Gerbes of chiral differential operators. III
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Introduction

This note is a sequel to [GII]. Its aim is to "switch on an exterior vector bundle" into the framework of \textit{op. cit.}

Let \(X\) be a smooth scheme over a fixed ground ring \(k\) containing \(1/2\) and \(E\) be a vector bundle (i.e. a locally free \(\mathcal{O}_X\)-module) of finite rank over \(X\). Consider the exterior algebra \(\Lambda E = \bigoplus_{i=0}^{\text{rk}(E)} \Lambda^i E\) (over \(\mathcal{O}_X\)); this is a sheaf of commutative superalgebras over \(X\), where by definition \(\mathcal{O}_X\) is purely even and the parity of a component \(\Lambda^i E\) is equal to the parity of \(i\).

In this note we study the chiral counterparts of the sheaf \(D_{\Lambda E}\) of superalgebras of differential operators acting on \(\Lambda E\). Similarly to \textit{op. cit.}, these \textit{chiral sheaves of differential operators on} \(\Lambda E\) exist locally and are by no means unique; the corresponding categories form a \textit{champ en groupoids} \(D_{\Lambda E}\) over \(X\), called the \textit{gerbe of chiral differential operators on} \(\Lambda E\).

Our first main result (see Theorem 5.9) says that the characteristic class \(c(D_{\Lambda E})\) lies in the second hypercohomology group \(H^2(X; \Omega^{2,3}_X)\) (i.e. in the same group where \(c(D_X)\) lies) and is equal to

\[
c(D_{\Lambda E}) = c(\Theta_{X/k}) - c(E) = c(\Omega^1_{X/k}) - c(E)
\]

(0.1)

where \(c(E)\) is the "Atiyah-Chern-Simons" class defined in [GII], 7.6. Here \(\Theta_{X/k}\) is the tangent bundle. Recall that \(\Omega^{2,3}_X\) denotes the length 1 complex of sheaves \(\Omega^2_{X/k} \rightarrow \Omega^{3,\text{closed}}_{X/k}\), with \(\Omega^2_{X/k}\) living in degree 0.

As usually, we obtain in fact a stronger statement, namely the equality (0.1) "on the level of cocycles". As a corollary of this, we conclude that for \(E = \Theta_{X/k}\) or \(E = \Omega^1_{X/k}\) our gerbes admit a canonical global section. In other words, there exist canonically defined \textit{the} sheaves of chiral do \(D_{\lambda \Theta_{X/k}}^h\) and \(D_{\Omega^1_{X/k}}^h\).

Section 6 is devoted to the study of the last sheaf, which is nothing but (the underlying sheaf of) \textit{chiral de Rham complex} from [MSV]. We obtain the transformation laws of 4 local generators of \(N = 2\) supersymmetry \(Q, J, G\) and \(L\), see Theorem 6.25. In particular, the component \(Q_0\) of the field \(Q(z)\) is a globally defined square zero derivation of \(D_{\Omega^1_{X/k}}^h\), which is the \textit{chiral de Rham differential} from \textit{op. cit.}

This completes an alternative construction of the chiral de Rham complex sketched in Section 6 of [MSV]. Its difference from the original construction is that it does not use Wick theorem and the arguments of "formal geometry".
In the last section we show that as a simple consequence of the Poincaré-Birkhoff-Witt theorem for $D_{1+1}$, and the Lefschetz fixed point theorem one gets a ”moonshine style” formula, cf. Theorem 7.9.

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§1. Preliminaries

1.1. We keep the assumptions of [GII]. We assume that the ground ring $k$ contains $1/2$. For a $k$-supermodule $M$, we denote by $M^{ev}$ (resp. $M^{odd}$) the submodule of even (resp. odd) elements, so that $M = M^{ev} \oplus M^{odd}$. For a homogeneous element $a \in M$, we denote by $p(a) \in \mathbb{Z}/2\mathbb{Z}$ its parity. When we speak about graded $k$-supermodules $M = \oplus_{i \in \mathbb{Z}/2\mathbb{Z}} M_i$ we imply that the $I$-grading is compatible with the parity, i.e. $M^x = \oplus_i M^{x_i}$ where $M^{x_i} = M^x \cap M_i$, $x = ev$ or odd.

Let $A$ be a commutative $k$-superalgebra. A Lie superalgebroid over $A$ is a Lie superalgebra over $k$ equipped with a structure of an $A$-module, such that the identities [GII] (0.2.1) and (0.2.2) hold true.

1.2. A $\mathbb{Z}_{\geq 0}$-graded vertex superalgebra (over $k$) is a $\mathbb{Z}_{\geq 0}$-graded $k$-superalgebra $V = \oplus_{i \geq 0} V_i$ equipped with a distinguished even vector $1 \in V_0$ (vacuum vector) and a family of bilinear operations

$$(a) : V \times V \longrightarrow V, \; n \in \mathbb{Z},$$

such that

$$p(a_{(n)} b) = p(a) + p(b); \; V_{i(n)} V_j \subset V_{i+j-n-1} \tag{1.2.1}$$

The following properties must hold:

$$1_{(n)} a = \delta_{n,-1} a; \; a_{(n)} 1 = 0 \; \text{for} \; n \geq 0, \; a_{(-1)} 1 = a \tag{1.2.2}$$

and

$$\sum_{j=0}^{\infty} \binom{m}{j} (a_{(n+j)} b_{(m+l-j)} c = \tag{1.2.3}$$

$$= \sum_{j=0}^{\infty} (-1)^j \binom{n}{j} \{ a_{(m+n-j)} b_{(l+j)} c - (-1)^{n+p(a)p(b)} b_{(n+l-j)} a_{(m+j)} c \}$$

for all $m, n, l \in \mathbb{Z}$. A particular case of (1.2.3) corresponding to $m = 0$:

$$(a_{(n)} b)_{(j)} c = \sum_{j=0}^{\infty} (-1)^j \binom{n}{j} \{ a_{(n-j)} b_{(l+j)} c - (-1)^{n+p(a)p(b)} b_{(n+l-j)} a_{(j)} c \} \tag{1.2.4}$$

Setting $n = l = -1$ we get

$$(a_{(-1)} b)_{(-1)} c = \sum_{j=0}^{\infty} \{ a_{(-1-j)} b_{(-1+j)} c + (-1)^{p(a)p(b)} b_{(-2-j)} a_{(j)} c \} \tag{1.2.5}$$
In the sequel we shall work only with $\mathbb{Z}_{\geq 0}$-graded vertex superalgebras, and call them simply vertex superalgebras. This $\mathbb{Z}_{\geq 0}$-grading will be called the grading by conformal weight.

1.3. Let $V$ be a vertex superalgebra. The even operators $\partial^{(j)} : V \to V$ of degree $j$ ($j \in \mathbb{Z}_{\geq 0}$) are defined in the same manner as in [GII], (0.5.5), and they satisfy [GII], (0.5.7), (0.5.8), (0.5.10) and (0.5.11). The ”supercommutativity” formula reads as

$$a_{(n)}b = (-1)^{n+p(a)p(b)+1} \sum_{j \geq 0} (-1)^j \partial^{(j)}(b_{(n+j)}a) \quad (1.3.1)$$

and we have the usual OPE formula

$$[a_{(m)}, b_{(n)}] = \sum_{j \geq 0} \binom{m}{j} (a_{(j)}b)_{(m+n-j)} \quad (1.3.2)$$

where in the left hand side stands the supercommutator

$$[a_{(m)}, b_{(n)}] := a_{(m)}b_{(n)} - (-1)^{p(a)p(b)}b_{(n)}a_{(m)} \quad (1.3.3)$$

§2. Vertex Superalgebroids

2.1. An extended Lie superalgebroid (over $A$) is a quintuple $\mathcal{T} = (A, T, \Omega, \partial, \langle , \rangle)$ where $T$ is a Lie superalgebra over $A$, $\Omega$ is an $A$-module equipped with a structure of a module over the Lie superalgebra $T$, $\partial : A \to \Omega$ an even $A$-derivation and a morphism of $T$-modules, $\langle , \rangle : T \times \Omega \to A$ an even $A$-bilinear pairing.

The following identities must be true ($a \in A, \tau, \nu \in T, \omega \in \Omega)$:

$$\langle \tau, \partial a \rangle = \tau(a) \quad (2.1.1)$$

$$\tau(a\omega) = \tau(a)\omega + (-1)^{p(\tau)p(a)}a\tau(\omega) \quad (2.1.2)$$

$$\langle a\tau(\omega), \nu \rangle = a\langle \tau(\omega), \nu \rangle + \langle \tau(\omega), \partial a \rangle \quad (2.1.3)$$

$$\tau(\langle \nu, \omega \rangle) = \langle \tau(\nu), \omega \rangle + (-1)^{p(\tau)p(\nu)}\langle \nu, \tau(\omega) \rangle \quad (2.1.4)$$

For example to a Lie superalgebroid $T$ one associates canonically an extended superalgebroid with $\Omega = Hom_A(T, A)$, as in [GII], 1.2.

2.2. De Rham - Chevalley complex. Let $\mathcal{T} = (A, T, \Omega, \ldots)$ be an extended Lie $A$-superalgebroid. Let us define $A$-modules $\Omega^i = \Omega^i(\mathcal{T})$, $i \in \mathbb{Z}_{\geq 0}$, as follows. Set $\Omega^0 = A$, $\Omega^1 = \Omega$. For $i \geq 2$, $\Omega^i$ is the submodule of the module of $A$-polylinear homomorphisms $h$ from $T^{i-1}$ to $\Omega$ such that the function $\langle \tau_1, h(\tau_2, \ldots, \tau_i) \rangle$ is skew symmetric (in the graded sense) with respect to all permutations of $(\tau_1, \ldots, \tau_i)$.

For example, if $\mathcal{T}$ is associated with a Lie superalgebroid $T$ as above then $\Omega^i = Hom_A(A_i^iT, A)$.
Let us define the even maps $d_{DR} = d_{DR}^i : \Omega^i \to \Omega^{i+1}$ as follows. For $i = 0$ we set $d_{DR}^0 = -\partial a$. For $i \geq 1$ we set

$$d_{DR}h(\tau_1, \ldots, \tau_i) = d_{\text{Lie}}h(\tau_1, \ldots, \tau_i) - (-1)^{|p(h)|p(\tau_1)}\partial(h(\tau_1, h(\tau_2, \ldots, \tau_i))) \tag{2.2.1}$$

where

$$d_{\text{Lie}}h(\tau_1, \ldots, \tau_i) = \sum_{j=1}^{i-1} (-1)^{|j|+1+p(\tau_j)}p(\tau_1)\cdots p(\tau_j)h(\tau_1, \ldots, \tau_j, \ldots, \tau_i) +$$

$$+ \sum_{1 \leq j < l \leq i} (-1)^{|j|+1+p(\tau_j)+p(\tau_l)+p(\tau_{j-l})}h(\tau_1, \ldots, \tau_j, \ldots, \tau_l, \ldots, \tau_i) \times$$

$$\times h([\tau_j, \tau_l], \tau_1, \ldots, \hat{\tau}_j, \ldots, \hat{\tau}_l, \ldots, \tau_i) \tag{2.2.2}$$

For example,

$$d_{DR}\omega(\tau) = (-1)^{|p(\omega)|p(\tau)}\{\tau(\omega) - \partial(\tau, \omega)\}, \tag{2.2.3}$$

for $\omega \in \Omega^1 = \Omega$; and

$$d_{DR}h(\tau_1, \tau_2) = -h([\tau_1, \tau_2]) + (-1)^{|p(h)|p(\tau_1)}\tau_1h(\tau_2) -$$

$$-(-1)^{|p(\tau_2)+p(\tau_1)}\tau_2h(\tau_1) - (-1)^{|p(h)|p(\tau_1)}\partial(\tau_1, h(\tau_2)), \tag{2.2.4}$$

for $h \in \Omega^2$.

Let us introduce the action of the Lie algebra $T$ on the modules $\Omega^i$ by

$$\tau(h)(\tau_1, \ldots, \tau_{i-1}) = \tau(h(\tau_1, \ldots, \tau_{i-1})) -$$

$$-\sum_{j=1}^{i-1} (-1)^{|j|+1+p(\tau_1)\cdots p(\tau_{j-1})}h(\tau_1, \ldots, [\tau, \tau_j], \ldots, \tau_i) \tag{2.2.5}$$

Let us define the convolution operators $\langle \tau, \cdot \rangle : \Omega^i \to \Omega^{i-1}$ by

$$\langle \tau, \cdot \rangle(\tau_1, \ldots, \tau_{i-2}) = (-1)^{|p(\tau)|p(h)}h(\tau, \tau_1, \ldots, \tau_{i-2}) \tag{2.2.6}$$

The maps $\{d^i_{DR}\}$ may be characterized as a unique collection of maps such that $d^0_{DR} = -\partial$ and the Cartan formula

$$\tau(h) = \langle \tau, d_{DR}h \rangle + d_{DR}(\tau, h) \tag{2.2.7}$$

holds true.

The maps $d_{DR}$ commute with the action of $T$. One checks that $d^2_{DR} = 0$, so we get a complex $(\Omega^\bullet(T), d_{DR})$ called the de Rham-Chevalley complex of $T$.

2.3. A vertex superalgebroid is a septuple $A = (A, T, \Omega, \partial, \gamma, \langle \cdot, \cdot \rangle, c)$ where $A$ is a supercommutative $k$-algebra, $T$ is a Lie superalgebroid over $A$, $\Omega$ is an $A$-module equipped with an action of the Lie superalgebra $T$, $\partial : A \to \Omega$ is an even derivation commuting with the $T$-action,

$$\langle \cdot, \cdot \rangle : (T \oplus \Omega) \times (T \oplus \Omega) \to A$$
is a supersymmetric even $k$-bilinear pairing equal to zero on $\Omega \times \Omega$ and such that 
$T_A = (A, T, \Omega, \partial, (, )|_{T \times T})$ is an extended Lie superalgebroid over $A$; $c : T \times T \rightarrow \Omega$ is a skew supersymmetric even $k$-bilinear pairing and $\gamma : A \times T \rightarrow \Omega$ is an even $k$-bilinear map.

The following axioms must hold $(a, b \in A; \tau, \tau_1 \in T)$:

$$\gamma(a, b\tau) = \gamma(ab, \tau) - a\gamma(b, \tau) -$$

$$(-1)^{p(\tau)p(a)}\gamma(a, b)\tau b - (-1)^{p(a)p(b)+p(\tau)p(a)}\gamma(a, b)\partial a$$  \hspace{1cm} (A1)

$$\langle a\tau_1, \tau_2 \rangle = a\langle \tau_1, \tau_2 \rangle + \langle \gamma(a, \tau_1), \tau_2 \rangle - (-1)^{p(\tau_1)p(\tau_2)}\tau_1 \tau_2 (a)$$  \hspace{1cm} (A2)

$$c(a\tau_1, \tau_2) = ac(\tau_1, \tau_2) + \gamma(a, [\tau_1, \tau_2]) -$$

$$(-1)^{p(\tau_1)p(\tau_2)+p(c)}\gamma(\tau_2(a), \tau_1) + (-1)^{p(\tau_2)p(\tau_1)+p(c)}\tau_2 (\gamma(\tau_1(a))) -$$

$$(-1)^{p(c)p(\tau_1)}\frac{1}{2}\langle \tau_1, \tau_2 \rangle \partial a + (-1)^{p(a)p(\tau_1)+p(\tau_2)}\frac{1}{2}\partial \tau_1 \tau_2 (a) -$$

$$(-1)^{p(\tau_2)p(\tau_1)}\frac{1}{2}\partial \tau_2 (\gamma(\tau_1(a)))$$  \hspace{1cm} (A3)

$$\langle \tau_1, \tau_2, \tau_3 \rangle + (-1)^{p(\tau_1)p(\tau_2)}\langle \tau_2, [\tau_1, \tau_3] \rangle = \tau_1 (\langle \tau_2, \tau_3 \rangle) -$$

$$(-1)^{p(\tau_1)p(\tau_2)}\frac{1}{2}\langle \tau_1, \tau_3 \rangle \partial a - (-1)^{p(\tau_2)p(\tau_1)\tau_2 (\gamma(\tau_1(a)))} +$$

$$+ (-1)^{p(\tau_1)p(\tau_2)}\langle \tau_2, c(\tau_1, \tau_3) \rangle + (-1)^{p(\tau_2)p(\tau_1)+p(\tau_2)}\langle \tau_3, c(\tau_1, \tau_2) \rangle$$  \hspace{1cm} (A4)

$$d_{Lie} c(\tau_1, \tau_2, \tau_3) = \frac{1}{2} \theta \{\langle \tau_1, \tau_2, \tau_3 \rangle + (-1)^{p(\tau_2)p(\tau_1)}[\langle \tau_1, \tau_3 \rangle, \tau_2] -$$

$$- (-1)^{p(\tau_1)p(\tau_2)+p(\tau_3)}\langle \tau_2, \tau_3 \rangle - \tau_1 (\langle \tau_2, \tau_3 \rangle) + (-1)^{p(\tau_2)p(\tau_1)}\tau_2 (\langle \tau_1, \tau_3 \rangle) -$$

$$- (-1)^{p(\tau_1)p(\tau_2)+p(\tau_3)}2\tau_3 (c(\tau_1, \tau_2)) \}$$  \hspace{1cm} (A5)

where $d_{Lie}$ is defined by (2.2.2).

**2.4.** All the constructions of [GHI] generalize to the $\mathbb{Z}/(2)$-graded case in an obvious manner.

**§3. Some formulas**

**3.1.** Let $A$ be a smooth $k$-algebra of relative dimension $n$, such that the $A$-module $T = Der_k(A)$ is free and admits a base $\{\tau_i\}$ consisting of commuting vector fields. Let $E$ be a free $A$-module of rank $m$, with a base $\{\phi_\alpha\}$. We shall call the set $\mathfrak{g} = \{\bar{\tau}_i; \phi_\alpha\} \subset A \oplus E$ a frame of $(A, E)$.

Consider a commutative $A$-superalgebra $\mathcal{A}_E = \bigoplus_{i=0}^m A^i_A(E)$ where the parity of $A^i_A(E)$ is equal to the parity of $i$. Each frame $\mathfrak{g}$ as above gives rise to a $\mathcal{A}_E$-base $\{\bar{\tau}_i; \psi_\alpha\}$ of the Lie superalgebroid $T_{\mathcal{A}_E} = Der_k(\mathcal{A}_E)$, defined as follows. We extend the fields $\bar{\tau}_i$ to derivations $\tau_i$ of the whole superalgebra $\mathcal{A}_E$ by the rule

$$\tau_i(a) = \bar{\tau}_i(a); \tau_i(\sum a_\alpha \phi_\alpha) = \sum \bar{\tau}_i(a_\alpha) \phi_\alpha$$  \hspace{1cm} (3.1.1)
(Note that this extension depends on a choice of a base \{\phi_\alpha\} of the module \(E\).
The fields \{\tau_i\} form a \(\Lambda E\)-base of the even part \(T^\Lambda_{\Lambda E} T^\Lambda_{\Lambda E}\).

We define the odd vector fields \(\psi_\alpha \in T^\Lambda_{\Lambda E}\) by

\[
\psi_\alpha \left( \sum a_\nu \phi_\nu \right) = a_\alpha; \quad \psi_\alpha (a) = 0
\]

These fields form a \(\Lambda E\)-base of \(T^\Lambda_{\Lambda E}\).

Let \{\omega_i; \rho_\alpha\} be the dual base of the module of 1-superforms \(\Omega_{\Lambda E} = Hom^\Lambda_{\Lambda E}(T_{\Lambda E}, \Lambda E)\), defined by

\[
\langle \tau_i, \omega_j \rangle = \delta_{ij}; \quad \langle \psi_\alpha, \rho_\beta \rangle = \delta_{\alpha\beta}; \quad \langle \psi_\alpha, \omega_i \rangle = 0
\]

3.2. Let us describe the effect of a change of frame. Let \(g' = \{\bar{\tau}'_i; \phi'_\alpha\}\) be another frame, with \(\bar{\tau}'_i = g'^{ij} \tau_j; \phi'_\alpha = A^{\alpha\beta} \phi_\beta, g = (g^{ij}) \in GL_n(A), A = (A^{\alpha\beta}) \in GL_m(A)\).

The corresponding new bases \(\tau'_i,\) etc. look as follows.

\[
\tau'_i = g^{ip} \tau_p + g^{i\alpha\gamma} \phi_\gamma \psi_\alpha
\]

where

\[
g^{i\alpha\gamma} = g^{iq}(A^{-1\mu})A^{\mu\gamma}
\]

Next,

\[
\psi'_\alpha = A^{-1\mu\alpha} \psi_\mu
\]

\[
\omega'_j = g^{-1p\iota} \omega_p
\]

\[
\rho'_\alpha = \tau_i (A^{\alpha\gamma}) \phi_\gamma \omega_i + A^{\alpha\mu} \rho_\mu
\]

Formulas for the inverse transformation:

\[
\tau_q = g^{-1\iota q} \tau'_i + \tau_q (A^{\alpha\gamma}) \phi_\gamma \psi'_\alpha
\]

\[
\psi'_\beta = A^{\alpha\beta} \psi'_\alpha
\]

\[
\omega'_j = g^{p\iota} \omega_p
\]

\[
\rho'_\beta = A^{-1\beta\iota} \rho'_\alpha + g^{p\beta\iota} \phi_\gamma \omega'_p
\]

These formulas show that \(T\) is canonically an \(A\)-module quotient of \(T_{\Lambda E}\) and \(\Omega = Hom_A(T, A)\) is canonically an \(A\)-submodule of \(\Omega_{\Lambda E}\). In fact the whole de Rham complex \(\Omega_A\) is canonically the subcomplex of \(\Omega^\Lambda_{\Lambda E}\).

3.3. Recall that

\[
g^{ip} \tau_p (g^{jq}) = g^{jp} \tau_p (g^{iq})
\]

\[
g^{ip} \tau_q (g^{jq}) = g^{jq} \tau_p (g^{ip})
\]

and

\[
\tau_p (g^{-1qr}) = \tau_q (g^{-1pr})
\]

see [GII], 5.4.
It is easy to see that
\[
tr\{\tau_p(A)\tau_q(A^{-1})\} = tr\{\tau_q(A)\tau_p(A^{-1})\} \quad (3.3.4)
\]

Using (3.3.1) and (3.3.4) one sees easily that
\[
g^{ip}\tau_p(g^{jνν}) = g^{jq}\tau_q(g^{iνν}) \quad (3.3.5)
\]

3.4. Let \( A = A_{ΛE}\) be the vertex superalgebroid corresponding to the frame \( g \).
We have the following identities in \( A \):

γ-formulas
\[
γ(a, bτ_i) = -τ_i(ab) - τ_i(b)a \quad (3.4.1)
\]
\[
γ(a, bφ_β, bτ_i) = δ_β a∂b \quad (3.4.2)
\]
\[
γ(aφ_β, bψ_α) = δ_β a∂b \quad (3.4.3)
\]

⟨,⟩-formulas
\[
⟨aτ_i, bτ_j⟩ = -bτ_iτ_j(a) - aτ_jτ_i(b) - τ_i(b)τ_j(a) \quad (3.4.4)
\]
\[
⟨aφ_αψ_β, bτ_i⟩ = δ_α bτ_i(a) \quad (3.4.5)
\]
\[
⟨aφ_αψ_β, φ_α′ψ_β′⟩ = abδ_α δ_β′ \quad (3.4.6)
\]

c-formulas
\[
c(aτ_i, bτ_j) = \frac{1}{2}\{τ_i(b)τ_j(a) - τ_j(a)τ_i(b)\} + \frac{1}{2}\{bτ_iτ_j(a) - aτ_jτ_i(b)\} \quad (3.4.7)
\]
\[
c(aφ_αψ_μ, bφ_βψ_ν) = \frac{δ_μ δ_ν}{2}\{a∂b - b∂a\} \quad (3.4.8)
\]
\[
c(aφ_αψ_μ, bτ_i) = -\frac{δ_ν a}{2}\{bτ_i(a)\} \quad (3.4.9)
\]
\[
c(aτ_i, bψ_α) = c(aφ_αψ_μ, bψ_ν) = 0 \quad (3.4.10)
\]

3.5. Let \( g' \) be another frame as in 3.2. We have
\[
γ(a, τ_p') = γ(a, g^{pq}τ_q + g^{pμν}φ_νψ_μ) =
\]
\[
=-τ_q(a)g^{pq} - τ_q(g^{pq})∂a + g^{pμν}∂a \quad (3.5.1)
\]
\[
γ(aφ_μ', ψ_α') = γ(aA^{μβ}φ_β, A^{-1αν}ψ_ν) = aA^{μβ}∂A^{-1βα} \quad (3.5.2)
\]

Next,
\[
⟨τ_i', τ_j⟩ = ⟨g^{ip}τ_p + g^{jμν}φ_αψ_μ, g^{jq}τ_q + g^{jνβ}φ_βψ_ν⟩ =
\]
\[ = -2g^p\tau_q(g^{jq}) - \tau_p(g^{jq})\tau_q(g^{jp}) + \\
+2g^p\tau_p(g^{jq}) + g^p g^q \tau_p(A^{-1\mu\beta}) A^{\beta\gamma} \tau_q(A^{-1\gamma\sigma}) A^{\sigma\nu} \] 

(3.5.3)

and

\[ \langle \tau_i', \psi_\alpha' \rangle = \langle \psi_\alpha', \psi_\beta' \rangle = 0 \] 

(3.5.4)

Finally,

\[ c(\tau_i', \tau_j') = \frac{1}{2} \left\{ \tau_p(g^{jq})\partial \tau_q(g^{jp}) - \tau_q(g^{jp})\partial \tau_p(g^{jq}) \right\} + \frac{1}{2} \left\{ g^{i\nu\sigma} \partial g^{j\nu\mu} - g^{i\nu\mu} \partial g^{j\nu\sigma} \right\} \] 

(3.5.5)

and

\[ c(\tau_i', \psi_\alpha') = c(\psi_\alpha', \psi_\beta') = 0 \] 

(3.5.6)

\[ §4. \text{Chern-Simons term} \]

This Section is parallel to [GII], Section 5.

4.1. We keep the setup of the previous section. Let \( \mathcal{A} = \mathcal{A}_{\Lambda E; g}, \mathcal{A}' = \mathcal{A}_{\Lambda E; g'} \) (resp. \( \mathcal{B} = \mathcal{B}_{\Lambda E; g}, \mathcal{B}' = \mathcal{B}_{\Lambda E; g'} \)) be the vertex superalgebroids (resp. prealgebroids) corresponding to our frames.

As in [GII], 5.5 we have a canonical isomorphism

\[ g = g_{\Phi; g'} = (Id_{\Lambda E}; Id_{T_{\Lambda E}}, Id_{\Omega_{\Lambda E}}, h) : \mathcal{B}' \xrightarrow{\sim} \mathcal{B} \] 

(4.1.1)

where

\[ h = h_{\Phi; g'} : T_{\Lambda E} \longrightarrow \Omega_{\Lambda E} \] 

(4.1.2)

is defined by the condition

\[ \langle x', h(y') \rangle = -\frac{1}{2} \langle x', y' \rangle, \quad x', y' \in \{ \tau_i' \} \cup \{ \psi_\alpha' \} \] 

(4.1.3)

Using (3.5.3) and (3.5.4) we find the following explicit formulas for \( h \):

\[ h(\tau' \iota) = h_{ij} \omega_j; \quad h(\psi' \alpha) = 0 \] 

(4.1.4)

where \( h_{ij} = h_{ij}^\Omega - h_{ij}^E \),

\[ h_{ij}^\Omega = \tau_p \tau_j(g^{ip}) + \frac{1}{2} \tau_q(g^{jp}) \tau_p(g^{rq}) g^{-1jr} \] 

(4.1.5)

cf. [GII], (5.7.2), and

\[ h_{ij}^E = \tau_j(g^{i\nu\sigma}) + \frac{1}{2} g^{i\nu\sigma} \tau_j(A^{-1\mu\beta}) A^{\beta\gamma} \tau_q(A^{-1\gamma\nu}) A^{\sigma\mu} \] 

(4.1.6)

The meaning of the notation \( h_{ij}^\Omega \) will become clear below, see §6.
4.2. We have
\[ A = g \cdot A' + b \] (4.2.1)
where the closed 3-form \( b \in \Omega^{3,cl}_{\Lambda_E} \) is defined by
\[ b(x', y') = c(x', y') - x'(h(y')) + (-1)^{p(x')p(y')}y'(h(x')), \] (4.2.2)
\( x', y' \in \{ \omega' \} \cup \{ \psi' \} \), cf. [GII], (5.7.3).

It is easy to see that \( \psi'_a(h(\tau'_j)) = 0 \); on the other hand we know already that \( h(\psi'_a) = 0 \) and \( c(\psi'_a, y') = 0 \). It follows that \( b \in \Omega^{3,cl} \subset \Omega^{3,cl}_{\Lambda_E} \).
Next, we have
\[ \tau'_i(h(\tau'_j)) = (g^{ip} \tau_p + g^{i\alpha \gamma} \phi_{\gamma} \psi_{\alpha})(h^j q \omega_q) = \]
(note that the second summand is zero)
\[ = (g^{ip} \tau_p)(h^j_{11} \omega_q) + (g^{ip} \tau_p)(h^j_{12} \omega_q) = g^{ip} \tau_p(h^j_{11}) \omega_q + h^j_{11} \partial g^{ip} - \]
\[ g^{ip} \tau_p(h^j_{12}) \omega_q - h^j_{12} \partial g^{ip} \] (4.2.3)
It follows that \( b = b_{\Omega} - b_E \) where \( b_{\Omega}, b_E \in \Omega^3 \) are given by
\[ b_{\Omega}(\tau'_i, \tau'_j) = \frac{1}{2} \{ \tau_p(g^{ip}) \partial \tau_q(g^{jq}) - \tau_q(g^{ip}) \partial \tau_p(g^{jq}) \} - \]
\[ -g^{ip} \tau_p(h^j_{11}) \omega_q - h^j_{11} \partial g^{ip} + g^{ip} \tau_p(h^j_{12}) \omega_q + h^j_{12} \partial g^{ip} \] (4.2.4)
and
\[ b_E(\tau'_i, \tau'_j) = -\frac{1}{2} \{ g^{i\mu \nu} \partial g^{j\mu \nu} - g^{i\mu \nu} \partial g^{j\mu \nu} \} - \]
\[ -g^{ip} \tau_p(h^j_{12}) \omega_q + g^{ip} \tau_p(h^j_{12}) \omega_q - h^j_{12} \partial g^{ip} + h^j_{12} \partial g^{ip} \] (4.2.5)
The form \( b_{\Omega} \) has already been computed in [GII], Magic Lemma 5.6 and Theorem 6.4 (b), and is equal to
\[ b_{\Omega}(\tau'_i, \tau'_j) = -\frac{1}{2} tr \{ g^{-1} \tau'_i(g) g^{-1} \tau'_j(g) g^{-1} \tau'_j(g) - g^{-1} \tau'_j(g) g^{-1} \tau'_j(g) g^{-1} \tau'_i(g) \} \omega_r' \] (4.2.6)
cf. loc. cit. (5.5.3) and (6.4.2). Note that \( b_{\Omega} \) is closed, hence \( b_E \) is closed.

4.3. Magic Lemma. We have
\[ b_E(\tau'_i, \tau'_j) = -\frac{1}{2} tr \{ A^{-1} \tau'_i(A) A^{-1} \tau'_j(A) A^{-1} \tau'_j(A) - A^{-1} \tau'_j(A) A^{-1} \tau'_i(A) A^{-1} \tau'_i(A) \} \omega_r' \] (4.3.1)

Proof. Let us denote the six terms in (4.2.5) by \( A, A', B, B', C \) and \( C' \). We have
\[ A = -\frac{1}{2} g^{iq} \tau_q(A^{-1} \mu \alpha) A^{i\nu} \tau_r \{ g^{ip} \tau_p(A^{-1} \nu \beta) A^{\beta \mu} \} \omega_r = \]
We have the corresponding new bases of $T$ where

$$\Lambda E,T = 10$$

which implies the Lemma.

Next,

$$B = -\frac{1}{2}g^{lp}\{g^q\tau_r(A^{-1\beta})A^\beta\tau_q(A^{-1\gamma})A^{\gamma\mu}\omega_r - g^{lp}\tau_r(g^{\mu\nu})\omega_r =
\frac{1}{2}g^{lq}g^{pq}\tau_r(A^{-1\beta})A^{\beta\gamma}\tau_q(A^{-1\gamma})A^{\gamma\mu}\omega_r - \frac{1}{2}g^{lq}g^{pq}\tau_r(A^{-1\beta})A^{\beta\gamma}\tau_q(A^{-1\gamma})A^{\gamma\mu}\omega_r -
-\frac{1}{2}g^{lq}g^{pq}\tau_r(A^{-1\beta})A^{\beta\gamma}\tau_q(A^{-1\gamma})A^{\gamma\mu}\omega_r - g^{lp}\tau_r(g^{\mu\nu})\omega_r$$

Finally,

$$C = -\frac{1}{2}g^{lq}g^{pq}\tau_r(A^{-1\beta})A^{\beta\gamma}\tau_q(A^{-1\gamma})A^{\gamma\mu}\tau_r(g^{\mu\nu})\omega_r - \tau_r(g^{lp})\tau_r(g^{\mu\nu})\omega_r$$

We see that $A1 = -C1', A2 = -B'2, B1 = -B'1$ (by (3.4.6)), $B2 = -A'2, B4 = -B'4$ and $C1 = -A'1$. Next,

$$B6 + C2 = -\tau_r\{g^{lp}\tau_r(g^{\mu\nu})\},$$

so $B6 + C2 = -B'6 - C'^2$ by (3.3.5).

So we are left with three terms: $A3, B3$ and $B5$ and their primed partners. It is easy to see that

$$A3 = B3 = -B5 = -\frac{1}{2}t_p\{A^{-1}\tau_i(A)A^{-1}\tau_j(A)A^{-1}\tau_r(A)\}\omega_r,$$

which implies the Lemma. \triangle

§5. Atiyah term

This section is parallel to [GII], Section 6.

5.1. We keep the setup of the previous section. Let us denote by $T_{\Lambda E} = (\Lambda E, T_{\Lambda E}, \Omega_{\Lambda E}, \theta)$ the extended vertex superalgebroid Lie corresponding to our data.

Let $g'' = \{\bar{\tau}^{i''}, \phi'^{\alpha''}\}$ be a third frame of $(A, E)$, with $\bar{\tau}^{i''} = g'^{ij}\bar{\tau}^{j}, \phi'^{\alpha''} = A^{\alpha\beta}\phi'^{\beta}$. We have the corresponding new bases of $T_{\Lambda E}$ and $\Omega_{\Lambda E}$ given by

$$\tau^{i''} = g'^{lp}\tau^{j''} + g'^{i\alpha}\phi'^{\beta}\psi^{\alpha}\hspace{1cm} (5.1.1)$$

where

$$g'^{i\alpha\gamma} = g'^{iq}\tau^{j''}(A'^{-1\alpha\mu})A'^{\mu\gamma} \hspace{1cm} (5.1.2)$$
\[ \psi''_\alpha = A'^{-1\mu} \psi'_{\mu} \]  \hspace{1cm} (5.1.3) \\
\[ \omega''_i = g'^{-1\mu_i} \omega'_{\mu} \]  \hspace{1cm} (5.1.4) \\
\[ \rho''_i = \tau(A'^{\alpha}) \phi'_i \omega'_i + A'^{\alpha} \rho'_{\mu} \]  \hspace{1cm} (5.1.5)

5.2. Let \( \mathcal{A}' = \mathcal{A}_{A, \mathcal{A}, \mathcal{B}} \) (resp. \( \mathcal{B}' = \mathcal{B}_{A, \mathcal{A}, \mathcal{B}} \)) be the vertex superalgebroid (resp. prealgebroid) corresponding to the third frame. We have canonical isomorphisms

\[ \mathcal{B}' \xrightarrow{\sim} \mathcal{B}' \xrightarrow{\sim} \mathcal{B} \]

as well as the morphism \( g_{\mathcal{B}, \mathcal{B}'}, \mathcal{B}' \xrightarrow{\sim} \mathcal{B} \) over \( \mathcal{F}_A \), given by functions \( h_{\mathcal{B}, \mathcal{B}'}, h_{\mathcal{B}, \mathcal{B}'}, h_{\mathcal{B}, \mathcal{B}''} \), and we are aiming to compute the discrepancy

\[ a = a_{\mathcal{B}, \mathcal{B}''} = h_{\mathcal{B}, \mathcal{B}'} + h_{\mathcal{B}, \mathcal{B}''} - h_{\mathcal{B}, \mathcal{B}''} \in \Omega^2_{A, \mathcal{A}} \]  \hspace{1cm} (5.2.1)

Note that

\[ \gamma(a, b, \psi_{\mu}) = -\psi_{\mu}(a) \partial b - \psi_{\mu}(b) \partial a = 0 \]  \hspace{1cm} (5.2.2)

(in \( \mathcal{A} \)), hence

\[ h_{\mathcal{B}, \mathcal{B}'}(a \psi'_{\mu}) = ah_{\mathcal{B}, \mathcal{B}'}(\psi'_{\mu}) - \gamma(a, \psi'_{\mu}) = 0 \]  \hspace{1cm} (5.2.3)

therefore

\[ h_{\mathcal{B}, \mathcal{B}'}(\psi''_{\alpha}) = h_{\mathcal{B}, \mathcal{B}'}(A'^{-1\mu} \psi'_{\mu}) = 0 \]  \hspace{1cm} (5.2.4)

It follows that

\[ a(\psi''_{\alpha}) = 0 \]  \hspace{1cm} (5.2.5)

5.3. Let us denote for brevity \( h := h_{\mathcal{B}, \mathcal{B}'}, h' = h_{\mathcal{B}, \mathcal{B}'}, h'' = h_{\mathcal{B}, \mathcal{B}''} \).

We have

\[ h(a \tau''_{p}) = ah(\tau''_{p}) - \gamma(a, \tau''_{p}) = \]
\[ = ah^{pr} \omega_{r} + \tau_{q}(a) \partial g^{pq} + \tau_{q}(g^{pq}) \partial a - g^{pr} A^{-1\alpha} \partial A^{-1\alpha} \]  \hspace{1cm} (5.3.1)

and

\[ h(a \phi'_{i} \psi'_{\alpha}) = -a A^{\gamma} \partial A^{\alpha} \]  \hspace{1cm} (5.3.2)

Thus we get

\[ h(\tau''_{i}) = h(g^{ip} \tau''_{p} + g^{i\alpha} \phi'_{i} \psi'_{\alpha}) = \]
\[ = g^{ip} h^{pr} \omega_{r} - g^{ip} h^{pr} \omega_{r} + \tau_{q}(g^{ip}) \partial g^{pq} + \tau_{q}(g^{pq}) \partial g^{ip} - g^{ip} A^{\gamma} \partial A^{\gamma} \]  \hspace{1cm} (5.3.3)

It follows that

\[ a = a_{\Omega} - a_{E} \]  \hspace{1cm} (5.3.4)

where

\[ a_{\Omega}(\tau''_{i}) = g^{ip} h^{pr} \omega_{r} + \tau_{q}(g^{ip}) \partial g^{pq} + \tau_{q}(g^{pq}) \partial g^{ip} + h_{\Omega}^{ip} \omega_{p} - h_{\Omega}^{ip} \omega_{r} \]  \hspace{1cm} (5.3.5)

and

\[ a_{E}(\tau''_{i}) = g^{ip} h^{pr} \omega_{r} + g^{ip} \partial g^{ip} + g^{i\alpha} A^{\gamma} \partial A^{\alpha} + h_{E}^{ip} \omega_{p} - h_{E}^{ip} \omega_{r} \]  \hspace{1cm} (5.3.6)
The form $a_\Omega$ has already been computed in [GII], Section 6. Namely, by Theorem 6.4 (a) from op. cit.,

$$a_\Omega(\tau_i'') = \frac{1}{2} \text{tr} \left\{ g'^{-1} \tau_i'' (g')^2 \tau_s (g) g^{-1} - g'^{-1} \tau_s'' (g')^2 \tau_i (g) g^{-1} \right\} \omega_s'' \quad (5.3.7)$$

5.4. Let us compute $a_E(\tau_i'')$. Let us denote the five terms in the rhs of (5.3.6) by $A, B, \mathcal{C}, \mathcal{D}$ and $\mathcal{E}$. Thus,

$$A = g'^{ip} g^{pq} \tau_r (A^{-1} \tau_{q}) A \tau_p (A^{-1} \tau_{q}) A \mu \omega_r + g'^{ip} \tau_r (g^\mu) \omega_r \quad (5.4.1)$$

$$B = g'^{ip} \tau_q (A^{-1} \mu) A \tau_p (g^\mu) \omega_r \quad (5.4.2)$$

$$C = g'^{ip} \tau_q \omega_r + g'^{ip} \tau_r (g^\mu) \omega_r \quad (5.4.3)$$

$$D = h^{ij} \omega_j = \frac{1}{2} g'^{ip} \tau_r (A^{-1} \mu) A \tau_p (A^{-1}) A \tau_p (A^{-1}) g^{-1} \omega_r + \tau_r (g^\mu) g^{-1} \omega_r = \frac{1}{2} g'^{ip} \tau_r (A^{-1} \mu) A \tau_p (A^{-1}) A \tau_p (A^{-1}) \omega_r + \tau_r (g^\mu) g^{-1} \omega_r \quad (5.4.4)$$

and

$$E = - h^{ir} \omega_r = - \frac{1}{2} (g')^q \tau_r ((A')^{-1} \tau_{q}) (A')^{-1} \tau_p (A')^{-1} \tau_{q} (A')^{-1} \tau_p \omega_r = - \tau_r (g^\mu) \quad (5.4.5)$$

We see that $2A = - E_1, C = - 2E_2, D = - E_4$. It is easy to see that $2A + D + E_1 = - B$.

Finally,

$$C + E_2 = \frac{1}{2} \text{tr} \left\{ A^{-1} \tau_i'' (A')^{-1} \tau_i'' (A) \right\} \omega_i'' \quad (5.4.6)$$

and

$$E_3 = - \frac{1}{2} \text{tr} \left\{ A^{-1} \tau_i'' (A')^{-1} \tau_i'' (A) \right\} \omega_i'' \quad (5.4.7)$$

So, we have proven

5.5. Lemma. The form $a_E$ is given by

$$a_E(\tau_i'') = \frac{1}{2} \text{tr} \left\{ A^{-1} \tau_i'' (A')^{-1} \tau_i'' (A) A^{-1} - A^{-1} \tau_i'' (A')^{-1} \tau_i'' (A) A^{-1} \right\} \omega_i'' \quad (5.5.1)$$
Combining 4.3 and 5.5 we get

**5.6. Theorem.** (a) The cocycle \( a_{g,g'} \) is given by

\[
a_{g,g'}(\tau''_i) = \frac{1}{2} tr \{ g'^{-1} \tau''_i(g') \tau''_i(g) g^{-1} - g'^{-1} \tau''_i(g') \tau''_i(g) g^{-1} \} \omega'' - \\
\frac{1}{2} tr \{ A'^{-1} \tau''_i(A') \tau''_i(A) A^{-1} - A'^{-1} \tau''_i(A') \tau''_i(A) A^{-1} \} \omega''
\]

(b) The 3-form \( b_{g,g'} \) is given by

\[
b_{g,g'}(\tau'_i, \tau'_j) = -\frac{1}{2} tr \{ g^{-1} \tau'_i(g) g^{-1} \tau'_j(g) g^{-1} \tau'_i(g) - g^{-1} \tau'_j(g) g^{-1} \tau'_i(g) g^{-1} \tau'_j(g) \} \omega'_r + \\
+\frac{1}{2} tr \{ A'^{-1} \tau'_i(A) A'^{-1} \tau'_j(A) A^{-1} \tau'_i(A) - A'^{-1} \tau'_j(A) A'^{-1} \tau'_i(A) A^{-1} \tau'_j(A) \} \omega'_r
\]

\[ (a_{E^*}, b_{E^*}) = (a_E, b_E) \]

Indeed, this follows from the easy identities

\[
tr \{ A'^{-1} \tau_i((A')^{-1}) A'^{-1} \tau_j((A')^{-1}) \} = -tr \{ A^{-1} \tau_r(A) A^{-1} \tau_j(A) A^{-1} \tau_i(A) \}
\]

and

\[
tr \{ A'^{-1} \tau_i((A')^{-1}) B^{-1} \} = tr \{ A^{-1} \tau_i(A) \tau_j(B) B^{-1} \}
\]

**5.7. Lemma.** Let \( E^* = \text{Hom}_A(E, A) \) be the dual module. We have

\[ (a_{E^*}, b_{E^*}) = (a_E, b_E) \]

Let us pass to the global situation. Let \( X \) be a smooth variety over \( k \) and \( E \) be a vector bundle over \( X \). As in [GII], we define the gerbe \( \mathcal{O}_{AE} \) of chiral differential operators on \( \Omega E \) over \( X \).

Its characteristic class \( c(\mathcal{O}_{AE}) \) will belong to the second hypercohomology \( H^2(X; \Omega^{(2,3)}_{\Omega E}) \) (in obvious notations). Recall that we have a canonical embedding of de Rham complexes

\[ \Omega_X \rightarrow \Omega_{AE} \]

In [GII], 7.6 we have defined the "Atiyah-Chern-Simons" characteristic class \( c(E) \in H^2(X; \Omega^{(2,3)}_X) \); let us denote by \( c(E)_{AE} \) its image in \( H^2(X; \Omega^{(2,3)}_{\Omega E}) \).

The theorem below is an immediate consequence Theorem 5.6 and Lemma 5.7.

**5.9. Theorem.** The class \( c(\mathcal{O}_{AE}) \) is equal to

\[ c(\mathcal{O}_{AE}) = c(\Theta_X) - c(E) = c(\Omega^1_X) - c(E) \]

where \( \Theta_X \) is the tangent bundle.
§6. Chiral de Rham complex

6.1. Let us return to the local situation 3.1, 4.1. Let $E$ be equal to the module of vector fields $T$. Given a base $\{\tilde{\tau}_i\}$ consisting of commuting vector fields, we get a frame $g = \{\tilde{\tau}_i; \phi_i := \tilde{\tau}_i\}$ of $(A,E)$. Let us call such frames *natural*.

Let $g, g'$ be two natural frames, with transition matrices as in 3.2. By definition, $(A^\ast) = (g^\ast)$. Therefore the coefficients $g^{i\alpha\gamma}$ (3.2.2) are given by

$$g^{i\alpha\gamma} = g^{i\nu}, \tau_q (g^{-1}g^{\nu\mu})g^{\mu\gamma} = -\tau_\alpha (g^{i\nu})g^{-1}g^{\mu\gamma}$$

(6.1.1)

where we have used (3.3.3). Consequently the function $h$ (4.1.6) is given by

$$h_{ij}^E = -\tau_\mu (g^{i\nu}) + \frac{1}{2} g^{i\nu}, \tau_j (g^{-1}g^{\nu\mu})g^{\mu\nu}$$

(6.1.2)

The second summand is equal to

$$-\frac{1}{2} g^{i\nu}, \tau_j (g^{-1}g^{\mu\nu})g^{\mu\gamma} = \frac{1}{2} g^{i\nu}, \tau_j (g^{-1}g^{\mu\nu})g^{\mu\nu}$$

(6.1.3)

We see that the first summand in (6.1.2) is equal to minus the first summand of (4.1.5), and second summand of (6.1.2) is equal to the second summand of (4.1.5). Thus

$$h_{ij}^E = 2\tau_\Gamma (g^{ij})$$

(6.1.4)

If $g, g', g''$ are natural frames of $(A,T)$ then Theorem 5.6 says that $a_{g,g',g''} = b_{g,g'} = 0$. This means that

the chiral superalgebrid $A_{\Lambda T; g}$ does not depend, up to a canonical isomorphism, on the choice of the base $\{\tilde{\tau}_i\}$. In other words, we have a canonically defined chiral algebroid $A_{\Lambda T}$.

Passing to chiral envelopes, we get a canonically defined chiral (vertex) superalgebra $D_{\Lambda T}^h$ of chiral differential operators on $T$.

It follows that

for each smooth variety $X$ we have a canonically defined sheaf of chiral superalgebras $D_{\Lambda \Theta X}^h$. These Zariski sheaves form in fact a sheaf in the étale topology.

The gluing functions for this sheaf are given explicitly by (6.1.4).

6.2. Lemma. In the situation 4.1, consider the function $h_E$

$$h_{ij}^E = \tau_\nu (g^{i\nu}) + \frac{1}{2} g^{i\nu}, \tau_j (A^{-1}g^{\mu\nu})A^{\beta\gamma}, \tau_q (A^{-1}g^{\gamma\nu})A^{\mu\nu}$$

(6.2.1)
The function \( h_{\mathcal{E}^*} \) associated with the dual module \( \mathcal{E}^* \) is given by

\[
h_{ij}^E = -\tau_j (g^{\nu\nu}) + \frac{1}{2} g^{ij} \tau_j (A^{-1})_{\mu\beta} A^{\beta\gamma} \tau_q (A^{-1})_{\gamma\nu} A^{\nu\mu}
\]  
(6.2.2)

This follows from the identities

\[
tr \{ \tau_i (A^i) A^{-1} \} = -tr \{ \tau_i (A^{-1}) A \}
\]  
(6.2.3)

and

\[
tr \{ \tau_i (A^i) A^{-1} \tau_j (A^j) A^{-1} \} = tr \{ \tau_i (A^{-1}) A \tau_j (A^{-1}) A \}
\]  
(6.2.4)

(cf. (5.7.2), (5.7.3)).

6.3. Let \( E \) be the module of 1-forms \( \Omega = \Omega^1_{A/k} \); its exterior algebra is the de Rham algebra of differential forms \( \Omega \). Frames of the form \( g = \{ \bar{\tau}_i, \phi_i := \omega_i \} \) will be called natural.

If \( g, g' \) are natural frames then formulas (6.1.2) and (6.1.3), together with the previous lemma, show that \( h_{ij}^E = h_{ij}^\Omega \) where \( h_{ij}^\Omega \) is given by (4.1.5). (Of course one easily checks this directly.) This explains the notation for \( h_{ij}^\Omega \).

In other words, we arrive at an interesting conclusion.

6.4. Theorem. The matrices \( h = (h_{ij}^E) \) defined in 4.1 are equal to 0 if \( E = \Omega \) and frames \( g, g' \) are natural.

6.4.1. Warning. The functions \( h_{g, g'} \) are nonzero since they are not linear.

6.5. On the other hand, Theorem 5.6 together Lemma 5.7 say that \( a_E \) and \( b_E \) are 0 for \( E = \Omega \) (for natural frames \( g, g', g'' \) of \( (A, \Omega) \)). This gives us a canonically defined chiral superalgebroid \( A^\Omega \). Its vertex envelope will be denoted \( D^\Omega_{\Omega} \) and called the chiral algebra of differential operators on \( \Omega \).

This implies

6.6. Theorem. For each smooth variety \( X \) the construction 6.2 - 6.5 gives a canonically defined sheaf of chiral superalgebras \( D^\Omega_{\Omega_X} \). These Zariski sheaves form a sheaf in the étale topology.

6.7. The de Rham differential may be considered as an odd first order differential operator acting on \( \Omega_X \) (and commuting with itself).

In coordinates, if \( g = \{ \bar{\tau}_i; \phi_i := \omega_i \} \) is a natural frame of \( (A, \Omega) \), it is given by

\[
Q_{g}^{\Omega} \phi_i = \phi_i \bar{\tau}_i
\]  
(6.7.1)

(\( \Omega \) is for "classical"). Let us check the independence of (6.7.1) on the choice of a frame. Let \( g' = \{ \bar{\tau}'_i; \omega'_i \} \) be another natural frame, \( \bar{\tau}'_i = g^{ij} \bar{\tau}_j; \omega'_i = g^{-1} \omega_i \) (cf. (3.2.4)).

Using (3.2.1) we have

\[
Q_{g'}^{\Omega} \phi_i = \phi_i \bar{\tau}'_i = g^{-1} \phi_i \{ g^{ij} \bar{\tau}_j + g^{ij} \phi_j \bar{\psi}_j \}
\]  
(6.7.2)
where
\[ g^{pq} = g^r \tau_r (g^p g)^{-1} q s = g^r \tau_r (g^ip g)^{-1} = \tau_q (g^ip) \]  \hspace{1cm} (6.7.3)

(we have used (3.3.1)). So,
\[ Q^c_{g'} = \phi_p \tau_p + g^{-1} q i g^q r s \phi_s \psi_r = Q^c_g + g^{-1} q i r s (g^q r) \phi_p \phi_s \psi_r \] \hspace{1cm} (6.7.4)

Note that
\[ g^{-1} q i \tau_s (g^q r) = -\tau_s (g^{-1}) q i \]
which is symmetric under the permutation of \( p \) with \( s \), due to (3.3.3); therefore the second summand in (6.7.4) is zero, i.e. \( Q^c_g = Q^{cl} g' \).

Thus, \( Q^{cl} \) is a correctly defined odd element of \( D_{\Omega} \). It is obvious from (6.7.1) that \([Q^{cl},Q^{cl}] = 0\).

6.8. Let us investigate the chiral counterpart of \( Q^{cl} \). Let us define an odd element \( Q_g \) (of conformal weight 1) of the vertex superalgebra \( D_{\Omega}^{ch} \) by
\[ Q_g = \phi_i(-1) \tau_i \] \hspace{1cm} (6.8.1)

Let \( g' \) be another natural frame as in 6.7. Due to Theorem 6.4, the element \( Q_{g'} \) goes under the canonical isomorphism \( D_{\Omega}^{ch} \to D_{\Omega}^{ch} \) to
\[ Q_{g'} = \phi'_i(-1) \tau'_i = \phi'_i \tau'_i - \gamma(\phi'_i, \tau'_i) \] \hspace{1cm} (6.8.2)

cf. [GII], (3.3.1).

6.9. Lemma. We have (in \( D_{\Omega}^{ch} \))
\[ \gamma(\phi'_i, \tau'_i) = -\partial \{ \text{tr}(\tau_r(g^{-1}) \phi_r) \} \] \hspace{1cm} (6.9.1)

6.10. Before the proof, let us write down useful formulas
\[ \gamma(a \phi_r, b \tau_i) = -\tau_i(a) \phi_r \partial b - \tau_i(b) \partial(a \phi_r) \] \hspace{1cm} (6.10.1)

and
\[ \gamma(a \phi_r, b \phi_s \psi_p) = -\delta_{rp} a \partial(b \phi_s) + \delta_{sp} b \partial(a \phi_r) \] \hspace{1cm} (6.10.2)

6.11. Proof of 6.9. We have
\[ \gamma(\phi'_i, \tau'_i) = \gamma(g^{-1} q i \phi_q, g^q r \tau_p + g^{i q r} \phi_r \psi_s) \]
where
\[ \gamma(g^{-1} q i \phi_q, g^q r \tau_p) = -\tau_p (g^{-1} q i) \phi_q \partial g^q r - \tau_p (g^q r) \partial(g^{-1} q i) \phi_q \] \hspace{1cm} (6.11.1)

and
\[ \gamma(g^{-1} q i \phi_q, g^{i q r} \phi_r \psi_s) = -g^{-1} q i \partial(g^{i q r} \phi_r) + g^{i q r} \partial(g^{-1} q i) \phi_q \] \hspace{1cm} (6.11.2)
Since $g^{ir} = \tau_r(g^{ir})$, the second summands in (6.11.1) and (6.11.2) cancel out. On the other hand, the first term in (6.11.1) is equal to

$$-\tau_q(g^{-1ri})\tau_s(g^{iq})\omega_s\phi_r = -\tau_r(g^{-1si})\tau_s(g^{iq})\omega_s\phi_r =$$

$$= g^{-1pq}\tau_s(g^{ab}) g^{-1ba} \tau_s(g^{ia}) \omega_s \phi_r = -\tau_r(g^{-1ba}) \tau_s(g^{ia}) \omega_s \phi_r = -\tau_r(g^{-1ba}) \partial(g^{-1ba}) \phi_r$$

Therefore

$$\gamma(\phi_i', \tau_i') = -\tau_r(g^{ab}) \partial(g^{-1ba}) \phi_r - g^{-1ba} \partial \{\tau_r(g^{ab}) \phi_r\} = -\partial \{\tau_r(g^{ab}) g^{-1ba} \phi_r\}.$$ QED.

6.12. From (6.8.1) we have $Q_\theta = \phi_i \tau_i$, and from 6.7 $\phi'_i \tau'_i = \phi_i \tau_i$. Therefore, (6.8.2) and Lemma 6.9 imply

6.13. Theorem. We have

$$Q_{g'} = Q_\theta + \partial \{\text{tr}(\tau_r(g) g^{-1}) \phi_r\} \quad (6.13.1)$$

6.14. Consider the field $Q_\theta(z)$ acting on the vertex algebra $D^{ch}_{11}$. Due to (6.13.1), its zeroth component $Q_{\theta 0}$ does not depend on the choice of the frame $g$. Therefore we get a canonical operator $Q_0$ acting on $D^{ch}_{11}$.

Since it is a zeroth component of a field, it is a derivation of the vertex algebra, and it is obvious from the local definition (6.8.1) that $[Q_0, Q_0] = 0$.

Consequently, for each smooth variety $X$ we get a canonical odd derivation $Q_{0X}$ of the sheaf $D^{ch}_{11X}$, such that $[Q_{0X}, Q_{0X}] = 0$. The pair $(D^{ch}_{11X}, Q_{0X})$ is the chiral de Rham complex from [MSV].

Our Theorem 6.13 is a version of op. cit., (4.1c).

6.15. In the situation 6.4, consider an even element $J_\theta$ of conformal weight 1 of the algebra $D^{ch}_{11\theta}$, given by

$$J_\theta = \phi_i(-1) \psi_i = \phi_i \psi_i \quad (6.15.1)$$

After a change of frame as in loc. cit., we get an element

$$J_{g'} = \phi'_i(-1) \psi'_i = \phi'_i \psi'_i - \gamma(\phi'_i, \psi'_i) \quad (6.15.2)$$

where we have again used Theorem 6.4. We have

$$\phi'_i \psi'_i = g^{-1pi} \phi_p g^{iq} \psi_q = \phi_p \psi_p = J_\theta$$

(see (3.2.3)). On the other hand, by (3.4.3)

$$\gamma(\phi'_i, \psi'_i) = \gamma(g^{-1pi} \phi_p, g^{iq} \psi_q) = \delta_{pq} g^{-1pi} \partial g^{iq} = g^{-1pi} \partial g^{ip} = \text{tr}(g^{-1} \partial g)$$

Thus

$$J_{g'} = J_\theta - \text{tr}(g^{-1} \partial g) \quad (6.15.3)$$
6.16. Consider an odd element $G_g$ of conformal weight 2 given by
\[ G_g = \psi_{i(-1)} \omega_i \] (6.16.1)

In the frame $g'$,
\[ G_{g'} = (g^{iq} \psi_q)(-1)(g^{-1s} \omega_s) \]

Note that $\psi_{q(j)a} = 0$ for $j \geq 0$ (everything happens in $D$), hence it follows from the commutativity formula (1.3.1) that
\[ a\psi_q = (a(-1)\psi_q = \psi_{q(-1)}a \] (6.16.2)

Therefore by "associativity" (1.2.5)
\[ (a\psi_q)(-1)\omega_s = \psi_{q(-1)}a(-1)\omega_s = \psi_{q(-1)}(ab\omega_s) \] (6.16.3)

Therefore
\[ G_{g'} = \psi_{q(-1)}g^{iq}g^{-1s} \omega_s = \psi_{s(-1)} \omega_s = G_g \] (6.16.6)

6.17. Let us investigate the Virasoro element. Define an even element $L_g$ of conformal weight 2 by
\[ L_g = L(b)_g + L(f)_g \] (6.17.1)

where
\[ L(b)_g = \omega_i(-1)\tau_i \] (6.17.2)

$((b)$ is for "bosonic") and
\[ L(f)_g = \rho_i(-1)\psi_i \] (6.17.3)

$((f)$ is for "fermionic"), cf. [MSV], (2.3a).

6.18. We have
\[ (a\omega_s)(-1)(b\tau_p) = \omega_s(-1)\{ab\tau_p + \tau_p(a)\partial b + \tau_p(b)\partial a\} - \partial \omega_s(-1)b\tau_p(a) \] (6.18.1)

Indeed, it follows from "associativity" (1.2.5) that
\[ (a\omega_s)(-1)(b\tau_p) = (\omega_s(-1)a)(-1)(b\tau_p) = \omega_s(-1)(a(-1)b\tau_p) + \omega_s(-2)a(0)(b\tau_p) + \\
+ a(-2)\omega_s(0)(b\tau_p) \]

Next,
\[ a(-1)(b\tau_p) = ab\tau_p - \gamma(a, b\tau_p) = ab\tau_p + \tau_p(a)\partial b + \tau_p(b)\partial a \]

(see [GII] (3.3.1));
\[ a(0)(b\tau_p) = -b\tau_p(0)a = -b\tau_p(a) \]

(see [GII] (3.3.2)). Finally
\[ \omega_s(0)(b\tau_p) = -(b\tau_p)(0)\omega_s + \partial(b\tau_p, \omega_s) \]
where
\[(br_p)_0(\omega_s) = (br_p)(\omega_s) = \langle \tau_p, \omega_s \rangle db = \delta_{p\mu}db\]
by (1.1.3), since
\[\tau_p(\omega_s) = 0, \quad (6.18.2)\]
and \[\langle br_p, \omega_s \rangle = b\delta_{p\mu} \] This implies
\[\omega_s(0)(br_p) = 0 \quad (6.18.3)\]
Formula (6.18.1) follows.

6.19. We have
\[(a\omega_s)(-1)(b\phi_\alpha\psi_\beta) = \omega_s(-1)\{ab\phi_\alpha\psi_\beta - \delta_{\alpha\beta}b\partial a\} \quad (6.19.1)\]
Indeed, by commutativity and "associativity" (1.2.5)
\[(a\omega_s)(-1)(b\phi_\alpha\psi_\beta) = (\omega_s(-1)a)(-1)(b\phi_\alpha\psi_\beta) = \omega_s(-1)a(-1)(b\phi_\alpha\psi_\beta) \]
On the other hand
\[a(-1)(b\phi_\alpha\psi_\beta) = ab\phi_\alpha\psi_\beta - \gamma(a, b\phi_\alpha\psi_\beta) = ab\phi_\alpha\psi_\beta - \delta_{\alpha\beta}b\partial a, \]
see (3.4.2). This implies (6.19.1).

6.20. We have
\[L_{(b)g'} = L_{(b)g} + \omega_s(-1)\{\tau_p(g^{-1si})\partial g^{ip} + g^{-1si}\tau_\beta(g^{i\alpha})\phi_\beta\psi_\alpha\} - \omega_s(-1)\partial g^{ip}\tau_p(g^{-1si}) \quad (6.20.1)\]
Indeed, due to Theorem 6.4
\[L_{(b)g'} = (g^{-1si}\omega_s)(-1)\{g^{ip}\tau_p + g^{i\alpha\beta}\phi_\beta\psi_\alpha\} \]
According to (6.18.1)
\[\langle g^{-1si}\omega_s \rangle(0)(g^{ip}\tau_p) = \omega_s(-1)\{g^{-1si}\tau^p p + \tau_p(g^{-1si})\partial g^{ip} + \tau_p(g^{ip})\partial g^{-1si}\} - \omega_s(-2)g^{ip}\tau_p(g^{-1si}) \]
By (6.18.2)
\[\langle g^{-1si}\omega_s \rangle(0)(g^{i\alpha\beta}\phi_\beta\psi_\alpha) = \omega_s(-1)\{\tau_\beta(g^{i\alpha})\phi_\beta\psi_\alpha\} - \omega_s(-1)\{g^{-1si}\tau_\beta(g^{i\alpha})\phi_\beta\psi_\alpha - \delta_{\alpha\beta}\tau_\beta(g^{i\alpha})\partial g^{-1si}\} \]
The third term in the first expression cancels out the second term in the second one, and we get (6.20.1).

6.21. We have
\[(a\rho_\mu)(-1)(b\psi_\nu) = \rho_{\mu(-1)}ab\psi_\nu \quad (6.21.1)\]
Indeed,
\[(a\rho_{\mu})_{(-1)}(b\psi_{\nu}) = (\rho_{\mu(-1)a})_{(-1)}(b\psi_{\nu}) = \rho_{\mu(-1)}ab\psi_{\nu} + a_{(-2)}\rho_{\mu(0)}b\psi_{\nu}\]
and by commutativity
\[\rho_{\mu(0)}b\psi_{\nu} = (b\psi_{\nu})(\rho_{\mu}) - \partial\{(b\psi_{\nu})(\mu)\} = \delta_{\nu\mu}\partial b - \delta_{\nu\mu}\partial b = 0,\]
\[\text{cf. (6.18.3). This implies (6.21.1).} \]

\[6.22.\text{ We have}\]
\[(a\phi_\gamma \omega_\iota)(-1)(b\psi_{\nu}) = \omega_{\iota(-1)}\{ab\phi_\gamma \psi_{\nu} - \delta_{\gamma\nu}a\partial b\} + \delta_{\nu\gamma}\omega_{\iota(-2)}ab \quad (6.22.1)\]

Indeed,
\[(a\phi_\gamma \omega_\iota)(-1)(b\psi_{\nu}) = (\omega_{\iota(-1)}a\phi_\gamma)(-1)(b\psi_{\nu}) = \omega_{\iota(-1)}(a\phi_\gamma)(-1)b\psi_{\nu} + \omega_{\iota(-2)}(a\phi_\gamma)(0)b\psi_{\nu}\]
where
\[(a\phi_\gamma)(-1)b\psi_{\nu} = a\phi_\gamma b\psi_{\nu} - \gamma(a\phi_\gamma, b\psi_{\nu}) = ab\phi_\gamma \psi_{\nu} - \delta_{\gamma\nu}a\partial b,\]
\[\text{cf. (3.4.3), and}\]
\[(a\phi_\gamma)(0)b\psi_{\nu} = (b\psi_{\nu})(a\phi_\gamma) = ab\delta_{\nu\gamma}\]

\[6.23.\text{ We have}\]
\[L(f)g = L(f)g + \omega_{\iota(-1)}\{\tau_\iota(g^{-1}\gamma)g^{\alpha\nu} \phi_\gamma \psi_{\nu} - \tau_\iota(g^{-1}\gamma)\partial g^{\alpha\gamma}\} + \omega_{\iota(-2)}\tau_\iota(g^{-1}\gamma)g^{\alpha\gamma} \quad (6.23.1)\]

Indeed,
\[L(f)g = \rho'_{\alpha(-1)}\psi_{\alpha} = \{g^{-1}\mu_\alpha \rho_\mu + \tau_\iota(g^{-1}\gamma)\phi_\gamma \omega_\iota\}_{(-1)}(g^{\alpha\nu} \psi_{\nu})\]
By (6.21.1)
\[(g^{-1}\mu_\alpha \rho_\mu)(-1)(g^{\alpha\nu} \psi_{\nu}) = \rho_{\mu(-1)}g^{-1}\mu_\alpha g^{\alpha\nu} \psi_{\nu} = L(f)g\]
and by (6.22.1)
\[(\tau_\iota(g^{-1}\gamma)\phi_\gamma \omega_\iota)(-1)(g^{\alpha\nu} \psi_{\nu}) = \omega_{\iota(-1)}\{\tau_\iota(g^{-1}\gamma)g^{\alpha\nu} \phi_\gamma \psi_{\nu} - \tau_\iota(g^{-1}\gamma)\partial g^{\alpha\gamma}\} + \omega_{\iota(-2)}\tau_\iota(g^{-1}\gamma)g^{\alpha\gamma}\]

\[6.24.\text{ Comparing (6.20.1) and (6.23.1) we see easily that}\]
\[L_{g'} = L_{(b)g'} + L_{(f)g'} = L_{(b)g} + L_{(f)g} = L_{g} \quad (6.24.1)\]

Let us collect our computations of transformation rules.

\[6.25. \text{Theorem. Let } g, g' \text{ be two natural frames of } (A, \Omega). \text{ Consider } 4 \text{ elements of the vertex superalgebra } D_{\Omega \cdot \Phi}^{ch}, \text{ given by}\]
\[Q_{g} = \phi_{i(-1)}\tau_{i} \quad (6.25.1)\]
(an odd element of conformal weight 1)

\[ J_g = \phi_{i(-1)}^i \psi_i \]  
(6.25.2)

(an even element of conformal weight 1)

\[ G_g = \psi_{i(-1)}^i \omega_i \]  
(6.25.3)

(an odd element of conformal weight 2) and

\[ L_g = \omega_{i(-1)}^i \tau_i + \rho_{i(-1)}^i \psi_i \]  
(6.25.4)

(an even element of conformal weight 2).

After the canonical identification \( D_{\Omega \cdot \Phi}^h = D_{\Omega \cdot \Phi}^h \) these elements are transformed as follows

\[ Q_{g'} = Q_g + \partial \{ \text{tr} (g^{-1} \tau_r(g)) \phi_r \} \]  
(6.25.5)

\[ J_{g'} = J_g - \text{tr} (g^{-1} \partial g) \]  
(6.25.6)

\[ G_{g'} = G_g \]  
(6.25.7)

and

\[ L_{g'} = L_g \]  
(6.25.8)

This is a version of [MSV], Theorem 4.2.

\section{Poincaré-Birkhoff-Witt}

\subsection{Let X be a smooth variety and \( D_{\Omega \cdot \Phi}^h \) be the sheaf discussed in the previous section, cf. Theorem 6.6. It is a sheaf of \( \mathbb{Z}_{\geq 0} \)-graded vertex algebras, so

\[ D_{\Omega \cdot \Phi}^h = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} D_{\Omega \cdot \Phi}^{h,n} \]  
(7.1.1)

where \( D_{\Omega \cdot \Phi}^{h,n} \) denotes the component of conformal weight \( n \).

According to Theorem 6.25 (see (6.25.8)) we have a canonical global section \( L \) of \( D_{\Omega \cdot \Phi}^{h,2} \). Let \( L(z) = \sum L_n z^{-n-2} \) be the corresponding field.

\subsection{Claim. A local section \( \alpha \in D_{\Omega \cdot \Phi}^h \) belongs to \( D_{\Omega \cdot \Phi}^{h,n} \) if and only if \( L_0(\alpha) = n\alpha \).

In a uniform notation \( L(z) = \sum L_n z^{-n-1} \) we have \( L_0 = L_{(1)} \). We shall check 7.1.1 simultaneously with

\subsection{Claim. The operator \( L_{-1} = L_{(0)} \) coincides with the canonical derivation \( \partial \) of the vertex algebra \( D_{\Omega \cdot \Phi}^h \).}
Both statements are local, so we may assume we are in the local situation 6.17. Note that the operator $L_{(0)}$ is a derivation with respect to the operation $(-1)$:

$$L_{(0)}(x_{(-1)}y) = (L_{(0)}x)_{(-1)}y + x_{(-1)}L_{(0)}y$$  (7.1.2)

Therefore it suffices to check 7.1.2 on the generators $a, \tau, \omega, \psi, \rho$ of the vertex algebroid $A_{\Omega, \partial}$, which is done by a simple explicit computation.

It follows from the associativity formula (1.2.4) that

$$L_{(1)}y_{(-1)}z = (L_{(1)}y)_{(-1)}z + y_{(-1)}L_{(1)}z + L_{(0)}y(0)z - y(0)L_{(0)}z =$$

$$= (L_{(1)}y)_{(-1)}z + y_{(-1)}L_{(1)}z$$  (7.1.3)

since

$$L_{(0)}y(0)z - y(0)L_{(0)}z = \partial(y(0)z) - y(0)\partial z = 0$$

In other words, $L_{(1)}$ is a derivation of $(-1)$. Therefore it suffices to check 7.1.1 on the generators of $A_{\Omega, \partial}$ as above, which is straightforward.

**7.2.** In the local situation 6.1, consider the local algebra $D_{\Omega, \partial}^{ch} = U\mathcal{A}_{\Omega, \partial}$. Let us introduce a $\mathbb{Z}$-grading

$$D_{\Omega, \partial}^{ch} = \oplus_{p \in \mathbb{Z}} D_{\Omega, \partial}^{ch,p}$$  (7.2.1)

to be called fermionic charge. For an element $x \in D_{\Omega, \partial}^{ch}$ let us denote its fermionic charge (to be defined) by $F(x) \in \mathbb{Z}$. It is defined uniquely by the following conditions:

(a) $F(a) = F(\tau_i) = F(\omega_i) = 0$; $F(\phi_i) = F(\rho_i) = -F(\psi_i) = 1$;

(b) $F(x_{(-1)}y) = F(x) + F(y)$.

Due to the transformation formulas (3.2.1) - (3.2.5) this grading is obviously preserved under a change of frames. Therefore in the situation of 7.1 the sheaf $D_{\Omega, \partial}^{ch}$ gets a canonical $\mathbb{Z}$-grading

$$D_{\Omega, \partial}^{ch} = \oplus_{p \in \mathbb{Z}} D_{\Omega, \partial}^{ch,p}$$  (7.2.2)

Note that parity is equal to fermionic charge modulo 2.

Here is another way to define the grading (7.2.2). First notice a simple

**7.2.1. Lemma.** Let $\mathcal{A} = (A, T, \Omega, \partial, \ldots)$ be a vertex (super)algebroid. For every invertible element $a \in A$ the operator $(a^{-1}\partial a)_{(0)}$ acting on $U\mathcal{A}$ is trivial.

**Proof.** Obviously this operator is trivial on $A = U\mathcal{A}_0$. Let $x \in U\mathcal{A}_1$ and $\tau \in T$ be its image under the canonical projection $U\mathcal{A}_1 \to T$. We have

$$(a^{-1}\partial a)_{(0)}x = -x(0)a^{-1}\partial a + \partial(x(1)a^{-1}\partial a) = -\tau(a^{-1}\partial a) + \partial(\tau, a^{-1}\partial a) =$$

$$= a^{-2}\tau(a)\partial a - a^{-1}\partial \tau(a) + \partial(a^{-1}\tau(a)) = 0,$$

so $(a^{-1}\partial a)_{(0)}$ is trivial on $U\mathcal{A}_1$. Therefore it is trivial on the whole algebra $U\mathcal{A}$ since $(?)_{(0)}$ is a derivation of the operation $(-1)$. $\triangle$
Applying this lemma to \( a = \det(g) \) in the formula (6.25.6) we see that the component \( J_{0;g} \) of the field \( J_g(z) = \sum J_{n;g}z^{-n-1} \) is preserved under the change of frames. Consequently it gives rise to a well defined endomorphism \( J_0 \) of the sheaf \( \mathcal{D}^{ch}_{\Omega_X} \).

7.2.2. Claim. A local section \( \alpha \in \mathcal{D}^{ch}_{\Omega_X} \) belongs to \( \mathcal{D}^{ch,p}_{\Omega_X} \) if and only if \( J_0(\alpha) = p\alpha \).

Indeed, the function \( F(\alpha) \) defined by \( J_0(\alpha) = F(\alpha)\alpha \) obviously satisfies the conditions (a) and (b) above.

7.3. The two gradings (7.1.1) and (7.2.2) are compatible: if we denote

\[
\mathcal{D}^{ch,p}_{\Omega_X;n} := \mathcal{D}^{ch}_{\Omega_X;n} \cap \mathcal{D}^{ch,p}_{\Omega_X}
\]

then

\[
\mathcal{D}^{ch}_{\Omega_X} = \oplus_{n \in \mathbb{Z} \geq 0; \ p \in \mathbb{Z}} \mathcal{D}^{ch,p}_{\Omega_X;n} \tag{7.3.2}
\]

For a fixed \( n \), only a finite number of sheaves \( \mathcal{D}^{ch,p}_{\Omega_X;n} \) are nonzero.

If the ground ring \( k \) is a field of characteristic 0 then the sheaves \( \mathcal{D}^{ch,p}_{\Omega_X;n} \) and \( \mathcal{D}^{ch,n-p}_{\Omega_X} \) are in a certain sense dual to each other, see [MS].

7.4. Starting from this point we assume that \( k \supset \mathbb{Q} \). According to a (superver- sion of) the PBW theorem, [GII], Theorem 9.18, the sheaf \( \mathcal{D}^{ch}_{\Omega_X} \) admits a canonical filtration such that the associated graded sheaf is canonically isomorphic to

\[
gr(\mathcal{D}^{ch}_{\Omega_X}) = \text{Sym}_{\Omega_X} \left\{ \left( \oplus_{n \geq 1} \Theta_{\Omega_X}(n) \right) \oplus \left( \oplus_{n \geq 1} \Omega_{\Omega_X}(n) \right) \right\} \tag{7.4.1}
\]

Here \( \Theta_{\Omega_X} \) (resp. \( \Omega_{\Omega_X} \)) denotes the tangent (resp. the cotangent) bundle of the supervariety \( (X, \Omega_X) \), and \( (?)_{(n)} \) means that this bundle is put into the conformal weight \( n \).

The endomorphisms \( L_0 \) and \( J_0 \) respect the canonical filtration; hence we get a canonical finite filtration \( F \) on each homogeneous component \( \mathcal{D}^{ch,p}_{\Omega_X;n} \). The graded quotients \( F_i \mathcal{D}^{ch,p}_{\Omega_X;n} / F_{i+1} \mathcal{D}^{ch,p}_{\Omega_X;n} \) are locally free \( \mathcal{O}_X \)-modules of finite rank (we shall see this in the course of computations below). This allows us to introduce the elements of the Grothendieck group \( K(X) \) of vector bundles

\[
[D^{ch,p}_{\Omega_X;n}] := \sum_i \left[ F_i \mathcal{D}^{ch,p}_{\Omega_X;n} / F_{i+1} \mathcal{D}^{ch,p}_{\Omega_X;n} \right] \in K(X) \tag{7.4.2}
\]

Here \([E]\) in the right hand side denotes the class of a vector bundle \( E \) in \( K(X) \). Consider the generating function

\[
cl(\mathcal{D}^{ch}_{\Omega_X})(y,q) := \sum_{p,n} [D^{ch,p}_{\Omega_X;n}] y^p q^n \in K(X)[y,y^{-1}][[q]] \tag{7.4.3}
\]

7.5. For a vector bundle \( E \) over \( X \) and an indeterminate \( x \) we introduce the notations

\[
[S_x E] = \sum_{i=0}^{\infty} [\text{Sym}^i_{\mathcal{O}_X} E] \in K(X)[[x]] \tag{7.5.1}
\]
and
\[ [\Lambda_x E] = \sum_{i=0}^{\infty} [\Lambda^i_{\Omega} x] \in K(X)[x] \] (7.5.2)

The following fact was noticed in [BL] (cf. also [W]).

7.6. Theorem (L. Borisov - A. Libgober) We have
\[
cl(D_{\psi}(y, q) = [\Lambda_y y \Omega_x] \cdot \prod_{n=1}^{\infty} \left\{ [S^{n\psi} \Theta_x] \cdot [S^{nq} \Omega^1_{\psi}] \cdot [\Lambda_{y-1} q^n \Theta_x] \cdot [\Lambda_{y+1} q^n \Omega^1_{\psi}] \right\} \] (7.6.1)

7.7. Proof. Let us understand the bundles \( \Theta_{\Omega_x} \) and \( \Omega^1_{\Omega_x} \) a little bit more attentively.

Let us consider the local situation 3.1, with \( E = \Omega \), so that \( \Lambda_x E = \Omega \). All our frames \( g \) will be natural. Let \( T_{\psi} \subset T_{\Omega} \) be the \( \Lambda \)-submodule with the base \( \{ \psi_i \} \). The coordinate change formula (3.2.3) shows that it is a well defined \( \Lambda \)-submodule of \( T \) independent on the choice of a frame, canonically isomorphic to \( T \).

We set
\[ T_{\psi} = \Omega \otimes_A T_{\psi} \subset T_{\Omega} \]
We denote by \( T_{\tau} \) the quotient \( \Omega \)-module \( T_{\Omega} / T_{\psi} \). Let \( T_{\tau} \subset T_{\Omega} \) be the \( \Lambda \)-submodule generated by all \( \tau_i \). The formula (3.2.1) shows that \( T_{\tau} \) is a well defined \( \Lambda \)-module canonically isomorphic to \( T \), and we have
\[ T_{\tau} \otimes_A \Omega \]

Returning to our variety \( X \), we see that we get two vector bundles \( \Theta_{\psi} \) and \( \Theta_{\tau} \) both isomorphic to \( \Theta_{\Omega_x} \) and a canonical short exact sequence
\[
0 \rightarrow \Theta_{\psi} \rightarrow \Theta_{\Omega_x} \rightarrow \Theta_{\tau} \rightarrow 0 \] (7.7.1)

with
\[ \Theta_{\psi} = \Omega \otimes \Theta_{\psi}; \quad \Theta_{\tau} = \Omega \otimes \Theta_{\tau} \] (7.7.2)

Note that \( \Theta_{\psi} \) has fermionic charge \(-1\) and \( \Theta_{\tau} \) has fermionic charge \(0\).

Dually, we have two vector bundles \( \Omega_{\psi}^1 \) and \( \Omega_{\tau}^1 \) both isomorphic to \( \Omega_{\tau}^1 \) and a canonical short exact sequence
\[
0 \rightarrow \Omega^1_{\psi} \rightarrow \Omega^1_{\Omega_x} \rightarrow \Omega^1_{\tau} \rightarrow 0 \] (7.7.3)

with
\[ \Omega^1_{\psi} = \Omega \otimes \Theta_{\psi}; \quad \Omega^1_{\tau} = \Omega \otimes \Theta_{\tau} \] (7.7.4)

The bundles \( \Omega_{\psi}^1 \), \( \Omega_{\tau}^1 \) have fermionic charges 1, 0 respectively.

Note that if \( E \) is a vector bundle then
\[ Sym_{\Omega_x} (\Omega \otimes E) = \Omega \otimes Sym_{\Omega_x} E \] (7.7.5)
Returning to PBW formula (7.4.1) we see that these remarks imply (7.6.1). \( \triangle \)

**7.8.** Starting from this point let us assume that \( k = \mathbb{C} \). Consider the formal power series

\[
\theta(y,q) = i^{-1}(y^{1/2} - y^{-1/2})q^{1/8} \prod_{n=1}^{\infty} \left\{ (1 - q^n)(1 - yq^n)(1 - y^{-1}q^n) \right\} \quad (7.8.1)
\]

It is nothing but the theta function \( \theta_1(h,z) \) as defined in [HC], II, 2, §10, formula (3), p. 204, with \( q = h^2 \) and \( y = z^2 \).

If \( f \in GL(V) \) is an automorphism of a \( d \)-dimensional vector space \( V \) with eigenvalues \( \lambda_1, \ldots, \lambda_d \), we shall denote by \( \theta_f(y,q) \) the power series

\[
\theta_f(y,q) = \frac{\prod_{i=1}^{d} \theta(\lambda_i y, q)}{\prod_{i=1}^{d} \theta(\lambda_i, q)} \quad (7.8.2)
\]

Let \( X \) be a proper smooth \( d \)-dimensional algebraic variety; let \( g : X \to X \) be a simple automorphism, which means by definition that the graph \( \Gamma_g \subset X \times X \) is transversal to the diagonal. This implies that the set \( X^0 \) if its fixed points is finite.

For each \( x \in X^0 \) denote by \( g_x \) the induced endomorphism of the cotangent space \( \Omega^1_{X,x} \). All eigenvalues of \( g_x \) are distinct from 1.

**7.9. Theorem.** Consider the power series

\[
T_{X,g}(y,q) := y^{-d/2} \sum_{a,b,n} (-1)^{a+b} \text{Tr}(g; H^a(X; \mathcal{D}^{ch;b}_{\Omega^1_{X,n}})) y^a q^n \quad (7.9.1)
\]

We have

\[
T_{X,g}(y,q) = \sum_{x \in X^0} \theta_{g_x}(y,q) \quad (7.9.2)
\]

**7.10. Proof.** Recall that according to the Atiyah-Bott holomorphic Lefschetz fixed point formula, if \( E \) is a \( g \)-equivariant vector bundle over \( X \) then

\[
\sum_{x} (-1)^i \text{Tr}(g; H^i(X; E)) = \sum_{x \in X^0} \frac{\text{Tr}(g; E_x)}{\text{det}(1 - g_x)} \quad (7.10.1)
\]

see [AB], Theorem 4.12. Note that if \( (V,f) \) are as in 7.8 then

\[
\text{Tr}(g; \text{Sym}_x(V)) = \prod_{i=1}^{d} (1 - \lambda_i x)^{-1} = \text{Tr}(g; \Lambda_x(V))^{-1} \quad (7.10.2)
\]

The proof of 7.6 shows that each sheaf \( \mathcal{D}^{ch;b}_{\Omega^1_{X,n}} \) carries a canonical filtration whose quotients are vector bundles and the associated graded sheaf is given by the formula (7.6.1). Therefore we may apply the Lefschetz formula (7.10.1).

Note that since in the expression (7.9.1) the fermionic charge \( a + b \) is taken into account, we should apply the Lefschetz formula to the element \( \text{cl}(\mathcal{D}^{ch}_{\Omega^1_{X}})(-y, q) \). Due to (7.10.2) each fixed point \( x \) gives a contribution

\[
y^{-d/2} \prod_{i=1}^{d} \left\{ \frac{1}{1 - \lambda_i} \prod_{n=1}^{\infty} \frac{(1 - \lambda_i y q^n)(1 - \lambda_i^{-1} y^{-1} q^n)}{(1 - \lambda_i q^n)(1 - \lambda_i^{-1} q^n)} \right\} = \theta_{g_x}(y,q)
\]
where $\lambda_i$ are the eigenvalues of $g_x$. This implies the theorem. $\triangle$.

The reader may wish to compare (7.9.2) with the explicit formulas for the trace of certain automorphisms of the Frenkel-Lepowsky-Meurman Monster vertex algebra, cf. [FLM].

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