Abstract. We give a characterization of $\tau$-rigid modules over Auslander algebras in terms of projective dimension of modules. Moreover, we show that for an Auslander algebra $\Lambda$ admitting finite number of non-isomorphic basic tilting $\Lambda$-modules and tilting $\Lambda^{op}$-modules, if all indecomposable $\tau$-rigid $\Lambda$-modules of projective dimension 2 are of grade 2, then $\Lambda$ is $\tau$-tilting finite.

1. Introduction

Recently Adachi, Iyama and Reiten [AIR] introduced $\tau$-tilting theory to generalize the classical tilting theory in terms of mutations. $\tau$-tilting theory is close to the silting theory introduced by [AiI] and the cluster tilting theory in the sense of [KR, IY, BMRRT].

Note that $\tau$-tilting theory depends on $\tau$-rigid modules. So it is very interesting to find all $\tau$-rigid modules for a given algebra. There are some works on this topic (See [A1, A2, AAC, IJY, IRRT, J, M, HuZh, AnMV, W, Z] and so on). In particular, Iyama and Zhang [IZ] classified all the support $\tau$-tilting modules and indecomposable $\tau$-rigid modules for the Auslander algebra $\Gamma$ of $K[x]/(x^n)$. They showed that the number of non-isomorphic basic support $\tau$-tilting $\Gamma$-modules is exactly $(n+1)!$. For an arbitrary Auslander algebra $\Lambda$, little is known on $\tau$-rigid $\Lambda$-modules. So a natural question is:

**Question 1.1. How to judge $\tau$-rigid modules over an arbitrary Auslander algebra?**

Our first goal in this paper is to give a partial answer to this question. Throughout this paper all algebras are finite-dimensional algebras over a field $K$ and all modules are finitely generated right modules.

For an algebra $\Lambda$, denote by $(-)^*$ the functor $\text{Hom}_\Lambda(-,\Lambda)$. For a $\Lambda$-module $M$, denote by $\text{pd}_\Lambda M$ (resp. $\text{id}_\Lambda M$) the projective dimension (resp. injective dimension) of $M$. Denote by $\text{grade} M$ the grade of $M$. Then we have the following theorem.

**Theorem 1.2.** (Theorems 3.3 and 3.10, Corollary 3.7) Let $\Lambda$ be an Auslander algebra and $M$ a $\Lambda$-module. Then we have the following:

1. Every simple module $S$ is $\tau$-rigid.
2. If $\text{pd}_\Lambda M = 1$, then $M$ is $(\tau)$-rigid if and only if $\text{Ext}^2_\Lambda(N, M) = 0$, where $N = M^{**}/M$.
3. If $\text{grade} M = 2$, then $M$ is $\tau$-rigid if and only if $\text{Tr} M$ is $\tau$-rigid with $\text{pd}_\Lambda \text{Tr} M = 1$.
4. If $\Lambda$ admits a unique simple module $S$ with $\text{pd}_\Lambda S = 2$, then
   a. Every indecomposable module $M$ with $\text{pd}_\Lambda M = 1$ is $(\tau)$-rigid.
   b. All indecomposable $\tau$-rigid $\Lambda$-modules $N$ with $\text{pd}_\Lambda N = 2$ are of grade 2.

On the other hand, Demonet, Iyama and Jasso gave a general description of algebras with finite number of support $\tau$-tilting modules in [DIJ] where they call the algebras $\tau$-tilting finite algebras. It is clear that an algebra $\Lambda$ is $\tau$-tilting finite if and only if so is its opposite algebra $\Lambda^{op}$. We should remark that an algebra is $\tau$-tilting finite implies that there are finite number of non-isomorphic basic tilting $\Lambda$-modules and tilting $\Lambda^{op}$-modules. It is natural to consider the following question.

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Question 1.3. When is an algebra admitting finite number of basic tilting $\Lambda$-modules and tilting $\Lambda^{op}$-modules $\tau$-tilting finite?

It is obvious that algebras of finite representation type are both tilting-finite and $\tau$-tilting finite. However, we need a non-trivial case. Our second goal of this paper is to give a more general answer to this question whenever $\Lambda$ is an Auslander algebra. We prove the following theorem in which the algebra is not necessary to be an Auslander algebra.

Theorem 1.4. (Theorem [2,4]) Let $\Lambda$ be an algebra of global dimension 2 admitting finite number of basic tilting $\Lambda$-modules and tilting $\Lambda^{op}$-modules. If all indecomposable $\tau$-rigid modules with projective dimension 2 are of grade 2, then $\Lambda$ is $\tau$-tilting finite.

The paper is organized as follows:

In Section 2, we recall some preliminaries. In Section 3, we prove the main results and give some examples to show the main results.

Throughout this paper, all algebras $\Lambda$ are basic connected finite dimensional algebras over an algebraic closed field $K$ and all $\Lambda$-modules are finitely generated right modules. Denote by $\text{mod}\Lambda$ the category of finitely generated right $\Lambda$-modules. For $M \in \text{mod}\Lambda$, denote by $\text{add}M$ the subcategory of direct summands of finite direct sum of $M$. We use $\text{Tr}$ to denote the Auslander transpose of $M$. Denote by $\tau$ the $AR$-translation and denote by $|M|$ the number of non-isomorphic indecomposable direct summands of $M$.

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2. Preliminaries

In this section we recall some basic preliminaries for later use. For an algebra $\Lambda$, denote by $\text{gl.dim}\Lambda$ the global dimension of $\Lambda$. We begin with the definition of Auslander algebras.

Definition 2.1. An algebra $R$ is called an Auslander algebra if $\text{gl.dim}\Lambda \leq 2$ and $I_i(R)$ is projective for $i = 0, 1$, where $I_i(R)$ is the $(i+1)$-th term in a minimal injective resolution of $R$.

Let $R$ be a representation-finite algebra and $A$ an additive generator of $\text{mod}\ R$. Auslander proved that there is a one to one correspondence between representation-finite algebras and Auslander algebras via $R \mapsto \text{End}_R(A)$. In this case, we call $\text{End}_R(A)$ the Auslander algebra of $R$. Furthermore, for $X \in \text{mod}\ R$ we denote by $P_X = \text{Hom}_R(A, X)$ and $S_X = P_X/\text{rad}P_X$. The following statement [AuRS] is essential in the proof of the main result.

Proposition 2.2. Let $X$ be an indecomposable $\Lambda$-module. Then

1. $\text{pd}_\Lambda S_X \leq 1$ if and only if $X$ is projective, and $0 \to P_{\text{rad}X} \to P_X \to S_X \to 0$ is a minimal projective resolution of $S_X$.
2. $\text{pd}_\Lambda S_X = 2$ if and only if $X$ is not projective, and the almost split sequence $0 \to \tau X \to E \to X \to 0$ gives a minimal projective resolution $0 \to P_{\tau X} \to P_E \to P_X \to S_X \to 0$ of $S_X$.

For a positive integer $k$, an algebra $\Lambda$ is called Auslander’s $k$-Gorenstein if $\text{pd}_\Lambda I_j(\Lambda) \leq j$ for $0 \leq j \leq k-1$. For a $\Lambda$-module $M$ and a positive integer $i$, we call grade $M \geq i$ if $\text{Ext}^i_\Lambda(M, \Lambda) = 0$ for $0 \leq j \leq i - 1$. We need the following result.

Lemma 2.3. Let $\Lambda$ be an Auslander algebra and $T \in \text{mod}\Lambda$. For $j = 1, 2$,

1. The subcategory $\{M|\text{grade}M \geq j\}$ is closed under submodules and factor modules.
2. Every simple $\Lambda$-module $S$ is either of grade 0 or of grade 2.
3. $\text{grade} \text{Ext}^j_\Lambda(T, \Lambda) \geq 2$. 


(4) The projective dimension of any composition factor of Ext$^1_{\Lambda}(T, \Lambda)$ is 2.

Proof. (1) is a straight result of [1] Proposition 2.4.

(2) follows from the fact Ext$^1_{\Lambda}(S, \Lambda) \simeq \text{Hom}_{\Lambda}(S, I(\Lambda))$ and $\Lambda$ is an Auslander algebra.

(3) By the definition of Auslander algebra, $\Lambda$ is Auslander’s 2-Gorenstein. Then by [FGR] $\Lambda$ is Auslander’s $k$-Gorenstein if and only if for each submodule $X$ of Ext$^1_{\Lambda}(T, \Lambda)$ with $T$ in mod$\Lambda$ and $i \leq k$, we have grade$X \geq i$. Then we have grade Ext$^1_{\Lambda}(T, \Lambda) \geq j$ for $j = 1, 2$. By (1) every composition factor $S$ of Ext$^1_{\Lambda}(T, \Lambda)$ has grade at least 1, and hence 2 by (2). Then by an induction on the length of Ext$^1_{\Lambda}(T, \Lambda)$, we get grade Ext$^1_{\Lambda}(T, \Lambda) \geq 2$.

(4) is a direct result of (1) and (3). □

In the following we recall some basic properties of $\tau$-rigid modules. We start with the following definition [AIR].

Definition 2.4. We call $M \in \text{mod}\Lambda$ $\tau$-rigid if $\text{Hom}_{\Lambda}(M, \tau M) = 0$. In addition, $M$ is called $\tau$-tilting if $M$ is $\tau$-rigid and $|M| = |\Lambda|$. Moreover, $M$ is called support $\tau$-tilting if there exists an idempotent $e$ of $\Lambda$ such that $M$ is a $\tau$-tilting $\Lambda/e$-module.

It is clear that any $\tau$-rigid $\Lambda$-module $M$ is rigid, that is, Ext$^1_{\Lambda}(M, M) = 0$. In general the converse is not true. But if pd$_{\Lambda}M = 1$, then $M$ is $\tau$-rigid if and only if $M$ is rigid. Recall that a $\Lambda$-module $T$ is called a (classical) tilting module if $T$ satisfies (1) pd$_{\Lambda}T \leq 1$, (2) Ext$_{\Lambda}^1(T, T) = 0$ and (3) $|T| = |\Lambda|$. It is showed in [AIR] that a tilting $\Lambda$-module is exactly a faithful support $\tau$-tilting $\Lambda$-module.

To judge $\tau$-rigid modules of dimension 2 over Auslander algebras, we also need the following lemma in [AIR].

Lemma 2.5. Let $\Lambda$ be an algebra and $M$ a $\Lambda$-module without projective direct summands. Then $M$ is $\tau$-rigid in $\text{mod}\Lambda$ if and only if Tr$M$ is $\tau$-rigid in $\text{mod}\Lambda^{op}$.

Recall that a morphism $f : M \to N$ is called right minimal (resp. left minimal) if $fg = f$ (resp. $gf = f$) implies that $g$ is an isomorphism, where $g$ is a homomorphism of the form $M \to M$ (resp. $N \to N$). The following properties of right minimal (resp. left minimal) morphisms in [HuZ] are useful for the proof of the main results.

Lemma 2.6. Let $0 \to A \xrightarrow{g} B \xrightarrow{f} C \to 0$ be a non-split exact sequence in $\text{mod}\Lambda$ with $B$ projective-injective. Then the following are equivalent:

(1) $A$ is indecomposable and $g$ is left minimal.
(2) $C$ is indecomposable and $f$ is right minimal.

3. Main results

In this section we give the main results of this paper and some examples to show the main results. Throughout this section, $\Lambda = \text{End}_{R}A$ is the Auslander algebra of a representation-finite algebra $R$ with an additive generator $A$.

It is showed by Igusa [19] that $S$ is rigid for any simple module $S$ over an algebra $\Gamma$ of finite global dimension. However, we give a new direct proof for the rigidity of simple modules whenever $\Gamma$ is an Auslander algebra.

Proposition 3.1. Let $\Lambda$ be an Auslander algebra and $S$ a simple $\Lambda$-module. Then Ext$_{\Lambda}^1(S, S) = 0$.

Proof. For a simple $\Lambda$-module $S$, we show the assertion by using the projective dimension of $S$.

If pd$_{\Lambda}S = 0$, there is nothing to show.

If pd$_{\Lambda}S = 1$, then we can get a minimal projective resolution $0 \to P_1(S) \to P_0(S) \to S \to 0$. Then the length of $P_1(S)$ is smaller than that of $P_0(S)$, and hence Ext$_{\Lambda}^1(P_1(S), S) = 0$. So one gets Ext$_{\Lambda}^1(S, S) \simeq \text{Hom}_{\Lambda}(P_1(S), S) = 0$.

If pd$_{\Lambda}S = 2$, then by Proposition 2.2, there is an AR-sequence $0 \to \tau X \to E \to X \to 0$ in mod$R$ such that $0 \to \text{Hom}_R(A, \tau X) \to \text{Hom}_R(A, E) \to \text{Hom}_R(A, X) \to S \to 0$ is a minimal projective
Remark 3.5. We should remark that the converse of Corollary 3.4 are not true in general (see the assertion (2)). □

Lemma 3.2. Let $\Lambda$ be an Auslander algebra, and let $M$ be a $\Lambda$-module with $\text{pd}_\Lambda M \leq 1$. Then the canonical map $M \xrightarrow{\sim} M^{**}$ is injective, and the projective dimension of any composition factor of $M^{**}/M$ is 2.

Proof. By [AuB], we get an exact sequence $0 \to \text{Ext}^1_{\Lambda^{\text{op}}}(\text{Tr}M, \Lambda) \to M \to M^{**} \to \text{Ext}^2_{\Lambda^{\text{op}}}(\text{Tr}M, \Lambda) \to 0$. To show the former assertion, it suffices to show that $\text{Ext}^1_{\Lambda^{\text{op}}}(\text{Tr}M, \Lambda) = 0$. In the following we show grade $\text{Tr}M = 2$. Since $\text{pd}_\Lambda M \leq 1$, then one gets $\text{Tr}M \simeq \text{Ext}^1_{\Lambda}(M, \Lambda)$ and hence by Lemma 2.3(3), grade $\text{Tr}M \geq 2$ holds, and hence $\text{Ext}^1_{\Lambda^{\text{op}}}(\text{Tr}M, \Lambda) = 0$ We get the desired injection. Then by Lemma 2.3(2), the latter assertion holds. □

Now we are in a position to state the following main result on the $(\tau)$-rigidity of modules with projective dimension 1.

Theorem 3.3. Let $\Lambda$ be an Auslander algebra and $M$ a $\Lambda$-module with $\text{pd}_\Lambda M = 1$. Then $\text{Ext}^1_{\Lambda}(M, M) = 0$ if and only if $\text{Ext}^2_{\Lambda}(N, M) = 0$ holds for $N = M^{**}/M$.

Proof. We show the assertion step by step.

1. For any $M \in \text{mod}\Lambda$, $M^{*}$ is projective. Here we only need the condition $\text{gl.dim} \Lambda = 2$.

   Let $P_1(M) \to P_0(M) \to M \to 0$ be a projective resolution of $M$. Applying the functor $(-)^*$, we get an exact sequence $0 \to M^{*} \to P_0(M)^{*} \to P_1(M)^{*}$. Since $\text{gl.dim} \Lambda \leq 2$, one gets that $M^{*}$ is a projective $\Lambda^{\text{op}}$-module. Thus $M^{**}$ is a projective $\Lambda$-module.

2. By Lemma [3.2], we get the exact sequence $0 \to M \to M^{**} \to \text{Ext}^2_{\Lambda^{\text{op}}}(\text{Tr}M, \Lambda)(= M^{**}/M) \to 0$. Applying the functor $\text{Hom}_\Lambda(-, M)$ to the exact sequence, we get the desired isomorphism since $M^{**}$ is projective by (1). □

Immediately, we have the following corollary.

Corollary 3.4. Let $\Lambda$ be an Auslander algebra and $M$ a $\Lambda$-module with $\text{pd}_\Lambda M = 1$.

1. If $\text{id}_\Lambda M = 1$, then $\text{Ext}^1_{\Lambda}(M, M) = 0$ holds.

2. If $\text{Ext}^1_{\Lambda}(S', M) = 0$ holds for any composition factor $S'$ of $M^{**}/M$, then $\text{Ext}^1_{\Lambda}(M, M) = 0$ holds.

Proof. (1) follows from Theorem 3.3 directly. By induction on the length of $M^{**}/M$, one can get the assertion (2). □

Remark 3.5. We should remark that the converse of Corollary [3.4] are not true in general (see Example [3.11(5)]).

Denote by $\text{ir-rig} \Lambda$ the set of isomorphism classes of indecomposable $\tau$-rigid $\Lambda$-modules. Similarly, one can define $\text{ir-rig} \Lambda^{\text{op}}$. Denote by $\mathcal{G}$ the subset of $\text{ir-rig} \Lambda$ consisting of isomorphism classes of $\tau$-rigid modules of grade 2 and denote by $\mathcal{S}$ the subset of $\text{ir-rig} \Lambda^{\text{op}}$ consisting of isomorphism classes of non-projective $\tau$-rigid submodules of $\text{add} \Lambda^{\text{op}}$. To judge $\tau$-rigid modules of projective dimension 2 over Auslander algebras, we need the following proposition.

Proposition 3.6. Let $\Lambda$ be an algebra of global dimension 2. There is a bijection between $\mathcal{G}$ and $\mathcal{S}$ via $\text{Tr}: M \mapsto \text{Tr}M$. 

Proof. By Lemma 2.5 \(M\) is \(\tau\)-rigid if and only if \(\text{Tr} M\) is \(\tau\)-rigid. Now it suffices to show that (a) \(M \in \mathcal{G}\) implies that \(\text{Tr} M \in \mathcal{S}\) and (b) \(M \in \mathcal{S}\) implies that \(\text{Tr} M \in \mathcal{G}\).

(a) Since \(M \in \mathcal{G}\), take the following minimal projective resolution of \(M\):

\[
\cdots \rightarrow P_{i}(M) \rightarrow P_{0}(M) \rightarrow M \rightarrow 0.
\]

Applying the functor \((-)^{*}\), we get an exact sequence

\[
0 = M^{*} \rightarrow P_{0}(M)^{*} \rightarrow P_{1}(M)^{*} \rightarrow \text{Tr} M \rightarrow 0,
\]

which is a minimal projective resolution of \(\text{Tr} M\). Then \(\text{pd}_{\Lambda} \text{Tr} M = 1\). On the other hand, since \(\text{grade} M = 2\), one gets the following sequences

\[
0 = M^{*} \rightarrow P_{0}(M)^{*} \rightarrow \Omega^{1} M^{*} \rightarrow \text{Ext}^{1}_{\Lambda}(M, \Lambda) = 0.
\]

and

\[
0 \rightarrow \Omega^{1} M^{*} \rightarrow P_{1}(M)^{*} \rightarrow P_{2}(M)^{*}
\]

Comparing exact sequences (3.1) with (3.2) and (3.3) one gets that \(\text{Tr} M\) is a submodule of \(P_{2}(M)^{*}\).

(b) Since \(M \in \mathcal{S}\) is non-projective and \(\text{gl.dim} \Lambda = 2\), then \(\text{pd}_{\Lambda} M = 1\). Take a minimal projective resolution of \(M\):

\[
0 \rightarrow P_{1}(M) \rightarrow P_{0}(M) \rightarrow M \rightarrow 0.
\]

Applying \((-)^{*}\), we get the following exact sequence

\[
0 \rightarrow M^{*} \rightarrow P_{0}(M)^{*} \rightarrow P_{1}(M)^{*} \rightarrow \text{Tr} M \rightarrow 0.
\]

Note that \(\text{Tr}\) is a duality and \(\text{pd}_{\Lambda} M = 1\), one gets that \(\text{Hom}_{\Lambda^{\text{op}}}(\text{Tr} M, \Lambda) = 0\). Since \(M\) can be embedded into a projective module, then \(M\) is torsionless, that is \(M \rightarrow M^{**}\) is injective. By \([\text{AuB}]\) there is an exact sequence

\[
0 \rightarrow \text{Ext}^{1}_{\Lambda^{\text{op}}}(\text{Tr} M, \Lambda) \rightarrow M \rightarrow M^{**} \rightarrow \text{Ext}^{2}_{\Lambda^{\text{op}}}(\text{Tr} M, \Lambda) \rightarrow 0
\]

which implies that \(\text{Ext}^{1}_{\Lambda^{\text{op}}}(\text{Tr} M, \Lambda) = 0\). Then \(\text{grade} \text{Tr} M = 2\).

As a corollary, we get the following.

Corollary 3.7. Let \(\Lambda\) be an Auslander algebra and \(M \in \text{mod} \Lambda\). If \(M\) is of grade 2, then \(M\) is \(\tau\)-rigid if and only if \(\text{pd}_{\Lambda} \text{Tr} M = 1\) in \(\text{mod} \Lambda^{\text{op}}\).

Proof. By Proposition 3.6, it is enough to show that \(\text{pd}_{\Lambda} M = 1\) if and only if \(M\) can be embedded into a projective module. Since \(\text{gl.dim} \Lambda = 2\), one gets that \(M\) can be embedded into a projective module implies that \(\text{pd}_{\Lambda} M = 1\). The converse follows from Lemma 3.2.

\[\blacksquare\]

Recall that from [DLJ] that an algebra \(\Lambda\) is called \(\tau\)-tilting finite if there are finite number of non-isomorphic indecomposable \(\tau\)-rigid modules in \(\text{mod} \Lambda\). It is clear that a \(\tau\)-tilting finite algebra admits finite number of tilting \(\Lambda\)-modules and tilting \(\Lambda^{\text{op}}\)-modules. To find a way from two-sided tilting finite to \(\tau\)-tilting finite, we have the following.

Theorem 3.8. Let \(\Lambda\) be an algebra of global dimension 2 admitting finite number of basic tilting \(\Lambda\)-modules and tilting \(\Lambda^{\text{op}}\)-modules. If all indecomposable \(\tau\)-rigid modules \(M\) with \(\text{pd}_{\Lambda} M = 2\) are of grade 2, then \(\Lambda\) is \(\tau\)-tilting finite.

Proof. By the assumption, there are finite number of tilting modules which implies that there are finite number of indecomposable \(\tau\)-rigid \(\Lambda\)-modules and \(\Lambda^{\text{op}}\)-modules of projective dimension less than or equal to 1. Then by Proposition 3.6 the number of indecomposable \(\tau\)-rigid \(\Lambda\)-module of grade 2 is equal to the number of indecomposable non-projective \(\tau\)-rigid submodules \(N\) of \(\Lambda^{\text{op}}\). Since \(\text{gl.dim} \Lambda = 2\), we get that \(\text{pd}_{\Lambda} N = 1\), and hence the number of this class of modules is finite. Note that all indecomposable \(\tau\)-rigid \(\Lambda\)-modules with projective dimension 2 are of grade 2, then the number of indecomposable \(\tau\)-rigid modules with projective dimension 2 is finite by Proposition 3.6.

\[\blacksquare\]

Immediately, we have the following corollary which confirms the \(\tau\)-tilting finiteness of the the Auslander algebra of \(K[x]/(x^{n})\) showed in [IZ].

Corollary 3.9. Let \(\Lambda\) be an Auslander algebra admitting finite number of basic tilting \(\Lambda\)-modules and tilting \(\Lambda^{\text{op}}\)-modules. If all indecomposable \(\tau\)-rigid modules \(M\) with \(\text{pd}_{\Lambda} M = 2\) are of grade 2, then \(\Lambda\) is \(\tau\)-tilting finite.
For a module $M$, denote by $\text{rad}M$ and $\text{soc}M$ the radical and the socle of $M$, respectively. Now we give the following classification of Auslander algebras admitting a unique simple module of projective dimension 2 which gives a support to Theorem 3.10 and Corollary 3.19.

**Theorem 3.10.** Let $\Lambda$ be an Auslander algebra. If $\Lambda$ admits a unique simple $\Lambda$-module $S$ with $\text{pd}_\Lambda S = 2$, then

1. $\Lambda$ is either the Auslander algebra of the path algebra $R = KQ$ with $Q : 1 \rightarrow 2$ or the Auslander algebra of the Nakayama local algebra $R$ of radical square zero.
2. Every indecomposable $\Lambda$-module $M$ with $\text{pd}_\Lambda M \leq 1$ is rigid, and hence $\tau$-rigid.
3. All indecomposable $\tau$-rigid $\Lambda$-modules $N$ with $\text{pd}_\Lambda N = 2$ are of grade 2.

**Proof.** Since (2) and (3) follow from (1) easily, we only show (1). By Proposition 2.2, there is a unique non-projective indecomposable $R$-module $X$ such that the $AR$-sequence $0 \rightarrow \tau X \rightarrow E \rightarrow X \rightarrow 0$ in $\text{mod}R$ induces a minimal projective resolution of $S$: $0 \rightarrow \text{Hom}_R(A, \tau X) \rightarrow \text{Hom}_R(A, E) \rightarrow \text{Hom}_R(A, X) \rightarrow S \rightarrow 0$. Then all indecomposable modules are projective except $X$. We claim that $X$ should be simple. Otherwise, there would be a simple factor module $Y$ of $X$ such that $Y \not\cong X$. By the proof above $Y$ would be projective and hence $X \cong Y$ is projective, a contradiction. Now we divide the proof in two parts.

(a) If $X$ is not injective, then all indecomposable injective $R$-modules are projective, and hence $R$ is self-injective. So we get that $R$ is local with a unique simple module $X$. Otherwise, there would be a simple projective-injective $R$-module. One gets a contradiction since $R$ is basic and connected. Taking a minimal projective resolution of $X$, we get the following exact sequence $0 \rightarrow \Omega^1 X \rightarrow P_0(X)(= R) \rightarrow X \rightarrow 0$. By Lemma 2.6, $\Omega(X)$ is indecomposable non-projective, and hence $\Omega^1 X \cong X$. Then $\text{rad}^2 R = 0$ holds. By [AuRS, IV, Proposition 2.16], $R$ is a Nakayama algebra.

(b) If $X$ is injective, then $X \not\cong \text{soc}P$ for any indecomposable projective $R$-module. Hence the injective envelope $I^0(R)$ is projective, that is, $R$ is Auslander’s 1-Gorenstein [FGR]. Then $P_0(X)$ is projective-injective since $X$ is injective. Taking a part of minimal projective resolution of $X$: $0 \rightarrow \Omega^1 X \rightarrow P_0(X) \rightarrow X \rightarrow 0$, one gets that $\Omega^1 X$ is indecomposable and projective by Lemma 2.6. Then we conclude that $R$ is a hereditary algebra.

In the following we show $R$ is a Nakayama algebra. One can show that $P_0(X)$ is the unique projective-injective module in $\text{mod}R$ since $R$ is a basic connected hereditary algebra. Then every indecomposable projective $R$-module is contained in $P_0(X)$ and admits a unique composition series. By [FGR], $R^{\text{op}}$ is also Auslander’s 1-Gorenstein. Similarly, every indecomposable projective $R^{\text{op}}$-module admits a unique composition series. So $R$ is a Nakayama algebra. By [AsSS, V, Theorem 3.2] and the fact all indecomposable $R$-modules are projective except one, we get that $R = KQ$ with $Q : 1 \rightarrow 2$. \hfill \Box

At the end of this paper we give another two examples to show our main results.

**Example 3.11.** Let $\Lambda$ be the Auslander algebra of $K[x]/(x^n)$. Then we have the following:

1. $\Lambda$ is given by

\[
\begin{array}{c}
1 \rightarrow \frac{a_1}{b_2} \rightarrow \frac{a_2}{b_3} \rightarrow \frac{a_3}{b_4} \rightarrow \ldots \rightarrow \frac{a_{n-2}}{b_{n-1}} \rightarrow \frac{a_{n-1}}{b_n} \rightarrow \frac{a_n}{b_{n+1}} \rightarrow n
\end{array}
\]

with relations $a_1b_2 = 0$ and $a_ib_{i+1} = b_{i-1}a_{i-1}$ for any $2 \leq i \leq n-1$. $\Lambda$ is of infinite representation type if $n \geq 5$.

2. All indecomposable module $M$ with $\text{pd}_\Lambda M = 1 = \text{id}_\Lambda M$ are direct summands of tilting modules, and hence $\tau$-rigid.

3. All indecomposable $\tau$-rigid modules of projective dimension 2 are of grade 2. (See [IZ] for details)

4. The number of tilting $\Lambda$-modules (resp. $\Lambda^{\text{op}}$-modules) is $n!$ ([IZ] [T]). By Theorem 3.8 $\Lambda$ is $\tau$-tilting finite.
(5) If $n = 4$, then the indecomposable module $M = \begin{pmatrix} 1 & 4 \\ 3 & 4 \end{pmatrix}$ is ($\tau$-)rigid with $\text{pd}_\Lambda M = 1$ and $\text{id}_\Lambda M = 2$ and $M^{\ast\ast} = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$. But $\text{Ext}_\Lambda^2(S(2), M) \neq 0$.

We should remark that there does exist an Auslander algebra $\Lambda$ such that an indecomposable $\tau$-rigid $\Lambda$-module with projective dimension 2 does not necessarily have grade 2.

Example 3.12. Let $\Lambda$ be the Auslander algebra of $KQ$ with $Q : 1 \to \begin{pmatrix} a_1 & 2 \\ a_2 & 3 \end{pmatrix}$. Then

(1) $\Lambda$ is given by the following quiver

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
a_1 & a_2 & a_3 & a_4 & a_5 & a_6
\end{array}
\]

with relations $a_2a_1 = 0$, $a_5a_3 = a_4a_2$ and $a_6a_4 = 0$.

(2) All indecomposable modules are $\tau$-rigid.

(3) The indecomposable module $M = \begin{pmatrix} 1 & 4 \\ 3 & 4 \end{pmatrix}$ is of projective dimension 2, but it is not of grade 2 since $\text{pd}_\Lambda S(4) = 1$.

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