Anderson’s considerations on the flow of superfluid helium: some offshoots

Eric Varoquaux ∗
CNRS-URM 2464 and CEA-IRAMIS,
Service de Physique de l’État Condensé,
Centre d’Études de Saclay,
91191 Gif-sur-Yvette Cedex (France)

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More than four decades have elapsed since the seminal 1966 paper of P.W. Anderson on the flow of superfluid helium, 4He at that time. But it is due to the discovery of superfluidity in the light isotope, 3He, that key advances on the experimental understanding of superflow, phase slippage, the critical velocities, macroscopic quantum effects and the superfluid analogue of the Josephson effects – pivotal concepts in superfluid physics – have been performed. In the aftermath of yet another evolutionary step, the achievement of Bose-Einstein condensation in cold atoms assemblies, this review surveys some of the experiments that have shed light on the more intimate effect of quantum mechanics on the hydrodynamics of the dense heliums.

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DRAFT

I. INTRODUCTION

Superfluids display quantum properties over large distance. Superfluid currents may persist indefinitely unlike those of ordinary fluids (Reppy and Lane, 1965); the circulation of flow velocity has been found quantised over meter-size paths (Verbeek et al., 1974). These manifestations of macroscopic quantum phenomena have constituted one of the early hallmarks of experimental condensed matter physics as reviewed over the years by a number of authors. Among others, Vinen (1963), Vinen (1966), Andronikashvili and Maladadze (1966), Vinen (1968), Khalatnikov (1965), Putterman and Rudnick (1971), Nozières and Pines (1990), Vollhardt and Wölfle (1990), Volovik (2003).

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only describes the hydrodynamics of superfluids [Khatrinikov, 1965] but has been extended to other fields of physics like superconductivity, the Bose-Einstein condensed gases. 

Hydrodynamics, and superfluid hydrodynamics in its wake, is expected to break down at small scale when the typical lengths of the problem at hand are no longer large compared to interatomic separation or other microscopic lengths characteristic of the internal structure of the fluid, the “size” of Cooper pairs in superfluid $^3$He for instance. It has however been known for a long time, notably from the ion propagation measurements of Rayfield and Reif (1964) that the relevant scale in superfluid $^4$He is surprisingly small, of the order of ångströms. In these experiments, the velocity of vortex rings could be measured in a direct way and compared to the usual formulae of classical hydrodynamics (see Sec. III). Rayfield and Reif found that, at first sight at least, hydrodynamics appears to remain valid down to atomic sizes. Their result holds for $^4$He, which is a dense fluid of bosons. The relevant scale is two to three orders of magnitude larger in superfluid $^3$He, which is a BCS-type $p$-wave superfluid in which the relevant scale is fixed by the “size” of the Cooper pair. This article aims at reviewing how the hydrodynamics of superfluid $^4$He and $^3$He evolves from large to small scale and ultimately breaks down at close distance, revealing the more intimate quantum properties of these fluids. This is no mean feat, as noted long ago by Uhlenbeck, who is quoted to have said “One must watch like a hawk to see Planck’s constant appear in hydrodynamics” (Puttermann, 1974).

The main object of study in the following is the time and space evolution of the phase of the macroscopic wavefunction, often simply referred to as “the phase”, in so-called aperture flow. This concept of “phase” with wave-mechanical properties governing the evolution of macroscopic quantities has become so well-spread that its meaning is, wrongly perhaps, taken for granted: an updated survey is given in Sec. II starting with a short description of the phenomenological two-fluid model.

A seminal concept was put forward by P. W. Anderson in 1966, following the lead of Feynman (1955), namely the recognition of the “phase” as the quantity commanding both the newly discovered Josephson effects and dissipation in superfluid flow as caused by vortex motion. A full understanding of these phenomena is of fundamental importance as they govern on the one hand the breakdown of viscousless flow – the most noteworthy feature of superflow – even in the low temperature limit and the appearance of entirely new classes of phenomena, namely the formation of quantised vorticity, vortex tangles and superfluid turbulence, the hydrodynamic analogues of the Josephson effects on the other: the latter phenomena, well known in superconductivity, have counterparts in fluid mechanics as described in Sec. VIII. More generally, they underpin the sort of interferometry that can be performed with the superfluid wavefunction. All of this depends on the phase of the macroscopic wavefunction of the superfluid.

These ideas were agitated in the mid-sixties, in particular at the Sussex Meeting in 1965, notably by Anderson (1966a), Nozières and Vinen. Reliable experimental observations were performed twenty years later only, as recalled in Sec. IV giving a host of new results and insights on superfluid hydrodynamics, notably an improved understanding of critical velocities and of the nucleation of vortices, discussed in Sec. V of the appearance of the Josephson regime of superflow through tiny apertures, described in Sec. VIII B.

This review on phase slippage in superfluids is intended to provide a gangway between the many excellent monographs [1] that provide the background material on this subject and the more specialised research publications in the literature that give the full, raw, sometimes arcane, coverage. As such, it does not constitute a comprehensive review – space and time constraining - but focuses on a few selected issues that provide the backbone of this subfield of superfluid hydrodynamics. Reviews with different flavours span over a quarter of a century and show how this field has evolved.

Also very worthy of notice are the reviews of closely related subjects, Sonin’s description of vortex dynamics Sonin (1987) and his hydrodynamic approach to the Josephson equation for the ac-effect, the Landau critical velocity in superfluid $^4$He by McClintock and Bowley (1995) and in superfluid $^3$He-B as summarised by Dobbs (2000), and vortex formation and dynamics in superfluid $^3$He by Eltsov et al. (2005).

II. THE BASIC SUPERFLUID: $^4$HE

Helium remains in the liquid phase down to absolute zero under its own vapour pressure owing to a large atomic zero point energy. Helium-4 undergoes an ordering transition toward a superfluid state at $T \sim 2$ K, which is now commonly viewed as a form of Bose-Einstein condensation. A similar transition occurs in helium-3 at $\sim 2$ K.

3 Among others, Nozières and Pines (1990), Tilley and Tilley (1990) Vollhardt and Wolfg (1990)
4 see Varoquaux et al. (1985), Varoquaux and Avenel (1987), Avenel and Varoquaux (1987), Varoquaux et al. (1990), Varoquaux et al. (1991), Varoquaux et al. (1992), Bowley et al. (1992), Avenel et al. (1993), Varoquaux and Avenel (1994), Zimmermann, Jr. (1996), Packard (1998), Varoquaux et al. (1999), Varoquaux (2000), Varoquaux et al. (2001), Varoquaux (2001), Davis and Packard (2002), Packard (2004) and Sato and Packard (2012).
5 p.12
$T \lesssim 2.7$ mK when Cooper pairing in a state with parallel spin $S = 1$ and relative orbital momentum $l = 1$ occurs.

A. The two-fluid hydrodynamics

The superfluid helium must obey some form of hydrodynamic equations given by the general conservation laws, Galilean invariance, and the thermodynamic equation of state that also describes its superfluid properties. These equations were written down for $^4$He by Landau (1941), Khalatnikov (1965), Landau and Lifshitz (1959) who made the key assumption that in order to describe the viscous-less fluid flow the independent hydro dynamical variables must include a velocity field $v_s$, to which is associated a fraction $\rho_s/\rho$ of the total density of the liquid. Ideal inviscid fluid motion is such that this velocity field is irrotational:

$$\nabla \times v_s = 0 \, .$$

The superfluid fraction velocity derives from a velocity potential, $v_s = \nabla \Phi$ and obeys the Euler equation for ideal fluid flow. The remainder of the fluid, the normal fraction $\rho_n = \rho - \rho_s$, $\rho = m_4/v_s$ being the $^4$He atomic mass divided by the volume occupied by one atom, to which is associated a “normal” velocity $v_n$, obeys an equation similar to the Navier-Stokes equation of viscous flow. The total momentum density of the helium liquid is the sum of the contributions of these two fluids:

$$j = \rho_s v_s + \rho_n v_n \, .$$

When $\rho_n$ and $\rho_s$ can be assumed incompressible – i.e., for small flow velocities, the separation between potential flow for $v_s$ and a Poiseuille flow for $v_n$ becomes exact. In this approximation, superflow is effectively decoupled from normal fluid motion. It has become customary to talk somewhat loosely of the motion of the superfluid as fully distinct from that of the normal fluid. This simplified view is adopted here but, occasionally, it fails (Idowu et al., 2000a,b).

Landau’s two-fluid model, preceded by the original suggestion of Tisza (1938), has been universally adopted (Enz, 1974). In addition to describing the dynamics of superfluids, it also gives a clear physical description of the normal fluid as a gas of thermally-excited elementary excitations, the phonons and the rotons, with a sharply defined energy spectrum $\epsilon(p)$ (Clyde, 1993; Griffin, 1987).

Landau attributed the non-viscous property of $^4$He flow to the sharpness of this spectrum. A momentum-exchange event involving the transfer of momentum $p$ between the fluid and an obstacle also involves a finite exchange of energy $\epsilon(p)$. If the fluid is moving at velocity $v_s$ with respect to the obstacle, this energy becomes, by a Galilean transformation:

$$\epsilon' = \epsilon + p \cdot v_s \, ;$$

the process becomes energetically favourable when $\epsilon' \sim 0$, that is, when the velocity approaches the minimum value reached by $\epsilon/p$. This condition defines the Landau critical velocity:

$$v_{L} = \frac{\epsilon(p)}{|p|} = \frac{\partial \epsilon}{\partial p} \, .$$

Such a limiting velocity has been observed in the propagation of ion waves where rotons are created in $^4$He under certain flow conditions as reviewed by McClintock and Bowley (1995).

The two-fluid hydrodynamics has been extended to the superfluid phases of $^3$He. A number of new features appear owing to the anisotropy introduced by the Cooper pairing in a $l = 1$ state of orbital momentum. Some of these features are discussed in Sect[8,6] but the separation of the hydrodynamics between a superfluid component and a “normal” component still holds as well as the existence of a Landau critical velocity.

Landau’s two-fluid hydrodynamics, based on early experiments on superfluids, accounts remarkably well for a whole class of thermodynamical and hydrodynamical properties, notably for the existence of second sound and for the non-classical rotational inertia in Andronikashvili oscillating-disks experiments (Andronikashvili and Mamaladze, 1966). It features full internal consistency; it assumes that the motion of $\rho_s$ is pure potential (irrotational) flow and carries no entropy, and it reaches the conclusion that, below $v_{L}$, this motion is indeed fully inviscid. However, it has a number of shortcomings pointed out in particular by London (1954). In particular, for the purpose of this review, it does not discuss the roots of the irrotationality condition, Eq. [1], which are to be found in the existence of a complex scalar order parameter arising from the transition to a Bose-condensed state, as recognised by London (1938). It gives no clue as to when the hydrodynamics of the superfluid fails at close distance as a macroscopic approach to hydrodynamic is bound to. Namely, it provides no way of estimating the superfluid coherence length - or healing length of the macroscopic wavefunction. Also, it completely disregards the existence of quantised vorticity, which, as recognised by Feynman

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6 see Landau and Lifshitz (1959), ch. XVI

7 see, for instance, Baym (1969) for a detailed proof

8 For a review see Hall and Hook (1986).

9 This question was nonetheless treated by the Landau school: Khalatnakov, 1965.
A change of gauge,\footnote{As implied in a footnote of a paper by Onsager \cite{onsager1949} and mentioned in the footnote in page 151 of London’s book.} is responsible for dissipation of the superfluid motion and for different, and more commonly met in practice, critical velocity mechanisms than that of $v_L$.

As already noted, the Landau criterion for critical velocities rests on the fact that the elementary excitation energy spectrum is sharply defined, that is, there are no excitations with low energy and small momenta that would be capable of exchanging momentum with the moving superfluid, thus causing dissipation. It turns out that $^3$He in the normal Fermi liquid state does display in neutron diffraction experiments an elementary excitation spectrum with a phonon and a roton part\footnote{A useful discussion of this topic can be found in Nozières and Pines \cite{nozierepin1990} Ch. 5 and also in Feynman \cite{feynman1972}.}\cite{griffith1987}\cite{stirling1976}. However, that spectrum is broad; $^4$He does not exhibit superfluid properties until Cooper pairs of atoms form and Bose-condense. As stressed by Feynman \cite{feynman1972}, it really is the lack – the scarcity in Feynman's own words – of low-lying energy levels at finite momenta, a property of the N-boson groundstate with a macroscopic number of particles in it and the existence of few excited states well-separated in energy, that results in superfluidity.

\section*{B. The superfluid order parameter}

A different description of superfluidity in which a central role was also attributed to the phase of the order parameter (assumed to describe this superfluidity) was introduced by Onsager in 1948, as reported by\footnote{Along the same lines, Leggett \cite{leggett1995} p. 453 may also be quoted: “Treatments of the concept of broken gauge symmetry are somewhat unsatisfactory…” – see the last Section.} London \cite{london1954} \cite{london1950} \cite{feynman1955} \cite{feynman1972}. The information on the localisation of the bosons at $r_1, r_2, \ldots, r_N$ and on their short-range correlations has been lost in Eq. \ref{eq:superfluid_order_parameter}. Expression \ref{eq:superfluid_order_parameter} is not the exact ground state wavefunction anymore; it is not strictly orthogonal to the excited state wavefunctions, that is, it contains a certain admixture of these excited states\footnote{See Feynman \cite{feynman1955}, Penrose and Onsager \cite{penroseonsager1956} or Landau and Lifshitz \cite{landaulifshitz1958} §61.}. However, considered as a “macroscopic matter field” in Anderson’s own words, it has provided a lot of mileage in describing the properties of superfluids.

\begin{equation}
\Psi_0 (r) = f(r) e^{i\psi(r)} . \tag{5}
\end{equation}

But the unifying power of quantum field gauge theories ultimately carried the day \cite{greiter2005} \cite{yang1962}. The fact that a droplet of superfluid randomly picks up a well-defined quantum phase when it nucleates out of vapour or out of normal fluid in a confined geometry, is referred to as the breaking of gauge symmetry. The term “Bose-symmetry”\footnote{In the words of C.N. Yang \cite{yang2003} “Weyl in 1929 came back with an important paper that really launched what was called, and is still called, gauge theory of electromagnetism, a misnomer. (It should have been called phase theory of electromagnetism.”}. It reduces to a complex scalar with a constant phase and a modulus that remains finite at every point in the sample. Atomic motion results in small scale, small amplitude fluctuations of this complex scalar. Averaging these fluctuations over finite, but still small, volume elements leaves a “coarse-grained” average wavefunction that simply reads $\Psi_0 = f \exp i\varphi$ with constant amplitude and phase, inasmuch as the fluid density remains constant throughout the sample. If the system is inhomogeneous on a scale much larger than the coarse-graining volume, the modulus and phase become slowly varying functions of the position $r$.

Indeed, as soon as wavefunctions are considered, the concept of quantum phase becomes relevant. Its early origin can be found in a formalisation of electrodynamics by Weyl \cite{weyl2003}, in which a gauge transformation explicitly introduces a factor $e^{i\theta}$ in the theory. A change of gauge, $A' \rightarrow A + \nabla \theta$, combined with a change of the wavefunction $\psi' \rightarrow \psi \exp[i\varphi/\hbar c]$ leaves the Schrödinger equation unchanged. The application of Weyl’s prescription to quantum-mechanical systems led F. London to turn the exponent $\theta$ into a purely imaginary quantity $i\varphi$, $\varphi$ then having the significance of a phase \cite{yang2003}. These historical developments explain the somewhat inadequate terminology that refers to changes of the phase as gauge transformations \cite{greiter2005}.\footnote{This in essence is the macroscopic wavefunction considered by Onsager \cite{onsager1949}, London \cite{london1950} and Feynman \cite{feynman1955} \cite{feynman1972}. The information on the localisation of the bosons at $r_1, r_2, \ldots, r_N$ and on their short-range correlations has been lost in Eq. \ref{eq:superfluid_order_parameter}. Expression \ref{eq:superfluid_order_parameter} is not the exact ground state wavefunction anymore; it is not strictly orthogonal to the excited state wavefunctions, that is, it contains a certain admixture of these excited states. However, considered as a “macroscopic matter field” in Anderson’s own words, it has provided a lot of mileage in describing the properties of superfluids.}
C. The superfluid velocity

The particle density \( n(r) \) at point \( r \) of the \( N \)-boson system is given in terms of this macroscopic wavefunction, Eq. (5), by

\[
n(r) = \int d^3r_1 \ldots d^3r_N \Psi_0^*(r) \Psi_0(r) \sum_{i=1}^{N} \delta(r-r_i), \tag{6}
\]

and the particle current density by

\[
j(r) = \int d^3r_1 \ldots d^3r_N \sum_{i=1}^{N} \frac{\hbar}{2m} \left[ \Psi_0^*(r) \delta(r-r_i) V \Psi_0(r) + \Psi_0(r) V \Psi_0^*(r) \right] \\
= \int d^3r_1 \ldots d^3r_N \sum_{i=1}^{N} \frac{\hbar}{2m} \delta(r-r_i) \left[ \Psi_0^*(r) V_i \Psi_0(r) - \Psi_0(r) V_i \Psi_0^*(r) \right] \\
= n(r) \frac{\hbar}{m} \nabla \phi(r). \tag{7}
\]

Equation (7) leads as a matter of course to the definition of the local mean velocity of the bosons as

\[
v_s = \frac{\hbar}{m} \nabla \phi(r). \tag{8}
\]

According to this definition, the quantity \( v_s \) derives from the velocity potential \( \Phi = (\hbar/m) \phi(r) \) and is identified to the quantity introduced in the two-fluid hydrodynamics under the same notation. The expression for the fluid velocity takes the same form as that for the probability density of a single particle in an external field as shown for instance in the textbook by Landau and Lifshitz (1958). The strong correlations between bosons in the dense system – in particular the hard core interactions – are averaged out in the coarse-graining procedure.

These results so far apply to a system of structureless bosons in a non-degenerate groundstate, that is, at \( T = 0 \) with the proviso made above that wavefunction \( \Phi \) stands for a coarse-grained average of a microscopic wavefunction that cannot be dealt with explicitly. At finite temperature, it can be assumed that the number of atoms involved in Eqs. (6) and (7) is proportional to \( \rho_s / \rho \). Such an extension rests on the commonly held view that the Bose-condensed fraction carries no entropy and behaves as if it remained at absolute zero. The effect of temperature is simply to reduce its occupation number. Definition (8) of the critical velocity thus appears as a convenient formal construction that reproduces, in the classical limit, the quantity postulated by Landau to construct the two-fluid hydrodynamic model.

The macroscopic (coarse-grained) wavefunction \( \Psi(r) = n_s(r) e^{i\phi(r)} \) can be expected to approximately obey the Gross-Pitaevskii equation (9). The relevance of the Gross-Pitaevskii equation to the description of Bose-condensed systems has been put in a bright new perspective for ultra-cold atoms in a trap, for which it constitutes an excellent approximation (Cohen-Tannoudji and Robilliard 2001; Dalfovo et al. 1999). For quasihomogeneous systems, Landau’s two-fluid hydrodynamics can then be recovered in this framework. This approach is not pursued here, except to mention that the Gross-Pitaevskii equation provides an estimate of the distance over which the two-fluid hydrodynamics needs to be supplemented by quantum corrections, as discussed in Sect. VIII. However, it only provides a coarse description of the dense superfluid, sometimes not even qualitatively correct, as discussed by Pomeau and Rica (1993) in the context of the breakdown of superflow.

D. A more microscopic approach

So far, the discussion has been based on the general properties of the groundstate wavefunction of \( N \)-boson systems, turned into a “macroscopic wavefunction” by coarse-grained averaging. No clue has been given as to the suspected relationship between Bose-Einstein condensation and superfluidity at the microscopic level. Off-diagonal long range order (ODLRO) represents the commonly acknowledged fundamental microscopic concept underlying both superconductivity and superfluidity. It represents the kind of order that prevails in a superfluid or a superconductor at \( T = 0 \), as put forward by Yang (1962), extending earlier work by Penrose and Onsager (1956) while a parallel route was taken by Bogolyubov and other representatives of the Russian school and in particular Beliaev (1958) for the system of interacting bosons (18).

ODLRO stands for the correlation that exists between atoms in a Bose-Einstein condensate. This correlation already exists in its simplest form in a gas of \( N \) non-interacting Bose particles in a box of volume \( V \), as ex-

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15 Sec. 15

16 Gross, 1961; Langer, 1968; Nozières and Pines, 1990; Pitaevskii, 1961.

17 See, e.g., Abrikosov et al. (1961).

18 See Kadanoff (2013) for an insightful account of the historical genesis of the idea of ODLRO and a discussion of the role of the condensate in superfluidity and superconductivity.
pressed by its single-particle density matrix

\[ \rho_1(r, r') = (N/V) \int \text{d}r_2 \cdots \text{d}r_N \Psi_0(r, r_2, \ldots, r_N) \Psi_0^*(r', r_2, \ldots, r_N), \]  

and by density matrices of higher rank. In Eq. (9), \( \Psi_0 \) is the eigenfunction of the ground state at \( T = 0 \) satisfying the boundary conditions at the box wall (rigid walls, or periodic). As the particles of an ideal gas do not interact, this wavefunction is simply the product of \( N \) identical single-particle wavefunctions \( \psi(r) \) evaluated at \( r = r_i \), the particle locations, suitably normalised and symmetrised. Upon integration over the \( N-1 \) particles \( r_2 \ldots r_N \), all what is left is the product

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This number density \( n_0 \) is of the same order of magnitude as the total density \( n \). In a usual fluid, the on-diagonal elements \( \rho_1(r, r) \) are of order of the particle number density \( n(r) \) and particle correlations decrease rapidly as \( r \rightarrow r' \) increases. The off-diagonal terms with \( r \neq r' \) decrease rapidly with particle spacing. By contrast, a superfluid can sustain a persistent current: large scale correlations should be strong so that, when a particle is deflected at \( r \) by an obstacle and kicked out of the condensate, a twin-sister particle is immediately relocated in the condensate at \( r' \) with no loss of order in momentum space. Such correlations are described in the density matrix by a term embodying the condensate of the same “structure” as the product in Eq. (10), supplemented by other terms for the part of the system that cannot be accommodated in the ground state because of interparticle collisions:

\[ \rho_1(r, r') = \Phi(r)\Phi(r') + \text{other matrix elements}. \]  

It should be noted that \( \Phi(r) \) is the single-particle wavefunction of the condensate of interacting particles but not that of free particles as for the ideal gas as in Eq. (10).

The (condensate) ground state is occupied by a macroscopic number of particles \( N_0 \lesssim N \), so that \( \Phi^*(r)\Phi(r) = N_0/V = n_0 \) is a sizeable fraction of the fluid density; \( \Phi(r) \) is equal to \( \sqrt{n_0} \) in absolute value and independent of \( r \) (to the extent that \( n_0 \) is constant in space). The term \( \Phi^*(r)\Phi(r') \) in Eq. (11) does not decay as the particle locations \( r \) and \( r' \) become far apart compared to interatomic distances: it describes the long-range correlations in the condensate, or ODLRO. The excited states with \( k \neq 0 \) and distribution \( n_k \) are not macroscopically populated and only have short range coherence. The summation over all these remaining contributions in Eq. (11) also amounts to a macroscopic term \( \sum_k n_k \), of order \( N \) of these terms decays as \( |r - r'| \) becomes large but there are a large number of them.

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\[ \text{Nozières and Pines} (1990) \] give in Ch. 10 a very transparent account of ODLRO using the notation of field theory, in which the density matrix reads

\[ \rho_1(r, r') = \sum_k \langle \phi | \psi^*(r) | \phi_k \rangle \langle \phi_k | \psi(r') | \phi \rangle, \]  

\( \psi^*(r) \) and \( \psi(r') \) being the boson creation and annihilation field operators, \( | \phi_k \rangle \) a complete set of eigenstates of the system and \( | \phi \rangle \) the state in which the average is expressed, which is taken as the ground state \( | \phi \rangle \). Among the intermediate states \( | \phi_k \rangle \), those of special relevance to the kicking-out and relocation processes discussed here are those which connect the ground state with \( N \) bosons to the ground state with \( N - 1 \) bosons. So attention must be focused on the following matrix element

\[ \Phi(r) = \langle \phi(N-1) | \psi(r) | \phi(N) \rangle \]  

that is taken to represent the condensate wavefunction.
Neither the $\Phi'(r)\Phi(r')$ term in Eq. (11) nor the incoherent terms can be expressed explicitly for the dense helium, contrarily to near-ideal BEC gases and the BCS theory (for Cooper pairs). In all these situations, ODLRO is found to be present and to constitute a unifying feature sufficient to ensure flux quantisation (BCS superconductors) or velocity circulation quantisation (dense superfluid helium). That the simple factorisation of the coherent part of the density matrix ensures superfluidity is a remarkable result. It has been established on general grounds by Yang (1962), Penrose and Onsager (1956); they have used various approximate forms for the groundstate wavefunction of dense helium-4 at $T = 0$ to illustrate the splitting of the density matrix, Eq. (11), and to evaluate the depletion of the condensate, \textit{i.e.,} the value of $n_0$. They have used in particular Feynman’s simple Ansatz for the superfluid wavefunction (Feynman, 1955), which assumes strong hard-core repulsion and weak 2-body attraction with a minor role in interparticle correlations. Only the former can be kept for an approximate evaluation of $n_0$. Building on this remark, Penrose and Onsager have noticed that this depletion can easily be derived from the (known) pair distribution for a classical gas of hard spheres such as the one pictured in Fig. 1; they found that collisions between hard spheres with diameter $2.6$ Å, $3.6$ Å apart, leave only about $8$ % of the helium atoms in the zero-momentum condensate. This value of $n_0/n$ in 4He has been confirmed by more elaborate theories and by experiment.

While this depletion is a small effect in low density atomic gases (Dalfovo et al., 1999), it is large in liquid helium.

![Fig. 1.](image)

The strong depletion of the condensate raises the following question: how is it that the condensate fraction is only $8$ % while the superfluid fraction in the two-fluid model is $100$ % at $T = 0$? Simply because these are not the same quantities. The superfluid density stands in fact for the inertia of the superfluid fraction, as measured for instance by a gyroscopic device sensitive to trapped superfluid currents (Reppy and Lane, 1965) or, less directly, by the decoupling of the superfluid component in an oscillating disk experiment (Andronikashvili and Mamaladze, 1966).

This quantity is different from the density of bosons in the macroscopically occupied quantum state seen as a hump in the neutron diffraction spectrum. When the superfluid is set into motion, the condensate enforces long-range order and drags the excited states along through the short-range correlations; there is entrainment of the atoms in the fluid by the condensate. Microscopic theory is needed to describe this process in detail, as reviewed for instance by Griffin (1987).

Deferring to the end, SLAB further discussion on the merits and differences of the microscopic approach – the ODLRO concept – of Penrose and Onsager (1956) and Yang (1962) and the macroscopic quantum field point of view – a discussion to be found in Appendix A1 of Anderson (1966b) – the superfluid order parameter in $^4$He will be taken in the following as the macroscopic wavefunction $\Psi_0(r) = f(r)e^{iq\theta}$, with $f = n_s^{1/2}$, $n_s$ being the superfluid number density. The condensate density $n_0$ remains half-buried in the formalism. But the phase, a variable of lesser relevance in the older quantum mechanics, is now given the important role of governing superfluid dynamics. A fundamental result, established early by Beliaev (1958), relates the phase of the condensate particles to the chemical potential, $\varphi = \mu t/\hbar$, $t$ being time, but the deep significance of this odd-looking relation became apparent later.

The role of the phase as a dynamical variable was put to the fore by Anderson (1964) (Anderson, 1965, 1966a, 1984) who noted that phase and particle number are canonically conjugate variables. This is a well-known property in quantum electrodynamics for photons in a cavity. The number of photons in a given mode and their phase, defined for coherent electromagnetic fields in the cavity, are non-commuting operators. As such, they obey an uncertainty relation (Heitler, 1954) that reads:

$$\delta N \delta \varphi \gtrsim 1.$$  

As remarked by Heitler, “if the number of quanta of a wave are given it follows from eq. (12) that the phase of this wave is entirely undetermined and vice versa. If for two waves the phase difference is given (but not the absolute phase) the total number of light quanta may be determined, but it is uncertain to which wave they belong”. This remark will bear implications throughout this review.

Superfluidity is more than simply the absence of viscosity supplemented by the condition that vortices have quantised circulation. The urge to observe the role of

\[20\] For a recent review, see Glyde (2013).

\[21\] See also Glyde (2013).
the phase in a Josephson-type effect – and the failure to
do so for a long period of time – became quite pressing
to confirm the picture drawn by Anderson of helium as
obeying quantum mechanics in a more profound way
than simply as an ideal inviscid fluid with quantised
velocity circulation.

E. Anderson’s phase slippage

Anderson’s famed considerations on the flow of su-
perfluid $^4$He (Anderson, 1966a) provided the conceptual
basis for this experimental search for Josephson-type ef-
effects in neutral matter. Their underlying aim was to
convoy a physical, laboratory-oriented, meaning to or-
der parameter (5) and, in particular, to its phase. These
“considerations” provided the groundwork for phase
slippage experiments in $^4$He; they were gradually fos-
tered in a series of Lectures Notes (Anderson, 1964, 1965,
1984) and built upon the ideas of London (1954), Feyn-
man (1955), and Penrose and Onsager (1956), and also on
the quantum field theoretic approach of Beliaev (1958).

In the absence of a fully-established microscopic the-
ory of dense boson systems, these considerations rest on
the following set of well-argumented conjectures:

1. Building on the properties of the coherent photon
fields in quantum electrodynamics recalled above, $N$
and $\phi$ are taken in dense liquid helium as canon-
ically conjugate dynamical (quantum) variables in
the sense that $N \leftrightarrow i(\partial/\partial \phi)$ and $\phi \leftrightarrow i(\partial/\partial N)$.

As such, they obey the uncertainty relation (12).
For a closed system with a fixed number of par-
ticles, the phase is completely undetermined. For
the phase to be determined within $\delta \phi \ll 1$, $N$
must be allowed to vary, that is, the condensate must be
able to exchange particles with other parts of the
complete physical system, which includes the non-
condensate fraction of the bosons and the eventual
measuring apparatus.

For the Josephson-effects experiments specifically
considered by Anderson, the two weakly-coupled
helium baths also exchange particles. For all these
reasons, $N$ is allowed to fluctuate locally so that $\delta N$
takes a non-zero value. It can be shown (Beliaev,
1958) that $\delta N$ is of order $N$ rather than unity, so that
$\delta \phi \sim O(1/N)$ and $\phi$ is well defined.

2. A Hamiltonian $\mathcal{H}$ should therefore exist such that,$N$ being allowed to vary,

$$
\hbar \frac{\partial N}{\partial t} = \frac{\partial \mathcal{H}}{\partial \phi},
$$

$$
\hbar \frac{\partial \phi}{\partial t} = -\frac{\partial \mathcal{H}}{\partial N}.
$$

Upon coarse-graining, the quantum operators be-
come quasi-classical and their coarse-grained aver-
age obey equations formerly identical to (13)
and (14). Eq. (13) defines the particle current $I =
\partial E/\partial \phi$ since $\partial \mathcal{H}/\partial \phi \Rightarrow \partial E/\partial \phi$
after averaging. Likewise with $\partial \mathcal{H}/\partial N \Rightarrow \partial E/\partial N = \mu + \frac{1}{2} m_4 v_s^2$
where $\mu$ is the chemical potential in the fluid at
rest, $\mu = m_4 P/p + m_4 gh + s_4 T$, Eq. (14) becomes

$$
\hbar \frac{\partial \phi}{\partial t} = -(\mu + \frac{1}{2} m_4 v_s^2).
$$

Eq. (15) states that, whenever there exists an electro-
chemical potential difference between two points
1 and 2 in a superfluid (or a superconductor), the
phase of the order parameter varies in time with a
rate proportional to $\mu_1 - \mu_2$: this $\alpha$-effect is quite
detectable and has nowadays many applications.
It was first discussed by Josephson (1962) (Joseph-
son, 1964) for the tunnelling current between su-
perconductors coupled through a thin barrier. A
full derivation in the case of superfluid helium
can be found in the monograph by Nozières and
Pines.\(^{22}\)

Upon taking the gradient of both left and right-
hand sides, Eq. (14) becomes, using the definition
of $v_s$,

$$
\frac{\partial v_s}{\partial t} + V \left( \frac{P}{\rho} + \frac{v_s^2}{2} \right) = 0,
$$

which is nothing else but the Euler equation for an
inviscid fluid with no vorticity ($\omega = \nabla \times v_s = 0$). Equation (16) is precisely the same as that for the
velocity of the superfluid component in Landau’s
two-fluid hydrodynamics.

3. Anderson assumed that Eq. (15) for the time vari-
ation of the order parameter phase holds with no
solution of continuity between the quantum tun-
nelling case and the classical inviscid fluid, i.e., that
it has universal applicability. This unifying ap-
proach is, in a broad sense, internally consistent
but details are missing of how the normal com-
ponent interacts with the superfluid component, which
brings dissipative terms into Eq. (16), and how the definition of $v_s$ as $(\hbar/m) \nabla \phi$ breaks down at small distances where coarse-graining cannot
be performed. These fine points have been raised
in the discussion at the end of Anderson’s com-
unication at the Sussex Symposium on Quan-
tum Fluids (Anderson, 1966b). His views are that

\(^{22}\) Nozières and Pines (1990) §5.7; the groundwork for Eq. (15) was laid
by Beliaev (1958) and Pitaevskii (1958).
the phase equation $\hat{j}(\mathbf{r})$, being more fundamental than $\hat{\rho}$, always hold. This equation describes both simple superfluid acceleration, expressed by $\hat{\rho}$, the ideal tunneling situation envisioned by Josephson (see Sect.VIII), and the case where the variation of the phase is caused by the motion of vorticity.

4. The last conjecture asserts that the dissipation of the kinetic energy of a superflow is, when averaged over time, proportional to the rate at which vorticity crosses the superflow streamlines. In fact, a stronger statement has been rigorously proved by Huggins, which governs the detailed transfer of energy between the potential flow of the superfluid and moving vorticity. This process is pivotal to the understanding of superflow decay and, more generally, of vortex dynamics as discussed in Sec.III.

Anderson’s ideas on phase slippage, linked to the motion of vortices, have provided the conceptual framework for the experiments on the onset of dissipation and the Josephson effects in superfluids, as discussed in Sect.IV and VIII.B. All facets of these experiments in superfluid $^4$He and $^3$He can be very well accounted for with the help of the macroscopic quantum phase $\Phi$. The same is true for the Bose-Einstein condensates formed by ultra-cold atomic gases but the approach from the point of view of atomic physics has brought new insights as discussed in Sec.IX.

In spite of these successes and other advances, notably the discovery of superfluidity in the fermionic light isotope $^3$He, the studies of Bose-Einstein condensation in ultra-cold atomic gases, and striking refinement of path integral simulations (see, e.g., Ceperley, 1995), these ideas are still surrounded by an aura of mystery that lingers on in spite of the facts that: (1) the formal theoretical groundwork has been put on a firmer basis, (2) the implications of the uncertainty relation to laboratory observations, as well as of the other conjectures of Anderson, have been clarified by the developments of the experiments in the past forty and so years since they were formulated.

This review will tackle some of these advances, in particular by showing what the phase slippage experiments really consist of, how phase slippage proceeds from a dissipative regime governed by vortex dynamics to a true dissipationless Josephson regime, and that this truly quantum behaviour manifests itself in matter waves interferometric measurements. All this stems from Anderson’s considerations on the flow of superfluid helium.

III. VORTEX DYNAMICS AND ENERGY DISSIPATION

Vortex filaments are extended quasi-one-dimensional defects in the superfluid, line vortices. At the core of these defects, the superfluid order parameter is either zero as in $^4$He or heavily distorted as in $^3$He. They form the prevalent topological defects in superfluids, vortex sheets in the bulk of the liquid being found only in $^3$He-A.

At distances larger than the core size, superfluid vortices behave according to the laws of ideal fluid hydrodynamics, that is, as classical vortices with a given, quantised, vorticity. Classical vortices have been studied for many decades (Lamb, 1945; Salamon, 1992). The properties of superfluid vortices have been the subject of detailed studies in recent years in order to clarify in a number of standing problems, their mass and impulse, the Magnus and Iordanski forces, the eigenmodes of isolated vortices - the Kelvin waves, the collective behaviour of vortex arrays - the Tkachenko waves, the reconnection of two vortices, superfluid vortex tangles, and lastly, the formation of vortices and their annihilation. This review is concerned mainly with the last topic but will be made of other properties of vortices, either single or few at a time, mostly without consideration to the normal fluid background. These properties bear a close resemblance to those of magnetic vortices in superconductors as pointed out by Sonin and Krusius (1994).

At temperatures well below 1 K, vortices in superfluid $^4$He experience negligible friction from the normal fluid, the fraction of which becomes very small. If they deform only little and slowly, they constitute stable fluid eddies: their velocity circulation is conserved (and furthermore, quantised!) and they cannot vanish to nothing. Their core radius, $a_0$, is of the order of the superfluid coherence length, a few Å in $^4$He (Donnelly, 1991; Glaberson and Donnelly, 1986), one to two orders of magnitude larger in $^3$He (Vollhardt and Wolke, 1990).

As the temperature increases, the scattering of phonons and rotons by the vortex cores causes dissipation. Mutual friction between the superfluid vortices and normal fluid sets in. Close to the superfluid transition temperature, the core size increases and eventually diverges.

Some of the properties of vortices that have a rele-
vance to phase slippage are summarised below. Extended coverage of this topic can be found in the monograph by [Donnelly (1991)]. Here, the dynamical properties of superfluid vortices are derived directly from the existence of a superfluid order parameter. Some simplifying approximations will be made in order to get a simpler physical description of a vortex element, treated more in the manner of a quasiparticle with mass, energy and impulse. The following discussion then rests on physical concepts such as energy conservation or the balance of forces. It follows largely the work of Sonin [1987]. It differs from the more traditional and rigorous fluid-mechanical approach, as can be found in the recent monograph by [Safran (1992)], but it provides a more intuitive feel for the behaviour of superfluid vortices that will prove useful in the description of phase slips.

A. Quantisation of circulation

Superfluid vortices have quantised circulation. This comes about because their core is non-superfluid and constitutes a topological defect in the superfluid. The circulation of the superfluid velocity \( \mathbf{v}_s \) on any path around such a defect,

\[
\oint \mathbf{v}_s \cdot d\mathbf{l} = \frac{\hbar}{m_4} \oint \mathbf{V} \cdot d\mathbf{l} = \frac{\hbar}{m_4} (\varphi_2 - \varphi_1) .
\]

amounts to \( \kappa_4 = 2\pi\hbar/m_4 \)\(^{25}\) because the phase \( \varphi \) of the order parameter can change only by multiples of \( 2\pi \) along any closed contour entirely located in the superfluid. This property holds for the true condensate wavefunction as a basic requirement of quantum mechanics. It is not altered in the coarse-graining average.

Consider the velocity circulation from point 1 to point 2 in Fig. 2 along a path \( \Gamma \) entirely located in the superfluid:

\[
\kappa = \int_1^2 \mathbf{v}_s \cdot d\mathbf{l} = \frac{\hbar}{m_4} \int_1^2 \mathbf{V} \cdot d\mathbf{l} = \frac{\hbar}{m_4} (\varphi_2 - \varphi_1) .
\]

Along another path \( \Gamma' \) also going from 1 to 2, as shown in Fig. 2, the circulation is \((\hbar/m_4)(\varphi_2 - \varphi_1 + 2n\pi)\). If \( \Gamma' \) can be deformed into \( \Gamma \) continuously while remaining in the superfluid, then \( n = 0 \). If this cannot be done, \( n \) may be a non-zero integer, 1 in the case under consideration.

If \( \Gamma \) crosses the core of a \(^4\)He vortex, in which superfluidity is destroyed and the order parameter amplitude is zero, \( n \) changes by 1 because \(^4\)He vortices carry a single quantum of circulation for reasons discussed below. Conversely, when a vortex crosses a superfluid path from 1 to 2, the circulation along that path changes by one quantum and the phase difference by \( 2\pi \). This simple property forms the basis of the phase slip phenomenon described in Sec.[LV]

Experiments have confirmed to a high accuracy the quantisation of hydrodynamic circulation both in \(^4\)He [Karn et al., 1980; Vinen, 1961; Whitmore and Zimmernann, Jr., 1968] and in \(^3\)He [Davis et al., 1991]. This feature constitutes a cornerstone of superfluid physics, and evidence for the reality of the superfluid quantum phase.

B. Vortex flow field and line energy

The flow velocity induced by a straight vortex filament, chosen along the unit vector \( \hat{\mathbf{z}} \), at a distance \( r \) measured in the plane perpendicular to \( \hat{\mathbf{z}} \) is easily expressed from the quantisation of the velocity circulation and the symmetry around the vortex axis as

\[
\oint \mathbf{v} \cdot d\mathbf{l} = \kappa_4 \implies \mathbf{v}_v = \frac{\kappa_4}{2\pi} \hat{\mathbf{z}} \times \frac{\hat{\mathbf{r}}}{r} ,
\]

provided that \( r \) is larger than \( a_0 \). The quantity \( \mathbf{v}_v \) is the vortical flow due to the vortex element. The superfluid velocity \( \mathbf{v}_s \) is the sum of an eventual potential flow \( \mathbf{v}_p \)

\[\text{FIG. 2 (a) } \Gamma \text{ can be deformed continuously into } \Gamma' \text{; both paths give the same phase difference between point 1 and point 2. (b) vortex V stands between the two paths; the phase differences along } \Gamma \text{ and } \Gamma' \text{ differ by } 2\pi.\]

\[\text{FIG. 3 The geometrical representation of a vortex loop, the surface spanning the loop with element } dS, \text{ and the solid angle } \Omega \text{ subtended by the loop from point } r, \text{ the tangent } \mathbf{t}, \text{ normal } \mathbf{n}, \text{ and binormal } \hat{\mathbf{b}} \text{ at point s. The flow lines of the vortex velocity field (dashed line) thread surface } S \text{ where the phase changes determination by } 2\pi. \text{ The choice of surface } S \text{ is immaterial as long as it subtends the vortex loop.}\]

\[\text{\footnotesize{\[25\]} The quantum of circulation in } ^4\text{He takes the value } 9.97 \times 10^{-4} \text{ cm}^2/\text{sec and in } ^3\text{He where the boson mass is } 2m_3, \kappa_3 = \hbar/m_3 = 6.65 \times 10^{-4} \text{ cm}^2/\text{sec.}\]
existing independently of the vortex, for instance applied externally, and of \( v_\omega \). The contribution of \( v_\omega \) to the loop integral in Eq. (19) is nil and leaves the circulation unchanged.

For \( r < a_0 \), the detailed structure of the core becomes important. A single straight vortex filament conveys to the helium sample an angular momentum of one Planck constant \( 2\pi\hbar \). Conversely, straight vortex filaments are created by rotating the helium container; they have been the object of very detailed and precise studies [26].

Equation (19) can be extended to curved vortices, provided that their radii of curvature are much larger than the core radius \( a_0 \), by noting that it bears a direct analogy with Ampère’s law, \( \mathbf{\nu} \) standing for the magnetic field and \( \kappa \) for the electric current carried by the conductor [27].

The velocity at point \( r \) induced by a closed vortex filament lying along the curve \( s \) is then given by the analogue of the Biot-Savart law in electrodynamics [28].

\[
v_\omega(r) = \frac{\kappa_4}{4\pi} \oint d\mathbf{l} \times \frac{r - \mathbf{s}(l)}{|r - \mathbf{s}(l)|^3}.
\]

The geometrical representation of the vortex loop by \( s \) is such that \( d\mathbf{l} = ds \mathbf{n} \) is a vector oriented along the tangent to the loop \( \mathbf{n} \) of infinitesimal length \( dl \), \( l \) being the arc length of the loop (see the sketch in Fig. 3). The tangent \( \mathbf{t} \) is the unit vector \( ds/dl = d\mathbf{l}/dl \). Its derivative with respect to \( l \) defines the normal to the loop \( \mathbf{n} \) and the radius of curvature \( R \): \( d\mathbf{l}/dl = d^2s/dl^2 = \mathbf{n}/R \). As noted above, the radius of curvature \( R \) should be large – and the change of orientation of the tangent \( d\mathbf{l}/dl \) small – with respect to the core radius for this representation of the vortex element as a one-dimensional line to be valid.

The integrand in Eq. (20) gives the contribution of the vortex element \( dl \) located at \( s \) on the loop to the full velocity field. An integration by parts yields

\[
v_\omega(r) = \frac{\kappa_4}{4\pi} \nabla \times \oint \frac{ds}{|r - \mathbf{s}|} = \nabla \times \mathbf{A}(r),
\]

which defines a vector potential for the vortex loop velocity field, \( v_\omega = \nabla \times \mathbf{A} \). The quantity \( \kappa_4 \mathbf{A} \) will be shown in the next paragraph to be the energy of the line element \( dl \).

Equation (21) fulfills the mantra of conventional mathematical physics according to which a vector field can be split into an irrotational part, which derives from a scalar potential, and a remainder, the solenoidal part, which is not curl-free and which derives from a vector potential. While utterly correct in mathematical terms, this point of view may be slightly misleading for the superfluid velocity fields. The latter are a subset only of the more general vector fields in the sense that vorticity is localised in space to the vortex cores and that the vortex line can be treated as a line singularity. By invoking Stokes’ theorem to transform the line integral in Eq. (20) into an integral over the surface spanned by the vortex loop,

\[
\oint dl \times \mathbf{a} = \iint (\mathbf{\nabla} \cdot \mathbf{a} - \mathbf{\nabla} \cdot \mathbf{a} \cdot d\mathbf{S})
\]

with \( \mathbf{a} = (r - \mathbf{R})/|r - \mathbf{R}|^3 \), the Biot-Savart law (20) can also be put under the following form [27]:

\[
v_\omega(r) = \frac{\kappa_4}{4\pi} \mathbf{v}_r \left\{ \iint_S \frac{r - \mathbf{R}}{|r - \mathbf{R}|^3} \cdot d\mathbf{S} \right\} = \frac{\hbar}{m_s} \nabla \varphi_v,
\]

the infinitesimal surface element \( d\mathbf{S} \) being located at position \( \mathbf{R} \). Thus the velocity fostered by the vortex derives from a scalar potential as well as a vector potential. This property rests on Eq. (20), which describes the velocity field of a line singularity of vorticity. Thus, everywhere in the superfluid but precisely at the vortex cores, the superfluid velocity \( v_s \) is indeed irrotational and derives from a scalar potential, the quantum phase [30].

The quantity in curly brackets in Eq. (22) is the solid angle under which the loop is seen from point \( r \), as pictured in Fig. 3. It amounts to twice the change of the order parameter phase \( \varphi_v \) due to the vortex flow field. The velocity induced by a vortex loop decreases at large distance from the loop as that of a dipole in the electromagnetic analogy, that is as \( 1/r^3 \), much faster than the \( 1/r \) dependence for straight vortex filaments (see Eq. (19)). The \( 1/r \) dependence can still be expected to hold at a distance away from the core smaller than the local radius of curvature of the vortex filament. At distances larger than the loop size, the velocity field rapidly dies away. This property is well known for magnetic fields fostered by electric current loops. It means, for practical purposes, that vortex loops far apart do not interfere and distant boundaries have no effect. These simplifying features will often be assumed in the following.

The flow around the core of a vortex element carries kinetic energy, obtained by integration of \( 1/2\rho_s v_s^2 d^3x \) over the volume \( V \) over which this flow extends. The quantity \( \rho_s \) is the superfluid density. This integral is evaluated with the help of the vector potential \( \mathbf{A} \), from which derives the vortical flow field, as follows:

\[
E_v = \frac{\rho_s}{2} \int_V d^3x \mathbf{v}_\omega \cdot \mathbf{v}_\omega = \frac{\rho_s}{2} \int_V d^3x \mathbf{v}_\omega \cdot \nabla \times \mathbf{A}.
\]

See [Dalfovo 1992], [Fetter 1976], [Salomaa and Volovik 1988], [Sonin 1987] [26]. See [Andronikashvili and Mamaladze 1961], [Hall 1966], [Andronikashvili et al. 1981], [Krussius et al. 1993], [Sonin 1987] [27]. See e.g. Lamb 1945, §47. See e.g. Saffman 1992 §2.3 [28]. See e.g. Saffman 1992 §2.3 [29]. The situation in superfluid \(^3\)He-A is more complicated, as discussed in [30].
From the vector identity \( \nabla \cdot (a \times b) = b \cdot (\nabla \times a) - a \cdot (\nabla \times b) \), the last expression for \( E_v \) can be transformed into the sum of two volume integrals, one of which can be changed into a surface integral over \( A \times \mathbf{v} \) with the divergence theorem. By taking the volume boundary sufficiently far from the vortex loop, the surface integral can be made negligible and there remains:
\[
E_v = \frac{\rho_s}{2} \int_V d\tau \mathbf{A} \cdot (\nabla \times \mathbf{v}_v) .
\]
The curl of \( \mathbf{v}_v \) is zero everywhere but on the vortex core, where it is singular \( \nabla \times \mathbf{v}_v = k_4 \hat{t} \delta_2 (r - s) \). Integration over the two-dimensional delta function \( \delta_2(x) \), defined in the plane normal to the tangent \( \hat{t} \) to the loop, reduces the volume integral for \( E_v \) to a line integral over the vortex loop:
\[
E_v = \frac{\rho_s k_4}{2} \int dl \cdot \mathbf{A} . \tag{23}
\]
The vortex kinetic energy is the circulation of the vector potential along the loop.

Substituting the expression of the vector potential, Eq.\( (21) \), into Eq.\( (23) \) leads to a standard expression of the vortex energy in terms of a double contour integral over the vortex loop:
\[
E_v = \frac{\rho_s k_4^2}{8\pi} \oint ds_1 \frac{ds_2}{dl_1} \left( \frac{ds_2}{dl_2} \left[ s_1(l_1) - s_2(l_2) \right] \right) . \tag{24}
\]
It can be noted on Eq.\( (24) \) that loops carrying two quanta of circulation would have four times the line energy of singly-charged ones. Vortices with multiple quanta of circulation are thus strongly disfavoured on energy grounds compared to that of separate singly-charged vortices; they are energetically unstable and decay spontaneously into several singly-charged entities. Only singly-charged loops and filaments are considered here.

For a circular ring of radius \( R \) the integral can be evaluated in terms of elliptic functions and expanded in the small parameter \( a_0 / R \). The kinetic energy associated with the ring velocity field is thus given by
\[
E_R = \frac{1}{2} \rho_s k_4^2 R \ln \frac{8R}{a_0} + O \left( \frac{a_0}{R} \right) . \tag{25}
\]

For a straight vortex filament, the integral for the kinetic energy in the volume comprised between two planes perpendicular to the filament can be calculated directly from Eq.\( (19) \). It is expressed for a unit length of vortex by:
\[
e_f = \frac{\rho_s k_4^2}{4\pi} \ln \left( \frac{r_m}{a_0} \right) . \tag{26}
\]
The logarithmic divergence is cut at short distance to \( a_0 \), taken as the definition of the core radius. Its value, of the order of one Å at low pressure, is obtained from experiment Rayfield and Reif \( 1964 \). The far distance cut-off \( r_m \) depends on the size of the container, and on other circumstances that perturb the flow pattern expressed by Eq.\( (19) \) like the flow field of other vortices, or of other parts of the same vortex if it is not quite straight. For instance, the ring energy, Eq.\( (25) \), stems from Eq.\( (26) \) if \( r_m \) is taken to be \( 8R \).

The line energy of the core, usually taken as
\[
e_{ab} = -\frac{7}{4} \frac{\rho_s k_4^2}{4\pi} , \tag{27}
\]
for a core rotating as a solid body, must be added to Eq.\( (26) \) to obtain the full vortex energy per unit length
\[
e_v = e_f + e_{ab} = \frac{\rho_s k_4^2}{4\pi} \ln \left( \frac{r_m}{a_0} \right) - \frac{7}{4} . \tag{28}
\]
This expression of the energy per unit length also holds for rings, filaments or general loops provided than \( r_m \gg a_0 \). It can be viewed as a force developing along the vortex axis, or line tension, that tends to shorten the vortex length. That is, it pulls on its ends: vortices left to themselves with a loose end would shrink to zero. Stable vortices in finite size containers either are closed on themselves in loops or connect to the container walls.

As stand-alone loops or pinned filaments, their length is constant as long as they cannot exchange energy with the rest of the fluid. If they connect to hard walls, their flow field must be such as to satisfy the condition that no fluid can penetrate into the wall. A convenient way of satisfying such a boundary condition is to continue the vortex filament into the wall, forming an imaginary image vortex. Such a continuation procedure can be shown to be possible and to yield a unique velocity field for vortices meeting with walls usually satisfy the condition of no flow through a solid boundary by standing perpendicular to it.

It thus stems from this discussion that finite length vortices always close on themselves or end at walls. In this latter case, they also form closed loops if their image is taken into account. The opposite view, namely that vortices are most of the time infinitely long as, for instance, vortices formed under rotation in a cylindrical helium bucket, is also held. The process of nucleation of vortices considered below obviously requires

\begin{itemize}
  \item \label{footnote1} Using the Gross-Pitaevskii equation, Roberts and Grant \( 1971 \) find that the prefactor \( 7/4 \) should be replaced by the not-so-different number \( 0.615 \).
  \item \label{footnote2} See e.g. Saffman \( 1992 \) §2.4.
  \item \label{footnote3} It is understood here that the boundary does not carry vorticity. A case of the contrary is discussed by Sonin \( 1994 \).
  \item \label{footnote4} See for instance Saffman \( 1992 \) §1.4.
\end{itemize}
that their size be finite (otherwise, the energy involved would be infinite): the isolated vortex loops dealt with in the following have a finite size, usually small.

It follows from the existence of a positive line tension that a vortex loop would tend to spontaneously reduce its length and minimise the line energy. However, the energy so released by the vortex loop in its motion is disposed of into the surrounding fluid only in certain conditions of flow. The line tension is opposed by other forces that arise from the vortex motion in the fluid or from its interaction with the boundaries, namely, the Magnus force and pinning forces. These forces govern the trajectory followed by the vortex, its change of shape and self-energy, and its interaction with the potential flow field in which it is immersed. These processes are discussed below and in §III.E.

C. Vortex line impulse and mass

If an external potential flow with velocity \( \mathbf{v}_p = (\hbar/m_4) \nabla \psi_p \) imposed by moving boundaries, a piston for instance, or nearby vortices, the kinetic energy of the combined flow \( \mathbf{v}_p + \mathbf{v}_v \) in a given volume \( V \) is the sum of the kinetic energy of the applied superflow \( \psi_p \), that of the vortex loop, obtained from Eq. (24), and the energy of interaction with the potential flow, \( E \). For a circular loop of radius \( R \)

\[
E = \frac{\hbar}{m_4} \int_s \psi_p \cdot \mathbf{v}_p \, dS = \frac{\hbar}{m_4} \int_s \nabla \psi_p \cdot \nabla \psi_v \, d\tau. \tag{29}
\]

Equation (29) represents the energy of interaction between the vortex and the applied flow. Making use of Green’s first identity expressed as

\[
\int_V \nabla \Phi \cdot \Psi \, d\tau = \int_S \Phi \nabla \Psi \cdot dS + \int_S \Phi \nabla^2 \Psi \, d\tau,
\]

\( S \) being the surface bounding volume \( V \) and \( dS \) being the outward pointing surface element, and taking into account mass conservation of the fluid in incompressible flow \( (\nabla^2 \psi = 0) \), \( \psi = (\hbar/m_4) \psi_p \) being the velocity potential of \( \psi_p \), the integral in Eq. (29) can be rewritten as

\[
E = \frac{\hbar}{m_4} \int_S \phi_v \mathbf{v}_p \cdot dS, \tag{30}
\]

where \( \phi_v \) is the phase change contributed by the vortex flow field.

The bounding surface \( S \) in fact yields two contributions to the integral in Eq. (30): the outer surface bounding \( V \) and, quite importantly, the cut spanning the vortex loop over which \( \phi_v \) changes discontinuously by \( 2\pi \) (see Fig. 3). If \( V \) can be chosen large enough, the velocity induced by the vortex on its surface is negligible and \( \phi_v \) is a constant: the contribution to (30) of the outer surface reduces to the net flux of \( \mathbf{v}_p \), which is zero. The contribution of the cut is \( 2\pi \) times the flux of \( \mathbf{v}_p \) through the vortex loop. Introducing the mass flux of the applied potential flow through the vortex loop, \( J_p \), the contribution of the cross term (29) takes the very simple form

\[
E = \frac{\hbar}{m_4} \int_{\text{loop}} \mathbf{v}_p \cdot dS = \kappa_4 J_p. \tag{31}
\]

Thus, an applied flow contributes to the vortex loop energy by the additional mass flux \( J_p \) threaded by the loop times the quantum of circulation.

In the event that \( \mathbf{v}_p \) can be considered homogeneous over the vortex loop, Eq. (31) becomes even simpler:

\[
E = \rho_s \kappa_4 \mathbf{v}_p, \tag{32}
\]

in which \( \mathbf{S} \) is the vectorial area of the loop, \( \int dS = (1/2) \oint r \times dl \), \( dl \) being the line element at point \( s \) of the oriented loop.

The total energy of the vortex is the sum of its energy in the rest frame, \( E_0 \), given by Eq. (24), and the energy of interaction with the potential flow, \( E_i \). For Eq. (32), this reads

\[
E_v = E_0 + P \cdot \mathbf{v}_p, \quad \text{with } P = \rho_s \kappa_4 \mathbf{S}, \tag{33}
\]

where \( P \) is defined as the impulse of the vortex loop.

It emerges from this derivation (and the various approximations made along the way) that, under a Galilean boost, vortex loops do behave as Landau quasiparticles, with an energy proportional to their length and an impulse proportional to their area. This approach puts some flesh on the bare bones of the conventional (and exact) fluid-mechanical vortex dynamics; it gives substance to the intuitive view than they can be treated as independent elementary entities. This physically meaningful manner of separating the vortical flow from the local potential superflow \( \psi_p \) will prove quite useful in the following.

For a circular loop of radius \( R \), a vortex ring, Eq. (33) gives the well-known result (Lamb, 1945):

\[
P_R = \pi \rho_s \kappa_4 R^2. \tag{34}
\]

The impulse is not simply a plain geometrical quantity as Eqs. (32) or (33) would let think. It is the resultant of the impulsive pressures that must be applied to the fluid at rest to create the vortex loop from rest. It possesses some of the properties of a true momentum. For instance, the propagation velocity of the vortex ring, Eq. (36), can be expressed as the group velocity associated
with the energy \( E_R \) and impulse \( P_R \) (Langer and Reppy, 1970; Roberts and Donnelly, 1970):

\[
v_R = \frac{dE_R}{dP_R} = \frac{\kappa_4}{4\pi R} \left( \ln \frac{8R}{a_0} - \frac{3}{4} \right),
\]

Expression (35) tends asymptotically to the actual velocity of a ring with a hollow core, which moves along its symmetry axis \( \hat{n} \) with velocity. In other words, a curved vortex element develops a self-velocity equal to that of a vortex ring with the same radius of curvature. This property makes perfect circular vortex rings a stable defect in the fluid. This appealingly simple result also applies to straight – or nearly straight – vortex pairs (Langer and Reppy, 1970).

However, these simple properties do not imply that a vortex has actual linear momentum. The vortical impulse is more elusive. For instance, it can be shown that a vortex ring moving freely under its own force at velocity \( v_R \) and impinging on a wall exerts no force on it (Fetter, 1972). This somewhat counter-intuitive result arises from the distribution of the flow around the vortex loop (Cross, 1974). The contribution of the flow that goes in the forward direction, and which causes the ring free motion, does contribute a momentum impulse into the wall equal to \( P_R \), but the returning fluid away from the ring, the backflow, yields an opposite contribution that leads to full cancellation of the momentum transfer recorded over an infinitely extended wall for the complete collision event. This push and pull action constitutes a reminder that actual momentum is carried by the individual fluid elements and that a vortex is a hydrodynamical object made up of many of those elements: the whole is greater than the sum of its parts.

The impulse of a vortex discussed above is in no way related to the vortex self-velocity as the product of this self-velocity by an inertial mass. The problem of the mass of a vortex has been a long lasting riddle, which has now been resolved in a satisfactory way in superfluid \(^4\)He.

This mass arises from several contributions. If it is assumed that the vortex has a hollow core of radius \( a_0 \) and that the compressibility of the surrounding superfluid in rapid rotation can be neglected, the vortex mass is simply the mass of the displaced fluid. For a cylindrical body, this amounts to \( \pi \rho_s a_0^2 \) per unit length. As \( a_0 \) is very small in \(^4\)He, \( 1 \sim 2 \, \text{Å} \), that is of the same order as the interparticle spacing, this contribution is very small.

However, the neglect of compressibility is unwarranted because the peripheral velocity around the vortex core turns out to be large. The pressure drop in the vicinity of the core is given by the Bernoulli equation, which reads for the potential flow of a compressible fluid:

\[
\frac{\delta P}{\rho_s} = \frac{1}{2} \frac{v^2}{c_s^2} = 0.
\]

The change in density at distance \( r \) from the core where the velocity is \( \kappa_4/2\pi r \) is then:

\[
\delta \rho_s = \frac{\delta P}{\kappa_4^2 \rho_s} = \frac{1}{8\pi^2} \frac{\rho_s}{c_s^2} \frac{1}{r^2},
\]

using the relation between the (first) sound velocity and the compressibility \( c_s = (\partial P/\partial \rho)^{-1/2} \) which is justified when the normal fluid fraction is small \( (\rho_s \approx \rho) \). The overall change of mass about a unit length of vortex filament arising from the fluid compressibility is obtained by integrating Eq.(38) over space:

\[
\mu_v = \int_{a_0}^{r_m} \int_0^{2\pi} \int_0^1 \delta \rho_s r \, dr \, d\theta \, dz = \frac{\kappa_4^2 \rho_s}{4\pi c_s^2} \ln \frac{r_m}{a_0}.
\]

The vortex mass diverges logarithmically with \( r_m \) and ranges from negligible for \( r_m/a_0 \sim 10^3 \) to important for large vortices, \( r_m/a_0 \sim 10^3 \). However, in most cases, the mass of the vortex remains small and can be neglected except for high frequency phenomena (Baym and Chandler, 1983; Sonin, 1987) and, possibly, for quantum tunnelling (Volovik, 1997).

The Bernoulli effect, Eq.(38), also causes \(^3\)He impurities and ions to be trapped on the vortex cores because their chemical potential decreases with the \(^4\)He density. They prefer to sit in low density regions of the fluid. Trapped impurities add their own inertial mass \( m_I \) to that of the core. In superfluid \(^3\)He, the core is large and yields the dominant contribution to the vortex mass (Duan and Leggett, 1992; Kopnin, 1997; Volovik, 1997).

D. The Magnus force and the Kelvin-Helmholtz theorem

Phase slippage by moving vorticity causes dissipation in superfluids and superconductors: if, referring for instance to the situation of Fig.1, there is not just one vortex as in Fig.2 but a constant stream of vortices crossing the path 1-2 at a rate of \( n \) per second, driven by some external force, a pressure difference develops in the superfluid according to the Josephson \( ac \)-relation (15). This is in direct analogy with the Kutta lift exerted...

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38 see Lamb, 1945, §163
39 Baym and Chandler, 1983; Duan, 1994; Sonin et al., 1998
40 see Landau and Lifshitz, 1959, §131
FIG. 4  Stream of vortices crossing a path between point 1 and point 2, moving from bottom to top. The vortices are represented by dots and are assumed perpendicular to the figure and nearly straight in the vicinity of 1-2. Each contributes angle $\theta_i$ to the phase difference recorded between 1 and 2. In its travel from far down to far up, each vortex contributes $2\pi$ to the phase difference. According to the dc Josephson effect, a pressure difference develops in the superfluid due to the stream of vortices.

by the surrounding fluid on a moving body to which vorticity is attached. If the superfluid is free to move, it is accelerated by the cross stream of vortices. Work is done onto the superfluid by the applied external force, for instance an electric field acting on charges trapped in the vortex cores. It is apparent in the simple case illustrated in Fig. 4 that the force develops perpendicular to the vortex displacement.

The same line of reasoning can be pursued by considering the interaction energy between a vortex loop and a potential flow $v_p$, Eq. (32). Under an infinitesimal displacement $\delta x$ of a small line element $\Delta l$, as shown in Fig. 5, the energy of the vortex loop changes according to

$$\delta \langle \Delta E_i \rangle = \kappa_4 \rho_s \delta x \cdot \Delta l \cdot v_i = \kappa_4 \rho_s \Delta l \cdot v_i \cdot \delta x,$$

where $v_i$ is the local superflow velocity as seen by the vortex element standing still. The local flow velocity $v_i$ is the sum of the applied superflow $v_p$ and the flow induced by the other parts of the vortex loop, $v_v$. Equation

$$(40)$$

expresses the functional derivative of the energy with respect to a deformation of the vortex $\delta \langle \Delta E_i \rangle / \delta x$.

If the vortex loop moves along at velocity $v_L$ together with the element under consideration in the rest frame of the observer, $v_i$ in Eq. (40) becomes $v_i - v_L$ and this force takes the same form as the Magnus force for a line vortex in classical hydrodynamics with a fluid density $\rho_s$:

$$\frac{\delta \langle \Delta E_i \rangle}{\delta x} = \Delta F_M = \kappa_4 \rho_s (v_L - v_i) \times \Delta l,$$

(41)

The Magnus force, Eq. (41), has a simple physical origin. It is due to the Bernoulli effect that arises from the rotational flow around the vortex core. As shown in Fig. 6, this flow adds to the potential flow in the lower half-plane and subtracts from it in the upper half-plane. The detailed calculation of the flow velocity past the cylindrical core has to be carried out, with the result that the tangential flow velocity at the surface of a rigid cylinder of radius $a_0$ is given by $2v_i \sin \theta$ at the point with polar angle $\theta$, to which must be added the vortex contribution, $\kappa_4 / 2\pi a_0$. Integrating the resulting pressure difference, given by Eq. (37), over the cylinder yields a downward force expressed by Eq. (41).

This result can be extended to a non-circular core (Batchelor, 1967) and leads to the Kutta-Joukovski lift on air plane wings, to refer to a well-known example. The cases where the core is surrounded by normal fluid,

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This result can be extended to a non-circular core (Batchelor, 1967) and leads to the Kutta-Joukovski lift on air plane wings, to refer to a well-known example. The cases where the core is surrounded by normal fluid,
contains trapped impurities, or can be traversed by the flow leads to complications, as discussed in particular by Sonin (1997) and Thouless et al. (2001). When no other force acts on the vortex core (such as, e.g., an electric field on charges trapped in the core, or friction from the normal fluid component, ... ) \( \mathbf{F}_m \) must be zero, hence \( \mathbf{v}_l = \mathbf{v}_i \): the vortex core moves with the local superfluid velocity. The velocity of the core at point \( \mathbf{s} \) is the sum of the velocity of the local potential flow \( \mathbf{v}_p \) at \( \mathbf{s} \) when there is no vortex, and of the velocity \( \mathbf{v}_v \) induced at \( \mathbf{s} \) by the other parts of the vortex. If no flow is applied, \( \mathbf{v}_p = 0 \), then \( \mathbf{v}_l = \mathbf{v}_v \): the vortex loop moves under its own flow field in the superfluid at rest. The vortex thus appears to behave as a quasiparticle in its own right although it stands only for the vortical part of the total flow. The physical picture that emerges from this (admittedly non-rigorous) approach rings a familiar bell to condensed matter physicists.

The free motion of vortex loops conforms to the Kelvin-Helmholtz theorem, which states that in a compressible, inviscid fluid vorticity is conserved when moving with the fluid particles. This result has been obtained here as a consequence of the quantisation of circulation, Eq. (19). The Kelvin-Helmholtz theorem is usually derived from the Euler equation and the implicit assumption that the motion of the fluid is isentropic (Landau and Lifshitz, 1959). A further implicit assumption is that the velocity field and the loop deformation are well-behaved analytically, that is, continuous in space and time. These remarks will be seen relevant in Sect. VI.A on vortex nucleation, which deals with the spontaneous appearance of vorticity, in other words, the violation of the Kelvin-Helmholtz theorem. The derivation given above does not hide these fine points under the rug; it explicitly rests on the quantisation of circulation, hence its conservation, and also implies isentropic and continuous superfluid motion. When this fails new phenomena occur: vortices may be nucleated.

E. Energy exchange between potential and vortical flows

Isolated circular rings propagate undistorted under their own velocity field in the superfluid at rest for symmetry reasons. Their velocity field is expressed by Eq. (36). Only a few vortex shapes propagate undistorted in their own velocity field. Straight vortex pairs and helical vortices are other examples.

For an arbitrarily curved vortex, the self-velocity of each curve element can be approximated by Eq. (36), \( R \) being replaced by the local radius of curvature, \( r_m = |\mathbf{d}^2 s/d\xi^2|^{-1} \), parameter \( \xi \) being the line length of the curve represented by \( s(\xi) \). The validity of this approximation, which requires that \( r_m \) be large with respect to the vortex core radius, has been discussed in particular by Schwarz (1978, 1985) who has used it in extensive numerical simulations of 3D vortex motion.

The vortex energy per unit length can be approximated by Eq. (28)

\[
\epsilon_v = \frac{\rho_p k^2}{4\pi} \left( \ln \left( \frac{r_m}{a_0} \right) - \frac{7}{4} \right),
\]

where \( r_m \) is the minimum distance over which the vortex flow field is undisturbed: it is the smallest of 1) the size of the container, 2) the average radius of curvature of the vortex, 3) the distance to neighbouring vortices. For ångström-size vortices, taking \( r_m/a_0 = 10 \), \( \epsilon_v \approx 2 \) kelvin.

The Magnus force on each element of the vortex line is thus approximated by \( \epsilon_v \) times its total length. Changes in \( r_m \) along the vortex line are disregarded because they involve logarithmic corrections. When the vortex deforms, its energy changes mostly as its length, and little with its shape.

The full energy of a curved vortex line is thus approximated by \( \epsilon_v \) times its total length. Changes in \( r_m \) along the vortex line are disregarded because they involve logarithmic corrections. When the vortex deforms, its energy changes mostly as its length, and little with its shape.

The Magnus force on each element of the vortex line arises ultimately from momentum conservation in the fluid and comes into play whenever the vortex trajectory differs from that of the local fluid particles (or coarse-grain averaged fluid cells). The effect of external forces and mutual friction has been set aside for simplicity, so that no work is done on the vortex itself except by the interaction with the local superflow. This ensures that any gain or loss of energy by the vortex balances that lost or gained by the potential flow. This is the essence of Anderson’s corollary on classical hydrodynamics (Anderson, 1966b, Appendix B), which was proved by Huggins (1970). The approach to the dynamics of superfluid vortices that has just been outlined lays the groundwork for a physically transparent derivation of this corollary.

If now \( dx \), used in Eq. (40) as a virtual displacement to compute the forces acting on \( \Delta l \), becomes a real displacement \( \mathbf{v}_l \Delta t \), actual work during the time \( \Delta t \) is done by the applied potential flow on the vortex loop. The energy balance is expressed by rewriting Eq. (40) as

\[
\delta(\Delta E_I) = \kappa_4 \rho_p \mathbf{A} \times (\mathbf{v}_p + \mathbf{v}_v) \cdot \mathbf{\Delta x} = \kappa_4 \rho_p \mathbf{A} \times \mathbf{v}_p \cdot \mathbf{v}_l \mathbf{\Delta t} + \kappa_4 \rho_p \mathbf{A} \mathbf{\times v}_v \mathbf{\times v}_l \mathbf{\Delta t}.
\]

Since in free motion \( \mathbf{v}_l = \mathbf{v}_i = \mathbf{v}_p + \mathbf{v}_v \), the energy increment is equal to zero: energy is conserved. The two terms in the last side of Eq. (42) are thus equal in magnitude and opposite in sign. The first term is readily

\[44 \text{ see } §8 \]

\[45 \text{ For a discussion, see Saffman (1992) } §1.6 \]

\[46 \text{ See also Greiter (2005), Zimmermann, Jr. (1993a) } \]
seen proportional to the rate at which the potential flow streamlines are crossed by the vortex element: it represents the contribution of the potential flow to the energy balance.

The second term requires a little more formal work to be recognised as a contribution to the vortex kinetic energy $E_v$. What needs to be shown is that the energy variation for a small, local, deformation of the vortex loop corresponds to the second term of the right-hand side of Eq. 12.

The energy variation under a change of the vortex loop $s(l)$ into $s(l) + \varepsilon \delta l(l - l_0)$ amounts to taking the functional derivative of $E_v[s]$, given by Eq. 24, with respect to the deformation $\delta s$ at $l_0$. This functional derivative is evaluated as follows:

$$\frac{\delta E[s]}{\delta s} \bigg|_{l_0} = \frac{\rho_s k_4^2}{8\pi} - \int \frac{ds(l_1)}{dl_1} \frac{ds(l_2)}{dl_2} \frac{s(l_1) - s(l_2)}{s(l_1) - s(l_2)^3} dl_2$$

$$+ \lim_{\varepsilon \to 0} \int \varepsilon \cdot \frac{ds(l_2)}{dl_2} \delta'(l_1 - l_0) \frac{dl_1}{s(l_1) - s(l_2)}$$

$$+ l_1 \iff l_2$$

(43)

The logarithmic divergences of the Dirac $\delta$-function, which comes from the differentiation of $ds(l_1)/dl_1$ yields the derivative of the integrand evaluated at $l_0$. The contribution of the integration over $l_2$ over the same contour with the same deformation is equal to that over $l_1$, expressed by the first two terms of the right hand side of Eq. 43, and yields a factor of 2 in the final result.\footnote{Further refined by Richards (1970)}

Using the notation $\hat{t}(l) = ds(l)/dl$, Eq. 43 can be written as

$$\frac{\delta E[s]}{\delta s} \bigg|_{l_0} = \frac{\rho_s k_4^2}{4\pi} \int \{ \hat{t}(l_0) \cdot \frac{s(l_0) - s(l_2)}{s(l_0) - s(l_2)^3} \hat{t}(l_2)$$

$$- \hat{t}(l_0) \cdot \left( \frac{s(l_0) - s(l_2)}{|s(l_0) - s(l_2)|} \right) \hat{t}(l_2) \} \cdot dl_2$$

$$= \rho_s k_4^2 \hat{t}(l_0) \times \left( \frac{s(l_0) - s(l_2)}{|s(l_0) - s(l_2)|} \right) \hat{t}(l_2)$$

(44)

The double cross product in the second equality of (44) appears because $(\mathbf{a} \cdot \mathbf{b}) \mathbf{c} - (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$. The last equality is obtained using the Biot-Savart law, Eq. 20, for $\mathbf{v}_v$. As in Eq. 42, $\mathbf{v}_v$ is the velocity induced by the vortex loop on itself at $l_0$. The logarithmic divergences for $l_0 = l_2$ are cut-off at the core radius $a_0$.

Equation 44 shows explicitly that the energy gained by the vortex in its small displacement $\delta x = v_v \delta t$ is equal to the quantum of circulation multiplied by the mass flow of the potential stream cut by the vortex. This important result was rigorously established by Huggins (1970) following a conjecture by Anderson (1966a). The gist of it is that whenever a vortex cuts potential flow streamlines, it reversibly exchanges energy with the potential flow. If the potential flow is divergent – for instance outward the mouth of a duct where the streamlines flare out, the vortex expands in length, collects energy from the flow and slows it down. If now the vortex runs away to a far off distance and never comes back, this energy is lost for the potential flow: energy has been dissipated by phase slippage. Reversing the flow direction, which then becomes convergent, results in the vortex shrinking and the potential flow picking up energy: a collapsing vortex dumps its energy into the potential flow. Both processes will be discussed in Sec. VI E on the phase slip mechanism using the simplified properties of superfluid vortices introduced here. But before turning to the inner details of the phase slips, their detection by experiments will be described in the following Section.

IV. PHASE SLIP PAGE EXPERIMENTS

As the $dc$ and $ac$ effects predicted in the early sixties by Bryan Josephson (Josephson, 1962, 1965, 1964) to take place between two suitably coupled superconductors were quickly observed (Anderson and Rowell, 1963; Shapiro, 1963), the search for analogous effects in superfluids also begun, with the tantalising goal of observing unique quantum-mechanical effects in hydrodynamics. This search for a long time gave inconclusive results\footnote{Gregory (1972), Guernsey (1971), Hulin et al. (1972), Khorana (1969). Khorana and Chandrasekhar (1967); Richards (1970); Richards and Anderson (1965).} or led to blind alleys\footnote{Cabrera et al. (1971), Musinski (1973), Musinski and Douglass (1972), Schofield et al. (1971).} It was only in the mid-eighties that decisive steps forward were taken.\footnote{Amar et al. (1988), Avenel and Varoquaux (1985, 1986). Varoquaux et al. (1982). Zimmermann et al. (1993), (1996).}

A. The Richards-Anderson experiment

The goal of Richards and Anderson (1965) was to observe the Josephson $ac$-effect by shining a sound wave at the same frequency as the effect and watch for some form of a beat pattern. In their experimental set up shown...
FIG. 7 The Richards-Anderson cell (1965).

in Fig. 7 two identical coaxial capacitors are suspended over a liquid helium bath cooled at a temperature below the lambda point (of the order of 1.15 K). One of the capacitors is open-ended, the other is partially closed at the bottom by nickel foil with a very small aperture. The foil is 25 micrometres thick, in which a 15 micrometre aperture had been punched with a sharp needle: the pinhole thus manufactured constitutes the “weak link”. Ultrasonic waves are shone by a quartz transducer immersed into the superfluid bath facing the micro-aperture in order to either maintain or create a pressure head in the partially closed capacitor. Sound waves of sufficient amplitude would create vortices in the aperture. The steady stream of these vortices flowing away from the aperture would, as in Fig. 4, induce a pressure head. The magnitude of this pressure head was expected to vary in steps with the amplitude of the sonic excitation, according to whether one, two, or more vortices are produced per cycle of sound wave.

A helium level difference between the two capacitors can be created externally and its return to equilibrium is then monitored with a capacitance bridge. Steps in the return to equilibrium were indeed observed by Richards and Anderson at level differences which were multiples and sub-multiples of the fundamental head difference expected from the Josephson ac relation:

$$\Delta z \left( \frac{n}{n'} \right) \frac{\hbar \omega}{m \omega g}$$

where $n$ and $n'$ are integers, and $g$ the acceleration of gravity. These results were reproduced by other researchers using similar setups (Hulin et al., 1972, 1971; Khorana, 1969; Khorana and Chandrasekhar, 1967). Other setups, involving rotating or oscillating toroidal cells (Gregory, 1972; Guernsey, 1971), vortices accelerated by ions (Carey et al., 1973), a two-orifice flow arrangement (Gamota, 1974) were also tried but with mixed success at best, suffering from lack of reproducibility and poised with numerous unexpected features.

By the time of the 13th International Conference on Low Temperature Physics, held in Boulder in August, 1972, it became clear that the early claims of observation of the Josephson ac effect by synchronisation of the pressure head on the sound frequency did not meet universal acceptance. On the contrary, an alternate explanation in terms of acoustic standing waves in the cell was put forward on experimental grounds by Leiderer and Pobell (1973), as well as Musinski and Douglass (1972), and on theoretical grounds by Rudnick (1973). It was nonetheless argued by Anderson and Richards (1975) that, although acoustic resonances in the cell could be a concern, they could not account for all of the features observed in their experiments.

These efforts directed toward the demonstration of the hydrodynamic Josephson effects, together with direct studies of the critical velocity itself (Gamota, 1973; Trela and Fairbank, 1967), did bring experimental confirmation of the views of Feynman and Anderson that vortices were associated with the appearance of dissipation in superfluid flow. However, quantitative studies leading to a clear picture of how these vortices were created and how they interacted with the superflow were lacking. A consensus grew that somehow their formation and evolution had a chaotic character, presumably due to random pre-existing vorticity in the superfluid and to a probable evolution toward some form of turbulent motion of the quantised vortices, a belief confirmed in part by the more recent studies described in Sec. VII. The flurry of activity stirred by the initial reports of observations related to the Josephson effects receded almost completely.

With the hindsight gained with the experiments described further on, it can now be concluded that the synchronisation envisioned by Richards and Anderson (1965) would be nigh impossible to achieve. A particularly clear exposition of this synchronisation experiment is given by Anderson et al. (1984) in terms of parametric effects due to the system non-linearities, of the same kind as frequency pulling in radiofrequency oscillators. These effects require that energy be stored reversibly in a non-linear element, here the Josephson junction or, for rf-devices, a non-linear inductance. In $^4$He far from the $\lambda$-point the relation between the current and the phase difference across the weak link shows no non-linearities. The energy that the vortices gather from the potential...
flow is carried irreversibly away from the orifice. It can be used to pull or push the flow in synchronism with the sound excitation for a very brief lapse of time only. Furthermore, for the comparatively large orifices used in the early experiments mentioned above, they appear in a rather irregular fashion, not individually but in lumps with a varying number of vortices, destroying the eventual synchronisation to a regular pattern of steps.

B. The hydromechanical resonator

In the early eighties, several groups went on striving to improve the detection techniques used in the search for the hydrodynamical Josephson effects. The use of a diaphragm-driven hydromechanical resonator fully immersed in the superfluid was proposed by Zimmermann, Jr., and his students. A similar device with two chambers was used for critical velocity measurements in superfluid $^3$He, and used by Lounasmaa et al. (1983) for the search of an ac Josephson effect in superfluid $^3$He, a topic that will be covered in Sect. VIII B. The expertise developed at Cornell on torsional oscillators was put to use in superfluid $^3$He by Reppy and his students (Crooker, 1984). Again, the hydrodynamic Josephson effects could not be observed in these various experiments, for one or several of the following reasons:

• the apertures used as weak links were too large;
• the mass flow sensitivity was marginally adequate only;
• the superfluid motion was driven from current sources that were too stiff to let the response of the weak link be seen;
• and, last but not least, the cells were too bulky and too sensitive to external mechanical vibrations to allow for non-invasive measurements.

The first condition was clearly perceived as essential. Efforts shifted from superfluid $^4$He to the newly discovered superfluid $^3$He because the coherence length is nearly three orders of magnitude larger, putting the fabrication of a genuinely-weak superfluid link closer to the reach of experimental low temperature physicists. Work was carried out in that direction by Wirth and Zimmermann, Jr. (1981), who were the first to use sub-micronic orifices in free-standing ultra-thin foils, and others (Amar et al. 1990, Avenel 1984, Sudraud et al. 1987).

The question of detection of the minute mass currents that would flow in micro-apertures could also be tackled in the early eighties as reliable rf-SQUIDs were becoming available. It became feasible to develop ultra-sensitive pressure and displacement gauges (Avenel 1984; Avenel and Varoquaux, 1986a; Beecken and Zimmermann, Jr. 1987a). In the for-

C. Early phase slippage experiments

The phase slippage experiments that were carried out starting from the mid-eighties (Avenel and Varoquaux 1985, Varoquaux et al. 1987) confirmed Feynman and Anderson’s views on dissipation in superflows and brought a large measure of clarification in the critical velocity problem (Varoquaux et al. 1991) and in the formation of vortices in superfluid $^3$He (Avenel et al. 1993). These experimental results and their interpretation have since been largely confirmed by other workers.

The design of the first weak link in which hydrodynamical Josephson effects were seen (Sudraud et al. 1987) struck a compromise between to conflicting requirements, that it be weak enough to effectively depress the wavefunction amplitude while preserving the macroscopic coherence of the superfluid, that it be big enough to let a measurable flow of liquid go through. More will be said on superfluid weak links later but it can be mentioned at this point that a slit geometry was chosen for the micro-aperture. The liquid flowed through a rectangular orifice, the smaller dimension of which was comparable with the coherence length in superfluid $^3$He, $\xi_0$, which is in the micron range. This orifice was micro-machined by ion-milling in a free standing foil the thickness of which was also comparable to $\xi_0$. The third dimension of the slit was made large to provide a substr-
Calling the device a “Helmholtz” resonator has been criticised as the compressibility of the fluid inside the chamber has a negligible effect at the low frequencies of the experiments, hence the little more convoluted appellation used here. The term “hydromechanical resonator” is also used in this paper.
The actual flow in the micro-aperture is the sum of the flow driven by the membrane and of the persistent flow threading the micro-aperture and the parallel channel, which depends on the quantum state of the loop \( j \). The amplitude drop \( \Delta A_1 \) caused by a single phase slip in a given half-cycle of the resonance corresponds to a change of \( \delta \phi \) by exactly \( 2\pi \). Normalising the membrane displacement \( A_m \) by \( \Delta A_1 \) provides a self-calibration of the data that is independent of less well known quantities such as the membrane stiffness, and the calibration factor of the displacement sensor.

In the following, aperture velocities are expressed by the number of turns \( \delta \phi / 2\pi \) by which the quantum-mechanical phase winds across the aperture. The actual flow velocity averaged over the cross-section of the micro-aperture is proportional to \( \hbar \delta \phi / m_\text{eff} \). The hydraulic length \( l_h \) of the micro-aperture is of the order of 1 \( \mu \text{m} \) in the experiments shown in Fig.9.

### D. Phase-slippage experimental results

The observation of phase slips in \(^4\text{He}\) has led to a number of quite significant results. They brought a confirmation of Anderson’s ideas, much welcome in view of the controversies about previous experiments. And quite importantly, they have shed light on the previously indecipherable problems of the critical velocity and of vortex nucleation. Below are summarised the most important qualitative features and some of their implications.

1. The critical velocity threshold, which can be seen on time charts such as that shown in Fig.10, is markedly temperature-dependent down to below 200 mK and reaches a well-defined plateau below 150 mK. These features can be seen in Fig.12 and will be discussed below. As the thermodynamic properties of superfluid \(^4\text{He}\) are very nearly independent of temperature below 1 K, this observation indicates that the critical process in action is not governed solely by hydrodynamics. It can be suspected that statistical mechanics plays the leading role.

2. Aperture size is not found to be a relevant factor, as long as it is “small enough”, roughly below a few \( \mu \text{m} \). This feature and the temperature dependence mentioned above are in sharp contrast with the Feynman critical velocity, which, as discussed below in \( \text{§V.B} \), exhibits a well-characterised dependence on size and none on temperature.

3. The actual velocity threshold for phase slips shows significant scatter from one slip to the next in a given sequence, as can be seen in Fig.23. This scatter lies much above the background noise level of detection of the peak amplitudes of the resonator motion. It represents a genuine stochastic property of the process at work, which turns out to display a temperature dependence similar to that of the critical velocity shown in Fig.13.

4. The phase slip pattern shows quite reproducible properties in the course of a given cool-down as long as the experimental cell is kept at a temperature below 10–15 K. If the temperature is cycled up to nitrogen temperature and back, small changes to the critical threshold and to the pattern itself can occur. This reveals the importance of minute changes in the surface state of the cell, e.g. contamination of the micro-aperture walls by solidified gases.

5. Quite importantly, phase slips are the signature that quantised vortices are created in aperture flow above a well-defined threshold of flow velocity. This statement arises from the highly reproducible phase change, which is measured to be \( 2\pi \) to the accuracy of the experiment and to amount to changes of precisely one quantum of circulation in the superfluid loop threading the micro-aperture and the long parallel channel. A detailed scenario for the occurrence and development of phase slips that shows how the phase difference by \( 2\pi \) develops has been described by [Burkhart et al., 1994] and is discussed below in \( \text{§V.I} \).

Critical velocities and phase slips in the superfluid phases of \(^3\text{He}\) show different features that will be briefly touched upon in Sect.\( \text{VII} \).
of different physical situations. This Section begins with an overview of the different brands of “critical” velocities that comply with this definition. It will end up by focusing on that which involves the phase slip phenomenon, namely, the nucleation of superfluid vortices.

Neither the problem of critical velocities in superfluids nor that of the nucleation of vortices are new. The former is as old as the discovery of superfluidity (see the monograph by Wilks [1967]). The latter, first discussed by Vinen in the early sixties (Vinen, 1963), has met an even more tortuous fate. It was first thought, still is in some quarters, to be nigh impossible (Vinen, 1963) on the grounds that such an extended hydrodynamical object as a vortex line with a finite circulation, involving the collective motion of a large number of helium atoms, would have a vanishingly small probability of occurring spontaneously.

For classical ideal fluids, this remark forms the essence of the Kelvin-Helmholtz theorem, which states that vorticity is conserved for isentropic motion of inviscid fluids. For superfluids, it even has more thrust since vorticity is quantised. More recent experiments probing superflow on a finer scale of length have shown otherwise. The short account of these problems that follows is based on the work of the author and his colleagues over the course of many years. More extended discussions as well as more complete bibliographies can be found in their publications and other articles, e.g. McClintock and Bowley (1995).

A. The Landau criterion

Landau [1941] explained the superfluidity of helium-4 by the sharpness of the dispersion curve for elementary excitations, phonons and rotons, shown in Fig. 10. This property is now associated with the existence of a Bose-Einstein condensate (Griffin [1987], as has long been suspected. Elementary excitation energy levels \( \epsilon(p) \) being well-defined, that is, having a negligible spread in energy, very low-lying states, energy-wise and momentum-wise, are extremely scarce. An impurity, or a solid obstacle, can only exchange an energy \( \epsilon(p) \) at momentum \( p \) that exactly matches the energy of an elementary excitation of the fluid.

If the superfluid moves at velocity \( v_s \), the energy of elementary excitations in the frame of reference at rest becomes \( \epsilon + v_s \cdot p \) (Baym [1969] Wilks [1967]). The same holds for a moving obstacle, by Galilean invariance. If this energy turns negative, elementary excitations proliferate and superfluidity is lost. The condition on the superfluid velocity for this to happen reads:

\[
 v_s \geq v_L = \frac{\epsilon(p)}{p_{\min}} \approx \frac{\epsilon(p)}{p_{\text{roton}}} .
\]  

Unless this condition can be met, there is no dissipative interaction between the fluid and its surroundings: the flow is viscousless.

The minimum value of \( \epsilon/p \) for helium is very close to the roton minimum, as shown in Fig. 10. In \(^4\text{He}\) at low pressure, \( v_L \approx 60 \text{ m/s} \). The Landau critical velocity \( v_L \) is smaller than the sound velocity \( c = 220 \text{ m/s} \) but larger than most critical velocities measured in various experiments. For the much less dense Bose-Einstein Condensed gases, which do not exhibit roton-like features, the minimum is the sound velocity, \( c = \epsilon(p)/|p|_{p=0} \).

B. Feynman’s approach

Feynman (Feynman [1955] Wilks [1967]) realised that, following Onsager, not only would vorticity be quantised in \(^4\text{He}\) in units of the quantum of circulation \( \kappa_4 = 2\pi\hbar/m_4 \) but, preceding Anderson, that these vortices would be responsible for the onset of dissipation and for a critical velocity in the superfluid. In Feynman’s views, vortices would be puffed out of the mouth of orifices much in the way of smoke rings – or von Karman alleys past obstacles in classical (Navier-Stokes) fluids.

A characteristic velocity associated with this process can be evaluated by considering the energy \( E_R \) and im-

---

60 It may be recalled here that the line energy of a \(^4\text{He}\) vortex is of the order of 2 kelvin per angstrom – see [IL].

61 Langer and Reppy 1970; Muirhead et al. 1984; Varoquaux et al. 1987;

62 Avenel et al. 1993; Varoquaux and Avenel 2003; Varoquaux et al. 2001, 1991.

63 See Khalatnikov 1965, Wilks 1967.
pulse \( P_R \) of a vortex ring, expressed by Eqs. (25) and (34):
\[
E_R = \frac{1}{2} \rho_s \kappa^2 R \left( \ln \frac{8R}{a_0} - \frac{7}{4} \right) + O\left(\frac{a_0}{R}\right),
\]
\[
P_R = \pi \rho_s \kappa^4 R^2.
\]

Such a vortex ring can be treated as an elementary excitation of the superfluid, which it rightfully is from the vantage point taken in Sec. III, and for which Landau’s criterion applies. The limiting velocity is reached for a radius \( R \) such that \( E_R/P_R \) is at a minimum, which occurs when \( R \) is as large as feasible, that is, of the order of the orifice size \( d \). This minimum value sets the velocity at which vortices can start to appear and defines the Feynman critical velocity:
\[
v_F \approx \frac{\kappa}{2nd} \ln \left( \frac{d}{a_0} \right).
\]

As discussed below, \( v_F \) is much closer to experimental values than the Landau critical velocity for rotons. Although this agreement is heartening, it also raises fresh questions: how do these vortices come about and evolve?

C. Several kinds of critical velocities

The compilation of the critical velocity data in various apertures and channels from various sources available at the time Exeter Meeting in 1990 is shown in Fig. 11. Two different critical velocity regimes appear clearly on the graph in Fig. 11, a fast regime for small apertures, of the phase-slip type, and a slower regime for larger channels, of the Feynman type. The data points from various sources (Varoquaux et al., 1991) for these two different types of critical velocity do not fall on well-defined lines as can be seen in Fig. 11 but merely bunch into clusters of points. As already stated, critical velocities in apertures and capillaries are not very reproducible from experiment to experiment, indicating that some less-well-controlled parameters, besides size, temperature and pressure, also exert an influence.

More recent data confirm this behaviour. In some occasions, switching between these two types of critical velocity has been observed in the course of the same cool-down (Hulin et al., 1974; Zimmermann, Jr., 1993a). The critical velocity that depends on channel size does follow on average relation (47) for the Feynman mechanism. The higher critical velocities, bunched around 5 to 10 m/s, faster than the Feynman \( v_F \) even for the smallest apertures but considerably slower than Landau’s \( v_L \) relate to the phase slip phenomenon and are discussed below.

As a basis for comparison, it is worthwhile to also mention the findings of the ion propagation studies in superfluid \(^4\)He at various pressures, which have been reviewed by McClintock and Bowley (McClintock and Bowley, 1991, 1995). Ions can be created in liquid helium and accelerated by electric field until they reach a critical velocity. The resulting drift velocities are measured by time-of-flight techniques. For negative ions, hollow bubbles 30 Å in diameter with an electron inside, two different behaviours are observed:

- Below about 10 bars, vortex rings are created, on the core of which the electron gets trapped: the drift velocity suddenly drops from that of the negatively charged bubble to the much slower vortex ring velocity (Rayfield and Reif, 1964).
- Above 10 bars, the accelerated ion runs into the roton creation barrier before vortex rings can be created. The Landau critical velocity is observed to decrease from about 60 m/s at SVP down to 46 m/s at 24 bars as the roton parameters change with pressure while the vortex creation velocity increases with pressure.
- Around 10 bars, both critical velocities, the Landau critical velocity for the formation of rotons and that for the formation of vortex rings can be observed to occur simultaneously because ions can be accelerated above the threshold for roton emission.

These ion propagation measurements provide a vivid illustration not only of the existence of a critical velocity obeying the Landau criterion but also that roton creation and vortex formation constitute different phenomena.
and can exist concurrently\textsuperscript{64} The vortex emission threshold displays other noteworthy features. It depends on temperature in a non-trivial way, comparable to that of the phase-slip and also shows the marked dependence on \textsuperscript{3}He impurity concentration observed for phase slips in micro-aperture flows but not in larger channels. In both ion propagation and aperture flow measurements, vortex formation displays similar features.

Altogether, a careful study of the experimental data in superfluid \textsuperscript{4}He reveals three different, well-defined, types of critical velocities, one which is the celebrated Landau critical velocity, \( v_L \), (observed only for ion propagation) another, \( v_F \), that appears to follow the Feynman criterion as shown in Fig\textsuperscript{11} with all the uncertainties on the hydrodynamical process of vortex creation in larger channels, and a third, \( v_c \), for phase slips, which is in want of an explanation: how are the vortices of phase slips in aperture flow created, and how does the situation differ from that in larger channels?

The short answer, based on qualitative evidence, is that the temperature dependence of \( v_c \) and its stochastic properties clearly point toward a process of nucleation by thermal activation above \( \sim 150 \) mK or so and by quantum tunnelling below. This conclusion contradicts the common-place daily observations of the formation of whirlpools and eddies, and also the widely held belief that large scale topological defects with a quantum charge of circulation cannot appear out of nowhere in the superfluid. It will be seen to hold in \textsuperscript{4}He only because the nucleated vortices have nanometric size, a fact that came to be appreciated because of the detailed analysis of phase slippage observations related below.

VI. PHASE SLIP CRITICAL VELOCITY: A STOCHASTIC PROCESS

A more firmly established answer to the question formulated above comes from a quantitative analysis of the experimental data for phase slips. These experiments do provide clues that, pieced together, conclusively show that, in small apertures, vortices are nucleated by thermal activation above about 150 mK, and by quantum tunnelling below.

Some preliminary remarks are in order. A glance at Fig\textsuperscript{9} reveals that the critical velocity is not a sharply defined quantity: it varies from slip to slip. Also, the local value of the velocity is not measured directly. Experiments record the mean value of the volume flow, which is assumed to be proportional to the local values of the flow field velocity. This assumption breaks down in the presence of vorticity and has to be somehow refined (see Sec\textsuperscript{VII}). The value of the critical threshold is not reproducible from one cool-down to the next with the same experimental cell. This lack of reproducibility in the measurements, both in micro-apertures and in larger channels, has obscured the critical velocity problem for a long time. It must however be considered as an integral part of this very problem.

Several steps are taken in the following\textsuperscript{65}

- The gross experimental features – the temperature dependence, an established pattern of randomness – suggest that the phase slip threshold is a stochastic property. So, one assumes on the basis of this temperature dependence that the underlying statistics is governed by Arrhenius’ law that describes the escape of a particle from a potential wall assisted by thermal fluctuations. Analytic expressions can then be derived for the critical velocity \( v_c \) and its deviations from the mean \( \Delta v_c \). Values for the model parameters, the height of the potential well and the frequency of escape attempts, that fit the data are found physically realistic. But this establishes only that the statistical interpretation of the \( T \)-dependent data as a thermally-assisted phenomenon makes sense.

- Then, in Sec\textsuperscript{VII} this Arrhenius process is extended to the \( T \)-independent plateau, which is assumed to arise from a transition to a macroscopic quantum tunnelling regime. Other analytic expressions can then be used to describe the plateau region with the same energy barrier and attempt frequency as in the first step. The value of crossover temperature between the two regimes and the inclusion of friction in the quantum regime provide additional valuable cross-checks.

- At this stage, a plausible underlying physical mechanism can be constructed that can provide an evaluation of the parameters obtained in the processing of the experimental information based on the previous assumptions.

- It can then be imagined how trapped \textsuperscript{3}He impurities affect this mechanism, providing more insight.

Now, on with the details. The first piece of evidence for the nucleation of vortices, that is their creation \textit{ex nihilo}, rests on the temperature dependence of the phase-slip critical velocity shown in Fig\textsuperscript{12} which increases in a near-linear manner when the temperature decreases from 2 K to \( \sim 0.2 \) K. That is, the functional dependence of \( v_c \) upon \( T \) goes as \( v_c = v_0(1 - T/T_0) \). As can be seen in

\textsuperscript{64} A noteworthy attempt to by-pass this experimental finding is that of Andreev and Melnikovsky\textsuperscript{[2004]}

\textsuperscript{65} More details can be found in Varoquaux and Avenel\textsuperscript{[2003], Varoquaux et al.\textsuperscript{[2001]}}.
Fig. 12: the data however depart from this linear dependence below 200 mK, where they reach a plateau, and above 2 K because the critical velocity goes to zero at $T_c$.

This temperature dependence, first observed in 1985 at Orsay [Varoquaux et al., 1987] and now a well-established experimental fact [Varoquaux et al., 2001] is very telling. It came as a surprise at first because the critical velocities observed before were temperature-independent below ~1 K. As the quantum fluid is nearly fully in its ground state below 1 K – the normal fluid fraction becomes less than 1 % – one is led to suspect that an Arrhenius-type process must come into play. If such is the case, that is, if a thermal fluctuation in the fluid with an energy of at most a few $k_B T$ can trigger the appearance of fully-formed vortex out of nowhere, the energy of this vortex must also be of the order of a few $k_B T$: it must be a very small vortex. But very small vortices require rather large superfluid velocities to sustain themselves – as seen on Eq.(36). A careful assessment of the situation is thus in order.

The nucleation rate for thermally activated process is given in terms of the activation energy by Arrhenius’ law:

$$\Gamma_K = \frac{\omega_0}{2\pi} \left( 1 + a^2 \right)^{1/2} - a \exp \left\{ - \frac{E_a}{k_B T} \right\}.$$  \hspace{1cm} (48)

where $\omega_0/2\pi$ is the attempt frequency and $E_a$ the activation energy of the process. The correction for dissipation in the square brackets has been introduced by Kramers [Kramers, 1940] to describe the escape of a particle trapped in a potential well and interacting with a thermal bath in its environment. The particle undergoes Brownian motion fluctuations and experiences dissipation. This dissipation is characterised by a dimensionless coefficient $a = 1/2\omega_0 \tau$, $\tau$ being the time of relaxation of the system toward equilibrium. In superfluid helium, dissipation is small. Although some dissipation is necessary for the system to reach equilibrium with its environment, its influence on the thermal activation rate is very small and will be neglected in the following. However, this will not be the case anymore in the quantum regime, considered below, because it causes decoherence.

The expression for the critical velocity that stems from the Arrhenius rate, Eq.(48), is derived as follows. In experiments such as those shown in Fig.9, the velocity varies periodically at the resonance frequency as $v_p \cos(\omega t)$, $v_p$ being the peak velocity of the potential flow. The probability that a phase slip takes place during the half-cycle $\omega t_i = -\pi/2$, $\omega t_i = \pi/2$ is

$$p = 1 - \exp \left\{ - \int_{t_i}^{t_i+\pi/\omega} \Gamma_K(P, T, v_p \cos(\omega t')) dt' \right\} \hspace{1cm} (49)$$

$$= 1 - \exp \left\{ - \frac{\omega_0}{2\pi \omega} \frac{\partial E_a}{\partial v} \mid_{v=0} \exp \left\{ - \frac{E_a}{k_B T} \right\} \right\}.$$

Equation 49 results from an asymptotic evaluation of the integral at the saddle point $t = 0$. The accuracy of the asymptotic evaluation 49 become questionable for $T \rightarrow 0$ as the energy barrier vanishes along with the thermal fluctuations. But, as will be seen below, quantum effects – namely, zero-point fluctuations – take over and the energy barrier never actually vanishes.

The critical velocity $v_c$ is defined as the velocity for which $p = 1/2$. This definition is independent of the experimental setup, except for the occurrence in Eq.(49) of the natural frequency of the hydromechanical resonator $\omega$. The implicit relation between $v_c$ and $E_a$ then reads:

$$\omega_0 \sqrt{\frac{-2\pi k_B T}{v_c \partial E_a / \partial v}} \exp \left\{ - \frac{E_a}{k_B T} \right\} = \ln 2. \hspace{1cm} (50)$$

In Eq.(50), the attempt frequency is normalised by the resonator drive frequency: the Brownian particle attempts to escape from the potential well at rate $\omega_0/2\pi$ but an escape event is likely only in the time window in a given half-cycle of the resonance during which the energy barrier stays close to its minimum value $E_a(v_c)$. This time interval is inversely proportional to $\omega$, which explains why an instrumental parameter gets its way into Eqs.(49) and (50).

The velocity at which each individual critical event takes place is a stochastic quantity. Its statistical spread can be characterised by the “width” of the probability distribution defined [Avenel et al., 1993; Zimmermann, Jr. et al., 1990] as the inverse of the slope of the distribution.
at $v_c$, $(\partial p/\partial v)|_{v_c}$). This critical width is found to be expressed by:

$$
\Delta v_c = -\frac{2}{\ln 2} \left\{ \frac{1}{v_c} + \frac{\partial^2 E_a}{\partial v^2} \bigg|_{v_c} \right\}^{-1}
$$

In the experiments, which are carried out at low temperatures and large critical velocities, the quantity in curly brackets in the right hand side of Eq. (51) is small with respect to the last term so that the width is given simply $\Delta v_c = -(2/\ln 2) k_B T \left( \partial E_a/\partial v \right)|_{v_c}$. Thus, the statistical width is an approximate measure of the inverse of the slope of $E_a$ in terms of $v$.

The quantities $v_c$ and $\Delta v_c$ are derived from $p$, itself obtained by integrating the histograms of the number of nucleation events ordered in velocity bins. The outcome of this procedure is illustrated in Fig. [14] $p$ shows an asymmetric-S shape characteristic of the double exponential dependence of $p$ on $v$, Eq. (49), a consequence of Arrhenius’ law, Eq. (48), being plugged into a Poisson probability distribution. The observation of this asymmetric-S probability distribution constitutes another experimental clue for the existence of a nucleation process. The quantities $v_c$ and $\Delta v_c$ are easily extracted from the probability curves $p(v)$, but the inverse path $v_c$ and $\Delta v_c = -(2/\ln 2) k_B T \left( \partial E_a/\partial v \right)|_{v_c}$ back to $E_a(v)$ and $\omega$ by numerical integration of the differential equation (50) requires more work and could introduce additional errors.

An improved procedure, discussed by Varoquaux and Avenel (2003), consists in obtaining directly the escape rate $\Gamma$ from the phase slip data. This rate is the ratio, for a given velocity bin, of the number of slips which have occurred at that velocity to the total time spent by the system at that given velocity. The outcome of this procedure is illustrated in Fig. [15]. The slope of $\ln \Gamma(v)$ directly yields $\partial E_a/\partial v$; the value of $\ln \Gamma$ at $v_c$ gives a combination of $\ln \omega$ and $E_a(v_c)$. As is well-known, it is still not so easy to solve this inverse problem and cleanly disentangle these two quantities (Varoquaux et al. 1986).

At this point experiment itself offers help as will now be described.

**A. Vortex nucleation: thermal vs quantum**

Below $\sim 0.15$ K, $v_c$ ceases abruptly to vary with $T$, as seen in Fig. [12]. For ultra-pure $^4$He (less than 1 part in $10^9$ of $^3$He impurities), $v_c(T)$ remains flat down to the lowest temperatures ($\sim 12$ mK) reached in the experiment. The crossover from one regime to the other is very sharp. At the same crossover temperature $T_{cr}$, $\Delta v_c$ also levels off sharply. It is believed on experimental grounds that this saturation is intrinsic and is not due to stray heating or parasitic mechanical vibrations (Avenel et al. 1993).

Even if all possibilities of an experimental artefact are cleared out, the mere observation of a plateau in $v_c$ is no sufficient proof for a crossover from the thermal regime to the quantum one: the effect of $^3$He impurities, shown in the insert of Fig. [12] also gives a levelling-off of $v_c$ vs $T$. 
This effect has been studied in detail (Varoquaux et al., 1993) and is well understood.

If the nucleation barrier were undergoing an abrupt change at $T_{qf}$ for instance because of a bifurcation toward a vortex instability of a different nature (Josserand and Pomeau, 1995), in all likelihood $\Delta v_c$ would jump to a different value characteristic of the new process (presumably small since $v_c$ reaches a plateau). Such a jump is not observed in Fig. 13. Furthermore, $v_c$ levels off below $T_{qf}$ which would imply through Eq. (50) that $E_q$ becomes a very steep function of $v_c$ but $\Delta v_c$ also levels off, which, through Eq. (51), would imply the contrary. This remark leads one to investigate the possibility that, below $T_{qf}$ thermally-assisted escape over the barrier gives way to quantum tunnelling under the barrier (Has et al., 1992). This switch from thermal to quantum does induce plateaus below $T_q$ for both $v_c$ and $\Delta v_c$.

During the course of these studies at Orsay-Saclay, the group of Peter McClintock at Lancaster was also reaching the conclusion from their ion propagation experiments of the existence of a crossover around 300 mK from a thermal to a quantum regime for the nucleation of vortices (Hendry et al., 1988), as predicted by Muirhead, Vinen, and Donnelly (Muirhead et al., 1984). There certainly are significant differences between the ion limiting drift velocity and aperture critical flow – in particular, the latter is nearly one order of magnitude smaller – but the qualitative similarities are strikingly telling. Thus, two completely different types of experiments point toward the fact that vortices do appear as a result of a nucleation process on a nanometric scale, both in a thermal regime and in a quantum one.

**B. The macroscopic quantum tunnelling rate**

To proceed with this line of thought, it is now assumed as a starting point that below $T_q$, zero point fluctuations do take over thermal fluctuations. The potential barrier is not surmounted with the assistance of a large thermal fluctuation, it is tunnelled under quantum-mechanically; non-conservation of energy is not a problem if it is brief enough, in accordance with the Heisenberg uncertainty principle for energy vs time resolution. The quantum-tunnelling event is “assisted” by the zero point fluctuations, so to speak (Martinis and Grabert, 1988), in the same manner as the Arrhenius process is assisted by thermal fluctuations. What is remarkable here, and not necessarily easily admitted, is that such an energy non-conserving process does affect a macroscopic number of atoms, as is that necessary to form a vortex of about 50 Å in length, as turns out to be the case.

Such “macroscopic quantum tunnelling” (MQT) processes have been the object of numerous experimental and theoretical studies, mainly in superconducting Josephson devices. The case for vortices in helium can be worked out in a very similar manner, as done by Varoquaux and Avenel (2003). Before turning to this brief account of MQT for vortices in $^4$He, some of the basic results of the extended body of theoretical studies that followed Caldeira and Leggett’s original work (Caldeira and Leggett, 1983) are introduced below.

The quantum tunnelling rate of escape out of a potential well $V(q)$ is a textbook problem. The rate is proportional to $\exp -S/h$, $S$ being, in the WKB approximation, the action of the escaping particle along the saddle-point trajectory at the top of the potential barrier, the so-called “bounce” (Coleman, 1977). For a particle of mass $m$ and energy $E$ escaping from a one-dimensional barrier $V(q)$, this action reads

$$S = 2 \int_{q_1}^{q_2} dq \sqrt{2m[V(q) - E]}.$$  

The determination of the bounce yields the points $q_1$ and $q_2$ at which the particle enters and leaves the barrier.

A discussion of the quantum tunnelling of vortices thus requires a Lagrangian formulation of vortex dynamics. Such a formulation has been carried out in particular by Sonin (1995) (see also Fischer (2000) for an extended discussion). However, analytic results can be obtained only at the cost of approximations and yield less than fair comparison with experiments (see the discussion by Varoquaux et al. (2001)).

A more productive approach (Varoquaux and Avenel, 2003) can be borrowed from the literature for Josephson devices (Caldeira and Leggett, 1983, Larkin et al., 1984).

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66 See for instance Landau and Lifshitz (1958) §50.
It consists in choosing for $V(q)$ a simple analytic form reduced to a sum of two terms, respectively parabolic and cubic term in $q$:

$$ V(q) = V_0 + \frac{1}{2} m \omega_0^2 q^2 \left(1 - \frac{2q}{3q_b}\right) , $$

where $\omega_0$ is the angular frequency of the lowest mode of the trapped particle (that will be found comparable to the attempt frequency) and $q_b$ the generalised coordinate of the barrier top location. The barrier height $E_b$ is equal to $m \omega_0^2 q_b^2 / 6$.

It can be checked that Eq. (53) is the expression of the vanishing potential barrier height in the “tilted washboard” model when the applied velocity reaches the limiting velocity, $v_0$, at which the system “runs away”, the so-called “lability” point. At this point where the system becomes labile, the critical velocity is reached even in the absence of thermal or quantum fluctuations. Such a hydrodynamic instability threshold at which vortices appear spontaneously has been shown to occur in numerical simulations of flows past an obstacle using the Gross-Pitaevskii equation by Frisch et al. (1992) and others.

The zero-temperature WKB tunnelling rate for the cubic-plus-parabolic potential $E_b$, Eq. (53), is found to be (Caldeira and Leggett, 1983)

$$ \Gamma_0 = \frac{\omega_0}{2 \pi} \left(120 \pi \frac{S_0}{\hbar}\right)^{1/2} \exp - \frac{S_0}{\hbar} , \tag{54} $$

the action $S_0$ being equal to $36E_b/5\omega_0$.

From this result, it can be anticipated that the crossover between the quantum and the thermal regime lies around a temperature close to that for which the exponents in Eqs. (48) and (54) are equal, namely $T = 5\omega_0/36k_B$ – assuming that the activation energy in Eq. (48), $E_{\alpha r}$, reduces to the simple cubic-plus-parabolic form, $E_b$. A more precise study of the mathematical properties of the quantum channel for escape leads to the following relation (Mel’nikov, 1991)

$$ \hbar \omega_0 = 2 \pi k_B T_q . \tag{55} $$

Once the crossover temperature has been determined from experiment, $\omega_0$ is fixed to pinpoint accuracy by Eq. (55). Its value agrees with that (less precisely determined) obtained from the analysis of the Arrhenious regime outlined in the previous paragraph: some degree of self-consistency has been achieved. The values of the barrier height $E_b$ at each given velocity then follow easily, using the full expressions for the rate in terms in terms of $E_b$, $\omega_0$ and, also, for the damping parameter $\alpha$ as discussed below.

### C. Friction in MQT

Damping turns out to matter significantly for quantum tunnelling of semi-macroscopic objects, contrarily to the thermal regime. The relevance and applicability of the concept of quantum tunnelling to macroscopic quantities such as the electric current through a Josephson junction or the flow of superfluid through a micro-aperture, although still sometimes questioned, have been checked in detail for the Josephson effect case (Martinis et al., 1987). One of the conceptual difficulties is that the macroscopic quantum system is coupled to an environment that acts as a thermal bath; this coupling gives rise to a source of classical fluctuations and friction, which leads to decoherence of the quantum process.

This issue was tackled by Caldeira and Leggett (1983), and a number of other authors. In the case of weak frequency-independent damping ($\alpha \ll 1$) and for the cubic-plus-parabolic potential, the tunneling rate takes the form:

$$ \Gamma_{qT} = \frac{\omega_0}{2 \pi} \left( \frac{864 \pi}{365} \frac{E_b}{\hbar \omega_0} \right)^{1/2} \times \exp \left\{ - \frac{36 E_b}{5 \hbar \omega_0} \left[ 1 + \frac{45 \zeta(3)}{\pi^3} \alpha \right] \right\} + \frac{18 \pi^2}{T^2_q} + O\left( \alpha^2, \alpha^4 \right) , \tag{56} $$

Thus, according to Eq. (56), damping depresses the MQT escape rate at $T = 0$ – a being a positive quantity – and introduces a temperature dependence that increases the rate as $T$ increases. These effects are large, even for weak damping, because they enter the exponent of the exponential factor in Eq. (56). Relation (55) between $T_q$ and $\omega_0$ is nearly unaffected by damping: $\omega_0$ is simply changed into $\omega_0(1 + a^2)^{1/2} - \alpha$ according to Eq. (48), a minor modification for $a \ll 1$.

Equation (56) is valid up to about $T_q/2$. From $T_q/2$ to $T_q$, one has to resort to numerical calculations (Grabert et al., 1987). In the thermal activation regime, $T \gg T_q$, quantum corrections affect the Kramers escape rate up to about $3T_q$ and can be evaluated analytically. These high-temperature quantum corrections depend only weakly on friction. A complete solution of the problem of the influence of friction, weak, moderate or strong, has first been worked out in the classical regime ($T \gg T_q$) by Grabert (1988) and extended to the temperature range $T \geq T_q$ by Kapu and Pollak (1989) who showed that the rate for arbitrary damping can be factorised in three terms,

$$ \Gamma = f_a \gamma \Gamma_k , \tag{57} $$

68 see, for instance, Mel’nikov (1991) and Varoquaux and Avenel (2003) for more references and details on this Section.
each term having a well-defined physical meaning: $\Gamma_K$ is the classical Kramers rate, $f_q$ the quantum correction factor, and $\Upsilon$ the depopulation factor. The high temperature limit of $f_q$ is

$$f_q = \exp\left\{ \frac{\hbar^2}{24} \left[ \omega_0^2 + \omega_b^2 \right] + O(\alpha/T^3, 1/T^4) \right\},$$  \hspace{1cm} (58)

in which $\omega_0$ and $\omega_b$ are the confining potential parameters depicted in Fig.16. Analytic results for $f_q$ are known to slightly below $T_\phi$ (Grabert et al. 1987; Hanggi et al. 1990).

The depopulation factor $\Upsilon$ arises from the depletion of the occupancy of the energy levels inside the potential well in the course of the escape process. This depletion occurs when the intermediate levels, if they exist, are not replenished fast enough by the thermal fluctuations. The depletion factor is unity at large $\alpha$ when the coupling of the Brownian particle with the thermal bath is large. It decreases to zero as $\alpha \to 0$ and the system becomes effectively decoupled from the environment. In the quantum tunnelling regime, dominated by zero point fluctuations, level depletion does not take place and $\Upsilon$ is unity. For the nucleation of vortices, friction turns out to always be both sufficient and not too large so that depopulation corrections remain small and $\Upsilon \approx 1$.

The escape rate calculated for three values of the damping parameter $\alpha$ over the full temperature range is shown in Fig.17. A hand sketch shows the influence of a temperature dependence in the damping coefficient. This unique situation is found in the nucleation of vortices in $^4$He as will now be described.

### D. Experimental energy barrier and damping coefficient

The knowledge of the analytic and numerical expressions for the rate $T$ makes it possible to extract from the measured nucleation rate and crossover temperature the values of the energy barrier in terms of $v_c$. The value of $\omega_0$ given by Eq. (55) ($\omega_0/2\pi = 2 \times 10^{10}$ Hz for $T_p=0.147$ K) is consistent with the attempt frequency appropriate to the thermally-activated regime (Varoquaux et al. 1986) and that found directly from the fits to the probability $p$ as shown in Fig.14. This agreement has been mentioned above. But furthermore, this value of $\omega_0$ corresponds well to the eigenfrequency of the highest Kelvin mode that a vortex filament in $^3$He can sustain.

The eigenmodes of a straight isolated vortex, the Kelvin modes, are helical waves with a dispersion relation expressed by

$$\omega^\pm = \frac{k_4}{\pi a_0^2} \left\{ 1 \pm \left[ 1 + k_0 \left( \frac{K_0(k_0)}{K_1(k_0)} \right) \right]^{1/2} \right\},$$  \hspace{1cm} (59)

where $K_0$ and $K_1$ are the modified Bessel functions of zeroth and first orders.

In the short wavelength limit,
The cylinder plays the role of the core of the vortex filament becomes:

\[ \omega^+ = \frac{\kappa_4}{\pi a_0^2}. \]  

(60)

By analogy with the 2D-motion of point charges subjected to a magnetic field (Muirhead et al., 1985), this frequency is sometimes called the “cyclotron” frequency. It sets the shortest time scale on which vortices can be expected to respond. Such a frequency is that of the cycloidal motion taken by a long hollow cylinder impulsively pulled sideways in an inviscid fluid (Donnelly, 1991). The cylinder plays the role of the core of the vortex, assumed to be hollow and with radius \(a_0\). The displaced mass per unit length of such a cylinder is \(\rho \pi a_0^2\), a standard result of classical fluid dynamics (Lamb, 1945).

For high frequency motions, the vortex mass is modified as discussed in II.C, and Eq. (60) is renormalised to \(\omega^+/2\pi = \kappa_4/[2\pi^2 a_0^2 \ln(r_m/a_0)]\), \(r_m\) being defined below Eq. 26 (Varoquaux et al., 1998). With \(a_0 = 2.5\,\text{Å}\) and \(r_m/a_0 \sim 10\), \(\omega^+/2\pi = 2.4 \times 10^{10}\,\text{Hz}\), a value comparable to the attempt frequency found with the help of Eq. (55) and quoted above. This value makes good physical sense.

The final step consists to extract the values for the energy barrier \(E_b\) from the measured escape rate. These values of \(E_b\) in the case of the experiments on ultra-pure \(^4\)He analysed in Varoquaux and Avenel (2003) are shown in Fig. 18.

The accuracy of the analysis can be checked by using the values of \(\dot{\omega}_0\) and \(E_b\) thus derived from of the nucleation rate to compute \(\bar{v}_c\) and \(\Delta \bar{v}_c\) using Eqs. (50) and (51) and compare with the experimentally determined values. This closure procedure provides a self-consistency check of the assumption made at the start, namely that vortices nucleate by quantum tunnelling below \(T_q\) and by a thermally-assisted process above.

The quantitative analysis can be carried out one step further using the variation of the barrier energy \(E_b\) in terms of \(v\) to construct an Arrhenius plot – the logarithm of the escape rate \(\Gamma\) vs the inverse temperature for a fixed potential well – from the experimental data and comparing directly the outcome to the results from theory. Arrhenius plots are thus drawn at constant \(E_b\) and varying temperature but the experimental results are obtained at velocities that vary with temperature, hence at varying \(E_b\)’s. As can be noted in Fig. 19, the raw experimental, velocity-dependent, rates exhibit little variation over the range of parameters: escape rates are only observed in a certain window determined by the measuring techniques.

At low temperatures, \(T < T_q\), the critical velocity is close to its zero temperature limit \(v_q\) and the corrections to \(\Gamma\) are small. As \(T\) increases above \(T_q\), \(v_c\) decreases and \(\Gamma\) has to be determined by piece-wise integration.
of $d\ln \Gamma/dv$. The high temperature extrapolation for $\Gamma$ obtained in such a manner does display the expected $1/T$ dependence, as seen in Fig. 19.

In the intermediate temperature range, the corrected $\Gamma$ shows, as can be seen in the inset of Fig. 19, a small but real drop below its zero temperature limit as the temperature is raised. As illustrated in Fig. 17, this drop reveals the influence of damping. A damping coefficient $\alpha$ that increases from 0 at $T = 0$ to $\sim 0.1$ around $T_\xi$ and more slowly above accounts for the observed drop (Varoquaux and Avenel, 2003). This $T$-dependent dissipation also makes the crossover between the thermal and the quantum regimes even sharper than for $\alpha = 0$, and closer to observations. The nucleation of vortices in $^4$He thus offers a rare observation of the effect of damping on MQT.

E. The vortex half-ring model

The case has been put so far for the nucleation of vortices, thermal or quantum. The nucleation barrier $E_\nu$ is of the order of a few kelvins (see Fig. 18) and the attempt frequency $\sim 2 \times 10^{10}$ Hz is of the order of the highest Kelvin waves mode.

A simple model accounts for these features. This model – the nucleation of vortex half-rings at a prominent asperity on the walls – finds its roots in the work of Langer and Reppy (1970), Langer and Fisher (1967), Volovik (1972), and Muirhead et al. (1984). It was further developed and put on the firm experimental findings described in the preceding paragraph by Avenel et al. (1993).

The model premises are quite simple. Consider, as done by Langer and Reppy (1970), the homogeneous nucleation of a vortex ring in a homogeneous flow $v_s$ extending over large distances. When the ring has grown to reach radius $R$ in a plane perpendicular to the flow, its energy in the laboratory frame, where the observer is a rest and sees the superfluid moving at velocity $v_s$, is expressed by

$$E_v = E_R - P_R v_s.$$  \hspace{1cm} (61)

The rest energy $E_R$ increases with vortex size as $R \ln R$ and the impulse $P_R$ as $R^2$: the impulse term becomes dominant at large radii and causes $E_v$ to become negative. The variation of $E_v$ in terms of $R$ has the shape of a confining well potential, which becomes shallower and shallower with increasing $v_s$, as depicted in Fig. 20. The barrier height can easily be computed numerically and substituted into the expression for $v_s$, Eq. (50). An analytic approximation for $v_s$, involving the neglect of logarithmic terms and valid for large vortices ($R \gg a_0$) has been given by Langer and Reppy (Langer and Reppy, 1970).

This critical process would yield a mist of vortices in the bulk of the superfluid. This sort of vorticity condensation does not take place for two reasons. Firstly, the velocity of potential flows, which follows from the Laplace equation, reaches its maximum value at the boundaries, not in the bulk. Secondly, the nucleation of a vortex half-ring at the boundary itself involves, for the same radius hence the same self-induced velocity, a half of the energy given by Eq. (61), for that reason alone, half-ring nucleation at the wall is always much more probable at the same velocity $v_s$ than full ring nucleation in the bulk. Halving the full-ring energy for the half-ring holds for classical hydrodynamics, the other half being taken care of by the image in the plane boundary. For a superfluid vortex, the actual energy of a half-ring is smaller than in the classical ideal fluid because the superfluid density is depleted at the solid wall and the core radius increases.

The half-ring model for the nucleation of vortices has been proposed for ion critical velocity by Muirhead et al. (1984) and for aperture flows by Burkhart et al. (1994) and Varoquaux and Avenel (1996). A variant, based on a different accounting of the vortex core energy, has been studied by Zimmermann, Jr. et al. (1998). Other mechanisms have been proposed by Andreev and Melnikovsky (2004) Josserand and Pomeau (1995) Josserand et al. (1995).
FIG. 21  Schematic views in 2D (left) and 3D (right) of the vortex half-ring trajectory over a point-like pinhole punched in an infinite plane. The dash-dash lines on the 2D plot are the potential flow streamlines that emerge from the orifice. The 3D view shows the vortex half-ring being first pushed by the potential flow to the right and then flying back over the pinhole to finally drift away to the left.

for which it is unclear that the end product of the nucleation process is actually a vortex.

The barrier height can easily be computed and substituted into the expressions for $v_c$ and $\Delta v_c$, Eqs.(50) and (51). Critical velocities $v_c$ and statistical widths $\Delta v_c$ computed in such a manner are shown as a function of temperature in Figs.12 and 13 for several values of the vortex core parameter $a_0$. A value of 4.5 Å gives near-quantitative agreement with the experimental observations over the entire temperature range. This value of $a_0$ is compatible not only with the temperature variations of $v_c$ and $\Delta v_c$ but also with the magnitude of the local $v_c$ found to be $20 \sim 22\;\text{m/s}$ using $^3\text{He}$ impurities as a local velocity probe (Varoquaux et al. 1993). It exceeds that in the bulk ($a_0 \simeq 2.5\;\text{Å}$), which is thought to reflect the proximity of the wall as discussed in greater details by Varoquaux et al. (2001). With this value, the nucleating half-ring has a radius of approximately 15 Å at the top of the barrier.

Once nucleated, the vortex half-ring floats away, carried out by the superfluid stream at the local superfluid velocity and by its own velocity, $v_R = \partial E_R/\partial P_R$. It can be noted that, at the top of the barrier, $\partial E_v/\partial R = 0$ and the vortex self-velocity $v_R$ exactly balances the applied $v_c$: the nucleating vortex is at a near standstill.

If the flow is uniform, with parallel streamlines, nothing much happens. The half-vortex wanders away. Interaction with the normal fluid, encounters with other vortices and friction on the solid boundaries cause a loss of vortex energy that eventually leads to its disappearance. If the flow is divergent, as in Fig.21, the vortex tends to follow the local streamlines and grow under the combined action of the potential flow and its own self-velocity: it then gains energy at the expense of the potential flow as explained in §III.E. In such a way, it can expand from nanometric to micrometric sizes and above.

The effect of $^3\text{He}$ impurities on the phase slip critical velocity at low temperature is striking, as seen in the inset of Fig.12. It is due to the condensation of these impurities on the vortex cores, which changes their line energy. This impurity dependence, studied in detail by Varoquaux et al. (1993), was used as a local probe of the superfluid velocity at which vortices nucleate. This velocity was found to be 22 m/s, an important piece of information in reasonable agreement with the value derived from the vortex half-ring model.$^7$ In all, the model parameters hang fairly well together.

$^7$ For a 15 Å radius vortex and $a_0 = 4.5\;\text{Å}$, $v_R = 13.5\;\text{m/sec}$ from Eq.35
VII. VORTEX PINNING, MILLS AND FLOW COLLAPSE

Single phase slips are observed in experimental situations that may be loosely characterised as “clean”, broadly speaking, for uncontaminated apertures of relatively small sizes (a few micrometres at the most), with low background of mechanical and acoustical interference’s, etc ..., and with probing techniques that do not manhandle the superfluid, namely, with low frequency hydromechanical resonators. When these conditions are not met, flow dissipation occurs in a more erratic manner in large bursts – multiple phase slips or ‘collapses’ of the superflow. Collapses of the superflow through an orifice were first observed by Sabo and Zimmermann, Jr. \(^{72}\) and by Hess (1977). \(^{73}\)

Multiple phase slips and collapses constitute an apparent disruption of the vortex nucleation mechanism described in the previous Section. Their properties have been studied in detail by Avenel et al. (1995) and are briefly mentioned below, together with possible mechanisms for their formation. It is likely that these events provide a bridge between the “clean” single phase slip case and the usual situation of the Feynman-type critical velocities that are temperature-independent below 1 K and dependent on the channel size. This problem, which is not fully resolved at present, almost certainly involves some form of preexisting vorticity.

A. Pinning, remanent vorticity and vortex mills

Remanent vorticity in \(^4\)He, which has long been assumed, has been shown directly to exist by Awschalom and Schwarz (1984) who looked at the propagation of ions in the presence of vortex lines. Ions can get trapped in the vortex cores and completely change course, revealing the presence of the vortex lines. These vortices, presumably nucleated at the \(\lambda\) transition where the critical velocity is very low, remain stuck in various places of the superfluid sample container. This trapped vorticity, according to Adams et al. (Adams et al., 1985), either is quite loosely bound to the substrate and disappears rapidly, or is strongly pinned and is dislodged only by strong perturbations.

To achieve a stable configuration, a pinned vortex has to take on a shape such that the local radius of curvature at each of its points results in a self-velocity which opposes exactly the local value of the applied flow. This dynamic equilibrium is what is meant here by pinning. Vortex pinning plays an important role in superfluids, bulk \(^4\)He as discussed here \(^{74}\) in films (Ellis and Li (1993), in \(^3\)He \(^{75}\) in neutron stars \(^{76}\) …

To account for laboratory observations and with the outcome of extensive numerical simulations of vortex dynamics, Schwarz has proposed the following formula for the velocity at which such strongly pinned vortices unpin \citep{Schwarz1981},

\[
v_u \leq \frac{\kappa b}{2\pi D} \ln \left(\frac{b}{a_0}\right),
\]

where \(D\) is the size of the pinned vortex and \(b\) is a characteristic size of the pinning asperity. Equation (62) bears a strong resemblance with that for the Feynman critical velocity, Eq. (47).

As a rule of thumb, the pinning energy of the vortex line on such an asperity with radius \(b\) is approximately equal to \(b\) times the line tension of the vortex given by Eq. (28). Long vortices unpin at very low velocities unless they are perched on a tall pedestal, but very small vortices pinned on microscopic defects at the cell walls can survive a wide range of superflow velocities; a 200 nm long vortex filament pinned at both ends on 20 Å asperities resists transverse flows of velocities up to 10 cm/s.

Fluctuations, which may be thermally activated, or may correspond to the zero point motion of the vortex \citep{Fetter1965}, have been seen to be important for the vortex nucleation process: so they are for unpinning processes \citep{Donev2001, Neumann2014}.

This trapped vorticity has long been suspected to be involved with the critical velocity problem. The long standing suggestion by Glaberson and Donnelly (1966) of vortex mills had its time of fame and still prevails in some circles. In these authors’ views, imposing a flow on a vortex pinned between the opposite lips of an aperture would induce deformations such that the vortex would twist on itself, undergo self-reconnections, and mill out free vortex loops. Upon scrutiny however, vortex mills are not so easy to set up.

The first thing to realise is that, according to Eq. (62), such a mill must involve a pinned vortex of submicrometric size for any flow velocity above a few cm/s in order for the pinned vortex not to be washed away. Thus, it cannot as such account for Feynman-type critical velocities found in large channels.

Less obviously, vortices are not prone to twist on themselves and foster loops. As shown by numerical simulations of 3D flows involving few vortices only \(^{77}\), vortex loops and filaments are found to be stable against

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\(^{72}\) as quoted by Hess (1977).

\(^{73}\) See Varoquaux et al. (1998) and Neumann and Zieve (2014) for more references.

\(^{74}\) See, among others, Hakonen et al. (1987), Krusius et al. (1993), Zieve et al. (1992).

\(^{75}\) See, e.g., Alpar et al. (1980), Langlois (2000), Packard (1972).

\(^{76}\) See also Neumann and Zieve (2014), Tsuboka and Maekawa (1994).

\(^{77}\) K.W. Schwarz: private communication to the author (1989)
FIG. 22 (a) Spiral-helix configuration of a streamwise vortex filament pinned at the centre of the bottom left section of the channel (Schwarz 1990b). The helical vortex self-velocity opposes the superflow while growing in size and spiralling on itself. (b) End-on view of the subsequent reconnection. The vortex is pinned near the upper left. The outwardly growing spiral periodically reconnects to the boundary and releases a new line segment, which then moves to the lower right. These numerical simulations are carried out in the presence of a sizeable mutual friction with the normal component in order to stabilise the numerical algorithms, which is why the generated half-rings decrease spontaneously in size.

B. The two types of large slips

Examples of multiple slips can be seen on the peak amplitude charts in Fig. 23 and in Fig. 24 for two different runs, which show rather different patterns. In Fig. 23, multiples slips are fairly frequent and the multiplicity involves moderate winding numbers. As the probability for a one-slip event per half-cycle is not large, that for a double slip is small, and it becomes negligible for higher multiples. A separate mechanism for their formation must be found.

Some degree of understanding of the formation of multiple slips can be gained by plotting the mean value of the phase slip sizes, expressed in number of quanta, against the flow velocity at which the slips take place

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large deformations: it takes the complex flow fields associated with fully developed vortex tangles to produce small rings (Tsubota et al. 2000, Svistunov 1995). And it takes some quite special vortex pinning geometry to set up a mill that actually works.

Schwarz has demonstrated the existence of such a mill by numerical simulations (Schwarz 1990b). Imagine a vortex filament pinned at one end in a region close to the aperture mouth or the channel entrance where the local \( v_s \), a near-stagnation point with its other end being carried away by the flow along the streamlines; the end that moves freely ends perpendicular to the wall. The filament develops a helical instability as depicted in Fig. 22, a sort of driven Kelvin wave, and reconnects sporadically to the wall when the amplitude of the helix grows large enough. The freed bit immediately stands against the flow and forms a vortex half-ring: such a helical vortex mill, which has to be of sub-micrometric size to withstand the near-by flow, does churn out fresh vortices. The occurrence of multiple slips, which can be seen in Fig. 23, is probably caused by some form of vortex mills on a microscopic size. Before coming to this topic, a description of multiple slips in greater details must be provided. But, at this point, the above remarks on the stability of vortex loops or half-loops in their course already make it unlikely that multiple slips be due to the production of small rings by the nucleating vortices twisting on themselves à la Glaberson-Donnelly, as suggested by Amar et al. (Amar et al. 1992). It is more likely that a working scheme involves wall reconnections à la Schwarz.

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See the review by Tsubota and Kobayashi (2009).
drodynamics in the bulk of the fluid but involves some complex interplay with the boundaries. As shown in Fig. 25, the velocity threshold for the appearance of multiple slips depends on hydrostatic pressure; in fact, the $P$-dependence of the upturn of $<n_+>$ vs $v$ exactly tracks that of the critical velocity for single phase slip nucleation. This indicates that multiple slips appear because of an alteration, or as a consequence, of the nucleation process of single slips.

The pattern of formation of multiple slips changes on cycling the cell from room temperature and back but remains stable during each given cool-down. It seems to depend on the degree of contamination of the cell, degree which cannot easily be controlled experimentally. The detailed microscopic configuration of the aperture wall where nucleation takes place probably plays an major role in multiple slip formation.

Another kind of very large drops in the resonance amplitude of the resonator was also observed, which sometimes resulted in a complete collapse of the resonance. These “singular” collapses were first studied by Hess (1977). An example of such an event is shown in Fig. 24 and in the insert. Under the conditions of this particular experiment (Avenel et al. [1995]) these events were rare (one in $10^4$ to $10^5$ slips); a striking feature is that they may occur at velocities much below the vortex nucleation threshold, down to at least a third of $v_c$, the critical velocity for phase slips. Is there yet another kind of critical velocity?

Multiple slips are different from these “singular” collapses and the underlying mechanisms responsible for both are bound to be different, as discussed below (Varoquaux et al., 2001).

C. In-situ contamination by atomic clusters

In a series of experiments conducted by Hakonen et al. (1998) [Varoquaux et al., 1998], in which the experimental cell was deliberately heavily contaminated by atomic clusters of air or H$_2$, numerous multiple slips and collapses of the “singular” type occurred. The peak amplitude charts of the resonator became mostly impossible to interpret, except in a few instances where two apparent critical velocities for single phase slips were observed. The higher critical velocity corresponded to the one observed in the absence of contamination. The lower critical velocity is thought to reveal the influence of a vortex pinned in the immediate vicinity of the nucleation site. This vortex induces a local velocity which adds to that of the applied flow and causes an apparent decrease in the critical velocity for phase slips. Because of this change, the presence of the pinned vortex could be monitored, the lifetime in the pinned state and the unpinning velocity could be measured, yielding precious information on the pinning process.

This observation, reported in detail by Hakonen et al.
FIG. 24 Absolute peak amplitudes at successive half-cycles of the resonator motion, normalised as in Fig. 23, vs half-cycle index in a $^4$He sample containing 100 ppb of $^3$He impurity, at 24.0 Bar and 12.5 mK (Avenel et al., 1995). The half-period is 31.8 msec. The top trace shows a succession of amplitude drops which correspond, for the largest part, to succession of phase slips by $2\pi$ of opposite sign, with occasional larger slips. The large feature around the 1000th half-cycle is a “singular” collapse, as defined in this Section. The insert shows the details of this collapse, (●) being for positive peaks, (◦) for negative peaks. It is preceded by a slip by $\times 2\pi$ and followed by a slow recovery of the peak amplitude caused by the applied drive, punctuated by single and double slips. The actual waveform corresponding to this collapse has actually been tracked (Avenel et al., 1995).

FIG. 25 Mean size of (positive) multiple slips vs velocity in phase winding number in nominal purity $^4$He (100 ppb $^3$He): (△) pressure sweep from 0.4 to 24 bars at 81.5 mK (for all even values of the pressure $P$, and 0.4, 1, 3, 5, 7 bars) – (⋄) temperature sweep at 16 bars – (◦) temperature sweep at 24 bars - (∗) drive level sweep at 24 bars, 81.5 mK – (■) temperature sweep at 0 bar (Varoquaux et al., 2001). For the temperature sweeps, from 14 to 200 mK approximately, $v$ first increases when the $^3$He impurities evaporate from the vortex core, reaches the quantum plateau and then decreases, as shown in the insert of Fig. 12. Lines connect successive data points in the temperature and pressure sweeps.

(1998) (Varoquaux et al., 1998), shows that pinned vorticity can contribute to the nucleation of new vortices at the walls of the experimental cell. Such pinned vortices as the one described above can, instead of interacting with the nucleation site, set up a transient vortex mill of the helical type and generate a burst of vortices. The existence of such pinned vortices is established; that they can form a micro-mill is highly plausible. Could this provide a possible explanation for multiple slip formation, as discussed by Varoquaux et al. (2001)? In such a scheme, the pinning event would take place immediately after nucleation when the velocity of the vortex relative to the boundary is still very small and the capture by a pinning site easy. The micro-mill remains in activity as long as the flow is sufficient to maintain the helical instability, which depends on the pinning stand geometry. As it is set up to withstand one flow direction, it is destroyed when the flow velocity reverses itself in the resonance motion. It eventually re-establishes itself during a subsequent resonance cycle, causing a new multiple slip. This process depends on the precise details of the pinning site configuration and of the primordial vortex trajectory, factors which allow for the variableness of multiple slips on contamination and pressure.

In the same experiments, a large number of unpinning events were also observed to take place at an “anomalously low” unpinning velocity. A parallel can be made with the singular collapses that also occur at “subcritical” velocities and that were also quite frequent in the same experiments, suggesting that the two effects have a common cause. Noting furthermore that pinning and unpinning processes were also quite frequent, releasing a fair amount of vagrant vorticity, it appears quite plausible that both singular collapses and low velocity unpinning events are caused by vagrant vortices hopping from pinning sites to pinning sites, eventually passing by close to a pinned vortex or a vortex nucleation site, and giving a transient boost to the local velocity, which pushes a pinned vortex off its perch or causes a burst of vortices to be shed.

These observations, albeit incidental, have important consequences for the critical velocity problem: existing vortices, either pinned or free-moving, can contribute to the nucleation of new vortices at the walls of the experimental cell at apparent velocities much lower than the critical velocity for phase slips. A mechanism is thus
provided by which superflow dissipation sets in at large scale for mean velocities much smaller than the velocity for vortex nucleation on the microscopic scale, possibly bridging the gap between phase slip and Feynman type critical velocities. Vortex nucleation at the walls is also quite likely to take part in the build-up of self-sustaining vortex tangles forming superfluid turbulence, attributed to reconnection mechanisms [Schwarz 1983].

To conclude this Section, the critical velocities in superfluids that are true and proven include the Landau critical velocity for roton creation in ion propagation (McClintock and Bowley, 1995), the formation of vortices by a hydrodynamical instability in BEC gases (Madison et al., 2001) and in \(^3\)He (Eltsov et al., 2005), the nucleation of vortices by thermal activation and quantum tunnelling in \(^4\)He, both for ion propagation and in aperture flow. Also, there is rather compelling experimental evidence for the interplay between vortex nucleation and pinned vorticity on a macroscopic scale; this evidence points toward the existence of helical vortex micro-mills that can generate vortices at fairly low applied velocities. Finally, vagrant vortices interacting with these mills, or with vortex nucleation sites, are found to generate enough vorticity to completely kill the applied superflow and explain singular collapses. How this can occur is illustrated in detail by the numerical simulations of the onset and decay of vortex tangles in large channels (Schwarz 1983; Schwarz and Rozen 1991), of the influence of surface roughness on the critical velocity for a self-sustaining vortex tangle (Schwarz 1992), and of the evolution of phase-slip cascade from a single remnant vortex as a function of channel size (Schwarz 1993b). A fair degree of understanding of the mechanisms behind the Feynman critical velocity has thus been achieved. The study of phase slippage has paved the way toward an explanation of the various critical velocities in superfluid helium-4.

VIII. JOSEPHSON-TYPE EFFECTS IN SUPERFLUIDS

Anderson's conjectures, seen in the previous Sections to be fully confirmed in the hydrodynamic (macroscopic) limit of quantised vortex dynamics, have also been carried over to the microscopic limit of quantum tunnelling, as described below.

The reasoning goes that Eqs. (13) and (14) are fundamental enough to carry the day both at large and short distances, namely when the coherence length is either small or large with respect to characteristic dimensions of the hydrodynamic weak link. The former case has been covered in the previous Sections. In the latter case - namely weak quantum coupling between two superfluid baths - the contention is that effects analogous to the famed Josephson effects between two weakly-coupled chunks of superconducting material must also exist between two loosely-connected pools of superfluids provided that superfluid coherence is not entirely lost through the connection. These Josephson-type effects in superfluids are dealt with below.

A. A simple model

The time evolution of \(\varphi\) and \(N\) is expressed quite generally by Eqs. (13) and (14), as discussed in Sec. II.B:

\[
i\hbar \dot{N} = [\mathcal{H}, N] = \frac{1}{\hbar} \frac{\partial \mathcal{H}}{\partial \varphi},
\]

\[
i\hbar \dot{\varphi} = [\mathcal{H}, \varphi] = -i \frac{\partial \mathcal{H}}{\partial N},
\]

\(\mathcal{H}\) being the Hamiltonian of the system.

These equations hold in fact for the operators \(\hat{N}\) and \(\hat{\varphi}\). However their coarse-grained averages can be treated as \(c\)-numbers to a very good approximation because their relative quantum uncertainties are very small. Averaging over a volume of superfluid that is small compared to the size of the sample but still contains a large number of atoms leads to Eq. (15):

\[
\frac{\hbar}{\partial t} \frac{\partial \varphi}{\partial t} = -(\mu + \hbar/2 m_4 v_s^2). 
\]

Equation (15) describes the Josephson \(ac\) effect. For superconductors (for which it was initially derived), \(\mu = 2eV, 2e\) being the electrical charge of the Cooper pair, \(V\) the applied electric potential. For superfluids, \(\mu = v_s P, v_s\) being the atomic volume for \(^4\)He, twice the atomic volume for \(^3\)He [81].

The \(ac\) Josephson relation applies more readily to phase and pressure differences. In particular, when applied to the gradient of the phase, it can be cast, using Eq. (8), into the Euler equation (16):

\[
\frac{\partial v}{\partial t} + \nabla(v_s P + \frac{1}{2} m_a v^2) = 0,
\]

\(m_a\) being the atomic mass of the effective boson.

Equations (15) and (16) look plainly classical enough. Quantum mechanics hides in the possible multi-valuedness of the overall phase \(\varphi(t, r)\) of the order parameter, yielding a quantised circulation of the fluid velocity.

[80] See the discussion following Anderson’s talk at the Sussex University Symposium in 1965 (Anderson, 1966b).

[81] The contribution of the entropy to the chemical potential, \(ST\), should also be taken into account in Eq. (15) if the temperature is not very low.
Also, and perhaps more importantly, how it provides a mechanism, as discussed in detail in the case of $^4$He in the previous Section, for this quantity to vary discontinuously from one determination to another, violating the Kelvin-Helmholtz theorem and foiling the derivation of the Euler equation, Eq. (16). The second Heisenberg equation of motion, that for $N$, expresses particle number conservation:

$$\hbar \frac{\partial N}{\partial t} = \frac{\partial E}{\partial \phi}.$$  \hspace{1cm} (63)

What has just been done is simply to reproduce the equations for the motion of the superfluid component in the two-fluid hydrodynamics from the fact that $N$ and $\phi$ are canonically conjugate. However, as stressed by Anderson (Anderson 1966a), the range of validity of Eq. (15) is quite wide and it will apply even when hydrodynamics is expected to break down as for tunnelling supercurrents. In the same kind of situations, the internal energy $E$ depends in a non-trivial way on $\phi$, as may be expected from Eq. (63).

When applied between two regions of the superfluid, Eqs. (15) and (63) describe the supercurrent flowing from one region to the other. This situation becomes especially interesting when the two regions, the two superfluid baths, are sufficiently well separated so that they only weakly coupled. Then, a well-defined phase difference between them, $\delta \phi$, can be sustained.

Such situation is schematised in Fig. 26. The partition separating the two baths presents a long, thin slit through which a trickle flow only of superfluid can leak. If the dimensions of the slit are comparable to the superfluid coherence length — the distance over which its wavefunction can heal — the amplitude of the wavefunction is reduced in the narrow passage, as pictured in the bottom panel of Fig. 26.

For superconductors, the case studied by Brian Josephson, the weak link can be provided by a thin layer of insulating oxide through which the Cooper pairs can tunnel quantum-mechanically. For superfluids, the only practical weak link so far is a microscopic aperture such as depicted in Fig. 26. As is well known in superconductivity, weak links, or micro-bridges, lead to the same kind of effects as tunnel junctions (Golubov et al. 2004; Likharev 1979).

For superflows through such a micro-aperture, the problem can be restricted to one dimension along $z$ and, to simplify further, the barrier (the weak link) can be taken as a square potential wall of height $U$ over length $l_b$. This is done in the following.

In the bulk of the fluid, the wave function corresponding to a state with energy $E$ is taken as a plane wave with identical amplitude $|\Phi| = (\rho_s/m_s)^{1/2}$ on both sides of the barrier ($m_s = 2m_n$ for superfluid $^3$He), but with phases that differ by $\delta \phi$: these are the boundary conditions at the weak link walls at $z = 0$ and $z = l_b$.

Inside the barrier $|\Phi(z)|$ is assumed to be severely depressed: the interactions within the fluid can be neglected. With this approximation of weak coupling, the tenuous fluid inside the weak link behaves as a simple non-interaction gas and the equation of motion reduces to the Schrödinger equation:

$$i\hbar \frac{\partial \Phi}{\partial t} = -\frac{\hbar^2}{2m_n} \nabla^2 \Phi + U \Phi, \quad U > E,$$

and also has a plane wave solution $\exp(-i(\epsilon t + kz))$. The energy $E$, or chemical potential, is taken to be same everywhere. The momentum takes two values corresponding to the two possible directions of (damped) propagation:

$$k_\pm = \pm (i/\hbar) \sqrt{2m_n(U - E)}.$$

Let $b_b = \hbar / \sqrt{2m_n(U - E)}$: the barrier height is characterised by a penetration length. The wave function inside the barrier is found by standard methods:

$$\Phi(z) = \frac{|\Phi|}{\sinh(b_b/\hbar_b)} \left\{ \sinh \left( \frac{z}{b_b} \right) e^{i \delta \phi} - \sinh \left( \frac{z - l_b}{b_b} \right) \right\}.$$
The modulus of $\Phi$ midway in the barrier, is expressed by

$$\Phi'(l_b/2) \Phi(l_b/2) = \frac{\rho_s/m_s}{2 \cosh^2(l_b/2b_b)} [1 + \cos \delta \phi], \quad (64)$$

and is a $2\pi$-periodic function, which vanishes for $\delta \phi = \pi \pm 2n\pi$. The weak coupling condition is satisfied in superfluid helium for $l_b \leq b_b$.

Knowing the wavefunction, the current density, Eq.(7), can be computed in a straightforward manner. The total current through a micro-aperture of effective cross section $s_b$ is found to be:

$$J = \frac{\hbar s_b}{2ib_b} \frac{\rho_s/m_s}{\sinh^2(l_b/b_b)} \left[ \sinh \frac{z}{b_b} e^{-i\delta \phi} - \sinh \left( \frac{z - l_b}{b_b} \right) \right] \left[ \cosh \frac{z}{b_b} e^{i\delta \phi} - \cosh \left( \frac{z - l_b}{b_b} \right) \right] - \text{complex conjugate}$$

$$= J_c \sin(\delta \phi), \quad \text{with} \quad J_c = \frac{\hbar}{m_s} \frac{s_b}{b_b} \frac{\rho_s}{\sinh(l_b/b_b)}. \quad (65)$$

Equation (65) describes the Josephson dc effect. Although this equation has been obtained here in a simplified manner, it is nearly identical to the result of more involved theories, each with its own set of approximations – the Ginzburg-Landau model (Monien and Tewordt, 1986), an ideal tunnel junction (Rainer and Lee, 1987), or a strictly point-like orifice (Kurkijärvi, 1988).

The supercurrent $J$ is periodic by $2\pi$ in $\delta \phi$ as it must be since changing the phase by $2\pi$ on one side of the barrier must leave the overall physical situation unchanged. It vanishes for $\delta \phi = \pm \pi$ not because the velocity, proportional to $i\phi$, goes to zero but because the superfluid density, which is proportional to $|\Phi|^2 \sin(\delta \phi)/\delta \phi$ inside the barrier, does; the modulus of the wave function at midpoint in the barrier, Eq.(64), vanishes: superfluidity is actually destroyed at that point, which is why the supercurrent goes to zero and the phase can slip by $2\pi$ (or lumps of $2\pi$).

If the coupling is not weak, a more elaborate calculation is necessary: the sine function is replaced by a general periodic function $f_{2n}(\delta \phi)$, the current-phase relation, or CPR for a “non-ideal” weak link. Often, this relation is not even single-valued and, when the phase is varied, the current may jump discontinuously from one determination to another: the weak link is then said to be hysteretic. This behaviour is due to the nucleation of vortices, as seen in the previous Section, and is accompanied by dissipation while the ideal Josephson case (when $f_{2n}(\delta \phi)$ is a sine function) is dissipation-less (Likharev, 1979; Huneberg, 2005; Viljas, 2005).

In the transition between the “ideal”, non-hysteretic, purely sinusoidal CPR’s and the mostly linear CPR seen, for example, in Fig.9, a slanted sine function is often observed. Part of this distortion arises from purely classical fluid flow in the vicinity of the micro-aperture. The full phase difference across the weak link, $\varphi_w$, includes, besides the phase difference across the barrier $\varphi_b$, the rather trivial velocity potential drop in the vicinity of the weak link where the superfluid velocity $v_s$, and the corresponding phase gradient, behave in accord with classical ideal fluid dynamics.

In order to account in a simple manner for this classical contribution, it is convenient to introduce the equivalent “hydraulic” length and cross-sectional area of these regions, $l_h$ and $s_h$, in such a way that the flow is described in a “rod-like” manner. The superfluid velocity is then expressed simply by $v_s = (\hbar/2m_s)\delta \phi_h/l_h$, and the current by $J = \rho_s s_h v_s$.

The total phase difference $\delta \phi$ is the sum of the phase drop through this “hydraulic” region and through the barrier acting as the weak link, assumed ideal, i.e. such that:

$$\delta \phi = \delta \phi_h + \delta \phi_b = \frac{2m_s l_h}{\rho_s s_h} J . \quad (66)$$

The same mass current also flows through the depletion region and varies, following Eq.(65), as a sine function of the phase difference $\delta \phi_b$ as long as the coupling is weak. Combining Eqs.(65) and (66), and renaming $\zeta$ the hydraulic part $\delta \phi_h$ of the phase difference to stress its ancillary role yields the relation between the current and the phase of a (slightly more) realistic micro-aperture:

$$\varphi = \zeta + a \sin \zeta , \quad J = J_c \sin \zeta , \quad (67)$$

with $a = (2m_s l_h/\rho_s s_h) J_c$ and $J_c$ expressed from Eq.(65). The non-ideality parameter $a$ and the critical current through the junction $J_c$ have a meaning in terms the geometrical details of the micro-aperture. They can be derived from experiments and compared with the expected values.

Since the healing length is of atomic dimensions for $^4$He, a near-ideal Josephson effect cannot be expected to be found in the micro-apertures that can be manufactured at present, except very close to the $\lambda$ point. Such experiments have been conducted successfully by Sukhatme et al. (2001) and Hoskinson et al. (2006), and are described in Chapter VIII. The experiments that have first shown the existence of the Josephson dc-effect in superfluids have been carried out in $^3$He (Avenel and Varoquaux, 1988), which forms a superfluid with more features than $^4$He.

84 Relations were first proposed by Deaver and Pierce (1972) for superconducting junctions. See also Likharev (1979).

85 See Avenel and Varoquaux (1988) for an example of this procedure and Viljas (2005) for a more complete analysis.
B. Current and phase in superfluid $^{3}$He

The helium-3 nucleus is made up of two protons and one neutron: $^{3}$He is a fermion. As for the abundant and heavier isotope, $^{4}$He, its zero-point energy in the condensed phase is large and it remains in the liquid phase down to absolute zero at pressures below about 35 bars. It thus forms a Fermi liquid with a Fermi sphere over which Landau quasiparticles float. Because the interatomic potential is attractive at large distance, these quasi-particles can form Cooper pairs and $^{3}$He was long suspected to become a BCS superfluid below some hard-to-predict temperature. The discovery by Osheroff, Richardson and Lee of the transition to not one but two superfluid phases (Osheroff et al., 1972) fixed the transition temperature to 2.49 mK on the melting curve, at a pressure of 34.34 bars and opened an exciting new chapter of low temperature physics.

As the experimental properties of these new superfluid phases were quickly unravelled (Lee and Richardson, 1978, Wheatley, 1975a,b), they were identified from their nuclear susceptibility properties observed by NMR as resulting from the formation of Cooper pairs in a spin-triplet state (Leggett, 1975). A new breed of superfluid was just born. The overall antisymmetry of the wavefunction under the exchange of two fermions then requires an odd angular momentum state $l = 1, 3 \ldots$. The available experimental data, mainly the phase diagram, the specific heat, and the nuclear susceptibility, led to the identification of the A and B phases as p-wave Cooper-pair superfluids with total spin $S = 1$ and total angular momentum $L = 1$.

The formalism describing the properties of these anisotropic superfluid phases was quickly developed. This formalism is an extension of the Bardeen-Cooper-Schrieffer (BCS) theory of s-wave superconductivity to the neutral triplet-spin-state superfluid. The most general pair wavefunction with three possible substates for the spin and the orbital parts is an arbitrary superposition of these 3x3 substates, involving nine complex parameters. Assuming weak coupling between the pairs – a surprisingly good assumption at low pressure – this extension of the BCS theory (Leggett, 1975) leads to a 3x3 order parameter for B-phase of the form

$$A_{\mu \nu} = \Delta(\mathbf{k}) e^{i\varphi} R_{\mu \nu}(\mathbf{n}, \theta).$$

The gap parameter $\Delta(\mathbf{k})$ and the phase factor $e^{i\varphi}$ have the same interpretation as in the case of s-wave superconductors. The B-phase contains the $S_z = 0, +1$, and -1 pairs ($|\uparrow\uparrow\rangle$, $|\uparrow\downarrow + \downarrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$) in equal amounts in zero applied magnetic field; $\Delta(\mathbf{k})$ is isotropic and independent of $\mathbf{k}$, the direction on the Fermi sphere.

The matrix $R_{\mu \nu}(\mathbf{n}, \theta)$ describes the rotation bringing the spin quantisation axis along the orbital quantisation axis. This rotation is characterised by a unit vector $\mathbf{n}$ and an angle $\theta$. Both the gap parameter $\Delta(\mathbf{k})$ and the rotation are real quantities independent of the overall phase $\varphi$. Therefore, the B-phase order parameter (68) thus takes the same form as that for $^{3}$He, namely the product of a phase factor $\exp(i\varphi)$ with a well-defined phase $\varphi$ and a phase independent modulus. The same reasoning as for the $^{4}$He case applies when performing a Galilean transformation: mass transport in the pseudo-isotropic B-phase is related to the gradient of $\varphi$.

The modulus of order parameter (68), or gap parameter $\Delta$, can be thought of as the binding energy of a Cooper pair at $T = 0$; it is of the order of $k_B T_c$, $T_c$ being the superfluid transition temperature. The smallest time lapse over which this energy can be defined is limited by the uncertainty relation for time–energy: $\delta t \sim \hbar/\Delta$. During that time, the pair spreads over a length $\zeta_0 = \hbar \nu_F / \Delta$, where $\nu_F$ is the velocity of the $^{3}$He quasiparticles over the Fermi surface. It can be seen from this somewhat heuristic argument (Davis and Packard (2002), Lounasmaa et al. (1983)) that properties of the superfluid are well-defined only over distances larger than the coherence length $\zeta_0$, of the order of 600 Å at $T = 0$ and low pressure and 120 Å at melting pressure. The prospect to observe quantum departures from classical hydrodynamics in $^{3}$He appears much more favourable than in the case of $^{4}$He, the coherence length is no longer very small compared to the size of apertures that can be micro-machined; departures from the kind of hydrodynamics that has led to phase slippage by vortex motion – namely, the genuine hydrodynamic Josephson effects – can be expected in the B-phase in submicron size apertures, or pinholes.

C. Weak links in p-wave superfluids

Weak links for superfluids come in two breeds, single micro-apertures in thin wall partitions and larger scale arrays of such apertures geometrically arranged a few microns apart, actual pinholes for the former, a mock-up for tunnel junctions for the latter. The flow patterns in

$^{86}$ As related by Leggett (1975) and Anderson and Brinkman (1975) – see the monograph by Vollhardt and Wölfle (1990).

$^{87}$ For $^{3}$He-B in weak coupling theory, $\Delta(0) = \alpha k_B T_c$ with $\alpha \leq 1.75$.

$^{88}$ The zero temperature coherence length of the B-phase is given by $\zeta_0 = \sqrt{|\zeta(3)|/4\pi^2} / [\hbar \nu_F / k_B T_c]$ (Vollhardt and Wölfle, 1990, §3.4). The temperature dependent coherent length diverges as $\zeta(T) = \zeta_0 / (1 - T/T_c)^{-1/2}$ in the Ginzburg-Landau regime.

$^{89}$ The state of the art in aperture manufacturing evolved with time from submicron slits (Sudraud et al., 1987) to nanometric holes (Pereverzev and Eska, 2001).
the vicinity of each kind lead to different behaviours of the weak link.

The first successful experiments were first carried out by Avenel and Varoquaux (1988), using the same micro-resonator and single aperture as for their experiments on phase-slippage in \( ^4\)He.

These experimental results triggered intense theoretical interest in the description of phase slippage in \( ^3\)He. Analytic calculations of the current through a pinhole orifice with all dimensions small compared to \( \xi_0 \) and with specular reflection of the quasiparticles on the walls by Kurkijärvi (1988) in the framework of quasiclassical theory, following earlier work by Kopnin (1986) and Monien and Tewordt (1986), lead to the following classical theory, following earlier work by Kopnin (1986) with specular reflection of the quasiparticles on the wall.

Analytic calculations of the current through a pinhole orifice by Avenel and Varoquaux (1988), using the same micro-hole, much like a soap bubble. The barrier parameters still exist, sustained by the quantum tunnelling of quasiparticles at the critical current through the weak link.

\[ J = \frac{\pi}{\sqrt{2}} N(E_F) \Delta(T) \sin \left( \frac{\varphi}{2} \right) \tan \left[ \frac{\Delta(T)}{k_B T} \cos \left( \frac{\varphi}{2} \right) \right], \]  

where \( N(E_F) \) is the density of states at the Fermi surface, \( v_F \) the Fermi velocity. For the B-phase \( a = \pi/2, b = 1/2 \), and Eq. (69) takes the same form as for a s-wave supercurrent through a superconducting micro-bridge.

A similar form also holds approximately for the A-phase with \( a = \pi/\sqrt{6} \) and \( b = (3/8) \sqrt{3}/2 \), and for the planar phase, a phase which may possibly be stabilised within the orifice by the walls.

Expression (69) reduces in the limit \( \Delta(T)/k_B T \ll 1 \) to the sinusoidal dependence of Eq. (65) for the current in terms of the phase difference across the barrier \( \varphi_b \). This result has been obtained by a number of authors using a variety of techniques. It is no real surprise that the details of the structure of the order parameter disappear when the dimensions of the orifice are small with respect to the coherence length and that s-wave-like results are found for both the A and B phases. Superfluid coherence is effectively weakened by the micro-orifice because the length over which it heals becomes larger than the physical size of the connecting duct; however, if the length of that duct is short enough, a sizeable supercurrent can still exist, sustained by the quantum tunnelling of quasiparticle pairs through the weak link.

At temperatures such that \( \Delta(T)/k_B T \) is no longer a small quantity, Eq. (69) becomes increasingly slanted with an abrupt slope close to \( \varphi = \pi \) when \( \cos \varphi/2 \) changes sign while retaining the periodicity by \( 2\pi \) in the phase difference. It displays a discontinuity for \( \varphi = \pi \) at \( T = 0 \). This behaviour of the weak link comes on top of the effect of the hydraulic inductance of the Deaver-Pierce model: the CPR is bound to become hysteretic and multivalued even for an extremely small pinhole in the limit \( T \to 0 \).

This simple theoretical description accounts well for the experiments of Avenel and Varoquaux (1988), whose results are reproduced in Eq. (69) single micro-apertures and arrays behave alike. The existence in superfluids of the analogues of Josephson effects in superconductors was thus established in 1988 on firm grounds, both experimentally and theoretically, but more features were soon revealed by further studies.

D. Multivalued CPR's, \( \pi \) states and \( \pi \) defects

While near-ideal Josephson behaviour prevails in \( ^3\)He-B at low pressure close to \( T_c \), departures from a sinusoidal current-phase relation were observed by Avenel and Varoquaux (1988) in a single orifice and later by Marchenkov et al. (1997) in an array of 0.1 \( \mu \)m–diameter apertures, evolving to a near-straight line relation below 0.6 \( T_c \). The latter case is reminiscent of the situation in \( ^4\)He, in which vortices are nucleated.

As the temperature is lowered further below \( T_c \), the superfluid coherence length becomes smaller than the aperture size used in present-day experiments. This trend is even more pronounced at higher pressure, where \( T_c \) is higher (and \( \xi_0 \) smaller). Room is thus left for a wall dominated order parameter texture within the weak link or its immediate proximity: the true p-wave nature of superfluid \( ^3\)He then unravels itself.

Very detailed numerical simulations based on the time-dependent Ginzburg-Landau equations have been carried out by Soininen et al. (1992) for a finite size aperture quite similar to the one used in the experiments.
FIG. 27 Staircase patterns (peak resonance amplitude versus applied drive level, both in arbitrary units) in $^3$He-B at a pressure of about 0.2 bar, from Avenel and Varoquaux (1988). The curves at various temperatures are shifted vertically for readability. Drive frequencies at the various temperatures are shifted from resonance so as to set comparable resonance conditions.

of Avenel and Varoquaux (1988) in which the structure of the p-wave order parameter can unfold. These simulations show in colourful figures the evolution of the components of the superfluid density tensor of $^3$He flowing through the micro-slit of the components of the tensor parallel to the two short dimensions of the slit. Both the mass and the spin degrees of freedom of the spin-triplet p-wave order parameter take part in the phase slippage process. The various components of the order parameter evolve separately in space and time and do not go to zero simultaneously at the same location in the micro-aperture. The regions of space, in which the order parameter is depressed and about which the phase slips, peel off from the walls and traverse the slit at right angle with the flow direction. Thus, phase slips in the p-wave superfluid exhibit a fairly complex spatial and temporal evolution both in the pseudo-isotropic B phase and in the anisotropic A phase. In addition, the A phase may exhibit core less phase slippage as suggested by Anderson and Toulouse (1977) and as discussed below.

The state of sophistication of the microscopic description of superfluid $^3$He makes it possible to obtain such a detailed information.

These simulations also illustrate the details of operation of a vortex mill (Soininen et al., 1992) in which phase slip avalanches and multiple vortex creation take place. These topological defects are found to evolve in a richer manner than in $^4$He. While near-ideal Josephson behaviour prevails in $^3$He-B at low pressure close to $T_c$, more complicated staircase patterns than those shown in Fig. (27) develop below 0.77 $T_c$ (Avenel and Varoquaux, 1989), which cannot be described by the Deaver-Pierce model. These patterns were not even reproducible from one cool-down through $T_c$ to the next, indicating that textural effects may be coming into play.

Among those features, two notable ones were reported by Backhaus et al. (1998), Marchenkov et al. (1999) and are shown in Fig.28. These authors used a two-hole micro-resonator with a weak link made of a 65x65 array of 100 nm round holes micro-machined in a 50 nm thick silicon nitride free-standing membrane. The 4225 holes in parallel offer a large enough flow path for the mass flow rate under an applied pressure head to be closely monitored. The phase is obtained from the measurement of the pressure head between the two sides of the weak link by integration of Eq.(15). They operated the resonator in free ringing mode to observe directly the current-phase relations in $^3$He-B at saturated vapour pressure. The first feature is the existence of two possible CPR’s at the same temperature, one with a larger critical current than the other, the second, the appearance in the $2\pi$-periodic CPR of an increasingly strong $\pi$-periodic admixture as the temperature is lowered, as shown in Fig.28. Avenel et al. (1998b) pointed out that this admixture could simply arise from the unavoidable dispersion between the sizes of the micro-holes in the array. This rather trivial explanation probably holds in part under all circumstances but is not the end of the story, as was

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95 More experimental details and further references can be found in the reviews by Davis and Packard (2002) and Sato and Packard (2012).

96 An alternate explanation for the existence of $\pi$-states is offered by Eska et al. (2010) and is based on the built-in non-linearities of the single-hole resonator used in the experiments of Backhaus et al. (1998), Marchenkov et al. (1999).
FIG. 28 Current-phase relations in $^3$He-B observed in an array weak link by Marchenkov et al. (1999). The two panels show CPR’s for the low current state (left panel) and the high current state (right panel) for temperatures ranging from $T/T_c = 0.850$ down to $0.450$ in steps of approximately $T/T_c$. The mass current through the weak link increases as the temperature is lowered. At temperatures close to $T_c$, the current-phase relations can be fit well with the Deaver-Pierce model. At the temperature decreases (and the critical current increases), this model becomes inadequate as a $\pi$-periodic component gradually sets in.

soon shown by Avenel et al. (2000)’s own observations using a single micro-aperture for which there is obviously no scatter in critical currents or transit times. Avenel et al. (2000) (Mukharsky et al., 2004) used the Sagnac effect (see §IX.A) to ramp up and down in a precise manner the macroscopic phase difference $\delta \phi$ applied across the weak link. They reported the observation of several different CPR branches, most with $\pi$-components, and with different critical velocities – usually more than two – at the same temperature but in different cooldowns through the superfluid transition temperature. Each of these several $I(\phi)$’s was usually robustly fixed in each run and the general trend was to go from $2\pi$ periodicity to $\pi$ periodicity as the temperature is lowered, although, occasionally, no $\pi$ periodicity was observed even at the lowest temperature. At higher pressure (10 bars), hysteretic behaviour in the single micro-aperture was prevalent and up to three simultaneous branches for the CPR were observed. Switching between these different branches could be triggered by applying strong transient drive voltage to the resonator, indicating that textural effects were most likely at play.

Some of these features were in fact predicted long before their observation by Thuneberg (1988) who worked out a numerical solution of the Ginzburg-Landau equations of the state of $^3$He-B confined inside a micro-aperture. This author found two different CPR’s according to whether the $\hat{n}$-vector of the B-phase order parameter, assumed to lie perpendicular to solid walls, is in a parallel or antiparallel configuration on both sides of the membrane carrying the micro-aperture. In the antiparallel configuration, the spin and mass currents are out of phase, resulting in a lower critical current. Eventually, the coupling between mass and spin currents leads to the admixture of a $\pi$-periodic component to the $2\pi$-periodic CPR.

The experimental discovery of these effects by Backhaus et al. (1998) and Marchenkov et al. (1999) triggered renewed theoretical interest. Thuneberg’s numerical findings were soon confirmed and sharpened by the analytic investigations of Yip (1999) and Viljas and Thuneberg (1999, 2002b) and extended numerical simulations for two-dimensional geometries by Viljas and Thuneberg (2002a). The upshots of these studies are the following:

• Following Yip (1999) and Viljas and Thuneberg (1999) the $\pi$ states in $^3$He-B are due to the interference of currents carried by quasiparticles with different spins which acquire different excess phases from the internal spin structure of the order parameter while travelling through the weak link. More specifically, the $|\uparrow \uparrow\rangle$ and $|\downarrow \downarrow\rangle$ Cooper pair populations may be viewed as independent superfluids the phases of which may be slightly

\footnote{see also Janne Viljas’ Thesis (Espoo 2004) available at http://lib.hut.fi/Diss/ Smerzi et al. (2001), Zhang and Wang (2001), Nishida et al. (2002) and the very clear review by Viljas and Thuneberg (2004). A related situation, that of “$\pi$-junctions”, has been much studied in electrodynamic junctions Golubov et al. (2004).}

\footnote{Also Zhang and Wang (2001).}
shifted with respect to one another because of a differing spin-orbit coupling. Summing the corresponding mass currents, given by Eq. 100, represented by slanted sine CPR’s shifted in phase by \( \pm \delta q \) leads, if the shift is large enough, to a positive-slope branch in the CPR at \( \pi \): this \( \pi \)-state mechanism relies on different spin-orbit orientations on both sides of the weak link and operates at the single pinhole level [Viljas and Thuneberg, 2002a].

- The Josephson coupling between two baths of \( {}^3 \text{He}-\text{B} \) mixes the phase difference to the spin-orbit texture of the order parameter: the equilibrium configuration of the texture then depends on the phase bias applied to (hence on the current carried by) the weak link. The texture is assumed fixed in the simpler calculations: this is the isotextural case, which offers only a coarse agreement with observations. If the texture is allowed to adjust to the local mass and spin currents by expressing the balance between its stiffness and its interactions with the walls and with the mass current, a \( \pi \)-state can also arise: this anisotextural effect requires a self-adjusting string of calculations and provides quantitative agreement with pinhole array experiments [Viljas and Thuneberg, 2002b].

- These refined calculations led to the realisation that multiple Andreev reflections and sub-gap structures also played a role in the transmission of the supercurrent through the weak link [Asano, 2001] and that a A-like phase inside the superfluid junction also leading to a \( \pi \)-state [Nishida et al., 2002].

- Dissipation in pressure-driven dc-supercurrents [Simmonds et al., 2000] could also be explained by multiple Andreev reflections [Mukharsky, 2004; Viljas, 2005] or by time-dependent anisotextural effects and spin-wave emission, [Viljas and Thuneberg, 2004a]: if a pressure difference is applied across the weak link, an ac-oscillation (at the ac-Josephson frequency) of the texture ensues, causing dissipation by spin-wave radiation. The two dissipation mechanisms, sub-gap processes and textural losses, can come on top of one another.

Observations related to these topics are those of Mukharsky et al. [2004] who, in the course of high-precision CPR measurements using the Sagnac effect [100] found the signature of a stable textural defect that sustains a change of the phase by \( \pi \) away from the weak link and is therefore different from the \( \pi \)-state discussed above. “Cosmic-like” solitons, proposed by Salomaa and Volovik [1988], could constitute such a defect but they are thought to be unstable in the bulk of the superfluid; it is not known whether they can be stabilised by the presence of walls.

In the perspective of the present article (see also the review by Davis and Packard [2002]), these complex features of the Josephson supercurrents illustrate quite vividly the nature of the superfluid order parameter and of the coherence it entails. But their detailed studies are complicated because they are entangled with order parameter textures, as mentioned above, but also because the state of the superfluid inside the micro-junction may not be precisely accounted for, as discussed in the next Section.

E. The peculiarities of the A-phase

The A-phase takes over from the B-phase at the superfluid transition temperature above a pressure of 21.2 bars because of strong coupling effects resulting from atomic localisation following the increase in density. Part of the enhanced interactions is mediated by spin-spin exchange, the so-called paramagnons. Because of these effects, the A-phase condensate only consists of \( S_z = +1 \) and -1 pairs, \(|\uparrow\downarrow\rangle\) and \(|\downarrow\downarrow\rangle\), and the energy gap above the Fermi surface \( \Delta(k) \) is strongly anisotropic while retaining the \( L = 1 \) symmetry: it vanishes at a node in the direction of \( \mathbf{\hat{I}} \), the orbital quantisation axis. As for its spin part, the A-phase behaves in some respect as an antiferromagnet with a spin quantisation axis \( \mathbf{\hat{d}} \). Its stability with respect to the B-phase is enhanced by an external magnetic field.

The A-phase order parameter in zero magnetic field is expressed in terms of the three unit vectors, the spin quantisation axis \( \mathbf{\hat{d}} \), and the orthonormal vectors \( \mathbf{\hat{m}} \) and \( \mathbf{\hat{n}} \) forming a triad with \( \mathbf{\hat{I}} \), the direction of the orbital angular momentum of the pairs. It is written in tensorial notations as:

\[
A_{\mu i} = \Delta A_{\mu i}(n\mathbf{\hat{m}}_i + i n\mathbf{\hat{n}}_i) .
\]

In \( {}^4 \text{He} \), the Bose order parameter is a simple complex number and the phase comes in quite naturally as it does for the BCS order parameter in s-wave superconductors, for ultra-cold atoms, and for the B-phase order parameter, Eq. 68, as discussed above.

No single phase factor appears spontaneously in expression (70) for the A-phase order parameter. It is still possible to invoke the fact that the single-particle wave function in the condensate possesses an indeterminate phase and that this phase goes over to the macroscopic Bose order parameter, which inherits of a global \( U(1) \) phase rotation broken symmetry.
FIG. 29 Staircase patterns in the A-phase at 28.4 bars and $T = 0.92T_c (T_c = 2.417 \text{ mK})$ (Avenel and Varoquaux, 1989). The left and right panels show the patterns slightly above and below the resonator resonance at $\omega_m = 1.783 \text{ Hz}$, as observed (upper curves) and as computed (lower curve) with the Deaver-Pierce model.

But this is not the whole story: a rotation of the triad $\hat{l}, \hat{m}, \hat{n}$ about the angular momentum directrix $\hat{l}$ by angle $\gamma$ also contributes an overall phase factor to the A-phase order parameter. This can be seen readily by considering the complex plane perpendicular to $\hat{l}$ containing the complex vector $\hat{m} + i \hat{n}$ that appears in Eq. (70): a rotation by angle $\gamma$ in the complex plane transforms this complex vector into $\exp(-i\gamma)(\hat{m} + i \hat{n})$. The Galilean invariance argument as used to derive the two-fluid model for $^4\text{He}$ leads, when everything is told, to the following expression for the velocity of superfluid mass transport:

$$v_s = -\frac{\hbar}{2m_3} (\nabla \gamma + \cos \beta \nabla \alpha). \tag{71}$$

The Euler angles $\alpha, \beta, \gamma$ fix the orientation of the orbital triad $\hat{l}, \hat{n}, \hat{m}$ in the chosen reference frame, $\gamma$ expressing a rotation about $\hat{l}$ as already mentioned.

Two independent phase gradients appear in Eq. (71). In one, $-\nabla \gamma$, the angle plays the role of the usual phase $\varphi$. This feature arises because the $U(1)$ phase rotation broken symmetry. The other stems from the bending of

heating of the vortex cores results in an energy flux carried by non-equilibrium quasiparticles and in a dissipation mechanism that can operate even at zero temperature.

The first contribution to the right hand side of Eq. (72) is recognised as the quantised velocity circulation around line singularities, as found in superfluid $^4\text{He}$, the second is expressed (Ho, 1978) as the area circumscribed on the unit sphere by unit vector $\hat{l}$ when carrying the loop integral along contour $\Gamma$. This last contribution is nil in the trivial case where $\hat{l}$ keeps pointing in a fixed direction. There exist other non-trivial cases with $\sigma(\Delta) = 0$ as discussed by Ho (1978), but, in general, this contribution is non-zero and the velocity circulation is non-quantised.

The interplay between superflow, vortices and textures of the order parameter becomes quite complex.

103 For a pointed but still gentle introduction to the intricacies of the A-
In particular, the A-phase persistent superflow can be relaxed by textural motion alone without the creation of topological singularities of the order parameter such as vortices. However, if large-scale motion of the texture is suppressed, persistent currents can be stabilised, as shown by the experiments of Gammel et al. [1985] designed to demonstrate the existence of such currents in the annular space of a torsional oscillator packed with 25 \( \mu \) silicon carbide powder. The effect of this confinement is to immobilise the \( \hat{I} \) vector. The (small) supercurrent was detected indirectly through its effect on the damping of the small-amplitude of the torsional oscillator. This crafty experiment showed that the A-phase possesses, if to a less convincing extent than the B-phase, the distinctive attribute of dissipationless flow.

The phase slippage concept can also be extended to the A-phase, as proposed by Anderson and Toulouse [1977] and Josephson-type experiments can be contemplated, with some uncertainty regarding their outcome because of the lack of quantisation of the velocity circulation, expressed by Eq. (71), and also because of the large dissipation associated with the motion of the order parameter gap in the direction of \( \hat{I} \), in which it has nodes.

These experiments were attempted by Avenel and Varoquaux [1989] using the same flexible wall Helmholtz resonator as for their B-phase experiments. Staircase patterns, shown in Fig. 29, at a temperature of 0.92\( T_c \) were observed in the A-phase both close to the superfluid transition temperature, where they revealed a rather non-ideal current-phase relation and further down in temperature where new features occurred, shown in Fig. 30.

The patterns in Fig. 29, obtained in the A-phase at \( T = 0.92 T_c \) at frequencies slightly above and below the resonance frequency of the flexible wall resonator (1.783 Hz) are quite well defined but differ markedly: dispersive effects are large, the resonance sharply peaked and dissipation low. Numerical simulations of the resonator response have been carried out using the Deaver-Pierce model, Eq. (67), and are shown below the experimental data curves. The resonance quality factor is high, \( Q = 80 \), the phase slips are hysteretic, with a non-ideality parameter \( \alpha = 5 \), and therefore dissipative.

Given this observation that the A-phase phase-slip pattern seemed to be following the ubiquitous Deaver-Pierce model at 0.92 \( T_c \), it could expected to become more and more ideal when raising the temperature closer to \( T_c \). This trend could unfortunately not be ascertained in these experiments because the operating frequencies, which decrease as \( \rho_s/\rho \), become too low and the useful signal gets lost in the background 1/f mechanical noise of the detection device.

Further down in temperature, an instructive direct comparison between the A-phase response and that of the B-phase at high pressure could be carried out by taking advantage of the following circumstance: at 28.4 bar, the A to B transition occurs at about 0.81 \( T_c \); it is signalled by a sudden drop in resonance frequency caused by a drop in superfluid density accompanying the first order transition. As the A-phase can be supercooled into the domain of stability of the B-phase, both phases can be studied at comparable frequencies.

The outcome of this comparison is shown in Fig. 30 which shows a remarkable similarity between the two phases. This observation reveals that the response of the superfluid in the weak link depends little on whether the bulk of the liquid is in the A or B phase. Thus, as discussed by Avenel and Varoquaux [1989], the observed behaviour inside the micro-slit correspond well to the situation described by Kurkijarvi [1988] who finds that the current-phase relations for the A and B phases differ only little. It may also happen that the state of the superfluid in the weak link remains the same irrespective of the state in the bulk. It has been predicted by Li and Ho [1988] and Fetter and Ullah [1988] that the A-polar phase would be favoured by the depletion of some of the components of the A-phase order parameter close to the aperture walls.

The sub-gap structure, shown in Fig. 30, which develops for resonance amplitudes below the critical threshold at which dissipative phase slips start to occur, is quite intriguing. It is interpreted by the authors as arising from possible (aniso)-textural effects inducing solid friction. It could also possibly be revealing the existence of sub-gap resonant levels.

These experiments establish very clearly that phase slippage takes place in the A phase, that persistent currents can be trapped in the loop threading the double-hole resonator, and that the velocity circulation along these trapped currents changes by multiples of the quantum of circulation in the same manner as in the B phase: it so turns out, as was the case in the persistent current experiments by Gammel et al. [1985], that the \( \hat{I} \)-texture is sufficiently well pinned in the regions where \( v_s \) is significant.
F. $^4$He close to the $\lambda$–point

The existence of Josephson-like effects was thus clearly established in superfluid $^3$He for the dc–effect and in both $^3$He and $^4$He superfluids for the ac–effect by Year 2000. The remaining question was the possible observation of a quasi-sinusoidal current-phase relation in $^4$He. The minuteness of the coherence length in $^4$He makes the fabrication of a suitable weak link a tall order except in the immediate vicinity of the $\lambda$-point, where it diverges as $\xi = \xi_0(1 - T/T_\lambda)^{-2/3}$ with $\xi_0 \sim 1$ to 2 Å (Langer and Reppy, 1970): superfluidity is killed by thermal fluctuations. Would the hydrodynamic Josephson effects not be even more readily washed out by the same token?

This concern was expressed by Zimmermann, Jr. (1987) whose argument runs approximately as follows. The Josephson coupling energy is obtained from the Josephson current by integration of Eq.(65) with respect to $(\hbar/m_4) \delta \varphi$. Its maximum value is therefore $(\hbar/m_4) E_J$:

$$ E_J = \left( \frac{\hbar}{m_4} \right)^2 \frac{s_b}{b_b} \frac{\rho_s}{\sinh(l_b/b_b)} . $$

(73)

In the weak coupling limit for which Eq.(65) holds, the penetration length is smaller than the length of the barrier, $b_b \leq l_b$: the wavefunction is strongly depleted within the barrier.

Making use of the scaling relation between $\rho_s(T)$ and $\xi(T)$ (Josephson, 1966),

$$ \rho_s \xi = (m_4/\hbar)^2 k_B T , $$

(74)

and since $\sinh(l_b/b_b) > l_b/b_b$, it stems from Eq.(73) that

$$ E_J \leq \frac{s_b}{b_b} \frac{k_B T}{\xi(T)} . $$

(75)

For the round pinhole of diameter $d$ considered by Zimmermann, Jr., $s_b = \pi d^2/4$ and $l_b > b_b$, the hydraulic length being $\pi d/4$ for a circular orifice, so that $E_J < k_B T d/\xi(T)$. Zimmermann concludes from this upper bound for the Josephson energy that, as $\xi(T)$ diverges upon approaching $T_\lambda$ from below, the Josephson coupling energy will end up being less than the thermal energy and that the Josephson dc-effect will be washed out by thermal fluctuations.

Similar concerns were spelled out by Ullah and Fetters (1989) for their calculations of weak link properties in $^3$He-B in the Ginzburg-Landau regime: “We do not address the important problem of thermal fluctuations destroying the superfluidity in the very small volume of the weak link. To our knowledge, there is no reliable, quantitative theory of the stability of the superfluid phase is severely confined geometries. We believe that this question can be convincingly

104 see Halperin et al. (1976) Hohenberg and Halperin (1977) Anderson (1966b), p.305 Anderson (1964), p. 120
answered only by experiment. ". This remark is even more relevant for superfluid $^4$He close to $T_\lambda$.

The first hint of a successful experimental observation was reported by Sukhatme et al. (2001) who used an array weak link of 24 micro-slits $3 \mu m \times 0.17 \mu m$ about 10 $\mu m$ apart in a 0.15 $\mu m$ thick membrane. and whose findings are summarised in Fig. 31 at 3.72 mK below $T_\lambda$, (the bottom curve in the figure, the scale of which is shrunk), the critical velocity is well-marked, as well as the staircase steps, indicating a dissipative phase slippage process. It thus seems that a phase-slip regime has been reached, as well as the staircase patterns of $^4$He close to the superfluid transition temperature $T_\lambda = 2.17$ K.

This conclusion raises questions, and possibly, some eyebrows as well:

- Would, for some reason, Zimmermann’s argument be invalid?
- How come that dissipative phase slippage, the mechanism for which seems to rely on the nucleation of a single vortex and its crossing of all streamlines of the superfluid flow through a single micro-slit as described in §VI.E, also operates for an extended array of them?

Zimmermann’s original argument, outlined above, was applied to a single round hole. Sukhatme et al. (2001)’s experiments involved 24 slits yielding an estimated enhancement factor of 500 in the superflow passage area $s_b$ that appears in Eq. (75), provided that the supercurrents in the apertures effectively sum up. The overall Josephson energy is increased by the same factor and the disruptive effect of thermal fluctuations is pushed back much closer to $T_\lambda$. This line of reasoning was pursued by Chui et al. (2003), but its soundness depends on the answer to the second question, which turns out to be trickier.

The Berkeley group carried a number of studies with aperture arrays that were composed of 65x65 = 4225 round pinholes, nominally 90 $\mu m$ in diameter, located on a square lattice $3 \mu m$ apart, micro-fabricated on a 50 $\mu m$ thick silicon nitride membrane. Two phase-slippage regimes were identified when the temperature was lowered below $T_\lambda$, as already observed by Sukhatme et al. (2001). Firstly, a reversible (non-dissipative) Josephson regime is observed at $\sim$ 50 to 100 $\mu K$ below $T_\lambda$, and slightly further down in temperature by Sato et al. (2006). In this regime, the phase-slips occur in a fully synchronous manner. Between approximately 0.3 to 15 mK below $T_\lambda$, a transition toward a dissipative phase-slips regime sets in as the synchronisation between the apertures gets lost. Further below $T_\lambda$, the phase-slip regime becomes asynchronous. The amplitude of the resultant phase slippage signal from the array does not sum up to what it should be. Also, it exhibits large slips and collapses somewhat similar to those described in §VII.B for a single orifice (but see below).

It is clear that inhomogeneities in aperture sizes and surface properties, the edge effects at the periphery of the array, and local critical fluctuations introduce a spread in the values of the critical current in the different apertures. Phase slips occur at different times during resonator motion. The summation of the currents through the various apertures, as attempted in the numerical simulations of

107 Approaching $T_\lambda$ from below, $\rho_s$ is known to vanish as $Langer$ and $Reppy$ (1970)

$$\rho_s \approx 2.4 \rho_\lambda (1 - T/T_\lambda)^{2/3},$$

where $\rho_s$ is the density at the $\lambda$ transition, 0.1459 g/cm$^3$. From relation (74), the temperature-dependent coherence length becomes

$$\xi(T) = 0.358/(1 - T/T_\lambda)^{2/3} \text{ in nm.}$$

108 Hoskinson et al. (2006b); Sato et al. (2006) see also Narayana and Sato (2010, 2011); Sato et al. (2008).
arrays of superfluid Josephson junctions by Avenel et al. (1998b), Pekker et al. (2007), Sato et al. (2007) has to be exercised with care.

It is generally agreed upon that the Josephson currents in the micro-apertures are small and perturb little the bulk quantum phases on both sides of the membrane carrying the weak link array. Phases are well defined below $T_\lambda$ – for instance, the quantisation of circulation is enforced – and so should their difference. As argued by Chui et al. (2003), this then would appear to leave only one degree of freedom to undergo fluctuations, with a thermal energy of $k_B T/2$ to be shared among the $N$ apertures of the array. The effect of fluctuations in each individual aperture is effectively quenched on taking the average over the whole array.

Another intriguing feature of aperture array dynamics is the large span in $T_\lambda - T$ over which the synchronous phase-slippage regime subsides both very close to the $\lambda$-point where quantum coherence should end up being killed by thermal fluctuations, and quite a way below it where it should be randomised by array imperfections. In other words, the robustness of the coherence effect mentioned above against dephasing by the imperfections of the environment appears quite remarkable. Perron et al. (2013) have pointed out that the superfluid onset in the micro-slits used by Sukhatme et al. (2001) is expected to be depressed by size effects to $T_\lambda - T_c \approx 430 \mu$K whereas the Josephson effect could be tracked to as close as $28\mu$K below $T_\lambda$. Similarly, for the pinholes used by Sato et al. (2006), $T_\lambda - T_c \approx 2.3$ mK while the Josephson effect could be tracked up to possibly 0.5 mK from $T_\lambda$. As concluded by Perron et al. (2013), “in both experiments one obtains superflow in a temperature region where the helium in the isolated weak links should be normal. Both of these experiments are thus relying on proximity effects, due to the surrounding bulk liquid, to maintain a non-zero order parameter in the weak links”.

Perron et al. (2013) draw their conclusions from studies of the interconnection of an array of $(2 \mu m)^2$ micro-pools linked through the film of superfluid $^4$He. They have found from measurements of the specific heat and the superfluid fraction in the vicinity of $T_\lambda$ that correlation effects are still effective at distances up to 100 times $\xi(T, L)$, the finite-size correlation length, suitably renormalised for confinement over distances of the order of the characteristic length $L$. This unexpectedly large extent of the correlation observed between micro-pools can be likened to the robustness of the coherent behaviour of Josephson junction arrays close to $T_\lambda$.

Pekker et al. (2007), besides their numerical studies, also treated the problem of aperture current summation in an irregular array as an order-disorder transition in a mean-field approximation approach. They introduce a distribution of aperture critical currents and an effective inter-aperture coupling parameter. They report qualitative agreement with the experiments of Sato et al. (2006) including “system-wide avalanches”.

The cross-aperture coupling may be seen to arise from a simple classical hydrodynamics scheme, which is an extension to arrays of the putative mechanism for single-aperture large slips discussed in §VII.B. Suppose that, during the surge of the superflow through the array, a (quantum) phase slip occurs early in one of the apertures, releasing a vortex half-ring that starts drifting (classically) sideways along the membrane supporting the array. Soon, this vortex half-ring runs into the flow lines emerging from a nearby aperture, gaining energy from it to proceed in its course and, possibly, triggering the nucleation of another vortex half-ring, and so on. This multiplication process may die by itself at the ebb of the flow, leaving only part of the array with slips by $2\pi$ or higher multiples. Or, if it overcomes the friction on the normal component, it may swell to the system-wide avalanches (Pekker et al. 2007) observed by Sato et al. (2006). These avalanches are thus intrinsic to aperture array dynamics and distinct from flow collapses in single apertures discussed in §VII.B.

Thus, even more so than in the case of single apertures, “macroscopic quantum coherence” manifests itself in the aperture array in a dual manner. First, the condensate acts as an ideal Euler fluid, maintaining orderly streamlines throughout the superflow in accordance to the Kelvin-Helmholtz theorem. Then, when a non-adiabatic process takes place, violating velocity circulation conservation, it does so in a quantum manner, allowing the phase to change by $2\pi$ multiples, for instance by the nucleation of a quantised vortex or by the current source or sink provided by Josephson tunnelling through a thin barrier of normal fluid.

IX. CONCLUDING COMMENTS

A. Matter waves and superfluid interferometry

The single-hole or two-hole hydromechanical resonators used in the phase slippage experiments described above have been presented so far as the analogues of rf or dc superconducting quantum devices (SQUIDs), a useful analogy to help understand the way they operate. Another analogy is used in this Section to illustrate the concept of coherent matter fields, or matter waves, introduced for superfluid helium by Anderson in 1965. These devices can also be considered as the likes of optical Sagnac interferometers; they can be used to measure absolute rotations with very high sensitivity.

Consider again a pool of superfluid in which the superflow velocity, expressed in the inertial frame of reference, is $v_s$, defined as proportional to the gradient of the superfluid order parameter phase: $v_s = (\hbar/m_s) V \varphi$. The quantity $m_s$ is the bare atomic mass $m_s$ in $^4$He or the Cooper pair mass $2m_3$ in $^3$He.
FIG. 32  Left panel: (a) Sketch of the $^4$He interferometer used by Hoskinson et al. [2006b]. The unshaded regions are filled with superfluid $^4$He. The upper chamber is closed on the top by a flexible metallised diaphragm which serves both as a microphone to detect the resonant oscillations and also as a pressure pump to maintain chemical potential differences across the arrays. The crosses indicate the position of the two aperture arrays that interrupt the superfluid channel enclosing the sense area. (b) Diagram of the interferometer equivalent circuit showing the analogy with the electrodynamic dc-SQUID. Basically, there are two superposed currents flowing through the Josephson junctions, one corresponding linked to the rotation flux, Eq.(77), the other being the read-out current from the diaphragm pump.

Right panel: Modulation of the amplitude of the diaphragm displacement on resonance as a function of the rotation flux picked up by the superfluid loop enclosing the sense area – see §IX.A for a detailed description of operation. In these experiments as well as those of Avenel and Varoquaux [1996] (Avenel et al. [1997]) and Mukharsky et al. [2004], the latter carried out in superfluid $^3$He, the Earth rotation is the source of rotation flux; the flux magnitude is varied by reorienting the interferometer with respect to the North-South axis as shown in Fig.(34). The measured data is shown by the symbols; the solid lines are fits of the data to the equation of motion of the interferometer as described by Hoskinson et al. [2006b]. The modulation curves were taken at temperatures $T = 12, 7.0, 4.0, 3.0, 2.0, 1.5, 0.9, 0.6, 0.4, \text{ and } 0.3 \text{ mK}$ from top to bottom. This temperature span covers the coherent Josephson regime in the array discussed in §VIII.F.

The law of inertia applies to this pool of helium in exactly the same manner as for Foucault’s pendulum, the motion of which becomes independent of that of the building that houses it as soon as it is set free to move. The circulation of the velocity is quantised in the inertial frame – the reference frame fixed with respect to the distant stars – along any closed contour $\Gamma$ located entirely in the superfluid:

$$\oint_\Gamma \mathbf{v_s} \cdot d\mathbf{l} = \frac{\hbar}{m_a} \oint_\Gamma \nabla \phi \cdot d\mathbf{l} = n \kappa_a ,$$

where $\kappa_a = 2\pi \hbar/2m_3$ for $^3$He, 3/2 times that quantity for $^4$He, and $n$ is an integer.

If the cryostat housing the pool is set into rotation with rotation vector $\Omega$, the velocity transforms in the new frame according to $\mathbf{v_s}' = \mathbf{v_s} - \Omega \times \mathbf{r}$ and the quantisation of circulation condition now reads

$$\oint_\Gamma \mathbf{v_s}' \cdot d\mathbf{l} = n \kappa_a - \oint_\Gamma \mathbf{r} \cdot d\mathbf{l} = n \kappa_a - 2 \Omega \cdot S_\Gamma , \quad (76)$$

where $S_\Gamma$ is the geometrical (oriented) area of the closed superfluid contour. For an actual conduit with finite cross section such as the one pictured in Figs.[32] and [34] there is a variety of choices for the contour $\Gamma$. The mean circulation of the velocity results from a suitable average over the various distinct superfluid contours threading the conduit. Taking the average of Eq.(76) over all the streamlines threading the conduit amidst stray thermal currents, pinned vortices, and textures, weighted according to the (infinitesimal) mass current that they carry, leads to (Avenel et al. [1997]):

$$\left\langle \oint_\Gamma \mathbf{v_s}' \cdot d\mathbf{l} \right\rangle = n \kappa_a + \kappa_b - 2 \Omega \cdot \langle S \rangle , \quad (77)$$

where $\langle S \rangle$ is the average of the contour areas over the conduit. The average of the quanta of circulation carried by the various streamlines, $\langle n \rangle \kappa_a$ has been written as $n \kappa_a + \kappa_b$ to separate explicitly the non-quantised phase bias $\delta \phi_b = 2\pi \kappa_b/\kappa_a$ from the strictly quantised contribu-
tion $2\pi n$.

The last term to the right of Eq. (7) also amounts to a non-quantised contribution to the phase bias, which varies with the flux of the rotation vector $\Omega$ through $\langle S \rangle$: the measurement of the corresponding phase difference with the interferometers depicted in Figs. 32 and 34.

$d\delta \phi = (m_\phi/\hbar)2\Omega \cdot \langle S \rangle$, gives access to the rotation velocity. Alternatively, changing the orientation with respect to the North axis of the superfluid loop picks up more or less of the rotation flux due to the Earth rotation $\Omega_\oplus$. This provides a way to couple a known phase difference to the weak link, providing the experimenter with a “gauge wheel” to steer the phase.

Exploiting the properties of superfluids to detect very slow rotations has been proposed even before the discovery of the Josephson effects in superfluids, understandably with some lack of accuracy as to how the experiment could be conducted. Cerdonio and Vitale clarified in 1984 the way in which inertial and gravitational fields could be detected with superfluid $^4$He analogues of the rf-SQUID [Bonaldi et al. 1990; Cerdonio and Vitale 1984]. A number of authors followed suit afterwards for superfluid $^4$He and $^3$He [Hess 1992; Packard and Vitale 1992; Varoquaux et al. 1992], and for the Bose-Einstein condensed gases (Stringari 2001).

Detailed schemes for the actual implementation of superfluid $^4$He gyros have been worked out with the help of numerical simulations [Avenel et al. 1994] and from the analysis of the operation of existing double-hole hydro-mechanical resonators. The first measurement of $\Omega_\oplus$ with a superfluid device was performed using a resonator operating in hysteretic mode in superfluid $^4$He with a rotation-sensing loop of 4.0 cm$^2$ by Avenel and Varoquaux (1996). Soon after, the Berkeley group reported the observation of the effect of the rotation of the Earth with a similar device operated in the staircase mode, in much the same way as conventional rf-SQUID magnetometers.

It should be mentioned, for the record, that early attempts to measure $\Omega_\oplus$ led to disappointing results to the dismay of experimenters [Avenel et al. 1998a; Schwab et al. 1996a,b]. It was soon realised that the currents in the bulk of the cell outside the resonator [Avenel et al. 1998a; Schwab et al. 1998], which are simply caused by the re-orientation of the cryostat were interfering with, and possibly overwhelming, the relatively weak $\Omega_\oplus$-induced Sagnac current in the pick-up loop. The influence of these stray currents can be made negligible by a proper design of the cell. A sheath on the port connecting the resonator to the main body of the cell was used to that effect by Avenel and Varoquaux (1996) (Avenel et al. 1997). The absence of such a decoupling device between the Sagnac current in the pick-up loop and the stray currents around the cell could cause uncontrolled inaccuracies of up to 30 % [Schwab et al. 1998, 1997, 1996a].

The potentialities of superfluid gyros as extremely sensitive and stable rotation sensors, able to track, e.g., General Relativity effects, have been considered by Avenel et al. (1998a), Chui and Penanen (2005), and Sato and Packard (2012). It appears that these gyros can compete with the most advanced rotation sensors, in particular because they are inherently driftless at very low temperatures.

These gyrometric devices are the direct superfluid analogues of the well-known Sagnac optical interferometers, as can be seen by inspection of the sketch of the latter in Fig. 33 and of the superfluid device in Fig. 34: the light source provides the incident light beam, the flexible membrane the supercurrent; counter-rotating waves travel along the square optical path, and along the coiled capillary, for the co-rotating part; the waves interfere in the beam splitter in the optical case, in the Josephson weak link in the superfluid case. The interferometer shown in Fig. 32 is closer to a Mach-Zehnder interferometer but the analogy goes along in the same way.

But is this reasoning by analogy, or the display of clear fringe patterns such as those shown in Fig. 32, sufficient proof that the Sagnac effect is involved in the operation of these Sagnac-like interferometers?

Apparently not; superfluid gyros are still sometimes mistaken for purely inertial devices such as spinning tops, as discussed by Varoquaux and Varoquaux (2008).

Clearly, the superfluid in a rotating bucket experiment is a dense medium. It can be weighed on a scale. For large enough rotation velocities, when enough vortex
Consider how these clocks can be synchronised, first when they are infinitely close to one another. The spacetime metric is characterised in the conventional notation by \( -\text{d}s^2 = g_{00}\text{d}(\text{x}^0)^2 + 2g_{0i}\text{d}\text{x}^i\text{d}x^i + g_{ij}\text{d}(x^i)\text{d}(x^j) \). The infinitesimal time interval \( \text{d}t \) between two nearly simultaneous events taking place at this given location in space is such that \( \text{d}s^2 = g_{00}\text{d}(\text{x}^0)^2 = -c^2\text{d}t^2 \), \( c \) being the velocity of light. If the clocks are now separated in space by an infinitesimal amount \( \text{d}x \) and the two events taken, e.g. at location \( A \), as the ticking of clock \( A \) for one and the signal transmitted by clock \( B \) of its ticking a small distance away for the second, the two ticks occur with a time lag given by \( \text{d}t = -g_{00}\text{d}x^i / g_{00} \), the repeated summation being on the space coordinates.

If clock \( B \) is now transported over a finite path \( \Gamma \) closing on itself in a frame rotating with velocity \( \Omega \), the total time shift results from an integration along path \( \Gamma \):

\[
\Delta t = \int_\Gamma \frac{g_{00}}{c^2} \text{d}\text{x}^i = \int_\Gamma \frac{\Omega \times \text{r} \cdot \text{dr}}{c^2} \Omega = \frac{2}{c^2} \Omega \cdot \text{S} ,
\]

\( \text{S} \) being the vector area subtended by the loop \( \Gamma \).

Time delay \( \Delta t \) between the reading of the transported clock and that of the clock standing still on the rotating platform lies at the root of the Sagnac effect. As it depends on the rotation velocity and the actual path \( \Gamma \), absolute clock synchronisation cannot be achieved. Sagnac corrections, Eq. (78), must be performed as done routinely for Global Positioning Systems (Ashby, 2004).

So much with clocks. For helium, a Lorentz invariant two-fluid model can be built over the usual Landau superfluid hydrodynamics as done by Carter and Khalatnikov (1992). The invariant velocity circulation, the generalisation of Eq. (17), reads

\[
\int_\Sigma [v'_0 \text{d}x^i + v'_i \text{d}x^i] = n\kappa ,
\]

where \( (v'_0, v'_i) \) is the four-velocity in the rotating frame \( (c^2 + v^2_n, -v^2_n) \). The normal fluid velocity \( v'_0 \) and the superfluid velocity \( v'_i \) are small compared to \( c \) so that the time-like component of the four-velocity reduces to \( c^2 \). The integration over \( \Sigma \) is an actual loop integral only for the space-like components. The corresponding world line is not closed because the time for synchronised clocks varies according to Eq. (78). Upon integration, Eq. (76) is recovered:

\[
\int_\Gamma v'_i \text{d}x^i = n\kappa + \int c^2 g_{00}\text{d}x^i / g_{00} \approx n\kappa - 2\Omega \cdot \text{S} ,
\]

\[\text{\tt 112} \text{Landau and Lifshitz (1971), §90.}\]

\[\text{\tt 113} \text{It may be worth recalling that this clock transportation experiment was actually performed by Hafele and Keating (1972) who boarded eastward and westward bound commercial jetliners taking as luggage a portable atomic clock.}\]

\[\text{\tt 114} \text{Also, Ho and Mermin (1980a).}\]
which establishes the link between superfluid physics and the relativistic clock approach: the true and honest Sagnac effect described by the transported clocks, Eq. (78) and the circulation quantisation condition in the rotating frame leading to Eq. (77) are one and the same.

Thus, Einstein-synchronised clocks provide the time standard by which phase differences can be kept track of in all the studied physical systems. As appropriately summarised by Greenberger (1983) for neutron interferometry experiments, “the phase shift (in the rotating interferometer) is seen to be caused by the different rates at which a clock ticks along each of the two beams”. The rate at which that clock ticks for helium depends on the chemical potential $\mu$, due to the molecular field of the condensate as shown by Beliaev, and on the Sagnac phase shift.

The helium Sagnac experiments illustrate convincingly the reality of matter wave interference in the superfluid heliums, a substantially massive coherent field. Coherent, as shown at great length here, means coherence of the quantum phase, giving a wave-like character to a bulky fluid. A few remarks on this follow.

B. Landau’s two fluids, ODLRO, and macrorealism

Anderson’s introductory words to the reprint of “Considerations on the Flow of Superfluid Helium” in his book of 1994 are the following: “I feel this is the clearest discussion of superfluidity available. Note that on many points this is contradictory or orthogonal to Landau orthodoxy as pronounced by Khalatnikov. Whether Landau would have agreed was never clarified because of his accident”. This remark poses the problem of the complementarity between the two-fluid model, the description of Bose condensation by the non-vanishing off-diagonal terms of the density matrix at long range, and the macroscopic wavefunction approach.

Landau’s legacy rests on two crucial aspects. Firstly, the two-fluid model has ruled once and for good on the separation between normal fluid and superfluid components, both for the thermodynamics and the hydrodynamics. As used in this review in its reduced form for incompressible flows, which treats both components as independent, it has allowed to basically disregard the normal component. There are obviously limits to this high-handed simplification, especially close to the $\lambda$-point, but it has provided the backbone of the simplified vortex dynamics of Sec. III and subsequent Sections on phase-slip processes.

The second pillar of Landau’s contributions to helium superflow is his criterion that many consider as the genuine intrinsic critical velocity in superfluids. This criterion rests on the sharpness of the excitation spectrum, which Landau implicitly postulated, but that was soon put on firm grounds by the work of Beliaev, Penrose, Onsager and others. The formal description of the Bose condensate correlations ended up in the concept of off-diagonal long range order (Yang, 1962) and a formal definition of the single wavefunction shared by the particles in the condensate. For not-too-complicated superfluids – namely $^4$He, the B-phase of $^3$He, BCS superconductors, and cold atoms BEC – this wavefunction, or order parameter, has a definite overall phase.

Anderson (1966a) has given dynamical variable status to this phase by extending its application to inhomogeneous situations. In Appendix A of his paper, “ODLRO vs macroscopic particle fields”, he states explicitly that “recognising that in principle the relative phase of any two (superfluid) systems may always be measured by a Josephson-type experiment, one immediately has a usable local description” (of those systems).

This local description has been put to good use. It has opened the way to a full understanding of the interaction of quantised vortices and superflow, put on firm classical hydrodynamics footing by Huggins (1970). Sonin (1995) and others have further expanded the vortex velocity field idea into a workable scheme for vortex dynamics, briefly recounted above. More importantly, it has bridged the gap between a predominantly “classical” two-fluid hydrodynamics and the more intimately quantum Josephson effects. The experimental observation of these effects, and in particular, the detailed way by which dissipative phase slippage, understood first as the nucleation and propagation of vortices, evolves into the purest brand of Josephson hydrodynamics effects, both for $^3$He and $^4$He superfluids, has brought fresh food for thought.

Detailed numerical simulations of vortex dynamics have been conducted by Schwarz and others, in particular for the problems of the vortex-in-an-aperture, or trapped on a thin wire, or else, pinned. Collision between vortices, and the resulting reconnection, their multiplication by vortex mills churning out fresh quantised vorticity, the formation of vortex tangles, superfluid turbulence, and the several ensuing critical velocities, are examples of the improved way of dealing with quantised vortices fostered for a great part by Anderson’s considerations.

Experimental observations have fully borne out over the years this central concept of a macroscopic quantum
phase governing the dynamical behaviour of the superfluid. At least in one given pool of superfluid. The idea that a given pool, or bucket, or droplet, of superfluid has its own phase has become so common place that the question of which value does this quantum phase take when no one has ever seen before seems a little pointless. As put by Anderson [1986]: “Do superfluids that have never seen each other have a well-defined relative phase”? Or, at an even more basic level, does an isolated droplet of superfluid have a phase? Before answering by a qualified “yes”, it may be useful to consider the new inputs on this question of the meaning, or reality, of the phase in Bose condensates that have emerged after 1996 from investigations in ultra-cold atomic systems. This is not the topic of this review, but interference experiments with BEC gases do bear on some aspects of it. Javanainen and Yoo [1996] studied by numerical simulations the setting up of an interference pattern between two condensates formed in separate cold atom traps are left to overlap with one another. They noted, by counting how non-interacting bosons were filling separate bins that the phases of the condensate wavefunctions were in no instance invoked to construct an interference process; Bose statistics sufficed. Castin and Dalibard [1997] tackled the same problem analytically. They studied the evolution of BE condensates, initially containing a given number of particles – that is, in a Fock state – when they are, at some instant, left to interact by leaking some atoms to a beam splitter. Two atom counters monitor the (+) and (-) output channels of the beam splitter. With only one condensate inputting atoms, the probability of detecting a second atom in – say – channel (+) once a first atom has been detected is 3/4 while that in the (-) channel is only 1/4. This well-known “bunching” tendency of bosons to crop together becomes rapidly overwhelming. The probability, after k counting events, to find all atoms in one channel and zero in the other, is given by

\[ P(k, 0) = \frac{1}{2^k} \frac{3}{4} \ldots \frac{2k - 1}{2k} , \]

which is much larger than the value 1/2^k for independent events. But, even more remarkably, this probability value for large k tends toward the average of the probabilities for the same sequence of events but obtained for a source of condensate atoms described as a “phase state” with a well defined phase while the initial Fock state had none. The value that this phase settles at is distributed randomly, namely

\[ \frac{1}{\pi} \int_{\pi/2}^{\pi} d\phi \cos^{2k} \phi = \frac{(2k)!}{(2^k k!)^2} \approx P(k, 0) + O(1/k) . \]

This particular process thus shows that the sequence of measurements brings a definite quantum phase to a state for which none existed to start with. When the dust has settled, one is left with a state containing N – k atoms and a phase known to order 1/k. No direct interaction between the atoms in the condensate has been assumed at any point; the gases involved are perfect gases. The effect described above purely originates from quantum-statistics.

Considering now two condensates A and B as for a real-life experiment, Castin and Dalibard [1997] proceed to show that the beams incident on the beam splitter from the combined Fock state |N_A, N_B⟩ again end up, after the sequence of measurements, being describable by the two fields |Ψ_A⟩ exp(iϕ_A) and |Ψ_B⟩ exp(iϕ_B) with explicit phases. The intensities of the two beams going out of the beam splitter build up as I_+ = 2|Ψ_A|^2 cos^2(δϕ/2) and I_- = 2|Ψ_A|^2 sin^2(δϕ/2): a measurable phase difference δϕ = ϕ_A – ϕ_B develops between the two condensates. This phase difference is an unpredictable random variable, which takes a different value for any realisation of the experiment. As concluded by Castin and Dalibard [1997], “the notion of phase-broken symmetry is therefore not indispensable in order to understand the beating of two condensates”.

This conclusion, which has gained wide acceptance, is beautifully illustrated by the experiments of Saba et al. [2005]. These authors dropped two condensates of like-species out of their traps and, during the course of their free fall and expansion, gently pushed with laser beams a trickle of atoms from one to the other. They dutifully observed the continuous emergence of a fringe pattern in a quintessential form, without beam splitters and interferometers, nor destruction of the condensate clouds, thus realising a nearly non-destructive measurement of their relative phase. These ultra-cold non-interacting gases certainly comply with the basic rules of quantum mechanics, as do photons in cavities for instance. Quantum statistical correlations between particles play the leading role. Superfluids differ in a number of respects.

The macroscopic wavefunction is defined on the premises that the particle number N and its possible variation ∆N are sufficiently large so that the uncertainty in ϕ, expressed by Eq. (12), ∆N ∆ϕ ∼ 1 is small in most instances. It is then neglected and the operators N and ϕ are “projected” on c-numbers. They have acquired a value once and for good: the phase “exists”.

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119 Such an experiment was successfully performed soon after by Andrews et al. [1997] and others.

120 See e.g. Horak and Barnett [1999], Nienhuis [2001].
[Leggett (1995)] first argues that, in order to attribute a well-defined meaning to an absolute phase, one has first to consider the relative phase of one bucket of superfluid with respect to another – presumably measured by performing a Josephson experiment. The various ways and constraints of such an experiment have been expounded in the previous Sections above: the “relative” phase between two weakly connected systems 1 and 2, \( \delta \varphi_{12} = \varphi_2 - \varphi_1 \), can be measured. Systems (1) and (2) are then separated and a third system brought in. System (3) is compared to (1), with the result \( \delta \varphi_{13} = \varphi_3 - \varphi_1 \). Is it possible to infer that the phase difference between (2) and (3) is \( \varphi_3 - \varphi_1 = \delta \varphi_{13} \)? If yes, then phases can be referred to a “standard” and acquire absolute meaning.

The not-so-trivial answer given by [Leggett (1995)] is “no” if the systems are let to settle to equilibrium with the environment and the two Josephson phase measurements are performed independently, but “yes” if they are done simultaneously. In other words, maintaining a superfluid phase standard across the various Standards Laboratories of the planet would require connecting them with a continuous superfluid duct. The phase information would have to be tapped from this standard at precisely the same time; a host of corrections such as that for local gravity or for the Sagnac effect – the Einstein clocks have to be synchronised – would have to be performed. If the phase readings are not simultaneous, the correlation between phase measurements between systems (1) and (2), and then (1) and (3) is upset by the act of measuring itself, with a part played by the environment; decoherence takes its toll and phases ultimately randomise [Sols, 1994; Zapata et al., 2003].

The situation of helium, where macroscopic quantum effects extend over lengths of tens of metres and more, and in which the Planck constant hides at the ångström scale provides a quite extreme example of macroscopic matter field. That the macroscopic field \( \Phi \) is a matter field. That the macroscopic field \( \Phi \) is

\[
\Phi = \frac{1}{\sqrt{2}} (\varphi_1 + \varphi_2)
\]

is a superposition of two different macroscopic states. However, the sort of coherence shown by ultra-cold atom condensates does not stand out readily. The macroscopic wavefunction cannot be described in terms of a linear superposition of two or more states with different macroscopic properties [Leggett (1980)].

The description of the superfluid dynamics in terms of the conjugate variables \( N \) and \( \varphi \) belongs as much to thermodynamics as to quantum mechanics. The correlation between atoms in the dense helium fluid relies more on their hard-core repulsion than on their boson statistics. Yet, the quantum interferences by quantum tunnelling in superfluid Josephson junctions, the quantum nucleation of vortices discussed in Sec VI.A, clearly reveal the importance of the latter. In these situations, the coarse-grained average loses part of the quantum information and some other procedure, more in line with the rules of quantum mechanics, and in particular, the principle of superposition, is probably in order. Would it be possible to envision experiments showing actual macroscopic quantum coherence as discussed by [Annett, 2003; Leggett, 2002]? The superfluid quantum phase would gain a dual acceptance, actual coherence in the superposition of different states at small scale on the one hand and, on the other, the “rigidity”, in the language of F. London, of the velocity potential of the ideal Euler fluid and the quantisation of the velocity circulation.

This “essay on criticism” of Anderson’s considerations is coming to a close. At this stage, but one hard conclusion can be drawn: many offshoots have already sprung and more are to come.

X. ACKNOWLEDGEMENT

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