On Dévissage for Witt groups

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1 Introduction

In this article, we develop a version of the dévissage theorem [BW, Theorem 6.1] for Witt groups of Cohen-Macaulay rings. To introduce this theorem, let $A$ be a commutative Noetherian domain of dimension $d$ with 2 invertible and $K$ be its quotient field. It is a classical question (known as purity) as to when the map $W(A) \rightarrow W(K)$ is injective. Purity is a conjecture when $A$ is a regular local ring and is affirmatively settled in ([CS, OP, OPSS]).

In general, we can extend the above map to the right for any regular scheme by considering the Gersten-Witt complex. Let $X = \text{Spec}(A)$ be of dimension $d$ with 2 invertible and $X^{(n)}$ denote the points of codimension $n$. A Gersten-Witt complex

$$0 \rightarrow W(A) \rightarrow \bigoplus_{x \in X^{(0)}} W(k(x)) \rightarrow \bigoplus_{x \in X^{(1)}} W(k(x)) \rightarrow \ldots \rightarrow \bigoplus_{x \in X^{(d)}} W(k(x)) \rightarrow 0$$

was first constructed by Pardon [Pa1]. He further conjectured the exactness of this sequence for regular local rings and affirmatively settled it in many cases.

Subsequently, with the introduction of triangular Witt groups by Balmer [B1, B2, B3], Witt groups could be viewed as a cohomology theory. Using this, another Gersten-Witt complex could be defined (though both complexes look similar, it seems unproven that the differentials match) [BW], similar to the one in K-theory. Once again it is an open question as to when the complex is exact. In particular, it is conjectured to be so when

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We describe below the generalized form of this theorem for a Cohen-Macaulay ring $A$ with dimension $d$ and this allows a suitable definition of Witt groups $W^n(D^{b}(A))$ for modules with finite projective dimension, and $W^{n}(D^{b}(A)) \cong 0$ otherwise. We establish a Gersten-Witt complex of coherent Witt groups.

In this article, we take the second approach. Let $\mathcal{MFPD}(A)$ be the category of finitely generated $A$–modules with finite projective dimension, $\mathcal{MFL}(A)$ be the category of finitely generated $A$–modules with finite length, and $\mathcal{A} = \mathcal{MFPD}_{fl}(A)$ be the full subcategory of finitely generated $A$–modules with finite projective dimension and finite length (the "intersection"). Note then that $\mathcal{A}$ is an exact category and has a natural duality given by $M \mapsto Ext^{d}_{A}(M, A)$ and so we can consider the Witt group $W(A)$. By Balmer $[B2]$, we already know that $W(A) \cong W^{d}(D^{b}(A))$.

We consider the category $D^{b}_{A}(\mathcal{A})$ with homologies in $\mathcal{A}$. Based on the fact that $\mathcal{A}$ actually has the 2-out-of-3 property for objects, we prove that the duality actually restricts to $D^{b}_{A}(\mathcal{A})$ and this allows a suitable definition of Witt groups $W^{i}(D^{b}_{A}(\mathcal{A}))$. Once defined, we prove that the above isomorphism actually factors through isomorphisms $W(A) \sim \rightarrow W^{d}(D^{b}_{A}(\mathcal{A})) \sim \rightarrow W^{d}(D^{b}(A))$.

We now consider the category $D^{b}_{A}(\mathcal{P}(A))$. Similar to $D^{b}_{A}(\mathcal{A})$, we establish that this category is stable under duality and define the Witt groups $W^{i}(D^{b}_{A}(\mathcal{P}(A)))$. Having done so, we prove our version of dévissage $[5.12]$ $[6.7]$:

$$W(A) \sim \rightarrow W^{d}(D^{b}_{A}(\mathcal{P}(A)))$$

$$W^{-}(A) \sim \rightarrow W^{d+2}(D^{b}_{A}(\mathcal{P}(A)))$$

$$W^{d+1}(D^{b}_{A}(\mathcal{P}(A))) \cong W^{d-1}(D^{b}_{A}(\mathcal{P}(A))) \cong 0.$$
or the existence of a natural dualizing complex, we do not have access to the equivalences of the derived categories with duality as in [BW] or [G1] (in particular we cannot use the powerful lemma of Keller [K §1.5, Lemma and Example(b)]). Our proof is thus necessarily more elementary and naïve than the one in [BW]. One of the key ingredients in the proof is the construction of a special sublagrangian (5.6) for symmetric forms in \( (\mathcal{D}^b_A(\mathcal{P}(A))) \).

In deed, the methods in this paper have much wider applications. This is the first of a series of articles ([MS2, MS3]) dedicated to apply the sublagrangian theorem of Balmer ([B3]), or otherwise, to singular varieties and provide further insight into nonsingular varieties. Our interest in these studies stems from the introduction of the Chow-Witt groups \( \widetilde{CH}^r(A) \) for \( 0 \leq r \leq d \), due to Barge and Morel [BM] and developed by Fasel [F1], as the obstruction groups for splitting of projective modules, which works best for nonsingular varieties. This also serves as a motivation for our interest in maintaining the category of projective modules in our statement of dévissage. Jean Fasel informs that, for singular varieties \( X \), Chow-Witt groups \( \widetilde{CH}^r(X) \) and obstruction classes can be defined in the same manner, using coherent Witt groups. However, it is not known whether vanishing of the obstruction classes would lead to splitting. We feel that, for the purpose of developing an obstruction theory for singular varieties, it would be more natural to consider some analogue of the derived Witt groups of the category of projective modules. This approach would be of its own independent interest for both computations and applications. The results in this article also give rise to the possibility of a Gersten-Witt complex for these Witt groups, parallel to that of coherent Witt groups.

A word about the layout of the article: in section 2 we establish the basic definitions and a key result on projective dimensions. In the section 3 we establish the important theorem that the categories \( \mathcal{D}^b_A(\mathcal{P}(A)) \) and \( \mathcal{D}^b_A(A) \) are closed under duality, and more specifically how the homologies of the dual look like. Once this is established, in section 4 we define the Witt groups of the above categories and expectedly, they are 4-periodic, i.e. \( W^n(\mathcal{D}^b_A(\mathcal{P}(A))) \overset{\sim}{\longrightarrow} W^{n+4}(\mathcal{D}^b_A(\mathcal{P}(A))) \). Finally, in sections 5 and 6 we prove our main theorems about dévissage.

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2 Basic Notations and Preliminaries

Notations 2.1. Throughout this article, \( A \) will denote a Cohen-Macaulay ring with \( \dim A_m = d \geq 2 \), for all maximal ideals \( m \) of \( A \). Further, \( 2 \) is always invertible in \( A \). We set up the notations:

1. \( \mathcal{M}(A) \): category of finitely generated \( A \)-modules.
2. \( \mathcal{MFDP}(A) \): full subcategory of finitely generated \( A \)-modules with finite projective dimension.

3. \( \mathcal{MFLL}(A) \): category of finitely generated \( A \)-modules with finite length.

4. \( \mathcal{A} = \mathcal{MFDP}_{fl}(A) \): category of finitely generated \( A \)-modules with finite length and finite projective dimension.

5. \( \mathcal{P}(A) \): category of finitely generated projective \( A \)-modules.

6. For any exact category \( \mathcal{C} \), \( Ch^b(\mathcal{C}) \) is the category of bounded chain complexes with objects in \( \mathcal{C} \), and \( D^b(\mathcal{C}) \) is its derived category.

7. For any two exact categories \( \mathcal{C}, \mathcal{D} \) in an ambient abelian category \( \mathcal{C}' \), \( Ch^b_D(\mathcal{C}) \) is the full subcategory of \( Ch^b(\mathcal{C}) \) consisting of complexes with homologies in \( \mathcal{D} \). \( D^b_D(\mathcal{C}) \) is its derived category, which is also the full subcategory of \( D^b(\mathcal{C}) \) consisting of objects from \( Ch^b_D(\mathcal{C}) \).

8. \( \mathcal{R} \): full subcategory of \( D^b_A(\mathcal{P}(A)) \) consisting of objects \( P_* \) such that \( P_i = 0 \) for \( i > d, i < 0 \) and \( H_i(P_*) = 0 \) for all \( i \neq 0 \) and \( H_0(P_*) \in \mathcal{A} \).

9. For objects \( M \) in \( \mathcal{A} \), let \( M^\vee = \text{Ext}^d_A(M, A) \) and \( \sim : M \sim M^{\vee \vee} \) be the identification by double ext. (but we make this more precise in diagram (5) and the explanation of \( \iota \).

10. We will denote complexes \( P_* \) by :

\[
\cdots 0 \longrightarrow P_m \overset{\partial_n}{\longrightarrow} P_{m-1} \longrightarrow \cdots \longrightarrow P_n \longrightarrow 0 \cdots
\]

11. A non-zero complex \( P_* \) is said to be supported on \([m, n]\) if \( P_i = 0 \) for all \( i < n \) and \( i > m \).

12. For a complex \( P_* \) of projective \( A \)-modules \( P_* \) will denote the usual dual induced by \( Hom(\ast, A) \) and \( \sim : P_* \sim P_*^{**} \) will denote the identification by evaluation. Note that the degree \( r \) component of the dual \( P_* \) is \((P_{-r})^* \).

13. The \textbf{length} of a non-zero complex \( P_* \) is defined as \( \ell(P_*) = u - l \) where \( P_u \neq 0, P_l \neq 0 \) and \( P_i = 0 \) for all \( i < l \) and \( i > u \).

14. Let \( B_r = B_r(P_*) := \partial_{r+1}(P_{r+1}) \subseteq P_r \) denote the module of \( r \)-boundaries and \( Z_r = Z_r(P_*) := \ker(\partial_r) \subseteq P_r \) denote the module of \( r \)-cycles (or the \( r^{th} \) syzygy).
15. The \( r \)th-homology of \( P_\bullet \) will be denoted by \( H_r = H_r(P_\bullet) := \frac{\ker(\partial_{r-1})}{\text{image}(\partial_r)} \). So, the \( r \)th-homology of the dual is \( H_r(P_\bullet^*) = \ker(\partial^*_{r-1}) = \frac{\text{image}(\partial^*_r)}{\ker(\partial^*_{r-1})} \).

16. A full exact subcategory \( C \) of an abelian category \( D \) is said to have the 2-out-of-3 property if for every short exact sequence in \( D \), whenever two of the objects are objects of \( C \), then so is the third.

Remark 2.2. The subcategory \( MF_{FD}(A) \) and \( A \) are both exact subcategories and in fact have the 2-out-of-3 property. The category \( R \) is also an exact category. Although it is a subcategory of \( D^b_A(\mathcal{P}(A)) \), it has no translation and is actually naturally equivalent to the category \( A \). The natural functor \( \eta : R \to A \) is given by sending a complex \( Q_\bullet \) to \( H_0(Q_\bullet) \). The inverse functor \( \iota \) is given by associating to objects \( M \in A \) a projective resolution of length \( d \).

Note further that when \( A \) is not regular, the categories \( D^b_A(\mathcal{P}(A)) \) and \( D^b_{MF_{PD}(A)}(\mathcal{P}(A)) \) are not closed under the cone operation as the following example demonstrates.

Example 2.3. Let \( (A, \mathfrak{m}) \) be a non-regular Cohen-Macaulay ring with \( \dim A = d \), such that \( \mathfrak{m} = (f_1, f_2, \ldots, f_d, z) \). We can assume, using prime avoidance, that \( f_1, f_2, \ldots, f_d \) is a regular sequence. Let \( U_\bullet = \text{Kos}_\bullet(f_1, f_2, \ldots, f_d) \) be the Koszul complex. Since the only nonzero homology of \( U_\bullet \) is \( H_0(U_\bullet) = \frac{A}{(f_1, f_2, \ldots, f_d)} \in A \), \( U_\bullet \) and all its translates are objects of both the categories above. Let \( C(z) \) denote the cone of the the chain complex map \( z : U_\bullet \to U_\bullet \). From the long exact homology sequence corresponding to the short exact sequence of chain complexes

\[
\begin{array}{cccccc}
0 & \longrightarrow & U_\bullet & \longrightarrow & C(z) & \longrightarrow & U_\bullet[1] & \longrightarrow & 0 \\
\end{array}
\]

it follows that

\[
H_0(\text{Cone}(z)) \cong \text{coker}(\frac{A}{(f_1, f_2, \ldots, f_d)} \to \frac{A}{(f_1, f_2, \ldots, f_d)}) \cong \frac{A}{\mathfrak{m} \notin A}.
\]

So, \( C(z) \) is not an object in \( D^b_A(\mathcal{P}(A)) \).

Next, we ask if \( CH^b_{MF_{PD}}(\mathcal{P}(A)) \) is closed under duality. We thank Sankar Dutta for providing the following example:

Example 2.4 (Dutta). Let \( (A, \mathfrak{m}, k) \) be any non-regular Cohen-Macaulay local ring, with \( \dim A = d \). Let

\[
\begin{array}{cccc}
\cdots & \longrightarrow & P_d & \xrightarrow{\partial_d} & P_{d-1} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & k & \longrightarrow & 0 \\
\end{array}
\]
be a projective resolution of $k$. Let $\ast$ denote $\text{Hom}(-, A)$ and $M = \text{cokernel}(\partial_d^\ast)$. Since $\text{Ext}^r(k, A) = 0$ for all $0 \leq r < d$, the sequence

$$
0 \longrightarrow P_0^\ast \longrightarrow \cdots \longrightarrow P_{d-1}^\ast \longrightarrow P_d^\ast \longrightarrow M \longrightarrow 0
$$

is a projective resolution of $M$. Dualizing this sequence, it follows that $\text{Ext}_d^A(M, A) \cong k$, which does not have finite projective dimension. In particular, $\text{Ch}_{\text{MF}P\text{D}}(\mathcal{P}(A))$ is not closed under duality.

Note however that in the above example, $M$ does not have finite length. Indeed, we will prove in section 3 that the category $\text{Ch}^b_A(\mathcal{P}(A))$ is closed under duality.

We mention a few standard results for the sake of completeness.

**Lemma 2.5.** Let $(A, m)$ be a Cohen-Macaulay local ring with $\dim A = d$. Let $M \in \mathcal{M}FL(A)$. Then Further, $\text{Ext}^i(M, A) = 0$ for all $i < d$, and $\text{Ext}^d(M, A) \neq 0$ is also in $\mathcal{M}FL(A)$. Note further that if $M \in A$, then so is $\text{Ext}^d(M, A)$.

**Lemma 2.6.** Let $A$ be a Cohen-Macaulay ring with $\dim A = d$. Let $M \in A$. There is a natural isomorphism

$$
\varpi : M \simto M^\vee.
$$

**Corollary 2.7.** $(A^\vee, \pm \varpi)$ are exact categories with duality.

**Proposition 2.8.** Let $A$ be Cohen-Macaulay with $d = \dim A_m \geq 2$ for all maximal ideals $m$. Let $P_{\bullet}$ be a complex in $\text{Ch}^b(\mathcal{P}(A))$. Assume that all the homologies $H_r := H_r(P_{\bullet}) \in A$. Then we claim:

1. The modules $B_r$ and $Z_r$ have finite projective dimension $\forall r$. In that case, $\text{proj dim}(Z_r) = \text{proj dim}(B_{r-1}) - 1$.

2. For all $r \in Z_r$, we have $\text{proj dim} B_r \leq d - 1$ and so $\text{proj dim} Z_r \leq d - 2$.

3. If $H_r \neq 0$ then $\text{proj dim} B_r = d - 1$.

**Proof.** Note that $P_{\bullet}$ is a bounded complex and so let it be supported on $[m, n]$. Then $Z_n = P_n$ and so has finite projective dimension. Since there are short exact sequences:

$$
0 \to B_r \to Z_r \to H_r \to 0 \quad 0 \to Z_r \to P_r \to B_{r-1} \to 0,
$$

it is clear that if $Z_r$ is of finite projective dimension, then so are $B_r$ and $Z_{r+1}$, and hence the proof follows by induction. The second part of (1) also follows from the above exact sequence.
Since $B_r$ is torsion free, it has depth at least 1 and hence, by the Auslander-Buchsbaum theorem, $\text{proj dim}(B_r) \leq d - 1$. So, $\text{proj dim}(Z_r) \leq d - 2$. So, (2) is established.

Now, assume $H_r \neq 0$. Choose a maximal ideal $\mathfrak{m}$ in the support of $H_r$. Then, consider the localized short exact sequence

$$0 \to (B_r)_\mathfrak{m} \to (Z_r)_\mathfrak{m} \to (H_r)_\mathfrak{m} \to 0.$$ 

Then, we get a long exact sequence of $\text{Tor}^A_\mathfrak{m}(-, A/\mathfrak{m})$, which gives us that

$$\text{Tor}^d_{A_\mathfrak{m}}((H_r)_\mathfrak{m}, A/\mathfrak{m}) \cong \text{Tor}^{d-1}_{A_\mathfrak{m}}((B_r)_\mathfrak{m}, A/\mathfrak{m}) \quad \text{and} \quad \text{Tor}^d_{A_\mathfrak{m}}((B_r)_\mathfrak{m}, A/\mathfrak{m}) \cong 0,$$

since we have already proved that $\text{proj dim}_A Z_r \leq d - 2$. Thus we obtain $\text{proj dim}_A(B_r)_\mathfrak{m} = d - 1$. Since we know that $\text{proj dim}_A(B_r) \leq d - 1$, this implies $\text{proj dim}_A(B_r) = d - 1$. This establishes (3).

The complexes in $Ch^b(P(A))$ with finite length homologies have at least $d$ nonzero components at the left where the homology is 0. This proposition plays a key role in sections 5 and 6.

**Proposition 2.9.** Let $A$ be Cohen-Macaulay with $d = \dim A_\mathfrak{m}$ for all maximal ideals $\mathfrak{m}$. Let $P_\bullet$ be a bounded complex of projective modules, such that $H_i = 0 \forall i > n$ and $H_n \neq 0$ is of finite length. Then $P_i \neq 0, n \leq i \leq n + d$.

**Proof.** Since $H_i = 0 \forall i > n$ and the complex is bounded, we get that $\frac{P_n}{B_n}$ is of finite projective dimension, since the components with indices $\geq n$ give a resolution. Now, let $\mathfrak{m}$ be a maximal ideal in the support of $H_n(P_\bullet)$, then $(H_n(P_\bullet))_\mathfrak{m} \subseteq \frac{(P_n)_\mathfrak{m}}{(B_n)_\mathfrak{m}}$ is of finite length, and hence $(P_n)_\mathfrak{m}$ has depth 0. By the Auslander-Buchsbaum theorem, $\text{proj dim}_A\frac{(P_n)_\mathfrak{m}}{(B_n)_\mathfrak{m}} = d$. But that means $\text{proj dim}_A\frac{P_n}{B_n} = d$. Hence, the resolution of $\frac{P_n}{B_n}$ given by the components of $P_\bullet$ with indices $\geq n$ must have length at least $d$. Hence, $P_i \neq 0, n \leq i \leq n + d$. The proof is complete.

### 3 Duality

As always, $A$ will denote a Cohen-Macaulay ring with $\dim A_\mathfrak{m} = d \geq 2$, for all maximal ideals $\mathfrak{m}$ and $A = \mathcal{MFPD}_{fl}(A)$. In this section, we prove that the category $Ch^b_A(P(A))$ is closed under duality and give a precise description of the homologies of the dual.

**Theorem 3.1.** Suppose $P_\bullet$ is a complex in $Ch^b(P(A))$ with homologies in $A$. Then we
have:

\[
\begin{align*}
\text{Ext}^i(Z_r, A) &\cong \begin{cases} 
\text{Ext}^d(H_{r+i-(d-1)}, A) & 1 \leq i \leq d-2 \\
0 & \text{for } i \geq d-1
\end{cases} 
\quad (1) \\
\text{Ext}^i(B_r, A) &\cong \begin{cases} 
\text{Ext}^d(H_{r+i-(d-1)}, A) & 1 \leq i \leq d-1 \\
0 & \text{for } i \geq d
\end{cases} 
\quad (2) \\
\text{Ext}^i\left(\frac{P_r}{B_r}, A\right) &\cong \begin{cases} 
\text{Ext}^d(H_{r+i-d}, A) & 1 \leq i \leq d \\
0 & i \geq d
\end{cases} 
\quad (3)
\end{align*}
\]

**Proof.** Since \( P_i = 0 \ \forall i \ll 0 \), the theorem is true for \( r \ll 0 \). So, we assume that the theorem is true for \( r-1 \) and prove it for \( r \).

Corresponding to the short exact sequence \( 0 \rightarrow Z_r \rightarrow P_r \rightarrow B_{r-1} \rightarrow 0 \), we get a long exact Ext-sequence which yields

\[
0 \rightarrow \text{Ext}^0(B_{r-1}, A) \rightarrow P_r^* \rightarrow \text{Ext}^0(Z_r, A) \rightarrow \text{Ext}^1(B_{r-1}, A) \rightarrow 0 
\quad (4)
\]

and for \( i \geq 1 \) we have \( \text{Ext}^i(Z_r, A) \cong \text{Ext}^{i+1}(B_{r-1}, A) \). Thus, the induction hypothesis yields that for \( i \geq 1 \),

\[
\text{Ext}^i(Z_r, A) = \text{Ext}^{i+1}(B_{r-1}, A) = \begin{cases} 
\text{Ext}^d(H_{r+i-(d-1)}, A) & 1 \leq i \leq d-2 \\
0 & \text{for } i \geq d-1
\end{cases}
\]

So, equation (1) is established.

Consider the long exact Ext-sequence corresponding to the short exact sequence \( 0 \rightarrow B_r \rightarrow Z_r \rightarrow H_r \rightarrow 0 \). By (2.5), \( \text{Ext}^i(H_r, A) = 0 \) for all \( i \neq d \) and since \( \text{Ext}^i(Z_r, A) = 0 \ \forall i > d-2 \) from equation (1), it follows that

\[
\text{Ext}^i(B_r, A) \cong \begin{cases} 
\text{Ext}^i(Z_r, A), & 0 \leq i \leq d-2 \\
\text{Ext}^d(H_r, A), & i = d-1 \\
0, & i \geq d
\end{cases}
\]

\[
\cong \begin{cases} 
\text{Ext}^0(Z_r, A), & i = 0 \\
\text{Ext}^d(H_{r+i-(d-1)}, A), & 1 \leq i \leq d-1 \\
0, & i \geq d
\end{cases}
\]

Now consider the short exact sequence \( 0 \rightarrow H_r \rightarrow \frac{P_r}{B_r} \rightarrow B_{r-1} \rightarrow 0 \). Again, \( \text{Ext}^i(H_r, A) = 0 \ \forall i \neq d \) from (2.5) and from equation (2) we get \( \text{Ext}^i(B_r, A) = 0 \ \forall i > d-1 \). So it follows
that

\[
\text{Ext}^i \left( \frac{P_r}{R}, A \right) \cong \begin{cases} 
\text{Ext}^i (B_{r-1}, A), & 0 \leq i \leq d - 1 \\
\text{Ext}^d (H_r, A), & i = d \\
0, & i > d
\end{cases}
\]

\[
\cong \begin{cases} 
\text{Ext}^i (B_{r-1}, A), & i = 0 \\
\text{Ext}^d (H_{r+i-d}, A), & 1 \leq i \leq d \\
0, & i > d
\end{cases}
\]

\[\square\]

**Corollary 3.2.** Suppose \( P \) is a complex in \( Ch^b (\mathcal{P}(A)) \) with homologies in \( A \). Then, for all \( r \in \mathbb{Z} \)

\[
\text{Ext}^i (B_r, A), \quad \text{Ext}^i (Z_r, A), \quad \text{Ext}^i \left( \frac{P_r}{R}, A \right)
\]

are in \( A \) for \( i \geq 1 \) and are in \( \mathcal{MFDP}(A) \) for \( i = 0 \).

**Proof.** By (2.5) and the preceding theorem (3.1), for \( i \geq 1 \), the statement is clear. For \( i = 0 \), we recall below equation (4) from the preceding proof:

\[
0 \rightarrow \text{Ext}^0 (B_{r-1}, A) \rightarrow P_r^* \rightarrow \text{Ext}^0 (Z_r, A) \rightarrow \text{Ext}^1 (B_{r-1}, A) \rightarrow 0.
\]

and that we also proved \( \text{Ext}^0 (B_r, A) \cong \text{Ext}^0 (Z_r, A) \) and \( \text{Ext}^0 \left( \frac{P_r}{R}, A \right) \cong \text{Ext}^0 (B_{r-1}, A) \).

Hence, it is enough to know that \( \text{Ext}^0 (B_{r-1}, A) \) satisfies the theorem. Once again induction saves the day! \[\square\]

This allows us to conclude our main theorem of the section.

**Theorem 3.3.** Let \( P \) be a complex as in theorem (3.1). Then, for \( t \in \mathbb{Z} \), we have

\[
H_t (P_r^*) \cong \text{Ext}^d (H_{t-d} (P_r), A) \cong H_{t-d} (P_r^*)^\vee.
\]

In particular, \( H_r (P_r^*) \in A \) and hence, \( \text{Ch}^b_A (\mathcal{P}(A)) \) is closed under duality.

**Proof.** Consider the dual complex: \( \cdots P_{t-1}^* \overset{(\partial_t)^*}{\rightarrow} P_t^* \overset{(\partial_{t+1})^*}{\rightarrow} P_{t+1}^* \cdots \). Note that \((\partial_{t+1})^* : P_t^* \rightarrow P_{t+1}^* \) factors through

\[
P_t^* \rightarrow B_t^* \hookrightarrow \left( \frac{P_{t+1}}{B_{t+1}} \right)^* \rightarrow P_{t+1}^*
\]

(recall \( d \geq 2 \)) and hence, \( \ker((\partial_{t+1})^*) = \ker(P_t^* \rightarrow B_t^*) = \left( \frac{P_t}{B_t} \right)^* \). Similarly, \( \ker((\partial_t)^*) = \left( \frac{P_{t-1}}{B_{t-1}} \right)^* \) and hence we obtain the exact sequence

\[
0 \rightarrow \left( \frac{P_{t-1}}{B_{t-1}} \right)^* \rightarrow (P_{t-1})^* \rightarrow \left( \frac{P_t}{B_t} \right)^* \rightarrow H_{-t} (P_r^*) \rightarrow 0.
\]

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Note also that there is an exact Ext-sequence
\[ 0 \to \left( \frac{P_{t-1}}{B_{t-1}} \right)^* \to (P_{t-1})^* \to (B_{t-1})^* \to \text{Ext}^1 \left( \frac{P_{t-1}}{B_{t-1}}, A \right) \to 0. \]

But since \( d \geq 2 \), we have
\[ 0 \to \left( \frac{P_{t-1}}{B_{t-1}} \right)^* \to (P_{t-1})^* \to (B_{t-1})^* \to \text{Ext}^1 \left( \frac{P_{t-1}}{B_{t-1}}, A \right) \to 0. \]

Hence, by (3.1), we get that \( H_{-t}(P^*_t) \cong \text{Ext}^d (H_{-d}(P^*_t), A) \). The rest follows from (2.5).

\[ \square \]

Remark 3.4. It is a straightforward diagram check that all the isomorphisms in (3.1) and (3.3) are natural. In particular, that means that if we have a morphism of complexes \( P \to Q \), then there is a commutative diagram:

\[
\begin{array}{ccc}
H_{-t}(Q^*_t) & \xrightarrow{H_{-t}(f^*)} & H_{-t}(P^*_t) \\
\downarrow & & \downarrow \\
H_{t-d}(Q^*_t) & \xrightarrow{H_{t-d}(f)} & H_{t-d}(P^*_t)
\end{array}
\]

Finally, as an easy consequence of the above theorem, we obtain (for free!) that \( D^b_A(A) \) is closed under duality \( M^\vee = \text{Ext}^d(M, A) \).

Theorem 3.5. The category \( \text{Ch}^b_A(A) \) is closed under the duality \( \vee \) induced by the duality \( \vee \) in \( A \).

Proof. Suppose \( M \) is a complex in \( \text{Ch}^b_A(A) \). Without loss of generality, we assume \( M \) is supported on \([n, 0]\). Each component \( M_i \) has a projective resolution of length \( d \), and putting them together with the induced maps, we get a double complex \( L_{\bullet \bullet} \), as in the left figure below:

\[
L_{\bullet \bullet} = \cdots \to \cdots \to \to L_{d-1} \to L_{d-1} \to 0 \to \to L_{d-2} \to L_{d-2} \to 0 \to \cdots \to \to L_0 \to L_0 \to 0 \to \cdots \to 0
\]

\[
M_{\bullet} = \cdots \to \cdots \to \to M_{n-1} \to M_{n-1} \to 0 \to \to M_{n-2} \to M_{n-2} \to 0 \to \cdots \to \to M_0 \to M_0 \to 0
\]

Dualizing, \( L_{\bullet \bullet}' \) gives a similar resolution of \( M_{\bullet}^\vee \), as shown on the right above (note that there are sign conventions on the differentials of the complexes here, in particular \( M_{\bullet}^\vee \) acquires a \((-1)^d\) factor on its differentials. We refer to \([W]\) for the conventions for total complexes.)

Now, the total complexes give quasi-isomorphisms
\[ \text{Tot}(L_{\bullet \bullet}) \to M, \quad \text{Tot}(L_{\bullet \bullet}') \to M_{\bullet}^\vee. \]
So, $H_i(Tot(L_{\bullet\bullet})) \in A$ for all $i \in \mathbb{Z}$. By (5.3), $H_i(Tot(L_{\bullet\bullet}')) \in A$ for all $i \in \mathbb{Z}$. Now after translating $Tot(L_{\bullet\bullet})$ $d$ components to the left, we observe that it is actually chain homotopy equivalent to $Tot(L_{\bullet\bullet}')$ and so we have $H_i(Tot(L_{\bullet\bullet})) \in A$. Finally, the above quasi-isomorphism yields that $H_i(M_{\bullet}^\vee) \sim H_i(Tot(L_{\bullet\bullet})) \in A$. This completes the proof. □

4 Definitions of Witt Groups

In this section, we define Witt groups of the categories we work with. In particular, we extend the definition of Witt groups from triangulated categories with duality to their additive subcategories which are closed under orthogonal sums, translations and isomorphisms. Since it is possible that there is cause for confusion about translation, we start by clearing the air.

Definition 4.1. In all the categories of complexes, there are two possible translations, $T_u$ and $T_s$. The complex $T_uP_\bullet$ is defined as $(T_uP_\bullet)_i = P_{i-1}$ and $\partial(T_uP_\bullet)_i = \partial(P_\bullet)_{i-1}$. The complex $T_sP_\bullet$ is defined as $(T_sP_\bullet)_i = P_{i-1}$ and $\partial(T_sP_\bullet)_i = -\partial(P_\bullet)_{i-1}$.

Note that $T_s$ seems to be the "standard" translation in literature and that is always the translation we use on any category of complexes.

However, given a duality $\ast$ on such a category (e.g. $D^b_A(A)$ and $D^b_A(\mathcal{P}(A))$), there are shifted dualities, $T^{n\ast}_s$ and $T^{n\ast}_u$. We work with the unsigned duality $T^{n\ast}_u$ until we reach section 6. Note however that $H_i(T^{n\ast}_sP_\bullet) = H_i(T^{n\ast}_uP_\bullet)$ and so much of what we will say is independent of the chosen duality.

Remark 4.2. We quickly review the situation for the categories $A$ and $\mathcal{R}$. First note that both of these categories are exact categories with duality and so the Witt groups are defined as in [QSS].

The functor $\iota$ induces duality preserving equivalences

$$\iota : (A,^\vee,\tilde{\omega}) \to (\mathcal{R}(A),T^d_u\ast,\omega)$$

$$\iota : (A,^\vee,-\tilde{\omega}) \to (\mathcal{R}(A),T^d_u\ast,-\omega)$$

of categories which then yield isomorphisms of the corresponding Witt groups

$$W(\iota) : W(A,^\vee,\tilde{\omega}) \sim W(\mathcal{R}(A),T^d_u\ast,\omega)$$

$$W(\iota) : W(A,^\vee,-\tilde{\omega}) \sim W(\mathcal{R}(A),T^d_u\ast,-\omega)$$

Finally, we get to our definitions of the Witt group. Given an exact category $\mathcal{E}$, its derived category will be denoted by $D^b(\mathcal{E})$. For a subcategory $\mathcal{C}$, $D^b_{\mathcal{C}}(\mathcal{E})$ will denote the full subcategory of $D^b(\mathcal{E})$ consisting of complexes with homologies in $\mathcal{C}$. 

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The derived categories which will be used in this article include:

\[ D^b_A(\mathcal{P}(A)) \subseteq D^b_{fl}(\mathcal{P}(A)) \subseteq D^b(\mathcal{P}(A)) \subseteq D^b(\mathcal{A}) \subseteq D^b(\mathcal{A}). \]

We have the following diagram of subcategories and functors:

\[ \begin{array}{ccc}
A & \xrightarrow{\mu} & D^b_A(\mathcal{A}) \\
\downarrow \scriptstyle{\iota} & & \downarrow \scriptstyle{\nu} \\
\mathcal{R}(A) & \xleftarrow{\mu'} D^b_A(\mathcal{P}(A)) & \xrightarrow{\nu'} D^b_{fl}(\mathcal{P}(A))
\end{array} \]

Here \( \mu(M) \) is the complex concentrated at degree zero. The functor \( \iota(M) \) is obtained by making a choice of projective resolution of length \( d \) and then defining \( M^\vee = \text{H}_0((\zeta(M))^*) \).

The functors \( \alpha \) and \( \beta \) are essentially induced by these ones, by taking the total complex (take a look at the proof of (3.5)).

We now move on towards the definitions of the Witt groups of the categories \( D^b_A(\mathcal{A}) \) and \( D^b_A(\mathcal{P}(A)) \). We once again remind the reader that this definition relies on the definitions in [B2].

**Definition 4.3.** Let \( \delta = \pm 1 \). Suppose \( K := (K, \#, \delta, \varpi) \) is a triangulated category with translation \( T \) and \( \delta \)-duality \( \# \). Suppose \( K_0 \) is a full subcategory of \( K \) that is closed under isomorphism, translation and orthogonal sum. We abuse notation and denote \( K_0 := (K_0, \#, \delta, \varpi) \) in order to keep track of the duality and canonical isomorphism in use.

1. Define the Witt monoid of \( MW(K_0) \) to be the submonoid

\[ MW(K_0) = \{(P, \varphi) \in MW(K) : P \in \text{Ob}(K_0)\}. \]

2. A symmetric space \( (P, \varphi) \in MW(K_0) \) will be called a **neutral space** in \( MW(K_0) \) if it has a Lagrangian \( (L, \alpha, w) \) in \( MW(K) \) such that \( L, L^\# \in \text{Ob}(K_0) \).

3. Let \( NW(K_0) \) be the submonoid of \( MW(K_0) \) generated by the isometry classes of neutral spaces in \( K_0 \).

4. Define the Witt group

\[ W(K_0) := \frac{MW(K_0)}{NW(K_0)}. \]

Note \( (Q, \chi) \in MW(K_0) \implies (Q, -\chi) \in MW(K_0) \). It is easy to check that \( (Q, \chi) \perp (Q, -\chi) \in NW(K_0) \). So, \( W(K_0) \) has a group structure. We use this definition in the context of derived categories of exact categories with duality.
5. Let \((\mathcal{C}, ^\vee, \varpi)\) be an exact subcategory with duality in an ambient abelian category \(\mathcal{C}'\) and let \(\mathcal{D}\) be any subcategory of \(\mathcal{C}'\) closed under orthogonal sum. Let \(K_0 = D^b_\mathcal{D}(\mathcal{C})\) \((\text{2.1})\). Then with the induced duality and natural isomorphism, the Witt group \(W\left(D^b_\mathcal{D}(\mathcal{C}), ^\vee, \delta, \varpi\right)\) is defined as above.

6. Accordingly, with \(T = T_u, T_s\), the Witt groups

\[
W\left(D^b_\mathcal{A}(\mathcal{P}(A)), T^{u_* \ast}, \pm 1, \pm \varpi\right), \quad W\left(D^b_\mathcal{A}(A), T^{u_* \ast}, \pm 1, \pm \tilde{\varpi}\right)
\]

are defined.

5 Isomorphisms of Witt Groups

All the functors above induce homomorphisms of Witt groups. As always, \(A\) denotes a Cohen-Macaulay ring with \(\dim A_m = d \geq 2\), for all maximal ideals \(m\) of \(A\). Let \(D^b_\mathcal{A}(A, ^\vee, \pm \tilde{\varpi})\) denote the duality structure, respectively, on \(D^b_\mathcal{A}(A)\) and \(D^b(\mathcal{A})\) induced by \((A, ^\vee, \pm \tilde{\varpi})\) and \(D^b_\mathcal{A}(\mathcal{P}(A))\) of the Witt groups. The goal of this section is to establish the following diagram of homomorphisms of Witt groups:

\[
\begin{array}{c}
\xymatrix{ W(\mathcal{A}, ^\vee, \pm \tilde{\varpi}) \ar[r]^{W(\mu)} & W\left(D^b_\mathcal{A}(\mathcal{A}, ^\vee, \pm \tilde{\varpi})\right) \ar[r]^{W(\nu)} & W\left(D^b(\mathcal{A}, ^\vee, \pm \tilde{\varpi})\right) \\
W(\mathcal{R}(A), T^{u_* \ast}, \pm \varpi) \ar[u]^{W(\iota)} \ar[r]_{W(\gamma)} & W\left(D^b_\mathcal{A}(\mathcal{P}(A))\right)_{\pm} \ar[u]_{W(\alpha)} }
\end{array}
\]

(6)

Note that we already know that \(W(\iota)\) is an isomorphism \((\text{4.2})\) and further, by \([\text{B3}, \text{Theorem 4.3}]\), \(W(\nu \circ \mu)\) is an isomorphism. The proof that

\[
W(\mu) : W(\mathcal{A}, ^\vee, \pm \tilde{\varpi}) \longrightarrow W\left(D^b_\mathcal{A}(\mathcal{A}, ^\vee, \pm \tilde{\varpi})\right)
\]

are isomorphisms follows from more abstract "general nonsense" which we prove in \([\text{A.1}]\) as part of the appendix \([\text{A}]\). Since \(W(\nu \circ \mu)\) and \(W(\mu)\) are isomorphisms, it is clear that so is \(W(\nu)\). The main result of this section is that \(W(\zeta)\) is an isomorphism. That being established, it is clear that \(W(\gamma)\) and \(W(\alpha)\) are isomorphisms.

For the rest of this section, we use the notation \# := \(T^{d_* \ast}\). First, we establish the following regarding the structure of symmetric forms.
Lemma 5.1. Suppose $\eta : X_\bullet \sim X_\bullet^\#$ is a symmetric form in $(D^b_A(\mathcal{P}(A))^\pm_u)$, such that

$$H_{-m}(X_\bullet) \neq 0, \quad \text{and} \quad H_i(X_\bullet) = 0 \quad \text{for all} \quad i < -m.$$ 

Then, there is a complex $P_\bullet$ in $Ch^b_A(\mathcal{P}(A))$ and a quasi-isomorphism $\varphi : P_\bullet \rightarrow P_\bullet^\#$ such that

1. $(P_\bullet, \varphi)$ is isometric to $(X_\bullet, \eta)$ in $(D^b_A(\mathcal{P}(A))^\pm_u)$.

2. $P_\bullet$ is supported on $[m + d, -m]$.

3. $H_{-m}(P_\bullet) \neq 0$.

Proof. Recall from (2.9) that since $H_{-m}(X_\bullet) \neq 0$, $X_\bullet$ has length at least $d$. By duality, we conclude that $m \geq 0$. By definition there is a complex $P_\bullet$ of projective modules and a quasi-isomorphism $t : P_\bullet \rightarrow X_\bullet$, a chain complex morphism $\varphi_0 : P_\bullet \rightarrow X_\bullet^\#$ such that $\eta = \varphi_0 t^{-1}$. Then, $\varphi = t^\# \varphi_0 = t^\# \eta t$ is a symmetric form on $P_\bullet$, and $(X_\bullet, \eta)$ is isometric to $(P_\bullet, \varphi)$. By including enough zeros on the two tails, we can assume $P_\bullet$ is supported on $[n + d, -n]$, for some $n \geq m$. If $m = n$ there is nothing to prove. So, assume $n > m$. We have, $H_{-n}(P_\bullet) = 0$. Inductively, we will cut down the support to $[m + d, m]$. We write $\nu : P_\bullet \rightarrow P_\bullet^\#$ as follows

$$
\begin{array}{cccccccc}
0 & \rightarrow & P_{n+d} & \rightarrow & P_{n+1+d} & \rightarrow & \cdots & \rightarrow & P_{n-1} & \rightarrow & P_{n} & \rightarrow & 0 \\
\varphi & \downarrow & \varphi & \downarrow & \varphi & \downarrow & \cdots & \downarrow & \varphi & \downarrow & \varphi & \downarrow & \\
0 & \rightarrow & P_{n}^\# & \rightarrow & P_{n+1}^\# & \rightarrow & \cdots & \rightarrow & P_{n-1}^\# & \rightarrow & P_{n}^\# & \rightarrow & 0 \\
\end{array}
$$

where $P_i$ are finitely generated projective $A$–modules. Since $n > m$, $H_{-n}(P_\bullet) \cong H_{-n}(P_\bullet^\#) \cong 0$. So, $\partial_{n-1}$ and $\partial_{n+1}$ are both split surjections. Thus there are homomorphisms $\epsilon_{n-1} : P_{n-1} \rightarrow P_{n-1}^\#$ and $\epsilon_{n+1} : P_{n+1} \rightarrow P_{n+1}^\#$ such that $\partial_{n-1} \circ \epsilon_{n-1} = Id$ and $\partial_{n+1} \circ \epsilon_{n+1} = Id$. Hence, $Z_{-n}$ and $P_{n+1}^\#$ are projective modules. Note that since $d \geq 2$, by (2.9), $\frac{P_{n+1}}{P_{n+1}^\#} = B_{n-2+d}$. Further, we obtain splittings $\sigma_{(n-1)} : P_{(n-1)} \rightarrow Z_{(n-1)}$ and $\sigma_{(n+1)} : B_{(n-1)+d} \rightarrow P_{(n-1)+d}$. This gives us a shorter complex $Q_\bullet$, naturally chain homotopic to $P_\bullet$ and an induced symmetric form on $Q_\bullet$:
Calling this map \( \varphi' \), \((Q_\bullet, \varphi')\) is obviously isometric to \((P_\bullet, \varphi)\) and hence to the original form \((X_\bullet, \eta)\). Since \(Q_\bullet\) is supported on \([(n-1)+d, -(n-1)]\) induction finishes the proof. □

Since \(\zeta\) is given by composing \(\nu\) and \(\alpha\) and there are maps \(W(\nu)\) and \(W(\alpha)\), it is clear that \(W(\zeta)\) is well-defined. However, we give an explicit proof which might also be somewhat illuminating considering the unsaid details about why duality-preserving functors induce maps of Witt groups. The proof essentially follows the proof in [B1, 2.11].

**Theorem 5.2.** The functor \(\zeta\) induces a well defined homomorphism

\[
W(\zeta) : W(A^\vee, \pm \widehat{\varpi}) \rightarrow W\left(\left(D^b_A(\mathcal{P}(A))\right)^\pm\right).
\]

**Proof.** We will only prove

\[
W(\zeta) : W(A^\vee, \widehat{\varpi}) \rightarrow W\left(D^b_A(A)^\pm\right).
\]

is well defined and the case of skew dualities follows similarly. It is clear that \(\zeta\) defines a well-defined map from \(MW(A^\vee, \widehat{\varpi})\) to \(MW\left(D^b_A(A)^\pm\right)\) since projective maps of modules can be lifted to a chain complex map of their resolutions (note that though the lift is not unique, it is unique up to homotopy and so gives the same morphism in \(D^b_A(\mathcal{P}(A))\)). So we need to check that the image of a neutral space in \(MW(A^\vee, \widehat{\varpi})\) is neutral in \(MW\left(D^b_A(A)^\pm\right)\).

Suppose \((M, \varphi_0)\) is a neutral space in \((A^\vee, \widehat{\varpi})\). Let \(\alpha_0 : N \rightarrow M\) be a lagrangian of \((M, \varphi_0)\). Then

\[
\begin{array}{c}
0 \longrightarrow N \xrightarrow{\alpha_0} M \xrightarrow{\alpha_0^\vee \varphi_0} N^\vee \longrightarrow 0
\end{array}
\]

is exact.

Suppose \(L_\bullet, P_\bullet\) are the chosen projective resolutions of \(N\) and \(M\) and \(\alpha : L_\bullet \rightarrow P_\bullet\) is the morphism induced from \(\alpha_0\). The above short exact sequence implies the composition \(L_\bullet \xrightarrow{\alpha} P_\bullet \xrightarrow{\alpha^\vee \varphi} L^\#\) is chain homotopic to 0 (hence the 0 map in \(D^b_A(\mathcal{P}(A))\)). Completing \(\alpha\) to an exact triangle, we get a morphism of exact triangles

\[\begin{array}{ccc}
L_\bullet & \xrightarrow{\alpha} & P_\bullet \xrightarrow{j} C_\bullet \xrightarrow{k} T(L_\bullet) \\
\downarrow \alpha^\# \varphi & & \downarrow s \\
L^\# & \xrightarrow{1} & L^\#
\end{array}\]

Note that \(H_0(C_\bullet) \cong N^\vee\) and \(\forall \ i \neq 0 \ H_i(C_\bullet) = 0\) and so \(C_\bullet\) is an object in \(D^b_A(\mathcal{P}(A))\). The map \(s\) is actually quite easy to describe, namely \(s = (0, \alpha^\# \varphi) : L_{n-1} \oplus P_n \longrightarrow L_n^\ast\) and it follows from the above morphism of triangles (or by direct checking) that \(s\) is a quasi-isomorphism. Hence,

\[\begin{array}{ccc}
L_\bullet & \xrightarrow{\alpha} & P_\bullet \xrightarrow{\alpha^\# \varphi} L^\# \xrightarrow{k \circ s^{-1}} T(L_\bullet)
\end{array}\]

is an exact triangle. Setting \(w = -T^{-1}(k \circ s^{-1})\), we get an exact triangle

\[T^{-1}(L^\#) \xrightarrow{w} L_\bullet \xrightarrow{\alpha} P_\bullet \xrightarrow{\alpha^\# \varphi} L^\#\]
Now all we require is that $T^{-1}w^# = w$.

$$T^{-1}w^# = w \iff T^{-1}(k \circ s^{-1}) \iff (T^{-1}(k \circ s^{-1}))^# = k \circ s^{-1} \iff T(s^{-1} \circ k^#) = k \circ s^{-1} \iff T(k^#) \circ s = T(s^#) \circ k.$$ 

A quick physical check of the maps in question yields that the first map is

\[
\begin{align*}
L_{n-1} \oplus P_n & \xrightarrow{(-1 \ 0)} L_{n-1} \\
& \xrightarrow{\varphi_{n-1}^* \circ \alpha_{d-n+1}} L_{d-n}^* \oplus P_{d-n+1}^*
\end{align*}
\]

while the second one is

\[
\begin{align*}
L_{n-1} \oplus P_n & \xrightarrow{0 - \varphi_{n-1}^* \circ \alpha_{d-n+1}} L_{d-n}^* \\
& \xrightarrow{(-1 \ 0)} L_{d-n}^* \oplus P_{d-n+1}^*
\end{align*}
\]

The matrices we thus obtain are

\[
\begin{pmatrix}
0 - \varphi_{n-1}^* \circ \alpha_{d-n+1} \\
0 0
\end{pmatrix}

\quad \text{and} \quad
\begin{pmatrix}
0 & 0 \\
\alpha_{d-n}^* \circ \varphi_n & 0
\end{pmatrix}
\]

which are homotopy equivalent using

\[
\tau = \begin{pmatrix} 0 & 0 \\ 0 & (-1)^n \varphi \end{pmatrix}.
\]

Therefore, $(L_\bullet, \alpha, w)$ is a lagrangian. Hence $W(\zeta)$ is a well defined homomorphism of groups.

We now proceed towards (5.11) which proves that $W(\zeta)$ is surjective. The main tool here is to construct a special sublagrangian and then use Balmer's sublagrangian construction [B2, Section 4 and Theorem 4.20] to reduce the length of $(P, \varphi)$.

Remark 5.3. Note that using (5.1), any symmetric form $(X_\bullet, \phi)$ in $(D^b_A (\mathcal{A})_u^+)$ with $X_\bullet$ not acyclic can be represented by

\[
\begin{align*}
P_\bullet &= \cdots \\
& \xrightarrow{\varphi} P_{n+d} \xrightarrow{\partial} P_{(n-1)+d} \xrightarrow{\partial} \cdots \xrightarrow{\partial} P_{-n-1} \xrightarrow{\partial} P_{-n} \xrightarrow{\partial} 0 \\
\varphi_n & \xrightarrow{\varphi_{n-1}^* \circ \alpha_{d-n+1}} P_{d-n}^* \xrightarrow{\partial^*} P_{d-n}^{*-1} \xrightarrow{\partial^*} \cdots \xrightarrow{\partial^*} P_{n+d}^* \xrightarrow{\partial^*} 0
\end{align*}
\]

with $H_{-n}(P_\bullet) \neq 0$.

Lemma 5.4. Let $(P_\bullet, \varphi)$ be as above. Then

1. $H_r(P_\bullet) = 0$ for $r = n+1, n+2, \ldots, n+d$. 

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2. $H_n(P_\bullet) \neq 0$.

**Proof.** The first point follows from (2.9). To prove (2), assume $H_n(P_\bullet) = 0$. Then, with $B_{n-1} = \text{image}(\partial_n)$ we have an exact sequence

$$0 \longrightarrow P_{n+d} \longrightarrow P_{(n-1)+d} \longrightarrow \cdots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \frac{B_{n-1}}{B_{n-1}} \longrightarrow 0$$

Since this is a projective resolution of the last term, if follows

$$H_{-n}(P_\bullet^*) = \text{Ext}^{d+1}(\frac{P_{n-1}}{B_{n-1}}, A) = 0.$$  

This is a contradiction to $H_{-n}(P_\bullet) \neq 0$. The proof is complete. \qed

Much of what follows is dependent on [B2] §4 and the interested reader is highly encouraged to take a look at it. We recall the definition of a sublagrangian of $(P_\bullet, \varphi)$:

**Definition 5.5.** A sublagrangian of a symmetric form $(P_\bullet, \varphi)$ is a pair $(L_\bullet, \alpha)$ with $L_\bullet \in \text{Ob}(D^b_A(\mathcal{P}(A)))$ and $\alpha : L_\bullet \rightarrow P_\bullet$, which satisfies that $\alpha^\# \circ \varphi \circ \alpha = 0$ in $D^b_A(\mathcal{P}(A))$.

For $(P_\bullet, \varphi)$ as above, (5.3) tells us that $H_n(P_\bullet) \neq 0$ and we already know it is in $A$. So it has a minimal projective resolution of length $d$. Let $L_\bullet$ be a projective resolution of $H_n(P_\bullet)$ of length $d$, shifted by $n$ places, as in the diagram below. Since $H_i(P_\bullet) \forall i > n$ by (2.9), the bottom line is a projective resolution of $\frac{P_n}{B_n}$ and so the inclusion $H_n(P_\bullet) \hookrightarrow \frac{P_n}{B_n}$ induces a map of complexes

$$0 \longrightarrow L_{n+d} \longrightarrow L_{(n-1)+d} \longrightarrow \cdots \longrightarrow L_n \longrightarrow 0$$

Note that since the composition $H_n(P_\bullet) \hookrightarrow \frac{P_n}{B_n} \rightarrow B_{n-1}$ is 0, we get a chain complex morphism $\nu : L_\bullet \rightarrow P_\bullet$.

**Lemma 5.6.** With the notations as above, for $n > 0, (L_\bullet, \nu)$ defines a sublagrangian of $(P_\bullet, \varphi)$.

**Proof.** Let $\alpha = \nu^\# \varphi \nu$ is as follows (the first line indicates the degrees):

\[
\begin{array}{cccccccc}
\cdots & 0 & L_n & \varphi & L_{n-1} & \varphi & \cdots & 0 \\
\alpha_n & \alpha_{n-1} & \alpha & \cdots & \alpha & \cdots & \alpha & \cdots \\
\cdots & L_{d-(n+2)}^* & \partial_{d-n}^* & \cdots & L_{d-n}^* & \partial_{d-n+1}^* & \cdots & L_{n+d}^* \\
\end{array}
\]
$L^\#$ is exact at all degrees except $-n$. Since $n > 0$, $H_i(L^\#) = 0 \forall i \geq n$. Hence, \( \text{image}(\alpha_n) \subseteq \ker(\partial_{d-n+1}) = \text{image}(\partial_{d-n}^*) \). So, \( \alpha_n \) lifts to a homomorphism \( h_n : L_n \rightarrow L_{d-(n+1)}^* \), i.e. \( \partial_{d-n}^*h_n = \alpha_n \). So \( \partial_{d-n}^*(\alpha_{n+1} - h_n\partial_{n+1}) = 0 \). Now we can inductively define a homotopy \( h_r : L_r \rightarrow L_{d-(r+1)}^* \) so that \( \alpha \) is homotopic to zero. The proof is complete. \( \Box \)

We intend to apply the sulagrangian construction of Balmer [B2, Theorem 4.20] to \( \nu \). Since \( \mathcal{D}_A(\mathcal{P}(A)) \) is not a triangulated category (in particular not closed under cones), we need to reprove some of the results in \[B2\, \text{Theorem 4.20}\]. The main (and only) thing we have to keep track of is that in all the constructions, our objects remain within the category \( \mathcal{D}_A(\mathcal{P}(A)) \). We start by checking that the cone of \( \nu \) constructed in (5.6) is an object of \( \mathcal{D}_A(\mathcal{P}(A)) \).

**Lemma 5.7.** With the notations of (5.6), let \( N_\bullet \) be the cone of \( \nu \). Then,

1. \( N_\bullet \) is in \( \mathcal{D}_A(\mathcal{P}(A)) \).

2. The homologies are given by

\[
H_i(N_\bullet) = \begin{cases} 
H_i(P_\bullet) & \text{if } n > i \geq -n \\
0 & \text{otherwise}
\end{cases}
\]

3. \( N_\bullet \) is supported on \([n + d + 1, -n]\).

**Proof.** The last point is obvious from the construction of the cone and (1) follows from (2). We prove (2). We have the exact triangle

\[
T^{-1}N_\bullet \rightarrow L_\bullet \xrightarrow{\nu} P_\bullet \rightarrow N_\bullet.
\]

By construction, \( H_n(L_\bullet) \xrightarrow{\sim} H_n(P_\bullet) \) and \( H_i(L_\bullet) = 0 \) for all \( i \neq n \). The long exact sequence of homologies

\[
\cdots \xrightarrow{\sim} H_{n+2}(N_\bullet) \rightarrow 0 \rightarrow H_{n+1}(N_\bullet) \rightarrow H_n(L_\bullet) \rightarrow H_n(P_\bullet) \rightarrow H_{n-1}(N_\bullet) \rightarrow \cdots
\]

establishes (2) and hence the lemma. \( \Box \)

Now we consider the dual \( N_\bullet^\# \) of the cone of \( \nu \).
Lemma 5.8. With the same notations as above (in (5.6)), consider the following morphism of exact triangles:

\[
\begin{array}{c}
T^{-1}N \xrightarrow{\nu_0} L \xrightarrow{\nu} P \xrightarrow{\nu_2} N \\
T^{-1}L^\# \xrightarrow{\nu_0^\#} N^\# \xrightarrow{\nu_2^\#} P^\# \xrightarrow{\nu^\#} L^\#
\end{array}
\]

(refer [B2, 4.3]...the existence of \(\mu_0\) is assured by combining axioms (TR1) and (TR3) of triangulated categories and using that \(2\) is invertible.)

Then,

1. \(N^\#\) is in \(D^b_A(\mathcal{P}(A))\).
2. \(N^\#\) is supported on \([n + d, -(n + 1)]\).
3. \(\mu_0\) induces an isomorphism of the \(n^{th}\) homology
   \[H(\mu_0) : H_n(L_\bullet) \xrightarrow{\sim} H_n(N^\#_\bullet).\]
4. \[H_i(N^\#_\bullet) \cong \begin{cases} H_i(P^\#_\bullet) & \text{if } n \geq i > -n \\ 0 & \text{otherwise} \end{cases}\]

Proof. (1) follows directly from (3.1). (2) follows because by (5.7) \(N_\bullet\) is supported on \([(n + 1) + d, -n]\). For (3), notice that the only nonzero homology of \(L^\#\) is at degree \(-n\). Since \(n > 0\), the long exact homology sequence of the second triangle gives us

\[H(\nu^\#_2) : H_n(N^\#_\bullet) \xrightarrow{\sim} H_n(P^\#_\bullet).\]

By choice of \(\nu\) and \(\varphi\), we know that

\[H(\nu) : H_n(L_\bullet) \xrightarrow{\sim} H_n(P_\bullet), \quad H(\varphi) : H_n(P_\bullet) \xrightarrow{\sim} H_n(P^\#_\bullet)\]

and hence, the commutative diagram

\[
\begin{array}{ccc}
H_n(L_\bullet) & \xrightarrow{H_n(\nu)} & H_n(P_\bullet) \\
\downarrow H_n(\mu_0) & & \downarrow H_n(\varphi) \\
H_n(N^\#_\bullet) & \xrightarrow{H_n(\nu^\#_2)} & H_n(P^\#_\bullet)
\end{array}
\]

gives us (3). We prove (4) now. Since the only nonzero homology of \(L^\#\) is at degree \(-n\), it is clear from the long exact homology sequence for the bottom exact triangle that

\[H_i(N^\#_\bullet) \cong H_i(P^\#_\bullet) \forall i \neq -n, -n - 1.\]
By (5.7), \( H_i(N_\ast) = 0 \) for all \( i \geq n \) and so

\[
H_{-(n+1)}(N^\#) = \text{Ext}^{d+2}\left( \frac{N_{n-1}}{B_{n-1}}, A \right) = 0, \quad H_{-n}(N^\#) = \text{Ext}^{d+1}\left( \frac{N_{n-1}}{B_{n-1}}, A \right) = 0.
\]

where \( B_{n-1} \subseteq N_{n-1} \) is the boundary submodule (the last part also follows directly because \( \text{Ext}^d(\frac{P_n}{B_n}, A) \approx \text{Ext}^d(H_n(P_\ast), A) \)). So, (4) is established. The proof is complete. \( \square \)

Now we consider the cone of \( \mu_0 \).

**Lemma 5.9.** With the notations in (5.6), (5.7) and (5.8), consider an exact triangle on \( \mu_0 \) as follows:

\[
L_\ast \xrightarrow{\mu_0} N^\#_\ast \xrightarrow{\mu_1} R_\ast \xrightarrow{\mu_2} T(L_\ast)
\]

where \( R_\ast \) is the cone of \( \mu_0 \). Then \( R_\ast \) is an object of \( D^b_A(P(A)) \). More precisely,

\[
H_i(R_\ast) = \begin{cases} 
H_i(N^\#_\ast) & \text{for } -(n-1) \leq i \leq n-1 \\
0 & \text{otherwise}
\end{cases}
\]

which tells us that \( R_\ast \) has exactly two nonzero homologies less than \( P_\ast \).

**Proof.** Note that the only nonzero homology of \( L_\ast \) is at degree \( n \). Using (5.8), the long exact homology sequence corresponding to the exact triangle is as follows:

\[
\cdots \xrightarrow{H_{n+2}(R_\ast)} 0 \xrightarrow{H_{n+1}(R_\ast)} H_n(L_\ast) \xrightarrow{\sim} H_n(N^\#_\ast) \xrightarrow{H_{n-1}(R_\ast)} H_{n-1}(N^\#_\ast) \xrightarrow{H_{n-1}(R_\ast)} 0 \cdots
\]

Therefore,

\[
H_i(R_\ast) = \begin{cases} 
H_i(N^\#_\ast) & \text{for } -(n-1) \leq i \leq n-1 \\
0 & \text{otherwise}
\end{cases}
\]

The proof is complete. \( \square \)

**Remark 5.10.** The readers are referred to [B2, 4.11] for the definition of a very good morphism \( L_\ast \to N^\#_\ast \). All we need in the sequel is that such morphisms exist [B2, 4.17] and that whenever they do, we obtain [B2, 4.20]

1. There is a symmetric form \( \psi : R_\ast \to R^\#_\ast \).
2. There is a lagrangian

\[
N^\#_\ast \to (P^\#_\ast, \varphi^{-1}) \perp (R_\ast, \psi).
\]
Theorem 5.11. Let \((P_\bullet, \varphi)\) be a symmetric form as in (5.3) with \(n > 0\). Then, there is a symmetric form \((Q_\bullet, \tau)\) such that

1. 
\[
[(Q_\bullet, \tau)] = [(P_\bullet, \varphi)] \quad \text{in} \quad W \left(D^b_\mathcal{A}(\mathcal{P}(A))^\pm_u \right).
\]

2. \(Q_\bullet\) has two less homologies than \(P_\bullet\) and it has support in \([k + d, -k]\) for some \(0 \leq k < n\).

**Proof.** Use the notations in (5.6), (5.7) and (5.8). Using the above remark (5.10), let \(\mu_0\) be a very good morphism and let \((R_\bullet, \psi)\) be the symmetric form obtained. Note that \(N^\#_\bullet, R_\bullet\) are objects in \(D^b_{\mathcal{MFP}, D''}(\mathcal{P}(A))\) by (5.7) and (5.9). Hence, by (2) of the above remark (5.10), \((R_\bullet, \psi)\) is Witt equivalent to \((P^\#_\bullet, -\varphi^{-1})\) in \((D^b_\mathcal{A}(\mathcal{P}(A))^\pm_u)\) by definition (4.3). Therefore in \(W \left(D^b_\mathcal{A}(\mathcal{P}(A))^\pm_u \right)\), we have

\[
[(R_\bullet, \psi)] = [(P^\#_\bullet, -\varphi^{-1})] = [(P_\bullet, -\varphi)] \quad \text{in} \quad W \left(D^b_\mathcal{A}(\mathcal{P}(A))^\pm_u \right).
\]

Now, \(H_i(R_\bullet) = 0\) if \(i < -(n - 1), i > (n - 1)\). By (5.1), \((R_\bullet, -\psi)\) is isometric to a form \((Q_\bullet, \tau)\) such that \(Q_\bullet\) is supported on \([k + d, -k]\) with \(0 \leq k \leq n - 1\). The proof is complete. 

Now we are ready to state and prove the main result of this article, which is our version of the dévissage theorem, i.e.

**Theorem 5.12.** The homomorphisms

\[
W(\zeta): W(\mathcal{A}, ^\vee, \pm \varpi) \longrightarrow W \left(\left(D^b_\mathcal{A}(\mathcal{P}(A))^\pm_u \right)\right)
\]

induced by the functor \(\zeta\) are isomorphisms.

**Proof.** First, we prove the surjectivity of the homomorphism \(W(\zeta)\). Suppose \(x \in W \left(\left(D^b_\mathcal{A}(\mathcal{P}(A))^\pm_u \right)\right)\). Then, by (5.1), we can write \(x = [(P_\bullet, \varphi)]\) of the form described in (5.3). Inductively, by (5.11), there is a form \((R_\bullet, \psi)\) in \((D^b_\mathcal{A}(\mathcal{P}(A))^\pm_u)\) such that

\[
[(R_\bullet, \psi)] = [(P_\bullet, \varphi)] = x \quad \text{in} \quad W \left(D^b_\mathcal{A}(\mathcal{P}(A))^\pm_u \right)
\]

and \(R_\bullet\) is supported in \([d, 0]\). By (2.9), \(R_\bullet\) is a projective resolution of \(M := H_0(R_\bullet) \in \mathcal{A}\). Further, \(\psi\) induces a form \(\psi_0 : M \sim \rightarrow M^\vee\) and clearly

\[
W(\zeta)([(M, \psi_0)]) = [(R_\bullet, \psi)] = x.
\]

So, \(W(\zeta)\) is surjective.

Now, we proceed to prove that \(W(\zeta)\) is injective. Suppose \((M, q)\) is a symmetric form in \((\mathcal{A}, ^\vee, \pm \varpi)\) and \(W(\zeta)([(M, q)]) = 0\). Write \((\zeta(M), \zeta(q)) = (P_\bullet, \varphi_0)\) where \(P_\bullet\) is a projective
resolution of $M$ of length $d$, and $\varphi_0$ is the induced symmetric form. So, $[(P_\bullet, \varphi_0)] = 0$ in $W\left((D^b_A(\mathcal{P}(A)))^\perp\right)$.

This means there is a neutral form $[(Q_\bullet, \varphi_1)]$ so that $[(P_\bullet, \varphi_0)] \perp [(Q_\bullet, \varphi_1)]$ is neutral in $W\left((D^b_A(\mathcal{P}(A)))^\perp\right)$. Since $[(Q_\bullet, \varphi_1)]$ is neutral, so is $[(Q_\bullet, -\varphi_1)]$. Hence, $[(P_\bullet, \varphi_0)] \perp [(Q_\bullet, \varphi_1)] \perp [(Q_\bullet, -\varphi_1)]$ is neutral. Using the usual isometry, we get that there is a hyperbolic form

$$\begin{pmatrix} Q_\bullet \oplus Q^\# \colon & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix} \quad \text{with} \quad Q_\bullet \in D^b_A(\mathcal{P}(A))$$

such that

$$(U_\bullet, \varphi) := \begin{pmatrix} P_\bullet \oplus Q_\bullet \oplus Q^\# \colon & \begin{pmatrix} \varphi_0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \end{pmatrix} \quad \text{is neutral in} \quad W\left((D^b_A(\mathcal{P}(A)))^\perp\right)$$

(we have so far followed the argument in [B2, 3.5])

So, $(U_\bullet, \varphi)$ has a lagrangian $(L_\bullet, \alpha)$. We have an exact triangle

$$T^{-1}L_\bullet \xrightarrow{w} L_\bullet \xrightarrow{\alpha} U_\bullet \xrightarrow{\varphi} L^\#_\bullet \quad T^{-1}w^\# = w.$$  

Before proceeding, we use (3.3) to make a startlingly simple observation about the homologies of $L^\#_\bullet$:

$$H_i(L^\#_\bullet) \cong H_{i-d}(L_\bullet) \cong Ext^d_i(H_{d-i-1}(L_\bullet), A) \cong Ext^d_i(H_{-i}(L_\bullet), A) \cong H_{-i}(L^\vee_\bullet).$$

Similarly, $H_i(U^\#_\bullet) \cong H_{-i}(U^\vee_\bullet)$ and further using the remark (3.4) about naturality of the maps, we get that the long exact homology sequences are

$$\cdots \to H_{-2}(L^\vee_\bullet) \xrightarrow{H_2(w)} H_1(L^\vee_\bullet) \xrightarrow{H_1(\alpha)} H_1(U^\vee_\bullet) \xrightarrow{H_1(\varphi)} H_{-1}(L^\vee_\bullet) \xrightarrow{H_{-1}(\alpha)} H_{-1}(L^\vee_\bullet) \cdots$$

$$\cdots \to H_{-2}(L^\#_\bullet) \xrightarrow{H_2(w)} H_1(L^\#_\bullet) \xrightarrow{H_1(\alpha)} H_1(U^\#_\bullet) \xrightarrow{H_1(\varphi)} H_{-1}(L^\#_\bullet) \xrightarrow{H_{-1}(\alpha)} H_{-1}(L^\#_\bullet) \cdots$$

Replacing the part of the top exact sequence in negative degree by the corresponding part of the bottom (dual) exact sequence, we get an exact sequence:

$$\cdots \to H_{-2}(L^\vee_\bullet) \xrightarrow{H_1(w)} H_1(L^\vee_\bullet) \xrightarrow{H_1(\alpha)} H_1(U^\vee_\bullet) \xrightarrow{H_1(\varphi)} H_{-1}(L^\vee_\bullet) \cdots$$
can apply \[B3, 4.1\] Lemma to this sequence. Since the sequence is exact, we have

\[ \text{is then also a triangulated category with the same translation } T \]

Notice that the complex above is very special, and is "symmetric" about \( H_0(U_\bullet) \). So we can apply \[B3, 4.1\] Lemma to this sequence. Since the sequence is exact, we have

\[ [(H_0(U_\bullet), H_0(\varphi))] = [(0, 0)] = 0 \quad \text{in } W(\mathcal{MFPD}^H(A)). \]

However,

\[(H_0(U_\bullet), H_0(\varphi)) = \left( H_0(P_\bullet) \oplus H_0(Q_\bullet) \oplus H_0(Q_\bullet^\vee), \begin{pmatrix} H_0(\varphi_0) & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right) \]

\[= (M, q) \perp \left( H_0(Q_\bullet) \oplus H_0(Q_\bullet^\vee), \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \]

So, we have

\[ [(M, q)] = \left[ (M, q) \perp \left( H_0(Q_\bullet) \oplus H_0(Q_\bullet^\vee), \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \right] = [(H_0(U_\bullet), H_0(\varphi))] = 0. \]

The proof is complete. \(\square\)

6 Shifted Witt Groups

In this section, we use the previous results to obtain our dévissage theorem for the Witt groups \( W^i(D^b_\Lambda(P(A))) \). We recall that \( A \) is a Cohen-Macauly ring with \( \dim A_m = d \geq 2 \) for all maximal ideals \( m \) and such that \( 2 \) is invertible in \( A \) and that \( A = \mathcal{MFPD}^H(A) \).

**Notations 6.1.** For integers \( j \geq 0 \) define the functor \( \zeta_j = T^{-j} \zeta \), which associates to an object \( M \) in \( A \) a projective resolution \( P_\bullet \) of \( M \) of length \( d \), such that \( H_{-j}(P_\bullet) = M \).

**Definition 6.2.** Suppose \( K := (K, \#, \delta, \varpi) \) is a triangulated category with translation \( T \) and \( \delta \)-duality \( \# \). We recall from \[B3\] that

\[ T^nK := (K, T^n\#, (-1)^n\delta, (-1)^{n(n+1)/2}\delta^n\varpi). \]

is then also a triangulated category with the same translation \( T \) but with \((-1)^n\delta\)-duality \( T^n\# \). If \( K_0 \) is a subcategory of \( K \) satisfying the conditions of \[L3\], we define \( T^nK_0 \) to be the same subcategory and translation with the induced duality structure from \( T^nK \). Using \[L3\], we define the shifted Witt groups by

\[ W^n(K) := W(T^nK) \quad W^n(K_0) := W(T^nK_0) \quad \forall \ n \in \mathbb{Z}. \]
Note that $T^2_s : T^nK \rightarrow T^{n+4}K$ is an equivalence of triangulated categories with duality, for all $n \in \mathbb{Z}$. Similarly, $T^*_s : T^nK_0 \rightarrow T^{n+4}K_0$ is an equivalence of categories with duality, for all $n \in \mathbb{Z}$ and so

$$W^n(K) \sim W^{n+4}(K) \quad W^n(K_0) \sim (T^nK_0) \quad \forall n \in \mathbb{Z}.$$  

**Definition 6.3.** Following [BW], by "standard" duality structure on $\mathcal{A}$, we mean the exact category $\left(\mathcal{A},^\vee, (−1)^{\frac{d(d−1)}{2}}\right)$. By "standard" skew duality structure on $\mathcal{A}$, we mean the exact category $\left(\mathcal{A},^\vee, (−1)^{\frac{d(d−1)}{2}}\right)$. We denote the Witt groups

$$W^+_\text{St}(A) = W\left(\mathcal{A},^\vee, (−1)^{\frac{d(d−1)}{2}}\right), \quad W^-\text{St}(A) = W\left(\mathcal{A},^\vee, (−1)^{\frac{d(d−1)}{2}}\right).$$

**Theorem 6.4.** Then, the functor $\zeta_0 : \mathcal{A} \rightarrow D^b_A(\mathcal{P}(A))$ induces an isomorphism

$$W(\zeta_0) : W^+\text{St}(A) \sim W^d\left(D^b_A(\mathcal{P}(A)), *, 1, \varpi\right).$$

**Proof.** Recall $\zeta_0$ was denoted by $\zeta$ in the previous sections. For notational convenience $\varpi_0 = (−1)^{\frac{d(d−1)}{2}}\varpi$. By theorem (5.12), we get the following isomorphism of Witt groups

$$\eta_0 : W^+\text{St}(A) \sim W\left(D^b_A(\mathcal{P}(A)), ^\#u, 1, \varpi_0\right).$$

There is a duality preserving equivalence [BW, Proof of Lemma 6.4]

$$\beta : \left(D^b(\mathcal{P}(A)), T^d, *, 1, \varpi_0\right) \rightarrow \left(D^b(\mathcal{P}(A)), T^d, *, (−1)^d, (−1)^{\frac{d(d−1)}{2}}\varpi\right).$$

Note that the later is the shifted category $T^d\left(D^b(\mathcal{P}(A)), *, 1, \varpi\right)$. Composing $\eta_0$ with the homomorphism induced by $\beta$, we get the result. \[\square\]

Now we prove the standard skew duality version of theorem (6.4).

**Theorem 6.5.** The functor $\zeta_1$ induces an isomorphism

$$W^+\text{St}(A) \sim W^{d−2}\left(D^b_A(\mathcal{P}(A)), *, 1, −\varpi\right).$$

**Proof.** By Theorem (5.12), we have an isomorphism

$$W^-\text{St}(A) \sim W\left(D^b_A(\mathcal{P}(A)), T^d, *, 1, −(−1)^{\frac{d(d−1)}{2}}\varpi\right).$$

Write $\varpi_0 = (−1)^{\frac{d(d−1)}{2}}\varpi$. There is an equivalence of categories [B2, 2.14]

$$T_s : \left(D^b_A(\mathcal{P}(A)), T^d, *, 1, \varpi_0\right) \rightarrow \left(D^b_A(\mathcal{P}(A)), T^d, *, 1, \varpi_0\right).$$

This induces an isomorphism

$$W\left(D^b_A(\mathcal{P}(A)), T^d, *, 1, \varpi_0\right) \sim W\left(D^b_A(\mathcal{P}(A)), T^d, *, 1, \varpi_0\right).$$

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As in the proof of [BW, Lemma 6.4], there is an equivalence of triangulated categories with duality

\[(D^b(P(A)), T_{u^2}^{-2} \ast, 1, \varpi_0) \rightarrow (D^b(P(A)), T_{s^2}^{-2} \ast, (-1)^{d-2}, (-1)^{d(d+1)} \varpi)\].

The latter category is \(T_{d-2}^{-2} \mathcal{B}(P(A)), \ast, 1, -\varpi\). The proof is complete. \(\square\)

Finally, we have the following regarding odd shifts.

**Theorem 6.6.** For \(n = d - 1, d - 3\), we have

\[W^n(D^b_A(P(A)), \ast, 1, \pm \varpi) = 0\].

**Proof.** First consider \(n = d - 1\). It would be enough to prove that

\[W(D^b_A(P(A)), T_{u^2}^{-1} \ast, 1, \pm \varpi) = 0\].

Suppose \((P_\ast, \varphi)\) is a form in \(D^b_A(P(A)), T_{u^2}^{-1} \ast, 1, \pm \varpi\). By a little tweak in (5.1), we can assume that \(P_\ast\) is supported on \([n + (d-1), -n]\) with \(n > 0\) and \(H_{-n}(P_\ast) \neq 0\). By imitating the arguments of theorem (5.11), we can keep shortening the length of the complexes which give our symmetric form. Eventually, we will be reduced to the case where the complex is \(P_\ast\) is supported on \([d - 1, 0]\). By theorem (2.9), \(P_\ast\) is exact. So, \([P, \varphi] = 0\). The same arguments apply when \(n = d - 3\). The proof is complete. \(\square\)

Using the 4-periodicity, we now obtain the theorem mentioned in the introduction:

**Theorem 6.7 (shiftFinal).** Let \(\mathcal{B} = (D^b_A(P(A)), T_s, 1, \varpi)\). Then, for \(n \in \mathbb{Z}\), we have

1. \(W^{d + 4n}(\mathcal{B}) = W^+(\mathcal{A})\),
2. \(W^{d + 4n+1}(\mathcal{B}) = 0\),
3. \(W^{d + 4n+2}(\mathcal{B}) = W^-(\mathcal{A})\),
4. \(W^{d + 4n+3}(\mathcal{B}) = 0\).

**A Some Formalism**

The purpose of this section is to prove the following theorem:
Theorem A.1. Suppose $\mathcal{E}$ is a full subcategory of a $\mathbb{Z}_{[}^{1}^{2}$ abelian category $\mathcal{B}$ with the 2 out of 3 property for short exact sequences, and has duality $(\mathcal{E}, \vee, \tilde{\varpi})$. Let $D^b(\mathcal{E}) := (D^b(\mathcal{E}), *, a, \tilde{\varpi})$ denote the derived category, with duality, of $(\mathcal{E}, \vee, \tilde{\varpi})$. Also, let $D^b_\mathcal{E}(\mathcal{E})$ denote the derived category, with duality, of objects in $D^b(\mathcal{E})$ with homologies in $\mathcal{E}$. Then the homomorphism

$$W(\mu) : W(\mathcal{E}, \vee, \tilde{\varpi}) \longrightarrow W(D^b_\mathcal{E}(\mathcal{E}))$$

induced by the functor $\mu : \mathcal{E} \longrightarrow D^b_\mathcal{E}(\mathcal{E})$ is an isomorphism.

In particular, with $\mathcal{E} = \mathcal{A}$ and $\mathcal{B} = \mathcal{M}(\mathcal{A})$, we obtain that

$$W(\mu) : W(\mathcal{A}, \vee, \pm \tilde{\varpi}) \longrightarrow W(D^b_\mathcal{A}(\mathcal{A}, \vee, \pm \tilde{\varpi}))$$

as promised in section 5.

The proof of the theorem is essentially the same as the proof of [B3, Theorem 3.2] with the extra check that all constructions yield complexes whose homologies are in $\mathcal{A}$. This boils down to using the most elementary of sublagrangians (concentrated in just one degree) and reducing length. In any case, we follow the proof in [B3, Theorem 3.2]. Since the category $\mathcal{E}$ has all the properties required in the results in [B3, Section 3]), we will freely borrow them.

To start with, injectivity of $W(\mu)$ follows directly because the isomorphism $W(\mathcal{E}) \xrightarrow{\sim} W(D^b(\mathcal{E}))$ (proven in [B3, Theorem 4.3]) factors as

$$W(\mathcal{E}) \xrightarrow{W(\mu)} W(D^b_\mathcal{E}(\mathcal{E})) \xrightarrow{\sim} W(D^b(\mathcal{E}))$$

We move to the proof of surjectivity which, as we mentioned above will require checking that we remain in $D^b_\mathcal{E}(\mathcal{E})$ through all the lemmas establishing [B3, Theorem 3.2]. To start with, we establish the following result regarding duality, which also provides an alternative proof of [3.5].

Lemma A.2. With the same notations as in (A.1), $D^b_\mathcal{E}(\mathcal{E})$ is closed under duality.

Proof. Let $P_\bullet$ be an object in the derived category $D^b_\mathcal{E}(\mathcal{E})$. Write

$$P_\bullet : \cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \longrightarrow P_{-1} \longrightarrow P_{-2} \longrightarrow \cdots$$

$$P_\bullet^\vee : \cdots \longrightarrow P_{-2}^\vee \longrightarrow P_{-1}^\vee \longrightarrow P_0^\vee \longrightarrow P_1^\vee \longrightarrow P_2^\vee \longrightarrow \cdots$$
Since the complexes are bounded and the homologies are objects of $\mathcal{E}$, all the kernels $Z_i = \ker(d_i)$, images $B_i = \text{image}(d_{i+1})$ and quotients $\frac{B_i}{Z_i}$ are also objects of $\mathcal{E}$. Hence, so are their duals. But we have an exact sequence

$$0 \longrightarrow \left( \frac{P_{t-1}}{B_{t-1}} \right)^\vee \longrightarrow P_{t-1}^\vee \longrightarrow \left( \frac{P_t}{B_t} \right)^\vee \longrightarrow H_t(P^\bullet) \longrightarrow 0$$

The first three terms in this sequence are in $\mathcal{E}$, hence so is $H_0(P^\bullet)$. The proof is complete. □

**Lemma A.3.** Let $x \in W(D^b_\mathcal{E}(\mathcal{E}))$. Then $x = (P^\bullet, s)$ such that

1. $P^\bullet$ is bounded and $s : P^\bullet \longrightarrow P^\bullet$ is a morphism of complexes, without denominator.
2. $s$ is quasi-isomorphism.
3. $s$ is strongly symmetric (i.e. $s_i^\vee = s_i \forall i \in \mathbb{Z}$).
4. $H_i(P^\bullet) \in \mathcal{E}$ for all $i \in \mathbb{Z}$.

**Proof.** Let the form $x$ be given by $(X^\bullet, \eta)$. By definition there is a complex $P^\bullet$ which is an object of $D^b_\mathcal{E}(\mathcal{E})$ and a chain complex quasi-isomorphisms $t : P^\bullet \longrightarrow X^\bullet$ and $\varphi_0 : P^\bullet \longrightarrow X^\bullet$ such that $\eta = \varphi_0 t^{-1}$. Then, $s = t^* \varphi_0 = t^* \eta t$ is a symmetric form on $P^\bullet$ and $(X^\bullet, \eta)$ is isometric to $(P^\bullet, \varphi)$. Clearly $s$ is an actual morphism of complexes, a quasi-isomorphism and $H_i(P^\bullet) \in \mathcal{E}$ for all $i \in \mathbb{Z}$. Finally, using that $\frac{1}{2}$ exists, we can make the map strongly symmetric. □

**Lemma A.4.** Let $(P^\bullet, s)$ be a symmetric form in $D^b_\mathcal{E}(\mathcal{E})$ as in (A.3), such that $P^\bullet$ is supported on $[m, n]$ with $m > n \geq 0$. Then $(P^\bullet, s)$ is isometric to a symmetric space $(Q^\bullet, t)$ such that $Q^\bullet$ is supported on $[n, −n]$ and $(Q^\bullet, t)$ has all the other properties of $(P^\bullet, s)$.

**Proof.** This is precisely [B3, Lemma 3.7] in our context. Note that since we can do this in the derived category without the homology condition $D^b(\mathcal{E})$, we use the same result to get $(Q^\bullet, t)$ isometric to $(P^\bullet, s)$ with the required condition. However, since the isometry gives in particular a quasi-isomorphism $P^\bullet \simto Q^\bullet$ and so $Q^\bullet$ is also an object in $D^b_\mathcal{E}(\mathcal{E})$. The proof is complete. □

**Lemma A.5.** Let $(P^\bullet, s)$ be a symmetric space, as in (A.3), with support on $[−n, n]$ and $n > 0$. Then there exists a symmetric space $(Q^\bullet, t)$ such that

1. $(Q^\bullet, t)$ is as in (A.3).
2. $(Q^\bullet, t)$ is supported in $[n, −(n − 1)]$.
3. $H_i(Q_\bullet) \in \mathcal{E}$ for all $i \in \mathbb{Z}$.

4. $[(P_\bullet, s)] + [(Q_\bullet, t)] = 0$ in $W(D^b_{\mathcal{E}}(\mathcal{E}))$.

**Proof.** Once again this is [B3] Lemma 3.9 in our context. We skim through the proof mentioning only the significant points and most important, checking the points where we need to check the extra homology condition. We begin by proving the lemma in the case $n \geq 2$. Write

$$
P_\bullet = \cdots 0 \xrightarrow{s} P_n \xrightarrow{d} P_{n-1} \xrightarrow{d} \cdots \xrightarrow{d} P_{n-1} \xrightarrow{d} \cdots \xrightarrow{d} P_{n-1} \xrightarrow{d} \cdots \xrightarrow{d} P_n \xrightarrow{0} 0
$$

$$
P^\bullet = \cdots 0 \xrightarrow{s} P^\bullet_{-n} \xrightarrow{d^\bullet} P^\bullet_{-n-1} \xrightarrow{d^\bullet} \cdots \xrightarrow{d^\bullet} P^\bullet_{-n-1} \xrightarrow{d^\bullet} \cdots \xrightarrow{d^\bullet} P^\bullet_{-n-1} \xrightarrow{d^\bullet} \cdots \xrightarrow{d^\bullet} P^\bullet_{-n} \xrightarrow{0} 0
$$

Define $(Q_\bullet, t)$ as follows, on the left side:

$$
Q_\bullet = \cdots 0 \xrightarrow{s} P_n \xrightarrow{d} P_{n-1} \xrightarrow{(0,d)} P_{n-2} \xrightarrow{d} \cdots P_{n-2} \xrightarrow{d} P_{n-1} \xrightarrow{d} \cdots \xrightarrow{d} P_{n-1} \xrightarrow{0} 0
$$

$$
Q^\bullet = \cdots 0 \xrightarrow{s} P^\bullet_{-n} \xrightarrow{d^\bullet} P^\bullet_{-n-1} \xrightarrow{-s} \cdots \xrightarrow{-s} P^\bullet_{-n-1} \xrightarrow{-s} \cdots \xrightarrow{-s} P^\bullet_{-n-1} \xrightarrow{-s} \cdots \xrightarrow{-s} P^\bullet_{-n} \xrightarrow{0} 0
$$

It was proved in [B3] Lemma 3.9 that $t$ is a quasi-isomorphism. So, it follows

$$
H_n(Q_\bullet) \cong 0, \quad H_{n-1}(Q_\bullet) \cong H_{n-1}(Q^\bullet) \cong \ker(d^\bullet) \in \mathcal{E}.
$$

Since $\text{image}(0, d_{n-1}) = \text{image}(d_{n-1})$ we have

$$
H_i(Q_\bullet) = H_i(P_\bullet) \in \mathcal{E} \quad \forall \quad i \leq n - 2.
$$

Therefore

$$
H_i(Q_\bullet) \in \mathcal{E} \quad \text{for all} \quad i \in \mathbb{Z}.
$$

So $Q_\bullet$ satisfies the last condition of (A.3). The other conditions of (A.3) are shown to be established in [B3] Lemma 3.9.

Therefore, $Q_\bullet$ satisfies (A.3). It was established in [B3] that $(P_\bullet, s) \perp (Q_\bullet, t)$ is neutral in $D^b(\mathcal{E})$, by showing that $(P_\bullet, s) \perp (Q_\bullet, t)$ is isometric (in $D^b(\mathcal{E})$) to the cone of the morphism $z : T^{-1}M_\bullet \longrightarrow M_\bullet$ defined as follows:

$$
T^{-1}M_\bullet = \cdots 0 \xrightarrow{z} P_n \xrightarrow{-d} P_{n-1} \xrightarrow{-d} \cdots P_{n-2} \xrightarrow{-d} P_{n-1} \xrightarrow{0} 0
$$

$$
M_\bullet = \cdots 0 \xrightarrow{-d^\bullet} P^\bullet_{-n} \xrightarrow{d^\bullet} P^\bullet_{-n-1} \xrightarrow{d^\bullet} \cdots \xrightarrow{d^\bullet} P^\bullet_{-n-1} \xrightarrow{d^\bullet} \cdots \xrightarrow{d^\bullet} P^\bullet_{-n} \xrightarrow{0} 0
$$

$$
degree = \begin{array}{ccc}
n & n-1 & -n \\
\hline
0 & 0 & 0
\end{array}
$$

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Since all the boundaries and cycles of $P_\bullet$ and $P_\bullet^\ast$ are objects of $\mathcal{E}$, so are $H_i(M_\bullet)$ and $H_i(M_\bullet)^\ast$. Therefore, $M_\bullet$ and $M_\bullet^\ast$ are objects of $D_{\mathcal{E}}^b(\mathcal{E})$.

Let $Z_\bullet = \text{cone}(z)$. In [B3, Lemma 3.9], it is further shown that there is a symmetric form $\chi : Z_\bullet \to Z_\bullet^\ast$ and that $(Z_\bullet, \chi)$ is isometric to $(P_\bullet, s) \perp (Q_\bullet, t)$ in $D_{\mathcal{E}}^b(\mathcal{E})$. But this tells us that $H_i(Z_\bullet) \cong H_i(P_\bullet) \oplus H_i(Q_\bullet)$ and hence $Z_\bullet$ is an object of $D_{\mathcal{E}}^b(\mathcal{E})$.

Now again following the proof of [B3, Lemma 3.9], it is shown that $T^{-1}z^\# = z$ in $D_{\mathcal{E}}^b(\mathcal{E})$ and that the form $\chi$ actually fits in to make $M_\bullet$ a lagrangian for $(Z_\bullet, \chi)$. Hence, this proves the lemma when $n \geq 2$.

In the case $n = 1$, $(P_\bullet, s)$ is given by

\[
P_{\bullet} = \begin{array}{c}
0 \\
P_{\bullet}^\ast
\end{array} \begin{array}{c}
P_1 \\
P_{\bullet}^0 \\
P_0 \\
P_{-1} \\
0
\end{array} \begin{array}{c}
P_0 \\
0
\end{array}
\]

Define $(Q_\bullet, s)$ as follows

\[
Q_\bullet = \begin{array}{c}
0 \\
0
\end{array} \begin{array}{c}
P_1 \\
P_{-1} \\
0
\end{array} \begin{array}{c}
P_{\bullet}^\ast \\
0
\end{array}
\]

The degree zero term is in the middle. In [B3, Lemma 3.9], it is established that $t$ is a quasi-isomorphism. It follows that $H_i(Q_\bullet) = 0$ for all $i \neq 0$ and

\[
H_0(Q_\bullet) = \frac{P_{\bullet}^{\vee} \oplus P_0}{P_1} \in \mathcal{E}.
\]

So, $Q_\bullet$ satisfies all the condition in (A.3), because the remaining three conditions are established in [B3, Lemma 3.9]. Now $(P, s) \perp (Q, t)$ has a lagrangian, namely

\[
T^{-1}M_\bullet = \begin{array}{c}
0 \\
0
\end{array} \begin{array}{c}
P_1 \\
P_0 \\
0
\end{array} \begin{array}{c}
P_1 \\
0
\end{array}
\]

Again,

\[
H_0(M_\bullet) = \ker(d^\vee), \quad H_1(M_\bullet) = \coker(d^\vee) \quad \text{are in} \quad \mathcal{E}.
\]
So, \( M_* \) and hence \( M_*^\tau \) are objects of \( \mathcal{D}^b_E(\mathcal{E}) \). The rest of the proof is exactly the same as in the case \( n \geq 2 \). The proof is complete. \( \square \)

**Finishing the proof of (A.1):**

We use (A.3) to represent any element \( x \) in \( W(D^b_A(\mathcal{A})) \) by a chain complex in \( D^b_A(\mathcal{A}) \) and a strongly symmetric quasi-isomorphism to its dual. Then, by alternate use of lemma (A.4) and (A.5), we reduce any element in \( W(D^b_E(\mathcal{E})) \) to a chain complex in \( D^b_E(\mathcal{E}) \), concentrated at degree zero. Of course that means the quasi-isomorphism is actually an isomorphism and hence \( x \) is the image of an element in \( W(\mathcal{A}) \) via \( W(\mu) \). So \( W(\mu) \) is also surjective. So, the proof of theorem (A.1) is complete. \( \square \)

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