EQUIVARIANT PICARD GROUPS AND LAURENT POLYNOMIALS

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ABSTRACT. Let $G$ be a finite group. For a $G$-ring $A$, let $\text{Pic}^G(A)$ denote the equivariant Picard group of $A$. We show that if $A$ is a finite type algebra over a field $k$ then $\text{Pic}^G(A)$ is contracted in the sense of Bass with contraction $H^1_{\text{et}}(G; \text{Spec}(A), \mathbb{Z})$. This gives a natural decomposition of the group $\text{Pic}^G(A[t, t^{-1}])$.

1. Introduction

Throughout the paper, $G$ is a finite group and a $G$-scheme will always mean a scheme which is separated, finite type over a field $k$ equipped with an action of $G$.

H. Bass introduced the notion of contracted functor to study $K$-theory of laurent polynomial rings (see chapter XII of [1]). In fact, Bass negative $K$-groups are defined using this notion. In [1], Bass also observed that the study of the notion of contracted functor for Picard groups may be useful to understand negative $K$-groups. In [10], Weibel studied the same for Picard groups and proved that the functor Pic is contracted on the category of schemes in the sense of Bass, i.e., for every scheme $X$, there is a natural decomposition

$$\text{Pic}(X[t, t^{-1}]) \cong \text{Pic}(X) \oplus N\text{Pic}(X) \oplus N\text{Pic}(X) \oplus H^1_{\text{et}}(X, \mathbb{Z}),$$

where $X[t, t^{-1}] = X \times \mathbb{G}_m$, $N\text{Pic}(X) = \ker[\text{Pic}(X \times \mathbb{A}^1) \to \text{Pic}(X)]$ and $\mathbb{Z}$ denotes the constant étale sheaf on $X$. The goal of this article is to prove an analogous result in the equivariant setting. The equivariant Picard group of a $G$-scheme $X$, denoted by $\text{Pic}^G(X)$, is the group of isomorphism classes of $G$-linearized line bundle on $X$. We show that the functor $\text{Pic}^G$ is contracted on the category of $G$-schemes in the sense of Bass. More precisely, we prove the following (Theorem 5.4 and Theorem 5.6):

**Theorem 1.1.** For a $G$-scheme $X$, there is a natural split exact sequence

$$0 \to \text{Pic}^G(X) \to \text{Pic}^G(X[t]) \oplus \text{Pic}^G(X[t^{-1}]) \to \text{Pic}^G(X[t, t^{-1}]) \to L\text{Pic}^G(X) \to 0$$

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with $\LPic^G(X) \cong H^1_{\et}(G; X; \mathbb{Z})$. Moreover,

$$\LPic^G(X) \cong \LPic^G(X[t]) \cong \LPic^G(X[t, t^{-1}]).$$

The organization of this article is as follows.

In section 2 we recall the notion of contracted functor for the category of $G$-rings (resp. $G$-schemes). We also discuss some known examples of contracted functors.

In section 3 we first recall few basics pertaining to group scheme action and quotients by group actions. Next, we discuss equivariant sheaves and cohomology theories which will play an important role to prove our results in the rest of the paper.

In section 4 we mainly study the homotopy invariance of the group $\Pic^G(X)$. A classical theorem of Traverso’s says that if $A$ is a seminormal ring then the natural map $\Pic(A) \to \Pic(A[t_1, t_2, \ldots, t_r])$ is an isomorphism for $r \geq 0$. We prove an equivariant version of Traverso’s theorem. The following is our main result of section 4 (see Theorem 4.5):

**Theorem 1.2.** Let $X$ be a seminormal $G$-scheme. Then the natural map

$$\Pic^G(X) \to \Pic^G(X[t_1, t_2, \ldots, t_r])$$

is an isomorphism for all $r \geq 0$.

In section 5 we study the contractibility of the functor $\Pic^G$. More explicitly, we prove (see Theorem 5.1) For every $G$-scheme $X$, let $N\Pic^G(X) = \ker[\Pic^G(X \times \mathbb{A}^1) \to \Pic^G(X)]$. Then there is a natural decomposition (by Theorem 5.1)

$$\Pic^G(X[t, t^{-1}]) \cong \Pic^G(X) \oplus N\Pic^G(X) \oplus N\Pic^G(X) \oplus H^1_{\et}(G; X, \mathcal{O}_X^\times).$$

We also deduce a general formula for the group $\Pic^G(X[t_1, t_1^{-1}, t_2, t_2^{-1}, \ldots, t_m, t_m^{-1}])$ (see Corollary 5.7). We observe that the cohomological interpretation of the term $\LPic^G(X)$ may fail for the Zariski topology (see Remark 5.8). Further, we study the kernel of the forgetful map $\eta_X : \Pic^G(X) \to \Pic(X)$. We show that $\ker(\eta_X) \cong H^1(G, H^0_{\zar}(X, \mathcal{O}_X^\times))$ and $\ker(\eta_X)$ is a contracted functor with contraction $H^1(G, H^0_{\et}(X, \mathbb{Z}))$ (see Theorem 5.9). Finally, we discuss a vanishing criterion for the term $\LPic^G(X)$. We show the following (see Corollary 5.11):

**Theorem 1.3.** Let $X = \text{Spec}(A)$ be a $G$-scheme. If $A$ is a hensel local ring then $\LPic^G(X) = 0$.

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2. Contracted functors

The notion of contracted functor from the category of rings to abelian groups was introduced by H. Bass (see chapter XII of [1]). This notion also makes sense from many categories (e.g., commutative rings, schemes, ring extensions etc.) to any abelian category (e.g., abelian groups, modules, sheaves etc.). Let us recall the notion of contracted categories (e.g., commutative rings, schemes, ring extensions etc.) to any abelian category (e.g., abelian groups, modules, sheaves etc.).

Recall that a ring $A$ is said to be $G$-ring if it has a left action of $G$ by ring automorphisms. Let $A$ and $B$ be two $G$-rings. A morphism from $A$ to $B$ is a ring homomorphism $f : A \rightarrow B$ such that $f(g.a) = g.f(a)$ for all $g \in G$ and $a \in A$.

Let $F$ be a functor from the category of $G$-rings to abelian groups. We define functors $NF$ and $LF$ as follows:

\[ NF(A) = N_1 F(A) = \ker[F(A[t]) \xrightarrow{t^{-1}} F(A)] \cong \text{coker}[F(A) \xrightarrow{F(t)} F(A[t])]; \]

\[ LF(A) = \text{coker}[F(A[t]) \oplus F(A[t^{-1}]) \xrightarrow{\text{add}} F(A[t, t^{-1}])]. \]

Here the $G$-action on $A[t]$ induces from $A$ with $g.t = t$ for all $g \in G$. Clearly, $F(A[t]) \cong F(A) \oplus NF(A)$ because the inclusion $A \xrightarrow{i_+} A[t]$ has a $G$-linear section $A[t] \xrightarrow{t^{-1}} A$. By iterating these functors, one can define $N^i F$ and $L^i F$ for $i > 0$. More generally, we get the formula

\[ F(A[t_1, t_2, \ldots, t_r]) \cong (1 + N)^r F(A) \text{ for } r \geq 0. \]

We say that $F$ is a acyclic functor if the following sequence

\[ 0 \rightarrow F(A) \xrightarrow{(+, -)} F(A[t]) \times F(A[t^{-1}]) \xrightarrow{\text{add}} F(A[t, t^{-1}]) \rightarrow LF(A) \rightarrow 0 \]

is exact for every $G$-ring $A$. Here $(+, -)(x) = (F(i_+)(x), -F(i_-)(x))$ for $x \in F(A)$. We say that $F$ is a contracted functor if (2.2) is naturally split exact, i.e., exact and there is a natural split map $LF(A) \rightarrow F(A[t, t^{-1}])$. Note that if $F$ is a contracted functor then there is a natural decomposition

\[ F(A[t, t^{-1}]) \cong F(A) \oplus N_1 F(A) \oplus N_{t-1} F(A) \oplus LF(A). \]

By iterating, we get

\[ F(A[t_1, t_1^{-1}, t_2, t_2^{-1}, \ldots, t_r, t_r^{-1}]) \cong (1 + 2N + L)^r F(A). \]

Note that the notion of contracted functor can also be defined in a similar way from the category of $G$-schemes to abelian groups. Here are few examples of known contracted functors.
**Example 2.1.** For a scheme $X$, let $U(X) = H^0_{zar}(X, \mathcal{O}_X)$. The global units functor $U$ is contracted on the category of schemes with contraction $LU(X) \cong H^0_{zar}(X, \mathbb{Z}) \cong H^0_{et}(X, \mathbb{Z})$. Moreover, $LU(X) \cong LU(X[t]) \cong LU(X[t, t^{-1}])$, i.e., $NLU(X) = L^2U(X) = 0$ (see Proposition 7.2 of [10]). The functor $\text{Pic}$ is also contracted on the category schemes with contraction $LPic(X) \cong H^1_{et}(X, \mathbb{Z}) \cong H^1_{nis}(X, \mathbb{Z})$ and $LPic(X) \cong LPic(X[t]) \cong LPic(X[t, t^{-1}])$ (see Theorem 7.6 and Proposition 7.7 of [10]).

**Example 2.2.** For each $n$, $K_n$ is a contracted functor on the category of quasi-projective scheme with contraction $LK_n = K_{n-1}$ (see Theorem V.8.3 of [12]).

**Example 2.3.** Given a ring extension $f : A \hookrightarrow B$, let $\mathcal{I}(f)$ be the multiplicative group of invertible $A$-submodule of $B$. One can check that $\mathcal{I}$ is a functor from the category of ring extensions to abelian groups. The functor $\mathcal{I}$ is contracted with contraction $L\mathcal{I}(f) \cong H^0_{nis}(\text{Spec}(A), f_*\mathbb{Z}/\mathbb{Z}) \cong H^0_{nis}(\text{Spec}(A), f_*\mathbb{Z}/\mathbb{Z})$ (see Theorem 5.1 of [8]). Write $f[t]$ (resp. $f[t, t^{-1}]$) for $A[t] \hookrightarrow B[t]$ (resp. $A[t, t^{-1}] \hookrightarrow B[t, t^{-1}]$). Then, we have $L\mathcal{I}(f) \cong L\mathcal{I}(f[t]) \cong L\mathcal{I}(f[t, t^{-1}])$ (see Proposition 3.4 of [8]). A map $f : X \rightarrow S$ of schemes is said to be faithful affine if it is affine and the structure map $\mathcal{O}_S \rightarrow f_*\mathcal{O}_X$ is injective. More generally, $\mathcal{I}$ can be thought as a functor from the category of faithful affine map of schemes to abelian groups. In fact, $\mathcal{I}$ is a contracted functor on the category of faithful affine map of schemes (see Theorem 5.2 of [8]).

3. Preliminaries on $G$-schemes and sheaves

Hereafter throughout this article, $k$ denotes a field and $G$ denotes a finite group unless otherwise stated. In this section, we briefly recall some basics on $G$-schemes and sheaves. The details can be found in [3, 7]. Let $\text{Sch}_k$ denote the category of separated, finite type schemes over $k$. Let $\text{Grp}$ denote the category of groups. For a group $G$, define a functor

$$G_k : (\text{Sch}_k)^{op} \rightarrow \text{Grp}, T \mapsto G_k(T) := \text{the group of locally constant maps } f : T \rightarrow G,$$

where $G$ has the discrete topology. Note that $G_k$ is represented by a scheme $\Pi_{g \in G}\text{Spec}(k)$. Hence, $G$ can be viewed as a group scheme over $k$ whose underlying scheme is $\Pi_{g \in G}\text{Spec}(k)$.

**The category of $G$-schemes:** Suppose $X \in \text{Sch}_k$. Then a morphism $\sigma : G \times_k X \rightarrow X$ is called an action of $G$ on $X$ if for all $T \in \text{Sch}_k$, the map $\sigma(T) : G(T) \times X(T) \rightarrow X(T)$ on $T$-valued points defines an action of the group $G(T)$ on the set $X(T)$. We simply write $g.x$ for $\sigma(g, x)$. Let $\text{Sch}^G_k$ denote the category of $G$-schemes. More explicitly, an object $X \in \text{Sch}^G_k$ is an object of $\text{Sch}_k$ equipped with an action of $G$ on $X$. A morphism between $G$-schemes $f : X \rightarrow Y$ is a $G$-linear morphism, i.e., it is a morphism in $\text{Sch}_k$ such that $f(g.x) = g.f(x)$. 
Quotients by group actions: Let $\sigma : G \times_k X \to X$ be a $G$-action on $X$. Write $\text{pr}_2$ for the projection map $G \times_k X \to X$. A morphism $q : X \to Y$ in $\text{Sch}/k$ is said to be $G$-invariant if $q \circ \text{pr}_2 = q \circ \sigma$, i.e., $q(g.x) = q(x)$ for all $x \in X(T)$ and $g \in G(T)$. By a quotients, we always mean a categorical quotients. We say that a morphism $q : X \to Y$ is a categorical quotient if $q$ is a $G$-invariant and $q$ is universal; which means that for any $G$-invariant morphism $q' : X \to Y'$ there is a unique $\alpha : Y \to Y'$ such that $q' = \alpha \circ q$. Note that if a quotient exists then it must be unique up to unique isomorphism. We usually denote the quotient by $q : X \to X/G$. If $X$ is quasi-projective over $k$ then the quotient $q : X \to X/G$ always exist (because in our case $G$ is a finite group). Moreover, we have the following properties:

1. $q$ is finite and surjective;
2. $\mathcal{O}_{X/G} = q_* (\mathcal{O}_X)^G$;
3. If $X = \text{Spec}(A)$ then $X/G = \text{Spec}(A^G)$, where $A^G = \{ a \in A \mid g.a = a \text{ for all } g \in G \}$.

$G$-sheaves and $G$-cohomology. Let $X$ be a $G$-scheme with an action map $\sigma$. Let $\mathcal{F}$ be a $\tau$-sheaf of abelian group. Here $\tau$ is any one of the Zariski, étale and Nisnevich Grothendieck topologies on $X$. We now recall few definitions for which $G$ is not necessarily a finite group. More generally, the following definition also makes sense for any algebraic group. A $G$-linearization of $\mathcal{F}$ is an isomorphism $\phi : \sigma^* \mathcal{F} \cong \text{pr}_2^* \mathcal{F}$ of sheaves on $G \times_k X$ with the following cocycle condition

$$\text{pr}_{23}^*(\phi) \circ (1 \times \sigma)^*(\phi) = (m \times 1)^*(\phi).$$

Here $m$ is the multiplication map $G \times_k G \to G$ and $\text{pr}_{23}$ is the projection to second and third factor $G \times_k G \times_k X \to G \times_k X$. As in our case $G$ is a finite group, a $G$-linearization of $\mathcal{F}$ is equivalent to a family of isomorphisms $\phi_g : g^* \mathcal{F} \cong \mathcal{F}$ for each $g \in G$ such that $\phi_e = id$ and $\phi_{gh} = \phi_h \circ h^* (\phi_g)$ for all $g, h \in G$. A $G$-sheaf in the $\tau$-topology is a pair $(\mathcal{F}, \phi)$, where $\mathcal{F}$ is a sheaf on $X$ and $\phi$ is a $G$-linearization of $\mathcal{F}$. A $G$-module on $X$ is a $G$-sheaf $(\mathcal{F}, \phi)$, where $\mathcal{F}$ is an $\mathcal{O}_X$-module and the $G$-linearization $\phi : \sigma^* \mathcal{F} \cong \text{pr}_2^* \mathcal{F}$ is an isomorphism of $\mathcal{O}_{G \times X}$-modules.

A morphism $f : (\mathcal{F}_1, \phi_1) \to (\mathcal{F}_2, \phi_2)$ between $G$-sheaves is a morphism $f : \mathcal{F}_1 \to \mathcal{F}_2$ of sheaves such that $\text{pr}_2^* f \circ \phi_1 = \phi_2 \circ \sigma^* f$. We call such a morphism as equivariant morphism. The set of equivariant morphisms from $(\mathcal{F}_1, \phi_1)$ to $(\mathcal{F}_2, \phi_2)$ is denoted by $\text{Hom}_G(\mathcal{F}_1, \mathcal{F}_2)$.

Let $\text{Ab}_\tau(G, X)$ denote the category of $G$-sheaves on $X$ in the topology $\tau$ whose objects are $G$-sheaves on $X$ in the topology $\tau$ and morphisms are the equivariant morphisms. The category $\text{Ab}_\tau(G, X)$ is abelian and it has enough injectives. If $\mathcal{F}$ is a $G$-sheaf then then the group $G$ acts naturally on the space of global sections $\Gamma(X, \mathcal{F})$. Let $\text{Ab}$ denote
the category of abelian groups. The invariant global section functor is defined by

$$\Gamma^G_X : Ab_\tau(G, X) \to Ab, \mathcal{F} \mapsto \Gamma(X, \mathcal{F})^G,$$

where $$\Gamma(X, \mathcal{F})^G = \{ x \in \Gamma(X, \mathcal{F}) | g.x = x \ \text{for all} \ g \in G \}.$$ Note that $$\Gamma^G_X = (-)^G \circ \Gamma(X, -)$$ is a left-exact functor. Then the $$\tau$$-$$G$$-cohomology groups $$H^p_\tau(G; X; \mathcal{F})$$ are defined as the right derived functors $$H^p_\tau(G; X; \mathcal{F}) := R^p\Gamma^G_X \mathcal{F}.$$ Consider the following commutative diagram

$$
\begin{array}{ccc}
Ab_\tau(G, X) & \xrightarrow{\Gamma(X, -)} & Ab \\
\downarrow{\Gamma^G_X} & & \downarrow{(-)^G} \\
Ab & & Ab
\end{array}
$$

of categories. The functor $$\Gamma(X, -)$$ sends injective $$G$$-sheaves to injective $$G$$-modules. Now the Grothendieck spectral sequence for the composition $$(-)^G \circ \Gamma(X, -)$$ gives the first quadrant convergent spectral sequence

$$E_2^{pq} = H^p(G, H^q_\tau(X, \mathcal{F})) \Rightarrow H^{p+q}_\tau(G; X; \mathcal{F}),$$

where $$H^*(G, -)$$ denotes the group cohomology. So, we have the five-term exact sequence (3.1)

$$0 \to H^1(G, H^0_\tau(X, \mathcal{F})) \to H^1_\tau(G; X; \mathcal{F}) \to H^0(G, H^1_\tau(X, \mathcal{F})) \to H^2(G, H^0_\tau(X, \mathcal{F})) \to H^2_\tau(G; X; \mathcal{F}).$$

4. Equivariant Picard groups and Seminormality

In this section, we show that for a seminormal $$G$$-scheme, the homotopy invariance of the equivariant Picard group holds. Let us begin by recalling the definition of equivariant Picard groups.

It is well known that the tensor product of two $$G$$-linearized line bundles over a $$G$$-scheme $$X$$ is also a $$G$$-linearized line bundle. The dual of any $$G$$-linearized line bundle is $$G$$-linearized as well (see p.32 of [3]). Thus, the isomorphism classes of $$G$$-linearized line bundles over $$X$$ form an abelian group with respect to the tensor product. We call it equivariant Picard group of $$X$$ and denote it by $$\text{Pic}^G(X)$$. A $$G$$-linear map $$f : X \to Y$$ always induces a group homomorphism $$f^* : \text{Pic}^G(Y) \to \text{Pic}^G(X)$$ by sending $$[(\mathcal{L}, \phi)]$$ to $$[(f^*\mathcal{L}, (id_G \times f)^*\phi)]$$. In fact, $$\text{Pic}^G$$ defines a functor from the category of $$G$$-schemes to abelian groups.

**Lemma 4.1.** Let $$X$$ be a $$G$$-scheme. There are natural isomorphism

$$\text{Pic}^G(X) \cong H^1_{zar}(G; X; \mathcal{O}_X^\times) \cong H^1_{et}(G; X; \mathcal{O}_X^\times) \cong H^1_{nis}(G; X; \mathcal{O}_X^\times).$$

*Proof.* See Theorem 2.7 of [3].
Thus, we have an exact sequence (by (3.1))
\[(4.1) \quad 0 \to H^1(G, H^0_p(X, O_X^*)) \to \text{Pic}^G(X) \to (\text{Pic}(X))^G \to H^2(G, H^0_p(X, O_X^*)) \to H^2_p(G; X; O_X^*).\]

We say that a ring $A$ is seminormal if the following holds: whenever $b, c \in A$ satisfy $b^3 = c^2$ there exists $a \in A$ such that $b = a^2$, $c = a^3$. For example, $\mathbb{C}[t^3 - t^2, t^2 - t]$ is a seminormal ring. A seminormal ring is necessarily reduced. Seminormality is a local property, i.e., $A$ is seminormal if and only if $A_p$ is seminormal for all $p \in \text{Spec}(A)$. A scheme $X$ is said to be seminormal if $\mathcal{O}_{X,x}$ is seminormal for all $x \in X$. Equivalently, $X$ is seminormal if $\Gamma(U, \mathcal{O}_X)$ is seminormal for each affine open subset $U$ of $X$. For more details, we refer to [5], [9].

**Proposition 4.2.** Let $X$ be an integral quasi-projective $G$-scheme. If $X$ is seminormal then so is $X/G$.

**Proof.** Since $X$ is quasi-projective, the quotient map $q : X \to X/G$ is finite with $\mathcal{O}_{X/G} = q_* (\mathcal{O}_X)^G$. We have to show that for every affine open subset $U$ of $X/G$, $\Gamma(U, \mathcal{O}_{X/G}) = \Gamma(X \times_{X/G} U, \mathcal{O}_X)^G$ is seminormal. Write $A = \Gamma(X \times_{X/G} U, \mathcal{O}_X)$. Note that $A$ is a seminormal domain. Let $b, c \in A^G \subset A$ such that $b^3 = c^2$. We may assume that $b, c \neq 0$. Since $A$ is seminormal, there exists $0 \neq a \in A$ such that $b = a^2$, $c = a^3$. Pick any $g \in G$. Let $d = g.a \neq 0$ because $b, c \neq 0$. Then $d^3 a^3 = d^2 a^4$. We get $g.a = d = a$. This means that $a \in A^G$. Hence, $A^G$ is seminormal. \hfill $\Box$

Next, we discuss the homotopy invariance of the functor $\text{Pic}^G$. Recall the notations from section 2. $\text{NPic}^G(X) = \text{ker}[\text{Pic}(X[t]) \to \text{Pic}(X)]$, where $X[t]$ denotes $X \times_k \mathbb{A}_k^1$. Similarly, we have $\text{NPic}^G(X)$. Since $\pi : X[t] \to X$ admits an equivariant section, we have the formula (2.1) for the functor $\text{Pic}^G$.

The affine version of the following lemma is well known (see [9]). For lack of a reference, we include a proof for non affine version.

**Lemma 4.3.** Let $X$ be a seminormal scheme. Then the natural map
\[\text{Pic}(X) \to \text{Pic}(X[t_1, t_2, \ldots, t_r])\]
is an isomorphism for all $r \geq 0$.

**Proof.** Since $X$ is seminormal, so is $X \times \mathbb{A}^r$ (see Corollary A.1 of [2]). So, it is enough to show that $\text{NPic}^G(X) = 0$. We know that a seminormal ring is necessarily reduced. Thus, $X$ is reduced. Let $\mathcal{N}^G \text{Pic}_{zar}$ be the Zariski sheafification of the presheaf $U \mapsto \text{NPic}(U)$. Since $X$ is seminormal, $\mathcal{N}^G \text{Pic}_{zar} = 0$ (see [9] or Theorem 2.4 of [2]). By Theorem 4.7 of [10], $\text{NPic}^G(X) = \text{NPic}(X_{red}) = H^0_{zar}(X, \mathcal{N}^G \text{Pic}) = 0$. \hfill $\Box$

**Lemma 4.4.** Let $X$ be a reduced $G$-scheme. Then $\text{NPic}^G(X) \cong (\text{NPic}(X))^G$. 

Proof. Since $X$ is reduced, $H^0(X, O^*_X) = H^0(X[t], O^*_X[t])$. By comparing the sequence \[\text{(5.1)}\] for $O^*_X$ and $O^*_X[t]$, we get $NPic^G(X) \cong (NPic(\mathcal{X}))^G$. \hfill \square

Theorem 4.5. Let $X$ be a seminormal $G$-scheme. Then the natural map

$$\text{Pic}^G(X) \to \text{Pic}^G(X[t_1, t_2, \ldots, t_r])$$

is an isomorphism for all $r \geq 0$.

Proof. The assertion follows from Lemmas 4.3 and 4.4. \hfill \square

5. Equivariant Picard group is contracted

The main goal of this section is to show that the functor Pic$^G$ is contracted in the sense of Bass.

Proposition 5.1. Let $X$ be a $G$-scheme. Suppose that $X = U \cup V$ for two open $G$-invariant subspaces $U$ and $V$ of $X$. Given a $G$-module $\mathcal{F}$, there is a long exact $\tau$-cohomology sequence (here $\tau \in \{\text{zar, et, nis}\}$)

$$0 \to H^0(G; X; \mathcal{F}) \to H^0(G; U; \mathcal{F}) \oplus H^0(G; V; \mathcal{F}) \to H^0(G; U \cap V; \mathcal{F}) \to H^1(G; X; \mathcal{F}) \to \ldots.$$  

Proof. Let $0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \ldots$ be an injective resolution on the category of $G$-sheaves. For an invariant open subspace $W$ of $X$, $j_W : W \hookrightarrow X$ denotes the natural $G$-linear inclusion. Write $\mathcal{Z}_U = j_U!j_U^*\mathcal{Z}$, $\mathcal{Z}_V = j_V!j_V^*\mathcal{Z}$ and $\mathcal{Z}_{U \cap V} = j_{U \cap V}!j_{U \cap V}^*\mathcal{Z}$, where $\mathcal{Z}$ is the constant sheaf with trivial action. We have an exact sequence of $G$-sheaves

\[\text{(5.1)}\]  

$$0 \to \mathcal{Z}_{U \cap V} \to \mathcal{Z}_U \oplus \mathcal{Z}_V \to \mathcal{Z} \to 0.$$  

Note that $\text{Hom}_G(\mathcal{Z}, \mathcal{I}) \cong (\mathcal{I}(X))^G$. Applying $\text{Hom}_G(-, \mathcal{I}^\bullet)$ to the exact sequence \[\text{(5.1)}\], we get a short exact sequence of complexes

$$0 \to (\mathcal{I}^\bullet(X))^G \to (\mathcal{I}^\bullet(U))^G \oplus (\mathcal{I}^\bullet(V))^G \to (\mathcal{I}^\bullet(U \cap V))^G \to 0.$$  

By taking the cohomology we get the assertion. \hfill \square

5.1. Fact: Suppose $f : \mathcal{F} \to \mathcal{F}'$ is an isomorphism $\tau$-sheaves on $X$ and $\phi : \sigma^*\mathcal{F} \cong Pr^*_2\mathcal{F}$ is a $G$-linearization of $\mathcal{F}$, where $\tau \in \{\text{zar, et, nis}\}$. Then $\phi' = \sigma^*(f)^{-1} \circ \phi \circ Pr^*_2(f)$ defines a $G$-linearization of $\mathcal{F}'$. In fact, $f : (\mathcal{F}, \phi) \to (\mathcal{F}', \phi')$ is a $G$-equivariant isomorphism.

Let $F$ be a functor from the category of schemes (resp. $G$-schemes) to abelian groups. Recall that $LF(X) = \text{coker}[F(X[t]) \oplus F(X[t^{-1}]) \to F(X[t, t^{-1}])]$ (see section 2).

Theorem 5.2. Let $X$ be a $G$-scheme. Then $\text{Pic}^G(\mathbb{P}^1_X) \cong \text{Pic}^G(X) \oplus (H^0(X, \mathcal{Z}))^G$ and there is a natural exact sequence

\[\text{(5.2)}\]  

$$0 \to \text{Pic}^G(X) \to \text{Pic}^G(X[t]) \oplus \text{Pic}^G(X[t^{-1}]) \to \text{Pic}^G(X[t, t^{-1}]) \to LPic^G(X) \to 0.$$
Proof. Let $\pi : \mathbb{P}^1_X \to X$ be the structure $G$-linear morphism. By the above fact [5.1] and the proof of Proposition 7.3 of [10], $\pi_* \mathcal{O}_{\mathbb{P}_X}^\times \cong \mathcal{O}_X^\times$ as $G$-sheaves. Then $H^0(X, \mathcal{O}_X^\times) \cong H^0(\mathbb{P}^1_X, \mathcal{O}_{\mathbb{P}^1_X}^\times)$ as $G$-modules. Consider the following commutative diagram (by (4.1))

$$
\begin{array}{cccc}
0 & \to & H^1(G, H^0(\mathcal{O}_X^\times)) & \to & \Pic^G(X) & \to & (\Pic(X))^G & \to & H^2(G, H^0(\mathcal{O}_X^\times)) \\
\downarrow \cong & & \downarrow & & \downarrow & & \downarrow \cong & & \\
0 & \to & H^1(G, H^0(\mathbb{P}^1_X, \mathcal{O}_{\mathbb{P}^1_X}^\times)) & \to & \Pic^G(\mathbb{P}^1_X) & \to & (\Pic(\mathbb{P}^1_X))^G & \to & H^2(G, H^0(\mathbb{P}^1_X, \mathcal{O}_{\mathbb{P}^1_X}^\times)).
\end{array}
$$

We know $\Pic(\mathbb{P}^1_X) \cong \Pic(X) \oplus H^0(X, \mathbb{Z})$ (see Proposition 7.3 of [10]). Thus, we get an exact sequence

$$0 \to \Pic^G(X) \to \Pic^G(\mathbb{P}^1_X) \to (H^0(X, \mathbb{Z}))^G \to 0.$$ 

Set $U(X) = H^0(X, \mathcal{O}_X^\times)$ for any scheme $X$. Further, by applying Proposition [5.1] to the $G$-sheaf $\mathcal{O}_{\mathbb{P}_X}^\times$ with the $G$-invariant subspaces $X[t]$ and $X[t^{-1}]$ of $\mathbb{P}^1_X$, we obtain

$$0 \to (U(\mathbb{P}^1_X))^G \to (U(X[t]))^G \oplus (U(X[t^{-1}]))^G \to (U(X[t, t^{-1}]))^G \to \Pic^G(\mathbb{P}^1_X) \to \Pic^G(X[t]) \oplus \Pic^G(X[t^{-1}]) \to \Pic^G(X[t, t^{-1}]) \to \ldots.$$ 

We have $U(\mathbb{P}^1_X) \cong U(X)$ and $U(X[t, t^{-1}]) \cong U(X) \oplus NU(X) \oplus NU(X) \oplus H^0(X, \mathbb{Z})$ (see Propositions 7.2 and 7.3 of [10]). Therefore, the above sequence (5.3) reduces to

$$0 \to (H^0(X, \mathbb{Z}))^G \xrightarrow{\partial} \Pic^G(\mathbb{P}^1_X) \to \Pic^G(X[t]) \oplus \Pic^G(X[t^{-1}]) \to \Pic^G(X[t, t^{-1}]) \to \ldots.$$ 

Note that the map $\partial$ is the right inverse of the map $\Pic^G(\mathbb{P}^1_X) \to (H^0(X, \mathbb{Z}))^G$. Hence the assertion. $\square$

Let $f : X \to S$ be a morphism between $G$-schemes, i.e., $G$-linear morphism. For an étale $G$-sheaf $\mathcal{F}$ on $X$, $f_* \mathcal{F}$ is a $G$-sheaf on $S$. In fact, $f_*$ sends injective $G$-sheaves on $X$ to injective $G$-sheaf on $S$ (see pp. 499 of [4]). Now the following commutative diagram

$$\begin{array}{ccc}
Ab_r(G, X) & \xrightarrow{f_*} & Ab_r(G, X) \\
\downarrow \gamma_X^G & & \downarrow \gamma_X^G \\
Ab & & Ab
\end{array}$$

of categories gives a first quadrant convergent spectral sequence

$$E_2^{pq} = H^p_{et}(G, S, R^q f_* \mathcal{F}) \Rightarrow H^{p+q}_{et}(G; X; \mathcal{F}).$$

Remark 5.3. The spectral sequence (5.4) also exists for the Zariski or Nisnevich topology. We stated the étale version as it will be used in Theorem 5.4 below.
Write $Y = X[t, t^{-1}], Y^+ = X[t]$, and $Y^- = X[t^{-1}]$. Let $\pi$ (resp. $\pi^+, \pi^-$) denote the structure $G$-linear map $Y \to X$ (resp. $Y^+ \to X, Y^- \to X$). We have the following isomorphisms of étale $G$-sheaves on $X$(see Proposition 7.2 of [10] and Fact (5.1))

$$\pi^+_* \mathcal{O}_{Y^+}^x \cong \mathcal{O}_X^x \times \mathcal{N}\mathcal{O}_X^x,$$

$$\pi_* \mathcal{O}_Y^x \cong \mathcal{O}_X^x \times \mathcal{N}\mathcal{O}_X^x \times \mathcal{N}\mathcal{O}_X^x \times \mathbb{Z}.$$

**Theorem 5.4.** $\text{Pic}^G$ is a contracted functor on the category $\text{Sch}^G/k$ with $L\text{Pic}^G(X) \cong H^1_{\text{et}}(G; X; \mathbb{Z})$. Moreover, the splitting map is given by:

$$L\text{Pic}^G(X) \cong H^1_{\text{et}}(G; X; \mathbb{Z}) \to H^1_{\text{et}}(G; X; \pi_* \mathcal{O}_X^x) \to \text{Pic}^G(X[t, t^{-1}]).$$

**Proof.** Let $\mathcal{P}_{\text{ic}}[T]$ (resp. $\mathcal{N}\mathcal{P}_{\text{ic}}$) denote the étale sheaf on $X$ associated to the presheaf $U \to \text{Pic}(U[t, t^{-1}])$ (resp. $U \to \mathcal{N}\text{Pic}(U)$). For a hensel local ring $A$, $L\text{Pic}(A) = 0$ (see Theorem 2.5 of [10]). Thus, $\mathcal{P}_{\text{ic}}[T] \cong \mathcal{N}\mathcal{P}_{\text{ic}} \oplus \mathcal{N}\mathcal{P}_{\text{ic}}$ as étale sheaves (see Proposition 5.1 of [10]). By using the Fact (5.1), we get

$$R^i \pi^+_* \mathcal{O}_{Y^+}^x \cong \mathcal{N}\mathcal{P}_{\text{ic}}$$

and $R^i \pi_* \mathcal{O}_Y^x \cong \mathcal{P}_{\text{ic}}[T] \cong \mathcal{N}\mathcal{P}_{\text{ic}} \oplus \mathcal{N}\mathcal{P}_{\text{ic}}$ as $G$-sheaves. We compare the spectral sequences (5.4) for $\pi, \pi^+$ and $\pi^-$, and get the following exact diagram (using Theorem 5.2)

\[
\begin{array}{cccccccc}
& & & & & & & \\
0 & \to & 0 & \to & \to & \to & 0 & \\
0 & \to & \text{Pic}^G(X) \oplus H^1_{\text{et}}(G; X, \mathcal{N}\mathcal{O}_X^x) \oplus H^1_{\text{et}}(G; X, \mathcal{N}\mathcal{O}_X^x) & \to & H^1_{\text{et}}(G; X, \pi_* \mathcal{O}_X^x) & \to & H^1_{\text{et}}(G; X, \mathbb{Z}) & \to & 0 \\
0 & \to & \text{Pic}^G(X) \oplus \mathcal{N}\text{Pic}^G(X) \oplus \mathcal{N}\text{Pic}^G(X) & \to & \text{Pic}^G(X[t, t^{-1}]) & \to & L\text{Pic}^G(X) & \to & 0 \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
0 & \to & H^1_{\text{et}}(G; X; \mathcal{O}_X^x \times \mathcal{N}\mathcal{O}_X^x \times \mathcal{N}\mathcal{O}_X^x) & \to & H^1_{\text{et}}(G; X; \mathcal{O}_X^x \times \mathcal{N}\mathcal{O}_X^x \times \mathcal{N}\mathcal{O}_X^x \times \mathbb{Z}) & \to & \to & \\
\end{array}
\]

A diagram chase gives the desired assertions. \(\square\)

**Remark 5.5.** The argument given above for the étale topology also works for the Nisnevich topology (see Remark 5.3). Therefore, we get $L\text{Pic}^G(X) \cong H^1_{\text{nis}}(G; X; \mathbb{Z})$. The result may fail for the Zariski topology. For example, consider the nodal curve $X = \text{Spec}(k[x, y]/(y^2 - x^2 - x^3))$ with $\mathbb{Z}_2$-action given by $(x, y) \mapsto (x, -y)$. Then $H^1_{\text{et}}(X, \mathbb{Z}) \cong L\text{Pic}(X) = \mathbb{Z}$ (see Remark 5.5.2 of [10]) and $H^1_{\text{zar}}(X, \mathbb{Z}) = 0$ because $X$ is an integral scheme. By considering the exact sequence (3.1) for the Zariski topology with constant sheaf $\mathbb{Z}$, we get

$$0 \to H^1(\mathbb{Z}_2, \mathbb{Z}) \to H^1_{\text{zar}}(\mathbb{Z}_2; X; \mathbb{Z}) \to H^0(\mathbb{Z}_2, H^1_{\text{zar}}(X, \mathbb{Z})) \to \ldots.$$
Note that $H^1(\mathbb{Z}_2, \mathbb{Z}) = 0$. Hence, $H^1_{zar}(\mathbb{Z}_2; X; \mathbb{Z}) = 0$. On the other hand, the exact sequence \([3.1] \) for the étale topology implies the following

$$0 \to H^1(\mathbb{Z}_2, \mathbb{Z}) \to H^1_{et}(\mathbb{Z}_2; X; \mathbb{Z}) \to H^0(\mathbb{Z}_2, \mathbb{Z}) \to H^2(\mathbb{Z}_2, \mathbb{Z}).$$

Therefore, $\text{LPic}^\mathbb{Z}_2(X) \cong H^1_{et}(\mathbb{Z}_2; X; \mathbb{Z}) \neq 0$ because $H^2(\mathbb{Z}_2, \mathbb{Z}) = \mathbb{Z}_2$ (see Example 6.2.3 of [11]).

**Theorem 5.6.** For a $G$-scheme $X$, $\text{LPic}^G(X) \cong \text{LPic}^G(X[t]) \cong \text{LPic}^G(X[t, t^{-1}])$. In other words, $\text{NLPic}^G(X) = \text{LPic}^G(X) = 0$.

**Proof.** We have (see Example 2.1)

$$H^0_{et}(X, \mathbb{Z}) \cong H^0_{et}(X[t], \mathbb{Z}) \cong H^0_{et}(X[t, t^{-1}], \mathbb{Z})$$

and

$$\text{LPic}(X) \cong \text{LPic}(X[t]) \cong \text{LPic}(X[t, t^{-1}]).$$

By comparing the sequence \([3.1] \) for $X$, $X[t]$ and $X[t, t^{-1}]$ with constant $G$-sheaf $\mathbb{Z}$, we get $H^1_{et}(G; X; \mathbb{Z}) \cong H^1_{et}(G; X[t]; \mathbb{Z}) \cong H^1_{et}(G; X[t, t^{-1}]; \mathbb{Z})$. Hence the assertion by Theorem 5.4. \( \square \)

**Corollary 5.7.** Let $X$ be a $G$-scheme. Then there is a natural decomposition

$$\text{Pic}^G(X[t_1, t_1^{-1}, t_2, t_2^{-1}, \ldots, t_m, t_m^{-1}]) \cong \text{Pic}^G(X) \oplus \prod_{k=1}^{m} \prod_{i=1}^{2k}\text{Pic}^{G}(X) \oplus \prod_{i=1}^{m} H^1_{et}(G; X; \mathbb{Z}).$$

**Proof.** By Theorem 5.4 $\text{Pic}^G$ is a contracted functor on $\text{Sch}^G/k$. The result is now clear from \([2.3] \) and Theorem 5.6. \( \square \)

Given a $G$-scheme, there is a natural homomorphism $\eta_X : \text{Pic}^G(X) \to \text{Pic}(X)$ sending $[(\mathcal{L}, \phi)]$ to $[\mathcal{L}]$. So, we get a natural transformation $\eta : \text{Pic}^G \Rightarrow \text{Pic}$ of functors on $\text{Sch}^G/k$. By Fact 5.1 the kernel of $\eta_X$ consists of the isomorphism classes of $G$-linearization of $\mathcal{O}_X$. The exact sequence \([4.1] \) implies that $\text{ker}(\eta_X) \cong H^1(G, H^0_{et}(X, \mathcal{O}_X^\tau))$, where $\tau \in \{\text{zar, et, nis}\}$. Next, we show that the kernel $\text{ker}(\eta)$ is a contracted functor on $\text{Sch}^G/k$.

Let $F$ and $F'$ be two contracted functors on some category $\mathcal{C}$ (e.g., rings or schemes). A morphism between contracted functors $F$ and $F'$ is a natural transformation $\eta : F \Rightarrow F'$ such that the following diagram commutes

$$
\begin{array}{ccc}
L_{\eta_X}(X) & \longrightarrow & F(X[t, t^{-1}]) \\
\downarrow_{\eta_X} & & \downarrow_{\eta_{X[t, t^{-1}]}} \\
L_{\eta_X}(X) & \longrightarrow & F'(X[t, t^{-1}])
\end{array}
$$

for all $X \in \text{ob}(\mathcal{C})$. 
Lemma 5.8. The natural transformation $\eta : \text{Pic}^G \Rightarrow \text{Pic}$ is a morphism of contracted functors on $\text{Sch}^G/k$.

Proof. Write $Y = X[t, t^{-1}]$ and $\pi : Y \to X$. The result follows from the following commutative diagram

$$
\begin{array}{ccc}
LPic^G(X) \cong H^1_{\text{et}}(G; X; \mathbb{Z}) & \longrightarrow & H^1_{\text{et}}(G; X, \pi_\ast O_Y^\times) \\
\downarrow & & \downarrow \\
H^0(G, H^1_{\text{et}}(X; \mathbb{Z})) & \longrightarrow & H^0(G, H^1_{\text{et}}(X, \pi_\ast O_Y^\times)) \\
\text{inclusion} & & \text{inclusion} \\
LPic(X) \cong H^1_{\text{et}}(X; \mathbb{Z}) & \longrightarrow & H^1_{\text{et}}(X, \pi_\ast O_Y^\times) \\
\downarrow & & \downarrow \\
H^0(G, H^1_{\text{et}}(Y; \mathbb{Z})) & \longrightarrow & H^0(G, H^1_{\text{et}}(Y, O_Y^\times)) \\
\text{inclusion} & & \text{inclusion} \\
\end{array}
$$

where the top and bottom rows are precisely the splitting maps for $\text{Pic}^G$ and $\text{Pic}$ respectively (see Theorem 5.4 and Theorem 7.6 of [10]), and for columns see (3.1). □

Theorem 5.9. The functor $\ker(\eta)$ is contracted on $\text{Sch}^G/k$. Moreover, the contraction term $L\ker(\eta_X)$ is isomorphic to $H^1(G, H^0_{\text{et}}(X; \mathbb{Z}))$.

Proof. By Lemma 5.8, $\eta$ is a morphism of contracted functors on $\text{Sch}^G/k$. Then $\ker(\eta)$ is a contracted functor with contraction $L\ker(\eta) = L\ker(\eta_X)$ (see Lemma 2.2 of [8]). The second statement follows from (3.1). □

We conclude by discussing a vanishing result of $LPic^G(X)$ for a hensel local ring.

Theorem 5.10. Let $X$ be a connected $G$-scheme. If $LPic(X) = 0$ then $LPic^G(X) = 0$.

Proof. The exact sequence (3.1) for the constant $G$-sheaf $\mathbb{Z}$ (using Example 2.1 and Theorem 5.4) gives that

$$
0 \to H^1(G, H^0_{\text{et}}(X, \mathbb{Z})) \to LPic^G(X) \to (LPic(X))^G \to \ldots
$$

Since $X$ is connected, $H^0_{\text{et}}(X, \mathbb{Z}) \cong H^0_{\text{zar}}(X, \mathbb{Z}) \cong \mathbb{Z}$. Thus, $H^1(G, \mathbb{Z}) = 0$ because $G$ is a finite group. Hence the assertion. □

Corollary 5.11. Let $X = \text{Spec}(A)$ be a $G$-scheme. If $A$ is a hensel local ring then $LPic^G(X) = 0$.

Proof. For a hensel local ring $A$, $LPic(A) = 0$ (see Theorem 2.5 of [10]). Hence the result by Theorem 5.10. □

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