Basic Approach to the Problem of Cosmological Constant and Dark Energy

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Most of the calculations done to obtain the value of the cosmological constant use methods of quantum gravity, a theory that has not been established as yet, and a variety of results are usually obtained. The numerical value of the cosmological constant is then supposed to be inserted in the Einstein field equations, hence the evolution of the Universe will depend on the calculated value. Here we present a fundamental approach to the problem. The theory presented here uses a Riemannian four-dimensional presentation of gravitation in which the coordinates are those of Hubble, i.e. distances and velocity rather than space and time. We solve these field equations and show that there are three possibilities for the Universe to expand but only the accelerating Universe is possible. From there we calculate the cosmological constant and find that its value is given by $1.934 \times 10^{-35} s^{-2}$. This value is in excellent agreement with the measurements obtained by the High-Z Supernova Team and the Supernova Cosmology Project. Finally it is shown that the three-dimensional space of the Universe is Euclidean, as the Boomerang experiment shows.

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1 Preliminaries

As in classical general relativity we start our discussion in flat space velocity which will then be generalized to curved space.

The flat-space velocity cosmological metric is given by

\[ ds^2 = \tau^2 dv^2 - (dx^2 + dy^2 + dz^2) \]

(1)

Here \( \tau \) is Hubble’s time, the inverse of Hubble’s constant, as given by measurements in the limit of zero distances and thus zero gravity. As such, \( \tau \) is a constant, in fact a universal constant (its numerical value is given in Section 8, \( \tau = 12.486 \) Gyr). Its role in cosmology theory resembles that of \( c \), the speed of light in vacuum, in ordinary special relativity. The velocity \( v \) is used here in the sense of cosmology, as in Hubble’s law, and is usually not the time-derivative of the distance.

The Universe expansion is obtained from the metric (1) as a null condition, \( ds = 0 \). Using spherical coordinates \( r, \theta, \phi \) for the metric (1), and the fact that the Universe is spherically symmetric \( d\theta = d\phi = 0 \), the null condition then yields \( dr/dv = \tau \), or upon integration and using appropriate initial conditions, gives \( r = \tau v \) or \( v = H_0 r \), i.e. the Hubble law in the zero-gravity limit.

Based on the metric (1) a cosmological special relativity (CSR) was presented in the text [1] (see Chapter 2). In this theory the receding velocities of galaxies and the distances between them in the Hubble expansion are united into a four-dimensional pseudo-Euclidean manifold, similarly to space and time in ordinary special relativity. The Hubble law is assumed and is written in an invariant way that enables one to derive a four-dimensional transformation which is similar to the Lorentz transformation. The parameter in the new transformation is the ratio between the cosmic time to \( \tau \) (in which the cosmic time is measured backward with respect to the present time). Accordingly, the new transformation relates physical quantities at different cosmic times in the limit of weak or negligible gravitation.

The transformation between the four variables \( x, y, z, v \) and \( x', y', z', v' \) (assuming \( y' = y \) and \( z' = z \)) is given by

\[ x' = \frac{x - tv}{\sqrt{1 - t^2/\tau^2}}, \quad v' = \frac{v - tx/\tau^2}{\sqrt{1 - t^2/\tau^2}}, \quad y' = y, \quad z' = z. \]

(2)

Equations (2) are the cosmological transformation and very much resemble the well-known Lorentz transformation. In CSR it is the relative cosmic time which takes the role of the relative velocity in Einstein’s special relativity. The transformation (2) leaves invariant the Hubble time \( \tau \), just as the Lorentz transformation leaves invariant the speed of light in vacuum \( c \).
2 Cosmology in spacevelocity

A cosmological general theory of relativity, suitable for the large-scale structure of the Universe, was subsequently developed [2-5]. In the framework of cosmological general relativity (CGR) gravitation is described by a curved four-dimensional Riemannian spacevelocity. CGR incorporates the Hubble constant $\tau$ at the outset. The Hubble law is assumed in CGR as a fundamental law. CGR, in essence, extends Hubble’s law so as to incorporate gravitation in it; it is actually a distribution theory that relates distances and velocities between galaxies. The theory involves only measured quantities and it takes a picture of the Universe as it is at any moment. The following is a brief review of CGR as was originally given by the author in 1996 in Ref. 2.

The foundations of any gravitational theory are based on the principle of equivalence and the principle of general covariance [6]. These two principles lead immediately to the realization that gravitation should be described by a four-dimensional curved spacetime, in our theory spacevelocity, and that the field equations and the equations of motion should be written in a generally covariant form. Hence these principles were adopted in CGR also. Use is made in a four-dimensional Riemannian manifold with a metric $g_{\mu\nu}$ and a line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. The difference from Einstein’s general relativity is that our coordinates are: $x^0$ is a velocitylike coordinate (rather than a timelike coordinate), thus $x^0 = \tau v$ where $\tau$ is the Hubble time in the zero-gravity limit and $v$ the velocity. The coordinate $x^0 = \tau v$ is the comparable to $x^0 = ct$ where $c$ is the speed of light and $t$ is the time in ordinary general relativity. The other three coordinates $x^k$, $k = 1, 2, 3$, are spacelike, just as in general relativity theory.

An immediate consequence of the above choice of coordinates is that the null condition $ds = 0$ describes the expansion of the Universe in the curved spacevelocity (generalized Hubble’s law with gravitation) as compared to the propagation of light in the curved spacetime in general relativity. This means one solves the field equations (to be given in the sequel) for the metric tensor, then from the null condition $ds = 0$ one obtains immediately the dependence of the relative distances between the galaxies on their relative velocities.

As usual in gravitational theories, one equates geometry to physics. The first is expressed by means of a combination of the Ricci tensor and the Ricci scalar, and follows to be naturally either the Ricci trace-free tensor or the Einstein tensor. The Ricci trace-free tensor does not fit gravitation in general, and the Einstein tensor is a natural candidate. The physical part is expressed by the energy-momentum tensor which now has a different physical meaning from that in Einstein’s theory. More important, the coupling constant that relates geometry to physics is now also different.

Accordingly the field equations are

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu},$$

exactly as in Einstein’s theory, with $\kappa$ given by $\kappa = 8\pi k/\tau^4$, (in general relativity it is
given by $8\pi G/c^4$), where $k$ is given by $k = G\tau^2/c^2$, with $G$ being Newton’s gravitational constant, and $\tau$ the Hubble constant time. When the equations of motion will be written in terms of velocity instead of time, the constant $k$ will replace $G$. Using the above equations one then has $\kappa = 8\pi G/c^2\tau^2$.

The energy-momentum tensor $T^{\mu\nu}$ is constructed, along the lines of general relativity theory, with the speed of light being replaced by the Hubble constant time. If $\rho$ is the average mass density of the Universe, then it will be assumed that $T^{\mu\nu} = \rho u^{\mu}u^{\nu}$, where $u^{\mu} = dx^{\mu}/ds$ is the four-velocity. In general relativity theory one takes $T_0^0 = \rho$. In Newtonian gravity one has the Poisson equation $\nabla^2 \phi = 4\pi G\rho$. At points where $\rho = 0$ one solves the vacuum Einstein field equations in general relativity and the Laplace equation $\nabla^2 \phi = 0$ in Newtonian gravity. In both theories a null (zero) solution is allowed as a trivial case. In cosmology, however, there exists no situation at which $\rho$ can be zero because the Universe is filled with matter. In order to be able to have zero on the right-hand side of Eq. (3) one takes $T_0^0$ not as equal to $\rho$, but to $\rho_{\text{eff}} = \rho - \rho_c$, where $\rho_c$ is the critical mass density, a constant in CGR given by $\rho_c = 3/(8\pi G\tau^2)$, whose value is $\rho_c \approx 10^{-29} g/cm^3$, a few hydrogen atoms per cubic meter. Accordingly one takes

$$T^{\mu\nu} = \rho_{\text{eff}} u^{\mu}u^{\nu}; \quad \rho_{\text{eff}} = \rho - \rho_c$$

for the energy-momentum tensor.

In the next sections we apply CGR to obtain the accelerating expanding Universe and related subjects.

### 3 Gravitational field equations

In the four-dimensional spacevelocity the spherically symmetric metric is given by

$$ds^2 = \tau^2 dv^2 - e^\mu dr^2 - R^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right),$$

where $\mu$ and $R$ are functions of $v$ and $r$ alone, and comoving coordinates $x^\mu = (x^0, x^1, x^2, x^3) = (\tau v, r, \theta, \phi)$ have been used. With the above choice of coordinates, the zero-component of the geodesic equation becomes an identity, and since $r, \theta$ and $\phi$ are constants along the geodesics, one has $dx^0 = ds$ and therefore

$$u^\alpha = u_\alpha = (1, 0, 0, 0).$$

The metric (5) shows that the area of the sphere $r = \text{constant}$ is given by $4\pi R^2$ and that $R$ should satisfy $R' = \partial R/\partial r > 0$. The possibility that $R' = 0$ at a point $r_0$ is excluded since it would allow the lines $r = \text{constants}$ at the neighboring points $r_0$ and $r_0 + \delta r$ to coincide at $r_0$, thus creating a caustic surface at which the comoving coordinates break down.
As has been shown in the previous sections the Universe expands by the null condition
ds = 0, and if the expansion is spherically symmetric one has \( d\theta = d\phi = 0 \). The metric
(5) then yields
\[
\tau^2 dv^2 - e^\mu dr^2 = 0,
\]
thus
\[
\frac{dr}{dv} = \tau e^{-\mu/2}.
\]
This is the differential equation that determines the Universe expansion. In the following
we solve the gravitational field equations in order to find out the function \( \mu (r,v) \).

The gravitational field equations (3), written in the form
\[
R_{\mu\nu} = \kappa \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right),
\]
where
\[
T_{\mu\nu} = \rho_{\text{eff}} u_\mu u_\nu + p (u_\mu u_\nu - g_{\mu\nu}),
\]
with \( \rho_{\text{eff}} = \rho - \rho_c \) and \( T = T_{\mu\nu} g^{\mu\nu} \), are now solved. Using Eq. (6) one finds that the
only nonvanishing components of \( T_{\mu\nu} \) are \( T_{00} = \tau^2 \rho_{\text{eff}}, T_{11} = c^{-1} \tau pe^\mu, T_{22} = c^{-1} \tau p R^2 \)
and \( T_{33} = c^{-1} \tau p R^2 \sin^2 \theta \), and that \( T = \tau^2 \rho_{\text{eff}} - 3c^{-1} \tau p \).

The only nonvanishing components of the Ricci tensor yield (dots and primes denote
differentiation with respect to \( v \) and \( r \), respectively), using Eq. (9), the following field
equations:
\[
R_{00} = -\frac{1}{2} \ddot{\mu} - \frac{2}{R} \dot{R} \ddot{R} - \frac{1}{4} \dot{\mu}^2 = \frac{\kappa}{2} \left( \tau^2 \rho_{\text{eff}} + 3c^{-1} \tau p \right),
\]
\[
R_{01} = \frac{1}{R} \dot{R} \ddot{\mu} - \frac{2}{R} \dot{R}^2 = 0,
\]
\[
R_{11} = e^\mu \left( \frac{1}{2} \ddot{\mu} + \frac{1}{4} \dot{\mu}^2 + \frac{1}{R} \dot{R} \ddot{R} \right) + \frac{1}{R} (\mu' R' - 2R')
= \frac{\kappa}{2} e^\mu \left( \tau^2 \rho_{\text{eff}} - c^{-1} \tau p \right),
\]
\[
R_{22} = R \ddot{R} + \frac{1}{2} R \dot{R} \ddot{\mu} + \ddot{R}^2 + 1 - e^{-\mu} \left( RR'' + \frac{1}{2} R R' \mu' + R'^2 \right)
= \frac{\kappa}{2} R^2 \left( \tau^2 \rho_{\text{eff}} - c^{-1} \tau p \right),
\]
\[
R_{33} = \sin^2 \theta R_{22} = \frac{\kappa}{2} R^2 \sin^2 \theta \left( \tau^2 \rho_{\text{eff}} - c^{-1} \tau p \right).
\]

The field equations obtained for the components 00, 01, 11, and 22 (the 33 component
contributes no new information) are given by
\[
-\ddot{\mu} - \frac{4}{R} \dot{R} - \frac{1}{2} \dot{\mu}^2 = \kappa \left( \tau^2 \rho_{\text{eff}} + 3c^{-1} \tau p \right),
\]

\[4\]
\[ 2\dot{R}' - R'\dot{\mu} = 0, \]
\[ \ddot{\mu} + \frac{1}{2}\dot{\mu}^2 + \frac{2}{R} \dot{R} \dot{\mu} + e^{-\mu} \left( \frac{2}{R} R' \mu' - \frac{4}{R} R'' \right) = \kappa \left( \tau^2 \rho_{eff} - c^{-1} \tau p \right) \]
\[ \frac{2}{R} \ddot{R} + 2 \left( \frac{\dot{R}}{R} \right)^2 + \frac{1}{R} \dot{R} \dot{\mu} + \frac{2}{R^2} + e^{-\mu} \left[ \frac{1}{R} R' \mu' - 2 \left( \frac{R'}{R} \right)^2 - \frac{2}{R} R'' \right] \]
\[ = \kappa \left( \tau^2 \rho_{eff} - c^{-1} \tau p \right). \]

It is convenient to eliminate the term with the second velocity-derivative of \( \mu \) from the above equations. This can easily be done, and combinations of Eqs. (12)–(15) then give the following set of three independent field equations:

\[ e^\mu \left( 2R\ddot{R} + \dot{R}^2 + 1 \right) - R^2 = -\kappa \tau c^{-1} e^\mu R^2 p, \]
\[ 2\ddot{R}' - R'\dot{\mu} = 0, \]
\[ e^{-\mu} \left[ \frac{1}{R} R' \mu' - \left( \frac{R'}{R} \right)^2 - \frac{2}{R} R'' \right] + \frac{1}{R} \dot{R} \dot{\mu} + \left( \frac{\dot{R}}{R} \right)^2 + \frac{1}{R^2} \]
\[ = \kappa \tau^2 \rho_{eff}, \]
other equations being trivial combinations of (16)–(18).

4 Solution of the field equations

The solution of Eq. (17) satisfying the condition \( R' > 0 \) is given by

\[ e^\mu = \frac{R^2}{1 + f(r)}, \]

where \( f(r) \) is an arbitrary function of the coordinate \( r \) and satisfies the condition \( f(r) + 1 > 0 \). Substituting (19) in the other two field equations (16) and (18) then gives

\[ 2R\ddot{R} + \dot{R}^2 - f = -\kappa c^{-1} \tau R^2 p, \]
\[ \frac{1}{RR'} \left( 2\dot{R} \dot{R}' - f' \right) + \frac{1}{R^2} \left( \dot{R}^2 - f \right) = \kappa \tau^2 \rho_{eff}, \]
respectively.

The simplest solution of the above two equations, which satisfies the condition \( R' = 1 > 0 \), is given by

\[ R = r. \]

Using Eq. (22) in Eqs. (20) and (21) gives

\[ f(r) = \kappa c^{-1} \tau pn^2, \]
and

\[ f' + \frac{f}{r} = -\kappa \tau^2 \rho_{\text{eff}} r, \quad (24) \]

respectively. The solution of Eq. (24) is the sum of the solutions of the homogeneous equation

\[ f' + \frac{f}{r} = 0, \quad (25) \]

and a particular solution of Eq. (24). These are given by

\[ f_1 = -\frac{2Gm}{c^2 r}, \quad (26) \]

and

\[ f_2 = -\frac{\kappa}{3} \tau^2 \rho_{\text{eff}} r^2. \quad (27) \]

The solution \( f_1 \) represents a particle at the origin of coordinates and as such is not relevant to our problem. We take, accordingly, \( f_2 \) as the general solution,

\[ f(r) = -\frac{\kappa}{3} \tau^2 \rho_{\text{eff}} r^2 = -\frac{\kappa}{3} \tau^2 (\rho - \rho_c) r^2 \]

\[ = -\frac{\kappa}{3} \tau^2 \rho_c \left( \frac{\rho}{\rho_c} - 1 \right) r^2. \quad (28) \]

Using the values of \( \kappa = 8\pi G/c^2 \tau^2 \) and \( \rho_c = 3/8\pi G\tau^2 \), we obtain

\[ f(r) = \frac{1 - \Omega}{c^2 \tau^2} r^2, \quad (29) \]

where \( \Omega = \rho/\rho_c \).

The two solutions given by Eqs. (23) and (29) for \( f(r) \) can now be equated, giving

\[ p = \frac{1 - \Omega}{\kappa c \tau^3} = \frac{c}{\tau} \frac{1 - \Omega}{8\pi G} = 4.544 (1 - \Omega) \times 10^{-2} \text{g/cm}^2. \quad (30) \]

Furthermore, from Eqs. (19) and (22) we find that

\[ e^{-\mu} = 1 + f(r) = 1 + \tau c^{-1} \kappa pr^2 = 1 + \frac{1 - \Omega}{c^2 \tau^2} r^2. \quad (31) \]

It will be recalled that the Universe expansion is determined by Eq. (8), \( dr/dv = \tau e^{-\mu/2} \). The only thing that is left to be determined is the signs of \( 1 - \Omega \) or the pressure \( p \).

Thus we have

\[ \frac{dr}{dv} = \tau \sqrt{1 + \kappa \tau c^{-1} pr^2} = \tau \sqrt{1 + \frac{1 - \Omega}{c^2 \tau^2} r^2}. \quad (32) \]

For simplicity we confine ourselves to the linear approximation, thus Eq. (32) yields

\[ \frac{dr}{dv} = \tau \left( 1 + \frac{\kappa}{2} \tau c^{-1} pr^2 \right) = \tau \left[ 1 + \frac{1 - \Omega}{2c^2 \tau^2} r^2 \right]. \quad (33) \]
5 Classification of Universes

The second term in the square bracket in the above equation represents the deviation due to gravity from the standard Hubble law. For without that term, Eq. (33) reduces to \(dr/dv = \tau\), thus \(r = \tau v + \text{const}\). The constant can be taken zero if one assumes, as usual, that at \(r = 0\) the velocity should also vanish. Thus \(r = \tau v\), or \(v = H_0 r\) (since \(H_0 \approx 1/\tau\)). Accordingly, the equation of motion (33) describes the expansion of the Universe when \(\Omega = 1\), namely when \(\rho = \rho_c\). The equation then coincides with the standard Hubble law.

The equation of motion (33) can easily be integrated exactly by the substitutions

\[
\sin \chi = \sqrt{\frac{(\Omega - 1)}{2}} \frac{r}{2c\tau}; \quad \Omega > 1, \quad (34a)
\]
\[
\sinh \chi = \sqrt{\frac{(1 - \Omega)}{2}} \frac{r}{2c\tau}; \quad \Omega < 1. \quad (34b)
\]

One then obtains, using Eqs. (33) and (34),

\[
dv = \frac{cd\chi}{(\Omega - 1)^{1/2} \cos \chi}; \quad \Omega > 1, \quad (35a)
\]
\[
dv = \frac{cd\chi}{(1 - \Omega)^{1/2} \cosh \chi}; \quad \Omega < 1. \quad (35b)
\]

We give below the exact solutions for the expansion of the Universe for each of the cases, \(\Omega > 1\) and \(\Omega < 1\). As will be seen, the case of \(\Omega = 1\) can be obtained at the limit \(\Omega \to 1\) from both cases.

**The case** \(\Omega > 1\). From Eq. (35a) we have

\[
\int dv = \frac{c}{\sqrt{(\Omega - 1)/2}} \int \frac{d\chi}{\cos \chi},
\]

where \(\sin \chi = r/a\), and \(a = c\tau \sqrt{(\Omega - 1)/2}\). A simple calculation gives [7]

\[
\int \frac{d\chi}{\cos \chi} = \ln \left| \frac{1 + \sin \chi}{\cos \chi} \right|.
\]

A straightforward calculation then gives

\[
v = \frac{a}{2\tau} \ln \left| \frac{1 + r/a}{1 - r/a} \right|. \quad (38)
\]

As is seen, when \(r \to 0\) then \(v \to 0\) and using the L’Hospital lemma, \(v \to r/\tau\) as \(a \to 0\) (and thus \(\Omega \to 1\)).

**The case** \(\Omega < 1\). From Eq. (35b) we now have

\[
\int dv = \frac{c}{\sqrt{(1 - \Omega)/2}} \int \frac{d\chi}{\cosh \chi},
\]
where \( \sinh \chi = r/b \), and \( b = c\tau \sqrt{1 - \Omega} / 2 \). A straightforward calculation then gives [7]

\[
\int \frac{d\chi}{\cosh \chi} = \arctan e^\chi.
\] (40)

We then obtain

\[
\cosh \chi = \sqrt{1 + \frac{r^2}{b^2}},
\] (41)

\[
e^\chi = \sinh \chi + \cosh \chi = \frac{r}{b} + \sqrt{1 + \frac{r^2}{b^2}}.
\] (42)

Equations (39) and (40) now give

\[
v = \frac{2c}{\sqrt{(1 - \Omega) / 2}} \arctan e^\chi + K,
\] (43)

where \( K \) is an integration constant which is determined by the requirement that at \( r = 0 \) then \( v \) should be zero. We obtain

\[
K = -\pi c / 2 \sqrt{(1 - \Omega) / 2},
\] (44)

and thus

\[
v = \frac{2c}{\sqrt{(1 - \Omega) / 2}} \left( \arctan e^\chi - \frac{\pi}{4} \right).
\] (45)

A straightforward calculation then gives

\[
v = \frac{b}{\tau} \left\{ 2 \arctan \left( \frac{r}{b} + \sqrt{1 + \frac{r^2}{b^2}} \right) - \frac{\pi}{2} \right\}.
\] (46)

As for the case \( \Omega > 1 \) one finds that \( v \rightarrow 0 \) when \( r \rightarrow 0 \), and again, using L’Hospital lemma, \( r = \tau v \) when \( b \rightarrow 0 \) (and thus \( \Omega \rightarrow 1 \)).

### 6 Physical meaning

To see the physical meaning of these solutions, however, one does not need the exact solutions. Rather, it is enough to write down the solutions in the lowest approximation in \( \tau^{-1} \). One obtains, by differentiating Eq. (33) with respect to \( v \), for \( \Omega > 1 \),

\[
d^2r/dv^2 = -kr; \quad k = \frac{(\Omega - 1)}{2e^2},
\] (47)

the solution of which is

\[
r (v) = A \sin \frac{v}{c} + B \cos \frac{v}{c},
\] (48)
where \( \alpha^2 = (\Omega - 1)/2 \) and \( A \) and \( B \) are constants. The latter can be determined by the initial condition \( r(0) = 0 = B \) and \( dr(0)/dv = \tau = A\alpha/c \), thus

\[
r(v) = \frac{c\tau}{\alpha} \sin \frac{\alpha v}{c}.
\]  
(49)

This is obviously a closed Universe, and presents a decelerating expansion.

For \( \Omega < 1 \) we have

\[
d^2r/dv^2 = \frac{(1 - \Omega)r}{2c^2},
\]  
(50)

whose solution, using the same initial conditions, is

\[
r(v) = \frac{c\tau}{\beta} \sinh \frac{\beta v}{c},
\]  
(51)

where \( \beta^2 = (1 - \Omega)/2 \). This is now an open accelerating Universe.

For \( \Omega = 1 \) we have, of course, \( r = \tau v \).

### 7 The accelerating Universe

We finally determine which of the three cases of expansion is the one at present epoch of time. To this end we have to write the solutions (49) and (51) in ordinary Hubble’s law form \( v = H_0 r \). Expanding Eqs. (49) and (51) into power series in \( v/c \) and keeping terms up to the second order, we obtain

\[
r = \tau v \left(1 - \frac{\alpha^2 v^2}{6c^2}\right),
\]  
(52a)

\[
r = \tau v \left(1 + \frac{\beta^2 v^2}{6c^2}\right),
\]  
(52b)

for \( \Omega > 1 \) and \( \Omega < 1 \), respectively. Using now the expressions for \( \alpha \) and \( \beta \), Eqs. (52) then reduce into the single equation

\[
r = \tau v \left[1 + (1 - \Omega) \frac{v^2}{6c^2}\right].
\]  
(53)

Inverting now this equation by writing it as \( v = H_0 r \), we obtain in the lowest approximation

\[
H_0 = h \left[1 - (1 - \Omega) \frac{v^2}{6c^2}\right],
\]  
(54)

where \( h = \tau^{-1} \). To the same approximation one also obtains

\[
H_0 = h \left[1 - (1 - \Omega) \frac{z^2}{6}\right] = h \left[1 - (1 - \Omega) \frac{r^2}{6c^2\tau^2}\right],
\]  
(55)

where \( z \) is the redshift parameter. As is seen, and it is confirmed by experiments, \( H_0 \) depends on the distance it is being measured; it has physical meaning only at the zero-distance limit, namely when measured locally, in which case it becomes \( h = 1/\tau \).
Figure 1: Hubble’s diagram describing the three-phase evolution of the Universe according
to cosmological general relativity theory. Curves (1) to (5) represent the stages of *decelerating* expansion according to \( r(v) = (ct/\alpha) \sin \alpha v/c \), where \( \alpha^2 = (\Omega - 1)/2 \), \( \Omega = \rho/\rho_c \), with \( \rho_c \) a constant, \( \rho_c = 3/8\pi G \tau^2 \), and \( c \) and \( \tau \) are the speed of light and the Hubble time in vacuum (both universal constants). As the density of matter \( \rho \) decreases, the Universe goes over from the lower curves to the upper ones, but it does not have enough time to close up to a big crunch. The Universe subsequently goes to curve (6) with \( \Omega = 1 \), at which time it has a *constant* expansion for a fraction of a second. This then followed by going to the upper curves (7), (8) with \( \Omega < 1 \) where the Universe expands with *acceleration* according to \( r(v) = (ct/\beta) \sinh \beta v/c \), where \( \beta^2 = (1 - \Omega)/2 \). Curve no. 8 fits the present situation of the Universe. (Source: S. Behar and M. Carmeli, Ref. 3)

It follows that the measured value of \( H_0 \) depends on the “short” and “long” distance scales [8]. The farther the distance \( H_0 \) is being measured, the lower the value for \( H_0 \) is obtained. By Eq. (55) this is possible only when \( \Omega < 1 \), namely when the Universe is accelerating. By Eq. (30) we also find that the pressure is positive.

The possibility that the Universe expansion is accelerating was first predicted using CGR by the author in 1996 [2] before the supernovae experiments results became known.

It will be noted that the constant expansion is just a transition stage between the decelerating and the accelerating expansions as the Universe evolves toward its present situation.

Figure 1 describes the Hubble diagram of the above solutions for the three types of expansion for values of \( \Omega_m \) from 100 to 0.245. The figure describes the three-phase evolution of the Universe. Curves (1)-(5) represent the stages of *decelerating expansion* according to Eq. (49). As the density of matter \( \rho \) decreases, the Universe goes over from the lower curves to the upper ones, but it does not have enough time to close up to a big crunch. The Universe subsequently goes over to curve (6) with \( \Omega_m = 1 \), at which time it has a constant expansion for a fraction of a second. This then followed by going to the upper curves (7) and (8) with \( \Omega_m < 1 \), where the Universe expands with *acceleration* according to Eq. (51). Curve no. 8 fits the present situation of the Universe. For curves (1)-(4) in the diagram we use the cutoff when the curves were at their maximum. In Table 1 we present the cosmic times with respect to the big bang, the cosmic radiation temperature and the pressure for each of the curves in Fig. 1.
Figures 2 and 3 show the Hubble diagrams for the distance-redshift relationship predicted by the theory for the accelerating expanding Universe at the present time, and Figures 4 and 5 show the experimental results.

Our estimate for $h$, based on published data, is $h \approx 80 \text{ km/sec-Mpc}$. Assuming $\tau^{-1} \approx 80 \text{ km/sec-Mpc}$, Eq. (55) then gives

$$H_0 = h \left[ 1 - 1.3 \times 10^{-4} \left(1 - \Omega\right) r^2 \right],$$

where $r$ is in Mpc. A computer best-fit can then fix both $h$ and $\Omega_m$.

To summarize, a theory of cosmology has been presented in which the dynamical variables are those of Hubble, i.e. distances and velocities. The theory describes the Universe as having a three-phase evolution with a decelerating expansion, followed by a constant and an accelerating expansion, and it predicts that the Universe is now in the latter phase. As the density of matter decreases, while the Universe is at the decelerating phase, it does not have enough time to close up to a big crunch. Rather, it goes to the constant-expansion phase, and then to the accelerating stage. As we have seen, the equation obtained for the Universe expansion, Eq. (51), is very simple.

Table 1: The Cosmic Times with respect to the Big Bang, the Cosmic Temperature and the Cosmic Pressure for each of the Curves in Fig. 1.
| Curve No.* | $\Omega_m$ | Time in Units of $\tau$ | Time (Gyr) | Temperature (K) | Pressure (g/cm$^2$) |
|-----------|------------|------------------------|------------|-----------------|---------------------|
| 1         | 100        | $3.1 \times 10^{-6}$   | $3.87 \times 10^{-5}$ | 1096            | -4.499               |
| 2         | 25         | $9.8 \times 10^{-5}$   | $1.22 \times 10^{-3}$ | 195.0           | -1.091               |
| 3         | 10         | $3.0 \times 10^{-4}$   | $3.75 \times 10^{-3}$ | 111.5           | -0.409               |
| 4         | 5          | $1.2 \times 10^{-3}$   | $1.50 \times 10^{-2}$ | 58.20           | -0.182               |
| 5         | 1.5        | $1.3 \times 10^{-2}$   | $1.62 \times 10^{-1}$ | 16.43           | -0.023               |
|           |            |                        |            |                 |                     |
|           |            | **DECELERATING EXPANSION** |            |                 |                     |
| 6         | 1          | $3.0 \times 10^{-2}$   | $3.75 \times 10^{-1}$ | 11.15           | 0                   |
|           |            | **CONSTANT EXPANSION**  |            |                 |                     |
| 7         | 0.5        | $1.3 \times 10^{-1}$   | $1.62$     | 5.538           | +0.023               |
| 8         | 0.245      | 1.0                    | 12.50      | 2.730           | +0.034               |
|           |            | **ACCELERATING EXPANSION** |            |                 |                     |

*The calculations are made using Carmeli’s cosmological transformation, Eq. (2), that relates physical quantities at different cosmic times when gravity is extremely weak.

For example, we denote the temperature by $\theta$, and the temperature at the present time by $\theta_0$, we then have

$$\theta = \frac{\theta_0}{\sqrt{1 - \frac{t^2}{\tau^2}}} = \frac{\theta_0}{\sqrt{1 - \frac{(\tau - T)^2}{\tau^2}}} = \frac{2.73K}{\sqrt{\frac{2\tau T - T^2}{\tau^2}}} = \frac{2.73K}{\sqrt{\frac{T}{\tau} \left(2 - \frac{T}{\tau}\right)}},$$

where $T$ is the time with respect to B.B.

The formula for the pressure is given by Eq. (30), $p = c(1 - \Omega)/8\pi G\tau$. Using $c = 3 \times 10^{10} \text{cm/s}$, $\tau = 3.938 \times 10^{17} \text{s}$ and $G = 6.67 \times 10^{-8} \text{cm}^3/\text{g s}^2$, we obtain $p = 4.544 \times 10^{-2} (1 - \Omega) g/\text{cm}^2$. 


8 Theory versus experiment

The Einstein gravitational field equations with the added cosmological term are [9]:

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}, \]  

(57)

where \( \Lambda \) is the cosmological constant, the value of which is supposed to be determined by experiment. In Eq. (57) \( R_{\mu\nu} \) and \( R \) are the Ricci tensor and scalar, respectively, \( \kappa = 8\pi G \), where \( G \) is Newton’s constant and the speed of light is taken as unity.

Recently the two groups (the Supernovae Cosmology Project and the High-Z Supernova Team) concluded that the expansion of the Universe is accelerating [10-16]. The two groups had discovered and measured moderately high redshift \((0.3 < z < 0.9)\) supernovae, and found that they were fainter than what one would expect them to be if the cosmos expansion were slowing down or constant. Both teams obtained

\[ \Omega_m \approx 0.3, \quad \Omega_\Lambda \approx 0.7, \]  

(58)

and ruled out the traditional \((\Omega_m, \Omega_\Lambda) = (1, 0)\) Universe. Their value of the density parameter \( \Omega_\Lambda \) corresponds to a cosmological constant that is small but, nevertheless, nonzero and positive,

\[ \Lambda \approx 10^{-52} \text{m}^{-2} \approx 10^{-35} \text{s}^{-2}. \]  

(59)
In previous sections a four-dimensional cosmological theory (CGR) was presented. Although the theory has no cosmological constant, it predicts that the Universe accelerates and hence it has the equivalence of a positive cosmological constant in Einstein’s general relativity. In the framework of this theory (see Section 2) the zero-zero component of the field equations (3) is written as

\[
R_0^0 - \frac{1}{2} \delta_0^0 R = \kappa \rho_{\text{eff}} = \kappa (\rho - \rho_c),
\]

where \( \rho_c = 3/\kappa \tau^2 \) is the critical mass density and \( \tau \) is Hubble’s time in the zero-gravity limit.

Comparing Eq. (60) with the zero-zero component of Eq. (57), one obtains the expression for the cosmological constant of general relativity,

\[
\Lambda = \kappa \rho_c = 3/\tau^2.
\]

(61)

To find out the numerical value of \( \tau \) we use the relationship between \( h = \tau^{-1} \) and \( H_0 \) given by Eq. (55) (CR denote values according to Cosmological Relativity):

\[
H_0 = h \left[ 1 - (1 - \Omega_m^{CR}) \frac{z^2}{6} \right],
\]

(62)

where \( z = v/c \) is the redshift and \( \Omega_m^{CR} = \rho_m/\rho_c \) with \( \rho_c = 3h^2/8\pi G \). (Notice that our \( \rho_c = 1.194 \times 10^{-29} \text{g/cm}^3 \) is different from the standard \( \rho_c \) defined with \( H_0 \).) The redshift parameter \( z \) determines the distance at which \( H_0 \) is measured. We choose \( z = 1 \) and take for

\[
\Omega_m^{CR} = 0.245,
\]

(63)

its value at the present time (see Table 1) (corresponds to 0.32 in the standard theory), Eq. (62) then gives

\[
H_0 = 0.874h.
\]

(64)

At the value \( z = 1 \) the corresponding Hubble parameter \( H_0 \) according to the latest results from HST can be taken [17] as \( H_0 = 70 \text{km/s-Mpc} \), thus \( h = (70/0.874) \text{km/s-Mpc} \), or

\[
h = 80.092 \text{km/s-Mpc},
\]

(65)

and

\[
\tau = 12.486 \text{Gyr} = 3.938 \times 10^{17} \text{s}.
\]

(66)

What is left is to find the value of \( \Omega_\Lambda^{CR} \). We have \( \Omega_\Lambda^{CR} = \rho_c^{ST}/\rho_c \), where \( \rho_c^{ST} = 3H_0^2/8\pi G \) and \( \rho_c = 3h^2/8\pi G \). Thus \( \Omega_\Lambda^{CR} = (H_0/h)^2 = 0.874^2 \), or

\[
\Omega_\Lambda^{CR} = 0.764.
\]

(67)

As is seen from Eqs. (63) and (67) one has

\[
\Omega_T = \Omega_m^{CR} + \Omega_\Lambda^{CR} = 0.245 + 0.764 = 1.009 \approx 1,
\]

(68)
which means the Universe is Euclidean.

As a final result we calculate the cosmological constant according to Eq. (61). One obtains

$$\Lambda = 3/\tau^2 = 1.934 \times 10^{-35} \text{s}^{-2}. \quad (69)$$

Our results confirm those of the supernovae experiments and indicate on the existance of the dark energy as has recently received confirmation from the Boomerang cosmic microwave background experiment [18,19], which showed that the Universe is Euclidean.

9 Concluding remarks

In this paper the cosmological general relativity, a relativistic theory in spacevelocity, has been presented and applied to the problem of the expansion of the Universe. The theory, which predicts a positive pressure for the Universe now, describes the Universe as having a three-phase evolution: decelerating, constant and accelerating expansion, but it is now in the latter stage. Furthermore, the cosmological constant that was extracted from the theory agrees with the experimental result. Finally, it has also been shown that the three-dimensional spatial space of the Universe is Euclidean, again in agreement with observations.

Recently [20,21], more confirmation to the Universe accelerating expansion came from the most distant supernova, SN 1997ff, that was recorded by the Hubble Space Telescope. As has been pointed out before, if we look back far enough, we should find a decelerating expansion (curves 1-5 in Figure 1). Beyond \( z = 1 \) one should see an earlier time when the mass density was dominant. The measurements obtained from SN 1997ff’s redshift and brightness provide a direct proof for the transition from past decelerating to present accelerating expansion (see Figure 6). The measurements also exclude the possibility that the acceleration of the Universe is not real but is due to other astrophysical effects such as dust.

Table 2 gives some of the cosmological parameters obtained here and in the standard theory.
Table 2: Cosmological parameters in cosmological general relativity and in standard theory

|                               | COSMOLOGICAL RELATIVITY | STANDARD THEORY |
|-------------------------------|--------------------------|-----------------|
| Theory type                   |                          |                 |
| Expansion type                |                          |                 |
| Spacevelocity                 | Tri-phase: deaccelerating, constant, accelerating | Spacetime One phase |
| Present expansion             | Accelerating (predicted) | One of three possibilities |
| Pressure                      | 0.034 g/cm²              | Negative Depends |
| Cosmological constant         | 1.934 × 10⁻³⁵ s⁻²        | Depends         |
| Ωₕ = Ωₘ + Ωₐ                  | (predicted)              | Depends         |
| Constant-expansion occurs at | 1.009                    | No prediction   |
| Constant-expansion duration   | 8.5 Gyr ago              |                 |
| Temperature at constant expansion | 146 K                  | No prediction   |

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Figure 6: Hubble diagram of SNe Ia minus an empty (i.e., “empty” $\Omega = 0$) Universe compared to cosmological and astrophysical models. (Source: A. Riess et al., Ref. 21)

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