Experimentally realizable control fields in quantum Lyapunov control

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I. INTRODUCTION

Quantum technologies, such as quantum information processing, offer many advantages over their classical counterparts, but it is challenging to make these new technologies function robustly due to the presence of noise. Quantum control—the application of control theory to quantum systems—provides a gateway to develop robust quantum technologies \cite{1,2}. Open-loop control and closed-loop control (feedback control) are by definition distinct but complementary control methodologies \cite{3}. Open-loop control is usually independent of measurement while feedback control, in contrast, involves measurement. Both types of control have been shown to be powerfully experimental and theoretical tools in classical and quantum contexts. For example, optimal gate synthesis by open-loop control \cite{5}, decoherence suppression by bang-bang control and dynamical decoupling \cite{4}, stabilization of pure and entangled states by feedback control \cite{3}, as well as the applications of feedback control in precision metrology and hypothesis testing \cite{6}.

In spite of some progress made, there are significant challenges facing the field, in particular, the measurement-induced decoherence in quantum feedback control and that stemming from experimental realizations of control fields in open-loop control. Lyapunov control is a hybrid of techniques from open-loop and feedback controls, it uses a feedback design to construct control fields but applies the fields into the system in an open-loop way. In other words, Lyapunov control is used to design a feedback law firstly which is then used to find the open-loop control by simulating the closed-loop system; next the control is applied to the quantum system in an open-loop way. From the above description of Lyapunov control, we find that the Lyapunov control includes two steps. For any initial states and a system Hamiltonian (assumed to be known exactly), the first step is to design a control law, i.e., to calculate the control field by simulating the dynamics of the closed-loop system. The second step is to apply the control law to the control system as an open-loop control.

Although the Lyapunov control is not limited by the measurement-induced decoherence, it is often confronted with time delays and realizability of the control fields in practical experiments. In this paper, we will shed light on these issues by examining Lyapunov control on a finite dimensional system. The following issues will be clarified: (1) The robustness of Lyapunov control against time-delay in the control fields; (2) the number of pulses necessary to simulate the control fields; (3) the possibility to replace the control fields with bang-bang signals, and (4) the efficiency of the Lyapunov control.

The paper is organized as follows. In Sec. II, we introduce the model and explore the convergency of the Lyapunov control. In Sec. III, we examine the effect of time-delay on the control fidelity. In Sec. IV, we introduce a pulse train to simulate the control field, and study the dependence of the fidelity on the number of pulses. The possibility to replace the pulse train with bang-bang signals and the efficiency of the Lyapunov control are also explored in this section. Finally we end in Sec.V with discussions.

II. MODEL DESCRIPTION AND CONVERGENCY OF THE CONTROL

Consider a quantum system governed by (\hbar = 1)

\[ i\frac{\partial}{\partial t}\ket{\psi(t)} = (H_0 + f(t)H_1)\ket{\psi(t)}, \]

where \(H_0\) and \(H_1\) are \(n \times n\) Hermitian matrices. Here \(H_0\) is a time independent Hamiltonian, corresponding to the free evolution of the system in absence of any external fields. The external interaction is taken into account as couplings \(f(t)H_1\) between a control field \(f(t)\) and the
system (through a time-independent Hermitian operator $H_1$). The state $|\psi(t)\rangle$ is normalized, $\langle |\psi(t)\rangle|\psi(t)\rangle\rangle = 1$. Our goal is to steer an arbitrary initial state $|\psi_0\rangle$ to the ground state of $H_0$, say $|\phi_g\rangle$. Although the ground state of $H_0$ is specified as the target state in this paper, the present method is not limited to this choice. In fact the method works for any target states that are eigenstates of $H_0$. Define the following real-value function

$$V(|\psi(t)\rangle, |\phi_g\rangle) = 1 - |\langle \psi(t)|\phi_g\rangle|^2,$$

we find $V = 2f(t)\Im(\langle \phi_g|\psi(t)\rangle\langle \psi(t)|H_1|\phi_g\rangle)$, where $\Im(...)$ denotes the imaginary part of $(...)$. By choosing

$$f(t) = -k\Im(\langle \phi_g|\psi(t)\rangle\langle \psi(t)|H_1|\phi_g\rangle),$$

with $k > 0$, $V$ decreases along a trajectory that leads the system from the initial state to the target state. Namely, any control field of the form given in Eq. (3) ensures $V < 0$. With such a control field $f(t)$, the distance between the actual state $|\psi(t)\rangle$ and the goal state $|\phi_g\rangle$ decreases.

We prove the convergence of $|\psi(t)\rangle$ to the target state $|\phi_g\rangle$ (i.e., asymptotic stability) by showing that the LaSalle invariant set $\mathcal{V}$ of states satisfying $\dot{V} = 0$ does not contain trajectories of the system, except the trajectories that lead the system to the target state. Clearly, $\dot{V} = 0$ is equivalent to $\Im(\langle \phi_g|\psi(t)\rangle\langle \psi(t)|H_1|\phi_g\rangle) = 0$. Notice that $\Im(ab) = 0$ if and only if there exists a real numbers $\lambda$ such that $\lambda a = b^\ast$. This is exactly the case for $\Im(\langle \phi_g|\psi(t)\rangle\langle \psi(t)|H_1|\phi_g\rangle) = 0$, if $\langle \psi(t)|\phi_g\rangle \neq 0$. Therefore if $\langle \psi(t)|\phi_g\rangle$ belongs to $\mathcal{V}$, it must satisfy $\langle \phi_g|\lambda I - H_1|\psi(t)\rangle = 0$ or $\langle \psi(t)|\phi_g\rangle = 0$, where $I$ denotes the identity operator. This means that the set of states orthogonal to $|\phi_g\rangle$ or being eigenstates of $H_1$ must include the invariant set $\mathcal{V}$. Namely, $\mathcal{V} \subseteq \{|\psi(t)\rangle : \langle \psi(t)|\phi_g\rangle = 0 \text{ or } H_1|\psi(t)\rangle = \lambda|\psi(t)\rangle \text{ for any } \lambda\rangle$. For $\mathcal{V}$ to be asymptotic invariant, the state in $\mathcal{V}$ must be an eigenstate of $H_0$, because the free Hamiltonian can usually not be turned off. In other words, if a state is not an eigenstate of $H_0$, it must evolve even if the control field $f(t)$ is zero, and in consequence $f(t)$ would get a non-zero value and thus the state would be outside the invariant set. All these together yield that the LaSalle invariant set $\mathcal{V} = \{ |\psi(t)\rangle : H_0|\psi(t)\rangle = E_0|\psi(t)\rangle \text{ for any } E_0\}$. We call the states in the invariant set critical points of $V$, which represent the maximum or minimum of the Lyapunov function $V$. For a $n$-dimensional non-degenerate quantum system, there are $n$ critical points, now we show that the maximal critical value occurs only when $|\psi(t)\rangle = |\phi_g\rangle$.

Define operator $A = I - |\phi_g\rangle\langle \phi_g|$, the Lyapunov function can be rewritten as $V(|\psi(t)\rangle, |\phi_g\rangle) = \langle \psi(t)|A|\psi(t)\rangle$. Obviously, $A$ and $H_0$ have the same eigenstates. We denote the eigenstates of $A$ as $|A_j\rangle$ ($j = 1, 2, ..., n$) with corresponding eigenvalues $A_j$. To determine the structure of $V(|\psi(t)\rangle, |\phi_g\rangle)$ around one of its critical points, say $|\psi_c\rangle = |A_j\rangle$ (may be one of $1, 2, ..., n$), we consider an infinitesimal variation $|\psi_c + \delta \psi\rangle$ of $|\psi_c\rangle$ such that $\langle \psi_c + \delta \psi|\psi_c + \delta \psi\rangle = 1$. Express $|\psi_c + \delta \psi\rangle$ in the basis of the eigenvectors of $A$.

$$|\psi_c + \delta \psi\rangle = |A_c\rangle + \sum_{j=1}^{n} \delta_j|A_j\rangle. \quad (4)$$

The normalization condition $\langle \psi_c + \delta \psi|\psi_c + \delta \psi\rangle = 1$ follows,

$$(\delta_c^* + \delta_c) + \sum_{j=1}^{n} \delta_j^*\delta_j = 0.$$  \quad (5)

Then

$$V(|\psi_c + \delta \psi\rangle) - V(|\psi_c\rangle) = \sum_{j \neq c} (A_j - A_c)\delta_j^* \delta_j. \quad (5)$$

Considering $\delta_j$ as variation parameters and noting $\delta_j^* \delta_j \geq 0$, we find that the structure of $V(|\psi(t)\rangle, |\phi_g\rangle)$ around the critical point $|\psi_c\rangle$ depends on the ordering of the eigenvalues of $A$. $|\psi_c\rangle$ is a local maximum as a function of the variations $\delta_j$ if and only if $A_c$ is the largest eigenvalue of $A$, and a local minimum iff $A_c$ is the smallest eigenvalue and a saddle point otherwise. This observation leads us to suspect that the minimum of $V$ is asymptotically attractive, in other words, the control field based on this Lyapunov function would drive the open system to the eigenstate of $A$ with the smallest eigenvalue, which is exactly the target state $|\phi_g\rangle$.

### III. Time-Delay Effect

For practical implementation of the Lyapunov-based control, a non-negligible (positive or negative) time delay is required to be taken into account. These delays may stem from actuators and electronic devices in the control loop. Before moving to the analytical discussions, we first examine the effect of time-delay on the fidelity of the Lyapunov control through an example.

Take a 5-dimensional system as an example, and choose

$$H_0 = \begin{pmatrix}
0.2 & 0 & 0 & 0 & 0 \\
0 & 1.0 & 0 & 0 & 0 \\
0 & 0 & 0.8 & 0 & 0 \\
0 & 0 & 0 & 0.5 & 0 \\
0 & 0 & 0 & 0 & 0.6
\end{pmatrix}, \quad (6)$$

as the free Hamiltonian. Let us use the previous Lyapunov control in order to trap our system in the ground state $|\phi_g\rangle = (1, 0, 0, 0, 0)^T$ of $H_0$. Setting

$$H_1 = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad (7)$$

as the control Hamiltonian and writing the actual state of the system as

$$|\psi(t)\rangle = (\psi_1(t), \psi_2(t), \psi_3(t), \psi_4(t), \psi_5(t))^T. \quad (8)$$
we obtain the control field by substituting $|\psi(t)\rangle$, $|\phi_g\rangle$, and $H_1$ into Eq. (3).

$$f(t) = -\Im[\psi_1(t)\psi_2^*(t) + \psi_3^*(t) + \psi_4^*(t) + \psi_5^*(t)].$$ (9)

Consider Eq. (1) without time-delay in $f(t)$, it has been proved that [13], (1) if the spectrum of $H_0$ is not degenerate, the invariant set of the closed loop system is spanned by the eigenstates $|\Phi\rangle$ of $H_0$ such that $\langle\Phi|H_1|\phi_g\rangle = 0$; and (2) when the invariant set reduces to $\{|\phi_g\rangle, -|\phi_g\rangle\}$, the state $|\phi_g\rangle$ is exponentially stable, while $-|\phi_g\rangle$ is unstable. This means that the system would converge to $|\phi_g\rangle$ under the control. Hence in the case of no time-delay, the control field $f(t)$ perfectly steers the system into the target state $|\phi_g\rangle$. The situation changes when the time delay is not zero. Note that the controlled dynamics Eq. (1) depends on the delay through the control field $f(t - \tau)$ significantly, the control problem would be much more different than that without time-delay. We shall quantify the effect of time delay on the controlled dynamics through control fidelity defined by,

$$F(t) = |\langle\phi_g|\psi(t)\rangle|^2.$$ (10)

Without time delay, the system would approach $|\phi_g\rangle$ as time tends to the convergence time. This is not the case when the time delay is not zero. Suppose that the (unnormalized) final state in the presence of time delay is $|\psi(\infty)\rangle = |\phi_g\rangle + \frac{\lambda}{\sqrt{1 + |\lambda|^2}}|\phi_g^\perp\rangle$, where $|\phi_g^\perp\rangle$ denotes a state orthogonal to $|\phi_g\rangle$ and $\lambda$ is a complex number. It is easy to show that $f(\infty) = -\Im(\langle\phi_g|H^2|\psi(\infty)\rangle)\langle\psi(\infty)|H_1|\phi_g\rangle = 0$, hence up to the first order in $\tau$, $|\phi_g^\perp\rangle$ does not depend on $\tau$. This is the reason why the fidelity is independent of $\tau$ when $\tau$ is very small. For large $\tau$, e.g., the terms with $\tau^2$ are not negligible, $|\phi_g^\perp\rangle$ can be calculated by $\frac{\partial f}{\partial \tau} = \frac{1}{2}\frac{\partial^2 f}{\partial \tau^2}\tau$ with $|\psi(\infty)\rangle$ in place of $|\psi(t)\rangle$. Clearly, $|\phi_g^\perp\rangle$ depends on the delay time $\tau$, and negative and positive time delays play different role.

IV. APPROACHING THE CONTROL FIELD BY A PULSE TRAIN

Observing Eq. (3), we find that the control fields are a continuous function of time. To realize such fields in experiment, we need to manipulate the control fields at each point of time. This is a challenging task in practical applications. The solution to this problem is to replace the control field by a pulse train. Then natural questions arise: How many pulses in the train are necessary to simulate the control fields? Is the control fidelity sensitive to the amplitude and duration of the pulse? If not, can we replace the Lyapunov control by a bang-bang control? In the following, we will answer these questions.

We numerically simulate the open-loop system Eqs. (1), (4), (7) and (9) with a pulse train to replace the control
field \( f(t) \). Numerical results are presented in Fig.2. Two observations can be made. (1) 50 sequence pulses are enough for simulating the control fields, i.e., most initial states can be driven into the target state with 50 sequence pulses in place of the control field. (2) The convergence time changes (prolongs) when we use the pulse train to simulate the control field. As expected, the fidelity depends on the number of pulse significantly as shown in the lower panel. Note that 50 pulses are taken to simulate the control fields as shown in the lower panel.

Before closing this section, we address the issue of efficiency of the Lyapunov control. It is believed that the

\[
f(t) = \begin{cases} f_0, & f(t) > 0, \\ -f_0, & f(t) < 0, \\ 0, & f(t) = 0, \end{cases}
\]

where \( f_0 \) is a constant. Fig. 4 shows the fidelity as well as the bang-bang signals and suggests us to arrange the control field in the following way:

\[
\text{FIG. 3: Fidelity as a function of pulse number. The fidelity is an averaged result taken over 10 randomly chosen initial states.}
\]

\[
\text{FIG. 4: Pulse and fidelity as a function of time. The initial state is randomly created, and in this plot, it is (0.4314, 0.3627, 0.5948, 0.3991, 0.4114)\%}.
\]

strictly in between the bounds) then that control is referred to as a bang-bang one. The bang-band controls are often implemented because of simplicity or convenience. The physical idea behind bang-bang control comes from refocusing techniques of NMR spectroscopy \cite{8}: control cycles are implemented in time via a sequence of strong and rapid pulses that provide a full decoupling from the environment. The decoherence suppression results in the increase of the NMR transversal relaxation time \( T_2 \), which is related to dephasing. Besides NMR, dynamical decoupling has been suggested for inhibiting the decay of unstable atomic states \cite{9, 10}, suppressing the decoherence of magnetic states \cite{11}, and reducing the heating in ion traps \cite{12}.

Noticing the simplicity and wide-range application of the bang-bang control, one may wonder whether we can use the bang-bang signal to replace the control field \( f(t) \). The answer is yes. Before going to the detail, let us first examine Eq. (3). The rate \( k \) in the control field \( f(t) \) implicates that the amplitude of the control field is not important, in contrast, its sign plays an exclusive role to guarantee \( \dot{V} < 0 \). This is reminiscent of the bang-bang signals and suggests us to arrange the control field in the following way:

\[
\text{FIG. 2: Fidelity versus time (upper panel), and pulse amplitude as a function of time (lower panel). The initial states for the upper panel are randomly chosen, while the control field shown in the lower panel corresponds to the red-dot line in the upper panel. Note that 50 pulses are taken to simulate the control fields as shown in the lower panel.}
\]
Accordingly, the fidelity degrades with the increase of the dimension of the control system. This finding suggests that Lyapunov-based control becomes difficult for high dimensional systems, but it is still an effective method to steer an open quantum system.

FIG. 5: The convergence time (upper panel) and the control fidelity (lower panel) versus dimension of the control system. The convergence time is defined as a time duration by which the averaged fidelity arrives at or larger than 0.95. The fidelity shown in the lower panel is taken at time 150. All fidelities used here are averaged over 100 randomly created initial states.

The Lyapunov control fields are traditionally required to manipulate at each point of time. In this paper, we have proposed a scheme to replace the Lyapunov control field by a pulse train or by a bang-bang signal. Our analysis and numerical simulations show that the control field can be well simulated by a pulse train, in which each pulse has the same time duration but different amplitude. We find that the control fidelity depends on the number of pulses in the train, and the dependence of the fidelity on the pulse number is numerically given. The possibility to replace the Lyapunov control field by a bang-bang signal is also explored. Surprisingly, the bang-bang signal works well in the Lyapunov control to replace the Lyapunov control fields. The physics behind the replacement is given and analyzed. The time-delay effect and the efficiency of the control under the effect of the time-delay are also examined in this paper. These results suggest that the Lyapunov control not only has the advantage combining the feedback and open-loop control, but it is also easy to realize for finite dimensional quantum systems.

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V. SUMMARY AND DISCUSSIONS

The Lyapunov control fields are traditionally required to manipulate at each point of time. In this paper, we have proposed a scheme to replace the Lyapunov control field by a pulse train or by a bang-bang signal. Our analysis and numerical simulations show that the control field can be well simulated by a pulse train, in which each pulse has the same time duration but different amplitude. We find that the control fidelity depends on the number of pulses in the train, and the dependence of the fidelity on the pulse number is numerically given. The possibility to replace the Lyapunov control field by a bang-bang signal is also explored. Surprisingly, the bang-bang signal works well in the Lyapunov control to replace the Lyapunov control fields. The physics behind the replacement is given and analyzed. The time-delay effect and the efficiency of the control under the effect of the time-delay are also examined in this paper. These results suggest that the Lyapunov control not only has the advantage combining the feedback and open-loop control, but it is also easy to realize for finite dimensional quantum systems.

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