SEMISIMPLICITY IN SYMMETRIC RIGID TENSOR CATEGORIES

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Dedicated to Susan Montgomery in honor of her 65th birthday

Abstract. Let $\lambda$ be a partition of a positive integer $n$. Let $C$ be a symmetric rigid tensor category over a field $k$ of characteristic 0 or $\text{char}(k) > n$, and let $V$ be an object of $C$. In our main result (Theorem 4.3) we introduce a finite set of integers $F(\lambda)$ and prove that if the Schur functor $S_\lambda V$ of $V$ is semisimple and the dimension of $V$ is not in $F(\lambda)$, then $V$ is semisimple. Moreover, we prove that for each $d \in F(\lambda)$ there exist a symmetric rigid tensor category $C$ over $k$ and a non-semisimple object $V \in C$ of dimension $d$ such that $S_\lambda V$ is semisimple (which shows that our result is the best possible). In particular, Theorem 4.3 extends two theorems of Serre for $C = \text{Rep}(G)$, $G$ is a group, and $S_\lambda V$ is $\wedge^n V$ or $\text{Sym}^n V$, and proves a conjecture of Serre (S1).

1. Introduction

Let $G$ be any group, let $k$ be a field and let $\text{Rep}(G)$ be the category of finite dimensional representations of $G$ over $k$. A classical result of Chevalley states that in characteristic 0, the tensor product $V \otimes W$ of any two semisimple objects $V, W \in \text{Rep}(G)$ is also semisimple [C]. Later on, Serre proved that this is also the case in positive characteristic $p$, provided that $\dim V + \dim W < p - 2$ [S2].

In [S1], Serre considered the “converse theorems”, and proved that $V \in \text{Rep}(G)$ is semisimple in each one of the following situations: there exists $W \in \text{Rep}(G)$ such that $\dim W \neq 0$ in $k$ and $V \otimes W$ is semisimple (Theorem 2.4, loc.cit), $V^\otimes n$ is semisimple for some $n \geq 1$ (Theorem 3.4, loc.cit), $\wedge^n V$ is semisimple for some $n \geq 1$ and $\dim V \neq 2, \ldots, n$ in $k$ (Theorem 5.2.5, loc.cit), or $\text{Sym}^n V$ is semisimple for some $n \geq 1$ and $\dim V \neq -n, \ldots, -2$ (Theorem 5.3.1, loc.cit).

Furthermore, Serre comments that it is easy to check that all the above mentioned results from [S1] extend to categories of linear representations of Lie algebras and restricted Lie algebras (when $p > 0$) (loc.cit, p. 510). Moreover, Serre explains how to extend his results Theorem 2.4 and Theorem 3.4 (loc.cit) to any symmetric rigid tensor category over $k$, and says on p.511 (loc.cit): “I have not managed to rewrite the proofs in tensor category style. Still, I feel that Theorem 5.2.5 on $\wedge^n V$ and Theorem 5.3.1 on $\text{Sym}^n V$ should remain true whenever $n! \neq 0$ in $k$, i.e., $p = 0$ or $p > n$.” This paper originated in an attempt to prove this conjecture of Serre.

A further natural generalization of Serre’s results would be to consider any Schur functor $S_\lambda$, and not only $\wedge^n$ and $\text{Sym}^n$. Namely, to look for an extension of Theorem 5.2.5 and Theorem 5.3.1 in [S1], where $C$ is any symmetric rigid tensor category.
over \( k \), and \( V \in \mathcal{C} \) is an object for which \( S_\lambda V \) is semisimple for some partition \( \lambda \) of \( n \). This is precisely the main purpose of this paper.

The paper is organized as follows. In Section 2 we note that in fact Theorem 2.4 from [S1] holds in a much more general situation than the symmetric one. More precisely, let \( \mathcal{C} \) be any rigid tensor category, and suppose that \( W \in \mathcal{C} \) is isomorphic to its double dual \( W^{**} \) via an isomorphism \( i \). This allows to define a scalar \( \dim_i(W) \) in \( k \), and we show that if \( \dim_i(W) \neq 0 \) and \( V \otimes W \) is semisimple, then \( V \) is semisimple (see Theorem 2.3). Examples, other than \( \mathcal{C} = \text{Rep}(G) \), are given by braided rigid tensor categories \( \mathcal{C} \) and by representation categories \( \mathcal{C} \) of Hopf algebras whose squared antipode is inner.

In Section 3 we note that Theorem 3.3 and Corollary 3.4 from [S1] hold in a much more general situation than the symmetric one, as well. More precisely, let \( \mathcal{C} \) be any rigid tensor category satisfying the commutativity condition, and let \( V \in \mathcal{C} \). We show that if \( V^\otimes n \otimes (V^\ast)^\otimes m \) is semisimple for some \( m, n \geq 0 \), not both equal to 0, then \( V \) is semisimple. In particular, if \( V^\otimes n \) is semisimple for some \( n \geq 1 \) then \( V \) is semisimple (see Theorem 3.1). Examples, other than \( \mathcal{C} = \text{Rep}(G) \), are given by braided rigid tensor categories \( \mathcal{C} \).

In Section 4 we state the main result of the paper (Theorem 4.3), and prove various results in preparation for its proof. Our main result extends Theorem 5.2.5 on \( \wedge^n \) and Theorem 5.3.1 on \( \text{Sym}^n \) in the group case \( \mathcal{C} = \text{Rep}(G) \) [S1], to any symmetric rigid tensor category \( \mathcal{C} \) over \( k \) and any Schur functor \( S_\lambda \) (so, in particular, it provides a proof to the conjecture of Serre ([S1], p.511)). More precisely, let \( \lambda \) be a partition of a positive integer \( n \), and assume that \( \text{char}(k) = 0 \) or \( \text{char}(k) > n \).

Let \( S_\lambda \) be the associated Schur functor (see [D2]) and let \( V \) be an object of \( \mathcal{C} \). In Theorem 4.3 we introduce a finite set of integers \( F(\lambda) \) and prove that if the dimension of \( V \) is not equal in \( k \) to an element of \( F(\lambda) \) and \( S_\lambda V \) is semisimple, then \( V \) is semisimple. Moreover, we prove that for each \( d \in F(\lambda) \) there exist a symmetric rigid tensor category \( \mathcal{C} \) over \( k \) and a non-semisimple object \( V \in \mathcal{C} \) of dimension \( d \) such that \( S_\lambda V \) is semisimple (which shows that our result is the best possible).

Section 5 is devoted to the proof of Theorem 4.3.

All tensor categories will be assumed to be rigid, \( k \)-linear Abelian, with finite dimensional Hom spaces, such that every object has a finite length, and \( \text{End}(1) = k \).

Acknowledgments. The author is grateful to Pavel Etingof for his encouragement, for his interest in the problem, and for stimulating and helpful discussions.

The author was supported by The Israel Science Foundation (grant No. 317/09).

2. FROM \( V \otimes W \) TO \( V \) IN RIGID TENSOR CATEGORIES

Let \( \mathcal{C} \) be a rigid tensor category. For an object \( V \in \mathcal{C} \) we let

\[
\text{coev}_V : 1 \rightarrow V \otimes V^* \quad \text{and} \quad \text{ev}_V : V^* \otimes V \rightarrow 1
\]

denote the coevaluation and evaluation maps associated to \( V \), respectively. Recall that

\[
(id_V \otimes \text{ev}_V) \circ (\text{coev}_V \otimes id_V) = id_V.
\]

The following two propositions were proved by Serre for \( \mathcal{C} := \text{Rep}(G) \), \( G \) any group [S1]. However, it is straightforward to verify that the same proofs work in any rigid tensor category \( \mathcal{C} \).
Proposition 2.1. ([S1, Proposition 2.1]) Let $V, W \in \mathcal{C}$, and let $V'$ be a sub-object of $V$. Assume that $\text{coev}_W: 1 \to W \otimes W^*$ and $V' \otimes W \to V \otimes W$ are split injections. Then $V' \to V$ is a split injection.

Proposition 2.2. ([S1, Proposition 2.3]) Assume $\text{coev}_W: 1 \to W \otimes W^*$ is a split injection and that $V \otimes W$ is semisimple. Then $V$ is semisimple.

One instance in which $\text{coev}_W: 1 \to W \otimes W^*$ is a split injection is the following. Assume that $W \in \mathcal{C}$ is isomorphic to its double dual $W^{**}$, and fix an isomorphism $i: W \to W^{**}$. This allows us to define the (quantum) dimension $\dim_i(W)$ of $W$ (relative to $i$) as the composition

$$\dim_i(W) := \text{ev}_W \circ (i \otimes \text{id}_{W^*}) \circ \text{coev}_W.$$ 

Note that $\dim_i(W) \in \text{End}(1) = k$. Now, clearly if $\dim_i(W) \neq 0$ in $k$, then $\text{coev}_W$ is a split injection. (See Remark 2.2 in [S1].)

As a consequence of Proposition 2.1 and Proposition 2.2 we have the following theorem, which generalizes Theorem 2.4 in [S1].

Theorem 2.3. Assume that $W \in \mathcal{C}$ is isomorphic to its double dual $W^{**}$ and let $i: W \to W^{**}$ be an isomorphism. If $V \otimes W$ is semisimple and $\dim_i(W) \neq 0$ in $k$ then $V$ is semisimple.

Remark 2.4. 1) It is known that if $\mathcal{C}$ is braided, any object $W$ is isomorphic to its double dual $W^{**}$. So in particular, if $H$ is a quasitriangular (quasi)Hopf algebra over $k$ and $V, W \in \text{Rep}(H)$ such that $V \otimes W$ is semisimple and $\dim W \neq 0$ in $k$, then $V$ is semisimple. The converse is not true.

2) If $H$ is a Hopf algebra whose squared antipode $S^2$ is inner (e.g., $S^2 = \text{id}$) then any $W \in \text{Rep}(H)$ is isomorphic to $W^{**}$. Therefore Theorem 2.3 holds for $\text{Rep}(H)$.

3) When $\mathcal{C}$ is symmetric, Serre already pointed out that Theorem 2.4 in [S1] holds for $\mathcal{C}$, with the same proof (see p. 510–511 in [S1]).

3. FROM $V^\otimes n \otimes V^\otimes m$ TO $V$ IN RIGID TENSOR CATEGORIES

The following theorem was proved by Serre for $\mathcal{C} := \text{Rep}(G)$, $G$ any group [S1]. Serre also explains that the same proof works in any symmetric rigid tensor category $\mathcal{C}$. In fact, the symmetry is used only to guarantee that for any $V \in \mathcal{C}$ the morphism

$$\text{id}_V \otimes \text{coev}_V: V \to V \otimes V \otimes V^*$$

is a split injection. We just note that in fact this is the case in any rigid tensor category $\mathcal{C}$ satisfying the following commutativity condition: there exists a functorial isomorphism $c: \otimes \to \otimes^{op}$ such that $c_{V \otimes 1} = c_{1 \otimes V} = \text{id}_V$ for any $V \in \mathcal{C}$ (e.g., $\mathcal{C}$ is braided, not necessarily symmetric). Indeed, let $\mathcal{C}$ be a rigid tensor category satisfying the coboundary condition. Then, using the naturality of $c$, one has

$$\text{(1)} \quad (\text{id}_V \otimes \text{ev}_V) \circ c_{V, V} \otimes \circ (\text{id}_V \otimes \text{coev}_V) = (\text{id}_V \otimes \text{ev}_V) \circ (\text{coev}_V \otimes \text{id}_V) = \text{id}_V.$$ 

Therefore we have the following result, which generalizes Theorem 3.3 and Corollary 3.4 in [S1].

Theorem 3.1. Let $\mathcal{C}$ be a rigid tensor category satisfying the commutativity condition, and let $V \in \mathcal{C}$. If $V^\otimes n \otimes V^\otimes m$ is semisimple for some $m, n \geq 0$, not both equal to 0, then $V$ is semisimple. In particular, if $V^\otimes n$ is semisimple for some $n \geq 1$ then $V$ is semisimple.
4. FROM $S\lambda V$ TO $V$ IN SYMMETRIC RIGID TENSOR CATEGORIES

In this section we assume that $C$ is a symmetric rigid tensor category over a field $k$, with a commutativity constraint $c$ (see e.g., [D1], [D2]).

4.1. Schur functors in $C$. Recall that given an object $X \in C$ and a nonnegative integer $m$, the symmetric group $S_m$ acts on $X^\otimes m$ via the symmetry $c$. Let $\beta$ be a partition of $m$, and assume that $\text{char}(k) > m$ if $\text{char}(k) \neq 0$. Let $V_\beta$ be the corresponding irreducible representation of $S_m$ and let $c_\beta \in k[S_m]$ be a Young symmetrizer associated with $V_\beta$. Then $c_\beta$ gives rise to a functor

$$c_\beta : C \to C, \ X \mapsto c_\beta (X^\otimes m).$$

Recall that the isomorphism type of the functor $c_\beta$ does not depend on the choice of $c_\beta$. We shall call $S_\beta X := c_\beta (X^\otimes m) \subseteq X^\otimes m$ the Schur functor of $X$ associated with $\beta$.

Schur functors in symmetric rigid tensor categories were introduced (more conceptually) and studied by Deligne in [D2]. Among many other things, it is proved there that for any object $X \in C$, $(S_\beta X)^*$ is canonically isomorphic to $S_\beta X^*$, a fact we shall use often in the sequel.

Example 4.1. Note in particular that $S_{(0)} X = 1$, $S_{(1)} X = X$, $S_{(m)} X = \text{Sym}^m X$ and $S_{(1^m)} X = \wedge^m X$.

4.2. The main result. Our goal is to generalize Theorem 5.2.5 and Theorem 5.3.1 from [S1] by replacing representations categories $\text{Rep}(G)$ of groups by any symmetric rigid tensor category $C$, and by replacing the Schur functors $\wedge^n$, $\text{Sym}^n$ by any Schur functor. More precisely, let $\lambda$ be a partition of a positive integer $n$ and let $V \in C$. Our goal is to find out when the semisimplicity of $S_\lambda V$ implies the semisimplicity of $V$, in terms of the dimension of $V$ only.

Fix a partition $\lambda$ of a positive integer $n$, with $p := p(\lambda)$ rows and $q := q(\lambda)$ columns, and let $(i, j)$ number the row and column of boxes for the Young diagram of $\lambda$. Let us introduce some notation.

- Let $R(\lambda)$ denote the integer interval $\{-q, \ldots, p\}$, and let $T(\lambda) \subseteq R(\lambda)$ include 0 if $\lambda$ is a hook (i.e., $(2, 2) \notin \lambda$), 1 if $(3, 2) \notin \lambda$, $-1$ if $(2, 3) \notin \lambda$, and $-p, -q$ if $\lambda$ is not a rectangle. Set $F(\lambda) := R(\lambda) \setminus T(\lambda)$.
- Let $G(\lambda)$ denote the set of all values $d$ in for which there exists a symmetric rigid tensor category $C$ over $k$ with a non-semisimple object $V$ of dimension $d$ such that $S_\lambda V$ is semisimple.

Remark 4.2. 1) We have that $F(\lambda) = -F(\lambda^*)$, where $\lambda^*$ is the conjugate of $\lambda$.

2) We have that $G(\lambda) = -G(\lambda^*)$. Indeed, if $(C, V)$ is a counterexample for $(\lambda, d)$ (i.e., $C$ is a symmetric rigid tensor category over $k$ with a non-semisimple object $V$ of dimension $d$ such that $S_\lambda V$ is semisimple) then $(C \boxtimes \text{Supervect}, V \otimes 1^{-1})$ is a counterexample for $(\lambda^*, -d)$, where Supervect is the category of finite dimensional super vector spaces over $k$ and $1^{-1} \in \text{Supervect}$ is the odd 1–dimensional space.

We can now state our main result concisely.

Theorem 4.3. Let $n$ be a positive integer, $n < \text{char}(k)$ in case $\text{char}(k) \neq 0$, and let $\lambda$ be a partition of $n$. Then the sets $F(\lambda)$ and $G(\lambda)$ coincide (where we view the relevant integers as elements of $k$ in an obvious way).
Example 4.4. Let $\mathcal{C}$ be a symmetric rigid tensor category over $k$, and let $V \in \mathcal{C}$.

1) Theorem 4.3 implies for $\lambda = (1^n)$ (respectively, $\lambda = (n)$), that if $S_{\lambda}V$ is semisimple and the dimension of $V$ is not equal in $k$ to an integer in the range $2, \ldots, n$ (respectively, $-n, \ldots, -2$), then $V$ is semisimple. For $\mathcal{C} = \text{Rep}(G)$, $G$ is any group, this is Theorem 5.2.5 from [S1] (respectively, Theorem 5.3.1 from [S1]).

2) Theorem 4.3 implies that if $S_{(2,1)}V$ is semisimple then so is $V$.

The proof of Theorem 4.3 is given in Section 5. The rest of this section is devoted to preparations for the proof.

4.3. Traces in $\mathcal{C}$. For an object $X \in \mathcal{C}$, let

$$\widetilde{ev}_X := ev_X \circ c_{X,X^*} : X \otimes X^* \to 1.$$ 

Recall that the dimension $\dim X \in k$ of $X$ is defined by

$$\dim X := \widetilde{ev}_X \circ coev_X : 1 \to 1.$$ 

In [JSV] it is explained that the family of functions

$$Tr_{A,B}^U : \text{Hom}(A \otimes U, B \otimes U) \to \text{Hom}(A, B), \ A, B, U \in \mathcal{C},$$

defined by

$$Tr_{A,B}^U(f) : A \xrightarrow{id_A \otimes coev_U} A \otimes U \otimes U^* \xrightarrow{f \otimes id_{U^*}} B \otimes U \otimes U^* \xrightarrow{id_B \otimes \widetilde{ev}_U} B,$$

is natural in $U$, $A$ and $B$, and satisfies the following property (among other properties)

$$Tr_{A,B}^{U \otimes W}(f) = Tr_{A,B}^U(Tr_{A \otimes U, B \otimes U}^W(f)).$$

Clearly, $Tr_{1,1}^U(id_U) = \dim U$.

We have the following two easy lemmas.

Lemma 4.5. Let $f : A \otimes U \to B \otimes W$ and $g : W \to U$ be morphisms. Then $Tr_{A,B}^U((id_B \otimes g)f) = Tr_{A,B}^W(f(id_A \otimes g))$.

Proof. Follows from the naturality of $Tr$ in $U$. \qed

Lemma 4.6. Let $f : A \otimes U \to B \otimes U$ and $g : W \to W$ be morphisms. Then $Tr_{A,B}^U(f) \otimes Tr_{1,1}^W(g) = Tr_{A,B}^{U \otimes W}(f \otimes g)$.

Proof. Follows easily from the definition of $Tr$, and the facts that $(U \otimes W)^* = W^* \otimes U^*$ with

$$coev_{U \otimes W} = (id_U \otimes c_{U^*, W \otimes U^*}) \circ (coev_U \otimes coev_W)$$

and

$$\widetilde{ev}_{U \otimes W} = (\widetilde{ev}_U \otimes \widetilde{ev}_W) \circ (id_U \otimes cw_{U^*, U^*})$$

(see e.g., [BK]). \qed
4.4. Traces of permutations. Fix a nonnegative integer $m$, and an object $X \in C$. In the sequel we shall identify the symmetric group $S_{m-1}$ with the stabilizer of 1 in $S_m$.

**Lemma 4.7.** For any $\sigma \in S_m$ and $\tau \in S_{m-1}$, $\text{Tr}_{X,X}^{X^\otimes m-1}(\sigma) = \text{Tr}_{X,X}^{X^\otimes m-1}(\tau \sigma \tau^{-1})$.

**Proof.** Follows easily from Lemma 4.5.\hfill\Box

**Lemma 4.8.** We have that $\text{Tr}_{X,X}^{X^\otimes m-1}((1 \cdots m)) = id_X$.

**Proof.** For any $i$ let us denote the cycle $(1 \cdots i)$ by $\sigma_i$. We are going to prove the lemma by induction on $m$ using the relation $\sigma_m = (12)\sigma_{m-1}$. We compute

\[
\text{Tr}_{X,X}^{X^\otimes m-1}(\sigma_m) \\
= \text{Tr}_{X,X}(\text{Tr}_{X,X}^{X^\otimes m-2}(\sigma_m)) \\
= \text{Tr}_{X,X}(\text{Tr}_{X,X}^{X^\otimes m-2}(((12) \otimes id) \circ (id \otimes \sigma_{m-1}))) \\
= \text{Tr}_{X,X}^{X}(12) \circ \text{Tr}_{X,X}^{X^\otimes m-2}(id \otimes \sigma_{m-1}) \\
= \text{Tr}_{X,X}^{X}(12) \circ (id_X \otimes \text{Tr}_{X,X}^{X^\otimes m-2}(\sigma_{m-1})) \\
= \text{Tr}_{X,X}^{X}(12) \circ (id_X \otimes id_X) \\
= id_X,
\]

where in the first equality we used (3), in the third equality we used the naturality of $\text{Tr}$ in $X \otimes X$, in the fifth equality we used the induction assumption, and in the last equality we used (1).\hfill\Box

**Lemma 4.9.** Let $\sigma_1 \sigma_2 \cdots \sigma_N \in S_m$ be a product of disjoint cycles, where reading from left to right the numbers $1, \ldots, m$ appear in an increasing order. Then $\text{Tr}_{X,X}^{X^\otimes m-1}(\sigma_1 \sigma_2 \cdots \sigma_N) = d^N \cdot id_X$, where $d$ is the dimension of $X$.

**Proof.** Lemma 4.8 is the case $N = 1$. Now use Lemma 4.6 to proceed by induction on $N$.\hfill\Box

**Proposition 4.10.** Let $\sigma \in S_m$, and let $N(\sigma)$ denote the number of disjoint cycles in $\sigma$. Then $\text{Tr}_{X,X}^{X^\otimes m-1}(\sigma) = d^{N(\sigma)-1} id_X$, where $d := \dim X$.

**Proof.** It is clear that for any $\sigma \in S_m$ there exists $\tau \in S_{m-1}$ such that $\tau \sigma \tau^{-1}$ decomposes into a product of disjoint cycles $\sigma_1 \sigma_2 \cdots \sigma_{N(\sigma)}$, where reading from left to right the numbers $1, \ldots, m$ are in an increasing order. Now, by Lemma 4.7, $\text{Tr}_{X,X}^{X^\otimes m-1}(\sigma) = \text{Tr}_{X,X}^{X^\otimes m-1}(\tau \sigma \tau^{-1})$, and hence the result follows from Lemma 4.9.\hfill\Box

4.5. The morphism $\theta_{X,m,\alpha,\beta}$. Given a partition $\alpha$ of a nonnegative integer $m - 1$, let $\alpha + 1$ denote the set of partitions of $m$ whose Young diagram is obtained by adding a single box to the Young diagram of $\alpha$.

Fix an object $X \in C$ of dimension $d := \dim X$, and partitions $\alpha$ of $m - 1$ and $\beta \in \alpha + 1$. We define the morphism

$$
\theta_{\alpha,\beta} = \theta_{X,m,\alpha,\beta} : X \to S_{\beta}X \otimes S_{\alpha}X^*
$$

as the following composition:

$$
\theta_{\alpha,\beta} : X \xrightarrow{id_X \otimes \text{coev}_{\alpha}} X \otimes S_{\alpha}X \otimes S_{\alpha}X^* \xrightarrow{\epsilon_{\beta} \otimes \epsilon_{\alpha}} S_{\beta}X \otimes S_{\alpha}X^*.
$$
Consider the morphism

\[ P_{\alpha, \beta} = P_{X, m, \alpha, \beta} : X \to X, \]

given as the composition

\[
(5) \quad P_{\alpha, \beta} : X \xrightarrow{\theta_{\alpha, \beta}} S_{\beta} X \otimes S_{\alpha} X^* \hookrightarrow X \otimes X^\otimes m^{-1} \otimes X^* \otimes m^{-1} \xrightarrow{id_X \otimes \overline{\epsilon} \otimes \overline{m^{-1}}} X. 
\]

In what follows we shall see that the morphism \( P_{\alpha, \beta} \) is a scalar multiple of the identity morphism \( id_X \) by some polynomial \( p_{\alpha, \beta}(d) \).

If we identify \( S_{m-1} \) with the stabilizer of 1 in \( S_m \), then clearly

\[ P_{\alpha, \beta} = Tr_{X, X}^{X^\otimes m^{-1}}((id_X \otimes c_\alpha) \circ c_\beta \circ (id_X \otimes c_\alpha)) : X \to X, \]

and hence, by Lemma 4.5,

\[ P_{\alpha, \beta} = Tr_{X, X}^{X^\otimes m^{-1}}((id_X \otimes c_\alpha) \circ c_\beta) : X \to X. \]

As an immediate consequence of Proposition 4.10, we get the following.

**Corollary 4.11.** Write \((id_X \otimes c_\alpha) \circ c_\beta \in k[ S_m]\) as a \( k \)-linear combination of group elements:

\[ (id_X \otimes c_\alpha) \circ c_\beta = \sum_{\sigma \in S_m} f_{\alpha, \beta}(\sigma) \sigma, \]

and set

\[ p_{\alpha, \beta}(d) := \sum_{\sigma \in S_m} f_{\alpha, \beta}(\sigma)d^{N(\sigma)-1}. \]

Then \( P_{\alpha, \beta} = p_{\alpha, \beta}(d)id_X \). In particular, if \( p_{\alpha, \beta}(d) \neq 0 \) in \( k \) then \( \theta_{\alpha, \beta} \) is a split injection. \( \square \)

Let \( \chi_\beta \) be the character of \( V_\beta \), and let

\[ e_\beta := \frac{\dim V_\beta}{m!} \sum_{\sigma \in S_m} \chi_\beta(\sigma) \sigma \]

be the primitive central idempotent in \( k[ S_m]\) associated with \( V_\beta \). Recall that \( e_\beta \) is equal to the sum of all the \((\dim V_\beta)\) Young symmetrizers \( c_\beta \) associated with \( V_\beta \).

In the following theorem we compute the polynomial \( p_{\alpha, \beta}(d) \) explicitly, in terms of \( \chi_\beta \).

**Theorem 4.12.** We have that

\[ Tr_{X, X}^{X^\otimes m^{-1}}((id_X \otimes e_\alpha) \circ e_\beta) = \left( \frac{\dim V_\alpha}{m!} \sum_{\sigma \in S_m} \chi_\beta(\sigma)d^{N(\sigma)-1} \right) id_X, \]

and hence

\[ p_{\alpha, \beta}(d) = \frac{1}{m! \dim V_\beta} \sum_{\sigma \in S_m} \chi_\beta(\sigma)d^{N(\sigma)-1}. \]

**Proof.** Clearly,

\[ (id_X \otimes e_\alpha) \circ e_\beta = \frac{\dim V_\alpha \dim V_\beta}{(m-1)! m!} \sum_{\sigma \in S_m} \left( \sum_{\tau \in S_{m-1}} \chi_\alpha(\tau) \chi_\beta(\tau^{-1} \sigma) \right) \sigma. \]
Therefore, by Proposition 4.10
\[
\text{Tr}_{X, X}^{\chi \otimes m^{-1}} ((id_X \otimes e_\alpha) \circ e_\beta)
\]
\[
= \left( \frac{\dim V_\alpha \dim V_\beta}{(m - 1)! m!} \sum_{\tau \in S_m} \left( \sum_{\sigma \in S_{m-1}} \chi_\alpha(\tau) \chi_\beta(\tau^{-1} \sigma) \right) d^{N(\sigma) - 1} \right) id_X
\]
\[
= \left( \frac{\dim V_\alpha \dim V_\beta}{(m - 1)! m!} \left( \sum_{\tau \in S_{m-1}} \chi_\alpha(\tau) \chi_\beta(\tau^{-1}) \right) \chi_\beta(z(d)) \right) id_X.
\]

Set \( z(d) := \sum_{\sigma \in S_m} d^{N(\sigma) - 1} \). Clearly, \( z(d) \) is a central element in \( k[S_m] \), hence it acts by the scalar \( \chi_\beta(z(d)) \) on \( V_\beta \). In particular, for any \( \tau \in S_m \),
\[
\chi_\beta(\tau^{-1} z(d)) = \chi_\beta(\tau^{-1}) \chi_\beta(z(d)) / \dim V_\beta.
\]
We therefore have
\[
\text{Tr}_{X, X}^{\chi \otimes m^{-1}} ((id_X \otimes e_\alpha) \circ e_\beta)
\]
\[
= \left( \frac{\dim V_\alpha}{(m - 1)! m!} \left( \sum_{\tau \in S_{m-1}} \chi_\alpha(\tau) \chi_\beta(\tau^{-1}) \right) \chi_\beta(z(d)) \right) id_X.
\]

Finally, recall that the multiplicity \( [\text{Res}^{S_m}_{S_{m-1}} \chi_\beta \circ \chi_\alpha] \) of \( V_\alpha \) in the restriction of \( V_\beta \) from \( S_m \) to \( S_{m-1} \) is equal to 1 (see e.g. [FH]), i.e.,
\[
\frac{1}{(m - 1)!} \sum_{\tau \in S_{m-1}} \chi_\alpha(\tau) \chi_\beta(\tau^{-1}) = [\text{Res}^{S_m}_{S_{m-1}} \chi_\beta \circ \chi_\alpha] = 1.
\]
We thus conclude that
\[
\text{Tr}_{X, X}^{\chi \otimes m^{-1}} ((id_X \otimes e_\alpha) \circ e_\beta)
\]
\[
= \left( \frac{\dim V_\alpha}{m!} \sum_{\sigma \in S_m} \chi_\beta(\sigma) d^{N(\sigma) - 1} \right) id_X,
\]
as claimed. \( \square \)

In fact, the polynomial \( p_{\alpha, \beta}(d) \) is closely related to a well known polynomial associated with the partition \( \beta \). Namely, let \( cp_\beta(d) := \prod_{(i,j) \in \beta} (d + j - i) \) be the content polynomial of \( \beta \), and recall that the polynomial (in \( d \)) \( \frac{1}{m! \dim V_\beta} \sum_{\sigma \in S_m} \chi_\beta(\sigma) d^{N(\sigma)} \) equals \( cp_\beta(d) \) (see e.g. [MacD]). Hence, by Theorem 4.12

\[
(6) \quad p_{\alpha, \beta}(d) = \frac{1}{m!} cp_\beta(d).
\]

Corollary 4.13. Let \( \alpha, \beta, X \) and \( d \) be as above, and let \( p(\beta), q(\beta) \) be the number of rows and columns in the diagram of \( \beta \), respectively.

1) If \( d \neq 1 - q(\beta), \ldots , p(\beta) - 1 \) in \( k \) then the morphism \( \theta_{\alpha, \beta} \) is a split injection.
2) Suppose \( \beta \) is a hook. If \( d \neq 1 - q(\beta), \ldots , -1, 1, \ldots , p(\beta) - 1 \) in \( k \) then the morphism \( \theta_{\alpha, \beta} \) is a split injection.

Proof. 1) Since \( d \neq 0 \) in \( k \), the result follows from (6) and Theorem 4.12

2) By Theorem 4.12 \( p_{\alpha, \beta}(0) = \frac{1}{m! \dim V_\beta} \sum_{\sigma} \chi_\beta(\sigma) \), where the sum is taken over all the \( m \)-cycles \( \sigma \) in \( S_m \). But it is well known (see e.g. [MacD]) that \( \chi_\beta \) vanishes on a \( m \)-cycle when \( \beta \) is not a hook, and that \( \chi_{(m-s,1^s)}(\sigma) = (-1)^s \) for any \( 0 \leq s \leq m \) and \( m \)-cycle \( \sigma \). Therefore \( \theta_{\alpha, \beta} \) is a split injection when \( d = 0 \) as well.

We are done. \( \square \)
Example 4.14. For the partition $\alpha = (1^{m-1})$, $S_\alpha X = \wedge^{m-1}X$ is the $(m-1)$th exterior power of $X$. Hence, by Corollary 4.13 if $\binom{d-1}{m-1} \neq 0$ in $k$, then the corresponding morphism $\theta_{(1^{m-1}),(1^{m-1})}$ is a split injection. This is a generalization of Lemma 5.1.12 in [S1] in the group case.

4.6. Extensions in $C$. Let $U, V, W \in C$ and let $f \in \text{Hom}(V, W)$, $g \in \text{Hom}(W, U)$. We shall denote by $f^*$ and $g^*$ the $k$–linear maps

$$f_* : \text{Ext}^1(U, V) \to \text{Ext}^1(U, W) \quad \text{and} \quad g^* : \text{Ext}^1(U, V) \to \text{Ext}^1(W, V)$$

induced by $f$ and $g$, respectively. Namely, given an extension

$$E : 0 \to V \to X \to U \to 0,$$

the extensions

$$f_*(E) : 0 \to W \to Y \to U \to 0 \quad \text{and} \quad g^*(E) : 0 \to V \to Z \to W \to 0$$

are obtained using the pushout

$$\begin{array}{cccc}
V & & X & \\
\downarrow f & & \downarrow & \\
W & & Y & \\
\end{array}$$

and the pullback

$$\begin{array}{cccc}
\downarrow & & \downarrow g & \\
X & & U & \\
\downarrow & & \downarrow & \\
Z & & W & \\
\end{array}$$

respectively.

We shall need the following two lemmas.

Lemma 4.15. For any objects $A, B, X \in C$, the $k$–linear spaces $\text{Ext}^1(B, A \otimes X)$ and $\text{Ext}^1(B \otimes X^*, A)$ are canonically isomorphic.

Proof. One associates to an element

$$0 \to A \otimes X \to W \to B \to 0$$

in $\text{Ext}^1(B, A \otimes X)$ an element in $\text{Ext}^1(B \otimes X^*, A)$ in the following way: since the functor $- \otimes X^* : C \to C$ is exact, tensoring our exact sequence with $X^*$ on the right yields the extension

$$E : 0 \to A \otimes X \otimes X^* \to W \otimes X^* \to B \otimes X^* \to 0.$$
Lemma 4.16. Let $\alpha$ be a partition of a nonnegative integer $m - 1$, let $\beta \in \alpha + 1$ and let $A, B, X \in \mathcal{C}$. Suppose that $\theta_{\alpha, \beta} = \theta_{X, m, \alpha, \beta}$ is a split injection. Then the $k$–linear map

$$(id_B \otimes \text{coev}_{S_n,X})_* : \text{Ext}^1(A, B \otimes X) \to \text{Ext}^1(A, B \otimes X \otimes S_{\alpha}X \otimes S_{\alpha}X^*)$$

is injective.

Proof. Indeed, since $\theta_{\alpha, \beta}$ is a split injection, we have that

$$(id_B \otimes \theta_{\alpha, \beta})_* : \text{Ext}^1(A, B \otimes X) \to \text{Ext}^1(A, B \otimes S_{\beta}X \otimes S_{\alpha}X^*)$$

is injective. But,

$$(id_B \otimes \theta_{\alpha, \beta})_* = (id_B \otimes c_{\beta} \otimes id_{S_n,X^*})_* \circ (id_B \otimes \text{coev}_{S_n,X})_*.$$  

We are done. \qed

4.7. The filtration on $S_\lambda V$ defined by a sub-object of $V$. Fix a sub-object $A$ of $V$ for the rest of the section, and consider the short exact sequence

$$(V) : 0 \to A \to V \to B \to 0;$$

it is an element in the $k$–linear space $\text{Ext}^1(B, A)$. Then $(V)$ defines a filtration on $S_\lambda V$ in the following way. For each $0 \leq i \leq n$ set

$$T_i := \bigoplus_{S \subseteq \{1, \ldots, n\}, |S| = i} V_{S(1)} \otimes \cdots \otimes V_{S(n)},$$

where $V_{S(j)} = V$ if $j \notin S$ and $V_{S(j)} = A$ if $j \in S$. Clearly, the $T_i$ define a $S_n$–equivariant filtration $T_\ast$ on $V^{\otimes n}$:

$$V^{\otimes n} = T_0 \supseteq T_1 \supseteq \cdots \supseteq T_n \supseteq T_{n+1} = 0,$$

whose composition factors are

$$T_i/T_{i+1} \cong \bigoplus_{S \subseteq \{1, \ldots, n\}, |S| = i} V_{S,1} \otimes \cdots \otimes V_{S,n}, 0 \leq i \leq n,$$

where $V_{S,j} = B$ if $j \notin S$ and $V_{S,j} = A$ if $j \in S$.

The filtration $T_\ast$ induces a filtration $F_\ast$ on $S_\lambda V$:

$$S_\lambda V = F_0 \supseteq F_1 \supseteq \cdots \supseteq F_n \supseteq F_{n+1} = 0,$$

where $F_i := c_\lambda(T_i)$ is the image of $T_i$ under the Schur functor $c_\lambda$. Let

$$(8) \quad V_i := F_i/F_{i+1}, \quad 0 \leq i \leq n,$$

be the composition factors of $F_\ast$, and let

$$(9) \quad V_i^2 := F_{i-1}/F_{i+1}, \quad 1 \leq i \leq n.$$

Since the filtration $T_\ast$ is $S_n$–equivariant, we have

$$(10) \quad V_i \cong c_\lambda(T_i/T_{i+1}) \cong \bigoplus_{\mu, \nu} N_{\mu, \nu}^\lambda(S_\mu A \otimes S_\nu B),$$

where $N_{\mu, \nu}^\lambda := [\text{Res}^S_{S_i \times S_{n-i}}(V_\lambda : V_\mu \otimes V_\nu]$ are the Littlewood-Richardson coefficients (see e.g., [FH]).

For each integer $0 \leq i \leq n$, let $\lambda - i$ denote the set of all partitions of $n - i$ whose Young diagram is obtained from that of $\lambda$ after deleting $i$ boxes (by convention,
\[ \lambda - n \text{ consists of one element } (0). \] By the Littlewood-Richardson rule (see e.g., [FH]), \( N_{\mu, \nu}^\lambda = 0 \) if \( \mu \not\in \lambda - (n - i) \) or \( \nu \not\in \lambda - i \). Therefore,

\[ (11) \quad V_i \cong \bigoplus_{\mu \in \lambda - (n - i), \nu \in \lambda - i} N_{\mu, \nu}^\lambda(S_\mu A \otimes S_\nu B). \]

(However, \( N_{\mu, \nu}^\lambda \) can still equal 0 for some pairs \( \mu \in \lambda - (n - i), \nu \in \lambda - i \), e.g., for \( \lambda = (2, 2) \), \( N_{(2,2)}^{(2,2)} = 0 \).)

Observe also that for any \( \mu' \in \lambda - (n - i + 1) \), \( \mu \in \lambda - (n - i) \) and \( \nu \in \lambda - i \), \( c_\mu \) defines a morphism

\[ c_\mu \otimes id_{S_\nu B} : S_{\mu'} A \otimes V \otimes S_\nu B \to V_i^2. \]

Since \( V_i^2 \) is a subquotient of \( S_\lambda V \), the following lemma is clear.

**Lemma 4.17.** If \( S_\lambda V \) is semisimple then the exact sequence

\[ (12) \quad (V_i^2) : 0 \to V_i \to V_i^2 \to V_{i-1} \to 0 \]

splits for any \( 1 \leq i \leq n \).

**4.8. The semisimplicity of \( V \).** Let \( 1 \leq i \leq n \) be an integer, \( \mu' \in \lambda - (n - i + 1) \) and \( \nu \in \lambda - i \). Tensoring our exact sequence \((V)\) by \( S_{\mu'} A \) on the left yields the extension

\[ (13) \quad E_1 : 0 \to S_{\mu'} A \otimes A \to S_{\mu'} A \otimes V \to S_{\mu'} A \otimes B \to 0. \]

Tensoring \( E_1 \) by \( S_\nu B \) on the right yields the extension

\[ (14) \quad E_2 : 0 \to S_{\mu'} A \otimes A \otimes S_\nu B \to S_{\mu'} A \otimes V \otimes S_\nu B \to S_{\mu'} A \otimes B \otimes S_\nu B \to 0. \]

Set

\[ (15) \quad \mu'_+ := \{ \mu \in \mu' + 1 \mid N_{\mu, \nu}^\lambda \neq 0 \}, \quad \nu_+ := \{ \nu' \in \nu + 1 \mid N_{\mu, \nu'}^\lambda \neq 0 \}. \]

The following lemma is clear.

**Lemma 4.18.** Let \( 1 \leq i \leq n \) be an integer, and let \( \mu' \in \lambda - (n - i + 1) \), \( \nu \in \lambda - i \). Then for any \( \mu \in \mu'_+ \) and \( \nu' \in \nu_+ \), the triple \((c_\mu \otimes id_{S_\nu B}, c_\mu \otimes id_{S_\lambda B}, id_{S_\mu A} \otimes c_{\nu'})\) defines a morphism of extensions \( E_2 \to (V_i^2)\):

\[
\begin{array}{cccccc}
0 & \longrightarrow & S_{\mu'} A \otimes A \otimes S_\nu B & \longrightarrow & S_{\mu'} A \otimes V \otimes S_\nu B & \longrightarrow & S_{\mu'} A \otimes B \otimes S_\nu B & \longrightarrow & 0 \\
& & \downarrow_{c_\mu \otimes id_{S_\nu B}} & & \downarrow_{c_\mu \otimes id_{S_\nu B}} & & \downarrow_{id_{S_\mu A} \otimes c_{\nu'}} & & \\
0 & \longrightarrow & V_i & \longrightarrow & V_i^2 & \longrightarrow & V_{i-1} & \longrightarrow & 0.
\end{array}
\]

Fix an integer \( 1 \leq i \leq n \), and \( \mu' \in \lambda - (n - i + 1), \nu \in \lambda - i \). For any \( \mu \in \mu'_+ \) and \( \nu' \in \nu_+ \), define the following two subsets of the ground field \( k \):

\[ (16) \quad A_i(\mu', \mu, \nu) := \{ d \mid p_{\mu', \mu}(d) = 0 \} \subseteq k \]
and

\[ (17) \quad B_i(\mu', \nu, \nu') := \{ d \mid p_{\nu, \nu'}(d) = 0 \} \subseteq k. \]

**Example 4.19.** By convention, \( \lambda - n = \{(0)\} \). Therefore, for any \( \mu', \nu \in \lambda - 1 \), we have that \( A_i((0), (1), \nu) = B_i((\mu', (0), (1)) = \emptyset \). On the other extreme, by Corollary 4.13 \( B_1((0), \nu, \lambda) = A_n(\mu', \lambda, (0)) = \{1 - q(\lambda), \ldots, p(\lambda) - 1\} \) if \( \lambda \) is not a hook, and \( B_1((0), \nu, \lambda) = A_n(\mu', \lambda, (0)) = \{1 - q(\lambda), \ldots, -1, 1, \ldots, p(\lambda) - 1\} \) if \( \lambda \) is a hook.
Let \( a := \dim A, b := \dim B \) for the rest of the paper.

**Lemma 4.20.** Let \( 1 \leq i \leq n \) be an integer, and let \( \mu' \in \lambda - (n-i+1), \nu \in \lambda - i \). Let \( \mu \in \mu'_+ \), and let \( \nu' \in \nu_+ \) be such that \( b \notin B_i(\mu', \nu, \nu') \). Then \((c_{\mu})_*(E_1) = 0 \) in \( \Ext^1(S_{\mu'}A \otimes B, S_\mu A) \).

**Proof.** By Lemma 4.18 and a standard fact on extensions (see e.g., [MacL]), we have that \((c_{\mu} \otimes id)_*(E_2) = (id \otimes c_{\nu'})^*(V_i^2)\). Since by Lemma 4.17 \((V_i^2) = 0\), we have that \((c_{\mu} \otimes id)_*(E_2) = 0 \) in \( \Ext^1(S_{\mu'}A \otimes B \otimes S_\mu B, S_\mu A \otimes S_\mu B) \).

Let \( f : \Ext^1(S_{\mu'}A \otimes B, S_\mu A) \rightarrow \Ext^1(S_{\mu'}A, S_\mu A \otimes B^*) \) be the isomorphism given by Lemma 4.15 (composed with the appropriate commutativity constraints). Then, it is straightforward to verify that

\[
0 = (c_{\mu} \otimes id)_*(E_2) = (g \circ (id \otimes coev_{S_\mu, B^*})_* \circ f)((c_{\mu})_*(E_1)).
\]

Now, by our assumption on \( b \) and Theorem 4.12, the morphism \( \theta_{B^*, n-i+1, \nu, \nu'} \) is a split injection. Therefore, by Lemma 4.16 \((id \otimes coev_{S_\mu, B^*})_* \) is injective, and the result follows.

We are now ready to prove the key proposition for the proof of Theorem 4.3.

**Proposition 4.21.** Assume there exist an integer \( 1 \leq i \leq n \), a pair of partitions \( \mu' \in \lambda - (n-i+1), \nu \in \lambda - i \) and a pair of partitions \( \mu \in \mu'_+ \), \( \nu' \in \nu_+ \), such that \( a \notin A_i(\mu', \mu) \) and \( b \notin B_i(\mu', \nu, \nu') \). Then \((V) = 0 \) in \( \Ext^1(B, A) \).

**Proof.** By Theorem 4.12, the morphisms \( \theta_{A, i-1, \mu', \mu} \) and \( \theta_{B, n-i+1, \nu, \nu'} \) are split injections. Consider now the following commutative diagram:

\[
\begin{array}{ccc}
\Ext^1(B, A) & \xrightarrow{f} & \Ext^1(B, A \otimes S_{\mu'} A \otimes S_\mu A^*) \\
\downarrow & & \downarrow \\
\Ext^1(B \otimes S_{\mu'} A, A \otimes S_\mu A) & \xrightarrow{(c_{\mu})_*} & \Ext^1(B \otimes S_{\mu'} A, S_\mu A),
\end{array}
\]

where \( f := (id_{A} \otimes coev_{S_\mu A})_* \), \( g := (c_{\mu} \otimes id_{S_{\mu'} A})_* \), and the two vertical isomorphisms are given by Lemma 4.15. Observe that \( gf = (\theta_{A, i-1, \mu', \mu})_* \) is injective. It is now clear that the proposition follows from Lemma 4.16 and Lemma 4.20.

5. **The proof of Theorem 4.3**

5.1. \( F(\lambda) \subseteq G(\lambda) \): We have to show that if \( d \in F(\lambda) \) then \( d \in G(\lambda) \), i.e., that there exists a symmetric rigid tensor category \( \mathcal{C} \) over \( k \) with a non-semisimple object \( V \) of dimension \( d \) such that \( S_\lambda V \) is semisimple. This follows from the following two observations.

1) Let \( r, s \) be nonnegative integers such that \( r + s \geq 2 \). One can introduce on the superspace \( V := \mathbb{C}^{r+s} \) a structure of a nonsemisimple representation of some supergroup (e.g., the supergroup of upper triangular matrices). On the other hand,
if $\lambda$ contains a box $(r+1, s+1)$ then $S_{\lambda} V = 0$ (see e.g., [D2]), so $S_{\lambda} V$ is automatically semisimple while $V$ is not.

2) Suppose $\lambda = (q^p)$ is a rectangle. If $V$ is a nonsemisimple group representation of dimension $p > 1$, then $S_{\lambda} V = (A^p V \otimes q)$ is 1-dimensional, so is automatically semisimple, while $V$ is not. Finally, for $-q$, use now Remark 4.2

We are done.

5.2. $G(\lambda) \subseteq F(\lambda)$: Let $C$ be any symmetric rigid tensor category over $k$, and let $V \in C$ be an object of $C$. We have to show that if $\dim V \notin F(\lambda)$ and $S_{\lambda} V$ is semisimple then so is $V$ (i.e., $\dim V \notin G(\lambda)$). To this end, it is enough to show that if $\dim V \notin F(\lambda)$ then there exist an integer $1 \leq i \leq n$, a pair of partitions $\mu' \in \lambda - (n-i+1)$, $\nu \in \lambda - i$, and a pair of partitions $\mu \in \mu', \nu' \in \nu$, satisfying the conditions of Proposition 4.2.

Let $\lambda^*$ denote the conjugate of $\lambda$. Write $\lambda = (\lambda_1, \ldots, \lambda_p)$ and $\lambda^* = (\lambda'_1, \ldots, \lambda'_q)$, where $q = \lambda_1 \geq \cdots \geq \lambda_p \geq 1$ and $p = \lambda'_1 \geq \cdots \geq \lambda'_q \geq 1$.

5.2.1. The general case. In this subsection we prove that if $\dim V \notin R(\lambda)$ then the exact sequence $(V)$ splits.

If $i = 1$, $A_1((0), (1), \nu) = \emptyset$ for any $\nu \in \lambda - 1$ (see Example 4.19), so there is no condition on $a$. Therefore, if $b$ is not equal in $k$ to an element of $B_1((0), (1), \nu)$ for some $\nu \in \lambda - 1$, we are done. So suppose $b$ is equal in $k$ to some element of

$$B_1((0), (1), \nu) = \{1 - q, \ldots, p - 1\},$$

which we shall continue to denote by $b$ (so now $b \in \mathbb{Z}$).

Subcase 1. Suppose that $b > 0$, and set $i := p - b + 1$; then $2 \leq i \leq p$. Let $\mu' := (\lambda_{p-i+1}, \ldots, \lambda_p - 1)$ be the last $i$ rows of $\lambda$ without the last box, and let $\nu := (\lambda_1, \ldots, \lambda_{p-i})$ be the first $p - i$ rows of $\lambda$. Let $\mu := (\lambda_{p-i+1}, \ldots, \lambda_p)$ and let $\nu' := (\lambda_1, \ldots, \lambda_{p-i}, 1)$. It follows easily from the Littlewood-Richardson rule that $\mu \in \mu'$, and $\nu' \in \nu$.

We now use (6) to find out that

\begin{equation}
A_n(\mu', \mu, \nu) = \{d \in k \mid p_{\mu', \mu}(d) = 0\} = \{1 - \lambda_{p-i+1}, \ldots, i - 1\}
\end{equation}

and

\begin{equation}
B_n(\mu', \nu, \nu') = \{d \in k \mid p_{\nu, \nu'}(d) = 0\} = \{1 - q, \ldots, p - i\},
\end{equation}

where $n_i := \sum_{j=p-i+1}^{p} \lambda_j$. We therefore see that $b = p - i + 1 \notin B_n(\mu', \nu, \nu')$. Now, if $a \notin A_n(\mu', \mu, \nu)$, we are done. Otherwise, we are done by our assumption on $\dim V$ (since $\dim V = a + b$).

Subcase 2. Suppose that $b = 0$. Then $\emptyset \notin B_n(\mu', (0), (1)) = \emptyset$ for any $\mu' \in \lambda - 1$. Now, if $a \notin A_n(\mu', (0), (1))$, we are done. Otherwise, we are done by our assumption on $\dim V$.

Subcase 3. Suppose that $b < 0$, and set $i := q + b + 1$; then $2 \leq i \leq q$. Let $\mu' := (\lambda_{q-i+1}, \ldots, \lambda'_q - 1)$ be the last $i$ columns of $\lambda$ without the last box, and let $\nu := (\lambda'_1, \ldots, \lambda'_{q-i})$ be the first $q - i$ columns of $\lambda$. Let $\mu := (\lambda'_{q-i+1}, \ldots, \lambda'_q)$ and let $\nu' := (\lambda'_1, \ldots, \lambda'_{q-i}, 1)$. It follows easily from the Littlewood-Richardson rule that $\mu \in \mu'$, and $\nu' \in \nu$.

We now use (6) to find out that

\begin{equation}
A_n(\mu', \mu, \nu) = \{d \in k \mid p_{\mu', \mu}(d) = 0\} = \{1 - i, \ldots, \lambda'_{q-i+1} - 1\}
\end{equation}
and
\[(21) \quad B_{n_i}(\mu', \nu, \nu') = \{d \in k \mid p_{\mu', \nu'}(d) = 0\} = \{i - q, \ldots, p - 1\},\]
where \(n_i := \sum_{j=q-i+1}^{q} \lambda_j\). We therefore see that \(b = i - q - 1 \notin B_{n_i}(\mu', \nu, \nu')\). Now, if \(a \notin A_{n_i}(\mu', \mu, \nu)\), we are done. Otherwise, we are done by our assumption on \(\dim V\).

5.2.2. The non-rectangle case. Assume \(\lambda\) is not a rectangle. We have to show that \(\dim V = p\) and \(\dim V = -q\) are allowed. Let \(b, i\) be as in Subcase 1 of 5.2.1.

Let \(\mu' := (\lambda_{p-i+2}, \ldots, \lambda_p)\) and \(\nu := (\lambda_1, \ldots, \lambda_{p-i+1} - 1)\) be the last \(i - 1\) rows of \(\lambda\) and the first \(p - i + 1\) rows of \(\lambda\) without the last box, respectively. Choose \(\mu \in \mu'_i\) with \(i - 1\) rows (it exists since \(\lambda\) is not a rectangle!) and let \(\nu' := (\lambda_1, \ldots, \lambda_{p-i+1})\).

It follows easily from the Littlewood-Richardson rule that \(\nu' \in \nu_+\). Moreover, we now have that \(A_{n_i}(\mu', \mu, \nu)\) and \(B_{n_i}(\mu', \nu', \nu')\). We thus conclude that \(\dim V = p\) is allowed in this case, as claimed.

The claim that \(\dim V = -q\) is allowed follows now from Remark 4.2.

5.2.3. The case \((3, 2) \notin \lambda\) or \((2, 3) \notin \lambda\). Suppose \((3, 2) \notin \lambda\). We have to show that \(\dim V = 1\) is allowed.

Subcase 1. Let \(b, i\) be as in Subcase 1 of 5.2.1.

First note that for \(2 \leq i \leq p - 2\), \(\lambda_{p-i+1} = 1\). Hence \(b \notin B_{n_i}(\mu', \mu, \nu)\) and \(1 - b \notin A_{n_i}(\mu', \mu, \nu)\) (see (15), (16)).

Now, for \(i = p - 1\) (so \(b = 2\)), take \(\mu' := (1^{p-1})\), \(\mu := (1^0)\), \(\nu := (q, 1, \lambda_2 - 1)\) and \(\nu' := (q, \lambda_2 - 1)\). It follows easily from the Littlewood-Richardson rule that \(\mu \in \mu'_i\) and \(\nu' \in \nu_+\). Moreover, \(-1 \notin A_p(\mu', \mu, \nu)\) and \(2 \notin B_p(\mu', \nu, \nu')\).

For \(i = p\) (so \(b = 1\)), take \(\mu' := (q, 1^{p-2})\), \(\mu := (q, 1^{p-1})\), \(\nu := (\lambda_2 - 1)\) and \(\nu' := (\lambda_2)\). It follows easily from the Littlewood-Richardson rule that \(\mu \in \mu'_i\) and \(\nu' \in \nu_+\). Moreover, \(0 \notin A_{p+q-1}(\mu', \mu, \nu)\) and \(1 \notin B_{p+q-1}(\mu', \nu, \nu')\).

Subcase 2. Let \(b, i\) be as in Subcase 2 of 5.2.1.

Take \(\mu' := (\lambda_2 - 1)\), \(\mu := (\lambda_2)\), \(\nu := (q, 1^{p-2})\) and \(\nu' := (q, 1^{p-1})\). It follows easily from the Littlewood-Richardson rule that \(\mu \in \mu'_i\) and \(\nu' \in \nu_+\). Moreover, \(1 \notin A_{\lambda_2}(\mu', \mu, \nu)\) and \(0 \notin B_{\lambda_2}(\mu', \nu, \nu')\).

Subcase 3. Let \(b, i\) be as in Subcase 3 of Subsection 5.2.1.

First note that for \(2 \leq i \leq q - 2\), \(\lambda_{q-i+1} = 1\). Hence \(b \notin B_{n_i}(\mu', \mu, \nu)\) and \(1 - b \notin A_{n_i}(\mu', \mu, \nu)\) (see (20), (21)).

Now, for \(i = q - 1\), \(q\) (so \(b = -2, -1\)), take \(\mu' := (\lambda_1, \lambda_2 - 1)\), \(\mu := (\lambda_1, \lambda_2)\), \(\nu := (1^{p-2})\) and \(\nu' := (1^{p-1})\). It follows easily from the Littlewood-Richardson rule that \(\mu \in \mu'_i\) and \(\nu' \in \nu_+\). Moreover, \(2, 3 \notin A_{n+2-p}(\mu', \mu, \nu)\) and \(-1, -2 \notin B_{n+2-p}(\mu', \nu, \nu')\).

We therefore conclude that \(\dim V = 1\) is allowed in this case, as claimed.

Finally, the claim that \(\dim V = -1\) is allowed in the case \((2, 3) \notin \lambda\) follows now from Remark 4.2.

5.2.4. The hook case. Assume \(\lambda\) is a hook. We have to show that \(\dim V = 0\) is allowed.

Subcase 1. Let \(b, i\) be as in Subcase 1 of 5.2.1.

Since \(\lambda_{p-i+1} = 1\) for \(i < p\), we get from (15) that \(-b \notin A_{n_i}(\mu', \mu, \nu)\). On the other hand, for \(i = p\) (so \(b = 1\)), take \(\mu' := (1^{p-1})\), \(\mu := (1^p)\), \(\nu := (q - 1)\) and
\( \nu' := (q) \). It follows easily from the Littlewood-Richardson rule that \( \mu \in \mu'_+ \) and \( \nu' \in \nu_+ \). Moreover, \(-1 \notin A_p(\mu', \mu, \nu) \) and \( 1 \notin B_p(\mu', \nu, \nu') \).

**Subcase 2.** Let \( b, i \) be as in Subcase 3 of 5.2.1.

Since \( \lambda'_{q-i+1} = 1 \) for \( i < q \), we get from (20) that \(-b \notin A_{n_i}(\mu', \mu, \nu) \). On the other hand, for \( i = q \) (so \( b = -1 \)), take \( \mu' := (q-1) \), \( \mu := (q) \), \( \nu := \left( 1^q - 1 \right) \) and \( \nu' := (1^q) \). It follows easily from the Littlewood-Richardson rule that \( \mu \in \mu'_+ \) and \( \nu' \in \nu_+ \). Moreover, \( 1 \notin A_q(\mu', \mu, \nu) \) and \( -1 \notin B_q(\mu', \nu, \nu') \).

We therefore conclude that \( \dim V = 0 \) is allowed in this case, as claimed.

This concludes the proof of the theorem. \( \square \)

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