THE VLASOV-MAXWELL-BOLTZMANN SYSTEM NEAR MAXWELLIANS WITH STRONG BACKGROUND MAGNETIC FIELD

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Abstract. In this paper, we are concerned with the construction of global-in-time solutions of the Cauchy problem of the Vlasov-Maxwell-Boltzmann system near Maxwellians with strong uniform background magnetic field. The background magnetic field under our consideration can be any given non-zero constant vector rather than vacuum in the previous results available up to now. Our analysis is motivated by the nonlinear energy method developed recently in [16, 24, 25] for the Boltzmann equation and the key point in our analysis is to deduce the dissipation estimates of the electronic field and strong background magnetic field.

1. Introduction. The motion of dilute ionized plasmas consisting of two-species particles (e.g., electrons and ions) under the influence of binary collisions and the self-consistent electromagnetic field can be modelled by the Vlasov-Maxwell-Boltzmann system (cf. [4, Chapter 19] as well as [21, Chapter 6.6])

\[
\begin{align*}
\partial_t F_+ + v \cdot \nabla_x F_+ + \frac{e_+}{m_+} \left( E + \frac{v}{c} \times B \right) \cdot \nabla_v F_+ &= Q(F_+, F_+) + Q(F_+, F_-), \\
\partial_t F_- + v \cdot \nabla_x F_- - \frac{e_-}{m_-} \left( E + \frac{v}{c} \times B \right) \cdot \nabla_v F_- &= Q(F_-, F_+) + Q(F_-, F_-). \quad (1)
\end{align*}
\]

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The electromagnetic field \([E, B] = [E(t, x), B(t, x)]\) satisfies the Maxwell equations
\[
\partial_t E - c \nabla_x \times B = -4\pi \int_{\mathbb{R}^3} v (e_+ F_+ - e_- F_-) \, dv,
\]
\[
\partial_t B + c \nabla_x \times E = 0,
\]
\[
\nabla_x \cdot E = 4\pi \int_{\mathbb{R}^3} (e_+ F_+ - e_- F_-) \, dv,
\]
\[
\nabla_x \cdot B = 0.
\]

Here \(\nabla_x = [\partial_{x_1}, \partial_{x_2}, \partial_{x_3}], \nabla_v = [\partial_{v_1}, \partial_{v_2}, \partial_{v_3}]\). The unknown functions \(F_\pm = F_\pm(t, x, v) \geq 0\) are the number density functions for the ions (+) and electrons (−) with position \(x = (x_1, x_2, x_3) \in \mathbb{R}^3\) and velocity \(v = (v_1, v_2, v_3) \in \mathbb{R}^3\) at time \(t \geq 0\), respectively, \(e_\pm\) and \(m_\pm\) are the magnitudes of their charges and masses accordingly, and \(c\) denotes the light speed.

Let \(F(v), G(v)\) be two number density functions for two types of particles with masses \(m_\pm\) and diameters \(\sigma_\pm\), then \(Q(F, G)(v)\) is defined as (cf. \([4, 31]\)):
\[
Q(F, G)(v) = \frac{(\sigma_+ + \sigma_-)^2}{4} \int_{\mathbb{R}^3 \times \mathbb{S}^2} \left| u - v \right|^2 \mathbf{b} \left( \frac{\omega \cdot (v - u)}{|u - v|} \right) \left\{ F(v')G(u') - F(v)G(u) \right\} d\omega du \equiv Q_{\text{gain}}(F, G) - Q_{\text{loss}}(F, G).
\]

Here \(\omega \in \mathbb{S}^2\) and \(\mathbf{b}\), the angular part of the collision kernel, satisfies the Grad angular cutoff assumption (cf. \([13]\))
\[
0 \leq \mathbf{b}(\cos \theta) \leq C|\cos \theta|
\]
for some positive constant \(C > 0\). The deviation angle \(\pi - 2\theta\) satisfies \(\cos \theta = \omega \cdot (v - u)/|v - u|\). Moreover, for \(m_1, m_2 \in \{m_+, m_-\}\),
\[
v' = v - \frac{2m_2}{m_1 + m_2} (v - u) \cdot \omega \omega, \quad u' = u + \frac{2m_1}{m_1 + m_2} (v - u) \cdot \omega \omega,
\]
which denote velocities \((v', u')\) after a collision of particles having velocities \((v, u)\) before the collision and vice versa. Notice that the above identities follow from the conservation of momentum \(m_1 v + m_2 u\) and energy \(\frac{1}{2}m_1 |v|^2 + \frac{1}{2}m_2 |u|^2\).

The exponent \(\gamma \in (-3, 1)\) in the kinetic part of the collision kernel is determined by the potential of intermolecular force, which is classified into the soft potential case when \(-3 < \gamma < 0\), the Maxwell molecular case when \(\gamma = 0\), and the hard potential case when \(0 < \gamma \leq 1\) which includes the hard sphere model with \(\gamma = 1\) and \(\mathbf{b}(\cos \theta) = C|\cos \theta|\) for some positive constant \(C > 0\). In the rest of this paper, we will focus on the hard sphere model.

### 1.1. The problem and reformulation of the VMB system

If we define
\[
\mu_+(v) = \frac{n_0}{e_+} \left( \frac{m_+}{2\pi \kappa_B T_0} \right)^{\frac{3}{2}} \exp \left( -\frac{m_+ |v|^2}{2\kappa_B T_0} \right),
\]
\[
\mu_-(v) = \frac{n_0}{e_-} \left( \frac{m_-}{2\pi \kappa_B T_0} \right)^{\frac{3}{2}} \exp \left( -\frac{m_- |v|^2}{2\kappa_B T_0} \right),
\]
\[
\mathfrak{B} = [\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3] \neq 0.
\]

where \([\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3]\) is any given non-zero constant vector, \(\kappa_B > 0\) is the Boltzmann constant, \(n_0 > 0\) and \(T_0 > 0\) are constant reference number density and temperature, respectively, and the reference bulk velocities have been chosen to be
zero, then it is easy to check that \([\mu_+(v), \mu_-(v), 0, \mathfrak{B}]\) is a trivial solution to the Vlasov-Maxwell-Boltzmann system (1), (2).

The main purpose of this paper is to construct global classical solutions to the Vlasov-Maxwell-Boltzmann system (1), (2) for the hard sphere model near \([\mu_+(v), \mu_-(v), 0, \mathfrak{B}]\) in the whole space \(\mathbb{R}^3\) with prescribed initial data

\[
F_\pm(0, x, v) = F_{0, \pm}(v, x), \quad E(0, x) = E_0(x), \quad B(0, x) = B_0(x), \tag{5}
\]

which satisfy the compatibility conditions

\[
\nabla_x \cdot E_0 = \int_{\mathbb{R}^3} (F_{0, +} - F_{0, -})dv, \quad \nabla_x \cdot B_0 = 0.
\]

To simplify the presentation, we assume in the rest of this paper that all the physical constants together with the generic constant \(4\pi\) appeared in (2) are chosen to be one. Under such an assumption, accordingly we can normalize the above Maxwellians as

\[
\mu = \mu_-(v) = \mu_+(v) = (2\pi)^{-3/2} e^{-|v|^2/2}.
\]

To study the stability problem around \([\mu(v), \mu(v), 0, \mathfrak{B}]\), we define the perturbation \(f_\pm = f_\pm(t, x, v)\) and \(B(t, x)\) by

\[
F_\pm(t, x, v) = \mu + \mu^{1/2} f_\pm(t, x, v), \quad \bar{B}(t, x) = B(t, x) - \mathfrak{B},
\]

then the Cauchy problem (1), (2), (5) is reformulated as

\[
\begin{align*}
\partial_t f_\pm + v \cdot \nabla_x f_\pm &+ E \cdot v^{1/2} + \frac{1}{2} E \cdot v f_\pm \mp (E + v \times \bar{B}) \cdot \nabla_v f_\pm = \Gamma_\pm(f, f), \\
\pm v \cdot \nabla_x f_\pm + L \pm f & = \Gamma_\pm(f, f), \\
\partial_t E - \nabla_x \times \bar{B} & = - \int_{\mathbb{R}^3} v \mu^{1/2} (f_+ - f_-)dv, \\
\partial_t \bar{B} + \nabla_x \times E & = 0, \\
\nabla_x \cdot E & = \int_{\mathbb{R}^3} \mu^{1/2} (f_+ - f_-)dv, \\
\nabla_x \cdot \bar{B} & = 0
\end{align*}
\]

(6)

with prescribed initial data

\[
F_\pm(0, x, v) = f_{0, \pm}(v, x), \quad E(0, x) = E_0(x), \quad \bar{B}(0, x) = B_0(x), \tag{7}
\]

which satisfy the compatibility conditions

\[
\nabla_x \cdot E_0 = \int_{\mathbb{R}^3} \mu^{1/2} (f_{0, +} - f_{0, -})dv, \quad \nabla_x \cdot B_0 = 0. \tag{8}
\]

Here, as in [15], for later use, we set \(f = [f_+, f_-]\) and the first equation of (6) can be rewritten as

\[
\begin{align*}
\partial_t f + v \cdot \nabla_x f - E \cdot v^{1/2} q_1 + Lf
= \frac{q_0}{2} E \cdot v f - q_0 \left( E + v \times \bar{B} \right) \cdot \nabla_v f - q_0 (v \times \mathfrak{B}) \cdot \nabla_v f + \Gamma(f, f),
\end{align*}
\]

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where \(q_0 = \text{diag}(1, -1)\), \(q_1 = [1, -1]\), and the linearized collision operator \(L = [L_+, L_-]\) and the nonlinear collision operator \(\Gamma = [\Gamma_+, \Gamma_-]\) are respectively defined by

\[
Lf = [L_+ f, L_- f], \quad \Gamma(f, g) = [\Gamma_+(f, g), \Gamma_-(f, g)].
\]
with

\[ L\pm f = -\mu^{-1/2} \left\{ Q \left( \mu, \mu^{1/2}(f_\pm + f_{\mp}) \right) + 2Q \left( \mu^{1/2}f_\pm, \mu \right) \right\}, \]

\[ \Gamma_\pm(f, g) = -\mu^{-1/2} \left\{ Q \left( \mu^{1/2}f_\pm, \mu^{1/2}g_\pm \right) + Q \left( \mu^{1/2}f_\pm, \mu^{1/2}g_{\mp} \right) \right\}. \]

For the linearized collision operator \( L \), it is well known (cf. [15]) that it is non-negative and the null space \( \mathcal{N} \) of \( L \) is spanned by

\[ \mathcal{N} = \text{span} \left\{ [1, 0]\mu^{3/2}, [0, 1]\mu^{3/2}, [v, v, v]\mu^{3/2} (1 \leq i \leq 3), [v|v|^2, |v|^2\mu] \right\}. \]

Moreover, for the hard sphere model, it is easy to see that \( L \) can be decomposed as

\[ Lf = \nu f - Kf \]

(10)

with the collision frequency \( \nu(v) \) and the nonlocal integral operator \( K = [K_+, K_-] \) being defined by

\[ \nu(v) = 2Q_{\text{loss}}(1, \mu) = 2 \int_{\mathbb{R}^3 \times S^2} |v - u|b \left( \frac{\omega \cdot (v - u)}{|v - u|} \right) \mu(u) du \sim 1 + |v| \]

(11)

and

\[
(K_\pm f)(v) = \mu^{-1/2} \left\{ 2Q_{\text{gain}} \left( \mu^{3/2}f_\pm, \mu \right) - Q \left( \mu, \mu^{1/2}(f_\pm + f_{\mp}) \right) \right\} \\
= \int_{\mathbb{R}^3 \times S^2} |u - v|b \left( \frac{\omega \cdot (v - u)}{|v - u|} \right) \mu^{1/2}(u) \]

\[
\times \left\{ 2\mu^{1/2}(u')f_\pm(u') - \mu^{1/2}(v')(f_\pm + f_{\mp})(u') + \mu^{1/2}(v)(f_\pm + f_{\mp})(u) \right\} d\omega du,
\]

(12)

respectively.

Define \( P \) as the orthogonal projection from \( L^2(\mathbb{R}^3_+) \times L^2(\mathbb{R}^3_+) \) to \( \mathcal{N} \), then for any given function \( f(t, x, v) \in L^2(\mathbb{R}^3_+) \times L^2(\mathbb{R}^3_+) \), one has

\[
Pf = a_+(t, x)[1, 0]\mu^{3/2} + a_-(t, x)[0, 1]\mu^{3/2} \]

\[
+ \sum_{i=1}^{3} b_i(t, x)[1, 1]v_i\mu^{1/2} + c(t, x)[1, 1]|v|^2 - 3)\mu^{1/2}
\]

(13)

with

\[ a_\pm = \int_{\mathbb{R}^3} \mu^{1/2}f_\pm dv, \]

\[ b_i = \frac{1}{2} \int_{\mathbb{R}^3} v_i\mu^{1/2}(f_+ + f_-) dv, \quad i = 1, 2, 3, \]

\[ c = \frac{1}{12} \int_{\mathbb{R}^3} (|v|^2 - 3)\mu^{1/2}(f_+ + f_-) dv. \]

Therefore, we have the following macro-micro decomposition with respect to the given global Maxwellian \( \mu(v) \), cf. [16],

\[ f(t, x, v) = Pf(t, x, v) + \{I - P\}f(t, x, v), \]

(14)

where \( I \) denotes the identity operator, and \( Pf \) and \( \{I - P\}f \) are called the macroscopic and the microscopic component of \( f(t, x, v) \), respectively.
For the hard sphere model, as shown in [15, Lemma 1], $L$ is locally coercive in the sense that
\[
- \langle f, Lf \rangle \geq \sigma_0 \| \{ I - P \} f \|_{\nu}^2 \equiv \sigma_0 \left\| \sqrt{\nu} \{ I - P \} f \right\|^2_{L^2(\mathbb{R}_3^3)}, \quad \nu(v) \sim 1 + |v| \tag{15}
\]
holds for some positive constant $\sigma_0 > 0$. Here and in the rest of this paper, we will use the following notations:

- $C$ is used to denote some positive constant (generally large) and $\kappa$, $\delta$, $\eta$, and $\lambda$ may take different values in different places;
- $A \lesssim B$ means that there is a generic constant $C > 0$ such that $A \leq CB$.
- $A \sim B$ means $A \lesssim B$ and $B \lesssim A$;
- The multi-indices $\alpha = [\alpha_1, \alpha_2, \alpha_3]$ and $\beta = [\beta_1, \beta_2, \beta_3]$ will be used to record spatial and velocity derivatives, respectively. And $\partial^\alpha \equiv \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3} \partial_{v_1}^{\beta_1} \partial_{v_2}^{\beta_2} \partial_{v_3}^{\beta_3}$.
- Similarly, the notation $\partial^\alpha$ will be used when $\beta = 0$ and likewise for $\partial_\beta$. The length of $\alpha$ is denoted by $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$. $\alpha' \leq \alpha$ means that no component of $\alpha'$ is greater than the corresponding component of $\alpha$, and $\alpha' < \alpha$ means that $\alpha' \leq \alpha$ and $|\alpha'| < |\alpha|$. And it is convenient to write $\partial_m = \partial_{x_m}$ ($m = 1, 2, 3$) and $\| \nabla_k f \| \equiv \sum_{|\alpha| = k} \| \partial^\alpha f \|$
- $\langle \cdot, \cdot \rangle$ is used to denote the $L^2_v \times L^2_v$ inner product in $\mathbb{R}_v^3$, with the $L^2_v$ norm $\| \cdot \|_{L^2_v}$. $(\cdot, \cdot)$ denotes the $L^2 \times L^2$ inner product either in $\mathbb{R}_v^3 \times \mathbb{R}_v^3$ or in $\mathbb{R}_v^3$ with the $L^2 \times L^2$ norm $\| \cdot \|$. While $(f | g)$ is used to denote $(f, g)$ with $g$ being the complex conjugate of $g$. For notational simplicity, we set $|f| \equiv |f|_{L^2_v}$ and $|f|_\ell \equiv \sum_{|\beta| \leq \ell} |\partial_\beta f|$ for $\ell \in \mathbb{Z}^+$. It is easy to see that $|f|_0 = |f|$;
- $\chi_\Omega$ is the standard indicator function of the set $\Omega$;
- $\| f(t, \cdot, \cdot) \|_{L_x^2 L_v^\beta} = \left( \int_{\mathbb{R}_v^3} \left( \int_{\mathbb{R}_v^3} |f(t, x, v)|^\beta dv \right)^\frac{2}{\beta} dx \right)^\frac{1}{2}$, and others like $\| f(t, \cdot, \cdot) \|_{L_x^\infty H_v^N}$ can be defined similarly;
- $B_C \subset \mathbb{R}^3$ denotes the ball of radius $C$ centered at the origin, and $L^2(B_C) \times L^2(B_C)$ stands for the space $L^2 \times L^2$ over $B_C$ and likewise for other spaces. Recall that $\nu(v) \sim 1 + |v|$ for hard sphere model, we set $|f|_{\nu} \equiv \int_{\mathbb{R}_v^3} |f|^2 \nu(v) dv$ and $\| f \|_{\nu}^2 \equiv \int_{\mathbb{R}_v^3} |f|^2 \nu(v) dv$;
- Assume that $N$ is a nonnegative integer and let $[f(t, x, v), E(t, x), B(t, x)]$ be a solution of the Cauchy problem of (6), (7), (8) of the Vlasov-Maxwell-Boltzmann system, we define the instant energy functionals $E_N(t)$ as

$$E_N(t) \sim \sum_{|\alpha| + |\beta| \leq N} \left\| \partial^\beta_\beta f(t) \right\|^2 + \left\| (E, B) (t) \right\|^2_{H_v^N},$$

while the corresponding energy dissipation rate functionals $D_N(t)$ and $\overline{D}_N(t)$ are given by

$$D_N(t) \sim \| (E + b \times \mathcal{B}, a_+ - a_-) (t) \|_\nu^2 + \sum_{1 \leq |\alpha| \leq N-1} \left\| \partial^\alpha \left( Pf, E, B \right) (t) \right\|^2$$

$$+ \sum_{|\alpha| = N} \| \partial^\alpha Pf(t) \|_\nu^2 + \sum_{|\alpha| + |\beta| \leq N} \left\| \partial^\beta_\beta \{ I - P \} f(t) \right\|^2_\nu,$$
1.2. Main result. We are now ready to state the main theorem in this paper.

Theorem 1.1. Take $N \geq 3$, assume that $\mathcal{B}$ is any given non-zero constant vector and $F_0(x,v) = \mu + \sqrt{\mu} f_0(x,v) \geq 0$. If we assume further that

$$Y_0 = \sum_{|\alpha|+|\beta| \leq N} \|\partial_\alpha^\beta f_0\| + \|E_0\|_{H^N} + \|B_0 - \mathcal{B}\|_{H^N}$$

is sufficiently small, then the Cauchy problem (6), (7), (8) of the Vlasov-Maxwell-Boltzmann system admits a unique global solution $[f(t,x,v), E(t,x), B(t,x)]$ satisfying $F(t,x,v) = \mu + \sqrt{\mu} f(t,x,v) \geq 0$.

**Remark 1.** Some remarks are listed as follows:

i). Note that the background magnetic field $\mathcal{B}$ under our consideration can be any non-zero vector rather than zero vector considered in the previous results obtained in [6, 8, 9, 15, 20, 27].

ii). Although the background magnetic field $\mathcal{B}$ is assumed to be a constant vector in our main result Theorem 1.1, it is not so difficult to see that a similar result still holds for those $x-$dependent background magnetic field $\mathcal{B}(x)$ satisfying

$$\nabla_x \times \mathcal{B}(x) = 0, \quad \nabla_x \cdot \mathcal{B}(x) = 0 \quad (16)$$

and

$$\mathcal{B}(x) \in L^\infty(\mathbb{R}^3), \quad \nabla_x \mathcal{B}(x) \in H^{N-1}(\mathbb{R}^3) \quad (17)$$

with $\|\nabla_x \mathcal{B}(x)\|_{H^{N-1}(\mathbb{R}^3)}$ being sufficiently small. The assumption (16) is to guarantee that $[\mu(v), \mu(v), 0, \mathcal{B}(x)]$ is a non-trivial solution of the Vlasov-Maxwell-Boltzmann system (1), (2), while the assumption (17) together with the fact that $\|\nabla_x \mathcal{B}(x)\|_{H^{N-1}(\mathbb{R}^3)}$ is assumed to be sufficiently small is used to close the analysis.

We note, however, that for $x \in \mathbb{R}^3$, it is not so easy to find a non-trivial background magnetic field $\mathcal{B}(x)$ satisfying (16) and (17). If we consider the problem in a periodic box $\mathbb{T}^3$, it is easy to see that $\mathcal{B}(x) = \frac{1}{2}[x_1, -2x_2, x_3]$ is a non-trivial solution of (16) satisfying $\mathcal{B}(x) \in L^\infty(\mathbb{T}^3), \nabla_x \mathcal{B}(x) \in H^{N-1}(\mathbb{T}^3)$. For these types of nontrivial background magnetic field $\mathcal{B}(x)$, although, generally speaking, $\|\nabla_x \mathcal{B}(x)\|_{H^{N-1}(\mathbb{T}^3)}$ can not be assumed to be small, our recent calculations show that if the system under consideration admits certain dissipation effect with respect to $v-$variable like the Vlasov-Maxwell-Landau system or the Vlasov-Maxwell-Boltzmann system without angular cutoff, it is still hopeful to obtain the nonlinear stability of the non-trivial solution $[\mu(v), \mu(v), 0, \mathcal{B}(x)]$. Such a problem is under our current study.

ii). In fact, if the background magnetic field $\mathcal{B}(x)$ depending on $x-$variable satisfy

$$\nabla_x \times \mathcal{B}(x) = 0, \quad \nabla_x \cdot \mathcal{B}(x) = 0 \quad (18)$$

from which $\mathcal{B}(x)$ is harmonic, furthermore,

$$\mathcal{B}(x) \in L^\infty(\mathbb{R}^3), \quad (19)$$

which implies that $\mathcal{B}(x)$ is a constant vector by Liouville’s Theorem.

To find a non-trivial background magnetic field $\mathcal{B}(x)$ satisfying (18). Instead, if we consider the problem in a periodic box $\mathbb{T}^3$, it is easy to see that $\mathcal{B}(x) = \frac{1}{2}[x_1, -2x_2, x_3]$ is a non-trivial solution of (18) satisfying $\mathcal{B}(x) \in L^\infty(\mathbb{T}^3)$. For these types of nontrivial background magnetic field $\mathcal{B}(x)$, our recent calculations show that if the system under consideration admits certain
dissipation effect with respect to \( v \)-variable like the Vlasov-Maxwell-Landau system or the Vlasov-Maxwell-Boltzmann system without angular cutoff, it is still hopeful to obtain the nonlinear stability of the non-trivial solution \([\mu(v), \mu(v), 0, B(x)]\). Such a problem is under our current study.

iii). Although only the hard sphere model is considered in this paper, we believe that, by combing the arguments used in this paper with those employed in [6] for the Vlasov-Maxwell-Boltzmann system (1), (2) with cutoff potentials and in [8, 11] for the Vlasov-Maxwell-Boltzmann system (1), (2) with non-cutoff potentials, similar results hold also for the Vlasov-Maxwell-Boltzmann system (1), (2) for the whole range of both cutoff potentials and non-cutoff potentials.

iv). Another interesting problem related for the Vlasov-Maxwell-Boltzmann system (1), (2) is to consider the global-in-time Vlasov-Poisson-Boltzmann limit of the Vlasov-Maxwell-Boltzmann system (1), (2) as the light speed \( c \) tends to infinity. We’re convinced that the arguments employed in [7, 10, 33, 34, 35, 36] to treat the global solvability of the Vlasov-Poisson-Boltzmann system together with the arguments used in this paper can be used to deal with such a problem, we will consider this problem in a forthcoming paper. For former results on the Vlasov-Poisson limit of the Vlasov-Maxwell system, those interested are referred to [3] and references cited therein.

v). Similar result holds for the relativistic Vlasov-Maxwell-Boltzmann system, cf. [18, 28] and the references cited therein for some former results related.

1.3. The background and the main strategy. The nonlinear energy method developed in [16, 24, 25] for the Boltzmann equation provides an effective approach to deal with the global solvability of the Cauchy problem of the Boltzmann type equations in the perturbative framework. In fact, based on such a method, the estimates on the linearized Boltzmann collision operator \( Lf \) and the nonlinear collision operator \( \Gamma(f, f) \) for both cutoff potentials [29] and non-cutoff potentials [2, 14], and some arguments developed recently to deal with the Vlasov-Poisson-Landau system [17, 23, 30, 32], the Vlasov-Maxwell-Landau system [5, 22], and the Vlasov-Poisson-Boltzmann system [7, 10, 33, 34, 35, 36], a satisfactory global well-posedness theory has already been established for the Cauchy problem (6), (7), (8) of the Vlasov-Maxwell-Boltzmann system near Maxwellians with zero background magnetic field, cf. [6, 8, 9, 15, 20, 27] for cutoff potentials and [8, 11] for non-cutoff potentials and the references cited therein.

For the Vlasov-Maxwell-Boltzmann system (6), an interesting problem is to consider the influence of strong magnetic field on the behavior of its solutions. For collisionless plasmas, such a problem is studied in [12], where the finite Larmor radius approximation is derived from the Vlasov equation, in the limit of large uniform magnetic field and with an external electric field.

As a first step to study the influence of strong magnetic field on the behavior of the solutions of the Vlasov-Maxwell-Boltzmann system (6), we consider the global solvability of the Cauchy problem (6), (7), (8) of the Vlasov-Maxwell-Boltzmann system near Maxwellians with strong background magnetic field. Note that for the Cauchy problem (6), (7), (8) with non-zero background magnetic field \( B \), to the best of our knowledge, no result is available up to now.

Now we outline the main ideas to deduce our main results. Compared with the case with zero background magnetic field considered in [6, 8, 9, 11, 15, 20, 27], the main technical problems in our analysis are caused by the term \( q_0(v \times B) \cdot \nabla_v f \). More precisely, main difficulties can be summarized as in the following:
i). How to deduce the dissipation estimates on the macroscopic components \( P \) \( f(t,x,v) \) and electromagnetic field \( \left[ E(t,x), \tilde{B}(t,x) \right] \), especially on the corresponding dissipation estimates on the electric field \( E(t,x) \)? In fact, due to the appearance of background magnetic field \( B \), \( E(t,x) \) solves

\[
2E = \partial_t G + \nabla_x (a_+ - a_-) + \nabla_x \cdot \mathcal{A}(I - P) f \cdot q_1
- E(a_+ + a_-) - 2b \times \vec{B} - 2b \times \mathfrak{B} - \left[ \langle v, -v \rangle \mu^{1/2}, Lf + \Gamma(f, f) \right]
\]

with

\[
\mathcal{A}(f) = (\mathcal{A}_{mj}(f))_{3 \times 3},
\]

\[
\mathcal{A}_{mj}(f) = \int_{\mathbb{R}^3} (v_m v_j - 1) \mu^{1/2} f dv, \quad 1 \leq m, j \leq 3,
\]

and the appearance of the term \( 2b \times \mathfrak{B} \) in (20) makes it difficult to yield the desired dissipation estimates of \( E(t,x) \). We note, however, that such an estimate plays an essential role in closing the energy type estimates when dealing with the term involving \( E \cdot v \mu^{1/2} q_1 \) since the resulting term

\[
\left( \partial_\beta \{ I - P \} \left( E \cdot v \mu^{1/2} q_1 \right), \partial_\beta \{ I - P \} f \right)
\]

can only be bounded as follows

\[
= \left( \partial_\beta \left( E \cdot v \mu^{1/2} q_1 \right), \partial_\beta \{ I - P \} f \right)
\leq \eta \| E \|^2 + \| \{ I - P \} f \|^2.
\]

ii). How to deal with the term involving \( \langle v \times \mathfrak{B} \rangle \cdot \nabla_v f \)? In fact, noticing that the microscopic component \( \{ I - P \} f(t,x,v) \) solves

\[
\partial_t \{ I - P \} f + v \cdot \nabla_x \{ I - P \} f - E \cdot v \mu^{1/2} q_1 + Lf
= -v \cdot \nabla_x Pf + P \{ v \cdot \nabla_x f \} + \{ I - P \} \{-q_0 \langle v \times \mathfrak{B} \rangle \cdot \nabla_v f \} + \{ I - P \} \tilde{G}(21)
\]

with

\[
\tilde{G} = \frac{q_0}{2} E \cdot v f - q_0 \left( E + v \times \vec{B} \right) \cdot \nabla_v f + \Gamma(f, f),
\]

the problem is how to estimate the corresponding terms involving the linear term \( \{ I - P \} \{-q_0 \langle v \times \mathfrak{B} \rangle \cdot \nabla_v f \} \) in the right hand side of (21) suitably?

Our main arguments to overcome the above difficulties are the following: For the first problem, to deduce the dissipation estimates on \( \partial^\alpha E(t,x) \) for each multi-index \( \alpha \) satisfying \( |\alpha| \geq 1 \), we mainly use the facts that for \( m, j = 1, 2, 3 \)

\[
\int_{\mathbb{R}^3} (|v|^2 - 5) v_j \mu^{1/2} \left( - (v \times \mathfrak{B}) \cdot \nabla_v \{ P_+ f \} + (v \times \mathfrak{B}) \cdot \nabla_v \{ P_- f \} \right) dv = 0,
\]

\[
\int_{\mathbb{R}^3} (v_m v_j - 1) \mu^{1/2} \left( - (v \times \mathfrak{B}) \cdot \nabla_v \{ P_+ f \} + (v \times \mathfrak{B}) \cdot \nabla_v \{ P_- f \} \right) dv = 0,
\]

which follow from the structure of the kernel space of the Boltzmann linearized collision operator \( L \), cf, Lemma 3.3 and Lemma 3.4 for details.

Besides, it is not easy to deduce the dissipation functionals \( \| E \|^2 \) due to the appearance of \( b \times \mathfrak{B} \). To do so, we combine \( E \) and \( b \times \mathfrak{B} \) to deduce the corresponding dissipation functionals \( \| E + b \times \mathfrak{B} \|^2 \) instead of \( \| E + b \times \mathfrak{B} \|^2 \) as previous results.
For the second problem, noticing that when $\mathcal{B}$ is a constant vector, one can deduce that
\[
\sum_{1 \leq |\beta| \leq N} \left( \partial_\beta^2 \left( (v \times \mathcal{B}) \cdot \nabla_v g(t, x, v) \right), \partial_\beta^2 g(t, x, v) \right) = 0,
\]
from which one can deduce a nice estimate on the corresponding terms involving the linear term $\{I - P\}\{-g_0(v \times \mathcal{B}) \cdot \nabla_v f\}$ in the right hand side of (21), cf. the analysis in step 2 of Lemma 3.9 for details.

The rest of this paper is organized as follows. In Section 2, we list some basic lemmas for later use. The proof of the main result will be given in Section 3.

2. Preliminary. In this section, we collect some basic lemmas for later use. For this purpose, we first list several basic estimates on the linearized Boltzmann collision operator $L$ and the nonlinear term $\Gamma(f, g)$ for the hard sphere model studied in this paper. The first one is on the coercivity estimate of the linearized collision operator $L$.

Lemma 2.1. (cf. [15]) For the hard sphere model, one has
\[
\langle Lf, f \rangle \gtrsim |\{I - P\}f|^2. \tag{22}
\]
Moreover, let $|\beta| \geq 1$, there exists $C > 0$ such that
\[
\langle \partial_\beta Lf, \partial_\beta f \rangle \gtrsim |\partial_\beta f|^2 - C |\chi_{|v| \leq 2Cv}|f|^2 \tag{23}
\]
holds.

For the estimate on the nonlinear collision operator $\Gamma(f, g)$, one has

Lemma 2.2. (cf. [15]) It holds for each multi-index $\alpha$ that
\[
\langle \partial^\alpha \Gamma(g_1, g_2), \partial^\alpha g_3 \rangle \lesssim \sum_{\alpha_1 \leq \alpha} |\partial^{\alpha_1} g_1|^2 |\partial^{\alpha - \alpha_1} g_2|^2 |\partial^\alpha \{I - P\}g_3|_v, \tag{24}
\]
while for each multi-index $\beta$ satisfying $|\beta| \geq 1$, one can also conclude that
\[
\langle \partial_\beta^2 \Gamma(g_1, g_2), \partial_\beta^2 g_3 \rangle \lesssim \sum_{\beta_1 + \beta_2 \leq \beta} |\partial_{\beta_1}^\alpha g_1|^2 |\partial^{\alpha - \alpha_1} g_2|^2 |\partial_\beta^2 g_3|^2. \tag{25}
\]

In what follows, we will collect some analytic tools which will be used in this paper.

The second is concerned with a Minkowski type integral inequality to exchange the orders of integration over $x \in \mathbb{R}^3$ and $v \in \mathbb{R}^3$.

Lemma 2.3. (cf. [1, 19]) For $1 \leq p \leq q \leq \infty$, we have
\[
\|f\|_{L^q_x L^p_v} \leq \|f\|_{L^q_x L^q_v}. \tag{26}
\]

3. A priori estimates and the proof of Theorem 1.1. In this section, we will deduce the desired a priori estimates and then prove Theorem 1.1. To make the presentation easy to follow, we divide this section into several subsections, the first one is concerned with the equations satisfied by the macroscopic component $Pf(t, x, v)$. 

3.1. Macro-structure for \([f(t,x,v), E(t,x), \vec{B}(t,x)]\). We list below some results concerning the macroscopic component of \(f(t,x,v)\). To this end, if we define

\[
\begin{align*}
    r_\pm &\equiv -v \cdot \nabla_x \{I_\pm - P_\pm\} f - L_\pm f, \\
g_\pm &\equiv \frac{1}{2} v \cdot Ef_\pm \mp \left( E + v \times \vec{B} \right) \cdot \nabla_v f_\pm \mp (v \times \mathcal{B}) \cdot \nabla_v f_\pm + \Gamma(f, f), \\
G &\equiv \left\langle v^{1/2}, \{I - P\} f \cdot q_1 \right\rangle, \\
A_{mj}(f) &\equiv \int_{\mathbb{R}^3} (v_m v_j - 1) \mu^{1/2} f dv, \ 1 \leq m, j \leq 3, \\
B_j(f) &\equiv \frac{1}{10} \int_{\mathbb{R}^3} (|v|^2 - 5) v_j \mu^{1/2} f dv, \ 1 \leq j \leq 3,
\end{align*}
\]

then one can deduce that the macroscopic component \(P f(t,x,v)\) satisfies the following equations

**Lemma 3.1.** One has by macro-projection that

\[
\begin{align*}
    \partial_t \left( \frac{a_+ + a_-}{2} \right) + \nabla_x \cdot b &= 0, \\
    \partial_t b_j + \partial_j \left( \frac{a_+ + a_-}{2} + 2c \right) + \frac{1}{2} \sum_{m=1}^{3} \partial_m A_{jm}(\{I - P\} f \cdot [1,1]) \\
    &= \frac{a_+ - a_-}{2} E_j + \left[ G \times \vec{B} \right]_j + [G \times \mathcal{B}]_j, \\
    \partial_t c + \frac{1}{3} \nabla_x \cdot b + \frac{5}{6} \sum_{m=1}^{3} \partial_m B_m(\{I - P\} f \cdot [1,1]) &= \frac{1}{6} G \cdot E, \\
    \partial_t (a_+ - a_-) + \nabla_x \cdot G &= 0, \\
    \partial_t G + \nabla_x (a_+ - a_-) - 2E + \nabla_x \cdot A(\{I - P\} f \cdot q_1) \\
    &= E(a_+ + a_-) + 2b \times \vec{B} + 2b \times \mathcal{B} + \left\langle [v, -v]^{1/2}, Lf + \Gamma(f, f) \right\rangle
\end{align*}
\]

and

\[
\begin{align*}
    \frac{1}{2} \partial_t A_{mj}(\{I - P\} f \cdot [1,1]) + \partial_j b_m + \partial_m b_j \\
    &= -\frac{2}{3} \delta_{mj} \nabla_x \cdot b - \frac{5}{3} \delta_{mj} \nabla_x \cdot B(\{I - P\} f \cdot [1,1]) \\
    &= \frac{1}{2} A_{mj}(r_+ + r_- + g_+ + g_-) - \frac{1}{3} \delta_{mj} G \cdot E, \\
    \frac{1}{2} \partial_t B_j(\{I - P\} f \cdot [1,1]) + \partial_j c \\
    &= \frac{1}{2} B_j(r_+ + r_- + g_+ + g_-).\end{align*}
\]

**Proof.** To this end, for any solution \(f(t,x,v)\) of the Vlasov-Maxwell-Boltzmann system (6), by applying the macro-micro decomposition (14), one can then derive
from (6) the following a fluid-type system of equations

\[
\begin{aligned}
\partial_t a_{\pm} + \nabla_x \cdot b + \nabla_x \cdot \langle vv^{1/2}, \{I_{\pm} - P_{\pm}\} f \rangle &= \langle \mu^{1/2}, g_{\pm} \rangle, \\
\partial_t (b_{j} + \langle v_j \mu^{1/2}, \{I_{\pm} - P_{\pm}\} f \rangle) + \partial_j (a_{\pm} + 2c) &\equiv E_j \\
+ \nabla_x \cdot \langle vv_j \mu^{1/2}, \{I_{\pm} - P_{\pm}\} f \rangle &= \langle v_j \mu^{1/2}, g_{\pm} + L f \rangle, \\
\partial_t (c + \frac{1}{6} \langle |v|^{2} - 3 \rangle \mu^{1/2}, \{I_{\pm} - P_{\pm}\} f) + \frac{1}{3} \nabla_x \cdot b \\
+ \frac{1}{6} \nabla_x \langle (|v|^{2} - 3) v \mu^{1/2}, \{I_{\pm} - P_{\pm}\} f \rangle &= \langle (|v|^{2} - 3) \mu^{1/2}, g_{\pm} - L f \rangle
\end{aligned}
\]  

and

\[
\begin{aligned}
\partial_t [A_{\mu\mu}(\{I_{\pm} - P_{\pm}\} f) + 2c] + 2\partial_m b_m &= A_{\mu\mu}(r_{\pm} + g_{\pm}), \\
\partial_j A_{\mu j}(\{I_{\pm} - P_{\pm}\} f) + \partial_j b_m + \partial_m b_j + \nabla_x \cdot \langle vv^{1/2}, \{I_{\pm} - P_{\pm}\} f \rangle \\
&= A_{\mu j}(r_{\pm} + g_{\pm}), \\
\partial_j B_j(\{I_{\pm} - P_{\pm}\} f) + \partial_j c &= B_j(r_{\pm} + g_{\pm}),
\end{aligned}
\]  

and noticing \(\mu^{1/2}, g_{\pm} = 0\), (28) and (29) follows by \(\langle \cdot_+, \cdot_+ \rangle + \langle \cdot_-, \cdot_- \rangle\) and \(\langle \cdot_+, \cdot_+ \rangle - \langle \cdot_-, \cdot_- \rangle\) of (30) and (31).

\[\square\]

The next lemma is concerned with some identities for \(A_{\mu j}(f)\) and \(B_j(f)\) defined in (27), which will play an essential role in deducing dissipation estimates on the macroscopic component \(P f(t,x,v)\) and the electromagnetic field \([E(t,x), B(t,x)]\), especially on the electronic field \(E(t,x)\).

**Lemma 3.2.** One has

\[
\begin{aligned}
A_{\mu j} \left( (v \times B) \cdot \nabla_v (P_{\pm} f) + (v \times B) \cdot \nabla_v (P_{-} f) \right) &= 0, \quad 1 \leq m, j \leq 3, \\
B_j \left( (v \times B) \cdot \nabla_v (P_{\pm} f) + (v \times B) \cdot \nabla_v (P_{-} f) \right) &= 0, \quad 1 \leq j \leq 3.
\end{aligned}
\]  

**Proof.** This proof can be easily verified by the definitions of \(P_{\pm} f\), we thus omit the details for brevity. \[\square\]

Now we are ready to deduce the desired dissipation estimates on the macroscopic component \(P f(t,x,v)\) and the electromagnetic field \([E(t,x), B(t,x)]\). To this end, we first have the following result on the macroscopic component \(P f(t,x,v)\).

**Lemma 3.3.** For \(N \geq 2\), there exists an interactive functional \(G_f(t)\) satisfying

\[
G_f(t) \lesssim \sum_{|\alpha| \leq N} \left\| \partial^\alpha f(t) \right\|^2
\]

such that

\[
\frac{d}{dt} G_f(t) + \sum_{1 \leq |\alpha| \leq N} \left\| \partial^\alpha P f(t) \right\|^2 + \left\| a_+(t) - a_-(t) \right\|^2 \lesssim \sum_{|\alpha| \leq N} \left\| \partial^\alpha (I - P) f(t) \right\|^2 + \mathcal{E}_N(t) \mathcal{D}_N(t)
\]

holds for all \(0 \leq t \leq T\).
Proof. Taking the $L^2_0 \times L^2_0$ inner product of (29) with $\partial_j c$, one can get that

$$\sum_{j=1}^{3} \|\partial_j c(t)\|^2 = \sum_{j=1}^{3} (\partial_j c(t), \partial_j c(t))$$

$$\lesssim -\sum_{j=1}^{3} \left( \frac{1}{2} \partial_j \mathbb{B}_j((I - P) f(t) \cdot [1, 1]), \partial_j c(t) \right)$$

$$+ \sum_{j=1}^{3} \left( \frac{1}{2} \mathbb{B}_j (r_+(t) + r_-(t) + g_+(t) + g_-(t)), \partial_j c(t) \right)$$

$$= -\frac{1}{2} \sum_{j=1}^{3} \frac{d}{dt} (\mathbb{B}_j((I - P) f(t) \cdot [1, 1]), \partial_j c(t))$$

$$+ \sum_{j=1}^{3} (\mathbb{B}_j((I - P) f(t) \cdot [1, 1]), \partial_j \partial_t c(t))$$

$$+ \sum_{j=1}^{3} \left( \frac{1}{2} \mathbb{B}_j (r_+(t) + r_-(t) + g_+(t) + g_-(t)), \partial_j c(t) \right).$$

(34)

For the second term on the right hand of the second equality in (34), we have from (30)$_3$ that

$$\frac{1}{2} \sum_{j=1}^{3} (\mathbb{B}_j((I - P) f(t) \cdot [1, 1]), \partial_j \partial_t c(t))$$

$$= \frac{1}{2} \sum_{j=1}^{3} (\mathbb{B}_j((I - P) f(t) \cdot [1, 1]),$$

$$\partial_j \left\{ \frac{1}{6} G(t) \cdot E(t) - \frac{1}{3} \nabla_x \cdot \mathbb{B}_j c(t) - \frac{5}{6} \sum_{m=1}^{3} \partial_m \mathbb{B}_m((I - P) f(t) \cdot [1, 1]) \right\}$$

$$\lesssim \eta \|\nabla_x c(t)\|^2 + \|\{I - P\} f(t)\|_{H^1_t L^2_x}^2 + E_2(t) D_2(t),$$

(35)

while for the last term on the right hand of (34), we can deduce that

$$\sum_{j=1}^{3} \left( \frac{1}{2} \mathbb{B}_j (r_+(t) + r_-(t) + g_+(t) + g_-(t)), \partial_j c(t) \right)$$

$$\lesssim \eta \|\nabla_x c(t)\|^2 + \|\{I - P\} f(t)\|_{H^1_t L^2_x}^2 + E_2(t) D_2(t),$$

(36)

where we have used (32).

Thus we can get by substituting the estimates (35) and (36) into (34) that

$$\frac{d}{dt} G_c^1(t) + \|\nabla_x c(t)\|^2 \lesssim \eta \|\nabla_x (h, c)(t)\|^2 + \|\{I - P\} f(t)\|_{H^1_t L^2_x}^2 + E_2(t) D_2(t).$$

(37)

Here

$$G_c^1(t) \equiv \frac{1}{2} \sum_{j=1}^{3} (\mathbb{B}_j((I - P) f(t) \cdot [1, 1]), \partial_j c(t)).$$
On the other hand, noticing

\[
\sum_{m,j=1}^{3} \int_{\mathbb{R}^3} \left| \partial_j b_m + \partial_m b_j - \frac{2}{3} \delta_{mj} \nabla_x \cdot b \right|^2 \, d\xi
= 2 \sum_{m,j=1}^{3} \| \partial_m b_j \|^2 + \frac{2}{3} \| \nabla_x \cdot b \|^2 = 2 \| \nabla_x b \|^2 + \frac{2}{3} \| \nabla_x \cdot b \|^2,
\]

we can get by repeating the argument used to deduce (37) and by using (28)\(_1\) and (28)\(_2\) that

\[
\frac{d}{dt} G_b^1(t) + 2 \| \nabla_x b(t) \|^2 + \frac{2}{3} \| \nabla_x \cdot b(t) \|^2 \\
\lesssim \eta \left( \| \nabla_x (a_+ + a_-)(t) \|^2 + \| \nabla_x c(t) \|^2 \right) + \| \{I - P\} f(t) \|_{H^1_x L^2_x}^2 + \mathcal{E}_2(t) \mathcal{D}_2(t).
\]

Here

\[
G_b^1(t) = \frac{1}{2} \sum_{m,j=1}^{3} \left( \partial_j b_m(t) - \partial_m b_j(t) - \frac{2}{3} \delta_{mj} \nabla_x \cdot b(t), \mathcal{A}(I - P) f(t) \cdot [1, 1] \right).
\]

Next, we estimate \(a_+ + a_-\). To this end, we have from (28)\(_2\), (28)\(_1\) and by employing the same argument to deduce (37) that

\[
\frac{d}{dt} G_a^1(t) + \| \nabla_x (a_+ + a_-)(t) \|^2 \\
\lesssim \| \nabla_x (b, c)(t) \|^2 + \| \{I - P\} f(t) \|_{H^1_x L^2_x}^2 + \mathcal{E}_2(t) \mathcal{D}_2(t)
\]

with

\[
G_a^1(t) = \sum_{j=1}^{3} \left( \partial_j (a_+ + a_-)(t), b_j(t) \right).
\]

Set

\[
G_{a,b,c}^1(t) = G_b^1(t) + \kappa_1 G_a^1(t) + \kappa_2 G_a^1(t), 0 < \kappa_2 \ll \kappa_1 \ll 1.
\]

we can deduce from (37), (38), and (39) that

\[
\frac{d}{dt} G_{a,b,c}^1(t) + \| \nabla (a_+ + a_-, b, c)(t) \|^2 \lesssim \| \{I - P\} f(t) \|_{H^1_x L^2_x}^2 + \mathcal{E}_2(t) \mathcal{D}_2(t).
\]


Finally, for the corresponding estimate on $a_+ - a_-$, we have from (28) that
\[
\|\nabla x (a_+ - a_-)(t)\|^2 + 2 \|(a_+ - a_-)(t)\|^2
\]
\[
= (\nabla x (a_+ - a_-)(t), \nabla x (a_+ - a_-)(t)) + 2 \langle a_+(t) - a_-(t), a_+(t) - a_-(t) \rangle
\]
\[
= (\nabla x (a_+ - a_-)(t), \nabla x (a_+ - a_-)(t)) + 2 (\nabla x \cdot E(t), a_+(t) - a_-(t))
\]
\[
= (\nabla x (a_+ - a_-)(t), \nabla x (a_+ - a_-)(t)) - 2 E(t) \nabla x (a_+ - a_-)(t)
\]
\[
= (-\partial_t G(t) - \nabla x \cdot \mathcal{A}(\{I - P\} f(t) \cdot q) + 2 b(t) \times \mathcal{B}, \nabla x (a_+ - a_-)(t))
\]
\[
+ (E(t)(a_+(t) + a_-(t)) + 2b(t) \times \tilde{B}(t), \nabla x (a_+ - a_-)(t))
\]
\[
+ \left( \left| v, -v \mu^{1/2} \right|, Lf(t) + \Gamma(f(t), f(t)) \right) \nabla x (a_+ - a_-)(t)
\]
\[
\lesssim - \frac{d}{dt} (G(t), \nabla x (a_+ - a_-)(t)) + (G(t), \nabla x \partial_t (a_+ - a_-)(t))
\]
\[
+ \eta \|a_+(t) - a_-(t)\|_{H^1_x}^2 + \|\nabla x b(t)\|^2 + \|\{I - P\} f(t)\|_{H^1_x L^2_t}^2 + \mathcal{E}_2(t) D_2(t)
\]
\[
\lesssim - \frac{d}{dt} (G(t), \nabla x (a_+ - a_-)(t)) + \eta \|a_+(t) - a_-(t)\|_{H^1_x}^2 + \|\nabla x b(t)\|^2
\]
\[
+ \|\{I - P\} f(t)\|_{H^1_x L^2_t}^2 + \mathcal{E}_2(t) D_2(t). \tag{41}
\]

Consequently
\[
\frac{d}{dt} (G(t), \nabla x (a_+ - a_-)(t)) + \|\nabla x (a_+ - a_-)(t)\|^2 + 2 \|(a_+ - a_-)(t)\|^2
\]
\[
\lesssim \|\nabla x b(t)\|^2 + \|\{I - P\} f(t)\|_{H^1_x L^2_t}^2 + \mathcal{E}_2(t) D_2(t). \tag{41}
\]

A suitable linear combination of (40) and (41) tells us that
\[
\frac{d}{dt} G'_f(t) + \|\nabla x (a_+ \pm a_-, b, c)(t)\|^2 + \|a_+(t) - a_-(t)\|^2
\]
\[
\lesssim \|\{I - P\} f(t)\|_{H^1_x L^2_t}^2 + \mathcal{E}_2(t) D_2(t) \tag{42}
\]
holds for some instant energy functional $G'_f(t)$ satisfying $G'_f(t) \lesssim \sum_{|\alpha| \leq 1} \|\partial^\alpha f(t)\|^2$
and the general case, i.e., the estimate (33), follows in the same spirit of the strategy used in deducing the above estimates (42), thus the proof of Lemma 3.3 is complete.

The next lemma focuses on the dissipative effect of the electromagnetic field $[E(t, x), \tilde{B}(t, x)]$.

**Lemma 3.4.** For $N \geq 2$, there exists an interactive functional $G_{E, \tilde{B}}(t)$ satisfying
\[
G_{E, \tilde{B}}(t) \lesssim \sum_{|\alpha| \leq N} \left\| \partial^\alpha \left[ f, E, \tilde{B} \right] (t) \right\|^2
\]
such that the following estimate
\[
\frac{d}{dt} G_{E, \tilde{B}}(t) + \sum_{1 \leq |\alpha| \leq N-1} \left\| \partial^\alpha \left[ E, \tilde{B} \right] \right\|^2 + \left\| \{E + b \times \tilde{B}\} \right\|^2
\]
\[
\lesssim \|\nabla_x [b, a_+ - a_-]\|_{H^N_x}^2 + \|\{I - P\} f\|_{H^N_x L^2_t}^2 + \mathcal{E}_N(t) D_N(t) \tag{43}
\]
holds for all $0 \leq t \leq T$. 
Proof. From (29)_1, one has

\[
2E + 2b \times \mathfrak{B} = \frac{\partial t}{\partial t} G + \nabla_x (a_+ - a_-) + \nabla_x \cdot \mathfrak{A}(\{I - P\} f \cdot q_1)
- E(a_+ + a_-) - 2b \times \bar{B} - \left[ [v, -v]^{1/2}, Lf + \Gamma(f, f) \right].
\]

Multiplying the resulting identity by \(\{E + b \times \mathfrak{B}\}\), and integrating the result with respect to \(x\) over \(\mathbb{R}^3\), one can get that

\[
\begin{align*}
2 \| \{E + b \times \mathfrak{B}\} \|^2 &= 2 \left( \{E + b \times \mathfrak{B}\}, \{E + b \times \mathfrak{B}\} \right) \\
&\lesssim \| (\partial_t G(t), \{E + b \times \mathfrak{B}\}) \| \\
&+ \left( \| \nabla_x (a_+ - a_-) + \nabla_x \cdot \mathfrak{A}(\{I - P\} f(t) \cdot q_1) \|, \{E + b \times \mathfrak{B}\} \right) \\
&- \left( E(a_+ + a_-) + 2b \times \bar{B} + \left[ [v, -v]^{1/2}, Lf + \Gamma(f, f) \right], \{E + b \times \mathfrak{B}\} \right).
\end{align*}
\]

Performing \(\partial^\alpha\) to the above equation, multiplying the resulting identity by \(\partial^\alpha \{E + b \times \bar{B}\}\) with \(|\alpha| \leq N - 1\), and integrating the result with respect to \(x\) over \(\mathbb{R}^3\), one can get that

\[
\begin{align*}
2 \| \partial^\alpha E(t) \|^2 &= 2 \left( \partial^\alpha E(t), \partial^\alpha E(t) \right) \\
&\lesssim \| (\partial_t \partial^\alpha G(t), \partial^\alpha E(t)) \| + \| \partial^\alpha \{\nabla_x (a_+ - a_-) + \nabla_x \cdot \mathfrak{A}(\{I - P\} f(t) \cdot q_1)\}, \partial^\alpha E(t) \| \\
&- \left( \partial^{\alpha} \left( E(t) (a_+ + a_-) (t) + 2b(t) \times \bar{B}(t) \right), \partial^{\alpha} E(t) \right) \\
&- \left( \partial^{\alpha} \left( [v, -v]^{1/2}, Lf(t) + \Gamma(f(t), f(t)) \right), \partial^{\alpha} E(t) \right) \\
&- \left( \partial^{\alpha} \{2b(t) \times \mathfrak{B}\}, \partial^{\alpha} E(t) \right) \\
&\lesssim \left( \frac{d}{dt} (\partial^\alpha G(t), \partial^\alpha E(t)) - (\partial^\alpha G(t), \partial^\alpha \partial_t E(t)) + \eta \| \partial^\alpha E(t) \|^2 \right) \\
&+ \| \partial^\alpha b(t) \|^2 + \| \partial^\alpha (a_+ (t) - a_- (t)) \|^2_{H^2} + \| \partial^\alpha \{I - P\} f(t) \|^2_{H^2 L^2} + \mathcal{E}_N(t) \mathcal{D}_N(t) \\
&\lesssim \left( \frac{d}{dt} (\partial^\alpha G(t), \partial^\alpha E(t)) - \left( \partial^{\alpha} \{G(t)\}, \partial^{\alpha} \left\{ \nabla \times \bar{B}(t) - G(t) \right\} \right) + \eta \| \partial^\alpha E(t) \|^2 \right) \\
&+ \| \partial^\alpha b(t) \|^2 + \| \partial^\alpha (a_+ (t) - a_- (t)) \|^2_{H^2} + \| \partial^\alpha \{I - P\} f(t) \|^2_{H^2 L^2} + \mathcal{E}_N(t) \mathcal{D}_N(t) \\
&\lesssim \left( \frac{d}{dt} (\partial^\alpha G(t), \partial^\alpha E(t)) + \eta \| \partial^\alpha E(t) \|^2 \right) \\
&+ \sum_{j=1}^{3} \left( \left\| \partial^{\alpha + \epsilon_j} \{G(t)\} \right\|^2 + \eta \left\| \partial^{\alpha - \epsilon_j} \nabla \times \bar{B}(t) \right\|^2 \right) \\
&+ \| \partial^\alpha b(t) \|^2 + \| \partial^\alpha (a_+ (t) - a_- (t)) \|^2_{H^2} + \| \partial^\alpha \{I - P\} f(t) \|^2_{H^2 L^2} + \mathcal{E}_N(t) \mathcal{D}_N(t),
\end{align*}
\]
Lemma 3.5. There exists an interactive functional $G_{f,E,B}(t)$ satisfying

$$G_{f,E,B}(t) \lesssim \sum_{|\alpha| \leq N} \left\| \partial^\alpha \left[ f, E, B \right] \right\|^2$$

Combing Lemma 3.3 with Lemma 3.4 gives

**Lemma 3.5.** There exists an interactive functional $G_{f,E,B}(t)$ satisfying

$$G_{f,E,B}(t) \lesssim \sum_{|\alpha| \leq N} \left\| \partial^\alpha \left[ f, E, B \right] \right\|^2$$
such that
\[
\frac{d}{dt} G_{f,E,\tilde{B}}(t) + \| [a_+ - a_-, \{ E + b \times \mathcal{B} \}] \|^2 \\
+ \sum_{1 \leq |\alpha| \leq N} \| \partial_\alpha P f(t) \|^2 + \sum_{1 \leq |\alpha| \leq N-1} \| \partial_\alpha [E, \tilde{B}] \|^2 \\
\lesssim \| \{ I - P \} f(t) \|^2_{L^2_{\mu,b}} + \mathcal{E}_N(t) \mathcal{D}_N(t)
\]
holds for all \(0 \leq t \leq T\).

Remark 2. Instead of \(\| E \|^2\) the case with zero background magnetic field considered in all the previous results, it is easy to see \(\| E + b \times \mathcal{B} \|^2\) is included in the corresponding energy dissipation rate functional, which is due to the appearance of the term containing the background magnetic fields \(\mathcal{B}\).

3.2. Some basic estimates. To close the desired energy type estimates, we need to deduce certain refined energy type estimates. The first one is on term \(q_0 E(t) \cdot vf(t)\).

Lemma 3.6. It holds for \(0 \leq t \leq T\) that
\[
(q_0 E(t) \cdot vf(t), f(t)) \\
\lesssim \| E(t) \|^2_{H^1_x} \left\{ \| a_+ - a_-(t) \|^2 + \| \nabla x P f(t) \|^2 \right\} \\
+ \| E(t) \|^2_{L^\infty_x} \| \{ I - P \} f(t) \|^2 + \eta \| \{ I - P \} f(t) \|^2_{\nu} + \eta \| \nabla_x b(t) \|^2.
\]

Proof. Firstly, the macro-micro decomposition (14) tells us that
\[
(q_0 E(t) \cdot vf(t), f(t)) \\
= \left( q_0 E(t) \cdot P f(t), P f(t) \right) + \left( q_0 E(t) \cdot v \{ I - P \} f(t), P f(t) \right) \\
+ \left( q_0 E(t) \cdot v \{ I - P \} f(t), \{ I - P \} f(t) \right).
\]

For the estimates on \(I_j (j = 1, 2, 3, 4)\), one can get by employing Holder’s inequality and Sobolev’s inequality that
\[
I_2 + I_3 + I_4 \lesssim \| E(t) \|^2_{H^1_x} \| \nabla x P f(t) \|^2 + \| E(t) \|^2_{L^\infty_x} \| \{ I - P \} f(t) \|^2_{\nu} + \eta \| \{ I - P \} f(t) \|^2_{\nu},
\]
while for the first term \(I_1\), recalling the definition of \(P f\) in (13) and \(q_0\), one can deduce that
\[
I_1 \\
= \left( E \cdot v \{ a_+ + b \cdot v + c(|v|^2 - 3) \} \right) \mu^{1/2}, \left\{ a_+ + b \cdot v + c(|v|^2 - 3) \right\} \mu^{1/2} \\
- \left( E \cdot v \{ a_- + b \cdot v + c(|v|^2 - 3) \} \right) \mu^{1/2}, \left\{ a_- + b \cdot v + c(|v|^2 - 3) \right\} \mu^{1/2} \\
= 2 \left( E \cdot v a_+ \mu, b \cdot v \right) - 2 \left( E \cdot v a_- \mu, b \cdot v \right) \\
= 2 \left( E \cdot v (a_+ - a_-) \mu, b \cdot v \right) \\
\lesssim \| E(t) \|^2_{L^2_x} \| a_+(t) - a_-(t) \|^2_{L^2_x} + \| b(t) \|^2_{L^2_x} \\
\lesssim \| E(t) \|^2_{H^1_x} \| a_+(t) - a_-(t) \|^2 + \eta \| \nabla_x b(t) \|^2.
\]
Thus (48) follows by substituting the estimates on $I_1 \sim I_4$ into (49), then the proof is complete.

The next lemma is concerned with an identity with respect to the term $(v \times \mathfrak{B}) \cdot \nabla_v g(t, x, v)$, which will play an essential role in our subsequent analysis.

**Lemma 3.7.** For $|\alpha| + |\beta| \leq N$ with $|\beta| \geq 1$ it holds that

$$\sum_{1 \leq |\beta| \leq N} \left( \partial^\beta_{\beta} \left( (v \times \mathfrak{B}) \cdot \nabla_v g(t, x, v) \right), \partial^\beta_{\beta} g(t, x, v) \right) = 0. \quad (50)$$

**Proof.** We only need to treat the case $|\beta| = 1$ since the other cases can be proved similarly. In such a case, one can get for $\beta = (1, 0, 0)$ that

$$\left( \partial^\alpha_\alpha \left\{ (v_2 \mathfrak{B}_3 - v_1 \mathfrak{B}_3) \partial_\alpha_1 g + (v_1 \mathfrak{B}_1 - v_1 \mathfrak{B}_3) \partial_\alpha_2 g + (v_1 \mathfrak{B}_3 - v_2 \mathfrak{B}_3) \partial_\alpha_3 g \right\}, \partial^\alpha_\alpha g \right)$$

$$= \int_{\mathbb{R}_x^2 \times \mathbb{R}_v^2} (\mathfrak{B}_2 \partial^\alpha_\alpha g \partial^\alpha_\alpha g - \mathfrak{B}_3 \partial^\alpha_\alpha g \partial^\alpha_\alpha g) \, dv \, dx.$$

Similarly, one can also get that

$$\left( \partial^\alpha_\alpha \left\{ (v_2 \mathfrak{B}_3 - v_3 \mathfrak{B}_3) \partial_\alpha_1 g + (v_1 \mathfrak{B}_1 - v_3 \mathfrak{B}_3) \partial_\alpha_2 g + (v_1 \mathfrak{B}_3 - v_2 \mathfrak{B}_3) \partial_\alpha_3 g \right\}, \partial^\alpha_\alpha g \right)$$

$$= \int_{\mathbb{R}_x^2 \times \mathbb{R}_v^2} (\mathfrak{B}_1 \partial^\alpha_\alpha g \partial^\alpha_\alpha g - \mathfrak{B}_3 \partial^\alpha_\alpha g \partial^\alpha_\alpha g) \, dv \, dx$$

and

$$\left( \partial^\alpha_\alpha \left\{ (v_2 \mathfrak{B}_3 - v_3 \mathfrak{B}_3) \partial_\alpha_1 g + (v_2 \mathfrak{B}_1 - v_3 \mathfrak{B}_3) \partial_\alpha_2 g + (v_1 \mathfrak{B}_3 - v_2 \mathfrak{B}_3) \partial_\alpha_3 g \right\}, \partial^\alpha_\alpha g \right)$$

$$= \int_{\mathbb{R}_x^2 \times \mathbb{R}_v^2} (\mathfrak{B}_1 \partial^\alpha_\alpha g \partial^\alpha_\alpha g - \mathfrak{B}_2 \partial^\alpha_\alpha g \partial^\alpha_\alpha g) \, dv \, dx.$$

Thus, one can easily deduce from the above identities that

$$\sum_{|\beta| = 1} \left( \partial^\beta_{\beta} \left( (v \times \mathfrak{B}) \cdot \nabla_v g \right), \partial^\beta_{\beta} g \right) = 0,$$

this verifies (50) for the case $|\beta| = 1$ and the case for general multi-index $\beta$ can be proved similarly, we omit the details for brevity.

For later use, we need the following result with respect to micro-projection.

**Lemma 3.8.** It holds that

$$\{I - P\} \left\{ E \cdot v \mu^{1/2} q_1 - q_0 (v \times \mathfrak{B}) \cdot \nabla_v f \right\} = \{E + b \times \mathfrak{B}\} \cdot v \mu^{1/2} q_1 \quad (51)$$

**Proof.** By applying the definition of $P$, one has

$$\{I - P\} E \cdot v \mu^{1/2} q_1 = E \cdot v \mu^{1/2} q_1 - P E \cdot v \mu^{1/2} q_1 = E \cdot v \mu^{1/2} q_1$$
Similarly, one also has
\[
\begin{align*}
\{I - P\} \left\{-q_0 (v \times \mathfrak{B}) \cdot \nabla_v Pf\right\} \\
= \{I - P\} \left\{-(v \times \mathfrak{B}) \cdot \nabla_v \left\{b \cdot v \mu^{1/2} q_1\right\}\right. \\
= \{I - P\} \left\{-(v \times \mathfrak{B}) \cdot b \mu^{1/2} q_1\right\} \\
= \{I - P\} \left\{(b \times \mathfrak{B}) \cdot b \mu^{1/2} q_1\right\} \\
= (b \times \mathfrak{B}) \cdot b \mu^{1/2} q_1 - P \left\{(b \times \mathfrak{B}) \cdot b \mu^{1/2} q_1\right\} \\
= (b \times \mathfrak{B}) \cdot b \mu^{1/2} q_1
\end{align*}
\]

Therefore, one has (51) from (52) and (53).

\[\square\]

3.3. Lyapunov-type energy inequalities for \(\mathcal{E}_N(t)\). With the above results in hand, we now turn to deduce the key a priori estimates on the solution

\[
\left[f_+(t, x, v), f_-(t, x, v), E(t, x), \vec{B}(t, x)\right]
\]

of the Cauchy problem (6), (7), (8) which has been defined on the time interval \([0, T]\) for some positive constant \(T > 0\).

**Lemma 3.9.** Let \(N \geq 3\), it holds that

\[
\frac{d}{dt} \mathcal{E}_N(t) + \mathcal{D}_N(t) \lesssim \left(\mathcal{E}_N(t) + \sqrt{\mathcal{E}_N(t)}\right) \mathcal{D}_N(t)
\]

for any \(0 \leq t \leq T\).

**Proof.** To simplify the presentation, the proof of this lemma is divided into the following two steps:

**Step 1:** The first step is concerned with the energy type estimates on \(\partial^\alpha f\). To this end, multiplying (9) by \(f\) and integrating the resulting identity with respect to \(v\) and \(x\) over \(\mathbb{R}^3 \times \mathbb{R}^3\), one can get first by employing the estimate (48) and by using Lemma 2.1 and Lemma 2.2 that

\[
\begin{align*}
\frac{d}{dt} \left\|f, E, \vec{B}\right\|_{L^2}^2 + \left\|\{I - P\} f(t)\right\|_\nu^2 \\
\lesssim \left\|E(t)\right\|_{H^2}^2 \left\\{\|a_+(t) - a_-(t)\|^2 + \left\|\nabla_x Pf(t)\right\|^2\right\} \\
+ \left\|E(t)\right\|_{L^\infty}^2 \left\\{\|\{I - P\} f(t)\|^2 + \eta\left\|\{I - P\} f(t)\right\|^2 + \eta\left\|\nabla_x b(t)\right\|^2 + \mathcal{E}_2(t) \mathcal{D}_2(t)\right\}. \tag{55}
\end{align*}
\]

Next for the corresponding high order estimates, one can get by applying \(\partial^\alpha\) with \(1 \leq |\alpha| \leq N\) to (9), multiplying the resulting identity by \(\partial^\alpha f\), integrating the final result with respect to \(v\) and \(x\) over \(\mathbb{R}^3 \times \mathbb{R}^3\), and by using Lemma 2.1 that

\[
\begin{align*}
\frac{d}{dt} \left\|\partial^\alpha \left(f, E, \vec{B}\right)\right\|_\nu^2 + \left\|\partial^\alpha \{I - P\} f(t)\right\|_\nu^2 \\
\lesssim \frac{1}{2} \left(\partial^\alpha \left\{q_0 E(t) \cdot v f(t)\right\}\right)_{J_1} + \left(\partial^\alpha \left\{q_0 \left(E(t) + v \times \vec{B}(t)\right) \cdot \nabla_v f\right\}\right)_{J_2} \\
- \left(\partial^\alpha \left\{q_0 (v \times \mathfrak{B}) \cdot \nabla_v f(t)\right\}\right)_{J_3} + \left(\partial^\alpha \Gamma (f(t), f(t))\right)_{J_4}.
\end{align*}
\]
For the estimates of $J_i (i = 1, 2, 3, 4)$, one can get from Lemma 2.2 that

$$J_1 = \sum_{\alpha_1 \leq \alpha} (q_0 \partial^{\alpha_1} E(t) \cdot v \partial^{\alpha - \alpha_1} f(t), \partial^{\alpha} f(t))$$

$$\lesssim (q_0 E(t) \cdot v \partial^{\alpha} f(t), \partial^{\alpha} f(t)) + \sum_{1 \leq |\alpha_1| \leq |\alpha| - 1} (q_0 \partial^{\alpha_1} E(t) \cdot v \partial^{\alpha - \alpha_1} f(t), \partial^{\alpha} f(t))$$

$$+ \sum_{\alpha_1 = \alpha} (q_0 \partial^{\alpha} E(t) \cdot v f(t), \partial^{\alpha} f(t))$$

$$\lesssim \left( \|E(t)\|_{H^N} + \eta \right) \left\langle (v)^{1/2} \nabla_x f(t) \right\rangle^2_{H^{N-1}} ,$$

$$J_2 \lesssim \left\| \left( E, \tilde{B} \right)(t) \right\|^2_{H^N} \left\| \nabla_x Pf(t) \right\|^2_{H^{N-1}} + \eta \left\langle (v)^{1/2} \nabla_x f(t) \right\rangle^2_{H^{N-1}}$$

$$+ \left\| \left( E, \tilde{B} \right)(t) \right\|^2_{H^N} \left\| \nabla_v \{I - P\} f(t) \right\|^2_{H^{N-1}},$$

and

$$J_4 \lesssim \left( \mathcal{E}_N(t) + \sqrt{\mathcal{E}_N(t)} \right) \mathcal{D}_N(t).$$

Noticing further that $J_3 = 0$ since $\mathfrak{B}$ is a constant vector, we can substitute the above estimates on $J_1 \sim J_4$ into (56) to get that

$$\frac{d}{dt} \left\| \partial^{\alpha} \left( f, E, \tilde{B} \right)(t) \right\|^2 + \left\| \partial^{\alpha} \{I - P\} f(t) \right\|^2_\nu$$

$$\lesssim \left\| \left( E, \tilde{B} \right)(t) \right\|^2_{H^N} \left\| \nabla_v \{I - P\} f(t) \left\langle (v)^{1/2} \right\rangle \right\|^2_{H^{N-1}} + \left( \mathcal{E}_N(t) + \sqrt{\mathcal{E}_N(t)} \right) \mathcal{D}_N(t)$$

holds for $1 \leq |\alpha| \leq N$.

**Step 2:** The second step is concerned with the estimates on the microscopic component $\{I - P\} f$. For this purpose, by applying $I - P$ to (9), one can get the following equations for the microscopic component $\{I - P\} f$ by employing the identity (51)

$$\partial_t \{I - P\} f + v \cdot \nabla_x \{I - P\} f + L f = \{E + b \times \mathfrak{B} \} \cdot v \mu^{1/2} q_1 - \{I - P\} \{q_0 (v \times \mathfrak{B}) \cdot \nabla_v \{I - P\} f\}$$

$$- v \cdot \nabla_x Pf + \{v \cdot \nabla_x f\} + \{I - P\} \tilde{G}$$

(57)

with

$$\tilde{G} = \frac{q_0}{2} E \cdot v f - q_0 \left( E + v \times \tilde{B} \right) \cdot \nabla_v f + \Gamma(f, f),$$

(58)

Applying $\partial_\beta^{\alpha}$ to (57), multiplying the result by $\partial_\beta^{\alpha} \{I - P\} f$ with $1 \leq |\beta| = n \leq N$ and integrating the final resulting identity with respect to $v$ and $x$ over $\mathbb{R}_0^3 \times \mathbb{R}_x^3$,
one has
\[
\frac{d}{dt} \sum_{|\beta| = n} \left\| \partial^\alpha_{\beta} (I - P) f(t) \right\|^2 + \sum_{|\beta| = n} \left\| \partial^\alpha_{\beta} (I - P) f(t) \right\|_\nu^2 \\
\lesssim \eta \sum_{|\beta'| < |\beta|} \left\| \partial^\alpha_{\beta'} (I - P) f(t) \right\|_\nu^2 + \sum_{|\beta| = n} \left\| \partial^\alpha_{\beta + \epsilon_i} (I - P) f(t) \right\|_\nu^2 \\
+ \left\| \nabla_x^{[\alpha + 1]} P f(t) \right\|^2 + \sum_{|\beta| = n} \left( \partial^\alpha_{\beta} \left\{ \{E + b \times \mathbb{B}\} \cdot v \mu^{1/2} q_1 \right\}, \partial^\alpha_{\beta} (I - P) f(t) \right).
\]

\[
\sum_{|\beta| = n} \left( \partial^\alpha_{\beta} (I - P) G(t), \partial^\alpha_{\beta} (I - P) f(t) \right).
\]

To estimate \( J_i (i = 5, 6, 7) \), we first get from Lemma 2.2 that \( J_7 \) can be estimated as in the following:
\[
J_7 \lesssim \left( \mathcal{E}_N(t) + \sqrt{\mathcal{E}_N(t)} \right) \mathcal{D}_N(t).
\]

For \( J_5 \), we can get by integration by parts and Cauchy’s inequality that
\[
J_5 = \sum_{|\beta| = n} (-1)^n \left( \partial^\alpha_{\beta} \left\{ \{E + b \times \mathbb{B}\} \cdot v \mu^{1/2} q_1 \right\}, \partial^\alpha_{\beta} (I - P) f(t) \right)
\]
\[
\lesssim \eta \left\| \partial^\alpha \{E + b \times \mathbb{B}\} \right\|^2 + \left\| \partial^\alpha (I - P) f(t) \right\|_\nu^2.
\]

\( J_6 = 0 \) follows by employing Lemma 3.7.

Plugging the above estimates on \( J_5, J_6 \) and \( J_7 \) into (60) gives
\[
\frac{d}{dt} \sum_{|\beta| = n} \left\| \partial^\alpha_{\beta} (I - P) f(t) \right\|^2 + \sum_{|\beta| = n} \left\| \partial^\alpha_{\beta} (I - P) f(t) \right\|_\nu^2 \\
\lesssim \eta \sum_{|\beta'| < |\beta|} \left\| \partial^\alpha_{\beta'} (I - P) f(t) \right\|_\nu^2 + \sum_{|\beta| = n} \left\| \partial^\alpha_{\beta + \epsilon_i} (I - P) f(t) \right\|_\nu^2 \\
+ \left\| \partial^\alpha (I - P) f(t) \right\|^2 + \left\| \nabla_x^{[\alpha + 1]} P f(t) \right\|^2 + \eta \left\| \partial^\alpha \{E + b \times \mathbb{B}\} \right\|^2 + \left( \mathcal{E}_N(t) + \sqrt{\mathcal{E}_N(t)} \right) \mathcal{D}_N(t).
\]

Taking the summation over \( \{ |\beta| = n, |\alpha| + |\beta| \leq N \} \) for each given \( 1 \leq n \leq |\alpha| + |\beta| \), and then taking the proper linear combination of those estimates with properly chosen constants, one has
\[
\frac{d}{dt} \sum_{|\alpha| + |\beta| \leq N, |\beta| \geq 1} \left\| \partial^\alpha_{\beta} (I - P) f(t) \right\|^2 + \sum_{|\alpha| + |\beta| \leq N, |\beta| \geq 1} \left\| \partial^\alpha_{\beta} (I - P) f(t) \right\|_\nu^2 \\
\lesssim \left\| (I - P) f(t) \right\|_{H^N_x L^2_t}^2 + \left\| \nabla_x P f(t) \right\|_{H^N_x L^2_t}^2 + \eta \left\| E + b \times \mathbb{B} \right\|_{H^N_x L^2_t}^2 + \left( \mathcal{E}_N(t) + \sqrt{\mathcal{E}_N(t)} \right) \mathcal{D}_N(t).
\]
A proper linear combination of (47), (55), (56) and (61) yields (54). This completes the proof of Lemma 3.9. □

3.4. The proof of main theorem. Now we are ready to deduce the desired uniform estimate which is sufficient to extend the local solutions step by step to a global one. To this end, we only need to close the a priori assumption

$$\sum_{|\alpha|+|\beta| \leq N} \left\| \partial_\beta^\alpha f(t) \right\|^2 + \left\| \left( E, \tilde{B}\right)(t) \right\|_{H^N_x}^2 \leq M, \quad 0 \leq t \leq T. \quad (61)$$

Here $M > 0$ is a suitably chosen sufficiently small positive constant.

In fact, under the a priori assumption (61), from the estimate (3.9), one can deduce that if $\mathcal{E}_N(0)$ is assumed to be sufficiently small, then

$$\mathcal{E}_N(t) \leq \mathcal{E}_N(0) \quad (62)$$

holds for all $0 \leq t \leq T$ and thus one can close the a priori assumption (61).

Having obtained (62), one can then deduce the global solvability result for the Cauchy problem (6), (7), (8) by the continuation argument. This completes the proof of Theorem 1.1.

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