SYMMETRY IN EXTENDED FORMULATIONS OF THE PERMUTAHEDRON

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ABSTRACT. It is well known that the permutahedron \( \Pi_n \) has \( 2^n - 2 \) facets. The Birkhoff polytope provides a symmetric extended formulation of \( \Pi_n \) of size \( \Theta(n^2) \). Recently, Goemans described a non-symmetric extended formulation of \( \Pi_n \) of size \( \Theta(n \log n) \). In this paper, we prove that \( \Omega(n^2) \) is a lower bound for the size of symmetric extended formulations of \( \Pi_n \).

1. INTRODUCTION

Extended formulations of polyhedra have gained importance in the recent past, because this concept allows to represent a polyhedron by a higher-dimensional one with a simpler description.

To illustrate the power of extended formulations we take a look at the permutahedron \( \Pi_n \subseteq \mathbb{R}^n \), which is the convex hull of all points obtained from \((1, 2, \cdots, n) \in \mathbb{R}^n \) by coordinate permutations. The minimal description of \( \Pi_n \) in the space \( \mathbb{R}^n \) looks as follows \([1]\):

\[
\sum_{v \in [n]} x_v = \frac{n(n + 1)}{2} \\
\sum_{v \in S} x_v \geq \frac{|S|(|S| + 1)}{2} \quad \text{for all } \emptyset \neq S \subset [n]
\]

Thus the permutahedron \( \Pi_n \) has \( n! \) vertices and \( 2^n - 2 \) facets. At the same time it is easy to derive an extended formulation of size \( \Theta(n^2) \) from the Birkhoff polytope \([1]\):

\[
\sum_{i \in [n]} iz_{i,v} = x_v \quad \text{for all } v \in [n] \\
\sum_{v \in [n]} z_{i,v} = 1 \quad \text{for all } i \in [n] \\
\sum_{i \in [n]} z_{i,v} = 1 \quad \text{for all } v \in [n] \\
z_{i,v} \geq 0 \quad \text{for all } i, v \in [n]
\]

The projection of the polyhedron described by \([1]\) to the \( x \)-variables gives the permutahedron \( \Pi_n \). Clearly, every coordinate permutation of \( \mathbb{R}^n \) maps \( \Pi_n \) to itself.

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The extended formulation (1) respects this symmetry in the sense that every such permutation of the $x$-variables can be extended by some permutation of the $z$-variables such that these two permutations leave (1) invariant (up to reordering of the constraints).

Also there exists a non-symmetric extended formulation of the permutahedron of size $\Theta(n \log(n))$ \cite{2}. This is the best one can achieve \cite{2} due to the fact that every face of $\Pi_n$ (including the $n!$ vertices) is a projection of some face of the extension. And since the number of faces of a polyhedron is bounded by 2 to the number of its facets, we can conclude that every extension of the permutahedron has at least $\log_2(n!) = \Theta(n \log(n))$ facets.

As we show in this paper the size of the extended formulation (1) is asymptotically optimal for symmetric formulations of the permutahedron. Thus there exists a gap in size between symmetric and non-symmetric extended formulations of $\Pi_n$. This situation appears in some other cases as well, e.g. the cardinality constrained matching polytopes and the cardinality constrained cycle polytopes \cite{3}. But even if the gaps observed in those cases are more substantial, the permutahedron is interesting because of the possibility to determine tight asymptotical lower bounds $\Omega(n^2)$ and $\Omega(n \log(n))$ on the sizes of symmetric and non-symmetric extended formulations.

The paper is organised as follows. Section 2 contains definitions of extensions, the crucial notion of section and some auxilary results. In section 3 we exploit some known techniques \cite{3}, \cite{4} and some new approaches to prove a lower bound on the number of variables and facets in symmetric extensions of the permutahedron.

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2. Extensions, Sections and Symmetry

Here we list some known definitions and results, which will be used later. For a broader discussion of symmetry in extended formulations of polyhedra we refer the reader to \cite{3}.

A polytope $Q \subseteq \mathbb{R}^d$ together with a linear map $p : \mathbb{R}^d \to \mathbb{R}^m$ is called an extension of a polytope $P \subseteq \mathbb{R}^m$ if the equality $p(Q) = P$ holds. Moreover, if $Q$ is the intersection of an affine subspace of $\mathbb{R}^d$ and the nonnegative orthant $\mathbb{R}_+^d$ then $Q$ is called a subspace extension. A (finite) system of linear equations and inequalities whose solutions are the points in an extension $Q$ of $P$ is an extended formulation for $P$.

Through the proof we mostly deal with such objects as sections $s : X \to Q$, which are maps that assign to every vertex $x \in X$ of $P$ some point $s(x) \in Q \cap p^{-1}(x)$. Such a section induces a bijection between $X$ and its image $s(X) \subseteq Q$, whose inverse is given by $p$.

For an arbitrary group $G \subseteq \mathbb{S}(m)$ acting on the set $X$ of vertices of $P$, an extension as above is symmetric (with respect to the action of $G$ on $X$), if for
every \( \pi \in G \) there is a permutation \( \pi_\pi \in \mathfrak{S}(d) \) with \( \pi_\pi . Q = Q \) and

\[
p(\pi_\pi . y) = \pi . p(y) \quad \text{for all } y \in p^{-1}(X).
\] (2)

We define an extended formulation \( A^\pi y = b^\pi, A^\leq y \leq b^\leq \) describing the polyhedron

\[
Q = \{ y \in \mathbb{R}^d : A^\pi y = b^\pi, A^\leq y \leq b^\leq \}
\]

extending \( P \subseteq \mathbb{R}^m \) as above to be symmetric (with respect to the action of \( G \) on the set \( X \) of vertices of \( P \)), if for every \( \pi \in G \) there is a permutation \( \pi_\pi \in \mathfrak{S}(d) \) satisfying (2) and there are two permutations \( \varphi^\pi_\pi \) and \( \varphi^\leq_\pi \) of the rows of \( (A^\pi, b^\pi) \) and \( (A^\leq, b^\leq) \), respectively, such that the corresponding simultaneous permutations of the columns and the rows of the matrices \( (A^\pi, b^\pi) \) and \( (A^\leq, b^\leq) \) leaves them unchanged. Clearly, in this situation the permutations \( \pi_\pi \) satisfy \( \pi_\pi . Q = Q \), which implies the following.

**Lemma 1.** Every symmetric extended formulation describes a symmetric extension.

We call an extension weakly symmetric (with respect to the action of \( G \) on \( X \)) if there is a section \( s : X \to Q \) such that for every \( \pi \in G \) there is a permutation \( \pi_\pi \in \mathfrak{S}(d) \) with \( s(\pi . x) = \pi_\pi . s(x) \) for all \( x \in X \). It can be shown that every symmetric extension is weakly symmetric.

Dealing with weakly symmetric extension we can define an action of \( G \) on the set \( S = \{ s_1, \ldots, s_d \} \) of the component functions of the section \( s : X \to Q \) with \( \pi . s_j = s_{\pi^{-1}(j)} \in S \) for each \( j \in [d] \). In order to see that this definition indeed yields a group action, observe that, for each \( j \in [d] \), we have

\[
(\pi.s_j)(x) = s_{\pi^{-1}(j)}(\pi(x)) = (\pi_{\pi^{-1}}.s(x))_j = s_j(\pi^{-1}.x) \quad \text{for all } x \in X,
\] (3)

from which one deduces \( \text{id}_m \cdot s_j = s_j \) for the identity element \( \text{id}_m \) in \( G \) as well as \( (\pi \pi').s_j = \pi_\pi.(\pi'.s_j) \) for all \( \pi, \pi' \in G \). The isotropy group of \( s_j \in S \) under this action is

\[
\text{iso}(s_j) = \{ \pi \in G : \pi . s_j = s_j \}.
\]

From (3) one sees that, \( \pi \) is in \( \text{iso}(s_j) \) if and only if we have \( s_j(x) = s_j(\pi^{-1}.x) \) for all \( x \in X \) or, equivalently, \( s_j(\pi.x) = s_j(x) \) for all \( x \in X \). In general, it will be impossible to identify the isotropy groups \( \text{iso}(s_j) \) without more knowledge on the section \( s \). However, for each isotropy group \( \text{iso}(s_j) \), one can at least bound its index \( (G : \text{iso}(s_j)) \) by the number of variables:

\[
(G : \text{iso}(s_j)) \leq d
\]

The next lemma could be used to transform an arbitrary symmetric extension to some subspace symmetric extension.

**Lemma 2.** If there is a symmetric extension in \( \mathbb{R}^d \) with \( f \) facets for a polytope \( P \), then there is also a symmetric subspace extension in \( \mathbb{R}^d \) with \( d \leq 2d + f \) for \( P \).

Using this lemma and having a lower bound on the number of variables in symmetric subspace extensions of the given polytope \( P \) one gets a lower bound on the total number of facets and variables in any symmetric extension of \( P \).
3. **Bound on Symmetric Subspace Extension of the Permutahedron**

Now we would like to establish a lower bound on the number of variables in symmetric subspace extensions of the permutahedron.

**Theorem 3.** For every \( n \geq 6 \) there exists no weakly symmetric subspace extension of the permutahedron \( \Pi_n \) with less than \( \frac{n(n-1)}{2} \) variables (with respect to the group \( G = \mathfrak{S}(n) \) acting via permuting the corresponding elements).

For the proof, we assume that \( Q \subseteq \mathbb{R}^d \) with \( d < \frac{n(n-1)}{2} \) is a weakly symmetric subspace extension of \( \Pi_n \). Weak symmetry is meant with respect to the action of \( G = \mathfrak{S}(n) \) on the set \( X \) of vertices of \( \Pi_n \) and we assume \( s : X \to Q \) to be a section as required in the definition of weak symmetry.

The operator \( \Lambda(\zeta) \) maps any permutation \( \zeta \) to the vector \( (\zeta^{-1}(1), \zeta^{-1}(2), \ldots, \zeta^{-1}(n)) \). Thus, we have

\[
X = \{ \Lambda(\zeta) : \zeta \in \mathfrak{S}(n) \}
\]

where \( \mathfrak{S}(n) \) is the set of all permutations on the set \([n] \), and

\[
(\pi.\Lambda(\zeta))_v = \Lambda(\zeta)_{\pi^{-1}(v)}
\]

holds for all \( \pi \in \mathfrak{S}(n), \zeta \in \mathfrak{S}(n) \).

In order to identify suitable subgroups of the isometry groups \( \text{iso}(s_j) \), we use the following result on subgroups of the symmetric group \( \mathfrak{S}(n) \) [4]. Here \( \mathfrak{A}(n) \) denotes the alternating group, i.e. the group consisting of all even permutations on the set \([n] \).

**Lemma 4.** For each subgroup \( U \) of \( \mathfrak{S}(n) \) with \( (\mathfrak{S}(n) : U) \leq \binom{n}{k} \) for \( k < \frac{n}{4} \), there is some \( W \subseteq [n] \) with \( |W| \leq k \) such that

\[
\{ \pi \in \mathfrak{A}(n) : \pi(v) = v \text{ for all } v \in W \} \subseteq U
\]

holds.

As we assumed \( d < \binom{n}{2} \), Lemma 4 implies that for all \( j \in [d] \)

\[
\{ \pi \in \mathfrak{A}(n) : \pi(v) = v \text{ for all } v \in V_j \} \subseteq \text{iso}(s_j)
\]

for some set \( V_j \subseteq [n], |V_j| \leq 2 \). We can prove that \( V_j \) can be chosen to contain not more than one element, which we denote by \( v_j \).

**Lemma 5.** For each \( j \in [d] \) there is some \( v_j \in [n] \) such that

\[
\{ \pi \in \mathfrak{A}(n) : \pi(v_j) = v_j \} \subseteq \text{iso}(s_j)
\]

This element \( v_j \) is uniquely determined unless \( \mathfrak{A}(n) \subseteq \text{iso}(s_j) \)

**Proof:** The statement of lemma is automatically true if the set \( V_j \) is empty or contains just one element. So let us assume the set \( V_j \) to consist of two elements \( \{v, w\} \). If the group \( \text{iso}(s_j) \) has two fixed blocks \( V_j \) and \([n] \setminus V_j \), the following inequality

\[
d < \frac{n(n-1)}{2} \leq (\mathfrak{S}(n) : \text{iso}(s_j))
\]

would contradict the assumed boundedness.
holds. Thus we can find some permutation \( \tau \in \text{iso}(s_j) \) such that \( \text{w.l.o.g. } \tau(v) \not\in \{v, w\} \).

Later it would be convenient to have \( \tau(w) = w \) and \( \tau \in \mathfrak{A}(n) \). If \( \tau(w) \neq w \) or \( \tau \not\in \mathfrak{A}(n) \) we regard a new permutation \( \tau' = \tau^{-1}\beta \tau \in \mathfrak{A}(n) \), where \( \beta \in \mathfrak{A}(n) \), \( \beta(v) = v, \beta(w) = w, \beta \tau(w) = \tau(w) \) and \( \beta \tau(v) \neq \tau(v) \). For every \( n \geq 6 \) such permutation \( \beta \) can be found since \( \tau(v) \not\in \{v, w, \tau(w)\} \). The construction of \( \tau' \) guarantees that \( \tau'(w) = w, \tau'(v) \neq v \) and \( \tau' \in \text{iso}(s_j) \). Hence we can assume that \( \tau(w) = w \) and \( \tau \in \mathfrak{A}(n) \).

To prove that the described in the lemma element \( v_j \) exists we can show that the element \( w \) is one of such possibilities

\[
\{ \pi \in \mathfrak{A}(n) : \pi(w) = w \} \subseteq \text{iso}(s_j)
\]

Any permutation \( \pi \in \mathfrak{A}(n), \pi(w) = w \) with the property \( \pi(v) \neq v \) can be represented as \( (\pi \tau \alpha)^{-1} \pi \alpha \) for any \( \alpha \in \mathfrak{S}(n) \). We choose a permutation \( \alpha \in \mathfrak{A}(n) \) such that \( \alpha(v) = v, \alpha(w) = w \) and \( \alpha \pi^{-1}(v) = \tau^{-1}(v) \). The existence of \( \alpha \) can be trivially proved for \( n \geq 6 \). Thus the permutation \( \pi \) belongs to \( \text{iso}(s_j) \) because all three permutations \( \tau, \alpha \) and \( \pi \tau \alpha^{-1} \) belong to \( \text{iso}(s_j) \) (note that \( \pi \tau \alpha^{-1} \) and \( \alpha \) are even permutations and fix elements \( v, w \)).

Any permutation \( \pi \in \mathfrak{A}(n), \pi(w) = w \) belongs to \( \text{iso}(s_j) \) whenever \( \pi(v) = v \).

Therefore we can conclude that (4) holds.

Having some other element \( u \in [n], u \neq w \) such that

\[
\{ \pi \in \mathfrak{A}(n) : \pi(u) = u \} \subseteq \text{iso}(s_j)
\]

we can prove that \( \mathfrak{A}(n) \subseteq \text{iso}(s_j) \), since every permutation \( \pi \in \mathfrak{A}(n) \) is a composition of not more than four permutations described by (4) and (5). \( \square \)

The next theorem has an important role in the proof because it describes the action of \( \mathfrak{A}(n) \) on the components \( s_j \).

**Theorem 6.** There exists a partition of the set \([d]\) into sets \( A_i \) and \( B_j \), such that each set \( B_j \) consists just of one element \( b_j \) and each set \( A_i \) consists of \( n \) elements \( a_1^i, a_2^i, \ldots, a_{r_i}^i \) elements with

\[
s_{a_1^i}(\pi.x) = s_{a_{r_i}^i}(x) \quad s_{b_j}(\pi.x) = s_{b_j}(x)
\]

for any vertex \( x \in X \) and all \( \pi \in \mathfrak{A}(n) \).

Let us consider, for \( v \in [n-2] \), permutations \( \rho_v \) consisting of just one cycle \( (v, v+1, v+2) \). We would like to show the existence of a partition \( A_i, \{b_j\} \) which satisfies all cardinality assumptions of Theorem 5 and satisfies equation (6) for all \( x \in X \) and all permutations \( \rho_v \). Such a partition satisfies Theorem 6 because every permutation \( \pi \in \mathfrak{A}(n) \) is a product of permutations \( \rho_v \).

Through the whole proof the action of \( \mathfrak{S}(d) \) is restricted to the action on vectors \( s(x) \) for \( x \in X \). It means that two permutation \( \pi' \) and \( \pi \) from \( \mathfrak{S}(d) \) are equivalent for us if \( s_{\pi'^{-1}(j)}(x) = s_{\pi^{-1}(j)}(x) \) for all \( x \) from \( X \) and for all \( j \) from \( [d] \). For example, we can take the identity permutation \( \text{id}_d \) instead of \( \pi \) if \( s_{\pi^{-1}(j)}(x) = s_j(x) \) for all \( x \in X \) and all \( j \in [d] \).
Lemma 7. For each $\pi = (w_1, w_2, w_3) \in A(n)$ there exists a permutation in $S(d)$, which is equivalent to $\pi$ such that all cycles of this permutation are of the form $(j_1, j_2, j_3)$ with $v_{j_1} = w_t$ and $A(n) \not\subseteq \text{iso}(s_{j_1})$ for all $t \in [3]$. 

Proof. The permutation $\pi^3$ is equivalent to the identity permutation $\text{id}_d$ since the permutation $\pi^3$ is the identity permutation $\text{id}_n$.

Thus any cycle $C$ of the permutation $\pi$ permutes indices of identical component functions of $s$ if the cycle length $|C|$ is not divisible by three. Hence, we can assume that every cycle $C$ of $\pi$ has length $|C| = 0 \mod 3$.

The same argument allows us to transform each cycle $C = (j_1, j_2, \cdots, j_3k)$ of the permutation $\pi$ into the following cycles $(j_1, j_2, j_3), \cdots, (j_{3k-2}, j_{3k-1}, j_{3k})$, offering an equivalent permutation to $\pi$. Thus we may assume that $\pi$ contains cycles of length three only.

Let us consider one of the cycles $(j_1, j_2, j_3)$ of the permutation $\pi$. We investigate two possible cases.

If the element $v_{j_1}$ does not belong to \{w_1, w_2, w_3\} or $A(n) \subseteq \text{iso}(s_{j_1})$ then we have $\pi \in \text{iso}(s_{j_1})$ and thus $\pi, \pi^2 \in \text{iso}(s_{j_1})$, which yields

$s_{j_1}(x) = s_{j_1}(\pi.x) = s_{\pi^{-1}(j_1)}(x) = s_{j_1}(x)$
$s_{j_1}(x) = s_{j_1}(\pi^2.x) = s_{\pi^{-2}(j_1)}(x) = s_{j_2}(x)$

This shows that the component functions $s_{j_1}, s_{j_2}, s_{j_3}$ are identical. Thus the cycle $(j_1, j_2, j_3)$ can be deleted.

Hence, we may w.l.o.g. assume $v_{j_1} = w_1$. For each $\tau' \in A(n)$ with $\tau'(w_3) = w_3$ and $\pi := \pi\tau'\pi^{-1} \in A(n)$ we have $\tau(w_1) = \pi\tau'\pi^{-1}(w_1) = \pi\tau'(w_3) = \pi(w_3) = w_1$. Since $\tau \in A(n)$ and $\tau(w_1) = w_1$ we have $\tau \in \text{iso} s_{j_1}$, and thus (see the fifth equation in the following chain) for all $x \in X$

$s_{j_1}(\pi^{-1}\pi.x) = s_{\pi^{-1}(j_1)}(\pi^{-1}\pi.x) = s_{j_1}(\pi^{-1}\pi.x) = s_{j_1}(\tau.x) = s_{\pi^{-1}(j_1)}(x) = s_{j_3}(x)$

Hence $\tau' \in \text{iso}(s_{j_3})$ holds. Thus we have $v_{j_3} = w_3$, unless $A(n) \subseteq \text{iso}(s_{j_3})$, which as in the treatment of the case $A(n) \not\subseteq \text{iso}(s_{j_1})$ would allow us to remove the cycle $(j_1, j_2, j_3)$. Similarly, one can establish $v_{j_2} = w_2$. \hfill $\square$

Now we want to prove the next useful lemma, which will help us to construct the desired partition.

Lemma 8. Let two permutations $\pi = (w_1, w_2, w_3)$ and $\sigma = (w_2, w_3, w_4)$ be given with $w_1 \neq w_4$ and suppose that the corresponding permutations $\pi_\pi$ and $\pi_\sigma$ satisfy the conditions from Lemma 7. If the permutation $\pi_\pi$ contains a cycle $(j_1, j_2, j_3)$ with $v_{j_1} = w_t$ for all $t \in [3]$ then one of the following is true:

a) The permutation $\pi_\pi$ contains a cycle $(j_2, j_3, j_4)$ with $v_{j_4} = w_4$.

b) The permutation $\pi_\sigma$ contains two cycles $(j_2, j_3, j_4)$ and $(j_2''', j_3, j_4')$ with $v_{j_2'''} = w_2, v_{j_3} = w_3$ and $v_{j_4'} = v_{j_4''} = w_4$.

Proof. Assume that permutation $\pi_\pi$ does not contain any cycle involving the index $j_2$. Every permutation $\mu \in A(n)$ can be represented as a combination $\tau'\sigma\tau$, where $\tau' \in A(n)$ and $\sigma \in S(d)$ have disjoint sets of fixed points.

We can apply the same argument as in the proof of Lemma 7, that is, if the cycle length $|C|$ is not divisible by three, then we can assume that every cycle $C$ of $\pi_\pi$ has length $|C| = 0 \mod 3$. The same argument allows us to transform each cycle $C = (j_1, j_2, j_3) \cdots, (j_{3k-2}, j_{3k-1}, j_{3k})$ of the permutation $\pi_\pi$ into the following cycles $(j_1, j_2, j_3), \cdots, (j_{3k-2}, j_{3k-1}, j_{3k})$, offering an equivalent permutation to $\pi_\pi$. Thus we may assume that $\pi_\pi$ contains cycles of length three only.
where \( \tau', \tau \) are even permutations with \( \tau'(w_2) = \tau(w_2) = w_2 \). Thus for any permutation \( \mu \in \mathfrak{S}(n) \) we have

\[
s_{j_2}(\mu.x) = s_{j_2}(\tau' \sigma \tau.x) = s_{j_2}(\sigma \tau.x) = s_{x_{\sigma^{-1}(j_2)}}(\tau.x) = s_{j_2}(\tau.x) = s_{j_2}(x)
\]

This contradicts conditions on \( x_\sigma \) from Lemma 7. We proceed in a similar way when no cycle in \( x_\sigma \) involves \( j_3 \).

If there are two different cycles \( (j_2', j_3', j_4') \) and \( (j_2'', j_3, j_4'') \) in the permutation \( x_\sigma \) we would like to prove that the component functions mentioned in the lemma are identical. For this let us consider the permutation \( \pi \sigma \) which could be written as a combination of two disjoint cycles \( (w_1,w_2)(w_3,w_4) \). From this we can conclude that \( (\pi \sigma)^2 \) is the identity permutation \( \id_n \), what implies that \( (x_\sigma x_\rho)^2 \) is equivalent to \( \id_n \).

For all \( x \in X \) we have

\[
s_{j_3}(x) = s_{j_3}(\pi^2 \sigma.x) = s_{x_{\pi^{-1}(j_3)}}(\pi \sigma \sigma \sigma.x) = s_{j_2}(\pi \sigma \sigma \sigma.x) = s_{x_{\pi^{-1}(j_2)}}(\pi \sigma \sigma.x)
\]

and we can continue this chain of equations using that \( v_{j_3'} = w_4 \) is not equal to any of the elements \( w_1, w_2 \) and \( w_3 \) in the following way:

\[
s_{x_{\pi^{-1}(j_2)}}(\pi \sigma \sigma.x) = s_{j_4'}(\pi \sigma \sigma.x) = s_{j_4'}(\sigma \sigma.x) = s_{x_{\pi^{-1}(j_1)}}(x) = s_{j_4}(x)
\]

Thus we proved that \( s_{j_3} \) and \( s_{j_3'} \) are identical component functions. Considering expression \( s_{j_2}(\pi^2 \sigma.x) \) we get that \( s_{j_2} \) and \( s_{j_2'} \) are identical as well.

\[
\square
\]

**Lemma 9.** For every cycle \( (j_1, j_2, j_3) \) in the permutation \( x_{\rho_1} \) we can find a set \( S_{(j_1, j_2, j_3)} = \{j_1, j_2, \ldots, j_n\} \) such that, for every \( v \in [n - 2] \), there is a permutation equivalent to \( \rho_v \), which contains the cycle \( (j_v, j_{v+1}, j_{v+2}) \) and has the properties required in Lemma 7.

**Proof:** Let us construct the set \( S_{(j_1, j_2, j_3)} \) in several steps. We start with \( S_{(j_1, j_2, j_3)} = \{j_1, j_2, j_3\} \), which satisfies the condition of the lemma for \( v = 1 \).

By Lemma 5 with \( \pi = \rho_1 \) and \( \sigma = \rho_2 \) we have to consider two possible cases concerning the cycle \( (j_1, j_2, j_3) \). In case 1(b) we can extend the set \( S_{(j_1, j_2, j_3)} \) to \( \{j_1, j_2, j_3, j_4\} \), such that \( S_{(j_1, j_2, j_3)} \) satisfies the conditions in the lemma for \( v = 1, 2 \). In case 1(c) we can update \( x_{\rho_2} \) by changing cycles \( (j_2, j_3, j_4'), (j_2'', j_3, j_4'') \) to \( (j_2, j_3, j_4'), (j_2', j_3', j_4') \), what will produce a permutation equivalent to \( x_{\rho_2} \). And we can choose \( S_{(j_1, j_2, j_3)} \) be equal to \( \{j_1, j_2, j_3, j_4'\} \).

In this manner we go through all \( v \in [n - 2] \) setting \( \pi = x_\rho_{v-1} \) and \( \sigma = x_\rho_v \) to extend the set \( S_{(j_1, j_2, j_3)} \) and, if necessary, to update the permutation \( x_{\rho_v} \).

Applying Lemma 5 to all cycles of \( x_{\rho_1} \) we get some disjoint sets \( S_{(j_1, j_2, j_3)} \) indexed by subsets of cycles of \( x_{\rho_1} \). Moreover, there is no cycles in \( x_{\rho_2}, \ldots, x_{\rho_{n-2}} \), which does not contain any index from the constructed sets \( S_{(j_1, j_2, j_3)} \). This is due to Lemma 8 applied to pairs \( \pi = x_\rho_v, \sigma = x_\rho_{v-1} \) for \( v \) ranging from \( 2 \) to \( n - 2 \) (in this order).

Now we can choose the sets \( A_i \) to be the sets \( S_{(j_1, j_2, j_3)} \) and the singletons \( \{b_j\} \) accordingly. Lemma 2 guarantees equation (6).
Now we have some understanding of how permutations $\mathfrak{A}(n)$ act on the component functions of $s$. Having this knowledge we can create an affine combination of points in the subspace extension $Q$ of $\Pi_n$, which has non-negative components, but does not project to the permutahedron $\Pi_n$.

Let us introduce the following subgroup of $\mathfrak{A}(n)$ defined by one element $w$ of the set $[n-1]$

$$H^*_w = \{ \pi \in \mathfrak{A}(n) : \pi([w]) = [w] \}$$

Thus $H^*_w$ is the set of all even permutations of $[n]$, which map $[w]$ to itself.

Let us consider the function $s^* : X \times [n-1] \to \mathbb{R}$ defined via

$$s^*(x, w) = \frac{\sum_{\pi \in H^*_w} s(\pi.x)}{|H^*_w|}$$

The value of $s^*$ is a convex combination of points from $Q$ and thus lies in $Q$ for any $x \in X$.

Let us consider a partition $A_i, \{b_j\}$ as in Theorem 5. First, we take a look at the sets $B_j$ and any vertex $x$ from $X$:

$$s^*_{b_j}(x, w) = \frac{\sum_{\pi \in H^*_w} s_{b_j}(\pi.x)}{|H^*_w|} = \frac{\sum_{\pi \in H^*_w} s_{b_j}(x)}{|H^*_w|}$$

Hence the value of $s^*_{b_j}$ depends just on $x$ and is equal to the value of $s_{b_j}(x)$ for any vertex $x \in X$.

For the components corresponding to $a^i_t$ from the set $A_i$, Theorem 6 gives us

$$s^*_t(x, w) = \frac{\sum_{\pi \in H^*_w} s_{a^i_t}(\pi.x)}{|H^*_w|} = \sum_{\pi \in H^*_w} \frac{s_{a^i_t}(x)}{|H^*_w|}$$

When $t$ belongs to the set $[w]$, we can calculate the component $s^*_t$ as

$$s^*_t(x, w) = \frac{\sum_{\pi \in H^*_w} 1}{|H^*_w|} \sum_{\pi \in H^*_w} s_{a^i_t}(x)$$

Moreover, we can estimate the number of permutations in the second sum by

$$|\{\pi \in H^*_w : \pi^{-1}(t) = u\}| = \frac{(w - 1)!(n - w)!}{2}$$

and thus conclude that

$$s^*_t(x, w) = \sum_{v \leq w} \frac{1}{|H^*_w|} \sum_{\pi \in H^*_w} s_{a^i_t}(x) = \sum_{v \leq w} \frac{s_{a^i_t}(x)}{w}$$

When $t$ does not belong to $[w]$, we can derive a similar equality

$$s^*_t(x, w) = \sum_{v > w} \frac{s_{a^i_t}(x)}{n - w}$$
Now we would like to outline the idea of the contradiction. Let us assume that we found some element \( w \) of the set \([n - 1]\) such that the statements

\[
\text{if } s_{a_i}^{i} (\Lambda(id_n)) > 0 \text{ then } \sum_{v > w} s_{a_i}^{i} (\Lambda(id_n)) > 0
\]  

and

\[
\text{if } s_{a_{w+1}}^{i} (\Lambda(id_n)) > 0 \text{ then } \sum_{v \leq w} s_{a_i}^{i} (\Lambda(id_n)) > 0
\]

hold for all sets \( A_i \) (we will establish the existence of such a \( w \) in Lemma \ref{lem:existence} below).

Let \( \zeta \) be some even permutation such that the conditions

\[
\zeta([w - 1]) \subset [w] \quad \text{and} \quad \zeta(w + 1) \in [w]
\]

are fulfilled. Since the permutation \( \zeta \) is even, by (6) we get

\[
s_{b_j} (\Lambda(\zeta)) = s_{b_j} (\Lambda(id_n)) \quad \text{and} \quad s_{a_i} (\Lambda(\zeta)) = s_{a_i} (\Lambda(id_n))
\]

The point \( y = (1 + \epsilon)s^{i} (\Lambda(id_n), w) - \epsilon s^{i} (\Lambda(\zeta), w) \) is an affine combination of points from \( Q \). The components \( b_j \) of the point \( y \) are equal to the non-negative value \( s_{b_j} (\Lambda(id_n)) \). The component \( a_i \) is equal to the value

\[
\frac{1}{w} ((1 + \epsilon) \sum_{v \leq w} s_{a_i}^{i} (\Lambda(id_n)) - \epsilon \sum_{v \leq w - 1} s_{a_i}^{i} (\Lambda(id_n)) - \epsilon s_{a_{w+1}}^{i} (\Lambda(id_n)))
\]

in the case \( t \leq w \) and to the value

\[
\frac{1}{n - w} ((1 + \epsilon) \sum_{v > w} s_{a_i}^{i} (\Lambda(id_n)) - \epsilon \sum_{v > w + 1} s_{a_i}^{i} (\Lambda(id_n)) - \epsilon s_{a_w}^{i} (\Lambda(id_n)))
\]

in the case \( t > w \). After simplification those values look like

\[
\frac{1}{w} \left( \sum_{v \leq w} s_{a_i}^{i} (\Lambda(id_n)) + \epsilon s_{a_{w+1}}^{i} (\Lambda(id_n)) - \epsilon s_{a_w}^{i} (\Lambda(id_n)) \right)
\]

and

\[
\frac{1}{n - w} \left( \sum_{v > w} s_{a_i}^{i} (\Lambda(id_n)) + \epsilon s_{a_{w+1}}^{i} (\Lambda(id_n)) - \epsilon s_{a_w}^{i} (\Lambda(id_n)) \right)
\]

From (8) and (9) follows that there exists \( \epsilon > 0 \) such that for all \( A_i \) all components \( a_i^{t} \) of the point \( y \) are non-negative. But the combination \( y \) gives a point in the projection to the original variables, which violates the inequality

\[
\sum_{v \in [w]} x_v \geq \frac{w(w + 1)}{2}
\]

This is true since the projection of \( s^{i} (\Lambda(id_n), w) \) belongs to the corresponding face and the projection of \( s^{i} (\Lambda(\zeta), w) \) does not. Thus for any \( \epsilon > 0 \) the point \( (1 + \epsilon)s^{i} (\Lambda(id_n), w) - \epsilon s^{i} (\Lambda(\zeta), w) \) can not belong to the permutahedron \( \Pi_n \).

The next lemma finishes the proof because of the above mentioned construction.

**Lemma 10.** There exists such element \( w \) from the set \([n - 1]\) which satisfies the conditions (8) and (9).
Proof. Since each set $A_i$ consists of $n$ components, we can conclude that number of sets $A_i$ is less than $\frac{n^2}{2}$ (recall $d < \frac{n(n-1)}{2}$).

For each set $A_i$ there can exist just one element $u$ from $[n-1]$, which violates the statement (8) for the mentioned index $i$ (it can just be the maximal element from $[n-1]$ for which $s_{\Lambda(id_n)}(\Lambda(id_n)) > 0$). Analogously, for each set $A_i$ there can exist just one element $u$ from $[n-1]$, which violates the statement (9).

Thus, for at least one element $w \in [n-1]$ both (8) and (9) are satisfied for all sets $A_i$.

Combining Lemma 2 and Theorem 3 we get the following theorem, which gives us a lower bound on the number of variables and facets in symmetric extensions of the permutahedron.

**Theorem 11.** For every $n \geq 6$ there exists no symmetric extended formulation of the permutahedron $\Pi_n$ with less than $\frac{n(n-1)}{4}$ variables and constraints (with respect to the group $G = S(n)$ acting via permuting the corresponding elements).

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