TRANSFER OF $A_{\infty}$-STRUCTURES TO PROJECTIVE RESOLUTIONS

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ABSTRACT. We show that an $A_{\infty}$-algebra structure can be transferred to a projective resolution of the complex underlying any $A_{\infty}$-algebra. Under certain connectedness assumptions, this transferred structure is unique up to homotopy. In contrast to the classical results on transfer of $A_{\infty}$-structures along homotopy equivalences, our result is of interest when the ground ring is not a field. We prove an analog for $A_{\infty}$-module structures, and both transfer results preserve strict units.

0. Introduction

It is a classical and motivating result in the theory of $A_{\infty}$-algebras that $A_{\infty}$-structures can be transferred along homotopy equivalences, i.e., if $f : A \sim \to B$ is a homotopy equivalence of complexes, and $B$ has an $A_{\infty}$-algebra structure, then $A$ has an $A_{\infty}$-algebra structure, and $f$ can be extended to a morphism of the $A_{\infty}$-algebras. Different versions of this result are proved in [12, 7, 9, 11, 8, 15, 18, 17]. It is most often used the in case $B$ is a dg-algebra, $A = H_{\ast}(B)$ is the homology algebra of $B$ (with zero differential), and the ground ring $k$ is a field, so there is a homotopy equivalence between $A$ and $B$.

In this paper we show that if $B$ has an $A_{\infty}$-algebra structure, and $q : A \to B$ is a projective resolution of the complex underlying $B$, over the ground ring $k$, then $A$ has an $A_{\infty}$-algebra structure such that $q$ is a strict morphism of $A_{\infty}$-algebras (by projective resolution we mean a cofibrant replacement in the projective model category structure on chain complexes over $k$). If $B$ satisfies $H_{i}(B) = 0$ for all $i < 0$, then the transferred structure on $A$ is unique up to homotopy. If the $A_{\infty}$-algebra structure on $B$ is strictly unital, then the transferred structure is also strictly unital, under a mild assumption on $A$. We prove analogous results for $A_{\infty}$-modules. Note that if $k$ is not a field, then a projective resolution is generally not a homotopy equivalence, so the earlier results do not apply.

These transfer results are a technical tool in developing Koszul duality relative to an arbitrary commutative base ring. By Koszul duality we mean in the generalized sense of using the bar construction, or some small replacement of it, to study the homological algebra of an algebra $B$ (associative, dg, or $A_{\infty}$), e.g., to construct canonical $B$-projective resolutions of $B$-modules. If $B$ is not projective as a module over $k$, then its bar construction can be nonsensical. For instance, if $B = k/I$ is a cyclic $k$-algebra, the bar construction of $B$ is an infinite sequence of copies of $B$ with zero differential. This tells us nothing about the structure of $B$ and seems to dash any hope of using the bar construction to construct resolutions. The transfer results proved here offer an alternative: resolve $B$ over $k$, transfer the algebra structure to the $k$-resolution, use the bar construction to construct resolutions there, and then
try to massage the result back down to $B$. This is carried out in detail in the case $B = k/I$ is cyclic in [3], and we hope to return to the general case in future work.

The proofs of our results use obstruction theory and the lifting properties of projective resolutions. Obstruction theory has been used in the construction of $A_{\infty}$-structures at least since [7]. Sullivan, in the context of topology, described obstruction theory as

\[ \ldots \text{much like being in a labyrinth with a weak miner's light attached to your forehead and being forced always to move forward. The light enables you to see if you may take your next step but it is not strong enough to tell you which fork to take when you must make a decision [6, \S 6.3].} \]

We can use this analogy to illustrate the relation of the present work to the classical results on transferring via homotopy equivalence. Our results show that there is always a path through the labyrinth of transferring an $A_{\infty}$-algebra structure to a projective resolution, and in many cases this path is unique up to homotopy, but we do not have global information, e.g., a map of the labyrinth. The classical homotopy transfer results give much more detailed information, formulated e.g., with SDR data, on the labyrinth of transferring an $A_{\infty}$-algebra structure via a homotopy equivalence. Additional information in specialized situations seems to be needed to better understand the $A_{\infty}$-structures that result in the case of resolutions.

Finally, we mention that the results here are easily dualized to give transfer results for injective resolutions, in the case of augmented or nonunital $A_{\infty}$-algebras. The way we handle the transfer of strictly unital (but potentially non-augmented) $A_{\infty}$-algebras assumes there is a free summand in degree zero. It is not clear at the moment how to adapt this to the injective case, since injective modules do not have free summands.

1. Notation and sign conventions

(1) Throughout, $k$ is a fixed commutative ring. By module, complex, map, etc. we mean $k$-module, $k$-linear map, etc. We place no boundedness or connectedness assumptions on complexes.

(2) For graded modules $M, N$, $\text{Hom}(M, N)$ and $M \otimes N$ are graded by:

\[
\text{Hom}(M, N)_n = \prod_{i \in \mathbb{Z}} \text{Hom}(M_i, N_{i+n}) \quad (M \otimes N)_n = \bigoplus_{i \in \mathbb{Z}} M_i \otimes N_{n-i}.
\]
If $(M, \delta_M)$ and $(N, \delta_N)$ are complexes, then $\text{Hom}(M, N)$ and $M \otimes N$ are complexes with differentials $\delta_{\text{Hom}}$ and $\delta_{\otimes}$ given by:

$$
\delta_{\text{Hom}}(f) = \delta_N f - (-1)^{|f|} f \delta_M \quad \delta_{\otimes} = \delta_M \otimes 1 + 1 \otimes \delta_N.
$$

A morphism of complexes is a cycle in the complex $(\text{Hom}(M, N), \delta_{\text{Hom}})$. A quasi-isomorphism is a morphism that induces an isomorphism in homology.

1. All elements of graded objects are assumed to be homogeneous. We write $\alpha \leq \beta$ for the degree of an element $x$. Set $\Pi$ to be the endofunctor of the category of graded modules defined by $(\Pi M)_n = M_{n-1}$ and $(\Pi f)[m] = [f(m)]$ for a morphism $f$, where $[m] \in (\Pi M)_n$ is the element corresponding to $m \in M_{n-1}$. There is a degree one natural transformation $1 \to \Pi$ that is the identity on every graded module, i.e., $s(x) = [x] \in \Pi M$. If $(M, \delta_M)$ is a complex, we set $\delta_{\Pi M} = -s \delta_M s^{-1}$, so $s : (M, \delta_M) \to (\Pi M, \delta_{\Pi M})$ is a degree one cycle in complex $(\text{Hom}(M, \Pi M), \delta_{\text{Hom}})$. We write $[x_1] \ldots [x_n] = sx_1 \otimes \ldots \otimes sx_n$.

2. Signs are introduced when applying a tensor product of morphisms as follows: $(f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x) \otimes g(y)$.

2. DEFINITIONS

In this section we collect various definitions we need to precisely state our main results. See e.g., [19, 13, 14] for an introduction to, and further context on, $A\infty$-objects, and [4] for an expanded version of the development below, using string diagrams. Throughout this section $A$ and $B$ denote graded modules.

Definition 2.1. For $l \in \mathbb{N}$, set $CC^l(A, B) = \text{Hom}(\Pi A)^{\otimes l}, \Pi B)$, and for $n \in \mathbb{N} \cup \{\infty\}$, set $CC^{\leq n}(A, B) = \prod_{1 \leq i \leq n} CC^i(A, B)$. We write $CC^\bullet(A, B)$ for $CC^{\leq \infty}(A, B)$.

Elements $f \in CC^\bullet(A, B)$ are denoted $f = (f^l)$, with $f^l \in CC^l(A, B)$ the $l$th tensor homogeneous component. Since $A$ and $B$ are graded, so is $CC^{\leq n}(A, B)$, using the convention on gradings of Hom and tensor products. We denote the $ith$ homogeneous component of this grading as $CC^{\leq n}(A, B)_i$.

1. The Gerstenhaber product of $\mu = (\mu^l) \in CC^\bullet(A, B)_i$ and $\nu = (\nu^l) \in CC^\bullet(A, A)_j$ is $\mu \odot \nu = ((\mu \odot \nu)^l) \in CC^\bullet(A, B)_{i+j}$, with

$$
(\mu \odot \nu)^l = \sum_{1 \leq i \leq l} \mu^j 1^{\otimes j} \otimes \nu^{l-j+1} 1^{\otimes i-j+1}.
$$

2. The $\ast$-product of $\nu = (\nu^l) \in CC^\bullet(B, B)_{-1}$ and $\alpha = (\alpha^l) \in CC^\bullet(A, B)_0$ is

$$
\nu \ast \alpha = ((\nu \ast \alpha)^l) \in CC^\bullet(A, B)_{-1}, \text{ with }
$$

$$
(\nu \ast \alpha)^l = \sum_{1 \leq i \leq l} \nu^j (\alpha^{i_1} \otimes \ldots \otimes \alpha^{i_{1+i}}).
$$

3. The homotopy $\ast$-product of $\nu \in CC^\bullet(B, B)_{-1}$ and $r \in CC^\bullet(A, B)_1$ with respect to $\alpha, \beta \in CC^\bullet(A, B)_0$ is $(\nu \ast_{\alpha, \beta} r) = ((\nu \ast_{\alpha, \beta} r)^l) \in CC^\bullet(A, B)_0$, with

$$
(\nu \ast_{\alpha, \beta} r)^l = \sum_{1 \leq i \leq l} \nu^j (\alpha^{j_1} \otimes \ldots \otimes \alpha^{j_{i-1}} \otimes r^{j_{i+1}} \otimes \beta^{j_{i+1}} \otimes \ldots \otimes \beta^{j_{i}}).
$$
For any $n \geq 1$, $CC^{\leq n}(A, A)$ is a submodule and quotient module of $CC^\bullet(A, A)$. The inclusion allows us to apply the three products defined above to elements of $CC^{\leq n}(A, A)$, but note that $CC^{\leq n}(A, A)$ is not closed under any of these operations. We write $(-)^{\leq n} : CC^\bullet(A, A) \to CC^{\leq n}(A, A)$ for the canonical projection.

**Definition 2.2.** Let $A$ and $B$ be graded modules and fix $n \in \mathbb{N} \cup \{\infty\}$.

1. A nonunital $A_n$-algebra structure on $A$ is an element $\nu \in CC^{\leq n}(A, A)_{-1}$ such that $(\nu \circ \nu)^{\leq n} = 0$. (The pair $(A, \nu)$ is then an $A_n$ algebra.)

2. A morphism between nonunital $A_n$-algebras $(A, \nu_A) \to (B, \nu_B)$ is an element $\alpha \in CC^{\leq n}(A, B)_0$ such that $(\nu_B \ast \alpha - \alpha \circ \nu_A)^{\leq n} = 0$.

3. A homotopy between morphisms of nonunital $A_n$-algebras $\alpha, \beta : (A, \nu_A) \to (B, \nu_B)$ is an element $r \in CC^{\leq n}(A, B)_1$ such that $\alpha - \beta = (\nu_B \ast r \circ \nu_A)^{\leq n}$.

The unital versions of the above are defined next. The adjective strict is used to distinguish from the weaker notion of a homotopy unit, see e.g., [17 §3.2].

**Definition 2.3.** Let $A, B$ be graded modules with fixed elements $1 \in A_0, 1 \in B_0$ and fix $n \in \mathbb{N} \cup \{\infty\}$.

1. An element $\nu^{\leq n} \in CC^{\leq n}(A, A)_{-1}$ is strictly unital (with respect to 1 in $A_0$) if $\nu^{\leq n}[1]a = [1]a = (-1)\nu[a][1]$ for all $a \in A$ and $\nu^k[a_1]\ldots[a_i][1]a_{i+1}\ldots[a_{k-1}] = 0$ for all $a_1, \ldots, a_{k-1} \in A$ with $k \neq 2$ and $k \leq n$. An $A_n$-algebra $(A, \nu^{\leq n})$ is strictly unital if $\nu^{\leq n}$ is strictly unital in the above sense.

2. An element $\alpha^{\leq n} \in CC^{\leq n}(A, B)_0$ is strictly unital if $\alpha^{\leq n}[1] = 1$ and for all $2 \leq k \leq n$ and $a_1, \ldots, a_{k-1} \in A$, we have $\alpha^k[a_1]\ldots[a_i][1]a_{i+1}\ldots[a_{k-1}] = 0$. A morphism of strictly unital $A_n$-algebras is strictly unital if it is strictly unital in the above sense.

3. An element $r^{\leq n} \in CC^{\leq n}(A, B)_1$ is strictly unital if for all $1 \leq k \leq n$ and all $a_1, \ldots, a_{k-1} \in A$, we have $r^k[a_1]\ldots[a_i][1]a_{i+1}\ldots[a_{k-1}] = 0$. A homotopy between strictly unital morphisms is strictly unital if it is strictly unital in the above sense.

We now recall definitions related to $A_n$-modules. It is a small but important point that $A_n$-module structures are naturally defined over $A_{n-1}$-algebras (as opposed to $A_n$-algebras).

**Definition 2.4.** Let $A$ be a graded module and $(M, \delta_M)$ a complex.

1. The graded module $\text{Hom}(M, M)$ is a canonically a dg algebra, with multiplication given by composition and differential $\delta_{\text{Hom}} = [\delta_M, -]$. We denote by $(\text{End} M, \nu_{\text{End}})$ the corresponding $A_\infty$-algebra, so $\nu^1_{\text{End}} = \delta_\text{Hom} = -s_\text{Hom}\delta_M^{-1}$ and $\nu^2_{\text{End}} = -s^\gamma(s^{-1})^{[2,2]}$, where $\gamma$ denotes composition. See also [4 Remark 2.10].

2. If $(A, \nu^{\leq n-1})$ is a nonunital $A_{n-1}$-algebra, an $A_n$-module structure on $(M, \delta_M)$ is a morphism of $A_{n-1}$-algebras $p_M \in CC^{\leq n-1}(A, End M)_0$. One can use the isomorphism $CC^{\leq l-1}(A, End M)_0 \cong \text{Hom}(\Pi A)^{\leq l-1} \otimes M, M)_{-1}$ to write an $A_n$-module structure $p_M = (p_M^l)$ as a sequence of maps $m^l : (\Pi A)^{\leq l-1} \otimes M \to M$, with $1 \leq l \leq n$, giving an idea why this is called an $A_n$-module structure.

It will be helpful to expand $CC^{\leq n}(A, B)$ by allowing tensor degree zero elements. Set $CC^{\leq n}(A, B) = \coprod_{0 \leq l \leq n} CC^{\leq l}(A, B) = \Pi B \otimes CC^{\leq n}(A, B)$. Given an $A_n$-module structure $p_M \in CC^{\leq n-1}(A, End M)_0$ on a complex $(M, \delta_M)$, we consider $\delta_M + p_M \in CC^\bullet(A, End M)_0$. Conversely, we can view $A_n$-module structures as elements $p_M \in$
CC_{\leq n}(A, \text{End } M)_0 \text{ such that } p^1_A \in \text{End } M \text{ is a differential and } p^2_{\geq 1}_M \text{ is an morphism of } A_{n-1} \text{-algebras from } A \text{ to the endomorphism } A_{\infty} \text{-algebra of the complex } (M, p^0_M).

To define morphisms of } A_n \text{-modules, we add to the list of products in } 2.1.

**Definition 2.5.** Let } A, M, N, P \text{ be graded modules.

1. Define the } \star \text{ product as follows,

\[
\star : CC^n_A(A, \text{Hom}(N, P))_k \otimes CC^n_A(A, \text{Hom}(M, N))_l \rightarrow CC^n_A(A, \text{Hom}(M, P))_{k+l-1}
\]

\[
\alpha \otimes \beta = (a^n) \otimes (b^n) \mapsto \left( s\gamma(s-1 \otimes s-1) \sum_{j=0}^n \alpha^j \otimes \beta^{n-j} \right) = \alpha \star \beta,
\]

where } \gamma \text{ is the composition map.

2. Let } (A, \nu^{\leq n-1}) \text{ be a nonunital } A_{n-1} \text{-algebra, for some } n \in \mathbb{N} \cup \{\infty\}. \text{ An element } f \in CC^{\leq n-1}(A, \text{Hom}(M, N))_1 \text{ is a morphism of } A_n \text{-modules } (M, p_M) \rightarrow (N, p_N) \text{ if }

\[
p_N \star f + (f^{\geq 1}) \circ \mu + f \star p_M = 0.
\]

The composition with a second morphism, } f \in CC^n_A(A, \text{Hom}(N, P))_1, \text{ is }

\[
\tilde{f} \star f \in CC^n_A(A, \text{Hom}(M, N))_1.
\]

3. An element } r \in CC^{\leq n-1}(A, \text{Hom}(M, N))_2 \text{ is a homotopy between morphisms of } A_n \text{-modules } f, g : (M, p_M) \rightarrow (N, p_N) \text{ if }

\[
f - g = r \circ \nu^{\leq n-1} + p_N \star r + r \star p_M.
\]

The unital version of } A_n \text{-modules is the following:

**Definition 2.6.** Let } (A, \nu^{\leq n-1}) \text{ be a strictly unital } A_{n-1} \text{-algebra. An } A_n \text{-module } (M, p_M) \text{ is strictly unital if } p_M \text{ is a strictly unital morphism of } A_{n-1} \text{-algebras. A morphism of strictly unital } A_n \text{-modules } (M, p_M) \rightarrow (N, p_N) \text{ is a morphism of } A_n \text{-modules } f^{\leq n-1} \text{ such that } f^{\leq n-1} \text{ is in } CC^{\leq n-1}(\mathcal{A}, \text{Hom}(M, N))_1, \text{ i.e., for all } 1 \leq j \leq n \text{ and } a_1, \ldots, a_j \in A, \text{ we have }

\[
f^j([a_1] \ldots |a_i \ldots |a_{i-1}|a_i \ldots |a_j]) = 0.
\]

A homotopy between morphisms of strictly unital } A_n \text{-modules } f^{\leq n}, g^{\leq n} : (M, p_M) \rightarrow (N, p_N) \text{ is a homotopy between morphisms of } A_n \text{-modules } r^{\leq n-1} \text{ such that } r^{\leq n-1} \text{ is in } CC^{\leq n-1}(\mathcal{A}, \text{Hom}(M, N))_2.

To transfer strictly unital } A_{\infty} \text{-structures, we need to place a further assumption on the pair } (A, 1).

**Definition 2.7.** A split element of a graded module } A \text{ is an element that generates a rank one free module. A graded module with split element is a pair } (A, 1) \text{ with } 1 \text{ a split element in } A_0, \text{ and a fixed (unlabeled) splitting } A \rightarrow k \text{ of the inclusion } k \rightarrow A, 1 \mapsto 1. \text{ Morphisms of graded modules with split elements } (A, 1) \rightarrow (B, 1) \text{ are always assumed to preserve the fixed splittings. An } A_n \text{ algebra with split unit is a triple } (A, 1, \nu^{\leq n}), \text{ where } (A, 1) \text{ is a graded module with split element and } \nu^{\leq n} \text{ is an } A_n \text{ algebra structure on } A \text{ that is strictly unital with respect to } 1.

If } (A, 1) \text{ is a graded module with split element, we set } \mathcal{A} = A/(k \cdot 1), \text{ and consider it as a submodule of } A \text{ via the fixed splitting of } 1. \text{ The projection } A \rightarrow \mathcal{A} \text{ identifies } CC^{\leq n}(\mathcal{A}, A) \text{ as a submodule of } CC^{\leq n}(A, A). \text{ The trivial strictly unital } A_{\infty} \text{-structure on } (A, 1) \text{ is a strictly unital element, denoted } \nu^{\leq n} \in CC^2(A, A)_{-1}, \text{ such that for } n \geq 2, \text{ every strictly unital element } \nu^{\leq n} \in CC^{\leq n}(A, A)_{-1} \text{ can be}
written \( \nu^{\leq n} = \mu^{\leq n} + \mu_{su} \) for a unique \( \mu^{\leq n} \in CC^{\leq n}(A,A)_1 \); see [3] Definition 4.3, Lemma 4.4 for details. Analogously, there is a trivial strictly unital morphism \( g_{su} \in CC^1(A,B)_0 \) and every strictly unital element \( \alpha^{\leq n} \in CC^{\leq n}(A,B)_0 \) can be written uniquely as \( \alpha^{\leq n} = \beta^{\leq n} + g_{su} \), for \( \beta^{\leq n} \in CC^n(A,B)_0 \). It is clear that an element \( r \in CC^n(A,B)_1 \) is strictly unital if and only if \( r \) is in \( CC^n(A,B)_1 \).

**Remark.** If \( k \) is a field, then every strictly unital \( A_n \) algebra is an \( A_\infty \) algebra with split unit, since every element of \( A \) is split. This is no longer the case if \( k \) is not a field: if \( I \) is a nonzero ideal of \( k \), then \( k/I \) has a strict, but not a split, unit.

We need the following homological algebra of complexes.

**Definition 2.8.** Let \((P,d_P)\) and \((A,d_A)\) be complexes, and recall the Hom-complex \((\text{Hom}(P,A), \delta_{\text{Hom}})\) has differential \( \delta_{\text{Hom}}(f) = d_A f - (-1)^{|f|} f d_P \). If \( f : (A,d_A) \rightarrow (B,d_B) \) is a morphism of complexes, we have the following morphisms of complexes,

\[
\begin{align*}
    f_* &= \text{Hom}(P,f) : (\text{Hom}(P,A), \delta_{\text{Hom}}) \rightarrow (\text{Hom}(P,B), \delta_{\text{Hom}}) \\
    f^* &= \text{Hom}(f,P) : (\text{Hom}(B,P), \delta_{\text{Hom}}) \rightarrow (\text{Hom}(A,P), \delta_{\text{Hom}}).
\end{align*}
\]

Thus \( \text{Hom}(P,-) \) and \( \text{Hom}(-,P) \) are endofunctors of the category of complexes.

A complex \((P,d_P)\) is semiprojective if for every surjective quasi-isomorphism \( f \), the morphism \( f_* = \text{Hom}(P,f) \) is also a surjective quasi-isomorphism. A semiprojective resolution of a complex \( M \) is a quasi-isomorphism \( P \xrightarrow{\sim} M \), with \( P \) semiprojective.

**Remark.** If \( k \) is a field, then every complex is semiprojective. If \( P_n = 0 \) for all \( n \ll 0 \), then \( P \) is semiprojective if and only if each \( P_n \) is a projective \( k \)-module. In particular, if \( M \) is concentrated in degree 0 (i.e., \( M \) is a module), then a projective resolution of \( M \) is a semiprojective resolution. If \( P \) is semiprojective, then \( P_n \) is a projective \( k \)-module for all \( n \), but not every complex of projective modules is semiprojective; for instance, \( \ldots \rightarrow \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \rightarrow \ldots \), where \( k = \mathbb{Z}/4\mathbb{Z} \). Every complex has a surjective semiprojective resolution.

Semiprojective complexes are the cofibrant objects in the projective model structure on the category of \( k \)-complexes [10] 2.3]. Semifree complexes, of which semiprojective are summands, were first defined in [2]. Semiprojective complexes are the \( K \)-projective complexes of projectives, using the terminology of [20], and cell \( k \)-modules, using the terminology of [16].

## 3. Statement of results

**Theorem 3.1.** Let \((B,\nu_B)\) be a strictly unital \( A_\infty \)-algebra.

1. Let \( q : (\Pi A, \nu^l_A) \rightarrow (\Pi B, \nu^l_B) \) be a surjective semiprojective resolution of the complex underlying \((B,\nu_B)\), and assume that \( A \) has a split element \( 1 \in A_0 \) with \( \nu^1_A([1]) = 0 \) and \( q([1]) = [1] \). There exists \( \nu_A \in CC^\bullet(A,A)_1 \), extending \( \nu^1_A \), such that \((A,1,\nu_A)\) is an \( A_\infty \)-algebra with split unit and \( q \) is a strict morphism\(^1\). If \((B,\nu_B)\) is augmented, then we can choose \( \nu_A \) such that \((A,\nu_A)\) is augmented\(^2\).

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1 A morphism of \( A_\infty \)-algebras \( f = (f^l) \in CC^\bullet(A,B)_0 \) is strict if \( f^l = 0 \) for all \( l \geq 2 \).
2 A strictly unital \( A_\infty \)-algebra \((B,\nu_B)\) is augmented if there is a strictly unital morphism \((B,\nu_B) \rightarrow (k,\mu_{su})\).
(2) Consider a diagram of strictly unital $A_\infty$-algebras,

$$
\begin{array}{ccc}
(A, \nu_A) & \xrightarrow{q} & (C, \nu_C) \\
& \downarrow{\alpha} & \downarrow{\beta} \\
(B, \nu_B). & & \end{array}
$$

Assume that $(C, 1, \nu_C)$ is an $A_\infty$-algebra with split unit such that $(\Pi C, \nu_C^1)$ is a semiprojective complex, and that $q = q^1$ is strict, with $q^1 : (\Pi A, \nu_A^1) \rightarrow (\Pi B, \nu_B^1)$ a surjective quasi-isomorphism of complexes. Assume further that there is a morphism of chain complexes $\delta^1 : (\Pi C, \nu_C^1) \rightarrow (\Pi A, \nu_A^1)$ such that $\delta^1[1] = [1]$ and $q^1 \delta^1 = \alpha^1$. Then there exists a strictly unital morphism of $A_\infty$-algebras $\delta : (C, \nu_C) \rightarrow (A, \nu_A)$ such that $q \delta = \alpha$.

(3) If $H_0(CC^{n+1}(C, A), d_{\overline{A}}) = 0$ for all $n \geq 1$, and $\delta'$ is another lifting of $\alpha$ through $q$, then $\delta$ and $\delta'$ are homotopic by a strictly unital homotopy:

$$
\begin{array}{ccc}
A & \xrightarrow{\delta} & C \\
\downarrow{\delta'} & & \downarrow{\gamma} \\
B. & & 
\end{array}
$$

In particular, if $A$ satisfies $H_0(CC^{n+1}(A, A), d_{\overline{A}}) = 0$ for all $n \geq 1$, then any two elements $\nu_A, \nu_B \in CC^\bullet(A, A)$, such that $(A, 1, \nu_A)$ and $(A, 1, \nu_B)$ are $A_\infty$-algebras with split units and $q$ is a strict morphism, are homotopy equivalent (via strictly unital homotopies).

Remark. In part 2 above, there is always a morphism of chain complexes $\delta^1 : (\Pi C, \nu_C^1) \rightarrow (\Pi A, \nu_A^1)$ such that $q^1 \delta^1 = \alpha^1$, using the lifting properties of the semiprojective complex $(\Pi C, \nu_C^1)$. But it is not clear that $\delta^1$ can be chosen such that $\delta^1[1] = [1]$. In certain situations such a $\delta^1$ always exists, e.g., if $A_0, B_0, C_0$ are all cyclic $k$-modules. It is also often the case, as in the corollary below, that $A = C$ and $\delta^1$ can be chosen to be the identity.

Corollary 3.2. Let $B$ be an associative $k$-algebra, and let $\pi : (A, d_A) \rightarrow B$ be a $k$-projective resolution. Assume that $A$ has a split element $1 \in A_0$ with $\pi(1) = 1 \in B$. Then $A$ has a strictly unital $A_\infty$-algebra structure such that $s \pi s^{-1}$ is a strict morphism of $A_\infty$-algebras, and this is unique up to strictly unital homotopy.

Proof of Corollary 3.2 We can consider $B$ as an $A_\infty$-algebra $(B, \nu_B)$ with $\nu_B^n = 0$ for all $n \neq 2$. The complex $(A, d_A)$ is semiprojective, so by [3.1](1) $\nu_A$ exists as claimed. To see the uniqueness statement, assume that $\nu_A$ and $\nu_B$ both satisfy the hypothesis. By [3.1](2), with $\delta^1 = 1$, there are strictly unital morphisms $\delta, \epsilon$ with $q \epsilon = q$ and $q \delta = q$, where $q = s \pi s^{-1}$,

$$
\begin{array}{ccc}
A & \xrightarrow{\delta} & A \\
\downarrow{\epsilon} & \downarrow{q} & \downarrow{q} \\
A & \rightarrow & B.
\end{array}
$$

It follows that $q \epsilon \delta = q = q \delta \epsilon$. Now note the canonical map $(CC^{n+1}(A, A), d_{\overline{A}}) \rightarrow (CC^{n+1}(A, B), d_{\overline{A}B})$ is a quasi-isomorphism, since $\overline{A}$ is semiprojective. Further, $CC^{n+1}(A, B)_0 \cong \text{Hom}((\overline{A} \otimes n+1)_{-n}, B) = 0$ for $n \geq 1$, since $\overline{A} \otimes n+1$ is concentrated.
in nonnegative degrees. Thus \( H_0(\text{CC}^{n+1}(A, A), d_\partial A) = 0 \) for all \( n \geq 1 \), and applying \ref{3.1}(3) shows that \( \delta \) and \( \delta e \) are homotopic to the identity (the definition of \((A, \nu_A)\) and \((A, \tilde{\nu}_A)\) being homotopy equivalent).

\[ \square \]

**Theorem 3.3.** Let \((A, 1, \nu)\) be an \( A_\infty \)-algebra with split unit such that \((\Pi A, \nu^1)\) is a semiprojective complex, and let \((M, p_M)\) be a strictly unital \( A_\infty \)-module over \((A, 1, \nu)\).

1. Let \( q: (G, p^0_G) \to (M, p^0_M) \) be a surjective semiprojective resolution of the complex underlying \((M, p_M)\). There exists \( p_G \in \text{CC}^*(A, \text{End} G)_0 \), extending \( p^0_G \), such that \((G, p_G)\) is a strictly unital \( A_\infty \)-module over \((A, 1, \nu_A)\) and \( q \) is a strict morphism of \( A_\infty \)-modules.

2. Consider a diagram of strictly unital \( A_\infty \)-modules over \((A, 1, \nu_A)\):

\[
\begin{array}{ccc}
(G, p_G) & \xrightarrow{q} & (M, p_M) \\
\downarrow & & \\
(N, p_N) & \xrightarrow{\alpha} & (M, p_M).
\end{array}
\]

Assume that \((N, p^0_N)\) is a semiprojective complex, and that \( q = q^1 \) is strict, with \( q^1: (G, p^0_G) \to (M, p^0_M) \) a surjective quasi-isomorphism of complexes. Then there exists a strictly unital morphism of \( A_\infty \)-modules \( \delta: (N, p_N) \to (G, p_G) \) such that \( q\delta = \alpha \).

3. If \( H_1(\text{CC}^n(A, \hom(N, G)), d_\hom A) = 0 \) for all \( n \geq 1 \), and \( \delta' \) is another lifting of \( \alpha \) through \( q \), then the strictly unital morphisms \( \delta \) and \( \delta' \) are homotopic (via a strictly unital homotopy):

\[
\begin{array}{ccc}
\delta & \sim & \delta' \\
\downarrow & & \\
N & \xrightarrow{\gamma} & M.
\end{array}
\]

In particular, if \( G \) satisfies \( H_1(\text{CC}^n(A, \hom(G, G)), d_\hom A) = 0 \) for all \( n \geq 1 \), then any two strictly unital \( A_\infty \)-module structures \( p_G, p_G' \in \text{CC}^*(A, \text{End} G)_0 \), such that \( q \) is a strict morphism, are homotopy equivalent (via strictly unital homotopies).

**Corollary 3.4.** Let \( B \) be an associative \( k \)-algebra, and \( \pi: (A, d_A) \to B \) a \( k \)-projective resolution such that \( A \) has a split element \( 1 \in A_0 \) with \( \pi(1) = 1 \in B \). Let \( \nu_A \) be a strictly unital \( A_\infty \)-structure on \( A \) such that \( s\pi s^{-1} \) is a strict morphism. (Such \( \nu_A \) exists by \ref{3.2}.)

Let \( M \) be a \( B \)-module and \( G \xrightarrow{\gamma} M \) a \( k \)-projective resolution. Then \( G \) has a strictly unital \( A_\infty \)-module structure over \((A, 1, \nu_A)\), and this is unique up to strictly unital homotopy.

The proof is similar to the proof of \ref{3.2}.

4. **Obstruction theory**

The main tool used in the proofs of Theorems \ref{3.1} and \ref{3.3} is obstruction theory. This is a way of extending an \( A_\infty \)-object (algebra, morphism, or homotopy) to an \( A_{n-1} \)-object, towards the goal of building an \( A_\infty \)-object. This general strategy is based on the following.

\[ \text{A morphism of } A_\infty \text{-modules } f = (f^n) \text{ is strict if } f^n = 0 \text{ for all } n \geq 1. \]
Lemma 4.1. Let $A$ and $B$ be graded modules.

(1) An element $\nu \in CC^\bullet(A, A)_{-1}$ is a nonunital $A_\infty$-algebra structure if and only if, for all $n \geq 1$, $\nu^\leq n \in CC^\leq n(A, A)_{-1}$ is a nonunital $A_n$-algebra structure.

(2) An element $\alpha \in CC^\bullet(A, B)_0$ is a morphism of nonunital $A_\infty$-algebras $(A, \nu_A) \rightarrow (B, \nu_B)$, if and only if, for all $n \geq 1$, $\alpha^\leq n \in CC^\leq n(A, B)_0$ is a morphism of $A_n$-algebras $(A, \nu_A^\leq n) \rightarrow (B, \nu_B^\leq n)$.

(3) An element $r \in CC^\bullet(A, B)_1$ is a homotopy between morphisms $\alpha, \beta : (A, \nu_A) \rightarrow (B, \nu_B)$ if and only if, for all $n \geq 1$, $r^\leq n \in CC^\leq n(A, B)_1$ is a homotopy between $\alpha^\leq n$ and $\beta^\leq n$.

Proof. Note first that $(\nu \circ \nu)^\leq n = (\nu^\leq n \circ \nu^\leq n)^\leq n$. Thus, if $\nu \circ \nu = 0$, then $(\nu \circ \nu)^\leq n = (\nu^\leq n \circ \nu^\leq n)^\leq n = 0$ for all $n \geq 1$, so $\nu^\leq n$ is an $A_n$-algebra structure for all $n \geq 1$. Conversely, if $\nu^\leq n$ is an $A_n$-algebra for all $n \geq 1$, then $(\nu \circ \nu)^\leq n = 0$ for all $n \geq 1$, and so $\nu \circ \nu = 0$. This proves part 1, and the other parts are proved in an analogous way. \qed

To pass from an $A_n$-object to $A_{n+1}$-object requires one to show that a certain cycle, called the obstruction, is a boundary. The complex where this occurs is the following.

Definition 4.2. Let $\nu_A^1$ and $\nu_B^1$ be $A_1$-algebra structures on $A$ and $B$ (i.e., $(\Pi A, \nu_A^1)$ and $(\Pi B, \nu_B^1)$ are complexes). For any $n \in \mathbb{N}$, we consider the complex $(CC^{n+1}(A, B), d_{AB})$, where $d_{AB}$ is the hom-complex differential $\delta_{AB}$ on $CC^{n+1}(A, B) = \text{Hom}(\Pi A)^{\otimes n}, \Pi B$ between the complexes $(\Pi A)^{\otimes n}, \delta_{\otimes} = \sum_j 1^{\otimes j} \otimes \nu_A^1 \otimes 1^{n-j-1})$ and $(\Pi B, \nu_B^1))$. Note that $d_{AB}(\alpha) = \nu_B^1 \alpha - (-1)^{|\alpha|} \alpha \circ \nu_A^1$.

Definition 4.3. Let $A$ and $B$ be graded modules.

(1) If $(A, \nu^\leq n)$ is an $A_n$-algebra, an element $\nu^{n+1} \in CC^{n+1}(A, A)_{-1}$ extends $\nu^\leq n$ if $(A, \nu^{n+1} = \nu^\leq n + \nu^{n+1})$ is an $A_{n+1}$-algebra. The obstruction (to extending $\nu^\leq n$)

$$o(\nu^\leq n) = -(\nu^\leq n \circ \nu^\leq n)^{\leq n+1} \in CC^{n+1}(A, A)_{-2}.$$ 

(2) If $(A, \nu_A^{n+1})$ and $(B, \nu_B^{n+1})$ are nonunital $A_{n+1}$-algebras, and $\alpha^\leq n \in CC^\leq n(A, B)_0$ is a morphism of nonunital $A_n$-algebras $(A, \nu_A^\leq n) \rightarrow (B, \nu_B^\leq n)$, an element $\alpha^{n+1} \in CC^{n+1}(A, B)_0$ extends $\alpha^\leq n$ if $\alpha^{n+1}$ is a morphism of $A_n$-algebras $(A, \nu_A^{n+1}) \rightarrow (B, \nu_B^{n+1})$. The obstruction (to extending $\alpha^\leq n$ to a morphism of $A_{n+1}$-algebras)

$$o(\alpha^\leq n) = -(\nu_B^{n+1} \star \alpha^\leq n - \alpha^\leq n \circ \nu_A^{n+1})^{\leq n+1} \in CC^{n+1}(A, B)_{-1}.$$ 

(3) If $\alpha^{n+1}, \beta^{n+1} : (A, \nu_A^{n+1}) \rightarrow (B, \nu_B^{n+1})$ are morphisms of nonunital $A_{n+1}$-algebras, and $r^{\leq n} \in CC^{\leq n}(A, B)_1$ a homotopy between $\alpha^\leq n$ and $\beta^\leq n$, an element $r^{n+1} \in CC^{n+1}(A, B)_1$ extends $r^{\leq n}$ if $r^{n+1}$ is a homotopy between $\alpha^{n+1}$ and $\beta^{n+1}$. The obstruction (to extending $r^{\leq n}$)

$$o(r^{\leq n}) = \alpha^{n+1} - \beta^{n+1} - (\nu_B^{n+1} \circ_\beta r^{\leq n} - r^{\leq n} \circ \nu_A^{n+1})^{\leq n+1} \in CC^{n+1}(A, B)_0.$$ 

The following is stated in [17, B.1] and a proof of the first part is given. We give full proofs of all three parts below for the ease of the reader, because they are essential to what follows, and because loc. cit. implicitly assumes that $k$ is a field.
(rather that every module is semisimple), though this hypothesis is not used there. Our proofs are based on the proof of part 1 given in loc. cit.

**Proposition 4.4.** Let $A$ and $B$ be graded modules.

1. If $(A, \nu^\leq)$ is a nonunital $A_n$-algebra, then the obstruction $o(\nu^\leq)$ is a cycle in $\langle CC(A, A), d_{AA} \rangle$. An element $\nu^{n+1}$ in $CC(A, A)_{-1}$ extends $\nu^\leq$ if and only if $d_{AA}(\nu^{n+1}) = o(\nu^\leq)$.

2. If $(A, \nu^\leq_A)$ and $(B, \nu^\leq_B)$ are nonunital $A_{n+1}$-algebras, and $\alpha^\leq \in CC(A, B)_0$ is a morphism of nonunital $A_{n}$-algebras $(A, \nu^\leq_A) \to (B, \nu^\leq_B)$, then the obstruction $o(\alpha^\leq)$ is a cycle in $\langle CC(A, B), d_{AB} \rangle$. An element $\alpha^{n+1} \in CC(A, B)_0$ extends $\alpha^\leq$ if and only if $d_{AB}(\alpha^{n+1}) = o(\alpha^\leq)$.

3. If $\alpha^{n+1}, \beta^{n+1} : (A, \nu^\leq_A) \to (B, \nu^\leq_B)$ are graded coalgebras and $r^\leq \in CC(A, B)_1$ is a homomorphism between $\alpha^\leq$ and $\beta^\leq$, then the obstruction $o(r^\leq)$ is a cycle in $\langle CC(A, B), d_{AB} \rangle$. An element $r^{n+1}$ extends $r^\leq$ if and only if $d_{AB}(r^{n+1}) = o(r^\leq)$.

For the proofs we will need the following technical material. If $V$ is a graded module and $n \in \mathbb{N} \cup \{\infty\}$, we set $T^r_n(V) = \bigoplus_{i \leq n} V^\otimes i$ to be the truncated tensor coalgebra on $V$. This is a nonunital graded coalgebra, with comultiplication the linear extension of $\Delta(v_1 \otimes \cdots \otimes v_i) = \sum_{1 \leq j \leq i-1} (v_1 \otimes \cdots \otimes v_j) \otimes (v_{j+1} \otimes \cdots \otimes v_i)$. Note that $CC_n(A, B) = \text{Hom}(T^r_n(\Pi A), \Pi B)$.

If $C, D$ are graded coalgebras and $\alpha, \beta : C \to D$ graded coalgebra morphisms, an $(\alpha, \beta)$-coderivation is a degree $-1$ map $r : C \to D$ satisfying $\Delta Dr - (r \otimes \alpha + r \otimes \beta)\Delta_C = 0$. We write $\text{Coder}^{\alpha, \beta}(C, D)$ for the set of $(\alpha, \beta)$-coderivations. It will often be the case that $\alpha = 1_C = \beta$; a coderivation is a $(1_C, 1_C)$-coderivation, and we will write $\text{Coder}(C, C)$ for the set of such.

**Lemma 4.5.** Let $A$ and $B$ be graded modules and fix $n \in \mathbb{N} \cup \{\infty\}$.

1. The canonical projection $\pi_i : T^r_n(\Pi B) \to \Pi B$ induces an isomorphism, $\Psi_n = (\pi_1) : \text{Hom}_{\text{coalg}_k}(T^r_n(\Pi A), T^r_n(\Pi B)) \xrightarrow{\cong} \text{Hom}(T^r_n(\Pi A), \Pi B)_0 = CC_n(A, B)_0$. The inverse applied to $\alpha \in CC_n(A, B)_0$ is given by:

\[
\pi_j \Psi_n^{-1}(\alpha)|_{(\Pi A)^{\otimes k}} = \sum_{i_1 + \cdots + i_j = k} \alpha^{i_1} \otimes \cdots \otimes \alpha^{i_j}.
\]

2. Let $\Psi_n^{-1}(\alpha), \Psi_n^{-1}(\beta) : T^r_n(\Pi A) \to T^r_n(\Pi B)$ be two coalgebra morphisms, with $\alpha, \beta \in CC_n(A, B)_0$. The canonical projection $\pi_i : T^r_n(\Pi B) \to \Pi B$ induces an isomorphism, $\Phi^\alpha,\beta_n = (\pi_1) : \text{Coder}^{\alpha, \beta}(T^r_n(\Pi A), T^r_n(\Pi B)) \xrightarrow{\cong} \text{Hom}(T^r_n(\Pi A), \Pi B) = CC_n(A, B)$. The inverse applied to $r \in CC_n(A, B)$ is given by:

\[
\pi_j(\Phi^\alpha,\beta_n)^{-1}(r)|_{(\Pi A)^{\otimes k}} = \sum_{i_1 + \cdots + i_j = k} \alpha^{i_1} \otimes \cdots \otimes \alpha^{i_{m-1}} \otimes r^{i_m} \otimes \beta^{i_{m+1}} \otimes \cdots \otimes \beta^{i_j}.
\]

**Proof.** Part 1 is [19, 2.19], and Part 2, in case both morphisms are the identity, is [19, 2.16]. The proof given in loc. cit. is easily modified to prove part 2 for arbitrary $\alpha$ and $\beta$. \hfill \qed

The isomorphisms above are related to the products defined in [21] as follows. (We write $\Phi_n$ for $\Phi_n^{1_C, 1_C}$ below and in the sequel.)
Lemma 4.6. Let $A$ and $B$ be graded modules and fix $n \in \mathbb{N} \cup \{\infty\}$.

1. For $\mu \in CC^{\leq n}(A,B)$ and $\nu \in CC^{\leq n}(A,A)$, we have $(\mu \circ \nu)^{\leq n} = \mu \Phi_{n}^{-1}(\nu)$.
2. For $\nu \in CC^{\leq n}(B,B)$ and $\alpha \in CC^{\leq n}(A,B)_0$, we have $(\nu \ast \alpha)^{\leq n} = \nu \Psi_{n}^{-1}(\alpha)$.
3. For $\nu \in CC^{\leq n}(B,B)_{-1}$, $r \in CC^{\leq n}(A,B)_1$, and $\alpha, \beta \in CC^{\leq n}(A,B)_0$, we have $\nu_{\alpha \beta} r = \nu(\Phi_{n}^{\beta})^{-1}(r)$.

Proof. Note that $\mu \Phi_{n}^{-1}(\nu)$ is a morphism $T_{co}^{n}(\Pi A) \to \Pi B$, i.e., is an element of $CC^{\leq n}(A,B)$. By the definition of $\Phi_{n}^{-1}$, this is given in tensor degree $1 \leq j \leq n$ by $\sum_{0 \leq j \leq n} \mu^{j}(1 \otimes \nu^{k-j+1} \otimes 1 \otimes \ldots \otimes 1)$, which agrees with $\mu \circ \nu$ in tensor degree $j$.

This proves part 1 and the others are proved in an analogous way. 

Proof of Proposition 4.4. We first prove part 1. Let $\nu^{n+1} \in CC^{n+1}(A,A)_{-1}$ be an arbitrary element, and set $\nu^{\leq n+1} = \nu^{\leq n} + \nu^{n+1}$. We have,

$$
(\nu^{\leq n+1} \circ \nu^{n+1})^{\leq n+1} = (\nu^{\leq n} \circ \nu^{n+1} + \nu^{n} \circ \nu^{n+1} + \nu^{n+1} \circ \nu^{1}),
$$

Thus to show $o(\nu^{\leq n})$ is a cycle, it is enough to show $((\nu^{\leq n+1} \circ \nu^{n+1})^{\leq n+1})$ is a cycle.

Since $\nu^{1}$ has tensor degree 1, $((\nu^{\leq n+1} \circ \nu^{n+1})^{\leq n} \circ \nu^{1}) = (((\nu^{\leq n+1} \circ \nu^{n+1}) \circ \nu^{1})^{\leq n+1}$.

Since $\nu^{\leq n+1} \circ \nu^{n+1}$ is concentrated in tensor degrees at least $n+1$ (because $\nu^{\leq n}$ is an $A_{n}$-algebra structure), $((\nu^{\leq n} \circ \nu^{n+1} \circ \nu^{1})^{\leq n+1} = ((\nu^{\leq n+1} \circ \nu^{n+1}) \circ \nu^{n+1})^{\leq n+1}$.

Analogously, $\nu^{1} \circ (\nu^{\leq n+1} \circ \nu^{n+1})^{\leq n+1} = (\nu^{\leq n+1} \circ (\nu^{\leq n+1} \circ \nu^{n+1}))^{\leq n+1}$.

Thus

$$
d_{AA}((\nu^{\leq n+1} \circ \nu^{n+1})^{\leq n+1}) = (\nu^{\leq n+1} \circ \nu^{n+1} - (\nu^{\leq n+1} \circ \nu^{n+1}) \circ \nu^{n+1})^{\leq n+1} = 0,
$$

since $f \circ (f \circ f) - (f \circ f) \circ f = 0$ for any element of $f \in CC^{*}(A,A)_{-1}$ by [4 §2], and thus $o(\nu^{n})$ is a cycle. By definition, $\nu^{n+1}$ extends $\nu^{e}$ if and only if $(\nu^{\leq n+1} \circ \nu^{n+1})^{\leq n+1} = 0$, and by (44) this is equivalent to $d_{AA}(\nu^{n+1}) = o(\nu^{n})$.

To prove part 2, let $\alpha^{n+1} \in CC^{n+1}(A,B)_{0}$ be an arbitrary element. We have

$$
(\nu^{\leq n+1} \ast \alpha^{n+1} - \alpha^{n+1} \circ \nu^{n+1})^{\leq n+1} = -o(\alpha^{\leq n}) + d_{AB}(\alpha^{n+1}),
$$

Thus to show $o(\alpha^{\leq n})$ is a cycle, it is enough to show (44) is a cycle. Set

$$
da = \Phi_{n+1}^{-1}(\nu^{n+1}) \in \text{Coder}(T_{co}^{n+1}(\Pi A), T_{co}^{n+1}(\Pi A))
$$

$$
db = \Phi_{n+1}^{-1}(\nu^{n+1}) \in \text{Coder}(T_{co}^{n+1}(\Pi B), T_{co}^{n+1}(\Pi B))
$$

$$
\zeta = \Psi_{n+1}^{-1}(\alpha^{n+1}) : T_{co}^{n+1}(\Pi A) \to T_{co}^{n+1}(\Pi B).
$$

By Lemma 4.6 we have

$$
(\nu^{\leq n+1} \ast \alpha^{n+1} - \alpha^{n+1} \circ \nu^{n+1})^{\leq n+1} = \nu^{\leq n+1} \zeta - \alpha^{n+1} \circ \nu^{n+1},
$$

and thus, we aim to show $d_{AB}(\nu^{\leq n+1} \zeta - \alpha^{n+1} \circ \nu^{n+1}) = 0$.

We first claim that $\nu^{1}_{B}(\nu^{\leq n+1} \zeta - \alpha^{n+1} \circ \nu^{n+1} \circ \nu^{n+1} \circ \nu^{1}) = -((\nu^{\leq n+1} \circ \nu^{n+1}) \circ \nu^{n+1} \circ \nu^{1})$. Indeed, because $\nu^{1}_{B}$ has tensor degree 1, $\nu^{1}_{B}(\nu^{\leq n+1} \zeta - \alpha^{n+1} \circ \nu^{n+1} \circ \nu^{1}) = \nu^{1}_{B}(\nu^{\leq n+1} \circ \nu^{n+1} \circ \nu^{1}) - \nu^{1}_{B}(\nu^{\leq n+1} \circ \nu^{n+1} \circ \nu^{1})$,

$$
(\nu^{\leq n+1} \circ \nu^{n+1} \circ \nu^{1}) - \nu^{1}_{B}(\nu^{\leq n+1} \circ \nu^{n+1} \circ \nu^{1}),
$$

and the claim follows from

$$
\nu^{n+1} \circ d_{B} = (\nu^{n+1} \circ \nu^{n+1} \circ \nu^{n+1} \circ \nu^{1}) = 0.
$$
which holds since $ν_B^{n+1}$ is an $A_{n+1}$ algebra structure.

We now claim that $(ν_B^{n+1}ζ - α^{n+1}d_A)ν_A^1 = (ν_B^{n+1}ζd_A)^{≤n+1}$. By tensor degree considerations $(ν_B^{n+1}ζ - α^{n+1}d_A)ν_A^1 = ((ν_B^{n+1}ζ - α^{n+1}d_A) ν_A^{n+1})^{≤n+1}$, and this is equal to $(ν_B^{n+1}ζ - α^{n+1}d_A)ν_A^{n+1}$ by Lemma 4.6. The claim follows since $d_Ad_A = 0$, since $ν_A^{n+1}$ is an $A_{n+1}$ algebra structure. Putting the two claims together, we have

$$d_{AB}(ν_B^{n+1}ζ - α^{n+1}d_A) = - (ν_B^{n+1}ζd_A)^{≤n+1} + (ν_B^{n+1}ζd_A)^{≤n+1} = 0.$$

This shows that $o(α^{≤n})$ is a cycle. By definition, $α^{n+1}$ is an extension of $α^{≤n}$ if and only if

$$(ν_B^{n+1}α^{n+1} - α^{n+1}ν_A^{n+1})^{≤n+1} = 0,$$

and this is equivalent to $d_{AB}(α^{n+1}) = o(α^{n})$ by (1.2).

To prove part 3, let $r^{n+1} ∈ CC^{n+1}(A,B)_1$ be an arbitrary element. We have

$$(ν_B^{n+1}α^{n+1} r^{n+1} - r^{n+1} ν_A^{n+1})^{≤n+1} + β^{n+1} - α^{n+1} = -o(r^{n}) + d_{AB}(r^{n+1}).$$

Thus, as above, we aim to show the left side of the above equation is a cycle. Using similar techniques as in the proof of part 2, one computes

$$ν_B^1(ν_B^{n+1}α^{n+1} r^{n+1} - r^{n+1} ν_A^{n+1})^{≤n+1} = -ν_B^1 r^{n+1} ν_A^{n+1}^{≤n+1} + 0,$$

$$ν_B^1(ν_B^{n+1}α^{n+1} r^{n+1} - r^{n+1} ν_A^{n+1})^{≤n+1} ν_A^1 = (ν_B^1 r^{n+1} ν_A^{n+1})^{≤n+1} + 0,$$

$$α^{n+1} ν_A^1 - ν_A^1 α^{n+1} = 0,$$

where the last two equalities use that $α^{≤n-1}$ and $β^{≤n+1}$ are morphisms of $A_{n+1}$-algebras. Combining the above four equations, we see that

$$d_{AB}(ν_B^{n+1}α^{n+1} r^{n+1} - r^{n+1} ν_A^{n+1})^{≤n+1} + β^{n+1} - α^{n+1} = 0.$$

The rest of the proof is analogous to part 2.

To use obstruction theory to construct strictly unital objects, we need to assume $(A, 1)$ is a graded module with split element (see Definition 2.1). It follows from Definition 2.3 that to extend a strictly unital $A_n$-algebra structure on $A$ to a strictly unital $A_{n+1}$-algebra structure, we need only an element of $CC^{n+1}(A,A)$, not $CC^{n+1}(A,A)$ (and analogously for morphisms and homotopies). If $ν_A^1$ is strictly unital, i.e., $ν_A^1[1] = 0$, the differential $d_{AB}$ preserves $CC^{n+1}(A,B)$. We write $d_{AB}$ for $d_{AB}|_{CC^{n+1}(A,B)}$, so $(CC^{n+1}(A,B), d_{AB})$ is a subcomplex of $(CC^{n+1}(A,B), d_{AB})$. If we denote by $ν_{AB} : ΠA → ΠB$ the map induced by $ν_A$, then $d_{AB}(α) = ν_{AB}α - (−1)^{|α|} α ν_{AB}$. The following shows we can work with the complex $(CC^{n+1}(A,B), d_{AB})$ to do strictly unital obstruction theory.

Lemma 4.7. Let $(A, 1)$ be a graded module with split element, $B$ a graded module with fixed element $1 ∈ B_0$, and fix $n ∈ N ∪ \{∞\}$.

(1) If $(A, 1, ν_A^{≤n} = ν_A^{≤n} + µ_A^{≤n})$ is an $A_n$-algebra with split unit, where $µ_A^{≤n} ∈ CC^{≤n}(A, A)$, then the obstruction $o(ν_A^{≤n})$ is in $CC^{n+1}(A,A) ⊆ CC^{n+1}(A,A)$. For $µ_A^{n+1} ∈ CC^{n+1}(A,A)_1$, $ν_A^{n+1} = ν_A^{n+1} + µ_A^{n+1}$ is a strictly unital $A_{n+1}$-algebra if $d_{AB}(µ_A^{n+1}) = o(ν_A^{n+1})$, when $n ≥ 2$, or $d_{AB}(µ_A^{n+1}) + d_{AA}(µ_A^{n+1}) = 0$ when $n = 1$. 


(2) If $(A, \nu_A^{\leq n+1})$ is an $A_{n+1}$-algebra with split unit, $(B, \nu_B^{\leq n+1})$ a strictly unital $A_{n+1}$-algebra, and $\alpha^{\leq n} = \beta^{\leq n} + g_{su} : (A, \nu_A^{\leq n}) \to (B, \nu_B^{\leq n})$ a strictly unital morphism of $A_n$-algebras, with $\beta^{\leq n} \in CC^{\leq n}(\mathbb{A}, B)$, then the obstruction $o(\alpha^{\leq n})$ is in $CC^{n+1}(\mathbb{A}, B) \subseteq CC^{n+1}(A, B)$. For $\beta^{n+1} \in CC^{n+1}(\mathbb{A}, B)_0$, $\alpha^{n+1} = \beta^{n+1} + g_{su}$ is a strictly unital morphism of $A_{n+1}$-algebras if $d_B(\beta^{n+1}) = o(\alpha^{\leq n})$.

(3) If $\alpha^{\leq n+1}, \beta^{\leq n+1} : (A, \nu_A^{\leq n+1}) \to (B, \nu_B^{\leq n+1})$ are strictly unital morphisms of strictly unital $A_{n+1}$-algebras, and $r^{\leq n} \in CC^{\leq n}(\mathbb{A}, B)_1$, a strictly unital homotopy between $\alpha^{\leq n}$ and $\beta^{\leq n}$, then the obstruction $o(r^{\leq n})$ is in $CC^{n+1}(\mathbb{A}, B)$. For $r^{n+1} \in CC^{n+1}(\mathbb{A}, B)_1$, $r^{\leq n+1}$ is a strictly unital homotopy between $\alpha^{n+1}$ and $\beta^{n+1}$ if $d_B(r^{n+1}) = o(r^{\leq n})$.

Proof. We prove part 1. By definition, $o(\nu^{\leq n}) = (\nu^{\leq n} \circ \nu^{\leq n})^{\leq n+1}$. If $n = 1$, then $o(\nu^{\leq n}) = 0 \in CC^{\leq n}(\mathbb{A}, A)$. If $n \geq 2$, we can write $\nu^{\leq n} = \mu^{\leq n} + \mu_{su}$, for $\mu^{\leq n} \in CC^{\leq n}(\mathbb{A}, A)$. Write $\mu^{\leq n} = \mu^{\leq n} + h^{\leq n} \in CC^{\leq n}(\mathbb{A}, A) \oplus CC^{\leq n}(\mathbb{A}, k)$, so $\nu^{\leq n} \circ \nu^{\leq n} = (\mu^{\leq n} + h^{\leq n} + \mu_{su}) \circ (\mu^{\leq n} + h^{\leq n} + \mu_{su})$. It is clear that $\mu^{\leq n} \circ h^{\leq n} = 0 = h^{\leq n} \circ h^{\leq n}$, and $\mu_{su} \circ h^{\leq n} = 0$ since it is an $A_{\infty}$-algebra structure. By [5] Lemma 4.7.1, $\mu^{\leq n} \circ \mu_{su} + \mu_{su} \circ \mu^{\leq n} = 0$, and by [5] Lemma 4.7.2, $h^{\leq n} \circ \mu_{su} + \mu_{su} \circ h^{\leq n} = \mu_{su} (h^{\leq n} \otimes 1 + 1 \otimes h^{\leq n})$. Thus, $\nu^{\leq n} \circ \nu^{\leq n} = \mu^{\leq n} \circ \mu^{\leq n} + h^{\leq n} \circ \mu^{\leq n} + \mu_{su} (h^{\leq n} \otimes 1 + 1 \otimes h^{\leq n})$, and this vanishes on any element that contains 1, so $o(\nu^{\leq n})$ is in $CC^{n+1}(\mathbb{A}, A)$.

Let $\mu^{\leq n+1} \in CC^{n+1}(\mathbb{A}, A)_{-1}$ be an arbitrary element. Assume first that $n \geq 2$. By [5] Lemma 4.7.1, $\nu^{\leq n} \circ \mu^{\leq n+1} + \mu_{su} \circ \mu^{\leq n+1} = \mu_{su} (\nu^{\leq n} \circ \mu^{\leq n+1} + 1 \otimes \nu^{\leq n})$. If $n = 1$, then $o(\nu^{\leq n}) = 0$, and so by [5] Lemma 4.7.2 again, $\nu^{1} \circ \mu^{1} + \mu_{su}$ is an $A_{n+1}$-algebra if $d_{AB}(\mu^{1} + \mu_{su}) = 0$. The other parts are proved analogously. □

We now formulate the module analogues of the definitions and results of this section. Proofs are not included, but are similar to their algebra analogues.

Lemma 4.8. Let $(A, \nu)$ be a nonunital $A_{\infty}$-algebra.

1. An element $p_M \in CC^{\leq n}(A, \text{End } M)_0$ is an $A_{\infty}$-module structure over $(A, \nu)$ if and only if $p_M^{\leq n-1}$ is an $A_n$-module structure over $(A, \nu^{\leq n-1})$ for all $n \geq 1$.

2. An element $f \in CC^{\leq n}(A, \text{Hom}(M, N))_1$ is a morphism of $A_{\infty}$-modules $(M, p_M) \to (N, p_N)$ if and only if $f^{\leq n-1}$ is a morphism of $A_n$-modules $(M, p_M^{\leq n-1}) \to (N, p_N^{\leq n-1})$ for all $n \geq 1$.

3. An element $f, g \in CC^{\leq n}(A, \text{Hom}(M, N))_2$ is a homotopy between morphisms of $A_{\infty}$-modules $f, g : (M, p_M) \to (N, p_N)$ if and only if $r^{\leq n-1}$ is a homotopy between $f^{\leq n-1}$ and $g^{\leq n-1}$ for all $n \geq 1$.

Definition 4.9. Let $(A, \nu^{\leq n})$ be a nonunital $A_n$-algebra.

1. If $(M, p^{\leq n-1})$ is an $A_n$-module over $(A, \nu^{\leq n-1})$, an element $p_n \in CC^{\leq n}(A, \text{End } M)_0$ extends $p^{\leq n-1}$ if $(M, p_n)$ is an $A_{n+1}$-module over $(A, \nu^{\leq n})$. The obstruction (to extending $p^{\leq n-1}$) is

$$o(p^{\leq n-1}) = - \left( \nu^{\leq n}_{\text{End}} \ast p^{\leq n-1} - p^{\leq n-1} \circ \nu^{\leq n} \right)^{\leq n} \in CC^{n}(A, \text{End } M)_{-1}.$$ 

2. If $(M, p_M^{\leq n})$ and $(N, p_N^{\leq n})$ are $A_{n+1}$-modules over $(A, \nu^{\leq n})$ and $f^{\leq n-1} : (M, p_M^{\leq n-1}) \to (N, p_N^{\leq n-1})$ a morphism of $A_n$-modules, an element $f_n \in CC^{n}(A, \text{Hom}(M, N))_1$ extends $f^{\leq n-1}$ if $f^{\leq n}$ is a morphism of $A_{n+1}$-modules.
Lemma 4.11. Let \((M, p_M^{≤n}) \rightarrow (N, p_N^{≤n})\). The obstruction (to extending \(f^{≤n-1}\)) is

\[
o(f^{≤n-1}) = \left( p^n_N \star f^{≤n-1} + f^{1≤i≤n-1} \circ \nu^{≤n} + f^{≤n-1} \star p^n_M \right)^{≤n}
\]

\(\in CC^n(A, \text{Hom}(M, N))_0\).

(3) If \(f^{≤n}, g^{≤n} : (M, p_M^{≤n}) \rightarrow (N, p_N^{≤n})\) are morphisms of \(A_{n+1}\)-modules, and \(r^{≤n-1}\) is a homotopy between \(f^{≤n-1}\) and \(g^{≤n-1}\), an element \(r^n \in CC^n(A, \text{Hom}(M, N))_2\) extends \(r^{≤n-1}\) if \(r^{≤n}\) is a homotopy between \(f^{≤n}\) and \(g^{≤n}\). The obstruction (to extending \(r^{≤n-1}\)) is

\[
o(r^{≤n-1}) = f^n - g^n - p^n_N \star r^{≤n-1} - r^{≤n-1} \circ \nu^{≤n} - r^{≤n-1} \star p^n_M
\]

\(\in CC^n(A, \text{Hom}(M, N))_1\).

Proposition 4.10. Let \((A, \nu^{≤n})\) be a nonunital \(A_n\)-algebra.

(1) If \((M, p_M^{≤n})\) is an \(A_n\)-module over \((A, \nu^{≤n-1})\), then the obstruction \(o(p^{≤n-1})\) is a cycle in \((CC^n(A, \text{End} M), d_{A, \text{End}})\). An element \(p^n\) in \(CC^n(A, \text{End} M)_0\) extends \(p^{≤n-1}\) if and only if \(d_{A, \text{End}}(p^n) = o(p^{≤n-1})\).

(2) If \((M, p_M^{≤n})\) and \((N, p_N^{≤n})\) are \(A_{n+1}\)-modules over \((A, \nu^{≤n})\) and \(f^{≤n-1} : (M, p_M^{≤n-1}) \rightarrow (N, p_N^{≤n-1})\) is a morphism of \(A_n\)-modules, then the obstruction \(o(f^{≤n-1})\) is a cycle in \((CC^n(A, \text{Hom}(M, N)), d_{A, \text{Hom}})\). An element \(f^n\) in \(CC^n(A, \text{Hom}(M, N))_1\) extends \(f^{≤n-1}\) if and only if \(d_{A, \text{Hom}}(f^n) = o(f^{≤n-1})\).

(3) If \(f^{≤n}, g^{≤n} : (M, p_M^{≤n}) \rightarrow (N, p_N^{≤n})\) are morphisms of \(A_{n+1}\)-modules, and \(r^{≤n-1}\) is a homotopy between \(f^{≤n-1}\) and \(g^{≤n-1}\), then the obstruction \(o(r^{≤n-1})\) is a cycle in \((CC^n(A, \text{Hom}(M, N)), d_{A, \text{Hom}})\). An element \(r^n\) in \(CC^n(A, \text{Hom}(M, N))_2\) extends \(r^{≤n-1}\) if and only if \(d_{A, \text{Hom}}(r^n) = o(r^{≤n-1})\).

Lemma 4.11. Let \((A, 1, \nu^{≤n})\) be an \(A_n\)-algebra with split unit.

(1) If \((M, p_M^{≤n}) = p^{≤n-1} + g^{≤n}\) is a strictly unital \(A_n\)-module over \((A, \nu^{≤n-1})\), with \(p^{≤n-1} \in CC^n(A, \text{End} M)\), then the obstruction \(o(p^{≤n-1})\) is in \(CC^n(A, \text{End} M)_0\). For \(p^n \in CC^n(A, \text{End} M)_0\), \(p^{≤n} + g^{≤n}\) is a strictly unital \(A_{n+1}\)-module over \((A, 1, \nu^{≤n})\) if \(d_{A, \text{End}}(p^n) = o(p^{≤n-1})\).

(2) If \((M, p_M^{≤n})\) and \((N, p_N^{≤n})\) are strictly unital \(A_{n+1}\)-modules over \((A, \nu^{≤n})\), and \(f^{≤n-1} : (M, p_M^{≤n-1}) \rightarrow (N, p_N^{≤n-1})\) is a morphism of strictly unital \(A_n\)-modules, then the obstruction \(o(f^{≤n-1})\) is in \(CC^n(A, \text{Hom}(M, N))\). For \(f^n \in CC^n(A, \text{Hom}(M, N))_1\), \(f^{≤n-1}\) is a morphism \((M, p_M^{≤n}) \rightarrow (N, p_N^{≤n})\) if \(d_{A, \text{Hom}}(f^n) = o(f^{≤n-1})\).

(3) If \(f^{≤n}, g^{≤n} : (M, p_M^{≤n}) \rightarrow (N, p_N^{≤n})\) are morphisms of strictly unital \(A_{n+1}\)-modules and \(r^{≤n-1}\) is a strictly unital homotopy between \(f^{≤n-1}\) and \(g^{≤n-1}\), then the obstruction \(o(r^{≤n-1})\) is in \(CC^n(A, \text{Hom}(M, N))\). For \(r^n \in CC^n(A, \text{Hom}(M, N))_2\), \(r^{≤n}\) is a homotopy between \(f^{≤n}\) and \(g^{≤n}\) if \(d_{A, \text{Hom}}(r^n) = o(r^{≤n-1})\).

5. PROOF OF MAIN RESULTS

Proof of Theorem [S1]. We first prove part 1. Assume that \(n \geq 1\), \((A, 1, \nu^{≤n})\) is an \(A_n\)-algebra with split unit, and the following diagram is commutative for all
1 \leq i \leq n:

\[
\begin{array}{c}
(\Pi A)^{\otimes i} \xrightarrow{\nu_i} \Pi A \\
\downarrow q^{\otimes i} \\
(\Pi B)^{\otimes i} \xrightarrow{\nu_i} \Pi B.
\end{array}
\]

(5.1)

This holds for \( n = 1 \) by hypothesis. We will construct \( \nu^{n+1}_A \) such that (5.1) is commutative for \( i = n + 1 \). The cases \( n = 1 \) and \( n \geq 2 \) require different proofs, but both use the following morphisms of chain complexes:

\[
\varphi = CC^{n+1}(\overline{A}, q) : (CC^{n+1}(\overline{A}, A), d_{\overline{A}}) \to (CC^{n+1}(\overline{A}, B), d_{\overline{B}}),
\]

\[
\phi = CC^{n+1}(q, B) : (CC^{n+1}(B, B), d_B) \to (CC^{n+1}(A, B), d_{AB}).
\]

Since \((\Pi \overline{A})^{\otimes n+1}, \delta_{\otimes} \) is a semiprojective complex and \( q \) is a surjective quasi-isomorphism, \( \varphi \) is also a surjective quasi-isomorphism. These maps fit into the following diagram, where the unlabeled morphism is inclusion,

\[
\begin{array}{c}
CC^{n+1}(\overline{A}, A) \\
\quad \cong \phi \\
CC^{n+1}(\overline{A}, B) \\
\quad \downarrow \\
CC^{n+1}(B, B) \xrightarrow{\phi} CC^{n+1}(A, B).
\end{array}
\]

Assume \( n = 1 \). We construct \( \nu_2^2 \) such that (5.1) is commutative. Consider the element \( \zeta = \phi(\nu_B^2) - q\mu_{su}^2 \in CC^2(A, B)_1 \). We have \( \zeta[a][1] = 0 = \zeta[1][a] \) for all \( a \in A \), since \( \nu_B \) is strictly unital and \( q[1] = [1] \). Thus \( \zeta \) is in \( CC^2(A, B)_1 \). Using the surjectivity of \( \varphi \), choose \( \mu^2 \in CC^2(A, A) \) such that \( \varphi(\mu^2) = \zeta \). We have, using \( \varphi \) is a morphism of chain complexes for the first equality,

\[
\varphi(d_{\overline{A}}(\mu^2)) = d_{\overline{B}}(\varphi(\mu^2)) = d_{\overline{B}}(\zeta)
\]

\[
= d_{AB}(\phi(\nu_B^2)) - qd_{AA}(\mu_{su}^2)
\]

\[
= \phi(d_{BB}(\nu_B^2)) - \varphi(\nu_B^2) - \varphi(\nu_B^2) + qd_{AA}(\mu_{su}^2),
\]

where we can write \( qd_{AA}(\mu_{su}^2) = \varphi(\nu_B^2) \) since \( d_{AA}(\mu_{su}^2) \) is in \( CC^2(A, A) \). Since \( \nu_B^{\leq 2} \) is an \( A_2 \)-algebra structure, and \( o(\nu_B^2) = 0 \), we have \( d_{BB} = 0 \) by [14](1). Rearranging the equation directly above shows \( \varphi(d_{\overline{A}}(\mu^2) + d_{AA}(\mu_{su}^2)) = 0 \), i.e., \( d_{\overline{A}}(\mu^2) + d_{AA}(\mu_{su}^2) \in \ker \varphi \). Since \( \varphi \) is a surjective quasi-isomorphism, \( \ker \varphi \) is acyclic, and thus there exists \( \mu_{su}^2 \) with \( d_{\overline{A}}(\mu_{su}^2) = d_{\overline{A}}(\mu^2) + d_{AA}(\mu_{su}^2) \). Set \( \mu_A^2 = \mu^2 - \mu_{su}^2 \). We then have \( d_{\overline{A}}(\mu_A^2) + d_{AA}(\mu_{su}^2) = 0 \), and so by [14](1), setting \( \nu_A^2 = \mu_A^2 + \mu_{su}^2 \) makes \( A, 1, \nu_A \) into an A2-algebra with split unit. Moreover,

\[
\phi(\nu_B^2) = q(\mu^2 + \mu_{su}^2)
\]

\[
= q(\mu^2 - \mu^2 + \mu_{su}^2) = q\nu_A^2.
\]

\[\text{To see that } (\Pi \overline{A})^{\otimes n+1}, \delta_{\otimes} \text{ is semiprojective, note that } \overline{A} = \ker((A, d_A) \to (k, 0)) \text{ is semiprojective, and the tensor product of two semiprojectives is semiprojective.}\]
where the second equality uses that \( \overline{\mu}^2 \) is in ker \( \varphi \). Thus (5.1) is commutative for \( i = 2 \).

Assume now that \( n \geq 2 \). We continue to use the morphisms \( \varphi \) and \( \psi \) defined above. We have, where the first equality is by definition,

\[
\phi(o(\nu_B^{n+1})) = (\nu_B^{\leq n} \circ \nu_B^{\leq n})^{\leq n+1} q^{\leq n+1} \varphi(\nu_B^{\leq n}) = q(\nu_A^{\leq n} \circ \nu_A^{\leq n})^{\leq n+1} = \varphi(o(\nu_A^{\leq n}))
\]

(The second equality follows from the commutativity of (5.1) for all \( 1 \leq i \leq n \), and the third from the fact that \( o(\nu_A^{\leq n}) \) is in \( CC^{n+1}(\overline{A}, A) \), by (4.4)(1).) Since \( \nu_B \) is strictly unital and \( q[1] = [1] \), the element \( \phi(\nu_B^{n+1}) \) is in \( CC^{n+1}(\overline{A}, B) \). Using the surjectivity of \( \varphi \), choose \( \mu^{n+1} \) in \( CC^{n+1}(\overline{A}, A) \) with \( \varphi(\mu^{n+1}) = \phi(\nu_B^{n+1}) \). We then have, where the first equality follows from (4.4)(1),

\[
0 = \phi(d_{BB}(\nu_B^{n+1}) - o(\nu_B^{\leq n})) = d_{AB}(\phi(\nu_B^{n+1})) - \varphi(o(\nu_B^{\leq n})) = d_{AB}(\varphi(\mu^{n+1})) - \varphi(o(\nu_B^{\leq n})) = \varphi(d_{AA}(\mu^{n+1}) - o(\nu_A^{\leq n}))
\]

i.e., \( d_{AA}(\mu^{n+1}) - o(\nu_A^{\leq n}) \) is in ker \( \varphi \). Since this element is a cycle, by (4.7)(1), and ker \( \varphi \) is acyclic, there exists \( \overline{\mu}^{n+1} \) in \( CC^{n+1}(\overline{A}, A) \) with \( d_{AA}(\overline{\mu}^{n+1}) = d_{AA}(\mu^{n+1}) - o(\nu_A^{\leq n}) \). Set \( \mu^{n+1}_A = \mu^{n+1} \overline{\mu}^{n+1} \), so \( d_{AA}(\mu^{n+1}_A) = o(\nu_A^{\leq n}) \). By (4.4)(1), \( \nu_B^{\leq n+1} = \mu^{\leq n+1}_A + \mu_{su} \) is a strictly unital \( \Lambda_{n+1} \)-algebra structure. Moreover, since \( \varphi(\mu^{n+1}_A) = \varphi(\mu^{n+1}) = \phi(\nu_B^{n+1}) \), and \( \nu_A^{\leq n+1} \), \( \nu_B^{\leq n+1} \) are both strictly unital, (5.1) is commutative for \( i = n + 1 \). It now follows by induction and Lemma (4.1) that there is a strictly unital element \( \nu_A \in CC^*(A, A)_{n-1} \) such that \( (A, 1, \nu_A) \) is an \( \Lambda_\infty \)-algebra with split unit and such that (5.1) is commutative for all \( i \geq 1 \). Thus \( q \) is a strict morphism \( (A, \nu_A) \to (B, \nu_B) \).

If \( (B, \nu_B) \) is augmented, then one can replace \( CC^{n+1}(B, B) \) with \( CC^{n+1}(\overline{B}, B) \), \( CC^{n+1}(A, B) \) with \( CC^{n+1}(A, \overline{B}) \), and \( CC^{n+1}(\overline{A}, A) \) with \( CC^{n+1}((\overline{A}, \overline{A}) \), mimicking the previous proof to construct \( \mu_A \in CC^*(\overline{A}, \overline{A}) \) such that \( \nu_A = \mu_A + \mu_{su} \) is an augmented \( \Lambda_\infty \)-structure and \( q \) is a strict morphism.

We now prove part 2. Write \( \alpha = \beta + g_{su} \), with \( \beta \in CC^*(\overline{C}, B)_0 \). Assume that \( n \geq 1 \), and \( \gamma^{\leq n} + g_{su} \) is a strictly unital morphism of \( \Lambda_n \)-algebras such that \( q(\gamma^{\leq n}) = \beta^{\leq n} \). This holds for \( n = 1 \) by hypothesis. Define \( \psi \) as follows,

\[
\psi = CC^{n+1}(\overline{C}, q) : (CC^{n+1}(\overline{C}, A), d_{\overline{C}A}) \to (CC^{n+1}(\overline{C}, B), d_{\overline{C}B}).
\]

Since \( q \) is a surjective quasi-isomorphism and \( (\Pi \overline{C})^{\otimes n+1} \) is semiprojective, \( \psi \) is a surjective quasi-isomorphism. We now calculate:

\[
\psi(o(\delta^{\leq n})) = q(\gamma^{\leq n} \circ \nu_C^{\leq n+1} - \nu_A^{\leq n+1} \circ \gamma^{\leq n})^{\leq n+1} \nu_B^{\leq n+1} - q(\nu_C^{\leq n+1} \circ \gamma^{\leq n})^{\leq n+1} = (\beta^{\leq n} \circ \nu_C^{\leq n+1} - \nu_B^{\leq n+1} \circ q) \circ \gamma^{\leq n})^{\leq n+1} = (\beta^{\leq n} \circ \nu_C^{\leq n+1} - \nu_B^{\leq n+1} \circ \beta^{\leq n})^{\leq n+1} = o(\alpha^{\leq n}).
\]

Using the surjectivity of \( \psi \), choose \( \tilde{\gamma} \in CC^{n+1}(\overline{C}, A)_0 \) with \( \psi(\tilde{\gamma}) = \beta^{n+1} \). Thus

\[
\psi(d_{\overline{C}A}(\tilde{\gamma}) - o(\delta^{\leq n})) = d_{\overline{C}B}(\tilde{\gamma}) - o(\alpha^{\leq n}) = d_{\overline{C}B}(\beta^{n+1}) - o(\alpha^{\leq n}) = 0,
\]
where the last equality holds by \(4.7\) (2). Thus \(d_{CA}(\gamma) - o(\delta^{\leq n})\) is in ker \(\psi\). By \(4.4\) (2) \(o(\delta^{\leq n})\) is a cycle, and thus \(d_{CA}(\gamma) - o(\delta^{\leq n})\) is also a cycle. Since \(\psi\) is a quasi-isomorphism, ker \(\psi\) is acyclic, and thus there exists \(\epsilon \in \ker \psi\) with \(d_{CA}(\epsilon) = d_{CA}(\gamma) - o(\delta^{\leq n})\). Set \(\gamma^{n+1} = \tilde{\gamma} - \epsilon\), so \(d_{CA}(\gamma^{n+1}) - o(\delta^{\leq n}) = 0\). By \(4.4\) (2), \(\delta^{\leq n} + g_{su}\) is a strictly unital morphism of \(\mathbb{A}_{n+1}\)-algebras. Also, \(p^{\gamma^{n+1}} = \psi(\gamma^{n+1}) = \psi(\tilde{\gamma}) = \beta^{n+1}\), since \(\psi(\epsilon) = 0\). Thus, by induction, there exists \(\delta \in CC^\bullet(C, A)\), such that \(\delta = \gamma + g_{su}\) is a strictly unital morphism with \(q\delta = \alpha\).

To prove part 3, let \(\delta' = \gamma' + g_{su}\) be another strictly unital morphism lifting \(\alpha\) through \(q\), and assume that \(H_0(CC^{n+1}(C, A), d_{CA}) = 0\) for all \(n \geq 1\). Since \(q\gamma' = q\gamma', \gamma' - \gamma\) is in ker \(\psi\). This is element is a cycle, since \(\gamma'\) and \(\gamma\) are cycles. Because ker \(\psi\) is acyclic, there exists \(r' \in CC^1(C, A)_1\) with \(d_{CA}(r') = \gamma' - \gamma\). Assume by induction \(r^{\leq n}\) is a strictly unital homotopy between \(\beta^{\leq n}\) and \(\gamma^{\leq n}\). Since \(o(r^{\leq n})\) is a cycle in \(CC^{n+1}(C, A)_0\), and \(H_0(CC^{n+1}(C, A), d_{CA}) = 0\), there exists \(r^{n+1}\) with \(d_{CA}(r^{n+1}) = o(r^{n+1})\). Thus by \(4.7\) (3), \(r^{\leq n+1}\) is a strictly unital homotopy between \(\gamma^{\leq n+1}\) and \(\gamma^{\leq n+1}\). Now by induction, and \(4.4\) (3), there exists a strictly unital homotopy between \(\delta\) and \(\delta'\).

\[\square\] Proof of Theorem 3.3. We first prove part 1. Write \(p_M = p_M + g_{su}\) with \(p_M \in CC^\bullet(C, A, \text{End} M)_0\). Assume that \(n \geq 1\) and \(p_M^{\leq n-1}\) is an element in \(CC^{\leq n-1}(\mathbb{A}, \text{End} G)_0\) such that \((G, p_M^{\leq n-1} + g_{su})\) is a strictly unital module over \((A, 1, \nu^{\leq n})\) and the following diagram is commutative for all \(0 \leq i \leq n - 1\):

\[
\begin{array}{ccc}
\Pi A & \rightarrow & \Pi \text{End} G \\
\Pi q \downarrow & & \downarrow \Pi q \\
\Pi \text{End} M & \rightarrow & \Pi \text{Hom}(G, M).
\end{array}
\]

This holds for \(n = 1\) by hypothesis (note that \(p^0 = p^0\)). The morphism \(q_* : (\text{End} G, \delta_{\text{Hom}}) \rightarrow (\text{Hom}(G, M), \delta_{\text{Hom}})\) is a surjective quasi-isomorphism, since \(q\) is a surjective quasi-isomorphism and \((G, p_M^0)\) is semiprojective. Consider the following morphisms of complexes:

\[
\varphi = CC^n(A, q_*) : (CC^n(A, \text{End} G), d_{A, \text{End}}) \rightarrow (CC^n(A, \text{Hom}(G, M)), d_{A, \text{Hom}})
\]

\[
\phi = CC^n(A, q^*) : (CC^n(A, \text{End} M), d_{A, \text{End}}) \rightarrow (CC^n(A, \text{Hom}(G, M)), d_{A, \text{Hom}})
\]

Since \(q_*\) is a surjective quasi-isomorphism and \(((\Pi A)^{\otimes n}, \delta_{\otimes})\) is semiprojective, \(\varphi\) is a surjective quasi-isomorphism. We have the following diagram:

\[
CC^n(A, \text{End} P) \overset{\varphi}{\rightarrow} CC^n(A, \text{Hom}(P, M)).
\]

Using the surjectivity of \(\varphi\), choose \(\mathfrak{p}^n\) in \(CC^n(A, \text{End} P)\) with \(\varphi(\mathfrak{p}^n) = \phi(\mathfrak{p}_M^n)\). Using that \(\varphi\) and \(\phi\) are morphisms of complexes, we have \(\varphi(d_{A, \text{End}}(\mathfrak{p}^n)) = \phi(d_{A, \text{End}}(\mathfrak{p}_M^n))\).

We also claim that

\[
\phi(o(p_M^{\leq n-1})) = \varphi(o(p_M^{\leq n-1})).
\]

Assuming the claim (we give a proof in the next paragraph), we then have

\[
\varphi(d_{A, \text{End}}(\mathfrak{p}^n) - o(p_M^{\leq n-1})) = \phi(d_{A, \text{End}}(\mathfrak{p}_M^n) - o(p_M^{\leq n-1})) = 0.
\]
where the last equality follows from \(1.11(1)\). Thus \(d_{\text{End}_{G}}(p^n) - o(p^n)\) is in \(\ker \phi\). Since \(\phi(p^n)\) is a cycle in \((CC^n, \text{End}_{G}), d_{\text{End}_{G}}\) and \(\ker \phi\) is acyclic (since \(\phi\) is a quasi-isomorphism), there exists \(\tilde{p}^n \in \ker \phi \subseteq CC^n(A, \text{End}_G)\) with \(d_{\text{End}_{G}}(\tilde{p}^n) = d_{\text{End}_{G}}(p^n) - o(p^n)\). Set \(\tilde{p}^n = \tilde{p}^n\), so \(d_{\text{End}_{G}}(\tilde{p}^n) = o(p^n)\). By \(4.11(1)\), \(\tilde{p}^n + g_m\) is a strictly unital \(A_n\)-module structure on \(G\). Moreover,

\[
\Pi q_*(\tilde{p}^n) = \phi(p^n) = \phi(\tilde{p}^n) = \Pi q^n_M = \Pi q^n_M,
\]

where the first equality is by the definition of \(\phi\) and \(p^n_G\), the second because \(\tilde{p}^n\) is in \(\ker \phi\), the third by choice of \(\tilde{p}^n\), and the fourth by definition of \(\phi\). Thus \((5.2)\) is commutative for \(i = n\). It now follows by induction and \(4.11(1)\) that there exists \(\tilde{p}_G \in CC^n(A, \text{End}_G)\) such that \((G, p^n_G + g_m)\) is a strictly unital \(A_n\)-module over \((A, 1, \nu)\). By induction \((5.2)\) is commutative for all \(i\), and thus \(q\) is a strict morphism of strictly unital modules.

We now prove that \((5.3)\) holds. First, we note:

\[
\phi(o(p^n_M)) = -\Pi q^*(\nu^\leq_n \circ p^n_M - p^n_M \circ \nu^\leq_n) \leq n
\]

and

\[
\varphi(o(p^n_G)) = -\Pi q^*(\nu^\leq_n \circ p^n_G - p^n_G \circ \nu^\leq_n) \leq n.
\]

Since \(\Pi q^*\) is a morphism of complexes and \(\nu^\leq_1 = \delta_{\text{Hom}}\), we have \(\Pi q^* \nu^\leq_1 = \delta_{\text{Hom}} \Pi q_*\). This gives the first equality below, the second follows from \((5.2)\), and the third is analogous to the first.

\[
\Pi q^* \nu^\leq_1 \circ p^n_M = \delta_{\text{Hom}} \Pi q_* \circ p^n_M = \delta_{\text{Hom}} \Pi q_* \circ p^n_G = \Pi q_* \circ p^n_G.
\]

Thus the first terms of \(\phi(o(p^n_M))\) and \(\varphi(o(p^n_G))\) agree. Using \((5.2)\), we have that \(\Pi q^* \circ p^n_G \circ \nu^\leq_n = \Pi q_* \circ p^n_G \circ \nu^\leq_n\), so the third terms also agree. Working on the second term, where we denote by \(\gamma\) various composition maps, we have

\[
\Pi q^* \nu^\leq_2 \circ p^n_M = -\Pi s \gamma(s^{-1} \otimes^2) \circ p^n_M
\]

\[
= -q^* \gamma(s^{-1} \otimes^2) \circ p^n_M
\]

\[
= -s \gamma(1 \otimes q^* (s^{-1}) \otimes^2) \circ p^n_M
\]

\[
= s \gamma(s^{-1}) \otimes^2 (1 \otimes q^* \circ q) \circ p^n_M.
\]

Analogously, \(\Pi q_* \circ p^n_G \circ \nu^\leq_n = s \gamma(s^{-1}) \otimes^2 (1 \otimes q_* \circ q) \circ p^n_G\). Using \((5.2)\) one checks these last two terms agree.

We now prove part 2. Since \((N, p^n_N)\) is semiprojective and \(q\) is a surjective quasi-isomorphism, there exists a morphism of chain complexes \(\delta^0 : (N, p^n_N) \to (G, p^n_G)\) such that \(\delta^0 = \alpha^0\). Assume by induction that there exists \(\delta^\leq_{n-1} \in CC^\leq_n(\overline{A}, \text{Hom}(N, G))\), a morphism of strictly unital \(A_n\)-modules \((N, p^n_N) \to (G, p^n_G)\) such that \(\delta^\leq_{n-1} = \alpha^\leq_{n-1}\). Set

\[
\psi = CC^n(A, q_* : (CC^n(A, \text{Hom}(N, M)), d_A \circ \text{Hom} = (CC^n(A, \text{Hom}(N, M)), d_A \circ \text{Hom})).
\]

Since \((N, p^n_N)\) is semiprojective and \(q\) is a surjective quasi-isomorphism, \(\Pi q_* : (\Pi \text{Hom}(N, G), \delta_{\text{Hom}}) \to (\Pi \text{Hom}(N, M), \delta_{\text{Hom}})\) is also a surjective quasi-isomorphism. Since \((\Pi \overline{A})^\otimes_n, \delta_{\otimes})\) is semi-projective, \(\psi\) is also a surjective quasi-isomorphism.
Using the surjectivity of $\psi$, choose $\delta \in CC^n(\mathbb{A}, \text{Hom}(N, G))$ with $\psi(\delta) = \alpha^n$. We claim $\psi(q(\delta^{\leq n-1})) = \alpha(\delta^{\leq n-1})$. This follows from the fact that $q$ is a strict morphism and that $q\delta^{\leq n-1} = \alpha^{n-1}$. We now have:

$$
\psi(d_{\text{Hom}}^\mathbb{A}(\delta) - \alpha(\delta^{\leq n-1})) = d_{\text{Hom}}(\alpha^n) - \alpha(\delta^{\leq n-1}) = 0,
$$

where the last equality holds by \ref{e:2.1} (2). Thus, $d_{\text{Hom}}^\mathbb{A}(\delta) - \alpha(\delta^{\leq n-1})$ is in $\ker \psi$. This element is a cycle by \ref{e:2.1} (2). Since $\psi$ is a quasi-isomorphism, ker $\psi$ is acyclic, and so there exists $\tilde{\delta}$ in $\ker \psi$ with $d_{\text{Hom}}^\mathbb{A}(\tilde{\delta}) = d_{\text{Hom}}(\tilde{\delta}) - \alpha(\delta^{\leq n-1})$. Set $\delta^n = \delta - \tilde{\delta}$. Since $d_{\text{Hom}}^\mathbb{A}(\delta^n) = 0(\delta^{\leq n-1})$, it follows from \ref{e:2.1} (2) that $\delta^n$ is a morphism of strictly unital $A_\infty$-modules. Also $q\delta^n = \psi(\delta^n) = \alpha^n$. Thus by induction and \ref{e:2.1} (2), there exists $\delta \in CC^n(\mathbb{A}, \text{Hom}(N, G))$ that is a morphism of strictly unital $A_\infty$-modules and $q\delta = \alpha$.

The proof of part 3 is analogous to the proof of part 3 of \ref{t:3.1}.

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