Fermionic realization of two-parameter quantum affine algebra $U_{r,s}(\mathfrak{sl}_n)$

Naihuan Jing and Honglian Zhang

Abstract. We construct all fundamental modules for the two parameter quantum affine algebra of type $A$ using a combinatorial model of Young diagrams. In particular we also give a fermionic realization of the two-parameter quantum affine algebra.

1. Introduction

Quantum enveloping algebras are one of the two main examples of quantum groups introduced by Drinfeld [Dr1] and Jimbo [Jb] in their study of the Yang-Baxter equation. In the most general definition the role of the quantum Yang-Baxter equation was given in the form of universal $R$-matrix. The first example was defined with the help of Yang-Baxter equation for a $2 \times 2$ $R$-matrix. In the physics literature and also in the work of Reshetikhin [R] the dependence on parameters can be more than one variable (see also [T, J2]).

The research on quantum enveloping algebras with two parameters have been revitalized by Benkart and Witherspoon in their work on Drinfeld double construction and Schur-Weyl duality [BW1, BW2, BW3]. For a review of early history the reader is referred to the introduction in [BW1]. After then generalizations to other simple Lie algebras were given by Bergeron, Gao and Hu in [BGH1, BGH2, BH] and their representations are studied accordingly. All these work show that the two parameter quantum groups have similar properties like the usual quantum groups but offer distinct features in two parameter cases. Recently by generalizing the vertex representations [FJ, J1] of quantum affine algebras, Hu, Rosso and the second author [HRZ] have further introduced a two parameter quantum affine algebra for the affine type $A$ and also obtained its Drinfeld realization.

It is well-known that quantum affine algebras also admit a fermionic realization [H]. The fermionic realization has played a fundamental role in quantum integrable systems and figured prominently in Kyoto school’s work on quantum affine algebras and their applications to statistical mechanics [DJKMO, JM]. The crystal basis [Ka1, Ka2] for the quantum general affine algebra was first constructed with help of Hayashi’s fermionic representation [H]. Our first motivation is to generalize...
this construction to construct all level one fundamental representations for two parameter quantum affine algebra $U_{r,s}(\widehat{\mathfrak{sl}}_n)$. In particular, the fermionic realization is also obtained, which is two parameter generalization of Misra-Miwa’s realization [MM] and a generalization of Leclerc and Thibon’s version of the combinatorial representation [LT].

Our second motivation is somewhat more fundamental to justify the study of two-parametric quantum groups. In the early days the true meaning of various parameters puzzled some researchers to question whether the introduction of other parameters is really necessary. In this paper we will explain the meaning of two parameters and obtain their combinatorial interpretation, and show how nicely many pieces of two-parameter quantum groups are patched together in our fermionic model. Roughly speaking, the two parameters correspond naturally to the left and right (multiplication) in our combinatorial model, and one further sees that the Serre relations are consequence of some combinatorial properties of our model.

2. Quantum Affine Algebra $U_{r,s}(\widehat{\mathfrak{sl}}_n)$

In this section, we will recall the structure of two-parameter quantum affine algebra $U_{r,s}(\widehat{\mathfrak{sl}}_n)$ defined in [HRZ]. Let $\mathbb{K} = \mathbb{Q}(r,s)$ denote the field of rational functions in two variables $r, s$ ($r \neq \pm s$). For $n \geq 2$ let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n-1$ be an orthonormal basis of $E = \mathbb{R}^n$ under the inner product $(, )$. Set $I = \{1, \ldots, n-1\}$, $I_0 = \{0\} \cup I$. Then $\Phi = \{\varepsilon_i - \varepsilon_j \mid i \neq j \in I\}$ is the set of roots for the simple Lie algebra $\mathfrak{sl}(n)$. We can take $\Pi = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid i \in I\}$ as the basis of simple roots. Let $\delta$ denote the primitive imaginary root of $\widehat{\mathfrak{sl}}_n$. Take $\alpha_0 = \delta - (\varepsilon_1 - \varepsilon_n)$, then $\Pi' = \{\alpha_i \mid i \in I_0\}$ is a base of simple roots of affine Lie algebra $\widehat{\mathfrak{sl}}_n$.

The following definition is an affinization of the two-parameter quantum groups for type $\mathfrak{sl}_n$ (see [BWI]).

**Definition 2.1.** Let $U = U_{r,s}(\widehat{\mathfrak{sl}}_n)$ ($n \geq 2$) be the unital associative algebra over $\mathbb{K}$ generated by the elements $e_j, f_j, \omega_j^{\pm 1}, \omega_j'^{\pm 1} (j \in I_0)$, $\gamma^{\pm \frac{1}{2}}, \gamma'^{\pm \frac{1}{2}}, D^{\pm 1}, D'^{\pm 1}$, satisfying the following relations:

(A1) $\gamma^{\pm \frac{1}{2}}, \gamma'^{\pm \frac{1}{2}}$ are central with $\gamma = \omega_0, \gamma' = \omega_0'$, such that $\omega_i \omega_i^{-1} = \omega'_i \omega'_i^{-1} = 1 = DD^{-1} = D'D'^{-1}$, and

$$\{\omega_i^{\pm 1}, \omega_j^{\pm 1}\} = [\omega_i^{\pm 1}, D^{\pm 1}] = [\omega_j^{\pm 1}, D^{\pm 1}] = [\omega_i^{\pm 1}, D'^{\pm 1}] = 0$$

$$= [\omega_i^{\pm 1}, \omega_j'^{\pm 1}] = [\omega_j'^{\pm 1}, D'^{\pm 1}] = [D'^{\pm 1}, D^{\pm 1}] = [\omega_i'^{\pm 1}, \omega_j'^{\pm 1}].$$

(A2) For $i, j \in I_0$,

$$De_i D^{-1} = r^{\delta_{0i}} e_i, \quad Df_i D^{-1} = r^{-\delta_{0i}} f_i,$$

$$\omega_j e_i \omega_j^{-1} = \langle \omega'_j, \omega_j \rangle e_i, \quad \omega_j f_i \omega_j^{-1} = \langle \omega'_j, \omega_j \rangle^{-1} f_i.$$

(A3) For $i, j \in I_0$,

$$D'e_i D'^{-1} = s^{\delta_{0i}} e_i, \quad D'f_i D'^{-1} = s^{-\delta_{0i}} f_i,$$

$$\omega'_j e_i \omega'_j^{-1} = \langle \omega'_j, \omega_j \rangle^{-1} e_i, \quad \omega'_j f_i \omega'_j^{-1} = \langle \omega'_j, \omega_j \rangle f_i.$$

(A4) For $i, j \in I_0$, we have

$$\{e_i, f_j\} = \frac{\delta_{ij}}{r-s} (\omega_i - \omega'_j).$$
(A5) For $i, j \in I_0$, but $(i, j) \notin \{ (0, n-1), (n-1, 0) \}$ with $a_{ij} = 0$, we have

$$e_i e_j = 0 = [f_i, f_j].$$

(A6) For $i \in I_0$, we have the $(r, s)$-Serre relations:

- $e_i^2 e_{i+1} - (r + s) e_i e_{i+1} e_i + (rs) e_i e_{i+1} e_i^2 = 0,$
- $e_i e_{i+1}^2 - (r + s) e_i e_{i+1} e_i + (rs) e_i^2 e_i = 0,$
- $e_{i-1}^2 e_0 - (r + s) e_{i-1} e_0 + (rs) e_0 e_{i-1} = 0,$
- $e_{i-1} e_0^2 - (r + s) e_0 e_{i-1} + (rs) e_0^2 e_{i-1} = 0.$

(A7) For $i \in I_0$, we have the $(r, s)$-Serre relations:

- $f_i^2 f_{i+1} - (r^{-1} + s^{-1}) f_i f_{i+1} f_i + (r^{-1} s^{-1}) f_i f_{i+1} f_i^2 = 0,$
- $f_i f_{i+1}^2 - (r^{-1} + s^{-1}) f_i f_{i+1} f_i + (r^{-1} s^{-1}) f_i^2 f_i = 0,$
- $f_{i-1}^2 f_0 - (r^{-1} + s^{-1}) f_{i-1} f_0 f_{i-1} + (r^{-1} s^{-1}) f_0 f_{i-1}^2 = 0,$
- $f_{i-1} f_0^2 - (r^{-1} + s^{-1}) f_0 f_{i-1} f_0 + (r^{-1} s^{-1}) f_0^2 f_{i-1} = 0,$

where $\langle \omega'_i, \omega_j \rangle$ is a skew-dual pairing defined as follows (more detail see [HRZ]):

$$\langle \omega'_i, \omega_j \rangle = \begin{cases} \rho(\epsilon_j, \alpha_i) S(\epsilon_{j+1}, \alpha_i), & (i \in I_0, \ j \in I) \\ \rho(-\epsilon_{j+1}, \alpha_0) S(-\epsilon_j, \alpha_i), & (i \in I_0, \ j = 0) \end{cases}$$

From now on, let us write briefly $\langle \omega'_i, \omega_j \rangle = (i, j)$.

It can be proved (see [HRZ]) that $U_{r,s}(\widehat{sl}_n)$ is a Hopf algebra with the coproduct $\Delta$, the counit $\varepsilon$ and the antipode $S$ defined below: for $i \in I_0$, we have

$$\Delta(\gamma^{\pm \frac{1}{2}}) = \gamma^{\pm \frac{1}{2}} \otimes \gamma^{\pm \frac{1}{2}}, \quad \Delta((\gamma')^{\pm \frac{1}{2}}) = (\gamma')^{\pm \frac{1}{2}} \otimes (\gamma')^{\pm \frac{1}{2}},$$

$$\Delta(D^{\pm 1}) = D^{\pm 1} \otimes D^{\pm 1}, \quad \Delta(D'^{\pm 1}) = D'^{\pm 1} \otimes D'^{\pm 1},$$

$$\Delta(w_i) = w_i \otimes w_i, \quad \Delta(w'_i) = w'_i \otimes w'_i,$$

$$\Delta(e_i) = e_i \otimes 1 + 1 \otimes e_i, \quad \Delta(f_i) = f_i \otimes w'_i + w'_i \otimes f_i,$$

$$\varepsilon(e_i) = \varepsilon(f_i) = 0, \quad \varepsilon((\gamma')^{\pm \frac{1}{2}}) = \varepsilon((\gamma')^{\mp \frac{1}{2}}) = \varepsilon(D^{\pm 1}) = \varepsilon(D'^{\pm 1}) = \varepsilon(w_i) = \varepsilon(w'_i) = 1,$$

$$S(\gamma^{\pm \frac{1}{2}}) = \gamma^{\mp \frac{1}{2}}, \quad S((\gamma')^{\pm \frac{1}{2}}) = (\gamma')^{\mp \frac{1}{2}}, \quad S(D^{\pm 1}) = D'^{\mp 1}, \quad S(D'^{\pm 1}) = D'^{\mp 1},$$

$$S(e_i) = -w'^{-1}_i e_i, \quad S(f_i) = -f_i w'^{-1}_i, \quad S(w_i) = w'^{-1}_i, \quad S(w'_i) = w'^{-1}_i.$$

**Remark 2.2.** The algebra $U_{r,s}(\widehat{sl}_2)$ is isomorphic to the quantum affine algebra $U_{q,q^{-1}}(\widehat{sl}_2)$ if set $rs^{-1} = q^2$, see [HRZ].

**Remark 2.3.** We remark that the two sets of generators $\omega_i, \omega'_i$ follow the original idea of [BW1, BW2] to naturally blend the second parameter into the relations. Roughly speaking when one identifies $\omega'_i$ to $\omega^{-1}_i$ the algebra specializes to the usual quantum affine algebra.

3. Fock space representations of $U_{r,s}(\widehat{sl}_n)$

In this section we construct a Fock space representation for the quantum affine algebra $U_{r,s}(\widehat{sl}_n)$ based on the fermionic representation of the usual quantum affine algebra.
The Fock space is modeled on the space of partitions. A partition is a decomposition of a natural number written in nondecreasing order. Let $\mathcal{P}(n)$ be the set of partitions of $n$. The generating function of partitions is given by

$$\sum_{n=0}^{\infty} |\mathcal{P}(n)|q^n = \prod_{n=1}^{\infty} (1 - q^n)^{-1}.$$ 

For each partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$ we associate the Young diagram consisting of $n$ nodes (or boxes) which are stacked in $l$ rows in the 4th quarter of the xy-coordinate system and aligned at the origin in such a way that the $i$th row occupies $\lambda_i$ nodes. The diagonal of the Young diagram is given by the nodes along the line $y = -x$. Another way to identify the Young diagram is the following: we specify the path starting at $(0, -\lambda'_1)$ and we move eastward by $\lambda_1$ steps and then we go north by $\lambda'_1 - \lambda'_2$ steps and so on. We say that a node or a box in $\lambda$ sitting at position $(a, -b)$ if its upper left corner is situated at the point $(a, -b)$. We will define the residue of any node at $(a, -b)$ to be $a - b \pmod{n}$. For instance the Young diagram $\lambda = (6, 4, 4, 2, 2)$ with residues is given in Figure 3. For convenience we allow the residues to take values in $\mathbb{Z}_n$. So in Figure 1 the residues shown in the lower left part will be 5, 4, 3 respectively for the quantum algebra $U_{r,s}(\hat{sl}_6)$.

A node $\gamma$ or box (convex corner) is called removable if $\lambda - \gamma$ is still a Young diagram. A contract corner is called intent if one node or box can be added at the corner to form another Young diagram. By abusing the terminology we will call a node $(a, b)$ of the border rim of the diagram of $\lambda$ indent if the box $(a - 1, b - 1)$, $(a + 1, b)$ or $(a, b - 1)$ can be added to $\lambda$ to form a new Young diagram. Note that the new diagram’s border does not contain the node except the node is a starting node or a final node of the border.

For example in Figure 1 the nodes at $(1, -1)$, $(3, 0)$, $(5, 0)$, $(0, -3)$ are indent nodes except that the nodes at $(5, 0)$, $(3, -1)$, $(1, -3)$ are removable nodes. By convention the trivial Young diagram $\phi$ has an imaginary indent node at $(-1, 1)$. For this reason we sometimes say that a point at $(a, -b)$ is removable or indented. In this sense the origin for the trivial Young diagram is indented.

For each $i \in \mathbb{Z}_n$ we define the Young diagram $|\lambda, i\rangle$ as the diagram that assigns the residues $i + a - b \pmod{n}$ to each node $(a, -b)$ in $\lambda$. The previous example is the Young diagram for $i = 0$. The following Figure 2 shows the residue of $(a, -b)$ for the configuration of $|\lambda, i\rangle$.
Let $K = \mathbb{Q}(r, s)$ be the field of rational functions in $r$ and $s$. For each $i \in \mathbb{Z}_n$ we define the Fock space $\mathcal{F}_i$ to be

$$\mathcal{F}_i = \bigoplus_{\lambda \in \mathcal{P}} K |\lambda, i\rangle.$$  

Let $\lambda$ and $\mu$ be two partitions such that $\mu$ is obtained from $\lambda$ by adding a node $\gamma$ of residue $i$, or $\mu/\lambda$ consists of a single $i$-node $\gamma$. We further let $I_i(\lambda)$ be the set of indent $i$-nodes in the boundary of $\lambda$, and $R_i(\lambda)$ be the set of its removable $i$-nodes in the boundary of $\lambda$. We also denote by $I_i(0)(\lambda)$ (resp. $R_i(0)(\lambda)$) the total number of indent (resp. removable) 0-nodes in diagram $\lambda$. We denote by $|I_i^0(\lambda)|$ and $|R_i^0(\lambda)|$ the number of the set $I_i^0(\lambda)$ and $R_i^0(\lambda)$ respectively.

For a finite set $S$, we use $|S|$ to denote its cardinality. We define the action of the simple generators as follows.

$$f_i | \lambda \rangle = \sum_{\mu} r^{|I_i^0(\lambda)|} s^{-|R_i^0(\lambda)|} |R_i^0(\lambda), \mu\rangle | \mu \rangle,$$

$$e_i | \lambda \rangle = \sum_{\mu} r^{|R_i^0(\mu, \lambda)|} s^{-|I_i^0(\mu, \lambda)|} |I_i^0(\mu, \lambda), \mu\rangle | \mu \rangle,$$

$$\omega_i | \lambda \rangle = r^{|I_i^0(\lambda)|} s^{-|R_i^0(\lambda)|} | I_i^0(\lambda), \lambda \rangle,$$

$$\omega'_i | \lambda \rangle = r^{|R_i^0(\lambda)|} s^{-|I_i^0(\lambda)|} | R_i^0(\lambda), \lambda \rangle,$$

$$D | \lambda \rangle = r^{-|R_i^0(\lambda)|} s^{-|I_i^0(\lambda)|} | -I_i^0(\lambda), \lambda \rangle,$$

$$D'_i | \lambda \rangle = r^{-|I_i^0(\lambda)|} s^{-|R_i^0(\lambda)|} | -R_i^0(\lambda), \lambda \rangle,$$

\[ \text{Figure 2. Residue for Young diagram } |\lambda, i\rangle \]
where the first sum runs through all \( \mu \) such that \( \mu / \lambda \) is a \( i \)-node, and the second sum runs through all \( \mu \) such that \( \lambda / \mu \) is a \( i \)-node. We remark that all the sums are obviously finite since it runs through the cells of the Young diagram. We also note that when \( r = s^{-1} = q \), the above action reduces to that of the quantum affine algebra \( U_q(\hat{sl}_n) \) formulated in [LT].

**Theorem 3.1.** For each \( i \in \mathbb{Z}_+ \) the Fock space \( \mathcal{F}_i \) is a level one representation of \( \mathcal{U}_q(\hat{sl}_n) \) and contains \( V(\Lambda_i) \) as a submodule and the highest weight vector is \( |\phi, i> \), where \( \phi \) is the empty diagram.

**Proof.** First of all we note the statement about the submodule is obtained by computing the action of the Heisenberg subalgebra. By symmetry it is enough to show the statement for the basic module \( V(\Lambda_0) \), so we will drop the second index \( i = 0 \) in \( |\lambda, 0 \rangle \). Namely, we take \( i = 0 \) and the residue for the node \((a, -b)\) is \( a - b \).

(i) From the construction we first have

\[
\omega_j e_i | \lambda > = \omega_j \sum_{\gamma \in R_i(\Lambda)} r_i^{I_j(\lambda - \gamma, \lambda)} | s_i^{I_j(\lambda - \gamma, \lambda)} | \lambda - \gamma > \\
= \sum_{\gamma \in R_i(\Lambda)} r_i^{I_j(\lambda - \gamma, \lambda)} | s_i^{I_j(\lambda - \gamma, \lambda)} | R_j(\lambda) | \lambda - \gamma > ,
\]

and similarly

\[
e_i \omega_j | \lambda > = \sum_{\gamma \in R_i(\Lambda)} r_i^{I_j(\lambda - \gamma, \lambda)} | s_i^{I_j(\lambda - \gamma, \lambda)} | R_j(\lambda) | \lambda - \gamma > ,
\]

Since for \( \gamma \in R_i(\lambda) \),

\[
|I_i(\lambda - \gamma)| = |I_i(\lambda)| + 1, \quad |R_i(\lambda - \gamma)| = |R_i(\lambda)| - 1,
\]

It follows immediately from (3.2) that

\[
\omega_j e_i = rs^{-1} e_i \omega_i = \langle i, i \rangle e_i \omega_i .
\]

On the other hand it follows from \( \gamma \in R_i(\lambda) \) \((i \neq j)\) that,

\[
|I_j(\lambda - \gamma)| = |I_j(\lambda)| + \begin{cases} \langle \varepsilon_j, \alpha_i \rangle, & (i \in I_0, j \in I) \\ -\langle \varepsilon_{i+1}, \alpha_0 \rangle, & (i \in I_0, j = 0) \end{cases}
\]

(3.3)

\[
|R_j(\lambda - \gamma)| = |R_j(\lambda)| + \begin{cases} \langle \varepsilon_{j+1}, \alpha_i \rangle, & (i \in I_0, j \in I) \\ \langle \varepsilon_1, \alpha_i \rangle, & (i \in I_0, j = 0) \end{cases}
\]

(3.4)

These combinatorial identities imply that

\[
\omega_j e_i = < i, j > e_i \omega_j ,
\]
(ii) To check the commutation relation (A4), we consider the actions of $e_i$ and $f_i$:

$$e_i f_i | \lambda \rangle = e_i \sum_{\gamma \in I_i(\lambda)} p |I_i^{(r)}(\lambda, \lambda + \gamma) \rangle s|R_i^{(r)}(\lambda, \lambda + \gamma)\rangle | \lambda + \gamma \rangle$$

$$= \sum_{\gamma \in I_i(\lambda)} p |R_i^{(r)}(\lambda, \lambda + \gamma)\rangle s|R_i^{(r)}(\lambda, \lambda + \gamma)\rangle$$

$$\times p |I_i^{(r)}(\lambda + \gamma, \lambda + \gamma)\rangle s|I_i^{(r)}(\lambda + \gamma - \gamma', \lambda + \gamma)\rangle | \lambda + \gamma - \gamma' \rangle$$

$$= \sum_{\gamma \in R_i(\lambda)} p |I_i^{(r)}(\lambda, \lambda + \gamma)\rangle s|R_i^{(r)}(\lambda, \lambda + \gamma)\rangle$$

$$\times s|I_i^{(r)}(\lambda + \gamma, \lambda + \gamma)\rangle + |I_i^{(r)}(\lambda + \gamma - \gamma', \lambda + \gamma)\rangle | \lambda + \gamma - \gamma' \rangle ,$$

where we used the result: $R_i(\lambda + \gamma) = R_i(\lambda) + \{ \gamma \}$.

Reversing the order of the product we have,

$$f_i e_i | \lambda \rangle = \sum_{\gamma \in I_i(\lambda)} p |R_i^{(r)}(\lambda - \gamma', \lambda)\rangle s|I_i^{(r)}(\lambda - \gamma', \lambda)\rangle$$

$$\times p |I_i^{(r)}(\lambda - \gamma', \lambda - \gamma' + \gamma)\rangle s|R_i^{(r)}(\lambda - \gamma', \lambda - \gamma' + \gamma)\rangle | \lambda - \gamma' + \gamma \rangle$$

$$= \sum_{\gamma \in R_i(\lambda)} p |R_i^{(r)}(\lambda - \gamma', \lambda)\rangle$$

$$\times s|I_i^{(r)}(\lambda - \gamma', \lambda)\rangle + |I_i^{(r)}(\lambda - \gamma', \lambda - \gamma' + \gamma)\rangle$$

$$+ \sum_{\gamma \in I_i(\lambda)} p |R_i^{(r)}(\lambda - \gamma', \lambda)\rangle$$

$$\times s|I_i^{(r)}(\lambda - \gamma', \lambda)\rangle + |R_i^{(r)}(\lambda - \gamma', \lambda - \gamma' + \gamma)\rangle | \lambda - \gamma' + \gamma \rangle .$$

The following fact is easily verified.

**Claim A** For all $\gamma \in I_i(\lambda), \gamma' \in R_i(\lambda)$

(3.5) $I_i^{(r)}(\lambda, \lambda + \gamma) - I_i^{(r)}(\lambda - \gamma', \lambda - \gamma' + \gamma)$

$$= R_i^{(r)}(\lambda - \gamma') - R_i^{(r)}(\lambda + \gamma - \gamma', \lambda + \gamma);$$

(3.6) $I_i^{(r)}(\lambda - \gamma', \lambda) - I_i^{(r)}(\lambda + \gamma - \gamma', \lambda + \gamma)$

$$= R_i^{(r)}(\lambda, \lambda + \gamma) - R_i^{(r)}(\lambda - \gamma', \lambda - \gamma' + \gamma).$$

Combining the above two expressions in Claim A, we get that

$$[e_i, f_i] | \lambda \rangle =$$

$$\sum_{\gamma \in I_i(\lambda)} p |I_i^{(r)}(\lambda + \gamma)\rangle + |R_i^{(r)}(\lambda, \lambda + \gamma)\rangle s|I_i^{(r)}(\lambda, \lambda + \gamma)\rangle + |R_i^{(r)}(\lambda, \lambda + \gamma)\rangle | \lambda \rangle$$

$$- \sum_{\gamma' \in R_i(\lambda)} p |I_i^{(r)}(\lambda - \gamma', \lambda)\rangle + |R_i^{(r)}(\lambda - \gamma', \lambda)\rangle s|I_i^{(r)}(\lambda - \gamma', \lambda)\rangle + |R_i^{(r)}(\lambda - \gamma', \lambda)\rangle | \lambda \rangle$$
The following Claim B is important for the further deduction.

**Claim B** For all $\gamma \in I_i(\lambda)$, $\gamma' \in R_i(\lambda)$

(3.7) \[ |I_i(\lambda - \gamma')| = |I_i^{(r)}(\lambda - \gamma', \lambda)| + |I_i^{(l)}(\lambda - \gamma', \lambda)| + 1; \]

(3.8) \[ |R_i(\lambda + \gamma)| = |R_i^{(r)}(\lambda, \lambda + \gamma)| + |R_i^{(l)}(\lambda, \lambda + \gamma)| + 1; \]

(3.9) \[ |I_i(\lambda)| = |I_i^{(r)}(\lambda, \lambda + \gamma)| + |I_i^{(l)}(\lambda, \lambda + \gamma)| + 1; \]

(3.10) \[ |R_i(\lambda)| = |R_i^{(r)}(\lambda - \gamma', \lambda)| + |R_i^{(l)}(\lambda - \gamma', \lambda)| + 1; \]

We note that the coefficient is given by

\[
\sum_{\gamma \in I_i(\lambda)} r |I_i^{(r)}(\lambda, \lambda + \gamma)| + |R_i^{(l)}(\lambda, \lambda + \gamma)| + |I_i^{(l)}(\lambda, \lambda + \gamma)| + 1
\]

\[
- \sum_{\gamma' \in R_i(\lambda)} r |I_i^{(r)}(\lambda - \gamma', \lambda)| + |R_i^{(l)}(\lambda - \gamma', \lambda)| + |I_i^{(l)}(\lambda - \gamma', \lambda)| + 1
\]

\[
= \sum_{\gamma \in I_i(\lambda)} r |I_i^{(r)}(\lambda, \lambda + \gamma)| - |R_i^{(r)}(\lambda, \lambda + \gamma)| - |R_i^{(l)}(\lambda, \lambda + \gamma)| - 1
\]

\[
- \sum_{\gamma' \in R_i(\lambda)} r |I_i^{(r)}(\lambda - \gamma', \lambda)| - |R_i^{(r)}(\lambda - \gamma', \lambda)| - |R_i^{(l)}(\lambda - \gamma', \lambda)| - 1
\]

where we used the relations (3.8) and (3.9) in the first term and the relations (3.7) and (3.10) in the second term. Then we have

The action of \[ [\varepsilon_i, f_i] \]

\[
= \sum_{\gamma \in I_i(\lambda)} r |R_i(\lambda + \gamma)| s |I_i(\lambda)| - 2 \left( r s^{-1} |I_i^{(r)}(\lambda, \lambda + \gamma)| - |R_i^{(r)}(\lambda, \lambda + \gamma)| - 1 \right)
\]

\[
- \sum_{\gamma' \in R_i(\lambda)} r |R_i^{(r)}(\lambda - \gamma', \lambda)| - 3 \left( r s^{-1} |I_i^{(r)}(\lambda - \gamma', \lambda)| - |R_i^{(r)}(\lambda - \gamma', \lambda)| - 2 \right)
\]

\[
= r |R_i(\lambda)| + 1 \sum_{\gamma \in I_i(\lambda)} s |I_i(\lambda)| - 2 \left( r s^{-1} |I_i^{(r)}(\lambda, \lambda + \gamma)| - |R_i^{(r)}(\lambda, \lambda + \gamma)| - 1 \right)
\]

\[
- \sum_{\gamma' \in R_i(\lambda)} (r s^{-1} |I_i^{(r)}(\lambda - \gamma', \lambda)| - |R_i^{(r)}(\lambda - \gamma', \lambda)| - 2)
\]

\[
= r |R_i(\lambda)| + 1 \sum_{\gamma \in I_i(\lambda)} s |I_i(\lambda)| - 2 \left( r s^{-1} |I_i^{(r)}(\lambda)| - |R_i(\lambda)| - 2 + (r s^{-1} |I_i^{(l)}(\lambda)| - |R_i(\lambda)| - 3 \right)
\]

\[
+ \ldots + (r s^{-1} |I_i^{(l)}(\lambda)| - |R_i(\lambda)| - |I_i(\lambda)| + |R_i(\lambda)| - 1)
\]

\[
= r |I_i(\lambda)| s |R_i(\lambda)| - r |R_i(\lambda)| s |I_i(\lambda)|
\]

\[
= \frac{r - s}{r - s}
\]

The action of \[ \omega_i - \omega_i' \]

Finally, we will check the \((r, s) - \) Serre relation:

\[
f_{i+1} f_i^2 - (r + s) f_i f_{i+1} f_i + rs f_i^2 f_{i+1} = 0.
\]
It follows from the definition that
\[
\sum_{\gamma_1, \gamma_2 \in I_i(\lambda) \atop \gamma_3 \in I_{i+1}(\lambda + \gamma_1 + \gamma_2)} \gamma^2 \left| \gamma \right> \left< \gamma \right|
\]
\[
= \sum_{\gamma_1, \gamma_2 \in I_i(\lambda) \atop \gamma_3 \in I_{i+1}(\lambda + \gamma_1 + \gamma_2)} \gamma_1 + \gamma_2 + \gamma_3 \gamma^3 \left| \gamma \right> \left< \gamma \right|
\]
For simplicity, we write
\[
I_{i+1}(\lambda + \gamma_1 + \gamma_2) = I_{i+1}(\lambda) \cup \left( I_{i+1}(\lambda + \gamma_1) - I_{i+1}(\lambda) \right) \cup \left( I_{i+1}(\lambda + \gamma_2) - I_{i+1}(\lambda) \right),
\]
which is derived since \( I_{i+1}(\lambda + \gamma_1 + \gamma_2) = I_{i+1}(\lambda) \cup I_{i+1}(\lambda + \gamma_1) \cup I_{i+1}(\lambda + \gamma_2). \)
Then the coefficient of the above expression becomes
\[
\sum_{\gamma_1 \in I_i(\lambda) \atop \gamma_2 \in I_{i+1}(\lambda + \gamma_1) \atop \gamma_3 \in I_{i+1}(\lambda + \gamma_2) - I_{i+1}(\lambda)} \gamma^3 \left| \gamma \right> \left< \gamma \right|
\]
\[
= \sum_{\gamma_1 \in I_i(\lambda) \atop \gamma_2 \in I_{i+1}(\lambda + \gamma_1) \atop \gamma_3 \in I_{i+1}(\lambda + \gamma_2) - I_{i+1}(\lambda) \atop \gamma_4 \in I_i(\lambda + \gamma_1 + \gamma_2)} \gamma^4 \left| \gamma \right> \left< \gamma \right|
\]
Furthermore we get
\[
\sum_{\gamma_1 \in I_i(\lambda) \atop \gamma_2 \in I_{i+1}(\lambda + \gamma_1) \atop \gamma_3 \in I_{i+1}(\lambda + \gamma_2) - I_{i+1}(\lambda)} \gamma^3 \left| \gamma \right> \left< \gamma \right|
\]
\[
= \sum_{\gamma_1 \in I_i(\lambda) \atop \gamma_2 \in I_{i+1}(\lambda + \gamma_1) \atop \gamma_3 \in I_{i+1}(\lambda + \gamma_2) - I_{i+1}(\lambda) \atop \gamma_4 \in I_i(\lambda + \gamma_1 + \gamma_2)} \gamma^4 \left| \gamma \right> \left< \gamma \right|
\]
Similarly, using
\[
I_{i+1}(\lambda + \gamma_1) = I_{i+1}(\lambda) \cup \left( I_{i+1}(\lambda + \gamma_1) - I_{i+1}(\lambda) \right),
\]
the coefficient of the second expression becomes
\[
\sum_{\gamma_1 \in I_i(\lambda) \atop \gamma_2 \in I_{i+1}(\lambda) \atop \gamma_3 \in I_{i+1}(\lambda + \gamma_1) \atop \gamma_4 \in I_i(\lambda + \gamma_1 + \gamma_2)} \gamma^4 \left| \gamma \right> \left< \gamma \right|
\]
\[
= \sum_{\gamma_1 \in I_i(\lambda) \atop \gamma_2 \in I_{i+1}(\lambda) \atop \gamma_3 \in I_{i+1}(\lambda + \gamma_1) \atop \gamma_4 \in I_i(\lambda + \gamma_1 + \gamma_2)} \gamma^4 \left| \gamma \right> \left< \gamma \right|
\]
\[
= \sum_{\gamma_1 \in I_i(\lambda) \atop \gamma_2 \in I_{i+1}(\lambda) \atop \gamma_3 \in I_{i+1}(\lambda + \gamma_1) \atop \gamma_4 \in I_i(\lambda + \gamma_1 + \gamma_2)} \gamma^4 \left| \gamma \right> \left< \gamma \right|
\]
\[
= \sum_{\gamma_1 \in I_i(\lambda) \atop \gamma_2 \in I_{i+1}(\lambda) \atop \gamma_3 \in I_{i+1}(\lambda + \gamma_1) \atop \gamma_4 \in I_i(\lambda + \gamma_1 + \gamma_2)} \gamma^4 \left| \gamma \right> \left< \gamma \right|
\]
Finally, using the definition again one has

$$f_i^2 f_{i+1} | \lambda \rangle = \sum_{\gamma_3 \in I_{i+1}^{(1)}(\lambda)} \sum_{\gamma_2 \in I_i^{(1)}(\lambda + \gamma_1 + \gamma_2)} \langle \gamma_3 | I_{i+1}^{(1)}(\lambda, \lambda + \gamma_2) + I_i^{(1)}(\lambda, \lambda + \gamma_1 + \gamma_3) | f_i f_{i+1} f_i + rs f_i^2 f_{i+1} | \lambda \rangle$$

Since for $\gamma_3 \in I_{i+1}^{(1)}(\lambda)$, one gets

$$I_i(\lambda + \gamma_3) = I_i(\lambda), \quad I_i(\lambda + \gamma_1 + \gamma_3) = I_i(\lambda + \gamma_1).$$

Thus the coefficient of the third expression becomes

$$\sum_{\gamma_3 \in I_{i+1}^{(1)}(\lambda)} \sum_{\gamma_2 \in I_i^{(1)}(\lambda + \gamma_1 + \gamma_3)} \langle \gamma_3 | I_{i+1}^{(1)}(\lambda, \lambda + \gamma_2) + I_i^{(1)}(\lambda, \lambda + \gamma_1 + \gamma_3) | f_i f_{i+1} f_i + rs f_i^2 f_{i+1} | \lambda \rangle$$

Combining the above three coefficients, we get the required result:

$$f_{i+1} f_i^2 - (r + s) f_i f_{i+1} f_i + rs f_i^2 f_{i+1} | \lambda \rangle = 0,$$

where we have used the following fact.

**Claim C** For $\gamma_2 \in I_i(\lambda + \gamma_1)$ and $\gamma_3 \in I_{i+1}(\lambda + \gamma_1 + \gamma_2)$, it follows that,

$$| I_{i+1}^{(r)}(\lambda + \gamma_2, \lambda + \gamma_2 + \gamma_3) | = | I_i^{(r)}(\lambda + \gamma_1, \lambda + \gamma_1 + \gamma_3) | + 1,$$

$$| R_{i+1}^{(r)}(\lambda + \gamma_2, \lambda + \gamma_2 + \gamma_3) | = | R_i^{(r)}(\lambda + \gamma_1, \lambda + \gamma_1 + \gamma_3) | - 1.$$

This completes the proof of Theorem 3.1.

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DEPARTMENT OF MATHEMATICS, NORTH CAROLINA STATE UNIVERSITY, RALEIGH, NC 27695-8205, USA

SCHOOL OF SCIENCES, SOUTH CHINA UNIVERSITY OF TECHNOLOGY, GUANGZHOU 510640, CHINA

E-mail address: jing@math.ncsu.edu

DEPARTMENT OF MATHEMATICS, SHANGHAI UNIVERSITY, SHANGHAI 200444, PR CHINA

E-mail address: hlzhangmath@shu.edu.cn