ALGEBRAIC ELLIPTIC COHOMOLOGY AND FLOPS II: SL-COBORDISM

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Abstract. In this paper, we study the algebraic Thom spectrum MSL in the motivic stable homotopy category of Voevodsky over an arbitrary perfect field \( k \). Using the motivic Adams spectral sequence, we compute the geometric part of the \( \eta \)-completion of MSL. As an application, we study the Krichever’s elliptic genus with integral coefficients, restricted to MSL. We determine its image, and identify its kernel as the ideal generated by differences of SL-flops. This was proved by B. Totaro in the complex analytic setting. In the appendix, we prove some convergence properties of the motivic Adams spectral sequence.

1. Introduction

1.1. Motivations. In this paper, we study the algebraic Krichever’s elliptic genus, when restricted to the SL-cobordism ring with integral coefficients.

Elliptic genera in topology have a renowned rigidity property, conjectured by Witten and proved by many others, including Bott-Taubes [BT89], Liu [Liu95], and Ando [An03], which says that for any Spin-manifold with \( S^1 \)-action, the \( S^1 \)-equivariant genus does not depend on the equivariant parameter. Similarly, there is a SU-rigidity theorem for certain elliptic genus with two parameters (referred to as the Krichever’s elliptic genus) proved by Krichever [K90] and Höhn [Höh91], which says the \( S^1 \)-equivariant Krichever’s elliptic genus of any SU-manifold is a constant. In [T00] Totaro proved that a genus has the SU-rigidity property if and only if the values of two birational manifolds related by a flop are equal. Further more, he proved that the Krichever’s elliptic genus is universal with respect to this property. The proof in [T00] uses topological constructions, which do not have direct counterparts for varieties over an arbitrary field.

In [LYZ13], the authors gave a purely algebraic proof of the fact that the kernel of Krichever’s elliptic genus coincides with the ideal generated by differences of flops, a proof which works for varieties over an arbitrary perfect field. Moreover, we obtained the existence of a corresponding motivic oriented cohomology theory representing elliptic cohomology. With rational coefficients, we have a description of the coefficient ring of the motivic elliptic cohomology. For a summary of the main results in [LYZ13], see § 6.1.

The rigidity property in topology suggests that after the restriction to SL-cobordism, this algebraic Krichever’s elliptic genus has better properties, which is the subject of the present paper. For example, it is natural to expect, as in topology, the image under the elliptic genus of the GL-cobordism is non-noetherian, but the image of SL-cobordism is.

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The coefficient ring of algebraic SL-cobordism has not been fully investigated, unlike its topological analogue which goes back to a classical result of Novikov in the study of Adams spectral sequences [Nov62]. In the present paper, following Novikov’s approach, we explore the motivic Adams spectral sequence to get necessary information about the coefficient ring of SL-cobordism for the study of elliptic genus. Along the way, we study convergence property of motivic Adams spectral sequence in Appendix A, with the results summarized in § 1.3.

1.2. Main theorems. Let $k$ be an arbitrary perfect field. Let $p$ be the exponential characteristic of $k$. That is, $p = 	ext{char} k$ if $k > 0$, $p = 1$ if $k = 0$. Let $SH(k)$ be the motivic stable homotopy category of $\mathbb{P}^1$-spectra (for the conventions we are following, see § 2). For any spectrum $X \in SH(k)$, let $X^\wedge_{\eta}$ be the completion of $X$ along the algebraic Hopf map $\eta : \mathbb{G}_m = S^{1,1} \to S^0 = S^{0,0}$. Let $MGL, MSL \in SH(k)$ be the Thom spectra of GL and SL respectively. There is a natural map $MGL^\wedge_{2p}[1/2p]^* \to MGL[1/2p]^*$ (Lemma 5.2), which we will prove to be an embedding.

Let $\text{Ell}$ be the ring $\mathbb{Z}[a_1, a_2, a_3, 1/2a_4]$, with an elliptic curves on it defined as the base-change of the Weierstrass curve $y^2 + \mu_1 xy + \mu_3 y = x^3 + \mu_2 x^2 + \mu_4 x + \mu_6$ along

$$
\mu_1 \mapsto 2a_1, \quad \mu_2 \mapsto 3a_2 - a_1^2, \quad \mu_3 \mapsto -a_3, \quad \mu_4 \mapsto -\frac{1}{2}a_4 + 3a_2^2 - a_4 a_3, \quad \mu_6 \mapsto 0.
$$

The local uniformizer $t = y/x$ of the elliptic curve induces a ring homomorphism $\phi : MGL^\wedge_{2p} \to \text{Ell}$, which is the algebraic Krichever’s elliptic genus [LYZ13, § 3.1]. The restriction of $\phi[1/2p] : MGL[1/2p]^* \to \text{Ell}[1/2p]^*$ to $MGL^\wedge_{2p}[1/2p]^*$ is denoted by $\overline{\phi}$.

Let $\mathcal{I}_{fl} \subseteq MGL[1/p]^*$ be the ideal generated by differences of flops. Define the ideal of SL-flops $\mathcal{I}_{fl}^\text{SL} \subseteq MGL^\wedge_{2p}[1/2p]^*$ to be $MGL^\wedge_{2p}[1/2p]^* \cap I_{fl}[1/2p]$. In this paper, we prove

**Theorem A.** The kernel of $\overline{\phi}$ in $MGL^\wedge_{2p}[1/2p]^*$ is $\mathcal{I}_{fl}^\text{SL}$; the image ring is a free polynomial ring $\mathbb{Z}[1/2p][3a_2, a_3, a_4]$, with $\text{deg}(a_i) = -i$.

Although this statement is similar to its topological analogue [T00, Theorem 6.1], the proof here is more involved. Not knowing the homotopy groups of motivic Thom spectrum $MSL$, we need to study the motivic Adams spectral sequence, which differs from its classical analogue in many aspects. One of the major differences is the non-unipotence of the algebraic Hopf map $\eta$.

Thanks to the general convergence properties of motivic Adams spectral sequence in Appendix A (summarized in § 1.3), we calculate of the $E_2$ page of the mod-$l$ motivic Adams spectral sequence converging to homotopy group of MSL, building up on [DI10]. We find degeneration of certain differentials (Proposition 5.6), hence, prove the following, which is the key ingredient in the proof of Theorem A.

**Theorem B.**

1. The ring $MGL^\wedge_{2p}[1/2p]^*$ is a free polynomial ring over $\mathbb{Z}[1/2p]$ with infinitely many generators, lying in degree 2, 3, 4, $\ldots$, which maps injectively into $MGL[1/2p]^*$.

2. An element $[X] \in MGL^\wedge_{2p,*}$ is a polynomial generator of degree $n$ of $MGL^\wedge_{2p}[1/2p]^*$ if and only if the following property holds:

$$
S^n(X) = \begin{cases} 
\pm l \cdot 2^b p^a & \text{if } n \text{ is a power of an odd prime } l; \\
\pm 2^b p^a & \text{if } n + 1 \text{ is a power of an odd prime } l; \\
\pm 2^b p^a & \text{otherwise},
\end{cases}
$$

where $S^n$ is the Chern number of degree $n$ associated to the symmetric polynomial $x_1^n + \cdots + x_n^n$ and $a, b$ are non-negative integers.
1.3. Convergence of motivic Adams spectral sequence. In topology, the convergence properties of the Adams spectral sequences were proven in [A58]. A simpler approach was later given by Bousfield in [Bous79], based on an idea of localization going back to Ravenel [Rav77].

In Appendix A, we study the convergence properties of the motivic Adams spectral sequence [Mor99], following similar method as in [Bous79]. The role of Postnikov tower in loc. cit. is replaced by the slice tower of Voevodsky [Voev02], which was studied recently in [RSØ16], on which our investigation relies.

Without going to the technical details, we state the motivic analogue of Bousfield’s theorem. For any spectrum $X \in \text{SH}(k)$, let $X^\wedge_l$ be the homotopy inverse limit of the system $X/(l^n) \to X/(l^{n-1})$, where $X/(l^n)$ is the homotopy cofiber of the multiplication map $l^n : X \to X$.

**Theorem C** (Theorem A.1). Let $l$ be a prime different than char($k$). Let $Y \in \text{SH}(k)$ be any motivic spectrum. Let $Y^\wedge_{HZ/l}$ be the homotopy inverse limit of Adams tower (14). If $Y$ has a cell presentation of finite type ([RSØ16, § 3.3]) and satisfies condition (Fin), then we have $Y^\wedge_{HZ/l} \cong Y^\wedge_{H/l}$.

**Organization of the paper.** In § 2, we recall the structure of motivic Steenrod algebra found by Voevodsky. In § 3, we study modules over the motivic Steenrod algebra coming from the Thom spectra MGL and MSL, which will be used in the calculation of the motivic Adams spectral sequence. In § 4, we construct an MSL class associated to a Calabi-Yau variety $X$ together with a trivialization of its canonical bundle, lifting the MGL-class of $X$. In § 5, we use the motivic Adams spectral sequence to prove Theorem B. In § 6, we use Theorem B to conclude the proof of Theorem A.

2. Preliminaries

Let $\text{Sm}_k$ be the category of smooth varieties over $k$. Let $\text{SH}(k)$ be the motivic stable homotopy category of $\mathbb{P}^1$-spectra. Recall that one has the infinite $\mathbb{P}^1$-suspension functor $\Sigma_\infty : \text{Sm}_k \to \text{SH}(k)$ and suspension functors

$$
\Sigma^n_S, \Sigma^n_{G_m}, \Sigma^n_{\text{et}} \cong \Sigma^n : \text{SH}(k) \to \text{SH}(k)
$$

for all $n \in \mathbb{Z}$. $\text{SH}(k)$ is a triangulated tensor category with translation $\Sigma_1$, tensor product $(\mathcal{E}, \mathcal{F}) \mapsto \mathcal{E} \wedge \mathcal{F}$ and unit the motivic sphere spectrum $\mathbb{S}_k := \Sigma^n_{\mathbb{Z}} \text{Spec} k$. An object $\mathcal{E}$ of $\text{SH}(k)$ and integers $n, m$ define a functor $\mathcal{E}^{n,m}$ from $\text{Sm}_k^{op}$ to the category of abelian groups by

$$
\mathcal{E}^{n,m}(X) := \text{Hom}_{\text{SH}(k)}(\Sigma^n_{\mathbb{P}^1} X, \Sigma^m_{\mathbb{S}} \Sigma^m_{G_m} \mathcal{E}).
$$

For simplicity we denote $\bigoplus_{n,m} \mathcal{E}^{n,m}(X)$ by $\mathcal{E}^{\ast,*}(X)$, and $\bigoplus_{n} \mathcal{E}^{2n,n}(X)$ by $\mathcal{E}^{\ast,*}(X)$. Recall that $\text{MGL}^{\ast,*}(k)$ is isomorphic to the Lazard ring $\mathbb{Laz}$.

2.1. The mod-$l$ motivic Steenrod algebra. In this section, we recall some basic facts of the motivic mod $l$ Steenrod algebra $A^{\ast,*}$ introduced by Voevodsky in [Voev03]. For the rest of the paper (with the exception of Appendix A), $l$ will be an odd prime different from char($k$). For such $l$, $A^{\ast,*}$ has similar behavior as its topological counterpart. Nevertheless, for the convenience of the readers, we collect the properties relevant to us.

Let $H^{\ast,*}(X, \mathbb{Z}/l) := \bigoplus_{p, q \in \mathbb{Z}} H^{p,q}(X, \mathbb{Z}/l)$ be the motivic cohomology of a smooth scheme $X$ with $\mathbb{Z}/l$-coefficients, and let $H^{\ast,*} := H^{\ast,*}(\text{Spec}(k), \mathbb{Z}/l)$. We have the following vanishing theorem.

**Theorem 2.1.** [MVW06] For every smooth variety $X$ and any abelian group $G$, we have $H^{p,q}(X, G) = 0$, if $p > q + \text{dim}(X)$ or $p > 2q$. 
Let $A^{*,*} := A^{*,*}(k, \mathbb{Z}/l)$ be the mod-$l$ motivic Steenrod algebra. By definition, $A^{*,*}$ is the subalgebra of End($H^{*,*}$) generated by the motivic Steenrod operator $P^i, i \geq 0$, the Bockstein homomorphism $\beta$, and operators of the form $u \mapsto au$, where $a \in H^{*,*}$. Recall the following Milnor's basis [M58], [Voev03, Section 13]

$$\{p(E, R) = Q(E)P^R \in A^{*,*} \mid R = (r_1, r_2, \cdots), E = (e_0, e_1, \cdots)\},$$

where $R$ and $E$ are two sequences of integers which are non-zero for finitely many $r_i, e_i$, such that $r_i \geq 0$, and $e_i = 0$ or $1$. For any $r \in \mathbb{N}$, denote

$$Q_r := Q(0, \cdots, 0, 1, 0, \cdots),$$

and $P^r := P(0, \cdots, 0, 1, 0, \cdots)$, where $1$ is on the $r$-th place.

In particular, $P^{(0, \cdots, 0)}$ is the operator of Steenrod powers $P^r$, and $Q_0 = \beta$ is the Bockstein homomorphism. These have similar properties as their topological counterparts.

**Lemma 2.2.** The motivic Steenrod algebra $A^{*,*}$ as a left $H^{*,*}$-algebra, has generators $Q_r$, for $r \geq 0$ and $P^r$, for $r \geq 1$. Furthermore, the following properties hold.

1. The bidegree of $Q_r$ is $(2l^r - 1, l^r - 1)$, and the bidegree of $P^r$ is $(2(l^r - 1), (l^r - 1))$.
2. The set of monomials in $\{P^r\}$ and $\{Q_r\}$ in a suitably defined order form a basis of the left module over $H^{*,*}$.
3. The elements $Q_r$ anti-commute.
4. (The Cartan formulas. See, e.g., [Hoy13, Lemma 5.3].) Let $\Delta$ be the coproduct of $A^{*,*}$, then

$$\Delta(Q_r) = Q_l \otimes 1 + 1 \otimes Q_i + \sum_{j=1}^i \sum_{E_1 + E_2 = [i-j+1]} \{p^jQ_{i-j}Q(E_1) \otimes Q_{i-j}Q(E_2),$$

$$\Delta(P^R) = \sum_{E= \{e_0, e_1, \cdots\}} \sum_{R_1 + R_2 = R - E} \tau^{\sum_{i=0}^r e_i}Q(E)P^R_1 \otimes Q(E)P^R_2.$$

Note that, when the field $k$ is algebraically closed, we have $H^{*,*} = \mathbb{Z}/l[\tau]$. In this case, $\Delta(Q_r) = Q_l \otimes 1 + 1 \otimes Q_i$.

### 3. Modules over the motivic Steenrod algebra

3.1. **A quotient of the Steenrod algebra.** Let $B \subset A^{*,*}$ be the $\mathbb{Z}/l[\rho]$-subalgebra generated by $\{Q_i\}_{i \geq 0}$. Let $M_B := A^{*,*}/A^{*,*}(Q_0, Q_1, \cdots)$ be the quotient of $A^{*,*}$ by the left ideal generated by $(Q_0, Q_1, \cdots)$. From Lemma 2.2 (2) we know that $M_B$, is isomorphic as a left $H^{*,*}$-module, to the polynomial ring $M_B = H^{*,*}[P^1, P^2, P^3, \cdots]$.

By the same argument as in [Nov62, Lemma 8], one has the following.

**Lemma 3.1.** There is an isomorphism of graded abelian groups

$$\bigoplus_{s,t,u} \text{Ext}^{s,(s+u)}_{A^{*,*}}(M_B, H^{*,*}) \cong \bigoplus_{s,t,u} \text{Ext}^{s,(s+u)}_{B}(\mathbb{Z}/l[\rho], H^{*,*}).$$

We now compute the right hand of the isomorphism in Lemma 3.1. By Lemma 2.2, $\{Q_r\}_{r \geq 0}$ are anti-commutative. Therefore, $B$ is an exterior algebra over $\mathbb{Z}/l[\rho]$ on $\{Q_r\}$. Let $V$ be the $\mathbb{Z}/l$-vector space spanned by $\{Q_r\}_{r \geq 0}$. We have the following Koszul resolution of $\mathbb{Z}/l[\rho]$ by free $B$-modules:

$$\cdots \rightarrow \text{Sym}^2 V^\vee \otimes_{\mathbb{Z}/l} B \rightarrow V^\vee \otimes_{\mathbb{Z}/l} B \rightarrow B \rightarrow \mathbb{Z}/l[\rho] \rightarrow 0.$$

**Lemma 3.2.** (1) For any $u$, we have $\text{Ext}^{s,(s+u)}_{B}(\mathbb{Z}/l[\rho], H^{*,*}) = 0$ if $t > 2u$. 
Lemma 3.3. The $A^*$-module homomorphism $M_B \rightarrow M_B \otimes_{H^*} M_B$ induces a ring structure on $\text{Ext}^{t,(s,t,u)}(M_B, H^*)$.

In particular, the isomorphism in Lemma 3.1 is an isomorphism of graded algebras.
Proposition 3.4. We have the following modules of \( A^{*,*} \).

\[ H^{*,*} := H^{*,*}(k, \mathbb{Z}/l), \quad H^{*,*}({\text{MSL}}) := H^{*,*}(\mathbb{MGL}, \mathbb{Z}/l), \quad \text{and} \quad H^{*,*}({\text{MGL}}) := H^{*,*}(\mathbb{MGL}, \mathbb{Z}/l). \]

In this section, we study in detail those modules. The main result of this section is the following.

The ring structure on \( \text{Ext}^{*,*} \) we are looking for is the composite of (2) and (3). The associativity follows from the coassociativity of \( \Psi^* \).

The product on \( \text{Ext}^{*,*}_A(M_B, H^{*,*}) \) we have the following modules of \( \mathbb{MGL} \) is a direct sum of cyclic modules over \( A^{top} \): \( H^{top}(\mathbb{MGL}) \equiv \bigoplus_{i \in \mathbb{Z}/l} M_B^{top} u_i \), where each summand \( M_B^{top} u_i \) is isomorphic to \( M_B^{top} \), with generator \( u_i \) in \( \deg(u_i) = 2|\iota| \). Similarly, we have the isomorphism \( H^{top}(\mathbb{MSL}) \equiv \bigoplus_{\iota \in \mathbb{Z}/l} M_B^{top} u_\iota \) of \( A^{top} \)-modules.
Recall that $H^{*,*}(\text{MGL})$ is generated by Chern classes. Hence, for any partition $\lambda$, there is a well-defined monomial symmetric function $u_{\lambda} \in H^{*,*}(\text{MGL})$. We have the $A^{*,*}$-module homomorphism

$$\Phi_{\lambda}: M_{\Lambda}u_{\lambda} \to H^{*,*}(\text{MGL}), \ a \cdot u_{\lambda} \mapsto a(u_{\lambda}),$$

where the element $u_{\lambda}$ has bidegree $(2|\lambda|, |\lambda|)$. Similarly, if $\lambda \in P_1$ is $l$-admissible, we have the module homomorphism $\Phi_{\lambda}: M_{\Lambda}u_{\lambda} \to H^{*,*}(\text{MSL})$. As in the topological setting, we have the following

**Lemma 3.5.** As $A^{*,*}$-modules, we have the following isomorphisms induced by $\Phi_{\lambda}$

$$H^{*,*}(\text{MGL}) \cong \bigoplus_{\lambda \in P_1} M_{\Lambda}u_{\lambda}, \ H^{*,*}(\text{MSL}) \cong \bigoplus_{\lambda \in P_1} M_{\Lambda}u_{\lambda}.$$

**Proof.** The arguments for the two isomorphisms are similar. For simplicity, we will only prove the second one and leave the first to the readers.

We check the map $\Phi: \bigoplus_{\lambda \in P_1} M_{\Lambda}u_{\lambda} \to H^{*,*}(\text{MSL})$ induced by $\{\Phi_{\lambda}\}$ is an isomorphism. Let $R^{\text{top}}: H^{2*,*}(\text{MSL}) \to H^{\text{top}}_{\text{MSL}}$ be the topological realization, which is an isomorphism. We have the following commutative diagram

$$\begin{array}{ccc}
\bigoplus_{\lambda \in P_1} (M_{\Lambda} \otimes H^{*,*} \mathbb{Z}/l)u_{\lambda} & \xrightarrow{\Phi \otimes H^{*,*} \mathbb{Z}/l} & \bigoplus_{\lambda \in P_1} (M_{\Lambda}^{\text{top}})u_{\lambda} \\
\Phi \otimes H^{*,*} \mathbb{Z}/l & \cong & \Phi \otimes H^{*,*} \mathbb{Z}/l \\
H^{2*,*}(\text{MSL}) & \xrightarrow{R^{\text{top}}} & H^{\text{top}}_{\text{MSL}}
\end{array}$$

In the diagram, all the other three morphisms are isomorphism. Therefore, $\Phi \otimes H^{*,*} \mathbb{Z}/l$ is also an isomorphism. By Nakayama Lemma, we deduce that $\Phi$ is surjective. Recall that $H^{*,*}(\text{MSL}) \cong H^{*,*}[c_2, c_3, \ldots]$, which in particular is a free $H^{*,*}$-module. Therefore, using Nakayama Lemma again, and taking into account the fact that $\ker(\Phi \otimes H^{*,*} \mathbb{Z}/l) = 0$, we see that $\ker \Phi = 0$. This completes the proof. \hfill $\square$

Similar to Lemma 3.3, the multiplication $\text{MSL} \wedge \text{MSL} \to \text{MSL}$ induces a map $H^{*,*}(\text{MSL}) \to H^{*,*}(\text{MSL}) \otimes H^{*,*}(\text{MSL})$ of $A^{*,*}$-modules. By Lemma 3.5, it induces the bottom map of the following diagram

$$\begin{array}{ccc}
H^{*,*}(\text{MSL}) & \xrightarrow{\Delta} & H^{*,*}(\text{MSL}) \otimes H^{*,*}(\text{MSL}) \\
\bigoplus_{\lambda \in P_1} M_{\Lambda}u_{\lambda} & \xrightarrow{\Delta} & \bigoplus_{\lambda \in P_1} (M_{\Lambda} \otimes H^{*,*}) (\bigoplus_{\lambda \in P_1} M_{\Lambda}u_{\lambda}),
\end{array}$$

where

$$\Delta(u_{\lambda}) = \sum_{(\lambda_0, \lambda_2) = \lambda, \lambda_0 \neq \lambda_2} [u_{\lambda_0} \otimes u_{\lambda_2} + u_{\lambda_2} \otimes u_{\lambda_0} + \sum_{(\lambda_1, \lambda_3) = \lambda} u_{\lambda_1} \otimes u_{\lambda_3}] + \sum_{(\lambda_0, \lambda_1) = \lambda} u_{\lambda_0} \otimes u_{\lambda_1}$$

modulo the augmentation ideal $\ker(H^{*,*} \to \mathbb{Z}/l)$.

Let $z_{\lambda} \in \text{Ext}^{(2|\lambda|, |\lambda|)}_{A^{*,*}}(H^{*,*}(\text{MSL}), H^{*,*}(\text{MSL}))$ be elements such that $(z_{\lambda}, u_{\lambda'}) = \delta_{\lambda, \lambda'}$. Hence, in the algebra $\bigoplus_{\lambda} \text{Ext}^{(2|\lambda|, |\lambda|)}_{A^{*,*}}(H^{*,*}(\text{MSL}), H^{*,*}(\text{MSL}))$, one has the relation $z_{\lambda}z_{\lambda'} = z_{\lambda + \lambda'}$ modulo the ideal $\ker(H^{*,*} \to \mathbb{Z}/l)$. Now Proposition 3.4 follows from the above lemmas.
4. The MSL-class of a Calabi-Yau variety

For $X$ a smooth projective scheme of dimension $d_X$ over our base-field $k$, there is a corresponding cobordism class $[X]_{\text{MGL}} \in \text{MGL}^{-2d_X,-d_X}(k)$. If $k \subset \mathbb{C}$, this can be constructed in a number of ways, for instance, one can use the canonical ring homomorphism $\phi_{\text{MGL}} : \text{Lat}^* \to \text{MGL}_{2*}(k)$ classifying the formal group law for MGL, compose with the similarly defined isomorphism $\phi_{\text{MU}} : \text{Lat}^* \to \text{MU}^{2*}$ and apply $\phi_{\text{MGL}} \circ \phi_{\text{MU}}^{-1}$ to the cobordism class of the compact complex manifold $X(\mathbb{C})$. Over a general field, one can use Merkurjev’s result [Merk02, Theorem 8.2] describing $\text{Lat}^*$ as the subring of $\mathbb{Z}[b_1, b_2, \ldots]$ defined via the Chern numbers of smooth projective varieties over $k$. For $I = (i_1 \geq \ldots \geq i_s) \in \mathbb{N}^s$ a partition, with $\sum i_j = d_X$, we have the Conner-Floyd Chern class, $c_I$, corresponding to the minimal symmetric sum $\sum \epsilon_1^{i_1} \cdots \epsilon_s^{i_s}$ containing the monomial $\epsilon_1^{i_1} \cdots \epsilon_s^{i_s}$. Defining $b_I = b_{i_1} \cdots b_{i_s}$, the $\mathbb{Z}$-bases $\{b_I\}$ of $\mathbb{Z}[b_1, b_2, \ldots] = H_{2*}(\text{BGL})$ and $\mathbb{Z}[c_1, c_2, \ldots] = H^{2*}(\text{BGL})$ are dual. For $X$ as above with tangent bundle $T_{X/k} := \Omega^\vee_{X/k}$, this gives us the Chern numbers $c_I(X) = \deg_k(c_I(T_{X/k}))$ and the corresponding element $b(X) := \sum c_I(X) \cdot b_I$ of $\mathbb{Z}[b_1, b_2, \ldots]$. Then the subgroup of $\mathbb{Z}[b_1, b_2, \ldots]$ generated by such elements is actually a subring and is equal to the image of $\text{Lat}^*$ in $\mathbb{Z}[b_1, b_2, \ldots]$ defined by a number of equivalent embeddings (for instance, using the logarithm of the universal formal group law). Thus we can consider $b(X)$ as an element of $\text{Lat}^{d_X}$ and one has the element $[X]_{\text{MGL}} := \phi_{\text{MGL}}(b(X)) \in \text{MGL}^{2d_X,d_X}(k)$; this agrees with the class defined above in the case $k \subset \mathbb{C}$.

Here we will describe how to lift the class $[X]_{\text{MGL}}$ to a class $[X, \theta_X]_{\text{MSL}} \in \text{MSL}^{-2d_X,-d_X}(k)$, where $\theta_X : \text{det}T_{X/k} \to O_X$ is a trivialization of the determinant bundle $\text{det}T_{X/k} := \Lambda^{d_X}T_{X/k}$. This is just an algebraic version of the classical Pontryagin-Thom construction, relying on constructions of Voevodsky, which we will briefly recall; we refer the reader to [Voe03(2), §2] and also to [Lev17, §1.1] for details.

Fix an integer $d \geq 1$ and consider the Segre embedding $i : \mathbb{P}^d \times \mathbb{P}^d \to \mathbb{P}^{d^2+2d}$. We fix homogeneous coordinates $X_0, \ldots, X_d$ for $\mathbb{P}^d$ in the first factor and $Y_0, \ldots, Y_d$ for $\mathbb{P}^d$ in the second factor. The embedding $i$ is defined by a choice of basis of $H^0(\mathbb{P}^d \times \mathbb{P}^d, O(1,1))$, and thus the divisor $H \subset \mathbb{P}^d \times \mathbb{P}^d$ defined by $\sum_{i=0}^d X_iY_i$ is $i(H_{\infty})$ for a unique hyperplane $H_{\infty} \subset \mathbb{P}^{d^2+2d}$. We let $j : \mathbb{P}^d \to \mathbb{P}^d \times \mathbb{P}^d$ be the open complement $\mathbb{P}^d \times \mathbb{P}^d \setminus H$, with projection $p : \mathbb{P}^d \to \mathbb{P}^d$ the restriction of the first projection $p_1 : \mathbb{P}^d \times \mathbb{P}^d \to \mathbb{P}^d$. Then $p : \mathbb{P}^d \to \mathbb{P}^d$ is an $\mathbb{A}^d$-bundle and $\mathbb{P}^d$ is an affine $k$-scheme, the so-called Jouanolou construction on $\mathbb{P}^d$. Voevodsky constructs a vector bundle $\nu_{\mathbb{P}^d}$ on $\mathbb{P}^d$ together with an isomorphism

$$p^*(T_{\mathbb{P}^d/k}) \oplus \nu_{\mathbb{P}^d} \cong O_{\mathbb{P}^d}^{d^2+2d}. \tag{5}$$

In addition, Voevodsky constructs a map

$$\eta_{\mathbb{P}^d} : T^{d^2+2d} \to \text{Th}(\nu_{\mathbb{P}^d})$$

in the pointed unstable motivic homotopy category $\mathcal{H}_*(k)$, satisfying:

$$\begin{align*}
\phi : \Sigma^\infty T \text{Th}(\nu_{\mathbb{P}^d}) &\cong (\Sigma^\infty X_+)^{\vee} \\
\text{i. There is a canonical isomorphism in } \text{SH}(k) \\
\phi : \Sigma^\infty T \text{Th}(\nu_{\mathbb{P}^d}) &\cong (\Sigma^\infty X_+)^{\vee} \\
\text{ii. The map } \phi \circ \Sigma^\infty T^{-2d} \eta : S_k \to (\Sigma^\infty X_+)^{\vee} \text{ is the dual of the map } \Sigma^\infty T \pi_X : \Sigma^\infty T X_+ \to S_k \text{ induced by the projection.}
\end{align*} \tag{6}$$

These properties follow, for example, from [Lev][Proposition 1.3].
Now let \( i_X : X \hookrightarrow \mathbb{P}^d \) be a projective embedding of a smooth projective \( k \)-scheme \( X \) of dimension \( d_X \) over \( k \). We have the Jouanolou construction on \( X \), \( \tilde{X} := X \times_{\mathbb{P}^d} \mathbb{P}^d \) with projections \( p_{1X} : \tilde{X} \to X \) and \( \tilde{i}_X : \tilde{X} \to \mathbb{P}^d \). Define the vector bundle \( \tilde{\nu}_X \) on \( \tilde{X} \) by

\[
\tilde{\nu}_X := N_{i_X} \oplus \tilde{i}^*(\nu_{\mathbb{P}^d}) = p_{1X}^*N_i \oplus \tilde{i}^*(\nu_{\mathbb{P}^d}),
\]

where \( N_{i_X} \) is the normal bundle of \( X \) in \( \mathbb{P}^d \) and \( N_i \) is the normal bundle of \( X \) in \( \mathbb{P}^d \). The isomorphism (5) and the canonical exact sequence

(7) \[ 0 \to T_X \to i^*T_{\mathbb{P}^d} \to N_i \to 0 \]

and the fact that \( \tilde{X} \) is affine induces an isomorphism

(8) \[ p_{1X}^*T_X \oplus \tilde{\nu}_X \cong \mathcal{O}_{\tilde{X}}^{d^2+2d}, \]

unique up to the choice of a splitting in the pull-back of the exact sequence to the affine scheme \( X \).

Let \( s_0 : \mathbb{P}^d \to \nu_{\mathbb{P}^d} \) be the 0-section and let \( s^X_0 : \tilde{X} \to \nu_{\mathbb{P}^d} \) be the morphism \( s_0 \circ \tilde{i} \). Let \( N_{s_0} \) be the normal bundle of \( s^X_0 \). We have the exact sequence

\[ 0 \to N_{i_X} \to N_{s_0} \to \tilde{i}^*\nu_{\mathbb{P}^d} \to 0, \]

with splitting given by the map \( N_{i_X} \to N_{s_0} \) induced by the projection \( \tilde{\nu}_{\mathbb{P}^d} \to \mathbb{P}^d \). This gives us a canonical isomorphism of \( N_{s_0} \) with \( \tilde{\nu}_X \). The Morel-Voevodsky homotopy purity isomorphism [MV99, Theorem 2.23]

\[ \text{Th}(N_{s_0}) \cong \tilde{\nu}_{\mathbb{P}^d}/(\tilde{\nu}_{\mathbb{P}^d} \setminus s^X_0(\tilde{X})) \]

in \( \mathcal{H}_s(k) \) gives us the sequence of maps in \( \mathcal{H}_s(k) \)

\[ \text{Th}(\tilde{\nu}_{\mathbb{P}^d}) = \tilde{\nu}_{\mathbb{P}^d}/(\tilde{\nu}_{\mathbb{P}^d} \setminus s^X_0(\tilde{X})) \to \tilde{\nu}_{\mathbb{P}^d}/(\tilde{\nu}_{\mathbb{P}^d} \setminus s^X_0(\tilde{X})) \cong \text{Th}(N_{s_0}) \cong \text{Th}(\tilde{\nu}_X). \]

Composing this composition with \( \tilde{\eta}_{\mathbb{P}^d} \) gives the map in \( \mathcal{H}_s(k) \)

\[ \tilde{\eta}_X : Td^2+2d \to \text{Th}(\tilde{\nu}_X). \]

The isomorphism (8) defines via projection a surjective bundle map

\[ \pi : \mathcal{O}_{\tilde{X}}^{d^2+2d} \to \tilde{\nu}_X \]

which in turn induces a map

\[ f_\pi : \tilde{X} \to \text{Gr}(r,d^2+2d) \]

classifying \( \pi \), where \( r \) is the rank of \( \tilde{\nu}_X \). If \( \mathcal{E}_{r,d^2+2d} \to \text{Gr}(r,d^2+2d) \) is the universal rank \( r \) quotient bundle, then \( f_\pi \) induces an isomorphism \( f_\pi^*\mathcal{E}_{r,d^2+2d} \cong \tilde{\nu}_X \), giving a commutative diagram

(9) \[
\begin{CD}
\tilde{\nu}_X @>{f_\pi}>> \mathcal{E}_{r,d^2+2d} \\
@VVV @VVV \\
\tilde{X} @>{f_\pi}>> \text{Gr}(r,d^2+2d)
\end{CD}
\]

This gives us the sequence of maps

(10) \[
Td^2+2d \xrightarrow{\tilde{\eta}_X} \text{Th}(\tilde{\nu}_X) \xrightarrow{\text{Th}(f_\pi)} \text{Th}(\mathcal{E}_{r,d^2+2d}) \to \text{Th}(\mathcal{E}_{r,\infty}) = \text{MGL}_r
\]
and thereby an element in \( \text{MGL}^{2(r-d^2-2d)r-d^2-2d}(k) \). Since \( r = d^2 + 2d - dx \), this gives us the class \([X]_{\text{MGL}}' \in \text{MGL}^{-2d_3,-2d_k}(k) \). Two choices of splitting in the exact sequence (8) are connected by an \( A^1 \)-homotopy, so the class \([X]_{\text{MGL}}' \) is independent of the choice of splitting.

Recall that we have denoted the exponential characteristic of \( k \) by \( p \).

**Proposition 4.1.** For \( X \) smooth and projective over \( k \), we have

\[
[X]_{\text{MGL}} = [X]_{\text{MGL}}' \in \text{MGL}^{-2d_3,-2d_k}(k)[1/p].
\]

**Proof.** Fix integers \( m \geq r \geq 1 \). Let \( \xi_{r,m} \to \text{Gr}(r,m) \) be the universal rank \( r \) quotient of \( O^m \); set \( \text{MGL}_{r,m} := \text{Th}(\xi_{r,m}) \). Taking the limit over \( m \) gives \( \text{Gr}(r,\infty) \cong \text{BGL}_r \), the universal rank \( r \) bundle \( \xi_r = \xi_{r,\infty} \to \text{BGL}_r \) and \( \text{MGL}_r := \text{MGL}_{r,\infty} = \text{Th}(\xi_r) \).

We have motivic cohomology \( H^{a,b}(-) \) defined as \( \text{Hom}_{\text{SH}(k)}((-), S^{a,b} \wedge H\mathbb{Z}) \), motivic homology \( H_{a,b}(-) := \text{Hom}_{\text{SH}(k)}(S^{a,b} \wedge \mathbb{S}_k, H\mathbb{Z} \wedge (-)) \) and the slant product

\[
H^{p,q}(X) \otimes H_{a,b}(X \wedge Y) \to H_{p+a-b-q}(Y);
\]

this latter is defined by sending \( f \otimes g, f : X \to S^{p,q} \wedge H\mathbb{Z}, g : S^{a,b} \wedge \mathbb{S}_k \to H\mathbb{Z} \wedge X \wedge Y \) to the composition

\[
S^{a,b} \wedge \mathbb{S}_k \xrightarrow{g} H\mathbb{Z} \wedge X \wedge Y \xrightarrow{r} X \wedge H\mathbb{Z} \wedge Y \xrightarrow{f \wedge \text{id}} S^{p,q} \wedge H\mathbb{Z} \wedge H\mathbb{Z} \wedge Y \xrightarrow{\text{id} \wedge \text{mu}_\mathbb{Z}} S^{p,q} \wedge H\mathbb{Z} \wedge Y
\]

and then desuspending by \( S^{p,q} \). For a space \( X \in \text{Spc}(k) \), we write \( H^{p,q}(X) \) for \( H^{p,q}(\Sigma_\infty^\infty X) \), etc.

We have the Thom class \( th(\xi_{r,m}) \in H^{2r+1}(\text{Th}(\xi_{r,m})) \). Taking \( X = \text{Th}(\xi_{r,m}), Y = \mathbb{P}(\xi_{r,m} \otimes \mathcal{O}_{\text{Gr}(r,m)}), \) the slant product with \( th(\xi_{r,m}) \) composed with pushforward on \( H_a \) by the diagonal map \( \Delta : \text{Th}(\xi_{r,m}) \to \text{Th}(\xi_{r,m}) \wedge \mathbb{P}(\xi_{r,m} \otimes \mathcal{O}_{\text{Gr}(r,m)}) \) and then the pushforward to \( \text{Gr}(r,m) \) gives us the map

\[
\pi_*^{a,m} \circ \theta(\xi_{r,m}) \cap (-) : H_{a+2r+b+r}(\text{Th}(\xi_{r,m})) \to H_{a,b}(\text{Gr}(r,m))
\]

which is an isomorphism, dual to the Thom isomorphism in motivic cohomology. Passing to the limit in \( m \), we have the isomorphism

\[
\pi_* \circ \theta(\xi_r) \cap (-) : H_{a+2r+b+r}(\text{MGL}_r) \to H_{a,b}(\text{BGL}_r)
\]

which is compatible with the structure maps \( \Sigma_r \text{MGL}_r \to \text{MGL}_{r+1} \) for the spectrum \( \text{MGL} \) on the one hand and the inclusions \( \text{BGL}_r \to \text{BGL}_{r+1} \) on the other, giving the Thom isomorphism on motivic homology

\[
\pi_* \circ \theta(\xi_r) \cap (-) : H_{a,b}(\text{MGL}) \to H_{a,b}(\text{BGL})
\]

Taking the product with the unit map \( \mathbb{S}_k \to H\mathbb{Z} \) defines the natural transformation

\[
\text{MGL}^{-p,-q}(\text{Spec } k) \to H_{p,q}(\text{MGL})
\]

so composing with the Thom isomorphism gives us the Hurewicz map

\[
h_{2s,*} : \text{MGL}^{-2s,-s}(\text{Spec } k) \to H_{2s,*}(\text{BGL}).
\]

Since each \( \text{Gr}(r,m) \) is cellular and \( H\mathbb{Z} \) is an oriented ring spectrum, the standard arguments from topology gives the isomorphism

\[
H_{2s,*}(\text{BGL}) \cong \mathbb{Z}[b_1, b_2, \ldots]
\]

with \( b_n \) in bi-degree \( (2n,n) \); the ring structure on \( H_{2s,*}(\text{BGL}) \) being induced by the maps \( \text{Gr}(r,m) \times \text{Gr}(s,n) \to \text{Gr}(r+s, n+m) \) given by direct sum of bundles. Similarly, we have

\[
H_{2s,*}(\text{BGL}) \cong \mathbb{Z}[c_1, c_2, \ldots]
\]
with $c_n$ restricted to $\text{Gr}(r, m)$ the $n$th Chern class of $\mathcal{E}_{r,m}$; we have the $\mathbb{Z}$-basis of $H^{2*,*}(\text{BGL})$ given by the Conner-Floyd classes, $c_I$, dual to the monomials $b_I$. This implies that the diagram

\[ \mathbb{L} \mathcal{L} \mathfrak{L}_{X}^{\infty} \xrightarrow{\phi_{\text{MGL}}} \text{MGL}^{-2*,*}(\text{Spec } k) \xrightarrow{\log} \mathbb{Z}[b_1, b_2, \ldots] \xrightarrow{h_{2*,*}} H_{2*,*}(\text{BGL}) \]

commutes: one can see this if $k$ admits an embedding in $\mathbb{C}$ by comparing with the analogous situation in topology. This implies the result for $k$ of characteristic zero, and the general result follows from this by passing to $\mathbb{Q}$-coefficients, working over the Witt vectors $W(k)$ and using the cellularity of $\text{BGL}$ and the isomorphism of rational motivic cohomology with rational $K$-theory to show that the restriction maps

\[ H_{2*,*}(\text{BGL} / W(k)) \otimes \mathbb{Q} \to H_{2*,*}(\text{BGL} / k) \otimes \mathbb{Q} \]

are isomorphisms. In addition, the map $\phi_{\text{MGL}}$ is an isomorphism after inverting $p$ [Hoy13, Proposition 8.2], hence the Hurewicz map is injective after inverting $p$.

We now use the properties (6)(i, ii). The projection $p_{1X} : \tilde{X} \to X$ is an isomorphism in $\mathcal{H}(k)$. The map $\tilde{\eta} : T^{d^2 + 2d} \to \text{Th}(\tilde{\nu}_X)$ thus defines by the Thom isomorphism an element

\[ [\tilde{\eta}] \in H_{2d_1, d_2}(\tilde{X}) \equiv H_{2d_1, d_2}(X) \equiv H^{0,0}(X) \]

the first isomorphism being given by $p_{X,*}$, the second by duality:

\[ H_{a,b}(X) \equiv H^{-a,-b}(\Sigma^\infty X_+) \equiv H^{2d_1, -d_2}(X), \]

with this last isomorphism given by the string of isomorphisms

\[ H\mathbb{Z} \wedge (\Sigma^\infty X_+)^Y \cong H\mathbb{Z} \wedge \Sigma^{-d^2 - 2d, -d_1 - d_2 - 2d_1} \Sigma^\infty T \text{Th}(\tilde{\nu}_X) \]

\[ \cong H\mathbb{Z} \wedge \Sigma^{-2d_1, -d_2} \Sigma^\infty X_+ \cong H\mathbb{Z} \wedge \Sigma^{-2d_1, -d_2} \Sigma^\infty X_. \]

In fact, since $\phi \circ \Sigma^{-d^2, 2d}\tilde{\eta}$ is the dual of $\Sigma^\infty \pi_X : \Sigma^\infty X_+ \to \mathbb{S}_k$, it follows that the image of $[\tilde{\eta}]$ in $H^{0,0}(X)$ is just $\pi_X(1)$, with $1 \in H^{0,0}(\text{Spec } k)$ the unit. Thus the image of $[\tilde{\eta}]$ in $H_{2d_1, d_2}(X) = \text{CH}_{d_2}(X)$ is the fundamental class $[X]$ of $X$.

For $Y \in \text{Sm} / k$, we let $K_0(Y)$ denote the reduced $K_0$ of vector bundles, that is, for $Y$ connected, $K_0(Y) = K_0(Y)/\mathbb{Z}[O_Y]$ and one extends to general $Y$ by taking the direct sum over the components of $Y$. By [MV99, Proposition 3.7, Proposition 3.9] sending a morphism $g : Y \to \text{Gr}(r, m)$ to the $K_0(Y)$-class of $g^* \mathcal{E}_{r,m}$ extends to an isomorphism

\[ K_0(Y) \equiv \text{Hom}_{\mathcal{H}(k)}(Y, \text{BGL}). \]

Considering the commutative diagram (9) used in the definition of $[X]^*_{\text{MGL}}$, we see that the map $f_\pi (\text{in } \mathcal{H}(k))$ corresponds to the class $-[T_{X/k}] \in K_0(X) = K_0(\tilde{X})$. Now take $I = (i_1, \ldots, i_s) \in \mathbb{N}^s$ with $\sum_i j \cdot i_j = d_X$ giving us the Conner-Floyd Chern class $c_I \in H^{2d_1, d_2}(\text{BGL})$. Since the map $f_\pi$ classifies $-[T_{X/k}]$ and the element of $H_{2d_1, d_2}(\tilde{X}) = H_{2d_1, d_2}(X)$ corresponding to $\tilde{\eta}$ is the fundamental class $[X]$, it follows that

\[ c_I(h_{2d_1, d_2}([X]^*_{\text{MGL}})) = \text{deg}_k(c_I([-T_{X/k}])). \]
Here, on the left-hand side we are considering $c_l$ as one of the additive generators of $H^{2, d_k} (BGL)$, dual to the monomial $h_l \in H^{2, d_k} (BGL)$, and on the right-hand side as the Conner-Floyd Chern class of the virtual bundle $- [T_{X/k}]$.

By the commutativity of the diagram (11) and the injectivity of $h_{2, *}$ after inverting $p$, it follows that

$$[X]_{MGL}' = \phi_{MGL}(b(X)) = [X]_{MGL} \text{ in } MGL^{-2d_k - d_k} (Spec \ k) [1/p]$$

\[ \square \]

Let $\tilde{Gr}(r, m) \to Gr(r, m)$ denote the $\mathbb{G}_m$-bundle

$$det \tilde{\mathcal{E}}_{r,m} \to 0_{det \mathcal{E}_{r,m}} \to Gr(r, m)$$

the $k$-scheme $\tilde{Gr}(r, m)$ is universal for pairs $(\pi, \theta)$, where $\pi : O_Y^m \to \mathcal{V}$ is a surjective map of vector bundles on some $Y \in Sch_k$ and $\theta : det \mathcal{V} \to O_Y$ is an isomorphism. We have the pull-back of $\tilde{\mathcal{E}}_{r,m}, \tilde{\mathcal{E}}_{r,m} \to \tilde{Gr}(r, m)$; let $BSL_r := colim_r \tilde{Gr}(r, m)$ and let $\tilde{\mathcal{E}}_r \to BSL_r$ be the colimit of the bundles $\tilde{\mathcal{E}}_{r,m} \to \tilde{Gr}(r, m)$, let $MSL_r := Th(\tilde{\mathcal{E}}_r)$. For $i_r : BSL_r \to BSL_{r+1}$ the evident inclusion, the isomorphism $i_r^* \mathcal{E}_{r+1} \cong \mathcal{E}_r$ gives the bonding map $\Sigma_r Th(\tilde{\mathcal{E}}_r) \to Th(\tilde{\mathcal{E}}_{r+1})$, defining the $T$-spectrum $MSL_r$; the evident maps $Th(\tilde{\mathcal{E}}_r) \to Th(\tilde{\mathcal{E}}_{r+1})$ give the map of $T$-spectra $MSL_r \to MGL$.

It is now easy to construct a lifting of $[X]_{MGL}'$ to a class $[X, \theta_X]_{MGL} \in MGL^{-2d_k - d_k} (k)$ corresponding to a smooth projective dimension $d_X$ $k$-scheme $X$ together with an isomorphism $\theta_X : det T_{X/k} \to O_X$. The isomorphism $\theta_X$ together with the isomorphism (8) induces a canonical isomorphism

$$det \tilde{\nu}_X \cong O_\tilde{X}$$

independent of the choice of splitting used to construct the isomorphism (8). This in turn gives us a canonical lifting of the pair $(f_\pi, \tilde{f}_\pi)$ in diagram (9) to a commutative diagram

\[
\begin{array}{ccc}
\tilde{\nu}_X & \xrightarrow{\tilde{g}_X} & \tilde{\mathcal{E}}_{r,d^2+d} \\
\tilde{X} \xrightarrow{\tilde{g}_X} & & \tilde{Gr}(r, d^2 + d)
\end{array}
\]

If we repeat the construction of $[X]_{MGL}'$, replacing $\tilde{f}_\pi$ with $\tilde{g}_\pi$, the analog of the sequence of maps (10) gives us an element $[X, \theta_X]_{MGL} \in MGL^{-2d_k - d_k} (k)$ lifting $[X]_{MGL}' = [X]_{MGL}$.

We collect the main results of this section in the following theorem.

**Theorem 4.2.** Let $X$ be a smooth projective $k$-scheme with a trivialization $\theta_X : det T_{X/k} \to O_X$ of the determinant bundle $det T_{X/k} := A^{d_k} T_{X/k}$. Then there is a class $[X, \theta_X]_{MGL} \in MGL^{-2d_k - d_k} (k)$ mapping to $[X]_{MGL}' \in MGL^{-2d_k - d_k} (k)$ under the projection $MSL \to MGL$. After inverting the exponential characteristic of $k$, $[X, \theta_X]_{MGL}$ lifts the element $[X]_{MGL} \in \mathbb{L}az$ defined via the Chern numbers of $- [T_{X/k}]$.

5. **The motivic Adams spectral sequence of $MSL$**

We remind the reader that $p$ denotes the exponential characteristic of $k$ and $l$ will be an odd prime different from $p$. 

5.1. Some completions of MGL. Let $X$ be any motivic spectrum, we construct a tower $C_X$ under $X$ in the following way. Let $E$ be any commutative ring spectrum. Let $E^s$ be the homotopy fiber of $\mathbb{S} \to E$. Hence, we have a triangle

$$E^s \to \mathbb{S} \to E. \tag{13}$$

Let $E^{s^k}$ be $E^s \wedge \cdots \wedge E^s$ $(s$-times). Smashing $(13)$ with $E^{s^k} \wedge X$, we get

$$E^{s^{k+1}} \wedge X \to E^{s^k} \wedge X \to E \wedge E^{s^k} \wedge X. \tag{14}$$

Write $X_s := E^{s^k} \wedge X$, and $W_s = E \wedge X_s$, then, the above triangle becomes

$$X_{s+1} \to X_s \to W_s. \tag{15}$$

Let $C_{s-1}$ be the homotopy cofiber of $X_s \to X_0$. Then, there are induced maps $C_s \to C_{s-1}$, and the cofiber of this map is $\sum_{(1,0)} W_s$. One gets a tower under $X$ of the form

$$X \to \cdots \to C_2 \to C_1 \to C_0. \tag{16}$$

The homotopy limit of the above tower is called the $E$-nilpotent completion of $X$, denoted by $X^\wedge_E$.

Let $X^\wedge I$ be the homotopy inverse limit of the system $X/(l^n) \to X/(l^{n-1})$, where $X/(l^n)$ is the homotopy cofiber of the multiplication map $l^n : X \to X$. Let $(X^\wedge)_l$ be the $l$-adic completion of the abelian group $X^\wedge$.

Let $E = H\mathbb{Z}/l$ be the Eilenberg-MacLane spectrum, and $X = MGL$. The main theorem of § 5.1 is the following.

**Proposition 5.1.** We have the isomorphism $(MGL^{2*,*})^\wedge I_l \cong (MGL^{\wedge}_{H\mathbb{Z}/l})^{2*,*}.$

By Theorem C, $MGL^\wedge_l$ is isomorphic to the completion $MGL^{\wedge}_{l,\eta}$ of $MGL$ at $l$ and $\eta$. Proposition 5.1 follows from the following lemmas.

**Lemma 5.2.** We have $(MGL^{\wedge}_\eta)^* = MGL^\wedge$. In particular, the map $MSL \to MGL$ induces a map $(MSL^{\wedge}_\eta)^* \to MGL^\wedge$.

**Proof.** This follows from the fact that the multiplication by $\eta$ is a zero map on MGL. \hfill $\Box$

**Lemma 5.3.** We have $(MGL^{2*,*})^\wedge I_l \cong (MGL^{\wedge}_{l})^{2*,*}.$

**Proof.** We need to show $\lim_n ((MGL/l^n)^{2*,*}) \cong (MGL^{2*,*})^\wedge$ and $(\lim_n MGL/l^n)^{2*,*} \cong \lim_n ((MGL/l^n)^{2*,*}).$

For simplicity, we write the $l^n$ torsion elements in any abelian group $A$ as $\overset{-}\otimes_{l^n} A$.

Multiplication by $l$ induces the following commutative diagram

$$\begin{array}{ccccccccc}
\cdots & \to & MGL & \overset{l}{\to} & MGL & \overset{\text{Id}}{\to} & MGL/l^n & \overset{\text{Id}}{\to} & \cdots \\
\downarrow & & \downarrow{\text{Id}} & & \downarrow{\text{Id}} & & \downarrow{\text{Id}} & & \downarrow{\text{Id}} \\
\cdots & \to & MGL & \overset{l^{n-1}}{\to} & MGL & \overset{\text{Id}}{\to} & MGL/l^{n-1} & \overset{\text{Id}}{\to} & \cdots \\
\end{array} \tag{17}$$

Applying the functor $[\mathbb{S}, \Sigma^k - ]$ to $(15)$, we get

$$\begin{array}{ccccccccc}
0 & \to & MGL^{s,l}/l^n & \overset{l^n}{\to} & (MGL/l^n)^{s,l} & \overset{\overset{-}\otimes_{l^n}}{\to} & MGL^{s+1,l}/l^n & \overset{-}\otimes_{l^n} & 0 \\
\downarrow & & \downarrow{\text{Id}} & & \downarrow{\text{Id}} & & \downarrow{\text{Id}} & & \downarrow{\text{Id}} \\
0 & \to & MGL^{s,l}/l^{n-1} & \overset{l^{n-1}}{\to} & (MGL/l^{n-1})^{s,l} & \overset{\overset{-}\otimes_{l^{n-1}}}{\to} & MGL^{s+1,l}/l^{n-1} & \overset{-}\otimes_{l^{n-1}} & 0 \\
\end{array} \tag{18}$$
Taking $\lim \limits_{\leftarrow n}$, we obtain the exact sequence

$$0 \rightarrow \lim \limits_{\leftarrow n} (\text{MGL}^{s,t} / \mathbb{P}^n) \rightarrow \lim \limits_{\leftarrow n}((\text{MGL} / \mathbb{P}^n)^{s,t}) \rightarrow \lim \limits_{\leftarrow n}((\text{MGL} / \mathbb{P}^n)^{s-1,t}) \rightarrow \lim \limits_{\leftarrow n}((\text{MGL} / \mathbb{P}^n)^{s+1,t}) \rightarrow \lim \limits_{\leftarrow n}((\text{MGL} / \mathbb{P}^n)^{s,t})$$

Using the fact that $\text{MGL}^{2s+1,t} = 0$, we conclude the isomorphism $\lim \limits_{\leftarrow n}((\text{MGL} / \mathbb{P}^n)^{2s+1,t}) \cong \lim \limits_{\leftarrow n}((\text{MGL} / \mathbb{P}^n)^{2s+1,t})$.  

Taking $(s, t) = (2m - 1, m)$ in (16), the third term, the $\mathbb{P}^s$-torsions in the Lazard ring $\text{MGL}^{2m,m}$, is zero. Therefore, the system $((\text{MGL} / \mathbb{P}^{2m-1,m})$ is isomorphic to $\{\text{MGL}^{2m-1,m} / \mathbb{P}^n\}$, which is a surjective system, and in particular has the Mittag-Leffler property. Hence, $\lim \limits_{\leftarrow n}((\text{MGL} / \mathbb{P}^{2m-1,m}) = 0$. Using the following short exact sequence

$$0 \rightarrow \lim \limits_{\leftarrow n}((\text{MGL} / \mathbb{P}^n)^{s-1,t}) \rightarrow \lim \limits_{\leftarrow n}((\text{MGL} / \mathbb{P}^n)^{s,t}) \rightarrow \lim \limits_{\leftarrow n}((\text{MGL} / \mathbb{P}^n)^{s,t}) \rightarrow 0,$$

we conclude that $(\lim \limits_{\leftarrow n}((\text{MGL} / \mathbb{P}^n)^{2s+1,t}) \cong \lim \limits_{\leftarrow n}((\text{MGL} / \mathbb{P}^{2s+1,t})$. This concludes the proof. \(\square\)

5.2. The motivic Adams spectral sequence. In § 5.2, we recall some facts from [DI10].

Let $X$ be any motivic spectrum. The (homological) mod-$l$ motivic Adams spectral sequence for $X$ is the spectral sequence of the tower (14). In [DI10], a cohomological mod-$l$ motivic Adams spectral sequence for $X$ was introduced, whose $E_2$-page is

$$E_2^{s,t,u} = \text{Ext}^{s,t+u}((H^s(X), H^t)), $$

and differential $d_r$ goes from $E_r^{s,t,u}$ to $E_r^{s+r,t+1,u}$. Here the degree $s$ is the homological degree, $t + s$ is the dimension, and $u$ is the motivic weight.

For any motivic symmetric ring spectrum $E$, let $(\mathcal{S})_E$ be the full subcategory $(\mathcal{S})_E$ of $E$-cellular spectra. By definition, $(\mathcal{S})_E$ is the smallest full subcategory containing $\mathcal{S}$, closed under taking homotopy cofibers and arbitrary wedges, and satisfies the property that if $A \in (\mathcal{S})_E$, then $A \wedge E \in (\mathcal{S})_E$.

**Proposition 5.4** ([DI10], Remark 6.11, Proposition 6.14). Assume $X$ is a $H\mathbb{Z}/l$-cellular spectrum satisfying

1. $H^s(X, \mathbb{Z}/l)$ is a free module over $H^s$;
2. all the homotopy cofibers of the “naive” Adams tower under $X$ ([DI10, 3]) are motivically finite type wedges of $E$ ([DI10, 2.11]);
3. the natural map $H_{s,t}(X, \mathbb{Z}/l) \rightarrow \text{Hom}_{H^{-t}}(H^s(X, \mathbb{Z}/l), H^*)$ is an isomorphism.

Then, the homological and cohomological motivic Adams spectral sequences are isomorphic from $E_2$-page onward. In particular, its $E_2$-page is given by (19).

If in addition $\lim \limits_{\leftarrow n}((E_r^{s,t,u})(X) = 0$ for each $(s, t, u)$, then the spectral sequence converges completely to $\pi_{s,t}(X_E)$. That is, there is a natural filtration $F_{s,t}$ such that the natural map

$$F_{s,t}(X_E) / F_{s+1,t}(X_E) \rightarrow E_{s,t}^{s,t}(X)$$

is an isomorphism.

5.3. Vanishing of differentials. Now we apply Proposition 5.4 to the case when $X$ is MGL or MSL.

It is easy to verify that $X = \text{MGL}$ or $\text{MSL}$ is a cellular spectrum, satisfying conditions of Proposition 5.4, and the Mittag-Leffler condition holds for the system $\{E_r^{s,t,u}(X)\}_{r \geq 2}$. Therefore, the $E_2$ page $\text{Ext}^{s,t,u}((H^s(X), H^t)$ coverages to $\pi_{s,t}(X_E) \equiv \pi_{s,t}(X_{\mathbb{P}^n})$ by Theorem A.1. In a special case if the degree is $(2*,*)$, and $X = \text{MGL}$, we have $(\text{MGL}^{2*,*})^l = (\text{MGL}^{2*,*})^l$ by Proposition 5.1.
The geometric part $MGL^{2s,t}$ of MGL has been studied. It is known that $MGL^{2s,t}(k)[\frac{1}{p}]$ is isomorphic to the localization of the Lazard ring at $p$, which is a free polynomial ring over $\mathbb{Z}[1/p]$.

Denote $E_r^{t,u}(X) := \bigoplus_{s \geq 0} E_r^{s,t,u}(X)$. We are interested in the case when $X = MGL$ or MSL, where $E_2^{s,t,u}(X) = \text{Ext}_{A^*}^{s,t,u}(H^{s,*}(X), H^{t,*})$. The differential $d_r$ of the motivic Adams spectral sequence sends $E_r^{t,u}(X)$ to $E_r^{t-1,u}(X)$. The restriction of $d_r$ to $E_r^{t,u}(X)$ is denoted by $d_r^{X,t,u}$, or simply by $d_r^{t,u}$ if $X$ is understood from the context.

**Proposition 5.5.** For $r \geq 2$, the differentials $d_r^{MGL,2n+1,u}$ and $d_r^{MGL,2n,u}$ vanish on the motivic Adams spectral sequence of MGL.

**Proof.** Comparing the $E_2$-page described in Proposition 3.4 and the geometric part $MGL^{2s,t}$, we get $E_2^{2n,u}(MGL) \cong E_\infty^{2n,u}(MGL)$. Therefore, any differentials going into $E_r^{2n,u}(MGL)$ and going from $E_r^{2n,u}(MGL)$ vanish for all $r \geq 2$. \qed

We have the following similar result for MSL.

**Proposition 5.6.** For $r \geq 2$, the differentials $d_r^{MSL,2n+1,u}$ and $d_r^{MSL,2n,u}$ vanish on the motivic Adams spectral sequence for MSL. As a consequence, $E_2^{2n,u}(MSL) \cong E_\infty^{2n,u}(MSL)$.

**Proof.** The morphism $\phi : MSL \to MGL$ induces the following commutative diagram

$$
\begin{array}{ccc}
E_r^{t,u}(MGL) & \xrightarrow{d_r^{MGL}} & E_r^{t-1,u}(MGL) \\
\phi \downarrow & & \downarrow \phi \\
E_r^{t,u}(MSL) & \xrightarrow{d_r^{MSL}} & E_r^{t-1,u}(MSL).
\end{array}
$$

Proposition 3.4 and Lemma 3.5 imply that $H^{s,*}(MSL)$ is a direct summand of $H^{s,*}(MGL)$ as modules over $A^{*,*}$. Therefore, the map $\phi : \text{Ext}_{A^{*,*}}^{s,t,u}(H^{s,*}(MSL), H^{t,*}) \to \text{Ext}_{A^{*,*}}^{s,t,u}(H^{s,*}(MGL), H^{t,*})$ is injective, for any $(s, t, u)$. In particular, $\phi : E_2^{2n,u}(MSL) \hookrightarrow E_2^{2n,u}(MGL)$ is injective. As the differential $d_r^{MSL,2n,u}$ vanishes, so is the differential $d_r^{MGL,2n,u}$.

The differential $d_r^{MSL,2n+1,u}$ vanishes by the vanishing line in Proposition 3.4, i.e., $E_2^{3(t,u)}(MSL) = 0$ for $t > 2u$. \qed

**Lemma 5.7.** For any $s \in \mathbb{Z}$, we have the following.

1. $(MSL_{\eta,1}^\wedge)^{2s,s} \hookrightarrow (MGL_{\eta}^\wedge)^{2s,s}$;
2. $(MSL_{\eta,1}^\wedge)^{2s,s} \cong \lim_{\to n} ((MSL_{\eta}^\wedge / p^n)^{2s,s})$;
3. there is a natural injective map $\Phi : ((MSL_{\eta}^\wedge)^{2s,s})_1 \to (MSL_{\eta,1}^\wedge)^{2s,s}$.

**Proof.** In the following commutative diagram

$$
\begin{array}{cccccc}
0 & \lim_{\to n} (MSL_{\eta,1}^\wedge / p^n)^{2s-1,s} & \xrightarrow{\text{lim}_{\to n}} & (MSL_{\eta}^\wedge / p^n)^{2s,s} & \xrightarrow{\text{lim}_{\to n}} & (MSL_{\eta}^\wedge / p^n)^{2s,s} & \xrightarrow{\text{lim}_{\to n}} & 0 \\
\downarrow & & | & & | & & | & & \\
0 & \lim_{\to n} (MGL / p^n)^{2s-1,s} & \xrightarrow{\text{lim}_{\to n}} & (MGL / p^n)^{2s,s} & \xrightarrow{\text{lim}_{\to n}} & (MGL / p^n)^{2s,s} & \xrightarrow{\text{lim}_{\to n}} & 0,
\end{array}
$$

the middle map $(MSL_{\eta,1}^\wedge)^{2s,s} \hookrightarrow (MGL_{\eta}^\wedge)^{2s,s}$ is injective by Propositions 5.5, 5.6, and Theorem C. This implies (1).
Also, by the proof of Lemma 5.3, \( \lim_{n \to \infty} (\text{MSL}/\ell^n)_{2s-1,s} = 0 \). Using the conclusion of (1), it implies \( \lim_{n \to \infty} (\text{MSL}^\wedge)_{2s-1,s} = 0 \). Therefore, we have \( (\text{MSL}^\wedge)_{2s,s} \cong (\text{MSL}^\wedge/\ell^n)_{2s,s} \). This implies (2).

Using the diagram (16) with MGL replaced by MSL, we have the exact sequence
\[
0 \to \lim_{n}(\text{MSL}_n^\wedge)_{2s} \to \lim_{n}(\text{MSL}_n^\wedge/\ell^n)_{2s} \to \lim_{n}(\text{MSL}_n^\wedge)^{s+1}_{d} \to \lim_{n}(\text{MSL}_n^\wedge)_{2s} \]
for any \( s, t \in \mathbb{Z} \). In particular, we have an injection \( \lim_{n}(\text{MSL}^\wedge)_{2s} \to \lim_{n}(\text{MSL}^\wedge/\ell^n)_{2s} \) for any \( s \in \mathbb{Z} \). As we know from (2), \( \lim_{n}(\text{MSL}^\wedge/\ell^n)_{2s} \) is isomorphic to \( (\text{MSL}^\wedge)_{2s} \). This implies (3). \( \square \)

**Lemma 5.8.** The map \( \Phi : ((\text{MSL}^\wedge)_{2s})^\wedge \to (\text{MSL}^\wedge)_{2s} \) is surjective. In particular \( (\text{MSL}^\wedge)_{2s} \) is a free module. 

**Proof.** By Proposition 5.6, we know \( \bigoplus_{t} E_{\infty}^{2u,a}(\text{MSL}) \) is a free polynomial ring over \( \mathbb{Z}/l \) whose generators are described in Proposition 3.4. By Propositions 5.4 and Theorem C, \( \bigoplus_{t} E_{\infty}^{2u,a}(\text{MSL}) \) is the associated graded ring of \( (\text{MSL}^\wedge)_{2s} \), with filtration induced by multiplication by \( l \). In particular, the polynomial generators over \( \mathbb{Z}_{(l)} \) of \( (\text{MSL}^\wedge)_{2s} \) have the same characterization as in the topological case. By [Nov62, Theorem 8], an element \([X] \in \text{MGL}_{2s}\) is a polynomial generator of degree \( n \) of \( (\text{MSL}^\wedge)_{2s} \) if and only if the following property holds: Let \( s^n(-) \) denote the Chern number of degree \( n \) associated to the symmetric polynomial \( x_1^n + \cdots + x_d^n \). Then there is a \( u \in \mathbb{Z} \) which is a unit in \( \mathbb{Z}_{(l)} \) such that
\[
\exists u \cdot l, \quad \text{if } n = l^i \text{ for some } i;
\]
\[
\exists u \cdot l, \quad \text{if } n + 1 = l^i \text{ for some } i;
\]
\[
\exists u, \quad \text{otherwise}.
\]

In other words, for any \( s \in \mathbb{Z} \), the \( \mathbb{Z}_{(l)} \)-module \( (\text{MSL}^\wedge)_{2s} \) is generated by the degree \( s \) monomials in the above polynomial generators.

We consider the following composition map
\[
(\text{MSL})^{2s} \xrightarrow{\pi} ((\text{MSL}^\wedge)_{2s})^\wedge \xrightarrow{\Phi} (\text{MSL}^\wedge)_{2s}.
\]

To prove the surjectivity of \( \Phi \), it suffices to show that \( \pi \circ \Phi \) is surjective.

By Theorem 4.2, there exists a class \([X, \theta]_{\text{MSL}} \in \text{MSL}^{-2n,-n}(k)\) associated to a dimension \( n \) Calabi-Yau manifold \( X \) together with an isomorphism \( \theta : \omega_X \cong O_X \), with \([X, \theta]_{\text{MSL}} \) lifting the class \([X]_{\text{MGL}} \in \text{MGL}^{-2d_X, -d_X}(k)\) under the projection \( \text{MSL} \to \text{MGL} \). In particular, the image of \([X, \theta]_{\text{MSL}} \) in \( \text{MGL}^{-2n,-n}(k) \) is the same as that given by the image of \( X \) in the Lazard ring (after inverting \( p = \text{char}(k) \)). Thus, it suffices to show that \( (\text{MSL}^\wedge)_{2s} \subset (\text{MGL}^\wedge)_{2s} \) is generated by classes \([X]_{\text{MGL}} \) of smooth projective Calabi-Yau manifolds \( X \) defined over \( k \).

Fix \( n \geq 2 \). For \( n_1, \ldots, n_r \) positive integers, let \( d_1 = n_1 + 1 \). Then each smooth hypersurface \( H \) of degree \( d_1, \ldots, d_r \) in \( \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r} \) is a Calabi-Yau manifold.

We consider the hypersurfaces constructed in [S68, Page 241] for each of the three cases in (\( \star \)). An elementary computation\(^1\) following the argument in [S68, loc. cit.] shows that for given \( n \geq 2 \) and this

\(^1\)One computes \( s \) from the formula on [S68, Page 241] by taking the logarithm of the total Chern class of the tangent bundle of \( H_n \), noting that the degree \( n \) term in \( \log(c(T_{H_n})) \) equals \( (-1)^n/n \sum \xi_i \), where the \( \xi_i \) are the Chern roots of \( T_{H_n} \).
choice of \( n_1, \ldots, n_r \), a smooth hypersurface \( H_n \) of \( \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r} \) of multi-degree \( d_1, \ldots, d_r \) satisfies Chern number condition (\(
abla\)).

For an arbitrary infinite field \( k \), such a smooth \( H_n \) always exists, by Bertini’s theorem. Thus, for each \( n \), there is a smooth projective Calabi-Yau manifold \( H_n \) defined over \( k \) whose Chern numbers satisfy (\(
abla\)). Thus the classes \([X]_{\text{MGL}}\) of smooth projective Calabi-Yau manifolds \( X \) defined over \( k \) give polynomial generators (over \( \mathbb{Z}_n \)) of \( \text{MSL}^{\wedge}_\eta \) and hence \( \Phi \circ \pi \) is surjective.

If \( k \) is a finite field, there is for each prime \( q \) (including \( q = p \)) \( \) a pro-\( q \)-power infinite extension \( L_q \) of \( k \) and thus a smooth \( H_{n,q} \) as above, defined over a finite extension \( L_q' \) of \( k \) of degree \( q^n e \). Taking norms from \( L_q' \) down to \( k \), we have the class \( \pi_{L_q'/k}[H_{n,q}, \theta]_{\text{MSL}} \) in \( \text{MSL}^{2n-2n}(k) \) that maps to \( q^n \) times a degree \( n \) polynomial generator of \( (\text{MSL}^{\wedge}_\eta)^{2s,s} \); since \( q \) was an arbitrary prime, this shows that \( \Phi \circ \pi \) is surjective in this case as well.

\[ \square \]

**Lemma 5.9.** For any \( s \in \mathbb{Z} \), we have \( ((\text{MSL}^{\wedge}_\eta)^{2s,s})^l \hookrightarrow (\text{MGL})^{2s,s} \).

**Proof.** From Lemma 5.7 (1), we have the embedding \( (\text{MSL}^{\wedge}_{\eta,l})^{2s,s} \hookrightarrow (\text{MGL}^d)^{2s,s} \). This, together with Lemma 5.8 and Lemma 5.3, implies this statement. \( \square \)

5.4. The coefficient rings.

**Proof of Theorem B.** By Proposition 5.6, we know \( \bigoplus_u E_{2n,u}^{\text{MSL}} \) is a free polynomial ring over \( \mathbb{Z}/l \) whose generators are described in Proposition 3.4. By Propositions 5.4 and Theorem C, \( \bigoplus_u E_{2n,u}^{\text{MSL}} \) is the associated graded ring of \( (\text{MSL}^{\wedge}_{\eta,l})^{2s,s} \equiv ((\text{MSL}^{\wedge}_{\eta})^{2s,s})^l \) with filtration induced by multiplication by \( l \). Theorem B (1) now follows from [Sw75, Lemma 20.29] (which still holds with coefficients in \( \mathbb{Z}[1/2p] \)).

As the polynomial generators of \( (\text{MSL}^{\wedge}_{\eta})^{2s,s} \) over \( \mathbb{Z}[1/2p] \) have the same characterization as in the topological case, Theorem B(2) follows from [Nov62, Theorem 8]. \( \square \)

6. Elliptic genus of MSL-varieties

In this section, we use [LYZ13] and Theorem B to prove Theorem A.

6.1. Summary of previous work. In [LYZ13], we studied the algebraic Krichever’s elliptic genus \( \phi : \text{Laz} \rightarrow \text{Ell} \), where Ell is \( \mathbb{Z}[a_1, a_2, a_3, 1/2a_4] \) with an explicit family of elliptic curves defined on it, and \( \phi \) is given by the Baker-Akhiezer function [LYZ13, (3.1)].

**Theorem 6.1 ([LYZ13]).** Let \( k \) be a perfect field of exponential characteristic \( p \). The oriented cohomology theory on \( \text{Sm}_k \) in the sense of [LM07] sending

\[ X \mapsto \text{MGL}^*(X) \otimes_{\text{Laz}} \text{Ell}[\Delta^{-1}], \]

is represented by a motivic oriented cohomology theory on \( \text{Sm}_k \) in the sense of [PS03]. Here \( \Delta \) is the discriminant.

This theorem gives a well-defined notion of the Krichever’s elliptic cohomology of a variety \( X \) with coefficients \( \mathbb{Z}[1/2] \).

Let \( \text{MGL}_k^* \) (resp. \( \text{Ell}_k^* \)) be the MGL-cobordism theory (resp. elliptic cohomology theory) with rational coefficients. The main focus of [LYZ13] is to study \( \phi_{\mathbb{Q}} : \text{MGL}_k^*(k) \to \text{Ell}_k^*(k) \) when \( k \) is an
arbitrary perfect field. Recall that two smooth projective $n$-folds $X_1$ and $X_2$ are related by a flop if we have the following diagram of projective birational morphisms:

\[(\overline{X})
\]

Here $Y$ is a singular projective $n$-fold with singular locus $Z$, such that $Z$ is smooth of dimension $n - 2k + 1$. We assume in addition that there exist rank $k$ vector bundles $A$ and $B$ on $Z$, such that the exceptional locus $F_1$ in $X_1$ is the $\mathbb{P}^{k-1}$-bundle $\mathbb{P}(A)$ over $Z$, with normal bundle $N_{F_1}X_1 = B \otimes O_A(-1)$. Similarly, the exceptional locus $F_2$ in $X_2$ is $\mathbb{P}(B)$, with normal bundle $N_{F_2}X_2 = A \otimes O_B(-1)$. Let $Q^3 \subset \mathbb{P}^4$ denote the $3$-dimensional quadric with an ordinary double point $v$, defined by the equation $x_1x_2 = x_3x_4$. We say that $X_1$ and $X_2$ are related by a classical flop if in addition $k = 2$, and along $Z$, $(Y, Z)$ is Zariski locally isomorphic to $(Q^3 \times Z, v \times Z)$.

Let $I_{fl} \subseteq \text{MGL}[1/p]^*$ be the ideal generated by differences of flops.

**Theorem 6.2 ([LYZ13]).** The kernel of the algebraic elliptic genus $\phi_\eta : \text{MGL}^*_{\eta}(k) \to \text{Ell}^*_{\eta}(k)$ is $I_{fl} \otimes \mathbb{Z}[1/p] \subset \mathbb{Q}$, and its image is the free polynomial ring $\mathbb{Q}[a_1, a_2, a_3, a_4]$.

In particular, $I_{fl} \subseteq \ker \phi$. It is shown that the ideal $I_{fl}$ is also generated by the differences of classical flops.

**6.2. Proof of Theorem A.** Recall the restriction $\phi : \text{MGL}[1/2p]^* \to \text{Ell}[1/2p]$ to $\text{MSL}^*_{\eta}[1/2p]^* \hookrightarrow \text{MGL}[1/2p]^*$ is denoted by $\overline{\phi}$. The ideal of SL-flops $I_{fl}^{SL} \subseteq \text{MSL}^*_{\eta}[1/2p]^*$ is $\text{MSL}^*_{\eta}[1/2p]^* \cap I_{fl}[1/2p]$. Then, Theorem 6.2 implies that $I_{fl}^{SL} \subseteq \ker \overline{\phi}$.

Theorem A is proved through the following lemmas.

**Lemma 6.3.** Assume $X_1$ and $X_2$ are related by a classical flop. We denote the Chern roots (in $\text{CH}^*$) of $A$ by $a_1, a_2$ and Chern roots of $B$ by $b_1, b_2$. Then, we have

\[
s_n([X_1 - X_2]) = \int_{\mathbb{P}(\mathbb{P}(A) \otimes O_{\mathbb{P}(A)}(-1) \oplus O)} \left( \sum_{n, i_1 + i_2 + i_3 + i_4 = n-3} a_1^{i_1} a_2^{i_2} a_3^{i_3} a_4^{i_4} \left[ (-1)^{i_1} \binom{n-1}{i_1} + (-1)^{i_2} \binom{n-1}{i_2} + (-1)^{i_3} \binom{n-1}{i_3} + (-1)^{i_4} \binom{n-1}{i_4} \right] \right).
\]

**Proof.** By the double point relation of MGL, we show in [LYZ13, Lemmas 4.2 and 4.4] that in MGL$(k)$,

\[
[X_1 - X_2] = \mathbb{P}_{\mathbb{P}(A)}(B \otimes O_{\mathbb{P}(A)}(-1) \oplus O) - \mathbb{P}_{\mathbb{P}(B)}(A \otimes O_{\mathbb{P}(B)}(-1) \oplus O).
\]

Note that the right hand side of (22) is the difference of two systems of iterated projective bundles.

For any $n$-dimensional vector bundle $V$ on a smooth quasi-projective variety $X$ with Chern roots $\{\lambda_i\}$, let $\pi : \mathbb{P}(V) \to X$ be the corresponding projective bundle. Take $f(t) \in \text{CH}^*(X)[[t]]$. Then it is well-known that

\[
\pi_*(f(c_1(O(1)))) = \sum_i \frac{f(-\lambda_i)}{\prod_{j \neq i}(\lambda_j - \lambda_i)}.
\]

A direct computation using (22) and the formula (23) shows the desired formula. \qed
Lemma 6.4. The polynomial generators (over \(\mathbb{Z}[1/2p]\)) of \(\text{MSL}_q^{\text{ad}}[1/2p]\) of degree greater than 4 lie in the ideal \(I_{f_1}\).

Proof. This follows directly from Theorem B (2), Lemma 6.3, and [T00, Lemma 6.2].

Lemma 6.5. The polynomial generators of \(\text{MSL}_q^{\text{ad}}[1/2p]\) with degree 2, 3, 4 have algebraically independent images under the map \(\phi\).

Proof. As in [Höh91], (see also [LYZ13, Proposition 5.2]), we have the following generators \(W_i\) of degree \(i\) of \(\text{MGL}^*\), characterized by their Chern numbers:

\[
\begin{align*}
c_1^2[W_2] &= 0, c_2[W_2] = 24; \\
c_1^3[W_3] &= 0, c_1c_2[W_3] = 0, c_3[W_3] = 2; \\
c_1^4[W_4] &= 0, c_1^2c_2[W_4] = 0, c_2^2[W_4] = 2, c_1c_3[W_4] = 0, c_4[W_4] = 6.
\end{align*}
\]

In particular, by Theorem B (2), we know that \(W_i, i = 2, 3, 4\), are polynomial generators (over \(\mathbb{Z}[1/2p]\)) of \(\text{MSL}_q^{\text{ad}}[1/2p]\) in the corresponding degrees.

In [LYZ13, § 5], we calculated that \(\phi(W_2) = 24a_2\), \(\phi(W_3) = a_3\), and \(\phi(W_4) = 6a_2^2 - a_4\). Hence, they are algebraically independent.

This finishes the proof of Theorem A.

We finish this paper by the following.

Remark 6.6. Note that the proof of SU-rigidity property of [K90] only uses properties of the Baker-Akhiezer function, hence, also applies to the algebraic genus \(\phi : \text{MGL}[1/2p]^* \to \text{Ell}[1/2p]\) studied in the present paper. Therefore, Theorem [LYZ13, Theorem B] implies that any genus factoring through \(I_{f_1}\) is SL-rigid. However, the argument in [T00] shows that the converse is also true in the topological setting, although at the present we do not know this in the algebraic setting.

Appendix A. Convergence of the motivic Adams spectral sequence

In the topological setting, in [Bous79], Bousfield proved that the nilpotent completion of any connective spectrum at the Eilenberg-MacLane spectrum of \(\mathbb{Z}/l\) is isomorphic to the Bousfield localization at the Moore spectrum. The goal of this appendix is to study the motivic analogues of Bousfield’s result.

Now let \(k\) be an arbitrary perfect field, and \(\text{SH}(k)\) the stable motivic homotopy category over \(k\). For any prime number \(l\), let \(S\mathbb{Z}/l \in \text{SH}\) be the motivic Moore spectrum, and \(E = H\mathbb{Z}/l \in \text{SH}\) the motivic Eilenberg-MacLane spectrum. For any motivic spectrum \(Y, Y^f\) is the completion of \(Y\) at \(E\) as in § 5.1, and \(Y^\wedge_E\) the nilpotent completion of \(Y\) at \(E\), i.e., the homotopy inverse limit of the Adams tower (14). Recall that for every integer \(q, r_q\) is the right adjoint of the inclusion \(i_q : \Sigma^{2q} \text{SH}^\text{eff} \subset \text{SH}\), and \(f_q\) is the composite \(i_q \circ r_q\). Any motivic spectrum \(Y\) fits into a distinguished triangle

\[
f_q(Y) \to Y \to f^{q-1}(Y) \to \Sigma^{1,0}f_q(Y).
\]

Following [RSQ16], define \(sc := \text{holim}_q f^{q-1}\) to be the slice completion endofunctor on \(\text{SH}(k)\).

We say a set of bi-degrees \(\{(p_i, q_i)\}_{i \in I}\) satisfies condition (Fin) if

(Fin) there exists some \(s \in \mathbb{Z}\), such that \(p_i - 2q_i \geq s\), for any \(i \in I\).

Let \(E = H\mathbb{Z}/l\) be the Eilenberg-MacLane spectrum. We say a spectrum \(Y \in \text{SH}\) satisfies condition (Fin), if \(E \wedge Y = \oplus_{i \in I} \Sigma^{p_i - q_i} E\), where the index set \(\{(p_i, q_i)\}_{i \in I}\) satisfies condition (Fin).
We have the following motivic analogue of the Bousfield isomorphism, which is a more precise version of Theorem C.

**Theorem A.1.** (1) Let $Y \in \text{SH}$ be any motivic spectrum, and let $E = H\mathbb{Z}/l$. Let $Y_E^{\wedge}$ be the completion of $Y$ at $l, \eta$. Then we have a weak equivalence

$$sc(Y_E^{\wedge}) \equiv sc(Y_E^{\wedge}).$$

If furthermore $Y$ has a cell presentation of finite type ([RS016, § 3.3]), then $sc(Y_E^{\wedge}) \equiv Y_{l,\eta}^{\wedge}$.

(2) Let $Y \in \text{SH}$ be any motivic spectrum satisfying condition (Fin), then $Y_E^{\wedge}$ is slice complete. If furthermore $Y$ has a cell presentation of finite type, then there is a weak equivalence $Y_E^{\wedge} \equiv Y_{l,\eta}^{\wedge}$.

**Remark A.2.** (1) If the field $k$ has characteristic 0, similar convergence property of the motivic Adams spectral sequence has been studied by Hu-Kriz-Ormsby in [HKO11]. The proof in the present paper has no restriction on the characteristic of the field.

(2) Theorem A.1 is false without any finiteness assumption. For an $E$-module $X$, we have $X_E^{\wedge} \equiv X$ (see e.g., [DI10, Remark 6.9]). Without condition (Fin), there are examples of non-zero $E$-modules whose slice completion is zero, one example being the étale cohomology spectrum $H^n\mathbb{Z}/l$.

(3) Examples of spectra having cell presentation of finite type and satisfying condition (Fin) include MGL, MSL, etc.

**A.1. Proof of Theorem A.1 (1).** In this subsection, we follow the approach of [Bous79] to prove Theorem A.1 (1).

We adapt the same notations as in [Bous79]. For each $s \geq 0$, let $E_s^\eta$ be as in § 5.1; Define $E_{s-1}$ by the triangle $E_s^\eta \to S \to E_{s-1} \to \Sigma^{1,0}E_s^\eta$. Recall we have the following sequence [Bous79, (5.1)],

$$E \wedge E_s^\eta \to E_s \to E_{s-1} \to \Sigma^{1,0}(E \wedge E_s^\eta).$$

For a spectrum $Y \in \text{SH}$, the tower $Y \to \{E_s \wedge Y\}$ under $Y$ has homotopy inverse limit $Y_E^{\wedge}$.

Let $M(E)$ be the collection of objects in $\text{SH}(k)$ consisting of finite extensions of $E$-modules, with the property that $W = f^nW$ for large enough $n \in \mathbb{N}$.

**Definition A.3.** An $E$-nilpotent resolution of $Y$ is a tower $Y \to \{W_s\}$ under $Y$, such that

1. $W_s \in M(E)$, for all $s$.
2. $\operatorname{colim}_s[W_s, W] \equiv [Y, W]$, for any $W \in M(E)$.

**Lemma A.4.** The tower $\{f^n(E_s \wedge Y)\}$ is an $E$-nilpotent resolution of $Y$.

**Proof.** Inductively, we can show that $E_s \wedge Y$ is a finite extension of $E$-modules. Indeed, $E \wedge E_s^\eta$ is an $E$-module, and $E_{s-1}$ is a finite extension of $E$-modules by induction hypothesis. Hence, $E_s \wedge Y$ is a finite extension of $E$-modules using (24).

For an $E$-module $M$, by [P11, Theorem 3.6.13(6)], the slice $s_n(M)$ is also an $E$-module. Using the homotopy cofiber sequence

$$s_n(M) \to f^{n+1}(M) \to f^n(M) \to \Sigma^{1,0}s_n(M)$$

and induction, it is easy to show that $f^n(E_s \wedge Y)$ is a finite extension of $E$-modules. Therefore, $f^n(E_s \wedge Y) \in M(E)$. 

For $N \in M(E)$, we have $f^n(N) = N$, for large enough $n \in \mathbb{N}$. Therefore, $\text{colim}_s [f^s(Y \times Y), N] \cong \text{colim}_s [E \wedge Y, N]$. To prove the claim, it suffices to show the isomorphism $\text{colim}_s [E \wedge Y, N] \cong [Y, N]$ for any $N \in M(E)$. We have the following diagram [Bous79, (5.2)],

$$
\begin{array}{c}
\cdots \\
\downarrow \\
E^{s+1} \longrightarrow S \longrightarrow E_s \longrightarrow \Sigma E^{s+1} \\
\downarrow \\
E^s \longrightarrow S \longrightarrow E_{s-1} \longrightarrow \Sigma E^s
\end{array}
$$

Applying $[- \wedge Y, N]$ and taking colimit, we get a long exact sequence

$$
\cdots \rightarrow \text{colim}_s [E^{s+1} \wedge Y, N] \rightarrow \text{colim}_s [E^s \wedge Y, N] \rightarrow [Y, N] \rightarrow \text{colim}_s [E^{s+1} \wedge Y, N] \rightarrow \cdots .
$$

It suffices to show that $\text{colim}_s [E^{s+1} \wedge Y, N] = 0$, or equivalently, the map $[E^s \wedge Y, N] \rightarrow [E^{s+1} \wedge Y, N]$ is a zero map.

It fits in the following long exact sequence

$$
\cdots \rightarrow [E \wedge E^s \wedge Y, N] \rightarrow [E^s \wedge Y, N] \rightarrow [E^{s+1} \wedge Y, N] \rightarrow \cdots .
$$

If $N = E \wedge X$, then, for any map $\phi : E^s \wedge Y \rightarrow N$, we can find an extension $\tilde{\phi} : E \wedge E^s \wedge Y \rightarrow N$, such that $\phi = \tilde{\phi} \circ u$, where $u$ is the unit map. This shows the claim when $N = E \wedge X$. For general $N \in M(E)$, $N$ is a finite extension of $E$-modules. By induction, we have $\text{colim}_s [E^{s+1} \wedge Y, N] = 0$. \qed

Similar to Lemma A.4, we have the following.

**Lemma A.5.** The tower $Y \rightarrow \{f^s(S\mathbb{Z}/l^s \wedge Y)\}$ is an $E$-nilpotent resolution of $Y$.

**Proof.** We first show that $\{f^s(S\mathbb{Z}/l^s \wedge Y)\}$ is $E$-nilpotent. We have the cofiber sequence

$$
s_n(M) \rightarrow f^{n+1}(M) \rightarrow f^n(M) \rightarrow \Sigma^{1,0} s_n(M)
$$

It is clear that $s_n(S\mathbb{Z}/l \wedge Y)$ is a module over $H\mathbb{Z}/l$, therefore, it is $E$-nilpotent. By induction, we have $f^n(S\mathbb{Z}/l \wedge Y)$ is $E$-nilpotent.

By definition, we have $N = f^n N$, for all $n > c(N)$, for some $c(N) \in \mathbb{N}$. Therefore, $[f^s(Y/l^s), N] = [Y/l^s, N]$, for $s > c(N)$. This implies the isomorphism $\text{colim}_s [f^s(Y/l^s), N] \cong [Y, N]$. To show the claim, it suffices to verify that $\text{colim}_s [S\mathbb{Z}/l^s \wedge Y, N] \cong [Y, N]$. We have the diagram

$$
\begin{array}{c}
Y \xrightarrow{p} Y \longrightarrow S\mathbb{Z}/l^s \wedge Y \\
\downarrow l \downarrow \\
Y \xrightarrow{p-1} Y \longrightarrow S\mathbb{Z}/l^{s-1} \wedge Y
\end{array}
$$

Applying the functor $[-, N]$, we get

$$
\begin{array}{c}
\cdots \rightarrow [Y, N] \rightarrow [Y/(p), N] \rightarrow [Y, N] \rightarrow [Y, N] \rightarrow \cdots \\
\downarrow l \downarrow \\
\cdots \rightarrow [Y, N] \rightarrow [Y/(p-1), N] \rightarrow [Y, N] \rightarrow [Y, N] \rightarrow \cdots 
\end{array}
$$

Taking the colimit of the above system, we get a long exact sequence. In order to show the claim, it suffices to show the colimit of the system $\{[Y, N], l\}$ is zero. This follows from the fact the multiplication map $l : N \rightarrow N$ by $l$ is a nilpotent map, for $N \in M(E)$. \qed

The same argument as [Bous79, 5.9] shows the following.
Lemma A.6. Two E-nilpotent resolutions have the same homotopy limit.

As a corollary, we have $\text{holim}_s f^s(\overline{E}_s \wedge Y) = \text{holim}_q f^q(S \mathbb{Z}/l^q \wedge Y)$. A diagonal argument as in [CS02, Theorem 24.9] shows that

$$\text{holim}_s f^s(\overline{E}_s \wedge Y) \equiv \text{holim}_s f^s \text{holim}_q(\overline{E}_q \wedge Y) = sc(Y_E^\wedge).$$

Using the diagonal argument, we have

$$\text{holim}_q f^q(S \mathbb{Z}/l^q \wedge Y) = \text{holim}_q f^q(\text{holim}_s S \mathbb{Z}/l^s \wedge Y) = sc(Y_I^\wedge).$$

As a consequence, $sc(Y_I^\wedge) \equiv sc(Y_E^\wedge)$.

Recall the following result of Röndigs-Spitzweck-Østvær.

Theorem A.7 ([RSØ16], Theorem 3.50). Suppose $Y$ has a cell presentation of finite type. There is a canonical weak equivalences between $sc(Y)$ and $Y_I^\wedge$.

Hence, if $Y$ has a cell presentation of finite type, we also have $sc(Y_I^\wedge) \equiv Y_I^\wedge$. This implies Theorem A.1 (1).

A.2. Proof of Theorem A.1 (2).

Lemma A.8. If $M \in \text{SH}$ satisfies condition (Fin). Then, $Y := E \wedge M$ is slice complete. That it, $Y$ is weakly equivalent to $\text{holim}_q f^q(Y)$.

Proof. We need to show the following isomorphism for any smooth variety $X \in \text{Sm}_k$, and any $a, b \in \mathbb{N}$,

$$[\Sigma^{a,b} \Sigma_T^c X, Y] \equiv [\Sigma^{a,b} \Sigma_T^c X, \text{holim}_q f^q Y].$$

By assumption, we have the decomposition $Y = \bigoplus_{\langle (p,q,i) \rangle \in I} \Sigma^{(p,q)} E$. Therefore, $f_q(Y) = \bigoplus_{q > 0, i} \Sigma^{p-q} E$.

The exact triangle $f_q Y \rightarrow Y \rightarrow f^q Y$ induces a long exact sequence

$$\cdots \rightarrow [\Sigma^{a,b} \Sigma_T^c X, f_q Y] \rightarrow [\Sigma^{a,b} \Sigma_T^c X, Y] \rightarrow [\Sigma^{a,b} \Sigma_T^c Y, f^q Y] \rightarrow [\Sigma^{a+1,b} \Sigma_T^c X, f_q Y] \rightarrow \cdots$$

The term $[\Sigma^{a,b} \Sigma_T^c X, f_q Y] = [\Sigma^{a,b} \Sigma_T^c X, \oplus_{q > 0, i} \Sigma^{p-q} E]$ embeds into the following

$$(25) \quad [\Sigma^{a,b} \Sigma_T^c X, \bigcap_{q > 0, i} \Sigma^{p-q} E] = \bigcap_{q > 0, i} [\Sigma^{a,b} \Sigma_T^c X, \Sigma^{p-q} E] = \bigcap_{q > 0, i} H^{p-q} (X, \mathbb{Z}/l).$$

By the vanishing Theorem 2.1, (25) vanishes if $q + s > a - b + \dim(X)$. Consequently, for $q \gg 0$, we have the isomorphism $[\Sigma^{a,b} \Sigma_T^c X, Y] \equiv [\Sigma^{a,b} \Sigma_T^c Y, f^q Y]$. In particular, $[\Sigma^{a,b} \Sigma_T^c X, Y] \equiv \text{colim}_q [\Sigma^{a,b} \Sigma_T^c X, f^q Y]$ and $R^1 \lim_q [\Sigma^{a,b} \Sigma_T^c X, f^q Y] = 0$. The desired isomorphism now follows from the short exact sequence

$$0 \rightarrow R^1 \text{lim}_q [\Sigma^{a+1,b} \Sigma_T^c X, f^q Y] \rightarrow [\Sigma^{a,b} \Sigma_T^c X, \text{holim}_q f^q Y] \rightarrow \text{colim}_q [\Sigma^{a,b} \Sigma_T^c X, f^q Y] \rightarrow 0.$$

Lemma A.9. If $M \in \text{SH}$ satisfies condition (Fin), then $E \wedge M$ satisfies condition (Fin).

Proof. We have

$$E \wedge E \wedge M \equiv E \wedge (\bigoplus_{i \in \Lambda} \Sigma^{p,q} E) = \bigoplus_{i \in \Lambda} \Sigma^{p,q} E \wedge E$$

$$= \bigoplus_{i \in \Lambda} \Sigma^{p,q} (\bigoplus_{l \in B} \Sigma^{p(l),q(l)} E) = \bigoplus_{i \in \Lambda} \bigoplus_{l \in B} \Sigma^{p(l),q(l)} E$$
Here the third equality $E \wedge E \cong \bigoplus_{I \in B} E^{p(I), q(I)}$ is due to Voevodsky (see also [Sp12, Theorem 11.24]), where the set of bidegrees $\{(p(I), q(I)) : I \in B\}$ are the same as in the Steenrod algebra. In particular, we have $p(I) \geq 2q(I)$ for any $I \in B$. This completes the proof. \hfill \Box

**Lemma A.10.** If $M$ satisfies condition $(\text{Fin})$, then for any $s \in \mathbb{N}$, $E_s \wedge M$ is a finite extension of spectra of the form $E \wedge M$ with $M$ satisfying condition $(\text{Fin})$.

**Proof.** We prove this by induction. For $s = 0$, $E_0 \wedge M = E \wedge M$, the assertion directly follows from Lemma A.9. For general $s > 0$, by induction hypothesis and the exact triangle $E \wedge E^s \wedge M \to E_s \wedge M \to E_{s-1} \wedge M$, it suffices to show that $E \wedge E^s \wedge M$ is a finite extension of spectra satisfying condition $(\text{Fin})$ for any $s$.

This, in turn, is proved by induction again. For $s = 1$, using the exact triangle $E \wedge E \wedge M \to E \wedge E \wedge M \to E \wedge E \wedge M$, we see that $E \wedge E \wedge M$ is an extension of $E \wedge M$ by $E \wedge E \wedge M$, both of which satisfy condition $(\text{Fin})$ by Lemma A.9. In general, consider the exact triangle

$$E \wedge E^s \wedge M \to E \wedge E^{s-1} \wedge M \to E \wedge E \wedge E^{s-1} \wedge M.$$  

By induction hypothesis and Lemma A.9, $E \wedge E \wedge E^{s-1} \wedge M$ and $E \wedge E \wedge E^{s-1} \wedge M$ are both a finite extension of spectra satisfying condition $(\text{Fin})$. Therefore, the claim follows from induction. \hfill \Box

Suppose $Y$ satisfies condition $(\text{Fin})$, then by Lemma A.8 and Lemma A.10, $E_s \wedge M$ is a finite extension of slice complete spectra. The slice complete spectra form a triangulated category. Therefore, $E_s \wedge Y$ is slice complete. As taking slice completion commutes with homotopy colimit, by the diagonal argument, $Y_E^s$ is slice complete.

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