Implicitization of the Vegter Yield Criterion

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Abstract. An advanced yield criterion plays a central role in rate-independent plasticity modeling of sheet metals in automotive and other industrial applications. Different from many non-quadratic yield criteria based on implicit polynomials, Vegter yield criterion is piecewise and quadratic as it is originally constructed by interpolation of several biaxial stress states using multiple second-order Bézier curves. Additional efforts in recent years have made Vetger yield criterion more user-friendly, flexible and robust for sheet metal forming applications. Nevertheless, Vetger yield criterion has only been presented in its parametric formulation in the literature. This work describes a method of reformulating the existing Vetger Lite yield criterion into the implicit form that is more commonly used in the conventional sheet metal plasticity modeling. The potential benefits of using both the original parametric and the new implicit forms of Vetger yield criterion for material parameter calibration, convexity certification, and computational simulations are discussed.

1. Introduction

A yield stress function $f(\sigma)$ plays an essential role in mathematical theory of anisotropic plasticity for sheet metal modeling since Hill’s seminal work [1, 2]. The yield stress function is used to establish the plastic yielding condition of a sheet metal subjected to a given Cauchy stress $\sigma$ and obtain the corresponding plastic strain increments via an associated flow rule. For example, a flat thin sheet may be modeled by Hill’s 1948 associated quadratic anisotropic plasticity [1] with its yield stress function given in plane stress $\sigma = (\sigma_x, \sigma_y, \tau_{xy})$ as

$$\Phi_h(\sigma_x, \sigma_y, \tau_{xy}) = \sigma^2 = f^2(\sigma) = A_1\sigma_x^2 + A_2\sigma_x\sigma_y + A_3\sigma_y^2 + A_4\tau_{xy}^2,$$

where polynomial coefficients $(A_1, A_2, A_3, A_4)$ are material constants calibrated from yield stress and/or plastic strain ratio data. The yield condition and plastic strain increments from the associated flow rule are given respectively as

$$f(\sigma) = \sigma_f(\dot{\varepsilon}^p), \quad \dot{\varepsilon}^p = \frac{\dot{\varepsilon}^p \partial f(\sigma)}{\partial \sigma} = \frac{\dot{\varepsilon}^p \partial \Phi_h(\sigma)}{2\sigma},$$

where $\sigma_f(\dot{\varepsilon}^p)$ is the equivalent flow strength of the sheet metal, $\dot{\varepsilon}^p$ is the plastic-work equivalent strain increment and $\dot{\varepsilon}^p = \int \ddot{\varepsilon}^p dt$.

Graphically, the yield condition represents a closed surface and the plastic strain increments form an outer vector normal to the convex yield surface (the flow rule is thus also called normality...
Figure 1. (a) The 3D yield surface given by a yield condition $\sigma_x^2 - 0.8636\sigma_x\sigma_y + 1.1397\sigma_y^2 + 1.9396\tau_{xy}^2 = \sigma_f^2$; (b) the 2D yield surface (a plane curve) given by a yield condition $\sigma_x^2 - 0.8636\sigma_x\sigma_y + 1.1397\sigma_y^2 = \sigma_f^2$.

rule). For example, the yield surface in tri-component plane stress ($\sigma_x, \sigma_y, \tau_{xy}$) and in biaxial stress ($\sigma_x, \sigma_y$) are shown in Figures 1(a) and 1(b) respectively for an aluminum sheet metal modeled by the yield condition $F(\sigma) = \Phi_h(\sigma) - \sigma_f^2 = 0$. In analytical geometry, the yield condition as presented is regarded as an implicit representation of a curve or surface [3]. A parametric curve or surface is better suited to generate points and thus plots in computer graphics. For example, the 2D yield surface in Figure 1(b) is created in fact by connecting many discrete points $(\sigma_x(\alpha), \sigma_y(\alpha))$ which are computed from the following parametric equations using polar coordinates for $\alpha \in [0, 2\pi]$

$$\sigma_x(\alpha) = r(\alpha) \cos \alpha, \quad \sigma_y(\alpha) = r(\alpha) \sin \alpha,$$

where

$$r(\alpha) = \frac{\sigma_f}{\sqrt{\cos^2 \alpha - 0.8636 \sin \alpha \cos \alpha + 1.1397 \sin^2 \alpha}}.$$

It is noted that the 2D yield surface (a plane curve or more precisely the boundary contour of a convex plane figure) in Figure 1(b) may also be written in the explicit form as $\sigma_y(\sigma_x) = 0.378872\sigma_x \pm \sqrt{0.877425\sigma_x^2 - 0.733881\sigma_x^2}$.

Although the subsequent development in advanced or non-quadratic yield criteria for sheet metal plasticity modeling has been primarily in the implicit formulation similar to Hill’s 1948 quadratic yield stress function [4, 5, 6, 7], a parametric formulation of advanced yield criteria has also been proposed in the literature [8, 9, 10]. In particular, the Vegter yield criterion has a more direct connection to computer graphics and uses multiple second-order Bézier curves to construct the entire closed 2D yield surface. Its methodology has also been called interpolation of biaxial stress states to emphasize its piecewise nature and its flexibility [11, 12, 13]. In terms of the commonly used terminology in analytical geometry and computer graphics, it belongs to the parametric instead of implicit formulation or representation of yield criteria. While there have been works in the field of computer graphics and computer-aided geometric design that address the conversion or transformation of parametric curves or surfaces into the implicit form [14, 15, 16], it appears that implicitization of the Vegter yield criterion has not been carried
out at all for sheet metal plasticity modeling. It is the aim of the current study to remedy this situation.

2. Implicitization of Quadratic Bézier Curves by Sylvester’s Matrix Method
As a preparation for describing implicitization of the Vegter yield criterion, a single second-order or quadratic parametric Bézier curve is first introduced in terms of its three control points. Then its implicitization by Sylvester’s matrix elimination method is presented which has its origin from the classical work in analytical geometry [17, 18].

2.1. Quadratic Bézier Plane Curves
Parametric curves are used predominately in computer graphics and Bézier curves are often adapted in computer-aided auto body design and many other applications [19]. A quadratic Bézier curve in a plane \((x, y)\) is specified by three control points \((x_1^r, y_1^r), (x_h, y_h)\) and \((x_2^r, y_2^r)\) (where the first and third points are called reference (end) points and the second or middle point is called a hinge point by Vegter and co-workers [10, 11, 12, 13]). It may be treated as a linear interpolation of two linear Bézier curves

\[
x(\lambda) = (1 - \lambda)[(1 - \lambda)x_1^r + \lambda x_h] + \lambda[(1 - \lambda)x_h + \lambda x_2^r] = (1 - \lambda)^2 x_1^r + 2(1 - \lambda)\lambda x_h + \lambda^2 x_2^r, \]

\[
y(\lambda) = (1 - \lambda)[(1 - \lambda)y_1^r + \lambda y_h] + \lambda[(1 - \lambda)y_h + \lambda y_2^r] = (1 - \lambda)^2 y_1^r + 2(1 - \lambda)\lambda y_h + \lambda^2 y_2^r,
\]

where the parameter \(0 \leq \lambda \leq 1\).

As illustrated in Figure 2(a), the two reference (first and last control) points are two endpoints of the quadratic Bézier curve shown in solid line while the hinge (the middle control) point does not lie on the curve at all. In fact, the hinge point is the intersection of two tangent lines (shown as dashed lines) passing through the two reference points, namely

\[
y^h = y_1^r + k_1(x_h - x_1^r), \quad y^h = y_2^r + k_2(x_h - x_2^r),
\]

where \((k_1, k_2)\) are tangents of the plane curve at these two reference points.

Although two quadratic Bézier curves with four control points can form a closed plane curve, at least three quadratic Bézier curves with six control points are needed to form a smooth closed plane curve. Using four quadratic Bézier curves with eight control points, Figure 2(b) gives an example similar to a planarly isotropic yield surface shown as Figure 1(a) in [8]. The continuity and smoothness are ensured as two adjacent curves share a common reference point with the same tangent. One can also observe in this example that the closed plane curve is enclosed by a convex polygon (in fact an isosceles trapezoid) in dashed lines formed by the hinge points.

2.2. Implicitization by the Matrix Elimination Method
The parametric Bézier curve given in Eq.(5) consists of two quadratic polynomials in the parameter \(\lambda\) and can be implicitized by Sylvester’s matrix elimination method [14, 15]. The two equations may be rewritten as

\[
\alpha_2\lambda^2 + \alpha_1\lambda + (\alpha_0 - x) = 0, \quad \beta_2\lambda^2 + \beta_1\lambda + (\beta_0 - y) = 0
\]

\[
\alpha_0 = x_1^r, \quad \alpha_1 = 2(x_h - x_1^r), \quad \alpha_2 = (x_1^r + x_2^r - 2x_h),
\]

\[
\beta_0 = y_1^r, \quad \beta_1 = 2(y_h - y_1^r), \quad \beta_2 = (y_1^r + y_2^r - 2y_h).
\]
Figure 2. (a) A single quadratic Bézier curve (solid line) defined by two reference (end) points and one hinge point; (b) a closed plane curve consisting of four quadratic Bézier curves (solid lines). Filled and open circles are respectively reference and hinge points.

The implicit expression of two polynomials of Eq.(7) is given by the determinant of Sylvester’s matrix

\[
\begin{vmatrix}
\alpha_2 & \alpha_1 & \alpha_0 - x & 0 \\
0 & \alpha_2 & \alpha_1 & \alpha_0 - x \\
\beta_2 & \beta_1 & \beta_0 - y & 0 \\
0 & \beta_2 & \beta_1 & \beta_0 - y \\
\end{vmatrix} = 0,
\]

that is,

\[
A_1x^2 + A_2xy + A_3y^2 + B_1x + B_2y + C = 0,
\]

where

\[
A_1 = \beta_2^2, \quad A_2 = -2\alpha_2\beta_2, \quad A_3 = \alpha_2^2, \quad B_1 = 2\alpha_2\beta_0\beta_2 + \alpha_1\beta_1\beta_2 - \alpha_2\beta_1^2 - 2\alpha_0\beta_2^2,
\]

\[
B_2 = \alpha_2\beta_1\alpha_1 + 2\alpha_2\beta_2\alpha_0 - 2\alpha_2^2\beta_0 - \alpha_1^2\beta_2,
\]

\[
C = (\alpha_2\beta_0 - \alpha_0\beta_2)^2 + (\alpha_1\beta_2 - \alpha_2\beta_1)(\alpha_1\beta_0 - \alpha_0\beta_1).
\]

The implicit form of a parametric quadratic Bézier curve is simply an equation of a conic curve in terms of a non-homogeneous quadratic polynomial with six coefficients \((A_1, A_2, A_3, B_1, B_2, C)\). It depicts actually a parabola as \(4A_1A_3 - A_2^2 = 0\).

In particular, the quadratic Bézier curve shown in Figure 2(a) are defined by the three control points \((x_1^r,y_1^r)=(0.6,1.0),(x^h,y^h)=(0.1,0.2)\) and \((x_2^r,y_2^r)=(1.0,0.1)\). Its parametric and implicit representations are given respectively as

\[
x(\lambda) = 1.4\lambda^2 - \lambda + 0.6, \quad y(\lambda) = 0.7\lambda^2 - 1.6\lambda + 1, \quad 0 \leq \lambda \leq 1,
\]

\[
0.49x^2 - 1.96xy + 1.96y^2 - 1.092x - 1.204y + 0.8988 = 0.
\]

The four quadratic Bézier curves that form the closed plane curve shown in Figure 2(b) are specified clockwise by the following eight control points \((x_1,y_1)=(-1.75,0), \quad (x_2,y_2)=(-1.75,0.619), \quad (x_3,y_3)=(0.1,2.281), \quad (x_4,y_4)=(1.419,1.722), \quad (x_5,y_5)=(1.419,0), \quad (x_6,y_6)=(1.419,-1.79), \quad (x_7,y_7)=(0,-1.22177)\) and \((x_8,y_8)=(-1.75,-0.521)\). Their parametric and implicit
representations are \((0 \leq \lambda \leq 1)\)

\[
\begin{align*}
  x(\lambda) &= 1.75\lambda^2 - 1.75, \quad y(\lambda) = -0.00989618\lambda^2 + 1.238\lambda, \\
  x(\lambda) &= -1.419\lambda^2 + 2.838\lambda, \quad y(\lambda) = -2.2159\lambda^2 + 0.987792\lambda + 1.2281, \\
  x(\lambda) &= -1.419\lambda^2 + 1.419, \quad y(\lambda) = 2.35823\lambda^2 - 3.58\lambda, \\
  x(\lambda) &= 1.75\lambda^2 - 3.5\lambda, \quad y(\lambda) = -0.179773\lambda^2 + 1.40155\lambda - 1.22177,
\end{align*}
\]

(13)

and

\[
\begin{align*}
  0.0000979344x^2 + 0.0346366xy + 3.0625y^2 - 2.68178x + 0.0606141y - 4.69342 &= 0, \\
  4.9102x^2 - 6.28871xy + 2.01356y^2 + 2.89582x + 8.92368y - 13.9961 &= 0, \\
  5.56123x^2 + 6.69265xy + 2.01356y^2 + 2.40369x - 9.49687y - 14.6087 &= 0, \\
  0.0323184x^2 + 0.629206xy + 3.0625y^2 - 1.78697x + 1.10111y - 3.22618 &= 0.
\end{align*}
\]

(14)

It is noted that the numerical values of coefficients for both parametric and implicit functions are given only up to six significant digits here. For example, \(\beta_2\) for the first quadratic Bézier curve of Eq.(13) is \(-0.009896181760807865\) and the corresponding \(A_1 = \beta_2^2\) of Eq.(14) is \(0.00009793441344294626\) in terms of 16 significant digits. Such approximation or slight inconsistency in precision is thus unavoidable due to numerical round-offs.

3. The Vegter Yield Criterion and Its Implicitization

We are now ready to present results of the implicit representation of the parametric Vegter yield criterion for applications used in sheet metal plasticity modeling. We consider here a variant of the original Vegter Lite yield criterion in biaxial loading [12]. The so-called Vegter Lite model uses a smaller number of reference points and thus reduces the total number of material constants from seventeen to eight for modeling a sheet metal under plane stress [11, 12].

![Figure 3. The Vegter yield loci in biaxial loading \((\tilde{\sigma} = \sigma/\sigma_f)\): (a) two quadratic Bézier curves for biaxial tension; (b) a set of six quadratic Bézier curves for the entire closed yield surface with tension-compression symmetry.](image-url)
3.1. The Vegter Lite yield criterion

As shown in Figure 3(a), only three reference points plus two hinge points are needed for the Vegter Lite model in biaxial tension. Assuming the weight \( w = 1 \) for the Vegter Lite yield criterion for simplicity, its second-order NURB interpolation is reverted back to the second-order Bézier interpolation. The two quadratic parametric Bézier curves for the yield criterion are counterclockwise from uniaxial tension \((\sigma_0, 0)\) along the rolling direction to equal biaxial tension \((\sigma_b, \sigma_b)\) and then to uniaxial tension \((0, \sigma_90)\) along the transverse direction

\[
\begin{align*}
\sigma_x &= (1 - \lambda)^2 \sigma_0 + 2(1 - \lambda) \lambda \sigma_{x1}^h + \lambda^2 \sigma_b, \quad \sigma_y = 2(1 - \lambda) \lambda \sigma_{y1}^h + \lambda^2 \sigma_b, \quad \sigma_x \geq \sigma_y \geq 0, \\
\sigma_x &= (1 - \lambda)^2 \sigma_b + 2(1 - \lambda) \lambda \sigma_{x2}^h, \quad \sigma_y = (1 - \lambda)^2 \sigma_b + 2(1 - \lambda) \lambda \sigma_{y2}^h + \lambda^2 \sigma_90, \quad 0 \leq \sigma_x \leq \sigma_y,
\end{align*}
\]

where \( 0 \leq \lambda \leq 1 \) and the biaxial stresses at the two hinge points are obtained from the yield stresses and plastic strain ratios per Eq.(6)

\[
\begin{align*}
\sigma_{x1}^h &= \frac{k_0 \sigma_0 + (1 - k_b) \sigma_b}{k_0 - k_b}, \quad \sigma_{y1}^h = \frac{k_0 k_b (\sigma_0 - \sigma_b) + k_0 \sigma_b}{k_0 - k_b}, \\
\sigma_{x2}^h &= \frac{(1 - k_b) \sigma_b - \sigma_90}{k_90 - k_b}, \quad \sigma_{y2}^h = \frac{k_0 k_b (\sigma_b - 1 - k_b) - k_0 \sigma_90}{k_90 - k_b}.
\end{align*}
\]

It is noted that the slope of a tangent line to the yield surface, the vector normal to the yield surface and the plastic strain ratios are related as follows

\[
(1, k) \cdot (\dot{\varepsilon}_p^x, \dot{\varepsilon}_p^y) = 0 \quad \text{or} \quad k = -\frac{\dot{\varepsilon}_p^x}{\dot{\varepsilon}_p^y} : \quad k_0 = 1 + \frac{1}{R_0}, \quad k_90 = \frac{R_90}{1 + R_90}, \quad k_b = -\frac{1}{R_b}.
\]

For the entire closed yield surface with tension-compression symmetry as shown in Figure 3(b), there is one additional unique quadratic parametric Bézier curve from uniaxial tension \((0, \sigma_90)\) along the transverse direction to uniaxial compression \((-\sigma_0, 0)\) along the rolling direction as

\[
\begin{align*}
\sigma_x &= 2(1 - \lambda) \lambda \sigma_{x3}^h - \lambda^2 \sigma_0, \quad \sigma_y = (1 - \lambda)^2 \sigma_90 + 2(1 - \lambda) \lambda \sigma_{y3}^h, \quad \sigma_x \leq 0, \quad \sigma_y \geq 0,
\end{align*}
\]

where

\[
\begin{align*}
\sigma_{x3}^h &= \frac{-k_0 \sigma_0 + \sigma_90}{k_0 - k_90}, \quad \sigma_{y3}^h = \frac{-k_0 k_b \sigma_0 + k_0 \sigma_90}{k_0 - k_90}.
\end{align*}
\]

Due to the tension-compression symmetry of the entire closed yield surface shown in Figure 3(b), the reference and hinge points with \( \sigma_y < 0 \) will just have negative values of coordinates of the reference and hinge points with \( \sigma_y > 0 \). So the other three quadratic parametric Bézier curves for the yield surface are readily obtained but not presented here for brevity.

3.2. Implicit Form of the Vegter Lite yield criterion for selected sheet metals

To illustrate its implicit representation, the simplified Vegter Lite yield criterion is applied to model an IF-steel and an Al-Mg-Si alloy as studied in [11]. The six experimental inputs in terms of three yield stresses \((\sigma_0, \sigma_90, \sigma_b)\) normalized by the equivalent yield stress \(\sigma_f\) and corresponding three plastic strain ratios \((R_0, R_{90}, R_b)\) for the two sheet metals are listed in Table 1. The corresponding biaxial stresses normalized by \(\sigma_f\) at the three hinge points are computed from Eqs.(16)-(19) and the results are given in Table 2.
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Table 1. Yield stresses and plastic strain ratios for two sheet metals [11]

|           | $\sigma_0/\sigma_f$ | $\sigma_{90}/\sigma_f$ | $\sigma_6/\sigma_f$ | $R_0$ | $R_{90}$ | $R_6$ |
|-----------|---------------------|------------------------|---------------------|-------|----------|-------|
| IF steel  | 1.004               | 0.997                  | 1.157               | 1.85  | 2.51     | 0.777 |
| Al alloy  | 1.021               | 1.009                  | 1.004               | 0.64  | 0.76     | 0.889 |

Table 2. Normalized stress values of three hinge points for two sheet metals

|           | $\sigma_0/\sigma_f$ | $\sigma_{90}/\sigma_f$ | $\sigma_6/\sigma_f$ | $\sigma_x$ | $\sigma_y$ | $\sigma_z$ |
|-----------|---------------------|------------------------|---------------------|-------------|-------------|-------------|
| IF steel  | 1.48283             | 0.73765                | 0.82366             | 1.58600     | -0.665950   | 0.520779    |
| Al alloy  | 1.28810             | 0.684432               | 0.722281            | 1.32089     | -0.754365   | 0.683251    |

The yield loci of the simplified Vegter Lite yield criterion are shown in Figures 4(a) and 4(b) in solid lines for IF steel and Al-Mg-Si aluminum alloy sheets respectively. Due to the center symmetry (see Figure 3(b)), only half of the entire closed yield surface with $\sigma_y \geq 0$ is shown and is defined by three quadratic parametric Bézier curves. The dashed lines form a convex polygon (again only half of the polygon is shown) that encloses the convex yield surface for each sheet metal. For IF steel, its three quadratic parametric Bézier curves are given in terms of the normalized stresses as:

\begin{align*}
\bar{\sigma}_x &= 1.004 + 0.95766\lambda - 0.80466\lambda^2, \quad \bar{\sigma}_y = 1.47531\lambda - 0.31831\lambda^2, \quad \bar{\sigma}_x \geq \bar{\sigma}_y \geq 0, \\
\bar{\sigma}_x &= 1.157 - 0.66667\lambda - 0.49033\lambda^2, \quad \bar{\sigma}_y = 1.157 + 0.85801\lambda - 1.01801\lambda^2, \quad 0 \leq \bar{\sigma}_x \leq \bar{\sigma}_y, \\
\bar{\sigma}_x &= -1.3319\lambda + 0.3279\lambda^2, \quad \bar{\sigma}_y = 0.997 - 0.95244\lambda - 0.04456\lambda^2, \quad \bar{\sigma}_x \leq 0, \quad \bar{\sigma}_y \geq 0.
\end{align*}

For the Al-Mg-Si alloy, its three quadratic parametric Bézier curves are given as

\begin{align*}
\bar{\sigma}_x &= 1.021 + 0.53419\lambda - 0.55119\lambda^2, \quad \bar{\sigma}_y = 1.36886\lambda - 0.36487\lambda^2, \quad \bar{\sigma}_x \geq \bar{\sigma}_y \geq 0, \\
\bar{\sigma}_x &= 1.004 - 0.56344\lambda - 0.44056\lambda^2, \quad \bar{\sigma}_y = 1.004 + 0.63379\lambda - 0.62879\lambda^2, \quad 0 \leq \bar{\sigma}_x \leq \bar{\sigma}_y, \\
\bar{\sigma}_x &= -1.50873\lambda + 0.48773\lambda^2, \quad \bar{\sigma}_y = 1.009 - 0.6515\lambda - 0.3575\lambda^2, \quad \bar{\sigma}_x \leq 0, \quad \bar{\sigma}_y \geq 0.
\end{align*}

Using Sylvester’s matrix elimination method detailed in Section 2.2, one can readily obtain the implicit form of the above Vegter yield loci in terms of three piecewise non-homogeneous quadratic polynomials. For IF steel, its three implicit yield functions are obtained as

\begin{align*}
F_{va}(\bar{\sigma}) &= -1.20472 + 1.0982\bar{\sigma}_x + 0.10132\bar{\sigma}_y^2 - 0.33062\bar{\sigma}_y - 0.51227\bar{\sigma}_x\bar{\sigma}_y + 0.64748\bar{\sigma}_y^2 = 0, \\
F_{vb}(\bar{\sigma}) &= -1.56662 - 0.29975\bar{\sigma}_x + 1.03633\bar{\sigma}_y + 1.33164\bar{\sigma}_y - 0.99832\bar{\sigma}_x\bar{\sigma}_y + 0.24042\bar{\sigma}_y^2 = 0, \\
F_{vc}(\bar{\sigma}) &= -0.38665 - 0.38311\bar{\sigma}_x + 0.00199\bar{\sigma}_y^2 + 0.28061\bar{\sigma}_y + 0.02922\bar{\sigma}_x\bar{\sigma}_y + 0.10752\bar{\sigma}_y^2 = 0.
\end{align*}

For the Al-Mg-Si alloy, its three implicit yield functions are obtained as

\begin{align*}
F_{va}(\bar{\sigma}) &= -0.64333 + 0.49417\bar{\sigma}_x + 0.13313\bar{\sigma}_y^2 + 0.11174\bar{\sigma}_y - 0.40222\bar{\sigma}_x\bar{\sigma}_y + 0.30381\bar{\sigma}_y^2 = 0, \\
F_{vb}(\bar{\sigma}) &= -0.72577 + 0.16385\bar{\sigma}_x + 0.39538\bar{\sigma}_y^2 + 0.52346\bar{\sigma}_y - 0.55404\bar{\sigma}_x\bar{\sigma}_y + 0.1941\bar{\sigma}_y^2 = 0, \\
F_{vc}(\bar{\sigma}) &= -1.06264 - 0.91029\bar{\sigma}_x + 0.12781\bar{\sigma}_y^2 + 0.81314\bar{\sigma}_y - 0.34873\bar{\sigma}_x\bar{\sigma}_y + 0.23788\bar{\sigma}_y^2 = 0.
\end{align*}
Figure 4. The Vegter yield loci in biaxial loading with $\sigma_y \geq 0$ defined by three quadratic parametric Bézier curves: (a) IF steel sheet (Eq.20); (b) Al-Mg-Si alloy sheet (Eq.21).

Similar to each parametric curve that is limited to $0 \leq \lambda \leq 1$, those implicit yield functions are applicable only within certain biaxial stress states, namely

$$
\begin{align*}
\sigma_x &\geq \sigma_y \geq 0 \quad \text{for} \quad F_{va}(\sigma) = \sigma_f^2 F_{va}(\tilde{\sigma}) = \Phi_{2a}(\sigma) - \sigma_f^2 = f_{2a}(\sigma) - \sigma_f^2 = 0, \\
\sigma_y &\geq \sigma_x \geq 0 \quad \text{for} \quad F_{vb}(\sigma) = \sigma_f^2 F_{vb}(\tilde{\sigma}) = \Phi_{2b}(\sigma) - \sigma_f^2 = f_{2b}(\sigma) - \sigma_f^2 = 0, \quad \text{(24)} \\
\sigma_y &\geq 0 \quad \text{and} \quad \sigma_x \leq 0 \quad \text{for} \quad F_{vc}(\sigma) = \sigma_f^2 F_{vc}(\tilde{\sigma}) = \Phi_{2c}(\sigma) - \sigma_f^2 = f_{2c}(\sigma) - \sigma_f^2 = 0.
\end{align*}
$$

The yield loci specified by those implicit yield functions $F_{va}(\sigma) = 0$, $F_{vb}(\sigma) = 0$ and $F_{vc}(\sigma) = 0$ are shown as dashed, dotted and dotted-and-dashed lines in Figures 5(a) and 5(b) for these two sheet metals respectively. As expected, within their applicability ranges, the yield loci by the implicit form of the Vegter yield criterion match completely those by its parametric form (solid lines). Due to the tension-compression central summery of the yield surface under consideration, three implicit yield conditions $F_{va}(-\sigma) = 0$, $F_{vb}(-\sigma) = 0$ and $F_{vc}(-\sigma) = 0$ completes the description of the remaining half of the entirely closed 2D yield surface for each sheet metal.

Figure 5. The Vegter yield loci in biaxial loading with $\sigma_y \geq 0$ defined by three implicit quadratic yield stress functions: (a) IF steel sheet (Eq.22); (b) Al-Mg-Si alloy sheet (Eq.23).
4. Discussion and Conclusions

The Vegter yield criterion is flexible and can be made to accept various numbers of experimental inputs for its calibration. The Lite version of the Vegter yield criterion uses only eight inputs \((\sigma_0, \sigma_{45}, \sigma_{90}, \sigma_b, R_0, R_{45}, R_{90}, R_b)\), identical to the requirements of YLD2000-2D [5, 6]. Just like YLD2000-2D, the Vegter Lite yield criterion needs only six out of those eight experimental inputs for modeling sheet metals with tension-compression symmetry under biaxial loading. In this study, it is shown that the parametric form of the biaxial Vegter Lite yield criterion with six quadratic Bézier curves can be converted into the more conventional implicit form with six piecewise non-homogeneous quadratic polynomials (only three of them are given due to tension-compression symmetry). As each quadratic polynomial stress function has six coefficients, the total number of polynomial coefficients for the entire closed 2D yield surface is 36 (identical to six times of the six total number of stress values from three control points per a quadratic Bézier curve shown in Figure 3(b)). Nevertheless, due to continuity, smoothness and symmetry conditions imposed on the yield surface, there are only 6 out of 36 coefficients are independent. That is, they can all be determined from six inputs \((\sigma_0, \sigma_{90}, \sigma_b, R_0, R_{90}, R_b)\) as given in this study.

Although the parametric form of Vegter yield criterion has been used over the years in various sheet metal forming analyses and numerical simulations without much problems at all [10, 11, 12, 13], there are certain advantages when its implicit form is also known. For example, the determination about the onset or switch-off of plastic yielding is crucial in almost all iterative algorithms for analyzing elastic-plastic deformation of a sheet metal. For a given (trial) stress state \(\sigma^* = (\sigma_x^*, \sigma_y^*)\) in a simulation, it is much easier to use the yield stress function \(f(\sigma) = \sqrt{\Phi_2(\sigma)}\) in the implicit form of the yield condition \(F(\sigma) = \Phi_2(\sigma) - \sigma_0^2(\varepsilon_p) = 0\) (Eq.24).

That is, it is very simple to check if \(F(\sigma^*) < 0\) (elastic) or \(F(\sigma^*) \geq 0\) (plastic). Completion of the same task will be a little bit more complicated by using the parameter form of the yield loci. That is, one needs to solve first \((\lambda^*, \eta^*)\) with \(0 \leq \lambda^* \leq 1\) and \(\eta^* > 0\) from two equations \(\sigma_x(\lambda^*) = \eta^* \sigma_x^*\) and \(\sigma_y(\lambda^*) = \eta^* \sigma_y^*\) to find out if \(\eta^* > 1\) (elastic) and \(\eta^* \leq 1\) (plastic). The complexity of this approach will increase if one deals with a parametric yield loci in tri-component plane stress or full 3D stresses.

Given an applied stress \(\sigma = (\sigma_x, \sigma_y)\) on the yield surface and its increment \(\dot{\sigma} = (\dot{\sigma}_x, \dot{\sigma}_y)\), plastic strain increments are more readily computed directly from the flow rule based on the implicit yield stress function \(f(\sigma_x, \sigma_y)\) as (assuming the equivalent stress-strain relation \(\sigma_f(\varepsilon_p)\) is known)

\[
\dot{\varepsilon}_x^p = \frac{\partial f}{\partial \sigma_x} \dot{\sigma}_x + \frac{\partial f}{\partial \sigma_y} \dot{\sigma}_y, \quad \dot{\varepsilon}_y^p = \frac{\partial f}{\partial \sigma_x} \dot{\sigma}_y - \frac{\partial f}{\partial \sigma_y} \dot{\sigma}_x, \quad \text{where} \quad \dot{\varepsilon}^p(\sigma, \dot{\sigma}) = \frac{1}{\sigma_f(\varepsilon_p)} \left( \dot{\sigma}_x \frac{\partial f}{\partial \sigma_x} + \dot{\sigma}_y \frac{\partial f}{\partial \sigma_y} \right). \tag{25}
\]

On the other hand, the computation of plastic strain increments from the parametric Vegter yield criterion is indirect by invoking the relation between tangent and normal vectors of the yield surface (Eq.17) and the plastic work rate equivalence [11], namely

\[
k = \frac{d\sigma_y(\lambda)}{d\sigma_x(\lambda)} = \frac{\sigma'_y(\lambda)}{\sigma'_x(\lambda)} \Rightarrow \quad \frac{\dot{\varepsilon}_x^p}{\sigma'_x(\lambda)} = \frac{\dot{\varepsilon}_y^p}{-\sigma'_y(\lambda)}, \quad \text{and} \quad \sigma_f \dot{\varepsilon}_p = \sigma_x \dot{\varepsilon}_x^p + \sigma_y \dot{\varepsilon}_y^p. \tag{26}
\]

This leads to the final parametric results on plastic strain increments as

\[
\dot{\varepsilon}_x^p = \dot{\varepsilon}_x^p \frac{\sigma_f \sigma'_y(\lambda)}{\sigma_x(\lambda) \sigma'_y(\lambda) - \sigma_y(\lambda) \sigma'_x(\lambda)}, \quad \dot{\varepsilon}_y^p = \dot{\varepsilon}_y^p \frac{-\sigma_f \sigma'_x(\lambda)}{\sigma_x(\lambda) \sigma'_y(\lambda) - \sigma_y(\lambda) \sigma'_x(\lambda)}, \tag{27}
\]

where the equivalent plastic strain increment \(\dot{\varepsilon}_p\) and the parameter increment \(\lambda\) are obtained from stress increments \(\dot{\sigma} = (\dot{\sigma}_x, \dot{\sigma}_y)\)

\[
\sigma_x = \sigma_f \dot{\sigma}_x(\lambda), \quad \sigma_y = \sigma_f \dot{\sigma}_y(\lambda) : \quad \dot{\sigma}_x = \sigma_f \dot{\sigma}_x(\lambda) \dot{\varepsilon}_x^p + \sigma_f \dot{\sigma}'_x(\lambda) \dot{\lambda}, \quad \dot{\sigma}_y = \sigma_f \dot{\sigma}_y(\lambda) \dot{\varepsilon}_y^p + \sigma_f \dot{\sigma}'_y(\lambda) \dot{\lambda} \tag{28}
\]
as
\[
\dot{\varepsilon}^p = \frac{1}{\sigma_f(\varepsilon^p)} \frac{\tilde{\sigma}_x'(\lambda) \sigma_x - \tilde{\sigma}_x'(\lambda) \sigma_y}{\tilde{\sigma}_y'(\lambda) - \tilde{\sigma}_y'(\lambda)} - \lambda = \frac{1}{\sigma_f(\varepsilon^p)} \frac{\tilde{\sigma}_x'(\lambda) \sigma_y - \tilde{\sigma}_y'(\lambda) \sigma_x}{\tilde{\sigma}_y'(\lambda) - \tilde{\sigma}_y'(\lambda)}.
\]

In summary, it has been shown in this study that implicitization of the parametric Vegter Lite yield criterion in biaxial loading [12] lead to the conventional (implicit) representation of the yield criterion as piecewise non-homogeneous quadratic polynomials in Cauchy stress components \((\sigma_x, \sigma_y)\). The methodology presented here can be readily extended to implicitization of the full parametric Vegter yield criterion [11] with six independent quadratic Bézier curves for accommodating additional inputs from both plane strain tension and pure shear tests. One may thus implement both Vegter yield criteria using more widely available iterative algorithms in computational anisotropic plasticity literature and finite element codes based on initial Hill’s 1948 (implicit) formulation of yield conditions. In addition, simple algebraic inequalities and numerical methods [20, 21] may be used to certify the convexity of a calibrated Vegter yield criterion in both biaxial loading and tri-component plane stress. Implicitization of the parametric Vegter yield criterion in tri-component plane stress \((\sigma_x, \sigma_y, \tau_{xy})\) and its convexity certification are currently under investigation and its outcomes will be reported in the near future.

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