Steady vortex patch solutions to the vortex-wave system

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Abstract
The vortex-wave system describes the motion of a two-dimensional ideal fluid in which the vorticity includes continuously distributed vorticity, which is called the background vorticity, and a finite number of concentrated vortices. In this paper we restrict ourselves to the case of a single point vortex in a planar bounded domain. We prove that there exists a family of steady solutions to this system whose background vorticity is a vortex patch with prescribed distribution. Moreover, we show that these vortex patches ‘shrink’ to a minimum point of the Kirchhoff–Routh function as the strength parameter of the background vorticity goes to infinity.

Keywords: vortex-wave system, vortex patch, Euler equation, Kirchhoff–Routh function, maximization, desingularization

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1. Introduction

The vortex-wave system was first introduced by Marchioro and Pulvirenti in [16] to describe the motion of a planar ideal fluid in which the vorticity consists of continuously distributed vorticity (wave part) and \( k \) concentrated vortices (vortex part). In the whole space \( \mathbb{R}^2 \) the system can be written as follows:

\[
\begin{align*}
\partial_t \omega + u \cdot \nabla \omega &= 0, \\
\frac{d}{dt} \omega &= J \nabla^* \ast \omega(x_i, t) + \sum_{j \neq i} \kappa_j J \nabla \Gamma(x_i - x_j), \quad i = 1, \cdots, k, \\
u &= J \nabla^* \ast \omega + \sum_{j=1}^{k} \kappa_j J \nabla \Gamma(x - x_j),
\end{align*}
\]

(1.1)
where \( \Gamma(x) = -\frac{1}{2\pi} \ln |x| \) is the fundamental solution of \(-\Delta \) in \( \mathbb{R}^2 \), \( J(x_1, x_2) = (x_2, -x_1) \) denotes clockwise rotation through \( \frac{\pi}{2} \), and \( \Gamma * \omega \) is the Newton potential of \( \omega \) defined by
\[
\Gamma * \omega(x, t) := -\frac{1}{2\pi} \int_{\mathbb{R}^2} \ln |x - y| \omega(y, t) \, dy.
\]
(1.2)

Let us explain system (1.1) briefly. The first equation is a transport equation for the background vorticity \( \omega(x, t) \), which means that the background vorticity is transported by the velocity ‘generated’ by itself (the term \( \mathcal{J} \nabla \Gamma * \omega \)), and \( k \) point vortices (the term \( \sum_{j=1}^{k} \kappa_j \mathcal{J} \nabla \Gamma(x_i - x_j) \)). The second equation expresses the fact that each point vortex \( x_i(t) \) moves by the velocity ‘generated’ by the background vorticity (the term \( \mathcal{J} \nabla \Gamma * \omega(x_i, t) \)) and the other \( k - 1 \) vortices (the term \( \sum_{j \neq i} \kappa_j \mathcal{J} \nabla \Gamma(x_i - x_j) \)). If \( \kappa_i = 0, i = 1, \ldots, k \), then the system reduces to the vorticity form of the Euler equation, which has been extensively studied; see [12, 15, 22] for example. If the background vorticity vanishes, then the system becomes the Kirchhoff–Routh equation, which is a model describing the motion of \( k \) concentrated vortices; see [11, 14, 19] for example.

The existence and uniqueness of solutions of the non-stationary vortex-wave system in the whole plane \( \mathbb{R}^2 \) have been extensively studied over the past decades; see [1, 5, 8, 9, 16, 17] for example. However, as far as we know, few results are known for steady solutions to this system. Our purpose here is to construct steady vortex patch solutions in the case of a single vortex in a bounded domain \( D \). More precisely, we will prove that for any patch rearrangement class \( \mathcal{N}^\mu \) defined by
\[
\mathcal{N}^\mu := \{ \omega \in L^\infty(D) \mid \omega = \mu I_A, A \text{ is a measurable set in } D, \mu |A| = 1 \},
\]
where \( \mu \in \mathbb{R}^+ \) is called the vorticity strength parameter, there exists a steady solution to the vortex-wave system, say \( (\omega^\mu, x^\mu) \), satisfying \( \omega^\mu \in \mathcal{N}^\mu \) and \( x^\mu \in D \). Moreover, \( \omega^\mu \) ‘shrinks’ to a minimum point of the Kirchhoff–Routh function as the strength parameter \( \mu \) goes to infinity.

The basic idea to prove the existence of \( (\omega^\mu, x^\mu) \) for fixed \( \mu \) is to construct a family of steady vortex patch solutions to the Euler equation, in which one part of the vorticity belongs to the rearrangement class \( \mathcal{N}^\mu \) while the other part ‘shrinks’ to a point, then we show the limit is in fact a steady solution to the vortex-wave system. We will use the result of Burton [2] on maximization of convex functionals on rearrangement classes to obtain approximate solutions, while the proof of the convergence is based on the idea of Turkington [20].

It is worthwhile to mention that our result is closely related to the desingularization of point vortices for the Euler equation, which has been studied by many authors; see [1, 4, 6, 13, 14, 19, 21] for example. Roughly speaking, desingularization of vortices for the Euler equation is to justify the Kirchhoff–Routh equation by approximation of the classical Euler equation. There are mainly two kinds of desingularization in the literature: the first kind is to consider a family of initial vorticity, which is sufficiently concentrated in \( k \) small regions, then the evolved vorticity according the Euler equation is also concentrated in \( k \) small regions all the time, and the limiting positions of these small regions can be approximated by the Kirchhoff–Routh equation; see [13, 14, 19] and the references therein; the second kind is to construct a sequence of steady solutions to the Euler equation that ‘shrinks’ to a critical point of the Kirchhoff–Routh function (or equivalently, a stationary solution to the Kirchhoff–Routh equation); see [4, 18, 20, 21] for instance.

Analogously, it is natural to consider the desingularization for the vortex-wave system. In [1], the author considered the first kind of desingularization, i.e. given a sequence of initial vorticity which is the sum of a given background vorticity and a concentrated vorticity ‘blob’, it was proved that the sequence of the evolved solutions according to the Euler
equation converges to the vortex-wave system in some sense. In contrast to [1], in this paper we are concerned with the second kind of desingularization, i.e. we construct a family of steady Euler solutions in which one part of the vorticity belongs to a given rearrangement class while the support of other part ‘shrinks’ to a point, and the limit is exactly a steady solution to the vortex-wave system.

We end this section by giving outline of this paper. In section 2, we introduce the vortex-wave system in bounded domains and state our main results. In section 3 we construct approximate solutions by solving a certain variational problem. In section 4 by comparing energy we show that the limit of approximate solutions is in fact a steady vortex patch solution to the vortex-wave system. In section 5 we consider the limit of the steady vortex solutions obtained in section 4 as the strength of the background vorticity goes to infinity.

2. Main results

2.1. Notations

Let $D \subset \mathbb{R}^2$ be a bounded and simply-connected domain with smooth boundary. The Green’s function for $-\Delta$ in $D$ with zero Dirichlet boundary condition is written as

$$G(x, y) = \frac{1}{2\pi} \ln \frac{1}{|x - y|} - h(x, y), \quad x, y \in D, \quad (2.1)$$

where $h(x, y)$ is the regular part of $G$. Note that $h(\cdot, \cdot)$ is bounded from below in $D \times D$. The Kirchhoff–Routh function of $D$ is defined to be

$$H(x) := \frac{1}{2} h(x, x), \quad x \in D, \quad (2.2)$$

and $\lim_{x \to \partial D} H(x) = +\infty$; see [19], lemma 2.2 for example. $2H$ is also called Robin function.

We shall use the following notations throughout this paper: $J(a, b) = (b, -a)$ denotes clockwise rotation through $\frac{\pi}{2}$ for any vector $(a, b) \in \mathbb{R}^2$, $|A|$ denotes the two-dimensional Lebesgue measure for any measurable set $A \subset \mathbb{R}^2$, $\bar{A}$ denotes the closure of some set $A \subset \mathbb{R}^2$ in the Euclidean topology, and $I_A$ denotes the characteristic function of some planar set $A$, that is, $I_A(x) = 1$ if $x \in A$, $I_A(x) = 0$ elsewhere. $\text{supp}(g)$ denotes the support of some function $g$, that is,

$$\text{supp}(g) = \{x | g(x) \neq 0\}. \quad (2.3)$$

$\text{dist}(\cdot, \cdot)$ denotes the distance between two sets,

$$\text{dist}(A, B) = \inf_{x \in A, y \in B} |x - y|. \quad (2.4)$$

For a given measurable function $g$ defined in $D$, the rearrangement class of $g$ is defined by

$$\mathcal{R}(g) = \{f | f : D \to \mathbb{R} \text{ is measurable and for any } a \in \mathbb{R}, \quad |\{f > a\}| = |\{g > a\}|\}. \quad (2.5)$$

For any $\omega \in L^\infty(D)$, we also define the stream function of $\omega$ by

$$G * \omega(x) := \int_D G(x, y) \omega(y) dy. \quad (2.6)$$

Note that since $\omega \in L^p(D)$ for any $p \in [1, +\infty]$, by $L^p$ estimate and Sobolev embedding $G * \omega \in W^{2p}(D) \cap C^1(\overline{D})$ for any $p \in [1, +\infty)$ and $\alpha \in (0, 1)$.
2.2. The vortex-wave system in bounded domains

We begin with a discussion on the Euler equation describing an ideal fluid with unit density moving in $D$,

\[
\begin{aligned}
\partial_t u + (u \cdot \nabla) u &= -\nabla P, \\
\nabla \cdot u &= 0, \\
u \cdot n|_{\partial D} &= 0,
\end{aligned}
\]  

(2.7)

where $u = (u_1, u_2)$ is the velocity field, $P$ is the scalar pressure, and $n$ is the outward unit normal of $\partial D$. Here we impose the impermeability boundary condition.

The scalar vorticity of the velocity $u$ is defined by $\omega := \partial_1 u_2 - \partial_2 u_1$. Since $D$ is simply connected, $u$ can be uniquely determined by $\omega$(see [15], section 1.2 for example), that is,

\[
u = J \nabla G \ast \omega.
\]  

(2.8)

So it suffices to consider the equation satisfied by $\omega$. Using the identity $\frac{1}{2} \nabla |u|^2 = (u \cdot \nabla) u + \omega J u$.

The first equation of (2.7) becomes

\[
u_t + \nabla (\frac{1}{2} |u|^2 + P) - \omega J u = 0.
\]  

(2.9)

Taking the curl on both sides of (2.9) we obtain the vorticity form of the Euler equation

\[
u_t + u \cdot \nabla \omega = 0,
\]  

(2.10)

which means that the vorticity is transported by the velocity $u$, while $u$ is ‘generated’ by $\omega$, i.e. $u = J \nabla G \ast \omega$.

When the vorticity is sufficiently concentrated at $k$ different points, equation (2.10) can be approximated by the following Kirchhoff–Routh equation in some sense:

\[
\frac{dx_i}{dt} = \sum_{j=1, j \neq i}^k a_i J \nabla x_j G(x_i, x_j) - a_i J \nabla H(x_i), \quad i = 1, \ldots, k,
\]  

(2.11)

where $x_i(t)$ represents the position of the $i$th point vortex, and $a_i$ is the corresponding vorticity strength. Equation (2.11) expresses the fact that each point vortex interacts with the others via the term $a_i J \nabla x_j G(x_i, x_j)$ and with the boundary via the term $-a_i J \nabla H(x_i)$. The approximation from the Euler equation to the Kirchhoff–Routh equation has been extensively studied; see [4, 6, 13, 14, 19] and the references therein.

Now we put the Euler equation and the Kirchhoff–Routh equation together, that is, we assume that the vorticity consists of both continuously distributed vorticity denoted by $\omega(x, t)$ and $k$ concentrated point vortices $x_i(t), i = 1, \ldots, k$. Then it is reasonable that the evolution of $\omega(x, t)$ and $x_i(t)$ obey the following system:

\[
\begin{aligned}
\partial_t \omega + u \cdot \nabla \omega &= 0, \\
\frac{d\omega}{dt} &= J \nabla G \ast \omega(x_i(t)) + \sum_{j=1, j \neq i}^k a_i J \nabla x_j G(x_i, x_j) - a_i J \nabla H(x_i), \\
u &= J \nabla G \ast \omega + \sum_{j=1}^k a_i J \nabla G(x_i, x_j),
\end{aligned}
\]  

(2.12)

which we call the vortex-wave system in bounded domains.

Let us now explain (2.12) briefly. The first equation in (2.12) means that evolution of the background vorticity $\omega(x, t)$ is influenced by the velocity field $J \nabla G \ast \omega$ ‘generated’ by itself and the velocity field $\sum_{j=1, j \neq i}^k a_i J \nabla x_j G(x_i, x_j)$ ‘generated’ by the $k$ point vortices with strength $a_i$, and the evolution of each $x_i(t)$ is influenced by the velocity field $J \nabla G \ast \omega(x_i(t))$ ‘generated’ by $\omega$ and the velocity $\sum_{j=1, j \neq i}^k a_i J \nabla x_j G(x_i, x_j)$ ‘generated’ by the other $k-1$ point vortices.
together with the boundary term $-a_i J \nabla H(x)$. If $\omega \equiv 0$, then (2.12) is exactly the Kirchhoff–Routh equation; if $a_i = 0$, $i = 1, \cdots, k$, then (2.12) becomes the vorticity form of the Euler equation.

### 2.3. Main results

In the rest of this paper we will restrict ourselves to the stationary vortex-wave system with a single point vortex, that is, $k = 1$ in (2.12). For simplicity we also assume that the point vortex has unit strength, namely, $a_1 = 1$.

More precisely, we will consider the following system:

$$\begin{cases}
J \nabla (G * \omega + G(x, \cdot)) \cdot \nabla \omega = 0, \\
\nabla G * \omega(x) - \nabla H(x) = 0.
\end{cases} \tag{2.13}$$

Since we are going to deal with vortex patch solutions which are discontinuous, it is necessary to introduce the weak formulation for the first equation in (2.13). To motivate the definition, let us assume that $\omega$ is a smooth solution. Then for any $\phi \in C_c^\infty (D)$, we have

$$\int_D \phi J \nabla (G * \omega + G(x, \cdot)) \cdot \nabla \omega \, dy = 0. \tag{2.14}$$

Now we claim that

$$\int_D \omega J \nabla (G * \omega + G(x, \cdot)) \cdot \nabla \phi \, dy = - \int_D \phi J \nabla (G * \omega + G(x, \cdot)) \cdot \nabla \omega \, dy = 0. \tag{2.15}$$

In fact, by the divergence theorem

$$\int_D \phi J \nabla G * \omega \cdot \nabla \omega \, dy = \int_D \phi \text{div}(\omega J \nabla G * \omega) \, dy$$

$$= \int_D \text{div}(\phi J \nabla G * \omega) \, dy - \int_D \omega J \nabla G * \omega \cdot \nabla \phi \, dy$$

$$= \int_{\partial D} \phi J \nabla G * \omega \cdot n \, dS - \int_D \omega J \nabla G * \omega \cdot \nabla \phi \, dy$$

$$= - \int_D \omega J \nabla G * \omega \cdot \nabla \phi \, dy, \tag{2.16}$$

where we have used the fact that $J \nabla G * \omega$ is a divergence-free vector field. To calculate the integral $\int_D \phi J \nabla G(x, \cdot) \cdot \nabla \omega \, dy$, the singularity of $\nabla G$ needs to be carefully dealt with. To this end we define $\Omega^a = \{ y \in D | G(x, y) > a \}$ and $D^a = D \setminus \Omega^a$ for $a > 0$. By the implicit function theorem, $\Omega^a$ is a simply connected domain with smooth boundary if $a > 0$ is sufficiently large. Again it follows from the divergence theorem that

$$\int_{D^a} \phi J \nabla G(x, \cdot) \cdot \nabla \omega \, dy = \int_{D^a} \phi \text{div}(\omega J \nabla G(x, \cdot)) \, dy$$

$$= \int_{D^a} \text{div}(\phi J \nabla G(x, \cdot)) \, dy - \int_{D^a} \omega J \nabla G(x, \cdot) \cdot \nabla \phi \, dy$$

$$= \int_{\partial D^a} \phi J \nabla G(x, \cdot) \cdot n \, dS - \int_{D^a} \omega J \nabla G(x, \cdot) \cdot \nabla \phi \, dy$$

$$= - \int_{D^a} \omega J \nabla G(x, \cdot) \cdot \nabla \phi \, dy. \tag{2.17}$$
On the other hand, since $\nabla G(x, \cdot) \in L^1(D)$, we have by Lebesgue’s dominated convergence theorem

$$
\lim_{a \to +\infty} \int_D \phi J \nabla G(x, \cdot) \cdot \nabla \omega \, dy = \int_D \phi J \nabla G(x, \cdot) \cdot \nabla \omega \, dy, \quad (2.18)
$$

and

$$
\lim_{a \to +\infty} \int_D \omega J \nabla G(x, \cdot) \cdot \nabla \phi \, dy = \int_D \omega J \nabla G(x, \cdot) \cdot \nabla \phi \, dy. \quad (2.19)
$$

Taking into account (2.17)–(2.19), we obtain

$$
\int_D \phi J \nabla G(x, \cdot) \cdot \nabla \omega \, dy = - \int_D \omega J \nabla G(x, \cdot) \cdot \nabla \phi \, dy. \quad (2.20)
$$

Hence we have proved (2.15).

In conclusion, if $\omega$ is a smooth solution to the system (2.13), then it must satisfy

$$
\int_D \omega J \nabla (G * \omega + G(x, \cdot)) \cdot \nabla \phi \, dy = 0, \quad \forall \phi \in C_0^\infty(D). \quad (2.21)
$$

Notice that the integral in (2.21) makes sense for any $\omega \in L^\infty(D)$ since $G * \omega \in C^1(D)$ and $\nabla G(x, \cdot) \in L^1(D)$, so we have the following definition:

**Definition 2.1.** $(\omega, x)$ is called a weak solution to (2.13) if $\omega \in L^\infty(D), x \in D$ and

$$
\begin{cases}
\int_D \omega J \nabla (G * \omega + G(x, \cdot)) \cdot \nabla \phi \, dy = 0, & \forall \phi \in C_0^\infty(D) \\
\nabla G * \omega(x) - \nabla H(x) = 0.
\end{cases} \quad (2.22)
$$

In this paper we focus on the steady vortex patch solution of (2.13), whose background vorticity $\omega$ has the form $\omega = a I_A$, where $a$ is a real number representing the strength of $\omega$ and $A \subset D$ is a Lebesgue measurable set.

The main result of this paper is as follows:

**Theorem 2.2.** Let $\mu$ be a positive real number satisfying $\frac{1}{\mu} > \frac{1}{|D|}$, and $\mathcal{N}^\mu$ be a rearrangement class defined by

$$
\mathcal{N}^\mu = \{\omega \in L^\infty(D) \mid \omega = \mu I_A, A \text{ is a measurable set in } D, |A| = 1\}. \quad (2.23)
$$

Then there exist $\omega^\mu \in \mathcal{N}^\mu$ and $x^\mu \in D$ such that $(\omega^\mu, x^\mu)$ is a weak solution to the stationary vortex-wave system (2.13). Moreover, there exists $b^\mu > 0$ such that

$$
\omega^\mu = \mu I_{G^\mu \omega^\mu + G(x^\mu, \cdot) > b^\mu}. \quad (2.24)
$$

**Remark 2.3.** If $\mu < \frac{1}{|D|}$, then $\mathcal{N}^\mu$ is empty. If $\mu = \frac{1}{|D|}$, then there is only one element in $\mathcal{N}^\mu$, that is, $\omega = \mu$. In this case $\omega$ is a smooth function and the first equation in (2.13) is satisfied for any $x \in D$, so we need only consider the second equation. Notice that $H|_{\partial D} = +\infty$, so we can always choose $x \in D$ such that $x$ is a maximum point, thus a critical point, for the function $G * \omega = H$.

As for the asymptotic behavior of $(\omega^\mu, x^\mu)$ as $\mu \to +\infty$, we can prove that, up to a subsequence, ‘most part’ of $\omega^\mu$ concentrates near a minimum point of $H$, say $x'$, and at the same time $x^\mu \to x'$. 
Theorem 2.4. Let \((\omega^\mu, x^\mu)\) be the weak solution to the stationary vortex-wave system (2.13) obtained in theorem 2.2, then up to a subsequence \(x^\mu \to x^*\) as \(\mu \to +\infty\), where \(x^*\) is a minimum point of \(H\). Moreover, there exists \(r_\mu\) such that as \(\mu \to +\infty\), \(r_\mu \to 0\) and

\[
\lim_{\mu \to +\infty} \int_{B_{r_\mu}(x^*)} \omega^\mu(x)dx = 1. \tag{2.25}
\]

Remark 2.5. Since \(\int_D \omega^\mu(x)dx = 1\), it is easy to check that \(\omega^\mu \to \delta(x^*)\) as \(\mu \to +\infty\) in the distributional sense, where \(\delta(x^*)\) is the Dirac measure located at \(x^*\). More precisely,

\[
\lim_{\mu \to +\infty} \int_D \omega^\mu(x)\phi(x)dx = \phi(x^*), \quad \forall \phi \in C_c^\infty(D). \tag{2.26}
\]

In fact,

\[
\left| \int_D \omega^\mu(x)\phi(x)dx - \phi(x^*) \right| \leq \int_D (\phi(x) - \phi(x^*))\omega^\mu(x)dx \leq \int_{B_{r_\mu}(x^*)} (\phi(x) - \phi(x^*))\omega^\mu(x)dx + \int_{D \setminus B_{r_\mu}(x^*)} (\phi(x) - \phi(x^*))\omega^\mu(x)dx \leq \sup_{x \in B_{r_\mu}(x^*)} |\phi(x) - \phi(x^*)| + 2\sup_{x \in D} |\phi(x)| \int_{D \setminus B_{r_\mu}(x^*)} \omega^\mu(x)dx \tag{2.27}
\]

which goes to 0 as \(\mu \to +\infty\), where we have used (2.25) and the continuity of \(\phi\) at \(x^*\).

Remark 2.6. When \(D\) is convex, \(H\) is a strictly convex function (see [3]), so there is only one minimum point for \(H\). In this case the phrase ‘up to a subsequence’ in theorem 2.4 can be removed.

3. Variational problem

Throughout this section we assume that \(\mu\) is a fixed positive real number. We will construct a family of steady vortex patch solutions to the Euler equation and analyze their asymptotic properties.

Let \(\lambda\) be a positive number. Define

\[
\mathcal{M}^\lambda = \{\omega \in L^\infty(D) \mid \omega = \omega_1 + \omega_2, \omega_1 \in \mathcal{N}^\mu, \omega_2 = \lambda \mathcal{I}_\lambda, \lambda |\mathcal{I}_\lambda| = 1, \text{supp}(\omega_1) \cap B = \emptyset\}, \tag{3.1}
\]

where \(\mathcal{N}^\mu\) is defined in theorem 2.2. For sufficiently large \(\lambda\), since \(\mu > \frac{1}{|D|}\), we know that \(\mathcal{M}^\lambda\) is not empty. Moreover, for any fixed \(\omega \in \mathcal{M}^\lambda\), we claim that \(\mathcal{M}^\lambda = \mathcal{R}(\omega)\). In fact, for any \(w \in \mathcal{M}^\lambda\), it is obvious that \(w\) and \(\omega\) have the same distributional function, that is, \(w \in \mathcal{R}(\omega)\), therefore \(\mathcal{M}^\lambda \subset \mathcal{R}(\omega)\). Conversely, for any \(w \in \mathcal{R}(\omega)\), \(|w = \lambda\) \cap \{w \neq \mu\} = |\{w = \lambda\} \cap \{w \neq \mu\}| = 0\), so we have \(w = \lambda I_{\{w = \lambda\}} + \mu I_{\{w = \mu\}}\), where \(|\{w = \lambda\}| = |\{w = \lambda\}| = 1/\lambda\) and \(|\{w = \mu\}| = |\{w = \mu\}| = 1/\mu\), which implies \(w \in \mathcal{M}^\lambda\). Consequently \(\mathcal{R}(\omega) \subset \mathcal{M}^\lambda\). Hereafter we will always assume \(\lambda\) to be sufficiently large.

The energy functional \(E\) on \(\mathcal{M}^\lambda\) is defined by

\[
E(\omega) = \frac{1}{2} \int_D \int_D G(x, y)\omega(x)\omega(y)dxdy, \quad \omega \in \mathcal{M}^\lambda, \tag{3.2}
\]

\[

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which represents the kinetic energy of an ideal fluid in $D$ with vorticity $\omega$.

Existence of a maximizer for $E$ relative to $\mathcal{M}^\lambda$ is an easy consequence of corollary 3.4 in [2]. Therein by choosing $L = -\Delta$, $E = \Psi$, $\mathcal{F} = \mathcal{M}^\lambda$ and $K$ as the Green’s operator, we have:

**Proposition 3.1.** There exists a maximizer for $E$ relative to $\mathcal{M}^\lambda$. Moreover, if $\omega^\lambda$ is a maximizer, then $\omega^\lambda = f(G * \omega^\lambda)$ a.e. in $D$ for some increasing function $f : \mathbb{R} \to \mathbb{R}$.

**Remark 3.2.** $\omega^\lambda$ is in fact a steady weak solution to the Euler equation, we refer the interested reader to [20] for a simple proof.

Let $\omega^\lambda \in \mathcal{M}^\lambda$ be a maximizer, then we can write $\omega^\lambda = \omega^\lambda_1 + \omega^\lambda_2$, where $\omega^\lambda_1 \in \mathcal{N}^\mu$, $\omega^\lambda_2 = \lambda f_B^\lambda \lambda |B^\lambda| = 1$, and $\text{supp}(\omega^\lambda_1) \cap B^\lambda = \emptyset$. For convenience we shall write $\psi^\lambda = G * \omega^\lambda$ and $\psi^\lambda_i = G * \omega^\lambda_i$, $i = 1, 2$.

**Lemma 3.3.** $\omega^\lambda_2 = \lambda I_{\psi^\lambda > c^\lambda}$, for some $c^\lambda > 0$.

**Proof.** Since $|\omega^\lambda_1| > 0$ and $\omega^\lambda = f(\psi^\lambda)$ a.e. in $D$, it follows that $\{t \in \mathbb{R} | f(t) = \lambda\}$ is not empty, then we can define $c^\lambda = \inf \{t \in \mathbb{R} | f(t) = \lambda\}$. By the fact that $f$ is an increasing function and $\psi^\lambda > 0$ in $D$ (by strong maximum principle), we have $c^\lambda > 0$.

By the definition of $\omega^\lambda$, $\omega^\lambda = f(\psi^\lambda) \equiv \lambda$ a.e. on $\{|x \in D | \psi^\lambda(x) > c^\lambda\}$, and $\omega^\lambda < \lambda$ a.e. on $\{|x \in D | \psi^\lambda(x) < c^\lambda\}$. Now we claim that $\omega^\lambda = 0$ a.e. on the set $\{|x \in D | \psi^\lambda(x) = c^\lambda\}$. Since $\psi^\lambda \in W^{2,p}(D)$ for any $p \in (1, +\infty)$, by the property of Sobolev functions (see [7], page 153, theorem 4.4) we obtain $D\psi^\lambda = 0$ a.e. on $\{|x \in D | \psi^\lambda(x) = c^\lambda\}$, then $D^2\psi^\lambda = 0$ a.e. on $\{|x \in D | \psi^\lambda(x) = c^\lambda\}$. Therefore we obtain $\omega^\lambda = -\Delta \psi^\lambda \equiv 0$ a.e. on $\{|x \in D | \psi^\lambda(x) = c^\lambda\}$.

In conclusion, we have proved that $\{|x \in D | \omega^\lambda(x) = \lambda\} = \{|x \in D | \psi^\lambda(x) > c^\lambda\}$, then by choosing $\lambda > \mu$ we have $B^\lambda = \{|x \in D | \psi^\lambda(x) > c^\lambda\}$, which is the desired result.

Now we begin to analyze the asymptotic behavior of $\omega^\lambda_2$ as $\lambda \to +\infty$. In this and the next section we shall use $C$ to denote various positive constants not depending on $\lambda$. First we give the lower bound of $E(\omega^\lambda)$.

**Lemma 3.4.** $E(\omega^\lambda) \geq -\frac{1}{\pi \varepsilon} \ln \varepsilon - C$, where $\varepsilon$ satisfies $\lambda \pi \varepsilon^2 = 1$.

**Proof.** We take the test function as follows: for any fixed $x_1 \in D$, define $\omega^\lambda = \bar{\omega}^\lambda_1 + \bar{\omega}^\lambda_2$, where $\bar{\omega}^\lambda_2 = \lambda f_{B^\lambda(x_1)}$, $\bar{\omega}^\lambda_1 \in \mathcal{N}^\mu$ and $\bar{\omega}^\lambda_1 = 0$ a.e. in $B_{\varepsilon}(x_1)$. It is easy to check that $\bar{\omega}^\lambda \in \mathcal{M}^\lambda$, so we have $E(\bar{\omega}^\lambda) \geq E(\omega^\lambda)$. By a simple calculation,

$$
E(\bar{\omega}^\lambda) = \frac{1}{2} \int_D \int_D G(x,y) \bar{\omega}^\lambda(x) \bar{\omega}^\lambda(y) dxdy
$$

$$
= \frac{1}{2} \int_D \int_D G(x,y) (\bar{\omega}^\lambda_1(x) + \bar{\omega}^\lambda_2(x)) (\bar{\omega}^\lambda_1(y) + \bar{\omega}^\lambda_2(y)) dxdy
$$

$$
= E(\bar{\omega}^\lambda_1) + E(\bar{\omega}^\lambda_2) + \frac{1}{2} \int_D \int_D G(x,y) \bar{\omega}^\lambda_1(x) \bar{\omega}^\lambda_2(y) dxdy + \frac{1}{2} \int_D \int_D G(x,y) \bar{\omega}^\lambda_2(x) \bar{\omega}^\lambda_1(y) dxdy
$$

$$
= E(\bar{\omega}^\lambda_1) + E(\bar{\omega}^\lambda_2) + \int_D \int_D G(x,y) \bar{\omega}^\lambda_1(x) \bar{\omega}^\lambda_2(y) dxdy, \tag{3.3}
$$

where we used the symmetry of the Green’s function, that is, $G(x,y) = G(y,x)$ for any $x, y \in D$. 

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Since $G \in L^1(D \times D)$, we have the following estimate for $E(\omega^1_\lambda)$:
\[
|E(\omega^1_\lambda)| = \frac{1}{2} \left| \int_D \int_D G(x,y)\omega^1_\lambda(x)\omega^1_\lambda(y) \, dx \, dy \right| \leq \frac{\mu^2}{2} \left| \int_D \int_D G(x,y) \, dx \, dy \right| \leq C.
\]  
(3.4)

For the term $\int_D \int_D G(x,y)\omega^1_\lambda(x)\omega^2_\lambda(y) \, dx \, dy$ in (3.3), by $L^p$ estimate we have
\[
\left| \int_D \int_D G(x,y)\omega^1_\lambda(x)\omega^2_\lambda(y) \, dx \, dy \right| = \left| \int_D G \ast \omega^1_\lambda(x)\omega^2_\lambda(y) \, dy \right| \leq C \int_D \omega^2_\lambda(y) \, dy = C.
\]  
(3.5)

It remains to estimate the lower bound of $E(\omega^2_\lambda)$,
\[
E(\omega^2_\lambda) = \frac{1}{2} \int_D \int_D G(x,y)\omega^2_\lambda(x)\omega^2_\lambda(y) \, dx \, dy \\
= -\frac{1}{4\pi} \int_D \int_D \ln |x - y|\omega^2_\lambda(x)\omega^2_\lambda(y) \, dx \, dy - \frac{1}{2} \int_D \int_D h(x,y)\omega^2_\lambda(x)\omega^2_\lambda(y) \, dx \, dy \\
= -\frac{\lambda^2}{4\pi} \int_{B_\epsilon(x_i)} \int_{B_\epsilon(x_i)} \ln |x - y| \, dx \, dy - \frac{1}{2} \int_D \int_D h(x,y)\omega^2_\lambda(x)\omega^2_\lambda(y) \, dx \, dy.
\]  
(3.6)

Since $|x - y| \leq 2\epsilon$ for $x, y \in B_\epsilon(x_i)$, we have
\[
-\frac{\lambda^2}{4\pi} \int_{B_\epsilon(x_i)} \int_{B_\epsilon(x_i)} \ln |x - y| \, dx \, dy \geq -\frac{\lambda^2}{4\pi} \int_{B_\epsilon(x_i)} \int_{B_\epsilon(x_i)} \ln |2\epsilon| \, dx \, dy \\
= -\frac{1}{4\pi} \ln \epsilon - \frac{1}{4\pi} \ln 2.
\]

On the other hand, by the continuity of $h(x,y)$ in $D \times D$, the integral $\int_{B_\epsilon(x_i)} \int_{B_\epsilon(x_i)} h(x,y) \, dx \, dy$ converges to $h(x_1, x_1)$, thus is uniformly bounded as $\lambda \to +\infty$. So we get
\[
E(\omega^2_\lambda) \geq -\frac{1}{4\pi} \ln \epsilon - C.
\]  
(3.7)

Using (3.3)–(3.5) and (3.7) we complete the proof.  

Now we define $T(\omega^\lambda) = \frac{1}{2} \int_D \omega^2_\lambda(x)(\psi^\lambda - c^\lambda)(x) \, dx$, which represents the kinetic energy of the fluid on $B^\lambda$. To simplify presentation we write $\zeta^\lambda = \psi^\lambda - c^\lambda$. By the fact that $\zeta^\lambda = 0$ on $\partial B^\lambda$, there holds
\[
T(\omega^\lambda) = \frac{1}{2} \int_{B^\lambda} \omega^2_\lambda(x)\zeta^\lambda(x) \, dx = \frac{1}{2} \int_{B^\lambda} |\nabla \zeta^\lambda(x)|^2 \, dx.
\]  
(3.8)

We have the following uniform upper bound estimate for $T$:

Lemma 3.5. $T(\omega^\lambda) \leq C$.

Proof. First by Hölder’s inequality, we have
\[
T(\omega^\lambda) = \frac{1}{2} \lambda \int_{B^\lambda} \zeta^\lambda(x) \, dx \leq \frac{1}{2} \lambda |B^\lambda|^{\frac{1}{2}} \left\{ \int_{B^\lambda} |\zeta^\lambda(x)|^2 \, dx \right\}^{\frac{1}{2}}.
\]
By the Sobolev embedding $W^{1,1}_0(D) \hookrightarrow L^2(D)$, we have
\[
\left\{ \int_{B^\lambda} |\zeta^\lambda(x)|^2 \,dx \right\}^{\frac{1}{2}} \leq \left\{ \int_B |(\zeta^\lambda)^+(x)|^2 \,dx \right\}^{\frac{1}{2}} \leq C \int_D |\nabla (\zeta^\lambda)^+(x)| \,dx,
\]
where $(\zeta^\lambda)^+(x) = \max\{0, \zeta^\lambda(x)\}$. It follows that
\[
T(\omega^\lambda) \leq C\lambda |B^\lambda|^{\frac{1}{2}} \int_D |\nabla (\zeta^\lambda)^+(x)| \,dx = C\lambda |B^\lambda|^{\frac{1}{2}} \int_R |\nabla \zeta^\lambda(x)| \,dx \leq C\lambda |B^\lambda| \left\{ \int_R |\nabla \zeta^\lambda(x)|^2 \,dx \right\}^{\frac{1}{2}}.
\]

Notice that $\lambda |B^\lambda| = \int_D \omega^\lambda_2(x) \,dx = 1$, we obtain
\[
T(\omega^\lambda) \leq C \left\{ \int_R |\nabla \zeta^\lambda(x)|^2 \,dx \right\}^{\frac{1}{2}}. \tag{3.9}
\]
By comparing (3.9) with (3.8) we get the desired result. \hfill \Box

**Lemma 3.6.** There exists $R_0 > 0$ such that $\text{diam}(\text{supp}(\omega^\lambda_1)) \leq R_0 \varepsilon$.

**Proof.** First we estimate the lower bound for $c^\lambda$. By the definition of $T(\omega^\lambda)$,
\[
E(\omega^\lambda) = T(\omega^\lambda) + \frac{1}{2} \int_D \omega^\lambda_1(x) \psi^\lambda(x) \,dx + \frac{c^\lambda}{2}. \tag{3.10}
\]
It is not difficult to check that $\int_D \omega^\lambda_1(x) \psi^\lambda(x) \,dx$ has a uniform upper bound. In fact,
\[
\int_D \omega^\lambda_1(x) \psi^\lambda(x) \,dx = \int_D \omega^\lambda_1(x) G*(\omega^\lambda_1 + \omega^\lambda_2)(x) \,dx
= \int_D \int_D G(x,y) \omega^\lambda_1(x) \omega^\lambda_1(y) \,dxdy + \int_D \int_D G(x,y) \omega^\lambda_1(x) \omega^\lambda_2(y) \,dxdy
\leq \mu^2 \int_D \int_D |G(x,y)| \,dxdy + |G*\omega^\lambda_1|_{L^\infty(D)}
\leq C. \tag{3.11}
\]

Now (3.10) together with lemmas 3.4 and 3.5 gives
\[
c^\lambda \geq -\frac{1}{2\pi} \ln \varepsilon - C. \tag{3.12}
\]
Now for any $x \in \text{supp}(\omega^\lambda_1)$, we have $\psi^\lambda(x) \geq c^\lambda$, that is,
\[
\int_D G(x,y) w^\lambda(y) \,dy \geq -\frac{1}{2\pi} \ln \varepsilon - C. \tag{3.13}
\]
Since $h(x, y)$ is bounded from below on $D \times D$, we have
\[
\int_D \ln \frac{1}{|x-y|} \omega^\lambda(y) \,dy + \ln \varepsilon \geq -C, \tag{3.14}
\]
or equivalently,
\[
\int_D \ln \frac{1}{|x-y|} \omega_1^\lambda(y)dy + \int_D \ln \frac{1}{|x-y|} \omega_2^\lambda(y)dy + \ln \varepsilon \geq -C. \tag{3.15}
\]

Notice that
\[
\left| \int_D \ln \frac{1}{|x-y|} \omega_1^\lambda(y)dy \right| \leq \mu \sup_{x \in D} \left| \int_D \ln |x-y|dy \right| \leq C, \tag{3.16}
\]
so we get
\[
\int_D \ln \frac{\varepsilon}{|x-y|} \omega_2^\lambda(y)dy \geq -C. \tag{3.17}
\]

Now let \( R > 1 \) be a positive number to be determined. We divide the integral in (3.17) into two parts, that is,
\[
\int_{B_{R\varepsilon}(x)} \ln \frac{\varepsilon}{|x-y|} \omega_2^\lambda(y)dy + \int_{D \setminus B_{R\varepsilon}(x)} \ln \frac{\varepsilon}{|x-y|} \omega_2^\lambda(y)dy \geq -C. \tag{3.18}
\]
By rearrangement inequality, the first integral in (3.18) can be estimated as follows:
\[
\int_{B_{R\varepsilon}(x)} \ln \frac{\varepsilon}{|x-y|} \omega_2^\lambda(y)dy \leq \lambda \int_{B_{R\varepsilon}(x)} \ln \frac{\varepsilon}{|y|}dy = \lambda \int_{B_{R}(0)} \ln \frac{\varepsilon}{|y|}dy = \frac{1}{2}. \tag{3.19}
\]
By comparing (3.18) with (3.19) we obtain
\[
\int_{D \setminus B_{R\varepsilon}(x)} \ln \frac{\varepsilon}{|x-y|} \omega_2^\lambda(y)dy \geq -C.
\]
We observe now that
\[
\int_{D \setminus B_{R\varepsilon}(x)} \ln \frac{\varepsilon}{|x-y|} \omega_2^\lambda(y)dy \leq \int_{D \setminus B_{R\varepsilon}(x)} \ln \frac{1}{R} \omega_2^\lambda(y)dy, \tag{3.20}
\]
therefore
\[
\int_{D \setminus B_{R\varepsilon}(x)} \omega_2^\lambda(y)dy \leq \frac{C}{\ln R}, \tag{3.21}
\]
which means
\[
\int_{B_{R\varepsilon}(x)} \omega_2^\lambda(y)dy \geq 1 - \frac{C}{\ln R}. \tag{3.22}
\]
Choosing \( R \) large such that \( 1 - \frac{C}{\ln R} > \frac{1}{2} \), we have
\[
\int_{B_{R\varepsilon}(x)} \omega_2^\lambda(y)dy > \frac{1}{2}. \tag{3.23}
\]
Since $x \in \text{supp}(\omega_1^\lambda)$ is arbitrary and $\int_D \omega_1^\lambda(y)dy = 1$, we get the desired result by choosing $R_0 = 2R$.

Up to now we have constructed a family of functions $\omega^\lambda_1$ and $\omega^\lambda_2$ and showed that $\text{diam}(\text{supp}(\omega_1^\lambda)) \to 0$ as $\lambda \to +\infty$. Now we are in a position to consider the limits of $\omega^\lambda_1$ and $\omega^\lambda_2$. To this end, define $x^\lambda$ to be the center of $\omega_1^\lambda$ by setting
\[
x^\lambda = \int_D x \omega_1^\lambda(x)dx.
\]
(3.24)

Up to a subsequence, we assume that for some $x^\mu \in \overline{D}$
\[
\lim_{\lambda \to +\infty} x^\lambda = x^\mu.
\]

On the other hand, since $\{\omega_1^\lambda\}$ is bounded in $L^\infty(D)$ (recall that $\mu$ is fixed in this section), up to a subsequence we assume that as $\lambda \to +\infty$
\[
\omega_1^\lambda \rightharpoonup \omega^\mu \quad \text{weakly star in } L^\infty(D)
\]
for some $\omega^\mu \in \overline{\mathcal{N}^\mu}$, where $\overline{\mathcal{N}^\mu}$ denotes the weak star closure of $\mathcal{N}^\mu$ in $L^\infty(D)$. By standard elliptic equation theory we also have as $\lambda \to +\infty$
\[
G \ast \omega_1^\lambda \to G \ast \omega^\mu \quad \text{in } C^{1,\alpha}(\overline{D}).
\]

We end this section by showing the following lemma which will be frequently used in the next section.

**Lemma 3.7.** We have

1. $|G \ast \omega_1^\lambda|_{L^\infty(D)} \leq C$, for some $C > 0$ not depending on $\lambda$.
2. $E(\omega_1^\lambda) = E(\omega^\mu) + o(1)$,
3. $\int_D G \ast \omega_1^\lambda(x) \omega_1^\lambda(x)dx = G \ast \omega^\mu(x^\mu) + o(1)$.

where $o(1)$ denotes quantities converging to 0 as $\lambda \to +\infty$.

**Proof.** To prove (1), it suffices to notice that $\omega_1^\lambda$ is bounded in $L^\infty(D)$, then the result follows from $L^\infty$ estimate and Sobolev embedding.

Now we turn to the proof of (2). By a simple calculation,
\[
\left|E(\omega_1^\lambda) - E(\omega^\mu)\right| = \frac{1}{2} \int_D \omega_1^\lambda G \ast \omega_1^\lambda dx - \frac{1}{2} \int_D \omega^\mu G \ast \omega^\mu dx
\]
\[
\leq \frac{1}{2} \int_D (G \ast \omega_1^\lambda - G \ast \omega^\mu)dx + \frac{1}{2} \int_D G \ast (\omega_1^\lambda - \omega^\mu)dx
\]
\[
\leq \frac{1}{2} \left|G \ast \omega_1^\lambda - G \ast \omega^\mu\right|_{L^\infty(D)} + o(1),
\]
(3.25)
which goes to 0 as $\lambda \to +\infty$.

To prove (2), noting that $\text{diam}(\text{supp}(\omega_2^\lambda)) \to 0$ and $x^\lambda \to x^\mu$, we can choose $r^\lambda$, $r^\mu \to 0$ as $\lambda \to +\infty$, such that $\text{supp}(\omega_2^\lambda) \subset B_{r^\lambda}(x^\mu)$. By the continuity of $G \ast \omega^\mu$ and the fact that $G \ast \omega_1^\lambda \to G \ast \omega^\mu$ in $L^\infty(D)$, it follows that
\[ \left| \int_D G * \omega^1_n(x) \omega^2_n(x) \, dx - G * \omega^\mu(x) \right| \]
\[ = \left| \int_D (G * \omega^1_n(x) - G * \omega^\mu(x)) \omega^2_n(x) \, dx \right| \]
\[ = \left| \int_{B_{\lambda}(x)} (G * \omega^1_n(x) - G * \omega^\mu(x)) \omega^2_n(x) \, dx \right| \]
\[ \leq \sup_{x \in B_{\lambda}(x)} |G * \omega^1_n(x) - G * \omega^\mu(x)| \]
\[ \leq \left| G * \omega^1_n(x) - G * \omega^\mu(x) \right| + \sup_{x \in B_{\lambda}(x)} |G * \omega^\mu(x) - G * \omega^\mu(x)| \]
\[ \to 0. \quad (3.26) \]

4. Proof of theorem 2.2

In this section we will give the proof of theorem 2.2. Before doing this we need to establish several preliminary lemmas first. We will show that the weakly star limit \( \omega^\mu \in \overline{N}^\mu \) of \( \omega^1_n \) actually belongs to \( \overline{N}^\mu \), \( x^\mu \in D \) actually lies in \( D \) and \((\omega^\mu, x^\mu)\) is a weak solution to the stationary vortex-wave system (2.13).

**Lemma 4.1.** Let \( \omega \in L^\infty(D), x \in D \), then \((\omega, x)\) is a weak solution of (2.13) if the following two conditions are satisfied

1. For any \( y \in D \), \( G * \omega(y) - H(y) \leq G * \omega(x) - H(x) \).
2. For any \( v \in \mathcal{R}(\omega) \),
   \[ E(v) + G * v(x) \leq E(\omega) + G * \omega(x). \quad (4.1) \]

**Proof.** Condition (1) in lemma 4.1 implies that \( x \) is a maximum point for the function \( G * \omega - H \) in \( D \), so \( \nabla G * \omega(x) - \nabla H(x) = 0 \).

In the following, for the sake of convenience we set
\[ F(v, y) = E(v) + G * v(y), \quad v \in \mathcal{R}(\omega), \quad y \in D. \quad (4.2) \]

For any given \( \phi \in C^\infty_0(D) \), define a family of \( C^1 \) transformations \( \Phi_t(x) : D \to D \) for \( t \in (-\infty, +\infty) \) by the following system of ordinary differential equations:

\[ \begin{cases} \frac{d\Phi_t(x)}{dt} = J \nabla \phi(\Phi_t(x)), & t \in \mathbb{R}, \\ \Phi_0(x) = x, \end{cases} \quad (4.3) \]

where \( J \) denotes the clockwise rotation through \( \frac{\pi}{2} \) as before. Note that (4.3) is solvable for all \( t \) since \( J \nabla \phi \) is a smooth vector field with compact support in \( D \). It is easy to see that \( J \nabla \phi \) is divergence-free, so by Liouville theorem (see [15], appendix 1.1) \( \Phi_t(x) \) is area-preserving, or equivalently for any measurable set \( A \subset D \)
\[ |\Phi_t(A)| = |A|. \quad (4.4) \]
Now define a family of test functions
\[ \omega^{(t)}(x) \triangleq \omega(t \Phi^{-1}(x)). \] (4.5)

Since \( \Phi_t \) is area-preserving, we have \( \omega^{(t)} \in \mathcal{R}(\omega) \), then condition (2) in lemma 4.1 implies that \( F(\omega^{(t)}, x) \) attains its maximum at \( t = 0 \), so \( \frac{\partial}{\partial t} F(\omega^{(t)}, x)|_{t=0} = 0 \). Expanding \( F(\omega^{(t)}, x_0) \) at \( t = 0 \) gives
\[
F(\omega^{(t)}, x) = \frac{1}{2} \int_D \int_D G(y, z) \omega(\Phi^{-1}(y)) \omega(\Phi^{-1}(z)) dydz + \int_D G(x, y) \omega(\Phi^{-1}(y)) dy \\
= \frac{1}{2} \int_D \int_D G(\Phi_t(y), \Phi_t(z)) \omega(y) \omega(z) dydz + \int_D G(x, \Phi_t(y)) \omega(y) dy \\
= \mathcal{E}(\omega) + t \int_D \omega(y) \nabla(G * \omega(y) + G(x, y)) \cdot J \nabla \phi(y) dy + o(t),
\]
as \( t \to 0 \). Here we used the symmetry of the Green’s function. So we have
\[
\int_D \omega(y) \nabla(G * \omega(y) + G(x, y)) \cdot J \nabla \phi(y) dy = 0, \ \forall \phi \in C_0^\infty(D),
\]
which completes the proof. \( \square \)

To apply lemma 4.1, we need to verify that \( (\omega^a, x^a) \) satisfies (1) and (2) in lemma 4.1. To this end, we need to obtain more information about \( (\omega^a, x^a) \). This is done by a series of lemmas to be given in the sequel.

**Lemma 4.2.** \( x^a \in D \).

**Proof.** By lemma 3.7 and the symmetry of the Green’s function,
\[
\mathcal{E}(\omega^a) = \frac{1}{2} \int_D \int_D G(x, y) (\omega^a_1 + \omega^a_2)(x)(\omega^a_1 + \omega^a_2)(y) dydx \\
= \mathcal{E}(\omega^a_1) + \mathcal{E}(\omega^a_2) + \int_D \int_D G(x, y) \omega^a_1(x) \omega^a_2(y) dydx \\
= \mathcal{E}(\omega^a) - \frac{1}{4\pi} \int_D \int_D \ln |x - y| \omega^a_1(x) \omega^a_2(y) dydx + G * \omega^a(x^a) - H(x^a) + o(1). \tag{4.6}
\]

To estimate the second term in (4.6), since by lemma 3.6 \( \text{supp}(\omega^a_1) \subseteq B_{\text{rec}}(x^a) \), we have
\[
- \frac{1}{4\pi} \int_D \int_D \ln |x - y| \omega^a_1(x) \omega^a_2(y) dydx = - \frac{1}{4\pi} \int_{B_{\text{rec}}(x^a)} \int_{B_{\text{rec}}(x^a)} \ln |x - y| \omega^a_1(y) \omega^a_2(x) dydx \\
= - \frac{1}{4\pi} \int_{B_{\text{rec}}(x^a)} \int_{B_{\text{rec}}(x^a)} \ln |x - y| \omega^a_1(x^a + y) \omega^a_2(x^a + x) dydx \\
\leq - \frac{\lambda^2}{4\pi} \int_{B_{\text{rec}}(x^a)} \int_{B_{\text{rec}}(x^a)} \ln |x - y| dydx \\
= - \frac{\lambda^2}{4\pi} \int_{B_{\text{rec}}(x^a)} \int_{B_{\text{rec}}(x^a)} \ln |x - y| dydx \\
\leq - \frac{\lambda^2}{4\pi} \ln \varepsilon + C, \tag{4.7}
\]
where $C > 0$ is a constant independent of $\varepsilon$ (independent of $\lambda$, equivalently). Here we have used Riesz’s rearrangement inequality (see [10], section 3.7) in deducing the first inequality and the fact $\ln |x - y| \in L^1(B_1(0) \times B_1(0))$ in deducing the second inequality. So we have

$$E(\omega^\lambda) \leq E(\omega^\mu) - \frac{1}{4\pi} \ln \varepsilon + C + G*\omega^\mu(x^\mu) - H(x^\mu) + o(1),$$

(4.8)

that is

$$E(\omega^\lambda) + \frac{1}{4\pi} \ln \varepsilon \leq E(\omega^\mu) + G*\omega^\mu(x^\mu) - H(x^\mu) + o(1).$$

(4.9)

If $x^\mu \in \partial D$, then $H(x^\mu) = +\infty$, which means that $E(\omega^\lambda) + \frac{1}{4\pi} \ln \varepsilon \to -\infty$ as $\lambda \to +\infty$, which is a contradiction to lemma 3.4.

**Lemma 4.3.** \(\sup_{v \in \mathcal{N}^\mu}(E(v) + G*\nu(x^\mu)) = \sup_{v \in \mathcal{N}^\mu}(E(v) + G*\nu(x^\mu))\).

**Proof.** First it is obvious that \(\sup_{v \in \mathcal{N}^\mu}(E(v) + G*\nu(x^\mu)) \leq \sup_{v \in \mathcal{N}^\mu}(E(v) + G*\nu(x^\mu))\).

On the other hand, for any $\omega \in \mathcal{N}^\mu$ we can choose a sequence $\{\omega^\mu_n\} \subset \mathcal{N}^\mu$ such that $\omega^\mu_n \to \omega$ weakly star in $L^\infty(D)$, then

$$E(\omega^\mu_n) + G*\omega^\mu_n(x^\mu) \to E(\omega) + G*\omega(x^\mu),$$

which gives \(\sup_{v \in \mathcal{N}^\mu}(E(v) + G*\nu(x^\mu)) \geq E(\omega) + G*\omega(x^\mu)\). Since $\omega \in \mathcal{N}^\mu$ is arbitrary, we have

$$\sup_{v \in \mathcal{N}^\mu}(E(v) + G*\nu(x^\mu)) \geq \sup_{v \in \mathcal{N}^\mu}(E(v) + G*\nu(x^\mu)).$$

This completes the proof.

**Lemma 4.4.** \(E(\omega^\mu) + G*\omega^\mu(x^\mu) = \sup_{\omega \in \mathcal{N}^\mu}(E(\omega) + G*\omega(x^\mu))\).

**Proof.** Recall that $\omega^\mu_1 = \lambda \mu^\mu$. By choosing $\nu^\lambda = \nu^1 + \omega^\mu_1$, such that $\nu^1 \in \mathcal{N}^\mu$, $\nu^1 \equiv 0$ a.e. on $B^\lambda$, it is obvious that $\nu^1 \in \mathcal{M}^\lambda$. As a consequence we have $E(\omega^\lambda) \geq E(\nu^\lambda)$, that is,

$$E(\omega^\lambda_1) + E(\omega^\mu_1) + \int_D G*\omega^\lambda_1(x)\omega^\mu_1(x)dx \geq E(\nu^1) + E(\omega^\mu_1) + \int_D G*\nu^1(x)\omega^\mu_1(x)dx,$$

(4.10)

which gives

$$E(\omega^\lambda_1) + \int_D G*\omega^\lambda_1(x)\omega^\mu_1(x)dx \geq E(\nu^1) + \int_D G*\nu^1(x)\omega^\mu_1(x)dx.$$  

(4.11)

By lemma 3.7 it follows that

$$E(\omega^\mu) + G*\omega^\mu(x^\mu) \geq E(\nu^1) + G*\nu^1(x^\mu) + o(1).$$

(4.12)

Since $\text{diam}(\text{supp}(\omega^\mu_1)) \to 0$ and $E$ is a continuous functional on $\mathcal{N}^\mu$, $\nu^1$ can be any element in $\mathcal{N}^\mu$ as $\lambda \to +\infty$, that is

$$E(\omega^\mu) + G*\omega^\mu(x^\mu) \geq E(v) + G*\nu(x^\mu), \ \forall v \in \mathcal{N}^\mu,$$

(4.13)
which, combined with lemma 4.3, leads to the desired result.

Lemma 4.5. $\omega^\mu \in \mathcal{N}^\mu$ and $\omega^\mu = \mu I_{\{G + \omega^\mu + G(l^\mu) > b^\mu\}}$ for some $b^\mu > 0$.

Proof. Define $\mathcal{F} = \{\omega \in L^\infty(D) | 1 \leq \omega \leq \mu, \int_D \omega(x)dx = 1\}$. Then for $\mathcal{F}$ we have the following two claims.

Claim 1. $\overline{\mathcal{N}^\mu} \subset \mathcal{F}$.

Proof of claim 1. By the definition of $\overline{\mathcal{N}^\mu}$ it suffices to show that $\mathcal{F}$ is closed in the weak star topology in $L^\infty(D)$. Let $\omega^\mu \in \mathcal{F}$, $\omega^\mu \rightarrow \omega^*$ weakly star in $L^\infty(D)$, that is,

$$\lim_{n \to +\infty} \int_D \omega^\mu(x)\phi(x)dx = \int_D \omega^*(x)\phi(x)dx, \forall \phi \in L^1(D),$$

(4.14)

it suffices to show that $\omega^* \in \mathcal{F}$.

First by choosing $\phi(x) \equiv 1$ we have

$$\lim_{n \to +\infty} \int_D \omega^\mu(x)dx = \int_D \omega^*(x)dx = 1.$$

Now we prove $0 \leq \omega^* \leq \mu$ by contradiction. Suppose that $|\{\omega^* > \mu\}| > 0$, then there exists $\varepsilon_0 > 0$ such that $|\{\omega^* > \mu + \varepsilon_0\}| > 0$. Denote $A = \{\omega^* > \mu + \varepsilon_0\}$, then for $\phi = I_A$ we have

$$0 = \lim_{n \to +\infty} \int_D (\omega^* - \omega^\mu)(x)\phi(x)dx = \lim_{n \to +\infty} \int_A \omega^*(x) - \omega^\mu(x)dx.$$

On the other hand

$$\lim_{n \to +\infty} \int_A (\omega^* - \omega^\mu)(x)dx \geq \varepsilon_0|A| > 0,$$

which is a contradiction. So we have $\omega^* \leq \mu$ a.e. on $D$.

A similar argument suggests that $\omega^* \geq 0$ a.e. on $D$, which completes the proof of claim 1.

Claim 2. There exists $\tilde{\omega} \in \mathcal{F}$ such that $E(\tilde{\omega}) + G(\tilde{\omega}) = \sup_{\omega \in \mathcal{F}} E(\omega) + G(\omega)$, moreover, any maximizer $\tilde{\omega}$ has the form $\tilde{\omega} = \mu I_{\{G + \omega^\mu + G(l^\mu) > b^\mu\}}$ for some $b^\mu > 0$.

Proof of claim 2. First we show that $\sup_{\omega \in \mathcal{F}} E(\omega) + G(\omega) < +\infty$. In fact, for any $\omega \in \mathcal{F}$,

$$E(\omega) + G(\omega) = \frac{1}{2} \int_D \int_D G(x,y)\omega(x)\omega(y)dy + \int_D G(x,\cdot)\omega(\cdot)dy \leq \frac{\mu^2}{2} \int_D \int_D |G(x,y)|dy + \mu \int_D G(x,\cdot)dy \leq C,$$

(4.15)

where $C$ is a positive number not depending on $\omega$ (may depending on $\mu$). Now we choose $\omega^\mu \in \mathcal{F}$ such that $\omega^\mu \rightarrow \tilde{\omega}$ and $E(\omega^\mu) + G(\omega^\mu) \rightarrow \sup_{\omega \in \mathcal{F}} E(\omega) + G(\omega)$. An argument similar to the one used in lemma 3.7 gives
\[
E(\tilde{\omega}) + G \ast \tilde{\omega}(x^\mu) = \sup_{\omega \in \mathcal{F}} (E(\omega) + G \ast \omega(x^\mu)).
\]

(4.16)

Now we prove that \( \tilde{\omega} \) is a vortex patch with the form \( \tilde{\omega} = \mu \lambda_{(G \ast \tilde{\omega} + G(x^\mu)) > b^\mu} \) for some \( b^\mu > 0 \). Define a family of test functions \( \omega^{(s)}(x) = \tilde{\omega} + s[z_0(x) - z_1(x)] \), \( s > 0 \), where \( z_0 \) and \( z_1 \) satisfy

\[
\begin{cases}
z_0, z_1 \in L^\infty(D), z_0, z_1 \geq 0, \int_D z_0 \, dx = \int_D z_1 \, dx, \\
z_0 = 0 & \text{in } D \setminus \{\tilde{\omega} \leq \mu - \delta\}, \\
z_1 = 0 & \text{in } D \setminus \{\tilde{\omega} \geq \delta\},
\end{cases}
\]

(4.17)

here \( \delta \) is any positive number. Note that for fixed \( z_0, z_1 \) and \( \delta, \omega^{(s)} \in \mathcal{F} \) provided \( s \) is sufficiently small (depending on \( \delta, z_0, z_1 \)). So we have

\[
\frac{d}{ds} [E(\omega^{(s)}) + G \ast \omega^{(s)}(x^\mu)] \bigg|_{s=0^+} \leq 0,
\]

(4.18)

which gives

\[
\sup_{\{\tilde{\omega} < \mu\}} (G \ast \tilde{\omega} + G(x^\mu, \cdot)) \leq \inf_{\{\tilde{\omega} > 0\}} (G \ast \tilde{\omega} + G(x^\mu, \cdot)).
\]

(4.19)

Now it is obvious that there exists \( r > 0 \) such that \( \tilde{\omega} \equiv \mu \) a.e. in \( B_r(x^\mu) \) (otherwise the left hand side of (4.19) equals \(+ \infty\)). Moreover, we can choose \( r \) sufficiently small such that

\[
\inf_{\{\tilde{\omega} > 0\} \cap D_r} (G \ast \tilde{\omega} + G(x^\mu, \cdot)) = \inf_{\{\tilde{\omega} > 0\} \cap D_r} (G \ast \tilde{\omega} + G(x^\mu, \cdot)),
\]

(4.20)

where \( D_r = D \setminus B_r(x^\mu) \). Then we have

\[
\sup_{\{\tilde{\omega} < \mu\} \cap D_r} (G \ast \tilde{\omega} + G(x^\mu, \cdot)) \leq \inf_{\{\tilde{\omega} > 0\} \cap D_r} (G \ast \tilde{\omega} + G(x^\mu, \cdot)).
\]

(4.21)

Since \( D_r \) is connected (for sufficiently small \( r \)) and \( \{\tilde{\omega} < \mu\} \cap D_r \cup \{\tilde{\omega} > 0\} \cap D_r = D_r \), we have \( \{\tilde{\omega} < \mu\} \cap D_r \cap \{\tilde{\omega} > 0\} \cap D_r \neq \emptyset \), then by the continuity of \( G \ast \tilde{\omega} + G(x^\mu, \cdot) \) on \( D_r \),

\[
\sup_{\{\tilde{\omega} < \mu\} \cap D_r} (G \ast \tilde{\omega} + G(x^\mu, \cdot)) = \inf_{\{\tilde{\omega} > 0\} \cap D_r} (G \ast \tilde{\omega} + G(x^\mu, \cdot)).
\]

(4.22)

Now define

\[
b^\mu = \sup_{\{\tilde{\omega} < \mu\} \cap D_r} (G \ast \tilde{\omega} + G(x^\mu, \cdot)) = \inf_{\{\tilde{\omega} > 0\} \cap D_r} (G \ast \tilde{\omega} + G(x^\mu, \cdot)).
\]

(4.23)

By maximum principle it is easy to see that \( \mu > 0 \), and it is also obvious that

\[
\begin{cases}
\tilde{\omega} = 0 & \text{a.e. in } \{G \ast \tilde{\omega} + G(x^\mu, \cdot) < b^\mu\} \cap D_r, \\
\tilde{\omega} = \mu & \text{a.e. in } \{G \ast \tilde{\omega} + G(x^\mu, \cdot) > b^\mu\} \cap D_r.
\end{cases}
\]

(4.24)

On \( \{G \ast \tilde{\omega} + G(x^\mu, \cdot) = b^\mu\} \cap D_r \), we have \( \nabla(G \ast \tilde{\omega} + G(x^\mu, \cdot)) = 0 \) a.e., which gives \( \tilde{\omega} = -\Delta(G \ast \tilde{\omega}) = -\Delta(G \ast \tilde{\omega} + G(x^\mu, \cdot)) = 0 \). Now it remains to show that \( G \ast \tilde{\omega} + G(x^\mu, \cdot) > b^\mu \) on \( B_r(x^\mu) \). This is an easy consequence of the maximum principle. In fact, by (4.20)
\[ b^\mu = \inf_{\{\omega > 0\} \cap \Omega} (G * \tilde{\omega} + G(x^\mu, \cdot)), \]
\[ = \inf_{\{\omega > 0\}} (G * \tilde{\omega} + G(x^\mu, \cdot)), \]
\[ \leq \inf_{B_r(x^\mu)} (G * \tilde{\omega} + G(x^\mu, \cdot)) \]
\[ \leq \inf_{\partial B_r(x^\mu)} (G * \tilde{\omega} + G(x^\mu, \cdot)), \]  
\[(4.25)\]

then by the strong maximum principle we have \( G * \tilde{\omega} + G(x^\mu, \cdot) > b^\mu \) on \( B_r(x^\mu) \).

In conclusion, we have proved that \( \tilde{\omega} \) has the form \( \tilde{\omega} = \mu_1^{(G + \tilde{\omega} + G(x^\mu, \cdot) > b^\mu)} \) for some \( b^\mu > 0 \), which completes the proof of claim 2.

Now we proceed to prove lemma 4.5. By claim 2 it is easy to see that
\[ \sup_{\omega \in N^\mu} (E(\omega) + G * \omega(x^\mu)) = \sup_{\omega \in F} (E(\omega) + G * \omega(x^\mu)), \]
\[(4.26)\]

therefore we obtain
\[ E(\omega^\mu) + G * \omega^\mu(x^\mu) = \sup_{\omega \in F} (E(\omega) + G * \omega(x^\mu)). \]

Using claim 2 again we get the desired result.

Remark 4.6. Lemma 4.5 is essential to this paper. We remark that corollary 3.4 in [2] can not be applied here anymore since \( \nabla^2 G \) is not a locally integrable function. The proof we give here is based on the idea of Turkington in [20] with modifications.

Now we are ready to prove theorem 2.2.

Proof of theorem 2.2. First by lemmas 4.4 and 4.5, \( \omega^\mu \) satisfies (2) in lemma 4.1 and has the form \( \omega^\mu = \mu_1^{(G + \tilde{\omega} + G(x^\mu, \cdot) > b^\mu)} \) for some \( b^\mu > 0 \). It suffices to show that \( x^\mu \) satisfies (1) in lemma 4.1.

Fix \( x_1 \in D \) and define \( \nu^\lambda = \nu_1^\lambda + \nu_2^\lambda \), where \( \nu_2^\lambda = \lambda I_{B_r(x_1)} \), \( \nu_1^\lambda \in N^\mu \) and \( \nu_1^\lambda = 0 \) a.e. on \( B_r(x_1) \). It is easy to check that \( \nu^\lambda \in \mathcal{M} \), so we have \( E(\omega^\lambda) \geq E(\nu^\lambda) \), that is,
\[ E(\omega_1^\lambda) + E(\omega_2^\lambda) + \int_D G * \omega_1^\lambda(x) \omega_2^\lambda(x) \, dx \geq E(\nu_1^\lambda) + E(\nu_2^\lambda) + \int_D G * \nu_1^\lambda(x) \nu_2^\lambda(x) \, dx, \]
\[(4.27)\]

then by lemma 3.7 for \( \lambda \) sufficiently large we have
\[ E(\omega^\mu) - \frac{1}{4\pi} \int_D \int_D \ln |x - y| \omega_2^\lambda(x) \omega_2^\lambda(y) \, dx \, dy - H(x^\mu) + G * \omega^\mu(x^\mu) + o(1) \]
\[ \geq E(\nu_1^\lambda) - \frac{1}{4\pi} \int_D \int_D \ln |x - y| \nu_2^\lambda(x) \nu_2^\lambda(y) \, dx \, dy - H(x_1) + G * \nu_1^\lambda(x_1). \]
\[(4.28)\]

On the other hand, by Riesz’s rearrangement inequality (see [10], section 3.7),
\[ -\frac{1}{4\pi} \int_D \int_D \ln |x - y| \omega_2^\lambda(x) \omega_2^\lambda(y) \, dx \, dy \leq -\frac{1}{4\pi} \int_D \int_D \ln |x - y| \nu_2^\lambda(x) \nu_2^\lambda(y) \, dx \, dy. \]
\[(4.29)\]

So we have
\[ E(\omega^\mu) - H(x^\mu) + G * \omega^\mu(x^\mu) + o(1) \geq E(\nu_1^\lambda) - H(x_1) + G * \nu_1^\lambda(x_1). \]
\[(4.30)\]
Again, since $E$ is a continuous functional on $N^\mu$ and $|B_\varepsilon(x_1)| \to 0$, $\nu^\lambda_1$ can be any element in $N^\mu$ as $\lambda \to +\infty$, that is,
\[ E(\omega^\mu) - H(x^\mu) + G * \omega^\mu(x^\mu) \geq E(v) - H(x_1) + G * v(x_1), \ \forall v \in N^\mu. \] 
(4.31)

In particular, we can choose $v = \omega^\mu$, then it follows
\[ -H(x^\mu) + G * \omega^\mu(x^\mu) \geq -H(x_1) + G * \omega^\mu(x_1), \ \forall x_1 \in D, \] 
(4.32)
which means that $x^\mu$ satisfies (1) in lemma 4.1. Therefore we complete the proof. \[ \Box \]

5. Proof of theorem 2.4

Up to now we have constructed $(\omega^\mu, x^\mu)$ as a steady vortex patch solution to the vortex-wave system for fixed $\mu$. Now we consider the asymptotic behavior of $(\omega^\mu, x^\mu)$ when $\mu \to +\infty$. As has been stated in theorem 2.4, we will show that both the support of $\omega^\mu$ and $x^\mu$ converge to a minimum point of $H$, which is a stationary solution to the Kirchhoff–Routh equation.

In this section we shall use $C$ to denote various positive numbers independent of $\mu$. We first establish several lemmas which can deduce theorem 2.4.

Lemma 5.1. For any $\omega \in N^\mu, x \in D$, we have
\[ E(\omega) + G * \omega(x) - H(x) \leq E(\omega^\mu) + G * \omega^\mu(x^\mu) - H(x^\mu). \]

Proof. For fixed $x \in D$, define a family of test functions $\nu^\lambda = \nu^\lambda_1 + \nu^\lambda_2, \nu^\lambda_2 \in M^\lambda, \nu^\lambda_1 \in N^\mu$ and $\nu^\lambda_1 = 0$ a.e. on $B_\varepsilon(x)$. It is easy to check $\nu^\lambda \in M^\lambda$, then by the definition of $\omega^\lambda$ we have $E(\nu^\lambda) \leq E(\omega^\lambda)$, that is,
\[ E(\nu^\lambda_1) + E(\nu^\lambda_2) + \int_D G * \nu^\lambda_1(y) \nu^\lambda_2(y) dy \leq E(\omega^\lambda_1) + E(\omega^\lambda_2) + \int_D G * \omega^\lambda_1(y) \omega^\lambda_2(y) dy. \]
(5.1)

Again by lemma 3.7
\[ E(\nu^\lambda_1) - \frac{1}{4\pi} \int_D \int_D \ln |y - z| \nu^\lambda_1(y) \nu^\lambda_2(z) dy dz - H(x) + G * \nu^\lambda_1(x) \]
\[ \leq E(\omega^\mu) - \frac{1}{4\pi} \int_D \int_D \ln |y - z| \omega^\mu_1(y) \omega^\mu_2(z) dy dz - H(x^\mu) + G * \omega^\mu(x^\mu) + o(1). \]
(5.2)

where $o(1) \to 0$ as $\lambda \to +\infty$. Using Riesz’s rearrangement inequality and taking into account (5.2) we have
\[ E(\nu^\lambda_1) - H(x) + G * \nu^\lambda_1(x) \leq E(\omega^\mu) - H(x^\mu) + G * \omega^\mu(x^\mu) + o(1). \]
(5.3)

As $\lambda \to +\infty, \nu^\lambda_1$ can be any element in $N^\mu$, so we obtain
\[ E(\omega) - H(x) + G * \omega(x) \leq E(\omega^\mu) - H(x^\mu) + G * \omega^\mu(x^\mu), \ \forall (\omega, x) \in (N^\mu, D). \]
(5.4)

Remark 5.2. One can also maximize $E(\omega) + G * \omega(x) - H(x)$ for $(\omega, x) \in (N^\mu, D)$ to obtain steady solution to the vortex-wave system, but it is much more interesting to construct solutions from the Euler equation, because the vortex-wave itself is an approximation of the Euler equation when a part of the vorticity is sufficiently concentrated.
In the following $s$ will be the positive number defined by $\mu \pi s^2 = 1$.

**Lemma 5.3.** There exists $\delta_0 > 0$, not depending on $\mu$, such that $\text{dist}(x^\mu, \partial D) > \delta_0$.

**Proof.** Fix $x_1 \in D$ and define $\bar{\omega}^\mu = \mu B_{x_1}$, then $\bar{\omega}^\mu \in \mathcal{N}^\mu$. By lemma 5.1

$$E(\bar{\omega}^\mu) + G * \bar{\omega}^\mu(x_1) - H(x_1) \leq E(\omega^\mu) + G * \omega^\mu(x^\mu) - H(x^\mu). \quad (5.5)$$

Using Riesz’s rearrangement inequality we get

$$- H(x_1) - 2H(x_1) - H(x_1) + o(1) \leq - \frac{1}{2} \int_D \int_D h(x, y) \omega^\mu(x) \omega^\mu(y) dx \ dy + \int_D h(x^\mu, y) \omega^\mu(y) dy - H(x^\mu), \quad (5.6)$$

since $h$ is bounded from below in $D \times D$, we have

$$H(x^\mu) \leq C, \quad (5.7)$$

then we get the desired result by the fact that $\lim_{x \to \partial D} H(x) = +\infty$. \hfill \square

**Lemma 5.4.** $G * \omega^\mu(x^\mu) \geq - \frac{1}{\pi} \ln s - C$.

**Proof.** Since $\text{dist}(x^\mu, \partial D) > \delta_0$, we can define $\bar{\omega}^\mu = \mu B_{x_1} \in \mathcal{N}^\mu$. Then by lemma 4.4

$$E(\bar{\omega}^\mu) + G * \bar{\omega}^\mu(x^\mu) \leq E(\omega^\mu) + G * \omega^\mu(x^\mu). \quad (5.8)$$

Again by Riesz’s rearrangement inequality we have

$$- \frac{1}{2} \int_D \int_D h(x, y) \bar{\omega}^\mu(x) \bar{\omega}^\mu(y) dx \ dy - \frac{1}{2\pi} \int_D \ln |x^\mu - y| \bar{\omega}^\mu(y) dy - \int_D h(x^\mu, y) \bar{\omega}^\mu(y) dy \leq - \frac{1}{2} \int_D \int_D h(x, y) \omega^\mu(x) \omega^\mu(y) dx \ dy + G * \omega^\mu(x^\mu), \quad (5.9)$$

since $h$ is bounded from below in $D \times D$ and $x^\mu$ is away from $\partial D$, we get

$$G * \omega^\mu(x^\mu) \geq - C - H(x^\mu) - \frac{1}{2\pi} \int_D \ln |x^\mu - y| \bar{\omega}^\mu(y) dy - 2H(x^\mu)$$

$$\geq - \frac{\mu}{2\pi} \int_{B(0)} \ln |y| dy - C$$

$$\geq - \frac{1}{2\pi} \ln s - C, \quad (5.10)$$

where we have used $\int_{B(0)} \ln |y| dy = \pi s^2 (\ln s - \frac{1}{2})$. \hfill \square

**Lemma 5.5.** There exists $\rho^\mu$ satisfying $\rho^\mu \to 0$ and $\int_{B_{\rho^\mu}(0)} \omega^\mu(x) dx \to 1$ as $\mu \to +\infty$.

**Proof.** By lemma 5.4,

$$- \frac{1}{2\pi} \int_D \ln |x^\mu - y| \omega^\mu(y) dy - \int_D h(x^\mu, y) \omega^\mu(y) dy \geq - \frac{1}{2\pi} \ln s - C, \quad (5.11)$$

since $h$ is bounded from below in $D \times D$, we get...
\int_D \ln \frac{s}{|x^\rho - y|} \omega^\mu (y) dy \geq -C. \quad (5.12)

Now let \( R > 1 \) be a positive real number to be determined. We have
\[
\int_{B_R(x')} \ln \frac{s}{|x^\rho - y|} \omega^\mu (y) dy + \int_{D \setminus B_R(x')} \ln \frac{s}{|x^\rho - y|} \omega^\mu (y) dy \geq -C. \quad (5.13)
\]

Observing that
\[
\int_{B_R(x')} \ln \frac{s}{|x^\rho - y|} \omega^\mu (y) dy \leq \mu \int_{B_1(x')} \ln \frac{s}{|x^\rho - y|} dy = \frac{1}{2},
\]
we get
\[
\int_{D \setminus B_R(x')} \ln \frac{s}{R^\rho} \omega^\mu (y) dy \geq \int_{D \setminus B_R(x')} \ln \frac{s}{|x^\rho - y|} \omega^\mu (y) dy \geq -C, \quad (5.15)
\]
which gives
\[
\int_{D \setminus B_R(x')} \omega^\mu (y) dy \leq \frac{C}{\ln R}. \quad (5.16)
\]

Since \( \int_D \omega^\mu (x) dx = 1 \), we obtain
\[
1 \geq \int_{B_R(x')} \omega^\mu (y) dy \geq 1 - \frac{C}{\ln R}. \quad (5.17)
\]

Now the lemma is proved by choosing \( R = s^{-\frac{1}{2}} \) and \( \rho^\mu = s^{\frac{1}{2}} \).

Since \( x^\mu \) is bounded and away from \( \partial D \), we assume that \( x^\mu \to x^* \in D \) (up to a subsequence) as \( \mu \to +\infty \). An argument similar to the one used in remark 2.5 shows that \( \omega^\mu \to \delta(x^*) \) in the sense of distribution.

**Lemma 5.6.** \( H(x^*) = \min_{x \in D} H(x) \).

**Proof.** Since \( H = +\infty \) on \( \partial D \), there exists \( x_1 \) such that \( H(x_1) = \min_{x \in D} H(x) \). It suffices to show \( H(x_1) \geq H(x^*) \). Define \( \tilde{\omega}^\mu = \mu_{B_{x_1}} \in \mathcal{N}^\mu \), then by lemma 5.1
\[
E(\tilde{\omega}^\mu) + G \ast \tilde{\omega}^\mu (x_1) - H(x_1) \leq E(\omega^\mu) + G \ast \omega^\mu (x^*) - H(x^*), \quad (5.18)
\]
that is,
\[
- \frac{1}{4\pi} \int_D \int_D \ln |x - y| \tilde{\omega}^\mu (x) \tilde{\omega}^\mu (y) dx dy - \frac{1}{2} \int_D h(x, y) \tilde{\omega}^\mu (x) \tilde{\omega}^\mu (y) dy dx \\
- \frac{1}{2\pi} \int_D \ln |x - y| \tilde{\omega}^\mu (y) dy - \int_D h(x, y) \tilde{\omega}^\mu (y) dy - H(x_1) \\
\leq - \frac{1}{4\pi} \int_D \int_D \ln |x - y| \omega^\mu (x) \omega^\mu (y) dx dy - \frac{1}{2} \int_D h(x, y) \omega^\mu (x) \omega^\mu (y) dy dx \\
- \frac{1}{2\pi} \int_D \ln |x^\rho - y| \omega^\mu (y) dy - \int_D h(x^\rho, y) \omega^\mu (y) dy - H(x^*). \quad (5.19)
\]

Taking the limit in (5.19) and using rearrangement inequality we obtain
\( H(x_1) \geq H(x^*) \), 

(5.20)

which is the desired result.

Having established the preliminary lemmas, we are now ready to prove theorem 2.4.

**Proof of theorem 2.4.** By choosing \( r_\mu = \rho_\mu + |x_\mu - x^*| \), theorem 2.4 is an easy consequence of lemmas 5.5 and 5.6.

**Remark 5.7.** There may be a better convergence for \( \omega_\mu \), that is, the support of \( \omega_\mu \) shrinks to \( x^* \) as \( \mu \to +\infty \), but we have not yet proved this. The main difficulty to estimate the size of \( \text{supp}(\omega_\mu) \) is that the mutual interaction energy between the background vorticity and the point vortex is very large, and energy estimate does not provide enough information anymore.

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