Multivariate exact and falsified sampling approximation *

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Abstract

Approximation properties of the expansions \( \sum_{k \in \mathbb{Z}^d} c_k \varphi(M^j x + k) \), where \( M \) is a matrix dilation, \( c_k \) is either the sampled value of a signal \( f \) at \( M^{-j}k \) or the integral average of \( f \) near \( M^{-j}k \) (falsified sampled value), are studied. Error estimations in \( L_p \)-norm, \( 2 \leq p \leq \infty \), are given in terms of the Fourier transform of \( f \). The approximation order depends on how smooth is \( f \), on the order of Strang-Fix condition for \( \varphi \) and on \( M \). Some special properties of \( \varphi \) are required. To estimate the approximation order of falsified sampling expansions we compare them with a differential expansions \( \sum_{k \in \mathbb{Z}^d} L f(M^{-j})(-k) \varphi(M^j x + k) \), where \( L \) is an appropriate differential operator. Some concrete functions \( \varphi \) applicable for implementations are constructed. In particular, compactly supported splines and band-limited functions can be taken as \( \varphi \). Some of these functions provide expansions interpolating a signal at the points \( M^{-j}k \).

Keywords scaling approximation, Strang-Fix condition, approximation order, sampling and differential expansions, falsified sampling expansions.

AMS Subject Classification: 41A58, 41A25, 41A63

1 Introduction

The well-known sampling theorem (Kotel’nikov’s or Shannon’s formula) states that

\[
f(x) = \sum_{k \in \mathbb{Z}} f(2^{-j}k) \frac{\sin \pi(2^j x - k)}{\pi(2^j x - k)}
\]

for band-limited to \([-2^j-1, 2^j-1]\) signals (functions) \( f \). This formula is very useful for engineers. It was just Kotel’nikov [17] and Shannon [21] who started to apply this formula for signal processing, respectively in 1933 and 1949. Up to now, an overwhelming diversity of digital signal processing applications and devices are based on it and more than successfully use it. Without sampling theorem it would be impossible to make use of internet, make photos and videos. However, mathematicians knew this formula much earlier, actually, it can be found in the papers by Ogura [20] (1920), Whittaker [27] (1915), Borel [3] (1897), and even Cauchy [11] (1841).

Nowadays (1) is also an important and interesting formula for mathematicians. Butzer with co-authors recently published several papers [6], [7], [8], where they analyze sampling theorem, its applications and development. In particular, the equivalence of sampling theorem to some other

*This research was supported by Grants from RFBR (9.38.62.2012) and St.Petersburg State University (9.38.62.2012).
classical formulas was established for a classes of band-limited functions. Also in [2], [11] they studied a generalization of sampling decomposition replacing the sinc-function \( \text{sinc}(x) := \frac{\sin \pi x}{\pi x} \) by certain linear combinations of B-splines.

Equality (1) holds only for functions \( f \in L^2(\mathbb{R}) \) whose Fourier transform is supported on \([-2^{j-1}, 2^{j-1}]\). However the right hand side of (1) (the sampling expansion of \( f \)) has meaning for every continuous \( f \) with a good enough decay. The problem of approximation of \( f \) by its sampling expansions as \( j \to +\infty \) was studied by many mathematicians. We mention only some of such results.

Brown [4] proved that for every \( x \in \mathbb{R} \)

\[
\left| f(x) - \sum_{k \in \mathbb{Z}} f(-2^{-j}k) \text{sinc}(2^j x + k) \right| \leq C \int_{|\xi| > 2^{j-1}} |\hat{f}(\xi)| d\xi, \quad (2)
\]

whenever the Fourier transform of \( f \) is summable on \( \mathbb{R} \). It is known that the pointwise approximation by sampling expansions does not hold for arbitrary continuous \( f \), even compactly supported. Moreover, Trynin [25] proved that there exists a continuous function vanishing outside of \((0, \pi)\) such that its deviation from the sampling expansion diverges at every point \( x \in (0, \pi) \).

Approximation by sampling expansions in \( L^p \)-norm was actively studied. Bardaro, Butzer, Higgins, Stens and Vinti [1], [5], proved that

\[
\Delta_p := \left\| f - \sum_{k \in \mathbb{Z}} f(-2^{-j}k) \text{sinc}(2^j x + k) \right\|_{L^p_{\mathbb{R}}} \to 0 \quad 1 \leq p < \infty,
\]

for \( f \in C \cap \Lambda_p \), where \( \Lambda_p \) consists of \( f \) such that

\[
\sum_{k} |f(x_k)|^p (x_k - x_{k-1}) < \infty
\]

for some class of admissible partition \( \{x_k\}_k \) of \( \mathbb{R} \). Also they proved that the Sobolev spaces \( W^n_p \), \( n \in \mathbb{N} \), are subspaces of \( \Lambda_p \), and that for every \( f \in W^n_p \)

\[
\Delta_p \leq C\omega(f(\cdot), 2^{-j}p) \quad (3)
\]

where \( \omega(\cdot)_p \) is the modulus of continuity in \( L_p \). The author of [22], [23] investigated approximation by sampling expansions

\[
\sum_{k \in \mathbb{Z}} f(-2^{-j}k) \varphi(2^j x + k)
\]

for a wide class of band-limited functions \( \varphi \). For \( p \geq 2 \), the error analysis was given in terms of the Fourier transform of \( f \). In particular, the approximation order was found for functions \( f \) in Sobolev spaces \( W^1_p \) with \( f' \in \text{Lip } \alpha \), \( \alpha > 0 \). In the case \( \alpha < 1 - 1/p \), the order of approximation is less than \( 2^{-j} \), which cannot be obtained from (3). Similar results were proved for the generalized sampling expansions (differential expansions)

\[
\sum_{k \in \mathbb{Z}} Lf(2^{-j} \cdot)(-k) \varphi(2^j x + k), \quad (4)
\]

where \( Lf := \sum_{l=0}^m \alpha_l f(l) \). Also an analog of Brown’s estimate [2] was prove for such expansions in [23].

Note that differential expansions [4] can be useful for engineering applications. Indeed, engineers do not deal with functions, they only have some discrete information about the function. If values of function at equidistributed nodes are known, then sampling expansion is very good for recovering the function. However, sometimes the values are known approximately. Assume that some device
gives the average value of a function $f$ on the interval $[2^{-j}k, 2^{-j}(k+h)]$ instead of $f(2^{-j}k)$, i.e., one knows the values
\[
\frac{1}{2^{-j}h} \int_{2^{-j}k}^{2^{-j}(k+h)} f(t) \, dt \approx \frac{1}{2^{-j}h} \int_0^{2^{-j}h} \sum_{l=0}^m \frac{1}{l!} f(l)(2^{-j}k + l) \, dt = \sum_{l=0}^m \frac{1}{(l+1)!} h^l f(2^{-j}k + l).\]

But the latter sum is nothing as $Lf(2^{-j}k)(k)$, where $a_l = \frac{1}{(l+1)!} h^l$.

Falsified sampling expansions (with the integral averages instead of the exact sampled values), were studied by Butzer and Lei in [10]. They compared such generalized expansions with the corresponding usual sampling expansions. Some error estimates in the $L_\infty$-norm were presented.

The aim of the present paper is to extend the results of [22], [23] to the multivariate case. However the class of functions $\varphi$ we are considering in the present paper is different. We assume that $\varphi$ is in $L_p \varphi$ has some appropriate decay. (the space $L_p$ is a subspace of $L_p$ introduced by Jia in [15], see Section 3). The function sinc does not belong to this class, but the class includes some compactly supported splines, which have an important advantage in applied aspect because the corresponding sampling expansions are finite. We study convergence and approximation order of differential sampling expansions in $L_p$-norm, $2 \leq p \leq \infty$. Also we analyze error estimation for the deviation of falsified sampling expansions (with the integral average sampled values) from the corresponding differential expansion. It appeared that choosing compactly supported $\varphi$ properly, it is possible to provide almost the same order of the error as the order of differential sampling approximation. Our error analysis of falsified sampling expansions is new even for the one-dimensional case.

The paper is organized as follows. Basic notations are given in Section 2. Some auxiliary lemmas are stated in Section 3. Section 4 is devoted to the scaling approximation. Differential and falsified sampling approximation in $L_p$-norm are discussed in Sections 5 and 6 respectively. In Section 7 some examples are given.

2 Notations and basic facts

$\mathbb{N}$ is the set of positive integers, $\mathbb{R}$ is the set of real numbers, $\mathbb{C}$ is the set of complex numbers. $\mathbb{R}^d$ is the $d$-dimensional Euclidean space, $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$ are its elements (vectors), $(x)_j = x_j$ for $j = 1, \ldots, d$, $(x, y) = x_1y_1 + \ldots + x_dy_d$, $|x| = \sqrt{(x, x)}$, $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^d$; $B_r = \{x \in \mathbb{R}^d : |x| \leq r\}$, $T_d = [-\frac{1}{2}, \frac{1}{2}]^d$; $\mathbb{Z}^d$ is the integer lattice in $\mathbb{R}^d$, $\mathbb{Z}_+^d := \{x \in \mathbb{Z}^d : x \geq \mathbf{0}\}$. If $\alpha, \beta \in \mathbb{Z}_+^d$, $a, b \in \mathbb{R}^d$, we set $\lfloor \alpha \rfloor = \sum_{j=1}^d \lfloor \alpha_j \rfloor$, $\lfloor \beta \rfloor = \sum_{j=1}^d \lfloor \beta_j \rfloor$, $\Delta_n := \{\alpha : \alpha \in \mathbb{Z}_+^d, |\alpha| < n\}$.

An integer $d \times d$ matrix $M$ whose eigenvalues are bigger than 1 in modulus is called a dilation matrix. Throughout the paper we consider that such a matrix $M$ is fixed and $m = |\text{det} M|$, $M^*$ denote the conjugate matrix to $M$. Since the spectrum of the operator $M^{-1}$ is located in $B_r$, where $r = r(M^{-1}) := \lim_{j \rightarrow +\infty} \|M^{-j}\|^{1/j}$ is the spectral radius of $M^{-1}$, and there exists at least one point of the spectrum on the boundary of $B_r$, we have
\[
\|M^{-j}\| \leq C_{M, \theta} \theta^{-j}, \quad j \geq 0, \tag{5}
\]
for every positive number $\theta$ which is smaller in module than any eigenvalue of $M$. In particular, we can take $\theta > 1$, then
\[
\lim_{j \rightarrow +\infty} \|M^{-j}\| = 0. \tag{6}
\]
A matrix $M$ is called isotropic if it is similar to a diagonal matrix such that numbers $\lambda_1, \ldots, \lambda_d$ are placed on the main diagonal and $|\lambda_1| = \cdots = |\lambda_d|$. Thus, $\lambda_1, \ldots, \lambda_d$ are eigenvalues of $M$ and the spectral radius of $M$ is equal to $|\lambda|$, where $\lambda$ is one of the eigenvalues of $M$. Note that if matrix $M$ is isotropic then $M^T$ is isotropic and $M^j$ is isotropic for all $j \in \mathbb{Z}$. It is well known that for an isotropic matrices $M$ and for any $j \in \mathbb{Z}$ we have

$$C_1^M |\lambda|^j \leq ||M^j|| \leq C_2^M |\lambda|^j, \quad \text{(7)}$$

where $\lambda$ is one of the eigenvalues of $M$.

If $\varphi$ is a function defined on $\mathbb{R}^d$, we set

$$\varphi_{jk}(x) := m^{j/2}_j \varphi(M^j x + k), \quad j \in \mathbb{Z}, k \in \mathbb{R}^d.$$ 

$L_p$ denotes $L_p(\mathbb{R}^d)$, $1 \leq p \leq \infty$. We use $W^n_p$, $1 \leq p \leq \infty$, $n \in \mathbb{N}$, to denote the Sobolev space on $\mathbb{R}^d$, i.e. the set of functions whose derivatives up to order $n$ are in $L_p(\mathbb{R}^d)$, with usual Sobolev norm.

If $f, g$ are functions defined on $\mathbb{R}^d$ and $\mathbf{f} \in L_1$, then $\langle f, g \rangle := \int_{\mathbb{R}^d} f(x) g(x) dx$.

For any function $f$, we set $f^-(\xi) := f(-\xi)$.

If $F \in L_1(\mathbb{T}^d)$, then $\hat{F}(k) = \int_{\mathbb{T}^d} F(x) e^{-2\pi i (k \cdot x)} dx$ is its $k$-th Fourier coefficient. If $f \in L_1$, then its Fourier transform is $\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i (x \cdot \xi)} dx$.

Denote by $S$ the Schwartz class of functions defined on $\mathbb{R}^d$. The dual space of $S$ is $S'$, i.e. $S'$ is the space of tempered distributions. The basic facts from distribution theory can be found, e.g., in [20]. Suppose $f \in S$, $\varphi \in S'$, then $\langle \varphi, f \rangle := \int \varphi(x) f(x) dx$. If $\varphi \in S'$, then $\hat{\varphi}$ denotes its Fourier transform defined by $\langle \hat{\varphi}, \varphi \rangle = \langle f, \varphi \rangle$, $f \in S$. If $\varphi \in S'$, $j \in \mathbb{Z}, k \in \mathbb{Z}^d$, then we define $\varphi_{jk}$ by $\langle f, \varphi_{jk} \rangle = \langle f \circ M^{-j}, \varphi \rangle, \forall f \in S$.

### 3 Auxiliary results

Here and in what follows $M$ denotes a dilation matrix. Given $\delta > 0$, we introduce a special notation for the following integrals if they make sense

$$T_{\gamma, q}^{l_n} (g) = T_{\gamma, q}^{l_n} (g, M, \delta) = \int_{|M^{-j} \xi| < \delta} |\xi|^{q_7} |g(\xi)|^q d\xi, \quad T_{\gamma, q}^{\text{Out}} (g) = T_{\gamma, q}^{\text{Out}} (g, M, \delta) = \int_{|M^{-j} \xi| \geq \delta} |\xi|^{q_7} |g(\xi)|^q d\xi.$$ 

**Lemma 1** Let $1 \leq q < \infty$, $1/p + 1/q = 1$, $j \in \mathbb{Z}^d$, $\varphi$ be a tempered distribution whose Fourier transform $\hat{\varphi}$ is a function on $\mathbb{R}^d$ such that $|\hat{\varphi}(\xi)| \leq C_\varphi |\xi|^N$ for almost all $\xi \notin \mathbb{T}^d$, $N = N(\varphi) \geq 0$, and $|\hat{\varphi}(\xi)| \leq C_\varphi' \text{ for almost all } \xi \in \mathbb{T}^d$. Suppose $g \in L_q$, $g(\xi) = O(|\xi|^{-N-d-\varepsilon})$ as $|\xi| \to \infty$, where $\varepsilon > 0$; $\gamma \in (N + \frac{d}{p}, N + \frac{d}{q} + \varepsilon)$ for $q \neq 1$, $\gamma = N$ for $q = 1$, and set

$$G_j(\xi) = G_j(\varphi, g, \xi) := \sum_{l \in \mathbb{Z}^d} g(M^{j+1} (\xi + l)) \hat{\varphi}(\xi + l).$$

Then $G_j$ is a 1-periodic function in $L_q(\mathbb{T}^d)$. \( g(\varphi_{jk}) = m^{j/2}_j G_j(k) \), and for every $\delta \in (0, \frac{1}{2})$

$$\left\| G_j - g(M^{j+1}) \varphi \right\|_{L_q(\mathbb{T}^d)}^q \leq m^{-j}(C, \varphi)^q \| M^{j+1} \|^{q-q_7} T_{\gamma, q}^{\text{Out}} (g). \quad \text{(8)}$$
Proof. First we will prove (8). For \( q > 1 \), using Hölder’s inequality, we have

\[
\left\| G_j - g(M^{j^*})z \right\|_{L^q(T^d)}^q \leq C_p^q \int_{T^d} \left\| \frac{\left| \xi + l \right|^\gamma |g(M^{j^*}(\xi + l))|}{\left| \xi + l \right|^{\gamma - N}} \right\|_{T^d}^q d\xi \leq C_{\varphi} \sup_{\xi \in T^d} \left( \sum_{l \in \mathbb{Z}^d, l \neq 0} \frac{1}{\left| \xi + l \right|^{\gamma - N}} \right)^{\frac{q}{\gamma}} \int_{T^d} \sum_{l \in \mathbb{Z}^d, l \neq 0} \left| \left( \xi + l \right)^\gamma |g(M^{j^*}(\xi + l))| \right|^q d\xi =
\]

\[
(C_{\gamma, \varphi})^q \int_{\mathbb{R}^d \setminus T^d} |\xi|^\gamma |g(M^{j^*}(\xi))|^q d\xi \leq (C_{\gamma, \varphi})^q \int_{|\xi| \geq \delta} |\xi|^\gamma |g(M^{j^*}(\xi))|^q d\xi \leq m^{-j}(C_{\gamma, \varphi})^q \|M^{j^{-}}\| \tau_{j, \gamma, \varphi}(g).
\]

If \( q = 1 \), then

\[
\left\| G_j - g(M^{j^*})z \right\|_{L^1(T^d)} \leq C_{\varphi} \int_{T^d} \left\| \frac{\left| \xi + l \right|^\gamma |g(M^{j^*}(\xi + l))|}{\left| \xi + l \right|^{\gamma - N}} \right\|_{T^d} d\xi =
\]

\[
C_{\varphi} \int_{\mathbb{R}^d \setminus T^d} |\xi|^\gamma |g(M^{j^*}(\xi))| d\xi \leq C_{\varphi} \int_{|\xi| \geq \delta} |\xi|^\gamma |g(M^{j^*}(\xi))| d\xi \leq m^{-j}C_{\varphi} \|M^{j^{-}}\| \tau_{j, \gamma, \varphi}(g).
\]

Combining (8) with

\[
\int_{T^d} |g(M^{j^*}(\xi))z| d\xi \leq (C_{\varphi})^q \int_{T^d} |g(M^{j^*}(\xi))|^q d\xi \leq (C_{\varphi})^q m^{-j} \|g\|_q^q,
\]

using Minkowski’s inequality and taking into account that the integral \( \tau_{j, \gamma, \varphi}(g) \) is convergent, we conclude that \( G_j \in L_q(T^d) \).

It remains to show that \( \langle g, \varphi_{jk} \rangle = m^{j/2}G_j(k) \), \( k \in \mathbb{Z}^d \), but this is true because

\[
\langle g, \varphi_{jk} \rangle = m^{-j/2} \int_{\mathbb{R}^d} g(\xi)\varphi(M^{-j}\xi)e^{-2\pi i (k, M^{-j}\xi)} d\xi = m^{j/2} \int_{T^d} G_j(\xi)e^{-2\pi i (k, \xi)} d\xi = m^{j/2}G_j(k),
\]

where the last but one equality is justified by Lebesgue’s dominated convergence theorem with taking into account the summability of the dominating function \( \sum_{l \in \mathbb{Z}^d} |g(M^{j^*}(\xi + l))\varphi(\xi + l)| \) on \( T^d \), which follows from the proof of (8) for \( q = 1 \). \( \Diamond \)

Let \( 1 \leq p \leq \infty \). Denote by \( L_p \) the set

\[
L_p := \left\{ f \in L_p(\mathbb{R}^d) : \|f\|_{L_p} := \left\| \sum_{k \in \mathbb{Z}^d} |f(\cdot + k)| \right\|_{L_p(T^d)} < \infty \right\}.
\]

With the norm \( \| \cdot \|_{L_p} \), \( L_p \) is a Banach space. The simple properties are: \( L_1 = L_1, \|f\|_p \leq \|f\|_\infty \), \( \|f\|_q \leq \|f\|_p \) for \( 1 \leq q \leq p \leq \infty \). Therefore, \( L_\infty \subset L_p \subset L_q \) for \( 1 \leq q \leq p \leq \infty \). If \( f \in L_p \) and compactly supported then \( f \in L_p \) for \( p \geq 1 \). If \( f \) decays fast enough, i.e. there exist constants \( C > 0 \) and \( \varepsilon > 0 \) such that \( |f(x)| \leq C(1 + |x|)^{-d-\varepsilon} \) \( \forall x \in \mathbb{R}^d \), then \( f \in L_\infty \). Also we need the following statement (see [15] Theorem 2.1)
**Theorem 4** Let \( k \) and \( \varphi \) be as in Lemma 1. Suppose \( 2 \leq p < \infty \), then for every finite subset \( \Omega \) of \( \mathbb{Z}^d \), the series \( \sum_{k \in \mathbb{Z}^d} |\langle g, \hat{\varphi}_{jk} \rangle|^p \) converges unconditionally in \( L_p(\mathbb{R}^d) \).

**Proof.** Suppose \( 2 \leq p < \infty \). Because of Lemma 1 and the Hausdorff-Young inequality, we have

\[
\left( \sum_{k \in \mathbb{Z}^d} |\langle g, \hat{\varphi}_{jk} \rangle|^p \right)^{\frac{1}{p}} = m^{\frac{1}{p}} \left( \sum_{k \in \mathbb{Z}^d} |\hat{G}_j(k)|^p \right)^{\frac{1}{p}} \leq m^{\frac{1}{p}} \|G_j\|_q < \infty,
\]

where \( G_j \) is a function from Lemma 1. By Proposition 2 we can state that for every finite subset \( \Omega \) of \( \mathbb{Z}^d \)

\[
\left\| \sum_{k \in \Omega} \langle g, \hat{\varphi}_{jk} \rangle \varphi \right\|_p \leq m^{\frac{1}{p}} \left( \sum_{k \in \Omega} |\langle g, \hat{\varphi}_{jk} \rangle|^p \right)^{\frac{1}{p}}.
\]

The series \( \sum_{k \in \mathbb{Z}^d} |\langle g, \hat{\varphi}_{jk} \rangle|^p \) is convergent, which yields that \( \sum_{k \in \mathbb{Z}^d} \langle g, \hat{\varphi}_{jk} \rangle \varphi \) converges unconditionally.

Similarly, if \( p = \infty \), then for every finite subset \( \Omega \) of \( \mathbb{Z}^d \)

\[
\left\| \sum_{k \in \Omega} \langle g, \hat{\varphi}_{jk} \rangle \varphi \right\|_{L_\infty} = m^{\frac{1}{p}} \left\| \sum_{k \in \Omega} |\langle g, \hat{\varphi}_{jk} \rangle|^p \right\|_{L_\infty} \leq m^{\frac{1}{p}} \|\varphi\|_{L_\infty} \sup_{k \in \Omega} |\langle g, \hat{\varphi}_{jk} \rangle| = m^1 \|\varphi\|_{L_\infty} \sup_{k \in \Omega} |\hat{G}_j(k)|,
\]

and the unconditional convergence follows from the Riemann-Lebesgue theorem. \( \diamond \)

## 4 Scaling Approximation

Scaling operator \( \sum_{k \in \mathbb{Z}^d} \langle f, \hat{\varphi}_{jk} \rangle \varphi_{jk} \) is a good tool of approximation for many appropriate pairs of functions \( \varphi, \bar{\varphi} \). We are interested in such operators, where \( \bar{\varphi} \) is a tempered distribution, e.g., the delta-function or a linear combination of its derivatives. In this case the inner product \( \langle f, \bar{\varphi}_{jk} \rangle \) has meaning only for functions \( f \) in \( S' \). To extend the class of functions \( f \) one can replace \( \langle f, \bar{\varphi}_{jk} \rangle \) by \( \langle \hat{f}, \bar{\varphi}_{jk} \rangle \) or by \( \langle g, \bar{\varphi}_{jk} \rangle \), where \( f = g^{-} \), and in this case we set

\[
Q_j(\varphi, \bar{\varphi}, f) = \sum_{k \in \mathbb{Z}^d} \langle g, \hat{\varphi}_{jk} \rangle \varphi_{jk}, \quad j \in \mathbb{Z}^+.
\]

In this section we study approximation properties of such operators for a large class of distributions \( \bar{\varphi} \).

A function \( f \in L_1(\mathbb{R}^d) \) is said to satisfy the **Strang-Fix condition of order** \( n \) if \( D^\beta f(k) = 0 \), for all \( k \in \mathbb{Z}^d \setminus \{0\} \) and \( \beta \in \Delta_n \).

**Theorem 4** Let \( 2 \leq p \leq \infty \), \( 1/p + 1/q = 1 \). Suppose

* \( \varphi \in \mathcal{L}_p \) and there exists \( B_\varphi > 0 \) such that

\[
\sum_{k \in \mathbb{Z}^d} |\hat{\varphi}(|x| + k)|^q < B_\varphi \quad \forall x \in \mathbb{R}, \quad (9)
\]
Let us fix some \( \delta \) such that \( |\hat{\varphi}(\xi)| \leq C_{\varphi}|\xi|^N \) for almost any \( \xi \notin T^d \), \( N = N(\varphi) \geq 0 \), and \( |\hat{\varphi}(\xi)| \leq C_{\varphi}' \) for almost all \( \xi \in T^d \);

- \( f = g^- \) where \( g \in L_q \), \( g(\xi) = O(|\xi|^{-N-d-\varepsilon}) \) as \( |\xi| \to \infty \), \( \varepsilon > 0 \).

Then

- if \( \hat{\varphi} \) is continuous at zero and \( \hat{\varphi}(0)\hat{\varphi}(0) = 1 \), then the Strang-Fix condition of order 1 for \( \varphi \) is necessary and sufficient for the convergence of \( Q_j(\varphi, \hat{\varphi}, f) \) to \( f \) in the \( L_p \)-norm as \( j \to +\infty \);
- if there exist \( n \in \mathbb{N} \) and \( \delta \in (0, 1/2) \) such that \( \hat{\varphi} \) is boundedly differentiable up to order \( n \) on \( \{|\xi| < \delta\} \), \( \hat{\varphi} \) is boundedly differentiable up to order \( n \) on \( \{|\xi| + l < \delta\} \) for all \( l \in \mathbb{Z}^d \setminus \{0\} \); the function \( \sum_{l \in \mathbb{Z}^d, l \neq 0} |D^\beta \hat{\varphi}(\xi + l)| \) is bounded on \( \{|\xi| < \delta\} \) for \( \beta = n \); \( D^\beta (1 - \hat{\varphi}^2)(0) = 0 \) for all \( \beta \in \Delta_n \); the Strang-Fix condition of order \( n \) holds for \( \varphi \); \( \gamma \in (N + \frac{2}{p}, N + \frac{2}{p} + \varepsilon) \) for \( p \neq \infty \), and \( \gamma = N \) for \( p = \infty \), then

\[
\|f - Q_j(\varphi, \hat{\varphi}, f)\|_p^p \leq C_1 M^{s-j} \|g_T^{out}(\xi, \gamma, q, g)\| + C_2 M^{s-j} \|g_T^{out}(\xi, \gamma, q, g)\|,
\]

where \( C_1 \) and \( C_2 \) do not depend on \( j \) and \( f \).

**Proof.** First of all we note that \( \hat{\varphi} \in L_q \) due to (9), and \( \hat{\varphi} \) is continuous because \( \varphi \in L_1 \). Let \( G_j(\xi) = G_j(\varphi, g, \xi) \) be defined as in Lemma 11. This function is 1-periodic and in \( L_q(\mathbb{R}^d) \). The function \( G_j(M^{s-j} \cdot \hat{\varphi}(M^{s-j} \cdot)) \) is in \( L_q \) because, due to (9),

\[
\int_{\mathbb{R}^d} |G_j(M^{s-j} \cdot \hat{\varphi}(M^{s-j} \cdot)|^q \, d\xi = m^j \int_{\mathbb{T}^d} |G_j(\xi)|^q \sum_{k \in \mathbb{Z}^d} |\hat{\varphi}(\xi + k)|^q \, d\xi \leq m^j B_{\varphi}{\|G_j\|}_{L_q(\mathbb{T}^d)}^q.
\]

Hence, its Fourier transform is in \( L_p \) due to the Hausdorff-Young inequality. On the other hand, \( Q_j(f) \in L_p \) by Lemma 10 and, due to Carleson's theorem (with the convergence of Fourier series over the cubic partial sums),

\[
G_j(M^{s-j} \xi)\hat{\varphi}(M^{s-j} \xi) = \sum_{k \in \mathbb{Z}^d} G_j(k)e^{2\pi i (k, M^{s-j} \xi)} \hat{\varphi}(M^{s-j} \xi) = \sum_{k \in \mathbb{Z}^d} \langle g, \hat{\varphi}(jk) \rangle \hat{\varphi}(jk) \text{ a.e.}
\]

The latter function coincides with \( Q_j(f) \) as a tempered distribution. It follows from the du Bois-Reymond lemma that the Fourier transform of \( G_j(M^{s-j} \cdot \hat{\varphi}(M^{s-j} \cdot)) \) coincides with \( Q_j(f) \) almost everywhere. Applying the Hausdorff-Young inequality we obtain

\[
\|f - Q_j(f)\|_p = \|f^* - Q_j(f)^*\|_p \leq \|g - G_j(M^{s-j} \cdot \hat{\varphi}(M^{s-j} \cdot))\|_q \leq \|g(1 - \hat{\varphi}(M^{s-j} \cdot)\hat{\varphi}(M^{s-j} \cdot))\|_q + \|\hat{\varphi}(M^{s-j} \cdot) \sum_{l \in \mathbb{Z}^d, l \neq 0} g(\xi + M^{s-j}l)\|_q. \quad (11)
\]

Let us fix some \( \delta \in (0, \frac{1}{2}) \) and consider the first summand in (11). Changing the variable we obtain

\[
\|g(1 - \hat{\varphi}(M^{s-j} \cdot)\hat{\varphi}(M^{s-j} \cdot))\|_q = m^j \|g(M^{s-j} \cdot)(1 - \hat{\varphi}(\hat{\varphi}))\|_q = m^j \int_{|\xi| \geq \delta} |g(M^{s-j} \xi)(1 - \hat{\varphi}(\xi)\hat{\varphi}(\xi))|^q \, d\xi + m^j \int_{|\xi| < \delta} |g(M^{s-j} \xi)(1 - \hat{\varphi}(\xi)\hat{\varphi}(\xi))|^q \, d\xi =: I_1 + I_2. \quad (12)
\]
Thus, \( I \) is

\[
I_1 \leq (C'_{\varphi, \bar{\varphi}})^q m^j \int_{\mathbb{R}^d \setminus \mathbb{T}^d} |\xi|^q |g(M^{*j} \xi)|^q d\xi + (C'_{\varphi, \bar{\varphi}})^q \delta^{-\gamma q} m^j \int_{\mathbb{T}^d \setminus \{ |\xi| < \delta \}} |\xi|^q |g(M^{*j} \xi)|^q d\xi \leq C_1 \| M^{*j} \| \gamma q T_{\text{out}}^{\text{out}}(g).
\]

(13)

Thus, \( I_1 \to 0 \) as \( j \to +\infty \). The second integral \( I_2 \) is

\[
I_2 = m^j \int_{|\xi| < \delta} |g(M^{*j} \xi)(1 - \varphi(\xi)\bar{\varphi}(\xi))|^q d\xi = \int_{\mathbb{R}^d} \chi_{M^{*j} \{ |\xi| < \delta \}}(\xi) |g(M^{*j} \xi)|^q d\xi.
\]

By the additional assumptions of the first statement, the integrand tends to zero as \( j \to +\infty \) for each \( \xi \in \mathbb{R}^d \). Since the integrand has a summable majorant, by Lebesgue’s dominated convergence theorem, we conclude that \( I_2 \to 0 \) as \( j \to +\infty \). Thus, by (12),

\[
\left\| g(1 - \varphi(M^{*j} \bar{\varphi}(M^{*j} \cdot)) \right\|_q \to 0 \quad \text{as} \quad j \to +\infty.
\]

(14)

Next, the second summand in (11) is

\[
\left\| \varphi(M^{*j} \cdot) \sum_{l \in \mathbb{Z}^d, l \neq 0} g(l + M^{*j} l)\varphi(M^{*j} \cdot + l) \right\|_q^q = m^j \left\| \varphi \sum_{l \in \mathbb{Z}^d, l \neq 0} g(M^{*j} \cdot + l)\varphi(\cdot + l) \right\|_q^q = m^j \sum_{k \in \mathbb{Z}^d} \int_{\mathbb{T}^d} |\varphi(\xi + k)|^q \left| \sum_{l \in \mathbb{Z}^d, l \neq 0} g(M^{*j} (\xi + l + k))\varphi(\xi + l + k) \right|^q d\xi = m^j \sum_{k \in \mathbb{Z}^d} \int_{\mathbb{T}^d} |\varphi(\xi + k)|^q \left| \sum_{l \in \mathbb{Z}^d, l \neq k} g(M^{*j} (\xi + l))\varphi(\xi + l) \right|^q d\xi + m^j \int_{\mathbb{T}^d} |\varphi(\xi)|^q \left| \sum_{l \in \mathbb{Z}^d, l \neq 0} g(M^{*j} (\xi + l))\varphi(\xi + l) \right|^q d\xi =: J_1 + J_2.
\]

(15)

To estimate \( J_1 \) we use (9), and then, by (8),

\[
J_1 \leq B_\varphi m^j \left\| G_j - g(M^{*j} \cdot)\varphi \right\|_{L_q(\mathbb{T}^d)}^q \leq C_1 \| M^{*j} \| \gamma q T_{\text{out}}^{\text{out}}(g).
\]

(16)

Thus, \( J_1 \to 0 \) as \( j \to +\infty \). Using Minkowski’s inequality, the second summand \( J_2 \) can be estimated
Thus, since 

By the additional assumptions of the first statement, the integrand tends to as follows

To estimate \( J_{21} \) again we use (9), and then, by (8),

\[
J_{21} \leq B_{\varphi,n} m^j\|G_j - g(M^{*j}.)\|_{L_d(T^d)}^q \leq C_{J_{21}} \|M^{* - j}\|^q \mathcal{T}^{Out}_{j,\gamma,q}(g).
\]

Since \( \hat{\varphi} \) is bounded, similarly to (13), we have

\[
J_{22} = m^j \int_{\mathbb{R}^d \setminus T^d} |\hat{\varphi}(\xi)g(M^{*j}X\xi)|^q|d\xi| \leq C_{J_{22}} \|M^{* - j}\|^q \mathcal{T}^{Out}_{j,\gamma,q}(g).
\]

Thus, \( J_{21} \to 0, J_{22} \to 0 \), as \( j \to +\infty \).

The third summand \( J_{23} \) can be represented as

\[
J_{23} = \int_{\mathbb{R}^d} \chi_{M^{*},T^d}(\xi) \sum_{k \in \mathbb{Z}^d,k \neq 0} |\hat{\varphi}(M^{* - j} \xi + k)|^q|g(\xi)\hat{\varphi}(M^{* - j} \xi)|^q|d\xi|
\]

By the additional assumptions of the first statement, the integrand tends to

\[
\sum_{k \in \mathbb{Z}^d,k \neq 0} |\hat{\varphi}(k)|^q|g(\xi)\hat{\varphi}(0)||d\xi|
\]

as \( j \to +\infty \) for each \( \xi \in \mathbb{R}^d \). Since the integrand has a summable majorant, by Lebesque's dominated convergence theorem, we conclude that

\[
J_{23} \to \sum_{k \in \mathbb{Z}^d,k \neq 0} |\hat{\varphi}(k)|^q|\hat{\varphi}(0)||g||_q^q, \quad j \to +\infty.
\]

Thus,

\[
\left\| \hat{\varphi}(M^{* - j}.) \sum_{l \in \mathbb{Z}^d,l \neq 0} g(\xi + M^{*j}l)\overline{\varphi}(M^{* - j} \cdot + l) \right\|^q_{q} \to \sum_{k \in \mathbb{Z}^d,k \neq 0} |\hat{\varphi}(k)|^q|\hat{\varphi}(0)||g||_q^q, \quad j \to +\infty,
\]

which together with (11) and (14) yields that the Strang-Fix condition for \( \varphi \) is necessary and sufficient for \( \|f - Q_j(f)\|_p \to 0 \) as \( j \to +\infty \).

Assume now that all assumptions of the second statement are satisfied. Set

\[
B_{\varphi,\hat{\varphi},n} := \sup_{|\xi| < \delta} \left( \sum_{|\beta| = n} \frac{|D^{|\beta|}\hat{\varphi}(\xi)|}{\beta!} \right)^q, \quad B_{\varphi,n} = \sup_{|\xi| < \delta} \left( \sum_{l \in \mathbb{Z}^d, l \neq 0} \frac{\sum_{|\beta| = n} D^{|\beta|}\hat{\varphi}(\xi + l)}{\beta!} \right)^q.
\]

If \( |\xi| < \delta, l \neq 0 \), then, by the Taylor formula with the remainder in Lagrange's form, we get

\[
\hat{\varphi}(\xi + l) = \sum_{|\beta| = n} \frac{\xi^{|\beta|}D^{|\beta|}\hat{\varphi}(l + \xi)}{\beta!}, \quad 1 - \hat{\varphi}(\xi)\overline{\varphi}(\xi) = -\sum_{|\beta| = n} \frac{\xi^{|\beta|}D^{|\beta|}\hat{\varphi}(l + \xi)}{\beta!}.
\]
for some \( t, r \in (0, 1) \), and hence
\[
\sum_{l \in \mathbb{Z}^d, l \neq 0} |\hat{\varphi}(\xi + l)|^q \leq B_{\varphi, n} |\xi|^{n_q}, \quad |1 - \hat{\varphi}(\xi)\hat{\varphi}(\xi)|^q \leq B_{\varphi, \varphi, n} |\xi|^{n_q}.
\]

It follows that
\[
m^j \int_{|\xi| < \delta} |g(M^{*j} \xi)(1 - \hat{\varphi}(\xi)\hat{\varphi}(\xi))|^q d\xi \leq B_{\varphi, \varphi, n} ||M^{*j}||^{n_q} \mathcal{T}^I_{j, n, q}(g),
\]
and
\[
m^j \int_{|\xi| < \delta} \sum_{k \in \mathbb{Z}^d, k \neq 0} |\hat{\varphi}(\xi + k)|^q |g(M^{*j} \xi)\hat{\varphi}(\xi)|^q d\xi \leq B_{\varphi, n} (C^{*j}_\varphi)^q ||M^{*j}||^{n_q} \mathcal{T}^I_{j, n, q}(g).
\]

Combining these relations with (13) and (9) respectively, we obtain
\[
m^j \int_{\mathbb{R}^d} |g(M^{*j} \xi)(1 - \hat{\varphi}(\xi)\hat{\varphi}(\xi))|^q d\xi \leq B_{\varphi, \varphi, n} ||M^{*j}||^{n_q} \mathcal{T}^I_{j, n, q}(g) + C_{11} ||M^{*j}||^{\gamma_q} \mathcal{T}^{Out}_{j, \gamma, q}(g)
\]
and
\[
m^j \int_{\mathbb{R}^d} \sum_{k \in \mathbb{Z}^d, k \neq 0} |\hat{\varphi}(\xi + k)|^q |g(M^{*j} \xi)\hat{\varphi}(\xi)|^q d\xi \leq B_{\varphi, n} (C^{*j}_\varphi)^q ||M^{*j}||^{n_q} \mathcal{T}^I_{j, n, q}(g) + B_{\varphi} m^j \int_{|\xi| < \delta} |g(M^{*j} \xi)\hat{\varphi}(\xi)|^q d\xi \leq (C^{*j}_\varphi)^q (B_{\varphi, n} ||M^{*j}||^{n_q} \mathcal{T}^I_{j, n, q}(g) + B_{\varphi} \delta^{-\gamma_q} ||M^{*j}||^{\gamma_q} \mathcal{T}^{Out}_{j, \gamma, q}(g)).
\]

These estimations together with (11)-(13) and (14)-(15) complete the proof of (10). \( \diamond \)

Note that the above proof is based on the technique employed by Jetter and Zhou [13], [14], and developed in [15].

Unfortunately it is complicated to estimate the approximation order of \( Q_j(\varphi, \varphi, f) \) from (10) in Theorem 4 for the general case. But we now will give such estimations for some special cases.

**Theorem 5** Let \( M \) be an isotropic matrix dilation and \( \lambda \) be its eigenvalue, \( j \in \mathbb{Z}_+ \). Suppose all conditions of Theorem 4 are fulfilled. Then
\[
\|f - Q_j(\varphi, \varphi, f)\|_p \leq \begin{cases} C|\lambda|^{-j(N + d_{-\varepsilon})} & \text{if } n > N + \frac{d}{p} + \varepsilon \\ Cj^{1/q}|\lambda|^{-jn} & \text{if } n = N + \frac{d}{p} + \varepsilon \\ C|\lambda|^{-jn} & \text{if } n < N + \frac{d}{p} + \varepsilon \end{cases}
\]

where \( C \) does not depend on \( j \).

**Proof.** Throughout the proof we denote by \( C \) and \( C' \) different constants which do not depend on \( j \). Since \( g(\xi) = O(|\xi|^{-N - d_{-\varepsilon}}) \), there exists a big enough number \( A \in \mathbb{R} \) such that \( |g(\xi)| \leq C|\xi|^{-N - d_{-\varepsilon}} \) for any \( |\xi| > A \). By Theorem 4 inequality (10) is valid. Let us consider the first term in (10). Since the set \( \{ \|M^{*j}\| \geq \delta \} \) is a subset of \( \{ |\xi| \geq \delta / ||M^{*j}|| \} \), we have
\[
\|M^{*j}\|^{\gamma_q} \mathcal{T}^{Out}_{j, \gamma, q}(g) = \|M^{*j}\|^{\gamma_q} \int_{|\xi| \geq \delta} |\xi|^{\gamma_q} |g(\xi)|^q d\xi \leq C \|M^{*j}\|^{\gamma_q} \int_{|\xi| \geq \delta / ||M^{*j}||} \frac{d\xi}{|\xi|^{(N + d_{-\varepsilon} - \gamma_q)n}},
\]
for all \( j > j_0 \), where \( j_0 \in \mathbb{Z} \) is such that \( \frac{\delta}{\|M^{*}\|} > A \). Using general polar coordinates with \( \rho := |\xi| \) and taking into account that \((N + d + \varepsilon - \gamma)q > d\), we obtain

\[
\int_{|\xi| > \delta / \|M^{*}\|} \frac{d\xi}{|\xi| \left( |N + d + \varepsilon - \gamma\right)q} \leq C \int_{\delta / \|M^{*}\|}^{+\infty} \frac{1}{\rho(N + d + \varepsilon - \gamma)q - d + 1} d\rho \leq C' \|M^{*}\|^{q(N + \frac{\varepsilon}{p} + \delta)}
\]

and, by \((7)\),

\[
\|M^{*}\|^{nqT_{j,n,q}^{Out}}(g) \leq C \|M^{*}\|^{q(N + \frac{\varepsilon}{p} + \delta)} \leq C' |\lambda|^{-jq(N + \frac{\varepsilon}{p} + \delta)}.
\]

Next, let us consider the second term in \((10)\). Since the set \( \{|M^{*}\xi| < \delta\} \) is a subset of \( \{|\xi| < \|M^{*}\|\delta\} \), we get

\[
\|M^{*}\|^{nqT_{j,n,q}^{In}}(g) = \|M^{*}\|^{nq} \int_{|M^{*}\xi| < \delta} |\xi|^{nq} |g(\xi)|^q d\xi \leq C \|M^{*}\|^{nq} \int_{|\xi| < \|M^{*}\|\delta} |\xi|^{nq} |g(\xi)|^q d\xi \leq C' \left( \|M^{*}\|^{nq} + \|M^{*}\|^{nq} \int_{A < |\xi| < \|M^{*}\|\delta} |\xi|^{(n - N - d - \varepsilon)q} d\xi \right)
\]

for all \( j > j_1 \), where \( j_1 \in \mathbb{Z} \) is such that \( A < \|M^{*}\|\delta \). Using general polar coordinates with \( \rho := |\xi| \) we obtain

\[
\|M^{*}\|^{nqT_{j,n,q}^{In}}(g) \leq C \left( \|M^{*}\|^{nq} + \|M^{*}\|^{nq} \int_{A}^{+\infty} \rho(n - N - d - \varepsilon)q + d - 1 d\rho \right).
\]

Let \( J := \max\{j_0, j_1\}, j > J \). If \( n < N + \frac{2}{p} + \varepsilon \), then the integral

\[
\int_{A}^{+\infty} \rho(n - N - d - \varepsilon)q + d - 1 d\rho
\]

is convergent, and using \((7)\), we have \( \|M^{*}\|^{nqT_{j,n,q}^{In}}(g) \leq C |\lambda|^{-jnq} \). Therefore, together with \((20)\) the final estimate in this case is \( \|f - Q_j(\varphi, \varphi, f)\|_p \leq C |\lambda|^{-jn} \).

Let \( n = N + \frac{2}{p} + \varepsilon \). Again using \((7)\), we have

\[
\|M^{*}\|^{nqT_{j,n,q}^{In}}(g) \leq C \left( \|M^{*}\|^{nq} + \|M^{*}\|^{nq} \ln \|M^{*}\| \right) \leq C' j |\lambda|^{-jnq}.
\]

Together with \((20)\) the final estimate in this case is \( \|f - Q_j(\varphi, \varphi, f)\|_p \leq C j^{1/q} |\lambda|^{-jn} \).

Similarly, if \( n > N + \frac{2}{p} + \varepsilon \), then

\[
\|M^{*}\|^{nqT_{j,n,q}^{In}}(g) \leq C \left( |\lambda|^{-jnq} + |\lambda|^{-jq(N + \frac{\varepsilon}{p} + \delta)} \right) \leq C' |\lambda|^{-jq(N + \frac{\varepsilon}{p} + \delta)}
\]

Therefore, together with \((20)\) the final estimate in this case is \( \|f - Q_j(\varphi, \varphi, f)\|_p \leq C |\lambda|^{-j(N + \frac{\varepsilon}{p} + \delta)} \). Observing the proof of Theorem 5 for the case \( n < N + \frac{2}{p} + \varepsilon \), we see that the estimate does not depend on \( \|M^{*}\| \). So, in this case, we can repeat the proof for arbitrary matrix dilation \( M \) with using \((5)\) instead of \((7)\), which leads to the following statement.
Theorem 6 Suppose all conditions of Theorem 4 are fulfilled, and \( n < N + \frac{d}{p} + \varepsilon \). Then
\[
\|f - Q_J(\varphi, \tilde{\varphi}, f)\|_p \leq C\theta^{-Jn}
\] (21)
for every positive number \( \theta \) which is smaller in module than any eigenvalue of \( M \) and some \( C \) which does not depend on \( J \).

The latter theorem does not provide approximation order of \( Q_J(\varphi, \tilde{\varphi}, f) \) better than \( \theta^{-Jn} \) even for very smooth functions \( f \). This drawback can be fixed under stronger restrictions on \( \varphi \).

Theorem 7 Let \( 2 \leq p \leq \infty, 1/p + 1/q = 1 \). Suppose

\begin{itemize}
    \item \( \tilde{\varphi} \) be a tempered distribution whose Fourier transform \( \hat{\tilde{\varphi}} \) is a function on \( \mathbb{R}^d \) such that \( |\hat{\tilde{\varphi}}(\xi)| \leq C_{\tilde{\varphi}}|\xi|^{N_2} \) for almost any \( \xi \not\in \mathbb{T}^d, N_2 > 0 \), and \( |\hat{\varphi}(\xi)| \leq C_{\varphi}' \) for almost all \( \xi \in \mathbb{T}^d; \)
    \item \( \varphi \in L_p, \) the function \( \sum_{k \in \mathbb{Z}^d} |\hat{\varphi}(\xi + k)|^q \) is bounded, and there exists \( \delta \in (0, 1/2) \) such that \( \hat{\varphi}(\xi)\hat{\varphi}(\xi) = 1 \) a.e. on \( \{|\xi| < \delta\} \), \( \hat{\varphi}(\xi) = 0 \) a.e. on \( \{|l - \xi| < \delta\} \) for all \( l \in \mathbb{Z} \setminus \{0\}; \)
    \item \( f = \hat{g}^{-}, \) where \( g \in L_q, g(\xi) = O(|\xi|^{-N-d-\varepsilon}) \) as \( |\xi| \to \infty, \varepsilon > 0. \)
\end{itemize}

Then
\[
\|f - Q_J(\varphi, \tilde{\varphi}, f)\|_p \leq C\theta^{-j(N + \frac{d}{p} + \varepsilon)}
\] (22)
for every positive number \( \theta \) which is smaller in module than any eigenvalue of \( M \) and some \( C \) which does not depend on \( j \).

Proof. Let \( \gamma \in (N + \frac{d}{p}, N + \frac{d}{p} + \varepsilon) \) for \( p \neq \infty \), and \( \gamma = N \) for \( p = \infty \). First we prove that
\[
\|f - Q_J(\varphi, \tilde{\varphi}, f)\|_p \leq C'\|M^{-J\gamma}\|^qT_{\text{Out}}^{\gamma,q}(g),
\] (23)
where \( C' \) does not depend on \( j \) and \( f \). For any compact set \( K \subset \mathbb{R}^d \), function \( g \) can be approximated in \( L_q(K) \) by infinitely smooth functions supported on \( K \). So, given \( j \), one can find a function \( \hat{F}_j \in C^\infty(\mathbb{R}^d) \) such that \( \text{supp} \hat{F}_j \subset \{|M^{-J}\xi| < \delta\} \) and\\
\[
\int_{|M^{-j}\xi|<\delta} |g(\xi) - \hat{F}_j(\xi)|^q d\xi \leq \left( \sup_{|M^{-j}\xi|<\delta} |\xi|^{Nq} \right)^{-1} \|M^{-j\gamma}\|^qT_{\text{Out}}^{\gamma,q}(g).
\] (24)
This yields\\
\[
T_{j,N,q}^{\text{In}}(g - \hat{F}_j) = \int_{|M^{-j}\xi|<\delta} |\xi|^{Nq} |(g(\xi) - \hat{F}_j(\xi))|^q d\xi \leq \|M^{-j\gamma}\|^qT_{\text{Out}}^{\gamma,q}(g).
\] (25)
Let \( F_j \) be a function whose Fourier transform is \( \hat{F}_j \). Evidently \( F_j^- = \hat{F}_j \), and, due to Carleson’s theorem and Lemma 11 we have
\[
\sum_{k \in \mathbb{Z}^d} \langle \hat{F}_j, \tilde{\varphi}jk \rangle \tilde{\varphi}jk(\xi) = \sum_{l \in \mathbb{Z}^d} \hat{F}_j(\xi + M^jl)\tilde{\varphi}(M^{-j}\xi + l)\hat{\varphi}(M^{-j}\xi)
\]
If \( l \neq 0 \) and \( F_j(\xi + M^jl) \neq 0 \), then \( |M^{-j}\xi + l| < \delta \) and hence \( \hat{\varphi}(M^{-j}\xi) = 0 \). So,
\[
\sum_{k \in \mathbb{Z}^d} \langle \hat{F}_j, \tilde{\varphi}jk \rangle \tilde{\varphi}jk(\xi) = \hat{F}_j(\xi),
\]
which yields that \( Q_J(\varphi, \tilde{\varphi}, F_j) = F_j \). It follows that
\[
\|f - Q_J(\varphi, \tilde{\varphi}, f)\|_p = \|f - F_j - Q_J(\varphi, \tilde{\varphi}, f - F_j)\|_p.
\]
Since the assumptions of the second statement in Theorem 4 is satisfied for every \( n \), in particular, for \( n = N \), we obtain
\[
\|f - F_j - Q_j(\varphi, \bar{\varphi}, f - F_j)\|_p \leq C_1 \|M^{s-j}\|^N \mathcal{I}_{j,N,q}^I(g - \hat{F}) + C_2 \|M^{s-j}\|^\gamma \mathcal{I}_{j,\gamma,q}^O(g - \hat{F}).
\]
To complete the proof of (23) it remains to use (25) and take into account that \( \mathcal{I}_{j,\gamma,q}^O(g - \hat{F}) = \frac{\gamma + \varepsilon}{\|M^{s-j}\|^\gamma} \mathcal{I}_{j,\gamma,q}^O(g) \).

Using the first inequality in (20), we have
\[
\|M^{s-j}\|^\gamma \mathcal{I}_{j,\gamma,q}^O(g) \leq C_3 \|M^{s-j}\|^{q(N + \frac{k}{p} + \varepsilon)},
\]
which yields (21) due to (5). \( \diamond \)

5 Differential expansions

The above theorems are proved for a wide class of distributions \( \bar{\varphi} \). The \( \delta \)-function can be taken as \( \bar{\varphi} \) with \( N = 0 \). If \( f = \hat{g} \) and \( g \) as in Theorem 4 then \( g \in L_1 \) and
\[
f(-M^{-j}k) = \hat{g}(M^{-j}k) = \int_{\mathbb{R}^d} g(\xi)e^{-2\pi ik \cdot M^{-j} \xi} d\xi = m^{j/2} \langle g, \hat{\delta}_{jk} \rangle, \quad k \in \mathbb{Z}^d.
\]
Hence
\[
Q_j(\varphi, \bar{\varphi}, f) = m^{-j/2} \sum_{k \in \mathbb{Z}^d} f(-M^{-j}k) \varphi_{jk},
\]
i.e. \( Q_j \) is a sampling differential of \( f \) in this case.

Consider next a differential operator \( L \) defined by
\[
Lf := \sum_{\beta \in \Delta_{N+1}} a_\beta D^\beta f, \quad a_\beta \in \mathbb{C}, a_0 \neq 0,
\]
where \( N \in \mathbb{Z}_+ \). The action of operator \( D^\beta \) is associated with the action of the corresponding derivative of the \( \delta \)-function. In more detail, let \( f = \hat{g}^- \), \( \int_{\mathbb{R}^d} (1 + |\xi|)^{N+\alpha} |g(\xi)| d\xi < \infty, \alpha > 0 \).

Therefore, \( f \) is continuously differentiable on \( \mathbb{R}^d \) up to the order \( N \). Then for \( \beta \in \Delta_{N+1} \)
\[
D^\beta f(M^{-j} \cdot)(-k) = D^\beta \hat{g}(M^{-j} \cdot)(k) = (-1)^{[\beta]} m^{j} D^\beta \hat{g}(M^{-j} \cdot)(k) = \int_{\mathbb{R}^d} g(M^{s-j} \xi)(-2\pi i \xi)^\beta e^{-2\pi i k \cdot \xi} d\xi = \int_{\mathbb{R}^d} g(\xi)(2\pi i M^{s-j} \xi)^\beta e^{2\pi i k \cdot M^{s-j} \xi} d\xi = (1)^{[\beta]} m^{j/2} \langle g, \hat{\delta}_{jk} \rangle
\]
If now \( \bar{\varphi} = \sum_{\beta \in \Delta_{N+1}} a_\beta (-1)^{[\beta]} D^\beta \delta \) (we say that \( \bar{\varphi} \) is associated with \( L \)), then
\[
m^{-j/2} Lf(M^{-j} \cdot)(-k) = \langle g, \bar{\varphi}_{jk} \rangle, \quad k \in \mathbb{Z}^d.
\]
Hence
\[
Q_j(\varphi, \bar{\varphi}, f) = m^{-j/2} \sum_{k \in \mathbb{Z}^d} Lf(M^{-j} \cdot)(-k) \varphi_{jk}.
\]
Thus, choosing appropriate function \( \varphi \) we can provide all conditions of Theorem 4 which together with Theorems 5 and 6 gives the following statement.
Theorem 8 Let $2 \leq p \leq \infty$, $1/p + 1/q = 1$, $N \in \mathbb{Z}_+$, $\gamma \in (N + \frac{d}{p}, N + \frac{d}{p} + \epsilon)$ for $p \neq \infty$, and $\gamma = N$ for $p = \infty$, a differential operator $L$ be defined by (29). Suppose

- $\widehat{\varphi}$ is the distribution associated with $L$;
- $\varphi \in L_p$ and there exists $B_\varphi > 0$ such that $\sum_{k \in \mathbb{Z}^d} |\hat{\varphi}(\xi + k)|^q < B_\varphi \quad \forall \xi \in \mathbb{R}^d$; there exist $n \in \mathbb{N}$ and $\delta \in (0, 1/2)$ such that $\hat{\varphi}^{\#}$ is boundedly differentiable up to order $n$ on $\{|\xi| < \delta\}$, $\hat{\varphi}$ is boundedly differentiable up to order $n$ on $\{|\xi + l| < \delta\}$ for all $l \in \mathbb{Z}^d \setminus \{0\}$; the function $\sum_{\ell \in \mathbb{Z}^d, \ell \neq 0} |D^\ell \hat{\varphi}(\xi + l)|$ is bounded on $\{|\xi| < \delta\}$ for $|\beta| = n$; $D^\beta(1 - \hat{\varphi}^{\#})(0) = 0$ for all $\beta \in \Delta_n$; the Strang-Fix condition of order $n$ holds for $\varphi$;
- $f = \hat{g}^\circ$, where $g \in L_q$, $g(\xi) = O(|\xi|^{-N - d - \varepsilon})$ as $|\xi| \to \infty$, $\varepsilon > 0$.

Then

$$\left\| f - m^{-j/2} \sum_{k \in \mathbb{Z}^d} Lf(M^{-j} \cdot)(-k)\varphi_{jk} \right\|_p^q \leq C_1 M^{* - j} \| q T^\text{Out}_{j, \gamma, q}(g) \|_p + C_2 M^{* - j} \| q T^\text{In}_{j, n, q}(g) \|_p, \quad (27)$$

where $C_1$ and $C_2$ do not depend on $j$ and $f$.

If, in addition, $n < N + \frac{d}{p} + \varepsilon$, then

$$\left\| f - m^{-j/2} \sum_{k \in \mathbb{Z}^d} Lf(M^{-j} \cdot)(-k)\varphi_{jk} \right\|_p \leq C \theta^{-jn}$$

for every positive number $\theta$ which is smaller in module than any eigenvalue of $M$ and some $C$ which does not depend on $j$.

If, in addition, $M$ an isotropic matrix dilation and $\lambda$ is its eigenvalue then

$$\left\| f - m^{-j/2} \sum_{k \in \mathbb{Z}^d} Lf(M^{-j} \cdot)(-k)\varphi_{jk} \right\|_p \leq \begin{cases} C |\lambda|^{-j(N + \frac{d}{p} + \varepsilon)} & \text{if } n > N + \frac{d}{p} + \varepsilon \\ C j^{1/q} |\lambda|^{-jn} & \text{if } n = N + \frac{d}{p} + \varepsilon \\ C |\lambda|^{-jn} & \text{if } n < N + \frac{d}{p} + \varepsilon \end{cases}, \quad (28)$$

where $C$ does not depend on $j$.

Similarly, the following statement follows from Theorem 7.

Theorem 9 Let $2 \leq p \leq \infty$, $1/p + 1/q = 1$, a differential operator $L$ be defined by (29). Suppose

- $\widetilde{\varphi}$ is the distribution associated with $L$;
- $\varphi \in L_p$, the function $\sum_{k \in \mathbb{Z}^d} |\tilde{\varphi}(\xi + k)|^q$ is bounded, and there exists $\delta \in (0, 1/2)$ such that $\tilde{\varphi}(\xi)\tilde{\varphi}(\xi) = 1$ a.e. on $\{|\xi| < \delta\}$, $\tilde{\varphi}(\xi) = 0$ a.e. on $\{|l - \xi| < \delta\}$ for all $l \in \mathbb{Z}^d \setminus \{0\}$;
- $f = \hat{g}^\circ$, where $g \in L_q$, $g(\xi) = O(|\xi|^{-N - d - \varepsilon})$ as $|\xi| \to \infty$, $\varepsilon > 0$.

Then

$$\left\| f - m^{-j/2} \sum_{k \in \mathbb{Z}^d} Lf(M^{-j} \cdot)(-k)\varphi_{jk} \right\|_p \leq C \theta^{-j(N + \frac{d}{p} + \varepsilon)} \quad (29)$$

for every positive number $\theta$ which is smaller in module than any eigenvalue of $M$ and some $C$ which does not depend on $j$.
6 Falsified sampling expansions

If exact sampled values of a signal \( f \) are known then sampling expansions are very useful for applications. Theorems 8 and 9 provide error estimates for this case. We now discuss what happens if exact sampled values at the points \( M^{-j}k \) are replaced by the following average values

\[
\frac{1}{V_h} \int_{B_h} f(M^{-j}k + M^{-j}t) dt = \frac{m^j}{V_h} \int_{M^{-j}B_h} f(M^{-j}k + t) dt =: \mathbf{A}v_h(f, M^{-j}k), \tag{30}
\]

where \( V_h \) is the volume of the ball \( B_h, h > 0 \). In the case \( d = 1, M = 2 \), it easily follows from the Tailor formula (see Introduction) that

\[
\frac{1}{2^{j-h}} \int_{2^{-j}B} f(t) dt \approx Lf(2^{-j} \cdot (k),
\]

where \( L \) is the differential operator (26) with \( \alpha_i = \frac{1}{(1 + j)^{1/2}} \).

We need the following lemma to prove a similar statement for the averages (30).

**Lemma 10** Let \( N \in \mathbb{N} \), function \( f \) be continuously differentiable up to order \( N \), \( A \) be a real-valued \( d \times d \) matrix. Then for all \( t, x \in \mathbb{R}^d \)

\[
\sum_{\beta \in \Delta_{N+1}} \frac{D^\beta f(Ax)}{\beta!} (At)\beta = \sum_{\beta \in \Delta_{N+1}} \frac{D^\beta f(Ax)}{\beta!} t^\beta.
\]

**Proof.** Firstly, we introduce some additional notations. Let \( p \in \mathbb{Z}_+ \), \( O_p = \{ \beta \in \mathbb{Z}_+^d : [\beta] = p \} \). Assume that the set \( O_p \) is ordered by lexicographic order. Namely, \((\beta_1, \ldots, \beta_d)\) is less than \((\alpha_1, \ldots, \alpha_d)\) in lexicographic order if \( \beta_j = \alpha_j \) for \( j = 1, \ldots, i-1 \) and \( \beta_i < \alpha_i \) for some \( i \). Let \( S(A, p) \) be a \((\#O_p) \times (\#O_p)\) matrix which is uniquely determined by

\[
\frac{(At)^\alpha}{\alpha!} = \sum_{\beta \in O_p} [S(A, p)]_{\alpha, \beta} t^\beta, \quad \tag{31}
\]

where \( \alpha \in O_p, t \in \mathbb{R}^d \). It can be verified that

\[
\alpha! [S(A, p)]_{\alpha, \beta} = \beta! [S(A^*, p)]_{\beta, \alpha}. \quad \tag{32}
\]

The above notations and the latter fact was borrowed [12].

Fix \( \beta \in \mathbb{Z}_+^d \). Let \( \mathcal{E} \) be the set of ordered samples with replacement of size \( [\beta] \) from the set \( \{e_1, \ldots, e_d\} \), where \( e_k \) is the \( k \)-th ort in \( \mathbb{R}^d \). An element \( e \in \mathcal{E} \) is a set \( \{e_{i_1}, \ldots, e_{i_{[\beta]}}\} \), where \( i_l \in \{1, \ldots, d\}, l = 1, \ldots, [\beta] \), \#\( \mathcal{E} \) = \( d^{[\beta]} \). For \( e \in \mathcal{E} \) denote by \( (e)_l := e_{i_l}, l = 1, \ldots, [\beta] \). Let \( T \) be a function defined on \( \mathcal{E} \) by \( T(e) := \sum_{i=1}^{[\beta]} (e)_i \). Note that \( T(e) \in \mathbb{Z}_+^d \) and \( [T(e)] = [\beta] \). Denote by \( b \) an element of \( \mathcal{E} \) so that \( T(b) = \beta \). Such \( b \) is unique up to a permutation. Using the higher chain rule, we have

\[
D^\beta[f(Ax)] = \frac{\partial^{[\beta]}f(Ax)}{\partial x^{[\beta]}}(x) = \sum_{e \in \mathcal{E}} \frac{\partial^{[\beta]}f(y)}{\partial y^{T(e)}} \bigg|_{y = Ax} \prod_{i=1}^{[\beta]} \frac{\partial (Ax)^{(e)_i}}{\partial x^{(b)_i}}, \quad x \in \mathbb{R}^d,
\]

where \( \prod_{i=1}^{[\beta]} \frac{\partial (Ax)^{(e)_i}}{\partial x^{(b)_i}} \) does not depend on \( x \) and \( D^\beta[f(Ax)] \) does not depend on the choice of \( b \). For different elements \( e, h \in \mathcal{E} \), we may have \( T(e) = T(h) \). Thus, we can group terms in the sum with equal values of \( T(\cdot) \). Namely,

\[
D^\beta[f(Ax)] = \sum_{\alpha \in \mathbb{Z}_+^d, [\alpha] = [\beta]} \frac{\partial^{[\beta]}f(y)}{\partial y^{[\alpha]}} \bigg|_{y = Ax} \sum_{e \in \mathcal{E}, T(e) = \alpha} \prod_{i=1}^{[\beta]} \frac{\partial (Ax)^{(e)_i}}{\partial x^{(b)_i}}. \tag{33}
\]
Thus, due to (31) with matrix $A$, we have
\[ D^\beta [f(Ax)] = e^{2\pi i (t, Ax)} \sum_{\alpha \in \mathbb{Z}_+^d, |\alpha| = |\beta|} t^\alpha \sum_{e \in E(T(e) = \alpha)} \prod_{i=1}^{[\beta]} \frac{\partial (Ax)^{(e)}}{\partial x^{(b)}}. \]

Thus, due to (31) with matrix $A$ replaced by $A^*$, we obtain
\[ \sum_{e \in E, T(e) = \alpha} \prod_{i=1}^{[\beta]} \frac{\partial (Ax)^{(e)}}{\partial x^{(b)}} = [S(A^*, p)]_{\beta, \alpha} \frac{\beta!}{\alpha!}. \tag{34} \]

Let now $f$ be an arbitrary function continuously differentiable up to order $N$, $0 \leq p \leq N$, $p \in \mathbb{Z}_+$. It follows from (32) and (34) that
\[ \sum_{\alpha \in \mathcal{O}_p} \frac{D^\alpha f(Ax)}{\alpha!} (At)^\alpha = \sum_{\alpha \in \mathcal{O}_p} D^\alpha f(Ax) \sum_{\beta \in \mathcal{O}_p} [S(A, p)]_{\alpha, \beta} t^\beta \frac{\beta!}{\alpha!} = \sum_{\beta \in \mathcal{O}_p} t^\beta \sum_{\alpha \in \mathcal{O}_p} D^\alpha f(Ax) \frac{[S(A^*, p)]_{\beta, \alpha}}{\alpha!} = \sum_{\beta \in \mathcal{O}_p} t^\beta \frac{\beta!}{\alpha!} D^\beta [f(Ax)]. \]

It remains to sum the latter expression over $p$ from 0 to $N$. Let
\[ Lf(M^{-j} \cdot )(k) = \sum_{\beta \in \Delta_{N+1}} a_\beta D^\beta [f(M^{-j} \cdot )](k), \quad a_\beta = \frac{1}{\beta! V_h} \int_{B_h} t^\beta dt. \tag{35} \]

By Lemma (10) we have
\[ Lf(M^{-j} \cdot )(k) = \frac{1}{V_h} \int_{B_h} \sum_{\beta \in \Delta_{N+1}} \frac{D^\beta [f(M^{-j} \cdot )](k)}{\beta!} t^\beta dt = \frac{1}{V_h} \int_{B_h} \sum_{\beta \in \Delta_{N+1}} \frac{D^\beta [f(M^{-j} \cdot )](k)}{\beta!} (M^{-j} t)^\beta dt = \frac{m^j}{V_h} \int_{M^{-j} B_h} \sum_{\beta \in \Delta_{N+1}} \frac{D^\beta [f(M^{-j} \cdot )](k)}{\beta!} t^\beta dt, \tag{36} \]
and, due to the Tailor formula,
\[ \frac{m^j}{V_h} \int_{M^{-j} B_h} f(M^{-j} k + t) dt \approx Lf(M^{-j} \cdot )(k). \]

We now are interested in error analysis for falsified sampling expansions
\[ m^{-j/2} \sum_{k \in \mathbb{Z}^d} \mathbf{A}^h(f, M^{-j} k) \varphi_{jk} = \sum_{k \in \mathbb{Z}^d} \frac{m^{-j/2}}{V_h} \int_{M^{-j} B_h} f(M^{-j} k + t) dt \varphi_{jk}. \]

To investigate the convergence and approximation order of falsified sampling expansions we can use Theorem (8) or Theorem (9) and estimate the sum $m^{-j/2} \sum_{k \in \mathbb{Z}^d} \varepsilon_j(-k) \varphi_{jk}$, where
\[ \varepsilon_j(k) := \frac{m^j}{V_h} \int_{M^{-j} B_h} f(M^{-j} k + t) dt - Lf(M^{-j} \cdot )(k). \tag{37} \]
Theorem 11 Let \( d < p \leq \infty, h > 0, N \in \mathbb{N}, \varphi \in \mathcal{L}_p \). Suppose \( f \in W^{N+1}_p \), operator \( L \) is defined by \((\mathcal{L}_p)\), \( \varepsilon_j(k) \) is defined by \((\mathcal{L}_p)\). Then

\[
\left\| m^{-j/2} \sum_{k \in \mathbb{Z}^d} \varepsilon_j(-k) \varphi_{jk} \right\|_p \leq C \theta^{-j(N+1)},
\]

for every positive number \( \theta \) which is smaller in modulus than any eigenvalue of \( M \) and some \( C \) which does not depend on \( j \).

**Proof.** Let fix \( j \in \mathbb{N} \). Due to Proposition 2,

\[
\left\| m^{-j/2} \sum_{k \in \mathbb{Z}^d} \varepsilon_j(-k) \varphi_{jk} \right\|_p = m^{-j/p} \left\| \sum_{k \in \mathbb{Z}^d} \varepsilon_j(-k) \varphi_{jk} \right\|_p \leq m^{-j/p} \| \varphi \|_{\mathcal{L}_p} \left( \sum_{k \in \mathbb{Z}^d} |\varepsilon_j(k)|^p \right)^{1/p}. \tag{39}
\]

By the Taylor formula with integral remainder, we have

\[
f(M^{-j}k + t) = \sum_{\beta \in \Delta_{N+1}} \frac{D^\beta f(M^{-j}k)}{\beta!} t^\beta + \sum_{\beta \in \mathbb{Z}^d_+ |\beta| = N+1} \frac{N+1}{\beta!} t^\beta \int_0^1 (1 - \tau)^N D^\beta f(M^{-j}k + t\tau) d\tau.
\]

It follows from \((\mathcal{L}_p)\) that

\[
\varepsilon_j(k) = \frac{m^j}{V_h} \int_{M^{-j}B_h} dt \sum_{\beta \in \mathbb{Z}^d_+ |\beta| = N+1} \frac{N+1}{\beta!} t^\beta \int_0^1 (1 - \tau)^N D^\beta f(M^{-j}k + t\tau) d\tau.
\]

Hence, taking into account that \( |t^\beta| \leq |t|^{|\beta|} \), we get

\[
|\varepsilon_j(k)| \leq \sum_{\beta \in \mathbb{Z}^d_+ |\beta| = N+1} \frac{N+1}{\beta!} \frac{m^j}{V_h} \int_{M^{-j}B_h} |t|^{|\beta|} dt \int_0^1 |D^\beta f(M^{-j}k + t\tau)| d\tau \leq 
\]

\[
\sum_{\beta \in \mathbb{Z}^d_+ |\beta| = N+1} \frac{N+1}{\beta!} \frac{m^j}{V_h} \int_0^1 dt \int_{\tau M^{-j}B_h} |t|^{|\beta|} |D^\beta f(M^{-j}k + t\tau)| d\tau d\tau = 
\]

\[
\sum_{\beta \in \mathbb{Z}^d_+ |\beta| = N+1} \frac{N+1}{\beta!} \frac{m^j}{V_h} \int_0^1 dt \int_{\tau M^{-j}B_h} |M^{-j}t|^{|\beta|} |D^\beta f(M^{-j}k + M^{-j}t)| d\tau d\tau.
\]

The latter integration is taken over the set \( \{ \tau \in [0,1], t \in \tau B_h \} = \{ \tau \in [0,1], |t| \leq \tau h \} \), or equivalently \( \{ |t| \in [0, h], \frac{|t|}{h} \leq \tau \leq 1 \} \). Changing the order of integration, we obtain

\[
|\varepsilon_j(k)| \leq \sum_{\beta \in \mathbb{Z}^d_+ |\beta| = N+1} \frac{N+1}{\beta! V_h} \int_{B_h} |M^{-j}t|^{|\beta|} |D^\beta f(M^{-j}k + M^{-j}t)| dt \int_0^1 \frac{d\tau}{\tau^{N+|\beta|+1}} \leq 
\]

\[
\sum_{\beta \in \mathbb{Z}^d_+ |\beta| = N+1} \frac{2}{\beta! V_h} \int_{B_h} |M^{-j}t|^{N+1} |D^\beta f(M^{-j}k + M^{-j}t)| \left( \frac{h}{|t|} \right)^{N+d} dt \leq 
\]

\[
\sum_{\beta \in \mathbb{Z}^d_+ |\beta| = N+1} \frac{2}{\beta! V_h} \int_{B_h} |M^{-j}t|^{N+1} |D^\beta f(M^{-j}k + M^{-j}t)| \left( \frac{h}{|t|} \right)^{N+d} dt \leq 
\]
\[
\begin{align*}
\sum_{\beta \in \mathbb{Z}_d^d, |eta| = N+1} & \frac{2h^{N+d}||M^{-j}||^{N+1}}{\beta!V_h} \int_{B_h} \frac{|D^\beta f(M^{-j}k + M^{-j}t)|}{|t|^{d-1}} dt = \\
\sum_{\beta \in \mathbb{Z}_d^d, |eta| = N+1} & \frac{2h^{N+d}m^j ||M^{-j}||^{N+1}}{\beta!V_h} \int_{M^{-j}B_h} \frac{|D^\beta f(M^{-j}k + t)|}{|M^j t|^{d-1}} dt.
\end{align*}
\]

Using Hölder’s inequality, we have
\[
\int_{M^{-j}B_h} \frac{|D^\beta f(M^{-j}k + t)|}{|M^j t|^{q(d-1)}} dt \leq \left( \int_{M^{-j}B_h} |D^\beta f(M^{-j}k + t)|^p dt \right)^{1/p} \left( \int_{M^{-j}B_h} \frac{dt}{|M^j t|^{q(d-1)}} \right)^{1/q},
\]
where \( q = \frac{p}{p-1} \). Since
\[
\int_{M^{-j}B_h} \frac{dt}{|M^j t|^{q(d-1)}} \leq m^{-j} \int_{B_h} \frac{dt}{|t|^{q(d-1)}}
\]
and \( q(d-1) < d \), the latter integral is finite. Summarizing the above estimates we obtain
\[
|\varepsilon_j(k)|^p \leq C_1 m^j ||M^{-j}||^{p(N+1)} \int_{M^{-j-k}B_h} \sum_{\beta \in \mathbb{Z}_d^d, |eta| = N+1} |D^\beta f(t)|^p dt,
\]
where \( C_1 \) does not depend on \( f \) and \( j \). It follows that
\[
\sum_{k \in \mathbb{Z}^d} |\varepsilon_j(k)|^p \leq C_1 m^j ||M^{-j}||^{p(N+1)} ||f||^{p}_{W^{N+1}_p},
\]
Combining this with (33) and (34), we get (35). \( \diamond \)

**Remark 12** If \( M \) is an isotropic matrix for which \( \lambda \) is an eigenvalue, then in the proof of Theorem 11 we can use inequality (7) instead of (5). Hence inequality (38) can be replaced by
\[
\left\| m^{-j/2} \sum_{k \in \mathbb{Z}^d} \varepsilon_j(-k) \varphi_{jk} \right\|_p \leq C|\lambda|^{-j(N+1)}.
\]

Using the above theorem we can state the convergence and approximation order of falsified sampling expansions.

**Theorem 13** Let \( d < p \leq \infty \), \( 1/p + 1/q = 1 \), \( h > 0 \), \( N \in \mathbb{Z}_+ \), \( M \) be an isotropic matrix dilation and \( \lambda \) be its eigenvalue. Suppose \( \tilde{\varphi} \) is the distribution associated with the differential operator \( L \) given by (38), \( \varphi \) and \( n \) are as in Theorem 5, \( f = \tilde{g}^- \), where \( g \in L_q \), \( g(\xi) = O(|\xi|^{-N-d-\varepsilon}) \) as \( |\xi| \to \infty \), \( \varepsilon > 1 \). Then
\[
\left\| f - m^{-j/2} \sum_{k \in \mathbb{Z}^d} \text{Av}_h(f, M^{-j}k) \varphi_{jk} \right\|_p \leq \begin{cases} C|\lambda|^{-j(N+1)} & \text{if } n > N + 1, \\ C|\lambda|^{-jn} & \text{if } n \leq N + 1. \end{cases} \tag{40}
\]
where \( C \) does not depend on \( j \).

**Theorem 14** Let \( d < p \leq \infty \), \( 1/p + 1/q = 1 \), \( h > 0 \), \( N \in \mathbb{Z}_+ \). Suppose \( \tilde{\varphi} \) is the distribution associated with the differential operator \( L \) given by (38), \( \varphi \) is as in Theorem 5, \( f = \tilde{g}^- \), where \( g \in L_q \), \( g(\xi) = O(|\xi|^{-N-d-\varepsilon}) \) as \( |\xi| \to \infty \), \( \varepsilon > 1 \). Then
\[
\left\| f - m^{-j/2} \sum_{k \in \mathbb{Z}^d} \text{Av}_h(f, M^{-j}k) \varphi_{jk} \right\|_p \leq C\theta^{-j(N+1)}, \tag{41}
\]
for every positive number \( \theta \) which is smaller in module than any eigenvalue of \( M \) and some \( C \) which does not depend on \( j \).
Observing the proof of Theorem 11, one can see that in the one-dimensional case an analog of (18) holds true for a wider class of functions \( f \) and any \( p \geq 1 \). Indeed, in this case \( M \) is a dilation factor, let \( M > 0 \). Then we have

\[
|\varepsilon(k)| \leq \frac{2}{(N+1)! h} \int_{B_h} |M^{-j}t|^{N+1} |f^{(N+1)}(M^{-j}k + M^{-j}t)| \left( \frac{h}{|t|} \right)^{N+1} dt \leq \frac{2h^{N+1}M^{-j(N+1)}}{(N+1)! h} \int_{B_h} |f^{(N+1)}(M^{-j}k + M^{-j}t)| dt = \frac{2h^{N+1}M^{-jN}}{(N+1)! h} \int_{M^{-j}B_h + M^{-j}k} |f^{(N+1)}(t)| dt.
\]

It follows that

\[
\left\| m^{-j/2} \sum_{k \in \mathbb{Z}} \varepsilon(-k) \varphi_{jk} \right\|_p \leq m^{-j/p} \| \varphi \|_p \sum_{k \in \mathbb{Z}} |\varepsilon(k)| \leq C_1 \| f \|_{W_1^{N+1} M^{-j(N+1)}},
\]

where \( C_1 \) does not depend on \( f \) and \( j \). This yields the following statements.

**Theorem 15** Let \( d = 1, p \geq 1, h > 0, N \in \mathbb{N} \), \( \varphi \in L_p \), Suppose \( f \in W_1^{N+1} \), \( \varepsilon(k) \) is defined by (37). Then

\[
\left\| m^{-j/2} \sum_{k \in \mathbb{Z}} \varepsilon(-k) \varphi_{jk} \right\|_p \leq CM^{-j(N+1/2)}.
\]

where \( C \) does not depend on \( j \).

**Theorem 16** Let \( d = 1, 2 \leq p \leq \infty, h > 0, N \in \mathbb{Z}_+ \),. Suppose \( \tilde{\varphi} \) is the distribution associated with a differential operator \( L \) given by (39), \( \varphi \) and \( n \) are as in Theorem 8. \( f \in W_1^{N+1} \), \( f^{(N+1)} \in L_1 \), \( \varepsilon > 0 \). Then

\[
\left\| f - m^{-j/2} \sum_{k \in \mathbb{Z}^d} \mathbf{Av}_h(f, M^{-j}k) \varphi_{jk} \right\|_p \leq \begin{cases} C|\lambda|^{-j(N+1/2)} & \text{if } n > N + \frac{1}{p} \\ C|\lambda|^{-jn} & \text{if } n \leq N + \frac{1}{p} \end{cases},
\]

where \( C \) does not depend on \( j \).

### 7 Examples

In this section some examples will be given to illustrate the obtained results.

Firstly we discuss construction of band-limited functions \( \varphi \). Theoretically, for every differential operator \( L \) one can easily construct compactly supported \( \tilde{\varphi} \in C^d(\mathbb{R}^d) \) such that \( \tilde{\varphi} \tilde{\varphi} = 1 \) on a small neighborhood of zero. All assumptions of Theorem 9 are satisfied. The approximation order of the corresponding expansions depends on how smooth is the function \( f \). However such expansions are not good from the computational point of view. We will not able to derive explicit formulas for \( \varphi \) which is needed for implementations.

For arbitrary differential operator \( L \) and arbitrary \( n \), we can construct \( \varphi \) satisfying all assumptions of Theorem 8 as follows. Let \( \tilde{\varphi} \) be the distribution associated with \( L \), \( T \) be a trigonometric polynomial such that \( T \) and all its derivatives up to order \( d + 1 \) vanish on the boundary of \( \frac{1}{2} \mathbb{T}^d \) and \( D^\beta(1 - T \tilde{\varphi})(0) = 0, |\beta| < n \). Define \( \tilde{\varphi} \) as the restriction of \( T \) onto \( \frac{1}{2} \mathbb{T}^d \). The approximation order of corresponding expansions depends on how smooth is the function \( f \) but cannot be better than \( n \). Deriving explicit formulas for \( \varphi \) is possible in this case, but they will be too bulky. Probably such a way is also not appropriate for applications.

We now present examples which can be useful in practice.
I. For sampling expansions we can take \( \tilde{\varphi} = \delta \) and \( \varphi(x) = \prod_{k=1}^{d} \left( \frac{\sin \pi x_k}{\pi x_k} \right)^2 \). The corresponding expansion of \( f \) interpolates \( f \) at the points \( M^{-j}k, k \in \mathbb{Z}^d \). Since \( \tilde{\varphi}(\xi) = \prod_{k=1}^{d} (1 - |\xi_k|) \chi_{[-1,1]}(\xi) \), the function \( \sum_{k \in \mathbb{Z}^d} |\tilde{\varphi}(\xi + k)|^q \) is bounded and \( \varphi \in \mathcal{L}_p \). The Strang-Fix condition for \( \varphi \) of order 1 is valid. So, all assumptions of Theorem 8 are satisfied, but the approximation order cannot be better than \( n = 1 \) by this theorem.

II. Now we illustrate Theorem 9 by improving the previous example. Let \( \psi(x) = \prod_{k=1}^{d} \left( \frac{\sin \pi x_k}{\pi x_k} \right)^2 \). Define \( \tilde{\varphi}(\xi) = 2\tilde{\varphi}(2\xi) - \tilde{\varphi}(4\xi) \). Thus, \( \tilde{\varphi} \) is continuous, has its support inside \( \mathbb{T}^d \) and equal to 1 for all \( \xi \in [1/4, 1/4]^d \), the function \( \sum_{k \in \mathbb{Z}^d} |\tilde{\varphi}(\xi + k)|^q \) is bounded. Clearly, \( \varphi(x) = \frac{1}{2\pi} \tilde{\varphi}(\frac{x}{2}) - \frac{1}{4\pi} \tilde{\varphi}(\frac{x}{4}) \) and therefore \( \varphi \in \mathcal{L}_p \). Taking again \( \tilde{\varphi} = \delta \) we obtain a sampling expansion with approximation order depending on how smooth is \( f \), according to \( (29) \).

Note that the same function \( \tilde{\varphi} \) is associated with the differential operator \( Lf = f + \sum_{|\beta| = 1} a_{\beta} D^{\beta} f \), where \( a_{\beta} = 0 \). Hence the functions \( \varphi, \tilde{\varphi} \) satisfy all conditions of Theorem 13 with \( n = 2, N = 1 \) and arbitrary \( h > 0 \), and, according to \( (41) \), or \( (40) \) in the case of isotropic \( M \), the approximation order of the corresponding falsified sampling expansions is 2 for smooth enough functions \( f \).

We now are interested in compactly supported functions \( \varphi \) which have an advantage for applications because the corresponding sampling and differential expansions are finite.

III. Let

\[
\tilde{\varphi}(\xi) = \prod_{k=1}^{d} \left( \frac{\sin \pi x_k}{\pi x_k} \right)^2 .
\]

Since \( \varphi(x) = \prod_{k=1}^{d} (1 - |\xi_k|) \chi_{[-1,1]}(\xi) \), \( \varphi \) is compactly supported and in \( \mathcal{L}_p \). Also, the function \( \sum_{k \in \mathbb{Z}^d} |\tilde{\varphi}(\xi + k)|^q \) is bounded, \( \tilde{\varphi} \) is continuously differentiable up to any order, the function \( \sum_{l \in \mathbb{Z}^d, l \neq 0} |D^{\beta} \tilde{\varphi}(\xi + l)| \) is bounded near the origin for \( |\beta| = 2 \). Also, the Strang-Fix condition of order 2 holds for \( \varphi \). The values of \( \tilde{\varphi} \) and its derivatives at the origin are

\[
\tilde{\varphi}(0) = 1, \quad D^{\beta} \tilde{\varphi}(0) = 0, \quad |\beta| = 1.
\]

So, if \( \tilde{\varphi} = \delta \), then all assumptions of Theorem 8 are satisfied. The corresponding sampling expansion of a signal \( f \) interpolates \( f \) at the points \( M^{-j}k, k \in \mathbb{Z}^2 \), the approximation order depends on how smooth is \( f \), but, according to \( (28) \), it cannot be better than 2. Again \( \tilde{\varphi} \) is associated with the differential operator \( Lf = f + \sum_{|\beta| = 1} a_{\beta} D^{\beta} f \), where \( a_{\beta} = 0 \). Hence the functions \( \varphi, \tilde{\varphi} \) satisfy all conditions of Theorem 13 with \( n = 2, N = 1 \) and arbitrary \( h > 0 \), and, according to \( (40) \), the approximation order of the corresponding falsified sampling expansions is 2 for smooth enough functions \( f \).

IV. Let \( d = 2 \),

\[
\tilde{\varphi}(\xi_1, \xi_2) = \frac{1}{(\pi^2 \xi_1^2 \xi_2)^2} \sin^3 \pi \xi_1 \sin^3 \pi \xi_2 + b_1 \sin^3 \pi \xi_1 \sin^4 \pi \xi_2 + b_2 \sin^4 \pi \xi_1 \sin^3 \pi \xi_2
\]

Again \( \varphi \) is compactly supported and in \( \mathcal{L}_p \). Also, the function \( \sum_{k \in \mathbb{Z}^d} |\tilde{\varphi}(\xi + k)|^q \) is bounded, \( \tilde{\varphi} \) is continuously differentiable up to any order. Since the trigonometric polynomial in the numerator of \( \tilde{\varphi} \) is bounded, the function \( \sum_{l \in \mathbb{Z}^d, l \neq 0} |D^{\beta} \tilde{\varphi}(\xi + l)| \) is bounded near the origin for \( |\beta| = 3 \). Also, the Strang-Fix condition of order 3 holds for \( \varphi \). The values of \( \tilde{\varphi} \) and its derivatives at the origin are

\[
\tilde{\varphi}(0,0) = 1, \quad D^{(1,0)} \tilde{\varphi}(0,0) = D^{(0,1)} \tilde{\varphi}(0,0) = 0.
\]

\[
D^{(2,0)} \tilde{\varphi}(0,0) = \pi^2 (2b_1 - 1), \quad D^{(0,2)} \tilde{\varphi}(0,0) = \pi^2 (2b_2 - 1), \quad D^{(1,1)} \tilde{\varphi}(0,0) = 0.
\]

Now, we choose an appropriate differential operator \( L \) in the form \( Lf = f + a_{(0,2)} D^{(0,2)} f + a_{(2,0)} D^{(2,0)} f \), or equivalently, the associated distribution \( \tilde{\varphi} = \delta + a_{(2,0)} D^{(2,0)} \delta + a_{(0,2)} D^{(0,2)} \delta \). Since

\[
D^{(2,0)} \tilde{\varphi}(0,0) = \pi^2 (2b_1 - 1), \quad D^{(0,2)} \tilde{\varphi}(0,0) = \pi^2 (2b_2 - 1), \quad D^{(1,1)} \tilde{\varphi}(0,0) = 0.
\]
\[ \hat{\varphi}(\xi) = 1 - 4\pi^2 \varphi_{(2,0)} \xi_1 - 4\pi^2 \varphi_{(2,0)} \xi_2, \] we have
\[ \hat{\varphi}(0, 0) = 1, \quad D^{(2,0)} \hat{\varphi}(0, 0) = -4\pi^2 \varphi_{(2,0)}, \quad D^{(0,2)} \hat{\varphi}(0, 0) = -4\pi^2 \varphi_{(2,0)}. \]
\[ D^{(1,0)} \hat{\varphi}(0, 0) = D^{(0,1)} \hat{\varphi}(0, 0) = D^{(1,1)} \hat{\varphi}(0, 0) = 0. \]
To satisfy condition \( D^\beta (1 - \hat{\varphi} \hat{\varphi})(0, 0) = 0 \) for \( \beta \in \Delta_3 \) we have to provide
\[ b_1 = \frac{1}{2} (1 - 4\pi^2), \quad b_2 = \frac{1}{2} (1 - 4\pi^2/2). \]
Finally, all conditions of Theorem 8 are satisfied. The approximation order depends on how smooth is \( f \), but, according to (35), it cannot be better than \( n = 3 \).

We now show that the coefficients \( b_1, b_2 \) can be chosen such that all conditions of Theorem 13 are satisfied with \( n = 3, N = 2 \) and arbitrary \( h > 0 \). In this case the differential operator \( L \) is given by (35). The coefficients \( a_\beta, \beta \in \Delta_3 \) are as follows
\[ a_{0,0} = 1, \quad a_{1,0} = a_{0,1} = a_{1,1} = 0, \quad a_{2,0} = a_{0,2} = \frac{1}{8} h^2. \]
Thus, we set \( b_1 = b_2 = \frac{1}{2} (1 - \frac{1}{8} h^2) \). According to (40), the approximation order of the corresponding falsified sampling expansions is 3 for smooth enough functions \( f \).

V. Let \( d = 1 \),
\[ \hat{\varphi}(\xi) = \frac{\sin^4 \pi \xi + b_1 \sin^5 \pi \xi + b_2 \sin^6 \pi \xi + b_3 \sin^7 \pi \xi}{(\pi \xi)^4}. \]
Since \( \varphi \) is bounded and compactly supported, it is in \( L_p \), and \( \hat{\varphi} \) is continuously differentiable up to any order. Also, \( \sum_{k \in \mathbb{Z}^d} |\hat{\varphi}(\xi + k)|^q \) is bounded. Since the trigonometric polynomial in the numerator of \( \hat{\varphi} \) is bounded, the function \( \sum_{l \in \mathbb{Z}^d, l \neq 0} |D^\beta \hat{\varphi}(\xi + l)| \) is bounded near the origin for \( \beta = 4 \). Also the Strang-Fix condition of order 4 holds for \( \varphi \). The values of \( \hat{\varphi} \) and its derivatives at the origin are
\[ \hat{\varphi}(0) = 1, \quad \hat{\varphi}'(0) = b_1 \pi, \quad \hat{\varphi}''(0) = \frac{2}{3} \pi^2 (3b_2 - 2), \quad \hat{\varphi}'''(0) = \pi^3 (6b_3 - 5b_1). \]

Now, we choose the appropriate differential operator \( L \) in the form \( Lf = f + a_1 f' + a_2 f'' + a_3 f''' \), or equivalently, the associated distribution \( \hat{\varphi} = \delta - \pi \varphi_{\pi} \delta' - \pi \varphi_{\pi} \delta'' - \pi \varphi_{\pi} \delta''' \). Since \( \hat{\varphi}(\xi) = 1 - 2\pi i \varphi_{\pi} \xi - 4\pi^2 \varphi_{\pi}^2 - 8\pi^3 \varphi_{\pi}^3 \), we have
\[ \hat{\varphi}(0) = 1, \quad \hat{\varphi}'(0) = -2\pi i \varphi_{\pi}, \quad \hat{\varphi}''(0) = -8\pi^2 \varphi_{\pi}^2, \quad \hat{\varphi}'''(0) = 48\pi^3 \varphi_{\pi}^3. \]
To satisfy condition \( D^\beta (1 - \hat{\varphi} \hat{\varphi})(0, 0) = 0 \) for \( \beta = 0, 1, 2, 3 \) we have to provide
\[ (1 - \hat{\varphi} \hat{\varphi})'(0) = \pi (b_1 - 2a_1) = 0, \]
\[ (1 - \hat{\varphi} \hat{\varphi})''(0) = \frac{2}{3} \pi^2 (-2 + 3b_2 - 12a_2 - 6i\varphi_{\pi}b_1) = 0, \]
\[ (1 - \hat{\varphi} \hat{\varphi})'''(0) = -\pi^3 (5b_1 - 6b_3 + 4i(3b_2 - 2)a_1 + 24b_1a_2 - 48a_3) = 0. \]

Thus, the coefficients of the function \( \hat{\varphi} \) can be easily successively found using the coefficients of the differential operator \( L \). Finally, all conditions of Theorem 8 are satisfied. The approximation order depends on how smooth is \( f \), but, according to (35), it cannot be better than \( n = 4 \).

We now show that the coefficients \( b_1, b_2, b_3 \) can be chosen such that all conditions of Theorem 13 are satisfied with \( n = 4, N = 3 \) and arbitrary \( h > 0 \). In this case the differential operator \( Lf = a_0 f + a_1 f' + a_2 f'' + a_3 f''' \) is given by (35) and its coefficients are defined as
\[ a_0 = 1, a_1 = 0, a_2 = \frac{h^2}{6}, a_3 = 0. \]
Using (44), we set $b_1 = 0, b_2 = -\frac{2}{3}h^2 - \frac{2}{3}h, b_3 = 0$. According to (39), the approximation order of the corresponding falsified sampling expansions is 4 for smooth enough functions $f$.

Note that all conditions of Theorem 16 are also satisfied, which provides approximation order for a wider class of functions $f$. Namely, according to (43), the approximation order is $3 + \frac{1}{p}$, whenever $f \in W^4_1, f^{(IV)} \in \text{Lip}_\epsilon, \epsilon > 0$. □

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