Free surface flows with large slopes: beyond lubrication theory

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(Dated: March 31, 2022)

The description of free surface flows can often be simplified to thin film (or lubrication) equations, when the slopes of the liquid-gas interface are small. Here we present a long wavelength theory that remains fully quantitative for steep interface slopes, by expanding about Stokes flow in a wedge. For small capillary numbers, the variations of the interface slope are slow and can be treated perturbatively. This geometry occurs naturally for flows with contact lines: we quantify the difference with ordinary lubrication theory through a numerical example and analytically recover the full Cox-Voinov asymptotic solution.

Free surface flows are encountered in many everyday life and industrial situations, ranging from soap films, sliding drops to paints and coatings. The hydrodynamic description of these 'free boundary problems' still provides a challenge of great fundamental and technological interest. The difficulty lies in the intricate coupling between the liquid-gas interface and the flow inside the film, which gives rise to a broad variety of instabilities and interface morphologies. In the case of 'thin' films, for which horizontal and vertical length scales are well separated, the problem is greatly reduced through a long wavelength expansion. At low Reynolds numbers this so-called lubrication approximation yields a single nonlinear equation for the evolution of the interface profile $h(x, y, t)$, and forms the accepted theoretical framework for free surface flows. This reduction is possible whenever surface tension ($\gamma$) dominates over viscosity ($\eta$), so that the capillary number $Ca = \eta U^* / \gamma$ serves as a small parameter; $U^*$ denotes the velocity scale of the problem.

The standard formulation of lubrication theory, however, has a severe drawback: it is only valid for small interface slopes, i.e. $|\nabla h| \ll 1$. While it is generally believed that the lubrication equation yields good qualitative predictions for larger slopes as well, one has to be careful with quantitative comparisons. This is particularly important for the problem of moving contact lines for which viscous forces tend to diverge as $h \to 0$. The microscopic mechanisms that release this singularity are highly disputed, and call for a fully quantitative description of experiments that often involve large contact angles. As it is practically infeasible to resolve the full hydrodynamic problem on all relevant length scales, ranging from molecular to millimetric, a simplified theory for finite slopes would be extremely valuable.

In this paper we present a generalization of the lubrication theory for free surface flows at low Reynolds numbers that remains exact for large slopes. The crucial observation is that in the limit of small $Ca$, capillary driving requires slow variations of the interface curvature, but there are no restrictions to the steepness of the interface.

We therefore consider the flow in a wedge with a finite, but slowly varying opening angle $\theta$ – see Fig. 1b. This geometry naturally occurs for contact lines. Treating the variation as a perturbation around a straight wedge then yields the equation for the interface profile $h(x, y, t)$.

This theory, summarized by Eqs. (14,15), remains fully quantitative for large slopes when the curvature $\kappa \ll 1/h$, while it enjoys the same mathematical structure as the usual lubrication theory. We furthermore show that the equation reproduces the asymptotics for $\theta(x)$ as computed by Voinov and Cox, in a relatively straightforward manner. However, the present work reaches beyond asymptotic relations: it describes all intermediate length scales as well, and allows to incorporate other forces such as a disjoining pressure or gravity.

**FIG. 1:** (a) The usual lubrication approximation is valid whenever the liquid-gas interface is slowly varying along the horizontal coordinates, and is thus restricted to small slopes. (b) Considering wedge-like profiles with slowly varying slope $\theta(x)$, we derive the long wavelength theory for steep slopes.
Reynolds number, the flow of incompressible Newtonian liquids is described by Stokes equations

\[ \nabla \cdot \mathbf{v} = 0, \]
\[ -\nabla p + \eta \Delta \mathbf{v} - \nabla \Phi = 0 \quad \implies \quad \nabla \times \Delta \mathbf{v} = 0, \]

where \( p(x, y, z) \) and \( \mathbf{v}(x, y, z) \) represent the pressure and velocity field respectively, while we consider body forces that derive from a potential \( \Phi \). (Time dependent profiles will be discussed below.) The equation is complemented with the boundary condition of Laplace pressure at the free surface

\[ p(z = h) = p_0 - \gamma \kappa, \]

where \( \kappa \) is the interface curvature. This gives rise to an intricate nonlinear coupling between the shape of the interface and the flow inside the film, which has to be resolved self-consistently. We limit the discussion to the case where the gas is hydrodynamically passive and we take a zero shear stress condition at the free interface.

In the limit where \( \text{Ca} \ll 1 \), the free boundary problem can be greatly reduced through the lubrication approximation: surface tension is sufficiently strong to drive the viscous flow through only minor variations of the shape of the free surface (Fig. 1a). One thus expects that the interface profile and the velocity field are slow functions of the horizontal coordinates, so that

\[ h(x, y) / h^* = \tilde{h} \left( \epsilon \frac{x}{h^*}, \epsilon \frac{y}{h^*} \right), \]

while we write

\[ \mathbf{v}(x, y, z) = \tilde{\mathbf{v}} \left( \epsilon \frac{x}{h^*}, \epsilon \frac{y}{h^*}, \frac{z}{h^*} \right) \]
\[ = \tilde{\mathbf{v}}_0 + \epsilon \tilde{\mathbf{v}}_1 + \epsilon^2 \tilde{\mathbf{v}}_2 + \cdots. \]

All lengths have been rescaled by the typical film thickness \( h^* \), and \( \epsilon = \text{Ca}^{1/3} \) is the small parameter of the expansion. The strategy is to solve Eqs. (1-3) order by order in \( \epsilon \). Here we briefly sketch the approach; for a more detailed derivation we refer to [2].

If we let \( \epsilon \) go to zero, the departure of \( h(x, y) \) from a horizontal interface becomes increasingly small. Hence, the velocity profile converges towards the parabolic (Poiseuille-like) profile in this limit, so that

\[ \tilde{\mathbf{v}}_0 = \frac{3U}{2} \left( 1 - 1 - \frac{z}{h} \right)^2. \]

Formally, this dominant flow can be obtained from Eq. (2) at order \( \epsilon^0 \). Note that \( \mathbf{v}_0 \) still evolves on a long scale through its dependence on \( h(x, y) \).

In deriving Eq. (4) we used the boundary conditions of no-slip at \( z = 0 \), and zero shear stress at \( z = h \). The prefactor has been chosen such that \( \mathbf{U} = 1/h \int_0^h dz \mathbf{v}_0 \) represents the depth-averaged velocity in the frame attached to the plate.

Since the dominant viscous forces in Eq. (2) arise from \( \mathbf{v}_0 \), we do not need to solve for the higher order velocities to obtain the equation for \( h(x, y) \). At leading order, Eqs. (2) and (3) reduce to the celebrated lubrication equation

\[ \nabla \Delta h = 3Ca \frac{U/U^*}{h^2} + \frac{1}{\gamma} \nabla \Phi|_{z=h}. \]

Combined with the depth-averaged continuity equation,

\[ \partial_t h + \nabla \cdot (h \mathbf{U}) = 0, \]

it provides the common theoretical framework for free surface flows, both in the scientific community as well as for industrial purposes. The lowest order terms that are neglected are of order \( \epsilon^2 \), so the expansion is valid in the limit of small slopes, \( |\nabla h|^2 \ll 1 \).

Theory for large slopes – We now perform a similar long wavelength expansion for wedge-like geometries, such as depicted in Fig. 1b. The crucial physical ingredient underlying the expansion is that surface tension dominates over viscosity, i.e. \( \text{Ca} \ll 1 \), so that variations of interface curvature are slow with respect to the relevant length scales. In principle there is no restriction to the slope of the interface: the only requirement is that the opening angle \( \theta \) is slowly varying (Fig. 1b). We therefore consider profiles with

\[ \theta(x) = \tilde{\theta} \left( \frac{x}{x^*} \right), \]

and perform an expansion in \( \epsilon \). Note that \( \tilde{\theta} \) itself is of order unity. Here we introduced the length scale \( x^* \), which is a typical distance to the ‘origin’ of the wedge – we show below that the analysis remains self-consistent as long as \( h \partial_x \theta \ll 1 \). For simplicity we discuss two-dimensional profiles so we omit the \( y \)-dependence.

In the spirit of the lubrication approximation, we expand the velocity as

\[ \mathbf{v}(x, z) = \tilde{\mathbf{v}}_0 + \epsilon \tilde{\mathbf{v}}_1 + \epsilon^2 \tilde{\mathbf{v}}_2 + \cdots, \]

and solve for the dominant flow \( \tilde{\mathbf{v}}_0 \). Again, \( \tilde{\mathbf{v}}_0 \) is obtained in the limit \( \epsilon \to 0 \), which in this case corresponds to a wedge of constant opening angle. Hence, the problem for \( \tilde{\mathbf{v}}_0 \) reduces to Stokes flow inside a straight wedge, which is easily solved analytically [10, 11].

To make this more explicit, we introduce local cylindrical coordinates, \( r(x, z) \) and \( \phi(x, z) \), which are defined by the locally tangent wedge (Fig. 2).
Combined with the Laplace pressure condition $\Phi$ this yields the generalized lubrication equation:

$$\partial_z \kappa = 3 \text{Ca} \frac{U/U^*}{h^2} F(\theta) + \frac{1}{\gamma} \partial_z \Phi|_{z=h}, \quad (14)$$

where

$$F(\theta) = \frac{2}{3} \frac{\tan \theta \sin^2 \theta}{\theta - \cos \theta \sin \theta}. \quad (15)$$

Since $\partial_z \kappa \sim (\epsilon/x^*)^2$, the expansion is self-consistent when taking $\epsilon = \text{Ca}^{1/2}$.

Comparing this result to the lubrication equation $\Phi$, one observes two differences. First, the function $F(\theta)$ can be seen as a correction factor for the viscous term: indeed, one recovers $F(\theta) \to 1$ for small slopes. Second, the left hand side of Eq. (14) now involves the full curvature

$$\kappa = \frac{\partial_z h}{(1 + \partial_z h^2)^{3/2}}. \quad (16)$$

The expansion thus provides an equation for the interface profile that has the same mathematical structure as the usual lubrication approximation, but which remains exact for large slopes. The lowest order terms that are neglected are of order $\epsilon = O(x^* \partial_x \theta)$, so the expansion is valid when $h \partial_x \theta \ll 1$. This does not mean that the description is limited to small heights: the height only appears as a reference scale to quantify the dimensionless curvature.

*Asymptotics for $\theta(x)$* – To illustrate the strength of the approach, we now show that Eq. (14) correctly reproduces the nontrivial asymptotic solution for $\theta(x)$, as obtained by Voinov [12] and Cox [13] for advancing contact lines. Anticipating the well known result, we express the relation between $\theta$ and $x$ as

$$g(\theta) = \text{Ca} \ln(x/x_0), \quad (17)$$

and solve for $g(\theta)$ using Eq. (14) with $\Phi = 0$. Differentiating $g(\theta)$ with respect to $x$ one finds $\partial_x \theta = \text{Ca}/(x g')$, so that the curvature can be written as $\kappa = \text{Ca} \cos \theta/(x g')$. Combining this with the expansion $h(x) = x \tan \theta(x)$ + $O(\text{Ca})$, Eq. (14) becomes to lowest order

$$\frac{\text{Ca} \cos \theta}{x^2 g'} = \frac{3 \text{Ca}}{x^2 \tan^2 \theta} F(\theta). \quad (18)$$

Here we used the fact that advancing contact lines move along the negative $x$-direction in the frame attached to the plate, so that $U_{\text{adv}} = -U^*$. Eliminating $g'$, we readily recover the famous result of [12, 13]

$$g(\theta) = \int_0^\theta du \frac{u - \cos u \sin u}{2 \sin u}. \quad (19)$$
which introduces the capillary length scale \( l \). The results of Eq. (14) (solid) provide significant corrections with respect to lubrication theory Eq. (6) (dashed).

**Discussion** – We have derived the long wavelength expansion for free surface flows in the case of steep interface profiles. This provides a significant improvement with respect to the usual lubrication approximation, whose validity is restricted to small slopes. The resulting theory has the same mathematical structure as the lubrication equation and is thus easily adapted to existing codes and methods. The most natural application of our work is found in wetting flows that involve large contact angles. This is illustrated in Fig. 3, where we computed the thickness of a flat film flowing down a vertical plate in the presence of a receding contact line. The effect of gravity is accounted for through \( \Phi = -\gamma \rho g x \), which introduces the capillary length scale \( l_c = \sqrt{\gamma/\rho g} \). For a given thickness \( h_{\text{film}} = l_c \sqrt{3Ca} \), there is a unique solution that connects to the contact line \( \theta_0 \). By numerically solving the interface profile down to a microscopic (molecular) height \( h_0 \), we can thus identify the slope very close to the contact line, denoted by the angle \( \theta_0 \). This yields a unique relation between the slope imposed at a microscopic distance from the contact line and the macroscopic film thickness. The results obtained from numerical integration of Eq. (14) (solid) displays significant quantitative differences with the prediction by lubrication theory (dashed) at angles \( \theta \gtrsim 30^\circ \).

Although in principle the expansion is not limited to contact lines, steep slopes are typically attained through curvatures \( hh'' \sim 1 \) in flows without contact lines. This lies beyond the strict validity of the expansion and one expects corrections due to curvature of the interface. These corrections can in principle be treated perturbatively as well, since \( Ca \ll 1 \) implies weak variations of curvature.

It is a pleasure to thank Bruno Andreotti, Jens Eggers, Marc Fermigier, Alexander Morozov and Julien Tailleur. This work was supported by a Marie Curie Intra-European Fellowships (MEIF-CT2003-502006) within the 6th European Community Framework Programme.

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[19] The problem of predicting the effective boundary condition \( \theta_0 \) at a microscopic scale \( h_0 \) lies at the heart of the contact line problem, and involves new microscopic mechanisms beyond classical hydrodynamics. The curves of Fig. 3 depend only weakly (logarithmically, see Eq. (14)) on the value of \( h_0 \).