UNIFORM UPPER BOUND
FOR A STABLE MEASURE OF A SMALL BALL

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Abstract
The authors of [1] stated the following conjecture: Let $\mu$ be a symmetric $\alpha$-stable measure on a separable Banach space and $B$ a centered ball such that $\mu(B) \leq b$. Then there exists a constant $R(b)$, depending only on $b$, such that $\mu(tB) \leq R(b)t\mu(B)$ for all $0 < t < 1$. We prove that the above inequality holds but the constant $R$ must depend also on $\alpha$.

Recently, the authors of [1] proved the following (Theorem 6.4 in [1]): Let $\mu$ be a symmetric $\alpha$-stable measure, $0 < \alpha \leq 2$, on a separable Banach space, fix $b < 1$, and let $B$ denote a centered ball such that $\mu(B) \leq b$. Then there exists a constant $R(b) = \frac{3}{b\sqrt{1-b}}$, depending only on $b$, such that for all $0 \leq t \leq 1$

$$\mu(tB) \leq R(b)t^{\alpha/2}\mu(B).$$

(1)

Of course, for small values of $t$, the quantity $t^{\alpha/2}$ is much larger than $t$. The authors of [1] stated in their Conjecture 7.4 that (1) is true for all symmetric $\alpha$-stable measures with $t$ instead of $t^{\alpha/2}$ and some $R(b)$ depending only on $b$.

In our earlier paper [3], we also gave an estimate of a stable measure of a small ball. Namely, we proved the following.

Let $\mu$ be a symmetric $\alpha$-stable measure, $0 < \alpha \leq 2$, on a separable Banach space, put $B = \{x : \|x\| \leq 1\}$, let $0 < r < \alpha$ and suppose that $\mu$ is so normalized that $\int \|x\|^r \mu(dx) = 1$. Then there exists a constant $K = K(\alpha, r)$ such that for all $0 \leq t \leq 1$

$$\mu(tB) \leq K(\alpha, r)t.$$

(2)

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Some estimates of $K(\alpha, r)$ were also given in [3], we recall one of them in the final Remark.

Some normalization of $\mu$ is needed, as we will show in the sequel (see Example), in the paper [3] we chose the normalizing condition $\int \|x\|^r \mu(dx) = 1$. But proving the inequality (2), we also obtained the inequality

$$\mu(tB) \leq K(\alpha, r)[1 - \mu(B)]^{-1/r} t.$$  \hspace{1cm} (3)

In this note we will show that using (3) we can prove an estimate that is very close to the above-mentioned conjecture, however, the constant $R(b)$ must depend also on $\alpha$.

The following is a generalization of (1).

**Theorem 1** Let $\mu$ be a symmetric $\alpha$-stable measure, $0 < \alpha \leq 2$, on a separable Banach space $F$. Then for every closed, symmetric, convex set $B \subset F$ and for each $b < 1$ there exists $R(\alpha, b)$ such that for all $0 \leq t \leq 1$

$$\mu(tB) \leq R(\alpha, b) t \mu(B), \text{ if } \mu(B) \leq b.$$  \hspace{1cm} (4)

First we show that the constant $R$ must depend on $\alpha$.

**Example.** Suppose that there exists positive function $R(b)$ that fulfills (4), does not depend on $\alpha$ and is bounded on every closed subinterval of $(0, 1)$ Let $X_\alpha$ be an $\alpha$-stable random variable with the characteristic function $e^{-|t|^\alpha}$. It is known (see e.g. [4]) that

$$|X_\alpha|^\alpha \overset{d}{\to} \frac{1}{W}, \text{ as } \alpha \to 0+,$$  \hspace{1cm} (5)

where $W$ is a random variable having the exponential distribution with mean 1. Consider one-dimensional ball $B = [-1, 1]$. From (5) we infer that

$$b_\alpha = P(X_\alpha \in B) = P(-1 \leq X_\alpha \leq 1) = P(|X_\alpha|^\alpha \leq 1) \to_{\alpha \to 0} P\left(\frac{1}{W} \leq 1\right) = \frac{1}{e}.$$  

Denote by $\mu$ the distribution of $X_\alpha$. It is easy to compute the value of the density of $\mu$ at zero:

$$p_\alpha(0) = \frac{1}{\pi} \int_0^\infty e^{-t^\alpha} dt = \frac{1}{\pi} \Gamma\left(\frac{1}{\alpha}\right).$$

Now

$$\lim_{\alpha \to 0} \lim_{t \to 0^+} \frac{1}{t} \mu(tB) = \lim_{\alpha \to 0} \lim_{t \to 0^+} \frac{1}{t} \int_0^t p(x) dx = \lim_{\alpha \to 0} p_\alpha(0) = \lim_{\alpha \to 0} \frac{1}{\pi} \Gamma\left(\frac{1}{\alpha}\right) = \infty,$$

and

$$\lim_{\alpha \to 0} R(b_\alpha) b_\alpha = R\left(\frac{1}{e}\right) \frac{1}{e},$$

contradicting the inequality (4).
This implies that $R(b)$ must also depend on $\alpha$.

The proof of the theorem is almost the same as the proof of (1) in the paper [1], the difference is that instead of Kanter inequality we use our estimate (3). For the sake of completeness we repeat this proof.

We start with two lemmas.

**Lemma 1.** Let $\mu$ be a symmetric $\alpha$-stable measure, $0 < \alpha \leq 2$, on a separable Banach space $F$. Fix $0 < r < \alpha$. Then there exists a constant $K(\alpha, r) \geq 2$ such that for every convex, symmetric, closed set $B \subset F$, every $y \in F$ and all $t \in [0, 1]$ there holds

$$\mu(tB + y) \leq K(\alpha, r) R t \mu(2B + y),$$

where $R = (\mu(B))^{-1} (1 - \mu(B))^{-1/r}$.

**Proof.** It is well-known that symmetric stable measures are conditionally Gaussian [2], hence they satisfy the Anderson property.

Case 1. If $y \in B$ then $B \subset 2B + y$ so that $\mu(B) \leq \mu(2B + y)$, hence by the Anderson property and (3)

$$\mu(tB + y) \leq \frac{K(\alpha, r)}{(1 - \mu(B))^{1/r}} t \leq \frac{K(\alpha, r) \mu(B)}{\mu(B)(1 - \mu(B))^{1/r}} t \mu(2B + y).$$

Case 2. If $y \notin B$ then take $r = [t^{-1} - 2^{-1}]$. Then for $k = 0, 1, ..., r$ the balls $\{y + kB\}$ are disjoint and contained in $y + 2B$, where $y_k = (1 - 2kt\|y\|^{-1})y$. By the Anderson property $\mu(y_k + tB) \geq \mu(y + tB)$ for $k = 0, 1, ..., r$. Therefore

$$\mu(tB + y) \leq (r + 1)^{-1} \mu(2B + y) \leq \frac{2t}{2 - t} \mu(2B + y) \leq \frac{K(\alpha, r)}{(1 - \mu(B))^{1/r}} \mu(2B + y) t,$$

because we assumed that $K(\alpha, r) > 2$ and $2 - t \geq 1 > (1 - \mu(B))^{1/r}$.

**Lemma 2.** With the same assumptions as in Lemma 1, we have for all $0 \leq \kappa \leq t \leq 1$

$$\mu(\kappa tB) \leq R' t \mu(\kappa B),$$

where $R' = \frac{2K(\alpha, r)}{\mu(B/2)(1 - \mu(B/2))^{1/r}}$.

**Proof.** For $0 \leq t \leq 1$ define a measure $\mu_t$ by the formula $\mu_t(C) = \mu(tC) = P(X/t \in C)$, where $X$ is a symmetric $\alpha$-stable random variable with the distribution $\mu$. Then $\mu_t$ is also $\alpha$-stable and we have the following equality:

$$\mu * \mu_s(C) = P(X + X'/s \in C) = P((1 + s^{-\alpha})^{1/\alpha} X \in C) = \mu_t(C),$$

where $t = (1 + s^{-\alpha})^{-1/\alpha}$ and $X'$ is an independent copy of $X$. Now by Lemma 1

$$\mu(\kappa(tB)) = \mu(t(\kappa B)) = P(X/t \in \kappa B) = \mu * \mu_s(\kappa B) = \int_F \mu \left( \frac{2\kappa B}{2} + y \right) \mu_s(dy).$$
\[ \leq \frac{K(\alpha, r)2\kappa}{\mu(B/2)(1 - \mu(B/2))^{1/r}} \mu_t(B) = \frac{2K(\alpha, r)}{\mu(B/2)(1 - \mu(B/2))^{1/r}} \kappa \mu(tB). \]

**Proof of the Theorem.** Fix \( B \) with \( \mu(B) \leq b \) and take \( s \geq 1 \) such that \( \mu(sB) = b \). Now, in Lemma 2, put \( \kappa = t \) and \( t = 1/2s \). Then

\[
\mu(tB) = \mu(t \cdot \frac{1}{2s} \cdot (2sB)) \leq t \frac{K(\alpha, r)^2}{\mu(sB)(1 - \mu(sB))^{1/r}} \mu \left( \frac{1}{2s} \cdot 2sB \right) \leq \frac{K(\alpha, r)^2}{\mu(sB)(1 - \mu(sB))^{1/r}} \kappa \mu(B) = R(b)K(\alpha, r) \mu(B),
\]

where \( R(b) = 2b^{-1}(1 - b)^{-1/r} \). Taking different values of \( r \in (0, \alpha) \) we get different values of \( K(\alpha, r) \). If, for simplicity, we take \( r = \alpha/2 \) we get \( R(\alpha, b) = K(\alpha, \alpha/2) 2^{\alpha/2} \left( \frac{1 - \Phi(x)}{\alpha} \right)^{1/\alpha} \). This ends the proof of the theorem.

**Remark.** Let us recall some estimates of \( K(\alpha, r) \) which were given in the paper [3]. If we take \( r = \alpha/2 \) then

\[ K(\alpha, \frac{\alpha}{2}) = \frac{1}{2^{1/\alpha} \sqrt{\pi}} \Gamma^2 \left( \frac{\alpha}{4} + \frac{1}{2} \right) \Gamma(1 + \frac{2}{\alpha}) \inf_{x > 0} \frac{1}{x^{2/\alpha}(1 - \Phi(x))}, \]

where \( \Phi \) is the distribution function of a standard normal variable. For different values of \( r \) other estimates are possible, it could be interesting to find the least value of \( K(\alpha, r) \). Of course, if we consider \( \alpha \geq \varepsilon > 0 \) then we can find

\[ R(b) = \sup_{\varepsilon \leq \alpha \leq 2} R(\alpha, b) < \infty \]

and then for all \( 0 \leq t \leq 1 \) and \( \alpha \geq \varepsilon \)

\[ \mu(tB) \leq R(b) t \mu(B), \quad \text{if} \quad \mu(B) \leq b. \]

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