SCATTERED $P$-SPACES OF WEIGHT $\omega_1$

WOJCECH BIELAS, ANDRZEJ KUCHARSKI, AND Szymon Plewik

Abstract. We examine dimensional types of scattered $P$-spaces of weight $\omega_1$. Such spaces can be embedded into $\omega_2$. There are established similarities between dimensional types of scattered separable metric spaces and dimensional types of $P$-spaces of weight $\omega_1$ with Cantor–Bendixson rank less than $\omega_1$.

1. Introduction

A topological space is said to be a $P$-space, whenever $G_\delta$ subsets are open. A topological space is scattered (dispersed) if every non-empty subspace of it contains an isolated point. If $X$ is a topological space and $\alpha$ is an ordinal number, then $X^{(\alpha)}$ denotes the $\alpha$-th derivative of $X$, compare [9, p. 261] or [15, p. 64]. If $X$ is a scattered space, then Cantor–Bendixson rank of $X$ is the least ordinal $N(X)$ such that the derivative $X^{(N(X))}$ is empty, see [7, p. 34]. Thus, if $X^{(N(X))} = \emptyset$ and $\beta < N(X)$, then $X^{(\beta)} \neq \emptyset$, also if $X$ is a scattered space of cardinality $\omega_1$, then $N(X) < \omega_2$.

This paper is a continuation of [1], where we have investigated crowded $P$-spaces of cardinality and weight $\omega_1$. Here, we examine scattered $P$-spaces of weight $\omega_1$. Following the idea that some proofs on $P$-spaces are similar to proofs concerning (scattered) metric spaces, compare [2, Lemma 2.2.], the readers can modify our argumentation to obtain results stated in [5], and also contained in [11] and [17].

It will be convenient to use the notation from [3] and [6]. A scattered $P$-space is assumed to be regular and of weight $\omega_1$, nevertheless, we shall repeat these assumptions in the statements of facts. For brevity, we write $\gamma \in Lim$ instead of $\gamma < \omega_2$ is an infinite limit ordinal. Also, a closed and open set will be called clopen. The sum of a family of $\kappa$ many homeomorphic copies of a space $X$ we denote $\bigoplus_\kappa X$. Basic facts about sums can be found in [3, pp. 74–76]. If topological spaces $X$ and $Y$ are homeomorphic, then we write $X \cong Y$. Following [4], [15] p.

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or \[9\), p. 112\], if \(X\) is homeomorphic to a subspace of \(Y\), then we write \(X \subset_h Y\). If \(X \subset_h Y\) and \(Y \subset_h X\), then we write \(X =_h Y\) and say that \(X\) and \(Y\) have the same dimensional type.

The paper is organised as follows. First, we observe that any scattered space of weight \(\omega_1\) has to be of cardinality \(\omega_1\) and then we establish a lemma on embeddings of spaces with a point together with a decreasing base consisting of clopen sets, Lemma 2. In Section 3, we are concerned with properties of elementary sets, i.e. clopen sets with the last non-empty derivative of cardinality 1. Lemma 3 says that a scattered \(P\)-space of weight \(\omega_1\) can be represented as the sum of a family of elementary sets. Theorem 4 generalises a result of B. Knaster and K. Urbanik, see \[8\] and \[17, Theorem 9\], that each scattered metric space is homeomorphic to a subspace of an ordinal number with the order topology. To be more precise, dimensional types of scattered \(P\)-spaces of weight \(\omega_1\) are represented by dimensional types of subspaces of \(\omega_2\). Corollary 5 states that any scattered \(P\)-space of weight \(\omega_1\) has a scattered compactification. The notion of a stable set enables us to reduce dimensional types of scattered \(P\)-spaces with countable Cantor–Bendixson rank to those of finite ranks. In Section 6, we examine spaces \(J(\alpha)\) for any \(\alpha < \omega_2\), in particular, we have established that the space \(J(\alpha)\) is maximal among elementary sets with Cantor–Bendixson rank not greater than \(\alpha\). Our main results are contained in Section 7. Theorem 30 and Corollary 31 are counterparts of \[5, Theorem 19\] and \[5, Corollaries 29 and 31\]. Finally, we add some remarks concerning \(P\)-spaces with uncountable Cantor–Bendixson ranks. We think that a more detailed description of such spaces requires new tools, therefore it seems to be troublesome.

2. Preliminaries

One can readily check the following properties of a \(P\)-space, see \[1\]. A regular \(P\)-space has a base consisting of clopen subsets, hence it is completely regular, \[1, Proposition 1\]. For a countable family of open covers, there exists an open cover which refines each member of this family. If a regular \(P\)-space is of cardinality \(\omega_1\), then any open cover has a refinement consisting of clopen sets, \[1, Lemma 14\], and also a countable union of clopen sets is clopen, \[1, Corollary 15\].

Note that, there exist \(P\)-spaces of cardinality \(\omega_1\) and of weight greater than \(\omega_1\). Indeed, let \(X = \omega_1 + 1\) be equipped with the topology such that countable ordinal numbers are isolated points and

\[B(\omega_1) = \{U \cup \{p\} : U \text{ is a club}\}\]

is a base at the point \(\omega_1 \in X\), where a closed unbounded subset of \(\omega_1\) is called a club, compare \[6, Definition 8.1\]. The intersection of
countably many clubs is a club and any base for filter generated by the family of all clubs is of cardinality greater than \(\omega_1\), which follows from \([6, \text{Lemma 8.4}].\) Therefore \(X\) is a \(P\)-space of cardinality \(\omega_1\) and of weight greater than \(\omega_1\).

**Proposition 1.** A scattered space of weight at most \(\omega_1\) is of cardinality at most \(\omega_1\).

**Proof.** If \(X\) is a scattered \(P\)-space of weight at most \(\omega_1\), then

\[
X = \bigcup \{ X^{(\alpha) \setminus X^{(\alpha+1)} : \alpha < \beta} ,
\]

where \(\beta < \omega_2\). The inherited topology of each \(X^{(\alpha) \setminus X^{(\alpha+1)}}\) is discrete and of cardinality at most \(\omega_1\), hence \(|X| \leq \omega_1\).

Suppose \(f : \omega_1 \to \omega_1\) is an injection. Clearly, we have the following.

\((\ast)\) \hspace{1cm} \forall \beta < \omega_1 \exists \alpha < \omega_1 f[(\alpha, \omega_1)] \subseteq (\beta, \omega_1).

\((\ast\ast)\) \hspace{1cm} \forall \alpha < \omega_1 \exists \beta < \omega_1 f^{-1}[(\beta, \omega_1)] \subseteq (\alpha, \omega_1).\)

The next lemma looks to be known, but for the readers convenience, we present it with a proof.

**Lemma 2.** Let \(X\) and \(Y\) be topological spaces such that the families of clopen subsets

\(B_x = \{ V^x_\alpha : \alpha < \omega_1 \} \) and \(B_y = \{ U^y_\alpha : \alpha < \omega_1 \} \)

are decreasing bases at points \(x \in X\) and \(y \in Y\), respectively. If

\(X = \{x\} \cup \bigcup \{ V^x_\alpha \setminus V^x_{\alpha+1} : \alpha < \omega_1 \},\)

and \(f : \omega_1 \to \omega_1\) is an injection, and \(\{ F_\alpha : \alpha < \omega_1 \} \) is a family of embeddings

\(F_\alpha : V^x_\alpha \setminus V^x_{\alpha+1} \to U^y_{f(\alpha)} \setminus U^y_{f(\alpha)+1}, \) for \(\alpha < \omega_1,\)

then \(X \subset h Y\).

**Proof.** We obtain an injection \(F : X \to F[X] \subseteq Y\), putting

\[
F(t) = \begin{cases} y, & \text{if } t = x; \\ F_\alpha(t), & \text{if } t \in V^x_\alpha \setminus V^x_{\alpha+1}. \end{cases}
\]

The sets \(V^x_\alpha \setminus V^x_{\alpha+1}\) and \(U^y_\alpha \setminus U^y_{\alpha+1}\) are clopen, hence \(F[X \setminus \{x\}] \subseteq Y\) is a homeomorphic copy of \(X \setminus \{x\}\). It remains to show that the function \(F\) is continuous at the point \(x \in X\) and \(F^{-1}\) is continuous at the point \(y \in Y\).

Fix a set \(U^y_\beta \ni F(x)\). By \((\ast)\) there exists \(\alpha < \omega_1\) such that \(f[(\alpha, \omega_1)] \subseteq (\beta, \omega_1)\). Therefore

\[
F[V^x_{\alpha+1}] = \{y\} \cup \bigcup_{\gamma > \alpha} F[V^x_\gamma \setminus V^x_{\gamma+1}] \subseteq \{y\} \cup \bigcup_{\gamma > \alpha} U^y_{f(\gamma)} \setminus U^y_{f(\gamma)+1} \subseteq U^y_\beta,
\]
hence $F$ is continuous at $x \in X$.

Now, fix $V^x_\alpha \ni x$. There exists $\beta < \omega_1$ such that $(\beta, \omega_1) \cap f[[0, \omega_1]] \subseteq f[(\alpha, \omega_1)]$, because of (**) We have

$$U^y_{\beta+1} \cap F[X] = \{y\} \cup \bigcup \{F[X] \cap U^y_\xi \setminus U^y_{\xi+1} : \xi > \beta\} \subseteq \{y\} \cup \bigcup \{F[X] \cap U^y_\gamma \setminus U^y_{f(\gamma)+1} : \gamma > \alpha\} = \{y\} \cup \bigcup_{\gamma > \alpha} F[V^x_\gamma \setminus V^x_{\gamma+1}] = F[V^x_\alpha].$$

Therefore $F^{-1}[U^y_{\beta+1}] \subseteq V^x_\alpha$, hence $F^{-1}$ is continuous at $y \in Y$. □

Let $X$ be a $P$-space. A base $B_x = \{V_\alpha : \alpha < \omega_1\}$ at a point $x \in X$ is called a $P$-base whenever

- $V_0 = X$ and sets $V_\alpha$ are clopen,
- $V_\beta \supseteq V_\alpha$ for $\beta < \alpha < \omega_1$,  
- $V_\alpha = \bigcap \{V_\beta : \beta < \alpha\}$ for a limit ordinal number $\alpha < \omega_1$.

Moreover, the sets $V_\alpha \setminus V_{\alpha+1}$ will be called slices. Also, we have

$$X \setminus \{x\} = \bigcup \{V_\alpha \setminus V_{\alpha+1} : \alpha < \omega_1\}.$$ 

Note that if $X$ is a $P$-space and $x \in X$, then there exists a $P$-base at point $x \in X$. Indeed, let $\{V_\alpha : \alpha < \omega_1\}$ be a base at a point $x \in X$, which consists of clopen sets. Putting $W_\alpha = \bigcap_{\gamma < \alpha} V_\gamma$, we obtain the family $\{W_\alpha : \alpha < \omega_1\}$ which is a desired $P$-base.

For the purpose of Theorem 15 we will need the following notions and Lemma 3. Let $(P, \leq)$ be an ordered set. An antichain in $P$ is a set $A \subseteq P$ such that any two distinct elements $x, y \in A$ are incomparable, i.e., neither $x \leq y$ nor $y \leq x$. A nonempty $C \subseteq P$ is a chain in $P$ if $C$ is linearly ordered by $\leq$. Now, assume that $(P, \leq)$ is a well-ordered set. If $1 \leq n < \omega$, then let $\leq$ be the coordinate-wise order on the product $P^n$, i.e. $(a_1, \ldots, a_n) \preceq (b_1, \ldots, b_n)$, whenever $a_i \leq b_i$ for $0 < i \leq n$. The following variant of Bolzano–Weierstrass theorem seems to be known, it can be deduced from [5, Lemma 28].

**Lemma 3.** If $(P, \leq)$ is a well-ordered set, then any infinite subset of $(P^n, \preceq)$ contains an infinite increasing sequence. In particular, any antichain and any decreasing sequence in $(P^n, \preceq)$ should be finite. □

### 3. On elementary sets

A clopen subset $E$ of a $P$-space is elementary, whenever the derivative $E^{(N(E)^{-1})}$ is a singleton. Clearly, a singleton is an elementary set and if $E$ is an elementary set, then $N(E)$ is not a limit ordinal.
Lemma 4. If $X$ is a regular scattered $P$-space of weight $\omega_1$, then any open cover of $X$ can be refined by a partition consisting of elementary sets.

Proof. Let \( \{U_\gamma : \gamma < \omega_1\} \) be an open cover of $X$. If $N(X) = 1$, then $X$ is a discrete space, so there is nothing to do. Assume that the hypothesis is fulfilled for each scattered $P$-space $Y$ with $N(Y) < \alpha$. If $N(X) = \alpha$ is a limit ordinal number, then the family \( \{X \setminus X(\gamma) : \gamma < \alpha\} \) is an open cover of $X$. So, there exists a partition \( \{V_\gamma : \gamma < \omega_1\} \) which refines both covers \( \{U_\gamma : \gamma < \omega_1\} \) and \( \{X \setminus X(\gamma) : \gamma < \alpha\} \). By the induction hypothesis, we can assume that each $V_\gamma$ is the union of elementary subsets, since $N(V_\gamma) \leq \gamma < \alpha$. In the case $N(X) = \beta + 1$, the derivative $X(\beta)$ is a discrete space. Let \( \{V_\gamma : \gamma < \omega_1\} \) be a partition of $X$ which refines \( \{U_\gamma : \gamma < \omega_1\} \) and such that each $V_\gamma \cap X(\beta)$ is a singleton or $V_\gamma \cap X(\beta) = \emptyset$. If $V_\gamma \cap X(\beta)$ is a singleton, then $V_\gamma$ is an elementary subset. But if a set $V_\gamma \subseteq X \setminus X(\beta)$, then, by the induction hypothesis, it is the union of a family of elementary subsets. □

Proposition 5. If $X$ is an elementary set and $\alpha < N(X)$, then there exists an elementary subset $E \subseteq X$ such that $N(E) = \alpha + 1$. Moreover, if $\alpha + 1 < N(X)$, then there exists uncountable many pairwise disjoint elementary subsets $E \subseteq X$ such that $N(E) = \alpha + 1$.

Proof. Assume that $\alpha < N(X)$ and $X^{(\alpha)} \setminus X^{(\alpha+1)} \neq \emptyset$. Each point of $X^{(\alpha)} \setminus X^{(\alpha+1)}$ is isolated in $X^{(\alpha)}$, hence there exists $x \in X^{(\alpha)}$ and a clopen set $E$ such that $E \cap X^{(\alpha)} = \{x\}$. Clearly, $E$ is an elementary set and $N(E) = \alpha + 1$.

If $\alpha + 1 < N(X)$, then fix an elementary set $E \subseteq X \setminus X^{(\alpha+2)}$ with a point $x \in X^{(\alpha+1)} \cap E$. We have $N(E \setminus \{x\}) = \alpha + 1$. By Lemma 4 the set $E \setminus \{x\}$ contains an uncountable family of pairwise disjoint elementary subsets, each one with the Cantor–Bendixson rank $\alpha + 1$. □

B. Knaster and K. Urbanik showed that a scattered separable metric space can be embedded in a sufficiently large countable ordinal number, see [8]. Later, R. Telgársky removed the assumption of separability, showing that each metrizable scattered space can be embedded in a sufficiently large ordinal number, see [17].

Theorem 6. Any regular scattered $P$-space of weight $\omega_1$ can be embedded into $\omega_2$.

Proof. We proceed inductively with respect to the rank $N(Y) < \omega_2$ of scattered $P$-spaces $Y$. If $N(Y) = 1$, then $Y$ is discrete, hence it is homeomorphic to the family of all non-limit countable ordinals. First,
we present the second step of the induction. Let $Y$ be a scattered $P$-space with $N(Y) = 2$. The derived set $Y^{(1)}$ is discrete and closed, so we can choose an open cover $P$ such that if $V \in P$, then $V \cap Y^{(1)}$ is a singleton or $V = Y \setminus Y^{(1)}$. Let $P^*$ be a partition which refines $P$. Thus each member of $P^*$ has at most one accumulation point and also $|P^*| \leq \omega_1$. Members of $P^*$ should be homeomorphic to $J(2)$, $i(2)$, $i(2) \oplus D$, or $D$, where $D$ is a discrete space and $J(2) = \text{succ}([0, \omega_1]) \cup \{\omega_1\}$, and $i(2) = \text{succ}([0, \omega_1]) \cup \{\omega_1\}$. Thus, one can embed members of $P^*$ into successive disjoint intervals of $\omega_2$.

We inductively assume that if $Z$ is a $P$-space with $N(Z) < \alpha$, then $Z$ is homeomorphic to a subspace of a successor interval of $\omega_2$. Let $Y$ be a $P$-space such that $N(Y) = \alpha$ is a limit ordinal, so $P = \{Y \setminus Y^{(\tau)} : \tau < \alpha\}$ is an open cover of $Y$. Let $P^*$ be a partition which refines $P$. For each $V \in P^*$, we have $N(V) < \alpha$, so one can embed members of $P^*$ into successive disjoint intervals of $\omega_2$.

Let $Y$ be a $P$-space such that $N(Y) = \gamma + 1 < \omega_2$. If $Y^{(\gamma)} = \{z\}$, then fix a $P$-base $\{W_\mu : \mu < \omega_1\}$ at $z \in Y$. Clearly,

$$P = \{W_\mu \setminus W_{\mu+1} : \mu < \omega_1\},$$

is a partition of the subspace $Y \setminus \{z\}$, consisting of clopen sets in $Y$. Let $Y_1 \subseteq [0, \tau_1)$ be a subspace homeomorphic to $Y \setminus W_1$ and $Y_\mu \subseteq (\tau_\mu, \tau_{\mu+1})$ be homeomorphic to $W_\mu \setminus W_{\mu+1}$, where $(\tau_\mu, \tau_{\mu+1})$ are successive disjoint intervals of $\omega_2$. The subspace

$$\bigcup\{Y_\mu : 0 < \mu < \omega_1\} \cup \{\sup_{0<\mu<\omega_1} \tau_\mu \subseteq [0, \sup_{0<\mu<\omega_1} \tau_\mu] \subseteq \omega_2$$

is homeomorphic to $Y$, where the ordinal number $\sup\{\tau_\mu : 0 < \mu < \omega_1\}$ is assigned to $z$. If $Y^{(\gamma)}$ is not a singleton, then $Y$ is the sum of elementary sets with Cantor–Bendixson rank $\gamma + 1$. As previously, we embed these elementary sets into successive disjoint intervals of $\omega_2$. □

Theorem 6 shows that all scattered $P$-spaces of weight $\omega_1$ share topological properties of the generalised ordered spaces, compare [10]. We omit a detailed discussion of this kind and confine ourselves to a counterpart of the Knaster–Urbanik result, see [8].

**Corollary 7.** A regular scattered $P$-space of weight $\omega_1$ has a scattered compactification of cardinality $\omega_1$.

**Proof.** Any regular scattered $P$-space of weight $\omega_1$ has a homeomorphic copy contained in a initial interval of $\omega_2$, thus the closure of this copy is the desired compactification. □

Clearly, among regular $P$-spaces only finite ones are compact, so any compactification of an infinite $P$-space is not a $P$-space.
4. Stable sets with finite Cantor–Bendixson rank

Assume that \( J(0) \) is the empty set and \( J(1) \) is a singleton. But \( J(2) = \text{succ}([0, \omega^2]) \cup \{ \omega^2 \} \), thus \( J(2) \) is a \( P \)-space with exactly one accumulation point \( x \) such that there exists a \( P \)-base \( \{ V_\alpha : \alpha < \omega_1 \} \) at \( x \in J(2) \) with all slices \( V_\alpha \setminus V_{\alpha + 1} \) of cardinality \( \omega_1 \), being discrete as subspaces. Assume that the \( P \)-space \( J(n - 1) \) is defined, then \( J(n) \) is the \( P \)-space with \( J(n)^{(n-1)} = \{ x \} \) such that there exists \( P \)-base \( \{ V_\alpha : \alpha < \omega_1 \} \) at the point \( x \in J(n) \) with \( V_\alpha \setminus V_{\alpha + 1} \cong \bigoplus_{\omega_1} J(n - 1) \), for each \( \alpha < \omega_1 \). Analogously, let \( i(0) = J(0) \). If the \( P \)-space \( i(n - 1) \) is defined, then \( i(n) \) is the \( P \)-space with \( i(n)^{(n-1)} = \{ x \} \) and a \( P \)-base at \( x \) such that slices are homeomorphic to the sum of \( \omega \) many copies of \( i(n - 1) \).

Adapting the idea from [13, p. 248], we change the definition of a stable set. Namely, among the elementary sets we shall single out stable sets as follows. Let \( E \) be an elementary set such that \( E^{(n)} = \{ g \} \), where \( n < \omega \). Considering \( E \) as a \( P \)-space, we say that \( E \) is a stable set, whenever there is a \( P \)-base at \( g \in E \) such that any two slices are homeomorphic. A singleton is a stable set. Let \( X \) be a \( P \)-space such that \( X^{(1)} = \{ g \} \). If there exists a \( P \)-base at \( g \in X \) with countably infinite slices, then \( X \) is a stable set. By Lemma 2, such a space \( X \) is unique up to homeomorphism, in fact it is \( i(2) \). The space \( J(2) \) is a stable set, but the elementary set \( i(2) \oplus D \) is not stable, whenever \( D \) is uncountable and discrete. Obviously, \( J(2) \) and \( i(2) \) are the only stable sets in the class of all \( P \)-spaces with Cantor–Bendixson rank 2. This class consists of spaces which have three different dimensional types: \( i(2) \), \( i(2) \oplus D \) and \( J(2) \); and moreover

\[
i(2) \subset_h i(2) \oplus D \subset_h J(2).
\]

If \( E \) is a stable set and \( N(E) = n + 1 \), then the set \( E \) is sometimes called \( n \)-stable. By Lemma 2 we have the following.

\[ \] **Corollary 8.** Suppose that \( E \) and \( F \) are \( n \)-stable sets with \( P \)-bases \( \{ V_\alpha : \alpha < \omega_1 \} \) and \( \{ W_\alpha : \alpha < \omega_1 \} \), respectively, witnessing stability. If \( V_0 \setminus V_1 \cong W_0 \setminus W_1 \), then \( E \cong F \). \[ \]

In the class of all elementary sets with Cantor–Bendixson rank 3 there is countably many elementary sets having different dimensional types. For example, spaces \( i(3) \oplus \bigoplus_n J(2) \) are of different dimensional types, depending on \( n \).

\[ \] **Lemma 9.** For each \( n < \omega \), there exist only finitely many non-homeomorphic \( n \)-stable sets. Also, any elementary set with finite Cantor–Bendixson rank is the sum of a family of stable sets.
Proof. We proceed by induction on \( n \). If \( n = 1 \) and \( E \) is an elementary set with \( N(E) = 1 \), then there is nothing to do. Let \( \mathcal{F}_k \) be a family which consists of all, up to homeomorphism, \( k \)-stable sets. Suppose that the family \( \mathcal{F}_k \) is finite, for each \( k < n \), and any elementary set with Cantor–Bendixson rank \( < n \) is the sum of a family of stable sets. Suppose \( E \) is an elementary set with \( E^{(n)} = \{ g \} \). Fix a \( P \)-base \( \mathcal{B} = \{ V_\alpha : \alpha < \omega_1 \} \) at \( g \in E \). By the induction hypothesis, assume that each slice \( V_\alpha \setminus V_\alpha+1 \) is the sum of a family \( \mathcal{U}_\alpha \) of elements from \( \bigcup \{ \mathcal{F}_k : k < n \} \). Let us define a subsequence of \( \mathcal{B} \) as follows.

- If \( Y \in \bigcup \{ \mathcal{F}_m : m < n \} \) appears only in countably many families \( \mathcal{U}_\alpha \), there exists \( \beta_Y < \omega_1 \) such that there is no \( Y \) in \( \mathcal{U}_\alpha \), where \( \beta_Y < \alpha < \omega_1 \).
- If \( Y \in \bigcup \{ \mathcal{F}_m : m < n \} \) appears uncountable many times only in countably many \( \mathcal{U}_\alpha \), there exists \( \gamma_Y < \omega_1 \) such that each \( \mathcal{U}_\alpha \) contains at most countably many copies of \( Y \), where \( \gamma_Y < \alpha < \omega_1 \).
- For the rest of \( Y \in \bigcup \{ \mathcal{F}_k : k < n \} \), put \( \gamma_Y = \beta_Y = 0 \).

Having defined \( \beta_Y \) and \( \gamma_Y \) for each \( Y \in \bigcup \{ \mathcal{F}_k : k < n \} \), choose an increasing function \( f : \omega_1 \to \omega_1 \) such that

\[
\begin{align*}
\max \{ \max \{ \gamma_Y, \beta_Y \} : Y \in \bigcup \{ \mathcal{F}_k : k < n \} \}
\end{align*}
\]

and there exists \( \tau \in \{ 0, \omega, \omega_1 \} \) such that \( Y \) appears \( \tau \)-many times in each slice \( V_{f(\alpha)} \setminus V_{f(\alpha+1)} \).

Clearly, the set \( V_{f(0)} \) is \( n \)-stable. Since the family \( \bigcup \{ \mathcal{F}_k : k < n \} \) is finite, it follows that the family of all \( n \)-stable sets is finite. By the induction hypothesis and Lemma 4, the set \( E \setminus V_{f(0)} \) is the sum of a family of stable sets.

For technical reasons, the sum \( \bigoplus_0 X \) is understood as the empty set.

**Lemma 10.** If \( X \) is a scattered \( P \)-spaces with finite Cantor–Bendixson rank, then there exists a partition

\[
X \cong \bigoplus_{\kappa_1} F_1 \oplus \ldots \oplus \bigoplus_{\kappa_m} F_m,
\]

where \( F_1, \ldots, F_m \) are all stable sets with Cantor–Bendixson rank not greater than \( N(X) \) and \( \kappa_i \in \omega \cup \{ \omega, \omega_1 \} \).

**Proof.** Let \( X \) be a scattered \( P \)-space with \( N(X) = n \). By Lemma 4, the space \( X \) is the sum of a family \( \mathcal{F} \) of elementary sets. By Lemma 9, each \( E \in \mathcal{F} \) is the sum of a family of stable sets. Thus, \( X \) is the sum of a family of \( k \)-stable sets, where \( k < n \). Therefore (again by Lemma 9), if \( F_1, \ldots, F_m \) is a sequence of all (up to homeomorphism) \( k \)-stable sets, where \( k < n \), then

\[
X \cong \bigoplus_{\kappa_1} F_1 \oplus \ldots \oplus \bigoplus_{\kappa_m} F_m,
\]
where and $\kappa_i \in \omega \cup \{\omega, \omega_1\}$.

**Theorem 11.** There are at most countably many non-homeomorphic scattered $P$-spaces with finite Cantor–Bendixson rank.

**Proof.** If $X$ is a scattered $P$-space with $N(X) < n$, then $X$ is homeomorphic to the sum as in Lemma [10] where $N(F_i) < n$ for each $F_i$. There are at most finitely many $k$-stable sets with $k < n$, hence there exist at most countably many such sums determined by the number of occurrences of $k$-stable sets with $k < n$, which suffices to finish the proof. □

**Corollary 12.** There exist countably many dimensional types of scattered $P$-spaces with finite Cantor–Bendixson rank. □

5. **Dimensional type of $P$-spaces with finite Cantor–Bendixson rank**

Some technical problems of $P$-spaces with countable Cantor–Bendixson rank can be reduced to studying $P$-spaces with finite Cantor–Bendixson rank.

**Proposition 13.** If $Y$ is an elementary $P$-space of weight $\omega_1$ with $N(Y) = n$, then $Y \subseteq h J(n)$.

**Proof.** If $n < 2$, then there is nothing to do. If $N(Y) = 2$, then check that the hypothesis is fulfilled for each $k < n$. Fix a $P$-base $\{W_\gamma : \gamma < \omega_1\}$ at the point $y \in Y^{(n-1)}$ and fix a $P$-base $\{V_\gamma : \gamma < \omega_1\}$ at the point $x \in J(n-1)$ such that $V_\gamma \setminus V_{\gamma+1} = \bigoplus_{\omega_1} J(n-1)$ for each $\gamma < \omega_1$. By Lemma [4]

$$W_\gamma \setminus W_{\gamma+1} \cong \bigoplus \{E_\mu : \mu < \omega_1\},$$

where subsets $E_\mu \subseteq Y$ are elementary and with $N(E_\mu) \leq n - 1$. So, by the induction hypothesis, there exist embeddings

$$f|_{W_\gamma \setminus W_{\gamma+1}} : W_\gamma \setminus W_{\gamma+1} \to V_\gamma \setminus V_{\gamma+1}.$$ 

Putting $f(y) = x$, we are done. □

**Proposition 14.** If $X$ is a scattered $P$-space of weight $\omega_1$ such that $X^{(2n)} = \{g\}$, then $J(n+1) \subseteq h X$.

**Proof.** If $n = 1$, then $J(2) \cong (X \setminus X^{(1)}) \cup \{g\}$. Assume that the hypothesis is fulfilled for any scattered $P$-space $Y$ such that $Y^{(2n-2)} = \{y\}$. Let

$$X^* = \{g\} \cup (X \setminus X^{(2n-1)}).$$

The point $g \in X^*$ has a $P$-base $\{V_\alpha : \alpha < \omega_1\}$ such that each slice $V_\alpha \setminus V_{\alpha+1}$ contains $\omega_1$ many pairwise disjoint elementary sets, each
with Cantor–Bendixson rank $2n - 1$. By the induction hypothesis, any such elementary set contains a copy of $J(n)$, hence $J(n + 1) \subseteq h X^*$.

If $k \geq 2n$ and $X^{(k)} = \{g\}$, then there exists $Y \subseteq X$ such that the derivative $Y^{(2n)}$ is a singleton, hence $J(n + 1) \subseteq h Y \subseteq X$. The assumption $k = 2n$ is minimal. Indeed, if $n = 2$ and $k = 3$, then $i(4)$ and $J(3)$ have incomparable dimensional types.

**Theorem 15.** Let $(F, \subseteq h)$ be an ordered set, where $F$ is a family of scattered $P$-spaces of weight $\omega_1$ with Cantor–Bendixson ranks $\leq n$. Then every antichain is finite and every strictly decreasing chain is finite.

**Proof.** Let $X$ be a scattered $P$-space with $N(X) = n$ and $F_1, \ldots, F_m$ be all $k$-stable sets with Cantor–Bendixson rank $k \leq n$. Applying Lemma 10, fix a partition $X = h \bigoplus_{\kappa_i^X} F_1 \oplus \ldots \oplus \bigoplus_{\kappa_i^X} F_m$; where $\kappa_i^X \in \omega \cup \{\omega, \omega_1\} = A$. Thus we have defined a function $\varphi$ such that

$$X \mapsto \varphi(X) = (\kappa_1^X, \ldots, \kappa_m^X) \in A^m.$$  

Consider the coordinate-wise order $(A^m, \preceq)$. Using elementary properties of the sum of spaces, we have the following implications:

1. $\varphi(X) \preceq \varphi(Y) \Rightarrow X \subseteq h Y$;
2. $X \subseteq h Y$ and $X \neq h Y \Rightarrow \exists \kappa_i^X < \kappa_i^Y$.

Condition (1) implies that if $U \subseteq F$ is an antichain with respect to $\subseteq h$, then $\{\varphi(X) : X \in U\}$ is an antichain in $(A^m, \preceq)$. Therefore, by Lemma 3 there is no infinite antichain in $F$.

Suppose that $(X_n)$ is a strictly decreasing sequence with respect to $\subseteq h$. Put

$$B_i = \{\{k, n\} : \kappa_i^{X_{\max}(k, n)} < \kappa_i^{X_{\min}(k, n)}\}.$$  

We have $[\omega]^2 = B_1 \cup \ldots \cup B_m$, by condition (2). According to Ramsey’s theorem [14, Theorem A], there exist an infinite subset $N \subseteq \omega$ and $i$ such that $\kappa_i^{X_n} < \kappa_i^{X_k}$ for each $k, n \in N$, where $k < n$. This contradicts $A$ being well-ordered.

Observe that our proof of Theorem 15 uncovers hidden usage of Ramsey’s theorem in the proof [5, Lemma 30].

**Corollary 16.** Let $(F, \subseteq h)$ be an ordered set, where $F$ is a family of scattered $P$-spaces of weight $\omega_1$ with finite Cantor–Bendixson rank. Then every antichain is finite and every strictly decreasing chain is
finite. However, among spaces of $\mathcal{F}$, there is $\omega$-many but not $\omega_1$-many different dimensional types.

Proof. Let $\mathcal{A}$ be an antichain of scattered $P$-spaces with finite Cantor–Bendixson rank. If $X, Y \in \mathcal{A}$, then $2N(X) < N(Y)$ is impossible. Suppose otherwise and put $n = N(X) + 1$, then $X \subset_h J(n)$ by Proposition 13. The inequality $N(Y) \geq 2n - 1$ and Proposition 14 imply $J(n) \subset_h Y$, a contradiction. Thus $\mathcal{A}$ has to be finite.

Suppose $X_1 \supset_h X_2 \supset_h \ldots$ is a strictly decreasing sequence of scattered $P$-spaces with finite Cantor–Bendixson rank. Then all spaces $X_n$ have Cantor–Bendixson rank $\leq N(X_1)$. By Theorem 15, the sequence is finite.

Observe that if $m \neq n$, then spaces $\bigoplus_m J(2)$ and $\bigoplus_n J(2)$ have different dimensional types. By Lemma 4 and 9, there is at most countably many different dimensional types among spaces with Cantor–Bendixson rank $n$, hence no family of dimensional types of spaces with finite Cantor–Bendixson rank can be uncountable. □

6. Maximal elementary sets

Proposition 13 states that $J(n)$ is maximal with respect to $\subset_h$ in the class of all $P$-spaces with Cantor–Bendixson rank $\leq n$. We proceed to a definition of maximal $P$-spaces with infinite Cantor–Bendixson ranks. Namely, let $J(\omega)$ be the sum of the family $\{J(n) : n < \omega\}$, i.e.

$$J(\omega) = \bigoplus \{J(n) : n < \omega\}.$$ 

If $\beta < \omega_2$ and the $P$-space $J(\beta)$ is already defined, then let $J(\beta + 1)$ be a $P$-space with $J(\beta + 1)^{(\beta)} = \{x\}$ such that there exists a $P$-base $\{V_\alpha : \alpha < \omega_1\}$ at $x \in J(\beta + 1)$ with all slices $V_\alpha \setminus V_{\alpha + 1}$ homeomorphic to the sum $\bigoplus_{\omega_1} J(\beta)$. If $\beta > \omega$ is a limit ordinal and the space $J(\gamma)$ is defined for each $\gamma < \beta$, then put

$$J(\beta) = \bigoplus \{J(\gamma) : \gamma < \beta\}.$$ 

Now, we establish some properties of $J(\beta)$.

Lemma 17. If $\gamma < \omega_2$, then $J(\gamma + 1) \cong J(\gamma + 1) \oplus \bigoplus_{\omega_1} J(\gamma)$.

Proof. Let $\{x\} = J(\gamma + 1)^{(\gamma + 1)}$ and let $\{V_\alpha : \alpha < \omega_1\}$ be a $P$-base at the point $x \in J(\gamma + 1)$. We have

$$V_0 \setminus V_1 \cong \bigoplus_{\omega_1} J(\gamma) \text{ and } V_1 \cong J(\gamma + 1) = V_0,$$

since $V_0 \setminus V_1$ is a clopen set, we are done. □

Lemma 18. If $\beta \in \text{Lim}$, then $J(\beta) \cong \bigoplus_{\omega_1} J(\beta)$.
Proof. Since \( J(\omega) = \bigoplus \{ J(n): n \in \omega \} \), using Lemma 17 we get
\[
J(\omega) \cong \bigoplus \{ J(n) \oplus \bigoplus_{\omega_1} J(n - 1): n > 0 \},
\]
therefore \( J(\omega) \cong \bigoplus_{\omega_1} J(\omega) \).

Assume that if \( \gamma \in \beta \cap \lim \), then the hypothesis of the lemma is fulfilled. According to Lemma 17, we get
\[
J(\omega) = \bigoplus \{ J(\gamma + 1): \gamma < \beta \} \oplus \bigoplus \{ J(\gamma): \gamma \in \beta \cap \lim \} \cong
\bigoplus \{ J(\gamma + 1) \oplus \bigoplus_{\omega_1} J(\gamma): \gamma < \beta \} \oplus \bigoplus \{ J(\gamma): \gamma \in \beta \cap \lim \} \cong
\bigoplus \{ J(\gamma + 1): \gamma < \beta \} \oplus \bigoplus_{\omega_1} \bigoplus \{ J(\gamma): \gamma < \beta \} \oplus \bigoplus \{ J(\gamma): \gamma \in \beta \cap \lim \}.
\]
Therefore \( J(\beta) \cong \bigoplus_{\omega_1} \bigoplus \{ J(\gamma): \gamma < \beta \} \cong \bigoplus_{\omega_1} J(\beta) \).

In fact, for each \( \beta \in \lim \) we have
\[
J(\beta) \cong \bigoplus \{ \bigoplus_{\omega_1} J(\gamma): \gamma < \beta \text{ and } \gamma \notin \lim \}.
\]

7. Dimensional type of \( P \)-spaces with countable and infinite Cantor–Bendixson rank

Note that \( J(2) \) cannot be embedded as a clopen subset of \( i(n) \). Indeed, if \( U \subseteq i(n) \) is a non-discrete clopen subset, then \( U \) contains clopen homeomorphic copy of \( i(2) \), but \( J(2) \) does not contain a clopen homeomorphic copy of \( i(2) \). Consequently no \( i(n) \), for \( n > 1 \), can be homeomorphic to a clopen subset of \( J(\omega) \). Analogously, no \( J(n) \), for \( n > 1 \), can be homeomorphic to a clopen subset of \( \bigoplus \{ i(n): n < \omega \} \). So, one can readily check that \( J(\omega) \) is not homeomorphic to \( \bigoplus \{ i(n): n < \omega \} \).

Nevertheless, we have the following.

Proposition 19. If \( X \) is a scattered \( P \)-space of weight \( \omega_1 \) such that \( N(X) = \omega \), then \( X =_h J(\omega) \).

Proof. According to Lemma 3 let \( X = \bigcup \{ E_\gamma: \gamma < \omega_1 \} \) be a partition such that each \( E_\gamma \) is an elementary set. For each \( n \), fix \( \gamma \) such that \( N(E_\gamma) > 2n \). By Proposition 13, we have \( J(n) \subseteq_h E_\gamma \), consequently \( J(\omega) \subseteq_h X \). The inequality \( N(E_\gamma) < \omega \) and Proposition 13 imply \( E_\gamma \subseteq_h J(\omega) \). Hence \( X \subseteq_h J(\omega) \), since \( J(\omega) \cong \bigoplus_{\omega_1} J(\omega) \) by Lemma 18.

Inductively one can check that \( J(n + 1) \cong J(n) \) for each \( n \in \omega \), but \( J(\omega) \cong J(\omega) \) and therefore, by induction, one readily checks \( J(\alpha) \cong J(\alpha) \) for any infinite \( \alpha \).

Proposition 20. If \( Y \) is an elementary set of weight \( \omega_1 \) with Cantor–Bendixson rank \( \alpha + 1 \), then \( Y \subseteq_h J(\alpha + 1) \).
Let simply mimic the proof of Proposition 13, for other ordinal numbers. According to Proposition 13, the thesis is fulfilled for $\beta < \alpha$. To finish the proof, put $E_\omega$. If Corollary 21.

Proof. Assume that $Y \subset J(\omega + 1)$. We have $J(\omega + 1)^{(\omega)} = \{x\}$ and $Y^{(\omega)} = \{y\}$. Let $\{V_\alpha : \alpha < \omega_1\}$ be a $P$-base at the point $y \in Y$ such that each slice $V_\alpha \setminus V_{\alpha + 1}$ is the sum of a family $R_\alpha$ consisting of elementary subsets and let $\{U_\alpha : \alpha < \omega_1\}$ be a $P$-base at the point $x \in J(\omega + 1)$ such that each slice $U_\alpha \setminus U_{\alpha + 1}$ is homeomorphic to $J(\omega) \cong \bigoplus_{\omega_1} J(\omega)$. Therefore, by the induction hypothesis, there exist embeddings $f|_{W_\gamma \setminus W_{\gamma + 1}} : W_\gamma \setminus W_{\gamma + 1} \to V_\gamma \setminus V_{\gamma + 1}$. To finish the proof, put $f(y) = x$. □

Corollary 21. If $X$ and $Y$ are elementary sets with Cantor–Bendixson rank $\omega + 1$, both of the weight $\omega_1$, then $X =_h Y$.

Proof. Assume that $X \subset J(\omega + 1)$. We have $J(\omega + 1)^{(\omega)} = \{x\}$ and $Y^{(\omega)} = \{y\}$. Let $\{V_\alpha : \alpha < \omega_1\}$ be a $P$-base at the point $y \in Y$ such that each slice $V_\alpha \setminus V_{\alpha + 1}$ is the sum of a family $R_\alpha$ consisting of elementary subsets and let $\{U_\alpha : \alpha < \omega_1\}$ be a $P$-base at the point $x \in J(\omega + 1)$ such that each slice $U_\alpha \setminus U_{\alpha + 1}$ is homeomorphic to $J(\omega) \cong \bigoplus_{\omega_1} J(\omega)$. Therefore, $J(\omega) \subset h V_\beta \setminus V_\beta$ for some $\beta < \omega_1$. Repeating this procedure $\omega_1$ many times, we obtain an unbounded subset $\{\beta_\gamma : \gamma < \omega_1\} \subseteq \omega_1$ such that $J(\omega) \subset h V_{\beta_\gamma} \setminus V_{\beta_\gamma + 1}$. Let $W_\gamma \subset V_{\beta_\gamma} \setminus V_{\beta_\gamma + 1}$ be a copy of $J(\omega) \cong \bigoplus_{\omega_1} J(\omega)$. Hence $J(\omega + 1) \cong \bigoplus\{W_\gamma : \gamma < \omega_1\} \cup \{g\} \subseteq Y$. □

Proposition 22. If $\beta \in \text{Lim}$, then any $P$-space $X$ of weight $\omega_1$ such that $N(X) \leq \beta$ is homeomorphic to a subset of $J(\beta)$, i.e. $X \subset h J(\beta)$.

Proof. By Proposition 13, the hypothesis is fulfilled for $\beta = \omega$. Assume that the hypothesis is fulfilled for each limit ordinal number $\gamma < \beta$. Let $X = \bigoplus\{E_\mu : \mu < \lambda\}$, where $\lambda \leq \omega_1$ and each $E_\mu$ is an elementary set with $N(E_\mu) < \beta$. By the induction hypothesis, fix embeddings $f_\mu : E_\mu \to J(\gamma_\mu)$, where $\gamma_\mu < \beta$. Since $J(\beta)$ has a representation as the sum $\bigoplus\{\bigoplus_{\omega_1} J(\gamma) : \gamma < \beta \text{ and } \gamma \notin \text{Lim}\}$, one readily checks that

$$\bigcup\{f_\mu : \mu < \lambda\} : X \to \bigoplus\{J(\gamma_\mu) : \mu < \lambda\} \cong J(\beta)$$

is a desired embedding. □
Corollary 23. If $X$ is a crowded $P$-space of weight $\omega_1$ and $Y$ is a scattered $P$-space of weight $\omega_1$, then $Y \subset_h X$.

Proof. According to Proposition 22, it suffices to prove inductively that $J(\alpha) \subset_h X$ for each $\alpha < \omega_2$. Recall that, by [11, Propositions 1 and 2], any open subset of a crowded $P$-space contains $\omega_1$-many clopen pairwise disjoint clopen subsets, i.e. $X = \bigoplus \{X_\alpha : \alpha < \omega_1\}$, where each $X_\alpha$ is crowded.

We have $J(1) \subset_h X$, since $J(1)$ is a one-point space. Assume that $J(\gamma) \subset_h Z$ for each $\gamma < \alpha$ and any crowded $P$-space $Z$ of weight $\omega_1$. If $\alpha \in \text{Lim}$, then $J(\alpha) = \bigoplus_{\gamma < \alpha} J(\gamma)$. By the induction hypothesis, we have $J(\gamma) \subset_h X_{\gamma}$, therefore $J(\alpha) \subset_h X$. If $\alpha = \beta + 1$, then fix $x \in X$ and a $P$-base $\{V_\gamma : \gamma < \omega_1\}$ at $x \in X$. Let $J(\beta + 1) = \{g\}$ and $\{W_\gamma : \gamma < \omega_1\}$ be a $P$-base at $g \in J(\beta + 1)$ such that $W_\gamma \setminus W_{\gamma + 1} \cong \bigoplus \omega_1 J(\beta)$. By the induction hypothesis, there exist embeddings $F_\gamma : W_\gamma \setminus W_{\gamma + 1} \to V_\gamma \setminus V_{\gamma + 1}$. By Lemma 2 we get $J(\beta + 1) \subset_h X$, putting $F(g) = x$.

By Proposition 19 and Corollary 21, $P$-spaces with the Cantor–Bendixon rank $\omega$ or $\omega + 1$ have dimensional type of $J(\omega)$ or $J(\omega + 1)$, respectively, and similarly for countable limit ordinals.

Theorem 24. If $\beta \in \text{Lim} \cap \omega_1$, then $J(\beta) =_h X$ for any $P$-space $X$ of weight $\omega_1$ with $N(X) = \beta$. Moreover, if $n > 0$ and $Z$ is a $P$-space of weight $\omega_1$ with $N(Z) = \beta + 2n - 1$, then $J(\beta + n) \subset_h Z$.

Proof. If $\beta = \omega$ and $n = 1$, then there is nothing to do, as it is observed just before this theorem. Assume that

\begin{equation}
(*)_\beta \text{ If } \omega \leq \gamma < \beta, \text{ then there exists } \lambda < \beta \text{ such that } \gamma < \lambda \text{ and if } E \text{ is an elementary set with } N(E) = \lambda, \text{ then } J(\gamma) \subset_h E.
\end{equation}

Conditions $(*)_\beta$ suffice to show that if $N(X) = \beta$, then $J(\beta) \subset_h X$. Indeed, $X$ is the sum of a family of elementary sets $E_\mu$ such that $\beta$ is the supremum of ordinal numbers $N(E_\mu)$, hence if $\gamma < \beta$, then one can choose an elementary set $E_\mu$ such that $J(\gamma)$, for each $\gamma$ a different one. Having chosen sets $E_\mu$, we get $\bigoplus \{J(\gamma) : \gamma < \beta\} \subset_h X$. Therefore, by virtue of Proposition 22, we get $J(\beta) =_h X$.

Now, we prove that if $n > 0$ and $N(Z) = \beta + 2n - 1$, then $J(\beta + n) \subset_h Z$. If $N(X) = \beta$, then $J(\beta) =_h X$ as it has been proved above. Using Lemma 13 one can readily check that $J(\beta + 1) \subset_h X$ whenever $N(X) = \beta + 1$. The inductive steps mimic the proof of Proposition 14. Namely, assume that the hypothesis is fulfilled for any $P$-space $Y$ such that $Y^{(\beta + 2n - 2)} = \{y\}$, i.e. $J(\beta + n) \subset_h Y$. Now fix a $P$-space $Z$ with
The point \( g \in Z^* \) has a \( P \)-base \( \{ V_\alpha : \alpha < \omega_1 \} \) such that each slice \( V_\alpha \setminus V_{\alpha+1} \) contains \( \omega_1 \) many pairwise disjoint elementary sets, each with a one-point \( (\beta + 2n - 2) \)-derivative. By the induction hypothesis, any such elementary set contains a copy of \( J(\beta + n) \), hence \( J(\beta + n + 1) \subseteq h \).

Observe that if \((*)_\beta\) is fulfilled, then \((*)_{\beta+\omega}\) is also fulfilled, which finishes the proof. \( \square \)

**Proposition 25.** If \( \beta \in \text{Lim} \cap \omega_1 \) and \( X \) is an elementary set with \( N(X) = \beta + 1 \), then \( J(\beta + 1) =_h X \).

**Proof.** Let \( X^{(\beta)} = \{ g \} \) and \( \{ V_\alpha : \alpha < \omega_1 \} \) be a \( P \)-base at \( g \in X \). By Proposition 20, we have \( X \subseteq h J(\beta + 1) \). Fix an uncountable subset \( \{ \gamma_\alpha : \alpha < \omega_1 \} \subseteq \omega_1 \), which is enumerated increasingly and such that \( N(V_{\gamma_\alpha} \setminus V_{\gamma_{\alpha+1}}) = \beta \) for each \( \alpha \). Putting \( W_\alpha = V_{\gamma_\alpha} \) and bearing in mind that \( \bigoplus_{\omega_1} J(\beta) = J(\beta) \), we have

\[
\bigoplus_{\omega_1} J(\beta) \subseteq h W_\alpha \setminus W_{\alpha+1},
\]

by Theorem 24. Applying Lemma 2, we get \( J(\beta + 1) \subseteq h X \). \( \square \)

The relation \( =_h \) is an equivalence, where

\[
[X]_h = \{ Y : Y =_h X \}
\]

is the equivalence class of \( X \). If \( \lambda \in \text{Lim} \cup \{ 0 \} \), then let \( \mathcal{P}_\lambda \) be the family of equivalence classes of \( P \)-spaces \( X \) such that \( \lambda < N(X) \leq \lambda + \omega \). Putting \( [X]_h <_h [Y]_h \), whenever \( X \subseteq h Y \), we obtain the ordered set, which we denote \( (\mathcal{P}_\lambda, <_h) \). By Theorem 24, classes \( [J(\lambda + 1)]_h \) and \( [J(\lambda + \omega)]_h \) are the least element and the greatest element of \( \mathcal{P}_\lambda \), respectively. If \( [X]_h \in \mathcal{P}_\lambda \), then \( X \setminus X^{(\lambda+1)} \) is the sum of a family of elementary sets with Cantor–Bendixson rank \( \lambda + 1 \). If \( [X]_h \in \mathcal{P}_\lambda \), then \( [X^{(\lambda)}]_h \in \mathcal{P}_0 \) and by Proposition 25 we have

\[
[X \setminus X^{(\lambda+1)}]_h = \bigoplus_\kappa J(\lambda + 1)]_h,
\]

where \( \kappa = |X^{(\lambda)} \setminus X^{(\lambda+1)}| \), i.e. \( \kappa \in \omega \cup \{ \omega, \omega_1 \} \), but if \( N(X) > \lambda + 1 \), then \( \kappa = \omega_1 \). Note that, the class \( [J(\omega^2)]_h \) does not belong to any family \( \mathcal{P}_\lambda \), despite the fact that if \( X \) is a \( P \)-space with \( N(X) < \omega^2 \), then \( [X]_h <_h [J(\omega^2)]_h \). A similar statement holds when \( \omega^2 \) is replaced by \( \gamma \in \text{Lim} \) such that \( \gamma \neq \lambda + \omega \) for each \( \lambda \in \text{Lim} \).

The lemma below is probably well known.

**Lemma 26.** If \( f : X \to Y \) is a continuous injection, then we have \( f[X^{(\alpha)}] \subseteq Y^{(\alpha)} \), for any ordinal number \( \alpha \). Moreover, if \( X \subseteq h Y \), then \( X^{(\alpha)} \subseteq h Y^{(\alpha)} \).
Proof. If $f: X \to Y$ is a continuous injection and $x \in X^{(1)}$, then we have
\[ f(x) \in \text{cl} f[X \setminus \{x\}] = \text{cl}(f[X] \setminus \{f(x)\}), \]
hence $f(x) \in f[X]^{(1)} \subseteq Y^{(1)}$. By induction on $\alpha$, assume that $\beta < \alpha$ implies $f[X^{(\beta)}] \subseteq Y^{(\beta)}$. Thus, if $\alpha = \beta + 1$, then
\[ f[X^{(\beta+1)}] \subseteq f[X^{(\beta)}]^{(1)} \subseteq (Y^{(\beta)})^{(1)} = Y^{(\beta+1)}. \]
If $\alpha$ is a limit ordinal, then
\[ f[\bigcap_{\beta < \alpha} X^{(\beta)}] \subseteq \bigcap_{\beta < \alpha} f[X^{(\beta)}] \subseteq \bigcap_{\beta < \alpha} Y^{(\beta)} = Y^{(\alpha)}. \]

It appears that Lemma 26 can be reversed.

**Lemma 27.** If $Z$ is a $P$-space such that $0 < N(Z) \leq \omega$ and $\lambda \in \text{Lim}$, then there exists a $P$-space $Z^*$ such that $Z^{*(\lambda)} \cong Z$. Moreover, $Z^* \setminus Z^{*(\lambda+1)}$ is the sum $\bigoplus_{\kappa} J(\lambda + 1)$, where $\kappa = |Z \setminus Z^{(1)}|$.

**Proof.** Fix a $P$-space $Z$ such that $0 < N(Z) \leq \omega$. Let $Z^* = Z^{(1)} \cup \bigcup \{J_x : x \in Z \setminus Z^{(1)}\}$, where $\{J_x : x \in Z \setminus Z^{(1)}\}$ are disjoint copies of $J(\lambda + 1)$. Equip the set $Z^*$ with a topology as follows. Each subset of the form $J_x$ is clopen in $Z^*$ and it is homeomorphic to $J(\lambda + 1)$. If $z \in Z^{(1)}$ and $V$ is a neighbourhood of $z$ in $Z$, then let
\[ V^* = V^{(1)} \cup \bigcup \{J_x : x \in V \setminus Z^{(1)}\}. \]
Let the family $B_z = \{V^* : V$ is a neighbourhood of $z \in Z\}$ constitute a base at the point $z \in Z^*$. We leave the reader to check that the space $Z^*$ is as desired. \hfill $\square$

**Lemma 28.** If $\lambda \in \text{Lim} \cap \omega_1$ and $E$ is a $P$-space with $E^{(\lambda+n)} = \{g\}$, then $E$ contains a subspace $Y =_h J(\lambda + 1)$ such that $Y^{(\lambda)} = \{g\}$.

**Proof.** Fix a $P$-space $E$ such that $E^{(\lambda+n)} = \{g\}$. If $n = 0$, then $E =_h J(\lambda + 1)$ by Proposition 25. If $n > 0$, then fix a $P$-base $\{V_\alpha : \alpha < \omega_1\}$ at $g \in E$ such that each slice $N(V_\alpha \setminus V_{\alpha+1}) = \lambda + n - 1$. By Theorem 24, we can choose a subspace $Y_\alpha \subseteq V_\alpha \setminus V_{\alpha+1}$ such that $Y_\alpha =_h J(\lambda)$. Therefore $\{g\} \cup \{Y_\alpha : \alpha < \omega_1\} \cong J(\lambda + 1)$. \hfill $\square$

**Lemma 29.** Let $X,Y$ be scattered $P$-spaces and $\lambda \in \text{Lim} \cap \omega_1$. If $X^{(\lambda)} \subsetneq_h Y^{(\lambda)}$, then $X \subsetneq_h Y$.

**Proof.** Fix an embedding $f : X^{(\lambda)} \to Y^{(\lambda)}$. For each $x \in X^{(\lambda)} \setminus X^{(\lambda+1)}$ there exists an elementary $U_x \subseteq X$ such that $U_x \cap X^{(\lambda)} = \{x\}$. Without loss of generality, we can assume that sets $U_x$ are pairwise disjoint.
and, by Theorem 24, each \( U_x = h(J(\lambda + 1)) \). Similarly, choose a family  
\( \{V_x \subseteq Y : x \in X^{(\lambda)} \setminus X^{(\lambda+1)}\} \) of pairwise disjoint elementary subsets of  
\( Y \) such that \( V_x \cap f[X^{(\lambda)} \setminus X^{(\lambda+1)}] = \{f(x)\} \). For each \( x \in X^{(\lambda)} \setminus X^{(\lambda+1)} \), we have \( N(V_x) \geq \lambda + 1 \) and \( U_x = h(J(\lambda + 1)) \). Thus we can define an embedding \( g_x : U_x \to V_x \) as in Lemma 28. If \( F : X \to Y \) is such that \( F|_{X^{(\lambda)}} = f \) and \( F|_{U_x} = g_x \), for each \( x \in X^{(\lambda)} \setminus X^{(\lambda+1)} \), then \( F \) is a desired embedding. \( \square \)

**Theorem 30.** If \( \lambda \) is a countable limit ordinal, then ordered sets \( (\mathcal{P}_\lambda, <_h) \) and \( (\mathcal{P}_0, <_h) \) are isomorphic. 

**Proof.** By Lemma 26, if \( [X]_h, [Y]_h \in \mathcal{P}_\lambda \) and \( [X]_h = [Y]_h \), then \( [X^{(\lambda)}]_h = [Y^{(\lambda)}]_h \). Thus we can define \( \psi : \mathcal{P}_\lambda \to \mathcal{P}_0 \) by \( \psi([X]_h) = [X^{(\lambda)}]_h \). It remains to show that \( \psi \) is a desired isomorphism. 

By Lemma 27 the function \( \psi \) is a surjection. Again, by Lemma 26 we have  
\[ [X]_h <_h [Y]_h \Rightarrow \psi([X]_h) <_h \psi([Y]_h), \]  
whenever \( [X]_h, [Y]_h \in \mathcal{P}_\lambda \). 

By Lemma 29 if \( [X]_h, [Y]_h \in \mathcal{P}_\lambda \) and \( X^{(\lambda)} \preceq_h Y^{(\lambda)} \), then \( X \preceq_h Y \), which implies injectivity of \( \psi \). So, the proof is finished. \( \square \)

**Corollary 31.** Let \( \mathcal{F} \) be a family of scattered \( P \)-spaces of weight \( \omega_1 \) with countable Cantor–Bendixson ranks. In \( (\mathcal{F}, \preceq_h) \), any antichain and any strictly decreasing chain are finite. 

**Proof.** Assume that \( \mathcal{A} \) is an antichain in \( (\mathcal{F}, \preceq_h) \) and \( X \in \mathcal{A} \), and \( N(X) = \lambda + n \), where \( \lambda \in \text{Lim} \) and \( n < \omega \). By Theorem 24 and Proposition 22 if \( Y \in \mathcal{A} \), then \( N(Y) = \lambda + m \), where \( m < \omega \) and \( \lambda < \omega_1 \). By Corollary 16 and Theorem 30 the family \( \mathcal{A} \) has to be finite. Analogously, we proceed in the case \( \mathcal{A} \) is a decreasing chain. \( \square \)

8. **A few remarks on \( P \)-spaces with Cantor–Bendixson rank \( \geq \omega_1 + 1 \)**

As we have learned there is only one sensible way of defining an elementary set with Cantor–Bendixson rank \( \beta + 1 \) for \( \beta \in \text{Lim} \), since any two such elementary sets have the same dimensional type by Proposition 29. This is not the case for elementary sets with Cantor–Bendixson rank \( > \omega_1 \). Namely, let \( Y(\omega_1) \) be an elementary set such that \( Y(\omega_1)^{(\omega_1)} = \{g\} \) and each slice \( V_\alpha \setminus V_{\alpha+1} = J(\alpha) \).

**Proposition 32.** If \( X \) is a \( P \)-space of the weight \( \omega_1 \) such that \( |X^{(\omega_1)}| = 1 \), then \( X = h(Y(\omega_1)) \) or \( X = h(J(\omega_1 + 1)) \), or \( X = h(Y(\omega_1) \oplus J(\omega_1)) \).
Proof. Let $X^{(\omega_1)} = \{x\}$. Let $\{V_\alpha : \alpha < \omega_1\}$ be a $P$-base at the point $x \in X$.

If each slice $V_\alpha \setminus V_{\alpha+1}$ has Cantor–Bendixson rank $\omega_1$, then, by Proposition 20

\[ X \subset h J(\omega_1 + 1) \]

and

\[ V_\alpha \setminus V_{\alpha+1} = h \bigoplus_{\beta < \omega_1} J(\beta) = J(\omega_1), \]

for each $\alpha < \omega_1$. Therefore $X = h J(\omega_1 + 1)$.

If each slice $V_\alpha \setminus V_{\alpha+1}$ has Cantor–Bendixson rank $< \omega_1$, then, by Proposition 22 for any slice $V_\alpha \setminus V_{\alpha+1}$ there exists $\gamma < \omega_1$ such that $V_\alpha \setminus V_{\alpha+1} \subset h Y(\omega_1)$, by Lemma 2. By Theorem 24 for any $\gamma < \omega_1$ there exists $\alpha < \omega_1$ such that $J(\gamma) \subset h V_\alpha \setminus V_{\alpha+1}$, therefore we get $X = h Y(\omega_1)$.

Without loss of generality, it remains to consider the case when $N(V_0 \setminus V_1) = \omega_1$ and $N(V_\alpha \setminus V_{\alpha+1}) < \omega_1$ for $\alpha \geq 1$. Then we have

\[ X = V_1 \cup (V_0 \setminus V_1) = h Y(\omega_1) \oplus J(\omega_1), \]

which completes the proof. \qed

One can readily check that

\[ J(\omega_1) \subset h Y(\omega_1) \subset h Y(\omega_1) \oplus J(\omega_1) \subset h J(\omega_1 + 1) \]

and no two of these four spaces have the same dimensional type.

9. Conclusions

Compare cardinal characteristics of some classes of dimensional types with classes of non-homeomorphic spaces.

By Corollary 12 and Theorems 23 and 30 if $\lambda < \omega_1$, then there are only countably many dimensional types of $P$-spaces with Cantor–Bendixson ranks $\leq \lambda$. Therefore, there is exactly $\omega_1$-many dimensional types of $P$-spaces with countable Cantor–Bendixson ranks.

There exist $2^{\omega_1}$-many non-homeomorphic $P$-spaces $X$ with $N(X) = \omega_1$, but if $\lambda \in Lim \cap \omega_1$, then there are continuum many non-homeomorphic $P$-spaces $X$ with $N(X) = \lambda$. Indeed, for each

\[ A \subseteq \{\alpha + 2k : \alpha \in Lim \cap \lambda \text{ and } k < \omega\}, \]

let $X$ be a $P$-space such that

\[ X^{(\alpha)} \setminus X^{(\alpha+2)} = h \begin{cases} \bigoplus_{\omega_1} i(2), & \text{if } \alpha \in A, \\ \bigoplus_{\omega_1} J(2), & \text{if } \alpha \notin A; \end{cases} \]

constructing such a $P$-space needs some extra work which we leave to the reader. We get that different subsets of $\lambda$ are assigned to
non-homeomorphic $P$-spaces. Hence, there are continuum many non-homeomorphic scattered $P$-spaces with countable Cantor–Bendixson rank. Similarly, one can prove that if $\lambda = \omega_1$, then there exist $2^{\omega_1}$-many non-homeomorphic scattered $P$-spaces.

It seems that examination of $P$-spaces with uncountable Cantor–Bendixson ranks needs several new ideas and extra efforts. As it has been noted earlier, the readers would easily conclude results concerning scattered separable metric spaces mimicking our argumentation. The same can be said about the so-called $\omega_\mu$-additive spaces, introduced by R. Sikorski [16], compare [12, p. 1]. Indeed, consider the family of all scattered $\omega_\mu$-additive spaces of weight $\omega_\mu$. Replacing $\omega_1$ by $\omega_\mu$, one can adapt our results to this family.

References

[1] W. Bielas, A. Kucharski, Sz. Plewik, Inverse limits which are $P$-spaces. Topology Appl. 312 (2022), Paper No. 108064, 18.
[2] A. Dow, More topological partition relations on $\beta\omega$. Topology Appl. 259 (2019), 50–66.
[3] R. Engelking, General topology. Mathematical Monographs, Vol. 60, PWN–Polish Scientific Publishers, Warsaw, (1977).
[4] M. Fréchet, Les dimensions d’un ensemble abstrait. Math. Ann. 68 (1910), no. 2, 145–168.
[5] W. D. Gillam, Embeddability properties of countable metric spaces. Topology Appl. 148 (2005), no. 1–3, 63–82.
[6] T. Jech, Set theory. Springer Monographs in Mathematics. Springer-Verlag, Berlin, (2003).
[7] A. S. Kechris, Classical descriptive set theory. Graduate Texts in Mathematics 156, Springer-Verlag, New York, (1995).
[8] B. Knaster, K. Urbanik, Sur les espaces complets séparables de dimension 0. Fund. Math. 40 (1953), 194–202.
[9] K. Kuratowski, Topology. Vol. I. Academic Press, New York-London; Państwowe Wydawnictwo Naukowe, Warsaw (1966).
[10] D. J. Lutzer, On generalized ordered spaces. Dissertationes Math. (Rozprawy Mat.) 89 (1971), 32 pp.
[11] S. Mazurkiewicz, W. Sierpiński, Contribution à la topologie des ensembles dénombrables. Fundamenta Mathematicae 1 (1920), 17–27.
[12] A. K. Misra, A topological view of $P$-spaces. General Topology and Appl. 2 (1972), 349–362.
[13] Sz. Plewik, M. Walczyńska, On metric $\sigma$-discrete spaces. Algebra, logic and number theory, 239–253, Banach Center Publ., 108, Polish Acad. Sci. Inst. Math., Warsaw, (2016).
[14] F. P. Ramsey, On a Problem of Formal Logic. Proc. London Math. Soc. (2) 30 (1929), no. 4, 264–286.
[15] W. Sierpiński, General topology. Translated by C. Cecilia Krieger. Mathematical Expositions, No. 7, University of Toronto Press, Toronto, (1952).
[16] R. Sikorski, Remarks on some topological spaces of high power. Fund. Math. 37 (1950), 125–136.
[17] R. Telgársky, *Total paracompactness and paracompact dispersed spaces*. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 16 (1968), 567–572.

**Institute of Mathematics, University of Silesia in Katowice, Bankowa 14, 40-007 Katowice, Poland**

*Email address*: wojciech.bielas@us.edu.pl

*Email address*: andrzej.kucharski@us.edu.pl

*Email address*: szymon.plewik@us.edu.pl