LAGRANGIAN FIBRATIONS FROM FANO VARIETIES OF CUBIC FOURFOLDS

COREY BROOKE

Abstract. We show that for a general cubic fourfold $X$ containing a plane $P$, the Mukai flop of its Fano variety of lines $F$ along the dual plane $P^*$ admits a map to $\mathbb{P}^2$ fibered in geometrically abelian surfaces. The explicit construction of this fibration affords an analysis of the torsor structure of the smooth fibers when the ground field $k$ is not closed.

1. Introduction

Let $X \subset \mathbb{P}^5$ be a smooth hypersurface of degree three, called a cubic fourfold. The variety $F \subset Gr(2,6)$ parametrizing lines contained in $X$ is called the Fano variety of lines on $X$ and is also smooth of dimension four. In [3], Beauville and Donagi show that over the base field $\mathbb{C}$, $F$ is a hyperkähler manifold of K3[2]-type, i.e. deformation equivalent to the Hilbert scheme of two points on a K3 surface.

Hyperkähler manifolds are higher-dimensional generalizations of K3 surfaces. For example, as with K3 surfaces, the only possible fiber structure on a hyperkähler manifold $M$ of dimension $2n$ is an abelian $n$-fold fibration over $\mathbb{P}^m$, called a Lagrangian fibration [13] [10]. If $M$ is birational to another hyperkähler manifold admitting a Lagrangian fibration, one says $M$ admits a rational Lagrangian fibration. The K3 surfaces admitting Lagrangian fibrations are called elliptic and are well-understood: the existence of a divisor class on $S$ squaring to zero under the intersection form determines whether the surface is elliptic [9]. Similarly, if $M$ is a hyperkähler manifold of K3[2]-type, then $M$ admits a rational Lagrangian fibration if and only if there is a divisor class $D \in \text{Pic}(M)$ isotropic with respect to a quadratic form $q_F$ called the Beauville-Bogomolov form [14].

This criterion implies that the Fano variety $F$ of lines on a complex cubic fourfold $X$ admits a rational Lagrangian fibration if and only if $X$ contains a special algebraic surface satisfying certain numerical criteria, as explained in Proposition 4. The first example is when $X$ contains a plane. Motivated by this intuition from hyperkähler geometry, we
depart from the complex setting and fix a general cubic fourfold $X$ containing a plane, proving the following via a direct construction:

**Theorem 1.** Let $X$ be a general cubic fourfold containing a plane $P$ over a field $k$ of characteristic not 2, and let $F$ be the Fano variety of lines on $X$. Then $F$ contains the dual plane $P^*$, and the Mukai flop $M$ of $F$ along $P^*$ admits a fibration of geometrically abelian surfaces $\rho : M \to \mathbb{P}^2$.

This paper describes the fibration $\rho$ explicitly, allowing an analysis of the fibers of $\rho$ as torsors when $k$ is not algebraically closed. Given a smooth fiber $T$ of $\rho$, there is a genus 2 curve $C$ and a simply transitive action of $\text{Pic}_C^0$ on $T$, all defined over the base field $k$. The following results describe the torsor structure of $T$.

**Theorem 2.** There is a commutative algebraic group scheme structure on the disconnected variety

$$G = \text{Pic}_C^0 \sqcup T \sqcup \text{Pic}_C^1 \sqcup T$$

defined over $k$.

**Corollary 3.** In the Weil-Châtelet group $H^1(k, \text{Pic}_C^0)$, $2[T] = [\text{Pic}_C^1]$ and $4[T] = 0$.

The arithmetic resembles analogous results for the Fano variety of maximal linear spaces contained in an intersection of two quadrics, described by Wang in [17].

Section 2 outlines numerical conditions governing which complex cubic fourfolds have Fano varieties of lines admitting rational Lagrangian fibrations. In Section 3, we review geometric constructions related to a cubic fourfold $X$ containing a plane $P$ and define a rational map $\pi : F \dashrightarrow \mathbb{P}^2$ from the Fano variety of lines $F$ on $X$. In Section 4, we analyze the Fano variety of a general cubic threefold containing a plane, yielding information about the (closures of) fibers of the rational map $\pi$. In Section 5, we resolve $\pi$ by a Mukai flop, proving Theorem 1, and in Section 6, we analyze the torsor structure of the smooth fibers of $\rho$, proving Theorem 2 and Corollary 3.

**Acknowledgments:** I am very grateful for the help of my advisor, Nicolas Addington, who suggested this project, guided my progress, and offered improvements for the paper. I would also like to thank the faculty and fellow students at the University of Oregon for their support. I was partially supported by NSF grants DMS-1902213 and DMS-2039316.
2. Hyperkähler Background

Let $F$ be the Fano variety of lines on a complex cubic fourfold $X$. As mentioned prior, $F$ is hyperkähler and admits a rational Lagrangian fibration if and only if the Beauville-Bogomolov form $q_F$ vanishes on a nontrivial divisor class $D \in H^2(F, \mathbb{Z})$ in the birational Kähler cone. By results from [12, Sect. 6] summarized in [15, Cor. 7.3], the existence of any nontrivial isotropic integral divisor class implies the existence of one in the birational Kähler cone. This section explains when such a class exists.

Let $W \subset X \times F$ be the incidence correspondence with $\pi_1$ and $\pi_2$ its projections. The Abel-Jacobi map $\alpha : H^4(X, \mathbb{Z}) \to H^2(F, \mathbb{Z})$, $\alpha(S) = \pi_2^*\pi_1^*(S)$ respects the Hodge filtration and restricts to an isomorphism $H^4(X, \mathbb{Z})_{\text{prim}} \sim \to H^2(F, \mathbb{Z})_{\text{prim}}$, proved in [3]. If $X$ is very general, meaning that $X$ is in the complement of countably many codimension one families in the moduli space of cubic fourfolds, then $H^{2,2}(X, \mathbb{Z}) = \mathbb{Z}h^2$ where $h$ denotes the hyperplane class, so $\text{Pic}(F) = \mathbb{Z}\alpha(h^2)$. In [7], Hassett proved $q_F(\alpha(h^2)) = 6$, from which we deduce $F$ admits no rational Lagrangian fibration.

Therefore, for $F$ to admit a rational Lagrangian fibration, the rank of $H^{2,2}(X, \mathbb{Z})$ must be greater than 1, meaning $X$ contains an algebraic surface $S$ not homologous to a complete intersection. Such a cubic is called special, and its discriminant is the discriminant of the intersection form on a rank two primitive sublattice of $H^{2,2}(X, \mathbb{Z})$ containing $h^2$. If $d \geq 8$ and $d \equiv 0, 2 \mod 6$, then there exist cubic fourfolds of discriminant $d$, forming an irreducible divisor $C_d$ in the moduli space of all cubic fourfolds. In this framework, we can describe precisely which Fano varieties of complex cubic fourfolds admit rational Lagrangian fibrations.

**Proposition 4.** $F$ admits a rational Lagrangian fibration if and only if $X \in C_d$ for some $d$ which is twice a square.

**Proof.** We have already seen that a necessary condition for $F$ to admit a rational Lagrangian fibration is that $X$ be special. Choose a primitive rank two sublattice $h^2 \in L \subset H^{2,2}(X, \mathbb{Z})$ and a class $S \in L \cap H^4(X, \mathbb{Z})_{\text{prim}}$. Let $i$ be the index of the sublattice $L' = \mathbb{Z}h^2 + \mathbb{Z}S$ in $L$. The discriminant of the intersection form on $L'$ is $3S^2$, so the discriminant of the intersection form on $L$ is $3S^2/i^2$, and $X \in C_d$ for $d = 3S^2/i^2$. There is a class $D \in \alpha(L)$ isotropic with respect to $q_F$ if
and only if there is such a class in $\alpha(L')$. We will show that such a class exists in $L'$ if and only if $d$ is twice a square.

In [3], it is shown that the Abel-Jacobi map $\alpha$ is compatible with the intersection form and Beauville-Bogomolov form in the sense that

$$q_F(\alpha(x), \alpha(y)) = -x.y.$$ 

for primitive classes $x, y$. Applying this yields

$$q_F(\alpha(mh^2 + nS)) = 6m^2 - n^2S^2 = 6m^2 - i^2 n^2 d/3.$$

Thus there is a class $D \in \alpha(L')$ satisfying $q_F(D) = 0$ if and only if there are integers $m, n$ with $2(3m/(in))^2 = d$. $\square$

**Remark 5.** In fact, it is proved in [14] that if an isotropic class $D \in \text{Pic}(F)$ is nef, then the complete linear system of $D$ defines a rational Lagrangian fibration.

**Example 6.** When $X$ contains a plane $P$, the intersection form on the primitive lattice $\mathbb{Z}h^2 + \mathbb{Z}P \subset H^4(X, \mathbb{Z})$ is

$$\begin{array}{c|cc}
  & h^2 & P \\
$h^2$ & 3 & 1 \\
P & 1 & 3 \\
\end{array}$$

In this case, $X \in \mathcal{C}_8$, so the Fano variety $F$ of $X$ is birational to some hyperkähler fourfold $M$ admitting a Lagrangian fibration. The integral nef divisor class isotropic with respect to the Beauville-Bogomolov form is $\alpha(h^2 - P)$. Note that $h^2 - P$ is the class of a quadric surface in $X$: taking hyperplanes $H_1, H_2 \supset P$, the intersection $X \cap H_1 \cap H_2 = Q \cup P$ where $\deg(Q) = 2$. Lemma 9 verifies that the complete linear system $|\alpha(Q)|$ induces the rational Lagrangian fibration $\pi : F \to \mathbb{P}^2$ defined in Section 3.

**Remark 7.** One can also describe the rational Lagrangian fibration $F \to \mathbb{P}^2$ from a moduli perspective, using Macrì and Stellari’s result from [11] that for all complex cubic fourfolds $X$ containing a plane, the Fano variety $F$ is birational to a smooth projective moduli space $M$ of twisted sheaves on a K3 surface $K$. The twisted sheaves are all supported on curves in a two-dimensional linear system on $K$, and one can obtain a Lagrangian fibration $M \to \mathbb{P}^2$ by sending a twisted sheaf to its support.

**Example 8.** If $X \in \mathcal{C}_{18}$, then $X$ contains a sextic del Pezzo surface $S$, as explained in [1]. The the classes $\alpha(3h^2 - S)$ and $\alpha(S - h^2)$ are Kähler and isotropic with respect to the Beauville-Bogomolov form; however, even with these divisor classes in hand, it is unclear how to describe the associated rational Lagrangian fibrations from $F$ geometrically.
3. The Quadric Surface Fibration

For the remainder of the paper, $X$ is a general cubic fourfold of discriminant 8 and $P \subset X$ is a plane; here, the term *general* means that $X$ is smooth and contains no other plane meeting $P$. Let $F$ be the Fano variety of lines in $X$. The ground field $k$ need not be closed but has characteristic not equal to 2. Our foundational geometric ingredient is the quadric surface fibration $q: Bl_P X \to P^\perp$ obtained via projection to a complementary plane $P^\perp$. A sextic curve $\Delta \subset P^\perp$ called the discriminant of $q$ parametrizes the singular fibers of $q$; it is here that we use the assumption on the characteristic of $k$. The assumption that $X$ contains no plane meeting $P$ guarantees that the singular fibers of $q$ are no worse than cones and, by Lemma 2 in [16], that $\Delta$ is smooth.

A double cover $K$ of $P^\perp$ branched over $\Delta$ parametrizes rulings of the fibers of $q$, and since $\Delta$ is a smooth sextic, $K$ is a smooth K3 surface. The lines in the fibers of $q$ form a conic fibration over $K$ defining a Brauer class $\beta \in \Br(K)[2]$. This is the twist on a K3 surface appearing in Macrì and Stellari’s previously-mentioned result.

Given a line $L \subset X \setminus P$, the projection $q(L) \subset P^\perp$ is again a line, so the formula $\pi(L) = q(L)$ defines a rational map $\pi: F \dashrightarrow (P^\perp)^*$. In light of Remark 5, the following lemma shows that when $k = \mathbb{C}$, $\pi$ is the rational Lagrangian fibration detected by the Beauville-Bogomolov form.

**Lemma 9.** The linear system $|\alpha(Q)|$ induces $\pi$, where $Q \subset X$ is a quadric surface and $\alpha: H^1(X, \mathbb{Z}) \to H^2(F, \mathbb{Z})$ is the Abel-Jacobi map.

**Proof.** Let $p \in P^\perp$ and $Q = q^{-1}(p)$. The pencil of lines in $P^\perp$ through $p$ represents $\mathcal{O}_{(P^\perp)^*}(1)$. Thus $\pi^*\mathcal{O}_{(P^\perp)^*}(1)$ is represented by the divisor on $F$ parametrizing lines $L \subset X$ for which $p \in \pi(L)$ or, equivalently, $L \cap Q \neq \emptyset$. This divisor is precisely $\alpha(Q)$. □

If $\ell \subset P^\perp$ is a line and $H$ is the hyperplane spanned by $\ell$ and $P$, then $\pi^{-1}(\ell)$ contains the lines in $X \cap H \setminus P$. That is, $\pi^{-1}(\ell)$ is birational to some component of the Fano variety of lines on a cubic threefold $Y$ containing $P$, obtained as a hyperplane section of $X$. The following lemma leverages a description of these threefolds.

**Lemma 10.** Let $X$ be a general cubic fourfold containing a plane $P$, and let $P^\perp \subset \mathbb{P}^5$ be a plane disjoint from $P$.

(i) For each $p \in P$, there is a unique hyperplane $H \supset P$ for which $Y = X \cap H$ is singular at $p$.

(ii) $Y$ contains only one plane.

(iii) $Z = \text{Sing}(Y) \cap P$ is a complete intersection of two conics in $P$. 

(iv) Each \( k \)-point of \( Z \) gives a section of the quadric surface fibration \( Bl_p Y \to \ell \) where \( \ell = (H \cap P^\perp) \).

(v) \( Y \) has isolated singularities.

(vi) If \( \ell \) is transverse to \( \Delta \), then \( \text{Sing}(Y) = Z \).

Proof. For (i), note that \( T_p(X \cap H) = T_p X \cap H \), so \( \dim(T_p(X \cap H)) = 4 \) if and only if \( H = T_p(X) \). To see (ii), recall that \( X \) contains no plane meeting \( P \), whereas any two planes in \( H \) meet.

Now, fix coordinates so that \( P = V(x_0, x_1, x_5) \) and \( H = V(x_5) \), and write the equation for \( Y \subset P^4 \) as \( x_0 Q_0 + x_1 Q_1 = 0 \) where the \( Q_i \) are quadratic in \( x_0, \ldots, x_4 \). Let \( q_i = Q_i(0, 0, x_2, x_3, x_4) \). The Jacobian criterion shows that \( \text{Sing}(Y) \) is cut out of \( Y \) by the equations

\[
Q_0 + x_0 \frac{dQ_0}{dx_0} + x_1 \frac{dQ_1}{dx_0} = Q_1 + x_0 \frac{dQ_0}{dx_1} + x_1 \frac{dQ_1}{dx_1} = x_0 \frac{dQ_0}{dx_i} + x_1 \frac{dQ_1}{dx_i} = 0
\]

for \( i = 2, 3, 4 \). Along \( P \), the equations become \( q_0 = q_1 = 0 \). In particular, \( Z \) is nonempty. To prove (iii), we will show that \( \dim Z = 0 \).

Define a morphism \( g : P \to (P^\perp)^* \) sending \( p \mapsto T_p X \cap P^\perp \). Notice that if \( \ell \subset P^\perp \) and \( H \) is the hyperplane spanned by \( \ell \) and \( P \), then

\[
g^{-1}(\ell) = \text{Sing}(H \cap X) \cap P,
\]

We proved that \( H \cap X \) has singularities along \( P \) for each hyperplane \( H \supset P \), so \( g \) is surjective. Any surjection \( \mathbb{P}^2 \to \mathbb{P}^2 \) is finite, and \( Z \) is a fiber of \( g \), so \( \dim Z = 0 \).

To show (iv), note that each quadric surface in \( Y \) is cut out by

\[
(sx_1 - tx_0 = sQ_0 + tQ_1 = 0
\]

for \( (s : t) \in \mathbb{P}^1 \), and each point in \( Z \) is a root of \( Q_i = 0 \) so satisfies these equations. Hence each quadric surface in \( Y \) contains \( Z \), and each point in \( Z \) gives a section of \( Bl_p Y \to \ell \).

For (v), \( Bl_p Y \) is smooth if and only if \( q : Bl_p X \to \mathbb{P}^2 \) is transverse to \( \ell \), meaning

\[
q_*T_y(Bl_p Y) + T_{q(y)}\ell = T_{q(y)}\mathbb{P}^2
\]

for all \( y \in Y \). Since

\[
\dim(q_*T_y(Bl_p Y)) = \begin{cases} 2, & \text{y is a smooth point of } q^{-1}(y) \\ 1, & \text{otherwise,} \end{cases}
\]

the only points where \( Bl_p Y \) might fail to be smooth are the cone points, of which there are finitely many. As \( Y \) also has finitely many singularities along \( P \), the singularities of \( Y \) are isolated.

Let \( S \subset Bl_p X \) be the curve tracing out the cone points in singular fibers of \( q \). Then \( q|_S : S \to \Delta \) has degree 1, and \( \Delta \) is smooth, so \( q|_S \) is
an isomorphism. Hence $q_\ast T_y(Bl_P X) = T_{q(y)} \Delta$ for $y \in S$, and $Bl_P Y$ is smooth at $y$ if
\[ T_{q(y)} \Delta + T_{q(y)} \ell = T_{q(y)} \mathbb{P}^2, \]
which holds when $\Delta$ and $\ell$ are transverse. To finish proving (vi), note that $Y$ is smooth away from $P$ if $Bl_P Y$ is smooth. \qed

Remark 11. There are cubic threefolds containing exactly one plane $P$ whose singular locus intersects $P$ in dimension 1, and these threefolds may even be nonsingular away from $P$. However, Lemma 10 shows that these threefolds do not embed in smooth cubic fourfolds.

4. Lines on cubic threefolds containing a plane

Let $Y$ be a cubic threefold containing exactly one plane $P$ and having isolated singularities. Let $F(Y)$ be the Fano scheme of lines on $Y$, and let $Z = \text{Sing}(Y) \cap P$. Projection away from $P$ gives a quadric surface fibration $q : Bl_P Y \to \mathbb{P}^1$ degenerating over a divisor of degree 6. Since $Y$ contains only one plane, the degenerate fibers of $q$ are no worse than cones.

The hypothesis on $Y$ is potentially more general than requiring that $Y$ be a hyperplane section of a general cubic fourfold containing a plane, but results analogous to those in Lemma 10 still hold via the same proofs, summarized below.

Lemma 12. With $Y$ as above, the following hold.
\begin{enumerate}
  
  \item $Z = \text{Sing}(Y) \cap P$ is a complete intersection of two conics in $P$.
  
  \item Each $k$-point of $Z$ gives a section of the quadric surface fibration $q : Bl_P Y \to \mathbb{P}^1$.
  
  \item $Y$ has isolated singularities.
  
  \item If the discriminant of $q$ is reduced, then $\text{Sing}(Y) = Z$.
\end{enumerate}

The lines contained in fibers of $q$ are parametrized by a surface $\mathcal{F}$, which is a subvariety of $F(Y)$ since a line in $Y$ belongs to at most one quadric surface. The Stein factorization of the projection $\mathcal{F} \to \mathbb{P}^1$ consists of a conic bundle $p : \mathcal{F} \to C$ and a double cover $C \to \mathbb{P}^1$ branched over the discriminant; the curve $C$ parametrizes rulings on fibers of $q$. By the Riemann-Hurwitz formula, the arithmetic genus of $C$ is 2.

Remark 13. It is worth describing the curve $C$ more explicitly, though this will not be used later in the paper. If $P = \{x_0 = x_1 = 0\}$, write the equation for $Y$ as
\begin{align*}
  f(x_0, ..., x_4) &= a_{11} x_2^2 + 2a_{12} x_2 x_3 + 2a_{13} x_2 x_4 + a_{22} x_3^2 + 2a_{23} x_3 x_4 \\
  &\quad + a_{33} x_4^2 + 2a_{14} x_2 + 2a_{24} x_3 + 2a_{34} x_4 + a_{44}
\end{align*}
with \(a_{ij} \in k[x_0, x_1]\). Let \(A\) be the symmetric matrix with entries \(a_{ij}\) for \(i \leq j\). Then the quadric surface \(Q = q^{-1}(a : b)\) is cut out by the quadratic form obtained by evaluating \(A\) at \(x_0 = a, x_1 = b\), denoted \(A|_{(a:b)}\). In particular, the equation for \(C\) in weighted projective space is \(z^2 = c \cdot \det A\) for some \(c \in k^\times\). By [17, footnote on p. 366], \(Q\) has a ruling defined over \(k\) if and only if \(\det A|_{(a:b)}\) is a square; on the other hand, \(C\) has a \(k\)-point lying over \((a : b)\) if and only if \(c \det A|_{(a:b)}\) is a square. Hence \(c \in k^\times 2\), and \(c = 1\) after a change of variables.

There is a bijection between sections of \(q : Bl_P Y \to \mathbb{P}^1\) and sections of \(p : \mathcal{F} \to C\) since choosing a point in a smooth quadric surface amounts to choosing a line in each of its rulings, explained further in [8, Sect. 3]. By Lemma 12, each (geometric) point \(z \in Z\) gives a section of \(q\), and \(\tau_z\) denotes the corresponding (geometric) section of \(p\). In a slight abuse of notation, for \(c \in C\) we typically regard \(\tau_z(c)\) as a line in \(\overline{Y}\) rather than as the corresponding point in \(\mathcal{F}\).

A line that meets \(P\) once belongs to \(\mathcal{F}\), and a line contained in \(P\) belongs to the dual plane \(P^*\); let \(\mathcal{U} \subset F(Y)\) be the open subscheme composed of lines in \(\overline{Y}\) not meeting \(P\).

**Theorem 14.** \(F(Y)\) contains the following components:

1. the plane \(P^*\) dual to \(P\),
2. a ruled surface \(\mathcal{F}\) over a curve \(C\) of arithmetic genus 2,
3. and a singular surface \(\overline{\mathcal{U}}\), geometrically birational to \(\text{Sym}^2 C\).

Moreover, each of the components above is irreducible if \(C\) is irreducible.

**Proof.** All that requires proof is the birational geometry of \(\overline{\mathcal{U}}\) over the separable closure \(k^s\).

Choose a point \(z \in Z\). For a line \(L \subset Y \setminus P\), let \(P'\) be the plane spanned by \(L\) and \(z\). Since \(Y\) is singular at \(z\), so too is the curve \(P' \cap Y\). Thus \(P' \cap Y\) consists of three lines: \(L\) and two other lines \(M\) and \(N\) (possibly equal), both containing \(z\). Note \(M, N \in \mathcal{F}\) since \(M\) and \(N\) meet but are not contained in \(P\). The equation \(\varphi_z(L) = \{p(M), p(N)\}\) defines a morphism \(\varphi_z : \mathcal{U} \to \text{Sym}^2 C\).

To show \(\varphi_z\) is a birational equivalence, we construct its rational inverse. Let \(V \subset \text{Sym}^2 C\) be the open set consisting of pairs \(\{c, d\}\) of distinct points such that \(\tau_z(c)\) and \(\tau_z(d)\) are not both contained in \(P\). For such a pair, the plane \(P'\) spanned by \(\tau_z(c)\) and \(\tau_z(d)\) meets \(Y\) in a third line \(L := \psi_z(c, d)\), defining a morphism \(\psi_z : V \to F(Y)\). Clearly \(\psi_z \circ \varphi_z = \text{id}_{\mathcal{U}}\), as needed.

**Remark 15.** Let \(S_z \subset Y\) be the union over \(c \in C\) of the lines \(\tau_z(c)\), a cone over \(C\). An unreduced length-2 subscheme of \(C\) specifies a line
τ_z(c) ⊂ S and a normal direction to that line in S, and together these span a plane P' meeting Y in a third line. In this way, one can instead define ψ_z as a rational map Hilb^2 C → U. The domain of ψ_z under this new definition is the complement of three points, i.e. the pairs \{c, d\} of distinct points with τ_z(c), τ_z(d) ⊂ P.

A line L ∈ U gives a section of q and hence a section σ_L of p. When C is smooth, so too is F, and the following lemma describes the numerics of the two types of sections of p mentioned so far.

**Lemma 16.** If C is smooth, then under the intersection pairing on F, 
\[ σ_L.τ_z = 2 \] and \[ σ_L.σ_M = 3 \] for \( L, M \in U \) and \( z \in Z \).

**Proof.** The first equation follows from the proof of Theorem 14: there are two lines meeting z and L. For the second equation, it is enough to calculate \( σ_L.σ_M \) for a generic choice of L and M since Theorem 14 implies U is irreducible when C is smooth. Suppose the span of L and M intersects Y in a smooth cubic surface S: such a pair of lines exists since a generic hyperplane section of Y is smooth and contains skew lines not meeting P.

The span of L and M intersects P in a line, say ℓ, which is one of the twenty-seven lines on S. The intersection number \( σ_L.σ_M \) counts how many lines in Y meet L, M, and P, each of which is a line in S meeting L, M, and ℓ. Given three disjoint lines in a cubic surface, there are three other lines meeting each of those. □

**Remark 17.** One can regard U as an open subscheme of the space of sections of \( p : F \to C \). Hassett and Tschinkel prove in Proposition 2 of [8] that the space of sections of a fixed large enough height (a measure of the degree of the normal bundle of a section) is an open subscheme of a projective bundle over Pic^d C for some d. The numerics from Lemma 16 show that the height of a section \( σ_L \) for \( L \in U \) is not large enough for Hassett and Tschinkel’s theorem to apply, yet Theorem 14 gives a similar result since Sym^2 C is a blowup of Pic^2 C when C is smooth.

The section finishes with an analysis of the pairwise intersections of the components of \( F(Y) \) and a description of the boundary of \( \overline{U} \).

**Lemma 18.** \( F \cap P^* \) is zero-dimensional.

**Proof.** If \( L \in F \cap P^* \), then L is contained in a quadric surface Q meeting P in a degenerate conic (one component of which is L). There are finitely many degenerate conics containing Z, so there are finitely many quadrics Q for which \( Q \cap P \) is degenerate by by Lemma 12. Each of these quadrics contributes two points to \( F \cap P^* \). □
It is well-known that $F(Y)$ is smooth at $L$ if and only if $Y$ is smooth along $L$; for a proof, apply the proof of [5, Prop. 6.24] to cubics or adapt [2] to an arbitrary field. The following two lemmas also use the fact that any $F(Y)$ is singular along the intersection of any two irreducible components.

**Lemma 19.** $\mathcal{U} \cap P^*$ consists of the pencils $z^* \subset P^*$ of lines in $P$ through each of the points $z \in Z$.

**Proof.** A line in $P$ is a singular point of $F(Y)$ if and only if it intersects $Z$ nontrivially, so

$$P^* \cap \text{Sing}(F(Y)) = \bigcup_{z \in Z} z^*.$$ 

Moreover, $P^*$ is smooth, so any point in $P^* \cap \text{Sing}(F(Y))$ belongs to a second component of $F(Y)$. Then

$$\bigcup_{z \in Z} z^* = (P^* \cap \mathcal{U}) \cup (P^* \cap \mathcal{F}).$$

The left-hand side is purely 1-dimensional, and $P^* \cap \mathcal{F}$ is 0-dimensional by Lemma 18, so $\cup_{z \in Z} z^* = \mathcal{U} \cap P^*$.

**Lemma 20.** The intersection $\mathcal{U} \cap \text{Sing}(\mathcal{F})$ is zero-dimensional.

**Proof.** If $C$ is smooth, then so is $\mathcal{F}$ and there is nothing to prove, so assume $C$ is singular. We also assume $Z$ is reduced, a totally analogous argument works when $Z$ is nonreduced. Working over $k^*$, let $z \in Z$.

Recall the rational map $\psi_z : \text{Hilb}^2 C \dashrightarrow \mathcal{U}$ defined in Theorem 14 and Remark 15. The domain of $\psi_z$ is the complement of three points: the pairs $\{c, d\}$ of distinct points for which $\tau_z(c)$ and $\tau_z(d)$ are contained in $P$. For such a pair, $\tau_z(c)$ and $\tau_z(d)$ each contain length-two subschemes of $Z$.

It is easy to see that $\psi_z$ is injective on its domain, so the graph $\Gamma$ of $\psi_z$ has only three positive-dimensional fibers over $\text{Hilb}^2 C$. It follows that there are only three curves in $\mathcal{U}$ that do not have dense open sets contained in the image of $\psi_z$, and all of these are accounted for by the pencils $w^*$ with $z \neq w \in Z$.

Note that $\text{Sing}(\mathcal{F})$ consists of a conic over each singular point of $C$. To show that these conics do not belong to $\mathcal{U}$, it remains to check that there is a nontrivial open set of each not contained in the image of $\psi_z$. Let $c$ be a singular point of $C$, let $Q = q^{-1}(c)$ be the quadric lying over $c$, and let $L \subset Q$ be a line intersecting neither $Z$ nor any line containing a length two subscheme of $Z$. Such a line $L$ is general in $Q$. The plane $P'$ spanned by $L$ and $z$ intersects $Y$ in two other lines: $\tau_z(c)$
and another line \( L' \subset P \) joining \( z \) to \( L \cap P \). Since \( L' \) does not belong to the image of \( \tau_z \), \( L \) is not in the image of \( \psi_z \). \( \square \)

Let \( C_z \subset \mathcal{F} \) denote the image of \( \tau_z \).

**Lemma 21.** \( \overline{U} \cap \mathcal{F} = \bigcup_{z \in Z} C_z \).

**Proof.** First, we show \( C_z \subset \overline{U} \cap \mathcal{F} \) for each point \( z \in Z \). Note \( C_z \subset \text{Sing}(F(Y)) \) since each line in \( C_z \) meets \( Z \). Hence

\[
C_z \subset \mathcal{F} \cap \text{Sing}(F(Y)) = (\mathcal{F} \cap \overline{U}) \cup (\mathcal{F} \cap P^*) \cup \text{Sing}(\mathcal{F}).
\]

But \( \mathcal{F} \cap P^* \) is finite by Lemma 18, \( \text{Sing}(\mathcal{F}) \) consists of finitely many conics, and the genus of \( C_z \) is 2, so there is no embedding

\[
C_z \to (\mathcal{F} \cap P^*) \cup \text{Sing}(\mathcal{F}).
\]

Therefore \( C_z \subset \overline{U} \cap \mathcal{F} \).

For the reverse inclusion, note that if \( L \in \overline{U} \cap \mathcal{F} \), then \( L \) is a singular point of \( F(Y) \) so passes through a singular point of \( Y \). The lines in \( \mathcal{F} \) passing through singular points of \( Y \) are parametrized by the curves \( C_z \) for \( z \in Z \) and by \( \text{Sing}(\mathcal{F}) \), so

\[
\overline{U} \cap \mathcal{F} \subset \bigcup_{z \in Z} C_z \cup \text{Sing}(\mathcal{F}).
\]

Applying Lemma 20, it only remains to show \( \overline{U} \cap \mathcal{F} \) is equidimensional. Indeed, \( F(Y) \) is Gorenstein, and \( P^* \) is Cohen-Macaulay, so

\[
\overline{F(Y)} \setminus P^* = \mathcal{F} \cup \overline{U}
\]

is Cohen-Macaulay by \([4, \text{Thm. 21.23}]\). By \([4, \text{Cor. 18.11}]\), \( \mathcal{F} \cap \overline{U} \) is purely 1-dimensional. \( \square \)

**Proposition 22.**

\[
\overline{U} = U \cup \bigcup_{z \in Z} (z^* \cup C_z).
\]

**Proof.** This follows directly from Lemmas 19 and 21. \( \square \)

Note that since \( \mathcal{F} \) and \( P^* \) meet in codimension 2, Hartshorne’s Connectedness Theorem \([4, \text{Thm. 18.12}]\) implies \( \mathcal{F} \cup P^* \) is not Cohen-Macaulay. Again using \([4, \text{Thm. 21.23}]\), \( \overline{U} \) is not Cohen-Macaulay, and in particular, \( \overline{U} \) is not smooth. The rational involutions defined below are useful for analyzing the singular locus of \( \overline{U} \) and will play a crucial role in Sections 5 and 6.

**Definition 23.** For \( c \in C \), define a rational map \( i_c : \overline{U} \rightarrow F(Y) \) as follows: a general line \( L \in \overline{U} \) meets a unique line \( M \) in the ruling of a quadric \( Q \) specified by \( c \). The third line (counting multiplicity) in \( \text{span}(L, M) \cap Y \) is \( i_c(L) \).
Lemma 24. For all $c$, the domain of $i_c$ includes the following:

(i) $L \in \mathcal{U}$;
(ii) $\tau_z(d)$ for $z \in Z$ and $c \neq d \in C$ if $\tau_z(d) \notin P$.

If also $\tau_z(c) \notin P$ for all $z$, then the domain of $i_c$ also includes

(iii) any line $\ell \in P^* \cap \mathcal{U}$ not of the form $\tau_z(d)$.

Proof. Let $Q \subset Y$ be the quadric surface with ruling $c$. It is clear that $i_c$ is defined on $\mathcal{U}$ and on $\tau_z(d)$ meeting the conditions in (ii): each of these lines $L$ meets a unique line $M$ in the ruling $c$ of $Q$, and the span of $L$ and $M$ is not $P$ so meets $Y$ in dimension one. A line $\ell$ of the form in (iii) also meets a unique line in the ruling $c$ of $Q$, namely $\tau_z(c)$. If $\tau_z(c) \notin P$, then the plane spanned by $\ell$ and $\tau_z(c)$ is not $P$ so meets $Y$ in dimension one.

Lemma 25. Each $i_c$ is a rational involution whose image is contained in $\mathcal{U}$.

Proof. It is easy to see that $i_c^2 = \text{id}_{\mathcal{U}}$. To show that the image of $i_c$ is contained in $\mathcal{U}$, it is enough to show that the image contains $\mathcal{U}$.

Let $L \in \mathcal{U}$ and take $M$ to be any line in $\mathcal{U}$ with $L \cap M \neq \emptyset$. The plane $P'$ spanned by $L$ and $M$ intersects $P$ at some point $y$ and intersects $Y$ in the union of $L$, $M$, and a third line $N$. The point $y \in P$ must lie on $N$, so $N \in \mathcal{F}$. Taking $c \in C$ to be the ruling $N$ belongs to, $L = i_c(M)$.

Lemma 26. If $Y \setminus P$ is smooth, then $\mathcal{U} \setminus P^*$ is smooth.

Proof. Since $Y$ is smooth away from $P$, $\mathcal{U}$ is smooth; it remains to check that $\mathcal{U}$ is smooth at each point in $C_z \setminus P^*$ for $z \in Z$. Suppose $\tau_z(c) \notin P^*$. Each rational map $i_d$ sends smooth points to smooth points since it is an involution, so it suffices to show that we can reach $\tau_z(c)$ by applying a sequence of maps $i_d$ to some line $L \in \mathcal{U}$.

We first prove that $\tau_z(c)$ is a smooth point of $\mathcal{U}$ if it meets some line $L \subset Y \setminus P$. Then $\varphi_z(L) = \{c, d\}$ for some $d \in C$, meaning that $L$, $\tau_z(c)$, and $\tau_z(d)$ are coplanar. By definition, $i_d(\tau_z(c)) = L$.

Now, consider the case that $\tau_z(c)$ meets some line $L \subset Y \setminus P$. If not, then the image of the embedding $\psi_z(c, -) : C \to \mathcal{U}$ must be $C_z$. Choose any line $L \subset Y \setminus P$, and let $\varphi_z(L) = \{e, e'\}$, meaning $\tau_z(e)$ and $\tau_z(e')$ meet $L$. As $\tau_z(e) \in \text{im}(\psi_z(c, -))$, there is some $d \in C$ with $\psi_z(c, d) = \tau_z(e)$, and

$$i_d(\tau_z(c)) = \tau_z(e).$$

Hence $\tau_z(c)$ is a smooth point of $\mathcal{U}$ if and only if $\tau_z(e)$ is, and the latter meets the line $L \subset Y \setminus P$. Applying the previous case, we are done.
The following observation about the rational involutions $i_c$ is important in later sections.

**Lemma 27.** For $c, d \in C$ and $z \in Z,$

$$i_c(\tau_z(d)) = i_d(\tau_z(c))$$

whenever both sides of the equation are defined.

**Proof.** It suffices to check that the two rational maps $C \times C \to \mathcal{U}$ defined by sending $(c, d)$ to $i_c(\tau_z(d))$ and $i_d(\tau_z(c))$ agree on a dense open set. For the open set, take

$$V = \{(c, d) \mid d \neq \bar{c} \text{ and } \tau_z(c), \tau_z(d) \not\subset P\} \subset C \times C.$$

For $(c, d) \in V$, the lines $\tau_z(c)$ and $\tau_z(d)$ meet at $z$, and the third line in $\text{span}(\tau_z(c), \tau_z(d))$ is, by definition, both $i_c(\tau_z(d))$ and $i_d(\tau_z(c))$. □

**Remark 28.** Lemma 27 affords another description of the birational maps $\psi_z : \text{Sym}^2 C \dashrightarrow \mathcal{U}$ from the proof of Theorem 14:

$$\psi_z(\{c, d\}) = i_c(\tau_i(d)).$$

As a consequence, when $\tau_z(c) \not\subset P$, $\tau_z(\bar{c})$ is in the domain of $i_c$ since $\psi_z(\{c, \bar{c}\})$ is well-defined. Explicitly, $i_c(\tau_z(\bar{c})) = T_z(Q \cap P) \in z^*$ where $Q$ is the quadric surface with rulings $c$ and $\bar{c}$.

5. Resolution via the Mukai flop

The goal of this section is to resolve the indeterminacy of $\pi$ by a Mukai flop, defined below.

**Definition 29.** The normal bundle of $P^* \subset F$ is isomorphic to the cotangent bundle to $P^*$, identifying the exceptional divisor of the blowup $\text{Bl}_{P^*} F \to F$ with

$$E = \{(l, p) : p \in l \subset P\} \subset P^* \times P.$$

The contraction of $E \subset \text{Bl}_{P^*} F$ by the second projection is the Mukai flop $M$ of $F$ along $P^*$.

Note that $M$ is smooth and contains the plane $P$. In the complex setting, $M$ is also hyperkähler. Abusing notation slightly, we identify $F \setminus P^*$ with $M \setminus P$ in what follows.

The rational map $\pi$ produces a rational map $\rho : M \dashrightarrow \mathbb{P}^2$ making the following diagram commute:

$$
\begin{array}{ccc}
\text{Bl}_{P^*} F & \longrightarrow & M \\
\downarrow & & \downarrow \rho \\
F & \longrightarrow & (P^\perp)^*
\end{array}
$$
Let $\ell \subset P \perp$, let $T$ be the closure of $\rho^{-1}(\ell)$, and let $Y$ be the intersection of $X$ with the hyperplane spanned by $\ell$ and $P$, a cubic threefold containing a plane. Let $C$, $Z$, and $\bar{U}$ be the schemes associated to $Y$ as in Section 4. Notice that

$$T \setminus P \cong \bar{U} \setminus P^* = \bar{U} \setminus \bigcup_{z \in Z} z^*.$$

**Lemma 30.** $Z \subset T$.

**Proof.** The point $(\ell, z) \in E$ parametrizes the normal direction to $\ell$ in $P$ as it moves in the pencil $z^*$. As $z^* \subset \bar{U}$, the pencil

$$\{(\ell, z) : z \in \ell\} \subset E \subset \text{Bl}_{P^*} F$$

lies in the proper transform of $\bar{U}$. The projection $\text{Bl}_{P^*} F \to M$ contracts this pencil to $z$, so $z \in T$, and $Z \subset T \cap P$. \qed

The embeddings $\tau_z : C \to \bar{U}$ determine rational maps $\upsilon_z : C \dasharrow T$ defined away from the three points $c \in C$ for which $\tau_z(c) \subset P$. As with $\tau_z$, denote by $C_z$ the closure of the image of $\upsilon_z$. Similarly, the rational involutions $i_c$ on $\bar{U}$ yield rational involutions $j_c$ on $T$.

**Lemma 31.** For $c \in C$, the domain of $j_c$ contains $(T \setminus P) \cup Z$.

**Proof.** Let $Q$ be the quadric surface with a ruling specified by $c$. The following cases describe all points on $(T \setminus P) \cup Z$:

(i) $L \in U$;
(ii) $\upsilon_z(d)$ for $z \in Z$ and $\bar{c} \neq d \in C$ if $\tau_z(d) \not\subset P$;
(iii) $\upsilon_z(\bar{c})$ for $z \in Z$ if $\tau_z(\bar{c}) \not\subset P$;
(iv) $z \in Z$.

On each of these types of points, $j_c$ is defined as follows.

(i) Note $i_c(L) \not\subset P^*$, so $i_c(L)$ can be identified with a point of $T$, and $j_c(L) = i_c(L)$.

(ii) The line in the $c$-ruling of $Q$ that $\tau_z(d)$ meets is $\tau_z(c)$. The third line $\ell$ in the plane spanned by $\tau_z(c)$ and $\tau_z(d)$ is $i_c(\tau_z(d))$ and can be identified with $j_c(\tau_z(d))$ so long as it does not lie in $P$. The rational map $\psi_z : \text{Sym}^2 C \dasharrow \bar{U}$ sending $\{a, b\}$ to $i_a(\tau_z(b))$ is injective, and it is straightforward to check that the pencil of lines in $P$ through $z$ is the image of the pencil of pairs $\{a, \bar{a}\}$. Thus $i_c(\tau_z(d)) \not\subset P$ since $\bar{c} \neq d$, as needed.

(iii) As explained in Remark 28, $i_c(\tau_z(\bar{c})) \in z^*$. Using Lemma 30, $j_c(\upsilon_z(\bar{c})) = z$.

(iv) Since $j_c$ is an involution, applying $j_c$ to both sides of (iii) gives $j_c(z) = \upsilon_z(\bar{c})$. \qed
Lemma 32. The rational maps $v_z : C \dashrightarrow T$ extend to morphisms. Explicitly, if $\tau_z(c) \subset P$ and $\{z, w\}$ is the degree 2 subscheme of $Z$ contained in $\tau_z(c)$, then $v_z(c) = w$.

Proof. First, we justify that $\tau_z(c)$ intersects $Z$ in degree 2. Choosing some smooth conic $B$ containing $Z$, we have
$$\deg(Z \cap L) \leq \deg(B \cap L) = 2$$
for any line $L \subset P$. On the other hand, $\tau_z(c)$ belongs to a degenerate conic containing $Z$ (namely the intersection of the quadric surface containing $\tau_z(c)$ with $P$), so $\deg(Z \cap \tau_z(c)) = 2$.

Now, let $d \in C$ be such that $\tau_z(d) \not\subset P$. Then
$$i_c(\tau_z(d)) = \tau_w(\bar{d}),$$
so
$$j_c(v_z(d)) = v_w(\bar{d}).$$
Also, Lemma 27 implies
$$j_d(v_z(c)) = j_c(v_z(d))$$
so
$$j_d(v_z(c)) = v_w(\bar{d}).$$
Applying $j_d$ to both sides of the equation above yields
$$v_z(c) = j_d(v_w(\bar{d})) = w,$$
as desired. $\square$

Lemma 33. $T \cap P = Z$.

Proof. Work over $k^s$ and fix some point $z \in Z$. Recall from Remark 28 the rational map
$$\psi_z : \text{Hilb}^2 C \dashrightarrow \overline{U}, \quad \{c, d\} \mapsto i_c(\tau_z(d)).$$
This gives rise to a rational map
$$\Psi_z : \text{Hilb}^2 C \dashrightarrow T, \quad \{c, d\} \mapsto j_c(v_z(d)).$$
In fact, $\Psi_z$ is a morphism since Lemmas 31 and 32 show that each involution $j_c$ is defined on the entire image of $v_z$. Moreover, the image of $\Psi_z$ is contained in $(T \setminus P) \cup Z$, so $(T \setminus P) \cup Z$ is closed. Since $T$ is irreducible, $T \cap P = Z$. $\square$

Corollary 34. Each rational involution $j_c$ extends to a morphism.

Proof. This follows immediately from Lemmas 31 and 33. $\square$

Proposition 35. The rational map $\rho : M \dashrightarrow \mathbb{P}^2$ is a morphism.
Proof. Since $M$ is normal, $\rho$ is a morphism if each point in $M$ lies in the closure of a unique fiber of $\rho$. To verify this, consider three cases:

(i) if $L \in M$ determines a line $L \subset X \setminus P$, then $L$ belongs only to the fiber of $\rho$ over $q(L)$ and lies in the closure of no other fibers;
(ii) if $L \in M$ is a line in $X$ meeting $P$ at a single point $p$, then Lemma 21 shows that $L$ is only in the closure of the fiber of $\rho$ over $T_pX \cap P^\perp$;
(iii) similarly, if $z \in P \subset M$, then Lemma 33 shows that $z$ is only in the closure of the fiber of $\rho$ over $T_pX \cap P^\perp$.

□

Proposition 36. $T$ is smooth when $Y \setminus P$ is smooth.

Proof. One can argue exactly as in Lemma 26, showing that any point on $T$ can be reached by applying a sequence of involutions of the form $j_c$ to a smooth point on $T$. □

By Lemma 12, $Y$ is smooth away from $P$ if $\rho(T) = \ell$ is transverse to $\Delta$. Thus the dual curve $\Delta^*$, which has degree 30 and geometric genus 10, parametrizes the singular fibers of $\rho$.

When a fiber $T$ is smooth, it becomes isomorphic to an abelian surface over at worst a quartic extension of the base field, proved below. Together with Proposition 35, this proves Theorem 1.

Proposition 37. If $T$ is a smooth fiber of $\rho$, then $T_K \cong \text{Pic}^2_{C,K}$ where $K/k$ is an extension of the base field over which one of the points $z \in Z$ is defined.

Proof. We suppress the base change from the notation. Since $C$ is a smooth curve of genus 2, $\text{Hilb}^2 C$ is isomorphic to the blowup of $\text{Pic}^2_C$ at the canonical divisor. For each $c \in C$, the morphism $\Psi_c : \text{Sym}^2 C \to T$ sends $\{c, \bar{c}\}$ to $z$ so factors through the blowdown $\text{Sym}^2 C \to \text{Pic}^2_C$, giving a birational morphism $\text{Pic}^2_C \to T$. Any rational map from a smooth variety to an abelian variety extends to a morphism; applying this fact to the rational inverse $T \dashrightarrow \text{Pic}^2_C$ yields $\text{Pic}^2_C \cong T$. □

6. Arithmetic of the smooth fibers

Let $T$ be a smooth fiber of $\rho : M \to \mathbb{P}^2$, and let $Z$ and $C$ be as before. The goal of this section is to prove Theorem 2 analyzing the torsor structure of $T$ when the ground field $k$ is not closed. Our approach mirrors Wang’s in [17].

The proofs of Lemmas 39 and 41 below use information about the incidence relations between lines on cubic surfaces. We fix the following
notation for the lines on a smooth cubic surface over an algebraically closed field, similar to the notational scheme used in [6, V.4].

Notation 38. A smooth cubic surface $S$ over an algebraically closed field is the blowup of $\mathbb{P}^2$ at six points $p_1, \ldots, p_6$ in general position and contains 27 lines:

(i) the six exceptional divisors $E_i$;
(ii) for each pair $\{i, j\}$, the proper transform $\ell_{ij}$ of the line joining $p_i$ to $p_j$;
(iii) and for each $i$, the proper transform $F_i$ of the conic passing through all the $p_j$ except $p_i$.

The incidence relations are as follows:

(i) $E_i$ meets $\ell_{ij}$ and $f_j$ for $i \neq j$;
(ii) $\ell_{ij}$ meets $E_i, E_j, F_i, F_j$, and $\ell_{hk}$ for $\{i, j\} \cap \{h, k\} = \emptyset$;
(iii) $F_i$ meets $E_j$ for $i \neq j$ and $\ell_{ij}$ for all $j$.

Moreover, given six pairwise skew lines $L_1, \ldots, L_6 \subset S$, the blowdown of the $L_i$ is isomorphic to $\mathbb{P}^2$, so one may assume $L_i = E_i$.

Let $T'$ be the variety isomorphic to $T$ whose points are denoted $-L$ for $L \in T$.

Lemma 39. The rules

\[ L + (c) = -j_c(L) \]

and

\[ -L + (c) = j_c(L) \]

define an action of $\text{Div}(C)$ on $T \sqcup T'$.

Proof. For the action to be well-defined, $(c) + (d)$ must act the same as $(d) + (c)$, so we need to check $j_c \circ j_d = j_d \circ j_c$ for any $c, d \in C$.

Work over $k^s$. As $T$ is irreducible, it suffices to show that the two maps $C \times C \times T \to T$ sending $(c, d, L)$ to $j_d \circ j_c(L)$ and $j_c \circ j_d(L)$ agree on a nonempty open set. In particular, we can choose $L \in U$ and $c, d \in C$ such that the $\mathbb{P}^3$ spanned by the lines $\sigma_L(\bar{c})$ and $\sigma_L(\bar{d})$ meets $Y$ in a smooth cubic surface $S$.

Label $\sigma_L(\bar{c}) = E_1$, $\sigma_L(\bar{d}) = E_2$, and $L = \ell_{12}$, using the notation for the lines on a smooth cubic surface. Notice that $S \cap P$ is a line meeting $\sigma_L(\bar{c})$ and $\sigma_L(\bar{d})$ but not $L$. Label this line $F_6$.

Now we calculate $j_c(L) = F_2$; indeed, only the line $F_2$ meets $E_1$ and $\ell_{12}$. The line $\ell_{26}$ meets $E_2$ and $F_6$ hence $\sigma_L(\bar{d})$ and $P$, so it is a line in the ruling of a quadric surface specified by $d$. As $j_c(L) = F_2$ meets $\ell_{26}$, $j_d \circ j_c(L)$ is the line meeting $F_2$ and $\ell_{26}$, namely $E_6$. One calculates $j_c \circ j_d(L) = E_6$ similarly. \qed
Proposition 40. The principal divisors act trivially on $T \sqcup T'$, so the action of $\text{Div}(C)$ on $T \sqcup T'$ descends to an action by $\text{Pic}_C$.

Proof. First, notice that $T$ and $T'$ are in different orbits of the action by $\text{Div}^0(C)$. So, it suffices to show that principal divisors act trivially on $T$.

Every nontrivial divisor class in $\text{Pic}_C^0$ can be represented as a difference of two effective divisors of degree 1 in exactly two ways: if $(c) - (d)$ represents $D$, then so does $[\bar{d}] - [\bar{c}]$. By Lemma 39, \( j_c \circ j_d = j_d \circ j_c \), so there is a well-defined morphism $\text{Pic}_C^0 \to \text{Aut}(T)$ sending $[c] - [d]$ to $j_d \circ j_c$.

The map is automatically a homomorphism: its image is a commutative, projective subgroup scheme, and any unital morphism of abelian varieties is a homomorphism. Moreover, $j_d \circ j_c(L) = L + [\bar{c}] - [\bar{d}]$, so the homomorphism $\text{Div}^0(C) \to \text{Aut}(T)$ coming from the group action described earlier factors through the morphism $\text{Pic}_C^0 \to \text{Aut}(T)$, i.e. principal divisors are in the kernel. □

Moreover, $\omega_C$ acts trivially on $T \sqcup T'$ since $L + [c] + [\bar{c}] = j_c^2(L) = L$, so the action by $\text{Pic}_C$ descends to an action by $\text{Pic}_C^0 \sqcup \text{Pic}_C^1$. Note that $\text{Pic}_C^0$ acts on each of $T$ and $T'$, and an element of $\text{Pic}_C^1$ exchanges the components $T$ and $T'$. The following lemma shows $T \sqcup T'$ is a torsor over $\text{Pic}_C^0 \sqcup \text{Pic}_C^1$.

Proposition 41. The action described above makes $T \sqcup T'$ a torsor over $\text{Pic}_C^0 \sqcup \text{Pic}_C^1$.

Proof. To show that the action of $\text{Pic}_C^0 \sqcup \text{Pic}_C^1$ on $T \sqcup T'$ is transitive, we must check that for each $L, M \in T$,

(i) there is a divisor class $[D] \in \text{Pic}_C^1$ so that $L + [D] = -M$;

(ii) there is a divisor class $[E] \in \text{Pic}_C^0$ so that $L + [E] = M$.

Moreover, we verify

(iii) the action of $\text{Pic}_C^0$ on $T$ is free;

(iv) the action of $\text{Pic}_C^0 \sqcup \text{Pic}_C^1$ on $T \sqcup T'$ is free.

(i) First, we show that when $L, M \in \mathcal{U}$ span a $\mathbb{P}^3$ meeting $Y$ in a smooth cubic surface, there exists $[D] \in \text{Pic}_C^1$ so that $L + [D] = -M$.

Using Lemma 16, let $c, d, e \in C$ be the points lying below $\sigma_L \cap \sigma_M \subset \mathcal{F}$. No two of $\sigma_L(c), \sigma_L(d), \sigma_L(e)$ meet: if $\sigma_L(c)$ met $\sigma_L(d)$, then they would span a plane containing $L$ and $M$ which would not meet $X$ in degree three. Working over $k^s$ and using the standard notation for
lines on the cubic surface $S$, assume $L = E_1$, $M = E_2$, $S \cap P = E_3$, $\sigma_L(c) = F_4$, $\sigma_L(d) = F_5$, and $\sigma_L(e) = F_6$. One calculates 
$$j_d \circ j_e(L) = j_d(\ell_{14}) = \ell_{26} = j_e(M),$$ 
so 
$$[c] - M = L - [c] - [d],$$
and 
$$L + [\bar{c}] + [\tilde{d}] + [\tilde{e}] = M.$$ 
Thus $L + [D] = -M$ where $[D] = [\bar{c}] + [\tilde{d}] + [\tilde{e}] - \omega_C$.

For $L \in \mathcal{U}$, define a morphism $s_L : \text{Pic}_C^1 \to \text{T}'$ by $s_L([D]) = L + [D]$. By the above, the image of $s_L$ contains the dense open set $V = \{-M \mid M \in \mathcal{U} \text{ and span}(L, M) \cap Y \text{ is a smooth cubic surface}\}$, so $s_L$ is surjective.

Next, consider the summation map 
$$\Sigma : \text{T} \times \text{Pic}_C^1 \to \text{T}', \ (L, [D]) \mapsto L + [D].$$
Let $-M \in \text{T}'$. As $s_L$ is surjective for each $L \in \mathcal{U}$, we get 
$$\mathcal{U} \subset \pi_1(\Sigma^{-1}(-M))$$
where $\pi_1$ is the projection $\text{T} \times \text{Pic}_C^1 \to \text{T}$. Since $\pi_1$ is closed, 
$$\pi_1(\Sigma^{-1}(-M)) = \text{T}.$$ 
That is, for each $L \in \text{T}$ there exists $[D] \in \text{Pic}_C^1$ so that $L + [D] = -M$.

(ii) Apply (i) to obtain $[D], [E] \in \text{Pic}_C^1$ with $L + [D] = -M$ and $M + [E] = -M$. Then $L + [D - E] = M$.

(iii) Work over $k^*$. It is enough to check that if $[c] + [d]$ fixes a point $z \in Z$, then $[c] + [d] = \omega_C$. Indeed, if $z = z + (c) + (d)$, then 
$$z + [\tilde{d}] = z + [c]$$
$$j_d(z) = j_e(z)$$
$$v_z(\tilde{d}) = v_z(c),$$
so $\tilde{d} = c$ since $v_z$ is an embedding.

(iv) The argument for (iii) also shows $\text{Pic}_C^0$ acts freely on $\text{T}'$. Moreover, if $L + [D] = L + [E]$ for $L \in \text{T} \sqcup \text{T}'$ and $[D], [E] \in \text{Pic}_C^1$, then $[D] - [E] \in \text{Pic}_C^1$, fixes $L$, so $[D] - [E] = 0$ by (iii). $\square$

We now prove Theorem 2 and Corollary 3. Let 
$$G = \text{Pic}_C^0 \sqcup \text{T} \sqcup \text{Pic}_C^1 \sqcup \text{T}'.$$

**Theorem 2.** There is a commutative group law on $G$ extending the group law on $\text{Pic}_C^0 \sqcup \text{Pic}_C^1$ such that for $L, M \in \text{T} \sqcup \text{T}'$ and $c \in C$,

(1) $L + [c] = -j_e(L)$ and $-L + [c] = j_e(L)$;
(2) $L + M$ is the unique divisor class $[D]$ such that $-L + [D] = M$ under the action of $\text{Pic}_C^0 \sqcup \text{Pic}_C^1$ on $T \sqcup T'$ defined prior.

Proof. All that remains to check is that the group law is associative, i.e. the following all hold for for $L, M, N \in T \sqcup T'$ and $[D], [E], [F] \in \text{Pic}_C^0 \sqcup \text{Pic}_C^1$.

\begin{align*}
(i) \quad ([D] + [E]) + [F] & = [D] + ([E] + [F]) \\
(ii) \quad (L + [D]) + [E] & = L + ([D] + [E]) \\
(iii) \quad (L + M) + [D] & = L + (M + [D]) \\
(iv) \quad (L + M) + N & = L + (M + N)
\end{align*}

The first is inherited from the associativity of $\text{Pic}_C^0 \sqcup \text{Pic}_C^1$, and the second follows from the fact that $\text{Div}(C)$ acts on $T \sqcup T'$. For (iii), let $[E] = L + M$, so $[E]$ is the unique divisor class such that $-L + [E] = M$.

Using (ii),

$$-L + ((L + M) + [D]) = M + [D].$$

Similarly, $L + (M + [D])$ is defined by the equation

$$-L + (L + (M + [D])) = M + [D],$$

so

$$-L + ((L + M) + [D]) = -L + (L + (M + [D])).$$

Since $\text{Pic}_C^0 \sqcup \text{Pic}_C^1$ acts freely on $T \sqcup T'$, equation (iii) holds. For (iv), let $K_1$ be the left-hand side and $K_2$ be the right-hand side. Using (iii) and commutativity,

$$M + K_1 = M + (N + (L + M))$$
$$= (M + N) + (L + M)$$
$$= (M + L) + (M + N)$$
$$= M + (L + (M + N))$$
$$= M + K_2,$$

so again using (iii), the divisor class $M + K_1$ sends $-K_1$ and $-K_2$ both to $M$. As $\text{Pic}_C^0 \sqcup \text{Pic}_C^1$ acts invertibly, $K_1 = K_2$. \hfill \Box

Corollary 3. In $H^1(k, \text{Pic}_C^0)$, $4[T] = 0$ and $2[T] = [\text{Pic}_C^1]$.

Proof. The short exact sequence of group schemes

$$0 \to \text{Pic}_C^0 \to G \to \mathbb{Z}/4\mathbb{Z} \to 0$$

yields a short exact sequence

$$0 \to \text{Pic}_C^0(k^s) \to G(k^s) \to \mathbb{Z}/4\mathbb{Z} \to 0$$

of Galois modules. In the long exact sequence in Galois cohomology, the connecting homomorphism $H^0(k, \mathbb{Z}/4\mathbb{Z}) \to H^1(k, \text{Pic}_C^0)$ sends $1 \mapsto [\pi^{-1}(1)] = [T]$ and $2 \mapsto [\text{Pic}_C^1]$. \hfill \Box
References

[1] Nicolas Addington, Brendan Hassett, Yuri Tschinkel, and Anthony Várilly-Alvarado. Cubic fourfolds fibered in sextic del Pezzo surfaces. Amer. J. Math., 141(6):1479–1500, 2019.
[2] W. Barth and A. Van de Ven. Fano varieties of lines on hypersurfaces. Arch. Math. (Basel), 31(1):96–104, 1978/79.
[3] Arnaud Beauville and Ron Donagi. La variété des droites d’une hypersurface cubique de dimension 4. C. R. Acad. Sci. Paris Sér. I Math., 301(14):703–706, 1985.
[4] David Eisenbud. Commutative algebra, volume 150 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1995. With a view toward algebraic geometry.
[5] David Eisenbud and Joe Harris. 3264 and all that—a second course in algebraic geometry. Cambridge University Press, Cambridge, 2016.
[6] Robin Hartshorne. Algebraic geometry. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977.
[7] Brendan Hassett. Special cubic fourfolds. Compositio Math., 120(1):1–23, 2000.
[8] Brendan Hassett and Yuri Tschinkel. Spaces of sections of quadric surface fibrations over curves. Compact moduli spaces and vector bundles, 564:227–249, 2012.
[9] Daniel Huybrechts. Lectures on K3 surfaces, volume 158 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2016.
[10] Jun-Muk Hwang. Base manifolds for fibrations of projective irreducible symplectic manifolds. Invent. Math., 174(3):625–644, 2008.
[11] Emanuele Macrì and Paolo Stellari. Fano varieties of cubic fourfolds containing a plane. Math. Ann., 354(3):1147–1176, 2012.
[12] Eyal Markman. A survey of Torelli and monodromy results for holomorphic-symplectic varieties. In Complex and differential geometry, volume 8 of Springer Proc. Math., pages 257–322. Springer, Heidelberg, 2011.
[13] Daisuke Matsushita. On fibre space structures of a projective irreducible symplectic manifold. Topology, 38(1):79–83, 1999.
[14] Daisuke Matsushita. On isotropic divisors on irreducible symplectic manifolds. Higher dimensional algebraic geometry—in honour of Professor Yujiro Kawamata’s sixtieth birthday, 74:291–312, 2017.
[15] Giovanni Mongardi and Antonio Rapagnetta. Monodromy and birational geometry of O’grady’s sixfolds. Journal de mathématiques pures et appliquées, 146(1):31–68, 2021.
[16] Claire Voisin. Théorème de Torelli pour les cubiques de $\mathbf{P}^5$. Invent. Math., 86(3):577–601, 1986.
[17] Xiaoheng Wang. Maximal linear spaces contained in the based loci of pencils of quadrics. Algebr. Geom., 5(3):359–397, 2018.

Department of Mathematics, University of Oregon, Eugene, OR 97403-1222, USA

Email address: cbrooke@uoregon.edu