BOUNDDED RATIOS OF PRODUCTS OF PRINCIPAL MINORS
OF POSITIVE DEFINITE MATRICES

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ABSTRACT. Considered is the multiplicative semigroup of ratios of products of principal minors bounded over all positive definite matrices. A long history of literature identifies various elements of this semigroup, all of which lie in a sub-semigroup generated by Hadamard-Fischer inequalities. Via cone-theoretic techniques and the patterns of nullity among positive semidefinite matrices, a semigroup containing all bounded ratios is given. This allows the complete determination of the semigroup of bounded ratios for 4-by-4 positive definite matrices, whose 46 generators include ratios not implied by Hadamard-Fischer and ratios not bounded by 1. For $n \geq 5$ it is shown that the containment of semigroups is strict, but a generalization of nullity patterns, of which one example is given, is conjectured to provide a finite determination of all bounded ratios.

1. Introduction

For an $n$-by-$n$ matrix $A = (a_{ij})$ and an index set $S \subseteq N = \{1, 2, \ldots, n\}$, $A[S]$ is the principal submatrix of $A$ lying in the rows and columns indicated by $S$. Given a collection $\alpha : \alpha_1, \alpha_2, \ldots, \alpha_p \subseteq N$ of index sets and exponents $\eta_1, \ldots, \eta_p \in \mathbb{R}^+$, let

$$\alpha(A) = \prod_{i=1}^{p} (\det A[\alpha_i])^{\eta_i}.$$ 

At times we also abbreviate $\alpha(A)$ as $\alpha_{\eta_1} \cdots \alpha_{\eta_p}$, omitting any exponents equal to 1. We are interested in understanding which ratios

$$\frac{\alpha}{\beta} = \frac{\alpha(A)}{\beta(A)}$$

have an upper bound that is independent of $A$, provided that $A$ is positive definite (PD). We call such ratios bounded, and absolutely bounded if 1 is an upper bound. The Hadamard-Fischer inequality \cite{Hadamard} provides a classical family of absolutely bounded ratios in which $\alpha_1 = S \cup T$ and $\alpha_2 = S \cap T$, while $\beta_1 = S$ and $\beta_2 = T$, for any two index sets $S, T \subseteq N$, with all exponents 1. This classical family is in fact bounded over any class of invertible matrices with positive principal minors and weakly sign-symmetric non-principal minors \cite{Koteljanskii}, and we will call such ratios Koteljanskii ratios \cite{K}. Historical information on determinantal inequalities for PD matrices may be found in \cite[ch. 7]{Hadamard}. Over real symmetric matrices, bounded ratios are equivalent to multiplicative inequalities on the face volumes of a parallelotope.

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A ratio is determined, up to equivalence, by the overall exponent (numerator minus denominator) of each nonempty subset of \( N \). The exponent of the empty set is chosen to make the sum of exponents 0, and the collection of \( 2^n \) exponents is called the formal logarithm of \( \frac{\alpha}{\beta} \), written \( \log(\frac{\alpha}{\beta}) \). Formal logarithms span a hyperplane in \( \mathbb{R}^{(2^n)} \) orthogonal to the all-1’s vector \( e \).

Let \( B_n \) be the set of bounded ratios, \( A_n \) the set of absolutely bounded ratios, and \( K_n \) the set of products of positive powers of Koteljanskii ratios. Then, for all \( n \), \( K_n \subseteq A_n \subseteq B_n \). These are multiplicative semigroups, and there are corresponding cones \( \log(K_n) \subseteq \log(A_n) \subseteq \log(B_n) \). All previously known multiplicative determinantal inequalities lie in \( K_n \). In [6], necessary conditions for boundedness were given, implicitly defining a semigroup \( E_n \) with \( B_n \subseteq E_n \), and it was shown that \( K_3 = E_3 \) but \( A_4 \neq E_4 \). Here, we define a new semigroup \( D_n \) by imposing linear inequalities on \( \log(D_n) \) coming from rank deficient positive semidefinite (PSD) matrices. We show that \( B_n \subseteq D_n \subseteq E_n \) for all \( n \). We also show that \( A_4 \neq B_4 = D_4 \neq E_4 \), and give explicit generators for \( B_4 \). Going further, we show that \( B_5 \neq D_5 \), but we conjecture that for each \( n \) there exists some finite system of linear inequalities, generalizing those coming from rank deficient PSD matrices, that is sufficient to characterize \( \log(B_n) \).

2. Cone theoretic determination of \( D_n \)

We call a ratio \( \frac{\alpha}{\beta} \) homogeneous if \( \alpha(D)/\beta(D) = 1 \) for every PD diagonal matrix \( D \), and call the set of all such ratios \( H_n \). All bounded ratios are homogeneous [6], and \( \log(H_n) \) is the subspace of \( \mathbb{R}^{(2^n)} \) that is orthogonal to the following set of \( n+1 \) vectors: the all-1’s vector \( e \) and, for each \( i \in N \), the vector whose entry \( T \) (as \( T \) ranges over \( T \subseteq N \)) is 1 whenever \( i \in T \), and 0 otherwise. A homogeneous ratio is uniquely determined by the exponents of subsets with cardinality at least 2. Since the dimension of \( \log(K_n) \) is \( 2^n-n-1 \), the cone \( \log(B_n) \) has full dimension within the subspace \( \log(H_n) \).

Let \( M \) be a matrix with \( n \) columns. We define \( \text{null}(M) \in \mathbb{R}^{(2^n)} \), the nullity type of \( M \), as the vector of null space dimensions for subsets of the columns of \( M \). (By convention the empty set of columns has nullity 0.) The nullity type of \( M \) also gives the multiplicity of 0 as an eigenvalue for any principal submatrix of \( M^*M \). For any \( n \) there exist finitely many distinct nullity types (corresponding to representations of labeled matroids on \( n \) elements).

Our first result is that the inner product of the formal logarithm of a ratio with any nullity type gives a necessary condition for boundedness.

**Theorem 1.** Let \( \frac{\alpha}{\beta} \) be a homogeneous ratio, let \( M \) be a matrix with \( n \) columns, and define a family of matrices \( A_\varepsilon \), \( 0 < \varepsilon < 1 \), by

\[
A_\varepsilon = M^*M + \varepsilon I.
\]

Then \( \frac{\alpha}{\beta} \) is bounded over the family \( A_\varepsilon \) if and only if \( \log(\frac{\alpha}{\beta})^T \text{null}(M) \geq 0 \).

**Proof.** Let \( s = \log(\frac{\alpha}{\beta})^T \text{null}(M) \), and let \( r \) represent the sum of the exponents in \( \alpha \), which is also the sum of the exponents in \( \beta \). Let \( \Lambda \) be the set of all nonzero eigenvalues of \( M^*M \) and its principal submatrices, and let \( \lambda_{\min} \) and \( \lambda_{\max} \) denote respectively the minimum and maximum elements of \( \Lambda \cup \{1\} \), so that \( \lambda_{\min} \leq 1 \leq \lambda_{\max} \). For any \( 0 < \varepsilon < 1 \), the matrix \( A_\varepsilon \) is PD, and the quantity \( \langle \frac{\alpha}{\beta}, A_\varepsilon \rangle \) can be expressed entirely in terms of the eigenvalues of principal submatrices of \( A_\varepsilon \). Each of
these eigenvalues is $\varepsilon$ or lies within the range $\lambda_{\min} + \varepsilon$ to $\lambda_{\max} + \varepsilon$. The total exponent of $\varepsilon$ in the eigenvalue product is exactly the inner product $\log(\mathcal{D})^T \nu \varrho(M) = s$. The contribution of all remaining eigenvalues either to the numerator $\alpha(A_\varepsilon)$ or to the denominator $\beta(A_\varepsilon)$ lies within the range $\lambda_{\min}^{nr}$ to $(\lambda_{\max} + 1)^{nr}$. Thus we have

$$\varepsilon^s \left( \frac{\lambda_{\min}}{\lambda_{\max} + 1} \right)^{nr} \leq \alpha(A_\varepsilon) \leq \varepsilon^s \left( \frac{\lambda_{\max} + 1}{\lambda_{\min}} \right)^{nr}.$$ 

If $s < 0$ the left inequality implies that $\frac{\mathcal{D}}{\delta}(A_\varepsilon)$ is unbounded, and if $s \geq 0$ the right inequality implies that $\frac{\mathcal{D}}{\delta}(A_\varepsilon)$ is bounded by a constant.

Given $n$, let $\nu_1, \ldots, \nu_\ell$ be a complete list of the nullity types of matrices with $n$ columns. The dual nullity semigroup $\mathcal{D}_n$ is defined as the set of homogeneous ratios $\mathcal{D}$ such that $\log(\mathcal{D})^T \nu_i \geq 0$ for all $1 \leq i \leq \ell$.

**Corollary 2.** For all $n$, $B_n \subseteq \mathcal{D}_n$. Moreover, if each extreme ray of $\log(\mathcal{D}_n)$ represents a bounded ratio, then $B_n = \mathcal{D}_n$.

The dual nullity cone $\log(\mathcal{D}_n)$ is a finite intersection of half-spaces and thus its set of extreme rays can (in principle, and sometimes in practice) be explicitly calculated. Some nullity types are redundant to this calculation: $\nu_1$ is trivial; $\nu_2(M_1[0]) - \nu(M_1 \oplus I)$ is orthogonal to $\mathcal{H}_n$, so one of these may be omitted; and $\nu(M_1 \oplus M_2) = \nu(M_1 \oplus I) + \nu(I \oplus M_2)$. Given a matrix $M$ with $n$ columns, let $\rho(M)$ be the vector in $\mathbb{R}^{(2^n)}$ whose entry for each $T \subseteq N$ is the rank of the subset $T$ of the columns of $M$. Since $\nu(M) + \rho(M)$ is the vector $[|T|]$ of cardinalities, which is orthogonal to $\mathcal{H}_n$, $\nu(M)$ and $-\rho(M)$ are interchangeable for the purposes of calculating $\mathcal{D}_n$. This fact can be useful computationally when $M$ has low rank.

**Example 1.** Let $n = 3$, and let $\nu_1, \ldots, \nu_5$ be the nullity types of the matrices

$$\left[\begin{array}{ccc}1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{array}\right], \left[\begin{array}{ccc}1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right].$$

Let $\mathcal{E}_3$ be the intersection of the five sets $\{ \mathcal{D} \in \mathcal{H}_3 : \log(\mathcal{D})^T \nu_i \geq 0 \}$. The six extreme rays of $\log(\mathcal{E}_3)$ are the formal logarithms of the Koteljanskii ratios

$$\begin{align*}
\{12\} & \oplus \{13\} \oplus \{23\} \oplus \{123\} \{1\} \{2\} \{3\} & \{1\} \{2\} \{3\} & \{1\} \{2\} \{3\} \{12\} \{13\} \{23\} \{123\} \{\emptyset\}.
\end{align*}$$

We thus have $\mathcal{K}_3 \subseteq \mathcal{A}_3 \subseteq \mathcal{B}_3 \subseteq \mathcal{D}_3 \subseteq \mathcal{E}_3 \subseteq \mathcal{K}_3$ and in particular $\mathcal{B}_3 = \mathcal{K}_3$, as was shown in [6].

The symmetric group on $n$ elements acts naturally on $\mathbb{R}^{(2^n)}$ and preserves every set we have defined. Set complementation $S^c = N \setminus S$ also acts naturally and preserves $\log(\mathcal{K}_n)$ and $\log(\mathcal{H}_n)$. Since $\frac{s}{n}(A) = \frac{s}{n}(A^{-1})$ by Jacobi’s identity [5], $\log(\mathcal{A}_n)$ and $\log(\mathcal{B}_n)$ are also invariant under the action of complementation. To show that $\log(\mathcal{D}_n)$ is invariant under complementation requires some matroid theory [9]: the complement of a nullity type is equivalent, up to vectors orthogonal to $\log(\mathcal{H}_n)$, to the nullity type of the dual matroid.

Theorem [11] generalizes the known set-theoretic necessary conditions ST0, ST1, and ST2.

**Theorem 3.** [6] For any bounded ratio $\mathcal{D}$ and any given index set $S \subseteq N$, the sum of exponents in $\mathcal{D}$ for supersets of $S$ is nonnegative (ST1), and likewise for subsets of $S$ (ST2). Any ratio satisfying ST1 and ST2 also satisfies homogeneity (ST0).
We define $\mathcal{E}_n$ as the set of ratios satisfying the elementary necessary conditions ST1 and ST2 for every $S \subseteq N$. These conditions can be restated in terms of nullity types. Given an index set $S \subseteq N$ with $|S| \geq 3$, define $M_S$ to be the matrix of size $(n - 1) \times n$ that is a permutation of $[I|e] \oplus I$, with the $[I|e]$ summand occurring in the columns of $S$ and the $I$ summand occurring in the complement of $S$. Then for $T \subseteq N$, entry $T$ of $\text{mul}(M_S)$ is 1 if $S \subseteq T$ and 0 otherwise. Given an index set $S \subseteq N$ with $|S| \leq n - 2$, define $M^S$ to be the matrix of size $1 \times n$ whose entries are 0 in the columns of $S$ and 1 elsewhere. Then for $T \subseteq N$, entry $T$ of $\rho(M^S)$ is 0 if $T \subseteq S$ and 1 otherwise. For example, with $n = 3$, the five matrices listed in Example 1 are $M^0$, $M^{(1)}$, $M^{(2)}$, $M^{(3)}$, and $M_{(123)}$.

**Proposition 4.** A homogeneous ratio $\frac{a}{j}$ satisfies conditions ST1 and ST2 if and only if $\log\left(\frac{a}{j}\right)^T \text{mul}(M_S) \geq 0$ for $|S| \geq 3$ and $\log\left(\frac{a}{j}\right)^T \text{mul}(M^S) \geq 0$ for $|S| \leq n - 2$.

**Proof.** The cases $|S| \leq 1$ for ST1 and $|S| \geq n - 1$ for ST2 are equivalent to the assumed homogeneity. The case ST1 with $|S| = 2$ will follow from ST2 with $|S| = n - 2$: for a homogeneous ratio, ST1 is satisfied for $\{i, j\}$ if and only if ST2 is satisfied for $\{i, j\}^c$, as can be seen by partitioning $2^N$ into four groups depending on the membership of $i$ and $j$, and comparing the total exponents of these groups under homogeneity. For $|S| \geq 3$, $\log\left(\frac{a}{j}\right)^T \text{mul}(M_S)$ is nonnegative if and only if $S$ occurs as a subset in $\alpha$ with a total exponent at least as great as in $\beta$. For $|S| \leq n - 2$, $\log\left(\frac{a}{j}\right)^T (e - \rho(M^S))$ is nonnegative if and only if $S$ occurs as a superset in $\alpha$ with a total exponent at least as great as in $\beta$. Since $\frac{a}{j}$ is homogeneous and $\text{mul}(M^S) + \rho(M^S) - e$ is orthogonal to $\log(H_n)$, this is equivalent to $\log\left(\frac{a}{j}\right)^T \text{mul}(M^S) \geq 0$. \hfill $\Box$

**Corollary 5.** For all $n$, $\mathcal{D}_n \subseteq \mathcal{E}_n$.

The name $\mathcal{E}_3$ in Example 1 is now justified, and we have shown, as in [6], that $\mathcal{K}_3 = \mathcal{E}_3$. Since the vectors $\text{mul}(M^S)$ for $|S| \leq n - 2$ and the vectors orthogonal to $\log(H_3)$ together span $\mathbb{R}(2^N)$, $\log(\mathcal{B}_n)$ is a pointed cone: no nontrivial ratio is both bounded and bounded away from 0.

3. **Bounding ratios not in $\mathcal{K}_n$**

Given a ratio $\frac{a}{j}$ in $\mathcal{D}_n \setminus \mathcal{K}_n$, it may be difficult to determine whether $\frac{a}{j}$ is bounded. We introduce a technique that can sometimes prove boundedness, starting with a known observation whose first nontrivial case is $n = 3$:

**Lemma 6.** [4] Let $A = (a_{ij})$ be an $n$-by-$n$ PD matrix with inverse $B = (b_{ij})$. Then for each $i = 1, \ldots, n$

\begin{equation}
2\sqrt{a_{ii}b_{ii}} + (n - 2) \leq \sum_{j=1}^{n} \sqrt{a_{jj}b_{jj}}.
\end{equation}

Eliminating a copy of $i$ on each side, replacing the sum with $(n - 1)$ times the maximum, and using the fact that $b_{ii} = \frac{(i)}{N}(A)$, we conclude:

**Corollary 7.** For $n \geq 3$ and $A$ PD, and for any $i \in N$,

\[
\min_{j \neq i} \left\{ \frac{(\{i\})}{(\{j\})^c}(A) \right\} \leq (n - 1)^2.
\]
Example 2. In \cite{6} it is shown that $A_4 \neq E_4$ using a permutation of
\[
R_1 = \{124\} \{134\} \{23\} \{1\} \{4\} \{12\} \{13\} \{14\} \{24\} \{34\}.
\]
Observe that this ratio is unchanged when indices 2 and 3 are interchanged; we may thus assume without loss of generality that $\{1\} \{23\} (A) \leq \{1\} \{2\} \{3\} (A)$. By applying Corollary \cite{7} to $A[\{123\}]$ with $i = 1$ we then obtain, factoring out a pair of Koteljanskii ratios,
\[
R_1 = \{124\} \{2\} \{134\} \{4\} \{1\} \{23\} \{12\} \{24\} \{14\} \{34\} \{2\} \{13\} \leq 4,
\]
which shows that $R_1$ belongs to $B_4$.

4. The extreme rays of $\log(B_4)$

Let $\mu_1, \ldots, \mu_{23}$ represent the distinct nullity types of column permutations of the following seven matrices: $M^0$, $M^{(1)}$, $M^{(12)}$, $M_{\{123\}}$, $M_{\{1234\}}$.

\[
M_6 = \begin{bmatrix}
1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1
\end{bmatrix}, \quad \text{and} \quad M_7 = \begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & 2 & 3
\end{bmatrix}.
\]

The intersection of $H_4$ with the 23 half-spaces $\{x \in \mathbb{R}^{16} : x^T \mu_i \geq 0\}$ contains $D_4$ and thus $B_4$. There are various software packages available for computing the extreme rays of an intersection of half-spaces in exact arithmetic; in this case the calculation was done with custom-written software, whose results were later verified with the program \texttt{lrs} \cite{1} (available at \url{http://cgm.cs.mcgill.ca/~avis/C/lrs.html}).

The 46 extreme rays include the 24 extreme rays of $\log(K_4)$ (those with $|S| = |T| = |S \cap T| + 1$) and the 6 permutations of $\log(R_1)$ from Example 2. The ratios corresponding to the remaining 16 extreme rays are permutations of
\[
R_2 = \{1234\} \{134\} \{23\} \{24\} \{34\} \{1\} \{12\} \{13\} \{14\} \{2\} \{3\} \{4\},
\]
\[
R_3 = \{1234\} \{23\} \{24\} \{34\} \{1\} \{1\} \{12\} \{13\} \{14\} \{2\} \{3\} \{4\},
\]
and their complements. Since $R_2$ and $R_3$ are both symmetric in indices 2 and 3, we may make the same assumption as in Example 2 and conclude from Corollary \cite{7} that
\[
R_2 = \{1234\} \{24\} \{124\} \{234\} \{13\} \{3\} \{4\} \{2\} \{13\} \leq 4
\]
and
\[
R_3 = \{1234\} \{24\} \{124\} \{1\} \{3\} \{4\} \{2\} \{13\} \leq 4.
\]

Although the proven bound is 4, experimental evidence suggests that the ratios $R_2$ and $R_3$ are absolutely bounded, which would imply $K_4 \neq A_4$, and that the correct upper bound for $R_1$ is $\frac{\pi}{2}$, which value it is known to approach \cite{8}. This is the first known case showing $A_n \neq B_n$ for any $n$. It has been shown \cite{9} that bounded ratios on M-matrices or inverse M-matrices are absolutely bounded, and preliminary work shows the same for (invertible) totally nonnegative matrices as far as $n < 6$.

The listed nullity types are thus sufficient to define $D_4$, and indeed $B_4$. 

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Theorem 8. The semigroup $\mathcal{B}_4$ is generated, up to permutation and complementation, by positive powers of $(\{12\},\emptyset,\{13\},\{12\})$, $R_1$, $R_2$, and $R_3$.

Theorem 9. The sets $\mathcal{B}_4$ and $\mathcal{D}_4$ are equal: a ratio $\frac{\alpha}{\beta} \in \mathcal{H}_4$ is bounded if and only if $\log(\frac{\alpha}{\beta})^T \mu_i \geq 0$ for all $i = 1, \ldots, 23$, with $\mu_i$ as listed above.

We end the section with an example showing that $\mathcal{D}_4 \neq \mathcal{E}_4$. The ratio

$$\alpha \beta = \{1234\} \{134\} \{12\} \{14\} \{23\} \{24\} \{3\} \emptyset \{123\} \{124\} \{234\} \{13\} \{34\} \{1\} \{2\} \{4\}$$

satisfies ST1 and ST2 with respect to every $S \subseteq \{1, 2, 3, 4\}$. However, the inner product $\log(\frac{\alpha}{\beta})^T \text{nul}(M_6)$ gives $-1$, and $\frac{\alpha}{\beta}$ is not bounded.

5. The case $n = 5$

A ratio in the semigroup $\mathcal{D}_5$ is bounded near $\varepsilon = 0$ for any matrix family of the form $(B + \varepsilon C)^r (B + \varepsilon C)$, where $B$ is singular and $C$ is generic in the sense that $Cx = By$ and $Bx = 0$ imply $x = 0$. We have shown that for $n = 4$ such matrix families, whose behavior depends only on the nullity type of $B$, are sufficient to determine $\mathcal{B}_4$. The following example shows that for $n \geq 5$, more general matrix families must be considered.

Example 3. Let $n = 5$ and consider the ratio

$$Q = \{12345\} \{1345\} \{2345\} \{123\} \{125\} \{345\} \{1\} \{2\} \emptyset \{1234\} \{1235\} \{145\} \{245\} \{345\} \{12\} \{13\} \{23\} \{4\} \{5\}.$$

We claim that $Q \in \mathcal{D}_5$. Since $Q$ is self-complementary, it is only necessary to check against matrices of rank up to $[\frac{5}{2}]$, i.e. $M$ of size $2 \times 5$. Uniformly deleting a specified index element in the sets of $Q$ yields a product of Koteljanskii ratios in all five cases, which eliminates the case in which $M$ has a zero column (using homogeneity). The nullity type of $M$ now depends only on a partition of the columns $\{1, 2, 3, 4, 5\}$ into parallel classes. Partitions into exactly two blocks give the nullity type of a direct sum and are thus redundant, leaving 37 cases to check.

Now let

$$P = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & \varepsilon & 0 & 1 & 2 \\ 0 & 0 & \varepsilon & 1 & 2 \\ 0 & 0 & 0 & \varepsilon & 0 \\ 0 & 0 & 0 & 0 & \varepsilon \end{bmatrix},$$

define $A_\varepsilon$ as $P^* P$, and consider the polynomial $\det A_\varepsilon[S]$ for some $S \subseteq N$. This is a nonnegative function of $\varepsilon$, and so for every $S$ the dominating term for $\varepsilon$ small must be an even power $C S \varepsilon^{2d_S}$ for $C_S > 0$ and $d_S$ a nonnegative integer. Call the vector $[d_S] \in \mathbb{R}^{[2S]}$ the *asymptotic nullity type* of $P$, or $\text{asn}(P)$. The fact that $\log(Q)^T \text{asn}(P) = -1$ means that the ratio $Q$ is not bounded, and $\mathcal{B}_5 \neq \mathcal{D}_5$.

We expect that any one-parameter matrix family making a particular ratio unbounded can be adequately approximated by polynomials in $\varepsilon$, and further that the maximum necessary degree of the approximation depends only on $n$. We thus expect $\log(B_n)$ to be a polyhedral cone for all $n$, as it is for $n \leq 4$. 


Conjecture 1. Given any $n$, define the asymptotic nullity type $\text{asn}(P)$ of an invertible $n \times n$ matrix $P$ of polynomials in $\varepsilon$ as in Example 3. Then there exists a finite list of polynomial matrices $P_1, \ldots, P_\ell$ such that a homogeneous ratio $\frac{\alpha}{\beta}$ is bounded if and only if $\log\left(\frac{\alpha}{\beta}\right)^T \text{asn}(P_i) \geq 0$ for all $1 \leq i \leq \ell$.

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