Abstract. We show that a gentle algebra over a field is \( \tau \)-tilting finite if and only if it is representation-finite. The proof relies on the “brick-\( \tau \)-tilting correspondence” of Demonet–Iyama–Jasso and on a combinatorial analysis.

Contents

1. Introduction and main result 1
2. The “brick-\( \tau \)-tilting correspondence” and two reduction results 2
3. Two classes of examples 3
4. Proof of the main Theorem 4

1. Introduction and main result

The theory of \( \tau \)-tilting was introduced in [AIR14] as a far-reaching generalization of classical tilting theory for finite-dimensional associative algebras. One of the main classes of objects in the theory is that of \( \tau \)-rigid modules: a module \( M \) over an algebra \( \Lambda \) is \( \tau \)-rigid if the space of morphisms \( \text{Hom}_\Lambda(M, \tau M) \) vanishes, where \( \tau \) is the Auslander–Reiten translation. In [DIJ17], conditions were established for an algebra \( \Lambda \) to admit only finitely many isomorphism classes of indecomposable \( \tau \)-rigid modules. Such an algebra is called \( \tau \)-tilting finite.

An obvious sufficient condition for an algebra to be \( \tau \)-tilting finite is for it to be representation-finite. This is not a necessary condition: for instance, if \( k \) is a field, then the algebra \( k\langle x, y \rangle/(x^2, y^2, xy, yx) \) is representation-infinite (since it is a string algebra in the sense of [BR87] and admits at least one band, namely \( xy^{-1} \)), but it is \( \tau \)-tilting finite (since it is local). Our aim in this note is to prove that, for a certain class of algebras called gentle algebras, representation-finiteness and \( \tau \)-tilting finiteness are equivalent conditions.

Gentle algebras form a subclass of the class of string algebras. They enjoy a simple definition in terms of generators and relations: a gentle algebra is a finite-dimensional algebra isomorphic to a quotient of a path algebra of a finite quiver \( Q \) by an ideal \( I \) generated by paths of length two, satisfying the condition that for every
vertex $v$ of $Q$, the minimal full subquiver with relation of $\tilde{Q} = (Q, I)$ containing $v$ and all arrows attached to $v$ is a full subquiver with relations of the one depicted below, where dotted line indicate relations.

\[
\begin{array}{c}
\vdots \\
v \\
\vdots \\
\end{array}
\]

Despite their simple definition, gentle algebras are encountered in many different contexts. They were introduced in \cite{AS87} in the study of iterated tilted algebras of type $\tilde{A}_m$, but have recently appeared in connection with dimer models \cite{Boc12, Bro12}, enveloping algebras of some Lie algebras \cite{HK06}, cluster algebras and categories arising from triangulated surfaces \cite{LP09, ABCJP10}, $m$-Calabi–Yau tilted algebras \cite{GE17, GE18}, non-kissing complexes of grids and associated objects \cite{McC17, GM18, PPP17, BDM17}, non-commutative nodal curves \cite{BD18}, and partially wrapped Fukaya categories \cite{HKK17, LP18}. Surface models have been introduced to study the category representations of a gentle algebra and associated categories \cite{BCS18, OPS18, PPP18}.

In this note, we prove the following theorem on gentle algebras. It is proved for Schurian gentle algebras in \cite{DIP} (see also \cite{Dem17}).

\begin{theorem} \label{thm:main}
A gentle algebra is $\tau$-tilting finite if and only if it is representation-finite.
\end{theorem}

The proof of the theorem uses the “brick-$\tau$-tilting correspondence” of \cite{DM17} (recalled in Section \ref{sec:prelim}), and applies a reduction of any gentle algebra to two classes of examples (studied in Section \ref{sec:examples}).

\section*{Acknowledgements}
I am grateful to Laurent Demonet and Yann Palu for informing me of their work in preparation \cite{DIP}, and to Vincent Pilaud for several discussions on $\tau$-tilting finite gentle algebras. I also thank Kaveh Mousavand for showing me families of examples of representation-infinite string algebras which are $\tau$-tilting finite, for letting me know of his ongoing work on special biserial algebra regarding questions similar to those treated in this note, and for pointing out a mistake in the proof of Proposition \ref{prop:tau} in an earlier version of this note.

\section*{Conventions and a note on terminology}
All algebras in this paper are finite-dimensional over a base field $k$, which is arbitrary. We compose arrows in quivers from left to right: if $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$ is a quiver, then $\alpha \beta$ is a path, while $\beta \alpha$ is not. For any arrow $\alpha$ of a quiver, we denote its source by $s(\alpha)$ and its target by $t(\alpha)$; we extend this notation naturally to the formal inverse $\alpha^{-1}$ of $\alpha$.

In order to keep this note short, the notions of strings, bands and string modules from \cite{BR87} will be used freely and without further introduction.
2. The “brick-τ-tilting correspondence” and two reduction results

We will be using two results, Corollaries 2.3 and 2.4, allowing us to reduce the number of vertices of our bound quivers. They will both follow from the following consequence of the brick-τ-tilting correspondence. Recall that a brick is a module whose endomorphism algebra is a division algebra.

**Theorem 2.1** (Theorem 1.4 of [DLJ17]). A finite-dimensional algebra \( \Lambda \) is τ-tilting finite if and only if there are only finitely many isomorphism classes of \( \Lambda \)-modules which are bricks.

**Corollary 2.2.** Let \( \Lambda \) and \( \Lambda' \) be two finite-dimensional algebra, and assume that there exists a fully faithful functor \( F : \text{mod } \Lambda \rightarrow \text{mod } \Lambda' \). If \( \Lambda' \) is τ-tilting finite, then so is \( \Lambda \).

**Proof.** If \( B \) is a brick over \( \Lambda \), then \( FB \) is a brick over \( \Lambda' \), since \( \text{End}_\Lambda(B) \) is isomorphic to \( \text{End}_{\Lambda'}(FB) \). Moreover, two bricks \( B \) and \( B' \) over \( \Lambda \) are isomorphic if and only if \( FB \) and \( FB' \) are isomorphic. Therefore, if \( \Lambda \) admits infinitely many bricks, then so does \( \Lambda' \). The result then follows from Theorem 2.1. \( \square \)

**Corollary 2.3** (First reduction, Theorem 5.12(d) of [DIR+18]). If \( \Lambda \) is τ-tilting finite and \( I \) is an ideal in \( \Lambda \), then \( \Lambda/I \) is τ-tilting finite.

**Proof.** Apply Corollary 2.2 to \(- \otimes_{\Lambda/I} \Lambda/I : \text{mod } \Lambda/I \rightarrow \text{mod } \Lambda. \) \( \square \)

**Corollary 2.4** (Second reduction). If \( \Lambda \) is τ-tilting finite and \( e \in \Lambda \) is an idempotent, then \( e\Lambda e \) is τ-tilting finite.

**Proof.** Apply Corollary 2.2 to \( \text{Hom}_{e\Lambda e}(\Lambda e, ?) : \text{mod } e\Lambda e \rightarrow \text{mod } \Lambda. \) Alternatively, apply [PPS18, Theorem 1.1] for a proof which does not use the brick-τ-tilting correspondence. \( \square \)

**Remark 2.5.** In practice, Corollary 2.3 implies that erasing arrows or vertices from a τ-tilting finite algebra yields another τ-tilting finite algebra, and Corollary 2.4 implies that erasing an arrow and replacing the paths of length 2 that went through it by “shortcut” arrows also preserves τ-tilting finiteness.

3. Two classes of examples

Our strategy to prove Theorem 1.1 will be to reduce any gentle algebra to the two classes of examples presented in this section.

**Example 3.1** (Type \( \tilde{A}_m \)). Let \( Q \) be a quiver of type \( \tilde{A}_m \), that is to say, an orientation of the following diagram with \( m + 1 \) vertices, where \( m \geq 1 \).

```
  • ——— • ——— • ——— •
  \  / \  / \  / \\
  • ——— • ——— • ——
  /     /     /   \\
  • ——— • ——— •——
```

The representation theory of the path algebra \( \Lambda = kQ \) is very well understood in this case, see for instance [ASS06, Section VIII.2]. In particular, this algebra is τ-tilting infinite.
Example 3.2. The second class of examples that we will consider will be given by the path algebras of quivers $Q$ defined by any orientation of the diagram below,

modulo the relations $\alpha_r \alpha_1$ and $\gamma_t \gamma_1$ (note that the orientations of $\alpha_1, \alpha_r, \gamma_1$ and $\gamma_t$ are imposed, while those of other arrows can be arbitrary). We allow $r = 1$ (in which case $\alpha_1 = \alpha_r$ is a loop whose square vanishes); we allow the same for $t$. We also allow $s = 0$; in this case, we require that the cycle on the left and the cycle on the right are not both oriented cycles, otherwise the algebra would be infinite-dimensional.

Proposition 3.3. The algebras defined in Example 3.2 are $\tau$-tilting infinite.

Proof. Let $\Lambda$ be an algebra in the class defined in Example 3.2.

We first deal with the case where $s = 0$. In this case, let $v$ be the vertex common to both cycles. Let $e = 1 - e_v$. Then the algebra $e \Lambda e$ is of type $A_m$, so it is $\tau$-tilting infinite. By Corollary 2.4, $\Lambda$ is $\tau$-tilting infinite.

Assume, therefore, that $s \geq 1$. We will construct an infinite family of bricks for $\Lambda$; by the brick-$\tau$-tilting correspondence (see Theorem 2.1), this will imply that $\Lambda$ is $\tau$-tilting infinite.

Let $b' = \alpha_1 \alpha_2 \ldots \alpha_{r-1} \alpha_r$ be the string corresponding to the cycle on the left, $b'' = \gamma_1 \gamma_2 \ldots \gamma_{t-1} \gamma_t$ be the one corresponding to the cycle on the right, and $\omega = \beta_1^{s_1} \ldots \beta_s^{s_s}$ be the middle string followed from left to right, where the $\delta_i, \varepsilon_i$ and $\zeta_i$ are the appropriate signs.

Define $b = (b')^{s_1} \omega (b'')^{s_1} \omega^{-1}$. We claim that the string module defined by $b$ is a brick. To prove this, we need to prove that the only substring of $b$ appearing both on top of and at the bottom of $b$ is $b$ itself, so that the endomorphism ring of the string module is isomorphic to the base field $k$ (using the description of all morphisms between string modules obtained in [CB89]). Here, we say that a substring $\sigma'$ of a string $\sigma$ is on top of $\sigma$ if the arrows in $\sigma$ adjacent to $\sigma'$ are leaving $\sigma'$, and that it is at the bottom of $\sigma$ if the arrows in $\sigma$ adjacent to $\sigma'$ are entering $\sigma'$.

Note first that the middle copy of $\omega$ is neither on top of nor at the bottom of $b$. Indeed, if $\varepsilon_1 = 1$, then $\omega$ is not at the bottom, since the first arrow of $(b'')^{s_1}$ is direct, and $\omega$ is not on top, since the last arrow of $(b')^{s_1}$ is direct; if $\varepsilon_1 = -1$, then $\omega$ is not at the bottom, since the last arrow of $(b')^{s_1}$ is inverse, and $\omega$ is not on top, since the last arrow of $(b'')^{s_1}$ is inverse.

Next, let us deal with the substrings of length 0 of $b$. The starting point of $b$ is on top if $\varepsilon_1 = 1$ or at the bottom if $\varepsilon_1 = -1$. It appears twice more in $b$: at the end of $b'$ and the end of $b$. At the end of $b'$ it is neither on top nor at the bottom, and at the end of $b$ it is on top if $\varepsilon_1 = 1$ or at the bottom if $\varepsilon_1 = -1$. Therefore this vertex does not occur both on top and at the bottom of $b$. The other vertices appearing several times in $b$ are the vertices of $\omega$ and the starting/ending point of $b''$. The former appear either twice at the top or twice at the bottom of $b$. The
latter cannot be both on top and at the bottom, since it appears in the middle of paths of length 2 of the following form: if $\varepsilon_1 = 1$, then these paths are $\beta_s^+ \gamma_1$ and $\gamma_1 \beta_s^- \varepsilon_1$, and if $\varepsilon_1 = -1$, then they are $\beta_s^- \gamma_1$ and $\gamma_1^\varepsilon_1 \beta_s^+$. In both cases the middle vertices is either on top of $b$ or at the bottom of $b$, but not both. Thus no substring of length 0 appears both at the top and the bottom of $b$.

Assume that there is a substring $\rho$ of length at least 1, different from $b$, which appears both on top and at the bottom of $b$. Since the only arrows of $b$ that are used twice are those of $\omega$, $\rho$ has to be a substring of $\omega$ and of $\omega^{-1}$. Since $\omega$ does not go twice through the same vertex, the only substring both on top and at the bottom of $\omega$ is $\omega$ itself. Hence $\rho = \omega$. But we saw above that the middle substring $\omega$ is neither on top nor at the bottom of $b$. This is a contradiction.

Thus the string module defined by $b$ is a brick.

Using the above arguments, one can also check that all powers of $b$ define bricks as well. Thus $\Lambda$ admits infinitely many pairwise non-isomorphic bricks, and by the brick-$\tau$-tilting correspondence (see Theorem 2.1), it is $\tau$-tilting infinite. □

4. Proof of the main Theorem

We now prove Theorem 1.1. Let $\tilde{Q} = (Q, I)$ be a gentle bound quiver. Assume that the algebra $\Lambda = kQ/I$ is of infinite representation type. Let us show that it is $\tau$-tilting infinite.

By [BR87], there exists a band $b$ on $\tilde{Q}$. Our strategy will be to reduce to one of the two cases in the following lemma.

Lemma 4.1. The algebra $\Lambda$ is $\tau$-tilting infinite if $\tilde{Q}$ admits a band $b$ satisfying one of the following conditions:

1. $b$ does not go through the same vertex twice (except for the starting and ending points of $b$); or
2. $b$ has the form $b = b' \omega b'' \omega^{-1}$, where
   - $b'$ and $b''$ are strings such that $s(b') = t(b')$ and $s(b'') = t(b'')$;
   - $(b')^2$ and $(b'')^2$ are not strings;
   - $\omega$ is a possibly trivial string;
   - none of $b'$, $b''$ and $\omega$ go through the same vertex twice (except for the endpoints of $b'$ and $b''$);
   - the only vertices that $b'$, $b''$ and $\omega$ may have in common are their ending points.

Proof. Let $I$ be the ideal generated by all arrows and vertices through which $b$ does not go. If we are in case (1), then $\Lambda/I$ is isomorphic to the path algebra of a quiver of type $\tilde{A}_m$. These algebras are $\tau$-tilting infinite, so the result follows from Corollary 2.3. If we are in case (2), then $\Lambda/I$ is isomorphic to an algebra in the class defined in Example 3.2. By Proposition 3.3, these are $\tau$-tilting infinite, so the result follows again from Corollary 2.3. □

To prove Theorem 1.1 it is therefore sufficient to show that a band as in Lemma 4.1 always exists.

Let $b$ be a band of minimal length on $\tilde{Q}$. If $b$ does not go through the same vertex twice (except at its endpoints), then by Lemma 4.1 (1), the theorem is proved.

Assume, therefore, that there is a vertex $u$ through which $b$ passes twice. Up to cyclic reordering of $b$, we can assume that this vertex is the starting point of $b$. Up
to choosing another such vertex \( u \), we can also assume that \( b = b'b'' \), with \( b' \) and \( b'' \) non-trivial strings starting and ending at \( u \) and such that \( b' \) does not go through the same vertex twice (except at its endpoints). Note that, by minimality of \( b \), the string \( b' \) cannot be a band; that is, \((b')^2 \) cannot be a string. The same is true for \( b'' \).

**Lemma 4.2.** Let \( b = b'b'' \) be as above. Then \( b' \) and \( b'' \) cannot have a vertex in common apart from their starting and ending points.

*Proof.* Assume that \( b' \) and \( b'' \) have another vertex in common, and let \( v \) be such a vertex. Write \( b' = \alpha_1^{j_1} \cdots \alpha_r^{j_r} \) and \( b'' = \beta_1^{j_1} \cdots \beta_s^{j_s} \), and assume that \( s(\alpha_j^{j_j}) = v = t(\beta_k^{j_k}) \), with \( i \neq 1, r \) and \( j \neq 1, s \). By minimality of \( b \), \( \alpha_1^{j_1} \alpha_2^{j_2+1} \cdots \alpha_r^{j_r} \beta_1^{j_1} \cdots \beta_s^{j_s} \) cannot be a band. This implies that \( \delta_j = \varepsilon_k \) and that the composition of \( \alpha_j^{j_j} \) and \( \beta_k^{j_k} \) is a relation in \( \hat{Q} \). But then \( \alpha_j^{- \delta_j - 1} \alpha_j^{- \delta_j - 2} \cdots \alpha_1^{- \delta_1} \beta_1^{j_1} \cdots \beta_s^{j_s} \) is a band, since \( \beta_k^{j_k} \) cannot be in a relation both with \( \alpha_j^{j_j} \) and with \( \alpha_j^{- \delta_j - 1} \). This again contradicts the minimality of \( b \).

Now, if \( b'' \) does not go twice through the same vertex (except for its endpoints), then we are in case (2) of Lemma [4.1] with \( \omega \) trivial, and the theorem is proved. Assume thus that \( b'' \) does go twice through a vertex \( v \) outside its endpoints (or three times through \( v \) if it is the endpoint of \( b'' \)). Up to choosing another vertex \( v \), we can assume that there is a substring \( b'' = \beta_j^{j_j} \beta_{j+1}^{j+1} \cdots \beta_{j_k}^{j_k} \) of \( b'' \) which starts and ends in \( v \), does not go through the same vertex twice outside its endpoints, and such that \( \omega := \beta_1^{j_1} \cdots \beta_{j_k - 1}^{j_k} \) does not go through the same vertex twice.

If \( \beta_k \) and \( \beta_{j-1} \) form a relation of \( \hat{Q} \), then \( \beta_j \) and \( \beta_k \) cannot form a relation, since there is at most one involving \( \beta_k \) and \( \beta_{j-1} \), \( \beta_j \) are distinct. Thus \( b'' \) is a band with no repeated vertices (except for its endpoints), and we have reduced to case (1) of Lemma [4.1] proving the theorem.

If, on the other hand, \( \beta_k \) and \( \beta_{j-1} \) do not form a relation, then \( b'' \omega b'' \omega^{-1} \) is a band satisfying the conditions of case (2) of Lemma [4.1].

This finishes the proof of Theorem [4.1].

---

**References**

[ABCJP10] Ibrahim Assem, Thomas Brüstle, Gabrielle Charbonneau-Jodoin, and Pierre-Guy Plamondon. Gentle algebras arising from surface triangulations. *Algebra Number Theory*, 4(2):201–229, 2010.

[AIR14] Takahide Adachi, Osamu Iyama, and Idun Reiten. \( \tau \)-tilting theory. *Compos. Math.*, 150(3):415–452, 2014.

[AS87] Ibrahim Assem and Andrzej Skowroński. Iterated tilted algebras of type \( \tilde{A}_n \). *Math. Z.*, 195(2):269–290, 1987.

[ASS06] Ibrahim Assem, Daniel Simson, and Andrzej Skowroński. *Elements of the representation theory of associative algebras. Vol. 1*, volume 65 of London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 2006. Techniques of representation theory.

[BCS18] Karin Baur and Raquel Coelho Simões. A geometric model for the module category of a gentle algebra. Preprint, [arXiv:1803.05802] 2018.

[BD18] Igor Burban and Yuriy Drozd. Non-commutative nodal curves and derived tame algebras. Preprint, [arXiv:1805.05174 [math.AG]] 2018.

[BDM+17] Thomas Brüstle, Guillaume Dubville, Kaveh Mousavand, Hugh Thomas, and Emine Yıldırım. On the combinatorics of gentle algebras. Preprint, [arXiv:1707.07665] 2017.

[Boc12] Raf Bocklandt. Consistency conditions for dimer models. *Glasg. Math. J.*, 54(2):429–447, 2012.
τ-tilting finite gentle algebras are representation-finite

[BR87] M. C. R. Butler and Claus Michael Ringel. Auslander-Reiten sequences with few middle terms and applications to string algebras. *Comm. Algebra*, 15(1-2):145–179, 1987.

[Bro12] Nathan Broomhead. Dimer models and Calabi-Yau algebras. *Mem. Amer. Math. Soc.*, 215(1011):viii+86, 2012.

[CB89] William W. Crawley-Boevey. Maps between representations of zero-relation algebras. *J. Algebra*, 126(2):259–263, 1989.

[Dem17] Laurent Demonet. Combinatorics of Mutations in Representation Theory. Thesis, Habilitation à diriger les recherches, 2017.

[DIJ17] Laurent Demonet, Osamu Iyama, and Gustavo Jasso. τ-tilting finite algebras, bricks, and g-vectors. *International Mathematics Research Notices*, page rnx135, 2017.

[DIP] Laurent Demonet, Osamu Iyama, and Yann Palu. τ-tilting modules over Schurian algebras. In preparation.

[DIR+18] Laurent Demonet, Osamu Iyama, Nathan Reading, Idun Reiten, and Hugh Thomas. Lattice theory of torsion classes. Preprint, [arXiv:1711.01785 [math.RT]] 2018.

[GE17] Ana Garcia Elsener. Gentle m-Calabi-Yau tilted algebras. Preprint, [arXiv:1701.07968 [math.RT]] 2017.

[GE18] Ana Garcia Elsener. Monomial Gorenstein algebras and the stably Calabi–Yau property. Preprint, [arXiv:1807.07018 [math.RT]] 2018.

[GM18] Alexander Garver and Thomas McConville. Oriented flip graphs of polygonal subdivisions and noncrossing tree partitions. *J. Combin. Theory Ser. A*, 158:126–175, 2018.

[HK06] Ruth Stella Huerfano and Mikhail Khovanov. Categorification of some level two representations of quantum $sl_n$. *J. Knot Theory Ramifications*, 15(6):695–713, 2006.

[HKK17] Fabian Haiden, Ludmil Katzarkov, and Maxim Kontsevich. Flat surfaces and stability structures. *Publ. Math. Inst. Hautes Études Sci.*, 126:247–318, 2017.

[LF09] Daniel Labardini-Fragoso. Quivers with potentials associated to triangulated surfaces. *Proc. Lond. Math. Soc. (3)*, 98(3):797–839, 2009.

[LP18] Yanki Lekili and Alexander Polishchuk. Derived equivalences of gentle algebras via Fukaya categories. Preprint, [arXiv:1801.06370 [math.SG]] 2018.

[McC17] Thomas McConville. Lattice structure of Grid-Tamari orders. *J. Combin. Theory Ser. A*, 148:27–56, 2017.

[OPS18] Sebastian Opper, Pierre-Guy Plamondon, and Sibylle Schroll. A geometric model for the derived category of gentle algebras. Preprint, [arXiv:1801.09659] 2018.

[PPP17] Yann Palu, Vincent Pilaud, and Pierre-Guy Plamondon. Non-kissing complexes and tau-tilting for gentle algebras. Preprint, [arXiv:1707.07574] 2017.

[PPP18] Yann Palu, Vincent Pilaud, and Pierre-Guy Plamondon. Non-kissing and non-crossing complexes for locally gentle algebras. Preprint, [arXiv:1807.04750 [math.CO]] 2018.

[PPS18] Vincent Pilaud, Pierre-Guy Plamondon, and Salvatore Stella. A τ-tilting approach to dissections of polygons. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 14:Paper No. 045, 8, 2018.

Laboratoire de Mathématiques d’Orsay, Université Paris-Sud, CNRS, Université Paris-Saclay, 91405 Orsay, France

E-mail address: pierre-guy.plamondon@math.u-psud.fr