A Survey on Springer theory (over $\mathbb{C}$)

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Abstract

A Springer map is for us a union of collapsings of (complex) homogeneous vector bundles and a Steinberg variety is just the cartesian product of a Springer map with itself. Ginzberg constructed on the (equivariant) Borel-Moore homology and on the (equivariant) $K$-theory of a Steinberg variety a convolution product making it an associative algebra, we call this a Steinberg algebra. The decomposition theorem for perverse sheaves gives the indecomposable, projective graded modules over the Steinberg algebra. Also Ginzberg’s convolution yields a module structure on the respective homology groups of the fibres under the Springer maps, which we call Springer fibre modules. In short, for us a Springer theory is the study of a Steinberg algebra together with its graded modules.

We give two examples: Classical Springer theory and quiver-graded Springer theory.

(1) Definitions and basic properties.
(2) Examples
   (a) Classical Springer Theory.
   (b) Quiver-graded Springer Theory.
(3) We discuss literature on the two examples.

Definition of a Springer Theory

Roughly, following the introduction of Chriss and Ginzburg’s book ([CG97])¹, Springer Theory is a uniform geometric construction for a wide class of (non-commutative) algebras together with families of modules over these algebras. Examples include

(1) Group algebras of Weyl groups together with their irreducible representations,
(2) affine Hecke algebras together with their standard modules and irreducible representations,
(3) Hecke algebras with unequal parameters,
(4) KLR-algebras (= Quiver Hecke algebras)
(5) Quiver Schur algebras

To understand this construction, recall for any algebraic group $G$ and closed subgroup $P$ (over $\mathbb{C}$) we call the principal bundles $G \rightarrow G/P$ homogeneous. For any $P$-variety $F$ given we have the associated bundle defined by the quotient

$$G \times^P F := G \times F/ \sim, \quad (g, f) \sim (g', f'): \iff \text{there is } p \in P: (g, f) = (g'p, p^{-1}f')$$

and $G \times^P F \rightarrow G/P, (g, f) \mapsto gP$. Given a representation $\rho: P \rightarrow \text{Gl}(F)$, i.e. a morphism of algebraic groups, we call associated bundles of the form $G \times^P F \rightarrow G/P$ homogeneous vector bundles.

Definition 1. The uniform geometric construction in all cases is given by the following: Given $(G, P_i, V, F_i)_{i \in I}$ with $I$ some finite set,

\[
\begin{cases}
\text{(1) } G \text{ a connected reductive group with parabolic subgroups } P_i, \\
\text{We also assume there exists a maximal torus } T \subset G \text{ which is contained in every } P_i, \\
\text{(2) } V \text{ a finite dimensional } G\text{-representation, } F_i \subset V \text{ a } P_i\text{-subrepresentation of } V, \ i \in I.
\end{cases}
\]

¹We take a more general approach, what usually is considered as Springer theory you find in the example classical Springer theory. Nevertheless, our approach is still only a special case of [CG97], chapter 8.
We identify $V, F_i$ with the affine spaces having the vector spaces as $\mathbb{C}$-valued points and consider the following morphisms of algebraic varieties\(^2\), let $E_i := G \times^P F_i, i \in I$

\[
E := \bigsqcup_{i \in I} E_i \quad \quad \quad \quad \pi := \bigrightrightarrow \quad \mu := \bigrightharpoonup
\]

Then, $E \to V \times \bigsqcup_{i \in I} G/P_i, [(g, f_i)] \mapsto (gf_i, g P_i)$ is a closed embedding (see [Slo80b], p.25,26), it follows that $\pi$ is projective. We call the algebraic correspondence\(^3\) $(E, \pi, \mu)$ Springer triple, the map $\pi$ Springer map, its fibres Springer fibres. Via restriction of $E \to V \times \bigsqcup_{i \in I} G/P_i$ to $\pi^{-1}(x) \to \{x\} \times \bigsqcup_{i \in I} G/P_i$ one sees that all Springer fibres are via $\mu$ closed subschemes of $\bigsqcup_{i \in I} G/P_i$.

We also have another induced roof-diagramm

\[
Z := E \times_V E \quad \quad \quad \quad V \quad \quad \quad \quad \quad \quad \quad \quad Z := (\bigsqcup_{i \in I} G/P_i) \times (\bigsqcup_{i \in I} G/P_i)
\]

with $p: E \times_V E' \xrightarrow{pr_E} E \quad \pi : V$ projective and $m: E \times_V E' \xrightarrow{(pr_E, pr_E)} E \times E \xrightarrow{\mu \times \mu} (\bigsqcup_{i \in I} G/P_i) \times (\bigsqcup_{i \in I} G/P_i)$. Observe, by definition

$Z = \bigsqcup_{i,j \in I} Z_{i,j}$, \quad $Z_{i,j} = E_i \times_V E_j$.

We call the roof-diagram $(Z, p, m)$ Steinberg triple, the scheme $Z$ Steinberg variety (even though as a scheme $Z$ might be neither reduced nor irreducibel). But in view of our (co-)homology choice below we only study the underlying reduced scheme and look at its $\mathbb{C}$-valued points endowed with the analytic topology.

If all parabolic groups $P_i$ are Borel groups, the Steinberg variety $Z$ is an iterated cellular fibration over $\bigsqcup_{i \in I} G/P_i$ (for an apropiate definition of iterated cellular fibration), for the precise statement see the next lemma. We choose a (co-)homology theory which can be calculated for spaces with cellular fibration property and which has a localization to the $T$-fixpoint theory. Let $H^A_*, A \in \{ pt, T, G \}$ be (A-equivariant) Borel-Moore homology. We could also choose (equivariant) $K$-theory, but we just give some known results about it.

There is a natural product $\ast$ on $H^A_*(Z)$ called convolution product constructed by Chriss and Ginzburg in [CG97].

\[
\ast : H^A_*(Z) \times H^A_*(Z) \to H^A_*(Z) \quad (c_{1,2}, c_{2,3}) \mapsto c_{1,2} \ast c_{2,3} := (q_{1,3})_*(p_{1,2}^*(c_{1,2}) \cap p_{2,3}^*(c_{2,3}))
\]

where $\cap$: $H^A_*(X) \times H^A_*(Y) \to H^A_{p+q-2d}(X \cap Y)$ is the intersection pairing which is induced by the $\cup$-product in relative singular cohomology for $X, Y \subset M$ two $A$-equivariant closed subsets of a $d$-dimensional complex manifold $M$ (cp. [CG97], p.98, (2.6.16)) and where $p_{a,b}: E \times E \times E \to E \times E$ is the projection on the $a, b$-th factors, $q_{a,b}$ is the restriction of $p_{a,b}$ to $E \times_V E \times_V E$. It holds

\[
H^A_*(Z_{i,j}) \ast H^A_*(Z_{k,l}) \subset \delta_{j,k} H^A_{p+q-e_k}(Z_{i,l}), \quad e_k = \dim \mathbb{C} E_k.
\]

We call $(H^A_*(Z), \ast)$ the (A-equivariant) Steinberg algebra for $(G, P_i, V, F_i)_{i \in I}$.

There is a following identification.

\(^2\)algebraic variety = separated integral scheme of finite type over a field

\(^3\)two scheme morpisms $X \xrightarrow{p} Z \xrightarrow{q} Y$ are called algebraic correspondence, if $p$ is proper and $q$ is flat
Theorem 0.1. ([CG97], chapter 8) Let $A \in \{pt, T, G\}$ we write $e_i = \dim_C E_i$. There is an isomorphism of $\mathbb{C}$-algebras

$$H^*_A(Z) \rightarrow \text{Ext}^{D^b_A(V)}(\bigoplus_{i \in I} (\pi_i)_* \mathbb{C}[e_i], \bigoplus_{i \in I} (\pi_i)_* \mathbb{C}[e_i]).$$

If we set

$$H^A_{[p]}(Z) := \bigoplus_{i,j \in I} H^A_{e_i+e_j-p}(Z_{i,j})$$

then $H^A_{[p]}(Z)$ is a graded $H^A_{[pt]}$-algebra and the isomorphism is an isomorphism of graded algebras. Furthermore, the Verdier duality on $D^b_A(V)$ induces an anti-involution on this algebra.

The proof is only given for $A = pt$, but as Varagnolo and Vasserot in [Var09] observed, the same proof can be rewritten for the $A$-equivariant case.

Convolution Modules (see [CG97], section 2.7)

Given two subsets $S_{1,2} \subset M_1 \times M_2$, $S_{2,3} \subset M_2 \times M_3$ the set-theoretic convolution is defined as

$$S_{1,2} \circ S_{2,3} := \{(m_1, m_3) \mid \exists m_2 \in M_2 : (m_1, m_2) \in S_{1,2}, (m_2, m_3) \in S_{2,3}\} \subset M_1 \times M_3.$$

Now, let $S_{i,j} \subset M_i \times M_j$ be $A$-equivariant locally closed subsets of smooth complex $A$-varieties, let $p_{i,j} : M_1 \times M_2 \times M_3 \rightarrow M_i \times M_j$ be projection on the $(i, j)$-th factors and assume $q_{1,3} := p_{1,3}|_{p_{1,2}(S_{1,2}) \cap p_{2,3}(S_{2,3})}$ is proper. Then we get a map

$$*: H^A_{[p]}(S_{1,2}) \times H^A_{[q]}(S_{2,3}) \rightarrow H^A_{[p+q-2 \dim_C M_2]}(S_{1,2} \circ S_{2,3})$$

$$c_{1,2} \ast c_{2,3} := (q_{1,3})_* (p_{1,2} c_{1,2} \cap p_{2,3} c_{2,3}).$$

This way we defined the algebra structure on the Steinberg algebra, but it also gives a left module structure on $H^A_{[p]}(S)$ for any $A$-variety $S$ with $Z \circ S = S$ and a right module structure when $S \circ Z = Z$.

(a) $M_1 = M_2 = M_3 = E$, embed $Z = E \times V \subset E \times E$, $E = E \times pt \subset E \times E$, then it holds $Z \circ E = E$. If we regrade the Borel-Moore homology (and the Poincaré dual $A$-equivariant cohomology) of $E$ as follows

$$H^A_{[p]}(E) := \bigoplus_{i \in I} H^A_{e_i-p}(E_i) \quad (= \bigoplus_{i \in I} H^A_{e_i+p}(E_i) =: H^A_{[p]}(E))$$

then $H^A_{[p]}(E)$ and $H^A_{[q]}(E)$ carry the structure of a graded left $H^A_{[p]}(Z)$-module.

(b) $M_1 = M_2 = M_3 = E$, embed $E \subset E \times E$ diagonally, then $Z \circ E = E$, it holds $H^A_{[p]}(E) = H^A_{[q]}(E)$ as graded algebras where $H^A_{[p]}(E) := \bigoplus_i H^A_{2e_i-p}(E_i)$ and the ring structure on the cohomology is given by the cup product. If we take now $Z = E \times V \subset E \times E$ then $E \circ Z = Z$ and we get a structure as graded left $H^A_{[p]}(E)$-module on $H^A_{[p]}(Z)$.

(c) $M_1 = M_2 = M_3 = E$, $A = pt$ embed $Z = E \times V \subset E \times E$, $\pi^{-1}(x) = \pi^{-1}(x) \times pt \subset E \times E$, then it holds $Z \circ \pi^{-1}(x) = E$. If we regrade the Borel-Moore homology and singular cohomology of $\pi^{-1}(x)$ as follows

$$H^A_{[p]}(\pi^{-1}(x)) := \bigoplus_{i \in I} H^A_{e_i-p}(\pi^{-1}(x)) \quad H^A_{[p]}(\pi^{-1}(x)) := \bigoplus_{i \in I} H^A_{e_i+p}(\pi^{-1}(x))$$

then $H^A_{[p]}(\pi^{-1}(x))$ and $H^A_{[q]}(\pi^{-1}(x))$ are graded left $H^A_{[p]}(Z)$-module.

We call these the Springer fibre modules.

Similarly in all examples one can obtain right module structure (the easy swaps are left to the reader). Independently, one can define the same graded module structure on $H_*(\pi^{-1}(x)), H^*(\pi^{-1}(x))$ using the description of the Steinberg algebra as Ext-algebra and a Yoneda operation (for this see [CG97], 8.6.13, p.448 ).

There is also a result of Joshua (see [Jos98]) saying that all hypercohomology groups $H^*_A(Z, F^\bullet), F^\bullet \in D^b_A(Z)$ carry the structure of a left (and right) $H^A_{[p]}(Z)$-module.
The Steinberg algebra

The Steinberg algebra \( H^A_{[w]}(Z) \) as module over \( H^*_A(pt) \).

We set \( \tilde{W} := \bigsqcup_{i,j \in I} W_{i,j} \) with \( W_{i,j} := W_i \setminus W/W_i \) where \( W \) is the Weyl group for \((G,T)\) and \( W_i \subset W \) is the Weyl group for \((L_i,T)\) with \( L_i \subset P_i \) is the Levi subgroup. We will fix representatives \( w \in G \) for all elements \( w \in \tilde{W} \).

Let \( C_w = G \cdot (eP_i,wP_i) \) be the \( G \)-orbit in \( G/P_i \times G/P_i \) corresponding to \( w \in W_{i,j} \).

**Lemma 1.**

1. \( p: C_w \subset G/P_i \times G/P_j \xrightarrow{pr_1} G/P_i \) is \( G \)-equivariant, locally trivial with fibre \( p^{-1}(eP_i) = P_i w P_j / P_j \).

2. \( P_i w P_j / P_j \) admits a cell decomposition into affine spaces via Schubert cells \( x B_j x^{-1} v P_j / P_j, v \in W_i \) (and for a fixed \( x \in W \) such that \( x P_j = P_i, B_j \subset P_j \) the Borel subgroup). In particular, \( H_{\text{odd}}(P_i w P_j / P_j) = 0 \) and

\[
H_*(P_i w P_j / P_j) = \bigoplus_{v \in W_i} C_{b_{i,j}}(v), \quad b_{i,j}(v) := [x B_j x^{-1} v P_j / P_j].
\]

It holds \( \deg b_{i,j}(v) = 2 \ell_{i,j}(v) \) where \( \ell_{i,j}(v) \) is the length of a minimal coset representative in \( W \) for \( x^{-1} v W_j \in W/W_j \).

3. For \( A \in \{pt,T,G\} \) it holds \( H^A_{\text{odd}}(C_w) = 0 \) and since \( G/P_i \) is simply connected

\[
H^A_n(C_w) = \bigoplus_{p+q=n} H^p_A(G/P_i) \otimes H_q(C_w), \quad H^A_*(C_w) = \bigoplus_{u \in W_i,v \in W_i} C_{b_i}(u) \otimes b_{i,j}(v),
\]

where \( b_i(u) = [B_u P_i / P_i]^t \) is of degree \( 2 \dim G/P_i - 2 \ell_i(u) \) with \( \ell_i(u) \) is the length of a minimal coset representative for \( u \in W/W_i \) and \( b_{i,j}(v) \) as in (2).

**proof:** Is left out because it is standard techniques. This implies the following properties for the homology of \( Z \).

**Lemma 2.**

1. \( Z \) has a filtration by closed \( G \)-invariant subvarieties such that the successive complements are \( Z_w := m^{-1}(C_w), w \in \tilde{W} \) and the restriction of \( m \) to \( Z_w \) is a vector bundle over \( C_w \) of rank \( d_w \) (as complex vector bundle). Furthermore,

\[
H^A_n(Z) = \bigoplus_{w \in \tilde{W}} H^A_n(Z_w) = \bigoplus_{w \in \tilde{W}} H^A_{n-2d_w}(C_w)
\]

\[
= \bigoplus_{i,j \in I} \bigoplus_{w \in W_{i,j}} \bigoplus_{u \in W/W_i,v \in W_j} C_{b_i}(u) \otimes b_{i,j}(v)
\]

where the last direct sum goes over the \( u,v \) with the property \( 2 \dim G/P_i - 2 \ell_i(u) + 2 \ell_{i,j}(v) = n - 2d_w \).

2. \( H_{\text{odd}}(Z) = 0, H^\text{odd}(Z) = 0. \)

3. \( Z \) is equivariantly formal (for \( T \) and \( G \), for Borel-Moore homology and cohomology). In particular, for \( A \in \{T,G\} \) the following forgetful maps are surjective \( H^*_A(Z) \to H_*(Z) \) and \( H^*_A(Z) \to H^*(Z) \) algebra homomorphisms. It even holds the stronger isomorphism of \( \mathbb{C} \)-algebras

\[
H_*(Z) = H^A_*(Z) / H^A_{0}(pt) H^A_*(Z)
\]

\[
H^*(Z) = H^A_*(Z) / H^A_{0}(pt) H^A_*(Z)
\]

As a consequence we get the following isomorphisms.

1. \( H^A_*(Z) = H_*(Z) \otimes_{\mathbb{C}} H^A_*(pt) \) of \( H^A_*(pt) \)-modules

2. \( H^*_A(Z) = H^*(Z) \otimes_{\mathbb{C}} H^*_A(pt) \) of \( H^*_A(pt) \)-modules

We can see that \( H^A_{[w]}(Z) \) as finite dimensional graded pieces and the graded pieces are bounded from below in negative degrees.
The Steinberg algebra $H^*_A(Z)$ and $H^*_A(E)$

Recall from a previous section that $H^*_A(E)$ is a graded left (and right) $H^*_A(Z)$-module and that $H^*_A(E)$ has a $H^*_A(pt)$-algebra structure with respect to the cup product, the $H^*_A(Z)$-operation is $H^*_A(pt)$-linear.

**Remark.** Let $q_i: E_i \to pt, \ i \in I$, there is an isomorphism of algebras

$$\text{End}_{H^*_A(pt)}(H^*_A(E)) = H^*_A(E \times E) = \text{Ext}^*_D(\bigoplus_{i \in I} (q_i)_* \mathbb{C}[e_i], \bigoplus_{i \in I} (q_i)_* \mathbb{C}[e_i]),$$

the first equality follows from [CG97], Ex. 2.7.43, p.123, for the second first use the Thom isomorphism to replace $E \times E$ by a union of flag varieties, then use theorem 0.1 from Chriss and Ginzburg for the Springer map given by the projection to a point.

Furthermore, under the identifications, the following three algebra homomorphisms are equal.

1. The map $H^*_A(Z) \to \text{End}_{H^*_A(pt)}(H^*_A(E)), c \mapsto (e \mapsto c * e)$.
2. $i_*: H^*_A(Z) \to H^*_A(E \times E)$ where $i: Z \to E \times E$ is the natural embedding.
3. $\text{Ext}^*_D(\bigoplus_{i \in I} (\pi_i)_* \mathbb{C}[e_i], \bigoplus_{i \in I} (\pi_i)_* \mathbb{C}[e_i]) \to \text{Ext}^*_D(\bigoplus_{i \in I} (\pi_i)_* \mathbb{C}[e_i], \bigoplus_{i \in I} (\pi_i)_* \mathbb{C}[e_i]),$

$$f \mapsto a_*(f)$$

where $a: V \to pt$.

**Lemma 3.** ([VV11], remark after Prop.3.1, p.12) Assume that $T \subset \bigcap_i P_i$ is a maximal torus and $Z^T = E^T \times E^T, E^T = \bigcup_{i \in I} (G/P_i)^T$. Let $A \in \{T, G\}$. There is an injective homomorphism of $H^*_A(pt)$-algebras

$$H^*_A(Z) \to \text{End}_{H^*_A(pt)}(H^*_A(E)),$$

Let $t$ be the Lie algebra of $T$, then it holds $H^*_G(E) \cong \mathbb{C}[t]^\oplus I$, where $\mathbb{C}[t]$ is the ring of regular functions on the affine space $t$.

**proof:** For $G$-equivariant Borel-Moore homology we claim that the following diagram is commutative

$$\begin{array}{ccc}
H^*_T(Z^T) & \longrightarrow & H^*_T(E^T \times E^T) \\
\uparrow & & \uparrow \\
H^*_T(Z) & \longrightarrow & H^*_T(E \times E) \\
\uparrow & & \uparrow \\
H^*_G(Z) & \longrightarrow & H^*_G(E \times E)
\end{array}$$

The commutativity of the lowest square uses functoriality of the forgetful maps. By assumption $Z^T = (E \times E)^T$, the highest horizontal map is an isomorphsim. Now, by the GKM-localization theorem the two vertical maps in the upper square are injective. That implies that the middle horizontal map has to be injective, together with (2) from the previous remark it implies the claim for $T$-equivariant Borel-Moore homology. But by the splitting principle, i.e. the identification of the $G$-equivariant Borel-Moore homology with the $W$-invariant subspace in the $T$-equivariant Borel-Moore homology, the forgetful maps become the inclusion of the $W$-invariant subspace. This means the two vertical maps in the lower square are injective. This implies that the lowest horizontal map is injective. Together, with (2) of the previous remark the claim follows for $A = G$. \qed

The main ingredient to the previous lemma is Goresky’s,Kottwitz’ and MacPherson’s localization theorem (see [GKM98]). Similar methods are currently developed by Gonzales for $K$-theory in [Gon].

The previous lemma is wrong for not equivariant Borel-Moore homology as the following example shows.
Example. Let $G$ be a reductive group with a Borel subgroup $B$ and $u$ be the Lie algebra of its unipotent radical. $Z := (G \times B u) \times_9 (G \times B u)$, then it holds that the algebra $H_\ast(Z)$ can under the isomorphism in Kwon (see [Kwo09]) be identified with $\mathbb{C}[t]/I_W \# \mathbb{C}[W]$ which is defined as the $\mathbb{C}$-vector space $\mathbb{C}[t]/I_W \otimes \mathbb{C}[W]$ with the multiplication $(f \otimes w) \cdot (g \otimes v) := fw(g) \otimes vw$. Furthermore, we can identify $\text{End}_\mathbb{C}(H_\ast(E))$ via the Thom-isomorphism and the Borel map with $\text{End}_{\mathbb{C}-\text{lin}}(\mathbb{C}[t]/I_W)$. The canonical map identifies with
\[
\mathbb{C}[t]/I_W \# \mathbb{C}[W] \rightarrow \text{End}_{\mathbb{C}-\text{lin}}(\mathbb{C}[t]/I_W)
\]
\[f \otimes w \mapsto (p \mapsto f(w(p)))
\]
This map is neither injective nor surjective. For example $\sum_{w \in W} 1 \otimes w \neq 0$ in $\mathbb{C}[t]/I_W \# \mathbb{C}[W]$ but its image $(p \mapsto \sum_{w \in W} w(p))$ is zero because $\sum_{w \in W} w(p) \in I_W$. Because both spaces have the same $\mathbb{C}$-vector space dimension, it is clear that it is also not surjective.

Furthermore, $H_\ast^A(Z)$ is a naturally a $H_\ast^A(E)$-module. It holds In fact, $H_\ast^A(E) \cong H_\ast^A(\bigcup_{v \in W, j} Z^{v,j})$ is even a subalgebra of $H_\ast^A(Z)$.

Corollary 0.1. In the situation of the previous lemma, i.e. $T \subset \bigcap_i P_i$ is a maximal torus and $Z^T = E^T \times E^T$, $E^T = \bigcup_i (G/P_i)^T$ and let $A \in \{T, G\}$. There are injective homomorphism of $H_\ast^A(\text{pt})$-algebras
\[H_\ast^A(\text{pt}) \subset H_\ast^A(E) \subset H_\ast^A(Z) \rightarrow \text{End}_{H_\ast^A(\text{pt})}(H_\ast^A(E)),\]
where the first inclusion is given by the pullback along the map $E \rightarrow \text{pt}$. In particular, $H_\ast^A(\text{pt})$ is contained in the centre of $H_\ast^A(Z)$ (we only know examples where it is equal to the centre).

Let $w \in \tilde{W}$. Observe, that $H_\ast^A(E)$ already operates on $H_\ast^A(Z^w)$ and the composition $H_\ast^A(Z) = \bigoplus_w H_\ast^A(Z^w)$ is a direct sum composition of $H_\ast^A(E)$-modules. Using the Thom-isomorphism, up to a degree shift we can also study $H_\ast^A(G^w)$ as module over $H_\ast^A(\bigcup_i G/P_i)$. Now, let $e_i$ be the idempotent in $H_\ast^A(\bigcup_i G/P_i) = \bigoplus_{i \in I} H_\ast^A(G/P_i)$ which corresponds to the projection on the $i$-th direct summand. Since for $w \in W_{i,j}$ it holds $H_\ast^A(C_w) = H_\ast^A(G/P_i) \otimes_C H_\ast^A(P_j w P_j/P_j)$ also as $H_\ast^A(G/P_i)$-module, we conclude that $H_\ast^A(C_w)$ is always a projective module over $H_\ast^A(\bigcup_i G/P_i)$.

Lemma 4. (1) Let $w \in W_{i,j}$. Each $H_\ast^A(Z^w)$ is a projective graded $H_\ast^A(E)$-module of the form
\[
\bigoplus_{v \in W_i} (H_\ast^A(E)e_i)[2d_v + \deg b_{i,j}(v)],
\]
where $[d]$ denotes the degree shift by $d$. In particular, $H_\ast^A(Z)$ is a projective graded $H_\ast^A(E)$-module.

(2) If all $P_i = B_i$ are Borel subgroups of $G$, then $H_\ast^A(Z) = \bigoplus_{w, j} W \times I (\bigoplus_{i \in I} (H_\ast^A(E)e_i)[d_{w, i,j}]$ as graded $H_\ast^A(E)$-module for certain $d_{w, i,j} \in \mathbb{Z}$. In particular, if we forget the grading $H_\ast^A(Z)$ is a free $H_\ast^A(E)$-module of rank $\#W \cdot \#I$.

Indecomposable projected graded modules over $H_\ast^A[Z]$ and their tops for a different grading
Let $X$ be an irreducible algebraic variety, we call a decomposition $X = \bigcup_{a \in A} S_a$ into finitely many irreducible smooth locally closed subsets a weak stratification. Since $\pi: E = \bigcup_{i \in I} E_i \rightarrow V$ is a $G$-equivariant projective map, there exists (and we fix it) a weak stratification into $G$-invariant subsets $V = \bigcup_{a \in A} S_a$ such that $\pi^{-1}(S_a) \rightarrow S_a$ is a locally trivial fibration with constant fibre $F_a := \pi^{-1}(s_a)$ where $s_a \in S_a$ one fixed point, for every $a \in A$. (For projective maps of complex algebraic varieties one can always find such a weak stratification, see [Ara01], 4.4.1-4.4.3)
Recall that for any $G$-equivariant projective map of complex varieties, the decomposition theorem (compare [BBD82] for the non-equivariant version and [BL94] for the equivariant version) holds. Let $A \in \{ pt, T, G \}$, we denote by $D^b_A(V)$ the $A$-equivariant derived category defined by Bernstein and Lunts in [BL94]. Let $r$ run over all irreducible $G$-equivariant local systems $L_t$ on some stratum $S_t = S_{a_t}$, $a_t \in A$, we write $IC^A_t := ([\tau^S_t], (IC^A(S_t, L_t))[d_{S_t}])$ with $d_{S_t} = \dim_{\mathbb{C} - \text{var}} S_t$ for the simple perverse sheaf in the category of $A$-equivariant perverse sheaves $Perv_A(V) \subset D^b_A(V)$, see again [BL94], p. 41. Let $e_i = \dim_{\mathbb{C}} E_i, i \in I$, then $\bigoplus E_i[e_i]$ is a simple perverse sheaf in $D^b_A(E)$. For a graded vector space $L = \bigoplus_{d \in \mathbb{Z}} L_d$ we define $L(n)$ to be the graded vector space with $L(n)_d := L_{n+d}$, we see $\mathbb{C}$ as the graded vector space concentrated in degree zero, $n \in \mathbb{Z}$. For any graded vector space concentrated in degree zero, $n \in \mathbb{Z}$, $L$ we define $\mathbb{C}(L) = \bigoplus_{d \in \mathbb{Z}} L_d \otimes_{\mathbb{C}} \mathbb{C}$.

For any graded simple module $X$ we write $F^i[X]$, for the (class of the) complex $(F^i[X])_d := F^{d+n}, n \in \mathbb{Z}$. Now given $F^i \in D^b_A(X)$ and a finite dimensional graded vector space $L := \bigoplus_{i=1}^\tau \mathbb{C}(d_i)$ we define

$$L \otimes_{gr} F^i := \bigoplus_{i=1}^\tau F^i[d_i] \in D^b_A(X)$$

The $A$-equivariant decomposition theorem applied to $\pi$ gives

$$\bigoplus_{i \in I} (\pi_i)_* \mathbb{C}[E_i] = \bigoplus_{t} L_t \otimes_{gr} IC^A_t \in D^b_A(V)$$

where the $L_t := \bigoplus_{d \in \mathbb{Z}} L_{t,d}$ are complex finite dimensional graded vector spaces.

Let $\mathbb{D}$ be the Verdier-duality on $V$, it holds $\mathbb{D}(\pi_* \mathbb{C}[d]) = \pi_* \mathbb{C}[d], \mathbb{D}(IC^A_t) = IC^A_t$ where for $t = (S, L)$ it holds $t^* := (S, L^*)$, $L^* := \text{Hom}(L, \mathbb{C})$. This implies $L_t = L_{t^*}$ for all $t$.

Indecomposable projectives in the category of graded left $H^A_{[s]}(Z)$-modules

We set

$$P^A_t := \text{Ext}^{n+d}_{D^b_A(V)}(IC^A_t, \bigoplus_{i \in I} (\pi_i)_* \mathbb{C}[E_i]).$$

It is a graded (left) $H^A_{[s]}(Z)$-module. It is indecomposable because $IC^A_t$ is simple. Clearly it holds as left graded $H^A_{[s]}(Z)$-modules

$$H^A_{[s]}(Z) = \bigoplus_{d \in \mathbb{Z}, t} L_{t,d} \otimes \left[ \bigoplus_{n \in \mathbb{Z}} \text{Ext}^{n+d}_{D^b_A(V)}(IC^A_t, \bigoplus_{i \in I} (\pi_i)_* \mathbb{C}[E_i]) \right]$$

$$= \bigoplus_{d \in \mathbb{Z}, t} L_{t,d} \otimes \mathbb{C} P^A_t[d]$$

$$= \bigoplus_{t} L_t \otimes_{gr} P^A_t$$

that implies that $P^A_t$ is a projective module and that $(P^A_t)_t$ is a complete set of isomorphism classes up to shift of indecomposable projective graded $H^A_{[s]}(Z)$-modules.

Assume that $H^A_{[s]}(pt)$ is a graded subalgebra of the centre of $H^A_{[s]}(Z)$, compare corollary 0.1.

Lemma 5. The elements $H^A_{[s]}(pt)$ operate on any graded simple $H^A_{[s]}(Z)$-module $S$ by zero. In particular, by lemma 2 we see that $S$ is a graded simple modules over $H^A_{[s]}(Z)$. Any graded simple module is finite-dimensional and there exists up to isomorphism and shift only finitely many graded simple modules.

For any graded simple module $S$ there is no nonzero degree zero homomorphism $S \to S(a), a \neq 0$.

proof: Since $S$ is simple it holds $H^A_{[s]}(pt) \cdot S$ is zero or $S$. Assume it is $S$, pick a non-nilpotent element $x \in H^d_A(pt)$ and $y \in S, y \neq 0$, homogeneous. Then, it holds $S = H^A_{[s]}(Z) \cdot y = H^A_{[s]}(Z) \cdot x^n y$ contradicting the fact that $S$ has to have a minimal degree generator. Therefore $H^A_{[s]}(pt) \cdot S = 0$. 

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By [NO82], II.6, p.106, we know that the graded simple modules considered as modules over the ungraded rings $H^A_s(Z), H_s(Z)$ are still simple modules. Since the finite-dimensional algebra $H_s(Z)$ has up to isomorphism only finitely many simples, the claim follows.

Any nonzero degree 0 homomorphism $\phi: S \to S(a)$ has to be an isomorphism. Let $S = H^A_s(Z) \cdot y$ as before, set $\deg y = m$. Then $S(a) = H^A_s(Z) \cdot \phi(y), \deg \phi(y) = m$ which gives a contradictions when considering the minimal nonzero degrees of $S$ and $S(a)$. \hfill \Box

**Corollary 0.2.** There is a bijection between isomorphism classes up to shift of

1. indecomposable projective graded $H^A_s(Z)$-modules
2. indecomposable projective graded $H_s(Z)$-modules
3. simple graded $H_s(Z)$-modules

The bijection between (1) and (2) is clear from the decomposition theorem, it maps $P \mapsto P/H^A_s(pt)P$. We pass from (3) to (2) by taking the projective cover and we pass from (2) to (3) by taking the top (which is graded because for a finite dimensional graded algebra the radical is given by a graded ideal).

**Example.** (due to Khovanov and Lauda, [KL09]) Let $G \supset B \supset T$ be a reductive group containing a Borel subgroup containing a maximal torus, $Z = G/B \times G/B$. Then, it is known that $H^G(Z) = \operatorname{End}_{\mathbb{C}[t]^W} (\mathbb{C}[t]) := NH$ where $W$ is the Weyl group associated to $(G,T)$ and $t = \operatorname{Lie}(T)$.

The $G$-equivariant pushforward (to the point) of the shift of the constant sheaf is a direct sum of copies of the constant sheaves on the point, therefore there exist precisely one indecomposable projective graded $H^G_s(Z)$-module up to isomorphism and shift. It is easy to see that $P := \mathbb{C}[t]$ is an indecomposable projective module and $P/H^G_s(pt)P = \mathbb{C}[t]/I_W$ is the only graded simple $NH$-module which is the top of $P$. Also, one checks that $H_s(Z) = \operatorname{End}_{\mathbb{C}(\mathbb{C}[t]/I_W)}$ is a semi-simple algebra which has up to isomorphism and shift only the one graded simple module $\mathbb{C}[t]/I_W$.

Now we equip the Steinberg algebra with a grading by positive integers which leads to a description of graded simple modules in terms of the multiplicity vector spaces $L_t$ in the BBD-decomposition theorem.

**Simple objects in the category of graded finitely generated left $H^A_{<\ast>} (Z)$-modules**

Given a graded vector space $L$, we write $\langle L \rangle := \bigoplus_{d \in \mathbb{Z}} L_d$ for the underlying (ungraded) vector space. If we regrade $H^A_s(Z)$ as follows

$$H^A_{<\ast>} (Z) := \bigoplus_{s,t} \operatorname{Hom}_{\mathbb{C}}(\langle L_t \rangle, \langle L_s \rangle) \otimes \operatorname{Ext}^n_{D_A(V)}(IC^A_t, IC^A_s),$$

in other words

$$H^A_{<\ast>} (Z) = \operatorname{Ext}^*(\bigoplus_t \langle L_t \rangle \otimes_{\mathbb{C}} IC^A_t, \bigoplus_t \langle L_t \rangle \otimes_{\mathbb{C}} IC^A_t)$$

as graded algebra. It holds as an algebra this one is isomorphic to $H^A_s(Z)$. With the same arguments as in the previous section one sees that $P^A_t := \operatorname{Ext}^*_{D_A(V)}(IC^A_t, \pi_A \mathbb{C})$ are a complete representative system for the isomorphism classes of the indecomposable projective graded $H^A_{<\ast>} (Z)$-modules.

We claim that there is a graded $H^A_{<\ast>} (Z)$-module structure on the (multiplicity-)vector space $\langle L_t \rangle$ such that the family $\{\langle L_t \rangle\}_{t}$ is a complete set of the isomorphism classes up to shift of graded simple modules. Using $\operatorname{Hom}(IC^A_t, IC^A_s) = \mathbb{C} \delta_{s,t}$, $\operatorname{Ext}^n(IC^A_t, IC^A_s) = 0$ for $n < 0$ we get

$$H^A_{<\ast>} (Z) = \bigoplus_{s,t} \operatorname{End}(\langle L_t \rangle) \otimes \operatorname{Hom}(\langle L_t \rangle, \langle L_s \rangle) \otimes \operatorname{Ext}^0(IC^A_t, IC^A_s).$$

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Now, the second summand is the graded radical, i.e. the elements of degree $> 0$ (with respect to the new grading). It follows

$$H^A_{<>}(Z) \twoheadrightarrow H^A_{<>}(Z)/(H^A_{<>}(Z))_{>0} = \bigoplus_t \text{End}_\mathbb{C}([L_t]).$$

This gives $(L_t)$ a natural graded $H^A_{<>}(Z)$-modul structure concentrated in degree zero (the positive degree elements in $H^A_{<>}(Z)$ operate by zero). Observe, that $(L_t)$ does not depend on $A$, i.e. in fact they are modules over $H^A_{<>}(Z)$ via the forgetful morphism $H^A_{<>}(Z) \twoheadrightarrow H_{<>}(Z)$.

That means we can instead look for the simple graded modules of $H_{<>}(Z)$.

**Remark.** Let $H$ is a finite dimensional positively graded algebra such that

$$H_0 = H_s/H_{>0} = \bigoplus_t \text{End}(L_t)$$

is a semi-simple algebra. Then $H_{>0}$ is the set of nilpotent elements, i.e. Jacobsen radical of $H_s$. Furthermore all simple and projective $H_s$-modules are graded modules.

* $(L_t)_t$ is the tuple of (pairwise distinct isomorphism classes of all) simple modules.

* For each $t$ pick an $e_t \in \text{End}(L_t) \subset H_0$ which corresponds to projection and then inclusion of a one dimensional subspace of $L_t$.

$(P_t := H_s \cdot e_t)_t$ is the tuple of (pairwise distinct isomorphism classes of all) indecomposable projective modules.

We can apply this remark to $H = H_{<>}(Z)$. As a consequence we see that up to shift $(L_t)_t$ is the tuple of (pairwise distinct isomorphism classes of all) simple graded $H^A_{<>}(Z)$-modules.

From now on, the case where the two gradings coincide will play a special role.

**Remark.** The following conditions are equivalent

1. $H^A_{[a]}(Z) = H^A_{<>}(Z)$ as graded algebra for every (at least one) $A \in \{pt, T, G\}$.
2. $(\pi_i)_i \in \mathbb{C}[e_i]$ is $A$-equivariant perverse for every $i \in I$ for every (at least one) $A \in \{pt, T, G\}$.
3. $\pi_i : E_i \to V$ is semi-small for every $i \in I$, this means by definition $\dim Z_{i,i} = e_i$ for every $i \in I$.

In this case, we say the Springer map is semi-small. Also, $\pi$ semi-small is equivalent to $H_{\text{top}}(Z_{i,i}) = H_{[0]}(Z_{i,i}), i \in I$. Observe, that $H_{[0]}(Z)$ is always a subalgebra and in the semi-small case isomorphic to the quotient algebra $H_{[a]}(Z)/(H_{[a]}(Z))_{>0}$. Assume $\pi$ semi-small, then it holds $2 \dim \pi_i^{-1}(x) \leq e_i - d_S$ where $x \in S$ belongs to the stratification and $H_{\text{top}}(\pi^{-1}(x)) := \bigoplus_i : 2 \dim \pi_i^{-1}(x) = e_i - d_S H_{2 \dim \pi_i^{-1}(x)}(\pi_i^{-1}(x))$ is a left $H_{[0]}(Z)$-modules via the restriction of the convolution construction. If $I$ consists of a single element, $H_{\text{top}}(Z) = H_{[0]}(Z)$ and $H_{2 \dim \pi_i^{-1}(x)}(\pi_i^{-1}(x))$ is an $H_{[0]}(Z)$-module independent of the condition $2 \dim \pi_i^{-1}(x) = \dim E - d_S$.

**Remark.** If one applies the decomposition theorem to $\pi_i, i \in I$ one gets that $L_t = \bigoplus_{i \in I} L_t^{(i)}$ (as graded vector space) where $L_t^{(i)}$ is the multiplicity vector space for $IC_t$ in $(\pi_i)_i \mathbb{C}[e_i]$. It holds $\{L_t^{(i)} | L_t^{(i)} \neq 0\}$ is the complete set of isomorphism classes of simple $H_s(Z_{i,i})$-modules.

**Remark.** In fact, Syu Kato pointed out that the categories of finitely generated graded modules over $H^A_{[a]}(Z)$ and $H^A_{<>}(Z)$ are equivalent. This has been used in [Kat13].

**Remark.** Now, we know that the forgetful (=forgetting the grading) functor from finite dimensional graded $H_{[a]}(Z)$-modules to finite dimensional $H_s(Z)$-modules maps graded simple modules to simple modules. We can use the fact that we know that simples and graded simples are parametrized by the same set to see: Every simple $H_s(Z)$-module $L_t$ has a grading such that it becomes a graded simple $H_{[a]}(Z)$-module and every graded simple is of this form.
Recall that Springer fibre modules in the category of graded \( H^∗ \) can be refined to a Whitney stratification see [Ara01], thm 1.9.10, p.30, which is totally ordered by stratification of IC-sheaves, see [Ara01], section 4.1, p.41), let 
\[
\sum_{k} (-1)^k \dim H^k(S, (L')^s \otimes L)
\]

where all are matrices indexed by \( s = (S, L), t = (S', L') \) such that \( L_t \neq 0, L_s \neq 0 \) and \((t)^t\) denote the transposed matrix.

Theorem 0.2. ([CG97], thm 8.7.5) Assume \( H_{\text{odd}}(\pi^{-1}(x)) = 0 \) for all \( x \in V \). Then, the following matrix multiplication holds
\[
[P : L] = IC \cdot D \cdot IC^t
\]

According to Kato in [Kat13], the whole theory of these algebras is reminiscent of quasi-hereditary algebras (but we have infinite dimensional algebras). He introduces standard and costandard modules for \( H^G_{<s>}(Z) \) in [Kat13], thm 1.3, under some assumptions. He shows that under these assumptions, \( H^G_{<s>}(Z) \) has finite global dimension (see [Kat13], thm 3.5).

Springer fibre modules in the category of graded \( H^A(Z)^s\)-modules

Recall, that Springer fibre modules \( H_{\text{ss}}(\pi^{-1}(x)), H^{|s|}(\pi^{-1}(x)), x \in V \) are naturally graded modules over \( H_{|s|}(Z) \), but if we forget about the grading and we can show that they are actually semi-simple or projective in \( H_s(Z)\)-mod, then, we can see them as semi-simple graded \( H^{|s|}_{<s>}(Z)\)-modules for \( A \in \{ G, T, pt \} \) by the previous section.

Let \( A = pt \). Since the map \( \pi \) is locally trivial over \( S := S_n \) we find that
\[
i_s^* (\bigoplus_{i \in I} R^k(\pi_i)_{*} \mathbb{C}[e_i]), \quad i_s^* (\bigoplus_{i \in I} R^k(\pi_i)_{*} \mathbb{C}[e_i])
\]

are local systems on \( S \), via monodromy they correspond to the \( \pi_1(S, s)\)-representations
\[
H^k(\pi^{-1}(s)) = \bigoplus_{i \in I} H^{e_i + k}(\pi_i^{-1}(s)), \quad \bigoplus_{i \in I} H^{e_i - k}(\pi_i^{-1}(s)) = H^k(\pi^{-1}(s))
\]

with \( e_i := \text{dim}_C E_i \) respectively (for a fixed point \( s = s_a \in S \), cp. [CG97], Lemma 8.5.4).

Now, let us make the extra assumption that the image of the Springer map is irreducible and the stratification \( \{ S_a \}_{a \in A} \) is a Whitney stratification (every algebraic stratification of an irreducible variety can be refined to a Whitney stratification see [Ara01], thm 1.9.10, p.30), which is totally ordered by inclusion into the closure. Let \( S \subset S' \) for two strata \( S, S' \), we write \( \text{Ind}_{S'}(L) := i_S^* \circ H^k(\mathcal{IC}(S', \mathcal{L})) \), i.e. we consider the functors for \( k \in [-d_{S'}, -d_S] \)
\[
\text{Ind}_{S'}(L)(\mathcal{L}) := \text{Ind}_{S'}(L)_{k} := i_S^* \circ H^k(\mathcal{IC}(S', \mathcal{L})),
\]

where \( \text{LocSys}(S) \) is the category of local systems on \( S \), i.e. locally constant sheaves on \( S \) of finite dimensional vector spaces. (for other \( k \in \mathbb{Z} \) this is the zero functor). If we apply the functor \( i_S^* \circ H^k \) on the right hand side of the decomposition theorem we notice the following (for the cohomology groups of IC-sheaves, see [Ara01], section 4.1, p.41), let \( t = (S', \mathcal{L}) \).

\[
i_s^* H^k(\mathcal{IC}_t) = \begin{cases} \mathcal{L}, & \text{ if } d_S = d_{S'}, k = -d_S \\ \text{Ind}_{S'}(\mathcal{L})_{k}, & \text{ if } d_S < d_{S'}, k \in [-d_{S'}, -d_S - 1] \\ 0, & \text{ else.} \end{cases}
\]

\( a = \) finitely many orbits with connected stabilizer groups in the image of the Springer map, pure of weight zero for \( H^G_{<s>}(Z) \) and of the \( IC_t \) in the decomposition theorem.
and
\[ i_S^* \mathcal{H}^k(\mathcal{IC}_t[d]) = \mathcal{H}^{k+d}(\nabla_S i_S^* \mathcal{IC}_t^s) = i_S^* \mathcal{H}^{-k-d-2d_s}(\mathcal{IC}_t^s) \]

implies
\[
i_S^* \mathcal{H}^k(\mathcal{IC}_t[d]) = \begin{cases} \mathcal{L}^s, & \text{if } d_S = d_S', k - d = -d_s \\ \text{Ind}^S_S(\mathcal{L}^s)_{-k-d-2d_s}, & \text{if } d_S < d_S', -k - d - 2d_s \in [-d_S', -d_S - 1] \\ 0 & \text{else.} \end{cases}
\]

where \( d_S = \dim \mathcal{S} \). This implies
\[
H^{[k]}(\pi^{-1}(s)) = \bigoplus_{t \in \mathbb{Z}} \bigoplus_{d \in \mathbb{C}} L_{t,d} \otimes \mathcal{L}^s \oplus \bigoplus_{t=(S',\mathcal{L})} \bigoplus_{d < d_S, d' \in \mathbb{Z}} L_{t,-d_s-k} \otimes \text{Ind}^S_S(\mathcal{L}^*)_{r}^s
\]

and
\[
H^{[k]}(\pi^{-1}(s)) = \bigoplus_{t \in \mathbb{Z}} \bigoplus_{d \in \mathbb{C}} L_{t,d} \otimes \mathcal{L}^s \oplus \bigoplus_{t=(S',\mathcal{L})} \bigoplus_{d < d_S, d' \in \mathbb{Z}} L_{t,-d_s-k} \otimes \text{Ind}^S_S(\mathcal{L}^*)_{r}^s
\]
as \( \pi_1(S,s) \)-representations. We call the direct summands isomorphic to \( \text{Ind}^S_S(\mathcal{L}^*)_{r}^s \), \( r \in [-d_S', -d_S - 1] \) the unwanted summands. Now we can explain how you can recover from the \( \pi_1(S,s) \)-representations
\( H^{[k]}(\pi^{-1}(s)), k \in \mathbb{Z} \) the data for the decomposition theorem (i.e. the local systems and the graded multiplicity spaces). If \( d_S \) is the maximal one, it holds
\[
H^{[s]}(\pi^{-1}(s)) = \bigoplus_{t \in \mathbb{Z}} L_{t,-d_s-k} \otimes \mathcal{L}
\]
and we can recover the graded multiplicity spaces \( L_t \) with \( t = (S,?) \) for the dense stratum occurring in the decomposition theorem. For arbitrary \( S \) we consider
\[
H^{[s]}(\pi^{-1}(s)) = H^{[s]}(\pi^{-1}(s))_{>S} \cong \bigoplus_{k \in \mathbb{Z}} L_{t,-d_s-k} \otimes \mathcal{L}
\]
and by induction hypothesis we know the \( \pi_1(S,s) \)-representation \( H^{[s]}(\pi^{-1}(s))_{>S} \), therefore we can recover the \( L_t \) with \( t = (S,?) \) from the above representation.

Now assume that \( \pi \) is semi-small. Then, we know that \( L_{t,d} = 0 \) for all \( t = (S,\mathcal{L}) \) whenever \( d \neq 0 \). We can also restrict our attention on a direct summand \( (\pi_1)_* \mathbb{C}[e_i] \) for one \( i \in I \) and find the decomposition into simple perverse sheaves. That means we only need \( H^{e_i-d_s}(\pi^{-1}_i(s)) \) to recover the data for the decomposition theorem. It also holds \( 2 \dim \pi^{-1}_i(s) \leq e_i - d_s, i \in I \) and since \( H^{e_i-d_s}(\pi^{-1}_i(s)) = 0 \) whenever \( 2 \dim \pi^{-1}_i(s) < e_i - d_s \), we only need to consider the strata \( S \) with \( 2 \dim \pi^{-1}_i(s) = e_i - d_s \), then
\[
H^{e_i-d_s}(\pi^{-1}_i(s)) = H^{top}(\pi^{-1}_i(s)) \neq 0
\]
and we call \( S \) a relevant stratum for \( i \in I \). We call a stratum relevant if it is relevant for at most one \( i \in I \).
Analogously, one can replace \( H^{[k]}(\pi^{-1}(s)) \) by \( H_{[-k]}(\pi^{-1}(s)) \) and stalk by costalk.
Let \( x \in V \) be arbitrary. By a previous section we know that \( H_{\ast}[\pi^{-1}(x)] \) and \( H^{[\ast]}[\pi^{-1}(x)] \) are left (and right) graded \( H_{\ast}(Z) \)-modules. The following lemma explains their special role. Unfortunately, the following statement is only known if all strata \( S \) contain a \( G \)-orbit \( s \in \mathcal{O} \subset S \) such that \( \pi_1(\mathcal{O}, s) = \pi_1(S, s) \). For local systems on the strata this is by monodromy the same as the assumption that all strata are \( G \)-orbits. Let \( C \) be a finite group, we write \( \text{Simp}(C) \) for the set of isomorphism classes of simple \( CC \)-modules and denote by \( 1 \in \text{Simp}(C) \) the trivial representation.\(^7\)

**Lemma 6.** ([CG97], Lemma 8.4.11, p.436, Lemma 3.5.3, p.170) Assume that the image of the Springer map contains only finitely many \( G \)-orbits.

(a) Let \( \mathcal{O} = Gx \subset V \) be a \( G \)-orbit. There is an equivalence of categories between

\[
\{ G \text{-equivariant local systems on } \mathcal{O} \} \leftrightarrow C(x)\text{-mod}
\]

where \( C(x) = \text{Stab}_G(x)/(\text{Stab}_G(x))^{\circ} \) is the component group of the stabilizer of \( x \). In particular, via monodromy also the \( \pi_1(\mathcal{O}, s) \)-representations which correspond to \( G \)-equivariant local systems on \( \mathcal{O} \) are equivalent to \( C(x)\text{-mod} \).

(b) The \( C(x) \)-operation and the \( H_{\ast}[\pi^{-1}(x)] \)-operation on \( H_{\ast}[\pi^{-1}(x)] \) (and on \( H^{[\ast]}[\pi^{-1}(x)] \)) commute.

The semi-simplicity of \( C(x)\text{-mod} \) implies that

\[
H_{\ast}[\pi^{-1}(x)] = \bigoplus_{k \in \mathbb{Z}} \bigoplus_{\chi \in \text{Simp}(C(x))} (H_{[k]}[\pi^{-1}(x)])_\chi \otimes C \chi
\]

where \( \text{Simp}(C(x)) \) is the set of isomorphism classes of simple \( C(x) \)-modules and for any \( C(x) \)-module \( M \) we call \( M_\chi := \text{Hom}_{C(x)\text{-mod}}(\chi, M) \) an isotypic component. Since the two operation commute it holds \( (H_{\ast}[\pi^{-1}(s)])_\chi \) naturally has the structure of a graded \( H_{\ast}(Z) \)-module. But we will from now just see it as a module over \( H_\ast(Z) \). As \( H_\ast(Z) - C(x) \)-bimodule decomposition we can write the previous decomposition as

\[
H_{\ast}[\pi^{-1}(s)] = \bigoplus_{\chi \in \text{Simp}(C(x))} H_\ast(\pi^{-1})_\chi \boxtimes \chi
\]

where \( H_\ast(\pi^{-1})_\chi \boxtimes \chi \) is the obvious bimodule \( H_\ast(\pi^{-1}(x))_\chi \otimes \chi \). As an immediate consequence of this we get, if \( Gx \) is a dense orbit in the image of the Springer map, then

\[
L_{t,-s}(-d_{Gx}) = H_{\ast}[\pi^{-1}(x)]_\chi, \text{ for } t = (x, \chi), \chi \in \text{Simp}(C(x)),
\]

in particular, \( H_{\ast}[\pi^{-1}(x)] \) is a semisimple \( H_\ast(Z) \)-module (graded and not graded), even a semisimple \( H_\ast(Z) - C(x) \)-bimodule. For more general orbits, we do not know if it is semi-simple. In the case of a semi-small Springer map we have the following result.

**Theorem 0.3.** Assume the image of the Springer map \( \pi \) has only finitely many orbits and \( \pi \) is semi-small. There is a bijection between the following sets

1. \( \{(x, \chi) \mid \mathcal{O} = Gx, \chi \in \text{Simp}(C(x)), H_{d_{\mathcal{O}}}[\pi^{-1}(x)]_\chi \neq 0 \} \) where the \( x \) in \( V \) are in a finite set of points representing the \( G \)-orbits in the image of the Springer map.

2. \( \text{Simp}(H_{<0>}[Z])\text{-mod} := \text{simple } H_{<0>}[Z]\text{-modules up to isomorphism} \)

3. \( \text{Simp}(H_{\lambda_\ast}(Z))\text{-mod} := \text{simple graded } H_{\lambda_\ast}(Z)\text{-modules up to isomorphism and shift for any } A \in \{pt, T, G\} \).

\(^7\)In the literature this is called \( \text{Irr}(C) \), we use the word irreducible only for a property of topological spaces.
Between (1) and (2), it is given by \((x, \chi) \mapsto H_{|d_\mathcal{O}|}(\pi^{-1}(x))_\chi\). We call this bijection the Springer correspondence.

For a relevant orbit \(\mathcal{O}\) (for at least one \(i \in I\)) it holds
\[
H_{|d_\mathcal{O}|}(\pi^{-1}(x))_1 = \bigoplus_{i: 2 \dim \pi^{-1}(x) = -d_\mathcal{O}} H_{\text{top}}(\pi^{-1}(x))_C(x) \neq 0
\]
and \(C(x)\) operates on the top-dimensional irreducible components of \(\pi^{-1}(x)\) by permutation. This implies we get an injection
\[
\{\text{relevant } G\text{-orbits in } \text{Im}(\pi)\} \hookrightarrow \text{Simp}(H_{<0>}(Z) - \text{mod})
\]
\[
\mathcal{O} = Gx \mapsto H_{|d_\mathcal{O}|}(\pi^{-1}(x))_C(x)
\]

**sketch of proof:** For \(k = d_\mathcal{O}\) look at the decomposition for \(H_{|k|}(\pi^{-1}(x))\) and use that \(L_{t,d} = 0\) whenever \(d \neq 0\) to see that the unwanted summands vanish. Then show that the decomposition coincides with the second decomposition (with respect to the irreducible characters of \(C(x)\)) of \(H_{|k|}(\pi^{-1}(x))\) which gives the identification of the \(L_t\) with the \(H_{|d_\mathcal{O}|}(\pi^{-1}(x))_\chi\).

It is an open question to understand Springer fibre modules more generally. Also, Springer correspondence hints at a hidden equivalence of categories. This functorial point of view we investigate in the next subsection.

**The Springer functor**

We consider \(H^A_{[s]}(Z)\) again with the grading from the thm 0.1. Let \(\text{proj}^Z H^A_{[s]}(Z)\) be the category of finitely generated projective \(\mathbb{Z}\)-graded left \(H^A_{[s]}(Z)\)-modules, morphisms are the module homomorphisms which are homogeneous of a degree 0. Let \(\mathcal{P}^A \subset D^b_A(X)\) be the full subcategory closed under direct sums and shifts generated by \(IC^A_t\), \(t = (S, \mathcal{L})\) be the tuple of strata with simple local system on it which occur in the decomposition theorem.

The following lemma is in a special case due to Catharina Stroppel and Ben Webster, see [SW11].

**Lemma 7.** The functor
\[
\text{proj}^Z H^A_{[s]}(Z) \to \mathcal{P}^A
\]
\[
M \mapsto \bigoplus_{\pi \in I} (\pi)_{s} \bigotimes_{E_i} [e_i] \otimes H^A_{[s]}(Z) M
\]
is an equivalence of semisimple categories mapping \(P^A_t \mapsto IC^A_t\). We call this the Springer functor

**proof:** By thm 0.1 we know \(H^A_{[s]}(Z) = \text{Ext}^*_D(V) (\bigoplus_{\pi \in I} (\pi)_{s} \bigotimes_{E_i} [e_i], \bigoplus_{\pi \in I} (\pi)_{s} \bigotimes_{E_i} [e_i])\) is an isomorphism of graded algebras. This makes the functor well-defined. The direct sum decomposition of \(\bigoplus_{\pi \in I} (\pi)_{s} \bigotimes_{E_i} [e_i]\) by the decomposition theorem in \(\mathcal{P}^A\) corresponds to idempotent elements in \(H^A_{[s]}(Z)\), which correspond (up to isomorphism and shift) to the indecomposable projective graded modules, let for example \(P_t = H^A_{[s]}(Z) e_t\). Shifts of graded modules are mapped to shifts in \(\mathcal{P}^A\), therefore the functor is essentially surjective. It is fully faithful because of the mentioned isomorphism
\[
\text{Hom}_{\text{proj}^Z H^A_{[s]}(Z)}(P_t, P_s(n)) = e_s H^A_{[s]}(Z) e_t = \text{Hom}_{D^b_A(V)}(IC_t, IC_s[n])
\]
\[
\square
\]

Let \(\mathcal{P}^A(V) \subset D^b_A(V)\) be the category of \(A\)-equivariant perverse sheaves on \(V\). Assume for a moment that the map \(\pi\) is semi-small. Then, we know that \(\bigoplus_{\pi \in I} (\pi)_{s} \bigotimes_{E_i} [e_i]\) is an object of \(\mathcal{P}^A(V)\). In this situation the two gradings of the Steinberg coincide. The top-dimensional Borel-Moore homology \(H_{\text{top}}(Z_{i,i})\) coincides with the degree zero subalgebra \(H_{[0]}(Z_{i,i})\). We want the Springer functor to go

\[\footnote{This name is due to Dustin Clausen in his thesis.}\]
to a category of perverse sheaves, i.e. we do not want to allow shifts of the grading for modules. Therefore, we pass to
\[ H_{[0]}(Z) = H_{<e>}(Z)/(H_{<e>}(Z))_{>0} = H^A_{<e>}(Z)/(H^A_{<e>}(Z))_{>0}, \quad A \in \{pt, T, G\} \]
and replace projective graded modules over \( H^A_{[i]}(Z) \) by the additive category of simple modules over \( H_{[0]}(Z) \), this equals the category \( H_{[0]}(Z) - \text{mod} \) of finite dimensional (ungraded) modules over \( H_{[0]}(Z) \) because the algebra is semi-simple.

In particular, it holds
\[ H_{[0]}(Z) = \text{Ext}^0_{A(V)}(\bigoplus_{i \in I}(\pi_i)_*\mathbb{C}[e_i], \bigoplus_{i \in I}(\pi_i)_*\mathbb{C}[e_i]) = \text{End}_{A(V)}(\bigoplus_{i \in I}(\pi_i)_*\mathbb{C}[e_i]), \quad A \in \{pt, T, G\}. \]

The following lemma is for classical Springer Theory due to Dustin Clausen, cp. Thm 1.2 in [Cla08].

**Lemma 8.** If the Springer map \( \pi \) is semi-small, we have the following version of the Springer functor
\[ \mathcal{S}: H_{[0]}(Z) - \text{mod} \to \mathcal{P}^G(V) \]
\[ M \mapsto \bigoplus_{i \in I}(\pi_i)_*\mathbb{C}[e_i] \otimes_{H_{[0]}(Z)} M \]

It holds that \( \mathcal{S} \) is an exact functor (between abelian categories) and it is fully faithful. If \( e_i + e_j \) is even for all \( i, j \in I \) the \( \mathcal{S} \) identifies \( H_{[0]}(Z) - \text{mod} \) with a semi-simple Serre subcategory of \( \mathcal{P}^G(V) \) (i.e. it is an exact subcategory which is also extension closed). Furthermore it is invariant under Verdier duality on \( \mathcal{P}^G(V) \).

**Remark.** Assume that the Springer map is semi-small, the image of the Springer map contains only finitely many \( G \)-orbits and each \( G \)-orbit is relevant and simply connected, then the Springer functor from above is an equivalence of categories. (The only known example for this is the classical Springer map for \( G = \text{GL}_n \), see later.)

**proof:** A similar proof as in the lemma above shows that the Springer functor induces an equivalence on the full subcategory of \( \mathcal{P}^G(V) \) generated by finite direct sums of direct summands of \( \bigoplus_{i \in I}(\pi_i)_*\mathbb{C}[e_i] \). This is a semi-simple category. Assume that \( e_i + e_j \) is even for all \( i, j \in I \), we have to see that it is extension closed. By composition with the forgetful functor we get a functor
\[ H_{[0]}(Z) - \text{mod} \xrightarrow{\mathcal{S}} \mathcal{P}^G(V) \xrightarrow{F} \mathcal{P}^{pt}(V) =: \mathcal{P}(V), \]
by [Cla08] the forgetful functor \( F \) is fully faithful. Now, by [Ara01], 4.2.10 the category \( \mathcal{P}(V) \) of \( D^b(V) \) is closed under extensions and admissible because it is the heart of a \( t \)-structure. By the Riemann Hilbert correspondence there exists an abelian category \( \mathcal{A} (= \text{regular holonomic } D\text{-modules on } V) \) and an equivalence of triangulated categories (= the de Rham functor)
\[ DR_V: D^b(\mathcal{A}) \to D^b(V) \]
such that the standard \( t \)-structure on \( D^b(\mathcal{A}) \) is mapped to the perverse \( t \)-structure and it restricts to an equivalence of categories \( \mathcal{A} \to \mathcal{P}(V) \). This implies that for \( X \cong DR_V(X'), Y \cong DR_V(Y') \) in \( \mathcal{P}(V) \) and \( n \in \mathbb{N}_0 \)
\[ \text{Ext}^n_{\mathcal{P}(V)}(X, Y) = \text{Ext}^n_{\mathcal{A}}(X', Y') = \text{Hom}_{D^b(\mathcal{A})}(X', Y'[n]) = \text{Hom}_{D^b(V)}(X, Y[n]) \]
where the first and the third equality follows from the de Rham functor and the second equality holds because it is the standard \( t \)-structure, cp. for example [GM03], p.286.

Now, since we know
\[ \text{Hom}_{D^b(V)}(\bigoplus_{i \in I}(\pi_i)_*\mathbb{C}[e_i], \bigoplus_{i \in I}(\pi_i)_*\mathbb{C}[e_i]) = H^{i>0}(Z) = \bigoplus_{i,j \in I} H_{e_i + e_j - 1}(Z) = 0 \]
because \( H_{odd}(Z) = 0 \) by lemma 2 and the assumption that \( e_i + e_j \) is even for every \( i, j \in I \). We obtain that
\[ \text{Ext}^1_{\mathcal{P}(V)}(\bigoplus_{i \in I}(\pi_i)_*\mathbb{C}[e_i], \bigoplus_{i \in I}(\pi_i)_*\mathbb{C}[e_i]) = 0, \]
i.e. the semi-simple category generated by the direct image of the Springer map is extension closed. □
**What is Springer Theory?**

One possible definition

Springer theory (for \((G, P_i, V, F_i)_{i \in I}\) and a choice of \(H\)) is to understand the Steinberg algebra together with its graded modules.

But I think today it is sensible to say Springer theory is the study of all categories and algebras (and modules over it) which have a construction originating in some Springer Theory data \((G, P_i, V, F_i)_{i \in I}\). Then, this includes

1. Monoidal categories coming from multiplicative families of Steinberg algebras and their Grothendieck ring. In particular, this includes Lusztig’s categories of perverse sheaves (see [Lus91] and the example quiver-graded Springer Theory later).

2. Noncommutative resolutions\(^9\) corresponding to the Springer map. In particular, this includes Bezrukavnikov’s noncommutative counterparts of the Springer map in [Bez06] and Buchweitz, Leuschke and van den Bergh’s articles [BLB10] and [BLB11].

3. Categories of flags of \((KQ-)\)-submodules for given quivers because their isomorphism classes parametrize orbits of (quiver-graded) Springer fibres. This includes for example Ringel’s and Zhang’s work on submodule categories and preprojective algebras [RZ12]. Also certain \(\Delta\)-filtered modules studied in [BHRR99], [BH00].

An (of course) incomplete overview can be found in the flowchart at the end of this article. We would also like you to observe that in the two examples we explore connections between objects roughly related to the following triangle

```
Steinberg algebras

Quantum groups

Perverse sheaves
```

**Classical Springer Theory**

This is the case of the following initial data

\[
\begin{align*}
\begin{cases}
(\ast) & G \text{ an arbitrary reductive group}, \\
(\ast) & P = B \text{ a Borel subgroup of } G, \text{ denote its Levi decomposition by } B = TU \\
& \text{ with } T \text{ maximal torus, } U \text{ unipotent}, \\
(\ast) & V = \mathfrak{g} \text{ the adjoint representation}, \\
(\ast) & F = \mathfrak{n} := \text{Lie}(U).
\end{cases}
\end{align*}
\]

We set \(\mathcal{N} := G\mathfrak{n}\), i.e. the image of the Springer map, and call it the nilpotent cone. We consider the Springer map as \(\pi: E = G \times \mathcal{B} \mathfrak{n} \rightarrow \mathcal{N}\). Explicitly, we can write the Springer triple as

\[
E = \{(n, gB) \in \mathcal{N} \times G/B \mid n \in \mathfrak{g}\mathfrak{b} := \text{Lie}(gBg^{-1})\}
\]

\[
\begin{array}{c}
\mathcal{N} \\
\pi = pr_1
\end{array} \quad \begin{array}{c}
\pi = pr_2
\end{array} \quad \begin{array}{c}
G/B
\end{array}
\]

\(\ast\) *here*: This means just a tilting vector bundle on \(E\), because this gives \(t\)-structures in the category of coherent sheaves on \(E\).
For $G = \text{GL}_n$ we identify $\text{GL}_n/B$ with the variety $Fl_n$ if complete flags in $\mathbb{C}^n$ and

$$E = \{(A, U^*) \in \text{End}_\mathbb{C}(\mathbb{C}^n) \times Fl_n \mid A^n = 0, A(U^k) \subset U^k, 1 \leq k \leq n\}.$$ 

It turns out, $\pi$ can be identified with the moment map of $G$, in particular, $E \cong T^*(G/B)$ is the cotangent bundle over $G/B$ and $\pi$ is a resolution of singularities for $\mathcal{N}$. But most importantly, this makes the Springer map a symplectic resolution of singularities and one can use symplectic geometry (see for example [CG97]).

The Steinberg variety is given by

$$Z = \{(n, gB, hB) \in \mathcal{N} \times G/B \times G/B \mid n \in gB \cap hB\}$$

for $G = \text{GL}_n$ we can write it as

$$Z = \{(A, U^*, V^*) \in \text{End}_\mathbb{C}(\mathbb{C}^n) \times Fl_n \times Fl_n \mid A^n = 0, A(U^k) \subset U^k, A(V^k) \subset V^k, 1 \leq k \leq n\}.$$

Recall, that we had the stratification by relative position $Z^w := m^{-1}(G \cdot (eB, wB)), w \in W$ where $W$ is the Weyl group of $G$ with respect to a maximal torus $T \subset B$. Since $Z^w \rightarrow G \cdot (eB, wB)$ is a vector bundle, we can easily calculate its dimension

$$\dim Z^w = \dim G \cdot (eB, wB) + \dim n \cap w^n$$

$$= \dim G - \dim B \cap wB + \dim n \cap w^n = \dim G - \dim T$$

$$= \dim E.$$

We conclude that $Z$ is equidimensional of dimension $e := \dim E$, in particular the Springer map is semi-small. Also we see that the irreducible components of $Z$ are given by $Z^w, w \in W$, that implies that the top-dimensional Borel-Moore homology group $H_{\text{top}}(Z)$ has a $\mathbb{C}$-vector space basis given by the cycles $[Z^w]$. In the semi-small case we know $H_{[0]}(Z) = H_{<0>}(Z) = H_{\text{top}}(Z)$ is a sub- and quotient algebra of $H_*(Z)$.

**Example.** $G = \text{SL}_2$, $B = \{(a \ b \ b \ a^{-1}) \mid b \in \mathbb{C}, a \in \mathbb{C} \setminus \{0\}\}$. Then $\mathcal{N} = \{(x, y, z) \in \mathbb{C}^3 \mid x^2 + zy = 0\}$ and $E = \{(A, L) \in \mathcal{N} \times \mathbb{P}^1 \mid L \subset \ker A\} = \{(\begin{pmatrix} x & y \\ z & -x \end{pmatrix}, [a : b]) \in M_2(\mathbb{C}) \times \mathbb{P}^1 \mid x^2 + zy = 0, xa + yb = 0, za - yb = 0\}$, the Springer map can be seen as the following picture

![Diagram](image)

This is well-known to be the crepant resolution of the $A_2$-singularity from the MacKay correspondence. In general, if $G$ is semi-simple of type $ADE$, then there exists a slice of the nilpotent cone such that the restricted map is the crepant resolution of the corresponding type singularity, see [Slo80b] for more details.
Theorem 0.4. (roughly Springer [Spr76]) There is an isomorphism of \( \mathbb{C} \)-algebras

\[
H_{\text{top}}(Z) \cong \mathbb{C}[W]
\]

\[
[Z^x] \mapsto s - 1
\]

The Springer functor (due to Clausen, [Cla08]) takes the form

\[
\mathbb{C}W-\text{mod} \to \mathcal{P}^G(N)
\]

\[
M \mapsto \pi_s \mathbb{C}[e] \otimes_{\mathbb{C}W} M
\]

and identifies \( \mathbb{C}W-\text{mod} \) with a semi-simple Serre subcategory of \( \mathcal{P}^G(N) \). This implies an injection on simple objects which are in \( \mathcal{P}^G(N) \) the intersection cohomology complexes associated to \( (\mathcal{O}, \mathcal{L}) \) with \( \mathcal{L} \) a simple \( G \)-equivariant local system on an \( G \)-orbit \( \mathcal{O} \subset \mathcal{N} \). As a consequence we get the bijection called Springer correspondence from thm 0.3

\[
\text{Simp}(W) \leftrightarrow \{ t = (\mathcal{O}, \mathcal{L}) \mid \text{certain (= occurring in the decomp. thm) } \}
\]

\[
= \{ (x, \chi) \mid x \in \mathcal{N} \text{ rep of } G \text{-orbits}, \chi \in \text{Simp}(C(x)), (H_{\text{top}}(\pi^{-1}(x))) \chi \neq 0 \}
\]

where \( \text{Simp}(W) \) is the set of isomorphism classes of simple objects in \( \mathbb{C}W-\text{mod} \). The inverse of the map is given by \( (x, \chi) \mapsto (H_{\text{top}}(\pi^{-1}(x))) \chi \). For this Springer map all orbits in \( \mathcal{N} \) are relevant, i.e. we also have an injection

\[
\{ G\text{-orbits in } \mathcal{N} \} \to \text{Simp}(W)
\]

\[
x \mapsto H_{\text{top}}(\pi^{-1}(x))^{C(x)}
\]

Remark. We remark that there are several alternative constructions of the group operation of \( W \) on the Borel-Moore homology/ singular cohomology of the Springer fibres. In [Ara01], section 5.5 you find an understandable treatment of Lusztig’s approach to this operation using intermediate extensions for perverse sheaves and Arabia provides a list of other authors and approaches to this (first Springer [Spr76],[Spr78], then Kazhdan-Lusztig [KL80], Slodowy [Slo80a], Lusztig [Lus81], Rossmann [Ros91]) and these operations differ between each other by at most by multiplication with a sign character (see [Hot81]).

Also, Springer proves with taking (co)homology of Springer fibres with rational coefficients that the simple \( \mathbb{C} \)-modules are all even defined over \( \mathbb{Q} \), a result which our approach does not give because the simple \( C(x) \)-modules are not necessarily all defined over \( \mathbb{Q} \) (cp. [CG97], section 3.5, p.170). In Carter’s book [Car85], p. 388, you find for simple \( g \)-representations the component groups \( C(x), x \in \mathcal{N} \) are one of the following list \( (\mathbb{Z}/2\mathbb{Z})^r, S_3, S_4, S_5, r \in \mathbb{N}_0 \) as a consequence he gets that the simple modules over the group ring are already defined over \( \mathbb{Q} \).

In the introduction of the book [BBM89] you find for a semisimple group \( G \) a triangle

```
\[ \begin{array}{ccc}
\text{simple } \mathbb{C}W\text{-modules} & \xrightarrow{\text{primitive ideals in } U(\mathfrak{g})} & G\text{-orbits in the nilpotent cone} \\
\end{array} \]
```

They explain it as follows (i.e. this is a summary of a their summary).

* There is an injection of \( G \)-orbits in \( \mathcal{N} \) into simple \( \mathbb{C}W \)-modules by the Springer correspondence.

* A primitive ideal in \( U(\mathfrak{g}) \) is a kernel of some simple \( U(\mathfrak{g}) \)-representation. The classification of primitive ideals is achieved as a result of the proof of the Kazhdan-Lusztig conjectures (see Beilinson-Bernstein [BB81], Brylinski-Kashiwara [BK81]). Any ideal in \( U(\mathfrak{g}) \) has an associated subvariety of \( \mathfrak{g} \). The associated variety of a primitive ideal is the closure of an orbit in \( \mathcal{N} \), this was first conjectured by Borho and Jantzen.

* Joseph associated to a primitive ideal a \( W \)-harmonic polynomial in \( \mathbb{C}[t] \) (=Goldie rank polynomial) which is a basis element of one of the simple \( \mathbb{C}W \)-modules.

We also have to mention the following important results which use \( K \)-theory instead of Borel-Moore homology.
**Parametrizing simple modules over Hecke algebras.** This field goes back to the work of Kazhdan and Lusztig on the proof of the Deligne-Langlands conjecture for Hecke algebras, see [KL87]. They realize irreducible representations of Iwahori Hecke algebra as Grothendieck groups of equivariant (with respect to certain groups) coherent sheaves on the Springer fibres. This is now known as Deligne-Langlands correspondence and we call similar results which come later for different Hecke algebras still DL-correspondence.

Let $G$ be an algebraic group and $X$ a $G$-variety, let $K^G_0(X) := K_0(coh^G(X))$ be the Grothendieck group of the category of $G$-equivariant coherent sheaves on $X$. The group $\mathbb{C}^*$ operates on the (classical) Steinberg variety $Z$ via $(n, gB, hB) \cdot t := (t^{-1}nt, gB, hB)$, the convolution product construction gives a ring structure on $K^{G \times \mathbb{C}^*}_0(Z)$.

Recall, for a reductive group we fix a maximal torus and a Borel subgroup $T \subset B \subset G$ and call $(W, S)$ the associated Weyl group with set of simple roots. We write $G = (G, S, W, 0)$.

**Theorem 0.5.** ([CG97], thm 7.2.5, thm 8.1.16 - DL correspondence for affine Hecke algebras)

Let $G$ be a connected, simply connected semi-simple group over $\mathbb{C}$.

(a) It holds $K^{G \times \mathbb{C}^*}_0(Z) \cong H$ where $H$ is the affine Hecke algebra associated to $(W, S)$, i.e., the $\mathbb{Z}[q, q^{-1}]$-algebra generated by $e^\lambda T_w \mid w \in W, \lambda \in X(T), e^0 = 1$ with relations

1. $(T_a + 1)(T_a - q) = 0, s \in S$, and $T_x T_y = T_y T_x$ for $x, y \in W$ with $\ell(xy) = \ell(x) + \ell(y)$.
2. The $\mathbb{Z}[q, q^{-1}]$-subalgebra spanned by $e^\lambda$ is isomorphic to $(\mathbb{Z}[q^\pm])[X^+_1, \ldots, X^+_n]$, $n = rk(T)$.
3. For $\lambda, \alpha^*_s = 0$ it holds $T_s e^\lambda = e^\lambda T_s$.

(b) The operation of $\mathcal{H}$ on a simple module factors over $H_\ast(Z^a)$, with $a = (s, t) \in G \times \mathbb{C}^*$ a semisimple element, in particular $H_\ast((\pi^{-1}(x))^\ast)$ via the convolution construction a $\mathcal{H}$-module. The operation of the component group $C(a) = Stab_{G \times \mathbb{C}^*}(a)/(Stab_{G \times \mathbb{C}^*}(a))^0$ on $H_\ast(\pi^{-1}(x))^\ast$ commutes with the $H_\ast(Z^a)$-operation and gives $H_\ast(\pi^{-1}(x))^\ast = \bigoplus_{\chi \in \text{Simp}(C(a))} K_{a, x, \chi} \otimes \chi$ for some $H_\ast(Z^a)$-modules $K_{a, x, \chi}$ (standard modules).

If $t \in \mathbb{C}$ is not a root of unity, then there is a (n explicit) bijection between

1. $\{G - \text{conj. cl. of } (s, x, \chi) \mid s \in G \text{ semisimple, } sxs^{-1} = tx, \chi \in \text{Simp}(C(s, t)), K_{(s, t), x, \chi} \neq 0\}$

and

2. Simple $\mathcal{H}$-modules where $q$ acts by multiplication with $t$.

All simples are constructed from the standard modules, in general it is difficult to determine when the candidates are nonzero. For $t$ a root of unity there is an injection of the set (2) in (1).

**Quiver-graded Springer Theory**

Let $Q$ be a finite quiver with set of vertices $Q_0$ and set of arrows $Q_1$. Let us fix a dimension vector $d \in \mathbb{N}^{Q_0}_0$ and a sequence of dimension vectors $d := (0 = d^0, \ldots, d^r =: d)$, $d^0 \leq d^{k+1}$. Quiver-graded
Springer Theory arises from the following initial data

\[
(*) G = \text{GL}_d := \prod_{i \in Q_0} \text{GL}_{d_i},
\]
\[
(*) P = P(d) := \prod_{i \in Q_0} P(d_i^*) \text{ where } P(d_i^*) \text{ is the parabolic in } \text{GL}_{d_i} \text{ fixing a (standard) flag } V_i^* \text{ in } \mathbb{C}^{d_i} \text{ with dimensions given by } d_i^*,
\]
\[
(*) V = R_Q(d) := \prod_{(i,j) \in Q_1} \text{Hom}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j}) \text{ with the operation } (g_i)(M_{i \to j}) = (g_i M_{i \to j} g_i^{-1})
\]
is called representation space.
\[
(*) F = F(d) := \{ (M_{i \to j}) \in R_Q(d) \mid M_{i \to j}(V_j^k) \subset V_j^k, \ 0 \leq k \leq \nu \}
\]

Given \(d\) and an (arbitrary) finite set \(I := \{ d = (0 = d_0^0, \ldots, d_\nu^\nu) \mid \nu \in \mathbb{N}, d_\nu^\nu = d\}\), we can describe the quiver-graded Springer correspondence explicitly via for \(d \in I\)

\[
E_d = \{ (M, U^*) \in R_Q(d) \times \text{Fl}_d \mid i \prec j \in Q_1: M_\alpha(U_i^k) \subset U_j^k, \ 1 \leq k \leq \nu \}
\]

where \(\text{Fl}_d = \prod_{i \in Q_0} \text{Fl}_{d_i}\) and \(\text{Fl}_d\) is the variety of flags of dimension \((0, d_1^1, d_2^2, \ldots, d_\nu^\nu = d)\) inside \(\mathbb{C}^d\) and we set \(E := \bigsqcup_{d \in I} E_d\).

\[
Z_{d,d'} := E_d \times_{R_Q(d)} E_{d'}
\]

\[
\{ (M, U^*, V^*) \in R_Q(d) \times \text{Fl}_d \times \text{Fl}_{d'} \mid i \prec j \in Q_1: M_\alpha(U_i^k) \subset U_j^k, M_\alpha(V_i^k) \subset V_j^k \}
\]

and the Steinberg variety is \(Z := \bigsqcup_{d,d' \in I} Z_{d,d'}\). This description goes back to Lusztig (cp. for example [Lus91]). It holds

\[
\dim E_d = \dim \text{Fl}_d + \dim F(d) = \sum_{i \in Q_0} \sum_{k=1}^{\nu-1} d_i^k (d_i^{k+1} - d_i^k) + \sum_{(i,j) \in Q_1} \sum_{k=1}^{\nu} (d_i^k - d_i^{k-1}) d_j^k,
\]

We define \(\langle d, d \rangle := \dim \text{GL}_d - \dim E_d\) and when \(Q\) is without oriented cycles this is the tits form for the algebra \(\mathbb{C}Q \otimes \mathbb{C}A_{\nu+1}\) (cp. [Wol09], Appendix)

\[
\langle d, d \rangle = \sum_{k=0}^{\nu} \langle d_i^k, d_i^{k+1} \rangle_{\mathbb{C}Q} - \sum_{k=0}^{\nu-1} \langle d_i^k, d_i^{k+1} \rangle_{\mathbb{C}Q}.
\]

Let us take \((d)_{i \in I}\) be the set of complete dimension filtrations of a given dimension vector \(d\). The \((\text{GL}_d\text{-equivariant})\) Steinberg algebra is the quiver Hecke algebra (for \(Q, d\)). If the quiver \(Q\) has no loops, the image of the injective map from lemma 3 has been calculated by Varagnolo and Vasserot in [Var09]. With generators and relations of the algebra they check that this is the same algebra as has been introduced by Khovanov and Lauda in [KL09] (and which was previously conjectured by Khovanov and Lauda to be the Steinberg algebra for quiver-graded Springer theory with complete dimension vectors). Independently, this has been proven by Rouquier in [Rou11].
Theorem 0.6. (quiver Hecke algebra, [Var09], [Rou11]) Let $Q$ be a quiver without loops and $\mathbf{d} \in \mathbb{Z}^d$ a fixed dimension vector. The $(\text{GL}_d$-equivariant) quiver-graded Steinberg algebra for complete dimension filtrations $R^G_d := H^G_d(Z)$ for $(Q, \mathbf{d})$ is as graded $\mathbb{C}$-algebra generated by

$$1_{i,i}, i \in I, \quad z_i(k), i \in I, 1 \leq k \leq d, \quad \sigma_i(s), i \in I, s \in \{(1,2),(2,3),\ldots,(d-1,d)\} =: \mathcal{S},$$

where $d := \sum_{a \in Q_0} da$, $I := I_d := \{(i_1,\ldots,i_d) \mid i_k \in Q_0, \sum_{k-1}^d i_k = 0\}$ and we see $\mathcal{S} \subset \mathcal{S}_d$ as permutations of $\{1,\ldots,d\}$, we also define

$$h_i((\ell,\ell+1)) = h_{i_{\ell+1},i_{\ell}} = \#\{\alpha \in Q_1 \mid \alpha: i_{\ell+1} \to i_{\ell}\}$$

and let

$$\deg 1_i = 0, \quad \deg z_i(k) = 2, \quad \deg \sigma_i((\ell,\ell+1)) = \begin{cases} 2h_i((\ell,\ell+1)) - 2, & \text{if } i_\ell = i_{\ell+1} \\ 2h_i((\ell,\ell+1)), & \text{if } i_\ell \neq i_{\ell+1} \end{cases}$$

subject to relations

1. (orthogonal idempotents)

$$1_i = \delta_{i,i}, \quad 1_i \sigma_i(s) 1_i = \sigma_i(s) \quad \quad 1_i z_i(k) 1_i = z_i(k)$$

2. (polynomial subalgebras)

$$z_i(k) z_i(k') = z_i(k') z_i(k)$$

3. For $s = (k,k+1), \; i = (i_1,\ldots,i_d)$ we write $i_s := (i_1,\ldots,i_{k+1},i_k,\ldots,i_d)$ and set

$$\alpha_s := \alpha_{i_s} := z_i(k) - z_i(k+1)$$

if it is clear from the context which $i$ is meant. We denote by $h_i(s) := \#\{\alpha \in Q_1 \mid \alpha: i_k \to i_{k+1}\}$.

$$\sigma_i(s) \sigma_{i,s}(s) = \begin{cases} 0, & \text{if } i = i_s \\ (-1)^{h_i(s)} \alpha_{i,s}^{h_i(s) + h_{i,s}(s)}, & \text{if } i \neq i_s \end{cases}$$

4. (straightening rule)

For $s = (\ell,\ell+1)$ we set

$$s(z_i(k)) = \begin{cases} z_i(k+1), & \text{if } k = \ell \\ z_i(k-1), & \text{if } k = \ell + 1 \\ z_i(k), & \text{else}. \end{cases}$$

$$\sigma_i(s) z_{i,s}(k) - s(z_{i,s}(k)) \sigma_i(s) = \begin{cases} -1_i, & \text{if } i = i_s, s = (k,k+1) \\ 1_i, & \text{if } i = i_s, s = (k-1,k) \\ 0, & \text{if } i \neq i_s \end{cases}$$

5. (braid relation)

Let $s,t \in \mathcal{S}, st = ts$, then

$$\sigma_i(s) \sigma_{i,t}(s) = \sigma_i(t) \sigma_{i,t}(s).$$

Let $i \in I, \; s = (k,k+1), t = (k+1,k+2)$. We set $s(\alpha_t) := (z_i(k) - z_i(k+2)) =: t(\alpha_s)$.

$$\sigma_i(s) \sigma_{i,s}(t) \sigma_{i,t}(s) - \sigma_i(t) \sigma_{i,s}(t) \sigma_{i,t}(s) = \begin{cases} P_{s,t}, & \text{if } ists = i, is \neq i, it \neq i \\ 0, & \text{else}. \end{cases}$$

where

$$P_{s,t} := \frac{\alpha_{i,s}^{h_i(s)} - (-1)^{h_{i,s}(s)} \alpha_{i,t}^{h_i(s)}}{\alpha_s + \alpha_t} - \frac{\alpha_{i,s}^{h_i(s)} - (-1)^{h_{i,s}(s)} \alpha_{i,t}^{h_i(s)}}{\alpha_s + \alpha_t}$$

is a polynomial in $z_i(k), z_i(k+1), z_i(k+2)$. 

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We call this the **quiver Hecke algebra** for $Q, d$.

using the degeneration of the spectral sequence argument from lemma 2 we get

**Corollary 0.3.** Let $Q$ be a quiver without loops and $d \in \mathbb{N}_0^{Q_0}$. The not-equivariant Steinberg algebra $R_d := H_{[x]}(Z)$ is the graded $\mathbb{C}$-algebra generated by

$$1, i \in I, \quad z_i(k), i \in I, 1 \leq k \leq d \quad \sigma_i(s), i \in I, s \in \{(1, 2), (2, 3), \ldots, (d - 1, d)\}$$

with the same degrees and relations as $R_d^G$ and the additional relations

$$P(z_i(1), \ldots, z_i(n)) = 0, \quad i \in I, \quad P \in \mathbb{C}[x_1, \ldots, x_d]^{S_d}.$$ 

**What about Springer fibre modules and the decomposition theorem?** This is not investigated yet. We make some remarks on it.

**Remark.**

1. If $Q$ is a Dynkin quiver\textsuperscript{10}, the images of all quiver-graded Springer maps have finitely many orbits. For all quiver $Q$ and dimension vector $d \in \mathbb{N}_0^{Q_0}$ all $\text{Gl}_d$-orbits in $R_Q(d)$ are connected, i.e. $C(x) = \{e\}$ for all $x \in R_Q(d)$.

2. In the case of finitely many orbits in the image of the Springer map, semi-smallness of the Springer map (associated to a dimension filtration $d$ of a dimension vector $d$) is equivalent to for every $x \in R_Q(d)$ it holds

$$2 \dim \pi^{-1}_d(x) \leq \dim \text{Ext}^1_{\mathbb{C}Q}(x, x) = \text{codim}_{R_Q(d)} Gx.$$ 

It is very rarely fulfilled.

3. If $Q$ is a Dynkin quiver and $d \in \mathbb{N}_0^{Q_0}$ a complete set of the isomorphism classes of simple modules for the quiver Coxeter algebra $R_d$ is parametrized by the $G := \text{Gl}_d$-orbits in $R_Q(d)$. For $x \in R_Q(d)$ we have a simple module of the form

$$L_{Gx} := \bigoplus_d L_{Gx}^{(d)}$$

where $d$ runs over all complete dimension filtrations of $d$ and $L_{Gx}^{(d)}$ is the multiplicity vector space occurring in the decomposition of $(\pi_d)_* \mathbb{C}[e_d]$. By the work of Reineke (see [Rei03]) there exists for every $x \in R_Q(d)$ a complete dimension filtration $d$ such that $Gx$ is dense in the image of $\pi_d$. This implies by the considerations on page 11 that

$$L_{Gx, -d}^{(d)}(-d_{Gx}) = H_{[x]}(\pi^{-1}_d(x)) \quad (\neq 0),$$

as graded vector spaces, where $d_{Gx} = \dim Gx$. In fact, Reineke even shows that there exists a $d$ for every $x$ such that the Springer map is a bijection over $Gx$, in which case $\dim L_{Gx}^{(d)} = 1$.

For $Q$ Dynkin, there are parametrizations of indecomposable graded projective modules in terms of Lyndon words, see [HMM12], which are not yet understood in the context of the decomposition theorem.

\textsuperscript{10}i.e. the underlying graph is a Dynkin diagram of type $A_n, D_n, E_6/7/8$. 

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Monoidal categorifications of the negative half of the quantum group

Again let \( Q \) be a finite quiver without loops. First Lusztig found the monoidal categorification of the negative half of the quantum group via perverse sheaves, then Khovanov and Lauda did the same with (f.g. graded) projective modules over quiver Hecke algebras. In the following theorem we are explaining the following diagrams of isomorphisms of twisted Hopf algebras over \( \mathbb{Q}(q) \).

\[
\begin{array}{c}
K_0(proj) \oplus H^G_s(Z) \otimes \mathbb{Q}(q) \\
\Rightarrow \\
\Rightarrow \\
U^- := U_q^-(Q) \\
K_0(P) \otimes \mathbb{Q}(q)
\end{array}
\]

In all three algebras there exists a notion of canonical basis which is mapped to each other under the isomorphisms. Also, there is a triangle diagram with isomorphisms defined over \( \mathbb{Z}[q, q^{-1}] \) which gives the above situation after applying \(- \otimes \mathbb{Z}[q, q^{-1}]\mathbb{Q}(q)\).

The negative half of the quantum group. The negative half \( U^- := U_q^-(Q) \) of the quantized enveloping algebra (defined by Drinfeld and Jimbo) associated to the quiver \( Q \) is defined via: Let \( a_{i,j} := \#\{ \alpha \in Q_1 \mid \alpha: i \rightarrow j, \text{ or } \alpha: j \rightarrow i \}, i \neq j \in Q_0 \). It is the \( \mathbb{Q}(q) \)-algebra generated by \( F_i, i \in Q_0 \) with respect to the (quantum Serre relations)

\[
\sum_{p=0}^{N+1} \left[ p, N + 1 - p \right] F_j^p F_i F_j^{N+1-p} = 0, \quad N = a_{i,j}, i \neq j
\]

where

\[
[n] := \prod_{k=1}^{n} \frac{q^k - q^{-k}}{q - q^{-1}}, \quad [n, m] := \frac{[n + m]}{[n][m]}.\]

Lusztig calls this \( \mathfrak{g} \)

A Hopf algebra is a bialgebra (i.e. an algebra which also has the structure of a coalgebra such that the comultiplication and counit are algebra homomorphisms) which also has an antipode, i.e. an anti-automorphism which is uniquely determined by the bialgebra through commuting diagrams. A twisted Hopf algebra differs from the Hopf algebra by: The comultiplication and the antipode are only homomorphisms if you twist the algebra structure by a bilinear form (see the example below). For more details on the definition see [LZ00]. The twisted \( \mathbb{Q}(q) \)-Hopf algebra structure is given by the following, it is by definition a \( \mathbb{Q}(q) \)-algebra which is \( \mathbb{N}^{Q_0} \)-graded and it has

1. (comultiplication)
   
   If we give \( U^- \otimes_{\mathbb{Q}(q)} U^- \) the algebra structure

   \[
   (x_1 \otimes x_2) (x'_1 \otimes x'_2) := q^{|x_2|} |x_1'| x_1 x'_1 \otimes x_2 x'_2
   \]

   where for \( x \in U^- \) we write \( |x| \in \mathbb{N}^{Q_0} \) for its degree and the symmetric bilinear form

   \[
   : \mathbb{Z}_0 \times \mathbb{Z}_0 \rightarrow \mathbb{Z}, \quad i \cdot i := 2, \quad i \cdot j := -a_{i,j} \text{ for } i \neq j
   \]

   Then the comultiplication is the \( \mathbb{Q}(q) \) algebra homomorphism

   \[
   U^- \rightarrow U^- \otimes U^-, \quad F_i \mapsto F_i \otimes 1 + 1 \otimes F_i
   \]

2. (counit) \( \epsilon: U^- \rightarrow \mathbb{Q}(q), \quad F_i \mapsto 0 \)

3. (antipode)
   
   Let \( U^-_{tw} \) be the algebra with the multiplication \( x \ast y := q^{|y|} |x| xy \)

   The antipode is the algebra anti-homomorphism

   \[
   U^- \rightarrow U^-_{tw}, \quad F_i \mapsto -F_i
   \]
Lusztig’s category of perverse sheaves. Lusztig writes complete dimension filtrations as words in the vertices \( i = (i_1, \ldots, i_d), \) \( i_t \in Q_0, \) set \( d := \sum_{i=1}^d i_t \) and defines

\[
L_i := (\pi_i)_{*}\mathbb{C}[e_i]
\]

where \( \pi_i : E_i := \text{GL}_d \times_{P} F_i \rightarrow RQ(d) \) is the quiver-graded Springer map and \( e_i = \dim_{\mathbb{C}} E_i. \) Let us call \( \mathcal{P}_{Q_0} \) the additive category generated by shifts of the \( L_i, i = (i_1, \ldots, i_d), i_t \in Q_0. \) The homomorphisms \( \text{Hom}(L_i, L_j[n]) \) in this category is zero unless \( d = \sum i_t = \sum j_k \) and then they are given by \( 1_j * H_{[n]}^{\text{GL}_d}(Z) * 1_i. \) It can be endowed with the structure of a monoidal category via

\[
L_i * L_j := L_{ij}
\]

where \( ij \) is the concatenation of the sequence \( i \) and then \( j. \)

Lemma 9. (Lusztig, [Lus91], Prop. 7.3) Let \( \mathcal{P} \) be the idempotent completion of \( \mathcal{P}_{Q_0} \) (i.e. we take the smallest additive category generated by direct summands of the \( L_i \) in \( D^{b}_{\text{GL}_d}(RQ(d)) \) and their shifts). It carries a monoidal structure and the inclusion induces

\[
K_0(\mathcal{P}_{Q_0}) = K_0(\mathcal{P})
\]

where the Grothendieck group has the ring structure from the monoidal categories and a \( \mathbb{Z}[q, q^{-1}] \)-module structure via the shift, i.e. \( q \cdot [M] := [M[1]], M \) an object in \( \mathcal{P}. \)

Remark. We call the monoidal category \( \mathcal{P} \) Lusztig’s category of perverse sheaves. Even though these are not perverse sheaves since we allow shifts of them and Lusztig originally defined them inside \( \bigcup D^{b}(RQ(d)) \) which of course gives a different category (for example in this category \( \text{Hom}(L_i, L_j[n]) = 1_j * H_{[n]}(Z) * 1_i. \) Nevertheless the two categories have the same Grothendieck group. In the view of the context here we think it is more appropriate to define it in the equivariant derived categories.

Remark. The previous lemma is no longer true if you allow your quiver to have loops.

For example if \( Q \) is the quiver with one loop. Then, let \( Z_n \) be the Steinberg algebra associated to \( (G = \text{GL}_n, B_n, \mathfrak{gl}_n, \mathfrak{n}_n) \) with \( B_n \subset \text{GL}_n \) the upper triangular matrices, \( \mathfrak{n}_n \) the Lie algebra of the unipotent radical of \( B_n. \) We claim

\[
K_0(\mathcal{P}) = \bigoplus_{n \in \mathbb{N}_0} K_0(\text{f.d. proj. graded } H_{[1]}(Z_n) - \text{modules})
\]

\[
= \bigoplus_{n \in \mathbb{N}_0} K_0(\text{f.d. simple graded } H_{[0]}(Z_n) - \text{modules})
\]

\[
= \bigoplus_{n \in \mathbb{N}_0} K_0(\text{f.d. graded } \mathbb{C} S_n - \text{modules}) = \left( \bigoplus_{n \in \mathbb{N}_0} K_0(\text{f.d. graded } \mathbb{C} S_n - \text{modules}) \right) \otimes_{\mathbb{Z}} \mathbb{Z}[q, q^{-1}]
\]

\[
= (\text{Symmetric functions}) \otimes_{\mathbb{Z}} \mathbb{Z}[q, q^{-1}]
\]

The first isomorphism is implied by the Corollary 0.2. The second equality is implied by semi-smallness of the classical Springer maps. For the third result see the section on classical Springer theory. The last equality is well-known, it maps the simple module \( S_{\lambda} (= \text{Specht module}) \) corresponding to a partition \( \lambda \) to the Schur function corresponding to \( \lambda. \)

But the category \( \mathcal{P}_{Q_0} \) corresponds to the submonoidal category given by finite direct sums of shifts of finite-dimensional free modules. This is a monoidal category generated by direct sums of shifts of one object \( E = S_1 \) and an arrow \( s : E^2 := E \otimes E \rightarrow E^2 \) of degree 0 with the relation \( (sE) \circ (Es) \circ (sE) = (Es) \circ (sE) \circ (Es) \) (see also [Rou11]). In this case \( K_0(\mathcal{P}_{Q_0}) = \mathbb{Z}[q, q^{-1}, T], [E] \rightarrow T \) which is much smaller than \( K_0(\mathcal{P}). \)

Now, let again be \( Q \) without loops. \( K_0(\mathcal{P}) \) has the structure of a twisted \( \mathbb{Z}[q, q^{-1}] \)-Hopf algebra. The algebra structure is given by the monoidal structure on \( \mathcal{P} \) which is defined by induction functors. A restriction functor for the category \( \mathcal{P} \) defines the structure of a coalgebra. For the geometric construction of these functors see [Lus91].
Theorem 0.7. (Lusztig, [Lus91], thm 10.17) The map
\[ \lambda_Q: \mathcal{U}^- \to K_0(\mathcal{P}) \otimes \mathbb{Z}_{[q,q^{-1}]} \mathbb{Q}(q) \]
\[ F_i \mapsto [L_i] \otimes 1, \quad i \in Q_0 \]

here we see \( i \in Q_0 \) as a sequence in the vertices of length 1. This defines an isomorphism of twisted \( \mathbb{Q}(q) \)-Hopf algebras.

Definition 2. We call \( B := \{ [L_i] \otimes 1 \mid i = (i_1, \ldots, i_d), i_t \in Q_0 \} \) canonical basis for \( K_0(\mathcal{P}) \otimes \mathbb{Q}(q) \).

We also call \( \lambda_Q^1(\mathcal{B}) \) canonical basis in \( \mathcal{U}^- \).

Also the image in \( K_0(\text{proj} \mathbb{Z} \oplus R^G_{\mathcal{L}}) \otimes \mathbb{Q}(q) \) is called canonical basis.

There are two intrinsic alternative definitions of the canonical basis for \( \mathcal{U}^- \) given by again Lusztig in [Lus90] for the finite type case and in general by Kashiwara’s crystal basis, see [Kas91].

Generators and relations for \( \mathcal{P}_{Q_0} \). This is due to Rouquier (cp. [Rou11]), it is the observation that the generators and relations of the quiver Hecke algebra rather easily give generators and relations for the monoidal category \( \mathcal{P}_{Q_0} \). In the category, we use the convention instead of \( E \to E' \) we write \( E \to E' \) is a morphism of degree \( n \). A composition \( g \circ f \) of a morphism \( f: E \to E' \) of degree \( n \) and \( g: E' \to E'' \) of degree \( m \) is the homomorphism \( E \to E'' \) of degree \( n + m \) given by \( E \overset{f}{\to} E'(n) \overset{g}{\to} E''(n + m) \).

Let \( Q \) be a quiver without loops. Let \( \mathcal{B} \) be the monoidal category generated by finite direct sums of shifts of objects \( E_a =: E_a(0), a \in Q_0 \) and arrows
\[ z_a: E_a \to E_a, \quad \sigma_{a,b}: E_aE_b \to E_bE_a, \quad a, b \in Q_0 \]
of degrees
\[ \deg z_a = 2, \quad \deg \sigma_{a,b} = \begin{cases} -2, & \text{if } a = b \\ 2h_{b,a}, & \text{if } a \neq b \end{cases} \]
where as before \( h_{a,b} := \#\{ \alpha \in Q_1 \mid \alpha: a \to b \}, \quad a, b \in Q_0 \), and assume relations

1. \( (s^2 = 1) \)
\[ \sigma_{ab} \circ \sigma_{ba} = \begin{cases} (-1)^{h_{b,a}}(E_bz_a - z_bE_a)^{h_{a,b} + h_{b,a}}, & \text{if } a \neq b \\ 0, & \text{if } a = b \end{cases} \]

2. \( \) (straightening rule)
\[ \sigma_{ab} \circ z_aE_b - E_bz_a \circ \sigma_{ab} = \begin{cases} 0, & \text{if } a \neq b, \\ E_aE_a, & \text{if } a = b, \end{cases} \]
\[ \sigma_{ab} \circ E_a z_b - z_bE_a \circ \sigma_{ab} = \begin{cases} 0, & \text{if } a \neq b, \\ -E_aE_a, & \text{if } a = b, \end{cases} \]

3. \( \) (braid relations) for \( a, b, c \in Q_0 \) we have the following inclusion of \( \mathbb{C} \)-algebras. Let \( \mathbb{C}[\alpha_s, \alpha_t] \) be the set of polynomials in \( \alpha_s, \alpha_t \).
\[ J_{a,b,c}: \mathbb{C}[\alpha_s, \alpha_t] \to \text{End}_{\mathcal{B}}(E_aE_bE_c) \]
\[ \alpha_s \mapsto E_a z_bE_c - z_aE_bE_c \]
\[ \alpha_t \mapsto E_aE_b z_c - E_a z_b E_c, \]

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we set \( t(\alpha_s^h) := (\alpha_s + \alpha_i)^h =: s(\alpha_s^h) \in \mathbb{C}[\alpha_s, \alpha_i], h \in \mathbb{N}_0 \). Then, the relation is

\[
\begin{align*}
\sigma_{ab} E_c \circ E_a \sigma_{eb} \circ \sigma_{ca} E_b - E_b \sigma_{ca} \circ \sigma_{da} E_d \circ E_{ab} &= J_{bab}(\alpha_s^h, \alpha_i) - J_{bab}(-\alpha_s^h, \alpha_i) \\
&= \begin{cases} 
J_{bab}(\alpha_s^h, \alpha_i) & \text{if } a = c, a \neq b, \\
0 & \text{else},
\end{cases}
\end{align*}
\]

for \( i = (i_1, \ldots, i_n), i \in Q_0 \) we set \( E_i := E_{i_1} E_{i_2} \cdots E_{i_n} \). Let \( I_d := \{ i = (i_1, \ldots, i_n) \mid \sum_i i_t = d \} \).

Then, by construction there is an isomorphism of algebras

\[
R_d^{Gl} \rightarrow \bigoplus_{i,j \in I_d} \text{Hom}(E_i, E_j)
\]

\[
1_i \mapsto id_{E_i},
\]

\[
z_i(t) \mapsto E_{i_1} E_{i_2} \cdots E_{i_{t-1}} z_i E_{i_{t+1}} \cdots E_{i_n},
\]

\[
\sigma_i(s) \mapsto E_{i_1} \cdots E_{i_{t-1}} \sigma_i E_{i_{t+1}} \cdots E_{i_n}, \quad \text{if } s = (t, t + 1) \in S_n
\]

**Theorem 0.8.** ([Rou11]) There is an equivalence of monoidal categories

\[
P_{Q_0} \rightarrow \mathcal{B}
\]

\[
L_i \mapsto E_i
\]

which is on morphisms the isomorphism of algebras from above.

Since we have not more knowledge on the decomposition theorem for quiver-graded Springer maps, we can not expect to find a similar easy description for the category \( \mathcal{P} \).

**Khovanov and Lauda’s catgorification of the negative half of the quantum group.** Many years later Khovanov and Lauda have a different approach to the same monoidal categorification as Lusztig. Instead the category \( \mathcal{P} \) they consider the category of projective graded (f.g.) modules over quiver Hecke algebras \( R_d := R_d^{Gl}, d \in \mathbb{N}_0 \)

\[
\text{proj}^2 \bigoplus_{d \in \mathbb{N}_0} R_d
\]

It is easy to see that we have natural injective maps \( \mu: R_d \otimes R_e \rightarrow R_{d+e} \) compatible with the algebra multiplication. We write \( 1_{d,e} := \mu(1 \otimes 1) \). From this there are (well-defined see [KL09],section 2.6) induction and restriction functors

\[
\text{Ind}_{d,e}^d: \text{proj}^2(R_d \otimes R_e) \rightarrow \text{proj}^2(R_d \otimes R_e)
\]

\[
\text{Res}_{d+e}^d: \text{proj}^2(R_d \otimes R_e) \rightarrow \text{proj}^2(R_d \otimes R_e)
\]

The induction functor gives \( \text{proj}^2 \bigoplus_{d \in \mathbb{N}_0} R_d \) the structure of a monoidal category via \( X \circ X' := \text{Ind}_{d,e}^d X \otimes X' \) where \( X \otimes X' \) is the natural graded \( R_d \otimes R_e \)-module structure.

The twisted \( \mathbb{Z}[q, q^{-1}]-\text{Hopf algebra structure on } K_0(\text{proj}^2 \bigoplus_{d \in \mathbb{N}_0} R_d^{Gl}) \) is given by:

Obviously, it is a \( \mathbb{Z}[q, q^{-1}]-\text{algebra with } q \text{ operating as the shift (1) on the graded modules, i.e. } q \cdot [M] := [M(1)]. \)

The comultiplication is given by \([\text{Res}]\cdot[P] := \sum_{d,e} d+e = f [\text{Res}_{d,e}^d(P)]\).

**Theorem 0.9.** ([Khovanov, Lauda, [KL09]) The map

\[
\kappa_Q: \mathcal{U} \rightarrow K_0(\text{proj}^2 \bigoplus_{d \in \mathbb{N}_0} R_d^{Gl}) \otimes \mathbb{Q}(q)
\]

\[
F_i \mapsto [\text{Res}_{d}^i] \otimes 1, \quad i \in Q_0
\]

where we consider \( i \in Q_0 \) as an element in \( \mathbb{N}_0 \), is an isomorphism of twisted \( \mathbb{Q}(q) \) Hopf algebras.
We want to point out: Khovanov and Lauda invented the quiver Hecke algebra, which later turned out (by Varagnolo and Vasserot’s result) to be the same as the Steinberg algebra of quiver-graded Springer theory. Their explicit description (generators and relations for the algebra) and diagram calculus (which we leave out in this survey) are a major step forward from Lusztig’s description. Their work sparked a big interest in this subject.

Remark. Let $Q$ be a Dynkin quiver. Then, the objects of the category $\mathcal{P}$ are direct sums of shifts of $IC_{O}$ where $O \subset R_{Q}(d)$ is a $\text{Gl}_{d}$-orbit (we do not write a local system if the trivial local system is meant). These are in bijection with isomorphism classes of $\mathbb{C}Q$-modules. The monoidal structure on $\mathcal{P}$ is constructed such that $K_{0}(\mathcal{P}) \otimes \mathbb{Q}(q)$ is the twisted Ringel-Hall algebra (over $\mathbb{Q}(q)$).

The isomorphism between the twisted Ringel-Hall algebra and the negative half of the quantum group associated to the underlying graph of the quiver is a theorem of Ringel, see for example [Rin93].
| (G, B, g, u) classical ST | $H_{\text{top}}(Z, \mathbb{C})$ | $H_\ast(Z, \mathbb{C})$ | $H^G_\ast(Z, \mathbb{C})$ | $K_0^{G \times \mathbb{C}^\ast}(Z) \otimes_{\mathbb{Z}} \mathbb{C}$ |
|--------------------------|----------------|----------------|-----------------|-----------------------------|
| (G, B, {0}, {0}) nil ST | CW | $\mathbb{C}[t] / I_W \# \mathbb{C}[W]$ | $\mathbb{C}[t] \# \mathbb{C}[W]$ | affine Hecke algebra |
| i.e. $Z = G/B \times G/B$ | $\mathbb{C}$ | End$_{\mathbb{C}-\text{lin}}(H^\ast(G/B))$ | End$_{H^G_\ast(G/B)}(H^G_\ast(G/B))$ | affine nil Hecke algebra |
| quiver-graded ST (complete dim filtrations) | ? | $R_d$ | quiver Hecke algebra (= KLR-algebra) | ? |

Further known examples are:

1. There is an exotic Springer theory (by Kato [Kat09], [Kat11], Achar and Henderson [AH08]). The Steinberg algebra $K_0^{G \times \mathbb{C}^\ast}(Z) \otimes_{\mathbb{Z}} \mathbb{C}$ is isomorphic to the Hecke algebra with unequal parameters of type $C_n^{(1)}$. Also Kato gave an exotic Deligne-Langlands correspondence.

2. Quiver-graded Springer theory for the oriented cycle quiver (allowing only nilpotent representations) gives that $H^G_\ast(Z)$ is isomorphic to the quiver Schur algebra (compare the work of Stroppel and Webster, [SW11].)
Figure 1: Springer Theory and related fields
0.1 Literature review

Collapsings of homogeneous vector bundles are quite ubiquitous (for example see [Kem76]).

(1) Classical Springer Theory (cp remark):

Classical Springer Theory is usually defined for semi-simple algebraic groups and goes back to first Springer [Spr76],[Spr78], then Kazhdan-Lusztig [KL80], Slodowy [Slo80a], Lusztig [Lus81], Rossmann [Ros91]) and the defined convolution operations differ between each other by at most multiplication with a sign character (see [Hot81]).

Also relevant is the earlier work on the topology of Springer fibres of Spaltenstein (see [Spa76], [Spa77]) and Vargas (see [Var79]) and the Springer map already occurs in Steinberg’s work (for example [Ste74]). A book on classical Springer Theory is written by Borho, Brylinski and Mac Pherson [BBM89]. A comprehensive treatment can be found in chapter 3 of [CG97] and a short one using perverse sheaves in [Ara01] if you speak French. I apologize to the many other authors who I do not mention.

(2) Quiver-graded Springer Theory:

First considered by Lusztig, see [Lus91]. Later, Reineke started to look at it as an analogue of the classical Springer theory, see [Rei03], also see [Wol09].

The quiver Hecke algebras as Steinberg algebras first occurred in the work of Varagnolo and Vasserot, cp. [Var09], and independently also in Rouquier’s article [Ron11].

Open problems/ wild speculations:

(O1) Are Springer fibre modules always semi-simple modules over the Steinberg algebra?

(O2) Which Steinberg algebras are affine cellular algebras?

Which have finite global dimension?

Partial answers: Brundan, Kleshchev and McNamara showed that KLR-algebras for Dynkin quivers are affine cellular (see [BKM12]).

Certain Steinberg algbras (including KLR-algebras for Dynkin quivers) have been shown to have finite global dimension (see [Kat13]). In [BKM12], the authors write that they expect that KLR-algebras have finite global dimension if and only if the quiver is Dynkin.

(O3) Are there Kazhdan-Lusztig polynomials and even a theory of canonical basis for Steinberg algebras?

Do we have Standard modules for Steinberg algebras?

Partial answers: Standard modules have been defined in [Kat13] under some assumptions (finitely many orbits in the image of the Springer map, ...).

The original definition of Kazhdan-Lusztig polynomials has been inspired by studying a base change between two bases in the Steinberg variety associated to classical Springer theory.

(O4) Can we describe noncommutative resolutions of singularities corresponding to Springer maps?

Can we adapt the notion of a noncommutative resolution of singularities using constructible instead of coherent sheaves?

Partial answers exists for the coherent sheaf theory: Bezrukavnikov studied it for classical Springer theory (see [Bez06]) and for quiver-graded Springer theory with $Q = A_2$ Buchweitz, Leuschke and van den Bergh studied noncommutative resolutions (see [BLB10], [BLB11]).

(O5) Does there exist a Schur-Weyl theory relating classical and quiver-graded Springer theory (for example via Morita equivalences of the associated Steinberg algebras)?

Partial answers only for type A-situations (so maybe it only exists in this case): due to Brundan, Kleshchev [BK09], see also for example [Web13].
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