Solutions in the scalar-tensor theory with nonminimal derivative coupling

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We present black hole type solutions in the scalar-tensor theory with nonminimal derivative coupling to the Einstein tensor. The effects of the nonminimal derivative coupling appear in the large scales, while the solutions approach those in the Einstein gravity in the small scales. For the particular coupling constant tuned to the inverse of the cosmological constant, the scalar field becomes trivial and the solutions in the Einstein gravity are recovered. For the other coupling constant, more general solutions can be obtained. If the two-dimensional maximally symmetric space is a two-sphere, the spacetime structure approaches anti-de Sitter spacetime in the large scales. On the other hand, if the two-dimensional space is a two-hyperboloid, the spacetime approaches de Sitter (AdS) spacetime. If it is a flat space, the cosmological constant affects only the amplitude of the scalar field. The extension to the higher-dimensional case is also straightforward. For a certain range of the negative cosmological constant, thermodynamic properties of a black hole are very similar to those of the Schwarzschild-AdS black hole in the Einstein gravity.

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I. INTRODUCTION

Recently, various modified gravity theories have been proposed in the context of presenting successful cosmological models for inflation and/or dark energy (see e.g.,[1] and references therein). Many of these theories can be described by the so-called (generalized) Galileon scalar-tensor theories[2-9]. Among these theories, we focus on the scalar-tensor theory with nonminimal derivative coupling to the Einstein tensor

\[ S = \frac{1}{2} \int d^4x \sqrt{-g} \left[ m_p^2 (R - 2\Lambda) - (g^{\mu\nu} - \frac{z}{m_p^2} G^{\mu\nu}) \partial_{\mu} \phi \partial_{\nu} \phi \right], \tag{1} \]

where the metric \( g_{\mu\nu} \) is the metric, \( g = \det(g_{\mu\nu}) \), and \( R \) and \( G_{\mu\nu} \) are the Ricci scalar and the Einstein tensor for the metric \( g_{\mu\nu} \), respectively. \( z \) parametrizes the nonminimal derivative coupling and \( \Lambda \) is the cosmological constant. \( m_p \) is the reduced Planck mass and we assume \( z = O(1) \). Despite the higher derivative coupling in the action, the highest derivative terms in the equations of motion are of the second order because of the contracted Bianchi identity \( \nabla_\mu G^{\mu\nu} = 0 \), where \( \nabla_\mu \) is the covariant derivative with respect to \( g_{\mu\nu} \). In the rest, we will set \( m_p = 1 \) unless it should be shown explicitly. We will focus on the vacuum solutions in the scalar-tensor theory (1), and will not consider solutions with the matter fields except for the scalar field.

The theory given by the action (1) is equivalent to the following class of the generalized Galileon scalar-tensor theory which is constituted of the Lagrangians \( L_i \) (\( i = 2, 3, 4, 5 \)) [6,7] with

\[ K(\phi, X) = X - \Lambda, \quad G_3(\phi, X) = 0, \]
\[ G_4(\phi, X) = \frac{1}{2}, \quad G_5(\phi, X) = -\frac{z}{2} \phi, \]

where we have defined \( X := -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \) and followed the definition of (2.1)-(2.4) in [7]. This can be explicitly confirmed by the partial integration of the nonminimal derivative coupling term in (1) with the use of the contracted Bianchi identity \( \nabla_\mu G^{\mu\nu} = 0 \), and by neglecting the total derivative term which does not contribute to the equations of motion. If we choose \( G_5 = \text{const} \), the \( L_5 \) term just reduces to the total derivative term. Hence, \( G_5 \propto \phi \) in (2) corresponds to one of the simplest choices of the Lagrangian \( L_5 \) which can provide the nontrivial contribution to the equations of motion. Therefore, the investigation of the theory (1) would be the very important first step to understand the properties of the gravitational physics in the scalar-tensor theory with the Lagrangian \( L_5 \). Taking the nonminimal derivative coupling to the Einstein tensor into consideration is also motivated by the low energy effective action of string theory [10-12] and the ghost-free nonlinear massive gravity [13]. On the other hand, the cosmological constant \( \Lambda \) is equivalently seen as the constant potential term of the scalar field. The potential term would affect the dynamics of the scalar field and the spacetime metric, as it modifies the asymptotic structure of the black hole spacetime and the expansion law of the universe in the Einstein gravity. In our present case, it is very interesting to investigate how the cosmological constant \( \Lambda \) interacts with the nonminimal derivative coupling term at the nonlinear level. In general, in order to understand the properties of the given gravitational theory, the investigation of exact solutions is particularly important. From the above points of view, in this paper we will investigate the exact solutions in the theory (1).

This theory has attracted much interest from the cosmological points of view [14,15]. The accelerating cosmological solutions with the nonminimal derivative coupling to the curvature were first considered in [14,15].
Exact cosmological solutions with derivative coupling to the Einstein tensor have been obtained in [10, 17, 21, 28]. Cosmological dynamics in this theory has been investigated in [18, 20]. Slow-roll inflationary models have been developed in [22, 23] as an extension of the nonminimal Higgs inflation model [30]. Reheating [22, 23] and curvaton [27] mechanisms have also been discussed recently. The linear cosmological perturbation theory in the cosmological scenario with nonminimal derivative coupling to the Einstein tensor has been introduced recently in [23].

Black holes would provide another interesting arena to probe the aspects of the modified gravity theories. References [31–33] investigated the behavior of a scalar field derivative coupled to the Einstein tensor on a charged black hole. Some exact solutions were obtained via dimensional reduction from the higher-dimensional Einstein-Gauss-Bonnet gravity [9]. A no-hair theorem was argued in [34]. A spherically symmetric, static solution with a vanishing cosmological constant [35]. In this solution, the spacetime becomes dynamical. We also assume that 

$$\Lambda = 0$$

was obtained in [35]. In this solution, the spacetime is asymptotically anti-de Sitter (AdS) where

$$\Lambda < 0$$

and those with the two-dimensional hyperbolic space. In this work, we explore the solutions in the five-dimensional extension of (1). In Sec. VII, we close the paper after giving a brief summary.

II. THE EQUATIONS OF MOTION

Let us derive the equations of motion by varying the action (1). Varying the action (1) with respect to the metric $g_{\mu\nu}$, the gravitational equation is given by

$$G_{\mu\nu} = T_{\mu\nu} - z L_{\mu\nu} - \Lambda g_{\mu\nu},$$

where

$$T_{\mu\nu} := \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla^\lambda \phi \nabla_\lambda \phi,$$

and

$$L_{\mu\nu} := -\nabla_\mu \nabla_\lambda \phi \Box \phi + \nabla_\lambda (\nabla_\mu \phi) \nabla^\lambda (\nabla_\nu \phi)$$

$$+ R_{\alpha\beta\mu\nu} \nabla^\alpha \phi \nabla^\beta \phi - \frac{1}{2} R \nabla_\mu \phi \nabla_\nu \phi$$

$$+ 2 \nabla^\lambda \phi R_{\lambda(\mu} \nabla_{\nu)} \phi - \frac{1}{2} G_{\mu\nu} \nabla_\lambda \phi \nabla^\lambda \phi$$

$$+ g_{\mu\nu} \left( - R^\alpha_\beta \nabla_\alpha \phi \nabla_\beta \phi + \frac{1}{2} (\Box \phi)^2 ight)$$

$$- \frac{1}{2} \nabla_\alpha \nabla_\beta \phi \nabla^\alpha \nabla^\beta \phi).$$

Similarly, varying the action (1) with respect to $\phi$, the equation of motion of the scalar field is given by

$$(g_{\mu\nu} - z G_{\mu\nu}) \nabla^\mu \nabla_\nu \phi = 0.$$

Note that the highest derivatives are still of the second order because of $\nabla_\mu G^{\mu\nu} = 0$. In this paper, we will consider the vacuum solutions in the scalar-tensor theory (1). Hence, the energy-momentum tensor does not contain the contribution of the matter fields except for the scalar field.

We look for solutions of Eqs. (3) and (4) under the metric ansatz

$$ds^2 = -f(r)dt^2 + g(r)dr^2 + r^2 d\Omega^2,$$

where $K = +1, 0, -1$ denotes the constant curvature of the two-dimensional maximally symmetric space. For $f(r) > 0$ and $g(r) > 0$, $r$ is spacelike and the spacetime is static, while for $f(r) < 0$ and $g(r) < 0$, $r$ is timelike and the spacetime becomes dynamical. We also assume that
\( \phi = \phi(r) \). The nontrivial components of the gravitational equation (3) are given by

\[
\begin{align*}
\frac{r}{2} (3z(\phi')^2 + 2g) \left( \frac{f'}{f} - \frac{g'}{g} \right) &= 2g(Kg - 1) - 2z(\phi')^2 - zr(\phi')^2 - 2\Lambda r^2 g^2, \\
r(3z(\phi')^2 + 2g) \frac{f'}{f} &= 2g(Kg - 1) + z(\phi')^2 (Kg - 3) + r^2 g(\phi')^2 - 2\Lambda r^2 g^2,
\end{align*}
\]

where a “prime” denotes the derivative with respect to \( r \). The scalar field equation of motion (4) is given by

\[
\frac{d}{dr} \left[ \frac{rf}{g} \right] \left( \frac{rf'}{f} - (Kg - 1) - \frac{r^2 g}{z} \right) \phi' = 0.
\]

We integrate Eq. (4) with respect to \( r \) and set the integration constant to be zero along the same line of (3). Then the equation (4) can further reduce to

\[
r \frac{f'}{f} = Kg - 1 + \frac{r^2 g}{z}.
\]

If we choose a nonzero integration constant, \( f \) is not decoupled from the other variables and hence to find analytic solutions becomes a more complicated problem. Note that because in the theory (1) there is the shift symmetry under the transformation \( \phi \rightarrow \phi + \text{const} \), we will present the derivative of the scalar field \( \phi' \) which can be uniquely determined, instead of \( \phi \) itself.

### III. Solutions with a Vanishing Cosmological Constant

We then solve Eqs. (3) and (10). In this section, we look for the solutions of the vanishing cosmological constant \( \Lambda = 0 \). In the next section, we look for solutions for \( \Lambda \neq 0 \).

Note that the solutions for \( K = +1 \) have been obtained in \(^{35}\), while those for \( K = -1 \) and \( K = 0 \) can also be obtained in the similar way. An explicit comparison of them with the solutions with a cosmological constant will be done in the next section.

#### A. The solutions for \( K = +1 \)

The solutions for \( K = +1 \) are given as follows.

For \( z > 0 \), the general solution is given by

\[
\begin{align*}
f(r) &= \frac{3}{4} \frac{2m}{r} + \frac{r^2}{12z} + \frac{\sqrt{z}}{4r} \arctan \left( \frac{r}{\sqrt{z}} \right), \\
g(r) &= \frac{(r^2 + 2z)^2}{4(r^2 + z)^2} f(r) \\
(\phi'(r))^2 &= -\frac{(r^3 + 2rz)^2}{4(r^2 + z)^3} f(r),
\end{align*}
\]

This solution was obtained in \(^{35}\). The overall normalization of \( f(r) \) is chosen to recover the Schwarzschild black hole solution in the small \( r \) limit. We obtain

\[
f(r) = \frac{-2m}{r} + 1 + \frac{r^4}{20z^2} + O(r^6),
\]

and \( g(r) \approx 1/f(r) \). The explicit dependence on the coupling constant \( z \) appears in the \( O(r^4) \) term, by which we could distinguish the present model from the Einstein gravity without a cosmological constant. The Schwarzschild solution can also be obtained in the limit of \( z \rightarrow \infty \). On the other hand, in the large \( r \) limit, the effects of the nonminimal derivative coupling become more important and we obtain

\[
f(r) = \frac{r^2}{12z} + \frac{3}{4} + \frac{-16m + \pi \sqrt{z}}{8r} + O(r^{-2}).
\]

Thus the asymptotic structure in the large \( r \) limit is AdS spacetime with the effective cosmological constant

\[ -\frac{1}{4z} (< 0) \]

\(^{35}\). The point \( f(r) = 0 \) corresponds to the horizon. The derivative of the scalar field with respect to the proper length, \( \frac{1}{\sqrt{|g|}} |\phi'(r)| \), remains finite at the horizon. Note that \( (\phi'(r))^2 \) is negative and the scalar field becomes ghostlike outside the horizon because \( f(r) > 0 \) for \( r > r_h \), where \( r_h \) is the position of the horizon so that \( f(r_h) = 0 \). In the next section, we will see that this property can be improved by adding a negative cosmological constant.

Following \(^{35, 36}\), the temperature of the black hole, \( \beta^{-1} \), is related to the surface gravity \( \kappa := \frac{1}{2} \sqrt{-g_{rr}} \big|_{r=r_h} \) as \( \beta = \frac{2\pi}{\kappa} \), and hence

\[
\beta = \frac{8\pi z r_h}{r_h^2 + 2z}.
\]

For a large \( z \), it reproduces the temperature of the Schwarzschild black hole \( \beta = 4\pi r_h \).
2. $z < 0$

For $z < 0$, the general solution is given by
\[
\begin{align*}
  f(r) &= 3 - \frac{2m}{r} + \frac{r^2}{12z} - \sqrt{-z} \arctan \left( \frac{r}{\sqrt{-z}} \right), \\
g(r) &= \frac{(r^2 - 2z)^2}{4(r^2 - z)^2f(r)}, \\
(\phi'(r))^2 &= -\frac{(r^3 - 2rz)^2}{4(r^2 - z)^3z f(r)}. 
\end{align*}
\]
where the domain of the coordinate $r$ is given by $0 < r < \sqrt{-z}$. The overall normalization of $f(r)$ is chosen to recover the Schwarzschild solution in the small $r$ limit \[12\]. The Schwarzschild solution can also be obtained in the limit of $z \to -\infty$. At the boundary $r = \sqrt{-z}$, the spacetime is regular but the proper distance to the boundary is infinite.

B. The solutions for $K = -1$

The solutions for $K = -1$ are given as follows.

1. $z < 0$

For $z < 0$, the general solution is given by
\[
\begin{align*}
  f(r) &= \frac{3}{4} - \frac{2m}{r} + \frac{r^2}{12z} - \sqrt{-z} \arctan \left( \frac{r}{\sqrt{-z}} \right), \\
g(r) &= \frac{(r^2 - 2z)^2}{4(r^2 - z)^2f(r)}, \\
(\phi'(r))^2 &= -\frac{(r^3 - 2rz)^2}{4(r^2 - z)^3z f(r)}. 
\end{align*}
\]

The overall normalization of $f(r)$ is chosen to recover the solution of $K = -1$ in the Einstein gravity in the small $r$ limit. We obtain
\[
  f(r) = 2m_r - 1 - \frac{r^4}{20z^2} + O(r^6),
\]
and $g(r) \approx 1/f(r)$. The explicit dependence on the coupling constant $z$ appears in the $O(r^4)$ term, by which we could distinguish the present model from the Einstein gravity without a cosmological constant. The solution in the Einstein gravity can also be obtained in the limit of $z \to -\infty$. In the large $r$ limit, the effects of the nonminimal derivative coupling become more important and we obtain
\[
\begin{align*}
  f(r) &= -\frac{r^2}{12(\sqrt{-z})} - \frac{3}{4} - \frac{16m + \pi \sqrt{-z}}{8r} + O(r^{-2}). 
\end{align*}
\]

Thus the asymptotic structure in the large $r$ limit is the de Sitter (dS) spacetime with the effective cosmological constant $\frac{1}{4(\sqrt{-z})}(>0)$. The derivative of the scalar field with respect to the proper length, $\frac{1}{\sqrt{|g|}}|\phi'(r)|$, remains finite at the horizon. Note that $(\phi'(r))^2$ is negative and the scalar field becomes ghostlike in the large-$r$ region because of $f(r) < 0$. In the next section, we will see that this property can be improved by adding a positive cosmological constant.

2. $z > 0$

For $z > 0$, the general solution is given by
\[
\begin{align*}
  f(r) &= \frac{3}{4} - \frac{2m}{r} + \frac{r^2}{12z} - \sqrt{z} \arctan \left( \frac{r}{\sqrt{z}} \right), \\
g(r) &= \frac{(r^2 - 2z)^2}{4(r^2 - z)^2f(r)}, \\
(\phi'(r))^2 &= -\frac{(r^3 - 2rz)^2}{4(r^2 - z)^3z f(r)}, 
\end{align*}
\]

where the domain of the coordinate $r$ is given by $0 < r < \sqrt{z}$. The overall normalization of $f(r)$ is chosen to recover the solution of $K = -1$ in the Einstein gravity in the small $r$ limit \[17\]. The solution in the Einstein gravity can also be obtained in the limit of $z \to 0$. At the boundary $r = \sqrt{z}$, the spacetime is regular.

C. The solutions for $K = 0$

The solution for $K = 0$ is given by
\[
\begin{align*}
  f(r) &= -\frac{2m}{r} + \frac{r^2}{12z^2}, \\
  g(r) &= \frac{1}{4f(r)}, \\
  (\phi'(r))^2 &= -\frac{1}{4z f(r)}. 
\end{align*}
\]

This solution is singular only at $r = 0$.

IV. SOLUTIONS WITH A COSMOLOGICAL CONSTANT

We then present the solutions with a nonzero cosmological constant $\Lambda \neq 0$.

A. The solutions for $K = +1$

The solutions for $K = +1$ are given as follows.
1. $z > 0$ ($z \neq -\frac{1}{4}$)

For $z > 0$, the general solution is given by

$$f(r) = \frac{1}{12\pi z} \left( -24mz + r^3(1 - \Lambda z)^2 \right)$$

$$- 3rz(-1 + \Lambda z)(3 + \Lambda z)$$

$$+ 3z^{3/2}(1 + \Lambda z)^2 \arctan\left( \frac{r}{\sqrt{z}} \right),$$

$$g(r) = \frac{(-2z + r^2(-1 + \Lambda z))^2}{4(r^2 + z)^2f(r)},$$

$$\left( \phi'(r) \right)^2 = -\frac{(1 + \Lambda z)(-2rz + r^3(-1 + \Lambda z))^2}{4(r^2 + z)^2zf(r)},$$

(21)

where the domain of the coordinate $r$ is given by $0 < r < \infty$. The overall normalization of $f(r)$ is chosen to recover the Schwarzschild (A)dS solution in the small $r$ limit. We obtain

$$f(r) = -\frac{2m}{r} + 1 - \frac{\Lambda r^2}{3} + \frac{(1 + \Lambda z)^2 r^4}{20z^2} + O(r^6)$$

(22)

and $g(r) \approx 1/f(r)$. The explicit dependence on the coupling constant $z$ appears in the $O(r^4)$ term, by which we could distinguish the present model from the Einstein gravity with a cosmological constant. Note that the z-dependent correction vanishes in the limit $z \to -\frac{1}{4}$ which corresponds to the case discussed in Sec. IV A 3. In the large $z$ limit, we obtain

$$f(r) = 1 - \frac{2m}{r} - \frac{\Lambda}{3} r^2 + \frac{\Lambda^2 r^4}{20} + O(\frac{1}{z}),$$

$$f(r)g(r) = \left( 1 - \frac{r^2\Lambda}{2} \right)^2 + O(\frac{1}{z}).$$

(23)

In the large $r$ limit, the effects of the nonminimal derivative coupling become more important and we obtain

$$f(r) = \frac{r^2(\Lambda z - 1)^2}{12z} - \frac{1}{4}(\Lambda z - 1)(\Lambda z + 3)$$

$$+ \frac{-16m + \pi \sqrt{z}(1 + \Lambda z)^2}{8r} + O(r^{-2}).$$

(24)

Thus the asymptotic structure in the large $r$ limit is AdS spacetime with the effective cosmological constant $\Lambda = -\frac{1}{4z^2}(< 0)$. The horizon is formed at the place where $f(r) = 0$. The event horizon is always formed but no cosmological horizon is formed even for $\Lambda > 0$, because of the asymptotically AdS property. Another interesting property obtained by adding a cosmological constant $\Lambda$ is that $(\phi'(r))^2$ can be positive outside the horizon for $\Lambda < -\frac{1}{4}$ and the scalar field does not become ghostlike, because $f(r) > 0$ for $r > r_h$, where $r_h$ is the position of the horizon so that $f(r_h) = 0$.

The derivative of the scalar field with respect to the proper length, $\frac{1}{\sqrt{|g|}} \partial_r \phi'(r)$, remains finite at the horizon. As discussed in the previous section, the temperature of the black hole $\beta^{-1}$ is given by

$$\beta = \frac{8\pi z r_h}{r_h^2(1 - \Lambda z) + 2z}.$$  

(25)

For a large $z$ and a fixed finite $\Lambda z$, it reproduces the temperature of the Schwarzschild black hole $\beta = 4\pi r_h$. On the other hand, for both large $z$ and $-\Lambda z$ ($\Lambda < 0$), $\beta = \frac{4\pi r_h}{1 - \Lambda z}$, which does not agree with the temperature of the Schwarzschild AdS black hole $\beta = \frac{4\pi r_h}{1 - \Lambda z}$, because (23) is not precisely the same as the metric of the Schwarzschild AdS. The point of $g(r) = 0$, $r = \sqrt{\frac{2z}{\Lambda z - 1}}$, becomes a singularity because the invariant $R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu}$ diverges there. This point appears at a finite coordinate position for $\Lambda > \frac{1}{z}$, while for $\Lambda < \frac{1}{z}$ there is no such singularity.

2. $z < 0$ ($z \neq -\frac{1}{4}$)

For $z < 0$, the general solution is given by

$$f(r) = \frac{1}{12\pi z} \left( -24mz + r^3(1 - \Lambda z)^2 \right)$$

$$- 3rz(-1 + \Lambda z)(3 + \Lambda z)$$

$$+ 3z^{3/2}(1 + \Lambda z)^2 \arctan\left( \frac{r}{\sqrt{z}} \right),$$

$$g(r) = \frac{(-2z + r^2(-1 + \Lambda z))^2}{4(r^2 + z)^2f(r)},$$

$$\left( \phi'(r) \right)^2 = -\frac{(1 + \Lambda z)(-2rz + r^3(-1 + \Lambda z))^2}{4(r^2 + z)^2zf(r)},$$

(26)

where the domain of the coordinate $r$ is given by $0 < r < \sqrt{-z}$. The overall normalization of $f(r)$ is chosen to recover the Schwarzschild-(A)dS solution in the small $r$ limit 22. In the large $(-z)$ limit, we reproduce 23. At the boundary $r = \sqrt{-z}$, the spacetime is regular but the proper distance to the boundary is infinite. For $\Lambda > \frac{1}{(-z)}$, a singularity appears at $r = \sqrt{\frac{2(-z)}{1 + \Lambda(-z)}(< \sqrt{-z})}$. 23

3. $z = -\frac{1}{4}$

For $z = -\frac{1}{4}$, the solution is given by

$$f(r) = 1 - \frac{2m}{r} - \frac{\Lambda}{3} r^2,$$

$$g(r) = \frac{1}{f(r)},$$

$$\phi'(r) = 0.$$  

(27)

The scalar field becomes trivial and the Schwarzschild-(A)dS solution is recovered.

B. The solutions for $K = -1$

The solutions for $K = -1$ are given as follows.
For $z < 0$, the general solution is given by

$$f(r) = \frac{1}{12rz} (-24mz + r^3(1 - \Lambda z)^2) + 3rz(-1 + \Lambda z)(3 + \Lambda z) - 3z(1 + \Lambda z)^2(-z)^{1/2}\arctan\left(\frac{r}{\sqrt{-z}}\right),$$

$$g(r) = \frac{(2z + r^2(-1 + \Lambda z))^2}{4(r^2 - z)^2f(r)},$$

$$(\phi'(r))^2 = -\frac{(1 + \Lambda z)(2rz + r^3(-1 + \Lambda z))^2}{4(r^2 - z)^3zf(r)},$$

(28)

where the domain of the coordinate $r$ is given by $0 < r < \infty$. The overall normalization of $f(r)$ is chosen to recover the solution of $K = -1$ in the Einstein gravity in the small $r$ limit. We obtain

$$f(r) = \frac{2m}{r} - 1 - \frac{\Lambda r^2}{3} - \frac{(1 + \Lambda z)^2r^4}{20z^2} + O(r^6),$$

(29)

and $g(r) \approx 1/f(r)$. The explicit dependence on the coupling constant $z$ appears in the $O(r^4)$ term, by which we could distinguish the present model from the Einstein gravity with a cosmological constant. Note that the $z$-dependent correction vanishes in the limit $z \to -\frac{1}{2}$ which corresponds to the case discussed in Sec. IV B 3. In the large $(-z)$ limit, we obtain

$$f(r) = -1 - \frac{2m}{r} - \frac{\Lambda r^2}{3} - \frac{\Lambda^2}{20r^4} + O\left(\frac{1}{(-z)}\right),$$

$$f(r)g(r) = \left(1 + \frac{r^2\Lambda}{2}\right)^2 + O\left(\frac{1}{(-z)}\right).$$

(30)

In the large $r$ limit, the effects of the nonminimal derivative coupling become more important and we obtain

$$f(r) = \frac{r^2(\Lambda z - 1)^2}{12z} + \frac{1}{4}(\Lambda z - 1)(\Lambda z + 3) - \frac{16m + \pi\sqrt{-z}(1 + \Lambda z)^2}{8r} + O(r^{-2}).$$

(31)

Thus the asymptotic structure in the large $r$ limit is dS spacetime with the effective cosmological constant $(1 - \Lambda z)^2(> 0)$. For $m < 0$, $f(r)$ is always negative, and $t$ and $r$ are always spacelike and timelike, respectively.

The derivative of the scalar field with respect to the proper length, $\frac{1}{\sqrt{|g|}}|\phi'(r)|$, remains finite at the horizon.

On the other hand, the point $g(r) = 0, r = \sqrt{\frac{2(-z)}{(-\Lambda z) - 1}}$, becomes singularity because the invariant $R_{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu}$ diverges there. This point appears at a finite coordinate position for $\Lambda < -\frac{1}{2}$ while for $\Lambda > -\frac{1}{2}$, there is no such singularity. Another interesting property obtained by adding a cosmological constant $\Lambda$ is that $(\phi'(r))^2$ can be positive in the large-$r$ region and the scalar field does not become ghostrlike if $\Lambda > -\frac{1}{2}$ because of $f(r) < 0$.

2. $z > 0$ ($z \neq -\frac{1}{2}$)

For $z > 0$, the general solution is given by

$$f(r) = \frac{1}{12rz} (-24mz + r^3(1 - \Lambda z)^2) + 3rz(-1 + \Lambda z)(3 + \Lambda z) - 3z^{3/2}(1 + \Lambda z)^2\arctanh\left(\frac{r}{\sqrt{z}}\right),$$

$$g(r) = \frac{(2z + r^2(-1 + \Lambda z))^2}{4(r^2 - z)^2f(r)},$$

$$(\phi'(r))^2 = -\frac{(1 + \Lambda z)(2rz + r^3(-1 + \Lambda z))^2}{4(r^2 - z)^3zf(r)},$$

(32)

where the domain of the coordinate $r$ is given by $0 < r < \sqrt{z}$. The overall normalization of $f(r)$ is chosen to recover the solution in the Einstein gravity in the large $r$ limit. In the large $z$ limit, we reproduce (30). At the boundary $r = \sqrt{z}$, the spacetime is regular. For $\Lambda < -\frac{1}{2}$, a singularity appears at $r = \sqrt{\frac{2z}{\Lambda z}} (\approx \sqrt{z})$.

3. $z = -\frac{1}{2}$

For $z = -\frac{1}{2}$, the solution is given by

$$f(r) = -1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}, \quad g(r) = \frac{1}{f(r)},$$

$$\phi'(r) = 0.$$  

(33)

Thus the scalar field becomes trivial.

C. The solutions for $K = 0$

1. $z \neq -\frac{1}{2}$

The solution for $K = 0$ with $z \neq -\frac{1}{2}$ is given by

$$f(r) = -1 - \frac{2m}{r} + r^2, \quad g(r) = \frac{1}{4f(r)},$$

$$(\phi'(r))^2 = -\frac{1 + \Lambda z}{4zf(r)}.$$  

(34)

Thus the dependence on the cosmological constant does not explicitly appear in the metric functions [17]. This solution is singular only at $r = 0$.

For $m = 0$ and $z < 0$, the solution [24] can be rewritten into the form of the flat Friedmann-Lemaître-Robertson-Walker metric

$$ds^2 = -dt^2 + e^{2H_z t}dx^2, \quad \phi(\tau) = \sqrt{1 + \frac{\Lambda z}{z}}\tau.$$  

(35)

where $H_z := \sqrt{-\frac{1}{z}}$ and $\tau$ is the proper time coordinate defined as $r = e^{H_z \tau}$, which agrees with the corresponding solution obtained in [17].
For $m < 0$ and $z < 0$, by introducing the new coordinates $r = \frac{2|m|}{H_z} \cosh^{\frac{3}{2}} H_z \tau$ and $t = \frac{2}{H_z} (x^3)$ into the form of the Bianchi-I metric
\[
ds^2 = -dt^2 + \frac{2^{\frac{3}{2}} |m|^\frac{3}{2}}{H_z^2} \sinh^{\frac{3}{2}} (3 H_z \tau) \times \left[ \frac{1}{\tanh^{\frac{3}{2}} (\frac{3}{2} H_z \tau)} \frac{d\Omega^2_{K=0}}{\sinh^{\frac{3}{2}} (\frac{3}{2} H_z \tau)} \right]
\]
The initial behavior around $\tau \sim 0$ is the same as the regular branch of the Kasner solution [37], and then the universe approaches dS spacetime in the late time limit [38, 39].

2. $z = -\frac{1}{3}$

For $z = -\frac{1}{3}$, the derivative of the scalar field becomes trivial. In case of $m = 0$, depending on the sign of $\Lambda$, the solution becomes either dS or AdS.

D. Comparison with the case without a cosmological constant

In this subsection, we compare the solutions with a cosmological constant with those without it. In Tables I and II, we have listed the basic properties of the solutions. “Domain”, “Singularities” and “Asymptotic behavior” in these tables correspond to the domain of the $r$ coordinate, the position of the curvature singularities, and the asymptotic behavior of spacetime in the $r \to \infty$ limit, respectively. The modifications due to a finite cosmological constant appear in the various properties of the spacetime.

| TABLE I: The properties of solutions without a cosmological constant. |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| $K = +1$        | $K = -1$        | $K = 0$         |
| $z > 0$         | $z < 0$         | $z > 0$         | $z < 0$         | $z > 0$         |
| Domain          | $0 < r < \infty$| $0 < r < \sqrt{1-z}$| $0 < r < \infty$| $0 < r < \infty$| $0 < r < \infty$| $0 < r < \infty$|
| Singularities   | $r = 0$         | $r = 0$         | $r = 0$         | $r = 0$         | $r = 0$         | $r = 0$         |
| Asymptotic behavior | AdS            | -               | -               | dS              | AdS              | dS              |

| TABLE II: The properties of solutions with a cosmological constant |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| $K = +1$        | $K = -1$        | $K = 0$         |
| $z > 0$         | $z < 0$         | $z > 0$         | $z < 0$         | $z > 0$         |
| Domain          | $0 < r < \infty$| $0 < r < \sqrt{1-z}$| $0 < r < \infty$| $0 < r < \infty$| $0 < r < \infty$|
| Singularities   | $r = \sqrt{\frac{2z}{\Lambda + 1}}$| $r = \sqrt{\frac{2(1-z)}{1 + \Lambda - z}}$| $r = \sqrt{\frac{2z}{1 - \Lambda}}$| $r = \sqrt{\frac{2(1-z)}{\Lambda - 1}}$| $r = \sqrt{\frac{2z}{\Lambda - 1}}$|
| Asymptotic behavior | AdS            | -               | -               | dS              | AdS              | dS              |

One of the most important properties which is absent in the case without a cosmological constant is that the black hole solutions in the Einstein gravity are exactly recovered if we choose the particular coupling constant $z = -\frac{1}{3}$. This is the case that the scalar field becomes trivial $\phi(r) = 0$ at any position $r$, as it is clear from Eqs. (21), (28), and (34). This is surprising, because the solutions in the Einstein gravity are recovered for the finite coupling constant $z = -\frac{1}{3}$, not for the vanishing coupling constant $z = 0$ as we naively guess. In addition to the fact that if $\Lambda = 0$, from Eqs. (11), (10), and (20), the scalar field becomes trivial and the solutions in the Einstein gravity are recovered in the limit of $|z| \to \infty$, this also gives another indication that $z$ should be a nonperturbative parameter along the line argued in [35].

For the black hole case $K = +1$ and $z > 0$, adding a negative cosmological constant can also prevent the scalar field from being ghostlike outside the horizon. We also find that adding a negative cosmological constant can make the effective energy-momentum tensor of the scalar field including the contribution of the nonminimal derivative coupling obtained from the right-hand side of [43] (but without the contribution of the cosmological constant)
\[
\tilde{T}^{\mu \nu} := T^{\mu \nu} - z L^{\mu \nu} = \text{diag}(\tilde{\rho}, \tilde{p}_r, \tilde{p}_a),
\]
where \( \tilde{\rho} \) is the effective energy density, and \( \tilde{p}_r \) and \( \tilde{p}_\theta \) are the effective pressures in the radial direction and in the direction of the two-sphere, respectively, to satisfy the weak energy condition for which any timelike observer measures a local positive energy density:

\[
\tilde{\rho} \geq 0, \quad \tilde{\rho} + \tilde{p}_r \geq 0, \quad \tilde{\rho} + \tilde{p}_\theta \geq 0, \tag{38}
\]

outside the horizon. In the case of \( \Lambda < \frac{1}{3} \), the solution is regular except for \( r = 0 \). From (21) and (11), for \( \Lambda = 0 \) the scalar field becomes ghostlike \((\phi')^2 < 0\) outside the horizon where \( f(r) > 0 \). In this case the weak energy condition for the effective energy-momentum tensor is also violated outside the horizon. But, because of the overall factor \((1 + \Lambda z)\) in (21), if \( \Lambda < -\frac{1}{3} \), \((\phi')^2\) becomes positive outside the horizon and then the weak energy condition is satisfied there. Thus, adding a negative cosmological constant can make the black hole solution healthy.

It should also be emphasized that adding a cosmological constant does not modify the asymptotic behaviors of the spacetime. What remains unchanged even if we add a cosmological constant is as follows:

1. The domain of the \( r \) coordinate for a given set of \( z \) and \( K \).
2. The asymptotic structure of the spacetime for the domain \( 0 < r < \infty \).
3. The metric for the solution of \( K = 0 \), shown in (20) and (31).

The above properties indicate that the asymptotic behaviors of the spacetime are determined by the sign of the nonminimal derivative coupling constant \( \Lambda \) and the curvature of the two-dimensional maximally symmetric space \( K \), irrespective of the value of the cosmological constant \( \Lambda \). This is also the unexpected result, from our intuitions in the Einstein gravity, where the asymptotic behavior in the large \( r \) limit crucially depends on the value of the cosmological constant. For example, if a cosmological constant is sufficiently positive, we might expect that the spacetime always becomes asymptotically de Sitter but our analysis has revealed that in model this is not the case. For a nonzero \( K \), although the cosmological constant \( \Lambda \) appears in the large \( r \) limit, the way of appearance is different from the case of the Einstein gravity and the coefficients of the leading \( O(r^2) \) terms in (21) and (31) are proportional to \( \frac{K - 3\Lambda}{128\Lambda m_p^2}r^2 \). Thus for \( K = +1 \) (and \( z > 0 \)) the spacetime is always asymptotically AdS even if \( \Lambda \) is positive. Similarly, for \( K = -1 \) (and \( z < 0 \)) it is always asymptotically dS even if \( \Lambda \) is negative. For \( K = 0 \), remarkably, the spacetime structure is also independent of \( \Lambda \) and purely determined by the sign of \( z \).

Moreover, another modification due to the existence of a nonzero cosmological constant is the appearance of the curvature singularity except for \( r = 0 \) (for a nonzero \( K \)), where \( g_{rr} \) vanishes. For example, for \( K = +1 \) a curvature singularity appears at the finite coordinate position if the cosmological constant is positive. Similarly, for \( K = -1 \) a curvature singularity appears at the finite coordinate position if the cosmological constant is negative. In addition, when the singularity appears at the finite coordinate position, \((\phi')^2 < 0\) in the large \( r \) limit and there is the violation of the weak energy condition in the large-\( r \) limit, indicating an instability at least at the quantum level. These properties suggest that in order to obtain the regular spacetime (except for the center at \( r = 0 \)), a choice of a too large positive (negative) cosmological constant is practically forbidden for \( K = +1 \) (\( K = -1 \)).

V. THERMODYNAMIC PROPERTIES OF THE BLACK HOLE

After giving the black hole solutions, we discuss the thermodynamic properties of our black hole solutions. In this section, we will show the dependence on the reduced Planck mass \( m_p \) explicitly. We consider the black hole solutions of \( K = +1 \) and \( z > 0 \) given in (21). We also focus on the case \( m_p^2 + \Lambda z < 0 \) for which \((\phi')^2 > 0\) outside the horizon, as argued in the previous section. In this section, we will work on the Euclidean frame, \( ds_E^2 = f(r) d\tau^2 + \frac{dr^2}{g(r)} + rz d\Omega^2_{K=+1} \), where \( r \) represents the Euclidean time, and \( f(r) \) and \( g(r) \) are given in (21), which by giving the \( m_p \) dependence back explicitly can be rewritten as

\[
f(r) = \frac{1}{12rzm_p} \left\{ -24m_z + m_p^3 r^3 \left( 1 - \frac{\Lambda z}{m_p^2} \right)^2 - 3m_prz \left( 1 + \frac{\Lambda z}{m_p^2} \right) \left( 3 + \frac{\Lambda z}{m_p^2} \right) + 3z^{3/2} \left( 1 + \frac{\Lambda z}{m_p^2} \right)^2 \arctan \left( \frac{m_p r}{\sqrt{z}} \right) \right\},
\]

\[
g(r) = \frac{-2z + r^2 (-m_p^2 + \Lambda z)^2}{4(m_p^2 r^2 + z)^2 f(r)},
\]

\[
(\phi'_E(r))^2 = \frac{-m_p^4 (m_p^2 + \Lambda z)(-2rz + r^3(-m_p^2 + \Lambda z))^2}{4(m_p^2 r^2 + z)^2 f(r)}.
\]

Note that there will be the difference in thermodynamic quantities shown below, by the overall factor \((8\pi)\) from those in Ref. (33). This difference comes from the definition of the reduced Planck mass \( m_p^2 = \frac{1}{8\pi G_N} \), where \( G_N \) represents Newton’s constant.

Following (32) and (40) for the Schwarzschild-AdS black hole, we compute thermodynamic quantities of our black hole solutions. The temperature of the black hole, \( \beta^{-1} \), is given in (25), where \( \beta \) represents the periodicity...
of the $\tau$ coordinate. Giving the reduced Planck mass $m_p$ back explicitly, it is given by

$$\beta = \frac{8\pi r_h^2}{m_p^2 r_h^2 + x (2 - \Lambda r_h^2)} = \frac{8\pi m_p x \sqrt{x}}{2 m_p^2 + x^2 (m_p^2 - \Lambda z)} \quad (41)$$

where we have introduced the dimensionless horizon position by $x := m_p r_h / \sqrt{z}$. For $\Lambda z < -m_p^2$, the temperature $\beta^{-1}$ is always positive, decreases for $x < x_{\text{min}}$ and increases for $x > x_{\text{min}}$, where $x_{\text{min}} := \sqrt{m_p^2 - \Lambda z} m_p$ is the value of $x$ for which the temperature $\beta^{-1}$ takes the minimal value. Note that the temperature $\beta^{-1}$ diverges in both the limits of $x \to 0$ and $x \to \infty$, and hence for a given temperature there are always two possible horizon sizes: one of them is a large black hole $x > x_{\text{min}}$ and the other is the small black hole $x < x_{\text{min}}$, as for the Schwarzschild-AdS black hole. In the discussions below, by “small” and “large”, we basically mean that $x < x_{\text{min}}$ and $x > x_{\text{min}}$, respectively.

As for the Schwarzschild AdS black hole in the Einstein gravity $^{40}$, the Euclidean action

$$S_E = \int d^4 x \sqrt{-g} L_E,$$

$$L_E = -\frac{1}{2} \left\{ m_p^2 (R_E - 2\Lambda) - (g_E^{\mu\nu} - \frac{2}{m_p^2} G_E)^{\mu\nu} \partial_\mu \phi_0 \partial_\nu \phi_E \right\}, \quad (42)$$

is divergent. Hence we need to regularize it by subtracting the Euclidean action of the background of $m = 0$ with the given values of $z$ and $\Lambda$, as done in $^{33,40}$. The regularized Euclidean action is defined by

$$S_{\text{reg}} := S_E - S_E^{(0)}, \quad (43)$$

where

$$S_E = 4\pi \beta \int_{r_h}^\infty dr r^2 \sqrt{f(r)} g(r) L_E(r),$$

$$S_E^{(0)} = 4\pi \beta_{(0)} \int_0^\infty dr r^2 \sqrt{f_0(r)} g_0(r) L_E^{(0)}(r), \quad (44)$$

and the quantities with “(0)” are defined for the background of $m = 0$, $r_h$ is the position of the horizon and $\bar{r}$ is the position of the boundary as the regulator of the Euclidean action. $\beta_{(0)}$ is determined by requiring that the periodicity of the $\tau$ direction and the geometry at the section of $\bar{r}$ in the two backgrounds of $m \neq 0$ and $m = 0$ should be identical, namely,

$$\beta_{(0)} \sqrt{f_0(\bar{r})} = \beta \sqrt{f(\bar{r})}. \quad (45)$$

The explicit computation of Eqs. (43) and (44) with the relation (45) shows that the divergent parts with positive power of $\bar{r}$ are canceled and the resultant regularized action contains only the finite value

$$S_{\text{reg}} = \frac{8\pi^2 x z}{3m_p^2 (2m_p^2 + x^2 (m_p^2 - \Lambda z))} \left\{ x (m_p^2 (3 - 2x^2) + m_p^2 (-6 + x^2) \Lambda z + (-3 + x^2) \Lambda^2 z^2) + 3 (m_p^2 + \Lambda z)^2 \arctan(x) \right\} \quad (46)$$

For $\Lambda = 0$, we recover the result in $^{33}$

$$S_{\text{reg}} = \frac{8\pi^2 x z}{3(2 + x^2)} \left[ 3x - 2x^3 + 3 \arctan(x) \right], \quad (47)$$

except for the overall ($8\pi$) difference which arises because of the reason shown in the first paragraph of this section.

$$E = \frac{\pi \sqrt{z}}{3m_p^2 (x^2 + 1)(2m_p^2 - x^2 (m_p^2 - \Lambda z))} \left\{ x \left( m_p^4 (18 - x^2 - 20x^4 - 4x^6) - m_p^4 (12 + 13x^2 - 14x^4 - 6x^6) \Lambda z \right) 
- m_p^2 (6 + 7x^2 - 8x^4) \Lambda^2 z^2 - x^2 (3 + 2x^2 + 2x^4) \Lambda^3 z^3 \right\} 
+ 3 (1 + x^2) (m_p^2 + \Lambda z) (2m_p^2 - x^2 (m_p^2 - \Lambda z)) \arctan(x) \right\},$$

$$S_{\text{ent}} = \frac{8\pi^2 x z}{m_p^2 (1 + x^2)(2m_p^2 - x^2 (m_p^2 - \Lambda z))} \left\{ m_p^4 (2 - x^2 - 2x^4) + m_p^2 x^2 (-1 + x^2) \Lambda z + x^4 \Lambda^2 z^2 \right\}. \quad (48)$$

The partition function $Z$ is related to the regularized Euclidean action $S_{\text{reg}}$ and the Helmholtz free energy $F$ by $\ln Z = -S_{\text{reg}} = -\beta F$. Then we can compute the energy and entropy of the black hole by $E = \frac{\partial S_{\text{ent}}}{\partial \beta}$ and $S_{\text{ent}} = \beta E - S_{\text{reg}}$, respectively, which are explicitly written as
It is straightforward to check that the first law of thermodynamics, \(dE = \beta^{-1}dS_{\text{ent}}\), is satisfied. In the limit of \(\Lambda = 0\), the above thermodynamic quantities in \([35]\) agree with the results in \([33]\) [again except for the overall \((8\pi)\) difference. Thermodynamic properties of the solutions with \(\Lambda = 0\) have been investigated in \([35]\). Here we focus on the case \(\Delta z < -m_p^2\) for which the scalar field is not ghostlike \((\phi')^2 > 0\) outside the horizon.

First, we discuss the properties of the energy \(E\). Figure 1 shows the region where \(E > 0\) on the \((x, \frac{\Delta z}{m_p^2})\) plane, which corresponds to the region above the drawn curve. Note that for \(\Delta z = -m_p^2\), the energy \(E\) is always positive. If \(-2m_p^2 \leq \Delta z < -m_p^2\), for a black hole with the intermediate horizon size \(x_{\text{min}} < x < x_1\), where \(x_1(> x_{\text{min}})\) is the value of \(x\) for which \(E = 0\), the energy \(E\) becomes negative. If \(\Delta z < -2m_p^2\), the energy \(E\) becomes negative for a large black hole \(x > x_{\text{min}}\).

We then discuss the behavior of the entropy \(S_{\text{ent}}\). In the small \(r_h\) limit \((x \ll 1)\),

\[
S_{\text{ent}} = 8\pi^2 m_p^2 r_h^2 - \frac{8\pi^2 r_h m_p^2}{z} (m_p^2 + \Delta z) + O(r_h^6),
\]

where the leading order term follows the ordinary area law of the black hole entropy \(S_{\text{ent}} = \frac{A_h}{4G_\mathcal{N}} = 2\pi m_p^2 A_h\) with the horizon area \(A_h = 4\pi r_h^2\) and \(m_p^2 = \frac{1}{4G_\mathcal{N}}\). In the large \(z\) limit, we also obtain the same leading order behavior as Eq. (49) following the area law. On the other hand, in the large \(r_h\) limit \((x \gg 1)\),

\[
S_{\text{ent}} = 8\pi^2 (2m_p^2 + \Delta z) r_h^2 + O(r_h^6).
\]

Thus in order for a large black hole to have the positive entropy, we have to impose \(2m_p^2 + \Delta z > 0\). Fig. 2 shows the region where \(S_{\text{ent}} > 0\) on the \((x, \frac{\Delta z}{m_p^2})\) plane, which corresponds to the region above the drawn curve. Although there is the similarity between Figs. 1 and 2, the regions of \(E > 0\) and \(S_{\text{ent}} > 0\) on the \((x, \frac{\Delta z}{m_p^2})\) plane do not precisely coincide for \(x > 1\), while they coincide for \(x < 1\). Note that for \(\Delta z = -m_p^2\), the area law \(S_{\text{ent}} = \frac{A_h}{4G_\mathcal{N}}\) is recovered, as expected from the discussions in Sec. IV A 3. If \(-2m_p^2 \leq \Delta z < -m_p^2\) only for the intermediate horizon sizes \(x_{\text{min}} < x < x_2\), where \(x_2(> x_{\text{min}})\) is the value of \(x\) for which \(S_{\text{ent}} = 0\), the entropy \(S_{\text{ent}}\) becomes negative. If \(\Delta z < -2m_p^2\), the entropy \(S_{\text{ent}}\) becomes always negative for a large black hole \(x > x_{\text{min}}\).

Next, the heat capacity \(C := -\beta^2 \frac{\partial E}{\partial T}\) is explicitly given by

\[
C = \frac{16\pi^2 x^2(z(2m_p^2 + x^2(m_p^2 - \Delta z)))}{m_p^2(1 + x^2)(2m_p^2 - x^2(m_p^2 - \Delta z))} \left\{ -m_p^2(4 - 4x^2 - 11x^4 - 4x^6 + 2x^8) + m_p^2 x^2(4 - 2x^2 + 2x^4 + 3x^6)\Delta z \right\}.
\]

In the limit of \(\Lambda = 0\), the heat capacity \([31]\) again agrees with the result in \([33]\). Let us focus on the case of \(\Lambda < 0\). For \(\Delta z = -m_p^2\), we recover the heat capacity for the Schwarzschild-AdS black hole

\[
C = \frac{16\pi^2 x^2}{r_h^2 \Lambda + 1}. \tag{52}
\]

In Fig. 3, we show the behavior of \(C\) (divided by \(z\)) as the function of \(x\) for the various choices of \(\frac{\Delta z}{m_p^2}\). The heat capacity \([51]\) diverges at \(x = x_{\text{min}}\) and changes its sign across this point. There is the clear difference in the behavior of the heat capacity across \(\Delta z = -2m_p^2\). In Fig. 4 we show the region of \(C > 0\), which corresponds to that between two drawn curves. The left curve in Fig. 4 corresponds to \(x = x_{\text{min}}\). If \(-2m_p^2 \leq \Delta z < -m_p^2\), \(C > 0\) for \(x > x_{\text{min}}\), while \(C < 0\) for \(0 < x < x_{\text{min}}\), which is very similar to the case of the Schwarzschild-AdS black hole in the Einstein gravity, in the sense that a large black hole \((x > x_{\text{min}})\) is thermodynamically stable while a small black hole \((x < x_{\text{min}})\) is unstable. On the other hand, if \(\Delta z < -2m_p^2\), \(C < 0\) for \(0 < x < x_{\text{min}}\) and \(x > x_3\), where \(x_3(> x_{\text{min}})\) is the value of \(x\) at which \(C = 0\), while \(C > 0\) for \(x_{\text{min}} < x < x_3\), indicating that only a black hole with an intermediate horizon size \(x_{\text{min}} < x < x_3\) can be thermodynamically stable. There is the difference from the case of \(\Lambda = 0\) discussed in \([33]\): For \(\Lambda = 0\), there is the region of \(x\) where the heat capacity \(C\) becomes positive even for \(x < x_{\text{min}}\), while in our case the heat capacity is always negative for \(x < x_{\text{min}}\).

Finally, let us investigate the behavior of the free energy \(F = \beta^{-1} S_{\text{reg}}\). In Fig. 5, the free energy \(F\) (divided by \(\sqrt{z}\)) is shown as the function of \(x\) for the various choices of \(\Delta z < -m_p^2\). In Fig. 6, the region where the free energy \(F > 0\) is shown on the \((x, \frac{\Delta z}{m_p^2})\) plane, which corresponds to the region below the drawn curve. As Fig. 5 shows, the free energy \(F\) vanishes at \(x = 0\) and takes a positive value for a smaller \(x\), irrespective of \(\Delta z\). There is the clear difference in the behavior of the free energy \(F\) for a larger \(x\), depending on whether \(\Delta z\) is greater than \(-2m_p^2\) or not. If \(-2m_p^2 < \Delta z < -m_p^2\) as \(x\) increases, the free energy \(F\) increases, but decreases after reaching a maximum, crosses zero and then becomes negative,
which is very similar to the case of the Schwarzschild-AdS black hole. The free energy takes the maximal value at

\[
x = x_{\text{peak}} \equiv m_p \sqrt{m_p^2 + \Lambda z - \sqrt{17m_p^4 - 6m_p^2\Lambda z - 7(\Lambda z)^2}}
\]

(53)

Thus, a large black hole \(x > x_4(> x_{\text{peak}})\), where \(x_4\) is the value of \(x\) at which \(F = 0\), has the negative free energy \(F < 0\). If \(\Lambda z < -2m_p^2\), as \(x\) increases, the free energy \(F\) is always positive and increasing. Thus, in contrast to the above case, the state of the thermal radiation \((x = 0)\) always has the lowest free energy \(F = 0\). Therefore, we expect that for \(-2m_p^2 < \Lambda z < -m_p^2\), the behavior of thermodynamic quantities is very similar to those of the Schwarzschild-AdS black hole and the Hawking-Page phase transition \[40\] would take place as in the Einstein gravity, while for \(\Lambda z \leq -2m_p^2\) the state of the thermal radiation \((x = 0)\) always has the minimal free energy and the Hawking-Page phase transition would not take place. Note that the behavior of the free energy \(F\) and the other thermodynamic quantities for \(\Lambda z = -m_p^2\) is precisely the same as in the case of the Schwarzschild-AdS black hole, and the Hawking-Page phase transition can take place \[40\]. Our results may be also useful to obtain the holographic interpretation of the scalar-tensor theory with nonminimal derivative coupling to the Einstein tensor, along the line of \[41\].

VI. SOLUTIONS IN THE FIVE-DIMENSIONAL MODEL

Before closing, we present the corresponding solutions in the five-dimensional version of the theory \[1\]. The metric ansatz is the same as \[7\] except that \(d\Omega_K^2\) denotes the three-dimensional maximally symmetric space. In this section, we again set \(m_p = 1\).

The main features of the solutions are the same as those of the corresponding solutions of the four-dimensional model. The extension to the higher-dimensional case is also straightforward, although we do not explicitly present the solutions of the models with more than six spacetime dimensions.
FIG. 3: The heat capacity $C$ (divided by $z$) is shown as the function of $x$. The top-left, top-right, bottom-left and bottom-right panels show the cases of $\frac{\Delta z}{m_p^2} = -1.5, -2, -2.5, -5$, respectively. In all panels, $C$ diverges at $x = x_{\text{min}}$. For $0 < x < x_{\text{min}}$, we always have $C < 0$. On the other hand, there is the clear difference on the behavior for $x > x_{\text{min}}$, depending on the value of $\Delta z$. In the case of $-2m_p^2 \leq \Delta z < -m_p^2$, $C$ is positive for $x > x_{\text{min}}$. On the other hand in the case of $\Delta z < -2m_p^2$, $C$ is positive only for the intermediate region $x_{\text{min}} < x < x_3$, where $x_3(> x_{\text{min}})$ is the value of $x$ at which $C = 0$.

FIG. 4: The region between two curves on the $(x, \frac{\Delta z}{m_p^2})$ plane corresponds to that of $C > 0$. The left and right curves show $x = x_{\text{min}}$ and $x = x_3$, respectively. For $-2m_p^2 \leq \Delta z < -m_p^2$, a large black hole has a positive heat capacity.

A. The solutions for $K = +1$

1. $z \neq -\frac{1}{3}$

The solution for $K = +1$ is given by

\[
 f(r) = \frac{1}{24r^2z} \left( -48mz + r^4(1 - \Lambda z)^2 
 - 6r^2z(-1 + \Lambda z)(3 + \Lambda z) 
 + 18z^2(1 + \Lambda z)^2 \ln \left( \frac{r^2}{3z} + 1 \right) \right),
\]

\[
 g(r) = \frac{(-6z + r^2(-1 + \Lambda z))^2}{4(r^2 + 3z)^2f(r)},
\]

\[
 (\phi'(r))^2 = -\frac{(1 + \Lambda z)(-6rz + r^3(-1 + \Lambda z))^2}{4(r^2 + 3z)^3zf(r)}, \quad (54)
\]
where the domain of the coordinate $r$ is given by $0 < r < \infty$ for $z > 0$ and by $0 < r < \sqrt{-3z}$ for $z < 0$. The overall normalization of $f(r)$ is chosen to recover the asymptotic structure of the Schwarzschild-(A)dS solution in the small $r$ limit. We obtain

$$f(r) = -\frac{2m}{r^2} + 1 - \frac{\Lambda r^2}{6} + \left(1 + \Lambda z\right)^2r^4 \cdot \frac{108z^2}{108z^2} + O(r^6),$$

and $g(r) \approx 1/f(r)$. The explicit dependence on the coupling constant $z$ appears in the $O(r^4)$ term, by which we could distinguish the present model from the Einstein gravity with a cosmological constant. Note that the $z$-dependent correction vanishes in the limit $z \to -1$, which corresponds to the case discussed in Sec. VI A 2. In the large $|z|$ limit, we obtain

$$f(r) = 1 - \frac{2m}{r} - \frac{\Lambda}{6}r^2 + \frac{\Lambda^2}{108}r^4 + O\left(\frac{1}{|z|}\right),$$

$$f(r)g(r) = \left(1 - \frac{r^2\Lambda}{6}\right)^2 + O\left(\frac{1}{|z|}\right).$$

For $z > 0$, in the large $r$ limit, the effects of the nonminimal derivative coupling become more important and we obtain

$$f(r) = \frac{r^2(\Lambda z - 1)^2}{24z} - \frac{1}{4}(\Lambda z - 1)(\Lambda z + 3) + O(r^{-2}).$$

FIG. 5: The free energy $F$ (divided by $\sqrt{z}$) is shown as the function of $x$. The top-left, top-right, bottom-left and bottom-right panels show the cases of $\frac{\Lambda z}{m^2} = -1.5, -2, -2.5, -5$, respectively.

FIG. 6: The region below the curve on the $(x, \frac{\Lambda z}{m^2})$ plane corresponds to the region of the positive free energy $F > 0$. 

Thus the asymptotic structure in the large $r$ limit is AdS spacetime with the effective cosmological constant \(\frac{(1-\Lambda z^2)}{4z}(<0)\). As for the four-dimensional solution, for $z > 0$, $(\phi'(r))^2$ can be positive outside the horizon for $\Lambda < \frac{1}{z}$ and the scalar field does not become ghostlike, because $f(r) > 0$ for $r > r_h$, where $r_h$ is the position of the horizon so that $f(r_h) = 0$.

The derivative of the scalar field with respect to the proper length, $\frac{1}{\sqrt{|g|}}|\phi'(r)|$, remains finite at the horizon. As discussed in the previous section, the temperature of the black hole $\beta^{-1}$ is given by

$$\beta = \frac{12\pi z r_h}{r_h^2(1-\Lambda z) + 6z}.$$  \hfill (58)

For a large $z$ and fixed $\Lambda z$, the temperature of the five-dimensional Schwarzschild black hole $\beta = 2\pi r_h$ is recovered. On the other hand, for both large $z$ and $(-\Lambda)z$ ($\Lambda < 0$), $\beta = \frac{2\pi z}{\sqrt{1-\Lambda z}}$, which slightly does not agree with the temperature of the five-dimensional Schwarzschild-AdS black hole $\beta = \frac{2\pi r_h}{1-\frac{2}{\Lambda} r_h}$, because [53] is not precisely the same as the metric of the Schwarzschild-AdS.

For $z > 0$, the point of $g(r) = 0$, $r = \sqrt{\frac{6z}{1+\Lambda z}}$, becomes a curvature singularity because the invariant $R^\alpha\beta\mu\nu R_{\alpha\beta\mu\nu}$ diverges there. This point appears at a finite coordinate position for $\Lambda > \frac{1}{z}$, while for $\Lambda < \frac{1}{z}$ there is no such singularity.

For $z < 0$ and $\Lambda > \frac{1}{z}$, a singularity appears at $r = \sqrt{\frac{6z}{1+\Lambda z}}(<\sqrt{3z})$ other than $r = 0$.

$$ 2. \quad z = -\frac{1}{\Lambda} $$

For $z = -\frac{1}{\Lambda}$, the solution is given by

$$ f(r) = -1 - \frac{2m}{r^2} - \frac{\Lambda}{6} r^2, \quad g(r) = \frac{1}{f(r)}, \quad \phi'(r) = 0. $$  \hfill (59)

Thus the scalar field becomes trivial.

B. The solutions for $K = -1$

1. $z \neq -\frac{1}{\Lambda}$

The solution for $K = -1$ is given by

$$ f(r) = \frac{1}{24r^2z} \left( -48mz + r^4(1 - \Lambda z)^2 ight. $$
$$ + 6r^2z(-1 + \Lambda z)(3 + \Lambda z) $$
$$ + 18z^2(1 + \Lambda z)^2 \ln \left( -\frac{r^2}{2z} + 1 \right), $$
$$ g(r) = \frac{6z + r^2(-1 + \Lambda z)^2}{4(r^2 - 3z)^2 f(r)}, $$
$$ (\phi'(r))^2 = \frac{(1 + \Lambda z)(6rz + r^3(-1 + \Lambda z))^2}{4(r^2 - 3z)^3 f(r)}. $$  \hfill (60)

where the domain of the coordinate $r$ is given by $0 < r < \infty$ for $z < 0$ and $0 < r < \sqrt{3z}$ for $z > 0$. The overall normalization of $f(r)$ is chosen to recover the solution in the Einstein gravity in the small $r$ limit. We obtain

$$ f(r) = -\frac{2m}{r^2} - 1 - \frac{\Lambda r^2}{6} - \frac{(1 + \Lambda z)^2 r^4}{108z^2} + O(r^6), $$  \hfill (61)

and $g(r) \approx 1/f(r)$. The explicit dependence on the coupling constant $z$ appears in the $O(r^4)$ term, by which we could distinguish the present model from the Einstein gravity with a cosmological constant. Note that the $z$-dependent correction vanishes in the limit $z \rightarrow -\frac{1}{\Lambda}$ which corresponds to the case discussed in Sec. VI B 2. In the large $|z|$ limit, we obtain

$$ f(r) = -1 - \frac{2m}{r^2} - \frac{\Lambda}{6} r^2 - \frac{\Lambda^2}{108} r^4 + O\left(\frac{1}{|z|}\right), $$
$$ f(r)g(r) = \left(1 + \frac{r^2\Lambda}{6}\right)^2 + O\left(\frac{1}{|z|}\right). $$  \hfill (62)

For $z < 0$, in the large $r$ limit, the effects of the nonminimal derivative coupling become more important and we obtain

$$ f(r) = \frac{r^2(\Lambda z - 1)^2}{24z} + \frac{1}{4}(\Lambda z - 1)(\Lambda z + 3) $$
$$ + O(r^{-2}). $$  \hfill (63)

Thus the asymptotic structure in the large $r$ limit is dS spacetime with the effective cosmological constant $\frac{(1-\Lambda z^2)}{4z}(>0)$.

The derivative of the scalar field with respect to the proper length, $\frac{1}{\sqrt{|g|}}|\phi'(r)|$, remains finite at the horizon. For $z < 0$, the point of $g(r) = 0$, $r = \sqrt{\frac{6z}{1+\Lambda z}}$, becomes a curvature singularity because the invariant $R^\alpha\beta\mu\nu R_{\alpha\beta\mu\nu}$ diverges there. This point appears for $\Lambda < \frac{1}{z}$, while for $\Lambda > \frac{1}{z}$ there is no such singularity. For $z > 0$ and $\Lambda < -\frac{1}{z}$, a singularity appears at
TABLE III: The properties of solutions in the five-dimensional model

| Domain | $K = +1$ | $K = -1$ | $K = 0$ |
|--------|---------|---------|--------|
| $z > 0$ | $z < 0$ | $z < 0$ | $z > 0$ | $z < 0$ |
| Singularity | $r = 0$ | $r = 0$ | $r = 0$ | $r = 0$ |
| $r = \sqrt{\frac{6z}{\Lambda}}$ (\(\Lambda > \frac{1}{2}\)) | $r = \sqrt{\frac{6(z-2)}{\Lambda}}$ (\(\Lambda < -\frac{1}{2}\)) | $r = \sqrt{\frac{6z}{\Lambda}}$ (\(\Lambda < -\frac{1}{2}\)) | $r = \sqrt{\frac{6(z-2)}{\Lambda}}$ (\(\Lambda > \frac{1}{2}\)) |
| Asymptotic behavior | AdS | - | AdS | dS |

$r = \sqrt{\frac{6z}{1 - \Lambda^2}} (< \sqrt{3z})$ other than $r = 0$. As for the four-dimensional solution, for $z < 0$, $(\phi'(r))^2$ can be positive in the large-$r$ region for $\Lambda > \frac{1}{1 - z}$ and the scalar field does not become ghostlike, because of $f(r) < 0$.

2. $z = -\frac{1}{\Lambda}$

For $z = -\frac{1}{\Lambda}$, the solution is given by

$$f(r) = -1 - \frac{2m}{r^2} - \frac{\Lambda}{6} r^2, \quad g(r) = \frac{1}{f(r)},$$

$$\phi'(r) = 0.$$  \hspace{1cm} (64)

Thus the scalar field becomes trivial.

C. The solutions for $K = 0$

The solution for $K = 0$ is given by

$$f(r) = -\frac{2m}{r^2} + \frac{r^2}{24z}, \quad g(r) = \frac{1}{4f(r)},$$

$$(\phi'(r))^2 = -\frac{1 + \Lambda z}{4z f(r)}.$$  \hspace{1cm} (65)

D. Summary

In Table III, the properties of the five-dimensional solutions are summarized. The essential properties remain the same as those in the four-dimensional ones. “Domain”, “Singularities” and “Asymptotic behavior” correspond to the domain of the $r$ coordinate, the position of the curvature singularities and the asymptotic behavior of spacetime in the $r \to \infty$ limit, respectively.

VII. CONCLUSIONS

We have obtained the black hole type solutions in the scalar-tensor theory with nonminimal derivative coupling to the Einstein tensor whose action is given by (68). Although we have discussed both the four- and five-dimensional models, here we mainly summarize the properties of the four-dimensional solutions because the behaviors of the spacetime and scalar field are not sensitive to the spacetime dimensionality. Adding a cosmological constant modifies the features of the solutions in some degree. The effect of the nonminimally derivative coupling becomes more important in the large-$r$ regions, while the solutions are approximately the same as those in the Einstein gravity in the small-$r$ regions, where the coordinate $r$ given in (67) becomes either timelike or spacetime, depending on the sign of $f(r)$ and $g(r)$. For the special choice of nonminimal coupling constant $z = -\frac{1}{\Lambda}$, only the solutions with trivial scalar field configurations $\phi = \text{const}$ were obtained and the solutions in the Einstein gravity with the same cosmological constant were reproduced.

For the other choices of the coupling constant, $z \neq -\frac{1}{\Lambda}$, more nontrivial solutions were obtained. The asymptotic structure of the spacetime depends on the sign of $z$ and $K$. For a two-sphere ($K = +1$), for a positive coupling constant ($z > 0$) with a cosmological constant smaller than the inverse of the coupling constant ($\Lambda < \frac{1}{2}$) the spacetime is regular except for $r = 0$, while for $\Lambda > \frac{1}{2}$ the spacetime also becomes singular at a finite coordinate position. For a two-hyperboloid ($K = -1$), for a
negative coupling constant \((z < 0)\) with a cosmological constant larger than the inverse of the coupling constant \((\Lambda > -\frac{1}{(z^2)})\) the spacetime is regular except for \(r = 0\), while for \(\Lambda < -\frac{1}{(z^2)}\) the spacetime also becomes singular at a finite coordinate position. For a two-dimensional flat space \((K = 0)\), we recover the Bianchi I universe approaching the dS spacetime. For all the above cases, the domain of the coordinate \(r\) is given by \(0 < r < \infty\). On the other hand, in the cases of \(K > 0\) with \(z < 0\) and \(K < 0\) with \(z > 0\), the spacetime is bounded at a finite \(r\) given by the coupling constant, but the proper distance to the boundary becomes infinite. We have also found that in the case of the ordinary black hole spacetime for \(K = +1\) and \(z > 0\), \((\phi'(r))^2\) can be positive outside the horizon and the weak energy condition for the effective energy-momentum tensor of the scalar field can be satisfied only for \(\Lambda < \frac{1}{K}\). This indicates the importance of adding a nonzero cosmological constant in our model to get the black hole solutions which are healthy outside the horizon. Note that for the same values of the cosmological constant no curvature singularity is formed except for that at the center \(r = 0\).

We then summarize the asymptotic properties of the solutions where the domain of the coordinate \(r\) is given by \(0 < r < \infty\). In the large \(r\) region, the effective cosmological constant is proportional to \(-\frac{2}{z^2}\). Thus, irrespective of the value of the cosmological constant \(\Lambda\), for a two-sphere \((K = +1)\) the effective cosmological constant is negative and hence the spacetime approaches AdS, while for a two-hyperboloid \((K = -1)\), the effective cosmological constant is positive and hence the spacetime approaches dS. For a two-dimensional flat space \((K = 0)\), the metric form does not explicitly depend on \(\Lambda\), while it appears in the amplitude of the scalar field. In all the cases, the spacetime is not asymptotically flat for a nonzero \(z\).

Thermodynamic properties of our black hole solutions have also been investigated. We have computed the energy, entropy, heat capacity and Helmholtz free energy of the black holes, along the same way for the case of the Schwarzschild-AdS black hole. The expressions of these thermodynamic quantities have generalized the results obtained without a cosmological constant \(\Lambda = 0\) \([32]\) to the case with a cosmological constant. We have found that the entropy for a large black hole with \(r_h \gg \sqrt{\frac{2}{z}}\) (setting \(m_p = 1\), where \(r_h\) represents the horizon radius, can become positive only for \(\Lambda z \geq -2\), while the leading behavior of the entropy for the small black hole \(r_h \ll \sqrt{\frac{2}{z}}\) follows the ordinary area law of the black hole entropy. For \(\Lambda z = -1\), we have completely recovered thermodynamic properties of the Schwarzschild-AdS black holes. For \(-2 \leq \Lambda z < -1\), thermodynamic quantities behave in the similar way as that of the Schwarzschild-AdS black hole. For \(\Lambda z < -2\), the behavior of thermodynamic quantities is different from those of the Schwarzschild-AdS black hole and also of the case of \(\Lambda = 0\), for example, the heat capacity is positive only for the black hole with an intermediate size.

Before closing this paper, we would like to mention the subjects related to our solutions, especially focusing on the case of the black hole solutions \((K = +1)\). The asymptotically AdS property of the five-dimensional black hole solution would be useful for constructing the Randall-Sundrum-type cosmological braneworld models \([32, 33]\) in the given class of the scalar-tensor theory. The modification of the cosmological brane dynamics may appear in the following two ways: One is the modification of the bulk spacetime metric, as explicitly investigated in this paper. The other is the modification of the junction conditions because of the existence of the nonminimal derivative coupling to the Einstein tensor (see \([14]\) for the junction conditions in the generalized Galileon scalar-tensor theory). The effective four-dimensional cosmological dynamics on the brane may involve the energy exchange process between the ordinary matter localized on the brane and the scalar field. On the other hand, the investigation of the stability and the holographic interpretation of our asymptotically AdS black hole solutions would also tell us the other essential aspects of the scalar-tensor theory with nonminimal derivative coupling to the Einstein tensor.

Another important issue is whether it is possible for us to obtain the asymptotically flat black hole solutions in the theory \([1]\), which may be useful for testing our model in astrophysical environments. At least, it is quite difficult to obtain the analytic asymptotically flat black hole solutions in the present model. In integrating the scalar field equation of motion \((9)\) and obtaining \((10)\) in the four-dimensional case, we set the integration constant to be zero, which is equivalent to assume \(G_{rr} = \frac{1}{2}g_{rr}\). The situation in the five-dimensional case is also similar to that in the four-dimensional case. Thus for such a choice it is impossible to obtain the asymptotically flat solution where all components of the Einstein tensor approach zero at the asymptotic infinity. One approach to the asymptotically flat solution might be to allow for a nonzero integration constant in integrating \((9)\) but the explicit integration of the remaining equations of motion would be too involved analytically. If we assume that both \(\phi\) and \(\phi'\) are regular at the horizon, it is possible to see that for a nonzero cosmological constant a new asymptotically flat black hole solution cannot be obtained. Integrating the scalar field equation of motion \((9)\), we find

\[
\frac{f^2}{g^2} \left( \frac{r f'}{f} - \left( K g - 1 \right) - \frac{r^2 g}{z} \right) \phi' = D, \tag{66}
\]

where \(D\) is an integration constant. If we assume the existence of a new black hole solution in which \(\phi'\) is regular at the horizon and expect that in the near horizon limit it behaves as \(f \sim 0, g \sim \frac{1}{z}\) and \(f' \sim \text{const.}\), the left-hand side of \((66)\) at the horizon vanishes, leading to \(D = 0\). Therefore, within the assumption of the regular \(\phi'\) it does not allow for the solution with \(D \neq 0\) \([32]\). Moreover, \(D = 0\) gives the conclusion that the left-hand side of \((66)\) must vanish for any \(r\) and hence only the solution must be of \(\phi' = 0\) (unless \(G_{rr} = \frac{1}{2}g_{rr}\), which
The very similar expansion history of the universe, as has been investigated for the various scalar-tensor/modified gravity theories (see e.g., Refs. 44–51). We hope to come back to these issues in our future publications.

Note Added: While this paper was being completed, Refs. 51, 52 on the similar topics have appeared on arXiv. A part of the results presented in this paper have overlap with those in these papers.

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