DYNAMICAL ESTIMATION OF A NOISY INPUT IN A SYSTEM WITH A CAPUTO FRACTIONAL DERIVATIVE. THE CASE OF CONTINUOUS MEASUREMENTS OF A PART OF PHASE COORDINATES

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Abstract. The problem of estimating (reconstructing) an unknown input for a system of nonlinear differential equations with the Caputo fractional derivative is considered. Information on the position of the system is available for observations and only a part of system’s parameters can be measured. The case of measuring all phase coordinates is also presented. The measurements are continuous and the data obtained in them are noisy. The considered problem is ill-posed and, to solve it, we use the method of dynamic inversion. It is based on regularization methods and constructions of positional control theory. In particular, we use the Tikhonov regularization method also known as the smoothing functional method and the Krasovskii extremal aiming method. The approach to estimating an unknown input implies introducing an auxiliary system (a model) with an appropriate rule of forming a control. The proposed estimation algorithm gives approximations of an unknown input and is stable under informational noises and computational errors. As an example illustrating the elaborated technique, a biological model of human immunodeficiency virus disease is used for simulation. The simulation results demonstrate the importance of the approach to on-line estimating unobservable parameters in real processes.

1. Introduction. Problem formulation. Different statements of identification problems arise in applied research and often play a key role. For example, if, during the evolutionary genesis of the process, not all parameters and characteristics are available for observation, which is a fairly common situation in reality, then the missing information needs to be filled. This issue is widely represented in studies related to mathematical modeling in information technologies (image restoration), ecology (assessing the impact of pollution sources on the environment), medicine (the process of spreading and treating viral diseases in one organism and the population as a whole) and other areas. Let us note some of monographs on this topic, without detracting from the contributions of a wide range of other authors. The

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foundations of identification theory were laid in [25, 41], and applied algorithms for the estimation and identification of non-stationary objects and external influences were described in [2, 6, 12, 19, 29].

The most common are a posteriori statements of identification problems, which have a rich history of study and a significant variety of solution methods, see, for example, [18, 39]. The approach often used in applied research is based on the use of splines, finite-difference approximation with an appropriate grid, and other types of approximations. Algorithms for solving specific problems in most cases use a discrete partitioning of the time interval and measurements of the function values at nodes. In this context, the problem arises that an increase in the number of mesh points in the presence of errors does not refine the result, and also leads to a significant increase in computations, especially if the solution of auxiliary problems at the nodes is additionally required. Therefore, in order to neutralize the occurrence of such situations, it is reasonable to use adaptive (combining the feedback principle) algorithms, one of which will be presented in this paper.

The main motivation of the paper is expanding the scope of the dynamic inversion method discussed below to a more general class of dynamical systems described by fractional differential equations. Fractional analysis is currently one of the actively developing areas of mathematics. While a large number of studies is mostly theoretical in nature, the reason for this is the complexity of the physical interpretation of fractional derivatives, in contrast to the whole analogue, which has a transparent interpretation.

Fortunately, various scientists have filled this gap, see [7, 30, 33, 38] and bibliography therein. An increasing number of papers have been devoted to different control problems for fractional order systems, including related problems of optimal and positional feedback control [10, 23], and reconstruction problems [4].

Let us present the definitions necessary for the problem formulation.

**Definition 1.1.** [13, p. 69]. A fractional integral of order $\gamma$ of an arbitrary function $x \in L_1(T, \mathbb{R}_n)$ with origin at a point $\sigma$ is defined by the formula
\[
[I^\gamma x](t) = \frac{1}{\Gamma(\gamma)} \int_\sigma^t (t-s)^{\gamma-1} x(s) \, ds, \quad \gamma \in (0, 1), \quad t \in T := [\sigma, \theta], \quad \theta < +\infty,
\]
where $\Gamma(\cdot)$ is the Euler gamma function [13, p. 24].

**Definition 1.2.** [13, p. 91]. For a function $x: T \rightarrow \mathbb{R}_n$ and an arbitrary real number $\gamma \in (0, 1)$, the expression
\[
[D^\gamma x](t) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dt} \int_\sigma^t (t-s)^{-\gamma} x(s) \, ds = \frac{d}{dt} [I^{1-\gamma} x](t),
\]
is called the Riemann–Liouville fractional derivative [13, p. 70], and the expression
\[
[D^* x](t) = \frac{d}{dt} [I^{1-\gamma}(x-x(\sigma))](t),
\]
is called the Caputo fractional derivative [13, p. 91].

In the paper, we consider a dynamical controlled system of nonlinear fractional order differential equations of the form
\[
[D^\gamma x](t) = f(t, x(t)) + Bu(t),
\]
where $x \in \mathbb{R}_n$ is the phase vector, $u(t) \in \mathbb{R}_m$ is an unknown external impact, which may be interpreted as a disturbance, $B$ is an $n \times m$-dimensional matrix, $f$ is an
$n \times n$-dimensional matrix function, which is continuous with respect to $t$. System (1) is operating on a finite time interval $T$.

The trajectory $x(\cdot)$ of system (1) depends on the time-varying input $u(\cdot)$. Its realization as well as $u(\cdot)$ are not known in advance. However, simultaneously with the motion of system (1), $\kappa_1 \leq n$ coordinates are available for measuring with some error. As a result of these measurements, we obtain vectors $\xi^h(t) \in \mathbb{R}^{\kappa_1}$ satisfying the relations

$$|\xi^h(t) - \bar{x}(t)|_{\kappa_1} \leq h, \quad t \in T,$$

where $h \in (0, 1)$ is the measurement error, $\bar{x}(t)$ is the vector composed of the first $\kappa_1$ coordinates of the vector $x(t)$, and $|\cdot|_n$ is the norm in the Euclidean space $\mathbb{R}^n$. The initial state of system (1) is assumed to be known exactly and

$$x(\sigma) = x_\sigma. \quad (3)$$

The problem considered in the paper is to find a special estimation procedure that works according to the feedback principle; the output of this procedure should approximate an unknown input in the system and should be formed using the results of inaccurate continuous measurements of available phase coordinates. The approximation constructed in this way must be insensitive to informational noises and computational errors.

Let us emphasize a number of distinctive features of the problem under consideration. First, the information about the entire position of the system or a part of the phase coordinates is available for the observation exactly during the process, that is, on-line. This feature follows from the property of non-anticipation of control structures for dynamical systems, which originates from Newtonian determinism. In other words, in practice, the behavior of a studied object or process is unknown a priori. Secondly, the fact that the same trajectory can be generated by different inputs, together with the presence of informational noises in the observation channel and computational errors, makes it impossible to find the input precisely. These moments explain the ill-posedness of the problem in the sense of Hadamard and require the use of special solving methods.

Therefore, in this paper, we adhere to an approach based on the concept of inverse problems for dynamical systems, closely related to the theory of ill-posed problems [11, 39], the methods of solving which are successfully implemented in the form of computational algorithms. The theoretical foundations of the developed approach were laid in [16]. The approach consists in a combination of the Krasovskii extremal aiming principle from the theory of guaranteed control [15] with one or two (may be different) regularization methods. Among which the most common are the classical Tikhonov regularization method (with a smoothing functional) and the residual method.

This approach has become widespread in [3, 21, 26, 27, 31], where systems with delay, with distributed parameters and stochastic systems have been considered. General trends in the development of this approach are reflected in review [22]. In the problem of reconstructing the right-hand side with a priori constraints for fractional-order systems, this method was used in [34], when measuring a part of the coordinates, in [35], and was also successfully applied to the approximate calculation of the Caputo fractional derivative for discrete measurements of the phase vector in [36] and for continuous measurements in [37]. In this paper, we consider the case of continuous measurements of a part of the coordinates, whereas discrete measurements are considered in the most of papers mentioned above. These
aspects lead to the use of a special technique to establish the procedure of estimating an unknown input in the present paper.

Research in the field of dynamical systems has historically been divided into two directions: systems with discrete time and systems with continuous time. Each of these assumptions has both positive and negative sides see, e.g., [40]. Systems with continuous time, methods for solving identification problems for them and their differences from systems with discrete time are reviewed in [42].

The on-line deconvolution algorithms described in [8, 28] have constructions similar to the estimation procedure proposed in this paper. They are based on a modification of the Lavrent’ev regularization method and are used in input identification and disturbance reduction problems. An approach based on ellipsoidal estimates is being actively developed, see, for example, [14, 17] and bibliography therein.

The structure of the paper is as follows. In Section 2, the case of measuring all coordinates of the state vector is considered. In this case, the description of an estimating procedure is given in Section 2.1, and its error rate, in Section 2.2. The more general case of measuring a part of coordinates of the state vector is considered in Section 3. Estimates of input approximation are obtained in Sections 3.1 and 3.2. The results of numerical simulation are presented in Section 4, where we use the biological model of HIV (human immunodeficiency virus) disease for illustrating the operation of the proposed technique and its possible application. Finally, we make conclusions in Section 5.

2. The case of measurement of all coordinates. First, we consider the basic situation corresponding to the measurement of all phase coordinates of system (1). Thus, the dimension of the vectors $\xi^h(t)$ is equal to $n$, i.e., $\kappa_1 = n$, and then relation (2) takes the form

$$|\xi^h(t) - x(t)|_n \leq h, \quad t \in T.$$  (4)

We assume that the function $f$ satisfies the Lipschitz condition with constant $L > 0$:

$$|f(t, x_1) - f(t, x_2)|_n \leq L|x_1 - x_2|_n \quad \forall t \in T, \quad \forall x_1, x_2 \in \mathbb{R}^n.$$  (5)

**Condition 1.** The inclusion $u(\cdot) \in L_\infty(T, \mathbb{R}^m)$ is fulfilled.

**Remark 1.** Let Condition 1 be fulfilled. Then there exists a constant $p > 0$ such that $|u(t)|_m \leq p$ for a.e. $t \in T$.

**Definition 2.1.** [13]. We denote by $AC^\gamma(T, \mathbb{R}^n)$ the class of functions $x: T \rightarrow \mathbb{R}^n$ representable by an integral of fractional order $\gamma \in (0, 1)$ of a summable function:

$$x(t) = [I^\gamma \varphi](t), \quad t \in T, \quad \varphi \in L_\infty(T, \mathbb{R}^n).$$

**Remark 2.** As follows from [13, Theorem 3.6], the operator $I^\gamma: L_\infty(T, \mathbb{R}^n) \rightarrow C(T, \mathbb{R}^n)$ is linear and continuous.

**Definition 2.2.** A function $x: T \rightarrow \mathbb{R}^n$ is called a solution to Cauchy problem (1), (3), starting at a point $(\sigma, x_\sigma)$, if $x(\cdot) \in AC^\gamma(T, \mathbb{R}^n)$, $x(\sigma) = x_\sigma$ and equality (1) holds for a.e. $t \in T$.

2.1. Description of the estimation procedure. The input estimation procedure is based on the method of dynamic inversion with a smoothing functional [26]. It implies two key points. The first is the choice of an auxiliary system (a model), which in our case is chosen in the form

$$[D^*_\gamma y^h](t) = f(t, \xi^h(t)) + Bv^h(t), \quad t \in T,$$  (6)
with the initial conditions

\[ y_h^b(\sigma) = y_{\sigma} := x_{\sigma}. \]

Without loss of generality, to simplify the calculations, we assume that the quantities \( y_h(t), \ t \in T, \) are measured exactly. The second key point is the rule for constructing the control \( v^h \) in model (6). Toward this aim, we fix some function \( \alpha = \alpha(h), \ \alpha : (0, 1) \to (0, 1) \). Then, we define control \( v^h(t) \) by the feedback law according to the formula

\[ v^h(t) = (\alpha(h))^{-1}B^T(\xi^h(t) - y^h(t)), \quad t \in T, \]  

(7)

where \( B^T \) is the transposed matrix of \( B \).

**Condition 2.** Let the functions \( \alpha : (0, 1) \to (0, 1), \ \delta : (0, 1) \to (0, 1) \) and constants \( \beta_1, \beta_2 \in (0, 2) \) and \( \beta_3, \beta_4 \in (0, \mu) \) be such that

(a) \( \alpha(h) \to 0, \ (\alpha(h))^{-1}h^{\beta_2} \to 0 \) as \( h \to 0; \)
(b) \( (\alpha(h))^{-1}h^\lambda \to 0 \) as \( h \to 0, \) where \( \lambda := \min\{\beta_1, 2 - \beta_1, 2 - \beta_2\}; \)
(c) \( \delta(h) \to 0, \ (\delta^2(h))^{-1}\pi_2^x(h) \to 0 \) as \( h \to 0, \) where \( \pi(h) := h^\rho + \alpha(h)h^{-\beta_3} + (\alpha(h))^{-1}h^\lambda, \ \rho := \min\{\beta_1, \beta_3, \mu - \beta_3\}; \)
(d) \( \delta(h) \to 0, \ (\delta^2(h))^{-1}\pi_2^x(h) \to 0 \) as \( h \to 0, \) where \( \eta(h) := h + \pi_2^x(h) + (\delta(h))^{-2}\pi_2^x(h); \)
(e) \( \delta(h) \to 0, \ (\delta^2(h))^{-1}\pi_2^x(h) \to 0 \) as \( h \to 0, \) where \( \eta(h) := h + (\delta(h))^{-2}\pi_2^x(h). \)

**Remark 3.** The set of parameters \( \alpha = h^\frac{1}{4}, \ \beta_1 = 1, \ \beta_2 = \frac{6}{5}, \ \beta_3 = \frac{1}{4}, \ \beta_4 = \frac{1}{5} \) and \( \delta = h^\frac{1}{2} \) satisfies Condition 2, herewith \( \lambda = \mu = \frac{3}{4}, \ \rho = \frac{1}{4}. \)

**Remark 4.** Let Condition 2(a) be satisfied. Then there exists a number \( h_* \in (0, 1) \) such that, for all \( h \in (0, h_*), \) the following inequality holds:

\[ c_2B(\alpha(h))^{-1}h^{\beta_2} \leq \frac{1}{2}, \]

where \( c_2 \) is the Euclidean norm of the matrix \( B. \)

**Lemma 2.3.** Let Conditions 1 and 2(a, b) be satisfied. Then one can write explicitly constants \( a_1 \) and \( a_2, \) independent of \( h \) and \( \alpha, \) such that, for all \( h \in (0, h_*), \) the following inequalities hold:

\[ |x(t) - y_h^b(t)|_n^2 \leq a_1(\alpha(h) + h^\mu), \]  

(8)

\[ |I^n|v_h^b|^2_m(t) \leq |I^n|u_m^2(t) + a_2(h^{\beta_1} + (\alpha(h))^{-1}h^\lambda), \quad t \in T. \]  

(9)

**Proof.** Let us prove inequality (8). Considering the difference between (1) and (6), we have

\[ |D_1^\gamma \epsilon^h(t)| = f(t, x(t)) - f(t, \xi^h(t)) + B(u(t) - v^h(t)) \quad \text{for a.e.} \quad t \in T, \]  

(10)

where \( \epsilon^h(t) := x(t) - y_h^b(t). \) Let us use the estimate from [9, Lemma 4.1] rewriting it in more convenient terms as

\[ \frac{1}{2}|D_1^\gamma |\epsilon^h|^2_m(t)| \leq |D_1^\gamma \epsilon^h|(t), \epsilon^h(t) \quad \text{for a.e.} \quad t \in T. \]  

(11)

Here \( (\cdot, \cdot) \) is the inner product in the Euclidean space \( \mathbb{R}^n. \) Since we assume that \( x(\sigma) = y(\sigma), \) then \( \epsilon(\sigma) = 0, \) and, consequently, the equality of the fractional derivatives of Riemann–Liouville and Caputo is true:

\[ |D_1^\gamma \epsilon^h|(t) = |D_1^\gamma \epsilon^h|(t) \quad \text{for a.e.} \quad t \in T. \]
To estimate formula (11) we get
\[ \frac{1}{2}[D^*_n|z^h|^2](t) \leq ([D^*_t z^h](t), z^h(t)) \quad \text{for a.e. } t \in T. \] (12)

By virtue of system (10) the latter inequality is transformed to the form
\[ \frac{1}{2}[D^*_n|z^h|^2](t) \leq g_f(t) + g_u(t) \quad \text{for a.e. } t \in T, \] (13)

where \( g_f(t) := (f(t, x(t)) - f(t, \xi^h(t)), z^h(t)) \), \( g_u(t) := (B(u(t) - v^h(t)), z^h(t)) \). Let us estimate \( g_f(t) \). Taking into account the Cauchy–Bunyakovsky–Schwartz inequality, Lipschitz condition (5) and formula (4) and using the Cauchy inequality with \( \beta_1 \in (0, 2) \), we derive
\[ g_f(t) \leq L|x(t) - \xi^h(t)|_m|z^h(t)|_m \]
\[ \leq Lh|\xi^h(t)|_n \]
\[ \leq \frac{1}{2}Lh^{2-\beta_1} + \frac{1}{2}Lh^{\beta_1}|z^h(t)|^2 \quad \text{for a.e. } t \in T. \] (14)

To estimate \( g_u(t) \), using the Cauchy–Bunyakovsky–Schwartz inequality and formula (4), we get
\[ g_u(t) \leq (u(t) - v^h(t), B^T(\xi^h(t) - y^h(t))) + c_B(u(t) - v^h(t), x(t) - \xi^h(t)) \]
\[ \leq (u(t) - v^h(t), B^T(\xi^h(t) - y^h(t))) \]
\[ + c_B h(|u(t)|_m + |v^h(t)|_m) \quad \text{for a.e. } t \in T. \] (15)

Substituting (14) and (15) into (13), we obtain
\[ [D^*_n|z^h|^2](t) \leq Lh^{2-\beta_1} + Lh^{\beta_1}|z^h(t)|^2 + 2(u(t) - v^h(t), B^T(\xi^h(t) - y^h(t))) \]
\[ + 2c_B h(|u(t)|_m + |v^h(t)|_m). \] (16)

This inequality can be transformed to the form
\[ [D^*_n|z^h|^2](t) = Lh^{2-\beta_1} + Lh^{\beta_1}|z^h(t)|^2 + 2(u(t) - v^h(t), B^T(\xi^h(t) - y^h(t))) \]
\[ + 2c_B h(|u(t)|_m + |v^h(t)|_m) + r(t). \] (17)

Here \( r(t) \) is some non-positive function. Differential equation (17) with the condition \( \varepsilon(\sigma) = 0 \) is equivalent to the following integral equation (see, e.g., [13, p. 141])
\[ |\xi^h(t)|^2 = c_1 h^{2-\beta_1}(t - \sigma)^\gamma + Lh^{\beta_1}|[\Gamma]| |\xi^h|^2 |(t) \]
\[ + 2|\Gamma|(u - v^h, B^T(\xi^h - y^h))(t) \]
\[ + 2c_B h([\Gamma]|u|_m(t) + [\Gamma]|v^h|_m(t)) + [\Gamma]|r|(t). \] (18)

The last term on the right-hand side of (18) satisfies the estimate
\[ [\Gamma]|r|(t) \leq 0, \quad t \in T. \]

Then equality (18) can be represented as
\[ |\xi^h(t)|^2 \leq c_2 h^{2-\beta_1} + Lh^{\beta_1}|[\Gamma]| |\xi^h|^2 |(t) + 2|\Gamma|(u - v^h, B^T(\xi^h - y^h))(t) \]
\[ + 2c_B h([\Gamma]|u|_m(t) + [\Gamma]|v^h|_m(t)). \] (19)
Adding the same terms with the factor $\alpha(h)$ to the left-hand and right-hand sides of (19), we get
\[
|\epsilon_h(t)|_n^2 + \alpha(h)|I^\gamma|\epsilon_h^2|u_n^2(t) - \alpha(h)|I^\gamma|u_n^2(t)\leq \alpha(h)|I^\gamma|\epsilon_h^2|u_n^2(t) - 2|I^\gamma|\epsilon_h|B^\top(\xi^h - y^h)|(t)
\]
\[
+ c_2h^{2-\beta_1} + Lh^{\beta_1}|I^\gamma||\epsilon_h^2|u_n^2(t) + 2c_Bb(|I^\gamma|u_n^2(t) + |I^\gamma|\epsilon_h^2|u_n^2(t)).
\]

Note that control (7) is a consequence of the formula
\[
v_h(t) = \arg\min\{\alpha(h)|u_n^2 - 2(B^\top(\xi^h(t) - y^h(t)), v), v \in \mathbb{R}^m\} \text{ for a.e. } t \in T,
\]
on the basis of which we can conclude that the inequality
\[
\alpha(h)|I^\gamma|\epsilon_h^2|u_n^2(t) - 2|I^\gamma|\epsilon_h|B^\top(\xi^h - y^h)|(t)
\]
\[
(20)
\]
holds $\forall u \in \mathbb{R}^m, \forall t \in T$. Due to this inequality, estimate (20) is reduced to the form
\[
|\epsilon_h(t)|_n^2 \leq \alpha(h)|I^\gamma|\epsilon_h^2|u_n^2(t)
\]
\[
\leq \alpha(h)|I^\gamma|u_n^2(t) + c_2h^{2-\beta_1} + Lh^{\beta_1}|I^\gamma||\epsilon_h^2|u_n^2(t)
\]
\[
+ 2c_Bb(|I^\gamma|u_n^2(t) + |I^\gamma|\epsilon_h^2|u_n^2(t)).
\]

It is easy to see that the inequality
\[
\alpha(h)|I^\gamma|\epsilon_h^2|u_n^2(t) \geq 0, \quad t \in T;
\]
is valid; therefore, transforming formula (21) taking into account Remark 1 and the Cauchy inequality with $\beta_2 \in (0, 2)$, we have
\[
|\epsilon_h(t)|_n^2 \leq c_2h^{2-\beta_1} + c_3p^2\alpha(h) + 2c_Bc_3ph + 2c_Bc_3h^{2-\beta_2} + 2c_Bh^{\beta_2}|I^\gamma|\epsilon_h^2|u_n^2(t)
\]
\[
+ Lh^{\beta_1}|I^\gamma||\epsilon_h^2|u_n^2(t).\]

(22)

To estimate $|I^\gamma|\epsilon_h^2|u_n^2(t), t \in T$, first, taking into account (7), we derive
\[
|\epsilon_h(t)|_m^2 = (\alpha(h))^{-2}B^\top(\xi^h(t) - y^h(t))_n^2
\]
\[
\leq c_B(\alpha(h))^{-2}(\epsilon_h(t) - x(t) + x(t) - y^h(t), \xi^h(t) - x(t) + x(t) - y^h(t))
\]
\[
\leq 2c_B(\alpha(h))^{-2}(h^2 + |\epsilon_h(t)|_m^2).
\]

Further, calculating the fractional-order integral $I^\gamma$ of both sides of the latter inequality, we obtain
\[
|I^\gamma|\epsilon_h^2|u_n^2(t) \leq 2c_Bc_3(\alpha(h))^{-2}h^2 + 2c_B(\alpha(h))^{-2}|I^\gamma|\epsilon_h^2|u_n^2(t), \quad t \in T.
\]

(23)

Given formula (23) in (22), we find
\[
|\epsilon_h(t)|_n^2 \leq c_2h^{2-\beta_1} + c_3p^2\alpha(h) + 2c_Bc_3ph + 2c_Bc_3h^{2-\beta_2} + 4c_Bc_3(\alpha(h))^{-2}h^{2+\beta_2}
\]
\[
+ (4c_B(\alpha(h))^{-2}h^{\beta_2} + Lh^{\beta_1})|I^\gamma||\epsilon_h^2|u_n^2(t).
\]

Since Condition 2(a) is fulfilled, the latter estimate is converted to the form
\[
|\epsilon_h(t)|_n^2 \leq c_0\alpha(h) + c_3h^{\mu} + (c_0(\alpha(h))^{-2}h^{\beta_2} + Lh^{\beta_1})|I^\gamma||\epsilon_h^2|u_n^2(t).
\]

The Gronwall–Bellman inequality [32, Theorem 6] gives us for $\gamma \in (\frac{1}{2}, 1)$:
\[
|\epsilon_h(t)|_n^2 \leq (c_0 \alpha + c_3h^{\mu}) \exp(K_1(c_0(\alpha(h))^{-2}h^{\beta_2} + Lh^{\beta_1})^2(t - \sigma) + t),
\]
\[
(24)
\]
where $K_1 := 2^{2\gamma-2}\Gamma(2\gamma - 1)$, and, for $\gamma \in (0, \frac{1}{2})$:
\[
|\epsilon_h(t)|_n^2 \leq 2^{1+\gamma\epsilon}e^{-\epsilon}(c_1 \alpha + c_3h^{\mu}) \exp(K_2(c_0(\alpha(h))^{-2}h^{\beta_2} + Lh^{\beta_1})^\theta(t - \sigma) + t),
\]
\[
(25)
\]
where $K_2 := 2^{\frac{q}{2}} (1 + \gamma)^{-\gamma} q^{-1} (\Gamma(\gamma)^2) \frac{1}{\gamma}$ and $q := 1 + \frac{1}{\gamma}$. In both cases, formulas (24) and (25) lead to the inequality

$$|\varepsilon^h(t)|_m^2 \leq c_7 \alpha(h) + c_8 h^\mu, \quad t \in T,$$

with the corresponding constants $c_7$ and $c_8$. Its consequence is (8).

Let us prove inequality (9). We rewrite formula (21) in the form

$$|\varepsilon^h(t)|_m^2 + \alpha(h)|\Gamma^h|v^h|_m^2(t) \leq \alpha(h)|\Gamma^h|u^h|_m^2(t) + c_2 h^{2-\beta_1} + L h^{\beta_1} |\varepsilon^h|_m^2(t)$$

$$+ 2c_B h(|\Gamma^h|u^h|_m^2(t) + |\Gamma^h|v^h|_m^2(t)).$$

Taking into account the inequality $|\varepsilon^h(t)|_m^2 \geq 0$ and Condition 1, applying the Cauchy inequality with $\beta_2 \in (0, 2)$, we have

$$\alpha(h)|\Gamma^h|v^h|_m^2(t) \leq \alpha(h)|\Gamma^h|u^h|_m^2(t) + c_2 h^{2-\beta_1} + L h^{\beta_1} |\varepsilon^h|_m^2(t)$$

$$+ 2c_1 c_B h + c_1 c_B h^{2-\beta_2} + c_B h^{\beta_2} |\Gamma^h|v^h|_m^2(t).$$

Transforming (26) with use of (8), we get

$$\alpha(h) \left(1 - c_B h^{\beta_2} \frac{h^{\beta_2}}{\alpha(h)}\right) |\Gamma^h|v^h|_m^2(t) \leq \alpha(h)|\Gamma^h|u^h|_m^2(t) + c_9 h^\mu + c_1 L h^{\beta_1} (a_1 \alpha(h) + a_2 h^\mu).$$

Adding and subtracting terms containing $|\Gamma^h|u^h|_m^2(t)$ on the right-hand side of the latter formula, we have

$$\alpha(h)(1 - c_B(\alpha(h))^{-1} h^{\beta_2})|\Gamma^h|v^h|_m^2(t) \leq \alpha(h)(1 - c_B(\alpha(h))^{-1} h^{\beta_2})|\Gamma^h|u^h|_m^2(t) + c_B h^{\beta_2} |\Gamma^h|u^h|_m^2(t)$$

$$+ c_9 h^\mu + c_1 L h^{\beta_1} (a_1 \alpha(h) + a_2 h^\mu). \quad (27)$$

The estimate $1 - c_B(\alpha(h))^{-1} h^{\beta_2} > 0$ follows from Remark 4 providing (27) to be rewritten as

$$|\Gamma^h|v^h|_m^2(t) \leq |\Gamma^h|u^h|_m^2(t) + (\alpha(h))^{-1} \left(1 - c_B \frac{h^{\beta_2}}{\alpha(h)}\right)^{-1} (c_{10} h^{\beta_2} + c_{11} h^\mu + c_{12} \alpha(h) h^{\beta_1}). \quad (28)$$

This formula, due to Condition 2(b), leads to inequality (9). □

**Lemma 2.4.** Under the conditions of Lemma 2.3, one can write out explicitly a constant $a_3$, independent from $h$ and $\alpha$, such that the inequality

$$|I^1|v^h|_m^2(t) \leq |I^1|u^h|_m^2(t) + a_3(h^{\beta_1} + (\alpha(h))^{-1} h^\lambda), \quad t \in T, \quad (29)$$

holds, where $[I^1 x](t) := \int_0^1 x(s) \, ds$.

**Proof.** Calculating the integral $I^1$ of the left-hand and right-hand sides of (16) and adding terms with the factor $\alpha(h)$, we obtain

$$|I^1-\gamma|\varepsilon^h|_m^2(t) + \alpha(h)|I^1|v^h|_m^2(t) - \alpha(h)|I^1|u^h|_m^2(t) \leq \alpha(h)|I^1|v^h|_m^2(t) - \alpha(h)|I^1|u^h|_m^2(t) - 2|I^1(v^h - u, B^T (\xi^h - y^h))|.$$
Further, repeating the transformations for formulas (26) and (28), only with the replacement of $I^1$ by $I^1$, we arrive at estimate (29).

Let us introduce the set $U(x(\cdot))$ of all controls generating the motion $x(\cdot)$ by the formula

$$U(x(\cdot)) = \{ u(\cdot) \in L_\infty(T, \mathbb{R}^m) : x(t) = x_\sigma + [I^1(f(\cdot, x(\cdot)) + Bu(\cdot))] (t), \forall t \in T \}.$$

We define the element $u_\ast(\cdot) = u_\ast(\cdot; x(\cdot))$ as an element of the minimal $L_2(T, \mathbb{R}^m)$-norm in $U(x(\cdot))$. It should be noted that the set $U(x(\cdot))$ is convex and closed in $L_2(T, \mathbb{R}^m)$. Therefore, the element $u_\ast(\cdot)$ exists and is unique.

**Theorem 2.5.** Let Conditions 1 and 2(a, b) be satisfied. Then we have the convergence

$$v^h(\cdot) \to u_\ast(\cdot) \text{ in } L_2(T, \mathbb{R}^m) \text{ as } h \to 0.$$

**Proof.** Let us show that, for an arbitrary sequence $h_j \to 0+$ as $j \to \infty$ and any measurements $\xi^{h_j}$ $([\xi^{h_j}](t) - x(t))_{m \leq h_j, \ t \in T}$, the convergence $v^{h_j}(\cdot) \to u_\ast(\cdot)$ in $L_2(T, \mathbb{R}^m)$ takes place as $h_j \to 0+$ ($j \to \infty$). We will prove this by contradiction. Suppose there exists a sequence $\{v^{h_j}\}$ such that

$$v^{h_j}(\cdot) \to v_\ast(\cdot) \text{ weakly in } L_2(T, \mathbb{R}^m) \text{ as } j \to \infty,$$

and at the same time

$$v_\ast(\cdot) \neq u_\ast(\cdot). \quad (31)$$

Then, choosing the subsequence $\{h_{j_k}\}$ from $\{h_j\}$, we obtain

$$y^{h_{j_k}}(\cdot) \to y_\ast(\cdot) \text{ in } C(T, \mathbb{R}^m) \text{ as } k \to \infty,$$

where $y^{h_{j_k}}(\cdot) = y(\cdot; \sigma, y_\sigma, v^{h_{j_k}})$. Following [34, Theorem 1], in view of the weak convergence of the controls $\{v^{h_{j_k}}\}$, we have the uniform convergence $\{y^{h_{j_k}}\}$; therefore, $y_\ast(\cdot)$ is a solution to the Cauchy problem

$$[D^2_y v^h](t) = f(t, v^h(t)) + Bv_\ast(t), \quad t \in T, \quad y(\sigma) = y_\sigma.$$

By Lemma 2.3, relation (8) implies the uniform on $T$ convergence $y_\ast(t)$ to $x(t)$, i.e., to the solution of problem (1). Thus, $y_\ast(t) \equiv x(t)$, $t \in T$, and, consequently, $u_\ast(\cdot) = u_\ast(\cdot)$. Then from (30) we conclude that

$$v^{h_j}(\cdot) \to u_\ast(\cdot) \text{ weakly in } L_2(T, \mathbb{R}^m) \text{ as } j \to \infty.$$

In view of the properties of the weak limit, this convergence yields the inequality

$$\lim_{k \to \infty} \|v^{h_{j_k}}\|_{L_2(T, \mathbb{R}^m)} \geq \|u_\ast\|_{L_2(T, \mathbb{R}^m)}. \quad (32)$$

From Lemma 2.4, where we have $[I^1|v^{h_{j_k}}|^2_m](\theta) = \|v^{h_{j_k}}\|^2_{L_2(T, \mathbb{R}^m)}$ for $t = \theta$, it follows that

$$\|v^{h_{j_k}}\|^2_{L_2(T, \mathbb{R}^m)} \leq \|u_\ast\|^2_{L_2(T, \mathbb{R}^m)} + b_3 h_{j_k}^{\beta_1} + b_4 (\alpha(h_{j_k}))^{-1} h_{j_k}^{\lambda_1}.$$ 

Since the last two terms of the obtained estimate are positive, we have

$$\lim_{k \to \infty} \|v^{h_{j_k}}\|_{L_2(T, \mathbb{R}^m)} \leq \|u_\ast\|_{L_2(T, \mathbb{R}^m)}. \quad (33)$$

As a result, combining (32) and (33), we get

$$\lim_{k \to \infty} \|v^{h_{j_k}}\|_{L_2(T, \mathbb{R}^m)} \leq \|u_\ast\|_{L_2(T, \mathbb{R}^m)}$$

This formula implies $u_\ast(\cdot) = u_\ast(\cdot)$ which contradicts (31). \qed
2.2. Estimation error. It is possible to estimate the error of the proposed approximation under some additional conditions.

**Condition 3.** The function \( u_*(\cdot) \) has a bounded variation on the interval \( T \), i.e. \( V(u;T) < \infty \).

**Theorem 2.6.** Let Conditions 1, 2(a)–(c), 3 hold, \( m = n \) and \( \det B \neq 0 \). Then one can explicitly write a constant \( a_4 \), independent of \( h \) and \( \alpha \), such that the following inequality holds:

\[
\|v^h(\cdot) - u_*(\cdot)\|^2_{L^2(T,\mathbb{R}^m)} \leq a_4\pi(h).
\]  

(34)

**Proof.** In view of equations (1) and (6) we have the inequality

\[
\left| \int_\sigma^{t} B(v^h(s) - u_*(s)) \, ds \right| \leq P_1 + P_2, \quad t \in T,
\]

(35)

where \( P_1 := \left| \int_\sigma^{t} [D_z^2(y^h - x)](s) \, ds \right|, \quad P_2 := \left| \int_\sigma^{t} (f(s,x^h(s)) - f(s,x(s))) \, ds \right|. \) To estimate \( P_1 \), we use the Cauchy inequality with \( \beta_3 \in (0,\mu) \) and formula (8), then we derive

\[
P_1 = \left| [I^{1-\gamma}(y^h - x)](t) \right|_n \\
\leq [I^{1-\gamma}|y^h - x|]_n(t) \\
\leq \frac{1}{2} [I^{1-\gamma}(h^{\beta_3} + h^{-\beta_3}|y^h - x|^2)](t) \\
\leq \frac{1}{2} [I^{1-\gamma}(h^{\beta_3} + a_1\alpha(h)h^{-\beta_3} + a_2\mu^{-\beta_3})](t) \\
\leq \bar{c}_1 h^{\beta_3} + \bar{c}_2 \alpha(h)h^{-\beta_3} + \bar{c}_3 h^{\mu-\beta_3}.
\]

(36)

Let us estimate \( P_2 \). Taking into account inequalities (4) and (5), we have

\[
P_2 \leq \int_\sigma^{t} L|\xi^h(s) - x(s)|_n \, ds \\
\leq \bar{c}_4 h.
\]

(37)

Using inequalities (36) and (37) in (35), we get

\[
\left| \int_\sigma^{t} B(v^h(s) - u_*(s)) \, ds \right| \leq \bar{c}_1 h^{\beta_3} + \bar{c}_2 \alpha(h)h^{-\beta_3} + \bar{c}_3 h^{\mu-\beta_3} + \bar{c}_4 h, \quad t \in T.
\]

(38)

Let \((\cdot,\cdot)\) denote the inner product in \( L^2(T,\mathbb{R}^n) \), then, taking into account relations \( \|u\|_{L^2(T,\mathbb{R}^n)}^2 = \int [I^1|u|^2](\theta), \quad \|v^h\|_{L^2(T,\mathbb{R}^n)}^2 = \int [I^1|v^h|^2](\theta) \) and inequality (29), we find

\[
\|u - v^h\|_{L^2(T,\mathbb{R}^n)}^2 \leq \|u\|_{L^2(T,\mathbb{R}^n)}^2 - 2(u,v^h) + \|v^h\|_{L^2(T,\mathbb{R}^n)}^2 \\
\leq 2\|u\|_{L^2(T,\mathbb{R}^n)}^2 - 2(u,v^h) + b_3 h^{\beta_1} + b_4(\alpha(h))^{-1}h^\lambda \\
= 2(u - v^h, u) + b_3 h^{\beta_1} + b_4(\alpha(h))^{-1}h^\lambda.
\]

(39)

By using the Lemma [20, p. 47] for estimates (38) and (39), we obtain

\[
\|u - v^h\|_{L^2(T,\mathbb{R}^n)}^2 \leq 2\bar{c}_B(V(u;T) + p)(\bar{c}_1 h^{\beta_3} + \bar{c}_2 \alpha(h)h^{-\beta_3} + \bar{c}_3 h^{\mu-\beta_3} + \bar{c}_4 h) \\
+ b_3 h^{\beta_1} + b_4(\alpha(h))^{-1}h^\lambda,
\]

where \( \bar{c}_B \) is the Euclidean norm of the matrix \( B^{-1} \). This inequality is reduced to the form (34).
Remark 5. The proof of Theorem 2.6 can be obtained for \( m < n \) if the rank of matrix \( B \) equals to \( m \).

Remark 6. For the set of parameters specified in Remark 3, estimate (34) takes the form
\[
\|u(\cdot) - v^h(\cdot)\|_{L_2(T, \mathbb{R}^m)}^2 \leq R_1 h^\frac{1}{2},
\]
where the constant \( R_1 > 0 \) is independent of \( h \) and \( \alpha \).

Remark 7. Theorems 2.5 and 2.6 remain valid if we take the system
\[
[D^*_t y^h](t) = f(t, y^h(t)) + Bu^h(t) + L(\xi^h(t) - y^h(t)), \quad t \in T,
\]
as a model.

3. The case of measuring a part of coordinates. Let \( \kappa_1 < n \). We denote by the symbol \( x_1 \in \mathbb{R}^{\kappa_1} \) the vector consisting of the first (measured) coordinates of the vector \( x \), then, respectively, the vector \( x_2 \in \mathbb{R}^{\kappa_2} \) (\( n = \kappa_1 + \kappa_2 \)) consists of the unobservable coordinates of \( x \). As a consequence, we have \( x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^n \). In the general case, the problem is intractable, therefore, we narrow the class of functions in the right-hand side of (1), namely, we restrict ourselves to functions of the form
\[
f(t, x) = f(t, x_1, x_2) = \begin{pmatrix} g_1(t, x_1) + C_1 x_2 \\ g_2(t, x_1, x_2) \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 \\ C_2 \end{pmatrix}.
\]
Thus, we consider the system
\[
[D^*_t x_1](t) = g_1(t, x_1(t)) + C_1 x_2(t), \quad t \in T, \tag{40}
\]
\[
[D^*_t x_2](t) = g_2(t, x_1(t), x_2(t)) + C_2 u(t), \quad t \in T, \tag{41}
\]
with the initial conditions
\[
x_1(\sigma) = x_{1\sigma}, \quad x_2(\sigma) = x_{2\sigma}.
\]
We assume that the functions \( g_1 : T \times \mathbb{R}^{\kappa_1} \to \mathbb{R}^{\kappa_1} \) and \( g_2 : T \times \mathbb{R}^{\kappa_1} \times \mathbb{R}^{\kappa_2} \to \mathbb{R}^{\kappa_2} \) satisfy the Lipschitz condition with constant \( L > 0 \) with respect to the variables \((t, x_1, x_2)\), i.e., the inequality
\[
\max\{|g_1(t_1, x_1) - g_1(t_2, y_1)|_{\kappa_1} ; |g_2(t_1, x_1, x_2) - g_2(t_2, y_1, y_2)|_{\kappa_2}\} \leq L|t_1 - t_2| + |x_1 - y_1|_{\kappa_1} + |x_2 - y_2|_{\kappa_2}
\]
holds for any \((t_1, x_1, x_2), (t_2, y_1, y_2) \in T \times \mathbb{R}^{\kappa_1} \times \mathbb{R}^{\kappa_2}\). The constant matrices \( C_1 \) and \( C_2 \) are of dimensions \( \kappa_1 \times \kappa_2 \) and \( \kappa_2 \times m \), respectively. As a result of observing \( \kappa_1 \) coordinates of the trajectory of system (40), vectors \( \xi^h(t) \in \mathbb{R}^{\kappa_1} \) are recorded with a given error:
\[
|\xi^h(t) - x_1(t)|_{\kappa_1} \leq h, \quad t \in T. \tag{43}
\]
Let at the initial time all the phase coordinates of the system be measured precisely.

Thus, we set the problem as follows. The trajectory of the system \( x(t) = \{x_1(t), x_2(t)\} \) is unknown in advance and is determined by an input \( u(\cdot) \), which is also not specified but satisfying Condition 1. It is required to find the input \( u(\cdot) \) generating \( x(\cdot) \).
3.1. Description of the estimation procedure. In this case, the estimation procedure is represented by two reconstruction units working simultaneously. Their action is described by the procedure from the Section 2.1. The first unit is for reconstructing the unknown coordinate \( x_2 \). The information obtained as a result of its work passes to the second unit, which in turn forms a certain approximation of \( u(\cdot) \) according to the feedback law.

Along with system (40), (41), we introduce an auxiliary system (a model)

\[
\begin{align*}
[D_1^*w_1](t) &= g_1(t, \xi^h(t)) + C_1\psi^h(t), \\
[D_2^*w_2](t) &= g_2(t, \xi^h(t), \psi^h(t)) + C_2v^h(t), \quad t \in T,
\end{align*}
\]

with the initial conditions

\[
w_1(\sigma) := x_{1\sigma}, \quad w_2(\sigma) := x_{2\sigma}.
\]

We assume that the coordinates of \( w_1 \) are also measured accurately. Thus, the work of the first reconstruction unit based on system (44) is to form an approximation of the unobservable coordinate \( w_2 \) according to the rule

\[
\psi^h(t) = (\alpha(h))^{-1}C_1^T(\xi^h(t) - w_1(t)), \quad t \in T.
\]

The second reconstruction unit corresponds to system (45) and uses the rule for generating a control of the form

\[
v^h(t) = (\delta(h))^{-1}C_2^T(\psi^h(t) - w_2(t)), \quad t \in T.
\]

Lemma 3.1. Let Conditions 1, 2(a)–(c) and 3 be satisfied, \( \kappa_1 = \kappa_2 \), \( \det C_1 \neq 0 \). Then one can explicitly write constants \( a_1 \) and \( a_4 \) independent of \( h \) and \( \alpha \), such that the following inequalities hold:

\[
\sup_{t \in T} |\epsilon^h_1(t)|^2_1 \leq a_1(\alpha(h) + h^\alpha),
\]

\[
\|\omega^h_2(\cdot)\|^2_{L_2(T; \mathbb{R}^2)} \leq a_4\pi(h),
\]

where \( \epsilon^h_1(t) := w_1(t) - x_1(t), \omega^h_2(\cdot) := \psi^h(\cdot) - x_2(\cdot) \).

Proof. The statement of the lemma follows directly from Lemma 2.3 and Theorem 2.6.

Lemma 3.2. Let Conditions 1, 2(a)–(d) and 3 be satisfied, \( \kappa_1 = \kappa_2 \), \( \det C_1 \neq 0 \). Then one can explicitly write a constant \( a_5 \), independent of \( h, \alpha \) and \( \delta \), such that the following inequality holds:

\[
[T^{1-\gamma}|\epsilon^{h^2}_{2}|_{\kappa_2}](t) \leq a_5(\delta(h) + \eta(h)), \quad t \in T,
\]

where \( \epsilon^h_2(t) := w_2(t) - x_2(t) \).

Proof. Considering the difference between (41) and (45), we have

\[
[D_1^*\epsilon^2_2](t) = g_2(t, \xi^h(t), \psi^h(t)) - g_2(t, x_1(t), x_2(t)) + C_2(v^h(t) - u(t)).
\]

Likewise, following the path of the proof of Lemma 2.4, using formula (12) for \( \epsilon^2_2 \), by virtue of system (50), we obtain

\[
\frac{1}{2}[D_4^*|\epsilon^{h^2}_{2}|_{n_1}](t) \leq z_g(t) + z_v(t), \quad t \in T,
\]

where \( z_g(t) := (g_2(t, \xi^h(t), \psi^h(t)) - g_2(t, x_1(t), x_2(t)), \epsilon^h_2(t)), z_v(t) := (C_2(v^h(t) - u(t)), \epsilon^2_2(t)) \).
Let us estimate \( z_g(t) \). Taking into account the Cauchy–Bunyakovsky–Schwartz inequality, Lipschitz condition \((42)\) and formula \((43)\), we derive

\[
|z_g(t)| \leq L(|\xi^h(t) - x_1(t)|_{\kappa_1} + |\psi^h(t) - x_2(t)|_{\kappa_2})|z^h_2(t)|_{\kappa_2} \\
\leq Lh + L|\omega^h_2(t)|_{\kappa_2}|z^h_2(t)|_{\kappa_2}.
\]

\( (52) \)

For the expression of \( z_u(t) \), by analogous transformations as in the derivation of \((15)\), we find the estimate

\[
|z_u(t)| \leq (v^h(t) - u(t), C_2^T(w_2(t) - \psi^h(t))) + c_2((v^h(t) - u(t), \psi^h(t) - x_2(t)) \\
\leq (v^h(t) - u(t), C_2^T(w_2(t) - \psi^h(t))) \\
+ c_2|\omega^h_2(t)|_{\kappa_2}(|u(t)|_m + |v^h(t)|_m) \quad \text{for a.e.} \quad t \in T,
\]

\( (53) \)

where \( c_2 \) is the Euclidean norm of the matrix \( C_2 \). Inequality \((51)\) in view of \((52)\) and \((53)\) is rewritten as

\[
[D^*_2|\varepsilon^h_2|_{\kappa_2}](t) \leq 2L(h + |\omega^h_2(t)|_{\kappa_2})|\varepsilon^h_2(t)|_{\kappa_2} \\
- 2(v^h(t) - u(t), C_2^T(\psi^h(t) - w_2(t))) \\
+ 2c_2|\omega^h_2(t)|_{\kappa_2}(|u(t)|_m + |v^h(t)|_m) \quad \text{for a.e.} \quad t \in T.
\]

\( (54) \)

Adding the same terms with the factor \( \delta(h) \) to the left-hand and right-hand sides of \((54)\), we get

\[
[D^*_2|\varepsilon^h_2|_{\kappa_2}](t) + \delta(h)|v^h(t)|^2_m - \delta(h)|u(t)|^2_m \\
\leq 2(Lh + |\omega^h_2(t)|_{\kappa_2})|\varepsilon^h_2(t)|_{\kappa_2} + \delta(h)|v^h(t)|^2_m - \delta(h)|u(t)|^2_m \\
- 2(v^h(t) - u(t), C_2^T(\psi^h(t) - w_2(t))) \\
+ 2c_2|\omega^h_2(t)|_{\kappa_2}(|u(t)|_m + |v^h(t)|_m) \quad \text{for a.e.} \quad t \in T.
\]

\( (55) \)

Since the control \( v^h \) defined by rule \((47)\) can also be deduced from the formula

\[
v^h(t) = \text{argmin} \{\delta(h)|v|^2_m - 2(C_2^T(\psi^h(t) - w_2(t)), v), \quad v \in \mathbb{R}^m\}, \quad t \in T,
\]

we have the inequality

\[
\delta(h)(|v^h(t)|^2_m - |u(t)|^2_m) - 2(v^h(t) - u(t), C_2^T(\psi^h(t) - w_2(t))) \leq 0,
\]

which is valid \( \forall u(t) \in \mathbb{R}^m, \quad t \in T \). Taking into account this formula, inequality \((55)\) is rewritten in the form

\[
[D^*_2|\varepsilon^h_2|_{\kappa_2}](t) + \delta(h)|v^h(t)|^2_m \\
\leq \delta(h)|u(t)|^2_m + 2L(h + |\omega^h_2(t)|_{\kappa_2})|\varepsilon^h_2(t)|_{\kappa_2} \\
+ 2c_2|\omega^h_2(t)|_{\kappa_2}(|u(t)|_m + |v^h(t)|_m) \quad \text{for a.e.} \quad t \in T.
\]

\( (56) \)

Obviously, the estimate

\[
\delta(h)|v^h(t)|^2_m \geq 0, \quad t \in T,
\]

holds. Then, discarding this term and calculating the integral \( I^1 \) of the left-hand and right-hand sides of \((56)\), we obtain

\[
[I^1(D^*_2|\varepsilon^h_2|_{\kappa_2})](t) \\
\leq \delta(h)[I^1|v^h_m|(t) + 2Lh[I^1|\varepsilon^h_2|_{\kappa_2}](t) + 2L[I^1(|\omega^h_2|_{\kappa_2}|\varepsilon^h_2|_{\kappa_2})](t) \\
+ 2c_2[I^1(|\omega^h_2|_{\kappa_2}(|u|_m + |v^h|_m))](t), \quad t \in T.
\]

\( (57) \)
Next, we estimate the terms of the right-hand side of (57). Using the inequality $2ab \leq a^2 + b^2$, we have

$$2Lh[I^1|\varepsilon_h^0|_2](t) \leq Lh(\theta - \sigma) + Lh[I^1|\varepsilon_h^0|_2](t).$$  

(58)

Applying the Cauchy–Bunyakovsky–Schwartz inequality and estimate (48), we deduce that

$$[I^1(\omega^0_2|_2\varepsilon^0_2)](t) \leq \left([I^1|\omega^0_2|_2^2](t)\right)^{\frac{1}{2}} \left([I^1|\varepsilon^0_2|_2^2](t)\right)^{\frac{1}{2}}$$

$$\leq \frac{1}{2}\left([I^1|\omega^0_2|_2^2](t)\right)^{\frac{1}{2}} + \frac{1}{2}\left([I^1|\varepsilon^0_2|_2^2](t)\right)^{\frac{1}{2}}[I^1|\varepsilon^0_2|_2^2](t)$$

$$\leq \frac{1}{2}\|\omega^0_2(\cdot)\|_{L_2(T,R^n_2)} + \frac{1}{2}\|\varepsilon^0_2(\cdot)\|_{L_2(T,R^n_2)}[I^1|\varepsilon^0_2|_2^2](t)$$

$$\leq \frac{1}{2}\pi^\frac{1}{2}(h) + \frac{1}{2}\pi^\frac{1}{2}(h)[I^1|\varepsilon^0_2|_2^2](t), \quad t \in T.$$  

(59)

Performing the same transformations as in the derivation of (59), we get

$$[I^1(\omega^0_2|_2|v^h|_m)](t) \leq \left([I^1|\omega^0_2|_2^2](t)\right)^{\frac{1}{2}} \left([I^1|v^h|_m^2](t)\right)^{\frac{1}{2}}$$

$$\leq \frac{1}{2}\pi^\frac{1}{2}(h) + \frac{1}{2}\pi^\frac{1}{2}(h)[I^1|v^h|_m^2](t), \quad t \in T,$$  

(60)

and additionally, when Condition 1 is fulfilled, we have

$$[I^1(\omega^0_2|_2|u|_m)](t) \leq \left([I^1|\omega^0_2|_2^2](t)\right)^{\frac{1}{2}} \left([I^1|u|_m^2](t)\right)^{\frac{1}{2}}$$

$$\leq \bar{c}_3\pi^\frac{1}{2}(h), \quad t \in T.$$  

(61)

Inequality (57), in view of the properties of fractional integrals and derivatives [13, Lemma 2.3] and estimates (59)–(61), can be rewritten as

$$[I^{1-\gamma}|\varepsilon^0_2|_2^2](t) \leq Lh + \bar{c}_4\delta(h) + L\pi^\frac{1}{2}(h) + L(h + \pi^\frac{1}{2}(h))[I^1|\varepsilon^0_2|_2^2](t)$$

$$+ c_2(2\bar{c}_3 + 1)\pi^\frac{1}{2}(h) + c_2\pi^\frac{1}{2}(h)[I^1|v^h|_m^2](t), \quad t \in T.$$  

(62)

We obtain the estimate necessary for further transformations of the last term on the right-hand side of (62). Taking into account formula (47), we have

$$|v^h(t)|_m^2 \leq c^2_2(\delta(h))^{-2}|\psi^h(t) - w_2(t)|^2_{\kappa_2}$$

$$= c^2_2(\delta(h))^{-2}|\psi^h_x(t) - x_2(t) + x_2(t) - w_2(t)|^2_{\kappa_2}$$

$$\leq 2c^2_2(\delta(h))^{-2}(\omega^0_2(t)|_{\kappa_2}^2 + |\varepsilon^0_2|_2^2(t)).$$

Calculating the integral $I^1$ of both sides of the resulting inequality and taking into account estimate (48), we arrive at the expression

$$[I^1|v^h|_m^2](t) \leq 2c^2_2(\delta(h))^{-2}\pi(h) + 2c^2_2(\delta(h))^{-2}[I^1|\varepsilon^0_2|_2^2](t).$$  

(63)

Transforming inequality (62) using (63), we find

$$[I^{1-\gamma}|\varepsilon^0_2|_2^2](t)$$

$$\leq Lh + \bar{c}_4\delta(h) + L\pi^\frac{1}{2}(h) + L(h + \pi^\frac{1}{2}(h))[I^1|\varepsilon^0_2|_2^2](t) + c_2(2\bar{c}_3 + 1)\pi^\frac{1}{2}(h)$$

$$+ 2c^2_2(\delta(h))^{-2}\pi^\frac{1}{2}(h) + 2c^2_2(\delta(h))^{-2}\pi^\frac{1}{2}(h)[I^1|\varepsilon^0_2|_2^2](t), \quad t \in T.$$
Applying the semigroup property of the fractional integral [13, Lemma 2.3], we rewrite the latter formula in the form
\[ I^{1-\gamma} e_h^2 (t) \leq \tilde{c}_5 (\delta (h) + \eta (h)) + (L h + (L + 2 c_0^2 (\delta (h))^{-2})^2 \pi h^2 (h)) [I^{1-\gamma} e_h^2 (t)], \quad t \in T. \]

Since Condition 2 is assumed to be satisfied, using the Gronwall–Bellman inequality [32, Theorem 6] for \( \gamma \in (\frac{1}{2}, 1) \), we have
\[ I^{1-\gamma} e_h^2 (t) \leq \tilde{c}_5 (\delta (h) + \eta (h)) \exp (K_1 (L + 2 c_0^2 (\delta (h))^{-2})^2 \pi h^2 (h) (t - \sigma) + t), \quad (64) \]
and, in the case \( \gamma \in (0, \frac{1}{2}) \),
\[ I^{1-\gamma} e_h^2 (t) \leq \tilde{c}_5 (\delta (h) + \eta (h)) + 2 \frac{v}{y} e^{-\sigma} \tilde{c}_5 (\delta (h) + \eta (h)) \exp (K_2 (L + 2 c_0^2 (\delta (h))^{-2})^2 y \pi h^2 (h) (t - \sigma) + t). \quad (65) \]

Here constants \( K_1, K_2 \) and \( q \) are introduced in the proof of Lemma 2.3. Both formulas (64) and (65) are reduced to the form
\[ I^{1-\gamma} e_h^2 (t) \leq \tilde{c}_6 (\delta (h) + \eta (h)), \quad t \in T, \quad (66) \]
with the corresponding constant \( \tilde{c}_6 \). A consequence of inequality (66) is (49).

**Lemma 3.3.** Let Conditions 1, 2(a)–(c) and 3 be satisfied, \( \kappa_1 = \kappa_2 \), and \( C_1 \neq 0 \). Then one can explicitly write a constant \( a_0 \) independent of \( h, \alpha \) and \( \delta \), such that for all \( h \in (0, h_*) \) the following inequality holds:
\[ I^1 [v_h^1 (t)] \leq I^1 [u_h^2 (t)] + a_0 (\delta (h))^{-1} \eta (h), \quad t \in T. \quad (67) \]

**Proof.** If we integrate the left-hand and right-hand sides of inequality (56), we get
\[ I^1 (D^1 [v_h^2 (t)]) + \delta (h) I^1 [v_h^1 (t)] \leq \delta (h) I^1 [u_h^2 (t)] + 2 L h (I^1 [v_h^2 (t)] + 2 L |I^1 (|\omega_{12}^2 |_{K_2} |e_h^2 |_{K_2})| (t) + 2 c_2 (I^1 (|\omega_{12}^2 |_{K_2} |v_h^1 |_{m_2}))) (t), \quad t \in T. \]

Further, repeating the transformations for formulas (58)–(63), we arrive at the inequality
\[ I^{1-\gamma} e_h^2 (t) + \delta (h) I^1 [v_h^2 (t)] \leq \delta (h) I^1 [u_h^2 (t)] + \eta (h) + L (h + \pi h^2 (h)) [I^{1-\gamma} e_h^2 (t)] (t) + 2 c_2 (\delta (h))^{-2} \pi h^2 (h) [I^{1-\gamma} e_h^2 (t)] (t), \quad t \in T. \]

Since the inequalities
\[ |v_h^2 (t)|_{K_2} \geq 0, \quad |I^{1-\gamma} e_h^2 (t)|_{K_2} \geq 0, \quad t \in T, \]
hold, then taking into account (49), we find
\[ \delta (h) I^1 [v_h^2 (t)] \leq \delta (h) I^1 [u_h^2 (t)] + \eta (h) + L \pi h^2 (h) [I^{1-\gamma} e_h^2 (t)] (t) + \tilde{c}_6 (\delta (h))^{-2} \pi h^2 (h) [I^{1-\gamma} e_h^2 (t)] (t), \quad t \in T. \]

Calculating the integrals on the right-hand side of this inequality, we have (67).

**Theorem 3.4.** Let the conditions of Lemma 2.3 be satisfied. Then the convergence
\[ v^h (\cdot) \rightarrow u_s (\cdot) \quad \text{in} \quad L_2 (T, \mathbb{R}^m) \quad \text{as} \quad h \rightarrow 0 \]
holds.
Proof. The proof of the theorem based on Lemmas 3.2 and 3.3 is performed in the same way as for Theorem 2.5.

3.2. Estimation error. For the case of measuring a part of coordinates, one can also write an error rate for the proposed estimating procedure.

Theorem 3.5. Let Conditions 1, 2 and 3 be fulfilled, \( m = \kappa_1 = \kappa_2 \), \( \det C_1 \neq 0 \) and \( \det C_2 \neq 0 \). Then one can explicitly write a constant \( a_7 \) independent of \( h \), \( \alpha \) and \( \delta \), such that the inequality

\[
\|v^h(\cdot) - u^*(\cdot)\|_{L_2(T,\mathbb{R}^m)} \leq a_7(\delta(h) + \eta(h))^{\frac{3}{2}}
\]

holds for all \( h \in (0, h_*) \).

Proof. The validity of this theorem can be established following the scheme of the proof of Theorem 2.6. Considering formulas (41) and (45), we find

\[
\left| \int_{\sigma} C_2(v^h(s) - u^*(s)) \right|_m \leq Q_1 + Q_2, \quad t \in T, \quad (69)
\]

where \( Q_1 := \left| \int_{\sigma} ([D^s w_2^h](s) - [D^s x_2](s)) \right|_m \), \( Q_2 := \left| \int_{\sigma} (g_2(t, \xi^h(t), \psi^h(t)) - g_2(t, x_1(t), x_2(t))) \right|_m \).

To estimate \( Q_1 \), we use the Cauchy–Bunyakovsky–Schwartz inequality. Then we get

\[
Q_1 = ||[I^{1-\gamma}(w^h_2 - x_2)](t)||_m \\
= ||[I^{1-\gamma}[\varepsilon^h_2]|_m](t)||_m \\
\leq \tilde{c}_1( ||[I^{1-\gamma}[\varepsilon^h_2]|_m]|(t)||^{\frac{1}{2}} \),
\]

and further, using inequality (49), we derive

\[
Q_1 \leq \tilde{c}_2(\delta(h) + \eta(h))^{\frac{3}{2}}. \quad (70)
\]

Let us estimate \( Q_2 \). In view of (42), (43), and also applying the Cauchy–Bunyakovsky–Schwartz inequality, we have

\[
Q_2 \leq L \int_{\sigma} (||\xi^h(s) - x_1(s)||_m + ||\psi^h(s) - x_2(s)||_m) \, ds \\
\leq L \int_{\sigma} (h + |\omega^h_2(s)|_m) \, ds \\
\leq \tilde{c}_3 h + \tilde{c}_4 \left( ||[I^{1}[\omega^h_2]|_m]|(t)||^{\frac{1}{2}} \right). \quad (71)
\]

Transforming (71) with the use of (48), we have

\[
Q_2 \leq \tilde{c}_3 h + \tilde{c}_5 \pi^{\frac{3}{2}}(h). \quad (72)
\]

Inequalities (70) and (72) are used in (69). Then we find

\[
\left| \int_{\sigma} C_2(v^h(s) - u^*(s)) \right|_m \leq \nu(h), \quad t \in T, \quad (73)
\]
where $\nu(h) := \tilde{c}_1(\delta(h) + \eta(h))^\frac{\gamma}{2} + \tilde{c}_2 h + \tilde{c}_3 \pi^h(h)$. Using the formula for the inner product in $L_2(T, \mathbb{R}^m)$ and inequality (67), we find
\[
\|u - v^h\|^2_{L_2(T, \mathbb{R}^m)} = \|u\|^2_{L_2(T, \mathbb{R}^m)} - 2(u, v^h) + \|v^h\|^2_{L_2(T, \mathbb{R}^m)} \\
\leq 2\|u\|^2_{L_2(T, \mathbb{R}^m)} - 2(u, v^h) + (\delta(h))^{-1}\eta(h) \\
= 2(u - v^h, u) + (\delta(h))^{-1}\eta(h).
\]

Based on estimates (73) and (74), we apply the result of Lemma in [20, p. 47]. Then we get
\[
\|u - v^h\|^2_{L_2(T, \mathbb{R}^m)} \leq 2\hat{c}_2(V(u; T) + p)\nu(h) + (\delta(h))^{-1}\eta(h),
\]
where $\hat{c}_2$ denotes the Euclidean norm of the matrix $C^{-1}_2$. This inequality is reduced to the form (68).

**Remark 8.** For the set of parameters specified in Remark 3, estimate (68) takes the form
\[
\|u(\cdot) - v^h(\cdot)\|^2_{L_2(T, \mathbb{R}^m)} \leq R_2 h^\frac{1}{4},
\]
where the constant $R_2 > 0$ is independent of $h$, $\alpha$ and $\delta$.

**Remark 9.** Theorems 3.4 and 3.5 remain valid if the system
\[
[D^*_1 w^h_1](t) = f(t, w^h_1(t)) + C_1 \psi^h(t) + L(\xi^h(t) - w^h_1(t)), \\
[D^*_1 w^h_2](t) = f(t, w^h_1(t), w^h_2(t)) + C_2 \psi^h(t) + \frac{3}{2}L(\psi^h(t) - w^h_2(t)), \\
\]
t $\in T$, is chosen as a model.

4. Simulation.

4.1. Example 1. Consider the application of the proposed approximation technique for a biological model of HIV disease. One of such models with the fractional order Caputo derivative was proposed in [1]. Its further developments are reflected in [24]. The following system is chosen as basic
\[
[D^*_1 x](t) = \beta - kx - dx + by, \\
[D^*_1 y](t) = kx - (b + \delta)x, \\
[D^*_1 z](t) = N\delta x - cz,
\]
where uninfected cells, infected cells in the eclipse phase and free HIV virus particles are denoted by $x$, $y$ and $z$, respectively. Description of the parameters $\beta$, $\delta$, $b$, $c$, $d$, $k$ and $N$ can be found in the mentioned papers.

For our illustrative purposes, we consider the reduced system
\[
[D^*_1 x](t) = \beta - kx - dx + bu, \\
[D^*_1 z](t) = N\delta x - cz,
\]
where $u$ is assumed to be an unknown input action, and only $z$ coordinate can be inaccurately measured. We use the following values of parameters of system (75):
\[
\beta = 1, \quad \gamma = 0.8, \quad b = 0.2, \quad c = 3.4, \quad d = 0.01, \quad k = 0.000027, \quad N\delta = 0.016.
\]
Thus, we have the system
\[
[D^*_1^{0.8} x](t) = 1 - 0.027xz - 0.01x + 0.2u, \\
[D^*_1^{0.8} z](t) = 0.016x - 3.4z.
\]
Let the system act on the time interval $T = [0, 0.5]$. Let us define the solution $z(t)$ in the form

$$z(t) = 0.1 t^2 + 0.01. \quad (77)$$

Then we find $x(t)$ from the second equation of system (76):

$$x(t) = 2.125 + 11.345 t^{1.2} + 21.25 t^2. \quad (78)$$

Therefore, system (76) is complemented by the initial conditions

$$x_0 = 2.125, \quad z_0 = 0.01.$$

Using formulas (77) and (78), we have the input action $u$ as follows:

$$u(t) = -0.978176 + 14.0883 t^{0.4} + 38.6897 t^{1.2} + 0.223975 t^2 + 0.0306316 t^{3.2} + 0.057375 t^4, \quad t \in T.$$

Next, following the proposed estimation procedure, we define the reconstruction blocks. Then model (44), (45) takes the form

$$[D_{0.8}^h w_1](t) = -3.4 \xi^h(t) + 0.016 \psi^h(t),$$
$$[D_{0.8}^h w_2](t) = 1 - 0.027 \xi^h(t) \psi^h(t) - 0.01 \psi^h(t) + 0.2 \psi^h(t),$$

$$w_1(0) = 0.01, \quad w_2(0) = 2.125. \quad (79)$$

The rules for forming controls (46), (47) are represented by

$$\psi^h(t) = 0.016 (\alpha(h))^{-1} (\xi^h(t) - w_1(t)),$$
$$\psi^h(t) = 0.2 (\delta(h))^{-1} (\psi^h(t) - w_2(t)), \quad t \in T.$$

Parameters of the algorithm are chosen as follows:

$$\alpha(h) = h^{0.8}, \quad \delta(h) = h^{0.3}, \quad \beta_1 = 0.1, \quad \beta_2 = 0.3, \quad \beta_3 = 0.2.$$

The simulation results are presented in Figures 1, 2 and 3. The computations were performed for $h = 10^{-6}$ and for $h = 10^{-8}$ and correspond to letters (a) and (b). Solid lines depict the components $x, z$ and unknown input action $u$ of original system (76) and dashed lines image the components $w_1, w_2$ and reconstructed action $v^h$ of model (79).

![Figure 1. The component $z$ of system (76) and the component $w_1$ of model (79)]
4.2. Example 2. This example demonstrates the work of the dynamical estimation algorithm in the case when the different inputs generate the same trajectory. Let us consider the system

\[ [D_{\gamma}^\alpha] x(t) = u_1 + 0.125 u_2, \quad t \in T = [0, 1], \quad x(0) = 0, \]  

(80)

i.e. comparing (80) with system (1), we have \( \gamma = 0.8, B = (1, 0.125) \in \mathbb{R}^{1 \times 2} \) and \( u = (u_1, u_2)^\top \in \mathbb{R}^2 \). We choose the components of the input \( u \) in system (80) as follows:

\[
\begin{align*}
u_1(t) &= \begin{cases} 
 t, & t \in [0, 0.5], \\
 0.25, & t \in (0.5, 1], 
\end{cases} \\
u_2(t) &= \sin(8t), \quad t \in T.
\end{align*}
\]

Obviously, there exists at least one input \( \tilde{u} = (\tilde{u}_1, \tilde{u}_2)^\top \) such as

\[
\tilde{u}_1(t) = \begin{cases} 
 t + 0.125 \sin(8t), & t \in [0, 0.5], \\
 0.125 \sin(8t), & t \in (0.5, 1], 
\end{cases} \\
\tilde{u}_2(t) = \begin{cases} 
 0, & t \in [0, 0.5], \\
 2, & t \in (0.5, 1], 
\end{cases}
\]

generating a trajectory which is the same as the input \( u \) does. This follows directly from the equality \( Bu(t) = B\tilde{u}(t), \quad t \in T \). Thus, the trajectory of system (80) is defined by the formula

\[
x(t) = x(0) + [I \gamma Bu](t), \quad t \in T.
\]

The noisy measured data are modeled in the form

\[
\xi^h(t) = x(t) + h \sin(2\pi t), \quad t \in T, \quad h \in (0, 1).
\]
Next, we get the model of the form

\[ [D^2 y](t) = Bv^h, \quad t \in T, \quad y(0) = 0, \quad (81) \]

in which the control \( v^h \) is chosen according to rule (7). Substituting (7) into (81) and following the Cauchy formula for fractional differential equations [5, Theorem 6], we obtain

\[
y(t) = (a(h))^{-1}BB^T \int_0^t (t-s)^{\gamma-1}E_{\gamma,\gamma}
\left(-\frac{BB^T}{a(h)}(t-s)^{\gamma}\right)\xi_h(t)ds, \quad t \in T,
\]

where \( E_{\alpha,\beta}(z) \) denotes the Mittag–Leffler function [13, p. 42]. The latter formula is used for calculating \( v^h \) by rule (7). Let the function \( a(h) \) be defined as \( a(h) = h^\gamma \) and \( \beta_1 = 0.1, \beta_2 = 0.3 \). The simulation results are presented in Figures 4 and 5. The computations were performed for \( h = 10^{-3} \) in Figure 4 and for \( h = 10^{-2} \) and for \( h = 10^{-3} \) corresponding to letters (a) and (b) in Figure 5. Solid lines represent the input \( u \) and its components and dashed lines the reconstructed action \( v^h \). As one can see in Figure 4, the components \( u_1 \) and \( u_2 \) differ significantly from the components \( v^h_1 \) and \( v^h_2 \), whereas the cumulative effect of theirs on systems (80) and (81), \( Bu \) and \( Bv^h \), respectively, is the same. It can be seen in Figure 5. It should be noted that, in this complicated case, according to Theorem 2.5, \( v^h \) approximates some input \( u_\ast \) of the minimal \( L_2 \)-norm.

![Figure 4](image1.png)

**Figure 4.** The components \( u_1, u_2 \) and the components \( v^h_1, v^h_2 \)

![Figure 5](image2.png)

**Figure 5.** The input action \( Bu \) and its estimation \( Bv^h \)
5. **Conclusions.** The on-line estimation problem of a noisy input in a system with the Caputo fractional derivative is considered. The problem is complicated by two aspects. First, the coordinates (a part of coordinates) are available for measuring with an error. Second, measurements are continuously taken on-line. We propose the estimating procedure of the unknown input resistant to informational noises and computational errors. The procedure is based on a combination of the Krasovskii extremal aiming method with the local modified Tikhonov regularization method. As a result of applying the estimating procedure, the stable approximation of the input is obtained according to the feedback principle. The orders of estimates in Theorems 2.6, 3.5, as well as in Remarks 6, 9 can be improved. The continuous measurements are undoubtedly important for a number of problems in the mathematical modeling of real processes. In the numerical example, the operation of the constructed estimating procedure is demonstrated for the model of HIV disease. The simulation presents a useful application of the proposed procedure to the on-line reconstruction (approximation) of an unobservable component and an unknown input of the model. Also, a regularizing approximation can be given in the case when the desired input is non-unique. As results of computations, Figures 1–3, 5 show that, with a decrease in error rate, the input is approximated more accurately. A further development of the suggested approach is seen both in applying other types of regularization (e.g., the Lavrent’ev regularization or the Tikhonov regularization with a non-differentiable stabilizing functional) and considering the infinite time interval.

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