Wellposedness and exponential stability for Boussinesq systems on real hyperbolic Manifolds

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Abstract. We investigate the global existence and exponential decay of mild solutions for the Boussinesq systems in $L^p$-phase spaces on the framework of real hyperbolic manifold $\mathbb{H}^d(\mathbb{R})$, where $d \geq 2$ and $1 < p \leq d$. We consider a couple of Ebin-Marsden’s Laplace and Laplace-Beltrami operators associated with the corresponding linear system which provides a vectorial matrix semigroup. Primarily, we show the existence and the uniqueness of the bounded mild solution for the linear system by using dispersive and smoothing estimates of the vectorial matrix semigroup. Next, using the fixed point arguments, we can pass from the linear system to the semilinear system to establish the existence of the bounded mild solutions. Due to Gronwall’s inequality, we will clarify the exponential stability of such solutions. Finally, we give an application of stability to the existence of periodic mild solutions for the Boussinesq systems.

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1. Introduction

In the present paper, we are concerned with the incompressible Boussinesq system in the hyperbolic space \((\mathbb{H}^d(\mathbb{R}), g)\), where the dimension \(d \geq 2\) and \(g\) is the hyperbolic metric

\[
\begin{aligned}
    &u_t + (u \cdot \nabla)u - Lu + \nabla p = \kappa \theta h + \text{div} F \\
    &\text{div} u = 0 \\
    &\theta_t - \tilde{L}\theta + (u \cdot \nabla)\theta = \text{div} f \\
    &u(x, 0) = u_0(x) \\
    &\theta(x, 0) = \theta_0(x)
\end{aligned}
\]

(1.1)

where \(L = -(d - 1) + \vec{\Delta}\) is Ebin-Marsden’s Laplace operator, \(\tilde{L} = \Delta_g\) is Laplace-Beltrami operator associated with metric \(g\), the constant \(\kappa > 0\) is the volume expansion coefficient. The field \(h\) is a generalized function of gravitational field satisfying Assumption 2.1 below, and the constant \(\kappa > 0\) is the volume expansion coefficient. The unknowns \(u\) is the velocity field, \(p\) is the scalar pressure, and \(\theta\) is the temperature. The vector field \(f\) is given such that \(\text{div} f\) represents the reference temperature and the second order tensor \(F\) is given such that \(\text{div} F\) represents the external force. Considering the zero-temperature case, i.e., \(\theta = 0\), then system (1.1) becomes the Navier-Stokes equations.

We now recall briefly some results on the Boussinesq system in Euclidean space \(\mathbb{R}^d\). Fife and Joseph [27] provided one of the first rigorous mathematical results for the convection problem by constructing analytic stationary solutions for the Boussinesq system with the bounded field \(h\), as well as analyzing some stability and bifurcation properties. After, Cannon and DiBenedetto [9] established the local-in-time existence in the class \(L^p(0, T; L^q(\mathbb{R}^n))\) with suitable \(p, q\). Hishida [30] (see also [48]) obtained the existence and exponential stability of global-in-time strong solutions for the Boussinesq system near to the steady state in a bounded domain of \(\mathbb{R}^3\). Later, by using the \(L^{p,\infty}-L^{q,\infty}\)-dispersive and smoothing estimates in weak-\(L^p\) spaces of the semigroup \(e^{-tL}\) associated with the corresponding linear equations of the Boussinesq system, Hishida [31] showed the existence and large-time behavior of global-in-time strong solutions in an exterior domain of \(\mathbb{R}^3\) under smallness assumptions on the initial data \((u_0, \theta_0)\). Well-posedness of time-periodic and almost periodic small solutions in exterior domains were proved in [35, 49] by employing frameworks based on weak-\(L^p\) spaces. The existence and stability of global small mild solutions for the Boussinesq system were studied in weak-\(L^p\) spaces in [22, 24] and in Morrey spaces in [2]. A result of stability in \(B^{3/2}_{2,1} \times B^{-1/2}_{2,1}\), under small perturbations, for a class of global large \(H^1\)-solutions was
proved by [44]. Brandolese and Schonbek [7] obtained results on the existence and time-decay of weak solutions for the Boussinesq system in whole space $\mathbb{R}^3$ with initial data $(u_0, \theta_0) \in L^2 \times L^2$. Li and Wang [43] analyzed the Boussinesq system in the torus $\mathbb{T}^3$ and obtained an ill-posedness result in $\dot{B}^{-1}_{\infty, \infty} \times \dot{B}^{-1}_{\infty, \infty}$ by showing the so-called norm inflation phenomena. Komo [41] analyzed the Boussinesq system in general smooth domains $\Omega \subset \mathbb{R}^3$ and obtained uniqueness criteria for strong solutions in the framework of Lebesgue time-spatial mixed spaces $L^p(0, T; L^q(\Omega))$ by assuming $(u_0, \theta_0) \in L^2 \times L^2$ and $g \in L^{8/3}(0, T; L^4(\Omega))$. Considering the case of a constant field $h$, Brandolese and He [8] showed the uniqueness of mild solutions in the class $(u, \theta) \in C([0, T], L^3(\mathbb{R}^3) \times L^1(\mathbb{R}^3))$ with $\theta \in L^\infty_{loc}((0, T); L^{q, \infty}(\mathbb{R}^3))$. The existence and uniqueness results in the partial inviscid cases of the Boussinesq system were studied in [15, 16], where the authors explored different kinds of conditions on the initial data $(u_0, \theta_0)$ involving $L^p$, $L^{p, \infty}$ (weak-$L^p$) and Besov spaces. Recently, the unconditional uniqueness of mild solutions for Boussinesq systems in Morrey-Lorentz spaces has established by Ferreira and Xuan [26]. Additionally, The well-posedness and stability of periodic mild solutions for Boussinesq systems in weak-Morrey spaces has studied by Xuan et al. [60].

We present in the following some related works which concerne the Navier-Stokes equations and generalized evolution equations on non-compact manifolds with negative Ricci curvatures. On these manifolds, Ebin-Marsden [19] introduced the notion of vectorial laplace operator by the mean of deformation tensor formula (today, it is known as Ebin and Marsden’s laplace operator), then they reformulated the Navier-Stokes equations on Einstein manifolds that have negative Ricci curvatures. Since then, this notion has been used in the works of Czubak and Chan [12, 13] and also Lichtenfelz [42] to prove the non-uniqueness of weak Leray solution of Navier-Stokes equation on the three-dimensional hyperbolic manifolds. Furthermore, Pierfelice [52] has proved the dispersive and smoothing estimates for Stokes semigroups on the generalized non-compact manifolds with negative Ricci curvature then combines these estimates with Kato-iteration method to prove the existence and uniqueness of strong mild solutions to Navier-Stokes equations. The existence and stability of periodic and asymptotically almost periodic mild solutions to the Navier-Stokes equations and generalized parabolic evolution equations on noncompact manifolds with negative curvature tensors have have been established in some recent works [33, 34, 38, 58, 59]. In the related works, the Navier-Stokes equations associated with Hodge-Laplace operator has been studied in several manifolds, e.g., on two sphere [10, 39], on compact Riemannian manifolds [20, 21, 40, 47, 54], or on the connected sums of $\mathbb{R}^3$ in [61].

In this paper, we consider the wellposedness and exponential stability of mild solutions for Boussinesq system (1.1) with initial data $(u(0), \theta(0))$ in $L^p(M; \Gamma(TM)) \times L^p(M; \mathbb{R})$ for the case $1 < p \leq d$. We will also revisit the existence of periodic mild solutions by using the stability result. This method is known as Serrin principle on
non-compact Riemannian manifolds (for detailed method see [36, 37] for the case of Navier-Stokes equations and see [55] for original method).

In particular, we first represent system (1.1) under the matrix intergral equation (see equation (2.8) below). Then, we use the estimates for the semigroups generating by Ebin-Marsden’s Laplace and Laplace-Beltrami operators (obtained in [52]) to prove the $L^p - L^q$-dispersive and smoothing estimates for the matrix semigroup associated with the Boussinesq system (see Lemma 3.3). Using these estimates we prove the existence of bounded mild solution for the linear equation corresponding Boussinesq system (see Lemma 3.2). After that, we establish the estimates for the bilinear operator associated with Boussinesq system, i.e., bilinear estimates (3.16), (3.18). Combining these estimates with the existence for the linear equation and fixed point arguments we establish the existence of bounded mild solution for the Boussinesq system in Theorem 3.4. We use cone inequality to prove the exponential stability of the Boussinesq system (see Theorem 4.1). Finally, we give an application of exponential stability to the existence of periodic mild solution (see Theorem 4.2).

This paper is organized as follows: Section 2 presents the real hyperbolic space and the Boussinesq system, Section 3 gives the $L^p - L^q$-dispersive and smoothing estimates and the proofs of the global existence of linear and semilinear equations, Section 4 provide the exponential stability and the application to existence of periodic mild solution for the Boussinesq system. Our main theorems are Theorem 3.2, 3.4, 4.1 and 4.2.

Notations. For the sake of convenience in presenting, through this paper we will utilize the following notations

- $(L^p \cap L^d)(X) := L^p(X) \cap L^d(X)$;
- $(L^p \cap L^d \cap L^r)(X) := L^p(X) \cap L^d(X) \cap L^r(X)$;
- $\|\cdot\|_r := \|\cdot\|_{L^r(X)}$ on the space $L^r(X)$;
- $\|\cdot\|_{\infty, r} := \|\cdot\|_{C_b(\mathbb{R}_+, L^r(X))}$ on the space of the bounded and continuous functions from $\mathbb{R}_+$ to the space $L^r(X)$;
- $\left\| \begin{bmatrix} u \\ \theta \end{bmatrix} \right\|_r := \max \left\{ \|u\|_{L^r(M, \Gamma(TM))}, \|\theta\|_{L^r(M, \mathbb{R})} \right\}$ on the product space $L^r(M; \Gamma(TM)) \times L^r(M; \mathbb{R})$.

2. Boussinesq system on the real hyperbolic manifold

Let $(M := \mathbb{H}^d(\mathbb{R}), g)$ be a real hyperbolic manifold of dimension $d \geq 2$ which is realized as the upper sheet

$$x_0^2 - x_1^2 - x_2^2 \ldots - x_d^2 = 1 \ (x_0 \geq 1),$$
of hyperboloid in $\mathbb{R}^{d+1}$, equipped with the Riemannian metric

$$g := -dx_0^2 + dx_1^2 + \ldots + dx_d^2.$$  

In geodesic polar coordinates, the hyperbolic manifold is

$$\mathbb{H}^d(\mathbb{R}) := \{(\cosh \tau, \omega \sinh \tau), \tau \geq 0, \omega \in S^{d-1}\}$$

with the metric

$$g := d\tau^2 + (\sinh \tau)^2 d\omega^2$$

where $d\omega^2$ is the canonical metric on the sphere $S^{d-1}$. A remarkable property on $\mathbb{M}$ is the Ricci curvature tensor: $\text{Ric}_{ij} = -(d-1)g_{ij}$. We refer readers to the reference [51] for more details about the hyperbolic geometry.

In order to define Laplace operator on manifolds, Ebin and Marsden introduced the vectorial laplace $L$ on vector field $u$ by using the deformation tensor (see [19] and more details in [56, 52]):

$$Lu := \frac{1}{2} \text{div}(\nabla u + \nabla u^i)^i,$$

where $\omega^i$ is a vector field associated with the 1-form $\omega$ by $g(\omega^i, Y) = \omega(Y) \forall Y \in \Gamma(TM)$. Since $\text{div} u = 0$, $L$ can be expressed as

$$Lu = \nabla^2 u + R(u),$$

where $\nabla^2 u = -\nabla^* \nabla u = \text{Tr}_g(\nabla^2 u)$ is the Bochner-Laplace and $R(u) = (\text{Ric}(u, \cdot))^i_i$ is the Ricci operator. Since $\text{Ric}(u, \cdot) = -(d-1)g(u, \cdot)$, we have $R(u) = -(d-1)u$ and

$$Lu = \nabla^2 u - (d-1)u.$$  

By using the Weitzenböck formula on 1-form $u^i$ (which is associated with $u$ by $g(u, Y) = u^i(Y), Y \in \Gamma(TM)$):

$$\Delta_H u^i = \nabla^* \nabla u^i + \text{Ric}(u, \cdot),$$

where $\Delta_H = d^*d + dd^*$ is the Hodge-Laplace on 1-forms, we can also relate $L$ to the Hodge-Laplace

$$Lu = (-\Delta_H u^i + 2\text{Ric}(u, \cdot))^i.$$

For simplicity we consider the incompressible Boussinesq system on the real hyperbolic manifold $\mathbb{M}$ with the volume expansion coefficient $\kappa = 1$:

$$\begin{cases}
  u_t + (u \cdot \nabla) u - Lu + \nabla p = \theta h + \text{div} F, \\
  \nabla \cdot u = 0, \\
  \theta_t - \tilde{L}\theta + (u \cdot \nabla) \theta = \text{div} f, \\
  u(0) = u_0, \\
  \theta(0) = \theta_0,
\end{cases} \quad (2.1)$$

where \( L = -(d-1) + \tilde{\Delta} \) is Ebin-Marsden’s Laplace operator, \( \tilde{L} = \Delta_g \) is Laplace-Beltrami operator associated with metric \( g \). The functions \( f : M \times \mathbb{R} \to \Gamma(TM) \) is given such that \( \text{div} f \) represents the reference temperature and \( F : M \times \mathbb{R} \to \Gamma(TM \otimes TM) \) is a second order tensor fields such that \( \text{div} F \) represents the external force. The unknowns are \( u(x,t) : M \times \mathbb{R} \to \Gamma(TM), p(x,t) : M \times \mathbb{R} \to \mathbb{R} \) and \( \theta(x,t) : M \times \mathbb{R} \to \mathbb{R} \) representing respectively, the velocity field, the pressure and the temperature of the fluid at point \((x,t) \in M \times \mathbb{R} \). Normally, the gravitational field \( h \) does not depend on time (see [5] for the formula of gravitational field on hyperbolic spaces). However, in this paper, we will consider a more general case, where \( h : M \times \mathbb{R}_+ \to \Gamma(TM) \) depends on time and satisfies the following assumption (see Appendix 5.1 for the discussion of gravitational field) which guarantees the regularity for elliptic problem to determine the pressure \( p \):

**Assumption 2.1.** Assume that function \( h(\cdot, t) \) satisfies

\[
h \in C_b(\mathbb{R}_+, L^\infty(\Gamma(TM))) \text{ and } h \in C_b(\mathbb{R}_+, L^{\frac{d}{d-1}}(\Gamma(TM))). \tag{2.2}
\]

Taking divergence to the first equation of system (2.6), we get

\[
\Delta_g p = \text{div}[-\text{div}(u \otimes u) + \theta h + \text{div} F]. \tag{2.3}
\]

If we consider \( u(\cdot, t) \in L^p(M; \Gamma(TM)), \theta(\cdot, t) \in L^p(M; \mathbb{R}), h(\cdot, t) \in L^\infty(M; \Gamma(TM)) \) and \( F(\cdot, t) \in L^{p/2}(M; \Gamma(TM \otimes TM)) \), then we have \( -\text{div}(u \otimes u) + \theta h + \text{div} F \in L^p(M; \Gamma(TM)) \). Moreover, the spectral of \( \Delta_g \) on hyperbolic manifold \( M = \mathbb{H}^d(\mathbb{R}) \) is \((-\infty, -(d-1)^2)\) which does not contain 0, then operator \( \Delta_g : W^{2,q}(M; \mathbb{R}) \to L^q(M; \mathbb{R}) \) is an isomorphism for \( 2 \leq q < \infty \). Therefore, for \( p > 1 \), we can choose the solution of elliptic equation (2.3) by

\[
p = \Delta_g^{-1}\text{div}[-\text{div}(u \otimes u) + \theta h + \text{div} F]. \tag{2.4}
\]

Hence

\[
\nabla p = \nabla(-\Delta_g)^{-1}\text{div}[\text{div}(u \otimes u) - \theta h - \text{div} F]. \tag{2.5}
\]

Since Riesz transforms are \( L^p \)-bounded on real hyperbolic manifolds (see [45]), we obtain that the operator \( \nabla(-\Delta_g)^{-1}\text{div} : L^p(M; \Gamma(TM)) \to L^p(M; \Gamma(TM)) \) is bounded. Therefore, we have \( \nabla p \in L^p(M; \Gamma(TM)) \).

Since we have the following identity

\[
\text{div}(\theta u) = \theta \text{div} u + (\nabla \theta) \cdot u = (u \cdot \nabla) \theta,
\]

the Boussinesq system (2.1) can be rewritten as

\[
\begin{cases}
  u_t + \text{div}(u \otimes u) - Lu + \nabla p = \theta h + \text{div} F, \\
  \nabla \cdot u = 0, \\
  \theta_t - \tilde{L} \theta + \text{div}(\theta u) = \text{div} f, \\
  u(0,x) = u_0(x) \in \Gamma(TM), \\
  \theta(0,x) = \theta_0(x) \in \mathbb{R},
\end{cases} \tag{2.6}
\]
Applying the Kodaira-Hodge operator $\mathbb{P} := I + \nabla(-\Delta_g)^{-1} \text{div}$ to the system (2.6), by the same manner to Navier-Stokes equation (see [52]) we get

$$
\begin{cases}
\partial_t u = Lu + \mathbb{P}(\theta h) + \mathbb{P}\text{div}(-u \otimes u + F), \\
\partial_t \theta = \widetilde{L}\theta + \text{div}(-\theta u) + \text{div} f, \\
\text{div} u = 0, \\
\text{div} \theta = 0,
\end{cases}
$$

(2.7)

We can consider system (2.7) with $(u, \theta)$ in the product space $C_b(\mathbb{R}_+, L^p(M; \Gamma(TM))) \times C_b(\mathbb{R}_+, L^p(M; \mathbb{R}))$. We set $\mathcal{A} := \begin{bmatrix} -L & 0 \\ 0 & -\widetilde{L} \end{bmatrix}$ acting on the Cartesian product space $L^p(M; \Gamma(TM)) \times L^p(M; \mathbb{R})$. Therefore, using Duhamel’s principle in a matrix form, we arrive at the following integral formulation for (2.7)

$$
\begin{bmatrix} u(t) \\ \theta(t) \end{bmatrix} = e^{-t\mathcal{A}} \begin{bmatrix} u_0 \\ \theta_0 \end{bmatrix} + B \left( \begin{bmatrix} u \\ \theta \end{bmatrix}, \begin{bmatrix} u \\ \theta \end{bmatrix} \right) (t) + T_h(\theta)(t) + T \left( \begin{bmatrix} F \\ f \end{bmatrix} \right) (t),
$$

(2.8)

where the bilinear, linear-coupling and external forced operators used in the above equation are given respectively by

$$
B \left( \begin{bmatrix} u \\ \theta \\ v \\ \xi \end{bmatrix}, \begin{bmatrix} u \\ \theta \end{bmatrix} \right) (t) := -\int_0^t e^{-(t-s)\mathcal{A}} \text{div} \begin{bmatrix} \mathbb{P}(u \otimes v) \\ w \xi \end{bmatrix} (s) ds,
$$

(2.9)

$$
T_h(\theta)(t) := \int_0^t e^{-(t-s)\mathcal{A}} \begin{bmatrix} \mathbb{P}(\theta h) \\ 0 \end{bmatrix} (s) ds
$$

(2.10)

and

$$
T \left( \begin{bmatrix} F \\ f \end{bmatrix} \right) (t) := \int_0^t e^{-(t-s)\mathcal{A}} \text{div} \begin{bmatrix} \mathbb{P}(F) \\ f \end{bmatrix} (s) ds.
$$

(2.11)

3. The global existence

3.1. Bounded mild solution for the linear equation. We first work to the following linear equation corresponding to the integral matrix equation (2.8):

$$
\begin{bmatrix} u(t) \\ \theta(t) \end{bmatrix} = e^{-t\mathcal{A}} \begin{bmatrix} u_0 \\ \theta_0 \end{bmatrix} + T_h(\eta)(t) + T \left( \begin{bmatrix} F \\ f \end{bmatrix} \right) (t),
$$

(3.1)

where

$$
T_h(\eta)(t) := \int_0^t e^{-(t-s)\mathcal{A}} \begin{bmatrix} \mathbb{P}(\eta h) \\ 0 \end{bmatrix} (s) ds.
$$

(3.2)

It is well-known that the $L^p - L^q$-dispersive and smoothing properties of the matrix semigroup $e^{-t\mathcal{A}}$ are really useful in order to estimate the bilinear, linear-coupling and external forced operators ($B(\cdot, \cdot)$, $T_h(\cdot)$ and $T(\cdot)$, respectively) in the equation (2.8). Concretely, we will show more clearly for this properties in what follows.
Lemma 3.1.

(i) For $t > 0$, and $p$, $q$ such that $1 \leq p \leq q \leq \infty$, the following dispersive estimates hold:

$$
\left\| e^{-t\Delta} \begin{bmatrix} u_0 \\ \theta_0 \end{bmatrix} \right\|_{L^q \times L^q} \leq [h_d(t)]^\frac{1}{p} - \frac{1}{q} e^{-t(\gamma_{p,q})} \left\| \begin{bmatrix} u_0 \\ \theta_0 \end{bmatrix} \right\|_{L^p \times L^p}
$$

(3.3)

for all $(u_0, \theta_0) \in L^p(\mathbf{M}; \Gamma(TM)) \times L^p(\mathbf{M}; \mathbb{R})$, where

$$
h_d(t) := C \max \left( \frac{1}{td/2}, 1 \right), \gamma_{p,q} := \frac{\delta_d}{2} \left[ \left( \frac{1}{p} - \frac{1}{q} \right) + \frac{8}{q} \left( 1 - \frac{1}{p} \right) \right]
$$

and $\delta_d$ are positive constants depending only on $d$.

(ii) For $p$ and $q$ such that $1 < p \leq q < \infty$ we obtain for all $t > 0$ that

$$
\left\| e^{-t\Delta} \text{div} \begin{bmatrix} T^g_0 \\ U^g_0 \end{bmatrix} \right\|_{L^q \times L^q} \leq [h_d(t)]^\frac{1}{p} - \frac{1}{q} e^{-t(\gamma_{p,q})} \left\| \begin{bmatrix} T^g_0 \\ U^g_0 \end{bmatrix} \right\|_{L^p \times L^p}
$$

(3.4)

for all tensor $T^g_0 \in L^p(\mathbf{M}; \Gamma(TM \otimes TM))$ and vector field $U_0 \in L^p(\mathbf{M}; \Gamma(TM))$.

Proof. We use the fact that

$$
e^{-t\Delta} = \begin{bmatrix} e^{tL} & 0 \\ 0 & e^{t\tilde{L}} \end{bmatrix}
$$

(3.5)

and the $L^p - L^q$-dispersive and smoothing estimates of the semigroup $e^{tL}$ (associated with Ebin-Marsden’s Laplace operator) and the heat semigroup $e^{t\tilde{L}}$ (associated with Laplace-Beltrami operator $\Delta_g$) which are proved by Pierfelice [52]. In particular, assertion $i)$ is valid since the fact that: for $t > 0$, and $p$, $q$ such that $1 \leq p \leq q \leq \infty$, the following $L^p - L^q$-dispersive estimates hold (see [52, Theorem 4.1] and its proof):

$$
\left\| e^{tL}u_0 \right\|_{L^q} \leq [h_d(t)]^\frac{1}{p} - \frac{1}{q} e^{-t(d-1+\gamma_{p,q})} \left\| u_0 \right\|_{L^p} \leq [h_d(t)]^\frac{1}{p} - \frac{1}{q} e^{-t(\gamma_{p,q})} \left\| u_0 \right\|_{L^p}
$$

(3.6)

for all $u_0 \in L^p(\mathbf{M}; \Gamma(TM))$ and

$$
\left\| e^{t\tilde{L}}\theta_0 \right\|_{L^q} \leq [h_d(t)]^\frac{1}{p} - \frac{1}{q} e^{-t(\gamma_{p,q})} \left\| \theta_0 \right\|_{L^p}
$$

(3.7)

for all $\theta_0 \in L^p(\mathbf{M}; \mathbb{R})$, where $h_d(t) := C \max \left( \frac{1}{td/2}, 1 \right), \gamma_{p,q} := \frac{\delta_d}{2} \left[ \left( \frac{1}{p} - \frac{1}{q} \right) + \frac{8}{q} \left( 1 - \frac{1}{p} \right) \right]$ and $\delta_d$ are positive constants depending only on $d$.

Assertion $ii)$ comes from the following $L^p - L^q$-smoothing estimates: for $1 < p \leq q < \infty$ and $t > 0$ we have (see [52, Corollary 4.3] and its proof):

$$
\left\| e^{tL} \text{div} T^g_0 \right\|_{L^q} \leq [h_d(t)]^\frac{1}{p} - \frac{1}{q} e^{-t(d-1+\gamma_{p,q})} \left\| T^g_0 \right\|_{L^p}
$$

(3.8)
3.1

\[ \| \tau \|^p \leq [h_d(t)]^{1 - \frac{1}{p} + \frac{1}{q} - \frac{3}{2} + \frac{3}{2} e^{-t\left(\frac{\gamma - \alpha_\nu}{2}ight)}} \| \tau_0^* \|_{L^p} \] (3.8)

and

\[ \left\| e^{\int_0^t \text{div} \tau_0^*} \right\|_{L^q} \leq [h_d(t)]^{1 - \frac{1}{p} + \frac{1}{q} - \frac{3}{2} + \frac{3}{2} e^{-t\left(\frac{\gamma - \alpha_\nu}{2}\right)}} \| \tau_0^* \|_{L^p} \] (3.9)

for all tensor \( \tau_0^* \in L^p(M; \Gamma(TM \otimes TM)) \) and vector field \( \tau_0 \in L^p(M; TM) \).

Let now \( 0 < \delta < 1 \), we will investigate the existence and uniqueness of bounded (in time) mild solution on the half time-line axis to Equation (2.7) on the following Banach spaces

\[ \mathcal{X} = \left\{ v \in C_b(\mathbb{R}_+, L^p(M; \Gamma(TM) \cap L^d(M; \Gamma(TM)))) \cap L^{d/\delta}(M; \Gamma(TM)) \right\} \]

such that \( \sup_{t > 0} \| v(t) \|_p + \| v(t) \|_d + [h_d(t)]^{-\frac{1}{\delta}} e^{\alpha t} \| v(t) \|_{\frac{d}{\delta}} < +\infty \),

equipped with the norm

\[ \| v \|_{\mathcal{X}} = \sup_{t > 0} \left( \| v(t) \|_p + \| v(t) \|_d + [h_d(t)]^{-\frac{1}{\delta}} e^{\alpha t} \| v(t) \|_{\frac{d}{\delta}} \right) ; \] (3.10)

\[ \mathcal{S} = \left\{ \theta \in C_b(\mathbb{R}_+, L^p(M; \mathbb{R}) \cap L^d(M; \mathbb{R}) \cap L^{d/\delta}(M; \mathbb{R})) \right\} \]

such that \( \sup_{t > 0} \| \theta(t) \|_p + \| \theta(t) \|_d + [h_d(t)]^{-\frac{1}{\delta}} e^{\alpha t} \| \theta(t) \|_{\frac{d}{\delta}} < +\infty \),

equipped with the norm

\[ \| \theta \|_{\mathcal{S}} = \sup_{t > 0} \left( \| \theta(t) \|_p + \| \theta(t) \|_d + [h_d(t)]^{-\frac{1}{\delta}} e^{\alpha t} \| \theta(t) \|_{\frac{d}{\delta}} \right) . \] (3.11)

And then, on the product space \( \mathcal{X} \times \mathcal{S} \), we define the norm

\[ \left\| \left[ \begin{array}{c} u \\ \theta \end{array} \right] \right\|_{\mathcal{X} \times \mathcal{S}} := \sup_{t > 0} \left\| \left[ \begin{array}{c} u \\ \theta \end{array} \right] (t) \right\|^{\bullet} , \]

where

\[ \left\| \left[ \begin{array}{c} u \\ \theta \end{array} \right] (t) \right\|^{\bullet} := \left\| \left[ \begin{array}{c} u \\ \theta \end{array} \right] (t) \right\|_p + \left\| \left[ \begin{array}{c} u \\ \theta \end{array} \right] (t) \right\|_d + [h_d(t)]^{-\frac{1}{\delta}} e^{\alpha t} \left\| \left[ \begin{array}{c} u \\ \theta \end{array} \right] (t) \right\|_{\frac{d}{\delta}} . \]

We primarily point out the existence of the bounded mild solution of the linear equation (3.1) in the following theorem.
Theorem 3.2. Let \((M, g)\) be a \(d\)-dimensional real hyperbolic manifold with \(d \geq 2\). Let \(1 < p \leq d\), \(0 < \delta < 1\) and \(0 < \alpha \leq \min\{\gamma_{d(d-d)/d}, \gamma_{d/2d-d/d}, \frac{\gamma_{d(d-d)/d}+\gamma_{d/2d-d/d}}{2}\}\). Suppose that \((u_0, \theta_0) \in (L^p \cap L^d)(M; \Gamma(TM)) \times (L^d \cap L^d)(M; R), \eta \in \mathcal{S}, \) the external forces \(h \in C_b(\mathbb{R}_+, L^d(M; \Gamma(TM))), \) \(F \in \mathcal{F} := C_b(\mathbb{R}_+, (L^{d+dp}_{d+dp} \cap L^{1+\delta}_{1+\delta} \cap L^{2+\delta}_{2+\delta})(M; \Gamma(TM \otimes TM))), \) \(f \in \mathcal{G} := C_b(\mathbb{R}_+, (L^{d+dp}_{d+dp} \cap L^{1+\delta}_{1+\delta} \cap L^{2+\delta}_{2+\delta})(M; \mathbb{R} \times \mathbb{R}))\) such that

\[
\|F\|_{\mathcal{F}} := \sup_{t \geq 0} \left( \|F(t)\|_{d+dp} + \|F(t)\|_{d+dp} + \|\eta(t)\|_{1+\delta} e^{\alpha t} \|F(t)\|_{2+\delta} \right) < +\infty,
\]

and

\[
\|f\|_{\mathcal{G}} := \sup_{t \geq 0} \left( \|f(t)\|_{d+dp} + \|f(t)\|_{d+dp} + \|\eta(t)\|_{1+\delta} e^{\alpha t} \|f(t)\|_{2+\delta} \right) < +\infty.
\]

Then, the Problem (3.1) with the initial value \((u_0, \theta_0)\) has one and only one mild solution \((u, \theta) \in \mathcal{X} \times \mathcal{S}\) satisfying

\[
\left\| \begin{bmatrix} u \\ \theta \end{bmatrix} \right\|_{\mathcal{X} \times \mathcal{S}} \leq 2 \left\| \begin{bmatrix} u_0 \\ \theta_0 \end{bmatrix} \right\| + N \|h\|_{\infty, \frac{d}{2}} \|\eta\|_{\mathcal{S}} + M \left\| \begin{bmatrix} F \\ f \end{bmatrix} \right\|_{\mathcal{F} \times \mathcal{G}},
\]

where the positive constants \(M\) and \(N\) are independent to \(h, \eta, F\) and \(f\); and

\[
\left\| \begin{bmatrix} u_0 \\ \theta_0 \end{bmatrix} \right\| := \left\| \begin{bmatrix} u_0 \\ \theta_0 \end{bmatrix} \right\|_p + \left\| \begin{bmatrix} u_0 \\ \theta_0 \end{bmatrix} \right\|_d.
\]

Proof. By Assumption 2.1 and interpolation inequality (see inequality (2.7) in [31, Lemma 2.1]), we obtain that \(h \in C_b(\mathbb{R}_+, L^q(\Gamma(TM)))\) for \(\frac{d}{2} < q \leq \infty\). Hence, we have \(h \in C_b(\mathbb{R}_+, L^d(M; \Gamma(TM)))\) for \(0 < \delta < 1\).

First step. Using Lemma 3.1 and the boundedness of the operator \(P\) (see more details in example [45]), we give the estimate for \(\left\| \begin{bmatrix} u(t) \\ \theta(t) \end{bmatrix} \right\|_p\). It is clear that

\[
\left\| \begin{bmatrix} u(t) \\ \theta(t) \end{bmatrix} \right\|_p \leq \left\| e^{-tA} \begin{bmatrix} u_0 \\ \theta_0 \end{bmatrix} \right\|_p + \|T_h(\eta)(t)\|_p + \left\| \begin{bmatrix} F \\ f \end{bmatrix} \right\|_{\mathcal{F}}(t)
\leq \left\| \begin{bmatrix} u_0 \\ \theta_0 \end{bmatrix} \right\|_p + \int_0^t \left\| e^{-(t-\tau)A} \begin{bmatrix} P(h\eta)(\tau) \\ 0 \end{bmatrix} \right\|_p d\tau
+ \int_0^t \left\| e^{-(t-\tau)A} \nabla \left( \begin{bmatrix} P(F) \\ f \end{bmatrix} \right)(\tau) \right\|_p d\tau
\leq \left\| \begin{bmatrix} u_0 \\ \theta_0 \end{bmatrix} \right\|_p + \int_0^t \left\| \begin{bmatrix} h_d(t-\tau) \frac{d}{2} e^{-\beta_1(t-\tau)} \right\|_p d\tau
\leq \int_0^t \left\| \begin{bmatrix} h_d(t-\tau) \frac{d}{2} e^{-\beta_1(t-\tau)} \right\|_p d\tau
\]
\[ \| \begin{bmatrix} u(t) \\ \theta(t) \end{bmatrix} \|_d \leq e^{-tA} \| \begin{bmatrix} u_0 \\ \theta_0 \end{bmatrix} \|_d + \| T_h(\eta)(t) \|_d + \| T \begin{bmatrix} F \\ f \end{bmatrix} \|_d \]
\[ \leq \| \begin{bmatrix} u_0 \\ \theta_0 \end{bmatrix} \|_d + \int_0^t e^{-(t-\tau)A} \| \begin{bmatrix} \mathbb{P}(h\eta)(\tau) \\ \mathbb{P}(F) \end{bmatrix} \|_d d\tau \]
\[ + \int_0^t e^{-(t-\tau)A} \text{div} \begin{bmatrix} \mathbb{P}(F) \end{bmatrix} (\tau) \|_d d\tau \]

where \( \beta_1 = \gamma_{dp/(1+\delta p), p}; \beta_2 = \frac{\gamma_{dp/(1+\delta p), p}}{2}; \gamma_{N1} := C^\frac{\delta-1}{\delta} \Gamma \left( \frac{1}{2} - \frac{\delta}{2} \right) + \frac{1}{\beta_1} \) and \( M_1 := C^\frac{\delta+1}{\delta} \left[ \beta_2 \Gamma \left( \frac{1}{2} - \frac{\delta}{2} \right) + \frac{1}{\beta_2} \right] \).

**Second step.** We continue to estimate \( \| \begin{bmatrix} u(t) \\ \theta(t) \end{bmatrix} \|_d \) by using again Lemma 3.1 and the boundedness of the operator \( \mathbb{P} \). Concretely, we see that
\begin{align}
\left\| \left[ u_0 \right] \right\|_d & + \int_0^t \left[ h_d(t - \tau) \right] \frac{2}{\hat{\beta}_2} e^{-\hat{\beta}_1(t - \tau)} \left\| \left[ \left( \frac{h_\eta(\tau)}{t} \right) \right] \right\|_\frac{d}{d + \delta} d\tau \\
+ \int_0^t \left[ h_d(t - \tau) \right] \frac{2}{\hat{\beta}_2} e^{-\hat{\beta}_2(t - \tau)} d\tau & \left\| \left[ F \right] \right\|_{\mathcal{F} \times \mathcal{Q}} \\
\left\| \left[ u_0 \right] \right\|_d & + \int_0^t \left[ h_d(t - \tau) \right] \frac{2}{\hat{\beta}_2} e^{-\hat{\beta}_2(t - \tau)} \left\| \left[ \left( \frac{h_\eta(\tau)}{t} \right) \right] \right\|_\frac{d}{d + \delta} d\tau \\
+ \int_0^t \left[ h_d(t - \tau) \right] \frac{2}{\hat{\beta}_2} e^{-\hat{\beta}_2(t - \tau)} d\tau & \left\| \left[ F \right] \right\|_{\mathcal{F} \times \mathcal{Q}} \\
\left\| \left[ u_0 \right] \right\|_d & + \int_0^t \left[ C^{\frac{d}{d + \delta}} (t - \tau) \right] \frac{2}{\hat{\beta}_2} e^{-\hat{\beta}_2(t - \tau)} d\tau \left\| \left[ F \right] \right\|_{\mathcal{F} \times \mathcal{Q}} \\
+ \int_0^t \left[ C^{\frac{d}{d + \delta}} (t - \tau) \right] \frac{2}{\hat{\beta}_2} e^{-\hat{\beta}_2(t - \tau)} d\tau & \left\| \left[ F \right] \right\|_{\mathcal{F} \times \mathcal{Q}} \\
\left\| \left[ u_0 \right] \right\|_d & + \tilde{N}_1 \left\| \left[ u_0 \right] \right\|_d + \tilde{M}_1 \left\| \left[ F \right] \right\|_{\mathcal{F} \times \mathcal{Q}},
\end{align}

where $\hat{\beta}_1 = \gamma_{d/(1+\delta),d}$, $\hat{\beta}_2 = \frac{\gamma_{d,d} + \gamma_{d/(1+\delta),d}}{2}$.

$\tilde{N}_1 := C^{\frac{d}{d + \delta}} \left[ \beta_1^{\frac{d}{d + 1}} \Gamma \left( 1 - \frac{d}{2} \right) + \frac{1}{\beta_1} \right] \left\| \left[ u_0 \right] \right\|_d$ and $\tilde{M}_1 := C^{\frac{d}{d + \delta}} \left[ \beta_2^{\frac{d}{d + 1}} \Gamma \left( \frac{1}{2} - \frac{d}{2} \right) + \frac{1}{\beta_2} \right]$.

**Third step.** It remains to estimate the boundedness of the third term

$$[h_d(t)]^{-\frac{d}{d + \delta}} e^{\alpha t} \left\| \left[ \left( \frac{u(t)}{\theta(t)} \right) \right] \right\|_\frac{d}{d + \delta}.$$
Indeed, it is obvious that \( [h_d(t)]^{-\frac{1-a}{d}} \leq C^{\frac{1}{d-1}} \) for all \( t > 0 \) and \( [h_d(t)]^{-\frac{1-a}{d}} e^{\alpha t} \left\| e^{-tA} \left[ \begin{array}{c} u_0 \\ \theta_0 \end{array} \right] \right\|_d \leq \left\| \begin{array}{c} u_0 \\ \theta_0 \end{array} \right\|_d \) for all \( t > 0 \). Therefore, we imply that

\[
[h_d(t)]^{-\frac{1-a}{d}} e^{\alpha t} \left\| \begin{array}{c} u(t) \\ \theta(t) \end{array} \right\|_d \\
\leq [h_d(t)]^{-\frac{1-a}{d}} e^{\alpha t} \left\| e^{-tA} \left[ \begin{array}{c} u_0 \\ \theta_0 \end{array} \right] \right\|_d + \| T_h(\eta)(t) \|_\frac{d}{2} + \left\| \mathbb{T} \left( \begin{array}{c} F \\ f \end{array} \right) (t) \right\|_\frac{d}{2} \\
\leq \left\| \begin{array}{c} u_0 \\ \theta_0 \end{array} \right\|_d + C^{\frac{1}{d-1}} e^{\alpha t} \int_0^t \left\| e^{-(t-\tau)A} \left[ \begin{array}{c} P(h\eta)(\tau) \\ 0 \end{array} \right] \right\|_\frac{d}{2} d\tau \\
+ C^{\frac{1}{d-1}} e^{\alpha t} \int_0^t \left\| e^{-(t-\tau)A} \text{div} \left( \begin{array}{c} F \\ f \end{array} \right) (\tau) \right\|_\frac{d}{2} d\tau \\
\leq \left\| \begin{array}{c} u_0 \\ \theta_0 \end{array} \right\|_d + C^{\frac{1}{d-1}} e^{\alpha t} \int_0^t \left\| h_d(t-\tau) \right\|_\frac{d}{2} e^{-\bar{\beta}_1(t-\tau)} \left\| (h\eta)(\tau) \right\|_\frac{d}{2} d\tau \\
+ C^{\frac{1}{d-1}} \sup_{t>0} [h_d(t)]^{-\frac{1-a}{d}} e^{\alpha t} \left\| \left[ \begin{array}{c} F(t) \\ f(t) \end{array} \right] \right\|_\frac{d}{2} \int_0^t [h_d(t-\tau)]^{-\frac{1-a}{d}} [h_d(\tau)]^{-\frac{1-a}{d}} e^{-(\bar{\beta}_2-\alpha)(t-\tau)} d\tau \\
\leq \left\| \begin{array}{c} u_0 \\ \theta_0 \end{array} \right\|_d + C^{\frac{1}{d-1}} \| h \|_{\infty, \frac{d}{4}} \| \eta \|_s \int_0^t [h_d(t-\tau)]^{-\frac{1-a}{d}} [h_d(\tau)]^{-\frac{1-a}{d}} e^{-(\bar{\beta}_1-\alpha)(t-\tau)} d\tau \\
+ C^{\frac{1}{d-1}} \left\| \begin{array}{c} F \\ f \end{array} \right\|_{\mathcal{E} \times \mathcal{E}} \int_0^t [h_d(t-\tau)]^{-\frac{1-a}{d}} [h_d(\tau)]^{-\frac{1-a}{d}} e^{-(\bar{\beta}_2-\alpha)(t-\tau)} d\tau \\
\leq \left\| \begin{array}{c} u_0 \\ \theta_0 \end{array} \right\|_d + \bar{N}_1 \| h \|_{\infty, \frac{d}{4}} \| \eta \|_s + \tilde{M}_1 \left\| \begin{array}{c} F \\ f \end{array} \right\|_{\mathcal{E} \times \mathcal{E}} , \tag{3.15}
\]

where \( \bar{\beta}_1 = \gamma_d/\delta, \delta/\delta, \bar{\beta}_2 = \frac{\delta_d/\delta + \gamma_d/\delta}{2} \); 
\[
\bar{N}_1 := C^{\frac{1}{d-1}} \int_0^t [h_d(t-\tau)]^{-\frac{1-a}{d}} [h_d(\tau)]^{-\frac{1-a}{d}} e^{-(\bar{\beta}_1-\alpha)(t-\tau)} d\tau < +\infty,
\]
and 
\[
\tilde{M}_1 := C^{\frac{1}{d-1}} \int_0^t [h_d(t-\tau)]^{-\frac{1-a}{d}} [h_d(\tau)]^{-\frac{1-a}{d}} e^{-(\bar{\beta}_2-\alpha)(t-\tau)} d\tau < +\infty.
\]
Finally, setting \( N := N_1 + \hat{N}_1 + \tilde{N}_1, M := M_1 + \hat{M}_1 + \tilde{M}_1 \) and combining the inequalities (3.13), (3.14) and (3.15), we obtain the boundedness (3.12) and our proof is completed. \( \Box \)

3.2. **Bounded mild solution for Boussinesq system.** In order to study bounded mild solutions for the equation (2.8), we need to estimate the bilinear operator \( B(\cdot, \cdot) \) given by the formula (2.9).

**Lemma 3.3.** Let \((M, g)\) be a \(d\)-dimensional real hyperbolic manifold with \(d \geq 2\) and \(1 < p \leq d\). There exists a universal constant \(K > 0\) such that

(i) for all \(t > 0\),

\[
\left\| B \left( \begin{bmatrix} u \\ \theta \\ \xi \end{bmatrix}, \begin{bmatrix} v \\ \tau \end{bmatrix} \right)(t) \right\|_p \leq K \left\| \begin{bmatrix} u \\ \theta \\ \xi \end{bmatrix} \right\|_{X \times S} \left\| \begin{bmatrix} v \\ \tau \end{bmatrix} \right\|_{X \times S},
\]

(3.16)

(ii) for all \(t > 0\),

\[
\left\| B \left( \begin{bmatrix} u \\ \theta \\ \xi \end{bmatrix}, \begin{bmatrix} v \\ \tau \end{bmatrix} \right)(t) \right\|_d \leq K \left\| \begin{bmatrix} u \\ \theta \\ \xi \end{bmatrix} \right\|_{X \times S} \left\| \begin{bmatrix} v \\ \tau \end{bmatrix} \right\|_{X \times S},
\]

(3.17)

(iii) and for all \(t > 0\),

\[
[h_d(t)]^{-\frac{1}{d}} e^{\alpha t} \left\| B \left( \begin{bmatrix} u \\ \theta \\ \xi \end{bmatrix}, \begin{bmatrix} v \\ \tau \end{bmatrix} \right)(t) \right\|_{\frac{p}{4}} \leq K \left\| \begin{bmatrix} u \\ \theta \\ \xi \end{bmatrix} \right\|_{X \times S} \left\| \begin{bmatrix} v \\ \tau \end{bmatrix} \right\|_{X \times S},
\]

(3.18)

where the constant \(K\) is not dependent on \(u, v, \theta, \xi\).

**Proof.** The proof is similarly the one of boundedness of the linear operator \(\mathbb{T}(\cdot)\) but we need to go further by estimating the tensor product \(u \otimes u\) and \(u \xi\). Using the boundedness of operator \(\mathbb{P}\), the \(L^p - L^q\)-smoothing estimates in assertion (ii) of Lemma 3.1 and H"older’s inequality we have that

\[
\left\| B \left( \begin{bmatrix} u \\ \theta \\ \xi \end{bmatrix}, \begin{bmatrix} v \\ \tau \end{bmatrix} \right)(t) \right\|_p \\
\leq \int_0^t \left\| e^{-(t-\tau)A_{\div}} \mathbb{P}(u \otimes v)(\tau) \right\| \frac{\| u \xi \|_{(\tau)} \|}{\| u \xi \|_{(\tau)}} d\tau \\
\leq \int_0^t [h_d(t-\tau)]^{-\frac{1}{d}} e^{-\beta_2(t-\tau)} \left\| \left[ \begin{bmatrix} u(\tau) \\ \theta(\tau) \end{bmatrix}, \begin{bmatrix} v(\tau) \\ \xi(\tau) \end{bmatrix} \right] \right\|_{\frac{p}{4}} d\tau \\
\mid 0^t [h_d(t-\tau)]^{-\frac{1}{d}} e^{-\beta_2(t-\tau)} \left\| u(\tau) \right\|_{\frac{p}{4}} \left\| v(\tau) \right\|_{\frac{p}{4}} d\tau.
\]
\begin{align*}
\leq \left\| \begin{bmatrix} u \\ \theta \end{bmatrix} \right\|_{X \times S} \left\| \begin{bmatrix} v \\ \xi \end{bmatrix} \right\|_{X \times S} \int_0^t [h_d(t - \tau)] \frac{\hat{\beta}_2(t-\tau)}{2} e^{-\frac{1+\delta}{\alpha} d \tau} [h_d(\tau)]^{\frac{1-\delta}{\alpha}} e^{-\alpha \tau} d\tau \\
\leq \hat{K}_1 \left\| \begin{bmatrix} u \\ \theta \end{bmatrix} \right\|_{X \times S} \left\| \begin{bmatrix} v \\ \xi \end{bmatrix} \right\|_{X \times S},
\end{align*}

where \( \hat{\beta}_2 = \frac{\gamma_{d,p} + \gamma d/(1+\delta) \gamma}{2} \) and

\[K_1 := \int_0^t [h_d(t - \tau)] \frac{\hat{\beta}_2(t-\tau) e^{-\frac{1+\delta}{\alpha} d \tau}}{2} e^{-\alpha \tau} d\tau < +\infty.\]

By the same way, we receive the estimate for \( \left\| B \left( \begin{bmatrix} u \\ \theta \end{bmatrix}, \begin{bmatrix} v \\ \xi \end{bmatrix} \right)(t) \right\|_d \) as follows.

\begin{align*}
\left\| B \left( \begin{bmatrix} u \\ \theta \end{bmatrix}, \begin{bmatrix} v \\ \xi \end{bmatrix} \right)(t) \right\|_d \\
\leq \int_0^t e^{-(t-\tau)A} \left[ \mathcal{P} \left( (u \otimes v)(\tau) \right) \right] d\tau \\
\leq \int_0^t [h_d(t - \tau)] \frac{\hat{\beta}_2(t-\tau)}{2} \left\| \begin{bmatrix} u(\tau) \\ \theta(\tau) \end{bmatrix} \right\|_d \left\| \begin{bmatrix} v(\tau) \\ \xi(\tau) \end{bmatrix} \right\|_d \, d\tau \\
\leq \hat{K}_1 \left\| \begin{bmatrix} u \\ \theta \end{bmatrix} \right\|_{X \times S} \left\| \begin{bmatrix} v \\ \xi \end{bmatrix} \right\|_{X \times S} \int_0^t [h_d(t - \tau)] \frac{\hat{\beta}_2(t-\tau)}{2} e^{-\alpha \tau} d\tau \\
\leq \hat{K}_1 \left\| \begin{bmatrix} u \\ \theta \end{bmatrix} \right\|_{X \times S} \left\| \begin{bmatrix} v \\ \xi \end{bmatrix} \right\|_{X \times S},
\end{align*}

where \( \hat{\beta}_2 = \frac{\gamma_{d,d} + \gamma d/(1+\delta) \gamma}{2} \) and

\[K_1 := \int_0^t [h_d(t - \tau)] \frac{\hat{\beta}_2(t-\tau) e^{-\frac{1+\delta}{\alpha} d \tau}}{2} e^{-\alpha \tau} d\tau < +\infty.\]

On the other hand, we need to estimate \( \left[ h_d(t) \right]^{-\frac{1-\delta}{\alpha}} e^{\alpha t} \left\| \begin{bmatrix} u(t) \\ \theta(t) \end{bmatrix} \right\|_d \). Indeed, since

\[ \left[ h_d(t) \right]^{-\frac{1-\delta}{\alpha}} \leq C^{-\frac{1}{\alpha}} \text{ for all } t > 0, \]

we then see that

\[ \left[ h_d(t) \right]^{-\frac{1-\delta}{\alpha}} e^{\alpha t} \left\| B \left( \begin{bmatrix} u \\ \theta \end{bmatrix}, \begin{bmatrix} v \\ \xi \end{bmatrix} \right)(t) \right\|_d \]
\[\leq C^\frac{\delta+1}{d} e^{\alpha t} \int_0^t \left\| e^{-(t-\tau)\mathcal{A}} \text{div} \left[ \mathcal{P}(u \otimes v)(\tau) \right] \frac{d\tau}{\delta}\right.\]

\[\leq C^\frac{\delta+1}{d} e^{\alpha t} \int_0^t \left[ h_d(t-\tau) \right]^\frac{\delta+1}{d} e^{-\tilde{\beta}_2(t-\tau)} \left\| \left[ \frac{(u \otimes v)(\tau)}{(u\xi)(\tau)} \right] \frac{d\tau}{\delta}\right.\]

\[\leq C^\frac{\delta+1}{d} e^{\alpha t} \int_0^t \left[ h_d(t-\tau) \right]^\frac{\delta+1}{d} e^{-\tilde{\beta}_2(t-\tau)} \left\| \left[ \frac{u(\tau)}{\theta(\tau)} \right] \frac{d\tau}{\delta} \right\| \left\| \frac{v(\tau)}{\xi(\tau)} \right\| \frac{d\tau}{\delta}\right.\]

\[\leq \tilde{K}_1 \left\| \frac{u}{\theta} \right\|_{\mathcal{X} \times \mathcal{S}} \left\| \frac{v}{\xi} \right\|_{\mathcal{X} \times \mathcal{S}} \int_0^t \left[ h_d(t-\tau) \right]^\frac{\delta+1}{d} \left[ h_d(\tau) \right]^\frac{2(\delta-d)}{d} e^{-(\tilde{\beta}_2-\alpha)(t-\tau)} e^{-\alpha\tau} d\tau < +\infty.\]

where \(\tilde{\beta}_2 = \frac{\gamma d/\delta + \gamma d/\delta}{2}\)

Hence, by setting \(K = \max\{K_1, \tilde{K}_1, \bar{K}_1\}\), we receive the inequalities (3.16), (3.17) and (3.18), and our proof is completed. \(\square\)

We are now able to clarify the existence and uniqueness of the bounded mild solution of the equation (2.8) in the following theorem.

**Theorem 3.4.** (Global in time mild solution) Let \((\mathcal{M}, g)\) be a \(d\)-dimensional real hyperbolic manifold with \(d \geq 2\). For \(1 < p \leq d\), suppose that the external forces \(h \in C_b(\mathbb{R}_+, L^\frac{2p}{d} (\mathcal{M}; \Gamma(TM)), F \in \mathcal{F} := C_b(\mathbb{R}_+, (L^\frac{2p}{d} \cap L^\frac{d}{d+\delta})(\mathcal{M}; \Gamma(TM \otimes TM)), f \in \mathcal{G} := C_b(\mathbb{R}_+, (L^\frac{2p}{d} \cap L^\frac{d}{d+\delta})(\mathcal{M}; \mathbb{R} \times \mathbb{R})). If the norms \(\|(u_0, \theta_0)\|, \|h\|_{\infty, \mathcal{F}}\)

\[\left\| \begin{bmatrix} F \\ f \end{bmatrix} \right\|_{\mathcal{F} \times \mathcal{G}} \]

are sufficiently small, then equation (2.8) has one and only one bounded mild solution \((\hat{u}, \hat{\theta})\) on a small ball of \(\mathcal{X} \times \mathcal{S}\).

**Proof.** In order to start, we denote

\[\mathcal{B}_\rho : = \{ (v, \eta) \in \mathcal{X} \times \mathcal{S} \text{ such that } \|(v, \eta)\|_{\mathcal{X} \times \mathcal{S}} \leq \rho \}.\]

For each \((v, \eta) \in \mathcal{B}_\rho\), we consider the linear equation

\[\begin{bmatrix} u(t) \\ \theta(t) \end{bmatrix} = e^{-t\mathcal{A}} \begin{bmatrix} u_0 \\ \theta_0 \end{bmatrix} + B \left( \begin{bmatrix} v \\ \eta \end{bmatrix}, \begin{bmatrix} v \\ \eta \end{bmatrix} \right)(t) + T_h(\eta)(t) + T \left( \begin{bmatrix} F \\ f \end{bmatrix} \right)(t). \quad (3.22)\]
Hence, for \((u, \theta)\) because of inequality (3.20) and the linear estimate for \(T(\cdot)\) as in the proof of Theorem 3.2, we obtain that for \((v, \eta) \in B_\rho\) there exists a unique bounded mild solution \((u, \theta)\) to (3.22) satisfying
\[
\left\|\begin{bmatrix} u \\ \theta \end{bmatrix}\right\|_{\mathcal{X} \times \mathcal{S}} \leq 2 \left\|\begin{bmatrix} u_0 \\ \theta_0 \end{bmatrix}\right\| + 3K \left\|\begin{bmatrix} v \\ \eta \end{bmatrix}\right\|_{\mathcal{X} \times \mathcal{S}}^2 + N \left\|h\right\|_{\infty, \frac{4}{5}} \left\|\eta\right\|_{\mathcal{S}} + M \left\|\begin{bmatrix} F \\ f \end{bmatrix}\right\|_{\mathcal{F} \times \mathcal{O}}.
\]
Therefore, we can define a map \(\Phi : \mathcal{X} \times \mathcal{S} \rightarrow \mathcal{X} \times \mathcal{S}\) as follows
\[
\Phi \begin{bmatrix} v \\ \eta \end{bmatrix} = \begin{bmatrix} u \\ \theta \end{bmatrix}.
\]
(3.24)

If \(\left\|\begin{bmatrix} u_0 \\ \theta_0 \end{bmatrix}\right\|_{\mathcal{O}}, \rho, \left\|h\right\|_{\infty, \frac{4}{5}}\) and \(\left\|\begin{bmatrix} F \\ f \end{bmatrix}\right\|_{\mathcal{F} \times \mathcal{O}}\) are small enough, then
\[
\left\|\begin{bmatrix} u \\ \theta \end{bmatrix}\right\|_{\mathcal{X} \times \mathcal{S}} \leq \rho
\]

because of inequality (3.23). Therefore, the map \(\Phi\) acts from \(B_\rho\) into itself.

Furthermore, it is obvious that
\[
\Phi \begin{bmatrix} v \\ \eta \end{bmatrix}(t) = \begin{bmatrix} u_0 \\ \theta_0 \end{bmatrix} + B \begin{bmatrix} v \\ \eta \end{bmatrix}(t) + T_h(\eta)(t) + \mathcal{T} \left(\begin{bmatrix} F \\ f \end{bmatrix}\right)(t).
\]
(3.25)

Hence, for \((u_1, \eta_1), (u_2, \eta_2) \in B_\rho\), applying again the bilinear estimates in Lemma 3.3 and the linear estimate for \(T(\cdot)\) as in the proof of Theorem 3.2, we have
\[
\left\|\Phi \begin{bmatrix} u_1 \\ \eta_1 \end{bmatrix} - \Phi \begin{bmatrix} u_2 \\ \eta_2 \end{bmatrix}\right\|_{\mathcal{X} \times \mathcal{S}}
\leq \sup_{t > 0} \left\|\int_0^t e^{-(t-\tau)\mathcal{A}} \text{div} \left[u_2(u_2 - u_1) + u_1(u_2 - u_1) \right] u_2(u_2 - u_1) \right\|_{\mathcal{X} \times \mathcal{S}} + \left\|T_h(\eta_1 - \eta_2)\right\|_{\mathcal{X} \times \mathcal{S}}
\leq \left\|\begin{bmatrix} u_1 - u_2 \\ \eta_1 - \eta_2 \end{bmatrix}\right\|_{\mathcal{X} \times \mathcal{S}} \left(3K \left\|\begin{bmatrix} u_1 \\ \eta_1 \end{bmatrix}\right\|_{\mathcal{X} \times \mathcal{S}} + 3K \left\|\begin{bmatrix} u_2 \\ \eta_2 \end{bmatrix}\right\|_{\mathcal{X} \times \mathcal{S}} + N \left\|h\right\|_{\infty, \frac{4}{5}}\right)
\leq \left\|\begin{bmatrix} u_1 - u_2 \\ \eta_1 - \eta_2 \end{bmatrix}\right\|_{\mathcal{X} \times \mathcal{S}} \left(6K \rho + N \left\|h\right\|_{\infty, \frac{4}{5}}\right).
\]
(3.26)
Therefore, when \( \rho \) and \( \|h\|_{\infty} \) are small enough such that \( 6K\rho + N\|h\|_{\infty} < 1 \), then the map \( \Phi \) becomes a contraction on \( \mathcal{B}_p \).

Therefore, by fixed point arguments there exists a unique fixed point \((\hat{u}, \hat{\theta})\) of \( \Phi \), and by the definition of \( \Phi \), this fixed point \((\hat{u}, \hat{\theta})\) is a bounded mild solution to equation (2.8). The uniqueness of \((\hat{u}, \hat{\theta})\) in the small ball \( \mathcal{B}_p \) is clearly by using inequality (3.26).

\[ \Box \]

4. ASYMPTOTICAL BEHAVIOUR AND APPLICATION

4.1. Exponential stability. In this section, we are going to apply Gronwall’s inequality in order to investigate exponential stability of the mild solution to equation (2.8).

**Theorem 4.1.** (Exponential stability). Let \((M, g)\) be a \( d\)-dimensional real hyperbolic manifold with \( d \geq 2 \). For \( 1 < p \leq d \), assume that the external force \( h \in C^p(\mathbb{R}_+, L^\frac{d}{\delta}(M; \Gamma(TM))) \) with the norm \( \|h\|_{\infty} \) small enough. Then, the mild solution \((u, \theta)\) of the equation (2.8) obtained in Theorem 3.4 is exponentially stable in the sense that for any other mild solution \((\tilde{u}, \tilde{\theta})\) of the equation (2.8) with initial data \((\tilde{u}_0, \tilde{\theta}_0)\) such that the norm \( \|(u_0 - \tilde{u}_0, \theta_0 - \tilde{\theta}_0)\| \) is sufficiently small, we have

\[
\left\| \begin{bmatrix} u - \tilde{u} \\ \theta - \tilde{\theta} \end{bmatrix} (t) \right\| < \left\| \begin{bmatrix} u_0 - \tilde{u}_0 \\ \theta_0 - \tilde{\theta}_0 \end{bmatrix} \right\| e^{-\Theta t} \text{ for } t > 0,
\]

where \( \Theta = \min\{\gamma_p, \beta_1, \beta_2, \hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_1 - \alpha, \hat{\beta}_2 - \alpha\} \) with \( \beta_i, \hat{\beta}_i, \tilde{\beta}_i, i = 1, 2 \) given in the previous sections.

**Proof.** For \((u, \theta)\) and \((\tilde{u}, \tilde{\theta})\) are two solutions of equation (2.8) with initial data \((u_0, \theta_0)\) and \((\tilde{u}_0, \tilde{\theta}_0)\), respectively; we see that \((u - \tilde{u}, \theta - \tilde{\theta})\) is solution of

\[
\begin{align*}
\begin{bmatrix} u(t) - \tilde{u}(t) \\ \theta(t) - \tilde{\theta}(t) \end{bmatrix} &= e^{-tA} \begin{bmatrix} u_0 - \tilde{u}_0 \\ \theta_0 - \tilde{\theta}_0 \end{bmatrix} + B \begin{bmatrix} u \\ \theta \end{bmatrix} (t) - B \begin{bmatrix} \tilde{u} \\ \tilde{\theta} \end{bmatrix} (t) \\
&\quad + T_h(\theta - \tilde{\theta}) \\
&= e^{-tA} \begin{bmatrix} u_0 - \tilde{u}_0 \\ \theta_0 - \tilde{\theta}_0 \end{bmatrix} + \int_0^t e^{-(t-\tau)A} \text{div} \begin{bmatrix} \mathbb{P}[\tilde{u}(\tilde{u} - u) + u(\tilde{u} - u)] \\ \tilde{u}(\theta - \tilde{\theta}) - \theta(\tilde{u} - u) \end{bmatrix} (\tau) d\tau \\
&\quad + \int_0^t e^{-(t-\tau)A} \begin{bmatrix} \mathbb{P}[h(\theta - \tilde{\theta})] \\ 0 \end{bmatrix} (\tau) d\tau.
\end{align*}
\]

(4.2)

Similar to Theorem 3.4, for the norms \( \|(u_0 - \tilde{u}_0, \theta_0 - \tilde{\theta}_0)\| \), \( \|h\|_{\infty} \) small enough, equation (4.2) has a unique mild solution \((u - \tilde{u}, \theta - \tilde{\theta})\) in the small ball \( \mathcal{B}_p \) of \( \mathcal{X} \times \mathcal{S} \) (here, we can chose \( \|(u_0, \theta_0)\| \leq \frac{\rho}{2}, \|\tilde{u}_0, \tilde{\theta}_0\| \leq \frac{\rho}{2} \) for a given \( \rho > 0 \)).
Now we establish the exponential decay of \((u(t) - \tilde{u}(t), \theta(t) - \tilde{\theta}(t))\) for large time \(t\). Using again the techniques as in the proof of the bilinear estimates in Lemma 3.3 and the linear estimate for \(T(\cdot)\) in Theorem 3.2, we imply that

\[
\begin{align*}
&\left\| \begin{bmatrix} u(t) - \tilde{u}(t) \\ \theta(t) - \tilde{\theta}(t) \end{bmatrix} \right\|_p \\
&\leq e^{-tA} \begin{bmatrix} u_0 - \tilde{u}_0 \\ \theta_0 - \tilde{\theta}_0 \end{bmatrix} \left\| + \int_0^t e^{-(t-\tau)A} \text{div} \begin{bmatrix} \mathbb{P}[\tilde{u}(\tilde{u} - u) + u(\tilde{u} - u)] \\ \tilde{u}(\theta - \tilde{\theta}) - \theta(\tilde{u} - u) \end{bmatrix}(\tau) \right\|_p d\tau \\
&+ \int_0^t e^{-(t-\tau)A} \begin{bmatrix} \mathbb{P}[h(\theta - \tilde{\theta})] \\ 0 \end{bmatrix}(\tau) \right\|_p d\tau \\
&\leq e^{-\gamma_{p,p}t} \left\| \begin{bmatrix} u_0 - \tilde{u}_0 \\ \theta_0 - \tilde{\theta}_0 \end{bmatrix} \right\|_p \\
&+ \int_0^t [h_d(t - \tau)]^{\frac{1}{2} + \frac{1}{4} + \frac{1}{4} - \beta_2(t-\tau)} \left\| \begin{bmatrix} u(\tilde{u} - u) + u(\tilde{u} - u) \\ \tilde{u}(\theta - \tilde{\theta}) - \theta(\tilde{u} - u) \end{bmatrix}(\tau) \right\|_p d\tau \\
&+ \int_0^t [h_d(t - \tau)]^{\frac{1}{2} + \frac{1}{4} - \beta_1(t-\tau)} \left\| \begin{bmatrix} h(\theta - \tilde{\theta}) \\ 0 \end{bmatrix}(\tau) \right\|_p d\tau \\
&\leq e^{-\gamma_{p,p}t} \left\| \begin{bmatrix} u_0 - \tilde{u}_0 \\ \theta_0 - \tilde{\theta}_0 \end{bmatrix} \right\|_p \\
&+ \int_0^t [h_d(t - \tau)]^{\frac{1}{2} + \frac{1}{4} + \frac{1}{4} - \beta_2(t-\tau)} \left\| \begin{bmatrix} 0 \\ 0 \end{bmatrix}(\tau) \right\|_p d\tau \\
&\leq e^{-\gamma_{p,p}t} \left\| \begin{bmatrix} u_0 - \tilde{u}_0 \\ \theta_0 - \tilde{\theta}_0 \end{bmatrix} \right\|_p \\
&+ \int_0^t [h_d(t - \tau)]^{\frac{1}{2} + \frac{1}{4} - \beta_2(t-\tau)} \left\| \begin{bmatrix} 0 \\ 0 \end{bmatrix}(\tau) \right\|_p d\tau \\
&\leq e^{-\gamma_{p,p}t} \left\| \begin{bmatrix} u_0 - \tilde{u}_0 \\ \theta_0 - \tilde{\theta}_0 \end{bmatrix} \right\|_p \\
&+ \int_0^t [h_d(t - \tau)]^{\frac{1}{2} + \frac{1}{4} - \beta_2(t-\tau)} \left\| \begin{bmatrix} 0 \\ 0 \end{bmatrix}(\tau) \right\|_p d\tau \\
&\leq e^{-\gamma_{p,p}t} \left\| \begin{bmatrix} u_0 - \tilde{u}_0 \\ \theta_0 - \tilde{\theta}_0 \end{bmatrix} \right\|_p \\
&+ \int_0^t [h_d(t - \tau)]^{\frac{1}{2} + \frac{1}{4} - \beta_2(t-\tau)} \left\| \begin{bmatrix} 0 \\ 0 \end{bmatrix}(\tau) \right\|_p d\tau, \quad (4.3)
\end{align*}
\]

where \(\beta_1 = \gamma_{dp/(1+dp),p}, \beta_2 = \gamma_{p,p} + \gamma_{dp/(1+dp),p} \).
Next step, by setting $y(\tau) = e^{\Theta t} \left\| \begin{bmatrix} u(\tau) - \bar{u}(\tau) \\ \theta(\tau) - \bar{\theta}(\tau) \end{bmatrix} \right\|_p$ for $\Theta < \min\{\gamma_{p,p}, \beta_1\}$, it is not difficult to point out that

$$y(t) \leq e^{-(\gamma_{p,p} - \Theta)t} \left\| \begin{bmatrix} u_0 - \bar{u}_0 \\ \theta_0 - \bar{\theta}_0 \end{bmatrix} \right\|_p + P_0 \int_0^t [h_d(t - \tau)]^{\frac{1}{\delta}} e^{-(\beta_2 - \Theta)(t-\tau)} [h_d(\tau)]^{\frac{1}{\alpha}} e^{-\alpha \tau} z(\tau) d\tau$$

$$+ \|h\|_{\infty, \frac{4}{\delta}} \int_0^t [h_d(t - \tau)]^{\frac{1}{\delta}} e^{-(\beta_1 - \Theta)(t-\tau)} z(\tau) d\tau,$$

(4.4)

where $P_0 := \left\| \begin{bmatrix} u \\ \theta \end{bmatrix} \right\|_{\mathcal{X} \times S} + \left\| \begin{bmatrix} \bar{u} \\ \bar{\theta} \end{bmatrix} \right\|_{\mathcal{X} \times S} \leq \frac{\rho}{2} + \frac{\rho}{2} = \rho$. By some simple computations, we receive that

$$\int_0^t [h_d(t - \tau)]^{\frac{1}{\delta}} e^{-(\beta_2 - \Theta)(t-\tau)} [h_d(\tau)]^{\frac{1}{\alpha}} e^{-\alpha \tau} d\tau \leq P < +\infty \text{ (see Appendix)},$$

and

$$\|h\|_{\infty, \frac{4}{\delta}} \int_0^t [h_d(t - \tau)]^{\frac{1}{\delta}} e^{-(\beta_1 - \Theta)(t-\tau)} d\tau$$

$$= C^{\frac{4}{\delta}} \|h\|_{\infty, \frac{4}{\delta}} \left( (\beta_1 - \Theta)^{\frac{1}{\delta} - 1} \Gamma \left( 1 - \frac{\delta}{2} \right) + \frac{1}{(\beta_1 - \Theta)} \right) \leq \bar{P} < +\infty.$$

Hence, we are able to apply Gronwall’s inequality to obtain

$$|y(t)| \leq \left\| \begin{bmatrix} u_0 - \bar{u}_0 \\ \theta_0 - \bar{\theta}_0 \end{bmatrix} \right\|_p e^{\rho P + \bar{P}} \text{ for all } t > 0.$$

This leads to the fact that

$$\left\| \begin{bmatrix} u(t) - \bar{u}(t) \\ \theta(t) - \bar{\theta}(t) \end{bmatrix} \right\|_p \leq e^{-\Theta t} \left\| \begin{bmatrix} u_0 - \bar{u}_0 \\ \theta_0 - \bar{\theta}_0 \end{bmatrix} \right\|_p \text{ for all } t > 0.$$

(4.5)

By the analogous arguments, we are also able to deduce easily that

$$\left\| \begin{bmatrix} u(t) - \bar{u}(t) \\ \theta(t) - \bar{\theta}(t) \end{bmatrix} \right\| \leq e^{-\Theta t} \left\| \begin{bmatrix} u_0 - \bar{u}_0 \\ \theta_0 - \bar{\theta}_0 \end{bmatrix} \right\| \text{ for all } t > 0.$$

(4.6)

Furthermore, we are going to estimate $z(\tau) := e^{\Theta t} [h_d(\tau)]^{-\frac{1}{\delta}} e^{\alpha \tau} \left\| \begin{bmatrix} u(\tau) - \bar{u}(\tau) \\ \theta(\tau) - \bar{\theta}(\tau) \end{bmatrix} \right\|_{\frac{4}{\delta}}$. Indeed, also note that $[h_d(t)]^{-\frac{1}{\delta}} \leq C^{\frac{1}{\delta}}$ for all $t > 0$, we then see that

$$z(t) \leq e^{\Theta t} [h_d(t)]^{-\frac{1}{\delta}} e^{\alpha t} \left\| \begin{bmatrix} u_0 - \bar{u}_0 \\ \theta_0 - \bar{\theta}_0 \end{bmatrix} \right\|_{\frac{4}{\delta}}.$$
(4.7)
This also leads to the relation
\[
[h_d(t)]^{-\frac{1}{d}} e^{\alpha t} \left\| \begin{bmatrix} u(t) - \tilde{u}(t) \\ \theta(t) - \tilde{\theta}(t) \end{bmatrix} \right\|_d \lesssim e^{-\Theta t} \left\| \begin{bmatrix} u_0 - \tilde{u}_0 \\ \theta_0 - \tilde{\theta}_0 \end{bmatrix} \right\|_d \quad \text{for all } t > 0. \tag{4.8}
\]
By the above inequalities (4.5), (4.6) and (4.8), one is able to assert that
\[
\left\| \begin{bmatrix} u - \tilde{u} \\ \theta - \tilde{\theta} \end{bmatrix} (t) \right\|_\Theta \lesssim e^{-\Theta t} \left\| \begin{bmatrix} u_0 - \tilde{u}_0 \\ \theta_0 - \tilde{\theta}_0 \end{bmatrix} \right\|_\beta \quad \text{for all } t > 0.
\]

4.2. Application to the existence of periodic solutions. In the following, we provide an application of the stability obtained in Theorem 4.1 by establish the existence of periodic mild solutions for Boussinesq system. Namely, we use the local exponential stability of solutions to the Boussinesq system on the real hyperbolic manifold to prove the existence and local uniqueness of a mild $T$-periodic solution to the Boussinesq equation (2.8) under the action of $T$-periodic external forces. The method is extended from [36, 37] and is called Serrin principle (see [55]). Our main result of this section reads as follows.

**Theorem 4.2.** Let $(M, g)$ be a $d$-dimensional real hyperbolic manifold with $d \geq 2$. For $1 < p \leq d$, suppose that the external forces $h \in C_b(\mathbb{R}, L^\frac{d}{d-mp}(M; \Gamma(TM)))$, $F \in \mathcal{F} := C_b(\mathbb{R}, (L^\frac{dp}{d-p} \cap L^\frac{d}{d-mp} \cap L^\frac{d}{d-mp})(M; \Gamma(TM \otimes TM)))$, $f \in \mathcal{G} := C_b(\mathbb{R}, (L^\frac{dp}{d-p} \cap L^\frac{d}{d-mp} \cap L^\frac{d}{d-mp})(M; \mathbb{R} \times \mathbb{R}))$ are sufficiently small as in theorem 3.4. Then, if the functions $F$, $f$ and $h$ are $T$-periodic, equation (2.8) has a $T$-periodic mild solution $(\tilde{u}, \tilde{\theta})$ in a small ball of $\mathcal{X} \times \mathcal{S}$. Moreover, the $T$-periodic mild solution to (2.8) is locally unique in the sense that: Two $T$-periodic mild solutions $(u, \theta)$ and $(v, \xi)$ starting sufficiently near each other (i.e., $\|(u(0) - v(0), \theta(0) - \xi(0))\|_\beta$ is sufficiently small) are identical.

**Proof.** For each sufficiently small initial data $(x, y) \in (L^p \cap L^d)(M; \Gamma(TM)) \times (L^p \cap L^d)(M; \mathbb{R})$, Theorem 3.4 follows that there exists a unique bounded mild solution $(v, \xi) \in \mathcal{X} \times \mathcal{S}$ to equation (2.8) with $(v(0), \xi(0)) = (x, y)$ in a small ball $B_\rho$ of $\mathcal{X} \times \mathcal{S}$, if $\|(F, f)\|_{\mathcal{F} \times \mathcal{G}}$ and $\|h\|_{\infty, \frac{d}{d-mp}}$ are also small enough. More precisely, the facts that
\[
\rho < \frac{1}{12K}, \quad \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\|_\beta < \frac{\rho}{4},
\]
\[
\|h\|_{\infty, \frac{d}{d-mp}} < \frac{1}{8N}, \quad \left\| \begin{bmatrix} F \\ f \end{bmatrix} \right\|_{\infty, \mathcal{F} \times \mathcal{G}} < \frac{\rho}{8M}
\]
guarantee the existence and uniqueness of such $(v, \xi)$ (see the proof of Theorem 3.4).
In fact, we are able to take an even smaller initial vector field \((u_0, \theta_0)\) such that 
\[
\rho < \frac{1}{48K}, \quad \|(u_0, \theta_0)\|_\otimes \leq \frac{\rho}{42}, \quad \|h\|_{\infty, \mathbb{F}} \leq \frac{1}{32N}, \quad \left\| \begin{bmatrix} F \\ f \end{bmatrix} \right\|_{\infty, \mathbb{F} \times \Theta} < \frac{\rho}{32M} \text{ (actually, } (u_0, \theta_0) \text{ may be taken to be } (0, 0)) \text{. This leads to the fact that } (u, \theta) \in \mathcal{B}_{\frac{\rho}{4}}, \text{ where } (u, \theta) \text{ is the unique bounded mild solution to equation (2.8). That means}
\[
\left\| \begin{bmatrix} u(t) \\ \theta(t) \end{bmatrix} \right\|_\otimes \leq \frac{\rho}{4}, \quad \forall t \geq 0.
\]

We now need to point out that the sequence \(\{(u(nT), \theta(nT))\}_{n \in \mathbb{N}}\) is a Cauchy sequence in the space 
\((L^p \cap L^d \cap L^{d/\delta})(\mathcal{M}; \Gamma(T\mathcal{M}))) \times (L^p \cap L^d \cap L^{d/\delta})(\mathcal{M}; \mathbb{R})\) with the norm 
\[
\left\| \begin{bmatrix} u(t) \\ \theta(t) \end{bmatrix} \right\| := \left\| \begin{bmatrix} u(t) \\ \theta(t) \end{bmatrix} \right\|_p + \left\| \begin{bmatrix} u(t) \\ \theta(t) \end{bmatrix} \right\|_d + [h_d(t)]^{-\frac{1}{d}} e^{at} \left\| \begin{bmatrix} u(t) \\ \theta(t) \end{bmatrix} \right\|_4.
\]

Indeed, for arbitrary fixed natural numbers \(m > n \in \mathbb{N}\), by putting \((z_1(t), z_2(t)) = (u(t + (m - n)T), \theta(t + (m - n)T))\), and using the periodicity of \(F, f\) and \(h\), we are easy to see that \((z_1, z_2)\) is also a mild solution to equation (2.8). Of course, \((z_1, z_2) \in \mathcal{B}_{\rho/4}.\) Therefore, Theorem 4.1 implies that
\[
\left\| \begin{bmatrix} u(t) - z_1(t) \\ \theta(t) - z_2(t) \end{bmatrix} \right\|_\otimes \leq \left\| \begin{bmatrix} u_0 - z_1(0) \\ \theta_0 - z_2(0) \end{bmatrix} \right\|_\otimes e^{-\Theta t} < K_0 e^{-\Theta t}, \quad (4.9)
\]
for all \(t \geq 0\), where the constant \(K_0\) independent of \(m, n\).

Thence, by \(t := nT\) in the above inequality and noting that \((z_1(t), z_2(t)) = (u(t + (m - n)T), \theta(t + (m - n)T))\), we imply that
\[
\left\| \begin{bmatrix} u(nT) - u(mT) \\ \theta(nT) - \theta(mT) \end{bmatrix} \right\|_\otimes \leq K_0 e^{-\Theta(nT)}, \quad (4.10)
\]
for all \(m > n \in \mathbb{N}\).

This follows that \(\left\{u(nT)\right\}_{n \in \mathbb{N}} \subset (L^p \cap L^d \cap L^{d/\delta})(\mathcal{M}; \Gamma(T\mathcal{M})))\) and \(\left\{\theta(nT)\right\}_{n \in \mathbb{N}} \subset (L^p \cap L^d \cap L^{d/\delta})(\mathcal{M}; \mathbb{R})\) are Cauchy sequences. Thus, the sequence \(\left\{(u(nT), \theta(nT))\right\}_{n \in \mathbb{N}}\) is convergent in 
\((L^p \cap L^d \cap L^{d/\delta})(\mathcal{M}; \Gamma(T\mathcal{M}))) \times (L^p \cap L^d \cap L^{d/\delta})(\mathcal{M}; \mathbb{R})\) with
\[
\left\| \begin{bmatrix} u(nT) \\ \theta(nT) \end{bmatrix} \right\|_\otimes \leq \frac{\rho}{4},
\]
and we then put
\[
\begin{bmatrix} u^* \\ \theta^* \end{bmatrix} := \lim_{n \to \infty} \begin{bmatrix} u(nT) \\ \theta(nT) \end{bmatrix} \in (L^p \cap L^d \cap L^{d/\delta})(\mathcal{M}; \Gamma(T\mathcal{M}))) \times (L^p \cap L^d \cap L^{d/\delta})(\mathcal{M}; \mathbb{R}).
\]
Obviously, \( \| (u^*, \theta^*) \|_\Phi \leq \frac{\rho}{4} \).

Taking now \((u^*, \theta^*)\) as the initial data, by Theorem 3.4, we obtain that there exists a unique bounded mild solution \((\hat{u}(t), \hat{\theta}(t))\) of the equation (2.8) in \( B_\rho \). We then prove that the mild solution \((\hat{u}(t), \hat{\theta}(t))\) is \( T \)-periodic. To do this, for each fixed \( n \in \mathbb{N} \) we put \((v(t), \xi(t)) = (u(t + nT), \theta(t + nT))\) for \( t \geq 0 \). Again, by the periodicity of \( F, f \) and \( h \) we have that \((v(t), \xi(t))\) is also a mild solution of equation (2.8) with \((v(0), \xi(0)) = (u(nT), \theta(nT))\).

Since inequality (4.1) with \((v, \xi)\) instead of \((u, \theta)\), we have

\[
\left\| \left[ \frac{\hat{u}(T)}{\theta(T)} - v(T) \right] \right\| \lesssim \left\| \left[ \frac{\hat{u}(0) - v(0)}{\theta(0) - \xi(0)} \right] \right\| e^{-\Theta T} \tag{4.11}
\]

This means that

\[
\left\| \left[ \frac{\hat{u}(T) - u((n + 1)T)}{\theta(T) - \theta((n + 1)T)} \right] \right\| \lesssim \left\| \left[ \frac{u^* - u(nT)}{\theta^* - \theta(nT)} \right] \right\| e^{-\Theta T} \tag{4.12}
\]

Taking now \( n \to \infty \) and utilizing the fact that

\[
\lim_{n \to \infty} \left[ \begin{array}{c} u(nT) \\ \theta(nT) \end{array} \right] = \left[ \begin{array}{c} u^* \\ \theta^* \end{array} \right] = \left[ \begin{array}{c} \hat{u}(0) \\ \hat{\theta}(0) \end{array} \right] \in (L^p \cap L^d \cap L^{d/\delta})(\mathbf{M}; \Gamma(TM)) \times (L^p \cap L^d \cap L^{d/\delta})(\mathbf{M}; \mathbb{R}),
\]

we obtain \((\hat{u}(T), \hat{\theta}(T)) = (\hat{u}(0), \hat{\theta}(0))\). Consequently, \((\hat{u}(t), \hat{\theta}(t))\) is \( T \)-periodic.

The uniqueness of the \( T \)-periodic solution follows from inequality (4.1). Namely, if \((u, \theta)\) and \((v, \xi)\) are two \( T \)-periodic mild solutions to equation (2.8) with initial values \((u_0, \theta_0)\) and \((v_0, \xi_0)\) with \( \| (u_0 - v_0, \theta_0 - \xi_0) \|_\Phi \) sufficiently small, respectively, then inequality (4.1) implies that

\[
\lim_{t \to \infty} \left\| \left[ \begin{array}{c} u(t) - v(t) \\ \theta(t) - \xi(t) \end{array} \right] \right\|_\Phi = 0. \tag{4.13}
\]

Due to periodicity and continuity of \((u, \theta)\) and \((v, \xi)\), this then yields that \((u(t), \theta(t)) = (v(t), \xi(t))\) for all \( t \in \mathbb{R}_+ \).

5. Appendix

5.1. Gravitational field on the hyperbolic spaces. In this subsection, we verify that the gravitational field on the hyperbolic space \( \mathbf{M} = \mathbb{H}^d \) with dimension \( d \geq 3 \) satisfying Assumption 2.1. First, we recall the formula of gravitational field in the case \( d \geq 3 \) as (in detail, see [5, Section 2]):

\[
\tilde{h}(r) = \frac{d \Phi(r)}{dr} = -\frac{(d - 2)GM \coth^{d-3} r}{\sinh^2 r}, \tag{5.1}
\]
where \( r \) is the radius of a geodesic ball centered at the origin \( O = (1, 0, 0...0) \) in hyperbolic manifold \( M \), \( G \) is the gravitational constant and \( \Phi(r) = GM \coth^{d-2}(r) \) is the gravitational potential acting on the test mass \( M \). Here, we choose the curvature radius \( R = 1 \) (in [5], if \( R \neq 1 \), then \( r \) is replaced by \( \frac{r}{R} \) in equation (5.1)). Observe that, for \( d \geq 3 \), we have the following equivalence

\[
\tilde{h}(r) \simeq \begin{cases} 
-GMr^{-2}, & \text{as } r \to \infty, \\
-GMe^{-2r}, & \text{as } r \to 0.
\end{cases}
\] (5.2)

We observe that the gravitational field \( \tilde{h} \) given by (5.1) does not depend on time, then Assumption 2.1 reduces to \( \tilde{h} \in L^\infty(\Gamma(TM)) \cap L^{\frac{d}{2}}(\Gamma(TM)) \). This condition is not valid on the whole space \( M \), but it can be valid on an exterior domain \( \Omega = M - B(O, \varepsilon) \), where \( B(O, \varepsilon) \) is a geodesic ball in \( M \) centered at the origin \( O = (1, 0, 0...0) \) with geodesic radius \( \varepsilon \). In this context, we have \( \tilde{h} \in L^\infty(\Gamma(T\Omega)) \cap L^{\frac{d}{2}}(\Gamma(T\Omega)) \). This condition is similar to the one given by Hishida (see conditions (3.1) and (3.2), page 61 in [31]) when he considered the Boussinesq equation on exterior domain in Euclid space \( \mathbb{R}^3 \).

Now, we verify \( \tilde{h} \in L^\infty(\Gamma(T\Omega)) \cap L^{\frac{d}{2}}(\Gamma(T\Omega)) \), for \( h \) given by (5.1). Clearly, on the exterior domain \( \Omega \), the condition \( \tilde{h} \in L^\infty(\Gamma(T\Omega)) \) is valid. Moreover, the norm of interpolation space defined on hyperbolic manifold \( M \) is given by (see notions in the proof of Corollary 3.3 in [1]):

\[
\|f\|_{L^q, \infty} = \sup_{0<r<1} r^\frac{d}{2} |f(r)| + \sup_{r \geq 1} e^{\frac{d-2}{2}r} |f(r)|.
\]

Using equivalence (5.2), we can show that the gravitational field \( \tilde{h} \) on \( M \) (with dimension \( d \geq 3 \)) satisfies \( \|\tilde{h}\|_{L^{\frac{d}{2}}(\Gamma(T\Omega))} < +\infty \), hence \( \tilde{h} \) belongs to \( L^{\frac{d}{2}}(\Gamma(T\Omega)) \).

In order to study the Boussinesq system (2.1) on the whole space \( M \), we extend to consider a generalized gravitational field \( h : M \times \mathbb{R}_+ \to \Gamma(TM) \) given by \( h(x,t) = \alpha(x,t)\tilde{h}(x), \) where \( \alpha : M \times \mathbb{R}_+ \to \mathbb{R} \), is a bounded and continuous function and has support outside the geodesic ball \( B(O, \varepsilon) \). Since \( \tilde{h} \in L^\infty(\Gamma(T\Omega)) \cap L^{\frac{d}{2}}(\Gamma(T\Omega)) \) and the properties of \( \alpha(x,t) \), the generalized gravitational field \( h(x,t) = \alpha(x,t)\tilde{h}(x) \) satisfies Assumption 2.1.

5.2. Some calculations. In this section, we give some detailed calculations which are used in the previous sections. For convenience, we recall some constants given in the paper.

\[
\beta_1 = \gamma dp/(1+\delta p), \beta_2 = \frac{\gamma dp/(1+\delta p) + \gamma dp}{2}, \hat{\beta}_1 = \gamma d/(1+\delta), \hat{\beta}_2 = \gamma d/(1+\delta),
\]

\[\text{If we consider the Boussinesq system (2.1) on the exterior domain } \Omega \text{ in hyperbolic space } M, \text{ we need boundary conditions } u(\cdot,t)|_{\partial \Omega} = 0 \text{ and } \theta(\cdot,t)|_{\partial \Omega} = \omega(\cdot,t). \text{ This condition is similar to the one on an exterior domain in Euclid space (see [31]).} \]
\[ \hat{\beta}_2 = \frac{\gamma_{d,d} + \gamma_{d/(1+\delta),d}}{2}, \quad \hat{\beta}_1 = \frac{\gamma_{d,2\delta,d/\delta}}{2}. \]

In this part, we will clarify the boundedness of the following integrals which are used in the previous sections.

\[ \tilde{N}_1 = C^{\frac{d-1}{d}} \int_0^t [h_d(t - \tau)]^{\frac{1}{d}} e^{-\left(\hat{\beta}_1 - \alpha\right)(t-\tau)} d\tau < +\infty; \]
\[ M_1 = C^{\frac{d-1}{d}} \int_0^t [h_d(t - \tau)]^{\frac{1}{d} + \frac{1}{d}} e^{-\left(\hat{\beta}_2 - \alpha\right)(t-\tau)} d\tau < +\infty; \]
\[ K_1 = \int_0^t [h_d(t - \tau)]^{\frac{1}{d} + \frac{1}{d}} e^{-\left(\hat{\beta}_2 - \alpha\right)(t-\tau)} [h_d(\tau)]^{\frac{1}{d}} e^{-\alpha\tau} d\tau < +\infty; \]
\[ \tilde{K}_1 = C^{\frac{d-1}{d}} \int_0^t [h_d(t - \tau)]^{\frac{1}{d} + \frac{1}{d}} e^{-\left(\hat{\beta}_2 - \alpha\right)(t-\tau)} [h_d(\tau)]^{\frac{2(1-\delta)}{d}} e^{-\alpha\tau} d\tau < +\infty, \]

and

\[ \int_0^t [h_d(t - \tau)]^{\frac{1}{d} + \frac{1}{d}} e^{-\left(\hat{\beta}_2 - \Theta\right)(t-\tau)} [h_d(\tau)]^{\frac{1}{d}} e^{-\alpha\tau} d\tau \leq P < +\infty; \]
\[ [h_d(t)]^{\frac{1}{d} + \frac{1}{d}} \int_0^t [h_d(t - \tau)]^{\frac{1}{d} + \frac{1}{d}} e^{-\left(\hat{\beta}_2 - \Theta\alpha\right)(t-\tau)} e^{-\alpha\tau} d\tau \leq Q < +\infty; \]
\[ [h_d(t)]^{\frac{1}{d} + \frac{1}{d}} ||h||_{\infty,d} \int_0^t [h_d(t - \tau)]^{\frac{1}{d} + \frac{1}{d}} e^{-\left(\hat{\beta}_1 - \Theta\alpha\right)(t-\tau)} d\tau \leq \hat{Q} < +\infty. \]

The boundedness of the integrals \( \tilde{N}_1, M_1, K_1, \tilde{K}_1, P \) and \( \hat{Q} \) are proved similarly. We prove only the boundedness of \( \tilde{N}_1 \). Indeed, we consider the following cases of \( t \):

**For the case: \( 0 < t < 1 \).** It is clear that

\[ \tilde{N}_1 = C^{\frac{d-1}{d}} \int_0^t [h_d(t - \tau)]^{\frac{1}{d}} e^{-\left(\hat{\beta}_1 - \alpha\right)(t-\tau)} d\tau \]
\[ \leq C^{\frac{d-1}{d}} \int_0^t (t - \tau)^{-\frac{1}{d} + \frac{1}{d}} \tau^{\frac{1}{d}} d\tau \]
\[ \leq C^{\frac{d-1}{d}} B \left( \frac{1 - \delta}{2}, \frac{1 + \delta}{2} \right) < +\infty, \]

where \( B(\cdot, \cdot) \) is the beta function.

**For the case: \( 1 \leq t \).** We imply that

\[ \tilde{N}_1 = C^{\frac{d-1}{d}} \int_0^t [h_d(t - \tau)]^{\frac{1}{d}} e^{-\left(\hat{\beta}_1 - \alpha\right)(t-\tau)} d\tau \]
\[ = C^{\frac{d-1}{d}} \int_0^1 [h_d(t - \tau)]^{\frac{1}{d}} e^{-\left(\hat{\beta}_1 - \alpha\right)(t-\tau)} d\tau \]
\[ + C^{\frac{d-1}{d}} \int_1^t [h_d(t - \tau)]^{\frac{1}{d}} e^{-\left(\hat{\beta}_1 - \alpha\right)(t-\tau)} d\tau \]
where \( \Gamma(\cdot) \) is the gamma function.

The boundedness of integral \( \tilde{K}_1 \) and \( Q \) are pointed out similarly, we prove only for \( \tilde{K}_1 \) as follows.

**In case:** \( 0 < t < 1 \). We have

\[
\tilde{K}_1 = C^{\frac{\delta-1}{\alpha}} \int_0^t \left[ h_d(t - \tau) \right]^{1 + \delta} [h_d(\tau)] \frac{2(1 - \delta)}{\alpha} e^{-(\tilde{\beta}_2 - \alpha)(t - \tau)} e^{-\alpha \tau} d\tau
\]
\[
\leq C^{\frac{\delta-1}{\alpha}} \int_0^t (t - \tau)^{-\frac{1 + \delta}{2}} \tau^{-(1 - \delta)} d\tau
\]
\[
\leq C^{\frac{\delta-1}{\alpha}} t^{1 + \delta} \int_0^1 \tau^{-(1 - \delta)} d\tau < +\infty, \quad \text{because } 0 < t < 1.
\]

**In case:** \( 1 \leq t \). It is not hard to get following estimates.

\[
\tilde{K}_1 = C^{\frac{\delta-1}{\alpha}} \int_0^1 \left[ h_d(t - \tau) \right]^{1 + \delta} [h_d(\tau)] \frac{2(1 - \delta)}{\alpha} e^{-(\tilde{\beta}_2 - \alpha)(t - \tau)} e^{-\alpha \tau} d\tau
\]
\[
= C^{\frac{\delta-1}{\alpha}} \int_0^1 \left[ h_d(t - \tau) \right]^{1 + \delta} [h_d(\tau)] \frac{2(1 - \delta)}{\alpha} e^{-(\tilde{\beta}_2 - \alpha)(t - \tau)} d\tau
\]
\[
+ C^{\frac{\delta-1}{\alpha}} \int_1^t \left[ h_d(t - \tau) \right]^{1 + \delta} [h_d(\tau)] \frac{2(1 - \delta)}{\alpha} e^{-(\tilde{\beta}_2 - \alpha)(t - \tau)} d\tau
\]
\[
\leq C^{\frac{\delta-1}{\alpha}} \int_0^1 \left( (1 - \frac{1 + \delta}{2} + 1) \right) \tau^{-(1 - \delta)} e^{-(\tilde{\beta}_2 - \alpha) \tau} d\tau
\]
\[
+ C^{\frac{\delta-1}{\alpha}} \int_1^t (t - \tau)^{-\frac{1 + \delta}{2}} e^{-(\tilde{\beta}_2 - \alpha)(t - \tau)} d\tau
\]
\[
\leq C^{\frac{\delta-1}{\alpha}} \left[ \left( \frac{2 + \delta}{2} + 1 \right) \int_0^1 \tau^{-(1 - \delta)} d\tau + 2^{1 - \delta} \int_0^{1/2} (t - \tau)^{-\frac{1 + \delta}{2}} + 1 \right] e^{-(\tilde{\beta}_2 - \alpha) \tau} d\tau
\]
\[
+ C^{\frac{\delta-1}{\alpha}} \int_1^t (t - \tau)^{-\frac{1 + \delta}{2}} e^{-(\tilde{\beta}_2 - \alpha)(t - \tau)} d\tau
\]
\[
\leq C^{\frac{\delta-1}{\alpha}} \left[ 2 \left( 2 + \delta \right) + 1 \right] \frac{1}{\delta 2^\delta} + \frac{2^{1 - \delta}}{\tilde{\beta}_2 - \alpha} \left( e^{\frac{\tilde{\beta}_2 - \alpha}{2}} - e^{-(\tilde{\beta}_2 - \alpha)} \right)
\]
\[
+ C^{\frac{\delta-1}{\alpha}} (2^{1 - \delta} + 1)(\tilde{\beta}_2 - \alpha)^{-\delta} \Gamma \left( \frac{1 - \delta}{2} \right) < +\infty.
\]
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