Effective gravitational couplings of higher-rank supersymmetric gauge theories

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ABSTRACT: When placed on four-manifolds, \( \mathcal{N} = 2 \) gauge theories couple to topological invariants of the background via two functions \( A \) and \( B \). General considerations allow for these functions to be fixed in terms of the Coulomb moduli and other parameters in the theory, but only up to multiplicative factors about which little is known. We extend earlier work on the microscopic study of these functions in the \( \Omega \)-background to \( \mathcal{N} = 2^* \) gauge theories with higher-rank \( U(N) \) gauge groups. We complement this analysis by carrying out a perturbative study of these functions. This allows us to determine the manner in which these multiplicative factors scale with the rank of the gauge group and the mass of the adjoint hypermultiplet.

KEYWORDS: Supersymmetric Gauge Theory, Extended Supersymmetry, Nonperturbative Effects

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1 Introduction

Gauge theories on four-manifolds. The quest to understand the strong-coupling dynamics of four-dimensional gauge theories has been a steady source of motivation for many theoretical physicists over the last half-century. Significant among the numerous developments in this area of research is the seminal work of Seiberg and Witten [1, 2], where holomorphy and duality were used to essentially solve a strongly coupled (supersymmetric) gauge theory.

Let us briefly recall the form that this solution takes. At low energies, the Coulomb moduli space of these theories is parametrised by a set of gauge-invariant order parameters \( \{ u_i \} \) and its dimension is the rank \( r \) of the gauge group.\(^1\) A generic point on the Coulomb moduli space finds the gauge group broken down to its maximal torus, and the low-energy

\(^1\)Throughout this paper, any sums or products over lowercase Latin indices will be understood to run over the set \( \{ 1, \cdots, r \} \).
effective theory is one of \( r \) Abelian vector multiplets controlled by a holomorphic function of the \( \mathcal{N} = 2 \) superfield called the prepotential. This prepotential receives 1-loop exact perturbative corrections and an infinite series of corrections due to instantons, whose contributions may be computed using an algebraic curve. In essence, the Seiberg-Witten solution established an equivalence between the quantum moduli space of a supersymmetric gauge theory and the moduli space of an algebraic curve, which in turn paved the way for a complete solution specifying the low-energy dynamics of the theory.

Much subsequent attention has been paid to \( \mathcal{N} = 2 \) theories living at fixed points of renormalisation group flows, i.e. superconformal field theories with eight supercharges, henceforth abbreviated \( \mathcal{N} = 2 \) SCFTs. Of particular interest in the study of these theories is the determination of their conformal central charges \( a \) and \( c \). Cognisant of the limitations of earlier (more indirect) methods that leveraged S-duality or holography to compute these quantities, Shapere and Tachikawa [3] proposed a more direct and general method: since the conformal central charges characterise the response of an \( \mathcal{N} = 2 \) SCFT to a metric perturbation, they proposed to put these theories on curved four-manifolds using a topological twist and take advantage of the relationship between the \( U(1)_R \) anomalies of \( \mathcal{N} = 2 \) gauge theories and their topologically twisted counterparts.

Once again, it will be useful to — if only briefly — recall some details. (We refer the reader to [3] for a more comprehensive discussion.) On one hand, we have the topological twisting of an \( \mathcal{N} = 2 \) theory [4], which in the language of supergravity corresponds to switching on an \( SU(2)_R \) gauge field and setting it equal to the self-dual part of the spin connection. This has the happy consequence of leaving one component of the supercharge to transform as a scalar, whose cohomology may subsequently be used to define physical operators. On the other hand, the theories we consider have an \( \mathcal{N} = 2 \) superconformal algebra, whose \( R \)-current anomaly equation is specified in terms of the conformal central charges \( a \) and \( c \). These two facts put together [5–7] can be used to relate the integrated \( U(1)_R \) anomaly \( \Delta R \) of the vacuum to a linear combination of topological invariants of the four-manifold with coefficients that are functions of the conformal central charges:

\[
\Delta R = 2(2a - c) \chi + 3c \sigma.
\]  

(1.1)

In terms of the curvature 2-form \( R \) and its dual \( \tilde{R} \), the topological invariants \( \chi \) and \( \sigma \) above are the Euler characteristic

\[
\chi = \frac{1}{32\pi^2} \int \text{tr} \, R \wedge \tilde{R},
\]  

(1.2)

and the signature

\[
\sigma = \frac{1}{24\pi^2} \int \text{tr} \, R \wedge R.
\]  

(1.3)

The path integral of topologically twisted \( \mathcal{N} = 2 \) gauge theories on four-manifolds appears, too, in the physical approach to Donaldson invariants [4, 8]. Here, the generating function of Donaldson invariants receives contributions from the so-called \( u \)-plane integral, an integral over the Coulomb branch.\(^2\) Let \( X \) be a four-manifold and \( b_p \) be the dimension

\(^2\) There is also a contribution from points in the Coulomb moduli space associated to extra massless particles, which we will not discuss here.
of the space of harmonic $p$-forms on $X$. When referring to forms/spaces associated to (anti-)self-dual forms we will use the superscripts $(\pm)$. Now, for the case of $b_1(X) = 0$ and $b_2^+(X) = 1$, the $u$-plane integral takes the form

$$Z_u \sim \int [da \, d\bar{a}] \, A(u)^\chi B(u)^\sigma \, \Psi.$$  

(1.4)

up to some normalisation that renders the integral dimensionless. We have written the $u$-plane integral with reference to a specific choice of electric special coordinates. The (underlined) measure factor tells us how the low-energy effective theory couples to the topological invariants of the four-manifold we saw in eqs. (1.2) and (1.3). In terms of the dimensions of spaces of harmonic forms, we recall that these are simply $\chi = \sum_i (-1)^i b_i$ and $\sigma = b_2^+ - b_2^-$. Finally, the term $\Psi$ is associated to the photon partition function of the low-energy effective theory and takes the form of a lattice theta function and can depend on insertions.

The $u$-plane integral is significant not just to the study of four-manifold invariants, but also to $\mathcal{N} = 2$ SCFTs, as one of the key results established by [3] was the direct relationship between the central charges $a$ and $c$ on one hand, and the functions $A$ and $B$ on the other. The argument here relies on $R$-symmetry restoration at superconformal fixed points and goes as follows: say that the functions $A$ and $B$ are (somehow) already determined, and imagine sitting at a point on the Coulomb branch near a superconformal fixed point. The low-energy effective theory is one of $r$ vector multiplets and (possibly, say) $h$ neutral hypermultiplets, which contribute $(\chi + \sigma)/2$ and $\sigma/4$ respectively to the integrated $R$-charge. If we now imagine moving to the nearby superconformal fixed point, the contribution to the $U(1)_R$ anomaly of fields that go massless as we do so must be added in — this is precisely the $R$-charge of the measure factor in eq. (1.4). The total integrated $U(1)_R$ anomaly is now

$$\Delta R = \chi R(A) + \sigma R(B) + r \left( \frac{\chi + \sigma}{2} \right) + h \frac{\sigma}{4}.$$  

(1.5)

Together, eqs. (1.1) and (1.5) determine $a$ and $c$ for any $\mathcal{N} = 2$ SCFT that lives on the Coulomb branch of some $\mathcal{N} = 2$ gauge theory in four dimensions. Given the central importance of the functions $A$ and $B$ to the above argument, our attention in this paper will be focused on them.

The functions $A$ and $B$. The requirement of modular invariance of the path integral coupled with the observation that the $u$-plane integral measure has an anomaly under change of electric coordinates that must be compensated for by the measure factor is in fact sufficient to constrain the form of the $A$ function. Noting further that all free massless states contribute to $c$ allows us to conclude that $B$ should vanish on the loci corresponding to them. For a generic $\mathcal{N} = 2$ gauge theory, then, the functions $A$ and $B$ are expected [8–12] to take the form:

$$A = \alpha \left( \det \frac{du_i}{da_j} \right)^{1/2} \quad \text{and} \quad B = \beta \Delta^{1/8}.$$  

(1.6)
In the above expression, $\Delta$ is the “physical discriminant” which may in general be different from the usual “mathematical discriminant” of the Seiberg-Witten curve, for reasons clarified in [3]. First, the form of the Seiberg-Witten curve is not unique, and its different forms (with different mathematical discriminants as normally defined) may still yield the same low-energy physics. Second, the $B$ function is sensitive to massless states, which do not necessarily correspond to cycles of a Seiberg-Witten curve.

On general grounds, eq. (1.6) is as good as one can do, and the constants $\alpha$ and $\beta$ cannot be fixed from such general considerations. They may in general depend on the masses of hypermultiplets, or the cut-off scale $\Lambda$. While there are conjectures for the $N$-dependence of these coefficients for SU($N$) super-Yang-Mills due to [11], and arguments that $\alpha$ and $\beta$ are mass-independent for the case of asymptotically free theories due to [12], almost nothing else is known about the coefficients $\alpha$ and $\beta$, except in the case of SU(2) super-Yang-Mills theory, where one can compare with Donaldson invariants.

In particular, to determine $\alpha$ and $\beta$ one needs to specialise to a particular theory and a specific choice of gravitational background. This specialisation will also provide us with an opportunity to locate our motivations in recent and interesting work on these questions.

**Gravitational background.** Our work is a continuation and extension of the work of [13] in two respects. Like them, our choice of gravitational background will be the $\Omega$-background of $\mathbb{C}^2$ (equivalently, $\mathbb{R}^4$), familiar from the equivariant localisation computations of [14, 15]. Here, the scalar supercharge constructed via the topological twisting procedure we described earlier is not nilpotent but squares instead to an isometry of the $\Omega$-background, which in this case is independent rotations in the two 2-planes that are parametrised by the complex numbers $(\epsilon_1, \epsilon_2)$. The path integral of an $\mathcal{N} = 2$ gauge theory on such a background reduces to the computation of equivariant integrals on the moduli space of instantons. In the flat-space limit, one recovers using this technology the low-energy effective prepotential, allowing for a first-principles verification of the Seiberg-Witten solution.

The $\Omega$-deformed partition function contains much more information than just the prepotential; since the Euler characteristic and signature of this background are

$$\chi \left( \mathbb{C}^2 \right) = \epsilon_1 \epsilon_2 \quad \text{and} \quad \sigma \left( \mathbb{C}^2 \right) = \frac{\epsilon_1^2 + \epsilon_2^2}{3}, \quad (1.7)$$

one can read off the functions $A$ and $B$ from the exact deformed partition function as coefficients in the expansion

$$\epsilon_1 \epsilon_2 \log Z = -F + (\epsilon_1 + \epsilon_2) H + \epsilon_1 \epsilon_2 \log A + \frac{\epsilon_1^2 + \epsilon_2^2}{3} \log B + \cdots, \quad (1.8)$$

where $\cdots$ are terms that are of higher order in the $\Omega$-deformation parameters.

To summarise, our strategy will be very similar to [13]: there are two independent ways of computing the functions $A$ and $B$ — the first is using the predictions in eq. (1.6), while the second is using eq. (1.8) — and in comparing the two we will determine the coefficients $\alpha$ and $\beta$. In this manner [13] not only showed that the $A$ and $B$ functions are determined by eq. (1.6), but also computed the coefficients $\alpha$ and $\beta$ for a number of
rank-1 gauge theories. Our focus will be on higher-rank gauge theories, in particular the so-called \( \mathcal{N} = 2^* \) theories (or mass-deformed \( \mathcal{N} = 4 \) super-Yang-Mills theories) with gauge group \( U(N) \).

A novel aspect of our analysis will be our use of earlier work on the resummation of chiral correlators in this class of theories into quasimodular forms and, in particular, results stemming from the use of modular anomaly equations in \cite{16}. While we focus (in the interest of being explicit and concrete) on the case of \( N = 3 \), it is clear that similar methods may in principle be employed for any \( N \). We also carry out a perturbative analysis to determine the scaling dependence of the constants \( \alpha \) and \( \beta \) on the mass of the adjoint hypermultiplet and the rank of the gauge group. We also carefully clarify the role of \( U(1) \) factors in ensuring that there is consistency and perfect agreement between our results and those of \cite{13}.

**Organisation.** This paper is organised as follows. In section 2 we study the deformed partition function of \( \mathcal{N} = 2^* \) gauge theories with \( U(N) \) gauge group at a generic point on the Coulomb branch, first in general, and then specialising to the case of \( N = 3 \). Here, we read off the \( A \) and \( B \) functions from the flat-space expansion of the deformed partition function, writing them in an expansion in the mass \( m \) of the adjoint hypermultiplet, with coefficients resummed into products of quasimodular forms of the S-duality group of the theory and functions of the classical vacuum expectation values of the adjoint scalar reconstituted into root- and weight-lattice sums. In section 3 we compute the \( A \) and \( B \) functions from predictions using the Seiberg-Witten curve. We then compare the two to determine the multiplicative coefficients \( \alpha \) and \( \beta \). In section 4 we clarify the role of \( U(1) \) factors, which are crucial in ensuring perfect agreement between predictions and microscopic computations. Finally, in section 5 we perform a perturbative analysis of \( \mathcal{N} = 2^* \) theories for all \( N \), taking care to show that our results are consistent with earlier findings. We conclude in section 6 with a brief summary and some future directions of study. We collect a few technical details in two appendices.

## 2 Deformed partition functions and localisation

In this section we discuss the \( \Omega \)-deformed partition function \( Z \) of the \( \mathcal{N} = 2^* \) gauge theory with gauge group \( U(N) \). We will be studying this theory on the Coulomb branch, where the scalar field \( \phi \) in the vector multiplet acquires a vacuum expectation value (vev) that we will parametrise as

\[
a = \langle \phi \rangle = \text{diag} (a_1, \ldots, a_N) .
\]

The Yang-Mills coupling constant \( g^2 \) and the \( \theta \)-angle are packaged, as usual, into the complexified gauge coupling

\[
\tau = \frac{\theta}{2\pi} + \frac{4\pi}{g^2} ,
\]

and, with a view towards its appearance in the instanton sector, it will be useful to define the elliptic nome

\[
q = e^{2\pi i \tau} .
\]
The partition function of our theory admits a neat factorisation [14] into the product of three contributions: classical (c), perturbative (p) and instanton (np) contributions:

\[ Z = Z_c \times Z_p \times Z_{np} \]  

(2.4)

2.1 Classical

The classical contribution is given by

\[ Z_c = q^{-\frac{1}{\epsilon_1 \epsilon_2} \sum_i a_i^2} \]  

(2.5)

and this form follows from the classical prepotential that is quadratic in the \( N = 2 \) superfield. The appearance of the \((\epsilon_1 \epsilon_2)^{-1}\) is to ensure that when we do extract the prepotential \( F \) from the deformed partition function as

\[ F = -\lim_{\epsilon_1, \epsilon_2 \to 0} \epsilon_1 \epsilon_2 \log Z \]  

(2.6)

the classical piece contributes as it ought to. Since the \( A \) and \( B \) functions appear at higher orders in the expansion eq. (1.8) of the deformed partition function, it is clear at this stage that \( Z_c \) does not contribute to the functions we are interested in. Consequently, it will not feature very much in subsequent discussions except for section 4, where we will modify eq. (2.5) in a simple but important way.

2.2 Perturbative

The perturbative contribution is exact at one-loop [17, 18]. In terms of the shorthand \( \epsilon = \epsilon_1 + \epsilon_2 \), the contributions from the vector multiplet \( (v) \) and the adjoint hypermultiplet \( (h) \) are given by [14, 15]

\[ Z_{p,v} = \prod_{i<j} \exp \left[ -\gamma_{\epsilon_1, \epsilon_2} (a_i - a_j; \Lambda) - \gamma_{\epsilon_1, \epsilon_2} (a_i - a_j - \epsilon; \Lambda) \right] \]  

\[ Z_{p,h} = \prod_{i,j} \exp \left[ \gamma_{\epsilon_1, \epsilon_2} (a_i - a_j + m - \epsilon/2; \Lambda) \right] \]  

(2.7)

Here \( \Lambda \) is a cut-off scale and the function \( \gamma_{\epsilon_1, \epsilon_2}(x; \Lambda) \) is defined as

\[ \gamma_{\epsilon_1, \epsilon_2}(x; \Lambda) = \frac{d}{ds} \bigg|_{s=0} \frac{\Lambda^s}{\Gamma(s)} \int_0^\infty dt \, t^{s-1} \frac{e^{-xt}}{(e^{x\Lambda t} - 1)(e^{x\Lambda t} - 1)} \]  

(2.8)

It will be useful to, at this point, take note of a subtlety regarding the difference between \( U(N) \) and \( SU(N) \) theories. In the former, the tensor product of fundamental and anti-fundamental representations is simply the adjoint, whereas in the latter we get the trivial representation in addition to the adjoint. Consequently, in addition to imposing the tracelessness condition on \( a \) in eq. (2.1) we must account for this crucial difference. It may be verified that the perturbative contributions from the adjoint hypermultiplet in theories with these two gauge groups are related as

\[ Z_{p,h} [U(N)] = \exp \left[ \gamma_{\epsilon_1, \epsilon_2} (m - \epsilon/2) \right] Z_{p,h} [SU(N)] \]  

(2.9)

This will be crucial in ensuring that our results for the dependence of the constants \( \alpha \) and \( \beta \) on \( N \) and \( m \) are consistent with those previously derived in the literature, a point we will return to in section 5.
2.3 Nonperturbative

Let us now discuss the instanton contribution to the partition function. The instanton partition function is given by the following contour integral \[14, 19–22\]

\[
Z_{np} = 1 + \sum_{k=1}^{\infty} \frac{q^k}{k!} \oint \prod_{I=1}^{k} \frac{d\chi_I}{2\pi i} z_k,
\]

where the \(k\)-instanton integrand \(z_k\) receives contributions from the gauge and matter sectors depending on the theory under consideration. In our case, we have a vector multiplet and an adjoint hypermultiplet, so

\[
z_k = z_{k,v} \times z_{k,h},
\]

where

\[
z_{k,v} = (-1)^k \prod_{I,J} \left( \frac{\chi_{IJ} + \delta_{IJ}}{\chi_{IJ} + \epsilon_1} \right) \left( \chi_{IJ} + \epsilon_2 \right) \prod_{I=1}^{N} \prod_{j=1}^{\epsilon_1 \epsilon_2} \left[ -(\chi_I - a_j)^2 + \frac{\epsilon_1 \epsilon_2}{4} \right]^{-1},
\]

\[
z_{k,h} = \prod_{I,J} \left( \frac{\chi_{IJ} + \epsilon_1 + m}{\chi_{IJ} + \epsilon_2 + m} \right) \left( \chi_{IJ} + m \right) \prod_{I=1}^{N} \prod_{j=1}^{\epsilon_1 + 2m} \left[ -(\chi_I - a_j)^2 + \frac{(\epsilon_2 + 2m)^2}{4} \right].
\]

In the above equations, \(\chi_I\) denote the instanton moduli and \(\chi_{IJ} \equiv \chi_I - \chi_J\). The contour integral is computed by closing the contours in the upper-half \(\chi_I\) planes and the residues are picked using the following pole prescription \[21, 23\]

\[1 \gg \text{Im} \epsilon_1 \gg \text{Im} \epsilon_2 \gg 0 .\]

It was emphasised by \[13\] that the \(A\) and \(B\) functions are defined in the Donaldson-Witten twist, and so we must work with the shifted masses \(m \rightarrow m - \epsilon/2\). This shift in the mass parameter retains its significance for us as well. Note that we have already implemented this in the formulas for the perturbative contributions to the deformed partition function in eq. (2.7). For the instanton sector, all the results that we report henceforth are obtained after performing such a mass shift in eq. (2.12).

Finally, it is important to keep in mind the global \(U(1)\) factors that distinguish the partition functions of \(U(N)\) and \(SU(N)\) theories. These differences were first identified in \[24\] in the case of rank-1 theories and for \(U(N)\) theories \[25–27\] take the following form:

\[
Z_{np} [U(N)] = Z_{U(1)} Z_{np} [SU(N)],
\]

where

\[
Z_{U(1)} = \left[ \prod_{k=1}^{\infty} \left( 1 - q^k \right) \right] ^{-\frac{N}{\epsilon_1 \epsilon_2} \pi^2 (m^2 + \epsilon_1 \epsilon_2 - \epsilon^2/4)}.
\]

We mention this in the interest of completeness, should the reader wish to compare our own results with their own, and to point out that the Fourier expansion in eq. (2.15) will contribute nontrivially to the \(A\) and \(B\) functions. It should also be noted that the dynamics of the low-energy effective theory is insensitive to this difference, since eq. (2.15)
is independent of the Coulomb vevs. Having said that, this contribution will appear in the microscopic computation of the $A$ and $B$ functions, and we will have more to say about it in the following sections, both in the context of specific examples in section 2.4, and in general for arbitrary $N$ in section 4.

We now move onto a discussion of the $A$ and $B$ functions computed via localisation. In the interest of being concrete, we will present results for the case of $U(3)$ gauge theory. Nevertheless, should one wish to, it is in principle straightforward to extend these computations to higher $N$. Further, the checks coming from curve computations treat all $U(N)$ theories on equal footing, so once again explicit checks for arbitrary $N$ are in principle possible using the methodology we follow.

2.4 Localisation results: $N = 3$

In this section we focus on the gauge group $U(3)$ and compute $A$ and $B$ via localisation.

The prepotential $F$ which describes the low-energy effective theory on the Coulomb branch can in principle be computed to any order in the instanton expansion from the partition function using eq. (2.6). The resummation of the prepotential into quasimodular forms of the S-duality group has been extensively studied; in particular, modular anomaly equations that recursively determine the prepotential of $N = 2$ gauge theories order-by-order in a mass expansion have been worked out for all simply laced gauge algebras in [28] and non-simply laced gauge algebras in [29].

At the next order in the expansion eq. (1.8) we get the $H$ function and this turns out to be zero as in the $U(2)$ theory [13]

$$H = 0.$$ (2.16)

As we have noted earlier, the classical contribution to the deformed partition function $Z_c$ as defined in eq. (2.5) does not contribute to either $A$ or $B$. Consequently, we will only focus on the quantum (i.e. perturbative and instanton) contributions to both.

2.4.1 The $A$ function

Let us now consider the contribution to $\log A$ from the perturbative (p) sector. We make use of the perturbative contribution to the partition function from the vector multiplet as well as the hypermultiplet in the adjoint representation as given in eq. (2.7). This yields the following 1-loop contribution to $\log A$:

$$\log A_p = \frac{1}{2} \sum_{i<j} \log \frac{a_i - a_j}{\Lambda}.$$ (2.17)

As for the instanton (np) contributions to $\log A$, we organise them in a mass expansion:

$$\log A_{np} = \sum_{n=0} A_n(a_i, \tau) m^{2n}.\quad (2.18)$$

The instanton contributions can in fact be resummed into quasimodular forms of the S-duality group, which in the present case is $SL(2, \mathbb{Z})$. These forms multiply functions of the
Coulomb vevs, which are most compactly presented in terms of the following root- and weight-lattice sums, defined in \cite{16, 28, 29} as

\[
C_{\nu;m_1\ldots m_\ell}^p = \sum_{\lambda \in W} \sum_{\alpha \in \Psi_\lambda} \sum_{\beta_1 \neq \ldots \neq \beta_\ell \in \Psi_\alpha} \frac{(\lambda \cdot a)^p}{(\alpha \cdot a)^n (\beta_1 \cdot a)^{m_1} \ldots (\beta_\ell \cdot a)^{m_\ell}}. \tag{2.19}
\]

Here, $W$ is the set of weights of the fundamental representation of $U(N)$, and $\Psi_\lambda$ and $\Psi_\alpha$ are the subsets of the root system $\Psi$ defined, respectively, as

\[
\Psi_\lambda = \{ \alpha \in \Psi | \lambda \cdot \alpha = 1 \} \quad \text{for any} \quad \lambda \in W, \tag{2.20}
\]

and

\[
\Psi_\alpha = \{ \beta \in \Psi | \alpha \cdot \beta = 1 \} \quad \text{for any} \quad \alpha \in \Psi. \tag{2.21}
\]

In the interest of keeping the notation as light as possible, we will also adopt the convention that when $p = 0$ in eq. (2.19) we do not explicitly write the same, i.e. $C_{\nu;m_1\ldots m_\ell}^0 \equiv C_{\nu;m_1\ldots m_\ell}$.

We refer the reader to appendix A for more details.

With these definitions and conventions in place, the results of equivariant localisation can be resummed order-by-order in the mass of the adjoint hypermultiplet. Below, we show the first few orders in the mass expansion of the nonperturbative contribution to the $A$ function:

\[
\log A_{\text{np}} = \left( \frac{3}{2} q + \frac{9}{4} q^2 + 2 q^3 + \frac{21}{8} q^4 + O(q^5) \right) + \frac{m^4}{1152} (8C_4 - C_{2;11}) (E_2^3 - E_4) + \frac{m^6}{1080} \left[ \left( 5E_2^3 - 3E_2E_4 - 2E_6 \right) C_6 - \frac{1}{32} \left( 5E_2^3 - 27E_2E_4 + 22E_6 \right) (2C_{4;2} + C_{3;3}) \right] + O(m^8). \tag{2.22}
\]

We have not displayed the full Fourier expansions since they are unwieldy and not particularly enlightening. However, the above resummed expression in the second and third lines of eq. (2.22) has been verified to $k = 4$ instantons.

What about the first line above, written out explicitly as a Fourier expansion? The origin of this term is precisely the global $U(1)$ factor we encountered in eq. (2.15) with $N = 3$, as can be easily verified by computing the contribution of $Z_{U(1)}$ to the $A$ function. We will have more to say about this in section 4. For now, we note that this factor won’t persist if we were studying the SU(3) theory instead.

2.4.2 The $B$ function

Let us now consider the contribution to $\log B$ from the perturbative sector. We make use of the perturbative contribution to the partition function as given in eq. (2.7) and obtain the following:

\[
\log B_p = \frac{1}{2} \sum_{i < j} \log \frac{a_i - a_j}{\Lambda} + \frac{1}{8} \sum_{i,j} \log \frac{m + a_i - a_j}{\Lambda}. \tag{2.23}
\]
For the instanton sector, let us consider the following mass expansion for \( \log B \)

\[
\log B_{np} = \sum_{n=0} B_n(a_i, \tau) m^{2n},
\]

(2.24)

and, as with the \( A \) function, we find that it is possible to reconstitute the \( B_n(a_i, \tau) \) into quasimodular forms of the S-duality group multiplying combinations of lattice sums. Taking both perturbative and nonperturbative terms into account, we find

\[
\log B_{p+np} = -\left( \frac{9}{4} q + \frac{27}{8} q^2 + 3 q^3 + \frac{63}{16} q^4 + O(q^5) \right) + \frac{3}{8} \log \frac{m}{\Lambda} + \frac{3}{4} \sum_{i<j} \log \frac{a_i - a_j}{\Lambda} + \frac{m^2}{64} C_{2;11} \left( 3C^2 - (C^1)^2 \right) E_2
\]

\[
- \frac{m^4}{256} \left[ 4 (E_2^2 + E_4) C_4 + (E_2^2 - E_4) C_{2;11} \right]
\]

\[
- \frac{m^6}{34560} \left[ 8 (25 E_2^3 + 48 E_2 E_4 + 17 E_6) C_6 + 24 \left( 5 E_2^3 + 3 E_2 E_4 - 8 E_6 \right) C_{5;1}
\]

\[
- 5 \left( 5 E_2^3 - 3 E_2 E_4 - 2 E_6 \right) C_{2;22} \right] + O(m^8).
\]

(2.25)

Once again, we have elected not to explicitly write out the Fourier series.\(^3\) Nevertheless, as in the case of \( \log A \), we have verified the above resummed expression up to \( k = 4 \) instantons. Further, the first line in the above equation comes from the \( U(1) \) factor, as can be verified by computing the contribution of \( Z_{U(1)} \) in eq. (2.15) to \( \log B \). We will have more to say about this factor in section 4. For now, we note that had we elected to study the \( SU(3) \) theory instead, this factor would not have been present.

### 3 Seiberg-Witten curves and testing predictions

We’d like to compute the \( A \) and \( B \) functions using eq. (1.6). For this, we’ll need to compute the chiral correlators \( u_k = \langle \text{Tr} \Phi^k \rangle \) and the physical discriminant \( \Delta \). We compute these quantities using the corresponding Seiberg-Witten curve. On comparing these results with those of the previous section, we should be able to fix the constants \( \alpha \) and \( \beta \).

Before specialising to the case of \( N = 3 \), we briefly discuss (for general \( N \)) what the Seiberg-Witten curve of these theories looks like. For our purposes, the Donagi-Witten [31] form of the curve is most suitable, but it will be useful to understand different parametrisations and how they are related. In the Donagi-Witten formulation, the Seiberg-Witten curve of an \( U(N) \) gauge theory with a massive adjoint hypermultiplet is given by an \( N \)-fold cover of an elliptic curve of the form

\[
y^2 = (x - \varepsilon_1)(x - \varepsilon_2)(x - \varepsilon_3),
\]

(3.1)

\(^3\)Note that eq. (2.25) contains both perturbative and nonperturbative contributions. Therefore, at weak coupling (i.e. \( q \to 0 \)) it reduces to the small \( m \) expansion of eq. (2.23).
where the $e_i$ sum to zero and their pairwise differences are proportional to Jacobi $\theta$-constants (see appendix B for more details):

\begin{align}
  e_2 - e_3 &= \frac{1}{4} \theta_2(\tau)^4, \\
  e_2 - e_1 &= \frac{1}{4} \theta_3(\tau)^4, \\
  e_3 - e_1 &= \frac{1}{4} \theta_4(\tau)^4.
\end{align}

Expanding out eq. (3.1) and making use of the relation between the Jacobi $\theta$-constants and the Eisenstein series (reviewed in appendix B), one finds that the elliptic curve can also be written as

\begin{equation}
  y^2 = x^3 - \frac{E_4}{48} x + \frac{E_6}{864}.
\end{equation}

Now, under modular transformations we know that $E_{2k}$ has weight $2k$. For consistency, we must assign modular weights to $x$ (2) and $y$ (3) as well.

The $N$-fold cover of this curve takes the form

\begin{equation}
  F(t, x, y) = \sum_{n=0}^{N} (-1)^n A_n P_{N-n}(t, x, y) = 0.
\end{equation}

For consistency, $t$ is assigned unit modular weight. The quantities $A_n$ are chiral correlators that transform in a simple manner (covariantly, as weight-$n$ objects) under S-duality transformations and parametrise the Coulomb moduli space, and the $P_n$ are polynomials that are determined recursively using

\begin{equation}
  \frac{dP_n}{dt} = nP_{n-1},
\end{equation}

together with conditions on the growth of these polynomials near infinity. We refer the reader to [16] for a more detailed discussion of the curve and its different forms.

Before proceeding, we must clarify the manner in which the coefficients $A_k$ that appear in the Seiberg-Witten curve are related to the gauge invariant Coulomb moduli $u_k$, since we have closed-form expressions for the former (in a mass expansion) whereas the latter are related to the $A$ function via eq. (1.6).

### 3.1 Parametrising the Coulomb branch

Different forms of the Seiberg-Witten curve entail different parametrisations of the Coulomb moduli space. For example, we might choose to work with the D’Hoker-Phong [32] curve:

\begin{equation}
  H(t) = \prod_{i=1}^{N} (t - e_i) = \sum_{n=0}^{N} (-1)^n W_n t^{N-n},
\end{equation}

where the $e_i$ are understood to be quantum corrected Coulomb vevs in that at very weak coupling they reduce to $a_i$. In terms of these, the $W_n$ are

\begin{equation}
  W_n = \sum_{i_1 < \cdots < i_n} e_{i_1} \cdots e_{i_n},
\end{equation}

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and it is easy to see that such quantities can be used to build up our object of interest (the chiral correlators $u_k$) since

$$u_k = \left\langle \frac{1}{k} \text{Tr} \Phi^k \right\rangle = \frac{1}{k} \sum_{i=1}^{N} e_i^k. \quad (3.8)$$

Our task in this section will be to review the manner in which these three different parametrisations of the Coulomb branch are related to each other.

First, the D’Hoker-Phong ($W_n$) and Donagi-Witten ($A_n$) parametrisations are related linearly, and the map between the two is determined solely by quasimodular forms [16].

The relation between the two is

$$A_n = \sum_{\ell=0}^{\lfloor n/2 \rfloor} \left( \frac{N - n + 2\ell}{2\ell} \right) (2\ell - 1)!! \left( \frac{m^2 E_2}{12} \right)^{\ell} W_{n-2\ell}, \quad (3.9)$$

which admits a simple inversion:

$$W_n = \sum_{\ell=0}^{\lfloor n/2 \rfloor} (-1)^{\ell} \left( \frac{N - n + 2\ell}{2\ell} \right) (2\ell - 1)!! \left( \frac{m^2 E_2}{12} \right)^{\ell} A_{n-2\ell}. \quad (3.10)$$

Next, let us review the relation between elementary symmetric polynomials constructed out of the quantum corrected Coulomb vevs in eq. (3.7) and their corresponding power sums in eq. (3.8). The relation between the two can be written as a logarithmic generating function:

$$\sum_{k=1}^{\infty} (-1)^{k-1} t^k u_k = \log \left[ 1 + \sum_{k=1}^{\infty} t^k W_k \right], \quad (3.11)$$

which, on comparing terms, gives

$$u_1 = W_1,$$
$$u_2 = \frac{1}{2} \left( W_1^2 - 2W_2 \right), \quad (3.12)$$
$$u_3 = \frac{1}{3} \left( W_1^3 - 3W_1 W_2 + 3W_3 \right),$$

and so on. These relations will be crucial for us, since the prediction for the $A$ function in eq. (1.6) is given in terms of the $u_k$ while earlier work on resummation and S-duality naturally finds closed-form expressions (in a mass expansion) for the $W_k$ or $A_k$. To proceed, we will use eq. (3.11) to arrive at closed-form expression for the $u_k$ before computing $A$ via eq. (1.6).

For the rest of this section, in order to compare with the results of section 2.4, we specialise to the case of $N = 3$.

### 3.2 Chiral correlators and the $A$ function

Setting $N = 3$ in eq. (3.4) and using the expressions for the $P_n$ from [16], the curve for the $\mathcal{N} = 2^*$ theory with gauge group $U(3)$ can be rearranged as

$$F(t, x, y) = t^3 - A_1 t^2 + \left( A_2 - 3m^2 x \right) t - \left( A_3 - m^2 A_1 x - 2m^3 y \right) = 0. \quad (3.13)$$
where $A_k$ are gauge invariant coordinates on the Coulomb moduli space of the theory. The three independent $A_k$ for the $U(3)$ theory [16] are given in terms of the root- and weight-lattice sums in eq. (2.19) as

\begin{align}
A_1 &= C^1, \\
A_2 &= \frac{1}{2!} \left( (C^1)^2 - C^2 \right) + \frac{m^2}{4} E_2 + \frac{m^4}{288} (E_2^2 - E_4) C_2 + \frac{m^6}{4320} (5E_2^3 - 3E_2E_4 - 2E_6) C_4 \\
&\quad + \frac{m^6}{3456} (E_2^3 - 3E_2E_4 + 2E_6) C_{2;11} + O(m^8), \\
A_3 &= \frac{1}{3!} \left( (C^1)^3 - 3C^2 C^1 + 2C^3 \right) + \frac{m^2}{12} E_2 C^1 + \frac{m^4}{288} (E_2^2 - E_4) \left( C_2 C^1 - 2C_2^1 \right) \\
&\quad + \frac{m^6}{4320} (5E_2^3 - 3E_2E_4 - 2E_6) \left( C_1 C^1 - 2C_1^1 \right) \\
&\quad + \frac{m^6}{3456} (E_2^3 - 3E_2E_4 + 2E_6) \left( C_{2;11} C^1 - 2C_{2;11}^1 \right) + O(m^8),
\end{align}

(3.14)

As we reviewed in the Introduction, the $A$ function is expected to be [3, 8–11]

\begin{equation}
A = \alpha \left( \det \frac{du_j}{d\alpha_j} \right)^{1/2}.
\end{equation}

(3.15)

We can now use eq. (3.14) together with eqs. (3.10) and (3.11) to compute the $u_k$ and subsequently the $A$ function according to eq. (3.15). We computed the logarithm of this function and checked that the resulting expression matches the results for the same obtained via localisation in section 2.4 provided

\begin{equation}
\alpha = f_{\alpha} \Lambda^{-3/2},
\end{equation}

(3.16)

where $f_{\alpha}$ is some as-yet-undetermined function of the complexified gauge coupling $\tau$ that will ensure that the nonperturbative contribution from the $U(1)$ factor matches. Said differently, without $f_{\alpha}$ in the above equation there is a mismatch between log $A$ computed using eq. (3.15) and the same function computed microscopically as in section 2.4, and this mismatch is essentially due to the $U(1)$ factor in eq. (2.15). We will fix $f_{\alpha}$ in section 4.

Note that the analysis of [16] treats all $U(N)$ gauge theories on equal footing, and it is in principle possible to perform a similar computation for any $N$. Indeed, in section 5 we will perform a perturbative analysis that will determine the scaling of $\alpha$ with $N$.

We now move onto a discussion of the $B$ function.

### 3.3 Discriminants and the $B$ function

Having demonstrated how one might go about computing the $A$ function for a higher-rank gauge theory, we now turn to the computation of the discriminant. The strategy we adopt is to eliminate the variables $x$ and $y$ in eq. (3.13) and write down a polynomial equation in $t$. After taking care of a few subtleties, the discriminant of this polynomial is the relevant “physical” discriminant.
We eliminate $y$ from eq. (3.13) using eq. (3.1). This substitution leaves us with the following polynomial in $(t, x)$,

$$Q = (t^3 - A_1 t^2 + t(A_2 - 3m^2 x) - A_3 + m^2 A_1 x)^2 - 4m^8(x - \epsilon_1)(x - \epsilon_2)(x - \epsilon_3). \quad (3.17)$$

The discriminant of the Seiberg-Witten curve captures, as usual, the singular locus of the curve. Singularities arise when the following conditions are satisfied

$$\frac{\partial Q}{\partial t} = \frac{\partial Q}{\partial x} = Q = 0. \quad (3.18)$$

The first of these requirements leads to

$$x = \frac{3t^2 - 2tA_1 + A_2}{3m^2} \quad (3.19)$$

and we use this to eliminate $x$ from $Q$ in eq. (3.17). This yields a polynomial $H(t)$. The singularities of the curve should satisfy the conditions $H = \partial_t H = 0$. This, however, does not imply that all the zeroes of these equations correspond to singularities of the curve. For example, in this case we see that $t = A_1/3$, which is not a singularity of the curve, is a solution of $\partial_t H = 0$. Thus, when we compute the discriminant of the polynomial $H(t)$, we discard this factor and refer to the remaining part as the “physical” discriminant $\Delta$ that is sensitive only to the singularities associated to massless particles. The factor that ought to be discarded is an overall multiplicative factor in the “mathematical” discriminant and hence can be removed easily. Further, as in [31], the explicit form of $\Delta$ is quite complicated but in the weak coupling limit and with tracelessness condition $\sum_i a_i = 0$ it reduces to

$$(4A_2^2 + 27A_3^2)^2 \left(4(A_2 - m^2) (A_2 - 4m^2)^2 + 27A_3^2\right). \quad (3.20)$$

The $B$ function is expected to be related to the discriminant as [3, 8–11]

$$B = \beta \Delta^{1/8}. \quad (3.21)$$

Once we have the correct discriminant $\Delta$, we can easily compute the $B$ function according to eq. (3.21). We computed the logarithm of this function and checked that we have agreement between the two computations provided we set

$$\beta = f_\beta m^3 \Lambda^{-\frac{21}{8}}, \quad (3.22)$$

where, just as in the case of the $A$ function, the factor $f_\beta$ in the above equation is some as-yet-undetermined function of the complexified gauge coupling $\tau$ that will ensure that the nonperturbative contribution from the U(1) factor matches. We will fix $f_\beta$ in the next section.

In principle, as with the previous section, the procedure followed in this section can be repeated for higher-rank gauge theories.

---

\footnote{This could also imply $t^3 - 3m^2 tx - t^2 A_1 + x^2 A_1 + t A_2 - A_3 = 0$. However, this condition is not compatible with $Q = \partial_t Q = 0$ when all the $\epsilon_i$ are distinct.}
4 U(1) factors

In this section we discuss in greater detail the U(1) factors and the manner that they affect determination of coefficients $\alpha$ and $\beta$ associated to the $A$ and $B$ functions through the as-yet-unspecified functions $f_\alpha$ and $f_\beta$ we saw in the previous section. Our discussion of these factors is influenced by the treatment of U(1) factors for gauge theories with rank-1 gauge groups in [30] and our goal in this section is to carry out a similar program for the higher-rank gauge theories we discuss in this paper.\footnote{We are grateful to Jan Manschot for emphasising this.}

Let us begin by rewriting the U(1) factor in eq. (2.15) in terms of the Dedekind $\eta$-function (see appendix B for its definition) as

$$Z_{U(1)} = \left[ q^{1/24} \frac{N^{\frac{2}{3}}}{\eta(\tau)} \right]^{\frac{N}{2} m^2 + e_1 e_2 - e^2/4}.$$  \hspace{1cm} (4.1)

We can compensate for the explicit appearance of the elliptic nome $q$ in the above expression by starting with a classical partition function that is slightly different. That is, had we started with

$$Z_c = Z_c \times q^{-N} \left( \frac{N}{2} m^2 + e_1 e_2 - e^2/4 \right),$$  \hspace{1cm} (4.2)

where $Z_c$ is in eq. (2.5), we will be able to express the effect of the U(1) factor on the constants $\alpha$ and $\beta$ solely in terms of the Dedekind $\eta$-function, as in [30]. Note that this redefinition does not affect the dynamics of the theory, since it is independent of the Coulomb vevs.

Assuming that the classical piece chosen is $Z_c$ in eq. (4.2), we can now expand the logarithm of the product of modified classical ($c'$) and U(1) factors around flat space as we did with eq. (1.8), and from this we can read off its contribution to the prepotential $F$, as well as the $A$ and $B$ functions. Focusing solely on the non-dynamical pieces, i.e. pieces that are independent of the Coulomb vevs, we find:

$$\epsilon_1 \epsilon_2 \log \left[ Z' Z_{U(1)} \right] \bigg|_{\epsilon_1 = 0} = -N m^2 \log \eta(\tau) - N \left( \epsilon_1 e_2 - \frac{e^2}{4} \right) \log \eta(\tau).$$  \hspace{1cm} (4.3)

The first term on the right-hand side in the above equation is precisely the $\alpha$-independent factor that was neglected in [28]. From the second term we can read off the contribution of the U(1) factor to the prepotential $F$ of the gauge theory. For $N = 2$, the power of $\eta$ corresponding to $C$ matches too, as can be read off from the first term on the right-hand side of eq. (4.3).

\footnote{The results of [30] include a third coupling $C$; this is essentially the contribution of the U(1) factor to the prepotential $F$ of the gauge theory. For $N = 2$, the power of $\eta$ corresponding to $C$ matches too, as can be read off from the first term on the right-hand side of eq. (4.3).}
5 Perturbative analysis of $\mathcal{N} = 2^*$ U($N$) theories

In this section, we will compute the perturbative contributions to the $A$ and $B$ functions in $\mathcal{N} = 2^*$ theories with U($N$) gauge group where $N$ is arbitrary. We will then use this to determine the dependence of the constants $\alpha$ and $\beta$ on the rank $N$ of the gauge group, the ultraviolet cut-off scale $\Lambda$, and the mass $m$ of the adjoint hypermultiplet.\footnote{It was argued in \[12\] that for asymptotically free theories $\alpha$ and $\beta$ are independent of mass parameter. This doesn’t apply, however, to the theory we are presently discussing.} Little is known about these constants for $\mathcal{N} = 2^*$ theory except for the SU(2) gauge group \[13, 33\] and we will use their results as a consistency check.

The perturbative contribution to the partition function is given in eq. (2.7). If we expand eq. (2.7) around the flat space limit i.e. $\epsilon_{1,2} \to 0$, we obtain the following perturbative contribution

$$
\log A_p = \frac{1}{2} \sum_{i<j} \log \left( \frac{a_i - a_j}{\Lambda} \right), \\
\log B_p = \frac{1}{2} \sum_{i<j} \log \left( \frac{a_i - a_j}{\Lambda} \right) + \frac{1}{8} \sum_{i,j} \log \left( \frac{m + a_i - a_j}{\Lambda} \right).
$$

(5.1)

Let us now compute the perturbative contribution to the $A$ and $B$ functions from the SW curve. The perturbative contribution to the $u_k$ variables is given by eq. (3.8) in the limit $q \to 0$, which is the limit in which the quantum corrected Coulomb vevs match their classical values, so

$$
|u_k|_p = \frac{1}{k} \sum_i a_i^k.
$$

(5.2)

The entries of the matrix appearing in eq. (3.15) are then straightforwardly found to be

$$
\left( \frac{du_i}{da_j} \right)_p = a_j^{i-1}.
$$

(5.3)

This matrix is well-known — it is the Vandermonde matrix — and so its determinant takes the familiar form

$$
\det \left( \frac{du_i}{da_j} \right)_p = \prod_{i<j} (a_i - a_j).
$$

(5.4)

Next, the perturbative discriminant receives a contribution due to the singularities coming from both the vector and adjoint hypermultiplet \[31\] in the form

$$
\Delta_p = \Delta_{p,v}^2 \Delta_{p,h},
$$

(5.5)

where

$$
\Delta_{p,v} = \prod_{i<j} (a_i - a_j)^2, \\
\Delta_{p,h} = \prod_{i \neq j} (a_i - a_j + m) = (-1)^{N(N+1)/2} \prod_{i<j} \left( (a_i - a_j)^2 - m^2 \right).
$$

(5.6)
Using eqs. (5.4) and (5.5) we can compute $\log A$ and $\log B$ using eqs. (3.15) and (3.21). We then compare this with eq. (5.1), which allows us to determine the dependence of the constants $\alpha$ and $\beta$ on the rank $N$ of the gauge group and the mass $m$ of the adjoint hypermultiplet as

$$
\alpha_{U(N)} = \Lambda^{\frac{N(N-1)}{4}} \quad \text{and} \quad \beta_{U(N)} = m^{\frac{N}{8}} \Lambda^{\frac{N(N-2)}{8}}.
$$

(5.7)

As a consistency check, we can ask if the above expressions match the results of [13, 33] for the case of gauge group $SU(2)$. To see this, recall that in the case of $SU(N)$ gauge groups the perturbative contribution to the $\Omega$-deformed partition function from the adjoint hypermultiplet in eq. (2.7) should be divided by a factor of $\exp[\gamma_{1,2}(m - \epsilon/2; \Lambda)]$, as was discussed in section 2.2. This leaves $\log A$ unchanged but affects $\log B$ and correspondingly we find

$$
\beta_{SU(N)} = m^{\frac{N}{8}} \Lambda^{\frac{(3N+1)(N-1)}{8}}.
$$

(5.8)

For the case of $N = 2$, the above formula correctly reproduces the scaling for $\beta$ in the $\mathcal{N} = 2^*$ theory with $SU(2)$ gauge group (upto some numerical factors) as given in [13].

6 Conclusions

In this paper, we studied the low-energy effective couplings $A$ and $B$ of $\mathcal{N} = 2$ supersymmetric gauge theories to the topological invariants of a four-manifold, which in our case was the $\Omega$-background. As in [13], we used the deformed partition function on this background to explicitly calculate the quantities $A$ and $B$. We then compared this with results from the corresponding Seiberg-Witten curves, taking particular care to treat $U(1)$ factors. Our findings generalise the results of [13] to $\mathcal{N} = 2^*$ theories with higher-rank $U(N)$ gauge groups. While we have explicitly worked out the case of $N = 3$, it is clear that the methodology we have employed will continue to work for higher $N$. To complement our findings, we have also presented a perturbative study of the $A$ and $B$ functions in these theories and determined the manner in which the multiplicative constants $\alpha$ and $\beta$ scale with the rank of the gauge group and the mass of the adjoint hypermultiplet.

Looking ahead, it would be interesting to explore the possibility of using alternative methods — such as the Eynard-Orantin topological recursion [34] or the theory of qq-characters [25, 27, 35–37] — to derive all-instanton results. We also note that [13] considered the case of the $SU(2)$ theory with $N_f = 4$ fundamental hypermultiplets. The extension of this analysis to superconformal SQCD theories with gauge group $SU(N)$ and $N_f = 2N$ fundamental hypermultiplets is more complicated. Nevertheless, there are results for $N = 3$ in [38, 39] and results for general $N$ in [40], which may provide a suitable starting point for such an analysis. Another possible direction would be the extension to other gauge algebras, where accidental isomorphisms at low rank will provide consistency checks. We hope to return to some of these questions in the future.
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A Lattice sums

For the unitary gauge group $U(N)$ the root system of the corresponding gauge algebra $u(N)$ is given in terms of an set of vectors $\{e_i\}$ in $\mathbb{R}^N$ where $i$ runs over $\{1, \cdots , N\}$ such that $e_i \cdot e_j = \delta_{ij}$, i.e. the basis is orthonormal. In terms of this, the roots are the set

$$\{ \pm (e_i - e_j) \mid 1 \leq i < j \leq N \}.$$ (A.1)

This basis can be used to expand out the lattice sums in eq. (2.19).

For example:

$$C_n = \sum_{i \neq j} \frac{1}{(a_i - a_j)^n},$$ (A.2)

which, as is easily verified, is identically zero for $n$ odd. As another example, for $N = 3$ we have:

$$C_{2,11} = 4 \frac{a_1 a_2 + a_1 a_3 + a_2 a_3 - a_1^2 - a_2^2 - a_3^2}{(a_1 - a_2)^2 (a_1 - a_3)^2 (a_2 - a_3)^2}.$$ (A.3)

B Modular forms

In this appendix, we give some details of the modular forms that appeared in the main text of the paper.

The Dedekind $\eta$-function is defined in terms of an infinite product as

$$\eta(\tau) = q^{1/24} \prod_{k=1}^{\infty} (1 - q^k).$$ (B.1)

The Jacobi $\theta$-functions are defined as

$$\theta \left[ \begin{array}{c} a \\ b \end{array} \right] (v|\tau) = \sum_{n \in \mathbb{Z}} e^{\pi i v(n - \frac{a}{2})^2 + 2\pi i (n - \frac{a}{2}) (v - \frac{b}{2})},$$ (B.2)

for $a, b = 0, 1$. We also have Jacobi $\theta$-constants that are defined as follows

$$\theta_2(\tau) \equiv \theta \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] (0|\tau), \quad \theta_3(\tau) \equiv \theta \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] (0|\tau), \quad \theta_4(\tau) \equiv \theta \left[ \begin{array}{c} 0 \\ 1 \end{array} \right] (0|\tau).$$ (B.3)
The first few terms in the Fourier expansion of these $\theta$-constants are as follows

$$\begin{align*}
\theta_2(\tau) &= 2q^{1\over 8}\left(1 + q + q^3 + q^6 + \cdots\right), \\
\theta_3(\tau) &= 1 + 2q^{1\over 8} + 2q^{2\over 8} + 2q^{9\over 8} + \cdots, \\
\theta_4(\tau) &= 1 - 2q^{1\over 8} + 2q^{2\over 8} - 2q^{9\over 8} + 2q^{8\over 8} + \cdots.
\end{align*}$$

(B.4)

where $q = e^{2\pi i \tau}$.

The Eisenstein series $E_{2k}$ are holomorphic functions on the upper-half plane, defined as

$$E_{2k}(\tau) = \frac{1}{2\zeta(2n)} \sum_{m,n \in \mathbb{Z}^2 \setminus \{0,0\}} \frac{1}{(m+n\tau)^{2k}},$$

(B.5)

which makes explicit that under an SL(2, Z) transformation of the argument, the Eisenstein series transform covariantly with weight $2k$. In terms of the generators $T$ and $S$ of the modular group that act on the upper-half plane coordinate $\tau$ via fractional linear transformations, the Eisenstein series (for $k \geq 2$) transform as

$$\begin{align*}
T & : \quad E_{2k}(\tau + 1) = E_{2k}(\tau), \\
S & : \quad E_{2k}(\tau + \tau') = \tau'^{2k} E_{2k}(\tau).
\end{align*}$$

(B.6)

In particular, the $T$-invariance of the Eisenstein series tells us that they admit a Fourier series; the Fourier expansion of the first few Eisenstein series are

$$\begin{align*}
E_4(\tau) &= 1 + 240q + 2160q^2 + 6720q^3 + 17520q^4 + \cdots, \\
E_6(\tau) &= 1 - 504q - 16632q^2 - 122976q^3 - 532728q^4 + \cdots.
\end{align*}$$

(B.7)

The case of $k = 1$ is special: under an $S$ transformation, it transforms anomalously as

$$E_2(-1/\tau) = \tau^2 E_2(\tau) + \frac{6}{4\pi} \tau,$$

(B.8)

and is therefore referred to as a quasimodular form. It, too, has a Fourier expansion:

$$E_2(\tau) = 1 - 24q - 72q^2 - 96q^3 - 168q^4 - \cdots.$$ 

(B.9)

The Eisenstein series of weights 4 and 6 can be written as polynomials in the Jacobi $\theta$-constants:

$$\begin{align*}
E_4(\tau) &= \frac{1}{2} \left( \theta_2(\tau)^8 + \theta_3(\tau)^8 + \theta_4(\tau)^8 \right), \\
E_6(\tau) &= \frac{1}{2} \left( \theta_3(\tau)^4 + \theta_4(\tau)^4 \right) \left( \theta_2(\tau)^4 + \theta_3(\tau)^4 \right) \left( \theta_4(\tau)^4 - \theta_2(\tau)^4 \right).
\end{align*}$$

(B.10)
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