Large covariance matrix estimation
via penalized log-det heuristics

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Abstract

This paper provides a comprehensive estimation framework for large covariance matrices via a log-det heuristics augmented by a nuclear norm plus $l_1$ norm penalty. We develop the model framework, which includes high-dimensional approximate factor models with a sparse residual covariance. The underlying assumptions allow for non-pervasive latent eigenvalues and a prominent residual covariance pattern. We prove that the aforementioned log-det heuristics is locally convex with a Lipschitz-continuous gradient, so that a proximal gradient algorithm may be stated to numerically solve the problem while controlling the threshold parameters. The proposed optimization strategy recovers with high probability both the covariance matrix components and the latent rank and the residual sparsity pattern, and performs systematically not worse than the corresponding estimators employing Frobenius loss in place of the log-det heuristics. The error bounds for the ensuing low rank and sparse covariance matrix estimators are established, and the identifiability condition for the latent geometric manifolds is provided. The validity of outlined results is highlighted by means of an exhaustive simulation study and a real financial data example involving euro zone banks.

Keywords: covariance matrix; log-det heuristics; local convexity; nuclear norm; high dimension
1 Introduction

Estimating high-dimensional covariance or precision matrices has become a crucial task nowadays, due to the increasing availability of datasets composed of a large number of variables $p$ compared to the sample size $n$ in many fields, like economics, finance, biology, genetics, health, climatology, and social sciences. The consistency of estimated covariance matrices is a prerequisite to perform several statistical procedures in high dimensions like principal component analysis, cluster analysis, graphical model inference, among others. Recent books on this relevant topic are Pourahmadi (2013) and Zagidullina (2021), while recent comprehensive reviews include Fan et al. (2016), Wainwright (2019), Lam (2020), Ledoit and Wolf (2021).

The amplitude of techniques developed to overcome the estimation issues in high dimensions now provides several state-of-the-art solutions, but also leaves some room to further improve the estimation process in many directions.

Although the theory of covariance matrix estimation for low-dimensional Gaussian data was developed in the fifties by pioneeristic contributions (Anderson, 1958), it became soon apparent that the sample covariance matrix $\Sigma_n$ is not a reliable estimator of the true covariance matrix $\Sigma^*$ when $p/n \to 1^-$, and is not a consistent estimator of $\Sigma^*$ when $p/n \geq 1$. As explained in Ledoit and Wolf (2004), in fact, sample eigenvalues may be severely overdispersed when $p/n \to 1^-$, as they follow the Marčenko-Pastur law (Marčenko and Pastur, 1967). This leads $\Sigma_n$ to be more and more numerically unstable under that condition. When $p/n \geq 1$, instead, $p - n$ sample eigenvalues are null, thus irremediably affecting the consistency of $\Sigma_n$. In addition, a large $p$ easily leads to identifiability issues for $\Sigma^*$, as the number of parameters to be recovered grows quadratically in $p$ in absence of further assumptions.

A first relevant idea to approach the shortcomings of $\Sigma_n$ dealt with exploiting Stein’s loss (Stein (1975, 1986); Dey et al. (1985)). Charles Stein’s seminal idea was to keep the sample eigenvectors fixed, and to recondition sample eigenvalues by applying some shrinkage function to them, in order to achieve consistency. This approach requires the adoption of a ‘double asymptotics’ framework, where both $p$ and $n$ are allowed to vary, to be well founded. In this respect, we are indebted to Wigner (1955), who first developed a comprehensive random ma-
Matrix theory in high dimensions. At the same time, it was proved only in [El Karoui (2008)] that the individual eigenvalues of a large covariance matrix can be successfully estimated. Eigenvalue shrinkage has been later extended by Olivier Ledoit and Micheal Wolf, who proposed linear and nonlinear shrinkage of sample eigenvalues, and derived a unified high-dimensional asymptotic framework for large covariance matrices ([Ledoit and Wolf, 2004, 2012, 2015]). A later eigenvalue shrinkage methodology based on sample splitting has been proposed by Lam (2016). Eigenvalue shrinkage is able to improve sample eigenvalues as much as possible, while [Ledoit and Péché (2011)] shows that sample eigenvectors may be far from the corresponding true ones, in spite of [Ledoit and Wolf (2021)] providing a measure that quantifies the discrepancy between sample and true eigenvectors.

A second relevant approach to large covariance matrix estimation passes by the assumption of a specific structure for $\Sigma^*$, in order to drastically reduce the number of parameters. One option is to assume a factor model structure. The spiked covariance model, introduced in [Johnstone (2001)] and recovered in [Johnstone and Lu (2009)], is a successful attempt of this kind. Linear eigenvalue shrinkage under the same model has been proposed in [Donoho et al. (2018)]. Another option is to assume some type of sparsity. Under that assumption, with respect to the underlying structure, it has been proposed to recover $\Sigma^*$ by applying hard-thresholding ([Bickel and Levina, 2008b]), soft-thresholding ([Bickel and Levina, 2008a]), generalized thresholding ([Rothman et al., 2009]), or adaptive thresholding ([Cai and Liu, 2011]).

These two types of algebraic structure enforced in $\Sigma^*$ may both be too restrictive. In fact, as pointed out by [Giannone et al. (2021)] for economic data, it is likely that the sparsity assumption is too strong in high dimensions, as the interrelation structure among the variables is actually more dense than sparse. At the same time, a strict factor model does not allow for any idiosyncratic covariance structure that may catch specific pairs of extra-correlated variables beyond the factors. As a consequence, it became clear that conditional sparsity with respect to an underlying factor model, that is the approximate factor model ([Chamberlain and Rothschild, 1983]), can effectively merge factor model and sparsity assumptions, being a parsimonious and flexible approach at the same time. A covariance matrix es-
timator assuming conditional sparsity is POET (Fan et al., 2013), that proposes to threshold the principal orthogonal complement to obtain a consistent solution.

Conditional sparsity can be imposed by assuming for $\Sigma^*$ a low rank plus sparse decomposition, that is

$$\Sigma^* = L^* + S^* = BB' + S^*,$$  \(1\)

where $L^* = BB' = U_L \Lambda_L U_L'$, with $U_L$ $p \times r$ semi-orthogonal matrix and $\Lambda_L$ $r \times r$ diagonal positive definite matrix, and $S^*$ is a positive definite and element-wise sparse matrix, containing only $s \ll \frac{p(p-1)}{2}$ off-diagonal non-zero elements. Structure (1) has become the reference model under which several high-dimensional covariance matrix estimators work. It can be recovered by nuclear norm plus $l_1$ penalization, that is by solving

$$\left(\hat{L}, \hat{S}\right) = \arg \min_{L,S} \mathcal{L}(L, S) + \Psi(L, S),$$  \(2\)

where $\Psi(L, S) = \psi\|L\|_* + \rho\|S\|_1$, $\|L\|_* = \sum_{i=1}^p \lambda_i(L)$ is the nuclear norm of $L$, i.e. the sum of the eigenvalues of $L^*$, $\|S\|_1 = \sum_{i=1}^p \sum_{j=1}^p |S_{ij}|$, $\psi$ and $\rho$ are non-negative threshold parameters, and $\mathcal{L}(L, S)$ is a smooth loss function.

The nuclear norm was first proposed in Fazel et al. (2001) as an alternative to PCA. Fazel (2002) furnishes a proof that $\psi\|L\|_* + \rho\|S\|_1$ is the tightest convex relaxation of the original non-convex penalty $\psi \text{rk}(L) + \rho\|S\|_0$. Donoho (2006) proves that the $l_1$ norm minimization provides the sparsest solution to most large underdetermined linear systems, while Recht et al. (2010) proves that the nuclear norm minimization provides guaranteed rank minimization under a set of linear equality constraints. Candès and Plan (2009) shows that $l_1$ norm minimization selects the best linear model in a wide range of situations. The nuclear norm has instead been used to solve large matrix completion problems, like in Srebro et al. (2005), Candès and Tao (2010), Mazumder et al. (2010), and Hastie et al. (2015). Nuclear norm plus $l_1$ norm minimization was first exploited in Candès et al. (2011) to provide a robust version of PCA under grossly corrupted or missing data.

Heuristics (2) has generated a stream of literature where new covariance matrix estimators
in high dimensions are derived. Agarwal et al. (2012) ensures optimal rates for the solutions of (2) under model (1) via a purely analytical approach, providing the approximate recovery of $\Sigma^*$ and a bounded non-identifiability radius for $L^*$. In Chandrasekaran et al. (2012), a latent graphical model structure, based on a sparse minus low rank decomposition for $\Sigma^*$, is learnt by solving a problem of type (2), providing both parametric and algebraic consistency, where the latter one holds if (i) the low rank estimate $\hat{L}$ is positive semidefinite having the true rank $r$, (ii) the residual estimate $\hat{S}$ is positive definite having the true sparsity pattern, and (iii) $\hat{\Sigma} = \hat{L} + \hat{S}$ is positive definite. The key for this result is to control the algebraic features of the low rank and sparse matrix varieties containing $L^*$ and $S^*$ respectively, because the low rank variety is locally curve, and so it may be very sensitive to small perturbations in $\Sigma_n$. The nuclear norm plus $l_1$ norm penalty $\Psi(L, S)$ had previously been exploited in Chandrasekaran et al. (2011) to provide exact recovery for structure (1) in absence of noise.

Building over Chandrasekaran et al. (2012), Luo (2011) derives LOREC estimator under model (1) for $\Sigma^*$ via objective (2) with $\mathcal{L}(L, S) = \mathcal{L}^{(F)}(L, S)$, $\mathcal{L}^{(F)}(L, S) = \frac{1}{2}\|\Sigma_n - (L+S)\|^2_F$.

In Farnè and Montanari (2020), UNALCE estimator is derived. UNALCE is based on problem (2) with $\mathcal{L}(L, S) = \frac{1}{2}\|\Sigma_n - (L+S)\|^2_F$, like LOREC, but overcomes LOREC deficiencies by developing a random matrix theory result that holds under a wide range of approximate factor models and accounts for the high-dimensional case $p \geq n$. UNALCE is both algebraically consistent in the sense of Chandrasekaran et al. (2012) and parametrically consistent in Frobenius and spectral norm. Although UNALCE is the optimal estimator in finite sample when $\mathcal{L}(L, S) = \mathcal{L}^{(F)}(L, S)$ in terms of Frobenius loss, there is room to further improve it by replacing $\mathcal{L}^{(F)}(L, S)$ by a different loss. The Frobenius loss optimizes in fact the entry by entry performance of $\hat{\Sigma}$. A loss able to explicitly control the spectrum estimation quality may be desirable, in order to ensure the algebraic consistency of $\left(\hat{L}, \hat{S}\right)$, while simultaneously optimizing the estimated spectrum in terms of distance from the true one. The loss $\mathcal{L}^{(ld)}(L, S) = \frac{1}{2} \log \det(I_p + \Delta_n \Delta'_n)$, $\Delta_n = \Sigma - \Sigma_n$, $\Sigma = L + S$, is a possible one satisfying
these needs, because it is controlled by the individual singular values of $\Delta_n$, since
\[
\log \det(I_p + \Delta_n \Delta'_n) = \log \prod_{i=1}^{p} (\lambda_i(I_p + \Delta_n \Delta'_n)) \leq \sum_{i=1}^{p} (1 + \lambda_i(\Delta_n)^2) = p + \sum_{i=1}^{p} (\lambda_i(\Delta_n)^2),
\]
thus providing an intrinsic eigenvalue correction. Inequality (3) trivially holds when $\Delta_n \Delta'_n$ is diagonal, and it can be proved in the general case by diagonalizing the positive semi-definite symmetric matrix $\Delta_n \Delta'_n$. Our estimator pair therefore becomes
\[
\left( \hat{L}, \hat{S} \right) = \arg \min_{L, S} \frac{1}{2} \log \det(I_p + \Delta_n \Delta'_n) + \psi\|L\|_* + \rho\|S\|_1.
\]
Problem (4) minimizes a loss controlled by the Euclidean norm of the eigenvalues of the error matrix, while simultaneously minimizing the latent rank and the residual support size, whose the nuclear norm and the $l_1$ norm are the respective tightest convex relaxations (see Fazel (2002)).

The mathematical properties of $\mathcal{L}^{(ld)}(L, S)$ have been extensively studied in Bernardi and Farnè (2022). The relevant challenge is that $\mathcal{L}^{(ld)}(L, S)$ is locally convex, i.e. it is convex into a specific range for $\Delta_n$. In this paper, we exploit this fact to propose a proximal gradient algorithm to compute (4) and to prove algebraic and parametric consistency for the estimates of $L^*, S^*$, and $\Sigma^*$ obtained in this way. To this end, we recall the first and the second derivatives of $\mathcal{L}^{(ld)}(L, S)$, its local convexity region, and the Lipschitzianity of $\mathcal{L}^{(ld)}(L, S)$ and of its gradient.

The remainder of the paper is structured as follows. In Section 2 we define our model, detailing the necessary assumptions to prove consistency. Section 3 presents the recalled mathematical analysis results related to $\mathcal{L}^{(ld)}(L, S) = \frac{1}{2} \log \det(I_p + \Delta_n \Delta'_n)$. Section 4 presents how the adoption of $\mathcal{L}^{(ld)}(L, S)$ as smooth term in (4) impacts on algebraic consistency. Section 5 establishes the algebraic and parametric consistency of $\left( \hat{L}, \hat{S} \right)$. Section 6 presents a solution algorithm for problem (4). Section 7 reports a wide simulation study. Section 8 describes a real data example. In the end, some concluding remarks follow. The proofs of mathematical results are reported in Appendix A.
Notation

Given a $p \times p$ symmetric positive semi-definite matrix $M$, we denote by $\lambda_i(M), i \in \{1, \ldots, p\}$, the eigenvalues of $M$ in descending order. To indicate that $M$ is positive definite or semi-definite we use the notations: $M \succ 0$ or $M \succeq 0$, respectively. Then, we recall the following norm definitions:

1. Element-wise:
   (a) $l_0$ norm: $\|M\|_0 = \sum_{i=1}^{p} \sum_{j=1}^{p} 1(M_{ij} \neq 0)$, which is the total number of non-zeros;
   (b) $l_1$ norm: $\|M\|_1 = \sum_{i=1}^{p} \sum_{j=1}^{p} |M_{ij}|$;
   (c) Frobenius norm: $\|M\|_F = \sqrt{\sum_{i=1}^{p} \sum_{j=1}^{p} M_{ij}^2}$;
   (d) Maximum norm: $\|M\|_\infty = \max_{1 \leq i,j \leq p} |M_{ij}|$.

2. Induced by vector:
   (a) $\|M\|_{0,v} = \max_{i \leq p} \sum_{j \leq p} 1(M_{ij} \neq 0)$, which is the maximum number of non-zeros per row–column, defined as the maximum ‘degree’ of $M$;
   (b) $\|M\|_{1,v} = \max_{j \leq p} \sum_{i \leq p} |M_{ij}|$;
   (c) Spectral norm: $\|M\|_2 = \lambda_1(M)$.

3. Schatten:
   (a) Nuclear norm of $M$, here defined as the sum of the eigenvalues of $M$: $\|M\|_* = \sum_{i=1}^{p} \lambda_i(M)$.

The minimum nonzero off-diagonal element of $M$ in absolute value is denoted as $\|M\|_{\text{min,off}} = \min_{1 \leq i,j \leq p, i \neq j, M_{ij} \neq 0} |M_{ij}|$.

Given a $p$–dimensional vector $v$, we denote by $\|v\| = \sqrt{\sum_{i=1}^{p} v_i^2}$ the Euclidean norm of $v$, by $\|v\|_\infty = \max_{i=1,\ldots,p} |v_i|$ the maximum norm of $v$, and by $v_{k,i}$ the $i$–th component of the indexed vector $v_k$.

Given two sequences $A_\nu$ and $B_\nu, \nu \to \infty$, we write $A_\nu = O(B_\nu)$ or $A_\nu \preceq B_\nu$, if there exists a positive real $C$ independent of $\nu$ such that $A_\nu / B_\nu \leq C$, we write $B_\nu = O(A_\nu)$ or
$A_\nu \geq B_\nu$, if there exists a positive real $C$ independent of $\nu$ such that $B_\nu/A_\nu \leq C$, and we write $A_\nu \simeq B_\nu$ if there exists a positive real $C$ independent of $\nu$ such that $A_\nu/B_\nu \leq C$ and $B_\nu/A_\nu \leq C$. Similarly, we write $A_\nu = o(B_\nu)$ or $A_\nu \prec B_\nu$, if there exists a positive real $C$ independent of $\nu$ such that $A_\nu/B_\nu \leq C$, and we write $B_\nu = o(A_\nu)$ or $A_\nu \succ B_\nu$, if there exists a positive real $C$ independent of $\nu$ such that $B_\nu/A_\nu < C$.

2 The model

Definition

We assume for the data the following model structure:

$$x = Bf + \epsilon,$$

where $B$ is a $p \times r$ semi-orthogonal loading matrix such that $B'B = I_r$, $f \sim (0_r, I_r)$ is a $r \times 1$ random vector, and $\epsilon \sim (0_p, S^*)$ is a $p \times 1$ random vector. Assuming that $E(f\epsilon') = 0_{r \times p}$, we obtain that

$$\Sigma^* := E(xx') = BB' + S^* = L^* + S^*,$$

where $L^* = BB'$ is positive semi-definite with rank $r < p$, and $S^*$ is positive definite with $s < \frac{p(p-1)}{2}$ non-zero off-diagonal elements. We define the unbiased sample covariance matrix as $\Sigma_n = \frac{1}{n-1} \sum_{i=1}^n x_i x_i'$.

Factor model assumptions

Assumption 1. (i) The eigenvalues of the $r \times r$ matrix $p^{-\alpha_1} B'B$ are bounded away from 0 for all $p \in \mathbb{N}$ such that $\lambda_i(B'B) \simeq p^{\alpha_i}$, $i = 1, \ldots, r$, for some $\frac{1}{2} < \alpha_r \leq \ldots \leq \alpha_1 \leq 1$; (ii) $\|b_j\|_{\infty} = O(1)$ for all $j = 1, \ldots, p$ and $r$ is finite for all $p \in \mathbb{N}$.

Assumption 1 prescribes different speeds of divergence for latent eigenvalues, and imposes a finite latent rank $r$. The left limit $O(p^{\frac{1}{2}})$ is imposed on latent eigenvalues to preserve the factor model structure as $p \to \infty$. Part (ii) could actually be relaxed to cope with $r = O(\log p)$,
but we avoid it for the sake of simplicity. The maximum loading magnitude is imposed to be bounded as $p \to \infty$, in order to derive a bound for the maximum norm of $\Delta_n$.

**Assumption 2.** For all $p \in \mathbb{N}$: (i) there exist $\delta_1 \in (0, \frac{1}{2}]$ and $\delta_2 > 0$, such that $\|S^*\|_{0,v} = \max_{1 \leq i \leq p} \sum_{j=1}^{p} 1(S^*_{ij} = 0) \leq \delta_2 p^{\delta_1}$; (ii) $\|S^*\|_{\infty} = O(1)$.

Assumption 2 controls the maximum number of non-zeros per row in the residual covariance component $S^*$ and imposes its maximum element to be $O(1)$ as $p \to \infty$. This allows to establish the traditional eigengap between $\lambda_r(B'B)$ and $\lambda_1(S^*)$ as $p \to \infty$, because $\|S^*\|_2 \leq \|S^*\|_{1,v} \leq \|S^*\|_{0,v}\|S^*\|_{\infty} \leq \delta_2 p^{\delta_1}$, and $\delta_1 \leq \frac{1}{2} < \alpha_r$ by Assumption 1(i). Part (i) is also needed to ensure the identifiability of the sparsity pattern in $S^*$, together with Assumption 5 (see later).

**Assumption 3.** In model (5), $E(f) = 0_r$, $V(f) = I_r$, $E(\epsilon) = 0_p$, $V(\epsilon) = S^*$, $\lambda_p(S^*) > 0$, $E(f'\epsilon') = 0_{r \times p}$, and there exist $b_1, b_2, c_1, c_2 > 0$ such that, for any $l > 0$, $k \leq n$, $i \leq r$, $j \leq p$:

$$\Pr(|f_{k,i}| > l) \leq \exp\{-l/(b_1)^{c_1}\}, \quad \Pr(|\epsilon_{k,j}| > l) \leq \exp\{-l/(b_2)^{c_2}\}.$$

Assumption 3 completes the framework to make (5) an approximate factor model, and imposes sub-Gaussian tails to factor scores and residuals. This ensures that all the moments of $f$ and $\epsilon$ exist, and is crucial to apply to $f$ and $\epsilon$ large deviation probabilistic results. Note that the magnitude of residual covariances (i.e., the off-diagonal entries of $S^*$) is controlled by Assumption 2 and that $S^*$ is imposed to be positive definite as $p \to \infty$.

**Identifiability assumptions**

In order to ensure the effectiveness of the composite penalty $\psi\|L\|_* + \rho\|S\|_1$ in recovering the latent rank $\text{rk}(L^*) = r$ and the residual number of nonzeros $|\text{supp}(S^*)| = s$ (where $\text{supp}(S)$ is the orthogonal complement of $\text{ker}(S)$ and $|\text{supp}(S^*)|$ denotes its dimension), we need to control the geometric manifolds containing $L^*$ and $S^*$. As in Chandrasekaran et al. (2011),
we assume \( L^* \in \mathcal{L}(r) \) and \( S^* \in \mathcal{S}(q) \), where

\[
\mathcal{L}(r) = \{ L \mid L \succeq 0, L = UDU', U'U = I_r, D \in \mathbb{R}^{r \times r} \text{diagonal} \},
\]

\[
\mathcal{S}(s) = \{ S \in \mathbb{R}^{p \times p} \mid S > 0, |\text{supp}(S)| \leq s \}.
\]

\( \mathcal{L}(r) \) is the variety of matrices with at most rank \( r \), \( \mathcal{S}(s) \) is the variety of (element-wise) sparse matrices with at most \( s \) non-zero elements, and the two varieties \( \mathcal{L}(r) \) and \( \mathcal{S}(s) \) can be disentangled if \( L^* \) is far from being sparse, and \( S^* \) is far from being low rank. For this reason, Chandrasekaran et al. (2011) defines the tangent spaces \( T(L^*) \) and \( \Omega(S^*) \) to \( \mathcal{L}(r) \) and \( \mathcal{S}(s) \) respectively, and proposes the following rank-sparsity measures:

\[
\xi(T(L^*)) = \max_{L \in T(L^*), \|L\|_2 \leq 1} \|L\|_\infty,
\]

\[
\mu(\Omega(S^*)) = \max_{S \in \Omega(S^*), \|S\|_\infty \leq 1} \|S\|_2.
\]

**Assumption 4.** Define \( \psi_0 = \frac{1}{\xi(T(L^*))} \sqrt{\frac{\log p}{n}} \). There exist \( \delta_L, \delta_S > 0 \) such that (i) the minimum eigenvalue of \( L^* \), \( \lambda_{\min}(L^*) \), is greater than \( \delta_L \frac{\psi_0}{\xi(T(L^*))} \), (ii) the minimum absolute value of the non-zero off-diagonal entries of \( S^* \), \( \|S^*\|_{\min, \text{off}} \), is greater than \( \delta_S \frac{\psi_0}{\mu(\Omega(S^*))} \).

Assumption 4 is crucial for identifiability, as it guarantees that the solution pair of (4) lies on the “right” manifolds, i.e. that \( \hat{L} \in \mathcal{L}(r) \) and \( \hat{S} \in \mathcal{S}(s) \) with high probability as \( n \to \infty \).

Then, according to Chandrasekaran et al. (2012), the identifiability condition to be satisfied requires a bound on \( \xi(T(L^*)) \mu(\Omega(S^*)) \) (see Section 4 for more details). For this reason, recalling from Chandrasekaran et al. (2011) that

\[
inc(L^*) \leq \xi(T(L^*)) \leq 2inc(L^*),
\]

\[
deg_{\min}(S^*) \leq \mu(\Omega(S^*)) \leq deg_{\max}(S^*),
\]

with

\[ inc(L^*) = \max_{j=1,...,p} \|P_{L^*}e_j\|, \]

where \( e_j \) is the \( j \)-th canonical basis vector, and \( P_{L^*} \) is the projection operator onto the row–column space of \( L^* \);
\[ \text{deg}_{\min}(\mathbf{S}^*) = \min_{1 \leq i \leq p} \sum_{j=1}^{p} \mathbb{1}(S^*_{ij} = 0), \quad \text{deg}_{\max}(\mathbf{S}^*) = \max_{1 \leq i \leq p} \sum_{j=1}^{p} \mathbb{1}(S^*_{ij} = 0) = \|\mathbf{S}^*\|_{0,\nu}; \]

we can control the degree of transversality of \( \mathcal{L}(r) \) and \( \mathcal{S}(s) \) by the following assumption.

**Assumption 5.** For all \( p \in \mathbb{N} \), there exist \( \kappa_L, \kappa_S > 0 \) with \( k_L \geq \frac{\delta_1}{r} \) and \( \kappa_S \leq \delta_2 \), such that
\[ \xi(T(L^*)) = \frac{\sqrt{r}}{\kappa_L p^{\delta_1}} \text{ and } \mu(\Omega(S^*)) = \kappa_S p^{\delta_1}. \]

Assumption 5 states that the maximum degree of \( \mathbf{S}^* \) is \( O(p^{\delta_1}) \), where \( \delta_1 < \alpha_r \) by Assumptions 1 and 2. More, the incoherence of \( \mathbf{L}^* \) is assumed to scale to \( O(p^{-\delta_1}) \), in order to keep the product \( \xi(T(L^*))\mu(\Omega(S^*)) \) proportional to \( O(1) \), which is crucial to prove algebraic consistency (see Theorem 1). This assumption resembles in nature the approximate factor model of Chamberlain and Rothschild (1983), because \( \delta_1 < \alpha_r \), such that the number of residual nonzeros will become negligible with respect to latent eigenvalues, and the manifold underlying \( \mathbf{L}^* \) will be progressively easier to retrieve as \( p \to \infty \).

### 3 Analytic setup

Let us reconsider our optimization problem

\[
(\hat{\mathbf{L}}, \hat{\mathbf{S}}) = \min_{\mathbf{L}, \mathbf{S} : \mathbf{L} \succeq 0, \mathbf{S} > 0, \mathbf{L} + \mathbf{S} > 0} \phi(\mathbf{L}, \mathbf{S}),
\]

with \( \phi(\mathbf{L}, \mathbf{S}) = \mathcal{L}^{(ld)}(\mathbf{L}, \mathbf{S}) + \mathcal{P}(\mathbf{L}, \mathbf{S}) \), where \( \mathcal{L}^{(ld)}(\mathbf{L}, \mathbf{S}) = \frac{1}{2} \log \det(\mathbf{I}_p + \Delta_n \Delta_n^\top) \) and \( \mathcal{P}(\mathbf{L}, \mathbf{S}) = \psi\|\mathbf{L}\|_* + \rho\|\mathbf{S}\|_1 \). We can disentangle the objective \( \phi(\mathbf{L}, \mathbf{S}) \) in its smooth and nonsmooth components as follows:

\[ \phi(\mathbf{L}, \mathbf{S}) = \phi_D(\mathbf{L}, \mathbf{S}) + \phi_{ND}(\mathbf{L}, \mathbf{S}), \]

where \( \phi_D(\mathbf{L}, \mathbf{S}) = \mathcal{L}^{(ld)}(\mathbf{L}, \mathbf{S}) \) is the smooth component of \( \phi(\mathbf{L}, \mathbf{S}) \) and \( \phi_{ND}(\mathbf{L}, \mathbf{S}) = \psi\|\mathbf{L}\|_* + \rho\|\mathbf{S}\|_1 \) is the non-smooth component of \( \phi(\mathbf{L}, \mathbf{S}) \). As explained in Nesterov (2013), problem (13) can be numerically solved by applying proximal gradient methods (see Section 6). To implement them, we need to calculate the first and the second derivatives of \( \phi_D(\mathbf{L}, \mathbf{S}) \), to
prove the Lipschitzianity of its gradient, and then to derive the conditions that guarantee its local convexity. We refer to Appendix A for the proofs.

First and second derivative

We explicit the first and the second derivative of \( \phi_D(L, S) = \frac{1}{2} \log \det \varphi(\Sigma) \), with \( \varphi(\Sigma) = \Sigma - \Sigma_n \) and \( \Sigma = L + S \), wrt \( L \) and \( S \).

**Proposition 1.**

\[
\frac{\partial \phi_D(L, S)}{\partial L} = \frac{\partial \phi_D(L, S)}{\partial S} = (I_p + \Delta_n \Delta_n')^{-1} \Delta_n.
\]

**Proposition 2.**

\[
\frac{\partial^2}{\partial \sigma_{ij} \partial \sigma_{hk}} \frac{1}{2} \log \det \varphi(\Sigma) = \left( \frac{1}{2} \text{Hess } \log \det \varphi(\Sigma) \right)_{ijhk} = \delta_{jk} (\varphi^{-1}(\Sigma))_{ih} - \sum_{\mu,\sigma} (\varphi^{-1}(\Sigma))_{h\mu} \Delta_{\mu j} (\varphi^{-1}(\Sigma))_{i\sigma} \Delta_{\sigma k}.
\]

From Proposition 2 it follows that if \( \Sigma - \Sigma_n = 0_{p \times p} \), we get

\[
\left( \frac{1}{2} \text{Hess } \log \det \varphi(\Sigma) \right)_{ijhk} = \delta_{jk} \otimes \delta_{ih} = (I_p \otimes I_p)_{ijhk},
\]

that is,

\[
\frac{1}{2} \text{Hess } \log \det \varphi(\Sigma) = I_p \otimes I_p.
\]

Lipschitz-continuity

Let us recall the two-argument matrix function

\[
\phi(L, S) = \phi_D(L, S) + \phi_{ND}(L, S),
\]
where $\phi_D(L,S) = \frac{1}{2} \log \det (I_p + \Delta_n \Delta_n')$, $\phi_{ND}(L,S) = \psi \|L\|_s + \rho \|S\|_1$, and $\Delta_n = \Sigma - \Sigma_n$, with $\Sigma = L + S$. The gradient of $\phi_D(L,S)$ with respect to $L$ and to $S$ is the same, and corresponds to $\frac{\partial \phi_D(L,S)}{\partial L} = \frac{\partial \phi_D(L,S)}{\partial S} = (I_p + \Delta_n \Delta_n')^{-1} \Delta_n$.

We now consider the vectorized gradient $\text{vec} \left( \frac{\partial \phi_D(L,S)}{\partial L} \right)$, that is the $2p^2$-dimensional vector $\left[ \text{vec}((I_p + \Delta_n \Delta_n')^{-1} \Delta_n) \right]$. We define two $p \times p$ matrices $\Sigma_2 = L_2 + S_2$ and $\Sigma_1 = \Sigma_2 + \epsilon \mathbf{H}$, $\epsilon > 0$, such that $\Delta_{1,n} = \Sigma_1 - \Sigma_n$ and $\Delta_{2,n} = \Sigma_2 - \Sigma_n$. We set the difference vector $\mathbf{d}(\Sigma_1, \Sigma_2) = \text{vec} \left( \frac{\partial \phi_D(L_1,S_1)}{\partial (L_1,S_1)} \right) - \text{vec} \left( \frac{\partial \phi_D(L_2,S_2)}{\partial (L_2,S_2)} \right)$, which is a $2p^2$-dimensional vector, composed of two identical components of $p^2$ elements, stacked one below the other: $\mathbf{d}(\Sigma_1, \Sigma_2) = \left[ \text{vec}((I_p + \Delta_{1,n} \Delta_{1,n}')^{-1} \Delta_{1,n} - (I_p + \Delta_{2,n} \Delta_{2,n}')^{-1} \Delta_{2,n}) \right] \text{vec}((I_p + \Delta_{1,n} \Delta_{1,n}')^{-1} \Delta_{1,n} - (I_p + \Delta_{2,n} \Delta_{2,n}')^{-1} \Delta_{2,n})'$. It follows that

$$\|\mathbf{d}(\Sigma_1, \Sigma_2)\|_2 \leq 2\|\text{vec}((I_p + \Delta_{1,n} \Delta_{1,n}')^{-1} \Delta_{1,n} - (I_p + \Delta_{2,n} \Delta_{2,n}')^{-1} \Delta_{2,n})\|_2,$$

or

$$\|\mathbf{d}(\Sigma_1, \Sigma_2)\|_F \leq 2\|(I_p + \Delta_{1,n} \Delta_{1,n}')^{-1} \Delta_{1,n} - (I_p + \Delta_{2,n} \Delta_{2,n}')^{-1} \Delta_{2,n}\|_F. \quad (18)$$

Therefore, we need to ensure that the rhs of (18) is bounded. To this purpose, we study the matrix $(I_p + \Delta_{1,n} \Delta_{1,n}')^{-1} \Delta_{1,n} - (I_p + \Delta_{2,n} \Delta_{2,n}')^{-1} \Delta_{2,n}$, that is equal to

$$(I_p + \Delta_{2,n} + \epsilon \mathbf{H})(\Delta_{2,n} + \epsilon \mathbf{H})' - (I_p + \Delta_{2,n} \Delta_{2,n}')^{-1} \Delta_{2,n},$$

and we prove the Lipschitzianity of the smooth function $\phi_D(L,S) = \frac{1}{2} \log \det (I_p + \Delta_n \Delta_n')$, and of its gradient function, $\frac{\partial \phi_D(L,S)}{\partial L} = \frac{\partial \phi_D(L,S)}{\partial S} = F(\Delta_n) = (I_p + \Delta_n \Delta_n')^{-1} \Delta_n$.

**Lemma 1.** The function $\psi(L,S) = \frac{1}{2} \log \det \varphi(\Sigma)$, with $\varphi(\Sigma) = (I_p + \Delta_n \Delta_n')$, is Lipschitz continuous with Lipschitz constant equal to 1:

$$|\log \det \varphi(\Sigma_1) - \log \det \varphi(\Sigma_2)| \leq \|\Sigma_1 - \Sigma_2\|_2,$$

where $\Sigma_1 = \Sigma_2 + \epsilon \mathbf{H}$.

**Lemma 2.** The function $\frac{\partial \psi(L,S)}{\partial L} = \frac{\partial \psi(L,S)}{\partial S} = (I_p + \Delta_n \Delta_n')^{-1} \Delta_n$ is Lipschitz continuous with
Lipschitz constant equal to $\frac{5}{4}$:

$$\|F(\Delta_n + \epsilon H) - F(\Delta_n)\|_2 \leq \frac{5}{4}\epsilon\|H\|_2 + O(\epsilon^2),$$

(20)

with $F(\Delta_n + \epsilon H) = (I_p + (\Delta_n + \epsilon H)(\Delta_n + \epsilon H)')(\Delta_n + \epsilon H)^{-1}$, for any $\epsilon > 0$.

From (18), noting that Lemma 2 holds for the Frobenius norm as well and setting $\epsilon = 1$, it follows that

$$\|d(\Sigma_1, \Sigma_2)\|_F \leq 2\|(I_p + \Delta_{1,n}\Delta'_{1,n})^{-1}\Delta_{1,n} - (I_p + \Delta_{2,n}\Delta'_{2,n})^{-1}\Delta_{2,n}\|_F \leq \frac{10}{4}\|\Delta_{1,n} - \Delta_{2,n}\|_F.$$  (21)

Local convexity

In order to apply proximal gradient methods, we need to prove that $\phi_D(L, S) = \frac{1}{2}\log\det(\Delta_n\Delta'_n)$ is locally convex around $\Sigma^*$. In the univariate context, the function $\frac{1}{2}\log\det(1 + x^2)$ is convex if and only if $|x| < \frac{1}{\sqrt{2}}$. In the multivariate context, it is therefore reasonable to suppose that a similar condition on $\Delta_n\Delta'_n$ ensures local convexity. A proof can be given by showing the positive definiteness of the Hessian of the log-det function evaluated around $\Sigma^*$. In other words, we need to show that there exists a positive $\delta$ such that, whenever $\|\Delta_n\Delta'_n\| < \delta$, the function $\frac{1}{2}\log\det(\Delta_n\Delta'_n)$ is convex with high probability.

**Lemma 3.** Given $0 < \mu \leq \frac{1}{3p}$, we have that the function

$$\log\det(I_p + AA')$$

(22)

is convex on the set $C_\mu = \{A|A\text{ is a real } p \times p \text{ matrix , } ||A||_2 \leq \mu\}$ where $||A||_2$ denotes the spectral norm of $A$.

Changing variables in an obvious way, we have therefore proven the following.
Corollary 1. For any $\delta > 0$ the function

$$\log \det (\delta^{-2}I_p + AA')$$

is convex on the closed ball $C_\delta = \{A|A\text{ is a real } p \times p \text{ matrix }, \|A\|_2 \leq \frac{1}{3p}\}$.

In conclusion, even though the function $\log \det (I_p + A)$ is always concave, Corollary 1 shows that the matrix function $\log \det (\delta^{-2}I_p + AA')$ can be made locally convex in arbitrary ball near 0, choosing a suitable $\delta$.

Probabilistic guarantees

In what follows, we show the asymptotic behaviour of the first and the second derivative of $\phi_D(L, S) = \frac{1}{2} \ln \det (I_p + \Delta_n \Delta_n')$.

Lemma 4. Under Assumptions 1, 2, and 3, it holds

$$\frac{1}{p^\alpha} \| \frac{\partial \phi_D(L, S)}{\delta L} \| \rightarrow 0_{p \times p}, \quad n \rightarrow \infty$$

Lemma 5. Under Assumptions 1, 2, and 3, it holds

$$\frac{1}{p^\alpha} \left( \frac{1}{2} \text{Hess log det } \varphi(\Sigma^*) \right)_{ijhk} \rightarrow \delta_{jk} \otimes \delta_{ih} = (I_p \otimes I_p)_{ijhk}, \quad n \rightarrow \infty$$

that is,

$$\frac{1}{2p^\alpha} \text{Hess log det } \varphi(\Sigma^*) \rightarrow I_p \otimes I_p.$$

In order to ensure that the convexity region of Corollary 1 is respected, since $\phi_D(L, S)$ is a stochastic function of $L$ and $S$ because $\Sigma_n$ is a random matrix, it is necessary to assess the probability $P(\|\Delta_n\| \geq \frac{1}{3p})$. Lemma A.1 shows that the claim $\|\Delta_n\| \leq C\frac{\sqrt{n}}{\sqrt{n}}$ holds for some $C > 0$ with probability $1 - O(1/n^2)$, under Assumptions 1, 2, and 3. Therefore, solving the inequality $\frac{1}{3p} \geq \frac{p^{\alpha_1}}{\sqrt{n}}$, equivalent to $\frac{1}{3p} \geq \frac{p^{\alpha_1}}{\sqrt{n}}$, we can derive the condition $n \geq \frac{p^{2\alpha_1+2}}{\sqrt{n}}$ to ensure the convexity region. In other words, we just need that $\delta^2 \geq \frac{n}{p^{2\alpha_1+2}}$, i.e. $\delta \geq \frac{\sqrt{n}}{p^{\frac{1}{2}+\frac{1}{2}\alpha_1}}$. In general, the condition $\delta \simeq \frac{\sqrt{n}}{p}$ is a sufficient one, for any finite $p$. 

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Lemma 6. Under Assumptions 1, 2, and 3, Corollary holds if and only if $\delta \geq \frac{\sqrt{n}}{p^{\alpha+1}}$.

4 Algebraic setup

Let us define the following measure of transversality between two algebraic matrix varieties $T_1$ and $T_2$:

$$\varrho(T_1, T_2) = \max_{\|N\|_2 \leq 1} \|P_{T_1}N - P_{T_2}N\|_2,$$

where $P_{T_1}$ and $P_{T_2}$ are the projection operators onto $T_1$ and $T_2$, respectively. Given two matrices $M_1$ and $M_2$ of equal size, we call $A$ the addition operator, such that $A(M_1, M_2) = M_1 + M_2$, and $A^\dagger$ the adjoint operator, such that $A^\dagger(M_1) = (M_1, M_1)$.

Hereafter, let $\Omega = \Omega(S^*)$ and $T = T(L^*)$, where $\Omega$ is the space tangent to $S(s)$ (see (5)) at $S^*$ and $T$ is the space tangent to $L(r)$ (see (7)) at $L^*$. We define the Cartesian product $\mathcal{Y} = \Omega \times T'$, where $T'$ is a manifold such that $\varrho(T, T') \leq \xi(T)/2$.

In light of these definitions, the following identities hold:

$$A^\dagger A(S, L) = (S + L, S + L);$$
$$P_{\mathcal{Y}A^\dagger A}(S, L) = (S + P_{\Omega}L, P_{T'}S + L);$$
$$P_{\mathcal{Y}A^\dagger A}(S, L) = (S + P_{\Omega}L, P_{T'}S + L).$$

We consider the following norm $g_\gamma$:

$$g_\gamma(\hat{L} - L^*, \hat{S} - S^*) = \max \left( \frac{\|\hat{S} - S^*\|_\infty}{\gamma}, \frac{\|\hat{L} - L^*\|_2}{\|L^*\|_2} \right),$$

with $\gamma \in \mathbb{R}^+$, $\psi_0 = \frac{1}{\xi(T(L^*))} \sqrt{\frac{\log p}{n}}$, $\rho_0 = \gamma \psi_0$, $\psi = p^\alpha \psi_0$, and $\rho = \rho_0$, where $\psi$ and $\rho$ are the thresholds in (4). The norm (28) is the dual norm of the composite penalty $\psi_0 \cdot \|\cdot\|_s + \rho_0 \cdot \|\cdot\|_1$, with which the direct sum $L(r) \oplus S(s)$ is naturally equipped. Obviously this $g_\gamma$-consistency implies consistency in $\ell_2$ norm.
We now consider the solution of the following algebraic problem:

\[
(\hat{\mathbf{S}}, \hat{L}_T) = \arg \min_{\mathbf{L} \in \mathcal{T}, \mathbf{S} \in \mathcal{S}} \frac{1}{2p^{\alpha_1}} \log \det (\mathbf{I}_p + \Delta_n \Delta_n') + \psi_0 \| \mathbf{L} \|_* + \rho_0 \| \mathbf{S} \|_1.
\]  

(29)

This is a version of minimization (1) rescaled by \(p^{\alpha_1}\). We may regard the smooth function

\[-\mathcal{L}^{(id)}(\mathbf{L}, \mathbf{S}) = -\frac{1}{2} \log \det (\mathbf{I}_p + \Delta_n \Delta_n') = -\phi_D(\mathbf{L}, \mathbf{S}),\]

with \(\Delta_n = \Sigma_n - \Sigma, \Sigma = \mathbf{L} + \mathbf{S},\) as a nonlinear function of the squared sample covariance matrix, which we need to maximize. \(-\mathcal{L}^{(id)}(\mathbf{L}, \mathbf{S})\) is concave into the convexity range of Lemma 6. In this view, we may call \(\mathcal{I}^*\) the operator that associates to each pair \((\mathbf{L}, \mathbf{S})\) the Fisher information of \(-\phi_D(\mathbf{L}, \mathbf{S}),\) i.e.

\[
\mathcal{I}^*(\mathbf{L}, \mathbf{S}) = -\frac{\partial^2(-\phi_D(\mathbf{L}, \mathbf{S}))}{\partial \mathbf{L}^2} = \text{Hess} \log \det \varphi(\mathbf{L}, \mathbf{S}), \quad \text{with} \quad \varphi(\mathbf{L}, \mathbf{S}) = (\mathbf{I}_p + \Delta_n \Delta_n').
\]

Under the conditions of Lemma 5, \(\frac{1}{4p^{\alpha_1}} \text{Hess} \log \det \varphi(\mathbf{L}^*, \mathbf{S}^*) \xrightarrow{n \to \infty} \mathbf{I}_p \otimes \mathbf{I}_p.\)

\(\mathcal{I}^*(\mathbf{L}, \mathbf{S})\) may be regarded as a map. Thus, we can write

\[
\mathcal{A}^T \mathcal{I}^* \mathcal{A}(\mathbf{S}, \mathbf{L}) = \mathcal{I}^*(\mathbf{S} + \mathbf{L}, \mathbf{S} + \mathbf{L});
\]

\[
\mathcal{P}_\gamma \mathcal{A}^T \mathcal{I}^* \mathcal{A} \mathcal{P}_\gamma(\mathbf{S}, \mathbf{L}) = \mathcal{I}^*(\mathbf{S} + \mathcal{P}_\Omega \mathbf{L}, \mathcal{P}_\mathcal{T} \mathbf{S} + \mathbf{L});
\]

\[
\mathcal{P}_\gamma \mathcal{A} \mathcal{A}^T \mathcal{I}^* \mathcal{A} \mathcal{P}_\gamma(\mathbf{S}, \mathbf{L}) = \mathcal{I}^*(\mathbf{S} + \mathcal{P}_\Omega \mathbf{L}, \mathcal{P}_\mathcal{T} \mathbf{S} + \mathbf{L}).
\]

Let us consider the matrices \(\Sigma_S = \mathcal{P}_\mathcal{T} \mathbf{S} + \mathbf{L}, \Sigma_S^\perp = \mathcal{P}_\mathcal{T} \mathbf{S} + \mathbf{L}, \Sigma_L = \mathcal{P}_\Omega \mathbf{L} + \mathbf{S}, \Sigma_L^\perp = \mathcal{P}_\Omega \mathbf{L} + \mathbf{S}.\) Following Chandrasekaran et al. (2011), to ensure the algebraic consistency of (1), we need to estimate the quantities

\[
\alpha_\Omega = \min_{\mathbf{S} \in \mathcal{S}, \| \mathbf{S} \|_\infty = 1} \| \mathcal{I}^*(\phi(\Sigma_L))\|_2, \quad \delta_\Omega = \max_{\mathbf{S} \in \mathcal{S}, \| \mathbf{S} \|_\infty = 1} \| \mathcal{I}^*(\phi(\Sigma_L^\perp))\|_2, \quad \beta_\Omega = \max_{\mathbf{S} \in \mathcal{S}, \| \mathbf{S} \|_2 = 1} \| \mathcal{I}^*(\phi(\Sigma_L))\|_\infty,
\]

\[
\alpha_{\mathcal{T}^*} = \min_{\mathbf{L} \in \mathcal{T}^*, \| \mathbf{L} \|_2 = 1} \| \mathcal{I}^*(\phi(\Sigma_S))\|_\infty, \quad \delta_{\mathcal{T}^*} = \max_{\mathbf{L} \in \mathcal{T}^*, \| \mathbf{L} \|_2 = 1} \| \mathcal{I}^*(\phi(\Sigma_S^\perp))\|_\infty, \quad \beta_{\mathcal{T}^*} = \max_{\mathbf{L} \in \mathcal{T}^*, \| \mathbf{L} \|_\infty = 1} \| \mathcal{I}^*(\phi(\Sigma_S))\|_2.
\]

From Lemma 2 we recall that

\[
\| F(\Delta_n + \epsilon \mathbf{H}) - F(\Delta_n)\|_2 \leq \frac{5}{4} \epsilon \| \mathbf{H} \|_2 + O(\epsilon^2),
\]

(30)

with \(F(\Delta_n + \epsilon \mathbf{H}) = (\mathbf{I}_p + (\Delta_n + \epsilon \mathbf{H})(\Delta_n + \epsilon \mathbf{H})^\perp)^{-1}(\Delta_n + \epsilon \mathbf{H}),\) for any \(\epsilon > 0.\) From Lemma
we recall that
\[
\frac{1}{2p^\alpha} \text{Hess } \log \det \varphi(\Sigma^*) \xrightarrow{n \to \infty} \mathbf{I}_p \otimes \mathbf{I}_p.
\]

Therefore, we can define the matrices \( \Delta_n = \Sigma^* - \Sigma_n \), with \( \Sigma^* = L^* + S^* \), and \( H_{T'} = S^* - P_{T'}(S^*) \), \( H_{T'} = S^* - P_{T'}(S^*) \), \( H_{T'} = L^* - P_{T'}(L^*) \), \( H_{T'} = L^* - P_{T'}(L^*) \). Considering that \( I^*(L, S) = -\frac{\partial[\phi^*_D(L, S)]}{\partial L} = -\frac{\partial[\phi^*_D(L, S)]}{\partial S} \), in light of (30), we can write:

\[
\alpha_{T'} = \begin{cases} \min_{L \in T', \|L\|_2 = 1} \|\Sigma L \otimes \Sigma L\|_2 & = \|H_{T'} \otimes H_{T'}\|_2 = \|H_{T'}\|_2^2, \\
\|S^*\|_2^2 & = \|S^*\|_2^2, \\
\end{cases}
\]

\[
\delta_{T'} = \begin{cases} \max_{L \in T', \|L\|_2 = 1} \|\Sigma S \otimes \Sigma S\|_\infty & = \|H_{T'} \otimes H_{T'}\|_\infty = \|H_{T'}\|_\infty^2, \\
\|S^*\|_\infty & = \|S^*\|_\infty, \\
\end{cases}
\]

\[
\beta_{T'} = \begin{cases} \max_{L \in T', \|L\|_2 = 1} \|\Sigma S \otimes \Sigma S\|_2 & = \|H_{T'} \otimes H_{T'}\|_2 = \|H_{T'}\|_2^2, \\
\|S^*\|_2^2 & = \|S^*\|_2^2, \\
\end{cases}
\]

\[
\alpha_\Omega = \begin{cases} \min_{S \in \Omega, \|S\|_\infty = 1} \|\Sigma L \otimes \Sigma L\|_\infty & = \|H_{\Omega} \otimes H_{\Omega}\|_\infty = \|H_{\Omega}\|_\infty^2, \\
\|L^*\|_\infty^2 & = \|L^*\|_\infty^2, \\
\end{cases}
\]

\[
\delta_\Omega = \begin{cases} \max_{S \in \Omega, \|S\|_\infty = 1} \|\Sigma L \otimes \Sigma L\|_2 & = \|H_{\Omega} \otimes H_{\Omega}\|_2 = \|H_{\Omega}\|_2^2, \\
\|L^*\|_2^2 & = \|L^*\|_2^2, \\
\end{cases}
\]

\[
\beta_\Omega = \begin{cases} \max_{S \in \Omega, \|S\|_\infty = 1} \|\Sigma L \otimes \Sigma L\|_\infty & = \|H_{\Omega} \otimes H_{\Omega}\|_\infty = \|H_{\Omega}\|_\infty^2, \\
\|L^*\|_\infty^2 & = \|L^*\|_\infty^2, \\
\end{cases}
\]

These calculations show that the impact of the local curvature of \( \phi_D(L, S) \) on the identification of the underlying matrix varieties is linear, due to the linearity of \( \phi_D(L, S) \), proved in Lemma 2.

At this stage, analogously to Chandrasekaran et al. (2011), we can define \( \alpha = \min(\alpha_{\Omega}, \alpha_{T'}) \), \( \delta = \max(\delta_{\Omega}, \delta_{T'}) \), \( \beta = \max(\beta_{\Omega}, \beta_{T'}) \), and assume that, for some \( \nu \in (0, \frac{1}{2}] \), the following holds.

Assumption 6.

\[
\frac{\delta}{\alpha} \leq 1 - 2\nu.
\] (32)

**Proposition 3.** Suppose that Assumptions 3 and 6 hold. Let \( \sqrt{\kappa_S \kappa_L} \leq \frac{1}{6} \left( \frac{\nu_\alpha}{\beta_0^{(2-\nu)}} \right)^2 \), and \( \gamma \in \left( \frac{3\kappa_{(T')}^2}{\kappa_\gamma}, \frac{\nu_\alpha}{\beta_0^{(2-\nu)}} \right) \), with \( \alpha, \beta, \gamma, \nu \) as previously defined. Then, under Assumptions 7, for all \( (S, L) \in U \) such that \( U = \Omega \times T' \) with \( \varphi(T, T') \leq \xi(T)/2 \), as \( n \to \infty \) the following holds with high probability.
1. $g_\gamma(P_{\gamma^*}A^\dagger I^\ast AP_{\gamma^*}(S, L)) \geq \frac{\alpha}{2}g_\gamma(S, L)$;

2. $g_\gamma(P_{\gamma^*}A^\dagger I^\ast AP_{\gamma^*}(S, L)) \leq (1 - \nu)g_\gamma(P_{\gamma^*}A^\dagger I^\ast AP_{\gamma^*}(S, L))$.

Proof. Suppose that Assumption 6 holds. Since $L \in T'$, $S \in \Omega$, and

$\gamma \in \left[\frac{3\xi(T(L^*))(2 - \nu)}{\nu \alpha}, \frac{\mu \alpha}{2\mu(\Omega(S^*))(2 - \nu)}\right]$, it is enough that $\frac{\sqrt{\kappa S}}{\kappa L} \leq \frac{1}{6}\left(\frac{\nu \alpha}{\mu(2 - \nu)}\right)^2$ under Assumption 5 to ensure that $\xi(T(L^*))\mu(\Omega(S^*)) \leq \frac{1}{6}\left(\frac{\nu \alpha}{\mu(2 - \nu)}\right)^2$. Then, since Assumptions 1-3 ensure that Lemma 5 holds, such that

$\frac{1}{p^2\kappa}||T^*\xi(L^*, S^*)||_\infty \xrightarrow{n \to \infty} ||I_p \otimes I_p||_\infty = 1$,

the proof of Proposition 3.3 in Chandrasekaran et al. (2012) straightforwardly applies as $n \to \infty$.

Remark 1. If $\alpha = \beta = 1$ and $\delta = 0$ as in Luo (2011) and Farnè and Montanari (2020), it follows that $\nu = \frac{1}{2}$ and Assumption 6 is automatically satisfied. More, the identifiability condition of Proposition 3 simplifies to $\frac{\sqrt{\kappa S}}{\kappa L} \leq \frac{1}{54}$, and the two claims of Proposition 3 are simplified accordingly.

5 Consistency

Let us denote the spectral decomposition of the non-definite symmetric random error matrix $\Delta_n$ as $U_\Delta A_\Delta U'_\Delta$, such that the one of the semi-definite matrix $\Delta_n A'_n$ results to be $U_\Delta A^2_\Delta U'_\Delta$. Recalling Woodbury formula, we can write

$(I_p + \Delta_n A'_n)^{-1} = (I_p + U_\Delta A^2_\Delta U'_\Delta)^{-1} = I_p - U_\Delta(A^2_\Delta + U'_\Delta U_\Delta)^{-1}U'_\Delta = I_p - U_\Delta(A^2_\Delta + I_p)^{-1}U'_\Delta$,

because $U_\Delta$ is orthogonal.
Therefore,

\[(I_p + \Delta_n \Delta_n')^{-1}\Delta_n = \]
\[= (I_p - U_\Delta (\Lambda^{-2} + I_p)^{-1}U_\Delta')\Delta_n = \]
\[= \Delta_n - U_\Delta (\Lambda^{-2} + I_p)^{-1}U_\Delta' \Delta_n = \]
\[= U_\Delta \Lambda_\Delta U_\Delta' - U_\Delta (\Lambda^{-2} + I_p)^{-1}U_\Delta \Lambda_\Delta U_\Delta'. \]

Going on, since \(U_\Delta\) is orthogonal, we can write

\[\Delta_n - U_\Delta (\Lambda^{-2} + I_p)^{-1}U_\Delta' \Delta_n = \]
\[= U_\Delta \Lambda_\Delta U_\Delta' - U_\Delta (\Lambda^{-2} + I_p)^{-1}U_\Delta \Lambda_\Delta U_\Delta' = \]
\[= U_\Delta [\Lambda_\Delta - (\Lambda^{-2} + I_p)^{-1}\Lambda]U_\Delta' = \]
\[= U_\Delta [(I_p - (\Lambda^{-2} + I_p)^{-1})\Lambda]U_\Delta'. \]

The matrix \(D_\Delta = (I_p - (\Lambda^{-2} + I_p)^{-1})\Lambda\) is of extreme interest. It is a \(p \times p\) diagonal matrix, whose \(i\)-th element is \(D_{ii} = \Lambda_{\Delta,ii} \left(1 - \frac{1}{1+\Lambda_{\Delta,ii}}\right), i = 1, \ldots, p\). The matrix \(D_\Delta\) acts as an eigenvalue correction matrix: it shrinks down very large or very small sample eigenvalues, those most affecting matrix inversion. This is why our covariance matrix estimate can be called “eigenvalue-regularized”. Defining the matrix \(\Psi_\Delta = U_\Delta [\Lambda_\Delta - (\Lambda^{-2} + I_p)^{-1}\Lambda]U_\Delta\), we can thus write \(\mathcal{I}^*(\phi(L,S)) = \Psi_\Delta \otimes \Psi_\Delta\). Qualitatively, we can see that the curvature of the smooth loss is taken under control by this intrinsic eigenvalue regularization mechanism.

In Proposition 3, we have bounded the degree of transversality between the low rank and the sparse variety, to control the impact of \(\mathcal{I}^*(\phi(L,S))\) on matrix variety identification. Now, we want to bound the error norm (28), with the solution pair \((\hat{L}, \hat{S})\) defined in (4). To reach that goal, following Chandrasekaran et al. (2011), we need before to consider the error norm of the solution pair (29).
Proposition 4. Let $\varrho(T', T) \leq \xi(T)/2$ and define

$$
\bar{r} = \max \left\{ \frac{4}{\alpha} [g_\gamma(A^\dagger \Delta_n) + g_\gamma(A^\dagger T^* C_{T'}) + \psi_0], \|C_{T'}\|_2 \right\}
$$

where $C_{T'} = P_{T'\perp} (L^*)$ and $\Delta_n = \Sigma_n - \Sigma^*$. Then, under the conditions of Proposition 3 and Lemma 6, the solution of problem (29) $(\hat{S}_\Omega, \hat{L}_{T'})$ satisfies

$$
g_\gamma(\hat{S}_\Omega - S^*, \hat{L}_{T'} - L^*) \leq 2\bar{r}
$$

with high probability as $n \to \infty$.

Proof of Proposition 4

Proof. Based on Proposition 3 we know that the optimum $(\hat{S}_\Omega, \hat{L}_{T'})$ is unique, because $\phi_D(L, S)$ is strictly convex if $S \in \Omega$ and $L \in T'$, as Lemma 6 holds true. By the tangent space constraints, we know that there exist two Lagrangian multipliers in the spaces $T'\perp$ and $\Omega\perp$, say, $Q_{T'\perp} \in T'\perp$ and $Q_{\Omega\perp} \in \Omega\perp$, such that $\phi'(L, S) + Q_{T'\perp} \in -\psi_0 \delta \|\hat{L}_{T'}\|_*$, and $\phi'(L, S) + Q_{\Omega\perp} \in -\psi_0$, where $\delta \|\hat{L}_{T'}\|_*$ and $\delta \|\hat{S}_\Omega\|_1$ denote the sub-differentials of $\|\hat{L}_{T'}\|_*$ and $\|\hat{S}_\Omega\|_1$, respectively (see Watson (1992)).

More, we can write

$$
P_\Omega(\hat{S}_\Omega) = -\psi_0 \rho_0 \text{sgn}(S^*), \quad \text{and} \quad P_{T'}(\hat{L}_{T'}) = -\psi_0 U_L U'_L,
$$

because $\hat{S}_\Omega \in \Omega$ and $\hat{L}_{T'} \in T'$. It follows that $\|P_{T'}(\hat{L}_{T'})\| \leq 2\psi_0$, $\|P_\Omega(\hat{S}_\Omega)\|_\infty = \psi_0 \gamma$, and therefore, $g_\gamma \left( P_{T'}(\hat{L}_{T'}), P_\Omega(\hat{S}_\Omega) \right) \leq 2\psi_0$.

Recalling (14), we can write

$$
\phi'(L, S)^{(ld)} = (I_p + (\hat{S}_\Omega + \hat{L}_{T'} - \Sigma_n)(\hat{S}_\Omega + \hat{L}_{T'} - \Sigma_n)^{-1}(\hat{S}_\Omega + \hat{L}_{T'} - \Sigma_n)).
$$

At this stage, we know that $\Delta_n \Delta'_n$ is a positive semi-definite matrix, that can be written
as $U_{\Delta}A_{\Delta}^2U_{\Delta}'$, with $U_{\Delta}$ orthogonal matrix. As a consequence, we can write

$$\Psi = \sum_{j=0}^{\infty}(-\Delta_n\Delta_n')^j = \sum_{j=0}^{\infty}U_{\Delta}(-\Lambda_{\Delta})^{2j}U_{\Delta}'.$$  

It follows that $\|\Psi(L, S)\| = \sum_{j=0}^{\infty}(-\|\Lambda_{\Delta}\|)^{2j}$. Moreover, since Lemma 6 holds true under the conditions of Proposition 3, we are sure that $\|\Delta_n\Delta_n'\| < 1$, which leads to $\sum_{j=0}^{\infty}(-\|\Lambda_{\Delta}\|)^{2j} = \frac{1}{1+\|\Delta_n\Delta_n'\|} < 1$ with high probability. Consequently, the following inequality

$$\|\phi'(L, S)\| \leq \|\hat{S}_\Omega + \hat{L}_{T'} - \Sigma_n\|$$

and the following lemma descend.

**Lemma 7.** Under the conditions of Proposition 4

$$\|\phi_D'(L, S)^{(ld)}\| \leq \|\phi_D'(L, S)^{(F)}\|.$$  

**Proof.** It is sufficient to recall that, under the conditions of Proposition 4

$$\|\phi_D'(L, S)^{(ld)}\| \leq \|\Psi(\hat{L}_{T'}, \hat{S}_\Omega)||\hat{S}_\Omega + \hat{L}_{T'} - \Sigma_n||,$$

with $\Psi(L, S) = (I_p + \Delta_n\Delta_n')^{-1}$, and $\|\Psi(\hat{L}_{T'}, \hat{S}_\Omega)|| = \frac{1}{1+\|\Delta_n\Delta_n'\|} < 1$, because Lemma 6 holds.

Now, following Luo (2011), we may observe that

$$\hat{S}_\Omega + \hat{L}_{T'} - \Sigma_n = \Delta_n + AT^*\Delta_n - C_{T'}.$$  

We apply Brouwer’s fixed point theorem, to look for the fixed point of the function

$$F(M_L, M_S) = (M_L, M_S) - (P_{Y^*}A^*A^*P_{Y'})^{-1}P_{Y^*}A^*(\Delta_n - AT^*C_{T'}).$$  

We know that $(P_{\Omega}(\hat{S}_\Omega), P_{T'}(\hat{L}_{T'}))$ is a fixed point of (35), and it is unique by Lemma 6.
More, from Proposition 3 (part 1) and from $g_\gamma \left( \mathcal{P}_{T'}(\hat{L}_{T'}), \mathcal{P}_{\Omega}(\hat{S}_{\Omega}) \right) \leq 2\psi_0$, we know that

$$g_\gamma(F(M_L, M_S)) \leq \frac{2}{\alpha} g_\gamma(\mathcal{P}_y(A^\dagger \Delta_n - A^\dagger T^* C_{T'} - Z)) \leq \frac{4}{\alpha} g_\gamma(A^\dagger \Delta_n - A^\dagger T^* C_{T'} - \psi_0).$$

Finally, it is enough to observe that

$$g_\gamma(\hat{S}_{\Omega} - S^*, \hat{L}_{T'} - L^*) \leq g_\gamma(F(M_L, M_S)) + ||C_{T'}||_2,$$

from which the thesis follows.

We may now state the main results of this section.

**Theorem 1.** Suppose that Proposition 4, Assumptions 4 and 5 hold. Define $\psi_0 = \frac{1}{\xi(L^*)} \sqrt{\log p \frac{1}{n}}$ and $\rho_0 = \gamma \psi_0$. Suppose that $\delta_1 \leq \frac{\alpha}{3}$. Then, there exists a positive real $\kappa$ independent of $p$ and $n$ such that, as $p, n \to \infty$ the pair of solutions defined in (4) satisfies:

1. $\mathcal{P}(\frac{1}{p^{\delta_1}} ||\hat{L} - L^*||_2 \leq \kappa \psi_0) \to 1$;
2. $\mathcal{P}(||\hat{S} - S^*||_\infty \leq \kappa \rho_0) \to 1$;
3. $\mathcal{P}(\text{rk}(\hat{L}) = \text{rk}(L^*)) \to 1$;
4. $\mathcal{P}(\text{sgn}(\hat{S}) = \text{sgn}(S^*)) \to 1$.

**Corollary 2.** Under all the assumptions and conditions of Theorem 1 as $p, n \to \infty$ it holds with high probability:

1. $\mathcal{P}(\frac{1}{p^{\delta_1}} ||\hat{S} - S^*||_2 \leq \kappa \sqrt{\log p \frac{1}{n}}) \to 1$;
2. $\mathcal{P}(\frac{1}{p^{\delta_1+\delta_1}} ||\hat{\Sigma} - \Sigma^*||_2 \leq \kappa \sqrt{\log p \frac{1}{n}}) \to 1$;
3. $\mathcal{P}(\lambda_p(\hat{S}) > 0) \to 1$;
4. $\mathcal{P}(\lambda_p(\hat{\Sigma}) > 0) \to 1$.

In addition, supposing that $\lambda_p(S^*) = O(1)$ and $\lambda_p(\Sigma^*) = O(1)$, the following statements hold as $p, n \to \infty$:

5. $\mathcal{P}(\frac{1}{p^{\delta_1}} ||\hat{S}^{-1} - S^{-1}||_2 \leq \kappa \sqrt{\log p \frac{1}{n}}) \to 1$;
6. \( P( \frac{1}{p^{1+\alpha_1}} \| \hat{\Sigma}^{-1} - \Sigma^{*^{-1}} \|_2 \leq \kappa \sqrt{\frac{\log p}{n}} ) \rightarrow 1. \)

Theorem 1 and Corollary 1 establish the algebraic and parametric consistency of the estimator pair in (4). Their proofs can be found in Appendix A. The following remarks clarify the most relevant theoretical aspects.

Remark 2. Parts 3 and 4 of Theorem 1 and Corollary 1, jointly considered, ensure the algebraic consistency of (4). This result is established by adapting the results of Chandrasekaran et al. (2012), by means of Propositions 3 and 4, that take into account the random nature of the second derivative of the smooth loss \( \phi_D(L, S)^{(ld)} \). Another necessary technical key is to control the manifolds containing \( L^* \) and \( S^* \) by Assumption 5, which causes the term \( O(p^{\delta_1}) \) to appear in the rates of \( \hat{S} \) and \( \hat{\Sigma} \) (parts 1 and 2 of Corollary 1).

Remark 3. Requiring \( \delta_1 > 0 \) (Assumption 2) ensures that the equation \( \xi(T(L^*)) = \sqrt{r \kappa L p^{\delta_1}} \) (Assumption 5) leads to an increasingly small identifiability error as \( p \rightarrow \infty \). In this respect, a high dimension is rather a blessing than a curse for our method. Note that the prevalence of the latent factor structure versus the residual one is preserved as \( p \rightarrow \infty \) by the condition \( \delta_1 < \alpha_r \), resulting from Assumptions 1 and 2.

Remark 4. We note that Theorem 1 and Corollary 1 hold as well for the solution pair of (2) with \( L(L, S) = \frac{1}{2} \| \Sigma_n - (L + S) \|_F^2 \), that corresponds to ALCE estimator (Farnè and Montanari, 2020), since the second derivative of the smooth component \( \phi_D(L, S)^{(F)} = \frac{1}{2} \| \Sigma_n - (L + S) \|_F^2 \) is constant and equal to the null matrix, i.e., \( \phi_D(L, S)^{(F)} \) is globally convex.

Remark 5. Should we assume \( \alpha_1 = \ldots = \alpha_r = 1 \) and \( \delta_1 = 0 \), we re-obtain the rates in spectral norm of Fan et al. (2013) for \( \hat{L} \) (part 1 of Theorem 1), \( \hat{S} \) (part 1 of Corollary 2), and \( \hat{\Sigma} \) (part 2 of Corollary 2), and the rate in maximum norm of Bickel and Levina (2008a) for \( \hat{S} \) (part 2 of Theorem 1).

Remark 6. The error rate of \( \hat{S}^{-1} \) is similar to Fan et al. (2013) and Bickel and Levina (2008a). The error rate of \( \hat{\Sigma}^{-1} \) is instead worse than the corresponding one in Fan et al. (2013). It may be improved by explicitly estimating the factor scores via this method, which is left to future research.
Remark 7. The error bound is maximum when \( \nu = \frac{1}{2} \). In that case, by Assumption [10] we obtain \( \delta = 0 \). It follows that \( P_{T'}(S^*) = 0_{p \times p} \) and \( P_{1\Omega}(L^*) = 0_{p \times p} \), and then, \( \alpha = 1 \) and \( \beta = 1 \). This means that, in the case \( \nu < \frac{1}{2} \), the rate of [14] is tighter, and adapts to the underlying algebraic structure. However, the price to pay is that the identifiability condition \( \frac{\sqrt{T \kappa}}{\kappa L} \leq \frac{1}{6} \left( \frac{\nu \alpha}{n(2-\nu)} \right)^2 \) becomes more stringent in that case.

Suppose now that we explicitly compare the heuristics of [Farnè and Montanari 2020]

\[
\phi(F)(L, S) = \min_{L, S} \frac{1}{2} \| \Sigma_n - (L + S) \|_F^2 + \psi \| L \|_+ + \rho \| S \|_1
\]  

(36)

to the heuristics of this paper:

\[
\phi(L, S)^{(id)} = \min_{L, S} \frac{1}{2} \log \det (I_p + (\Sigma_n - (L + S))(\Sigma_n - (L + S))^T) + \psi \| L \|_+ + \rho \| S \|_1.
\]  

(37)

We write \( \phi(L, S)^{(F)} = \phi_D(L, S)^{(F)} + \phi_{ND}(L, S) \), with \( \phi_D(L, S)^{(F)} = \| \Sigma_n - (L + S) \|_F^2 \), and \( \phi(L, S)^{(id)} = \phi_D(L, S)^{(id)} + \phi_{ND}(L, S) \), with \( \phi_D(L, S)^{(id)} = \log \det (I_p + (\Sigma_n - (L + S))(\Sigma_n - (L + S))^T) \). We have shown in [14] that \( \phi'_D(L, S)^{(F)} = (I_p + \Delta_n \Delta_n^{-1})^{-1} \Delta_n \), where \( \Delta_n = \Sigma_n - (L + S) \). We can easily derive that \( \phi'_D(L, S)^{(F)} = \Delta_n = \Sigma_n - (L + S) \).

Let us define the pair of solutions \((\hat{L}^{(F)}, \hat{S}^{(F)})\) as \( \arg \min_{L, S} \phi^{(F)}(L, S) \) and \((\hat{L}^{(id)}, \hat{S}^{(id)})\) as \( \arg \min_{L, S} \phi^{(id)}(L, S) \), with \( \hat{\Sigma}^{(F)} = \hat{L}^{(F)} + \hat{S}^{(F)} \) and \( \hat{\Sigma}^{(id)} = \hat{L}^{(id)} + \hat{S}^{(id)} \). The following important theorem holds.

**Theorem 2.** Let \( \nu = \frac{1}{2} \). Then, Theorem [14] and Corollary [12] hold for \((\hat{L}^{(F)}, \hat{S}^{(F)})\). More, under all the conditions of Theorem [14], as \( p, n \to \infty \) it holds:

1. \( \| \hat{L}^{(id)} - L^* \|_{L, S} \leq 1 \) and \( \| \hat{\Sigma}^{(id)} - \Sigma^* \| \leq 1 \);
2. \( \| \hat{S}^{(id)} - S^* \|_{S, \infty} \leq 1 \) and \( \| \hat{\Sigma}^{(id)} - S^* \| \leq 1 \).

In addition, supposing that \( \lambda_p(S^*) = O(1) \) and \( \lambda_p(\Sigma^*) = O(1) \), as \( p, n \to \infty \) it holds:

3. \( \| \hat{S}^{(id)} - 1 \|_{S, \infty} \leq 1 \) and \( \| \hat{\Sigma}^{(id)} - 1 \|_{\Sigma, \infty} \leq 1 \).

**Proof.** The proof descends from Theorem [14] and Corollary [12] that also hold straightforwardly for \( \hat{L}^{(F)} \) and \( \hat{S}^{(F)} \), and because Lemma [7] holds under the conditions of Proposition [14] such that
the error bound in $g_\gamma$ norm of $(\hat{L}_{ld}, \hat{S}_{ld})$ is smaller than or equal to the corresponding error bound $\tilde{r}_F$ of $(\hat{L}^{(F)}, \hat{S}^{(F)})$.

**Remark 8.** Theorem 2 states that, if $\nu = \frac{1}{2}$ (see Remark 7), $(\hat{L}^{(F)}, \hat{S}^{(F)})$ and $(\hat{L}_{ld}, \hat{S}_{ld})$ are both algebraically and parametrically consistent, and the error bound in $g_\gamma$ norm of $(\hat{L}_{ld}, \hat{S}_{ld})$ is systematically not larger than the corresponding bound of $(\hat{L}^{(F)}, \hat{S}^{(F)})$. If $\nu < \frac{1}{2}$, the error bound in $g_\gamma$ norm of $(\hat{L}_{ld}, \hat{S}_{ld})$ is tighter than the corresponding bound of $(\hat{L}^{(F)}, \hat{S}^{(F)})$, but the identifiability condition $\frac{\sqrt{\kappa_S \kappa_L}}{\kappa_L} \leq \frac{1}{16} \left(\frac{\nu}{\beta(2-\nu)}\right)^2$ of Theorem 1 becomes more stringent (see Remark 7).

### 6 Solution algorithm

Exploiting the results of Section 3, we provide a solution algorithm for problem (4). Following Luo (2011), Nesterov (2013) and the supplement of Farnè and Montanari (2020), and setting the relevant step-size to apply as $\ell = \frac{10}{4}$ from (21), we derive the following optimization procedure.

In Algorithm 1, we first rescale, at step 1, by the trace of the input $\Sigma_n$, and we then restore the original scale at step 4. This approach differs from the original application in Farnè and Montanari (2020). It has the advantage to set the threshold grid and to perform threshold selection in a controllable way. By Algorithm 1 we derive $(\hat{L}_{A}^{(ld)}, \hat{S}_{A}^{(ld)})$, where the superscript A stands for ALCE (ALgebraic Covariance Estimator). Note that we derive the step-size $\ell = \frac{10}{4}$ from the Lipschitz constant $l = \frac{5}{4}$ derived in Section 3 (see (21)).

Algorithm 2 is the analog of Algorithm 1 for problem (2) with $L(L, S) = L^{(F)}(L, S)$.

Algorithms 1 and 2, unlike the algorithm in Farnè and Montanari (2020), allow to define the vector of initial thresholds $\psi_{\text{init}}$ as a function of $\frac{1}{p}$, and the vector of initial thresholds $\rho_{\text{init}}$ as a function of $\frac{1}{p\sqrt{p}}$, because $\sqrt{p}$ is the maximum allowed degree order in the residual component under Assumption 2. This is due to the fact that we rescale by the trace of the input in both algorithms (see step 1). In the end, for each threshold pair $(\psi, \rho)$ we can calculate $\hat{\Sigma}_{A}^{(ld)}(\psi, \rho) = \hat{L}_{A}^{(ld)}(\psi, \rho) + \hat{S}_{A}^{(ld)}(\psi, \rho)$, $\hat{\Sigma}_{A}^{(F)}(\psi, \rho) = \hat{L}_{A}^{(F)}(\psi, \rho) + \hat{S}_{A}^{(F)}(\psi, \rho)$, and
Algorithm 1 Pseudocode to solve problem (1) given any input covariance matrix $\Sigma_n$.

1. Set $(L_0, S_0) = \frac{1}{2tr(\Sigma_n)}(\text{diag}(\Sigma_n), \text{diag}(\Sigma_n))$, $\eta_0 = 1$.

2. Initialize $Y_0 = L_0$ and $Z_0 = S_0$. Set $t = 1$.

3. For $t \geq 1$, repeat:
   
   (i) calculate $\Delta_{t,n} = Y_{t-1} + Z_{t-1} - \Sigma_n$;
   
   (ii) compute $\frac{\partial}{\partial Y_{t-1}} \log \det \big( I_p + \Delta_{t,n} \Delta_{t,n}' \big) = \frac{\partial}{\partial Z_{t-1}} \log \det \big( I_p + \Delta_{t,n} \Delta_{t,n}' \big) = \big( I_p + \Delta_{t,n} \Delta_{t,n}' \big)^{-1} \Delta_{t,n}$;
   
   (iii) apply the singular value thresholding (SVT, Cai et al. (2010)) operator $T_{\psi_0}$ to $E_{Y,t} = Y_{t-1} - \frac{1}{\ell} \big( I_p + \Delta_{t,n} \Delta_{t,n}' \big)^{-1} \Delta_{t,n}$, with $\ell = \frac{10}{4}$, and set $L_t = T_{\psi}(E_{Y,t}) = \hat{U} \hat{D}_{\psi} \hat{U}^\top$;
   
   (iv) apply the soft-thresholding operator $T_{\rho}$ (Daubechies et al., 2004) $T_{\rho_0}$ to $E_{Z,t} = Z_{t-1} - \frac{1}{\epsilon} \big( I_p + \Delta_{t,n} \Delta_{t,n}' \big)^{-1} \Delta_{t,n}$, with $\ell = \frac{10}{4}$, and set $S_t = T_{\rho}(E_{Z,t})$;
   
   (v) set $(Y_t, Z_t) = (L_t, S_t) + \left\{ \frac{\eta_t-1}{\eta_t} \right\} \{ (L_{t-1}, S_{t-1}) - (L_{t-1}, S_{t-1}) \}$ where $\eta_t = \frac{1}{2} + \frac{1}{\ell} \sqrt{1 + 4\eta_{t-1}^2}$
   
   (vi) stop if the convergence criterion $\frac{\|L_t-L_{t-1}\|_F}{1+\|L_t-L_{t-1}\|_F} + \frac{\|S_t-S_{t-1}\|_F}{1+\|S_t-S_{t-1}\|_F} \leq \varepsilon$.

4. Set $\hat{L}^{(ld)}_A = \text{tr}(\Sigma_n)Y_t$ and $\hat{S}^{(ld)}_A = \text{tr}(\Sigma_n)Z_t$.

Algorithm 2 Pseudocode to solve problem (2) with $\mathcal{L}(L, S) = \mathcal{L}^{(F)}(L, S)$.

1. Set $(L_0, S_0) = \frac{1}{2tr(\Sigma_n)}(\text{diag}(\Sigma_n), \text{diag}(\Sigma_n))$, $\eta_0 = 1$.

2. Initialize $Y_0 = L_0$ and $Z_0 = S_0$. Set $t = 1$.

3. For $t \geq 1$, repeat:
   
   (i) calculate $\frac{\partial}{\partial Y_{t-1}} \|Y_{t-1} + Z_{t-1} - \Sigma_n\|_F^2 = \frac{\partial}{\partial Z_{t-1}} \|Y_{t-1} + Z_{t-1} - \Sigma_n\|_F^2 = Y_{t-1} + Z_{t-1} - \Sigma_n$;
   
   (ii) apply the singular value thresholding (SVT) operator $T_{\psi}$ to $E_{Y,t} = Y_{t-1} - \frac{1}{\ell} (Y_{t-1} + Z_{t-1} - \Sigma_n)$ and set $L_t = T_{\psi}(E_{Y,t}) = \hat{U} \hat{D}_{\psi} \hat{U}^\top$;
   
   (iii) apply the soft-thresholding operator $T_{\rho}$ to $E_{Z,t} = Z_{t-1} - \frac{1}{\ell} (Y_{t-1} + Z_{t-1} - \Sigma_n)$ and set $S_t = T_{\rho}(E_{Z,t})$;
   
   (iv) set $(Y_t, Z_t) = (L_t, S_t) + \left\{ \frac{\eta_t-1}{\eta_t} \right\} \{ (L_{t-1}, S_{t-1}) - (L_{t-1}, S_{t-1}) \}$ where $\eta_t = \frac{1}{2} + \frac{1}{\ell} \sqrt{1 + 4\eta_{t-1}^2}$
   
   (v) stop if the convergence criterion $\frac{\|L_t-L_{t-1}\|_F}{1+\|L_t-L_{t-1}\|_F} + \frac{\|S_t-S_{t-1}\|_F}{1+\|S_t-S_{t-1}\|_F} \leq \varepsilon$.

4. Set $\hat{L}^{(F)}_A = \text{tr}(\Sigma_n)Y_t$ and $\hat{S}^{(F)}_A = \text{tr}(\Sigma_n)Z_t$. 

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\[ \tilde{\Sigma}_A^{(F)}(\psi, \rho) = \tilde{L}_A^{(F)}(\psi, \rho) + \tilde{S}_A^{(F)}(\psi, \rho). \]

Following Farnè and Montanari (2020), we also perform the unshrinkage of estimated latent eigenvalues, as this operation improves the sample total loss as much as possible in the finite sample. We thus get the UNALCE (UNshrunken ALCE) estimates as:

\[ \tilde{L}_U = \tilde{U}_A(\tilde{A}_A + \psi I_r)\tilde{U}_A', \tag{38} \]
\[ \text{diag}(\tilde{S}_U) = \text{diag}(\tilde{\Sigma}_A) - \text{diag}(\tilde{L}_U), \tag{39} \]
\[ \text{off} - \text{diag}(\tilde{S}_U) = \text{off} - \text{diag}(\tilde{S}_A), \tag{40} \]

where \( \psi > 0 \) is any chosen eigenvalue threshold parameter.

By setting \( \tilde{r}_A = \text{rk}(\tilde{L}_A) \) and defining the spectral decomposition of \( \tilde{L}_A \) as \( \tilde{L}_A = \tilde{U}_A\tilde{D}_A\tilde{U}_A' \), with \( \tilde{U}_A \) a \( p \times \tilde{r}_A \) matrix such that \( \tilde{U}_A'\tilde{U}_A = I_{\tilde{r}_A} \), and \( \tilde{D}_A \) a \( \tilde{r}_A \times \tilde{r}_A \) diagonal matrix, it can be proved (Farnè and Montanari, 2020) that it holds

\[ \left( \tilde{L}_U, \tilde{S}_U \right) = \arg \min_{L \in \tilde{L}(\tilde{r}_A), S \in \tilde{S}_{\text{diag}}} \frac{1}{2} \| \Sigma_n - (L + S) \|_2, \tag{41} \]

where

\[ \tilde{L}(\tilde{r}_A) = \{ L \mid L \succeq 0, L = \tilde{U}_A D \tilde{U}_A', D \in \mathbb{R}^{r \times r} \text{diagonal} \}, \tag{42} \]
\[ \tilde{S}_{\text{diag}} = \{ S \in \mathbb{R}^{p \times p} \mid \text{diag}(L) + \text{diag}(S) = \text{diag}(\tilde{\Sigma}_A), \text{off} - \text{diag}(S) = \text{off} - \text{diag}(\tilde{S}_A), L \in \tilde{L}(\tilde{r}_A) \}. \tag{43} \]

For this reason, we calculate \( \tilde{L}_U \) and \( \tilde{S}_U \) as in (38), (39) and (40) by Algorithm 1 or 2 and we obtain, for each threshold pair \( (\psi, \rho) \), the pairs of estimates

\[ \left( \tilde{L}_U^{(ld)}(\psi, \rho), \tilde{S}_U^{(ld)}(\psi, \rho) \right) \text{ and } \left( \tilde{L}_U^{(F)}(\psi, \rho), \tilde{S}_U^{(F)}(\psi, \rho) \right). \]

As a consequence, we can derive the overall UNALCE estimates as \( \tilde{\Sigma}_U^{(ld)}(\psi, \rho) = \tilde{L}_U^{(ld)}(\psi, \rho) + \tilde{S}_U^{(ld)}(\psi, \rho) \) and \( \tilde{\Sigma}_U^{(F)}(\psi, \rho) = \tilde{L}_U^{(F)}(\psi, \rho) + \tilde{S}_U^{(F)}(\psi, \rho). \)

Then, given the latent variance proportions \( \tilde{\theta}(\psi, \rho)_A = \frac{\text{tr}(\tilde{L}(\psi, \rho)_A)}{\text{tr}(\tilde{\Sigma}(\psi, \rho)_A)} \) and \( \tilde{\theta}(\psi, \rho)_U = \frac{\text{tr} \tilde{L}(\psi, \rho)_U}{\text{tr}(\tilde{\Sigma}(\psi, \rho)_U)}, \)
we can select the optimal threshold pairs \((\psi_U, \rho_U)\) and \((\psi_A, \rho_A)\) by minimizing the MC criteria

\[
MC(\psi, \rho)_U = \max \left\{ \frac{\|\hat{L}(\psi, \rho)_U\|_2}{\hat{\theta}(\psi, \rho)_U}, \frac{\|\hat{S}(\psi, \rho)_U\|_{1,v}}{\gamma(1 - \hat{\theta}(\psi, \rho)_U)} \right\},
\]

\[
MC(\psi, \rho)_A = \max \left\{ \frac{\|\hat{L}(\psi, \rho)_A\|_2}{\hat{\theta}(\psi, \rho)_A}, \frac{\|\hat{S}(\psi, \rho)_A\|_{1,v}}{\gamma(1 - \hat{\theta}(\psi, \rho)_A)} \right\},
\]

where \(\gamma = \frac{\rho}{\psi}\) is the ratio between the sparsity and the latent eigenvalue threshold (see Farnè and Montanari (2020) for more details). In this way, we can select the optimal threshold pairs \((\psi_A, \rho_A) = \arg \min_{(\psi, \rho)} MC(\psi, \rho)_A\) and \((\psi_U, \rho_U) = \arg \min_{(\psi, \rho)} MC(\psi, \rho)_U\). This procedure is applied for Algorithms 1 and 2, where the possible threshold pairs are derived by the Cartesian product of the initial vectors \(\psi_{init}\) and \(\rho_{init}\) (see Section 8).

7 Simulation study

In this section, we test the theoretical results of previous sections on some data simulated with this purpose. Hereafter, we report the key simulation parameters:

1. the dimension \(p\) and the sample size \(n\);
2. the rank \(r\) and the condition number \(c = \text{cond}(L^*) = \lambda_1(L^*)/\lambda_r(L^*)\) of the low rank component \(L^*\);
3. the trace of \(L^*\), \(\tau \theta p\), where \(\tau\) is a magnitude parameter and \(\theta = \text{tr}(L^*)/\text{tr}(\Sigma^*)\) is the proportion of variance explained by \(L^*\);
4. the number of off-diagonal non-zeros \(s\) in the sparse component \(S^*\);
5. the minimum latent eigenvalue \(\lambda_r(L^*)\);
6. the minimum nonzero off-diagonal residual entry in absolute value \(\|S^*\|_{\text{min,off}}\);
7. the proportion of non-zeros over the number of off-diagonal elements, \(\pi_s = \frac{2s}{p(p-1)}\);
8. the proportion of (absolute) residual covariance \(\rho_{S^*} = \frac{\sum_{i=1}^p \sum_{j \neq i} |S^*_{ij}|}{\sum_{i=1}^p \sum_{j \neq i} |S_{ij}|}\);
Table 1: Simulated settings: parameters.

| Setting | p   | n   | p/n | r  | θ  | c   | \( \pi_s \) | \( \rho_s^* \) | spikiness | sparsity |
|---------|-----|-----|-----|----|----|-----|-----------|-------------|-----------|----------|
| 1       | 100 | 1000| 0.1 | 4  | 0.7| 2   | 0.0238    | 0.0045     | low       | high     |
| 2       | 100 | 1000| 0.1 | 3  | 0.8| 4   | 0.1172    | 0.0072     | high      | low      |
| 3       | 150 | 150 | 1   | 5  | 0.8| 2   | 0.0320    | 0.0033     | middle    | middle   |
| 4       | 200 | 100 | 2   | 6  | 0.8| 2   | 0.0366    | 0.0039     | middle    | middle   |

Table 2: Simulated settings: spectral norms and condition numbers.

| Setting | \( \| L^* \|_2 \) | \( \lambda_r(L^*) \) | c     | \( \| S^* \|_2 \) | \( \| S^* \|_{min,off} \) | \( \text{cond}(S^*) \) | \( \| \Sigma^* \|_2 \) | \( \text{cond}(\Sigma^*) \) |
|---------|-----------------|-----------------|-------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 1       | 23.33           | 11.67           | 2     | 3.78            | 0.0275          | 2.26e07         | 24.49           | 9.49e07         |
| 2       | 128             | 32              | 4     | 5.58            | 0.0226          | 2.53e05         | 130.14          | 4.07e06         |
| 3       | 32              | 16              | 2     | 2.56            | 0.0161          | 2.35e13         | 32.48           | 1.58e10         |
| 4       | 35.56           | 17.78           | 2     | 4.69            | 0.0138          | 1.17e13         | 36.39           | 3.09e09         |

9. \( K = 100 \) replicates for each setting.

The detailed simulation algorithm is reported in [Farnè (2016)](http://example.com).

The main parameters of simulated settings are reported in Tables 1 and 2. Setting 1 presents not so spiked eigenvalues and a very sparse residual component. Setting 2 has spiked eigenvalues and a far less sparse residual. Settings 3 and 4 are intermediately spiked and sparse but present a much lower \( p/n \) ratio. In particular, while Settings 1 and 2 have \( p/n = 0.1 \), Setting 3 has \( p/n = 1 \) and Setting 4 has \( p/n = 2 \). In each setting, the eigenvalues of \( L^* \) and \( \Sigma^* \) almost overlap, while the eigenvalues of \( S^* \) are much smaller. Note that the minimum allowed off-diagonal residual element in absolute value, \( \| S^* \|_{min,off} \), decreases from Setting 1 to Setting 4.

For each scenario, we simulate \( N = 100 \) replicates from model (5), thus getting 100 instances of the input sample covariance matrix \( \Sigma_n \). Then:

- we apply Algorithm 1 to each generated \( \Sigma_n \) to get the pair of estimates (4), that we call ALCE-ld pair: \( \left( \hat{L}^{(ld)}_A, \hat{S}^{(ld)}_A \right) \). Then, we apply the unshrinkage steps in (38), (39), (40), and we get the UNALCE-ld pair of estimates \( \left( \hat{L}^{(ld)}_U, \hat{S}^{(ld)}_U \right) \).
- we apply Algorithm 2 to each generated \( \Sigma_n \) to get the pair of estimates (2), that we call ALCE-F pair: \( \left( \hat{L}^{(F)}_A, \hat{S}^{(F)}_A \right) \). Then, we apply the unshrinkage steps in (38), (39), (40), and we get the UNALCE-F pair: \( \left( \hat{L}^{(F)}_U, \hat{S}^{(F)}_U \right) \).
Let us denote the generic low rank estimate as \( \hat{L} \), the generic sparse estimate as \( \hat{S} \), and the generic covariance matrix estimate \( \hat{\Sigma} = \hat{L} + \hat{S} \). The performance metrics to assess the quality of estimates are the Frobenius total loss \( TLF = \| \hat{\Sigma} - \Sigma^* \|_F \); the spectral total loss \( TL2 = \| \hat{\Sigma} - \Sigma^* \|_2 \); the spectral low rank loss \( LL2 = \| \hat{L} - L^* \|_2 \); the sparse maximum loss \( SLM = \| \hat{S} - S^* \|_\infty \). The proportion of wrongly recovered latent ranks is \( err(\hat{r}) = \frac{1}{N} \sum_{k=1}^{K} \mathbb{1}(\hat{r}_k = r) \).

The estimated proportion of latent variance \( \hat{\theta} \), of residual covariance \( \hat{\rho}S \), and of residual non-zeros \( \hat{\pi}s \) are also computed. Their estimation performance is measured by the estimation bias for each parameter, defined as \( bias(\hat{\theta}) = \hat{\theta}_\text{mean} - \theta, bias(\hat{\rho}S) = \hat{\rho}_{S,\text{mean}} - \rho^*, bias(\hat{\pi}s) = \hat{\pi}_{s,\text{mean}} - \pi^s \), where \( \hat{\theta}_\text{mean}, \hat{\rho}_{S,\text{mean}} \) and \( \hat{\pi}_{s,\text{mean}} \) are the mean estimates of \( \theta, \rho^S \) and \( \pi^s \) over the \( N \) replicates.

The performance in terms of eigen-structure recovery is measured for \( \Sigma^* \) by \( \lambda(\hat{\Sigma}) \), which is defined as the Euclidean distance between the estimated and true eigenvalues of \( \Sigma^* \):

\[
\lambda(\hat{\Sigma}) = \sqrt{\sum_{i=1}^{p} (\hat{\lambda}_i(\hat{\Sigma}) - \lambda_i(\Sigma^*))^2}.
\] (45)

Measure (45) is similarly defined for \( L^* \) and \( S^* \) as \( \lambda(\hat{L}) \) and \( \lambda(\hat{S}) \), respectively. All three measures are averaged over the \( N \) replicates.

In the end, we calculate the following metrics for sparsity pattern recovery:

- \( poserr = \frac{pos - wr}{pos} \), where \( pos - wr \) are the positive elements estimated as zero or negative by mistake;
- \( negerr = \frac{neg - wr}{pos} \), where \( neg - wr \) are the negative elements estimated as zero or positive by mistake;
- \( zeroerr = \frac{zero - wr}{pos} \), where \( zero - wr \) are the zero elements estimated as positive or negative by mistake.

These measures are averaged over the \( N \) replicates.

Tables 3, 4, 5, and 6 report simulation results with respect to UNALCE-ld, ALCE-ld, UNALCE-F, and ALCE-F, for Scenarios 1,2,3,4, respectively. First, we can note that the
latent rank is perfectly recovered by all methods. The UNALCE estimates have a consistent advantage over ALCE ones for what concerns the latent variance proportion $\theta$, which is an important parameter in factor modelling. This advantage increases as $\theta$ and $p$ increase. This fact reflects upon the metrics on eigenvalue estimation. Concerning the performance metrics, we can see that log-det based estimates tend to be slightly more accurate than Frobenius ones under Scenario 4, that presents a large $p/n$. Concerning the sparsity pattern metrics, we note that, as $p/n$ and $S_{\text{min,off}}$ increase, the sparsity pattern recovery gets increasingly worse.

Table 3: Simulation results for Scenario 1.

|                | UNALCE-ld | ALCE-ld | UNALCE-F | ALCE-F |
|----------------|-----------|---------|----------|--------|
| $\text{TLF}$  | 6.9833    | 6.9850  | 6.9826   | 6.9872 |
| $\text{TL2}$  | 4.7471    | 4.7623  | 4.7488   | 4.7869 |
| $\text{LL2}$  | 4.7198    | 4.7340  | 4.7165   | 4.7420 |
| $\text{SLM}$  | 0.2105    | 0.2096  | 0.2157   | 0.2078 |
| $\text{err}(\hat{r})$ | 0         | 0       | 0        | 0      |
| $\text{bias}(\hat{\theta})$ | -0.0039   | -0.0058 | -0.0023  | -0.0071 |
| $\text{bias}(\hat{\rho}_S)$ | -0.0001   | 0.0001  | -0.0009  | -0.0009 |
| $\text{bias}(\hat{\pi}_S)$ | 0.0172    | 0.0184  | 0.0069   | 0.0068 |
| $\lambda(\hat{\Sigma})$ | 5.5079    | 5.5066  | 5.5083   | 5.5115 |
| $\lambda(\hat{S})$ | 0.2856    | 0.2876  | 0.3013   | 0.2940 |
| $\lambda(\hat{L})$ | 7.7540    | 7.7861  | 7.8553   | 7.7487 |
| $\text{poserr}$ | 0.2058    | 0.2013  | 0.2953   | 0.2821 |
| $\text{negerr}$ | 0.2078    | 0.2058  | 0.2906   | 0.2832 |
| $\text{zeroerr}$ | 0.0225    | 0.0236  | 0.0141   | 0.0137 |

8 Real data analysis

In this section, we compute $(\hat{L}_U^{(ld)}, \hat{S}_U^{(ld)})$ and $(\hat{L}_U^{(F)}, \hat{S}_U^{(F)})$ on a selection of 361 macroeconomic indicators provided by the European Central Bank for 364 systematically important Euro Area banks. The indicators, taken in logarithms, mainly are financial items in the banks’ balance sheet, reported at a high level of granularity. All data refer to Q4-2014.

Table 7 reports estimation results. Algorithms 1 and 2 are applied on the input sample covariance matrix, setting the vector of eigenvalue thresholds as $\psi_{\text{init}} = \frac{i}{p}$, with $i =$
Table 4: Simulation results for Scenario 2.

|       | UNALCE-ld | ALCE-ld | UNALCE-F | ALCE-F |
|-------|-----------|---------|----------|--------|
| TLF   | 11.6898   | 11.6917 | 11.6895  | 11.6931|
| TL2   | 7.8031    | 7.8324  | 7.7983   | 7.8402 |
| LL2   | 7.7352    | 7.7552  | 7.7271   | 7.7601 |
| SLM   | 0.1659    | 0.157   | 0.1559   | 0.1593 |

\[ \text{err}(\hat{r}) \]
\[ \text{bias}(\hat{\theta}) \]
\[ \text{bias}(\hat{\rho}_S) \]
\[ \text{bias}(\hat{\pi}_s) \]
\[ \lambda(\hat{\Sigma}) \]
\[ \lambda(\hat{S}) \]
\[ \lambda(\hat{L}) \]

\[ \text{poserr} \]
\[ \text{negerr} \]
\[ \text{zeroerr} \]

Table 5: Simulation results for Scenario 3.

|       | UNALCE-ld | ALCE-ld | UNALCE-F | ALCE-F |
|-------|-----------|---------|----------|--------|
| TLF   | 13.0149   | 13.0182 | 13.0287  | 13.0098|
| TL2   | 8.8829    | 8.8599  | 8.8981   | 8.8598 |
| LL2   | 8.8756    | 8.8442  | 8.8823   | 8.8477 |
| SLM   | 0.4004    | 0.3935  | 0.3952   | 0.3952 |

\[ \text{err}(\hat{r}) \]
\[ \text{bias}(\hat{\theta}) \]
\[ \text{bias}(\hat{\rho}_S) \]
\[ \text{bias}(\hat{\pi}_s) \]
\[ \lambda(\hat{\Sigma}) \]
\[ \lambda(\hat{S}) \]
\[ \lambda(\hat{L}) \]

\[ \text{poserr} \]
\[ \text{negerr} \]
\[ \text{zeroerr} \]
Table 6: Simulation results for Scenario 4.

|       | UNALCE-ld | ALCE-ld | UNALCE-F | ALCE-F |
|-------|-----------|---------|----------|--------|
| TLF   | 20.9703   | 20.9719 | 20.9889  | 20.9712|
| TL2   | 13.1786   | 13.1007 | 13.2233  | 13.0961|
| LL2   | 13.1599   | 13.0811 | 13.2006  | 13.0717|
| SLM   | 0.6544    | 0.6840  | 0.6564   | 0.6556 |

|       | err(\hat{\tau}) | bias(\hat{\theta}) | bias(\hat{\rho}_S) | bias(\hat{\pi}_S) | \lambda(\hat{\Sigma}) | \lambda(\hat{S}) | \lambda(\hat{L}) | poserr | negerr | zeroerr |
|-------|------------------|--------------------|---------------------|-------------------|----------------------|----------------|----------------|--------|--------|---------|
|       | 0                | -0.0060            | -0.0011             | -0.0209           | 10.002               | 0.7847         | 10.2028       | 0.7888 | 0.7853 | 0.0073  |
|       |                  |                     |                     |                   | 10.0684              | 0.8050          | 10.2576       | 0.7405 | 0.7351 | 0.0110  |
|       |                  |                     |                     |                   | 9.9909               | 0.7699          | 10.219        | 0.7561 | 0.7491 | 0.0104  |
|       |                  |                     |                     |                   | 10.0903              | 0.8013          | 10.2874       | 0.7466 | 0.7453 | 0.0102  |

$\frac{1}{20}, \frac{1}{10}, \frac{1}{5}, \frac{1}{3}, \frac{1}{2}, 1, 2, 5, 10, 20,$ and the vector of initial thresholds $\rho_{init}$ as $\frac{1}{\sqrt{p}} \psi_{init}$.

The scree plot of sample eigenvalues (Figure 1) highlights the presence of only one latent eigenvalue.

It follows that the estimated latent rank is 1 in both cases. The latent variance proportion is a bit smaller for $\hat{L}_U^{(ld)}$ compared to $\hat{L}_U^{(F)}$. More, $\hat{S}_U^{(ld)}$ is a bit more selective than $\hat{S}_U^{(F)}$ for residual nonzeros, and this results in a lower presence of non-zeros.

From the estimates $\hat{L}$, $\hat{S}$, and $\hat{\Sigma}$, we get for each variable $i = 1, \ldots, p$ the estimated commonality as $\frac{\hat{L}_{ii}}{\hat{\Sigma}_{ii}}$ and the estimated idiosyncrasy as $\frac{\hat{S}_{ii}}{\hat{\Sigma}_{ii}}$. The estimated residual degree is obtained as $deg_{\hat{S},i} = \sum_{j=1}^{p} 1(\hat{S}_{ij} \neq 0)$. We obtain the spectral decomposition of $\hat{L}$ as $U_L D_L U_L'$, and we compute the vector of loadings $U_L D_L^2$. The estimated loadings are very similar for $\hat{L}_U^{(ld)}$ and $\hat{L}_U^{(F)}$, although for $\hat{L}_U^{(ld)}$ they are slightly more concentrated, and denote the contrast between loans and receivables and the rest of supervisory indicators.

Table 8 shows that the extracted factor is mainly connected to total assets, followed by variables representing available cash. Table 9 shows that the variables most connected with

\footnote{Whenever one of the thresholds selected by the MC criterion \cite{44} lies in the grid extremes, it is advisable to shift the vector $\psi_{init}$ to the left. In this case, the MC criterion selected the pair with positions (9, 2) for UNALCE-ld and (5, 3) for UNALCE-F.}
all the others are related to credit risk, deposits, and loans. Table 10 shows that the most marginal variables wrt the factor structure are related to equity instruments and derivatives.

Table 8: Supervisory data: this table reports the top six variables by estimated commonality, with respect to $\hat{L}_U^{(ld)}$. This measure provides a ranking of the variables by systemic importance in determining the latent structure.

| Supervisory indicator | Commonality |
|-----------------------|-------------|
| Total assets          | 0.6593      |
| Advances that are not loans - Other financial corporations | 0.3747 |
| Held for trading - Equity instruments - Carrying amount | 0.3657 |
| Loans and advances - Governments - Allowances for credit risk | 0.3606 |

9 Conclusions

In this paper, we provide a study on the estimation of large covariance matrices in high dimensions under the low rank plus sparse assumption by minimizing a log-det heuristics augmented by a nuclear norm plus $l_1$ norm penalty. In particular, we prove the local convexity and the Lipschitzianity of the proposed log-det heuristics, which allows to solve the
Table 9: Supervisory data: this table reports the top four variables by estimated degree, with respect to $\hat{S}^{(ld)}_U$. This measure provides a ranking of the most connected variables with all the others, conditionally on the latent factor.

NFC stands for Non-Financial Corporations.

| Supervisory indicator                                                                 | Degree |
|--------------------------------------------------------------------------------------|--------|
| Credit spread option - Notional amount - Sold                                        | 104    |
| Deposits - NFC - Fair value                                                          | 102    |
| Deposits - NFC - Repurchase agreements - Held for trading                             | 94     |
| Loan commitments - Non-performing - Nominal amount                                   | 92     |

Table 10: Supervisory data: this table reports the top three variables by estimated idiosyncracy, with respect to $\hat{S}^{(ld)}_U$. This measure provides a ranking of the variables by systemic irrelevance in determining the latent structure.

NFC stands for Non-Financial Corporations.

| Supervisory indicator                                                                 | Idiosyncracy |
|--------------------------------------------------------------------------------------|--------------|
| Derivatives: Trading - Credit - Sold                                                 | 1            |
| Held for trading - Equity instruments - NFC - Carrying amount                        | 1            |
| Financial assets - Fair value - Equity instruments - NFC - Carrying amount            | 1            |

optimization problem via a proximal gradient algorithm. We bound the curvature of the log-det heuristics under an appropriate random matrix theory framework. Then, by adapting the results of Chandrasekaran et al. (2012), we solve the algebraic identification problem behind the low rank and sparse component recovery, due to the linearity of the log-det heuristics, and we prove the algebraic and parametric consistency of the ensuing pair of low rank and sparse covariance matrix estimators. We also prove that the same pair of estimators performs systematically not worse than the corresponding estimator of Farnè and Montanari (2020) obtained by nuclear norm plus $l_1$ norm penalized Frobenius loss minimization. A new solution algorithm, that also allows to control for the input threshold parameters, is proposed. A wide simulation study proves the validity of our theoretical results, and an ECB supervisory data example shows the usefulness of our approach on a real dataset.

A Proofs

Lemma A.1. Let $\lambda_r(\Sigma_n)$ be the $r-th$ largest eigenvalue of the sample covariance matrix $\Sigma_n = \frac{1}{n} \sum_{k=1}^{n} x_k x_k'$. Under Assumptions 1, 2 and 3, $\lambda_r(\Sigma_n) \simeq n^{-\gamma r}$ with probability approaching 1.
Proof. On one hand, we note that, since \( r + p - p = r \leq p \), dual Weyl inequality \( \text{(see Tao (2011))} \) can be applied, leading to

\[
\lambda_r(\Sigma^*) \geq \lambda_r(L^*) + \lambda_p(S^*). \tag{46}
\]

From \( (46) \), we can write

\[
\lambda_r(\Sigma^*) \succeq O(p^{\alpha_r}) + O(p^{\delta_1}) = O(p^{\alpha_r}), \tag{47}
\]

because \( \lambda_r(L^*) \simeq p^{\alpha_r} \) by Assumption \( \Pi(i) \), and \( \lambda_p(S^*) = O(p^{\delta_1}) \) by Assumption \( \Pi \) with \( \delta_1 < \alpha_r \).

On the other hand, Lidskii inequality \( \text{(see Tao (2011))} \) leads to

\[
\lambda_r(\Sigma^*) \leq \lambda_r(L^*) + \sum_{j=1}^{r} \lambda_j(S^*). \tag{48}
\]

From \( (48) \), we can write

\[
\lambda_r(\Sigma^*) \preceq O(p^{\alpha_r}) + O(rp^{\delta_1}) = O(p^{\alpha_r}), \tag{49}
\]

because \( \lambda_r(L^*) \simeq p^{\alpha_r} \) by Assumption \( \Pi(i) \), \( \lambda_p(S^*) = O(p^{\delta_1}) \) with \( \delta_1 < \alpha_r \) by Assumption \( \Pi \) and \( r \) is finite for all \( p \in \mathbb{N} \) by Assumption \( \Pi(ii) \). It follows that \( \lambda_r(\Sigma^*) \simeq p^{\alpha_r} \).

Recalling that \( \Sigma_n = \frac{1}{n} \sum_{k=1}^{n} x_k x_k' \) and \( x_k = Bf_k + \epsilon_k \), where \( f_k \) and \( \epsilon_k, k = 1, \ldots, n, \) are respectively the vectors of factor scores and residuals for each observation, we can decompose the error matrix \( E_n = \Sigma_n - \Sigma^* \) in four components as follows \( \text{(cf. Fan et al. (2013))} \):

\[
E_n = \Sigma_n - \Sigma^* = \hat{D}_1 + \hat{D}_2 + \hat{D}_3 + \hat{D}_4,
\]
where:

\[
\hat{D}_1 = \frac{1}{n} B \left( \sum_{k=1}^{n} f_k f'_k - I_r \right) B',
\]

\[
\hat{D}_2 = \frac{1}{n} \sum_{k=1}^{n} \left( \epsilon_k \epsilon'_k - S^* \right),
\]

\[
\hat{D}_3 = \frac{1}{n} B \sum_{k=1}^{n} f_k \epsilon'_k,
\]

\[
\hat{D}_4 = \hat{D}'_3.
\]

Following Fan et al. (2013), we note that

\[
\| \hat{D}_1 \|_2 \leq \left\| \frac{1}{n} \left( \sum_{k=1}^{n} f_k f'_k - I_r \right) \right\|_2 \leq r \left\| \frac{1}{n} B \sum_{k=1}^{n} f_k f'_k - E(f_k f'_k) \right\|_\infty,
\]

since \( E(f) = 0_r \) and \( V(f) = I_r \), \( \| BB' \|_2 = O(p^{\alpha_1}) \) by Assumption 1(i), and

\[
\left\| \frac{1}{n} \left( \sum_{k=1}^{n} f_k f'_k - I_r \right) \right\|_2 \leq r \left\| \frac{1}{n} \left( \sum_{k=1}^{n} f_k f'_k - I_r \right) \right\|_\infty
\]

by Assumption 1(ii).

Under Assumption 3, we can apply Lemma 4 in Fan et al. (2013), which claims

\[
\max_{i,j \leq r} \left| \frac{1}{n} \sum_{k=1}^{n} f_{ik} f'_{jk} - E(f_{ik} f_{jk}) \right| \leq C' \frac{1}{\sqrt{n}}, \tag{50}
\]

with probability \( 1 - O(1/n^2) \) (\( C' \) is a real positive constant). Consequently, we obtain

\[
\| \hat{D}_1 \|_2 \leq C' \frac{r p^{\alpha_1}}{\sqrt{n}} \leq C' \frac{p^{\alpha_1}}{\sqrt{n}} \tag{51}
\]

by Assumption 1(ii).

Then, we note that the diagonal elements of the matrix \( S^* \) are bounded by a finite constant, due to Assumption 2(ii). Under Assumption 3 (12) in Bickel and Levina (2008a) thus holds
for the matrix $S^*$, leading to:

$$
\|\hat{D}_2\|_\infty = \max_{i,j \leq p} \left| \frac{1}{n} \sum_{k=1}^{n} \epsilon_{ik}\epsilon_{jk} - \mathbb{E}(\epsilon_{ik}\epsilon_{jk}) \right| \leq C' \sqrt{\frac{\log p}{n}}, \quad (52)
$$

that holds with probability $1 - O(1/n^2)$. Since by Assumption 2(i) $\|S^*\|_{0,v} \leq O(p^{\delta_1})$, we can write

$$
\|\hat{D}_2\|_2 \leq \|\hat{D}_2\|_{0,v} \|\hat{D}_2\|_\infty = C' p^{\delta_1} \sqrt{\frac{\log p}{n}}, \quad (53)
$$

where we used the fact that

$$
P(\|\hat{D}_2\|_{0,v} = \|S^*\|_{0,v}) \to 1
$$
as $n \to \infty$.

Now, we study the random term $\max_{i \leq r, j \leq p} \left| \frac{1}{n} \sum_{k=1}^{n} f_{ik}\epsilon_{jk} \right|$. We know from Lemma 3 in Fan et al. (2013) that this term has sub-exponential tails, due to Assumption 3. Thus, we only need to study how its standard deviation evolves in our context. We consider the following Cauchy-Schwarz inequality:

$$
\max_{i \leq r, j \leq p} \left| \frac{1}{n} \sum_{k=1}^{n} f_{ik}\epsilon_{jk} \right| \leq C' \max_{i \leq r} \sqrt{V(f_i)} \max_{j \leq p} \sqrt{V(\epsilon_j)}.
$$

From (50), we know that $\max_{i \leq r} \sqrt{V(f_i)} \leq C' \frac{1}{\sqrt{n}}$ with probability $1 - O(1/n^2)$. From (52), we know that $\max_{j \leq p} \sqrt{V(\epsilon_j)} \leq C' p^{\delta_1} \frac{1}{\sqrt{n}} \sqrt{\frac{\log p}{n}}$ with probability $1 - O(1/n^2)$. It follows that, with probability $1 - O(1/n^2)$, it holds

$$
\left\| \frac{1}{n} \sum_{k=1}^{n} f_{ik}\epsilon_{jk} \right\|_2 \leq \sqrt{\|S^*\|_{0,v}^2} \left\| \frac{1}{n} \sum_{k=1}^{n} f_{ik}\epsilon_{jk} \right\|_\infty = C' \sqrt{\frac{p^{\delta_1}}{n^2} p^{\frac{\delta_1}{2}} \frac{1}{\sqrt{n}} \sqrt{\frac{\log p}{n}}}, \quad (54)
$$

where the r.h.s is bounded by $C' p^{\delta_1} \frac{1}{\sqrt{n}} \sqrt{\frac{\log p}{n}}$, due to Assumption 1(ii).

Consequently, we obtain with probability $1 - O(1/n^2)$ the following claim

$$
\|\hat{D}_3\|_2 \leq \left\| \frac{1}{n} \sum_{k=1}^{n} f_{ik}\epsilon_{jk} \right\| \times \|B\| \leq C' \left( \frac{1}{p^\frac{1}{2}} \sqrt{\frac{\log p}{n}} \right) \left( p^{\frac{\alpha_1}{2}} \right) = C' p^{\frac{\alpha_1}{2} + \frac{\delta_1}{2}} \frac{1}{\sqrt{n}} \sqrt{\frac{\log p}{n}}.,
$$

(55)
because $\|B\| = O(p^{\alpha_2})$ by Assumption \(\Pi(i)\).

Putting (51), (53), and (55) together, the following bound is proved with probability $1 - O(1/n^2)$:

$$\|\Sigma_n - \Sigma^*\|_2 \leq C' p^{\alpha_1} \sqrt{n},$$

(56)

because $\delta_1 < \alpha_r \leq \alpha_1$ by Assumptions \(\Pi(i)\) and \(\Pi(i)\). It follows that

$$\frac{1}{p^{\alpha_1}} \|\Sigma_n - \Sigma^*\|_2 \xrightarrow{n \to \infty} 0,$$

(57)

which proves the thesis.

\[\square\]

**Lemma A.2.** Under Assumptions \(\Pi(ii), \Pi(ii)\) and \(\Pi\)

$$\|\Sigma_n - \Sigma^*\|_\infty \leq C' \sqrt{\log p n},$$

(58)

with probability approaching one as $n \to \infty$.

**Proof.** Under Assumptions \(\Pi(ii)\) and \(\Pi\) with probability $1 - O(1/n^2)$,

$$\|\hat{D}_1\|_\infty \leq \left\| \frac{1}{n} \left( \sum_{k=1}^{n} f_k f'_k - I_r \right) \right\|_\infty \|BB'\|_\infty \leq C' \sqrt{\frac{1}{n}},$$

(59)

because $\|BB'\|_\infty \leq (\max_{j=1,...,p} \|b_j\|^2 \leq r^2 \|B\|_\infty^2 = O(1)$ for all $p \in \mathbb{N}$.

Under Assumptions \(\Pi(ii)\) and \(\Pi\), (52) ensures that, with probability $1 - O(1/n^2)$,

$$\|\hat{D}_2\|_\infty = \max_{i,j \leq p} \left| \frac{1}{n} \sum_{k=1}^{n} \epsilon_{ik} \epsilon_{jk} - E(\epsilon_{ik} \epsilon_{jk}) \right| \leq C' \sqrt{\frac{\log p}{n}}.$$

(60)

Under Assumptions \(\Pi(ii), \Pi(ii)\) and \(\Pi\) from (59) and (60) we get

$$\|\hat{D}_3\|_\infty = \left\| \frac{1}{n} \sum_{k=1}^{n} f_k f'_k \right\|_\infty \leq C' \sqrt{\frac{\log p}{n}},$$

(61)

with probability $1 - O(1/n^2)$. Putting together (59), (60), (61), the thesis follows. \[\square\]
Proof of Proposition 1

The proof is analogous to the proof of equation (6) in Bernardi and Farnè (2022).

Proof of Proposition 2

The proof is analogous to the proof of equation (9) in Bernardi and Farnè (2022).

Proof of Lemma 1

The proof is analogous to the proof of Lemma 3 in Bernardi and Farnè (2022).

Proof of Lemma 2

The proof is analogous to the proof of Lemma 4 in Bernardi and Farnè (2022).

Proof of Lemma 3

The proof is analogous to the proof of Lemma 1 in Bernardi and Farnè (2022).

Proof of Lemma 4

Proof. It is sufficient to observe that, by triangular inequality,

$$
\|(I_p + \Delta_n \Delta_n')^{-1} \Delta_n\| \leq \|(I_p + \Delta_n \Delta_n')^{-1}\| \|\Delta_n\|,
$$

(62)

and that \( \frac{1}{p'} \|\Delta_n\| \xrightarrow{n \to \infty} 0_{p \times p} \) as \( n \to \infty \) by (57), under Assumptions 1, 2, and 3.

Proof of Lemma 5

Proof. We start by equation (20), where we set \( \Sigma = \Sigma^* \), \( \epsilon = 1 \), and \( H = \Delta_n = \Sigma_n - \Sigma^* \). Then, we note that \( \frac{1}{p'} \|H\| \xrightarrow{n \to \infty} 0_{p \times p} \) as \( n \to \infty \) by (57) under Assumptions 1, 2, and 3, such that \( \frac{1}{2p'} \|F(\Delta_n) - F(0)\| \xrightarrow{n \to \infty} 0_{p \times p} \) under those assumptions, which means that

$$
\frac{1}{2p'} \|\text{Hess log det } \varphi(\Sigma_n) - \text{Hess log det } \varphi(\Sigma^*)\| \xrightarrow{n \to \infty} 0_{p \times p}.
$$
Then, the thesis follows.

Proof of Lemma 6

Proof. In Lemma $A.1$, the claim $\|\Delta_n\| \leq C\frac{p^\alpha}{\sqrt{n}}$ holds for some $C > 0$ with probability $1 - O(1/n^2)$, under Assumptions 1, 2, and 3 (see (57)). Solving the inequality $\frac{1}{3p} \geq \frac{p^2}{n}$, that becomes $\frac{1}{93p^2} \geq \frac{p^2}{n}$, we can derive the condition $n \geq p^{2\alpha+2}\delta_1^2$, ensuring that $\|\Delta_n\| \leq \frac{1}{3p}$, as prescribed by Corollary 1.

Proposition 5. Let $\gamma$ be in the range of Proposition 3 and suppose that the minimum eigenvalue of $L^*$ is such that $\lambda_r(L^*) > \delta_L \psi_0 \xi(T)$ and $\|S^*\|_{\min, off} > \delta_S \psi_0 \mu(\Omega)$ with $\delta_L$ and $\delta_S$ finite positive reals. Suppose also that

$$g_\gamma(A^T \Delta_n) \leq \frac{\psi_0 \nu}{6(2 - \nu)}, \quad (63)$$

with $\Delta_n = \Sigma_n - \Sigma^*$. Then, under the conditions of Proposition 4 and Assumption 4 if $\delta_1 \leq \frac{\alpha}{\kappa}$, there exists a unique $\tilde{T}$ satisfying Proposition 3 when setting $\tilde{T} = T'$ therein, and a corresponding unique solution pair $(\hat{S}_\Omega, \hat{L}_{\tilde{T}})$ of (64), such that:

1. $g(T, \tilde{T}) \leq \frac{\psi_0}{4}, \quad \text{rk}(\hat{S}_\Omega, \hat{L}_{\tilde{T}}) = r, \quad \text{sgn}(\hat{S}_\Omega, \hat{L}_{\tilde{T}}) = \text{sgn}(S^*, \Sigma^*)$ for all $i, j = 1, \ldots, p$;

2. $g_\gamma(A^T \tilde{T}^* C_{\tilde{T}}) \leq \frac{\psi_0 \nu}{6(2 - \nu)}$, and $\|C_{\tilde{T}}\|_2 \leq \frac{16(3 - \nu)}{3\alpha(2 - \nu)} \psi_0$, with $C_{\tilde{T}} = \mathcal{P}_{\tilde{T}^*}(L^*)$;

3. $(\hat{S}_\Omega, \hat{L}_{\tilde{T}})$ is also the unique solution of problem (4), with high probability as $n \to \infty$.

Proof. First, we need to ensure that under Assumption 5, Assumptions 4(i) and 4(i) are compatible, i.e. that

$$\lambda_r(L^*) > \delta_L \psi_0 \xi(T) \geq \delta_L \left( \frac{\sqrt{T}}{\kappa_L} \right)^3 \frac{3p^{\delta_1}}{\sqrt{n}}$$

under $\lambda_r(L^*) \simeq p^{\alpha r}$, which holds true if $\delta_1 \leq \frac{\alpha}{\kappa}$, because $n \to \infty$. Assumptions 4(ii) and 4(i) are instead always compatible, as

$$0 < \delta_S \psi_0 \mu(\Omega(S^*)) \leq \frac{\delta_2}{\xi(T(L^*))} \frac{p^{\delta_1}}{\sqrt{n}} \leq \frac{54 \delta_2 p^{\delta_1}}{\sqrt{n}} < \|S^*\|_{1, v} \leq \delta_2 p^{\delta_1},$$
that is always verified as $n \to \infty$. Then, the proof is analogous to the proof of Proposition 5.3 in [1], by noticing that $\lambda_r(L^*) > \delta L \frac{\psi_0}{\xi^*(T)}$ and $\|S^*\|_{\text{min,off}} > \delta S \frac{\psi_0}{\mu(\Omega)}$ hold under Assumption 4. Propositions 3 and 4 hold under Assumption 1-3 with $\gamma$ in the range of Proposition 3 and $\delta$ satisfying Lemma 6.

### Proof of Theorem 1

Following [1], we need to derive the conditions that guarantee that $\text{rk}(\hat{L}_L^T) = r$ and $\text{sgn}(\hat{S}_\Omega) = \text{sgn}(S^*)$, and that $(\hat{L}_L^T, \hat{S}_\Omega)$ (see problem (29)) is a global solution.

Let us define the tangent space to $L(r)$ in a generic $\tilde{L} \neq L^*$:

$$\tilde{T}(\tilde{L}) = \{ M \in \mathbb{R}^{p \times p} \mid M = UY_1 + Y_2 U' \mid Y_1, Y_2 \in \mathbb{R}^{p \times r}, U \in \mathbb{R}^{p \times r}, U'U = I_r, U'\tilde{L}U \in \mathbb{R}^{r \times r} \text{ diagonal}, \tilde{L} \in L(r) \}.$$  

Consider the solution pair

$$(\hat{S}_\Omega, \hat{L}_L) = \arg \min_{L \in T} \sum_{\Omega} \min \{ \| \Sigma_n - (L + \hat{S}) \|_F^2 + \psi_0 \| L \|_* + \rho_0 \| S \|_1 \}. \quad (64)$$

We note that, in Proposition 3 part 1 ensures the identification of the latent rank and the residual sparsity pattern with high probability, and bounds the identification error; part 2 bounds the contribution of the orthogonal component to the overall error rate $\tilde{r}$ (see Proposition 3), and part 3 is the key to prove the following conditions:

- $\| P_{T^\perp} (\hat{S}_\Omega + \hat{L}_L - \Sigma^*) \| < \psi_0$
- $\| P_{\Omega^\perp} (\hat{S}_\Omega + \hat{L}_L - \Sigma^*) \| < \psi_0 \gamma$

that, together with the two previously proved conditions (see the proof of Proposition 4)

- $P_T (\hat{S}_\Omega + \hat{L}_L - \Sigma^*) = -\psi_0 U_L U_L'$
- $P_T (\hat{S}_\Omega + \hat{L}_L - \Sigma^*) = -\psi_0 \gamma \text{sgn}(S^*)$, 

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ensure the global optimality of the solution pair $(\hat{L}_T, \hat{S}_\Omega)$ (see Boyd et al. (2004)).

More, recalling the definition of the $\rho$ measure (27), and specializing it to the context of Proposition 5 part 1, we get

$$\rho(T, T') = \max_{\|N\|_2 \leq 1} \|P_T N - P_{T'} N\|_2,$$

(65)

where $P_T$ and $P_{T'}$ are the projection operators onto $T$ and $T'$, respectively. From (65) and $ho(T, T') \leq \xi(T)/4$, it follows that $\|P_T L^* - P_{T'} L^*\|_2 = \|L^* - P_{T'} L^*\|_2 = \|P_{T'} L^*\|_2 = \|C_T\|_2 \leq \frac{\xi(T)}{4}$. At this stage, from Assumption 5 we can recall that $\xi(T(L^*)) = \frac{\sqrt{\kappa}}{\eta L_{\gamma_1}}$, which means that, since $\delta_1 > 0$ by Assumption 2, we get $\xi(T) = \frac{\sqrt{\kappa}}{\eta L_{\gamma_1}} \xrightarrow{p \to \infty} 0$. Therefore, since $\delta_1 > 0$, as $p \to \infty$, the condition $\|C_T\|_2 \leq \frac{16(3-\nu)}{3\eta \kappa_{\gamma_1}} \psi_0$ is inactive. It follows that, as $p \to \infty$, the error bound $\tilde{r}$ of Proposition 3

$$\tilde{r} = \max \left\{ \frac{4}{\alpha} [g_\gamma(A^\dagger \Delta_n) + g_\gamma(A^\dagger T^* C_{T'}) + \psi_0], \|C_{T'}\|_2 \right\}$$

only depends on the first argument of the maximum, which in turn, if $\alpha = \beta = 1$, $\delta = 0$, $\nu = \frac{1}{2}$, according to Proposition 5 attains its maximum, equal to $\frac{40}{\sqrt{\log p}} \psi_0$. This occurs because (63) is maximum at $\nu = \frac{1}{2}$. Note that, in that case, the range for $\gamma$ in Proposition 3 collapses to $\gamma \in [9\xi(T), \frac{1}{\eta \mu(\Omega)}]$, and the identifiability condition becomes $\sqrt{\frac{\kappa_{\gamma_1}}{\eta L}} \leq \frac{1}{44}$.

Now, we note that under Assumptions 1 2 3 with probability tending to one as $n \to \infty$, it holds:

$$g_\gamma(A^\dagger \Delta_n) = g_\gamma(\Sigma_n - \Sigma^*, \Sigma_n - \Sigma^*)$$

$$\leq \max \left\{ \frac{\|\Sigma_n - \Sigma^*\|_\infty}{\gamma}, \frac{\|\Sigma_n - \Sigma^*\|_2}{\gamma} \right\} \leq \max \left\{ \frac{\|\Sigma_n - \Sigma^*\|_\infty}{\gamma}, \frac{\|\Sigma_n - \Sigma^*\|_2}{\gamma} \right\} \leq C' \frac{1}{9\xi(T)} \sqrt{\frac{\log p}{n}}.$$

This results descends from Lemma A.2 from the condition $\gamma \in \left[ \frac{3\xi(T(L^*))(2-\nu)}{\mu^\alpha}, \frac{\mu^\alpha}{2\mu(\Omega(S^*))\beta(2-\nu)} \right]$.
of Proposition 3, where the minimum for $\gamma$, $\frac{3\xi(T(L^*))(2-\nu)}{\mu_\alpha}$, is attained for $\alpha = 1$ and $\nu = \frac{1}{2}$, and from Lemma A.1 under Assumptions 1, 2, 3.

Since we have set $\psi_0 = \frac{1}{2} \xi(T) \sqrt{\log p}$, the condition of Proposition 5 can be written as $g_\gamma(A^{\dagger} \Delta_n) \leq \psi_0 \leq \frac{p^{1/2}}{18\sqrt{r} \sqrt{n}} \sqrt{\log p}$ by Assumption 5. Therefore, setting $C = \frac{k_1}{18\sqrt{r}}$ and $C' = \frac{1}{2}$, under Assumptions 1-5 Proposition 5 (parts 1, 3 and 4) ensures that the solution $(\hat{S}, \hat{L})$ of (4) satisfies

$$g_\gamma(\hat{S} - S^*, \hat{L} - L^*) \leq C \frac{80}{9} \psi_0 \leq \frac{\kappa p^{1/2}}{\sqrt{n}},$$

(66)

where $\kappa = \frac{80}{9} \frac{k_1}{18\sqrt{r}}$. Recalling the definition of $g_\gamma$ in (28), we can thus write

$$\|\hat{L} - L^*\|_2 \leq C' p^{\alpha_1} \frac{80}{9} \psi_0 \leq \kappa p^{\alpha_1 + \delta_1} \sqrt{\frac{\log p}{n}},$$

$$\|\hat{S} - S^*\|_1 \leq C \frac{80}{9} \gamma \psi_0 \leq \kappa \sqrt{\frac{\log p}{n}}.$$

This proves parts 1 and 2 of Theorem 1. Finally, Proposition 5 (parts 2 and 4) ensures that

$$\mathcal{P}(\text{rk}(\hat{L}) = r) \to 1,$$

$$\mathcal{P}(|\hat{S} - S^*)|_{0,v} = |S^*|_{0,v}) \to 1,$$

as $n \to \infty$. This proves parts 3 and 4 of Theorem 1.

**Proof of Corollary 2**

*Proof.* Suppose that all the assumptions and conditions of Theorem 1 hold, and recall that the pair $(\hat{S}, \hat{L})$ is the solution of (4). Then, part 1 holds true because of Theorem 1 part 2 and Assumption 2(i), as

$$\|\hat{S} - S^*\|_2 \leq \|\hat{S} - S^*\|_{0,v} \leq \|\hat{S} - S^*\|_1 \leq \kappa \|S^*\|_{0,v} \sqrt{\frac{\log p}{n}} \leq \kappa p^{1/2} \frac{\delta_1}{\sqrt{n}},$$

where we used the fact that $\mathcal{P}(\|\hat{S} - S^*\|_{0,v} = \|S^*\|_{0,v}) \to 1$ as $n \to \infty$. 

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Part 2 holds true under Assumptions $(\Pi i)$ and $(\Pi i)$ because

\[
\|\hat{\Sigma} - \Sigma^*\|_2 \leq \|\hat{L} - L^*\|_2 + \|\hat{S} - S^*\|_2 \leq
\leq \kappa p^{\alpha_1 + \delta_1} \sqrt{\frac{\log p}{n}} + \kappa \delta_2 p^{\delta_1} \sqrt{\frac{\log p}{n}}.
\]

Then, Proposition 5 (part 4) ensures part 3 of the Corollary, as \(\hat{S} > 0\) because \(\hat{S} \in S(s)\) as \(n \to \infty\). Part 4 of the Corollary descends by Proposition 5 (parts 2 and 4), because \(P(\text{rk}(\hat{L}) = r) \to 1\) as \(n \to \infty\) (part 3 of Theorem 1), and \(\hat{\Sigma} > 0\) because

\[
\lambda_p(\hat{\Sigma}) \geq \lambda_p(\hat{L}) + \lambda_p(\hat{S}) > 0 + \lambda_p(\hat{S}) > 0,
\]

by dual Lidksii inequality and part 3 of the Corollary.

Part 5 of the Corollary holds because

\[
\|\hat{S}^{-1} - S^*-1\| \leq \|\hat{S} - S^*\| \frac{1}{\lambda_p(S^*)} \frac{1}{\lambda_p(\hat{S})},
\]

\(\lambda_p(S^*) = O(1)\) by assumption, and \(\lambda_p(\hat{S})\) tends to \(\lambda_p(S^*)\) as \(n \to \infty\).

Analogously, part 6 holds because

\[
\|\hat{\Sigma}^{-1} - \Sigma^*-1\| \leq \|\hat{\Sigma} - \Sigma^*\| \frac{1}{\lambda_p(\Sigma^*)} \frac{1}{\lambda_p(\hat{\Sigma})},
\]

\(\lambda_p(\Sigma^*) = O(1)\) by assumption, and \(\lambda_p(\hat{\Sigma})\) tends to \(\lambda_p(\Sigma^*)\) as \(n \to \infty\).

\[\blacksquare\]

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