A NOTE ON JÖRGENS-CALABI-POGORELOV THEOREM

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1. Let $S_k(A)$ denote the $k$th principal symmetric function of the eigenfunctions of an $n \times n$ matrix $A$, i.e.

$$
\det(A + tI) = \sum_{k=0}^{n} S_k(A)t^{n-k}.
$$

The following classical result is well known.

**Theorem A** (Jörgens-Calabi-Pogorelov, [4], [2], [6]). Let $f(x)$ be a convex entire solution of

$$
S_n(\text{Hess } f) \equiv \det(\text{Hess } f) = 1, \quad x \in \mathbb{R}^n,
$$

where $\text{Hess } f$ is the Hessian matrix of $f(x) = f(x_1, \ldots, x_n)$. Then $f(x)$ is a quadratic polynomial, i.e.

$$
f(x) = a + \langle b, x \rangle + \langle x, Ax \rangle,
$$

(1)

where $A$ is an $n \times n$ matrix with constant real coefficients and $\langle \cdot, \cdot \rangle$ stands for the scalar product in $\mathbb{R}^n$.

Let us consider the operator

$$
L[f] \equiv \sum_{i=1}^{n} a_i(x)S_i(\text{Hess } f) = 0.
$$

(2)

In [1], A.A. Borisenko has established that affine functions $f(x) = a + \langle b, x \rangle$ are the only entire convex solutions of (2) with the linear growth (i.e. $f(x) = O(||x||)$ as $x \to \infty$) in the following special cases, namely, when

$$
L[f] = S_n(\text{Hess } f) - S_1(\text{Hess } f) = \det \text{Hess } f - \Delta f = 0
$$

(3)

and

$$
L[f] = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^kS_{2k+1}(\text{Hess } f) = 0.
$$

(4)

Notice that solutions to (3) and (4) describe special Lagrangian submanifolds given a non-parametric form.

Let us consider the following condition.

(Q) either $a_k(x) \equiv 0$ on $\mathbb{R}^n$, or there exist two positive constants $\mu_1 \leq \mu_2$ such that $\mu_1 \leq |a_k(x)| \leq \mu_2$.

Let us denote by $J = J(L)$ the set of indices $i$, $1 \leq i \leq n$ such that $a_i(x) \neq 0$. The main purpose of this note is to establish the following generalization of [1].

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**Theorem.** Let \( f(x) \) be an entire convex \( C^2 \)-solution of (2) and that the structural condition (Q) is satisfied. If
\[
\lim_{\|x\| \to \infty} \sup \frac{|f(x)|}{\|x\|^2} = 0
\]
then \( S_i(A(x)) \equiv 0 \) for any \( i \in J \), in particular, \( \det \text{Hess} f(x) = 0 \). If additionally \( a_1(x) \neq 0 \) then \( f(x) \) is an affine function.

**Remark 1.** We construct an example in paragraph 4 below which shows that (5) is optimal in the sense that there exist operators \( L \) satisfying the condition (Q) and possessing solutions growing quadratically \( f(x) \sim \|x\|^2 \) as \( x \to \infty \) and such that \( \text{Hess} f(x) \neq 0 \).

2. We use the standard convention to write \( A \geq B \) if \( A - B \) is a positive semi-definite matrix.

**Lemma 1.** Let \( A(x) \geq 0 \) be a continuous \( n \times n \) matrix solution of
\[
L(A(x)) = \sum_{i=1}^{n} a_i(x) S_i(A(x)) = 0, \quad x \in \mathbb{R}^n,
\]
where \( L \) is subject to the condition (Q). Then either \( S_i(A(x)) \equiv 0 \) for any \( i \in J \), or there exists \( k \in J \) and a constant \( \sigma_0 \) depending on \( \mu_1 \) and \( \mu_2 \) such that for all \( x \in \mathbb{R}^n \) the inequality holds
\[
S_k(A(x)) \geq \sigma_0 > 0.
\]

**Proof of Lemma 1.** Note that \( S_k(A(x)) \geq 0 \) in virtue of the positive semi-definiteness of \( A(x) \). Then, if all (non-identically zero) coefficients \( a_i \) have the same sign then \( S_k(A(x)) \equiv 0 \) holds for any \( i \in J \). Now suppose that there exists \( x_0 \in \mathbb{R}^n \) and a number \( k \in J \) such that \( S_k(A(x_0)) > 0 \). In that case, there exist two coefficients \( a_i \) having different signs. Observe that by the condition (Q) this also holds true in the whole \( \mathbb{R}^n \). Let us rewrite (6) as
\[
|a_{i_1}(x_0)|S_{i_1}(A(x_0)) + \ldots + |a_{i_m}(x_0)|S_{i_m}(A(x_0)) = |a_{j_1}(x_0)|S_{j_1}(A(x_0)) + \ldots + |a_{j_p}(x_0)|S_{j_p}(A(x_0)),
\]
where \( i_1 < \ldots < i_m, j_1 < \ldots < j_p \), and also \( i_1 < j_1 \). We claim that \( k = i_1 \) satisfies the conclusion of the lemma. Indeed, we have
\[
S_{i_1}(A(x_0)) \leq b_1 S_{j_1}(A(x_0)) + \ldots + b_p S_{j_p}(A(x_0)),
\]
where \( b_k = |a_{j_k}(x_0)|/|a_{i_1}(x_0)| \leq \mu_2/\mu_1 \). Now, using Proposition 3.2.2 in [5, p. 106], we have
\[
\left( \frac{S_k(A(x_0))}{\binom{n}{k}} \right)^m \leq \left( \frac{S_m(A(x_0))}{\binom{n}{m}} \right)^k,
\]
for any \( 1 \leq m \leq k \leq n \), therefore by (7)
\[
S_{i_1}(A(x_0)) \leq \frac{\mu_2}{\mu_1} \sum_{k=1}^{p} \alpha_k \cdot (S_{i_1}(A(x_0)))^{\nu_k},
\]
where \( \nu_k = j_k/i_1 > 1 \) and \( \alpha_k = \binom{n}{j_k} \cdot \binom{n}{i_1}^{-\nu_k} \). Observe that the left hand side of the equation
\[
\frac{\mu_2}{\mu_1} \sum_{k=1}^{p} \alpha_k \cdot \sigma_k^{\nu_k-1} = 1
\]
is an increasing function of \( \sigma \geq 0 \), and let \( \sigma = \sigma_0 \) denote its (unique) positive root. Then in virtue of the positiveness of \( S_1(A(x_0)) \) we conclude that \( S_1(A(x_0)) \geq \sigma_0 \). By the continuity assumption on \( A(x) \), the latter inequality also holds in the whole \( \mathbb{R}^n \) which proves the lemma.

\[
\]

**Corollary 1.** Let \( f(x) \in C^2(\mathbb{R}^n) \) be a convex solution of (2) under the condition (Q). Then either \( \det \text{ Hess } f \equiv 0 \) in \( \mathbb{R}^n \) or there exists \( k \in J \) such that the inequality
\[
S_k(\text{Hess } f(x)) \geq \sigma_0 > 0
\]
holds for all \( x \in \mathbb{R}^n \) with \( k, \sigma_0 \) chosen as in Lemma 4.

3. **Proof of the Theorem.** We claim that under the hypotheses of the theorem there holds \( S_i(\text{Hess } f(x)) \equiv 0 \) for any \( i \in J \). Indeed, arguing by contradiction we have by Lemma 4 that (3) holds in the whole \( \mathbb{R}^n \) for some \( k \in J \). One can assume without loss of generality, replacing if needed \( f(x) \) by \( f(x) + c + \langle a, x \rangle \), that \( f(x) \geq 0 \) in \( \mathbb{R}^n \). Given an arbitrary \( \epsilon > 0 \), the condition (3) yields the existence of a constant \( p \in \mathbb{R} \) such that \( f(x) \leq \frac{\epsilon}{2} \|x\|^2 + p \) for any \( x \in \mathbb{R}^n \). But \( g(x) = \frac{\epsilon}{2} \|x\|^2 - f(x) \to \infty \) uniformly as \( x \to \infty \), hence it attains its minimum value at some point, say \( x_0 \in \mathbb{R}^n \), and there holds
\[
\text{Hess } g(x_0) = \text{Hess}(\frac{\epsilon}{2} \|x\|^2 - f(x))|_{x_0} \geq 0,
\]
which yields \( \text{Hess } f(x_0) \leq \epsilon I \) with \( I \) being the unit matrix.

Since \( \text{Hess } f(x_0) \geq 0 \) we obtain applying the majorization principle (see, for instance Corollary 4.3.3 in [4]) that
\[
S_k(\text{Hess } f(x_0)) \leq S_k(\epsilon I) = \epsilon^k \binom{n}{k}.
\]
But the assumption \( S_k(\text{Hess } f(x)) \geq \sigma_0 \) yields easily a contradiction with the arbitrariness of the \( \epsilon \). This proves our claim. In particular, in virtue of the convexity of \( f \) we also have \( \text{Hess } f(x) \geq 0 \), hence \( \text{Hess } f(x) \) has zero eigenvalues for any \( x \in \mathbb{R}^n \) implying \( \det \text{ Hess } f(x) \equiv 0 \) in \( \mathbb{R}^n \) (see also Corollary 4). If, additionally, \( a_1(x) \neq 0 \) then \( 1 \in J \) and the claim implies \( S_1(\text{Hess } f(x)) = \Delta f(x) \equiv 0 \). Applying again the convexity of \( f(x) \) easily yields that \( \text{Hess } f(x) \equiv 0 \) in \( \mathbb{R}^n \), hence \( f(x) \) is an affine function and finishes the proof of the theorem.

4. **Example.** Let \( \alpha(t) \) be a positive function, non-identically constant and such that \( 0 < q \leq \alpha(t) \leq q^{-1} \) for some fixed \( 0 < q < 1 \). Let us consider the function
\[
f(x_1, \ldots, x_n) = \sum_{i=1}^{n} \int_{0}^{x_i} (x_i - t) \alpha(t) dt.
\]
Then \( \text{Hess } f(x) = (\alpha(x_i) \delta_{ij})_{1 \leq i, j \leq n} \), hence \( f(x) \) is convex and satisfies
\[
S_n(\text{Hess } f(x)) - \omega(x)S_1(\text{Hess } f) = 0
\]
with \( \omega(x) = \alpha(x_1) \ldots \alpha(x_n) / \sum_{i=1}^{n} \alpha(x_i) \). We have \( \frac{1}{2}q^{n+1} \leq a_1(x) \leq \frac{1}{n}q^{-n-1} \), which establishes that \( L \) satisfies the condition (Q). On the other hand,
\[
\frac{q}{2} \|x\|^2 \leq f(x) \leq \frac{1}{2q} \|x\|^2,
\]
thus \( f(x) \) has the quadratic growth at infinity.
References

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