CALABI-YAU METRICS ON CANONICAL BUNDLES OF FLAG VARIETIES

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Abstract. This note gives a simple formula for the unique asymptotically conical Calabi-Yau metrics on the canonical bundle of a flag variety known to exist by the work of R. Goto and others. This is done by generalizing the well known Calabi Ansatz to general Kähler classes. We give some examples of explicit families, in particular, a formula for the two dimensional family of asymptotically conical metrics on the canonical bundle of $F_{1,2}$.

1. Introduction

Much work has been done on manifolds with a Ricci-flat Kähler metrics asymptotic to a cone metric at infinity [11, 8, 4]. By a cone we mean a manifold $\mathbb{R}^+ \times M$ with a complex structure $J$ for which a metric $\mathbf{g} = dr^2 + r^2 g$ is Kähler, where $g$ is a metric on $M$ and $r$ is the radial coordinate on $\mathbb{R}^+$. If we add the vertex, such a manifold $Y$ is known to be a normal affine variety, with an isolated singularity. We say that $N$ with metric $\hat{g}$ is asymptotic to the cone metric $\mathbf{g}$ if there are compact sets $K \subset N$ and $L \subset Y$, and a diffeomorphism $\phi : Y \setminus L \rightarrow N \setminus K$ so that on $Y \setminus L$ the metrics satisfy

\begin{equation}
\| \nabla^j (\phi^* \hat{g} - \mathbf{g}) \| = O(r^{-\gamma - j})
\end{equation}

for some decay constant $\gamma > 0$, where $\nabla$ is the Riemannian connection on $Y$ and the norm is that induced by $\mathbf{g}$. One also requires a similar convergence of the complex structures, or one can consider the stronger case in which $\phi$ is as biholomorphism. Existence and uniqueness results for Ricci-flat (Calabi-Yau) asymptotically conical Kähler metrics are known in the latter case when $\mathbf{g}$ is Calabi-Yau. In this case such Calabi-Yau metrics are proved to exist in [3] in each Kähler class, and are proved to be unique provided they satisfy (1) with $\gamma > 0$. Furthermore, the metrics are known to satisfy (1) with $\gamma = 2n$, $\dim \mathbb{C} N = n$, when the Kähler class is a compactly supported class, and $\gamma = 2$ otherwise.

In this note we give a simple formula for these metrics when $N$ is the total space of the canonical bundle $K_Z$ of a simply connected homogeneous Kähler manifold $Z$. These manifolds are precisely the flag varieties, i.e. $Z = G^C/P$ where $G^C$ is a complex semisimple Lie group and $P \subset G^C$ is a parabolic subgroup. We suppose that $G \subset G^C$ is a real form, all of which act transitively on $Z$. The $G$-invariant Kähler forms on $Z$, and more generally the $G$-invariant closed $(1, 1)$-forms, can be described explicitly [1], and they are uniquely determined by their class in

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$H^2(Z)$. It follows that the invariant Kähler metric $\omega$ in $\pi^1_c(Z)$ is Einstein, since $\text{Ric}_\omega \in 2\pi c_1(Z)$ and is $G$-invariant. That is

\[ \text{Ric}_\omega = 2\omega. \]

It follows that $KZ$, $K_Z$ minus the zero section, has a Calabi-Yau cone metric. Note that the Remmert reduction of $K_Z$ gives the affine variety $Y$, mentioned above, and holomorphic map $\varpi : K_Z \to Y$. So $N = K_Z$ is a crepant resolution of the singularity of $Y$ with exceptional divisor $Z$.

We give an explicit Calabi Ansatz formula for the asymptotically conical Calabi-Yau metric on $N$ in each Kähler class.

**Theorem 1.1.** The unique asymptotically conical Ricci flat Kahler metric on $K_Z$, for each Kähler class, is given by

\[ \hat{\omega} = V(\rho)\pi^*\omega + 2\sqrt{-1}V'(\rho)d\varphi \wedge \overline{d\varphi}, \]

where $\varpi = h|v|$, $h$ the $G$-invariant Hermitian metric on $K_Z$, $\rho = r^2$, where $V(\rho)$ is smooth, and $\Theta$ is semi-positive $G$-invariant $(1,1)$-form on $Z$.

This metric is asymptotic to the Calabi-Yau cone metric with decay rate precisely $\gamma = 2n$ if $[\hat{\omega}]$ is proportional to $\pi^*c_1(Z)$, and precisely $\gamma = 2$ otherwise.

The Kähler class is class is compact, i.e. contained in cohomology with compact supports $H^2(N, \mathbb{R})$ precisely when it is proportional to $\pi^*c_1(Z)$. In this case (3) is the well known Calabi Ansatz [2], see also [6, 5].

2. Geometry of flag varieties

We fix some notation; see [9, 12] for background on Lie groups. Let $G^C$ be a complex semisimple Lie group, with Lie algebra $\mathfrak{g}^C$. Let $B \subset G^C$ be a Borel subgroup with Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}^C$, and Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b}$ and $\Pi$ the simple system of roots associated to the positive system of roots $\Delta^+$ defined by $\mathfrak{b}$. Let $G \subset G^C$ be the compact real form, with Lie algebra $\mathfrak{g}$ constricted from the Cartan-Weyl basis of $\mathfrak{g}^C$. Recall parabolic subgroups of $G^C$ are the subgroups which contain a Borel subgroup. Let $P \subset G^C$ be a parabolic subgroup, containing $B$. The Lie algebra $\mathfrak{p}$ of $P$ is well known to be characterized by a subset $S_p \subset \Pi$. Then with respect to the choice of Cartan subalgebra $\mathfrak{h}$ and simple system of roots $\Pi$ we have the following root space decompositions

\[ \mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha \]

\[ \mathfrak{p} = \mathfrak{h} \oplus \mathfrak{b} \oplus \bigoplus_{\alpha \in \langle S_p \rangle} \mathfrak{g}_{-\alpha} \]

And

\[ \mathfrak{g} = \mathfrak{p} \oplus \mathfrak{q} \]

where

\[ \mathfrak{q} = \bigoplus_{\alpha \in \Delta^+ \setminus \langle S_p \rangle} \mathfrak{g}_{-\alpha}. \]

Further, note that $\mathfrak{q} = \bigoplus_{\alpha \in \Delta^+ \setminus \langle S_p \rangle} \mathfrak{g}_\alpha \subset \mathfrak{p}$ is the nilradical of $\mathfrak{p}$. And

\[ \mathfrak{p} = \mathfrak{q} \oplus \mathfrak{l}. \]
where

$$1 = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{S}_h} (\mathfrak{g}_{\alpha} \oplus \mathfrak{g}_{-\alpha})$$

is a Levi compliment.

Note that $G \cap P = L$ is a compact Lie group, whose Lie algebra $\mathfrak{h}_0$ is a real form of $\mathfrak{h}$, and we have $G^C/P = G/L$. Furthermore, $G \cap B = T$ is a maximal torus of $G$ and a real form of the Cartan subgroup $H$ defined by $\mathfrak{h}$.

Let $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ with $\alpha_i \in \mathfrak{S}_p$ for $i = k + 1, \ldots, \ell$. And let $\varpi_1, \ldots, \varpi_\ell$ be the corresponding fundamental weights, that is the algebraically integral elements of $\mathfrak{h}^*$ with $\langle \varpi_i, \alpha_j \rangle = \delta_{ij}$, where $\delta_{ij}$ denote the coroots.

Given a flag variety $Z = G^C/P$, the affine variety $Y$, $\mathbb{K}_Z^\circ$ plus the singularity, with its Calabi-Yau cone metric, and its resolution $N = \mathbb{K}_Z$, $\varpi : N \to Y$, have a particularly elegant description. Let $\lambda \in \mathfrak{h}^*$ be dominant algebraically integral, with $\langle \lambda \Delta_i \rangle > 0$ for $i = 1, \ldots, k$, and $= 0$ for $i = k + 1, \ldots, \ell$. In other words, $\lambda = \sum_{i=1}^k n_i \varpi_i$ with each $n_i > 0$. The character $\chi^\lambda : H \to \mathbb{C}^*$ can be seen to extend to $P$. Denote by $\mathbb{C}(\lambda)$ to be the complex line with the action of $\chi^\lambda : P \to \mathbb{C}^*$. The associated line bundle $\mathcal{L}_\lambda := G^C \times_P \mathbb{C}(\lambda)$ is ample. Let $L(\lambda)$ be the irreducible representation of $G^C$ with highest weight $\lambda$. By the Borel-Weil theorem we have

$$L(\lambda)^* = H^0(Z, \mathcal{O}(\mathcal{L}_\lambda)).$$

One can further show that if $v_\lambda \in L(\lambda)$ is a highest weight vector, that the orbit $G^C v_\lambda \subset L(\lambda)$ gives an embedding $Z \subset \mathbb{P}(L(\lambda))$. In other words, $\mathcal{L}_\lambda$ is very ample.

The tangent bundle is $TZ = G^C \times_P \mathfrak{g}/\mathfrak{p}$. The anticanonical bundle $\mathbb{K}_Z^{-1} = \Lambda^m TZ = G^C \times_P \Lambda^m (\mathfrak{g}/\mathfrak{p})$, and is easily seen to be $\mathcal{L}_{\delta_P}$, where $\delta_P := \sum_{\alpha \in \Delta^+ \setminus \mathfrak{S}_p} \alpha$. Note that one can show the character $\chi^{\delta_P}$ extends to $P$ by showing that it is trivial on the commutator subgroup. We have an embedding

$$Z \subset \mathbb{P}(L(\delta_P))$$

so that $\mathbb{K}_Z$ is the restriction of the tautological line bundle on $\mathbb{P}(L(\delta_P))$. Each line bundle $\mathcal{L}_\lambda$ has a natural $G$-invariant Hermitian metric, since $\mathcal{L}_\lambda = G \times_L \mathbb{C}(-\lambda)$ and $\chi^{-\lambda}$ restricted to $L$ preserves the usual norm on $\mathbb{C}$. Provide $L(\delta_P)$ with the unique, up to scale, $G$-invariant Hermitian inner product, which provided a Hermitian metric $h$ on the tautological line bundle on $\mathbb{P}(L(\delta_P))$. This metric restricts to a $G$-invariant metric on $\mathbb{K}_Z$ in (7), since we have $G^C \cdot v_{\delta_P} = G \cdot v_{\delta_P}$. Thus the Fubini-Study metric $\omega_{FS} = \sqrt{-1} \partial \bar{\partial} \log h|z|^2$, in local holomorphic coordinates, restricts to the $G$-invariant Kähler-Einstein metric $\omega$ in (2).

In (1) it was observed any closed $(1,1)$-form $\omega$ on $Z$ has a quasi-potential, that is, if $\pi : G^C \to Z$ is the projection then $\pi^* \omega = \sqrt{-1} \partial \bar{\partial} \phi$ for a smooth real valued $G$-invariant function $\phi$. If $\rho$ is an irreducible representation of $G^C$ with a $G$-invariant Hermitian inner product and $v$ is a highest weight vector, then $\phi := \log \| \rho(g) v \|$ is a $G$-invariant quasi-potential. That is, there is a closed $(1,1)$-form $\omega_\phi$ on $Z$, so that $\pi^* \omega_\phi = \sqrt{-1} \partial \bar{\partial} \phi$. In particular, let $\rho^{\varpi_i}$ be the irreducible representation with highest weight $\varpi_i$ for $i = 1, \ldots, k$. Then

$$\phi_{\varpi_i} = \log \| \rho^{\varpi_i} v \|$$

is a quasi-potential. Note that by the $\partial \bar{\partial}$-lemma, all $G$-invariant closed $(1,1)$-forms are harmonic.
We summarize important properties of flag varieties. The second result is due to \[1\].

**Proposition 2.1.** The Picard group of \( Z = G^C/P \) is\[ \text{Pic}(Z) = \mathbb{Z}\{\varpi_1, \ldots, \varpi_k\}. \]
The ample, and very ample, line bundles correspond to elements \( \sum_{i=1}^{k} n_i \varpi_i \) with each \( n_i > 0 \).

The closed \( G \)-invariant \((1,1)\)-forms on \( Z \) are \( \omega_0 \) for a quasi-potential \[ \phi = \sum_{i=1}^{k} c_i \log \|\rho^{\varpi_i} v\| \] with \( c_i \in \mathbb{R} \), and \( \omega_0 \) is a Kähler form precisely when \( c_i > 0 \) for \( i = 1, \ldots, k \).

Thus the Kähler cone \( \mathcal{K}_Z \) of \( Z \) is identified with the open face of the Weyl chamber \( \mathcal{W}_p \) of the positive system of roots spanned by \( \{\varpi_1, \ldots, \varpi_k\} \), so
\[ \mathcal{K}_Z = \mathbb{R}_{>0} \varpi_1 + \cdots + \mathbb{R}_{>0} \varpi_k. \]

The Kähler cone \( \mathcal{K}_N \) of \( N = K_Z \), is the pull back of \( \mathcal{K}_Z \) under \( \pi : K_Z \to Z \).

We have the exact sequence
\[ 0 \to H^2_c(N, \mathbb{R}) \to H^2(N, \mathbb{R}) \to H^2(M, \mathbb{R}) \to 0 \]
where \( M \subset K_Z^\times \) is the \( S^1 \) subbundle. And \( H^2_c(Z, \mathbb{R}) \) is the line spanned by \( \pi^* c_1(K_X) \), which is identified with the line spanned by \( \delta_p \) in \( \mathcal{W}_p \).

Let \( h_{\omega_0} \) be the Hermitian metric with Kähler form \( \omega_0 \). For \( X, W \in T^{1,0}Z \) we have \( h_{\omega_0}(X, W) = -\sqrt{-1} \omega_0(X, \overline{W}) \). One can show that \( h_{\omega_0} \) has the following description \[1\] in terms of the Cartan-Weyl basis \( X_\alpha \in \mathfrak{g}_\alpha \) of \( \mathfrak{g} \). The tangent space of \( Z \) at \( eP \) can be identified with \( \mathfrak{q} \) by the infinitesimal acton of \( \mathfrak{q} \). For each root space \( \mathfrak{g}_\beta \subset \mathfrak{q} \), we have that \( X_\beta \) induces a tangent vector at \( eP \). Then on can show that the \( X_\beta \) are orthogonal and
\[ h_{\omega_0}(X_\beta, X_\beta) = \sum_{i=1}^{k} \frac{c_i}{2} \varpi_i(\beta) \]

3. Main theorem

In this section we fix \( Z = G^C/P_\ast \), \( \dim_\mathbb{C} Z = m \), with the \( G \)-invariant Kähler-Einstein metric \( \omega(\bar{\omega}) \), and provide \( K_Z \) with the \( G \)-invariant Hermitian metric \( h \). Then define a radial function \( r \) on the total space \( N \) of \( K_Z \) by \( r^2 = h(v, v) \) for \( v \in K_Z \). Then \( \omega = \sqrt{-1} \partial \bar{\partial} \log r \). Let \( \Theta \) be a semi-positive \( G \)-invariant \((1,1)\)-form on \( Z \). Set \( t = \log r \), so \( \omega = \sqrt{-1} \partial \bar{\partial} t \). We consider metrics on \( K_Z^\times \) of the form
\[ \hat{\omega}_\Theta = \Theta + \sqrt{-1} \partial \bar{\partial} F(t). \]

In this section we will write \( \Theta \) for \( \pi^* \Theta \) to simplify notation. We have
\[ \hat{\omega} = \Theta + V(t) \omega + \sqrt{-1} V'(t) \partial t \wedge \bar{\partial} t \]
where we set \( V(t) = F'(t) \). Set \( \eta = d^c \log r^2 \), where \( d^c = \frac{\sqrt{-1}}{2r} (\partial - \partial) \). Then a simple computation gives
\[ \sqrt{-1} \partial t \wedge \bar{\partial} t = \frac{\sqrt{-1}}{r^2} \partial r \wedge \bar{\partial} r = \frac{1}{2} \frac{dr}{r} \wedge \eta. \]
And the volume form

\[ \hat{\omega}_\Theta = \frac{m+1}{2} V'(V \omega + \Theta)^m \wedge \frac{dr}{r} \wedge \eta. \tag{11} \]

If we rescale the radial function \( \tilde{r} = r^{\frac{m+1}{m+2}} \), then

\[ \Omega_{CY} = \frac{1}{2} \sqrt{-1} \partial \bar{\partial} \tilde{r}^2 = \frac{1}{m+1} \tilde{r} d\tilde{r} \wedge \eta + \frac{\tilde{r}^2}{m+1} \omega \tag{12} \]

is a Ricci flat cone metric on \( K^2_Z \). This the Kähler form of the cone metric \( \mathcal{F} = d\tilde{r}^2 + \tilde{r}^2 g \) over the Sasakian manifold \( M = \{ \tilde{r} = 1 \} \subset K^2_Z \). This the \( S^1 \) subbundle of \( K^2_Z \) can be given a natural Sasakian structure, which makes \((M, g)\) is Sasakian-Einstein, by taking the lift of the Kähler-Einstein metric \( \tilde{\omega} = \frac{1}{m+1} \omega \) with Einstein constant \( 2m+2 \) on \( Z \). More precisely,

\[ g = \tilde{\eta} \otimes \tilde{\eta} + \pi^* \tilde{g} \]

where \( \tilde{g} \) is the Riemannian metric of \( \tilde{\omega} \), and \( \tilde{\eta} = d^c \log \tilde{r}^2 = \frac{1}{m+1} \eta \) is a contact form with \( \frac{1}{2} d\tilde{\eta} = \hat{\omega} \). The cone metric \( \mathcal{F} \) is Ricci flat because its Ricci curvature satisfies

\[ \text{Ric}_{\mathcal{F}} = \text{Ric}_{\hat{\omega}} - (2m+2) \hat{\omega}. \]

See [7, 10] for more details.

We also have the Kähler cone metric with respect to the potential \( \tilde{r}^2 \)

\[ \tilde{\omega} = \frac{1}{2} \sqrt{-1} \partial \bar{\partial} \tilde{r}^2 = r dr \wedge \eta + r^2 \omega, \]

and a straightforward computation gives

\[ \tilde{\omega}^{m+1} = (m+1)^{m+2} r^{2m} \tilde{\omega}_{CY}^{m+1}. \tag{13} \]

Let \( \mu = \omega^m \) be the unique, up to scale, \( G \)-invariant volume form on \( Z \). Diagonalize \( \Theta \) at a point with respect to \( \omega \) at a point, with eigenvalues \( b_i > 0, \ i = 1, \ldots, m \), then we have

\[ (V \omega + \Theta)^m = \sum_{k=0}^{m} \binom{m}{k} V^k \omega^k \wedge \Theta \]

\[ \sum_{k=0}^{m} V^k \sigma_{m-k} \mu \tag{14} \]

where \( \sigma_{m-i} \) is the elementary symmetric function in the \( b_i i = 1, \ldots, m \).

From (11) and (13) we have

\[ \hat{\omega}_\Theta = 2^{m-1} (m+1)^{(m+2)} r^{-2} \left( V' \sum_{k=0}^{m} V^k \sigma_{m-k} \right) \tilde{\omega}_{CY}^{m+1} \]

For convenience we will use the variable \( \rho = r^2 \), so \( V'(t) = V' (\rho) 2 \rho \). Then \( \hat{\omega}_\Theta \) is Ricci flat if

\[ V'(\rho) \prod_{j=1}^{m} (V + b_j) = C \tag{15} \]

for some constant \( C > 0 \). Since each \( b_i > 0 \), if we choose \( V_0 > 0 \), then

\[ \int_{V_0}^{V} \prod_{j=1}^{m} (x + b_j) \ dx = C \rho \tag{16} \]
defines an analytic increasing function $V(\rho)$.

The equation for (16) can be written as

$$
\sum_{k=0}^{m} \frac{1}{k+1} V^{k+1} \sigma_{m-k} = C \rho + C_0
$$

(17)

We rewrite (10) in the variable $\rho = r^2$ to get

$$
\hat{\omega} = \Theta + V(\rho) \omega + \sqrt{-1} 2V'(\rho) \partial r \wedge \overline{\partial r}
$$

which we observe extends to a non-singular metric on $\mathcal{N} = \mathbb{K} \mathbb{Z}$.

Note that when $\Theta = 0$ in equation (17) we have $\sigma_{m-k} = 0$ for $k \neq m$ and

$$
V = (m+1)^{\frac{m+1}{m+1}} (C \rho + C_0)^{\frac{m}{m+1}}
$$

(18)

which is the familiar Calabi Ansatz for the anti-canonical polarization.

We want to compare $\hat{\omega}$ on the end of $\mathcal{N}$ with the Ricci-flat cone metric (12). We consider the Puiseux series of $V$ in the variable $\tau = \rho - \frac{1}{m}$ from the polynomial equation

$$
\tau \left( \sum_{k=0}^{m} \frac{1}{k+1} V^{k+1} \sigma_{m-k} - C_0 \right) = C
$$

(19)

we get the series Puiseux series

$$
V(\rho) = \sum_{k=-1}^{\infty} c_k \rho^{-\frac{k}{m+1}}
$$

(20)

which converges for large $\rho$, and has first coefficients $c_{-1} = (m+1)^{\frac{1}{m+1}} C^{\frac{1}{m+1}}$ and $c_0 = \frac{c_1}{m}$. In the potential $\tilde{r}^2 = r^2 \tilde{\omega}$ we have

$$
V(\rho) = \sum_{k=-1}^{\infty} c_k \tilde{r}^{-2k}
$$

(21)

and

$$
V'(\rho) = \sum_{k=-1}^{\infty} c_k \frac{k}{m+1} \tilde{r}^{-2(m+1+k)}
$$

(22)

Using (21) and (22) we have, where $c = (m+1)c_{-1},$

$$
\omega_{\Theta} - c\overline{\omega} = V(\rho)(m+1)\omega + \Theta + V'(\rho)\tilde{r}^{2m} (m+1)^2 \tilde{r} d\tilde{r} \wedge \tilde{\eta} - c (\tilde{r} d\tilde{r} \wedge \tilde{\eta} + \tilde{r}^2 \tilde{\omega})
$$

$$
= (m+1)(c_0 + O(\tilde{r}^{-2})) \omega + \Theta + O(\tilde{r}^{-4}) \tilde{r} d\tilde{r} \wedge \tilde{\eta}.
$$

Taking the norm of this with the Calabi-Yau cone metric $g$ gives (1) with $\gamma = 2$. The asymptotics of the derivatives in (11) also follow from (21) and (22).

Note that the rate $\gamma = 2$ cannot be increased unless $[\omega_{\Theta}] \in H^2(Y, \mathbb{R})$. Suppose this is the case, so we may take $\Theta = 0$ and the potential $V(\rho)$ is given by (18). The Newton series gives

$$
V(\rho) = (m+1)^{\frac{1}{m+1}} C^{\frac{1}{m+1}} \tilde{r}^2 + (m+1)^{\frac{m}{m+1}} C^{\frac{m}{m+1}} C_0 \tilde{r}^{-2m} + O(\tilde{r}^{-4m-2}).
$$

And also

$$
V'(\rho) = (m+1)^{\frac{m}{m+1}} C^{\frac{m}{m+1}} \tilde{r}^{-2m} + (-m)(m+1)^{\frac{-2m-1}{m+1}} C^{\frac{-m}{m+1}} C_0 \tilde{r}^{-4m-2} + O(\tilde{r}^{-6m-4}).
$$
From these formulas we have
\[ \hat{\omega} - c_{\text{CY}} = (m + 1)V(\rho)\hat{\omega} + V'(\rho)\hat{\omega}^{2m}(m + 1)^2\hat{\omega}^2 \hat{d}\hat{r} \wedge \tilde{\eta} - c(\hat{d}\hat{r}^2 \wedge \hat{\omega}^2) \]
\[ = \left( (m + 1)\frac{m}{m + 2} C^{\frac{m}{m + 2}} C_0 r^{2m} + O(r^{-4m - 2}) \right) \hat{\omega} \]
\[ + \left( (-m)(m + 1)\frac{m - 2m + 4}{m + 3 - 2m - 2} C^{\frac{m}{m + 3 - 2m - 2}} C_0 r^{-2m - 2} + O(r^{-4m - 2}) \right) \hat{d}\hat{r} \wedge \tilde{\eta}. \]

When we take the norm of this with the Calabi-Yau cone metric \( \gamma \) we get (1) with \( \gamma = 2m + 2 \).

### 4. Examples

We consider examples in which the formula in Theorem 1 gives information, that is examples of \( Z = G^C / P \) in which the Kähler cone has dimension greater than one. We list the three cases in which the formula for \( V \) can be solved by radicals. The theory of flag varieties in Section 2 can be used to express the metrics to \( K \), which is very ample by Proposition 2.1, and \( Z = \mathbb{C}P^1 \times \mathbb{C}P^1 \) where we choose \( C = C + C_0 \). If \( [\epsilon] = \epsilon \cdot P \in G^C / P \) then \( Q \ni q \leftrightarrow q[\epsilon] \) defines a coordinate chart.

#### 4.1. Complete flag manifolds

Let \( Z = G^C / B \), so \( S = \emptyset \). Then \( m = \text{dim}_C Z = |\Delta^+| \) and the Kähler cone \( K_Z \) is given by the Weyl chamber \( \mathcal{W} \) fixed by the positive system. The compact Kähler classes correspond to the ray \( \mathbb{R}_{\geq 0} \delta \subset \mathcal{W} \) spanned by \( \delta = 1/2 \sum_{\alpha \in \Delta^+} \alpha \). Note that \( \delta \) is integral, \( \langle \delta, \alpha \rangle = 1 \) for each \( \alpha \in \Pi \), and corresponds to \( K_Z \), which is very ample by Proposition 2.1 and \( Z \) has Fano index 2.

#### 4.2. \( Z = \mathbb{C}P^1 \times \mathbb{C}P^1 \)

A \( G \)-invariant semi-positive \((1,1)\)-form on \( Z \) is \( \Theta = b_1 \theta_1 + b_2 \theta_2 \) where \( \theta_1 \) and \( \theta_2 \) are pullbacks of the invariant metrics on the \( \mathbb{C}P^1 \) factors, with radius \( \sqrt{b_1} \). We have \( \sigma_1 = b_1 + b_2 \) and \( \sigma_2 = b_1 b_2 \).

The potential \( V(\rho) \) is a solution to
\[ V^3 + \frac{3}{2} V^2 \sigma_1 + 3V \sigma_2 - f(\rho) = 0, \]
where \( f(\rho) = \rho + C_0 \). A formula for a solution to a cubic equation was first published in 1545 by Gerolamo Cardano [3].

Define
\[ \Delta_0(b_1, b_2) = \frac{9}{4} \sigma_1^2 - 9 \sigma_2 = \frac{9}{4}(b_1 - b_2)^2 \]
\[ \Delta_1(b_1, b_2, \rho) = \frac{27}{4} \sigma_1^3 - \frac{81}{2} \sigma_1 \sigma_2 - 27 f(\rho) \]
\[ = \frac{27}{4} (b_1^3 + b_2^3 - 3b_1^2 b_2 - 3b_1 b_2^2) - 27 f(\rho). \]

Then
\[ (23) \]
\[ V(\rho) = -\frac{1}{2} \sigma_1 + \frac{1}{3} \frac{1}{(2)^{1/3}} \left( -\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0} \right)^{1/3} + \frac{2}{3} \Delta_0 \left( -\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0} \right)^{-1/3} \]
where we choose \( C_0 > 0 \) large enough so that \( V(\rho) \) is positive. Then \( V(\rho) \) in (23) gives an explicit 2-dimensional family of Calabi-Yau metrics.
The anticanonical embedding of $Z = \mathbb{CP}^1 \times \mathbb{CP}^1 \hookrightarrow \mathbb{P}(W)$ is induced by the orbit of the highest weight vector in the representation $W$ with highest weight $2\delta = 2\sigma_1 + 2\sigma_2$. This representation is $W = \mathfrak{sl}(2, \mathbb{C}) \otimes \mathfrak{sl}(2, \mathbb{C})$. We have the coordinate system $U(z_1, z_2)$ on $Z$ given by the action of

$$Q = \left\{ \begin{array}{c} 1 \\ z_1 \\ 1 \end{array} \right\} \times \left\{ \begin{array}{c} 1 \\ z_2 \end{array} \right\} \in \mathbb{C}$$

on the highest weight vector

$$Q \cdot \left( \begin{array}{c} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{array} \right) = \left( \begin{array}{cc} -z_1^2 \\ z_1 \\ -z_2^2 \\ z_2 \end{array} \right).$$

Since the Hermitian metric $h$ on $K_Z$ is the restriction of that on $W$, if we denote by $w$ the fiber coordinate on $K_Z$ then over $U$ we have

$$\rho = r^2 = |w|^2(1 + |z_1|^2)(1 + 2|z_2|^2 + |z_2|^4).$$

The semi-positive $G$ invariant $(1, 1)$ forms on $Z$ are

$$\Theta = b_1 d\sigma_1 \wedge d\sigma_1 / (1 + |z_1|^2)^2 + b_2 d\sigma_2 \wedge d\sigma_2 / (1 + |z_2|^2)^2.$$  

Combining (23), (24) and (25) in formula (3) gives an explicit formula for this metric in $U$.

4.3. $Z = \mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$. Similar to the last example the semi-positive $G$-invariant forms are $\Theta = b_1 \sigma_1 + b_2 \sigma_2 + b_3 \sigma_3$, for $b_1, b_2, b_3 \geq 0$. The potential $V(\rho)$ is a solution to

$$V^4 + \frac{4}{3} V^3 \sigma_1 + 2V^2 \sigma_2 + 4V \sigma_3 - f(\rho) = 0,$$

where $f(\rho) = C \rho + C_0$. We have

$$\sigma_1 = b_1 + b_2 + b_3$$
$$\sigma_2 = b_1 b_2 + b_2 b_3 + b_1 b_3$$
$$\sigma_3 = b_1 b_2 b_3$$

The solution to a quartic equation is credited to Lodovico Ferrari in 1540, but a solution was first published in 1545 by Gerolamo Cardano [3]. We get an explicit, though admittedly unwieldy formula in this case. We first make a substitution converting (26) to the depressed quartic

$$x^4 + px^2 + qx + r = 0,$$

where

$$p = 2\sigma_2 - \frac{2}{3} \sigma_1^2 \leq 0$$
$$q = \frac{8}{27} \sigma_1^3 - \frac{4}{3} \sigma_1 \sigma_2 + 4 \sigma_3$$
$$r = -\frac{1}{27} \sigma_1^4 - f(\rho) - \frac{4}{3} \sigma_1 \sigma_3 + \frac{2}{9} \sigma_1^2 \sigma_2$$

and $V = x - \frac{\sigma_1}{2}$. We consider the resolvent cubic

$$R(y) = 8y^3 + 8py^2 + (2p^2 - 8r)y - q^2 = 0.$$
Notice that \( r \to -\infty \) as \( \rho \to \infty \). If \( r^2 \geq -2p \) then one can easily show that \( R(\frac{\sigma^2}{r^2}) > 0 \). So by Rolle’s theorem we have that \( R(y) \) has a real root \( y_0 \) with \( 0 < y_0 < \frac{a^2}{2} \). One can solve for \( y_0 \) in terms of radicals using Cardano’s formula in Example 4.2. To solve for \( y_0 \) we define

\[
\Delta_0 = 4\sigma_2^2 - 16\sigma_1\sigma_3 - 12f(\rho) \\
\Delta_1 = 16\sigma_2^3 - 96\sigma_1\sigma_2\sigma_3 - 12\sigma_1^2f(\rho) + 432\sigma_3^2 + 144\sigma_2f(\rho)
\]

Note that \( \Delta_0 < 0 \) for large \( \rho \). Then

\[
y_0 = -\frac{1}{3} + \frac{1}{6(2)^{1/3}} \left( \Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_3} \right)^{1/3} + \frac{2^{2/3}}{6} \Delta_0 \left( \Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_3} \right)^{-1/3}.
\]

Then the greatest root of (26) is

\[
V(\rho) = -\frac{\sigma_1}{3} - \frac{\sqrt{y_0}}{2} + \frac{1}{2} \sqrt{-2y_0 - 2p + q\sqrt{\frac{y_0}{2}}} \\
or \quad V(\rho) = -\frac{\sigma_1}{3} + \frac{\sqrt{y_0}}{2} + \frac{1}{2} \sqrt{-2y_0 - 2p - q\sqrt{\frac{y_0}{2}}},
\]

depending on whether \( q > 0 \) or \( q < 0 \).

4.4. \( Z = F_{1,2} = SU(3)/T^2 \). This flag manifold is well known as the twistor space of \( \mathbb{CP}^2 \) with the Fubini-Study metric with the opposite orientation, which is anti-self-dual. It is one of only two Kähler twistor spaces of anti-self-dual 4-manifolds, the other being \( \mathbb{CP}^3 \), the twistor space of \( S^2 \). The Ricci-flat cone \( Y = C(M) \cup \{o\} \) is actually hyperkähler. So \( N = K_Z \) with the Ricci flat metrics of Theorem 1.1 is a resolution of a singular hyperkähler metric.

The Kähler cone of \( Z \) is two dimensional, spanned by \( w_1, w_2 \). The eigenvalues of a \( G \)-invariant semi-positive form \( \Theta \), relative to \( \omega \), from (19) are \( b_1, b_2, \frac{1}{2}(b_1 + b_2) \). The potential \( V \) is again a solution to (26) but with

\[
\sigma_1 = \frac{3}{2}(b_1 + b_2) \\
\sigma_2 = 2b_1b_2 + \frac{1}{2}(b_1^2 + b_2^2) \\
\sigma_3 = \frac{1}{2}(b_1^2b_2 + b_1b_2^2)
\]

And again we make a substitutions converting (26) to the depressed quartic

\[
x^4 + px^2 + qx + r = 0,
\]
where straightforward computation gives

\[ p = 2\sigma_2 - \frac{2}{3}\sigma_1^2 \]
\[ = -\frac{1}{2}(b_1 - b_2)^2 \]
\[ q = \frac{8}{27}\sigma_1^3 - \frac{4}{3}\sigma_1\sigma_2 + 4\sigma_3 \]
\[ = 0 \]
\[ r = -\frac{1}{27}\sigma_4^4 - \frac{4}{3}\sigma_1\sigma_3 + \frac{2}{9}\sigma_2\sigma_1^2 - f(\rho) \]
\[ = \frac{1}{16}(b_1 - b_2)^4 - b_1^2b_2^2 - f(\rho) \]

The variables are related by

\[ V = x - \frac{a_1}{3} \]

Luckily \( q = 0 \) and the depressed quartic can be easily factored to give

\[ x = \sqrt{-\frac{p}{2} + \sqrt{p^2 - 4r}} \]

Substituting the above formulae into this give an expression for the potential \( V \) in terms of the parameters \( b_1, b_2 \)

\[ V(\rho) = \frac{-(b_1 + b_2)}{2} + \sqrt{\frac{(b_1 - b_2)^2}{4} + 2\sqrt{b_1^2b_2^2 + C\rho + C_0}} \]

where \( C, C_0 > 0 \).

We can express the family of Calabi-Yau metrics on \( \mathbf{K}_{F_{1,2}} \) explicitly in an open dense holomorphic coordinate chart. The anticanonical embedding of \( Z \) is given as an orbit of \( G^C \) in the representation of highest weight \( 2\varpi_1 + 2\varpi_2 \). The fundamental weight \( \varpi_1 \) is the highest weight of \( \mathfrak{sl}(3,\mathbb{C}) \) acting on \( \mathbb{C}^3 \), while \( \varpi_2 \) is the highest weight of \( \Lambda^2\mathbb{C}^3 \), which is the dual representation. By the Weyl dimension formula the representation \( W_{m_1,m_2} \) with highest weight \( m_1\varpi_1 + m_2\varpi_2 \) has

\[ \dim_{\mathbb{C}} W_{m_1,m_2} = \frac{1}{2}(m_1 + 1)(m_2 + 1)(m_1 + m_2 + 1). \]

The anticanonical embedding \( Z \hookrightarrow \mathbb{P}(W_{2,2}) \) is the orbit of a highest weight vector by \( \mathfrak{sl}(3,\mathbb{C}) \).

We have a coordinate system \( U(z_1, z_2, z_3) \) on \( Z \) given by the action of

\[ Q = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ z_1 & 1 & 0 \\ z_2 & 0 & 1 \end{bmatrix} \mid z_1, z_2, z_3 \in \mathbb{C} \right\} \]

It suffices to compute the orbit of the highest weight vector of \( W_{1,0}^2 \otimes W_{0,1}^2 \), since \( W_{2,2} \) is the irreducible component containing the highest weight vectors. If \( w \) is the fiber coordinate on \( \mathbf{K}_Z \) restricted to \( U \), then we compute

\[ (31) \quad \rho = r^2 = |w|^2(1 + |z_1|^2 + |z_3|^2)(1 + |z_2|^2 + |z_1z_2 - z_3|^2)^2 \]

The semi-positive \( G \)-invariant \((1,1)\)-forms are expressed in \( U \) by

\[ (32) \quad \Theta = \sqrt{-1}b_1\partial\overline{\partial}\log(1 + |z_1|^2 + |z_3|^2) + \sqrt{-1}b_2\partial\overline{\partial}\log(1 + |z_2|^2 + |z_1z_2 - z_3|^2) \]
Combining \( (30) \), \( (31) \) and \( (32) \) in formula \( (3) \) gives an explicit formula for this metric in \( U \).

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