ABSTRACT

We use the methods of group theory to reduce the equations of motion of two spin systems in (2+1) dimensions to sets of coupled ordinary differential equations. We present solutions of some classes of these sets and discuss their physical significance.
Les méthodes de la théorie des groupes sont utilisées pour réduire les équations du mouvement de deux systèmes de spins de dimensions (2+1) à des systèmes d’équations différentielles ordinaires. Les solutions de certaines classes de ces systèmes sont présentées et les aspects physics sont discutés.

1. Introduction

In this paper we look for solutions of the equations of the Landau-Lifshitz model (with, perhaps, nonvanishing anisotropy) and of a nonlinear vector diffusion equation. The equations are given, respectively, by

\[ \frac{\partial \vec{\phi}}{\partial t} = \vec{\phi} \times \vec{F} \]  \hspace{1cm} (1.1)

and

\[ \frac{\partial \vec{\phi}}{\partial t} = \vec{F} - \vec{\phi}(\vec{\phi} \cdot \vec{F}), \]  \hspace{1cm} (1.2)

where \( \vec{F} \) is given by

\[ \vec{F} = \Delta \vec{\phi} + (A\phi_3 + B)\vec{e}_3, \]  \hspace{1cm} (1.3)

where \( \vec{e}_3 \) is a unit vector in the 3\textsuperscript{rd} direction in the \( \vec{\phi} \) space and \( \vec{\phi} \) satisfies \( \vec{\phi} \cdot \vec{\phi} = 1 \). A and B are possible anisotropy coefficients.

The motivation for this work comes from the original observation made by Landau and Lifshitz\(^\text{[1]}\) in their study of the ferromagnetic continuum. They pointed out that for phenomena for which substantial spatial variations occur only over a large number of lattice spacings, we can use the continuum approximation. They showed that a ferromagnetic medium is characterised by the magnetisation vector \( \vec{M} \) (like the vector \( \vec{\phi} \) above) which precesses around the effective magnetic field and so obeys, what is now called the Landau-Lifshitz equation, namely (1.1). Since the original work of Landau and Lifshitz many papers have been written on the subject\(^\text{[2]}\) and the equation has been modified by the inclusion of various additional
terms to $\vec{F}$. It has been used to describe the dynamics of magnetic bubbles in a ferromagnetic continuum and also of vortices in HeII or in a superconductor\cite{2}. Various studies of the dynamics of such topological soliton-like structures have been performed both theoretically and experimentally\cite{3}\cite{4} and they have exhibited many interesting, and perhaps unexpected, phenomena - like the skew deflection of these structures under the influence of a magnetic field gradient which resembles the more familiar Hall motion of electrons in external magnetic and electric fields\cite{5}.

A recent work of Papanicolaou and Tomaras\cite{2}, as well as some earlier work of other people\cite{6} has shown that many experimentally observed facts can indeed be explained using the Landau-Lifshitz equation. Much of the work involved deriving various conserved quantities describing these structures and then using them to restrict the description of the dynamics. All this work has provided further evidence as to the relevance of the Landau-Lifshitz equation to the description of physical phenomena. However, as the Landau-Lifshitz equation is quite complicated, only some results were obtained in an analytical form. Most more recent studies\cite{7} involved numerical simulations.

The vector nonlinear diffusion equation (1.2) has less obvious physical applications but it has been used\cite{8} in the study of phase ordering kinetics where one investigates the time evolution of a system quenched from the disordered into an ordered phase. This topic has attracted considerable attention in recent years\cite{9}. In fact, it has been shown that many features of phase ordering in systems supporting topologically stable defects (for example, in systems described by the $O(N)$ vector model in $d$ dimensions with $d \leq N^{10}\cite{11}$, or in two and three-dimensional nematic liquid crystals\cite{12} can be understood theoretically by investigating the dynamics of the numerous topological defects generated during the quench. A special and interesting case is that of the $O(3)$ model system in 2 spatial dimensions. It supports topologically stable, but non-singular objects which, in the condensed matter community language, are called topological textures. Such systems were studied numerically in ref. \cite{8}.
Given the paucity of analytical results for both equations (1.1) and (1.2) (especially involving the dynamics) one of the aims of this paper is to see what time dependent solutions can be found using the group theoretical method of symmetry reduction\(^{13}\)\(^{14}\)\(^{15}\). This method exploits the symmetry of the original equations to find solutions invariant under some subgroup (the classic example one can give here involves seeking solutions in three dimensions which are rotationally invariant). The method puts all such attempts on a unified footing and it has been applied with success to many equations\(^{16}\). The method gives equations whose solutions represent specific solutions of the full equations; the solutions are determined locally and the method does not tell us whether these solutions are stable or not with respect to any perturbations.

In a recent paper\(^{17}\), two of us (PW and WJZ) together with M. Grundland, have applied this technique to looking for solutions of the relativistic $CP^1$ model.

In this paper we investigate solutions of (1.1) and (1.2). We are particularly interested in time dependent solutions; all time independent solutions of (1.1) and (1.2) (when there is no anisotropy) are also the time independent solutions of the relativistic model and so can be found in ref [17].

Like in the relativistic $CP^1$ model studied before, in order to perform the symmetry reductions, we have to decide what variables to use. To avoid having to use the constrained variables ($\vec{\phi}$) it is convenient to use the $W$ formulation of the model which involves the stereographic projection of the sphere $\vec{\phi} \cdot \vec{\phi} = 1$ onto the complex plane. In this formulation instead of using the $\vec{\phi}$ fields, we express all the dependence on $\vec{\phi}$ in terms of their stereographic projection onto the complex plane $W$. The $\vec{\phi}$ fields are then related to $W$ by

$$
\phi_1 = \frac{W + W^*}{1 + |W|^2}, \quad \phi_2 = i \frac{W - W^*}{1 + |W|^2}, \quad \phi_3 = \frac{1 - |W|^2}{1 + |W|^2}.
$$

To perform our analysis it is convenient to use the polar version of the $W$ variables; i.e. to put $W = R \exp iQ$ and then study the equations for $R$ and $Q$. The
advantage of this approach is that the equations become simple; the disadvantage comes from having to pay attention that \( R \) is real and \( Q \) should be periodic with a period of \( 2\pi \). (If the period is not \( 2\pi \) then the solution may become multi-valued)

Thus if we find solutions that do not obey these restrictions, then these solutions, however interesting they may be, cannot be treated as solutions of the original model.

In the case of the Landau-Lifshitz equation the equations for \( R \) and \( Q \) take the form

\[
\partial_t R - 2 \frac{(1 - R^2)}{(1 + R^2)} \left( \partial_x Q \partial_x R + \partial_y Q \partial_y R \right) - R(\partial_{xx} Q + \partial_{yy} Q) = 0
\]

and

\[
\partial_t Q = B + A \frac{1 - R^2}{1 + R^2} \frac{\partial_{xx} R + \partial_{yy} R}{R} + \frac{(1 - R^2)}{(1 + R^2)} \left( (\partial_x Q)^2 + (\partial_y Q)^2 \right) + \frac{2R}{(1 + R^2)} \left( (\partial_x R)^2 + (\partial_y R)^2 \right),
\]

while for the diffusion case they are respectively

\[
\partial_t Q - 2 \frac{(1 - R^2)}{R(1 + R^2)} \left( \partial_x Q \partial_x R + \partial_y Q \partial_y R \right) - (\partial_{xx} Q + \partial_{yy} Q) = 0
\]

and

\[
\partial_t R + BR + A R \frac{1 - R^2}{1 + R^2} \partial_{xx} R - \partial_{yy} R + \frac{(1 - R^2)R}{(1 + R^2)} \left( (\partial_x Q)^2 + (\partial_y Q)^2 \right) + \frac{2R}{(1 + R^2)} \left( (\partial_x R)^2 + (\partial_y R)^2 \right) = 0.
\]

Note, that, in the Landau-Lifshitz case, if we put \( R = 1 \) the equations become \( \Delta Q = 0 \) and \( \partial_t Q = B \) which have a very simple solution, and in the diffusion case, we have to set \( B = 0 \) and then we end up with \( \partial_t Q - \Delta Q = 0 \) as the equation for \( Q \). The latter case is the nonrelativistic analogue of what was found in the relativistic case where \( R = 1 \) reduced the equation for \( Q \) to the linear wave equation for the phase \( Q \).
In the next section we determine the symmetry group of our equations (1.6), (1.5) and of (1.7) and (1.8). In the following sections we solve the derived equations and discuss their solutions.

2. The Symmetry Group and its Two Dimensional Subgroups

The symmetry group of our systems of equations, respectively (1.6) and (1.5) and (1.7) and (1.8), can be calculated using the standard methods\(^\text{[13]}\)\(^\text{[14]}\)\(^\text{[15]}\)\(^\text{[16]}\). We actually made use of a MACSYMA package\(^\text{[18]}\) that provides a simplified and partially solved set of determining equations.

Solving the determining equations we find that three different cases must be distinguished:

1. \(A = B = 0\), i.e. the anisotropy is absent. The Landau-Lifshitz equation and the diffusion equation have isomorphic symmetry groups, consisting of translations in space and time directions, rotations in the \(x, y\) plane, dilations and a group of \(O(3)\) rotations between the components of the field \(\vec{\phi}\). The corresponding Lie algebra \(L_1\) has the structure of a direct sum

\[
L_1 = s(2, 1) \oplus O(3).
\]

(2.1)

Bases for these two algebras are given by the following vector fields, acting on space-time and on the fields in the \(\{R, Q\}\) realisation of eq. (1.5)-(1.8):

\[
s(2, 1) : \quad P_0 = \partial_t, \quad P_1 = \partial_x, \quad P_2 = \partial_y, \quad L = -x\partial_y + y\partial_x, \quad D = 2t\partial_t + x\partial_x + y\partial_y.
\]

\[
su(2), \quad X = \frac{1}{2} \left( sinQ \left( R - \frac{1}{R} \right) \partial_Q + cosQ \left( R^2 + 1 \right) \partial_R \right), \\
Y = \frac{1}{2} \left( cosQ \left( R - \frac{1}{R} \right) \partial_Q - sinQ \left( R^2 + 1 \right) \partial_R \right), \\
Z = \partial_Q.
\]

(2.3)

2. \(A \neq 0\).
The symmetry algebra for both equations is reduced to

\[ L_2 = \{P_0, P_1, P_2, L\} \oplus \{Z\}, \tag{2.4} \]

i.e. the dilations are absent and the only \( \vec{\phi} \) rotations left are those around the third axis (i.e. around \( \phi_3 \)).

3. \( A = 0, \quad B \neq 0. \)

The symmetry algebra for the dissipative equations (1.8) and (1.7) is still \( L_2 \), as in (2.4). That of the Landau-Lifshitz equation is

\[ L_3 = \{P_0, P_1, P_2, L, \tilde{D}\} \oplus \{X, Y, Z\}, \tag{2.5} \]

with

\[ \tilde{D} = 2t\partial_t + x\partial_x + y\partial_y + 2Bt\partial_Q. \tag{2.6} \]

In order to perform symmetry reduction we need to classify the subalgebras of the symmetry algebras \( L_1, L_2 \) and \( L_3 \). We wish to reduce equations (1.5)-(1.8) to ordinary differential equations. To do this, we will require that the solutions are invariant under a two-dimensional subgroup of the symmetry group. In order to do this systematically we need to derive a classification of the two-dimensional subalgebras of the symmetry algebra. Moreover, we can restrict ourselves to subalgebras, all elements of which act nontrivially on space-time, i.e. which do not contain any rotations in \( \vec{\phi} \) space.

The subalgebra classification can be done in an algorithmic way\(^{[14]}\); the results are quite simple and we present them without a proof.

1. \( A = B = 0 \). Every two-dimensional subalgebra of \( L_1 \), each element of which acts nontrivially on space-time, is conjugate under the action of the group of
inner automorphisms to one of the following ones

\[ A_{2,1} = \{ P_1 + aZ, P_2 + bZ \}, \]
\[ A_{2,2} = \{ L + aZ, P_0 + bZ \}, \]
\[ A_{2,3} = \{ P_0 + aZ, P_2 + bZ \}, \]
\[ A_{2,4} = \{ P_0 - vP_1 + aZ, P_2 + bZ \}, \quad v \neq 0, \]
\[ A_{2,5} = \{ D + bL + aZ, P_0 \}, \quad b \neq 0, \]
\[ A_{2,6} = \{ D + aZ, P_0 \}, \]
\[ A_{2,7} = \{ D + aZ, L + bZ \}, \]
\[ A_{2,8} = \{ D + aZ, P_2 \}. \]

The parameters \( a, b \) and \( v \) are arbitrary real numbers. In some cases their ranges can be further constrained but that is not important for our purposes.

2. \( A \neq 0 \).

Every two-dimensional subalgebra of the considered type is conjugate to one listed above as \( A_{2,1}, \ldots A_{2,4} \).

3. \( A = 0, \quad B \neq 0 \).

For the dissipative equations (1.8) and (1.7) the subalgebra classes are represented by \( A_{2,1}, \ldots A_{2,4} \). For the Landau-Lifshitz equations (1.5) and (1.6) they are represented by \( A_{2,1}, \ldots A_{2,8} \) with \( D \) replaced by \( \bar{D} \) and \( P_0 \) replaced by \( P_0 = P_0 + bZ \).

We can now proceed to perform various reductions. We are particularly interested in reductions that do not result in time independence as these were already studied in ref [17].
3. Solutions of the Landau-Lifshitz Equation

3.1. General Procedure

Our aim is to solve the Landau-Lifshitz equations (1.5) and (1.6), using the method of symmetry reduction. This involves assuming that a solution is invariant under a subgroup $G_0$ of the symmetry group $G$, namely one of the two dimensional groups corresponding to the algebras $A_{2,1}, \ldots, A_{2,8}$ of (2.7). The assumption makes it possible to reduce the partial differential equations (1.5) and (1.6) to a pair of coupled ordinary differential equations. Whenever possible, we decouple them and find explicit solutions for the functions $R$ and $Q$, hence for $W$, and finally for the vector $\vec{\phi}$ figuring in (1.1).

For all 8 algebras in (2.7) the invariant solution will have the form

$$R(x, y, t) = R(\xi), \quad Q(x, y, t) = \alpha(\xi) + \beta(x, y, t) \quad (3.1)$$

where $\xi$ and $\beta$ are explicitly given and $R(\xi)$ and $\alpha(\xi)$ satisfied coupled ordinary differential equations obtained by substituting (3.1) into (1.5) and (1.6).

The reduced equation (1.5) is

$$(\nabla\xi)^2 \alpha\xi + \left[2 \frac{(1 - R^2)}{R(1 + R^2)} (\nabla\xi)^2 R\xi + \Delta\xi\right] \alpha\xi = \frac{R\xi}{R} \xi_t - 2 \frac{(1 - R^2)}{R(1 + R^2)} R\xi (\nabla\xi, \nabla\beta) - \Delta\beta. \quad (3.2)$$

For algebra $A_{2,1}$ we have

$$\nabla\xi^2 = \Delta\xi = \Delta\beta = 0, \quad \xi_t = 1 \quad (3.3)$$

and so (3.2) reduces to $R\xi = 0$.

In all other cases we have $(\nabla\xi)^2 \neq 0$. Eq. (3.2) is a first order linear inhomogeneous equation for $\alpha\xi$. We can integrate it explicitly and obtain $\alpha\xi$ in terms of
R, whenever the functions \( \xi \) and \( \beta \) satisfy
\[
\frac{d}{d\xi} \left( \frac{h}{(\nabla \xi)^2} \right) = 0, \quad \frac{d}{d\xi} \left[ h \frac{(\nabla \xi, \nabla \beta)}{(\nabla \xi)^2} \right] - \frac{h \Delta \beta}{(\nabla \xi)^2} = 0 \tag{3.4}
\]
where
\[
h(\xi) = 1 \quad \text{for} \quad \Delta \xi = 0,
\]
\[
h'(\xi) = \frac{\Delta \xi}{(\nabla \xi)^2} \quad \text{for} \quad \Delta \xi \neq 0. \tag{3.5}
\]

Conditions (3.4) are always satisfied for the algebras \( A_{2,2} \ldots A_{2,6} \), not however for \( A_{2,7} \) and \( A_{2,8} \). When conditions (3.4) are satisfied, we can integrate eq. (3.2) once to obtain
\[
\alpha \xi = \frac{S(1 + R^2)^2}{h} \frac{1 + R^2}{R^2} + \mu \frac{1 + R^2}{R^2} + \nu \tag{3.6}
\]
where \( S \) is an arbitrary real integration constant and where we have
\[
\mu = -\frac{\nu}{2}, \quad \nu = 0, \quad \text{for} \quad A_{2,4}
\]
\[
\mu = 0, \quad \nu = -\frac{a}{\xi + 1}, \quad \text{for} \quad A_{2,5}
\]
\[
\mu = 0, \quad \nu = \frac{a \xi}{1 + \xi^2}, \quad \text{for} \quad A_{2,6}
\]
\[
\mu = 0, \quad \nu = 0, \quad \text{for} \quad A_{2,2, A_{2,3}}. \tag{3.7}
\]

Equation (1.6) for algebras \( A_{2,2}, \ldots, A_{2,8} \) is reduced to a second order differential equation for \( R(\xi) \), that also involves \( \alpha \xi(\xi) \). For reductions corresponding to Lie algebras \( A_{2,2}, \ldots, A_{2,6} \) we can substitute \( \alpha \xi \) from (3.6), to obtain an ordinary differential equation for \( R(\xi) \) alone. To transform this equation to a standard form we put
\[
R(\xi) = \sqrt{-U(\eta)}, \quad \eta = \int h^{-1}(\xi)d\xi. \tag{3.8}
\]

The equation for \( U(\eta) \) is then written as
\[
U_{\eta \eta} = \left( \frac{1}{2U} + \frac{1}{U - 1} \right) U^2 - \frac{2S^2}{U} (U + 1)(U - 1)^3 + p \frac{U(U + 1)}{U - 1} + q U + m(U - 1)^2. \tag{3.9}
\]
Equation (3.9) can be integrated in terms of elliptic functions if \( p, q \) and \( m \) are constants. This is always the case for algebras \( A_{2,3}, \ldots, A_{2,6} \). In the case of algebra \( A_{2,2} \) this is true if we set \( A = 0, B = b \).
Eq. (3.9) has a first integral that we can write as

$$U_\eta^2 = -4S^2U^4 + K_1U^3 + KU^2 + K_2U + K_3$$  \hspace{1cm} (3.10)

where $K$ is an integration constant, and the constants $K_1, K_2$ and $K_3$ are related to the coefficients $S, p, q$ and $m$ in (3.9).

In this article we restrict ourselves to solutions of the Landau-Lifshitz equation that are obtained by solving (3.10).

We shall first discuss solutions of (3.10) in general, then run through algebras $A_{2,1}, A_{2,2}, \ldots, A_{2,6}$ and specify the values of the coefficients in (3.10) in each case, as well as the independent variable $\eta$.

Algebra $A_{2,1}$ leading to a first order equation, will be treated separately.

3.2. Solutions of the elliptic function equation

We shall call (3.10) the “elliptic function equation”. Its solutions are of course well known\textsuperscript{[19]}. We shall however list those that are relevant in the context of solving (3.9), and more importantly, the Landau-Lifshitz equation.

Several comments are in order here:

1. The functions $R(\eta)$ must be real (and nonnegative), hence $U(\eta)$ must be real and nonpositive.

2. For $S \neq 0$ the coefficient of the highest power of $U$ in (3.10) is nonnegative. This means that all real solutions of (3.10) are nonsingular.

3. For $S = 0, K_1 \neq 0$ in (3.10) the real solutions of (3.10) can be singular. Since we are really interested in the fields $\phi_i$ we note that singular solutions of $U$ will give regular functions $\phi_i$.

4. In general, equation (3.10) is solved in terms of Jacobi elliptic functions. However, these reduce to elementary functions whenever the polynomial on the right hand side has multiple roots, or when $S = K_1 = 0$. 
Let us run through individual cases.

I. \( S \neq 0 \)

We rewrite (3.10) as

\[
U_\eta^2 = -4S^2(U - U_1)(U - U_2)(U - U_3)(U - U_4) \tag{3.11}
\]

1. \( U_1 \leq U \leq U_2 = U_3 = U_4 < 0 \)

\[
U(\eta) = U_2 - \frac{U_2 - U_1}{1 + S^2(U_2 - U_1)^2(\eta - \eta_0)^2} \tag{3.12}
\]

this is an algebraic solitary wave, equal to \( U_2 \) for \( \eta \to \pm \infty \), and dipping down to \( U_1 \) for \( \eta = \eta_0 \).

2. \( U_1 = U_2 = U_3 < U \leq U_4 \leq 0 \)

\[
U(\eta) = U_1 + \frac{U_4 - U_1}{1 + S^2(U_4 - U_1)^2(\eta - \eta_0)^2} \tag{3.13}
\]

Also an algebraic solitary wave, rising to \( U = U_4 \) for \( \eta = \eta_0 \), equal to \( U_1 \) for \( \eta \to \pm \infty \).

3. \( U_1 \leq U < U_2 = U_3 < U_4, \ U_2 \leq 0 \)

\[
U(\eta) = U_2 - \frac{(U_4 - U_2)(U_2 - U_1)}{(U_4 - U_1) \cosh^2 \mu(\eta - \eta_0) - (U_2 - U_1)} \tag{3.14}
\]

\[ \mu = S \sqrt{(U_4 - U_2)(U_2 - U_1)} \]

4. \( U_1 < U_2 = U_3 < U \leq U_4 \leq 0 \)

\[
U(\eta) = U_3 + \frac{(U_3 - U_1)(U_4 - U_3)}{(U_4 - U_1) \cosh^2 \mu(\eta - \eta_0) - (U_4 - U_3)} \tag{3.15}
\]

with \( \mu \) as in (3.14).

The last two solutions are solitons, the first one a well, the second a bump.
5. \( U_1 \leq U \leq U_2 < U_3 = U_4, \ U_2 \leq 0 \)

\[
U(\eta) = U_4 - \frac{(U_4 - U_2)(U_4 - U_1)}{(U_2 - U_1) \sin^2 \mu(\eta - \eta_0) + U_4 - U_2}
\]

\[
\mu = S \sqrt{(U_4 - U_2)(U_4 - U_1)}
\]

(3.16)

6. \( U_1 = U_2 < U_3 \leq U \leq U_4 \leq 0 \)

\[
U(\eta) = U_1 + \frac{(U_4 - U_1)(U_3 - U_1)}{(U_4 - U_3) \sin^2 \mu(\eta - \eta_0) + U_3 - U_1}
\]

\[
\mu = S \sqrt{(U_4 - U_1)(U_3 - U_1)}
\]

(3.17)

7. \( U_1 \leq U \leq U_2 < U_3 < U_4, \ U_2 \leq 0 \)

\[
U(\eta) = U_4 - \frac{(U_4 - U_2)(U_4 - U_1)}{(U_2 - U_1) \sin^2 \mu(\eta - \eta_0), k) + U_4 - U_2}
\]

\[
\mu = S \sqrt{(U_4 - U_2)(U_3 - U_1)}, \quad k^2 = \frac{(U_4 - U_3)(U_2 - U_1)}{(U_4 - U_2)(U_3 - U_1)}
\]

(3.18)

8. \( U_1 < U_2 < U_3 \leq U \leq U_4 < 0 \)

\[
U(\eta) = U_1 + \frac{(U_4 - U_1)(U_3 - U_1)}{(U_4 - U_3) \sin^2 \mu(\eta - \eta_0) + U_3 - U_1}
\]

(3.19)

with \( k^2 \) and \( \mu \) as in (3.18).

9. \( U_1 \leq U \leq U_2 \leq 0, \ U_3,4 = p \pm iq, \ q > 0 \)

\[
U(\eta) = \frac{(MU_1 - NU_2) \cn(\mu(\eta - \eta_0), k) + MU_1 + NU_2}{(M - N) \cn(\mu(\eta - \eta_0), k) + M + N}
\]

\[
M^2 = (U_2 - p)^2 + q^2, \quad N^2 = (U_1 - p)^2 + q^2
\]

\[
k^2 = \frac{(U_2 - U_1)^2 - (M - N)^2}{4MN}, \quad \mu = 2S \sqrt{MN}
\]

(3.20)

Solutions (3.16), \ldots, (3.20) are periodic. All the elementary solutions can be viewed as limits of solutions (3.18), (3.19) and (3.20).
II. $S = 0, \ K_1 \neq 0$

Set

$$\mu = \frac{1}{2} \sqrt{|K_1|(U_3 - U_1)}$$

(3.21)

1. $K_1 < 0, \ U_1 = U_2 < U \leq U_3 \leq 0$

$$U = U_3 - (U_3 - U_2) \tanh^2 \mu (\eta - \eta_0)$$

(3.22)

2. $K_1 < 0, \ U < U_1 = U_2 < U_3, U_1 \leq 0$

$$U = U_3 - \frac{(U_3 - U_1)}{\tanh^2 \mu (\eta - \eta_0)}$$

(3.23)

3. $K_1 < 0, \ U < U_1 = U_2 = U_3 \leq 0$

$$U = U_1 - \sqrt{\frac{2}{-K_1 (\eta - \eta_0)^2}}$$

(3.24)

4. $K_1 < 0, \ U \leq U_1 < U_2 = U_3, U_1 \leq 0$

$$U = U_3 - \frac{U_3 - U_1}{\sin^2 \mu (\eta - \eta_0)}$$

(3.25)

5. $K_1 < 0, \ U_1 < U_2 < U \leq U_3 \leq 0$

$$U = U_3 - (U_3 - U_2) \sn^2(\mu (\eta - \eta_0), k), \quad k^2 = \frac{U_3 - U_2}{U_3 - U_1}$$

(3.26)

6. $K_1 < 0, \ U \leq U_1 < U_2 < U_3, U_1 \leq 0$

$$U = U_3 - \frac{U_3 - U_1}{\sn^2(\mu (\eta - \eta_0), k)}$$

(3.27)

$k$ as in (3.26)
7. $K_1 > 0, U_1 < U < U_2 = U_3 = 0$

$$U = U_1 \frac{1}{\cosh^2 \mu(\eta - \eta_0)}$$ (3.28)

8. $K_1 > 0, U_1 < U < U_2 < 0 < U_3$

$$U = (U_2 - U_1) \text{sn}^2(\mu(\eta - \eta_0), k), \quad k^2 = \frac{U_2 - U_1}{U_3 - U_1}$$ (3.29)

9. $K_1 < 0, U_1 \leq 0, U_{2,3} = p \pm iq, q > 0$

$$U = U_1 + A - \frac{2A}{1 - \text{cn}(\mu(\eta - \eta_0), k)}$$

$$A^2 = (p - U_1)^2 + q^2, \quad k^2 = \frac{A - p + U_1}{2A}, \quad \mu = \sqrt{|K_1|A}$$ (3.30)

III. $S = 0, K_1 = 0, K \neq 0$

1. $K > 0, U \leq U_1 < 0 < U_2$

$$U = U_1 - (U_2 - U_1) \sinh^2 \frac{\sqrt{|K|}}{2}(\eta - \eta_0)$$ (3.31)

2. $K > 0, U < U_1 = U_2 = 0$

$$U = -\exp(-\sqrt{|K|}(\eta - \eta_0))$$ (3.32)

3. $K < 0, U_1 < U < U_2 \leq 0$

$$U = U_1 + (U_2 - U_1) \cos^2 \frac{\sqrt{-K}}{2}(\eta - \eta_0)$$ (3.33)

IV. $S = K_1 = K = 0, K_2 \neq 0$

$$U = -\frac{K_3}{K_2} + \frac{K_2}{4}(\eta - \eta_0)^2, \quad K_2 < 0, \quad K_3 < 0$$ (3.34)

V. $S = K_1 = K = K_2 = 0$

$$U = \sqrt{K_3}(\eta - \eta_0), \quad K_3 > 0$$ (3.35)
3.3. INDIVIDUAL REDUCTIONS

1. Algebra $A_{2.1}$.

This is an exceptional case when (3.2) implies $R_\xi = 0$. We find that the only solution for $W$ of (1.4) is

$$W = R_0 e^{iQ}, \quad Q = ax + by + \left( B + \frac{1 - R_0^2}{1 + R_0^2} (a^2 + b^2 + A) \right) t + \alpha_0$$

(3.36)

where $R_0$ and $\alpha_0$ are integration constants.

2. Algebra $A_{2.2}$.

We find

$$W = R(\rho) \exp i[\alpha(\rho) + a\phi + bt], \quad \xi = \rho$$

(3.37)

where $\rho$ and $\phi$ are polar coordinates. The singlevaluedness of $W$ requires $a$ to be an integer. The phase $\alpha(\rho)$ and variable $\eta$ satisfy

$$\alpha_\rho(\rho) = \frac{S(1 + R^2)^2}{\rho R^2}, \quad \eta = \ln \rho$$

(3.38)

(see (3.6)). For the function $U(\eta)$ of (3.8) we obtain the elliptic function equation if and only if we set

$$A = 0, \quad b = B$$

(3.39)

$(A$ and $B$ are defined in (1.6)).

We have

$$K_1 = K_2 = 2a^2 + 4S^2 - \frac{K}{2}, \quad K_3 = -4S^2$$

(3.40)

in (3.10).
For $S \neq 0$ eq. (3.40) implies that we can have two negative and two positive roots in eq. (3.11) or two negative roots and two complex conjugate ones. These cases lead to real solutions, namely (3.16), (3.18) and (3.20). Note that all of them are periodic. In particular, for $a = 0$ eq. (3.11) always has a double root $U_3 = U_4 = 1$ and reduces to

$$U^2 = -4S^2[U^2 + (1 + \frac{K}{8S^2})U + 1]$$

(3.41)

For $S = 0$, $K \neq 4a^2$ we obtain the equation

$$U^2 = 2(a^2 - \frac{K}{4})U(U - U_1)(U - U_2),$$

$$U_1U_2 = 1, \quad U_1 + U_2 = \frac{2K}{K - 4a^2}.$$  

(3.42)

The relevant solutions of (3.42) in this case are:

1. $2a^2 \leq K < 4a^2, U_1 < U_2 < 0$
   solution (3.29) (with $U_3 = 0$). For $K = 2a^2$ we have $U = -1$.

2. $K > 4a^2, U_1 \leq U_2 < 0$
   Solutions (3.26), (3.27), (3.22) and (3.23) (all with $U_3 = 0$).

3. $K > 4a^2, 0 < U_1 < U_2$
   Solutions (3.26) (with $U_3 \to U_2, U_2 \to U_1, U_1 = 0$).

4. $K > 4a^2, U_{1,2} = p \pm iq, q > 0$ solution (3.30).

For $S = 0, K = 4a^2$ (3.10) reduces to an elementary one and its solution is

$$R(\rho) = -R_0^2\rho^{\pm 2a}$$

(3.43)

where $R_0$ is an integration constant.

3. Algebra $A_{2,3}$ and $A_{2,4}$
The reduction formulas in both of these cases are

\[ W = R(\xi) \exp i[\alpha(\xi) - at - by], \quad \xi = x + vt, \quad \eta = \xi \quad (3.44) \]

with \( v = 0 \) and \( v \neq 0 \) for the algebras \( A_{2,3} \) and \( A_{2,4} \) respectively. Since the Landau-Lifshitz equation is not Galilei invariant, we cannot change the value of \( v \) by a group transformation. The transformation (3.8) leads to (3.10) with

\[
\begin{align*}
K_1 &= -\frac{K}{2} + 4S^2 + 2(A + b^2) - \frac{v^2}{2} + 2vS + 2(-B + a) \\
K_2 &= -\frac{K}{2} + 4S^2 + 2(A + b^2) + \frac{3v^2}{2} - 6vS + 2(B - a) \\
K_3 &= -(2S - v)^2. \\
\end{align*}
\quad (3.45)
\]

Eq. (3.6) in this case gives

\[ \alpha_\xi = S \frac{(1 + R^2)^2}{R^2} \quad (3.46) \]

For \( S \neq 0 \) we obtain (3.11) with the constraint

\[ U_1U_2U_3U_4 = \left(1 - \frac{v}{2S}\right)^2 \quad (3.47) \]

imposed on these roots. Hence, only even number of roots can be negative (0, 2 or 4). This however means that all solutions (3.12) . . . (3.20) can occur, though in some cases we must impose \( U_4 \leq 0 \) (\( U_4 = 0 \) is allowed for \( v = 2S \)).

For \( S = 0 \), all solutions (3.22) , . . . , (3.30) can occur.

4. Algebra \( A_{2,5} \)

The reduction formula is

\[ W = R(\xi) \exp i[\alpha(\xi) + \frac{a}{\bar{b}} \phi + Bt], \quad \xi = \ln \rho + \frac{1}{b} \phi \quad (3.48) \]

and we must set \( A = 0 \) in the Landau-Lifshitz equation. The phase \( \alpha(\xi) \) satisfies

\[ \alpha_\xi = S \frac{(1 + R^2)^2}{R^2} - \frac{a}{b^2 + 1} \quad (3.49) \]

(see (3.6) ).
The function $U = -R^2(\xi)$ satisfies (3.10) and for $S \neq 0$ we have

$$K_1 = K_2 = -\frac{K}{2} + 4S^2 - \frac{2a^2}{(b^2 + 1)^2}, \quad K_3 = -4S^2. \quad (3.50)$$

The solutions that can occur in this case are (3.14), . . . , (3.20). However, there are constraints between various parameters of the solution which follow from the requirement of singlevaluedness of $W$.

For $a = 0$ eq. (3.11) again reduces to (3.41) and we only obtain the elementary periodic solutions (3.16).

For $S = 0$, $K_1 \neq 0$ we obtain the equation

$$U_0^2 = K_1 U(U - U_1)(U - U_2),$$

$$U_1 U_2 = 1, \quad U_1 + U_2 = 2\left[1 + \frac{2a^2}{K_1(b^2 + 1)}\right]. \quad (3.51)$$

For $-\frac{a^2}{b^2 + 1} < K_1 < 0$ we have $U_1 \leq U_2 < 0$ and solutions (3.22), (3.23), (3.26) and (3.27) are obtained.

For $K_1 > 0$ we have $0 < U_1 \leq U_2$ but we obtain no real solutions.

For $K_1 < -\frac{a^2}{(b^2 + 1)}$ we obtain solutions (3.30).

Finally, for $S = 0$, $K_1 = 0$ the solution is $U = -R_0^2 \exp(\pm\sqrt{K}\xi)$ and hence

$$R = R_0 \exp\left[\pm\frac{1}{2}\sqrt{K}(\ln \rho + \frac{1}{b}\phi)\right] \quad (3.52)$$

For $S = 0$, $K_1 \neq 0$ equation (3.10) reduces to

$$U_0^2 = K_1 U(U^2 + \frac{K}{K_1} U + 1), \quad K_1 = -\frac{K}{2} - \frac{2a^2}{(b^2 + 1)^2} \quad (3.53)$$

Real solutions are obtained only for $K_1 < 0$. More specifically, solutions (3.27) and (3.30) can occur for any $K_1 < 0$. Solution (3.26) for $K_1$ in the range $-\frac{2a^2}{(b^2 + 1)^2} \leq K_1 < 0$, (3.22) for $K_1 = -\frac{a^2}{(b^2 + 1)^2}$ and (3.23) either for $a = 0$, or $K_1 = -\frac{a^2}{(b^2 + 1)^2}$. 

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5. Algebra $A_{2,6}$

The reduction formula is

$$W = R(\xi) \exp i[\alpha(\xi) + Bt + a \ln x], \quad \xi = \frac{y}{x}, \quad \eta = \arctan \frac{y}{x} = \phi$$

(3.54)

and $\alpha$ satisfies

$$\alpha \xi = \frac{S}{1 + \xi^2} \frac{(1 + R^2)^2}{R^2} + \frac{a \xi}{1 + \xi^2}$$

(3.55)

and $U(\phi) = -R^2(\xi)$ satisfies (3.11) with $K_1 = K_2 = -\frac{K}{2} + 4S^2 + 2a^2$, $K_3 = -4S^2$.

For $a = 0$ the equation again reduces to (3.41).

A real solution is obtained only for $K > 8S^2$, namely (3.16). It is periodic in $\phi$ and hence singlevalued when $\mu$ is an integer.

For $S = 0$ we get a real solution for $K_1 < 0$, namely solution (3.25) which in this case reduces to

$$U(\phi) = -\tan^2 \frac{1}{2} \sqrt{|K_1|}(\phi - \phi_0).$$

(3.56)

This is a singlevalued function whenever $\sqrt{|K_1|}$ is an integer.

6. Algebras $A_{2,7}$ and $A_{2,8}$

The corresponding reductions lead to equations that we cannot decouple without introducing higher derivatives, so we will not treat them here.
4. Solutions of the Nonlinear Diffusion Equation

4.1. General Procedure

Let us consider the system (1.7) and (1.8), the NDLE for short. We impose that the solution be invariant under one of the Lie groups generated by the algebra in eq. (2.7). The functions \( R \) and \( Q \) will then have the form (3.1) with \( \xi \) and \( \beta \) as in Section 3 (different for each subalgebra \( A_{2,1}, \ldots, A_{2,8} \)).

As for the LL equation, algebra \( A_{2,1} \) must be treated separately.

For \( A_{2,2}, \ldots, A_{2,8} \) we always have \( (\nabla \xi)^2 \neq 0 \). Eq. (1.7) and (1.8) reduce to

\[
\alpha_{\xi \xi} = -2 \frac{(1 - R^2) R_\xi}{1 + R^2} R \alpha_\xi + \frac{f_\xi}{f} \alpha_\xi - 2f m \frac{(1 - R^2) R_\xi}{1 + R^2} + hf
\]

\[
R_{\xi \xi} = \frac{f_\xi}{f} R_\xi + \frac{2R}{1 + R^2} R^2_\xi + \frac{1 - R^2}{1 + R^2} R [\alpha_\xi^2 + 2m f \alpha_\xi + g^2] + \frac{1}{(\nabla \xi)^2} [BR + AR \frac{1 - R^2}{1 + R^2}]
\]

The functions \( f(\xi), m(\xi), h(\xi) \) and \( g(\xi) \) are defined by the relations

\[
\frac{f_\xi}{f} = \frac{\xi_t - \Delta \xi}{(\nabla \xi)^2}, \quad m = \frac{(\nabla \xi, \nabla \beta)}{(\nabla \xi)^2 f}, \quad h = \frac{\beta_\xi - \Delta \beta}{(\nabla \xi)^2 f}, \quad g^2 = \frac{(\nabla \beta)^2}{(\nabla \xi)^2}
\]

In order to decouple equations (4.1) and (4.2), we impose a restriction on the functions defined in (4.3) namely

\[
m_\xi + h = 0
\]

Eq. (4.1) can then be integrated once to give

\[
\alpha_\xi = [S \frac{(1 + R^2)^2}{R^2} - m] f(\xi)
\]
where $S$ is an integration constant. We substitute (4.5) into (4.2) and put

$$R(\xi) = \sqrt{-U(\eta)}, \quad \eta = \int f(\xi) d\xi. \quad (4.6)$$

The equations are decoupled and the one for $U(\eta)$ is already in a standard form\textsuperscript{[20]} namely

$$U_{\eta\eta} = \left(\frac{1}{2U} + \frac{1}{U-1}\right)U^2 + 2S^2(1+U)(1-U) + M\frac{1+U}{1-U}U + NU \quad (4.7)$$

with

$$M = \frac{2}{f^2} \left(g^2 - m^2 f^2 + \frac{A}{(\nabla \xi)^2}\right)$$

$$N = \frac{2}{f^2} \frac{B}{(\nabla \xi)^2}. \quad (4.8)$$

We now make a further restriction, namely, that $M$ and $N$, defined in (4.8) are constants. Eq. (4.7) then has a first integral $K$ and we obtain the elliptic equation (3.10) with

$$K_1 = -\frac{K}{2} + 4S^2 + M + N$$

$$K_2 = -\frac{K}{2} + 4S^2 + M - N$$

$$K_3 = -4S^2. \quad (4.9)$$

In many cases we have $M = N = 0$ and the polynomial on the right hand side of (3.10) has a double root at $U_3 = U_4 = 1$. The solution we obtain for $S \neq 0$ is (3.16) with $U_4 = 1$ i.e:

$$U(\eta) = 1 - \frac{(1-U_2)(1-U_1)}{(U_2-U_1)\sin^2 \mu(\eta - \eta_0) + 1 - U_2} \quad (4.10)$$

$$\mu = S\sqrt{(1-U_2)(1-U_1)}, \quad U_1 \leq U \leq U_2 < 0$$

For $S = 0$, $K > 0$ we obtain solution (3.25) i.e

$$U = -\tan^2 \sqrt{\frac{K}{8}(\eta - \eta_0)} \quad (4.11)$$
4.2. Individual reductions

1. Algebra $A_{2,1}$

We have

\[ R = R(t), \quad Q = \alpha_0 + ax + by \quad (4.12) \]

where $\alpha_0$ is a constant and $R_t$ satisfies:

\[ R_t = -\frac{R(1 - R^2)}{1 + R^2}(A + a^2 + b^2) - BR \quad (4.13) \]

Equation (4.13) can easily be integrated (differently depending on whether $(A + B + a^2 + b^2)(A - B + a^2 + b^2)$ vanishes, or not) and we obtain a transcendental equation for $R(t)$.

2. Algebra $A_{2,2}$

The reduction formula is (3.37). We have $m = 0$ and (4.4) requires $b = 0$ so the solutions are static ones. The variable $\eta$ and constants involved satisfy

\[ \eta = \ln \rho, \quad M = 2a^2 \]
\[ A = B = N = b = 0 \quad (4.14) \]

Since we have $b = 0$, the solutions are static ones. All solutions (3.12), \ldots, (3.30) can occur.

3. Algebra $A_{2,3}$

We have

\[ \eta = x, \quad M = 2(b^2 + A), \quad N = 2B, \quad a = 0 \quad (4.15) \]

and again, the solutions are static ones, since the reduction formula is (3.44). All solutions of Section 3.2 can occur.
4. Algebra $A_{2,4}$

The reduction formula is (3.44) and we have

$$\eta = \frac{1}{v}(e^{v(x+vt)} - 1), \quad a = 0, \quad A = -b^2, \quad B = 0$$

$$M = N = 0, \quad v \neq 0$$

(4.16)

The obtained solutions are (4.10) and (4.11) and they are $t$-dependent.

5. Algebra $A_{2,5}$

We have eq. (3.48) with

$$\eta = \xi = \ln \rho + \frac{1}{b} \phi, \quad A = B = N = 0,$$

$$M = \frac{2a^2b^2}{(b^2+1)^2}, \quad b \neq 0$$

(4.17)

Since we have $M \geq 0$ we obtain the solutions (3.16), (3.18), (3.20), (3.22), (3.23), (3.26), (3.27) and (3.30). The parameters of these solutions must satisfy, however, certain constraints in order for the solutions to be single-valued. For $K = 2M$ we obtain

$$U = -R_0^2 \rho^{\frac{2al}{b^2+1}} e^{\frac{2al}{b^2+1} \phi}$$

(4.18)

a solution that is not singlevalued.

6. Algebra $A_{2,6}$

The reduction formula is (3.54) and we have

$$\eta = \int \frac{d\xi}{1 + \xi^2} = \phi, \quad A = B = a = M = N = 0$$

(4.19)

so the relevant solutions are (4.10) and (4.11) The solutions are static and they are singlevalued for $\mu$ or $\sqrt{K/8}$ being integers.
7. Algebra $A_{2,7}$

We put
\[ W = R(\xi) \exp i[\alpha(\xi) + \frac{a}{2} \ln t + b\phi] \]
\[ \xi = \frac{x^2 + y^2}{t}, \quad \eta = \int \frac{1}{\xi} e^{-\frac{\xi}{4}} d\xi = Ei\left(-\frac{1}{4}\xi\right) \] (4.20)

where $Ei(x)$ is the exponential integral function. Moreover, we have $b = a = A = B = M = N = 0$ and the relevant solutions are (4.10) and (4.11)

8. Algebra $A_{2,8}$

We have
\[ W = R(\xi) \exp i[\alpha(\xi) + \frac{a}{2} \ln t], \quad \xi = \frac{x}{\sqrt{t}} \]
\[ \eta = \int e^{-\xi^2/4} d\xi = \sqrt{\pi} \Phi(\xi) \] (4.21)

where $\Phi(x)$ is the probability integral. We have $M = N = A = B = a = 0$ and so we obtain solutions (4.10) and (4.11)

We see that time-dependent solutions are obtained for the algebras $A_{2,1}, A_{2,4}, A_{2,7}$ and $A_{2,8}$. For $A_{2,5}, A_{2,7}$ and $A_{2,8}$ the solutions are trigonometric ones.

The phases $\alpha(\xi)$ can be calculated by direct integration, since we have

\[ \frac{d\alpha}{d\eta} = -S \frac{(1 - U)^2}{U} \] (4.22)

and $U$ is already known.

Thus for $U$ given by (4.10) we get

\[ \alpha(\eta) = \sqrt{\frac{(U_1 - 1)(U_2 - 1)}{U_1 U_2}} \arctan\left\{ \sqrt{\frac{U_2(U_1 - 1)}{U_1(U_2 - 1)}} \cot \mu(\eta - \eta_0) \right\} \]
\[ - \sqrt{(U_1 - 1)(U_2 - 1)} \arctan\left\{ \sqrt{\frac{(U_1 - 1)}{(U_2 - 1)}} \cot \mu(\eta - \eta_0) \right\} + \alpha_0, \] (4.23)

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while for $U$ given by (4.11) we have

$$\alpha = 4\sqrt{\frac{2}{K}} S \cot\{\sqrt{2K}(\eta - \eta_0)\} + \alpha_0. \quad (4.24)$$

5. Conclusions

The Landau-Lifshitz equation (1.1) has received quite a bit of previous attention, mainly in the context of continuum Heisenberg ferromagnetic spin systems \cite{21} \cite{22} \cite{23} \cite{24} \cite{25} \cite{26} \cite{27}. The anisotropy coefficients $A$ and $B$ of eq. (1.3) were usually set equal to zero. Use was made of the fact that eq (1.1) is integrable, at least in the one-dimensional case, or in the two-dimensional, spherically symmetric one.

A one-soliton solution has been obtained \cite{23} \cite{25} a radially symmetric one. In our variables $R, Q$ of eq. (1.5), (1.6) this solution corresponds to

$$R = \frac{4t + \alpha_1}{4t + \alpha_1 + \cosh^2\left[\frac{4t + \alpha_1}{(4t + \alpha_1)^2 + \alpha_2^2}(x^2 + y^2)\right]} \quad (5.1)$$

where $\alpha_1$ and $\alpha_2$ are arbitrary real constants. As noted by Lakshamanan and Porsezian \cite{25} the soliton spreads in time.

The solution (5.1) is not among the invariant solutions obtained in this article, nor can it be obtained from such a solution by applying transformations from the symmetry group. As often happens \cite{28}, the method of symmetry reduction that does not rely on integrability, provides different solutions for integrable equations, than the use of Lax pairs, or Backlund transformations.

We note that eq. (1.1) with $A \neq 0$ is not integrable.

In general, we have reduced the LL equation to the ordinary differential equation (3.9). We have integrated eq. (3.9) in terms of elliptic functions whenever $p, q$ and $m$ are constants. For the algebras $A_{2,3}, \ldots, A_{2,6}$ this was always the case.
For algebra $A_{2,2}$ we obtained eq. (3.10) only for $A = 0, B = b$. Let us briefly consider the case when the anisotropy coefficient $A$ does not vanish. We then return to the original variable $\xi = \rho = \sqrt{x^2 + y^2} = \exp \eta$ and transform eq. (3.9) into

$$U_{\rho\rho} = \left( \frac{1}{2U} + \frac{1}{U-1} \right) U^2 - \frac{1}{\rho} U_\rho + \frac{2S^2}{\rho^2} (U-1)^2 (-U + \frac{1}{U}) - \frac{2(a^2 + A^2 \rho^2)}{\rho^2} + 2(B - b)$$

(5.2)

For $a = 0, b = B$ this is the equation for the fifth Painlevé transcendent $P_{V}$ [20]. However, for $B \neq b$ eq (5.2) does not have the Painlevé property. According to the Painlevé conjecture [29][30], eq. (1.1) is hence, in general, not integrable.

This has not stopped us from obtaining numerous solutions, both in integrable and nonintegrable cases. The algebra $A_{2,2}$ (cylindrical symmetry) for $A = 0, B = b$ leads to periodic solutions, as discussed in Section 3.3. The periodicity is in the radial variable $\rho$. The time dependence is restricted to the phase $Q$, as is seen in eq. (3.37). Moreover the time-dependence is entirely due to the presence of the external field $B$ (we have $b = B$) that generates a rotation between the components $\phi_1$ and $\phi_2$ of the original vector $\vec{\phi}$.

Some elementary nonperiodic solutions that we can extract from Section 3 are

$$R^2 = \frac{U_1 - U_2 S^2 (U_2 - U_1 (\ln \rho / \rho_0)^2}{1 + S^2 (U_2 - U_1)^2 (\ln \rho / \rho_0)^2}$$

(5.3)

$$R^2 = \frac{4R_0^2 U_1(U_2 - U_1) - U_2(U_4 - U_1)[2R_0^2 + \rho^2\mu + R_0^4 \rho^{-2\mu}] - 4(U_2 - U_1)R_0^2}{(U_4 - U_1)[2R_0^2 + \rho^2\mu + R_0^4 \rho^{-2\mu}] - 4(U_2 - U_1)R_0^2}$$

(5.4)

with $S, R_0, \rho_0, U_i$ constants and

$$Q = S \int \frac{(1 + R^2)^2}{\rho R^2} d\rho + a\phi + Bt$$

(5.5)

in both cases.
For $S = 0$ we have for instance

$$R = [-U_1 + \sqrt{\frac{2}{-K_1} (ln \frac{\rho}{\rho_0})^{-1}}]^{1/2}$$  \hspace{1cm} (5.6)

$$R = \frac{2\sqrt{-U_1 R_0}}{\rho^\mu + R_0^2 \rho^{-\mu}}$$  \hspace{1cm} (5.7)

with

$$Q = a\phi + Bt + Q_0$$  \hspace{1cm} (5.8)

For $A \neq 0$, as mentioned above, solutions are obtained in terms of $P_V(\rho)$. Their time dependence is again given by the term $Bt$ in the phase $Q$.

For algebras $A_{2,3}$ and $A_{2,4}$ we obtain eq. (3.10) and a multitude of explicit solutions for all values of $a, b, A$ and $B$. Note that for

$$S \neq 0, \quad B = a, \quad A = -b^2$$  \hspace{1cm} (5.9)

in particular for the one dimensional ($b = 0$), static ($a = 0$) with no external fields ($A = B = 0$), two of the roots in eq. (3.11) coincide and the equation reduces to

$$U_2^2 = -4S^2(U - 1)^2[U^2 + (-\frac{K_1}{4S^2} + 2)U - \frac{K_3}{4S^2}]$$  \hspace{1cm} (5.10)

Eq. (5.10) only allows elementary solutions like (3.12), ...(3.17), not however the elliptic function ones. These occur when the fields $A$ and $B$ are such that (5.9) is not satisfied.

To our knowledge, the NLDE (1.2) has not been investigated from the point of view of its integrability and we have no solutions to compare ours to.

We have derived many explicit exact solutions of both equations. Looking at them we note that most of them have infinite energy. They can describe coherent phenomena in various solid state and condensed matter applications.
Looking first at the solutions of the LL equation we note that some of our solutions have finite energy. In particular, this is the case for (3.43). This solution is obtained from the familiar static solution describing \( n \) solitons “on top of each other.” Its time dependence is given by the factor \( e^{iBt} \) which thus describes a rotation of this static solution in the \( \phi_1, \phi_2 \) plane with the angular frequency given by the anisotropy \( B \). The other solutions of this class correspond to the static elliptic solutions discussed in ref [17] again rotated by \( e^{iBt} \).

The solutions corresponding to algebras \( A_{2,1}, A_{2,3} \) and \( A_{2,4} \) have infinite energies. As such, they describe various waves in the medium (generalisations of plane waves). These can for instance be spin waves; the energy per period is finite.

An interesting class of solutions are those corresponding to algebras \( A_{2,5} \) and \( A_{2,6} \). Given the choice of parameters, these solutions can be of finite energy; however, due to their dependence on the variable \( \phi = \arctan \frac{y}{x} \), they may become singular when \( x \) and \( y \) vanish. They can be used to describe media with defects.

Most of the comments made above apply also to the solutions of the NLDE. The static solutions in both cases are of course the same. When we consider non-static solutions, the most interesting from the physical point of view, are perhaps solutions corresponding to algebras \( A_{2,1}, A_{2,7} \) and \( A_{2,8} \). All of them have infinite energies. The solution corresponding to \( A_{2,1} \) describes a structure shrinking towards the origin (or expanding to infinity - depending on the values of the parameters). The other solutions describe field configurations evolving in time. They can be used in the description of some physical phenomena in condensed matter or solid state physics.

Among the questions that we plan to return to, we mention the study of ”partially invariant” solutions\(^{[32]}\)\(^{[33]}\)\(^{[34]}\) of eq. (1.1)and (1.2), and also ”conditionally invariant” ones\(^{[35]}\). A study of solutions involving Painlevé transcendents is also warranted.
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