HIGHER FROBENIUS-SCHUR INDICATORS FOR SEMISIMPLE HOPF ALGEBRAS IN ARBITRARY CHARACTERISTIC

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Abstract. Let $H$ be a semisimple Hopf algebra over an algebraically closed field $k$ of arbitrary characteristic with a nonzero integral $\Lambda$. We show that the antipode $S$ of $H$ satisfies the equality $S^2(h) = uh^{-1}$, where $h \in H$ and $u = S(\Lambda(2))\Lambda(1)$. We use this formula of $S^2$ to generalize higher Frobenius-Schur indicators from characteristic 0 case to the context of arbitrary characteristic. Then we extend the indexes of these indicators from positive integers to all integers. Moreover, we show that all these indicators are gauge invariants for the tensor category $\text{Rep}(H)$ of finite dimensional representations of $H$.

1. Introduction

Linencko-Montgomery [10] generalized the classical Frobenius-Schur (FS) indicators from group-theoretic result to the setting of a semisimple involutory Hopf algebra $H$. They also defined higher FS indicators $\nu_n(V)$ by using idempotent integral $\Lambda$ of $H$, namely,

$$\nu_n(V) = \chi_V(\Lambda(1) \cdots \Lambda(n)) \quad \text{for} \quad n \geq 1,$$

where $\chi_V$ is the character afforded by finite dimensional representation $V$ of $H$. The higher FS indicators were later extensively studied by Kashina-Sommerhäuser-Zhu for semisimple Hopf algebras over an algebraically closed field of characteristic zero [7], and by Ng-Schauenburg for semisimple quasi-Hopf algebras over the field of complex numbers [13]. The notion of higher FS indicators has been generalized to objects of a pivotal category [14, 15].

Kaplansky’s fifth conjecture states that the antipode of a semisimple Hopf algebra is an involution. Over an algebraically closed field of characteristic zero, a semisimple Hopf algebra is also cosemisimple [8], otherwise stated, the dual Hopf algebra is semisimple. In this situation the antipode is an involution [9]. For a semisimple Hopf algebra over an algebraically closed field of positive characteristic, the antipode is not known to be an involution unless the positive characteristic is larger than a certain number (see [19, 4]).
Let $H$ be a finite dimensional semisimple Hopf algebra with the antipode $S$ over an algebraically closed field $\mathbb{k}$ of arbitrary characteristic. By comparing two associative symmetric and non-degenerate bilinear forms on $H$, we obtain a formula for $S^2$ as follows:

$$S^2(h) = uhu^{-1},$$

where $h \in H$ and $u = S(\Lambda_{(2)})\Lambda_{(1)}$ for a nonzero integral $\Lambda$ of $H$. We study some properties of the element $u$. According to the formula of $S^2$, we characterize the equality $S^2 = id$ in terms of the integral $\Lambda$ of $H$, namely, $S^2 = id$ if and only if the integral $\Lambda$ is cocommutative. This gives a statement of the Kaplansky’s fifth conjecture in another way: any nonzero integral of a semisimple Hopf algebra is cocommutative.

Ng-Schauenburg [14] defined the $n$-th FS indicator of an object of a pivotal category to be the trace of a certain linear operator. Based on the formula $S^2(h) = uh^{-1}$, we have an isomorphism of $H$-modules

$$j_{u,V} : V \rightarrow V^{**}, \quad j_{u,V}(v)(f) = f(uy) \text{ for } v \in V, f \in V^{**},$$

which is functorial in $V$. As the element $u = S(\Lambda_{(2)})\Lambda_{(1)}$ is not known to be a group-like element, the functorial isomorphism $j_u : id \rightarrow (-)^{**}$ is not known to be a tensor isomorphism. In other words, the category $\text{Rep}(H)$ of finite dimensional representations of $H$ is not known to be pivotal with respect to the structure $j_u$. Even though, using the functorial isomorphism $j_u$ we may still define the $n$-th FS indicator $\nu_n(V)$ of $V$ to be the trace of a certain linear operator as Ng-Schauenburg did in [14]. It is similar to the case of characteristic zero that $\nu_n(V)$ can be entirely described in terms of the integral $\Lambda$ of $H$, namely,

$$\nu_n(V) = \chi_V(u^{-1}\Lambda_{(1)} \cdots \Lambda_{(n)}) \text{ for } n \geq 1. \quad (1.2)$$

Moreover, the formula (1.2) does not depend on the choice of the nonzero integral $\Lambda$ and it recovers the original formula (1.1) when the characteristic of $\mathbb{k}$ is zero and $\Lambda$ is idempotent.

Note that the formula (1.2) can be written as $\nu_n(V) = \chi_V(u^{-1}P_n(\Lambda))$ for $n \geq 1$, where $P_n$ is the $n$-th Sweedler power map of $H$. As the $n$-th Sweedler power map $P_n$ can be defined for any $n \in \mathbb{Z}$, this enables us to extend the $n$-th FS indicator $\nu_n(V)$ from $n \geq 1$ to the case $n \in \mathbb{Z}$. In view of this, we define $\nu_n(V) = \chi_V(u^{-1}P_n(\Lambda))$ for all $n \in \mathbb{Z}$. More explicitly,

$$\nu_n(V) = \begin{cases} 
\chi_V(u^{-1}\Lambda_{(1)} \cdots \Lambda_{(n)}), & n \geq 1; \\
\chi_V(u^{-1}\epsilon(\Lambda)), & n = 0; \\
\chi_V(u^{-1}\Lambda_{(-n)} \cdots \Lambda_{(1)}), & n \leq -1.
\end{cases}$$

In particular, replacing $V$ by the regular representation $H$, we reconstruct the $n$-th indicator of $H$ defined by the trace of the map $S \circ P_{n-1}$ (see [6, 7]). We show that $V$ and its dual $V^*$ have the same FS indicators. We prove that the $n$-th FS indicator $\nu_n(V)$ is an invariant of the tensor category $\text{Rep}(H)$ for any $n \in \mathbb{Z}$ and any finite dimensional representation $V$ of $H$. The main argument of the proof
Throughout this paper, $H$ is a finite dimensional semisimple Hopf algebra over an algebraically closed field $\mathbb{k}$ of arbitrary characteristic with counit $\varepsilon$, antipode $S$, multiplication $m$ and comultiplication $\Delta$. The comultiplication $\Delta(a)$ will be written as $\Delta(a) = a_{(1)} \otimes a_{(2)}$ for $a \in H$, where we omit the summation sign. We denote by $\Lambda$ and $\lambda$ the left and right integrals of $H$ and $H^*$ respectively so that $\lambda(\Lambda) = 1$. Since the semisimple Hopf algebra $H$ is unimodular, the left and right integrals of $H$ are the same. The center of $H$ is denoted by $Z(H)$. We refer to [12] for the basic theory of Hopf algebras.

If $V$ is a finite dimensional $H$-module, then $V$ is also called a representation of $H$ via the algebra homomorphism $\rho_V : H \rightarrow \text{End}_\mathbb{k}(V)$ given by $\rho_V(h)(v) = h \cdot v$ for $h \in H$ and $v \in V$. We will make no distinction between the two notations. The character of $V$ is the map $\chi_V : H \rightarrow \mathbb{k}$ given by $\chi_V(h) = \text{tr}(\rho_V(h))$ for $h \in H$. The $\mathbb{k}$-linear dual space $V^*$ is also an $H$-module via $(h \cdot f)(v) := f(S(h)v)$ for $h \in H$, $f \in V^*$ and $v \in V$. In particular, the dual module $V^*$ has the character $\chi_{V^*} = \chi_V \circ S$.

The category $\text{Rep}(H)$ of finite dimensional representations of the Hopf algebra $H$ is a semisimple monoidal category, where the monoidal structure stems from the comultiplication $\Delta$.

Recall that the dual Hopf algebra $H^*$ has an $H$-bimodule structure given by

$$(a \rightarrow f)(b) = f(ba), \ (f \leftarrow a)(b) = f(ab) \text{ for } a, b \in H, \ f \in H^*.$$  

Moreover, $(H^*, \rightarrow)$ and $(\rightarrow, H^*)$ are free $H$-modules generated by $\lambda$, i.e., $H^* = \lambda \leftarrow H$ and $H^* = H \rightarrow \lambda$ (see [17, Corollary 2(b)]). This provides an associative and non-degenerate bilinear form $H \times H \rightarrow \mathbb{k}$ by $a \times b \mapsto \lambda(ab)$ for $a, b \in H$.

Moreover, the pair $(H, \lambda)$ is a Frobenius algebra with the Frobenius homomorphism $\lambda$ satisfying the equality (see [17, Eq.(1.1)]):

$$a = \lambda(a\Lambda(1))S(\Lambda(2)) = \lambda(S(\Lambda(2))a)\Lambda(1) \text{ for } a \in H.$$  

The pair $\Lambda(1) \otimes S(\Lambda(2))$ satisfying (2.1) is called the dual basis of $H$ with respect to the Frobenius homomorphism $\lambda$.

The paper is organized as follows: In Section 2, we present some basic results on semisimple Hopf algebras. In Section 3, we deduce the formula of $S^2$ by comparing two bilinear forms on $H$. We investigate some properties of the element $u = S(\Lambda(2))\Lambda(1)$ and show that the integral $\Lambda$ is cocommutative if and only if $S^2 = id$. In Section 4, we generalize higher FS indicators from characteristic zero to general characteristic and extend the indexes of these indicators from positive integers to all integers. In Section 5, we show that the $n$-th FS indicator $\nu_n(V)$ is a gauge invariant for any integer $n$ and any finite dimensional representation $V$ of $H$.
Since the right integral \( \lambda \) of \( H^* \) satisfies \( \lambda(ab) = \lambda(S^2(b)a) \) for all \( a, b \in H \) (see [17, Theorem 3(a)]), the Hopf algebra \( H \) is a symmetric algebra with a symmetric bilinear form given by
\[
H \times H \rightarrow \mathbb{k}, \quad a \times b \mapsto \lambda(uab) = (\lambda \leftarrow u)(ab) = (u \rightarrow \lambda)(ab),
\]
where \( u \) is a unit of \( H \) satisfying \( S^2(h) = uh^{-1} \) for all \( h \in H \) and the Frobenius homomorphism \( \lambda \leftarrow u = u \rightarrow \lambda \) holds because \( \lambda(au) = \lambda(S^2(u)a) = \lambda(ua) \) for all \( a \in H \). Using (2.1) we may see that the pair \( \Lambda(1) \otimes u^{-1}S(\Lambda(2)) \) is a dual basis of \( H \) with respect to \( \lambda \leftarrow u (= u \rightarrow \lambda) \) (see also [3, Lemma 1.4(2)]). The symmetry of the Frobenius homomorphism \( \lambda \leftarrow u (= u \rightarrow \lambda) \) means that
\[
\Lambda(1) \otimes u^{-1}S(\Lambda(2)) = u^{-1}S(\Lambda(2)) \otimes \Lambda(1). \tag{2.2}
\]

By Wedderburn’s theorem, the semisimple Hopf algebra \( H \) is isomorphic to a direct sum of full matrix algebras over \( \mathbb{k} \), namely,
\[
H \cong \bigoplus_{i \in I} M_{d_i}(\mathbb{k}).
\]
Let \( e_i \) be the idempotent of \( H \) satisfying that \( He_i = M_{d_i}(\mathbb{k}) \). Then \( \{e_i\}_{i \in I} \) forms a complete set of central primitive idempotents of \( H \). Let \( V_i \) be a simple left module (unique up to isomorphism) over the matrix algebra \( M_{d_i}(\mathbb{k}) \). Then \( \dim_k(V_i) = d_i \) and \( \{V_i\}_{i \in I} \) forms a complete set of simple left \( H \)-modules up to isomorphism. The regular representation \( H \) has the decomposition \( H \cong \bigoplus_{i \in I} V_i^{\text{end}} \) as \( H \)-modules, so the character \( \chi_H \) of \( H \) is equal to \( \sum_{i \in I} d_i \chi_i \), where each \( \chi_i \) is the character of \( V_i \).

For any simple \( H \)-module \( V_i \) and any \( \varphi \in \text{End}_k(V_i) \), we use the dual basis \( \Lambda(1) \otimes u^{-1}S(\Lambda(2)) \) with respect to the Frobenius homomorphism \( \lambda \leftarrow u \) to define the map \( \overline{I}(\varphi) \in \text{End}_k(V_i) \) by
\[
\overline{I}(\varphi)(v) = \Lambda(1)\varphi(u^{-1}S(\Lambda(2))v) \quad \text{for } v \in V_i.
\]
Note that \( \overline{I}(\varphi) \) lies in \( \text{End}_H(V_i) \cong \mathbb{k} \). There exists a unique element \( c_i \in \mathbb{k} \) such that
\[
\overline{I}(\varphi) = c_i \text{tr}(\varphi)id_{V_i} \quad \text{for all } \varphi \in \text{End}_k(V_i).
\]
Such an element \( c_i \), depending only on the isomorphism class of \( V_i \), is called the Schur element associated to \( V_i \) (see [5, Theorem 7.2.1]). Since \( H \) is semisimple, it follows from [5, Theorem 7.2.6] that the Schur element \( c_i \neq 0 \) in \( \mathbb{k} \) and the Frobenius homomorphism \( \lambda \leftarrow u \) can be written as follows:
\[
\lambda \leftarrow u = u \rightarrow \lambda = \sum_{i \in I} c_i \chi_i. \tag{2.4}
\]

3. A formula for the square of antipodes

In this section, we will provide a formula for \( S^2 \) by virtue of a nonzero integral \( \Lambda \) of \( H \). Then we study some properties of the element \( u := S(\Lambda(2))\Lambda(1) \). Especially, we will give a sufficient and necessary condition for \( S^2 = id \) via the integral \( \Lambda \).
Throughout this section, $H$ is a finite dimensional semisimple Hopf algebra over the field $\mathbb{k}$ of arbitrary characteristic. Let $u$ be a unit of $H$ satisfying $S^2(a) = uau^{-1}$ for all $a \in H$. We fix a left integral $\Lambda$ of $H$ and a right integral $\lambda$ of $H^*$ such that $\lambda(\Lambda) = 1$. We denote $\{V_i\}_{i \in I}$ the set of all simple left $H$-modules up to isomorphism. For each $V_i$ we denote $c_i$ the Schur element of $V_i$ associated to the dual basis $\Lambda(1) \otimes u^{-1} S(\Lambda(2))$ of $H$ with respect to the Frobenius homomorphism $\lambda \hookrightarrow u$. We denote $\{e_i\}_{i \in I}$ the set of all central primitive idempotents of $H$. We first establish a relationship between the elements $u$ and $u = S(\Lambda(2))\Lambda(1)$.

**Theorem 3.1.** With the notations above, we have

1. $\dim_{\mathbb{k}}(V_i) \neq 0$ in $\mathbb{k}$ for any $i \in I$.
2. $u = u \sum_{i \in I} \dim_{\mathbb{k}}(V_i)c_ie_i$, which is a unit of $H$.

**Proof.** (1) Note that each central primitive idempotent $e_i$ acts as the identity on $V_i$ and annihilates $V_j$ for $j \neq i$. It follows that $\chi_j(e_i) = \dim_{\mathbb{k}}(V_i)$ if $i = j$ and 0 otherwise. By (2.4) we have

$$\chi_i(a) = \chi_i(ae_i) = \sum_{j \in I} \frac{1}{c_j} \chi_j(c_iae_i) = (u \to \lambda)(c_iae_i) = (uc_ie_i \to \lambda)(a).$$

Thus, $\chi_i = uc_ie_i \to \lambda$ and hence

$$\chi_H = \sum_{i \in I} \dim_{\mathbb{k}}(V_i)\chi_i = u \sum_{i \in I} \dim_{\mathbb{k}}(V_i)c_ie_i \to \lambda.$$

This means that the form $\beta_1(a,b) := \chi_H(ab)$ and the form $\beta_2(a,b) := (u \to \lambda)(ab)$ is the same up to the central element $\sum_{i \in I} \dim_{\mathbb{k}}(V_i)c_i e_i$, namely,

$$\beta_1(a,b) = \beta_2(a,b \sum_{i \in I} \dim_{\mathbb{k}}(V_i)c_i e_i) \text{ for all } a,b \in H.$$  

Observe that the forms $\beta_1(-,-)$ and $\beta_2(-,-)$ are both associative and non-degenerate, where the non-degeneracy of $\beta_1(-,-)$ follows from the semisimplicity of $H$. It follows from [11, Section 1.2.5] that $\sum_{i \in I} \dim_{\mathbb{k}}(V_i)c_i e_i$ is a central unit of $H$. Thus, $\dim_{\mathbb{k}}(V_i) \neq 0$ for all $i \in I$.

(2) For any map $\varphi \in \text{End}_{\mathbb{k}}(H)$, the trace of $\varphi$ is $\text{tr}(\varphi) = \lambda(\varphi(S(\Lambda(2)))\Lambda(1))$ (see [17, Theorem 2]). Taking into account that $\varphi = L_a$, where $L_a$ is the left multiplication operator of $H$ by $a$, we have

$$\chi_H(a) = \text{tr}(L_a) = \lambda(aS(\Lambda(2))\Lambda(1)) = (S(\Lambda(2))\Lambda(1) \to \lambda)(a).$$

This implies that $\chi_H = S(\Lambda(2))\Lambda(1) \to \lambda$. Comparing it with (3.1) and using the non-degeneracy of the Frobenius homomorphism $\lambda$, we have

$$S(\Lambda(2))\Lambda(1) = u \sum_{i \in I} \dim_{\mathbb{k}}(V_i)c_ie_i.$$

Thus, the element $u$ satisfying $S^2(a) = uau^{-1}$ for $a \in H$ is the same as $S(\Lambda(2))\Lambda(1)$ up to a central unit $\sum_{i \in I} \dim_{\mathbb{k}}(V_i)c_i e_i$.  \qed
Remark 3.2. The result that each \( \dim_k(V_i) \neq 0 \) in \( k \) has been observed in [2, Theorem 3.6]. This result also appeared in [4, Corollary 3.2(ii)] in the case of \( H \) being a semisimple and cosemisimple Hopf algebra. If the field \( k \) has characteristic 0, we emphasize that \( u = \varepsilon(\Lambda) \) since in this case \( S^2 = \text{id} \) (see [8] or [9]) implying that \( u = S(\Lambda(2))\Lambda(1) = S(\Lambda(2))S^2(\Lambda(1)) = S(S(\Lambda(1))\Lambda(2)) = \varepsilon(\Lambda) \).

Theorem 3.1 gives a formula for \( S^2 \), namely,

\[
S^2(a) = uau^{-1} \quad \text{for} \quad a \in H,
\]

where \( u = S(\Lambda(2))\Lambda(1) \). In the sequel, we will replace \( u \) with \( u \). In this case, the equality (2.2) turns out to be

\[
\Lambda(1) \otimes u^{-1}S(\Lambda(2)) = u^{-1}S(\Lambda(2)) \otimes \Lambda(1),
\]

which is the dual basis of \( H \) with respect to the Frobenius homomorphism \( \lambda \leftarrow u \). The Schur element associated to the simple \( H \)-module \( V_i \) under the new dual basis \( \Lambda(1) \otimes u^{-1}S(\Lambda(2)) \) with respect to the Frobenius homomorphism \( \lambda \leftarrow u \) is \( \frac{1}{\dim_k(V_i)} \).

Therefore, the equality (2.4) turns out to be

\[
\lambda \leftarrow u = u \rightarrow \lambda = \sum_{i \in I} \dim_k(V_i)\chi_i = \chi_H.
\]

By applying [3, Theorem 1.5] and (3.2), we obtain the expression of each central primitive idempotent \( e_i \) of \( H \) as follows:

\[
e_i = \dim_k(V_i)\chi_i(\Lambda(1))u^{-1}S(\Lambda(2)) = \dim_k(V_i)\chi_i(u^{-1}S(\Lambda(2))\Lambda(1)).
\]

Let \( g \in G(H) \) and \( \alpha \in \text{Alg}(H,k) \) be the modular elements of \( H \) and \( H^* \) respectively. Recall that the Radford’s formula of \( S^4 \) has the form (see [16, Proposition 6]):

\[
S^4(a) = \alpha^{-1} \rightarrow (gag^{-1}) \leftarrow \alpha.
\]

Since \( H \) is unimodular, i.e., \( \alpha = \varepsilon \), the Radford’s formula of \( S^4 \) now becomes

\[
S^4(a) = gag^{-1}.
\]

The distinguished group-like element \( g \) and the integral \( \Lambda \) of \( H \) satisfy the following useful equality (see [17, Theorem 3(d)]):

\[
\Lambda(2) \otimes \Lambda(1) = \Lambda(1) \otimes S^2(\Lambda(2))g.
\]

After these preparations, we give some properties of the element \( u \) as follows:

**Proposition 3.3.** The element \( u = S(\Lambda(2))\Lambda(1) \) satisfies the following properties:

1. \( u = \chi_H(\Lambda(1))S(\Lambda(2)) \).
2. \( \Lambda(1)u^{-1}S(\Lambda(2)) = 1 \).
3. \( \lambda(e_i) = \dim_k(V_i)\chi_i(u^{-1}) \).
4. \( uS(u) = S(u)u = \varepsilon(\Lambda) \sum_{i \in I} \frac{\dim_k(V_i)^2}{\lambda(e_i)} e_i \).
5. The distinguished group-like element \( g = S(u^{-1})u = uS(u^{-1}) \).
Proof: (1) It follows from (3.4) that $e_i u = \dim_k (V_i) \chi_i (\lambda(1)) S(\lambda(2))$. Thus,
\[
\begin{align*}
u &= \sum_{i \in I} e_i u = \sum_{i \in I} \dim_k (V_i) \chi_i (\lambda(1)) S(\lambda(2)) = \chi_H (\lambda(1)) S(\lambda(2)).
\end{align*}
\]

(2) Since $\lambda(1) \otimes u^{-1} S(\lambda(2)) = u^{-1} S(\lambda(2)) \otimes \lambda(1)$ by (3.2), we obtain the desired result by multiplying the tensor factors together.

(3) Since $e_i = \dim_k (V_i) \chi_i (\lambda(1)) u^{-1} S(\lambda(2))$, it follows that
\[
e_i = u e_i u^{-1} = \dim_k (V_i) \chi_i (\lambda(1)) S(\lambda(2)) u^{-1}.
\]
Hence
\[
\lambda(e_i) = \dim_k (V_i) \chi_i (\lambda(1)) \lambda(S(\lambda(2)) u^{-1}) = \dim_k (V_i) \chi_i (u^{-1}),
\]
where the last equality follows from (2.1).

(4) For any $a \in H$, we have $S^3(a) = S(S^2(a)) = S(u a u^{-1}) = S(u^{-1}) S(a) S(u)$, we also have $S^3(a) = S^2(S(a)) = u S(a) u^{-1}$. It follows that $S(u) u$ is a central unit of $H$. The equality $u S(u) = S(u) u$ holds because $S(u) = S(S^2(u)) = S^2(S(u)) = u S(u) u^{-1}$. For the central unit $u S(u)$, we suppose that $u S(u) = \sum_{i \in I} k_i e_i$, where each scalar $k_i \neq 0$ in $\mathbb{K}$. Then $e_i u^{-1} = \frac{1}{k_i} e_i S(u)$. We have
\[
\lambda(e_i) = (u^{-1} \rightarrow \chi_H)(e_i) = \chi_H(e_i S(u)) = \frac{\dim_k (V_i)}{k_i} \chi_i (e_i S(u)) = \frac{\dim_k (V_i)}{k_i} \chi_i (S(u)) = \frac{\dim_k (V_i)}{k_i} (\chi_i \circ S)(u).
\]
Hence
\[
\lambda(e_i) = \frac{\dim_k (V_i)}{k_i} \chi_i (S(\lambda(1)) S(\lambda(2))) = \frac{\dim_k (V_i)^2 e(\Lambda)}{k_i} \neq 0.
\]
It follows that $k_i = \frac{\dim_k (V_i)^2 e(\Lambda)}{\lambda(e_i)}$ and $u S(u) = \sum_{i \in I} k_i e_i = e(\Lambda) \sum_{i \in I} \frac{\dim_k (V_i)^2}{\lambda(e_i)} e_i$.

(5) Note that $\Lambda(2) \otimes \Lambda(1) = \Lambda(1) \otimes S^2(\Lambda(2)) g$ by (3.5). Applying $S \otimes id$ to both sides of this equality and multiplying the tensor factors together, we have $u = S(u) g$ or $g = S(u^{-1}) u$.

As a consequence, we obtain the following result:

**Corollary 3.4.** For any central primitive idempotent $e_i$ of $H$, we have $\lambda(e_i) = \lambda(S(e_i))$.

**Proof:** We denote $S(e_i) = e_{i^*}$ for some $i^* \in I$, then $V_{i^*} = V_{i^*}$, or equivalently, $\chi_{i^*} = \chi_{i^*}$ (see [3, Lemma 1.8]). By Proposition 3.3 (3) we have
\[
\lambda(S(e_i)) = \lambda(e_{i^*}) = \dim_k (V_{i^*}) \chi_{i^*} (u^{-1}) = \dim_k (V_i) \chi_i (S(u^{-1})).
\]
Since \( uS(u) = \varepsilon(\Lambda) \sum_{i \in I} \frac{\dim_k(V_i)^2}{\dim_k(V_i)} e_i \), it follows that \( S(u^{-1}) = u^{-1} \sum_{i \in I} \frac{\lambda(e_i)}{\varepsilon(\Lambda) \dim_k(V_i)} e_i. \) Thus,

\[
\Lambda(S(e_i)) = \dim_k(V_i) \chi_i(S(u^{-1})) = \frac{\lambda(e_i)}{\varepsilon(\Lambda) \dim_k(V_i)} \chi_i(u) = \frac{\lambda(e_i)}{\varepsilon(\Lambda) \dim_k(V_i)} \chi_i(\Lambda(\Lambda(2))) = \lambda(e_i).
\]

We complete the proof. \( \square \)

If the field \( \mathbb{k} \) has characteristic 0, then the antipode \( S \) of \( H \) satisfies \( S^2 = \text{id} \) (see [8] or [9]). This further implies that the integral \( \Lambda \) of \( H \) is cocommutative (see [8, Proposition 2(b)]). The following gives a stronger result that \( \Lambda \) being cocommutative is equivalent to \( S^2 = \text{id} \) no matter what the characteristic of the field \( \mathbb{k} \) is.

**Theorem 3.5.** Let \( H \) be a finite dimensional semisimple Hopf algebra over the field \( \mathbb{k} \). The following statements are equivalent:

1. The nonzero integral \( \Lambda \) of \( H \) is cocommutative.
2. The nonzero integral \( \lambda \) of \( H^* \) is cocommutative.
3. \( S^2 = \text{id} \).

**Proof.** It can be seen from [17, Corollary 5] that Part (2) and Part (3) are equivalent. We next show that Part (1) and Part (3) are equivalent. If \( \Lambda \) is cocommutative, then \( u = S(\Lambda(2))\Lambda(1) = S(\Lambda(1))\Lambda(2) = \varepsilon(\Lambda) \). It follows from \( S^2(a) = uau^{-1} \) that \( S^2 = \text{id} \). Conversely, if \( S^2 = \text{id} \), then \( u = S(\Lambda(2))\Lambda(1) = S(\Lambda(2))S(\Lambda(1)) = S(\Lambda(1))\Lambda(2) = \varepsilon(\Lambda) \). By Proposition 3.3, we have \( g = S(u^{-1})u = 1 \). Since \( \Lambda(2) \otimes \Lambda(1) = \Lambda(1) \otimes S^2(\Lambda(2))g \) by (3.5), it follows that \( \Lambda(2) \otimes \Lambda(1) = \Lambda(1) \otimes \Lambda(2) \). We complete the proof. \( \square \)

### 4. Higher FS indicators

If the field \( \mathbb{k} \) has character 0, the \( n \)-th FS indicators of finite dimensional representations of the semisimple Hopf algebra \( H \) have been studied in [7]. In this section, we will generalize these indicators to the case that the field \( \mathbb{k} \) has arbitrary characteristic and describe them via a nonzero integral \( \Lambda \) of \( H \). Then we extend the \( n \)-th FS indicators from \( n \geq 1 \) to the situation \( n \in \mathbb{Z} \). We begin with the following preparations.

Let \( A \) be a finite dimensional unimodular Hopf algebra over the field \( \mathbb{k} \) with a nonzero integral \( \Lambda \). Since \( \Lambda \) is a left integral as well as a right integral of \( A \), the following two equalities hold for all \( a \in A \) (see [9, Lemma 1.2]):

\[
\Lambda(1) \otimes a\Lambda(2) = S(a)\Lambda(1) \otimes \Lambda(2), \tag{4.1}
\]

\[
\Lambda(1)a \otimes \Lambda(2) = \Lambda(1) \otimes \Lambda(2)S(a). \tag{4.2}
\]
Applying $\Delta \otimes id$ to both sides of the equality (4.2), we have

\begin{equation}
(4.3) \quad \Lambda(a_1) \otimes \Lambda(a_2) \otimes \Lambda(a_3) = \Lambda(a_1) \otimes \Lambda(a_2) \otimes \Lambda(a_3) S(a).
\end{equation}

Let $m_1$ and $\Delta_1$ be the identity map of $A$. For any natural number $n \geq 2$, the maps $m_n : A^{\otimes n} \to A$ and $\Delta_n : A \to A^{\otimes n}$ are defined respectively by $m_n = m \circ (m_{n-1} \otimes id)$ and $\Delta_n = (\Delta_{n-1} \otimes id) \circ \Delta$. For instance,

\[ \Delta_n(a) = a_1 \otimes \cdots \otimes a_n \quad \text{and} \quad (m_n \circ \Delta_n)(a) = a_1 \cdots a_n \quad \text{for} \ a \in A. \]

The following result will be used later:

**Proposition 4.1.** Let $A$ be a finite dimensional unimodular Hopf algebra over the field $\mathbb{k}$ with a nonzero integral $\Lambda$. For any natural number $n \geq 1$, we have

1. $\Lambda_1 \cdots \Lambda_n \in Z(A)$.
2. $\Lambda_1 \cdots \Lambda_n(a) \in Z(A)$.

**Proof.** (1) Applying $m_n \circ (id \otimes \Delta_{n-1})$ to both sides of (4.1), we have

\[ \Lambda(a_1) \Lambda(a_2) \cdots \Lambda(a_{n-1}) = S(a) \Lambda(a_1) \Lambda(a_2) \cdots \Lambda(a_n). \]

Applying $m_n \circ (\Delta_{n-1} \otimes id)$ to both sides of (4.2), we have

\[ \Lambda(a_1) \Lambda(a_2) \cdots \Lambda(a_{n-1}) = \Lambda(a_1) \Lambda(a_2) \cdots \Lambda(a_n) S(a). \]

The two equalities give the desired result that $\Lambda(a_1) \Lambda(a_2) \cdots \Lambda(a_n) \in Z(A)$.

(2) On the one hand, applying $\Delta_{n-1} \otimes id$ to both sides of (4.1), we have

\[ \Lambda(a_1) \Lambda(a_2) \cdots \Lambda(a_{n-1}) \otimes \Lambda(a_n) = S(a) \Lambda(a_1) \otimes \Lambda(a_2) \cdots \otimes \Lambda(a_{n-1}) \Lambda(a_n). \]

Reversing the order of these tensor factors and applying $m_n$ on both sides, we have

\[ a \Lambda(a_1) \Lambda(a_2) \cdots \Lambda(a_{n-1}) \Lambda(a_n) = \Lambda(a_1) S(a) \Lambda(a_2) \cdots \Lambda(a_{n-1}) \Lambda(a_n). \]

On the other hand, applying $id \otimes \Delta_{n-1}$ to both sides of (4.2), we have

\[ \Lambda(a_1) \otimes \Lambda(a_2) \cdots \otimes \Lambda(a_{n-1}) \otimes \Lambda(a_n) = \Lambda(a_1) \otimes \Lambda(a_2) \cdots \otimes \Lambda(a_{n-1}) S(a) \Lambda(a_n) \Lambda(a_1). \]

Also, reversing the order of these tensor factors and applying $m_n$, we have

\[ \Lambda(a_1) \Lambda(a_2) \cdots \Lambda(a_{n-1}) \Lambda(a_n) = \Lambda(a_1) \Lambda(a_2) \cdots \Lambda(a_{n-1}) \Lambda(a_n) S(a). \]

We obtain that $\Lambda(a_1) \Lambda(a_2) \cdots \Lambda(a_{n-1}) \Lambda(a_n) \in Z(A)$. \hfill $\square$

Let $H$ be a finite dimensional semisimple Hopf algebra over the field $\mathbb{k}$ with a nonzero integral $\Lambda$ and $u = S(\Lambda(a_2)) \Lambda(a_1)$. Applying $\Delta_{n-1} \otimes id$ to both sides of the equality: $\Lambda_2 \otimes \Lambda_1 = \Lambda_1 \otimes S^2(\Lambda_2) g$ (see (3.5)), we have

\[ \Lambda_2 \otimes \cdots \otimes \Lambda(a_n) \otimes \Lambda(a_1) = \Lambda_1 \otimes \cdots \otimes \Lambda(a_{n-1}) \otimes S^2(\Lambda(a_n)) g. \]

Since $g = u S(u^{-1})$ and $S^2(\Lambda(a_n)) = u \Lambda(a_n) u^{-1}$, the above equality induces the following useful equality:

\begin{equation}
(4.4) \quad \Lambda_2 \otimes \cdots \otimes \Lambda(a_n) \otimes u^{-1} \Lambda_1 = \Lambda_1 \otimes \cdots \otimes \Lambda(a_{n-1}) \otimes \Lambda(a_n) S(u^{-1}).
\end{equation}
Note that the category $\text{Rep}(H)$ of finite dimensional representations of $H$ is a semisimple tensor category. Let $j_u : \text{id} \to (-)^{**}$ be a natural isomorphism between the identity functor and the functor of taking the second dual. It is completely determined by a collection of $H$-module isomorphisms

$$j_{u,V} : V \to V^{**}, \quad j_{u,V}(v)(f) = f(uv) \text{ for } v \in V, f \in V^*.$$ 

The inverse of $j_{u,V}$ is

$$j_{u,V}^{-1} : V^{**} \to V, \quad \alpha \mapsto j_{u,V}(\alpha),$$

where $j_{u,V}(\alpha) \in V$ satisfies the equality $f(j_{u,V}(\alpha)) = \alpha(S^{-1}(u^{-1})f)$ for $f \in V^*$. Since $S^2(h) = uu^{-1}$ and $u$ is not known to be a group-like element, the natural isomorphism $j_u$ is not necessarily a tensor isomorphism. Although the representation category $\text{Rep}(H)$ with respect to the structure $j_u$ is not necessarily pivotal, we may still define higher FS indicators for any finite dimensional representation of $H$ using the structure $j_u$ of $\text{Rep}(H)$.

We denote $V^\otimes n$ the $n$-th tensor power of $V$ where $V^\otimes 0$ is the trivial $H$-module $\Bbbk$. For any natural number $n \geq 1$, we define the following $\Bbbk$-linear map

$$E^n_V : \text{Hom}_H(\Bbbk, V^\otimes n) \to \text{Hom}_H(\Bbbk, V^\otimes n), \quad f \mapsto E^n_V(f),$$

where $E^n_V(f)$ is an $H$-module morphism from $\Bbbk$ to $V^\otimes n$ given by

$$E^n_V(f) : \Bbbk \overset{\text{coev}_{V^*}}\longrightarrow V^* \otimes V^{**} = V^* \otimes \Bbbk \otimes V^{**} \overset{id \otimes f \otimes id}{\longrightarrow} V^* \otimes V^\otimes n \otimes V^{**} \overset{\text{ev}_V \otimes id}{\longrightarrow} V^\otimes (n-1) \otimes V^{**} \overset{id \otimes j_u^{-1}}{\longrightarrow} V^\otimes n.$$

Here the maps $\text{coev}_{V^*}$ and $\text{ev}_V$ are the usual coevaluation morphism of $V^*$ and evaluation morphism of $V$ respectively. If we set $f(1) = \sum v_1 \otimes \cdots \otimes v_n \in V^\otimes n$, the above definition of $E^n_V(f)$ shows that

$$(4.5) \quad E^n_V(f)(1) = \sum v_2 \otimes \cdots \otimes v_n \otimes u^{-1}v_1.$$

Similar to [14], we give the definition of the $n$-th FS indicator of $V$ to be the trace of the linear operator $E^n_V$ as follows:

**Definition 4.2.** Let $H$ be a finite dimensional semisimple Hopf algebra over the field $\Bbbk$. For any finite dimensional representation $V$ of $H$, the $n$-th FS indicator of $V$ is defined by

$$\nu_n(V) = tr(E^n_V) \text{ for } n \geq 1.$$ 

Similar to the characteristic 0 case, the $n$-th FS indicator of $V$ can be described by a nonzero integral $\Lambda$ of $H$:

**Theorem 4.3.** Let $\Lambda$ be a nonzero integral of $H$ and $u = S(\Lambda_{(2)})\Lambda_{(1)}$. Suppose $\chi_V$ is the character of a finite dimensional representation $V$ of $H$. We have

$$\nu_n(V) = \chi_V(u^{-1}\Lambda_{(1)} \cdots \Lambda_{(n)}) \text{ for } n \geq 1.$$
Proof. We first show the equality \( v_n(V) = \chi_V(u^{-1}A(1) \cdots A(n)) \) where \( A \) is an idempotent integral. Suppose that \( A \) is the following \( k \)-linear map

\[
\alpha : V^{\otimes n} \to V^{\otimes n}, \quad v_1 \otimes v_2 \otimes \cdots \otimes v_n \mapsto v_2 \otimes \cdots \otimes v_n \otimes v_1
\]

and \( \delta = \alpha \circ (u^{-1}A(1) \otimes A(2) \otimes \cdots \otimes A(n)). \) We have

\[
\delta(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = \alpha(u^{-1}A(1)v_1 \otimes A(2)v_2 \otimes \cdots \otimes A(n)v_n)
= A(v_1)v_2 \otimes \cdots \otimes A(n)v_n \otimes u^{-1}A(1)v_1
\]

by (4.4)

\[
= A(1)v_2 \otimes \cdots \otimes A(n-1)v_{n-1} \otimes A(n)v_n S(u^{-1})v_1.
\]

This shows that \( \delta(V^{\otimes n}) \subseteq A \cdot V^{\otimes n} = (V^{\otimes n})^H \). Note that the map

\[
\Phi : \text{Hom}_H(k, V^{\otimes n}) \to (V^{\otimes n})^H, \quad f \mapsto f(1)
\]

is an \( H \)-module isomorphism. We claim that the following diagram is commutative:

\[
\begin{array}{ccc}
\text{Hom}_H(k, V^{\otimes n}) & \xrightarrow{\Phi} & \text{Hom}_H(k, V^{\otimes n}) \\
\downarrow \Phi & & \downarrow \Phi \\
(V^{\otimes n})^H & \xrightarrow{\delta} & (V^{\otimes n})^H.
\end{array}
\]

Indeed, for any \( f \in \text{Hom}_H(k, V^{\otimes n}) \), we suppose that \( f(1) = \sum v_1 \otimes \cdots \otimes v_n \in V^{\otimes n} \). It follows from \( f(1) = f(A \cdot 1) = A \cdot f(1) \) that

\[
\sum v_1 \otimes \cdots \otimes v_n = \sum A(1)v_1 \otimes \cdots \otimes A(n)v_n.
\]

On the one hand, we have

\[
(\delta \circ \Phi)(f) = \delta(f(1)) = \delta(\sum v_1 \otimes \cdots \otimes v_n)
= A \cdot (\sum v_2 \otimes \cdots \otimes v_n \otimes S(u^{-1})v_1) \quad \text{by (4.6)}
\]

On the other hand, we have

\[
(\Phi \circ E_V^n)(f) = E_V^n(f)(1) = \sum v_2 \otimes \cdots \otimes v_n \otimes u^{-1}v_1 \quad \text{by (4.5)}
= \sum A(2)v_2 \otimes \cdots \otimes A(n)v_n \otimes u^{-1}A(1)v_1 \quad \text{by (4.7)}
= \sum A(1)v_2 \otimes \cdots \otimes A(n-1)v_{n-1} \otimes A(n)v_n S(u^{-1})v_1 \quad \text{by (4.4)}
= A \cdot (\sum v_2 \otimes \cdots \otimes v_n \otimes S(u^{-1})v_1).
\]

We obtain that \( \delta \circ \Phi = \Phi \circ E_V^n \), or equivalently, \( E_V^n = \Phi^{-1} \circ \delta \circ \Phi \). It follows that

\[
v_n(V) = \text{tr}(E_V^n) = \text{tr}_V \circ \delta
= \text{tr}_\otimes (\alpha \circ (u^{-1}A(1) \otimes A(2) \otimes \cdots \otimes A(n)))
= \text{tr}_V(u^{-1}A(1) \cdots A(n))
= \chi_V(u^{-1}A(1) \cdots A(n)).
\]
where the equality \( \text{tr}_V(\alpha \circ (u^{-1}\Lambda_1 \otimes \Lambda_2 \otimes \cdots \otimes \Lambda_\ell)) = \text{tr}_V(u^{-1}\Lambda_1 \cdots \Lambda_\ell) \) follows from [7, Lemma 2.3]. We have shown that \( \nu_n(V) = \chi_V(u^{-1}\Lambda_1 \cdots \Lambda_\ell) \)
where \( \Lambda \) is idempotent. Since \( u^{-1}\Lambda_1 \cdots \Lambda_\ell \) does not depend on the choice of the nonzero integral \( \Lambda \), the equality \( \nu_n(V) = \chi_V(u^{-1}\Lambda_1 \cdots \Lambda_\ell) \) holds for any nonzero integral \( \Lambda \) of \( H \).

**Remark 4.4.** If the field \( \mathbb{k} \) has characteristic 0 and \( \Lambda \) is idempotent, then \( u = \epsilon(\Lambda) = 1 \). In this case, the \( n \)-th FS indicator of \( V \) is \( \chi_V(\Lambda_1 \cdots \Lambda_\ell) \), which is the one defined in [7, Definition 2.3].

In the rest of this section, we will extend the \( n \)-th FS indicator \( \nu_n(V) \) of \( V \) from \( n \geq 1 \) to the case \( n \in \mathbb{Z} \). Let \( A \) be a finite dimensional Hopf algebra over the field \( \mathbb{k} \). The \( n \)-th Sweedler power map \( P_n : A \to A \) is defined by

\[
P_n(a) = \begin{cases} 
a_{(1)} \cdots a_{(n)}, & n \geq 1; \\
\epsilon(a), & n = 0; \\
S(a_{(1)}) \cdots S(a_{(-n)}), & n \leq -1.
\end{cases}
\]

Note that \( P_n \) is the \( n \)-th power of the identity map \( id_A \) under the convolution product \( * \) of \( \text{Hom}_\mathbb{k}(A, A) \). It follows that

\[P_n * P_l = P_{n+l} \quad \text{for all } n, l \in \mathbb{Z}.
\]

Since \( \text{Hom}_\mathbb{k}(A, A) \) is finite dimensional, the set of operators \( \{P_n\}_{n \in \mathbb{Z}} \) is \( \mathbb{k} \)-linearly dependent. There exist a minimal positive integer \( m \leq (\dim_\mathbb{k} A)^2 \) and scalars \( c_0, \ldots, c_{m-1} \in \mathbb{k} \) such that

\[c_0P_0 + \cdots + c_{m-1}P_{m-1} + P_m = 0.
\]

Multiplying the equality by \( P_l \) under the convolution product \( * \), we have

\[c_0P_l + \cdots + c_{m-1}P_{l+m-1} + P_{l+m} = 0
\]

for all \( l \in \mathbb{Z} \). Hence the sequence \( \{P_n\}_{n \in \mathbb{Z}} \) is linearly recursive and the following monic polynomial

\[\Phi(X) = c_0 + c_1X + \cdots + c_{m-1}X^{m-1} + X^m
\]

satisfying (4.8) is called the minimal polynomial of \( \{P_n\}_{n \in \mathbb{Z}} \) (see [18]). Note that \( c_0 = \Phi(0) \neq 0 \) by [18, Lemma 3.11 (3)].

Using the \( n \)-th Sweedler power map \( P_n \) of the semisimple Hopf algebra \( H \), we may see that

\[\nu_n(V) = \chi_V(u^{-1}P_n(\Lambda)) \quad \text{for } n \geq 1.
\]

However, this expression is well-defined for any integer \( n \). Thus, we may extend this formula from \( n \geq 1 \) to any integer \( n \) described as follows:

**Definition 4.5.** Let \( H \) be a finite dimensional semisimple Hopf algebra over the field \( \mathbb{k} \). For any finite dimensional representation \( V \) of \( H \) and any \( n \in \mathbb{Z} \), the \( n \)-th FS indicator of \( V \) is defined by

\[\nu_n(V) = \chi_V(u^{-1}P_n(\Lambda)),
\]

where \( u = S(\Lambda(2))\Lambda(1) \).
**Remark 4.6.**

1. Note that $S(\Lambda) = \Lambda$. The $n$-th FS indicator of $V$ can be written as

$$
\nu_n(V) = \begin{cases} 
\chi_V(u^{-1} \Lambda_1 \cdots \Lambda_{(n)}), & n \geq 1; \\
\chi_V(u^{-1} \varepsilon(\Lambda)), & n = 0; \\
\chi_V(u^{-1} \Lambda_{(-n)} \cdots \Lambda_{(1)}), & n \leq -1.
\end{cases}
$$

2. By Proposition 3.3 (4), we have

$$
u_n(V) = \nu(V) = \chi_V(u^{-1} \Lambda) = \chi_V(u^{-1} \Lambda) = \chi_V(u^{-1} \Lambda) = \lambda. \tag{4}
$$

It follows that

$$
\nu_0(V) = \varepsilon(\Lambda) \chi_V(u^{-1}) = \varepsilon(\Lambda) \chi_V(u^{-1} S(u^{-1}) S(u))
$$

$$
= \sum_{i \in I} \frac{\lambda(e_i)}{\dim_k(V_i)^2} \chi_V(e_i S(u))
= \sum_{i \in I} \frac{\lambda(e_i)}{\dim_k(V_i)^2} \chi_V(e_i S(\Lambda_{(1)} S(\Lambda_{(2)}))
$$

$$
= \sum_{i \in I} \frac{\lambda(e_i)}{\dim_k(V_i)^2} \chi_V(e_i S(\Lambda_{(2)} S(\Lambda_{(1)}))
= \varepsilon(\Lambda) \sum_{i \in I} \frac{\lambda(e_i)}{\dim_k(V_i)^2} \chi_V(e_i).
$$

3. $\nu_1(V) = \nu(V) = \chi_V(u^{-1} \Lambda) = \chi_V(u^{-1} \Lambda) = \lambda. \tag{5}
$$

4. By Proposition 4.1, $\Lambda_{(1)} \Lambda_{(2)}$ and $\Lambda_{(2)} \Lambda_{(1)}$ are both central elements of $H$, they are determined by the values that the characters $\chi_i$ for all $i \in I$ take on them. It follows from $\chi_i(\Lambda_{(1)} \Lambda_{(2)}) = \chi_i(\Lambda_{(2)} \Lambda_{(1)})$ that $\Lambda_{(1)} \Lambda_{(2)} = \Lambda_{(2)} \Lambda_{(1)}$. Therefore, $\nu_{-2}(V) = \nu_{2}(V)$.

The higher FS indicators of any simple module $V_i$ can be described as follows:

**Proposition 4.7.** For any $n \in \mathbb{Z}$ and any simple module $V_i$ with the character $\chi_i$, we have

$$
\nu_n(V_i) = \frac{\chi_i(P_n(\Lambda)) \lambda(e_i)}{\dim_k(V_i)^2}.
$$

**Proof:** Since $P_n(\Lambda) \in Z(H)$ for any $n \in \mathbb{Z}$, it follows that $P_n(\Lambda) = \sum_{i \in I} \frac{\chi_i(P_n(\Lambda))}{\dim_k(V_i)} e_i$.

The $n$-th FS indicator of $V_i$ is

$$
\nu_n(V_i) = \chi_i(u^{-1} P_n(\Lambda)) = \frac{\chi_i(P_n(\Lambda))}{\dim_k(V_i)} \chi_i(u^{-1}) = \frac{\chi_i(P_n(\Lambda)) \lambda(e_i)}{\dim_k(V_i)^2},
$$

where the last equality follows from Proposition 3.3 (3). \hfill \square

If the field $k$ has characteristic 0, any finite dimensional representation $V$ and its dual $V^*$ have the same $n$-th FS indicators for all $n \geq 1$ (see [7, Section 2.3]). The following result shows that it holds for all $n \in \mathbb{Z}$ over the field $k$ of arbitrary characteristic.

**Proposition 4.8.** Let $H$ be a finite dimensional semisimple Hopf algebra over the field $k$ and $V$ a finite dimensional representation of $H$ with the dual $V^*$. We have $\nu_n(V) = \nu_n(V^*)$ for all $n \in \mathbb{Z}$. 


Proof: Since $S(\Lambda) = \Lambda$, we have $S(P_n(\Lambda)) = P_n(\Lambda)$ for any $n \in \mathbb{Z}$. For $n \geq 1$, the $n$-th FS indicator of $V^*$ is

$$
\nu_n(V^*) = (\chi_{V^*})(u^{-1}P_n(\Lambda)) = (\chi_V \circ S)(u^{-1}P_n(\Lambda))
= \chi_V(\Lambda_1 \cdots \Lambda_n)S(u^{-1}) = \chi_V(\Lambda_1 \cdots \Lambda_nu^{-1})$$ by (4.4)

$$= \chi_V(u^{-1}\Lambda_1 \Lambda_2 \cdots \Lambda_n) = \nu_n(V).$$

For $n \leq -1$, the $n$-th FS indicator of $V^*$ is

$$\nu_n(V^*) = (\chi_{V^*})(u^{-1}P_n(\Lambda)) = (\chi_V \circ S)(u^{-1}P_n(\Lambda))
= \chi_V(\Lambda_{(n)} \cdots \Lambda_{(1)}S(u^{-1})) = \chi_V(S(u^{-1})\Lambda_{(n)} \cdots \Lambda_{(1)})$$

$$= \chi_V(S(u^{-1})u^{-1}\Lambda_{(1)} \cdots \Lambda_{(n)}) = \chi_V(u^{-1}\Lambda_{(n)} \cdots \Lambda_{(1)})$$

$$= \nu_n(V).$$

For the case $n = 0$, we denote $S(e_i) = e^{*i}$ for any $i \in I$, then $*$ is a permutation of $I$, $V^{*i} \cong V_i$ and $\lambda(e^{*i}) = \lambda(e_i)$ by Corollary 3.4. We have

$$\nu_0(V^*) = \varepsilon(\Lambda) \sum_{i \in I} \frac{\lambda(e_i)}{\dim_k(V_i)} \chi_V(S(e_i))$$ by Remark 4.6(2)

$$= \varepsilon(\Lambda) \sum_{i \in I} \frac{\lambda(e^{*i})}{\dim_k(V^{*i})} \chi_{V^*}(e^{*i}) = \varepsilon(\Lambda) \sum_{i \in I} \frac{\lambda(e_i)}{\dim_k(V_i)} \chi_V(e_i)$$

$$= \nu_0(V).$$

We complete the proof. □

Kashina-Sommerhäuser-Zhu has shown in [7, Proposition 2.5] that the $n$-th FS indicator of the regular representation of a semisimple Hopf algebra $H$ over the field $k$ of characteristic zero can be described as follows:

$$\nu_n(H) = \text{tr}(S \circ P_{n-1})$$ for $n \geq 1$.

The next result shows that this formula holds for the field $k$ of arbitrary characteristic and for any $n \in \mathbb{Z}$.

Proposition 4.9. Let $H$ be a finite dimensional semisimple Hopf algebra over the field $k$. For any $n \in \mathbb{Z}$, the $n$-th FS indicator of the regular representation of $H$ can be written as $\nu_n(H) = \text{tr}(S \circ P_{n-1})$, where $P_{n-1}$ is the $(n-1)$-th Sweedler power map of $H$.

Proof: We choose a left integral $\Lambda$ of $H$ and a right integral $\lambda$ of $H^*$ such that $\lambda(\Lambda) = 1$. For any $n \in \mathbb{Z}$, by Radford’s trace formula [17, Theorem 2], we have

$$\text{tr}(S \circ P_{n-1}) = \text{tr}(P_{n-1} \circ S) = \lambda(S(\Lambda_2))(P_{n-1} \circ S)(\Lambda_1))$$

$$= \lambda(S(\Lambda_2))P_{n-1}(S(\Lambda_1))) = \lambda(\Lambda_1)P_{n-1}(\Lambda_2))$$

$$= \lambda(P_n(\Lambda)) = \chi_H(u^{-1}P_n(\Lambda))$$ by (3.3)
\[ \nu_n(H). \]

We complete the proof. \[ \square \]

5. Gauge invariants

In this section, we will show that the \( n \)-th FS indicator \( \nu_n(V) \) is a gauge invariant of the tensor category \( \text{Rep}(H) \) for any \( n \in \mathbb{Z} \) and any finite dimensional representation \( V \) of the semisimple Hopf algebra \( H \).

Recall that a (normalized) twist for a finite dimensional Hopf algebra \( A \) is an invertible element \( J \in A \otimes A \) that satisfies \( (e \otimes id)(J) = (id \otimes e)(J) = 1 \) and

\[ (\Delta \otimes id)(J)(1 \otimes 1) = (id \otimes \Delta)(J)(1 \otimes J). \]

Taking the inverse of both sides of this equality, we have

\[ (J^{-1} \otimes 1)(\Delta \otimes id)(J^{-1}) = (1 \otimes J^{-1})(id \otimes \Delta)(J^{-1}). \]

We write \( J = J^{(1)} \otimes J^{(2)} \) and \( J^{-1} = J^{-(1)} \otimes J^{-(2)} \), where the summation is understood.

We also write \( J_{21} = J^{(2)} \otimes J^{(1)} \).

Given a twist \( J \) for \( A \) one can define a new Hopf algebra \( A' \) with the same algebra structure and counit as \( A \), for which the comultiplication \( \Delta' \) and antipode \( S' \) are given respectively by

\[ \Delta'(a) = J^{-1} \Delta(a) J, \]
\[ S'(a) = Q_j^{-1} S(a) Q_j, \quad \text{for } a \in A, \]

where \( Q_j = S(J^{(1)}J^{(2)}) \), which is an invertible element of \( A \) with the inverse \( Q_j^{-1} = J^{-(1)}S(J^{-2}) \).

The element \( Q_j \) satisfies the following identity (see [1, Eq (5)]):

\[ \Delta(Q_j) = (S \otimes S)(J_{21}^{-1})(Q_j \otimes Q_j)J^{-1}. \]

It follows from [1, Lemma 2.4] that

\[ (S \otimes S)(J_{21}^{-1})(Q_j \otimes 1) = \Delta(S(J^{(1)}))(J^{(2)} \otimes 1) = S(J_{(2)}^{(1)}J^{(2)} \otimes S(J_{(1)}^{(1)}). \]

Note that the integral \( \Lambda \) of \( A \) is also an integral of \( A' \). Suppose

\[ \Lambda_n(\Lambda) = \Lambda_{(1)} \otimes \cdots \otimes \Lambda_{(n)} \quad \text{and} \quad \Lambda'_n(\Lambda) = \Lambda_{(1)} \otimes \cdots \otimes \Lambda_{(n)}. \]

With the notations above, we have the following result:

**Proposition 5.1.** Let \( A \) be a finite dimensional unimodular Hopf algebra over the field \( \mathbb{k} \). Suppose \( J \) is a normalized twist of \( A \) and \( \Lambda \) is a nonzero integral of \( A \).

\[ \begin{align*}
(1) \quad & \text{We have } \Lambda_{(1)} \otimes \Lambda_{(2)} = Q_j^{-1}\Lambda_{(1)}(\otimes \Lambda_{(2)}Q_j. \\
(2) \quad & \text{For any natural number } n \geq 3, \text{ we have}
\Lambda_{(1)}\Lambda_{(2)} \cdots \Lambda_{(n-1)} \otimes \Lambda_{(n)} \\
&= Q_j^{-1}\Lambda^{(1)}(J_{(1)}^{(1)})^{-1}(J_{(2)}^{(1)})^{-1} \cdots \Lambda^{(n-2)}(J_{(n-2)}^{(1)})^{-1}(J_{(n-2)}^{(1)})^{-1} \Lambda^{(n-1)}(J^{(2)} \otimes J^{(2)} \otimes J^{(2)} \otimes \Lambda_{(n)}Q_j.
\end{align*} \]
Proof. (1) A direct calculation shows that
\[
\Lambda_1 \otimes \Lambda_2 = \Lambda'(\Lambda) = J^{-1} \Delta(\Lambda) J = J^{(1)} \Lambda_1 J^{(1)} \otimes J^{(2)} \Lambda_2 J^{(2)}
\]
\[
= J^{(1)} S(J^{(1)} \Lambda_1) J^{(1)} \otimes J^{(2)} \Lambda_2 J^{(2)} \quad \text{by (4.1)}
\]
\[
= J^{(1)} S(J^{(2)} \Lambda_1) \otimes J^{(1)} S(J^{(1)}) J^{(2)} \quad \text{by (4.2)}
\]
\[
= Q_J^{-1} \Lambda_1 \otimes \Lambda_2 Q_J.
\]

(2) The equality needed to prove is indeed the following equality:
\[
\Lambda_1 \Lambda_2 \cdots \Lambda_{n-1} \otimes \Lambda_n = Q_J^{-1} (m_{n-2} \circ \Delta_{n-2})(\Lambda_1 J^{(1)} J^{(1-1)}) \Lambda_2 J^{(2)} \otimes J^{(2)} \Lambda_3 Q_J.
\]

We proceed by induction on \( n \). We apply \( id \otimes \Lambda' \) to Part (1) and obtain that
\[
\Lambda_1 \otimes \Lambda_2 \otimes \Lambda_3 = Q_J^{-1} \Lambda_1 \otimes \Lambda_2 \otimes \Lambda_3
\]
\[
= Q_J^{-1} \Lambda_1 \otimes J^{-1} \Delta(\Lambda_2) \Delta(\Lambda_3) J
\]
\[
= Q_J^{-1} \Lambda_1 \otimes J^{-1} \Delta(\Lambda_2)(S \otimes S)(J_{21}^{-1})Q_J \quad \text{by (5.2)}
\]
\[
= Q_J^{-1} \Lambda_1 \otimes J^{-1} \Delta(\Lambda_2)(S(J_{21}^{-1})) J^{(2)} \otimes S(J_{11}^{-1}))(1 \otimes Q_J) \quad \text{by (5.3)}
\]
\[
= Q_J^{-1} \Lambda_1 \otimes J^{-1} \Delta(\Lambda_2)(S(J_{11}^{-1}))(J^{(2)} \otimes Q_J)
\]
\[
= Q_J^{-1} \Lambda_1 J^{(1)} \otimes J^{-1} \Delta(\Lambda_2) Q_J \quad \text{by (4.2)}
\]
\[
= Q_J^{-1} \Lambda_1 J^{(1)} \otimes J^{-1} \Delta(\Lambda_2) J^{(2)} \otimes J^{(2)} \Lambda_3 Q_J.
\]

It follows that
\[
\Lambda_1 \Lambda_2 \otimes \Lambda_3 = Q_J^{-1} \Lambda_1 J^{(1)} J^{(1-1)} \Lambda_2 J^{(2)} \otimes J^{(2)} \Lambda_3 Q_J
\]
\[
= Q_J^{-1} (m_1 \circ \Delta_1)(\Lambda_1 J^{(1)} J^{(1-1)}) \Lambda_2 J^{(2)} \otimes J^{(2)} \Lambda_3 Q_J.
\]

This proves the case \( n = 3 \). Suppose the equality holds for the case \( n \), namely,
\[
(5.4) \quad \Lambda_1 \Lambda_2 \cdots \Lambda_{n-1} \otimes \Lambda_n
\]
\[
= Q_J^{-1} (m_{n-2} \circ \Delta_{n-2})(\Lambda_1 J^{(1)} J^{(1-1)}) \Lambda_2 J^{(2)} \otimes J^{(2)} \Lambda_3 Q_J.
\]

To prove the case \( n + 1 \), we set \( J' = J \) and apply \( id \otimes \Lambda' \) to both sides of (5.4). It follows that
\[
\Lambda_1 \Lambda_2 \cdots \Lambda_{n+1} \otimes \Lambda_n \otimes \Lambda_{n+1}
\]
\[
= Q_J^{-1} (m_{n-2} \circ \Delta_{n-2})(\Lambda_1 J^{(1)} J^{(1-1)}) \Lambda_2 J^{(2)} \otimes \Delta_J J^{(2)} \Lambda_3 Q_J)
\]
\[
= Q_J^{-1} (m_{n-2} \circ \Delta_{n-2})(\Lambda_1 J^{(1)} J^{(1-1)}) \Lambda_2 J^{(2)} \otimes \Lambda_3 \Lambda_J \Delta(\Lambda_3) J
\]
\[
= Q_J^{-1} (m_{n-2} \circ \Delta_{n-2})(\Lambda_1 J^{(1)} J^{(1-1)}) \Lambda_2 J^{(2)} \otimes \Lambda_3 \Lambda_J \Delta(\Lambda_3)(S \otimes S)(J_{21}^{-1})(Q_J \otimes Q_J) \quad \text{by (5.2)}
\]
\[
= Q_J^{-1} (m_{n-2} \circ \Delta_{n-2})(\Lambda_1 J^{(1)} J^{(1-1)}) \Lambda_2 J^{(2)} \otimes \Lambda_3 \Lambda_J \Delta(\Lambda_3)(J^{(1)})(Q_J \otimes Q_J) \quad \text{by (5.3)}
\]
\[
= Q_J^{-1} (m_{n-2} \circ \Delta_{n-2})(\Lambda_1 J^{(1)} J^{(1-1)}) \Lambda_2 J^{(2)} \otimes \Lambda_3 \Lambda_J \Delta(\Lambda_3)(J^{(2)} \otimes Q_J) \quad \text{by (4.3)}
\]
\[
= Q_J^{-1} (m_{n-2} \circ \Delta_{n-2})(\Lambda_1 J^{(1)} J^{(1-1)}) \Lambda_2 J^{(2)} \otimes \Lambda_3 \Lambda_J \Delta(\Lambda_3)(J^{(2)} \otimes Q_J) \quad \text{by (5.1)}
\]
\[
= Q_J^{-1} (m_{n-2} \circ \Delta_{n-2})(\Lambda_1 J^{(1)} J^{(1-1)}) \Lambda_2 J^{(2)} \otimes \Lambda_3 \Lambda_J \Delta(\Lambda_3)(J^{(2)} \otimes Q_J).
\]
Using \( f^{(1)} J^{(1)} \otimes f^{(2)} J^{(2)} = 1 \otimes 1 \), we have
\[
\Lambda(1) \Lambda(2) \cdots \Lambda(n) A(n) \otimes A(n) = Q_J^{-1} \left( m_{n-2} \circ \Delta_{n-2} \left( \Lambda(1), J^{(1)} \right) A(2), J^{(2)} \right) J^{(2)} A(3) J^{(2)} \otimes J^{(2)} Q_J
\]
\[
= Q_J^{-1} \left( m_{n-2} \circ \Delta_{n-2} \left( \Lambda(1), J^{(1)} \right) A(2), J^{(2)} \right) J^{(2)} A(3) J^{(2)} \otimes J^{(2)} Q_J
\]
\[
= Q_J^{-1} \left( m_{n-1} \circ \Delta_{n-1} \left( \Lambda(1), J^{(1)} \right) A(2), J^{(2)} \right) J^{(2)} \otimes J^{(2)} Q_J
\]
We complete the proof. \( \square \)

As a consequence, we have the following invariants of \( A \) under twisting:

**Proposition 5.2.** Let \( A \) be a finite dimensional unimodular Hopf algebra over the field \( \mathbb{k} \) and \( J \) a normalized twist of \( A \). For any \( n \in \mathbb{Z} \) and any integral \( \Lambda \) of \( H \), we have \( P_n^J(\Lambda) = P_n(\Lambda) \), where \( P_n^J \) and \( P_n \) are the \( n \)-th Sweedler power maps of \( A^J \) and \( A \) respectively.

**Proof.** Suppose \( n \geq 3 \). Applying \( \Delta_{n-2} \otimes \text{id} \) to both sides of the equality \( f^{(1)} J^{(1)} \otimes f^{(2)} J^{(2)} = 1 \otimes 1 \), we have
\[
f^{(1)} J^{(1)} \otimes f^{(1)} J^{(1)} \cdots f^{(n-2)} J^{(1)} f^{(n-2)} J^{(2)} = 1 \otimes 1 \cdots 1 \otimes 1.
\]
This equality as well as Proposition 5.1 (2) shows that
\[
P_n^J(\Lambda) = \Lambda(1) \Lambda(2) \cdots \Lambda(n) Q_J
\]
\[
= Q_J^{-1} \left( \Lambda(1), J^{(1)} \right) A(2), J^{(2)} \right) J^{(2)} \cdots \left( \Lambda(n-2), J^{(n-2)} \right) J^{(n-2)} \Lambda(n) Q_J
\]
\[
= P_n(\Lambda).
\]

For the case \( n = 2 \), it follows from Proposition 5.1 (1) that
\[
P_2^J(\Lambda) = \Lambda(1) \Lambda(2) = Q_J^{-1} \Lambda(1) \Lambda(2) Q_J = \Lambda(1) \Lambda(2) = P_2(\Lambda).
\]
It is obvious that \( P_0^J(\Lambda) = \Lambda = P_0(\Lambda) \) and \( P_0^J(\Lambda) = \epsilon(\Lambda) = P_0(\Lambda) \). This proves the case \( n \geq 0 \). For the case \( n \leq 0 \), we proceed by induction on \( n \). We suppose that \( P_n^J(\Lambda) = P_n(\Lambda) \) hold for \( -l \leq n < 0 \). We will show that \( P_{-l-1}^J(\Lambda) = P_{-l-1}(\Lambda) \). Let \( \Phi(X) \) and \( \Phi^J(X) \) be the minimal polynomial of the sequences \( \{P_n\}_{n \in \mathbb{Z}} \) and \( \{P_n^J\}_{n \in \mathbb{Z}} \) respectively. Let \( \sum_{i=0}^m c_i X^i \) be the minimal multiple of \( \Phi(X) \) and \( \Phi^J(X) \). Then
\[
c_0 P_0 + c_1 P_1 + \cdots + c_m P_m = 0 \quad \text{and} \quad c_0 P_n^J + c_1 P_1^J + \cdots + c_m P_m^J = 0.
\]
Note that \( c_0 \neq 0 \) since \( \Phi(0) \neq 0 \) and \( \Phi^J(0) \neq 0 \). It follows that
\[
P_0 = -c_0^{-1} (c_1 P_1 + \cdots + c_m P_m) \quad \text{and} \quad P_0^J = -c_0^{-1} (c_1 P_1^J + \cdots + c_m P_m^J).
\]
Multiplying \( P_{-l-1} \) and \( P_{-l-1}^J \) on both sides respectively and applying them to \( \Lambda \), we have
\[
P_{-l-1}(\Lambda) = -c_0^{-1} (c_1 P_{-l}(\Lambda) + \cdots + c_m P_{m-1}(\Lambda))
\]
and
\[ P^J_{-l+1}(\Lambda) = -c_0^{-1}(c_1 P^J_{-l}(\Lambda) + \cdots + c_m P^J_{m-l+1}(\Lambda)). \]
By the induction hypothesis and the result that \( P^J_n(\Lambda) = P_n(\Lambda) \) for \( n \geq 0 \), we conclude that \( P^J_{-l+1}(\Lambda) = P_{-l+1}(\Lambda) \). The proof is finished. \( \square \)

**Remark 5.3.** Proposition 5.2 shows that the sequence \( \{P_n(\Lambda)\}_{n \in \mathbb{Z}} \) of a unimodular Hopf algebra \( A \) is invariant under twisting. More explicitly, for the case \( n \leq -1 \), \( P^J_n(\Lambda) = P_n(\Lambda) \) is exactly the equality
\[ \Lambda_{(1)} \cdots \Lambda_{(n)} = \Lambda_{(1)} \cdots \Lambda_{(n)}. \]
For \( n \leq -1 \), \( P^J_n(\Lambda) = P_n(\Lambda) \) is the equality \( S^J(\Lambda_{(1)}) \cdots S^J(\Lambda_{(n)}) = S(\Lambda_{(1)}) \cdots S(\Lambda_{(-n)}) \), which induces the following equality:
\[ \Lambda_{(-n)} \cdots \Lambda_{(1)} = \Lambda_{(-n)} \cdots \Lambda_{(1)}. \]

Applying Proposition 5.2 to higher FS indicators, we next show that \( \nu_n(V) \) for all \( n \in \mathbb{Z} \) are preserved under twisting.

**Proposition 5.4.** Let \( H \) be a finite dimensional semisimple Hopf algebra over the field \( \mathbb{C} \) and \( V \) a finite dimensional representation of \( H \). The \( n \)-th FS indicator \( \nu_n(V) \) of \( V \) is invariant under twisting for any \( n \in \mathbb{Z} \).

**Proof.** Let \( \Lambda \) be a nonzero integral of \( H \) and \( J \) a normalized twist for \( H \). Since
\[ \Delta^J(\Lambda) = \Lambda_{(1)} \otimes \Lambda_{(2)} = Q^{-1}(\Lambda_{(1)} \otimes \Lambda_{(2)})Q, \]
it follows that
\[
(5.5) \quad u^J := S^J(\Lambda_{(2)})\Lambda_{(1)} = S^J(\Lambda_{(2)})Q^{-1} \Lambda_{(1)} = Q^{-1}S(Q)u,
\]
where \( u = S(\Lambda_{(2)})\Lambda_{(1)}. \) For \( H \)-module \( V \) with the character \( \chi_V \), we denote \( V^J \) the same as \( V \) as \( \mathbb{C} \)-linear space but thought of as an \( H^J \)-module. We shall show that \( \nu_n(V^J) = \nu_n(V) \). Note that the character of \( V^J \) is also \( \chi_V \). For any \( n \in \mathbb{Z} \), we have
\[ \nu_n(V^J) = \chi_V(u^J)^{-1}P^J_n(\Lambda) \]
\[ = \chi_V(u^{-1}S(Q^{-1})Q_JP_n(\Lambda)) \text{ by (5.5)} \]
\[ = \chi_V(u^{-1}S(Q_J^{-1})Q_JP_n(\Lambda)) \text{ by Proposition 5.2} \]
\[ = \chi_V(u^{-1}S^2(J^{-2})S(J^{-1}))S(J^1)(J^{2})P_n(\Lambda)) \]
\[ = \chi_V(u^{-1}S(J^{-2})S(J^{-1}))S(J^1)(J^{2})P_n(\Lambda)) \]
\[ = \chi_V(u^{-1}S(J^{-1}))S(J^1)(J^{2})P_n(\Lambda))J^{-2}) \]
\[ = \chi_V(u^{-1}S(J^{-1}))S(J^1)(J^{2})J^{-2}P_n(\Lambda)) \text{ by Proposition 4.1} \]
\[ = \chi_V(u^{-1}P_n(\Lambda)) \]
\[ = \nu_n(V). \]
We complete the proof. \( \square \)

We are now ready to state the main result that higher FS indicators are gauge invariants of the tensor category Rep(\( H \)).
Theorem 5.5. Let $H$ and $H'$ be two finite dimensional semisimple Hopf algebras over the field $\mathbb{K}$. If $\mathcal{F} : \text{Rep}(H) \to \text{Rep}(H')$ is an equivalence of tensor categories, then $\nu_n(V) = \nu_n(\mathcal{F}(V))$ for any $n \in \mathbb{Z}$ and any finite dimensional representation $V$ of $H$.

Proof. Since the $\mathbb{K}$-linear equivalence $\mathcal{F} : \text{Rep}(H) \to \text{Rep}(H')$ is a tensor equivalence, it follows from [13, Theorem 2.2] that $H$ and $H'$ are gauge equivalent in the sense that there exist a twist $J$ of $H$ such that $H'$ is isomorphic to $H^J$ as bialgebras. Let $\sigma : H' \to H^J$ be such an isomorphism. Then $\sigma$ is automatically a Hopf algebra isomorphism. The isomorphism $\sigma$ induces a $\mathbb{K}$-linear equivalence $(-)^\sigma : \text{Rep}(H) \to \text{Rep}(H')$ as follows: for any finite dimensional $H$-module $V$, $V^\sigma = V$ as $\mathbb{K}$-linear space with the $H'$-module action given by $h'v = \sigma(h')v$ for $h' \in H'$, $v \in V$, and $f^\sigma = f$ for any morphism $f$ in $\text{Rep}(H)$. Moreover, the equivalence $\mathcal{F}$ is naturally isomorphic to the $\mathbb{K}$-linear equivalence $(-)^\sigma$ (see [6, Theorem 1.1]). Therefore,

$$\nu_n(\mathcal{F}(V)) = \nu_n(V^\sigma).$$

Let $\Lambda'$ be a nonzero integral of $H'$ and $S'$ the antipode of $H'$. Note that the map $\sigma : H' \to H^J$ is a Hopf algebra isomorphism. It follows that $\sigma(\Lambda') = \Lambda$ is a nonzero integral of $H^J$ and $\sigma(P'_{n}(\Lambda')) = P_{n}^{J}(\Lambda)$, where $P'_{n}$ and $P_{n}^{J}$ are the $n$-th Sweedler power maps of $H'$ and $H^J$ respectively. In particular,

$$\sigma((u')^{-1}P'_{n}(\Lambda')) = (u^{J})^{-1}P_{n}^{J}(\Lambda),$$

where $u' = S'(\Lambda'_{(2)})\Lambda'_{(1)}$ and $u^{J} = S^{J}(\Lambda_{(2)})\Lambda_{(1)}$. We have

$$\nu_n(V^\sigma) = \chi_{V^\sigma}(u')^{-1}P'_{n}(\Lambda'))
= \chi_{V^\sigma}(\sigma(u')^{-1}P'(\Lambda'))
= \chi_{V^\sigma}(u^{J})^{-1}P_{n}^{J}(\Lambda)
= \nu_n(V^J)
= \nu_n(V),$$

where the last equality follows from Proposition 5.4. We conclude that $\nu_n(\mathcal{F}(V)) = \nu_n(V)$ for any $n \in \mathbb{Z}$ and any finite dimensional representation $V$ of $H$. \quad \Box

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