Research Article

Classification of the Quasifiliform Nilpotent Lie Algebras of Dimension 9

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Received 3 November 2013; Accepted 2 January 2014; Published 6 March 2014

Academic Editor: Peter G. L. Leach

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On the basis of the family of quasifiliform Lie algebra laws of dimension 9 of 16 parameters and 17 constraints, this paper is devoted to identify the invariants that completely classify the algebras over the complex numbers except for isomorphism. It is proved that the nullification of certain parameters or of parameter expressions divides the family into subfamilies such that any couple of them is nonisomorphic and any quasifiliform Lie algebra of dimension 9 is isomorphic to one of them. The iterative and exhaustive computation with Maple provides the classification, which divides the original family into 263 subfamilies, composed of 157 simple algebras, 77 families depending on 1 parameter, 24 families depending on 2 parameters, and 5 families depending on 3 parameters.

1. Introduction

The interest in classifying nilpotent Lie algebras is broad both within the academic community and the industrial engineering community, since they are applied in classical mechanical problems and current research in scientific disciplines as modern geometry, solid state physics, or particle physics [1–5]. Lie algebras classification consists in determining equivalence relations that subdivide the original set in equivalence classes defined by at least one element in each set, and it is usual to classify the algebras except for isomorphisms. The solvable Lie algebras classification problem comes down in a sense to the nilpotent Lie algebras classification [6] and computer algebra has been indispensable. However, the more the dimension increases, the more and more complex is the determination of exhaustive lists of Lie algebras, so new computation methodologies are a present field of research [7, 8] with current symbolic manipulation programs such as Reduce, Mathematica, or Maple [9].

The classification of nilpotent Lie algebras over the complex numbers experimented an important advance based on the works of Ancochá-Bermúdez and Goze [10] introducing an invariant more potent than the previously existing: the characteristic sequence or Goze’s invariant (defined in Section 2.1). Those authors were able, by using the characteristic sequence as an invariant, to classify the nilpotent Lie algebras of dimension 7 [11] and the filiform Lie algebras of dimension 8 [12]. Later, by using that invariant, Gomez and Echarte [13] classify the filiform Lie algebras of dimension 9. Afterward, Castro et al. [14] develop an algorithm for symbolic language for finding the generic families of filiform Lie algebras in any dimension with the restrictions required to the parameters.

Subsequent works about quasifiliform Lie algebras classification were centered on specific types of families or subclasses, obtaining results applicable to higher dimensions. For instance, the classifications of naturally graded [15] and graded by derivations [16] quasifiliform Lie algebras. These works extended to other algebras, with a high nilindex, the classification of graded filiform Lie algebras, studied initially by Vergne [17, 18], obtained from the gradation related to the filtration produced in a natural way by the descending central sequence.

In this paper we focus on a method of identification of the invariants that completely classify the nilpotent Lie algebras of dimension 9 over the complex numbers except for isomorphisms. With this aim, the dimensions of the subalgebras of its derived series, of its descending central
series, and of its descending central series centralizers are used as class invariants. The exhaustive analysis has been developed with significant computational effort; the total code is 2820 pages in 37 files, summing more than 12000 lines of Maple code, and these programs have provided 3038 pages of results [19]. We strongly recommend the reading of Bäuerle and de Kerf [20], Benjumea et al. [21], and Sendra et al. [22] to become familiar with Lie algebras terminology and symbolic computation with Maple.

2. The Subfamilies of Laws

2.1. Preliminaries. Let \( g \) be a nilpotent Lie algebra; the characteristic sequence of \( \text{ad}(X) \) is denoted by \( c(X) = (c_1, \ldots, c_k, 1) \), and for the lexicographic order \( \text{c}(g) = \max_{x \in g \setminus [g, g]} c(X) \) is known as the Goze’s invariant or characteristic sequence [23]. Obviously \( \text{c}(g) \) is an invariant for the isomorphisms and, by construction, there is at least one vector \( X \in g \setminus [g, g] \) such that \( \text{c}(g) = c(X) \); all vector verifying this condition is called characteristic vector of the algebra.

The abelian algebra of dimension \( n \) is the only one with Goze’s invariant \((1, \ldots, 1)\); in abelian algebras the characteristic sequence is \((2, \ldots, 2, 1, \ldots, 1)\), in Heisenberg algebras it is \((n - 1, 1)\), and in quasifiliform algebras it is \((n - 2, 1, 1)\). A Lie algebra \( g \) is nilpotent if and only if the characteristic polynomial of the matrix \( \text{ad}(x) = \lambda^9 \), for every vector \( x \) of \( g \). Anyway this condition is often difficult to be applied, so the moment in the process, when the nilpotence condition should be applied or, much better, when the condition should be applied for each vector, has to be chosen carefully. The condition of being quasifiliform can be also interpreted in terms of matrices. Thus the vectors candidate to characteristic vectors, that is, the vectors in \( g \setminus [g, g] \), have to satisfy that the respective adjoint matrices do not have nonnull minors of order \( \leq 7 \). As in the case of the nilpotence, this condition has to be applied with caution and in several stages.

Every quasifiliform Lie algebra of dimension 9 can have an adapted base \( \{x_0, x_1, \ldots, x_8\} \) such that

\[
[x_0, x_i] = x_{i+1}, \quad 1 \leq i \leq 6; \quad [x_0, x_1] = 0, \quad 7 \leq i \leq 8.
\]

(1)

On the whole all the bracket products can be described by

\[
[x_j, x_k] = \sum_{i=0}^{n-1} c_{ij}^{nk} \cdot x_k, \quad 0 \leq i, j \leq n - 1,
\]

(2)

where \( c_{ij}^{nk} \) are the algebra structure constants.

The laws of every complex quasifiliform Lie algebra (QFLA) of dimension 9 can be described by the following family with 16 parameters and 17 polynomial restriction equations [19] derived from the Jacobi identity:

\[ [x_0, x_1] = x_{i+1}, \quad 1 \leq i \leq 6, \quad (3a) \]

\[ [x_1, x_2] = \alpha_1 x_4 + \alpha_2 x_5 + \alpha_3 x_6 + \alpha_4 x_7 + \alpha_5 x_8, \quad (3b) \]

\[ [x_1, x_3] = \alpha_1 x_5 + \alpha_2 x_6 + \alpha_3 x_7, \quad (3c) \]

\[ [x_1, x_5] = 2\alpha_6 x_6 + (2\alpha_7 - 1) x_7, \quad (3e) \]

\[ [x_1, x_6] = \alpha_8 x_7 + \alpha_9 x_8, \quad (3f) \]

\[ [x_1, x_8] = \alpha_{10} x_9 + \alpha_{11} x_{10}, \quad (3g) \]

\[ [x_2, x_3] = -\alpha_6 x_5 + (\alpha_1 - \alpha_7) x_6 + (\alpha_2 - \alpha_9) x_7 - \alpha_9 x_8, \quad (3h) \]

\[ [x_2, x_4] = -\alpha_6 x_6 + (\alpha_1 - \alpha_7) x_7, \quad (3i) \]

\[ [x_2, x_5] = (2\alpha_6 - \alpha_{10}) x_7 - \alpha_1 x_8, \quad (3j) \]

\[ [x_2, x_8] = \alpha_{12} x_9 + \alpha_{13} x_{10} + \alpha_{14} x_6 + \alpha_{15} x_7, \quad (3k) \]

\[ [x_3, x_4] = -3\alpha_6 + \alpha_10 x_7 + \alpha_1 x_8, \quad (3l) \]

\[ [x_3, x_5] = \alpha_{12} x_5 + \alpha_{13} x_6 + \alpha_{14} x_7, \quad (3m) \]

\[ [x_4, x_8] = \alpha_{12} x_5 + \alpha_{13} x_6 + \alpha_{14} x_7, \quad (3n) \]

\[ [x_5, x_8] = \alpha_{12} x_7 \]

subject to

\[ \alpha_9 \alpha_{12} = 0, \quad (4a) \]

\[ \alpha_9 \alpha_{12} = 0, \quad (4b) \]

\[ \alpha_9 \alpha_{13} = 0, \quad (4c) \]

\[ \alpha_9 \alpha_{12} = 0, \quad (4d) \]

\[ \alpha_9 \alpha_{13} = 0, \quad (4e) \]

\[ \alpha_9 \alpha_{14} = 0, \quad (4f) \]

\[ \alpha_{10} \alpha_{12} = 0, \quad (4g) \]

\[ \alpha_{11} \alpha_{12} = 0, \quad (4h) \]

\[ \alpha_{11} \alpha_{13} = 0, \quad (4i) \]

\[ \alpha_{11} \alpha_{14} = 0, \quad (4j) \]

\[ \alpha_{11} \alpha_{15} = 0, \quad (4k) \]

\[ \alpha_{11} \alpha_{16} = 0, \quad (4l) \]

\[ \alpha_{11} (3 \alpha_1 - \alpha_7) = 0, \quad (4m) \]

\[ \alpha_{12} (\alpha_1 - \alpha_7) = 0, \quad (4n) \]

\[ \alpha_5 \alpha_{12} - 2\alpha_6^2 - 3\alpha_6 \alpha_{15} = 0, \quad (4o) \]

\[ 2(\alpha_2 - \alpha_6) \alpha_{12} + 3(\alpha_1 - \alpha_7) \alpha_{13} + 2(\alpha_6 - \alpha_{10}) \alpha_{14} = 0, \quad (4p) \]

\[ \alpha_5 \alpha_{14} - 2(2\alpha_1 + \alpha_6) \alpha_6 - \alpha_6 \alpha_{16} + (3\alpha_1 - \alpha_7) \alpha_{10} = 0 \quad (4q) \]

with the application of the Jacobi identity to the 3-tuple \( (x_0, x_i, x_j) \), where \( x_i, x_j \) are base vectors different from
\[ x_0 \text{ vector. Table 1 shows the structure constants corresponding with the 16 parameters. From here forward the Lie Algebra Families will be denoted as } \mu(\alpha_1, \ldots, \alpha_{16}). \]

Our objective is to study exhaustively the case of dimension 9; therefore the coefficients identification is tackled in an iterative and interactive way by imposing the Jacobi identity. Maple programs have been developed so that all the equations resulting from the application of the above-mentioned conditions are obtained, the simplest conditions are applied, and the process is repeated until there are no restrictions of simple application.

The exhaustiveness of the classification is developed by analyzing all the possible combinations of values of the 16 parameters (\(\alpha_1, \ldots, \alpha_{16}\)), which is summarized within the cases shown in the following subsections: case A.1 (\(\alpha_{11} \neq 0\) and \(\alpha_1 \neq 0\)), A.2 (\(\alpha_{11} = 0\) and \(\alpha_1 = 0\)), B.1.1 (\(\alpha_{11} = 0\), \(\alpha_9 = 0\), and \(\alpha_8 = 0\)), B.1.2 (\(\alpha_{11} = 0\), \(\alpha_9 \neq 0\), and \(\alpha_8 = 0\)), B.2.1 (\(\alpha_{11} = 0\), \(\alpha_9 = 0\), and \(\alpha_8 \neq 0\)), and B.2.2 (\(\alpha_{11} = 0\), \(\alpha_9 = 0\), and \(\alpha_8 = 0\)). In all the cases the nonisomorphism is proved in the corresponding propositions.

2.2. General Case

**Proposition 1.** The nilpotent QFLA of dimension 9 and \(\alpha_{11} \neq 0\) are nonisomorphic to the algebras with \(\alpha_{11} = 0\).

**Proof.** For the family described by (3a)–(3o) and (4a)–(4q), its descending central series is \(\mathcal{D}^\infty g = \langle x_2, x_3, x_4, x_5, x_6, x_7, \alpha_8 x_5, \alpha_8 x_6, \alpha_8 x_7, x_1 x_2, \alpha_9 x_6, \alpha_9 x_7, \alpha_{11} x_2, x_1 x_3 \rangle, \mathcal{D}^2 g = \langle x_3, x_4, x_5, x_6, x_7, \alpha_8 x_7, \alpha_9 x_7, \alpha_{11} x_2, x_1 x_3, \alpha_9 x_7, \alpha_{11} x_2, x_1 x_3 \rangle, \mathcal{D}^3 g = \langle x_3, x_4, x_5, x_6, x_7, \alpha_8 x_7, \alpha_9 x_7, \alpha_{11} x_2, x_1 x_3, \alpha_9 x_7, \alpha_{11} x_2, x_1 x_3, \alpha_9 x_7, \alpha_{11} x_2, x_1 x_3 \rangle, \) and so forth. Thus \(\dim \mathcal{D}^2 g = 3\) if \(\alpha_{11} \neq 0\) and \(\dim \mathcal{D}^2 g = 2\) if \(\alpha_{11} = 0\). Therefore the nullity of \(\alpha_{11}\) constitutes the first classification criterion. \(\square\)

2.3. Case A. \(\alpha_{11} \neq 0\).

**Proposition 2.** The nilpotent QFLA of dimension 9 with \(\alpha_{11} \neq 0\) and \(\alpha_1 \neq 0\) are nonisomorphic to the algebras with \(\alpha_{11} \neq 0\) and \(\alpha_1 = 0\).

**Proof.** If \(\alpha_{11} \neq 0\), from restrictions (4a)–(4q) it can be deduced that \(\alpha_6 = \alpha_5 = \alpha_3 = \alpha_8 = 0\) and \(\alpha_7 = 3\alpha_4\). By computing the Jacobi equations, the family of laws is reduced to

\[
[x_0, x_i] = x_{i+1}, \quad 1 \leq i \leq 6, \quad (5a)
\]
\[
[x_1, x_2] = \alpha_4 x_4 + \alpha_5 x_5 + \alpha_4 x_6 + \alpha_4 x_7 + \alpha_4 x_8, \quad (5b)
\]
\[
[x_1, x_3] = \alpha_1 x_5 + \alpha_2 x_5 + \alpha_3 x_6 + \alpha_9 x_7, \quad (5c)
\]
\[
[x_1, x_4] = 3\alpha_1 x_6 + \alpha_6 x_7 + \alpha_9 x_8, \quad (5d)
\]
\[
[x_1, x_5] = 5\alpha_1 x_7, \quad (5e)
\]
\[
[x_1, x_6] = \alpha_3 x_7 + \alpha_6 x_7, \quad (5f)
\]
\[
x_2 = -2\alpha_1 x_6 + (\alpha_2 - \alpha_9) x_7 - \alpha_9 x_8, \quad (5g)
\]
\[
[x_2, x_3] = -2\alpha_1 x_6 + (\alpha_2 - \alpha_9) x_7 - \alpha_9 x_8, \quad (5h)
\]
\[
x_2, x_4 = -2\alpha_1 x_7, \quad (5i)
\]
\[
x_3 = \alpha_1 x_7 + \alpha_1 x_8, \quad (5j)
\]
without restrictions derived from the Jacobi identity (4a)–(4q). It can be observed that \(x_5\) and \(x_8\) are now central; thus the application of the elementary change of base

\[
y_j = x_j, \quad i \neq 8, \quad (6)
\]
\[
y_8 = \alpha_{10} x_7 + \alpha_{11} x_8 \quad (6)
\]
permits us to suppose that \(\alpha_{10} = 0\) and \(\alpha_{11} = 1\). Then (5f), (5i), and (5j) are simplified and the derived series is \(\mathcal{D}^2 g = \langle x_2, x_3, x_4, x_5, x_6, x_7, \alpha_8 x_7, \alpha_9 x_7, -2\alpha_1 x_6, (\alpha_2 - \alpha_9) x_7, -\alpha_9 x_8, -2\alpha_7 x_7, \alpha_9 x_8 \rangle\), and so forth. Thus \(\dim \mathcal{D}^2 g = 3\) if \(\alpha_1 \neq 0\) and \(\dim \mathcal{D}^2 g \leq 2\) if \(\alpha_1 = 0\). Therefore the nullity of \(\alpha_1\) constitutes a new classification criterion. \(\square\)

In this subsection (case A), the notation to describe the parameters of the subfamily \(i\) is reduced to \(\mu_i (\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_7, \alpha_6, \alpha_9, \alpha_8)\) for simplification. Figure 1 shows the classification in 19 subfamilies in the case A. They are classified with the criteria summarized in Figure 2 and detailed in the following cases.

2.3.1. Case A.1. One has \(\alpha_{11} \neq 0\) and \(\alpha_1 \neq 0\).

**Proposition 3.** Case A.1 permits us to suppose that \(\alpha_1 = 1\).

**Proof.** With the elementary change of base CB,

\[
y_0 = x_0, \quad (7)
\]
\[
y_j = \frac{x_j}{\alpha_1}, \quad 1 \leq j \leq 7,
\]
\[
|CB| = 1/\alpha_1 \neq 0 \text{ and then } \alpha_1 = 1. \quad \square
\]

The subfamilies of laws with the structure (5a)–(5j) are \(\mu_i (1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, 3, \alpha_9, \alpha_8)\) with \(i = 1, 2, 3\).

Let us denote, from here forward, by \(\delta\) the new parameters obtained from the changes of base and \(\tilde{\mu}\) the Lie algebra families depending on these new parameters \(\delta\) (which in general depend on the 16 parameters \(\mu\)), in order to differentiate the new representation \(\tilde{\mu} (\delta, \alpha)\) from the representation of the families depending in general on the 16 parameters \(\mu_i (\alpha_1, \ldots, \alpha_{16})\).
Proposition 4. The nilpotent QFLA of dimension 9 with $\alpha_{11} \neq 0$ and $\alpha_1 \neq 0$ can be classified in three nonisomorphic subfamilies $\mu_i$ with $i$ from 1 to 3, described in Figure 1, according to the conditions described in Figure 2.

Proof. Let us apply the change of base:

$$
y_0 = P_0 x_0 + P_1 x_1 + P_2 x_2 + P_3 x_3 + P_4 x_4
+ P_5 x_5 + P_6 x_6 + P_7 x_7 + P_8 x_8,
y_1 = Q_0 x_0 + Q_1 x_1 + Q_2 x_2 + Q_3 x_3 + Q_4 x_4
+ Q_5 x_5 + Q_6 x_6 + Q_7 x_7 + Q_8 x_8,
y_{i+1} = [y_0, y_i], \quad 1 \leq i \leq 6,
y_8 = [y_1, y_6].
$$

The subfamilies of laws are $\bar{\mu}_i (\delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_7)$. The determinant of the change matrix is $P_0^{18} (P_0 Q_0 - P_1 Q_0)^9$; thus $P_0 \neq 0$ and $(P_0 Q_0 - P_1 Q_0) \neq 0$. Since the coefficient of $y_3$ and $y_4$ in $[y_1, y_2]$ must be null, then $Q_0 = 0$ and $Q_1 = P_0^5$. Let us apply (8) again and the only restriction is $P_0 \neq 0$. Thus, with the coefficient identifications, the new parameters are

$\delta_2 = \frac{(-2P_1 + \alpha_2 P_0)}{P_0^2}$

$\delta_3 = \frac{(-2Q_0^2 + 4P_0^2 Q_3 - 7P_0 \alpha_2 P_0^3 + 7P_1^2 P_0^2 + \alpha_3 P_0^4)}{P_0^6}$

$\delta_4 = (\alpha_4 P_0^6 - 2\alpha_2^2 P_0^5 P_1 + 2\alpha_2 P_0^3 Q_3 + 2\alpha_2 P_0^2 Q_2^2 - \alpha_2 P_0^5 P_1 \alpha_8 + 2Q_0^3 - 12P_0 \alpha_3 P_0^5
- 42P_1 P_0^3 Q_3 - 24P_2 P_0^3 + 56P_4 P_1^2 \alpha_2 + 18P_1 Q_0^2 P_0 - 6P_2 Q_0^2 Q_3 + 6P_0 Q_4
+ 6P_1 P_0^2 Q_2 + 6P_0^2 P_1 \alpha_9 - 6P_0^5 P_3
+ 2\alpha_3 P_0^3 Q_3 - Q_0^2 P_0^2 \alpha_9) \times (P_0^5)^{-1},$

$\delta_5 = (2P_0 P_1 Q_0 Q_2 + 2P_0 P_2 P_1^2 P_0 Q_2 + 6P_4^2 \alpha_9 P_0^4
- 2P_2 P_0 Q_2^2 - 2P_0 Q_1 Q_2 + 6P_4 P_1 P_0 P_1
- 8P_0^2 Q_2^2 - 2P_0 P_1 Q_0^2 \alpha_2 - 4P_1 P_0^2 Q_2
+ 2P_2 P_0 Q_3 + 2P_0 P_0 Q_2 - 4P_0 Q_3 P_0
+ 20P_0^2 Q_3^2 - Q_0^2 P_0^2 \alpha_0 + 2Q_3 P_0 \alpha_9
+ P_1 Q_0^2 P_0 \alpha_8 - P_1 P_0^2 \alpha_4 + P_0^2 Q_3^2 + 2P_0^2 Q_3
+ P_0^6 \alpha_5 + 2P_1 P_0 Q_0 \alpha_5 - 2P_1 \alpha_9 P_0^3 Q_3
+ \alpha_2^2 P_0^5 P_1^2 \alpha_6 - \alpha_2 P_0^5 P_1 \alpha_9 - 2P_0^5 P_3 \alpha_2
- 2P_0^5 P_4 + 15P_0^2 Q_1 P_1^2 P_0 + P_0^2 P_1^2 \alpha_9
- 20P_0^3 P_1^3 \alpha_2 - 2P_0^3 P_1 \alpha_8) \times (P_0^5)^{-1},$

$\delta_6 = \frac{(-2P_0 + \alpha_4 P_0)}{P_0^2}$

$\delta_7 = \frac{(-2Q_0^2 + 4P_0^2 Q_3 - 7P_0 \alpha_2 P_0^3 + 7P_1^2 P_0^2 + \alpha_3 P_0^4)}{P_0^6}$

Let us select $P_1$, $Q_3$, $P_3$, and $P_4$ appropriately and the subfamilies of laws result in $\bar{\mu}_i(0,0,0,0,\delta_2,\delta_3,\delta_4,\delta_5,\delta_6,\delta_7)$:

$$[y_0, y_i] = y_{i+1}, \quad 1 \leq i \leq 6,$$

$$[y_1, y_2] = y_4,$$

$$[y_1, y_3] = y_5,$$

$$[y_1, y_4] = 3y_6 + \delta_6 y_7 + \delta_7 y_8,$$
Thus $\delta_9 = 1$, and the algebra is described by $\mu_3(1,0,0,0,0,3,0,0,0,1)$. Finally, the forth subcase is the algebra $\mu_3(1,0,0,0,0,0,3,0,0,0,1)$.

2.3.2. Case $A.2$. One has $\alpha_{11} \neq 0$ and $\alpha_3 = 0$.

**Proposition 5.** The nilpotent QFLA of dimension 9 with $\alpha_{11} \neq 0$, $\alpha_3 = 0$, and $\alpha_2 \neq \alpha_8$ are nonisomorphic to the algebras with $\alpha_{11} \neq 0$, $\alpha_3 = 0$, and $\alpha_2 \neq \alpha_8$.

**Proof.** The derived series is $\mathcal{D}^1g = \langle x_2, x_3, x_4, x_5, x_6, x_7, x_8 \rangle$, $\mathcal{D}^2g = \langle (\alpha_2 - \alpha_8)x_2, x_8 \rangle$, and so forth; thus $\text{Dim}[\mathcal{D}^2g] = 2$ if $\alpha_2 \neq \alpha_8$ and $\text{Dim}[\mathcal{D}^2g] = 1$ if $\alpha_2 = \alpha_8$. The nullity of $\alpha_2 - \alpha_8$ constitutes a new classification criterion.

**Proposition 6.** The nilpotent QFLA of dimension 9 with $\alpha_{11} \neq 0$, $\alpha_3 = 0$, and $\alpha_2 \neq \alpha_8$ can be classified in ten nonisomorphic subfamilies $\mu_j$ with $j$ from 4 to 13, described in Figure 1, according to the conditions described in Figure 2.

**Proof.** Let us apply the change of base (8). The subfamilies of laws are $\mathcal{P}(\delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_7)$ and the restrictions $P_0 \neq 0$ and $Q_1 \neq 0$. Thus, with the coefficient identifications, the new parameters are

$$\delta_2 = \frac{Q_1 \alpha_2}{P_0^2},$$

(16)
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\[ \delta_4 = -\left(2\alpha_2 Q_1 P_1 \alpha_2 + 2Q_1 P_0 Q_4 \alpha_2 - Q_2 P_0 \alpha_2 \right) + \alpha_2 Q_1^2 P_1 \alpha_2 - Q_2^2 P_0 \alpha_4 
\]
\[ -2Q_1 P_0 Q_3 \alpha_4 + P_0 Q_2^2 \alpha_4 \right) \times \left( Q_1 P_0^6 \right)^{-1}, \]
\[ \delta_5 = \left( -\alpha_2 Q_1^3 P_1 P_0 \alpha_0 - 2\alpha_2 Q_1^2 P_1 P_0 + \alpha_2 Q_1^3 P_1 \alpha_0 \right) + \left( 2P_2 Q_1 \alpha_2 Q_2 - P_2 P_0 \alpha_2^2 Q_2 + 2P_3 P_0 Q_4 \alpha_2 \right) 
\[ -2P_1 \alpha_2 Q_2^2 + P_1 P_0 Q_2 \alpha_2 + 2P_2 Q_0 \alpha_2 Q_3 + 2Q_2 P_0^3 Q_4 \alpha_3 
\[ -Q_2^2 P_0^3 \alpha_5 - 2Q_4 P_0^2 Q_2 + Q_2^2 P_0^3 \right) \times \left( Q_1 P_0^6 \right)^{-1}, \]
\[ \delta_6 = \left( \alpha_2 Q_1^4 P_1 \alpha_0 Q_2 - Q_2 Q_0 \alpha_0 - 2P_0 Q_3 \alpha_2 + 2Q_2^2 P_1 \alpha_2 - P_0 Q_4 \right) \]
\[ P_0^2 Q_2^2 \]
\[ \delta_9 = \left( -Q_2^2 P_0 \alpha_6 + Q_2^3 P_0 \alpha_0 + 2P_0 Q_3 \alpha_0 + 2Q_2^2 P_1 \alpha_2 - P_0 Q_4 \right) \]
\[ P_0^2 Q_2^2 \]

Selecting \( Q_2 \) and \( Q_3 \) appropriately the subfamilies of laws result in \( \mu_k(\delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_7, \delta_8, \delta_9) \) and the restrictions \( P_0 \neq 0 \) and \( Q_1 \neq 0 \). Thus, with the coefficient identifications, the new parameters are

\[ \delta_2 = \frac{Q_1 \alpha_2}{P_0^2}, \]
\[ \delta_3 = \frac{Q_1 \alpha_3}{P_0^2}, \]
\[ \delta_4 = \frac{Q_1 (-3 \alpha_4 P_4 + P_0 \alpha_4)}{P_0^2}, \]
\[ \delta_5 = \left( -\alpha_2 Q_1^3 P_1 P_0 \alpha_0 - 2\alpha_2 Q_1^2 P_1 P_0 + \alpha_2 Q_1^3 P_1 \alpha_0 \right) + \left( 2P_2 Q_1 \alpha_2 Q_2 - P_2 P_0 \alpha_2^2 Q_2 + 2P_3 P_0 Q_4 \alpha_2 \right) 
\[ -2P_1 \alpha_2 Q_2^2 + P_1 P_0 Q_2 \alpha_2 + 2P_2 Q_0 \alpha_2 Q_3 + 2Q_2 P_0^3 Q_4 \alpha_3 
\[ -Q_2^2 P_0^3 \alpha_5 - 2Q_4 P_0^2 Q_2 + Q_2^2 P_0^3 \right) \times \left( Q_1 P_0^6 \right)^{-1}, \]
\[ \delta_9 = \left( P_1 Q_1 \alpha_2 Q_2 + Q_2 Q_0 \alpha_0 - 2P_0 Q_3 \alpha_2 + 2Q_2^2 P_1 \alpha_2 - P_0 Q_4 \right) \]
\[ \left( P_0^2 Q_2^2 \right) \]

Let us select \( Q_2 \) and \( Q_3 \) appropriately and the subfamilies of laws result in \( \mu_k(\delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_7, \delta_8, \delta_9) \) and the restrictions \( P_0 \neq 0 \) and \( Q_1 \neq 0 \). Thus, with the coefficient identifications, the new parameters are

\[ \delta_2 = \frac{Q_1 \alpha_2}{P_0^2}, \]
\[ \delta_3 = \frac{Q_1 \alpha_3}{P_0^2}, \]
\[ \delta_4 = \frac{Q_1 (-3 \alpha_4 P_4 + P_0 \alpha_4)}{P_0^2}, \]
\[ \delta_5 = \left( -\alpha_2 Q_1^3 P_1 P_0 \alpha_0 - 2\alpha_2 Q_1^2 P_1 P_0 + \alpha_2 Q_1^3 P_1 \alpha_0 \right) + \left( 2P_2 Q_1 \alpha_2 Q_2 - P_2 P_0 \alpha_2^2 Q_2 + 2P_3 P_0 Q_4 \alpha_2 \right) 
\[ -2P_1 \alpha_2 Q_2^2 + P_1 P_0 Q_2 \alpha_2 + 2P_2 Q_0 \alpha_2 Q_3 + 2Q_2 P_0^3 Q_4 \alpha_3 
\[ -Q_2^2 P_0^3 \alpha_5 - 2Q_4 P_0^2 Q_2 + Q_2^2 P_0^3 \right) \times \left( Q_1 P_0^6 \right)^{-1}, \]
\[ \delta_9 = \left( P_1 Q_1 \alpha_2 Q_2 + Q_2 Q_0 \alpha_0 - 2P_0 Q_3 \alpha_2 + 2Q_2^2 P_1 \alpha_2 - P_0 Q_4 \right) \]
\[ \left( P_0^2 Q_2^2 \right) \]

2.4. Case B. One has \( \alpha_{11} = 0 \).

Proposition 8. The nilpotent QFLA of dimension 9 and \( \alpha_{11} = 0 \) and \( \alpha_0 \neq 0 \) are nonisomorphic to the algebras with \( \alpha_{11} = 0 \) and \( \alpha_0 = 0 \).

Proof. For the family described by (3a)–(3o) and (4a)–(4q), \([6^c g] = \langle x_4, x_5, x_6, x_7, x_8, x_9 \rangle\); therefore its dimension is \( \text{Dim}[6^c g] = 5 \), if \( \alpha_0 \neq 0 \) or \( \text{Dim}[6^c g] = 4 \), if \( \alpha_0 = 0 \), and the nullity of \( \alpha_0 \) constitutes a new classification criterion.
2.4.1. Case B.1. One has \( \alpha_{11} = 0 \) and \( \alpha_9 \neq 0 \).

**Proposition 9.** The nilpotent QFLA of dimension 9 with \( \alpha_{11} = 0 \), \( \alpha_9 \neq 0 \), and \( \alpha_6 \neq 0 \) are nonisomorphic to the algebras with \( \alpha_{11} = 0 \), \( \alpha_9 \neq 0 \), and \( \alpha_6 = 0 \).

**Proof.** If \( \alpha_9 \neq 0 \), it can be deduced that \( \alpha_{12} = \alpha_{13} = \alpha_{14} = 0 \). By computing the Jacobi equations, the family of laws is reduced to

\[
\begin{align*}
[x_0, x_i] &= x_{i+1}, & 1 \leq i \leq 6, \\
[x_1, x_2] &= \alpha_1 x_4 + \alpha_2 x_5 + \alpha_3 x_6 + \alpha_4 x_7 + \alpha_5 x_8 , \\
[x_1, x_3] &= \alpha_1 x_5 + \alpha_2 x_6 + \alpha_3 x_7, \\
[x_1, x_4] &= \alpha_6 x_5 + \alpha_7 x_6 + \alpha_8 x_7 + \alpha_9 x_8, \\
[x_1, x_5] &= 2 \alpha_6 x_6 + (2 \alpha_7 - \alpha_1) x_7, \\
[x_1, x_6] &= \alpha_{10} x_7, \\
[x_1, x_8] &= \alpha_{15} x_6 + \alpha_{16} x_7, \\
[x_2, x_3] &= -\alpha_6 x_5 + (\alpha_1 - \alpha_7) x_6 + (\alpha_2 - \alpha_8) x_7 - \alpha_9 x_8, \\
[x_2, x_4] &= -\alpha_6 x_6 + (\alpha_1 - \alpha_7) x_7, \\
[x_2, x_5] &= (2 \alpha_6 - \alpha_{10}) x_7, \\
[x_2, x_8] &= \alpha_{15} x_7, \\
[x_3, x_4] &= (-3 \alpha_6 + \alpha_{10}) x_7
\end{align*}
\]

with two restrictions

\[
\begin{align*}
-2 \alpha_6^2 - \alpha_9 \alpha_{15} &= 0, \\
-4 \alpha_1 \alpha_6 + 3 \alpha_1 \alpha_{10} - 2 \alpha_6 \alpha_7 - \alpha_7 \alpha_{10} - \alpha_9 \alpha_{16} &= 0.
\end{align*}
\]

Since \( \alpha_9 \neq 0 \), the application of the elementary change of base CB

\[
\begin{align*}
y_0 &= y_1, \\
y_i &= \frac{x_i}{\alpha_9}, & 1 \leq i \leq n - 1
\end{align*}
\]

with \( |\text{CB}| = \frac{1}{\alpha_9} \neq 0 \)

permits us to suppose that \( \alpha_9 = 1 \). Then from (39), \( \alpha_{15} = -2 \alpha_6^2 \) and \( \alpha_{16} = -4 \alpha_1 \alpha_6 + 3 \alpha_1 \alpha_{10} - 2 \alpha_6 \alpha_7 - \alpha_7 \alpha_{10} \). This implies that (33) and (37) are changed to \( [x_1, x_8] = -2 \alpha_6^2 x_6 + \alpha_1 (4 \alpha_6 + \alpha_{10}) - \alpha_7 (2 \alpha_6 + \alpha_{10}) x_7 \) and \([x_2, x_4] = -2 \alpha_6 \alpha_7\), respectively, and the subfamily of laws \( \mu_i(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, 0, 0, 0, 0, 0) \), with \( i \) from 20 to 63, has no restrictions (4a)–(4q). Its derived series is \( \mathcal{D} g = \langle x_2, x_3, x_4, x_5, x_6, x_7, x_8 \rangle \), \( \mathcal{D}^2 g = \langle -\alpha_6 x_5 + (\alpha_1 - \alpha_7) x_6, (\alpha_2 - \alpha_8) x_5 - \alpha_9 x_6 + (\alpha_1 - \alpha_7) x_7, (2 \alpha_6 - \alpha_{10}) x_7 - 2 \alpha_6^2 x_7 - (3 \alpha_6 + \alpha_{10}) x_7 \rangle \) and so forth. Thus \( \text{Dim}[\mathcal{D}^2 g] = 4 \) if \( \alpha_6 \neq 0 \) and \( \text{Dim}[\mathcal{D}^3 g] \leq 3 \) if \( \alpha_6 = 0 \). Therefore the nullity of \( \alpha_6 \) constitutes a new classification criterion.

**Case B.1.1:** \( \alpha_{11} = 0 \), \( \alpha_9 \neq 0 \) and \( \alpha_6 \neq 0 \). Figure 3 provides the classification in 18 subfamilies in this case.

**Case B.1.2:** \( \alpha_{11} = 0 \), \( \alpha_9 \neq 0 \), and \( \alpha_6 = 0 \). Figure 4 provides the classification in 26 subfamilies in this case.

2.4.2. Case B.2. One has \( \alpha_{11} = 0 \) and \( \alpha_9 = 0 \).

The restrictions in the family (4a)–(4q) are reduced to

\[
\begin{align*}
\alpha_5 \alpha_{12} &= 0, \\
\alpha_6 \alpha_{12} &= 0, \\
\alpha_6 \alpha_{13} &= 0, \\
\alpha_{10} \alpha_{12} &= 0, \\
\alpha_{12} (\alpha_1 - \alpha_7) &= 0, \\
\alpha_5 \alpha_{13} - 2 \alpha_6^2 &= 0, \\
2 (\alpha_2 - \alpha_8) \alpha_{12} + 3 (\alpha_1 - \alpha_7) \alpha_{13} + 2 (\alpha_6 - \alpha_{10}) \alpha_{14} &= 0, \\
\alpha_5 \alpha_{14} - 2 (2 \alpha_1 + \alpha_7) \alpha_6 - \alpha_5 \alpha_{16} + (3 \alpha_1 - \alpha_7) \alpha_{10} &= 0.
\end{align*}
\]

**Proposition 10.** The nilpotent QFLA of dimension 9 with \( \alpha_{11} = 0 \), \( \alpha_9 = 0 \), and \( \alpha_6 \neq 0 \) are nonisomorphic to the algebras with \( \alpha_{11} = 0 \), \( \alpha_9 = 0 \), and \( \alpha_6 = 0 \).

**Proof.** Equations (43) and (46) imply that \( \alpha_6 = 0 \). By computing the Jacobi equations, the subfamily of laws is \( \mu_m(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, 0, \alpha_7, 0, 0, 0, 0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, 0, \alpha_7, 0, 0, 0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \), with \( m \) from 64 to 263, and the restrictions are reduced to 6. Its descending central series is \( \mathcal{E}^1 g = \langle x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9 \rangle \), and so forth. Thus the nullity of \( \alpha_6 \) constitutes a new classification criterion.

An exhaustive and extensive process of analysis with the same methodology shown in the previous subsections leads to the final subclassification, which is summarized in the following Figures.

**Case B.2.1:** \( \alpha_{11} = 0 \), \( \alpha_9 = 0 \), and \( \alpha_6 \neq 0 \). Figure 5 provides the classification in 55 subfamilies in this case.

**Case B.2.2:** \( \alpha_{11} = 0 \), \( \alpha_9 = 0 \), and \( \alpha_6 = 0 \). Figures 6 and 7 provide the classification in 145 subfamilies in this case.

3. Concluding Remarks

Computational aid has been indispensable in this piece of research. A PC Pentium 4 of 2.4 GHz and the programming language Maple 6 have been used in the process. The library modules developed represent approximately 12,000 lines of code. In some cases, in this massive application of computational resources and looking for the simplification of some laws, procedures that perhaps can be considered of "inverse engineering" have been used in order to find some very complex changes of base, which have allowed us to
### Figure 3: Case B.1.1

\[ \alpha_{11} = 0, \alpha_9 \neq 0, \text{and } \alpha_6 \neq 0; \text{classification of the QFLA of dimension 9.} \]

### Figure 4: Case B.1.2

\[ \alpha_{11} = 0; \alpha_9 \neq 0, \text{and } \alpha_6 = 0; \text{classification of the QFLA of dimension 9.} \]
Figure 5: Case B.2.1: $\alpha_{11} = 0, \alpha_{9} = 0, \alpha_{5} \neq 0$; classification of the QFLA of dimension 9.
Additional result from previous cases

\[ \lambda_{11} = 0, \lambda_9 = 0, \text{and } \lambda_5 = 0. \]

Classification of the QFLA of dimension 9.

**Figure 6:** Case B.2.2. First part: \( \lambda_{11} = 0, \lambda_9 = 0, \text{and } \lambda_5 = 0. \) Classification of the QFLA of dimension 9.
Figure 7: Case B.2.2. Second part: $\alpha_{11} = 0$, $\alpha_{9} = 0$, and $\alpha_{5} = 0$. Classification of the QFLA of dimension 9.
eliminate some parameters in the laws involved. In any case, the massive application of changes of base and characteristic vector has allowed us to obtain the complete classification in 263 subfamilies of the QFLA laws of dimension 9.

The 263 families have been represented in the paper, consisting of 157 simple algebras, 77 families depending on 1 parameter, 24 families depending on 2 parameters, and 5 families depending on 3 parameters. The classification is complete since any couple of the obtained 263 families is nonisomorphic and any quasifiliform Lie algebra of dimension 9 is isomorphic to one of them. The nonisomorphism of the 263 Lie algebra families has been proved in the 10 propositions of the paper, and the completeness of the classification is proved by the "exhaustive" analysis of all the possible cases, depending on the combination of the values of the 16 parameters \((a_1 \cdots a_{16})\).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

The authors appreciate the aid of José Ramón Gómez Martín, Professor of the University of Seville, advisor of the Ph.D. thesis of one of the authors, which constitutes the basis of these works.

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