The spread of the spectrum of a nonnegative matrix with a zero diagonal element

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Abstract

Let $A = [a_{ij}]_{i,j=1}^n$ be a nonnegative matrix with $a_{11} = 0$. We prove some lower bounds for the spread $s(A)$ of $A$ that is defined as the maximum distance between any two eigenvalues of $A$. If $A$ has only two distinct eigenvalues, then $s(A) \geq \frac{n}{2(n-1)} r(A)$, where $r(A)$ is the spectral radius of $A$. Moreover, this lower bound is the best possible.

Keywords: nonnegative matrices, spectrum, spread

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1. Introduction

Let $A$ be a complex $n \times n$ matrix with the spectrum $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. The spectral radius and the trace of $A$ are denoted by $r(A)$ and $\text{tr}(A)$, respectively. The spread $s(A)$ of $A$ is the maximum distance between any two eigenvalues, that is, $s(A) = \max_{i,j} |\lambda_i - \lambda_j|$. This quantity was introduced by Mirsky [4], and it has been studied by several authors; see e.g. [3] and the references therein. Note that $s(\lambda A) = |\lambda|s(A)$ for every complex number $\lambda$ and that the spread of a nilpotent matrix is zero. Thus, when studying the

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spread of a matrix $A$, there is no loss of generality in assuming that $r(A) = 1$.

Let $C_n$ (with $n \geq 2$) be the collection of all nonnegative $n \times n$ matrices $A = [a_{ij}]_{i,j=1}^n$ such that $a_{11} = 0$ and $r(A) = 1$. It is not difficult to prove (see e.g. Proposition 2.1) that the spread of a matrix $A \in C_n$ cannot be zero, that is, the number 1 cannot be the only point in the spectrum of $A$. This motivates searching for lower bounds for the spread of $A$. If $A$ has only two distinct eigenvalues, we prove that $s(A) \geq \frac{n}{2(n-1)}$, and we provide a matrix for which this lower bound is achieved. Such a matrix is necessarily irreducible, that is, there exists no permutation matrix $P$ such that

$$P^T AP = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

where $A_{11}$ and $A_{22}$ are square matrices.

2. Results

We start with an easy observation.

**Proposition 2.1.** Let $A$ be a nonnegative $n \times n$ matrix with the spectral radius $r(A) = 1$. If $A$ has $k$ zero diagonal elements, then

$$s(A) \geq \frac{k}{n}.$$  

In particular, if $A \in C_n$ then

$$s(A) \geq \frac{1}{n}.$$  

**Proof.** Since $A$ is a nonnegative matrix, the spectral radius $r(A) = 1$ is its Perron eigenvalue. We denote it by $\lambda_1$, while the rest eigenvalues of $A$ are denoted by $\lambda_2, \lambda_3, \ldots, \lambda_n$. For every $i = 1, 2, \ldots, n$ we have

$$\text{Re}(1 - \lambda_i) \leq |1 - \lambda_i| = |\lambda_1 - \lambda_i| \leq s(A),$$

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and so $1 - s(A) \leq \Re \lambda_i$. It follows that

$$n(1 - s(A)) \leq \sum_{i=1}^{n} \Re \lambda_i = \sum_{i=1}^{n} \lambda_i = \text{tr} (A).$$

However, $\text{tr} (A) = \sum_{i=1}^{n} a_{ii} \leq n - k$, as $A$ has $k$ zero diagonal elements and $a_{ii} \leq r(A) = 1$ for all $i$. We thus obtain that $n(1 - s(A)) \leq n - k$, and so $ns(A) \geq k$ as asserted. \qed

Applying the known inequalities of Johnson, Loewy and London we will prove a better result for matrices in $C_n$. Let $A$ be a nonnegative $n \times n$ matrix and let $s_k := \text{tr} (A^k)$ for $k \in \mathbb{N}$. The JLL-inequalities (discovered independently by Loewy and London [2], and Johnson [1]) state that

$$s_k^m \leq n^{m-1} s_{km}$$

for all positive integers $k$ and $m$. A slight modification of their proof gives the following inequalities.

**Proposition 2.2.** Let $A$ be a nonnegative $n \times n$ matrix with $k$ zero diagonal elements. Then

$$s_1^m \leq (n - k)^{m-1} s_m$$

for all $m \in \mathbb{N}$. In particular, if $A \in C_n$ then

$$s_1^m \leq (n - 1)^{m-1} s_m$$

for all $m \in \mathbb{N}$.

**Proof.** Since $A$ is a nonnegative matrix, we have

$$s_m = \text{tr} (A^m) \geq \sum_{i=1}^{n} a_{ii}^m = \sum_{i \in J} a_{ii}^m,$$
where \( J = \{ i \in \{ 1, 2, \ldots , n \} : a_{ii} > 0 \} \). On the other hand, Hölder’s inequality gives

\[
s_1^m \leq \left( \sum_{i \in J} a_{ii} \right)^m \leq (n - k)^{m-1} \sum_{i \in J} a_{ii}^m,
\]

and so we conclude that \( s_1^m \leq (n - k)^{m-1} s_m \). \( \square \)

Using Proposition 2.2 we prove the following lower estimates for the spread of a matrix in \( \mathcal{C}_n \).

**Theorem 2.3.** If \( A \in \mathcal{C}_n \) then

\[
s(A) > \frac{2}{4 + \sqrt{2(n + 3)}}
\]

for \( n \geq 6 \),

\[
s(A) \geq \frac{5}{8 + \sqrt{74}}
\]

for \( n = 5 \), and

\[
s(A) \geq \frac{1}{3}
\]

for \( n = 4 \).

**Proof.** Since \( s(A) > 0 \) by Proposition 2.1 and since the result is true if \( s(A) \geq 1 \), we may assume that \( s := s(A) \in (0, 1) \), and consequently the eigenvalues of \( A \) have positive real parts. Let \( \lambda_1 = r(A) = 1, \lambda_2, \lambda_3, \ldots, \lambda_n \) be the spectrum of \( A \). By Proposition 2.2 we have

\[
\left( \sum_{i=1}^{n} \lambda_i \right)^2 = s_1^2 \leq (n - 1)s_2 = (n - 1)\sum_{i=1}^{n} \lambda_i^2.
\]

This inequality can be rewritten in the form

\[
\sum_{i=1}^{n} \lambda_i^2 \leq \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (\lambda_i - \lambda_j)^2.
\] (1)
The right-hand side of (1) is clearly at most $n(n-1)s^2/2$. To obtain a lower bound for the left-hand side of (1), we choose any eigenvalue $\lambda$ of $A$. Since $\lambda + \overline{\lambda} = 2 \text{Re} \lambda \geq 2(1 - s) > 0$, we have

$$\lambda^2 + \overline{\lambda}^2 = (\lambda + \overline{\lambda})^2 - 2|\lambda|^2 \geq (2(1 - s))^2 - 2 = 4s^2 - 8s + 2,$$

and so we obtain the following lower bound for the left-hand side of (1):

$$\sum_{i=1}^{n} \lambda_i^2 = 1 + \sum_{i=2}^{n} \lambda_i^2 \geq 1 + \frac{n-1}{2} (4s^2 - 8s + 2).$$

Therefore, the inequality (1) gives the inequality

$$\frac{n(n-1)}{2} s^2 \geq 1 + \frac{n-1}{2} (4s^2 - 8s + 2),$$

which leads to the inequality

$$(n-1)(n-4)s^2 + 8(n-1)s - 2n \geq 0. \quad (2)$$

For $n = 4$ we obtain that $s \geq \frac{1}{3}$, while for $n = 5$ we have

$$2s^2 + 16s - 5 \geq 0,$$

implying that

$$s \geq \frac{-8 + \sqrt{74}}{2} = \frac{5}{8 + \sqrt{74}}.$$

If $n \geq 6$ we rewrite the inequality (2) to the form

$$(n^2 - 5n)s^2 + 8ns - 2n \geq -4s^2 + 8s = 4s(2 - s) > 0,$$

and so

$$(n - 5)s^2 + 8s - 2 > 0.$$

It follows that

$$s > \frac{-4 + \sqrt{2(n+3)}}{n-5} = \frac{2}{4 + \sqrt{2(n+3)}}.$$

This completes the proof. \qed
For $n \in \{2, 3\}$ we can obtain sharp lower bounds for the spread of a matrix in $C_n$.

**Proposition 2.4.** If $A \in C_2$ then $s(A) \geq 1$; if $A \in C_3$ then $s(A) \geq \frac{3}{4}$. Both bounds are exact.

**Proof.** Let 1 and $\lambda$ be the eigenvalues of $A \in C_2$. By Proposition 2.2, we have

$$(1 + \lambda)^2 = s_1^2 \leq s_2 = 1 + \lambda^2,$$

and so $\lambda \leq 0$ proving that $s(A) \geq 1$. The diagonal matrix $\text{diag}(0, 1) \in C_2$ shows that this lower bound is exact.

In the case $n = 3$ we first suppose that a matrix $A \in C_3$ has real eigenvalues 1, $\lambda$ and $\mu$. We may assume that $0 \leq \lambda \leq \mu \leq 1$. Then the inequality (1) gives the inequality

$$1 + \lambda^2 + \mu^2 \leq (1 - \lambda)^2 + (1 - \mu)^2 + (\lambda - \mu)^2,$$

and so

$$2\lambda^2 \leq 2\lambda\mu \leq (1 - \lambda)^2 + (1 - \mu)^2 - 1 \leq 2(1 - \lambda)^2 - 1 = 2\lambda^2 - 4\lambda + 1.$$

It follows that $\lambda \leq \frac{1}{4}$, so that $s(A) \geq \frac{3}{4}$.

Assume now that a matrix $A \in C_3$ has eigenvalues 1, $\lambda = a + ib$ and $\overline{\lambda} = a - ib$, where $a \in \mathbb{R}$ and $b > 0$. By Proposition 2.2, we have

$$(1 + 2a)^2 = s_1^2 \leq 2s_2 = 2(1 + \lambda^2 + \overline{\lambda}^2) = 2 + 4a^2 - 4b^2 \leq 2 + 4a^2,$$

and so $a \leq \frac{1}{4}$. This implies that $s(A) \geq \frac{3}{4}$ as asserted.

The exactness of this lower bound is proved by the matrix

$$A = \frac{1}{4} \begin{bmatrix} 0 & 2 & 0 \\ 0 & 3 & 1 \\ 2 & 0 & 3 \end{bmatrix} \in C_3$$

the spectrum of which is $\{1, \frac{1}{4}, \frac{1}{4}\}$. □
For $n \geq 4$ it looks difficult to obtain exact lower bounds for the spread of matrices in $C_n$. We thus restrict our attention to a special subset of $C_n$. Proposition 2.1 trivially implies that every matrix in $C_n$ has at least two distinct eigenvalues, that is, 1 is not the only point in its spectrum. Let $D_n$ (with $n \geq 2$) be the collection of all matrices in $C_n$ having exactly two distinct eigenvalues. We now prove sharp lower bounds for the spread of matrices in $D_n$.

**Theorem 2.5.** If $A \in D_n$ then

$$s(A) \geq \frac{n}{2(n-1)}.$$ 

Moreover, this bound is the best possible, i.e., there is a (necessarily irreducible) matrix $A \in D_n$ such that $s(A) = \frac{n}{2(n-1)}$.

**Proof.** Assume first that a matrix $A \in D_n$ is irreducible. Then 1 is a simple eigenvalue of $A$ by the Perron-Frobenius theorem. Therefore, $A$ also has an eigenvalue $\lambda \in (-1, 1)$ of multiplicity $n - 1$. In this case the inequality reads as follows:

$$1 + (n - 1)\lambda^2 \leq (n - 1)(1 - \lambda^2).$$

Simplifying it, we obtain

$$\lambda \leq \frac{n - 2}{2(n - 1)}.$$ 

This implies that

$$s(A) = 1 - \lambda \geq \frac{n}{2(n-1)}.$$ 

Assume now that a matrix $A \in D_n$ is reducible. Then, up to similarity with a permutation matrix, we may assume that

$$A = \begin{bmatrix}
A_{11} & A_{12} & A_{13} & \cdots & A_{1m} \\
0 & A_{22} & A_{23} & \cdots & A_{2m} \\
0 & 0 & A_{33} & \cdots & A_{3m} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & A_{mm}
\end{bmatrix}$$

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where each of $A_{11}, A_{22}, \ldots, A_{mm}$ is either an irreducible (square) matrix or a $1 \times 1$ block. Let $A_{kk}$ be one of these diagonal blocks that has a zero diagonal element. Without loss of generality we may assume that $s(A) < 1$, so that 0 is not in the spectrum of $A$ implying that all $1 \times 1$ diagonal blocks are non-zero. Therefore, if $A_{kk}$ is an $r \times r$ matrix, then $r \geq 2$, and so

$$s(A) \geq s(A_{kk}) \geq \frac{r}{2(r-1)} > \frac{n}{2(n-1)}.$$  

This completes the proof of the first assertion of the theorem.

To show that the lower bound can be achieved, we define the matrix $A = [a_{i,j}]_{i,j=1}^n$ with nonzero elements: $a_{i,i+1} = n - i$ for $i = 1, 2, \ldots, n-1$, $a_{i,i} = n$ for $i = 2, 3, \ldots, n$, and $a_{i,j} = 2$ if $i - j$ is an even positive integer. We also introduce the upper triangular matrix $U = [u_{i,j}]_{i,j=1}^n$ with nonzero elements: $u_{i,i+1} = n - i$ for $i = 1, 2, \ldots, n-1, u_{1,1} = 2(n-1)$ and $u_{i,i} = n - 2$ for $i = 2, 3, \ldots, n$. For example, if $n = 5$ then

$$A = \begin{bmatrix} 0 & 4 & 0 & 0 & 0 \\ 0 & 5 & 3 & 0 & 0 \\ 2 & 0 & 5 & 2 & 0 \\ 0 & 2 & 0 & 5 & 1 \\ 2 & 0 & 2 & 0 & 5 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 8 & 4 & 0 & 0 & 0 \\ 0 & 3 & 3 & 0 & 0 \\ 0 & 0 & 3 & 2 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}.$$  

The proof is complete if we show that $A$ and $U$ are similar matrices, because then we have $r(A) = 2(n-1)$, $s(A) = n$, and $\frac{1}{2(n-1)}A \in \mathcal{D}_n$. Define two nilpotent matrices

$$N = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$  

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and

\[
M = \begin{bmatrix}
0 & n-1 & 0 & \cdots & 0 & 0 \\
0 & 0 & n-2 & \cdots & 0 & 0 \\
0 & 0 & 0 & n-3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0
\end{bmatrix}.
\]

Introduce also the matrix

\[
S = (I + N)(I - N)^{-1} = (I + N)(I + N + N^2 + N^3 + \ldots + N^{n-1}) =
\]

\[
= I + 2N + 2N^2 + 2N^3 + 2N^4 + \ldots + 2N^{n-1} = 
\]

\[
\begin{bmatrix}
1 & 0 & 0 & \cdots & 0 & 0 \\
2 & 1 & 0 & \cdots & 0 & 0 \\
2 & 2 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
2 & 2 & 2 & 2 & \cdots & 1 \\
2 & 2 & 2 & 2 & \cdots & 2 \\
\end{bmatrix}.
\]

Let \(e_1, \ldots, e_n\) be the standard basis vectors, and let \(e = e_1 + \ldots + e_n = (1, 1, \ldots, 1)^T\). Observe that

\[
A = M + nI - ne_1e_1^T + 2(N^2 + N^4 + N^6 + \ldots) =
\]

\[
= M + (n - 2)I - ne_1e_1^T + 2(I - N^2)^{-1}
\]

and

\[
U = M + (n - 2)I + ne_1e_1^T.
\]

Note also that \([N, M] := NM - MN = I - ne_1e_1^T\). By induction one can verify that \([N^k, M] = kN^{k-1} - ne_1e_1^T\) for \(k = 1, 2, \ldots, n\). Then the commutator of \(S\) and \(M\) is

\[
[S, M] = 2 \sum_{k=1}^{n} [N^k, M] = 2 \sum_{k=1}^{n} kN^{k-1} - 2n \sum_{k=1}^{n} e_k e_1^T = 2(I - N)^{-2} - 2nee_1e_1^T.
\]
Now we have

\[ SU - AS = [S, M] + n(Se_1)e_1^T + n e_1 e_1^T S - 2(I - N^2)^{-1} S = \]

\[ = 2(I - N)^{-2} - 2nee_1^T + n(2e - e_1)e_1^T + n e_1 e_1^T - 2(I - N^2)^{-1}(I + N)(I - N)^{-1} = \]

\[ = 2(I - N)^{-2} - 2(I - N)^{-2} = 0. \]

This proves that the matrices \( A \) and \( U \) are similar. \( \square \)

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