Dempster-Shafer Belief

Function - A New

Interpretation

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1 Introduction

Dempster-Shafer theory of evidence has been found by many researchers very attractive as a way of modeling reasoning behaviour under uncertainty stemming from ignorance. It provides a framework for representation of certainty of a logical formula without necessity of expressing commitment to any of its consequences. E.g. we can express our 100 % belief in fact that Tweedy’s wife is either Mary or Jane and at the same time express our total ignorance to the fact who of them is actually his wife (zero belief attached to the statement ”Mary is Tweedy’s wife” and zero belief in ”Jane is Tweedy’s
wife”).

If a theory is to become of practical importance in expert systems application - as foundation for knowledge representation and reasoning, at least the following conditions must be fulfilled:

- there must exist an efficient method for reasoning within this framework
- there must exist a clear correspondence between the contents of the knowledge base and the real world
- there must be a clear correspondence between the reasoning method and some real world process
- there must exist a clear correspondence between the results of the reasoning process and the results of the real world process corresponding to the reasoning process.

Only under such circumstances we can say that the expert system is helpful as it allows us either to predict or to follow retrospectively real world processes.

Dempster initiated the theory of evidence in his paper [4] and other works, and Shafer developed this theory in his book [21] and other publications. Though it became obvious that the DST (Dempster-Shafer Theory) captures many intuitions behind the human dealing with uncertainty (e.g. as mentioned above), it did not become a foundation for implementation of expert systems with uncertainty due to claimed high computational complexity [9].
In the recent years, however, a number of efficient methods for dealing with DS reasoning have been developed - see e.g. [23] and citations therein. So the first of the above mentioned conditions is met. Meeting of other conditions proved to be more complicated.

Smets [26] and also initially Shafer [21] insisted on Bels (measures of uncertainty in the DST) not being connected to any empirical measure (frequency, probability etc.) considering the domain of DST applications as the one where ”we are ignorant of the existence of probabilities”, and warn that the DST is ”not a model for poorly known probabilities” ([26], p.324). The question may be raised, however, what practically useful can be obtained from a computer reasoning on the basis of such a DST. It would have to be demonstrated that humans indeed reason as DST. Then the computer, if fed with our knowledge, would be capable to predict our conclusions on a given subject. However, to my knowledge, no experiment confirming that humans actually use internally DST for reasoning under uncertainty has been carried out. Under these circumstances the computer reasoning with DST would tell us what we have to think and not what we think. Hence, from the point of view of computer implementation the position of Smets and Shafer is not acceptable.

The other category of DST interpretations, described by Smets as approaches assuming existence of an underlying probability distribution, which is only approximated by the Bels (called by him PXMY models), is represented by early works of Dempster [4], papers of Kyburg [12], Fagin [7],
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[8], Halpern [10], Skowron [24], Grzymała-Busse [9] and others. Both Smets [26] and Shafer [22] consider such approaches as inadequate as most of them give rise to contradictions and counter intuitiveness. As Smets states, "Far too often, authors concentrate on the static component (how beliefs are allocated?) and discover many relations between TBM (transferable belief model of Smets) and ULP (upper lower probability) models, inner and outer measures (Fagin and Halpern [6]), random sets (Nguyen [16]), probabilities of provability (Pearl [17]), probabilities of necessity (Ruspini [19]) etc. But these authors usually do not explain or justify the dynamic component (how are beliefs updated?), that is, how updating (conditioning) is to be handled (except in some cases by defining conditioning as a special case of combination). So I (that is Smets) feel that these partial comparisons are incomplete, especially as all these interpretations lead to different updating rules.” ([26], pp. 324-325). Smets attributes this failure to the very nature of attempts of assigning a probabilistic interpretation. We disagree with Smets and will show in this paper that creation of a probabilistic interpretation of the DST incorporating the Dempster rule of combination is actually possible. However, this new interpretation indicates the need for a drastic change in viewing the Dempster rule: it does not accommodate evidence, but prejudices. How this statement is to be understood, will be visible later. Nonetheless our interpretation allows for assignment of an experimentally verifiable numerical meaning to a DS knowledge base, assigns a numerical meaning to the reasoning process (the DS rule of combination) and yields agreement between
numerical empirical interpretation of the results of DS reasoning and results of a real world process. This means that we have an interpretation fitting formal interpretation of the DS theory to the largest extent ever achieved.

Smets ([26],p.327) subdivided the DST into two categories: a closed world category (as if excluding the possibility of contradictions in the "evidence") and an open world category of DST (as if allowing for this). Let us assume that two independent experts elicited their beliefs concerning the event A: both assigned beliefs of 0.7 to the event A, and beliefs of 0.3 to the event \(\neg A\). The open world DST will lead us to a combined belief in A of 0.5 and in \(\neg A\) of 0.1. The closed world assumption on the other hand will assign a combined belief in A of 0.7 and in \(\neg A\) of 0.3. I find it a dismaying property of a theory if collecting agreeing information from independent expert shall decline my belief in the opinions of both experts. Hence only closed world theories are subject of this paper.

We first recall the formal definition of the DS-Theory, then introduce some notation used throughout the rest of the paper. Subsequently we develop our interpretation of the joint belief distribution and of evidential updating. We conclude with a brief comparison of our interpretation with other attempts.
2 Formal Definition of the Dempster-Shafer Theory of Evidence

Let us make the remark that if an object is described by a set of discrete attributes $X_1, X_2, ..., X_n$ taking values from their respective domains $\Xi_1, \Xi_2, ..., \Xi_n$ then we can think of it as being described by a complex attribute $X$ having vector values, that is the domain $\Xi$ of $X$ is equal:

$$\Xi = \{(x_1, x_2, ..., x_n) | x_i \in \Xi_i, i = 1, ..., n\}$$

So unless specified otherwise let us assume that we are talking of objects described by a single attribute $X$ taking its values from the domain $\Xi$. We say that $\Xi$, the domain of $X$ is our space of discourse spanned by the attribute $X$. We shall also briefly say that $X$ is our space of discourse instead.

For the purpose of this paper we define the Bel-function as follows (compare also [10], [26], [22]):

**Definition 1** The Belief Function in the sense of the DS-Theory is defined as $\text{Bel} : 2^\Xi \rightarrow [0, 1]$ with $\Xi = \Xi_1 \times \Xi_2 \times \ldots \times \Xi_n$ being the space spanned by the attribute $X = X_1 \times X_2 \times \ldots \times X_n$ with

$$\forall A, A \in \Xi \quad \text{Bel}(A) = \sum_{B \subseteq A} m(B)$$

where $m(A)$ is a Mass Function in the sense of the DS-Theory (see Def.2 below).
The function $m$ is defined as

**Definition 2** The Mass Function in the sense of the DS-Theory is defined as $m : 2^\Xi \rightarrow [0, 1]$ with

$$m(\emptyset) = 0$$

$$\sum_{A \in 2^\Xi} m(A) = 1$$

$$\forall A \in 2^\Xi \quad m(A) \geq 0$$

**Definition 3** Whenever $m(A) > 0$, we say that $A$ is the focal point of the Bel-Function.

Let us also introduce the Pl-Function (Plausibility) as:

**Definition 4** The Plausibility Function in the sense of the DS-Theory is defined as

$$\forall A, A \subseteq \Xi \quad Pl(A) = 1 - Bel(\Xi - A)$$

Beside the above definition a characteristic feature of the DS-Theory is the so-called DS-rule of combination of independent evidence:

**Definition 5** Let $Bel_{E_1}$ and $Bel_{E_2}$ represent independent information over the same space of discourse. Then:

$$Bel_{E_1 \cap E_2} = Bel_{E_1} \oplus Bel_{E_2}$$
defined as:
\[ m_{E_1,E_2}(A) = c \cdot \sum_{B,C: A=B\cap C} m_{E_1}(B) \cdot m_{E_2}(C) \]

(c - normalizing constant) represents the Combined Belief-Function of Two Independent Beliefs

3 Denotation

F. Bacchus in his paper [2] on axiomatization of probability theory and first order logic shows that probability should be considered as a quantifier binding free variables in first order logic expressions just like universal and existential quantifiers do. So if e.g. \( \alpha(x) \) is an open expression with a free variable \( x \) then \([\alpha(x)]_x\) means the probability of truth of the expression \( \alpha(x) \). (The quantifier \([\cdot]_x\) binds the free variable \( x \) and yields a numerical value ranging from 0 to 1 and meeting all the Kolmogoroff axioms). Within the expression \([\alpha(x)]_x\) the variable \( x \) is bound. See [2] on justification why other types of integration of probability theory and first order logic or propositional logic fail. Also for justification of rejection of the traditional view of probability as a function over sets. While sharing Bacchus’ view, we find his notation a bit cumbersome so we change it to be similar to the universal and existential quantifiers throughout this paper. Furthermore, Morgan [14] insisted that the probabilities be always considered in close connection with the population they refer to. Bacchus’ expression \([\alpha(x)]_x\) we rewrite as:

\[ \Pr_{\alpha(x)} \] - the probability of \( \alpha(x) \) being true within the population
P. The P (population) is a unary predicate with P(x)=TRUE indicating that the object x(∈ Ω, that is element of a universe of objects) belongs to the population under considerations. If P and P’ are populations such that ∀xP’(x) → P(x) (that is membership in P’ implies membership in P, or in other words: P’ is a subpopulation of P), then we distinguish two cases:

**case 1:** (Prob\(^{P'(x)}\)\(x, \alpha(x)\)) = 0 (that is probability of membership in P’ with respect to P is equal 0) - then (according to [14] for any expression α(x) in free variable x the following holds for the population P’: (Prob\(^{P'(x)}\)\(x, \alpha(x)\)) = 1

**case 2:** (Prob\(^{P'(x)}\)\(x, \alpha(x)\)) > 0 then (according to [14] for any expression α(x) in free variable x the following holds for the population P’:

\[
(\text{Prob}^{P'(x)} \alpha(x)) = \frac{\text{Prob}^{P'(x)} \alpha(x) \land P'(x)}{\text{Prob}^{P'(x)} P'(x)}
\]

We also use the following (now traditional) mathematical symbols:

∀\(x\)\(, \alpha(x)\) - always \(\alpha(x)\) (universal quantifier)

∃\(x\)\(, \alpha(x)\) - there exists an x such that \(\alpha(x)\) (existential quantifier)
4 A New Interpretation of Belief Functions

The empirical meaning of a new interpretation of the DS Belief function will be explained by means of the following example:

**Example 1** Let us consider a daily-life example. Buying a bottle of hair shampoo is not a trivial task from both the side of the consumer and the manufacturer. If the consumer arrives at the consciousness that the shampoos may fall into one of the four categories: high quality products (excellent for maintaining cleanliness and health of the consumer) (H), moderate quality products (keeping just all Polish industry standards) (M), suspicious products (violating some industry standards) (S) and products dangerous for health and life (containing bacteria or fungi or other microbes causing infectious or invasive diseases, containing cancerogenous or poisonous substances
etc.) (D), he has a hard time upon leaving his house for shopping. Clearly, precise chemical, biochemical and medical tests exist which may precisely place the product into one of those obviously exclusive categories. But the Citizen\(^1\) Coot\(^2\) usually neither has a private chemical laboratory nor enough money to make use of required services. Hence Citizen Coot coins a personal set of "quality" tests \(M^1\) mapping the pair (bottle of shampoo, quality) into the set \{TRUE, FALSE\} (the letter O - object - stands for bottle of shampoo, H, M, S, D indicate quality classes: high, moderate, suspicious, dangerous):

1. If the shampoo is heavily advertised on TV then it is of high quality \((M^1(O, \{H\}) = TRUE)\) and otherwise not \((M^1(O, \{H\}) = FALSE)\).

2. If the name of the shampoo was never heard on TV, but the bottle looks fine (pretty colours, aesthetic shape of the bottle), then the shampoo must be of moderate quality \((M^1(O, \{M\}) = TRUE)\) and otherwise not \((M^1(O, \{M\}) = FALSE)\).

3. If the packaging is not fine or the date of production is not readable on the bottle or the product is out of date, but the shampoo smells acceptably otherwise then it is suspicious \((M^1(O, \{S\}) = TRUE)\) and otherwise not \((M^1(O, \{S\}) = FALSE)\).

\(^1\)The term "Citizen" was a fine socialist time descriptor allowing to avoid the cumbersome usage of words like "Mr.", "Mrs." and "Miss"

\(^2\)This family name was coined as abbreviation for "Citizen Of Our Town"
4. If either the packaging is not fine or the date of production is not readable on the bottle or the product is out of date, and at the same time the shampoo smells awfully, then it is dangerous \( M_1(O, \{D\}) = TRUE \) and otherwise not \( M_1(O, \{D\}) = FALSE \).

Notice that the criteria are partially rational: a not fine looking bottle may in fact indicate some decaying processing of the shampoo or at least that the product remains for a longer time on the shelf already. Bad smell is usually caused by development of some bacteria dangerous for human health. Notice also that test for high and moderate quality are enthusiastic, while the other two are more cautious.

Notice that the two latter tests are more difficult to carry out in a shop than the leading two (the shop assistant would hardly allow to open a bottle before buying). Also, there may be no time to check whether the shampoo was actually advertised on TV or not (as the son who carefully watches all the running advertisements stayed home and does his lessons). Hence some simplified tests may be quite helpful:

- \( M_1(O, \{S, D\}) \): If the packaging is not fine or the product is out of date or the production date is not readable then the product is either suspicious or dangerous \( M_1(O, \{S, D\}) = TRUE \) and otherwise not \( M_1(O, \{D, S\}) = FALSE \).

- \( M_1(O, \{H, M\}) \): If the packaging looks fine, then the product is either of high or moderate quality \( M_1(O, \{M, H\}) = TRUE \) and otherwise
not \( (M^1(O, \{M, H\}) = FALSE) \).

Clearly these tests are far from being precise ones, but for the Citizen Coot no better tests will be ever available. What is more, they are not exclusive: if one visits a dubious shop at a later hour, one may buy a product meeting both \( M^1(O, \{H\}) \) and \( M^1(O, \{D\}) \) as defined above!

Let us assume we have two types of shops in our town: good ones (G) and bad ones (B). (Let \( M^2 : \Omega \times 2\{G, B\} \rightarrow \{TRUE, FALSE\} \) indicate for each shampoo in which shop type it was available. Further, let \( M^3 : \Omega \times 2\{H, M, S, D\} \times \{G, B\} \rightarrow \{TRUE, FALSE\} \) indicate for each shampoo both its quality and the type of shop it was available from. Let clearly \( M^1(O, Quality) \land M^2(O, Shop) = M^3(O, Quality \times Shop) \).

The good shops are those with new furniture, well-clothed shop assistants. Bad ones are those with always dirty floor or old furniture, or badly clothed shop assistants. Clearly, again, both shop categories may be considered (nearly) exclusive as seldom well clothed shop assistants do not care of floors.

Let us assume we have obtained the statistics of shampoo sales in our town presented in Table 1:

|       | \( G \) | \( B \) |
|-------|--------|--------|
| \( H \) | 20     | 0      |
| \( M \) | 0      | 30     |
| \( S \) | 10     | 10     |
| \( D \) | 5      | 20     |

Rows and columns are marked with those singleton tests which were passed (e.g. in the left upper corner there are 20 shampoo bottles sold in an undoubtedly bad shop and having exclusively high quality, that is for all those bottles (O) \( M^1(O, \{H\}) = TRUE, M^1(O, \{M\}) = FALSE, M^1(O, \{S\}) = FALSE, M^1(O, \{D\}) = FALSE, \) and \( M^2(O, \{B\}) = TRUE, M^2(O, \{G\}) = FALSE).
Table 1: Sold shampoos statistics

| Quality true for | Shop type | B   | G   | B,G  | Total |
|------------------|-----------|-----|-----|------|-------|
| H                | 20        | 100 | 70  |      | 190   |
| M                | 80        | 100 | 110 |      | 290   |
| S                | 50        | 5   | 15  |      | 70    |
| D                | 10        | 1   | 3   |      | 14    |
| H,S              | 15        | 10  | 14  |      | 39    |
| M,S              | 30        | 20  | 25  |      | 75    |
| H,D              | 8         | 2   | 3   |      | 13    |
| M,D              | 15        | 7   | 10  |      | 32    |
| total            | 228       | 245 | 250 |      | 723   |
FALSE.) The measurement of $M^1(O, \{H\})$ would yield TRUE for $190+39+13 = 242$ bottles and FALSE for the remaining 581 bottles, the measurement of $M^1(O, \{D\})$ would yield TRUE for $14+13+32 = 59$ bottles, and FALSE for the remaining 664 bottles. The measurement $M^1(O, \{S, D\})$ will turn true in $70+14+39+75+13+12 = 343$ cases and FALSE in the remaining 480 cases.

In general let us assume that we know that objects of a population can be described by an intrinsic attribute $X$ taking exclusively one of the $n$ discrete values from its domain $\Xi = \{v_1, v_2, ..., v_n\}$. Let us assume furthermore that to obtain knowledge of the actual value taken by an object we must apply a measurement method (a system of tests) $M$

**Definition 6** $X$ be a set-valued attribute taking as its values non-empty subsets of a finite domain $\Xi$. By a measurement method of value of the attribute $X$ we understand a function:

$$M : \Omega \times 2^{\Xi} \to \{TRUE, FALSE\}$$

where $\Omega$ is the set of objects, (or population of objects) such that

- $\forall_{\omega \in \Omega} M(\omega, \Xi) = TRUE$ ($X$ takes at least one of values from $\Xi$)
- $\forall_{\omega \in \Omega} M(\omega, \emptyset) = FALSE$
- whenever $M(\omega, A) = TRUE$ for $\omega \in \Omega$, $A \subseteq \Xi$ then for any $B$ such that $A \subseteq B$ $M(\omega, B) = TRUE$ holds,
• whenever \( M(\omega, A) = TRUE \) for \( \omega \in \Omega, \ A \subseteq \Xi \) and if \( \text{card}(A) > 1 \) then there exists \( B, \ B \subset A \) such that \( M(\omega, B) = TRUE \) holds.

• for every \( \omega \) and every \( A \) either \( M(\omega, A) = TRUE \) or \( M(\omega, A) = FALSE \) (but never both).

\( M(\omega, A) \) tells us whether or not any of the elements of the set \( A \) belong to the actual value of the attribute \( X \) for the object \( \omega \).

The measuring function \( M(O,A) \), if it takes the value \( TRUE \), states for an object \( O \) and a set \( A \) of values from the domain of \( X \) that the \( X \) takes for this object (at least) one of the values in \( A \).

It makes sense to talk of such measuring function assigning truth values to sets of values of an attribute if it is possibly cheaper to measure \( M(O,A) \) than to measure \( M(O,B) \) whenever \( B \subset A \) and we are interested in avoiding measuring \( M(O,B) \) whenever possible, that is whenever measuring \( M(O,A) \) suffices. For example, measuring pH-value with a pH-meter may turn out to be more expensive than one with litmus paper, at the advantage of a higher precision.

The above definition assumes that this measurement method is superset- and subset-consistent that is: Whenever \( M(\text{object}, A) = TRUE \) then

\[ \forall_{B: A \subset B} \ M(\text{object}, B) = TRUE \]

holds, and if \( \text{card}(A) > 1 \) then

\[ \exists_{B: B \subset A} \ M(\text{object}, B) = TRUE \]
holds. The superset consistency means that if a test for larger set of values indicates FALSE then it is not necessary to test its subsets as they will not contribute to our knowledge of the value of X (cost savings). The subset consistency means that if the M-test for a given value set gives true than in end effect at least of its singleton subsets would yield TRUE for the respective M-test. It is clearly the matter of convention: we assume that we can always provide the answer YES or NO, and whenever we are in doubt we still answer YES.

Such a convention is not an unusual one: in various legal systems "anything, that is not forbidden by law, is permitted"; in the default logics if a default statement cannot be proven wrong, it is assumed correct.

In any case, this means that from the universe of all possible objects, a concrete measurement method selects a population for which its assumptions are satisfied. E.g. if we have a measurement method for measuring pH-values, we surely consider an aqueous sodium solution as a member of our universal population, but never a car as such (because then pH-value has no meaning at all).

Furthermore we consider this measurement method a stable one that is whenever the same object is presented, the results are the same. However, let us assume that the measurement method is not completely reliable: it measures only quantities related to the quantity X and not X itself. So it is conceivable that e.g. $M(object, \{v_1\}) = TRUE$ and at the same time $M(object, \{v_2\}) = TRUE$ though both values of X are deemed to be exclu-
sive. For practical reasons however it may not bother us at all as either the true value of X may not be accessible at all (e.g. the true event of killing or not killing a person by the suspect belongs to the past and can never be recalled as such), may be too expensive to access (e.g. if the most reliable method of checking whether a match can inflame or not it to inflame it, but thereafter it would be useless, so we check only for its color, dryness etc.) or it may be prohibitive to access it for other reasons, e.g. social (sex may be treated with extremely high precision as an exclusive attribute taking values male, female, but we would reluctantly check the primary features before deciding to call someone Mr, Miss or Mrs). Beside this it may prove too expensive to check all the elementary hypotheses (e.g. in the medical diagnosis) so that after stating $M(\text{object}, \{v_1\}) = \text{TRUE}$ we do not bother of other alternatives, that is of the degree of imprecision of the relationship between the measured quantities and the real values of X. We assume that the approximations of X achieved by the measurement method are in most cases sufficient for our decision making (whatever its nature), so we do not insist on closer knowledge of X itself.

So though we wish X to take singleton values only, we actually live with the fact that for our practical purposes X is possibly set-valued.

Let us make at this point some remarks on practical relevance.

**Example 2** If we are making statistical tests on equality or non-equality of two quantities (means, variances, distributions), we can purely logically say
that the quantities are either equal or not equal but never both. However, the available indirect measurement method (by sampling) may lead to a statement that there is neither evidence to reject equality nor to reject non-equality. So we say that in those cases both equity and inequity holds. We still enjoy statistical inference because in sufficiently many other cases statistics provides us with more precise results.

Example 3 Similarly if we consider components of a chemical substance, the measurement methods for absence and presence of a component may be different from one another depending whether or not we should be more sensitive to its presence or absence and hence in some cases applying both may lead to apparently contradicting results.

Let us furthermore assume that with each application of the measurement procedure some costs are connected, increasing roughly with the decreasing size of the tested set A so that we are ready to accept results of previous measurements in the form of pre-labeling of the population. So

Definition 7 A label $L$ of an object $\omega \in \Omega$ is a subset of the domain $\Xi$ of the attribute $X$.

A labeling under the measurement method $M$ is a function $l : \Omega \rightarrow 2^\Xi$ such that for any object $\omega \in \Omega$ either $l(\omega) = \emptyset$ or $M(\omega, l(\omega)) = TRUE$.

Each labelled object (under the labeling $l$) consists of a pair $(O_j, L_j)$, $O_j$ - the $j^{th}$ object, $L_j = l(O_j)$ - its label.

By a population under the labeling $l$ we understand the predicate $P : \Omega \rightarrow \{0, 1\}$.
\{TRUE, FALSE\} of the form \( P(\omega) = TRUE \text{ iff } l(\omega) \neq \emptyset \) (or alternatively, the set of objects for which this predicate is true)

If for every object of the population the label is equal to \( \Xi \) then we talk of an unlabeled population (under the labeling \( l \)), otherwise of a pre-labelled one.

Let us assume that in practice we apply a modified measurement method \( M_l \) being a function:

**Definition 8** Let \( l \) be a labeling under the measurement method \( M \). Let us consider the population under this labeling. The modified measurement method

\[
M_l : \Omega \times 2^\Xi \rightarrow \{TRUE, FALSE\}
\]

where \( \Omega \) is the set of objects, is is defined as

\[
M_l(\omega, A) = M(\omega, A \cap l(\omega))
\]

(Notice that \( M_l(\omega, A) = FALSE \) whenever \( A \cap l(\omega) = \emptyset \).)

For a labeled object \((O_j, L_j)\) (\(O_j\) - proper object, \(L_j\) - its label) and a set \( A \) of values from the domain of \( X \), the modified measurement method tells us that \( X \) takes one of the values in \( A \) if and only if it takes in fact a value from intersection of \( A \) and \( L_j \). Expressed differently, we discard a priori any attribute not in the label.

Please pay attention also to the fact, that given a population \( P \) for which the measurement method \( M \) is defined, the labeling \( l \) (according to its definition) selects a subset of this population, possibly a proper subset, namely
the population $P'$ under this labeling. \( P'\(\omega\) = P(\omega) \land M(\omega, l(\omega)) \). Hence also $M_l$ is defined possibly for the "smaller" population $P'$ than $M$ is.

**Example 4** In practice, we frequently have to do with pre-labelled population. The statistics of illnesses based on poly-clinical data are based on a population pre-labelled by financial status (whether or not they are ready to visit a physician with less serious disease due to economical background), educational background (whether or not they estimate properly the seriousness of the disease, whether or not they care of symptoms) etc. Similarly in chemical analysis knowledge of substrates pre-labels the tests on composition of the product (not relevant measurements are a priori discarded) etc. ◇

**Example 5** To continue Citizen Coot example, we may believe that in good shops only moderate and high quality products are available, that is we assign to every shampoo $\omega$ the label $l(\omega) = \emptyset$ (we discard it from our register) if $\omega$ denies our belief that there are no suspicious nor dangerous products in a good shop, and $l(\omega) = \{H, M\}$ if it is moderate or high quality product in a good shop and $l(\omega) = \Xi$ to all the other products. After this rejection of shampoos not fitting our beliefs we have to do with (a bit smaller) sold-shampoos-population from Table 5:

Please notice the following changes: Suspicious and dangerous products encountered in good shops were totally dropped from the statistics (their
Table 2: Modified sold shampoos statistics

| Quality true for | Shop type B  | G  | B,G | Total |
|------------------|--------------|----|-----|-------|
| H                | 20 112       | 70 | 202 |
| M                | 80 127       | 110| 317 |
| S                | 65 0         | 0  | 65  |
| D                | 13 0         | 0  | 13  |
| H,S              | 15 0         | 14 | 29  |
| M,S              | 30 0         | 25 | 55  |
| H,D              | 8 0          | 3  | 11  |
| M,D              | 15 0         | 10 | 25  |
| total            | 246 239      | 232| 717 |
existence was not revealed to the public). Suspicious and dangerous products from shops with unclear classification (good/bad shops) were declared to come from bad shops. Products from good shops which obtained both the label high quality and dangerous were simply moved into the category high quality products (the bad smell was just concealed) etc. This is frequently the sense in which our beliefs have impact on our attitude towards real facts and we will see below that the Dempster-Shafer Theory reflects such a view of beliefs. ◇

Let us now define the following function:

Definition 9

\[ Bel^M_P(A) = \frac{\text{Prob}^P_O(O(\neg M(O, \Xi - A)))}{\text{Prob}^P_O(O)} \]

which is the probability that the test \( M \), while being true for \( A \), rejects every hypothesis of the form \( X = v_i \) for every \( v_i \) not in \( A \) for the population \( P \). We shall call this function "the belief exactly in the result of measurement".

Let us define also the function:

Definition 10

\[ Pl^M_P(A) = \frac{\text{Prob}^P_O(O(M(O, A)))}{\text{Prob}^P_O(O)} \]

which is the probability of the test \( M \) holding for \( A \) for the population \( P \). Let us refer to this function as the "Plausibility of taking any value from the set \( A \)".
Last not least be defined the function:

Definition 11

\[ m^M_P(A) = \underset{O}{\prod_{O}} \left( \bigwedge_{B;B=\{v_i\} \subseteq A} M(O,B) \land \bigwedge_{B;B=\{v_i\} \subseteq \Xi \setminus A} \neg M(O,B) \right) \]

which is the probability that all the tests for the singleton subsets of \( A \) are true and those outside of \( A \) are false for the population \( P \).

Let us illustrate the above concepts with Citizen Coot example:

Example 6 For the belief function for sold-bottles-population and the measurement function \( M^3 \), if we identify probability with relative frequency, we have the focal points given in the Table 4:

It is easily seen that:

**THEOREM 1** \( m^M_P \) is the mass Function in the sense of DS-Theory.

**PROOF:** We shall recall the definition and construction of the DNF (Disjunctive Normal Form). If, given an object \( O \) of a population \( P \) under the measurement method \( M \), we look at the expression

\[ expr(A) = \bigwedge_{B;B=\{v_i\} \subseteq A} M(O,B) \land \bigwedge_{B;B=\{v_i\} \subseteq \Xi \setminus A} \neg M(O,B) \]

for two different sets \( A_1, A_2 \subseteq \Xi \) then clearly \( expr(A_1) \land expr(A_2) \) is never true - the truth of the one excludes the truth of the other.
Table 3: Mass and Belief Function under Measurement Method $M^3$

| Set                      | $m_p^{M^3}$ | $Bel_p^{M^3}$ |
|--------------------------|-------------|---------------|
| {(H,B) }                 | 20/723      | 20/723        |
| {(H,G) }                 | 100/723     | 100/723       |
| {(H,B),(H,G) }           | 70/723      | 190/723       |
| {(M,B) }                 | 80/723      | 80/723        |
| {(M,G) }                 | 100/723     | 100/723       |
| {(M,B),(M,G) }           | 110/723     | 290/723       |
| {(S,B) }                 | 50/723      | 50/723        |
| {(S,G) }                 | 5/723       | 5/723         |
| {(S,B),(S,G) }           | 15/723      | 70/723        |
| {(D,B) }                 | 10/723      | 10/723        |
| {(D,G) }                 | 1/723       | 1/723         |
| {(D,B),(D,G) }           | 3/723       | 14/723        |
| {(H,B),(S,B) }           | 15/723      | 85/723        |
| {(H,G),(S,G) }           | 10/723      | 115/723       |
| {(H,B),(S,B),(H,G),(S,G) } | 14/723   | 299/723       |
| {(M,B),(S,B) }           | 30/723      | 160/723       |
| {(M,G),(S,G) }           | 20/723      | 125/723       |
| {(M,B),(S,B),(M,G),(S,G) } | 25/723   | 435/723       |
| {(H,B),(D,B) }           | 8/723       | 38/723        |
| {(H,G),(D,G) }           | 2/723       | 103/723       |
| {(H,B),(D,B),(H,G),(D,G) } | 3/723   | 217/723       |
| {(M,B),(D,B) }           | 15/723      | 105/723       |
| {(M,G),(D,G) }           | 7/723       | 108/723       |
| {(M,B),(D,B),(M,G),(D,G) } | 10/723  | 336/723       |
They represent mutually exclusive events in the sense of the probability theory. On the other hand:

$$\bigvee_{A;A \subseteq \Xi} expr(A) = TRUE$$

hence:

$$\left( \text{Prob}^{P(O)}_O \left( \bigvee_{A;A \subseteq \Xi} expr(A) \right) \right) = \left( \text{Prob}^{P(O)}_O TRUE \right) = 1$$

and due to mutual exclusiveness:

$$\sum_{A;A \subseteq \Xi} \text{Prob}^{P(O)}_O expr(A) = 1$$

which means:

$$\sum_{A;A \subseteq \Xi} m^M_M(A) = 1$$

Hence the first condition of Def.2 is satisfied. Due to the second condition of Def.6 we have

$$\left( \text{Prob}^{P(O)}_O expr(\emptyset) \right) = 1 - \left( \text{Prob}^{P(O)}_O (M(O, \Xi)) \right) =$$

$$= 1 - \left( \text{Prob}^{P(O)}_O TRUE \right) = 1 - 1 = 0$$

Hence

$$m^M_M(\emptyset) = 0$$

The last condition is satisfied due to the very nature of probability: Probability is never negative. So we can state that $m^M_M$ is really a Mass Function in the sense of the DS-Theory.

Q.e.d.
THEOREM 2 \( B_{LP}^{M} \) is a Belief Function in the sense of DS-Theory corresponding to the \( m_{LP}^{M} \).

PROOF: Let \( A \) be a non-empty set. By definition

\[
M(O, \Xi - A) = \bigvee_{C = \{v_i\} \subseteq \Xi - A} M(O, C)
\]

hence by de-Morgan-law:

\[
\neg M(O, \Xi - A) = \bigwedge_{C = \{v_i\} \subseteq \Xi - A} \neg M(O, C)
\]

On the other hand, \( \neg M(O, \Xi - A) \) implies \( M(O, A) \).

But:

\[
M(O, A) = \bigvee_{B \subseteq A} \left( \bigwedge_{C : C = \{v_i\} \subseteq B} M(O, C) \land \bigwedge_{C : C = \{v_i\} \subseteq A - B} \neg M(O, C) \right)
\]

So:

\[
\neg M(O, \Xi - A) = \neg M(O, \Xi - A) \land M(O, A) =
\]

\[
= \bigwedge_{C : C = \{v_i\} \subseteq \Xi - A} \neg M(O, C) \land M(O, A) =
\]

\[
= \bigwedge_{C : C = \{v_i\} \subseteq \Xi - A} \neg M(O, C) \land \left( \bigvee_{B \subseteq A} \left( \bigwedge_{C : C = \{v_i\} \subseteq B} M(O, C) \land \right.ight.
\]

\[
\left. \bigwedge_{C : C = \{v_i\} \subseteq A - B} \neg M(O, C) \right) =
\]

\[
= \bigvee_{B \subseteq A} \left( \bigwedge_{C : C = \{v_i\} \subseteq B} M(O, C) \land \bigwedge_{C : C = \{v_i\} \subseteq \Xi - A} \neg M(O, C) \right)
\]
\[
\bigwedge_{C; C=\{v_i\}\subseteq A-B} \neg M(O, C) = \\
= \bigvee_{B\subseteq A} \left( \bigwedge_{C; C=\{v_i\}\subseteq B} M(O, C) \land \bigwedge_{C; C=\{v_i\}\subseteq \Xi-B} \neg M(O, C) \right)
\]

Hence

\[
\neg M(O, \Xi - A) = \bigvee_{B\subseteq A} \text{expr}(B)
\]

and therefore:

\[
\left( \text{Prob}^P(O) \neg M(O, \Xi - A) \right) = \left( \text{Prob}^P(O) \bigvee_{B\subseteq A} \text{expr}(B) \right)
\]

\text{expr}(A) being defined as in the previous proof. As we have shown in the proof of the previous theorem, expressions under the probabilities of the right hand side are exclusive events, and therefore:

\[
\left( \text{Prob}^P(O) \neg M(O, \Xi - A) \right) = \sum_{B\subseteq A} \left( \text{Prob}^P(O) \text{expr}(B) \right)
\]

that is:

\[
\text{Bel}^M_P (A \in 2^\Xi) = \sum_{B\subseteq A} m^M_P (B)
\]

As the previous theorem shows that \(m^M_P\) is a DS Theory Mass Function, it suffices to show the above. Q.e.d. □

**THEOREM 3** \(P^M_P\) is a Plausibility Function in the sense of DS-Theory and it is the Plausibility Function corresponding to the \(\text{Bel}^M_P\).
**PROOF:** By definition:

\[ P_l^M(A) = \text{Prob}^{M(O)}_{O}(O, A) \]

hence

\[ P_l^M(A) = 1 - (\text{Prob}^{P(O)}_{O}(\neg M(O, A))) \]

But by definition:

\[ (\text{Prob}^{P(O)}_{O}(\neg M(O, A))) = (\text{Prob}^{P(O)}_{O}(\neg M(O, \Xi-(\Xi-A)))) = Bel^M_{P}(\Xi-A) \]

hence

\[ P_l^M(A) = 1 - Bel^M_{P}(\Xi-A) \]

Q.e.d. \(\square\)

Two important remarks must be made concerning this particular interpretation:

- Bel and Pl are both defined, contrary to many traditional approaches, as THE probabilities and NOT as lower or upper bounds to any probability.

- It is Pl(A) (and not Bel(A) as assumed traditionally) that expresses the probability of A, and Bel(A) refers to the probability of the complementary set \(A^C\).
Of course, a complementary measurement function is conceivable to revert the latter effect, but the intuition behind such a measurement needs some elaboration. We shall not discuss this issue in this paper.

Let us also define the following functions referred to as labelled Belief, labelled Plausibility and labelled Mass Functions respectively for the labeled population $P$:

**Definition 12** Let $P$ be a population and $l$ its labeling. Then

$$Bel^M_P(A) = \frac{\text{Prob}^{P(\omega)}_{\omega}(\neg M_l(\omega, \Xi - A))}{\text{Prob}^{P(\omega)}_{\omega}(M_l(\omega, A))}$$

$$Pl^M_P(A) = \frac{\text{Prob}^{P(\omega)}_{\omega}(M_l(\omega, A))}{\text{Prob}^{P(\omega)}_{\omega}(\neg M_l(\omega, \Xi - A))}$$

$$m^M_P(A) = \frac{\text{Prob}^{P(\omega)}_{\omega}(\bigwedge_{B: B = \{v_i\} \subseteq A} M_l(\omega, B) \land \bigwedge_{B: B = \{v_i\} \subseteq \Xi - A} \neg M_l(\omega, B))}{\text{Prob}^{P(\omega)}_{\omega}(\neg M_l(\omega, \Xi - A))}$$

Let us illustrate the above concepts with Citizen Coot example:

**Example 7** For the belief function for sold-bottles-population $P$ and the measurement function $M^3$, let us assume the following labeling:

$l(\omega) = \{(H,G),(H,B),(M,G),(M,B),(S,B),(D,B)\}$

for every $\omega \in \Omega$, which means that we are convinced that only high and moderate quality products are sold in good shops. For the population $P'$ under
this labeling, if we identify probability with relative frequency, we have the focal points given in the Table 4:

It is easily seen that:

**THEOREM 4** $m^M_{pl}$ is the mass Function in the sense of DS-Theory.

**PROOF:** To show this is suffices to show that the modified measurement method $M_l$ possesses the same properties as the measurement method $M$.

Let us consider a labeling $l$ and a population $P$ under this labeling.

Let $O$ be an object and $L$ its label under labeling $l$ ($L = l(O)$). Always $M_l(O, \Xi) = TRUE$ because by definition $M_l(O, \Xi) = M(O, \Xi \cap L) = M(O, L)$ and by definition of a labeled population for the object’s $O$ label $L$ $M(O, L) = TRUE$.

Second, the superset consistency is satisfied, because if $A \subseteq B$ then if $M_l(O, A) = TRUE$ then also $M_l(O, A) = M(O, A \cap L) = TRUE$, but because $A \cap L \subseteq B \cap L$ then also $M(O, B \cap L) = TRUE$, but by definition $M(O, B \cap L) = M_l(O, B) = TRUE$ and thus it was shown that $M_l(O, A) = TRUE$ implies $M_l(O, B) = TRUE$ for any superset $B$ of the set $A$.

Finally, also the subset consistency holds, because if $M(O, L \cap A) = TRUE$ then there exists a proper subset $B$ of $L \cap A$ such that $M(O, B) = TRUE$. But in this case $B = L \cap B$ so we can formally write $M(O, L \cap
Table 4: Mass and Belief Function under Modified Measurement Method $M_i^3$

| Set                              | $m_{P_i}^{M_i^3}$ | $Bel_{P_i}^{M_i^3}$ |
|----------------------------------|--------------------|---------------------|
| {\(H,B\)}                      | 20/717             | 20/717              |
| {\(H,G\)}                      | 112/717            | 112/717             |
| \(\{H,B\},\{H,G\}\)            | 70/717             | 202/717             |
| {\(M,B\)}                      | 80/717             | 80/717              |
| {\(M,G\)}                      | 127/717            | 127/717             |
| \(\{M,B\},\{M,G\}\)           | 110/717            | 317/717             |
| {\(S,B\)}                      | 65/717             | 65/717              |
| {\(D,B\)}                      | 13/717             | 13/717              |
| \(\{H,B\},\{S,B\}\)           | 15/717             | 100/717             |
| \(\{H,B\},\{S,B\},\{H,G\}\)  | 14/717             | 184/717             |
| \(\{M,B\},\{S,B\}\)           | 30/717             | 175/717             |
| \(\{M,B\},\{S,B\},\{M,G\}\)  | 25/717             | 387/717             |
| \(\{H,B\},\{D,B\}\)           | 8/717              | 41/717              |
| \(\{H,B\},\{D,B\},\{H,G\}\)  | 3/717              | 114/717             |
| \(\{M,B\},\{D,B\}\)           | 15/717             | 108/717             |
| \(\{M,B\},\{D,B\},\{M,G\}\)  | 10/717             | 228/717             |
$B) = TRUE$. Hence we see that $M_l(O, A) = TRUE$ implies the existence of a proper subset $B$ of the set $A$ such that $M_l(O, B) = TRUE$.

Hence considering analogies between definitions of $m^M_P$ and $m^M_{P_l}$ as well as between the respective Theorems we see immediately that this Theorem is valid.

Q.e.d.

**THEOREM 5** $Bel^M_{P_l}$ is a Belief Function in the sense of DS-Theory corresponding to the $m^M_{P_l}$.

**PROOF:** As $M_l$ is shown to be a DS Theory Mass Function and considering analogies between definitions of $Bel^M_P$ and $Bel^M_{P_l}$ as well as between the respective Theorems we see immediately that this Theorem is valid.

Q.e.d.

**THEOREM 6** $Pl^M_{P_l}$ is a Plausibility Function in the sense of DS-Theory and it is the Plausibility Function corresponding to the $Bel^M_P$.

**PROOF:** As $M_l$ is shown to be a DS Theory Mass Function and considering analogies between definitions of $Pl^M_P$ and $Pl^M_{P_l}$ as well as between the respective Theorems we see immediately that this Theorem is valid.

Q.e.d.

This does not complete the interpretation.

Let us now assume we run a "(re-)labelling process" on the (pre-labelled or unlabeled) population $P$. 

Definition 13 Let \( M \) be a measurement method, \( l \) be a labeling under this measurement method, and \( P \) be a population under this labeling (Note that the population may also be unlabeled). The (simple) labelling process on the population \( P \) is defined as a functional \( LP : \mathcal{2}^\Xi \times \Gamma \rightarrow \Gamma \), where \( \Gamma \) is the set of all possible labelings under \( M \), such that for the given labeling \( l \) and a given nonempty set of attribute values \( L \) (\( L \subseteq \Xi \)), it delivers a new labeling \( l' \) (\( l' = LP(L, l) \)) such that for every object \( \omega \in \Omega \):

1. if \( M_l(\omega, L) = \text{FALSE} \) then \( l'(\omega) = \emptyset \) (that is \( l' \) discards a labeled object \( (\omega, l(\omega)) \) if \( M_l(\omega, L) = \text{FALSE} \)

2. otherwise \( l'(\omega) = l(\omega) \cap L \) (that is \( l' \) labels the object with \( l(\omega) \cap L \) otherwise.

Remark: It is immediately obvious, that the population obtained as the sample fulfills the requirements of the definition of a labeled population.

The labeling process clearly induces from \( P \) another population \( P' \) (a population under the labeling \( l' \)) being a subset of \( P \) (hence perhaps ”smaller” than \( P \)) labelled a bit differently. Clearly if we retain the primary measurement method \( M \) then a new modified measurement method \( M_{l'} \) is induced by the new labeling. The (re-)labelling process may be imagined as the diagnosis process made by a physician. A patient ”labelled” with symptoms observed by himself (many symptoms remain hidden for the physician, like the body temperature curve over last few days) is relabeled by the physician when being ill (labelled with the diseases suspected) or rejected (declared
healthy due to symptoms not matching physician’s diagnostic procedure).

Let us define the following

**Definition 14** "labelling process function" $m^{LP:L} : 2^\Xi \to [0, 1]$ is defined as:

$$m^{LP:L}(L) = 1$$

$$\forall B : B \in 2^\Xi, B \neq L m^{LP:L}(B) = 0$$

It is immediately obvious that:

**THEOREM 7** $m^{LP:L}$ is a Mass Function in sense of DS-Theory.

Let $Bel^{LP:L}$ be the belief and $Pl^{LP:L}$ be the Plausibility corresponding to $m^{LP:L}$. Now let us pose the question: what is the relationship between $Bel_{P'}^{M'}$, $Bel_P^{M}$, and $Bel^{LP:L}$. It is easy to show that

**THEOREM 8** Let $M$ be a measurement function, $l$ a labeling, $P$ a population under this labeling. Let $L$ be a subset of $\Xi$. Let $LP$ be a labeling process and let $l' = LP(L, l)$. Let $P'$ be a population under the labeling $l'$. Then $Bel_{P'}^{M'}$ is a combination via DS Combination rule of $Bel_P^{M}$, and $Bel^{LP:L}$, that is:

$$Bel_{P'}^{M'} = Bel_P^{M} \oplus Bel^{LP:L}$$

**PROOF:** Let us consider a labeled object $(O_j, L_j)$ from the population $P$ (before re-labeling, that is $L_j = l(O_j)$) which passed the relabeling and
became \((O_j, L_j \cap L)\), that is \(L_j \cap L = l'(O_j)\). Let us define \(\text{expr}_B\) (before relabeling) and \(\text{expr}_A\) (after labeling) as:

\[
\text{expr}_B((O_j, L_j), A) = \bigwedge_{B; B = \{v_i\} \subseteq A} M_l(O, B) \land \\
\quad \longland_{B; B = \{v_i\} \subseteq \Xi - A} \neg M_l(O, B)
\]

and

\[
\text{expr}_A((O_j, L_j), A) = \bigwedge_{B; B = \{v_i\} \subseteq A} M_{l'}(O, B) \land \\
\quad \longland_{B; B = \{v_i\} \subseteq \Xi - A} \neg M_{l'}(O, B)
\]

Let \(\text{expr}_B((O_j, L_j), C) = \text{TRUE}\) and \(\text{expr}_A((O_j, L_j), D) = \text{TRUE}\) for some \(C\) and some \(D\). Obviously then for no other \(C\) and no other \(D\) the respective expressions are valid. It holds also that:

\[
\text{expr}_B((O_j, L_j), C) = \bigwedge_{B; B = \{v_i\} \subseteq C} M(O_j, L_j \cap B) \land \\
\quad \longland_{B; B = \{v_i\} \subseteq \Xi - D} \neg M((O_j, L_j \cap B)
\]

and

\[
\text{expr}_A((O_j, L_j), D) = \bigwedge_{B; B = \{v_i\} \subseteq D} M(O_j, L_j \cap L \cap B) \land \\
\quad \longland_{B; B = \{v_i\} \subseteq \Xi - D} \neg M(O_j, L_j \cap L \cap B)
\]

In order to get truth on the first expression, \(C\) must be a subset of \(L_j\), and for the second we need \(D\) to be a subset of \(L_j \cap L\). Furthermore, for a singleton \(F \subseteq \Xi\) either \(M(O_j, L_j \cap F) = \text{TRUE}\), \(M(O_j, L_j \cap L \cap F) = \text{TRUE}\), and then it belongs to \(C, L\) and \(D\), or \(M(O_j, L_j \cap F) = \)
\( TRUE, M(O_j, L_j \cap L \cap F) = FALSE, \) and then it belongs to \( C, \) but not to \( L \) and hence not to \( D, \) or \( M(O_j, L_j \cap F) = FALSE, \) so due to superset consistency also \( M(O_j, L_j \cap L \cap F) = FALSE, \) and then it belongs neither to \( C \) nor to \( D \) (though membership in \( L \) does not need to be excluded). So we can state that \( D = C \cap L, \)

So the absolute expected frequency of objects for which \( expr_A(D) \) holds, is given by:

\[
\sum_{C; D = C \cap L} \text{samplecardinality} \cdot m_{P}^{M_i}(C)
\]

that is:

\[
\sum_{C; D = C \cap L} \text{samplecardinality} \cdot m_{P}^{M_i}(C) \cdot m_{LP;L}(L)
\]

which can be easily re-expressed as:

\[
\sum_{C, G; D = C \cap G} \text{samplecardinality} \cdot m_{P}^{M_i}(C) \cdot m_{LP;L}(G)
\]

So generally:

\[
m_{P}^{M_i}(D) = c \cdot \sum_{C, G; D = C \cap G} m_{P}^{M_i}(C) \cdot m_{LP;L}(G)
\]

with \( c \) - normalizing constant.

Q.e.d. \( \square \)

**Example 8** To continue Citizen Coot example let us recall the function \( Bet_{P}^{M_i} \) from Example 6 which is one of an unlabeled population. Let us
define the label

\[ L = \{(H, G), (H, B), (M, G), (M, B), (S, B), (D, B)\} \]

as in Example 7. Let us define the labeling process function as

\[ m^{LP:L}(L) = 1 \]

\[ \forall B; B \in 2^\Xi, B \neq L m^{LP:L}(B) = 0 \]

. Let us consider the function \( Bel^{M}_{P'} \) from Example 7. It is easily seen that:

\[ Bel^{M}_{P'} = Bel^{M}_P \oplus Bel^{LP:L} \]

\[ \diamond \]

Let us try another experiment, with a more general (re-)labeling process. Instead of a single set of attribute values let us take a set of sets of attribute values \( L^1, L^2, ..., L^k \) (not necessarily disjoint) and assign to each one a probability \( m^{LP,L^1,L^2,...,L^k}(A) \) of selection.

**Definition 15** Let \( M \) be a measurement method, \( l \) be a labeling under this measurement method, and \( P \) be a population under this labeling (Note that the population may also be unlabeled). Let us take a set of (not necessarily disjoint) nonempty sets of attribute values \( \{L^1, L^2, ..., L^k\} \) and let us define the probability of selection as a function \( m^{LP,L^1,L^2,...,L^k}_P : 2^\Xi \rightarrow [0, 1] \) such that

\[ \sum_{A_i; A_i \subseteq \Xi} m^{LP,L^1,L^2,...,L^k}_P (A) = 1 \]
\[ \forall A; A \in \{L^1, L^2, ..., L^k\} m^{LP, L^1, L^2, ..., L^k}(A) > 0 \]

\[ \forall A; A \notin \{L^1, L^2, ..., L^k\} m^{LP, L^1, L^2, ..., L^k}(A) = 0 \]

The (general) labelling process on the population \( P \) is defined as a (randomized) functional \( LP : 2^\Xi \times \Delta \times \Gamma \rightarrow \Gamma \), where \( \Gamma \) is the set of all possible labelings under \( M \), and \( \Delta \) is a set of all possible probability of selection functions, such that for the given labeling \( l \) and a given set of (not necessarily disjoint) nonempty sets of attribute values \( \{L^1, L^2, ..., L^k\} \) and a given probability of selection \( m^{LP, L^1, L^2, ..., L^k} \) it delivers a new labeling \( l'' \) such that for every object \( \omega \in \Omega \):

1. a label \( L \), element of the set \( \{L^1, L^2, ..., L^k\} \) is sampled randomly according to the probability distribution \( m^{LP, L^1, L^2, ..., L^k} \); This sampling is done independently for each individual object,

2. if \( M_l(\omega, L) = FALSE \) then \( l''(\omega) = \emptyset \) (that is \( l'' \) discards an object \( (\omega, l(\omega)) \) if \( M_l(\omega, L) = FALSE \))

3. otherwise \( l''(\omega) = l(\omega) \cap L \) (that is \( l'' \) labels the object with \( l(\omega) \cap L \) otherwise.)

Again we obtain another ("smaller") population \( P'' \) under the labeling \( l'' \) labelled a bit differently. Also a new modified measurement method \( M'' \) is induced by the "re-labelled" population. Please notice, that \( l'' \) is not derived deterministically. Another run of the general (re-)labeling process LP may result in a different final labeling of the population and hence a different subpopulation under this new labeling.
Example 9 The (re-)labelling process may be imagined as the diagnosis process made by a team of physicians in a poly-clinic. A patient "labelled" with symptoms observed by himself is (a bit randomly) directed by the ward administration to one of the available internists each of them having a bit different educational background and/or a different experience in his profession, hence taking into consideration a bit different set of working hypotheses. The patient is relabeled by the given physician being ill (labelled with the diseases suspected) or rejected (declared healthy) according to the knowledge of this particular physician. The final ward statistics of illnesses does not take into account the fact that a physician may have had no knowledge of a particular disease unit and hence qualified the patient either healthy or ill of another, related disease unit. And it reflects the combined processes: of random allocations of patients to physicians and of belief worlds of the physicians rather then what the patients were actually suffering from. (We are actually satisfied with the fact that both views of ward statistics usually converge).

Clearly:

THEOREM 9 \( m^{LP,L_1,...,L_k} \) is a Mass Function in sense of DS-Theory.

Let \( Bel^{LP,L_1,...,L_k} \) be the belief and \( Pl^{LP,L_1,...,L_k} \) be the Plausibility corresponding to \( m^{LP,L_1,...,L_k} \). Now let us pose the question: what is the relationship between \( Bel^{M_i}_{P^+} \), \( Bel^{M_i}_P \), and \( Bel^{LP,L_1,...,L_k} \). It is easy to show that
THEOREM 10 Let $M$ be a measurement function, $l$ a labeling, $P$ a population under this labeling. Let $LP$ be a generalized labeling process and let $l''$ be the result of application of the $LP$ for the set of labels from the set $\{L^1, L^2, ..., L^k\}$ sampled randomly according to the probability distribution $m^{LP,L^1,L^2,...,L^k}$. Let $P''$ be a population under the labeling $l''$. Then the expected value over the set of all possible resultant labelings $l''$ (and hence populations $P''$) (or, more precisely, value vector) of $\text{Bel}_{P''}^M$ is a combination via DS Combination rule of $\text{Bel}_{P}^M$ and $\text{Bel}_{LP,L^1,...,L^k}$, that is:

$$E(\text{Bel}_{P''}^M) = \text{Bel}_{P}^M \oplus \text{Bel}_{LP,L^1,...,L^k}.$$

PROOF: By the same reasoning as in the proof of Theorem 8 we come to the conclusion that for the given label $L^i$ and the labeling $l''$ (instead of $l'$ the absolute expected frequency of objects for which $expr_A(D)$ holds, is given by:

$$\sum_{C:D=C \cap L^i} \text{samplecardinality} \cdot m_P^{M_i}(C) \cdot m^{LP,L^1,...,L^k}(L^i)$$

as the process of sampling the population runs independently of the sampling the set of labels of the labeling process.

But $expr_A(D)$ may hold for any $L^i$ such that $C \subseteq L^i$, hence in all the $expr_A(D)$ holds for as many objects as:

$$\sum_{i=1}^{i=k} \sum_{C:D=C \cap L^i} \text{samplecardinality} \cdot m_P^{M_i}(C) \cdot m^{LP,L^1,...,L^k}(L^i)$$
which can be easily re-expressed as:

\[
\sum_{C,G:D=C \cap G} \text{samplecardinality} \cdot m_P^{M_i}(C) \cdot m^{L_1;\ldots;L_k}(G)
\]

So generally:

\[
E(m_P^{M_i}(D)) = c \cdot \sum_{C,D=C \cap G} m_P^{M_i}(C) \cdot m^{L_1;\ldots;L_k}(G)
\]

with c - normalizing constant. Hence the claimed relationship really holds.

Q.e.d. 

**Example 10** The generalized labeling process and its consequences may be realized in our Citizen Coot example by randomly assigning the sold bottles for evaluation to two "experts", one of them - considering about 30% of the bottles - is running the full M test procedure, and the other - having to consider the remaining 70% of checked bottles - makes it easier for himself by making use of his belief in the labeling l of Example 7.

4.1 Summary of the New Interpretation

The following results have been established in this Section:

- concepts of measurement and modified measurement methods have been introduced
- a concept of labelled population has been developed
• it has been shown that a labelled population with the modified measurement method can be considered in terms of a Joint Belief Distribution in the sense of DS-Theory,

• the process of ”relabeling” of a labelled population has been defined and shown to be describable as a Belief Distribution.

• it has been shown that the relationship between the Belief Distributions of the resulting relabeled population, the basic population and the relabeling process can be expressed in terms of the Dempster-Rule-of-Independent-Evidence-Combination.

This last result can be considered as of particular practical importance. The interpretation schemata of DS Theory made by other authors suffered from one basic shortcoming: if we interpreted population data as well as evidence in terms of their DS schemes, and then combine the evidence with population data (understood as a Dempster type of conditioning) then the resulting belief function cannot be interpreted in terms of the population data scheme, with subsequent updating of evidence making thinks worse till even the weakest relation between the belief function and the (selected sub)population is lost.

In this paper we achieve a break-through: data have the same interpretation scheme after any number of evidential updating and hence the belief function can be verified against the data at any moment of DS evidential reasoning.
The above definition and properties of the generalized labeling process should be considered from a philosophical point of view. If we take one by one the objects of our domain, possibly labelled previously by an expert in the past, and assign a label independently of the actual value of the attribute of the object, then we cannot claim in any way that such a process may be attributed to the opinion of the expert. Opinions of two experts may be independent of one another, but they cannot be independent of the subject under consideration. This is the point of view with which most people would agree, and should the opinions of the experts not depend on the subject, then at least one of them may be considered as not expert.

This is exactly what we want to point at with our interpretation: the precise pinpointing at what kind of independence is assumed within the Dempster-Shafer theory is essential for its usability. Under our interpretation, the independence relies in trying to select a label for fitting to an object independently of whatever properties this object has (including its previous labeling). The distribution of labels for fitting is exactly identical from object to object. The point, where the dependence of object’s labeling on its properties comes to appearance, is when the measurement method states that the label does not fit. Then the object is discarded. From philosophical point of view it means exactly that we try to impose our philosophy of life onto the facts: cumbersome facts are neglected and ignored. We suspect that this is exactly the justification of the name ”belief function”. It expresses not what we see but what we would like to see.
Our suspicion is strongly supported by the quite recent statement of Smets that "authors (of multiple interpretations in terms of upper lower probability models, inner and outer measures, random sets, probabilities of provability, probabilities of necessity etc.) usually do not explain or justify the dynamic component, that is, how updating (conditioning) is to be handled (except in some cases by defining conditioning as a special case of combination. So I (that is Smets) feel that these partial comparisons are incomplete, especially as all these interpretations lead to different updating rules. " Our interpretation explains both the static and dynamic component of the DST, and does not lead to any other but to the Dempster Rule of Combination, hence may be acceptable from the rigorous point of view of Smets. As in the light of Smets’ paper [26] we have presented the only correct probabilistic interpretation of the DS theory so far, we feel to be authorized to claim that our philosophical assessment of the DST is the correct one.

We have seen from the proofs of the theorems of this paper, that our interpretation may be called a true one. The paper of Smets [26] permits us to claim that we have found the true interpretation.

5 Belief from Data

As the DS-belief function introduced in this paper is defined in terms of frequentist measures, there exists a direct possibility of calculating the belief
function from data.

It has to be assumed that we have a data set for which the measurements of type $M_l$ have been carried out for each singleton subset of the space of discourse $\Xi$. The results of these measurements may be available for example as a set-valued attribute associated with each object in such a way that the values actually appearing are those for which the singleton set tests were positive (i.e. TRUE). In this case if for an object the attribute $X$ has the value $X = A$ with $A \subseteq \Xi$ then this object increases the count for the DS-Mass Function $m(A)$ (and for no other $m$).

Whenever any statistical quantity is estimated from data, there exists some risk (uncertainty) about unseen examples. If we assume some significance levels, we can complete the estimation by taking the lower bounds as actual estimates of $m$’s and shifting the remaining burden (summing up to 1) onto the $m(\Xi)$ just taking for granted that doubtful cases may be considered as matching all the measurements.

6 Discussion

In the past, various interpretations have been sought for the Dempster-Shafer Bel-Functions. Two main steams of research were distinguished by Smets [26]: probability related approaches and probability discarding approaches
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(the former disguised, the latter welcome by Smets). Let us make some comparisons with our interpretation and its underlying philosophy.

6.1 Shafer and Smets

Shafer [22] and Smets [26] have made some strong statements in defense of the Dempster-Shafer theory against sharp criticism of this theory by its opponents as well as unfortunate users of the DST who wanted to attach it to the dirty reality (that is objectively given databases). Smets [26] and also initially Shafer [21] insisted on Bel not being connected to any empirical measure (frequency, probability etc.) considering the domain of DST applications as the one where ”we are ignorant of the existence of probabilities”, and not one with ”poorly known probabilities” ([26], p.324). The basic property of probability, which should be dropped in the DST axiomatization, should be the additivity of belief measures. Surely, it is easily possible to imagine situations where in the real life additivity is not granted: imagine we have had a cage with 3 pigs, we put into it 3 hungry lions two hours ago, how many animals are there now ? (3 + 3 < 6). Or ten years ago we left one young man and one young woman on an island in the middle of the atlantic ocean with food and weapons sufficing for 20 years. How many human beings are there now ? (1 + 1 > 2).

The trouble is, however, that the objects stored in databases of a computer behave usually (under normal operation) in an additive manner. Hence the DST is simply disqualified for any reasoning within human collected data on
real world, if we accept the philosophy of Smets and Shafer.

The question may be raised at this point, what else practically useful can be obtained from a computer reasoning on the basis of such a DST. If the DST models, as Smets and Shafer claim, human behaviour during evidential reasoning, then it would have to be demonstrated that humans indeed reason as DST. We take e.g. 1000 people who never heard of Dempster-Shafer theory, briefly explain the static component, provide them with two opinions of independent experts and expect of them to answers what are their final beliefs. Should their answers correspond to results of the DST (at least converge toward them), then the computer, if fed with our knowledge, would be capable to predict our conclusions on a given subject. However, to my knowledge, no experiment like this has ever been carried out. Under these circumstances the computer reasoning with DST would tell us what we have to think and not what we think. But I don’t suspect that anybody would be happy about a computer like this.

Hence, from the point of view of computer implementation the philosophy of Smets and Shafer is not acceptable. Compare also Discussion in [10] on the subject.

Both of them felt a bit uneasy about a total loss of reference to any scientific experiment checking practical applicability of the DST and suggested some probabilistic background for decision making (e.g. the pigeonistic probabilities of Smets), but I am afraid that by these interpretations they fall precisely into the same pitfalls they claimed to avoid by their highly abstract
As statistical properties of Shafer's [21] notion of evidence are concerned, sufficient criticism has been expressed by Halpern and Fagin ([10] in sections 4-5). Essentially it is pointed there at the fact that "the belief that represents the joint observation is equal to the combination is in general not equal to the combination of the belief functions representing the individual (independent) observations" (p.297). The other point raised there that though it is possible to capture properly in belief functions evidence in terms of probability of observations update functions (section 4 of [10]), it is not possible to do the same if we would like to capture evidence in terms of beliefs of observations update functions (section 5 of [10]).

As Smets probabilistic interpretations are concerned, let us "continue" the killer example of [26] on pages 330-331. "There are three potential killers, A, B, C. Each can use a gun or a knife. I shall select one of them, but you will not know how I select the killer. The killer selects his weapon by a random process with p(gun)=0.2 and p(knife)=0.8. Each of A, B, C has his own personal random device, the random devices are unrelated. ...... Suppose you are a Bayesian and you must express your "belief" that the killer will use a gun. The BF (belief function) solution gives Bel(gun) = 0.2 × 0.2 × 0.2 = 0.008. ..... Would you defend 0.2 ? But this applies only if I select a killer with a random device ...... But I never said I would use a random device; I might be a very hostile player and cheat whenever I can. .... . So you could interpret bel(x) as the probability that you are sure to win whatever Mother Nature
(however hostile) will do."

Yes, I will try to continue the hostile Mother Nature game here. For completeness I understand that \( Bel(knife) = 0.8^3 = 0.512 \) and \( Bel(\{gun, knife\}) = 1 \). But suppose there is another I, the chief of gangster science fiction physicians, making decisions independently of the chief I of the killers. The chief I of physicians knows of the planned murder and has three physicians X, Y, Z. Each can either rescue a killed man or let him die. I shall select one of them, but you will not know how I select the physician. The physician, in case of killing with a gun, selects his attitude by a random process with \( p(\text{rescue}|\text{gun}) = 0.2 \) and \( p(\text{let die}|\text{gun}) = 0.8 \) and he lets the person die otherwise. Each of X, Y, Z has his own personal random device, the random devices are unrelated. ...... Suppose you are a Bayesian and you must express your "belief" that the physician will rescue if the killer will use a gun. The BF (belief function) solution gives \( Bel_1(\text{rescue}|\text{gun}) = 0.2^3 = 0.008 \), \( Bel_1(\text{let die}|\text{gun}) = 0.8^3 = 0.512 \), \( Bel_1(\{\text{rescue, let die}\}|\text{gun}) = 1 \). Also \( Bel_2(\text{let die}|\text{knife}) = 1 \). As the scenarios for \( Bel_1 \) and \( Bel_2 \) are independent, let us combine them by the Dempster rule: \( Bel_{12} = Bel_1 \oplus Bel_2 \). We make use of the Smets' claim that "the de re and de dicto interpretations lead to the same results" ([26], p. 333), that is \( Bel(A|B) = Bel(\neg B \vee A) \). Hence

\[
m_{12}(\{(\text{gun, let die}), (\text{knife, let die}), (\text{knife, rescue})\}) = 0.480
\]

\[
m_{12}(\{(\text{gun, rescue}), (\text{knife, rescue})\}) = 0.008
\]
\[ m_{12}(\{(\text{knife, rescue}), (\text{gun, let die})\}) = 0.512 \]

Now let us combine \( Bel_{12} \) with the original \( Bel \). We obtain:

\[ m \oplus m_{12}(\{(\text{gun, let die})\}) = 0.008 \cdot 0.480 + 0.008 \cdot 0.512 = 0.008 \cdot 0.992 \]

But these two unfriendly chiefs of gangster organizations can be extremely unfriendly and in fact your chance of winning a bet may be as bad as 0.008 \cdot 0.512 for the event \( (\text{gun, let die}) \). Hence the "model" proposed by Smets for understanding beliefs functions as "unfriendly Mother Nature" is simply wrong. If the Reader finds the combination of \( Bel_2 \) with the other Bels a little tricky, then for justification He should refer to the paper of Smets and have a closer look at all the other examples.

Now returning to the philosophy of "subjectivity" of Bel measures: Even if a human being may possess his private view on a subject, it is only after we formalize the feeling of subjectiveness and hence ground it in the data that we can rely on any computer’s "opinion". We hope we have found one such formalization in this paper. The notion of labeling developed here substitutes one aspect of subjective human behaviour - if one has found one plausible explanation, one is too lazy to look for another one. So the process of labeling may express our personal attitudes, prejudices, sympathies etc. The interpretation drops deliberately the strive for maximal objectiveness aimed at by traditional statistical analysis. Hence we think this may be a promising path for further research going beyond the DS-Theory formalism.
Smets [26] views the probability theory as a formal mathematical apparatus and hence puts it on the same footing as his view of the DST. However, in our opinion, he ignores totally one important thing: The abstract concept of probability has its real world counterpart of relative frequency which tends to behave approximately like the theoretical probability in sufficiently many experimental settings as to make the abstract concept of probability useful for practical life. And a man-in-the-street will expect of the DST to possess also such a counterpart or otherwise the DST will be considered as another version of the theory of counting devils on a pin-head.

Let us also have a look at interpretations disguised by Shafer and Smets (i.e. all the mentioned below):

### 6.2 DST and Random Sets

The canonic random set interpretation [16] is one with a statistical process over set instantiations. The rule of combination assumes then that two such statistically independent processes are run and we are interested in their intersections. This approach is not sound as empty intersection is excluded and this will render any two processes statistically dependent. We overcome this difficulty assuming in a straightforward manner that we are "walking" from population to population applying the Rule of Combination. Classical DS theory in fact assumes such a walk implicitly or it drops in fact the
assumption that Bel() of the empty set is equal to 0. In this sense the random set approaches may be considered as sound as ours.

However, in many cases the applications of the model are insane. For example, to imitate the logical inference it is frequently assumed that we have a Bel-function describing the actual observed value of a predicate \( P(x) \), and a Bel-Function describing the implication "If \( P(x) \) then \( Q(x) \)" [13]. It is assumed further that the evidence on the validity of both Bel's has been collected independently and one applies the DS-rule of combination to calculate the Bel of the predicate \( Q(x) \). One has then to assume that there is a focal m of the following expression:  

\[
m(\{(P(x), Q(x)), (\neg P(x), Q(x)), (\neg P(x), \neg Q(x))\})
\]

which actually means that with non-zero probability at the same time \( P(x) \) and \( \neg P(x) \) hold for the same object as we will see in the following example:

Let \( Bel_1 \) represent our belief in the implication, with focal points:

\[
m_1(P(x) \rightarrow Q(x)) = 0.5, \quad m_1(\neg(P(x) \rightarrow Q(x))) = 0.5,
\]

Let further the independent opinion \( Bel_2 \) on \( P(x) \) be available in the form of focal points:

\[
m_2(P(x)) = 0.5, \quad m_2(\neg P(x)) = 0.5
\]

Let \( Bel_{12} = Bel_1 \oplus Bel_2 \) represent the combined opinions of both experts. The focal points of \( Bel_{12} \) are:

\[
m_{12}(\{(P(x), Q(x))\}) = 0.33, \quad m_{12}(\{(P(x), \neg Q(x))\}) = 0.33,
\]

\[
m_{12}(\{\neg(P(x), Q(x)), (\neg P(x), \neg Q(x))\}) = 0.33
\]
$m_{12}(\{(P(x), Q(x))\}) = 0.33$ makes us believe that there exist objects for which both $P(x)$ and $Q(x)$ holds. However, a sober (statistical) look at expert opinions suggests that all situations for which the implication $P(x) \rightarrow Q(x)$ holds, must result from falsity of $P(x)$, hence whenever $Q(x)$ holds then $\neg P(x)$ holds. These two facts combined mean that $P(x)$ and its negation have to hold simultaneously. This is actually absurdity overseen deliberately. The source of this misunderstanding is obvious: the lack of proper definition of what is and what is not independent. Our interpretation allows for sanitation of this situation. We are not telling that the predicate and its negation hold simultaneously. Instead we say that for one object we modify the measurement procedure (set a label) in such a way that it, applied for calculation of $P(x)$, yields true and at the same time for another object, with the same original properties we make another modification of measurement procedure (attach a label to it) so that measurement of $\neg P(x)$ yields also true, because possibly two different persons were enforcing their different beliefs onto different subsets of data.

Our approach is also superior to canonical random set approach in the following sense: The canonical approach requires knowledge of the complete random set realizations of two processes on an object to determine the combination of both processes. We, however, postpone the acquisition of knowledge of the precise instantiation of properties of the object by interleaving the concept of measurement and the concept of labeling process. This has a
close resemblance to practical processing whenever diagnosis for a patient is made. If a physician finds a set of hypotheses explaining the symptoms of a patient, he will usually not try to carry out other testing procedures than those related to the plausible hypotheses. He runs clearly at risk that there exists a different set of hypotheses also explaining the patients’s symptoms, and so a disease unit possibly present may not be detected on time, but usually the risk is sufficiently low to proceed in this way, and the cost savings may prove enormous.

6.3 Upper and Lower Probabilities

Still another approach was to handle Bel and Pl as lower and upper probabilities [4]. This approach is of limited use as not every set of lower and upper probabilities leads to Bel/Pl functions [12], hence establishing a unidirectional relationship between probability theory and the DS-theory. Under our interpretation, the Bel/Pl function pair may be considered as a kind of interval approximations to some ”intrinsic” probability distributions which, however, cannot be accessed by feasible measurements and are only of interest as a kind of qualitative explanation to the physical quantities really measured.

Therefore another approach was to handle them as lower/upper envelops to some probability density function realization [12], [8]. However, the DS
rule of combination of independent evidence failed.

6.4 Inner and Outer Measures

Still another approach was to handle Bels/Pl in probabilistic structures rather than in probabilistic spaces [7]. Here, DS-rule could be justified as one of the possible outcomes of independent combinations, but no stronger properties were available. This is due to the previously mentioned fact that exclusion of empty intersections renders actually most of conceivable processes dependent. Please notice that under our interpretation no such ambiguity occurs. This is because we not only drop empty intersecting objects but also relabel the remaining ones so that any probability calculated afterwards does not refer to the original population.

So it was tried to drop the DS-rule altogether in the probabilistic structures, but then it was not possible to find a meaningful rule for multistage reasoning [10]. This is a very important negative outcome. As the Dempster-Shafer-Theory is sound in this respect and possesses many useful properties (as mentioned in the Introduction), it should be sought for an interpretation meeting the axiomatic system of DS Theory rather than tried to violate its fundamentals. Hence we consider our interpretation as a promising one for which decomposition of the joint distribution paralleling the results for probability distributions may be found based on the data.
6.5 Rough Set Approach

An interesting alternative interpretation of the Dempster-Shafer Theory was found within the framework of the rough set theory [24], [9]. Essentially the rough set theory searches for approximation of the value of a decision attribute by some other (explaining) attributes. It usually happens that those attributes are capable only of providing a lower and upper approximation to the value of the decision attribute (that is the set of vectors of explaining attributes supporting only this value of the decision variable, and the set of vectors of explaining attributes supporting also this value of the decision variable resp.- for details see texts of Skowron [24] and Grzymała-Busse [9]).

The Dempster Rule of combination is interpreted by Skowron [25] as combination of opinions of independent experts, who possibly look at different sets of explanation attributes and hence may propose different explanations.

The difference between our approach and the one based on rough sets lies first of all in the ideological background: We assume that the ”decision attribute” is set-valued whereas the rough-set approach assumes it to be single-valued. This could have been overcome by some tricks which will not be explained in detail here. But the combination step is here essential: If we assume that the data sets for forming knowledge of these two experts are exhaustive, then it can never occur that these opinions are contradictory. But the DST rule of combination uses the normalization factor for dealing with cases like this. Also the opinions of experts may have only the form of a simple (that is deterministic) support function. Hence, rough-set in-
interpretation implies axioms not actually present in the DST. Hence rough set interpretation is on the one hand restrictive, and on the other hand not fully conforming to the general DST. From our point of view the DST would change the values of decision variables rather then recover them from expert opinions.

Here, we come again at the problem of viewing the independence of experts. The DST assumes some strange kind of independence within the data: the proportionality of the distribution of masses of sets of values among intersecting subsets weight by their masses in the other expert opinion. Particularly unhappy is the fact for the rough set theory, that given a value of the decision variable, the respective indicating vectors of explaining variables values must be proportionally distributed among the experts not only for this decision attribute value, but also for all the other decision attribute values that ever belong to the same focal point. Hence applicability of the rough set approach is hard to justify by a simple(, ”usual” as Shafer wants) statistical test. On the other hand, statistical independence required for Dempster rule application within our approach is easily checked.

To demonstrate the problem of rough set theory with recombination of opinions of independent experts let us consider an example of two experts having the combined explanatory attributes $E_1$ (for expert 1) and $E_2$ (for expert 2) both trying to guess the decision attribute $D$. Let us assume that $D$ takes one of two values: $d_1, d_2$, $E_1$ takes one of three values $e_{11}, e_{12}, e_{13}$, $E_2$ takes one of three values $e_{21}, e_{22}, e_{23}$. Furthermore let us assume that the
rough set analysis of an exhaustive set of possible cases shows that the value $e_{11}$ of the attribute $E_1$ indicates the value $d_1$ of the decision attribute $D$, $e_{12}$ indicates $d_2$, $e_{13}$ indicates the set $\{d_1, d_2\}$, Also let us assume that the rough set analysis of an exhaustive set of possible cases shows that the value $e_{21}$ of the attribute $E_2$ indicates the value $d_1$ of the decision attribute $D$, $e_{22}$ indicates $d_2$, $e_{32}$ indicates the set $\{d_1, d_2\}$, From the point of view of bayesian analysis four cases of causal influence may be distinguished (arrows indicate the direction of dependence).

$$E_1 \rightarrow D \rightarrow E_2$$
$$E_1 \leftarrow D \leftarrow E_2$$
$$E_1 \leftarrow D \rightarrow E_2$$
$$E_1 \rightarrow D \leftarrow E_2$$

From the point of view of bayesian analysis, in the last case attributes $E_1$ and $E_2$ have to be unconditionally independent, in the remaining cases: $E_1$ and $E_2$ have to be independent conditioned on $D$. Let us consider first unconditional independence of $E_1$ and $E_2$. Then we have that:

$$\left( \mathbb{P}_{\omega}^{P(\omega)} E_1(\omega) = e_{11} \land E_2(\omega) = e_{22} \right) =$$

$$= \left( \mathbb{P}_{\omega}^{P(\omega)} E_1(\omega) = e_{11} \right) \cdot \left( \mathbb{P}_{\omega}^{P(\omega)} E_2(\omega) = e_{22} \right) > 0$$

However, it is impossible that $\left( \mathbb{P}_{\omega}^{P(\omega)} E_1(\omega) = e_{11} \land E_2(\omega) = e_{22} \right) > 0$ because we have to do with experts who may provide us possibly with information not specific enough, but will never provide us with contradictory
information. We conclude that unconditional independence of experts is impossible.

Let us turn to independence of $E_1$ and $E_2$ if conditioned on $D$. We introduce the following denotation:

\[
\begin{align*}
\rho_1 &= \Prob_{\omega}^{P(\omega)} D(\omega) = d_1 \\
\rho_2 &= \Prob_{\omega}^{P(\omega)} D(\omega) = d_2 \\
\epsilon'_1 &= \Prob_{\omega}^{(D(\omega)=d_1) \land P(\omega)} E_1(\omega) = e_{11} \\
\epsilon'_3 &= \Prob_{\omega}^{(D(\omega)=d_1) \land P(\omega)} E_1(\omega) = e_{13} \\
f'_1 &= \Prob_{\omega}^{(D(\omega)=d_1) \land P(\omega)} E_2(\omega) = e_{21} \\
f'_3 &= \Prob_{\omega}^{(D(\omega)=d_1) \land P(\omega)} E_2(\omega) = e_{23} \\
e''_2 &= \Prob_{\omega}^{(D(\omega)=d_2) \land P(\omega)} E_1(\omega) = e_{12} \\
e''_3 &= \Prob_{\omega}^{(D(\omega)=d_2) \land P(\omega)} E_1(\omega) = e_{13} \\
f''_2 &= \Prob_{\omega}^{(D(\omega)=d_2) \land P(\omega)} E_2(\omega) = e_{22} \\
f''_3 &= \Prob_{\omega}^{(D(\omega)=d_2) \land P(\omega)} E_2(\omega) = e_{23}
\end{align*}
\]

Let $Bel_1$ and $m_1$ be the belief function and the mass function representing the knowledge of the first expert, let $Bel_2$ and $m_2$ be the belief function and the mass function representing the knowledge of the second expert. Let
Bel_{12} and \( m_{12} \) be the belief function and the mass function representing the knowledge contained in the combined usage of attributes \( E_1, E_2 \) if used for prediction of \( D \) - on the grounds of the rough set theory. It can be easily checked that:

\[
m_{1}(\{d_1\}) = e'_1 \cdot p_1, \quad m_{1}(\{d_2\}) = e''_2 \cdot p_2, \quad m_{1}(\{d_1, d_2\}) = e'_{3} \cdot p_1 + e'''_{3} \cdot p_2
\]

\[
m_{2}(\{d_1\}) = f'_1 \cdot p_1, \quad m_{2}(\{d_2\}) = f''_2 \cdot p_2, \quad m_{2}(\{d_1, d_2\}) = f'_{3} \cdot p_1 + f'''_{3} \cdot p_2
\]

and if we assume the conditional independence of \( E_1 \) and \( E_2 \) conditioned on \( D \), then we obtain:

\[
m_{12}(\{d_1\}) = e'_1 \cdot f'_1 \cdot p_1 + e'_1 \cdot f'_3 \cdot p_1 + e'_{3} \cdot f'_1 \cdot p_1
\]

\[
m_{12}(\{d_2\}) = e''_2 \cdot f''_2 \cdot p_2 + e''_2 \cdot f''_3 \cdot p_2 + e'''_{2} \cdot f''_2 \cdot p_2
\]

\[
m_{12}(\{d_1, d_2\}) = e'_{3} \cdot f'_3 \cdot p_1 + e'''_{3} \cdot f''_3 \cdot p_2
\]

However, the Dempster rule of combination would result in (\( c \) - normalization constant):

\[
m_{1} \oplus m_{2}(\{d_1\}) = c \cdot (e'_1 \cdot f'_1 \cdot p_1^2 + e'_1 \cdot f'_3 \cdot p_1^2 + e'_{3} \cdot f'_3 \cdot p_1 \cdot p_2 + e'_{3} \cdot f'_1 \cdot p_1^2 + e''_{3} \cdot f''_1 \cdot p_1 \cdot p_2)
\]

\[
m_{1} \oplus m_{2}(\{d_2\}) = c \cdot (e''_2 \cdot f''_2 \cdot p_2^2 + e''_2 \cdot f''_3 \cdot p_1 \cdot p_2 + e''_3 \cdot f''_3 \cdot p_2^2 + e''_3 \cdot f''_2 \cdot p_1 \cdot p_2 + e'''_{3} \cdot f'''_2 \cdot p_2^2)
\]

\[
m_{1} \oplus m_{2}(\{d_1, d_2\}) = c \cdot e'_{3} \cdot f'_3 \cdot p_1^2 + e'''_{3} \cdot f''_3 \cdot p_2^2 + e'_{3} \cdot f''_3 \cdot p_1 \cdot p_2 + e''_3 \cdot f'''_3 \cdot p_1 \cdot p_2)
\]

Obviously, \( Bel_{12} \) and \( Bel_{1} \oplus Bel_{2} \) are not identical in general. We conclude that conditional independence of experts is also impossible. Hence no usual
statistical independence assumption is valid for the rough set interpretation of the DST. This fact points at where the difference between rough set interpretation and our interpretation lies in: in our interpretation, traditional statistical independence is incorporated into the Dempster’s scheme of combination (labelling process).

By the way, lack of correspondence between statistical independence and Dempster rule of combination is characteristic not only of the rough set interpretation, but also of most of the other ones. The Reader should read carefully clumsy statements of Shafer about DST and statistical independence in [22].

6.6 General Remarks

The Dempster-Shafer Theory exists already over two decades. Though it was claimed to reflect various aspects of human reasoning, it has not been widely used in expert systems until recently due to the high computational complexity. Three years ago, however, an important paper of Shenoy and Shafer [23] has been published, along papers of other authors similar in spirit, which meant a break-through for application of both bayesian and Dempster-Shafer theories in reasoning systems, because it demonstrated that if joint (bayesian or DS) belief distribution can be decomposed in form of a belief network than it can be both represented in a compact manner and marginalized efficiently by local computations.

This fact makes them suitable as alternative fundamentals for represen-
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Reasoning in bayesian belief networks has been subject of intense research work also earlier [20], [23], [15], [17]. There exist methods of imposing various logical constraints on the probability density function and of calculating marginals not only of single variables but of complicated logical expressions over elementary statements of the type \( X = x \) \((x \text{ belonging to the domain of the variable } X)\) [17]. There exist also methods determining the decomposition of a joint probability distribution given by a sample into a bayesian belief network [3], [18], [1], [27].

It is also known that formally probability distributions can be treated as special cases of Dempster-Shafer belief distributions (with singleton focal points) [10].

However, for application of DS Belief-Functions for representation of uncertainty in expert system knowledge bases there exist several severe obstacles. The main one is the missing frequentist interpretation of the DS-Belief function and hence neither a comparison of the deduction results with experimental data nor any quantitative nor even qualitative conclusions can be drawn from results of deduction in Dempster-Shafer-theory based expert systems [13].

Numerous attempts to find a frequentist interpretation have been reported \(\text{e.g.} \ [7], [8], [9], [10], [12], [22], [24])\). But, as Smets [26] states, they failed either trying to incorporate Dempster rule or when explaining the nature of probability interval approximation. The Dempster-Shafer The-
ory experienced therefore sharp criticism from several authors in the past [17], [10]. It is suggested in those critical papers that the claim of DST to represent uncertainty stemming from ignorance is not valid. Hence alternative rules of combination of evidence have been proposed. However, these rules fail to fulfill Shenoy/Shafer axioms of local computation [23] and hence are not tractable in practice. These failures of those authors meant to us that one shall nonetheless try to find a meaningful frequentist interpretation of DST compatible with Dempster rule of combination.

We have carefully studied several of these approaches and are convinced that the key for many of those failures (beside those mentioned by Halpern in [10]) was: (1) treating the Bel-Pl pair as an interval approximation and (2) viewing combination of evidence as a process of approaching a point estimation. In this paper we claim that the most reasonable treatment of Bel’s Pl’s is to consider them to be POINT ESTIMATES of probability distribution over set-valued attributes (rather then Interval estimates of probability distribution over single valued attributes). Of course, we claim also that Bel-Pl estimates by an interval some probability density function but in our interpretation that ”intrinsic” probability density function is of little interest for the user. The combination of evidence represents in our interpretation manipulation of data by imposing on them our prejudices (rather then striving for extraction of true values).

Under these assumptions a frequentionistically meaningful interpretation to the Bel’s can be constructed, which remains consistent under combination
of joint distribution with "evidence", giving concrete quantitative meaning to results of expert system reasoning. Within this interpretation we were able to prove the correctness of Dempster-Shafer rule. This means that this frequentist interpretation is consistent with the DS-Theory to the largest extent ever achieved.

7 Conclusions

• According to Smets [26] there has existed no proper frequentist interpretation of the Dempster-Shafer theory of evidence so far.

• In this paper a novel frequentist interpretation of the Dempster-Shafer Theory has been found allowing for close correspondence between Belief and Plausibility functions and the real data.

• This interpretation fits completely into the framework of Bel/Pl definitions and into the Dempster rule of combination of independent evidence relating for the first time in DST history this rule to plain statistical independence just overcoming difficulties of many alternative interpretations of the Dempster-Shafer-Theory. Hence this interpretation dismisses the claim of Smets [26] that such an interpretation cannot exist.

• It is distinguished by the fact of postponing the moment of measuring object properties behind combination of evidence leading even to
dropping some costly measurements altogether.

- The interpretation allows for subjective treatment of Bel’s and Pl’s as some approximations to unknown probability distribution of an intrinsic, but not accessible, attribute.

- The introduced concept of labeled population may to some extent represent subjectivity in viewing probabilities.

- This interpretation questions the common usage of the DST as a mean to represent and to reason with uncertainty stemming from ignorance. This view has been already shaken by works of Pearl [17] and Halpern and Fagin [10]. What our interpretation states clearly is that the DST should be viewed as a way to express unwillingness to accept objective facts rather than as a mean to express ignorance about them. Hence it should be called a theory of prejudices rather than a theory of evidence.

Finally, I feel obliged to apologize and to say that all critical remarks towards interpretations of DST elaborated by other authors result from deviations of those interpretations from the formalism of the DST. I do not consider, however, a deviation from DST as a crime, because modifications of DST may and possibly have a greater practical importance than the original theory. The purpose of this paper was to shed a bit more light onto the intrinsic nature of pure DST and not to call for orthodox attitudes towards DST.
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