Reducibility mod $p$ of integral closed subschemes in projective spaces – an application of arithmetic Bézout

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Abstract. In [4], we showed that we can improve results by Emmy Noether and Alexander Ostrowski ([8]) concerning the reducibility modulo $p$ of absolutely irreducible polynomials with integer coefficients by giving the problem a geometric turn and using an arithmetic Bézout theorem ([2]). This paper is a generalization of [4], where we show that combining the methods of [4] with the theory of Chow forms leads to similar results for flat, equidimensional, integral, closed subschemes of arbitrary codimension in $\mathbb{P}^s_{\mathbb{Z}}$.

Introduction. Let $K$ be a number field with ring of integers $R$, and $Z$ a flat, equidimensional, integral, closed subscheme of dimension $r + 1$ and degree $d$ in $\mathbb{P}^s_R$ ($s, d \geq 2$), with absolutely irreducible generic fiber. One can show that the fiber $Z_{k(p)}$ is also absolutely irreducible for all but finitely many prime ideals $p$ of $R$ (e.g. [5, Theorem 9.7.7] and [6, Theorem 4.10]).

We would like to bound the (product of the) norms of the prime ideals $p$ of $R$ for which the fiber $Z_{k(p)}$ is not absolutely irreducible in terms of the projective height of $Z$, as defined in [2]. In this paper, using arithmetic intersection theory, we solve for any fixed $n < d$ the analogous problem obtained by replacing “absolutely irreducible” by “is not a union of two closed subschemes of degrees $n$ and $d - n$, respectively”. To prove this theorem, we use Chow forms, and translate the problem to bounding the height of an intersection in some projective space. Thus, the proof becomes a straightforward application of an arithmetic Bézout Theorem for non-proper intersections given in [2], which reduces it to bounding degrees and heights of specific cycles in terms of the data provided.

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**Some notation.** Given a ring $R$ as above, and a locally free $\mathcal{O}_{\text{Spec}(R)}$-module $\mathcal{E}$ of finite rank $s + 1$ ($s \geq 0$), let $\mathbf{P}(\mathcal{E}) = \text{Proj}_{\text{Spec}(R)}(\text{Sym}(\mathcal{E}^\vee))$ be the associated space of lines, where $\mathcal{E}^\vee$ denotes the dual sheaf of $\mathcal{E}$, and let $\pi$ denote its structural morphism. We suppose $\mathcal{E}$ endowed with a Hermitian metric $h$, and endow $\mathcal{E}^\vee$ with the dual metric.

Let $r$ be a positive integer. For $i = 0, \ldots, r$, let $P_i = \mathbf{P}(\mathcal{E})$, and $P_i^\vee = \mathbf{P}(\mathcal{E}^\vee)$.

We endow the canonical quotient line bundle $\mathcal{O}(1)$ on $P_i^\vee$ with the quotient metric (cf [2, 3.1.2.3]), and let $M_i$ be the pullback of the resulting Hermitian line bundle $\mathcal{O}(1)$ on $P_i^\vee$ to $\prod_{i=0}^r P_i^\vee$.

Finally, for $x \in \mathbb{N}$, let $F_{x,r}(\mathcal{E}) := \bigotimes_{i=0}^r \text{Sym}^x(\mathcal{E})$.

**Chow divisors and forms.** By [2, 4.3.1], we can associate to each non-zero algebraic cycle $Z \in \mathbb{Z}_{r+1}(\mathbf{P}(\mathcal{E}))$ a Chow divisor $\text{Ch}_1(Z)$ (where $1 = (1, \ldots, 1) \in \mathbb{Z}^{r+1}$) in $Z_1(\prod_{i=0}^r \mathbf{P}_i^\vee)$, which is effective (resp. flat, resp. flat and irreducible) if such is the case for $Z$.

Let $Z$ now be a non-zero effective cycle of degree $x$ in $\mathbb{Z}_{r+1}(\mathbf{P}(\mathcal{E}))$. Generically, the associated Chow divisor $\text{Ch}_1(Z)_K$ is the divisor of a non-zero multihomogeneous form $\phi_{1,Z}$ in 

$$H^0\left(\prod_{i=0}^r (\mathbf{P}_i^\vee)_K, \otimes M_i^x\right) \cong F_{x,r}(\mathcal{E})_K,$$

called the Chow form of $Z_K$. Thus we can associate a point of $\mathbf{P}(F_{x,r}(\mathcal{E}))(K)$ to each non-zero effective cycle of degree $x$ in $\mathbb{Z}_{r+1}(\mathbf{P}(\mathcal{E}))$. If the class number of $K$ is one, there exists a generalized Chow form $\phi_{1,Z}$ over $R$, for which $\text{Ch}_1(Z) = \text{div}(\phi_{1,Z})$ in $\prod \mathbf{P}_i^\vee$. Similarly, for every point $t$ of $\text{Spec}(R)$, we can define Chow divisors and forms for the cycles contained in the fibre above $t$ ([2, 4.3.2]).

If $Z$ is moreover flat over $\text{Spec}(R)$, we have the following result:

**Proposition.** Let $Z \in \mathbb{Z}_{r+1}(\mathbf{P}(\mathcal{E}))$ be a flat, integral, closed subscheme of $\mathbf{P}(\mathcal{E})$ of degree $x$, with Chow divisor $\text{Ch}_1(Z)$. Let $\phi_K$ be the Chow form of $Z_K$. Let $[\phi_K] \in \mathbf{P}(F_{x,r}(\mathcal{E}))(K)$ be the corresponding point, and $P_Z$ its Zariski closure in $\mathbf{P}(F_{x,r}(\mathcal{E}))$. Then for every point $t$ of $\text{Spec}(R)$, the fiber $P_{Z,t}$ is the point of $\mathbf{P}(F_{x,r}(\mathcal{E}))_t$ corresponding to the Chow form $\phi_t$ of $Z_t$.

**Proof.** It suffices to note that by construction, we have $\text{Ch}_1(Z_t) = \text{Ch}_1(Z)_t$ for every point $t$ of $\text{Spec}(R)$ ([2, 4.3.2]). In particular, as $Z$ is flat, the Zariski closure
of \text{div}(\phi_K) = \text{Ch}_1(Z_K) is \text{Ch}_1(Z).

\textbf{Components of degree } n. \text{ Let } d \in \mathbb{N}_{>0}, \text{ and fix integers } 1 \leq n \leq d - 1 \text{ and } 0 \leq r \leq s. \text{ Let us simplify the notation by setting } F_x := F_{x,r}(\mathcal{E}) \text{ for every } x. \text{ Consider the morphism }

\psi : \mathbb{P}(F_n) \times \mathbb{P}(F_{d-n}) \to \mathbb{P}(F_d)

defined by taking the product on sections (seen as multihomogeneous forms on } \prod \mathbb{P}^y). \text{ Let } \mathcal{W} \text{ denote the image of } \psi.

Let } Z \text{ be a flat, integral, closed subscheme of degree } d \text{ in } Z_{r+1}(\mathbb{P}(\mathcal{E})), \text{ and let } P_Z \text{ be as in the proposition. By dimension arguments and the proposition, the intersection of } P_Z \text{ and } \mathcal{W} \text{ is either } P_Z, \text{ if } Z_K \text{ has a component (irreducible or not) of degree } n, \text{ or a finite number of closed points whose images under the structural morphism } \pi : \mathbb{P}(F_d) \to \text{Spec } (R) \text{ are the prime ideals } q_1, \ldots, q_v \text{ above which the fiber of } Z \to \text{Spec } (R) \text{ has such a component.}

Before stating the theorem, we note that if } \mathcal{E} \text{ is isomorphic to } R^{s+1}, \text{ then each vector bundle } F_x \text{ is free, and can be endowed, in a natural way, with a basis } B_x ([2, p. 985]). \text{ Indeed, in this case, } F_x \text{ is a space of multihomogeneous forms as described in [2, 4.3.13], whose basis is formed by the monomials. We will use this basis to identify } F_x \text{ with } R^{N_x+1} (\text{where } N_x := \text{rk}(F_x) - 1).

The following theorem only deals with the trivial vector bundle, i.e. } \mathcal{E} = R^{s+1}, \text{ endowed with the standard Hermitian metric.

\textbf{Theorem.} \text{ Let } Z \in Z_{r+1}(\mathbb{P}^s_R) \text{ be a flat, integral, closed subscheme of } \mathbb{P}^s_R \text{ of dimension } r + 1 \text{ and degree } d (s, d \geq 2, r \geq 0), \text{ and } n \in \{1, \ldots, d - 1\} \text{ an integer such that } Z_K \text{ cannot be written as the union of two closed subschemes of degrees } n \text{ and } d - n, \text{ respectively. Let } q_1, \ldots, q_v \text{ be the distinct prime ideals of } R \text{ above which the geometric fiber of } Z \text{ can be written as such a union. Setting } N_{x,r,s} := \text{rk}(\bigotimes_{i=0}^r \text{Sym}^x(R^{s+1})) - 1, \text{ we have}

\begin{equation}
\log \prod_{j=1}^v N(q_j) \leq \frac{1}{1 + \delta_{n,d-n}} \left( \frac{N_{n,r,s} + N_{d-n,r,s}}{N_{n,r,s}} \right) h_K(Z) + O(1)
\end{equation}

when } h_K(Z) \text{ tends to infinity, where } h_K \text{ is the projective height associated to the standard Hermitian metric on } R^{s+1}, \text{ as defined in [2, 4.1.1] (see also [3, 2.1.5]), and } \delta \text{ is the Kronecker delta function. Moreover, we can replace the } O(1) \text{ by an explicit function of } s, d, r, \text{ and } n \text{ (see the proof).}
Remark. For the hypersurface case \( r = s \), we find a stricter bound in [4], due to the fact that horizontal hypersurfaces (which correspond to the flat integral closed subschemes here) are (directly) parametrized by a projective space, making it unnecessary to use Chow forms. The \( M_x \) used there correspond to the \( N_{x,r,s} \) for \( r = 0 \) in this paper.

Proof. As noted before, the set \( \{q_1, \ldots, q_v\} \) is the support of \( \pi(P_Z \cap W) \) in \( \text{Spec}(R) \). In particular, \( \log \prod N(q_i) = h_K(|P_Z \cap W|) \). By the arithmetic Bézout theorem [2, 5.5.1.iii], we have

\[
h_K(|P_Z \cap W|) \leq \deg_K(P_Z) h_K(W) + h_K(P_Z) \deg_K(W) + \frac{1}{2} [K : Q] \deg_K(P_Z) \deg_K(W)(M_d + 1) \log(2).
\]

By definition of \( P_Z \), its degree equals one. Using the further shortened, and somewhat misleading, notation \( N_x := N_{x,r,s} \), we find that the other terms on the right can be bounded as follows:

(1) \( h_K(P_Z) = h_K(Z) + d [K : Q] \sigma_r + (r + 1) \log(s + 1) \),

where \( \sigma_x = \frac{1}{2}(x + 1) \sum_{m=2}^{x+1} \frac{1}{1/m} \).

(2) \( \deg_K(W) = \frac{[K : Q]}{1+\delta_{n,d-n}} \left( \frac{N_{n+N_{d-n}}}{N_n} \right) \log \left( \frac{(d+1)^{3(r+1)(s+1)}}{2} \left( \frac{N_{n+N_{d-n}+1}}{N_n+N_{d-n}+1} \right) \right) \).

(3) \( h_K(W) \leq \frac{[K : Q]}{1+\delta_{n,d-n}} \left( \frac{N_{n+N_{d-n}+1}}{N_n+N_{d-n}+1} \right) \log \left( \frac{(d+1)^{3(r+1)(s+1)}}{2} \left( \frac{N_{n+N_{d-n}+1}}{N_n+N_{d-n}+1} \right) \right) \).

leading to the result of the theorem.

Proof of (1). Let \( \{a_I\} \) be the coefficients of \( P_{Z,K} \) (i.e. of the form \( \phi_{1,Z_K} \)) in the basis \( B_d \). Then

\[
h_K(P_Z) = \sum_{\sigma} \log \left( \sum |a_I|^2 \right)^{1/2} - \sum_p \min_I v_p(a_I) \log N(p).
\]

Another height associated to \( B_d \) ([2, 4.3.4.1]) is

\[
h_B(P_Z) := h_B(\text{Ch}_1(Z)) = \sum_{\sigma} \log \left( \sum |a_I| \right) - \sum_p \min_I v_p(a_I) \log N(p).
\]

Clearly, we have \( h_K(P_Z) \leq h_B(P_Z) \). By [2, Theorem 4.3.8, (4.3.33), and (4.1.2)],

\[
h_B(P_Z) \leq h_K(Z) + d [K : Q] \sigma_r + (r + 1) \log(s + 1)).
\]

In particular, we have \( h_K(P_Z) = h_K(Z) + \mathcal{O}(1) \).

Remark. Before giving the proofs of (2) and (3), let us note that the morphism \( \psi \) was used under the notation \( \phi_n \) in [4], where the degree and height of its image
were bounded explicitely. Here we give only sketches of the proofs of (2) and (3), the details can be found in [loc.cit.].

**Proof of (2).** Let \( f_n \) (resp. \( f_{d-n} \)) denote the projection from \( \mathbf{P}(F_n) \times \mathbf{P}(F_{d-n}) \) onto the first (resp. second) coordinate. Using intersection theory, we find

\[
\deg(\psi) \deg_K(W) = \deg(c_1\mathcal{O}_{\mathbf{P}(F_n)}(1)^N_{n+N_{d-n}} \cdot [\psi^*(\mathcal{O}(F_n) \times \mathcal{O}(F_{d-n}))]),
\]

where \( c_1\mathcal{O}_{\mathbf{P}(F_n)}(1) \) is the first Chern class of \( \mathcal{O}_{\mathbf{P}(F_n)}(1) \). By the projection formula, and the fact that \( \psi^*\mathcal{O}_{\mathbf{P}(F_n)}(1) = f_n^*\mathcal{O}_{\mathbf{P}(F_n)}(1) \otimes f_{d-n}^*\mathcal{O}_{\mathbf{P}(F_{d-n})}(1) \), this implies that

\[
\deg_K(W) = \frac{1}{1 + \delta_{n,d-n}} \left( \frac{N_n + N_{d-n}}{N_n} \right).
\]

**Proof of (3).** As in the proof of (2), we use intersection theory, but this time with metrics. By [2, 4.1.2 and Proposition 2.3.1], we have

\[
h_K(W) = [K : \mathbb{Q}] \sigma_{N_n+N_{d-n}} \deg_K(W)
+ \frac{1}{\deg(\psi)} \deg(h_1(\psi^*\mathcal{O}_{\mathbf{P}(F_n)}(1))_{N_n+N_{d-n}+1} | \mathbf{P}(F_n) \times \mathbf{P}(F_{d-n}))).
\]

The arithmetic degree on the right is the projective height of \( \mathbf{P}(F_n) \times \mathbf{P}(F_{d-n}) \) associated to the line bundle \( \mathcal{L} := \psi^*\mathcal{O}_{\mathbf{P}(F_n)}(1) \) endowed with the pullback \( \rho \) under \( \psi \) of the standard Hermitian metric on \( \mathcal{O}_{\mathbf{P}(F_n)}(1) \). We will bound this height in two steps, using a comparison of metrics on \( \mathcal{L} \). First, let \( (\mathcal{L}, \rho') \) denote the line bundle \( \mathcal{L} \) endowed with the product metric obtained by taking the standard Hermitian metrics on \( \mathcal{O}_{\mathbf{P}(F_n)}(1) \) and \( \mathcal{O}_{\mathbf{P}(F_{d-n})}(1) \). The associated projective height is

\[
h_{(\mathcal{L}, \rho')}(\mathbf{P}(F_n) \times \mathbf{P}(F_{d-n})) := \deg(h_1(\mathcal{L}, \rho'))_{N_n+N_{d-n}+1} | \mathbf{P}(F_n) \times \mathbf{P}(F_{d-n})),
\]

which, by the projection formula and the decomposition of the metrized line bundle \( (\mathcal{L}, \rho') \) as a (tensor)product, equals

\[
[K : \mathbb{Q}] \left( \left( \frac{N_n + N_{d-n}}{N_n} \right) \sigma_{N_n} + \left( \frac{N_n + N_{d-n}}{N_n} \right) \sigma_{N_n} \right).
\]

The second step in bounding the height that we want consists of comparing the norms \( || \cdot || \) and \( || \cdot ||' \) associated to \( \rho \) and \( \rho' \). Let \( \varphi : (\mathbf{P}(F_n) \times \mathbf{P}(F_{d-n}))(\mathbb{C}) \to \mathbb{R} \) be defined by \( \langle || \cdot ||, || \cdot ||' \rangle = \exp(\varphi)(|| \cdot ||^2 \| || \cdot \|^2) \). For each embedding \( \sigma : K \hookrightarrow \mathbb{C} \), and \( (a, b) = ((a_0 : \ldots : a_{N_n}), (b_0 : \ldots : b_{N_d-n}) \in (\mathbf{P}(F_n) \times \mathbf{P}(F_{d-n}))(\mathbb{C}) \), we let \( f_a,
resp. \( g_b \), be the corresponding multihomogeneous polynomial (in \((r + 1)(s + 1)\) variables). We have

\[
\exp(\varphi_\sigma(a, b)) = \left( \frac{L_2(f_ag_b)}{L_2(f_a)L_2(g_b)} \right)^2,
\]

where the \( L_2 \)-norm \( L_2(f) \) of a (multi)homogeneous polynomial \( f = \sum c_I X^I \) is \((\sum c_I c_I)^{1/2}\). From results of [7, 3.2], we can now deduce that

\[
\sup_{(a, b)}(\varphi_\sigma(a, b)) \leq 3(r + 1)(s + 1) \log(d + 1)
\]

for every \( \sigma : K \hookrightarrow \mathbb{C} \). The last step consists of combining this inequality with the results of [2, Proposition 3.2.2] and [1, Lemma 2.6.ii] (see also [4]) to obtain

\[
h_{(\mathcal{L}, \rho)}(\mathbb{P}(F_n) \times \mathbb{P}(F_{d-n})) \leq h_{(\mathcal{L}, \rho')}((\mathbb{P}(F_n) \times \mathbb{P}(F_{d-n})))
\]

\[
+ [K : \mathbb{Q}] \frac{N_n + N_{d-n} + 1}{2} \deg(\psi) \deg_K(W) 3(r + 1)(s + 1) \log(d + 1),
\]

which, after some simplification, leads to the bound for \( h_K(W) \) stated in (3).

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