ON THE DETERMINING WAVENUMBER FOR THE NONAUTONOMOUS SUBCRITICAL SQG EQUATION

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ABSTRACT. A time-dependent determining wavenumber was introduced in [5] to estimate the number of determining modes for the surface quasi-geostrophic (SQG) equation. In this paper we continue this investigation focusing on the subcritical case and study trajectories inside an absorbing set bounded in \( L^{\infty} \). Utilizing this bound we find a time-independent determining wavenumber that improves the estimate obtained in [5]. This classical approach is more direct, but it is contingent on the existence of the \( L^{\infty} \) absorbing set.

KEY WORDS: Subcritical quasi-geostrophic equation, determining modes, global attractor.
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1. INTRODUCTION

In this paper we estimate the number of determining modes for the forced subcritical surface quasi-geostrophic (SQG) equation (see [12])

\[
\begin{align*}
\frac{\partial \theta}{\partial t} + u \cdot \nabla \theta + \nu \Delta^{\alpha} \theta &= f, \\
u &= R^\perp \theta,
\end{align*}
\]

where \( x \in \mathbb{T}^2 = [0, L]^2 \), \( 1 < \alpha < 2 \), \( \nu > 0 \), \( \Lambda = \sqrt{-\Delta} \) is the Zygmund operator, and

\( R^\perp \theta = \Lambda^{-1}(-\partial_2 \theta, \partial_1 \theta) \).

The initial data \( \theta(0) \in L^2(\mathbb{T}^2) \) and the force \( f \in L^p \) for some \( p > 2/\alpha \) are assumed to have zero average. We will also consider time-dependent forces \( f \in L^\infty(0, \infty; L^p(\mathbb{T}^2)) \).

A time-dependent determining wavenumber \( \Lambda(t) \) was introduced in [5] in the case where \( \alpha \in (0, 2) \) and the force could be potentially rough. The determining wavenumber was defined based only on the structure of the equation and without any requirements on the regularity of solutions. It was shown that if two solutions coincide below \( \Lambda(t) \), the difference between them decay exponentially, even when they are far away from the attractor. Moreover, \( \Lambda(t) \) was shown to be uniformly bounded for all the solutions on the global attractor when \( \alpha \in (1, 2) \) and \( f \in L^p \), \( p > 2/\alpha \), in which case the attractor is bounded in \( L^{\infty} \). In this paper we investigate this situation further and present a different, more direct approach in the subcritical case \( \alpha \in (1, 2) \). Here we consider solutions that already entered an \( L^{\infty} \) absorbing set and take advantage of the \( L^{\infty} \) bound (which is proportional to the \( L^p \)-norm of the force) to define a time-independent determining wavenumber \( \Lambda \) and improve the final estimate for the number of determining modes that we had in [5]. The drawback of this method is that it is less general and works only for regular solutions in the \( L^{\infty} \) absorbing set. For a more complete background on the topic of finite dimensionality.

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of flows, we refer the readers to [5, 7, 10, 11, 15, 16, 17, 18, 19, 20, 22] and references therein.

For the critical SQG equation ($\alpha = 1$), due to the balance of the nonlinear term and the dissipative term, the global regularity problem was challenging. However it was solved by different authors using different sophisticated methods in [2, 9, 13, 25, 26]. For the subcritical SQG equation with $1 < \alpha < 2$, the dissipative term dominates. In this case the global regularity was obtained in [27].

In this paper, we will consider forces $f \in L^\infty(0, \infty; L^p(\mathbb{T}^2))$, $p > 2/\alpha$, such that

$$ \sup_{t > 0} \|f\|_p \leq F,$$

for some $F > 0$. Then $\{\theta \in L^2 : \|\theta\|_\infty \leq R_\infty\}$ is an absorbing set in $L^2$ (see [5]), where

$$ R_\infty \sim \lambda_0^{\frac{2}{\alpha} - \frac{\alpha}{\nu}} F. $$

Here $\lambda_0 = 1/L$. We prove the following.

**Theorem 1.1.** Let $\alpha \in (1, 2)$, $l > \frac{1}{\alpha - 1}$, and $Q \in \mathbb{N}$ be such that

$$ A := \lambda_0 2^Q \geq \left( \frac{C L^2 R_\infty}{\nu} \right)^{\frac{1}{\alpha - 1}}, $$

where $C$ is some absolute constant. Let $\theta_1(t)$ be a solution of (1.1) with $f = f_1$ and $\theta_2(t)$ be a solution to (1.1) with $f = f_2$. If

$$ \|\theta_1(t)\leq Q - \theta_2(t)\leq Q\|_{B^0_{l,1}} \to 0, \quad \text{and} \quad \|f_1 - f_2\|_{B_{l,0}} \to 0, \quad \text{as} \quad t \to \infty, $$

then

$$ \|\theta_1(t) - \theta_2(t)\|_{B^0_{l,1}} \to 0 \quad \text{as} \quad t \to \infty. $$

Moreover, if $\theta_1(t)$ and $\theta_2(t)$ are two complete (ancient) solutions of (1.1) with $f \in L^p$, $p > 2/\alpha$, such that $\theta_1, \theta_2 \in L^\infty((-\infty, \infty); L^2)$

then

$$ \theta_1(t)\leq Q = \theta_2(t)\leq Q, \quad \forall t < 0, $$

then

$$ \theta_1(t) = \theta_2(t), \quad \forall t \in \mathbb{R}. $$

Due to the second part of the theorem, for any two solutions $\theta_1(t), \theta_2(t)$ on the attractor $\mathcal{A}$, such that $(\theta_1)\leq Q \equiv (\theta_2)\leq Q$, we have $\theta_1 \equiv \theta_2$. Here

$$ \mathcal{A} = \{\theta(0) : \theta(t) \text{ is a complete bounded solution, i.e., } \theta \in L^\infty((-\infty, \infty); L^2)\}. $$

In the subcritical case $\alpha > 1$ it is easy to show that $\mathcal{A}$ is a global attractor by virtue of classical methods, or applying the evolutionary system framework [4] that requires the existence of an absorbing ball, energy inequality, and continuity of trajectories. This method does not require proving the existence of a compact absorbing set, and it was used in [4] to show that $\mathcal{A}$ is the global attractor in the critical case $\alpha = 1$ (see also [9] for the existence of the global attractor in $H^1$).

## 2. Preliminaries

### 2.1. Notations.

We denote by $A \lesssim B$ an estimate of the form $A \leq CB$ with some absolute constant $C$, and by $A \sim B$ an estimate of the form $C_1 B \leq A \leq C_2 B$ with some absolute constants $C_1, C_2$. We write $\| \cdot \|_p = \| \cdot \|_{L^p}$, and $(\cdot, \cdot)$ stands for the $L^2$-inner product.
2.2. **Littlewood-Paley decomposition.** We recall briefly the Littlewood-Paley decomposition theory, which is one of the main techniques used in the paper. For a more detailed description on this theory we refer readers to the books [1, 23].

Denote $\lambda_q = 2^q L$ for integers $q$. A nonnegative radial function $\chi \in C_0^\infty (\mathbb{R}^n)$ is chosen such that

\[(2.1) \quad \chi(\xi) = \begin{cases} 
1, & \text{for } |\xi| \leq \frac{3}{4} \\
0, & \text{for } |\xi| \geq 1.
\end{cases}\]

Let $\varphi(\xi) = \chi(\xi/2) - \chi(\xi)$ and

\[\varphi_q(\xi) = \begin{cases} 
\varphi(2^{-q} \xi) & \text{for } q \geq 0, \\
\chi(\xi) & \text{for } q = -1.
\end{cases}\]

For a tempered distribution vector field $u$, its Littlewood-Paley projection $u_q$ is defined as follows.

\[h_q = \sum_{k \in \mathbb{Z}^n} \varphi_q(k)e^{2\pi i k \cdot x} \]

\[u_q := \Delta_q u = \sum_{k \in \mathbb{Z}^n} \hat{u}_k \varphi_q(k)e^{2\pi i k \cdot x} = \frac{1}{L^2} \int_{\mathbb{R}^2} h_q(y) u(x - y) dy, \quad q \geq -1,
\]

where $\hat{u}_k$ is the $k$th Fourier coefficient of $u$. Then we have

\[u = \sum_{q=-1}^{\infty} u_q\]

in the distributional sense. We also denote

\[u_{\leq Q} = \sum_{q=-1}^{Q} u_q, \quad u_{(Q,R]} = \sum_{p=Q+1}^{R} u_p, \quad \hat{u}_q = \sum_{|p-q| \leq 1} u_p.
\]

The Besov $B_{l,l}^s$-norm is defined as

\[\|u\|_{B_{l,l}^s} = \left( \sum_{q=-1}^{\infty} \lambda_q^s \|u_q\|^l_l \right)^{\frac{1}{l}}.
\]

The following inequalities will be frequently used:

**Lemma 2.1.** *(Bernstein’s inequality)* Let $n$ be the space dimension and $r \geq s \geq 1$. Then for all tempered distributions $u$,

\[\|u_q\|_r \leq \lambda_q^{n\left(\frac{1}{r} - \frac{1}{s}\right)} \|u_q\|_s.
\]

**Lemma 2.2.** Assume $2 < l < \infty$ and $0 \leq \alpha \leq 2$. Then

\[l \int u_q \Lambda^\alpha u_q |u_q|^{l-2} dx \geq \lambda_q^\alpha \|u_q\|_l^\alpha.
\]

For a proof of Lemma 2.2 see [3, 14].
2.3. Bony’s paraproduct and commutator. Bony’s paraproduct formula will be used to decompose the nonlinear terms. We will use the same version as in [6]:

\[
\Delta_q(u \cdot \nabla v) = \sum_{|q-p| \leq 2} \Delta_q(u_{\leq p-2} \cdot \nabla v_p) + \sum_{|q-p| \leq 2} \Delta_q(u_p \cdot \nabla u_{\leq p-2}) + \sum_{p \geq q-2} \Delta_q(\tilde{u}_p \cdot \nabla v_p).
\]

Some terms in this decomposition will be estimated using commutators. Let

\[
[\Delta_q, u_{\leq p-2} \cdot \nabla]v_p := \Delta_q(u_{\leq p-2} \cdot \nabla v_p) - u_{\leq p-2} \cdot \nabla \Delta_q v_p.
\]

By definition of \(\Delta_q\) and Young’s inequality,

\[
|||\Delta_q, u_{\leq p-2} \cdot \nabla||v_p||_r \lesssim ||\nabla u_{\leq p-2}||_\infty ||v_p||_r,
\]

for any \(r > 1\).

3. Global attractor

First, we recall the \(L^\infty\) estimates from [5].

Lemma 3.1. Let \(\alpha \in (0, \infty)\) and \(\theta\) be a solution of (1.1) on \([0, \infty)\) with \(\theta(0) \in L^2\) and

\[
\sup_{t > 0} \|f(t)\|_p \leq F,
\]

for some \(F \geq 0\) and \(p \in (2/\alpha, \infty)\). Then, for every \(t > 0\),

\[
\|\theta(t)\|_{L^\infty} \lesssim \begin{cases} \frac{\|\theta(0)\|_2}{(vt)^{\frac{\alpha}{p}}} + \left( \frac{F}{\nu} \right)^{\frac{1}{p-\alpha}} \|\theta(0)\|_2^{\frac{p-\alpha}{p}}, & p < \infty, \\ \frac{\|\theta(0)\|_2}{(vt)^{\frac{\alpha}{p}}} + \left( \frac{F}{\nu} \right)^{\frac{1}{p-\alpha}} \|\theta(0)\|_2^{\frac{\alpha}{p}}, & p = \infty. \end{cases}
\]

Proof. It follows from the proof of Lemma 4.2 in [5] if \(\|f\|_p\) is replaced with \(F\). \(\square\)

Due to the energy equality

\[
\|\theta(t)\|_2^2 = \|\theta(t_0)\|_2^2 - \nu \int_{t_0}^t (\|A^{\frac{\alpha}{2}} \theta\|_2 + (f, \theta)) \, d\tau, \quad 0 \leq t_0 \leq t,
\]

we have

\[
\|\theta(t)\|_2^2 \leq \|\theta(0)\|_2^2 e^{-\nu(2\pi \lambda_0)^a t} + \frac{\|\Lambda^{-\frac{\alpha}{2}} f\|_2^2}{\nu^2(2\pi \lambda_0)^{a/2}} \left( 1 - e^{-\nu(2\pi \lambda_0)^a t} \right), \quad t > 0,
\]

which implies the existence of an absorbing ball in \(L^2\). Indeed, for any bounded set \(U \subset L^2\) there exists time \(t_L^2\), such that

\[
\theta(t) \in B_{L^2}, \quad \forall t \geq t_L^2,
\]

for any solution \(\theta(t)\) with \(\theta(0) \in U\). Here

\[
B_{L^2} = \{ \theta \in L^2 : \|\theta\|_2 \leq R_2 \}, \quad R_2 = \frac{\|\Lambda^{-\frac{\alpha}{2}} f\|_2}{\nu \lambda_0^{a/2}}.
\]

Moreover, there is a compact global attractor \(A \subset B_{L^2}\),

\[
A = \{ \theta(0) : \theta(t) \text{ is a complete bounded trajectory, i.e., } \theta \in L^\infty((-\infty, \infty); L^2) \},
\]
which is the $\omega$-limit of $B_{L^2}$, the minimal closed attracting set, and the maximal bounded invariant set. Moreover, $A$ is bounded in $L^\infty$ due to Lemma 3.1. More precisely,

$$A \subset B_{L^\infty} = \left\{ \theta \in B_{L^2} : \|\theta\|_\infty \lesssim \left( \frac{\|f\|_p}{\nu} \right)^{\frac{p}{p + p\alpha - 2}} \|\theta\|_2^{\frac{p\alpha - 2}{p + p\alpha - 2}} \right\},$$

where

$$R_\infty \sim \left( \frac{\|f\|_p}{\nu} \right)^{\frac{p}{p + p\alpha - 2}} \left( \frac{\|\Lambda_{-\alpha/2} f\|_2}{\nu\lambda_{\alpha/2}^0} \right)^{\frac{p\alpha - 2}{p + p\alpha - 2}} \leq \lambda_{\alpha - \alpha/2}^0 \frac{\|f\|_p}{\nu}. \quad (3.2)$$

In addition, $B_{L^\infty}$ is an absorbing set, i.e., for any bounded set $U \subset L^2$ there exists time $t_{L^\infty}$, such that

$$\theta(t) \in B_{L^\infty}, \quad \forall t \geq t_{L^\infty},$$

for any solution $\theta(t)$ with $\theta(0) \in U$.

## 4. Proof of the Main Result

First we recall a generalization of Grönwall’s lemma from [13].

**Lemma 4.1.** Let $\alpha(t)$ be a locally integrable real valued function on $(0, \infty)$, satisfying for some $0 < T < \infty$ the following conditions:

$$\liminf_{t \to \infty} \int_t^{T + t} \phi(\tau) \, d\tau > 0, \quad \limsup_{t \to \infty} \int_t^{T + t} \phi^-(\tau) \, d\tau < \infty,$$

where $\phi^- = \max\{-\phi, 0\}$. Let $\psi(t)$ be a measurable real valued function on $(0, \infty)$ such that

$$\psi(t) \to 0, \quad \text{as} \quad t \to \infty.$$

Suppose $\xi(t)$ is an absolutely continuous non-negative function on $(0, \infty)$ such that

$$\frac{d}{dt} \xi + \phi \xi \leq \psi, \quad \text{a.e. on } (0, \infty).$$

Then

$$\xi(t) \to 0 \quad \text{as} \quad t \to \infty.$$

Now we are ready to prove the main result.

**Proof of Theorem 1.1** Denote $u_1 = R^\perp \theta_1$ and $u_2 = R^\perp \theta_2$. Let $f = f_1 - f_2$ and $w = \theta_1 - \theta_2$, which satisfies the equation

$$w_t + u_1 \cdot \nabla w + \nu \Lambda^\alpha w + R^\perp w \cdot \nabla \theta_2 = f. \quad (4.1)$$
Projecting equation (4.1) onto the $q$th shell, multiplying it by $l w_q |w_q|^{-2}$, adding up for all $q \geq -1$, and applying Lemma 2.2 yields

$$\frac{d}{dt} \|w(t)\|_{B^0_{l,1}}^l + C \nu \|A^{\alpha/l} w\|_{B^0_{l,1}} - \left(\frac{2}{C \nu}\right)^{l-1} \|A^{-\alpha+1/l} f\|_{B^0_{l,1}}^l \leq$$

$$-l \sum_{q \geq -1} \int_{\mathbb{R}^3} \Delta_q (R^+ w \cdot \nabla \theta_2) w_q |w_q|^{-2} \, dx \, d\tau$$

$$-l \sum_{q \geq -1} \int_{\mathbb{R}^3} \Delta_q (u_1 \cdot \nabla w) w_q |w_q|^{-2} \, dx \, d\tau$$

$$= I + J,$$

for some absolute constant $C$. Using Bony’s paraproduct formula, $I$ is decomposed as

$$I = -l \sum_{q \geq -1} \sum_{|p| \leq 2} \int_{\mathbb{R}^3} \Delta_q (R^+ w_{\leq p-2} \cdot \nabla (\theta_2)_p) w_q |w_q|^{-2} \, dx$$

$$-l \sum_{q \geq -1} \sum_{|p| \leq 2} \int_{\mathbb{R}^3} \Delta_q (R^+ w_{\leq p-2} \cdot \nabla (\theta_2)_p) w_q |w_q|^{-2} \, dx$$

$$-l \sum_{q \geq -1} \sum_{p \geq q-2} \int_{\mathbb{R}^3} \Delta_q (R^+ w_{\leq p-2} \cdot \nabla (\theta_2)_p) w_q |w_q|^{-2} \, dx$$

$$= I_1 + I_2 + I_3.$$

Recall that $A = 2^Q/L$. To estimate $I_1$ we use Hölder’s inequality and split it as follows:

$$|I_1| \leq l \sum_{q \geq -1} \sum_{|p| \leq 2} \int_{\mathbb{R}^3} |\Delta_q (R^+ w_{\leq p-2} \cdot \nabla (\theta_2)_p) w_q| |w_q|^{-2} \, dx$$

$$\lesssim l \sum_{q > Q} \|w_q\|^{-1} |l_{i} \sum_{|q-p| \leq 2} \lambda_p \| (\theta_2)_p \|_{\infty} \sum_{Q < p' \leq p-2} \| R^+ w_{p'} \|_{l}$$

$$+ l \sum_{q > Q} \|w_q\|^{-1} \sum_{|q-p| \leq 2} \lambda_p \| (\theta_2)_p \|_{\infty} \| R^+ w_{\leq Q} \|_{l}$$

$$+ l \sum_{q < Q} \|w_q\|^{-1} \sum_{|q-p| \leq 2} \lambda_p \| (\theta_2)_p \|_{\infty} \| R^+ w_{\leq p-2} \|_{l}$$

$$\equiv I_{11} + I_{12} + I_{13}.$$
Then using Young’s inequality, Jensen’s inequality and the fact that \( \| R^l w_q \|_l \lesssim \| w_q \|_l \), we obtain

\[
|I_{11}| \lesssim R_\infty l \left( \sum \sum \lambda_p \sum_{|q-p| \leq 2} \| w_q \|_l^{l-1} \| R^l w_{p'} \|_l \right) \\
\lesssim R_\infty l \left( \sum \sum \lambda_p \| w_p \|_l^{l-1} \| R^l w_{p'} \|_l \right) \\
\lesssim A^{1-\alpha+\frac{\alpha}{n}} R_\infty l \left( \sum \sum \lambda_p^{\alpha(l-1)} \| w_p \|_l^{l-1} \| R^l w_{p'} \|_l \right) \\
\lesssim A^{1-\alpha} R_\infty l \left( \sum \sum \lambda_p^{\alpha(l-1)} \| w_p \|_l^{l-1} \| R^l w_{p'} \|_l \right) \\
\lesssim A^{1-\alpha} R_\infty l \left( \sum \sum \lambda_p^{\alpha(l-1)} \| w_p \|_l^{l-1} \| R^l w_{p'} \|_l \right)
\]

where we needed \( 1 - \alpha + \frac{\alpha}{n} < 0 \), i.e., \( l > \alpha/(\alpha - 1) \). Now we take small enough \( \epsilon > 0 \), such that \( 1 - \alpha + \frac{\alpha}{n} + \epsilon < 0 \), and use Hölder’s inequality, Young’s inequality, and Jensen’s inequality to infer

\[
I_{12} \lesssim R_\infty l \left( \sum \sum \lambda_p \| R^l w_{\leq Q} \|_l \right) \\
\lesssim R_\infty l \left( \sum \sum \lambda_q \| w_q \|_l^{l-1} \| R^l w_{\leq Q} \|_l \right) \\
\lesssim R_\infty l \left( \sum \sum \lambda_q^{1-\alpha+\frac{\alpha}{n}+\epsilon} \| w_q \|_l^{l-1} \| R^l w_{\leq Q} \|_l \right) \\
\lesssim A^{1-\alpha+\frac{\alpha}{n}+\epsilon} R_\infty l \left( \sum \lambda_q^{\alpha l^{-1}} \| w_q \|_l^{l-1} \| R^l w_{\leq Q} \|_l \right) \\
\lesssim A^{1-\alpha+\frac{\alpha}{n}+\epsilon} R_\infty l \left( \sum \lambda_q^{\alpha l^{-1}} \| w_q \|_l^{l-1} \| R^l w_{\leq Q} \|_l \right) \\
\lesssim A^{1-\alpha} \frac{R_\infty l}{|l|} \left( \sum \sum \lambda_q^{\alpha l^{-1}} \| w_q \|_l^{l-1} \| R^l w_{\leq Q} \|_l \right) \\
\lesssim A^{1-\alpha} R_\infty l \left( \sum \lambda_q^{\alpha l^{-1}} \| w_q \|_l^{l-1} \| R^l w_{\leq Q} \|_l \right) \\
\lesssim A^{1-\alpha} R_\infty l \left( \sum \sum \lambda_p \| R^l w_{\leq Q} \|_l \right) \\
\lesssim A^{1-\alpha} R_\infty l \left( \sum \lambda_p \| w_q \|_l + A^{1+\epsilon} R_\infty l \| w_{\leq Q} \|_l \right)
\]
and similarly,

\[ I_{13} \lesssim R_\infty l \sum_{q \leq Q} \| w_q \|_l^{l-1} \sum_{|q-p| \leq 2} \lambda_p \| R^l w \|_{l-2} \]

\[ \lesssim R_\infty l \sum_{q \leq Q} \lambda_q \| w_q \|_l^{l-1} \| R^l w \|_l \]

\[ \lesssim R_\infty l \sum_{q \leq Q} \lambda_q^{1+\epsilon} \lambda_q^{-\alpha(l-1)} \| w_q \|_l^{l-1} \| R^l w \|_l \]

\[ \lesssim A^{1-\alpha} R_\infty l \sum_{q \leq Q} \lambda_q^{-\alpha(l-1)} \| w_q \|_l^{l-1} A^{\alpha-1} \| R^l w \|_l \]

\[ \lesssim A^{1-\alpha} R_\infty l \left( \sum_{q \leq Q} \lambda_q^{-\alpha(l-1)} \| w_q \|_l^{l-1} \right) + A^{(l-1)(\alpha-1)} R_\infty l \| R^l w \|_l^l \]

\[ \lesssim A^{1-\alpha} R_\infty l \sum_{q \leq Q} \lambda_q^{\alpha} \| w_q \|_l^l + A^{(l-1)(\alpha-1)} R_\infty l \| w \|_l^l \]

For \( I_2 \), splitting the summation and using Hölder’s inequality, we obtain

\[ |I_2| \lesssim l \sum_{n \geq 1} \sum_{|q-p| \leq 2} \int_{\mathbb{R}^3} | \Delta_q (R^l w_p \cdot \nabla (\theta_2)_{p-2}) w_q | \| w_q \|_{l-2} dx \]

\[ \lesssim l \sum_{q \geq Q-2} \sum_{|q-p| \leq 2} \sum_{p' \leq p-2} \lambda_{p'} \| (\theta_2)_{p'} \|_{\infty} \| R^l w_p \|_l \| w_q \|_{l-1} \]

\[ + l \sum_{q \leq Q-2} \sum_{|q-p| \leq 2} \sum_{p' \leq p-2} \lambda_{p'} \| (\theta_2)_{p'} \|_{\infty} \| R^l w_p \|_l \| w_q \|_{l-1} \]

\[ \equiv I_{21} + I_{22} \]

The first term is estimated as

\[ I_{21} \lesssim R_\infty l \sum_{p \geq Q-4} \| w_p \|_l^l \sum_{p' \leq p-2} \lambda_{p'} \]

\[ \lesssim R_\infty l \sum_{p \geq Q-4} \lambda_p^\alpha \| w_p \|_l^l \sum_{p' \leq p-2} \lambda_{p'} \lambda_{1-\alpha} \]

\[ \lesssim A^{1-\alpha} R_\infty l \sum_{p \geq Q-4} \lambda_p^\alpha \| w_p \|_l^l \]

For the second term we have

\[ I_{22} \lesssim R_\infty l \sum_{q \leq Q-2} \sum_{|q-p| \leq 2} \sum_{p' \leq p-2} \lambda_{p'} \| R^l w_p \|_l \| w_q \|_{l-1} \]

\[ \lesssim R_\infty l \sum_{q \leq Q} \| w_q \|_l^l \sum_{p' \leq Q} \lambda_{p'} \]

\[ \lesssim A R_\infty l \sum_{q \leq Q} \| w_q \|_l^l \]
To estimate $I_3$, we first integrate by parts and then use Hölder’s inequality obtaining

$$|I_3| \leq l \sum_{q \geq -1} \sum_{p \geq q-2} \int_{\mathbb{R}^3} |\Delta_q (R^l \tilde{w}_p(\theta_2)_p) \nabla (w_q |w_q|^{l-2})| \, dx$$

$$\lesssim l^2 \sum_{q \geq -1} \sum_{p \geq q-2} \int_{\mathbb{R}^3} |\Delta_q (R^l \tilde{w}_p(\theta_2)_p) \nabla |w_q| |^{l-2} | \, dx$$

$$\lesssim l^2 \sum_{q > Q} \lambda_q \|w_q\|^{l-1} \sum_{p \geq q-2} \|R^l \tilde{w}_p\|_l \|\theta_2\|_p \|\theta_2\|_\infty$$

$$+ l^2 \sum_{q \leq Q} \lambda_q \|w_q\|^{l-1} \sum_{p \geq q-2} \|R^l \tilde{w}_p\|_l \|\theta_2\|_p \|\theta_2\|_\infty$$

$$\equiv I_{31} + I_{32}.$$

For the first term we use Jensen’s inequality:

$$I_{31} \lesssim R_{\infty}^2 \sum_{p > Q-3} \|R^l \tilde{w}_p\|_l \sum_{q < p \leq p+2} \lambda_q \|w_q\|^{l-1}$$

$$\lesssim R_{\infty}^2 \sum_{p > Q-3} \lambda_{\tilde{p}} \|w_p\|_l \sum_{q < p \leq p+2} \lambda_q^{(l-1)} \|w_q\|^{l-1} \lambda_q^{\tilde{p}} \lambda_{\tilde{p}}$$

$$\lesssim A^{1-\alpha} R_{\infty}^2 \sum_{q > Q-3} \lambda_q \|w_q\|_l.$$

For the second term, Hölder’s inequality, Young’s inequality, and Jensen’s inequality yield

$$I_{32} \lesssim R_{\infty}^2 \sum_{q \leq Q} \lambda_q \|w_q\|_l \sum_{p \geq q-2} \|R^l \tilde{w}_p\|_l$$

$$\lesssim R_{\infty}^2 \sum_{q \leq Q} \|w_q\|_l \sum_{p \geq q-2} \lambda_{\tilde{p}} \|R^l \tilde{w}_p\|_l \lambda^{-\tilde{p}} \lambda_{\tilde{p}}$$

$$\lesssim A^{1-\tilde{p}} R_{\infty}^2 \sum_{q \leq Q} \|w_q\|_l \sum_{p \geq q-2} \lambda_{\tilde{p}} \|R^l \tilde{w}_p\|_l \lambda^{-\tilde{p}}$$

$$\lesssim AR_{\infty}^2 \sum_{q \leq Q} \left( \|w_q\|_l + A^{-\alpha} \left( \sum_{p \geq q-2} \lambda_{\tilde{p}} \|R^l \tilde{w}_p\|_l \lambda^{-\tilde{p}} \right) \right)^{\frac{1}{l}}$$

$$\lesssim AR_{\infty}^2 \sum_{q \leq Q} \|w_q\|_l + A^{1-\alpha} R_{\infty}^2 \sum_{q \leq Q} \left( \sum_{p \geq q-2} \lambda_{\tilde{p}} \|R^l \tilde{w}_p\|_l \lambda^{-\tilde{p}} \right)^{\frac{1}{l}}$$

$$\lesssim A^{1-\alpha} R_{\infty}^2 \sum_{q \geq -1} \lambda_q \|w_q\|_l + AR_{\infty}^2 \sum_{q \leq Q} \|w_q\|_l.$$

Therefore, for $l$ such that $1 - \alpha + \tilde{p} < 0$ we have

$$|I| \lesssim A^{1-\alpha} R_{\infty}^2 \sum_{q \geq -1} \lambda_q \|w_q\|_l + \left( A^{(l-1)(\alpha-1)} + A^{1+\epsilon l} \right) R_{\infty}^2 \sum_{q \leq Q} \|w_q\|_l$$

$$\lesssim A^{1-\alpha} R_{\infty}^2 \sum_{q \geq -1} \lambda_q \|w_q\|_l + A^{(l-1)(\alpha-1)} R_{\infty}^2 \sum_{q \leq Q} \|w_q\|_l,$$

where $\epsilon$ is chosen small enough so that $1 - \alpha + \tilde{p} + \epsilon < 0$ and hence $(l-1)(\alpha-1) > 1 + \epsilon l$. 

(4.3)
We now estimate $J$, where we first apply Bony’s paraproduct formula:

$$J = \sum_{q \geq -1} \sum_{|q-p| \leq 2} \int_{\mathbb{R}^3} \Delta_q((u_1)_{\leq p-2} \cdot \nabla w_p) w_q |w_q|^{l-2} \, dx$$

$$- \sum_{q \geq -1} \sum_{|q-p| \leq 2} \int_{\mathbb{R}^3} \Delta_q((u_1)_p \cdot \nabla w_{\leq p-2}) w_q |w_q|^{l-2} \, dx$$

$$- \sum_{q \geq -1} \sum_{q \geq -2} \int_{\mathbb{R}^3} \Delta_q((u_1)_p \cdot \nabla \tilde{w}_p) w_q |w_q|^{l-2} \, dx$$

$$= J_1 + J_2 + J_3.$$ 

Observing that $\sum_{|p-q| \leq 2} \Delta_q w_p = w_q$, we then decompose $J_1$ using the commutator notation (2.2):

$$J_1 = \sum_{q \geq -1} \sum_{|q-p| \leq 2} \int_{\mathbb{R}^3} [\Delta_q, (u_1)_{\leq p-2} \cdot \nabla] w_p w_q |w_q|^{l-2} \, dx$$

$$- \sum_{q \geq -1} \int_{\mathbb{R}^3} (u_1)_{\leq q-2} \cdot \nabla w_q w_q |w_q|^{l-2} \, dx$$

$$- \sum_{q \geq -1} \sum_{|q-p| \leq 2} \int_{\mathbb{R}^3} ((u_1)_{\leq p-2} - (u_1)_{\leq q-2}) \cdot \nabla \Delta_q w_p w_q |w_q|^{l-2} \, dx$$

$$= J_{11} + J_{12} + J_{13}.$$ 

The term $J_{12}$ vanishes because $\text{div} (u_1)_{\leq q-2} = 0$. To estimate $J_{11}$ we will use (2.3),

$$\|[[\Delta_q, (u_1)_{\leq p-2} \cdot \nabla] w_p]_l \| \lesssim \|\nabla (u_1)_{\leq p-2}\|_\infty \|w_p\|_l.$$ 

Then splitting the summation we get

$$|J_{11}| \leq \sum_{q \geq -1} \sum_{|q-p| \leq 2} \|[[\Delta_q, (u_1)_{\leq p-2} \cdot \nabla] w_p]_l \|_l |w_q|^{l-1}$$

$$\leq \sum_{q \geq Q-2} \sum_{|q-p| \leq 2} \|\nabla (u_1)_{\leq p-2}\|_\infty \|w_p\|_l \|w_q\|_l^{l-1}$$

$$+ \sum_{q \leq Q-2} \sum_{|q-p| \leq 2} \|\nabla (u_1)_{\leq p-2}\|_\infty \|w_p\|_l \|w_q\|_l^{l-1}$$

$$\equiv J_{111} + J_{112}.$$
Now note that \( \| (u_1)_q \|_\infty \lesssim \| (\theta_1)_q \|_\infty \leq R_\infty \). So using Hölder’s and Bernstein’s inequalities, we obtain

\[
J_{111} \lesssim l \sum_{q > Q-2} \sum_{|q-p| \leq 2} \sum_{p' \leq q} \lambda_{p'} \| (u_1)_{p'} \|_\infty \| w_p \|_l \| w_q \|_{l-1}^{-1}
\]
\[
\lesssim R_\infty l \sum_{q > Q-2} \sum_{|q-p| \leq 2} \sum_{p' \leq q} \lambda_{p'} \| w_p \|_l \| w_q \|_{l-1}^{-1}
\]
\[
\lesssim R_\infty l \sum_{q > Q-2} \sum_{|q-p| \leq 2} \sum_{p' \leq q} \lambda_{p'} \| w_q \|_l^l
\]
\[
\lesssim R_\infty l \sum_{q > Q-2} \sum_{|q-p| \leq 2} \sum_{p' \leq q} \lambda_{p'} \| w_q \|_l^l \lambda^{1-\alpha}_q
\]
\[
\lesssim A^{1-\alpha} R_\infty l \sum_{q > Q-2} \lambda^{\alpha}_q \| w_q \|_l^l.
\]

Similarly,

\[
J_{112} \lesssim l \sum_{q \leq Q-2} \sum_{|q-p| \leq 2} \sum_{p' \leq p-2} \lambda_{p'} \| (u_1)_{p'} \|_\infty \| w_p \|_l \| w_q \|_{l-1}^{-1}
\]
\[
\lesssim R_\infty l \sum_{q \leq Q-2} \sum_{|q-p| \leq 2} \sum_{p' \leq q} \lambda_{p'} \| w_p \|_l \| w_q \|_{l-1}^{-1}
\]
\[
\lesssim R_\infty l \sum_{q \leq Q} \sum_{p' \leq Q-2} \lambda_{p'} \| w_q \|_l^l
\]
\[
\lesssim A R_\infty l \sum_{q \leq Q} \| w_q \|_l^l.
\]

To estimate \( J_{13} \), we first use Hölder’s inequality and split the summation as follows:

\[
|J_{13}| \leq l \sum_{q \geq -1} \sum_{|q-p| \leq 2} \int_{\mathbb{R}^3} \left|((u_1)_{p-2} - (u_1)_{q-2}) \cdot \nabla \Delta_q w_p\right| |w_q|^{l-1} dx
\]
\[
\lesssim l \sum_{q > Q-2} \sum_{|q-p| \leq 2} \sum_{q-3 \leq p' \leq q} \int_{\mathbb{R}^3} \left|((u_1)_{p'}) \nabla \Delta_q w_p \right| |w_q|^{l-1} dx
\]
\[
+ l \sum_{q \leq Q-2} \sum_{|q-p| \leq 2} \sum_{q-3 \leq p' \leq q} \int_{\mathbb{R}^3} \left|((u_1)_{p'}) \nabla \Delta_q w_p \right| |w_q|^{l-1} dx
\]
\[
= J_{131} + J_{132}.
\]

Now Jensen’s inequality yields

\[
J_{131} \lesssim l \sum_{q > Q-2} \| w_q \|_l^{l-1} \sum_{|q-p| \leq 2} \sum_{q-3 \leq p' \leq q} \lambda_p \| w_p \|_l \| (u_1)_{p'} \|_\infty
\]
\[
\lesssim R_\infty l \sum_{q > Q-2} \| w_q \|_l^{l-1} \sum_{|q-p| \leq 2} \lambda_p \| w_p \|_l
\]
\[
\lesssim R_\infty l \sum_{q > Q-2} \lambda_q^{\frac{(l-1)}{l}} \| w_q \|_l^{l-1} \sum_{|q-p| \leq 2} \lambda_p^{\frac{l}{p}} \| w_p \|_l \| (u_1)_{p'} \|_\infty \lambda^{1-\alpha}_q
\]
\[
\lesssim A^{1-\alpha} R_\infty l \sum_{q > Q-4} \lambda^{\alpha}_q \| w_q \|_l^l.
\]
And similarly, for the second term,

\[ J_{1,2} \lesssim l \sum_{q \leq Q} \| w_q \|_{q-1} \sum_{|q-p| \leq 2} \lambda_p \| w_p \|_t \sum_{q \leq p' \leq q} \| (u_1)_{p'} \|_\infty \]

\[ \lesssim R_\infty l \sum_{q \leq Q} \| w_q \|_{q-1} \sum_{|q-p| \leq 2} \lambda_p \| w_p \|_t \]

\[ \lesssim R_\infty l \sum_{q \leq Q} \lambda_q \| w_q \|_t \]

\[ \lesssim A R_\infty l \sum_{q \leq Q} \| w_q \|_t. \]

For \( J_2 \) we use Hölder’s inequality obtaining

\[ |J_2| \leq l \sum_{q \geq 1} \sum_{|q-p| \leq 2} \int_{\mathbb{R}^3} |\Delta_q((u_1)_p \cdot \nabla w_{\leq p-2})| |w_q|^{t-1} \, dx \]

\[ \lesssim l \sum_{q \geq Q} \| w_q \|_{q-1} \sum_{|q-p| \leq 2} \| (u_1)_p \|_\infty \sum_{p' \leq p-2} \lambda_{p'} \| w_{p'} \|_t \]

\[ + l \sum_{q \leq Q} \| w_q \|_{q-1} \sum_{|q-p| \leq 2} \| (u_1)_p \|_\infty \sum_{p' \leq p-2} \lambda_{p'} \| w_{p'} \|_t \]

\[ \equiv J_{21} + J_{22}. \]

Recall that \( \| (u_1)_q \|_\infty \lesssim \| (\theta_1)_q \|_\infty \leq R_\infty \). Hence we can use Jensen’s inequality to deduce that

\[ J_{21} \lesssim R_\infty l \sum_{q > Q} \| w_q \|_{q-1} \sum_{p' \leq q} \lambda_{p'} \| w_{p'} \|_t \]

\[ \lesssim R_\infty l \sum_{q > Q} \lambda_{q^{q-1}} \| w_q \|_{q-1} \sum_{p' \leq q} \lambda_{p'} \| w_{p'} \|_t \lambda_{p'-q}^{1-\alpha} \]

\[ \lesssim A^{1-\alpha} R_\infty l \sum_{q > Q} \lambda_q \| w_q \|_t, \]

where we needed \( l > \alpha \). While the second term is estimated as

\[ J_{22} \lesssim R_\infty l \sum_{q \leq Q} \| w_q \|_{q-1} \sum_{p' \leq q} \lambda_{p'} \| w_{p'} \|_t \]

\[ \lesssim A R_\infty l \sum_{q \leq Q} \| w_q \|_{q-1} \sum_{p' \leq Q} \| w_{p'} \|_t \]

\[ \lesssim A Q R_\infty l \sum_{q \leq Q} \| w_q \|_t. \]

Since \( \| (u_1)_q \|_r \leq \| (\theta_1)_q \|_r \), for any \( r \in (1, \infty) \), the term \( J_3 \) enjoys the same estimate as \( I_3 \). Hence we conclude that

\[ (4.4) \quad |J| \lesssim A^{1-\alpha} R_\infty l^2 \sum_{q \geq -1} \lambda_q \| w_q \|_t^r + A Q R_\infty l^2 \sum_{q \leq Q} \| w_q \|_t. \]
Thanks to (4.2)–(4.4), inequality (4.2) yields
\[
\frac{d}{dt}\|w(t)\|_{B^0_{1,1}} \leq -C\nu\|\Lambda^{\alpha/2}w\|_{B^0_{1,1}} + C_1 A^{1-\alpha} R_\infty l^2 \sum_{q \leq Q} \lambda_q^\alpha \|w_q\|_1
\]
\[
+ \left( \frac{2}{C\nu} \right)^{l-1} l^{l-2} \|\Lambda^{-\alpha(1-\frac{1}{l})}f\|_{B^0_{1,1}} + C_2 A^{(l-1)(\alpha-1)} R_\infty l^2 \sum_{q \leq Q} \|w_q\|_1,
\]
for some absolute constants \(C, C_1,\) and \(C_2.\) Thus we have
\[
\frac{d}{dt}\|w(t)\|_{B^0_{1,1}} + \phi\|w(t)\|_{B^0_{1,1}} \leq \psi(t),
\]
where
\[
\phi = \frac{1}{2}(2\pi\lambda_0)^\alpha C\nu,
\]
\[
\psi(t) = \left( \frac{2}{C\nu} \right)^{l-1} l^{l-2} \|\Lambda^{-\alpha(1-\frac{1}{l})}f\|_{B^0_{1,1}} + C_2 A^{(l-1)(\alpha-1)} R_\infty l^2 \sum_{q \leq Q} \|w_q\|_1,
\]
provided
\[
A = \left( \frac{2C_1 l^2}{C\nu R_\infty} \right)^{\frac{1}{\alpha-1}}.
\]
Note that
\[
\psi(t) \to 0 \quad \text{as} \quad t \to \infty,
\]
due to the assumption of the theorem. Since also \(\alpha > 0,\) the first part of the theorem follows from Lemma 4.1.

To prove the second part, where \(\psi(t) = 0\) for all \(t < 0,\) we note that
\[
\|w(t)\|_{B^0_{1,1}} \leq \|w(t_0)\|_{B^0_{1,1}} e^{-\alpha(t-t_0)}, \quad t_0 \leq t \leq 0,
\]
thanks to Grönwall’s inequality. Since \(\theta_1\) and \(\theta_2\) are on the global attractor, we have
\[
\|w(t)\|_{B^0_{1,1}} \lesssim \|w(t)\|_{B^0_{1,1}}^{1-\frac{\theta_2}{\theta_1}} \|w(t)\|_2^{\frac{\theta_2}{\theta_1}}
\]
\[
\lesssim R_\infty^{1-\frac{\theta_2}{\theta_1}} R_2^{\frac{\theta_2}{\theta_1}},
\]
for all \(t.\) Taking the limit as \(t_0 \to -\infty\) gives \(w(t) = 0\) for all \(t \leq 0,\) and hence \(w \equiv 0.\)

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