On the history of Levi-Civita’s parallel transport

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Abstract

In this historical note, we wish to highlight the crucial conceptual role played by the principle of virtual work of analytical mechanics, in working out the fundamental notion of parallel transport on a Riemannian manifold, which opened the way to the theory of connections and gauge theories. Moreover, after a detailed historical-technical reconstruction of the original Levi-Civita’s argument, a further historiographical deepening and a related critical discussion of the question, are pursued.

1 Introduction

The role of the principle of virtual works of analytical mechanics, in formulating Levi-Civita’s parallel transport of Riemannian geometry, has already been emphasized in [29], to which we refer for the first prolegomena to the question as well as for a wider and comprehensive historical contextualization of it. In this note, which falls into the intersection area among history of mechanics, history of differential geometry and history of theoretical physics, we would like to deepen this aspect regarding the genesis of one of the most important ideas of modern mathematics and its applications to theoretical physics (above all, field theory), enlarging the framework of investigation with the introduction and critical discussion of further, new historiographical elements of the question, after having reconstructed technically the original argument of Levi-Civita. In what follows, therefore, we briefly recall, as a minimal historical introduction, some crucial biographical moments of the Levi-Civita, which will help us to better lay out the question here treated.

To be concise, Tullio Levi-Civita, besides to have been one of the most important mathematicians of 20th century with wide interests in physics and mathematical physics (cf. [37]), was, in particular, a clever master in applying methods of absolute differential calculus to other subject-matters which go beyond pure mathematics, above all mechanics and mathematical physics, and vice versa, as emerges, for instance, from a detailed historical enquiry of the seminal paper [35] – whose first results have been exposed in [29] – which was conceived, by the Levi-Civita, in a well-determined moment of his academic and scientific career, that we rapidly sketch out as follows.

Soon after he graduated at Padua in 1892 with Gregorio Ricci-Curbastro, Levi-Civita was appointed, in 1895, as internal professor to the high school attached to the Faculty of Science of Pavia University. In this period, Levi-Civita started his academic career and research activity with some notable studies in analytical mechanics and higher mechanics, where, for the first time, he applied the new methods of absolute differential calculus to approach and solve certain

1 Keywords: Riemannian manifold, Levi-Civita’s parallel transport, virtual work law, linear connection.

2 2010 MSC: 01-08, 01A60, 01A85, 53-03, 53B05, 53B20, 70-03, 70F20.

3 2008 PACS: 02.40.Ky, 02.40.Yy, 01.65.+g, 45.20.Jj, 11.15.-q.
open problems of mathematical physics. In 1896, Levi-Civita moved to Padua, to the chair of rational mechanics (left uncovered by death of Ernesto Padova), who held for more than two decades. In this place, comforted by the quite life of his own home town, he continued with research in mathematical physics, with particular attention to higher mechanics and its applications. Just in this period, Levi-Civita conceived his famous work \[35\], here in historical quest.

The Levi-Civita’s research in Padua period, was fully devoted to mathematical physics arguments, with some occasional work on pure mathematics but ever motivated to answering to some technical issues inherent formal methodologies of treatment of mathematical physics questions and problems. However, this line of research was yet motivated by the cold reception of the new methods of absolute differential calculus by international mathematical community, a formal system conceived just by Ricci-Curbastro and Levi-Civita. In fact, last Levi-Civita’s work published on this argument, was the well-known report on absolute differential calculus, required by Felix Klein for the *Mathematische Annalen*, at the turn of 19th century (cf. \[48\]), and written with his schoolmaster Ricci-Curbastro.

It was only after absolute differential calculus turned out to be at the formal basis of general relativity theory that Levi-Civita willingly turned back to this argument, around the late of 1910s, starting just with the seminal paper\[35\], with which he opened as well new perspectives in the mathematical treatment of relativistic theories after having had a crucial correspondence just with Albert Einstein, at the beginnings of 1915, on some crucial, yet problematic, formal aspects of his well-known gravitational field equations, which Levi-Civita definitively clarified. In 1919, Levi-Civita moved to Roma, first to the chair of higher calculus, whose lecture notes, centred on absolute differential calculus, were recollected in \[10\], then to the chair of rational mechanics. The decennial experience in teaching the latter subject-matter, was concretized in the celebrated treatise \[38\], written in collaboration with Ugo Amaldi, whose first edition dates back to 1923.

In regard to the seminal paper\[35\], our main intention, here, is above all to show how a basic formal tool of analytical mechanics, like the principle of virtual works in the Lagrangian formulation, was ingeniously used first to sketch, then to develop formally, the geometric idea of parallelism on a Riemannian manifold, as originally conceived by Levi-Civita. With respect to what has been said in \[29\], in this note we would like technically to deepen just this founding moment of the pathway followed by Levi-Civita in developing his celebrated and fruitful idea, as well as to add further historiographical data with a related critical discussion and examination. What will emerge from that, will be the extreme simplicity and elegant formal style, together the high intuitive charge and clarifying powerfulness, of this typical method of study and research performed by Levi-Civita in almost all his works.

Nevertheless, just in regard to \[35\], in most of the related mathematical literature\[6\] we often find the statement according to which Levi-Civita’s parallel transport was motivated by the attempt to give a geometrical interpretation to the so-called covariant derivative of absolute differential calculus, a tool formally explained in \[48\] (on the basis of previous ideas of Elwin B. Christoffel), which was indeed provided later not by Levi-Civita, but rather by others (among whom are Hermann Weyl and Gerhard Hessenberg). Instead, if we read carefully his paper \[35\], one sees that Levi-Civita wasn’t never motivated by this end, but rather by the will to simplify the computation of the curvature of a Riemannian manifold, re-examining the

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4In the follows, when we quote \[35\], we always refer to the version contained in the Volume 4 of the collected papers of Levi-Civita \[37\].

5Strangely enough, this work seems have not been translated in any other foreign language.

6Of which we quote \[55\], p. 4; \[32\], Vol. I, p. 287. \[44\], p. 19. In particular, we also cite \[51\], App. A, Sect. 1, p. 358, which even says textually that: “Perhaps the motivating reason for Levi-Civita’s search for what is now called the Levi-Civita connection was the need to find an analog of the vector gradient of Euclidean geometry [...] in the context of Riemannian geometry. In fact, Levi-Civita called his discovery the absolute derivative”. 
covariant behavior of Riemann’s symbols and their occurrence in relativistic questions (cf. [35], Introduction).

On the other hand, even if Levi-Civita was aimed by the attempt to enquire on covariant behavior of those geometric entities involved in the computation of Riemannian curvature, mainly Riemann’s symbols, the covariant derivative was not yet the predominant, central formal tool of absolute differential calculus involved in his investigations which, as we shall see later, were merely carried out within a geometrical framework set up according to an analytical mechanics standpoint, as an intuitive guide for the next analytical calculation of curvature, independently of the covariant derivative algorithm. This conclusion may be also inferred by what Levi-Civita himself said at page 7 of the preface to [40] (to which we refer for more details), namely, that the close relationships between covariant derivative and parallelism were explicited later by others scholars, not by him. But, just to consider the numerous, new results and improvements coming from his notion of parallelism, Levi-Civita held two courses at Roma University in the years 1922-23, whose lecture notes were collected and drawn up by Enrico Persico in [40].

As Levi-Civita himself said in the Introduction to [35], initially he was motivated by the intention to simplify the methods of calculus of the curvature of a generic Riemannian manifold, involving Riemann’s symbols[7]. To this end, a preliminary geometric sight of this question, allowed him to work out an idea of parallelism on a Riemannian manifold which turned out to be a needful notion for the computation of the curvature of this manifold according to the usual formal methods of the time, basically centred on the possible vectorial circuituations along suitable infinitesimal closed contours (among which are the so-called geodesic parallelogrammoids) lying on the given manifold, hence considering the related commutation properties of certain geometrical parameters or entities closely related with the circuitating vectors[8].

In pursuing this research program for the computation of the curvature of a Riemannian manifold (which will led to the so-called Levi-Civita’s geometric characterization of Riemannian curvature[9]), Levi-Civita devoted the first fourteen sections of his memoir – i.e., most of the whole paper – just to introduce and explain the new concept of parallelism upon an arbitrary Riemannian manifold $V^n$ with dimension $n \geq 2$, embedded in some ordinary Euclidean space $\mathbb{R}^N$ (who he denotes with $S^n$). Therefore, he applied this new geometric idea for simplifying the computation of Riemannian curvature, hence exposed the achieved results in the sections 15 and 16 of [35]. Levi-Civita himself said textually, in the Introduction to [35], that such a paper originally sprung out only for this end, but that accordingly it was then enlarged consistently to provide as well the related geometrical interpretation, not to covariant derivative, but to this new geometric trick for calculating curvature. Likewise, in the contents of [40], parallelism is exposed (at Ch. V) before covariant derivative (at Ch. VI), not vice versa.

This latter geometrical interpretation, therefore, as again Levi-Civita pointed out in the Introduction to [35], was referred to the notion of curvature of a Riemannian manifold, as originally conceived, although quite implicitly, by Riemann in his celebrated 1854 inaugural lecture Über die Hypothesen, welche der Geometrie zu Grunde liegen, and not to other, like...
covariant derivative, this latter having been very far from the manifest intentions that really pushed Levi-Civita to drawing up his work. He, more times, in his memoir of 1917, made reference to this Riemann seminal paper, as well as to the related Heinrich Weber Commentatio mathematica (Part II) in which further formal remarks and details to Riemann’s work of 1854, are exposed. So, the main aim which led Levi-Civita to draw up his memoir of 1917, was essentially to deepen and clarify pioneering Riemann’s ideas on the curvature of metric manifolds, as sketched in the famous inaugural lecture of 1854. Levi-Civita succeeded in this difficult task, in a very elegant and intuitive fashion, by means of an analogical-conceptual transcription of the initial pure geometry question into a suitable analytical mechanics framework.

Levi-Civita, afterwards, devoted the last two sections 17 and 18 of [35], just to explicitate what he deemed were implicitly present in the original Riemann’s memoir, just through those suitable geometrical tools, primarily his notion of parallelism, which showed to be indispensable to accomplish this task, but not expressly considered by Riemann. In these sections of [35], Levi-Civita exactly used his notion of parallelism on a generic Riemannian manifold, as applied in the previous sections 15 and 16 of [35], to clarify and determine easier the covariant behavior of Riemann’s symbols as well as the curvature of a Riemannian manifold with a generic metric, by means of that usual method making reference to infinitesimal geodesic parallelogrammoids and related commutation properties of the first-order infinitesimal displacement operators $d$ and $\delta$.

Anyway, what we wish to highlight in the present note, is that, by examining technically the main passages of his construction of the notion of parallelism, one becomes even more aware that Levi-Civita’s strong training in analytical mechanics surely had a primary, central role in the related conceptual developments of this idea, influencing so deeply the related logical reasoning fashion in such a way that intuition and insight earned much; but, at the same time, all that guided Levi-Civita in his analytical and rigorous treatment of the question, easily reaching to the final local differential equations of parallelism. This may be also due to the fact that geometry and mechanics then had (and, in a certain sense, still have) evanescent boundaries, wide intertwine zones and a common language with very much similar traits and meaning analogies, with mutual benefit. In what follows, we shall try to reconstruct, explain and justify, in a detailed manner, the crucial points of the original Levi-Civita’s proceeding in which are masterly involved concepts and methods of analytical mechanics together differential geometry arguments.

2 A brief recall on the principle of virtual works

In this section, we spend only a very few words on the principle of virtual works, referring to [29] for a deeper historical discussion of it. To be closest to Levi-Civita, as well as for a more methodological coherence with the historical issue here treated, we shall mainly follow his celebrated treatise on rational mechanics [38] to briefly enunciate the principle of virtual works in its formal essence, to turn out enough for our aims.

The fundamental dynamical equations of a generic system of $N(\geq 1)$ material points of mass $m_i$, subjected to preassigned active forces $\vec{F}_i$ and constraint reactions $\vec{R}_i$, may be written as follows

\[(\vec{F}_i - m_i \vec{a}_i) + \vec{R}_i = \vec{0}, \quad i = 1, 2, ..., N,\]  

(1)

where $-m_i \vec{a}_i$ are said to be inertial forces. These equations are the formal statement of the so-called D’Alembert’s principle, which allows to reduce formally any dynamical problem to a static one. This principle, which may be enunciated in many, yet equivalent, fashions, is,
together Lagrange’s principle of virtual works (see later), one of the key principles of analytical mechanics.

The *principle of virtual works*, in its original Lagrangian formulation, states that the work of the reactions \( \vec{R}_i \), due to smooth constraints, is non-negative for any irreversible virtual displacement, while is zero for any reversible virtual displacement. In the case of bilateral, smooth constraints, as expressed, for example, by \( r \) equalities providing equations of a smooth manifold of codimension \( r \), all compatible virtual displacements are reversible, so we have that the virtual work of constraint reactions is zero, whence the principle of virtual works reads

\[
\delta L = \sum_i \vec{R}_i \cdot \delta \vec{P}_i = 0,
\]

where \( \delta \vec{P}_i \) is the first-order virtual displacement of the point of application of \( \vec{R}_i \). Relation (2) is also said to be the *symbolic equation of statics*.\(^{13}\)

If we accept, following Lagrange,\(^{14}\) that inertia is another force, then it should be add to the active ones. Therefore, from (1) and (2), it follows that Lagrange’s principle of virtual work states that a finite system of material points is balanced when the active forces \( \vec{F}_i \), to which it is subjected, satisfy

\[
\delta L = \sum_i \vec{F}_i \cdot \delta \vec{P}_i = 0,
\]

where \( \delta \vec{P}_i \) is the first-order infinitesimal displacement of the application point of \( \vec{F}_i \). Thus, if a system is at equilibrium, the virtual work of all active forces \( \vec{F}_i \) will vanish for any virtual displacement. Relation (3) is also said to be the *symbolic equation of dynamics*.\(^{16}\)

If the constraints are holonomic, then they are expressed as equalities in the intrinsic parameters of the system,\(^{17}\) and the vanishing of the virtual work of constraint reactions assumes an interesting geometrical meaning, as we shall see later. In particular, in the case of a material point constrained to lie upon a smooth surface (or a smooth curve), then we have a reaction which is perpendicular to the surface (or to the curve), while every first-order virtual displacement lies on the tangent plane (or on the tangent line) of the surface (or of the curve). Exactly in this case, the constraint reaction therefore spends no work.\(^{18}\)

This latter case-study will be a very emblematic one in the following historical enquiry, when we shall discuss how and why the symbolic equation of dynamics (3), was so crucial in developing Levi-Civita’s notion of parallel transport on a Riemannian manifold, bringing back, in doing this, to the consideration, following Levi-Civita, of a mechanical conceptual analogy just with this case-study.

### 3 On Levi-Civita’s parallel transport

Our historical method basically consists in a careful reading and in a detailed historical analysis primarily of the original sources related to the question here under investigation. In our

\(^{12}\)Cf. [38], Vol. I, Ch. XV; Vol. II, Part I, Ch. V, Sect. 3, Nos. 18-21; [11], Vol. I, Ch. XIV, Sect. 2, Nos. 4-8; Vol. II, Ch. V, Sect. 3, Nos. 17-19; [1], Vol. II, Ch. V, Sect. 1, No. 4; [2], Ch. I, Sects. 1-2; [23], Vol. 1, Ch. XIII, Sect. 4.

\(^{13}\)Also said to be *D’Alembert-Lagrange principle* as reformulated by Lagrange (cf. [3], Ch. IV), or *general equation of virtual work* (cf. [5], Vol. I, Ch. XV, Sect. 318; [33], Vol. I; [53]). See also the references quoted in the previous footnote.

\(^{14}\)Cf. [12].

\(^{15}\)Cf. [20], Ch. 1, Sect. 1.4, Eqs. (1.43)-(1.45); [54], Part I, Ch. 6, p. 210; Part III, Ch. 12, p. 441.

\(^{16}\)Cf. [38], Vol. II, Part I, Ch. V, Sect. 3, No. 20.

\(^{17}\)Cf. [38], Vol. I, Ch. VI, Sects. 1 and 3.

\(^{18}\)Cf. [38], Vol. I, Ch. XV, Sec. 1, No. 3-a).
case, therefore, we strictly follow first the original paper of Levi-Civita, i.e., [35], hence other possible related works of the same author. In the next section, then, we shall consider, historiographically, part of the secondary literature on the argument, whose sources, discussed and criticized from an historical standpoint, will allow us to do new, further remarks and hints on the question.

In [35], Sect. 1, Levi-Civita begins with the consideration of two arbitrary directions \( \vec{\alpha}, \vec{\alpha}' \) emerging from two infinitesimally near points \( P, P' \) of a generic Riemannian manifold \( V_n \), embedded in an \( N \)-dimensional Euclidean space \( S_N \) (i.e., \( \mathbb{R}^N \)) of suitable dimension \( N \). Thinking \( V_n \) as immersed into \( S_N \), we may start considering \( \vec{\alpha} \) and \( \vec{\alpha}' \) in \( S_N \), where the Euclidean geometry condition of parallelism implies that these two directions \( \vec{\alpha}, \vec{\alpha}' \) are parallel if and only if

\[
\text{angle}(\vec{\alpha}, \vec{f}) = \text{angle}(\vec{\alpha}', \vec{f})
\]

(4)

for any auxiliary direction \( \vec{f} \) emerging from \( P \) and \( P' \), according to equipollence relation in \( S_N \).

Hence, Levi-Civita highlights that this parallelism condition, in \( V_n \), a priori depends on the path joining \( P \) with \( P' \) and relying on \( V_n \), being independent of it only in ordinary Euclidean spaces (which are flat). Now, we have to specify condition (4), by analyzing the geometric behavior of \( \vec{\alpha} \) and \( \vec{\alpha}' \), supposed to be parallel between them, when \( P \) moves towards \( P' \) along a generic curve passing for \( P \) and \( P' \), and fully relying on \( V_n \). This geometrical sight will give rise to the notion of parallelism in \( V_n \).

Levi-Civita considers a generic metric on an arbitrary finite-dimensional manifold\(^{19} \) \( V_n \), of the type

\[
ds^2 = \sum_{i,j=1}^{n} a_{ij} dx_i dx_j,
\]

(5)

then, he embeds \( V_n \) in a Euclidean space \( S_N \) with sufficiently great dimension \( N \leq n(n+1)/2 \), so that it may be described by the system of equations\(^{20} \)

\[
y_\nu = y_\nu(x_1, ..., x_n), \quad \nu = 1, 2, ..., N,
\]

(6)

where the \( y_\nu \) are (Cartesian) coordinates in \( S_N \), while the \( x_n \) are intrinsic (or Lagrangian) coordinates on \( V_n \).

Now, we observe that the functional system (6) may be thought as the configuration space of a constrained mechanical system with \( n \) degrees of freedom subjected to \( N \) smooth holonomic bilateral constraints, which identify a differentiable manifold structure of dimension \( n \). This is the central point of a possible mechanical interpretation of Levi-Civita’s parallelism notion: the shift from a point on \( V_n \) to another infinitesimally nearby, undergoes\(^{21} \) in the corresponding conceptual analogy that refers to an holonomic mechanical system\(^{21} \) of material points with unitary mass, whose kinetic energy \( T \), with respect to (6), is such that \( 2T dt^2 = \sum_{i,j} a_{ij} dx_idx_j \).

For simplicity, Levi-Civita considers unit vectors, hence a direction of \( S_N \), arbitrarily fixed, identified by the unit vector \( \vec{\alpha} \), with direction cosines \( \alpha_\nu \), and an auxiliary direction of \( S_N \), identified by the unit vector \( \vec{f} \), with direction cosines \( f_\nu \), \( \nu = 1, 2, ..., N \); both are supposed emerging from an arbitrarily fixed point \( P \) of \( V_n \), but immersed into \( S_N \). Therefore, the direction cosines of both unit vectors \( \vec{\alpha} \) and \( \vec{f} \) are computed with respect to \( S_N \). All that is formally licit as \( V_n \) is embedded in the ambient space \( S_N \), so each direction belonging to \( V_n \) also belongs to \( S_N \).

The point \( P \) may be thought as a material point (with unitary mass) aimed by a certain movement along an arbitrary smooth curve \( C \) lying on \( V_n \), parameterized by the curvilinear (or

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\(^{19}\text{Cf. } \) [35], Ch. XXV.

\(^{20}\text{Cf. } \) [35], eq. (1), p. 4.

\(^{21}\text{Cf. } \) [38], Vol. II, Part I, Ch. V, Sect. 9, No. 63; Vol. II, Part II, Ch. XI, Sect. 4, No. 15.
natural) abscissa \( s \) as in \( \xi \), so that \( \alpha_\nu = \alpha_\nu(s), \nu = 1, 2, \ldots, N \). Let \( x_i = x_i(s), \ i = 1, 2, \ldots, n \) be the intrinsic parametric equations of \( C \). Then, \( C \) may be also represented by the parametric equations \( y_\nu = y_\nu(s), \ \nu = 1, 2, \ldots, N \), when it is thought embedded in \( S_N \) via \( \xi \). Indeed, since \( x_i = x_i(s), \ i = 1, 2, \ldots, n \), we have

\[
y_\nu = y_\nu(x_1(s), \ldots, x_n(s)), \quad \nu = 1, 2, \ldots, N. \tag{7}
\]

It is evident that, in the analog constrained system of above, \( C \) is a trajectory in the manifold of admissible configurations \( V_n \), parameterized by time \( t \) according to the parametric equations \( x_\nu = x_\nu(t), \ \nu = 1, 2, \ldots, N \), with \( t \in \mathbb{R}^+ \).

To find a generic unit direction emerging from an arbitrary point \( P \) of \( C \), Levi-Civita derives its parametric representation, given by \( \xi \), with respect to the natural abscissa \( s \)

\[
y'_\nu = \sum_{i=1}^{n} \frac{\partial y_\nu}{\partial x_i} x'_i, \quad \nu = 1, 2, \ldots, N, \tag{8}
\]

so obtaining the direction cosines with respect to \( S_N \), while \( x'_i \) are the direction cosines of the same unit direction but with respect to \( V_n \).

Hence, Levi-Civita considers, in a certain point \( P \) of \( C \), an arbitrarily fixed direction \( \vec{\alpha} \) of \( V_n \), whose direction cosines are \( \xi^{(i)}, \ i = 1, 2, \ldots, n \) with respect to \( V_n \), and \( \alpha_\nu, \ \nu = 1, 2, \ldots, N \) with respect to \( S_N \), so that, by the ansatz \( y'_\nu \to \alpha_\nu \) (in \( S_N \)), \( x'_i \to \xi^{(i)} \) (in \( V_n \)), from \( \xi \) it also follows.

\[
\alpha_\nu = \sum_{i=1}^{n} \frac{\partial y_\nu}{\partial x_i} \xi^{(i)}, \quad \nu = 1, 2, \ldots, N, \tag{9}
\]

which is a linear form on \( \xi^{(i)} \), i.e., the direction cosines of \( \vec{\alpha} \) with respect to \( V_n \).

When \( P \) varies along \( C \), imagined as aimed by a movement on \( V_n \), ordinary parallelism in \( S_N \) implies the equality of the angle between \( \vec{\alpha} \) and an auxiliary direction \( \vec{f} \) arbitrarily fixed in \( S_N \), according to Euclidean condition \( \xi \) (of synthetic geometry). Now, starting from this stance, Levi-Civita gradually introduces an intrinsic notion of parallelism in \( V_n \), considering two nearby points \( P \) and \( P' \) of \( C \), lying on \( V_n \), with \( P \) moving to \( P' \) along \( C \), never leaving out \( V_n \). Therefore, in considering the arbitrarily fixed direction \( \vec{f} \) of \( S_N \), with direction cosines \( f_\nu \) (in \( S_N \)), it follows that the cosine of the angle between \( \vec{f} \) and \( \vec{\alpha} \) in \( S_N \), is given by

\[
\cos \left( \vec{f}, \vec{\alpha} \right) = \sum_{\nu=1}^{N} \alpha_\nu f_\nu. \tag{10}
\]

Then, Levi-Civita considers an infinitesimal variation \( ds \) of the natural abscissa \( s \) on \( V_n \) along \( C \), which implies that the cosine provided by \( \xi \), undergoes the following first-order variation.

\[
d \cos \left( \vec{f}, \vec{\alpha} \right) = ds \sum_{\nu=1}^{N} \alpha'_\nu(s) f_\nu. \tag{11}
\]

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\(^{22}\)Cf. \[35\], (4), p. 5.

\(^{23}\)Cf. \[35\], Eqs. (7), p. 6.

\(^{24}\)In regard to this first-order variation, the arbitrariness with which \( \vec{f} \) may be fixed, implies that such an auxiliary direction, defined according to the equipollence relation in \( S_N \), may be considered as independent of \( s \), at least locally in \( P \in V_n \) (i.e., in \( T_P(V_n) \)) – see later). Instead, the direction \( \vec{\alpha} \) a priori depends on \( s \) as it varies with the movement of \( P \) along \( C \) on \( V_n \), even in a neighborhood of \( P \). The equipollence relation as settled in \( S_N \), locally restricted to a neighborhood of \( P \in V_n \) and defined at varying of all the curves \( C(\subseteq V_n) \) passing by \( P \), will led to the individuation (and, later, to the modern definition) of the so-called *tangent space* \( T_P(V_n) \) to \( V_n \) in \( P \). In \[35\], Levi-Civita shall consider \( \vec{f} \) as belonging only to \( T_P(V_n) \), and not to the whole \( S_N \) as in \[4\], for reaching his notion of parallelism on \( V_n \).
Now, the ordinary parallelism in $S_N$ between the two directions $\tilde{\alpha}(s)$ (in $P$) and $\tilde{\alpha}(s + ds)$ (in $P'$), as expressed by (41), would require (11) to vanish when $\tilde{f}$ varies arbitrarily in $S_N$, so implying $\alpha_n$ to be constant or uniform. But, just at this point, Levi-Civita introduces the key argument which will led to his notion of parallelism on $V_n$. Indeed, since the main aim of Levi-Civita was to determine the curvature of a Riemannian manifold $V_n$, he started to approach this problem from an intuitive viewpoint, that is to say, working out initially a geometrical setting of the problem, then carrying on with its analytical formulation within absolute differential calculus framework.

As has been already said above (Section 1), the usual way to determine the curvature of a Riemannian manifold $V_n$, consists in the circuitation of a given vector around a suitable infinitesimal closed path entirely lying on the given manifold, usually a "parallelogrammoid" whose sides are first-order infinitesimal geodetic traits, and drew around an arbitrary point $P$ of $V_n$. This vector should be rotated, all around this circuit, in a parallel manner to itself, in such a way that, after a complete circuitation, once reached the same point from which it departed, the possible deviation angle between initial and final directions of such a vector in this same final and initial point, will provide an estimate of the curvature of $V_n$.

It follows therefore the extreme importance to have a preliminary notion of parallelism on a generic Riemannian manifold before to determine the curvature of the latter, and, to this end, as already said, Levi-Civita gave a first geometrical sketch to this formal problem just making appeal to analytical mechanics. In this geometrical sight of the question, Levi-Civita felt the need to consider only circuitations of a generic applied vector (as, for instance, $\tilde{\alpha}$) whose application point $P$ always relies upon the manifold $V_n$, never leaving out it.

This circuitation, which therefore takes place exclusively upon $V_n$ and in a neighborhood of $P$, should have, according to this Levi-Civita’s geometrical framework, the main intuitive purpose, so to speak, "to feel the shape of $V_n"); its distortions, cambers, deformations, and so on. And, the only intuitive way to accomplish this end, is just the one warranting that such a circuitating vector, say $\tilde{\alpha}$, remains always in relationship with some intrinsic geometric entity characterizing $V_n$ – i.e., $T_P(V_n)$ – during its circuituation, which is the result of the composition of sequential shifts along infinitesimal traits of smooth curves on $V_n$ (usually, geodesic curves). Therefore, all that has been just said, should also hold along each of these infinitesimal (curve’s) traits.

Since a Riemannian manifold is characterized by the main property to be locally like some Euclidean space $S_N$, then it follows that the tangent space, say $T_P(V_n)$, at $V_n$ in some its point $P$, is just that geometric entity characterizing better the local differentiable structure (following the well-known Hermann Weyl work of 1913 on Riemann’s surfaces) underlying any Riemannian manifold. Therefore, the Euclidean condition (11) will have an intrinsic meaning related only to $V_n$ when the variability of $\tilde{f}$ is restricted from the whole $S_N$ to $T_P(V_n)$, so guaranteeing the above required reference of $\tilde{\alpha}$ to $V_n$ during its movement upon $V_n$. This intrinsic geometrical requirement, in particular, should hold too for (11).

At the same time, the variability’s restriction given by $\tilde{f} \in T_P(V_n)$, also guarantees that, during the infinitesimal movement of $\tilde{\alpha}$ along $C$ on $V_n$, this latter "smooths out" $V_n$, so "feeling" its local curvature while $\tilde{\alpha}$ moves. This is just the key geometrical intuition had by Levi-Civita in thinking how constructively define a possible notion of parallelism on a generic Riemannian manifold.

25 This procedure is deeply argued in [24], where, in discussing of Riemannian curvature, the author retakes some previous considerations made by Jan A. Schouten on absolute differential calculus (cf. [50]) and Levi-Civita’s parallelism, mainly exposed taking into account the possible mechanical interpretations most of which falling into the kinematics of rigid bodies. In this regard, we shall return on Schouten’s work in the next section.

26 Just according to the new definition of parallelism as introduced by Levi-Civita in [35].

27 Here, we use a modern notation for the tangent vector space to a Riemannian manifold, yet not used by Levi-Civita in his 1917 memoir, who simply speaks of "a lying of $S_N$ tangents in $P$ to $V_n"$ ([35], p. 2).
manifold. Afterwards, this geometrical idea had to be formulated in analytical fashion, and to this end, Levi-Civita made appeal to his wide and deep knowledge of analytical mechanics, to be precise, calling into question the principle of virtual works in its widest and pregnant geometrical meaning.

Levi-Civita, thus, claimed that, to this end, the directions \( \vec{f} \) must be exactly those compatible with the constraints, \( ^{28} \) once having assumed valid that pattern analogy which considers \( P \) as a material point (with unitary mass) subjected to the smooth constraints \( ^{6} \), by which \( \vec{f} \) must lie on \( T_P(V_n) \), that is to say, \( \vec{f} \) must be correlated, in this mechanical analogy, with first-order displacements compatible with constraints \( ^{6} \). In such a case, while \( P \) moves along \( C \) on \( V_n \), the direction \( \vec{\alpha} \) must be gradually transported always with respect to \( \vec{f} \in T_P(V_n) \), hence compatibly with the smooth constraints \( ^{6} \), if we wish to define a vectorial displacement (of \( \vec{\alpha} \)) which must be intrinsically (correlated with or) related to \( V_n \).

At this point, still inside this geometrical framework worked out in the analytical mechanics context, Levi-Civita, in looking at the formal aspect of \( ^{11} \), describes a kind of physical work in \( S_N \) made by some active forces \( ^{29} \) in a certain sense corresponding to \( \alpha_{\nu}^\prime(s) \) (in their Cartesian components), applied to the material point \( P \) (with unitary mass) moving along \( C \) with respect to the smooth constraints \( ^{6} \) identifying \( V_n \) as an holonomic mechanical system. So, he was legitimated to see in \( ^{11} \), within the above mechanical analogy, the formal expression of the principle of virtual works according to \( ^{3} \), when we replace \( f_{\nu} \in T_P(V_n) \) with quantities, say \( \delta y_{\nu} \), proportional to first-order virtual displacements compatible with smooth constraints \( ^{6} \), that is to say, \( \delta y_{\nu} \).

Therefore, with the ansatz \( ^{30} \) \( \vec{F} \to \vec{\alpha} \), \( \delta \vec{F} \to \vec{f} \), made in \( ^{3} \), hence with the further replacement of \( \vec{f} = (f_{\nu}) \in T_P(V_n) \) (in \( S_N \)) with \( \delta y_{\nu} \) (in \( V_n \)), from \( \delta L = 0 \), it follows that the Euclidean parallelism condition on \( V_n(\leftrightarrow S_N) \), given by the vanishing of \( ^{11} \), reduces to

\[
\sum_{\nu=1}^{N} \alpha_{\nu}^\prime(s) \delta y_{\nu} = 0, \tag{12}
\]

for any variation \( \delta y_{\nu} \), that is, "for any admissible first-order displacement compatible with the constraints" \( ^{6} \), as Levi-Civita himself said textually in \( ^{35} \), p. 7. With a suitable mechanical interpretation of the \( \alpha_{\nu}^\prime(s) \), for instance considering them as a kind of mechanical action in \( S_N \), \( ^{12} \) is a formulation of the virtual work principle in \( S_N \) related to the smooth bilateral holonomic system defined by \( ^{6} \), hence related to analytical mechanics on a Riemannian manifold \( ^{31} \).

The condition \( ^{12} \) is however related to \( S_N \). But Levi-Civita solved too this formal problem as follows \( ^{32} \). To get an intrinsic form, from \( ^{6} \), we have \( ^{33} \)

\[
\delta y_{\nu} = \sum_{k=1}^{n} \frac{\partial y_{\nu}}{\partial x_k} \delta x_k, \quad \nu = 1, 2, ..., N, \tag{13}
\]

\(^{28}\) Cf. \( ^{35} \), p. 7.

\(^{29}\) Cf. \( ^{10} \), p. 120.

\(^{30}\) We cannot consider the correspondence \( \vec{R} \to \vec{\alpha} \) instead of \( \vec{F} \to \vec{\alpha} \), because, a priori, not always \( \vec{\alpha} \) is normal to \( V_n \) as required by reactions to smooth constraints. Therefore, in applying the principle of virtual works in this pattern analogy of Levi-Civita, we should take into account the symbolic equation of dynamics (as involving active forces \( \vec{F} \)), rather than the symbolic equation of statics (as involving reactions \( \vec{R} \)).

\(^{31}\) Cf. \( ^{22} \), Ch. V; \( ^{85} \), Ch. V; \( ^{61} \), Ch. II; \( ^{12} \), Ch. VI, Sect. I, Nos. 87-89, 92; \( ^{3} \), Ch. IV; \( ^{6} \), Part II, Ch. V, Sect. 6; \( ^{27} \), Ch. 3, Sect. 2, No. 2.6.; \( ^{14} \), Ch. 15; \( ^{21} \), Ch. 1, Sects. 1.9-12, Ch. 4; \( ^{14} \), Part III, Ch. 12.

\(^{32}\) Therefore, devoid of any historical base are all those statements for which Levi-Civita’s procedure to get parallelism, was extrinsic (to some \( S_N \)) and not intrinsic. Levi-Civita, in the simplest and fastest way, got intrinsic conditions for parallelism (in \( V_n \)) by means of the intermediary use of an auxiliary space \( S_N \).

\(^{33}\) Cf. \( ^{35} \), unnumbered equation before eqs. (8), p. 7.
with \( \delta x_k \) arbitrary first-order virtual displacement on \( T_P(V_n) \) (in \( V_n \)), so that (12) now reduces to:

\[
\sum_{\nu=1}^{N} \alpha'_{\nu}(s) \frac{\partial y_\nu}{\partial x_k} = 0 \quad (k = 1, 2, ..., n),
\]

(14)

which are (intermediary) formal conditions, in the intrinsic variable (or Lagrangian coordinates) \( x_k \), for the parallelism of the direction \( \vec{\alpha} \) moving along \( C \) on \( V_n \). Nevertheless, in (14) there are still parameters regarding \( S_N \), so that, in order to have a full intrinsic relation in \( V_n \), it is needed to involve only parameters regarding \( V_n \), without any reference to \( S_N \).

To this end, we replace the direction cosines \( \alpha_\nu(s) \) (in \( S_N \)) with their expression given by (2), in such a way to involve only the intrinsic direction cosines \( \xi^{(i)}(s) \) (in \( V_n \)). So, developing related formal passages, we finally deduce:

\[
\frac{d\xi^{(i)}}{ds} + \sum_{j,l=1}^{n} \Gamma^i_{jl} \frac{dx_j}{ds} \xi^{(l)} = 0 \quad (i = 1, 2, ..., n),
\]

(15)

where \( \Gamma^i_{jl} \) are the Christoffel symbols of second kind with respect to the intrinsic coordinates \( x_k \) (of \( V_n \)), defined as follows:

\[
\Gamma^i_{jl} = \sum_{k=1}^{n} a^{ik}(\frac{\partial a_{kl}}{\partial x_j} + \frac{\partial a_{jk}}{\partial x_l} - \frac{\partial a_{lj}}{\partial x_k}) \quad (i, j, l = 1, 2, ..., n),
\]

(16)

with \( \|a^{ik}\| \) coefficient matrix of the reciprocal form of (5). The (15) are the so-called (intrinsic) Levi-Civita’s equations of parallelism on a Riemannian manifold \( V_n \), equipped with a generic metric of the type (5).

They are first-order ordinary differential equations on the direction cosines \( \xi^{(i)} \) of the arbitrary direction \( \vec{\alpha} \), emerging from \( P \), transported, along a curve \( C \) on \( V_n \), until up it reaches the infinitesimal nearby point \( P' \), from where a parallel direction \( \vec{\alpha}' \) (to \( \vec{\alpha} \)) emerges, with new direction cosines \( \xi^{(i)} + d\xi^{(i)} \) such that (by (15))

\[
d\xi^{(i)} + \sum_{j,l=1}^{n} \Gamma^i_{jl} dx_j \xi^{(l)} = 0 \quad (i = 1, 2, ..., n).
\]

(17)

The (15), identify a (regular) linear system of ordinary differential equations on \( \xi^{(i)} \), whose related theorems of existence and uniqueness allow to determine a (unique) direction parallel to every other preassigned one. It was the starting point for every possible notion of connection of differential geometry, whose so-called “coefficients” are just the \( \Gamma^i_{jl} \) of (17). We spend a few words on this last point, but in modern notation.

If the smooth curve \( C \) has parametric equation \( x: [0, 1] \to V_n \), then Levi-Civita’s local parallel transport along \( C \), as expressed by the differential forms (17), establishes a (linear) isomorphism between (linear) tangent spaces \( T_{x(t)}(V_n) \), \( t \in [0, 1] \) (if \( P \in C \) is identified by \( x(t) \)), of the tangent bundle \( T(V_n) = \bigcup_{P \in V_n} T_P(V_n) \) (with disjoint union), placed at infinitesimal nearby points of \( V_n \). Instead, Levi-Civita’s global parallel transport along \( C \), as expressed by the ODE system (13), is the isomorphism, say \( \nabla_c \), defined by

\[
\nabla_c : T_{x(0)}(V_n) \to T_{x(1)}(V_n)
\]

(18)

through which, via (15), from the initial direction \( (\xi^{(1)}(0), ..., \xi^{(n)}(0)) \), as initial conditions to (15), we get the final direction \( (\xi^{(1)}(1), ..., \xi^{(n)}(1)) \), as the corresponding unique solution to (15).

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34 Cf. [55], Eq. (8), p. 7.
35 Cf. [35], Eq. (14), p. 8.
36 Cf. [8], Ch. II.
by means of the well-known theorems of existence and uniqueness for the regular system of first-order ordinary differential equations \[^15\]. So, we say that the vector \(\xi^{(0)}\in T_{x^{(0)}}(V_n)\) is parallel (according to Levi-Civita) to the vector \(\xi^{(t)} = \nabla_C(\xi^{(0)})\in T_{x^{(t)}}(V_n)\), along \(C(\subseteq V_n)\), for every \(t\in[0,1]\) arbitrarily fixed. At varying of \(C\) in the set of all the possible smooth curves \(C\) of \(V_n\), from \(\nabla_C\) we may identify a so-called linear connection on \(V_n\), say \(\nabla\), which generalizes the usual notion of directional derivative of ordinary Euclidean spaces, to generic Riemannian manifolds\[^37\].

Thus, the formal deduction of the intrinsic conditions \[^15\] (or \[^17\]), characterizing Levi-Civita’s notion of parallel transport of the generic direction \(\alpha\) along an arbitrary curve \(C(\subseteq V_n)\) as a function of its directional parameters \(\xi_1, \ldots, \xi_n\) with respect to \(V_n\), basically relies on the symbolic equation of dynamics \([3]\), which has allowed to deduce the parallel conditions \[^14\], whence \[^15\]. Just in this, is the power of Levi-Civita’s discovery, that is to say, having put into relation infinitesimally nearby points of a Riemannian manifold \(V_n\) by means of a linear isomorphism (i.e., \(\nabla_C\)) ‘connecting’ the related (linear) tangent spaces at \(V_n\). This had never been considered before Levi-Civita’s work, which therefore became rightly a milestone of differential geometry.

Therefore, we can say that the greatness, together the clarity and simplicity, of Levi-Civita’s work, just rely on the preliminary geometrical setting given to this question, worked out, as seen, within analytical mechanics framework, and that always drove Levi-Civita’s reasoning with those intuitions and insights that only a geometrical preview may provide. So, analytical mechanics, with its pregnant geometrical meaning, was always a constant guide-pattern in all those Levi-Civita’s works in which such an ancient and noble doctrine could be directly applied or simply considered at a conceptual analogy level. Besides to what has been said above, for corroborating further this latter epistemological stance, at least in the case-study here discussed, we note that Levi-Civita also considered\[^38\] the covariant systems \(\xi^{(i)} = \sum_{k=1}^n a_{ik}\xi^{(k)}\), whose elements are said to be moments, associated to the contravariant system \(\xi^{(i)}\), hence he reformulated \[^17\] in terms of \(\xi_i\), so obtaining new equations whose analytical form resembles a particular expression\[^40\] of Lagrange’s equations of dynamics on a Riemannian manifold (Riemannian mechanics\[^41\]).

In conclusion, it is clear that, at least till to the definition of intrinsic parallelism on a Riemannian manifold, the deduction of Levi-Civita’s intrinsic equations of parallelism was mainly carried out within a formal framework highly characterized by a guiding geometrical meaning coming from analytical mechanics on a Riemannian manifold. In particular, \[^17\] are deduced from \[^14\], that is to say, the former are nothing but a particular reformulation of the symbolic equation of dynamics for a Riemannian manifold. In the conclusions of \[^29\], a deep historiographical investigation proving the constant allusion, more or less tacit, by Levi-Civita to the principle of virtual works in working out his intrinsic equations of parallelism, has been achieved. Only later, Levi-Civita was even more explicit in recognizing this, as, for instance, done in treating and arguing on his notion of parallelism respectively in \[^39\] and \[^40\].

In this note, we have stressed this aspect of the fundamental Levi-Civita’s memoir of 1917 because, from this moment onward, many other renowned mathematicians retaken the basic ideas here exposed, to open other, fruitful ways in mathematics, above all in differential geometry and its applications to physics. Here, we mention only the basic work of Élie Cartan in Riemannian geometry, which started just from this Levi-Civita’s memoir of 1917, above all with

\[^{37}\] Cf. \[^{14}\], Part I, Ch. 1, Sect. 1.1.

\[^{38}\] Levi-Civita also considered (in \[^{35}\], Sect. 7) the particular case in which \(C\) is a geodetic curve of \(V_n\), but this has not been neither the general case taken into consideration in \[^{35}\] in deducing the main equations \[^{17}\] nor the initial motivation to his 1917 memoir.

\[^{39}\] Cf. \[^{35}\], Sect. 5.

\[^{40}\] Cf. \[^{35}\], Sect. 5, Eqs. (\(I_i\)), p. 12; see also \[^{22}\], Ch. X. and \[^{2}\], Vol. I.

\[^{41}\] Cf. \[^{2}\], Vol. I.
the acquisition of that particular method (above discussed) used to deduce the intrinsic equa-
tions of parallelism \([15]\). To be precise, Cartan extensively used as well analytical mechanics
concepts and methods in pursuing his work on affine connections and holonomic spaces, quoting
frequently Levi-Civita’s work, taken as a guide-model together the consistent work of Gaston
Darboux on the geometry of surfaces, in whose celebrated four-volumes treatise \textit{Leçons sur la
Théorie Générale des Surfaces} of the 1880s, the fundamental elements of analytical mechanics
on a Riemannian manifold are exposed\([42]\).

4 Further historical remarks

Besides to what has been exposed in \([29]\), in this last section we wish to outline further histori-
cal remarks on the celebrated work of Levi-Civita on parallel transport, \([35]\). To be precise, we
would like to point out, another time, the extreme originality and uniqueness of the method
followed by Levi-Civita in pursuing his scopes in \([35]\). The customary use of analytical me-
chanics methods in his mathematical studies, was really original and an usual working praxis
which led him to reach the highest results, as the one treated above. Many influential historical
sources cite other possible analogous case-studies, just regarding the crucial question concern-
ing parallelism in a generic Riemannian manifold, but none of these may be compared with
the greatness and originality of Levi-Civita’s insight, formally carried on with an impeccable
stylistic elegance\([43]\) and with an amazing simplicity of method. We briefly discuss some of these
sources, which make manifest reference to \([35]\).

Marcel Berger stated\([44]\), for instance, that Levi-Civita is placeable into the so-called \textit{golden
triangle} of Riemannian geometry, whose vertices are: the \textit{curvature} (via the \textit{connection}), the
\textit{parallel transport} and the \textit{absolute differential calculus}. This triangle was first understood by
Ricci and Levi-Civita at the end of 19th century. Berger says also that the basic \textit{lemma}, which
is the key to everything, is just the existence and the uniqueness of a canonical connection,
called the \textit{Levi-Civita’s connection}, on any Riemannian manifold.

According to Berger, it is important to realize that this lemma is a "miracle": many people
have tried to understand it, with more or less sophisticated concepts, but we consider that it
remains a miracle. This might perhaps explain why there have been many variations in the
historical interpretations of this fundamental notion of mathematics. In this note, we have shed
some light on this "miracle", trying to clarify historically that has been the right place in which
such a "miracle" happened, finding in the analytical mechanics framework, as seen above, the
right context in which it occurred as such.

Doing reference to analytical mechanics has been a typical and fruitful praxis of Levi-Civita’s
working. Indeed, besides to what has been said above on \([35]\), another historiographical prove of
the predominance of geometrical sight in formulating and approaching a general formal problem
by Levi-Civita, and arising above all from the analytical mechanics framework, may be found
in a work immediately following \([35]\) but closely related to the same research program centred
on formal aspects of general relativity, namely \([36]\), where Levi-Civita gave an extremely inter-

\[\text{42Cf. [17], Tome II, Livre V, Chapitre VIII. Soon after Darboux’s treatise, also the celebrated many volumes}
\text{Paul E. Appell’s \textit{Traité de mécanique rationnelle} of the 1890s, deals with the first elements of the basic analytical}
\text{mechanics from a Riemannian viewpoint, taking the legacy of the previous works made by Rudolf Lipschitz,}
\text{Joseph Liouville, Joseph Bertrand, Edouard J.B. Goursat and others, on the analytical foundations of rational}
\text{mechanics (cf. [43]). One of the first textbooks devoted to a comprehensive treatment of analytical mechanics,}
\text{with first explicit recalls to Riemannian geometry, was [61] (see, in particular, Ch. II). In any case, in regard}
\text{to the history of analytical mechanics on Riemannian manifold contextual with the historical question treated}
\text{in the present paper, it might be useful, from an historiographical standpoint, to look at the few references}
\text{consulted by Levi-Civita and Amaldi in drawing up their treatise [38] (see, in particular, [38], Vol. I, p. VI).}
\text{43In this regard, see also the opinion expressed in [30], p. 73.}
\text{44Cf. [7], Ch. XV, Sect. 15.3.}\]
testing and clear interpretation of the Einstein’s field equations of general relativity consisting in giving a generalized form to D’Alembert principle of analytical mechanics in the relativistic context, got by means of a geometrical sight, ingeniously had by Levi-Civita, of certain geometrical relations involving first-order covariant derivatives of Riemann’s symbols, due to Luigi Bianchi and, therefore, said to be Bianchi’s identities. The final result is now a formally correct expression of Einstein’s field equations in terms of an extended D’Alembert principle of analytical mechanics to general relativity, so reaching new and proper physical interpretations of the field equations themselves.

Furthermore, in the authoritative Klein’s Encyklopädie der mathematischen Wissenschaften mit Einschlußihrer Anwendungen, which is one of the widest and richest bibliographical sources of the time, there are some other interesting references usually not quoted elsewhere, regarding Levi-Civita’s parallel transport and its possible links with mathematical physics. Among these, a work of Adrian D. Fokker, namely [24], which is even quoted, in this encyclopedia, as the only one, of that time, to have given a geometrical and mechanical interpretation of Levi-Civita’s parallelism.

In this Fokker’s paper, however, there is an interesting exposition of the usual geometrical method to calculate Riemann curvature by means of the circuitation of a vector around an infinitesimal closed path whose sides are infinitesimal geodesic traits, according to some ideas of Schouten (on the so-called geodesic displacement), where frequent recalls to kinematics of rigid body are done in regard to certain geodesic displacements considered by the author. The principle of the method, nevertheless, which surely makes reference often to rigid mechanics notions, seems to be very close to Cartan’s method of moving frame (which appears to anticipate this last in many of its respects), but does not have to do, in no way, with Levi-Civita’s one.

Likewise to the case of Fokker’s paper, in this encyclopaedia is also quoted the well-known William Thompson (Lord Kelvin) and Peter G. Tait two-volumes Treatise on Natural Philosophy of 1879, in regard to first notions of parallelism on a surface, built up by means of kinematical methods. Nevertheless, what is exposed in this popular treatise on mathematical physics about this argument, does not have any conceptual link with the original method of Levi-Civita which, as has been already said above, makes use, originally and fruitfully, of higher analytical mechanics methods and concepts.

Finally, also Vladimir I. Arnold, after having rightly recalled that, in Riemannian geometry as well as therefore in general relativity theory and in gauge theory of field theory, a fundamental role is played by Levi-Civita’s connection (defined through the coefficients $\Gamma^i_{jl}$ of (17)), which defines parallel transport of vectors along a manifold with a generic Riemannian metric, stated nevertheless that, the most physically natural definition of this (quite non-obvious) vector transport on a Riemannian manifold was first provided by Johann Radon in a work of 1918.

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45Cf. [35], Sect. 8, Eqs. (10'), p. 55. Among other, in this paper, Levi-Civita gave, for the first time, the real, formally correct expression of Einstein’s field equations which were initially set up, by Einstein himself, in a not properly correct form from the analytical standpoint. To this formal lack, Levi-Civita remedied, still again, through his powerful and ingenious method consisting in doing reference, whenever possible, to analytical mechanics, trying to lay out in its framework the given formal problem, to be better and easier approached and solved, with the further possibility to have an enlightening physical interpretation (cf. [46], Sect. 6). However, right criticisms to the formal correctness of first forms of Einstein’s field equations were moved by Levi-Civita to Einstein since the first months of 1915, with a thick correspondence between them (cf. [11], Vol. 8), which allowed Einstein to give finally the right and formally correct expression to his celebrated field equations, around the end of 1915, almost in the same period in which Hilbert wrote his known works on the related variational aspects of the question, following the analytical mechanics pattern of Hamilton’s variational principle (cf. [10]).

46Cf. [20], Band 3, Teilen 3, III.D.11-B.II.18.

47Cf. [20], Band 3, Teilen 3, III.D.11-B.II.18, p. 131.

48See also [17], Tome II, Livre V, Ch. VII, for the analogy of the method.

49Moreover, retaken later by [16].

50Cf. [4], Ch. 6, Sect. 6.4.3, pp. 331-32.

51Cf. [31], Part III, Ch. II. Strangely enough, the only, official source which reports this Radon’s work, is [31].
on the early theory of adiabatic invariants, which seems, according to Arnold, having been
motivated just from Levi-Civita’s memoir of 1917.

To be precise, Radon imagined a conceptual physical experiment\(^\text{52}\) (Gedankenexperiment) in
which is placed, at an arbitrary point of a given Riemannian manifold, some oscillatory system
as, for example, the one got suspending a Foucault’s pendulum over this point, or considering,
in the tangent space at this point, a Hooke’s elastic system with potential energy proportional
to the square of the distance from the original point. Then, under suitable initial conditions,
this oscillatory system is conceived to perform a so-called eigenoscillation in the direction
defined by some generic vector of the tangent space. Hence, if this oscillatory system is being
transported, slowly and smoothly, along some path lying on the given manifold, then it follows,
from adiabatic theory, that the oscillation will remain (in the adiabatic approximation) an
eigenoscillation. Furthermore, its direction (i.e., the polarization) will rotate somehow during
the motion of the point along the path, this rotation – which proves to be an orthogonal linear
transformation of the initial tangent space into the terminal one – Radon observed to be just
due to Levi-Civita’s parallel transport (or connection) in dependence on the curvature of the
manifold.

It is historically interesting to note that Radon’s theory on adiabatic invariants was not
understood by geometers of the time, mainly because they were not familiar with these latter,
so that it was unfairly forgotten. Arnold, however, above defined this Radon’s interpretation as
the first most physically natural definition of Levi-Civita’s parallel transport, which sounds us
a quite strange, seen the great and deep expertise in mathematical physics, even more in me-
chanics, owned by Arnold, who, maybe, has never read the original Levi-Civita’s memoir that,
strangely, was never translated in any foreign language. Indeed, everyone knows the minimal
requisites of analytical mechanics, in reading the original Levi-Civita’s memoir, immediately
recognizes the yet tacit use of founding principles of analytical mechanics. Nonetheless, Radon’s
interpretation is, surely, a very clever one, although quite cumbersome and, however, relegated
to a mere thought experiment.

There is, furthermore, another historical aspect of the history of Levi-Civita’s parallel trans-
port, concerning a question of discovery’s priority, that deserves here to be clarified briefly. Pre-
cisely, Schouten claimed\(^\text{53}\) the priority in discovering a notion of parallelism in a Riemannian
manifold, called geodesic displacement and exposed in \(^\text{50}\), which should include, according to
him, the Levi-Civita’s one\(^\text{54}\). He stated that this his paper had already been drawn up since
1915, but that it was yet published (with an unusual and strange delay) only in 1919 in the
proceedings of the Academy of Sciences of Amsterdam, inasmuch officially classified as "Ver-
schenen Februari 1919", i.e., published on February, 1919. In this regard, some oral testimonies
say that Luitzen J. Brouwer, who also contributed initially to the subject and was a colleague
of Schouten, opposed to the revindication of the latter, giving priority to Levi-Civita.

Anyway, what is really important in historiography, no matter any other testimony, is the
certainty and officiality\(^\text{55}\) in dating the related historical sources used as such, and, until up
now, there are no reliable official historical sources, except little reliable oral witnesses, that

\(^{52}\)Levi-Civita, instead, follows, from an epistemological viewpoint, a pattern analogy settled between mathe-

\(^{53}\)Cf. \[20\], Band 3, Teilen 3, III.D.11-B.II.18, p. 131; \[20\], \[57\], \[58\].

\(^{54}\)Cf. \[50\], footnote a), p. 46.

\(^{55}\)Like, for example, the registration of a possible preprint in some lawcourt or in a suitable inventory deposited
in some academic department office or library. On the other hand, most of the academic journals which host
publications of various type, often report the so-called publication history, i.e., date of submission or receipt
of the paper in its first version, dates of further revisions, date of acceptance, and so on. All that, allows, for
instance, to have objective historiographical data. This has not been possible for Schouten’s paper, to confirm
the priority claimed by him.
objectively corroborate what Schouten claimed. So, until proven otherwise, his work remains however dated to 1919. Likewise, some sources quote the work of Gerhard Hessenberg as contemporary to that of Levi-Civita, in introducing a notion of parallelism on a Riemannian manifold, following a vectorial method. In any case, this last work was realized fully within Ricci’s and Levi-Civita’s absolute differential calculus framework, without any useful geometrical consideration or insight, differently from Levi-Civita’s one, whose greatness relies just in its deep and immediate geometrical intuition of the crucial idea underlying the fundamental notion of connection.

Finally, according to a remark due to Jacques Hadamard, as early Darboux, in dealing with geodesic curvature of an ordinary surface computed by means of his method of moving frame (Darboux’s repère mobile), had already reached a similar notion of parallelism, yet without having been rightly recognized as such. But, even if surely Darboux’s oeuvre established a deep, wide and interesting intertwining between geometry and mechanics, he restricted the study only to ordinary surfaces in Euclidean spaces, while in , he effectively considered the case of the parallel displacement, upon a three-dimensional Riemannian manifold, of a trièdre mobile moving along a curve lying on a Riemannian manifold, but with methods, approaches and results which didn’t have that immediate and powerful geometric intuition owned by Levi-Civita’s work, which distinguishes for its higher contextual coherence, conceptual clearness and great potentiality, together methodological originality and generality, as those copious achievements, immediately later pursued in differential geometry and theoretical physics, manifestly proved. However, the right and large tribute to Darboux’s work, just in this context, will come from the next, pioneering work of Cartan on Riemannian geometry, which yet started from Levi-Civita’s memoir, following that fruitful method based on a preliminary intuitive sight (mainly, of geometric nature) which always accompanied and led Levi-Civita’s formal thought.

5 Conclusions

In short, we may say that, after what has been said so far, as well as after having consulted most of the main modern and past literature either in differential geometry and its applications to physics, above all to general relativity, the conclusion of this our further historical investigation (besides ), is that nobody has highlighted, even fleetingly, the crucial and ingenious

\[56\text{From a direct inspection of the original issue where Schouten’s work was published, it is turned out that, in the table of contents (Inhoud) of this issue (in which seven contributions are listed in chronological order – the Schouten’s one, is the 6th – and everyone with its own internal page enumeration), is reported the publication date of February, 1919 (Verschenen Februari 1919), while in the title-page of the contribution, is instead reported the year 1918. So, there is no certainty neither on this historical datum. Nevertheless, from an historiographical standpoint, we should consider the date of 1919, the only one to have been explicitly quoted as such, i.e., as a “publication date” (Verschenen Februari 1919). Instead, just at the end of , there is the date of final writing of the paper by Levi-Civita, that is, November, 1916, while at the beginning of , there is the date in which this memoir was presented, the 24th of December 1916, at the periodic monthly sessions of the Circolo Matematico di Palermo, before it were being sent to the related Rendiconti del Circolo Matematico di Palermo to be officially published. In any way, even in this further case, there is no right comparison with the great intuitive power and the wide prospective amplitude opened by Levi-Civita’s method.}

\[57\text{Mentioned in Ch. VI, p. 73, but without having further, more detailed, bibliographical indications neither to this Hadamard’s affirmation nor to the related Darboux’s work (cf. Tome II, Livre V, and, above all, Livre II, Ch. II).}

\[58\text{See, above all, Tome II, Livre V, Chapitres VI-VIII.}

\[59\text{See Livre II, Chapitre II, NN. 115-16, where he decomposes the infinitesimal quadratic element } ds^2 \text{ in the sum of three independent infinitesimal components related to rotations and translations of the repère mobile. A similar method, but extended to the } n\text{-dimensional case, will be then retaken by Anita Carpanese in to compute curvature of a generic Riemannian manifold, again starting from Levi-Civita’s result.}

\[60\text{Here, we have quoted only those references which have been strictly essential in drawing up this note.}
role played by the fundamental analytical mechanics principles in Levi-Civita’s construction, notwithstanding their manifest presence in his works. The lucky establishment of that conceptual pattern analogy between geometry and analytical mechanics, led Levi-Civita to work out a preliminary framework in which at first lay out the original geometrical question, then approach and solve it, achieving a so rich harvest of fruitful results that conferred an unusual success to this method intertwining geometry and analytical mechanics.

However, from a methodological standpoint, this reflects, as Ugo Amaldi said in the biographical introduction to [37], that traditional and usual methodological praxis of modern Italian mathematical school (between the end of 19th century, and the beginning of 20th century), fundamentally based on a preliminary geometric sight of any possible formal question before this is being treated in a pure analytical way. In the case-study here examined, this way turned out to be extremely fruitful in mathematics, as the next work of Elie Cartan masterfully witnessed as well as in theoretical physics, as the next results in field theory (above all, in general relativity) testified too. This reasoning’s fashion was, almost always, presents in all those Levi-Civita’s works where such a method could run: we have only considered here, in a detailed manner, [35], as well as, fleetingly, [36], as emblematic historical events.

As in most of his works, Levi-Civita was able not only to give a first, clarifying geometrical setting to every formal question had to be treated, but also in using every possible tools of mathematical physics to approach it, above all higher mechanics, maybe for its strong geometrical intuition. This was, as already said, a powerful and successful method which featured almost all the intellectual work of Levi-Civita, as well as other scholars, like Hadamard. Indeed, as pointed out in [25], both Levi-Civita and Hadamard, for instance in dealing with formal problems concerning wave propagations and their discontinuities to be treated by means of systems of hyperbolic partial differential equations, always put before these suitable physical arguments thanks to which outline easier the needful formal framework in which to lay out these questions to be then analytically handled.

To testify the related methodological importance and power, as we have seen above in discussing briefly also [36], this typical method of Levi-Civita was, for instance, so crucial in helping Einstein to reach finally the definitive, correct analytical form of his celebrated field equations, which initially were affected by some formal lacks. This was possible after having laid out the question concerning the analytical setting of the equations of gravitational field, within an appropriate analytical mechanics framework which allowed easy Levi-Civita to interpret these equations as a suitable extension of D’Alembert principle to relativity context, as well as to get new physical interpretations of them which, with a formidable intuition, Levi-Civita masterly used to guide his analytical investigation of gravitational field equations, so being able to suggest to Einstein what pathway follows to reach their correct, formal expression.

The main aim of this note has been therefore to put in evidence this historical-epistemological aspect of Levi-Civita’s method in approaching mathematical problems, mainly discussing an historical case-study (namely, [35]) whose idea there contained turned out to be so crucial for the next development of mathematics and theoretical physics, i.e., the one related to the discovery of parallelism in a generic Riemannian manifold. As regard then the latter context, seen the wideness of the related historical literature already existent, we think that enough is to recall only Hermann Weyl’s (cf. [60]) and Élie Cartan’s (cf. [14]) works on pure differential geometry and its applications to theoretical physics, to witness the remarkable prominence of the next developments of Levi-Civita’s idea. On the other hand, right lately, this notion has been also retaken by the current research in applied engineering, to show the extent relevance of the topic, still now: in this regard, here we quote only two recent, interesting works of structural

\[\text{Cf. [34], Ch. 2, Sect. 1.}\]
\[\text{Cf. [46], Sect. 6.}\]
\[\text{Cf. [19].}\]
mechanics applied to engineering sciences, just centred on Levi-Civita’s notion of parallel transport.

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