Chaos Suppression in a Pendulum Equation through Parametric Excitation with Phase Shift for Ultra-Subharmonic Resonance

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Authors’ contributions

This work was carried out in collaboration between all authors. XC designed the study, and wrote the first draft of the manuscript. XF and JT managed the analyses of the study and numerical simulations. XC and JT managed the literature searches. All authors read and approved the final manuscript.

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ABSTRACT

Under ultra-subharmonic resonance, we investigate the chaos suppression of pendulum equation by using Melnikov methods, and get the conditions of suppressing chaos for homoclinic and heteroclinic orbits, respectively. At the same time, we give some numerical simulations including the bifurcation diagrams of system and corresponding phase diagrams, and observe that the chaos behaviors of system may be suppressed to period-$n$ ($n \in \mathbb{Z}^+$) orbits by adjusting the value of $\Psi$. Although our results are only necessary, not sufficient. Numerical simulations show that our method is effective in suppressing chaos for this case.

Keywords: Pendulum equation; Parametric excitation; chaos; chaos control; Melnikov methods.

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1 INTRODUCTION

In next sections, we consider the following pendulum equation

\[ \dot{x} = y, \quad \dot{y} = -\alpha x - \delta y - \left(1 + f_1 \cos(\Omega t + \Psi)\right) x + f_1 \sin(\omega t + \theta), \]

where \( \delta \) is a damping constant, \( \alpha \) represents a spring constant, \( \theta \) denotes the phase shift, \( f_1 \cos(\Omega t + \Psi) \) represents the parametric excitation, \( f_1 \sin(\omega t + \theta) \) denotes the external excitations. When \( \Omega/\omega = p/q, \) \( p \geq 1, q > 1 \) and \( (p, q) = 1 \), the resonance generated by system (1) is called ultra-subharmonic resonance.

There is plenty of application background about pendulum equation, which can express practical models in real life, for example, super-conducting derive [1], shunted model of electrical rotator [2], synchronous electric motor models [3], Josephson junctions [4],[5],[6], and so on. Then, a lot of investigators are attracted by their nonlinear dynamic behavior, they do plenty of work in recent decades, we can see literature [7]-[26], and so on.

For the system (1), there are plenty of results about dynamic behavior. For example, Nayfeh et al. [27] and Wiggins [28] used Melnikov methods to investigate the chaos behavior of system (1) for \( f_0 = 0 \) and \( \theta = 0 \), they gave the existence condition of chaos in system (1). For \( \Psi = 0 \) and \( \theta = 0 \), Jing et al. [29],[30] considered the dynamic behavior of system (1) by using chaos theory under periodic perturbation and quasi-periodic perturbation, and showed the existence condition of chaos in system (1). As \( \Psi = 0 \), Chen et al. [31],[32] used bifurcation theory and chaos theory to investigate the existence condition of bifurcation and chaos in system (1), furthermore, used Melnikov methods to research the chaos suppression of system (1) under primary and sub-harmonic resonance. Yang et al. [33],[34] used Melnikov methods to consider the chaos suppression of system (1) when \( \theta = 0 \).

Although the dynamic behavior of system (1) has been extensively investigated, it is still worth research. In this paper, We further consider the chaos suppression of system (1) under ultra-subharmonic resonances. Although Chacón [35] investigated a similar equation, his conclusion hold only for \( q = 1 \), we will extend the results to more general situations by Melnikov methods. Our result is a necessary condition, but it holds not only for \( q = 1 \), but also for \( q > 1 \). Next, we give theoretical analysis and numerical simulation in the following sections, respectively.

2 FIXED POINTS AND PHASE PORTRAITS FOR UNPERTURBED SYSTEM

When \( \delta = f_1 = f_0 = 0 \), system (1) is an unperturbed system which can be expressed to

\[ \dot{x} = y, \quad \dot{y} = -(\alpha x + \sin x), \]

the corresponding Hamiltonian function is written as

\[ H(x, y) = 1 + \frac{\alpha}{2} x^2 - \cos x + \frac{1}{2} y^2. \]

When \( \alpha = 0.1 \), \( O(0, 0), \ C_1(3.49906, 0), \ C_2(5.67921, 0), \ C_3(-3.49906, 0) \) and \( C_4(-5.67921, 0) \) are equilibrium points of system (2), \( O, \ C_2 \) and \( C_4 \) are centers, \( C_1 \) and \( C_3 \) are saddles. Fig.1 gives the phase portraits of (2) at \( \alpha = 0.1 \). It indicates that \( C_1 \) and \( C_5 \) is linked by orbits \( \Gamma_{het}^{+} \) and \( \Gamma_{het}^{-} \), \( \Gamma_{hom}^{+} \), \( \Gamma_{hom}^{-} \) and \( \Gamma_{hom}^{c} \), where \( \Gamma_{het}^{+} \) and \( \Gamma_{het}^{-} \) are heteroclinic orbits, \( \Gamma_{hom}^{+} \) and \( \Gamma_{hom}^{-} \) are homoclinic orbits.

![Fig. 1. Phase portrait of system (2) for \( \alpha = 0.1 \).](image)
In next section, using Melnikov methods, we investigate the chaos suppression of system (1).

3 CHAOS INHIBITION CONDITIONS

Fig. 1. shows that there are two homoclinic orbits and two heteroclinic orbits for $\alpha = 0.1$. So, for ultra-subharmonic resonance, we can investigate chaos suppression of system (1) by Melnikov methods [35].

We can get the following Melnikov function of equations (1)

$$M(t_0) = \int_{-\infty}^{+\infty} y_0(t) \{ f_1 \sin[\omega(t + t_0) + \theta] - f_0 \cos[\Omega(t + t_0) + \Psi] \sin x_0(t) \} dt - \delta \int_{-\infty}^{+\infty} y_0^2(t) dt,$$

(3)

where $(x_0, y_0) = (x_0(t), y_0(t))$ is the heteroclinic or homoclinic orbit for unperturbed system.

When $(x_0, y_0)$ is homoclinic orbit, $x_0(t)$ and $y_0(t)$ is even and odd, respectively. Then, $M(t_0)$ becomes

$$M_1(t_0) = -2\delta \int_{0}^{\infty} y_0^2(t) dt + 2f_1 \cos(\omega t_0 + \theta) \int_{0}^{\infty} y_0 \sin(\omega t) dt + 2f_0 \sin(\Omega t_0 + \Psi) \int_{0}^{\infty} y_0(t) \sin(\Omega t) \sin x_0(t) dt$$

$$= -C_{hom} + A_{hom} \cos(\omega t_0 + \theta) + B_{hom} \sin(\Omega t_0 + \Psi),$$

(4)

where $A_{hom} = 2f_1 \int_{0}^{\infty} y_0(t) \sin(\omega t) dt$, $B_{hom} = 2f_0 \int_{0}^{\infty} y_0(t) \sin(\Omega t) \sin x_0(t) dt$ and $C_{hom} = 2\delta \int_{0}^{\infty} y_0^2(t) dt$.

When $(x_0, y_0)$ is heteroclinic orbits, $x_0(t)$ and $y_0(t)$ is odd and even, respectively. So $M(t_0)$ can be expressed as

$$M_2(t_0) = -C_{het} + A_{het} \sin(\omega t_0 + \theta) + B_{het} \sin(\Omega t_0 + \Psi),$$

(5)

where $A_{het} = 2f_1 \int_{0}^{\infty} y_0(t) \cos(\omega t) dt$, $B_{het} = 2f_0 \int_{0}^{\infty} y_0(t) \sin(\Omega t) \sin x_0(t) dt$ and $C_{het} = 2\delta \int_{0}^{\infty} y_0^2(t) dt$.

Let $t_1 = t_0 + \theta / \omega$ and $\phi = \Psi - \theta \Omega / \omega$, then Melnikov function (4) and (5) become

$$M_1(t_1) = -C_{hom} + A_{hom} \cos(\omega t_1) + B_{hom} \sin(\Omega t_1 + \phi)$$

(6)

and

$$M_2(t_1) = -C_{het} + A_{het} \sin(\omega t_1) + B_{het} \sin(\Omega t_1 + \phi),$$

(7)

respectively.

By the Melnikov method, the system (1) may be chaotic for $f_0 = 0$, $A_{het} - C_{het} \geq 0$ or $A_{hom} - C_{hom} \geq 0$, and the system (1) may be chaotic for $f_0 \neq 0$, $B_{het} \leq A_{het} - C_{het}$ or $B_{hom} \leq A_{hom} - C_{hom}$ (these inequalities ensure $M_1(t_1)$ or $M_2(t_1)$ change sign at some $t_0$). In order to $M_1(t_1)$ or $M_2(t_1)$ always having the same sign, we find a necessary condition, which is $B_{hom} > A_{hom} - C_{hom}$ for (6) or $B_{het} > A_{het} - C_{het}$ for (7). So, we can suppress chaos by using inequalities $B_{hom} > A_{hom} - C_{hom}$ or $B_{het} > A_{het} - C_{het}$, which can hold through adjusting the value of parameters $\Psi$ and $f_0$. Next, we first study the chaos suppression for homoclinic orbit, then for heteroclinic orbit.

3.1 For Homoclinic Orbits

For right homoclinic orbit, the corresponding Melnikov function is expressed as

$$M_1^+(t_1) = -C_{hom} + A_{hom} \cos(\omega t_1) + B_{hom} \sin(\Omega t_1 + \phi^+).$$

(8)

If $f_0 = 0$, the equality (8) becomes to

$$M_1(t_1) = -C_{hom} + A_{hom} \cos(\omega t_1).$$

(9)
We suppose Melnikov function (9) changes sign at some $t_1$, i.e., $C_{\text{hom}} < A_{\text{hom}}$. At the same time, we choose some $f_0 \neq 0$ such that $B_{\text{hom}} \leq A_{\text{hom}} - C_{\text{hom}}$. Because $B_{\text{hom}} \leq A_{\text{hom}} - C_{\text{hom}}$ is a sufficient condition for $M_1^+(t_1)$ to change sign at some $t_1$, then

$$B_{\text{hom}} > A_{\text{hom}} - C_{\text{hom}} \equiv B_{\text{min}}$$

is a necessary condition for $M_1^+(t_1) < 0$ for all $t_1$.

For the above inequality (10) to be a sufficient condition for $M_1(t_1) < 0$ for all $t_1$, the following inequality

$$A_{\text{hom}} \cos(\omega t_1) + B_{\text{hom}} \sin(\Omega t_1 + \phi^+) \leq A_{\text{hom}} - B_{\text{hom}}$$

must hold. When $m$ and $n$ are positive integers,

$$\frac{p}{q} = \frac{4m + 3 - 2\phi^+ / \pi}{4n},$$

is a sufficient condition for Eq. (11) holding for an infinity of $t_1$ values. $\phi_{\text{min}}^+$ denotes the suitable value of $\phi^+$ in this case, then $\psi_{\text{min}} = \phi_{\text{min}}^+ + \theta \Omega / \omega$ ($\psi_{\text{min}}^+$ denotes suitable value of $\psi^+$).

Equality (12) gives a good situations in which $M_1^+(t_1)$ is as near as possible to be tangency case for $B_{\text{hom}} = B_{\text{min}}$, so it provides us a good situation to suppression chaos. By computing, we get

$$f_{0\text{min}} = (1 - \frac{C_{\text{hom}}}{A_{\text{hom}}}) R,$$

which is the lower threshold of $f_0$, where

$$R = \int_0^{+\infty} y_0(t) \sin(\omega t) dt / \int_0^{+\infty} y_0 \sin(\Omega t) \sin[x_0(t)] dt$$

Next, we study the upper threshold of $f_0$. Don’t enhance the chaos of system (1), we add a necessary condition $B_{\text{hom}} < A_{\text{hom}} + C_{\text{hom}} \equiv B_{\text{max}}$ for $M_1^+(t_1) < 0$. In order to it be sufficient for $M_1^+(t_1) < 0, \forall t_1 \in R$, there must be

$$A_{\text{hom}} \cos(\omega t_1) + B_{\text{hom}} \sin(\Omega t_1 + \phi^+) \leq B_{\text{hom}} - A_{\text{hom}}.$$  \hspace{1cm} (14)

We find that the following equation

$$\frac{p}{q} = \frac{4m + 1 - 2\phi^+ / \pi}{4n + 2}$$

is a sufficient condition for inequality (14) holding for an infinity of $t_1$ values, where the integers $n$ and $m$ are positive. $\phi_{\text{max}}^+$ denotes the suitable value of $\phi^+$, then $\psi_{\text{max}}^+ = \phi_{\text{max}}^+ + \theta \Omega / \omega$ ($\psi_{\text{max}}^+$ denotes suitable value of $\psi^+$).

We find that equality (15) is necessary for (14) holding for all $t_1$, and also gives us a good situation in which chaos can be eliminated because $M_1^+(t_1)$ is as near as possible to tangency condition for $B_{\text{hom}} = A_{\text{hom}} + C_{\text{hom}} \equiv B_{\text{max}}$. For this situation, by computing, we get the following upper threshold

$$f_{0\text{max}} = (1 + \frac{C_{\text{hom}}}{A_{\text{hom}}}) R,$$

For left homoclinic orbit, we have

$$M_1^+(t_1) = -C_{\text{hom}} - A_{\text{hom}} \cos(\omega t_1) + B_{\text{hom}} \sin(\Omega t_1 + \phi^-).$$  \hspace{1cm} (17)

Let $t_1 = \tau_0 + \pi / \omega, \phi^- = \eta^- - \Omega \pi / \omega$, Eq. (17) becomes

$$M_1^+(\tau_0) = \frac{C_{\text{hom}} + A_{\text{hom}} \cos(\omega \tau_0) + B_{\text{hom}} \sin[\Omega \tau_0 + \eta^-].}$$  \hspace{1cm} (18)
Using the same way, we get the lower amplitude threshold \( f_{0 \text{min}} \) and upper amplitude threshold \( f_{0 \text{max}} \) for \( \frac{p}{q} = \frac{4m + 3 - 2\pi/\pi}{4n} \), respectively. Because of \( \phi^- = \eta^- - \Omega \pi/\omega \) and \( \phi = \Psi - \theta t/\omega \), we gain \( \eta_{\text{min,max}} = \phi_{\text{min,max}} + \Omega \pi/\omega \), \( \Psi_{\text{min,max}} = \eta_{\text{min,max}} - \Omega \pi/\omega + \theta t/\omega \). Then, \( \eta_{\text{min,max}} \equiv \phi_{\text{min,max}} (\text{mod} 2\pi) \) (19) and \( \Psi_{\text{min,max}} \equiv \frac{\Omega}{\omega} \pi (\text{mod} 2\pi) \). (20)

According to above analysis, we obtain result 1.

**Result 1.** Assume parameters \( \delta, \alpha, f_1, \theta, \Omega \) and \( \omega \) satisfy \( \frac{p}{q} = \frac{\pi}{2} \) \((p, q \in N^+, q > 1, (p, q) = 1)\), then the chaos of homoclinic orbit of equation (1) can be suppressed at some values of \( \Psi_{\text{min,max}} \) and \( \phi_{\text{min,max}} \), where \( \Psi_{\text{min,max}} = \alpha_{\text{min,max}} + \theta t/\omega - \Omega \pi/\omega \), \( \phi_{\text{min,max}} = \phi_{\text{min,max}} + \theta t/\omega \), and \( \phi_{\text{min}} \) satisfy (12), \( \phi_{\text{max}} \) satisfy (15), \( \eta_{\text{min}} \) and \( \eta_{\text{max}} \) satisfy (19).

# 3.2 For Heteroclinic Orbits

For upper heteroclinic orbits, the corresponding Melnikov function is expressed as

\[
M^+_2(t_1) = -C_{\text{het}} \sin(\omega t_1) + B_{\text{het}} \sin(\Omega t_1 + \phi^+). \tag{21}
\]

When \( f_0 = 0 \), the Melnikov function (21) becomes to

\[
M^+_2(t_1) = -C_{\text{het}} \sin(\omega t_1). \tag{22}
\]

Suppose Melnikov function (22) changes sign at some \( t_1 \), i.e. \( C_{\text{het}} \leq A_{\text{het}} \). We choose some \( f_0 \neq 0 \) such that \( B_{\text{het}} \leq A_{\text{het}} - C_{\text{het}} \). Because \( B_{\text{het}} \leq A_{\text{het}} - C_{\text{het}} \) is sufficient for \( M^+_2(t_1) \) changing sign at some \( t_0 \), then \( B_{\text{het}} > A_{\text{het}} - C_{\text{het}} \equiv B_{\text{min}} \) is necessary for \( M^+_2(t_1) < 0 \), \( (t_1 \in R) \). In order to it be sufficient for \( M^+_2(t_1) < 0 \), \( (t_1 \in R) \), there must be

\[
A_{\text{het}} - B_{\text{het}} \geq A_{\text{het}} \sin(\omega t_1) + B_{\text{het}} \sin(\Omega t_1 + \phi^+). \tag{23}
\]

When \( m, n \) are positive integers,

\[
\frac{p}{q} = \frac{4m + 3 - 2\phi^+}{4n + 1}, \tag{24}
\]

is sufficient for (23) holding for an infinity of \( t_1 \) values. In this case, \( \phi^+_{\text{min}} \) denotes the suitable value of \( \phi^+ \), then \( \Psi_{\text{min}} = \phi^+_{\text{min}} + \theta t/\omega \).

The equality (24) is necessary for inequality (23) holding for all \( t_1 \), but it shows us a good situation to suppressing chaos because \( M^+_2(t_1) \) is as near as possible to be tangency condition for \( B_{\text{het}} = B_{\text{min}} \) in this situation. By computing, we get

\[
f_{0 \text{min}} = (1 - \frac{C_{\text{het}}}{A_{\text{het}}}) R_1(25),
\]

which is the upper threshold of \( f_0 \), where

\[
R_1 = f_1 \int_0^{+\infty} y_0(t) \cos(\omega t) dt/ \int_0^{+\infty} y_0(t) \sin(\Omega t) dt. \tag{25}
\]

For studying the upper threshold of \( f_0 \), we add a necessary condition \( B_{\text{hom}} < A_{\text{hom}} + C_{\text{hom}} \equiv B_{\text{max}} \) for \( M^+_2(t_1) < 0 \) which don’t enhance the chaos of system (1). In order to it be sufficient for \( M^+_2(t_1) < 0, \forall t \in R \), there must be

\[
A_{\text{het}} \sin\omega t_1 + B_{\text{het}} \sin(\Omega t_1 + \phi^+) \leq B_{\text{het}} - A_{\text{het}}. \tag{26}
\]
We find that the following equation

$$\frac{p}{q} = \frac{4m + 1 - 2\phi^+/\pi}{4n + 3},$$

(27)

is necessary for inequality (26) holding, where integers $n$ and $m$ are positive, $\phi_{\text{max}}$ denotes the suitable value of $\phi^+$, then $\Psi^\text{max} = \phi^+_{\text{max}} + \theta\Omega/\omega$ is the suitable value of $\Psi^-$. Equality (27) again provides us a good situation in which $M^\pm_2(t_1)$ is as near as possible to be tangency case for $B_{\text{het}} = A_{\text{het}} + C_{\text{het}} \equiv B_{\text{max}}$, so we can suppress chaos at this situation. By calculating, we get

$$f_{\text{max}} = (1 + \frac{C_{\text{het}}}{A_{\text{het}}})R_1,$$

(28)

$f_{\text{max}}$ is an upper threshold.

For lower heteroclinic orbit, there is

$$M^\pm_2(t_1) = -C_{\text{het}} + A_{\text{het}}\sin\omega t_1 + B_{\text{het}}\sin(\Omega t_1 + \phi^-).$$

(29)

In the same way as the upper heteroclinic orbit, we obtain $\phi_{\text{min}, \text{max}}$ such that $\phi_{\text{min}, \text{max}} \equiv \phi_{\text{min}, \text{max}}(\text{mod}2\pi)$, i.e.

$$\Psi_{\text{min}, \text{max}}^- \equiv \Psi_{\text{min}, \text{max}}^+(\text{mod}2\pi).$$

(30)

From above analysis, the following conclusion can be obtained.

**Result 2.** Assume parameters $\delta, \alpha, f_1, \theta, \Omega$, and $\omega$ satisfy $\Omega = \frac{p}{q}$ ($p, q \in N^+, q > 1, (p, q) = 1$). When $f_0 \in ((1 - \frac{C_{\text{hom}}}{A_{\text{hom}}})R_1, (1 + \frac{C_{\text{hom}}}{A_{\text{hom}}})R_1)$, the heteroclinic chaos of system (1) may be eliminated at some values of $\Psi_{\text{min}, \text{max}}^\pm$, where $\Psi_{\text{min}, \text{max}}^- = \phi_{\text{min}, \text{max}}^+ + \theta\Omega/\omega$, $\phi_{\text{min}, \text{max}}^+$ and $\phi_{\text{min}, \text{max}}^-$ satisfy equation (24) and equation (27), respectively, $\phi_{\text{min}, \text{max}}^+$ satisfy $\phi_{\text{min}, \text{max}}^- \equiv \phi_{\text{min}, \text{max}}^+(\text{mod}2\pi)$, $\phi_{\text{min}, \text{max}}^+$ satisfy $\phi_{\text{min}, \text{max}}^- \equiv \phi_{\text{min}, \text{max}}^+(\text{mod}2\pi)$.

## 4 NUMERICAL SIMULATIONS

In order to verify our theoretical conclusions, Our algorithms include two steps: using the fourth Runge-Kutta technique to solve the solution of equation (2), then calculating $A_{\text{hom}}, A_{\text{het}}, B_{\text{hom}}, B_{\text{het}}, C_{\text{hom}}$ and $C_{\text{het}}$ by integrating. Fixing $\alpha = 0.1, f_1 = 0.519, \delta = 0.125, \theta = \frac{\pi}{2}$.

The homoclinic bifurcation curves for $A_{\text{hom}} = C_{\text{hom}}$ and heteroclinic bifurcation curve for $A_{\text{het}} = C_{\text{het}}$ in $(\omega, f_1)$ plane are showed in Fig. 2, respectively. Taking $\omega = f_1, f_1 = 0.519$; $\delta = 0.125$, we obtain $A_{\text{hom}} > C_{\text{hom}}$ and $A_{\text{het}} > C_{\text{het}}$, which indicates that the dynamics behavior of equation (1) expresses chaos at $f_1 = 0.519$. The chaotic attractor of system (1) is given in Fig. 3 for $\omega = 1, \alpha = 0.1, \delta = 0.125, f_0 = 0$ and $f_1 = 0.519$. Next, we would consider the chaos suppression through parametric excitation.

(1) $\Omega = \frac{1}{2}$.

Taking $\Omega = 0.5, \omega = 1$. We obtain $f_{\text{min}} = 0.3235$, $f_{\text{max}} = 0.7722$ and $\Psi_{\text{min}, \text{max}}^+ = \{\frac{\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{19\pi}{12}\}$ by the result 1. When $f_0 = 0.5$, the bifurcation diagram is showed in Fig. 4(a1), it means that the chaos of homoclinic orbit is converted to period-2 orbit at $\Psi = \frac{\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}$ and $\frac{19\pi}{12}$. Fig. 4(b1) is the corresponding period-2 orbits at $\Psi = \frac{\pi}{12}$. According to the result 2, we get $f_{\text{min}} = 0.1417$, $f_{\text{max}} = 1.8036$ and $\Psi_{\text{min}, \text{max}}^+ = \{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{4\pi}{3}, \frac{11\pi}{6}\}$. When $f_0 = 1.2$, Fig. 4(a2) is the corresponding bifurcation diagram, it indicates that the chaos of heteroclinic orbit is converted to period-2 orbit at $\Psi = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{4\pi}{3}$ and $\frac{11\pi}{6}$. Fig. 4(b2) is the corresponding period-2 orbits at $\Psi = \frac{\pi}{6}$. 

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Fig. 2. Homoclinic bifurcation curve for $A_{\text{hom}} = C_{\text{hom}}$ and heteroclinic bifurcation curve for $A_{\text{het}} = C_{\text{het}}$ in $(\omega, f_1)$ plane with $\alpha = 0.1$, $\delta = 0.125$.

Fig. 3. The chaotic attractor of system (1) for $\alpha = 0.1$, $\omega = 1$, $\delta = 0.125$, $f_1 = 0.519$ and $f_0 = 0$.

Fig. 4. Fixing $\alpha = 0.1$, $f_1 = 0.519$, $f_0 = 0.5$, $\delta = 0.125$, $\Omega = 0.5$, $\omega = 1$ and $\theta = \frac{\pi}{4}$. (a1) The bifurcation diagram of system (1) in $(\Psi, x)$ plane for $f_0 = 0.5$, (b1) the corresponding period-2 orbits at $\Psi = \frac{\pi}{3}$. (a2) The bifurcation diagram of system (1) in $(\Psi, x)$ plane for $f_0 = 1.2$, (b2) the corresponding period-2 orbits at $\Psi = \frac{\pi}{4}$. 
Taking $\Omega = \frac{1}{3}, \omega = 1$. By the result 1, we get $f_{0\min} = 0.3852$, $f_{0\max} = 0.9197$ and $\Psi_{\min,\max} = \{\frac{2\pi}{9}, \frac{5\pi}{8}, \frac{8\pi}{11}, \frac{11\pi}{14}, \frac{14\pi}{17}, \frac{17\pi}{20}\}$. The bifurcation diagram in $(\Psi,x)$ plane for $f_0 = 0.4$ is given in Fig. 5 (a1), which indicates that the chaos of homoclinic orbit is converted to period-3 orbit at $\Psi = \frac{2\pi}{9}, \frac{5\pi}{8}, \frac{8\pi}{11}, \frac{11\pi}{14}, \frac{14\pi}{17}, \frac{17\pi}{20}$. The corresponding period-3 orbits at $\Psi = \frac{2\pi}{9}$ is given in Fig. 5(b1). According to the result 2, we obtain $f_{0\min} = 0.2161$, $f_{0\max} = 2.7507$ and $\Psi_{\min,\max} = \{\frac{15\pi}{18}, \frac{18\pi}{18}, \frac{21\pi}{18}\}$. When $f_0 = 1$, Fig. 5(a2) gives the bifurcation diagram, which shows that the chaos of heteroclinic orbit is converted to period-3 orbit at $\Psi = \frac{15\pi}{18}, \frac{18\pi}{18}, \frac{21\pi}{18}$. The corresponding period-3 orbits at $\Psi = \frac{15\pi}{18}$ is given in Fig. 5(b2).

From cases (1)-(3), we find that the chaos of homoclinic orbit and heteroclinic orbit can be convert to periodic orbit when the results 1 and 2 hold. Our results is necessary, not sufficient, but the effect of suppressing chaos of homoclinic orbit and heteroclinic orbit is good when some positive integer values of $p, q$ are given.
Fig. 6. Fixing $\alpha = 0.1, f_1 = 0.519, \delta = 0.125, \Omega = \frac{2}{3}, \omega = 1$ and $\theta = \frac{\pi}{2}$. (a1) The bifurcation diagram of system (1) in $(\psi, x)$ plane for $f_0 = 0.73$, (b1) the corresponding period-3 orbits at $\psi = \frac{5\pi}{18}$. (a2) The bifurcation diagram of system (1) in $(\psi, x)$ plane for $f_0 = 1$, (b2) the corresponding period-3 orbits at $\psi = \frac{5\pi}{18}$.

5 CONCLUSIONS

In this article, we use Melnikov methods to investigate the chaos suppression of pendulum equation under ultra-subharmonic resonance, and give the necessary conditions of chaos suppression for heteroclinic and homoclinic chaos, respectively. Numerical simulations shows the chaos behaviors can be converted to periodic orbits. The conclusion in paper [35] is a sufficient condition, but holds only for $q = 1$. In this paper, the conclusion obtained is a necessary condition, but the result holds not only for $q = 1$, but also for $q > 1$. Although our results are only necessary, not sufficient. When some positive integer values of $p, q$ are given, this methods is effective in suppressing chaos under ultra-subharmonic resonance.

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COMPETING INTERESTS

We certify that this manuscript complies with Ethical Standards. It is original and has not been published and will not be submitted elsewhere for publication while being considered by Current Journal of Applied Science and Technology. The authors declare that they have no conflict of interest. This article does not contain any studies with human participants or animals performed by any of the authors. Informed consent was obtained from all individual participants included in the study.

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