Complete monotonicity of multinomial probabilities and its application to Bernstein estimators on the simplex

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Abstract

Let $d \in \mathbb{N}$ and let $\gamma_i \in [0, \infty)$, $x_i \in (0, 1)$ be such that $\sum_{i=1}^{d+1} \gamma_i = M \in (0, \infty)$ and $\sum_{i=1}^{d+1} x_i = 1$. We prove that

$$a \mapsto \frac{\Gamma(aM + 1)}{\prod_{i=1}^{d+1} \Gamma(a\gamma_i + 1)} \prod_{i=1}^{d+1} x_i^{a\gamma_i}$$

is completely monotonic on $(0, \infty)$. This result generalizes the one found by Alzer (2018) for binomial probabilities ($d = 1$). As a consequence of the log-convexity, we obtain some combinatorial inequalities for multinomial coefficients. We also show how the main result can be used to derive asymptotic formulas for quantities of interest in the context of statistical density estimation based on Bernstein polynomials on the $d$-dimensional simplex.

Keywords: multinomial probability, complete monotonicity, Gamma function, combinatorial inequalities, Bernstein polynomials, simplex

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1. Introduction

For any $d \in \mathbb{N}$, let $[d] \doteq \{1, 2, \ldots, d\}$. For any $\mathbf{v} = (v_1, v_2, \ldots, v_d) \in \mathbb{R}^d$, write $\|\mathbf{v}\| \doteq \sum_{i=1}^{d} |v_i|$. Denote the $d$-dimensional simplex and its interior by

$$\mathcal{S} \doteq \{ \mathbf{x} \in [0, 1]^d : \|\mathbf{x}\| \leq 1 \} \quad \text{and} \quad \text{Int(} \mathcal{S} \text{)} \doteq \{ \mathbf{x} \in (0, 1)^d : \|\mathbf{x}\| < 1 \}.$$

Given a random sample $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n$ on $\mathcal{S}$ from some unknown distribution $F$, define the Bernstein estimator on the simplex

$$\hat{F}_{m,n}(\mathbf{x}) \doteq \sum_{\mathbf{k} \in \mathbb{N}^d : \|\mathbf{k}\| \leq m} F_n(\mathbf{k}/m)P_{k,m}(\mathbf{x}), \quad \mathbf{x} \in \mathcal{S}, \quad (1.1)$$

where $m, n \in \mathbb{N}$, $F_n(\mathbf{y}) \doteq \frac{1}{n} \sum_{j=1}^{n} 1(\mathbf{y} \leq \mathbf{y}_j)$ is the empirical cumulative distribution function, $x_{d+1} \doteq 1 - \|\mathbf{x}\|$, $k_{d+1} \doteq m - \|\mathbf{k}\|$, and

$$P_{k,m}(\mathbf{x}) \doteq \frac{m!}{\prod_{i=1}^{d+1} k_i!} \prod_{i=1}^{d+1} x_i^{k_i}. \quad (1.2)$$

Our first goal is to prove that $a \mapsto P_{sk,am}(\mathbf{x})$ is completely monotonic on $(0, \infty)$, see Definition 1.1 below. In fact, we prove a slightly more general statement in Theorem 2.1. From the log-convexity, we deduce some combinatorial inequalities for multinomial coefficients in Section 3. The proof of the theorem and the combinatorial inequalities follow very closely, and generalize, the work of Alzer (2018). In Section 4, we show how Theorem 2.1 can be used to prove asymptotic formulas for quantities of interest related to (1.1). To our knowledge, the statistical properties (bias, variance, mean integrated squared error, etc.) of the estimator in (1.1) (and the associated density estimator, see e.g. Babu & Chaubey (2006); Leblanc (2010)) have never been studied when $d > 1$, except for the pointwise mean squared error of the density estimator in Tenbusch (1994) when $d = 2$. This was our motivation for this article.

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Definition 1.1 (Complete monotonicity). A non-constant function \( a \mapsto g(a) \) is said to be completely monotonic on \((0, \infty)\), if \( g \) has derivatives of all orders and satisfies
\[
(-1)^n g^{(n)}(a) > 0, \quad \text{for all } n \in \mathbb{N}_0, \ a \in (0, \infty).
\] (1.3)

Remark 1.2. Inequality (1.3) is usually not strict when defining complete monotonicity, but non-constant functions that satisfy the non-strict version of (1.3) automatically satisfy the strict version, see (Dubourdieu, Miller & Samko, 1996) for the original proof or (van Haeringen, 1996, p.395) for a simpler proof.

We will need the two following lemmas during the proof of Theorem 2.1.

Lemma 1.3. Let \( g : (0, \infty) \rightarrow (0, 1) \). If \((−\log g)′\) is completely monotonic on \((0, \infty)\), then \( g \) is completely monotonic on \((0, \infty)\).

Proof. Take \( f : (0, \infty) \rightarrow (0, 1) : y \mapsto e^{-y} \) and \( h : (0, \infty) \rightarrow (0, \infty) : x \mapsto -\log g(x) \). Since \( h \) is positive and \( h′ = (−\log g)′ \) is completely monotonic by hypothesis, then \( g = f \circ h \) is completely monotonic by Theorem 2 in Miller & Samko (2001).

Lemma 1.4. If \( \mathbf{u} = (u_1, u_2, \ldots, u_d) \in \text{Int}(S) \), \( u_{d+1} \doteq 1 - \|u\| > 0 \) and \( y > 1 \), then
\[
J_u(y) \doteq \frac{1}{y - 1} - \sum_{i=1}^{d+1} \frac{1}{y^{1/u_i} - 1} > 0.
\] (1.4)

Proof. Lemma 1 in Alzer (2018) proves (1.4) in the case \( d = 1 \). Fix \( d \geq 2 \) and assume that (1.4) is true for any smaller integer. Let \( y > 1 \). By Lemma 1 in Alzer (2018),
\[
\frac{1}{y - 1} - \frac{1}{y^{1/\|u\|} - 1} - \frac{1}{y^{1/(1-\|u\|)} - 1} > 0.
\] (1.5)

Therefore, (1.4) will follow if we can show that
\[
\frac{1}{y^{1/\|u\|} - 1} - \sum_{i=1}^{d} \frac{1}{y^{1/v_i} - 1} > 0.
\] (1.6)

Simply define \( z \doteq y^{1/\|u\|} \) and \( v_i \doteq u_i/\|u\| \), then (1.6) is equivalent to
\[
\frac{1}{z - 1} - \sum_{i=1}^{d} \frac{1}{z^{1/v_i} - 1} > 0,
\] (1.7)
which is true by the induction hypothesis. \(\Box\)

2. Main result

Below is a generalization of the theorem in Alzer (2018).

Theorem 2.1. For any \( d \in \mathbb{N}, \ M \in (0, \infty), \ x \in \text{Int}(S), \ x_{d+1} \doteq 1 - \|x\| > 0, \) and any \( \gamma \in [0, \infty)^d \) such that \( \|\gamma\| \leq M \) and \( \gamma_{d+1} \doteq M - \|\gamma\| \geq 0 \), the function
\[
g(a) \doteq \frac{\Gamma(aM + 1)}{\prod_{i=0}^{d+1} \Gamma(a\gamma_i + 1)} \prod_{i=1}^{d+1} x_i^{a\gamma_i}
\] (2.1)
is completely monotonic on \((0, \infty)\).

Remark 2.2. In the proof of Theorem 2.1, we will show that \((−\log g)′\) is completely monotonic on \((0, \infty)\), which is a stronger statement by Lemma 1.3.

Remark 2.3. Soon after the first version of the present paper was posted on arXiv.org, Qi et al. (2018) gave an alternative proof of the complete monotonicity of \((−\log g)′\) and rewrote the combinatorial inequalities of Section 3 in terms of multivariate beta functions.
Proof. Let $M \in (0, \infty)$, $x \in \text{Int}(S)$ and $a > 0$. The theorem in Alzer (2018) proves our statement in the case $d = 1$ (when the components of $\gamma$ are integers, but the adjustment is trivial). Therefore, fix $d \geq 2$ and assume that the theorem is true for any smaller integer. If there exists $i \in [d + 1]$ such that $\gamma_i = 0$, the theorem reduces to proving that (2.1) is completely monotonic for a $d$ that is smaller then the one that we previously fixed, which is true by the induction hypothesis. Thus, assume for the remainder of the proof that

\[
\gamma_i > 0, \quad \text{for all } i \in [d + 1].
\]

Define

\[
h(a) = - \log g(a) = - \log \Gamma(aM+1) + \sum_{i=1}^{d+1} \log \Gamma(a\gamma_i+1) - a \sum_{i=1}^{d+1} \gamma_i \log x_i.
\]

Then,

\[
h'(a) = -M \psi(aM+1) + \sum_{i=1}^{d+1} \gamma_i \psi(a\gamma_i+1) - \sum_{i=1}^{d+1} \gamma_i \log x_i,
\]

where $\psi(\log \Gamma') = \Gamma'/\Gamma$. Using the integral representation

\[
\psi'(z) = \int_0^\infty \frac{te^{-(z-1)t}}{e^t-1} dt, \quad z > 0,
\]

see (Abramowitz & Stegun, 1964, p.260), we obtain (take $t = s/M$ and $t = s/\gamma_i$)

\[
h''(a) = -M^2 \psi'(aM+1) + \sum_{i=1}^{d+1} \gamma_i^2 \psi'(a\gamma_i+1) = -M^2 \int_0^\infty \frac{te^{-aMt}}{e^t-1} dt + \sum_{i=1}^{d+1} \gamma_i^2 \int_0^\infty \frac{te^{-a\gamma_it}}{e^t-1} dt
\]

where $J_M(y)$ is defined in (1.4). Applying Lemma 1.4 gives

\[
(-1)^n h^{(n+1)}(a) = \int_0^\infty s^n e^{-as} J_{\gamma_i/M}(e^{s/M}) ds > 0, \quad n \in \mathbb{N}, \ a > 0.
\]

If we show that $h'(a) > 0$ for $a > 0$, then $h'$ will be completely monotonic under Definition 1.1 and we will be able to conclude that $g$ is completely monotonic by Lemma 1.3. Since $h'$ is decreasing (see (2.7) when $n = 1$), we show that $\lim_{a \to \infty} h'(a) \geq 0$ to conclude the proof.

If we apply the recurrence formula

\[
\psi(z+1) = \psi(z) + \frac{1}{z}, \quad z > 0,
\]

see (Abramowitz & Stegun, 1964, p.258), we obtain from (2.4) the representation

\[
h'(a) = \frac{d}{a} - MR(aM) + \sum_{i=1}^{d+1} \gamma_i R(a\gamma_i) + \sum_{i=1}^{d+1} \gamma_i \log \left(\frac{\gamma_i/M}{x_i}\right),
\]

where $R(z) \triangleq \psi(z) - \log z$. Using the asymptotic formula

\[
\psi(z) \sim \log z - \frac{1}{2z} - \ldots \quad (as \ z \to \infty)
\]

see (Abramowitz & Stegun, 1964, p.259), we conclude from (2.9) and Jensen’s inequality (for the convex function $-\log(\cdot)$ and the probability weights $P_i \triangleq \gamma_i/M$ and $Q_i \triangleq x_i$) that

\[
\lim_{a \to \infty} h'(a) = M \sum_{i=1}^{d+1} \frac{\gamma_i}{M} \log \left(\frac{\gamma_i/M}{x_i}\right) \geq -M \log \left(\sum_{i=1}^{d+1} x_i\right) = 0.
\]

This ends the proof. \qed

Remark 2.4. Interestingly, the sum on the left-end side of the inequality in (2.11) is the Kullback-Leibler divergence $D_{KL}(P||Q)$. It is well defined because of (2.2) and the fact that $x \in \text{Int}(S)$ by hypothesis (which implies $0 < x_i < 1$ for all $i \in [d+1]$).
3. Some combinatorial inequalities

In the context of Theorem 2.1, define
\[
C(a) = \frac{\Gamma(aM + 1)}{\prod_{i=1}^{d+1} \Gamma(a\gamma_i + 1)}, \quad a \in (0, \infty).
\] (3.1)

Below are three simple combinatorial inequalities for the multinomial coefficients in (3.1). They generalize the ones proved in Alzer (2018) for binomial coefficients.

**Corollary 3.1.** Let \( k \in \mathbb{N} \) and let \( a_j \in (0, \infty) \), \( \lambda_j \in (0, 1) \), \( j \in \{1, 2, \ldots, k\} \), be such that \( \sum_{j=1}^{k} \lambda_j = 1 \). The following inequalities hold:

(a) \( C(\sum_{j=1}^{k} \lambda_j a_j) \leq \prod_{j=1}^{k} C(a_j)^{\lambda_j} \), where equality holds if and only if all the \( a_j \)'s are the same.

(b) \( \prod_{j=1}^{k} C(a_j) < C(\sum_{j=1}^{k} a_j) \).

(c) If \( a_1 \leq a_3 \), then \( C(a_1 + a_2)C(a_3) \leq C(a_1)C(a_2 + a_3) \), where equality holds if and only if \( a_1 = a_3 \).

**Proof.** By (2.7) in the case \( n = 1 \), we know that \( g \) in the statement of Theorem 2.1 is strictly log-convex, which implies (a) by definition. Point (b) follows from Lemma 3 in Alzer (2018) because \( g \) is differentiable on \([0, \infty)\), \( g(0) = 1 \) and \( g \) is (strictly) positive, (strictly) decreasing and strictly log-convex on \((0, \infty)\). Point (c) follows from a trivial adaptation of the proof of Corollary 3 in Alzer (2018) using (2.7). \( \Box \)

4. Application to Bernstein estimators on the simplex

In recent years, there has been a sustained interest in the study of statistical properties of Bernstein estimators on the unit hypercube, whether we talk about the cumulative distribution function (cdf) estimators

\[
\hat{F}_{m,n}(x) = \sum_{k \in \mathbb{N}_0^d \cap [0,m]^d} F_n(k/m) \prod_{i=1}^{d} \binom{m}{k_i} x_i^{k_i} (1-x_i)^{k_i}, \quad x \in [0,1]^d,
\] (4.1)

where \( F_n \) denotes the empirical cdf (given a random sample \( y_1, y_2, \ldots, y_n \) from an unknown cdf \( F \)), or the density estimators

\[
\hat{f}_{m,n}(x) = m^d \sum_{k \in \mathbb{N}_0^d \cap [0,m-1]^d} \mathbb{P}_n \left( \binom{k_m + 1}{m} \prod_{i=1}^{d} \binom{m-1}{k_i} x_i^{k_i} (1-x_i)^{k_i}, \quad x \in [0,1]^d,
\] (4.2)

where \( \mathbb{P}_n \) denotes the empirical measure. For more information, the reader is referred to Babu et al. (2002), Babu & Chaubey (2006), Belalia (2016), Belalia et al. (2017), Ghosal (2001), Igarashi & Kakizawa (2014), Kakizawa (2011), Janssen et al. (2012, 2014, 2017), Leblanc & Johnson (2007), Leblanc (2009, 2010, 2012a,b), Lu (2015), Petrone (1999), Prakasa Rao (2005), Tenbusch (1994) and Vitale (1975).

One clear advantage of Bernstein estimators over kernel estimators (for example) is that they generally perform better near the boundary, see e.g. Leblanc (2012b). To our knowledge, the statistical properties of Bernstein estimators on the simplex (see (1.1)), and the associated density estimators, have never been studied in the literature, except in the univariate case where they coincide with (4.1) and (4.2) above, and except for the pointwise mean squared error of the density estimator in Tenbusch (1994) when \( d = 2 \). This subject is worth investigating because there are instances in practice where the distribution that we would like to estimate lives naturally on the \( d \)-dimensional simplex. One such example is the Dirichlet distribution, which is the conjugate prior of the multinomial distribution in Bayesian estimation, see e.g. Lange (1995) for an application in the context of allele frequency estimation in genetics. In those instances, we would expect that the estimators defined on the simplex perform better than the ones defined on the unit hypercube, especially near the boundary \( \|x\| = 1 \).

Following Leblanc & Johnson (2007) and Leblanc (2010), define

\[
S_{r,s,m}(x) \equiv \sum_{k \in \mathbb{N}_0^d \cap \|k\| \leq m} P_{r,k,m}(x)P_{s,k,m}(x), \quad x \in S,
\]

for \( r, s, m \in \mathbb{N} \). This family of polynomials would arise in the context of statistical density estimation based on the Bernstein estimators in (1.1) (see e.g. the appendix in Leblanc (2010)). Theorem 2.1 will be used to prove Proposition 4.2 below.
The following lemma generalizes Theorem 1.1 (iii) in Leblanc & Johnson (2007), and Lemma 3 (ii) and (iv) in Leblanc (2010) when \( j = 0 \).

**Lemma 4.1.** Let \( d, r, s, m \in \mathbb{N} \) and \( x \in \text{Int}(S) \), and define the covariance matrix

\[
\Sigma = rs(r + s)(\text{diag}(x) - xx^T).
\]  

(4.3)

We have

\[
m^{d/2} S_{r,s,m}(x) = \phi_{r,s}(x) + o_\star(1), \quad \text{as } m \to \infty,
\]

where

\[
\phi_{r,s}(x) \doteq \frac{(\gcd(r,s))^d}{(2\pi)^{d/2}(\det(\Sigma))^{1/2}}.
\]  

(4.4)

**Proof.** Let \( U_1, \ldots, U_m \) and \( V_1, \ldots, V_m \) be two (independent) sequences of independent random vectors such that \( U_i \sim \text{Multinomial}(r,x) \) and \( V_i \sim \text{Multinomial}(s,x) \) for each \( i \in [d] \). Now, let \( H = \gcd(r,s)I_d \) where \( I_d \) is the \( d \times d \) identity matrix, and define \( W_i = sU_i - rV_i \) so that the \( j \)-th component of \( W_i \) has a lattice distribution with span \( H_{jj} = \gcd(r,s) \). Note that \( W_i^\star = H^{-1} W_i \) has span 1 in all \( d \) directions. The covariance matrix of \( W_i \) is given by \( \Sigma \) in (4.3). We can write \( S_{r,s,m}(x) \) in terms of the \( W_i^\star \)’s as

\[
S_{r,s,m}(x) = \mathbb{P} \left( \sum_{i=1}^m sU_i = \sum_{i=1}^m rV_i \right) = \mathbb{P} \left( \sum_{i=1}^m W_i^\star = 0 \right).
\]

Therefore, using Theorem 3.1 of Athreya & Janicki (2016) (a local central limit theorem for random vectors with lattice distributions), \( \det(H) = (\gcd(r,s))^d \) and the fact that the covariance matrix of \( W_i^\star \) is equal to \( H^{-1} \Sigma H^{-1} \), we obtain the conclusion. \( \square \)

The following proposition generalizes Lemma 4 in Leblanc (2010) when \( j = 0 \).

**Proposition 4.2.** Let \( r, s, m \in \mathbb{N} \) and let \( h : S \to \mathbb{R} \) be any bounded measurable function. As \( m \to \infty \),

(a) \( m^{d/2} \int_S S_{r,s,m}(x)dx = \frac{2^{-d} \pi^m}{(d/2 + 1/2)} + O(m^{-1}) = \int_S \phi_{r,s}(x)dx + O(m^{-1}) \),

(b) \( \int_S h(x)(m^{d/2} S_{r,s,m}(x) - \phi_{r,s}(x))dx = o(1) \).

**Proof.** Assume for now that \( r = s = 1 \). We have

\[
\int_S S_{1,1,m}(x)dx = \sum_{\|k\| \leq m} \int_S (P_{k,m}(x))^2 dx = \sum_{\|k\| \leq m} \left( \frac{\Gamma(m+1)}{\prod_{i=1}^{d+1} \Gamma(k_i+1)} \right)^2 \int_S \prod_{i=1}^{d+1} x_{k_i}^2 dx
\]

\[
= \sum_{\|k\| \leq m} \left( \frac{\Gamma(m+1)}{\prod_{i=1}^{d+1} \Gamma(k_i+1)} \right)^2 \prod_{i=1}^{d+1} \frac{\Gamma(2k_i+1)}{\Gamma(2m+d+1)}
\]

\[
= \left( \frac{\Gamma(m+1)}{\Gamma(2m+d+1)} \right) \sum_{\|k\| \leq m} \prod_{i=1}^{d+1} \left( \frac{2k_i}{k_i} \right).
\]  

(4.5)

To obtain the third equality, we used the normalization constant for the Dirichlet distribution. Note that

\[
\sum_{\|k\| \leq m} \prod_{i=1}^{d+1} \left( \frac{2k_i}{k_i} \right) = (-4)^m \sum_{\|k\| \leq m} \prod_{i=1}^{d+1} \left( \frac{2k_i}{k_i} \right) = (-4)^m \sum_{\|k\| \leq m} \prod_{i=1}^{d+1} \left( \frac{1}{k_i} \right)
\]

\[
= (-4)^m \left( \frac{-(d+1)/2}{m} \right)
\]

\[
= \left( \frac{m + \frac{d+1}{2}}{m} \right)^4 m,
\]  

(4.6)

where the last three equalities follow, respectively, from (5.37), the Chu-Vandermonde convolution (p. 248), and (5.14) in Graham et al. (1994). By applying (4.6) and the duplication formula

\[
4^y = \frac{2 \sqrt{\pi} \Gamma(2y)}{\Gamma(y) \Gamma(y + 1/2)}, \quad y \in (0, \infty),
\]  

(4.7)
see (Abramowitz & Stegun, 1964, p.256), in (4.5), we get
\[
\int_S S_{1,1,m}(x)dx = \frac{(\Gamma(m+1))^2}{\Gamma(2m+1)} \cdot \frac{\Gamma(m+d/2 + 1/2)}{\Gamma(m+1)} \cdot 4^m \\
= \frac{2\sqrt{\pi} \Gamma(m+1)}{2\sqrt{\pi} \Gamma(m+1)} \cdot \frac{\Gamma(m+d/2 + 1/2)}{\Gamma(m+1)} \cdot 4^m \\
= \frac{\Gamma(d/2 + 1/2) \Gamma(m+d/2 + 1)}{\Gamma(d/2 + 1/2) \Gamma(m+1)} \cdot 4^m \\
= \frac{2^{-d} \sqrt{\pi}}{\Gamma(d/2 + 1/2)} \prod_{i=1}^{d/2+1/2} (m+i)^{-1}, \\
= \frac{2^{-d} \sqrt{\pi}}{\Gamma(d/2 + 1/2)} \prod_{i=1}^{d/2+1/2} (m+i)^{-1} \cdot \frac{\Gamma(m+1)}{\Gamma(m+1/2)}. \\
\]
Using the fact that
\[
\frac{\Gamma(m+1)}{m^{1/2} \Gamma(m+1/2)} = 1 + \frac{1}{8m} + O(m^{-2}), \\
\]
see (Abramowitz & Stegun, 1964, p.257), we obtain
\[
m^{d/2} \int_S S_{1,1,m}(x)dx = \frac{2^{-d} \sqrt{\pi}}{\Gamma(d/2 + 1/2)} + O(m^{-1}). \\
\] (4.9)
In the case \(r = s = 1\), the expression for \(\Sigma\) in (4.3) is equal to \(2(\text{diag}(x) - xx^T)\). Using the square-root-free symbolic Cholesky decomposition for covariance matrices of multinomial distributions (see Theorem 1 in Tanabe & Sagae (1992)), we deduce that \(\text{det}(\Sigma) = 2^d \text{det}(\text{diag}(x) - xx^T) = 2^d \prod_{i=1}^{d+1} x_i\). Therefore,
\[
\int_S \frac{1}{(2\pi)^{d/2} |\text{det}(\Sigma)|^{1/2}} dx = \frac{1}{2^d \pi^{d/2}} \int_S \prod_{i=1}^{d+1} x_i^{1/2 - 1} dx \\
= \frac{1}{2^d \pi^{d/2}} \frac{\Gamma(1/2)^{d+1}}{\Gamma(d/2 + 1/2)} \\
= \frac{2^{-d} \sqrt{\pi}}{\Gamma(d/2 + 1/2)}. \\
\] (4.10)
Together with (4.9) and (4.4), this proves (a) for \(r = s = 1\).

Now, the almost-everywhere convergence from Lemma 4.1 and the mean convergence from (a) imply that \(\{S_{1,1,m}(\cdot)\}_{m \in \mathbb{N}}\) is uniformly integrable, see (Shiryaev, 1996, p.189). By Theorem 2.1, \(a \mapsto P_{a_k,am}\) is decreasing on \((0, \infty)\), so
\[
S_{r,s,m}(x) \leq \sum_{\|h\| \leq m} (P_{k,m}(x))^2 = S_{1,1,m}(x), \\
\] (4.11)
which implies that \(\{S_{r,s,m}(\cdot)\}_{m \in \mathbb{N}}\) is also uniformly integrable. Hence, by Lemma 4.1, we must have (a) in the general case \(r, s \in \mathbb{N}\). Finally, the almost-everywhere convergence and the uniform integrability imply the \(L^1\) convergence, so (b) follows immediately from Jensen’s inequality and the fact that \(h\) is bounded. \(\square\)

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