Studying evolution of periodic wave regimes on a falling down liquid film

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Abstract. The numerical simulation of wavy regimes of the freely-falling liquid film was realized using the previously discovered symmetry of model equations. The calculation results are presented for the dynamics of amplitudes of significant harmonics and for the wave surface evolution. The paper has resulted in finding a specific regime, characterized by an alternate predominance of the first and second harmonics. At certain values of parameters this limiting cycle may have a very large period, during which the solution is attracted to the quasi-steady-state traveling regime, then suddenly transforms, and again tends to the quasi-steady-state with a phase shift of a half wave length.

1. Introduction
The flow of a thin layer of viscous liquid (film) over an inclined solid plane has been studied for over a hundred years [1]. Until the middle of the last century, the interest in the falling films was almost exclusively utilitarian. In 1941, Friedman and Miller [2] noted: “In the general field of diffusional processes, including absorption, extraction, heat transfer, humidification, and distillation, the flow of thin liquid films is often encountered”. In accordance with practical problems, researches were mainly experimental in nature and were focused on a turbulent flow regime. Precision measurements of the fluid velocity on the surface of very thin laminar liquid layers freely flowing over a vertical tube [2] have revealed its significant exceedence over the velocity calculated with the trivial Nusselt solution for a smooth film. A qualitative explanation of this consisted in the necessity to take into account wave formation on the liquid surface.

The impetus to a theoretical study of wave flows of liquid films was given in a series of brilliant studies of Kapitza [3]. Notable photos of the film surface showed the existence of many regular wave flow regimes that were much more stable than the plane-parallel Nusselt flow. The sustained efforts of subsequent researchers to describe such regimes have been rewarded by the profound results that reveal the general nature of the hydrodynamic instability and have direct application in various fields of physics of continuous media. It is suffice to mention the widely known equation of Kuramoto-Sivashinskii, derived for the first time to simulate the very wave evolution on the liquid film surface [4]. The interaction of effects of long-wave pumping, short-wave dissipation and perturbation nonlinearity (each of them is expressed by only one term of the K-S equation) leads to an unusually complex pattern of steady-state traveling wave modes, formed due to the instability of the trivial solution, as well as to an extremely complex dynamics of perturbations, demonstrating main features of the spatial-temporal deterministic chaos.
The full formulation of the problem of the wavy film flow includes the Navier – Stokes and the continuity equations with appropriate kinematic and dynamic boundary conditions. The major problem is the uncertainty of the moving boundary position, which is determined in the course of solving. If the effects of droplet entrainment and solid surface drying are excluded from consideration, the fluid flow area turns out to be simply connected. The presence of surface tension ensures the absence of sharp edges on the film surface. Under these conditions, the function determining the surface position is often unambiguous. Then, there exists a continuously differentiable coordinate transformation, mapping the area of the fluid flow on a band of constant thickness [5]:

\[
x = x, \quad \eta = y/h(x, t) - 1, \quad t = t.
\]

Governing equations of fluid dynamics were written in new coordinates in the following dimensionless form [6]:

\[
\begin{align*}
\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left( \frac{Q^2}{h} \right) + \frac{\partial}{\partial \eta} \left( \frac{QV}{h} \right) &= \frac{1}{\varepsilon \text{Re}} \frac{\partial^2 Q}{\partial \eta^2} + \frac{3h}{\varepsilon \text{Re}} + \frac{18}{5} \frac{\partial^3 h}{\partial x^3}, \\
\frac{\partial h}{\partial t} + \frac{\partial Q}{\partial x} + \frac{\partial V}{\partial \eta} &= 0, \\
Q = V = 0, & \text{ at } \eta = -1, \\
\frac{\partial Q}{\partial \eta} = 0, & V = 0, \text{ at } \eta = 0.
\end{align*}
\]

Here, \( h \) is a dimensionless film thickness, \( Q = hu, V = hv \), where \( u \) and \( v \) are dimensionless longitudinal and transversal contravariant components of velocity vector, \( \text{Re} = gh_0^3/3\nu^2 \) is the Reynolds number, and \( \varepsilon = h_0/l_0 \) is the wavelength parameter. In derived equations, perturbations are assumed to have long wavelengths \( \varepsilon \ll 1 \), and liquid flow rates are supposed to be moderate \( \varepsilon \text{Re} \sim 1 \). The selected characteristic scales are: the average thickness \( h_0 \) and the superficial velocity of a waveless Nusselt flow \( u_0 = gh_0^2/3\nu \). The scales of the length \( l_0 \) and time \( l_0/u_0 \) are determined from the relationship:

\[
h_0 \frac{l_0}{l_0} = \varepsilon = \sqrt{\frac{18 \text{Re}^{5/3}}{5 \left(3^{1/3} \text{Fi}^{1/3}\right)^{5/3}}},
\]

where \( \text{Fi} = \sigma^3/\rho^2g^4 \) is the film number, characterizing liquid properties. As it is shown in [7], this choice provides the value of neutral wave number \( \alpha_n \approx 1 \).

2. Calculation method

The problem (2) is solved by the pseudospectral (collocation) method. Functions depending on transverse coordinate \( \eta \) are expanded in a series on Chebyshev polynomials \( T_i \). The paper [7] showed that at moderate values of Reynolds numbers the steady-state traveling solutions of the problem (2) in the area extended over the transverse coordinate \( \eta \in [-1, 1] \) are characterized by the following symmetry:

\[
u(x, \eta, t) = -v(x, -\eta, t), \quad v(x, \eta, t) = -v(x, -\eta, t).
\]

Therefore, in this range one may choose only even Chebyshev polynomials for the function
Q, and only odd ones for the function $V$:

$$Q(x, \eta, t) = \sum_{j=1}^{M} Q_{2j}(x, t) \left( T_{2j}(\eta) - 1 \right) \equiv \sum_{j=1}^{M} Q_{2j}(x, t) \bar{T}_{2j}(\eta),$$

$$V(x, \eta, t) = \sum_{j=1}^{M} V_{2j+1}(x, t) \left( T_{2j+1}(\eta) - \eta \right) \equiv \sum_{j=1}^{M} V_{2j+1}(x, t) \bar{T}_{2j+1}(\eta).$$

(3)

Here, $M$ is a number of Chebyshev nodes.

The choice (3) provides an automatic satisfaction of boundary conditions on the free surface and on the solid wall.

So, for a set of Chebyshev nodes:

$$\eta_l = -\cos\left( \frac{l\pi}{2(M+1)} \right), \quad l = 1, ..., M,$$

we obtain the set of equations:

$$\sum_{j=1}^{M} \frac{\partial Q_{2j}}{\partial t} \bar{T}_{2j}(\eta_l) = \frac{1}{\varepsilon \text{Re} h^2} \sum_{j=1}^{M} Q_{2j} \bar{T}'_{2j}(\eta_l) + \frac{18}{5} \frac{\partial^3 h}{\partial x^3} + \frac{3}{\varepsilon \text{Re} h} -$$

$$- \sum_{j=1}^{M} \sum_{k=1}^{M} \frac{\partial}{\partial x} \left( \frac{Q_{2j} Q_{2k}}{h} \right) \bar{T}_{2j}(\eta_l) \bar{T}_{2k}(\eta_l) - \frac{1}{h} \sum_{j=1}^{M} \sum_{k=1}^{M} Q_{2j} V_{2k+1} \left( \bar{T}_{2j} \bar{T}_{2k+1} \right)'(\eta_l),$$

$$\frac{\partial h}{\partial t} + \sum_{j=1}^{M} \frac{\partial Q_{2j}}{\partial x} \bar{T}_{2j}(\eta_l) + \sum_{j=1}^{M} V_{2j+1} \bar{T}'_{2j+1}(\eta_l) = 0$$

(4)

Here apostrophe denotes a derivative with respect to $\eta$. Functions that depend on the longitudinal coordinate were expanded into spatial Fourier series:

$$[Q_{2j}, V_{2j+1}, h](x, t) = \sum_{k=-\infty}^{\infty} [Q_{2j,k}, V_{2j+1,k}, h_k](t) e^{i\alpha kx}. \quad (5)$$

Here $\alpha$ is the wave number of the periodic solution, $i$ is the imaginary unit. The functions $Q$, $V$ and $H$ are real-valued, so $Q_{2j,-k} = \bar{Q}_{2j,k}$, $V_{2j+1,-k} = \bar{V}_{2j+1,k}$, $H_{-k} = \bar{H}_k$. The overline denotes the operation of complex conjugation.

Substituting representations (5) in (4) and limiting to finite numbers of harmonics we obtain the system of ordinary differential equations which was solved by the Runge-Kutta method of the 4-th order.

3. The calculation results

As it is known, the plane-parallel Nusselt flow is unstable relative to linear periodic perturbations with wave number $\alpha < \alpha_n \approx 1$. The dynamics of small perturbations with $\alpha > \alpha_n$ is trivial: they exponentially attenuate with time. Amplitudes of unstable perturbations with wave numbers close to a neutral wave number first exponentially grow. The first harmonic of such perturbations obtains energy from the main flow, and due to nonlinear effects passes it to the second and higher harmonics, which wave number is located in the stable area and in such a way providing energy dissipation. As a result, the steady-state traveling mode is usually formed quickly. According
to numerical experiments, such a behavior is specific for the majority of perturbations from the range $0.5 < \alpha < 1$. Farther to the area of instability, the second harmonic (its wave number is $2\alpha$) also turns out to be unstable. This means that at certain conditions during evolution there is a possibility of appearance of quasi-steady-state states with predominance of only even harmonics. Such a state may be interpreted as a steady-state-traveling mode with a double wave number $2\alpha$, with imposed perturbations with nonzero odd harmonics (such perturbations have a double spatial period and a wave number $\alpha$). The research shows that at $0.41 < \alpha < 0.5$ this mode (the wave number $2\alpha$) is linearly unstable relative to such perturbations, so the evolution turns out to be more complicated.

The research numerically considers the dynamics of periodic perturbations of free surface ($|H_1| \neq 0$ and $|H_2| \neq 0$) in the range of wave numbers $0.41 < \alpha < 0.5$ and Reynolds numbers $1 < Re < 30$. Typical scenarios of the evolution of perturbations with small initial amplitude are presented in figure 1. Figure 1 presents the evolution of amplitudes of the first two harmonics ($H_1 = |H_1|e^{i\phi}, H_2 = |H_2|e^{i\psi}$) at the value $\alpha = 0.42$. Here, there are dependences of moduli of harmonics’ amplitudes (fig. 1a) and the “phase difference” $\phi = 2\phi - \psi$ (fig. 1b). It is seen that

![Figure 1. Evolution of the modulus of harmonics’ amplitudes (upper) and phase difference (lower) at $\alpha = 0.42$.](image-url)
Figure 2. Evolution of the amplitude of the first (line 1) and second (line 2) harmonics.

These functions change periodically. During a substantial part of the period, the amplitude of the second (and as the calculation has shown, the first-born even harmonics) are close to the values that correspond to the steady-state traveling solution with a double wave number. At that, the amplitude of the first and other odd harmonics exponentially grows. At the same time, there is an increase in nonlinear effects, the foremost of which turns out to be the energy transfer from the first to the second harmonic. As a result, the amplitude of the second harmonic passes through null (and its phase immediately changes to $\pi$), soon after that the growth of the first harmonic stops, and now the second harmonic grows for the account of the first harmonic.

As it is seen from fig. 1b, during the growth of the first harmonic the "phase difference" is negative and with good precision is equal to $-\pi/2$. Until $\phi > 0$ the first harmonic rapidly decreases, and the second one grows.

To better understand such pulsation regimes we will carry out their analysis on the basis of the simplified model of a film flow at small Reynolds numbers (equation of Nepomnyashchy):

$$H_t + 4HH_x + H_{xx} + H_{xxxx} = 0 \quad (6)$$

For these equations an interesting class of solutions was found in [8]. In them, such a "pulsating" character is more pronounced than in solutions of the equations (2). In case of initial perturbations evolution with $H_1 \neq 0$, $H_2 \neq 0$ and dimensionless wave numbers $0.41 < \alpha < 0.5$, an analogous quasiperiodical regime is set. An example of the evolution of moduli of the first two harmonics’ amplitudes is given in Fig. 2($\alpha = 0.45$). It may be seen that after the onset the amplitude of the first harmonic tends to null, whereas the amplitude of the second harmonic reaches some constant value. The system remains in such a state for quite a long time (time range 50 - 120). Then, for a short period of time, the flow parameters drastically change (time range 120 - 150), and again a quasi-stationary regime becomes established.

Presenting the solution to equation (6) as Fourier series, obtain an infinite system of ordinary differential equations for Fourier harmonics.

$$\dot{H}_k - (\alpha^2 - \alpha^4)H_k + 2i\alpha k \sum H_{k'}H_{k-k'} = 0 \quad (7)$$
Supposing that all harmoinics except for \( H_1, H_{-1}, H_2, H_{-2} \) equal null, obtain a low-dimensional approximation of the equations (7):

\[
\begin{align*}
\dot{H}_1 - (\alpha^2 - \alpha^4)H_1 + 2i\alpha H_2 H_{-1} &= 0 \\
\dot{H}_{-1} - (\alpha^2 - \alpha^4)H_{-1} - 2i\alpha H_1 H_{-2} &= 0
\end{align*}
\]  

(8)

Seeking for the solution to the equations (8) in the form:

\[
H_1 = |H_1(t)|e^{i\phi(t)}, \quad H_2 = H_{-2} = \text{const} > 0
\]  

(9)

In this case, the ”phase difference” is \( \phi = 2\phi_0 \), i.e. coincides with the double phase of the first harmonic. The equations (8) taking into account (9) is presented in the form:

\[
\begin{align*}
|H_1| - (\alpha^2 - \alpha^4)|H_1| + 2\alpha H_2|H_1|\sin(\phi) &= 0 \\
\phi + 4\alpha H_2 \cos(\phi) &= 0
\end{align*}
\]  

(10)

From the equation (10) it is seen that at small values of \( H_2 \) the first harmonic \( H_1 \) will always be growing. At rather large value of \( H_2 \), the function \( H_1(t) \) may become decreasing. From equation (11) it may be obvious that at large times the ”phase difference” \( \phi \to -\pi/2 \), and such a regime will be unstable that is seen from equation (10). The change in sign \( H_2 \) will lead to \( \phi \to \pi/2 \). The formed quasi-steady regime differs in phase from the previous one by \( \pi \). This means that wave maxima have changed to minima and vice versa.

4. Conclusion

The numerical simulation of wavy regimes of the freely-falling liquid film was realized using the previously discovered symmetry of model equations. The calculation results are presented for the dynamics of amplitudes of significant harmonics and for the wave surface evolution. The study has resulted in finding a specific regime, characterized by an alternate predominance of the first and second harmonics. At certain values of parameters this limiting cycle may have a very large period, during which the solution tends to the quasi-steady-state traveling regime, then suddenly transforms, and again is attracted to the quasi-steady-state with a phase shift of a half wave length.

Acknowledgments

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