On many-sorted $\omega$-categorical theories

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Abstract

We prove that every many-sorted $\omega$-categorical theory is completely interpretable in a one-sorted $\omega$-categorical theory. As an application, we give a short proof of the existence of non $G$-compact $\omega$-categorical theories.

1 Introduction

A many–sorted structure can be easily transformed into a one–sorted by adding new unary predicates for the different sorts. However $\omega$-categoricity is not preserved. In this article we present a general method for producing $\omega$-categorical one–sorted structures from $\omega$-categorical many–sorted structures. This is stated in Corollary 3.2, the main theorem in this paper. Our initial motivation was to understand Alexandre Ivanov’s example (in [4]) of an $\omega$-categorical non $G$-compact theory. In Corollary 3.3 we apply our results to offer a short proof of the existence of such theories.

Our method is based on the use of a particular theory $T_E$ of equivalence relations $E_n$ on $n$-tuples. The quotient by $E_n$ is an imaginary sort containing a predicate $P_n$ which can be used to copy the $n$-th sort of the given many-sorted theory. Since the complexity of $T_E$ is part of the complexity of the $\omega$-categorical one–sorted theory obtained by our method, it is important to classify $T_E$ from the point of view of stability, simplicity and related properties. It turns out that $T_E$ is non-simple but it does not have SOP2. A similar

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example of a theory with such properties has been presented by Shelah and Usvyatsov in \[7\]. Their proof, as ours, relies on Claim 2.11 of \[3\], which is known to have some gaps. A revised version of \[3\] will be posted in arxiv.org. In the meanwhile Kim and Kim have obtained a new proof of the same result: Proposition 2.3 from \[5\].

The one–sorted theory $T_E$ is interdefinable with some many–sorted theory $T^*$ which is presented and discussed in Section 2. In order to describe $T^*$ we need a version of Fraïssé’s amalgamation method that can be applied to the many–sorted case (see Lemma 2.1). In Section 3 some results on stable embeddedness from the third author (in \[8\]) are extended and used to prove Corollary 3.2. Section 4 is devoted to classify $T_E$ from the stability point of view.

A previous version of these results appeared in the second author’s Ph.D. Dissertation \[6\]. They have been corrected in some points and in general they have been elaborated and made more compact.

## 2 $T^*$ and Fraïssé’s amalgamation

Let $L$ be a countable many–sorted language with sorts $S_i$, $(i \in I)$, and let $\mathcal{K}$ be a class of finitely generated $L$–structures.\[1\] We call an $L$–structure $M$ a Fraïssé limit of $\mathcal{K}$ if the following holds:

1. $\mathcal{K} = \text{Age}(M)$, where Age$(M)$ is the class of all finitely generated $L$–structures which are embeddable in $M$.
2. $M$ is at most countable.
3. $M$ is ultra–homogeneous i.e., any isomorphism between finitely generated substructures extends to an automorphism of $M$.

By a well–known argument $\mathcal{K}$ can only have one Fraïssé limit, up to isomorphism.

**Lemma 2.1.** Let $\mathcal{K}$ be as above. Then the following are equivalent:

a) The Fraïssé limit of $\mathcal{K}$ exists and is $\omega$-categorical.

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\[1\]We allow empty sorts if $L$ has no constant symbols of that sort.
b) \( K \) has the amalgamation property AP, the joint embedding property JEP, the hereditary property HP (i.e., finitely generated \( L \)-structures which are embeddable in elements of \( K \) belong themselves to \( K \)) and satisfies
\[(*) \text{ for all } i_1 \ldots i_n \in I \text{ there are only finitely many quantifier-free types of tuples } (a_1, \ldots, a_n) \text{ where the } a_j \text{ are elements of sort } S_{i_j} \text{ in some structure } A \in K.\]

If the Fraïssé limit of \( K \) exists, it has quantifier elimination.

Proof. a) \( \Rightarrow \) b). It is well known that the age of an ultra–homogeneous structure has AP, JEP and HP. All quantifier–free types which occur in elements of \( K \) are quantifier–free types of tuples of the Fraïssé limit. So property (*) follows from the Ryll–Nardzewski theorem.

b) \( \Rightarrow \) a). The quantifier-free type qftp(\( \bar{a} \)) determines the isomorphism type of the structure generated by \( \bar{a} \). Hence (*) implies that \( K \) contains at most countably many isomorphism types. The existence of the Fraïssé limit \( M \) follows now from AP, JEP and HP.

If two sequences \( \bar{a} \) and \( \bar{b} \) have the same quantifier-free type in \( M \), there is an automorphism of \( M \) which maps \( \bar{a} \) to \( \bar{b} \) and so it follows that \( \bar{a} \) and \( \bar{b} \) have the same type in \( M \). Consider a formula \( \varphi(\bar{x}) \) and the set \( P_{\varphi(x)} = \{ \text{qftp}(\bar{a}) : M \models \varphi(\bar{a}) \} \). Then
\[M \models \varphi(\bar{a}) \iff \text{qftp}(\bar{a}) \in P_{\varphi} \iff M \models \bigvee_{p \in P_{\varphi}} p(\bar{a}).\]

Now, (*) implies that \( P_{\varphi} \) is finite and that in \( M \) all \( p = \text{qftp}(\bar{a}) \) are finitely axiomatisable, that is, \( p = \langle \chi_p \rangle \) for some quantifier-free \( \chi_p(x) \). Then \( M \models \varphi(\bar{a}) \iff \bigvee_{p \in P_{\varphi}} \chi_p(\bar{a}) \). So \( M \) has quantifier elimination and it is \( \omega \)-categorical since there are only finitely many possibilities for the \( \chi_p \), depending only on the number and the sorts of the free variables of \( \varphi \). \( \square \)

It is easy to see that the theory of the Fraïssé limit is the model–completion of the universal theory of \( K \).

**Definition 2.2.** Let \( L^* \) be the language with countably many sorts \( S, S_1, \ldots, \), function symbols \( f_i : S^i \to S_i \), and constants \( c_i \in S_i \) and \( T^0 \) the theory of all \( L^* \)-structures \( A \) with
\[f_i(\bar{a}) = c_i \iff \bar{a} \text{ has some repetition}\]
for all \( \bar{a} \in S^i(A) \). Furthermore let \( \mathcal{K}^* \) be the class of all finitely generated models of \( T^0 \).

**Lemma 2.3.** \( \mathcal{K}^* \) satisfies the conditions of Lemma 2.1.

**Proof.** The class of all models of \( T^0 \) has AP and JEP and therefore also \( \mathcal{K}^* \). (*) follows easily from the fact that \( f_i(a_{m_1}, \ldots, a_{m_i}) = c_i \) for all \( i > k \) and \( \{a_{m_1}, \ldots, a_{m_i}\} \subset \{a_1, \ldots, a_k\} \).

We define \( M^* \) to be the Fraïssé limit of \( \mathcal{K}^* \) and \( T^* \) to be the complete theory of \( M^* \). \( T^* \) is the model–completion of \( T^0 \).

Recall the following definition from [2]:

**Definition 2.4.** Let \( T \) be a complete theory and \( P \) a 0–definable predicate. \( P \) is called stably embedded if every definable relation on \( P \) is definable with parameters from \( P \).

**Remarks**

1. For many–sorted structures with sorts \((S_i)_{i \in I}\) this generalises to the notion of a sequence \((P_i)_{i \in I}\) of 0–definable \( P_i \subset S_i \) being stably embedded.

2. While the definition is meant in the monster model, an easy compactness argument shows that, if \( P(M) \) is stably embedded in \( M \) for some weakly saturated\(^2\) model \( M \), then this is true for all models.

3. If \( M \) is saturated then \( P \) is stably embedded if and only if every automorphism (i.e. elementary permutation) of \( P(M) \) extends to an automorphism of \( M \). This was claimed in [2] only for the case that \(|M| > |T|\). But the proof can easily be modified to work for the general case. One has to use the fact that if \( A \) has smaller size than \( M \), then any type over a subset of \( \text{dcl}^e(A) \) can be realized in \( M \).

4. If \( M \) is \( \omega \)–categorical, it can be proved that for every finite tuple \( a \in M \) there is a finite tuple \( b \in P \) such that every relation on \( P \) which is definable over \( a \) can be defined using the parameter \( b \).

**Lemma 2.5.** In \( T^* \) the sequence of sorts \((S_1, S_2, \ldots)\) is stably embedded.

\(^2\)M is weakly saturated if every type over the empty set is realized in \( M \).
Proof. Clear since \( \text{tp}(\bar{a}/S_1, \ldots) = \text{tp}(\bar{a}/f_1(\bar{a}), \ldots) \). See also the discussion in [2].

For a complete theory \( T \) and a 0–definable predicate \( P \) the induced structure on \( P \) consists of all 0–definable relations on \( P \). Note that the automorphisms of \( P \) with its induced structure are exactly the elementary permutations of \( P \) in the sense of \( T \).

**Lemma 2.6.** In \( T^* \) the induced structure on \( (S_1, S_2, \ldots) \) equals its \( L^*_{>0} \) structure, where \( L^*_{>0} \) is the sublanguage of \( L^* \) which has only the sorts \( S_1, S_2, \ldots \) and the constants \( c_1, c_2, \ldots \).

Proof. Quantifier elimination.

Let \( T^*_{>0} \) denote the theory of all \( L^*_{>0} \) structures, where all sorts \( S_i \) are infinite. Clearly \( T^*_{>0} \) is the restriction of \( T^* \) to \( L^*_{>0} \).

**Lemma 2.7.** Every model of \( T^*_{>0} \) can be expanded to a model of \( T^* \).

Proof. It is easy to see that the following amalgamation property is true:

Let \( N \) be a model of \( T^0 \) with infinite sorts \( S_i(N) \). Let \( A \) be a finitely generated substructure of \( N \) and \( B \in \mathcal{K}^* \) an extension of \( A \). Then \( B \) can be embedded over \( A \) in an extension \( N' \) of \( N \) which is a model of \( T^0 \) and such that \( S_i(N') = S_i(N) \) for all \( i \).

If a model of \( T^*_{>0} \) is given, we expand it arbitrarily to a model \( N \) of \( T^0 \) and apply the above amalgamation property repeatedly such that the union of the resulting chain is a model of \( T^* \) which has the same sorts \( S_i \) as \( N \).

**Corollary 2.8.** There is an \( \omega \)–categorical one-sorted theory \( T_E \) with a series of 0–definable infinite predicates \( P_1, P_2, \ldots \) in \( T^\text{res} \) such that

1. \( (P_1, P_2, \ldots) \) is stably embedded
2. The many-sorted structure induced on \( (P_1, P_2, \ldots) \) is trivial.
3. For every sequence \( \kappa_1, \kappa_2, \ldots \) of infinite cardinals there is a model \( N \) of \( T_E \) such that \( |P_i(N)| = \kappa_i \).
Proof. The language $L_E$ of $T_E$ will contain for each $i$ a symbol $E_i$ for an equivalence relation between $i$–tuples. Let $M = (S, S_1, S_2 , \ldots)$ be a model of $T^*$. For $a, b \in S^n$ define $E_i(a, b) \iff f_i(a) = f_i(b)$. $T_E$ is the theory of $M_E = (S, E_1, E_2, \ldots)$. The $S_i$ live in $M_E^{eq}$ as $M_E^1 / E_i$ and the $c_i$ are 0–definable in $M_E$. We set $P_i = S_i \setminus \{c_i\}$.

It is easy to see that $T_E$ as constructed in the proof is the model–completion of the theory of all structures $(M, E_1, E_2, \ldots)$ where $E_n$ is an equivalence relation on $M^n$ where one equivalence class consists of all $n$–tuples which contain a repetition. That $T_E$ has quantifier elimination can be proved as follows: Every formula $\varphi(\bar{x})$ of $L_E$ is equivalent to a quantifier–free $L^*$–formula $\varphi'(\bar{x})$. $\varphi'(\bar{x})$ is a boolean combination of formulas of the form $f_i(\bar{x}') = f_i(\bar{x}'')$ and $f_i(\bar{x}') = c_i$, which are equivalent to quantifier–free $L_E$–formulas: $f_i(\bar{x}') = f_i(\bar{x}'')$ is equivalent to $E_i(\bar{x}', \bar{x}'')$, $f_i(\bar{x}_1', \ldots, \bar{x}_l') = c_i$ is equivalent to $\bigvee_{1 \leq k < l \leq i} \bar{x}_k' = \bar{x}_l'$.

3 Expansions of stably embedded predicates

Let $T$ be complete theory with two sorts $S_0$ and $S_1$. We consider $S_1$ as a structure of its own carrying the structure induced from $T$ and denote by $T \upharpoonright S_1$ the theory of $S_1$.

Lemma 3.1. Let $T$ be complete theory with two sorts $S_0$ and $S_1$. Let $\bar{T}_1$ be a complete expansion of $T \upharpoonright S_1$. Assume that $S_1$ is stably embedded. Then we have

1. $\bar{T} = T \cup \bar{T}_1$ is complete \footnote{Actually we have: $S_1$ is stably embedded if and only if $\bar{T}$ is complete for all complete expansions $T_1$.} (\cite[Lemma 3.1]{8}]

2. $S_1$ is stably embedded in $\bar{T}$ and $T \uparrow S_1 = \bar{T}_1$.

3. If $T$ and $\bar{T}_1$ are $\omega$–categorical, then $\bar{T}$ is also $\omega$–categorical.

Proof. \footnote{Actually we have: $S_1$ is stably embedded if and only if $\bar{T}$ is complete for all complete expansions $T_1$.} Let $\bar{M} = (M_0, \bar{M}_1)$ and $\bar{M}' = (M'_0, \bar{M}'_1)$ be saturated models of $\bar{T}$ of the same cardinality and $M = (M_0, M_1)$ and $M' = (M'_0, M'_1)$ their restrictions to the language of $T$. Since $T$ and $\bar{T}_1$ are complete, there are isomorphisms $f : M \rightarrow M'$ and $g : \bar{M}_1 \rightarrow \bar{M}'_1$. $gf^{-1}$ is an automorphism of $M'_1$. Since $M'_1$ is stably embedded in $M'$, $gf^{-1}$ extends to an automorphism
$h$ of $M'$. $hf$ is now an isomorphism from $M$ to $M'$ which extends $g$.

2. We use the same notation as in the proof of 1. Let $\tilde{M}$ be a saturated model of $\tilde{T}$. We have to show that every automorphism $f$ of $\tilde{M}_1$ extends to an automorphism of $\tilde{M}$. But $f$ extends to an automorphism of $M$, which is automatically an automorphism of $\tilde{M}$.

3. Start with two countable models $\tilde{M}$ and $\tilde{M}'$ and proceed as in the proof of 1.

Corollary 3.2. Every many-sorted $\omega$-categorical theory is completely (the induced structure is exactly this) interpretable in a one-sorted $\omega$-categorical theory.

Proof. Let $T$ be a complete theory with countably many sorts $P_1, P_2, \ldots$. We consider $T$ as an expansion of $T_E \upharpoonright (P_1, P_2, \ldots)$ and set $\tilde{T} = T_E \cup T$. $\tilde{T}$ is a one-sorted complete theory. We have $\tilde{T} \upharpoonright (P_1, P_2, \ldots) = T$. If $T$ is $\omega$-categorical, $\tilde{T}$ is also $\omega$-categorical.

Corollary 3.3 (Ivanov). There is a one-sorted $\omega$-categorical theory which is not $G$-compact.

Proof. By 1 there is a many-sorted $\omega$-categorical theory $T$ which is not $G$-compact. Interpret $T$ in a one-sorted $\omega$-categorical theory $\tilde{T}$ as in Corollary 3.2. Then $T$ is also not $G$-compact. For this one has to check that if $\tilde{T}$ is $G$-compact, then every 0-definable subset with its induced structure is also $G$-compact. This follows from the following description of $G$-compactness: $a, b$ of length $\omega$ are in the relation $nc^\omega$ if $a$ and $b$ are the first two elements of an infinite sequence of indiscernibles. A complete theory is $G$-compact, if the transitive closure of $nc^\omega$ is type-definable. (Note that $(a, b)$ is in the transitive closure of $nc^\omega$ if and only if $a$ and $b$ have the same Lascar-strong type.)

4 Classification of $T_E$

Proposition 4.1. $T_E$ has $TP_2$, the tree property of the second kind, and therefore it is not simple.
Proof. We show that $\varphi(x; y, u, v) = E_2(xy, uv)$ has $\text{TP}_2$. Let $(b_i : i < \omega)$, $(c_i : i < \omega)$, and $(d_i : i < \omega)$ be pairwise disjoint sequences of different elements such that $\neg E_2(c_i d_i, c_j d_j)$ for $i \neq j$. For $i, j \in \omega$, let $\bar{a}_i = b_i c_j d_j$. By compactness we can see that for any $\eta \in \omega^\omega$, the set \{\varphi(x; \bar{a}_{\eta(i)}^i) : i < \omega\} is consistent, and since the $c_i d_i$'s are in different $E_2$-classes, for each $i < \omega$, the set \{\varphi(x; \bar{a}_i^i) : j < \omega\} is 2-inconsistent.

**Lemma 4.2 (Independence lemma).** Let $a, b, c, d', d''$ be tuples in the monster model of $T_E$ and $F$ a finite subset. Assume that $a$ and $c$ have only elements from $F$ in common. If $d'a \equiv_F d'b \equiv_F d''b \equiv_F d''c$, then there exists some $d$ such that $d'a \equiv_F da \equiv_F dc \equiv_F d''c$.

\[ \begin{array}{c}
 a \\
 \downarrow \\
 d \\
 \downarrow \\
 c \\
 \end{array} \quad \begin{array}{c}
 d' \\
 b \\
 \downarrow \\
 d'' \\
 \end{array} \]

Proof. Let $A, B, C, D'$ and $D''$ denote the set of elements of the tuples $a, b, c, d', d''$, respectively. We note first that we can assume that $F$ is contained in $A, B$ and $C$, since otherwise we can increase $a$, $b$ and $c$ by elements from $F$. Then we note that if $A$ and $D'$ intersect in a subtuple $f$, this tuple also belongs to $B$ and $C$ and therefore to $F$. So we have that $A \cap D'$ is contained in $F$ and similarly that $C \cap D''$ is contained in $F$.

It suffices to find an $L_E$-structure $M$ extending $AC$ and containing a new tuple $d$ with the same quantifier-free type as $d'$ over $A$ and of $d''$ over $C$. Take as $d$ a new tuple of the right length which intersects $A$ and $B$ in the subtuple $f$. We have then $d'a \equiv_F da \equiv_F dc \equiv_F d''c$, where $g \equiv_F h$ means that $g$ and $h$ satisfy the same equality-formulas over $F$, i.e. $g_i = g_j$ iff $h_i = h_j$ and $g_i = f_j$ iff $h_i = f_j$. If $D$ denotes the elements of $d$, it follows that the intersection of any two of $A, C$ and $D$ belongs to $F$.

It remains to define the relations $E_n$ on $ACD$. Let $E_n^0$ denote the part of $E_n$ which is already defined on $AC$. Let $E_n'$ be the relation $E_n$ transported from $AD'$ to $AD$ via the identification $d' \mapsto d$ and $E_n''$ the relation $E_n$ transported from $CD''$ to $CD$ via the identification $d'' \mapsto d$. Note that $d' \equiv_F d''$ implies that $E_n'$ and $E_n''$ agree on $DF$. We define $E_n$ on $ACD$ as the transitive closure of

\[ E_n^0 \cup E_n' \cup E_n'' \cup E_n^\text{rep} \cup \Delta, \]
where $E_{\text{rep}}^{n}$ is the set of all pairs of $n$–tuples from $ACD$ which contain repetitions and $\Delta$ is the identity on $(ACD)^n$.

We have to show that the new structure defined on $AC$ agrees with the original structure. Also we must check that the structure on $AD$ (and $CD$) agrees with the structure on $AD'$ (and $CD'$) via $d \mapsto d'$ (and $d \mapsto d''$). Using the fact that an $n$-tuple which e.g. belongs to $AC$ and $AD$ belongs already to $A$, it is easy to see that we have to show the following:

For all $n$–tuples $x \in A$, $y \in C$ and $z \in DF$

1. $E_n^{'}(x, z) \land E_n^{''}(z, y) \Rightarrow E_n^0(y, x)$
2. $E_n^{''}(z, y) \land E_n^0(y, x) \Rightarrow E_n^{'}(x, z)$
3. $E_n^0(y, x) \land E_n^{'}(x, z) \Rightarrow E_n^{''}(z, y)$

Let $z'$ and $z''$ be the subtuples of $DF$ and $D'F$ which correspond to $z$.

Proof of 1: Assume $E_n^{'}(x, z)$ and $E_n^{''}(z, y)$. We have then $E_n(x, z')$ and $E_n(z'', y)$. $d'a \equiv_F d'b$ implies $z'a \equiv_F z'b$, which implies that there is a tuple $x'$ in $B$ such that $z'x \equiv_F z'x'$. So we have $E_n(z', x')$. $d'b \equiv_F d''b$ implies $z'x' \equiv_F z''x'$ and whence $E_n(z'', x')$. Now we can connect $y$ and $x$ as follows: $y \ E_n \ z'' \ E_n \ x' \ E_n \ z' \ E_n \ x$.

Proof of 2: Assume $E_n^{''}(z, y)$ and $E_n^0(y, x)$. We have then $E_n(z'', y)$. As above we find a tuple $y' \in B$ such that $E_n(z'', y')$ and $E_n(z', y')$. The chain $x \ E_n \ y \ E_n \ z'' \ E_n \ y' \ E_n \ z'$ shows that $E_n^{'}(x, z)$.

Proof of 3: Symmetrical to the proof of 2.

In order to state [5, Proposition 2.3] we need the following terminology:

(1) A tuple $\bar{\eta} = (\eta_0, \ldots, \eta_{d-1})$ of elements of $2^{<\omega}$ is $\cap$-closed if the set $\{\eta_0, \ldots, \eta_{d-1}\}$ is closed unter intersection.

(2) Two $\cap$-closed tuples $\bar{\eta}$ and $\bar{\nu}$ are isomorphic if they have the same length and

(i) $\eta_i \mathbin{\triangleleft} \eta_j$ iff $\nu_i \mathbin{\triangleleft} \nu_j$

(ii) $\eta_i^t \mathbin{\triangleleft} \eta_j$ iff $\nu_i^t \mathbin{\triangleleft} \nu_j$ for $t = 0, 1$. 

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(3) A tree \((a_\eta : \eta \in 2^{<\omega})\) of tuples of the same length is \textit{modeled} by \((b_\eta : \eta \in 2^{<\omega})\) if for every formula \(\phi(\bar{x})\) and every \(\cap\)-closed \(\bar{\eta}\) there is a \(\cap\)-closed \(\bar{\nu}\) isomorphic to \(\bar{\eta}\) such that \(\models \phi(\bar{\eta}) \iff \models \phi(\bar{\nu})\).

(4) \((b_\eta : \eta \in 2^{<\omega})\) is \textit{indiscernible} if \(\models \phi(\bar{\eta}) \iff \models \phi(\bar{\nu})\) for all isomorphic \(\cap\)-closed \(\bar{\eta}, \bar{\nu}\).

\begin{lemma}[\textsection 4.3 (\textsection 2.3). See also \textsection 3]\textbf{Lemma 4.3} \end{lemma}

\textbf{Lemma 4.3} \textbf{(H. Adler).} The formula \(\varphi(x, y)\) has SOP\(_2\) in \(T\) if there is a binary tree \((a_\eta : \eta \in 2^{<\omega})\) such that for every \(\eta \in 2^{\omega}\), \(\{\varphi(x, a_{\eta|n}) : n < \omega\}\) is consistent and for every incomparable \(\eta, \nu \in 2^{<\omega}\), \(\varphi(x, a_\eta) \land \varphi(x, a_\nu)\) is inconsistent. The theory \(T\) has SOP\(_2\) if some formula \(\varphi(x, y)\in L\) has SOP\(_2\) in \(T\).

\begin{remark}[\textsection 4.5] \textbf{Remark 4.5} \end{remark}

\textbf{Remark 4.5} \textbf{(H. Adler).} The formula \(\varphi(x, y)\) has SOP\(_2\) in \(T\) if and only if \(\varphi(x, y)\) has the tree property of the first kind TP\(_1\): there is a tree \((a_\eta : \eta \in \omega^{<\omega})\) such that for every \(\eta \in \omega^{\omega}\), \(\{\varphi(x, a_{\eta|n}) : n < \omega\}\) is consistent and for every incomparable \(\eta, \nu \in \omega^{<\omega}\), \(\varphi(x, a_\eta) \land \varphi(x, a_\nu)\) is inconsistent.

\begin{proof} By compactness. \end{proof}

\begin{proposition}[\textsection 4.6] \textbf{Proposition 4.6.} \end{proposition}

\textbf{Proposition 4.6.} \textbf{\(T_E\) does not have SOP\(_2\).}

\begin{proof} We follow ideas from a similar proof in \textsection 7. Assume \(\varphi(x, y)\) has SOP\(_2\) in \(T_E\) and the tree \((a_\eta : \eta \in 2^{<\omega})\) witnesses it. Choose for every \(\eta\) a tuple \(d_\eta\) such that \(\models \phi(d_\eta, a_\nu)\) for all \(\nu \subseteq \eta\).

By Lemma 4.3 we can assume that the tree \((d_\eta a_\eta : \eta \in 2^{<\omega})\) is indiscernible. Let us now look at the elements \(a_{00}, a_{01}, a_{00}, d_{000}, d_{010}\). We have by indiscernibility

\[d_{000}a_{00} \equiv d_{000}a_\emptyset \equiv d_{010}a_\emptyset \equiv d_{010}a_{01} \equiv d_{010}a_{01} \equiv d_{010}a_{01}.\]

If the tuples \(a_{00}\) and \(a_{01}\) are disjoint, we can apply the Independence Lemma to \(a = a_{00}, b = a_{00}, c = a_{01}, d' = d_{000}, d'' = d_{010}\) to get a tuple \(d\) such that

\[d_{000}a_{00} \equiv da_{00} \equiv da_{01} \equiv da_{01} \equiv d_{010}a_{01}.\]

It follows that \(\models \varphi(d, a_{00}) \land \varphi(d, a_{01})\), which contradicts the SOP\(_2\) of the tree.

If \(a_{00}\) and \(a_{01}\) are not disjoint, we argue as follows: Assume that \(a_{00}\) and \(a_{01}\) have an element \(f\) in common, say \(f = a_{00,i} = a_{01,j}\). Then \(a_{00}a_{01} \equiv a_{00}a_{01}\) implies \(a_{00,i} = a_{01,j}\). So we have \(a_{000,i} = a_{00,i}\) and it follows from
indiscernibility that $f = a_{00,i} = a_{01,i} = a_{01,i}$. Let $F$ be the set of elements which occur in both $a_{00}$ and $a_{01}$. We have seen that the elements of $F$ occur in $a_{00}$, $a_{00}$ and $a_{01}$ at the same places. Therefore

$$d_{000}a_{00} \equiv_F d_{000}a_{00} \equiv_F d_{010}a_{00} \equiv_F d_{010}a_{01}$$

and we can again apply the Independence Lemma.

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