \in \mathcal{I}} \frac{e^T(Sym^d S^\vee|_{L_I})(c_1^T(Q|_{L_I}))^\delta}{e^T(N_{L_I})}
\]
\[
= (-1)^\delta \sum_{I \in \mathcal{I}} \prod_{v_i \in \mathbb{N}, \sum_{i \in I} v_i = d} \left( \sum_{i \in I} v_i h_i \right) \left( \sum_{j \notin I} h_j \right)^\delta
\]
\[
= \prod_{i \in I} \prod_{j \notin I} (h_i - h_j).
\]

Replacing $h_i$ by $\lambda_i$, we get the desired formula.

Another proof of Theorem 1.1 is as follows. Consider the action of $T = \mathbb{C}^\times$ on $\mathbb{P}^n$ given in coordinates by
\[
t \cdot (x_0 : \cdots : x_n) = (t^{\lambda_0} x_0 : \cdots : t^{\lambda_n} x_n).
\]

The induced action of $T$ on $\mathbb{G}(k, n)$ also has $\binom{n+1}{k+1}$ isolated fixed points $L_I$ as above. At each $L_I$, the torus action on the fibers $S|_{L_I}$ and $Q|_{L_I}$ have characters $\lambda_i \rho$ for $i \in I$ and $\lambda_j \rho$ for $j \notin I$ respectively, where $\rho$ is the character of $T$ defined by $\rho(t) = t$ for all $t \in T$. We denote the weight of $\rho$ by $h$. In this case, the $T$-equivariant Chow ring of a point
is $R_T = \mathbb{C}[h]$. With this set-up, we have
\[
e^T(N_{L_i}) = \prod_{i \in I} \prod_{j \notin I} (\lambda_j - \lambda_i) h^{(k+1)(n-k)}
= (-1)^{(k+1)(n-k)} T_I h^{(k+1)(n-k)}.
\]
Similarly, we also have
\[
e^T(\text{Sym}^d S^\vee|_{L_i}) = \prod_{v_i \in \mathbb{N}, \sum_{i \in I} v_i = d} \left( \sum_{i \in I} v_i (-\lambda_i h) \right)
= (-1)^{\binom{d+k}{d}} S_I h^{\binom{d+k}{d}},
\]
and
\[
c^T_1(Q|_{L_i}) = \sum_{j \notin I} \lambda_j h = Q_I h.
\]
By (1) and Bott’s residue formula, we obtain
\[
\deg(F_k(X)) = (-1)^\delta \sum_{I \in \mathcal{I}} S_I Q_I^\delta h^{(k+1)(n-k)} T_I h^{(k+1)(n-k)}.
\]
Cancelling $h^{(k+1)(n-k)}$, we get the desired formula.

**Acknowledgements**

This work is a part of my Ph.D. thesis at the University of Kaiserslautern. I would like to take this opportunity to express my profound gratitude to my advisor Professor Wolfram Decker. I would also like to thank Dr. Janko Böhm for his valuable suggestions.

**References**

[1] Allen B. Altman and Steven L. Kleiman, *Foundations of the theory of Fano schemes*, Compositio Math. 34 (1977), 3–47.
[2] David A. Cox and Sheldon Katz, *Mirror symmetry and Algebraic Geometry*, American Mathematical Society, 1999.
[3] Hiep Dang, *Intersection theory with applications to the computation of Gromov-Witten invariants*, Ph.D thesis, Technische Universität Kaiserslautern, 2013.
[4] Olivier Debarre and Laurent Manivel, *Sur la variété des espaces linéaires contenus dans une intersection complète*, Math. Ann. 312 (1998), 549–574.
[5] Dan Edidin and William Graham, *Equivariant intersection theory*, Invent. math. 131 (1998), 595–634.
[6] Dan Edidin and William Graham, *Localization in equivariant intersection theory and the Bott residue formula*, Amer. J. Math. 120 (1998), 619–636.
[7] William Fulton, *Intersection theory*, second edition. Springer, 1997.
[8] Joe Harris, *Galois groups of enumerative problems*, Duke Math. J. **46** (1979), 685–724.
[9] Adrian Langer, *Fano schemes of linear spaces on hypersurfaces*, Manuscripta Math. **93** (1997), 21–28.
[10] Laurent Manivel, *Symmetric functions, Schubert polynomials and degeneracy loci*, American Mathematical Society, 2001.
[11] William A. Stein et al. *SAGE Mathematics Software (Version 5.11)*, the SAGE Development Team, 2013. Available at [http://www.sagemath.org](http://www.sagemath.org).
[12] B. L. van der Waerden, *Zur algebraischen Geometrie. II. die geraden Linien auf den Hyperflächen des $\mathbb{P}_n$*, Math. Ann. **108** (1933), 253–259.

Department of Mathematics, University of Dalat, 01 Phu Dong Thien Vuong, Da Lat, Vietnam

E-mail address: hiepdt_tt@dlu.edu.vn