HERMITE–HADAMARD TYPE INEQUALITIES VIA DIFFERENTIABLE $h_ϕ$–PREINVEX FUNCTIONS FOR FRACTIONAL INTEGRALS

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Abstract. In this paper, we consider a new class of convex functions which is called $h_ϕ$–preinvex functions. We prove several Hermite–Hadamard type inequalities for differentiable $h_ϕ$–preinvex functions via Fractional Integrals. Some special cases are also discussed. Our results extend and improve the corresponding ones in the literature.

1. Introduction

The following inequality is well-known in the literature as Hermite–Hadamard inequality. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function with $a < b$ and $a, b \in I$. Then the following holds

\[
 f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}.
\]

Recently, Hermite-Hadamard type inequality has been the subject of intensive research. Hermite–Hadamard inequality can be considered as necessary and sufficient condition for a function to be convex. It provides estimates of the mean value of continuous convex function.

In recent years, several extensions and generalizations have been proposed for classical convexity (see [1 − 24]). A significant generalization of classical convex functions is that of $ϕ$–convex functions introduced by Noor [10]. Noor [10] has investigated the basic properties of $ϕ$–convex functions and showed that $ϕ$–convex functions are nonconvex functions. Noor [9] extended Hermite–Hadamard type inequalities for $ϕ$–convex functions.

Motivated by the ongoing research on generalizations and extensions of classical convexity, Varosanec [23] introduced the class of $h$–convex functions. Noor et al. [13] introduced another generalization of classical convexity which is called as $h_ϕ$–convex functions. Finally Noor et al. Several Hermite–Hadamard type inequalities are proved for $h_ϕ$–preinvex functions[20].

Weir and Mond [24] investigated the class of preinvex functions. It is well-known that the preinvex functions and invex sets may not be convex functions and convex sets. For recent investigation on preinvexity (see [1, 4, 5 − 8, 17, 22]).

Motivated and inspired by the recent research in this field, we consider a new class of convex functions, which is called $h_ϕ$–preinvex functions. This class includes

\textbf{Key words and phrases.} integral inequalities, fractional integrals, Hermite-Hadamard Inequality, Preinvex functions, Hölder’s inequality.

\textbf{2010 Mathematics Subject Classification} 26D15, 26A51, 26A33, 26A42.
several known and new classes of convex functions such as $\varphi$–convex functions [10], preinvex functions [21], $h$–convex functions [23] as special cases. We derive several new Hermite–Hadamard type fractional integral inequalities for $h_{\varphi}$–preinvex functions and their variant forms. Results obtained in this paper continue to hold for these special cases. Our results represent significant generalizations of the previous results. The interested readers are encouraged to find novel and innovative applications of $h_{\varphi}$–preinvex functions and fractional integrals.

2. Preliminaries

Let $\mathbb{R}^n$ be the finite dimensional Euclidean space. Also let $0 \leq \varphi \leq \frac{\pi}{2}$ be a continuous function.

**Definition 1.** ([12]) The set $K_{\varphi n}$ in $\mathbb{R}^n$ is said to be $\varphi$–invex at $u$ with respect to $\varphi(\cdot)$, if there exists a bifunction $\eta(\cdot, \cdot) : K_{\varphi n} \times K_{\varphi n} \to \mathbb{R}$, such that

$$u + te^{i\varphi} \eta(v, u) \in K_{\varphi n}, \quad \forall u, v \in K_{\varphi n}, \quad t \in [0, 1].$$

The $\varphi$–invex set $K_{\varphi n}$ is also called $\varphi n$–connected set. Note that the convex set with $\varphi = 0$ and $\eta(v, u) = v - u$ is a $\varphi$–invex set, but the converse is not true.

**Definition 2.** ([10]) Let $K_{\varphi}$ be a set in $\mathbb{R}^n$. Then the set $K_{\varphi}$ is said to be $\varphi$–convex with respect to $\varphi$, if and only if

$$u + te^{i\varphi} (v, u) \in K_{\varphi}, \quad \forall u, v \in K_{\varphi}, \quad t \in [0, 1].$$

For $\varphi = 0$, the set $K_{\varphi}$ reduces to the classical convex set $K$. That is,

$$u + t(v, u) \in K, \quad \forall u, v \in K, \quad t \in [0, 1].$$

**Definition 3.** ([24]) A set $K_n$ is said to be invex set with respect to bifunction $\eta(\cdot, \cdot)$, if

$$u + t\eta(v, u) \in K_n, \quad \forall u, v \in K_n, \quad t \in [0, 1].$$

The invex set $K_n$ is also called $\eta$–connected set.

**Definition 4.** ([20]) Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ be a nonnegative function. A function $f$ on the set $K_{\varphi n}$ is said to be $h_{\varphi}$–preinvex function with respect to $\varphi$ and bifunction $\eta$, if

$$f(u + te^{i\varphi} \eta(v, u)) \leq h(1 - t)f(u) + h(t)f(v), \quad \forall u, v \in K_{\varphi n}, \quad t \in [0, 1].$$

**Remark 1.** One can deduce several known concepts from Definition 4 as:

1. For $h(t) = t$ Definition 4 reduces to the definition for $\varphi$–preinvex functions (see [12]).
2. For $\varphi = 0$ Definition 4 reduces to the definition for $h$–preinvex functions (see [17]).
3. For $\eta(v, u) = v - u$ Definition 4 reduces to the definition for $h_{\varphi}$–convex functions (see [13]).
4. For $\varphi = 0$ and $\eta(v, u) = v - u$ Definition 4 reduces to the definition for $h$–convex functions (see [23]).

Now we discuss some special cases of Definition 4.

I. For $h(t) = t^s$ where $s \in (0, 1]$ in (2.2) we have the definition for $s_{\varphi}$–preinvex functions.
Definition 5. A function $f$ on the set $K_{\varphi n}$ is said to be $s_{\varphi}−$preinvex function with respect to $\varphi$ and $\eta$, if

$$f(u + te^{i\varphi}\eta(v, u)) \leq (1 - t)^s f(u) + t^s f(v), \quad \forall u, v \in K_{\varphi n}, \ t \in [0, 1]. \ (2.3)$$

II. For $h(t) = 1$ in (2.2) we have the definition for $P_{\varphi}−$preinvex functions.

Definition 6. A function $f$ on the set $K_{\varphi n}$ is said to be $s_{\varphi}−$preinvex function with respect to $\varphi$ and $\eta$, if

$$f(u + te^{i\varphi}\eta(v, u)) \leq f(u) + f(v), \quad \forall u, v \in K_{\varphi n}, \ t \in [0, 1]. \ (2.4)$$

Definition 7. ([25]) Let $f \in L^1[a, b]$. The Riemann-Liouville fractional integral $J^\alpha_{a+} f(x)$ and $J^\alpha_{b-} f(x)$ of order $\alpha > 0$ are defined by

$$J^\alpha_{a+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha - 1} f(t) \ dt \quad x > a \quad (2.5)$$

and

$$J^\alpha_{b-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t - x)^{\alpha - 1} f(t) \ dt \quad x < b \quad (2.6)$$

respectively, where $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha - 1} dt$ is Gamma function and $J^0_{a+} f(x) = J^0_{b-} f(x) = f(x)$.

3. Main results

In this section, we will discuss our main results for $h_{\varphi}−$preinvex functions and fractional integrals.

Using the technique of [1] and [20], we prove the following Lemma which play a key role in our study.

Lemma 1. Let $I \subseteq \mathbb{R}$ be a open invex set with respect to bifunction $\eta : I \times I \rightarrow \mathbb{R}$ where $\eta(b, a) > 0$. If $f' \in L^1[a, a + e^{i\varphi}\eta(b, a)]$ and $\alpha \geq 0$, then

$$[f(a) + f(a + e^{i\varphi}\eta(b, a))] - \frac{\Gamma(\alpha + 1)}{[e^{i\varphi}\eta(b, a)]} \left\{ J^\alpha_{a+} f(x) + J^\alpha_{(a+e^{i\varphi}\eta(b, a))} f(x) \right\}$$

$$= e^{i\varphi}\eta(b, a) \int_0^1 [t^\alpha - (1 - t)^\alpha] f'(a + te^{i\varphi}\eta(b, a)) dt. \ (3.1)$$
Proof. Let

\[
\int_0^1 [(1-t)^\alpha - t^\alpha] f'(a + te^{i\varphi}\eta(b,a)) dt = \frac{\left((1-t)^\alpha - t^\alpha\right) f(a + te^{i\varphi}\eta(b,a))}{e^{i\varphi}\eta(b,a)}
\]

+ \frac{\alpha}{e^{i\varphi}\eta(b,a)} \int_0^1 [(1-t)^{\alpha-1} + t^{\alpha-1}] f(a + te^{i\varphi}\eta(b,a)) dt

= - \frac{[f(a) + f(a + e^{i\varphi}\eta(b,a))] - \alpha}{e^{i\varphi}\eta(b,a)}\int_a^{a+e^{i\varphi}\eta(b,a)} \left[ (a + e^{i\varphi}\eta(b,a) - u)^{\alpha-1} + \left(\frac{u-a}{e^{i\varphi}\eta(b,a)}\right)^{\alpha-1}\right] f(u) du

= - \frac{[f(a) + f(a + e^{i\varphi}\eta(b,a))] - \alpha}{e^{i\varphi}\eta(b,a)}\int_a^{a+e^{i\varphi}\eta(b,a)} \left[ (a + e^{i\varphi}\eta(b,a) - u)^{\alpha-1} + \left(\frac{u-a}{e^{i\varphi}\eta(b,a)}\right)^{\alpha-1}\right] f(u) du

+ \frac{\alpha}{e^{i\varphi}\eta(b,a)} \int_0^a \left[ (a + e^{i\varphi}\eta(b,a) - u)^{\alpha-1} + \left(\frac{u-a}{e^{i\varphi}\eta(b,a)}\right)^{\alpha-1}\right] f(u) du

This complete the proof. \(\square\)

Theorem 1. Let \(I \subseteq \mathbb{R}\) be a open invex set with respect to bifunction \(\eta : I \times I \to \mathbb{R}\). Suppose \(f : I \to \mathbb{R}\) is a differentiable function such that \(f' \in L[a,a + e^{i\varphi}\eta(b,a)]\). Let \(f'\) be \(h_\varphi\)-preinvex function. Then, for \(\eta(b,a) > 0\) and \(\alpha > 0\),

\[
\left| f(a) + f(a + e^{i\varphi}\eta(b,a)) - \frac{\Gamma(\alpha+1)}{e^{i\varphi}\eta(b,a)^{\alpha+1}} \left[ J_a^\alpha f(x) + J_{(a+e^{i\varphi}\eta(b,a))}^\alpha f(x) \right] \right|
\]

\[
\leq e^{i\varphi}\eta(b,a) \left[ |f'(a)| + |f'(b)| \right] \int_0^1 (1-t)^\alpha - t^\alpha |h(t)| dt.
\]

Proof. Using Lemma 2 and \(h_\varphi\)-preinvexity of \(f'\), we have

\[
\left| f(a) + f(a + e^{i\varphi}\eta(b,a)) - \frac{\Gamma(\alpha+1)}{e^{i\varphi}\eta(b,a)^{\alpha+1}} \left[ J_a^\alpha f(x) + J_{(a+e^{i\varphi}\eta(b,a))}^\alpha f(x) \right] \right|
\]

\[
\leq e^{i\varphi}\eta(b,a) \int_0^1 (1-t)^\alpha - t^\alpha \left| f'(a + te^{i\varphi}\eta(b,a)) \right| dt
\]

\[
\leq e^{i\varphi}\eta(b,a) \int_0^1 (1-t)^\alpha - t^\alpha \left( h(1-t) |f'(a)| + h(t) |f'(b)| \right) dt
\]
\[
= e^{\varphi \eta(b, a)} \left[ \int_0^1 |(1 - t)^\alpha - t^\alpha| \, h(t) \, |f'(a)| \, dt + \int_0^1 |(1 - t)^\alpha - t^\alpha| \, h(t) \, |f'(a)| \, dt \right]
\]
\[
= e^{\varphi \eta(b, a)} \left[ \int_0^1 |t^\alpha - (1 - t)^\alpha| \, h(t) \, |f'(a)| \, dt + \int_0^1 |(1 - t)^\alpha - t^\alpha| \, h(t) \, |f'(b)| \, dt \right]
\]
\[
= e^{\varphi \eta(b, a)} \left[ |f'(a)| + |f'(b)| \right] \int_0^1 |(1 - t)^\alpha - t^\alpha| \, h(t) \, dt.
\]
This completes the proof. \(\square\)

Now we have some special cases.

I. If \(h(t) = t\), then we have the result for \(\varphi\)-preinvexity.

**Corollary 1.** Let \(I \subseteq \mathbb{R}\) be a open invex set with respect to \(\eta : I \times I \to \mathbb{R}\). Suppose that \(f : I \to \mathbb{R}\) is a differentiable function such that \(f' \in L_1[a, a + e^{\varphi \eta(b, a)}]\). If \(|f'|\) is \(\varphi\)-preinvex on \(I\), then, for \(\eta(b, a) > 0\),

\[
|f(a) + f(a + e^{\varphi \eta(b, a)}) - \frac{\Gamma(\alpha + 1)}{e^{\varphi \eta(b, a)}\eta^\alpha} \left[ J_{\alpha} f(x) + J_{(a + e^{\varphi \eta(b, a)})}^\alpha f(x) \right] | \leq e^{\varphi \eta(b, a)} \left[ |f'(a)| + |f'(b)| \right] \frac{1}{\alpha + 2} \left[ 1 - \frac{1}{2^{\alpha + 2}} \right].
\]

(3.3)

II. If \(h(t) = t^s\), then we have the result for \(s_\varphi\)-preinvexity

**Corollary 2.** Let \(I \subseteq \mathbb{R}\) be a open invex set with respect to \(\eta : I \times I \to \mathbb{R}\). Suppose that \(f : I \to \mathbb{R}\) is a differentiable function such that \(f' \in L_1[a, a + e^{\varphi \eta(b, a)}]\). If \(|f'|\) is \(s_\varphi\)-preinvex on \(I\), then, for \(\eta(b, a) > 0\) and \(s \in (0, 1)\),

\[
|f(a) + f(a + e^{\varphi \eta(b, a)}) - \frac{\Gamma(\alpha + 1)}{e^{\varphi \eta(b, a)}\eta^\alpha} \left[ J_{\alpha} f(x) + J_{(a + e^{\varphi \eta(b, a)})}^\alpha f(x) \right] | \leq e^{\varphi \eta(b, a)} \left[ |f'(a)| + |f'(b)| \right] \left[ B_{\frac{1}{s}}(s + 1, \alpha + 1) \frac{1}{s + 1} \left[ 1 - \frac{1}{2^{s + \alpha}} \right] \right].
\]

(3.4)

III. If \(h(t) = 1\), then we have the result for \(P_\varphi\)-preinvexity.

**Corollary 3.** Let \(I \subseteq \mathbb{R}\) be a open invex set with respect to \(\eta : I \times I \to \mathbb{R}\). Suppose that \(f : I \to \mathbb{R}\) is a differentiable function such that \(f' \in L_1[a, a + e^{\varphi \eta(b, a)}]\). If \(|f'|\) is \(P_\varphi\)-preinvex on \(I\), then, for \(\eta(b, a) > 0\),

\[
|f(a) + f(a + e^{\varphi \eta(b, a)}) - \frac{\Gamma(\alpha + 1)}{e^{\varphi \eta(b, a)}\eta^\alpha} \left[ J_{\alpha} f(x) + J_{(a + e^{\varphi \eta(b, a)})}^\alpha f(x) \right] | \leq e^{\varphi \eta(b, a)} \left[ |f'(a)| + |f'(b)| \right] \frac{2}{\alpha + 1} \left[ 1 - \frac{1}{2^\alpha} \right].
\]

(3.5)
Theorem 2. Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \to \mathbb{R}$. Suppose that $f : I \to \mathbb{R}$ is a differentiable function such that $f' \in L_1[a, a + e^{i\varphi} \eta(b, a)]$. Let $|f'|^p$ be $h_{\varphi}$-preinvex on $I$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\eta(b, a) > 0$. Then

$$|f(a) + f(a + e^{i\varphi} \eta(b, a)) - \frac{\Gamma(\alpha + 1)}{[e^{i\varphi} \eta(b, a)]^\alpha} \left[ J^\alpha_{a+} f(x) + J^\alpha_{a+e^{i\varphi} \eta(b, a)} - f(x) \right]| \leq \left( \frac{2}{\alpha p + 1} \right)^{\frac{1}{p}} e^{i\varphi} \eta(b, a) \left[ 1 - \frac{1}{2^{\alpha p}} \right] \left\{ \left[ f'(a) \right]^p + \left[ f'(b) \right]^p \right\}^{\frac{1}{p}} h(t) dt \right\}^{\frac{1}{p}}. $$

Proof. Using Lemma 2, we have

$$\left| f(a) + f(a + e^{i\varphi} \eta(b, a)) - \frac{\Gamma(\alpha + 1)}{[e^{i\varphi} \eta(b, a)]^\alpha} \left[ J^\alpha_{a+} f(x) + J^\alpha_{a+e^{i\varphi} \eta(b, a)} - f(x) \right] \right| \leq e^{i\varphi} \eta(b, a) \int_0^1 |(1 - t)\alpha - t\alpha| f'(a + te^{i\varphi} \eta(b, a)) dt \leq e^{i\varphi} \eta(b, a) \int_0^1 |(1 - t)\alpha - t\alpha| f'(a + te^{i\varphi} \eta(b, a)) dt. $$

From Holders inequality, we have

$$\left| f(a) + f(a + e^{i\varphi} \eta(b, a)) - \frac{\Gamma(\alpha + 1)}{[e^{i\varphi} \eta(b, a)]^\alpha} \left[ J^\alpha_{a+} f(x) + J^\alpha_{a+e^{i\varphi} \eta(b, a)} - f(x) \right] \right| \leq e^{i\varphi} \eta(b, a) \left( \int_0^1 |(1 - t)\alpha - t\alpha|^p dt \right)^{\frac{1}{p}} \left( \int_0^1 \left| f'(a + te^{i\varphi} \eta(b, a)) \right|^q dt \right)^{\frac{1}{q}} \leq e^{i\varphi} \eta(b, a) \left( \int_0^1 |(1 - t)\alpha - t\alpha|^p dt + \int_0^1 (t\alpha - (1 - t)\alpha)^p dt \right)^{\frac{1}{p}} \left( \int_0^1 \left| f'(a + te^{i\varphi} \eta(b, a)) \right|^q dt \right)^{\frac{1}{q}}. $$

Here $h_{\varphi}$-preinvexity of $|f'|^p$, we have following inequality

$$\left| f(a) + f(a + e^{i\varphi} \eta(b, a)) - \frac{\Gamma(\alpha + 1)}{[e^{i\varphi} \eta(b, a)]^\alpha} \left[ J^\alpha_{a+} f(x) + J^\alpha_{a+e^{i\varphi} \eta(b, a)} - f(x) \right] \right| \leq \left( \frac{2}{\alpha p + 1} \right)^{\frac{1}{p}} e^{i\varphi} \eta(b, a) \left[ 1 - \frac{1}{2^{\alpha p}} \right] \left\{ \left[ f'(a) \right]^p + \left[ f'(b) \right]^p \right\}^{\frac{1}{p}} h(t) dt \right\}^{\frac{1}{p}}. $$

This completes the proof.

Now we have some special cases for (3.6).
I. If \( h(t) = t \), then we have the result for \( \varphi \)-preinvexity.

Corollary 4. Let \( I \subseteq \mathbb{R} \) be a open invex set with respect to \( \eta : I \times I \rightarrow \mathbb{R} \). Suppose that \( f : I \rightarrow \mathbb{R} \) is a differentiable function such that \( f' \in L_1[a, a + e^{i\varphi}\eta(b, a)] \). Let \( |f'|^p \) be \( \varphi \)-preinvex on \( I, \alpha + \beta = 1 \) and \( \eta(b, a) > 0 \). Then

\[
|f(a) + f(a + e^{i\varphi}\eta(b, a)) - \frac{\Gamma(\alpha + 1)}{[e^{i\varphi}\eta(b, a)]^{\alpha}} \left[ J_{a+}^\alpha f(x) + J_{(a+e^{i\varphi}\eta(b,a))}^\alpha f(x) \right] | \\
\leq \left( \frac{2}{\alpha p + 1} \right)^\beta e^{i\varphi}\eta(b, a) \left[ 1 - \frac{1}{2^\alpha p} \right] \frac{1}{p} \left\{ \left| f'(a) \right|^{\frac{p}{p-1}} + \left| f'(b) \right|^{\frac{p}{p-1}} \right\}^{\frac{p-1}{p}}. 
\]

II. If \( h(t) = t^s \), then we have the result for \( s_\varphi \)-preinvexity.

Corollary 5. Let \( I \subseteq \mathbb{R} \) be a open invex set with respect to \( \eta : I \times I \rightarrow \mathbb{R} \). Suppose that \( f : I \rightarrow \mathbb{R} \) is a differentiable function such that \( f' \in L_1[a, a + e^{i\varphi}\eta(b, a)] \). Let \( |f'|^p \) be \( s_\varphi \)-preinvex on \( I, \alpha + \beta = 1 \) and \( \eta(b, a) > 0 \) and \( s \in (0, 1) \). Then

\[
|f(a) + f(a + e^{i\varphi}\eta(b, a)) - \frac{\Gamma(\alpha + 1)}{[e^{i\varphi}\eta(b, a)]^{\alpha}} \left[ J_{a+}^\alpha f(x) + J_{(a+e^{i\varphi}\eta(b,a))}^\alpha f(x) \right] | \\
\leq \left( \frac{2}{\alpha p + 1} \right)^\beta e^{i\varphi}\eta(b, a) \left[ 1 - \frac{1}{2^\alpha p} \right] \frac{1}{p} \left\{ \left| f'(a) \right|^{\frac{p}{p-1}} + \left| f'(b) \right|^{\frac{p}{p-1}} \right\}^{\frac{p-1}{p}}. 
\]

III. If \( h(t) = 1 \), then we have the result for \( P_\varphi \)-preinvexity.

Corollary 6. Let \( I \subseteq \mathbb{R} \) be a open invex set with respect to \( \eta : I \times I \rightarrow \mathbb{R} \). Suppose that \( f : I \rightarrow \mathbb{R} \) is a differentiable function such that \( f' \in L_1[a, a + e^{i\varphi}\eta(b, a)] \). Let \( |f'|^p \) be \( P_\varphi \)-preinvex on \( I, \alpha + \beta = 1 \) and \( \eta(b, a) > 0 \). Then

\[
|f(a) + f(a + e^{i\varphi}\eta(b, a)) - \frac{\Gamma(\alpha + 1)}{[e^{i\varphi}\eta(b, a)]^{\alpha}} \left[ J_{a+}^\alpha f(x) + J_{(a+e^{i\varphi}\eta(b,a))}^\alpha f(x) \right] | \\
\leq \left( \frac{2}{\alpha p + 1} \right)^\beta e^{i\varphi}\eta(b, a) \left[ 1 - \frac{1}{2^\alpha p} \right] \frac{1}{p} \left\{ \left| f'(a) \right|^{\frac{p}{p-1}} + \left| f'(b) \right|^{\frac{p}{p-1}} \right\}^{\frac{p-1}{p}}. 
\]

Theorem 3. Let \( I \subseteq \mathbb{R} \) be a open invex set with respect to \( \eta : I \times I \rightarrow \mathbb{R} \). Suppose that \( f : I \rightarrow \mathbb{R} \) is a differentiable function such that \( f' \in L_1[a, a + e^{i\varphi}\eta(b, a)] \). Let \( |f'|^q \), \( q > 1 \) is \( h_\varphi \)-preinvex on \( I \), then, for \( \eta(b, a) > 0 \),

\[\text{(10)}\]

\[
|f(a) + f(a + e^{i\varphi}\eta(b, a)) - \frac{\Gamma(\alpha + 1)}{[e^{i\varphi}\eta(b, a)]^{\alpha}} \left[ J_{a+}^\alpha f(x) + J_{(a+e^{i\varphi}\eta(b,a))}^\alpha f(x) \right] | \\
\leq \left( \frac{2}{\alpha + 1} \right)^\beta \left[ 1 - \frac{1}{2^\alpha} \right] e^{i\varphi}\eta(b, a) \left\{ \left| f'(a) \right|^{\frac{p}{p-1}} + \left| f'(b) \right|^{\frac{p}{p-1}} \right\}^{\frac{p-1}{p}} \int_0^1 (1-t)\eta h(t) dt 
\]
Proof. Using Lemma 2, we have

\[
|f(a) + f(a + e^{i\varphi}\eta(b, a)) - \frac{\Gamma(\alpha + 1)}{[e^{i\varphi}\eta(b, a)]^\alpha} \left[J_{a^\alpha} f(x) + J_{(a + e^{i\varphi}\eta(b, a))^\alpha} f(x)\right]| \\
= \left|e^{i\varphi}\eta(b, a) \int_0^1 |(1-t)^\alpha - t^\alpha| f' \left(a + te^{i\varphi}\eta(b, a)\right) dt\right| \\
\leq e^{i\varphi}\eta(b, a) \int_0^1 |(1-t)^\alpha - t^\alpha| |f' \left(a + te^{i\varphi}\eta(b, a)\right)| dt
\]

Using power-mean inequality, we have

\[
|f(a) + f(a + e^{i\varphi}\eta(b, a)) - \frac{\Gamma(\alpha + 1)}{[e^{i\varphi}\eta(b, a)]^\alpha} \left[J_{a^\alpha} f(x) + J_{(a + e^{i\varphi}\eta(b, a))^\alpha} f(x)\right]| \\
\leq e^{i\varphi}\eta(b, a) \left(\int_0^1 |(1-t)^\alpha - t^\alpha|^q dt\right)^{1/q} \left(\int_0^1 \left|f' \left(a + te^{i\varphi}\eta(b, a)\right)\right|^q dt\right)^{1/q} \\
= e^{i\varphi}\eta(b, a) \left(\int_0^1 ((1-t)^\alpha - t^\alpha) dt + \int_0^1 (t^\alpha - (1-t)^\alpha) dt\right)^{1/q} \\
\times \left(\int_0^1 \left|f' \left(a + te^{i\varphi}\eta(b, a)\right)\right|^q dt\right)^{1/q}
\]

now using \(h_\varphi\)-preinvexity of \(|f'|^q\), we have

\[
\left|f(a) + f(a + e^{i\varphi}\eta(b, a)) - \frac{\Gamma(\alpha + 1)}{[e^{i\varphi}\eta(b, a)]^\alpha} \left[J_{a^\alpha} f(x) + J_{(a + e^{i\varphi}\eta(b, a))^\alpha} f(x)\right]\right| \\
\leq \frac{2^{1/q}}{\alpha + 1})^{1/p} e^{i\varphi}\eta(b, a) \left[1 - \frac{1}{2^\alpha}\right]^p \left(\int_0^1 \left|h(1-t)\right|^{\frac{\alpha}{1-q}} + h(t)\right|^{\frac{\alpha}{1-q}} \right) dt \\
= \left(2^{1/q} e^{i\varphi}\eta(b, a) \left[1 - \frac{1}{2^\alpha}\right]\right)^{1/p} \left\{\left[f'(a)\right]^{\frac{1}{\alpha-q}} + \left[f'(b)\right]^{\frac{1}{\alpha-q}}\right\} \int_0^1 \left|(1-t)^\alpha - t^\alpha\right| h(t) dt
\]

This completes the proof. \(\square\)

Now we have some special cases for (3.10).

I. If \(h(t) = t\), then we have the result for \(\varphi\)-preinvexity.

Corollary 7. Let \(I \subseteq \mathbb{R}\) be a open invex set with respect to \(\eta : I \times I \to \mathbb{R}\). Suppose that \(f : I \to \mathbb{R}\) is a differentiable function such that \(f' \in L_1[a, a + e^{i\varphi}\eta(b, a)]\). If \(\left|f'\right|^q\), \(q > 1\) is \(\varphi\)-preinvex on \(I\), then, for \(\eta(b, a) > 0\),

(3.11)

\[
\left|f(a) + f(a + e^{i\varphi}\eta(b, a)) - \frac{\Gamma(\alpha + 1)}{[e^{i\varphi}\eta(b, a)]^\alpha} \left[J_{a^\alpha} f(x) + J_{(a + e^{i\varphi}\eta(b, a))^\alpha} f(x)\right]\right| \\
\leq \left(2^{1/q} e^{i\varphi}\eta(b, a) \left[1 - \frac{1}{2^\alpha}\right]\right)^{1/p} e^{i\varphi}\eta(b, a) \left\{\left[f'(a)\right]^{\frac{1}{\alpha-q}} + \left[f'(b)\right]^{\frac{1}{\alpha-q}}\right\} \left[\frac{1}{\alpha + 2} \left[1 - \frac{1}{2^\alpha+2}\right] \right]^{\frac{\alpha}{1-q}}.
\]
II. If $h(t) = t^s$, then we have the result for $s_{\varphi}$-preinvexity.

**Corollary 8.** Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \to \mathbb{R}$. Suppose that $f : I \to \mathbb{R}$ is a differentiable function such that $f' \in L_1[a, a + e^{\varphi \eta}(b, a)]$. If $\left| f' \right|^q$, $q > 1$ is $s_{\varphi}$-preinvex on $I$, then, for $\eta(b, a) > 0$ and $s \in (0, 1]$,

\[
(3.12)
\]

\[
\left| f(a) + f(a + e^{\varphi \eta}(b, a)) - \frac{\Gamma(\alpha + 1)}{e^{\varphi \eta}(b, a)\alpha} \left[ J_1^\alpha f(x) + J_2^\alpha f(x) \right] \right|
\]

\[
\leq \left( \frac{2^{\alpha + 2} - 2}{2^{\alpha + 1} (\alpha + 1)} \right) e^{\varphi \eta}(b, a) \left\{ \left| f'(a) \right|^{\frac{\alpha}{\alpha + 1}} + \left| f'(b) \right|^{\frac{\alpha}{\alpha + 1}} \right\}.
\]

III. If $h(t) = 1$, then we have the result for $P_{\varphi}$-preinvexity.

**Corollary 9.** Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \to \mathbb{R}$. Suppose that $f : I \to \mathbb{R}$ is a differentiable function such that $f' \in L_1[a, a + e^{\varphi \eta}(b, a)]$. If $\left| f' \right|^q$, $q > 1$ is $P_{\varphi}$-preinvex on $I$, then, for $\eta(b, a) > 0$,

\[
(3.13)
\]

\[
\left| f(a) + f(a + e^{\varphi \eta}(b, a)) - \frac{\Gamma(\alpha + 1)}{e^{\varphi \eta}(b, a)\alpha} \left[ J_1^\alpha f(x) + J_2^\alpha f(x) \right] \right|
\]

\[
\leq \left( \frac{2^{\alpha + 2} - 2}{2^{\alpha + 1} (\alpha + 1)} \right) e^{\varphi \eta}(b, a) \left\{ \left| f'(a) \right|^{\frac{\alpha}{\alpha + 1}} + \left| f'(b) \right|^{\frac{\alpha}{\alpha + 1}} \right\}.
\]

Using the technique of [1], we prove the following result which helps us in proving our next results.

**Lemma 2.** Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \to \mathbb{R}$ where $\eta(b, a) > 0$. If $f'' \in L_1[a, a + e^{\varphi \eta}(b, a)]$, then

\[
(3.14)
\]

\[
\left[ e^{\varphi \eta}(b, a) \right]^2 \int_0^1 \left[ \frac{1 - (1 - t)^{\alpha + 1} - t^{\alpha + 1}}{\alpha + 1} \right] f''(a + te^{\varphi \eta}(b, a)) dt.
\]

**Proof.** Let

\[
\int_0^1 \left[ \frac{1 - (1 - t)^{\alpha + 1} - t^{\alpha + 1}}{\alpha + 1} \right] f''(a + te^{\varphi \eta}(b, a)) dt
\]

\[
= \left[ \frac{1 - (1 - t)^{\alpha + 1} - t^{\alpha + 1}}{\alpha + 1} \right] f'(a + te^{\varphi \eta}(b, a)) \bigg|_0^1 - \frac{1}{e^{\varphi \eta}(b, a)} \int_0^1 [(1 - t)^{\alpha} - t^{\alpha}] f'(a + te^{\varphi \eta}(b, a)) dt
\]

\[
= - \frac{1}{e^{\varphi \eta}(b, a)} \int_0^1 [(1 - t)^{\alpha} - t^{\alpha}] f'(a + te^{\varphi \eta}(b, a)) dt
\]

\[
= \frac{1}{(e^{\varphi \eta}(b, a))^2} \left[ \left| f(a) + f(a + e^{\varphi \eta}(b, a)) \right| - \frac{\Gamma(\alpha + 1)}{e^{\varphi \eta}(b, a)\alpha} \left[ J_1^\alpha f(x) + J_2^\alpha f(x) \right] \right]
\]

This completes the proof. □
Theorem 4. Let \( I \subseteq \mathbb{R} \) be a open invex set with respect to \( \eta : I \times I \to \mathbb{R} \). Let \( f : I \to \mathbb{R} \) be a twice differentiable function such that \( f'' \in L_1[a, a + e^i\varphi \eta(b, a)] \). If \( f'' \) is \( \varphi \)-preinvex on \( I \), then, for \( \eta(b, a) > 0 \),

\[
(3.15) \quad \left| f(a) + f(a + e^i\varphi \eta(b, a)) - \frac{\Gamma(\alpha + 1)}{[e^{i\varphi \eta(b, a)}]^{\alpha}} \left\{ J_{a^+}^\alpha f(x) + J_{(a + e^i\varphi \eta(b, a))}^\alpha f(x) \right\} \right|
\leq \left| e^{i\varphi \eta(b, a)} \right|^2 \left\{ \left| f''(a) \right| + \left| f''(b) \right| \right\} \int_0^1 \left[ \frac{1 - (1 - t)^\alpha - t^{\alpha + 1}}{\alpha + 1} \right] h(t)dt.
\]

Proof. Using Lemma 3, we have

\[
\left| f(a) + f(a + e^i\varphi \eta(b, a)) - \frac{\Gamma(\alpha + 1)}{[e^{i\varphi \eta(b, a)}]^{\alpha}} \left\{ J_{a^+}^\alpha f(x) + J_{(a + e^i\varphi \eta(b, a))}^\alpha f(x) \right\} \right|
= \left| e^{i\varphi \eta(b, a)} \right|^2 \int_0^1 \left[ \frac{1 - (1 - t)^\alpha - t^{\alpha + 1}}{\alpha + 1} \right] f''(a + te^i\varphi \eta(b, a))dt
\leq \left| e^{i\varphi \eta(b, a)} \right|^2 \int_0^1 \left[ \frac{1 - (1 - t)^\alpha - t^{\alpha + 1}}{\alpha + 1} \right] f''(a + te^i\varphi \eta(b, a))dt
\leq \left| e^{i\varphi \eta(b, a)} \right|^2 \int_0^1 \left[ \frac{1 - (1 - t)^\alpha - t^{\alpha + 1}}{\alpha + 1} \right] \left( h(1 - t) f''(a) + h(t) f''(b) \right) dt
= \left| e^{i\varphi \eta(b, a)} \right|^2 \left\{ \left| f''(a) \right| + \left| f''(b) \right| \right\} \int_0^1 \left[ \frac{1 - (1 - t)^\alpha - t^{\alpha + 1}}{\alpha + 1} \right] h(t)dt.
\]

This completes the proof. \(\square\)

Now, we discuss some special cases for (3.15).

I. If \( h(t) = t \), then we have the result for \( \varphi \)-preinvexity.

Corollary 10. Let \( I \subseteq \mathbb{R} \) be a open invex set with respect to \( \eta : I \times I \to \mathbb{R} \). Suppose that \( f : I \to \mathbb{R} \) be a twice differentiable function such that \( f'' \in L_1[a, a + e^i\varphi \eta(b, a)] \). If \( f'' \) is \( \varphi \)-preinvex on \( I \), then, for \( \eta(b, a) > 0 \),

\[
(3.16) \quad \left| f(a) + f(a + e^i\varphi \eta(b, a)) - \frac{\Gamma(\alpha + 1)}{[e^{i\varphi \eta(b, a)}]^{\alpha}} \left\{ J_{a^+}^\alpha f(x) + J_{(a + e^i\varphi \eta(b, a))}^\alpha f(x) \right\} \right|
\leq \left| e^{i\varphi \eta(b, a)} \right|^2 \left\{ \frac{\alpha}{2(\alpha + 1)(\alpha + 2)} \left( \left| f''(a) \right| + \left| f''(b) \right| \right) \right\}.
\]

II. If \( h(t) = t^\alpha \), then we have the result for \( s_\varphi \)-preinvexity

Corollary 11. Let \( I \subseteq \mathbb{R} \) be a open invex set with respect to \( \eta : I \times I \to \mathbb{R} \). Let \( f : I \to \mathbb{R} \) be a twice differentiable function such that \( f'' \in L_1[a, a + e^i\varphi \eta(b, a)] \). If \( f'' \) is \( s_\varphi \)-preinvex on \( I \), then, for \( \eta(b, a) > 0 \) and \( s \in (0, 1) \),

\[
(3.17) \quad \left| f(a) + f(a + e^i\varphi \eta(b, a)) - \frac{\Gamma(\alpha + 1)}{[e^{i\varphi \eta(b, a)}]^{\alpha}} \left\{ J_{a^+}^\alpha f(x) + J_{(a + e^i\varphi \eta(b, a))}^\alpha f(x) \right\} \right|
\leq \left| e^{i\varphi \eta(b, a)} \right|^2 \left\{ \left( \frac{1}{(\alpha + s + 2)(s + 1)} - \frac{B_\varphi(s + 1, \alpha)}{\alpha + 1} \right) \right\}.
\]
Corollary 12. Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta: I \times I \to \mathbb{R}$. Let $f: I \to \mathbb{R}$ be a twice differentiable function such that $f'' \in L^1[a, a + e^{i\varphi}\eta(b, a)]$. If $f''$ is $P_{\varphi}$-preinvex on $I$, then, for $\eta(b, a) > 0$,

\begin{equation}
(3.18)
|f(a) + f(a + e^{i\varphi}\eta(b, a)) - \frac{\Gamma(\alpha + 1)}{|e^{i\varphi}\eta(b, a)|^{\alpha}} \left\{ J_{a+}^\alpha f(x) + J_{(a+e^{i\varphi}\eta(b,a))}^\alpha f(x) \right\} | \leq \left[ e^{i\varphi}\eta(b, a) \right]^2 \left( \left| f''(a) \right|^q + \left| f''(b) \right|^q \right)^{\frac{1}{q}} \left( \int_0^1 h(t)dt \right)^{\frac{1}{q}}
\end{equation}

Theorem 5. Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta: I \times I \to \mathbb{R}$. Let $f: I \to \mathbb{R}$ be a twice differentiable function such that $f'' \in L^1[a, a + e^{i\varphi}\eta(b, a)]$. If $f''$ is $P_{\varphi}$-preinvex on $I$, then, for $\eta(b, a) > 0$,

\begin{equation}
(3.19)
|f(a) + f(a + e^{i\varphi}\eta(b, a)) - \frac{\Gamma(\alpha + 1)}{|e^{i\varphi}\eta(b, a)|^{\alpha}} \left\{ J_{a+}^\alpha f(x) + J_{(a+e^{i\varphi}\eta(b,a))}^\alpha f(x) \right\} | \leq \left[ e^{i\varphi}\eta(b, a) \right]^2 \left( 1 - \frac{1}{2^\alpha} \left( \left| f''(a) \right|^q + \left| f''(b) \right|^q \right)^{\frac{1}{q}} \left( \int_0^1 h(t)dt \right)^{\frac{1}{q}} \right)
\end{equation}

Proof. Using Lemma 3, we have

\begin{align*}
&|f(a) + f(a + e^{i\varphi}\eta(b, a)) - \frac{\Gamma(\alpha + 1)}{|e^{i\varphi}\eta(b, a)|^{\alpha}} \left\{ J_{a+}^\alpha f(x) + J_{(a+e^{i\varphi}\eta(b,a))}^\alpha f(x) \right\} | \\
&= \left[ e^{i\varphi}\eta(b, a) \right]^2 \int_0^1 \left[ \frac{1 - (1 - t)^{\alpha+1} - t^{\alpha+1}}{\alpha + 1} \right] f''(a + te^{i\varphi}\eta(b, a))dt \\
&\leq \left[ e^{i\varphi}\eta(b, a) \right]^2 \left( \int_0^1 \left[ \frac{1 - (1 - t)^{\alpha+1} - t^{\alpha+1}}{\alpha + 1} \right] dt \right)^{\frac{1}{q}} \left( \int_0^1 \left| f''(a + te^{i\varphi}\eta(b, a)) \right|^q dt \right)^{\frac{1}{q}} \\
&\leq \left[ e^{i\varphi}\eta(b, a) \right]^2 \alpha + 1 \left( 1 - \frac{1}{2^\alpha} \right) \left( \int_0^1 h(t) \left( \left| f''(a) \right|^q + \left| f''(b) \right|^q \right)dt \right)^{\frac{1}{q}} \\
&= \left[ e^{i\varphi}\eta(b, a) \right]^2 \alpha + 1 \left( 1 - \frac{1}{2^\alpha} \right) \left( \left| f''(a) \right|^q + \left| f''(b) \right|^q \right) \int_0^1 h(t)dt^{\frac{1}{q}}
\end{align*}

This completes the proof. \qed

We have some special cases for (3.19).

I. If $h(t) = t$, then we have the result for $\varphi$-preinvexity.

Corollary 13. Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta: I \times I \to \mathbb{R}$. Suppose that $f: I \to \mathbb{R}$ be a twice differentiable function such that $f'' \in L^1[a, a + e^{i\varphi}\eta(b, a)]$. If $f''$ is $P_{\varphi}$-preinvex on $I$, then, for $\eta(b, a) > 0$,
Corollary 14. Let \( L_1[a, a + e^{i\varphi} \eta(b, a)] \). If \( f'' \) is \( \varphi \)-preinvex on \( I \), then, for \( \eta(b, a) > 0 \),

\[
(3.20) \quad \left| f(a) + f(a + e^{i\varphi} \eta(b, a)) - \frac{\Gamma(\alpha + 1)}{[e^{i\varphi} \eta(b, a)]^\alpha} \left( J_0^\alpha f(x) + J_0^\alpha \left( f(x) - f(a) \right) \right) \right| \\
\leq \frac{\left[ e^{i\varphi} \eta(b, a) \right]^2}{\alpha + 1} \left( 1 - \frac{1}{2^\alpha} \right) \left( \frac{1}{2} \right)^{\frac{s}{4}} \left( \left| f''(a) \right|^q + \left| f''(b) \right|^q \right)^{\frac{1}{q}}
\]

II. If \( h(t) = t^s \), then we have the result for \( s_\varphi \)-preinvexity.

Corollary 15. Let \( I \subseteq \mathbb{R} \) be a open invex set with respect to \( \eta : I \times I \to \mathbb{R} \). Let \( f : I \to \mathbb{R} \) be a twice differentiable function such that \( f'' \in L_1[a, a + e^{i\varphi} \eta(b, a)] \). If \( f'' \) is \( s_\varphi \)-preinvex on \( I \), then, for \( \eta(b, a) > 0 \) and \( s \in (0, 1] \),

\[
(3.21) \quad \left| f(a) + f(a + e^{i\varphi} \eta(b, a)) - \frac{\Gamma(\alpha + 1)}{[e^{i\varphi} \eta(b, a)]^\alpha} \left( J_0^\alpha f(x) + J_0^\alpha \left( f(x) - f(a) \right) \right) \right| \\
\leq \frac{\left[ e^{i\varphi} \eta(b, a) \right]^2}{\alpha + 1} \left( 1 - \frac{1}{2^\alpha} \right) \left( \frac{1}{s + 1} \right)^{\frac{s}{4}} \left( \left| f''(a) \right|^q + \left| f''(b) \right|^q \right)^{\frac{1}{q}}
\]

III. If \( h(t) = 1 \), then we have the result for \( P_\varphi \)-preinvexity.

Theorem 6. Let \( I \subseteq \mathbb{R} \) be a open invex set with respect to \( \eta : I \times I \to \mathbb{R} \). Let \( f : I \to \mathbb{R} \) be a twice differentiable function such that \( f'' \in L_1[a, a + e^{i\varphi} \eta(b, a)] \). If \( f'' \) is \( P_\varphi \)-preinvex on \( I \), then, for \( \eta(b, a) > 0 \),

\[
(3.23) \quad \left| f(a) + f(a + e^{i\varphi} \eta(b, a)) - \frac{\Gamma(\alpha + 1)}{[e^{i\varphi} \eta(b, a)]^\alpha} \left( J_0^\alpha f(x) + J_0^\alpha \left( f(x) - f(a) \right) \right) \right| \\
\leq \frac{\left[ e^{i\varphi} \eta(b, a) \right]^2}{\alpha + 1} \left( \frac{\alpha}{\alpha + 1} \right)^{\frac{1}{q}} \left( \left| f''(a) \right|^q + \left| f''(b) \right|^q \right)^{\frac{1}{q}} \\
\times \left[ \left| f''(a) \right|^q + \left| f''(b) \right|^q \right]^{\frac{1}{q}} \int_0^1 \left[ \frac{1 - (1 - t)^{\alpha + 1} - t^{\alpha + 1}}{\alpha + 1} \right] h(t) dt
\]
Proof. Using Lemma 3 and well-known power-mean inequality, we have
\[
|f(a) + f(a + e^{i\varphi} \eta(b, a)) - \frac{\Gamma(\alpha + 1)}{[e^{i\varphi} \eta(b, a)]^\alpha} \{ J^\alpha_{a+} f(x) + J^\alpha_{(a+e^{i\varphi} \eta(b,a))} - f(x) \}| \\
= \left[ e^{i\varphi} \eta(b, a) \right]^2 \left[ \frac{1}{\alpha + 1} \int_0^1 \left[ 1 - (1-t)^{\alpha+1} - t^{\alpha+1} \right] f''(a + te^{i\varphi} \eta(b, a)) dt \right] \\
\leq \left[ e^{i\varphi} \eta(b, a) \right]^2 \left( \int_0^1 \left[ \frac{1}{\alpha + 1} \int_0^1 \left[ 1 - (1-t)^{\alpha+1} - t^{\alpha+1} \right] dt \right] \right) \frac{1}{\alpha + 1} \\
\times \left[ \int_0^1 \left[ \frac{1}{\alpha + 1} \int_0^1 \left[ 1 - (1-t)^{\alpha+1} - t^{\alpha+1} \right] dt \right] h(1-t) \left| f''(a) \right|^q + h(t) \left| f''(b) \right|^q \right] dt \right) \frac{1}{\alpha + 1} \\
\leq \left[ e^{i\varphi} \eta(b, a) \right]^2 \left( \frac{\alpha}{\alpha + 1} \right) \frac{1}{\alpha + 1} \left\{ \left[ \int_0^1 \left[ 1 - (1-t)^{\alpha+1} - t^{\alpha+1} \right] dt \right] \\
\times \left( h(1-t) \left| f''(a) \right|^q + h(t) \left| f''(b) \right|^q \right) dt \right) \frac{1}{\alpha + 1} \\
= \left[ e^{i\varphi} \eta(b, a) \right]^2 \left( \frac{\alpha}{\alpha + 1} \right) \frac{1}{\alpha + 1} \left\{ \left\{ \left| f''(a) \right|^q + \left| f''(b) \right|^q \right\} \right\} \\
\times \int_0^1 \left[ \frac{1}{\alpha + 1} \int_0^1 \left[ 1 - (1-t)^{\alpha+1} - t^{\alpha+1} \right] h(t) dt \right] \frac{1}{\alpha + 1} \\
\] 
\]

This completes the proof. \(\square\)

We have some special cases for (3.23).

I. If \(h(t) = t\), then we have the result for \(\varphi\)-preinvexity.

Corollary 16. Let \(I \subseteq \mathbb{R}\) be a open invex set with respect to \(\eta : I \times I \rightarrow \mathbb{R}\). Suppose that \(f : I \rightarrow \mathbb{R}\) be a twice differentiable function such that \(f'' \in L_1[a, a + e^{i\varphi} \eta(b, a)]\). If \(f''\) is \(\varphi\)-preinvex on \(I\), then, for every \(\eta(b, a) > 0\),
\[
(3.24) \\
\left| f(a) + f(a + e^{i\varphi} \eta(b, a)) - \frac{\Gamma(\alpha + 1)}{[e^{i\varphi} \eta(b, a)]^\alpha} \{ J^\alpha_{a+} f(x) + J^\alpha_{(a+e^{i\varphi} \eta(b,a))} - f(x) \} \right| \\
\leq \left[ e^{i\varphi} \eta(b, a) \right]^2 \left( \frac{1}{2} \right) \frac{\alpha}{\alpha + 1} \left\{ \left\{ \left| f''(a) \right|^q + \left| f''(b) \right|^q \right\} \right\} \\
\]

II. If \(h(t) = t^e\), then we have the result for \(s_{\varphi}\)-preinvexity

Corollary 17. Let \(I \subseteq \mathbb{R}\) be a open invex set with respect to \(\eta : I \times I \rightarrow \mathbb{R}\). Let \(f : I \rightarrow \mathbb{R}\) be a twice differentiable function such that \(f'' \in L_1[a, a + e^{i\varphi} \eta(b, a)]\). If
| $f''$ | is $s_{\varphi}$–preinvex on $I$, then, for every $\eta(b, a) > 0$ and $s \in (0, 1]$, 

| (3.25) |

| $f(a) + f(a + e^{i\varphi}\eta(b, a)) - \frac{\Gamma(\alpha + 1)}{e^{i\varphi}\eta(b, a)} \left\{ J^\alpha_a f(x) + J^\alpha_{(a + e^{i\varphi}\eta(b, a))} f(x) \right\}$ |

| $\leq \left[ e^{i\varphi}\eta(b, a) \right]^2 \left( \frac{\alpha}{\alpha + 1} \right)^{\frac{1}{p}} \times \left( \left\{ |f''(a)|^q + |f''(b)|^q \right\} \left( \frac{1}{s + 1} - \frac{B_{\frac{1}{p}}(s + 1, \alpha)}{\alpha + 1} \right) \right)^{\frac{1}{q}}$ |

III. If $h(t) = 1$, then we have the result for $P_{\varphi}$–preinvexity.

**Corollary 18.** Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \to \mathbb{R}$. Let $f : I \to \mathbb{R}$ be a twice differentiable function such that $f'' \in L_1[a, a + e^{i\varphi}\eta(b, a)]$. If $f''$ is $P_{\varphi}$–preinvex on $I$, then, for every $\eta(b, a) > 0$,

| (3.26) |

| $f(a) + f(a + e^{i\varphi}\eta(b, a)) - \frac{\Gamma(\alpha + 1)}{e^{i\varphi}\eta(b, a)} \left\{ J^\alpha_a f(x) + J^\alpha_{(a + e^{i\varphi}\eta(b, a))} f(x) \right\}$ |

| $\leq \left[ e^{i\varphi}\eta(b, a) \right]^2 \left( \frac{\alpha}{\alpha + 1} \right)^{\frac{1}{p}} \times \left( \left\{ |f''(a)|^q + |f''(b)|^q \right\} \right)^{\frac{1}{p}}$ |

**Conclusion 1.** If we take $\alpha = 1$ in (3.3)–(3.5), (3.7)–(3.9), (3.11)–(3.13), (3.16)–(3.18), (3.20) – (3.22), (3.24) – (3.26), we obtain Corollary 3.3 – 3.26 in [20].

**Conclusion 2.** If we take $\varphi = 0$ in (3.1), we obtain Lemma 2.3 in [4].

**Conclusion 3.** If we take $\alpha = 1$ and $\varphi = 0$ in our results, we get some Hermite-Hadamard inequalities.
References

[1] Barani, A.; Ghazanfari, A.G.; Dragomir, S.S.: Hermite-Hadamard inequality for functions whose derivatives absolute values are preinvex. J. Inequal. Appl. 2012, 247 (2012)
[2] Dragomir, S.S.; Pearce, C.E.M.: Selected topics on Hermite–Hadamard inequalities and applications. Victoria University (2000)
[3] Dragomir, S.S.; Pecaric, J.; Persson, L.E.: Some inequalities of Hadamard type. Soochow J. Math. 21, 335–341, (1995)
[4] İşcan, İ.: Hermite–Hadamard’s inequalities for preinvex function via fractional integrals and related functional inequalities, Am. J. Math. Anal. 1 (3) (2013) 33–38.
[5] Latif, M.A.: Some inequalities for differentiable prequasiinvex functions with applications. Konuralp J. Math. 1(2), 17–29 (2013)
[6] Latif, M.A.; Dragomir, S.S.: Some Hermite-Hadamard type inequalities for functions whose partial derivatives in absolute value are preinvex on the co-ordinates. Facta Universitatis (NIS) Ser. Math. Inform. 28(3), 257–270, (2013)
[7] Latif, M.A.; Dragomir, S. S.; Momani, E.: Some weighted integral inequalities for differentiable preinvex and prequasiinvex functions. RGMIA (2014)
[8] Noor, M.A.: On Hadamard integral inequalities involving two log-preinvex functions. J. Inequal. Pure Appl. Math. 8, 1–6 (2007)
[9] Noor, M.A.: On Hermite-Hadamard integral inequalities for product of two nonconvex functions. J. Adv. Math. Studies 2(1), 53–62 (2009)
[10] Noor, M.A.: Some new classes of nonconvex functions. Noor, M.A.; Awan, M.U.; Noor, K.I.: On some inequalities for relative semi-convex functions. J. InequaI. Appl., 2013, 332 (2013)
[11] Noor, M.A.; Noor, K.I.: Generalized preinvex functions and their properties. Journal of Appl. Math. Stochastic Anal., 2006(12736), 1–13, doi:10.1155/JAMSA/2006/12736
[12] Noor, M.A.; Awan, M.U.; Bashir, B.: Hermite–Hadamard inequalities for $h_p$-convex functions. Nonl. Anal. Formum. 18, 65–76 (2013)
[13] Noor, M.A.; Awan, M.U.: Fractional Hermite-Hadamard inequalities for some new classes of Godunova-Levin functions. Appl. Math. Inf. Sci. 8(6), 2865–2872, (2014)
[14] Noor, M.A.; Awan, M.U.; Li, J.: On Hermite–Hadamard type Inequalities for h-preinvex functions. Filomat, in press
[15] Noor, M.A.; Awan, M.U.; Al-Said, E.: Iterative methods for solving nonconvex equilibrium variational inequalities. Appl. Math. Inf. Sci. 6(1), 65–69, (2012)
[16] Noor, M.A.; Awan, M.U.; Khan, S.: Hermite–Hadamard type inequalities for differentiable $h_p$–preinvex functions. Arab. J. Math.4:63-76 (2015)
[17] Sarikaya, M.Z.; Set, E.: Özdemir M.E.: On some new inequalities of Hadamard type involving h-convex functions. Acta Math. Univ. Comenianae. 2, 265–272 (2010)
[18] Varosanec, S.: On h-convexity. J. Math. Anal. Appl. 326, 303–311 (2007)
[19] Weir, T.; Mond, B.: Preinvex functions in multiple objective optimization. J. Math. Anal. Appl. 136, 29–38 (1998)
[20] Samko, S.G.; Kilbas, A.A.; Marichev, O.I.: Fractional Integrals and Derivatives, Theory and Applications, Gordon and Breach, Yverdon, Switzerland, 1993
