A parallel-plate waveguide antenna radiating through a perfectly conducting wedge

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Abstract
Parallel-plate waveguide antennas hosted in an electrically perfectly conducting wedge are studied. The analysis relies on an exact singular integral equation formulation. Asymptotic extraction techniques are employed to recast the kernel functions as the sum of two terms: one singular in closed form and another having the form of either a rapidly converging eigenfunctions expansion or a computationally efficient diffraction integral. The obtained integral equation is discretised in the framework of the Nyström method using rapidly converging quadratures that take into account both the singularities of the kernels and the geometrical singularities at the edges. Numerical examples and case studies demonstrate the high accuracy and efficiency of the proposed algorithms. The analysis covers the canonical problem of a parallel-plate waveguide antenna backed on a perfectly conducting wedge as a special case.

1 | INTRODUCTION

Flanged waveguide antennas usually are taken to radiate through slots on an infinite ground plane. The infinite-ground plane hypothesis is convenient as it facilitates the formulation of the relevant boundary value problem as an integral equation. However, it may be inadequate because it ignores truncation effects that can significantly affect the antenna performance. The impact of the finite-size ground plane on the radiation characteristics of rectangular and circular aperture antennas was studied in [1] using geometrical optics and uniform theory of diffraction methods.

To include the truncation effects into the analysis, in this article, we consider parallel-plate waveguide antennas with plate separation \( a \) hosted in a perfectly electrically conducting (PEC) wedge. The primary excitation is assumed to be a TM mode propagating in the waveguide region and incident on the formed radiation aperture. The boundary value problem is formulated in the framework of the field equivalence principle (Section 2) as a singular integral equation of the form

\[
\int_0^a [K_1(x, x') \ln|x - x'| + K_2(x, x')] M(x') dx' = y(x), \quad 0 < x < a
\]

having the equivalent surface magnetic current density \( M(x) \) across the slot as the unknown. The kernel of the integral equation is a linear combination of the wedge and the parallel-plate waveguide Green's functions.

Nevertheless, conventional wedge and parallel-plate waveguide Green's functions having the form of eigenfunction expansions exhibit slow convergence near a source point because of the singularity of the Green's functions at source points. To properly handle the kernel functions, we proceed as follows. First, we split the wedge Green's function into two parts, one containing the singularity and another having the form of a diffraction integral [2, 3] computable with high precision by algorithms of [4, 5]. Next, for the waveguide Green's function, in Section 3, we apply a suitable asymptotic technique [6, 7] to recast it as the sum of two parts, one singular in closed form and another having the form of a rapidly converging series expansion.

Using the new, improved form of the kernels, in Section 4, we discretise the integral equation in the context of the Nyström method. To achieve this, we apply, independently, an exponentially converging modified trapezoidal rule and a high-accuracy Gauss-type quadrature, both taking full account of the singularities of the kernel and the geometrical singularities at the edges. Numerical results and case studies presented in Section 5 demonstrate the high accuracy and efficiency of the proposed algorithms.
Our analysis includes, as a particular case, the canonical diffraction problem of a PEC half-plane positioned parallel to one of the faces of a perfectly conducting right-angled wedge. To the best of our knowledge, no study has addressed this problem so far.

Two remarks are in order regarding the techniques used in this paper, that fall under the Nyström method.

**Remark 1:** First-kind Fredholm integral equations of the form \( \int_\Omega K(x,x')f(x')dx' = y(x) \) are, in general, ill-posed problems \([8]\) in that small changes in the driving term \( y(x) \) can lead to very large changes to the response \( f(x) \). With regard to the Nyström and Galerkin methods, formally applicable to such equations as well, this instability is manifested by excessive condition numbers in the system matrices. We emphasize, however, that this is not the case with (1). Equation (1) is a logarithmically singular integral equation and, as such, can be converted to equivalent second-kind Fredholm equations by the method of analytical regularisation (MAR) \([9, 10]\). In \([11]\), where further results on connections between first-kind logarithmically singular and second-kind Fredholm equations are contained, it is shown that the Galerkin matrices of a weakly singular integral equation of the type (1) and its second-kind counterpart obtained by the method of analytical regularisation are scaled versions of one another. In other words, numerical methods applied to the initial, first-kind equation and its second-kind counterpart have the same behaviour.

**Remark 2:** In view of the previous remark, Nyström-type discretizations are, in fact, an alternative to MAR-based numerical solutions obtained after reducing the boundary value problem to a Fredholm second-kind matrix equation. As they both generate numerical algorithms with mathematically guaranteed convergence, and their accuracy is easily controlled by the order of discretisation, either of them can and should be used as a reference data source for the other.

The assumed \( \exp(iwt) \) time dependence is suppressed throughout the analysis that follows.

## 2 | Problem Formulation

In Figure 1a, we show the cross section in the \( x-y \) plane of a parallel-plate waveguide antenna hosted in a PEC wedge with faces at \( \varphi = 0 \) and \( \varphi = \psi \) and radiating in the open region \((\varepsilon_1, \mu_1)\) through the aperture \( S \) formed at \( y = 0, b < x < a + b \). Polar coordinates in the \( x-y \) plane are denoted by \((\rho, \varphi)\). The \( z \)-axis lies along the edge of the wedge. The feeding waveguide has plate separation \( a \) and is filled with a medium characterised by \((\varepsilon_\Delta, \mu_2)\). The primary excitation is an incident \( \text{TM}^\circ \) (also \( \text{TE}^\circ \)) mode propagating in the \( \hat{y} \)-direction:

\[
\tilde{H}^{\text{inc}} = 2H_0 \psi_m(x)e^{-\gamma_0 y}, \quad E^{\text{inc}} = \frac{1}{ik_2 Y_2} \nabla \times \tilde{H}^{\text{inc}},
\]

where \( H_0 = 1 / m, k_2 = \omega \sqrt{\varepsilon_\Delta \mu_2}, Y_2 = \sqrt{\varepsilon_\Delta / \mu_2} \),

\[\text{FIGURE 1} \quad \text{(a) Cross section in the } x-y \text{ plane of a parallel-plate waveguide antenna radiating through the upper face of a perfectly conducting wedge; the axis of the wedge coincides with the } z \text{-axis.} \]

\[\text{(b) Parallel-plate waveguide antenna backed on a right-angled PEC wedge}\]

\[\psi_m(x) = \cos \left[ \frac{m \pi}{a} (x - b) \right], \]

and

\[\gamma_m = \left( \frac{m \pi}{a} \right)^2 - k_2^2 \right)^{1/2}, \quad 0 \leq \arg(\gamma_m) \leq \frac{\pi}{2}. \]  

Our aim is to determine the field \((\tilde{E}_1, \tilde{H}_1)\) transmitted in the open space (region 1) and the total field \((\tilde{E}_2, \tilde{H}_2)\) generated within the waveguide (region 2).

The configuration of a PEC half-plane positioned parallel to one of the faces of a perfectly conducting right-angled wedge, shown in Figure 1b, results from that of Figure 1a in the special case when \( b = 0, \psi = 3 \pi / 2 \). As far as we know, this problem has not yet been addressed.

### 2.1 | Fields in the exterior region

Refer to the configuration of Figure 1a. According to the surface equivalence principle \([12]\), the field in region 1 can be obtained by shorting the aperture \( S \) and placing the magnetic surface current along side it, as illustrated in Figure 2a. In (4), \( \tilde{E} \) is the total electric field. By the use of this equivalence, we can obtain the integral representation.
\[ \bar{M}(x) = \bar{E}(x, 0) \times \bar{\gamma} = 2E_x(x, 0) = \bar{M}(x) \]  

(4)

\[
\begin{align*}
\bar{H}_1(\bar{\rho}) &= \frac{1}{ik_1Y_1} \nabla \times \bar{H}_1(\bar{\rho}), \\
\bar{E}_1(\bar{\rho}) &= \frac{1}{ik_1Y_1} \nabla \times \bar{H}_1(\bar{\rho}),
\end{align*}
\]

(5)

where \( C \) is the x-axis interval \([b, b + a]\), \( k_1 = \alpha / \sqrt{\varepsilon_1\mu_1} \), and \( Y_1 = \sqrt{\varepsilon_1 / \mu_1} \). In (5), \( \bar{\mathcal{H}}_{z1}(\bar{\rho}, \bar{\rho}') \) denotes the magnetic field that would have been generated at \( \bar{\rho} \) if a magnetic line source of unit strength had been placed at \( \bar{\rho}' \) with \( S \) short-circuited. For \( \psi > \pi \), which is the case of interest here, and for arbitrary values of \( \rho, \rho' \), we can write

\[
\mathcal{H}_{z1}(\rho; \rho') = \mathcal{H}_{z1}(\rho; \rho') = -\frac{k_1 Y_1}{4} G(\rho, \rho'),
\]

(6)

where \( G(\rho, \rho') \) is the scalar \( H \)-mode Green’s function of the perfectly conducting wedge [2].

**FIGURE 2** (a) Equivalent problem for the field in region 1.
(b) Equivalent problem for the field in region 2. The aperture \( S \) is short-circuited. Magnetic current densities \( \bar{M} \) and \( -\bar{M} \) are imposed on \( S \) at \( y = 0^+ \) and \( y = 0^- \), respectively.

General methods of constructing \( G(\rho, \rho') \) leading to the so-called angular transmission and radial transmission representations are discussed in Section 6.5 of Ref. [2]; variants can be found in [3]. Here, we adopt the following computationally efficient version of the angular transmission representation [3]:

\[
G(\rho, \rho') = G(\rho, \varphi; \rho', \varphi')
\]

\[
= H_0^2(\imath k_1 \mathcal{R}(\varphi'))u(\pi - |\varphi - \varphi'|)
\]

\[
+ H_0^2(\imath k_1 \mathcal{R}(-\varphi'))u(\pi - |\varphi - \varphi'|)
\]

\[
+ H_0^2(\imath k_1 \mathcal{R}(2\varphi - \varphi'))u(\pi - 2\varphi + \varphi + \varphi')
\]

\[
- D(\rho, \rho'),
\]

(7)

where \( H_0^2(\imath) \) is the second kind Hankel function of order 0, \( \mathcal{R}(\imath) \) (the unit step function) is 1 for \( \rho > 0 \), 0 for \( \rho < 0 \), and 1/2 for \( \rho = 0 \), while

\[
\mathcal{R}(\alpha) = \sqrt{\rho^2 + \rho'^2 - 2\rho \rho' \cos(\varphi - \alpha)}^{1/2}.
\]

The term \( D(\rho, \rho') \) in (7) stands for the diffraction integral [3]

\[
D(\rho, \rho') = \int_1^{\infty} Q(\rho, \rho'; \tau)d\tau,
\]

(8)

where

\[
Q(\rho, \rho'; \tau) = \frac{2\tau}{\pi \sqrt{\tau^4 - 1}} H_0^2(\imath k_1 \sqrt{\rho^2 + \rho'^2 + 2\rho \rho' \tau^2})
\]

\[
\times B^+ [\varphi, \varphi'; \tau \cosh^{-1}(\tau^2)]
\]

(9)

with

\[
B^+ (\varphi, \varphi'; \mu) = B(\varphi, \varphi'; \mu) + B(\varphi, -\varphi'; \mu),
\]

(10a)

\[
B(\varphi, \varphi'; \mu) = \frac{\pi}{2 \mu} \frac{\sin \pi (\mu + \varphi - \varphi')}{\cos \varphi - \cos (\mu + \varphi - \varphi')}
\]

\[
+ \frac{\pi}{2 \mu} \frac{\sin \pi (\mu - \varphi + \varphi')}{\cos \varphi - \cos (\mu - \varphi + \varphi')}.
\]

\[
(10b)
\]

2.1.1 | The simple case \( \psi = \pi \)

It is easy to show that formula (7) for the scalar \( H \)-mode Green’s function is consistent with the simple case when the wedge opens up to form a PEC half space. Let \( \psi = \pi \) to form a half space problem. The solution can be easily obtained using images:
\[ G(\bar{\rho}, \bar{\rho}') = G(x, y; x', y') \]
\[ = H_0^{(2)} \left[ k_1 \sqrt{(x-x')^2 + (y-y')^2} \right] + H_0^{(2)} \left[ k_1 \sqrt{(x-x')^2 + (y+y')^2} \right]. \] (11)

On the other hand, for \( \psi = \pi \), it follows from (10b) that \( \mathcal{B}(\psi, \phi'; x); \mathcal{T} = 0 \) after simple algebraic manipulations. By virtue of (9) and (10a), this implies that the diffraction integral \( D(\bar{\rho}; \bar{\rho}') \) in (8) vanishes. In addition,
\[ \mathcal{R}(2\psi - \phi') = \mathcal{R}(2\pi - \phi') = \mathcal{R}(-\phi') \]
and
\[ u(\pi - \phi - \phi') = 0, \]
\[ u(\pi - 2\psi + \phi + \phi') + u(\pi - \phi - \phi') = u(\pi + \phi + \phi') + u(\pi - \phi - \phi') = 1 \]
for \( 0 < \phi, \phi' < \pi \), which is the case here. Then (7) becomes,
\[ G(\bar{\rho}, \bar{\rho}') = H_0^{(2)}[k_1 \mathcal{R}(\phi')] + H_0^{(2)}[k_1 \mathcal{R}(-\phi')], \]
which coincides with (11) in virtue of the properties
\[ \mathcal{R}(\pm \phi') = [\bar{\rho}^2 + \rho'^2 - 2\bar{\rho}\rho' \cos(\phi \mp \phi')]^{1/2} \]
\[ = |\bar{\rho} \mp \rho'| = \sqrt{(x-x')^2 + (y \mp y')^2}. \]

**Remark 3**: In connection with the diffraction integral in (8), using the large-argument asymptotic formula
\[ H_0^{(2)}(z) \sim \frac{2i}{\pi z^{1/2}} e^{-iz}, \]
we can see that, for large \( r \), \( Q(\bar{\rho}, \bar{\rho}' ; r) \) varies as \( e^{-ik_1 \sqrt{2\rho\rho'} f(r)} \) where \( f(r) \) is smooth. Therefore, \( Q(\bar{\rho}, \bar{\rho}' ; r) \) oscillates with asymptotic period \( T_p = 2\pi / (k_1 \sqrt{2\rho\rho'}) \). This property is very important because it allows the highly accurate computation of \( D(\bar{\rho}, \bar{\rho}') \) by applying to (8) the sophisticated algorithms of [4, 5].

### 2.2 Waveguide field

The simplest way to obtain an integral representation for the field in region 2 is to consider the equivalent problem illustrated in Figure 2b. Note that, shorting the aperture \( \mathcal{S} \) and placing the magnetic surface current \( -\bar{M} \) alongside it, the continuity of the tangential electric field on \( \mathcal{S} \) is automatically satisfied. Using this model, we can write the field within the waveguide as the sum of two terms,
\[ (\underline{E}_2, \underline{H}_2) = (\underline{E}^{\text{ext}}, \underline{H}^{\text{ext}}) + (\underline{E}^{\text{cat}}, \underline{H}^{\text{cat}}). \]

The first term (excitation field) is the field produced by the incident modal wave with the aperture short-circuited. One may prove that
\[ \underline{H}^{\text{ext}}(x, y) = \frac{2\pi i}{c} \int_{\mathcal{C}} \mathcal{M}(x') \underline{H}_{2z}(x', 0^+; x, y) dx', \] (13)
where \( \mathcal{M}(x, y; x', y') = 2\mathcal{H}_{2z}(x', y'; x, y) \) stands for the magnetic field that would have been produced at \( (x, y) \) inside region 2 if a magnetic line source \( 2\mathcal{M}_0 \delta(x-x') \delta(y-y') \) of unit strength \( \{M_0 = 1\} \) had been placed at \( (x', y') \) with the aperture short-circuited. One may find [13] that
\[ \mathcal{H}_{2z}(x', y'; x, y) = \frac{-i k_2 Y_2}{2a} \sum_{n=0}^{\infty} \frac{\epsilon_n}{r_n} \psi_n(x) \psi_n(x') \] (14)
\[ \times \left( e^{-r_n |y-y'|} + e^{-r_n |y+y'|} \right), \]
where \( \epsilon_n = 2 - \delta_{m0} \) is the Neumann's factor.

### 2.3 Integral equation

By enforcing continuity on the total tangential magnetic field across \( \mathcal{S} \), namely \( \underline{H}_{2z}(x, 0) = \underline{H}_{2z}(x, 0) \) for \( x \in \mathcal{C} \), we obtain
\[ \int_{\mathcal{C}} \mathcal{M}(x') \left[ \mathcal{H}_{2z}(x', 0^+; x, 0^+) + \mathcal{H}_{2z}(x', 0^-; x, 0^-) \right] dx' \]
\[ = 2H_0 \psi_n(x), \quad x \in \mathcal{C}. \] (15)

This is a weakly (logarithmically) singular integral equation that requires specialised treatment in the way outlined in the next section.

## 3 Treatment of kernels

Here, we discuss the convergence and some analytical properties of the kernel functions encountered in the integral equation (15). We also suggest computationally efficient alternatives to the kernels.
3.1 | Kernel function $\mathcal{H}_{21}(x', 0^+; x, 0^+)$.

From (6) using (7) we obtain

$$\mathcal{H}_{21}(x', 0^+; x, 0^+) = \mathcal{H}_{21}(x, 0^+; x', 0^+) = -\frac{k_1 Y_1}{4} \left[ 2H_0^{(2)}(k_1 |x-x'|) - D(x, 0; x', 0) \right].$$

(16)

The property [14]

$$H_0^{(2)}(k_1 z) = \left[ 1 - \frac{2}{\pi} \left( \ln \frac{k_1 z}{2} + C \right) \right] J_0(k_1 z)$$

$$+ \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} J_{2n}(k_1 z)$$

(17)

(C the Euler constant) suggests in a natural way the decomposition

$$\mathcal{H}_{21}(x', 0^+; x, 0^+) = S_1(x, x') + \mathcal{H}_{21}(x, x'),$$

(18)

where the first part

$$S_1(x, x') = \frac{i k_1 Y_1}{\pi} J_0(k_1 |x-x'|) \ln |x-x'|$$

is logarithmically singular at $x = x'$, and where the second part

$$\mathcal{H}_{21}(x, x') = \mathcal{H}_{21}(x', 0^+; x, 0^+) - S_1(x, x')$$

(20)

is an analytic function of its arguments. For $x = x'$, in particular, we obtain

$$\mathcal{H}_{21}(x, x) = \frac{k_1 Y_1}{2} \left[ 1 - \frac{2}{\pi} \left( \ln \frac{k_1}{2} + C \right) \right]$$

$$+ \frac{k_1 Y_1}{4} D(x, 0; x, 0).$$

(21)

3.2 | Kernel function $\mathcal{H}_{22}(x', 0^-; x, 0^-)$.

While the series in (14) converges exponentially if $y$ is not too close to $y'$, the exponential convergence rate is lost when $y = y'$. If additionally $x = x'$, the series diverges. Of course, this behaviour is expected since Green’s functions are singular, that is, they become infinite when the source and observation points coincide. Consequently, the series expansion (14) is inappropriate for solving the singular integral equation (15) wherein values of the Green’s functions at the source point do appear inside the integral.

To overcome this difficulty, we apply here the singularity extraction technique proposed in [6] and also used in [7], which enables splitting the kernel into two parts. The first part contains the singularity and can be given in closed form. The second part is a series expansion with improved convergence for any combination of the source and observation points.

3.2.1 | Singularity extraction

We shall make use of the expansion

$$\frac{1}{\gamma_n} = \sum_{m=0}^{\infty} \frac{\mu_m}{n^{2m+1}},$$

(22)

where

$$\mu_m = \frac{(2m-1)!! k_1^{2m}}{(2m)!!} \frac{d^{2m+1}}{\pi^m},$$

(23)

and the property (see [15, (1.148)], [16, (5.2.3.3)])

$$\sum_{n=1}^{\infty} \frac{e^{nt}}{n^m} \cos(mt) = \Re \text{Li}_m(e^{t})^2, \ m \in \mathbb{Z}^+, \ t \in \mathbb{R},$$

(24)

where

$$\text{Li}_m(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^m}$$

is the polylogarithm function.

Let

$$g_n^{(K)} = \sum_{m=0}^{K} \frac{\mu_m}{n^{2m+1}},$$

(25)

$$\rho_n = \frac{1}{\gamma_n} - g_n^{(K)}$$

(26)

with $K$ a preassigned positive integer. If we subtract $g_n^{(K)}$ out of $1/\gamma_n$ in equation (14), add it back and use the summation formula (24), we obtain

$$\mathcal{H}_{22}(x', 0^-; x, 0^-) = -\frac{i k_2 Y_2}{a} \left[ \frac{1}{\gamma_0} + 2 \sum_{n=1}^{\infty} \rho_n \psi_n(x) \psi_n(x') \right.$$  

$$+ \sum_{m=0}^{K} \mu_m S_{2m+1}^+(x, x') \right],$$

(27)

where

$$S_{2m+1}^+(x, x') = 2 \sum_{n=1}^{\infty} \frac{1}{n^{2m+1}} \psi_n(x) \psi_n(x')$$

$$\Re \left( \text{Li}_{m+1} \left[ e^{x-x'} \right] + \text{Li}_{m+1} \left[ e^{x+x'-2b} \right] \right).$$

(28)
The infinite series over \( n \) in (27) converges as \( 1/n^{2K+1} \).

In view of the properties \([15, 16]\)

\[
\begin{align*}
\text{Li}_1(z) & = -\ln(1-z), \\
\text{Li}_{2m+1}(1) & = \zeta(2m+1), \quad m \geq 1, \\
\text{Li}_{2m+1}(\varepsilon^2) & = -\frac{\varepsilon^{2m+1}|\ln|\varepsilon|}{(2m)!} \quad \text{modulo smooth functions, } m > 0
\end{align*}
\] (29)

\( \zeta(m) \) denotes the Riemann zeta function, we deduce from (27) that the singularity of the Green’s function \( H_{22} \) is confined in the term \( \text{Li}_1 \left[ e^{\frac{2\pi i}{a} (x-x')} \right] \). In addition, we see that the polylogarithmic terms \( \text{Li}_{2m+1} \left[ e^{\frac{2\pi i}{a} (x-x')} \right] \) are bounded when \( m > 0 \), but their higher-order derivatives become singular beyond some order depending on \( m \). Consequently, any discretisation scheme incorporating form (27) of the kernel function \( H_{22} \) is bound to yield mediocre-accuracy results.

We can resolve this issue by realizing that, from the physical viewpoint, the singularity of the Green's function in (27) is attributed to both the source term \( H^{(2)}_0 \left( k_2 R' \right) \) and the image term \( H^{(2)}_0 \left( k_2 R \right) \), where \( R^2 \equiv [(x-x')^2 + (y \pm y')^2]^{1/2} \). When \( y = y' = 0 \), these terms simplify to \( H^{(2)}_0 \left( k_2 |x-x'| \right) \), that is, they become singular at \( x = x' \). Consequently, we can write:

\[
H_{22}(x', 0^-; x, 0^-) = S_2(x, x') + \tilde{H}_{22}(x, x')
\] (30)

in analogy with (18), where the first part

\[
S_2(x, x') = \frac{i k_2 Y_2}{\pi} j_0(|k_2(x-x')|) \ln |x-x'|
\] (31)

is logarithmically singular, and where the second part

\[
\tilde{H}_{22}(x, x') = H_{22}(x', 0^-; x, 0^-) - S_2(x, x')
\] (32)

is a smooth function for any combination of its arguments. For \( x = x' \), in particular, we can find after some manipulation that

\[
\tilde{H}_{22}(x, x) = \frac{-i k_2 Y_2}{a} \left[ \frac{1}{Y_0} + 2 \sum_{n=1}^{\infty} \left( \psi'_n(x) \right)^2 p_n \right]
+ \frac{a}{\pi} \left( \ln a + \Re \text{Li}_1 \left[ e^{\frac{2\pi i}{a} (x-b)} \right] \right)
+ \sum_{m=1}^{K} \eta_m S_{2m+1}(x, x).
\] (33)

3.3 | Final form of the integral equation

The manipulations developed so far in this section can be used to bring the underlying integral equation (15) into its final form. Specifically, taking into account (18)-(20) and (30)-(32), equation (15) is written

\[
\int_{C} M(x') \left[ \mathcal{K}_d(x, x') \ln|x-x'| + \mathcal{K}_b(x, x') \right] dx' = 2 H_0 \psi_m(x), \quad x \in C,
\] (34)

where

\[
\begin{align*}
\mathcal{K}_d(x, x') &= \frac{i}{\pi} k_1 Y_1 j_0 |k_1(x-x')| \\
&+ \frac{i}{\pi} k_2 Y_2 j_0 |k_2(x-x')|, \\
\mathcal{K}_b(x, x') &= \mathcal{T}_{21}(x, x') + \mathcal{T}_{22}(x, x').
\end{align*}
\] (35)

4 | DISCRETIZATION AND FINAL RESULTS

There are two families of algorithms suitable for discretising the integral equation (34).

The first family involves direct Gauss-Gegenbauer and Gauss-Jacobi quadratures, such as those developed in [17] and [18], respectively, tailored to the singularities of the field at the edges of the radiating aperture. For the configuration of Figure 1a, these singularities have the form

\[
\left(1 - \frac{2(x-(2b+a))^2}{a^2} \right)^{-\frac{1}{2}}.
\]

The direct Gauss-Gegenbauer discretisation scheme is briefly discussed in Section 4.3.

The quadratures of the second family rely on variable transformation methods [19, 20]. They offer the advantage of handling the endpoint singularities without the need to specify them explicitly. Two such algorithms based on the modified trapezoidal rule and the Gauss-Chebyshev rule are described in Sections 4.1 and 4.2, respectively.

4.1 | First discretisation scheme: modified trapezoidal rule

Let

\[
\begin{align*}
x &= x(t) = b + \frac{a}{2\pi} t, \\
x' &= x'(t) = b + \frac{a}{2\pi} t, \quad 0 \leq t, \tau \leq 2\pi.
\end{align*}
\] (37)

In addition, let \( t = \omega(s) \), \( 0 \leq s \leq 2\pi \), be an infinitely differentiable and strictly monotone function such that its derivatives satisfy
where \( p > 1 \) is a preassigned positive integer. An example of such a transformation is (see Eq. (3.81) of [19])

\[
\psi(s) = \frac{2\pi[v(s)]^p}{[v(s)]^p + [v(2\pi - s)]^p}, \quad 0 \leq s \leq 2\pi,
\]

(39)

where

\[
v(s) = \left(1 + \frac{1}{p}\right)\left(\frac{\pi - s}{\pi}\right)^3 + \frac{1}{p} - \frac{1}{2}.
\]

If we set \( t = \psi(s) \), \( \tau = \psi(\sigma) \)
in (37) and then substitute into (34), we obtain the following integral equation:

\[
\int_0^{2\pi} f(\sigma) \left[\frac{1}{2} N_a(s, \sigma) \ln \left(4 \sin^2 \frac{s - \sigma}{2}\right) + N_b(s, \sigma)\right] d\sigma = 2H_0 \psi_m(x(\psi(s))), \quad 0 \leq s \leq 2\pi,
\]

(40)

where

\[
f(\sigma) \equiv M(b + \frac{1}{2\pi} \psi(\sigma)),
\]

\[
N_a(s, \sigma) = \frac{a}{2\pi} \psi'(\sigma) K_a(x(\psi(s)), x'(\psi(\sigma))),
\]

and

\[
N_b(s, \sigma) = \frac{a}{2\pi} \psi'(\sigma) K_b(x(\psi(s)), x'(\psi(\sigma)))
\]

\[
+ N_a(s, \sigma) \left[\ln \frac{a}{2\pi} + g(s - \sigma) + v(s, \sigma)\right]
\]

(42)

(43)

with

\[
v(s, \sigma) = \begin{cases} 
\ln \frac{\psi(s) - \psi(\sigma)}{s - \sigma}, & s \neq \sigma \\
\ln |\psi'(s)|, & s = \sigma
\end{cases}
\]

(44)

and

\[
g(z) = -\ln \left|\frac{2\sin \frac{z}{2}}{z}\right|, \quad g(0) = 0.
\]

(45)

In obtaining (40) from (34), we have also made use of the obvious decompositions

\[
\ln |\psi(s) - \psi(\sigma)| = v(s, \sigma) + \ln |s - \sigma|,
\]

\[
\ln |s - \sigma| = \frac{1}{2} \ln \left(4 \sin^2 \frac{s - \sigma}{2}\right) + g(s - \sigma).
\]

We emphasize that the transformed equation (40) is equivalent to the original integral equation (34), therefore its solution coincides with the unique solution of the boundary value problem.

Remark 4: Under the smoothing conditions (38), the role of the variable transformation \( \tau = \psi(\sigma) \) is to convert the integrand in (34) into another with a smooth periodic extension. For such smooth integrands, the modified trapezoidal rule is recognized as one of the most efficient types of quadrature, as its discretisation error is exponentially small. We stress that the actual endpoint singularities have been eliminated and no longer appear in (40).

We can now discretise the transformed integral equation (40) using the quadratures [19, Sect. 3.5]:

\[
\int_0^{2\pi} f(\sigma) N_a(s, \sigma) d\sigma \approx \frac{\pi}{N} \sum_{k=0}^{2N-1} f(\sigma_k) N_a(s, \sigma_k),
\]

(46)

\[
\int_0^{2\pi} f(\sigma) N_b(s, \sigma) d\sigma \approx \frac{\pi}{N} \sum_{k=0}^{2N-1} f(\sigma_k) N_b(s, \sigma_k) R_k^{(N)}(s), \quad 0 \leq s \leq 2\pi,
\]

(47)

where

\[
\sigma_k = \frac{k\pi}{N},
\]

(48)

\[
R_k^{(N)}(s) = \frac{1}{N} \cos[N(s - \sigma_k)]
\]

\[
-2 \sum_{m=1}^{N-1} \frac{1}{m} \cos[m(s - \sigma_k)].
\]

(49)

Finally, if we apply the resulting equation at the nodes \( s = \sigma_0, \sigma_1, \ldots, \sigma_{2N-1} \), we obtain a system of \( 2N \) linear algebraic equations in \( 2N \) unknowns \( f(\sigma_0), f(\sigma_1), \ldots, f(\sigma_{2N-1}) \). In terms of \( \{f(\sigma_i)\}_{i=0}^{2N-1} \), we can compute any quantity of interest, such as reflection coefficients and the far-field as outlined below.

4.1.1 | Reflection coefficients

Using the change of variables \( x' = x'(\tau) = x'(\psi(\sigma)) \) in (13) and applying the modified trapezoidal rule to the resulting integral, we obtain the following expression for the total magnetic field in region 2:
where
\[ H_{z2}(x, y) = H_{z2}^{inc} + \sum_{k=0}^{\infty} \varepsilon_k \Gamma_{mk} \psi_k(x) e^{i\omega y}, \] (50)

with
\[ \Gamma_{mk} = \delta_{mk} + \frac{ik_k Y_2}{2N} \sum_{j=0}^{2N-1} w'(\sigma_j) f(\sigma_j) \psi_k(x_j), \] (51)

\[ x_j = b + \frac{a}{2\pi} w(\sigma_j), \]

is the reflection coefficient of the TM\(_m^y\) mode in response to the TM\(_m\) incident mode. For \( k = m \), in particular,
\[ R = \Gamma_{mm} \]
gives the self-reflection coefficient of the incident TM\(_m^y\) mode.

### 4.1.2 Far-zone magnetic field

The easiest way to evaluate the far-zone magnetic field \( H_{z2}(\hat{\rho}) \) radiated in region 1 is to apply (5) using the radial transmission representation of the wedge Green's function [2, Sect. 6.5b]:
\[ H_{z2}(x', 0^+: \hat{\rho}) = -\frac{\pi k_1 Y_1}{2\psi} \sum_{n=0}^{\infty} \varepsilon_n f(\sigma) H_{z2}^{(2)} (k_1 \rho) \cos \frac{n\pi \psi}{\psi}, \] (52)

in conjunction with the large-argument asymptotic formula
\[ H_{z2}^{(2)} (k_1 \rho) \sim \sqrt{\frac{2i}{\pi k_1 \rho}} e^{-ik_1 \rho} e^{-i \frac{\pi \rho}{2\psi}}. \]

Note that, in the far-field region, the series in (52) converges very strongly (exponentially).

Substituting (52) into (5), setting \( x' = x'(\sigma) = x'(w(\sigma)) \), and evaluating the resulting integral by the modified trapezoidal rule, we obtain
\[ H_{z2}(\rho, \psi) = -\frac{k_1 a Y_1}{2N} \sqrt{\frac{2i}{2k_1 \rho}} e^{-ik_1 \rho} F(\psi), \] (53)

where
\[ F(\psi) = \sum_{n=0}^{\infty} c_n \cos \frac{n\pi \psi}{\psi} \] (54)

with
\[ c_n = c_n e^{-\frac{2\pi n}{\psi}} \sum_{j=0}^{2N-1} w'(\sigma_j) f(\sigma_j) H_{z2}^{(2)} (k_1 x_j). \] (55)

### 4.2 Second discretisation scheme: Gauss-type rule

Let
\[ x = x(t) = b + \frac{a}{2} (1 + t), \]
\[ x' = x'(\tau) = b + \frac{a}{2} (1 + \tau), \]
\[ -1 \leq t, \tau \leq 1. \] (56)

In addition, let \( t = \omega(s), -1 \leq s \leq 1 \), be an infinitely differentiable and strictly monotone function such that
\[ \omega^{(n)}(-1) = \omega^{(n)}(1) = 0 \quad \text{for } i = 1, 2, \ldots, p, \] (57)

where \( p > 1 \) is a preassigned positive integer. An example of such a transformation is [20].

\[ \omega(s) = 2 \frac{B_s(s, p)}{B(p, p)} - 1, \quad -1 \leq s \leq 1, \] (58)

where
\[ B_s(s, p) = \int_0^s t^{s-1} (1 - t)^{p-s-1} dt, \]
\[ B(a, b) = \int_0^1 t^{a-1} (1 - t)^{b-1} dt \]

are the incomplete beta function and the Euler beta function, respectively.

If we make the change of variables \( t = \omega(s) \) and \( \tau = \omega(\sigma) \) in (56) and then substitute into equation (34), we obtain
\[ \int_{-1}^{1} f(\sigma) \left[ M_a(s, \sigma) \ln |s - \sigma| + M_b(s, \sigma) \right] d\sigma = 2H_0 \psi_m \left[ x(\omega(s)) \right], \quad -1 \leq s \leq 1, \] (59)

where
\[ f(\sigma) \equiv M \left( b + \frac{a}{2} \omega(\sigma) \right), \] (60)
\[ M_a(s, \sigma) = \frac{a}{2} \omega'(\sigma) K_a [x(\omega(s)), x(\omega(\sigma))], \] (61)

and
\[ M_b(s, \sigma) = \frac{a}{2} \omega'(\sigma) K_b [x(\omega(s)), x(\omega(\sigma))] \] (62)
with \( v(s, \sigma) \) given again by (44).

**Remark 5:** Just as with (40), a significant property of the new integral equation (59) resulting from the smoothing conditions (57) is the periodicity of the kernel function. We remind that, as a consequence of (57), the endpoint singularities have been eliminated and no longer appear in (59). The smoothness of the integrand at the endpoints \( \pm 1 \) allows us to discretise (59) via Gauss-type quadratures ([17, 18, 21]) employing orthogonal polynomial basis functions such as Chebyshev, Legendre, Gegenbauer, or others. 

Here, we make use of the following simple and powerful Chebyshev-based Gauss-type quadratures (see [21]):

\[
\int_{-1}^{1} f(\sigma) M_b(s, \sigma) d\sigma = \int_{-1}^{1} f(\sigma) \sqrt{1 - \sigma^2} M_b(s, \sigma) d\sigma \\
= \frac{\pi}{N} \sum_{k=1}^{N} f(\chi_k) \sqrt{1 - \chi_k^2} M_b(s, \chi_k),
\]

\[
\int_{-1}^{1} f(\sigma) M_a(s, \sigma) \ln|s - \sigma| d\sigma \\
= \int_{-1}^{1} f(\sigma) \sqrt{1 - \sigma^2} M_a(s, \sigma) \ln|s - \sigma| d\sigma \\
\approx \frac{\pi}{N} \sum_{k=1}^{N} f(\chi_k) \sqrt{1 - \chi_k^2} M_a(s, \chi_k) \beta(s, \chi_k)
\]

\((-1 \leq s \leq 1)\), where [21]

\[
\chi_k = \cos \left( \frac{(2k - 1)\pi}{2N} \right),
\]

\[
\beta(s, \chi_k) = \frac{1}{N} \sum_{k=0}^{N-1} \epsilon_k T_{\nu}(\chi_k) \xi_{\nu}(s)
\]

\((T_{\nu}(s)) \) is the first-kind Chebyshev polynomial of degree \( \nu \)) with

\[
\xi_{\nu}(t) = \int_{-1}^{1} \frac{T_{\nu}(\tau)}{\sqrt{1 - \tau^2}} \ln|\tau - t| d\tau \\
= -\pi \begin{cases} 
\ln 2, & \nu = 0 \\
T_{\nu}(t)/\nu, & \nu = 1, 2, \ldots 
\end{cases}
\]

Applying the resulting equation at the nodes \( s = \chi_1, \chi_2, \ldots, \chi_N \), we obtain a system of \( N \) linear algebraic equations in \( N \) unknowns \( f(\chi_1), f(\chi_2), \ldots, f(\chi_N) \). In terms of \( \{f(\chi_i)\}_{i=1}^{N} \), we can compute any quantity of interest. For example, using the change of variables \( x' = x'(t) = x'(\psi(\sigma)) \) in (13) and applying the Gauss-Chebyshev rule to the resulting integral, we arrive again at equation (50) for the total magnetic field in region 2. Now, the counterpart of formula (51) is

\[
\Gamma_{nm} = \delta_{nm} + \frac{i\pi k_2}{2} \sum_{j=1}^{N} \sqrt{1 - \chi_j^2} w(\chi_j) f(\chi_j) \psi_{\nu}(x_j),
\]

with

\[
x_j = b + \frac{d}{2} w(\chi_j),
\]

from that the reflection coefficients can be specified readily.

Similarly, from (5), we obtain again equation (53), where now

\[
F(\psi) = \sum_{n=0}^{\infty} d_n \cos \frac{n\pi \psi}{\psi}
\]

with

\[
d_n = \pi e_n \prod_{j=1}^{N} \sqrt{1 - \chi_j^2} w(\chi_j) f(\chi_j) \psi_{\nu}(x_j).
\]

**Remark 6:** Gauss-type rules based on first-kind Chebyshev polynomials pertain to integrals of the type \( \int_{-1}^{1} (1 - t^2)^{\nu/2} f(t) dt \). So, to enable their application, in the second step in each of (63) and (64), we artificially introduced the weight function \( q(\sigma) = 1/\sqrt{1 - \sigma^2} \) by simultaneously dividing and multiplying the integrand by \( q(\sigma) \). However, we emphasize that \( q(\sigma) \) has nothing to do with the actual endpoint singularities, referred to at the beginning of this section, and is not a substitute for it. In this connection, we also recall that, as a consequence of (57), the endpoint singularities have been eliminated and are no longer present in the integral equation (59).

### 4.3 Direct Gauss-Gegenbauer rule

This discretisation scheme fully takes account of the actual endpoint singularities. Specifically, if we make again the change of variables (56) and set

\[
M(x') = (1 - r^2)^{-\nu/2} M(\tau), \quad \nu = \frac{1}{6}
\]

then (34) becomes

\[
\text{To avoid unnecessary complications, in this subsection, we restrict ourselves to the configuration of Figure 1a. The structure of Figure 1b can be treated similarly by Jacobi-polynomial-based quadratures like those developed in Ref. [18].}
\[ f_1'(1 - \tau^2)^{-1/2} \mathcal{M}(\tau) [K_1(t, \tau) \ln |t - \tau| + K_2(t, \tau)]d\tau = 2H_0 \psi_m(x(t)), \quad -1 \leq t \leq 1, \]  
where
\[ K_1(t, \tau) = \frac{d}{2} \mathcal{K}_1[x(t), x(\tau)], \]
\[ K_2(t, \tau) = \frac{d}{2} \mathcal{K}_2[x(t), x(\tau)] + K_1(t, \tau) \ln \frac{d}{2}. \]

Next, to the integrals in (72), we apply the quadrature rules [17]:
\[
\int_{-1}^1 (1 - \tau^2)^{-1/2} \mathcal{M}(\tau) K_2(t, \tau)d\tau = \sum_{i=1}^{N} w_i \mathcal{M}(\tau_i) K_2(t, \tau_i),
\]
\[
\int_{-1}^1 (1 - \tau^2)^{-1/2} \mathcal{M}(\tau) \ln |\tau - t|d\tau = \sum_{k=0}^{N-1} b_k \sum_{i=1}^{N} w_i \mathcal{M}(\tau_i) K(t, \tau_i) C_k(\tau_i),
\]
where \( C_k(\tau) \) is the Gegenbauer polynomial of degree \( k \) and index \( \nu \), \( \{\tau_1, \tau_2, \ldots, \tau_N\} \) are the roots of \( C_N(\tau) = 0 \),
\[
\begin{align*}
\omega_i &= \frac{\pi^{\nu+1} \Gamma(\nu+1)}{N! [\Gamma(\nu+1)]^2} \frac{1}{(1 - \tau_i^2) [C_{N-1}^+(\tau_i)]^2}, \\
\beta_k &= \frac{\pi^{2\nu+1} \Gamma(\nu+2\nu)}{m! [\Gamma(\nu+1)]^2}
\end{align*}
\]
and
\[
J(t, k) = \int_{-1}^1 (1 - \tau^2)^{-1/2} C_k(\tau) \ln |\tau - t|d\tau.
\]

The computation of the integral in (79) is explained in [17].

Finally, applying the obtained equations at \( t = \tau_1, \tau_2, \ldots, \tau_N \), we obtain a linear algebraic system from which the unknowns \( \{\mathcal{M}(\tau_i)\}_{i=1}^N \) can be found and in terms of them any quantity of interest.

5 | NUMERICAL RESULTS

In this section, we present numerical examples demonstrating the efficiency and accuracy of the proposed algorithms. For testing the correctness of the implementation, we independently used the discretisation schemes based on the modified trapezoidal, Gaussian-type, and direct Gauss-Gegenbauer quadratures discussed in the preceding section. In all cases, the agreement between their corresponding results was excellent.

In what follows, we assume unless otherwise specified that \( \epsilon_1 = \epsilon_2 = \epsilon_0 \) and \( \mu_1 = \mu_2 = \mu_0 \). In obtaining the numerical results, we truncated the infinite series encountered in (27) and (33), retaining only \( n_{\text{max}} \) terms. In the modified trapezoidal and Gaussian-type rules, we employed the transformations (39) and (38), respectively.

Table 1 pertains to the case when \( \psi = \pi, m = 0 \) and \( a = \lambda / \pi \), where \( \lambda \) is the free-space wavelength; as mentioned in Section 2.1, in this case, the parallel-plate waveguide is radiating through a slot on the infinite ground plane \( y = 0 \). As we can see, both the modified trapezoidal rule and the Gaussian-type quadrature converge rapidly, with their corresponding results coinciding for sufficiently large quadrature order. The obtained results are confirmed from those of the direct Gauss-Gegenbauer algorithm and, also, agree with relevant results from [22].

In connection with the same two algorithms, based on the modified trapezoidal and Gaussian-type rules, in Table 2, we focus on their convergence versus the number of quadrature nodes when \( \psi = 3\pi/2 \) and \( m = 0 \). When the number of quadrature points \( (2N \text{ and } N) \) is sufficiently large, the reflection coefficients computed by the algorithms settle down to the same value, considered the reference (exact) value. As we can see, the convergence is exponential. The final results are accurate within the round-off errors. These results are also confirmed from those of the direct Gauss-Gegenbauer algorithm.

Let
\[
\epsilon = (R - R_{\text{ref}}) / R_{\text{ref}}
\]
be the relative error of \( R \), where \( R_{\text{ref}} \) denotes the reference value specified as described above. Figure 3, showing the logarithm (base 10) of \(|\epsilon| \) versus the number of quadrature points, pertains to the case \( \psi = 3\pi/2, \beta = 0, \text{ and } a = 0.4\lambda \). The configuration—shown in the inset—in this particular case consists of a semi-infinite perfectly conducting plate and a PEC wedge, a face of which is parallel to the plate. The excitation is the TM\(_0\) mode of the formed parallel plate waveguide. Again, we can observe the rapid convergence and high accuracy of the algorithms. In this example, for sufficiently large \( N \), the relative error of the reflection coefficient becomes very small, of the order of \(10^{-16}\), which is comparable to the round-off error.

Next, in Figure 4, the normalized surface magnetic current density \( \mathcal{M}(x)/\mathcal{Z}_0 \) \( (\mathcal{Z}_0 = \sqrt{\mu_0/\epsilon_0} = 1/\mathcal{Y}_0 \text{ is the impedance of free space}) \) is shown against \( x \). As seen, the results of the modified trapezoidal and Gaussian-type rules coincide with those of the direct Gegenbauer rule. This implies that the modified trapezoidal and Gaussian-type rules can reproduce the anticipated singular behaviour of the fields near the edges (edge condition) even though, in their context, we did not make explicit use of such a constraint. Of course, the confirmation of the edge condition is a consequence of the equivalence of the transformed equations (40) and (59) with the original integral equation (34).
TABLE 1 Magnitude and phase of the reflection coefficient when $\psi = \pi, a = \lambda/\pi, m = 0$

| $2N$ | $|R|$ | $\psi$ (in Deg) | $|\epsilon|$ | $\psi$ (in Deg) |
|------|------|----------------|---------|----------------|
| 8    | 0.2656508279095300 | $-96.96739575001973$ | 0.2656213934231817 | $-97.35969867628904$ |
| 16   | 0.2652802818494491 | $-97.17565994104122$ | 0.2652801364676119 | $-97.17572511168764$ |
| 32   | 0.2652799035838985 | $-97.17580703430660$ | 0.2652799029820420 | $-97.17580727176431$ |
| 64   | 0.2652799028237736 | $-97.17580733407726$ | 0.2652799028233099 | $-97.17580733426117$ |
| 128  | 0.2652799028232911 | $-97.17580733426878$ | 0.2652799028232915 | $-97.17580733426838$ |

From [22]: $R = (0.264, -97.6^\circ)$

| $2N$ | $|R|$ | $\psi$ (in Deg) | $|\epsilon|$ | $\psi$ (in Deg) |
|------|------|----------------|---------|----------------|
| 8    | 0.20862656508279095300 | $2.067 \times 10^{-3}$ | 0.208625328874176 | $2.115 \times 10^{-3}$ |
| 16   | 0.2082226684183292 | $2.066 \times 10^{-6}$ | 0.208222535841542 | $1.429 \times 10^{-6}$ |
| 32   | 0.2082223899386364 | $4.473 \times 10^{-9}$ | 0.208222382624641 | $9.338 \times 10^{-10}$ |
| 64   | 0.208222380686241 | $2.834 \times 10^{-12}$ | 0.2082223806808581 | $1.168 \times 10^{-12}$ |
| 128  | 0.208222380680338 | $1.333 \times 10^{-16}$ | 0.208222380680377 | $1.866 \times 10^{-14}$ |

In Figure 3, we show the magnitude $|R|$ of the reflection coefficient versus $b/\lambda$ when $m = 0$ and $a = 0.4\lambda$. We see that the graph oscillates about the horizontal dashed line $|R| = |R_{FW}|$, where $|R_{FW}| = 0.2075936433867771$ is the magnitude of the reflection coefficient of the flanged waveguide antenna (i.e., the waveguide antenna that radiates through an aperture $S$ on an infinite ground plane). In addition, we observe that the graph approaches the line $|R| = |R_{FW}|$ for sufficiently large $b$, as expected. The variation of $|R|$ with the wedge angle $\psi$ is shown in Figure 6 when $m = 0, a = 0.4\lambda$ both for $b = 0.1\lambda$ and for $b = 0.18\lambda$. We see that $|R|$ decreases monotonically as $\psi$ increases. All three algorithms yield indistinguishable results.

FIGURE 3 $\log_{10}|\epsilon|$ versus number of quadrature points when $m = 0, b = 0, a = 0.4\lambda, K = 4$ and either $\rho = 12$ (trapezoidal rule), or $\rho = 4$ (Gauss-type rule)

FIGURE 4 Normalized surface magnetic current density on the aperture when $a = 2.5\lambda, b = 0.15\lambda, \psi = 3\pi/2$, and $m = 0$. (a) Real part. (b) Imaginary part
In Figure 7, we show the magnitude of the reflection coefficients $\Gamma_{00}$ and $\Gamma_{02}$ (case $m = 0$) versus $a/\lambda$ when either $b = 0$ or $b = 0.15\lambda$. In addition, we show how these results compare with the results for an open-ended parallel-plate waveguide with plate separation $a$. The results in the latter case were obtained by the use of the formula (8.14b) of Ref. [23] based on the Wiener-Hopf technique.

In Figure 8, we show the snapshot of the total electric field at $\omega t = \pi/2$ when $m = 0, a = 0.4\lambda$, and $b = 0$. As expected, the incident and reflected waves, travelling in opposite directions, produce a standing wave pattern within the waveguide. In Figure 9, we show similar results in the case when $m = 1, a = 0.6\lambda$, and $b = 0$.

In Figure 10, we show the far-field pattern $F(\phi)$ when $m = 0, a = 0.6\lambda, b = 2\lambda$, and $\psi = 3\pi/2$ (solid line). For the sake of comparison, the magnetic field pattern of the flanged parallel-plate waveguide antenna —case $\psi = \pi$ (dashed line)—
the antenna in the presence of the wedge is a perturbation of the field pattern of the flanged antenna. This perturbation is shown in the inset in Figure 3) are presented in Figure 11. As seen from Figures 10 and 11, the field pattern of the antenna in the presence of the wedge is a perturbation of the field pattern of the flanged antenna. This perturbation is enhanced by an increase in $\phi$, especially for $\phi > \pi/2$.

6 CONCLUSION

We studied parallel-plate waveguide antennas radiating through a perfectly conducting wedge by combining field equivalence and Green's functions methods in the context of singular integral equation techniques with Nystrom-type discretisation. Several quadrature rules have been proposed with low-cost implementation and high accuracy. Use of the field equivalence principle eliminates the need to deal with the unwieldy natural (electric) current sources flowing on the wedge faces and the waveguide walls, which otherwise would require specialised treatment by methods of [24] and [25], for example.

Here, the primary excitation was an incident TM$^0$ mode. The TE$^0$ case could be treated along similar lines, now in terms of singular integral–integro-differential equations. The proposed method can also be used in cases when the radiating parallel-plate waveguide is either multilayered or inclined.

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REFERENCES

1. Abosrwal, N.A., Balanis, C.A., Birtcher, C.R.: Impact of finite ground plane edge diffractions on radiation patterns of aperture antennas. Prog Electromagn Res B. 55, 1–21 (2013)
2. Felsen, L.B., Marcuvitz, N.: Radiation and scattering of waves. Prentice-Hall, Inc. (1973)
3. Tsalamengas, J.L.: Rapidly convergent eigenfunction expansions of Green functions for a perfectly conducting wedge. IEEE Trans Antennas Propag. 61(3), 1334–1341 (2013)
4. Lyness, J.N.: Integrating some infinite oscillating tails. J Comp Appl Math, 109–117 (1985)
5. Espelid, T.O., Overholt, K.J.: DQAINF: an algorithm for automatic integration of infinite oscillating tails. pp. 83–101. Numerical Algorithms (1994)
6. Tsalamengas, J.L.: ‘Exponentially converging Nystrom's methods for systems of singular integral equations with applications to open/closed strip- or slot-loaded 2D structures’. IEEE Trans Antennas Propag. 54(5), 1549–1558 (2006)
7. Kakkavas, A.T., Tsalamengas, J.L.: Finite phased arrays of dielectric-loaded parallel plate-fed coplanar slot antennas: singular integral equation formulation with highly convergent kernels and Nystrom discretisation. IEEE Trans Antennas Propag. 62(8), 4031–4040 (2014)
8. Delves, L.M., Mohamed, J.L.: Computational methods for integral equations. Cambridge Univ. Press, Cambridge (1992)
9. Nosich, A.I.: The method of analytical regularisation in wave-scattering and eigenvalue problems: foundations and review of solutions. IEEE Antenn Propag Mag. 41(5), 34–49 (1999)
10. Nosich, A.I.: Method of analytical regularisation in computational photonics. Radio Sci. 51(3), 1421–1430 (2016)
11. Fikioris, G.: A note on the method of analytical regularisation. IEEE Antenn Propag Mag. 43(2), 4–40 (2001)
12. Balanis, C.A.: Advanced engineering electromagnetics. John Wiley and Sons, NJ (2012)
13. Eom, H.J.: Electromagnetic wave theory for boundary-value problems: an advanced course on analytical methods. Springer, Berlin (2004)
14. Abramowitz, M., Stegun, I.A.: Handbook of mathematical functions. Dover, New York (1972)
15. Gradshteyn, I.S., Ryzhik, I.M.: Table of integrals, series, and products. Academic, New York (1965)
16. Prudnikov, A.P., Yu, A.B., Marichev, O.I.: Integrals and series: Volume 1: Elementary functions. Gordon and Breach, New York (1986)
17. Tsalamengas, J.L.: Quadrature rules for weakly singular, strongly singular, and hypersingular integrals in boundary integral equation methods. J Comput Phys. 303, 498–513 (2015)
18. Tsalamengas, J.L.: Gauss-Jacobi quadratures for weakly, strongly, hyper- and nearly-singular integrals in boundary integral equation methods for domains with sharp edges and corners. J Comput Phys. 325, 338–357 (2016)
19. Colton, D., Kress, R.: Inverse Acoustic and Electromagnetic scattering theory. Springer, Berlin (1998)
20. Monegato, G., Scuderi, L.: Numerical integration of functions with boundary singularities. J Comput Appl Math. 112, 201–214 (1999)
21. Tsalamengas, J.L.: A direct method to quadrature rules for a certain class of singular integrals with logarithmic, Cauchy, or Hadamard-type singularities. Int J Numer Model: Electronic Networks, Devices and Fields. 25, 512–524 (2012)
22. Hong, K.: Diffraction by a flanged parallel-plate waveguide. Radio Sci. 7(10), 955–963 (1972)
23. Boersma, J.: Ray-optical analysis of reflection in an open-ended parallel-plane waveguide. I: TM case. SIAM J Appl Math. 29(1), 164–195 (1975)
24. Veliev, E.I.: Plane wave diffraction by a half-plane: a new analytical approach. J Electromagn Waves Appl. 13(10), 1439–1453 (1999)
25. Chien, D.N., Tanaka, K., Tanaka, M.: TEM radiation from a parallel-plate waveguide with an arbitrarily flanged surface of finite size. Radio Sci. 39(3), 1–8 (2004)

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