Explicit convex hull description of bivariate quadratic sets with indicator variables

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MIQCPs with indicator variables

- Consider a mixed-integer quadratically constrained optimization problem (MIQCP) with indicator variables of the form:

\[
\begin{align*}
\min & \quad x^T Q_0 x + c_0^T x \\
\text{s.t.} & \quad x_i (1 - z_i) = 0, \quad \forall i \in \{1, \ldots, n\} \\
& \quad x^T Q_j x + c_j^T x \leq b_j, \quad \forall j \in \{1, \ldots, m\} \\
& \quad Ax \leq \alpha, \quad Bz \leq \beta \\
& \quad x \in \mathbb{R}^n, \quad z \in \{0, 1\}^n.
\end{align*}
\]

- The binary variables \( z \) are often referred to as indicator variables.

- Problem \( \text{(QPI)} \) subsumes several classes of difficult optimization problems such as best subset selection, constrained portfolio optimization, optimal power flow with transmission switching, among others.

- We are not assuming that \( Q_j \) is positive semi-definite (PSD).

- We are interested in the quality of convex relaxations for Problem \( \text{(QPI)} \).
Convexification of MIQCPs with indicator variables

- Introduce auxiliary variables $X_{ij} := x_ix_j$ for all $i, j \in [n]$ to obtain a reformulation of Problem (QPI):

\[
\begin{align*}
\min & \quad \langle Q_0, X \rangle + c_0^T x \\
\text{s.t.} & \quad x_i(1 - z_i) = 0, \quad \forall i \in [n] \\
& \quad X = xx^T \\
& \quad \langle Q_j, X \rangle + c_j^T x \leq b_j, \quad \forall j \in [m] \\
& \quad Ax \leq \alpha, \ Bz \leq \beta \\
& \quad x \in \mathbb{R}^n, \ z \in \{0, 1\}^n, \ X \in S_n^+, \\
\end{align*}
\]

where $S_n^+$ denotes the set of $n \times n$ PSD matrices.

- **Assumption:** $Ax \leq \alpha$ implies $x \geq 0$.

- To construct strong convex relaxations for Problem (QPI'), we should effectively convexify the set:

\[
S_n = \left\{ (x, X, z) : \ X = xx^T, \ x(1 - z) = 0, \ x \geq 0, \ z \in \{0, 1\}^n \right\}
\]
**Convexification of \( S_n \)**

- The closure of the convex hull of \( S_1 = \{(x_1, X_{11}, z_1) : X_{11} = x_1^2, x_1(1 - z_1) = 0, x_1 \geq 0, z_1 \in \{0, 1\}\} \), is (Akturk et al 2009, Gunluk and Linderoth 2010):

  \[
  \overline{\text{conv}}(S_1) = \left\{ (x, X, z) : X_{11}z_1 \geq x_1^2, x_1 \geq 0, X_{11} \geq 0, z_1 \in [0, 1] \right\}.
  \]

- The Perspective relaxation of Problem \((\text{QPI}')\):

  \[
  \begin{aligned}
  \min & \quad \langle Q_0, X \rangle + c^T_0 x \\
  \text{s.t.} & \quad X \succeq xx^T \\
  & \quad X_{ii}z_i \geq x_i^2, \quad \forall i \in [n] \\
  & \quad \langle Q_j, X \rangle + c^T_j x \leq b_j, \quad \forall j \in [m] \\
  & \quad Ax \leq \alpha, Bz \leq \beta, \ x \in \mathbb{R}^n, \ z \in [0, 1]^n, \ X \in S^n_+.
  \end{aligned}
  \]

  (Persp)

- Very effective for problems with \( m = 0, Q_0 \in S^n_+ \) such that \( Q_0 \) is diagonal or diagonally dominant (Frangioni and Gentile 2006, Gunluk and Linderoth 2010, Dong et al 2015).
Convexification of $S_n$

- To improve the quality of convex relaxations for Problem (QP1'), it is natural to study the facial structure of $\text{conv}(S_2)$:

$$S_2 := \left\{ (x, X, z) : X_{11} = x_1^2, X_{12} = x_1 x_2, X_{22} = x_2^2, x_1 (1 - z_1) = 0, x_2 (1 - z_2) = 0, x_1, x_2 \geq 0, z_1, z_2 \in \{0, 1\} \right\}.$$ 

- Anstreicher and Burer (Mathematical Programming, 2021): $S_2$ with additional constraints $x_1, x_2 \leq 1$; an extended formulation, containing three additional variables, for the convex hull that is SDP representable.

- Convex hull of the epigraph of bivariate convex quadratic functions:
  - Atamturk et al (JMLR, 2021): $Z^-_2 := \left\{ (x, t, z) : t \geq d_1 x_1^2 - 2x_1 x_2 + d_2 x_2^2, x_i (1 - z_i) = 0, x_i \geq 0, z_i \in \{0, 1\}, i \in \{1, 2\} \right\}$ with $d_1, d_2 > 0, d_1 d_2 \geq 1$, 
  - Han et al (Mathematical Programming, 2023): $Z^+_2 := \left\{ (x, t, z) : t \geq d_1 x_1^2 + 2x_1 x_2 + d_2 x_2^2, x_i (1 - z_i) = 0, x_i \geq 0, z_i \in \{0, 1\}, i \in \{1, 2\} \right\}$ with $d_1, d_2 > 0, d_1 d_2 \geq 1$. Also propose an extended SDP relaxation.

- More interesting results on convex quadratic optimization with indicator variables (Atamturk and Gomez 2019, Wei et al 2021, Wei et al 2022).
We give an explicit characterization for the closure of the convex hull of $S_2$ in the space of original variables.

- A De Rosa and A Khajavirad, Explicit convex hull description of bivariate quadratic sets with indicator variables, arXiv:2208.08703, 2022.
Constructing the convex hull of $S_2$

- We can construct the convex hull of $S_2$ using

$$\overline{\text{conv}}(S_2) = \overline{\text{conv}}\left(\overline{\text{conv}}(P_1) \cup \overline{\text{conv}}(P_2) \cup \overline{\text{conv}}(P_3) \cup \overline{\text{conv}}(P_4)\right),$$

where

- $P_1 := \{(x, X, z) : z_1 = z_2 = 0, x_1 = x_2 = X_{11} = X_{12} = X_{22} = 0\}$,
- $P_2 := \{(x, X, z) : z_1 = 1, z_2 = 0, X_{11} = x_1^2, x_2 = X_{12} = X_{22} = 0, x_1 \geq 0\}$,
- $P_3 := \{(x, X, z) : z_1 = 0, z_2 = 1, x_1 = X_{11} = X_{12} = 0, X_{22} = x_2^2, x_2 \geq 0\}$,
- $P_4 := \{(x, X, z) : z_1 = z_2 = 1, X_{11} = x_1^2, X_{12} = x_1 x_2, X_{22} = x_2^2, x_1, x_2 \geq 0\}$.

and

- $\overline{\text{conv}}(P_1) = \{(x, X, z) : z_1 = z_2 = 0, x_1 = x_2 = X_{11} = X_{12} = X_{22} = 0\}$,
- $\overline{\text{conv}}(P_2) = \{(x, X, z) : z_1 = 1, z_2 = 0, X_{11} \geq x_1^2, x_2 = X_{12} = X_{22} = 0, x_1 \geq 0\}$,
- $\overline{\text{conv}}(P_3) = \{(x, X, z) : z_1 = 0, z_2 = 1, x_1 = X_{11} = X_{12} = 0, X_{22} \geq x_2^2, x_2 \geq 0\}$,
- $\overline{\text{conv}}(P_4) = \{(x, X, z) : z_1 = z_2 = 1, X_{11} \geq x_1^2, X_{22} \geq x_2^2,
\quad (X_{11} - x_1^2)(X_{22} - x_2^2) \geq (X_{12} - x_1 x_2)^2, x_1 \geq 0, x_2 \geq 0, X_{12} \geq 0\}.$
A convex extended formulation for $\overline{\text{conv}}(S_2)$

- Using disjunctive programming:

$$
\overline{\text{conv}}(S_2) = \text{cl}\left\{ (x, X, z) : \exists (x, X, z, \tilde{x}, \tilde{X}, \lambda) \in \Sigma \right\},
$$

where $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_4)$, $\tilde{X} = (\tilde{X}_1, \ldots, \tilde{X}_4)$, with $\tilde{x}^i = (\tilde{x}_1^i, \tilde{x}_2^i) \in \mathbb{R}^2$ and $\tilde{X}^i = (\tilde{X}_{11}^i, \tilde{X}_{12}^i, \tilde{X}_{22}^i) \in \mathbb{R}^3$ for all $i \in \{1, \ldots, 4\}$, and

$$
\Sigma := \left\{ (x, X, z, \tilde{x}, \tilde{X}, \lambda) : \\
x = \sum_{i=1}^{4} \tilde{x}^i, \quad X = \sum_{i=1}^{4} \tilde{X}^i, \quad z = \sum_{i=1}^{4} \lambda_i z^i, \quad \sum_{i=1}^{4} \lambda_i = 1, \quad \lambda_i \geq 0 \ \forall i \\
\left(\frac{\tilde{x}^i}{\lambda_i}, \frac{\tilde{X}^i}{\lambda_i}, z^i\right) \in \overline{\text{conv}}(P_i) \text{ if } \lambda_i > 0, \quad \tilde{x}^i = \tilde{X}^i = 0 \text{ if } \lambda_i = 0 \right\}.
$$
A convex extended formulation for $\overline{\text{conv}}(S_2)$

- Using disjunctive programming:

$$\overline{\text{conv}}(S_2) = \text{cl}\left\{(x, X, z) : \exists (x, X, z, \tilde{x}, \tilde{X}, \lambda) \in \Sigma\right\},$$

where

$$\Sigma = \left\{(x, X, z, \tilde{x}, \tilde{X}, \lambda) : \begin{array}{l}
\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1, \lambda_i \geq 0 \forall i \in \{1, \ldots, 4\}, z_1 = \lambda_2 + \lambda_4, z_2 = \lambda_3 + \lambda_4, \\
x_1 = \tilde{x}_1^2 + \tilde{x}_1^4, X_{11} = \tilde{X}_{11}^2 + \tilde{X}_{11}^4, x_2 = \tilde{x}_2^3 + \tilde{x}_2^4, X_{22} = \tilde{X}_{22}^3 + \tilde{X}_{22}^4, X_{12} = \tilde{X}_{12}^4
\end{array} \right\}.$$
A partial projection - first step

- Projecting out $\lambda_1, \lambda_2, \lambda_3, \tilde{x}_1^2, \tilde{x}_2^3, \tilde{X}_{12}^4, \tilde{X}_{11}^2, \tilde{X}_{22}^3$, we obtain:

\[
\begin{align*}
\max\{0, z_1 + z_2 - 1\} & \leq \lambda_4 \leq \min\{z_1, z_2\}, \quad X_{12} \geq 0, \\
\begin{cases}
(z_1 - \lambda_4)(X_{11} - \tilde{X}_{11}^4) \geq (x_1 - \tilde{x}_1^4)^2, & \tilde{x}_1^4 \leq x_1, \quad \text{if } \lambda_4 \neq z_1 \\
\tilde{x}_1^4 = x_1, & \tilde{X}_{11}^4 = X_{11}, \quad \text{otherwise}
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
(z_2 - \lambda_4)(X_{22} - \tilde{X}_{22}^4) \geq (x_2 - \tilde{x}_2^4)^2, & \tilde{x}_2^4 \leq x_2, \quad \text{if } \lambda_4 \neq z_2 \\
\tilde{x}_2^4 = x_2, & \tilde{X}_{22}^4 = X_{22}, \quad \text{otherwise}
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
\lambda_4\tilde{X}_{11}^4 \geq (\tilde{x}_1^4)^2, & \lambda_4\tilde{X}_{22}^4 \geq (\tilde{x}_2^4)^2, \quad (\lambda_4\tilde{X}_{11}^4 - (\tilde{x}_1^4)^2)(\lambda_4\tilde{X}_{22}^4 - (\tilde{x}_2^4)^2) \geq (\lambda_4\tilde{X}_{12}^4 - \tilde{x}_1^4\tilde{x}_2^4)^2, \\
\tilde{x}_1^4 \geq 0, & \tilde{x}_2^4 \geq 0, \tilde{X}_{12}^4 \geq 0, \quad \text{if } \lambda_4 \neq 0 \\
\tilde{x}_1^4 = \tilde{x}_2^4 = \tilde{X}_{11}^4 = \tilde{X}_{12}^4 = \tilde{X}_{22}^4 = 0, \quad \text{otherwise}
\end{cases}
\end{align*}
\]

- Replacing, $\tilde{X}_{11}^4 = X_{11}$ by $\tilde{X}_{11}^4 \leq X_{11}$, $\tilde{X}_{22}^4 = X_{22}$ by $\tilde{X}_{22}^4 \leq X_{22}$, and removing $X_{12} \geq 0$, we obtain the OptPairs relaxation of Han et al (MPA 2023):

\[
\begin{align*}
\begin{pmatrix}
z_1 - \lambda_4 & x_1 - \tilde{x}_1^4 \\
x_1 - \tilde{x}_1^4 & X_{11} - \tilde{X}_{11}^4
\end{pmatrix} \succeq 0, \quad \begin{pmatrix}
z_2 - \lambda_4 & x_2 - \tilde{x}_2^4 \\
x_2 - \tilde{x}_2^4 & X_{22} - \tilde{X}_{22}^4
\end{pmatrix} \succeq 0, \quad \begin{pmatrix}
\lambda_4 & \tilde{x}_1^4 & \tilde{x}_2^4 \\
\tilde{x}_1^4 & \tilde{X}_{11}^4 & \tilde{X}_{12}^4 \\
\tilde{x}_2^4 & \tilde{X}_{12}^4 & \tilde{X}_{22}^4
\end{pmatrix} \succeq 0
\end{align*}
\]

\[
z_1 + z_2 - 1 \leq \lambda_4, 0 \leq \tilde{x}_1^4 \leq x_1, 0 \leq \tilde{x}_2^4 \leq x_2
\]
A partial projection - second step

- Projecting out $\lambda_1, \lambda_2, \lambda_3, \tilde{x}_1^2, \tilde{x}_2^3, \tilde{X}_{12}^4, \tilde{X}_{11}^2, \tilde{X}_{22}^3, \tilde{X}_{11}^4, \tilde{X}_{22}^4$, we obtain:

$$\overline{\text{conv}}(S_2) = \text{cl}\left\{(x, X, z) : \exists (x, X, z, \tilde{x}_1^4, \tilde{x}_2^4, \lambda_4) \in \tilde{\Sigma}\right\},$$

$$\tilde{\Sigma} := \left\{(x, X, z, \tilde{x}_1^4, \tilde{x}_2^4, \lambda_4) : X_{12} > 0, z_1 + z_2 - 1 \leq \lambda_4 \leq \min\{z_1, z_2\}, \lambda_4 > 0, \right.$$  

$$X_{11} - \frac{(\tilde{x}_1^4)^2}{\lambda_4} - \frac{(x_1 - \tilde{x}_1^4)^2}{z_1 - \lambda_4} \geq 0, \quad X_{22} - \frac{(\tilde{x}_2^4)^2}{\lambda_4} - \frac{(x_2 - \tilde{x}_2^4)^2}{z_2 - \lambda_4} \geq 0,$$

$$\left(X_{11} - \frac{(\tilde{x}_1^4)^2}{\lambda_4} - \frac{(x_1 - \tilde{x}_1^4)^2}{z_1 - \lambda_4}\right) \left(X_{22} - \frac{(\tilde{x}_2^4)^2}{\lambda_4} - \frac{(x_2 - \tilde{x}_2^4)^2}{z_2 - \lambda_4}\right) \geq \left(\frac{X_{12}}{\lambda_4} - \frac{\tilde{x}_1^4 \tilde{x}_2^4}{\lambda_4}\right)^2,$$

$$0 \leq \tilde{x}_1^4 \leq x_1, \quad 0 \leq \tilde{x}_2^4 \leq x_2\right\}$$

- Whenever we write a function of the form $f = \frac{u^2}{v}$, $v > 0$, we refer to it closure defined as:

$$\hat{f}(u, v) := \begin{cases} 
\frac{u^2}{v}, & \text{if } v > 0 \\
0, & \text{if } u = v = 0 \\
+\infty, & \text{if } u \neq 0, \ v = 0.
\end{cases}$$
A projection strategy

• Consider a function \( f : (x, y) \in \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \cup \{\infty\} \) and two sets \( \mathcal{G} \subset \mathbb{R}^n \), \( \mathcal{D} \subset \mathbb{R}^n \times \mathbb{R}^m \). Define

\[
Q := \{ x \in \mathcal{G} : \exists y \in \mathbb{R}^m \text{ s.t. } f(x, y) \leq 0, (x, y) \in \mathcal{D} \}.
\]

For every \( x \in \mathcal{G} \), consider the following optimization problem:

\[
\begin{align*}
\min_{(x, y)} & \quad f(x, y) \\
\text{s.t.} & \quad (x, y) \in \mathcal{D}
\end{align*}
\]

and consider a set

\[
\mathcal{E} \subseteq \{ x \in \mathcal{G} : \text{Problem (Proj) admits a minimizer, denoted by } y_x \}.
\]

Then

\[
Q \cap \mathcal{E} = Q_{\mathcal{E}} := \{ x \in \mathcal{E} : f(x, y_x) \leq 0, (x, y_x) \in \mathcal{D} \}.
\]
Projection by convex optimization

• To characterize $\text{conv}(S_2)$, it suffices to solve the following parametric convex optimization problem for all $(x, X, z) \in G \supset \text{conv}(S_2)$:

$$
\begin{align*}
\min_{\tilde{x}_1^4, \tilde{x}_2^4, \lambda_4} & \quad \frac{(\tilde{x}_1^4)^2}{\lambda_4} + \frac{(x_1 - \tilde{x}_1^4)^2}{z_1 - \lambda_4} + \frac{(X_{12} - \frac{\tilde{x}_1^4 \tilde{x}_2^4}{\lambda_4})^2}{X_{22} - \frac{(\tilde{x}_2^4)^2}{\lambda_4} - \frac{(x_2 - \tilde{x}_2^4)^2}{z_2 - \lambda_4}}, \\
\text{s.t.} & \quad X_{22} - \frac{(\tilde{x}_2^4)^2}{\lambda_4} - \frac{(x_2 - \tilde{x}_2^4)^2}{z_2 - \lambda_4} \geq 0, \\
& \quad z_1 + z_2 - 1 \leq \lambda_4 \leq \min\{z_1, z_2\}, \quad \lambda_4 > 0 \\
& \quad 0 \leq \tilde{x}_1^4 \leq x_1, \quad 0 \leq \tilde{x}_2^4 \leq x_2.
\end{align*}
$$

(P_{x,X,z})
The Convex hull of $S_2$

- **Theorem:** There exists a convex set $\tilde{S}$ such that $\text{ri} (\text{conv}(S_2)) \subseteq \tilde{S} \subseteq \overline{\text{conv}}(S_2)$ and:

$$\overline{\text{conv}}(S_2) = \bigcup_{i=1}^{8} \text{cl} (\tilde{S} \cap R_i),$$

where the sets $R_i$, $i \in \{1, \ldots, 8\}$ satisfy $R_i \cap R_j = \emptyset$ for all $i \neq j$ and $\bigcup_{i=1}^{8} R_i \supseteq \overline{\text{conv}}(S_2)$, and

- if $i \in \{1, 2, 6\}$, then

$$\text{cl}(\tilde{S} \cap R_i) = \left\{ (x, X, z) : X_{11} \geq \frac{x_1^2}{z_1}, \ X_{22} \geq \frac{x_2^2}{z_2}, \ x_1, x_2 \geq 0, \ X_{12} \geq 0, \right\} \cap \text{cl}(R_i),$$

where

$R_1 := \left\{ (x, X, z) : x_1x_2(z_1 + z_2 - 1) \leq X_{12}z_1z_2, \ X_{12} \max\{z_1, z_2\} \leq x_1x_2, \ X_{12} > 0, \ z_1, z_2 > 0 \right\}$

$\cup \left\{ (x, X, z) : X_{12} = 0 \right\}$,
The Convex hull of $S_2$

$$\mathcal{R}_2 := \left\{ (x, X, z) : z_1 \leq z_2, X_{12}z_2 > x_1x_2, X_{12}z_1 \leq x_1x_2, X_{12} > 0, z_1 > 0, \right. \left. x_1^2(z_2 - z_1)(X_{22}z_2 - x_2^2) \geq z_1(X_{12}z_2 - x_1x_2)^2 \right\},$$

$$\mathcal{R}_6 := \left\{ (x, X, z) : X_{12}z_1z_2 < x_1x_2(z_1 + z_2 - 1), X_{12} > 0, z_1, z_2 > 0, \right. \left. (1 - z_1)(z_1 + z_2 - 1)x_1^2(X_{22}z_2 - x_2^2) \geq \left( X_{12}z_1z_2 - x_1x_2(z_1 + z_2 - 1) \right)^2 \right\}.$$ 

- if $i \in \{3, 4\}$, then

$$\text{cl}(\tilde{S} \cap \mathcal{R}_i) = \left\{ (x, X, z) : X_{22} \geq \frac{x_2^2}{z_2}, \left( X_{11} - \frac{x_1^2}{z_2} \right)(X_{22} - \frac{x_2^2}{z_2}) \geq \left( X_{12} - \frac{x_1x_2}{z_2} \right)^2, \right. \left. x_1, x_2 \geq 0, X_{12} \geq 0, z_1, z_2 \in [0, 1] \leq 1 \right\} \cap \text{cl}(\mathcal{R}_i),$$

$$\mathcal{R}_3 := \left\{ (x, X, z) : z_1 < z_2, X_{12}x_2 > X_{22}x_1, z_1(X_{12}z_2 - x_1x_2)^2 > x_1^2(z_2 - z_1)(X_{22}z_2 - x_2^2), \right. \left. x_1 \geq 0, X_{12} > 0, z_1 > 0 \right\},$$

$$\mathcal{R}_4 := \left\{ (x, X, z) : z_2 \leq z_1, X_{12}x_2 > X_{22}x_1, x_1 \geq 0, X_{12} > 0, z_2 > 0 \right\}.$$
The Convex hull of $S_2$

\[
\text{cl}(\tilde{S} \cap \mathcal{R}_5) = \left\{ (x, X, z) : X_{22} \geq \frac{x_2^2}{z_2}, \ (X_{11} - \frac{x_1^2}{z_1})(X_{22} - \frac{x_2^2}{z_1}) \geq \left( X_{12} - \frac{x_1 x_2}{z_1} \right)^2, \right. \\
x_1, x_2 \geq 0, \ X_{12} \geq 0, \ z_1, z_2 \in [0, 1] \left. \right\} \cap \text{cl}(\mathcal{R}_i),
\]

where

\[
\mathcal{R}_5 = \{(x, X, z) : X_{12} z_1 > x_1 x_2, \ X_{22} x_1 \geq X_{12} x_2, \ x_2 \geq 0, \ X_{12} \geq 0, \ z_1, z_2 > 0\},
\]

\[
\text{cl}(\tilde{S} \cap \mathcal{R}_7) = \left\{ (x, X, z) : X_{22} \geq \frac{x_2^2}{z_2}, \ (X_{11} - \frac{x_1^2}{z_1})(X_{22} - \frac{x_2^2}{z_1}) \geq \left( X_{12} - x_1 x_2 \right)^2, \right. \\
x_1, x_2 \geq 0, \ X_{12} \geq 0, \ z_1, z_2 \in [0, 1] \left. \right\} \cap \text{cl}(\mathcal{R}_i),
\]

where

\[
\mathcal{R}_7 := \{(x, X, z) : X_{12} z_1 z_2 < x_1 x_2 (z_1 + z_2 - 1), \ X_{12} > 0, \ z_1, z_2 > 0, \ x_1^2(x_2^2 - X_{22}(1 - z_1))(X_{22} z_2 - x_2^2) \right. \\
> 2x_1 x_2 X_{12} z_1 (X_{22} z_2 - x_2^2) - X_{12}^2 (X_{22}(z_1 + z_2 - 1) + x_2^2(1 - 2z_1 - z_2(1 - z_1)))\}.\]
The Convex hull of $S_2$

- $\text{cl}(\tilde{\mathcal{S}} \cap \mathcal{R}_8) =$

\[
\left\{ (x, X, z) : \begin{array}{l}
X_{11} \geq \frac{x_1^2}{z_1}, \quad X_{22} \geq \frac{x_2^2}{z_2}, \quad x_1, x_2 \geq 0, \quad X_{12} \geq 0, \quad z_1, z_2 \in [0, 1], \\
\quad z_1(1 - z_2)\left(X_{11} - \frac{x_1^2}{z_1}\right)x_2^2 \geq (z_1 + z_2 - 1)\left(X_{12} \frac{z_1 z_2}{W} - x_1 x_2\right)^2
\end{array} \right\} \cap \text{cl}(\mathcal{R}_i),
\]

where

\[
W := (z_1 + z_2 - 1) - \frac{1}{x_2} \sqrt{(X_{22}z_2 - x_2^2)(1 - z_1)(z_1 + z_2 - 1)}.
\]

and

\[
\mathcal{R}_8 := \left\{ (x, X, z) : X_{12}z_1z_2 < x_1 x_2(z_1 + z_2 - 1), \quad X_{12} > 0, \quad z_1, z_2 > 0 \right\} \setminus (\mathcal{R}_6 \cup \mathcal{R}_7),
\]
How to generate supporting hyperplanes of \( \text{conv}(S_2) \)?

• Consider an open set \( \mathcal{A} \subset \mathbb{R}^n \), four continuously differentiable functions \( f_1, f_2, h_1, h_2 : \mathcal{A} \to \mathbb{R} \), and two sets \( \mathcal{R}, \mathcal{C} \subseteq \mathcal{A} \) such that \( \mathcal{C} \) is an \( n \)-dimensional convex set and

\[
\mathcal{R} = \left\{ x : f_1(x) > 0, f_2(x) \geq 0 \right\},
\]

\[
\mathcal{C} \cap \mathcal{R} = \left\{ x : h_1(x) \geq 0, h_2(x) \geq 0, f_1(x) > 0, f_2(x) \geq 0 \right\}.
\]

• Consider a point \( \hat{x} \in \mathcal{R} \) such that \( h_1(\hat{x}) = 0, \nabla h_1(\hat{x}) \neq 0, h_2(\hat{x}) > 0, f_1(\hat{x}) > 0, f_2(\hat{x}) > 0 \), and for which there exists \( \hat{r} > 0 \) such that \( B_{\hat{r}}(\hat{x}) \subset \mathcal{A} \) and:

\[
\forall x \in \{ x \in \mathcal{C} \cap \mathcal{R} \cap B_{\hat{r}}(\hat{x}) : h_1(x) = 0 \} \text{ and } \forall s > 0 \text{ it holds } B_s(x) \setminus (\mathcal{C} \cap \mathcal{R}) \neq \emptyset.
\]

Then there exists \( r > 0 \) such that

\[
\{ x \in B_r(\hat{x}) : h_1(x) = 0 \} = \partial \mathcal{C} \cap B_r(\hat{x}).
\]
A simple separation algorithm

• Given a convex set \( C \), we say that the inequality \( q(x) \geq 0 \) supports \( C \) at \( \hat{x} \), or is a supporting inequality for \( C \) at \( \hat{x} \), if \( q(x) \geq 0 \) for all \( x \in C \) and \( \{x : q(x) = 0\} \) is a supporting hyperplane of \( C \) at \( \hat{x} \).

• Let \( \tilde{C} \) be a convex relaxation of \( S_2 \) defined by:

\[
\tilde{C} := \left\{ (x, X, z) : X_{11} \geq \frac{x_1^2}{z_1}, \quad X_{22} \geq \frac{x_2^2}{z_2}, \quad (X_{11} - x_1^2)(X_{22} - x_2^2) \geq (X_{12} - x_1x_2)^2, \quad X_{12} \geq 0, \quad x_1, x_2 \geq 0, \quad z_1, z_2 \in [0, 1] \right\}.
\]

• The separation problem: Given a point \( (\tilde{x}, \tilde{X}, \tilde{z}) \in \tilde{C} \), decide whether \( (\tilde{x}, \tilde{X}, \tilde{z}) \) is in \( \text{conv}(S_2) \) or not, and in the latter case, find a supporting inequality for \( \text{conv}(S_2) \) that is violated by \( (\tilde{x}, \tilde{X}, \tilde{z}) \).
A simple separation algorithm over \( \text{conv}(S_2) \)

**Input:** A point \((\tilde{x}, \tilde{X}, \tilde{z}) \in \tilde{C}\)

**Output:** A Boolean \( \text{Inside} \) together with a supporting inequality \( h(x, X, z) \geq 0 \) for \( \text{conv}(S_2) \), violated by \((\tilde{x}, \tilde{X}, \tilde{z})\) if \( \text{Inside} = \text{false} \).

Find the unique \( k \in \{1, \ldots, 8\} \) such that \((\tilde{x}, \tilde{X}, \tilde{z}) \in R_k\) if all inequalities defining \( \text{cl}(S \cap R_k) \) are satisfied by \((\tilde{x}, \tilde{X}, \tilde{z})\), then

\[
\text{Inside} = \text{true} \\
\text{return}
\]

else

\[
\text{Inside} = \text{false} \\
\text{if } k \in \{3, 4\}, \text{ then} \\
\text{Let } q(x, X, z) = \left( X_{11} - \frac{x_1^2}{z_2} \right) \left( X_{22} - \frac{x_2^2}{z_2} \right) - \left( X_{12} - \frac{x_1 x_2}{z_2} \right)^2
\]

else if \( k = 5 \), then

\[
\text{Let } q(x, X, z) = \left( X_{11} - \frac{x_1^2}{z_1} \right) \left( X_{22} - \frac{x_2^2}{z_1} \right) - \left( X_{12} - \frac{x_1 x_2}{z_1} \right)^2
\]

else if \( k = 8 \), then

\[
\text{Let } q(x, X, z) = z_1(1 - z_2) \left( X_{11} - \frac{x_1^2}{z_1} \right) x_2^2 - (z_1 + z_2 - 1) \left( X_{12} \frac{z_1 z_2}{W} - x_1 x_2 \right)^2
\]

Define the point \((\hat{x}, \hat{X}, \hat{z})\) with components equal to \((\tilde{x}, \tilde{X}, \tilde{z})\) except for \( \hat{X}_{11} \) which is chosen so that \( q(\hat{x}, \hat{X}, \hat{z}) = 0 \). Let \( h(x, X, z) \) be the first-order Taylor expansion of \( q(x, X, z) \) at \((\hat{x}, \hat{X}, \hat{z})\).

\[
\text{return } h(x, X, z) \geq 0
\]
A corollary of our convex hull characterization

• For any $i \neq j \in [n]$, any supporting inequality of

$$\left\{ (x_i, x_j, X_{ii}, X_{ij}, X_{jj}, z_i) : \begin{pmatrix} z_i & x_i & x_j \\ x_i & X_{ii} & X_{ij} \\ x_j & X_{ij} & X_{jj} \end{pmatrix} \succeq 0 \right\}, \quad (1)$$

at a boundary point of (1) satisfying

$$X_{ij}z_i > x_ix_j, \quad X_{jj}x_i > X_{ij}x_j, \quad x_i, x_j > 0, \quad z_i, z_j \in (0, 1),$$

is a supporting inequality for $\overline{\text{conv}}(S_2)$, and hence is valid inequality for $\overline{\text{conv}}(S_n)$.

• Atamturk and Gomez 2019 propose the following rank-one inequalities

$$\begin{pmatrix} z_i + z_j & x_i & x_j \\ x_i & X_{ii} & X_{ij} \\ x_j & X_{ij} & X_{jj} \end{pmatrix} \succeq 0,$$

and show the impact of the addition of such constraints to the perspective relaxation.