A LIMIT THEOREM FOR DIFFUSIONS ON GRAPHS WITH VARIABLE CONFIGURATION

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Abstract. A limit theorem for a sequence of diffusion processes on graphs is proved in a case when vary both parameters of the processes (the drift and diffusion coefficients on every edge and the asymmetry coefficients in every vertex), and configuration of graphs, where the processes are set on. The explicit formulae for the parameters of asymmetry for the vertices of the limiting graph are given in the case, when, in the pre-limiting graphs, some groups of vertices form knots contracting into a points.

Introduction

The main object considered in the present paper is diffusion processes on graphs; a graph is treated as a one-dimensional topological space with branching points, rather than as a discrete scheme. Such processes arise naturally, on the one hand, in a number of applied models (e.g., in a model describing the motion of nutrients in the root system of a plant, see [1]) and, on the other hand, in some theoretical constructions (e.g., in the study of small random perturbations of Hamiltonian dynamical systems, see [2], or in the study of the asymptotic geometric properties of discrete groups, see [3]). Such processes possess a number of interesting and nontrivial internal structural peculiarities. Let us mention one of them, that was revealed by B.S.Tsirelson and consists in the following (see detailed review in [4]): the typical diffusion on a graph generates a filtration, that cannot be obtained from the filtration generated by some (possibly, infinite-dimensional) Brownian motion, ”in real time”, i.e., by means of a morphism.

The full description of a diffusion process on a graph in terms of its infinitesimal operator is given in [5]. In a nonformal way, such a process can be described as a mixture of the motions ”along an edge” and ”in a neighborhood of a vertex”. A motion of the first type is described by a one-dimensional diffusion process and is defined by its coefficients of drift and diffusion. To describe a motion of the second type, it is necessary to set additionally the parameters playing the role of boundary conditions at a vertex, that define the behavior of a process in the vertex. In [5], these objects are called ”the gluing parameters”. We call them also ”the asymmetry parameters”, since the construction of the process, that we use as the basic one, differs from the one developed in [5]. The purpose of this paper consists in the description of the limit behavior of a diffusion process in the situation where both the above-mentioned parameters of the process and the graph itself vary. If the limiting lengths of edges of the graph are nonzero (i.e., the vertices of the graph do not ”glue together”, and the configuration of the graph, in

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fact, does not vary), then the required limiting result is very similar to the standard limit theorems of the theory of diffusion processes. The new specific problems requiring a separate analysis arise in the case where some groups of vertices form knots contracting into a points. In this case, the description of the limiting behavior of diffusions can be investigated in the framework of the approach developed in [5], and non-trivial problem consists in calculation of the asymmetry parameters of the limiting process. As an example of the expressions that can be finally obtained, we give the model, in which the graph consists of two vertices and three edges: the edge with length $\varepsilon$ joining the vertices and two half-lines beginning at these vertices. If the diffusion on each edge is a Brownian motion, then its distribution converges, as $\varepsilon \to 0$, to the distribution of a skew Brown motion on a real line ([6]). Its asymmetry parameters are set by a single “skewing parameter” $q (p_\pm = \frac{1±q}{2})$ which is calculated in the given model as follows: if $q_1, q_2$ are analogous parameters for the vertices of a prelimiting graph, then $q = \text{th}(\lambda_1 + \lambda_2)$, where $\lambda_{1,2} = \text{arcth} q_{1,2}$, $\text{th} c \equiv \frac{e^c - e^{-c}}{e^c + e^{-c}}$ is a hyperbolic tangent, and $\text{arcth}$ is the function inverse to $\text{th}$. This simple example shows that the determination of the asymmetry parameters for a vertex, that is obtained as a result of the contraction of a group of vertices into a single point, is a nontrivial problem. In the present paper, we propose a method of the solution of this problem.

Let us describe one possible application of the main result of the paper. Earlier, we have mentioned the Tsirelson’s result that states that if, for some vertex of a graph, at least 3 it’s asymmetry parameters are nonzero (i.e., the vertex of a graph has multiplicity $\geq 3$ and is a triple point), then the diffusion $X$ is essentially singular in the following sense: there does not exist any morphism of the filtration, generated by any Brownian motion, to the filtration generated by $X$ ([4]). We call further the singularity, related to the presence of a triple point (in the above-mentioned sense), the Tsirelson’s singularity.

The theory of diffusion processes includes a number of results, in which the diffusions, containing singularities of some type (such as an asymmetric semipermeable boundary at some point, or a partial reflection with delay at the boundary of a domain), are represented as the weak limits of nonsingular diffusions (see [7],[8]). In these results, typically, the coefficients of prelimiting diffusions, in a certain sense, model the singular terms that are present in the limiting process (such as the singular drift coefficient $a = q \delta_0$ of a skew Brownian motion). The Tsirelson’s singularity is related to the structure of the phase space (the presence of a nontrivial branching point), rather than to properties of the coefficients of the process. The main theorem of the present paper allows us to represent a process, possessing the Tsirelson’s singularity, as the weak limit of the processes without such singularities. We will construct such a representation, changing the phase space of the process and representing a vertex with multiplicity $\geq 3$ as a result of “the contraction into a point” of a knot, whose vertices have multiplicity $\leq 2$ (see Example 2 below). Such a trick gives, in perspective, the possibility to study the properties of stochastic flows (i.e., the processes describing a motion of a families of points) corresponding to the diffusion on a graph, that possesses the Tsirelson’s singularity and for which, by this reason, one cannot define the common law of motion of the family of points as the strong solution to a system of SDE’s.

1. Basic notation and constructions
1.1. Phase space. Everywhere in what follows, a graph means a connected metric space composed of a finite number of subsets (edges of the graph) homeomorphic to a segment or a half-line. It is assumed that the only intersection points for these subsets are the images of the ends of a segment or a half-line; all such points are the vertices of the graph.

It is convenient to consider the graph to be oriented, by assuming that if a point, whose motion is described by a diffusion process, is not positioned on a definite edge, then this point can move into this edge, only by passing through the separated vertex of the edge (its "beginning"). It is clear that this does not restrict the generality, because any non-oriented edge-"segment" can be represented as two oriented copies; for an edge-"ray", its beginning, obviously, is the single vertex belonging to it. For the vertex $i$ of a graph $G$, we denote, by $\mathcal{P}_i$, the family of vertices joined with it by edges beginning at the vertex $i$, and these edges are denoted as $L_{i,j}, j \in \mathcal{P}_i, r = 1, \ldots, R(i,j)$. The necessity to introduce the additional parameter $r$ is caused by that two vertices can be joined by several edges. Analogously, the edges that leave the vertex $i$ and are homeomorphic to a half-line are denoted as $L_{i,\infty}^r, r = 1, \ldots, R(i,\infty)$. The interior of the set $\bigcup_{j \in \mathcal{P}_i} \bigcup_r L_{i,j}^r \cup \bigcup_r L_{i,\infty}^r$ is denoted as $O_i$ and called the maximum neighborhood of the vertex $i$.

On each edge, we introduce a natural parametrization in the following way: for the edge $L_{i,\infty}^r$, we consider the homeomorphism with $[0, +\infty)$ to be fixed and define the coordinate of a point on the edge as the number corresponding to it via this homeomorphism. The coordinates of points on the edges $L_{i,j}^r, j \in \mathcal{P}_i$ are determined analogously, by considering the homeomorphism of the edge with $[0, l_{i,j}^r]$ to be fixed.

We make no assumptions that the graph is imbedded in any larger metric space (e.g., that it is planar). On the other hand, the natural parametrization allows us to homeomorphically imbed each of the maximum neighborhoods $O_i$ in $\mathbb{R}^2$, the corresponding image being a part of a bundle of half-lines in $\mathbb{R}^2$. If this does not cause misunderstanding, we will omit the corresponding homeomorphism, by considering that any $O_i$ is a part of a bundle of half-lines in $\mathbb{R}^2$ and that any edge is a part of $\mathbb{R}^+$.  

1.2. Construction of the process. In our consideration, we will use two constructions of a diffusion process on a graph. One construction sets the process in terms of its infinitesimal operator and is given in [5]. The other, more explicit construction describes the diffusion process in terms of its excursions. We cannot give a reference, where the required construction would be described in the generality sufficient for our purposes, therefore, we give its description here. Of course, we do not pretend for a priority, because various versions of such a construction were given earlier (see [9], §4.2 and [10],[11]), and its main idea is widely known. In the next subsection, we will show that both versions of the construction of a diffusion process on a graph lead to the same object.

First, let a graph $G$ be a bundle of half-lines, i.e., it has one vertex and $R$ edges-"rays". The *Walsh’s Brownian motion* is set on such a graph as follows ([10],[11]): it is a continuous Feller Markov process; on the start from a point on one of the rays, up to the first moment to hit of the vertex, this process propagates along this ray as the Wiener process. On the start from the vertex, the process is constructed as follows: one take the Brownian motion $B$ on $[0, +\infty)$ with reflection at zero, and construct a sequence of identically distributed random
values \( \{\varepsilon_n, n \geq 1\} \), that are independent of the process \( B \) and one another and such that \( P(\varepsilon_1 = k) = p_k, k = 1, \ldots, R \), where \( p_1, \ldots, p_R \) are the given numbers ("the asymmetry parameters" of the vertex), \( p_1 + \cdots + p_R = 1 \). Then one enumerate, in an arbitrary measurable way, all the excursions \( \{(a_m, b_m)\} \) of the process \( B \) at zero and denote, by \( m(t) \) for an arbitrary time moment \( t \), the random variable setting the number of a current excursion, i.e., such natural number that \( a_{m(t)} < t < b_{m(t)} \). Such a variable is well defined on the set \( \{B_t > 0\} \) having probability 1. Now the value of the required process \( X \) at a time moment \( t \) is determined in the following way: it is located on a ray with the number \( \varepsilon_{m(t)} \), and its coordinate on this ray is equal to \( B_t \).

Next, let the graph \( \mathcal{G} \) be the same one as above, and let, for each edge \( L_r \), the functions \( a^r, \sigma^r \) on the ray \((0, +\infty)\), which are interpreted as the coefficients of drift and diffusion of the process on the edge, be set. We assume that these functions are measurable, the functions \( \sigma^r, [\sigma^r]^{-1} \) are locally bounded, and \( a^r \) is locally integrable. In addition, in order to shorten the consideration and to exclude the possibility of an "explosion", we assume that the functions \( a^r, \sigma^r \) possess at most linear growth at infinity. We introduce a new parametrization

\[
\hat{x} = S^r(x) = \int_0^x \exp\left[-\int_0^y \frac{2a^r(z)}{[\sigma^r(z)]^2} \, dz\right] \, dy, \quad x \in L_r,
\]

and, for the given collection \( \varphi = \{p_1, \ldots, p_R\} \) of the asymmetry parameters of a vertex, construct a Walsh’s Brownian motion \( \hat{X} \) with these parameters. We introduce a process \( \{r(s), s \geq 0\} \) such that \( \hat{X}_s \in L^{r(s)}, s \geq 0 \), almost surely and put

\[
\theta_t = \int_0^t \left\{ \sigma^{r(s)}(\hat{X}_s) \cdot [S^{r(s)}]'(\hat{X}_s) \right\}^2 \, ds, \quad \tau_t = \theta_t^{-1} = \inf\{u|\theta_u \geq t\}, \quad t \geq 0.
\]

Now, we put \( X_t = S^{-1}(\hat{X}_t), t \geq 0 \), where \( S^{-1} \) is the change of a parametrization on the initial graph which is inverse to \( S \). By construction, \( X \) is the Feller Markov process with continuous trajectories. The proof of this assertion is exactly the same as that for an analogous proposition about the general one-dimensional diffusion (see [9], Chap. 3). We interpret process \( X \) as the diffusion process on a bundle of half-lines with the coefficients of drift and diffusion on rays \( \{a^r, \sigma^r\} \) and the asymmetry parameters of the vertex \( \varphi \).

At last, let a graph \( \mathcal{G} \) be arbitrary, let the collection of asymmetry parameters \( \varphi_i \) be set for its each vertex, and let the coefficients of drift and diffusion \( \{a^r_{i,j}, \sigma^r_{i,j}\} \) be set for each edge \( L^r_{i,j} \). Supposing, e.g., that \( a^r_{i,j}(x) = \sigma^r_{i,j}(x) = 1, j \neq \infty, x \geq l^r_{i,j} \) (or \( a^r_{i,j}(x) = a^r_{i,j}(l^r_{i,j}), \sigma^r_{i,j}(x) = \sigma^r_{i,j}(l^r_{i,j}), j \neq \infty, x \geq l^r_{i,j} \), if \( a^r_{i,j}, \sigma^r_{i,j} \) are continuous on \([0, l^r_{i,j}]\)\), we can consider that these functions are given on the whole \( \mathbb{R}^+ \). Now, the Feller Markov process with continuous trajectories is well defined on \( \mathcal{G} \) via the following convention: on the start from a point lying in the maximum neighborhood \( O_i \) of any vertex \( i \), it moves as a diffusion process on the bundle of rays with the parameters \( \varphi_i, \{a^r_{i,j}, \sigma^r_{i,j}, j \in \{1, \ldots, N\} \cup \{\infty\}, r = 1, \ldots, R(i, j)\} \) till the exit from \( O_i \). At the moment of exit from \( O_i \), it is located at some vertex \( j \in \mathcal{P}_i \), and, after this moment and up to the next exit time (from \( O_j \), it moves as a diffusion process on the bundle of rays in the maximum neighborhood \( O_j \), and so on. The process, constructed in the way described above, is the diffusion process on \( \mathcal{G} \) with the parameters \( \{\varphi_i, a^r_{i,j}, \sigma^r_{i,j}\} \).

Let us note that the constructed process spends zero time at every point of the phase space with probability 1. A wider class of the processes with "sticky" points can be constructed by
means of a time change (see the details, e.g., in [12], Section 3.3). In order to shorten the exposition, we exclude such processes from consideration.

1.3. Martingale description and infinitesimal characteristics of the process.

To describe the infinitesimal characteristics of the process constructed in the previous section, it is sufficient to consider the case where the graph $G$ is a bundle of half-lines. For such $G$, let the diffusion $X$ with the parameters $\varphi = \{p_r\}, \{a^r, \sigma^r\}$ be given. We assume that the functions $a^r, \sigma^r$, in addition to the above-imposed conditions, are continuous on $[0, +\infty)$. By $X^r$, we denote a coordinate process on the $r$-th edge: $X^r_t$ is equal to the coordinate $X_t$ on the edge $L_r^r$, if $X_t$ lies on this edge at a time moment $t$, and $X^r_t$ equals zero otherwise. A vertex of the graph is denoted by the letter $O$. The process $X$ can be described in terms analogous to those of the Skorokhod problem for a process on a half-line with reflection at a point of the boundary.

**Proposition 1.** There exists a nondecreasing process $V_t$ such that

(i) it increases only when the vertex is visited by the process $X$, i.e.,

$$\int_0^t I_{\{X_s \neq O\}} dV_s = 0 \quad a.s., \quad t \geq 0;$$

(ii) for an arbitrary edge $L^r$, the process

$$M^r_t \equiv X^r_t - p_r V_t - \int_0^t a^r(X^r_s) I_{\{X_s \in L^r\}} ds$$

is a continuous martingale with the quadratic variation

$$\langle M^r \rangle_t = \int_0^t [\sigma^r(X^r_s)]^2 I_{\{X_s \in L^r\}} ds.$$

**Proof.** For a Walsh’s Brownian motion, this assertion follows from the reasoning analogous to that given in [4], Section 3: each of the processes $X^r$ belongs to the class $\Sigma_+$ ([13], VI.4.4.), i.e., can be represented as a sum $M^r + V^r$, where $M^r$ is a local martingale, and $V^r$ is a nondecreasing process such that $\int_0^t I_{\{X^r_s > 0\}} dV^r_s \equiv 0$. It is easy to verify that the quadratic characteristic of $M^r$ is equal to $\int_0^t I_{\{X^r_s > 0\}} ds$. On the other hand (see [13], VI.4.4),

$$V^r_t = \lim_{\epsilon \to 0^+} \frac{1}{2\epsilon} \int_0^t I_{\{X^r_s \in (0, \epsilon)\}} ds.$$

Let $V_t$ be the local time of the process $X$ at a vertex $O$, i.e.,

$$V_t = \lim_{\epsilon \to 0^+} \frac{1}{2\epsilon} \int_0^t I_{\{\text{dist}(X_s, O) < \epsilon\}} ds.$$

It is easy to see that both $V^r$ and $V$ are $W$-functionals of the process $X$. Moreover, their characteristics $f^r$ and $f$ are connected by the relation $f^r = p^r \cdot f$ by construction. Using Theorem 6.3 in [14], we get $V^r = p^r \cdot V$. That is, Proposition 1 is valid for a Walsh’s Brownian motion, and the process $V$ is the local time of the process $X$ at zero.

Now let $X_t = S^{-1}(\hat{X}_t)$, where $\hat{X}$ is a Walsh’s Brownian motion (we use the construction of the previous section), and $V_t$ is its local time at the vertex. Then Proposition 1 is valid with $V_t = \hat{V}_t$, that follows from the Itô formula (note that $[S^r]'(0) = 1$).
Assertions (i), (ii) can be naturally interpreted as a version of the martingale problem for the pair of processes \((X, V)\). Another version of the martingale problem for the process \(X\) is given below.

**Proposition 2.** Let a continuous function \(\phi\) on \(\mathbb{S}\) be equal, on every edge \(L^r\), to some function \(\phi^r \in C^2((0, +\infty))\). Denote \(A\phi(x) = a^r(x)[\phi^r]'(x) + \frac{1}{2}\sigma^r(x)[\phi^r]''(x), x \in L^r\) and \(\Delta_O(\phi) = \sum_r p_r[\phi^r]'(0)\).

Then, for an arbitrary \(\phi\) satisfying the above-indicated condition and such that \(\Delta_O(\phi) = 0\), the process

\[
M^\phi_t = \phi(X_t) - \int_0^t A\phi(X_s) \, ds
\]

is a continuous martingale.

This assertion follows immediately via the Itô formula from Proposition 1 and the fact that the process \(X\) spends zero time at \(O\).

It is important that, as the following theorem shows, both the martingale problems given in Propositions 1, 2 are well-posed. By \(\mathcal{D}_A\), denote the set of continuous bounded functions \(\phi\) on \(\mathbb{S}\), such that the function \(A\phi\) is well defined, continuous, and bounded on \(\mathbb{S}\), and the condition \(\Delta_O(\phi) = 0\) holds.

**Theorem 1.** 1) The operator \((A, \mathcal{D}_A)\) is an infinitesimal operator of the process \(X\) constructed in the previous subsection.

2) The process \(X\) is the unique solution of the martingale problem posed in Proposition 2, endowed by the given initial distribution \(P(X_0 \in \cdot)\).

3) Let \(V\) be the process constructed in the proof of Proposition 1. Then the pair of processes \((X, V)\) is the unique solution of the martingale problem posed in Proposition 1, endowed by the given initial distribution \(P(X_0 \in \cdot)\), that satisfies the following conditions:

(i) \(P(V_0 = 0) = 1\);

(ii) with probability 1, the process \(X\) spends zero time at the vertex \(O\).

**Proof.** The fact that \((A, \mathcal{D}_A)\) is a pre-generator of the process \(X\) follows from Proposition 2. Theorem 3.1 in [5] ensures the fact that \(\mathcal{D}_A\) is the whole domain of definition of the generator of the process \(X\). The second assertion follows from Theorems 3.1 and 2.2 in [5]. The third assertion is a consequence of the second one and the Itô formula.

According to Theorem 1, the constructive description of a diffusion process on a graph, presented in the previous section, and the semigroup description presented in Section 3 in [5] are equivalent.

**1.4. Limit theorem for a graph with constant configuration.** The construction of Section 1.2 yields directly the following limiting result for the sequence of diffusion processes \(\{X^n\}\) on a graph \(\mathbb{S}\) with the parameters \(\{\phi^n_i, a^n_{i,j}, \sigma^n_{i,j}\}\).

**Theorem 2.** Let

\[
a^n_{i,j} \to a^r_{i,j}, \quad L^1_{\text{loc}}(\mathbb{R}^+), \quad \sigma^n_{i,j} \to \sigma^r_{i,j}
\]

locally uniformly on \([0, +\infty)\), and let

\[
\phi^n_i = \{p^n_{r,i,j}, j \in \mathcal{P}_i \cup \{\infty\}, r = 1, \ldots R(i, j)\} \to \phi_i = \{p^r_{i,j}, j \in \mathcal{P}_i \cup \{\infty\}, r = 1, \ldots R(i, j)\}
\]
componentwise (recall that we suppose that the coefficients $a_{i,j}^n, \sigma_{i,j}^n$ are given on $[0, +\infty)$, by setting them, if necessary, to a constant on $(t_{i,j}^n, +\infty)$).

Then the sequence of the distributions of processes $\{X^n\}$ converges weakly in $C(\mathbb{R}^+, \mathcal{G})$ to the distribution of a diffusion process $X$ with the parameters $\{\varphi_i, a_{i,j}^*, \sigma_{i,j}^*\}$.

Proof. In each of the maximum neighborhoods $\mathcal{O}_i$, we consider the changes of the phase variable $S^n, [S^n]^{-1}$, that are defined by the coefficients $X^n$ on the corresponding edges. By virtue of the imposed conditions, $S^n, [S^n]^{-1}$ converge locally uniformly to $S, [S]^{-1}$, defined by the coefficients of $X$. By performing the change of a phase variable for $X^n$ in each of the neighborhoods $\mathcal{O}_i$, that is inverse to (1), and the change of the time that is inverse to (2), we obtain the process $\hat{X}^n$ being a composition of the Walsh’s Brownian motions, switching themselves at the time moment of the transition from one neighborhood to another one. The integrands $\{\sigma^n[S^n]\}'^2$ in the time change (2) time also converge on each edge locally uniformly to $\{\sigma[S]\}'^2$. The probability of the event, that at least $M$ transitions from one neighborhood to another one occur for the process $\hat{X}^n$, can be estimated uniformly in both $n$ and the starting point by a term of the form $C\alpha^M$, where $\alpha \in (0, 1), C$ are some constants (this follows from the estimate $P(\sup_{t \leq t} |W(t)| > c) \leq e^{-\frac{c^2}{4t}}$ for the Wiener process $W$; see also an analogous estimate for the Itô processes in [15], Lemma 8.5). Thus, by virtue of the strong Markov property for $X^n, X$, the proof of Theorem 2 is reduced to the following. There exists the sequence of the Walsh’s Brownian motions $\{Z^n\}$ on a given bundle of rays starting from the vertex, and the corresponding collections $\psi^n$ converge componentwise to the collection $\psi$ for the Walsh’s Brownian motion $Z$. There is also a family of neighborhoods of the vertex $\mathcal{O}^n, \lim_n \mathcal{O}^n = \emptyset$, and $\tau_{\emptyset}^Z$ and $\tau_0^Z$ are the moments of the exit of the processes from $\mathcal{O}^n$ and $\emptyset$, respectively. We have to show that the distributions of the pair $(Z^n, \tau_{\emptyset}^Z)$ converge weakly to the distribution $(Z, \tau_0^Z)$. This can be proved, by leaning on three following simple assertions. First, $Z^n$ converges weakly to $Z$ (here, we can explicitly write the transition probabilities for $Z^n, Z$). Secondly, $\tau_{\emptyset}^Z$, for an arbitrary neighborhood $\mathcal{O}$, is an almost surely continuous functional of the trajectory $Z$. At last, $\tau_0^Z$ is a monotonous and, for almost all trajectories $Z$, continuous function of $\emptyset$. Two last facts are a consequence of analogous assertions for the Brownian motion. Theorem 2 is proved.

Remarks. 1. The assertion of Theorem 2 could be proved, by using the general limit theorem 4.1 in [5]. However, the reasoning presented in the proof is important for us by themselves, because it is, in essence, a part of the proof of the main result of the present paper, Theorem 3.

2. The assertion of Theorem 2 remains valid if we assume that, for each $\{X^n\}$, its diffusion coefficients are written w.r.t. its own parametrization $\Psi^n$, and $\Psi^n \circ \varphi^{-1} \rightarrow \text{id}, \Psi \circ [\Psi^n]^{-1} \rightarrow \text{id}$ locally in $C^2(\mathbb{R}^+)$ on each edge. Informally, this means that the lengths of edges can vary, not tending to zero.

2. Limit theorem for graphs with variable configuration

2.1. Statement. Further, we assume that the diffusion processes $X^n$ are set on the graphs $\mathcal{G}^n$ with identical combinatorial configuration (i.e., with the identical procedure to join edges), but with different metrics. Formally, this corresponds to the setting of different parameterizations
\( \Psi^n \) on the same graph \( \mathcal{G} \). Nonformally, this means that the family of edges is not changed, but their lengths vary. We assume that the characteristics \( \{ \varphi_i^n, a_{ij}^n, \sigma_{ij}^n \} \) of the processes \( X^n \) (relative to the corresponding parametrizations \( \Psi^n \)) satisfy the conditions of Theorem 2 and consider the question about the limiting behavior of the distributions of the processes \( X^n \).

We assume that the lengths \( l_{ij}^{r,n} \) of some edges tend to zero. For \( i, j \) such that \( \exists r : l_{ij}^{r,n} \rightarrow 0, p_{ij}^{r,n} \not\rightarrow 0 \), we write \( i \rightarrow j \). For \( i, j \) such that \( \exists j_1, \ldots, j_m : i \rightarrow j_1, j_1 \rightarrow j_2, \ldots, j_m \rightarrow j \), we write \( i \leadsto j \). We require that the following symmetry condition be satisfied:

\[ i \leadsto j \iff j \leadsto i. \tag{3} \]

If condition (3) fails then the limiting process can fail to be a diffusion. Namely, the limiting process (in the sense of the convergence of finite-dimensional distributions) can possess discontinuous trajectories if condition (3) does not hold.

By construction, "\( \leadsto \)" is the equivalence relation; the collection of vertices connected by this relation will be called "a knot". It is natural to define a new graph \( \mathcal{G} \), on which a limiting process will be defined at last, as a graph, in which the vertices are the knots of the initial graph, and the edges are those edges of the initial graph that have not contracted into a point. We impose the following natural condition:

**A.** On each of the edges \( L_{ij}^{r,n} \) not contracting into a point, the parameterizations \( \Psi^n \) converge in the sense of the convergence of the \( C^2 \)-diffeomorphisms of \( \mathbb{R}^+ \) to a certain parametrization \( \Psi \), and \( l_{ij}^{r,n} \rightarrow l_{ij}^r > 0 \) as \( n \rightarrow \infty \).

The phase space of the limiting process is the graph \( \mathcal{G} \) with the parametrization \( \Psi \). To formulate the limiting result, it is necessary to set the projection \( \{ X^n \} \) of the initial sequence \( \{ X^n \} \) on this space (see [5], Section 4). This can be performed in the following way: if \( X_i^n \) lies on a non-contracting edge of the graph \( \mathcal{G} \), then \( \hat{X}_i^n \) lies on the corresponding edge of the graph \( \mathcal{G} \), and its coordinate on this edge is obtained from the coordinate of \( X_i^n \) by the transformation \( \Psi \circ [\Psi^n]^{-1} \). If \( X_i^n \) lies on a contracting edge, whose ends belong to the knot \( \hat{i} \), then \( \hat{X}_i^n = \hat{i} \). It is clear that the trajectories of the process \( X^n \) are continuous.

The asymmetry parameters of the limiting process will be determined by the internal structure of prelimiting knots, let us introduce the necessary objects and assumptions. The main assumption consists in that each knot is homogeneous, i.e., the lengths of all internal edges in it tend to zero with the same rate,

\[ \forall \hat{i} \ \exists \phi_i(\cdot) : \phi_i(n) \rightarrow 0 \text{ and } \forall i, j \in \hat{i}, r = 1, \ldots, R(i, j) \quad \frac{l_{ij}^{r,n}}{\phi_i(n)} \rightarrow l_{ij}^r > 0, \quad n \rightarrow +\infty, \tag{4} \]

where the numbers \( \{ l_{ij}^r \} \) are given. This assumption can be weakened, but it cannot be removed at all. To shorten the consideration, we omit the details here. We note only that if condition (4) is not imposed, we can faced with situations where the limiting process spends a positive time at a knot with a nonzero probability.

We recall that we assume that, for each vertex \( i \), the collections of the parameters \( \psi_i^n \) converge to a certain collection \( \psi_i = \{ p_{ij}^r, j \in P_i \cup \{ \infty \}, r = 1, \ldots, R(i, j) \} \).
Let the knot \( \hat{t} \) be fixed. We set \( N_1 = \# \{ i \in \hat{t} \} \),
\[
\alpha^\hat{t}_{i,j} = \sum_{r \leq R(i,j)} \frac{P^r_{i,j}}{\pi_{i,j}}, \quad i, j \in \hat{t}, \quad \beta^\hat{t}_i = \sum_{j \in \pi_i \cap \hat{t}} \alpha^\hat{t}_{i,j}, \quad i \in \hat{t}.
\]
Consider the \( N_1 \times N_1 \)-matrix \( A^\hat{t} \) defined by
\[
A^\hat{t}_{i,j} = \frac{\alpha^\hat{t}_{i,j}}{\beta^\hat{t}_i}, \quad i, j \in \hat{t}.
\]
The matrix \( A^\hat{t} \) is the matrix of transition probabilities for some Markov chain. By virtue of condition (3), all the states of this chain form unique class of essential states. Therefore, there exists the unique invariant distribution for the chain. We denote this distribution by \( \pi^\hat{t} \); and set the collection \( \hat{\phi}_t \) in the following way: each edge \( L^\hat{t}_{i,j} \) of the graph \( \hat{\mathcal{G}} \) is represented by some edge \( L^\hat{t}_{i,j} \) of the graph \( \mathcal{G} \) with \( i \in \hat{t}, j \notin \hat{t} \) (it is possible that \( j = \infty \), then \( j = \infty \)). For this edge, we put
\[
\hat{p}^\hat{t}_{i,j} = P^i \frac{\pi^\hat{t}_i}{\beta^\hat{t}_i} \frac{p^r_{i,j}}{\pi_{i,j}}, \quad (5)
\]
where \( P^\hat{t} \) is the normalizing factor which is defined by the condition
\[
\sum_{j \in \pi_i \cup \{ \infty \}, r \leq R(i,j)} \hat{p}^\hat{t}_{i,j} = 1.
\]
At last, we set the functions \( \hat{a}^\hat{t}_{i,j}, \hat{\sigma}^\hat{t}_{i,j} \) as the limits of the functions \( a^{r,m}_{i,j}, \sigma^{r,m}_{i,j} \) on each edge \( L^\hat{r}_{i,j} \) corresponding to the edge \( L^\hat{r}_{i,j} \). Now we can formulate the main result of the present paper.

**Theorem 3.** Let the characteristics \( \{ \phi^n_i, a^n_{i,j}, \sigma^n_{i,j} \} \) of the processes \( X^n \) satisfy the conditions of Theorem 2, and let conditions (3) and (4) be satisfied. Let also the sequence of distributions \( \mu^n(\cdot) = P(X^n(0) = \cdot) \) converge weakly to some measure \( \mu \). Then the sequence of distributions of the processes \( \{ \hat{X}^n \} \) in \( C(\mathbb{R}^+), \hat{\mathcal{G}} \) converges weakly to the distribution of the diffusion process \( \hat{X} \) with the above-set parameters \( \{ \hat{\phi}_i, \hat{a}^\hat{r}_{i,j}, \hat{\sigma}^\hat{r}_{i,j} \} \) and with the initial distribution \( \mu \).

**2.2. Proof.** The reasoning analogous to those used in the proof of Theorem 2 allows us to restrict our consideration to the case where \( a^{r,m}_{i,j} \equiv 0, \sigma^{r,m}_{i,j} \equiv 1 \), all the vertices of the initial graph \( \mathcal{G} \) form a single knot, and only nontrivial (i.e., not contracting into a point) edges are the edges-“rays” that are homeomorphic to half-line.

**Remark 3.** For such a reduction, it is significant that the conditions of convergence of the coefficients \( a^{r,m}_{i,j}, \sigma^{r,m}_{i,j} \), which were formulated in Theorem 2, hold also for edges contracting into a point. Otherwise, the assertion of Theorem 3 can be violated. For example, if for some (not all) edges, contracting into a point, \( a^{r,m}_{i,j} \) are constant functions tending to \( +\infty \) as \( n \to +\infty \), then the homogeneity condition (4) will be broken after a change of the phase variable. If for some (not all) edges, contracting into a point, \( \sigma^{r,m}_{i,j} \) are constant functions, which tend sufficiently rapidly to 0 as \( n \to +\infty \), then the time spent by the process \( \hat{X}^n \) at the vertex \( \hat{t} \), will not tend to zero.

Let us proceed with the proof of Theorem 3 in the above-indicated case. First of all, we note that the sequence of distributions of the processes \( \{ \hat{X}^n \} \) is weakly compact in \( C(\mathbb{R}^+, \hat{\mathcal{G}}) \). The
simplest way to prove this, is to use the criterion for weak compactness given in [16], Theorem 8.2. For an arbitrary $\varepsilon > 0$, the process $\{\hat{X}^n\}$ outside the vertex neighborhood $B(O, \varepsilon)$ with radius $\varepsilon$ is the Brownian motion. This easily implies that, on every finite time interval $[0, T]$, for the continuity modulus $w_T(\hat{X}^n, \delta) \equiv \sup_{s,t \leq T, |t-s| < \delta} \text{dist}(\hat{X}^n_t, \hat{X}^n_s)$, the following convergence holds true:

$$\sup_n P(w_T(\hat{X}^n, \delta) \geq 2\varepsilon) \to 0, \quad \delta \to 0.$$  

This allows one to apply the above-mentioned theorem.

Our aim is to show that any limiting point of a sequence of the distributions of the processes $\{\hat{X}^n\}$ gives a solution of the martingale problem for a Walsh’s Brownian motion, that was formulated in Proposition 2. Since, by assertion 2) of Theorem 1, this problem is correctly posed, this yields the assertion of the main theorem. To prove the required martingale characterization of the limiting point, we will study the limiting behavior of resolvents of the processes $\{\hat{X}^n\}$ (or, more exactly, the Laplace transforms of their distributions; note that each of the processes $\hat{X}^n$ is not Markov) in detail.

Let the vertex $i$, the edge-”ray” with the number $r \leq R(i, \infty)$ which leaves this vertex, and the function $\phi \in C_{bL}([0, +\infty))$ be fixed. Consider the quantities

$$E^n_i(t) = E(\phi(\hat{X}^n_i), \hat{X}_i^n \in L_1^{t, \infty}|X^n_0 = i) = E(\phi(\hat{X}^n_i), X^n_i \in L_1^{t, \infty}|X^n_0 = i), \quad t > 0.$$  

By $\tau_i$, denote the moment of the first exit of the process $X^n$ from the neighborhood $O_i$. For $E^n_i(\cdot)$, an analog of the renewal equation, written at the moment $\tau_i$, looks as

$$E^n_i(t) = Q^n_i(t) + \sum_{k \in P_i} \int_0^t E^n_i(t-s)P(X^n_{\tau_i} = k, \tau_i \in ds), \quad (6)$$

where $Q^n_i(t) = E(\phi(\hat{X}^n_i), X^n_i \in L_1^{t, \infty}, \tau_i > t|X^n_0 = i)$ (it is clear that $Q^n_i(t) > 0$ only if $i = i$). Considering (6) for all $i$, we get the convolutional equation for the vector $E^n(t)$ composed of the components of $E^n_i(t)$. Let us introduce the Laplace transformations

$$U^n_i(\lambda) = \int_0^\infty e^{-\lambda t} E^n_i(t)dt, \quad V^n_i(\lambda) = \int_0^\infty e^{-\lambda t} Q^n_i(t)dt, \quad C^n_{ik}(\lambda) = \int_0^\infty e^{-\lambda t} P(X^n_k = k, \tau_i \in dt),$$

and let $U^n(\lambda)$, $V^n(\lambda)$, and $C^n(\lambda)$ be, respectively, two vectors and a matrix composed of the components of $U^n_i(\lambda)$, $V^n_i(\lambda)$, and $C^n_{ik}(\lambda)$. Equation (6) yields

$$U^n(\lambda) = V^n(\lambda) + C^n(\lambda)U^n(\lambda), \quad U^n(\lambda) = [I - C^n(\lambda)]^{-1} V^n(\lambda). \quad (7)$$

To describe the limiting behavior of $U^n$, we need the following lemma allowing us to write $V^n(\lambda), C^n(\lambda)$ explicitly. Denote $\Phi(\lambda) = \int_0^\infty e^{-\lambda t} \phi(B_t)dt$, where $B$ is the Brownian motion with reflection on $[0, +\infty)$, starting from zero.

**Lemma 1.** Let a Walsh’s Brownian motion $Z$ with $M$ rays and the asymmetry parameters $p_1, \ldots, p_M$ be given. Let the points $z_1, \ldots, z_m$ be marked on the rays $L_1, \ldots, L_m$ ($m < M$) at the distances $l_1, \ldots, l_m$ from the vertex, and let $\tau$ be the first moment when $Z$ hits one of these points.
We now introduce the Laplace transformations.

Then, on the start of the process $Z$ from the vertex $O$,

$$
\int_0^{+\infty} e^{-\lambda t} E(\phi(Z_t), Z_t \in L_k, \tau > t) dt = p_k \Phi(\lambda) \left\{ \sum_{j=1}^{m} p_j \cth[l_j \sqrt{2\lambda}] + \sum_{j=m+1}^{M} p_j \right\}^{-1}, \lambda > 0, k > m,
$$

$$
\int_0^{+\infty} e^{-\lambda t} P(Z_{\tau} = z_k, \tau \in dt) = \frac{p_k}{\sh[l_k \sqrt{2\lambda}]} \left\{ \sum_{j=1}^{m} p_j \cth[l_j \sqrt{2\lambda}] + \sum_{j=m+1}^{M} p_j \right\}^{-1}, \lambda > 0, k \leq m,
$$

where $\cth \equiv \frac{\cosh}{\sinh}, \sh \equiv \frac{\cosh - \sinh}{2}$.

Proof. It is sufficient to consider the case where, for some $\gamma > 0$, $\phi(u) = \phi(0)$, $u \in [0, \gamma]$ (the general case can be obtained from it by approximation of general $\phi \in C^b([0, +\infty)$ by functions, that are constant in some neighborhood of 0). Let $B$ be the Brownian motion with reflection, whose excursions have been used in the construction of $Z$ (see Section 1.2). For $x > 0$, we denote, by $\tau_x$, the time moment of the first passage of the level $x$ by the process $B$. Then, for $x < \min(\gamma, l_1, \ldots, l_m)$, $k > m$,

$$
E(\phi(Z_t), Z_t \in L_k, \tau > t) = \phi(0) P(Z_t \in L_k, \tau_x > t) +
$$

$$
+ \sum_{j=1}^{M} \int_0^{t} E_{j,x}(\phi(Z_{t-s}), Z_{t-s} \in L_k, \tau > t-s) P(Z_{\tau_x} \in L_j, \tau_x \in ds),
$$

where $E_{j,x}(\cdot)$ means the averaging over the distribution of the process $Z$ on the start from the point located on $L_j$ at the distance $x$ from the vertex. We set $P(\tau_x \leq t) = T_x(t)$. By the construction of the process $Z$, we have

$$
P(Z_t \in L_k, \tau_x > t) = p_r[1 - T_x(t)], \quad P(Z_{\tau_x} \in L_j, \tau_x \in ds) = p_j T_x(ds).
$$

For $j \neq k$,

$$
E_{j,x}(\phi(Z_t), Z_t \in L_k, \tau > t) = \int_0^{t} E(\phi(Z_{t-s}), Z_{t-s} \in L_k, \tau > t-s) Q_{j,x}(ds),
$$

$Q_{j,x}(s) \equiv P(W_{\theta_j}^x = 0, \theta_j \leq s)$, where $W^x$ is the Brownian motion starting from the point $x$, and $\theta_j$ is the moment of its exit from the interval $(0, l_j)$ ($l_j \equiv +\infty$ for $j > m$). At last,

$$
E_{k,x}(\phi(Z_t), Z_t \in L_k, \tau > t) = \int_0^{t} E(\phi(Z_t), Z_{t-s} \in L_k, \tau > t-s) Q_{j,x}(ds) + F_x(t),
$$

where $F_x(t) = E(\phi(W_t^x), \theta > t)$, $\theta$ is the moment of exit of $W^x$ from the interval $(0, +\infty)$. Thus, we have the convolutional equation for the function $H_k(t) = E(\phi(Z_t), Z_t \in L_k, \tau > t)$:

$$
H_k(t) = p_k \left\{ \phi(0)[1 - T_x(t)] + \int_0^{t} F_x(t-s) T_x(ds) \right\} + \sum_{j=1}^{M} p_j \int_0^{t} \int_0^{t-s} H_r(t-s-u) Q_{j,x}(du) T_x(ds).
$$

We now introduce the Laplace transformations

$$
G_k(\lambda) = \int_0^{\infty} e^{-\lambda t} H_k(t) dt, \quad S_x(\lambda) = \int_0^{\infty} e^{-\lambda t} T_x(dt),
$$

$$
R_{j,x}(\lambda) = \int_0^{\infty} e^{-\lambda t} Q_{j,x}(dt), \quad \Psi_x(\lambda) = \int_0^{\infty} e^{-\lambda t} F_x(t) dt.
$$
and rewrite (10) in the following form:

\[ G_k(\lambda) = \frac{p_k \phi(0)}{\lambda} [1 - S_x(\lambda)] + p_k S_x(\lambda) \Psi_x(\lambda) + \sum_{j=1}^{M} p_j S_x(\lambda) R_{j,x}(\lambda) G_k(\lambda). \]  

(11)

Let us differentiate (11) with respect to \( x \) at the point \( x = 0 \). Taking into account that (see [7], §1.7)

\[ S_x(\lambda) = \frac{1}{\text{ch}[x \sqrt{2\lambda}]}, \quad R_{j,x}(\lambda) = \begin{cases} \frac{\text{sh}[(i - j) \sqrt{2\lambda}]}{\text{sh}[j \sqrt{2\lambda}]}, & j \leq m \\ \exp[-x \sqrt{2\lambda}], & j > m \end{cases}, \]

we have

\[ [S_x(\lambda)]'_{x=0} = 0, \quad [R_{j,x}(\lambda)]'_{x=0} = \begin{cases} -\sqrt{2\lambda} \cdot \frac{\text{ch}[j \sqrt{2\lambda}]}{\text{sh}[j \sqrt{2\lambda}]}, & j \leq m \\ -\sqrt{2\lambda}, & j > m \end{cases}, \]

and, hence,

\[ p_k[\Psi_x(\lambda)]'_{x=0} - G_k(\lambda) \sqrt{2\lambda} \left\{ \sum_{j=1}^{m} p_j \text{ch}[j \sqrt{2\lambda}] + \sum_{j=m+1}^{M} p_j \right\} = 0. \]  

(12)

Repeating literally the performed calculations for the process \( B \) (which can be interpreted now as a Walsh’s Brownian motion on a graph with only one edge), we arrive at the equality

\[ [\Psi_x(\lambda)]'_{x=0} - \Phi(\lambda) \sqrt{2\lambda} = 0. \]  

(12′)

Relations (12) and (12′) yield (8). Analogously, for \( H_{k}^1(dt) \equiv P(Z_{\tau} = z, \tau \in dt), k \leq m \), we get

\[ H_{k}^1(dt) = p_k \int_{0}^{t} Q_{k,x}^1(dt - s)T_x(ds) + \sum_{j=1}^{M} p_j \int_{0}^{t} \int_{0}^{t-s} H_{k}^1(dt - s - u)Q_{j,x}(du)T_x(ds), \]

where \( Q_{k,x}^1(dt) \equiv P(W_{\theta k}^x = l_k, \theta_k \in dt) \). This equation in terms of the Laplace transform \( G_k^1(\lambda) = \int_{0}^{\infty} e^{-\lambda t} H_{k}^1(dt) \) takes the form

\[ G_k^1(\lambda) = p_k S_x(\lambda) R_{k,x}^1(\lambda) + \sum_{j=1}^{M} p_j S_x(\lambda) R_{j,x}(\lambda) G_k^1(\lambda), \]  

(13)

where \( R_{k,x}^1(\lambda) \equiv \int_{0}^{\infty} e^{-\lambda t} Q_{k,x}^1(dt) = \frac{\text{sh}[x \sqrt{2\lambda}]}{\text{sh}[k \sqrt{2\lambda}]}. \) By differentiating (13) with respect to \( x \) at the point \( x = 0 \), we get equality (9). Lemma 1 is proved.

We can suppose that \( \phi(n) = \frac{1}{n} \) in (4). Since \( \text{ch} x = 1 + o(x), \text{sh} x = x + o(x^2), x \to 0 \), relation (8) yields (for a fixed \( \lambda > 0 \))

\[ V_{1,n}(\lambda) = \Phi(\lambda) \sqrt{2\lambda} \frac{p_{1,k}^r}{n} \left\{ \sum_{k \in P_{1,1}} \frac{p_{1,k}^r}{p_{1,k}^r} \right\}^{-1} + o\left(\frac{1}{n}\right), \quad n \to +\infty, \quad V_{1,n}(\lambda) = 0, \quad i \neq \mathbf{i}. \]  

(14)

In an analogous way, relation (9) yields \( C_{n}(\lambda) = A_{n} - B_{n}(\lambda) \), where

\[ A_{i,k}^n = \frac{\sum_{r \leq R(i,k)} p_{i,k}^r}{n}, \quad B_{i,k}^n(\lambda) = \frac{\sqrt{2\lambda}}{n} \left\{ \sum_{h \in P_{i,1}} \frac{p_{i,h}^r}{p_{i,h}^r} \sum_{r \leq R(i,k)} \frac{p_{i,h}^r}{p_{i,h}^r} \right\} + o\left(\frac{1}{n}\right), \quad n \to +\infty. \]  

(15)
The matrices $A^n$ converge to the matrix $A$ introduced before the formulation of Theorem 3. This matrix is the transition probability matrix for a Markov chain that is ergodic by virtue of condition (3). Hence, the matrices $A^n$ possess the same property, beginning from some $n$. This allows us to rewrite (7) in the form more convenient for the asymptotic analysis, by performing a suitable change of the basis. Namely, we write the decomposition $\mathbb{R}^N = (e_1) + (\pi)^\perp$, $[\mathbb{R}^N]^* = (e_1) + (\pi)^\perp$, where $N$ is the number of vertices in the initial graph (knot), $\mathbb{R}^N$ and $[\mathbb{R}^N]^*$ are, respectively, the spaces of column-vectors and row-vectors of dimension $N$, $e_1 = (1, \ldots, 1)^\top$, and $\pi = (\pi_1, \ldots, \pi_N)$ is the invariant distribution corresponding to $A$. Next, we choose the basis $\{e_2, \ldots, e_N\}$ in $(\pi)^\perp$ and the basis $\{b_2, \ldots, b_N\}$ in $(e_1)^\perp$ in such a way that $\langle b_i, e_j \rangle = \delta_{ij}, i, j = 1, \ldots, N$, where $\delta_{ij}$ is the Kronecker delta, $b_1 \equiv \pi$, and, for $y = (y_1, \ldots, y_N) \in [\mathbb{R}^N]^*$ and $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, $\langle y, x \rangle \equiv \sum_{k=1}^N x_k y_k$. Writing now the matrix $(I - \tilde{A})_{i,k} = \delta_{ik} - \langle b_i, A_{i,k} e_k \rangle$ (i.e., we write the matrix $I - \tilde{A}$ w.r.t. the bases $\{b_i\}, \{e_i\}$), we get a block matrix of the form $I - \tilde{A} = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}$, where $D_{i,k} = (I - \tilde{A})_{i,k}, i, k = 2, \ldots, N$. The ergodicity of the matrix $A$ yields that

$$[y(I - A) = 0, \quad y \in (e_1)^\perp] \Rightarrow y = 0, \quad [(I - A)x = 0, \quad x \in (\pi)^\perp] \Rightarrow x = 0,$$

that means that the matrix $D$ is invertible.

We perform an analogous operation for all $n$, by introducing the bases $\{e_i^n\}, \{b_i^n\}$ in such a way that $\langle b_i^n, e_j^n \rangle = \delta_{ij}, e_1^n = (1, \ldots, 1)^\top$, and $\pi = (\pi_1, \ldots, \pi_N)$, $b_1^n = \pi^n$ being the invariant distribution for $A^n$. It is clear that this can be performed in such a way that $e_i^n \rightarrow e_i, b_i^n \rightarrow b_i, n \rightarrow +\infty$. Then, writing the matrix $I - A^n$ w.r.t. the bases $\{b_i^n\}, \{e_i^n\}$, we get the matrix $I - \tilde{A}^n = \begin{pmatrix} 0 & 0 \\ 0 & D^n \end{pmatrix}$, where $D^n \rightarrow D, [D^n]^{-1} \rightarrow D^{-1}$. Making now the change of variables

$$\tilde{V}^n(\lambda) = (b_i^n, V^n(\lambda)), \tilde{U}^n(\lambda) = (b_i^n, U^n(\lambda)) \text{ in (7), we obtain}$$

$$\tilde{U}^n(\lambda) = \begin{pmatrix} \tilde{B}_{1,1}^n(\lambda) \\ \tilde{B}_{1,1}^n(\lambda) \\ D^n + \tilde{B}_{1,1}^n(\lambda) \end{pmatrix}^{-1} \tilde{V}^n(\lambda),$$

(16) where $\tilde{B}_{i,k}^n(\lambda) = \langle b_i^n, B^n(\lambda) e_k^n \rangle$, $\tilde{B}_{i,j}^n(\lambda) = (\tilde{B}_{i,j}^n(\lambda))_{i,j=2}^N$, $\tilde{B}_{i,1}^n(\lambda) = (\tilde{B}_{i,2}^n(\lambda), \ldots, \tilde{B}_{i,N}^n(\lambda))^\top$, $\tilde{B}_{1,1}^n(\lambda) = (\tilde{B}_{2,1}^n(\lambda), \ldots, \tilde{B}_{N,1}^n(\lambda))^\top$. To invert the block matrix in (16), we will use a suitable modification of the Gauss method. We note that, at sufficiently great $n$, the matrix $D^n + \tilde{B}_{1,1}^n(\lambda)$ is invertible. Next, by virtue of (15), $\tilde{B}_{1,1}^n(\lambda) = O(\frac{1}{n})$ and

$$n\tilde{B}_{1,1}^n(\lambda) \rightarrow \sqrt{2\lambda} \sum_{i=1}^N \left\{ \frac{\pi_i}{r \leq R(i,\infty)} \frac{p_{i,\infty}}{p_{i,k}} \right\} > 0, \quad n \rightarrow +\infty.$$

Therefore, the number $\gamma_n(\lambda) = \tilde{B}_{1,1}^n(\lambda) - \tilde{B}_{1,1}^n(\lambda)[D^n + \tilde{B}_{1,1}^n(\lambda)]^{-1}\tilde{B}_{1,1}^n(\lambda)$ is nonzero for great $n$. Then, denoting $\tilde{V}^n(\lambda) = (\tilde{V}_2^n(\lambda), \ldots, \tilde{V}_N^n(\lambda))^\top$, $\tilde{U}^n(\lambda) = (\tilde{U}_2^n(\lambda), \ldots, \tilde{U}_N^n(\lambda))^\top$, we get (see [17], Lemma 2.3, and, in particular, formula (12)) that, for such $n$,

$$\tilde{U}^n(\lambda) = [\gamma_n(\lambda)]^{-1} \left( \tilde{V}^n(\lambda) - \langle \tilde{B}_{1,1}^n(\lambda), [D^n + \tilde{B}_{1,1}^n(\lambda)]^{-1}\tilde{V}^n(\lambda) \rangle \right),$$

$$\tilde{U}^n(\lambda) = [D^n + \tilde{B}_{1,1}^n(\lambda)]^{-1} (-\tilde{U}^n(\lambda)\tilde{B}_{1,1}^n(\lambda) + \tilde{V}^n(\lambda)).$$
These relations and the relations

\[ \hat{V}_i^n(\lambda) = O\left(\frac{1}{n}\right), \quad n\hat{V}_i^n(\lambda) \to \Phi(\lambda)\sqrt{2\lambda} \cdot \pi_l p_{l,\infty} \left\{ \sum_{k \in \mathcal{P}, r \leq R(l,k)} \frac{p_{l,k}^r}{\tilde{p}_{l,\infty}^r} \right\}^{-1}, \quad n \to +\infty, \]

that are valid by virtue of (14), imply that, as \( n \to +\infty, \)

\[ \hat{U}_i^n(\lambda) \to \Phi(\lambda) \sum_{i=1}^{N} \sum_{r \leq R(i,\infty)} \frac{\tilde{p}_{l,\infty}}{\tilde{p}_{l,\infty}^r}, \quad \hat{U}_i^n(\lambda) \to 0, \quad i = 2, \ldots, N. \]

We recall that \( \beta_i = \sum_{k \in \mathcal{P}, r \leq R(i,k)} \frac{p_{l,k}^r}{\tilde{p}_{l,\infty}^r}. \) Performing the inverse change of the variables \( \hat{U} \to U, \)
we get that, for an arbitrary vertex \( i, \)

\[ U_i^n(\lambda) = \int_{0}^{\infty} e^{-\lambda t} E(\phi(\hat{X}_t^n), \hat{X}_t^n \in L_{t,\infty}^{l} \mid X_0^n = i) dt \to \Phi(\lambda) \cdot \tilde{p}_{l,\infty}^r, \]  

(17)

where \( \tilde{p}_{l,\infty}^r \) is given by equality (5). For an arbitrary point \( x \in \mathcal{S}, \)

\[ E(\phi(\hat{X}_t^n), \hat{X}_t^n \in L_{t,\infty}^{l} \mid X_0^n = x) = E(\phi(\hat{X}_t^n), \tau > t \mid X_0^n = x) I_{x \in L_{t,\infty}^{l}} + \]

\[ + \sum_{i=1}^{N} \int_{0}^{t} E(\phi(\hat{X}_{t-s}^n), \hat{X}_{t-s}^n \in L_{s,\infty}^{l} \mid X_0^n = i) P(\tau \in ds, X_s^n = i), \]  

(18)

where \( \tau \) is the moment when the process \( X^n \) hits one of the vertices. For \( x \) lying on one of the rays, the distribution of \( \tau \) coincides with the distribution of the same moment for a Walsh’s Brownian motion (or a Wiener process on a half-line). On the other hand, it is easy to verify

\[ \tau \xrightarrow{P} 0 \quad \text{for} \ x \in \mathcal{S}_0 \ (\mathcal{S}_0 = \mathcal{G} \setminus \mathcal{G} \text{ is the joint of all the edges contracting into a point}), \]  

and this convergence is uniform in \( x \in \mathcal{S}_0. \) Then, for an arbitrary continuous function \( \psi \) on the limiting bundle of rays \( \mathcal{G}, \) relations (17) and (18) yield

\[ \int_{0}^{\infty} e^{-\lambda t} E(\psi(\hat{X}_t^n)) \mid X_0^n = x) dt \to R_\lambda \psi(O), \quad x \in \mathcal{S}_0, \]

\[ \int_{0}^{\infty} e^{-\lambda t} E(\psi(\hat{X}_t^n)) \mid X_0^n = x) dt \to R_\lambda \psi(x), \quad x \in \mathcal{G}, \]  

(19)  

(19’)

where \( R_\lambda \psi(x) = \int_{0}^{\infty} e^{-\lambda t} E(\psi(\hat{X}_t)) \mid X_0 = x) \) is the resolvent of the Walsh’s Brownian motion \( \hat{X}, \)
whose asymmetry parameters are given by equality (5). Moreover, the convergence in (19) and (19’) is uniform on \( \mathcal{S}_0 \) and every bounded subset \( \mathcal{G}, \) respectively.

We now can present, at last, a martingale characterization of the limiting points of the sequence \( \{\hat{X}_n\}. \) Let \( \hat{X} \) be one of such points which is the limit of the subsequence \( \{\hat{X}_n\}. \) By \( A, \) we denote the infinitesimal operator of the above-mentioned Walsh’s Brownian motion \( \hat{X}, \) whose domain has been described in Proposition 2. For arbitrary \( \lambda > 0, \phi \in D_A, t_0 > 0, t_1, \ldots, t_m \subset [0, t_0], G \in C_b(\mathbb{R}^m), \) by using the Markov property of the processes \( \hat{X}_n \) and relations (19) and (19’) for \( \psi = 0 \) and \( \psi = \phi A, \) we get the equality

\[ E \int_{t_0}^{\infty} e^{-\lambda(t-t_0)} G(\hat{X}_{t_1}, \ldots, \hat{X}_{t_m}) [\lambda \phi(\hat{X}_t) - A \phi(\hat{X}_t)] dt = \]

\[ = \lim_{k \to \infty} E \int_{t_0}^{\infty} e^{-\lambda(t-t_0)} G(\hat{X}_{n_{t_1}}, \ldots, \hat{X}_{n_{t_m}}) [\lambda \phi(\hat{X}_t) - A \phi(\hat{X}_t)] dt \]
\[ = \text{EG}(\hat{X}_1, \ldots, \hat{X}_{t_m})[\lambda R_\lambda \phi(\hat{X}_{t_0}) - R_{\lambda A}\phi(\hat{X}_{t_0})] = \text{EG}(\hat{X}_1, \ldots, \hat{X}_{t_m})\phi(\hat{X}_{t_0}). \]  
(20)

We now repeat the arguments given in [5], Section 2: relation (20) yields

\[ \int_{t_0}^{\infty} \lambda e^{-\lambda t} \text{EG}(\hat{X}_1, \ldots, \hat{X}_{t_m}) \left[ \phi(\hat{X}_t) - \phi(\hat{X}_{t_0}) - \int_{t_0}^{t} \phi'(\hat{X}_s) \, ds \right] \, dt = 0. \]

Since a continuous function is uniquely determined by its Laplace transform, we have that \( \text{EG}(\hat{X}_1, \ldots, \hat{X}_{t_m})[\phi(\hat{X}_t) - \phi(\hat{X}_{t_0}) - \int_{t_0}^{t} \phi'(\hat{X}_s) \, ds] = 0. \) Since \( G \) and \( t_1, \ldots, t_m \) are arbitrary, this implies that the process \( \phi(\hat{X}_t) - \int_{t_0}^{t} \phi'(\hat{X}_s) \, ds \) is a martingale. That is, \( \hat{X} \) is a solution of the martingale problem presented in Proposition 2. In addition, the distribution of \( \hat{X}_0 \) is known and coincides with \( \mu \). Then, by virtue of Theorem 1, the distribution of \( \hat{X} \) coincides with the distribution of a Walsh’s Brownian motion with the initial distribution \( \mu \). Because we took, as \( \hat{X} \), an arbitrary limiting point of the sequence of distributions \( \{\hat{X}^n\} \), this means that the whole sequence converges to the indicated distribution. Theorem 3 is proved.

3. Examples

In this section, we present two examples of the application of Theorem 3. First, consider the example, in which the limiting graph \( \mathcal{G} \) consists of one vertex and two half-lines. In this case, the limiting process is the well known skew Brownian motion (see [9], [18]).

**Example 1.** Let \( \mathcal{G} \) consist of two vertices \( \{1, 2\} \) and four edges \( L_{1,2}, L_{2,1}, L_{1,\infty}, \) and \( L_{2,\infty} \). Let the lengths of the edges \( L_{1,2} \) and \( L_{2,1} \) be \( l_{1,2}^n = n^{-1}\theta \) and \( l_{2,1}^n = n^{-1}\kappa \), where \( \theta, \kappa > 0 \) are some constants. We suppose that the drift coefficient is zero and the diffusion coefficient is unity on all the edges. We write the asymmetry parameters for the vertices \( \{1, 2\} \) in the form

\[ p_{1,\infty}^n = \frac{1 + q_1^n}{2}, \quad p_{1,2}^n = \frac{1 - q_1^n}{2}, \quad p_{2,\infty}^n = \frac{1 + q_2^n}{2}, \quad p_{2,1}^n = \frac{1 - q_2^n}{2}, \quad q_i^n \to q_i \in (-1, 1), \]  
(21)

\( i = 1, 2. \) Then, by virtue of Theorem 3 (the calculations are simple and omitted), the limiting process possesses the asymmetry parameters \( p_\pm = \frac{1 \pm q}{2} \), where

\[ \frac{1 + q}{1 - q} = \frac{\theta}{\kappa}, \quad \frac{1 + q_1}{1 - q_1} = \frac{1 - q_2}{1 + q_2}. \]  
(22)

Consider two special cases. First, if \( l_{1,2} = l_{2,1} \), we have, in essence, the graph with one non-oriented edge contracting into a point or, from the other viewpoint, the diffusion process with two singular points (“membranes”) contracting into a single point. We can rewrite new formula (22) as \( q = \theta h(\lambda_1 - \lambda_2), \lambda_i = \arcth q_i, \) which reproduces the result mentioned in Introduction. The difference in signs is caused by the fact that the notation of the asymmetry parameters used in [6] differs from (21).

Another interesting special case arises if \( l_{1,2}^n \equiv 0 \). In this situation, the prelimiting process is a mixture of two Brownian motions on the half-lines \( (-\infty, -n^{-1}\theta], [-n^{-1}\kappa, +\infty) \), for which the switch from one motion to another one happens at the moments of hitting of the point \( n^{-1}\theta \) (for the first motion) or \( -n^{-1}\kappa \) (for the second one). After the switch, a new motion starts from the point 0. Such a process is naturally interpreted as the Brownian motion with buffer zones on a straight line.
It is known that the asymmetry of a skew Brownian motion at zero can be interpreted in different ways: as the result of "a gambling of excursions" with unequal probabilities ([9]), the presence of a singular drift coefficient \( a = q \delta_0 \) ([13]), or, what is close to the previous, the presence of a semipermeable membrane at the point 0 ([12]). Theorem 3 allows us to propose one more, apparently completely new interpretation of the skew Brownian motion. Let us consider the ordinary Brownian motion (without any singularity!) and introduce the buffer zones in the neighborhood of zero with a fixed ratio of the lengths of the zones \( \vartheta = \frac{\theta}{\kappa} \). Then, "from the macroscopic viewpoint" (i.e., when the size of the zones tends to zero), this motion has the form of a skew Brownian motion with the parameter \( q = \frac{\vartheta - 1}{\vartheta + 1} = \frac{\theta - \kappa}{\theta + \kappa} \) (see Fig. 1).

\[
\begin{align*}
\ell_{i,2}^n &= \frac{\theta}{n} \\
\ell_{i,i}^n &= \frac{\kappa}{n} \\
q &= \frac{\theta - \kappa}{\theta + \kappa}
\end{align*}
\]

Fig. 1.

Example 2. Let \( \hat{G} \) be a bundle consisting of \( m \) half-lines, and let the diffusion process \( \hat{X} \) on \( \hat{G} \) have the coefficients of drift and diffusion \( a^r \) and \( \sigma^r \), \( r = 1, \ldots, m \) and the asymmetry parameters \( p^r \), \( r = 1, \ldots, m \). Additionally, we assume that the functions \( a^r, \sigma^r \), \( r = 1, \ldots, m \) satisfy the local Lipschitz condition on \( \mathbb{R}^+ \), so that, from the start from some point on the edge till the moment of hitting of the vertex, the process \( \hat{X} \) can be represented as the strong solution of an SDE driven by a one-dimensional Wiener process. At the same time, if at least three numbers from the collection \( \{ p^r \} \) are nonzero, then the vertex of the graph \( \hat{G} \) is a triple point, and the diffusion \( \hat{X} \) contains the Tsirelson’s singularity and, in particular, cannot be presented as the strong solution to any system of SDEs driven by a (multidimensional or even infinite-dimensional) Wiener process.

We now construct an approximation of this diffusion (in the sense of the convergence in distribution) by the sequence of diffusions not containing such a singularity. Consider the graph \( G \) consisting of \( m \) vertices enumerated by the numbers 1, \ldots, \( m \) and 2\( m \) edges. One edge-"ray" and one edge-"segment" leave each vertex. The edge-"segment" joins the \( i \)-th vertex with the \( i + 1 \)-th one (to shorten the notation, we use the agreement \( m + 1 \equiv 1 \)). We put

\[
a^n_{i,\infty} = a^i, \quad \sigma^n_{i,\infty} = \sigma^i, \quad a^n_{i,i+1} \equiv a^i(0), \quad \sigma^n_{i,i+1} \equiv \sigma^i(0),
\]

\[
p^n_{i,\infty} = p^n_{i+1}, \quad l^n_{i,i+1} = \frac{1}{n} p^i,
\]

\( i = 1, \ldots, m \) (we drop the superscript \( r \), because each two vertices are connected by at most one edge, and exactly one edge-"ray" leaves each vertex). Let us consider a sequence of diffusion processes \( \{ X^n \} \) on \( G \) with parameters (23),(24) and apply Theorem 3 to it. The limiting graph has the single vertex, therefore, we drop the superscript \( \hat{\cdot} \) in the notations given below. We have
\( \alpha_{i,i+1} = 1 \) and \( \alpha_{i,j} = 0 \) for \( j \neq i + 1 \). Therefore,

\[
\beta_i = 1, \quad A_{i,j} = \begin{cases} 
1, & j = i + 1, \\
0, & j \neq i + 1
\end{cases}, \quad \pi_i = \frac{1}{m}, \quad i = 1, \ldots, m.
\]

Then, as far as \( \sum_r p^r = 1 \), the normalizing constant \( P \) in formula (5) is \( P = \frac{1}{m} \). Hence, the limiting asymmetry parameters are equal to \( p_1, \ldots, p_m \). The coefficients \( a^n_{i,j}, \sigma^n_{i,j} \), satisfy, obviously, the conditions of Theorem 2. Hence, the projections \( \hat{X}^n \) of the processes \( X^n \) on \( \hat{G} \) converge in distribution in \( C(\mathbb{R}^+, \hat{G}) \) to the process \( \hat{X} \) (see Fig. 2).

![Fig. 2.](image)

By construction, the process \( X^n \) has no triple points. Let us show that this process can be presented as a mixture (in the same sense as that in Example 1) of the collection of strong solutions of one-dimensional SDEs. This means that \( X^n \) can be obtained from a one-dimensional Wiener process "in real time".

Consider \( m \) diffusion processes \( Y^n_1, \ldots, Y^n_m \), and let the \( i \)-th process be defined on a half-line \( L^n_i = [\frac{1}{n}, \frac{1}{n}, \infty) \) and have the coefficients

\[
a_i(x) = a^i(0)I_{[\frac{1}{n}, \frac{1}{n}, 0]}(x) + a^i(x)I_{\mathbb{R}}(x), \quad \sigma_i(x) = \sigma^i(0)I_{[\frac{1}{n}, \frac{1}{n}, 0]}(x) + \sigma^i(x)I_{\mathbb{R}}(x).
\]

By virtue of the Lipschitz property of the coefficients, each of the processes \( Y^n_i \) can be represented as the strong solution of a one-dimensional SDE prior to its exit on the boundary of a half-line. The process \( X^n \) can be represented now as follows: at every time moment, the point \( X^n \) is located on one of the half-lines \( L^n_i \) and moves along it by the law defined by the process \( Y^n_i \) till the moment of its exit on the boundary of a half-line. Then the point passes to the point with the coordinate 0, located on the half-line \( L^n_{i+1} \), and proceed moving along \( L^n_{i+1} \) via the law of \( Y^n_{i+1} \), etc. The presented interpretation implies that the filtration generated by the process \( X^n \) can be obtained by a morphism from the filtration generated by a one-dimensional Wiener process. That is, the process \( X^n \) does not contain Tsirelson's singularity.

Let us summarize: "untwisting" a vertex with multiplicity \( \geq 3 \), i.e., representing a vertex with multiplicity \( \geq 3 \) as a result of the "contraction into a point" of a knot, all the vertices of which have multiplicity \( \leq 2 \), we have obtained the approximation of a process containing the Tsirelson’s singularity by processes without such a singularity. It is clear that one can apply the same trick to graphs of arbitrary configuration, "untwisting" each vertex with multiplicity \( \geq 3 \).

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