Large time behavior of the heat kernel

Yehuda Pinchover
Department of Mathematics
Technion - Israel Institute of Technology
Haifa 32000, Israel
pincho@techunix.technion.ac.il

Abstract
In this paper we study the large time behavior of the (minimal) heat kernel $k^M_P(x, y, t)$ of a general time independent parabolic operator $L = u_t + P(x, \partial_x)u$ which is defined on a noncompact manifold $M$. More precisely, we prove that

$$\lim_{t \to \infty} e^{\lambda_0 t} k^M_P(x, y, t)$$

always exists. Here $\lambda_0$ is the generalized principal eigenvalue of the operator $P$ in $M$.

2000 Mathematics Subject Classification. Primary 35K10; Secondary 35B40, 58J35, 60J60.

Keywords. Heat kernel, ground state, principal eigenvalue, recurrence.

1 Introduction
Let $k^M_P(x, y, t)$ be the (minimal) heat kernel of a time independent parabolic operator $Lu = u_t + P(x, \partial_x)u$ which is defined on a noncompact Riemannian manifold $M$. Denote by $\lambda_0$ the generalized (Dirichlet) principal eigenvalue of the operator $P$ in $M$.

Over the past three decades, there have been a large number of works devoted to large time estimates of the heat kernel in various settings (see for example the following monographs and survey articles [2, 4, 6, 8, 9, 14, 15, 16, 17, 19, 20, 21], and the references therein). Despite the wide diversity of the results in this field, the following basic question has not been fully answered.

Question 1.1 Does $\lim_{t \to \infty} e^{\lambda_0 t} k^M_P(x, y, t)$ always exist?
The aim of this paper is to give a complete answer to Question 1.1 for arbitrary \( P \) and \( M \). The following theorem \([12]\) gives only a partial answer to the above question (see also \([3, 9, 14, 18, 19]\)).

**Theorem 1.2** Let \( P \) an elliptic operator on \( M \).

(i) If \( P - \lambda_0 \) is subcritical in \( M \) (i.e. \( \int_0^\infty e^{\lambda_0 t} k_P^M(x, y, t) dt < \infty \)), then

\[
\lim_{t \to \infty} e^{\lambda_0 t} k_P^M(x, y, t) = 0.
\]

(ii) If \( P - \lambda_0 \) is positive-critical in \( M \) (i.e. \( \int_0^\infty e^{\lambda_0 t} k_P^M(x, y, t) dt = \infty \), and the ground states \( \varphi \) and \( \varphi^* \) of \( P - \lambda_0 \) and \( P^* - \lambda_0 \) respectively, satisfy \( \varphi^* \varphi \in L^1(M) \)), then

\[
\lim_{t \to \infty} e^{\lambda_0 t} k_P^M(x, y, t) = \frac{\varphi(x) \varphi^*(y)}{\int_M \varphi^*(z) \varphi(z) \, dz}.
\]

(iii) If \( P - \lambda_0 \) is null-critical in \( M \) (i.e. \( \int_0^\infty e^{\lambda_0 t} k_P^M(x, y, t) dt = \infty \), and the ground states \( \varphi \) and \( \varphi^* \) of \( P - \lambda_0 \) and \( P^* - \lambda_0 \) respectively, satisfy \( \varphi^* \varphi \notin L^1(M) \)), then

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T e^{\lambda_0 t} k_P^M(x, y, t) dt = 0.
\]

Moreover, if one assumes further that \( P \) is a formally symmetric operator \((P = P^*)\), then in the null-critical case \( \lim_{t \to \infty} e^{\lambda_0 t} k_P^M(x, y, t) = 0 \).

The main result of the present paper is the following theorem which answers the author’s conjecture \([12, \text{Remark 1.4}]\) about the existence of the limit in the null-critical nonsymmetric case.

**Theorem 1.3** Assume that \( P - \lambda_0 \) is a (nonsymmetric) null-critical operator in \( M \). Then

\[
\lim_{t \to \infty} e^{\lambda_0 t} k_P^M(x, y, t) = 0.
\]

Thus, theorems 1.2 and 1.3 indeed solve Question 1.1. More precisely, with the aid of Theorem 1.1 of \([12]\), we have

**Corollary 1.4** The \( \lim_{t \to \infty} e^{\lambda_0 t} k_P^M(x, y, t) \) exists for all \( x, y \in M \), and the limit is positive if and only if the operator \( P - \lambda_0 \) is positive-critical.

Moreover, let \( G_{P-\lambda}^M(x, y) \) be the minimal positive Green function of the elliptic operator \( P - \lambda \) on \( M \). Then

\[
\lim_{t \to \infty} e^{\lambda_0 t} k_P^M(x, y, t) = \lim_{\lambda \to \lambda_0} (\lambda_0 - \lambda) G_{P-\lambda}^M(x, y). \quad (1.1)
\]
The proof of Theorem 1.3 hinges on Lemma 4.1 which is a slight extension of a lemma of Varadhan (see, [13, Lemma 9, page 259] or [14, pp. 192–193]). Varadhan proved his lemma for positive-critical operators on $\mathbb{R}^d$ using a purely probabilistic approach. Our key observation is that the assertion of Varadhan’s lemma is valid under the weaker assumption that the skew product operator $\bar{P} = P \otimes I + I \otimes P$ is critical in $\bar{M} = M \times M$, where $I$ is the identity operator on $M$. We note that if $\bar{P}$ is subcritical in $\bar{M}$, then by Theorem 1.2, the heat kernel of $\bar{P}$ on $\bar{M}$ tends to zero as $t \to \infty$. Since the heat kernel of $\bar{P}$ is equal to the product of the heat kernels of its factors, it follows that if $\bar{P}$ is subcritical in $\bar{M}$, then $\lim_{t \to \infty} k^{M}_{\bar{P}}(x, y, t) = 0$.

In Section 4, we formulate and give a purely analytic proof of Lemma 4.1. Our proof of the lemma is in fact the translation of Varadhan’s proof to the analytic apparatus. It uses the large time behaviors of the parabolic capacitory potential and of the heat content (see Section 3).

The proof of Theorem 1.3 is given in Section 5. We conclude the paper with some open problems which are closely related to the large time behavior of the heat kernel (see Section 6).

Remark 1.5 In the null-recurrent case, the heat kernel may decay very slowly as $t \to \infty$, and one can construct a complete Riemannian manifold $M$ such that all its Riemannian products $M^j, j \geq 1$ are null-recurrent (see [8]).

Remark 1.6 We would like to point out that the results of this paper, are also valid for an elliptic operator $P$ in divergence form and also for a strongly elliptic operator $P$ with locally bounded coefficients.

2 Preliminaries

Let $P$ be a linear, second order, elliptic operator defined in a noncompact, connected, $C^3$-smooth Riemannian manifold $M$ of dimension $d$. Here $P$ is an elliptic operator with real, Hölder continuous coefficients which in any coordinate system $(U; x_1, \ldots, x_d)$ has the form

$$P(x, \partial_x) = -\sum_{i,j=1}^{d} a_{ij}(x) \partial_i \partial_j + \sum_{i=1}^{d} b_i(x) \partial_i + c(x),$$

(2.1)

where $\partial_i = \partial/\partial x_i$. We assume that for every $x \in M$ the real quadratic form

$$\sum_{i,j=1}^{d} a_{ij}(x) \xi_i \xi_j, \quad \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$$

(2.2)
is positive definite. The formal adjoint of $P$ is denoted by $P^*$. We consider the parabolic operator

$$Lu = u_t + Pu \quad \text{on } M \times (0, \infty).$$

Let $\{M_j\}_{j=1}^{\infty}$ be an exhaustion of $M$, i.e. a sequence of smooth, relatively compact domains such that $M_1 \neq \emptyset$, $\text{cl}(M_j) \subset M_{j+1}$ and $\bigcup_{j=1}^{\infty} M_j = M$. For every $j \geq 1$, we denote $M_j^* = M \setminus \text{cl}(M_j)$. Let $M_\infty = M \cup \{\infty\}$ be the one-point compactification of $M$. By the notation $x \to \infty$, we mean that $x \to \infty$ in the topology of $M_\infty$.

Denote the cone of all positive (classical) solutions of the equation $Pu=0$ in $M$ by $\mathcal{C}_P(M)$. The generalized principal eigenvalue is defined by

$$\lambda_0 = \lambda_0(P,M) := \sup\{\lambda \in \mathbb{R} : \mathcal{C}_{P-\lambda}(M) \neq \emptyset\}.$$ 

Throughout this paper we always assume that $\lambda_0 \geq 0$.

For every $j \geq 1$, consider the Dirichlet heat kernel $k_P^{M_j}(x,y,t)$ of the parabolic operator $L = \partial_t + P$ in $M_j$. So, for every continuous function $f$ with a compact support in $M$, $u(x,t) = \int_{M_j} k_P^{M_j}(x,y,t)f(y)\,dy$ solves the initial-Dirichlet boundary value problems

$$
\begin{align*}
Lu &= 0 \quad \text{in } M_j \times (0, \infty), \\
u &= 0 \quad \text{on } \partial M_j \times (0, \infty), \\
u &= f \quad \text{on } M_j \times \{0\}.
\end{align*}
$$

By the maximum principle, $\{k_P^{M_j}(x,y,t)\}_{j=1}^{\infty}$ is an increasing sequence which converges to $k_P^M(x,y,t)$, the minimal heat kernel of the parabolic operator $L$ in $M$. If

$$\int_0^{\infty} k_P^M(x,y,t)\,dt < \infty \quad \text{(respectively, } \int_0^{\infty} k_P^M(x,y,t)\,dt = \infty),$$

then $P$ is said to be a subcritical (respectively, critical) operator in $M$, [14].

It can be easily checked that for $\lambda \leq \lambda_0$, the heat kernel $k_{P-\lambda}^M$ of the operator $P - \lambda$ is equal to $e^{\lambda t}k_P^M(x,y,t)$. Since we are interested in the asymptotic behavior of $e^{\lambda_0 t}k_P^M(x,y,t)$, we assume throughout the paper (unless otherwise stated) that $\lambda_0 = 0$.

It is well known that if $\lambda_0 > 0$, then $P$ is subcritical in $M$. Clearly, $P$ is critical (respectively, subcritical) in $M$, if and only if $P^*$ is critical (respectively, subcritical) in $M$. Furthermore, if $P$ is critical in $M$, then $\mathcal{C}_P(M)$ is a one-dimensional cone. In this case, $\varphi \in \mathcal{C}_P(M)$ is called a
ground state of the operator $P$ in $M$ \[12, 14\]. We denote the ground state of $P^*$ by $\varphi^*$.

The ground state $\varphi$ is a global positive solution of the equation $Pu = 0$ of minimal growth in a neighborhood of infinity in $M$. That is, if $v \in C(M^*_j)$ is a positive solution of the equation $Pu = 0$ in $M^*_j$ such that $\varphi \leq v$ on $\partial M^*_j$, then $\varphi \leq v$ in $M^*_j$ \[12, 14\].

In the critical case, the ground state $\varphi$ (respectively, $\varphi^*$) is a positive invariant solution of the operator $P$ (respectively, $P^*$) in $M$ (see for example \[12, 14\]). That is,

\[
\int_M k^M_P(x, y, t)\varphi(y) \, dy = \varphi(x), \quad \text{and} \quad \int_M k^M_P(x, y, t)\varphi^*(x) \, dx = \varphi^*(y). \quad (2.5)
\]

**Definition 2.1** A critical operator $P$ is said to be positive-critical in $M$ if $\varphi^*\varphi \in L^1(M)$, and null-critical in $M$ if $\varphi^*\varphi \notin L^1(M)$.

**Remark 2.2** Let $1$ be the constant function on $M$, taking at any point $x \in M$ the value 1. Suppose that $P1 = 0$. Then $P$ is subcritical (respectively, positive-critical, null-critical) in $M$ if and only if the corresponding diffusion process is transient (respectively, positive-recurrent, null-recurrent) \[14\]. In fact, since we are interested in the critical case, it is natural to use the $h$-transform with $h = \varphi$. So,

\[
P^\varphi u = \frac{1}{\varphi} P(\varphi u) \quad \text{and} \quad k^M_P(x, y, t) = \frac{1}{\varphi(x)} k^M_P(x, y, t)\varphi(y).
\]

Note that $P^\varphi$ is null-critical (respectively, positive-critical) if and only if $P$ is null-critical (respectively, positive-critical), and the ground states of $P^\varphi$ and $(P^\varphi)^*$ are 1 and $\varphi^*\varphi$, respectively. Moreover,

\[
\lim_{t \to \infty} k^M_P(x, y, t) = 0 \quad \text{if and only if} \quad \lim_{t \to \infty} k^M_P(x, y, t) = 0.
\]

Therefore, throughout the paper (unless otherwise stated), we assume that

(A) \[ P1 = 0, \quad \text{and} \quad P \text{ is a critical operator in } M. \]

It is well known that on a general noncompact manifold $M$, the solution of the Cauchy problem for the parabolic equation $Lu = 0$ is not uniquely determined (see for example \[10\] and the references therein). On the other hand, under Assumption (A), there is a unique minimal solution of the Cauchy problem and of certain initial-boundary value problems for bounded initial and boundary conditions. More precisely,
**Definition 2.3** Let $f$ be a bounded continuous function on $M$. By the minimal solution $u$ of the Cauchy problem

\[
Lu = 0 \quad \text{in } M \times (0, \infty),
\]

\[
u = f \quad \text{on } B^* \times \{0\},
\]

we mean the function

\[
u(x, t) := \int_M k^M_P(x, y, t)f(y) \, dy.
\] (2.6)

**Definition 2.4** Let $B \subset M$ be a smooth bounded domain such that $B^* := M \setminus \text{cl}(B)$ is connected. Assume that $f$ is a bounded continuous function on $B^*$, and $g$ is a bounded continuous function on $\partial B \times (0, \infty)$. By the minimal solution $u$ of the initial-boundary value problem

\[
Lu = 0 \quad \text{in } B^* \times (0, \infty),
\]

\[
u = g \quad \text{on } \partial B \times (0, \infty),
\]

\[
u = f \quad \text{on } B^* \times \{0\},
\] (2.7)

we mean the limit of the solutions $u_j$ of the following initial-boundary value problems

\[
Lu = 0 \quad \text{in } (B^* \cap M_j) \times (0, \infty),
\]

\[
u = g \quad \text{on } \partial B \times (0, \infty),
\]

\[
u = 0 \quad \text{on } \partial M_j \times (0, \infty),
\]

\[
u = f \quad \text{on } (B^* \cap M_j) \times \{0\}.
\]

**Remark 2.5** It can be easily checked that the sequence $\{u_j\}$ is indeed a converging sequence which converges to a solution of the initial-boundary value problem (2.7).

### 3 Auxiliary results

**Lemma 3.1** Assume that $P1 = 0$ and that $P$ is critical in $M$. Let $B := B(x_0, \delta) \subset M$ be the ball of radius $\delta$ centered at $x_0$, and suppose that
$B^{*} = M \setminus \text{cl}(B)$ is connected. Let $w$ be the heat content of $B^{*}$, i.e. the minimal nonnegative solution of the following initial-boundary value problem

$$
Lu = 0 \quad \text{in } B^{*} \times (0, \infty),
$$

$$
u = 0 \quad \text{on } \partial B \times (0, \infty),
$$

$$
u = 1 \quad \text{on } B^{*} \times \{0\}. \quad (3.1)
$$

Then $w$ is a decreasing function of $t$, and \(\lim_{t \to \infty} w(x, t) = 0\) locally uniformly in $M$.

**Proof:** Clearly,

$$
w(x, t) = \int_{B^{*}} k^{B^{*}}_{P}(x, y, t) \, dy < \int_{M} k^{M}_{P}(x, y, t) \, dy = 1. \quad (3.2)
$$

It follows that $0 < w < 1$ in $B^{*} \times (0, \infty)$. Let $\varepsilon > 0$. By the semigroup identity and (3.2),

$$
w(x, t + \varepsilon) = \int_{B^{*}} k^{B^{*}}_{P}(x, y, t + \varepsilon) \, dy =
$$

$$
\int_{B^{*}} \left( \int_{B^{*}} k^{B^{*}}_{P}(x, z, t) k^{B^{*}}_{P}(z, y, \varepsilon) \, dz \right) \, dy =
$$

$$
\int_{B^{*}} k^{B^{*}}_{P}(x, z, t) \left( \int_{B^{*}} k^{B^{*}}_{P}(z, y, \varepsilon) \, dy \right) \, dz < \int_{B^{*}} k^{B^{*}}_{P}(x, z, t) \, dz = w(x, t). \quad (3.3)
$$

Hence, $w$ is a decreasing function of $t$, and therefore, \(\lim_{t \to \infty} w(x, t)\) exists.

We denote the limit function by $v$. So, $0 \leq v < 1$ and $v$ is a solution of the elliptic equation $Pu = 0$ in $B^{*}$ which satisfies $u = 0$ on $\partial B$. Therefore, $1 - v$ is a positive solution of the equation $Pu = 0$ in $B^{*}$ which satisfies $u = 1$ on $\partial B$. On the other hand, it follows from the criticality assumption that $1$ is the minimal positive solution of the equation $Pu = 0$ in $B^{*}$ which satisfies $u = 1$ on $\partial B$. Thus, $1 \leq 1 - v$, and therefore, $v = 0$.  \(\square\)

**Definition 3.2** Let $B := B(x_0, \delta) \subset \subset M$. Suppose that $B^{*} = M \setminus \text{cl}(B)$ is connected. The nonnegative (minimal) solution

$$
v(x, t) = 1 - \int_{B^{*}} k^{B^{*}}_{P}(x, y, t) \, dy
$$

is called the parabolic capacitory potential of $B^{*}$. Note that $v$ is indeed the minimal nonnegative solution of the initial-boundary value problem

$$
Lu = 0 \quad \text{in } B^{*} \times (0, \infty),
$$

$$
u = 1 \quad \text{on } \partial B \times (0, \infty),
$$

$$
u = 0 \quad \text{on } B^{*} \times \{0\}. \quad (3.4)
$$
Corollary 3.3 Under the assumptions of Lemma 3.1, the parabolic capacitory potential $v$ of $B^*$ is an increasing function of $t$, and $\lim_{t \to \infty} v(x, t) = 1$ locally uniformly in $M$.

Proof: Clearly, 

$$v(x, t) = 1 - \int_{B^*} k_P^M(x, y, t) dy = 1 - w(x, t)$$  \hspace{1cm} (3.5)$$
where $w$ is the heat content of $B^*$. Therefore, the corollary follows directly from Lemma 3.1. \qed

4 Varadhan’s lemma

In this section, we give a purely analytic proof of a lemma of Varadhan [19, Lemma 9, page 259] for a slightly more general case. We consider the Riemannian product manifold $\bar{M} := M \times M$. A point in $\bar{M}$ is denoted by $\bar{x} = (x_1, x_2)$. Let $P_{x_i}$, $i = 1, 2$ denote the operator $P$ in the variable $x_i$, and let $\bar{P} = P_{x_1} + P_{x_2}$ be the skew product operator defined on $\bar{M}$. We denote by $\bar{L}$ the corresponding parabolic operator. Note that if $\bar{P}$ is critical in $\bar{M}$, then $P$ is critical in $M$. Moreover, if $P$ is positive-critical in $M$, then $\bar{P}$ is positive-critical in $\bar{M}$.

Lemma 4.1 Assume that $P1 = 0$. Suppose further that $\bar{P}$ is critical on $\bar{M}$. Let $f$ be a continuous bounded function on $M$, and let 

$$u(x, t) = \int_M k_P^M(x, y, t)f(y) dy$$

be the minimal solution of the Cauchy problem with initial data $f$ on $M$. Fix $K \subset \subset M$. Then 

$$\lim_{t \to \infty} \sup_{x_1, x_2 \in K} |u(x_1, t) - u(x_2, t)| = 0.$$ 

Proof: Denote by $\bar{u}(\bar{x}, t) := u(x_1, t) - u(x_2, t)$. Recall that the heat kernel $\bar{k}(\bar{x}, \bar{y}, t)$ of the operator $\bar{L}$ on $\bar{M}$ satisfies 

$$\bar{k}^M_{\bar{P}}(\bar{x}, \bar{y}, t) = k_P^M(x_1, y_1, t)k_P^M(x_2, y_2, t).$$  \hspace{1cm} (4.1)$$

8
By (2.5) and (4.1), we have
\[
\bar{u}(\bar{x}, t) = u(x_1, t) - u(x_2, t) = \\
\int_M k_P^M (x_1, y_1, t) f(y_1) dy_1 - \int_M k_P^M (x_2, y_2, t) f(y_2) dy_2 = \\
\int_M \int_M k_P^M (x_1, y_1, t) k_P^M (x_2, y_2, t) (f(y_1) - f(y_2)) dy_1 dy_2 = \\
\int_M k_P^M (\bar{x}, \bar{y}, t) (f(y_1) - f(y_2)) d\bar{y}.
\]

Hence, \( \bar{u} \) is the minimal solution of the Cauchy problem for the equation \( \bar{L} \bar{u} = 0 \) with initial data \( f(x_1) - f(x_2) \) on \( M \).

Fix a compact set \( K \subset M \) and \( x_0 \in M \setminus K \), and let \( \varepsilon > 0 \). Let \( B := B((x_0, x_0), \delta) \subset M \setminus K \), where \( K = K \times K \), and \( \delta \) will be determined below. We may assume that \( B^* = M \setminus \text{cl}(B) \) is connected. Then \( \bar{u} \) is a minimal solution of the following initial-boundary value problem
\[
\begin{align*}
\bar{L} \bar{u} &= 0 \quad \text{in } B^* \times (0, \infty), \\
\bar{u}(\bar{x}, t) &= u(x_1, t) - u(x_2, t) \quad \text{on } \partial B \times (0, \infty), \\
\bar{u}(\bar{x}, 0) &= f(x_1) - f(x_2) \quad \text{on } B^* \times \{0\}.
\end{align*}
\]

We need to prove that \( \lim_{t \to \infty} \bar{u}(\bar{x}, t) = 0 \).

By the superposition principle (which obviously holds for minimal solutions), we have
\[
\bar{u}(\bar{x}, t) = u_1(\bar{x}, t) + u_2(\bar{x}, t) \quad \text{on } B^* \times [1, \infty),
\]
where \( u_1 \) solves the initial-boundary value problem
\[
\begin{align*}
\bar{L} u_1 &= 0 \quad \text{in } B^* \times (1, \infty), \\
u_1(\bar{x}, t) &= u(x_1, t) - u(x_2, t) \quad \text{on } \partial B \times (1, \infty), \\
u_1(\bar{x}, 0) &= 0 \quad \text{on } B^* \times \{1\},
\end{align*}
\]
and \( u_2 \) solves the initial-boundary value problem
\[
\begin{align*}
\bar{L} u_2 &= 0 \quad \text{in } B^* \times (1, \infty), \\
u_2(\bar{x}, t) &= 0 \quad \text{on } \partial B \times (1, \infty), \\
u_2(\bar{x}, 0) &= u(x_1, 1) - u(x_2, 1) \quad \text{on } B^* \times \{1\}.
\end{align*}
\]
Clearly, \( |\bar{u}(\bar{x}, t)| \leq 2\|f\|_\infty \) on \( M \times (0, \infty) \). Note that if \( \bar{x} = (x_1, x_2) \in \partial B \), then on \( M \), \( \text{dist}_M(x_1, x_2) < 2\delta \). Using Schauder’s parabolic interior estimates on \( M \), it follows that if \( \delta \) is small enough, then
\[
|\bar{u}(\bar{x}, t)| = |u(x_1, t) - u(x_2, t)| < \varepsilon \quad \text{on } \partial B \times (1, \infty).
\]
By comparison of \( u_1 \) with the parabolic capacitory potential of \( B^* \), we obtain that
\[
|u_1(\bar{x}, t)| \leq \varepsilon \left(1 - \int_{B^*} \bar{k}_{\bar{P}}^{B^*}(\bar{x}, \bar{y}, t - 1) d\bar{y}\right) < \varepsilon \quad \text{in } B^* \times (1, \infty). \tag{4.5}
\]
On the other hand,
\[
|u_2(\bar{x}, t)| \leq 2\|f\|_\infty \int_{B^*} \bar{k}_{\bar{P}}^{B^*}(\bar{x}, \bar{y}, t - 1) d\bar{y} \quad \text{in } B^* \times (1, \infty). \tag{4.6}
\]
It follows from (4.6) and Lemma 3.1 that there exists \( T > 0 \) such that
\[
|u_2(\bar{x}, t)| \leq \varepsilon \quad \text{for all } \bar{x} \in \bar{K} \text{ and } t > T. \tag{4.7}
\]
Combining (4.5) and (4.7), we obtain that
\[
|u(x_1, t) - u(x_2, t)| \leq 2\varepsilon \quad \text{for all } x_1, x_2 \in K \text{ and } t > T. \]
Since \( \varepsilon \) is arbitrary, the lemma is proved.

5 Proof of Theorem 1.3

Without loss of generality, we may assume that \( P1 = 0 \), where \( P \) is a null-critical operator in \( M \). We need to prove that \( \lim_{t \to \infty} k_{\bar{P}}^{M}(x, y, t) = 0 \).

Consider again the Riemannian product manifold \( \bar{M} := M \times M \) and let \( \bar{P} = P_{x_1} + P_{x_2} \) be the corresponding skew product operator which is defined on \( \bar{M} \). If \( \bar{P} \) is subcritical on \( \bar{M} \), then by Theorem 1.2, \( \lim_{t \to \infty} \bar{k}_{\bar{P}}^{M}(x, y, t) = 0 \).

Since
\[
\bar{k}_{\bar{P}}^{M}(\bar{x}, \bar{y}, t) = k_{\bar{P}}^{M}(x_1, y_1, t)k_{\bar{P}}^{M}(x_2, y_2, t),
\]
it follows that \( \lim_{t \to \infty} k_{\bar{P}}^{M}(x, y, t) = 0 \).

Therefore, there remains to prove the theorem for the case where \( \bar{P} \) is critical in \( \bar{M} \). Fix a nonnegative, bounded, continuous function \( f \neq 0 \) such that \( \varphi^*f \in L^1(M) \), and consider the solution
\[
v(x, t) = \int_{M} k_{\bar{P}}^{M}(x, y, t)f(y) dy.
\]
Let \( t_n \to \infty \). Then by subtracting a subsequence, we may assume that for any \( t \in \mathbb{R} \) the function \( v(x, t + t_n) \) converges to a nonnegative solution \( u \in \mathcal{H}_+(M \times \mathbb{R}) \), where
\[
\mathcal{H}_+(M \times \mathbb{R}) := \{u \geq 0 | Lu = 0 \text{ in } M \times \mathbb{R}\}.
\]

Invoking Lemma 4.1 (Varadhan’s lemma), we see that \( u(x, t) = \alpha(t) \). Since \( u \) solves the parabolic equation \( Lu = 0 \), it follows that \( \alpha(t) \) is a nonnegative constant \( \alpha \).
We claim that $\alpha = 0$. Suppose to the contrary that $\alpha > 0$. The assumption that $\varphi^* f \in L^1(M)$ and (2.3) imply that for any $t > 0$

$$\int_M \varphi^*(y)v(y,t)\,dy = \int_M \varphi^*(y) \left( \int_M k^M_P(y,z,t)f(z)\,dz \right)\,dy = \int_M \left( \int_M \varphi^*(y)k^M_P(y,z,t)\,dy \right) f(z)\,dz = \int_M \varphi^*(z)f(z)\,dz < \infty. \quad (5.1)$$

On the other hand, by the null-criticality, Fatou’s lemma, and (5.1) we have

$$\int_M \varphi^*(z)\alpha\,dz = \int_M \varphi^*(z) \lim_{n \to \infty} v(z, t_n)\,dz \leq \liminf_{n \to \infty} \int_M \varphi^*(z)v(z, t_n)\,dz = \int_M \varphi^*(z)f(z)\,dz < \infty.$$

Hence $\alpha = 0$, and therefore

$$\lim_{t \to \infty} \int_M k^M_P(x, y, t)f(y)\,dy = \lim_{t \to \infty} v(x, t) = 0. \quad (5.2)$$

Using the parabolic Harnack inequality and (2.3), we obtain that

$$k^M_P(x, y, t + t_n) \leq c_2(y)\varphi(x), \quad k^M_P(x, y, t + t_n) \leq c_1(x)\varphi^*(y) \quad (5.3)$$

for all $x, y \in M$ and $t + t_n > 1$ (see [12]). Now let $t_n \to \infty$ be a sequence such that $\lim_{n \to \infty} k^M_P(x, y, t + t_n)$ exists for all $(x, y, t) \in M \times M \times \mathbb{R}$. We denote the limit function by $u(x, y, t)$. It is enough to show that any such $u$ is the zero solution. Recall that as a function of $x$ and $t$, $u \in \mathcal{H}_+(M \times \mathbb{R})$ (see [12]). Moreover, (5.3), the semigroup identity, and the dominated convergence theorem imply that

$$u(x, z, t + 1) = \int_M u(x, y, t)k^M_P(y, z, 1)\,dy.$$

It follows that either $u = 0$, or $u$ is a strictly positive function. On the other hand, Fatou’s lemma and (5.2) imply that

$$\int_M u(x, y, 0)f(y)\,dy \leq \lim_{n \to \infty} \int_M k^M_P(x, y, t_n)f(y)\,dy = 0.$$

Since $f \geq 0$, it follows that $u = 0$.

Let $P$ be an elliptic operator of the form (2.1) such that $\lambda_0 \geq 0$, and let $v \in \mathcal{C}_P(M)$ and $v^* \in \mathcal{C}_P^+(M)$. It is well known [13] that

$$\int_M k^M_P(x, y, t)v(y)\,dy \leq v(x), \quad \text{and} \quad \int_M k^M_P(x, y, t)v^*(x)\,dx \leq v^*(y). \quad (5.4)$$
The parabolic Harnack inequality and (5.4) imply that
\[ k_P^M(x, y, t) \leq c_1(y)v(x), \quad k_P^M(x, y, t) \leq c_2(x)v^*(y) \] (5.5)
for all \( x, y \in M \) and \( t > 1 \) (see [12]). Recall that in the critical case, \( v \) and \( v^* \) are in fact the ground states \( \varphi \) and \( \varphi^* \) of \( P \) and \( P^* \) respectively, and by (2.5), we have equalities in (5.4).

We now use theorems 1.2 and 1.3, estimate (5.5), and the dominated convergence theorem to strengthen Lemma 4.1 for initial conditions which satisfy a certain integrability condition.

**Corollary 5.1** Let \( P \) be an elliptic operator of the form (2.1) such that \( \lambda_0 \geq 0 \). Let \( f \) be a continuous function on \( M \) such that \( v^*f \in L^1(M) \) for some \( v^* \in C_P^*(M) \). Let
\[ u(x, t) = \int_M k_P^M(x, y, t)f(y)\,dy \]
be the minimal solution of the Cauchy problem with initial data \( f \) on \( M \). Fix \( K \subset \subset M \). Then
\[ \lim_{t \to \infty} \sup_{x \in K} |u(x, t) - F(x)| = 0, \]
where
\[ F(x) = \begin{cases} 
\varphi(x) \int_M \varphi^*(y)f(y)\,dy & \text{if } P \text{ is positive-critical in } M, \\
0 & \text{otherwise.}
\end{cases} \]
Suppose now that \( P1 = 0 \) and \( \int_M k_P^M(\cdot, y, t)\,dy = 1 \) (i.e. \( 1 \) is a positive invariant solution of the operator \( P \) in \( M \)). Corollary 5.1 implies that for any \( j \geq 1 \) and all \( x \in M \) we have
\[ \lim_{t \to \infty} \int_{M_j^*} k_P^M(x, y, t)\,dy = \begin{cases} 
\int_{M_j^*} \varphi^*(y)\,dy & \text{if } P \text{ is positive-critical in } M, \\
1 & \text{otherwise.}
\end{cases} \]
Therefore, if \( P \) is not positive-critical in \( M \), and \( f \) is a bounded continuous function such that \( \lim \inf_{x \to \infty} f(x) = \varepsilon > 0 \), then
\[ \lim \inf_{t \to \infty} \int_M k_P^M(x, y, t)f(y)\,dy \geq \varepsilon. \] (5.6)
Hence, if the integrability condition of Corollary 5.1 is not satisfied, then the large time behavior of the minimal solution of the Cauchy problem may be complicated. The following example of W. Kirsch and B. Simon [13] demonstrates this phenomenon.
Example 5.2 Consider the heat equation in \( \mathbb{R}^d \). Let \( R_j = e^{e^j} \) and let
\[
f(x) = 2 + (-1)^j \quad \text{if} \quad R_j < \sup_{1 \leq i \leq d} |y_i| < R_{j+1}, \quad j \geq 1.
\]
Let \( u \) be the minimal solution of the Cauchy problem with initial data \( f \). Then for \( t \sim R_jR_{j+1} \) one has that \( u(0, t) \sim 2 + (-1)^j \), and thus \( u(0, t) \) does not have a limit. Note that by Lemma 4.1, for \( d = 1 \), \( u(x, t) \) has exactly the same asymptotic behavior as \( u(0, t) \) for all \( x \in \mathbb{R} \).

6 Remarks and open problems

In this section, we mention some general open problems that are related to the large time behavior of the heat kernel. The first conjecture deals with the exact long time asymptotics of the heat kernel.

**Conjecture 6.1 (E. B. Davies [7])** Let \( L = u_t + P(x, \partial_x) \) be a parabolic operator which is defined on a Riemannian manifold \( M \). Fix a reference point \( x_0 \in M \). Then the limit
\[
\lim_{t \to \infty} \frac{k_M(x, y, t)}{k_M(x_0, x_0, t)}
\]
exists and is positive for all \( x, y \in M \).

The answer to this conjecture seems to be closely related to the question of the existence of a \( \lambda_0 \)-invariant positive solution (see [7, 13]).

The second conjecture was posed by the author [12, Conjecture 3.6].

**Conjecture 6.2** Suppose that \( P \) is a critical operator in \( M \), then the ground state \( \varphi \) is a minimal positive solution in the cone \( H_+(M \times \mathbb{R}) \) of all non-negative solutions of the parabolic equation \( Lu = 0 \) in \( M \times \mathbb{R} \).

As noticed in [12], if the conjecture is true, then Theorem 1.3 would follow from (5.3).

Recall also that by the parabolic Martin representation theorem, the minimal positive solutions in \( H_+(M \times \mathbb{R}) \) are all parabolic Martin functions. Note that in the positive-critical case, the ground state is clearly a parabolic Martin function \( K \) which corresponds to a fundamental sequence of the form \( \{(t_n, y_0)\} \), where \( t_n \to -\infty \) and \( y_0 \) is a fixed point in \( M \). Indeed, by the definition of a Martin function and Theorem 1.2, we have
\[
K(x, y_0, t) = \lim_{n \to \infty} \frac{k_P^M(x, y_0, t - t_n)}{k_P^M(x_0, y_0, -t_n)} = \frac{\varphi(x)}{\varphi(x_0)}.
\]
On the other hand, if Conjecture 6.1 is true, then it can be easily checked that (6.2) is valid also in the null-critical case and therefore, the ground state is always a Martin function.

Recently K. Burdzy and T. S. Salisbury [1] raised the following more general problem

**Question 6.3** Determine which minimal harmonic functions are minimal in $H_+(M \times \mathbb{R})$, the cone of all parabolic functions.

For more details see [1].

**Acknowledgments**

The author wishes to thank A. Grigor’yan and R. Pinsky for valuable discussions. This work was partially supported by the Fund for the Promotion of Research at the Technion.

**References**

[1] K. Burdzy and T. S. Salisbury, On minimal parabolic functions and time-homogeneous parabolic $h$-transforms, *Trans. Amer. Math. Soc.* 351 (1999), 3499–3531.

[2] I. Chavel, “Isoperimetric Inequalities. Differential Geometric and Analytic Perspectives”, Cambridge Tracts in Mathematics 145, Cambridge University Press, Cambridge, 2001.

[3] I. Chavel and L. Karp, Large time behavior of the heat kernel: the parabolic $\lambda$-potential alternative, *Comment. Math. Helv.* 66 (1991), 541–556.

[4] T. Coulhon, Heat kernels on non-compact Riemannian manifolds: a partial survey, in “Séminaire de Théorie Spectrale et Géométrie”, pp. 167–187, Sém. Théor. Spectr. Géom. 15, Univ. Grenoble I, Saint-Martin-d’Hères, 1997.

[5] T. Coulhon and A. Grigor’yan, On-diagonal lower bounds for heat kernels on non-compact manifolds and Markov chains, *Duke Math. J.* 89 (1997), 133-199.

[6] E. B. Davies, “Heat Kernels and Spectral Theory”, Cambridge Tracts in Mathematics 92, Cambridge University Press, Cambridge, 1990.

[7] E. B. Davies, Non-Gaussian aspects of heat kernel behaviour, *J. London Math. Soc. (2)* 55 (1997), 105–125.

[8] A. Grigor’yan, Estimates of heat kernels on Riemannian manifolds, in “Spectral Theory and Geometry”, pp. 140–225, London Math. Soc. Lecture Note Ser. 273, Cambridge Univ. Press, Cambridge, 1999.
[9] R. Z. Has’minski˘i, “Stochastic Stability of Differential Equations”, Monographs and Textbooks on Mechanics of Solids and Fluids: Mechanics and Analysis, 7, Sijthoff & Noordhoff, Alphen aan den Rijn, 1980.

[10] K. Ishige and M. Murata, Uniqueness of nonnegative solutions of the Cauchy problem for parabolic equations on manifolds or domains, Ann. Scuola Norm. Sup. Pisa Cl. Sci.(4) 30 (2001), 171–223.

[11] W. Kirsch and B. Simon, Approach to equilibrium for a forced Burgers equation, J. Evol. Equ. 1 (2001), 411–419.

[12] Y. Pinchover, Large time behavior of the heat kernel and the behavior of the Green function near criticality for nonsymmetric elliptic operators, J. Funct. Anal. 104 (1992), 54–70.

[13] Y. Pinchover, On nonexistence of any $\lambda_0$-invariant positive harmonic function, a counter example to Stroock’s conjecture, Comm. Partial Differential Equations 20 (1995), 1831–1846.

[14] R. G. Pinsky, “Positive Harmonic Function and Diffusion”, Cambridge University Press, Cambridge, 1995.

[15] F. O. Porper and S. D. `Eidel’man, Two-sided estimates of the fundamental solutions of second-order parabolic equations and some applications of them, Uspekhi Mat. Nauk 39 (1984), 107–156.

[16] D. W. Robinson, “Elliptic Operators and Lie Groups”, Oxford Mathematical Monographs, Oxford University Press, New York, 1991.

[17] B. Simon, Schrödinger semigroups. Bull. Amer. Math. Soc. 7 (1982), 447–526.

[18] B. Simon, Large time behavior of the heat kernel: on a theorem of Chavel and Karp, Proc. Amer. Math. Soc. 118 (1993), 513–514.

[19] S. R. S. Varadhan, “Lectures on Diffusion Problems and Partial Differential Equations”, Tata Institute of Fundamental Research 64, Springer-Verlag, Berlin, 1980.

[20] N. Th. Varopoulos, L. Saloff-Coste and T. Coulhon, “Analysis and Geometry on Groups”, Cambridge Tracts in Mathematics 100, Cambridge University Press, Cambridge, 1992.

[21] W. Woess, “Random Walks on Infinite Graphs and Groups”, Cambridge Tracts in Mathematics 138, Cambridge University Press, Cambridge, 2000.