Quantum impossible differential and truncated differential cryptanalysis

Huiqin Xie$^{1,2,3}$, Li Yang$^{1,2,*}$

1. State Key Laboratory of Information Security, Institute of Information Engineering, Chinese Academy of Sciences, Beijing 100093, China
2. Data Assurance and Communication Security Research Center, Chinese Academy of Sciences, Beijing 100093, China
3. School of Cyber Security, University of Chinese Academy of Sciences, Beijing 100049, China

Abstract

We study applications of BV algorithm and present quantum versions of impossible differential cryptanalysis and truncated differential cryptanalysis based on it. Afterwards, we analyze their efficiencies and success probabilities rigorously. In traditional impossible differential attack or truncated differential attack, it is difficult to extend the differential path, which usually limits the number of rounds that can be attacked. By contrast, our approach treats the first $r - 1$ rounds of the cipher as a whole and applies BV algorithm on them directly. Thus extending the number of rounds is not a problem for our algorithm.

Keywords: post-quantum cryptography, quantum cryptanalysis, differential cryptanalysis, impossible differential

1. Introduction

The development of quantum computing has greatly impacted classical cryptography. Due to Shor’s algorithm [1], most currently used public-key cryptosystems are known to be insecure against adversaries in possession of quantum computers, such as RSA, ELGamal and any other schemes based on factorization or discrete logarithms. This motivated the advent of

*Corresponding author email: yangli@iie.ac.cn
post-quantum cryptography, which studies classical systems resistant against quantum adversaries.

In light of the fact that quantum algorithms cause a severe threat for public-key cryptography, it is natural to study the impact of quantum attacks on symmetric cryptosystems. A representative example is Grover’s search algorithm \cite{2}. It can provide a quadratic speedup for brute-force attacks. In addition, Simon’s algorithm \cite{3} has also been applied to cryptanalysis. Kuwakado and Morri use it to construct a quantum distinguisher for 3-round Feistel scheme \cite{4} and recover partial key of Even-Mansour construction \cite{5}. Santoli and Schaffiner extend their result and present a quantum forgery attack on CBC-MAC scheme \cite{6}. In \cite{7}, Kaplan et al. use Simon’s algorithm to attack various symmetric cryptosystems, such as CBC-MAC, PMAC, CLOC and so on. They also study how differential and linear cryptanalysis behave in the post-quantum world \cite{8}. In addition to Simon’s algorithm, Bernstein-Vazirani (BV) algorithm \cite{9} is also applied to cryptanalysis. Li and Yang proposed two methods to execute quantum differential cryptanalysis based on BV algorithm in \cite{10}. Inspired by their idea, Xie and Yang solved the flaws in their method and proposed quantum version of differential cryptanalysis and impossible differential cryptanalysis based on BV algorithm \cite{11}. In this paper, we apply BV algorithm to truncated differential cryptanalysis and impossible differential crypanalysis. Unlike the quantum impossible differential attack, which finds impossible differential directly, our method applies the miss-in-the-middle technique.

2. Preliminaries

In this section, we briefly discuss the notations and definitions used in this paper. Let \( F_2 = \{0, 1\} \) denote a finite field of characteristic 2. \( F_2^n \) is a vector space over \( F_2 \). We denote \( B_n \) as the set of all functions from \( F_2^n \) to \( F_2 \).

2.1. The linear structure of Boolean functions

Linear structures of Boolean functions have been investigated for their cryptanalytic significance, which are defined as following:

**Definition 1.** Let \( f \) be a function in \( B_n \). A vector \( a \in F_2^n \) is called a linear structure of \( f \), if

\[
f(x \oplus a) + f(x) = f(a) + f(0), \quad \forall x \in F_2^n,
\]

where \( \oplus \) denotes the bitwise exclusive-or.
Let $U_f$ be the set of all linear structures of $f$, and
\[ U_f^i = \{ a \in F_2^n | f(x \oplus a) + f(x) = i, \forall x \in F_2^n \} \]
for $i = 0, 1$. Obviously $U_f = U_f^0 \cup U_f^1$.

The linear structures of a Boolean function is related to its Walsh spectrum closely, which is defined as below:

**Definition 2.** The Walsh spectrum of a function $f \in \mathcal{B}_n$ is also a function in $\mathcal{B}_n$, defined as
\[ S_f(\omega) = \frac{1}{2^n} \sum_{x \in F_2^n} (-1)^{f(x) + \omega \cdot x}. \]

The following lemma demonstrates the link between the linear structure and Walsh spectral.

**Lemma 1** ([12], Corollary 1). For $f \in \mathcal{B}_n$, let $N_f = \{ \omega \in F_2^n | S_f(\omega) \neq 0 \}$. Then $\forall i \in \{0, 1\}$, it holds that
\[ U_f^i = \{ a \in F_2^n | \omega \cdot a = i, \forall \omega \in N_f \}. \]

According to Lemma 1, if we have a sufficiently large subset $H$ of $N_f$, we can obtain $U_f^i$ by solving the equation $x \cdot H = i$. (Here $x \cdot H = i$ denotes the system of linear equations $\{ a \cdot \omega = i | \omega \in H \}$.)

For positive integers $m, n$, let $\mathcal{C}_{m,n}$ denote the set of all functions from $F_2^m$ to $F_2^n$. The linear structure of a vector function in $\mathcal{C}_{m,n}$ can be defined similarly.

**Definition 3.** Let $F$ be a function in $\mathcal{C}_{m,n}$. A vector $a \in F_2^m$ is called a linear structure of $F$, if there exists a vector $\alpha \in \{0, 1\}^n$ such that
\[ F(x \oplus a) \oplus F(x) = \alpha, \ \forall x \in \{0, 1\}^m. \]

Suppose $F = (F_1, F_2, \cdots, F_n)$, we can find the linear structures of $F$ by first searching for the linear structures of each component function $F_j$ respectively, and then taking the intersection. Let $U_F$ be the set of the linear structures of $F$, and $U_F^\alpha = \{ a \in F_2^n | F(x \oplus a) \oplus F(x) = \alpha, \forall x \}$. It is obvious that $U_F = \bigcup_{\alpha} U_F^\alpha$. 

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2.2. Bernstein-Vazirani algorithm

BV algorithm was proposed in 1993 [9]. It solves the following problem: given a function \( f(x) = a \cdot x \), where \( a \in \{0, 1\}^n \) is a secret string, find \( a \). With the access to a quantum oracle which computes \( f \) in superposition, BV algorithm works as follows:

1. Perform the Hadamard transform \( H^{(n+1)} \) on the initial state \( |\psi_0\rangle = |0\rangle^\otimes n |1\rangle \) to get
   \[
   |\psi_1\rangle = \sum_{x \in F_2^n} \frac{|x\rangle}{\sqrt{2^n}} \cdot \frac{|0\rangle - |1\rangle}{\sqrt{2}}.
   \]

2. Query the given oracle which computes \( f \) results in the state
   \[
   |\psi_2\rangle = \sum_{x \in F_2^n} \frac{(-1)^{f(x)} |x\rangle}{\sqrt{2^n}} \cdot \frac{|0\rangle - |1\rangle}{\sqrt{2}}.
   \]

3. Apply the Hadamard gates \( H^{(n)} \) to the first \( n \) qubits, giving
   \[
   |\psi_3\rangle = \sum_{y \in F_2^n} \frac{1}{2^n} \sum_{x \in F_2^n} (-1)^{f(x)+y \cdot x} |y\rangle,
   \]
   where we omit the last qubit for the simplicity. If \( f(x) = a \cdot x \), we have
   \[
   |\psi_3\rangle = \sum_{y \in F_2^n} \frac{1}{2^n} \sum_{x \in F_2^n} (-1)^{(a \oplus y) \cdot x} |y\rangle = |a\rangle.
   \]

Measuring \( |\psi_3\rangle \) in the computational basis will gives \( a \) with probability 1.

If we run BV algorithm on a general function \( f \) in \( B_n \), the resulting state before measurement will be
\[
\sum_{y \in F_2^n} S_f(y) |y\rangle,
\]
where \( S_f(\cdot) \) is the Walsh spectrum of \( f \). Measuring this state in the computational basis, we will obtain \( y \) with probability \( S_f(y)^2 \). Therefore, running BV algorithm on \( f \) always gives a vector in \( N_f \). Based on this fact, a quantum algorithm for finding nonzero linear structures of \( f \) was proposed in [13]:

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Algorithm 1
Let $p(n)$ be an arbitrary polynomial function of $n$. $\Phi$ denotes the null set. Initialize the set $H := \Phi$.
1 For $p = 1, 2, \cdots, p(n)$, do
2 Run BV algorithm with a query on the quantum oracle of $f$ to get an $n$-bit output $\omega \in \mathcal{N}_f$.
3 Let $H = H \cup \{\omega\}$.
4 end
5 Solve the equations $x \cdot H = i$ to get solution $A^i$ for $i = 0, 1$ respectively.
6 If $A^0 \cup A^1 \subseteq \{0\}$, then output “No” and halt.
7 Else, output $A^0$ and $A^1$.

For arbitrary function $f \in \mathcal{B}_n$, we let

$$
\delta_f' = \frac{1}{2^n} \max_{a \in \mathcal{F}_2} \max_{i \in \mathcal{F}_2} |\{x \in \mathcal{F}_2^n|f(x \oplus a) + f(x) = i\}|. \tag{1}
$$

It’s obviously that $\delta_f' < 1$. The smaller $\delta_f'$ is, the better for ruling out the vectors that are not linear structures of $f$ when running Algorithm 1. The following two theorems justify the validity of Algorithm 1.

Theorem 1 ([13], Theorem 4.1). If running Algorithm 1 on a function $f \in \mathcal{B}_n$ gives sets $A^0$ and $A^1$, then for all $a \in A^i$ ($i = 0, 1$), all $\epsilon$ satisfying $0 < \epsilon < 1$, we have

$$
Pr\left[1 - \frac{|\{x \in \mathcal{F}_2^n|f(x \oplus a) + f(x) = i\}|}{2^n} < \epsilon\right] > 1 - e^{-2p(n)\epsilon^2}. \tag{2}
$$

Theorem 2 ([11], Theorem 2). Suppose $\delta_f' \leq p_0 < 1$. If Algorithm 1 with $p(n) = cn$ queries returns $A^0$ and $A^1$, then for any $a \notin U^i_f$ ($i = 0, 1$),

$$
Pr[a \in A^i] \leq p_0^c. \tag{3}
$$

These two theorems are proved in [13] and [11] respectively, and we present the proofs in Appendix A for the paper to be self-contained.

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3. Impossible differential cryptanalysis

Impossible differential cryptanalysis is a chosen-plaintext attack introduced by Biham, Biryukov and Shamir [14]. We consider a r-round block cipher. Suppose $K$ and $S$ are the key space of the first $r - 1$ rounds and the key space of the last round, respectively. Let $F_k$ be the function which maps the plaintext $x$ to the input $y$ of the last round, where $k \in K$ is the right key of the first $r - 1$ round. Suppose $F_k(x) = y$, $F_k(x') = y'$, then $\Delta x = x \oplus x'$ and $\Delta y = y \oplus y'$ are called input difference and output difference respectively. $(\Delta x, \Delta y)$ is called a differential. Impossible differential cryptanalysis exploits the fact that there exists a differential $(\Delta x, \Delta y)$ of $F_k$ such that $F_k(x \oplus \Delta x) \oplus F_k(x) \neq \Delta y$, $\forall x \in F_2^n$.

This kind of differential is called impossibility differential. Impossible differential cryptanalysis is composed by two phases: (I) Find some impossible differential $(\Delta x, \Delta y)$ of $F_k$; (II) Sieve the subkey of the last round based on the differential found in the first phase. In this section, we will focus on the first phase and propose a quantum algorithm for finding impossible differentials. Unlike the algorithm proposed in [11], which finds impossible differentials directly, our one applies the miss-in-the-middle technique.

3.1. A quantum algorithm to find impossible differentials

The miss-in-the-middle technique is generally used to find impossible differentials in classical cryptanalysis. The basic idea is to connect two differential paths of probability one, whose corresponding input and output differences do not match, to obtain an impossible differential. Specially, for $v \in \{1, \cdots, r - 2\}$, we divide $F_k$ into two parts: $F_k = \hat{F}_k^{(v)} \cdot \hat{F}_k^{(1)}$, where $\hat{F}_k^{(v)}$ corresponds to the first $v$ rounds of $F_k$ and $\hat{F}_k^{(1)}$ the last $r - v$ rounds. The key space is accordingly divided into $K = K_1 \otimes K_2$, and $k = (k_1, k_2)$. If $(\Delta x_1, \Delta y_1), (\Delta x_2, \Delta y_2)$ are differentials with probability one of $\hat{F}_k^{(v)}$ and $\hat{F}_k^{(1)}$ respectively and $\Delta y_1 \neq \Delta x_2$, then $(\Delta x_1, \Delta y_2)$ will be an impossible differential of $F_k$. Therefore, the main problem is how to find a differential with probability one of a given vector function. Intuitively, we can use BV algorithm since a linear structure of a vector function can induce a differential with probability one. For example, let $\hat{F}_k^{(v)} = (\hat{F}_k^{(v)}(1), \hat{F}_k^{(v)}(2), \cdots, \hat{F}_k^{(v)}(n))$. We can obtain a differential with probability one of $\hat{F}_k^{(v)}$ by first applying Algorithm 1 to each component function $\hat{F}_k^{(v)}$ respectively, and then choosing a public linear structure as the input differential.
However, there still exists a problem: the attacker has no access to the oracle of $F_k$. He can only query the whole encryption function. In traditional impossible differential cryptanalysis, the attackers also has no the oracle access of $F_k$. Thus he analyzes the properties of the encryption algorithm and tries to find the impossible differentials that are independent of the key, i.e. the differentials that are always impossible no matter what the key is. Inspired by the method used in [11], we treat the key as a part of the input of the function and run Algorithm 1 on the new function. Specifically, suppose $K_1 = \{0, 1\}^m$ and $K_2 = \{0, 1\}^l$. We define the following two functions:

$\hat{G}^{(v)}: \{0, 1\}^n \times \{0, 1\}^m \rightarrow \{0, 1\}^n$

$(x, k_1) \rightarrow \hat{F}^{(v)}_{k_1}(x)$

$\tilde{G}^{(v)}: \{0, 1\}^n \times \{0, 1\}^l \rightarrow \{0, 1\}^n$

$(x, k_2) \rightarrow \tilde{F}^{(v)}_{k_2}(x)$

Let $\hat{G}^{(v)} = (\hat{G}_1^{(v)}, \cdots, \hat{G}_n^{(v)})$, $\tilde{G}^{(v)} = (\tilde{G}_1^{(v)}, \cdots, \tilde{G}_n^{(v)})$. Each $\hat{G}_j^{(v)}$ or $\tilde{G}_j^{(v)}$ is deterministic and known to the attack, so the oracle access to them is available. As analyzed previously, by running Algorithm 1 on each $\hat{G}_j^{(v)}$ respectively, one is expected to obtain a differential with probability one of $\hat{G}_j^{(v)}$. But to make it also a differential of $\tilde{F}^{(v)}_{k_1}$ (no matter what $k_1$ is), the last m bits of the input difference, which corresponds to the difference of the key, need to be zero. To do this, we will make a slight modification when calling Algorithm 1 as a subroutine. Based on the above analysis, we present a quantum algorithm to find impossible differentials of $F_k$ as following:

### Algorithm 2

The oracle accesses of each $\hat{G}_j^{(v)}$ and $\tilde{G}_j^{(v)}$ for $v = 1, \cdots, r-2, j = 1, \cdots, n$ are available. Let $p(n)$ be an arbitrary polynomial function of $n$. All appearing sets are initialized to the null set $\Phi$.

1. **For** $v = 1, 2, \cdots, r-2$, **do**
2.     **For** $j = 1, \cdots, n$, **do**
3.         **For** $p = 1, 2, \cdots, p(n)$, **do**
4.             Run the BV algorithm on $\hat{G}_j^{(v)}$ to get a $(n+m)$-bit output $\omega = (\omega_1, \cdots, \omega_n, \omega_{n+1}, \cdots, \omega_{n+m}) \in N_{\hat{G}_j^{(v)}}$.
5.             Let $H = H \cup \{ (\omega_1, \cdots, \omega_n) \}$.
6.         **End**
7.     **End**
8. **End**
9. Solve the equation $x \cdot H = t_{u,j}$ to get the set $A_{u,j}^{t_{u,j}}$ for $t_{u,j} = 0, 1$, respectively. And let $A_{u,j} = A_{u,j}^0 \cup A_{u,j}^1$. 

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If $A_{i,j} \subseteq \{0\}$, then break. (Exit current loop.)

Else, let $H = \Phi$.

End.

If $A_{u,1} \cap \cdots \cap A_{u,n} \subseteq \{0\}$, then continue. (Jump to the next iteration of current loop.)

Else choose an arbitrary nonzero vector $a \in A_{v,1} \cap \cdots \cap A_{v,n}$ and output $(\Delta x_1, \Delta y_1) = (a, t_{v,1}, \cdots, t_{v,n})$, where $t_{v,1}, \cdots, t_{v,n}$ is the superscripts such that $a \in A_{t_{v,1}} \cap A_{t_{v,2}} \cap \cdots \cap A_{t_{v,n}}$.

For $j = 1, \cdots, n$, do

For $p = 1, 2, \cdots, p(n)$, do

Run the BV algorithm on $\tilde{G}_v(j)$ to get a $(n + l)$-bit output $\omega = (\omega_1, \cdots, \omega_n, \omega_{n+1}, \cdots, \omega_{n+l}) \in N_{\tilde{G}_v(j)}$.

Let $H = H \cup \{(\omega_1, \cdots, \omega_n)\}$.

End

Solve the equation $x \cdot H = s_{v,j}$ to get the set $B_{v,j}^{s_{v,j}}$ for $s_{v,j} = 0, 1$, respectively.

Let $B_{v,j} = B_{v,j}^0 \cup B_{v,j}^1$ and $H = \Phi$.

End.

If $B_{v,1} \cap \cdots \cap B_{v,n} = \Phi$, then continue.

Else choose an arbitrary vector $b \in B_{v,1} \cap \cdots \cap B_{v,n}$ and let $(\Delta x_2, \Delta y_2) = (b, s_{v,1}, \cdots, s_{v,n})$, where $s_{v,1}, \cdots, s_{v,n}$ is the superscripts such that $b \in B_{s_{v,1}}^{s_{v,1}} \cap B_{s_{v,2}}^{s_{v,2}} \cap \cdots \cap B_{s_{v,n}}^{s_{v,n}}$.

If $\Delta y_1 \neq \Delta x_2$, then output $(\Delta x_1, \Delta y_2, 0)$ and halt.

Else output $(\Delta x_1, \Delta y_2, 1)$ and halt.

End.

The lines 2-12 of Algorithm 2 is for finding a linear structure $(\Delta x_1, \Delta y_1)$ of $\tilde{G}^{(v)}$, while lines 13-22 is for finding a linear structure $(\Delta x_2, \Delta y_2)$ of $\tilde{G}^{(v)}$. Once the linear structures of these two functions are found for some $v$, the algorithm halts. When the attacker executes impossible differential attack, he first constructs the quantum oracles and then runs Algorithm 2. If the attacker gets the output $(\Delta x_1, \Delta y_2, 0)$, he gets an impossible differential of $F_k$ that is independent of the key. Afterwards, he uses it to sieve the subkey of the last round as classical impossible differential cryptanalysis. If he gets $(\Delta x_1, \Delta y_2, 1)$, then he gets a differential with probability one of $F_k$. This situation is actually more conducive to sieve the subkey of the last round,
but the probability of it happening is usually very small since a general encryption scheme does not have linear structures.

3.2. Analysis of the algorithm

In this subsection, we analyze the success probability and complexity of Algorithm 2. Let

$$\delta'_G = \max \{ \delta'_{G^{(i,j)}} \mid 1 \leq i \leq r - 2, 1 \leq j \leq n \},$$

where $\delta'_{G^{(i,j)}}$ is defined as in Eq. (1). It is obviously that $\delta'_G < 1$. The smaller $\delta'_G$ is, the better for ruling out the vectors that are not linear structures of $\hat{G}^{(v)}$ or $\tilde{G}^{(v)}$ when running Algorithm 2. The following theorem justifies the validity of Algorithm 2:

**Theorem 3.** Suppose $\delta'_G \leq p_0 < 1$. If running Algorithm 2 with $p(n) = n$ gives the output $(\Delta x_1, \Delta y_2, 0)$, then $(\Delta x_1, \Delta y_2)$ is an impossible differential of $F_k$ for any $k \in K$ except a negligible probability. If it gives $(\Delta x_1, \Delta y_2, 1)$, then $(\Delta x_1, \Delta y_2)$ is a differential with probability 1 of $F_k$ for any $k \in K$ except a negligible probability.

**Proof.** Since $a \cdot (\omega_1, \ldots, \omega_n) = 0$ indicates $(a\|0, \cdots, 0) \cdot (\omega_1, \cdots, \omega_{n+m}) = 0$, the vector $(a\|0, \cdots, 0)$ can be seen as an output when we execute Algorithm 1 on $\hat{G}^{(v)}$. According to Theorem 2, it holds that

$$Pr[(a\|0, \cdots, 0) \notin U_{\hat{G}^{(v)}}^{t_v, j}] \leq p_0^n.$$

Therefore,

$$Pr[(a\|0, \cdots, 0) \notin U_{\hat{G}^{(v)}}^{t_v, j}] \leq p_0^n.$$

When $(a\|0, \cdots, 0) \in U_{\hat{G}^{(v)}}^{t_v, j}$, we have

$$\hat{G}^{(v)}((x, k_1) \oplus (a\|0, \cdots, 0)) \oplus \hat{G}^{(v)}(x, k_1) = (t_v, 1, \cdots, t_v, n), \forall x \in F_2^n, \forall k_1 \in K_1.$$

That is,

$$\hat{F}_{k_1}^{(v)}(x \oplus a) \oplus \hat{F}_{k_1}^{(v)}(x) = (t_v, 1, \cdots, t_v, n), \forall x \in F_2^n, \forall k_1 \in K_1.$$

Thus $(a, t_v, 1, \cdots, t_v, n)$ is a differential with probability 1 of $\hat{F}_{k_1}^{(v)}$ for any $k_1 \in K_1$ except a negligible probability. Similarly, $(b, s_v, 1, \cdots, s_v, n)$ is a differential
with probability 1 of \( \hat{F}_{k_2}^{(v)} \) for any \( k_2 \in \mathcal{K}_2 \) except a negligible probability.

Since \( F_k = \hat{F}_{k_2}^{(v)} \cdot \hat{F}_{k_1}^{(v)} \), the conclusion holds.

\[ \square \]

As long as the block cipher satisfies that \( \delta'_G \leq p_0 < 1 \) and has at least one impossible differential that is composed of two differential paths of probability one, Algorithm 2 will find a impossible differential of it with overwhelming probability. About the complexity of Algorithm 2, since \( p(n) = O(n) \) is enough, \( O(2(r-2)n^2) \) quantum queries are needed. Algorithm 2 executes BV subroutine for \( O(2(r-2)n^2) \) times and solves \( O(2(r-2)n) \) systems of linear equations. In addition, Algorithm 2 needs to find the intersection of \( A'_{v,j} \)s or \( B'_{v,j} \)s. The corresponding complexity depends on the size of these sets. Suppose \( q = \max_{v,j} \{|A_{v,j}|, |B_{v,j}|\} \), then the complexity of taking intersection for \( 2(r-2) \) times is \( O(2(r-2)qn \log q) \). The value of \( q \) depends on the property of the encryption algorithm. Generally speaking, \( q \) will not be large since a well constructed encryption algorithm usually does not have many linear structures. Moreover, one can also choose a greater \( p(n) \) to decrease the value of \( q \). Therefore, the complexity of Algorithm 5 is \( O(n^2) \) in total.

In traditional impossible differential attack, it is difficult to extend the differential path, which usually limits the number of rounds that can be attacked. By contrast, our approach treats \( \hat{G}^{(v)} \) and \( \hat{G}^{(v)} \) as a whole and applies BV algorithm on them directly. Thus extending the number of rounds is not a problem for our algorithm.

4. Truncated differential cryptanalysis

Truncated differential cryptanalysis was introduced by Knudsen \[15\]. In a conventional differential attack, the attacker analyzes the full difference between two texts. While the truncated variant considers differences that are only partially determined. That is, the attack makes predictions of only some of the bits instead of the full block. We still consider a \( r \)-round block cipher. Suppose \( F_k \) is the function which maps the plaintext \( x \) to the input \( y \) of the last round. \( \mathcal{K} \) and \( \mathcal{S} \) are the key space of the first \( r-1 \) rounds and the key space of the last round, respectively. Let \((\Delta x, \Delta y)\) be a differential of \( F_k \). If \( \Delta x' \) is a subsequence of \( \Delta x \) and \( \Delta y' \) is a subsequence of \( \Delta y \), then \((\Delta x', \Delta y')\) is a truncated differential of \( F_k \). In this paper, we only consider the case where \( \Delta x' = \Delta x \) is a full difference. The bits that appear in \( \Delta y' \) are called predicted bits, and the others are called unpredicted bits. If \( \Delta z \) is a full output differential of \( F_k \) and the predicted bits of \( \Delta y' \) are
equal to the corresponding bits of $\Delta z$, then we say $\Delta z$ matches $\Delta y'$, denoted $\Delta z = \Delta y'$. Truncated differential cryptanalysis is composed by two phases. In the first phase, the attacker tries to find a high probability truncated differential $(\Delta x, \Delta y')$ of $F_k$. And in the second phase, he uses the found truncated differential to recover the key of the last round. Specifically, the attacker fixes the input difference $\Delta x$ and makes queries on whole encryption function to get $2^N$ ciphers. Then for each $s \in S$, he decrypts the last round to obtain $N$ output differences, and counts the number of them that match $\Delta y'$. The right key is likely to be the one with maximum count.

The success probability and the number of pairs needed in counting scheme are related to the signal to noise ratio $[16]$, which is defined as:

$$S/N = \frac{|S| \times p}{\gamma \times \lambda},$$

where $p$ is the probability of the truncated differential used in the attack, $|S|$ is the number of possible candidate keys of the last round, $\gamma$ is the average number of keys suggested by each pair of plaintexts and $\lambda$ is the ratio of non-discarded pairs to all pairs. In this paper we only consider the case where $\lambda = 1$. If $S/N \leq 1$, then a differential attack will not succeed. The larger $S/N$ is, the better for executing truncated differential attack. See Appendix B for further details.

The quantum algorithm that we will propose is applied in the first phase of truncated differential cryptanalysis, i.e. finding a high probability truncated differential. However, unlike classical truncated differential cryptanalysis, the truncated differential found by our algorithm has high probability only for partial keys in $\mathcal{K}$. Specifically, suppose $q(n)$ is an arbitrary polynomial. The attacker can execute the algorithm properly so that it works for at least $(1 - \frac{1}{q(n)})$ of the keys in $\mathcal{K}$.

4.1. A quantum algorithm to find truncated differentials

Suppose the length of the keys in $\mathcal{K}$ is $m$. As before, we define the vector function

$$G : \{0, 1\}^{n+m} \rightarrow \{0, 1\}^n$$

$$(x, k) \rightarrow F_k(x).$$

Let $G = (G_1, \cdots, G_n)$. The oracle access of each $G_j$ is available. A algorithm to find a high probability truncated differential of $F_k$ is as follows:
Algorithm 3
The oracle access of each $G_j$ ($1 \leq j \leq n$) is given. $l(n)$ and $q(n)$ are two arbitrary polynomials chosen by the attacker. Let $p(n) = \frac{1}{2}l(n)^2q(n)^2n^3$ and initialize the set $H := \Phi$.

1. For $j = 1, 2, \cdots, n$, do
2.   For $p = 1, \cdots, p(n)$, do
3.     Run the BV algorithm on $G_j$ to get a $(n + m)$-bit output $\omega = (\omega_1, \cdots, \omega_n, \omega_{n+1}, \cdots, \omega_{n+m}) \in N_{G_j}$.
4.     Let $H = H \cup \{(\omega_1, \cdots, \omega_n)\}$.
5.   End
6. Solve the equation $x \cdot H = i_j$ to get the set $A_{j}^{i_j}$ for $i_j = 0, 1$, respectively.
7. Let $A_j = A_j^0 \cup A_j^1$ and $H = \Phi$.
8. End.
9. If there exists $t \in \{1, 2, \cdots, n\}$ that satisfies: (1) there are $t$ numbers $j_1, \cdots, j_t \in \{1, \cdots, n\}$ such that $A_{j_1} \cap \cdots \cap A_{j_t} \supseteq \{0\}$; (2) $S/N = 2^t(1 - \frac{1}{l(n)}) > 1$, then choose the value of $t$ as large as possible, and choose an arbitrary nonzero vector $a \in A_{j_1} \cap \cdots \cap A_{j_t}$. Let

$$b_j = \begin{cases} i_j, & j \in \{j_1, \cdots, j_t\} \\ \times, & j \notin \{j_1, \cdots, j_t\}, \end{cases}$$

where $i_j$ is the superscript such that $a \in A_{j}^{i_j}$. Output $(a, b)$.
10. Else output “No”.

The output $(a, b)$ of Algorithm 3 is a truncated differential of $F_k$. The symbol “$\times$” in vector $b$ stands for the unpredicted bits. When the attacker executes truncated differential attack, he first chooses proper polynomials $l(n), q(n)$ and then runs Algorithm 3 to get $(a, b)$. According to Theorem 4 presented in last subsection, $(a, b)$ is a truncated differential of $F_k$ that has high probability for most $k$ in $\mathcal{K}$. Specifically, there exists a subset $\mathcal{K}'$ of $\mathcal{K}$ such that $|\mathcal{K}'|/|\mathcal{K}| \geq 1 - \frac{1}{q(n)}$, and for any $k \in \mathcal{K}'$, $(a, b)$ is a truncated differential of $F_k$ with a probability greater than $1 - \frac{1}{l(n)}$. To justify the feasibility of recovering the key of the last round using $(a, b)$, we need to compute the signal to noise ratio $S/N$. We first estimate the value of $\gamma$, i.e. the average number of keys suggested by each pair of plaintexts. Since
$t$ bits of $b$ are predicted, there are $2^{n-t}$ output differences matching $b$ in total. In counting scheme, each pair of plaintexts will be decrypted by $|S|$ keys respectively. The corresponding $|S|$ output differences obtained by this process can be seen as random. Thus for each pair of plaintexts, there are $\gamma = \frac{2^{n-t}}{2^n} \times |S| = \frac{|S|}{2^t}$ keys counted in average. Therefore,

$$N/S = \frac{|S| \times (1 - \frac{1}{l(n)})}{|S| \times 1} = 2^t (1 - \frac{1}{l(n)}) > 1.$$  

The attacker can use truncated differential $(a, b)$ to recover the subkey of the last round as in classical truncated differential attack. This attack works for at least $\left(1 - \frac{1}{q(n)}\right)$ of keys in $\mathcal{K}$. Even if Algorithm 3 outputs “No”, the attacker can adjust the values of $q(n)$, $l(n)$ and try again.

4.2. Analysis of the algorithm

In this section, we analyze the success probability and complexity of Algorithm 3. To do this, we first give following theorem:

**Theorem 4.** If running Algorithm 3 with $p(n) = \frac{1}{2}l(n)q(n)^2n^3$ quantum queries on $G$ gives a truncated differential $(a, b)$, then there exists a subset $\mathcal{K}' \subseteq \mathcal{K}$ such that $|\mathcal{K}'|/|\mathcal{K}| \geq 1 - \frac{1}{q(n)}$, and for any $k \in \mathcal{K}'$, it holds that

$$Pr\left[\left|\{x \in \mathbb{F}_2^n \mid F_k(x \oplus a) + F_k(x) = b\}\right| > 1 - \frac{1}{l(n)}\right] > 1 - ne^{-n}.$$  

That is, except a negligible probability, the output $(a, b)$ is a truncated differential of $F_k$ with a probability greater than $1 - \frac{1}{l(n)}$. (Here “=” means $F_k(x \oplus a) + F(x)$ matches $b$.)

**Proof.** Since $a \cdot (\omega_1, \cdots, \omega_n) = 0$ indicates $(a \parallel 0, \cdots, 0) \cdot (\omega_1, \cdots, \omega_{n+m}) = 0$, the vector $(a \parallel 0, \cdots, 0)$ can be seen as an output when we execute Algorithm 3 on $G_j$, $s = 1, \cdots, t$. According to Theorem 1, we have

$$\left|\{z \in \mathbb{F}_2^{n+m} \mid G_j(z \oplus (a \parallel 0, \cdots, 0) \oplus G_{j_s}(z) = b_{j_s})\}\right| > 1 - \epsilon, \ \forall s = 1, 2 \cdots, t$$  

(3)

holds with a probability greater than $(1 - e^{-2p(n)c^2})^n$. If Eq.(3) holds, then the number of $z$ satisfying

$$G_{j_s}(z \oplus (a \parallel 0, \cdots, 0) \oplus G_{j_s}(z) = b_{j_s})$$  

(4)
for both $s = 1$ and $s = 2$ is at least $2^{n+m}(2(1-\epsilon)-1) = 2^{n+m}(1-2\epsilon)$. Similarly, the number of $z$ satisfying Eq.(4) for all $s = 1, 2, 3$ is at least $2^{n+m}(1-3\epsilon)$. By induction, the number of $z$ satisfying Eq.(4) for all $s = 1, 2, \ldots, t$ is at least $2^{n+m}(1-t\epsilon)$. Thus with a probability greater than $(1-e^{-2p(n)\epsilon^2})^n$, it holds that
\[
\frac{\left|\{z \in F_2^{n+m} | G(z \oplus (a\|0, \ldots, 0) \oplus G(z) = b)\} \right|}{2^{n+m}} > 1 - t\epsilon.
\]
Eq.(5) indicates $E_k(V(k)) > 1 - t\epsilon$, where $E_k(\cdot)$ means the expectation when the key $k$ is chosen uniformly at random from $\mathcal{K}$. Therefore, if Eq.(5) holds, then for any polynomial $q(n)$, we have
\[
Pr_k[V(k) > 1 - q(n)t\epsilon] > 1 - \frac{1}{q(n)}.
\]
That is, for at least $(1 - \frac{1}{q(n)})$ of keys in $\mathcal{K}$, it holds that $V(k) > 1 - q(n)t\epsilon$. Let $\mathcal{K}'$ be the set of these keys, then $|\mathcal{K}'|/|\mathcal{K}| \geq 1 - \frac{1}{q(n)}$ and for all $k \in \mathcal{K}'$, it holds that
\[
Pr[V(k) > 1 - q(n)t\epsilon] > (1-e^{-2p(n)\epsilon^2})^n.
\]
Let $\epsilon = \frac{1}{l(n)q(n)\epsilon}$. Noticing that $p(n) = \frac{1}{2}l(n)^2q(n)^2n^3$, we have
\[
Pr\left[\frac{\left|\{x \in F_2^n | F_k(x \oplus a) + F_k(x) = b\} \right|}{2^n} > 1 - \frac{1}{l(n)}\right] > 1 - ne^{-n},
\]
which completes the proof.

When executing truncated differential cryptanalysis, the attacker first runs Algorithm 3. Except a negligible probability, the truncated differential $(a, b)$ he obtains has probability greater than $1 - \frac{1}{l(n)}$ for at least $(1 - \frac{1}{q(n)})$ of keys in $\mathcal{K}$. Then the attacker uses it to recover the subkey of the last round as in classical truncated differential attack. This attack works for at least $(1 - \frac{1}{q(n)})$ of keys in $\mathcal{K}$. The number of pairs needed by the counting scheme is related to the signal to noise ratio. (See [16] or Appendix B for more details.) The higher $S/N$ is, the less pairs of plaintexts are needed. For example, it is observed experimentally that when $S/N$ is about $1 - 2$. 

\[\square\]
about 20-40 occurrences of right pairs are sufficient. Thus $O\left(\frac{40}{1 - \frac{1}{l(n)}}\right)$ pairs of plaintexts are enough.

To analyze the complexity of Algorithm 3, we divide it into two parts: running BV algorithm to obtain the sets $A_j$'s; and finding truncated differential $(a, b)$ by taking the intersection of some of these sets. In the first part, Algorithm 3 needs to run BV algorithm for $np(n) = \frac{1}{2}l(n)^2q(n)^2n^4$ times. Thus $O(l(n)^2q(n)^2n^4)$ quantum queries are needed. In the second part, the attacker first let $t = n$, if $A_1 \cap \cdots \cap A_n \supseteq \{0\}$, then chooses an arbitrary nonzero vector $a$ in the intersection. Otherwise, the attacker considers $t = n - 1$. If there exist $n - 1$ sets $A_{i_1}, \ldots, A_{i_{n-1}}$ such that $A_{i_1} \cap \cdots \cap A_{i_{n-1}} \supseteq \{0\}$, then he chooses an arbitrary nonzero vector $a$ in the intersection. Otherwise, let $t = n - 2$. The attacker continues this process until the value of $t$ is too small to satisfy the condition $\left(\frac{S}{N} = 2^t(1 - \frac{1}{l(n)}) > 1\right)$ or the amount of calculation for finding intersection is too large. Specifically, suppose $\alpha = \max_j |A_j|$, then finding $t$ sets whose intersection has nonzero vector needs $O({n \choose t}\alpha \log \alpha)$ times calculation. The value of $\alpha$ depends on the properties of the block cipher. Generally speaking, $\alpha$ will not be large since a well constructed encryption algorithm usually does not have many approximate linear structures. In addition, the attacker can reduce the value of $\alpha$ by choosing a larger $p(n)$. Anyway, the attacker tries from $t = n, n - 1, n - 2 \cdots$ until $2^t(1 - \frac{1}{l(n)}) \leq 1$ or $n \choose t\alpha \log \alpha$ is so large that it exceeds the attacker’s computational power.

One of the advantages of our algorithm is that it can find the high probability truncated differential directly. While in classical case, the attacker needs to analyze the properties of each round of the cipher to find a truncated differential path with a high probability, which may be much more complicated with the increase of the number of rounds.

5. Discussion and conclusion

In this paper, we propose quantum algorithms to execute impossible differential cryptanalysis and truncated differential cryptanalysis, respectively. Two algorithms both take BV algorithm as subroutine. Afterwards, we analyze the succeed probability and complexity of them. We believe our work provides some helpful and inspirational methods for quantum cryptanalysis.
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Appendix A. Proofs of Theorem 1 and Theorem 2

In this section, we present the proofs of Theorem 1 and Theorem 2, which can be found in [13] and [11], respectively. Before that, we first give the following lemma:

Lemma 2 ([13], Theorem 3.1). Let \( f \in B_n \), then \( \forall a \in F_2^n, \forall i \in F_2 \),

\[
\sum_{\omega \cdot a} S^2_f(\omega) = \frac{|\{x \in F_2^n | f(x \oplus a) + f(x) = i\}|}{2^n}.
\]

Now we give the proof of Theorem 1, which is stated as following:

Theorem 1 ([13], Theorem 4.1) If running Algorithm 1 on a function \( f \in B_n \) gives sets \( A^0 \) and \( A^1 \), then for all \( a \in A^i \) \((i = 0, 1)\), all \( \epsilon \) satisfying \( 0 < \epsilon < 1 \), we have

\[
Pr\left[1 - \frac{|\{x \in F_2^n | f(x \oplus a) + f(x) = i\}|}{2^n} < \epsilon \right] > 1 - e^{-2p(n)x^2}.
\]
Proof. For all \( a \in A^i \) (\( i = 0, 1 \)),
\[
Pr_x[f(x \oplus a) + f(x) = i] = \frac{|\{x \in F^n_2 \mid f(x \oplus a) + f(x) = i\}|}{2^n} = \frac{|V^i_{f,a}|}{2^n}.
\]
Let \( p = |V^i_{f,a}|/2^n \) and \( q = 1 - p \), then \( p, q \in [0, 1] \). We define a random variable \( Y \) as following:
\[
Y(\omega) = \begin{cases} 0, & \omega \cdot a = i; \\ 1, & \omega \cdot a \neq i. \end{cases}
\]
According to Lemma 2, the expectation of \( Y \) is \( E(Y) = 1 \cdot q = 1 - p \). The \( p(n) \) times of running the BV algorithm produce \( p(n) \) independent identical random variables \( Y_1, \ldots, Y_{p(n)} \). By Hoeffding’s inequality,
\[
Pr\left[q - \frac{1}{p(n)} \sum_{j=1}^{p(n)} Y_j \geq \epsilon \right] \leq e^{-2p(n)\epsilon^2}.
\]
Note that \( a \in A^i \), we have \( \sum_j Y_j \) must be 0 (otherwise there exists some \( Y_j = 1 \), then \( a \notin A^i \)). Thus \( Pr[q \geq \epsilon] \leq e^{-2p(n)\epsilon^2} \). This indicates
\[
Pr[1 - p < \epsilon] = Pr[q < \epsilon] > 1 - e^{-2p(n)\epsilon^2},
\]
which completes the proof.

\[\square\]

Theorem 2 ([11], Theorem 2) If running Algorithm 1 on a function \( f \in B_n \) gives sets \( A^0 \) and \( A^1 \), then for all \( a \in A^i \) (\( i = 0, 1 \)), all \( \epsilon \) satisfying \( 0 < \epsilon < 1 \), we have
\[
Pr\left[1 - \frac{|\{x \in F^n_2 \mid f(x \oplus a) + f(x) = i\}|}{2^n} < \epsilon \right] > 1 - e^{-2p(n)\epsilon^2}. \quad (A.1)
\]
Proof. Without loss of generality, we suppose \( i = 0 \). The case where \( i = 1 \) can be proved by similar way. If \( a \notin U^0_{f} \), then according to Lemma 1 there exists a vector \( \omega \in N_f \) such that \( \omega \cdot a = 1 \). Let \( K = \{\omega \in N_f \mid \omega \cdot a = 1\} \). If the \( cn \) times of running BV algorithm ever gives a vector \( \omega \in K \), then \( a \notin A^0 \). Let \( W \) denote the random variable obtained by running BV algorithm, then
\[
Pr[W \in K] = \sum_{a, \omega=1} S^2_f(\omega) = 1 - \frac{|V^0_{f,a}|}{2^n} \geq 1 - p_0.
\]
The second formula holds due to Lemma 2. Therefore,

\[ P[a \in A^0] = [1 - P(W \in K)]^n \leq p_0^n, \]

which completes the proof.

\[ \square \]

Appendix B. The signal to noise ratio

In this section we recall the notion of the signal to noise ratio \[16\], which gives us a tool to evaluate the usability of a counting scheme based on a differential.

**Definition 4.** The ratio between the number of times the right key is counted and the number of times a random key is counted is called the signal to noise ratio of the counting scheme and is denoted by \( S/N \).

Suppose the key space of the last round of the encryption algorithm is \( S \), then \( |S| \) is the number of possible values of the subkey. Let \( \gamma \) be average number of keys suggested by each pair of plaintexts and \( \lambda \) be the ratio of non-discarded pairs to all pairs. (There may be process to discarded the wrong pairs before they are actually counted.) The number of times a random key is counted is \( m \cdot \alpha \gamma \lambda / |S| \), where \( m \) is the number of pairs. Suppose \( p \) is the probability if the differential (or truncated differential) used in the attack, then the number of times the right key is counted is about \( m \cdot p \). The signal to noise ratio of a counting scheme is therefore

\[ S/N = \frac{m \cdot p}{m \cdot \gamma \lambda / |S|} = \frac{|S| \cdot p}{\gamma \lambda}. \]

If \( S/N \leq 1 \), then a differential attack will not succeed. In counting scheme, if decrypting a pair of ciphertexts will gives a output difference that matches the truncated output difference of the given high probability truncated differential, then this pair of ciphertexts is called right pair. The number of pairs needed by a counting scheme is usually related to the number of right pairs needed, which is mainly a function of the signal to noise ratio. When \( S/N \) is high enough, only a few occurrences of right pairs are needed to determine the value of the subkey. It’s observed experimentally \[16\] that
when $S/N$ is about $1 - 2$, about $20 - 40$ right pairs are usually enough. When $S/N$ is much higher even $3 - 4$ right pairs are sufficient. On average, about $O\left(\frac{1}{p}\right)$ pairs of ciphertexts gives a right pair. Thus when $S/N$ is $1 - 2$, the number of pairs needed is $O\left(\frac{40}{p}\right)$.

We present a example in [15] here to explain the relation between $S/N$ and the complexity of the counting scheme. A 5-round cipher uses round function

$$f(x, k) = (x \oplus k)^{-1}$$

in $GF(2^n)$ for an odd $n$. This cipher is high resistant against differential attack using full differentials since any 3-round differential has a probability at most $2^{3-2n}$. Considering the counting scheme on the round key of the last round, the signal to the noise ratio is

$$S/N < \frac{2^n \times 2^{3-2n}}{1 \times 1} < 1$$

for $n > 3$. The differential attack will not succeed. In a attack counting on the round keys of the last two rounds, the maximum probability of 2-round differentials is $2/2^n$. Thus

$$S/N = \frac{2^{2n} \times 2^{1-n}}{1 \times 1} = 2^{n+1}.$$  

The attack will succeed with $O(2^n)$ chosen plaintexts. If we consider truncated differential attack, there are a 2-round differential of probability 1, where only one bit of the differential is predicted. Therefore,

$$S/N = \frac{2^{2n} \times 1}{2^{2(n-1)} \times 1} = 2.$$  

The attack will succeed with sufficiently many pairs. The author in [15] used 18 pairs to attack on a 5-round 18 bit cipher successfully.