On isometric reflexions in Banach spaces

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Dedicated to Professor S. G. Krein

Abstract

We obtain the following characterization of Hilbert spaces. Let $E$ be a Banach space whose unit sphere $S$ has a hyperplane of symmetry. Then $E$ is a Hilbert space iff any of the following two conditions is fulfilled:

a) the isometry group $\text{Iso} E$ of $E$ has a dense orbit in $S$;

b) the identity component $G_0$ of the group $\text{Iso} E$ endowed with the strong operator topology acts topologically irreducible on $E$.

Some related results on infinite dimensional Coxeter groups generated by isometric reflexions are given which allow to analyse the structure of isometry groups containing sufficiently many reflexions.

INTRODUCTION

Let $E$ be a real Banach space, $S = S(E)$ the unit sphere in $E$, $\text{Iso} E$ the isometry group of $E$ endowed with the strong operator topology, and $G_0 = G_0(E)$ the identity component of $\text{Iso} E$. A reflexion in $E$ is an operator of the form $s_{e,e^*} = 1_E - 2e^* \otimes e$, where $e \in E, e^* \in E^*$ and $e^*(e) = 1$. If $s = s_{e,e^*} \in \text{Iso} E$, then one may assume also that $||e||_E = ||e^*||_{E^*} = 1$; in this case we will call $e$ the reflexion vector and $e^*$ the reflexion functional; regarding as sphere points, $e$ and $-e$ are called reflexion points. The unit sphere $S$ is symmetric with respect to the mirror hyperplane $\text{Ker} e^*$ of $s$. It turns out that this imposes strong restrictions on the isometry group $\text{Iso} E$.

We say that a proper subspace $H \subset E$ is biorthogonally complemented in $E$ if there exists a bicontractive projection $p$ of $E$ onto $H$, i.e. such that $||p||_E = ||1_E - p||_E = 1$.

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Theorem 1. Let $s_{e,e^*}$ be an isometric reflection in $E$. Let $H = \overline{\text{span}}(G_0e)$ be the minimal closed subspace of $E$ containing the orbit $G_0e$. Then

a) $H$ is a Hilbert space and $H$ is biorthogonally complemented in $E$ or $H = E$;

b) furthermore, there exists a projection $p$ of $E$ onto $H$ such that

i) $1_E - 2p \in \text{Iso } E$,

ii) $(1_E - p) + \bar{u}p \in \text{Iso } E$ for any $\bar{u} \in O(H) = \text{Iso } H$, and

iii) any $g \in \text{Iso } E$ such that $g|H = \bar{u} \in O(H)$ has the form $g = \bar{v}(1_E - p) + \bar{u}p$, where $\bar{v} = g|\text{Ker } p \in \text{Iso } \text{Ker } p$;

c) the orbit $G_0e$ coincides with the unit sphere $S(H)$ of $H$.

This subject is related to the following Banach - Mazur rotation problem ([3], p.242):

Let $E$ be a separable Banach space such that the group $\text{Iso } E$ acts transitively on the unit sphere $S$. Is it true that $E$ is a Hilbert space?

Recall (see [23, Ch.IX], §6) that the group $\text{Iso } L_p$, where $L_p = L_p[0; 1]$ and $1 \leq p \neq 2 < \infty$, has exactly two orbits on the unit sphere $S_p = S(L_p)$. One of them consists of the functions in $S_p$ with the zero set of a positive measure, and the other one contains the rest. Thus, both orbits are dense in $S_p$. One says that the group $\text{Iso } E$ acts almost transitively on $S$ if it has a dense orbit in $S$. This is the case in the above examples and also in the anisotropic spaces $L_{pq}$. In a non-separable $L_p$-space the second of the above two orbits is empty, and thus it is a non-Hilbert Banach space with the isometry group acting transitively on the unit sphere. This shows that the assumption of separability in the Banach - Mazur problem is essential.

Observe that $\text{Iso } E$ is a Banach-Lie group. If this group is transitive on the unit sphere $S$, then $S$ is a homogeneous space of $\text{Iso } E$. If in addition $S$ has a hyperplane of symmetry $L$, it should be a symmetric space. Indeed, $L$ is a mirror hyperplane of an isometric reflection. The unit sphere $S$ having a reflexion point, by transitivity each point $x \in S$ should be a reflexion point of an isometric reflexion $s = s_{x,x^*}$. Furthermore, $x$ is an isolated fixed point of the involution $-s|S$ which acts as $-1$ at the supporting hyperplane $x^* = 1$ to $S$ at $x$ (we are grateful to J. Arazy for this remark). From Theorem 2 below it follows that $S$ being a symmetric space of the group $\text{Iso } E$, $E$ should be a Hilbert space. In fact, in Theorem 2 more strong criteria for $E$ to be a Hilbert space are done. They hold without the separability assumption.

Theorem 2. Let the group $\text{Iso } E$ contains a reflection $s_{e,e^*}$ along the vector $e \in S$. Then $E$ is a Hilbert space iff either of the following two conditions is fulfilled:

a) $\text{Iso } E$ acts almost transitively on $S$;

b) $e$ is a cyclic vector of the strong identity component $G_0$ of $\text{Iso } E$ (i.e. $E = \overline{\text{span}}(G_0e)$).
The second statement is a corollary of Theorem 1; the first one, being much simpler, is proven along the same lines. By a theorem of Godement [9] any isometric operator in a Banach space has a non-trivial invariant subspace (see also [28] for a more general fact). From Theorem 1 one obtains the following

**Corollary.** Let $E$ be a non-Hilbert Banach space. If there is an isometric reflexion in $E$, then all operators in $G_0(E)$ have a common non-trivial invariant Hilbert subspace $H$, biorthogonally complemented in $E$. Moreover, if $G_0(E)$ is a non-trivial group, then $\dim H > 1$. In particular, in this case there is an orthogonally complemented euclidean plane in $E$.

Note that by a theorem of Yu. Lyubich [20] if a finite dimensional Banach space has an infinite isometry group, i.e. if the group $G_0(E)$ is non-trivial, then $E$ has a euclidean plane $L$ with a contractive projection $p : E \to L$ (in this case $L$ is called *orthogonally complemented* in $E$) (see also [16], [21]). From the other hand, there are Banach spaces of infinite dimension with big isometry groups, but without any orthogonally complemented euclidean subspace of dimension greater than 1. Indeed, $L_p = L_p[0;1]$, where $1 < p \neq 2 < \infty$, contains no such a subspace, while the group $G_0$ is non-trivial. Furthermore, there is no bicontractive projection of $L_p$ ($p \neq 2$) onto a hyperplane [13, 14]; in particular, there is no isometric reflexion. The same is true in general for rearrangement-invariant (r.i.) ideal Banach lattices, or symmetric spaces, of (classes of) measurable functions different from $L_2$ [14, Theorem 4.4]. Recall [17, 19] that a r.i. (or symmetric) space $E$ on the interval $[0;1]$ satisfies the following axioms:

1) $1 \in E$ and $||1||_E = 1$.
2) For any measure preserving transformation $\alpha$ of the interval $[0;1]$ the shift operator $T_\alpha : x(t) \to x(\alpha(t))$ acts isometrically in $E$.
3) If $x(t) \in E$ and $|y(t)| \leq |x(t)|$ a.e., then $y(t) \in E$ and $||y(t)||_E \leq ||x(t)||_E$.

If $E$ is a r.i. space different from $L_2$, then every $g \in \text{Iso} E$ has a weighted shift representation $g : x(t) \to h(t)x(\phi(t))$, where $h = g(1) \in E$ and $\phi$ is a transformation of $[0;1]$ preserving measurability (see [30, 31] for the complex case and [13,14] for the real one; see also [1], [18], [22], [29]. As for symmetric sequence spaces, see [23, Ch.IX], [2], [6], [8]). Furthermore, $\phi$ should be measure–preserving except in the case where $E$ coincides with some of the $L_p$, probably endowed with a new equivalent norm [30] (see also [14], [18], [22]). In particular, this shows that $L_p$ are the only r.i. spaces where the orbits of the isometry group are dense in the unit sphere.

The content of the paper is the following. Section 1 contains a preliminary finite dimensional version of Theorem 1. The proofs of Theorems 1 and 2 are given
in section 2. Besides, section 2 contains a version of Theorem 1 where no operator topology is prescribed (see Theorem 2.10). In section 3 we classify the Coxeter groups in infinite dimensional case (probably, this classification is not new). In sections 4 and 5 we consider Banach spaces possessing total families of isometric reflexions. A kind of a structure theorem for isometry groups is proven (Theorem 5.7). It applies the notions of Hilbert and Coxeter partial orthogonal subspace decompositions, introduced earlier in this section. In the last section we give an application to isometry groups of the ideal generalized sequence spaces.

The main results of this paper were announced in [26]; see [27] for their proofs. Somehow, the proofs have never been published before. The present article contains some new facts, and the exposition of the old ones is quite different.

1 Isometric reflections in finite dimensional Banach spaces

Let $A$ be a set of reflexions in a real vector space $E$ and $W$ be the group generated by the reflexions in $A$. Denote by $\Gamma_{W,A}$ the Coxeter graph of $W$. Recall [5] that $\Gamma_{W,A}$ has $A$ as the set of vertices; two vertices are connected by an edge iff the corresponding reflexions do not commute. By $\Gamma_W$ we denote the full Coxeter graph of $W$, i.e. $\Gamma_W = \Gamma_{W,R}$, where $R = R(W)$ is the set of all the reflexions in $W$.

1.1. Lemma ([5, Ch. V, 3.7]). A group $W$ generated by a set $A$ of orthogonal reflexions in $\mathbb{R}^n$ is irreducible iff the origin is the only fixed point of $W$ and the Coxeter graph $\Gamma_{W,A}$ is connected. In particular, $\Gamma_W$ is connected iff its subgraph $\Gamma_{W,A}$ is connected.

Let $E$ be a finite dimensional Banach space. Then $\text{Iso} E$ is a compact Lie group, and there exists a scalar product in $E$ invariant with respect to $\text{Iso} E$. It can be defined, for instance, by averaging of any given scalar product over the Haar measure on $\text{Iso} E$. In general, such an invariant scalar product is not unique. Being orthogonal, two isometric reflexions in $E$ along the vectors $e_1, e_2 \in S(E)$ commute iff either $e_1 = \pm e_2$ or $e_1 \perp e_2$.

The proof of the following lemma is simple and can be omitted.

1.2. Lemma. Let a connected submanifold $M$ of $\mathbb{R}^n$ be invariant under a reflexion $s_{e, e^*}$ which fixes a point $x \in M$ and acts identically on the tangent space $T_x M$. Then $M$ is contained in the mirror hyperplane $\text{Ker} e^*$. 

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The main result of this section is the following

1.3. Proposition. Let $E$ be a real Banach space of dimension $n$. Let $G \subset \text{Iso } E$ be a closed subgroup of a positive dimension which contains reflexions $t_1, \ldots, t_n$ along linearly independent vectors $e_1, \ldots, e_n$. Then there exists a subspace $H \subset E$ such that

a) $\dim H \geq 2$, $H$ is euclidean and biorthogonally complemented in $E$;
b) the unit sphere $S(H)$ of $H$ coincides with an orbit of the identity component $G_0$ of $G$;
c) there exists a projection $p$ of $G$ with any reflexion $e$ linearly independent vectors $b$.

Proof. Fix an invariant scalar product in $E$ and identify $E$ with $\mathbb{R}^n$ in such a way that $\text{Iso } E \subset O(n)$. Let $u_1, \ldots, u_n$ be the system of vectors in $\mathbb{R}^n$ biorthogonal to the system $e_1, \ldots, e_n$. Since $\dim G > 0$, the orbit $Gu_i$ has a positive dimension for at least one value of $i$, say for $i = 1$. We may also assume that $u_1 \in S^{n-1}$, where $S^{n-1}$ is the euclidean unit sphere in $\mathbb{R}^n$. Let $M$ be the connected component of the orbit $Gu_1$ which contains $u_1$. Since $u_1$ is fixed by any of the reflexions $t_i, i = 2, \ldots, n$, $M$ is invariant under these reflexions, an hence the tangent space $T = Tu_1 M$ is invariant, too. Thus for each $i = 2, \ldots, n$ either $e_i \in T$ or $e_i \perp T$. Put

$$A = \{i \in \{2, \ldots, n\} \mid e_i \in T\}$$

and

$$B = \{i \in \{2, \ldots, n\} \mid e_i \perp T\}.$$

Since $G \subset O(n)$ and $u_1 \in S^{n-1}$, we have $M \subset S^{n-1}$, and so $T \subset Tu_1 S^{n-1}$. Therefore $T \perp u_1$. It follows that $T \subset L$, where $L = \text{span}(e_2, \ldots, e_n)$, and therefore $T = \text{span}(e_i \mid i \in A)$ (hereafter $\text{span}$ means the linear span).

Thus, $\dim M = \dim T = \text{card}A$. Since $M$ is $t_i$-invariant for $i \in B$, by Lemma 1.2, $M$ is contained in the subspace $H = \{v \in E \mid v \perp e_i, i \in B\}$. It is easily seen that $k = \dim H = \text{card}A + 1 = \dim M + 1$. Thus $M$ is a closed submanifold of each of the unit spheres $S_r(H) = S_r(E) \cap H$, where $r = ||u_1||_E$, and $S^{k-1} = S^{n-1} \cap H$, of the same dimension $\dim M = \dim H - 1 = k - 1$. Hence $M$ coincides with both of them. At the same time, being connected $M$ coincides with the orbit $G_0 u_1$. Here $k \geq 2$, since $\dim M > 0$. Therefore, $H$ is euclidean and the unit sphere $S(H)$ coincides with the orbit $G_0(u_1/r)$.

Since $T \subset H$ and $e_i \in T$ for each $i \in A$, where $\text{card}A = k - 1 > 0$, there exists $i_0 \in A$ such that $e_{i_0} \in S(H)$, and thus $S(H) = G_0 e_{i_0}$. Let $w_1, \ldots, w_k \in H$ be an orthogonal basis in $H$ with $||w_i||_E = 1$, $i = 1, \ldots, k$, and $g_1, \ldots, g_k \in G$ be such that $g_i(e_{i_0}) = w_j$. Then $s_j = g_j t_{i_0} g_j^{-1} \in G$ is the orthogonal reflexion along the vector
\(w, j = 1, \ldots, k\). By the same reasoning as above, for any vector \(w \in S(H)\) the orthogonal reflexion \(s_{w,w^*}\) along \(w\) belongs to \(G\).

The reflexions \(s_j, j = 1, \ldots, k\), pairwise commute, and so \(p = \frac{1}{2}(1_E - \Pi_{i=1}^k s_i)\) is the orthogonal projection of \(E\) onto \(H\) such that \(\tau = 1_E - 2p = \Pi_{i=1}^k s_i \in G \subset \text{Iso} E\). Thus, \(||p||_E = \frac{1}{2}||1_E - \tau||_E = 1\) and either \(E = H\) or \(||1_E - p||_E = \frac{1}{2}||1_E + \tau||_E = 1\). Therefore, \(H\) is a biorthogonally complemented subspace of \(E\).

Any orthogonal reflexion \(\bar{s}\) in \(H\) coincides with the restriction to \(H\) of some reflexion \(s \in G\), where in fact \(s = (1_E - p) + \bar{s}p\). The same is true for any orthogonal operator \(\bar{u} \in O(H)\); indeed, the group \(O(H)\) is generated by orthogonal reflexions.

Let \(t \in G\) be a reflexion. The mirror hyperplane of \(t\) intersects with \(H\) by a subspace of \(H\) of dimension \(k - 1 > 0\). Therefore, \(t\) has a fixed point on the sphere \(M = S^{k-1} \subset H\), and so \(t(M) \cup M\) is connected and contained in the orbit \(Gu_1\). It follows that \(t(M) = M\), \(H\) is invariant with respect to \(t\) and so \(t\) and \(p\) commute. This completes the proof. \(\Box\)

1.4. Corollary. Let \(W\) be a group generated by isometric reflexions in a finite dimensional Banach space \(E\). If \(W\) is irreducible and infinite, then \(E\) is euclidean and \(W\) is dense in the orthogonal group \(\text{Iso} E \approx O(n), n = \dim E\).

Proof. Let \(G\) be the closure of \(W\) in \(\text{Iso} E\) and \(G_0\) be the identity component of \(G\). Since \(W\) is irreducible, by Lemma 1.1, it contains \(n\) reflexions along linearly independent vectors, and the Coxeter graph \(\Gamma_W\) is connected. Let \(H\) be the euclidean subspace of \(E\) constructed in Proposition 1.3. Since by (c), \(H\) is invariant with respect to the reflexions from \(W\), for each \(s_{e,e^*} \in W\) either \(e \in H\) or \(e \perp H\). If \(A\) resp. \(B\) is the set of reflexions from \(W\) of the first resp. second type, then each element of \(A\) commutes with every element of \(B\). By the connectedness of the graph \(\Gamma_W\) one of the sets \(A\) and \(B\) should be empty. This shows that \(H = E\). By (c), \(\bar{u} \in G\) for any \(\bar{u} \in O(H)\). Therefore, \(G = O(H)\) and we are done. \(\Box\)

Remark. Related results can be found in [6], [10, (1.7)], [23], [25].

2 Proofs of Theorems 1 and 2

2.1. Definition. Let \(E\) be a real Banach space, and let \(s_1, s_2\) be two isometric reflexions in \(E\) along linearly independent vectors \(e_1, e_2 \in S = S(E)\). Denote by \(\alpha(s_1, s_2)\) the minimal positive angle between the lines containing \(e_1\) and \(e_2\), measured with respect to an invariant inner product in the plane \(L = \text{span}(e_1, e_2)\). Put \(\alpha(s_1, s_2) = 0\) iff \(e_1 = \pm e_2\).

2.2. Remarks. a) It is easily seen that the above definition does not depend on the choice of an invariant scalar product in \(L\).
b) An isometric reflection \( s = s_{e,e^*} \) in \( E \) is uniquely defined by the reflexion point \( e \in S(E) \). Indeed, this is true for the restriction of \( s \) to any finite dimensional subspace \( F \) containing \( e \), since the mirror hyperplane \( \text{Ker} e^* \cap F \) of \( s|_F \) is orthogonal to \( e \) with respect to an invariant scalar product on \( F \). Thus, this is true for \( s \) itself.

c) Two isometric reflections \( s_1 \) and \( s_2 \) commute iff either \( \alpha(s_1, s_2) = 0 \), i.e. \( e_1 = \pm e_2 \) or \( \alpha(s_1, s_2) = \frac{\pi}{2} \), i.e. \( e_1 \perp e_2 \) in \( L \).

2.3. Lemma. Let \( s_i = s_{e_i,e_i^*} \), \( i = 1, 2 \), be two isometric reflexions in \( E \). Then

\[
\cos^2 \alpha(s_1, s_2) = e_1^*(e_2)e_2^*(e_1).
\]

Proof. This is evidently true if \( e_1 = \pm e_2 \). Assume, further, that \( e_1 \) and \( e_2 \) are linearly independent. Let an invariant scalar product in the plane \( L = \text{span}(e_1, e_2) \) be given by the bilinear form \( B = \begin{pmatrix} b & a \\ a & c \end{pmatrix} \) with respect to the basis \( (e_1, e_2) \) in \( L \). Consider the orthogonal projection \( p_i = \frac{1}{2}(1_L + s_i|L) \) of \( L \) onto the mirror line \( l_i \) of the axial reflexion \( s_i|L, i = 1, 2 \). Since \( p_i(e_j) \perp e_i \) for \( j \neq i \), we have

\[
0 = B(p_1(e_2), e_1) = B(e_2 - e_1^*(e_2)e_1, e_1) = a - e_1^*(e_2)b
\]

and

\[
0 = B(p_2(e_1), e_2) = a - e_2^*(e_1)c.
\]

Thus

\[
a^2 = e_1^*(e_2)e_2^*(e_1)bc,
\]

and so

\[
\cos^2 \alpha(s_1, s_2) = \frac{a^2}{bc} = e_1^*(e_2)e_2^*(e_1).
\]

\( \Box \)

2.4. Corollary.

\[
\cos \alpha(s_1, s_2) \geq 1 - \|e_1 - e_2\|_E.
\]

In particular, if \( e_1 \neq e_2 \) and \( \|e_1 - e_2\|_E < 1 \), then \( s_1 \) and \( s_2 \) do not commute.

Proof. Since \( s_i \in \text{Iso} E \), and so \( \|e_i\|_E = \|e_i^*\|_{E^*} = e^*(e) = 1 \), we have

\[
|1 - e_1^*(e_2)| = |e_1^*(e_1 - e_2)| \leq \|e_1 - e_2\|_E
\]

and

\[
|1 - e_2^*(e_1)| \leq \|e_1 - e_2\|_E.
\]
We can assume that $||e_1 - e_2||_E < 1$. Then from the above inequalities we obtain

$$|e^*_1(e_2)| \geq 1 - ||e_1 - e_2||_E$$

and

$$|e^*_2(e_1)| \geq 1 - ||e_1 - e_2||_E.$$ 

The desired inequality follows from the latter two by multiplying them and making use of Lemma 2.3.

2.5. Lemma. Let $s = s_{e,e^*} \in Iso E$. Consider the function on $Iso E \times Iso E$

$$\phi_s(g_1, g_2) = \sin^2 \alpha(s_1, s_2)$$

where $s_i = g_is_i^{-1}, i = 1, 2$. Then

a) $\phi_s$ is left invariant, i.e.

$$\phi_s(g_1, g_2) = \phi_s(gg_1, gg_2) = \phi_s(1_E, g_1^{-1}g_2)$$

for each $g, g_1, g_2 \in Iso E$.

b) 

$$\phi_s(g_1, g_2) = 1 - e^*(g_1^{-1}g_2(e))e^*(g_2^{-1}g_1(e)).$$

Therefore, $\phi_s$ is continuous on $(Iso E)^2$ in the strong operator topology.

c) For any two elements $g', g'' \in G_0$ such that $\phi_s(g', g'') > 0$, and for any $\epsilon, 0 < \epsilon < 1$, one can find a finite chain of elements $h_0 = g', h_1, \ldots, h_n = g''$ with the property $0 < \phi_s(h_i, h_{i+1}) < \epsilon$, so that the reflexions $t_i = h_i s_i^{-1}$ and $t_{i+1}$ do not commute for all $i = 0, 1, \ldots, n - 1$.

Proof. (a) is evident. The identity in (b) easily follows from the equality

$$\phi_s(g_1, g_2) = 1 - e^*(g_1^{-1}g_2(e))e^*(g_2^{-1}g_1(e)),$$

which follows from (a) and Lemma 2.3. The second statement of (b) is true since $Iso E$ is a topological group with respect to the strong operator topology. To prove (c), consider the covering of $G_0$ by the open subsets

$$U_\epsilon(g) = \{ h \in G_0 | \phi_s(g, h) < \epsilon \}.$$

Since $G_0$ is connected, any two of them $U_\epsilon(g')$ and $U_\epsilon(g'')$ can be connected by a finite chain of such subsets, and the assertion follows.

2.6. Proposition. Let $s = s_{e,e^*} \in Iso E$, $g_1, \ldots, g_n \in G_0$ and $H' = \text{span}(e_1, \ldots, e_n)$, where $e_i = g_i(e), i = 1, \ldots, n$. Then
a) $H'$ is euclidean;  

b) there exists a unique projection $p'$ of $E$ onto $H'$ such that $1_E - 2p \in \text{Iso } E$;  

c) the unit sphere $S(H')$ of $H'$ is contained in the orbit $G_0 e$, and for each vector $v \in S(H')$ there exists a reflexion $s_{v,v^*} \in \text{Iso } E$ along $v$ commuting with $p'$.

\textbf{Proof.} First we construct a finite dimensional subspace $F$ containing $H'$ which satisfies all the properties of (a), (b), (c) above.

Put $g_0 = 1_E$ and for each pair $(g_i, g_{i+1}), i = 0, \ldots, n - 1$, find a chain $\{h_{ij}\}_{j=0}^n$ as in Lemma 2.5.c above. The proposition is evident in the case when $\dim H' = 1$, and so we may assume that $g_i(e) \neq e$ for at least one value $i_0$ of $i$. Since the continuous function $\phi_s$ takes all its intermediate values on $G_0$, we can also choose the element $h = h_{i_0,1}$ in such a way that the angle $\alpha(s, hsh^{-1})$ is irrational modulo $\pi$, and thus the group generated by the reflections $s$ and $hsh^{-1}$ is infinite.

Put $F = \text{span}(h_{ij}(e) | j = 0, \ldots, n, i = 0, \ldots, n)$. Let $W$ be the group generated by the reflexions $(t_{ij}|F)$ in $F$, where $t_{ij} = h_{ij}sh_{ij}^{-1}$, $j = 0, \ldots, n$, $i = 0, \ldots, n$. It is clear that the origin is the only fixed point of $W$ in $F$. Since, by the construction the Coxeter graph, $\Gamma_W$ of $W$ is connected, by Lemma 1.1, $W$ is irreducible. $W$ being infinite, by Corrolary 1.4, the subspace $F$ is euclidean and the closure of $W$ coincides with the group $\text{Iso } E = O(F)$.

Let $v_1, \ldots, v_l$, where $l = \dim F$, be a basis of $F$ chosen from the system $(h_{ij}(e))$, and $t_k = t_{v_k,v_k^*}, k = 1, \ldots, l$, be the corresponding reflexions from the system $(t_{ij})$. Put $M = \bigcap_{k=1}^l \text{Ker } v_k^*$. It is easily seen that $E = M \oplus F$. Let $F'$ be a finite dimensional subspace of $E$ containing $F$, endowed with an invariant scalar product. Then for each $k = 1, \ldots, l$ the restriction $t_k' = t_k|F'$ is an orthogonal reflexion in $F'$, and so $v_k \perp (\text{Ker } v_k^* \cap F')$. Therefore, $F \perp (M \cap F')$. It follows that each of the vectors $h_{ij}(e) \in F$ is orthogonal to $M \cap F'$, too, so that the restriction $t_{ij}(M \cap F')$ is the identity mapping. This gives the representation $t_{ij} = (1_E - p) + t_{ij}p$, where $p$ is the projection of $E$ onto $F$ along $M$. Thus, each element $\tilde{g} \in W$ can be represented as the restriction to $F$ of the isometry $g = (1_E - p) + \tilde{g}p \in \text{Iso } E$. If a sequence $\tilde{g}_i \in W$ converges to an element $\tilde{h} \in O(F)$, then the sequence of extensions $g_i$ converges to the extension $h = (1_E - p) + \tilde{h}p$ of $\tilde{h}$, where $\tilde{h} \in \text{Iso } E$. In particular, in this way each orthogonal reflexion in $F$ extends to a unique isometric reflexion in $E$, and each element $\bar{u} \in O(F)$ extends to the unique isometry $u = (1_E - p) + \bar{u}p \in \text{Iso } E$. It follows that $S(F) \subset G_0(e)$.

Let $f_1, \ldots, f_l$ be an orthogonal basis in $F$ and $\bar{s}_1, \ldots, \bar{s}_l$ be the orthogonal reflexions in $F$ along these vectors. It is easily seen that then $p = \frac{1}{2}(1_E - \prod_{i=1}^l s_i)$, and thus $1_E - 2p = \prod_{i=1}^l s_i \in \text{Iso } E$. If $s' \in \text{Iso } E$ is a reflexion along a vector $v' \in S(F)$, then as above $s' = (1_E - p) + s'p = (1_E - p) + ps'p$, and so $s'$ and $p$ commute.

It is evident that the subspace $H' \subset F$ has the same properties as $F$ itself, and therefore (a), (b), (c) are fulfilled. $\square$
2.7. Remark. It is easily seen that if $H' \subset H''$ are two subspaces as in Proposition 2.6, then for the corresponding projections $p', p''$ we have $p' \prec p''$, i.e. $p'p'' = p' (= p''p')$.

2.8. Proof of Theorem 1.a. Let $x, y$ be two arbitrary vectors in $H$. Then for any $\epsilon > 0$ in the linear span of the orbit $G_0e$ there exist two vectors $x_\epsilon, y_\epsilon$ such that $||x - x_\epsilon||_E < \epsilon, ||y - y_\epsilon||_E < \epsilon$. Let $x_\epsilon = \sum_{i=1}^n a_i g_i(e)$ and $y_\epsilon = \sum_{i=1}^n b_i g_i(e)$, where $g_i \in G_0, i = 1, \ldots, n$. Put $H' = \text{span}(g_i(e)|i = 1, \ldots, n)$. By Proposition 2.6, the subspace $H'$ is euclidean, and therefore the norm in $H'$ satisfies the four squares identity. In particular,

$$||x_\epsilon + y_\epsilon||^2 + ||x_\epsilon - y_\epsilon||^2 = 2(||x_\epsilon||^2 + ||y_\epsilon||^2).$$

Passing to the limit we see that the same identity holds for $x, y \in H$. It follows that $H$ is a Hilbert space (see [7], Ch.7, §3).

Consider further the family of all finite dimensional subspaces $H'$ which belong to the linear span of the orbit $G_0e$. Let $\mathcal{P} = \{p'\}$ be the corresponding partially ordered family of finite dimensional projections $E \to H'$ such that $1_E - 2p' \in \text{Iso} E$. For a fixed vector $v \in S(E)$ and for each $p' \in \mathcal{P}$ consider the subset

$$Y_{p'} = \omega\{p''(v) \mid p'' \in \mathcal{P}, p' \prec p'' \},$$

where $\omega$ denotes the closure with respect to the weak topology in $E$. The family $\{Y_{p'}\}$ has the property that for each finite system of projections $p'_1, \ldots, p'_{n} \in \mathcal{P}$ the intersection $\cap_{i=1}^n Y_{p'_i}$ is non-empty. Indeed, let $H_0 = \text{span}(\text{Im} p'_1, \ldots, \text{Im} p'_{n})$, and let $p'_0 \in \mathcal{P}$ be the corresponding projection of $E$ onto $H_0$. Then $p'_i \prec p'_0$, and hence $p'_0(v) \in Y_{p'_i}$ for each $i = 1, \ldots, n$. Since $H$ is a Hilbert space, the unit ball $B(H)$ of $H$ is weakly compact. It follows that the centralized family $\{Y_{p'}\}_{p' \in \mathcal{P}}$ of weakly closed subsets of $B(H)$ has a non-empty intersection. By the Barry’s Theorem [4] the generalized sequence of projections $\mathcal{P} = \{p'\}$ converges in the strong operator topology to its upper bound $p$ which is a projection of $E$ onto $H$, and which satisfies the condition (i) $1_E - 2p \in \text{Iso} E$. In particular, $H$ is biorthogonally complemented in $E$. This proves (a).

b, c. Let $s' = s_{x,x^*} \in \text{Iso} E$, where $v \in G_0e$. Then $s'$ commutes with any projection $p' \in \mathcal{P}$ such that $p'(x) = x$, which means that $\text{Ker} p' \subset \text{Ker} x^*$. Passing to the limit we see that $p$ commutes with $s'$ and Ker $p \subset \text{Ker} x^*$, too. It follows that $s' = (1_E - p) + s'p$.

Let $x_0 \in S(H)$ be the limit of a generalized sequence of vectors $x_\alpha \in G_0e \cap S(H)$. Then the corresponding sequence of isometric reflections $s_\alpha = s_{x_\alpha,x_\alpha^*} = g_\alpha s_\alpha^{-1}$, where $g_\alpha \in G_0$ and $g_\alpha(e) = x_\alpha$, is strongly convergent to the reflexion $s_0 = s_{x_0,x_0^*} \in \text{Iso} E$. Indeed, from the representation $s_\alpha = (1_E - p) + s_\alpha p$ it easily follows that the generalized sequence $\{s_\alpha\}$ converges to $s_0 = (1_E - p) + s_0 p$ on each of the complementary subspaces Ker $p$ and $H$. 

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Let $\bar{u} \in O(H)$ and $x \in E$. Consider the extension $u = (1_E - p) + \bar{u}p$ of $\bar{u}$ to $E$. Since $\|p(x)\|_E = \|up(x)\|_E$, there exists an orthogonal reflexion $s_0$ in $H$ such that $s_0p(x) = up(x)$. Let $s_0 = (1_E - p) + s_0p \in IsoE$. Then we have $\|x\|_E = \|s_0(x)\|_E = \|(1_E - p)(x) + s_0p(x)\|_E = \|u(x)\|_E$. Therefore, $u \in IsoE$, and thus (ii) is fulfilled.

Now it is clear that the orbit $G_0e \subset S(H)$ contains the orbit of the strong identity component of the orthogonal group $O(H)$, and so it coincides with the sphere $S(H)$. This proves (e).

Let $g \in IsoE$ be such that $\bar{u} = g|H \in O(H)$. We will show that $g$ leaves the subspace Ker$p$ invariant and thus $\bar{v} = g|Ker p \in Iso(Ker p)$ and $g = \bar{v}(1_E - p) + \bar{u}p$.

Suppose that $g(Ker p) \not\subset Ker p$. Consider the operator $g_1 = gu^{-1} \in IsoE$. We have $g_1|H = 1_H$ and $g_1|Ker p = g|Ker p$. By our assumption there exists a vector $x \in Ker p$ such that $g_1(x) \notin Ker p$, and so $pg_1(x) \neq 0$. Denote $x_1 = (1_E - p)g_1(x)$ and $x_2 = pg_1(x)$. Then $g_1(x_1) = x_1 + x_2$, hence $g_1^{-1}(x_1) = g_1^{-1}(g_1(x_1) - x_2) = x_1 - x_2$.

Consider two functions $\phi(t) = ||x_1 + tx_2||_E$ and $\psi(t) = ||x + tx_2||_E$. Since $1_E - 2p \in IsoE, x_1 \in Ker p$ and $x_2 \in Im p$, we have $||x_1 + x_2|| = ||x_1 - tx_2||$ and $||x + tx_2|| = ||x - tx_2||$. Thus, both $\phi$ and $\psi$ are even functions. From the equalities

$$g_1^{-1}(x_1 + tx_2) = x_1 - x_2$$

and

$$g_1(x + tx_2) = x_1 + (1 + t)x_2$$

and the fact that $g_1 \in IsoE$ we obtain that $\phi(t) = \psi(1 - t)$ and $\psi(t) = \phi(1 + t)$. It follows that $\phi(t) = \phi(-t) = \psi(1 + t) = \phi(t + 2)$. Therefore, being convex and periodic function on $R$, $\phi(t)$ should be constant. This is possible only if $x_2 = 0$, i.e. $g_1(x) \in Ker p$, which is a contradiction. Thus, (iii) is fulfilled as well. This completes the proof of Theorem 1.

It has been already noted that the statement of Theorem 2.b is a direct corollary of Theorem 1. Thus, it is enough to prove Theorem 2.a.

2.9. Proof of Theorem 2.a. It is enough to show that the four squares identity holds in $E$. For this it is enough, as it was done in the proof of Theorem 1, to approximate an arbitrary pair of vectors $x, y \in E$ by a sequence $\{(x_\alpha, y_\alpha)\}$ of pairs of vectors belonging to finite dimensional euclidean subspaces $H_\alpha$ of $E$. In turn, it is enough to show that any pair of vectors in the linear span of the orbit $Ge$ of the group $G = IsoE$ belongs to a finite dimensional euclidean subspace $H'$ of $E$. Indeed, it is easily seen that under our assumptions any orbit of $G$ in the unit sphere $S = S(E)$ is dense in $S$. In particular, the orbit $Ge$ is dense in $S$.

Fix such a pair $x, y \in span(Ge)$ and consider a subspace $H' = span(g_1(e), \ldots , g_n(e))$, $g_i \in G, i = 1, \ldots , n$, containing this pair. Since the orbit $Ge$ is dense in $S$, for any
two vectors $g'(e)$ and $g''(e) \in Ge$ one can find a finite chain of vectors $h_j(e) \in Ge, j = 0, \ldots, k$, such that $h_0(e) = g'(e), h_k(e) = g''(e)$ and $\|h_{j+1}(e) - h_j(e)\|_E < \frac{1}{10}, j = 0, 1, \ldots, k - 1$. Find such a chain $\{h_{ij}(e)\}_{j=1}^{k-1}$ for each of the pairs $(g(e), g+1(e), i = 1, \ldots, n - 1$, and put $F = \text{span}(h_{ij}(e), j = 0, \ldots, k, i = 1, \ldots, k - 1)$. We may assume that $E$ is infinite dimensional (otherwise the proof is simple), and that $\dim L > 8$. Let $\{t_{ij} = h_{ij}sh_{ij}^{-1}\}$ be the system of isometric reflexions along the vectors $h_{ij}(e), j = 1, \ldots, k, i = 1, \ldots, n - 1, and let W be the group generated by the restrictions $t_{ij}|F$. By Corollary 2.4, the Coxeter graph $\Gamma_W$ is connected, and since the system of vectors $(h_{ij}(e))$ is complete in $F$, by Lemma 1.1, the group $W$ is irreducible. For a pair of vectors $(v' = h_{ij}(e), v'' = h_{ij+1}(e))$ we have $0 < \|v' - v''\| < \frac{1}{10}$, and so by Corollary 2.4,$$
abla^\alpha(t_{ij}, t_{ij+1}) < \arccos \frac{9}{10}.$$From the classification of Coxeter groups [5, Ch.VI, sect.4] it follows that the group $W$ is infinite. Thus, by Corollary 1.4, the subspaces $F$ and $H' \subset F$ are euclidean. The theorem is proven. $\Box$

A priori, the strong operator topology could be still too strong in order that the identity component $G_0$ be big enough to apply Theorem 1 in an efficient way. Next we give a version of Theorem 1 which does not involve any operator topology.

Recall that a group $G$ is locally finite if every finitely generated subgroup of $G$ is finite.

2.10. Theorem. Let $s = s_{e,e^*}$ be an isometric reflexion in a Banach space $E$. Denote

$$U = \{g \in \text{Iso } E \mid [s, g^{-1}sg] \neq 1_E\}.$$ Let $G_1$ be the subgroup of $\text{Iso } E$ generated by $U$, and $H = \text{span}(G_1e)$. If any of the following two conditions (i), (ii) is fulfilled, then all the conclusions (a), (b), (c) of Theorem 1 hold:

(i) The group $W_1$ generated by the set of reflexions $\text{IR}_1 = \{g^{-1}sg\}_{g \in G_1}$ is not locally finite.

(ii) The orbit $G_1e$ contains three linearly independent vectors $e_1, e_2, e_3$, where $\|e_1 - e_2\|_E < 1 - \cos \pi/5$.

Proof. Repeating the arguments used in the proofs of Theorems 1 and 2, it is enough to show that for each finite subset $\sigma \subset \text{IR}_1$ there exists another finite subset $\sigma_1 \subset \text{IR}_1$ such that $\sigma \subset \sigma_1$ and the group generated by the reflexions from $\sigma_1$, as well as its restiction to the subspace $\text{span}(v \mid s_{e,v^*} \in \text{IR}_1)$, is infinite. In other words, each finite subgraph $\gamma$ of the Coxeter graph $\Gamma_{W_1}$ should be contained in a finite connected
subgraph $\gamma_1 \subset \Gamma_{W_1}$ with the following property: the group $W(\gamma_1)$ generated by the reflexions which correspond to the vertices of $\gamma_1$, is infinite. The latter holds as soon as the Coxeter graph $\Gamma_{W_1}$ is connected and contains a finite subgraph $\gamma_0$ such that the group $W(\gamma_0)$ is infinite. If the first of these conditions is fulfilled, than the second follows from each of the assumptions (i) and (ii) above. Indeed, it is clear for (i).

As for (ii), by the connectedness of the graph $\Gamma_{W_1}$, one can find a finite connected subgraph $\gamma_0 \subset \Gamma_{W_1}$ which contains three vertices corresponding to the reflexions $s_1, s_2, s_3 \in IR_1$ with the reflexion vectors $e_1, e_2, e_3$, resp. From the classification of finite Coxeter groups [5, Ch.VI, sect.4] it follows that the group $W(\gamma_0)$ is infinite.

Indeed, if $V(\gamma_0)$ is the subspace generated by the reflexion vectors of the reflexions from $W(\gamma_0)$, then $\dim V(\gamma_0) \geq 3$ and by Corollary 2.3, the order of the rotation $s_1s_2 \in W(\gamma_0)$ is greater than 5.

Thus, it remains to check that the graph $\Gamma_{W_1}$ is connected. Let the vertices $v, v'$ of $\Gamma_{W_1}$ correspond to the reflexions $s, s' = h^{-1}sh$ resp., where $h = g_ng_{n-1} \cdots g_1 \in G_1$ is arbitrary and $g_i \in U, i = 1, \ldots, n$. Put $h_i = g_ig_{i-1} \cdots g_1$ and $s_i = h_i^{-1}sh_i, i = 0, \ldots, n$, so that $h_0 = 1_E, h_1 = g_1, h_n = h$ and $s_0 = s, s_n = s'$. Since $g_1 \in U$, the reflexions $s_0 = s$ and $s_1$ do not commute, and thus $0 < \phi(1_E, h_1) < 1$. By Lemma 2.4.a, $0 < \phi(1_E, h_1) = \phi(g_1, g_1h_1) = \phi(h_1, h_2) < 1$, and therefore the reflexions $s_1$ and $s_2$ do not commute, as well. By induction, we see that $s_i$ does not commute with $s_{i+1}$ for all $i = 0, \ldots, n - 1$, and so the vertices $v, v'$ of the graph $\Gamma_{W_1}$ are connected by a path. This concludes the proof.  

3 Infinite Coxeter groups

In sections 4, 5, 6 below we will use the classification of infinite Coxeter groups. Although it should be well known, in view of the lack of references we reproduce it here in all details.

By an infinite Coxeter group we mean an infinite locally finite group $W$ generated by reflexions in a real vector space $V$ which is algebraically irreducible in $V$. We fix the following notation and conventions.

3.1. Notation. Denote by $\mathbf{R}^\Delta$ the linear space of all the real functions with finite support defined on a given set $\Delta$, and by $\mathbf{R}_0^\Delta$ the subspace of functions with the zero mean value. Let

$A_\Delta$ be the group of finite permutations of elements of $\Delta$ acting in $\mathbf{R}_0^\Delta$;

$B_\Delta$ be the group of finite permutations of $\Delta$ and changes of sign of values at the points of finite subsets of $\Delta$ acting in $\mathbf{R}^\Delta$;

$D_\Delta$ be the subgroup of $B_\Delta$ which consists of finite permutations and changes of signs of even numbers of coordinates acting in $\mathbf{R}^\Delta$.  

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If $\Delta$ is infinite then $A_\Delta$, $B_\Delta$, $D_\Delta$ are infinite Coxeter groups. Let $\epsilon_\delta$ be the characteristic function of the one-point subset $\{\delta\}$ of $\Delta$, so that $(\epsilon_\delta | \delta \in \Delta)$ is the standard Hamel basis of $R^\Delta$. Let $R^\Delta$ be endowed with the standard scalar product. Then $A_\Delta$ (resp. $B_\Delta$, $D_\Delta$) is generated by orthogonal reflexions along vectors from the infinite root system $(\epsilon_\delta - \epsilon_\delta')$ (resp. $(\pm \epsilon_\delta \pm \epsilon_\delta')$, $(\pm \epsilon_\delta \pm \epsilon_\delta')$) $(\delta, \delta' \in \Delta, \delta \neq \delta')$.

In the category of pairs $(W, V)$, where $W$ is a group generated by reflexions in a real vector space $V$, there is a natural notion of isomorphism. We will also use a notion of subpair. Namely, we will say that $(W', V')$ is a subpair of $(W, V)$ if $W'$ is the restriction of a subgroup of $W$ generated by reflexions to its invariant subspace $V'$. An embedding of pairs is an isomorphism with a subpair. In the proposition below isomorphism of Coxeter groups means isomorphism of pairs, rather then isomorphism of abstract groups.

### 3.2. Proposition

Any infinite Coxeter group $W$ is isomorphic to one and only one of the groups $A_\Delta$, $B_\Delta$, $D_\Delta$.

**Proof.** In the sequel $\gamma$ denotes a finite connected subgraph of the Coxeter graph $\Gamma_W$, $G(\gamma)$ denotes the finite subgroup of $W$ generated by reflexions $s_i = s_{e_i, e_i^*} \in \gamma, i = 1, \ldots, \text{card } \gamma$, $V(\gamma) = \text{span}(e_i, i = 1, \ldots, \text{card } \gamma)$ and $n(\gamma) = \dim V(\gamma)$. By Lemma 1.1, the restriction $G(\gamma)|V(\gamma)$ is irreducible, so it is a finite Coxeter group. The full Coxeter graph $\Gamma_{G(\gamma)}$ can be naturally identified with a finite connected subgraph $\tilde{\gamma}$ of $\Gamma_W$ containing $\gamma$; in fact, $\tilde{\gamma}$ is the maximal subgraph of $\Gamma_W$ with the properties that $V(\tilde{\gamma}) = V(\gamma)$ and $G(\tilde{\gamma}) = G(\gamma)$ (but the first one alone does not determine $\tilde{\gamma}$). If $n = n(\gamma) > 8$, then $G(\gamma)$ is one of the Coxeter groups $A_n, B_n, D_n$ [5, Ch.VI, sect. 4].

Let $\Delta$ be a set with card $\Delta = \dim V$, where $\dim V$ is the cardinality of a Hamel basis in $V$. The proposition follows from the assertions (i) - (iii) below.

1. $(W, V) \approx (B_\Delta, R^\Delta)$ if $(G(\gamma_0), V(\gamma_0)) \approx (B_n, R^n)$ for some $\gamma_0 \subset \Gamma_W$;
2. $(W, V) \approx (D_\Delta, R^\Delta)$ if there is no $\gamma \subset \Gamma_W$ for which $(G(\gamma), V(\gamma)) \approx (B_n, R^n)$ and $(G(\gamma_0), V(\gamma_0)) \approx (D_n, R^n)$ for some $\gamma_0 \subset \Gamma_W$ with $n = n(\gamma_0) \geq 4$;
3. $(W, V) \approx (A_\Delta, R^\Delta_0)$ in the other cases.

From now on we consider Coxeter graphs as weighted graphs. As usual, the weight of an edge $(s', s'')$ is the order of the product $s' s''$. Since $W$ is a locally finite group, the weights on $\Gamma_W$ take only finite values. Recall that the Coxeter graphs of types $A_n$ and $D_n$ have only edges of weight 3, while in any of the Coxeter graphs of type $B_n$ there are edges of weight 4. Thus, if the assumption of (i) holds, then $(G(\gamma'), V(\gamma')) \approx (B_{n'}, R^{n'})$ for any $\gamma' \supset \gamma_0$ with $n' = n(\gamma') > 8$, and thus $(W, V)$ is the inductive limit of some net of pairs $(B_{n'}, R^{n'})$.

Next we show that for $\gamma \subset \gamma'$, where $\gamma \supset \gamma_0$ and $n(\gamma) > 8$, the embedding $(G(\gamma), V(\gamma)) \subset (G(\gamma'), V(\gamma'))$ is coordinatewise. This means that under the isomorphisms $(G(\gamma), V(\gamma)) \approx (B_n, R^n)$ and $(G(\gamma'), V(\gamma')) \approx (B_{n'}, R^{n'})$ the pair $(B_n, R^n)$
is a coordinate subpair of \((B_{n'}, \mathbb{R}^{n'})\). Indeed, identify \((G(\gamma'), V(\gamma'))\) with \((B_{n'}, \mathbb{R}^{n'})\). Then \(V(\gamma) \subset \mathbb{R}^{n'}\) is spanned by a subsystem of the root system \((\pm \epsilon_i, \pm \epsilon_i \pm \epsilon_j, 1 \leq i < j \leq n', \text{ and } G(\gamma)\) is generated by the corresponding orthogonal reflexions. A plane in \(\mathbb{R}^{n'}\) spanned by roots may contain reflexion root vectors of 2, 3 or 4 different reflexions from \(B_{n'}\). It is a coordinate plane precisely when it contains 4 reflexions.

Since \((G(\gamma), V(\gamma)) \approx (B_n, \mathbb{R}^n)\), the subspace \(V(\gamma)\) contains \(\binom{n}{2}\) planes of the latter type which span it. At the same time, these planes should be coordinate planes of \(\mathbb{R}^{n'}\). Therefore, \(V(\gamma)\) is a coordinate subspace \(\mathbb{R}^n \subset \mathbb{R}^{n'}\), and so \(G(\gamma)\) coincides with the group \(B_n\) generated by reflexions along those roots of the above root system which belong to \(V(\gamma)\).

For any of the graphs \(\gamma \supset \gamma_0\) with \(n = n(\gamma) > 8\) all the vertices of the full Coxeter graph \(\tilde{\gamma}\) (see the notation above) are divided in two types: those which correspond to sign change reflexions, i.e. reflexions along the roots of the form \(\epsilon_i, i = 1, \ldots, n\), and others. Being coordinatewise, embeddings of pairs respect this division. Thus, it is well defined in the inductive limit \(\Gamma_W\). Note that the vertices of change sign type in \(\Gamma_W\) are those which incident only with edges of weight 4. Denote by \(\Delta\) the set of all the vertices of \(\Gamma_W\) of change sign type. Fix \(\delta_0 \in \Delta \cap \gamma_0\), and let \(\epsilon_0 = \epsilon_{\delta_0}\) be one of the two opposite roots in \(V(\gamma_0)\) which correspond to the reflexion \(\delta_0\). It is easily seen that for any \(\gamma \supset \gamma_0\) with \(n = n(\gamma) > 8\) the orbit \(G(\gamma) \epsilon_0\) consists of the roots of coordinate type \(\pm \epsilon_i\) in \(V(\gamma)\), and so the class of conjugates of \(\delta_0\) in \(W\) coincides with \(\Delta\). Choosing one of any two opposite root vectors in the orbit \(W(\epsilon_0)\) we obtain a Hamel basis of the \(W\)-invariant subspace \(\text{span}(W(\epsilon_0))\) which coincides with \(V\) since \(W\) is assumed to be irreducible in \(V\). Thus, we obtain a Hamel basis of \(V\) formed by roots of coordinate type. This yields an isomorphism \(V \approx \mathbb{R}^\Delta\). The root system of \(W\), which consists of the vectors of the two orbits \(W(\epsilon_0)\) and \(W(\epsilon_0 + \epsilon_1)\), where \(\epsilon_1 \neq \epsilon_0\) is another coordinate vector in \(V(\gamma_0)\), corresponds under this isomorphism to the root system \((\pm \epsilon_\delta, \pm \epsilon_\delta \pm \epsilon_{\delta'}, |\delta, \delta' \in \Delta, \delta \neq \delta'|\) of the group \(B_\Delta\). Therefore, \((W, V) \approx (B_\Delta, \mathbb{R}^\Delta)\). This proves (i).

Next we consider the case (ii), where there is no subgroup \(G(\gamma) \subset W\) of type \(B_n\), but at least one of them, say \(G(\gamma_0)\), has type \(D_n\) for some \(n = n(\gamma_0) \geq 4\). First we show that any subgroup \(G(\gamma) \supset G(\gamma_0)\) is of type \(D_n(\gamma)\), and all the embeddings \(G(\gamma) \hookrightarrow G(\gamma')\) are coordinatewise.

The group \(D_4\) contains 4 pairwise commuting reflexions along the root vectors \(\epsilon_1 \pm \epsilon_2, \epsilon_3 \pm \epsilon_4\). If \(v_1, \ldots, v_4\) are 4 mutually orthogonal root vectors from the root system \((\pm (\epsilon_i - \epsilon_j), 1 \leq i < j \leq n')\) of type \(A_{n'}\) and \(L = \text{span}(v_1, \ldots, v_4)\), then the only reflexions in \(A_{n'} \setminus L\) are the orthogonal reflexions along \(v_1, \ldots, v_4\), and so \(A_{n'} \setminus L\) does not contain \(D_4\). Therefore, the Coxeter group \(G(\gamma) \supset G(\gamma_0)\) is not of type \(A_{n(\gamma)}\), and thus it must be of type \(D_{n(\gamma)}\).

Let \(F\) be a subspace of dimension 4 of \(\mathbb{R}^{n'}\) generated by 4 mutually orthogonal
root vectors from the root system \((\pm \epsilon_i \pm \epsilon_j), 1 \leq i < j \leq n',\) of type \(D_{n'},\) and 
\(G(F) \subset D_{n'}\) be the subgroup generated by the orthogonal reflexions along the roots
in \(F.\) Then \(G(F)\) is irreducible (and of type \(D_4\)) iff \(F\) is a coordinate subspace of \(\mathbb{R}^{n'}.\)
Thus, if \((G(\gamma), V(\gamma)) \subset (D_{n'}, \mathbb{R}^{n'})\) is of type \(D_n,\) where \(n = n(\gamma) \geq 4,\) then \(V(\gamma)\) is
a coordinate subspace of \(\mathbb{R}^{n'}.\)

Fix a reflexion vector \(v_0\) of a reflexion from \(G(\gamma_0).\) If \(G(\gamma) \supseteq G(\gamma_0),\) then the
orbit \(G(\gamma) v_0\) is a root system of type \(D_{n(\gamma)}\) of the Coxeter group \(G(\gamma).\) Consider
the infinite root system \(W(v_0)\) in \(V.\) Since \(W\) is irreducible, this system is complete in
\(V.\) Note that two roots \(v', v''\) of \(D_n\) are contained in the same coordinate plane in \(\mathbb{R}^n\)
iff the sets of their neighborhood vertices in the full Coxeter graph \(\Gamma_{D_n}\) coincide.
In this case the vectors \((\pm \epsilon_i \pm \epsilon_j) = (\pm \epsilon_i, \pm \epsilon_j), i \neq j,\) are contained in the coordinate
axes which are the intersections of coordinate planes. The same pairing is defined
on the above root system of \(W.\) In this way, fixing one of any two opposite vectors
\((\pm \epsilon_i \pm \epsilon_j)\) arbitrarily, we obtain a Hamel basis \(\Delta\) in \(V,\) which in turn provides us with
an isomorphism \((W, V) \approx (D_\Delta, \mathbb{R}^{\Delta}).\) This proves (ii).

Assume further that any subgroup \(G(\gamma') \subset W\) with \(n' = n(\gamma') > 8\) is of type
\(A_{n'},\) where \(A_{n'}\) acts by permutations in \(\mathbb{R}^{n'+1}.\) Let \((G(\gamma), V(\gamma)) \subset (A_{n'}, \mathbb{R}^{n'+1})\) be of
the Coxeter graph \(\Gamma_{D_n}\) coincide. We will show that \(V(\gamma)\) is a coordinate subspace of \(\mathbb{R}^{n'+1}.\) Let \(s_{ij}\)
be the orthogonal reflexions (transpositions) along the roots \(\pm (\epsilon_i - \epsilon_j), 1 \leq i < j \leq n' + 1.\)
Put
\[I = \{i \in \{1, \ldots, n' + 1\} | s_{ij} \in G(\gamma) \text{ for some } j \in \{1, \ldots, n' + 1\}\} .\]
Thus, if \(s_{kl} \in G(\gamma),\) then \(k, l \in I.\) Vice versa, \(s_{kl} \in G(\gamma)\) for any pair \(k, l \in I, k \neq l.\)
This follows from the connectedness of the Coxeter graph \(\Gamma_{G(\gamma)}\) and the following
remark: if \(s_{ij} \in G(\gamma)\) and \(s_{jk} \in G(\gamma),\) then \(s_{ik} \in G(\gamma).\) Indeed, \(s_{jk}(\epsilon_i - \epsilon_j) = \epsilon_j - \epsilon_k\)
and thus \(s_{ik} = s_{jk}s_{ij}s_{jk}.\) Now we see that \(V(\gamma) = \mathbb{R}^I = \operatorname{span} (\epsilon_i - \epsilon_j | i, j \in I)\) is a
coordinate subspace of \(\mathbb{R}^{n'+1},\) and \(G(\gamma) \subset A_{n'}\) is a subgroup of permutations of the
set \(I.\)

On the set of edges of the full Coxeter graph \(\Gamma_{D_n}\) consider the following equivalence
relation: \((s_{i_1,j_1}, s_{i_2,j_2}) \sim (s_{k_1,l_1}, s_{k_2,l_2})\) iff these four transpositions have an index in
common. Then this index is the same for the whole equivalence class, so that the
set of classes is \(\{1, \ldots, n\}.\) Since this equivalence relation is compatible with the
embedding of pairs of \((G(\gamma), V(\gamma)) \leftrightarrow (G(\gamma'), V(\gamma')),\) it can be defined as well on
the whole graph \(\Gamma_{W}.\) Let \(\Delta\) be the set of the equivalence classes. Let \(v \in \Gamma_{W}\) be a vertex.
Then all the edges incident with \(v\) belong to two different classes \(\delta, \delta' \in \Delta,\) where
each class \(\delta \in \Delta\) consists of the edges of a complete subgraph of \(\Gamma_{W},\) and each pair
of these complete subgraphs which correspond to some \(\delta, \delta' \in \Delta, \delta \neq \delta',\) has exactly
one vertex \(v(\delta, \delta')\) in common. It is easily seen that the action of \(W\) on \(\Gamma_{W}\) by inner
automorphisms is locally finite and compatible with the equivalence relation, and so
it induces the action of \(W\) on \(\Delta\) by finite permutations, such that the reflexions in

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W act as transpositions.

Fix a reflexion \( s_0 = s(\delta, \delta') \in W \) which corresponds to the transposition \((\delta, \delta')\), with the reflexion vector \( e_0 = e(\delta, \delta') \). Then the orbit \( W(e_0) \) is a root system of \( W \) which spans \( V \). Fixing one of the each two opposite roots, we obtain a Hamel basis of \( V \) which corresponds to the basis of \( \mathbb{R}\delta' \) consisting of the root vectors \( \epsilon_\delta - \epsilon_{\delta'} (\delta, \delta' \in \Delta, \delta \neq \delta') \) of \( A_\Delta \). This gives an isomorphism \((W,V) \approx (A_\Delta, \mathbb{R}\delta') \). The proof is complete. \( \Box \)

4 Total families of isometric reflexions

Denote by \( \text{IR}(E) \) the set of all the reflexions in \( \text{Iso } E \), and let \( W = W(E) \) be the subgroup of \( \text{Iso } E \) generated by the reflexions from \( \text{IR}(E) \). In this section we assume that \( \text{IR}(E) \) contains a total subset of reflexions \( \{s_\alpha = s_{\epsilon_\alpha, \epsilon^*_\alpha}\}_{\alpha \in A} \), which means that the family of linear functionals \( T = \{\epsilon^*_\alpha\}_{\alpha \in A} \subset E^* \) is a total family.

4.1. Lemma. Let \( g_1, g_2 \in \text{Iso } E \). In the notation as above assume that \( g_1(e_\alpha) = g_2(e_\alpha) \) for all \( \alpha \in A \). Then \( g_1 = g_2 \).

Proof. Put \( g_0 = g_1^{-1}g_2 \). Then \( g_0(e_\alpha) = e_\alpha \) for all \( \alpha \in A \). Since \( s_\alpha \) is the only isometric reflexion in the direction of \( e_\alpha \) (see Remark 2.2.b), it coincides with \( s'_\alpha = g_0s_\alpha g_0^{-1} = s_{\epsilon_\alpha, \epsilon^*_\alpha}^{-1}(\epsilon^*_\alpha) \), and so \( g_0^{-1}(\epsilon^*_\alpha) = \epsilon^*_\alpha \), i.e. \( g_0(\epsilon^*_\alpha) = \epsilon^*_\alpha \) or, in other words, \( \epsilon^*_\alpha(g_0(v) - v) = 0 \) for all \( \alpha \in A \). Since \( T \) is total, it follows that \( g_0 = 1_E \). \( \Box \)

Let, as before, \( G_0 \) be the strong identity component of \( \text{Iso } E \), and let \( W \) be the group generated by the reflexions in \( \text{Iso } E \).

4.2. Lemma. \( W \) is locally finite iff \( G_0 \) is trivial.

Proof. Assume that \( G_0 \) is trivial. To prove that \( W \) is locally finite it is enough to show that each subgroup \( W' \) of \( W \) generated by a finite number of reflexions \( \{s_i = s_{\epsilon_i, \epsilon^*_i}\}_{i=1,...,n} \subset \text{IR}(E) \) is finite. Suppose that \( W' \) is an infinite group. Put \( F' = \text{span}(\epsilon_i \mid i = 1,\ldots,n) \). Let \( G' \) be the closure of \( W' \) in \( \text{Iso } E \) in the strong operator topology. It is easily seen that the closed subspace \( M' = \bigcap_{i=1}^n \text{Ker } \epsilon^*_i \) is a complementary subspace of \( F' \), i.e. \( E = M' \oplus F' \), and it coincides with the fixed point subspace of \( W' \). Hence, it also coincides with the fixed point subspace of \( G' \). It follows that \( G' = 1_{M'} \oplus \bar{G}' \), where \( \bar{G}' \subset O(F') \) is the closure of \( W' \mid F' \) in \( \text{Iso } F' \). Thus, \( G' \) is a compact Lie group, and being infinite it has a non-trivial identity component. This is a contradiction.

Assume now that \( G_0 \) is non-trivial. Then, as it was shown in the proof of Proposition 2.6, there exist reflexions \( s', s'' \in \text{IR}(E) \) such that the angle \( \alpha(s', s'') \) is irrational.

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modulo $\pi$, and so the subgroup of $W$ generated by these two reflexions is infinite.

Remind that a group $G$ of operators in a Banach space $E$ is called topologically irreducible if it has no nontrivial closed invariant subspace.

4.3. Lemma. Let $W'$ be a group generated by a set of reflexions $\{s_\alpha = s_{e_\alpha,e_\alpha^*}\}_{\alpha \in A'} \subset \text{IR}(E)$. Then $W'$ is topologically irreducible iff the following two conditions are fulfilled:

i) The system of vectors $(e_\alpha | \alpha \in A')$ is complete, i.e. $E = \text{span}(e_\alpha | \alpha \in A')$.

ii) The Coxeter graph $\Gamma_{W',A'}$ is connected.

Proof. Since the closed subspace $E' = \text{span}(e_\alpha | \alpha \in A')$ is invariant with respect to $W'$, the first condition (i) is necessary for $W'$ being irreducible. Let $\Gamma'$ be a connected component of $\Gamma_{W',A'}$. It is easily seen that the closed subspace $F' = \text{span}(e | s_{e,e^*} \in \Gamma')$ is invariant, too. Thus, the second condition (ii) is necessary, too.

Assume further that (i) and (ii) are fulfilled. Let $F'$ be a closed invariant subspace of $W'$. Put $A = \{\alpha \in A' | e_\alpha \in F'\}$ and $B = \{\alpha \in A' | e_\alpha \notin F'\}$. Being invariant $F'$ is contained in $\text{Ker} e_{\beta}^*$ for each $\beta \in B$. It follows that $\alpha(s_\alpha,s_\beta) = \pi/2$, and so $[s_\alpha,s_\beta] = 1_E$ for any $\alpha \in A, \beta \in B$. By (ii) this implies that either $A = \emptyset$ or $B = \emptyset$. From (i) it easily follows that the system of linear functionals $(e_\alpha^* | \alpha \in A) \subset E^*$ is total. Thus, if $A = \emptyset$, then $F' \subset \cap\{\text{Ker} e_\alpha^* | \alpha \in A'\} = \{0\}$, and if $B = \emptyset$, then $F' \supset \text{span}(e_\alpha | \alpha \in A') = E$. In any case, $F'$ is not a proper subspace. This shows that $W'$ is topologically irreducible.

Let things be as in Lemma 4.3. Consider the algebraic linear subspace $V' = \text{span}(e_\alpha | \alpha \in A')$. The group $W'$ is algebraically irreducible in $V'$ iff the Coxeter graph $\Gamma_{W',A'}$ is connected. In this case $W'$ is topologically irreducible in the closed subspace $E' = \overline{V'} = \text{span}(e_\alpha | \alpha \in A')$. If $W'$ is finite and the Coxeter graph $\Gamma_{W'}$ is connected, then $\dim V' = n < \infty$ and $W'|V'$ is a finite Coxeter group, i.e. a finite irreducible group generated by orthogonal reflexions in $\mathbb{R}^n$ (here we identify $V'$ with $\mathbb{R}^n$ by choosing an orthonormal basis with respect to an invariant scalar product in $V'$). Let the group $\text{Iso}E$ be discrete in the strong operator topology, i.e. $G_0 = \{1_E\}$. Then by Lemma 4.3, $W'$ is a locally finite group. If $\dim V' = \infty$ and the Coxeter graph $\Gamma_{W'}$ is connected, then $W'$ is an infinite Coxeter group, and by Proposition 3.2, it is isomorphic to one of the groups $A_\Delta, B_\Delta, D_\Delta$. The next proposition shows that if the pair $(W',V')$ is maximal, it can not be of type $D_\Delta$.

4.4. Proposition. Let the notation be as above. If $\dim V' = \infty$ and $(W',V') \approx (D_\Delta, \mathbb{R}^\Delta)$, then the group $W'$ can be extended to a subgroup $W'' \subset \text{Iso}E$ generated by reflexions along vectors in $V'$ and such that $(W'',V') \approx (B_\Delta, \mathbb{R}^\Delta)$. 

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For the proof we need the following lemma on partial orthogonal decompositions in Banach spaces.

**4.5. Lemma.** Let \( \{p_i\}_{i=1,2,...} \) be a sequence of projections in a Banach space \( E \) such that

a) \( 1_E - 2p_i \in \text{Iso}E \) for all \( i = 1, 2, ...; \)
b) the projections \( p_i \) are mutually orthogonal, i.e. \( p_ip_j = 0 \) for all \( i \neq j. \)

Then \( \limsup_{i \to \infty} \|(1_E - p_i)(x)\|_E = \|x\|_E \) for all \( x \in E. \)

**Proof.** By (a), we have \( \|p_i\|_E = \|1_E - p_i\|_E = 1 \) for all \( i = 1, 2, ... \). From (a) and (b) it follows that \( \prod_{i=1}^k (1_E - 2p_i) = 1_E - 2\sum_{i=1}^k p_i \in \text{Iso}E, \) and so \( \| \sum_{i=1}^k p_i \|_E = \| 1_E - \sum_{i=1}^k p_i \|_E = 1, \) as well.

Assume that there exist \( x_0 \in E \) and \( \epsilon_0 > 0 \) such that

\[
\|(1_E - p_i)(x_0)\|_E \leq \|x_0\|_E - \epsilon_0 \text{ for all } i = 1, 2, ... .
\]

Then

\[
\frac{1}{k}\sum_{i=1}^k (1_E - p_i)(x_0)\|_E \leq \frac{1}{k}\sum_{i=1}^k \|(1_E - p_i)(x_0)\|_E \leq \|x_0\|_E - \epsilon_0 .
\]

Therefore

\[
\epsilon_0 \leq \|x_0\|_E - \frac{1}{k}\sum_{i=1}^k (1_E - p_i)(x_0)\|_E \leq \|x_0\|_E - \frac{1}{k}\sum_{i=1}^k (1_E - p_i)(x_0)\|_E = \frac{1}{k}\|(\sum_{i=1}^k p_i)(x_0)\|_E \leq \frac{1}{k}\|x_0\|_E .
\]

This is a contradiction. \( \Box \)

**Proof of Proposition 4.4.** Identify \( V' \) with \( \mathbb{R}^\Delta \) via an isomorphism \( (W', V') \approx (D_\Delta, \mathbb{R}^\Delta), \) and consider in \( V' \) the root system \( \{ \pm \epsilon_{\delta'}, \pm \epsilon_{\delta''} \} \) of type \( D_\Delta. \) Denote by \( s_{\delta', \delta''}^+ \) the isometric reflexion along the vector \( v_{\delta', \delta''} = \epsilon_{\delta'} + \epsilon_{\delta''}, \delta', \delta'' \in \Delta, \delta' \neq \delta'', \) and by \( s_{\delta', \delta''}^- \) the isometric reflexion along the vector \( v_{\delta', \delta''} = \epsilon_{\delta'} - \epsilon_{\delta''}. \) Put \( d_{\delta', \delta''} = s_{\delta', \delta''}^+ s_{\delta', \delta''}^- \in W', \) so that \( d_{\delta', \delta''} \) is the operator of change of signs of the coordinates \( \delta' \) and \( \delta''. \)

Choose a countable subset \( \{\delta_i\}_{i=1,2,...} \subset \Delta \) and put \( d_{i,j} = d_{\delta_i, \delta_j}. \) Then the involutions \( d_{i,j} \) pairwise commute and \( d_{n,k}d_{k,m} = d_{n,m}. \) The orthogonal projections \( p_{i,j} = \frac{1}{2}(1_E - d_{i,j}) \) onto planes also pairwise commute. For each triple of different indices \( n, m, k \) consider the one-dimensional projection \( p_{n,m}^k = p_{n,k}p_{n,m}. \) Since \( p_{n,m}^k \) and \( p_{n,m}^{k,j} \) commute and have the same image, they coincide; indeed, \( p_{n,m}^k = p_{n,m}^{k,j} = p_{n,m}^{k,j} = p_{n,m}^{k,j}. \)

Denote by \( p_{n} \) their common value, and consider the corresponding reflexion \( s_{n} = 1_E - 2p_{n} \) along the coordinate vector \( \epsilon_{\delta_n}. \) It is easily seen that
\[ s_n s_m = d_{n,m} \text{ and } s_m(1_E - p_{m,k}) = 1_E - p_{m,k}. \] By Lemma 4.5, for a fixed \( n \in \mathbb{N} \) and for any \( x \in E, \epsilon > 0 \) there exist \( k, m \in \mathbb{N} \) such that
\[
\| s_n(x) \|_E \leq \| (1_E - p_{k,m})s_n(x) \|_E + \epsilon = \| s_n(1_E - p_{k,m})(x) \|_E + \epsilon =
\| s_n s_m(1_E - p_{k,m})(x) \|_E + \epsilon = \| d_{n,m}(1_E - p_{k,m})(x) \|_E + \epsilon =
\| (1_E - p_{k,m})(x) \|_E + \epsilon \leq \| x \|_E + \epsilon .
\]
It follows that \( s_n \in \text{Iso}_E \). Since \( \delta_n \in \Delta \) is taken as arbitrary, this implies that for any \( \delta \in \Delta \) there exists an isometric reflexion along the vector \( \epsilon_\delta \). Thus, the group \( \text{Iso}_E \) contains the subgroup \( W'' \) generated by reflexions along vectors of the root system \( \{ \pm \epsilon_\delta, \pm \epsilon_\delta' \pm \epsilon_\delta'' \} \) of type \( B_\Delta \).

4.6. **Corollary.** Let \( \dim E = \infty, G_0 = G_0(E) = \{ 1_E \} \), and the group \( W = W(E) \) generated by all the isometric reflexions in \( E \) be topologically irreducible. Then \( W \) is an infinite Coxeter group of type \( A_\Delta \) or \( B_\Delta \).

4.7. **Remark.** Let \( \dim E = n < \infty \). Then Proposition 4.4 still holds in the case when \( n \) is odd. Indeed, in this case \( B_n = W'' \) is the subgroup of the group \( \text{Iso}_E \) generated by the subgroup \( W' = D_n \) and the element \( -1_E \). But for \( n \) even the statement of Proposition 4.4 in general is not valid. As an example, consider \( E = \mathbb{R}^n \), where \( n = 2k \geq 4 \), with the unit ball \( B(E) \) being the convex hull of the \( D_n \)-orbit of the point \( v_0 = (1, 2, \ldots, n) \). Then the image \( s_n(v_0) = (1, 2, \ldots, n-1, -n) \) of \( v_0 \) by the reflexion \( s_n = s_{\epsilon_n, \epsilon_n^*} \) does not belong to \( B(E) \) (indeed, it is separated from \( B(E) \) by the hyperplane \( -x_n + \sum_{i=1}^{n-1} x_i = \frac{n(n+1)}{2} \)). Hence \( B(E) \) is not invariant with respect to the action of the Coxeter group \( B_n \) on \( \mathbb{R}^n = E \), and so \( B_n \) is not a subgroup of \( \text{Iso} E \).

5 **Hilbert and Coxeter decompositions**

Let, as before, \( E \) be a Banach space with a total family of isometric reflexions. In this section we construct a partial orthogonal decomposition of \( E \) which consists of two parts: **Hilbert decomposition** into a direct sum of biorthogonally complemented Hilbert subspaces, and it Coxeter decomposition into a direct sum of closed subspaces endowed with topologically irreducible Coxeter groups generated by isometric reflexions. In a sense, this decomposition is orthogonal (see Lemma 5.4 and Proposition 5.6). Both of these decompositions are stable under the action of the isometry group \( \text{Iso}_E \), and the second one is fixed under the action of the identity component \( G_0 \).
The main result of the section, Theorem 5.7, is a kind of a structure theorem for the isometry group $\text{Iso}E$.

5.1. Notation. As above, by $\text{IR}(E)$ we denote the set of all the isometric reflexions in $E$ which is assumed to be total. To each subspace $V$ of $E$ we attach two closed subspaces, the kernel

$$V_0 = \overline{\text{span}} \{ e \in V \mid s_{e,e^*} \in \text{IR}(E) \}$$

and the hull

$$\hat{V} = \bigcap \{ \text{Ker} e^* \mid s_{e,e^*} \in \text{IR}(E), V \subset \text{Ker} e^* \};$$

we put $\hat{V} = E$ if there is no $s_{e,e^*} \in \text{IR}(E)$ such that $V \subset \text{Ker} e^*$. It is easily seen that

i) $V_0 \subset V \subset \hat{V}$,

ii) $V_0 = V_0$, $\hat{V} = \hat{V}$, and

iii) if $V \subset V'$, then $V_0 \subset V_0'$ and $\hat{V} \subset \hat{V}'$.

Observe that possibly $V_0$ resp. $\overline{V}$ is a proper subspace of $V$ resp. $\hat{V}$. For instance, this is the case when $E = l_\infty$ and $V = c$ (the subspace of convergent sequences); indeed, then $\hat{V} = E$ and $V_0 = c_0$.

Denote also $\text{IR}_V = \{ s_{e,e^*} \in \text{IR}(E) \mid e \in V \}$. Let $W_V$ be the group generated by the reflexions from $\text{IR}_V$.

5.2. Coxeter decomposition. This is a partial subspace decomposition defined on the fixed point subspace $F = \text{Fix} G_0$ of the group $G_0 = G_0(E)$. Let $\Gamma_F = \Gamma_{W_F}$ be the full Coxeter graph of the group $W_F$, and let $\mathcal{A}$ be the set of the connected components of $\Gamma_F$. For $\alpha \in \mathcal{A}$ denote by $\text{IR}_\alpha$ the set of reflexions in $\text{IR}_F$ which correspond to vertices of the component $\alpha$ of $\Gamma_F$. Put $V_\alpha = \overline{\text{span}} \{ e \mid s_{e,e^*} \in \text{IR}_\alpha \}$; so, $\text{IR}_\alpha = \text{IR}_{V_\alpha}$. Put also $W_\alpha = W_{V_\alpha}$. Then $V_\alpha$ is a closed subspace of the kernel $F_0$, and the group $W_\alpha \mid V_\alpha$ is topologically irreducible. By the discussion after Lemma 4.3, $W_\alpha$ is a Coxeter group. If $\dim V_\alpha = \infty$, then by Corollary 4.6, $W_\alpha$ has type $A_\Delta$ or $B_\Delta$.

The set $\mathcal{A}$ can be devided into equivalence classes which correspond to the isomorphism types of the Coxeter pairs $(W_\alpha, V_\alpha)$. Since $G_0$ is a normal subgroup of the group $\text{Iso} E$, its fixed point subspace $F$ is invariant with respect to $\text{Iso} E$; the same is true for the kernel $F_0$ and the hull $\hat{F}$. Each isometry $g \in \text{Iso} E$ acts (by conjugation) on the set $\text{IR}_F$ and also on the graph $\Gamma_F$, and so on the set $\mathcal{A}$. It is clear that the above partition of $\mathcal{A}$ is stable under this action and its equivalence classes are invariant.

5.3. Hilbert decomposition. Consider the following equivalence relation defined on the set $\text{IR}(E) \setminus \text{IR}_F$:

$$s_{e,e^*} \sim s_{e',e'^*} \text{ iff } e' \in G_0 e.$$

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Let \( \mathcal{B} \) be the set of its equivalence classes. By Theorem 1, to each \( \beta \in \mathcal{B} \) there corresponds the unique Hilbert subspace \( H_\beta = \text{span} (G_0 e \mid s_{e,e^*} \in \beta) \) and the unique bicontractive projection \( p_\beta : E \to H_\beta \) satisfying all the properties of Theorem 1.b. An isometry \( g \in \text{Iso } E \) induces the action \( g_\ast \) on the set \( \mathcal{B} \) which is defined as follows: \( g_\ast \beta = \beta' \) iff \( g(H_\beta) = H_{\beta'} \). In particular, the orthogonal bases in \( H_\beta \) and in \( H_{\beta'} \) are of the same cardinality. The following lemma shows that this partial decomposition into Hilbert subspaces is orthogonal; moreover, all of the subspaces \( H_\beta \) are orthogonal to the fixed point subspace \( F \).

Let \( \text{IR}_\beta = \text{IR}_{H_\beta} \). Note that \( \text{IR}(E) = \text{IR}_A \cup \text{IR}_B \), where \( \text{IR}_A = \text{IR}_F = \bigcup_{\alpha \in A} \text{IR}_\alpha \) and \( \text{IR}_B = \bigcup_{\beta \in B} \text{IR}_\beta \).

5.4. Lemma. a) Let \( s,s' \in \text{IR}(E) \). If \( [s,s'] \neq 1_E \), then \( s,s' \) belong either to the same subset \( \text{IR}_A \), where \( \alpha \in A \), or to the same subset \( \text{IR}_\beta \), where \( \beta \in \mathcal{B} \).
b) The projection \( p_\beta \) commutes with any reflexion \( s \in \text{IR}(E) \) for any \( \beta \in \mathcal{B} \).
c) Furthermore, \( p_\beta p_\beta' = 0 \) for any \( \beta, \beta' \in \mathcal{B}, \beta \neq \beta' \), and \( p_\beta \mid F = 0 \) for any \( \beta \in \mathcal{B} \).

Proof. a. Let \( s_i = s_{e_i,e_i^*}, i = 1,2 \), be two arbitrary distinct reflections from \( \text{IR}_\beta \), where \( \beta \in \mathcal{B} \). Being restricted to the Hilbert subspace \( H_\beta \) the rotation \( r = s_1 s_2 \in \text{Iso } E \) in the plane \( \text{span } (e_1,e_2) \) belongs to the connected component \( G_0(H_\beta) \) of the orthogonal group, and so by Theorem 1.c, \( r \in G_0 \). Since \( F = \text{Fix } G_0 \subset \text{Fix } r = \text{Ker } e_1^* \cap \text{Ker } e_2^* \), we have that \( e_i^*(e) = 0 \) for each \( e \in F \). Therefore, if \( s = s_{e,e^*} \in \text{IR}_F \) then by Lemma 2.3, \( \alpha(s_i,s) = \frac{\pi}{2} \), and thus \( [s_i,s] = 1_E, i = 1,2 \) (see Remark 2.2.c).

If \( \beta' \in \mathcal{B} \) and \( \beta' \neq \beta \), then the subspace \( H_{\beta'} \) is invariant with respect to the rotation \( r = s_1 s_2 \in G_0 \). One may assume that \( r \mid L \neq -1_L \), and so either \( H_{\beta'} \subset \text{Fix } r \) or \( L \subset H_{\beta'} \). The second case is impossible (indeed, otherwise by the construction, we would have \( H_{\beta} = H_{\beta'} \) and so \( \beta = \beta' \)). Thus, \( H_{\beta'} \subset \text{Fix } r = \text{Ker } e_1^* \cap \text{Ker } e_2^* \). As above, it follows that \( [s_i,s] = 1_E \) for each \( s \in \text{IR}_\beta \).

To prove (a) it remains to note that the definition of the set \( A \) (5.2) yields that \( [s,s'] = 1_E \) if \( s \in \text{IR}_\alpha, s' \in \text{IR}_{\alpha'} \), where \( \alpha, \alpha' \in A \) and \( \alpha \neq \alpha' \).

Now, (b) and (c) easily follow from (a) by the construction of the projections \( p_\beta \) as in 2.8. \( \circ \)

5.5. Lemma. a) \( \hat{F}_0 = \hat{F} = F \).
b) \( (H_\beta)_0 = \hat{H}_\beta = H_\beta \) for any \( \beta \in \mathcal{B} \).

Proof. a. Let \( s_{e,e^*} \in \text{IR}(E) \) be such that \( F_0 \subset \text{Ker } e^* \). Then \( e \notin F_0 \), and hence \( e \in H_\beta \) for some \( \beta \in \mathcal{B} \). As in the proof of Lemma 5.4, it follows that \( F \subset \text{Ker } e^* \), and so \( F \subset \hat{F}_0 = \bigcap_{\beta \in \mathcal{B}} \text{Ker } p_\beta \). Since the subspace \( \bigcup_{\beta \in \mathcal{B}} H_\beta \) is \( G_0 \)-invariant, it is clear that \( \hat{F}_0 \) is \( G_0 \)-invariant, too.
If $\hat{F}_0 \neq F$, then there exists $g_0 \in G_0$ such that $g_0|\hat{F}_0 \neq 1_{F_0}$, and so $g_0(x) \neq x$ for some $x \in \hat{F}_0$. Note that both $g_0(x)$ and $x$ belong to $\ker e^*$ for each $e^*$ such that $s_{e,e^*} \in \mathrm{IR}(E) \setminus \ker F$ (indeed, in this case $g_0(e) = e$, and thus $g_0(e^*) = e^*$). Therefore, $e^*(g_0(x) - x) = 0$ for each $e^*$ as above, and also for each $e^*$ such that $s_{e,e^*} \in \mathrm{IR}(E)$. Since the system of functionals $(e^*|s_{e,e^*} \in \mathrm{IR}(E))$ is total, we have $g_0(x) - x = 0$, which is a contradiction. This proves (a).

b. If $\hat{H}_\beta \neq H_\beta$ for some $\beta \in \mathcal{B}$, then $(1_E - p_\beta)(x) \neq 0$ for some vector $x \in \hat{H}_\beta$. By Lemma 5.4.b, the projection $p_\beta$ commutes with any reflexion $s = s_{e,e^*} \in \mathrm{IR}(E)$. Thus, if $H_\beta \subset \ker e^*$, then also $\hat{H}_\beta \subset \ker e^*$. Therefore, $s(y) = y$ for all $y \in \hat{H}_\beta$, $p_\beta s(y) = p_\beta(y)$ and $s(1_E - p_\beta)(y) = (1_E - p_\beta)(y)$. The latter means that $(1_E - p_\beta)(y) \in \ker e^*$. Hence, $(1_E - p_\beta)(\hat{H}_\beta) \subset \hat{H}_\beta$.

Now, we have $(1_E - p_\beta)(x) \in \ker e^*$ for any $e^*$ such that $s_{e,e^*} \in \mathrm{IR}(E)$. This contradicts to the assumption that the system $\mathrm{IR}(E)$ is total, since $(1_E - p_\beta)(x) \neq 0$.

Put $R_0 = \text{Span}(\bigcup_{\beta \in \mathcal{B}} H_\beta)$ and $\hat{R} = \hat{R}_0$.

5.6. Proposition. a) The subspace $R_0 + F$ is closed, and if $p_{R_0,F} : R_0 + F \to R_0$ is the first projection, then $1_{R_0 + F} - 2p_{R_0,F} \in \text{Iso}(R_0 + F)$. Therefore, the projection $p_{R_0,F}$ is bicontractive.

b) For any $\alpha \in \mathcal{A}$ there exists a projection $p_\alpha : F_0 + \hat{R} \to V_\alpha$ such that

i) $p_\alpha$ commutes with any reflexion $s \in \mathrm{IR}(E)$ and $p_\alpha p_\alpha = p_\alpha = 0$ for all $\alpha' \in \mathcal{A}$, $\alpha' \neq \alpha$, $\beta \in \mathcal{B}$;

ii) $\|p_\alpha\|_{F_0 + \hat{R}} \leq 2$ and $\|1_{F_0 + \hat{R}} - p_\alpha\|_{F_0 + \hat{R}} = 1$, if the latter projection is non-zero;

iii) moreover, if the Coxeter group $W_\alpha$ is a group of type $B_\Delta$, then $1_{F_0 + \hat{R}} - 2p_\alpha \in \text{Iso}(F_0 + \hat{R})$, and so $\|p_\alpha\|_{F_0 + \hat{R}} = 1$, too.

c) The subspace $F_0 + \hat{R}$ is closed, and if both subspaces $F_0$ and $\hat{R}$ are non-trivial and $p_{F_0,\hat{R}} : F_0 + \hat{R} \to F_0$ is the first projection, then $\|1_{F_0 + \hat{R}} - p_{F_0,\hat{R}}\|_{F_0 + \hat{R}} = 1$ and $\|p_{F_0,\hat{R}}\|_{F_0 + \hat{R}} \leq 2$.

Proof. a. Let $x = x_1 + x_2$, where $x_1 \in R_0$ and $x_2 \in F$. For any $\epsilon > 0$ there exists a finite subset $\sigma \subset \mathcal{B}$ and a vector $x_1^\sigma \in \bigoplus_{\beta \in \sigma} H_\beta$ such that $\|x_1 - x_1^\sigma\|_E < \epsilon$. Since $u_\sigma = \prod_{\beta \in \sigma}(1_E - 2p_\beta) \in \text{Iso}E$ and $u_\sigma(x_1^\sigma) = -x_1^\sigma$, $u_\sigma(x_2) = x_2$, we have $\|x_1^\sigma + x_2\|_E = \|x_1 + x_2\|_E$. Thus, if $R_0 \neq \{0\}$ and $F \neq \{0\}$, then $\|1_{R_0 + F} - 2p_{R_0,F}\|_{R_0 + F} = 1$, and therefore also $\|p_{R_0,F}\|_{R_0 + F} = \|1_{R_0 + F} - p_{R_0,F}\|_{R_0 + F} = 1$, if both subspaces $R_0$ and $F$ are non-trivial. By the closed graph theorem, this implies that the subspace $R_0 + F$ is closed.

b. If $\dim V_\alpha < \infty$, put $p'_\alpha = (1/\text{card}W_\alpha) \sum_{g \in W_\alpha} g$. Then $p'_\alpha$ is a projection on the fixed point subspace $F_\alpha$ of the group $W_\alpha$, which coincides with $\cap_{e,e^*} \ker e^*$, and
thus it is a complementary subspace to \( V_\alpha = \text{Ker} p'_\alpha \). It is clear that \( ||p'_\alpha||_E = 1 \), and so \( ||1_E - p'_\alpha||_E \leq 2 \). From Lemma 5.4 and the definition of \( p'_\alpha \) it follows that the projection \( p_\alpha = (1_E - p'_\alpha) | (R_0 + F) \) satisfies (i); by the above inequalities, it also satisfies (ii).

Next consider the case when \( \text{dim} V_\alpha = \infty \) and the Coxeter group \( W_\alpha \) is of type \( A_\Delta \). For a finite subset \( \sigma \subset \Delta \) denote by \( V_\sigma \) the subspace generated by the root vectors \( \epsilon_\delta - \epsilon_{\delta'} \), where \( \delta, \delta' \in \sigma, \delta \neq \delta' \), and by \( W_\sigma \) the Coxeter group of type \( A_n \), where \( n = \text{dim} V_\sigma \), generated by the isometric reflexions along these vectors. Define the projections \( p'_\sigma \) resp. \( p_\sigma \) in the same way as \( p'_\alpha \) resp. \( p_\alpha \) above. It is clear that \( p_\sigma \) commutes with any reflexion \( s \in \text{IR}(E) \setminus \text{IR}_\alpha \) and satisfies all the other properties in (i), (ii). It is easily seen that the net \( (p_\sigma) \) is strongly convergent to the identity on the subspace \( V_\alpha \), and that all the projections \( p_\sigma \) vanish on the subspace \( \hat{R} + V'_\alpha \) where \( V'_\alpha = \text{span}(\bigcup_{\alpha' \in A_\setminus \{\alpha\}} V_{\alpha'}) \). As in (a) above it follows that \( F_0 = V_\alpha + V'_\alpha \). Therefore, this net is strongly convergent on the subspace \( F_0 + \hat{R} \) to the projection \( p_\alpha \) which has the properties (i) and (ii).

Finally, assume that \( W_\alpha \) is a Coxeter group of type \( B_\Delta \). Then for any finite subset \( \sigma \subset \Delta \) the product \( u_\sigma = \prod_{\delta \in \sigma} s_\delta \) of pairwise commuting reflexions \( s_\delta = s_{\epsilon_\delta - \epsilon_{\delta'}} \in \text{IR}_\alpha \) is an isometric involution with the fixed point subspace \( \cap_{\delta \in \sigma} \text{Ker} \epsilon_\delta^* \supset \hat{R} + V'_\alpha \). Similarly as above, the net of the restrictions \( \left( u_\sigma | (F_0 + \hat{R}) \right) \) is strongly convergent to an isometric involution \( u_\alpha \) which has \( V_\alpha \) and \( \hat{R} + V'_\alpha \) as its spectral subspaces. It is easily seen that the projection \( p_\alpha = (1_{F_0 + \hat{R}} + u_\alpha)/2 \) possesses all the properties mentioned in (i), (ii) and (iii).

c. By the closed graph theorem it is enough to check the second statement. For a finite subset \( \sigma \subset \text{IR}_A \) let \( W_\sigma \) be a finite group generated by reflexions from \( \sigma \), and let \( V_\sigma \) be the linear span of the reflexion vectors of these reflexions. Then the action of \( W_\sigma \) in \( V_\sigma \) is fixed point free, and so the projection \( p'_\sigma = (1/\text{card} W_\sigma) \sum_{g \in W_\sigma} g \) onto the fixed point subspace \( F_\sigma \supset \hat{R} \) of \( W_\sigma \) vanishes on \( V_\sigma \). Consider the net of finite dimensional projections \( (p_\sigma = 1_{F_0 + \hat{R}} - p'_\sigma | (F_0 + \hat{R})) \) onto the subspaces \( V_\sigma \). Observe that \( \bigcup_\sigma V_\sigma \) is dense in the subspace \( F_0 \). Since all of \( p_\sigma \) vanish on \( \hat{R} \) and satisfy the norm inequalities of (ii), this net is strongly convergent to the projection \( p_{F_0, \hat{R}} \) which also satisfies these inequalities. This completes the proof. \( \Box \)

Remark. For further information on the Hilbert decomposition, see Proposition 6.2 and examples 6.8 below.

5.7. Theorem. The subspaces \( F, F_0, \hat{R} \) and \( R_0 \) are invariant with respect to the group \( \text{Iso} E \), and there are natural monomorphisms \( \text{Iso} E \hookrightarrow \text{Iso} R_0 \times \text{Iso} F_0 \), \( G_0(E) \hookrightarrow \prod_{\beta \in B} \text{G}_0(H_\beta) \) and \( \prod_{\beta \in B} \text{O}(H_\beta) \hookrightarrow \text{Iso} (F + R_0) \).

Proof. The invariance of the subspaces \( F \) and \( F_0 \) was already established in 5.2; the
invariance of $R_0$ follows from the remark in 5.3. Similar arguments applied to the conjugate action of $\text{Iso } E$ on $E^*$ provide the invariance of $\hat{R}$.

Since the set $\{ e \in S(E) \mid s_{e,e^*} \in IR(E) \}$ is contained in $F_0 \cup ( \bigcup_{\beta \in B} H_{\beta} ) \subset \hat{R} \uparrow F_0$, the latter summands being invariant, it follows from Lemma 4.1 that the restriction mappings $g \mapsto g \upharpoonright R_0$, $g \mapsto g \upharpoonright F_0$, $g \mapsto g \upharpoonright H_{\beta}$ induce the monomorphisms $\text{Iso } E \hookrightarrow \text{Iso } R_0 \times \text{Iso } F_0$ and $G_0(E) \hookrightarrow \prod_{\beta \in B} G_0(H_{\beta})$.

As for the last statement, fix arbitrary $g = \prod_{\beta \in B} \bar{u}_{\beta} \in \prod_{\beta \in B} O(H_{\beta})$. For any finite subset $\sigma \subset B$ put $u_{\sigma} = \prod_{\beta \in \sigma} u_{\beta}$, where $u_{\beta} = \bar{u}_{\beta} p_{\beta} + (1_E - p_{\beta}) \in \text{Iso } E$ (see Theorem 1.b). We will show that the net $\{ u_{\sigma} \mid (F + R_0) \} \subset \text{Iso } (F + R_0)$ strongly converges to an element $u \in \text{Iso } (F + R_0)$ such that $u \upharpoonright H_{\beta} = \bar{u}_{\beta}$. Therefore, the correspondence $\prod_{\beta \in B} O(H_{\beta}) \ni g \mapsto u \in \text{Iso } (F + R_0)$ yields the desired monomorphism.

By the Banach - Steinhaus Theorem, it is enough to show that for any $x \in F + R_0$ the generalized sequence $(u_{\sigma}(x))$ is convergent. Let $x = x_1 + x_2$, where $x_1 \in F$ and $x_2 \in R_0$. For any $\epsilon > 0$ there exists a finite subset $\sigma \subset B$ such that $||(1_E - \sum_{\beta \in \sigma} p_{\beta})(x_2)||_E < \epsilon / 2$. If $\sigma'$ and $\sigma''$ are two finite subsets of $B$ containing $\sigma$, then $u_{\sigma'} - u_{\sigma''} = (u_{\sigma'} - u_{\sigma''})(1_E - \sum_{\beta \in \sigma} p_{\beta})$, and so

$$||(u_{\sigma'} - u_{\sigma''})(x_2)||_E \leq ||u_{\sigma'}(1_E - \sum_{\beta \in \sigma} p_{\beta})(x_2)||_E + ||u_{\sigma''}(1_E - \sum_{\beta \in \sigma} p_{\beta})(x_2)||_E < \epsilon .$$

Thus, $(u_{\sigma}(x))$ is a generalized Cauchy sequence, and hence it is convergent. This proves the theorem. ◯

Remark. In general, the monomorphisms in Theorem 5.7 are not surjective; see Example 6.8.2 below.

6 An application: Isometry groups of ideal generalized sequence spaces

6.1. Definitions. Recall the following notions (see e.g. [17], [19]). Let $(e_\alpha)_{\alpha \in \Delta}$ be a system of vectors in a Banach space $E_0$. It is called a generalized Schauder basis of $E_0$ if each vector $e \in E_0$ has a unique, up to permutations, decomposition $e = \sum_{i=1}^{\infty} a_i e_{\alpha_i}$, where $(\alpha_i)_{i=1,...}$ is a sequence of pairwise distinct indices from $\Delta$. If this series is still convergent to $e$ after any permutation of its members, then this basis is called unconditional. In this case for any choices of signes $\theta = (\theta_\alpha)_{\alpha \in \Delta}$, where $\theta_\alpha = \pm 1$, the linear operators $M_\theta(e) = \sum_{i=1}^{\infty} \theta_{\alpha_i} a_i e_{\alpha_i}$ are uniformly bounded. The
number \( \sup_{\theta} \| M_{\theta} \|_{E_0} \) is called the unconditional constant of the basis \((e_\alpha)_{\alpha \in \Delta}\). For instance, any complete orthonormal system in a Hilbert space is an unconditional basis with the unconditional constant 1. If the index set \( \Delta \) is countable, we have the usual notion of an unconditional basis.

The generalized unconditional basis \((e_\alpha)_{\alpha \in \Delta}\) is called symmetric if for any bijection \( \pi : \Delta \to \Delta \) the linear operator

\[
\pi^* : E_0 \ni e = \sum_{i=1}^{\infty} a_i e_{\alpha_i} \mapsto \sum_{i=1}^{\infty} a_i e_{\pi(\alpha_i)} = \pi^*(e) \in E_0
\]

is bounded, and so the infinite symmetric group \( S_\Delta = \text{Biject}(\Delta) \) acts in \( E_0 \), being uniformly bounded there. The constant \( \sup_{\theta,\pi} \| M_{\theta} \pi^* \|_{E_0} \) is called the symmetric constant of the basis \((e_\alpha)_{\alpha \in \Delta}\).

For instance, in the classical Banach space \( c_0(\Delta) \) of generalized sequences convergent to zero, with \( \Delta \) as a set of indices, the system of the standard basis vectors \((\epsilon_{\delta})_{\delta \in \Delta}\) form a symmetric basis with the symmetric constant 1 (remark that each vector in \( c_0(\Delta) \) has a countable support). Fixing a generalized unconditional basis \((e_\alpha)_{\alpha \in \Delta}\) in \( E_0 \) we obtain a representation of \( E_0 \) as a generalized sequence space contained in \( c_0(\Delta) \). If the unconditional constant of this basis is 1, then \( E_0 \) is an ideal Banach lattice.

Recall that an ideal generalized sequence space \( E \) is a Banach space of sequences defined on an index set \( \Delta \) such that if \( x = (x_\alpha)_{\alpha \in \Delta} \in E \), then for any sequence \( y = (y_\alpha)_{\alpha \in \Delta} \) with \( |y_\alpha| \leq |x_\alpha| \) for all \( \alpha \in \Delta \) one has \( y \in E \) and \( \| y \|_E \leq \| x \|_E \). It is called a symmetric generalized sequence space if \( E \) is an ideal generalized sequence space, where the symmetric group \( S_\Delta \) of all bijections of \( \Delta \) acts isometrically.

The next simple lemma should be well known; by the lack of references we give a proof. We say that a family of reflexions is orthogonal if the reflexions from the family pairwise commute.

**6.2. Lemma.** Let \( E \) be a Banach space with a total orthogonal family of isometric reflexions \((s_\delta = s_{\epsilon_\delta, \epsilon_\delta^*})_{\delta \in \Delta}\). Identify \( E \) with a generalized sequence space with the index set \( \Delta \) by posing \( \bar{x} = (\epsilon_\delta^*(x))_{\delta \in \Delta} \) for \( x \in E \). Let \( E_0 = \text{span} (\epsilon_\delta | \delta \in \Delta) \). Then we have

a) The system \((\epsilon_\delta)_{\delta \in \Delta}\) is a generalized unconditional basis in \( E_0 \) with the unconditional constant 1, and so \( E_0 \) is an ideal generalized sequence space.

b) If the Coxeter group \( B_\Delta \) of permutations and sign changes of finite number of coordinates acts isometrically in \( E_0 \), then \((\epsilon_\delta)_{\delta \in \Delta}\) is a symmetric basis in \( E_0 \) with the symmetric constant 1, and so \( E_0 \) is a symmetric generalized sequence space.
Proof. \( a \). Let \( \sigma \) be a finite subset of \( \Delta \). Consider the coordinate subspace

\[
E_\sigma = \{ x = (x_\delta)_{\delta \in \Delta} \in E_0 \mid x_\delta = 0 \text{ for all } \delta \notin \sigma \}.
\]

Let \( p_\sigma = \frac{1}{2}(1_{E_0} - u_\sigma) \), where \( u_\sigma = \prod_{\delta \in \sigma} s_\delta \), be the coordinate projection \( E_0 \to E_\sigma \). Since \( u_\sigma \in \text{Iso } E_0 \), we have \( ||p_\sigma||_{E_0} = ||1_{E_0} - p_\sigma||_{E_0} = 1 \).

Fix an arbitrary vector \( x \in E_0 \). For any \( n \in \mathbb{N} \) there exists a finite subset \( \sigma_n \) of \( \Delta \) and \( y_n \in E_{\sigma_n} \) such that \( ||x - y_n||_E < 1/n \). Then also \( ||p_{\sigma_n}(x) - y||_{E_0} < 1/n \), and so \( ||(1_{E_0} - p_{\sigma_n})(x)||_{E_0} < 2/n \). It follows that \( x \) has at most countable support contained in \( \Omega = \bigcup_{i=1}^\infty \sigma_i = \{\delta_1, \ldots, \delta_k, \ldots\} \), and \( ||x - \sum_{i=1}^k x_i \epsilon_\delta_i||_{E_0} \to 0 \). Thus the system \( (\epsilon_\delta)_{\delta \in \Delta} \) is a generalized Schauder basis in \( E_0 \). It is easily seen that for any fixed subset \( \Omega \subset \Delta \) the net of isometric involutions \( (u_{\sigma} \mid \sigma \subset \Omega, \text{card } \sigma < \infty) \) strongly converges on \( E_0 \) to the isometric involution \( u_\Omega \), and therefore the basis \( (\epsilon_\delta)_{\delta \in \Delta} \) of \( E_0 \) is unconditional with the unconditional constant 1.

\( b \). Fix a permutation \( \pi \in S_\Delta \), a vector \( x \in E_0 \) and \( \epsilon > 0 \) arbitrarily. Let \( \sigma \) be a finite subset of \( \Delta \) such that \( ||(1_{E_0} - p_\sigma)(x)||_{E_0} < \epsilon \). There exists a finite permutation \( \pi' \in S_\Delta \) such that \( \pi'|\sigma = \pi|\sigma \). Since the Coxeter group \( B_\Delta \) acts isometrically on \( E_0 \), we have

\[
||\pi^* p_\sigma(x)||_{E_0} = ||\pi'^* p_\sigma(x)||_{E_0} = ||p_\sigma(x)||_{E_0}.
\]

Thus, the linear operator \( \pi^* \) is well defined and isometric on the dense subspace \( \mathbf{R}^\Delta \) of \( E_0 \). Therefore, it can be extended isometrically onto \( E_0 \), and since the same is true for \( (\pi^{-1})^* \), this extension does belong to the group \( \text{Iso } E_0 \). This proves (b).

Remark. It is not true, in general, that under the assumption of this lemma \( E \) itself should be an ideal space if all single sign changes are isometries of \( E \). As an example, consider the space \( c \) of convergent sequences, which is not an ideal lattice.

We return to the Hilbert decomposition, keeping all the notation and the conventions of Section 5.

6.3. Proposition. There exists an ideal generalized sequence space \( X \) with \( \mathcal{B} \) as an index set such that the subspace \( R_0 \) of \( E \) is isometric to the Banach sum \( (\bigoplus_{\beta \in \mathcal{B}} H_\beta)_X \).

Proof. For each \( \beta \in \mathcal{B} \) fix a vector \( e_\beta \in S(H_\beta) \). Consider the subspace \( X = \text{span}(e_\beta \mid \beta \in \mathcal{B}) \subset F + R_0 \). Since the system of functionals \( (e_\beta^* \mid \beta \in \mathcal{B}) \) is biorthogonal to the system \( (e_\beta \mid \beta \in \mathcal{B}) \) and the reflexions \( s_{e_\beta,e_\beta^*} \mid X \) are isometric, by Lemma 5.9.a, the latter system is an unconditional basis in \( X \) with the unconditional constant 1, and so \( X \) can be identified with an ideal generalized sequence space on \( \mathcal{B} \).
Put $R_1 = \bigoplus_{\beta \in B} H_{\beta}$. We will show that the correspondence

$$\tau : R_0 \ni x \mapsto (p_{\beta}(x))_{\beta \in B} \in R_1$$

is a linear isometry of $R_0$ onto $R_1$. Put $e_\beta' = \frac{p_{\beta}(x)}{||p_{\beta}(x)||_E} \in S(H_\beta)$ if $p_{\beta}(x) \neq 0$. Let $u_\beta \in O(H_\beta)$ be such that $u_\beta(e_\beta') = e_\beta$ if $p_{\beta}(x) \neq 0$ and $u_\beta = 1_{H_\beta}$ otherwise. As follows from Theorem 5.7, there exists $u \in \text{Iso} R_0$ such that $u \mid H_\beta = u_\beta$ for all $\beta \in B$. Since $u(x) \in X$ and $u$ is an isometry, it is clear that $\tau(x) \in R_1$ and $||\tau(x)||_{R_1} = ||x||_{R_0}$.

To show that $\tau$ is surjective, fix arbitrary vector $\tilde{x} \in R_1$, $\tilde{x} = (x_\beta \in H_\beta)_{\beta \in B}$. Then $x' = \sum_{\beta \in B} ||x_\beta||_E e_\beta \in X \subset R_0$. For each $\beta \in B$ let $\bar{u}_\beta \in O(H_\beta)$ be such that $\bar{u}_\beta(x_\beta) = ||x_\beta||_E e_\beta$. As above, there exists $u \in \text{Iso} R_0$ such that $u \mid H_\beta = \bar{u}_\beta$ for all $\beta \in B$. If $x_0 = u^{-1}(x')$, then we have $\tau(x_0) = \tilde{x}$. Thus, $\tau$ is an invertible isometry. This completes the proof.

6.4. Notation. Consider again an ideal generalized sequence space $E$ with an index set $\Delta$. Without lost of generality one may assume that $||e_\delta||_E = 1$ for all $\delta \in \Delta$. For a subset $\Omega \subset \Delta$ let $E(\Omega)$ be a strip $E(\Omega) = \{x = (x_\delta) \in E \mid x_\delta = 0 \text{ for all } \delta \in \Delta \setminus \Omega\}$. Any such strip is biorthogonally complemented in $E$; indeed, the operator of multiplication by the characteristic function of $\Omega$ is a bicontractive projection $p_\Omega : E \to E(\Omega)$ with $1_E - p_\Omega = p_{\Delta \setminus \Omega}$.

Put $E_0(\Omega) = \overline{\text{span}}(e_\delta)_{\delta \in \Omega}$, so that $E = E(\Delta)$, $E_0 = E_0(\Delta)$ and $E_0(\Omega) = E_0 \cap E(\Omega)$. We also preserve in this particular case all the other notation introduced in section 5. The next proposition shows that the Hilbert and Coxeter decompositions of an ideal generalized sequence space yield an orthogonal decomposition into strips.

A reflexor vector $e_\delta$ of the single sign change $s_\delta = s_{e_\delta} e_\delta^* \in \text{IR}(E)$ belongs to a certain subspace $V_\alpha$ or $H_\beta$. Putting $\Delta_\alpha = \{\delta \in \Delta \mid s_\delta \in V_\alpha\}$ and $\Delta_\beta = \{\delta \in \Delta \mid s_\delta \in H_\beta\}$ we obtain a disjoint partition of $\Delta$ by the subsets $\{\Delta_\alpha, \Delta_\beta\}_{\alpha \in A, \beta \in B}$. Put also $\Delta_A = \bigcup_{\alpha \in A} \Delta_\alpha$ and $\Delta_B = \bigcup_{\beta \in B} \Delta_\beta$, so that $\Delta = \Delta_A \cup \Delta_B$.

Proposition 6.5. In the notation as above one has

a) i) $V_\alpha = E_0(\Delta_\alpha)$, $\hat{V}_\alpha = E(\Delta_\alpha)$ for each $\alpha \in A$ and
ii) $H_\beta = E(\Delta_\beta) = E_0(\Delta_\beta) = l_2(\Delta_\beta)$ for each $\beta \in B$;
iii) if card $(\Delta_\alpha) = \infty$, then $W_\alpha$ is a Coxeter group of type $B_{\Delta_\alpha}$;

b) i) $F_0 = E_0(\Delta_A)$ and $R_0 = E_0(\Delta_B)$, so that $E_0 = R_0 + F_0$;
ii) $F = E(\Delta_A)$ and $R = E(\Delta_B)$, so that $E = R + F$;

c)
i) $p_{\alpha} = p_{\Delta_{\alpha}} \mid (F_0 + \hat{R})$ and $p_{\beta} = p_{\Delta_{\beta}}$ for all $\alpha \in \mathcal{A}$, $\beta \in \mathcal{B}$;

ii) $p_{R_0,F} = p_{\Delta_{S}} \mid (R_0 + F)$ and $p_{F_0,R} = p_{\Delta_{A}} \mid (F_0 + \hat{R})$ (see Proposition 5.6).

Proof. a. Put $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$ and $V_\gamma = H_\gamma$ for $\gamma \in \mathcal{B}$. By the above definitions, $E_0(\Delta_\gamma) \subset V_\gamma$ for all $\gamma \in \mathcal{C}$. Let $\gamma \in \mathcal{C}$, $\delta \in \Delta_\gamma$ and $\delta' \in \Delta \setminus \Delta_\gamma$. By Lemma 5.4, $s_{\delta'} = s_{\epsilon_{\delta'},\epsilon_{\delta'}} \in \text{IR}(E) \setminus \text{IR}_\gamma$ commutes with any reflexion $s \in \text{IR}_\gamma$, and so $V_\gamma \subset \text{Ker} \epsilon_{\delta'}^*$. Therefore, $\tilde{V}_\gamma \subset \text{Ker} \epsilon_{\delta'}^*$, too, and hence $\tilde{V}_\gamma \subset \bigcap_{\delta' \in \Delta \setminus \Delta_\gamma} \text{Ker} \epsilon_{\delta'}^* = E(\Delta_\gamma)$. In particular, each reflexion vector $e$ of a reflexion $s = s_{\epsilon_{\delta',\epsilon_{\delta'}}} \in \text{IR}(E)$ belongs to one of the strips $E(\Delta_\gamma)$, where $\gamma \in \mathcal{C}$. Namely, $e = (x_\delta) \in E(\Delta_\gamma)$ if $x_\delta \neq 0$ for at least one $\delta \in \Delta_\gamma$. Furthermore, in the latter case either $e = \pm \epsilon_{\delta}$ or the reflexions $s$ and $s_\delta$ do not commute.

Let $\gamma = \alpha \in \mathcal{A}$. From the classification of the infinite Coxeter groups in section 4 it follows that if $W$ is such a group and $s \in W$, then the set of all reflexions in $W$ that do not commute with $s$ contains not more than a finite subset of pairwise commuting reflexions. This means that the reflexion vector $e$ of any given reflexion $s \in \text{IR}_\alpha$ has only a finite number of non-zero coordinates, i.e. $e \in \text{span}(\epsilon_{\delta} \mid \delta \in \Delta_\alpha) \subset E_0(\Delta_\alpha)$. Thus, $V_\alpha \subset E_0(\Delta_\alpha)$, and therefore, $V_\alpha = E_0(\Delta_\alpha)$, which is the first statement of (a.i). In particular, the reflexion vectors $(\epsilon_{\delta})_{\delta \in \Delta_\alpha}$ of sign change reflexions $(s_{\delta})_{\delta \in \Delta_\alpha} \subset \text{IR}_\alpha$ form a complete orthogonal system in $V_\alpha$.

If $\text{card}(\Delta_\alpha) < \infty$, then clearly $E(\Delta_\alpha) = E_0(\Delta_\alpha) = V_\alpha = \tilde{V}_\alpha$. If $\text{card}(\Delta_\alpha) = \infty$, then by Corollary 4.6, the group $W_\alpha$ generated by reflexions from $\text{IR}_\alpha$ is a Coxeter group of type $A_\Delta$ or $B_\Delta$. But the Coxeter group $A_\Delta$ does not contain a complete set of pairwise commuting reflexions, i.e. there is no orthogonal subsystem of the root system $(\epsilon_{\delta} - \epsilon_{\delta'} \mid \delta,\delta' \in \Delta', \delta \neq \delta')$ which would be a Hamel basis of $\mathbf{R}_0^{\Delta'}$. This excludes the first case, and so the group $W_\alpha$ should be a Coxeter group of type $B_\Delta$. It is clear that $\text{card}(\Delta') = \text{card}(\Delta_\alpha)$. This proves (a.iii).

Let, further, $\gamma = \beta \in \mathcal{B}$. Then $E_0(\Delta_\beta)$ is a subspace of the Hilbert space $H_\beta$, and the system $(\epsilon_{\delta})_{\delta \in \Delta_\beta}$ is an orthonormal basis of $E_0(\Delta_\beta)$. Thus, $E_0(\Delta_\beta) = l_2(\Delta_\beta)$. Assume that $H_\beta \neq E_0(\Delta_\beta)$. Let $x \in H_\beta$ be a non-zero vector orthogonal to $E_0(\Delta_\beta)$.

It is easily seen that $\epsilon_{\delta}^*(x) = 0$ for all $\delta \in \Delta_\beta$. This is impossible, since $H_\beta \subset E(\Delta_\beta)$ and $x \neq 0$. Therefore, $H_\beta = E_0(\Delta_\beta) = l_2(\Delta_\beta)$.

Let $p_\beta : E \to H_\beta$ be the projection as in Theorem 1.b. Suppose that $H_\beta \neq E(\Delta_\beta)$. Then the restriction $p_\beta \mid E(\Delta_\beta)$ is a non-identical projection, so that there exists a non-zero vector $x \in \text{Ker} p_\beta \cap E(\Delta_\beta)$. Fixing $\delta \in \Delta_\beta$, consider the plane $L = \text{span}(x,\epsilon_{\delta})$. There are two commuting isometric reflexions in $L$, namely $(1_L - 2p_\beta) \mid L$ and $s_\delta \mid L$. Therefore, $x \in \text{Ker} \epsilon_{\delta}^*$ for all $\delta \in \Delta_\beta$, and so $x = 0$, which is a contradiction. Hence, $H_\beta = E(\Delta_\beta) = E_0(\Delta_\beta) = l_2(\Delta_\beta)$. This proves (a), besides the second equality in (a.i), which is proven below.

b. For any $\gamma \in \mathcal{C}$ consider the isometric involution $u_\gamma = 1_E - 2p_{\Delta_\gamma}$ with the spectral
subspaces $E(\Delta_\gamma)$ and $E(\Delta \setminus \Delta_\gamma)$. It is easily seen that for any $s = s_{e,e^*} \in IR(E)$ the isometries $su_\gamma$ and $u_\gamma s$ coincide on the total system of reflexion vectors $(s_\delta)_{\delta \in \Delta}$. From Lemma 4.1 it follows that they coincide on $E$. Thus, the involution $u_\gamma$ commutes with each reflexion $s_{e,e^*} \in IR(E)$. Therefore, one of its spectral subspaces contains the vector $e$ and another one is contained in the mirror hyperplane $\text{Ker } e^*$. Hence, for any $s_{e,e^*} \in IR_\gamma$, one has $\text{Ker } e^* \supset E(\Delta \setminus \Delta_\gamma)$.

Let the set $C$ be divided into two disjoint parts $C = C' \cup C''$. Put $\Omega' = \bigcup_{\gamma \in C'} \Delta_\gamma$, $\Omega'' = \bigcup_{\gamma \in C''} \Delta_\gamma$, so that $\Omega', \Omega''$ consist of some parts of the disjoint partition $\Delta = \bigcup_{\gamma \in C} \Delta_\gamma$. We are going to show, more generally, that $E_0(\Omega') = E(\Omega')$, which easily implies the equalities in (b.ii) and (a.i).

By the considerations above, we have $E(\Omega') \subset \bigcap(\text{Ker } e^* | s_{e,e^*} \in IR_{C''})$, where $IR_{C''} = \bigcup_{\gamma \in C''} IR_\gamma$. On the other hand, $E(\Omega') = \bigcap_{\delta \in \Omega''} \text{Ker } e^*_\delta \supset \bigcap(\text{Ker } e^* | s_{e,e^*} \in IR_{C''})$. Therefore, $E(\Omega') = \bigcap(\text{Ker } e^* | s_{e,e^*} \in IR_{C''}) = E_0(\Omega')$. The last equality is clear from the definition of the envelope, because $s_{e,e^*} \in IR_{C''}$ iff $E_0(\Omega') \subset \text{Ker } e^*$. This proves (b) and the second equality in (a.i).

(c). The isometric involutions $u_\beta = 1_E - 2p_{\Delta_\beta}$ and $1_E - 2p_\beta$ coincide on vectors of the system $(s_\delta)_{\delta \in \Delta}$, so by Lemma 4.1, they coincide on $E$. This proves the second equality in (c.i). By the same reasoning (see Proposition 5.6.b.iii) the first equality in (c.i) holds. The equalities (c.ii) follow from (b), just by the definition of the projections involved. By Proposition 5.6.b.i, the projection $p_\alpha$ commutes with any sign change reflexion $s_\delta$. By the same type of arguments as those used in the proof of (b), it follows that $\text{Ker } p_\alpha = \text{Ker } (p_{\Delta_\alpha} | (F_0 + R))$. Since the images also coincide, we have the first equality in (c.i). This proves the proposition.

This proposition, together with Theorem 5.7 and the remark that the union of the subspaces $H_\beta, \beta \in \mathcal{B}$, is invariant with respect to the group $\text{Iso } E$, leads to the following

**6.6. Corollary.** a) $$G_0(E) \subset \bigoplus_{\beta \in \mathcal{B}} G_0(l_2(\Delta_\beta)) \quad \text{and} \quad \text{Iso } E \subset \text{Iso } E(\Delta_A) \bigoplus \text{Iso } E(\Delta_B);$$

b) each element of the group $(\text{Iso } E) | E(\Delta_B)$ is of the form $(x_\beta) \mapsto (u_\beta(x_\beta))$, where $x_\beta \in l_2(\Delta_\beta)$, $u_\beta : l_2(\Delta_\beta) \to l_2(\Delta_{\pi(\beta)})$ is an isometry of Hilbert spaces for each $\beta \in \mathcal{B}$, and $\pi$ is a permutation of the set $\mathcal{B}$.

**6.7. Remark.** Let $\alpha \in \mathcal{A}$ be such that $\text{card } \Delta_\alpha = \infty$. Then $\text{Iso } V_\alpha$ contains a Coxeter subgroup of type $B_{\Delta_\alpha}$. It is not true in general that it contains also the symmetric
group $S_{\Delta_\alpha}$ of shift operators. In fact, this latter group is contained in $\text{Iso} \ V_\alpha$, but probably in some other representation of $V_\alpha$ as an ideal generalized sequence space. Indeed, consider any symmetric generalized sequence space $M$ on $\Delta_\alpha$, such that the system $(\epsilon_\delta)_{\delta \in \Delta}$ is a Schauder basis in $M$. Fix a disjoint partition of $\Delta_\alpha$ into pairs $(\delta, \delta')$. Then by Lemma 6.2, the corresponding subsystem $(\epsilon_\delta \pm \epsilon_{\delta'})$ of the root system is an unconditional Schauder basis in $M$ with the unconditional constant 1, and this basis is not symmetric. Thus, using the dual system of functionals, one can represent the strip component $M = V_\alpha$ as an ideal generalized sequence space which is not symmetric and such that the isometry group does not act as permutations and sign changes (the image of a basis vector under an isometric reflexion might be a vector with 4 non-zero coordinates!). Recall that a symmetric basis is unique; moreover, a basis which in a sense is symmetric enough, is unique [11]. Thus, here we have an unconditional basis with a relatively small group of symmetries.

Conversely, if $g \in \text{Iso} E$ is such that $g(V_\alpha) = V_{\alpha'}$, where $\alpha \in A$ is as above, then one can represent $V_\alpha$ resp. $V_{\alpha'}$ as a symmetric generalized sequence space on $\Delta_\alpha$ resp. $\Delta_{\alpha'}$, and then $g$ should be an operator of the form $(x_\alpha) \mapsto (\pm x_{\pi(\alpha)})$, where $\pi : \Delta_\alpha \to \Delta_{\alpha'}$ is a bijection (indeed, $\pi$ must transfer the sign change reflexions from $\text{IR}_\alpha$ into sign change reflexions from $\text{IR}_{\alpha'}$).

In [24, 25] certain conditions on an ideal generalized sequence space are given which guarantee that its isometry group acts by permutations and sign changes. This is always the case in a symmetric sequence spaces different from $l_2$ [23, ch. IX], [6] (see also [2], [8] for the complex field).

Next we give several examples related to the results of Sect. 5 and 6.

6.8. Examples.

1) ([24], [25]) Fix a sequence of real numbers $p_k \geq 1$, $k = 1, \ldots$. The Orlicz–Nakano space $E = l_\{\{p_k\}\}$ consists of all sequences of real numbers $x = (\xi_k)_{k=1}^\infty$ such that the following norm is finite:

$$||x||_E = \inf \{\lambda > 0 | \sum_{k=1}^\infty |\xi_k/\lambda|^{p_k} \leq 1\}.$$ 

It is an ideal sequence space.

Put $\Delta_q = \{i \in \mathbb{N} | p_i = q\}$, where $q = q_1,q_2,\ldots$ are pairwise distinct. Then $A = \{q_i | i \neq 2\}$ and $\Delta_A = \{i | p_i \neq 2\}$, $B = \{2\}$ if $\Delta_2 \neq \emptyset$ and $B = \emptyset$ otherwise; $E(\Delta_q) = l_q(\Delta_q)$. The group $\text{Iso} E$ is the direct product of the groups $O(l_2(\Delta_2))$ and $\text{Iso} \Delta_A$, where $\text{Iso} \Delta_A$ is the group of all permutations of coordinates $\xi_i$, $i \in \Delta_A$, preserving the partition $\Delta_A = \bigcup \Delta_q$, and arbitrary sign changes of these coordinates. Indeed, this direct product evidently is a subgroup of $\text{Iso} E$; the converse inclusion follows from the results of [24], [25] in view of the decomposition from Corollary 6.6.
In a similar way one can describe the isometry groups of more general modular sequence spaces or of Banach sums of (symmetric) ideal sequence spaces.

2) Let $E$ be the space of all convergent complex sequences with the supremum norm. Then $E$ is a Banach sum of the real euclidean planes $H_i$, $i = 1, \ldots$. We have that $E = \hat{R}$, and $E_0 = R_0$ is the subspace of sequences in $E$ convergent to zero. The group $\text{Iso } E_0$ is the semi-direct product of $O(2)^\omega$ and the infinite symmetric group $S_\omega$, while $\text{Iso } E$ is its proper subgroup (indeed, if $g \in \text{Iso } E$, then the corresponding sequence of orthogonal plane transformations from $O(2)^\omega$ is convergent). This shows that all the inclusions in Theorem 5.7 are strict. Observe that here $E$ is not an ideal sequence space.

3) Consider $E = R^4$ with the norm

$$||x||_E = ||(\xi_1, \xi_2, \eta_1, \eta_2)||_E = [(\xi_1^2 + \xi_2^2)^{1/2} + |\eta_1|^2 + |\eta_2|^2]^{1/2}.$$ 

It is easily seen that here $\hat{R} = R_0 = H = \{x \in E \mid \xi_1 = \xi_2 = 0\}$ and $F = F_0 = \{x \in E \mid \xi_1 = \xi_2 = 0\}$. Furthermore, $E = F_0 \oplus R_0$ is an ideal space, and both strips $F_0$ and $R_0$ are euclidean planes. Thus, $G_0(E) \neq G_0(F_0) \oplus G_0(R_0)$, and so $\text{Iso } E \neq \text{Iso } F_0 \oplus \text{Iso } R_0$ (cf. Corollary 6.6.a).

4) Slightly modifying example 4, consider $E \cong R^5$ with the norm

$$||x||_E = ||(\xi_1, \xi_2, \eta_1, \eta_2, \varsigma_1 )||_E = [(\xi_1^2 + \xi_2^2)^{1/2} + |\eta_1|^2 + |\eta_2|^2 + |\varsigma_1|^2]^{1/2}.$$ 

Being the direct sum of two euclidean planes $H_1$ and $H_2$, which are strips invariant under $G_0(E)$, the subspace $R_0$ itself is euclidean. Thus, $G_0(R_0) \neq G_0(H_1) \oplus G_0(H_2) = G_0(E)$ (cf. Corollary 6.6.a).

5) Let, further, $\tilde{E} = c \oplus c \oplus c$ with the norm $||x,y,z||_E = \sup_{i=1,\ldots} \{(\xi_i^2 + \eta_i^2)^{1/2} + |\varsigma_i|\}$, where $x = (\xi_i)_{i=1}^\infty \in c$, $y = (\eta_i)_{i=1}^\infty \in c$, $z = (\varsigma_i)_{i=1}^\infty \in c$. Consider the hyperplane $E = \{(x,y,z) \in \tilde{E} \mid \lim_{i \to \infty} \eta_i = \lim_{i \to \infty} \varsigma_i\}$. Here we have $\hat{R} \approx c \oplus c_0$, $F \approx c_0$. Thus, $\hat{R} + F$ is a hyperplane in $E$, and there is no contractive projection of $E$ onto $\hat{R}$ and onto $F$, in contrary to the case of ideal sequence spaces (cf. Propositions 5.6 and 6.5.b, c).

The following questions are directly related to the subject of this paper.

For a given Banach space $E$, consider the constant

$$c(E) = \inf_{e \in E, e^* \in E^*, e^*(e) = 1} \{||s_{e,e^*}||_E\}.$$ 

It is clear that $1 \leq c(E) \leq 3$, and $c(E) = 1$ in the case when there exists an isometric reflexion is $E$. It is easily seen that $c(L_p)$ is a convex function of $p$ which takes the
value 1 only for \( p = 2 \) and the value 3 only for \( p = 1 \) and \( p = \infty \). For any given finite set of reflexions in \( E \) one can find an equivalent norm \( \| \cdot \|' \) on \( E \) in such a way that the group generated by these reflexions will be a subgroup of the isometry group of the new norm. In particular, \( c(E, \| \cdot \|') = 1 \).

Consider, further, the constant

\[
\varsigma(E) = \sup_{\| \cdot \|' \sim \| \cdot \| \in E} \{ c(E, \| \cdot \|') \}.
\]

By the definition, \( \varsigma(E) \in [1, 3] \) is a numerical invariant of isomorphism. Is it nontrivial?

Let \( \varsigma(n) = \varsigma(\mathbb{R}^n) = \varsigma(l_2^n) \). Let \( M_n \) be the Minkowski compact of classes of isometric norms in \( \mathbb{R}^n \), endowed with the Banach–Mazur distance. Denote by \( A_n \) the subset of \( M_n \) which consists of the classes of norms having an isometric reflexion (or, the same, a hyperplane of symmetry). It is easy to show that \( \log \varsigma(n) \) coincides with the radius of the metric factor space \( M_n/A_n \) with respect to the distinguish point which corresponds to \( A_n \). It is known that the radius of the \( M_2 \) centred at the class of euclidean norms is \( \log \sqrt{2} \) (F. Behrend, 1937; see [12, sect. 7] for this and for some further information). Thus, \( \varsigma(2) \leq \sqrt{2} \).

6.9. Problem. Is it true that

\[
\varsigma(3) < 3 ?
\]
\[
\varsigma(n) < 3 \text{ for any } n ?
\]
\[
\limsup_{n \to \infty} \varsigma(n) < 3 ?
\]
\[
\varsigma(l_2) < 3 ?
\]

If the answer to any of the above questions is “yes”, which seems to be less plausible, then, of course, the exact value of the corresponding constant \( \varsigma \) would be worthwhile to find.

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