Transverse Invariant Higher Spin Fields

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Abstract

It is shown that a symmetric massless bosonic higher-spin field can be described by a traceless tensor field with reduced (transverse) gauge invariance. The Hamiltonian analysis of the transverse gauge invariant higher-spin models is used to control a number of degrees of freedom.

1 Introduction

It was shown recently in [1], [2] that a free massless spin two field (i.e. linearized gravity) can be consistently described by a traceless rank-2 tensor field with transverse gauge symmetry that corresponds to linearized volume-preserving diffeomorphisms. We extend this result to massless fields of arbitrary spin by showing that a spin-s symmetric massless field can be described by a rank-s traceless symmetric tensor. This formulation is in some sense opposite to the approach developed in [3, 4, 5] where a massless field is described by a traceful tensor. Recall that the standard Fronsdal’s formulation of a spin-s massless field operates with a rank-s double traceless tensor [6]. For recent reviews on higher-spin (HS) gauge theories see [7].

Although, like in the case of gravity, the obtained model is a gauge fixed version of the original Fronsdal model [6] the equivalence is not completely trivial. Actually, the standard counting of degrees of freedom is that each gauge parameter in the gauge transformations with first order derivatives kills two degrees of freedom [9]. Therefore one can expect that the invariance under reduced gauge symmetry may be not sufficient to compensate all extra degrees of freedom. As we show this is not the case. The reason

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is that the remaining gauge symmetry parameters satisfy the differential transversality conditions \( \partial^{\nu} \xi_{\nu \mu_2 \ldots \mu_{s-1}} = 0 \).

Generally, as explained in this paper, a partial gauge fixing at the Lagrangian level can give rise to a model which, if treated independently of the original gauge model, may differ from the latter. In particular, the Hamiltonian interpretation of the gauge fixed Lagrangian model may differ from that of the original model. This can happen in the case where the gauges and constraints on gauge parameters are differential. For example, as shown in Section 5, this does happen in electrodynamics in the temporary gauge. Since the transversality condition on the gauge parameter is also of this type, a more careful analysis of the counting of the number of degrees of freedom in the model under consideration is needed. The Hamiltonian analysis of Section 5 shows that the transverse gauge invariant HS model has as many degrees of freedom as the original Fronsdal model in the topologically trivial situation.

Note, that the original Lagrangian and field/gauge transformations content for a massless field of an arbitrary spin were derived by Fronsdal in [6] by taking the zero rest mass limit \( m^2 \to 0 \) in the Lagrangian of Singh and Hagen for a massive HS field of [8]. To the best of our knowledge, it has not been analyzed in the literature what is a minimal field content appropriate for the description a massless HS field. The proposed formulation operates in terms of an irreducible Lorentz tensor field, thus being minimal. It is equivalent to the Fronsdal’s one in the topologically trivial situation but may differ otherwise. Also let us note that since it has a relaxed gauge symmetry compared to that of the Fronsdal formulation, it may in principle have more freedom at the interaction level.

The layout of the rest of the paper is as follows. In Section 2 we recall the standard description of massive and massless fields of arbitrary spin. In Section 3 transverse and Weyl invariant Lagrangian is constructed and a generating action is given. The equivalence of transverse and Weyl invariant Lagrangian to the Fronsdal’s Lagrangian is checked in Section 4. Hamiltonian analysis and examples are given in Section 5.

2 Free Massless Higher-Spin Fields

A spin-\( s \) bosonic totally symmetric massive field in Minkowski space can be described on shell [10] by a totally symmetric tensor field \( \varphi_{\mu_1 \ldots \mu_s} \) that satisfies the conditions

\[
\begin{align*}
&\Box + m^2 \varphi_{\mu_1 \ldots \mu_s} = 0, \\
&\partial^{\nu} \varphi_{\nu \mu_2 \ldots \mu_s} = 0, \\
&\varphi_{\nu \mu_3 \ldots \mu_s} = 0.
\end{align*}
\]

Greek indices \( \mu, \nu, \lambda, \rho = 0, \ldots, d-1 \) are vector indices of \( d \)-dimensional Lorentz algebra \( o(d-1,1) \). The indices are raised and lowered by mostly minus invariant tensor \( \eta_{\mu \nu} \) of \( o(d-1,1) \). A group of indices to be symmetrized is denoted by placing them in brackets or, shortly, by the same letter. For example, \( \partial_{\mu} \phi_{\mu} \equiv \partial_{\mu_1} \phi_{\mu_2} + \partial_{\mu_2} \phi_{\mu_1} \).
These form the complete set of local Poincare-invariant conditions on $\varphi_{\mu_1...\mu_s}$. In the massless case $m^2 = 0$ a gauge invariance with an on-shell traceless rank-$(s - 1)$ tensor gauge parameter reduces further the number of physical degrees of freedom.

As pointed out by Fierz and Pauli in [11], for (2.1) to be derivable from a Lagrangian a set of auxiliary fields has to be added for $s > 1$ (in the case of spin two considered by Fierz and Pauli this is a scalar auxiliary field $\varphi$, which together with a traceless $\varphi_{\mu_1\mu_2} = \varphi_{\mu_1\mu_2} + \eta_{\mu_1\mu_2}\varphi$). Auxiliary fields are zero on shell, thus carrying no physical degrees of freedom. For totally symmetric massive fields of integer spins, the Lagrangian formulation with a minimal set of auxiliary fields was worked out by Singh and Hagen in [8]. For a spin-$s$ field they introduced a set of auxiliary fields, which consists of symmetric traceless tensors of ranks $s - 2, s - 3, \ldots, 0$. An elegant gauge invariant (Stueckelberg) formulation was proposed by Zinoviev in [12]. (For alternative approaches to massive fields see also [13, 14, 15] and references therein.)

The Lagrangian of a spin-$s$ massless field can be obtained [6] in the limit $m^2 \to 0$. The auxiliary fields of ranks from 0 to $(s - 3)$ decouple while the residual rank-$(s - 2)$ traceless auxiliary field $\varphi_{\mu_1...\mu_{s-2}}$ and the physical rank-$s$ traceless field $\varphi_{\mu_1...\mu_s}$ form the symmetric field

$$\varphi_{\mu_1...\mu_s} = \varphi_{\mu_1...\mu_s} + \eta_{(\mu_1\mu_2}\varphi_{\mu_3...\mu_{s-2})}$$

that satisfies the double tracelessness condition

$$\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4} \varphi_{\mu_1...\mu_s} = 0,$$

which makes sense for $s \geq 4$. The resulting Lagrangian possesses gauge invariance with a traceless rank-$(s - 1)$ gauge parameter $\xi_{\mu_1...\mu_{s-1}}$,

$$\delta \varphi_{\mu_1...\mu_s} = s \partial_{(\mu_1} \xi_{\mu_2...\mu_s)}, \quad \xi^\nu_{\nu\mu_3...\mu_{s-1}} = 0.$$

In the spin two case of linearized gravity, the gauge law (2.3) corresponds to linearized diffeomorphisms.

Let us write down a most general bilinear action and Lagrangian (modulo total derivatives) of a double traceless field with at most two derivatives as

$$\mathcal{L} = (-)^s \sum_{\alpha = a, b, c, f, g} \mathcal{L}_\alpha, \quad S = \int d^d x \mathcal{L},$$

where

$$\mathcal{L}_a = \frac{a}{2} \partial_\nu \phi_{\mu_1...\mu_s} \partial^\nu \phi^{\mu_1...\mu_s},$$

$$\mathcal{L}_b = - \frac{bs(s - 1)}{4} \partial_\nu \phi^\rho_{\rho\mu_3...\mu_s} \partial^\nu \phi^\lambda_{\lambda\mu_3...\mu_s},$$

$$\mathcal{L}_c = - \frac{cs}{2} \partial^\nu \phi_{\nu\mu_2...\mu_s} \partial_\rho \phi^{\rho\mu_2...\mu_s},$$

$$\mathcal{L}_f = \frac{fs(s - 1)}{2} \partial_\nu \phi^\rho_{\rho\mu_3...\mu_s} \partial_\lambda \phi^{\lambda\mu_3...\mu_s},$$

$$\mathcal{L}_g = - \frac{gs(s - 1)(s - 2)}{8} \partial^\nu \phi^\rho_{\rho\mu_4...\mu_s} \partial_\lambda \phi^\sigma_{\sigma\mu_4...\mu_s}.$$
with arbitrary coefficients $a, b, c, f, g$. For $\mathcal{L}$ to describe a spin-$s$ field, the coefficient $a$ has to be nonzero (so, we set $a = 1$).

The variation of (2.4) is

$$
\delta \mathcal{L} = \left( G_{\mu_1...\mu_s} - \frac{s(s-1)}{2(\Upsilon - 2)} \eta(\mu_1\mu_2 G^\rho_{\rho\mu_3...\mu_s}) \right) \delta \phi^{\mu_1...\mu_s}, \tag{2.6}
$$

where $\Upsilon = d + 2s - 4$ and

$$
G_{\mu_1...\mu_s} = \Box \phi_{\mu(s)} - b \frac{s(s-1)}{2} \eta_{\mu\nu} \phi_{\lambda(\mu(2)} - cs \partial_{\mu} \partial^\sigma \phi_{\nu(\mu(s-1) + f} \frac{s(s-1)}{2} \eta_{\mu\nu} \phi_{\lambda(\mu(2)} \right) - g \frac{s(s-1)(s-2)}{4} \eta_{\mu\nu} \partial_{\mu} \partial^\sigma \phi_{\lambda(\mu(s-3).} \tag{2.7}
$$

The requirement that the action is invariant under (2.3) fixes the coefficients

$$
a = b = c = f = g. \tag{16}
$$

### 3 Transverse and Weyl Invariant Massless Higher-Spin Fields

Let us consider a weaker condition on the action imposed by the reduced gauge symmetry (2.3) with the transverse gauge parameter $\xi_{\mu_1...\mu_{s-1}}$

$$
\delta \phi_{\mu_1...\mu_s} = s \partial_{(\mu_1} \xi_{\mu_2...\mu_{s-1)}}, \quad \partial^\nu \xi_{\mu_2...\mu_{s-1}} = 0, \quad \xi_{\nu\mu_3...\mu_{s-1}} = 0. \tag{3.1}
$$

The invariance of action (2.4) under (3.1) fixes only the ratio $a/c = 1$ while the rest of the coefficients remains free. This ambiguity can be used to look for another symmetry to kill extra degrees of freedom. Taking into account the double tracelessness condition (2.2), a use of rank-$(s-2)$ symmetric traceless gauge parameter $\zeta_{\mu_1...\mu_{s-2}}$ is a natural option

$$
\delta \phi_{\mu_1...\mu_s} = s \frac{(s-1)}{2} \eta(\mu_1\mu_2 \zeta_{\mu_3...\mu_{s-2}}), \quad \zeta_{\nu\mu_3...\mu_{s-2}} = 0. \tag{3.2}
$$

The requirement for (2.4) to be invariant under the additional (Weyl) symmetry (3.2) fixes the rest of the coefficients

$$
b = \Upsilon + 2, \quad f = \frac{2}{\Upsilon}, \quad g = \frac{-2(\Upsilon - 4)}{\Upsilon^2}. \tag{3.3}
$$

Note that, not too surprisingly, the resulting Lagrangian (2.4) can be obtained from the Fronsdal’s Lagrangian (i.e. that with $a = b = c = f = g = 1$) via the substitution

$$
\tilde{\phi}_{\mu_1...\mu_s} = \phi_{\mu_1...\mu_s} - \frac{1}{\Upsilon} \frac{s(s-1)}{2} \eta(\mu_1\mu_2 \phi^\nu_{\nu\mu_3...\mu_s}), \quad \tilde{\phi}_{\nu\mu_3...\mu_s} = 0. \tag{3.4}
$$

There is a generating action $S^{gen}$ that gives rise both to the Fronsdal and to the Weyl invariant actions in particular gauges. $S^{gen}$ results from the Fronsdal action by introducing a traceless Stueckelberg field $\chi_{\mu_1...\mu_{s-2}}$ of rank-$(s-2)$ via the substitution

$$
\phi_{\mu_1...\mu_s} \rightarrow \phi_{\mu_1...\mu_s} + \frac{s(s-1)}{2} \eta(\mu_1\mu_2 \chi_{\mu_3...\mu_s}). \tag{3.5}
$$
where $\phi_{\mu_1...\mu_s}$ is the double traceless field. The gauge transformations are
\[
\delta \phi_{\mu_1...\mu_s} = s \partial(\mu_1, \xi_{\mu_2...\mu_s}) + \frac{s(s-1)}{2} \eta(\mu_1, \mu_2, \xi_{\mu_3...\mu_s}), \quad \delta \chi_{\mu_1...\mu_{s-2}} = -\varepsilon_{\mu_1...\mu_{s-2}} \tag{3.6}
\]
with $\varepsilon_{\mu_1...\mu_{s-2}}$ being a traceless rank-$(s-2)$ gauge parameter. Fixing $\chi_{\mu_1...\mu_{s-2}}$ to zero by the gauge parameter $\varepsilon_{\mu_1...\mu_{s-2}}$, we obtain the spin-$s$ Fronsdal’s Lagrangian. Alternatively, we can gauge fix the trace of $\phi_{\mu_1...\mu_s}$ to zero by the same Stueckelberg parameter $\varepsilon_{\mu_1...\mu_{s-2}}$. The leftover symmetry is with
\[
\varepsilon_{\mu_1...\mu_{s-2}} = \frac{\Gamma}{2} \partial^\nu \xi_{\nu \mu_1...\mu_{s-2}}. \tag{3.7}
\]
Then, gauge fixing the field $\chi_{\mu_1...\mu_{s-2}}$ to zero gives the Lagrangian (4.1) and constraint
\[
\partial^\nu \xi_{\nu \mu_2...\mu_{s-1}} = 0. \tag{3.8}
\]
Thus $S^{gen}$ reduces to the transversely invariant action (4.1) and Fronsdal action in particular gauges. Note that a generating action of this type naturally appears in the BRST analysis as discussed by Pashnev and Tsulaia in [17].

Now we are in a position to check whether this theory is unitary and describes the correct number of physical degrees of freedom of a spin-$s$ massless representation of $iso(d-1,1)$, thus being equivalent to the conventional spin-$s$ Fronsdal massless theory.

4 Spectrum

Having fixed pure algebraic gauge symmetry with parameter $\zeta_{\mu_1...\mu_{s-2}}$ to eliminate the trace of $\phi_{\mu_1...\mu_s}$ one gets the Lagrangian
\[
\mathcal{L} = (-)^s \left( \frac{1}{2} \partial^\nu \phi_{\mu_1...\mu_s} \partial^\nu \phi_{\mu_1...\mu_s} - \frac{s}{2} \partial^\nu \phi_{\nu \mu_2...\mu_s} \partial^\rho \phi^{\rho \mu_2...\mu_s} \right) \tag{4.1}
\]
with, respectively, the equations of motion, gauge transformation law and constraints
\[
\Box \phi_{\mu_1...\mu_s} = s \partial(\mu_1, \xi_{\mu_2...\mu_s}) + \frac{s(s-1)}{2} \eta(\mu_1, \mu_2, \xi_{\mu_3...\mu_s}) = 0, \quad \delta \phi_{\mu_1...\mu_s} = s \partial(\mu_1, \xi_{\mu_2...\mu_s}), \quad \partial^\nu \xi_{\nu \mu_2...\mu_{s-1}} = 0, \quad \xi^\nu_{\nu \mu_3...\mu_{s-1}} = 0, \quad \phi^\nu_{\nu \mu_3...\mu_s} = 0. \tag{4.2}
\]

To analyze the physical meaning of these equations and gauge transformations it is convenient to use the standard momentum frame
\[
p^\mu = (E/\sqrt{2}, 0, ..., 0, E/\sqrt{2}), \quad p^\mu p_\nu = 0 \tag{4.3}
\]
and light-cone coordinates\[4\]
\[
x^\pm = (x^0 \pm x^d)/\sqrt{2}, \quad x^i - \text{unchanged}, \tag{4.4}
\]
\[\delta_{ij} = \text{diag}(+...+).\]
in which the metric $\eta_{\mu\nu}$ has the form

$$\eta_{+-} = \eta_{-+} = 1, \quad \eta_{ij} = -\delta_{ij}. \quad (4.5)$$

We use the following notation for components of $\phi_{\mu_1...\mu_s}$

$$\phi^{+}(k),-(m),i(s-k-m) \equiv \phi^{+}_{++...-},_{m}^{i_1...i_{s-k-m}}. \quad (4.6)$$

$\phi^{'}_{+}(k),-(m),i(s-k-m-2)$ denotes the trace $\phi^{+}_{+},-(m),i(s-k-m-2),j\delta^{jl}$ of $o(d-2)$ indices.

The system (4.2) reduces to

$$k(1-\frac{2m}{r})\phi^{+}_{+},-(m+1),i(s-k-m) + \frac{(s-k-m)(s-k-m-1)}{r} \delta_{ii}\phi^{+},-(m+2),i(s-k-m-2) = 0,$$

$$\delta\phi^{+}_{+},-(m),i(s-k-m) = k\xi^{+}_{+},-(m),i(s-k-m), \quad (4.7)$$

$$\xi^{+}_{+},-(m+1),i(s-k-m) = 0, \quad 2\xi^{+}_{+},-(m+1),i(s-k-m-2) = \xi_{+}^{'}(k),-(m),i(s-k-m-2);$$

$$2\phi^{+}_{+},-(m+1),i(s-k-m-2) = \phi^{'}_{+},-(m),i(s-k-m-2).$$

The first equation of (4.7) implies that $\phi^{+}_{+},-(m+1),i(s-k-m)$ is a pure trace for $m = 0...s-1$, $k = 1...(s-1)$. As a result the on shell non-zero components are $o(d-2)$ traceless components $\phi^{+}_{+},-(0),i(s-k), \ k = 0...s$. However, those with $k = 1...s$ are pure gauge. Thus, only the traceless component of $\phi^{+}_{+},-(0),i(s) \equiv \phi_{i(s)}$ is physical, describing a spin-s symmetric representation of the massless little group $o(d-2)$. Unitarity of the theory follows from the equivalence of the transverse-invariant and Fronsdal’s Lagrangians in the sector of physical degrees of freedom.

A less trivial question not answered by this analysis is whether the leftover gauge symmetries in a partially gauge fixed model remain gauge symmetries of the latter model treated independently (say, if the original model was not known). Complete answer to this question is provided by the Hamiltonian analysis. To illustrate what could happen let us start with the spin one example.

5 Hamiltonian analysis

5.1 Example of spin one in the temporary gauge

An instructive example is provided by Maxwell electrodynamics formulated in terms of a gauge potential $A_{\mu}$

$$\mathcal{L} = \frac{1}{2} \left( \partial^{\mu} A_{\mu} \partial^{\nu} A_{\nu} - \partial_{\mu} A_{\nu} \partial^{\mu} A^{\nu} \right),$$

$$\Box A_{\mu} - \partial_{\mu} \partial^{\nu} A_{\nu} = 0, \quad (5.1)$$

$$\delta A_{\mu} = \partial_{\mu} \xi(x).$$
Imposing the temporary gauge \( A_0 = 0 \) at the Lagrangian level we obtain the gauge fixed Lagrangian\(^5\)

\[
\mathcal{L} = \frac{1}{2} \left( \dot{A}_K \dot{A}^K + \partial^I A_I \partial^J A_J - \partial_I A_J \partial^J A^I \right).
\] (5.2)

Expressing all velocities via momenta we arrive at the unconstrained dynamics with the Hamiltonian

\[
H = \frac{1}{2} \left( \Pi_K \Pi^K + \partial^I A_I \partial^J A_J - \partial_I A_J \partial^J A^I \right).
\] (5.3)

Clearly, the gauge fixed Lagrangian (5.2) describes \( d-1 \) degrees of freedom (2\((d-1)\) in the phase space). This is to be compared with the \( d-2 \) degrees of freedom (2\((d-2)\) in the phase space) of the original model. This mismatch has the following origin. One additional phase space degree of freedom comes from the leftover gauge symmetry parameter that solves

\[
\partial_0 \xi = 0.
\] (5.4)

Another one is due to the loss of the Gauss law constraint in the theory.

Indeed, in electrodynamics, the Gauss law \( \text{div} E = 0 \) results from the variation of the action (5.1) over \( A_0 \) (for simplicity we set electric current equal to zero). This equation is lost in the gauge fixed theory (5.2). From the gauge invariance a weaker condition follows

\[
\partial_0 \text{div} E = 0.
\] (5.5)

In the dynamical system (5.2), the equation (5.5) is indeed one of the field equations. But it is not a constraint any more, thus bringing another phase space degree of freedom into the game.

The naive equivalence argument might be that once some gauge is reachable by a gauge transformation it can be imposed at the Lagrangian level because any variation over a gauge fixed variable can be expressed as a combination of a gauge symmetry variation and a local variation of the unfixed variables. Generically, this argument is wrong however because it neglects the issue of locality. Namely it is not guaranteed that the compensating gauge symmetry transformation is local in terms of the variation of the gauge fixed variable because it may require resolution of some differential equation on the gauge symmetry parameter with respect to the time variable. In our example, this is manifested by the condition (5.4). This is why the Gauss law in electrodynamics is not reproduced in the model (5.2) treated independently of the underlying model from which it has been derived.

The conclusion is that, if treated independently, a gauge fixed model (i.e. forgetting the symmetries and field equations of the original model) is guaranteed to be equivalent to the original one in the gauges that impose algebraic (i.e., free of time derivatives)

\(^5\)Capital Latin indices \( I, J, K, \ldots \) are vector indices of \( o(d - 1) \), e.g. \( \mu = (0,1) \). Dot denotes the time derivative, i.e. \( \dot{\phi} = \partial_0 \phi \)}
constraints on the gauge symmetry parameters. This is the case of Stueckelberg fields and gauge symmetries.

The situation with the spin two field considered in [1, 2] and with partially fixed HS gauge fields discussed in this paper is somewhat analogous to the temporary gauge example discussed in this section because it involves the differential constraint (3.1) on the gauge symmetry parameter. It is therefore instructive to reanalyze the models by the Hamiltonian methods.

5.2 Spin two

Let us consider the spin two case in more detail. (The Hamiltonian analysis of nonlinear gravity was originally given in [18] (see also the textbook [9])). The gauge fixed Lagrangian is

$$
\mathcal{L} = \frac{1}{2} \left( \partial_\mu \phi_{\nu\lambda} \partial^\mu \phi^{\nu\lambda} - 2 \partial^\mu \phi_\mu \partial_\lambda \phi^{\lambda\nu} \right), \quad \phi^{\nu\nu} = 0. \tag{5.6}
$$

Using notation $Q_{IK} \equiv \phi_{IK}$, $Q_I \equiv \phi_{0I}$, the corresponding momenta are

$$
\Pi_{IJ} = \dot{Q}_{IJ} - \delta_{IJ} \dot{Q}_K^K + 2 \delta_{IJ} \partial^K Q_K, \\
\Pi^I = -2 \partial_K Q^{KI}. \tag{5.7}
$$

As velocities $\dot{Q}^I$ do not contribute to $\Pi^I$, the primary constraints are

$$
\psi^K_1 = \Pi^K + 2 \partial_I Q^{KI}. \tag{5.8}
$$

The Hamiltonian is

$$
H = \frac{1}{2} \left( \Pi_{IJ} \Pi^{IJ} - \frac{1}{d-2} \Pi^I \Pi^I_J \right) + \frac{2}{d-2} \Pi^I J \partial_J Q^I + \beta_K (\Pi^K + 2 \partial_I Q^{KI}) + \\
+ \frac{1}{2} \left( \partial_I Q_J K \partial^I Q^{JK} + \partial_I Q^I J \partial^I Q^K_K \right) - \partial_I Q_K \partial^I Q_K + \\
- \frac{d}{d-2} \partial_I Q_I \partial^I Q_J - \partial^I Q_K \partial_I Q^{KJ}, \tag{5.9}
$$

where $\beta_K$ are Lagrange multipliers.

Secondary and ternary constraints $\psi^K_2, \psi^K_3$ result from Poisson brackets $[ \ , \ ]$ with the Hamiltonian (5.9)

$$
[\psi^K_1, H] = \psi^K_2 = -\Delta Q^K - \partial^K \partial_J Q^I + \partial_J \Pi^{JK}, \tag{5.10}
$$

$$
[\psi^K_2, H] = \partial^K \psi_3, \quad \psi^K_3 = \Delta Q^I_J - \partial_I \partial_J Q^{IJ}, \tag{5.11}
$$

where $\Delta \equiv \partial^A \partial_A$.

The further commutation of the ternary constraints produces no new constraints, so that $\psi_A (A=1,2,3)$ form the complete list. All constraints are first class

$$
[\psi_A, \psi_B] = 0, \quad A, B = 1, 2, 3. \tag{5.12}
$$
In this analysis the kernel of the operator $\partial K$ is assumed to be trivial so that $\partial K \psi_3 = 0$ is equivalent to $\psi_3 = 0$. Note however that the two conditions may be different in a topologically nontrivial situation (say, for the torus compactification) differently accounting some discrete degrees of freedom. Note also that $\psi_3$ is just the linearized first-class constraint associated with $g^{00}$ in usual Hamiltonian gravity [18].

The number of physical degrees of freedom (PDoF) is

$$
PDoF = \frac{(d+2)(d-1)}{2} - 2(d-1) - 1 = \frac{d(d-3)}{2} = R(2, d-2) = 2|_{d=4}, \quad (5.13)$$

where $R(s, d)$ is the dimension of a rank-$s$ symmetric traceless tensor

$$
R(s, d) = \frac{(d+2s-2)(d+s-3)!}{(d-2)!s!}. \quad (5.14)
$$

Let us stress that the reason why the gauge fixed model under consideration turns out to be equivalent to the Pauli-Fierz model is just that the ternary constraint $\psi_3$ appears in (5.11) under the operator $\partial K$, which, in turn, is the consequence of the Lorentz invariance of the chosen gauge. In the temporary gauge electrodynamics example the equation analogous to (5.11) is (5.5) which is not a constraint however.

### 5.3 Spin three

Let us consider a spin three massless field, as a simplest HS example. We use the following notation for the space-like projections of $\phi_{\mu\nu\lambda}$: $Q_{ABC} \equiv \phi_{ABC}$; $Q_{AB} \equiv \phi_{AB0}$, $Q_{BA}^B \equiv \phi_{000}$ and $\Pi_{ABC}, \Pi_{AB}$ for the corresponding momenta.

The Hamiltonian that results from the action (4.1) has the form

$$
H = \frac{1}{2} \Pi_{ABC}^2 - \frac{3}{2d} \Pi_{BA}^2 - \frac{1}{4(d-1)^2} \Pi_{B}^B + \frac{6d}{d} \Pi_{B}^{BA} \partial^C Q_{AC} + \frac{3d}{2(d-1)^2} \Pi_{B}^{B} \partial A Q_{C}^{CA} + \\
+ \frac{1}{2} (\partial_{A} Q_{BCD})^2 - \frac{3}{2} (\partial_{A} Q_{ABC})^2 + \frac{3}{2} (\partial_{A} Q_{B}^{BC})^2 + \frac{3(d^2 + 4d - 2)}{4(d-1)^2} (\partial_{A} Q_{BA})^2 + \\
- \frac{3}{2} (\partial_{A} Q_{BC})^2 - \frac{3(d+2)}{d} (\partial_{A} Q_{AB})^2 - \frac{1}{2} (\partial_{A} Q_{B}^{B})^2 + \beta_{AB}(\Pi_{AB} + 3\partial^C Q_{ABC}),
$$

(5.15)

where tilde $\sim$ denotes the traceless part; for example $\widetilde{Q}_{AB} \equiv Q_{AB} - \frac{1}{(d-1)^2} \delta_{AB} Q_{C}^{C}$. There are four generations of constraints in this case:

$$
\widetilde{\psi}_{1}^{AB} = \Pi_{AB} + 3 \partial_{C} Q_{ABC},
$$

(5.16)

$$
\widetilde{\psi}_{2}^{AB} = \frac{1}{3} [\widetilde{\psi}_{1}^{AB}, H] = \partial_{C} \Pi_{ABC} - \Delta Q_{AB} - \partial^{(A} \widetilde{Q}^{B)C},
$$

(5.17)

$$
[\widetilde{\psi}_{2}^{AB}, H] = \partial^{(A} \widetilde{\psi}_{3}^{B)},
$$

(5.18)

$$
\psi_{3}^{A} = 2 \partial_{B} \partial_{C} Q_{ABC} - 2 \Delta Q_{B}^{BA} - \frac{1}{(d-1)^2} \partial^{A} \Pi_{B}^{B} - \frac{(d+2)}{(d-1)^2} \partial^{A} \partial_{C} Q_{B}^{BC},
$$
\[ [\psi^A, H] - \partial_B \psi^{AB} = \frac{1}{(d-1)} \partial^A \psi_4, \]
\[ \psi_4 = -d \Delta Q^B_B + \partial_A \Pi_B^{BA} - 2 \partial_A \partial_B Q^{AB}. \]

Being first-class, the constraints \( \tilde{\psi}^{AB}_1, \tilde{\psi}^{AB}_2, \psi^A_3, \psi_4 \) along with the tracelessness conditions imply \( R(3, d-2) \) physical degrees of freedom, which is two in \( d = 4 \).

Let us compare these results with the Hamiltonian analysis of the Fronsdal’s formulation of spin three which goes as follows. The primary constraints of Fronsdal’s theory of massless spin three \( \psi^{AB}_1, \psi^A_1, \psi_1 \) are associated with those components of the spin three field that carry index 0 thus having a time derivative of the gauge parameter in their transformation law. These generate the secondary first-class constraints \( \tilde{\psi}^{AB}_2, \psi^A_2, \psi_2 \) that results in \( R(3, d) + R(1, d) - 2R(2, d-1) - 2R(1, d-1) - 2R(0, d-1) = R(3, d-2) = 2|_{d=4} \) degrees of freedom.

In the traceless formulation considered here, the constraints \( \psi^A_1 \) and \( \psi_1 \) are absent, whereas \( \psi^A_2 \) and \( \psi_2 \) re-appear as the constraints of third and fourth generation, \( \psi^A_3 \) and \( \psi_4 \), respectively. As a result, compared to the Fronsdal formulation, the deficit of first-class constraints equals exactly to the deficit of field components so that the number of degrees of freedom remains unchanged. The counting of degrees of freedom for higher spins is analogous.

### 5.4 Higher spins

It is well-known (see [19] for the formal proof) that a number of first-class constraints equals to the number of gauge parameters independent on a Cauchy surface assuming that different time derivatives \( \xi, \dot{\xi}, \ddot{\xi} \) are independent on a Cauchy surface. Let us use this fact to count a number of first-class constraints for a massless spin-s field. Having decomposed \( \xi_{\mu_1...\mu_{s-1}} \) as

\[ \xi_{0...0}^{A(s-1-k-m)} = \sum_{k=0}^{i=s-k-1} \frac{(s-k-1)!}{(i-k-i-1)!} \partial (A_1...A_k) \xi_{A_{i+1}...A_{s-k-i-1}}^{k,i}, \]

with \( \partial^R \xi_{BA_2...A_{s-k-1}}^{k,i} = 0 \), the tracelessness condition \( \xi^\nu_{\nu\mu_2...\mu_{s-1}} = 0 \) and (3.8) acquire the form

\[ \partial_{\nu} \xi_{A(s-k-1-2)}^{k,i+1} = \Delta \xi_{A(s-k-1-2)}^{k,i+1}, \]
\[ \xi_{A(s-k-1-3)}^{k,i+2} = \Delta \xi_{A(s-k-1-3)}^{k,i+2} + s_{A(s-k-1-3)BB} \delta^{B}B. \]

The first equation allows us to express \( \xi_{A(s-k-m-2)}^{k,m} \) with \( m > 0 \) via time-derivatives of \( \xi^{k+m,0} \) as

\[ \xi_{A(s-k-m-2)}^{k,m} = \frac{(\partial_{\nu})^m}{\Delta^m} \xi_{A(s-k-m-2)}^{k+m,0}, \]

\(^{6}\)We acknowledge with gratitude that the idea of this analysis was communicated to us by I.Tyutin
whereas the second one states that the trace of $\xi^{k,0}$ is expressed via $\xi^{k+2,0}$ as

$$
\left(1 - \frac{(\partial_0)^2}{\Delta}\right) \xi^{k+2,0}_{A(s-k-3)} = \xi^{k,0}_{A(s-k-3)BB5BB}.
$$

(5.23)

As a result, the traceless components of $\xi^{k,0}_{A_1...A_{s-k-1}}$ with $k = 0...(s-1)$ remain the only independent parameters. The number of independent gauge parameters that appear in \((2.3)\) with \((\partial^0)^r\) is $R(s-r,d-1)$ for $r \geq 1$ and $R(s-1,d-1)$ for $r = 0$. This gives the correct number of physical degrees of freedom

$$
PDoF = R(s, d) - 2R(s - 1, d - 1) - \sum_{k=0}^{k=s-2} R(k, d - 1) \equiv R(s, d - 2),
$$

(5.24)

which is the dimension of the spin-$s$ irreducible representation of the massless little group $o(d - 2)$, which is two in $d = 4$.

Since first-class constraints associated with the gauge transformations generated by the parameters carrying $r$ time derivatives appear as constraints of $r^{th}$ generation, this analysis also explains why in the transverse formulation of a spin-$s$ field, first-class constraints appear up to the $(s+1)^{th}$ generation.

Note also that the two models are equivalent in the topologically trivial situation with invertible space-like derivatives simply because the partial gauge fixing that reduces the Fronsdal model to the transverse gauge invariant can be interpreted as being of Stueckelberg type with respect to the components of the gauge parameters contracted with the space-like derivatives in the transversality condition \((3.8)\). The reason why this is not true in the example of electrodynamics in the temporary gauge is that the condition \((5.4)\) contains only time derivative.

To conclude, the formulation of massless fields in terms of traceless tensors is equivalent to the original Fronsdal formulation in the topologically trivial situation although it may be different otherwise.

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