Free Fermions at Finite Temperature: An Application of the Non-Commutative Algebra

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Abstract

Charret et. al. applied the properties of the Grassmann generators to develop a new method to calculate the coefficients of the high temperature expansion of the grand canonical partition function of self-interacting fermionic models in any \( d \)-dimensions \( (d \geq 1) \). The method explores the anti-commuting nature of fermionic fields and avoids the calculation of the fermionic path integral. We apply this new method to the relativistic free Dirac fermions and recover the known results in the literature.

PACS numbers: 02.90.+p, 05.30.Fk

Keywords: Fermionic System at Finite Temperature, Non-Commutative Algebra, Mathematical Methods in Physics

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1 Introduction

The path integral approach has been extensively applied to calculate the thermodynamic properties of the quantum field theories\cite{1, 2, 3}. Through this approach, it has been calculated the leading contribution, in the high temperature limit ($\beta \ll 1$, where $\beta = \frac{1}{kT}$), to the effective potential of these theories\cite{2, 4}. We also have a standard high temperature perturbation theory derived from the path integral expression of the partition function\cite{5, 6}. In the calculation of the Helmholtz free energy, via the path integral approach, we do not keep track of its contribution that are $\beta$-independent. In general this contribution is irrelevant to the thermodynamic properties of the model, but even we are not going to deal here with supersymmetric models, we remember that this contributions are crucial to verify if the supersymmetry is broken or not at finite temperature\cite{7, 8, 9}, since for a theory to be supersymmetric the value of its vacuum energy has to be zero.

For theories involving bosons, in the path integral we first integrate over the conjugate momenta. Bernard\cite{1} showed that in this case we have to be careful in constructing the path integral to get the overall $\beta$-dependent constant of the grand canonical partition function. This constant does not appear in pure fermionic models since in these cases the momentum conjugate to the fermionic fields are their own hermitian conjugate. Even for these fermionic models, along the calculation of the partition function done via the path integral approach, we get $\beta$-independent terms that are dropped out (see the free fermion case that is fully discussed in reference\cite{3}).

The partition function of free fermions models is calculated directly from its path integral expression since it is equal to the determinant of its dynamical operator\cite{3, 10}. When the lagrangean density of model has self-interacting fermionic terms, it is not possible any more to calculate exactly the non-gaussian non-commutative path integral. One common approach is to get the bosonized version of the model\cite{11}. Another standard way to handle the path integral over non-commutative function is to do a perturbation theory taking care of the signs coming from fermion loops\cite{1, 2, 3}.

The interesting properties of the non-commutative Grassmann algebra has been applied to get the contributions from spin configurations to the partition function of the classical bidimensional Ising model\cite{12}. In reference\cite{13} Charret et al. proposed a new way to calculate the coefficients of the high temperature expansion of the grand canonical partition function of self-interacting fermionic models in $d$-dimensions ($d \geq 1$). They applied the method to the Hatsugay-Kohmoto model\cite{13}, that is an exactly solvable model. The approach was also applied to the unidimensional Generalized Hubbard model to get the coefficients up to order...
\(\beta^3\) of the high temperature expansion of its grand canonical partition function. Differently from other approaches, these coefficients calculated by this method are analytical and exact.

Up to now the approach of Charret et al. has only been applied to fermionic models already regularized on a lattice with space unit one. The aim of the present paper is to use this method to calculate the Helmholtz free energy of the free fermion Dirac, that is a continuous theory and whose exact result is already known in the literature. It is also important to check its application to a fermionic lattice model.

In section 2 we give a summary of the results of the method of Charret et al. fully described in reference [13]. In section 3 we apply the approach to the free Dirac fermion using two expansions: in subsection 3.1 we first expand the fermionic field operators in the basis of the eigenstates of energy and in subsection 3.2 we consider the naive fermion model on a three dimensional space lattice. In both ways, we show that is possible to re-sum the high temperature expansion of the grand canonical partition function of the model and compare our results with the known ones in the literature. In appendix A we present the formulae related to the lagrangean density of the free Dirac fermion. In appendix B we have a dictionary of some formulae showing their continuum and discrete expressions.

2 A Survey of the Approach of Charret et al.

The grand canonical partition function of any quantum system can be written as a trace over all quantum physical states:

\[
\mathcal{Z}(\beta) = \text{Tr}[e^{-\beta K}],
\]

where \(\beta = \frac{1}{kT}\), \(k\) is the Boltzmann’s constant and \(T\) is the absolute temperature. The operator \(K\) is defined as \(K = H - \mu N\), \(H\) being the hamiltonian of the system, \(\mu\) is the chemical potential and \(N\) is a conserved operator. In the high temperature limit (\(\beta \ll 1\)), \(\mathcal{Z}(\beta)\) has the expansion

\[
\mathcal{Z}(\beta) = \text{Tr}[\mathbb{1}] + \sum_{n=1}^{\infty} \frac{(-\beta)^n}{n!}\text{Tr}[K^n].
\]

The specatation value of any fermionic operator can be written as a multivariable integral over Grassmann variables. The mapping among the fermionic operators that appear in the fermionic model and the grassmannian generators is such that they satisfy the same algebra. Let \(a_i^\dagger\) and \(a_j\) be the hermitian conjugate fermionic operators that satisfy the anti-commuting relations.
\{a_i, a_j^\dagger\} = 1_\delta_{ij} \quad \text{and} \quad \{a_i, a_j\} = 0, \quad (3) 

where \(i = 1, \ldots, \mathcal{N}\). The generators of the associated Grassmann algebra has dimension \(2^{2\mathcal{N}}\), and can be written explicitly as \(\{\bar{\eta}_1, \cdots, \bar{\eta}_{\mathcal{N}}; \eta_1, \cdots, \eta_{\mathcal{N}}\}\). They satisfy the following anti-commutation relations:

\{\eta_i, \eta_j\} = 0, \quad \{\bar{\eta}_i, \bar{\eta}_j\} = 0 \quad \text{and} \quad \{\bar{\eta}_i, \eta_j\} = 0. \quad (4)

The mapping

\[ a_i^\dagger \rightarrow \bar{\eta}_i \quad \text{and} \quad a_j \rightarrow \frac{\partial}{\partial \bar{\eta}_j}, \] \quad (5)

preserves the algebra (3) due to the algebra (4) satisfied by the Grassmannian generators.

For any self-interacting fermionic model in a \(d\)-dimension lattice (\(d \geq 1\)), the coefficients of high temperature expansion (2) can be written as the multivariable Grassmann integral \[13\]

\[ \text{Tr}[K^n] = \int d\eta_I d\bar{\eta}_I \exp \sum_{I,J=1}^{2\mathcal{N}^d} \bar{\eta}_I A_{I,J} \eta_J \times \prod_{\nu=0}^{n-1} \mathcal{K}^{\nu}(\bar{\eta}, \eta, \nu = 0) \mathcal{K}^{\nu}(\bar{\eta}, \eta, \nu = 1) \cdots \mathcal{K}^{\nu}(\bar{\eta}, \eta, \nu = n-1), \quad (6) \]

\(N^d\) is the number of sites in the \(d\)-dimensional lattice. Matrix \(A\) is independent of the operators \(H\) and \(N\) and is equal to

\[ A = \begin{pmatrix} A^{\uparrow\uparrow} & 0 \\ 0 & A^{\downarrow\downarrow} \end{pmatrix}. \]

Each element of matrix \(A\) is a matrix of dimension \(nN^d \times nN^d\) and \(\Phi\) is the null matrix in this dimension. The indices \(I, J\) in matrix \(A\) vary in the interval, \(I, J = 1, 2, \ldots, 2nN^d\). The matrices \(A^{\uparrow\uparrow}\) and \(A^{\downarrow\downarrow}\) are equal and

\[ A^{\uparrow\uparrow} = A^{\downarrow\downarrow} = \begin{pmatrix} 1_{N^d \times N^d} & -1_{N^d \times N^d} & \Phi_{N^d \times N^d} & \cdots & \Phi_{N^d \times N^d} \\ \Phi_{N^d \times N^d} & 1_{N^d \times N^d} & -1_{N^d \times N^d} & \cdots & \Phi_{N^d \times N^d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Phi_{N^d \times N^d} & \Phi_{N^d \times N^d} & \Phi_{N^d \times N^d} & \cdots & 1_{N^d \times N^d} \end{pmatrix}, \]

where \(1_{N^d \times N^d}\) and \(\Phi_{N^d \times N^d}\) are the identity and null matrices of dimension \(N^d \times N^d\) respectively. Due to the fact that the submatrices \(A^{\uparrow\downarrow}\) and \(A^{\downarrow\uparrow}\) are null, the result of the multivariable integral (6) is equal to the product of the independent contributions of the sectors \(\sigma\sigma = \uparrow\uparrow\) and \(\sigma\sigma = \downarrow\downarrow\).
The grassmannian function $K^\otimes(\bar{\eta}, \eta)$ in eq.(3) is obtained from the normal ordered operator $K$ by doing the naive mapping: $a^\dagger_{i,\sigma} \rightarrow \bar{\eta}_I$ and $a_{i,\sigma} \rightarrow \eta_I$, where $i = 1, 2, ..., N^d$ and $\sigma = \uparrow, \downarrow$.

Finally, the result of the integrals that contribute to $\text{Tr}[K_n]$ are independent of the $\sigma\sigma$ sector since $A_{\uparrow\uparrow} = A_{\downarrow\downarrow}$. We present here only the results of the integrals of the sector $\sigma\sigma = \uparrow\uparrow$. All those integrals are of the following type:

$$M(L, K) = \int \prod_{i=1}^{nN^d} d\eta_i d\bar{\eta}_i \bar{\eta}_{l_1} \eta_{k_1} \ldots \bar{\eta}_{l_m} \eta_{k_m} e^{\sum_{i,j=1}^{nN^d} \bar{\eta}_i A_{ij}^\dagger \eta_j},$$

(9)

where $L \equiv \{l_1, \ldots, l_m\}$ and $K \equiv \{k_1, \ldots, k_m\}$. We should remember that the grassmannians functions $K^\otimes$ are polynomials of the generators of the algebra. The products $\eta \bar{\eta}$ are ordered such that $l_1 < l_2 < ... < l_m$ and $k_1 < k_2 < ... < k_m$. The results of the integrals of type (9) are\[18\]

$$M(L, K) = (-1)^{\sum_{i=1}^{m} (l_i + k_i)} A(L, K),$$

(10)

where $A(L, K)$ is the determinant obtained from matrix $A^{\uparrow\uparrow}$ by the cut of the lines $l_1, l_2, ..., l_m$ and the columns $k_1, k_2, ..., k_m$. $M(L, K)$ is a co-factor of matrix $A^{\uparrow\uparrow}$. Result (10) reduce the calculation of the multivariable integrals over anti-commuting variables to the calculation of co-factors of a well defined matrix whose elements are commuting numbers.

In general it should not be easy to calculate the co-factors of matrix $A^{\sigma\sigma}$ since each of its elements is a matrix of dimension $N^d \times N^d$. However, due to the block structure of matrix $A^{\sigma\sigma}$ Charret et. al showed in reference [13] that it is possible to diagonalize it for any value of $n$ and $N$. The results are analytical and allow us to take the thermodynamic limit.

3 Grand Canonical Partition Function for Free Relativistic Fermions

The lagrangean density of the free Dirac fermion is:

$$\mathcal{L} = \bar{\Psi}(\vec{x}, t)(\not{\partial} - m)\Psi(\vec{x}, t),$$

(11)

whose fermionic field operators satisfy the anti-commutation relations

$$\{\Psi_\mu(\vec{x}, t), \Pi_\nu(\vec{x}', t)\} = i\delta_\mu\nu\delta(\vec{x} - \vec{x}'), \quad \mu, \nu = 1, 2, 3, 4$$

(12)

and

$$\{\Psi_\mu(\vec{x}, t), \Psi_\nu(\vec{x}', t)\} = \{\Pi_\mu(\vec{x}, t), \Pi_\nu(\vec{x}', t)\} = 0,$$

(13)
where $\Pi^\nu(\vec{x}, t)$ is the canonical momentum of $\Psi^\nu(\vec{x}, t)$ and $\Pi^\nu(\vec{x}, t) = i\Psi^\dagger_\nu(\vec{x}, t)$.

From the lagrangean density (11) and the equation of motion satisfied by the fermionic field operators, the total hamiltonian operator of the system can be written as

$$H = \int_V d^3\vec{x} i\Psi^\dagger(\vec{x}, t)\partial_0\Psi(\vec{x}, t).$$

We use the natural units where $\hbar = c = e = 1$. Our metric is $\text{diag}(g_{\mu\nu}) = (1, -1, -1, -1)$.

### 3.1 Expansion in the Basis of Eigenstates of Energy

The fermionic operator $\Psi(\vec{x}, t)$ written in the basis of eigenstates of energy of the free Dirac fermion is

$$\Psi(\vec{x}, t) = \frac{1}{\sqrt{2}} \sum_{\vec{k}} \sum_{r=1}^{2} [a_r(\vec{k})u_r(\vec{k})e^{-ik_\mu x^\mu} + b_r^\dagger(\vec{k})v_r(\vec{k})e^{ik_\mu x^\mu}],$$

where $k_\mu x^\mu = k_0 x_0 - \vec{k}.\vec{x}$, and $k_0 = \sqrt{|\vec{k}|^2 + m^2} > 0$. The destruction fermionic operators $a_r$, $b_r$ and their respective hermitian conjugate satisfy the anti-commutation relations

$$\{a_r(\vec{k}), a^\dagger_s(\vec{k}')\} = \delta_{rs}\delta_{\vec{k},\vec{k}'} \quad \text{and} \quad \{b_r(\vec{k}), b^\dagger_s(\vec{k}')\} = \delta_{rs}\delta_{\vec{k},\vec{k}'}.$$  

All others anti-commutation relations of these operators are null. The spinors components $u_r(\vec{k})$ and $v_r(\vec{k})$ are given by eqs. (49) and (50).

The hamiltonian operator written in terms of creation and destruction operators of defined energy becomes

$$H = -\mathbb{1}\sum_{\vec{k}}\sum_{r=1}^{2} k_0 + \sum_{\vec{k}}\sum_{r=1}^{2} k_0[a_r^\dagger(\vec{k})a_r(\vec{k}) + b_r^\dagger(\vec{k})b_r(\vec{k})],$$

$\mathbb{1}$ been the identity operator. We define $E_0$ as the vacuum energy, $E_0 \equiv \langle 0|H|0 \rangle = -2\sum_{\vec{k}} k_0$, $|0\rangle$ being the vacuum state of the fermionic model.

Our aim is to calculate the grand canonical partition function of the free Dirac fermion in contact with a reservoir of heat and electric charge. Eqs. (19), (20) and (21) allow us to calculate the coefficients of the high temperature expansion of this function. For the present case, the operator $K$ in expression (1) is: $K = H + E_0\mathbb{1} - \mu Q$, where $\mu$ is the chemical potential and $Q$ is the total electric charge operator of the free relativistic fermions. Operator $Q$ written in terms of the creation and destruction operators is
\[ Q = 2 \sum_{\vec{k}} \mathbb{1} + \sum_{\vec{k}} \sum_{r=1}^{2} [a_r^\dagger(\vec{k})a_r(\vec{k}) - b_r^\dagger(\vec{k})b_r(\vec{k})]. \]  

(18)

\[ \equiv Q_0 \mathbb{1} + :Q: . \]

Finally, we have that \( K \) operator can be written as

\[ K = 2 \sum_{\vec{k}} \sum_{r=1}^{2} (k_0 - \mu) a_r^\dagger(\vec{k})a_r(\vec{k}) + \sum_{\vec{k}} \sum_{r=1}^{2} (k_0 + \mu) b_r^\dagger(\vec{k})b_r(\vec{k}) + \mathbb{1}(E_0 - \mu Q_0) \]

\[ \equiv K_a + K_b + \mathbb{1}(E_0 - \mu Q_0). \]  

(19)

Due to the anti-commutation relations (16), we have that \([K_a, K_b] = 0\), and therefore \( \mathcal{Z}(\beta) \) can be written as

\[ \mathcal{Z}(\beta) = e^{-\beta(E_0 - \mu Q_0)} Z_a(\beta) Z_b(\beta), \]

(20)

where

\[ Z_a(\beta) \equiv Tr_a[e^{-\beta K_a}] \quad \text{and} \quad Z_b(\beta) \equiv Tr_b[e^{-\beta K_b}]. \]

(21)

The calculation of the functions \( Z_a(\beta) \) and \( Z_b(\beta) \) are equivalent. We present here only the details of the calculation of \( Z_a(\beta) \). For a free Dirac fermion, the traces for \( r = 1 \) and \( r = 2 \) are equal. Then,

\[ Z_a(\beta) = Tr_a[e^{-\beta K_a}] = [Tr_a^{(1)}[e^{-\beta K_a^{(1)}]}]^2 \]

\[ = \left[ \prod_{\vec{k}} Tr_a^{(1)}[e^{-\beta(k_0 - \mu)n_{a}^{(1)}(\vec{k})}] \right]^2 . \]

(22)

In \( Tr_a^{(1)} \) we have the vector \( \vec{k} \) fixed. Since \( n_{a}^{(1)}(\vec{k}) \) is a commuting operator, we apply the Newton multinomial to write

\[ Tr_a^{(1)}[e^{-\beta(k_0 - \mu)n_{a}^{(1)}(\vec{k})}] = Tr_a^{(1)}[\mathbb{1}] + \sum_{n=1}^{\infty} \frac{(-\beta)^n}{n!}(k_0 - \mu)^n Tr_a^{(1)}[n_{a}^{(1)}(\vec{k})^n]. \]

(23)

We should note that for fixed \( \vec{k} \) the anti-commutation relations (16) are identical to the relations (3), therefore, \( Tr_a^{(1)}[n_{a}^{(1)}(\vec{k})^n] \) can be written as a Grassmann multivariable integral (8) with an equivalent mapping to eq.(3) for the associated generators of the non-commutative algebra. The traces that contribute to eq.(23) are written as the following anti-commuting integrals:
\[ T_{r_a^{(1)}}[\mathbb{1}] = \int \prod_{l=1}^{n} d\eta_{\mathbf{k}}(I) d\bar{\eta}_{\mathbf{k}}(I) e^{\sum_{l,j=1}^{n} \eta_{\mathbf{k}}(I) A_{IJ}^{(11)} \eta_{\mathbf{k}}(J)}, \]  

(24)

and

\[ T_{r_a^{(1)}}[\mathbf{n}_a^{(1)}(\mathbf{k})^n] = \int \prod_{l=1}^{n} d\eta_{\mathbf{k}}(I) d\bar{\eta}_{\mathbf{k}}(I) e^{\sum_{l,j=1}^{n} \eta_{\mathbf{k}}(I) A_{IJ}^{(11)} \eta_{\mathbf{k}}(J)} \times \]
\[ \times \bar{\eta}_{\mathbf{k}}(1) \eta_{\mathbf{k}}(1) \ldots \bar{\eta}_{\mathbf{k}}(n) \eta_{\mathbf{k}}(n), \quad \text{for } (n > 0). \]  

(25)

The grassmannian expression for \( T_{r_a^{(1)}}[\mathbf{n}_a^{(1)}(\mathbf{k})^n] \) is obtained from eq.(3) with \( N = 1 \) and \( d = 1 \). Therefore the elements of matrix \( \mathbf{A}^{(11)} \) in eq.(3) are just numbers. From eqs.(3) and (11), we conclude that having the \( \bar{\eta} \)'s in the integrand corresponds to deleting the first \( n \) lines in matrix \( \mathbf{A}^{(11)} \) and, in the same way, having the \( \eta \)'s corresponds to deleting the first \( n \) columns of the same matrix. Therefore, from the expression of matrix \( \mathbf{A}^{(11)} \) (see eq.(3)), for arbitrary \( n \), we realize that the matrix \( \mathbf{A}_n^{(11)} \) obtained after the cuts of \( n \)-first lines and \( n \)-first columns is an upper triangular matrix whose determinant is equal to 1. Beside this, we have that \( \det[\mathbf{A}^{(11)}] = 2 \), for any value of \( n \). In summary, for arbitrary \( n \) we have that

\[ T_{r_a^{(1)}}[\mathbb{1}] = 2 \quad \text{and} \quad T_{r_a^{(1)}}[\mathbf{n}_a^{(1)}(\mathbf{k})^n] = 1. \]  

(26)

Substituting results (26) in expression (23) and resumming it we get

\[ T_{r_a^{(1)}}[e^{-\beta \mathbf{K}^{(1)}}] = \prod_{\mathbf{k}} (1 + e^{-\beta(k_0-\mu)}), \]  

(27)

that returning to eq.(24) gives us

\[ \mathcal{Z}(\beta) = \prod_{\mathbf{k}} e^{2\beta(k_0+\mu)}(1 + e^{-\beta(k_0-\mu)})(1 + e^{-\beta(k_0+\mu)})^2. \]  

(28)

The relation between the Helmholtz free energy and the grand canonical partition function is\[15\]

\[ \mathcal{W}(\beta) = -\frac{1}{\beta} \ln(\mathcal{Z}(\beta)). \]  

(29)

Substituting result (28) in eq.(23) the expression derived for the Helmholtz free energy is

\[ \mathcal{W}(\beta) = -\frac{2}{\beta} \sum_{\mathbf{k}} [\beta(k_0 + \mu) + \ln(1 + e^{-\beta(k_0-\mu)}) + \ln(1 + e^{-\beta(k_0+\mu)})]. \]  

(30)
3.2 Free Dirac Fermions on the Lattice

The lattice calculation of models including fermions has been an important tool to learn the properties of these models. The aim of this subsection is to show that the method of Charre et al. can be equally well applied to the lattice version of fermionic models. To do so, we consider the most naive lattice realization of the free Dirac fermion. We remember that the crucial point to apply the method of Charre et al. is to work with fermionic field operators that satisfy the anti-commutation relations (3).

The Hamiltonian operator of the free Dirac fermion is

\[ H = \int_V d^3\vec{x} \ i\Psi^\dagger(\vec{x}, t)[\gamma^0\vec{\gamma} \cdot \vec{\nabla} - m\gamma^0]\Psi(\vec{x}, t), \tag{31} \]

where \( \vec{\gamma} = (\gamma^1, \gamma^2, \gamma^3) \). From appendix B, we get that the operator \( H \) written on the lattice becomes

\[ H = \sum_{n_1,n_2,n_3=-\frac{N}{2}}^{\frac{N}{2}} \sum_{\alpha,\beta=1}^{4} \sum_{j=1}^{3} \left\{ i\frac{a^2}{2}\Psi^\dagger_\alpha(\vec{n}a, t)(\gamma^0\vec{\gamma} \cdot \vec{\nabla})_{\alpha\beta} \Psi_\beta(\vec{n}a + j\vec{a}, t) - \Psi^\dagger_\alpha(\vec{n}a, t)(\gamma^0)_{\alpha\beta} \Psi_\beta(\vec{n}a, t) \right\} - ia^3m\Psi^\dagger_\alpha(\vec{n}a, t)(\gamma^0)_{\alpha\beta} \Psi_\beta(\vec{n}a, t), \tag{32} \]

where \( a \) is the distance between the nearest sites in each space direction. The space point \( \vec{x} \) on the lattice is written as: \( \vec{x} = \vec{n}a \). \( N \) is the total number of space sites in each direction. For simplicity we take to be \( N \) even.

Operator \( K \) in expression (1) is: \( K = H - \mu Q \), where \( \mu \) is the chemical potential and \( Q \) is the total electric charge operator. The discrete expression of \( Q \) is

\[ Q = a^3 \sum_{n_1,n_2,n_3=-\frac{N}{2}}^{\frac{N}{2}} \Psi^\dagger_\alpha(\vec{n}a, t)\Psi_\alpha(\vec{n}a, t). \tag{33} \]

Imposing the space periodic boundary condition for the fermionic field operators, their Fourier decompositions are

\[ \Psi_\alpha(\vec{n}a, t) = \frac{1}{\sqrt{V}} \sum_{k_1,k_2,k_3=-\frac{2\pi}{L}}^{a} \tilde{\psi}_\alpha(\frac{\pi\vec{k}}{L}, t)e^{i\frac{\pi\vec{k}}{L}\cdot\vec{n}a}, \tag{34} \]

where \( \alpha = 1, 2, 3, 4 \) and \( L = \frac{Na}{2} \). The fermionic field operators \( \Psi_\alpha(\vec{x}, t) \) and \( \Psi^\dagger_\alpha(\vec{x}, t) \) satisfy the anti-commutation relations (12) and (13) which imply that the Fourier components of the fermionic field operators obey the relations:
\begin{align}
\{\hat{\psi}_\alpha(\frac{\pi \vec{l}}{L},t),\hat{\psi}_\beta(\frac{\pi \vec{k}}{L},t)\} &= \delta_{\alpha\beta} \delta^{(3)}_{\vec{l},\vec{k}} \quad \text{and} \quad \{\hat{\psi}_\alpha(\frac{\pi \vec{l}}{L},t),\hat{\psi}_\beta(\frac{\pi \vec{k}}{L},t)\} = 0, \tag{35}
\end{align}

that are identical to relations (3), the necessary algebra to apply the method of Charret et al.

In momentum space, operator $K$ is written as

$$K \equiv \frac{1}{a} \sum_{l_1,l_2,l_3=-\frac{N}{2}}^{N-1} K_{\vec{l}},$$ \tag{36}

where

$$K_{\vec{l}} \equiv \tilde{\Psi}^\dagger\left(\frac{\pi \vec{l}}{L},t\right) R\left(\frac{\pi \vec{l}}{L}\right) \tilde{\Psi}\left(\frac{\pi \vec{l}}{L},t\right),$$ \tag{37}

and

$$\tilde{\Psi}\left(\frac{\pi \vec{l}}{L},t\right) \equiv \begin{pmatrix}
\tilde{\psi}_1 \\
\tilde{\psi}_2 \\
\tilde{\psi}_3 \\
\tilde{\psi}_4
\end{pmatrix}. \tag{38}$$

The matrix $R\left(\frac{\pi \vec{l}}{L}\right)$ is defined as

$$R\left(\frac{\pi \vec{l}}{L}\right) = \begin{pmatrix}
a(m-\mu)I & \sin\left(\frac{\pi}{L}l_j a\right) \sigma_j \\
\sin\left(\frac{\pi}{L}l_j a\right) \sigma_j & -a(m+\mu)I
\end{pmatrix}. \tag{39}$$

We have an implicit sum over $j$ in the off-diagonal elements of matrix $R$ and $I$ is the identity matrix of dimension $2 \times 2$. Due to the anti-commutation relations (33) we have that $[K_{\vec{l}}, K_{\vec{k}}] = 0$. Therefore, the grand canonical partition function of the model becomes

$$Z(\beta) = \prod_{l_1,l_2,l_3=-\frac{N}{2}}^{\frac{N}{2}-1} Tr_{\vec{l}} \left[ e^{-\frac{\beta}{2} K_{\vec{l}}} \right], \tag{40}$$

where $Tr_{\vec{l}}$ means that the trace is calculated for fixed $\vec{l}$. We should notice that the operator $K_{\vec{l}}$ is not diagonal in momentum space. We make the similarity transformation: $PRP^{-1} = D$, where the diagonal matrix $D$ is

$$D = \begin{pmatrix}
\lambda_+ I & 0 \\
0 & \lambda_- I
\end{pmatrix}, \tag{41}$$

where $\lambda_+$ and $\lambda_-$ are the eigenvalues of matrix $R$, and

$$\lambda_{\pm} = -a(\mu \pm \tilde{\omega}(\vec{l})) \tag{42}$$
with
\[ \tilde{\omega}(\vec{l}) \equiv \sqrt{m^2 + \tilde{p}_1^2 + \tilde{p}_2^2 + \tilde{p}_3^2}, \]
and \( \vec{l} \equiv (l_1, l_2, l_3) \). We followed reference [10] and made the change of variables
\[ \tilde{p}_i = \frac{1}{a} \sin(p_i a), \] (44)
where \( p_i = \frac{\pi}{a} l_i \). The new fermionic field operators \( \Psi' = P \tilde{\Psi} \) and \( \Psi'^\dagger = \tilde{\Psi'}^\dagger P^{-1} \) also preserves the anti-commutation relations (35).

The function \( Z(\beta) \) written in terms of the new fermionic field operators has the same form as the r.h.s. of eq.(22). Following similar steps, we get the Helmholtz free energy for the model on the lattice, that is,
\[
\mathcal{W}(\beta) = -2 \sum_{l_1, l_2, l_3 = -\frac{N}{2}}^{\frac{N}{2}} \left( \mu + \tilde{\omega}(\vec{l}) \right) - \\
-2 \beta \sum_{l_1, l_2, l_3 = -\frac{N}{2}}^{\frac{N}{2}} \left[ \ln \left( 1 + e^{-\beta(\tilde{\omega}(\vec{l}) + \mu)} \right) + \ln \left( 1 + e^{-\beta(\tilde{\omega}(\vec{l}) - \mu)} \right) \right].
\]
(45)

In the limit \( a \to 0 \) the function \( \mathcal{W}(\beta) \) agrees with the result of reference [10] and is equal to twice the result (30). Here as well in the usual lattice calculation with fermions, the doubling problem is lifted by including the Wilson term
\[
H^{(W)} = i \frac{ra}{2} \int_V d^3 \vec{x} \, \Psi'^\dagger(\vec{x}, t) \gamma^0 \nabla^2 \Psi(\vec{x}, t),
\]
(46)
in hamiltonian (31) and \( r \) is the Wilson’s constant.

4 Conclusions

Recently Charret et al. proposed a new way to calculate the coefficients of the high temperature expansion of the grand canonical partition function \( Z(\beta) \) of any self-interacting fermionic model in \( d \)-dimension \((d \geq 1)\) [13]. To apply this method to calculate these coefficients, it is enough to write the second quantization expression of the hamiltonian and a conserved operator in terms of operators that satisfy the anti-commutation relations (3). In this approach, at each order \( \beta^n \) of the high temperature expansion of the function \( Z(\beta) \), the calculation of the coefficients is reduced to get the co-factors of a matrix with commuting entries (see matrix A in eqs. (7) and
For a fixed number of sites on the lattice, all the mathematical objects in the calculation are well-defined. This approach avoids to calculate the fermionic path integral of \( Z(\beta) \).

We applied the method of Charret et al. to the free Dirac fermion by first expanding it in the basis of the eigenstates of energy of the free fermions. The Hamiltonian and the total electric charge operators (eqs. \((\ref{eq:hamiltonian})\) and \((\ref{eq:charge})\) respectively) include the respective vacuum contribution. Result \((\ref{result})\) gives two divergent terms to the Helmholtz free energy \( W(\beta) \): the vacuum energy and the electric charge of the vacuum. This last divergent term does not appear when we do the calculation via the usual path integral approach\([3]\), while the \( \beta \)-dependent terms in \( W(\beta) \) are identical in both methods. In the path integral calculation\([3]\) divergent terms that are \( \beta \) and \( \mu \)-independent are dropped out. This does not happens in the present approach.

Since the calculation of fermionic lattice models are an important tool, we also applied the approach of Charret et al. to a naive version of the free Dirac fermion on the lattice\([10]\). In this case we also get the doubling problem that is lifted by the Wilson term, as happens in the usual fermion lattice calculations.

In summary, we can affirm that the method of Charret et al. can also be applied to continuous fermionic models. It is an analytical approach that allows without ambiguity to calculate all terms of the function \( Z(\beta) \) for the free Dirac fermion, including the divergent terms coming from the vacuum contribution. The next step is to study in this approach the renormalization scheme associated to physical quantities for fermionic models with self-interacting terms.

5 Acknowledgements

O.R.S. and M.T.T. are in debt with H. Rothe for valuable discussions about lattice fermionic free models. O.R.S. thanks CAPES for total financial support. S.M.de S. and M.T.T. thank CNPq for partial financial support. M.T.T. also thanks FINEP and S.M.de S. thanks FAPEMIG for partial financial support.

Appendix

A Useful Formulae

In the lagrangean density for the free Dirac fermion (see eq.\((\ref{lagrangian})\)),

\[
\mathcal{L} = \bar{\Psi}(\vec{x}, t)(\not{\partial} - m)\Psi(\vec{x}, t),
\]

the Dirac matrices \( \gamma^\mu, \mu = 0, \ldots, 3 \), are:
\[ \gamma_0 = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \gamma^i = i \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad i = 1, 2, 3. \] (48)

The matrices \( \sigma^i \), \( i = 1, 2, 3 \) are the Pauli matrices, and \( \mathbb{1} \) and \( \mathbb{0} \) are the identity and null matrices of dimension \( 2 \times 2 \) respectively. We have \( \overline{\Psi}(\vec{x}, t) = -i \Psi^\dagger(\vec{x}, t) \gamma_0 \).

The spinors components \( u_r(\vec{k}) \) and \( v_r(\vec{k}) \) in eq.(15) are:

\[
\begin{align*}
\hat{u}_1(\vec{k}) &= \sqrt{m + k_0} \begin{pmatrix} 1 \\ 0 \\ \frac{k_3 - ik_2}{m + k_0} \\ \frac{k_1 + ik_2}{m + k_0} \end{pmatrix}, \\
\hat{u}_2(\vec{k}) &= \sqrt{m + k_0} \begin{pmatrix} 0 \\ \frac{k_3 - ik_2}{m + k_0} \\ \frac{k_1 + ik_2}{m + k_0} \\ 1 \end{pmatrix}, \\
\hat{v}_1(\vec{k}) &= \sqrt{m + k_0} \begin{pmatrix} \frac{k_3 - ik_2}{m + k_0} \\ \frac{k_1 + ik_2}{m + k_0} \\ 0 \\ 1 \end{pmatrix}, \\
\hat{v}_2(\vec{k}) &= \sqrt{m + k_0} \begin{pmatrix} \frac{k_1 + ik_2}{m + k_0} \\ 1 \end{pmatrix}.
\end{align*}
\] (49, 50)

**B Summary of Expressions on the Lattice**

We assume that each direction in the three dimensional space has \( N \) sites. For simplicity \( N \) is assumed to be even. The distance between nearest points along each direction is \( a \). In the space lattice, each point is written as \( \vec{x} = \vec{n}a \), where \( \vec{n} = (n_1, n_2, n_3) \), \( n_i = -\frac{N}{2}, -\frac{N}{2} + 1, \cdots, \frac{N}{2} \), and \( i = 1, 2, 3 \).

We have the following dictionary between the continuous to the discrete representations of the formulae:

\[
\begin{align*}
\int_V d^3 \vec{x} & \rightarrow a^3 \sum_{n_1,n_2,n_3=-\frac{N}{2}}^{\frac{N}{2}} \\
\delta(\vec{x} - \vec{x}') & \rightarrow \frac{1}{a^3} \delta_{\vec{n},\vec{m}}^{(3)} \equiv \frac{1}{a^3} \delta_{n_1, m_1} \delta_{n_2, m_2} \delta_{n_3, m_3},
\end{align*}
\] (51, 52)

where \( \vec{x} = \vec{n}a, \vec{x}' = \vec{m}a \) with \( \vec{n} = (n_1, n_2, n_3) \) and \( \vec{m} = (m_1, m_2, m_3) \) respectively.

Finally, the first derivative defined on the lattice is defined as\[10\]

\[
\partial_i \Psi_\alpha(\vec{x}, t) \rightarrow \frac{1}{2a} [\Psi_\alpha(\vec{n}a + \hat{i}a, t) - \Psi_\alpha(\vec{n}a - \hat{i}a, t)],
\] (53)

where \( \hat{i} \) is the unit vector in the direction \( \hat{i} \).
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