Relative-locality phenomenology on Snyder spacetime

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Abstract

We study the effects of relative locality dynamics in the case of the Snyder model. Several properties of this model differ from those of the widely studied $\kappa$-Poincaré models: for example, in the Snyder case the action of the Lorentz group is preserved, and the composition law of momenta is deformed by terms quadratic in the inverse Planck energy. From the investigation of time delay and dual curvature lensing we deduce that, because of these differences, in the Snyder case the properties of the detector are essential for the observation of relative locality effects. The deviations from special relativity do not depend on the energy of the particles and are much smaller than in the $\kappa$-Poincaré case, so are beyond the reach of present astrophysical experiments. However, these results have a conceptual interest, because they show that relative-locality effects can occur even if the action of the Lorentz group on phase space is not deformed.

Keywords: quantum gravity phenomenology, Snyder spacetime, relative locality, deformed relativistic symmetries, curved momentum space

(Some figures may appear in colour only in the online journal)

1. Introduction

The possibility of testing quantum spacetime scenarios via Planck scale effects in the kinematics of point-like particles became of primary relevance for quantum gravity phenomenology over the last 15 years [1–7]. In this perspective the ‘doubly special relativity’ (or, as some authors prefer to call it, the ‘deformed relativistic symmetries’) (DSR) scenario [8–10], in which the Planck scale enters in the laws of motion as an observer-independent scale deforming the algebra of relativistic symmetries, has played a prominent role (see for instance [11, 12]).
As the understanding of DSR advanced it has been realized that the introduction in a relativistic theory of a second invariant scale (in addition to the speed of light) with dimensions of an inverse momentum, i.e. proportional to the inverse of the Planck energy $E_p \sim 10^{19}$ GeV, enforces to abandon the concept of absoluteness of locality: the locality of a process is relative to the distance from the observer, in such a way that a process remains local only for an observer local to the process itself [11, 13, 14]. The relative locality framework [15] (see also [16–18]) has been proposed as a formulation of DSR capable of taking into account relative locality effects, focusing on the (curved) momentum space associated to the quantum spacetime deformation, and taking into account the kinematical properties of particle processes by means of a Lagrangian formulation with suitable boundary terms.

So far the relative locality framework has been investigated extensively only for the well-known case of $\kappa$-Poincaré symmetries [19–22], while some work has been done for a model of momentum space based on a quantum spacetime with non-commutativity of SL(2,R) type, called ‘spinning spacetime’ by the authors in [23]. The purpose of the present paper is to study the implications of the relative locality framework for another well-known model of deformed momentum space, based on a quantum spacetime with Snyder type non-commutativity [24–26]. The peculiarity of this model is the preservation of the linear action of the Lorentz group on phase space. This can help to single out the role of the deformation of Lorentz invariance on the effects connected to relative locality. In fact, it has been shown that no such effects are present in the case of the propagation of free particles [27] (see also [28]). However, when considering interacting particles, one must take into account that the composition law of momenta is deformed [26, 29], and hence one can expect relative-locality effects in more complicated settings involving interactions. However, these are presumably greatly suppressed and far from the reach of present experimental sensibility. It is important to notice indeed that while the $\kappa$-deformation implies modifications to kinematical laws of linear order in $1/E_p$, in Snyder momentum space the deformation scale is proportional to the square of the inverse Planck energy, $\beta \propto 1/E_p^2$. We thus expect that the possible phenomenological outcomes, if any, should be suppressed by a further Planck-scale factor with respect to the already tiny ones arising from $\kappa$-deformation. However, it is interesting at least from a theoretical point of view to show the existence of some effects associated with Snyder spacetime arising from the relative locality formalism. We will devote the last part of this manuscript to the study of some relevant processes that could provide this kind of effects.

Another peculiarity of Snyder spaces is that the composition law of momenta is not only noncommutative, as in most models of DSR, but also nonassociative [26], determining further complications in the investigation of interactions. To some extent, at the classical level, the nonassociativity is less disturbing than at the quantum level, since a natural ordering can be assumed, at least for three-particle interactions. Still, when one considers more than one process, as for causally connected interactions, a further ambiguity arises in the choice of how to group summations of more than two momenta, in addition to the one due to noncommutativity. We will discuss this issue in the course of our analysis.

Finally, we recall that the geometrical properties of relative locality momentum space introduced in [15] have been investigated in [30] in the case of Snyder space and its generalizations. It turns out that the momentum space associated to these models is a maximally symmetric one, with constant curvature and vanishing torsion and nonmetricity. However, for our investigations these results are irrelevant, so we refer to [30] the interested reader.

Throughout the manuscript we use units such that the speed of light $c$ is set to 1.
2. The Snyder model

The Snyder model [24–26] is defined by the deformed Heisenberg algebra
\[
\{x^\mu, p_\nu\} = \delta_\mu^\nu - \beta p^\mu p_\nu, \quad \{x^\mu, x^\nu\} = -\beta J^{\mu\nu}, \quad \{p_\mu, p_\nu\} = 0,
\]
where \(J^{\mu\nu} = x^\mu p^\nu - p^\mu x^\nu\) are the generators of the Lorentz transformations. The parameter \(\beta\) has dimension of inverse mass square and is usually assumed to be of order \(1/E^2\). The Lorentz algebra and its action on phase space are undeformed. Snyder space can be also described in terms of a curved momentum space given by a hyperboloid of equation \(\eta_\Lambda^2 = -1/\beta\) embedded in flat five-dimensional space of coordinates \(\eta_\mu\), with parametrization \(p_\mu = \eta_\mu/\sqrt{\beta \eta_4}\).

Since the Lorentz transformations are undeformed, the dispersion relations maintain the same form as in special relativity, and the Hamiltonian of a free particle can be chosen as
\[
H = \frac{p \cdot p}{2}.
\]

The equations of motion following from (1) and (2) are
\[
\dot{x}^\mu = \{x^\mu, H\} = (1 - \beta p \cdot p) p^\mu, \quad \dot{p}_\mu = \{p_\mu, H\} = 0.
\]

They can be obtained varying the action [31]
\[
S = -\int_{-\infty}^{\infty} ds \left[ x^\mu \left( \eta_{\mu\nu} + \beta \frac{P_\mu P_\nu}{1 - \beta p \cdot p} \right) \dot{p}^\nu + \frac{N}{2} \left( p \cdot p - m^2 \right) \right]
\]
where \(N\) is a Lagrange multiplier enforcing the Hamiltonian constraint \(p \cdot p = m^2\).

Starting from (1), one can define a Hopf algebra [26]. The coproduct of this algebra entails the deformed addition law for momenta,
\[
(p \oplus q)_\mu = \frac{1}{1 + \beta p \cdot q} \left[ \left( 1 + \frac{\beta p \cdot q}{1 + \sqrt{1 - \beta p \cdot p}} \right) p_\mu + \sqrt{1 - \beta p \cdot p} q_\mu \right].
\]
Notice that this law is noncommutative, \(p \oplus q \neq q \oplus p\), as in other well-known models, but also nonassociative, \(k \oplus (p \oplus q) \neq (k \oplus p) \oplus q\).

The antipode of the element \(p\) of the Hopf algebra, i.e. the element \(\ominus p\) such that \(p \oplus (\ominus p) = 0\), is given by
\[
\ominus p = -p.
\]

3. Interactions in relative locality

We start now the discussion of the dynamics on Snyder space in accordance with the relativistic description of distant observers given by the framework of relative locality [15]. The action for a noninteracting particle is of course given by (4). The definition of an action for interacting particles requires instead some discussion. In this section, we consider a simple interaction with one incoming and two outgoing particles, and use a Lagrangian formalism, following the treatment given in [19] for the case of the \(\kappa\)-Poincaré model.

In this formalism, the action for three free particles of momenta \(k_\mu, p_\mu, q_\mu\) and masses \(m_k, m_p, m_q\) with positions \(z^\mu, x^\mu, y^\mu\), respectively, interacting at parameter time \(s_0\) (see figure 1),

\[3\] We denote \(a^\mu b_\mu \equiv a \cdot b\).
can be written adding to the terms describing the propagation of the noninteracting particles an interaction term

\[ -\xi^\mu_{[0]} K^\mu_{[0]} \] [15], as

\[
S = -\int_{-\infty}^{s_0} ds \left[ \left( z^\mu + \beta \frac{z \cdot k k^\mu}{1 - \beta k \cdot k} \right) \dot{k}_\mu + \frac{N_k}{2} (k \cdot k - m^2_k) \right] - \int_{s_0}^{\infty} ds \left[ \left( x^\mu + \beta \frac{x \cdot p p^\mu}{1 - \beta p \cdot p} \right) \dot{p}_\mu + \frac{N_p}{2} (p \cdot p - m^2_p) \right] - \int_{s_0}^{\infty} ds \left[ \left( y^\mu + \beta \frac{y \cdot q q^\mu}{1 - \beta q \cdot q} \right) \dot{q}_\mu + \frac{N_q}{2} (q \cdot q - m^2_q) \right] + \xi^\mu_{[0]} K^\mu_{[0]} ,
\]

where \( K^\mu_{[0]} = 0 \) is the conservation law at the interaction, while \( N_k, N_p, N_q \) and \( \xi^\mu_{[0]} \) are Lagrange multipliers. However, the \( \xi^\mu_{[0]} \) can be interpreted as interaction coordinates, which vanish for an observer local to the interaction, but not for distant observers [15]. The coordinates \( z^\mu, x^\mu, y^\mu \) are instead the position coordinates measured by generic observers. This interpretation is enforced by the crucial requirement that translations of parameter \( b^\mu \) are generated by the momentum conservation law \( K^\mu_{[0]} \) [4, 19]. From the boundary equation (15) below, we can in fact see that \( z^\mu, x^\mu \) and \( y^\mu \) all vanish when \( \xi^\mu_{[0]} = 0 \), but in general they are different when \( \xi^\mu_{[0]} \neq 0 \). This is the principle of relative locality: interactions that are local for an observer at the interaction point are seen by distant observers as nonlocal.

Several possibilities exist for the definition of \( K^\mu_{[0]} \). The most natural one, \( K^\mu_{[0]} = k_\mu \ominus (p \oplus q)_\mu \), does not satisfy the relativity requirement for the \( \kappa \)-Poincaré model [19]. The same problem arises for the Snyder model. The problem emerges when one considers finite worldlines with two endpoints. In this case, requiring that the equations of motion for two observers at rest \( A \) and \( B \) be the same, as required by the relativity principle [15], implies that the translations generated by the 'total momentum' (conservation laws) at the two ends of the worldlines give rise to the same relation between the position measured by the two observer, as described in detail in section 4. With the previous definition of \( K^\mu_{[0]} \) it is not possible to satisfy this request. However, because of (6), the conservation law can be equivalently written as

\[
K^\mu_{[0]} = k_\mu - (p \oplus q)_\mu = 0 ,
\]

and this expression permits to overcome the problem, as it was observed in [19] for the case of the \( \kappa \)-Poincaré model. This form of the conservation law expresses the requirement that the

\[ 4 \] This prescription is not standard, however for undeformed translations, it is equivalent to the usual one that the translations are generated by the momentum of the particle.
total momentum $k_\mu$ before the interaction be equal to the one after the interaction, $(p \oplus q)_\mu$. In particular, in our case we define

$$K_{\mu}^{[0]} = k_\mu - \frac{1}{1 + \beta p \cdot q} \left[ \left( 1 + \frac{\beta p \cdot q}{1 + \sqrt{1 - \beta p \cdot p}} \right) p_\mu + \sqrt{1 - \beta p \cdot p} q_\mu \right].$$

(9)

Note that this expression, due to the noncommutativity of the addition law of momenta, is not symmetric under the exchange of the outgoing particles. This feature has been already discussed in [19], where it has been shown that different orderings for the momenta appearing in the boundary terms produce in general different predictions for the physical observables. One can interpret the different ordering choices as different channels for the process. Moreover, due to the nonassociativity of the addition law, (8) is also not invariant if one changes the grouping of the momenta in the sum. However, at the classical level it seems natural to choose an expression like (8) that distinguishes the incoming particles from the outgoing ones. At the quantum level, or in more complicated processes, one may however be forced to consider also different orderings of the sums (see section 4).

For simplicity, from now on we consider the linearized theory, although it is not difficult to reproduce the calculation in the nonlinear case. The linearization of the action (7) yields

$$S = - \int_{-\infty}^{0} ds \left[ \left( \xi^{\mu} + \beta z \cdot k \xi^{\mu} \right) \dot{k}_\mu + \frac{N_k}{2} (k \cdot k - m_k^2) \right]$$

$$- \int_{s_0}^{\infty} ds \left[ \left( \xi^{\mu} + \beta x \cdot p \xi^{\mu} \right) \dot{p}_\mu + \frac{N_q}{2} (p \cdot p - m_q^2) \right]$$

$$- \int_{s_0}^{\infty} ds \left[ \left( \xi^{\mu} + \beta y \cdot q \xi^{\mu} \right) \dot{q}_\mu + \frac{N_q}{2} (q \cdot q - m_q^2) \right] + \xi^{\mu}_0 K_{\mu}^{[0]} + O(\beta^2),$$

(10)

with

$$K_{\mu}^{[0]} = k_\mu - p_\mu - q_\mu + \frac{\beta}{2} (p \cdot q p_\mu + 2 p \cdot q q_\mu + p \cdot p q_\mu) + O(\beta^2).$$

(11)

The equations of motion derived from (10) are

$$\dot{k}_\mu = \dot{p}_\mu = \dot{q}_\mu = 0, \quad k \cdot k - m_k^2 = p \cdot p - m_p^2 = q \cdot q - m_q^2 = 0,$$

$$\ddot{x}_\mu = N_q (1 - \beta k \cdot k) k_\mu, \quad \ddot{p}_\mu = N_q (1 - \beta p \cdot p) p_\mu, \quad \ddot{y}_\mu = N_q (1 - \beta q \cdot q) q_\mu,$$

(12)

together with $K_{\mu}^{[0]}(s_0) = 0$.

The boundary terms at $s = s_0$ yield

$$(\eta_{\mu\nu} + \beta k_k k_\nu) \xi^{\nu} (s_0) = \frac{\partial K_{\mu}^{[0]}}{\partial k_\mu} \xi^{\nu}_0 = \xi^{\mu}_0,$$

$$(\eta_{\mu\nu} + \beta p_k p_\nu) \xi^{\nu} (s_0) = \frac{\partial K_{\mu}^{[0]}}{\partial p_\mu} \xi^{\nu}_0,$$

$$(\eta_{\mu\nu} + \beta q_k q_\nu) \xi^{\nu} (s_0) = \frac{\partial K_{\mu}^{[0]}}{\partial q_\mu} \xi^{\nu}_0,$$

(13)

where
\[
\frac{\partial \mathcal{K}^{[0]}_\mu}{\partial p_\nu} \xi'_\nu^0 = - \xi'^0_\mu + \beta \left( q^\mu q_\nu + p^\mu q_\nu + \frac{1}{2} p_\nu q^\mu + \frac{1}{2} q \cdot q \delta^\mu_\nu \right) \xi'^0_\nu, \\
\frac{\partial \mathcal{K}^{[0]}_\mu}{\partial q_\nu} \xi'^0_\nu = - \xi'^0_\mu + \beta \left( \frac{1}{2} p^\mu p_\nu + p^\mu q_\nu + \frac{1}{2} p \cdot p \delta^\mu_\nu + p \cdot q \delta^\mu_\nu \right) \xi'^0_\nu.
\] (14)

Inverting the matrices at the left hand side, (13) can be put in the form

\[
\begin{align*}
\dot{z}^\mu(s_0) &= (\delta^\mu_\nu - \beta k^\mu k_\nu) \xi'^\nu_0, \\
\dot{x}^\mu(s_0) &= (\delta^\mu_\nu - \beta p^\mu p_\nu) \frac{\partial K^{[0]}_A}{\partial p_\nu} \xi'^\nu_0, \\
\dot{y}^\mu(s_0) &= (\delta^\mu_\nu - \beta q^\mu q_\nu) \frac{\partial K^{[0]}_A}{\partial q_\nu} \xi'^\nu_0.
\end{align*}
\] (15)

As discussed before, the boundary conditions establish that if the observer is local to the interaction, i.e. \( \xi'^0_0 = 0 \), the endpoints of the worldlines of all the particles are at the origin of the observer, while if \( \xi'^0_0 \neq 0 \), the endpoints of the worldlines of different particles do not coincide. Under infinitesimal deformed translations of parameter \( b^\nu \) between two observers \( A \) and \( B \), generated by the total momentum \( \mathcal{K}^{[0]}_\mu \), the spacetime coordinates transform as

\[
\begin{align*}
\dot{z}^\mu(s) &= \dot{z}^\mu_A + b^\nu \{ \mathcal{K}^{[0]}_\nu, z^\mu_A \} = \dot{z}^\mu_A(s) - (\delta^\mu_\nu - \beta k^\mu k_\nu) b^\nu, \\
\dot{x}^\mu(s) &= \dot{x}^\mu_A + b^\nu \{ \mathcal{K}^{[0]}_\nu, x^\mu_A \} = \dot{x}^\mu_A(s) - (\delta^\mu_\nu - \beta p^\mu p_\nu) \frac{\partial K^{[0]}_A}{\partial p_\nu} b^\lambda, \\
\dot{y}^\mu(s) &= \dot{y}^\mu_A + b^\nu \{ \mathcal{K}^{[0]}_\nu, y^\mu_A \} = \dot{y}^\mu_A(s) - (\delta^\mu_\nu - \beta q^\mu q_\nu) \frac{\partial K^{[0]}_A}{\partial q_\nu} b^\lambda.
\end{align*}
\] (16)

The equations of motion and the boundary conditions are invariant under these transformations if

\[
\xi'^\mu_{[0]|B} - \xi'^\mu_{[0]|A} = b^\mu.
\] (17)

In fact, the equations of motion (12) only contain the momenta and the time derivatives of the coordinates. The momenta are obviously invariant under translations, while differentiating (16) one sees that also \( \dot{z}, \dot{x} \) and \( \dot{y} \) are invariant, because the momenta are conserved. Moreover, using (16) it is easy to check that the boundary conditions (15) are invariant if (17) holds.

Of course one could choose a different ordering for the outgoing momenta, \( \mathcal{K}^{[0]}_\mu = k_\nu - (q \odot p)_\mu \). Although the momenta \( p \) and \( q \) are interchanged everywhere, the main conclusions are unaltered. As mentioned above, one may interpret this fact as the existence of two different channels for the interaction.

4. Causally connected interactions

We can now consider two causally connected interactions, occurring at \( s_0 \) and \( s_1 \), as depicted in figure 2.
The linearized action is the obvious generalization of (10):

\[
S = -\int_{-\infty}^{\infty} ds \left( \epsilon^\mu + \beta x \cdot k k^\mu \right) \dot{K}_\mu + \frac{N_k}{2} (k \cdot k - m_k^2) - \int_{-\infty}^{s_1} ds \left( \epsilon^\mu + \beta x \cdot p p^\mu \right) \dot{p}_\mu + \frac{N_p}{2} (p \cdot p - m_p^2)
- \int_{-\infty}^{s_1} ds \left( \epsilon^\mu + \beta x \cdot q q^\mu \right) \dot{q}_\mu + \frac{N_q}{2} (q \cdot q - m_q^2)
- \int_{-\infty}^{s_1} ds \left( \epsilon^\mu + \beta x^\prime \cdot p^\prime p^\prime_\mu \right) \dot{p}^\prime_\mu + \frac{N_{p'}}{2} (p' \cdot p' - m_{p'}^2)
- \int_{-\infty}^{s_1} ds \left( \epsilon^\mu + \beta x^\prime \cdot q^\prime q^\prime_\mu \right) \dot{q}^\prime_\mu.
\]

The equations of motion read

\[
\dot{K}_\mu = \dot{p}_\mu = \dot{q}_\mu = \dot{p}_\mu = \dot{p}'_\mu = 0,
\]

\[
k \cdot k - m_k^2 = p \cdot p - m_p^2 = q \cdot q - m_q^2 = p' \cdot p' - m_{p'}^2 = p'' \cdot p'' - m_{p''}^2 = 0,
\]

\[
\dot{x}_\mu = N_k (1 - \beta k \cdot k) x_\mu, \quad \dot{p}_\mu = N_p (1 - \beta p \cdot p) p_\mu, \quad \dot{q}_\mu = N_q (1 - \beta q \cdot q) q_\mu,
\]

\[
\dot{x}'_\mu = N_{p'} (1 - \beta p' \cdot p') p'_\mu, \quad \dot{p}'_\mu = N_{p'} (1 - \beta p'' \cdot p'') p''_\mu.
\]

The boundary conditions at \( s_0 \) still yield (15), while those at \( s_1 \) give

\[
x^\mu (s_1) = (\delta^\mu_\nu - \beta p^\mu_\nu p_\nu) \frac{\partial K_{[1]}^\lambda}{\partial p^\nu_\nu} \zeta^\lambda_{[1]},
\]

\[
x''^\mu (s_1) = (\delta^\mu_\nu - \beta p''^\mu_\nu p'\nu) \frac{\partial K_{[1]}^\lambda}{\partial p''^\nu_\nu} \zeta^\lambda_{[1]},
\]

\[
x'''^\mu (s_1) = (\delta^\mu_\nu - \beta p'''^\mu_\nu p''\nu) \frac{\partial K_{[1]}^\lambda}{\partial p'''^\nu_\nu} \zeta^\lambda_{[1]}.
\]

Under infinitesimal translation generated by \( K_{[1]}^\mu \), one has

\[
x''_B = x''_A + b'' \{ K_{[1]}^\mu, x''_A \} = x''_A - (\delta''^\mu_\nu - \beta p''^\mu_\nu p_\nu) \frac{\partial K_{[1]}^\lambda}{\partial p''^\nu_\nu} b^\lambda,
\]

\[
x'''_B = x'''_A + b''' \{ K_{[1]}^\mu, x'''_A \} = x'''_A - (\delta'''^\mu_\nu - \beta p'''^\mu_\nu p''_\nu) \frac{\partial K_{[1]}^\lambda}{\partial p'''^\nu_\nu} b^\lambda.
\]

As mentioned in the previous section, the fulfillment of the relativity principle implies some conditions on the form of the boundary terms associated to causally connected interactions. In
particular, a crucial requirement for the choice of the conservation laws at the interactions is that the following relation holds \[19\]

$$\frac{\partial K^{[0]}_\mu}{\partial p^\mu} = - \frac{\partial K^{[1]}_\mu}{\partial p^\mu}. \quad (22)$$

This is necessary in order to get the same equations of motion for both observers \(A\) and \(B\): in fact,

$$x^a_B(s_0) = x^a_A(s_0) + b_\lambda (\delta^a_{\mu} - \beta^a p^\mu) \frac{\partial K^{[0]}_\mu}{\partial p^\nu} \frac{\partial K^{[1]}_\mu}{\partial p^\nu}, \quad x^a_B(s_1) = x^a_A(s_1) - b_\lambda (\delta^a_{\mu} - \beta^a p^\mu) \frac{\partial K^{[1]}_\mu}{\partial p^\nu} \frac{\partial K^{[0]}_\mu}{\partial p^\nu}, \quad (23)$$

and \(\dot{x}_\mu = N_\mu (1 - \beta p \cdot p) p_\mu\) can hold for both observers only if (22) is verified. This implies that the translation parameters do not depend on \(s\). In order to satisfy (22), the authors of [19] propose for \(K^{[1]}\), in the case of \(\kappa\)-Poincaré

$$K^{[1]} = (p \oplus q) - (p' \oplus p'' \oplus q). \quad (24)$$

In our case, the second expression is not well defined, because of nonassociativity, but one may set

$$K^{[1]} = (p \oplus q) - ((p' \oplus p'') \oplus q), \quad (25)$$

as is natural from a physical point of view. In this case, the problems related to nonassociativity are more evident, since one may choose for example \(K^{[1]} = (p \oplus q) - (p' \oplus (p'' \oplus q))\), but our choice seems to be the most natural, at least at the classical level. However, for more complicated processes, this simple prescription may not be consistent with the principle of relativity (see next section), and one may be forced to consider several different choices compatible with the nonassociative and noncommutative character of the composition law. Again, one may interpret the alternative choices as different channels for the interaction.

As in the previous section, it can be checked that all the equations of motion are invariant under translations generated by the choice (25) of the conservation laws. In fact, the time derivative of the coordinates are unchanged under translations, because the nontrivial terms in the transformations depend only on the conserved momenta. Moreover, substituting (15), (23) into (16) and (21), one can see that also the boundary conditions maintain their form if \(\xi^{[1]}_\mu |_A - \xi^{[1]}_\mu |_B = b_\mu\) and (17) hold.

It must be noted that both the introduction of \(q\) in \(K^{[1]}\) and the ordering of the momenta are rather ad hoc assumptions and are justified by the fact that these conservation laws give the correct result. A drawback is that they give rise to a sort of entanglement, since with these prescriptions the interaction at a point depends on the past history of the particle (this recalls the spectator problem in DSR).

It appears therefore that in the Snyder case one can obtain a relativistic description of the interaction between particles using the same prescriptions for the interaction terms as in the \(\kappa\)-Poincaré case. It is likely that this recipe is valid for any relative locality model.

5. Time delay and dual curvature lensing in Snyder spacetime with interactions

We want to consider now, in our Snyder framework, a typical process suitable to highlight the effects of symmetry deformations on the time delay in the detection of ultra-high energy particles emitted by astrophysical sources. These effects have been proved to be of phenomenological relevance within the most studied relative-locality scenario with \(\kappa\)-Poincaré momentum
space [19], as for instance for the (in-vacuo) propagation of gamma ray bursts (GRB) or astrophysical neutrinos (see [6] for an up-to-date analysis), providing an example where the Planck-scale effects manifest themselves in a linear dependence of the times of arrival on the energies of the detected particles.

In the case of non-interacting particles, time delay effects have been investigated for the Snyder model in [31], where it has been shown that they are absent, due to the preservation of the linear action of the Lorentz transformations on phase space. However, in the case of interacting particles, the composition law of momenta is deformed [26, 29], and time-delay effects may arise.

Besides time delays, another kind of effect, involving the direction of particle propagation, has been studied in the relative locality literature [12, 23, 32, 33], where it has been named ‘dual gravity’ or, more appropriately, ‘dual curvature’ lensing. The effect is similar in its manifestation to the more standard gravitational lensing: in both cases the angle of observation of signals produced by some astrophysical sources is deflected so that their apparent position is different from their true position. But, while in the case of gravitational lensing this is due to the bending of light by massive objects between the source and the observer, dual curvature lensing is caused only by the properties of curvature of momentum space characterizing particle kinematics, and thus occurs also in in-vacuo propagation. As for in-vacuo dispersion, frameworks that provide a dual curvature lensing depending linearly on the propagating particle energies can be tested with presently available experimental observations [34]. We will set the framework of our analysis so to be able to take into account also this kind of effects.

We consider the process depicted in figure 3. This process can be used to describe schematically both the GRB-photon and GRB-neutrino scenarios. In the first case the particle \((p', x')\) can be interpreted as a highly boosted neutral pion, which decays at the source \((s_0)\) into a ‘hard’ (high energy) GRB photon \((p, x)\) and a second photon \((k, z)\). The GRB photon \((p, x)\) propagates freely and is detected through its interaction at \(s_1\) with a particle \((q, y)\) (for instance in the excitation and de-excitation of an atom of the detector) with which it exchanges a small amount of its momentum. In the second case the particle \((p', x')\) can be interpreted as a highly boosted charged pion, that decays at \((s_0)\) into a muon \((k, z)\) and a muon-neutrino \((p, x)\). The latter propagates freely until it is detected through its interaction with a particle \((q, y)\) (for instance a deep inelastic scattering with a nucleon of an atom at the detector) at the detector at \(s_1\).

The process of figure 3 can be described by an action of the form of (18), where the interactions are encoded in the boundary terms

\[
\begin{align*}
K^{[0]}_0 &= (q \oplus p') - (q \oplus p) \oplus k, \\
K^{[1]}_0 &= (q \oplus p) \oplus k - (p'' \oplus q') \oplus k.
\end{align*}
\]

(26)

Notice that in this case it is not possible to respect the naturalness condition stated in the previous section. In fact, the request of relativity, equation (22), forces one to choose the same ordering for the double sum \(q \oplus p \oplus k\) in the two interaction terms, preventing the definition of any intuitive criterion for this choice. Due to the nonassociativity, different orderings are not equivalent, and then one can define alternative interaction terms, as for example

\[
\begin{align*}
K^{[0]}_0 &= (q \oplus p') - q \oplus (p \oplus k), \\
K^{[1]}_0 &= q \oplus (p \oplus k) - (p'' \oplus q') \oplus k.
\end{align*}
\]

(27)

\[5\] Since the effect is due only to the curvature of momentum space, and does not depend on the dynamics of geometry, the latter choice appears more appropriate [12].
or even further ones obtained by permutations of the momenta. These alternative choices would give rise to results analogous to those following from (26), but with different numerical coefficients. Again, these could be considered as different channels of the interaction.

For simplicity, we shall restrict our analysis to \((2+1)\) dimensions, which turns out to be sufficient for our discussion, and consider only terms up to first order in \(\beta \sim \frac{1}{M_{\text{pl}}^2}\).

We want to compare the arrival time and direction of the hard GRB-photon (or the GRB-neutrino) \((p', x')\) at the detector with the arrival time (and direction) of a second ‘soft’ photon \((p_t, x_t)\), of much lower energy, emitted at the source simultaneously to the hard \((p, x)\) photon, but in an uncorrelated process. We consider a first observer \(A\) local to the emission event \(s_0\) and a second observer \(B\), at rest with respect to \(A\), local to the detector. Assuming the distance\(^6\) between the emission event and the detector to be \(T\), we define \(B\)’s coordinates to be obtained from \(A\)’s coordinates by a pure translation with parameters

\[
 b_{\mu} \equiv (b_0, b_1, b_2) = (T, T, 0) \, .
\]

This amounts to define the origin of \(B\)’s frame at what would be the point of the detector reached by a standard special relativistic photon emitted at \(A\)’s in the direction of its \(x_0^A\) axis.

We assume now that the ‘trigger’ photon \((p_t, x_t)\) is emitted at the source in a process similar to the one of figure 3, but not correlated to the former. We can picture schematically the two processes as in figure 4. We will first analyze the process of figure 3 generically and then impose the energy conditions on the particles specifying the high (hard GRB photon/GRB-neutrino) and low energy (trigger) processes.

Integrating with respect to \(s\) equations (3) (or (12)), we see that \(A\) and \(B\) describe a generic particle \((p, x)\) to travel along the worldlines

\[
 x_{A,B}^j \left( x_{A,B}^0 \right) = x_{A,B}^j + \left. \frac{x_{A,B}^j}{\partial x_{A,B}^0} \right|_{p_0 = p_0(p)} \left( x_{A,B}^0 - x_{A,B}^0 \right) \\
 = x_{A,B}^j + \frac{p^j}{p^0} \left( x_{A,B}^0 - x_{A,B}^0 \right) \, .
\]

---

\(^6\)The notion of distance here is subtle, since it needs to be operatively defined, for instance by considering the actual exchange of signals between the two observers at the emission and at the detector, so that its definition may be affected by the non-standard spacetime framework. Here we refer to the notion of distance one would have in ordinary special relativity. This assumption will be justified in the course of the analysis.
where \( x^\mu_{A,B} \equiv (x^0_{A,B}, x^1_{A,B}, x^2_{A,B}, x^3_{A,B}) \) are constants of motion specifying the worldline initial conditions. \( p^0(\vec{p}) = \sqrt{\vec{p}^2 + m^2} \) is the on-shell relation \((\vec{p} \equiv (p_1, p_2))\), and the momentum of the particle is identical for observers connected by a pure translation. With the definition given in the previous sections, translations act rigidly on the worldline coordinates so that each point of the worldline changes by the the same amount. If the relation between the coordinates of A and B after a translation is given by

\[
x^\mu_B = x^\mu_A + \delta x^\mu,
\]

using the boundary terms (26) the shifts \( \delta x^\mu \) are given by the Poisson brackets:

\[
\delta x^\mu = b^\nu \left\{ \left( \{ q \oplus p \} \right)_\mu, x^\nu \right\},
\]

where (see (1))

\[
\{ p^\mu, x^\nu \} = -\delta^\mu_\nu + \beta p^\mu p^\nu, \quad \{ q^\mu, x^\nu \} = \{ k^\mu, x^\nu \} = 0.
\]

From (5) we get, at first order in \( \beta \)

\[
(q \oplus p)_\mu \simeq q_\mu + p_\mu - \frac{\beta}{2} \left( q \cdot p q_\mu + q \cdot q p_\mu + 2q \cdot p p_\mu \right).
\]

and

\[
\left( (q \oplus p) \oplus k \right)_\mu \simeq (q \oplus p)_\mu + k_\mu - \frac{\beta}{2} \left( (q \cdot p) \cdot k (q \oplus p)_\mu + (q \cdot p) \cdot (q \oplus p) k_\mu + 2(q \cdot p) \cdot k k_\mu \right)
\]

\[
\simeq q_\mu + p_\mu + k_\mu - \frac{\beta}{2} \left( q \cdot p + q \cdot k + p \cdot k \right) q_\mu
\]

\[
- \frac{\beta}{2} \left( q \cdot q + 2q \cdot p + q \cdot k + p \cdot k \right) p_\mu - \frac{\beta}{2} \left( q \cdot q + p \cdot p + 2q \cdot p + 2q \cdot k + 2p \cdot k \right) k_\mu.
\]

so that we find

\[
\delta x^\mu \simeq -b^\mu + \frac{\beta}{2} \left( q \cdot q + 2q \cdot p + q \cdot k + p \cdot k \right) b^\mu + \beta (b \cdot p + b \cdot k) p^\mu
\]

\[
+ \frac{\beta}{2} \left( b \cdot q + 2b \cdot p + 2b \cdot k \right) q^\mu + \frac{\beta}{2} \left( b \cdot q + b \cdot p + 2b \cdot k \right) k^\mu.
\]
We now specify the energy conditions for both processes. Consider first the process for the trigger photon. This is described by the above equations by setting \( p, q, k \rightarrow p_t, q_t, k_t \). In order to take into account possible dual curvature effects \cite{23}, we allow the photon to be emitted at \( A \) with a (small) angle \( \alpha_t \) between the \( x^t_A \) and \( x^t_B \) axis, to be determined by the condition that the photon is detected at \( B \)'s spatial origin \( \vec{x}_B = (z_B, \vec{x}_B) = (0, 0) \). The presence of this angle can be implemented by defining the photon momentum as

\[
\vec{p}_t = (p_t^0, p_t^1, p_t^2) = (|\vec{p}_t| \cos \alpha_t, |\vec{p}_t| \sin \alpha_t).
\]  

(36)

In \( A \)'s frame the equations of motion for the photon become

\[
\begin{align*}
(x^t_A) &= (x^t_A) \cos \alpha_t, \\
(x^t_A) &= (x^t_A) \sin \alpha_t,
\end{align*}
\]  

(37)

where we have imposed the initial conditions \( x^t_A \equiv (0, 0, 0) \) enforcing the photon to be emitted at \( A \)'s spacetime origin. Imposing \( x_B = (0, 0) \) in (29) we find the equations of motion in \( B \)'s frame

\[
\begin{align*}
(x^t_B) &= \left( (x^t_B) - (x^t_B) \right) \cos \alpha_t, \\
(x^t_B) &= \left( (x^t_B) - (x^t_B) \right) \sin \alpha_t.
\end{align*}
\]  

(38)

Substituting (30) in (38), and using (37), we get

\[\tan \alpha_t = \frac{\delta x^2}{\delta x^t}, \quad (x^t_B) = \delta x^0 - \delta x^1 \sqrt{1 + \tan^2 \alpha_t}.\]  

(39)

It follows from (35) and (28) that

\[\alpha_t \simeq \tan \alpha_t \simeq -\beta \left( \left( p^0 - p^1 \right) + \left( k^0 - k^1 \right) \right) p^2 + \frac{\beta}{2} \left( k^0 - k^1 \right) q^2,\]  

\[= \frac{\beta}{2} \left[ \left( q^0 - q^1 \right) + \left( p^0 - p^1 \right) + 2 \left( k^0 - k^1 \right) \right] k^2 + \frac{\beta}{2} \left[ \left( q^0 - q^1 \right) + 2 \left( p^0 - p^1 \right) + 2 \left( k^0 - k^1 \right) \right] q^2.\]  

(40)

where we are neglecting terms of order \( \beta^2 \). Notice now that, since the angle \( \alpha_t \) is of order \( \beta \), as can be seen from (36), the difference between \( p^0 \) and \( p^1 \) can be neglected in (40). Similarly, assuming\(^7\) the direction of the \( (k, z) \) worldline to be collinear to the \( (p, x) \) one, so that \( k^t_A \equiv (|\vec{k}_A|, \bar{k}_A \cos \alpha, |\bar{k}_A| \sin \alpha) \), then \( k^0_A - k^1_A \simeq O(\beta) \). Moreover \( p^2_t \simeq \alpha_t |\vec{p}_t| = O(\beta) \) as well as \( k^2_A \simeq \alpha_t |\vec{k}_A| = O(\beta) \) and, at linear order in \( \beta \), we are left with

\[\alpha_t \simeq -\frac{\beta}{2} \left( q^0 - q^1 \right) q^2.\]  

(41)

Before discussing the interpretation of this angle, let us calculate the time of detection of the trigger photon, given by the second of equation (39). Using again (35) and (28) we find

\[
\left( x^t_B \right) = \beta T \left[ \left( p^0 - p^1 \right) + \left( k^0 - k^1 \right) \right] \left( p^0_t - p^1_t \right) + \frac{\beta}{2} T \left[ 2 \left( p^0_t - p^1_t \right) + \left( q^0 - q^1 \right) + 2 \left( k^0 - k^1 \right) \right] \left( q^0_t - q^1_t \right) + \frac{\beta}{2} T \left[ \left( p^0 - p^1 \right) + \left( q^0_t - q^1_t \right) + 2 \left( k^0 - k^1 \right) \right] \left( k^0_t - k^1_t \right).
\]  

(42)

\(^7\)Beware that \( p^2_t, k^2, q^2 \) represent the second component of the momentum vectors, and not the squares of \( p, k, q \), to which we reserve the notation \( p \cdot p, q \cdot q, k \cdot k \).

\(^8\)The pion \( (\vec{p}', \vec{x}') \) decaying at the source is highly boosted, so that the product particles \( (p, x) \) and \( (k, z) \) can be taken to be collinear. We keep this assumption also for the trigger photon, which we consider also to be a GRB photon (of lower energy).
The same considerations that led us to (41) reduce the last equation to
\[
\left(\bar{x}_l^0\right)_B \simeq \frac{\beta}{2} T \left(q_l^0 - q_l^1\right)^2.
\]

It emerges from equations (41) and (43) that the angle and time with which the trigger photon is detected at \(B\) depend on the details of the interaction at the detector. In particular we can set
\[
q_l^\mu \equiv (E_q t, |\vec{q}_l| \cos \theta_t, |\vec{q}_l| \sin \theta_t)
\]
and obtain
\[
\alpha_t \simeq -\frac{\beta}{2} (E_q - |\vec{q}_l| \cos \theta_t) |\vec{q}_l| \sin \theta_t,
\]
\[
\left(\bar{x}_l^0\right)_B \simeq \frac{\beta}{2} T (E_q - |\vec{q}_l| \cos \theta_t)^2.
\]

We can repeat exactly the same procedure that has led us to (45) to derive the angle \(\alpha\) and the arrival time for the hard GRB-photon (or the GRB-neutrino) \((p, x)\), and find
\[
\alpha \simeq -\frac{\beta}{2} (E_q - |\vec{q}| \cos \theta) |\vec{q}| \sin \theta,
\]
\[
\left(\bar{x}_l^0\right)_B \simeq \frac{\beta}{2} T (E_q - |\vec{q}| \cos \theta)^2.
\]

We thus finally find that the hard particle (the hard GRB-photon or the GRB-neutrino) is detected, with respect to the trigger photon, at an angle and time delay
\[
\Delta \alpha \simeq -\frac{\beta}{2} \left( (E_q - |\vec{q}| \cos \theta) |\vec{q}| \sin \theta - (E_q - |\vec{q}_l| \cos \theta_l) |\vec{q}_l| \sin \theta_l \right),
\]
\[
\Delta t \simeq \frac{\beta}{2} T \left( (E_q - |\vec{q}| \cos \theta)^2 - (E_q - |\vec{q}_l| \cos \theta_l)^2 \right).
\]

If the particles \(q_l\) and \(q\) at the detector are non relativistic and particularly such that their momentum is much smaller than their mass \((|\vec{q}|/m \ll 1)\), the last expressions reduce to
\[
\Delta \alpha \simeq -\frac{\beta}{2} (m_q |\vec{q}| \sin \theta_m - m_{q_l} |\vec{q}_l| \sin \theta_{l_m}),
\]
\[
\Delta t \simeq \frac{\beta}{2} T \left( m_q^2 - m_{q_l}^2 \right).
\]

The results are pictured in figure 5.

Equation (47) shows that both time-delay and dual-curvature lensing effects are present, depending on the details of the interaction between the \((p, x)\) (or \((p_t, x_t)\)) particle and the detector. The time-delay effect is extremely tiny, since it is proportional to the square of the ratio between the particle masses at the detector, like for instance atoms or nucleons composing the detector, and the Planck mass \((\beta \propto 1/M_{P}^2)\), if the Snyder deformation has to be understood as generated by some quantum gravity effect. The only amplifying factor for the time delay is the distance \(T\) traveled by the photon from the source to the detector. Notice however that the effect does not depend on the photon (or neutrino) energies. This means that in principle the time delay induced by the Snyder deformation could be investigated considering also low-energy particles, the drawback being of course that the particle energy does not act as an amplifier for the Planckian effect. Thus the effect is far beyond the reach of present astrophysical experiments, but our analysis shows that, at least within the relative-locality scenario, one could in principle devise a detector capable of testing a deformation of spacetime symmetries of Snyder type. Similar considerations hold for the lensing effect, but this is much fainter, and has only a theoretical relevance.
An important remark is that the effects we have found mostly depend on the properties of the detector, rather than of the incoming particles. This is a distinctive feature of the Snyder phenomenology.

Finally, it is interesting to see if the same effects are obtained also for a different choice of the interaction term, like (27), instead of (26). In this case, one has

\[
(q \oplus (p \oplus k))_\mu \simeq q_\mu + p_\mu + k_\mu - \frac{\beta}{2} (q \cdot p + q \cdot k) q_\mu
- \frac{\beta}{2} (q \cdot q + 2q \cdot p + 2q \cdot k + p \cdot k) p_\mu
- \frac{\beta}{2} (q \cdot q + p \cdot p + 2q \cdot p + 2q \cdot k + 2p \cdot k) k_\mu,
\]

and (35) becomes

\[
\delta x^\mu \simeq - b^\mu + \frac{\beta}{2} (q \cdot q + 2q \cdot p + 2q \cdot k + p \cdot k) b^\mu + \beta (b \cdot p + b \cdot k) p^\mu
+ \frac{\beta}{2} (b \cdot q + 2b \cdot p + 2b \cdot k) q^\mu + \frac{\beta}{2} (b \cdot p + 2b \cdot k) k^\mu.
\]

Repeating the same steps as before, the calculation of the time delay $\Delta t$ and of the dual curvature lensing $\Delta \alpha$ again reproduces the result (47). At this order of approximation, the nonassociativity is therefore not relevant for the experimental predictions.

Due to the noncommutativity, one may however also modify the ordering of the momenta in (26) or (27). In this case, a factor of 2 may appear in (47) for some permutations, but the qualitative results are not modified. It is reasonable to assume that an actual measurement would average among all the possible outcomes predicted by modifying the interaction term.
6. Discussion

In this paper we have extended the investigation of the dynamics of relative locality to the case of the Snyder model. Formally, this generalization does not introduce new features in comparison with previously studied models, except for the nonassociativity of the momenta addition law, that entails some ambiguities in the definition of the interaction terms. However, phenomenological prediction can be rather different.

More specifically, we have found that in the relative locality framework Snyder momentum space predicts, under certain conditions, a non-null time delay in the arrival of photons emitted simultaneously from a distant source. It is worth comparing the leading term characterizing the time delay effect for the Snyder case with the one for the $\kappa$-Poincaré case obtained in [19]:

$$\text{Snyder} : \Delta t_S \simeq T_\gamma \frac{\Delta m_{\text{det}}^2}{E_p}, \quad \kappa\text{-Poincaré} : \Delta t_\kappa \simeq T_\gamma \frac{\Delta E_\gamma}{E_p}. \quad (51)$$

In both instances the effect depends on the distance (in time) $T_\gamma$ traveled by the photons from the source to the detector. However, for the Snyder case it does not depend on the photon energies $E_\gamma$, as expected, since in Snyder momentum space the on-shell relation for a massless particle is undeformed (and thus there is no in-vacuo dispersion for photons) so that, besides being of second order in $1/E_p$, it lacks one of the two sources of amplification that balance the smallness of the Planckian effect. As a matter of fact, the Snyder effect is of a different nature with respect to the $\kappa$-Poincaré one, since it depends on the mass difference $\Delta m_{\text{det}}$ of the particles at the detector with which two photons emitted simultaneously at the source respectively interact, rather than on the energy difference $\Delta E_\gamma$ between the traveling photons, as for the $\kappa$-Poincaré case. This means that the effect we have found for the Snyder model affects the propagation of photons independently of their energies, but only depending on the details of the processes by which they interact at the detector, so that it could be in principle investigated using astrophysical events emitting low-energy photons. Obviously the effect is too tiny to be realistically taken into consideration experimentally.

We notice that an effect similar to the one we found for Snyder, with the time delay driven by the ratio $\Delta m_{\text{det}}/E_p$, is present also for the $\kappa$-Poincaré case, even if it is subleading with respect to the term in (51). Indeed, if one repeats the analysis of time delay in the framework of [19] (we skip the details of the derivation and leave it to the reader), and sets the energy conditions we used in section 5, one obtains the time delay

$$\Delta t_\kappa \simeq T_\gamma \frac{\Delta E_\gamma}{E_p} + T_\gamma \frac{\Delta m_{\text{det}}}{E_p}. \quad (52)$$

We understand this effect as a feature of the formalism of relative locality, where the nontrivial summation law of the momenta of the particles entering the interaction manifests itself in a dependence of the detection times on the kinematical details of the process.

We have also found that in Snyder framework an effect of dual curvature lensing [12, 23, 32, 33] is present. The effect is similar to the one discussed in discussed in [23] for a different framework\(^9\), where the magnitude of the corrections depends only on the details of the interaction, and not on the energy of the propagating particles. Again the effect is extremely tiny, since not even the propagation distance $T$ can provide a source of amplification, and has therefore only a theoretical relevance.

To conclude, we recall that the Snyder model differs from $\kappa$-Poincaré models in several respects: it preserves the linear action of the Lorentz group and by consequence the leading

\(^9\)In the case studied in [23] only the dual curvature effect is present, while time delay is absent.
correction to special relativity are of order $1/E_p^2$ [29], and moreover the law of addition of momenta is nonassociative. The former properties imply that relative locality effects can only be detected in many-particle interactions and are highly suppressed with respect to the corresponding $\kappa$-Poincaré corrections. The latter one gives rise to much more involved problems in the definition of the interactions, but does not affect phenomenology significantly.

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References

[1] Amelino-Camelia G 2013 Living Rev. Relativ. 16 5
[2] Mattingly D 2005 Living Rev. Relativ. 8 5
[3] Amelino-Camelia G, Ellis J R, Mavromatos N E, Nanopoulos D V and Sarkar S 1998 Nature 393 763
[4] Alfaro J, Morales-Tecotl H A and Urrutia L F 2000 Phys. Rev. Lett. 84 2318
[5] Jacob U and Piran T 2007 Nat. Phys. 3 87
[6] Amelino-Camelia G, D’Amico G, Rosati G and Loret N 2017 Nat. Astron. 1 0139
[7] Amelino-Camelia G, D’Amico G, Fiore F, Puccetti S and Ronco M 2017 arXiv:1707.02413 [astro-ph.HE]
[8] Amelino-Camelia G 2002 Int. J. Mod. Phys. D 11 35
Amelino-Camelia G 2001 Phys. Lett. B 510 255
[9] Kowalski-Glikman J 2001 Phys. Lett. A 286 391
[10] Magueijo J and Smolin L 2003 Phys. Rev. D 67 044017
[11] Rosati G, Amelino-Camelia G, Marciano A and Matassa M 2015 Phys. Rev. D 92 124042
[12] Amelino-Camelia G, Barcaroli L, Bianco S and Pensato L 2017 Adv. High Energy Phys. 2017 6075920
[13] Amelino-Camelia G, Matassa M, Mercati F and Rosati G 2011 Phys. Rev. Lett. 106 071301
[14] Amelino-Camelia G, Loret N and Rosati G 2011 Phys. Lett. B 700 150
[15] Amelino-Camelia G, Freidel L, Kowalski-Glikman J and Smolin L 2011 Phys. Rev. D 84 084010
Amelino-Camelia G, Freidel L, Kowalski-Glikman J and Smolin L 2011 Gen. Relativ. Gravit. 43 2547
[16] Carmona J M, Cortes J L, Mazon D and Mercati F 2011 Phys. Rev. D 84 085010
[17] Kowalski-Glikman J 2013 Int. J. Mod. Phys. A 28 1330014
[18] Amelino-Camelia G, Gubitosi G and Palmasano G 2016 Int. J. Mod. Phys. D 25 1650027
[19] Amelino-Camelia G, Arzano M, Kowalski-Glikman J, Rosati G and Trevisan G 2012 Class. Quantum Grav. 29 075007
[20] Gubitosi G and Mercati F 2013 Class. Quantum Grav. 30 145002
[21] Amelino-Camelia G, Bianco S, Brighenti F and Buonocore R J 2015 Phys. Rev. D 91 084045
[22] Kowalski-Glikman J and Rosati G 2015 Phys. Rev. D 91 084061
[23] Amelino-Camelia G, Arzano M, Bianco S and Buonocore R J 2013 Class. Quantum Grav. 30 065012
[24] Snyder H S 1947 Phys. Rev. 71 38
[25] Gol’fand Y 1960 J. Exp. Theor. Phys. 10 356
Kadyshevskii V 1963 J. Exp. Theor. Phys. 14 1340
Jaroszkiewicz G 1995 J. Phys. A: Math. Gen. 28 L343
Romero J M and Zamora A 2004 Phys. Rev. D 70 105006
Banerjee R, Kulkarni S and Samanta S 2006 J. High Energy Phys. JHEP05(2006)077
Mignemi S 2015 Int. J. Mod. Phys. D 24 1550043
[26] Battisti M V and Meljancic S 2009 Phys. Rev. D 79 067505
Battisti M V and Meljancic S 2010 Phys. Rev. D 82 024028
Girelli F and Livine E R 2011 J. High Energy Phys. JHEP03(2011)132

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[27] Mignemi S and Samsarov A 2017 Phys. Lett. A 381 1655
[28] Amelino-Camelia G and Astuti V 2015 Int. J. Mod. Phys. D 24 1550073
[29] Ivetić B, Mignemi S and Samsarov A 2016 Phys. Rev. D 94 064064
[30] Ivetić B and Mignemi S 2017 arXiv:1711.07438 [hep-th]
[31] Mignemi S and Štrajn R 2016 Phys. Lett. A 380 1714
[32] Freidel L and Smolin L 2011 arXiv:1103.5626 [hep-th]
[33] Amelino-Camelia G, Barcaroli L and Loret N 2012 Int. J. Theor. Phys. 51 3359
[34] Amelino-Camelia G, Barcaroli L, D’Amico G, Loret N and Rosati G 2017 Int. J. Mod. Phys. D 26 1750076