On the Embedding of Space-Time Symmetries into Simple Superalgebras

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Abstract

We explore the embedding of Spin groups of arbitrary dimension and signature into simple superalgebras in the case of extended supersymmetry. The R-symmetry, which generically is not compact, can be chosen compact for all the cases that are congruent mod 8 to the physical conformal algebra so($D - 2,2$), $D \geq 3$. An so(1,1) grading of the superalgebra is found in all cases. Central extensions of super translation algebras are studied in this framework.
1 Introduction

Supersymmetry algebras in higher dimensions [1, 2] have been the subject of investigation in recent years in connection with their relation to string and M theories [3]. The AdS-CFT correspondence (for a review, see Ref. [4]), which describes a duality between world-volume brane theories and AdS supergravities, has renewed the interest in superconformal field theories and their underlying superalgebras [5, 6, 7, 8]. It has also been conjectured [3, 5, 6, 7, 9, 10] that brane superalgebras are closely related to the extension of superconformal algebras to larger algebras including antisymmetric tensor generators.

The most remarkable example is osp(1|32, R), with 32 real spinor charges. By contraction [11] it gives the M-theory superalgebra (super Poincaré in an 11-dimensional space of signature (10, 1)), with the maximal central extension (two and five brane charges). It also can be seen as the super conformal algebra in a space of dimension $D = 10$ and signature (9, 1) [5, 6]. The conformal group in $D = 10$ acts linearly on a space of dimension 12, (10, 2), and the spinor charge is a real Weyl spinor in this space. The (1,0) chiral $D = 10$ super Poincaré algebra with a five-brane central charge is a subalgebra of osp(1|32, R).

Moving to one dimension more, the M-theory super Poincaré algebra (with two and five-brane charges) can be regarded as a subalgebra of the superconformal algebra of a space time of dimension 11. This superalgebra is osp(1|64, R), whose odd part (64 spinorial charges) forms a real spinor in a space of dimension 13, (11, 2) [12, 13]. This example is relevant because it has been shown that M-theory can be regarded as a phase of an osp(1|64, R) theory [14]. The full symmetry is spontaneously broken, so it is realized non linearly. Some attempts to derive M-theory from a gauge theory of the group Sp(32, R) have been made in Ref. [10].

In this paper we generalize the classification of space-time superalgebras given in Ref. [13], in terms of the dimension $D$ and signature $\rho$ of space time, to extended supersymmetry algebras with an R-symmetry group acting on the conformal spinors.

Other than brane superalgebras, the present results are in relation with different versions of supersymmetric theories with $D \leq 11$, in particular with the theories introduced by Hull [15], M, M* and M’, which are formulated in 11 dimensional space-times with signatures $(s, t) = (10, 1), (9, 2), (6, 5)$ respectively. Notice that the signature for all these theories is $\rho = s - t = \pm 1$. 
mod 8. Our analysis implies that the superalgebras appearing in them can be seen as a contraction of the very same osp(1|32, R) algebra or as a subalgebra of osp(1|64, R). This suggests that they are all different phases of the same theory [15, 16].

The present study evidences the deep relation between the R-symmetry algebra and the space-time algebra (the Spin(s,t)-algebra which embeds so(s,t), according to the notation of [13]) that appear as simple factors in the bosonic part of the superconformal algebra. For the signature of the physical conformal group, (s,t) = (D − 2, 2) a compact R-symmetry group is allowed. For the Euclidean superconformal algebra, that is, for signature (s,t) = (D − 1, 1) only non compact R-symmetry groups are allowed. This implies that a proper treatment of Euclidean supersymmetric theories require superalgebras with non compact R-symmetry, as observed long ago by Zumino [17], and recently discussed in [18, 19, 20].

The paper is organized as follows. In Section 2 we extend the analysis of ref. [13] to N supersymmetries. In Section 3, we show how, in any dimension, the superconformal algebra has an so(1,1) grading. The super Poincaré algebra emerges as a non semisimple subalgebra of the superconformal algebra. In Section 4, we consider N-extended super Poincaré algebras with maximal central extension.

2 Conformal superalgebras with N supersymmetries

We recall here briefly the formalism used in Ref. [13]. Let S be a complex vector space. A conjugation of S is an antilinear map $\sigma : S \rightarrow S$,

$$\sigma(av) = a^*\psi(v), \quad a \in \mathbb{C}, \ v \in S,$$

such that $\sigma^2 = 1$. The set

$$S^\sigma = \{v \in S|\sigma(v) = v\}$$

is a real vector space. If instead $\sigma^2 = -1$ we say that $\sigma$ is a pseudoconjugation of S, and the condition $\sigma(v) = v$ is inconsistent.

Suppose now that S is a $\mathcal{G}$-module, for a complex Lie algebra $\mathcal{G}$. Let $\sigma : S \rightarrow S$ be an antilinear map satisfying $\sigma^2 = \pm 1$ (that is, $\sigma$ is a conjugation
or a pseudoconjugation on $S$). Then the map

$$\psi(X) = \sigma \circ X \circ \sigma^{-1}$$

is a conjugation of $G$. If $S$ is irreducible one can prove that $G^\psi$ is a real form of $G$. The action of $G^\psi$ on $S$ commutes with $\sigma$.

The real forms of classical Lie algebras arise in this way, except for the unitary algebras $\text{su}(p,q)$. The algebras $\text{sl}(n, \mathbb{R})$, $\text{so}(p,q)$, $\text{sp}(2p, \mathbb{R})$ correspond to conjugations on the space of the fundamental representation and the algebras $\text{su}^*(2p)$, $\text{so}^*(2p)$ and $\text{usp}(2p, 2q)$ correspond to pseudoconjugations.

Let $V$ be a real vector space of dimension $D = s + t$ with a non degenerate symmetric bilinear form of signature $\rho = s - t$. We denote by Spin$(s,t)$ the corresponding spin group. Spinor representations are linear, orthogonal or symplectic depending on the existence of a Spin$(s,t)$-invariant bilinear form, which can be symmetric or antisymmetric. The existence and symmetry of the bilinear form depend exclusively on the dimension $D$ modulo 8 \[21\]. The complexified group Spin$(s,t)^\mathbb{C}$ is then embedded into a classical Lie group, Sl$(n, \mathbb{C})$, SO$(n, \mathbb{C})$ or Sp$(n, \mathbb{C})$ where $n$ is the dimension of the spinor representation. Denoting by $D = D_0 \mod(8)$ and $\rho = \rho_0 \mod(8)$, we have that for $D_0 = 2, 6$ the spinors are linear, for $D_0 = 0, 1, 7$ they are orthogonal and for $D_0 = 3, 4, 5$ they are symplectic.

The existence of a conjugation or a pseudoconjugation in the space $S$ of the spinor representation (commuting with the action of the orthogonal algebra) depends on the signature $\rho \mod 8$. We have then that $o(s,t)$ is embedded into a real form of the above algebra, which will be determined in terms of $\rho$. This real algebra is called the Spin$(s,t)$ algebra. In Table 1 the real form for each case is listed \[13\]. The odd and even cases are treated separately.

We consider now superalgebras containing $\text{so}(s,t)$ in the even part and whose odd part consists on one or more copies of the spinor representation of $\text{so}(s,t)$. If the superalgebra is required to be simple, then the anticommutators of the odd charges generate the whole even subalgebra. In particular, the generators of the orthogonal group in the spinor representation must have the appropriate symmetry to appear in the right hand side of the anticommutator of two spinors. This analysis was carried out in Ref. \[13\] for $N = 1$ supersymmetry. The smallest simple superalgebra containing $\text{so}(s,t)$ has even part equal to the Spin$(s,t)$ algebra, listed in Table 1, times an R-symmetry algebra which is usp$(2)$ or so$^*(2)$ (quaternionic case) or $\text{u}(1)$
This superalgebra is called Spin(\(s,t\)) superalgebra. The same structure appears in the case of extended supersymmetry, although the generalization is not completely straightforward since new cases appear due to the presence of the internal index space.

The odd part of the superalgebra is, as a vector space, a tensor product \(S \otimes W\) of the spinor representation space \(S\) with the R-symmetry space \(W\). If the spin algebra is a symplectic algebra, the R-symmetry is an orthogonal algebra and they build an orthosymplectic algebra. If the spin algebra is orthogonal, then the R-symmetry is symplectic and they give an orthosymplectic algebra with the roles of the orthogonal and symplectic groups interchanged.

\footnote{Except for \(D = 7, \rho = 3\), where a smaller simple superalgebra, the exceptional superalgebra \(f(4)\), contains the orthogonal group in its even part.}

### Table 1: Spin(\(s,t\)) algebras.

| Type       | \(D_0\) | \(\rho_0\) | Algebra                                                                 | Algebra                                                                 |
|------------|---------|------------|------------------------------------------------------------------------|------------------------------------------------------------------------|
| Orthogonal | \(D_0 = 1, 7\) | \(\rho_0 = 1, 7\) | \(\text{so}(2^{(D-1)/2}, \mathbb{R})\) if \(D = \rho\) \(\text{so}(2^{(D-1)/2}, 2^{(D-1)/2})\) if \(D \neq \rho\) | \(\text{so}^*(2^{(D-1)/2})\)                                           |
| Quaternionic | \(\rho_0 = 3, 5\) | Quaternionic | \(\text{so}^*(2^{(D-1)/2})\)                                           | \(\text{so}^*(2^{(D-1)/2})\)                                           |
| Symplectic  | \(D_0 = 3, 5\) | \(\rho_0 = 1, 7\) | \(\text{sp}(2^{(D-1)/2}, \mathbb{R})\) if \(D = \rho\) \(\text{sp}(2^{(D-1)/2}, 2^{(D-1)/2})\) if \(D \neq \rho\) | \(\text{sp}(2^{(D-1)/2}, \mathbb{C})\) \(\mathbb{R}\) |
| Quaternionic | \(\rho_0 = 3, 5\) | Quaternionic | \(\text{so}(2^{(D-1)/2}, \mathbb{R})\) if \(D = \rho\) \(\text{so}(2^{(D-1)/2}, 2^{(D-1)/2})\) if \(D \neq \rho\) | \(\text{so}(2^{(D-1)/2}, \mathbb{C})\) \(\mathbb{R}\) |
| Complex    | \(\rho_0 = 2, 6\) | Complex    | \(\text{so}(2^{(D-1)/2}, \mathbb{C})\) \(\mathbb{R}\)                  | \(\text{so}(2^{(D-1)/2}, \mathbb{C})\) \(\mathbb{R}\)                  |
| Linear     | \(D_0 = 0\) | \(\rho_0 = 0\) | \(\text{sl}(2^{(D-1)/2}, \mathbb{R})\) if \(D = \rho\) \(\text{su}^*(2^{(D-1)/2})\) if \(D \neq \rho\) | \(\text{sl}(2^{(D-1)/2}, \mathbb{R})\) if \(D = \rho\) \(\text{su}^*(2^{(D-1)/2})\) if \(D \neq \rho\) |
| Quaternionic | \(\rho_0 = 4\) | Quaternionic | \(\text{su}(2^{(D-1)/2}, \mathbb{R})\) if \(D = \rho\) \(\text{su}(2^{(D-1)/2}, 2^{(D-1)/2})\) if \(D \neq \rho\) | \(\text{su}(2^{(D-1)/2}, \mathbb{R})\) if \(D = \rho\) \(\text{su}(2^{(D-1)/2}, 2^{(D-1)/2})\) if \(D \neq \rho\) |
| Complex    | \(\rho_0 = 2, 6\) | Complex    | \(\text{su}(2^{(D-1)/2}, \mathbb{C})\) \(\mathbb{R}\)                  | \(\text{su}(2^{(D-1)/2}, \mathbb{C})\) \(\mathbb{R}\)                  |
terchanged. If the spin algebra is a linear (unitary) algebra, then the R-
symmetry is also a linear (unitary) algebra and they build a superalgebra
form the linear (unitary) series.

In order to obtain a real superalgebra, a conjugation commuting with the
action of the even part of the superalgebra must exist in the total space $S \otimes W$. If the spinor is real then there is a conjugation $\sigma_S$ of $S$ commuting with the
action of the Spin($s, t$) algebra. The R-symmetry factor which acts on $W$ also
commutes with a conjugation, $\sigma_W$. If the spinor is quaternionic then there
is a pseudoconjugation $\sigma_S$ on $S$, and there is also a pseudoconjugation on $W$
commuting with the R-symmetry. Then $\sigma_S \otimes \sigma_W$ is a conjugation in the total
space. If the spinor is complex, the Spin($V$) algebra is either su($p, q$) or a
complex group (symplectic or orthogonal). In the first case the R-symmetry
is also su($m, n$) and in the second case the R-symmetry is a complex group
(orthogonal or symplectic respectively). The real representation is obtained
by taking the complex vector space as a real one of twice the dimension.

As an example, we compute the case $D_0 = 1, 7$, $\rho_0 = 1, 7$. The spinors
are orthogonal and real. The anticommutator of two odd generators is of the
form

$$\{Q^i_\alpha, Q^j_\beta\} = \sum_k A^{ij}_{\alpha\beta} [\mu_1 \cdots \mu_k] Z_{[\mu_1 \cdots \mu_k]},$$

(1)

The generators of the orthogonal group are $Z_{[\mu_1 \mu_2]}$. For $N = 1$ the factor $A^{ij}$
is not present. Then, in order to have $Z_{[\mu_1 \mu_2]}$ in the right hand side of (1),
the morphism $\gamma^{[\mu_1 \mu_2]}_{\alpha\beta}$ must be symmetric. Since it is antisymmetric, there
is no superconformal algebra in this case. For $N > 1$, one can choose an
antisymmetric matrix $A^{ij} = \epsilon^{ij}$ and the orthogonal generators are allowed in
the right hand side of (1). If follows that the R-symmetry group is Sp(2$N, \mathbb{R}$).

In Table 2 we list the R-symmetry groups and the Spin($s, t$) superalgebras.
The compact cases $D = \rho$ are not listed but they are immediate. We mark
with the symbol “◦” the cases that do not arise in the non extended case.

The cases marked with a symbol “∗” allow the possibility of a compact R-
symmetry group. They correspond to so($s, 2$), that is the physical conformal
groups.
Table 2: Spin($s,t$) superalgebras.

| $D_0$ | $\rho_0$ | R-symmetry     | Spin($s,t$) superalgebra                                      |
|-------|-----------|----------------|-------------------------------------------------------------|
| ⨇     | 1,7       | $sp(2N,\mathbb{R})$ | $osp(2^{\frac{D-1}{2}}, 2^{\frac{D-1}{2}}|2N,\mathbb{R})$ |
| ⨊     | 1,7       | $usp(2N-2q,2q)$    | $osp(2^{\frac{D-4}{2}}|2N-2q,2q)$                          |
| ⨊     | 3,5       | $so(N-q,q)$       | $osp(N-q,q|2^{\frac{D-1}{2}})$                            |
|       | 1,7       | $so^*(2N)$        | $osp(2^{\frac{D-1}{2}}, 2^{\frac{D-1}{2}}|2N)$           |
|       | 0, 2,6    | $sp(2N,\mathbb{C})_{\mathbb{R}}$ | $osp(2^{\frac{D-4}{2}}|2N,\mathbb{C})_{\mathbb{R}}$ |
| ⨊     | 4, 2,6    | $sl(N,\mathbb{R})$ | $sl(2^{\frac{D-4}{2}}|N,\mathbb{R})$                      |
| ⨊     | 4, 2,6    | $su(N-q,q)$       | $su(2^{\frac{D-4}{2}}, 2^{\frac{D-4}{2}}|N-q,q)$          |
|       | 2,6       | $su^*(2N,\mathbb{R})$ | $su(2^{\frac{D-4}{2}}|2N)^*$                               |
| ⨊     | 0, 4      | $so(N-q,q)$       | $osp(N-q,q|2^{\frac{D-1}{2}})$                            |
|       | 4, 2,6    | $so(N,\mathbb{C})_{\mathbb{R}}$ | $osp(N|2^{\frac{D-4}{2}},\mathbb{C})_{\mathbb{R}}$ |
|       | 4, 4      | $so^*(2N)$        | $osp(2^{\frac{D-4}{2}}|2N^*,2^{\frac{D-4}{2}})$           |

3 \textbf{so}(1, 1) grading of the Spin($s,t$) superalgebra

Let $G^k$ be a compact semisimple Lie algebra, its complexification being $G^c$. Let $\theta : G^k \rightarrow G^k$ be an involutive automorphism, $\theta^2 = 1$. $G^k$ splits into two eigenspaces,

$$G^k = K + P.$$ 

$K$ is the eigenspace with eigenvalue $+1$ of $\theta$, and $P$ the eigenspace with eigenvalue $-1$. The vector space

$$G = L_0 + iP$$

is a non compact real form of $G^c$. (2) is called a Cartan decomposition of $G$ and the procedure is known as the Weyl unitary trick. All the Cartan decompositions are listed in Ref.[22]. $L_0$ is the maximal compactly embedded subalgebra of $G$, and $P$ carries an irreducible representation of $L_0$.

If $G$ is simple, then the algebra $K$ is either semisimple or is a semisimple algebra plus a $u(1)$ factor. For example, let $G^k = so(p + q)$ and $\theta_{p,q}(X) =
$I_{p,q}X_{p,q}$, where

$$I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}.$$ 

Then $\mathcal{K} = \text{so}(p) \oplus \text{so}(q)$. $\mathcal{P}$ is the bifundamental representation of $\text{so}(p) \oplus \text{so}(q)$, $(p,q)$.

One can apply the Weyl unitary trick to $G$ with respect to a Cartan involution $\theta'$ that commutes with $\theta$. $G$ splits into four eigenspaces of the simultaneous eigenvalues of $\theta$ and $\theta'$,

$$G = G^{++} + G^{+-} + iG^{-+} + iG^{--}.$$ 

The first sign is the eigenvalue of $\theta$ and the second is the eigenvalue of $\theta'$. So

$$\mathcal{K} = G^{++} + G^{-+}, \quad \mathcal{P} = G^{+-} + G^{--}.$$ 

The Lie algebra

$$G' = G^{++} + iG^{+-} + iG^{-+} + G^{--}$$

is another non compact form of $G^c$ (corresponding to the Cartan involution $\theta' \circ \theta$). The maximal compact subalgebra is $G^{++} + G^{--}$. We take the decomposition

$$L_0 = G^{++} + iG^{+-}, \quad \mathcal{P}' = iG^{-+} + G^{--}.$$ 

$L_0$ is a non compact real form of the complexification of $\mathcal{K}$.

In the example of $\text{so}(d)$, we can take $\theta = \theta_{d-2,2}$ and $\theta' = \theta_{s,t}$, $s + t = d$. Then

$$G' = \text{so}(s,t), \quad L_0 = \text{so}(s-1,t-1) \oplus \text{so}(1,1)$$

and the bifundamental representation $(d-2,2)$ splits into two irreducible representations, which are irreducible with respect to $\text{so}(1,1)$, $(d-2)^{t+1} \oplus (d-2)^{-1}$. This is in fact the splitting of the conformal algebra with respect to the Lorentz subalgebra times the dilatation. The vector space $\mathcal{P}'$ contains the translations $P_\mu$ and the conformal boosts $K_\mu$.

The Lie algebra $G' = \text{so}(s,t)$ has a Lie algebra grading,

$$G' = L_{-1} + L_0 + L_{+1},$$

being the grade the charge with respect to $\text{so}(1,1)$. $L_{\pm 1}$ are abelian subalgebras and $L_0 \otimes L_{\pm 1}$ is the Poincaré algebra.

The same procedure can be applied to all the Spin$(s,t)$ algebras in Table II, and a grading is always found with respect to a $\text{so}(1,1)$ subalgebra, which
can always be identified with the dilatation of the conformal group embedded into the Spin(s, t) algebra. \( L_0 \otimes L_{\pm 1} \) is a semidirect sum of algebras that generalizes the super Poincaré algebra. \( L_{\pm 1} \) are abelian algebras containing the translations and central charges. These decompositions are listed in [23] for all the simple Lie algebras. In Table 3 we give \( L_0 \) for all the Spin(s, t) algebras.

| Spin(s, t) algebra | \( L_0 \) | Fundamental Representation |
|---------------------|---------|---------------------------|
| so(n, n)            | sl(n, \( \mathbb{R} \)) \( \oplus \) so(1, 1) | \( 2n = (n)^{1/2} \oplus (n')^{-1/2} \) |
| so\(^\ast\)(4n)     | su\(^\ast\)(2n) \( \oplus \) so(1, 1) | \( 4n = (2n)^{1/2} \oplus (2n')^{-1/2} \) |
| so(2n, \( \mathbb{C} \)) | gl(n, \( \mathbb{C} \)) | \( 2n = (n)^{1/2} \oplus (n')^{-1/2} \) |
| sp(2n, \( \mathbb{R} \)) | sl(n, \( \mathbb{R} \)) \( \oplus \) so(1, 1) | \( 2n = (n)^{1/2} \oplus (n')^{-1/2} \) |
| usp(2n, 2n)         | su\(^\ast\)(2n) \( \oplus \) so(1, 1) | \( 4n = (2n)^{1/2} \oplus (2n')^{-1/2} \) |
| sp(2n, \( \mathbb{C} \)) | gl(2n, \( \mathbb{C} \)) | \( 2n = (n)^{1/2} \oplus (n')^{-1/2} \) |
| sl(2n, \( \mathbb{R} \)) | sl(n, \( \mathbb{R} \)) \( \oplus \) sl(n, \( \mathbb{R} \)) \( \oplus \) so(1, 1) | \( 2n = (n, 1)^{1/2} \oplus (1, n')^{-1/2} \) |
| su\(^\ast\)(4n)     | su\(^\ast\)(2n) \( \oplus \) su\(^\ast\)(2n) \( \oplus \) so(1, 1) | \( 4n = (2n, 1)^{1/2} \oplus (1, 2n')^{-1/2} \) |
| su(n, n)            | sl(n, \( \mathbb{C} \)) \( \oplus \) so(1, 1) | \( 2n = (n)^{1/2} \oplus (n')^{-1/2} \) |

Table 3: so(1, 1)-grading.

When the Spin(s, t) algebra is an orthogonal algebra, \( L_{\pm 1} \) are in the two-fold antisymmetric representation of \( L_0 \). When the Spin(s, t) algebra is a symplectic algebra, then \( L_{\pm 1} \) are in the two-fold symmetric representation of \( L_0 \). When the Spin(s, t) algebra is a linear algebra, then \( L_{\pm 1} \) are in the bifundamental representation of \( L_0 \).

The spinor representation of so(s, t), \( S_{(s,t)} \) decomposes as

\[
S_{(s,t)} \xrightarrow{so(s-1,t-1) \oplus so(1,1)} S_{(s-1,t-1)}^{1/2} \oplus S_{(s-1,t-1)}^{-1/2}.
\]

If \( D \) is even, then a chiral representation decomposes into two representations with opposite chirality.

so(s − 1, t − 1) is embedded into \( L_0 \). When promoting the spinor representation of so(s, t) to the fundamental representation of the Spin(s, t) algebra,
the splitting (3) corresponds to the splitting under $L_0$, which is given also in Table 3. $n'$ denotes the dual representation of $n$.

In fact, one can check that $L_0$ contains in each case, the full Spin($s-1, t-1$)-algebra. The embeddings are given in Table 4.

| Real case               | Quaternionic case         | Complex case               |
|-------------------------|---------------------------|----------------------------|
| $\mathfrak{sl}(2n, \mathbb{R}) \supset \mathfrak{so}(n, n)$ | $\mathfrak{su}^*(2n) \supset \mathfrak{so}^*(2n)$ | $\mathfrak{sl}(2n, \mathbb{C}) \supset \mathfrak{so}(2n, \mathbb{C})$ |
| $\mathfrak{sl}(2n, \mathbb{R}) \supset \mathfrak{sp}(2n, \mathbb{R})$ | $\mathfrak{su}^*(2n) \supset \mathfrak{usp}(n, n)$ | $\mathfrak{sl}(2n, \mathbb{C}) \supset \mathfrak{sp}(2n, \mathbb{C})$ |

Table 4: Embedding of Spin($s, t$) algebras.

The superalgebras of Table 4 have also a so($1, 1$) grading

$$S = L_{-1} + S_{-1/2} + L_0 + S_{1/2} + L_{1}.$$  

In the case of extended supersymmetry, $L_0$ contains also the R-symmetry factor. The spaces $L_{\pm 1}$ have a simple meaning in terms of the $\gamma$-matrices of so($s-1, t-1$). From the grading properties and the simplicity of the superalgebra, it is clear that

$$\{S_{\pm 1/2}, S_{\pm 1/2}\} = L_{\pm 1}.$$  

In fact, $L_{\pm}$ are irreducible representations of $L_0$. The anticommutator

$$\{Q_{\pm 1/2}, Q_{\pm 1/2}\}$$

can in general be written as in (1), and the term $\gamma^\mu$ ($\mu = 1, \ldots, s + t - 2$) appears since it corresponds to the momentum (with grade 1). It follows that in the r.h.s of (1) will appear only terms corresponding to matrices $\gamma^{[\mu_1 \ldots \mu_m]}$ with the same symmetry properties as $\gamma^\mu$. In fact, from dimensional considerations, all of these terms appear. For $s + t = 7, 8, 9$ mod 8, the $\gamma^\mu$ are antisymmetric and dim($L_{\pm}$) = $n(n-1)/2$. For $s + t = 3, 4, 5$ mod 8, they are symmetric and dim($L_{\pm}$) = $n(n+1)/2$. For $s + t = 2, 6$ mod 8, the relevant anticommutator is a left spinor with a right spinor. In this case the morphisms (coefficients in the r.h.s. of (1)) have no definite symmetry and dim($L_{\pm}$) = $n^2$.  

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4 The orthosymplectic algebra and the maximal central extension of Poincaré supersymmetry

Let $2n$ be the real dimension of a spinor representation and $N$ the number of such spinors present in the superalgebra. The spinor charges are denoted $Q^i_\alpha, \quad i = 1, \ldots N, \alpha = 1, \ldots 2n$.

Since the anticommutator $\{Q^i_\alpha, Q^j_\beta\}$ is symmetric, the biggest simple superalgebra containing only these odd generators (maximal $\text{Spin}(s,t)$ algebra in the language of Ref. [13]) is $\text{osp}(1|2nN, \mathbb{R})$, with bosonic part $\hat{\mathcal{L}} = \text{sp}(2nN, \mathbb{R})$. It is clear that $\mathcal{L}$ contains as a subalgebra the $\text{Spin}(s,t)$ algebra plus the $R$-symmetry,

$$\mathcal{L} \oplus R\text{–symmetry} \subset \hat{\mathcal{L}}.$$ 

We want to show that the $\text{so}(1,1)$ grading of $\mathcal{L}$ extends to $\hat{\mathcal{L}}$ and to $\text{osp}(1|2nN)$. We have that

$$\hat{\mathcal{L}} = \hat{\mathcal{L}}_+ + \hat{\mathcal{L}}_0 + \hat{\mathcal{L}}_-,$$

where $\hat{\mathcal{L}}_0 = \text{sl}(nN, \mathbb{R}) \oplus \text{so}(1,1)$ and $\hat{\mathcal{L}}_{\pm 1}$ are in the two-fold symmetric representation of $\text{sl}(nN, \mathbb{R})$ with charges $\pm 1$ with respect to $\text{so}(1,1)$. To show that this grading is compatible with the one of the $\text{Spin}(s,t)$ algebra, we have to show that

$$\mathcal{L}_0 \oplus R\text{–symmetry} \subset \hat{\mathcal{L}}_0.$$ 

We consider the complex linear algebra

$$\text{sl}(nN, \mathbb{C}) \cong \text{gl}(n, \mathbb{C}) \otimes \text{gl}(N, \mathbb{C})/\mathbb{C}^*.$$ 

(The bracket in the tensor product of algebras is defined as

$$[a \otimes a', b \otimes b'] = [a, b] \otimes [a', b'].$$

One has that

$$\text{sl}(n, \mathbb{C}) \simeq \text{sl}(n, \mathbb{C}) \otimes \text{Id} \subset \text{sl}(nN, \mathbb{C})$$

$$\text{sl}(N, \mathbb{C}) \simeq \text{Id} \otimes \text{sl}(N, \mathbb{C}) \subset \text{sl}(nN, \mathbb{C}),$$

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and that
\[
\text{sl}(n, \mathbb{C}) \oplus \text{sl}(N, \mathbb{C}) \subset \text{sl}(nN, \mathbb{C}).
\]

Notice that the fundamental representation of \(\text{sl}(nN, \mathbb{C})\) becomes the bifundamental \((\mathbb{C}^n \otimes \mathbb{C}^N)\) of \(\text{sl}(n, \mathbb{C}) \oplus \text{sl}(N, \mathbb{C})\). The adjoint of \(\text{sl}(nN, \mathbb{C})\) decomposes under \(\text{sl}(n, \mathbb{C}) \oplus \text{sl}(N, \mathbb{C})\) in the adjoint of \(\text{sl}(n, \mathbb{C}) \oplus \text{sl}(N, \mathbb{C})\) plus the tensor products of the adjoints of \(\text{sl}(n, \mathbb{C})\) and \(\text{sl}(N, \mathbb{C})\).

Let \(\mathcal{G}_n \subset \text{sl}(n, \mathbb{C})\) and \(\mathcal{G}_N \subset \text{sl}(N, \mathbb{C})\). Then we have
\[
\mathcal{G}_n \oplus \mathcal{G}_N \subset \text{sl}(nN, \mathbb{C}).
\]

In particular, \(\mathcal{G}_n\) and \(\mathcal{G}_N\) can be orthogonal, symplectic, or the linear algebras themselves. We have just to check that the appropriate real forms are contained in \(\text{sl}(nN, \mathbb{R})\). We recall that the real form \(\text{sl}(nN, \mathbb{R})\) is obtained from a conjugation in the fundamental representation space. Moreover, \(\text{sl}(nN, \mathbb{R})\) is the set of all traceless matrices in \(\text{sl}(nN, \mathbb{C})\), commuting with a conjugation in \(\mathbb{C}^{nN}\) (which can be brought by an isomorphism to the standard conjugation on \(\mathbb{C}^{nN}\)).

The direct sum of real forms \(\mathcal{G}_n^\prime \oplus \mathcal{G}_N^\prime\) is contained in \(\text{sl}(nN, \mathbb{R})\) if the bifundamental representation of \(\mathcal{G}_n^\prime \oplus \mathcal{G}_N^\prime\) commutes with a conjugation. This happens when both, \(\mathcal{G}_n^\prime\) and \(\mathcal{G}_N^\prime\), commute with a conjugation in their respective spaces, \(\mathbb{C}^n\) and \(\mathbb{C}^N\), and when they both commute with a pseudoconjugation.

If the algebras are complex, then one can see \((\mathcal{G}_n^\prime)_{\mathbb{R}}\) and \((\mathcal{G}_N^\prime)_{\mathbb{R}}\) inside a real group
\[
\begin{align*}
(\mathcal{G}_n^\prime)_{\mathbb{R}} & \subset \text{sl}(n^\prime, \mathbb{C})_{\mathbb{R}} \subset \text{sl}(2n^\prime, \mathbb{R})) \subset \text{sp}(4n^\prime, \mathbb{R}), \\
(\mathcal{G}_N^\prime)_{\mathbb{R}} & \subset \text{sl}(N^\prime, \mathbb{C})_{\mathbb{R}} \subset \text{sl}(2N^\prime, \mathbb{R})) \subset \text{sp}(4N^\prime, \mathbb{R})
\end{align*}
\]
with \(n = 2n^\prime, N = 2N^\prime\).

The unitary algebras are embedded into complex algebras that one can see as real, and then embedded in a linear complex group as above (3).

We give the embeddings for all dimensions and signatures in Table 5. The real dimension of the spinor representation. is given in terms of \(D\) and depends on the reality properties of the spinor. For clarity, it is given in Table 6.

The grading extends to the orthosymplectic superalgebra \(\hat{\mathcal{S}} = \text{osp}(1|2m, \mathbb{R})\), where \(m\) is the number appearing in the third column of Table 5.

\[
\hat{\mathcal{S}} = \hat{\mathcal{L}}_{-1} + \hat{\mathcal{S}}_{-1/2} + \hat{\mathcal{L}}_0 + \hat{\mathcal{S}}_{1/2} + \hat{\mathcal{L}}_1.
\]
Table 5: Graded embeddings

| $D_0$ | $\rho_0$ | $\mathcal{L}_0$ | $\mathcal{L}_0 \oplus \mathbb{R}$–symmetry |
|-------|-------|----------------|---------------------------------|
| 1,7   | 1,7   | $\text{sl}(2N^{D-2}/2, \mathbb{R})$ | $\text{sl}(2^{(D-2)/2}, \mathbb{R}) \oplus \text{sp}(2N, \mathbb{R})$ |
| 0     | 0     | $\text{sl}(2N^{D-2}/2, \mathbb{R})$ | $\text{sl}(2^{(D-2)/2}, \mathbb{R}) \oplus \text{sp}(2N, \mathbb{R})$ |
| 1,7   | 3,5   | $\text{sl}(2N^{D-2}/2, \mathbb{R})$ | $\text{su}^*(2^{(D-2)/2}) \oplus \text{usp}(2N-2q, 2q)$ |
| 3,5   | 1,7   | $\text{sl}(N^{D-3}/2, \mathbb{R})$ | $\text{su}^*(2^{(D-3)/2}) \oplus \text{so}^*(2N)$ |
| 4     | 0     | $\text{sl}(N^{D-3}/2, \mathbb{R})$ | $\text{su}^*(2^{(D-3)/2}) \oplus \text{so}^*(2N)$ |
| 3,5   | 3,5   | $\text{sl}(N^{D-3}/2, \mathbb{R})$ | $\text{su}^*(2^{(D-3)/2}) \oplus \text{so}^*(2N)$ |
| 4     | 4     | $\text{sl}(2N^{D-2}/2, \mathbb{R})$ | $\text{su}^*(2^{(D-2)/2}) \oplus \text{so}^*(2N)$ |
| 0     | 2,6   | $\text{sl}(2N^{D-2}/2, \mathbb{R}) \supset \text{sl}(2N^{D-2}/2, \mathbb{C})$ | $\text{sl}(2^{(D-2)/2}, \mathbb{C}) \oplus \text{sp}(2N, \mathbb{C})$ |
| 2,6   | 0     | $\text{sl}(N^{D-1}/2, \mathbb{R})$ | $\text{sl}(2^{(D-1)/2}, \mathbb{R}) \oplus \text{sl}(2^{(D+1)/2}, \mathbb{R}) \oplus \text{sl}(N, \mathbb{R})$ |
| 2,6   | 2,6   | $\text{sl}(N^{D-1}/2, \mathbb{R}) \supset \text{sl}(N^{D-1}/2, \mathbb{C})$ | $\text{sl}(2^{(D-1)/2}, \mathbb{C}) \oplus \text{su}(N-q, q)$ |
| 2,6   | 4     | $\text{sl}(N^{D-1}/2, \mathbb{R})$ | $\text{su}^*(2^{(D-1)/2}) \oplus \text{su}^*(2^{(D+1)/2}) \oplus \text{su}^*(2N)$ |
| 4     | 2,6   | $\text{sl}(N^{D-1}/2, \mathbb{R}) \supset \text{sl}(N^{D-1}/2, \mathbb{C})$ | $\text{sl}(2^{(D-1)/2}, \mathbb{C}) \oplus \text{so}(N, \mathbb{C})$ |
| 0     | 4     | $\text{sl}(2N^{D-2}/2, \mathbb{R})$ | $\text{su}^*(2^{(D-2)/2}, \mathbb{C}) \oplus \text{usp}(2N-2q, 2q)$ |

To see this, it is enough to give the decomposition of the fundamental representation of $\text{sp}(2m, \mathbb{R})$ with respect to $\text{sl}(m, \mathbb{R}) \oplus \text{so}(1, 1)$,

$$
(2m) \xrightarrow{\text{sl}(m, \mathbb{R}) \oplus \text{so}(1, 1)} (m)^{1/2} \oplus (m')^{-1/2}.
$$

Finally, $\hat{S}_{+1/2} + \hat{L}_{+1}$ is a superalgebra which is the maximal central extension of the supertranslation algebra.

Table 6: Real dimensions of spinor representations

| $\rho_0$ (odd) | real dim ($S$) | reality | $\rho_0$ (even) | real dim ($S^\pm$) | reality |
|---------------|---------------|---------|-----------------|-------------------|---------|
| 1             | $2^{(D-1)/2}$ | $\mathbb{R}$ | 0               | $2^{D/2}$         | $\mathbb{R}$ |
| 3             | $2^{(D+1)/2}$ | $\mathbb{H}$ | 2               | $2^{D/2}$         | $\mathbb{C}$ |
| 5             | $2^{(D+1)/2}$ | $\mathbb{H}$ | 4               | $2^{D/2}$         | $\mathbb{H}$ |
| 7             | $2^{(D-1)/2}$ | $\mathbb{R}$ | 6               | $2^{D/2}$         | $\mathbb{C}$ |
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