Solutions to the complex Korteweg-de Vries equation: Blow-up solutions and non-singular solutions

Ying-ying Sun\textsuperscript{1}, Juan-ming Yuan\textsuperscript{2}; Da-jun Zhang\textsuperscript{1†}

\textsuperscript{1}Department of Mathematics, Shanghai University, Shanghai 200444, P.R. China
\textsuperscript{2}Department of Financial and Computational Mathematics, Providence University, Shalu, Taichung 433, Taiwan

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Abstract

In the paper two kinds of solutions are derived for the complex Korteweg-de Vries equation, including blow-up solutions and non-singular solutions. We derive blow-up solutions from known 1-soliton solution and a double-pole solution. There is a complex Miura transformation between the complex Korteweg-de Vries equation and a modified Korteweg-de Vries equation. Using the transformation, solitons, breathers and rational solutions to the complex Korteweg-de Vries equation are obtained from those of the modified Korteweg-de Vries equation. Dynamics of the obtained solutions are illustrated.

Keywords: blow-up, non-singular solutions, complex Korteweg-de Vries equation

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1 Introduction

The famous Korteweg-de Vries (KdV) equation,

\[ u_t + 6uu_x + u_{xxx} = 0, \] (1.1)

is referred to as the complex KdV (cKdV) equation if \( u = u(x,t) \) is a complex-valued function. This equation provides an example admitting wave blow-up in finite time \cite{1}, and also serves as a model describing irrotational flows in a shallow water channel \cite{2}. Blow-up of solutions of (complex) PDEs has been largely investigated numerically \cite{3,4,5,6}, while exact solutions with blow-up are interesting as well. In \cite{7}, blow-up from 2-soliton solutions of the cKdV equation was considered by Bona and Weissler under the assumption of the space variable \( x \) being complex. In \cite{8} Li made use of Darboux Transformation to get exact blow-up solutions to the cKdV equation (1.1). In his treatment \( x \) is kept real while the wave number \( k \) is complex. In fact, (1.1) can be viewed as a complex-valued integrable system, and then it naturally admits \( N \)-complex soliton solutions. We will show that solutions obtained in \cite{8} can also be derived from 1-complex soliton.

\footnote*{E-mail: jmyuan@pu.edu.tw}

\footnote{Corresponding author. E-mail: djzhang@staff.shu.edu.cn}
Furthermore, the same technique will be applied to another solution of the cKdV equation and derive possible blow-up points.

Besides, we will also derive exact solutions without singularity for the cKdV equation, via Miura transformation, from solutions of the modified KdV (mKdV) equation. In fact, there are two mKdV equations,

\[ \text{mKdV}^+ : \quad v_t + 6v^2 v_x + v_{xxx} = 0, \quad (1.2) \]

and

\[ \text{mKdV}^- : \quad v_t - 6v^2 v_x + v_{xxx} = 0. \quad (1.3) \]

These two equations exhibit different aspects \[9, 10, 11\]. On the real-value level, (1.1) and (1.3) are related by the well-known Miura transformation (MT) \[12\]

\[ u = -v^2 \pm v_x \]  

which provides an approach to the solutions of the KdV equation \[1.1\] from those of the mKdV- equation \[1.3\]. It is hard to reverse the MT to get solutions of the mKdV- equation (cf.\[11\]). In a recent paper \[13\], we gave a complete investigation on exact solutions of the mKdV+ equation \[1.2\] in terms of Wronskians. We have got solitons, breathers as well as non-singular rational solutions (by means of a Galilean transformation) for \[1.2\]. Note that the MT between \[1.1\] and \[1.2\] is of complex-valued form (cf.\[14\])

\[ u = v^2 \pm iv_x. \]  

This means for the real-valued solution \( v(t, x) \) of the mKdV+ equation \[1.2\], the MT provides a complex-valued solution \( u(t, x) \) to the cKdV equation \[1.1\]. This fact enables us to list non-singular solutions to the cKdV equation \[1.1\], including solitons, breathers and rational solutions.

The paper is arranged as follows. In Section 2, we find out blow-up solutions from a complex soliton solution and a double-pole solution. In Section 3, we derive non-singular solutions to the cKdV equation. Dynamics of some obtained solutions are also illustrated.

## 2 Blow-up solutions

### 2.1 Some exact solutions to the cKdV equation

First, we note that no matter \( u \) is real or complex in \[1.1\], it is invariant under the Galilean transformation

\[ u(x, t) \rightarrow \alpha + u(x + 6\alpha t, t), \quad \alpha \in \mathbb{C}, \]  

which may provides non-zero asymptotics. Besides, if we set

\[ u = u_1 + iu_2, \quad u_1 = \text{Re}[u], \quad u_2 = \text{Im}[u], \]

then the cKdV equation \[1.1\] is split into

\[ u_{1,t} + 6u_1u_{1,x} - 6u_2u_{2,x} + u_{1,xxx} = 0, \quad (2.2a) \]

\[ u_{2,t} + 6u_1u_{2,x} + 6u_2u_{1,x} + u_{2,xxx} = 0. \quad (2.2b) \]

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If we consider equation (1.1) as a real-valued equation, then it is a typical integrable system and it has variety of solutions. A well-known result is that through the transformation
\[ u = \frac{2(fff_{xx} - f_{xx}^2)}{f^2} \]  
(2.3)
equation (1.1) is written into the bilinear form
\[ (D_t D_x + D_x^4)f \cdot f = 0, \]  
(2.4)where the \( D \) is the Hirota’s bilinear operator defined as \[ D_{m,t}D_{n,x}a(t,x) \cdot b(t,x) = (\partial_{t} - \partial_{t'})^m(\partial_{x} - \partial_{x'})^n a(t,x)b(t',x')|_{t'=t,x'=x}. \]  
(2.5)\( N \)-soliton solution is expressed through (2.3) and
\[ f = \sum_{\mu=0,1} \exp\left(\sum_{j=1}^{N} \mu_j \xi_j + \sum_{1 \leq j < l}^{N} \mu_j \mu_l A_{jl}\right), \]  
(2.6a)where
\[ \xi_j = k_j x - k_j^3 t + h_j, \quad e^{A_{jl}} = \left(\frac{k_j - k_l}{k_j + k_l}\right)^2, \quad j, l = 1, 2, \ldots, N, \]  
(2.6b)\( k_j, h_j \) are constants, and the summation of \( \mu \) takes over all possible combinations of \( \mu_j (j = 1, 2, \ldots, N) \). Particularly, for 1-soliton (\( N = 1 \)) and 2-soliton (\( N = 2 \)) solutions, \( f \) can be written respectively in the following
\[ f_1(x,t) = 1 + e^{\xi_1}, \]  
(2.7)\[ f_2(x,t) = 1 + e^{\xi_1} + e^{\xi_2} + \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 e^{\xi_1 + \xi_2}. \]  
(2.8)Then 1-soliton solution is given through the transformation (2.3) as
\[ u = \frac{2ff_{1,xx} - f_{1,xx}^2}{f_1^2} = \frac{2k_1^2 e^{\xi_1}}{(1 + e^{\xi_1})^2} \]  
(2.9)if \( k_1, h_1 \in \mathbb{R} \). If taking \( k_j \in \mathbb{C} \) in (2.6), one will have complex solutions to the cKdV equation (1.1).

2.2 Blow-up solution from 1-soliton
The idea of finding blow-up points is as the following. We consider \( k_1 \) and \( h_1 \) in \( \xi_1 \) are complex and for convenience we write
\[ k_1 = k_{11} + ik_{12}, \quad h_1 = h_{11} + ih_{12}, \quad k_{11}, k_{12}, h_{11}, h_{12} \in \mathbb{R}. \]  
In this case, \( f_1 \) is rewritten as
\[ f_1 = 1 + e^{\eta}(\cos \theta_1 + i \sin \theta_1) \]  
(2.10)
with
\[ \eta_1 = k_{11}x - k_{11}^3 t + 3k_{11}k_{12}^2 t + h_{11}, \quad \theta_1 = k_{12}x + k_{12}^3 t - 3k_{12}k_{11}^2 t + h_{12}. \] (2.11)

Singularity of \( u \) then appears when \( f_1 = 0 \), i.e.,
\[
\begin{cases}
1 + e^{\eta_1} \cos \theta_1 = 0, \\
e^{\eta_1} \sin \theta_1 = 0,
\end{cases}
\] (2.12)

which holds if and only if
\[
\begin{cases}
\eta_1 = 0, \\
\theta_1 = (2s + 1)\pi, \quad s \in \mathbb{Z}.
\end{cases}
\] (2.13)

In more details, this is a linear system of \((x, t)\):
\[
\begin{pmatrix}
k_{11} & k_{11}(3k_{12}^2 - k_{11}^2) \\
k_{12} & k_{12}(k_{11}^2 - 3k_{12}^2)
\end{pmatrix}
\begin{pmatrix}
x \\
t
\end{pmatrix}
= \begin{pmatrix}
-h_{11} \\
h_{12} + (2s + 1)\pi
\end{pmatrix}.
\] (2.14)

Blow-up points correspond to separated zeros of the above system, which requires the coefficient matrix is non-singular, i.e.,
\[
\begin{vmatrix}
k_{11} & k_{11}(3k_{12}^2 - k_{11}^2) \\
k_{12} & k_{12}(k_{11}^2 - 3k_{12}^2)
\end{vmatrix} \neq 0,
\]
say \( k_{11}k_{12}(k_{11}^2 + k_{12}^2) \neq 0 \), or,
\[ k_{11}k_{12} \neq 0. \] (2.15)

Separated blow-up points \{\((x, t)\)\} are then given by
\[
x = -\frac{h_{11}}{k_{11}} + \frac{(3k_{12}^2 - k_{11}^2)(k_{12}h_{11} - k_{11}h_{12} + k_{11}\pi + 2k_{11}s\pi)}{2k_{11}k_{12}(k_{11}^2 + k_{12}^2)},
\]
(2.16a)
\[
t = \frac{k_{11}h_{12} - k_{12}h_{11} - k_{11}(2s + 1)\pi}{2k_{11}k_{12}(k_{11}^2 + k_{12}^2)},
\] (2.16b)

with \( s \in \mathbb{Z} \). The coordinates of the above blow-up points are the same as derived in [8], up to the Galilean transformation (2.1). Obviously, the blow-up points are on a straight line related to given \( k \) and \( h \), written as
\[ h_{11} + k_{11}x - k_{11}(k_{11}^2 - 3k_{12}^2)t = 0. \] (2.17)

As an example to illustrate, we take \( k_{11} = \frac{1}{2}, \ k_{12} = -\frac{1}{2}, \ h_{11} = 0, \ h_{12} = \pi \), then the blow-up points are \((x, t) = (-2s\pi, 4s\pi)\) which are on the line \(2x + t = 0\). The corresponding blow-up solution is
\[
u = \frac{i e^{\frac{i}{2}(t+2x)}}{(e^{\frac{i}{2}(t+2x)} - e^{\frac{i}{2}t})^2} = u_1 + iu_2,
\] (2.18a)

where
\[
u_1 = \frac{(-1 + e^{\frac{i}{2}t+x})e^{\frac{i}{2}(t+2x)} \sin \frac{1}{2}(t - 2x)}{[1 + e^{\frac{i}{2}t+x} - 2e^{\frac{i}{2}(t+2x)} \cos \frac{1}{4}(t - 2x)]^2},
\] (2.18b)
\[
u_2 = \frac{e^{\frac{i}{4}(t+2x)}(-2e^{\frac{i}{4}(t+2x)} + (1 + e^{\frac{i}{4}t+x}) \cos \frac{1}{4}(t - 2x))}{[1 + e^{\frac{i}{2}t+x} - 2e^{\frac{i}{2}(t+2x)} \cos \frac{1}{4}(t - 2x)]^2},
\] (2.18c)

The equation is depicted in Fig[1].
2.3 Blow-up solution from double-pole solution

A double-pole solution to the KdV equation (1.1) is given by (2.3) with

\[ f(x,t) = 1 + (x - 3k_1^2t)e^{k_1 x - k_1^3 t} - \frac{1}{4k_1^2}e^{2k_1 x - 2k_1^3 t}. \] (2.19)

This corresponds to take the limit \( k_2 \to k_1 \) in (2.8) with redefined

\[ e^{h_1} = \frac{1}{k_1 - k_2}, \quad e^{h_2} = -\frac{1}{k_1 - k_2}. \]

To analyze possible isolated blow-up points, we consider the case of \( k_1 = ik_{12} \). Note that in this case \( f \) can be written as

\[ f = 1 - \frac{1}{4k_{12}^2} + (x + 3k_{12}^2t + \frac{1}{2k_{12}} \cos \theta) \cos \theta + i(x + 3k_{12}^2t + \frac{1}{2k_{12}} \cos \theta) \sin \theta, \] (2.20)

where

\[ \theta = k_{12}x + k_{12}^3 t. \]

Restricting \( f = 0 \) and from the imaginary part one has either

\[ \sin \theta = 0, \] (2.21a)

or

\[ 2k_{12}^2(x + 3k_{12}^2t) + \cos \theta = 0. \] (2.21b)

In the first case, (2.21a) together with the real part of \( f \) requires

\[
\begin{align*}
    k_{12}x + k_{12}^3 t &= s\pi, \\
    x + 3k_{12}^2 t &= (-1)^{s+1}(1 + \frac{1}{4k_{12}^2}),
\end{align*}
\] (2.22)

with solution

\[
\begin{align*}
    x &= \frac{(-1)^s + 4(-1)^s k_{12}^2 + 12k_{12}s\pi}{8k_{12}^2}, \\
    t &= -\frac{(-1)^s + 4(-1)^s k_{12}^2 + 4k_{12}s\pi}{8k_{12}^2},
\end{align*}
\] (2.23)
which provides isolated blow-up points \((x, t)\). It is worth noting that the points \((2.23)\) of \(k_{12}^2 = 1/4\) are not isolated, which belongs to the second case, i.e., \((2.21b)\).

The second case does not lead to any isolated blow-up points. In fact, in the light of \((2.21b)\), one has \(k_{12}^2 = 1/4\) and then \((2.21b)\) reduces to
\[
G(t, x) = \frac{1}{2} \left( x + \frac{3t}{4} \right) + \cos \left( \frac{x}{2} + \frac{t}{8} \right) = 0. \tag{2.24}
\]
Noting that
\[
G_t(t, x) = \frac{3}{8} - \frac{1}{8} \sin \left( \frac{x}{2} + \frac{t}{8} \right) \neq 0, \quad \forall x, t \in \mathbb{R},
\]
\((2.24)\) determines an implicit function \(t = t(x), \ x \in \mathbb{R},\) on which \(f = 0\). That means in the case of \((2.21b)\) there is no isolated blow-up point.

Let us sum up this subsection.

- When \(k_{11} = 0\) and \(k_{12}^2 = \frac{1}{4}\), the corresponding solution \(u\) is real, and there always exists a moving singular point on the curve \((2.24)\).

- When \(k_{11} = 0\) and \(k_{12}^2 \neq \frac{1}{4}\), there exist blow-up points \((x, t)\) coordinated by \((2.23)\) and located on two parallel straight lines
\[
\begin{align*}
12k_{12}^4t + 4k_{12}^2x + 4k_{12}^2 + 1 &= 0, \tag{2.25a} \\
12k_{12}^4t + 4k_{12}^2x - 4k_{12}^2 - 1 &= 0. \tag{2.25b}
\end{align*}
\]
Solution \(u\) of this case is given by \((2.3)\) with \(f\) defined in \((2.20)\).

For illustration we take \(k_1 = i\) and \(u = u_1 + iu_2\). In this case, we have
\[
\begin{align*}
u_1 &= \frac{B}{A^2}, \tag{2.26a} \\
u_2 &= \frac{C}{A^2}. \tag{2.26b}
\end{align*}
\]
where
\[
\begin{align*}
A &= 17 + 16(3t + x)^2 + 40(3t + x) \cos(t + x) + 8 \cos(2t + 2x), \\
B &= -8[(3t + x)(20 \cos(3t + 3x) + (5(85 + 16(3t + x)^2) \\
&- 256 \sin(t + x)) \cos(t + x) + 32(3t + x)(2 \cos(2t + 2x) - 5 \sin(t + x))] \\
&- 16[68 \cos(2t + 2x) + 25 \sin(t + x) + 4(8 + 33(3t + x)^2 - 5 \sin(3t + 3x))], \\
C &= 24[2(29 + 16(3t + x)^2) \cos(t + x) - 8 \cos(3t + 3x) \\
&+ (3t + x)[80 + ((12t + 4x)^2 - 81 - 8 \cos(2t + 2x)) \sin(t + x)] - 40 \sin(2t + 2x)].
\end{align*}
\]
The corresponding blow-up points are
\[
(x, t) = \left( \frac{(-1)^s5 + 12s\pi}{8}, -\frac{(-1)^s5 + 4s\pi}{8} \right), \quad s \in \mathbb{Z}. \tag{2.27}
\]
We plot \(u_1\) and \(u_2\) in Fig.2.
3 Non-singular solutions

By means of the MT (1.5), solutions of the cKdV equation (1.1) can be derived from those of the mKdV\(^+\) equation (1.2).

3.1 Exact solutions of the mKdV\(^+\) equation

Let us list out solutions of the mKdV\(^+\) equation (1.2) classified in [13]. We make use of Wronskians \(\hat{N}^{-1}\) and lower-triangular Toeplitz matrices set \(\tilde{G}_N\) for which one can refer to Appendices A and B.

3.1.1 Solitons and breathers

Soliton and breather solutions of the cKdV equation (1.1) can be described by

\[ v = 2 \left( \arctan \frac{F_2}{F_1} \right)_x = \frac{-2(F_{1,x}F_2 - F_1F_{2,x})}{F_2^2 + F_1^2}, \]

where

\[ f = f(\varphi) = |\hat{N} - 1| = F_1 + iF_2, \quad F_1 = \text{Re}[f], \quad F_2 = \text{Im}[f]. \]

Wronskian vectors corresponding to different kinds of solutions are the following.

- **For soliton solutions:**
  \[ \varphi = \varphi_N^{[s]} = (\varphi_1, \varphi_2, \cdots, \varphi_N)^T, \]
  with
  \[ \varphi_j = a_j^+ e^{\xi_j} + i a_j^- e^{-\xi_j}, \quad \xi_j = k_j x - 4k_j^3 t + \xi_j^{(0)}, \quad a_j^+, a_j^-, k_j, \xi_j^{(0)} \in \mathbb{R}. \]

- **For limit solutions of solitons:**
  \[ \varphi = \varphi_N^{[ls]}(k_1) = A^+ Q_0^+ + i A^- Q_0^-, \quad A^\pm \in \tilde{G}_N(\mathbb{R}), \]
with
\[ Q_0^\pm = (Q_0^+, Q_0^-) \]
\[ Q_{0,s}^\pm = \frac{1}{s!} \partial_{\xi_1} e^{\pm \xi_1}, \]
where \( \xi_1 \) is defined in (3.2b).

- **For breather solutions:**
  \[ \varphi = \varphi_{2N}^{[b]} = (\varphi_{11}, \varphi_{12}, \varphi_{21}, \varphi_{22}, \cdots, \varphi_{N1}, \varphi_{N2})^T, \]
  with
  \[ \varphi_{j1} = a_j e^{\xi_j} + b_j e^{-\xi_j}, \quad \varphi_{j2} = \bar{a}_j e^{\xi_j} - \bar{b}_j e^{-\xi_j}, \]
  \[ \xi_j = k_j x - 4k_j^3 t + \xi_j(0), \quad a_j, b_j, \xi_j(0) \in \mathbb{C}. \]

- **For limit solutions of breathers:**
  \[ \varphi = \varphi_{2N}^{[lb]}(k_1) = (\varphi_{11}^+, \varphi_{12}^-, \varphi_{21}^+, \varphi_{22}^-, \cdots, \varphi_{N1}^+, \varphi_{N2}^-)^T, \]
  and the elements are given through
  \[ \varphi^+ = (\varphi_{11}^+, \varphi_{21}^+, \cdots, \varphi_{N1}^+)^T = A Q_0^+, \]
  \[ \varphi^- = (\varphi_{12}^-, \varphi_{22}^-, \cdots, \varphi_{N2}^-)^T = \bar{A} Q_0^-, \]
  where \( A, B \in \tilde{G}_N(\mathbb{C}), \)
  \[ Q_0^\pm = (Q_{0,0}^\pm, Q_{0,1}^\pm, \cdots, Q_{0,N-1}^\pm)^T, \]
  and \( \xi_1 \) is defined in (3.4c).

- **Mixed solutions:**
  Mixed solutions can be obtained by arbitrarily combining the above vectors to be a new Wronskian vector. For example, take
  \[ \varphi = \left( \begin{array}{c} \varphi_{N1}^{[a]} \\ \varphi_{N2}^{[a]}(k_{N1+1}) \end{array} \right), \]
  then the related solution corresponds to the interaction between \( N_1 \)-soliton and a \((N_2 - 1)\)-order limit-soliton solutions.

### 3.1.2 Rational solutions

To get rational solutions, one has to make use of the Galilean transformation
\[ v(x, t) = v_0 + V(X, t), \quad X = x - 6v_0^2 t, \]
by which the mKdV equation (1.2) is transformed to a mixed equation
\[ V_t + 12v_0 V V_X + 6V^2 V_X + V_{XXX} = 0, \]

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where \( v_0 \) is a real parameter. (3.8) admits non-trivial rational solutions which can be transformed back to those for the the mKdV\(^+\) equation (1.2). These rational solutions to (1.2) are given by

\[
v(x, t) = v_0 - \frac{2(F_1 F_2 - F_1^2 F_2)}{F_2^2 + F_1^2}, \quad X = x - 6v_0^2 t, \quad v_0 \neq 0 \in \mathbb{R},
\]  

(3.9a)

where

\[
f = f(\psi) = |N - 1| = F_1 + iF_2, \quad F_1 = \text{Re}[f], \quad F_2 = \text{Im}[f],
\]  

(3.9b)

and the Wronskian is composed by

\[
\psi = (\psi_1, \psi_2, \cdots, \psi_N)^T,
\]  

(3.10a)

with

\[
\psi_{j+1} = \frac{1}{(2j)!} \frac{\partial^{2j}}{\partial k_1^{2j}} \varphi_1 \bigg|_{k_1 = 0}, \quad (j = 0, 1, \cdots, N - 1),
\]  

(3.10b)

and

\[
\varphi_1 = \sqrt{2v_0 + 2ik_1 e^{\eta_1} + \sqrt{2v_0 - 2ik_1 e^{-\eta_1}}}, \quad \eta_1 = k_1 X - 4k_3 t, \quad X = x - 6v_0^2 t, \quad k_1 \in \mathbb{R}. \quad (3.10c)
\]

### 3.2 Miura transformation and solutions to the cKdV equation

There is the following complex Miura transformation

\[
u = v^2 \pm iv_x
\]  

(3.11)

to relate the cKdV equation (1.1) and real mKdV\(^+\) equation (1.2) through

\[
u_t + 6\nu \nu_x + \nu_{xxx} = (2v \pm i\partial_x)(v_t + 6v^2 v_x + v_{xxx}).
\]  

(3.12)

Noting that in (3.11) \( u = u_1 + iu_2 \) but \( v \) is real, solutions to the cKdV equation (2.2) are then given as the following

\[
u_1 = v^2, \quad u_2 = \pm v_x,
\]  

(3.13)

where \( v \) is a solution to the mKdV\(^+\) equation (1.2) and has been listed out in Sec. 3.1. Using (3.13), the equations (2.2a) and (2.2b) can be transformed into

\[
u_t + 6\nu^2 v_x + v_{xxx} = 0
\]  

(3.14a)

\[
(v_t + 6v^2 v_x + v_{xxx})_x = 0
\]  

(3.14b)

then we can see the equation (3.14a) shares all its solutions with the equation (3.14b). (3.13) also provides a possible transformation to bilinearize the cKdV equation (2.2). Taking

\[
u_1 = -\left(\ln \frac{\bar{f}}{f} \right)_x, \quad u_2 = \pm i\left(\ln \frac{\bar{f}}{f} \right)_{xx}
\]  

(3.15)

then we find both (2.2a) and (2.2b) can be bilinearized as

\[
(D_t + D_x^2) \bar{f} \cdot f = 0,
\]  

(3.16a)

\[
D_x^2 \bar{f} \cdot f = 0,
\]  

(3.16b)

which is nothing but the bilinear mKdV\(^+\) equation.
3.3 Examples and illustration

Now let us have a look at some examples of solutions of the cKdV equation (1.1) or (2.2).

- One-soliton solution:

\[
\begin{align*}
 u_1 &= \frac{16a_1^2 - k_1^2}{(a_1^2 e^{8k_1^2 t-2k_1 x} + a_1^2 e^{-8k_1^2 t+2k_1 x})^2}, \\
 u_2 &= \pm \frac{8a_1^2 a_2^2 e^{8k_1^2 t+2k_1 x}(a_1^2 e^{4k_1 x} - a_1^2 e^{16k_1^3 t})}{(a_1^2 e^{4k_1 x} + a_1^2 e^{16k_1^3 t})^2}, \\
 & \quad a_1^+, a_1^-, k_1, \in \mathbb{R}. \\
\end{align*}
\]

(3.17a, 3.17b)

- One-breather solution:

\[
\begin{align*}
 u_1 &= 4 \left( \frac{\arctan \left( \frac{F_2}{F_1} \right)}{x} \right)^2 = \frac{4(F_1 x F_2 - F_1 F_2 x)^2}{(F_2^2 + F_1^2)^2}, \\
 u_2 &= \pm 2 \left( \frac{\arctan \left( \frac{F_2}{F_1} \right)}{x} \right)^2 = \pm 2 \left( -\frac{F_1 x F_2 + F_1 F_2 x}{F_2^2 + F_1^2} \right)^2, \\
 & \quad \text{where} \\
 F_1 &= 4k_{11}(a_{11} b_{11} + a_{12} b_{12}) \cos(24k_{11}^2 k_{12} t - 8k_{12}^3 t - 2k_{12} x) \\
 & \quad \quad + 4k_{11}(a_{12} b_{11} - a_{11} b_{12}) \sin(24k_{11}^2 k_{12} t - 8k_{12}^3 t - 2k_{12} x), \\
 F_2 &= -2k_{12} e^{-2k_{11}(4k_{11}^2 t + 12k_{12}^2 t + x)} \left[ (b_{11}^2 + b_{12}^2) e^{16k_{11}^3 t} + (a_{11}^2 + a_{12}^2) e^{4k_{11}(12k_{12}^2 t + x)} \right]. \\
\end{align*}
\]

(3.18a, 3.18b, 3.18c, 3.18d)

- Rational solution:

\[
\begin{align*}
 u_1 &= \left( v_0 - \frac{4v_0}{1+4v_0^2(x - 6v_0^2 t)^2} \right)^2, \\
 u_2 &= \pm \frac{32v_0^3(x - 6v_0^2 t)}{(1 + 4v_0^2(x - 6v_0^2 t)^2)^2}, \\
 & \quad v_0 \neq 0 \in \mathbb{R}. \\
\end{align*}
\]

(3.19a, 3.19b)

These solutions are illustrated as the following, where the sign for \( u_2 \) we have taken “+”.

4 Conclusions and discussions

For the cKdV equation we have derived its two kinds of solutions: solutions with isolated blow-up points and solutions without any singularities. The real KdV equation admits variety of solutions. By enlarging the field of wave numbers \( \{ k_j \} \) from \( \mathbb{R} \) to \( \mathbb{C} \), we naturally get complex solutions for the cKdV equation. We analyzed 1-soliton solution and the obtained isolated blow-up points are the same as those derived from Darboux transformation [3]. Besides, we considered a double-pole solution. The solution admits isolated blow-up points located on two parallel straight lines, which is a new feature for the waves with blow-up. For the solutions
without any singularities, we classified them according to the classification of solutions of the mKdV$^+$ equation. The connection of the two equations is a complex Miura transformation. Some numerical simulation results \cite{17} cope with our exact solutions. Besides, compared with the known exact solutions in \cite{18}, our bilinerization for the cKdV equation and classification for its solutions are quite neat.
For further discussion, let us back to the blow-up solutions. For a solution with two complex wave numbers $k_1, k_2$, e.g. the 2-soliton solution (2.3) with (2.8), i.e.

$$f_2(x, t) = 1 + e^{\xi_1} + e^{\xi_2} + \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 e^{\xi_1 + \xi_2},$$  

(4.1)

the analysis will become more complicated. For this moment we can not give a systematic analysis to write out its isolated blow-up points, as we have done in our paper for the cases of 1-soliton and double-pole solution. However, for some special case we find the following dynamical picture (Fig.6), which shows more straight lines for isolated blow-up points.

![Figure 6: Shape and motion of the 2-soliton solution given by (2.3) with (4.1) for $k_1 = 2 + i$, $k_2 = 1 - i$, $h_1 = h_2 = 0$. (a) $u_1 = \text{Re}[u]$. (b) $u_2 = \text{Im}[u]$.

It is known that the cKdV equation is invariant under some transformations, such as the Galilean transformation (2.1) and shift transformation $x \rightarrow x + x_0$. Our isolated blow-up points of 1-soliton case cope with the results of [8] in light of the Galilean transformation (2.1). In [7] Bona and Weissler showed that 2-soliton solution of the cKdV equation admits blow-ups when $x_0 \in \mathbb{C}$. Their approach is quite different from ours. Finally, let us revisit the double-pole solution generated from (2.19). For the real KdV equation such a solution has a singular point moving along two logarithm curves (cf.[19]), while for the complex KdV equation, the blow-ups are located on two parallel straight lines, not logarithm curves. Actually, (2.19) can be generalized to

$$f(x, t) = 1 + a(x - 3k_1^2t + b)e^{k_1x - k_3^3t + h} - \frac{1}{4k_1^2}e^{2(k_1x - k_3^3t + h)}, \quad k_1, a, b, h \in \mathbb{C},$$  

(4.2)

which can be derived either via Hirota’s method(cf.[16]) or via some shift transformations $x \rightarrow x + x_0, \quad t \rightarrow t + t_0$. Note that, as we have shown in Sec. 3.2, $f(x, t)$ given in (2.19) for the double-pole solution can be viewed as a limiting result of (4.1) with $k_2 \rightarrow k_1$. It would be interesting to investigate the blow-up phenomenon of a general complex 2-soliton solution (eg. with complex $k_j$) and its double-pole limit.

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A Wronskians

A \(N \times N\) Wronskian is defined as

\[
W(\phi_1, \phi_2, \cdots, \phi_N) = |\phi_1, \phi^{(1)}_1, \cdots, \phi^{(N-1)}_1| = \begin{vmatrix}
\phi^{(0)}_1 & \phi^{(1)}_1 & \cdots & \phi^{(N-1)}_1 \\
\phi^{(0)}_2 & \phi^{(1)}_2 & \cdots & \phi^{(N-1)}_2 \\
\vdots & \vdots & \ddots & \vdots \\
\phi^{(0)}_N & \phi^{(1)}_N & \cdots & \phi^{(N-1)}_N \\
\phi^{(0)}_1 & \phi^{(1)}_1 & \cdots & \phi^{(N-1)}_1 \\
\phi^{(0)}_2 & \phi^{(1)}_2 & \cdots & \phi^{(N-1)}_2 \\
\vdots & \vdots & \ddots & \vdots \\
\phi^{(0)}_N & \phi^{(1)}_N & \cdots & \phi^{(N-1)}_N \\
\end{vmatrix}, \quad (A.1)
\]

where \(\phi^{(l)}_j = \partial^l \phi_j / \partial x^l\) and \(\phi = (\phi_1, \phi_2, \cdots, \phi_N)^T\) is called the entry vector of the Wronskian. Usually we use the compact form\[20\]

\[
W(\phi) = |\phi, \phi^{(1)}, \cdots, \phi^{(N-1)}| = |0, 1, \cdots, N - 1| = |N - 1|, \quad (A.2)
\]

where \(\overline{N - j}\) indicates the set of consecutive columns \(0, 1, \cdots, N - j\). A Wronskian provides simple forms for its derivatives and this advantage admits direct verification of solutions that are expressed in terms of Wronskians.

B Lower-triangular Toeplitz matrices

A \(N\)th-order lower triangular Toeplitz matrix is a matrix in the following form

\[
\mathcal{A} = \begin{pmatrix}
a_0 & 0 & 0 & \cdots & 0 & 0 \\
a_1 & a_0 & 0 & \cdots & 0 & 0 \\
a_2 & a_1 & a_0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
a_{N-1} & a_{N-2} & a_{N-3} & \cdots & a_1 & a_0
\end{pmatrix}_{N \times N}, \quad a_j \in \mathbb{C}. \quad (B.1)
\]

All such matrices compose a commutative semigroup \(\tilde{G}_N(\mathbb{C})\) with identity with respect to matrix multiplication and inverse, and the set \(G_N(\mathbb{C}) = \{ \mathcal{A} | \mathcal{A} \in \tilde{G}_N(\mathbb{C}), |\mathcal{A}| \neq 0 \}\) makes an Abelian group.

For more details, please refer to Ref.\[21\].

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