VELOCITY CENTROIDS AS TRACERS OF THE TURBULENT VELOCITY STATISTICS

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ABSTRACT

We use the results of magnetohydrodynamic (MHD) simulations to emulate spectroscopic observations and use maps of centroids to study their statistics. In order to assess under which circumstances the scaling properties of the velocity field can be retrieved from velocity centroids, we compare two-point statistics (structure functions and power spectra) of velocity centroids with those of the underlying velocity field and analytic predictions presented by us in a previous paper. We tested a criterion for recovering velocity spectral index from velocity centroids derived in our previous work and propose an approximation of the earlier criterion using only the variances of “unnormalized” velocity centroids and column density maps. It was found that both criteria are necessary, but not sufficient, to determine if the centroids recover velocity statistics. Both criteria are well fulfilled for subsonic turbulence. We find that for supersonic turbulence with sonic Mach numbers \( M_s \gtrsim 2.5 \), centroids fail to trace the spectral index of velocity. Asymptotically, however, we claim that recovery of velocity statistics is always possible provided that the density spectrum is steep and the observed inertial range is sufficiently extended. In addition, we show that velocity centroids are useful for anisotropy studies and determining the direction of the magnetic field, even if the turbulence is highly supersonic, but only if it is sub-Alfvénic. This provides a tool for mapping the magnetic field direction and for testing whether the turbulence is sub-Alfvénic or super-Alfvénic.

Subject headings: ISM: general — ISM: structure — MHD — radio lines: ISM — turbulence

1. INTRODUCTION

It is well established that the interstellar medium (ISM) is turbulent. From the theoretical point of view this arises from the very large Reynolds numbers present in the ISM. (The Reynolds number is defined as the inverse ratio of the dynamical timescale—i.e., eddy turnover time—to the viscous damping timescale.) From an observational standpoint there is also plenty of evidence that supports a turbulent ISM, where the turbulence spans over scales that range from astronomical units to kiloparsecs (Larson 1981, 1992; Scalo 1984, 1987). Velocity centroids have been around as a measure of the velocity field for a long time now (von Hoerner 1951; Munch 1958) and have been widely used to study turbulence in molecular clouds (Kleiner & Dickman 1985; Dickman & Kleiner 1985; Miesch & Bally 1994). Power spectra, correlation, and structure functions have been traditionally, and still are, the most widely used tools to characterize the statistics of emissivity maps. These statistical tools have been used to study the scaling properties of turbulence, e.g., to determine its spectral index. Recently, more elaborate techniques have been proposed to analyze observational data, such as the “\( \Delta \)-variance” wavelet transform (Stutzki et al. 1998; Mac Low & Ossenkopf 2000; Ossenkopf & Mac Low 2002). Such techniques can be used to obtain velocity information from spectral data and have some advantages, like being less sensitive to the effects of edges or noise. Regardless of the method used, it is usually assumed that the map traces the velocity fluctuations, which as we show below is not always true. To separate the velocity from the density contribution, “modified velocity centroids”

Studies of statistics of turbulence using interstellar scintillations have been fruitful (Narayan & Goodman 1989; Spangler & Gwinn 1990). However, this technique is restricted to the study of ionized media and, in particular, to density fluctuations alone (see Cordes 1999). Nowadays, radio spectroscopic observations of neutral media provide us with an enormous amount of data containing information about interstellar turbulence, including a more direct physical quantity for studying turbulence: velocity. But the emissivity of a spectral line depends on both the velocity and density fields simultaneously, and the separation of their individual contributions is not trivial. Much effort has been put into this difficult task, and several statistical measures have been proposed to extract velocity information from spectroscopic data (see the review by Lazarian 1999). Line widths are one of the simplest measures (Larson 1981, 1992; Scalo 1984, 1987). Velocity centroids have been around as a measure of the velocity field for a long time now (von Hoerner 1951; Munch 1958) and have been widely used to study turbulence in molecular clouds (Kleiner & Dickman 1985; Dickman & Kleiner 1985; Miesch & Bally 1994).

How to compare interstellar turbulence with the results of numerical simulations and theoretical expectations is an important question that must be addressed. After all, theoretical constructions involve necessary simplifications, while numerical simulations of turbulence involve Reynolds and magnetic Reynolds numbers that are very different from those in the ISM. Are numerical simulations of the ISM of value? To what extent do they reproduce interstellar turbulence? These are the sort of questions one attempts to answer with observations.

Substantial advances in the understanding of the scaling of compressible MHD turbulence have been made in recent years (see review by Lazarian & Cho 2005 and references therein) allowing a direct comparison of the theoretical expectations with observations. How reliable are the turbulence spectra obtained via observations?

\(^1\) This scaling was used to solve important astrophysical problems, for instance, finding the rates of scattering of cosmic rays (Yan & Lazarian 2004) and acceleration of cosmic dust (Yan et al. 2004).
(MVCs) were derived in Lazarian & Esquivel (2003, hereafter LE03).

There have been parallel efforts to develop new statistics that trace velocity fluctuations. Here we mention the “spectral correlation function” (SCF; Rosolowsky et al. 1999; Padoan et al. 2001), “velocity channel analysis” (VCA; Lazarian & Pogosyan 2000; Lazarian et al. 2001; Esquivel et al. 2003), MVCs (LE03), and “velocity coordinate spectrum” (VCS; Lazarian 2004). Both VCA and VCS are good for studies of supersonic turbulence. Although SCF was introduced as an empirical tool, its properties that are concerned with the statistics of turbulence can be derived using the general theory presented in Lazarian & Pogosyan 2004.

The synergy of these different techniques is very advantageous for studies of interstellar turbulence. Miville-Deschênes et al. (2003b) attempted to test the results obtained with VCA using velocity centroids. However, as we show in the paper, without a reliable criterion of whether velocity centroids reflect the velocity statistics, such studies deliver rather limited insight.

Numerical simulations provide us with an ideal testing ground for the statistical tools available for application to observational data. However, we must note that the situation is rather complex. On one hand, real observations depend critically on the physical properties of the object under study, such as variations in the excitation state of the tracer and the radiation transfer within it (see Lazarian & Pogosyan 2004). In addition observational limitations, like finite signal-to-noise ratio and map size, gridding effects, beam pattern, beam error, etc., are also present. On the other hand, numerical simulations have their own limitations, such as finite box size and resolution, numerical viscosity, and the physics available to a particular code. This paper is mostly concerned about the projection effects and the impact of density fluctuations on centroid maps, which are shared by observations and simulations.

In LE03 we studied the maps of velocity centroids as tracers of the turbulent velocity statistics. We derived analytical relations between the two-point statistics of velocity centroids and those of the underlying velocity field. We also identified an important term in the structure function of centroids that includes density information and that can be extracted from observables. Subtraction of that term can better isolate the velocity contribution, and this yielded a new measure that we termed “modified velocity centroids” (MVCs). In LE03 we proposed a criterion for determining whether velocity centroids reflect the scaling properties of the underlying turbulent velocity (e.g., structure functions or spectra of velocity). A major goal of this paper is to test the predictions in LE03 using synthetic maps obtained via MHD simulations and to determine when velocity centroids indeed reflect the velocity statistics.

Earlier on, in Lazarian et al. (2002) we showed how velocity centroids can be used to reveal the anisotropy of MHD turbulence and how this anisotropy can be used for studies of the plane-of-sky magnetic field. This technique was further discussed by Vestuto et al. (2003). In this paper we show how Mach number and Alfvén Mach number affect the anisotropy of velocity centroid statistics.

The results of LE03 obtained in terms of structure functions are trivially recasted in terms of spectra and correlation functions. Therefore we use structure, correlation functions, and spectra interchangeably through our paper, depending on what measure is more convenient. While being interchangeable, for practical statistical data handling, different measures have their own advantages and disadvantages. We discuss these in the example of the power-law scalar field and thus benchmark our further velocity centroid study. We also deal with a potentially pernicious issue of nonuniformity of notations and normalizations that plague the relevant literature by having details of our derivations in the appendices, which constitute an important part of the paper.

In this work we perform a detailed numerical study of the ability of velocity centroids to extract turbulent velocity statistics. We study the issues of velocity-density correlations and outline the relation of velocity centroids to other techniques. In § 2 we review the basic problem of the density and velocity contributions to spectroscopic observations. We summarize LE03 in § 3. (We include in this work appendices with mathematical derivations omitted in our earlier short communication.) In § 4 we test the analytical predictions and the spectral indices from our numerical data. In § 5 we show how centroids can be used for turbulence anisotropy studies and determination of the plane-of-sky direction of the magnetic field. A discussion of the results can be found in § 6, and a summary in § 7.

2. TURBULENCE STATISTICS AND SPECTRAL-LINE DATA

Due to the stochastic nature of turbulence, it is best described by statistical measures. Among these we have two-point statistics such as structure functions, correlation functions, and power spectra (see, for instance, Monin & Yaglom 1975). Their definition and a more comprehensive discussion can be found in Appendix A. Structure and correlation functions depend in general on a “lag” $r$, the separation between two points $x_1$ and $x_2$, such that $r = x_2 - x_1$. The power spectrum is defined as the Fourier transform of the correlation function and is a function of the wave-number vector $k$, with amplitude $k = |k| \sim 2\pi/r$, where $r = |r|$. An additional simplification is achieved if the turbulent field is isotropic, in which case the structure and correlation functions depend only on the magnitude of the separation $r$ (and not on the direction), and similarly if the power spectrum is only a function of $k$. This is not strictly true for magnetized media, as the presence of a magnetic field introduces a preferential direction for motion. In fact, MHD turbulence becomes axisymmetric in a system of reference defined by the direction of the local magnetic field (see reviews by Cho & Lazarian 2005 and Cho et al. 2003a), thus breaking the isotropy. However, since the local direction changes from one place to another, the anisotropy is rather modest, and it is still possible to characterize the turbulence with isotropic statistics (see Esquivel et al. 2003).

2.1. Three-dimensional Power-Law Statistics

In the simplest realization of turbulence we have injection of energy at the largest scales. The energy cascades down without losses to the small scales, at which viscous forces become important and turbulence is dissipated. At intermediate scales, between the injection and the dissipation scales, the turbulent cascade is self-similar. This range constitutes the so-called inertial range. There the physical variables are proportional to simple powers of eddy sizes, and the two-point statistics can be described by power laws. For power-law statistics Lazarian & Pogosyan (2000) discussed two regimes, a short-wave-dominated regime, corresponding to a shallow spectrum, and a long-wave-dominated regime, corresponding to a steep spectrum.

When dealing with numerical data one encounters a few non-trivial effects that we find advantageous to discuss below. The insight into the limitations of numerical procedures that involve
Conversion from mathematically equivalent statistics helps us for the rest of the paper.

2.1.1. Steep (Long-Wave-dominated) Spectrum

Consider an isotropic, power-law, one-dimensional power spectrum of the form

\[ P_{1D}(k) = Ck^{\gamma_{1D}}, \]  

(1)

where \( C \) is a constant. A steep spectrum corresponds to spectral indices \( \gamma_{1D} < -1 \). The structure function \( D(r) \) can be written in terms of the spectrum as

\[ D(r) = 4 \int_0^\infty [1 - \cos(kr)]P_{1D}(k) \, dk. \]  

(2)

For a power-law, steep power spectrum (substituting eq. [1] into eq. [2]), the structure function also follows a power law:

\[ D(r) = Ar^\xi = Ar^{-1-\gamma_{1D}} \quad 0 < \xi < 2, \]  

(3)

where \( A = C2\pi/[\Gamma(1+\gamma)\sin(\pi\gamma/2)] \) and \( \Gamma(\gamma) \) is the Euler gamma function. The relation between the spectral index of the structure function and the power spectrum can be generalized for isotropic fields to \( P_{ND} \propto k^{\gamma_{ND}} \), with

\[ \gamma_{ND} = -N - \xi, \]  

(4)

where \( N \) is the number of dimensions (see Appendix A). For instance, the velocity \( v \) in Kolmogorov turbulence scales as \( v \propto r^{1/3} \), which corresponds to a spectral index for the structure function of \( \xi = 2/3 \), and therefore to a three-dimensional power spectrum \( P_{3D} \) index \( \gamma_{3D} = -11/3 \), a two-dimensional power spectrum \( P_{2D} \) index \( \gamma_{2D} = -8/3 \), and a one-dimensional power spectrum \( P_{1D} \) index \( \gamma_{1D} = -5/3 \). Note that the Kolmogorov spectrum falls into the steep-spectra category.

Structure functions given by equation (3) are well defined only for \( \xi > 0 \), which satisfies \( D(0) = 0 \), and \( \xi < 2 \), so that the representation in terms of Fourier integrals is possible (see Monin & Yaglom 1975).\(^3\)

To illustrate the relation between the two-point statistics and power-law spectrum, as well as the difficulties associated with handling numerical data, we produced a three-dimensional Gaussian cube with a prescribed (three-dimensional) spectral index of \( \gamma_{3D} = -11/3 \), as described in Esquivel et al. (2003). This type of data cube is somewhat similar to the fractional Brownian motion (fBms) fields used by Brunt & Heyer (2002) or the de-phased fields used in Brun et al. (2003). However, as in real observations, they do not have perfect power-law spectrum for a particular realization, but only in a statistical sense (see Esquivel et al. 2003). In Figure 1 we show the calculated three-dimensional power spectrum, structure, and correlation functions of our steep Gaussian cube and compare them with the prescribed scaling properties. The power spectrum is computed using a fast Fourier transform (FFT) in three dimensions and then averaged in wavenumber \( k \). Ideally one can compute directly in real space the three-dimensional structure or correlation functions; however, a three-dimensional field of the dimensions used here \((216^3)\) is already quite expensive computationally. It would require looping over all the points in the data cube to do the average and repeat for all the possible values for the lag (in three dimensions as well). Fortunately, since the data cubes were produced using FFT, we can safely compute the correlation function with spectral methods. The correlation function can be expressed as a convolution integral \( B(r) \propto \int dr' f(r)f(r + r') \), which can be calculated as a simple product of the Fourier-transformed fields. That is, \( B(r) \propto \mathcal{F} \{ f(k) f^*(k) \} \), where \( \mathcal{F} \{ f(k) \} \) is the Fourier transform of \( f(r) \) and \( f^*(k) \) its complex conjugate. Then, with the use of equation (A4), we can obtain the structure function. The resulting correlation and structure functions in three dimensions are then averaged in \( r \). Note that the Gaussian cubes have wraparound

\[^3\] Correlation functions in this regime are maximal and finite at \( r = 0 \). It is important to notice also that for a power-law steep spectrum the structure function is a power law, but the correlation function is not (see eq. [A4]). The correlation function is a constant minus a growing (positive index) power law; therefore in a log-log scale it is flat at small scales and drops at large scales.
periodicity and that the largest variation available corresponds to scales of $L/2$, where $L$ is the size of the computational box. In fact, we only plot the structure and correlation functions up to such separations. We see a fair agreement between the prescribed spectral index $\gamma_{3D} = -11/3$ and the measured scaling properties. The power spectrum in Figure 1a shows departures from a strict power law that are more evident for small wavenumbers. This is natural for this type of data cube, where random deviations from a strict power law are expected at all scales. But at large scales (small $k$) we have fewer points for the statistics, and the departures do not average to zero, while at small scales they almost do.

2.1.2. Shallow (Short-Wave-dominated) Spectrum

When the energy spectrum is shallow (i.e., $\gamma_{1D} > -1$), the fluctuations of the field are dominated by small scales, so this spectrum is termed short-wave-dominated. The density at high Mach numbers is an example of such a shallow spectrum (see Beresnyak et al. 2005). In this case neither the structure nor the correlation functions can be strictly represented by power laws. In fact, in order for the Fourier transforms to converge in this case, we need to introduce a cutoff for small wavenumbers, such that the power spectrum is only a power law for $k > k_0$, in other words,

$$P_{1D}(k) = C'(k_0^2 + k^2)^{\eta/2} = C'(k_0^2 + k^2)^{-(\eta - 1)/2}. \quad (5)$$

For a power spectrum of this form, the corresponding correlation function of the fluctuations is

$$\tilde{B}(r) = A' \left( \frac{r}{r_c} \right)^{\eta/2} K_{\eta/2} \left( \frac{2\pi r}{r_c} \right), \quad (6)$$

where $r_c = 2\pi/k_0$, $K_\eta(x)$ is the $\eta$th-order, modified Bessel function of the second kind (also sometimes referred as the hyperbolic Bessel function), and $A' = C'^{-1-\eta/2} \pi^{(\eta + 1)/2} \Gamma(\eta + 1)/2$. The $\eta$th-dimensional power spectrum index for a correlation function of the form $B \propto (r/r_c)^{\eta/2} K_{\eta/2}(2\pi r/r_c)$ can also be generalized to $P_{ND} \propto (k_0^2 + k^2)^{\eta_{ND}/2}$, with

$$\eta_{ND} = -N - \eta. \quad (7)$$

This relation is very similar to that for the long-wave-dominated case (eq. [4]). Actually for $r < r_c$, $K_{\eta/2}$ can be expanded as $\sim(r/r_c)^{\eta/2}$; thus, the three-dimensional correlation function (as opposed to the structure function as in the long-wave-dominated regime) goes as a power law $\tilde{B}(r) \sim (r/r_c)^{\eta}$ for small separations. Note that for a shallow spectrum the structure function grows rapidly at the smallest scales and then flattens. Similarly to the steep spectrum, we produced a short-wave-dominated Gaussian three-dimensional field, with a prescribed index of $\gamma_{3D} = -2.5$. In Figure 2 we present the expected and calculated two-point statistics. Here the critical scale $r_c$ is determined by the smallest wavenumber ($k_0 = 2\pi/L$); in our case it corresponds to the size of the computational box ($r_c = L$). We see again a fair agreement with the prescribed and calculated spectra. However, Figure 2 reveals a significant departure of the calculated correlation function from the prediction of equation (6) for large lags. The explanation of such a difference is that the analytical relations of spectra (eqs. [1] and [5]) with the structure and correlation functions (eqs. [3] and [6]) is exact only in the limit of continuous integrals over infinite wavenumbers. The data sets presented in this section are constructed in Fourier space and then translated to real space by means of discrete Fourier transforms of the form

$$\tilde{u}(x) = \sum_{k=1}^{L-1} |P_{1D}(k)|^{1/2} \exp(2\pi ikx/L). \quad (8)$$

The sum runs from $k = 1$, ensuring that $\langle \tilde{u}(x) \rangle = 0$. In practice, we evaluate the Fourier transforms via FFT and explicitly set the $k = 0$ component of the spectrum to zero to guarantee that the average of the fluctuating part of the field is zero. The resulting field has a limited range of available harmonics, which are determined basically by the computational grid size ($L$). We have constructed large one-dimensional fields, in which we see that the gap between the analytical and the computed correlation functions gets smaller as the resolution increases. Thus, it is not surprising that the spectra show a much better correspondence than the correlation functions. At the same time, the structure functions do not deal with the lowest harmonics, which introduce the largest errors (see Monin & Yaglom 1975). Therefore
they are less affected by the lack of lower harmonics, as can be confirmed by the fact that they are closer to the analytical prediction than the correlation functions. It is important to always keep in mind the issues that can arise from the discrete nature of the data. However, we must note that this particular problem lies with the generation of the data sets in frequency space and not with the computation of correlation or structure functions via spectral methods. We obtain identical results using FFT and looping in directly in real space to do the required average. In real life, the limitation is likely to be in the opposite direction: the finite wave-numbers available would show up as an uncertainty in determining the power spectra, while the structure and correlation functions should be estimated with smaller errors (if measured directly in real space).

2.2. Structure Functions of Quantities Projected along the Line of Sight

From spectroscopic observations we can not obtain either the density or velocity fields in real space \((x, y, z)\), but we have to deal with projections along the line of sight (LOS). Despite the fact that our main goal is to extract velocity statistics from the centroids of velocity, we discuss in this section the statistics of density integrated along the LOS (column density). It becomes clear below that the same procedure can be applied to obtain velocity statistics. The issue of projection has been previously discussed in Lazarian (1995); here we briefly state some results that are relevant to this work. In Appendix B we show the projection effects of structure functions for the particular case of power-law statistics, and in Appendix C we show the power spectrum of a homogeneous field that has been integrated along the LOS.

In what follows we assume that the emissivity of our medium is proportional to the first power of the density. (This is true, for instance, in the case of H I.) We consider an isothermal medium and neglect the effects of self-absorption. In this case the integrated intensity of the emission (integrated along the velocity coordinate) is proportional to the column density (see Appendix A for more details):

\[
I(X) = \int \alpha \rho_s(x, v_z) dv_z = \int \alpha \rho(x) dz,
\]

where \(\alpha\) is a constant and \(\rho(x)\) is the mass density. The density of emitters \(\rho_s(x, v_z)\) can be identified as the column density per velocity interval, commonly referred as \(dN/dv\). To distinguish between two-dimensional and three-dimensional vectors, we use capital letters to denote the former and lower case for the latter (i.e., \(X = [x, y], x = [x, y, z]\)). Our assumption is satisfied for observational data in which the medium is optically thin, thermalized, and has constant excitation conditions. However, for any observed map the applicability of this assumption has to be examined carefully. Even for H I, widespread self-absorption has been detected (for example, Jackson et al. 2002; Li & Goldsmith 2003).

Consider the structure function of the integrated intensity described in equation (9)

\[
\left\langle (I(X_1) - I(X_2))^2 \right\rangle = \left\langle \left( \int_0^{z_{\text{tot}}} \alpha \rho(x_1) dx - \int_0^{z_{\text{tot}}} \alpha \rho(x_2) dx \right)^2 \right\rangle,
\]

where we have written explicitly the limits of integration, with \(z_{\text{tot}}\) being the size of the object (in the LOS direction). Clearly \(z_{\text{tot}}\) does not necessarily have to coincide with the transverse size (in the plane of the sky) of the object under study; however, that is the case in our data sets. As described in Lazarian (1995), we can expand the square in equation (10), combining

\[
\left( \int \chi(x) dx \right)^2 = \int \int \chi(x_1) \chi(x_2) dx_1 dx_2,
\]

and the elementary identity

\[
(a - b)(c - d) = \frac{1}{2} [(a - d)^2 + (b - c)^2 - (a - c)^2 - (b - d)^2],
\]

to obtain

\[
\left\langle (I(X_1) - I(X_2))^2 \right\rangle = \alpha^2 \int_0^{z_{\text{tot}}} \int_0^{z_{\text{tot}}} dz_1 dz_2 \left[ d_s(r) - d_s(0) \right] X_{x_1} X_{x_2},
\]

where \(d_s(r)\) is the three-dimensional structure function of the density,

\[
d_s(r) = \left\langle [\rho(x_1) - \rho(x_2)]^2 \right\rangle.
\]

This definition is general and does not require any particular functional form of the three-dimensional structure functions (i.e., power-law statistics). The problem of formally inverting equation (13) to obtain the underlying statistics, allowing for anisotropic turbulence, with an arbitrary spectrum (i.e., not a power law), has been presented in Lazarian (1995), but it is somewhat mathematically involved. For three-dimensional fields with a homogeneous and isotropic power-law spectrum, the structure functions of the integrated fields (two-dimensional maps) can be simply approximated by two power laws, one at small and the other at large separations (see Minter 2002 and also Appendix B). For instance, if the density has a power spectrum \(P_{3D, \rho} \propto k^{\gamma_3}\), the structure function of the column density will have the form

\[
\left\langle (I(X_1) - I(X_2))^2 \right\rangle \propto R^\mu,
\]

where \(R\) is the separation in the plane of the sky \((R = |x_2 - x_1|)\) and

\[
\mu = \begin{cases} 
-\gamma_3 - 2 & \text{for } R \ll z_{\text{tot}} \text{ (either steep or shallow spectrum)}, \\
-\gamma_3 - 3 & \text{for } R \gg z_{\text{tot}} \text{ and } \gamma_3 < -3 \text{ (steep spectrum)}, \\
0 & \text{for } R \gg z_{\text{tot}} \text{ and } \gamma_3 > -3 \text{ (shallow spectrum)}. 
\end{cases}
\]

In contrast, the power spectrum of a field integrated along the LOS corresponds to selecting only the \(k_{\perp, \text{LOS}} = 0\) components of the underlying three-dimensional spectrum, or more precisely only the solenoidal part (see Appendix C). Thus, for isotropic and homogeneous power-law statistics, the two-dimensional power spectrum will reflect the three-dimensional spectral index. If, for instance, the density has a power spectrum \(P_{3D, \rho} \propto k^{\gamma_3}\), the spectrum of the column density will scale as

\[
P_{2D, I} \propto k^{\gamma_3}.
\]

We computed spectra and structure functions for the Gaussian cubes used before, integrated along the \(z\)-direction, and presented
the second-order structure functions. The stars correspond to a steep spectrum with a prescribed three-dimensional power spectrum index of $\gamma_{3D} = -11/3$, the diamonds correspond to a shallow spectrum with a prescribed three-dimensional power spectrum index of $\gamma_{3D} = -2.5$. In the solid and dotted lines, respectively, we plotted for reference the expectations for both long-wave- or short-wave-dominated cases. The vertical scales in panel (b) have been modified arbitrarily for visual purposes.

Fig. 3.—Two-point statistics for Gaussian fields integrated along the $z$-direction. In panel (a) we show the two-dimensional power spectra; panel (b) corresponds to the second-order structure functions. The functions correspond to a steep spectrum with a prescribed three-dimensional power spectrum index of $\gamma_{3D} = -11/3$, the diamonds correspond to a shallow spectrum with a prescribed three-dimensional power spectrum index of $\gamma_{3D} = -2.5$. In the solid and dotted lines, respectively, we plotted for reference the expectations for both long-wave- or short-wave-dominated cases. The vertical scales in panel (b) have been modified arbitrarily for visual purposes.

them in Figure 3, along with the expected behavior from equations (16) and (17). For the structure functions we show only the asymptotic behavior for small separations ($R \ll z_{tot}$), as the $R \gg z_{tot}$ scales are unavailable. (The maximum scale not affected by wraparound periodicity is $L/2 = z_{tot}/2$.)

If the LOS size of the object under study is much smaller than its size in the plane of the sky (so $R \gg z_{tot}$ is possible), we would see the underlying three-dimensional spectral index of the structure functions for large separations (i.e., no projection effect). With enough resolution we could use the two-dimensional spectral index of the column density map for $R \ll z_{tot}$ to infer the underlying three-dimensional index of the density. However, for the resolution used here (216$^2$ pixels), the projected structure function for small lags is already in the transition between the two asymptotes in equation (16). Thus, the measured index of the projected map is always shallower (smaller) than the actual index $\mu \approx -\gamma_{3D} \approx 2$. Another subtle, yet interesting, point is that the power spectrum applied to the map of the integrated velocity field, such as

$$V_z(X) = \int v_z(x) dz,$$  

will recover only the incompressible (solenoidal) component of the field. This could be potentially used to study the role of compressibility in turbulence statistics by combining velocity centroids and VCA (LE03).

3. MODIFIED VELOCITY CENTROIDS REVISITED

Velocity centroids have been widely used to relate their statistics with velocity; their conventional form is (Munch 1958)

$$C(X) = \frac{\int v_z \rho_0(X, v_z) dv_z}{\int \rho_0(X, v_z) dv_z}.$$  

We refer to this definition as “normalized” centroids. The denominator in equation (19) introduces an extra algebraic complica-

tion for a direct analytical treatment of the two-point statistics. For the sake of simplicity we start considering “unnormalized” velocity centroids

$$S(X) = \int \alpha v_z \rho_0(X, v_z) dv_z.$$  

(Note that they have units of density times velocity as opposed to velocity alone.)

Like the expression in equation (9), with emissivity proportional to the first power of the density and no self absorption, the structure function of the unnormalized velocity centroids is

$$\langle [S(x_1) - S(x_2)]^2 \rangle =$$

$$\langle \left( \alpha \int v_z(x_1) \rho(x_1) dz - \alpha \int v_z(x_2) \rho(x_2) dz \right)^2 \rangle.$$  

(21)

Presenting the density and velocity as a sum of a mean value and a fluctuating part: $\rho = \rho_0 + \tilde{\rho}$, and $v_z = v_0 + \tilde{v}_z$, where $\rho_0 = \langle \rho \rangle$, $v_0 = \langle v_z \rangle$, and the fluctuations satisfy $\langle \tilde{\rho} \rangle = 0$ and $\langle \tilde{v}_z \rangle = 0$. Analogous to equation (13), the structure function of the unnormalized centroids can be written as

$$\langle [S(x_1) - S(x_2)]^2 \rangle = \alpha^2 \int dz_1 dz_2 \left[ D(r) - |D(r)|_{x_1 = x_2} \right],$$  

(22)

with

$$D(r) = \left[ \langle v_z(x_1) \rho(x_1) - v_z(x_2) \rho(x_2) \rangle \right]^2.$$  

(23)
The velocity dispersion $D(r)$ can be approximated as

$$D(r) \approx \langle v_z^2 \rangle d_r(r) + \langle \rho^2 \rangle d_v(r) - \frac{1}{2} \langle d_v(r) d_r(r) \rangle + c(r),$$

(24)

which includes the underlying three-dimensional structure function of the LOS velocity

$$d_v(r) = \left\langle \left( v_z(x_1) - v_z(x_2) \right)^2 \right\rangle$$

(25)

and cross-correlations of velocity and density fluctuations

$$c(r) = 2 \langle \tilde{v}_z(x_1) \tilde{\rho}(x_1) \rangle - 4 \rho_0 \langle \tilde{\rho}(x_1) \tilde{v}_z(x_1) \tilde{v}_z(x_2) \rangle.$$

(26)

Because the derivation of equations (22), (24), and (26) involves some tedious algebra, we place it in Appendix D. With all of this, the structure function of unnormalized velocity centroids can be decomposed as

$$\left\langle [S(x_1) - S(x_2)]^2 \right\rangle = I1(R) + I2(R) + I3(R) + I4(R),$$

(27)

where

$$I1(R) = \alpha^2 \langle v_z^2 \rangle \int dz_1 dz_2 \left[ d_r(r) - \left| d_r(r) \right|_{x_1 = x_1} \right],$$

(28a)

$$I2(R) = \alpha^2 \langle \rho^2 \rangle \int dz_1 dz_2 \left[ d_v(r) - \left| d_v(r) \right|_{x_1 = x_1} \right],$$

(28b)

$$I3(R) = -\frac{1}{2} \alpha^2 \int dz_1 dz_2 \left[ d_r(r) d_v(r) - \left| d_r(r) \right|_{x_1 = x_1} d_v(r) \right],$$

(28c)

$$I4(R) = \alpha^2 \int dz_1 dz_2 \left[ c(r) - \left| c(r) \right|_{x_1 = x_1} \right].$$

(28d)

With this new decomposition, the definition of the structure function of “modified” velocity centroids (MVCs) becomes clearer and is

$$M(R) = \left\langle [S(x_1) - S(x_2)]^2 - \langle v_z^2 \rangle \left[ I(x_1) - I(x_2) \right]^2 \right\rangle$$

$$= \left\langle [S(x_1) - S(x_2)]^2 \right\rangle - I1(R)$$

$$= I2(R) + I3(R) + I4(R).$$

(29)

The velocity dispersion $\langle v_z^2 \rangle$ can be obtained directly from observations using the second moment of the spectral lines

$$\langle v_z^2 \rangle = \frac{\int v_z^2 \rho_s(X, v_z) dv_z}{\int \rho_s(X, v_z) dv_z}.$$

(30)

Thus, $I1(R)$ can also be related to the structure function of column density by

$$I1(R) = \langle v_z^2 \rangle \left[ I(x_1) - I(x_2) \right]^2.$$ Similarly, the power spectrum of centroids can be decomposed as (details in Appendix F)

$$P_{2D,v}(K) = \langle \rho^2 \rangle \langle v_z^2 \rangle \left( \alpha z_{tot} \right)^2 \delta(K) + \rho_0^2 P_{2D,i}(K) + \alpha^2 \rho_0^2 P_{2D,i}(K) + \mathcal{F}\{B3(R)\} + \mathcal{F}\{B4(R)\}.$$  

(31)

The term $\langle \rho^2 \rangle \langle v_z^2 \rangle \left( \alpha z_{tot} \right)^2 \delta(K)$ has no effect in the slope of the power spectrum because it only has power at $K = 0$. The expressions $P_{2D,i}(K)$ and $P_{2D,v}(K)$ are the spectra of the column density and the integrated velocity, respectively. They can be used to obtain the three-dimensional spectral index of density (or velocity), as shown in Appendix C. The expression $\mathcal{F}\{B3(R)\}$ is the Fourier transform of $B3(R)$, a cross term analogous to $I3(R)$, but in terms of the correlation functions. Similarly, $\mathcal{F}\{B4(R)\}$ includes the same density-velocity cross-correlations as $I4(R)$.

The power spectrum of MVCs can be obtained by subtracting $\langle v_z^2 \rangle P_{2D,i}(K)$ from the spectrum of the centroids. We derive in Appendix G a criterion for MVCs to trace the statistics of velocity better than unnormalized centroids. It was found that, with very little dependence on the spectral index, MVCs are advantageous compared to unnormalized centroids at small lags. This result is general, and we tested it using analytical expressions for the structure functions. For simplicity we considered only the two physically motivated cases, steep density and shallow density with steep velocity. The latter was not included explicitly in Appendix G because we obtain almost identical results in both cases.

If $S = 0$ and $\mathcal{F}\{B3(R)\} + \mathcal{F}\{B4(R)\}$ can be neglected, the spectrum of unnormalized centroids will trace the solenoidal component of the underlying velocity spectrum ($F_N(k)$, see Appendix C). But if the turbulent velocity field is mostly solenoidal, as supported by numerical simulations (Matthaeus et al. 1996; Porter et al. 1998; Cho & Lazarian 2003), the power spectrum is uniquely defined, assuming isotropy [$E(k) = \int F_N(k) dk \approx 4 \pi^2 F_N(k)$]. In the same way if $I1(R)$ can be eliminated (either for being small compared to the structure function of centroids or by subtraction—MVCs) and if $I2(R) \gg I3(R) + I4(R)$, the structure function of the remaining map will trace the structure function of a map of integrated turbulent velocity, and we can, in principle, recover the underlying three-dimensional velocity statistics (see Appendix B). With this background [and disregarding the cross terms $I3(R)$ and $I4(R)$] we arrived in LE03 at a criterion for safe use of (unnormalized) velocity centroids: if

$$\left\langle [S(x_1) - S(x_2)]^2 \right\rangle \gg \langle v_z^2 \rangle \left[ I(x_1) - I(x_2) \right]^2,$$

(32)

then the structure function of velocity centroids will mostly trace the turbulent velocity statistics, otherwise the density fluctuations are important and will be reflected in the centroid measures. When the structure function of velocity centroids is shallower or at least not much steeper than that of the column density, which can be verified by the power spectrum or by computing the structure function directly in three-dimensions for a few values for the lag, then the criterion proposed in LE03 can be simplified to use only the variances of the two maps (and the velocity dispersion):

$$\left\langle \tilde{S}^2 \right\rangle \gg \langle v_z^2 \rangle \langle \tilde{I}^2 \rangle.$$

(33)

If any of these two criteria is violated, one could, in principle, subtract the contribution of density, and the MVCs would trace the velocity structure function, provided that we could neglect the cross terms.

The contribution of velocity-density cross-correlations ($c(r)$) has been studied earlier. For VCA it has been shown to be marginal (Lazarian et al. 2001; Esquivel et al. 2003). However, a more detailed discussion of their effect in the context of MVCs is necessary and is provided below.

The expressions $I3(R)$ and $\mathcal{F}\{B3(R)\}$ are in some sense “cross terms”; $I4(R)$ and $\mathcal{F}\{B4(R)\}$ are related to correlations

\footnote{Note that eq. (26) is somewhat different from LE03, in which there was a misprint, which has no effect on the results presented.}
between density and velocity. We expect both pairs to grow as we increase the “interrelation” between density and velocity. We refer to \( I_3(R) \) (or \( F^1(B_3(R)) \)) simply as “cross term” and to \( I_4(R) \) (or \( F^2(B_4(R)) \)) as “cross-correlations” of density-velocity. The latter should be zero for uncorrelated data. At the same time the cross term can be studied analytically for power-law statistics, as presented in Appendix E, and will not be zero, even in the case of uncorrelated velocity and density fields. Before computing them directly, in order to get a feeling of how important the cross term could become, one could consider the structure functions. First of all, note that \( \langle |S(x_1) - S(x_2)|^2 \rangle \) is positive defined, as are \( I_1(R) \) and \( I_2(R) \). The remaining terms can be negative, in which case they must be smaller than the sum of \( I_1(R) \) and \( I_2(R) \). Let us focus for the moment on the contribution of \( I_3(R) \) and disregard cross-correlations between density and velocity. Its magnitude is maximal at large scales, as are \( I_1(R) \) and \( I_2(R) \). At such scales \( I_3(R) \) is on the order of \( I_1(R) \) and \( I_2(R) \). However, in \( I_2(R) \) the structure function of velocity is weighted by \( \rho^2 = \rho_0^2 + (\overline{\rho}^2) \) instead of only \( \overline{\rho}^2 \), enhancing the velocity statistics compared to the cross term. The importance of the cross term at the small scales (in which we are most interested) will depend on details such as how steep the underlying structure functions are (see Appendix E), and the zero levels of density and velocity (see Ossenkopf et al. 2005).

This is easy to understand for a particular case of power-law statistics of the form \( d_i(r) \propto r^m \) and \( d_o(r) \propto r^n \), with \( m, n > 0 \), i.e., both fields steep. Here the cross term scales as \( \propto r^{m+n} \), which is steeper than both the velocity and density. If at large scales \( I_1(R) \) and \( I_2(R) \) are on the order of \( I_3(R) \), provided that the latter falls more rapidly toward small scales, its contribution will be smaller than both the velocity and density structure functions at those scales. But if the density or the LOS velocity (or both) have a shallow spectrum, the cross term can be larger than \( I_1(R) \) or \( I_2(R) \) and can affect significantly the statistics of centroids. Measured spectral indices of density in the literature range from \( \gamma_{3D} \approx -2.5 \) to \(-4.0 \), which include both shallow and steep spectra. This is true for observations in different environments in the ISM (for instance, Deshpande et al. 2000; Bensch et al. 2001; Stanimirović & Lazarian 2001; Ossenkopf & Mac Low 2002), as well as for numerical simulations (see Cho & Lazarian 2002; Brun & Mac Low 2004; Beresnyak et al. 2005). The velocity spectral index is less known from observations but has been measured to be only in the steep regime (for example, using VCA in Stanimirović & Lazarian 2001), also in agreement with simulations. From the theoretical standpoint, at small scales when self-gravity is important, we might expect clumping that results in enhanced small-scale structure (yielding a shallow spectrum). On the other hand there are no clear physical grounds, to our knowledge, that will produce a small-scale-dominated (shallow) velocity field. However, even in the simple case of steep density and steep velocity spectra, it is not clear beforehand how important density-velocity cross-correlations \( I_4(R) \) could be. Later, we analyze the contribution of the cross terms and density-velocity cross-correlations in more detail, including spectra.

### 4. TESTING VELOCITY CENTROIDS NUMERICALLY

In LE03 we performed some preliminary tests of the modified velocity centroids using numerical simulations and compared the power spectrum with that of the velocity field, with normalized (eq. [19]) and with unnormalized centroids (eq. [20]). In this section we provide a more detailed test to investigate under what conditions velocity centroids can be used to recover the velocity statistics.

#### 4.1. The Data

We took compressible MHD data cubes from the numerical simulations of Cho & Lazarian (2003). These data cubes correspond to fully developed (driven) turbulence. The turbulence is driven in Fourier space (solenoidally) at wavenumbers \( 2 \leq k_{driving}L/(2\pi) \leq 3.4 \). The data cubes have a resolution of \( 216^3 \) pixels. We use four sets of simulations; the parameters for each run are summarized in Table 1. The models include various values of the plasma \( \beta \) (ratio of gas to magnetic pressures), sonic Mach numbers \( M_s \), and Alfvén Mach number \( M_A \). All of these parameters can be found in the ISM under different situations. For more details about the simulations we refer the reader to Cho & Lazarian (2003). The outcome of the simulations are density and velocity data cubes that we use to compute the centroids. We refer to these data sets as “original,” and the existing correlations between density and velocity (consistent with MHD evolution) are left intact. The numerical simulations have a limited inertial range. We have power-law statistics (i.e., inertial range) neither at the largest scales (smallest wavenumbers) due to the driving of the turbulence nor at the smallest scales (largest wavenumbers) because of numerical dissipation. Thus, it is very difficult to estimate spectral indices because the measured log-log slope is quite sensitive to the range in wavenumbers (or lags) used. This poses a problem of obtaining quantitative results. For that reason, we created another data set by modifying the original fields to have strict power-law spectra, following the procedure in Lazarian et al. (2001). The procedure consists in modifying the amplitude of the Fourier components of the data so they follow a power law, while keeping the phases intact. This way we preserve most of the spatial information. By keeping the phases we also minimize the effect of the modification to the density-velocity correlations. In addition, these new fields have the same mean value as the original data sets, and the magnitude of their power spectra (vertical offset) was fixed to match the original variances as well. We refer to these data sets as “reformed.”

#### 4.2. Results

Column density and centroids of velocity are two-dimensional maps, and therefore it is not computationally restrictive to obtain their correlation or structure function directly in real space. The power spectrum is often computationally cheaper because FFT can be used. However, the inherent difficulties of applying Fourier analysis to real data, for instance, instrumentation response and the lack of periodic boundary conditions, make power spectrum often unreliable. To alleviate this problem more elaborate techniques like wavelet transforms have been proposed (Zielinski & Stutzki 1999). Moreover, if the observed maps are not naturally arranged in a Cartesian grid, one would need to smear the data onto that kind of grid to use FFT, which is not necessary for structure or correlation functions if they are computed by directly averaging in configuration space. In spite of this, because the simulations we used are in a Cartesian grid and indeed have periodic boundary conditions, we can compute spectra, correlation,
and structure functions using FFT (see § 2) with as good accuracy as doing the average in real space. The relation between the structure function, correlation function, and power spectrum of unnormalized centroids can be found in Appendix F.

4.2.1. Three-dimensional Statistics

Before we study the two-dimensional maps and try to extract from those the underlying three-dimensional statistics, we start by computing the three-dimensional statistics. This is shown in Figure 4. It is noticeable that only for the case in which the turbulence is subsonic ($M_s/C_0^2 < 5$) the level of the velocity fluctuations is larger than that of the density. The limited inertial range in the original simulations (i.e., not perfect power-law statistics) is also evident in the figure. Spectral indices (log-log slopes), both for power spectra and structure functions from Figure 4, are given in Table 2. For the original data sets the indices for power spectra were obtained in a range of wavenumbers $kL/(2\pi) \sim [5, 15]$ (between the scale of injection and the scales at which dissipation is dominant). The structure function’s spectral indices were calculated with the corresponding values for spatial separations, $r/L \sim [5, 15]/C_{138}$.

The reformed data sets were constructed using the power spectra indices estimated for the original MHD simulations. We can see in Figure 4 the idealized power-law spectra of the reformed data sets. At the same time, although structure functions do not have such perfect power-law behavior (see discussion at the end of § 2.1), they show improvement compared to the original simulations. The range of scales used to measure the spectral indices for the reformed data sets is $kL/(2\pi) \sim [3, 5]; 100$, or $r/L \sim [100; 3, 5]$ much wider than for the original sets. Notice that power spectra for the density become shallower with increasing Mach number. At the same time the spectral index of the velocity is always steep.

4.2.2. Statistics of Projected Quantities (Two-dimensional)

A natural way to study how the velocity centroids trace the statistics of velocity is to compare their two-point statistics to those of an integrated velocity map (eq. [18]). This map can be used to obtain the velocity spectral index in the same way as column density can be used to obtain that of density. Therefore it is a direct measure of the underlying velocity statistics (see Appendices B and C). However, such a map is not observable, while velocity centroids are. We computed power spectra and structure functions of two-dimensional maps of the various centroids (normalized, unnormalized, and “modified”), integrated velocity, and integrated density. The results for power spectra are shown in Figures 5 and 6 for the original and modified data sets, respectively. Similarly, Figures 7 and 8 show the results for structure functions. The most noticeable difference in the figures is, of course, the larger inertial range of the reformed data set. We also present in Table 3 a summary

| TABLE 2 | THREE-DIMENSIONAL SPECTRAL INDICES |
|---------|-----------------------------------|
|         | DENSITY                          | LOS VELOCITY |
|         | $\gamma_{3D}$ | $m$ | $\gamma_{3D}$ | $m$ |
| A........ | (3.5) 3.5 | (0.4) 0.6 | (3.8) 3.8 | (0.5) 0.8 |
| B........ | (3.3) 3.3 | (0.3) 0.4 | (3.6) 3.6 | (0.5) 0.6 |
| C........ | (3.1) 3.1 | (0.1) 0.3 | (4.0) 4.0 | (0.8) 0.9 |
| D........ | (2.6) 2.6 | . . . . . | (3.8) 3.8 | (0.6) 0.8 |

Note.—The values in parentheses correspond to the original simulations; the other values correspond to the reformed data sets.

a The measured power-spectrum index in this case corresponds to a shallow spectrum. Thus, the correlation function is expected to follow a power law, not the structure function.
of spectral indices (log-log slope) measured over $5 \leq kL/(2\pi) \leq 25$ (or $\frac{1}{25} \leq R/L \leq \frac{1}{2}$) for the original data sets and $kL/(2\pi) \sim [3.5, 100]$ (or $r/L \sim [\frac{1}{100}, \frac{1}{25}]$) for the reformed sets.

Comparing the spectral indices derived in three dimensions (Table 3) with those of column density and integrated velocity in Table 3, one can notice a better correspondence for the reformed data sets. This is true for the power-spectra index $\gamma_{3D}$, as well as for $m$, and $\mu$ for structure functions (related by eq. [16]). Directly from Figures 5–8 one can see that only for the case of subsonic turbulence (model A, $M_s \sim 0.5$) does the spectrum of centroids clearly scale with that of integrated velocity. In this case the power spectra of all the variations of centroids recover the spectral index of the velocity, within 10% error for the original simulations and <3% for the reformed data sets. For the cases of mildly supersonic turbulence (models B and C, $M_s \sim 2.5$) it is clear neither from the figures nor from the measured slopes if centroids trace velocity. For the strongly supersonic case (model D, $M_s \sim 7$), it is obvious that velocity centroids fail to recover the velocity scaling.

Due to the finite width effects discussed at the end of § 2.2, it is more difficult to determine quantitatively the spectral index from centroid maps using structure functions. According to equation (16), one should restrict to measure the spectral index for lags either much smaller, or much larger, than the LOS extent of the object under study. The latter is not feasible with our simulations because the maximum lag available, unaffected by wrap-around periodicity, is $L/2 = z_{tot}/2$. The other case ($R \ll z_{tot}$) is not strictly possible with the resolution used here. For the MHD data cubes we have to avoid the smallest scales because they are not within the inertial range (i.e., they are already dominated by dissipation). Actually, the lags used to measure $\mu$ in the original data sets are well within the transition between the two asymptotic power laws of equation (16). Thus, if we obtain the three-dimensional spectral indices from $\gamma_{3D} \simeq -\mu - 2$ (taking $\mu$ from Table 3), we would only get upper limits for the actual $\gamma_{3D}$. Keeping in mind that our definition of the index includes the minus sign, this means that the real index is going to be a larger negative number than the $\gamma_{3D}$ inferred from structure functions. A lower limit on the spectral index can be obtained as $\gamma_{3D} \simeq -\mu - 3$; unfortunately this provides a rather wide range of possible indices and is not useful for practical purposes (for instance, distinguishing between two models of turbulence). For the reformed data sets the situation is better, because we can use the smallest scales. This can be verified from the better agreement of $\gamma_{3D} \simeq -\mu - 2$ with $\gamma_{3D}$ measured directly from power spectra. However, $R \ll z_{tot}$ is still not strictly fulfilled, and thus we get only lower limits of $\mu$. We recognize this projection effect as an important drawback for the structure functions compared with spectra. Power spectra do not suffer so strongly from finite width projection effects and could be used to obtain more accurately the spectral index of the underlying three-dimensional index field from integrated quantities (see also Ossenkopf et al. 2005). This is true for synthetic data, but for real observations the power spectrum is not as reliable (see Bensch et al. 2001). At the same time, the problem of recovering quantitatively the spectral index from structure functions might be overcome for real observations with a larger inertial range. With enough spatial resolution one can have
easily over two decades of inertial range in the plane of the sky (below the size of a given object along the LOS direction). In the spirit of keeping the description in terms of structure functions, which might be better suited for some type of real observations, and using the limited range we have, it is still possible to do a comparative analysis between the spectral index of structure functions of centroids to that of integrated velocity or density.

From Figures 5–8 we see that the relative importance of the density and velocity statistics on the centroids changes with the Mach number of turbulence. For the low-Mₙ model (panels [a] in Figures 5–8) the level of the density fluctuations is very small compared to the velocity fluctuations (weighted by \( h v^2 \) and \( h / C^2_i \), respectively), and all the centroids trace remarkably well the velocity statistics. In this case the simplified criterion \( h \tilde{S}^2_i / (h v^2) \approx k^{90} \) for the original data set and \( k^{30} \) for the reformed one. This result also agrees with the notion that velocity centroids trace the statistics of velocity in the case of subsonic turbulence (where density fluctuations are expected to be negligible, as the turbulence is almost incompressible). For the rest of the models density has an increasing impact, which is reflected in the centroids, and our criteria are either not entirely met or violated. We can also see for supersonic turbulence the slope of the structure function of unnormalized centroids is almost always steeper than that of the column density. This means that one need to use the full criterion as proposed in LE03 to judge whether centroids will trace velocity or not. We see however that our simplification of the criterion seems to discriminate when centroids trace the statistics of integrated velocity, at least for the data sets employed here. For all the supersonic cases the measured power-spectrum index from the different centroids fails to give the index of velocity, and in general, their centroid indices are found to lie between those of velocity and density, with more scatter for the original simulations compared to the reformed fields. For the original data sets of models B, C, and D, the ratio \( \tilde{S}^2_i / (\langle v^2 \rangle) \) is \( \approx 5, 4, \) and 3, respectively. For the corresponding reformed data cubes our simplified criterion gives \( \tilde{S}^2_i / (\langle v^2 \rangle) \approx 2, 4, \) and 3, respectively. In all of these cases \( \tilde{S}^2_i / (\langle v^2 \rangle) \approx 1 \) is not strictly true, and indeed there is an important contribution of density fluctuations. This is consistent with the results in LE03 in which we see evidence of a density-dominated regime at small scales (rather a density-contaminated regime, as centroids do not show the same index as density either). Our results agree with those presented in Brunt & Mac Low (2004), in which they found that centroids do not provide good velocity representation for the supersonic turbulence they studied (\( M_M > 1.9 \)).

4.2.3. Cross Terms and Density-Velocity Cross-Correlations

The cross terms, \( I_3(R), \) \( F(\langle B_3 \rangle) \), as well as the cross-correlations in \( I_4(R) \) and \( F(\langle B_4 \rangle) \), have contributions of velocity and density that we cannot disentangle entirely from observables. Furthermore, they cannot be expressed in terms of integrated quantities, and they have to be computed from three-dimensional statistics. We used Fourier transforms to obtain the structure and correlation functions in three dimensions that are needed to produce independently all the terms in equations (27) and (31). These three-dimensional statistics were integrated numerically to get \( F(\langle B_3 \rangle), I_3(R), F(\langle B_4 \rangle), \) and \( I_4(R) \). To check the accuracy of the three-dimensional quantities obtained and our numerical integration to two dimensions, we verified with the cases...
in which the statistics can be also obtained directly in two dimensions (e.g., $I_1[R]$ and $I_2[R]$) and always found a good agreement. The results of the decomposition in terms of power spectra (eq. [31], after $K$ averaging) are shown in Figure 9 for the original simulations and in Figure 10 for the reformed data sets. The decomposition for $R$-averaged structure functions is presented in Figures 11 and 12 for the original and modified data sets, respectively. Note that because structure functions span fewer decades in the vertical axis, the separation of all the terms is generally clearer than for the power spectra. Note also that for the subsonic model (A), the spectrum and structure function of the centroids are almost unaffected by density fluctuations, cross terms, or density-velocity cross-correlations. Cross-correlations are found to be larger for supersonic turbulence compared to the subsonic case, but from this limited data set we can not conclude that they scale in some particular way with the sonic Mach number. It is also significant that the cross term increases relative to the other terms with $M_s$. For spectra, in all the supersonic models (B, C, and D) the cross term $F_{B3(R)}$ is dominant. This would mean that the log-log slope measured from spectra in all of this cases is not a direct reflection of the velocity spectral index. At the same time we observe that the magnitude of density-velocity cross-correlations can only be entirely ignored for model A (subsonic turbulence). For structure functions $I_3(R)$ is comparable to or larger in magnitude than the contribution of column density. At the same time it gets closer but does not become larger than $I_2(R)$. In fact, it was found to be always steeper than the velocity term; therefore its contribution at small scales could be neglected. Velocity-density cross-correlations in $I_4(R)$ for all the cases we considered here$^5$ never had an important impact on the statistics of centroids. Anyway, $I_3(R) + I_4(R)$ as a whole (especially at the smallest separations, which we are most interested in) has been found to be smaller than the integrated velocity term, suggesting that there could be cases in which $I_3(R) + I_4(R)$—the cross term and cross-correlations—can be neglected. However, since the indices from power spectra of centroids failed to give the velocity spectral index for models B, C, and D, we conclude that the retrieval of spectral indices from velocity centroids over the entire inertial range should be restricted to very low sonic Mach numbers ($M_s < 2.5$). Furthermore, the relative importance of the cross terms grow together with the column density term as the strength of turbulence increases. For $M_s > 7$ certainly the contribution of the column density is considerable. In this case MVCs could be used to remove the contribution from the column density term in the structure function of centroids. However at such high Mach number the cross terms cannot be neglected either.

At a first glance it might seem surprising that the velocity centroids could be dominated by velocity even when the three-dimensional structure functions of the density had a zero point (offset) larger than those of the velocity. This effect arises primarily from the fact that the density is positive defined and necessarily has a nonzero mean, whereas the velocity field can have a zero mean. This results in the factors that multiply the density...
and velocity structure functions in equation (24). The factor \( h \) multiplies the density structure function, and we can minimize the undesired density contribution by shifting our velocity axis in such a way that \( v_0 = 0 \). For a more detailed discussion about the velocity and density zero levels, see Ossenkopf et al. (2005).

5. VELOCITY CENTROIDS AND ANISOTROPY STUDIES IN SUPERSONIC TURBULENCE

We have seen how density fluctuations in supersonic turbulence affect our ability to determine the velocity spectral index from observations. But is there something else velocity centroids could be used for, even in supersonic turbulence? The presence of a magnetic field introduces a preferential direction of motion, thus breaking the isotropy in the turbulent cascade. In a turbulent magnetized plasma, eddies become elongated along the direction of the local magnetic field. Velocity statistics have been suggested to study this anisotropy (Lazarian et al. 2002; Esquivel et al. 2003; Vestuto et al. 2003). For instance, isocontours of two-point statistics of velocity centroids, instead of being circular, as in the isotropic case, are ellipses with symmetry axes that reveal the direction of the mean magnetic field. We present in Figure 13 contours of equal correlation of velocity centroids (unnormalized) from our simulations. The magnitude of the magnetic field determines how much anisotropy will be present. We can see from Figure 13 that the anisotropy is very clear for models A, B, and D, regardless of the large differences in sonic Mach

![Diagram](image-url)

**Fig. 8.**—Same as Fig. 7, but for the reformed data sets.

| Table 3: Spectral Indices (Two-dimensional) of Quantities Integrated along the LOS |
|-----------------------------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|
| Model                            | \( -\gamma_{3D} \) | \( \mu^* \)       | \( -\gamma_{3D} \) | \( \mu^* \)       | \( -\gamma_{3D} \) | \( \mu^* \)       | \( -\gamma_{3D} \) | \( \mu^* \)       |
| A..........................   | (4.0) 3.5          | (0.5) 1.4         | (3.6) 3.8         | (0.9) 1.5         | (3.6) 3.8         | (0.8) 1.5         | (3.5) 3.8         | (0.8) 1.5         |
| B..........................   | (3.8) 3.3          | (0.4) 1.2         | (3.9) 3.6         | (1.0) 1.4         | (3.5) 3.3         | (0.8) 1.1         | (3.5) 3.2         | (0.9) 1.1         |
| C..........................   | (3.3) 3.1          | (0.6) 1.1         | (4.5) 4.0         | (1.3) 1.6         | (3.8) 3.4         | (1.1) 1.3         | (3.9) 3.5         | (1.2) 1.3         |
| D..........................   | (2.8) 2.7          | (0.2) 0.7         | (4.7) 3.8         | (0.8) 1.5         | (3.4) 2.8         | (0.5) 0.9         | (3.8) 2.9         | (0.7) 1.0         |

Note.—The values in parentheses correspond to the original simulations; the other values correspond to the reformed data sets.

a Since this index is not measured at scales corresponding to \( R \ll z_{\text{iso}} \) but rather in the transition between the two asymptotic regimes in eq. (16), these are only lower limits on the actual \( \mu \) for small lags.

6 Both velocity centroids and spectral correlation functions were demonstrated to trace the magnetic field in Lazarian et al. (2002); channel maps were studied for the same purpose in Esquivel et al. (2003), and velocity centroids in Vestuto et al. (2003).
number and plasma $\beta$. The only case in which the anisotropy is not evident (model C) is our only super-Alfve\'nic simulation. The concept of super-Alfve\'nic turbulence is advocated, for instance, by Padoan et al. (2004a) for molecular clouds. Another way to visualize the anisotropy is to plot the correlation functions of centroids in the parallel or perpendicular directions relative to $B_0$; this is shown in Figure 14. In our simulations this corresponds to plotting the value at the intercepts in Figure 13; in observations it would mean plotting the correlation function only along the major or minor axis of the contours of equal correlation. The anisotropy is evident as a different scale length (correlation length) for correlations parallel or perpendicular to the mean magnetic field. For subsonic turbulence the degree of anisotropy reflects the ratio $(\tilde{B}/B_0)^2$. However for supersonic turbulence, as the contributions of density get important and as the density at high Mach number gets more isotropic (see Beresnyak et al. 2005), one could expect the anisotropy to decrease while the ratio $\tilde{B}/B_0$ stays the same. We found very little evidence of this decrease, as can be verified from models A, B, and D in Figure 14; all three runs have the same $\tilde{B}/B_0$ and a large contrast in sonic Mach number. In model C, even though the magnetic pressure is larger than the gas pressure ($\beta = 0.2$), it corresponds to a weak mean field $\tilde{B} > B_0$. In this case the magnetic field has very chaotic structure at large scales (see Cho & Lazarian 2003). As the fluctuations at large scales determine the anisotropy of the projected data, the MHD anisotropy is erased after LOS averaging.

Whether the ISM is sub-Alfvenic is still up for debate. The fact that centroids are only anisotropic for sub-Alfvenic turbulence gives us an opportunity to study the conditions under which that is the case. It is certainly encouraging that for sub-Alfvenic turbulence the degree of anisotropy does not seem to be strongly affected by density fluctuations. However, we should add a word of warning in trying to determine the ratio $\tilde{B}/B_0$ from velocity centroids in supersonic turbulence. The fact that centroids do not represent velocity for $M_s > 2.5$ calls for some caution as to what extent the ratio $\tilde{B}/B_0$ can be obtained from observations. Nevertheless, the result for the direction of the mean field is rather robust, including for highly supersonic turbulence, provided a relatively large ordered component $B_0$. Anisotropies of turbulence measured by different techniques, including centroids, provide a promising way of measuring the direction of the component of the magnetic field perpendicular to the LOS. For practical purposes the technique can be tested using polarized radiation arising from aligned dust grains (see Lazarian 2003 for a discussion of grain alignment) or CO polarization arising from the Goldreich-Kylatis effect (see Lai et al. 2003).

For any particular observational data one should bear in mind that only the plane-of-sky components of the mean and fluctuating magnetic field are available. Therefore, for instance, a cloud with sub-Alfvenic turbulence but with a mean magnetic field directed along the line of sight would look like a cloud with super-Alfvenic turbulence. The distinction between the two cases may...
be done, however, using an ensemble of clouds. It is unlikely to have the mean magnetic field always directed toward the observer. If, indeed, the turbulence is typically sub-Alfvénic, the anisotropy should show up in centroid maps.

6. DISCUSSION

We used MHD data cubes to produce centroid maps and explored the limitations of velocity centroids for studying interstellar turbulence. We investigated numerically the analytical predictions in our previous study of velocity centroids (LE03). In that work, we decomposed the structure function of velocity centroids into three main contributions, namely integrated (or column) density, integrated velocity, and “cross terms.” By calculating separately all the terms in the structure function of unnormalized centroids, we showed that the decomposition works well.

We found two important restrictions on the procedure accuracy, both related to the finite resolution in the numerical simulations. First, structure functions of integrated quantities can, in principle, be used to retrieve the underlying three-dimensional spectral index if calculated for lags either much greater or much smaller than the LOS extent of the object. We showed, however, that with the resolution used here (2163), the structure function of integrated quantities can only give an upper limit on the actual spectral index. Second, the limited inertial range in our simulations presents an additional constraint on the accuracy of estimating the spectral index, mainly because the measured log-log slope is very sensitive to the range of scales used.

In regards to the finite width projection effects one can potentially still do a comparative study of the centroid statistics with the statistics of velocity and density. That is, we can compare the spectral index of the centroid maps to that of column density or integrated density and see which is closer. However, for real observational data with the possibility of sample lags $R \ll z_{\text{tot}}$, obtaining the three-dimensional spectral index directly from structure functions is not a problem.

The power spectrum is an alternative for obtaining the scaling properties of the turbulent velocity field that does not suffer from such projection effects (i.e., the spectral index can be measured at any wavenumbers within the inertial range). In order to use power spectra also, we constructed a power-spectra decomposition analogous to that in LE03, and by calculating each term separately, we tested successfully the validity of this decomposition.

We found that the scaling properties of the underlying velocity field (i.e., spectral index) can be reliably retrieved for subsonic turbulence. However, the contamination from cross terms clearly showed up in all the supersonic cases for spectra, while for structure functions it was only evident for $M_s \sim 7$.

To alleviate the other problem (not enough inertial range), we introduced reformed versions of the original simulations. The power spectra of the new data sets are strict power laws, with almost the same spatial structure and density-velocity cross-correlations. The spectral indices obtained with these reform data sets are in better agreement with the analytical predictions.

We also tested successfully a criterion for which velocity centroids give trustworthy information (eq. [32]). The criterion can be obtained entirely from observations. For practical purposes we suggest an approximation of the criterion in LE03 in terms only of the variances of the maps of column density and velocity centroids, instead of computing all the structure

![Fig. 10.—Same as Fig. 9, but for the reformed data sets.](image-url)
functions (eq. [33]). If this ratio is less than unity, velocity centroids are not trustworthy. When the ratio in equation (33) is small but larger than unity, we recommend using the full criterion, as proposed in LE03. In our only case of subsonic turbulence (model A) centroids traced the integrated velocity structure function extremely well, and our simplified criteria was fulfilled:

$$h \tilde{S}_i^2 / (h v_z^2) I_i^2 k^{90} \text{and } 30 \text{ for the original and reformed data sets, respectively. For the rest of the models (B, C, and D) the criteria was not satisfied, } (h \tilde{S}_i^2 / (h v_z^2) I_i^2 k^{5/2})^{1/2} C_{138}/ C_{24}, \text{ and the centroids failed to trace the statistics of the velocity field. In this models one can see a better correspondence between centroids and integrated velocity for the } M_s/C_{24}/5 \text{ runs, compared to the highly supersonic case } M_s/C_{24}/7. \text{ Thus, one might hope that centroids are able to trace the spectral index of velocity for supersonic turbulence, but only for } M_s < 2.5. \text{ We must recognize that this study was done using a limited data set. A more complete exploration of parameter space is desirable to determine better the range of applicability of velocity centroids, including MVCs.}

We have seen that velocity centroids are only reliable at low Mach numbers. At the same time there are other techniques available that work for strongly supersonic turbulence, such as VCA and VCS. The techniques are complementary and can be used simultaneously. For subsonic turbulence, VCA and VCS can be used with higher mass species. Note, that the different techniques pick up different components of the velocity tensor. This can potentially allow a separation of the compressional and solenoidal parts of the velocity field (LE03). Determining these components is crucial for understanding properties of interstellar turbulence, its sources, and its dissipation.

We stress that every technique has its own advantages. Velocity centroids can reproduce velocity better than VCA and VCS when the velocity statistics is not a straight power law. Therefore, velocity centroids can better pick up the dissipation and injection energy scales.

Nevertheless, one have to bear in mind that spectral indices derived from all of the techniques above do not provide a complete description of turbulence. Anisotropy is another parameter that can be studied. For instance, we showed that velocity centroids are useful for studying the anisotropy of the turbulent cascade, even for highly supersonic turbulence. We showed, however, that this is restricted to a relatively large mean field. For super-Alfvénic turbulence, a model favored by some researchers (Padoan et al. 2004b), the anisotropy of centroids is marginal, which allows us to test these theories. Anisotropy is not only present in velocity centroid maps, but in other statistics as well, such as the spectral correlation function (see Lazarian et al. 2002). Combining the two measures can improve the confidence in the results.

In addition, we need tools to study the turbulence variations in space (i.e., intermittency). Higher order velocity structure functions (those described in this paper are second order) have been shown to be promising for such studies (see Müller & Biskamp 2000; Cho et al. 2003b). Velocity centroids can be easily recasted in terms of higher order statistics, opening a new window for intermittency studies. Note that higher moments can provide the

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**Fig. 11.—Decomposition of the structure function of unnormalized velocity centroids (eq. [27]), for the original data sets. The solid line is the structure function of the two-dimensional map of velocity centroids. The dotted line is $I_1(R)$, and the dashed line is $I_2(R)$; both were obtained from two-dimensional maps. The cross term $I_3(R)$ is plus signs, while the density-velocity cross-correlations $I_4(R)$ are the crosses. This last two were computed from three-dimensional statistics and then integrated along the LOS. The diamonds show the sum of all the terms in the decomposition for comparison with the solid line. The $I_3(R)$ and $I_4(R)$ can be negative; for that reason we plot only their magnitude.**
directions of the intermittent structures if they get oriented parallel to the magnetic field. Our results indicate that such studies can be carried out even for high Mach number turbulence. In addition, studies of intermittency with centroids can be incorporated into other techniques. For instance, the interpretation of results of the principal component analysis technique suggested as a tool for turbulence studies by Heyer & Schloerb (1997) depends on the degree of intermittency of the turbulence (Brunt et al. 2003; M. Heyer 2004, private communication).

Velocity centroids have been used for studies of turbulence for many decades. How can we comment on these studies from the point of view of our present theoretical understanding? Let us glance at the available studies. Turbulence in H ii regions is usually subsonic. Therefore, velocity centroids could be probably trustworthy there (see O'Dell & Castaneda 1987). Supersonic turbulence in molecular clouds (see Miesch & Bally 1994) is a field for which velocity centroids might be in error.7 The same is probably true for H i studies (see Miville-Desches et al. 2003a)8 in which turbulence is highly supersonic in cold H i.

We feel that a more careful analysis of particular conditions present in the regions under study is necessary, however. On one hand, while the turbulence is supersonic for molecular clouds, the cores of molecular clouds are mostly subsonic (see Tafalla et al. 2004; Myers & Lazarian 1998). For such cores velocity centroids provide a reliable tool for obtaining velocity statistics, if the resolution of the cores is adequate. On the other hand, our results are based on the analysis of isothermal numerical simulations. In the presence of substantial density contrasts caused by the coexistence of different phases, the velocity centroids may get unreliable even for subsonic turbulence. Therefore testing of the necessary criteria discussed in this paper may be advantageous not only for the molecular and H i data but also for data obtained for H ii regions.

We also note that for the present analysis we assumed that the emission is optically thin and that the emissivity is proportional to the first power of density. Discussion of more complex, but still observationally valuable, cases will be done elsewhere. The work started in this direction (see Lazarian & Pogosyan 2004) is encouraging.

Recent work on centroids includes papers by Ossenkopf & Mac Low (2002) in which they noticed that centroids poorly reproduce velocity statistics for supersonic turbulence. Our work above confirms their finding and also establishes a regime ($M_s < 2.5$) when the results by centroids can be reliable.

A quite different and optimistic conclusion about centroids was obtained in Miville-Desches et al. (2003b). They used Brownian-noise artificial data and obtained an excellent correspondence between the centroids and the underlying velocity. From the point of view of our analysis the origin of this correspondence is in the choice of the mean density level. In these simulations, in order to make the density positive defined, the authors added a substantial mean density to the fluctuating density. It is clear from equation (28b) that adding a mean density increases

---

7 Both in H ii regions and in molecular clouds, there are spectral lines in which the emissivity is proportional to the square of the density. Without going into the details, we can say that density is interchangeable with the emissivity in our treatment, allowing our results to be directly applied to such species. There are, however, some caveats in relation to the change of the spectral index of emissivity while the density has the same spectral index, but that will be discussed elsewhere.

8 An interesting feature found in Miville-Desches et al. (2003a) is that the statistics of velocity centroids is almost identical to the statistics of the density field. This could be due to the dominance of the $I_1$ term, which can be tested.
the contribution of the $I_2(R)$ term, which is the term responsible for the velocity contribution.

This work is complementary to the work on $\Delta$-variance of centroids that we are doing in collaboration with V. Ossenkopf and J. Stutzki (Ossenkopf et al. 2005). There Brownian-noise simulations are used, but extra care is taken to avoid being misled by the effect of adding mean density. An iterative procedure is proposed there, which allows for correction of the contribution of the cross terms when density-velocity correlations are negligible.

In this paper we are dealing mostly with unnormalized centroids. The MVCs allow a different outlook on the problem of obtaining velocity statistics from the small-scale asymptotes. Indeed, our analysis in the paper shows that the term $I_4(R)$ is unimportant for the cases that we studied. In addition, we believe that the velocity has steep spectra. Therefore, if the density is also steep, asymptotically the term $I_3(R)$, which is then steeper than both velocity and density, should be negligible (see Appendix E.1). As the result, if we see from the integrated intensity maps that density indeed is steep, we can use MVCs as suggested in LE03, which for sufficiently small lags are bound to represent the velocity statistics. The critical scale at which this is true can be obtained using the analytical expressions found in

![Figure 13](image-url)
Appendix B.1. Formally, to find such a critical scale, one should know the mean density. However, if the inertial range is sufficiently long, one should not be worried about the exact value for that critical scale.

We have not seen much advantage of such a use in our numerical runs because of the limited inertial range available. However, unlike numerical simulations, astrophysical conditions provide us with a substantially larger inertial range. For instance, Stanimirović & Lazarian (2001) showed that the turbulent spectrum for velocity spans from the size of the Small Magellanic Cloud, which is ~4 kpc, to the minimal scale resolved, i.e., ~40 pc. In partially ionized gas we expect this spectrum to proceed to subparsec scales. In fully ionized gas (see also the discussion in Lazarian et al. 2004) the scale of Alfvén and slow-mode damping may be only hundreds of kilometers.

The limitations of this asymptotic approach stem from the fact that for supersonic turbulence the density field tends to become shallower. But, as we can see from the MHD simulations used here, at $M_s \sim 2.5$ we find that it is still steep, and therefore the MVCs should be reliable asymptotically. Therefore, for handling observational data we can provide another prescription: if the inertial range is sufficiently long and the density is steep, then MVCs asymptotically represent the velocity field. For instance, steep density was reported in Miville-Deschênes et al. (2003b). Thus, for such density fields, asymptotic use of MVCs should be advantageous.

In the cases when the underlying statistics is not a power law, the interpretation of centroids is more involved. We have seen this in our analysis of non-power-law data from MHD simulations. While centroids do represent injection and damping scales, the integration along the LOS does interfere with a more finely detailed study. In this situation, one should use the full technique of inversion as proposed in Lazarian (1995). However, we expect that in most cases astrophysical turbulence is self-similar over large expanses of scales, and therefore the power-law representation is adequate.

We have made use of the analytical expressions for unnormalized velocity centroids (LE03). Our work, as well as Levrier (2004), show that normalization of centroids improves them rather marginally, while it makes the analytical insight essentially impossible. In this paper we tested that major conclusions reached for unnormalized centroids are applicable to the normalized ones.

7. SUMMARY

In this paper we have provided a systematic study of velocity centroids as a technique of retrieving velocity statistics from observations. We used both results of MHD simulations and of “reformed” data sets that have larger inertial range. We found

1. Centroids of velocity can be successfully used to retrieve the scaling properties of the underlying three-dimensional velocity field for subsonic turbulence. For supersonic turbulence with sonic Mach number $\geq 2.5$, velocity centroids failed to trace the spectral index of velocity.

Fig. 14.—Correlation functions of velocity centroids taken in directions parallel and perpendicular to the mean magnetic field. The anisotropy shows in the different scale lengths for the distinct directions and is noticeable in the little differences between panels (a), (b), and (d), which correspond to the same ratio $B/\bar{B}$, but very different sonic Mach numbers. Panel (c) corresponds to super-Alfvénic ($\beta > B_0$) turbulence, and the anisotropy is not evident in the centroid maps.
2. Our numeric simulations confirmed the expression for centroid statistics obtained in LE03. In particular, we tested successfully the criterion in LE03 for the reliable use of velocity centroids. We showed that it reflects a necessary condition for centroids to reproduce the velocity statistics.

3. We studied the modified velocity centroids (MVCs) proposed in LE03. It is shown that MVCs reflect the statistics of velocity better than unnormalized centroids for small lags. This result is valid for both steep and shallow density fields with steep velocity. Combined with the fact that products of density and velocity structure functions get subdominant for steep velocity and density at small lags, this provides a way to reliably study turbulence using MVCs.

4. We showed that velocity-density cross-correlations are marginally important for our data set, at least for small lags. Combined with the fact that products of density and velocity structure functions get subdominant for steep velocity and density at small lags, this provides a way to reliably study turbulence using MVCs.

5. We demonstrated that velocity centroids can be used for both subsonic and supersonic turbulence to study the anisotropy introduced by the magnetic field. Even when they fail to retrieve the velocity spectral index in supersonic turbulence, for up to at least $M_A \sim 7$ and for sub-Alfvénic turbulence, they provide reliably the direction of the component of the magnetic field perpendicular to the LOS.

6. If turbulence is super-Alfvénic, it results in a marginal anisotropy of velocity centroids. This provides a good way to test whether the Alfven Mach number of turbulence in molecular clouds is less or greater than unity.

7. Within their domain of applicability, velocity centroids provide a good tool for turbulence studies that should be used in conjunction with other tools, e.g., VCA and VCS.

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APPENDIX A

TURBULENCE STATISTICS IN THREE DIMENSIONS

A1. TURBULENCE STATISTICS IN REAL (XYZ) SPACE

The two-point correlation function of a vector field $\mathbf{u}(x) = [u_i(x), u_j(x), u_k(x)]$ is defined as

$$B_{ij}(r) = \langle u_i(x_1) u_j(x_2) \rangle,$$  \hspace{1cm} (A1)

where $r = x_2 - x_1$ is the separation or “lag” in Cartesian coordinates $i,j = x,y,z$ and $\langle \ldots \rangle$ denotes an ensemble average over all space. An additional definition can be made in terms of the fluctuations of the field. This can be obtained formally by replacing $u(r)$ by $\hat{u}(x) = u(x) - \langle u(x) \rangle$ in equation (A1). Note that this necessarily implies $\langle \hat{u} \rangle = 0$. We refer to this variation simply as the correlation function of fluctuations and denote it by

$$\tilde{B}_{ij}(r) = \langle \hat{u}_i(x_1) \hat{u}_j(x_2) \rangle = B_{ij}(r) - \langle u_i \rangle \langle u_j \rangle.$$  \hspace{1cm} (A2)

In the same manner, it is customary to define the structure function of the same vector field $\mathbf{u}(x)$ as

$$D_{ij}(r) = \langle [u_i(x_1) - u_j(x_2)] [u_i(x_1) - u_j(x_2)] \rangle$$

$$= \langle \hat{u}_i(x_1) - \hat{u}_i(x_2) \rangle [\hat{u}_i(x_1) - \hat{u}_i(x_2)].$$  \hspace{1cm} (A3)

Note that structure functions, by definition, depend only on the fluctuating part of the field and are insensitive to the mean value of the field. Combining equations (A1) and (A3), it is trivial to see that

$$D_{ij}(r) = 2 [\tilde{B}_{ij}(0) - \tilde{B}_{ij}(r)] = 2 [B_{ij}(0) - B_{ij}(r)],$$  \hspace{1cm} (A4)

where $\tilde{B}_{ij}(0)$ is also known as the variance. The correlation of fluctuations must vanish at infinity; that is, $\tilde{B}_{ij}(r) \rightarrow 0$, as $r \rightarrow \infty$. Then from equation (A4), it is clear that $D_{ij}(\infty) = 2\tilde{B}_{ij}(0)$. Thus, if we know the structure function of a field, we can obtain the correlation function of fluctuations and vice versa. However, to get the correlation function in general (as in eq. [A1]), we also need to know the mean value of the field ($\langle u \rangle$). The correlation and structure functions are equivalent in theory. In practice, however, where we have a restricted averaging space, it is easier to determine $D_{ij}(r)$ more accurately than $B_{ij}(r)$. At the same time, $\tilde{B}_{ij}(0)$ is usually better determined than $D_{ij}(\infty)$ (see discussion in Monin & Yaglom 1975).

Alternatively, we can use a spectral representation to describe turbulence. The spectral density tensor, or N-dimensional power spectrum, $F_{ij}(k) = P_{ND}(k)$, is given through a Fourier transform of the correlation function of fluctuations:

$$F_{ij}(k) = P_{ND}(k) \equiv \frac{1}{(2\pi)^3} \int e^{-i\mathbf{k} \cdot \mathbf{r}} \tilde{B}_{ij}(r) d^3r,$$  \hspace{1cm} (A5)

where $k$ is the wavenumber vector. The definitions presented above are general and also apply to scalar fields.
The power spectrum of velocity (in the incompressible regime) has an important physical interpretation as the distribution of kinetic energy (per unit mass) as a function of scale. If \( \mathbf{u}(r) \) is the velocity field, the power spectrum is the energy in scales between \( k \) and \( k + \delta k \), and thus the total energy is proportional to \( \langle (\mathbf{u}(r))^2 \rangle = \int P_{xy}(k) \, dk \). Note, however, that the physical interpretation is quite different if we talk about the spectra of another quantity (e.g., the power spectra of density fluctuations). For isotropic fields the correlation, structure, or spectral tensors can be expressed via longitudinal (parallel to \( r \), denoted by subscript “LL”) or transverse (normal to \( r \), denoted by subscript “NN”) components (Monin & Yaglom 1975):

\[
\begin{align*}
B_{ij}(r) &= [B_{LL}(r) - B_{NN}(r)] \frac{k_j^2}{k^2} + B_{NN}(r) \delta_{ij}, \\
D_{ij}(r) &= [D_{LL}(r) - D_{NN}(r)] \frac{k_j^2}{k^2} + D_{NN}(r) \delta_{ij}, \\
F_{ij}(k) &= [F_{LL}(k) - F_{NN}(k)] \frac{k_j k_i}{k^2} + F_{NN}(k) \delta_{ij},
\end{align*}
\]

where \( \delta_{ij} \) is the Kronecker delta (\( \delta_{ij} = 1 \) for \( i = j \), and \( \delta_{ij} = 0 \) for \( i \neq j \)). Solenoidal motions (divergence-free, therefore incompressible) correspond to the transverse components, whereas potential motions (curl-free, therefore compressible) correspond to the longitudinal components.

**A2. TURBULENCE STATISTICS AS OBSERVED (POSITION-POSITION-VELOCITY SPACE)**

Spectroscopic observations do not provide the distribution of gas in real-space coordinates \( (x \equiv [x, y, z]) \). Instead we observe the intensity of emission of a given spectral line at a position \( \mathbf{X} \) in the sky and at a given velocity \( v \) along the LOS. (We use capital letters for two-dimensional vectors and lowercase for three-dimensional vectors). Observational data are usually arranged in matrices with coordinates \( (\mathbf{X}, v) \), also called position-position-velocity (or simply PPV) cubes. We identify the LOS with the coordinate \( z \). Thus, in the plane-parallel approximation the relation between real space and PPV space is that of a map \( (\mathbf{X}, z) \rightarrow (\mathbf{X}, v) \), with \( \mathbf{X} = (x, y) \).

At any point the LOS velocity can be decomposed into a regular flow, a thermal, and a turbulent component \( v_z(x) = v_{z,\text{reg}}(x) + v_{z,\text{thermal}} + v_{z,\text{turb}}(x) \). This way, the distribution of the Doppler-shifted atoms follows a Maxwellian of the form

\[
\phi(x) dv_z = \frac{1}{(2\pi\beta)^{3/2}} \exp\left\{ -\frac{\left[ v_z - v_{z,\text{reg}}(x) - v_{z,\text{turb}}(x) \right]^2}{2\beta} \right\} dv_z,
\]

where \( \beta = \kappa_B T/m \), \( \kappa_B \) is the Maxwell-Boltzmann constant, \( T \) is the temperature, and \( m \) is the atomic mass. In PPV space, the density of emitters \( \rho(\mathbf{X}, v_z) \) can be obtained integrating along the LOS

\[
\rho(\mathbf{X}, v_z) d\mathbf{X} dv_z = \int d\mathbf{z} \rho(\mathbf{x}) \phi(\mathbf{x}) d\mathbf{X} dv_z,
\]

where \( \rho(\mathbf{x}) \) is the mass density of the gas in spatial coordinates. The density of emitters \( \rho(\mathbf{X}, v_z) \) can be identified as the column density per velocity interval, commonly referred as \( dN/dv \). Equation (A8) simply counts the number of atoms at a position in the plane of the sky \( \mathbf{X} \), with a \( z \)-component of velocity in the range \( [v_z, v_z + dv_z] \), and the limits of integration are defined by the LOS extent. The integrated intensity of the emission (integrated along the velocity coordinate) corresponds to the column density under the assumptions of optically thin media and emissivity linearly proportional to the density:

\[
I(\mathbf{X}) \equiv \int \alpha \rho(\mathbf{X}, v_z) dv_z = \int \alpha \rho(\mathbf{x}) dz.
\]

**APPENDIX B**

**PROJECTION OF STRUCTURE FUNCTIONS OF A POWER-LAW SPECTRUM FIELD**

In this section we exemplify the long-wave-dominated (steep) case as if coming from a velocity field, while the shallow case as if from a density field. The results are interchangeable, depending on the specific spectral index they have (although there is no physical motivation to consider a shallow velocity field).

**B1. PROJECTION OF A FIELD WITH A STEEP POWER SPECTRUM**

Consider a homogeneous and isotropic velocity field with a steep power-law spectrum. The LOS (chosen to correspond with the \( z \)-direction) velocity structure function will be of the form

\[
\left\langle \left[ v_z(\mathbf{x}_1) - v_z(\mathbf{x}_2) \right]^2 \right\rangle = C_1 \rho^n,
\]

where \( C_1 \) is a constant.
where \( r = \left( R^2 + (z_2 - z_1)^2 \right)^{1/2} \), \( R \) is the separation in the plane of the sky \( R^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 \), and \( (z_2 - z_1) \) is the separation along the LOS. This way we can rewrite the three-dimensional power-law structure function as

\[
\langle [v_z(x_1) - v_z(x_2)]^2 \rangle = C_1 \left( R^2 + (z_2 - z_1)^2 \right)^{m/2}.
\]  

(B2)

The projection of the velocity field, as described in §2.2, results in

\[
\langle [V_z(x_1) - V_z(x_2)]^2 \rangle = C_1 \int_0^{z_{\text{los}}} \int_{z_{\text{los}} - z_1}^{z_{\text{los}} - z_2} \left\{ \left( R^2 + (z_2 - z_1)^2 \right)^{m/2} - (z_2 - z_1)^m \right\} dz_2 dz_1,
\]  

(B3)

where \( z_{\text{los}} \) is the largest scale along the LOS. Since the field is isotropic, it is possible to evaluate this integral changing variables from \( z_1 \) and \( z_2 \) to \( z_+ = (z_1 + z_2)/2 \) and \( z_- = z_2 - z_1 \). With these new variables, equation (B3) becomes

\[
\langle [V_z(x_1) - V_z(x_2)]^2 \rangle = 2C_1 \int_0^{z_{\text{los}}} \int_{z_2}^{z_2 - z_2/2} \left( R^2 + z_2^2 \right)^{m/2} - z_2^m dz_+ dz_- = 2C_1 \int_0^{z_{\text{los}}} \left( R^2 + z_2^2 \right)^{m/2} - z_2^m (z_{\text{los}} - z_-) dz_-
\]

(B4)

The remaining integral can be solved analytically in terms of hypergeometric functions and converges only for \( m > -1 \) (which is automatically satisfied since the spectrum is steep, therefore with \( m > 0 \)), yielding (a similar formula can be found in Stutzki et al. [1998])

\[
\langle [V_z(x_1) - V_z(x_2)]^2 \rangle = 2C_1 z_{\text{los}}^2 \times \left\{ R_m^2 \right\}
\]

(B5)

where \( _2F_1 \) is the hypergeometric function, with a series expansion (hypergeometric series) of the form

\[
_2F_1(a, b, c; x) = y(x) = 1 + \frac{ab}{c} \frac{x}{1!} + \frac{a(a + 1)b(b + 1)x^2}{c(c + 1)} \frac{x}{2!} + \ldots
\]  

(B6)

However, for all practical purposes, we can calculate this numerically, directly from the definition in equation (B4), yet the zero point (determined by \( C_1 \)) has to be estimated. At small separations \( (R \ll z_{\text{los}}) \), the projected structure functions are well approximated by a power law of the form

\[
\langle [V_z(x_1) - V_z(x_2)]^2 \rangle \approx C_1 R^{m+1} \quad R \ll z_{\text{los}},
\]  

(B7)

where \( C_1' \) is a constant that can be related to \( C_1 \) by matching the zero point of the two-dimensional structure function from the data to a numerical computation using equation (B3). For large separations \( (R \gg z_{\text{los}}) \) equation (B5) will follow a power law of the form

\[
\langle [V_z(x_1) - V_z(x_2)]^2 \rangle \approx C_1'' R^m \quad R \gg z_{\text{los}},
\]  

(B8)

where \( C_1'' \) is another constant that will depend on the zero point of the velocity fluctuations and on \( z_{\text{los}} \) as well. If we have enough inertial range below \( z_{\text{los}} \), the spectral index \( m \) can be obtained from the two-dimensional structure function and verified with the power spectrum. An example of a numerically integrated structure function using equation (B4) and the asymptote in equations (B7) and (B8) is shown in Figure 15 for a structure function index of \( m = 2/3 \). Figure 15a is almost equivalent to Figure 3, where we calculate the structure function of integrated data cubes (Gaussian), but in Figure 15a only theoretical expressions have been used, so agreement of the two results provides us with a good verification.

B2. PROJECTION OF A FIELD WITH A SHALLOW POWER SPECTRUM

Consider a shallow density field (also isotropic) with a three-dimensional structure function of the form (combining eqs. [A4] and [6])

\[
\langle (\rho(x_1) - \rho(x_2))^2 \rangle = 2 \left[ B(0) - C_2 \left( \frac{r}{r_c} \right)^{\eta/2} K_{\eta/2} \left( \frac{r}{r_c} \right) \right].
\]  

(B9)
The structure function of integrated density (column density) for this case can be written as

\[
\frac{I(x_1)}{C_0} \frac{I(x_2)}{C_1}^{1/2} \frac{DE}{C_2} = \frac{C_0^{1/2} R_{m}^{1/2}}{R^2 + z^2} + \frac{C_0}{r_c} \frac{z}{r_c} \frac{z}{r_c} \frac{z}{r_c},
\]

where for a shallow spectrum \( \eta < 0 \). This integral is more difficult to evaluate analytically than the long-wave-dominated case, but for practicality we can solve it numerically. We also find that for small scales, the result can be well approximated by a simple power law:

\[
\frac{I(x_1)}{I(x_2)} \approx C_2 R^{m+1} \quad R \ll z_{\text{tot}},
\]

Similarly, the zero point can be found by matching the numerical result of equation (B11) to the calculated (two-dimensional) structure function from the data. For large separations \( R \gg z_{\text{tot}} \) the resulting structure function of the integrated field will follow the asymptotic scaling of the three-dimensional structure function:

\[
\frac{I(x_1)}{I(x_2)} \approx \text{constant} = C'_2 R^0 \quad R \gg z_{\text{tot}}.
\]

In Figure 15 we also show an example of the numerical calculation of an integrated structure function using (B10) and the asymptotes in equations (B11) and (B12) for an index \( \gamma_{3D} = -5 \) (corresponding to a three-dimensional power spectrum with an index \( \gamma_{3D} = -2.5 \)).

APPENDIX C

POWER SPECTRA OF A FIELD INTEGRATED ALONG THE LINE OF SIGHT

The procedure presented here can also be repeated analogously for a scalar field (e.g., density). Consider the correlation function of velocity, projected along the LOS direction (chosen here to be \( z \)):

\[
\langle V_z(R) V_z(X + R) \rangle = \int dz_1 V_z(x) \int dz_2 V_z(x + r) = \int dz_1 dz_2 V_z(x) V_z(x + r) = \int dz_1 dz_2 B_{z z}(r),
\]

or in terms of the underlying three-dimensional spectrum (inverting eq. [A5])

\[
\langle V_z(R) V_z(X + R) \rangle = \int dz_1 dz_2 \left[ \int d^3k e^{i k \cdot r} F_{z z}(k) \right].
\]
The two-dimensional power spectrum can be obtained taking the Fourier transform of the last expression

$$ P_{2D,r}(K) = \frac{1}{4\pi^2} \int d^2R e^{-iK\cdot R} \left\{ \int d^2z dz_2 \left[ \int d^3k e^{iK\cdot R + ikz_1} F_{zz}(k, k_z) \right] \right\}. $$

(C3)

Using $k = (K, k_z)$ and $r = (R, z_2 - z_1)$,

$$ P_{2D,r}(K) = \frac{1}{4\pi^2} \int d^2R e^{-iK\cdot R} \left\{ \int d^2z dz_2 \left[ \int d^3k e^{iK\cdot R + ik(z_2 - z_1)} F_{zz}(K, k_z) \right] \right\}. $$

(C4)

We can interchange the order of integration:

$$ P_{2D,r}(K) = \frac{1}{4\pi^2} \int d^2z_1 dz_2 \left\{ \int \int d^3k' e^{iK'(z_2 - z_1)} \delta(K' - K) F_{zz}(K', k'_z) \right\}. $$

(C5)

After performing the integral over $R$, one has

$$ P_{2D,r}(K) = \int d^2z_1 dz_2 \left\{ \int d^3k' \left[ e^{iK'(z_2 - z_1)} \delta(K' - K) F_{zz}(K', k'_z) \right] \right\}. $$

(Integrating over $K'$,

$$ P_{2D,r}(K) = \int d^2z_1 dz_2 \left[ \int dk_z e^{ik(z_2 - z_1)} F_{zz}(-K, k'_z) \right]. $$

(C6)

Changing variables to $z_+ = (z_1 + z_2)/2$ and $z_- = z_2 - z_1$, using $F_{zz}(-k) = F_{zz}(k)$, after integration in $z_-$,

$$ P_{2D,r}(K) = 4\pi \int dz_+ \left[ \int dk_z \delta(k_z) F_{zz}(K, k'_z) \right]. $$

(C7)

Now we do the integral over $k'_z$ and then finally over $z_+$:

$$ P_{2D,r}(K) = 4\pi \int dz_+ \left[ F_{zz}(K, 0) \right] = 4\pi z_0 F_{zz}(K, 0). $$

(C8)

Finally, replacing $k_z = 0$, in equation (A6c) $F_{zz}(K, 0) = F_{NN}(K)$, we recover from the integrated structure function the spectrum of the solenoidal component of the velocity:

$$ P_{2D,r}(K) = 4\pi z_0 F_{NN}(K). $$

(C9)

APPENDIX D

SECOND-ORDER STRUCTURE FUNCTION OF UNNORMALIZED VELOCITY CENTROIDS

Consider the structure function of the unnormalized velocity centroids (eq. [21]):

$$ \langle |S(x_1) - S(x_2)|^2 \rangle = \left\langle \alpha \int v_2(x_1)\rho(x_1) \, dz - \alpha \int v_2(x_2)\rho(x_2) \, dz \right\rangle^2. $$

(D1)

As described in § 2.2 (and in Lazarian 1995), we can rewrite the previous equation, as in equations (22) and (23), as

$$ \langle |S(x_1) - S(x_2)|^2 \rangle = \alpha^2 \int \int dz_1 dz_2 \left[ D(r) - |D(r)|_{x_1=x_2} \right], $$

(D2)

where

$$ D(r) = \left\langle (v_2(x_1)\rho(x_1) - v_2(x_2)\rho(x_2))^2 \right\rangle. $$

(D3)
Using the definitions $\nu_2 = \nu_0 + \tilde{\nu}_2$ and $\rho = \rho_0 + \tilde{\rho}$ in (D3), we have

$$
D(r) = \left\langle \left[ v_0 \rho(x_1) - \rho(x_2) \right] + \left[ \nu_0 \tilde{\rho}(x_1) - \tilde{\rho}(x_2) \right] + \left[ \nu_2 \rho(x_1) - \nu_2(x_2) \right] \right\rangle
$$

Expansion of the last expression yields

$$
D(r) = \left\langle v_0^2 \left[ \rho(x_1) - \rho(x_2) \right]^2 + \rho_0^2 \left[ \nu_2(x_1) - \nu_2(x_2) \right]^2 + \left[ \nu_2 \rho(x_1) - \nu_2(x_2) \tilde{\rho}(x_2) \right]^2 \right\rangle
$$

Similarly to the derivation of equation (A6a), it can be shown that for an isotropic (four-dimensional) field,

$$
\text{Evidently, this hypothesis is identically true for Gaussian fields; strongly non-Gaussian fields may show deviations.}
$$

Considering the cross terms one by one, the third term in equation (D5) is

$$
\left\langle \left[ \tilde{\nu}_2(x_1) \tilde{\rho}(x_1) - \tilde{\nu}_2(x_2) \tilde{\rho}(x_2) \right] \right\rangle = \left\langle \left[ \tilde{\nu}_2(x_1) \tilde{\rho}(x_1) \right] \right\rangle - 2 \left\langle \tilde{\nu}_2(x_1) \tilde{\rho}(x_2) \right\rangle + \left\langle \tilde{\nu}_2(x_2) \tilde{\rho}(x_2) \right\rangle.
$$

Relating the fourth-order moments to the second order using the Millionshikov hypothesis (see Monin & Yaglom 1975), $\langle h_1 h_2 h_3 h_4 \rangle \approx \langle h_1 h_2 \rangle \langle h_3 h_4 \rangle + \langle h_1 h_3 \rangle \langle h_2 h_4 \rangle + \langle h_1 h_4 \rangle \langle h_2 h_3 \rangle$, we have

$$
\left\langle \tilde{\nu}_2^2(x_1) \tilde{\rho}^2(x_1) \right\rangle = \left\langle \tilde{\nu}_2(x_1) \tilde{\rho}(x_1) \tilde{\rho}(x_1) \right\rangle \approx \left\langle \tilde{\nu}_2^2(x_1) \right\rangle \left\langle \tilde{\rho}^2(x_1) \right\rangle + 2 \left\langle \tilde{\nu}_2(x_1) \tilde{\rho}(x_1) \right\rangle^2.
$$

If we combine the last three lines using

$$
\left\langle \left[ \tilde{\nu}_2(x_1) - \tilde{\nu}_2(x_2) \right]^2 \right\rangle = \left\langle \tilde{\nu}_2^2(x_1) \right\rangle - 2 \langle \tilde{\nu}_2(x_1) \tilde{\nu}_2(x_2) \rangle + \left\langle \tilde{\nu}_2^2(x_2) \right\rangle
$$

$$
\langle \tilde{\nu}_2(x_1) \tilde{\nu}_2(x_2) \rangle = \left\langle \tilde{\nu}_2^2 \right\rangle - \frac{1}{2} \left\langle \left[ \tilde{\nu}_2(x_1) - \tilde{\nu}_2(x_2) \right]^2 \right\rangle.
$$

To treat terms of the form $\langle \tilde{\nu}_2 \tilde{\rho} \rangle$, we can generalize the correlation function defined in equation (A1), considering a four-dimensional vector of the form $[v_0(x), v_2(x), v_2(x), \rho(x)]$. The resulting cross-correlation between the $z$-component of the velocity and the density is

$$
B_{z\rho}(r) = \langle v_2(x_1) \rho(x_2) \rangle.
$$

Similarly to the derivation of equation (A6a), it can be shown that for an isotropic (four-dimensional) field, $B_{z\rho}(r) = B_{z\rho}(r) r_j / r$ and

$$
B_{\rho z}(r) = B_{\rho z}(r) r_j / r = -B_{z\rho}(r) r_j / r. \text{ Thus, } \langle \tilde{\nu}_2(x_1) \tilde{\rho}(x_1) \rangle = -\langle \tilde{\nu}_2(x_2) \tilde{\rho}(x_1) \rangle, \text{ and equation (D9) simplifies to}
$$

$$
\left\langle \left[ \tilde{\nu}_2(x_1) \tilde{\rho}(x_1) - \tilde{\nu}_2(x_2) \tilde{\rho}(x_2) \right]^2 \right\rangle \approx 2 \langle \tilde{\nu}_2^2(x_1) \tilde{\rho}^2(x_1) \rangle - 2 \langle \tilde{\nu}_2(x_1) \tilde{\rho}(x_2) \rangle + \langle \tilde{\nu}_2^2(x_2) \tilde{\rho}^2(x_2) \rangle
$$

$$
+ \left\langle \tilde{\nu}_2^2 \right\rangle \left\langle \tilde{\rho}^2 \right\rangle - \frac{1}{2} \left\langle \left[ \tilde{\nu}_2(x_1) - \tilde{\nu}_2(x_2) \right]^2 \right\rangle \left\langle \rho(x_1) \right\rangle + \rho_0^2 \left\langle \tilde{\nu}_2(x_1) - \tilde{\nu}_2(x_2) \right\rangle \left\langle \rho(x_1) \right\rangle - \rho_0^2 \left\langle \tilde{\nu}_2(x_1) - \tilde{\nu}_2(x_2) \right\rangle \left\langle \rho(x_2) \right\rangle.
$$

$$
Evidently, this hypothesis is identically true for Gaussian fields; strongly non-Gaussian fields may show deviations.
The next term of equation (D5)

$$2v_0\rho_0\langle[\rho(x_1) - \rho(x_2)] [v_z(x_1) - v_z(x_2)] \rangle = 0,$$

as shown explicitly in Monin & Yaglom (1975). The fifth term in equation (D5) is

$$2\rho_0 \langle [\tilde{v}_z(x_1) - \tilde{v}_z(x_2)] [\tilde{\rho}(x_1) - \tilde{\rho}(x_2)] \rangle = 2\rho_0 \langle \tilde{v}_z(x_1) \tilde{\rho}(x_1) \rangle + 2\rho_0 \langle \tilde{v}_z(x_2) \tilde{\rho}(x_2) \rangle - 2\rho_0 \langle \tilde{v}_z(x_1) \tilde{\rho}(x_2) \tilde{\rho}(x_1) \rangle - 2\rho_0 \langle \tilde{v}_z(x_2) \tilde{\rho}(x_2) \tilde{\rho}(x_1) \rangle \tag{D13}$$

For the terms with third-order correlations, we need to introduce the so-called two-point third-order moments

$$B_{ji}(r) = \langle u_i(x_1) u_j(x_2) \rangle,$$

which can be generalized to include cross-correlations of density and velocity as explained for equation (D10). Analogously to the second-order cross-correlations, considering an isotropic (four-dimensional) field and decomposing it in terms of longitudinal and lateral components, it can be shown (for more details we refer the reader to Monin & Yaglom [1975]) that

$$\langle \tilde{v}_z(x_1) \tilde{\rho}(x_2) \tilde{\rho}(x_1) \rangle = \langle \tilde{\rho}(x_1) \tilde{\rho}(x_2) \tilde{\rho}(x_1) \rangle,$$

$$\langle \tilde{v}_z(x_1) \tilde{\rho}(x_2) \tilde{\rho}(x_1) \rangle = -\langle \tilde{v}_z(x_2) \tilde{\rho}(x_1) \tilde{\rho}(x_1) \rangle \tag{D15a}$$

Equation (D15a), combined with $\langle \tilde{\rho}(x_1) \tilde{v}_z^2(x_1) \rangle = \langle \tilde{\rho}(x_2) \tilde{v}_z^2(x_2) \rangle$, reduces equation (D13) to

$$2\rho_0 \langle [\tilde{v}_z(x_1) - \tilde{v}_z(x_2)] [\tilde{\rho}(x_1) - \tilde{\rho}(x_2)] \tilde{\rho}(x_2) \rangle = 4\rho_0 \langle \tilde{\rho}(x_1) \tilde{v}_z^2(x_1) \rangle - 4\rho_0 \langle \tilde{v}_z(x_1) \tilde{v}_z(x_2) \tilde{\rho}(x_1) \rangle \tag{D16}$$

Similarly the last term in equation (D5) can be written

$$2v_0 \langle [\rho(x_1) - \rho(x_2)] [\tilde{v}_z(x_1) \tilde{\rho}(x_1) - \tilde{v}_z(x_2) \tilde{\rho}(x_2)] \rangle = 2v_0 \langle \tilde{v}_z(x_1) \tilde{\rho}(x_1) \rangle + 2v_0 \langle \tilde{v}_z(x_2) \tilde{\rho}(x_2) \rangle - 2v_0 \langle \tilde{v}_z(x_1) \tilde{\rho}(x_2) \tilde{\rho}(x_1) \rangle - 2v_0 \langle \tilde{v}_z(x_2) \tilde{\rho}(x_2) \tilde{\rho}(x_1) \rangle \tag{D17}$$

But in this case, $\langle \tilde{v}_z(x_1) \tilde{\rho}_z^2(x_1) \rangle = \langle \tilde{v}_z(x_2) \tilde{\rho}_z^2(x_2) \rangle = 0$, combined with equation (D15b), yields

$$2v_0 \langle [\rho(x_1) - \rho(x_2)] [\tilde{v}_z(x_1) \tilde{\rho}(x_1) - \tilde{v}_z(x_2) \tilde{\rho}(x_2)] \rangle = 0 \tag{D18}$$

Finally, combining equations (D5), (D11), (D12), (D16), and (D18),

$$D(r) \approx v_0^2 \left\langle \rho(x_1) - \rho(x_2) \right\rangle + \rho_0^2 \left\langle v_z(x_1) - v_z(x_2) \right\rangle^2 + 2 \langle \tilde{v}_z(x_1) \tilde{\rho}(x_1) \rangle^2 + \langle \tilde{v}_z^2 \rangle \left\langle \rho(x_1) - \rho(x_2) \right\rangle^2 \tag{D19}$$

After grouping some terms,

$$D(r) \approx \rho_0^2 \left\langle v_z(x_1) - v_z(x_2) \right\rangle + \langle \tilde{v}_z^2 \rangle \left\langle \rho(x_1) - \rho(x_2) \right\rangle^2 \tag{D20}$$

Finally, with $\langle \rho^2 \rangle = \langle \tilde{\rho}^2 + \rho_0^2 \rangle$ and $\langle v^2 \rangle = \langle \tilde{v}^2 + v_0^2 \rangle$, we arrive at equations (24) and (26):

$$D(r) \approx \rho_0^2 \left\langle v_z(x_1) - v_z(x_2) \right\rangle + \rho_0^2 \left\langle \rho(x_1) - \rho(x_2) \right\rangle^2 - \frac{1}{2} \left\langle [v_z(x_1) - v_z(x_2)]^2 \right\rangle \left\langle [\rho(x_1) - \rho(x_2)]^2 \right\rangle + c(r), \tag{D21}$$

with

$$c(r) = 2 \langle \tilde{v}_z(x_1) \tilde{\rho}(x_1) \rangle^2 + 4\rho_0 \langle \tilde{\rho}(x_1) \tilde{v}_z^2(x_1) \rangle - 4\rho_0 \langle \tilde{v}_z(x_1) \tilde{v}_z(x_2) \tilde{\rho}(x_1) \rangle \tag{D22}$$

We should note that the second term in equation (D22) has no contribution to the structure function of centroids because $\langle \tilde{\rho}(x_1) \tilde{v}_z(x_1)^2 \rangle = \langle \tilde{\rho}(x_1) \tilde{v}_z(x_1)^2 \rangle_{|z=x_1}$, so it cancels out when substituted into equation (D2). Therefore we omitted this term in the body of the paper. Moreover, this term does not appear in the correlation function of unnormalized centroids; not including it in the definition of $c(r)$ shows better the symmetry between structure functions with correlation functions and spectra.
APPENDIX E

CENTROIDS AND THE CROSS TERM FOR POWER-LAW STATISTICS

We have already discussed the differences between power-law long-wave- and short-wave-dominated fields (steep and shallow spectra, respectively), in particular, the fact that although in all cases power spectra are power laws, but structure functions are only power laws for steep spectra whereas correlation functions are only power laws (at small scales) for shallow spectra. In this section we sketch the expectations for the cross term \( I3[R] \) for the idealized case of infinite power-law spectra, considering all the different shallow and steep combinations.

E1. STEEP DENSITY AND STEEP VELOCITY

In this case the structure function of both fields is well described by a power law. Consider power-law isotropic underlying statistics of the form \( d_s(r) = C_s r^m \) and \( d_v(r) = C_v r^n \), where \( m \) and \( n > 0 \), and \( C_s \) and \( C_v \) are constants. Here the cross term will be

\[
I3(R) = -\frac{1}{2} \alpha^2 C_s C_v \int \int dz_1dz_2 \left( r^{m+n} - |r^{m+n}|_{x_1=x_2} \right),
\]

which is analogous to the projection of a steep field with a spectral index \( \gamma_3 \) and \( \gamma_2 \) is large, the resulting cross term is negative and steeper than both density and velocity. Thus, at sufficiently small scales, its contribution can be safely neglected compared to the integrated density and velocity.

E2. SHALLOW DENSITY AND STEEP VELOCITY

Here the structure function of velocity is a power law \( d_v \) and \( d_s \) is constant. For small separations (relative to a critical scale \( r_c \) as discussed in the main body of the paper), the density fluctuations will have a power-law correlation function \( \langle \rho(x_1) \rho(x_2) \rangle \approx C_s r^m \), where \( C_s \) is constant. Therefore, the structure function of density can be presented as \( d_s(r) = 2 \left( \langle \rho^2 \rangle - |\rho^{m+n}|_{x_1=x_2} \right) \). This combination of indices, the cross term becomes

\[
I3(R) \approx -\langle \rho^2 \rangle \left( [V(x_1) - V(x_2)]^2 \right) + \alpha^2 C_s C_v \int \int dz_1dz_2 \left( r^{m+n} - |r^{m+n}|_{x_1=x_2} \right).
\]

The exact contribution of the terms inside the double integrals to \( I3(R) \) and, in general, the importance of the cross term to the centroid statistics will depend on the constants \( C_s \) and \( C_v \). If at large scales the magnitude of the term containing the structure function of integrated velocity and the cross terms are comparable, then at small scales the centroids could fail in tracing velocity.

E3. STEEP DENSITY AND SHALLOW VELOCITY

This case is somewhat analogous to that described above. Consider for the density a power-law structure function \( d_s(r) \) with \( n > 0 \) and \( C_s \) constant, and for the velocity, a power-law correlation function that translates into a structure function \( d_v(r) \) with \( m < 0 \) and \( C_v \) constant. We have already discussed that a shallow velocity is not physically motivated. However, if we have a steep velocity field without infinite power-law behavior, we could have a structure function that resembles the shallow case. For instance, if the structure function “saturates” at large separations, it will be similar to that of a shallow field, which grows rapidly at small scales and then flattens at large scales. The corresponding cross term is

\[
I3(R) \approx -\langle \tilde{z}_2 \rangle \left( [I(x_1) - I(x_2)]^2 \right) + \alpha^2 C_s C_v \int \int dz_1dz_2 \left( r^{m+n} - |r^{m+n}|_{x_1=x_2} \right).
\]

\(^{10}\) Eqs. (E3), (E5), and (E7) coincide with the decomposition we derived in Ossenkopf et al. (2005).
It has a component that scales as the column density (first term on the right). The remaining component could be positive (if \(m + n > 0\)) or negative (if \(m + n < 0\)). Unnormalized centroids for these indices can be expressed as

\[
\left\langle \left[ S(x_1) - S(x_2) \right]^2 \right\rangle \approx \nu_0^2 \left\langle \left| I(x_1) - I(x_2) \right|^2 \right\rangle + \left( \rho_0^2 + \left\langle \tilde{p}^2 \right\rangle \right) \left\langle \left| V(x_1) - V(x_2) \right|^2 \right\rangle + \alpha^2 C_r \rho \int dz_1 dz_2 \left( r^{m+n} - |r^{m+n}|_{x_1 = x_2} \right) + I4(R). \quad (E5)
\]

**E4. SHALLOW DENSITY AND SHALLOW VELOCITY**

For this combination of indices we can consider power-law correlation functions (for small separations) yielding structure functions of the form \(d_2(r) \approx 2 \left| \langle \tilde{p}^2 \rangle - C_r \rho^n \right|\) and \(d_3(r) \approx 2 \left| \langle \tilde{v}^2 \rangle - C_v r^m \right|\), with \(m + n < 0\), and \(C_r\) and \(C_v\) are constants. As a result, the cross term becomes

\[
I3(R) \approx - \left\langle \nu_0^2 \right\rangle \left\langle \left| I(x_1) - I(x_2) \right|^2 \right\rangle - \left\langle \tilde{p}^2 \right\rangle \left\langle \left| V(x_1) - V(x_2) \right|^2 \right\rangle - 2\alpha^2 C_r \rho \int dz_1 dz_2 \left( r^{m+n} - |r^{m+n}|_{x_1 = x_2} \right). \quad (E6)
\]

Now we have a term that scales as the column density, a term that scales as the integrated velocity, and the term inside the integrals (now negative). Note also that for large values of the velocity or density dispersion, the contribution of column density or integrated velocity, respectively, is increased within \(I3(R)\). But such contributions will be canceled for the unnormalized centroids that will have a structure function

\[
\left\langle \left| S(x_1) - S(x_2) \right|^2 \right\rangle \approx \nu_0^2 \left\langle \left| I(x_1) - I(x_2) \right|^2 \right\rangle + \rho_0^2 \left\langle \left| V(x_1) - V(x_2) \right|^2 \right\rangle - 2\alpha^2 C_r \rho \int dz_1 dz_2 \left( r^{m+n} - |r^{m+n}|_{x_1 = x_2} \right) + I4(R). \quad (E7)
\]

**APPENDIX F**

**CORRELATION FUNCTION AND POWER SPECTRA OF UNNORMALIZED CENTROIDS**

Structure functions are considered to be preferable statistics according to Monin & Yaglom (1975). They are subjected to fewer errors related to averaging. In any case, correlation functions are trivially related to each other by the formula in equation (A4). Using the expression for the structure function of unnormalized centroids from LE03, one gets (see also Levrier [2004] for a direct derivation)

\[
\left\langle S(x_1)S(x_2) \right\rangle \approx \left\langle \rho^2 \right\rangle \left\langle v_z^2 \right\rangle \left( \alpha z_{med} \right)^2 + \alpha^2 \nu_0^2 \int \left\langle \tilde{\rho}(x_1) \tilde{\rho}(x_2) \right\rangle dz_1 dz_2 \\
+ \alpha^2 \rho_0^2 \int \left\langle \tilde{v}_z(x_1) \tilde{v}_z(x_2) \right\rangle dz_1 dz_2 + \alpha^2 \int \left\langle \tilde{\rho}(x_1) \tilde{\rho}(x_2) \right\rangle \left\langle \tilde{v}_z(x_1) \tilde{v}_z(x_2) \right\rangle dz_1 dz_2 \\
- \alpha^2 \int \left\langle \tilde{\rho}(x_1) \tilde{v}_z(x_2) \right\rangle \left\langle \tilde{v}_z(x_1) \tilde{\rho}(x_2) \right\rangle dz_1 dz_2. \quad (F1)
\]

In analogy with equation (27) we can rewrite (F1) as

\[
\left\langle S(x_1)S(x_2) \right\rangle \approx \left\langle \rho^2 \right\rangle \left\langle v_z^2 \right\rangle \left( \alpha z_{med} \right)^2 + B1(R) + B2(R) + B3(R) + B4(R), \quad (F2)
\]

where

\[
B1(R) = \alpha^2 \nu_0^2 \int \left\langle \tilde{\rho}(x_1) \tilde{\rho}(x_2) \right\rangle dz_1 dz_2, \quad (F3a)
\]

\[
B2(R) = \alpha^2 \rho_0^2 \int \left\langle \tilde{v}_z(x_1) \tilde{v}_z(x_2) \right\rangle dz_1 dz_2, \quad (F3b)
\]

\[
B3(R) = \alpha^2 \int \left\langle \tilde{\rho}(x_1) \tilde{\rho}(x_2) \right\rangle \left\langle \tilde{v}_z(x_1) \tilde{v}_z(x_2) \right\rangle dz_1 dz_2, \quad (F3c)
\]

\[
B4(R) = -\alpha^2 \int \left\langle \tilde{\rho}(x_1) \tilde{v}_z(x_2) \right\rangle \left\langle \tilde{v}_z(x_1) \tilde{\rho}(x_2) \right\rangle dz_1 dz_2 + 2\alpha^2 \rho_0 \int \left\langle \left| \tilde{\rho}(x_1) \tilde{v}_z(x_1) \tilde{v}_z(x_2) \right| \right\rangle dz_1 dz_2. \quad (F3d)
\]

Here \(B1(R)\) is \(\nu_0^2\) times the correlation function of fluctuations of column density, \(B2(R)\) is \(\alpha^2 \rho_0^2\) the correlation function integrated velocity, \(B3(R)\) is different from \(2B1(R)\) only by a constant, and \(B4(R)\) contains the density-velocity cross-correlations found in \(I4(R)\).
The power spectrum of unnormalized centroids is the Fourier transform of equation (F1):

\[ P_{2D,s}(K) = \langle \rho^2 \rangle \langle v_z^2 \rangle (\alpha \sigma_{\text{int}})^2 \delta(K) + \nu_0^2 P_{2D,\lambda}(K) + \alpha^2 \rho_0^2 P_{2D,v,r}(K) + \mathcal{F}\{B3(R)\} + \mathcal{F}\{B4(R)\}. \]  

(F4)

It is interesting that the weighting factors are no longer \( \langle \rho_0^2 + \tilde{\rho}^2 \rangle \) and \( \langle \rho_0^2 + \tilde{\rho}^2 \rangle \) but only \( \rho_0^2 \) and \( \rho_0^2 \). In fact the contribution of the column density alone can be eliminated by removing the mean LOS velocity \( \nu_0 \). However, the cross term and density-velocity cross-correlations do affect the scaling properties of the centroids of velocity, and as the turbulence increases in strength and the ratio \( \langle \tilde{\rho}^2 \rangle / \rho_0 \) grows, the “contamination” will also be stronger (see Ossenkopf et al. 2005).

The correlation function of MVCs can be obtained by subtracting \( \langle v_z^2 \rangle \) times the correlation function of column density from the correlation function of normalized centroids, or equivalently

\[ \text{CF}_{\text{MVC}}(R) = \langle S(x_1)S(x_2) \rangle - \langle I(x_1)I(x_2) \rangle \approx \langle \rho^2 \rangle \langle v_z^2 \rangle (\alpha \sigma_{\text{int}})^2 + B2(R) + B3'(R) + B4(R), \]

with

\[ B3'(R) = -\alpha^2 \int \int \langle \tilde{\rho}(x_1)\tilde{\rho}(x_2) \rangle \left[ \langle \tilde{v}_z(x_1)\tilde{v}_z(x_2) \rangle - \langle \tilde{v}_z(x_1)\tilde{v}_z(x_2) \rangle \right] dz_1dz_2 \]

\[ = -\frac{\alpha^2}{2} \int \int \langle \tilde{\rho}(x_1)\tilde{\rho}(x_2) \rangle d_n(r)dz_1dz_2. \]

(F6)

Thus, the power spectrum of MVCs is

\[ P_{2D,MVC}(K) = \langle \rho^2 \rangle \langle v_z^2 \rangle (\alpha \sigma_{\text{int}})^2 \delta(K) + \alpha^2 \rho_0^2 P_{2D,v,r}(K) + \mathcal{F}\{B3'(R)\} + \mathcal{F}\{B4(R)\}. \]

(F7)

Since the three measures (spectra, correlation, and structure functions) are trivially related, they represent the same statistics. They can be used interchangeably as dictated by practical convenience.

APPENDIX G

UNNORMALIZED CENTROIDS VERSUS MVC

When are MVCs better than regular unnormalized centroids? To answer this question, let us consider MVCs in terms of correlation functions. First, we can minimize the contribution of density fluctuations to the unnormalized centroids by setting \( \nu_0 = 0 \). In this case we can fold the difference of unnormalized and modified centroids into the cross terms \( B3(R) \) and \( B3'(R) \), respectively. Thus, the criterion for MVCs to be an improvement over unnormalized centroids would be \( |B3(R)| > |B3'(R)| \). In other words,

\[ \frac{1}{2} \left| \int \int \langle \tilde{\rho}(x_1)\tilde{\rho}(x_2) \rangle \langle \tilde{v}_z(x_1)\tilde{v}_z(x_2) \rangle d_0(r)dz_1dz_2 \right| > 1. \]

(G1)

Let us consider a more realistic steep velocity field, with a power-law structure function that saturates at some cutoff scale \( r_{\lambda,0} \) (see Lazarian & Pogosyan 2000, 2004):

\[ d_n(r) = 2\langle \tilde{v}_z^2 \rangle \frac{r^m}{r^m + r_{\lambda,0}^m}. \]

(G2)

We use \( r_{\lambda,0} \) to avoid confusion with the critical scale for the short-wave-dominated spectra and because this cutoff scale can also be related to a correlation length for the correlation function:

\[ \langle \tilde{v}_z(x_1)\tilde{v}_z(x_2) \rangle = \langle \tilde{v}_z^2 \rangle \frac{r_{\lambda,0}^m}{r^m + r_{\lambda,0}^m}. \]

(G3)

If we combine this steep velocity field with a steep density field, i.e., \( \langle \tilde{\rho}(x_1)\tilde{\rho}(x_1) \rangle = \langle \tilde{\rho}^2 \rangle r_{\rho,0}^n / (r^m + r_{\rho,0}^n) \), with the same change of variables as before \( (z^- = z_2 - z_1) \), the criterion in equation (G1) becomes

\[ \frac{\int \int \langle \tilde{v}_z^2 \rangle \left[ \left( R^2 + z_2^2 \right)^{m/2} + r_{\rho,0}^n \right] \left[ \left( R^2 + z_2^2 \right)^{m/2} + r_{\lambda,0}^m \right] dz^-}{\int \int \langle \tilde{v}_z^2 \rangle \left[ \left( R^2 + z_2^2 \right)^{m/2} + r_{\rho,0}^n \right] \left[ \left( R^2 + z_2^2 \right)^{m/2} + r_{\lambda,0}^m \right] dz^-} > 1. \]

(G4)

In Figure 16 we plotted isocontours of this ratio for various combinations of parameters. Within the parameter space corresponding to isocontours above 1.0, MVCs are expected to trace the statistics of velocity better than unnormalized centroids. In Figures 16a–16c, we present, for \( r_{\rho,0} = r_{\lambda,0} \) how the ratio in equation (G4) varies with the lag \( R \) over a extreme contrast of possible spectral indices.
In Figures 16–16f we choose a particular set of spectral indices and relax the condition $r/C_26;0 = v/0$. In all cases $z_{tot} = 100$. We find that the criterion for MVCs to be better than unnormalized centroids is satisfied for lags smaller than the cutoff for the velocity field, with very little sensitivity to the particular spectral index or the cutoff scale of density.

We performed the same exercise for the other physically motivated case, that is, steep velocity and shallow density. A shallow density can be approximated by the same expression for the correlation function of density, but with a small $r/C_0$. We tested for spectral indices from 0.1 to 1.0 for the velocity and from $r/C_0$ to 0.1 for the density. The same behavior as in the steep-steep case was found. That is, MVCs are expected to trace the statistics of velocity better than unnormalized centroids for lags smaller than the velocity correlation scale ($r_v;0$) and smaller than the LOS extent of the object ($z_{tot}$).

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![Fig. 16.](image_url)
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