ON OPTIMAL PERIODIC DIVIDEND STRATEGIES FOR LÉVY RISK PROCESSES

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ABSTRACT. In this paper, we revisit the optimal periodic dividend problem, in which dividend payments can only be made at the jump times of an independent Poisson process. In the dual (spectrally positive Lévy) model, recent results have shown the optimality of a periodic barrier strategy, which pays dividends at Poissonian dividend-decision times, if and only if the surplus is above some level. In this paper, we show the optimality of this strategy for a spectrally negative Lévy process whose dual has a completely monotone Lévy density. The optimal strategies and value functions are concisely written in terms of the scale functions. Numerical results are also provided.

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1. INTRODUCTION

In the classical de Finetti’s optimal dividend problem, the expected total discounted dividends accumulated until ruin are maximized. To model the surplus of an insurance company that increases by premium and decreases by insurance payments, a compound Poisson process with downward jumps or more generally a spectrally negative Lévy process is used. Nowadays, fluctuation theory and scale functions are known to be useful, particularly if the optimal strategy is guessed to be a barrier strategy reflecting the underlying process at an upper barrier. Numerous computations are possible for the reflected Lévy process, and these can be used to solve the problem in a straightforward manner.

Despite the analytical tractability of the classical continuous-time model, the barrier strategies are unfortunately not implementable in practice. On the other hand, while the models with deterministic discrete payment times are ideal, they lack analytical tractability, and numerical methods are required to solve them. Recently, with the aim of developing a more realistic yet analytically tractable model, random discrete payment times were considered. For example, in the research by Albrecher et al. [1, 2], if

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the random times are suitably chosen, analytical approaches can be used to compute various identities of interest. Random observation times are also suggested in various economic literatures. See, for example, the discussion in the introduction of [27] motivated by rational inattention [29] in macroeconomics literature. See also the discussions given in [20] and [28] for real option problems with random intervention times.

In this paper, we focus on the periodic barrier strategy and its optimality when dividend payments can only be made at the jump times of an independent Poisson process. In this context, Avanzi et al. [5] solved the case with positive hyperexponential jumps; this case was later generalized by Pérez and Yamazaki [25] and Zhao et al. [31] for a general spectrally positive Lévy process. By assuming that the intervals are independent exponential random variables, we can still formulate it as a one-dimensional Markovian problem. It is also known (see, e.g., [21] in the context of finance) that this can give an approximation for the case of constant intervals. See also the Erlang(2) interarrival time case recently considered by Avanzi et al. [6].

We consider the spectrally negative Lévy case, which is suitable in the insurance context. In the spectrally negative case, the surplus process can instantaneously jump to the liquidation region where the value function flattens suddenly, and for this reason the analysis is sensitive to the choice of the Lévy measure. On the other hand, this never happens in the spectrally positive case and the liquidation region can be ignored. For these reasons, the proof of optimality is significantly more difficult in the spectrally negative model than in the spectrally positive model. In particular, in order to show the variational inequalities, the spectrally positive Lévy case can be handled using known general results on the scale function. On the other hand, for the spectrally negative case, these are not sufficient and further properties of the scale functions that hold only for a subset of spectrally negative Lévy processes need to be considered (see Theorem 2 and Corollary 1 in [23]).

One currently known sufficient condition is the completely monotone assumption on the Lévy density, under which the scale function can be written as the difference of an exponential function and a completely monotone function (see Remark 3.2). Accordingly, Loeffen [22] (see also Yin et al. [30] for an analytic approach) and Kyprianou et al. [19] showed the optimality of a barrier strategy in the classical case and that of a threshold strategy under the absolutely continuous assumption on the dividend strategy, respectively. In this paper, we show that the completely monotone assumption is again a sufficient condition for the optimality of a periodic barrier strategy in the considered problem.

The class of Lévy risk processes with completely monotone Lévy densities include a variety of important processes. To name a few, we have the spectrally negative $\alpha$-stable process used in [16], the (one-sided) gamma process considered in [12], the (one-sided) inverse Gaussian process used in [11], and finally Crámer-Lundberg processes with heavy-tailed Weibull, Pareto, and hyperexponential jumps (see [3] Chapter 1.2).
By Bernstein’s theorem, a completely monotone function has the form
\[ f(t) = \int_0^\infty e^{-tx} \lambda(dx), \]
for a possibly infinite measure \( \lambda \). From this, it can be seen that a Lévy measure with a completely monotone density is roughly a mixture of exponential distributions, implying that larger jumps are less frequent. We refer the reader to, e.g., Feldmann and Whitt [15] regarding the approximation of a completely monotone distribution using mixtures of exponential distributions. The empirical results shown in [9] suggest that financial models should be modeled using Lévy processes with completely monotone Lévy densities.

Under the periodic barrier strategy, the surplus is pushed down to a given barrier at each Poisson arrival time at which it is above the barrier. The controlled process is precisely the Parisian-reflected Lévy process considered in [8, 26]. Its fluctuation identities can be used efficiently to conduct the following “guess and verify” procedure:

1. The expected net present value (NPV) of dividends under the periodic barrier strategy can be written in terms of generalizations of the scale functions. The candidate barrier, which we call \( b^* \), is chosen so that the corresponding (candidate) value function, if \( b^* > 0 \), becomes smoother at the barrier. In particular, this candidate barrier is chosen so that it becomes \( C^2(0, \infty) \) (resp. \( C^3(0, \infty) \)) for the case of bounded (resp. unbounded) variation.

2. We then analyze the existence of \( b^* \) such that the expected NPV satisfies the smoothness condition. To this end, we use the special property of the scale function under the completely monotone assumption that its derivative first decreases and then increases. We will achieve \( b^* > 0 \) or \( b^* = 0 \), where, for the case \( b^* = 0 \), the future prospect is negative and it should be liquidated as quickly as possible.

3. By using the selected candidate barrier \( b^* \geq 0 \), its optimality is confirmed through a verification lemma requiring the analysis of the harmonic property and the slope of the candidate value function. By using the known property of the scale functions, the harmonic property can be analyzed easily. In contrast, the analysis of the slope above the barrier \( b^* \) is a great challenge. Motivated by the technique used in Kyprianou et al. [19], we manage this by using the decomposition of the scale function to an exponential function and a completely monotone function.

To observe the link with the classical case, we also analyze the convergence as the Poisson arrival rate increases to infinity. In particular, we show that the optimal barrier \( b^* \) and the value function converge to those in the classical case [7, 22]. The analytical results are further confirmed through a sequence of numerical experiments. By using a simple case with i.i.d. exponentially distributed jumps, we confirm the optimality and conduct sensitivity analysis with respect to the parameters describing the problem.

The remainder of the paper is organized as follows. In Section 2, we review the spectrally negative Lévy process and present the mathematical model. In Section 3, we review the periodic barrier strategy and obtain the expected NPV of dividends by using the scale functions. Section 4 states a condition of
the candidate barrier \( b^* \) and shows its existence. The optimality of the selected strategy is confirmed in Section 5. Section 6 shows the analysis of the convergence as the Poisson arrival rate goes to infinity. Finally, Section 7 concludes the paper with numerical results.

2. Preliminaries

2.1. Spectrally negative Lévy processes. Let \( X = (X(t); t \geq 0) \) be a Lévy process defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). For \( x \in \mathbb{R} \), we denote the law of \( X \) by \( \mathbb{P}_x \) when it starts at \( x \) and, for convenience, write \( \mathbb{P} \) in place of \( \mathbb{P}_0 \). Accordingly, we write \( \mathbb{E}_x \) and \( \mathbb{E} \) for the associated expectation operators. Throughout the paper, we assume that \( X \) is spectrally negative, meaning here that it has no positive jumps and that it is not the negative of a subordinator. We define the Laplace exponent

\[
\psi(\theta) := \log \mathbb{E}[e^{\theta X(1)}] = \gamma \theta + \frac{\eta^2}{2} \theta^2 + \int_{(-\infty,0)} (e^{\theta z} - 1 - \theta z 1_{\{z>1\}}) \Pi(dz), \quad \theta \geq 0,
\]

where \( \gamma \in \mathbb{R}, \eta \geq 0, \) and \( \Pi \) is a measure on \((-\infty,0)\) called the Lévy measure of \( X \) that satisfies

\[
\int_{(-\infty,0)} (1 \wedge z^2) \Pi(dz) < \infty.
\]

It is well-known that \( X \) has paths of bounded variation if and only if \( \eta = 0 \) and \( \int_{(-1,0)} |z| \Pi(dz) \) is finite. In this case, its Laplace exponent is given by

\[
\psi(\theta) = c \theta + \int_{(-\infty,0)} (e^{\theta z} - 1) \Pi(dz), \quad \theta \geq 0,
\]

where

\[
c := \gamma - \int_{(-1,0)} z \Pi(dz).
\]

Note that necessarily \( c > 0 \), since we have ruled out the case that \( X \) has monotone paths.

2.2. The optimal dividend problem with Poissonian dividend-decision times. We assume that the dividend payments can only be made at the arrival times \( T_r := (T(i); i \geq 1) \) of a Poisson process \( N^r = (N^r(t); t \geq 0) \) with intensity \( r > 0 \), which is independent of \( X \). Let \( \mathbb{F} := (\mathcal{F}(t); t \geq 0) \) be the filtration generated by the processes \((X, N^r)\).

In this setting, a strategy \( \pi := (L^\pi(t); t \geq 0) \) is a nondecreasing, right-continuous, and \( \mathbb{F} \)-adapted process such that the cumulative amount of dividends \( L^\pi \) admits the form

\[
L^\pi(t) = \int_{[0,t]} \nu^\pi(s) dN^r(s), \quad t \geq 0,
\]

for some \( \mathbb{F} \)-adapted càglàd process \( \nu^\pi \). For a more detailed description of the problem, we refer the reader to the spectrally positive case studied in [25].
The surplus process $U^\pi$ after dividends are deducted is such that
\[ U^\pi(t) := X(t) - L^\pi(t) = X(t) - \sum_{i=1}^{\infty} \nu^\pi(T(i))1_{\{T(i) \leq t\}}, \quad 0 \leq t \leq \sigma_0^\pi, \]
where
\[ \sigma_0^\pi := \inf\{t > 0 : U^\pi(t) < 0\} \]
is the corresponding (continuously-monitored) ruin time. Here and throughout, let $\inf \emptyset = \infty$. As in [25], the payment cannot exceed the available surplus and hence
\[ 0 \leq \nu^\pi(s) \leq U^\pi(s-), \quad s \geq 0. \tag{2.4} \]

Over the set of all admissible strategies $A_r$ that satisfy all the constraints described above, we need to maximize, for $q > 0$, the expected NPV of dividends paid until ruin:
\[ v^\pi(x) := \mathbb{E}_x \left( \int_{[0,\sigma_0^\pi]} e^{-qt} dL^\pi(t) \right) = \mathbb{E}_x \left( \int_{[0,\sigma_0^\pi]} e^{-qt}\nu^\pi(t)dN^r(t) \right), \quad x \geq 0. \]
Hence, the problem is to compute the value function
\[ v(x) := \sup_{\pi \in A_r} v^\pi(x), \quad x \geq 0, \]
and obtain the optimal strategy $\pi^*$ that attains it, if such a strategy exists.

Hereafter, we mean, by $e_p$, an exponential random variable with parameter $p > 0$, independent of the process $X$ so that we can write $T(1) = e_p$.

### 3. Periodic barrier strategies

As in the spectrally positive case [25], our objective is to show the optimality of the periodic barrier strategy, say $\pi^b = (L^b_r(t); t \geq 0)$, with the resulting controlled process $U^b_r$ being the Lévy process with Parisian reflection above given as follows. We have
\[ U^b_r(t) = X(t), \quad 0 \leq t < T^+_{b^+}(1) \tag{3.1} \]
where
\[ T^+_{b^+}(1) := \inf\{T(i) : X(T(i)) > b\}. \tag{3.2} \]
The process then jumps downward by $X(T^+_{b^+}(1)) - b$ so that $U^b_r(T^+_{b^+}(1)) = b$. For $T^+_{b^+}(1) \leq t < T^+_{b^+}(2) := \inf\{S \in T_r : S > T^+_{b^+}(1), U^b_r(S-) > b\}$, we have $U^b_r(t) = X(t) - (X(T^+_{b^+}(1)) - b)$. The process $U^b_r$ can be constructed by repeating this procedure.

Suppose $L^b_r(t)$ is the cumulative amount of (Parisian) reflection until time $t \geq 0$. Then we have
\[ U^b_r(t) = X(t) - L^b_r(t), \quad t \geq 0, \]
(3.3) \[ L_r^b(t) = \sum_{T_r^+(i) \leq t} (U_r^b(T_r^+(i) - b), \quad t \geq 0, \]

where \((T_r^+(n); n \geq 1)\) can be constructed inductively by using (3.2) and \(T_r^+(n + 1) := \inf\{S \in \mathcal{T}_r : S > T_r^+(n), U_r^b(S -) > b\}, \quad n \geq 1.\)

It is clear that the strategy \(\pi^b := (L_r^b(t); t \geq 0), \quad b \geq 0,\) is admissible with \(\nu_{\pi^b}(t) = (U_r^b(t) - b) \vee 0.\)

We denote its expected NPV of dividends by

(3.4) \[ v_b(x) := \mathbb{E}_x \left( \int_{[0, \sigma^b_0]} e^{-\theta t} dL_r^b(t) \right), \quad x \geq 0, \]

where \(\sigma^b_0 := \inf\{t > 0 : U_r^b(t -) < 0\}.\)

3.1. Computation of the expected NPV (3.4). The expected NPV of dividends as in (3.4) can be computed directly by using the fluctuation theory. Toward this end, we first review the scale functions.

Fix \(q > 0.\) We use \(W^{(q)} : \mathbb{R} \to [0, \infty)\) for the scale function of the spectrally negative Lévy process \(X,\) which takes the value zero on the negative half-line, while on the positive half-line, it is a continuous and strictly increasing function such that

(3.5) \[ \int_0^\infty e^{-\theta x} W^{(q)}(x) dx = \frac{1}{\psi(\theta) - q}, \quad \theta > \Phi(q), \]

where \(\psi\) is as defined in (2.1) and

(3.6) \[ \Phi(q) := \sup\{\lambda \geq 0 : \psi(\lambda) = q\}. \]

We also define, for \(x \in \mathbb{R},\)

\[ \breve{W}^{(q)}(x) := \int_0^x W^{(q)}(y) dy, \]

\[ \overline{W}^{(q)}(x) := \int_0^x \overline{W}^{(q)}(y) dy, \]

\[ Z^{(q)}(x) := 1 + q \overline{W}^{(q)}(x), \]

\[ \underline{Z}^{(q)}(x) := \int_0^x Z^{(q)}(z) dz = x + q \int_0^\infty \int_0^z W^{(q)}(w) dw dz. \]

Because \(W^{(q)}(x) = 0\) for \(-\infty < x < 0,\) we have

(3.7) \[ W^{(q)}(x) = 0, \quad \overline{W}^{(q)}(x) = 0, \quad Z^{(q)}(x) = 1, \quad \text{and} \quad \underline{Z}^{(q)}(x) = x, \quad x \leq 0. \]
If we define \( \tau_0^- := \inf\{t \geq 0 : X(t) < 0\} \) and \( \tau_0^+ := \inf\{t \geq 0 : X(t) > b\} \) for any \( b > 0 \), then

\[
\mathbb{E}_x \left( e^{-q \tau_0^+} 1_{\{\tau_0^+ < \tau_0^-\}} \right) = \frac{W(q)(x)}{W(q)(b)},
\]

(3.8)

\[
\mathbb{E}_x \left( e^{-q \tau_0^-} 1_{\{\tau_0^- > \tau_0^+\}} \right) = Z(q)(x) - Z(q)(b) \frac{W(q)(x)}{W(q)(b)}.
\]

Remark 3.1. Regarding the asymptotic behaviors near zero, as in Lemmas 3.1 and 3.2 of [17].

\[
W(q)(0) = \begin{cases} 0 & \text{if } X \text{ is of unbounded variation}, \\ \frac{1}{e} & \text{if } X \text{ is of bounded variation}, \end{cases}
\]

(3.9)

\[
W(q)(0^+) := \lim_{x \downarrow 0} W(q)(x) = \begin{cases} \frac{2}{\eta^2} & \text{if } \eta > 0, \\ \infty & \text{if } \eta = 0 \text{ and } \Pi(-\infty, 0) = \infty, \\ \frac{\eta + \Pi(-\infty, 0)}{2} & \text{if } \eta = 0 \text{ and } \Pi(-\infty, 0) < \infty. \end{cases}
\]

In addition, according to Lemma 3.3 of [17],

\[
e^{-\Phi(q)x} W(q)(x) \xrightarrow{\psi'(\Phi(q))^{-1}} x \uparrow \infty.
\]

(3.10)

For the expression of (3.4), we also use the scale function \( W(q+r) \) and \( \Phi(q+r) \) defined by (3.5) and (3.6), respectively, with \( q \) replaced with \( q+r \). Note that

\[
\Phi(q+r) > \Phi(q), \quad r > 0
\]

(3.11)

and from identity (5) in [24]

\[
W(q+r)(x) - W(q)(x) = r \int_0^x W(q+r)(u)W(q)(x-u)du, \quad x \in \mathbb{R}.
\]

(3.12)

We also define, for \( q, r > 0 \) and \( x \in \mathbb{R} \),

\[
Z(q)(x, \Phi(q+r)) := e^{\Phi(q+r)x} \left( 1 - r \int_0^x e^{-\Phi(q+r)z} W(q)(z)dz \right)
\]

\[
= r \int_0^\infty e^{-\Phi(q+r)z} W(q)(z + x)dz > 0,
\]

(3.13)

where the second equality holds because (3.5) gives \( \int_0^\infty e^{-\Phi(q+r)x} W(q)(x)dx = r^{-1} \). By differentiating this with respect to the first argument,

\[
Z(q)(x, \Phi(q+r)) := \frac{\partial}{\partial x} Z(q)(x, \Phi(q+r)) = \Phi(q+r) Z(q)(x, \Phi(q+r)) - r W(q)(x), \quad x > 0.
\]

(3.14)

The following results, related to the computation of the expected NPV under a periodic barrier strategy at the level \( b \), are immediate applications of Corollary 3.1 (ii) in [26]. Note that only the case \( b > 0 \) is covered in Corollary 3.1 (ii) in [26], but can be extended to the case \( b = 0 \) by monotone convergence by taking a decreasing sequence of down-crossing times.
Lemma 3.1. For all $b \geq 0$ and $x \in \mathbb{R}$,
\[
    v_b(x) = r \frac{W^{(q)}(x) + r \int_0^b W^{(q+r)}(x - b - y)W^{(q)}(y + b)dy - rW^{(q)}(b)W^{(q+r)}(x - b)}{\Phi(q + r)Z^{(q+r)}(b, \Phi(q + r))} - rW^{(q+r)}(x - b).
\] (3.15)

It is noted that the expression (3.15) also hold for $-\infty < x \leq b$ with
\[
    v_b(x) = \frac{r}{\Phi(q + r)Z^{(q+r)}(b, \Phi(q + r))} W^{(q)}(x).
\]

Moreover, for the case $b = 0$, by (3.12) the expression (3.15) is simplified as follows:
\[
    v_0(x) = \frac{r \left( W^{(q+r)}(x) - rW^{(q)}(0)W^{(q+r)}(x) \right)}{\Phi(q + r)(\Phi(q + r) - rW^{(q)}(0))} - rW^{(q+r)}(x).
\] (3.16)

3.2. Completely monotone case. In the remainder of the paper, we assume the following.

Assumption 3.1. The Lévy measure $\Pi$ of the dual process $-X$ has a completely monotone density. That is, $\Pi$ has a density $\pi$ whose $n^{th}$ derivative $\pi^{(n)}$ exists for all $n \geq 1$ and satisfies
\[
    (-1)^n \pi^{(n)}(x) \geq 0, \quad x > 0.
\]

This assumption is known to be a sufficient condition of optimality for the classical spectrally negative case by Loeffen [22] and for the absolutely continuous case by Loeffen et al. [19].

Remark 3.2. Under Assumption 3.1, we have the following.

(1) As in Theorem 2 of [23], the scale function $W^{(q)}(x)$ is infinitely differentiable and can be written as
\[
    W^{(q)}(x) = \Phi'(q)e^{\Phi(q)x} - \int_0^\infty e^{-xt}\mu^{(q)}(dt), \quad x > 0,
\]
for some finite measure $\mu^{(q)}$.

(2) As in the proof of Theorem 3 of [22], we have $W^{(q+p)}(x) > 0$ for all $x > 0$, and hence there exists $\bar{b} \in [0, \infty)$ such that $W^{(q+p)} < 0$ on $(0, \bar{b})$ and $W^{(q+p)} > 0$ on $(\bar{b}, \infty)$.

(3) As shown in Loeffen [22], the optimal solution for the classical case is to reflect (in the classical sense) from above at $\bar{b}$; the value function is given by
\[
    \bar{v}(x) := \sup_{\pi \in A_\infty} \mathbb{E}_x \left( \int_{[0,\sigma^\Pi_\bar{b}]} e^{-q t}dL^\pi(t) \right) = \begin{cases} 
        \frac{W^{(q)}(x)}{W^{(q+p)}(\bar{b})} & x \leq \bar{b}, \\
        \frac{W^{(q)}(\bar{b})}{W^{(q+p)}(\bar{b})} + x - \bar{b} & x > \bar{b}, 
    \end{cases}
\] (3.17)

where $A_\infty$ is the set of nondecreasing, right-continuous, and $\mathbb{F}$-adapted processes, as a relaxation of $A_r$. 

4. Selection of optimal barrier $b^*$

In this section, we focus on the above-mentioned periodic barrier strategy and choose a candidate barrier $b^*$, and show its existence.

4.1. Smooth fit. Motivated by many relevant studies, we choose the barrier $b^*$ so that the degree of smoothness of $v_{b^*}$ at $b^*$ increases by one (if $b^* > 0$). Unlike the classical model in [7] and [22], we see in our model that $v_{b^*}$ becomes $C^2(0, \infty)$ (resp. $C^3(0, \infty)$) for the case $X$ is of bounded (resp. unbounded) variation.

Here, we shall show that the desired smoothness at $b > 0$ is satisfied on condition that

$$\mathcal{C}_b : W^{(q)\prime}(b) = \frac{\Phi(q + r)}{r} Z^{(q)\prime}(b, \Phi(q + r)),$$

where $Z^{(q)\prime}(b, \Phi(q + r))$ is given as in (3.14).

To this end, we first compute the derivatives of (3.15). Recall the smoothness of the scale function as in Remark 3.2.

Lemma 4.1. For $b > 0$ and $x \in (0, \infty) \setminus \{b\}$,

$$v_b'(x) = r \left( \frac{W^{(q)\prime}(x) + r \int_0^{x-b} W^{(q+r)}(y)W^{(q)\prime}(x-y)dy}{\Phi(q + r)Z^{(q)\prime}(b, \Phi(q + r))} - W^{(q+r)}(x-b) \right),$$

$$v_b''(x) = r \left( \frac{W^{(q)\prime}(x) + r W^{(q+r)}(x-b)W^{(q)\prime}(b) + r \int_0^{x-b} W^{(q+r)}(y)W^{(q)\prime}(x-y)dy}{\Phi(q + r)Z^{(q)\prime}(b, \Phi(q + r))} - W^{(q+r)}(x-b) \right),$$

$$v_b'''(x) = r \frac{W^{(q)\prime}(x) + r W^{(q+r)}(x-b)W^{(q)\prime}(b)}{\Phi(q + r)Z^{(q)\prime}(b, \Phi(q + r))} \left( W^{(q)\prime}(x) + r W^{(q+r)}(x-b)W^{(q)\prime}(b) \right)
+ r W^{(q+r)}(x-b) W^{(q)\prime}(b) + r \int_0^{x-b} W^{(q+r)}(y)W^{(q)\prime}(x-y)dy - r W^{(q+r)}(x-b).$$

Proof. (i) By $\int_0^{x-b} W^{(q+r)}(x-b-y)W^{(q)}(y)dy = \int_0^{x-b} W^{(q+r)}(y)W^{(q)}(x-y)dy$ and dominated convergence, we have

$$\frac{\partial}{\partial x} \left( W^{(q)}(x) + r \int_0^{x-b} W^{(q+r)}(y)W^{(q)}(x-y)dy - r W^{(q)}(b)W^{(q+r)}(x-b) \right)$$

$$= W^{(q)\prime}(x) + r \lim_{\epsilon \to 0} \frac{\int_0^{x+\epsilon-b} W^{(q+r)}(y)W^{(q)}(x+\epsilon-y)dy - \int_0^{x-b} W^{(q+r)}(y)W^{(q)}(x-y)dy}{\epsilon}$$

$$- r W^{(q+r)}(x-b)W^{(q)}(b)$$

$$= W^{(q)\prime}(x) + r \int_0^{x-b} W^{(q+r)}(y)W^{(q)\prime}(x-y)dy.$$
Hence, by differentiating (3.15), we obtain (4.2). (ii) By differentiating (4.5), the dominated convergence theorem gives

\[ \frac{\partial^2}{\partial x^2} \left( W^{(q)}(x) + r \int_0^{x-b} W^{(q+r)}(x - b - y)W^{(q)}(y + b)dy - rW^{(q)}(b)\overline{W}^{(q+r)}(x - b) \right) \]

\[ = W^{(q)''}(x) + rW^{(q+r)'}(x - b)W^{(q)'}(b) + r \int_0^{x-b} W^{(q+r)'}(y)W^{(q)'}(x - y)dy. \]

Hence, by using the aforementioned identity and by differentiating (4.3), we obtain (4.4). (iii) By further differentiating (4.6), the dominated convergence theorem gives

\[ \frac{\partial^3}{\partial x^3} \left( W^{(q)}(x) + r \int_0^{x-b} W^{(q+r)}(x - b - y)W^{(q)}(y + b)dy - rW^{(q)}(b)\overline{W}^{(q+r)}(x - b) \right) \]

\[ = W^{(q)'''}(x) + rW^{(q+r)''}(x - b)W^{(q)''}(b) + rW^{(q+r)''}(x - b)W^{(q)''}(b) \]

\[ + r \int_0^{x-b} W^{(q+r)''}(y)W^{(q)''}(x - y)dy. \]

Hence, by using the aforementioned identity and by differentiating (4.3), we obtain (4.4).

By the smoothness of the scale function on \( \mathbb{R} \setminus \{0\} \) as in Remark 3.2, the derivatives (4.2), (4.3), and (4.4) are continuous on \((0, \infty) \setminus \{b\} \). Now, we analyze their continuity at \( b \) for the cases of bounded and unbounded variation. Recall the behaviors of the scale function around zero as in Remark 3.1. Based on (3.15) and (4.2) we determine that \( v_b \) and \( v'_b \) are continuous functions on \((0, \infty) \) regardless of the choice of \( b \). In addition, we note that

\[ v''_b(b^+) - v''_b(b^-) = rW^{(q+r)}(0) \left( \frac{rW^{(q)'}(b)}{\Phi(q + r)Z^{(q)'}(b, \Phi(q + r))} - 1 \right). \]

a) Let us assume that \( X \) has bounded variation paths. By (4.7) and the fact that \( W^{(q+r)}(0) > 0 \) as in (3.9), \( v''_b \) is continuous on \((0, \infty) \) if and only if \( \mathcal{C}_b \) in (4.1) holds.

b) On the other hand let us assume that \( X \) has unbounded variation paths. Then using, in (4.7), the fact that \( W^{(q+r)}(0) = 0 \) as in (3.9), \( v''_b \) is continuous on \((0, \infty) \) regardless of the choice of \( b \). In addition, if \( \mathcal{C}_b \) is satisfied, we note by (4.4) that

\[ v'''_b(x) = \frac{r}{\Phi(q + r)Z^{(q)'}(b, \Phi(q + r))} \left( W^{(q)'''}(x) \right. \]

\[ + \left. rW^{(q+r)}(x - b)W^{(q)''}(b) + r \int_0^{x-b} W^{(q+r)''}(y)W^{(q)''}(x - y)dy \right), \]

and hence \( v'''_b \) is continuous on \((0, \infty) \). In the following, we summarize the obtained results.

Lemma 4.2. Suppose \( b > 0 \) satisfies \( \mathcal{C}_b \) in (4.1). Then, \( v_b \) is \( C^2(0, \infty) \) for the case \( X \) is of bounded variation, while it is \( C^3(0, \infty) \) for the case \( X \) is of unbounded variation.
Remark 4.1. The verification lemma (Lemma 5.1) requires only the $C^1(0, \infty)$ and $C^2(0, \infty)$ conditions for the cases of bounded and unbounded variation, respectively. The extra smoothness obtained in Lemma 4.2 will not be directly used for the proof of optimality. However, it will be shown in Proposition 5.1, that the barrier selected by this smoothness criteria satisfies the desired slope condition of the value function.

4.2. Existence of $b^*$. Here we pursue the existence of $b^* > 0$ such that the condition $C_b$ for $b = b^*$ holds.

Define, for $b > 0$,

$$h(b) := e^{-\Phi(q+r)b} \left[ rW'(q)(b) - \Phi(q+r)Z^{(q)}(b, \Phi(q+r)) \right]$$

(4.8)

$$= e^{-\Phi(q+r)b} \left[ rW'(q)(b) - \Phi(q+r) \{ \Phi(q+r)Z^{(q)}(b, \Phi(q+r)) - rW'(q)(b) \} \right],$$

with its initial value

$$h(0+) = r[W'(q)(0+) + \Phi(q+r)W(q)(0)] - \Phi^2(q+r).$$

It is clear that $C_b$ is satisfied if and only if $h(b) = 0$.

By differentiating (4.8), we get

$$h'(b) = re^{-\Phi(q+r)b}W''(q)(b), \quad b > 0.$$  

(4.9)

In addition, by (3.10), (3.11), and (3.13) and because $\int_0^b e^{-\Phi(q+r)z} W'(q)(z) dz \xrightarrow{b \to \infty} \int_0^\infty e^{-\Phi(q+r)z} W(q)(z) dz = r^{-1}$ in view of (3.5),

$$h(b) = re^{-\Phi(q+r)b}W'(q)(b) - \Phi(q+r) \left( \Phi(q+r) \left( \Phi(q+r) - r \int_0^b e^{-\Phi(q+r)z} W'(q)(z) dz \right) \right)$$

(4.10)

$$\xrightarrow{b \to \infty} 0.$$  

Hence, we can write

$$h(b) = -r \int_b^\infty e^{-\Phi(q+r)y} W'(q)(y) dy, \quad b > 0.$$  

(4.11)

Based on (4.9) and (4.11) and Remark 3.2 (2), the function $h$ decreases on $(0, \bar{b})$ with $h(\bar{b}) < 0$. It then increases on $(\bar{b}, \infty)$ and converges to $0$. (See the plots in Figure 1 in Section 7.)

The above argument and the continuity of $h(b)$ imply that there exists $0 < b^* < \bar{b}$ such that $h(b^*) = 0$ (or equivalently $C_b$ for $b = b^*$) if and only if

$$h(0+) > 0 \iff \left( W'(q)(0+) + \Phi(q+r)W(q)(0) \right) > \Phi^2(q+r).$$  

(4.12)

For the case $h(0+) \leq 0$, we set $b^* = 0$.

We summarize the results in the following proposition.
Proposition 4.1. (i) If (4.12) holds, then there exists
\begin{equation}
0 < b^* < \bar{b} \tag{4.13}
\end{equation}
such that $\mathfrak{C}_b$ for $b = b^*$ is satisfied and $h(b) \geq 0$ if and only if $b \in (0, b^*]$. 
(ii) Otherwise, $h(b) \leq 0$ for all $b > 0$.

Remark 4.2. By using (3.9), $b^* = 0$ if and only if one of the following holds: 
(i) $\eta > 0$ and $r \leq \frac{\eta^2}{2} \Phi^2(q + r)$ or 
(ii) $\eta = 0$, $\Pi(-\infty, 0) < \infty$, and $\Phi^2(q + r) \geq r \left( \frac{q + \Pi(-\infty, 0)}{c^2} + \frac{\Phi(q + r)}{c} \right)$.

Remark 4.3. In view of (i) of Remark 4.2 and by using (2.1), 
\begin{equation}
r - \frac{\eta^2}{2} \Phi^2(q + r) = -q + \Phi(q + r) \gamma + \int_{(-\infty,0)} (e^{\Phi(q+r)z} - 1 - \Phi(q+r)z1_{\{z>-1\}}) \Pi(dz). \tag{5.1}
\end{equation}
For the case $\eta > 0$, we have $\Phi(q + r) \sim \frac{\sqrt{2r}}{\eta}$ as $r \to \infty$. Hence,
\begin{equation}
\frac{\eta}{\sqrt{2r}} \left( r - \frac{\eta^2}{2} \Phi^2(q + r) \right) = \frac{\eta}{\sqrt{2r}} \left( -q + \Phi(q + r) \gamma + \int_{(-\infty,0)} (e^{\Phi(q+r)z} - 1 - \Phi(q+r)z1_{\{z>-1\}}) \Pi(dz) \right) \\
\xrightarrow{r \to \infty} \gamma - \int_{(-\infty,0)} z1_{\{z>-1\}} \Pi(dz) = c.
\end{equation}
Hence, we obtain $b^* > 0$ for a large enough $r$, if $c > 0$. This is consistent with the classical case as given in Theorem 1 in [23].

5. Verification of Optimality

With $b^* \geq 0$ defined above, we now show the optimality of the obtained periodic barrier strategy $\pi^{b^*}$. For the case $b^* > 0$, because $\mathfrak{C}_b$ holds for $b = b^*$, the expected NPV (3.15) can be succinctly written as
\begin{equation}
v_{b^*}(x) = \frac{W(q)(x) + r \int_{0}^{x-b^*} W^{(q+r)}(x - b^* - y)W^{(q)}(y + b^*)dy - rW^{(q)}(b^*)W^{(q+r)}(x - b^*)}{W^{(q+r)}(b^*)} \\
- r\bar{W}^{(q+r)}(x - b^*), \quad \text{for } x \in \mathbb{R}, \tag{5.1}
\end{equation}
where, in particular, for $x < b^*$,
\begin{equation}
v_{b^*}(x) = \frac{W(q)(x)}{W^{(q+r)}(b^*)}. \tag{5.2}
\end{equation}
In contrast, for the case $b^* = 0$, the expected NPV is given by (3.16).
5.1. Verification lemma. Let $\mathcal{L}$ be the infinitesimal generator associated with the process $X$ applied to a $C^1$ (resp. $C^2$) function $f$ for the case $X$ is of bounded (resp. unbounded) variation:

\begin{equation}
\mathcal{L}f(x) := \gamma f'(x) + \frac{1}{2}\sigma^2 f''(x) + \int_{(-\infty,0)} \left[f(x + z) - f(x) - f'(x)z 1_{\{z < 0\}}\right] \Pi(dz), \quad x > 0.
\end{equation}

We now provide a verification lemma. The proof is essentially the same as Lemma 4.3 in [25] (which deals with the spectrally positive case with a terminal payoff/penalty), and is hence omitted.

**Lemma 5.1** (Verification lemma). Suppose $\hat{\pi} \in \mathcal{A}$ is such that $v_{\hat{\pi}}$ is $C^1(0,\infty)$ (resp. $C^2(0,\infty)$) for the case $X$ is of bounded (resp. unbounded) variation, and satisfies

\begin{equation}
(\mathcal{L} - q)v_{\hat{\pi}}(x) + r \max_{0 \leq t \leq x} \{l + v_{\hat{\pi}}(x - l) - v_{\hat{\pi}}(x)\} \leq 0, \quad x > 0.
\end{equation}

Then, $v_{\hat{\pi}}(x) = v(x)$ for all $x \geq 0$, and hence $\hat{\pi}$ is an optimal strategy.

Note that $W^{(q)}$ is infinitely differentiable on $(0, \infty)$, as pointed out in Remark 3.2. By the proof of Lemma 4 of [7], for any $B > 0$, $(e^{-q(t \wedge \tau_0 \wedge \tau_B^+)}W^{(q)}(X(t \wedge \tau_0 \wedge \tau_B^+)); t \geq 0)$ is a martingale, and hence

\begin{equation}
(\mathcal{L} - q)W^{(q)}(y) = 0, \quad y > 0.
\end{equation}

Similarly, by Proposition 2 of [7],

\begin{equation}
(\mathcal{L} - q)Z^{(q)}(y) = 0, \quad y > 0.
\end{equation}

Note that these identities also hold when $q$ is replaced with $q + r$. By using these identities, we show the following.

**Lemma 5.2.** For $y > 0$,

\begin{equation}
(\mathcal{L} - q)W^{(q+r)}(y) = rW^{(q+r)}(y),
\end{equation}

\begin{equation}
(\mathcal{L} - q)\overline{W}^{(q+r)}(y) = 1 + r\overline{W}^{(q+r)}(y),
\end{equation}

\begin{equation}
(\mathcal{L} - q)\overline{W}^{(q+r)}(y) = y + r\overline{W}^{(q+r)}(y).
\end{equation}

**Proof.** (i) By (5.5), we have $(\mathcal{L} - q)W^{(q+r)}(y) = (\mathcal{L} - q - r)W^{(q+r)}(y) + rW^{(q+r)}(y) = rW^{(q+r)}(y)$.

(ii) By using (5.6), we have

\begin{equation}
(\mathcal{L} - q)\overline{W}^{(q+r)}(y) = \frac{1}{r + q}(\mathcal{L} - q)\left(Z^{(q+r)}(y) - 1\right) = \frac{r}{r + q}Z^{(q+r)}(y) + \frac{q}{r + q} = 1 + r\overline{W}^{(q+r)}(y).
\end{equation}

(iii) By integration by parts and the proof of Lemma 4.5 in [13], we have

\begin{equation}
(\mathcal{L} - q - r)\overline{W}^{(q+r)}(y) = (\mathcal{L} - q - r)\int_y^y zW^{(q+r)}(y - z)dz = y.
\end{equation}

Hence, we have (5.9).
Lemma 5.3. For \( b^* \geq 0 \), we have

\[
(\mathcal{L} - q)v_{b^*}(x) = \begin{cases} 
0 & \text{if } x \in (0, b^*], \\
r \{(x - b^*) + v_{b^*}(b^*) - v_{b^*}(x)\} & \text{if } x \in (b^*, \infty).
\end{cases}
\]  

Proof. (i) Suppose \( b^* > 0 \). For \( 0 < x < b^* \), by (5.2) and (5.5), we have

\[
(\mathcal{L} - q)v_{b^*}(x) = \frac{1}{W(q^r(b^*))(\mathcal{L} - q)W(q)(x)} = 0.
\]

Now suppose \( x > b^* \). By the proof of Lemma 4.5 in [13], we obtain

\[
(\mathcal{L} - (q + r))\int_0^{x-b^*} W^{(q+r)}(x - b^* - y)W^{(q)}(y + b^*)dy = (\mathcal{L} - (q + r))\int_{b^*}^{x} W^{(q+r)}(x - u)W^{(q)}(u)du
\]

which together with (5.5) implies

\[
(\mathcal{L} - q)
\left(W^{(q)}(x) + r \int_0^{x-b^*} W^{(q+r)}(x - b^* - y)W^{(q)}(y + b^*)dy\right)
\]

\[
= r \left(W^{(q)}(x) + r \int_0^{x-b^*} W^{(q+r)}(x - b^* - y)W^{(q)}(y + b^*)dy\right).
\]

Hence, by applying (5.8), (5.9), and (5.11) in (5.1), we have

\[
(\mathcal{L} - q)v_{b^*}(x) = \frac{r}{W(q^r(b^*))} \left(W^{(q)}(x) + r \int_0^{x-b^*} W^{(q+r)}(x - b^* - y)W^{(q)}(y + b^*)dy\right)
\]

\[
- r \frac{W(q^r(b^*))}{W(q^r(b^*))} \left(rW^{(q+r)}(x - b^*) + 1\right) - \frac{r}{W(q^r(b^*))} \left((x - b^*) + rW^{(q+r)}(x - b^*)\right)
\]

\[
= -r \left\{(x - b^*) + v_{b^*}(b^*) - v_{b^*}(x)\right\}.
\]

Finally, this can be extended to the case in which \( x = b^* \) by taking limits as \( x \to b^* \).

(ii) Suppose \( b^* = 0 \). By applying (5.7), (5.8), and (5.9) in (3.16), we have

\[
(\mathcal{L} - q)v_0(x) = \frac{r(\mathcal{L} - q)W^{(q+r)}(x) - rW^{(q)}(0)(\mathcal{L} - q)W^{(q+r)}(x)}{\Phi(q + r)(\Phi(q + r) - rW^{(q)}(0))} - r(\mathcal{L} - q)W^{(q+r)}(x)
\]

\[
= r \left[rW^{(q+r)}(x) - rW^{(q)}(0)(1 + rW^{(q+r)}(x)) - \Phi(q + r)(\Phi(q + r) - rW^{(q)}(0)) - (x + rW^{(q+r)}(x))\right],
\]

which equals \(-r(x + v_0(0) - v_0(x))\) as desired. \(\square\)

We now prove the following.

Proposition 5.1. For \( b^* \geq 0 \), we have \( v'_{b^*}(x) \geq 1 \) for \( x \in (0, b^*) \) and \( 0 \leq v'_{b^*}(x) \leq 1 \) for \( x \in (b^*, \infty) \).
To prove this proposition, we first rewrite the derivative (4.2) by using the decomposition of the scale function given in Remark 3.2 (1).

**Lemma 5.4.** For \( x \geq b \geq 0 \), we have

\[
v'_b(x) = K + r \int_0^\infty e^{-tx} g(t, b) \mu^{(q+r)}(dt) \frac{\Phi(q + r)Z(q')b, \Phi(q + r))}{\Phi(q + r)Z(q')(b, \Phi(q + r))},
\]

where

\[
K := r \frac{\Phi'(q + r)}{\Phi(q + r)} + r \int_0^\infty \frac{\mu^{(q+r)}(dt)}{t},
\]

\[
g(t, b) := t + rW'(0) + r \int_0^b e^{ut} W'(u)du - e^{bt} \Phi(q + r)Z(q')(b, \Phi(q + r))
\]

\[
= t + rW'(0) + r \int_0^b e^{ut} \left( W'(u) - \frac{\Phi(q + r)}{r} Z(q')(b, \Phi(q + r)) \right)du - \frac{\Phi(q + r)}{t} Z(q')(b, \Phi(q + r)).
\]

Note that by considering \( \int_0^\infty e^{-xt} \mu^{(q+r)}(dt) \) as the density of the \((q + r)\)-resolvent measure of \(-X\) at \( x > 0 \) as in Theorem 2.7 (iv) of [17],

\[
\frac{1}{q + r} \mathbb{P}(X(e_{q+r}) < 0) = \int_0^\infty \int_0^\infty e^{-xt} \mu^{(q+r)}(dt)dx = \int_0^\infty \int_0^\infty e^{-xt} dx \mu^{(q+r)}(dt) = \int_0^\infty \frac{\mu^{(q+r)}(dt)}{t}.
\]

**Proof.** By differentiating both sides of (3.12), for \( x > 0 \),

\[
W^{(q+r)}(x) - rW^{(q+r)}(x)W^{(q)}(0) - r \int_0^b W^{(q+r)}(x - u)W^{(q)}(u)du
\]

\[
= W^{(q)}(x) + r \int_0^x W^{(q+r)}(x - u)W^{(q)}(u)du = W^{(q)}(x) + r \int_0^{x - b} W^{(q+r)}(y)W^{(q)}(x - y)dy.
\]

Hence, the equality (4.2), for \( b > 0 \), reduces to

\[
(5.12) \quad \frac{1}{r} v'_b(x) = \frac{W^{(q+r)}(x) - rW^{(q+r)}(x)W^{(q)}(0) - r \int_0^b W^{(q+r)}(x - u)W^{(q)}(u)du}{\Phi(q + r)Z(q')(b, \Phi(q + r))} - W^{(q+r)}(x - b).
\]

The same expression is obtained for \( b = 0 \) by differentiating (3.16). Further, by using Remark 3.2 (1), we can write

\[
(5.13) \quad \frac{1}{r} v'_b(x) = \Phi'(q + r) \frac{\Phi(q + r)e^{(q+r)x} - re^{(q+r)x}W^{(q)}(0) - r \int_0^b e^{(q+r)(x - u)}W^{(q)}(u)du}{\Phi(q + r)Z(q')(b, \Phi(q + r))}
\]

\[
- \Phi'(q + r) \int_0^{x - b} e^{\Phi(q+r)u}du + \int_0^{x - b} \int_0^\infty e^{-tu} \mu^{(q+r)}(dt)du
\]

\[
+ \int_0^\infty e^{-tx} \mu^{(q+r)}(dt) + r \int_0^\infty e^{-tx} W^{(q)}(0) \mu^{(q+r)}(dt) + r \int_0^b \int_0^\infty e^{-t(x-u)} \mu^{(q+r)}(dt)W^{(q)}(u)du.
\]
Integration by parts gives
\[
\int_0^b e^{\Phi(q+r)(x-u)} W^{(q)\prime}(u) du = e^{\Phi(q+r)x} \left[ e^{-\Phi(q+r)b} W^{(q)}(b) - W^{(q)}(0) + \Phi(q+r) \int_0^b e^{-\Phi(q+r)u} W^{(q)}(u) du \right],
\]
and hence
\[
\Phi(q+r)e^{\Phi(q+r)x} - re^{\Phi(q+r)x} W^{(q)}(0) - r \int_0^b e^{\Phi(q+r)(x-u)} W^{(q)\prime}(u) du = e^{\Phi(q+r)(x-b)} \left( e^{\Phi(q+r)b} \Phi(q+r) \left( 1 - r \int_0^b e^{-\Phi(q+r)u} W^{(q)}(u) du \right) - r W^{(q)}(b) \right) = e^{\Phi(q+r)(x-b)} Z^{(q)\prime}(b, \Phi(q+r)).
\]
Now, by using the above expression in (5.13), we obtain
\[
\frac{1}{r} \upsilon_{b^*}'(x) = \frac{\Phi'(q+r)}{\Phi(q+r)} \left( e^{\Phi(q+r)(x-b)} - (e^{\Phi(q+r)(x-b)} - 1) \right) + \int_0^{x-b} \int_0^\infty e^{-tu} \mu^{(q+r)}(dt) du + \int_0^\infty te^{-tx} \mu^{(q+r)}(dt) + r \int_0^\infty e^{-tx} W^{(q)}(0) \mu^{(q+r)}(dt) + r \int_0^\infty e^{-t(x-u)} \mu^{(q+r)}(dt) W^{(q)\prime}(u) du \frac{\Phi(q+r) Z^{(q)\prime}(b, \Phi(q+r))}{\Phi(q+r) Z^{(q)\prime}(b, \Phi(q+r))}
\]
Hence, the result is obtained.

We are now ready to prove the proposition.

Proof of Proposition 5.1. (i) Suppose \( b^* > 0 \).

1. Suppose \( x \leq b^* \). Because \( b^* \leq b \) and \( W^{(q)\prime} \) is decreasing on \((0, b)\) as mentioned in Remark 3.2 (2),
\[
\upsilon_{b^*}'(x) = \frac{W^{(q)\prime}(x)}{W^{(q)\prime}(b^*)} \geq 1.
\]
(2) Suppose \( x > b^* \). First, because the strategy \( \pi^{b^*} \) pushes the process to \( b^* \) at the first exponential time \( T(1) = e_r \) at which the process is above \( b^* \), we can write
\[
\upsilon_{b^*}(x) = \mathbb{E}[e^{-q e_r} [(X(e_r) + x - b^*) \lor 0] 1_{\{X(e_r) + x \geq 0\}}] + \mathbb{E}[e^{-q e_r} \upsilon_{b^*}((X(e_r) + x)^\wedge b^*) 1_{\{X(e_r) + x < 0\}}],
\]
where \( X \) is the running infimum process of \( X \). Because \( \upsilon_{b^*} \) is nonnegative and increasing on \((0, b^*)\) according to (1), this is increasing in \( x > b^* \). This shows that \( \upsilon_{b^*}'(x) \geq 0 \).

By Lemma 5.4 and because the condition \( \mathcal{E}_b \) holds for \( b = b^* \),
\[
\upsilon_{b^*}'(x) = K + \int_0^\infty e^{-t x} g(t, b^*) \mu^{(q+r)}(dt) \frac{W^{(q)\prime}(b^*)}{W^{(q)\prime}\prime(b^*)},
\]

(5.14)
where by $\mathfrak{C}_b$ for $b = b^*$,

$$ g(t, b^*) = t + rW^{(q)}(0) + r \int_0^{b^*} e^{ut}(W^{(q)}(u) - W^{(q)}(b^*))du - \frac{r^2}{t} W^{(q)}(b^*). $$

We have $g(0+, b^*) = -\infty$ and because (4.13) and Remark 3.2 (2) give $W^{(q)}(u) \geq W^{(q)}(b^*)$ for $u \leq b^*$,

$$ \frac{\partial}{\partial t} g(t, b^*) = 1 + r \int_0^{b^*} ue^{ut}(W^{(q)}(u) - W^{(q)}(b^*))du + \frac{r}{t^2} W^{(q)}(b^*) > 1. $$

Hence, there exists $p > 0$ such that

$$ g(t, b^*) \leq 0 \iff t \leq p. $$

By monotone convergence,

$$ \int_0^{\infty} e^{-tx} g(t, b^*) \mu^{(q+r)}(dt) = -\int_0^{p} e^{-tx} |g(t, b^*)| \mu^{(q+r)}(dt) + \int_p^{\infty} e^{-tx} g(t, b^*) \mu^{(q+r)}(dt) \xrightarrow{x \to \infty} 0. $$

Hence, in view of (5.15), $\lim_{x \to \infty} v_{b^*}(x) = K$.

To show that

$$ 0 \leq K \leq 1, $$

by using (5.14) and denoting $\overline{X}$ as the running supremum process of $X$,

$$ v_{b^*}(x) \leq \mathbb{E} \left[ e^{-qer} (x + \overline{X}(e_r) - b^*) \right] + \mathbb{E} \left[ e^{-qer} \sup_{0 \leq y \leq b^*} v_{b^*}(y) \right] \leq x \mathbb{E} \left[ e^{-qer} \right] + \Phi(r)^{-1} + \mathbb{E} \left[ e^{-qer} \sup_{0 \leq y \leq b^*} v_{b^*}(y) \right], $$

where in the last inequality we used $\mathbb{E}[e^{-qer}\overline{X}(e_r)] \leq \mathbb{E}[\overline{X}(e_r)] = \Phi(r)^{-1} < \infty$ (see Exercise 3.6 in [18]). Hence, $K = \lim_{x \to \infty} v_{b^*}(x)/x \leq \mathbb{E}\left[e^{-qer}\right] < 1$. Moreover, based on the aforementioned argument according to which $v_{b^*}$ is nondecreasing, we must have $K \geq 0$.

Because $v_{b^*}'(b^*) = 1$ according to (5.2) and the smoothness at $b^*$ as stated in Lemma 4.2,

$$ 1 = v_{b^*}'(b^*) = K + \int_0^{\infty} \frac{e^{-tb^*} g(t, b^*) \mu^{(q+r)}}{W^{(q)}(b^*)} (dt). $$

In addition, (5.16) gives $e^{-(x-b^*)p} g(t, b^*) \geq e^{-(x-b^*)t} g(t, b^*)$ for all $t > 0$. Therefore, by using these and (5.17),

$$ v_{b^*}'(x) = K + \int_0^{\infty} \frac{e^{-(x-b^*)t} e^{-tb^*} g(t, b^*) \mu^{(q+r)}}{W^{(q)}(b^*)} (dt) \leq K + e^{-(x-b^*)p} \int_0^{\infty} \frac{e^{-tb^*} g(t, b^*) \mu^{(q+r)}}{W^{(q)}(b^*)} (dt) = K + e^{-(x-b^*)p}(1 - K) \leq 1. $$
(ii) Suppose $b^* = 0$. First, we can write
\[
v_0(x) = \mathbb{E}\left[e^{-q_0r}(X(e_x) + x)1_{\{X(e_x) + x \geq 0\}}\right] + \mathbb{E}\left[e^{-q_0r}v_0(0)1_{\{X(e_x) + x \geq 0\}}\right].
\]
Because $v_0$ is nonnegative, this increases in $x$, showing that $v_0'(x) \geq 0$.

By Lemma 5.4 and the fact that $Z^{(q)}(0, \Phi(q + r)) = \Phi(q + r) - rW^{(q)}(0)$, we have
\[
v_0'(x) = K + r\frac{\int_0^\infty e^{-tx}g(t, 0)\mu^{(q+r)}(dt)}{\Phi(q + r)(\Phi(q + r) - rW^{(q)}(0))},
\]
where
\[
g(t, 0) = t + rW^{(q)}(0) - \frac{1}{t}\Phi(q + r)(\Phi(q + r) - rW^{(q)}(0)).
\]
Recall that $b^* = 0$ if and only if (i) or (ii) of Remark 4.2 holds. For case (i), we have $\Phi(q + r) - rW^{(q)}(0) = \Phi(q + r) > 0$ by (3.9), while in case (ii), by Remark 4.2 (ii), we have
\[
\Phi(q + r) - rW^{(q)}(0) = \Phi(q + r) - r c \geq r q + \Pi(-\infty, 0) > 0.
\]
This implies that $g(0+, 0) = -\infty$ and
\[
\frac{\partial}{\partial t} g(t, 0) = 1 + \frac{\Phi(q + r)}{t^2}(\Phi(q + r) - rW^{(q)}(0)) > 1.
\]
Hence, there exists $p > 0$ such that $g$ is negative on $(0, p)$ and positive on $(p, \infty)$. Then, $e^{-xp}g(t) \geq e^{-xt}g(t)$ for all $t > 0$.

By (5.12),
\[
v_0'(0+) = r\frac{W^{(q+r)'}(0+) - rW^{(q+r)}(0)W^{(q)}(0)}{\Phi(q + r)(\Phi(q + r) - rW^{(q)}(0))} = \frac{rW^{(q+r)'}(0+)}{\Phi(q + r)(\Phi(q + r) - rW^{(q)}(0))},
\]
where, under (ii) of Remark 4.2, the second equality holds by (3.9). Noting that $b^* = 0$ if and only if $\Phi^2(q + r) \geq rW^{(q+r)'}(0+) + r\Phi(q + r)W^{(q)}(0)$ (in view of (4.12) and Proposition 4.1), we have
\[
v_0'(0+) = \frac{rW^{(q+r)'}(0+)}{\Phi(q + r)(\Phi(q + r) - rW^{(q)}(0))} \leq 1.
\]
On the other hand, (5.18) gives
\[
v_0'(0+) = K + r\frac{\int_0^\infty g(t, 0)\mu^{(q+r)}(dt)}{\Phi(q + r)(\Phi(q + r) - rW^{(q)}(0))},
\]
and hence
\[
r\frac{\int_0^\infty g(t, 0)\mu^{(q+r)}(dt)}{\Phi(q + r)(\Phi(q + r) - rW^{(q)}(0))} \leq 1 - K.
\]
Using these and (5.17), for \( x > 0, \)
\[
v_0'(x) = K + r \frac{\int_0^\infty e^{-xt} g(t,0) \mu(q+r)(dt)}{\Phi(q+r)(\Phi(q+r) - rW(q)(0))} \leq K + e^{-xp}r \frac{\int_0^\infty g(t,0) \mu(q+r)(dt)}{\Phi(q+r)(\Phi(q+r) - rW(q)(0))} \leq K + e^{-xp}(1 - K) \leq 1.
\]

Next, by the application of Proposition 5.1 the following result is immediate.

**Lemma 5.5.** For \( b^* \geq 0 \) we have
\[
\max_{0 \leq t \leq x} \{ l + v_{b^*}(x - l) - v_{b^*}(x) \} = \begin{cases} 0 & \text{if } x \in [0, b^*], \\ x - b^* + v_{b^*}(b^*) - v_{b^*}(x) & \text{if } x \in (b^*, \infty). \end{cases}
\]

By Lemmas 5.3 and 5.5, \( v_{b^*} \) satisfies the variational inequality (5.4). Hence, by Lemma 5.1, we have the optimality of the periodic barrier strategy \( \pi^{b^*} \), and the value function is given by \( v = v_{b^*} \).

6. **Convergence to the classical case**

In this section, we analyze the behavior of the optimal barrier \( b^* \) and the value function \( v_{b^*} \) with respect to the parameter \( r \). Solely in this section, we write \( h^{(r)}, b^*_r, \) and \( v^{(r)} = v^{(r)}_{b^*_r} \), to stress the dependence on \( r > 0 \).

**Lemma 6.1.**
1. The optimal periodic barrier \( b^*_r \) is increasing in \( r \).
2. We have \( b^*_r \to \bar{b} \) as \( r \to \infty \).
3. When \( W^{(q)'(0+)} < \infty \), \( b^*_r \) is zero for sufficiently small \( r \). When \( W^{(q)'(0+)} = \infty \), \( b^*_r \to 0 \) as \( r \to 0 \).

**Proof.**
1. For \( b > 0 \), integration by parts applied to (4.11) and the use of Remark 3.2 (which implies \( e^{-\Phi(q+r)y}W^{(q)'(y)} \to 0 \) as \( y \to \infty \)) give
\[
\hat{h}^{(r)}(b) = -r \frac{\Phi(q+r)}{\Phi(q+r)b} \left[ e^{-\Phi(q+r)b}W^{(q)'(b)} + \int_b^\infty e^{-\Phi(q+r)y}W^{(q)'(y)}dy \right].
\]

Because the third derivative \( W^{(q)'(y)} \) is always positive as described in Remark 3.2 (2),
\[
\tilde{h}^{(r)}(b) := \frac{\Phi(q+r)}{r} e^{\Phi(q+r)b} \hat{h}^{(r)}(b) = \left[ W^{(q)'(b)} + \int_b^\infty e^{-\Phi(q+r)y}W^{(q)'(y)}dy \right] - W^{(q)'(y+b)}dy < 0,
\]
\( b > 0 \).

By Remark 3.2 (2) and because \( r \mapsto \Phi(q+r) \) is increasing, \( r \mapsto \tilde{h}^{(r)}(b) \) is increasing for all \( b > 0 \). This directly implies that \( b^*_r = \sup\{ b > 0 : \tilde{h}^{(r)}(b) > 0 \} \) (with \( \sup\emptyset = 0 \)) is increasing in \( r \).

2. When \( \bar{b} = 0 \), the convergence is immediate because \( b^*_r = 0 \) for all \( r > 0 \). Hence, we assume that \( \bar{b} > 0 \).
By considering (1) and because \( b^*_r < \bar{b} \) for all \( r > 0 \) as in (4.13), there exists
\[
0 \leq b^*_\infty := \lim_{r \to \infty} b^*_r \leq \bar{b}.
\]
To show \( b^*_\infty = \bar{b} \), assume, to derive a contradiction, that \( b^*_\infty < \bar{b} \). This and Remark 3.2 (2) imply that \( W^{(q)''}(b^*_\infty) < 0 \).

(i) Suppose \( b^*_\infty > 0 \). By Remark 3.2 (1) and an application of Fubini’s Theorem, we have
\[
\int_0^\infty e^{-\Phi(q+r)y} W^{(q)''}(y + b^*_\infty) dy = \Phi'(q) \Phi^3(q) \int_0^\infty e^{-\Phi(q+r)y} e^{\Phi(q)(y + b^*_\infty)} dy
\]
\[+ \int_0^\infty \int_0^\infty t^3 e^{-t(y+b^*_\infty)} e^{-\Phi(q+r)y} \mu(q)(dt) dy
\]
\[= \frac{\Phi'(q) \Phi^3(q)}{(\Phi(q+r) - \Phi(q))} e^{\Phi(q)b^*_\infty} + \int_0^\infty \frac{t^3}{t + \Phi(q+r)} e^{-tb^*_\infty} \mu(dt),
\]
which is finite for any \( r > 0 \) and vanishes in the limit as \( r \to \infty \), because \( b^*_\infty \in (0, \infty) \) and \( \mu \) is a finite measure. Hence, we can take a sufficiently large \( r' \), such that
\[
\tilde{h}(r')(b^*_\infty) = - \left[ W^{(q)''}(b^*_\infty) + \int_0^\infty e^{-\Phi(q+r)y} W^{(q)''}(y + b^*_\infty) dy \right]
\]
is positive. However, this contradicts with \( \tilde{h}(r')(b^*_\infty) \leq 0 \) for all \( r > 0 \) (which is implied by \( b^*_\infty \geq b^*_r \) and Proposition 4.1 (i)). Hence, we must have \( b^*_\infty = \bar{b} \) for the case \( b^*_\infty > 0 \).

(ii) Suppose \( b^*_\infty = 0 \). In this case, \( h(r) \) (and hence \( \tilde{h}(r) \) as well) is uniformly negative for all \( r > 0 \) by Proposition 4.1 (ii). Then, the assumption \( 0 = b^*_\infty < \bar{b} \) implies \( W^{(q)''}(x) < 0 \) on \((0, \bar{b})\) in view of Remark 3.2 (2). Take any \( 0 < \epsilon < \bar{b} \), then we have
\[
\tilde{h}(r')(\epsilon) = - \left[ W^{(q)''}(\epsilon) + \int_0^\infty e^{-\Phi(q+r)y} W^{(q)''}(y + \epsilon) dy \right],
\]
which can be shown to be positive for a sufficiently large \( r' \), using the same argument as that in (i); this contradicts with the uniform negativity of \( \tilde{h}(r') \). Hence, we must have \( \bar{b} = 0 = b^*_\infty \).

(3) In the case \( W^{(q)''}(0+) < \infty \), by taking \( r \to 0 \) in (4.12), we see that \( h(r)(0+) < 0 \), and hence \( b^*_r = 0 \) for small enough \( r > 0 \).

In the case \( W^{(q)''}(0+) = \infty \), by using the first equality of (4.10), we have, for \( b > 0 \),
\[
\lim_{r \to 0} h(r)(b) = -\Phi(q)^2 < 0.
\]
Hence, for any fixed \( b > 0 \), we have \( h(r)(b) < 0 \) (and hence \( b^*_r < b \) by the form of \( h(r) \)) for sufficiently small \( r > 0 \); this shows that \( b^*_r r^{10} \to 0 \).

We now show the convergence of \( v(r) \) to the value function in the classical case \( \bar{v} \) as described in (3.17).

**Proposition 6.1.** As \( r \to \infty \), \( v(r)(x) \) converges to \( \bar{v}(x) \) for all \( x \geq 0 \).
proof. (i) Suppose \( \bar{b} > 0 \).

Fix \( 0 \leq x < \bar{b} \). Because \( b_r^* \) increases to \( \bar{b} \) by Lemma 6.1, we can choose a sufficiently large \( \bar{r} \) such that for all \( r > \bar{r} \), \( b_r^* > x \), and hence \( v^{(r)}(x) = W^{(q)}(x)/W^{(q)}(b_r^*) \). This converges to \( \bar{v}(x) = W^{(q)}(x)/W^{(q)}(\bar{b}) \) by Lemma 6.1 and because \( W^{(q)} \) is continuous by Remark 3.2.

Fix \( x = \bar{b} \). Then we have \( v^{(r)}(b_r^*) = W^{(q)}(b_r^*)/W^{(q)}(b_r^*) \), which is increasing in \( r \) by the monotonicity of \( b_r^* \) (as in Lemma 6.1) and that of the mapping \( y \mapsto W^{(q)}(y)/W^{(q)}(y) \) (by (8.22) of [18]). This together with the monotonicity of \( v^{(r)} \) in \( x \) (by Proposition 5.1) gives

\[
v^{(r)}(\bar{b}) > v^{(r)}(b_r^*) \xrightarrow{r \to \infty} \frac{W^{(q)}(\bar{b})}{W^{(q)}(b_r^*)} = \bar{v}(\bar{b}).
\]

On the other hand, \( v^{(r)}(\bar{b}) \leq \bar{v}(\bar{b}) \) for all \( r > 0 \) (because \( A_r \subset A_\infty \)), and thus \( v^{(r)}(\bar{b}) \xrightarrow{r \to \infty} \bar{v}(\bar{b}) \).

Fix \( x > \bar{b} > 0 \) and \( r \) sufficiently large such that \( b_r^* > 0 \). First, we can write (5.1) as

\[
v^{(r)}(x) = \frac{rA^{(r)}(x, b_r^*)}{\Phi(q + r)Z^{(q)}(b_r^*, \Phi(q + r))} + B^{(r)}(x - b_r^*) = \frac{A^{(r)}(x, b_r^*)}{W^{(q)}(b_r^*)} + B^{(r)}(x - b_r^*)
\]

where

\[
A^{(r)}(y, b) := W^{(q)}(y) + r \int_{y-b}^{y-b} W^{(q+r)}(y - b - z)W^{(q)}(z + b)dz
\]

\[
- rW^{(q)}(b)W^{(q+r)}(y - b) - W^{(q+r)}(y - b) \frac{Z^{(q)}(b, \Phi(q + r))}{\Phi(q + r)},
\]

\[
B^{(r)}(y) := r \left( \frac{1}{\Phi^2(q + r)} W^{(q+r)}(y) - \frac{W^{(q+r)}(y)}{\Phi^{(q+r)}}(y) \right).
\]

We also set \( \tilde{A}^{(r)}(y, b) := A^{(r)}(y, b)/W^{(q)}(b) \).

Fix \( 0 < b_0 < \bar{b} < x \). By using the limiting case of Lemma 5.1 in [26] (where the limit can be easily obtained by Lemma B.3 of [26] and monotone convergence), we have

\[
(6.1) \quad \tilde{A}^{(r)}(x, b) = \mathbb{E}_{x-b}(e^{-q\tau^r}; \tau^r < \tau_0^-) + \mathbb{E}_{x-b} \left( e^{-(q+r)\tau^-_0} \frac{W^{(q)}(X(\tau^-_0) + b)}{W^{(q)}(b)} ; \tau^-_0 < \infty \right).
\]

This gives a bound:

\[
\mathbb{E}_{x-b}(e^{-q\tau^r}; \tau^r < \tau_0^-) \leq \tilde{A}^{(r)}(x, b) \leq \mathbb{E}_{x-b_0}(e^{-q\tau^r}; \tau^r < \tau_0^-) + \mathbb{E}_{x-b} \left( e^{-(q+r)\tau^-_0} \right), \quad b_0 < b < \bar{b}.
\]

Notice that the dominated convergence theorem gives

\[
(6.2) \quad \mathbb{E}_y(e^{-q\tau^r}; \tau^r < \tau^-_0) \xrightarrow{r \to \infty} 1, \quad y > 0,
\]

and \( \mathbb{E}_{x-b}(e^{-(q+r)\tau^-_0}) \xrightarrow{r \to \infty} \mathbb{P}_{x-b}(\tau^-_0 = 0) = 0 \). Hence, \( \sup_{b_0 \leq b \leq \bar{b}} \tilde{A}^{(r)}(x, b) \xrightarrow{r \to \infty} 1 \).

In contrast, by (the limiting case of) Lemma 5.2 in [26], which holds for any spectrally negative Lévy process, we have

\[
B^{(r)}(x - b) = \mathbb{E}_{x-b} \left( e^{-q\tau^r} X(\tau^r); \tau^r < \tau^-_0 \right).
\]
This gives
\[ B_r(x - b) - (x - b) = \mathbb{E}_{x-b} \left( e^{-qe_r} (X(e_r) - (x - b)); e_r < \tau_0^- \right) - (x - b) \left( 1 - \mathbb{E}_{x-b} \left( e^{-qe_r}; e_r < \tau_0^- \right) \right) \]
\[ = \mathbb{E} \left( e^{-qe_r} X(e_r); e_r < \tau_{-(x-b)}^- \right) - (x - b) \left( 1 - \mathbb{E}_{x-b} \left( e^{-qe_r}; e_r < \tau_0^- \right) \right). \]

Hence,
\[ \inf_{b_0 \leq b \leq \bar{b}} (B_r(x - b) - (x - b)) \]
\[ \geq \mathbb{E} \left( e^{-qe_r} X(e_r); X(e_r) < 0, e_r < \tau_{-(x-b_0)}^- \right) - (x - b_0) \left( 1 - \mathbb{E}_{x-b} \left( e^{-qe_r}; e_r < \tau_0^- \right) \right) \]
\[ \overset{r \to \infty}{\longrightarrow} 0, \]
where the convergence holds by (6.2) and because the fact that \( |X(e_r)| \leq (x - b_0) \) on the event \( \{ X(e_r) < 0, e_r < \tau_{-(x-b_0)}^- \} \), implies, by dominated convergence, that
\[ \mathbb{E} \left( e^{-qe_r} X(e_r); X(e_r) < 0, e_r < \tau_{-(x-b_0)}^- \right) \overset{r \to \infty}{\longrightarrow} 0. \]

On the other hand,
\[ \sup_{b_0 \leq b \leq \bar{b}} (B_r(x - b) - (x - b)) \leq \mathbb{E} \left( e^{-qe_r} \overline{X}(e_r); e_r < \tau_{-(x-b_0)}^- \right) \leq \mathbb{E} \left( \overline{X}(e_r) \right) \overset{r \to \infty}{\longrightarrow} 0, \]
where the last limit holds because \( \overline{X}(e_r) \) is exponentially distributed with parameter \( \Phi(r) \). Hence,
\[ \sup_{b_0 \leq b \leq \bar{b}} |B_r(x - b) - (x - b)| \overset{r \to \infty}{\longrightarrow} 0. \]
Now, by using these uniform convergence results together with \( b_r^* \to \bar{b} \),
\[ v_r(x) = \frac{W(q)(b_r^*)}{W(q)(b_r^*)} \overline{A}(x, b_r^*) + B_r(x - b_r^*) \overset{r \to \infty}{\longrightarrow} \frac{W(q)(\bar{b})}{W(q)(\bar{b})} + x - \bar{b}, \]
as desired.

(ii) Suppose \( \bar{b} = 0 \). Notice in this case that \( b_r^* = 0 \) for all \( r > 0 \), and thus it suffices to show the (pointwise) convergence of
\[ v_0(r)(x) := \frac{r A_r(x, 0)}{\Phi(q + r)(\Phi(q + r) - r W(q)(0))} + B_r(x). \]
Here, \( B_r(x) \overset{r \to \infty}{\longrightarrow} x \) as a special case of (6.3) when \( x > 0 \). This also holds for \( x = 0 \) because
\[ 0 \leq B_r(0) = \mathbb{E} \left( e^{-qe_r} X(e_r); e_r < \tau_0^- \right) \leq \mathbb{E} \left( \overline{X}(e_r) \right) \overset{r \to \infty}{\longrightarrow} 0. \]
On the other hand, by using (3.12), we have
\[
A^{(r)}(x, 0) = W^{(q)}(x) + r \int_0^x W^{(q+r)}(x - y) W^{(q)}(y) dy - r W^{(q)}(0) W^{(q+r)}(x) \\
- W^{(q+r)}(x) \frac{\Phi(q + r) - r W^{(q)}(0)}{\Phi(q + r)} \\
= r W^{(q)}(0) \left( - \frac{W^{(q+r)}(x)}{\Phi(q + r)} + \frac{W^{(q+r)}(x)}{\Phi(q + r)} \right)
\]
Suppose \( X \) is of unbounded variation. Then, \( A^{(r)}(x, 0) = 0 \), and hence it is clear that \( v^{(r)}_0(x) \xrightarrow[r \to \infty]{} x = \bar{v}(x) \), as desired.

Suppose \( X \) is of bounded variation (then by Remark 4.2, we have \( \Pi((-\infty, 0) < \infty) \). Since \( q + r = \psi(\Phi(q + r)) = c \Phi(q + r) + \int_{(-\infty,0)} (e^{\Phi(q+r)z} - 1) \Pi(dz) \) by (2.2), we have by monotone convergence
\[
\Phi(q + r) - \frac{r}{c} = \frac{q}{c} + c^{-1} \int_{(-\infty,0)} (1 - e^{\Phi(q+r)z}) \Pi(dz) \xrightarrow[r \to \infty]{} \frac{q + \Pi(-\infty, 0)}{c}.
\]
By this and because \( \Phi(q + r)c \sim r \) as \( r \to \infty \) (see also Remark 3.1), we have
\[
\Phi(q + r)(\Phi(q + r) - r W^{(q)}(0)) = \Phi(q + r)(\Phi(q + r) - \frac{q}{c}) \xrightarrow[r \to \infty]{} 1 \frac{W^{(q+r)}(0)}{W^{(q+r)(0+)}},
\]
On the other hand, by noting that \( W^{(q)}(0) > 0 \), shifting the process by \( b \) in (6.1) and taking \( b \to 0 \), together with the dominated convergence theorem, give
\[
\mathbb{E}_x(e^{-qe_F}; e_F < \tau^-_0) + \mathbb{E}_x \left( e^{-(q+r)\tau^-_0} \lim_{k \to 0} \frac{W^{(q)}(X(\tau^-_0))}{W^{(q)}(0)}; \tau^-_0 < \infty \right) = \frac{A^{(r)}(x, 0)}{W^{(q)}(0)}.
\]
Because \( X \) does not creep downward (\( \mathbb{P}_x(X(\tau^-_0) = 0, \tau^-_0 < \infty) = 0 \) for all \( x \geq 0 \)) for the case of bounded variation (see Exercise 7.6 of [18]), the second expectation on the left hand side is zero. Now taking \( r \to \infty \) on both sides, we have \( A^{(r)}(x, 0)/W^{(q)}(0) \xrightarrow[r \to \infty]{} 1 \). In conclusion, we have \( v^{(r)}_0(x) \xrightarrow[r \to \infty]{} W^{(q)}(0)/W^{(q)(0+)} + x = \bar{v}(x) \), as desired.

\[\square\]

7. Numerical Examples

We conclude this paper with a sequence of numerical experiments. Here, to better understand the sensitivity with respect to each parameter describing the underlying process, we consider a simple case using a (drifted) compound Poisson process with i.i.d. exponential-size jumps, which satisfies Assumption 3.2. Both cases with and without Brownian motions are considered.

More specifically, we assume, for some \( c \in \mathbb{R} \) and \( \sigma \geq 0 \),
\[
X(t) - X(0) = ct + \sigma B(t) - \sum_{n=1}^{N(t)} Z_n, \quad 0 \leq t < \infty,
\]
\[
\sum_{n=1}^{N(t)} Z_n = \int_0^t \gamma(t) dW(t),
\]
where \( \gamma(t) \) is the intensity function.
where \( B = (B(t); t \geq 0) \) is a standard Brownian motion, \( N = (N(t); t \geq 0) \) is a Poisson process with arrival rate \( \kappa \), and \( Z = (Z_n; n = 1, 2, \ldots) \) is an i.i.d. sequence of exponential random variables with parameter \( \lambda \). The processes \( B, N, \) and \( Z \) are assumed mutually independent. This is a special case of the spectrally negative version of the phase-type Lévy process in [4], which admits an analytical form of the scale function, as in [14]. We refer the reader to [14, 17] for the forms of the corresponding scale functions.

7.1. Computation of the value function. We first illustrate the computation scheme of the optimal barrier \( b^* \) and the value function \( v = v_{b^*} \). Here, for \( X \) in (7.1), we consider the following sets of parameters:

\[
\text{Case 1: } \sigma = 0.2, c = 1.5, \quad \text{Case 2: } \sigma = 0.2, c = 0.1, \quad \text{Case 3: } \sigma = 0.2, c = 0,
\]

and

\[
\text{Case 1': } \sigma = 0, c = 1.5, \quad \text{Case 2': } \sigma = 0, c = 1.15, \quad \text{Case 3': } \sigma = 0, c = 0.1.
\]

For other parameters, we set \( \kappa = \lambda = 1, r = 0.5, \) and \( q = 0.05 \). These parameters are chosen so that \( b^* > 0 \) for Cases 1 and 1', \( 0 = b^* < b \) for Cases 2 and 2', and \( 0 = b^* = \bar{b} \) for Cases 3 and 3', where we recall that \( \bar{b} \) is the optimal barrier in the classical case, as defined in Remark 3.2.

Recall that the optimal barrier \( b^* \) is the unique root of \( h = 0 \) if \( h(0^+) > 0 \) and zero otherwise. Figure 1 plots the function \( h \) along with the points at \( b^* \) and \( \bar{b} \). For Cases 1 and 1', \( h \) starts at a strictly positive value, decreases until \( \bar{b} \), and then increases to zero; \( b^* \) becomes the unique point at which \( h \) vanishes. For Cases 2 and 2' (where \( \bar{b} > 0 \)), \( h \) starts at a negative value and then behaves similarly to Cases 1 and 1'; because \( h \) is uniformly negative, we set \( b^* = 0 \). For Cases 3 and 3' (where \( \bar{b} = 0 \)), \( h \) starts at a negative value and monotonically increases to zero; again we set \( b^* = 0 \).

With the computed values of \( b^* \), the value function \( v = v_{b^*} \) is obtained using (5.1) and (3.16) for the cases \( b^* > 0 \) and \( b^* = 0 \), respectively. To confirm the optimality, in Figure 2, we plot \( v_{b^*} \) along with suboptimal NPVs \( v_b \) with \( b \neq b^* \). It can be confirmed in all cases that \( v_{b^*} \) dominates \( v_b \) for \( b \neq b^* \), uniformly in \( x \). As shown in Proposition 5.1, \( v_{b^*} \) is smooth and its slope is larger than 1 if and only if \( x < b^* \).

Regarding the comparison between the unbounded and bounded variation cases, the main differences include the degree of smoothness at \( b^* \) and the behavior of \( v_{b^*} \) in the vicinity of zero. The degree of smoothness is not visually clear as it is at least twice continuously differentiable in both cases. On the other hand, the difference in the vicinity of zero can be observed: as the starting value \( x \) decreases to zero, \( v \) converges to zero for the unbounded variation case (Cases 1, 2, and 3), but not for the bounded variation case (Cases 1', 2', and 3').

7.2. Sensitivity analysis. We now numerically study the behaviors of the optimal barrier \( b^* \) and the value function \( v_{b^*} \) with respect to the parameters describing the problem. In the remaining numerical
results, we set $\kappa = \lambda = 1$, $c = 1.5$, $r = 0.5$, and $q = 0.05$, unless stated otherwise. Both unbounded and bounded variation cases with $\sigma = 0.2$ and $\sigma = 0$ are considered.
Figure 2. The corresponding value function $v_{b^*}(x)$ (solid) along with suboptimal expected NPVs $v_b$ (dotted) for $b = 0, b^*/4, b^*/2, 3b^*/4, (b^* + \bar{b})/2, \bar{b},$ and $\bar{b} + (\bar{b} - b^*)/2$ for Cases 1 and 1'; $b = \bar{b}/2, \bar{b},$ and $3\bar{b}/2$ for Cases 2 and 2'; and $b = 1/3, 2/3, \text{ and } 1$ for Cases 3 and 3'. The values at $b^*$ are indicated by circles. Those at the suboptimal barriers $b > b^*$ (resp. $b < b^*$) are indicated by up-pointing (resp. down-pointing) triangles when $b \neq \bar{b}$ and those at $b = \bar{b}$ are indicated by squares.
ON OPTIMAL PERIODIC DIVIDEND STRATEGIES FOR LÉVY RISK PROCESSES

Figure 3. Plots of $v_{b^*}$ for $c = 1, 1.1, \ldots, 4.9$, and 5 with $\sigma = 0.2$ (left) and $\sigma = 0$ (right). The values at $b^*$ are indicated by circles. The function $v_{b^*}$ is increasing in $c$ uniformly in $x$.

Figure 3 plots $v_{b^*}$ and the points at $b^*$ for various values of the drift parameter $c$. Naturally, the value function $v_{b^*}$ is increasing in $c$ uniformly in $x$. It is also observed that $b^* = 0$ for sufficiently small values of $c$, and increases in $c$ to some finite limit.

Figure 4. Plots of $v_{b^*}$ for $\kappa = 0.001, 0.002, \ldots, 0.008, 0.009, 0.01, 0.02, \ldots, 0.08, 0.09, 0.1, 0.2, \ldots, 2.9$, and 3 with $\sigma = 0.2$ (left) and $\sigma = 0$ (right). The values at $b^*$ are indicated by circles. The function $v_{b^*}$ is decreasing in $\kappa$ uniformly in $x$.

Figures 4 and 5 plot the results for various values of the jump rate $\kappa$ and the jump-size parameter $\lambda$, respectively. It is confirmed that $v_{b^*}$ decreases in $\kappa$ and increases in $\lambda$ (uniformly in $x$). Interestingly, $b^*$ is not monotone in these parameters (contrary to what we observed in Figure 3). When $\kappa$ is sufficiently large, the future aspect is negative, and hence $b^* = 0$. As $\kappa$ decreases, $b^*$ departs from zero and starts increasing. However, the lower the value of $\kappa$, the easier it is to avoid ruin. Therefore, with sufficiently
small $\kappa$, $b^*$ can be set low. Owing to these tradeoffs, $b^*$ is not monotone in $\kappa$. The same observation applies to the analysis for $\lambda$.

![Figure 5](image1.png)

**Figure 5.** Plots of $v_{b^*}$ for $\lambda = 0.1, 0.2, \ldots, 2.9, 3, 4, \ldots, 9, 15, 20, \ldots, 95$, and 100 with $\sigma = 0.2$ (left) and $\sigma = 0$ (right). The values at $b^*$ are indicated by circles. The function $v_{b^*}$ is increasing in $\lambda$ uniformly in $x$.

![Figure 6](image2.png)

**Figure 6.** Plots of $v_{b^*}$ (dotted) for $r = 0.001, 0.002, \ldots, 0.01, 0.02, \ldots, 0.09, 0.1, 0.2, \ldots, 0.9, 1, 2, \ldots, 99$, and 100 along with the classical value function $\bar{v}$ (solid) with $\sigma = 0.2$ (left) and $\sigma = 0$ (right). The values of $v_{b^*}$ at $b^*$ are indicated by circles and those of $\bar{v}$ at $\bar{b}$ are indicated by squares. The function $v_{b^*}$ is increasing in $r$ uniformly in $x$.

Finally, we study the behaviors of $v_{b^*}$ and $b^*$ with respect to the rate of dividend payment opportunities $r$. Figure 6 plots $v_{b^*}$ and the points at $b^*$ for various values of $r$ along with those in the classical case (3.17). It is confirmed that $v_{b^*}$ is monotonically increasing in $r$ (uniformly in $x$) to the classical case. As studied in Lemma 6.1, $b^*$ is monotone in $r$, and converges to zero as $r \downarrow 0$ and to $\bar{b}$ as $r \uparrow \infty$; this
confirms the results in Lemma 6.1. While the convergence to zero is relatively fast, we find that the convergence to $\bar{b}$ is rather slow. In [25], the same numerical analysis was obtained for the spectrally positive case; in their case $b^*$ was shown to accurately approximate the optimal barrier for the classical case even for a moderate value of $r$. We conjecture that this difference is due to the chance of jumping to ruin between Poisson observation times (which can be made negligible in the absence of downward jumps). With downward jumps, $b^*$ is more sensitive to the choice of $r$.

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