BOOTSTRAP PERCOLATION ON THE HAMMING TORUS WITH THRESHOLD 2

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Abstract. This paper analyzes various questions pertaining to bootstrap percolation on the \(d\)-dimension Hamming torus where each node is open with probability \(p\). We find the critical exponent for the event that a \(2j\)-dimension sublattice becomes open and compute exact limiting values for probabilities for the existence of such a \(2j\)-sublattice. We use Stein-Chen’s method to show that the number of \(2j\)-dimension sublattices that become open can be approximated by a Poisson random variable.

Bootstrap percolation first appeared in a paper by Chalupa et al [6] as a model for ferromagnetism. Adler [1] provides a wonderful introduction to the subject.

In general we take a graph \(G = (V, E)\) with vertex set \(V\) and edge set \(E\) and a parameter \(\theta\) which we call the threshold. Each vertex in the graph is initially in one of two possible states, either open or closed. At each subsequent step a vertex can become open if at least \(\theta\) of its neighbors are open. Once open, a vertex remains open for all eternity.

We describe the increasing evolution of the configuration formally. Let \(\omega_t \in \{0, 1\}^V\) denote the configuration of the vertices at time \(t \geq 0\). If a vertex \(v\) is open at step \(t\) we say \(\omega_t(v) = 1\) and similarly if the vertex is closed at time \(t\), \(\omega_t(v) = 0\). For bootstrap percolation with threshold \(\theta\), \(\omega_t\) evolves as follows for \(t \geq 0\):

\[
\omega_{t+1}(v) = \begin{cases} 
1, & \omega_t(v) = 1 \\
1, & \sum_{v' \sim v} \omega_t(v') \geq \theta \\
0, & \text{otherwise}
\end{cases}
\]

where \(v' \sim v\) if there is an edge in \(E\) connecting \(v\) and \(v'\).

Many natural questions can be asked. Given some initial configuration, we can ask what the evolved configuration look like after some time. In particular we care about the steady state, \(\omega_\infty\). Typically this is viewed probabilistically. Given a distribution on \(\omega_0\) what can we say about \(\omega_\infty\).

The first major progress came from van Enter [13] and later Schonmann [12]. They proved a \(0 - 1\) law for configurations on \(V = Z^d\) with edges connecting each vertex to its \(2d\) nearest neighbors. In their model, initially each vertex is independently open with probability \(p\). For \(\theta \leq d\), if \(p > 0\) then the entire grid becomes open with probability 1. If \(\theta > d\) then everything becomes completely open only if \(p = 1\).

The next big step in the history of bootstrap percolation was to view the process on an increasing family of graphs \(G = \{G_n = (V_n, E_n)\}\) where the initial probability

2010 Mathematics Subject Classification. 60K35.
Key words and phrases. Bootstrap Percolation.
that a vertex is open is given by a function of \( n, p = p(n) \). As each graph is finite, 
\( f^n(p) := \mathbb{P}_p(\omega_\infty \equiv 1) \) can be viewed as an increasing polynomial in \( p \) with \( f^n(0) = 0 \) and \( f^n(1) = 1 \). By continuity there is a critical value, \( p_c \), such that \( f^n(p_c) = 1/2 \). Much work centers around finding bounds on \( p_c \) as a function of \( n \).

Aizenman and Lebowitz \([2]\) showed for the finite \( d \) dimensional grid, \([n]^d\), and threshold \( \theta = 2 \), there exists constants \( c_1, c_2 \) such that \( c_1 < (\log n)^{d-1} p_c < c_2 \). Moreover, they show that the transition for \( f^n(p) \) from 0 to 1 is sharp near \( p_c \).

In a widely celebrated paper Holroyd \([9]\) showed that
\[
p_c \sim \frac{\pi^2}{18} \log n
\]
when \( d = \theta = 2 \). Later this result was expanded on by Holroyd, Ligget, and Romik \([10]\) to \( d = 2, \theta = k + 1 \) where the neighborhood of a vertex is the \( k \) closest vertices in each of the cardinal directions. They show \( p_c \sim \frac{\pi^2}{(3(k + 2)(k + 1))} \log n \) for this graph. These types of results have been extended to higher dimensions by \([3]\), random graphs \([4]\), and more geometric settings \([5]\). It is a very active area of research.

Our graph of interest is the \( d \)-dimensional Hamming torus. The Hamming torus has the same vertex set as the lattice, \( V = [n]^d \), but the edge set is modified to connect every vertex that can be connected with a straight path on the grid. In terms of the coordinates edge set is
\[
E = \{(v, w) : v \text{ differs from } w \text{ in exactly one coordinate }\}.
\]
Gravner et al. introduced the study of bootstrap percolation on the Hamming torus \([4]\). In their paper they focus on evolution thresholds greater than 2. They find threshold functions of the critical probability, \( p_c(\theta, d) \), for the event \( C = \{\omega_\infty \equiv 1\} \), where \( \mathbb{P}_{p_c(\theta, d)}(C) = 1/2 \). They also consider finer structure:
\[
C_i = \{\exists W \subset [n]^d \text{ with } \dim(W) = i \text{ s.t. } \omega_{\infty}|W = 1\},
\]
and they find bounds for the critical exponent of threshold functions \( p_c(\theta, i, d) \), where \( \mathbb{P}_{p_c(\theta, i, d)}(C_i) = 1/2 \). For \( i = 0 \) we slightly alter the definition to be
\[
C = \{\exists v \text{ s.t. } \sum_{w \sim v} \omega(w) \geq \theta\}.
\]

In many cases they were able to show the critical probability is of the form \( p_c(\theta, i, d) = (1 + o(1))an^{-\beta} \) for some \( \beta < 0 \). We call \( \beta \) the critical exponent. For \( d = \theta = 3 \) they showed the threshold function for \( C_0 \), and \( C_1 \) is \( (1 + o(1))an^{-2} \) and the threshold function for \( C_2, C_3 \) is \( (1 + o(1))an^{-5/3} \), for some constant \( a > 0 \). They showed for \( d \geq 3 \), \( C_1 \) and \( C_2 \) have different critical exponents. For \( d = 2 \), and all values of \( \theta \), the critical exponent is the same for \( C_1 \) and \( C_2 \).

We consider the case \( \theta = 2 \) and \( d > 2 \). Let \( j < \sqrt{d} \). We show that the equivalently defined threshold functions for \( C_2, C_4, \ldots, C_{2j} \) have distinct exponents. We will also show for \( i < j \), the threshold functions for \( C_{2i-1} \) and \( C_{2i} \) have the same exponent. For \( i > \sqrt{d} \), we show that \( C_i \) all have the same critical exponent. After we have determined the critical exponent for these events, we will give a precise description of the asymptotics of \( p_c(\theta, i, d) \). Unlike the threshold functions for the grid \( \mathbb{Z}^d \) found in Holroyd \([9]\), \( p_c(\theta, i, d) \) is not sharp. For the remainder of this paper we drop the parameter \( \theta \) as it will always be 2.
1. Statements

First we need a few definitions.

**Definition 1.1.** A subset $V \subset [n]^d$ is a **sublattice** if there exists a set of indices $I(V)$ and constants $\{a_l|l \in I(V)\}$ such that $v_l = a_l$ for all $l \in I(V)$ if and only if $v \in V$. We say $V$ has dimension $i$ if $|I(V)| = d - i$. 

**Definition 1.2.** For a set of open nodes, $S = \{v : \omega_t(v) = 1\}$ at some time $t$, we denote the open nodes of the evolved configuration by

$$\langle S \rangle = \{v : \omega_{\infty}(v) = 1 \text{ where } \omega_0(v) = 1 \iff v \in S\}.$$ 

We say a sublattice $V$ is **internally spanned** if there exists a subset $S \subset V$ of open nodes at time 0, such that $V = \langle S \rangle$.

**Definition 1.3.** A sublattice $V$ is **maximal** in $\langle S \rangle$ if no other sublattice in $\langle S \rangle$ contains $V$.

Our results center around the following events:

- $I_V = \{\exists S \subset V \text{ such that } \omega_0(v) = 1 \forall v \in S \text{ and } \langle S \rangle = V\}$. In other words $V$ is internally spanned.
- $I_l = \{\exists V \text{ such that } \dim(V) = l \text{ and } I_V \text{ occurs }\} = \bigcup_{\dim(V) = l} I_V$.
- $C_l = \{\exists V \text{ such that } \dim(V) = l \text{ and } \omega_{\infty}(v) = 1 \text{ for all } v \in V\}$.

Note the slight difference in the definitions of $I_l$ and $C_l$. For $C_l$ the only thing that matters is the final state $\omega_\infty$ where for $I_l$ it is important how one gets to $\omega_\infty$.

Lastly we prove a statement about the random variable $D = \sup_{0 \leq l \leq d} \{|1_{I_l}|\}$. This gives the dimension of the largest sublattice that is internally spanned.

**Theorem 1.1.** If $p = f(n)n^{-d/(j+1)-j}$, then

$$(\mathbb{P}_p(I_{2j}), \mathbb{P}_p(C_{2j}) \to \begin{cases} 0 & \text{ if } f(n) \to 0 \\ 1 & \text{ if } f(n) \to \infty \\ 1 - e^{-\lambda(j,d,a)} & \text{ if } f(n) \to a \end{cases}$$

where $\lambda(j,d,a) = \binom{d}{2j} (2j)! 2^{-j} a^{j+1}$.

This implies

$$p_c = (1 + o(1)) a \frac{1}{2} n^{-d/(j+1)-j}$$

where $1 - e^{-\lambda(j,d,a)} = 1/2$.

The next result comes from the application of the Stein-Chen method [11]. For two non-negative integer valued random variables $Y$ and $Z$ the total variation is defined as

$$d_{TV} = \frac{1}{2} \sum_{k=0}^{\infty} |\mathbb{P}(Y = k) - \mathbb{P}(Z = k)|.$$
For the remaining results we will assume \( p = an^{-d/(j+1)-j} \) and \( \lambda(j, d, a) = \left(\frac{d}{2j}\right)(2j)!2^{-j}a^{j+1} \).

To simplify the statement of results we let \( \lambda = \lambda(j, d, a) \) where no confusion will arise.

**Theorem 1.2.** For \( j(j+1) < d \), let \( Y \) denote the number of sublattices \( V \subset [n]^d \) such that both \( \dim(V) = 2j \) and \( I_V \) occurs, and let \( Z \) denote a Poisson(\( \lambda \)) random variable. Then

\[
\lim_{n \to \infty} d_{TV}(Y, Z) \to 0.
\]

The precision given by Theorem 1.2 leads to the following result:

**Theorem 1.3.** Let \( j(j+1) = d > 6 \), \( c = \left(\frac{d-2j+2}{2}\right) \), and \( \lambda' = \left(\frac{d}{2j-2}\right)(2j-2)!2^{-j+1}a^j \).

Then

\[
\mathbb{P}_p(I_{2j}) \to \sum_{k=1}^{\infty} \frac{e^{-\lambda'}}{k!} \lambda'^k (1 - e^{-\lambda'^k}).
\]

For \( l \geq j \),

\[
\mathbb{P}_p(I_{2l}) \to \mathbb{P}_p(I_{2j}).
\]

Lastly we have a useful corollary for the random variable \( D \). We assume \( p, \lambda, \lambda', \) and \( c \) are as above.

**Corollary 1.4.** Largest Sublattices \( (D = 2l) \)

If \( j(j+2) < d \), then

\[
\mathbb{P}_p(D = 2j - 2) \to e^{-\lambda'}
\]

\[
\mathbb{P}_p(D = 2j) \to 1 - e^{-\lambda'}.
\]

If \( j(j+1) < d < (j+1)(j+2) \), then

\[
\mathbb{P}_p(D = 2j - 2) \to e^{-\lambda'}
\]

\[
\mathbb{P}_p(D = d) \to 1 - e^{-\lambda'}.
\]

If \( j(j+1) = d > 6 \), then

\[
\mathbb{P}_p(D = 2j - 4) \to e^{-\lambda'}
\]

\[
\mathbb{P}_p(D = 2j - 2) \to \sum_{k=1}^{\infty} \frac{e^{-\lambda'}}{k!} \lambda'^k (e^{-\lambda'^k}),
\]

\[
\mathbb{P}_p(D = d) \to 1 - \sum_{k=0}^{\infty} \frac{e^{-\lambda'}}{k!} \lambda'^k e^{-\lambda'^k}.
\]

In Section 2, we prove lemmas that describe the evolution \( \omega_t \) when \( \theta = 2 \). In Section 3, we prove both upper and lower bounds for the critical exponent for the events \( C_{2j} \) and \( I_{2j} \). In Section 4 we use the Stein-Chen method \([11]\) to describe precisely the asymptotics of \( p_c(l, d) \). In Section 5 we combine everything to prove our statements.
2. Growth for $\theta = 2$

We begin with the simplest case. Suppose $u \neq v$ are the only nodes which are initially open. Define the distance between to nodes as $\text{dis}(u, v) = \sum_{i=1}^{d} 1_{u_i \neq v_i}$, the number of coordinates where $u$ and $v$ differ. If $\text{dis}(u, v) > 2$ then no new nodes become open $\langle u, v \rangle = \{u, v\}$. If $\text{dis}(u, v) \leq 2$ then $u$ and $v$ must agree for all but at most 2 indices.

Without loss of generality, we may choose a basis so that $u_i = v_i$ for $i > 2$. Suppose first that $u_2 = v_2$ as well (i.e. $\text{dis}(u, v) = 1$), the line $\{(t, u_2, \cdot \cdot \cdot) : t \in [n]\}$ has two nodes initially on, and after one step every node in that line becomes open. Every node not on the line has at most one neighbor on the line, so growth stops.

If $\text{dis}(u, v) = 2$, then after one step the co-neighbors of $u$ and $v$, $u' = (u_1, v_2, \cdot \cdot \cdot)$ and $v' = (v_1, u_2, \cdot \cdot \cdot)$, become open. The nodes $u$ and $u'$ are two different open neighbors for every closed node in the line $\{(u_1, s, \cdot \cdot \cdot) : s \in [n]\}$, so after two steps the entire line becomes open. The same is true for the lines containing both $u$ and $v'$, both $v$ and $u'$, and both $v'$ and $v$. Once those lines are open every other node in the plane $\{(t, s, \cdot \cdot \cdot) : (t, s) \in [n]^2\}$ has a at least two (in fact four) open neighbors, so the entire plane becomes open.
Growth for higher dimension sublattices is a bit more involved. First we generalize the distance function to subsets $S_1, S_2$ as follows,

$$\text{dis}(S_1, S_2) = \inf_{u \in S_1, v \in S_2} \text{dis}(u, v).$$

For a $2^j$-dimension sublattice, $V_{2^j}$, to become open, most of the time there are $j + 1$ distinct points $\{v_1, \ldots, v_{j+1}\} \subset V_{2^j}$ such that $\langle v_1, \ldots, v_{j+1} \rangle = V_{2^j}$ and that occurs as follows. Let $V_0 = \langle v_1 \rangle$. For $1 \leq i \leq j$, $\text{dis}(v_{i+1}, V_{2^i-2}) = 2$.

$$V_0 = \{v_1\}$$

$$V_2 = \langle v_1, v_2 \rangle$$

$$V_{2i} = \langle v_{i+1}, V_{2i-2} \rangle = \langle v_1, \ldots, v_{i+1} \rangle$$

$$V_{2j} = \langle v_{j+1}, V_{2j-2} \rangle = \langle v_1, \ldots, v_{j+1} \rangle.$$

There exist exactly two points that are distance 1 away from both $v_{j+1}$ and some point in $V_{2j-2}$. After one step, those nodes become open. Then the lines connecting those points with $v_{j+1}$ and $V_{2j-2}$ become open, followed by the planes containing those new lines. This continues until eventually the whole $2^j$ dimension sublattice becomes open.

We will state and prove a few necessary lemmas. The key point in the next few lemmas is growth continues only if there are two sets of open nodes within distance 2 of each other.

**Lemma 2.1.** For $S \subset [n]^d$, let $S$ denote the smallest sublattice that contains $S$. If $V$ is a sublattice and $u$ is a node with $\text{dis}(V, u) \leq 2$ then

$$\langle V, u \rangle = \{V \cup u\}.$$

**Lemma 2.2.** If $V, W$ are open sublattices and $\text{dis}(V, W) \leq 2$ then $\langle V, W \rangle = \{V \cup W\}$.

The next two lemmas give conditions for when and how a sublattice is internally spanned.

**Lemma 2.3.** For an initial configuration of open nodes $S$, let $V$ be a maximal sublattice in $\langle S \rangle$. Then $V$ is internally spanned with $V = \langle S \cap V \rangle$.

**Lemma 2.4.** Let $S$ be a set of open nodes in $[n]^d$ with $V \subset \langle S \rangle$ a maximal open sublattice. There exist disjoint subsets $S_1, S_2 \subset S$ and sublattices $V_1, V_2 \subset V$ such that $\langle S_1 \rangle = V_1$, $\langle S_2 \rangle = V_2$, and $\langle S_1 \cup S_2 \rangle = V$.

**Proof of Lemma 2.4.** (By induction) We have shown that the lemma holds if $V$ has dimension 0 (is a single node). Suppose the lemma holds for all sublattices $W$ with $\text{dim}(W) < i$. Let $V$ be a sublattice with $\text{dim}(V) = i$ and let $u$ be a node with $\text{dis}(V, u) \leq 2$. Without loss of generality we assume the last $d-i$ coordinates are fixed, i.e. $\{I(V)\} = [i+1, d]$. Again without loss of generality we may also assume that $u = \{(u_1, \ldots, u_i) : u_l = \alpha_l(V) \text{ for } l > i + 2\}$. 


Let \( V_k \) denote the sublattice of \( V \) that fixes the \( k^{th} \) coordinate to the value \( u_k \). Then \( V_k \) has dimension \( i - 1 \) and \( \text{dis}(V_k, u) \leq 2 \). By the induction hypothesis, \( \langle V_k, u \rangle = \{ V_k, u \} \).

For \( a = (a_1, \cdots, a_d) \in \{ V, u \} \), there are two neighbors

\[
a_{u_1} = (u_1, a_2, \cdots, a_{i+2}, a_{i+3}, \cdots, a_d) \in \{ V_1, u \}
\]

and

\[
a_{u_2} = (a_1, a_2, \cdots, a_{i+2}, a_{i+3}, \cdots, a_d) \in \{ V_2, u \},
\]

so \( a \) becomes open and we can conclude \( \{ V, u \} \subseteq \{ V_1, V_2, u \} \subseteq \{ V, u \} \). Trivially \( \langle V, u \rangle \subseteq \{ V, u \} \) so we have equality for the two sets. Moreover, if \( u \not\in V \) then \( i+1 \leq \text{dim}(\{ V, u \}) \leq i + 2 \).

**Proof of Lemma 2.2** This is a natural extension of Lemma 2.1. Trivially we have \( \langle V, W \rangle \subseteq \{ V, W \} \). Let \( V^0 = V \). We define \( V^l \) based on \( V^{l-1} \). Let \( W^{l-1} \) denote the subset of \( W \) that satisfies \( 0 < \text{dis}(V^{l-1}, u) \leq 2 \) for every \( u \in W^{l-1} \). For \( l > 0 \) if \( W \cap (V^{l-1})^c \) is non-empty there exists a \( w_l \in W^{l-1} \). We then define \( V^l = \{ V^{l-1}, w_l \} \) for some choice of \( w_l \). By the previous this is the sublattice \( \{ V^{l-1}, w_l \} \). Its dimension is strictly greater than \( \text{dim}(V^{l-1}) \). If \( W \cap (V^{l-1})^c \) is empty then \( V^l = V^{l-1} \). Since \( \{ V^l \} \) is an increasing sequence of sublattices bounded by \( \{ V, W \} \) it must stabilize to some sublattice \( V^m \) in a finite number of steps. By definition \( V \subseteq V^m \), and more importantly, \( W \cap (V^m)^c = \emptyset \) so \( W \subseteq V^m \). We also have that \( V^m \subseteq \{ V, \cup_{i \not\in l}w_l \} \subseteq \{ V, W \} \). Combining everything we get

\[
\{ V, W \} \subseteq V^m \subseteq \langle V, W \rangle \subseteq \{ V, W \}
\]

and the lemma holds.

**Proof of Lemma 2.3** Let \( S_1 = S \cap V \) and \( S_2 = \{ S \backslash S_1 \} \). If \( \langle S_1 \rangle = V \) then we are done. Suppose that \( \langle S_1 \rangle \neq V \). Since \( V \) eventually becomes open, there must be some node \( u \in \langle S_2 \rangle \) such that \( \text{dis}(\langle S_1 \rangle, \langle u \rangle) \leq 2 \), otherwise evolution would stop and \( V \) could not be contained in \( \langle S \rangle \). In particular, there is a node \( u \in \langle S_2 \rangle \) such that \( u \not\in V \) yet \( \text{dis}(V, u) \leq 2 \). By Lemma 2.2, the smallest sublattice that contains both \( u \) and \( V \) becomes open eventually. However \( V \) is maximal so no such \( u \) can exists and \( \langle S \rangle = V \).

**Proof of Lemma 2.4** \( V \) is maximal so we may assume \( \langle S \rangle = V \). Consider the sequence of nested collections of sublattices contained in \( \langle S \rangle \),

\[
\{ W^0_i \} \subset \{ W^1_i \} \subset \cdots \subset \{ W^k_i \} \subset V
\]

where \( S = \{ W^0_i \} \) and \( \{ W^{k+1}_i \} \) is formed by finding two sublattices \( W^k_{i_1} \) and \( W^k_{i_2} \) within Hamming distance 2 of each other and setting \( W^{k+1}_i = \langle W^k_{i_1} \cup W^k_{i_2} \rangle \) and reindexing the others appropriately. Since \( S \) is finite, eventually we will have two sublattices \( W^k_{i_1}, W^k_{i_2} \neq V \) such that \( \langle W^k_{i_1} \cup W^k_{i_2} \rangle = V \). Each \( W^k_{i_l} \) had a unique set \( S_l \) such that \( \langle S_l \rangle = W^k_{i_l} \) for \( l = 1, 2 \).
2.1. Growth Heuristics. To get an idea of the exponent of \( p_c(2j, d) \) we make a heuristic argument for the exponent of \( p_T(2j, d) \). For a 2\( j \)-dimension sublattice \( V \), we get an estimate on the probability \( \mathbb{P}_p(\mathcal{I}_V) \). The probability for \( \mathcal{I}_V \) only depends on the dimension of \( V \). From Lemma 2.4 we know that \( V \) is internally spanned if there exist two internally spanned sublattices, \( V_1 \) and \( V_2 \), with \( \langle V_1, V_2 \rangle = V \). We assume that \( \dim(V_1) = 0 \) and \( \dim(V_2) = 2j - 2 \). There are roughly \( n^{2j} \) choices for \( V_1 \) and roughly \( n^2 \) choices for \( V_2 \). We estimate the probability that at least one \( V_1 \) and one \( V_2 \) are internally spanned by assuming independence and using expectation. We let \( M_i(p) = M_i \) denote the probability that a particular \( i \)-dimension sublattice is internally spanned. In particular, \( \mathbb{P}_p(\mathcal{I}_V) = M_{2j} \).

This implies

\[
M_{2j} \approx \mathbb{P}_p(\exists \ V_1 \subset V \text{ s.t. } \mathcal{I}_{V_1})\mathbb{P}_p(\exists \ V_2 \subset V \text{ s.t. } \mathcal{I}_{V_2}) \approx n^{2j}pn^2M_{2j-2}.
\]

We approximate the \( M_{2j-2} \) in the same manner and get the estimate:

\[
M_{2j} \approx \prod_{i=0}^{j} n^{2i-2j}pn^2 = n^{j(j+1)+2j}p^{j+1}.
\]

There are roughly \( n^{d-2j} \) choices for \( V \) so

\[
\mathbb{P}_p(\mathcal{I}_{2j}) \approx n^{d-2j}M_{2j} \approx n^{d-2j+j(j+1)+2j}p^{j+1}.
\]

Setting this equal to 1 and solving for \( p \) gives the appropriate estimate:

\[
p_T(2j, d) \approx n^{-d/(j+1)-j}.
\]

The next few sections will show that this estimate is reasonably accurate (up to a constant factor). We will also show that \( p_T(2j, d) \) has the same asymptotics as \( p_c(2j, d) \).

3. Critical Probability

To find the asymptotics of \( p_c(2j, d) \), we will first prove upper and lower bounds for the exponent of \( p_T(2j, d) \). Since \( \mathcal{I}_{2j} \subset C_{2j} \) any upper bound for \( p_T(2j, d) \) will hold for \( p_c(2j, d) \). With a little more work, we then prove the lower bound for the exponent of \( p_T(2j, d) \) will also be a lower bound for the exponent of \( p_c(2j, d) \).

For odd dimension sublattices we will show that \( \mathbb{P}_p(\mathcal{I}_{2j-1}) \leq \mathbb{P}_p(\mathcal{I}_{2j}) \) hence \( p_c(2j - 1, d) = p_c(2j, d) \). This is apparent in the case of a line and a plane. For a line to be internally spanned, two nodes need to be co-linear, whereas for a plane to be internally spanned, two nodes only need to be co-planar.

First the upper bound.

3.1. Upper Bound.

**Proposition 3.1.** For any \( f(n) \to \infty \), and for all \( d > j(j+1) \), if \( p = f(n)n^{-d/(j+1)-j} \), then

\[
\mathbb{P}_p(C_{2j}), \mathbb{P}_p(\mathcal{I}_{2j}) \to 1.
\]

This implies

\[
p_c(2j - 1, d) \leq p_c(2j, d) \leq p_T(2j, d) < f(n)n^{-d/(j+1)-j}.
\]
We will prove this proposition with the caveat that $f(n)$ does not grow too fast. Since $P_p(I_{2j})$ and $P_p(C_{2j})$ are increasing in $p$ the proposition will still be true for faster growing $f(n)$.

**Proof.** First we define a sufficient event $E_{2j} \subset I_{2j} \subset C_{2j}$. If we can show $P_p(E_{2j}) \to 1$ for some value of $p$ then we can conclude $P_p(C_{2j}), P_p(I_{2j}) \to 1$ as well.

For a fixed set of constants $\alpha = \{\alpha_{2j+1}, \cdots, \alpha_d\}$, let $V_\alpha$ denote the sublattice given by

$$V_\alpha = \{ v \in [n]^d : v_i = \alpha_i \text{ for } 2j + 1 \leq i \leq d \}.$$  

There are $n^{d-2j}$ such sublattices. For $\alpha \neq \alpha'$, $V_\alpha \cap V_{\alpha'} = \emptyset$. Each event $I_{V_\alpha}$ will depend only on the nodes in $V_\alpha$ so the events are independent. The nodes in each $V_\alpha$ are all i.i.d. random $\{0,1\}$-variables so the events will all have the same probability $P_p(I_{V_\alpha}) = P_p(I_{V_{\alpha'}})$. We now define the sufficient event

$E_{2j} = \bigcup_{\alpha} I_{V_\alpha}$. 

We will show that $P_p(E_{2j}) \to 1$ for sufficiently large $p$ that satisfy the conditions of the proposition. Since $E_{2j} \subset I_{2j} \subset C_{2j}$ this implies $P_p(I_{2j}), P_p(C_{2j}) \to 1$ as well.

**Lemma 3.2.** Let $j, d$ and $p$ be as defined in Proposition 3.1 and $2i \leq 2j$.

$$M_{2i} \geq (2i)!2^{-i-1}n^{i(i+3)p^{i+1}(1-o(1)).}$$

**Proof of Lemma 3.2.** Let $V$ be a sublattice with dimension $2i$. Suppose we have a collection of nodes $\alpha = \{v_1, \cdots, v_{i+1}\} \subset V$ such that $\langle\{v_1, \cdots, v_{i+1}\}\rangle = V$. The probability that only these nodes are open is exactly $p^{i+1}(1-p)^{n^{2i-i-1}}$. Let $L_V$ be the set of all such collections. Then

$$M_{2i} = P_p(I_V) \geq \sum_{L_V} p^{i+1}(1-p)^{n^{2i-i-1}} = |L_V|p^{i+1}(1-o(1)).$$

We call a collection, $\alpha = \{v_1, \cdots, v_{i+1}\}$ perfect in $V$ if $\text{dis}(v_L, v_l) = 2(l-1)$ for $l' < l$ and $v_1 < v_2$ in lexicographical ordering. For $l' \leq i$, $\alpha_{l'} = \{v_{l'}, \cdots, v_{i+1}\}$ is perfect in $\langle\alpha_{l'}\rangle = V'$ and $\dim(V') = 2i'$. Note that a non-trivial rearrangement of a perfect collection is not a perfect collection. This makes counting them easier.

Let $L^*_V \subset L_V$ denote the subset of perfect collections for $V$. We will compute a lower bound for $|L^*_V|$ inductively. Suppose for any sublattice $W$ with $\dim(W) = 2i - 2$ we have

$$|L^*_W| \geq (2i - 2)!2^{-i}n^{(1-i)(i+2)}.$$ 

By induction we have,
\[ |\mathcal{L}_V^*| = \sum_{\dim(W)=2i-2} \sum_{\alpha' \in \mathcal{L}_W^*} \sum_{v \in V} 1_{\alpha' \cup v \text{ is perfect}} \]

\[ \geq \sum_{\dim(W)=2i-2} \sum_{\alpha' \in \mathcal{L}_W^*} (n - i - 1)^{2i} \]

\[ \geq \sum_{\dim(W)=2i-2} (2i - 2)!2^{-i} n^{i(i-1)(i+2)} n^{2i} (1 - o(1)) \]

\[ \geq \left( \frac{2i}{2} \right)^2 (2i - 2)!2^{-i} n^{i(i-1)(i+2)} (1 - o(1)) \]

\[ = (2i)!2^{-i-1} n^{i(i+3)} (1 - o(1)) \]

Since this formula holds if \( V \) is a plane then it will hold for all \( V \) with dimension small enough that the approximation \( (1 - p)^{n^2 - i - 1} = 1 - o(1) \) is valid. This approximation is valid exactly when \( d > i(i+1) \).

Therefore \( M_{2i} = P_p(\mathcal{I}_V) \geq |\mathcal{L}_V^*|^p i^1 \geq |\mathcal{L}_V^*|^p i^1 = (2i)!2^{-i-1} n^{i(i+3)} p^i (1 - o(1)). \]

Initially we will assume that \( f(n) \) grows slowly enough that \( O(n^{j+3}) p^i = O(n^{2j-d} f(n)^{j+1}) \)

is much less than 1.

The event \( E_{2j} \) is equivalent to the event that there exists an \( \alpha \) such that \( \mathcal{I}_{V_\alpha} \) occurs. There are \( n^{d-2j} \) choices for \( \alpha \). The events \( \{\mathcal{I}_{V_\alpha}\} \) are independent, so the probability that no \( \mathcal{I}_{V_\alpha} \) occurs is bounded by

\[ \left( 1 - O(n^{2j-d} f(n)^{j+1}) \right)^{n^{d-2j}} \leq e^{-O(f(n)^{j+1})} = o(1). \]

Therefore the probability that \( \mathcal{I}_{V_\alpha} \) occurs for some \( \alpha \), and hence \( E_{2j} \) occurs, is bounded below by \( P_p(E_{2j}) \geq 1 - o(1) \rightarrow 1 \). Since \( E_{2j} \) is increasing in \( p \) we can remove the restriction on the growth of \( f(n) \). Therefore for any \( f(n) \rightarrow \infty \) we have \( P_p(E_{2j}) \rightarrow 1 \).

We now have shown

\[ P_p(E_{2j}) \leq P_p(\mathcal{I}_V) \leq P_p(C_{2j}) \rightarrow 1 \]

and we conclude

\[ p_c(2j, d) \leq p(2j, d) < f(n)n^{-d/(j+1)-j}. \]

3.2. **Lower Bound.** In this section we prove the lower bound for for the critical exponent of \( p(2j, d) \). Again we assume \( j(j+1) < d \).

**Proposition 3.3.** For any \( f(n) \rightarrow 0 \), if \( p = f(n)n^{-d/(j+1)-j} \), then

\[ P_p(\mathcal{I}_{2j}) \rightarrow 0. \]
This implies $p_I(2j, d) > f(n)n^{-d/(j+1)−j}$.

Proof. Let $V_{2j}$ denote the set of all sublattices of $[n]^d$ that have dimension $2j$. The union bound gives:

$$
\Pr_p(I_{2j}) \leq \sum_{\text{dim}(V) = 2j} \Pr_p(I_V) \leq \binom{d}{2j} n^{d-2j} M_{2j}(p).
$$

The majority of the proof is showing $M_{2j}(p) = O(f(n)^{j+1} n^{2j-d})$ when $p = f(n)n^{-d/(j+1)−j}$. Then $\Pr_p(I_{2j}) = O(f(n)^{j+1}) \to 0$ which implies $p_I(2j, d) > f(n)n^{-d/(j+1)−j}$.

First let’s start with the simplest possibilities for $V$: a single node, a line, and a plane.

- For a single node $u$,
  $$
  \Pr_p(I_u) = p.
  $$

- For a single line $l$,
  $$
  \Pr_p(I_l) = \Pr(\text{Bin}(n, p) \geq 2) = \binom{n}{2} p^2 (1-p)^{n-2} = O(n^2 p^2).
  $$

- For a single plane $P$,
  $$
  \Pr_p(I_P) \leq \Pr(\text{Bin}(n^2, p) \geq 2) = (1 + o(1)) 2^{-1} n^4 p^2.
  $$

Note that a plane is more likely to be internally spanned than as an internally spanned line requires at least two collinear points.

Keeping the conditions of Proposition 3.3 and $p = f(n)n^{-d/(j+1)−j}$, we have the following lemma:

**Lemma 3.4.** For $1 \leq i \leq j$,

\begin{align*}
M_{2i-1}(p) &= O(n^{i(i+3)−2} p^{i+1}). \\
M_{2i}(p) &= (1 + o(1))(2i)! 2^{-i-1} n^{i(i+3)} p^{i+1}.
\end{align*}

*Proof. (By induction)*

We assume the lemma holds for all $0 \leq l \leq 2i - 2$ and show by induction that the formulas hold for dimensions $2i$ and $2i - 1$. Note the lemma holds for a point, a line and a plane, so our base case is covered.

First let’s assume a sublattice $V$ is internally spanned. By Lemma 2.3 there exists proper sublattices $V_1, V_2 \subset V$ both internally spanned by disjoint subsets $S_1$ and $S_2$ such that $V = \langle V_1, V_2 \rangle$. Let $D_V$ denote the set of possible pairs of such sublattices of $V$ with $\dim(V_1) \leq \dim(V_2)$. For $V$ with dimension $2i$ or $2i - 1$, let $D'_V$ denote the subset of $D_V$ where $\dim(V_1) = 0$, and $\dim(V_2) = 2i - 2$.

$I_V$ can be expressed as a union over $D_V$ of events of the form $I_{V_1} \circ I_{V_2}$, where $\circ$ denotes the events occur disjointly.

We will show the probability $I_V$ occurs is almost entirely determined by the probability there exists a pair $(V_1, V_2) \in D'_V$ such that $I_{V_1} \circ I_{V_2}$ occurs. Let $E_{D'_V} = \{ \exists (V_1, V_2) \in D'_V \text{ s.t. } I_{V_1} \circ I_{V_2} \}$. We have the following lemma.
Lemma 3.5. \( \mathbb{P}_p(\mathcal{I}_V) = \mathbb{P}_p(E_{D'_V})(1 + O(n^{-1})). \)

Suppose this lemma is true. If \( \dim(V) = 2i - 1 \), then there are at most \( O(n^{2i-1}n) \) pairs in \( D'_V \). The union bound gives

\[
\mathbb{P}_p(E_{D'_V}) \leq \sum_{D'_V} M_0 M_2i-2 \leq O(n^{2i})M_0 M_2i-2 = O(n^{i(i+3)-2}p^{i+1}),
\]

proving the first part of Lemma 3.4.

If \( \dim(V) = 2i \), there are at most \( \binom{2i}{2}(n^{2i}n^2) \) pairs in \( D'_V \). The union bound gives

\[
\mathbb{P}_p(E_{D'_V}) \leq \left( \frac{2i}{2} \right) n^{2i}n^2 M_0 M_2i-2 = (1 + o(1))(2i)!2^{-i-1} n^{i(i+3)}p^{i+1},
\]

proving the second part of Lemma 3.4.

Proof of Lemma 3.5 (By induction). We may assume Lemma 3.4 is true for up to dimension \( 2i - 2 \). This will allow us to prove Lemma 3.5 for dimension \( 2i - 1 \) and \( 2i \) hence proving Lemma 3.4 for those values as well. We can then proceed inductively to prove the lemmas are true for all values up to \( 2j \).

The union bound gives:

\[
\mathbb{P}_p(\mathcal{I}_V) \leq \mathbb{P}_p\left( \bigcup_{D_V} \mathcal{I}_{V_1} \circ \mathcal{I}_{V_2} \right).
\]

\( \mathcal{I}_{V_1} \) are increasing events. By the BK-inequality [8]

\[
\mathbb{P}_p(\mathcal{I}_{V_1} \circ \mathcal{I}_{V_2}) \leq \mathbb{P}_p(\mathcal{I}_{V_1}) \mathbb{P}_p(\mathcal{I}_{V_2})
\]

and

\[
\mathbb{P}_p(\mathcal{I}_V) \leq \sum_{D_V} \mathbb{P}_p(\mathcal{I}_{V_1} \circ \mathcal{I}_{V_2}) \leq \sum_{D_V} \mathbb{P}_p(\mathcal{I}_{V_1}) \mathbb{P}_p(\mathcal{I}_{V_2}).
\]

For now we assume the lemma is true for \( \dim(V) = 2i - 1 \). Then we prove the lemma is true for \( \dim(V) = 2i \) and let \( D_V(a,b) \) denote the subset of \( D_V \) where \( \dim(V_1) = a \) and \( \dim(V_2) = b \) and \( a \leq b < 2i \). \( \{V_1, V_2\} \) has at most dimension \( a + b + 2 \) if it is in fact a subspace. Therefore if \( a + b + 2 < 2i \), then \( D_V(a,b) \) is empty. Otherwise \( |D_V(a,b)| \) is at most \( O(n^{2i-a}n^{2i-b}) \). Assume \( a + b + 2 = 2i + \delta \) for some \( \delta > 0 \).

\[
\mathcal{M}(a,b) = \sum_{D_V(a,b)} M_a M_b = O\left( n^{4i-a-b} M_a M_b \right).
\]

If \( a = 2i + 1 \), then \( n^{2i-(2i-1)} M_{2i-1} \leq n^{-1}n^{2i-2i} M_{2i} \) so we may assume that \( a \) (and \( b \)) are both even. Let \( a = 2i_1 \), and \( b = 2i_2 \), with \( i_1 + i_2 + 1 = i + k \). We know the values of \( M_{2i_1} \) and \( M_{2i_2} \). Therefore

\[
\mathcal{M}(a,b) \leq O(n^{4i-2i_1-2i_2} M_{2i_1} M_{2i_2}) \leq O(n^{4i-2i_1-2i_2} n^{i_1(i_1+3)+i_2(i_2+3)} p^{i_1+1+i_2+1}) \leq O(n^{i(i+3)} p^{i+1} n^{k(k-1)-2i_1i_2}).
\]
Lemma 3.6. If $l_1 > 0$, then $k(k - 1) - 2i_1 i_2 < -1$ and Lemma 3.5 is true for $\dim(V) = 2i$. The proof for $\dim(V) = 2i - 1$ follows a similar approach.

With Lemma 3.5 proved, we can conclude from Lemma 3.4

$$M_{2i} \leq (2!)2^{-i_1}n^{i_1+1}(1 + o(1)).$$

This with Lemma 3.2 gives

$$M_{2i} = (2!)2^{-i_1}n^{i_1+1}(1 + o(1)).$$

Combining these results we get

$$\mathbb{P}_p(I_{2j}) \leq O(n^{d-2j})\mathbb{P}_p(I_V) \leq O(n^{d-2j})o(n^{2j-d}) = o(1).$$

This implies

$$p_{I}(2j, d) > f(n)n^{d/(j+1) - j}$$
as desired.

3.3. Bounds for $p_c(2j, d)$. In this section we will show the bounds for the exponent of $p_{I}(2j, d)$ hold for the exponent of $p_c(2j, d)$. We know the upper bound holds from previous arguments. We need to show the lower bound for $p_{I}(2j, d)$ from Proposition 3.3 also holds for $p_c(2j, d)$.

If $C_{2j}$ occurs then there exists some sublattice with dimension greater than or equal to $2j$ that is internally spanned. The next lemma will show that for any dimension $b > 2j$, $\mathbb{P}_p(I_b) \to 0$ if $\mathbb{P}_p(I_{2j}) \to 0$. This implies that $\mathbb{P}_p(C_{2j}) \to 0$ as well.

Lemma 3.6. If $p = f(n)n^{-d/(j+1) - j}$ as in Proposition 3.5 then $\mathbb{P}_p(C_{2j}) \to 0$. Therefore $p_c(2j, d) > f(n)n^{-d/(j+1) - j}$ for large $n$.

Proof of Lemma 3.6. If $C_{2j}$ occurs, then there must be some $b$-dimensional sublattice $V_b$ such that $I_{V_b}$ occurs and $b \geq 2j$. We know $I_{2j} \to 0$ by the choice of $p$, so we need to show that $\mathbb{P}_p(I_b) \to 0$ for $b > 2j$. By Lemma 2.1 if $I_{V_b}$ occurs for some $V_b$, there exist $V_1$ and $V_2 \subset V_b$ with $\dim(V_1) \leq \dim(V_2) < b$ such that $I_{V_1} \circ I_{V_2}$ occurs and $\langle V_1, V_2 \rangle = V_b$. We may assume $\dim(V_2) \leq 2j$. For simplicity we assume the following

$$\dim(V_1) = 2j - 2a_1$$

$$\dim(V_2) = 2j - 2a_2$$

$$\dim(V_b) = 2j + 2i$$

with $a_1 \geq a_2 \geq 0$, and $0 < 2k < d - 2j$.

Since $\langle V_1, V_2 \rangle = V_b$ we have the that $j - a_1 - a_2 + 1 = i + k$ for some $k > 0$. Let

$$E_{i,k,a_1,a_2} = \{ V_1, V_2 \subset V_b \ s.t. \ V_1, V_2 = V_b \}$$

that satisfy all the requirements above. Using $j(j + 1) < d$ and $j - i - k + 1 = a_1 + a_2$ we get the following bound:
The number of sublattices with dimension 2 then each subspace although some dependency exists, if \( j \)

\[ p \]

\( I \)

...random variable for the event \( X \)

\( d \)

\( (5) \)

\( E(R) \)

\( (6) \)

\[ \sum \]

\[ n \]

\[ a, b \]

\( \) sublattices of dimension \( b \). Therefore the probability there exists an internally spanned sublattice of dimension greater than \( 2j \) tends to zero.

Now we can conclude that \( p_c(2j, d) \) is also bounded below \( f(n)n^{-d(j+1)−j} \) for any \( f(n) \rightarrow 0 \).

\[ 3.4. \] \( p_c(2j, d) \) for \( j(j + 1) > d \). Let \( j' = \sup\{i : i(i + 1) < d\} \). For \( j(j + 1) > d \) and \( f(n) \rightarrow \infty \), suppose \( p > n^{-d(j'+1)−j'} \). For any \( 2j \)-dimension sublattice, the expected number of nodes that are initially open is \( n^{2j}p \geq n^{2j−2j'} \rightarrow \infty \) and hence

\[ \mathbb{P}_p(I_{2j}|I_{2j−2}) \rightarrow 1. \]

Therefore \( p_c(2j, d) \rightarrow p_c(2j−2, d) \). The case where \( j(j + 1) = d \) will require a little more work and will be presented in Section 6.

4. Poisson Approximation

We use the Stein-Chen method for approximation by a Poisson distribution. We will use the version found in [11]:

**Theorem 4.1.** Let \( X_1, \ldots, X_m \) be indicator variables with \( \mathbb{P}(X_i = 1) = p_i, Y = \sum_{i=1}^m X_i \), and \( \lambda = \mathbb{E}[Y] = \sum p_i \). For each \( i \in [m] \), let \( N_i \subset [m] \) where \( i \in N_i \) and \( X_i \) is independent of \( \{X_j : j \notin N_i\} \). If \( p_{ij} := \mathbb{E}[X_iX_j] \) and \( Z \sim Po(\lambda) \), then

\[ (5) \]

\[ d_{TV}(Y, Z) \leq \sum_{i=1}^m \left( \sum_{j \in N_i} p_i p_j + \sum_{j \in N_i \setminus \{i\}} p_{ij} \right). \]

Let \( \Gamma \) denote the set of all \( 2j \)-dimension sublattices in \([n]^d\). Each sublattice \( V \) has a dependency set \( \Gamma_V \) where \( \mathcal{I}_W \) depends on \( \mathcal{I}_V \) for \( W \in \Gamma_V \). When \( p = \alpha n^{-d(j+1)−j} \) then each subspace \( V \in \Gamma \) is internally spanned with probability \( (2j)!2^{−j−1}a^{j+1}n^{2j} \). Although some dependency exists, if \( j(j + 1) < d \), we will show that the distribution of the number of sublattices with dimension \( 2j \) which are internally spanned approaches a Poisson distribution.

To fit our random variables with that of the theorem, we let \( 1_V \) denote the indicator random variable for the event \( \mathcal{I}_V \). For all \( V, W \in \Gamma \),

\[ (6) \]

\[ p_V = p_W = M_{2j} = (2j)!2^{−j−1}a^{j+1}n^{2j} (1 + o(1)) \]
and
\[ p_{VW} = \mathbb{E}[1_V 1_W] = \mathbb{P}(\mathcal{I}_V \cap \mathcal{I}_W). \]

Let \( Y = \sum_\Gamma 1_V \). Then
\[ \lambda = \mathbb{E}[Y] = (1 + o(1)) \sum_\Gamma (2j)! 2^{-j-1}n^{2j-d} = \left( \frac{d}{2j} \right) (2j)! 2^{-j-1}n^{j+1}. \]

Finally we let \( Z \sim \text{Po}(\lambda) \), a Poisson random variable with parameter \( \lambda \).

Plugging everything into (5) we get
\[ d_{TV}(Y, Z) \leq \sum_{V \in \Gamma} \left( \sum_{W \in \Gamma \setminus \{V\}} p_{VW} + \sum_{W \in \Gamma \setminus \{V\}} p_{VW} \right). \]

Since
\[ \sum_{W \in \Gamma \setminus \{V\}} p_{VW} + \sum_{W \in \Gamma \setminus \{V\}} p_{VW} \]

does not depend on the choice of \( V \), we can approximate the right-hand side of (9) by
\[ |\Gamma||\Gamma \setminus M_{2j}| \sum_{W \in \Gamma \setminus \{V\}} p_{VW} \]

The quantity \( p_{VW} \) depends on the dimension \( \text{dim}(V \cap W) = l \). We break up \( \Gamma_V \) into subsets \( \Gamma_{V}^l \) where \( W \in \Gamma_V^l \) if \( \text{dim}(V \cap W) = l \).

For each \( l, |\Gamma_V^l| = O(n^{2j-l}) \) so \( |\Gamma_V^l| = O(n^{2j}) \). This gives the bound
\[ |\Gamma||\Gamma_V^l| M_{2j}^2 = O(n^{4j-d}) \rightarrow 0 \]

for the left half of (11). The remaining portion of the (11) requires a bit more work. We compute upper bounds for \( p_{VW} \) that depend on \( l \).

As before \( j(j+1) < d \) and \( p = an^{-d/(j+1)-j} \). We state a slightly more general lemma in that we have \( \text{dim}(V) = \text{dim}(W) = 2i \leq 2j \).

**Lemma 4.2.** Let \( \text{dim}(V \cap W) = 2i - 2k \) with \( 0 \leq k \leq i \). Then,
\[ p_{VW} = \mathbb{P}_p(\mathcal{I}_V \cap \mathcal{I}_W) = O(n^{4ik-2k(k-1)+(i-k)(i-k+3)p^{i+k+1}}). \]

In particular, if \( i = j \) then for some \( \epsilon > 0 \),
\[ \mathbb{P}_p(\mathcal{I}_V \cap \mathcal{I}_W) = O(n^{2j-2k-d-\epsilon}). \]

This upper bound also holds for \( \text{dim}(V \cap W) = 2i - 2k + 1 \) though we will always assume even dimension intersection for simplicity.

**Proof of Lemma 4.2.** (By induction). In this proof we will use induction on both \( i \) and \( k \). Our base case of \( i = k = 0 \) is satisfied. To continue we state two useful sublemmas.
Sublemma 4.2.1. Let $V' \subset V$ be sublattices that satisfy $\dim(V) = 2i$ and $\dim(V') = 2i - 2k$ or $2i - 2k + 1$. Then

$$\Pr(I_V | I_{V'}) = O(n^{2ki - k(k - 1)}p^k).$$

Sublemma 4.2.2. Let $V$ and $W$ be sublattices with non-trivial intersection. Then

$$\Pr(I_V \cap I_W) = O \left( \Pr(I_V | I_{V \cap W}) \Pr(I_W | I_{V \cap W}) \Pr(I_{V \cap W}) \right).$$

Combining Lemma 3.4, Sublemma 4.2.1 where $V' = V \cap W$, and Sublemma 4.2.2 we get

$$\Pr(I_V \cap I_W) = O(n^{4ki - 2k(k - 1) + (i - k)(i - k + 3)p^i + k + 1}).$$

When $i = j$ we get for some $\epsilon > 0$,

$$\Pr(I_V \cap I_W) = O(n^{i(j + 3)}p^{j + 1}(n^{2j - k - 1}p^k))$$
$$= O(n^{2j - d_n - k(k + 1)}n^{-\epsilon})$$
$$= O(n^{2j - 2k - d_n - \epsilon})$$

where we use the simplification $n^{2j}p \leq n^{-\epsilon}$. Assuming both Sublemmas 4.2.1 and 4.2.2 are true, this proves Lemma 4.2.

Proof of Sublemma 4.2.1 (By induction). Let $V' \subset V$ satisfy $\dim(V) = 2i$ and $\dim(V') = 2i - 2k$ where $0 \leq k \leq i$. We assume the lemma is true for all pairs of values $(i', k')$ such that if $i' < i$, then $0 \leq k' \leq i'$ and if $i' = i$, $0 \leq k' < k$. The statement holds for all $i$ if $k = 0$, covering our base case.

If $I_V$ occurs, then from Lemma 2.4 there exists $(V_1, V_2) \in D_V$ such that $I_{V_1} \cap I_{V_2}$ occurs. Similar to the proofs of Lemma 3.4 and Lemma 4.3, the union bound for the conditional probability will be dominated by pairs $(V_1, V_2) \in D_V$ that satisfy $\dim(V_1) = 0$ and $\dim(V_2) = 2i - 2$. We denote this subset of $D_V$ by $D_V'$. Then

$$\Pr(I_V | I_{V'}) \leq \sum_{(V_1, V_2) \in D_V} \Pr(I_{V_1} | I_{V'}) \Pr(I_{V_2} | I_{V'}) \Pr(I_{V_1} \cap I_{V_2}).$$
$$\leq O(1) \sum_{(V_1, V_2) \in D_V'} \Pr(I_{V_1} | I_{V'}) \Pr(I_{V_2} | I_{V'}) \Pr(I_{V_1} \cap I_{V_2}).$$

If $v_1 \notin V'$ then there are $O(1)$ choices of $V_2$ such that $V' \subset V_2$ and $O(n^2)$ choices such that $\dim(V_2 \cap V') < \dim(V')$. When $v_1 \in V'$ all $O(n^2)$ choices of $V_2$ have $\dim(V_2 \cap V') < \dim(V')$. There are less than $n^{2i}$ choices of $v_1 \notin V'$ and $n^{2i} - 2k$ choices of $v_1 \in V'$. Let $V_2^*$, $V_2^{**}$, and $V_2^{***}$ denote representatives from each of these choices of $V_2$. Expectation gives us the upper bound
\[ P(I_V | I_{V'}) = O(1)n^{2i}pP(I_{V_2} | I_{V'}) \]

\[ + n^{2i}pO(n^2)P(I_{V_2'} \cap I_{V'}) \]

\[ + n^{2i-2k}O(n^2)P(I_{V_2''} \cap I_{V'}). \]

Here we apply the inductive hypothesis to each of these terms. The contribution from
\[ (23) \quad \text{and} \quad (24) \]
will be negligible compared to the right-hand side of \[ (22) \]. This gives
\[ P(I_V | I_{V'}) = O(n^{2i}p^{2(i-1)(k-1)-(k-1)(k-2)}p^{k-1}) \]
\[ = O(n^{2ik-k(k-1)}p^k). \]

\[ \square \]

**Proof of Sublemma 4.2.2 (By induction).** The direction of the induction is the reverse
of Sublemma 4.2.1. We have \( \dim(V) = 2i, \dim(W) = 2i', \dim(V \cap W) = 2i - 2k \)
and will assume the sublemma is true if either \( i' < i \) or \( k' > k \).

\[ P(I_V \cap I_W) \leq \sum_{(W_1,W_2) \in D_W} P(I_V \cap \{I_{W_1} \circ I_{W_2}\}) \]
\[ = O(1) \sum_{(w_1,w_2) \in D'_W} P(I_V \cap \{I_{w_1} \circ I_{W_2}\}) \]
\[ = O(1) \sum_{(w_1,w_2) \in D'_W} P(I_V \mid \{I_{w_1} \circ I_{W_2}\})P(\{I_{w_1} \circ I_{W_2}\}) \]

The terms where \( w_1 \notin V \) and \( V \cap W = V \cap W_2 \) dominate this sum. There are at most
\( n^{2i} \) such \( w_1 \) and \( O(1) \) such \( W_2 \).

\[ P(I_V \cap I_W) = O(1) \sum_{w_1 \notin V \cap W \subset W_2} P(I_V \mid I_{W_2})P(I_{w_1})P(I_{W_2}) \]
\[ = O(1)n^{2k}pP(I_V \cap I_{W_2}) \]

for some choice of \( W_2 \). Since \( \dim(W_2) < 2i \) we can apply now use the inductive hypothesis
to get

\[ P(I_V \cap I_W) = O(1)n^{2i}pP(I_V \mid I_{V \cap W_2})P(I_{W_2} \mid I_{V \cap W_2})P(I_{V \cap W_2}) \]
\[ = O(1)P(I_V \mid I_{V \cap W})P(I_W \mid I_{V \cap W})P(I_{V \cap W}) \]

as desired. \[ \square \]
A slightly modified argument will show the same upper bound holds when \( \dim(V \cap W) = 2i - 2k + 1 \). Also the base case where \( \dim(V \cap W) = 0 \) holds. We conclude that

\[
|\Gamma_{V}^{2j-2k+1}|_{pVW} \leq |\Gamma_{V}^{2j-2k}|_{pVW} \leq O(n^{2k - 2j - d - 2k + 1} n^{d-2j}) \leq O(n^{2j-d-2k}).
\]

Plugging this into the Stein-Chen bound we conclude that \( dTV(Y,Z) \to 0 \) and the number of internally spanned sublattices are approximately Poisson with parameter \( \lambda = \binom{d}{2j}(2j)!2^{-j}(a^{j+1}) \). This proves Theorem 1.2.

5. Proofs of Theorems

The proof of Theorem 1.1 follows from Theorem 1.2 and Lemma 3.6. These combine to show \( \Pr_p(C_{2j} \setminus I_{2j}) \to 0 \).

For Theorem 1.3 we have to do a little work. If \( d = j(j + 1) \) then

\[
d/(j + 1) + j = d/j + j - 1 = 2j.
\]

Let \( p = an^{-2j} \), \( \lambda' = \lambda_{2j-2} = \binom{d}{2j-2}(2j - 2)!2^{-j+1}a^{j} \), and \( Y_{2j-2} \) denote the number of sublattices of dimension \( 2j - 2 \) that are internally spanned. By Theorem 1.2

\[
dTV(Y_{2j-2}, Z(\lambda')) \to 0,
\]

where \( Z(\lambda') \) is a Poisson(\( \lambda' \)) random variable. For each \( k \geq 0 \) we have

\[
\Pr_p(Y_{2j-2} = k) \to \frac{e^{-\lambda'}}{k!}\lambda'^{k}.
\]

(25)

For each of these \( k \) open sublattices, there are exactly \( c = \binom{d-2j+2}{2} \) distinct sublattices with dimension \( 2j \). The number of nodes \( u \) with distance exactly 2 away from one of the open sublattices is \( cn^{2j}(1 - o(1)) \). Although it is possible for two open sublattices of dimension \( 2j - 2 \) to exist in the same \( 2j \)-dimension sublattice, this event has probability tending to zero. The probability that there exists some \( 2j \)-dimension sublattice with two disjoint open \( (2j-2) \)-dimension sublattices is bounded by

\[
O(n^{d-2j}n^{4j-4-2d}) = O(n^{j-2}).
\]

This tends to zero if \( j > 1 \). When \( j = 1, \ d = 2 \) and we are dealing with a plane, which is well understood.

Otherwise, there are in total \( ckn^{2j}(1-o(1)) \) that, if open, would lead to a sublattice of dimension \( 2j \) that is internally spanned. The probability that none of these are open is given by \( (1-p)^{ckn^{2j}(1-o(1))} \). Hence

\[
\Pr(I_{2j}) = \sum_{k=1}^{\infty} \Pr_p(Y_{2j-2} = k) \left( 1 - (1 - an^{-2j})^{ckn^{2j}(1-o(1))} \right),
\]

which for large \( n \) gives

\[
\Pr_p(I_{2j}) \to \sum_{k=1}^{\infty} \frac{e^{-\lambda'}}{k!}\lambda'^{k}(1 - e^{-ack}),
\]
proving the theorem.

For Corollary 1.4 we look at the three cases. In each case we assume $p = an^{-d/(j+1) - j}$.

- $(j + 1)(j + 2) < d$. By Theorem 1.2 $\mathbb{P}_p(I_{2j+2}) \to 0$, and $\mathbb{P}_p(I_{2j-2}) \to 1$. Therefore the largest sublattice has either dimension $2j - 2$ with probability $e^{-\lambda}$ or dimension $2j$ with probability $1 - e^{-\lambda}$. In terms of the random variable $D$, we have

$$\mathbb{P}(D = 2j - 2) \to e^{-\lambda},$$

$$\mathbb{P}(D = 2j) \to 1 - e^{-\lambda}.$$  

- $j(j+1) < d < (j+1)(j+2)$. Similar to the previous case, there are no internally spanned sublattices with dimension $2j$ with probability $e^{-\lambda}$ leaving the maximal sublattice to have dimension $2j - 2$. However if there is an open $2j$-dimension sublattice, then the expected number of open nodes exactly distance 2 away is $O(n^{2j+2}p)$. Since this expectation tends to infinity, an open $2j$-dimension sublattice will become an open $(2j + 2)$-dimension sublattice $a.a.s.$ In this case all of $[n]^d$ will open $a.a.s.$ Hence with probability tending to $1 - e^{-\lambda}$, the maximal sublattice will be the entire space. This gives

$$\mathbb{P}_p(D = 2j - 2) \to e^{-\lambda},$$

$$\mathbb{P}_p(D = 2j) \to 1 - e^{-\lambda}.$$  

- $j(j+1) = d > 6$. With probability $e^{-\lambda'}$ no sublattice with dimension $2j - 2$ or greater is open, and with probability tending to one there is a open sublattice with dimension $2j - 4$. Therefore

$$\mathbb{P}_p(D = 2j - 4) \to e^{-\lambda'}.$$  

For $2j - 2$ to be the maximum dimension of open sublattices, there must be some positive number $k$ of such sublattices open, but no neighbors distance 2 away from those sublattices can be open. Hence

$$\mathbb{P}_p(D_{2j-2}) = \sum_{k=1}^{\infty} \frac{e^{-\lambda'}}{k!} \lambda^k e^{-ack}.$$  

If there exists an open $2j$-dimension sublattice, then by the arguments in the previous case the entire lattice, $[n]^d$ becomes open. The limiting value for $\mathbb{P}_p(I_{2j})$ is given in Theorem 1.3. Altogether we have

$$\mathbb{P}_p(D = 2j - 4) \to e^{-\lambda'},$$

$$\mathbb{P}_p(D = 2j - 2) \to \sum_{k=1}^{\infty} \frac{e^{-\lambda'}}{k!} \lambda^k e^{-ack},$$

$$\mathbb{P}_p(D = d) \to 1 - \sum_{k=0}^{\infty} \frac{e^{-\lambda'}}{k!} \lambda^k e^{-ack}.$$
• $d = 6$ ($j = 2$). We still have with probability $e^{-\lambda^j}$ that no plane (dimension $2j - 2$) is open. The big difference is that if $k \geq 2$ planes are open, there is a non-trivial probability that two of the planes are exactly distance two from each other, in which case the entire space would become open and $D = d$. A plane embedded in a 6-dimension space is determined by the values of the 4 fixed coordinates. The other two coordinates we call the free coordinates as they take on all values in $[n]$. If the free coordinates of the two planes do not overlap, then the planes are exactly distance 2 apart. Let $d_k$ denote the probability that for $k$ distinct planes, at least two have free coordinates that do not overlap.

For a plane $P$, let $N(P)$ denote the set of at most $cn^4$ possible nodes, $u \in [n]^6$, such that $\dim(\langle u, P \rangle) = 4$. With probability tending to 1, the number of nodes in $N(P_s) \cap N(P_t)$ is at most $o(n^4)$ for all $1 \leq s < t \leq k$. Hence the total number of nodes that cause at least one of $k$ planes to evolve into an internally spanned 4-dimensional sublattice is at least $ckn^4 - o(n^4)$. The number of nodes that determine $d_k$ is only $O(n^2)$ so if we remove those we still have at least $ckn^4 - o(n^4)$ nodes remaining that would cause a 4-dimensional sublattice to be internally spanned. This occurs with probability at least $(1 - an^{-4})^{ckn^4 - o(n^4)} = e^{-ack}(1 - o(1))$. Therefore

$$\mathbb{P}_p(D = 0) \to e^{-\lambda^j},$$

$$\mathbb{P}_p(D = 2) \to \sum_{k=1}^{\infty} \frac{e^{-\lambda^j}}{k!} \lambda^k (1 - d_k) e^{-ack},$$

$$\mathbb{P}_p(D = 6) \to 1 - e^{-\lambda^j} - \sum_{k=1}^{\infty} \frac{e^{-\lambda^j}}{k!} \lambda^k (1 - d_k) e^{-ack}.$$

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