Robust Model Predictive Control via System Level Synthesis
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Abstract—In this paper, we consider the robust model predictive control (MPC) of a linear time-varying (LTV) system with norm bounded disturbances and model uncertainty, wherein a series of constrained optimal control problems (OCPs) are solved. Guaranteeing robust feasibility of these OCPs is challenging, due to both disturbances perturbing the predicted states, and model uncertainty which can render the closed-loop system unstable. As such, a trade-off between the numerical tractability and conservativeness of the solutions is often required. We use the System Level Synthesis (SLS) framework to reformulate these constrained OCPs over closed-loop system responses, and show that this allows us to transparently account for norm bounded additive disturbances and LTV model uncertainty by computing robust state feedback policies. We further show that by exploiting the underlying linear-fractional structure of the resulting robust OCPs, we can significantly reduce the conservativeness of existing SLS-based robust control methods. We conclude with numerical examples demonstrating the effectiveness of our methods.

I. INTRODUCTION

Model predictive control (MPC) has achieved remarkable success in solving multivariable constrained control problems across a wide range of application areas, such as process control [1], automotive systems [2], aerospace [3], power networks [4], robot locomotion [5], and biomedical applications [6, 7]. In MPC, a control action is computed by solving a finite-horizon constrained optimal control problem (OCP) at each sampling time and then applying the first control action. Guarantees of closed-loop system stability are provided by the assumption that the OCPs are solved over a closed-loop system response. However, the stability and performance of MPC depends on the accuracy of the model being used, and indeed robustness to both additive disturbances and model uncertainty must be considered. Although MPC using a nominal model (i.e., one ignoring uncertainty) offers some level of robustness [8], it has been shown that the closed-loop system achieved by nominal MPC can be destabilized by an arbitrarily small disturbance [9]. As a result, robust MPC, which explicitly deals with uncertainty, has received much attention [10].

When only additive disturbances are present, open-loop MPC, which optimizes over a sequence of control actions \( \{u_0, \ldots, u_{N-1}\} \) when solving the sequence of online OCPs, subject to suitably robust constraints, can be applied but tend to be overly conservative or even infeasible [11]. On the other hand, closed-loop MPC, which optimizes over the control policies \( \pi = \{\pi_0(\cdot), \ldots, \pi_{N-1}(\cdot)\} \), can reduce the conservativeness of the solutions. However, the policy space is infinite dimensional and renders the online OCPs intractable. The problem can be rendered tractable by restricting the policies \( \pi \) to lie in a function class that admits a finite dimensional parameterization. For example, policies of the form \( \pi_i(x) = K x + v_i \) are considered in [13–15], where \( K \) is a pre-stabilizing feedback gain that is fixed beforehand, thus reducing the decision variables to the vectors \( \{v_1, \ldots, v_{N-1}\} \). To reduce conservativeness, an affine feedback control law \( \pi(x) = K_i x + v_i \) can be applied with decision variables \( K_i \) and \( v_i \); however, the resulting OCP is non-convex. In [25], it was observed that by re-parameterizing the OCPs to be over disturbance based feedback policies of the form \( \pi_i(w) = \sum_{j=0}^{i-1} M_{ij} w_j + v_i \), the resulting OCPs convex. In [16, 17], the authors propose an alternative (tube-MPC) invariant set based approach, which bounds the system trajectories within a tube that robustly satisfy constraints.

The more challenging problem considering model uncertainty is tackled in [13–15, 12]. When polytopic or structured feedback model uncertainty occurs, a linear matrix inequality (LMI) based robust MPC method is proposed in [12]. When both model uncertainty and additive disturbances are present, [17] designs tubes containing all possible trajectories under polytopic uncertainty assumptions. Alternative approaches based on dynamic programming [18] are shown to obtain tight solutions, but the computation quickly becomes numerically intractable. Adaptive robust MPC, which considers estimation of the parametric uncertainty while implementing robust control, is proposed in [19–21].

As described above, there is a rich body of work addressing the robust MPC problem, and it remains an active area of research for which no definitive solution exists. The recently developed System Level Synthesis (SLS) parameterization [22] provides approach to tackling the robust MPC problem. The SLS framework transforms the OCP from one over control laws to one over closed-loop system responses, i.e., the linear maps taking the disturbance process to the states and inputs in closed loop. Robust variants of the SLS parameterization allow for a transparent mapping of model uncertainty on system behavior, allowing for the joint effects of additive disturbances and model errors to be transparently characterized when solving OCPs [23]. In this paper, we apply SLS to solve the robust MPC problem under both additive disturbance and model uncertainty. Our contributions are summarized as follows:

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We use SLS to solve robust MPC problems with model uncertainty and additive disturbances for LTV systems, and show that it allows a favorable tradeoff between conservativeness and computational complexity.

We reduce the conservativeness of existing SLS-based robust control methods by exploiting the underlying linear fractional structure of the resulting robust OCPs.

The remainder of the paper is organized as follows. Section III states the robust MPC problem formulation. After introducing SLS preliminaries in Section III we develop the SLS formulation of robust MPC in two cases: (i) additive disturbances with no model uncertainty (Section IV) and (ii) additive disturbances with model uncertainty (Section V). Section VI compares our method with other methods with a numerical example, and Section VII concludes the paper.

Notation: $x_{i,j}$ is shorthand for the set $\{x_i, x_{i+1}, \ldots, x_j\}$. We use bracketed indices to denote the time of the real system and subscripts to denote the prediction time indices within an MPC loop, i.e., the system state at time $t$ is $x(t)$ and the $t$-th prediction is $x_t$. If not explicitly specified, the dimensions of matrices are compatible and can be inferred from the context. For two vectors $x$ and $y$, $x \leq y$ is component-wise comparison. For a matrix $Q$, $Q \succeq 0$ means $Q$ is positive semidefinite. A linear, causal operator $R$ defined over a horizon of $T$ has the matrix representation

$$R = \begin{bmatrix}
R^{0,0} & R^{1,0} \\
R^{1,1} & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
R^{T,T} & \cdots & R^{T,1} & R^{T,0}
\end{bmatrix}$$

where $R^{i,j}$ is a matrix of compatible dimension. $R(i,:)$ denotes the $i$-th block row of $R$ and $R(:,j)$ denotes the $j$-th block column of $R$, both indexing from 0.

II. PROBLEM FORMULATION

In this paper, we consider robust MPC of an underlying discrete-time LTV system

$$x(k+1) = A(k)x(k) + B(k)u(k) + w(k), k \geq 0,$$

where $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{R}^m$ is the control input, $w(k) \in \mathbb{R}^n$ is the disturbance at time $k$. The matrices $(A(k), B(k))$ denote the real system matrices at time $k$, but are unknown. Instead, the nominal model $(\hat{A}(k), \hat{B}(k))$ is available and the model errors are bounded in terms of the $\infty \to \infty$ induced norm as:

$$\|\Delta_A(k)\|_{\infty \to \infty} \leq \epsilon_A, \|\Delta_B(k)\|_{\infty \to \infty} \leq \epsilon_B, \forall k \geq 0$$

where $\Delta_A(k) = A(k) - \hat{A}(k)$, $\Delta_B(k) = B(k) - \hat{B}(k)$. Further, we assume the disturbance is also norm-bounded by:

$$w(k) \in \mathcal{W} = \{w \in \mathbb{R}^n | \|w\|_{\infty} \leq \sigma_w\}.$$  

In robust MPC, a series of finite horizon robust constrained OCPs are solved, with the current state $x(k)$ set as the initial condition. Motivated by the norm-bounded uncertainty in (3), in each MPC loop, we consider the general form of uncertainty described by linear, causal operators and formulate our problem as follows:

**Problem 1.** Consider the robust MPC of an LTV system with additive disturbance and model uncertainty. In each loop, consider the uncertainty operators $\Delta_A$ and $\Delta_B$ of the form (1) with horizon $T$. Let $\Delta_{A_t} = \Delta_A(t,:)$, $\Delta_{B_t} = \Delta_B(t,:)$ and $\Delta_{A_t}(x_0, t) = \sum_{i=0}^{T} \Delta_{A_t} x_i$, $\Delta_{B_t}(u_0, t) = \sum_{i=0}^{T} \Delta_{B_t} u_i$. At time $k$, denote the current state as $x(k)$. Solve the following robust optimal control problem:

$$\min_{\pi} \ J(x(k), \pi)$$

$$x_{t+1} = \hat{A}_t x_t + \Delta_{A_t}(x_0, t) + \hat{B}_t u_t + \Delta_{B_t}(u_0, t) + w_t$$

s.t. $x_t \in \mathcal{X}, t \in \mathcal{U}, \mathcal{X}_T \in \mathcal{X}$

$$\forall u_t \in \mathcal{W}, \forall t = 0, 1, \cdots, T-1$$

$$\forall \|\Delta_A\|_{\infty \to \infty} \leq \epsilon_A, \forall \|\Delta_B\|_{\infty \to \infty} \leq \epsilon_B$$

$$x_0 = x(k)$$

where $\pi = \pi_0 \in \mathbb{R}^{T-1}$ is the set of causal control policies, $J(x(k), \pi)$ is the cost function to be specified. $(\hat{A}_t, \hat{B}_t)$ is the known LTV nominal model, $\Delta_A$, $\Delta_B$ are the norm-bounded uncertainty operators and $\mathcal{X}, \mathcal{U}, \mathcal{X}_T$ are polytopic state, input and terminal constraints defined as:

$$\mathcal{X} = \{x \in \mathbb{R}^n | F_x x \leq b_x\}, \mathcal{U} = \{u \in \mathbb{R}^m | F_u u \leq b_u\},$$

$$\mathcal{X}_T = \{x \in \mathbb{R}^n | F_T x \leq b_T\}.$$  

Additionally, we define the disturbance set $\mathcal{W}$ as in (4).

**Remark 1.** The robust OCP formulation in (5) contains the uncertainty model (3) as a special case when the predictive model is of the form

$$x_{t+1} = (\hat{A}_t + \Delta_A^{(0)} x_t + (\hat{B}_t + \Delta_B^{(0)}) u_t + w_t$$

where we restrict $\Delta_A, \Delta_B$ to be block diagonal.

In Problem (1) the objective $J(x(k), \pi)$ is a function of the initial condition and the applied control policies. For simplicity, we apply the nominal cost $J_{\text{nom}}(x(k), \pi)$:

$$J_{\text{nom}}(x(k), \pi) = \sum_{t=0}^{T-1} (\pi_t^T Q_t \pi_t + u_t^T R_t u_t) + \pi_T^T Q_T \pi_T$$

s.t. $x_{t+1} = \hat{A}_t x_t + \hat{B}_t u_t, u_t = \pi_t(x_t)$

$$x_0 = x(k), \forall t = 0, 1, \cdots, T-1$$

for $x_t$ the nominal trajectory and $Q_t \geq 0, R_t > 0, Q_T \geq 0$ the state, input and terminal cost matrices, respectively.

We remark that in Problem (1) we optimize over the closed-loop control policies $\pi = \pi_0 \in \mathbb{R}^{T-1}$ instead of open-loop control sequences $u_0, \cdots, u_{T-1}$, as open-loop policies are overly conservative (10). Since the space of all causal control policies is infinite dimensional and renders the optimization (5) intractable, we restrict our search to causal linear time-variant state feedback controllers (11).

1. Affine feedback control policies can be implemented as a linear feedback controllers by augmenting the state to $\hat{x} = [x; \pi]$. As such, we restrict our discussion to linear feedback policies without loss of generality.
III. SYSTEM LEVEL SYNTHESIS

In this section, we review relevant concepts of SLS, and show how it can be used to reformulate OCPs over closed-loop system responses. A more extensive review of SLS can be found in [22].

A. Finite Horizon System Level Synthesis

For the dynamics in (2) and a fixed horizon \( T \), we define \( x, u \) as the column concatenations of all states \( x_{0:T} \) and inputs \( u_{0:T} \) up to time \( T \), i.e.,

\[
x^T = [x_0^T \ x_1^T \ \cdots \ x_T^T], \quad u^T = [u_0^T \ u_1^T \ \cdots \ u_T^T].
\]

In particular, we let \( w = [w_0^T \ w_1^T \ \cdots \ w_{T-1}^T]^T \) by considering \( x_0 \) as the first component of disturbance. Let \( u_t \) be a causal time-variant state-feedback controller, i.e., \( u_t = K_t(x_{0:t}) = \sum_{i=0}^t K_{t-i} x_i. \) Then we have \( u = Kx \) with \( K \in \mathbb{R}^{(T+1)n \times (T+1)n} \) of the form (1). We can concatenate the dynamics matrices as \( A = \text{blkdiag}(A_0, \cdots, A_{T-1}, 0), B = \text{blkdiag}(B_0, \cdots, B_{T-1}, 0). \) Let \( Z \) be the block-downshift operator, i.e., a matrix with identity matrix on the first block subdiagonal and zeros elsewhere. Under the feedback controller \( K \), the closed-loop behavior of the system (2) over the horizon of length \( T \) can be represented as

\[
x = Z(A + BK)x + w
\]

and the closed-loop transfer functions describing \( w \mapsto (x, u) \) are given by

\[
x_u = \begin{bmatrix} I - Z(A + BK) & ZK(1 - Z(A + BK))^{-1} \end{bmatrix} w. \tag{7}
\]

These maps are called system responses: as \( Z \) is the block-downshift operator, the matrix inversion in (7) always exists. Instead of optimizing over feedback controllers \( K \), SLS allows for a direct optimization over system responses \( \Phi_x, \Phi_u \) defined as

\[
x_u = \begin{bmatrix} \Phi_x \ 
\Phi_u \end{bmatrix} w \tag{8}
\]

where \( \Phi_x \in \mathbb{R}^{(T+1)n \times (T+1)n}, \Phi_u \in \mathbb{R}^{(T+1)m \times (T+1)n} \) are two block-lower-triangular matrices of the form (1). The next theorem shows that the two parameterizations (7) and (8) are equivalent for an LTV \( K \) and a pair \( \{\Phi_x, \Phi_u\} \) satisfying suitable affine constraints:

**Theorem 1.** [22, Theorem 2.1] Over the horizon \( \tau = 0, 1, \cdots, T, \) for the system dynamics (2) with the block-lower-triangular state feedback control law \( K \) defining the control action as \( u = Kx \), the following are true:

1. The affine subspace defined by

\[
[I - ZA \ -ZB] \begin{bmatrix} \Phi_x \\ \Phi_u \end{bmatrix} = I \tag{9}
\]

parameterizes all possible system responses (8).

2. For any block-lower-triangular matrices \( \{\Phi_x, \Phi_u\} \) satisfying (9), the controller \( K = \Phi_u \Phi_x^{-1} \) achieves the desired response.

Eq. (9) is called the *achievability constraint*. With Theorem 1 we can equivalently transform the problem of controller synthesis into closed-loop system responses design by setting \( x = \Phi_x w, u = \Phi_u w \), and constraining that the system responses lie in the affine space (9). The map (8) characterizes the effects of disturbances \( w \) on the states \( x \) and inputs \( u \) and thus allows a direct translation of the state and input constraints into constraints on \( \Phi_x, \Phi_u. \)

B. Robustness in System Level Synthesis

A useful result in characterizing the robustness of the closed-loop system responses is presented here. When the achievability constraint (9) is not exactly satisfied, the system responses can be characterized explicitly in the following theorem:

**Theorem 2.** [22, Theorem 2.2] Let \( \Delta \) be an arbitrary block-lower-triangular matrix of the form (1). Suppose \( \{\Phi_x, \Phi_u\} \)

\[
[I - ZA \ -ZB] \begin{bmatrix} \Phi_x \\ \Phi_u \end{bmatrix} = I - \Delta. \tag{10}
\]

If \( (I - \Delta(1) : \cdots : \Delta(T))^{-1} \) exists for \( i = 0, \cdots, T \), then the controller \( K = \Phi_u \Phi_x^{-1} \) achieves the system response

\[
x_u = \begin{bmatrix} \Phi_x \\ \Phi_u \end{bmatrix} (I - \Delta)^{-1} w \tag{11}
\]

Theorem 2 describes how the perturbations in the achievability constraint affect the closed-loop responses. It can be seen as a robust variant of Theorem 1. As will be seen in the sequel, we will use \( \Delta \) to capture the uncertainty in the system dynamics, and the expression (11) affords us a transparent and explicit mapping of its effects on system behavior. Next we show how SLS can be applied to the robust MPC problem. In Section V we investigate robust MPC with no model uncertainty as a motivating example. In Section V we extend our analysis to the general setting of additive disturbance and model uncertainty, and show that a suitable relaxation of the resulting robust SLS OCP results in a quasi-convex optimization problem.

IV. ROBUST MPC WITH NO MODEL UNCERTAINTY

In this section, we assume that the system dynamics are exactly known, i.e., that \( \epsilon_A = \epsilon_B = 0 \), and \( A_t = A, B_t = B_t, \forall t \in 5 \). We can formulate this variant of (5) as

\[
\min_{\pi} J_{\text{nom}}(x(0), \pi) \substack{x_{t+1} = A_t x_t + B_t u_t + w_t \\ s.t. \ u_t = \pi(x_t, 0, 0, 0, t-1) \\ x_t \in \mathcal{X}, u_t \in \mathcal{U}, x_T \in \mathcal{X}_T \\ \forall t = 0, 1, \cdots, T - 1, \forall w_t \in \mathcal{W} \\ x_0 = x(0)}
\]

assuming the current state is \( x(0) \) without loss of generality. As described in Section III we restrict our search of feedback policies to LTV state-feedback controllers \( K \) using the SLS parameterization.
Proposition 1. Proposition 1] The following convex optimization problem solves the robust optimal control problem (12):

\[
\begin{aligned}
\min_{\Phi_x, \Phi_u} & \left\| \begin{bmatrix} Q^2 & 0 \\ R & 0 \end{bmatrix} \begin{bmatrix} \Phi_x(:,0) \\ \Phi_u(:,0) \end{bmatrix} \right\|_2^2 \\
\text{s.t.} & \begin{bmatrix} I - ZA & -ZB \\ \Phi_x & \Phi_u \end{bmatrix} = I \\
& G_x(\Phi_x;k) \leq b_x, G_u(\Phi_u;k) \leq b_u, \forall k = 0, 1, \ldots, T.
\end{aligned}
\]  

(13)

where \( Q = \text{blkdiag}(Q_0, \ldots, Q_{T-1}, Q_T) \), \( R = \text{blkdiag}(R_0, \ldots, R_{T-1}, 0) \), \( \Phi_x(:,0) \) and \( \Phi_u(:,0) \) denote the first block columns of \( \Phi_x \) and \( \Phi_u \), respectively, and

\[
\begin{aligned}
G_x(\Phi_x;k) &= F_{x,j}^T \Phi_x^{k:k} x_0 + \sum_{i=1}^k \| F_{x,j}^T \Phi_x^{k-i} \|_1 \sigma_w, \\
G_u(\Phi_u;k) &= F_{u,j}^T \Phi_u^{k:k} x_0 + \sum_{i=1}^k \| F_{u,j}^T \Phi_u^{k-i} \|_1 \sigma_w,
\end{aligned}
\]

(14)

with \( j \) indexing the row vectors of \( F_x \) and \( F_u \) and the entries of \( G_x \) and \( G_u \). The linear state feedback controller is synthesized as \( \hat{K} = \Phi_u \Phi_x^{-1} \).

Proof. See Appendix.

Remark 2. The previous results can be generalized to disturbances \( w \) lying in a polytopic set, i.e., \( w_t \in \mathcal{W} = \{ w \in \mathbb{R}^n \mid F_w w \leq b \} \). For each \( k \) in (13), the robust state constraint in row \( j \) can be written as

\[
F_{x,j}^T \Phi_x^{k:k} x_0 + \sum_{i=1}^k \max_{w_i \in \mathcal{W}} F_{x,j}^T \Phi_x^{k-i} w_i \leq b_{x,j}.
\]

(15)

We can rewrite (15) as a set of linear constraints using the dual of each maximization term in (15). For each row \( j \), we can rewrite (15) as

\[
\begin{aligned}
F_{x,j}^T \Phi_x^{k:k} x_0 + \sum_{i=1}^k \lambda_{t,i} & \leq b_{x,j} \\
F_{w}^T \lambda_{t,j} & = \Phi_x^{k-k+t} F_{x,j} \\
\lambda_{t,j} & \geq 0
\end{aligned}
\]

(16)

where \( \lambda_{t,j} \) is the dual variable of compatible dimensions – the robust constraints on the control inputs can be handled in a similar way. Using a similar duality argument, general convex norm bounds on the disturbance \( w_t \), e.g., the \( p \)-norms for \( p \geq 1 \), can be treated in a similar manner as shown in (14), with the resulting constraints being formulated in terms of the corresponding dual norm.

V. ROBUST MPC WITH MODEL UNCERTAINTY

As in the previous section, we optimize a nominal performance metric subject to robust constraint satisfaction with respect to both disturbance and model uncertainty.

A. Robustness in Nominal Control Synthesis

With the nominal model \( \hat{A} = \text{blkdiag}(\hat{A}_0, \ldots, \hat{A}_{T-1}, 0) \) and \( \hat{B} = \text{blkdiag}(\hat{B}_0, \ldots, \hat{B}_{T-1}, 0) \), we can synthesize the controller \( \hat{K} = \Phi_u \Phi_x^{-1} \) with \( (\Phi_x, \Phi_u) \) satisfying the nominal achievability constraint (18):

\[
[I - Z \hat{A} - Z \hat{B}] \begin{bmatrix} \Phi_x \\ \Phi_u \end{bmatrix} = I.
\]

(17)

On the other hand, \( (\hat{\Phi}_x, \hat{\Phi}_u) \) satisfy the approximate achievability constraint with respect to the real dynamics \( \hat{A} = \hat{A} + \Delta \hat{A}, \hat{B} = \hat{B} + \Delta \hat{B} \) where the uncertainty operators \( \{\Delta \hat{A}, \Delta \hat{B}\} \) are defined in Problem 1. This can be seen by rewriting (17) as

\[
[I - Z \hat{A} - Z \hat{B}] \begin{bmatrix} \hat{\Phi}_x \\ \hat{\Phi}_u \end{bmatrix} = I - Z \left[ \Delta \hat{A} \quad \Delta \hat{B} \right] \begin{bmatrix} \hat{\Phi}_x \\ \hat{\Phi}_u \end{bmatrix} = I - \Delta \hat{\Phi}
\]

with the notation

\[
\hat{\Phi} = \begin{bmatrix} \hat{\Phi}_x \\ \hat{\Phi}_u \end{bmatrix}, \quad \Delta = Z \left[ \Delta \hat{A} \quad \Delta \hat{B} \right].
\]

(19)

Thus, \( (\hat{\Phi}_x, \hat{\Phi}_u) \) satisfy an inexact achievability constraint (18) of the system \( (\hat{A}, \hat{B}) \). Let \( \Delta = \Delta \hat{\Phi} \) in (10). Since \( \Delta \) now is a strict block-lower-triangular matrix and \( (I - \Delta \times 0)^{-1} \) exists for all \( i \), by Theorem 2 the closed-loop system response of the system \( (A, B) \) with the controller \( \hat{K} \) is given by

\[
\begin{bmatrix} x \\ u \end{bmatrix} = \hat{\Phi} (I - \Delta \hat{\Phi})^{-1} w
\]

(20)

where the second equality is an application of the Woodbury identity \( 26 \). In the next step, we will construct sufficient conditions on the approximate system responses \( (\Phi_x, \Phi_u) \) in (17) such that the synthesized controller \( \hat{K} \) guarantees the robust satisfaction of all state and input constraints. Compared with existing SLS-based techniques in solving robust OCPs, our method reduces conservativeness in the solutions by exploiting the linear fractional transform (LFT) structure of the system responses \( 20 \) which is discussed in the next subsection \( 24 \).

B. Upperbound through Robust Performance

In this section, we provide an easily-verified sufficient condition for the norm of the system response in (20) to be upper bounded by a constant. This condition exploits the LFT structure of (20) and is inspired by the algebraic robust performance conditions in \( 27 \).

Let \( \mathcal{L}_{TV} \) be the space of all bounded linear causal operators mapping \( \ell_{\infty} \) to \( \ell_{\infty} \). Define the perturbation set \( \Delta_v = \{ \Delta \in \mathcal{L}_{TV} \mid \| \Delta \|_{\infty} \leq 1, \Delta \text{ is strictly causal} \} \).

Proposition 2. \( 28 \) Consider System I and System II in Fig. 7 where

\[
M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \quad \Delta = \begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix}
\]

(21)
with $\Delta_1, \Delta_2 \in \Delta_c$. The matrices are of compatible dimensions. Then the following statements are equivalent:

1) In System I, $(I - M_{22}\Delta_1)^{-1}$ exists and

$$\|M_{11} + M_{12}\Delta_1(I - M_{22}\Delta_1)^{-1}M_{21}\|_{\infty \rightarrow \infty} < 1 \quad (22)$$

for all $\Delta_1 \in \Delta_c$.

2) In System II, $(I - M\tilde{\Delta})^{-1}$ exists for all $\tilde{\Delta} = \text{blkdiag}(\Delta_1, \Delta_2), \Delta_1, \Delta_2 \in \Delta_c$.

We state our sufficient condition in the following theorem:

**Theorem 3.** Consider the matrices $\Phi$ and $\Delta$ in (20). Let $\|\Delta\|_{\infty \rightarrow \infty} \leq \epsilon$ for some $\epsilon > 0$. For any $\beta > 0$ and matrix $P$ of compatible dimensions,

$$\|P\Phi + P\Delta(I - \Phi\Delta)^{-1}\Phi\|_{\infty \rightarrow \infty} \leq \beta, \forall \|\Delta\|_{\infty \rightarrow \infty} \leq \epsilon \quad (23)$$

if

$$\|P\Phi\|_{\infty \rightarrow \infty} + \beta\epsilon\|\Phi\|_{\infty \rightarrow \infty} \leq \beta.\quad (24)$$

**Proof.** See Appendix.

Theorem 3 provides a sufficient condition (24) which is convex for the robust norm bound constraint on the system responses (23) to be satisfied.

**C. Robust Optimization Formulation**

We formulate a convex optimization problem in $\{\Phi_x, \Phi_u\}$ for robust control synthesis in this section. First we concatenate all the constraints $x_t \in X, u_t \in U, x_T \in X_T$ in (5) on $x_t$ and $u_t$ over the horizon $t = 0, 1, \ldots, T$ as

$$F_1 \hat{x} \leq b,$$

where

$$F = \text{blkdiag}(F_x, \ldots, F_x, F_T, F_u, \ldots, F_u),$$

$$b = [b^T_x \cdots b^T_x b^T_T b^T_u \cdots b^T_u].$$

We decompose $\Phi, \Delta$ and $w$ as follows to further exploit their structure:

$$\Phi = \begin{bmatrix} \Phi_x & \Phi_u \end{bmatrix} = \begin{bmatrix} \Phi^0_x & \Phi^w_x \\ \Phi^0_u & \Phi^w_u \end{bmatrix}, \quad w = \begin{bmatrix} x_0 \\ w \end{bmatrix},$$

$$\Delta = \begin{bmatrix} \Delta^0_A & \Delta^0_B \\ \Delta_A & \Delta_B \end{bmatrix} = \begin{bmatrix} \Delta^0_A \Delta^0_B \\ \Delta_A \Delta_B \end{bmatrix},$$

where $\Phi^0$ is the first block column of $\Phi$, $\Delta^0$ is the first block row of $\Delta$ and all other block matrices are of compatible dimensions. The main result of this paper is presented as follows:

**Theorem 4.** Consider the convex optimization problem formulated below:

$$\min_{\Phi} \left\| \begin{bmatrix} Q \Phi^T \\ \mathcal{R} \Phi^T \end{bmatrix} \begin{bmatrix} \Phi_x(:,0) \\ \Phi_u(:,0) \end{bmatrix} x_0 \right\|_2^2$$  \quad (27a)

s.t.  \quad $$(I - Z\hat{\Delta})^{-1}g = I$$  \quad (27b)

$$F_j^T \Phi^0 x_0 + \|F_j^T \Phi^w\|_2 \frac{1 - \sigma^T}{1 - \tau} \gamma + \beta \sigma_w \leq b_j, \forall j$$  \quad (27c)

$$\|F_j^T \Phi^0\|_{\infty \rightarrow \infty} + \beta\|\Phi^w\|_{\infty \rightarrow \infty} \leq \beta, \forall j$$  \quad (27d)

$$\|\epsilon_A \Phi^0\|_{\infty \rightarrow \infty} \leq \tau/2$$  \quad (27e)

$$\|\epsilon_B \Phi^w\|_{\infty \rightarrow \infty} \leq \gamma/2$$  \quad (27f)

with constants $\epsilon = \epsilon_A + \epsilon_B, \tau > 0, \gamma > 0$ and $\beta > 0$. Any controller synthesized from a feasible solution $\Phi$ to (27) for any $(\tau, \gamma, \beta)$ guarantees the constraints satisfaction under all possible model uncertainty and disturbances in (5).

**Proof.** See Appendix.

Despite the existence of three hyperparameters $(\tau, \gamma, \beta)$, in practice (27) can be effectively solved by grid searching over $(\tau, \gamma, \beta)$. To allow efficient grid search, the upper and lower bounds on $(\tau, \gamma, \beta)$ can be found by bisection which has $O(\log(1/\epsilon_{tol}))$ complexity in terms of the tolerance $\epsilon_{tol}$.

**Algorithm 1** describes this procedure using the standard bisection algorithm bisect. When applying bisect, a variant of (27) is constructed with a smaller set of selected constraints whose feasibility guides the search of the lower or upper bounds on the hyperparameters. Since (27c), (27d) are independent of $x_0$, the lower bounds on $\tau, \beta$ obtained from Algorithm 1 are valid for all $x_0$ and only need to be computed once in receding horizon control.

**Algorithm 1** Find lower and upper bounds on $(\tau, \gamma, \beta)$

**Input:** $\epsilon_{tol} > 0, \text{range}(\tau, \gamma, \beta)$

**Output:** $lb(\tau, \gamma, \beta), ub(\tau, \gamma, \beta)$

1. **procedure** LOWERUPPERSOUNDS$(\tau, \gamma, \beta)$
2. $lb(\tau) = \text{bisection}(\tau, \{\text{min}_A, 1\text{st} \text{ s.t.} \text{ (27b)}\}, \text{ (27c)}\}$
3. $lb(\gamma) = \text{bisection}(\gamma, \{\text{min}_B, \text{ (27b)}\}, \text{ (27d)}\}$
4. $lb(\beta) = \text{bisection}(\beta, \{\text{min}_B, \text{ (27d)}\}, \text{ (27e)}\}$
5. In (27c), let $\tau = lb(\tau), \gamma = lb(\gamma)$
6. $ub(\beta) = \text{bisection}(\beta, \{\text{min}_B, \text{ (27d)}\}, \text{ (27f)}\}$
7. In (27c), let $\gamma = lb(\gamma), \beta = lb(\beta)$
8. $ub(\tau) = \text{bisection}(\tau, \{\text{min}_A, \text{ (27b)}\}, \text{ (27c)}\}$
9. $ub(\gamma) = \frac{1 - ub(\tau)^2}{1 - lb(\tau)^2}$
10. return $lb(\tau, \gamma, \beta), ub(\tau, \gamma, \beta)$

\[2\] All the bisection algorithms search over range$(\tau, \gamma, \beta)$ and use $\epsilon_{tol}$ for termination as specified in the input.
Simulation shows that a coarse grid search (e.g., $3 \times 3 \times 3$) of the hyperparameters between their lower and upper bounds can usually find feasible tuples of $(\tau, \gamma, \beta)$ in the early iterations. Furthermore, with Algorithm 1, we can verify the infeasibility of the relaxation (29) for all choices of hyperparameters if the contradiction $l_2(\cdot) > u_2(\cdot)$ is observed for $\tau, \gamma$ or $\beta$. With fixed hyperparameters, problem (29) is a quadratic program which has complexity $O(T^6(n + m)^3)$.

Compared with the previous SLS-based robust optimization relaxations (23), problem (27) significantly reduces the conservativeness by decomposing the effects of the disturbance $w$ (27e) and the initial condition $x_0$ (27f) on the feasibility of the problem through the hyperparameters $\tau$ and $\gamma$, respectively, and applying the robust performance argument in Section VI-B. In Section VI-B, we provide a numerical example for conservativeness comparison.

VI. NUMERICAL EXAMPLES

We consider robust MPC of a two-dimensional system as an illustrating example where Problem (5) is solved recursively in a receding horizon manner. Although the proposed framework deals with time-varying dynamics, for illustration purposes we investigate an LTI system in this section and use SLS to solve two robust MPC problems: one with only disturbance and the other with both model uncertainty and disturbance.

A. Example: Robust MPC with No Model Uncertainty

Consider the robust MPC problem (5) when $\epsilon_A = \epsilon_B = 0$ (or equivalently problem (29)) with the system dynamics

$$x_{t+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_t + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u_t + w_t$$

and the nominal cost function $J_{	ext{nom}}$ defined by the state weight $Q = I_2$, the input weight $R = 0.1$ and the terminal weight $Q_T = Q$. The horizon is chosen to be $T = 5$. We choose the initial condition as $x_0 = [-7, -2]^\top$ and the state and input constraints are

$$\mathcal{X} = \left\{ x \in \mathbb{R}^2 \mid \begin{bmatrix} -10 \\ -10 \end{bmatrix} \leq x \leq \begin{bmatrix} 10 \\ 10 \end{bmatrix} \right\},$$

$$\mathcal{U} = \left\{ u \in \mathbb{R} \mid -2 \leq u \leq 2 \right\}.$$  

(29)

The disturbance is norm-bounded by $w_t \in \mathcal{W} = \left\{ w \in \mathbb{R}^2 \mid \|w\|_\infty \leq 0.1 \right\}$. The terminal set $\mathcal{X}_T$ is found to be the maximal robust control invariant set of system (28) through Algorithm 2 with a tuning horizon of length 2. $\mathcal{X}_T$ is a polytope (Fig. 2a) and the corresponding piecewise affine local control law that renders $\mathcal{X}_T$ robustly steered into $\mathcal{X}_T$ under any admissible control actions.

Using the SLS formulation in (13), a linear state-feedback controller $K$ is synthesized. The open-loop prediction under $K$ is shown in Fig. 2b. In the prediction, both the state constraint (purple) and the terminal constraint (green) are robustly satisfied. The closed-loop trajectory of running MPC for 7 steps is shown in Fig. 2c.

B. Robust MPC with Model Uncertainty

We apply the same problem setup in Section VI-A except that we now assume $\epsilon_A = 0.02$, $\epsilon_B = 0.02$. The nominal system is defined as

$$x_{t+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_t + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u_t + w_t,$$

and the real dynamics of the system is

$$A = \begin{bmatrix} 1.02 & 1 \\ 0 & 1.02 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 \\ 1.02 \end{bmatrix}.$$  

(30)

The terminal set $\mathcal{X}_T$ is again found by Algorithm 2 taking both model uncertainty and disturbance into consideration. Then we solve Problem 1 through the SLS relaxation (27) together with Algorithm 1 guiding the search for hyperparameters. The results are shown in Fig. 3 where the closed-loop trajectories with different initial conditions are plotted for 10 steps.

Complexity analysis: Since the terminal set $\mathcal{X}_T$ (the green region in Fig. 3) applied in this example is the maximum robust invariant set, there exists a robust (piecewise affine) controller for each $x_0 \in \mathcal{X}_T$ and no solution to the robust OCP 5 exists for $x_0 \notin \mathcal{X}_T$. However, obtaining such robust piecewise affine feedback controller for $x_0 \in \mathcal{X}_T$ requires the use of dynamic programming and multi-parametric quadratic programming with robustified constraints. The underlying problem has a worst-case complexity that is \textit{exponent-}
with YALMIP [31] and MOSEK [32] on an Intel i7 CPU. The logarithms of the running time \( t \) for a grid search of \( x \) \( \in \mathbb{R}^8 \) with varying horizon \( T \). The red curve has slope 6.

The SLS relaxation is polynomial in the problem size, we want to know how conservative the relaxation \( (27) \) is in terms of the feasibility of the initial condition \( x_0 \). In Fig. 6a, we do a grid search of \( x_0 \) inside \( \mathcal{X}_T \). For each \( x_0 \), Algorithm 1 is applied first, followed by a \( 3 \times 3 \times 3 \) grid search of \( (\tau, \gamma, \beta) \) to determine the hyperparameters. We say \( x_0 \) is feasible (yellow square) if problem \( (27) \) is feasible with some tuple of hyperparameters and infeasible (red dots) if \( (27) \) is verified infeasible by Algorithm 1. Unverified \( x_0 \)’s (white asterisk) are those for which a feasible tuple of hyperparameters have not been found during the grid search. Fig. 6a shows that the feasible initial conditions cover a large portion of \( \mathcal{X}_T \). Compared with the dynamic programming approach, the SLS relaxation \( (27) \) achieves similar region of feasible initial conditions while being computationally efficient.

**Compare with existing SLS relaxations:** In existing SLS literature [23], a relaxation similar to \( (27) \) has been used to solve robust OCPs, which we denote as the coarse SLS relaxation in this paper. In the coarse SLS relaxation, there exists only one hyperparameter \( \tau \) and the conservativeness of the coarse relaxation is evaluated in Fig. 6a following the same procedure shown above. The SLS relaxation \( (27) \) in our paper considerably reduces the conservativeness compared with the coarse SLS relaxation with paying only a small amount of computational cost for the extra hyperparameters through Algorithm 1.

**VII. Conclusion**

We proposed an SLS based approach for robust MPC with LTV dynamics and norm bounded additive disturbances and model uncertainty. A robust LTV state feedback controller is synthesized through SLS by optimizing over the closed-loop system responses. Computationally efficient convex relaxations of the robust optimal control problems are formulated. Simulation results indicate that the proposed framework achieves a favorable balance between conservativeness and computational complexity as compared with dynamic programming based approaches. The proposed approach is also significantly less conservative than existing SLS based robust control methods in the literature.

**APPENDIX A**

**Proof of Proposition 1.** By Theorem 1, the achievability constraint in \( (13) \) implies \( x = \Phi_w x_0 + u = \Phi_w u_0 \). Then the nominal state and control trajectories are given by \( x_0, u_0 \). As a sanity test, we solved \( (27) \) with \( x_0, u_0 \) for the robust MPC problem considered in this section with varying horizon \( T \). The problems were solved in MATLAB R2018a with YALMIP [31] and MOSEK [32] on an Intel i7-6700K CPU. The logarithms of the running time \( t \) (in seconds) and the horizon \( T \) are plotted in Fig. 5. It is observed that the running time curve (blue) is always below the worst-case one (red).

**Conservativeness evaluation:** Although the complexity of the SLS relaxation is polynomial in the problem size, we want to know how conservative the relaxation \( (27) \) is in terms of the feasibility of the initial condition \( x_0 \). In Fig. 6a, we do a grid search of \( x_0 \) inside \( \mathcal{X}_T \). For each \( x_0 \), Algorithm 1 is applied first, followed by a \( 3 \times 3 \times 3 \) grid search of \( (\tau, \gamma, \beta) \) to determine the hyperparameters. We say \( x_0 \) is feasible (yellow square) if problem \( (27) \) is feasible with some tuple of hyperparameters and infeasible (red dots) if \( (27) \) is verified infeasible by Algorithm 1. Unverified \( x_0 \)’s (white asterisk) are those for which a feasible tuple of hyperparameters have not been found during the grid search. Fig. 6a shows that the feasible initial conditions cover a large portion of \( \mathcal{X}_T \). Compared with the dynamic programming approach, the SLS relaxation \( (27) \) achieves similar region of feasible initial conditions while being computationally efficient.

**Compare with existing SLS relaxations:** In existing SLS literature [23], a relaxation similar to \( (27) \) has been used to solve robust OCPs, which we denote as the coarse SLS relaxation in this paper. In the coarse SLS relaxation, there exists only one hyperparameter \( \tau \) and the conservativeness of the coarse relaxation is evaluated in Fig. 6a following the same procedure shown above. The SLS relaxation \( (27) \) in our paper considerably reduces the conservativeness compared with the coarse SLS relaxation with paying only a small amount of computational cost for the extra hyperparameters through Algorithm 1.
Proof of Theorem 3. \(\Delta\) in problem (13) are equivalent to the constraints in (12).

\[ \Delta_y = \Delta_x \times X \]

Next we want to show that \(\Delta \in \{ \Delta \in \mathcal{L}_{TV} | \|\Delta\|_{\infty} \leq 1, \Delta \text{ is causal} \} \). Consider matrices

\[ M = \begin{bmatrix} X & X \\ Y & Y \end{bmatrix}, \quad \tilde{\Delta} = \begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} \]

where \( X \in \mathbb{R}^{p \times q}, Y \in \mathbb{R}^{p_2 \times q}, \Delta_1 \in \mathbb{R}^{q \times p_1}, \Delta_2 \in \mathbb{R}^{q_2 \times p_2}, \Delta_1, \Delta_2 \in \Delta_c \). Next we want to show that

\[ \|X\|_{\infty} + \|Y\|_{\infty} < 1 \Rightarrow (I - M\tilde{\Delta})^{-1} \text{ exists } \forall \tilde{\Delta}. \]

For all \( d_1, d_2 > 0 \), define

\[ D_p = \begin{bmatrix} d_1I_{p_1} & d_2I_{p_2} \end{bmatrix}, \quad D_{q}^{-1} = \begin{bmatrix} d_1I_{q_1} & d_2I_{q_2} \end{bmatrix}. \]

It follows that \( D_q^{-1} \tilde{\Delta} = \tilde{\Delta} D_q^{-1} \). Since \( D_p(I - M\tilde{\Delta})D_q^{-1} = I - D_p M \tilde{\Delta} D_q^{-1} \), we have \( I - M\tilde{\Delta} \) is invertible if and only if \( I - D_p M \tilde{\Delta} D_q^{-1} \) is invertible for some \( d_1, d_2 > 0 \). Substitute \( D_p, D_q^{-1} \) and we have

\[ D_p M \tilde{\Delta} D_q^{-1} = \begin{bmatrix} X & \frac{d_1}{d_2} X \\ Y & \frac{1}{d_2} Y \end{bmatrix} = \begin{bmatrix} X & \lambda X \\ Y & \lambda Y \end{bmatrix}. \]

with \( \lambda = \frac{d_1}{d_2} > 0 \). Then

\[ \|D_p M \tilde{\Delta} D_q^{-1}\|_{\infty} \leq \max\{\|X\|_{\infty} + \lambda \|X\|_{\infty} + \frac{1}{\lambda} \|Y\|_{\infty} + \|Y\|_{\infty}\} \]

Taking the infimum over \( \lambda \) (or equivalently \( d_1, d_2 \)), we have

\[ \inf_{\lambda} \|D_p M \tilde{\Delta} D_q^{-1}\|_{\infty} \leq \max\{\|X\|_{\infty} + \lambda \|X\|_{\infty} + \frac{1}{\lambda} \|Y\|_{\infty} + \|Y\|_{\infty}\} \]

where the infimum is achieved by \( \lambda = d_1/d_2 = \|Y\|_{\infty}/\|X\|_{\infty} \). Thus

\[ \|X\|_{\infty} + \|Y\|_{\infty} < 1 \Rightarrow \inf_{d_1, d_2} \|D_p M \tilde{\Delta} D_q^{-1}\|_{\infty} < 1 \]

\[ \Rightarrow (I - M\tilde{\Delta})^{-1} \text{ exists } \forall \tilde{\Delta} \]
and the term $\mathcal{E} = F_j \Phi \Delta^w (I - \Phi \Delta^w)^{-1} \Phi^0 x_0$ separately.

Bounding $1$: By the submultiplicativity of the $\infty \rightarrow \infty$ matrix induced norm, we can upperbound term $1$ by

$$F_j^T (\Phi^w + \Phi \Delta^w (I - \Phi \Delta^w)^{-1} \Phi^w) \Phi^w \leq F_j^T (\Phi^w + \Phi \Delta^w (I - \Phi \Delta^w)^{-1} \Phi^w) \Phi^w \|_{\infty \rightarrow \infty} \leq \| F_j^T (\Phi^w + \Phi \Delta^w (I - \Phi \Delta^w)^{-1} \Phi^w) \|_{\infty \rightarrow \infty} \sigma_w.$$

It can be easily checked that $\| \Delta^w \|_{\infty \rightarrow \infty} \leq \epsilon_A + \epsilon_B = \epsilon$ and $(I - \Phi \Delta^w)^{-1}$ exists. By Theorem 5, we have that

$$\| F_j^T (\Phi^w + \Phi \Delta^w (I - \Phi \Delta^w)^{-1} \Phi^w) \|_{\infty \rightarrow \infty} \leq \beta_j$$

is a sufficient condition for

$$\| F_j^T (\Phi^w + \Phi \Delta^w (I - \Phi \Delta^w)^{-1} \Phi^w) \|_{\infty \rightarrow \infty} \leq \beta_j,$$

and $1$ $\leq \beta_j \sigma_w$ if $\| F_j^T (\Phi^w + \Phi \Delta^w (I - \Phi \Delta^w)^{-1} \Phi^w) \|_{\infty \rightarrow \infty} \leq \beta_j$. To reduce the number of hyperparameters, we replace $\beta_j$ by a uniform $\beta$ for all $j$. This still guarantees the robustness of the constraints in (22) and can be seen as a tradeoff between computational complexity and conservativeness.

Bounding $2$: Since $\Delta = Z [\Delta_A | \Delta_B]$ and $Z \Delta_A, Z \Delta_B$ are strict block lower-triangular matrices, we have $(\Phi^w \Delta^w)^{T+1} = 0$ and $(I - \Phi \Delta^w)^{-1} = \sum_{k=0}^T (\Phi^w \Delta^w)^k$. Then we can bound term $2$ as:

$$F_j^T (\Phi^w \Delta^w (I - \Phi \Delta^w)^{-1} \Phi^0 x_0) = F_j^T \sum_{k=1}^T (\Phi^w \Delta^w)^k \Phi^0 x_0$$

$$= F_j^T \sum_{k=0}^{T-1} (\Phi^w \Delta^w)^k \Delta^w \Phi^0 x_0$$

$$\leq \| F_j^T (\Phi^w) \|_1 \sum_{k=0}^{T-1} (\Delta^w \Phi^w)^k \Delta^w \Phi^0 x_0 \|_{\infty \rightarrow \infty}$$

$$\leq \| F_j^T (\Phi^w) \|_1 \sum_{k=0}^{T-1} (\Delta^w \Phi^w)^k \|_{\infty \rightarrow \infty} \| \Delta^w \Phi^0 x_0 \|_{\infty \rightarrow \infty}$$

$$\leq \| F_j^T (\Phi^w) \|_1 \sum_{k=0}^{T-1} \| \Delta^w \Phi^w \|_{\infty \rightarrow \infty} \| \Delta^w \Phi^0 x_0 \|_{\infty \rightarrow \infty}$$

$$\leq \| F_j^T (\Phi^w) \|_1 \frac{1 - T \tau}{1 - \tau} \gamma$$

for any $\tau, \gamma > 0$ such that $\| \Delta^w \Phi^w \|_{\infty \rightarrow \infty} \leq \tau$ and $\| \Delta^w \Phi^0 x_0 \|_{\infty \rightarrow \infty} \leq \gamma$. The first inequality applies the Hölder’s inequality while the rest inequalities are from the triangular inequality and the submultiplicativity of the $\infty \rightarrow \infty$ induced norm.

Bounds with $\tau, \gamma$: First we bound $\| \Delta^w \Phi^w \|_{\infty \rightarrow \infty}$ by:

$$\| \Delta^w \Phi^w \|_{\infty \rightarrow \infty} \leq \left\| \left[ \frac{\Delta^w}{\epsilon_A} \right] \frac{\Delta^w}{\epsilon_B} \right\|_{\infty \rightarrow \infty} \| \Phi^w \|_{\infty \rightarrow \infty}$$

$$\leq \left[ \frac{\epsilon_A}{\epsilon_B} \right] \frac{\epsilon_B}{\epsilon_A} \| \Phi^w \|_{\infty \rightarrow \infty}$$

$$\leq \frac{\epsilon_B}{\epsilon_A} \left[ \frac{\epsilon_A}{\epsilon_B} \right] \| \Phi^w \|_{\infty \rightarrow \infty} \leq \gamma / 2$$

for any $\alpha \in (0, 1)$. The first inequality follows from submultiplicativity and the second inequality holds because the induced infinity-norm is the maximum absolute row sum matrix. For $\| \Delta^w \Phi^0 x_0 \|_{\infty \rightarrow \infty}$ a similar upperbound can be obtained. Then by setting $\alpha = 0.5$, we obtain that the following conditions

$$\| \epsilon_A \Phi^w \|_{\infty \rightarrow \infty} \leq \tau / 2, \quad \| \epsilon_B \Phi^w \|_{\infty \rightarrow \infty} \leq \gamma / 2$$

guarantee that $\| \Delta^w \Phi^w \|_{\infty \rightarrow \infty} \leq \tau$ and $\| \Delta^w \Phi^0 x_0 \|_{\infty \rightarrow \infty} \leq \gamma$ hold for all $\| \Delta_A \|_{\infty \rightarrow \infty} \leq \epsilon_A, \| \Delta_B \|_{\infty \rightarrow \infty} \leq \epsilon_B$.

Putting everything together, we prove Theorem 4.

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