General relativistic time dilation and increased uncertainty in generic quantum clocks

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The theory of relativity associates a proper time with each moving object via its world line. In quantum theory however, such well-defined trajectories are forbidden. After introducing a general characterisation of quantum clocks, we demonstrate that, in the weak-field, low-velocity limit, all “good” quantum clocks experience time dilation as dictated by general relativity when their state of motion is classical (i.e. Gaussian). For nonclassical states of motion, on the other hand, we find that quantum interference effects may give rise to a significant discrepancy between the proper time and the time measured by the clock. We also show how ignorance of the clock’s state of motion leads to a larger uncertainty in the time measured by the clock — a consequence of entanglement between the clock time and its center-of-mass degrees of freedom. We demonstrate how this lost precision can be recovered by performing a measurement of the clock’s state of motion alongside its time reading.

I. INTRODUCTION

One of the most important programs in theoretical physics is the pursuit of a successful theory unifying quantum mechanics and general relativity. Arguably, many of the difficulties arising in this pursuit stem from a lack of understanding of the nature of time [1–3], particularly the conflict between how it is conceived in the two theories [4]. For example, in general relativity a given object’s proper time is defined geometrically according to that object’s world line, but according to quantum mechanics such well-defined trajectories are impossible. How do we then assign a proper time to the delocalised objects described by quantum theory?

Confusion over the nature of time is not limited to the context of quantum gravity. Indeed, even in non-relativistic quantum mechanics, there is much to clarify. In the latter, time is not treated on the same footing as other observable quantities in the theory — it is a parameter, rather than being represented by a self-adjoint operator. This aspect of the theory of quantum mechanics troubled its founders; Wolfgang Pauli, for example, famously noted the impossibility of a self-adjoint operator corresponding to time [5]. Specifically, a quantum observable with outcome \( t \in \mathbb{R} \) equal to the time parameterising the system’s evolution, can only be achieved in the limiting case of infinite energy, and is therefore unphysical, as we discuss in Sec. II. It is however possible to construct self-adjoint operators whose outcome is approximately \( t \), with an error that can in principle be made arbitrarily small, in systems of finite energy (see e.g. [6]).

If one takes an operationalist viewpoint, one can draw conclusions about time by discussing the behaviour of clocks. This underlies Einstein’s development of special relativity [7], the prototypical example of operationalism [8]. In a quantum setting, the clocks must of course be quantised, leading naturally to a time operator. This operational approach has revealed fundamental limitations to time-keeping in non-relativistic quantum systems [6, 9–11].

The present work concerns slowly-moving generic quantum clocks embedded in a weakly-curved spacetime. We are interested in the phenomenon of time dilation, and how quantizing the clock may result in predictions distinct from those of general relativity alone. We take the operational viewpoint noted above, associating time with the outcomes of measurements performed on generic quantum clocks. In this setting, in contrast with the study of clocks in non-relativistic quantum mechanics, we must consider the position and momentum of the clock. Quantum theory dictates that it be subject to some degree of spatial delocalisation in addition to its “temporal delocalisation” (i.e. the indeterminacy of its time-reading), as depicted in Fig. 1.

Relativistic effects manifest in this quantum setting via a coupling between the clock’s kinematic and internal (i.e. time-measuring) degrees of freedom. The form of this coupling has been derived in a number of different ways: via the relativistic dispersion relation and mass-energy equivalence [12], via the Klein-Gordon equation in curved spacetime [13], and as a consequence of classical time dilation between the clock and the laboratory frame of reference [14]. It has been used to predict a relativistically-induced decoherence effect [15], and a corresponding reduction in the visibility of quantum interference experiments [14]. This approach treats spacetime classically, making no reference to a quantum theory of gravity. It is applicable only in the low-energy, weak-gravity limit, so that we may consider relativistic effects as perturbative corrections to non-relativistic quantum mechanics.

We begin by introducing a way of characterising the accuracy of a generic quantum clock, before describing how to incorporate the phenomenon of time dilation into...
II. NON-RELATIVISTIC QUANTUM CLOCKS

Before we discuss relativistic effects, it is convenient to introduce a characterisation of the extent to which a given quantum clock measures time accurately. We associate the relevant degree of freedom with a Hilbert space $\mathcal{H}_c$, and consider an initial clock state $\rho_c(0)$ which evolves according to Schrödinger equation, with Hamiltonian $\hat{H}_c$. We use the term laboratory time to refer to the parameter appearing in the Schrödinger equation. At laboratory time $t$, the measurement corresponding to a self-adjoint operator $\hat{T}_c$ is performed, whose outcome we refer to as the clock time. A quantum clock is then defined by the tuple $\{\hat{T}_c, \hat{H}_c, \rho_c(0)\}$.

Given a quantum clock satisfying the Heisenberg form of the canonical commutation relation $[\hat{T}_c, \hat{H}_c] = i \mathbb{1}_c$,\(^1\) the mean clock time will exactly follow the laboratory time, i.e. $\langle \hat{T}_c \rangle = t \mod T_0 \forall t$ for some (possibly infinite) clock period $T_0 > 0$. Such a clock is referred to as idealised. Demanding only this commutation relation, one can construct an idealised clock with finite $T_0$ and with $\hat{H}_c$ bounded below, on the condition of strict confinement of the wavefunction, and thus an infinitely strong potential [16]. If we further impose canonical commutation relations of the Weyl form, we necessarily have that $\hat{H}_c$ is unbounded below (Pauli’s theorem) [17], and $T_0 = \infty$.

Since idealised clocks are desirable, we seek to quantify the extent to which a realistic clock, with finite energy, deviates from an idealised one. To this end, we introduce a clock’s error operator $E(t)$: for a given clock characterised by $\{\hat{T}_c, \hat{H}_c, \rho_c(0)\}$, we define $\tilde{E}(t)$ via

$$-i[\hat{T}_c, \hat{H}_c] \rho_c(t) = \rho_c(t) \ast \tilde{E}(t)$$

where $\rho_c(t) = e^{-it\hat{H}_c} \rho_c(0) e^{it\hat{H}_c}$. The error operator $\tilde{E}(t)$ allows us to quantify the unavoidable discrepancy between the clock time and the laboratory time $t$ via

$$\langle \hat{T}_c \rangle_{NR}(t) = t + \int_0^t dt' tr[\tilde{E}(t')$$

where the subscript $NR$ denotes that the clock does not account for relativistic effects (i.e. it evolves under the action of $\hat{H}_c$ alone), and we have assumed for convenience that $\langle \hat{T}_c \rangle_{NR}(0) = 0$. For the unphysical, idealised clock, we have $-i[\hat{T}_c, \hat{H}_c] = \mathbb{1}_c$, and therefore $\tilde{E}(t) = 0 \forall t$. We therefore call a clock good when $\tilde{E}(t)$ is small in norm relative to the other quantities involved.

An interesting case is when the clock is a $d$-dimensional spin system, with evenly-spaced energy eigenvalues (corresponding to the spin-projection states). Constructing $\hat{T}_c$ such that its eigenvectors form an appropriate mutually- unbiased basis with respect to the spin-projection states results in a periodic clock — a quantum version of the common dial clock (see Fig. 1). Choosing $\rho_c(0)$ to be one of the eigenvectors of $\hat{T}_c$, one arrives at the well-known Salecker-Wigner-Peres clock [18, 19]. In this case, $E(t)$ is not always small; one has $tr[\hat{E}(t)] = -1$ at regular intervals of laboratory time, regardless of the dimension $d$ or energy. This arises due to the clock states’ lack of coherence in the eigenbasis of $\hat{T}_c$; specifically $[\hat{T}_c, \rho_c(t)]$ is zero at regular intervals of laboratory time. This issue can be removed by choosing $\rho_c(0)$ to have some quantum coherence in this basis, as in the Quasi-Ideal clock [6]. In the latter case, $\|\hat{E}(t)\|_1$ is exponentially small in both the dimensionality $d$ and the mean energy of the clock $\forall t$. The Salecker-Wigner-Peres and Quasi-Ideal clocks are discussed in Appendix C.

III. RELATIVISTIC TIME DILATION

Classically, in the post-Newtonian approximation of general relativity, the proper time $\tau$ experienced by an

\[^1\] Here and throughout this manuscript, we use units such that $\hbar = 1$. The symbol $\mathbb{1}_c$ denotes the identity operator on $\mathcal{H}_c$. 

\[\text{FIG. 1. Depiction of a spatially-delocalised quantum clock exhibiting temporal indeterminacy. In the theory of relativity, the time measured by a clock is defined geometrically by integrating along a well-defined spacetime path, and the clock may be arbitrarily precise and accurate. This contrasts with time in quantum mechanics, where a fully quantum treatment requires clocks to be delocalised in space and experience an indeterminacy in their time. They cannot be meaningfully assigned a single trajectory. Relativity dynamically couples the temporal and kinematic quantum degrees of freedom, resulting in time measurements which are influenced by the delocalised spatial trajectory.}\]
observer is determined by [20]
\[
dτ = d\hat{t} \left( 1 - \frac{v^2}{c^2} + \frac{\Phi(r)}{c^2} \right),
\] (3)
to first order in $v^2/c^2$ and $\Phi/r^2$, where $v$ is the observer’s velocity, $r$ is the distance from a gravitating body of mass $M$, $\Phi(r) = -GM/r$ is the Newtonian gravitational potential, and $\hat{t}$ is the proper time of a fictional observer at rest at $r = \infty$. To that end, we consider a total Hilbert space quantum mechanics, we must describe how its temporal and kinematic spaces (which can be understood as a consequence of mass-energy equivalence [21]) are related to first order in $v^2/c^2$.

If we now consider the clock to be subject to the laws of quantum mechanics, we must describe how its temporal degree of freedom interacts with its kinematic degrees of freedom. To that end, we consider a total Hilbert space $\mathcal{H} = \mathcal{H}\text{c} \otimes \mathcal{H}_k$, where the space $\mathcal{H}_k$ corresponds to the kinematic variables in one dimension, namely position $\hat{x}$ and momentum $\hat{p}$, with $[\hat{x}, \hat{p}] = i\mathbb{1}_k$. The coupling between the temporal and kinematic spaces (which can be understood using the distribution of a more nonclassical kinematic state can result in a radically modified time-dilation effect. We consider a kinematic state constructed by superposing two Gaussian wavepackets with different mean initial positions, i.e. $|\psi\rangle_k \propto \sqrt{\alpha} |\psi_1\rangle_k + \sqrt{1-\alpha} |\psi_2\rangle_k$ for some $0 < \alpha < 1$, where $|\psi_1\rangle_k$ and $|\psi_2\rangle_k$ are Gaussian states differing only in the value of $\langle \hat{x} \rangle$. Specifically, $|\psi_1\rangle_k$ and $|\psi_2\rangle_k$ have mean positions $\bar{x}_0$ and $\bar{x}_0 + \Delta x_0$ respectively, standard deviation in position $\sigma_x$, standard deviation in momentum $\sigma_p$, and for simplicity we take both wavepackets to have the same initial mean momentum. This is illustrated in Fig. (2). We was defined according to the non-relativistic evolution of the clock (i.e. under $H_c$ alone). We now consider two possibilities for $\rho_k(0)$ — a classical (i.e. Gaussian) and a nonclassical state.

A. A Gaussian state of motion

Taking the initial kinematic state $\rho_k(0)$ to be a pure Gaussian wavepacket with mean momentum $\bar{p}$, standard deviation of the momentum $\sigma_p$, and mean position $\bar{x}_0$, one finds
\[
R(t) = -\frac{\bar{p}_0^2 + \sigma_p^2}{2mc^2} + \frac{\bar{x}_0 g}{c^2} + \frac{\bar{p}_0 g \bar{t}}{mc^2} - \frac{1}{3} \left( \frac{gt^2}{c^2} \right)^2.
\] (7)

In this case, setting $E(t) = 0$ and using Eqs. (2) and (6), one finds that the average time measured by the idealised quantum clock is identical to the expectation value of the classical proper time (Eq. (4)) for an observer with a momentum following the same (i.e. Gaussian) probability distribution. As noted in Sec. II, however, the idealised clock is unphysical, and a physical clock will then necessarily exhibit a non-relativistic quantum correction (the second term in Eq. (2)) as well as a contribution arising from both relativity and quantum mechanics (the final term in Eq. (6)). However, since both these effects are proportional to $\text{tr}[\hat{E}(t)]$, they can be made arbitrarily small, for example by using the Quasi-Ideal clock discussed in Sec. II with an appropriately high dimensionality and mean energy [6]. Interestingly, for the Salecker-Wigner-Peres clock discussed Sec. II, $\text{tr}[\hat{E}(t)] = -1$ whenever the clock is in an eigenstate of $\hat{T}_c$, exactly cancelling the usual classical relativistic effect in Eq. (6) (see Appendix B). This property of the Saleker-Wigner-Peres clock states has also been identified as the cause for suboptimal performance in other areas, related to quantum error correction with clocks [22].

B. A nonclassical state of motion

The use of a Gaussian wavepacket for the kinematic part of the initial state was the most classical choice in the sense that such states are the only ones with a non-negative Wigner function [23], and saturate the position-momentum uncertainty relation. A consideration of a more nonclassical kinematic state can result in a radically modified time-dilation effect. We consider a kinematic state constructed by superposing two Gaussian wavepackets with different mean initial positions, i.e. $|\psi\rangle_k \propto \sqrt{\alpha} |\psi_1\rangle_k + \sqrt{1-\alpha} |\psi_2\rangle_k$ for some $0 < \alpha < 1$, where $|\psi_1\rangle_k$ and $|\psi_2\rangle_k$ are Gaussian states differing only in the value of $\langle \hat{x} \rangle$. Specifically, $|\psi_1\rangle_k$ and $|\psi_2\rangle_k$ have mean positions $\bar{x}_0$ and $\bar{x}_0 + \Delta x_0$ respectively, standard deviation in position $\sigma_x$, standard deviation in momentum $\sigma_p$, and for simplicity we take both wavepackets to have the same initial mean momentum. This is illustrated in Fig. (2). We
denote the average clock time in this case by \( \langle T_c \rangle_{\text{mix}}(t) \). To extract the part of the effect arising from quantum coherence, we contrast this with the case of a classical mixture of two such states according to probabilities \( \alpha \) and \( 1 - \alpha \). The average clock time in the latter case, which we denote \( \langle T_c \rangle_{\text{sup}}(t) \), can easily be calculated via Eq. (6) by taking the corresponding weighted sum, i.e.

\[
\langle \hat{T}_c \rangle_{\text{mix}}(t) = \alpha \langle \hat{T}_c \rangle_{\psi_1}(t) + (1 - \alpha) \langle \hat{T}_c \rangle_{\psi_2}(t),
\]

where \( \langle \hat{T}_c \rangle_{\psi}(t) \) is the expectation value for state \( |\psi_i\rangle \). For simplicity of expression, we consider good clocks in the sense discussed in Sec. II, so that the contribution arising from a nonzero \( E(t) \) is negligible. One then has

\[
\langle \hat{T}_c \rangle_{\text{sup}}(t) = \langle \hat{T}_c \rangle_{\text{mix}}(t) + T_{\text{coh}}(t),
\]

where \( T_{\text{coh}}(t) \), the contribution due to coherence between the two constituent Gaussian states, is given by

\[
T_{\text{coh}}(t) := t \left( \frac{\Delta x_0}{\sigma_x} \right)^2 \frac{\sigma_x^2}{2} - \frac{g \Delta x_0}{c^2} (1 - 2\alpha) \exp \left( -\frac{\left( \frac{\Delta x_0}{\sigma_x} \right)^2}{\sqrt{1 - \alpha}} \right) + 2
\]

where \( \sigma_v := \sigma_p/m \) is their standard deviation in their initial “velocity”. We recall that, since the wavepackets saturate the uncertainty relation, we have that \( \sigma_p = 1/(2\sigma_x) \).

In Eq. (10) we see two contributions to \( T_{\text{coh}}(t) \): one proportional to \( \sigma_v^2 \) and one due to the average difference in gravitational potential experienced by the wavepackets (\( g \Delta x_0 \)). In other words, we see separately how motional and gravitational time dilation act on the wavefunction to generate quantum coherence effects. Noting that we can write \( g \Delta x_0 = v_g^2/2 \), where \( v_g \) is the velocity accrued by a classical object falling a distance of \( \Delta x_0 \) from rest, we see that for a given \( \Delta x_0/\sigma_x \) and \( \alpha \), the relative strength of the motional and gravitational parts is determined by the comparative magnitudes of \( \sigma_v \) and \( v_g \).

Further noting that the exponential term in the denominator of Eq. (10) equals \( 1/(\psi_1|\psi_2) \) (c.f. the grey region in Fig. (2)), we see that the exponential decrease in \( T_{\text{coh}}(t) \) with increasing \( \Delta x_0 \) is a result of the decreasing overlap between the two wavepackets. From this we infer that \( T_{\text{coh}}(t) \) arises due to interference between \( |\psi_1\rangle \) and \( |\psi_2\rangle \). On the other hand, we evidently have \( T_{\text{coh}}(t) = 0 \) when \( \Delta x_0 = 0 \), as in that case \( |\psi_1\rangle = |\psi_2\rangle \), and we return to the classical-motion scenario described in Sec. IV A.

Consequently there exists, for a given \( \sigma_v \) and \( \alpha \), an intermediary value of \( \Delta x_0/\sigma_x \) which maximises \( T_{\text{coh}}(t) \) by finding a balance between the wavefunction overlap and the “nonclassicality” of the kinematic state.

Given that modern-day atomic clocks are accurate and precise enough to observe classical relativistic time dilation in tabletop experiments [24], it is natural to ask whether the quantum contribution to the time dilation might also be observable. The classical contribution in the present case is given by \( \langle \hat{T}_c \rangle_{\text{mix}}(t) - \langle \hat{T}_c \rangle_{\text{NR}}(t) \), and therefore the relative magnitude of the quantum contribution is \( |T_{\text{coh}}(t)/\langle \hat{T}_c \rangle_{\text{mix}}(t) - \langle \hat{T}_c \rangle_{\text{NR}}(t)\rangle| \). As an example, we find that this ratio is approximately 0.13 after evolution for \( t = 0.5 \) s for a particle with the mass of an electron, a separation of \( \Delta x_0 = \sigma_x = 10 \pm \mu \text{m} \), at one metre above sea level, i.e. \( g = 9.81 \text{ m s}^{-2} \), \( \Delta x_0 = 1 \text{ m} \), and with \( \alpha = 1/3 \), \( \sigma_v = 5.8 \text{ m s}^{-1} \). This appears promising, though we stress that this example is illustrative. A careful analysis concerning potential experimental platforms is required before conclusions can be drawn about the prospect of observing the effect.

V. CLOCK PRECISION AND THE COUPLING OF TEMPORAL AND KINEMATIC DEGREES OF FREEDOM

In this section we show how the clock’s precision is modified by the relativistic coupling between its kinematic and temporal degrees of freedom. In particular we show how the entanglement generated by the interaction Hamiltonian in Eq. (5) increases the uncertainty associated with measurements of the temporal variable.

We quantify the clock’s precision via the standard deviation of the clock time, which we denote \( \sigma_T(t) \). For simplicity, we ignore the contribution due to gravity. In order to calculate the relativistic contribution to \( \sigma_T(t) \) leading order, we now include Hamiltonian terms up to order \( \hat{p}^2/(mc)^2 \) (see Appendix D). Let \( \sigma_{T,\text{NR}}(t) \) denote the standard deviation of the clock time in the absence of the relativistic coupling. Then, assuming the clock’s kinematic and temporal degrees of freedom to be uncorrelated at \( t = 0 \), one finds that \( \sigma_T(t) \) separates into

\[
\sigma_T(t) = \sigma_{T,\text{NR}}(t) + \sigma_{T,\text{R}}(t) + \sigma_{T,\text{NL}}(t),
\]

where the term \( \sigma_{T,\text{R}}(t) \) is a contribution that remains finite in the case of an idealised clock, given by

\[
\sigma_{T,\text{R}}(t) := \frac{t^2}{8 \sigma_{T,\text{NR}}(t)} \frac{(\hat{p}^2) + \sigma_{\hat{p}}^2}{m^4 c^4},
\]

where \( \sigma_{\hat{p}}^2 \) is the standard deviation of the observable \( \hat{p}^2 \), and the term \( \sigma_{T,\text{NL}}(t) \) is the contribution arising entirely due the clock’s non-idealised nature. Its full form
is given in Appendix D. Note that $\sigma_{T,1}(t) > 0$, and the
effect of the relativistic coupling on good clocks (in the
sense discussed in Sec. II) is therefore to increase the
clock’s temporal uncertainty. In the case of an idealised
clock, $\sigma_{T,NI}(t)$ is constant in laboratory time and can be
made arbitrarily small by an appropriate choice of
$\rho_c(0)$. Furthermore, $\sigma_{T,NI}(t) = 0$. Consequently, according
to Eq. (12), the standard deviation of the clock time
increases quadratically with laboratory time.

The decrease in precision is a consequence of the clock’s
temporal state losing information via its entanglement
with the kinematic degrees of freedom. We now show
how this effect can be reduced by recovering some of
the lost information via a measurement of the clock’s mo-
momentum. We consider course-grained momentum mea-
surements, and show how the uncertainty in the clock
time decreases as the measurement is made more pre-
cise. We again choose a classical (i.e. Gaussian) state
of motion, as in Sec. IV A, and we consider an idealised
clock for simplicity, as this can be approximated arbitrar-
ily well (see the discussion in Sec. II). We define a set of
projection operators $\{\hat{\Pi}_{n,\delta p}\}_n$ acting on $\mathcal{H}_k$, with $n \in \mathbb{Z}$
and $\delta p > 0$, which correspond to a partition of the range
of momentum values into bins of width $\delta p$, with bin $n$
centred on momentum value $n\delta p$, i.e.

$$\hat{\Pi}_{n,\delta p} := \int_{(n,\delta p)} dp |p\rangle \langle p|_k.$$  \hspace{1cm} (13)

The bin width $\delta p$ therefore characterises the amount of knowledge that we gain from the measurement, and
the degree of localisation in momentum space of the post-measurement clock state. As $\delta p$ approaches zero,
$\{\hat{\Pi}_{n,\delta p}\}_n$ approaches the set of projectors onto momen-
tum eigenstates. As $\delta p \to \infty$, on the other hand, the pro-
jectors tend to the identity operator, i.e. the case where
no measurement is performed. Fig. 3 shows examples of
$\sigma_T(t)$ conditioned on a given outcome of the measure-
ment, at different laboratory times $t$, and for different
values of $\delta p$. Quantifying the coarseness of the measure-
ment by $q := \delta p/\sigma_p$, we find that for $q \to 0$, we recover
all of the information leaked into the kinematic state, i.e. 
after the measurement, $\sigma_T(t) = \sigma_T(0) = \sigma_{T,NI}(t)$. For
$q \to \infty$, on the other hand, no information is recovered,
as no measurement had been performed.

VI. DISCUSSION

Throughout this work, we have made reference to the
laboratory time $t$, which can be interpreted as the time
marked by a classical clock in the laboratory frame. One
may however wish to consider a fully quantum set-up.
One way to achieve this, is a straightforward extension
of the framework given here. One may simply treat $t$ as a
bookkeeping coordinate, to be used to calculate the aver-
age clock time experienced by multiple quantum clocks,
and therefore their average clock time relative to each
other. In doing so, all our derived results can be written
in terms of the average readings of quantum clocks with-
out any reference to a classical idealised clock or extrinsic
time.

Our results are in apparent contrast with the recent
claim that quantum clocks do not witness classical time
dilation [25]. For the specific clock used by those authors,
the error operator $\hat{E}(t)$ (defined via Eq. (1)) acts to ex-
actly cancel the relativistic time dilation on average to first
order for classical states of motion (see Appendix B).
Our results regarding time dilation were obtained for ar-
bitrary clocks, and we find a relativistic time dilation to
lowest order for all clocks other than for a few special
cases such as the Salecker-Wigner-Peres.

The present work has been concerned with the effect
of relativity in a stopwatch scenario, that is, we are inter-
ested in the time elapse between two events. This con-
trasts with the ticking clocks discussed in [10, 11, 26, 27].
The incorporation of relativity into the latter is an open
problem.

The reduction in clock precision that we predict is both
a quantum and relativistic effect, as the general theory
of relativity allows for clocks which are always perfectly
precise, and clocks in non-relativistic quantum theory are
generally uncoupled from their kinematic degrees of free-
dom. This leads to the interesting statement that rela-
tivity requires us to ask how a clock is moving as well
as the time it measures, or we must pay a penalty in its
precision. Though our explicit calculation of the de-
creased clock precision was carried out neglecting the ef-
effect of gravity, the same principle will hold for a nonzero
gravitational field. A general initial clock state which is
uncorrelated between temporal and kinematic degrees of
freedom will only typically remain in the set of separable states if either the eigenstates of the total Hamiltonian are separable (which they are not), or if the initial state is particularly mixed [28]. One can therefore only escape the relativistic decrease in precision by increasing the mixedness of the initial state (and thus decreasing its precision anyway). The process of recovering the precision via measurements of the kinematic state will likely be modified by gravity, however. In particular, the gravitational coupling between the temporal degree of freedom and the clock’s spatial position will mean that a perfectly accurate measurement of the momentum degree of freedom will no longer totally restore the clock’s precision.

The decreased clock precision is related to the effective decoherence of quantum systems discussed in [15], though, contrary to our work, it is the decoherence of the kinematic degrees of freedom which is studied in that case. Other work has predicted a limit of the ability of multiple clocks to jointly measure time due to a gravity-mediated interaction between them [29], and a tendency for which-path information in a matter interferometer to be encoded in clock degrees of freedom, leading to relativity-induced decrease in quantum interferometric visibility [14]. We use the same model as those authors, namely that relativity couples the internal degrees of freedom of a system to its kinematic degrees of freedom, leading to the interaction Hamiltonian in Eq. (5) in the limit of low velocities and a weak gravitational field. A discussion and rebuttal of some of the criticisms of this model can be found in [30]. Since a full theory of quantum gravity is yet to be experimentally established, all results at the interface between quantum theory and gravity are naturally controversial. Ultimately, future experiments will determine the full scope of validity for these results.

VII. CONCLUSION

In the non-relativistic limit, we characterised a generic quantum clock by an initial state \( \rho_0(0) \), a Hamiltonian \( \hat{H}_c \) and an operator \( \hat{T}_z \) corresponding to the observable used to measure time. Not all such systems can serve as clocks; for example, a clock where \( \rho_0(0) \) is an energy eigenstate will not evolve in time. An unavoidable consequence of quantum theory is that any finite-energy clock cannot be perfectly accurate (a constraint which is distinct from the uncertainty principle) and we quantified this inaccuracy by defining an error operator. We call a clock good, when its error operator is small in norm.

Setting this framework into a relativistic context, we considered the average time dilation according to a generic quantum clock, finding that for Gaussian states of motion, any quantum clock will experience the usual classical relativistic time dilation with some quantum error (given by the trace of the error operator) which can be made arbitrarily small. Taking a nonclassical state of motion, namely a superposition of Gaussian states, we found that interference results in an average clock time which differs from the prediction of classical relativity, even when the error operator of a clock is zero (the idealised clock).

We then discussed how the incorporation of relativity leads to entanglement between the clock’s time-measuring and motional degrees of freedom. We calculated the corresponding reduction in clock precision for classical states of motion, showing how it increases over time. We demonstrated how the information lost from the clock’s time-measuring degrees of freedom can be recovered by performing a measurement of the clock’s motion in addition to its time reading.

It is our hope that these results will fuel interest from the experimental community, so that experiments might be devised to observe combined quantum relativistic effects, perhaps shedding light on the obscure relationship between the two theories.

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Appendix A: Relativistic coupling of the clock's temporal and kinematic degrees of freedom

Here we sketch a derivation of the coupling between the clock’s kinematic and time-measuring degrees of freedom. A similar derivation can be found in [12]. We consider two reference frames in which we give a classical description of the clock: one frame in which the clock is at rest, with energy $E_{\text{rest}}$, and another (the laboratory frame) in which it has energy $E_{\text{lab}}$ and momentum $p_j$. The norm of the clock’s four-momentum is a scalar quantity, and therefore the same in both frames, and thus (assuming a static spacetime) one has

$$E_{\text{lab}} = \sqrt{-g_{00} (E_{\text{rest}}^2 + p_j^2 c^2)},$$

(A1)
where we use the Einstein summation convention, with Latin indices denoting a sum over spatial coordinates, and $p_j p^j = g^{ij} p_i p_j$. We consider only a single spatial dimension, and use the post-Newtonian metric to approximate a weak gravitational field:

$$g_{00} = -\left[1 + \frac{2\Phi(x)}{c^2} + \frac{2\Phi(x)^2}{c^4}\right] + \mathcal{O}\left(\left(\frac{\Phi(x)}{c^2}\right)^3\right) \quad \text{and} \quad g_{ij} = \delta_{ij} \left[1 - \frac{2\Phi(x)}{c^2}\right] + \mathcal{O}\left(\left(\frac{\Phi(x)}{c^2}\right)^3\right). \quad (A2)$$

Furthermore, we take the common approximation of the Newtonian potential around a point $r_0$, i.e. $\Phi(x) \approx \Phi_0 + gx$, where $\Phi_0$ and $g$ are respectively the gravitational field and acceleration at the point $x = 0$. Now, we note that in order to mark the passage of time the clock must have some evolving internal structure, which is associated with some internal energy. Denoting the clock’s rest mass by $m$ and its internal energy by $E_c$, then by mass-energy equivalence,

$$E_{\text{rest}} = mc^2 + E_c. \quad (A3)$$

After substituting this into Eq. (A1), making a Taylor expansion in $E_c/mc^2$, $p/mc$, and $\Phi(x)/c^2$, and quantising the result, one finds that the clock evolves subject to the Hamiltonian

$$\hat{H} = \hat{H}_c + \hat{H}_k + \hat{H}_{ck}, \quad (B1)$$

where upon quantisation $E_{\text{lab}} \rightarrow \hat{H}$ (which acts upon the space $\mathcal{H}_c \otimes \mathcal{H}_k$) and $E_c \rightarrow \hat{H}_c$ (which acts upon the space $\mathcal{H}_c$),

$$\hat{H}_k = mc^2 + mg\hat{x} + \frac{\hat{p}^2}{2m} - \frac{\hat{p}^4}{8m^3c^2} \quad (B2)$$

represents the rest-mass-energy, gravitational potential energy and kinetic energy (with a lowest-order relativistic correction) of the clock, and

$$\hat{H}_{ck} = \hat{H}_c \otimes \left(-\frac{\hat{p}^2}{2mc^2} + \frac{g\hat{x}}{c^2}\right). \quad (B3)$$

is a relativistic coupling of the temporal and kinematic degrees of freedom. We have neglected terms of order $(\mathcal{H}_c/mc^2)^2$, $(\hat{p}/mc)^4$, $(g\hat{x})^2/c^4$ and $g\hat{x}\hat{p}^2/m^2c^4$ and higher. In the following, we refer to such terms with the notation $\mathcal{O}(1/c^4)$. Note that $\hat{H}$ generates evolution with respect to the proper time of a classical observer at $x = 0$, i.e. the laboratory time $t$.

**Appendix B: Time dilation in generic quantum clocks**

We now calculate the average time dilation experienced by a generic quantum clock evolving under the Hamiltonian given in Eq. (B1). We begin by employing perturbation theory to calculate the evolution of the clock state in the low-velocity, weak-gravity limit described in Appendix A (i.e. neglecting $\mathcal{O}(1/c^4)$ terms). As in the main text, we choose units such that $\hbar = 1$.

Defining a *free* Hamiltonian by $\hat{H}_0 := \hat{H}_c + \hat{H}_k$, one can find the evolution of the clock’s state via the interaction picture, giving

$$\rho(t) = e^{-i\hat{H}_0 t} \rho(0) e^{i\hat{H}_0 t} - i e^{-i\hat{H}_0 t} \left\{ \int_0^t dt' [\hat{H}_{ck}(t'), \rho(0)] \right\} e^{i\hat{H}_0 t}. \quad (B4)$$

where $\hat{H}_{ck}(t) := e^{i\hat{H}_0 t} \hat{H}_{ck} e^{-i\hat{H}_0 t}$ is the interaction Hamiltonian in the interaction picture. Tracing over the kinematic space $\mathcal{H}_k$, we find the reduced state corresponding to the temporal degree of freedom, which we write as

$$\rho_c(t) = \rho_{c,NR}(t) + \rho_c^{(1)}(t), \quad (B5)$$

where $\rho_{c,NR}(t) := e^{-i\hat{H}_c t} \rho_c(0) e^{i\hat{H}_c t}$ is the non-relativistic evolution of the reduced state, and the lowest-order relativistic correction to it is given by

$$\rho_c^{(1)}(t) = -i \int_0^t dt' \text{tr}_k \left[ e^{-i\hat{H}_0 t'} [\hat{H}_{ck}(t'), \rho(0)] e^{i\hat{H}_0 t'} \right]. \quad (B6)$$
Now, we consider a clock whose temporal and kinematic degrees of freedom are initially uncorrelated, writing

\[ \rho(0) = \rho_c(0) \otimes \rho_k(0), \]  

for some initial reduced states \( \rho_c(0) \) and \( \rho_k(0) \). Then, writing \( \hat{H}_{ck} = \hat{H}_c \otimes \hat{V}_k \), i.e. \( \hat{V}_k := -\frac{\hat{p}^2}{2m^2c^2} + \frac{\Phi(x)}{c^2} \), we calculate the integrand of Eq. (B3) to find

\[ \text{tr}_k\left[e^{-i\hat{H}_{ot}}[\hat{H}_{ck}(t'), \rho(0)]e^{i\hat{H}_{ot}}\right] = \text{tr}_k\left[\hat{H}_{ck}(t'), \rho_{c,nr}(t)\right] \text{tr}_k\left[e^{i\hat{H}_k t} \hat{V}_k e^{-i\hat{H}_k t} \rho_k(0)\right]. \]  

Using the relation

\[ e^{\hat{X}Y}e^{-\hat{X}} = \hat{Y} + \frac{1}{2!}[\hat{X}, [\hat{X}, \hat{Y}]] + \frac{1}{3!}[[\hat{X}, [\hat{X}, [\hat{X}, \hat{Y}]]) + \ldots \]  

for some \( \hat{X} \) and \( \hat{Y} \) (see Prop. 3.35 in [31]), one finds by induction that

\[ e^{i\hat{H}_k t} \hat{V}_k e^{-i\hat{H}_k t} = \frac{1}{c^2} \left( -\frac{\hat{p}^2}{2m^2} + \frac{g\hat{x}}{c^2} + \frac{2gt}{m^2} - g^2\hat{t}^2 \right). \]  

Performing the integration in Eq. (B3), we find

\[ \rho_c^{(1)}(t) = -i \left[ \hat{H}_c, \rho_{c,nr}(t) \right] tR(t), \]  

where \( R(t) := \text{tr} \left[ \left( -\frac{\hat{p}^2}{2m^2c^2} + \frac{g\hat{x}}{c^2} + \frac{\hat{p}gt}{mc^2} - \frac{g^2\hat{t}^2}{3c^2} \right) \rho_k(0) \right] \). Having obtained the evolution of the clock’s temporal state, one can calculate the average clock time, giving

\[ \langle \hat{T}_c(t) \rangle = \langle \hat{T}_c \rangle_{nr}(t) + tR(t) \{1 + \text{tr}[\hat{E}(t)]\} \]  

where \( \langle \hat{T}_c \rangle_{nr} := \text{tr}[\hat{T}_c \rho_{c,nr}(t)] \) is the clock time in the absence of relativistic effects, and where the error operator \( \hat{E}(t) := -i[\hat{T}_c, \hat{H}_c] \rho_{c,nr}(t) - \rho_{c,nr}(t) \) was introduced in Sec. II of the main text.

**Appendix C: Examples of relativistic time dilation in quantum clocks**

1. **An idealised quantum clock**

Recall that an idealised clock is one satisfying the canonical commutation relation, \( [\hat{T}_c, \hat{H}_c] = i \), and consequently \( \hat{E}(t) = 0 \). Let \( \hat{T}_c = \hat{x}_c/c + a\hat{1}_c \) and \( \hat{H}_c = c\hat{p}_c \), where \( \hat{x}_c \) and \( \hat{p}_c \) are position and momentum operators and \( a \in \mathbb{R} \) is chosen such that \( \langle \hat{T}_c \rangle(0) = 0 \). We first consider the case of “classical” motion, i.e. where the initial kinematic state of the clock is a pure Gaussian wavepacket with mean momentum \( \bar{p}_0 \), standard deviation of the momentum \( \sigma_p \) and mean position \( \bar{x}_0 \), i.e. \( \rho_k(0) = |\psi\rangle\langle\psi|_k \) with

\[ |\psi\rangle_k = \int dp \psi(p) |p\rangle_k; \quad \psi(p) := \frac{1}{(2\pi\sigma_p^2)^{1/4}} e^{-\left(\frac{p-\bar{p}_0}{2\sigma_p}\right)^2} e^{-i\bar{x}_0(p-\bar{p}_0)}, \]  

We then have that \( R(t) = -\frac{\bar{p}_0^2 + \sigma_p^2}{2mc^2} + \frac{g\bar{x}_0}{c^2} + \frac{\bar{p}_0gt}{mc^2} - \frac{1}{3} \left( \frac{gt}{c} \right)^2 \) and therefore

\[ \langle \hat{T}_c(t) \rangle = \left[ 1 - \frac{\bar{p}_0^2 + \sigma_p^2}{2mc^2} + \frac{g\bar{x}_0}{c^2} + \frac{\bar{p}_0gt}{mc^2} - \frac{1}{3} \left( \frac{gt}{c} \right)^2 \right] t. \]  

This expression is the average proper time of a classical clock (see Eq. (4) in the main text) with position \( \bar{x}_0 \) and a random momentum following a Gaussian distribution with mean \( \bar{p}_0 \) and standard deviation \( \sigma_p \). To summarise, an idealised clock in a classical (i.e. Gaussian) state of motion experiences the usual classical relativistic time dilation.

Let us now consider a nonclassical initial kinematic state, specifically a superposition of two Gaussian states, sometimes referred to as a cat state. We will contrast this with the case of comparable classical probabilistic mixture of the two states. Let

\[ |\psi_j\rangle_k := \int dp \psi_j(p) |p\rangle_k; \quad \psi_j(p) := \frac{1}{(2\pi\sigma_p^2)^{1/4}} e^{-\left(\frac{p-\bar{p}_j}{2\sigma_p}\right)^2} e^{-i\bar{x}_j(p-\bar{p})}, \quad j \in \{1, 2\}. \]
We now take the initial kinematic state to be in the superposition $|ψ⟩_k = \sqrt{α}|ψ_1⟩_k + e^{iθ}\sqrt{1-α}|ψ_2⟩_k$ for some $α \in (0, 1)$ and $θ \in \mathbb{R}$, and find the average clock time, which we denote $⟨T_c⟩_\text{mix}$, using Eq. (B9). We then repeat this procedure for the initial kinematic state in the probabilistic mixture, $ρ_0(0) = α|ψ_1⟩⟨ψ_1| + (1-α)|ψ_2⟩⟨ψ_2|$, and denote the average clock time in that case by $⟨T_c⟩_\text{mix}(t)$. From the linearity of the trace, we see immediately that

$$⟨\hat{T}_c⟩_\text{mix}(t) = α⟨\hat{T}_c⟩_ψ(t) + (1-α)⟨\hat{T}_c⟩_ψ_2(t),$$

where $⟨\hat{T}_c⟩_ψ(t)$ is the average clock time of a clock with initial kinematic state $|ψ_j⟩_k$. One then finds that

$$⟨\hat{T}_c⟩_\text{sup}(t) = ⟨\hat{T}_c⟩_\text{mix}(t) + T_{\text{coh}}(t),$$

where

$$T_{\text{coh}}(t) = t \left( \frac{Δx_0}{2σ_σ} \right)^2 \frac{σ_x^2}{c^2} − \frac{gΔx_0}{c^2}(1-2α) \frac{e^{2iθ}}{\sqrt{(1-α)σ_x^2 + 2}}$$

is a contribution to the clock time arising due to interference between the two Gaussian states constituting the coherent superposition, and where $σ_x = σ_σ/m$ and $σ_σ = 1/2σ_p$ are respectively the standard deviation of the velocity and position of the Gaussian wavepackets used to construct the initial state, and $Δx_0 = \bar{x}_2 - \bar{x}_1$.

2. Salecker-Wigner-Peres clock

Consider a $d$-dimensional Hilbert space and a clock Hamiltonian with equally-spaced energy levels, i.e. $H_c = \sum_{j=0}^{d-1} j\omega |e_j⟩⟨e_j|$, for some $ω > 0$. This corresponds, for example, to the case where the clock is a spin-$j$ system, and then $d = 2j + 1$. One can obtain a so-called time basis $\{|θ_m⟩\}$ by taking the discrete Fourier transform of the energy eigenstates,

$$|θ_m⟩ = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} e^{-2iπjm/d} |e_j⟩, \quad m = 0, 1, \ldots, d-1$$

which are used to construct a clock-time operator $\hat{T}_c = \sum_{m=0}^{d-1} mT_0 |θ_m⟩⟨θ_m|$, with $T_0 := 2\pi$. When the clock is in the state $|θ_m⟩$, we have $⟨\hat{T}_c⟩ = mT_0$. Choosing an initial state $ρ_0(0) = |θ_0⟩⟨θ_0|$, then the clock state will at regular time intervals $t_m := mT_0/d$, $m \in \mathbb{Z}$ “focus” into one of the time eigenstates, i.e. $⟨\hat{T}_c⟩_{θ_k}(t_m) = mT_0/d (m \mod d)$. At intermediate times $t \neq t_m$, on the other hand, the clock will be in a superposition of its time eigenstates (see e.g. [6], Appendix B). This setup is known as the Salecker-Wigner-Peres clock (or alternatively, the Larmor clock) [18, 19].

A straightforward calculation gives $tr[⟨\hat{T}_c⟩_{θ_m}|θ_m⟩⟨θ_m|] = 0$ for all $m = 0, 1, \ldots, d-1$, and therefore from the definition of the error operator $\hat{E}(t)$, we have $tr[\hat{E}(t_m)] = -1$ for all $m \in \mathbb{Z}$ and for all clock dimensions $d \in \mathbb{N}^+$. Consequently, when the Salecker-Wigner-Peres clock is in a time eigenstate, Eq. (B9) tells us that $⟨\hat{T}_c⟩_{θ_k}(t_m) = ⟨\hat{T}_c⟩_{θ_k}(t_m)$ for any state of the clock’s kinematic degrees of freedom. To reiterate, this clock is a special case where the average relativistic time-dilation is exactly cancelled by the quantum error of the clock at regular time intervals regardless of how large $d$ is and regardless of the motional state. In this regard, we note that the Salecker-Wigner-Peres clock was used in [25], where it was concluded that “quantum clocks do not witness classical time dilation”.

3. The Quasi-Ideal clock

Consider the same setting as for the Salecker-Wigner-Peres clock described above, namely the clock Hilbert space, Hamiltonian $H_c$ and clock-time operator $T_c$, but now with the initial temporal state $ρ_0(0) = |Ψ(m_0)⟩⟨Ψ(m_0)|$, with

$$|Ψ(m_0)⟩ = \sum_{m=0}^{d-1} g_{m_0}(m) |θ_m⟩,$$

where $g_{m_0}(m)$ is a complex Gaussian distribution centred on $m_0$ and with standard deviation $σ_σ \in (0, d)$. The choice $σ_σ = \sqrt{d}$ corresponds to the case in which the standard deviation in both the time basis $\{|θ_m⟩\}$ and energy basis time
basis \(|e_j\rangle\) is approximately the same. For other choices, one obtains a clock whose behaviour tends to that of the idealised clock when \(\bar{\sigma}_c \to \infty\) and \(d/\bar{\sigma}_c \to \infty\) as \(d \to \infty\). This is then the \textit{Quasi-Ideal} clock, introduced in [6]. We now show that the contribution of the quantum error to the relativistic time dilation, i.e. \(\text{tr}[\hat{E}(t)]\) (see Eq. (B9)), tends exponentially to zero with increasing clock dimensionality. In other words, we show that the relativistic time dilation experienced by the idealised clock can be approximated arbitrarily well by a Quasi-Ideal clock of high enough dimension.

From the definition of the error operator \(\hat{E}(t)\), we have

\[
\text{tr}[\hat{E}(t)] = -i\text{tr}[(\hat{T}_c, \hat{H}_c)\rho_{c, NR}(t)] - 1
\]

Choosing the initial clock state \(|\Psi(0)\rangle\langle\Psi(0)|\), the non-relativistic evolution of the state \(\rho_{c, NR}(t)\) is given by,

\[
\rho_{c, NR}(t) = e^{-i\hat{H}_c t}|\Psi(0)\rangle\langle\Psi(0)| e^{i\hat{H}_c t},
\]

Now, Theorem 8.1 in [6] states that

\[
e^{-i\hat{H}_c t}|\Psi(0)\rangle = |\Psi(t/T_0)\rangle + |\varepsilon\rangle,
\]

where the specific form of \(|\varepsilon\rangle\) is irrelevant for the present purpose, only that as \(d \to \infty\) and \(d/\bar{\sigma}_c \to \infty\),

\[
\sqrt{|\varepsilon|} = \mathcal{O}\left(\text{poly}(d)\left(e^{-\frac{\pi^2}{2\bar{\sigma}_c^2}} + e^{-\frac{\pi^2}{2\bar{\sigma}_c^2}}\right)\right),
\]

and furthermore Theorem 11.1 in [6] states that

\[
[\hat{T}_c, \hat{H}_c]|\Psi(m)\rangle = \varepsilon|\Psi(m)\rangle + |\varepsilon_{\text{comm}}\rangle \quad \forall m \in \mathbb{R}
\]

where, again, the specific form of \(|\varepsilon_{\text{comm}}\rangle\) is irrelevant, only that as \(d \to \infty\),

\[
\sqrt{|\varepsilon_{\text{comm}}|} = \mathcal{O}\left(\text{poly}(d)\left(e^{-\frac{\pi^2}{2\bar{\sigma}_c^2}} + e^{-\frac{\pi^2}{2\bar{\sigma}_c^2}}\right)\right).
\]

Using these two theorems, one finds that

\[
|\text{tr}[\hat{E}(t)]| = |\text{tr}\left[|\varepsilon_{\text{comm}}\rangle \langle\Psi(t/T_0)| + [\hat{T}_c, \hat{H}_c]||\varepsilon\rangle\langle\Psi(t/T_0)| + |\Psi(t/T_0)\rangle\langle\varepsilon| + |\varepsilon\rangle\langle\varepsilon|\right]|
\leq |\text{tr}[|\varepsilon_{\text{comm}}\rangle \langle\Psi(t/T_0)|]| + |\text{tr}[\hat{T}_c, \hat{H}_c]| |\varepsilon\rangle\langle\Psi(t/T_0)| + |\Psi(t/T_0)\rangle\langle\varepsilon| + |\varepsilon\rangle\langle\varepsilon|\right]|
\]

We can now bound the terms on the r.h.s. of Eq. (C16). To start with

\[
|\text{tr}[|\varepsilon_{\text{comm}}\rangle \langle\Psi(t/T_0)|]| = |\langle\varepsilon_{\text{comm}}|\Psi(t/T_0)\rangle| \leq \sqrt{|\varepsilon_{\text{comm}}|\varepsilon_{\text{comm}}} = \mathcal{O}\left(\text{poly}(d)\left(e^{-\frac{\pi^2}{2\bar{\sigma}_c^2}} + e^{-\frac{\pi^2}{2\bar{\sigma}_c^2}}\right)\right)
\]

Using the Cauchy-Schwarz inequality, \(|\text{tr}[\hat{A}\hat{B}]| \leq \text{tr}[\hat{A}^\dagger \hat{A}]\text{tr}[\hat{B}^\dagger \hat{B}]\) for bounded linear operators \(\hat{A}, \hat{B}\); one finds that for two kets \(|a\rangle, |b\rangle\)

\[
|\text{tr}[\hat{T}_c, \hat{H}_c]|a\rangle\langle b| \leq 2\left|\text{tr}[(\hat{T}_c, \hat{H}_c)^2]\right| a\rangle\langle b| \leq \frac{16\pi^2 e^d}{\bar{\sigma}_c^2} a\rangle\langle b|
\]

Therefore, using Eqs. (C17), (C20), and (C12); one finds from Eq. (C16)

\[
|\text{tr}[\hat{E}(t)]| \leq \mathcal{O}\left(\text{poly}(d)\left(e^{-\frac{\pi^2}{2\bar{\sigma}_c^2}} + e^{-\frac{\pi^2}{2\bar{\sigma}_c^2}}\right)\right)
\]

for all \(t \in \mathbb{R}\). We therefore find that for any initial clock width of the form \(\bar{\sigma}_c = d^\eta\) for some fixed \(\eta > 0\), \(\text{tr}[\hat{E}(t)]\) is upper-bounded by a quantity which decreases exponentially with dimension. Furthermore, note that for \(0 < \eta < 1/2\), the standard deviation of \(\hat{T}_c\) for the initial Quasi-Ideal clock state approaches zero as \(d \to \infty\). Thus in the limit of large dimension, the relativistic time dilation (Eq. (B9)) is that of the idealised clock, up to errors that decay exponentially in the dimension.
Appendix D: The effect of the coupling on a clock’s precision

We now show how a clock’s precision, as quantified by the standard deviation \( \sigma_T := \sqrt{\langle \dot{T}^2 \rangle - \langle \dot{T} \rangle^2} \), changes over time as a result of the relativistic coupling between temporal and kinematic degrees of freedom. For simplicity, we now consider flat (Minkowski) spacetime. Here we work to a precision one order higher than before, i.e. neglecting only \( \mathcal{O}\left(\left(\frac{\hat{p}}{mc}\right)^6\right) \) terms. This is because, as we shall see, relativistic effects on the precision of the idealised clock vanish below \( \mathcal{O}(1/c^4) \). Repeating the procedure outlined in Appendix A, one finds that the relativistic coupling between temporal and kinematic degrees of freedom is now

\[
\hat{H}_{ck} = \hat{H}_c \otimes \hat{W}_k + \mathcal{O}\left(\left(\frac{\hat{p}}{mc}\right)^6\right)
\]

with \( \hat{W}_k := -\hat{p}^2/2m^2c^2 + 3\hat{p}^4/8m^4c^4 \).

Now, since in this case \( [\hat{H}_{ck}, \hat{W}_k] = [\hat{H}_{ck}, \hat{H}_c] = 0 \), we can simply calculate the relevant observables without going into the interaction picture. Using Eq. (B6), we have

\[
\langle \hat{T}_c^n \rangle(t) = \text{tr} \left[ \sum_{q=0}^\infty \frac{1}{q!} [i(\mathbb{1}_k + \hat{W}_k)t]^q \hat{H}_c, \hat{T}_c^n]_q \right]
\]

where \([\cdot, \cdot]_q\) denotes the \( q \)-th order nested commutator, i.e. \([\hat{A}, \hat{B}]_q := \hat{B} \text{ and } [\hat{A}, [\hat{A}, \hat{B}]_{q-1}]\) for \( q > 0 \). Applying the Binomial theorem to \( (\mathbb{1}_k + \hat{W}_k)t \) and truncating at the appropriate order, one finds, after some algebra

\[
\langle \hat{T}_c^n \rangle(t) = \langle \hat{T}_c^n \rangle_{\text{NR}}(t) + \text{tr} \left[ \left\{ it[\hat{H}_c, \hat{T}_c^n] \otimes \hat{W}_k - t^2[\hat{H}_c, [\hat{H}_c, \hat{T}_c^n]] \otimes \hat{W}_k^2 \right\}\rho_{\text{NR}}(t) \right]
\]

where \( \rho_{\text{NR}}(t) := e^{-i(\hat{H}_c + \hat{W}_k)t} \rho(0) e^{i(\hat{H}_c + \hat{W}_k)t} \) is the non-relativistic evolution of the clock’s total state and \( \langle \hat{T}_c^n \rangle_{\text{NR}}(t) := \text{tr}[\hat{T}_c^n \rho_{\text{NR}}(t)] \), and where it is understood that \( \hat{W}_k^2 \) contains an \( \mathcal{O}\left(\left(\frac{\hat{p}}{mc}\right)^6\right) \) term to be neglected. Assuming an uncorrelated initial state as in Appendix B, i.e. \( \rho(0) = \rho_c(0) \otimes \rho_k(0) \), and further defining \( \langle \hat{W}_k^n \rangle_0 := \text{tr}[\hat{W}_k^n \rho_k(0)] \), we have

\[
\langle \hat{T}_c^n \rangle(t) = \langle \hat{T}_c^n \rangle_{\text{NR}}(t) + \text{tr}[\hat{W}_k]_0 \text{tr}[\hat{H}_c, [\hat{H}_c, \hat{T}_c^n]] \rho_{\text{NR}}(t).
\]

Then by a straightforward, if lengthy, calculation, one finally finds that \( \sigma_T(t) \), separates into

\[
\sigma_T(t) = \sigma_{T,\text{NR}}(t) + \sigma_{T,\text{I}}(t) + \sigma_{T,\text{NI}}(t),
\]

where \( \sigma_{T,\text{NR}}(t) \) is the clock’s standard deviation in the absence of relativistic effects, \( \sigma_{T,\text{I}}(t) \) is a contribution which remains in the case of an idealised clock, given by

\[
\sigma_{T,\text{I}}(t) := \frac{\langle \hat{p}^2 \rangle}{8\sigma_{T,\text{NR}}(t)} + \sigma_p^2
\]

where \( \sigma_p^2 \) is the standard deviation of \( \hat{p}^2 \), and \( \sigma_{T,\text{NI}}(t) \) is a contribution arising due to the non-idealised nature of the clock, given by

\[
\sigma_{T,\text{NI}}(t) := \frac{\langle \hat{W}_k \rangle_0 t}{2\sigma_{T,\text{NR}}(t)} \{ \text{tr}[\hat{E}(t) + \hat{E}^\dagger(t)] \hat{T}_c] - 2\langle \hat{T}_c \rangle_{\text{NR}}(t) \text{tr}[\hat{E}(t)] \}
\]

\[
- \left( \langle \hat{W}_k \rangle_0 t \right)^2 \left[ \text{tr}[\hat{E}(t) + \hat{E}^\dagger(t)] \hat{T}_c] - 2\langle \hat{T}_c \rangle_{\text{NR}}(t) \text{tr}[\hat{E}(t)] \right]^2 - \langle \hat{W}_k \rangle_0 t \left[ 2\text{tr}[\hat{E}(t)] + \text{tr}[\hat{E}(t)] \right] \}
\]

\[
- \frac{\langle \hat{W}_k \rangle_0 t^2}{2\sigma_{T,\text{NR}}(t)} \left[ 2\text{tr}[\hat{E}(t)] + i[t[\hat{H}_c + \hat{W}_k, \hat{T}_c - \hat{T}_c^\dagger \hat{H}_c] \rho_{\text{NR}}(t) + \hat{H}_c \hat{T}_c \hat{E}(t) - \hat{E}^\dagger(t) \hat{T}_c \hat{H}_c] 
\]

\[
+ 2i\langle \hat{T}_c \rangle_{\text{NR}}(t) \text{tr}[\hat{H}_c (\hat{E}(t) - \hat{E}^\dagger(t))] \}
\]

where \( \hat{E}(t) \) was defined above, and \( \hat{e} := i[\hat{H}_c, \hat{T}_c] - \mathbb{1}_c \). Note that \( \sigma_{T,\text{I}}(t) = \mathcal{O}\left(\left(\frac{\hat{p}}{mc}\right)^4\right) \), which necessitated that we work to one order higher of precision than when we calculated the clock time (Appendix B).