The contact angle in inviscid fluid mechanics

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Abstract. We show that in general, the specification of a contact angle condition at the contact line in inviscid fluid motions is incompatible with the classical field equations and boundary conditions generally applicable to them. The limited conditions under which such a specification is permissible are derived; however, these include cases where the static meniscus is not flat. In view of this situation, the status of the many ‘solutions’ in the literature which prescribe a contact angle in potential flows comes into question. We suggest that these solutions which attempt to incorporate a phenomenological, but incompatible, condition are in some, imprecise sense ‘weak-type solutions’; they satisfy or are likely to satisfy, at least in the limit, the governing equations and boundary conditions everywhere except in the neighbourhood of the contact line. We discuss the implications of the result for the analysis of inviscid flows with free surfaces.

Keywords. Free surface flows; inviscid contact angle; finite amplitude motions.

1. Introduction

Consider an inviscid liquid, under a passive, inert gas, partially filling a smooth walled container. A gravitational field acts on the liquid. The liquid–gas interface is subject to surface tension. The interface meets the walls of the container at a line called the contact line. At any point on the contact line, the angle between the normal to the gas–liquid interface and the normal to the solid wall is called the contact angle, \( \alpha \). In the quiescent state, for many pure materials and smooth solid surfaces, the contact angle for many gas/liquid/solid systems is a function of the materials alone. Here, we will assume this to be true and call this contact angle, the static contact angle \( \alpha_s \).

The surface tension at the gas–liquid interface, from now on called the interface, requires that there be a jump in the normal stress across it if it is not flat. In the inviscid case the jump is in the pressure and for a static interface the liquid pressure is just the hydrostatic pressure. Thus the shape of the static interface depends on a Bond number \( Bo \), the ratio of a measure of the gravitational force to a measure of the force due to surface tension. But this shape also depends on the static contact angle \( \alpha_s \), which provides a boundary condition for the differential equation that determines the static interface shape. The static contact angle is therefore a parameter that influences the static meniscus shape and needs to be prescribed in order to calculate the meniscus shape.

When the interface is in motion, the surface tension again requires that a pressure jump, proportional to the interface curvature and to the coefficient of surface tension, exist across it. Of course, the pressure in the liquid will now be determined by the unsteady...
Bernoulli equation. The situation as regards the contact angle, however, is more complicated and confused. If one examines the dynamical equations for the interface it is not at all obvious that one needs to prescribe a condition on the contact angle. Moreover, as is well-known, for the case of linearized disturbances between plane, vertical walls where the initial interface is plane, the classical solution can be obtained without any specification of the contact angle, which turns out to be constant and is equal to $\pi/2$ throughout the motion. On the other hand, there are many examples in the literature where analyses and calculations have been made of unsteady potential motions where a condition on the contact angle has been prescribed as a boundary condition at the contact line. Just a short list of these could include Miles [9], Billingham [3] and Shankar [15]. It appears that the motivation to prescribe the contact angle comes from the apparent behaviour of real, viscous interfaces and the need to tailor inviscid models so that they lead to realistic results for real interfaces. The question that we raise here is: are we really free to prescribe the contact angle in inviscid potential flows and if not, what is the status of the ‘solutions’ that have appeared in the literature that purport to model ‘real’ contact angle behaviour?

Our interest in this question is recent and followed an investigation of contact angle behaviour in viscous flows with pinned contact lines [17]. We had been aware that Benjamin and Ursell [2] had shown that the contact angle would remain constant in linearized, inviscid, potential motions about a flat interface, i.e. one corresponding to a contact angle of $\pi/2$. Without carrying out an analysis, we had assumed the result to be generally true and in [17] had, perhaps influenced by all the work employing a constant contact angle, even asserted this. When, however, we recently tried to prove the result, it began to be clear that there was a problem here: analysis, following [2], seemed to show that the contact angle cannot, in general, be prescribed. Our purpose here is to demonstrate this and to try to place the existing literature (including ours!), in which a contact angle condition is employed in potential motions, in proper perspective. We believe that this is an important issue because even if an inviscid ‘solution’ is a good model of reality in some sense, we should be clear in what sense, or approximate sense, the ‘solution’ is a solution.

2. Analysis

We consider the inviscid motion of a liquid in an arbitrary, three-dimensional smooth walled container. The motion is generated by the translational motion of the container. The restriction to translational motions is essential to ensure potential flow in the moving frame, a necessary condition for some of the results that will be derived. The fluid is initially in static equilibrium with the gas above it which is at uniform pressure\(^1\). The motion is assumed to start and continue with a uniform pressure over the interface; we will assume the gas to be passive, i.e. it only exerts a constant pressure on the liquid interface. In §2.1, we write down the equations governing the motion. In §2.2, we first consider planar motions; motions in a cylinder of arbitrary cross-section are examined in §2.3. Finally, in §2.4, we summarize the main results.

2.1 Governing equations

We write the equations in a reference frame attached to the container (please, see figure 1). The container wall is given by $f(x,y,z) = 0$. Rectangular cartesian coordinates\(^1\)The condition of static equilibrium can be relaxed and, in fact, has to be in the case of time periodic wave motions.
Figure 1. A schematic showing a liquid in a container with a free surface making contact with the boundary of the container at the contact line.

are employed with gravity generally in the negative $z$-direction. Our analysis is restricted to the case when the interface is representable by a single valued smooth function, e.g. by $z = \zeta(x, y, t)$. The interface motion can be of finite amplitude however. The equations governing the liquid motion are the continuity and Euler equations:

\begin{align}
\nabla \cdot \mathbf{u} &= 0, \\
\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} &= -\nabla p + \mathbf{F},
\end{align}

where $\mathbf{u}$ is the liquid velocity, $p$ is the pressure and $\mathbf{F}$ is the net body force, both real and fictitious. $\mathbf{F}$ can be an arbitrary function of time. These have to be solved subject to the boundary conditions on (a) the container wall and (b) the interface. The condition on the wall is the no-penetration condition $\mathbf{u} \cdot \nabla f = 0$. The conditions on the interface $z = \zeta(x, y, t)$ are

\begin{align}
\zeta_t + u \zeta_x + v \zeta_y &= w, \\
p - p_a &= -\frac{1}{Bo} \kappa
\end{align}

which respectively are the kinematic condition on the interface and the normal stress condition on it. $Bo$ is a suitably defined Bond number, $p_a$ is the constant ambient pressure over the interface and $\kappa$ is the local interface curvature. The interface $z = \zeta(x, y, t)$ is assumed to intersect the container wall $f(x, y, z) = 0$ in a smooth curve called the contact line. It is further assumed that $\zeta$, $\mathbf{u}$ and $p$ are analytic in all variables and that the smoothness of the solutions is up to and including the boundary. It must be pointed out that the existence of such solutions is not known at present. The possibly time dependent angle $\alpha$ made by the contact line with the wall is given by

\begin{equation}
\frac{f_z - f_x \zeta_x - f_y \zeta_y}{|\nabla f|\left[1 + \zeta_x^2 + \zeta_y^2\right]^{1/2}} = \cos \alpha(t),
\end{equation}

where (4) is to be evaluated at any point $(x_c(t), y_c(t), \zeta(x_c(t), y_c(t), t))$ on the contact line. We will show, over the next two sub-sections, that no \textit{a priori} prescription of $\alpha(t)$ is possible though in a few cases, it turns out to be a constant equal to its initial value $\alpha_s$. 

\[\text{The contact angle in inviscid fluid mechanics}\]
2.2 Planar motions

The container is of arbitrary cross-section in the \( x \)-\( z \) plane and the motion is 2D. The contact line in this case consists of just two points (A and B, say). During the course of the motion, let the contact line traverse regions \( R \) of the wall surface. We will distinguish two cases depending on whether \( R \) is locally flat or not. We designate the former the ‘flat or straight-wall’ case and the latter as the ‘curved-wall’ case. Examples of the former are rectangular and wedge-shaped containers (with the \( z \)-axis along the wedge); a cylindrical channel is curved-walled. We first consider the straight-wall case.

2.2.1 The flat or straight-wall case. The \( x \) - and \( z \) -axes are chosen such that, locally, the body surface is a line of constant \( x \), say \( x = 0 \). There will be an \( x \) component of gravity in this case; the gravity direction has no bearing on the analysis however. The no-penetration condition implies \( u(0, z, t) = 0 \) from which it follows that \( u_{\zeta(0)}(0, z, t) = 0 \) where \( u_{\zeta(0)} \) is the \( 0 \)th derivative of \( u \) with respect to \( z \).

Now, differentiate (3a) with respect to \( x \) to obtain an equation valid on the interface and hence on the contact line

\[
\xi_{tt}(x,t) + u(x, \xi(x,t), t) \xi_{xx}(x,t) + [u_x(x, \xi(x,t), t) \xi_x(x,t) + u_x(x, \xi(x,t), t)]\xi_x(x,t) = w_x(x, \xi(x,t), t) + w_x(x, \xi(x,t), t) \xi_x(x,t). \tag{5a}
\]

Using the fact that an inviscid flow starting from rest has to be irrotational both in an inertial frame and a frame translating with respect to the inertial, we get \( u_z - w_x = 0 \), which with the continuity equation \( u_x + w_x = 0 \) allows (5a) to be written in the form

\[
\xi_{tt} = u_z - (2u_x + \xi_x u_x) \xi_x - u \xi_{xx}. \tag{5b}
\]

where the arguments in (5a) have been dropped to reduce the clutter. When (5b) is applied at the contact line we obtain

\[
\xi_{tt}(0,t) = -2u_x(0, \xi(0,t), t) \xi_x(0,t). \tag{6}
\]

Note that, if the container were to be rotating as well, \( u_z - w_x \neq 0 \) and the above analysis would not apply. Equation (6) shows that \( \xi_{tt} \) at the contact line cannot be specified arbitrarily; it certainly is not zero in general\(^2\). This means that the contact angle changes with time in a manner that cannot be prescribed beforehand. However, for \( \alpha = \pi/2 \), not only is \( \xi_{tt}(0,0) = 0 \) but also \( \xi_{tt(m)}(0,0) = 0 \) for all \( k = 2, 3, \ldots \). We will show this by mathematical induction\(^3\). Let the induction proposition be

\[
P(k): \xi_{tt(m)}(0,0) = 0 \ \forall \ m = 0, \ldots, k.
\]

\( P(0) \) is true because \( \alpha_0 = \pi/2 \). Assume \( P(k) \) is true; we will show that \( P(k+1) \) is true. Rewriting (5b) as \( \xi_{tt} = u_z - a(u, \xi) \xi_x - u \xi_{xx} \), the \( k \)th time derivative of this equation can be written as

\[
\xi_{tt(k+1)} = u_{z(k)} - [a \xi_x]_{x(k)} - (u \xi_{xx})_{x(k)}. \]

\(^2\)It should be noted that we cannot set \( u_z(0, \xi(0,t), t) = 0 \) and argue that the contact angle is constant; the reason is that one cannot specify two conditions at the boundary in a potential flow and \( u(0, z, t) \) has already been set to 0.

\(^3\)It is not possible to integrate (6) directly in time to establish the result since all that we know is that \( \xi_{tt}(0,0) = 0 \) and hence from (6) that \( \xi_{tt}(0,0) = 0 \).
With respect to the above equation, we observe the following:

1. \( \partial \{ u_c \} / \partial \tau^k = 0 \) at \( x = 0 \forall k = 0, 1, 2, \ldots \)
2. \( \partial \{ a \xi \} / \partial \tau^k = \sum_{m=0}^{k} b_m a_{(k-m)} \xi_{m} \) where \( b_m \) is the binomial coefficient \( C(k,m) \). By the truth of \( P(k) \), this sum is zero at \( x = 0 \).
3. \( \partial \{ u_{xx} \} / \partial \tau^k = \sum_{m=0}^{k} C(k,m) u_{(k-m)} \xi_{xx(m)} \). This sum is zero \( \forall k = 0, 1, \ldots \) as \( u \) and all its time derivatives vanish on the contact line.

Thus we have \( \xi_{u(k+1)}(0,0) = 0 \) which means \( P(k+1) \) is true. Thus, by mathematical induction, \( P(n) \) is true \( \forall n = 0, 1, \ldots \) and so \( \xi_u(0,t) = 0 \). This in turn implies that \( \xi_s(0,t) = 0 \) for all time and the contact angle remains at \( \pi/2 \) for all time. Some observations are noteworthy:

1. Irrotationality of the motion is necessary but not sufficient.
2. This is a nonlinear result, i.e. it holds irrespective of the perturbation amplitude and the shape of the static meniscus as long as the initial contact angle \( \alpha_s \) is \( \pi/2 \). In particular, the static meniscus need not be flat.
3. The result holds as long as the region of the body surface over which the contact line moves is flat. The shape of the body elsewhere is immaterial.

### 2.2.2 The curved wall case

While (5b) is still true, the observations following it are not and nothing can be said about the behaviour of the contact angle for nonlinear motions even for the case of \( \alpha_s = \pi/2 \). It turns out that the contact angle is not constant even for linearized motions with \( \alpha_s = \pi/2 \). This is counter-intuitive as locally one would expect the curved wall to look ‘straight’ and it is instructive to see where the straight-wall analysis breaks down in this case. Representing the body surface by \( z = g(x) \) the no-penetration condition is written as

\[
u g' - w = 0. \tag{7a}\]

Differentiating (7a) on the wall yields

\[
g'' u(x, g(x)) + g' \{ u_x + u_{x} g' \} = w_x + g' w_z \tag{7b}\]

\( \alpha_s = \pi/2 \) yields the relations \( \eta_s''(x_c) = -1/g' \) and \( g''(x_c) = \eta_s''(x_c) \), where \( \eta_s(x) \) is the static meniscus and the primes denote differentiation with respect to \( x \). Using these relations in (7b) yields

\[
\eta_s''(x_c, z_c) = \eta_s''(x_c) w_x + \eta_s' u_s - u_z. \tag{7c}\]

The initial interface may be flat or curved depending on the value of \( g' \) (infinite or finite) at the contact line. If (3a) is linearized about this initial interface we have

\[
\eta_t + w \eta_s' = w. \tag{8a}\]

where \( \eta(x,t) \) is the perturbation, caused by the container motion, of the static meniscus \( \eta_s \). We need to show that \( \eta_s(x_c, t) = \eta_s(x_c, 0) \) for all time. To this end, differentiate (8a) with respect to \( x \) to obtain

\[
\eta u_x + (u_x + \eta_s u_z) \eta_s' + w_x = w. \tag{8b}\]
which on using (7c) becomes
\[ \eta_{xt} = (1 - \eta'^2)(w_x + u_c) + \eta'_{tx}(w_z - 2u_x). \]  
(8c)

Thus \( \eta_{xt}(x_c,0) = 0 \) as the fluid is at rest initially. However, the time derivatives of \( u \) and \( w \) are in general not zero which means that \( \eta_{xt}(x_c,t) \neq 0 \) for arbitrary time \( t \).

2.3 Three-dimensional motions

An analysis similar to the one in §2.2 shows that the contact angle cannot be prescribed arbitrarily in the three-dimensional case as well. However, following Benjamin and Ursell [2], we will now show that a contact angle of \( \pi/2 \) remains constant for linear (infinitesimal amplitude) wave motions in a right cylinder of arbitrary but smooth cross-section standing on one of its ends. Let the body cross-section be given by \( g(x,y) = 0 \) and the interface by \( \xi(x,y,z,t) = z - \zeta(x,y,t) = 0 \), then we have the body and interface unit normals as
\[ \hat{n}_b = \frac{-g_x\hat{i} + g_y\hat{j}}{|\nabla g|}, \]
\[ \hat{n}_i = \frac{-\xi_x\hat{i} - \xi_y\hat{j} + \hat{k}}{\sqrt{1 + \xi_x^2 + \xi_y^2}}, \]
whence the contact angle \( \alpha_s \) is given by
\[ \cos \alpha_s = \hat{n}_b \cdot \hat{n}_i = \hat{n}_b \cdot \nabla \xi = \frac{\partial \xi}{\partial n_b}. \]

Since \( \alpha_s = \pi/2 \), we have initially \( \partial \zeta / \partial n_b = 0 \). Linearizing (3a) and making use of the irrotationality of the motion, the kinematic condition is written as
\[ \zeta_t + u \frac{\partial \zeta}{\partial r} + v \frac{\partial \zeta}{\partial \theta} = w, \]
(9)
where \( \phi \) is the velocity potential governing the motion. Note that (9) is applied on \( z = 0 \). Differentiating (9) in the direction of \( \hat{n}_b \), we have
\[ \zeta_{nt} = \frac{\partial^2 \phi}{\partial n_b \partial z}, \]
(10)
which on using the no-penetration condition on the contact line \( \partial \phi / \partial n_b = 0 \), leads to \( \zeta_{nt} = 0 \) on the contact line. Since this is true for all time, this means \( \partial \zeta / \partial n_b = 0 \) for all time, i.e., the contact angle remains \( \pi/2 \).

The same result does not hold in the nonlinear case. We show this for the simplest case of a right circular cylinder. Employing cylindrical coordinates \( (r, \theta, z) \), the kinematic condition is
\[ \zeta_t + u \frac{\partial \zeta}{\partial r} + v \frac{\partial \zeta}{\partial \theta} + \frac{\zeta_\theta}{r} \theta = w, \]
(11)
applied on \( z = \zeta(r, \theta, t) \) and \( u, v \) and \( w \) are the \( r, \theta \) and \( z \) components of velocity. \( \alpha_s = \pi/2 \) translates to \( \partial \zeta / \partial r = 0 \) on \( r = a \) where \( a \) is the radius of the cylinder. Differentiating (11) with respect to \( r \), we have
\[ \zeta_{rr} + u \zeta_{rr} + (u_r + u \zeta_r) \zeta_r + \frac{v}{r} \zeta_{r\theta} - \frac{v}{r^2} \zeta_\theta + \frac{\zeta_\theta}{r} = (v_r + v \zeta_r) \zeta_r = w_r + w \zeta_r. \]
(12a)
Using irrotationality and continuity, the above equation can be written as

\[
\zeta_{rt} = -u\zeta_{rr} - 2u_r\zeta_r + u_z(1 - \zeta_r^2) + \frac{v}{r} \left( \frac{\zeta_\theta}{r} - \zeta_{r\theta} \right)
- \frac{v_r\zeta_\theta}{r} - \frac{v_\theta\zeta_r}{r} - \frac{v_z\zeta_\theta\zeta_r}{r}.
\]  

(12b)

Since the motion starts from rest, \(u = v = w = 0\) initially. So are all the spatial derivatives of velocity. Initially \(\zeta_r\) and \(\zeta_{r\theta}\) are both zero on the contact line. Finally, \(\zeta_{rt}(a, \theta, 0) = 0\).

It can be shown that \(\zeta_{rt}(a, \theta, 0) = 0\). However the time derivative of (12c) will contain terms like \(\zeta_{\theta r}v_r/r\) which are not necessarily zero and hence \(\zeta_{rt}(a, \theta, t) \neq 0\) for arbitrary time \(t\). It can be shown however that a contact angle of \(\pi/2\) is preserved by the class of axisymmetric motions (see Appendix C). This is consistent with the result obtained in the two-dimensional case.

2.4 A summary of the results

We summarize, for the convenience of the readers, the main points of the last two sections. The most important point is that in inviscid fluid motions starting from rest in a container, the contact angle cannot be prescribed in advance and neither is there need for such a prescription. However, if one insists on prescribing the contact angle, this would necessarily result in a ‘weak-type’ solution – one that is in violation of the actual behaviour of the contact angle. However, in the special case of \(\alpha_s = \pi/2\), there exist situations where the contact angle remains constant throughout the motion. These situations include cases of curved static menisci. Prescription of conditions other than this in this case will again lead to ‘weak-type’ solutions.

3. Discussion

Our discussion will center on the considerable confusion that exists in the literature for the last five decades, on the role of the contact angle in inviscid fluid motions. First we should make clear that there is no confusion whatsoever in the classical literature on wave motion in liquids, for example as given in Lamb [8]. In the classical literature, capillarity is considered only in situations where the liquids are not bounded by solid boundaries; in such situations there is no contact line and so the difficulty we are considering does not arise.
3.1 Examples from the literature

We will substantiate our statements above by presenting below a small sample of the literature dealing with inviscid contact lines. We hasten to add that we have no wish to criticize any particular worker or group; indeed we cite our own work as manifestations of this confused state of affairs. The following is in rough chronological order:

(a) The article by Reynolds and Satterlee [14] appears in NASA SP-106, which was the bible for many aerospace engineers for almost three decades. This excellent article deals with all aspects of the low-gravity behaviour of liquid propellants. In dealing with low-gravity sloshing, they indicate the boundary conditions to be imposed, ‘plus a contact angle condition’. For the linearized problem they suggest that this takes the form ‘$h_r = \gamma h$’ where $h$ is the perturbation to the interface height and $\gamma$ is a constant to account for ‘contact angle hysteresis’. After the formulation of the general linearized problem, they indicate that they are still faced with a ‘formidable problem’; then they just deal with the special flat interface case in a cylindrical tank. For this special case they point out “Note that we were unable to enforce any contact point condition” and further note that the contact angle is unchanged for this solution. It is to be remarked that the authors wished to impose a contact angle condition but were unable to do so and found the contact angle to remain constant ($= \pi/2$). The reason is that they were working with the ‘classical’ spatial eigenfunctions and could only recover the ‘classical’ solution, which our analysis showed will lead to a constant contact angle in the flat interface case.

(b) The papers by Moore and Perko [10] and Perko [12] are important not only because they are among the first to deal with large-amplitude motions and surface instabilities leading to breakers, but also because they suggest new methods of dealing with the liquid sloshing problem. Both papers deal with the initial value problem for curved interfaces under the influence of capillarity and are based on expanding the velocity potential in a series of harmonic functions with time varying coefficients. The solution method in [10] is such that the evolution of the interface, at each increment of time, does not involve the current contact angle. In fact the only role of the contact angle is in determining the initial interface shape. Consequently, we find from their interesting figure 3, which shows breakers at the wall, that the contact angle appears to change with time. These developments are entirely consistent with the analysis in §2 which showed that if $\alpha_s \neq \pi/2$, the contact angle will vary in general. Perko [12] extends the earlier analysis to the general axisymmetric case. However, the author also points out that it “includes the constant contact angle boundary condition necessary to have a well-posed problem in computation” (emphasis added). This is puzzling since computations were possible in the earlier paper without any condition on the contact angle, and in fact it was not possible to impose such a condition. The author points out that the new figure 4, with a constant contact angle, is for the same conditions as for the earlier figure 3, where it was not constant; there is no suggestion as to which one we should prefer or why. It appears from §2 that the ‘solutions’ in Perko [12] are not classical ones.

(c) Chu [4] is representative of a large class of papers in the 1960s and 1970s which suggested ways in which slosh frequencies, forces and moments could be calculated for axisymmetric containers under low-gravity conditions. This particular paper suggests the use of a Galerkin type of procedure. As regards the contact line, Chu says “In
addition, there is an interface contact point condition which takes the form . . .” and gives the condition given in (a) above.

(d) It was pointed out in §2 that if the initial contact angle is $\pi/2$ and the side walls are straight, even a non-linear, two-dimensional motion would maintain the contact angle at $\pi/2$. This implies that if non-linear, capillary-gravity standing waves exist, every such wave would be a solution of a problem where straight walls are located at the nodal points of the wave motion. An example of this possibility is the solution found in Concus [5] where such a solution is found to be third order in the amplitude of the waves.

(e) Myshkis et al. [11] is an encyclopaedic book on low-gravity fluid mechanics. However, in Chapter 5 when they formulate the small oscillation problem, after writing down eq. (5.2.14) all they have to say on the contact angle is “…(5.2.14) is the linearized condition for the conservation of the contact angle”. In fact, the classical theory tells us it will not be conserved in general.

(f) A totally different aspect is presented by Benjamin and Scott [1], who appear to have been the first to consider the natural frequencies of a confined liquid with a flat interface, but whose contact line is pinned. The inviscid modelling of this situation is immediately seen to be problematic because the no-slip condition seems to be required at the contact line, a condition that cannot in general be satisfied by a potential flow. In other words, there is no classical solution to this problem. In light of this, Benjamin and Scott using the framework of functional analysis, define various function spaces, operators and other tools to formulate a ‘weak solution’ to the key equation (7b) of their paper and finally get estimates for the frequencies using Rayleigh’s principle. Moreover, they show that their theoretical estimates agree well with measured values of the wave periods.

(g) Hocking [7] is a widely cited paper because a new model for the contact angle condition, \( \partial \eta'/\partial t' = \lambda \partial \eta'/\partial n' \), is introduced for the \( \alpha_s = \pi/2 \) case; here \( \eta' \) is the surface elevation of the small disturbance above the flat static interface. The model is introduced to account in some way for the ‘wetting property’ of the fluid and includes both the free and pinned edge conditions as limiting cases. In his introduction Hocking says “The presence of capillarity adds an extra term to the free-surface pressure condition. .......... The increase in the order of the pressure condition, however, requires extra conditions to be imposed when the solution is sought in a finite region” (emphasis added). This is a common misconception. While it is true that capillarity increases the order of the equation governing the static meniscus and hence the number of boundary conditions needed in this case, this does not apply in the dynamical situation. This can be seen by just considering the flat interface case between vertical walls. In any case, the analysis of §2 shows that no extra contact angle condition can be prescribed for classical solutions.

(h) It suffices now to mention that the old confusions persist into the new millenium, typical samples being [3,15,16,19].

3.2 The current status of the contact angle in inviscid flows and how it has come about

We will now summarize how the dynamical, inviscid contact angle appears to be viewed by most workers and why it has come to be this way. Recall that in the classical works the
question of the contact angle never arose because confined flows with boundaries and capillarity were not normally studied. Early studies on the latter were confined to linearized, two-dimensional flows between vertical walls; here the classical, exact solutions correspond to a contact angle of $\pi/2$, which is maintained throughout the motion. In the 1960s the space programmes required solutions to more general problems involving curved static interfaces and static contact angles other than $\pi/2$. The difficulty posed by these problems forced approaches that were either semi-analytical or numerical and some like Moore and Perko [10] did not attempt to impose a dynamical contact angle condition; $\alpha_0$ affected the initial conditions alone through the static meniscus. The imperative to impose a contact angle condition at the contact line, not permitted in general by the classical inviscid formulation, appears to have come from experimental observations of real, viscous contact lines. It is well-known [6,16,13,18] that real, dynamic viscous contact lines display complex behaviour and are not at all well-understood with many parameters playing a role. It is in attempting to model this complicated behaviour in an inviscid framework that the need for contact angle conditions began to be felt and then applied. The earlier models of a constant contact angle and the one used in [14] (essentially to model contact angle hysteresis) are in a sense non-dynamic. Hocking’s [7] model is a dynamic one, attempting in an inviscid framework to account for contact line hysteresis or viscous wetting effects at the contact line. In any case, the purpose is to account for viscous and other real effects in an inviscid, potential model of the flow.

Thus the need to more realistically model the dynamic contact line appears to require the freedom to impose a condition at the inviscid contact line. But as was shown in §2 the classical field equations and boundary conditions do not in general provide this freedom. Then the natural question is, what is the status of the very large and important body of work in which a contact angle condition is imposed, in violation of the classical formulation? Let us call solutions which are obtained without such a condition ‘classical’. Then this body of work referred to does not deal with classical solutions. This means that these ‘solutions’ will be found to violate at least some of the boundary conditions or assumptions at the contact line.

3.3 The importance of the present result for inviscid free-surface flows

A natural question that arises is: how important is the present result that the contact angle cannot, in general, be prescribed in a potential flow? Before we attempt to answer this question we would like to consider the situations shown in figure 2.

The configurations in (a) and (b) are planar while (c) is axisymmetric and $\alpha_0 = \pi/2$ in all three cases. According to our theory, finite amplitude motions in (a) preserve the contact angle, while the contact angle is not preserved in (b) and (c) even for infinitesimal motions. This is not a result that is obvious at all and shows that we must be very careful when dealing with the contact angle in inviscid flows.

Both from general considerations and from the above example it should be clear that the result is of real importance in basic inviscid flow theory. But one may still inquire whether the result is of any practical importance, for example, in the calculation of the natural frequencies of surface waves or in sloshing calculations. This is not easy to answer because we do not know how to calculate such things without making any assumptions about the contact angle. For example, say we wish to find the natural frequency in a case where the static contact angle $\alpha \neq \pi/2$. It appears that the frequency cannot come out of an eigenvalue problem by assuming a solution harmonic in time and an expansion in terms
The contact angle in inviscid fluid mechanics

Figure 2. The figure shows three examples of liquids in containers which have gas–liquid interfaces above them. In all the three cases the static contact angle \( \alpha_s = \pi/2 \). The geometry is planar in (a) and (b), while in (c) the container is a cone with the same half angle as in (a), which too is flat walled. The present theory predicts that the contact angle will be preserved in finite amplitude motions in (a) but will not be preserved even in infinitesimal motions in (b) and (c).

of spatial modes \( \phi(x) \) because the latter would imply some assumption about the contact angle at the boundary. It appears that the only way out would be to do the initial value problem with the correct static meniscus and let the field evolve freely. But it is unlikely that this can be done easily and certainly not analytically. However, we should point out that Moore and Perko [10] solve the initial value problem without making any assumption on the contact angle and apparently without making any assumption on a functional form for the interface. On the other hand, Perko [12] solves the same problem holding the contact angle fixed at the static value. His figure 4 can be compared directly with the earlier figure 8 of Moore and Perko. While there are differences, fortunately and as might be expected, the qualitative overall pictures are similar. This suggests that the violation of the correct condition at the contact line will lead to violation of the interface conditions at the contact line and except in extraordinary circumstances will not have much of an effect on the overall field. Thus it appears that natural frequencies and sloshing modes will not be greatly affected provided the containers are sufficiently large. This must be especially true of the natural frequencies which are related to integrals over the whole field. Indirect evidence in support of this position is that most frequency calculations done by different methods, which presumably violate the correct conditions differently, agree well with one another.
4. Conclusion

We have shown that the classical field equations governing the motion of a confined inviscid liquid under a passive gas do not, in general, permit the independent specification of a contact angle condition at the contact line. In fact the only cases where such a condition may be permissible are when the static contact angle is \( \pi/2 \), the container walls are flat, at least in the neighbourhood of the contact line, and (i) the motion is two-dimensional or (ii) the motion is a small three-dimensional disturbance from a flat initial interface. The restrictions are indeed surprising as is the difference between two- and three-dimensional motions.

These results have a somewhat serious bearing on the vast literature that exists in which ‘solutions’ have been found to inviscid motions in which various contact angle conditions have been imposed. It is our contention that these cannot be classical solutions to the classical field equations since classical solutions do not permit the imposition of a contact angle condition. It is suggested that these ‘solutions’ belong to an improperly defined class of ‘weak-type solutions’, in the sense that they attempt to solve the field equations in an approximate sense, with some of the equations being solved exactly. The need for such ‘solutions’ is driven by the compulsion to try to model in an inviscid framework, the complicated behaviour of moving viscous contact lines. Examples were given of other cases where a similar situation exists.

Finally, we have shown that while the present result is of basic importance in the theory of inviscid free-surface flows, it is unlikely to seriously affect the practical calculation of natural frequencies and sloshing modes in containers.

Appendix A

In §2, the initial contact angle \( \alpha_s = \pi/2 \) was shown to remain constant under certain conditions. One of these conditions was that the initial interface be the static meniscus itself and that the fluid be quiescent initially, i.e., we start from a static equilibrium. This would be the normal procedure of posing an initial value problem. On the other hand, we can seek special solutions such as periodic motions where though initially the fluid is at rest and the pressure over the interface constant, the state is not one of static equilibrium and the interface is not the static meniscus. Examples are the well-known linear and nonlinear periodic oscillations \([8,5]\) between two parallel vertical walls. The initial contact angle \( \alpha_s = \pi/2 \) would remain constant in this case as well.

Appendix B

We consider here the case of \( \alpha_s = 0 \); at first glance, it seemed the contact angle would be preserved in this case as well. Since \( \eta_s''(0,t) = \infty \), we reorient the coordinates such that \( x \) is along the wall, \( z \) normal to it and \( z = \zeta(x,t) = \eta_s(x) + \eta(x,t) \) gives the interface, as before. The initial interface, close to \( z = 0 \) can also be described as \( x = \xi_s(z) \), a description that will be needed below.

The linearized kinematic condition is written as

\[
\eta_{zt} = (1 - \eta_s^2)u_z - \eta_s''u - \eta_s'u_x, \quad (B1)
\]
where \( \eta_s'(0) = 0 \). \( \eta_s(0,0) \) is zero as the initial state is one of rest. The \( k \)th derivative of (B1) is

\[
\eta_{xt}(k+1) = (1 - \eta_s'^2)u_{xt}(k) - \eta_s''u_{x}(k) - \eta_s'u_{xt}(k).
\]

(B2)

It is easy to see that the first and third terms vanish at \( x = 0 \) as \( u_z = \eta_s' = 0 \) there. If we can show that the second term vanishes as well, we would be done. In the second term, \( u(0,0) \neq 0 \) so we want \( \eta_s''(0) = 0 \).

It is convenient to write the equation for the static meniscus with gravity along the \( x \)-axis. The equation would be

\[
\xi_s - \frac{1}{Bo} \frac{\xi''_s}{(1 + \xi_s'^2)^{3/2}} = \lambda,
\]

where the prime denotes differentiation with respect to \( z \) and \( \lambda \) is a non-zero constant equal to \(-\kappa(0)/Bo\) where \( \kappa(0) \) is the curvature of the interface at the contact line and is given by

\[
\kappa(0) = \frac{\xi''_s(0)}{(1 + \xi_s(0)^2)^{3/2}}.
\]

(B4)

Now, we can write, by chain differentiation,

\[
\eta_s''(x) = -\frac{\xi''_s(z)}{\xi_s'(z)}.
\]

(B5)

Since \( \xi'_s(0) \) is infinite but \( \kappa(0) \) finite and non-zero, we have from (B4) that \( \xi''_s(0)/\xi'_s(0) \) is a non-zero finite quantity. But by (B5), this means \( \eta_s''(0) \) is non-zero as well. Hence, \( \eta_{xt}(k+1) \neq 0 \) in general and the contact angle will vary during the motion.

Appendix C

In §2, it was shown that the contact angle will, in general, change for nonlinear motions in a right circular cylinder. However, we prove here that the contact angle \( \alpha_s = \pi/2 \) is preserved for the special class of axisymmetric nonlinear motions in a right circular cylinder. The proof by induction follows the general pattern of the proof in §2.2.1. Noting that axisymmetric motions mean that the interface is given by \( z = \zeta(r,t) \), let the induction proposition be

\[
P(k): \ \zeta_{rt}(m)(0,0) = 0 \forall m = 0, \ldots, k.
\]

P(0) is true because \( \alpha_s = \pi/2 \). Assume \( P(k) \) is true; we will show that \( P(k+1) \) is true. Writing (12b) for this case, we have

\[
\zeta_{rt} = -u\zeta_{rr} - 2u_r\zeta_r + u_z(1 - \zeta_r^2)
\]

(C1)

as the rest of the terms are identically zero due to axisymmetry. The \( k \)th derivative of (C1) will have the following terms:

1. \( \partial\{u\zeta_{rr}\}/\partial t^k = \sum_{m=0}^{k} C(k,m)u_{j(k-m)} \zeta_{rt}(m) \). This sum is zero \( \forall k = 0, 1, \ldots \) as \( u \) and all its time derivatives vanish on the contact line.
2. $\partial \{ u_r \zeta \}/\partial t^k = \sum_{m=0}^{k} C(k,m) u_{r(k-m)} \zeta_{r(m)}$. By the truth of $P(k)$, this sum is zero at $r = a$.

3. $\partial \{ u_z(1 - \zeta^2_r) \}/\partial t^k = \sum_{m=0}^{k} C(k,m) u_{r(k-m)} (1 - \zeta^2_r)_{r(m)}$. This sum is zero $\forall k = 0, 1, \ldots$ as $u_z$ and all its time derivatives vanish on the contact line.

Thus we have $\zeta_{r(k+1)}(0,0) = 0$ which means $P(k + 1)$ is true. Thus, by mathematical induction, $P(n)$ is true $\forall n = 0, 1, \ldots$ and so $\zeta_t(0,t) = 0$. This in turn implies that $\zeta_r(0,t) = 0$ for all time and the contact angle remains at $\pi/2$ for all time.

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