ON D-MODULES OF CATEGORIES I

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1. INTRODUCTION

A theorem of Goodwillie [11] shows that the periodic cyclic homology has homotopy invariance in characteristic zero. Getzler [9] constructs, under a suitable condition, a flat connection on the relative periodic cyclic homology of an associative algebra over a base commutative ring of characteristic zero. The connection can be thought of as a noncommutative analogue of Gauss-Manin connection.

Morita invariant property is of fundamental importance in noncommutative algebraic geometry. Unfortunately, to the knowledge of the author, Morita invariance of the original connection remains open. Nowadays, in view of several motivations including homological mirror symmetry, the significance of Morita invariance of the associated connection becomes clear. Equivariant contexts (that is, categories with group actions) naturally arise from interesting situations: for example matrix factorizations [22], schemes with group actions, etc. However, explicit formulas often destroy symmetries, and it is ill-suited for equivariant contexts.

Let us move to the world of higher categories. In the present paper, we give constructions of D-module structures on the periodic cyclic homology of a stable ∞-category. The theory is applicable to equivariant contexts and has favorable features such as Morita invariance. Also, a generalization from the setting of algebras to stable ∞-categories or the likes is useful. For example, it enables us to use exact sequences associated to exact sequences of categories. Many operations for ∞-categories are intolerant of explicit methods so that they often require conceptual or geometric understandings. We need to give a conceptual way of constructing connections.

We propose two different methods. Here is a brief overview of them.

(I) Canonical extensions of factorization homology to mapping stacks. The first approach is based on a simple observation on factorization homology. Let M be a symmetric monoidal ∞-category satisfying some condition on colimits. Factorization homology has been extensively developed in last decade, see [1], [2], [3], [13], [19] for the foundation. We also refer to the survey [10]. The factorization homology theory takes a manifold M and a structured algebra B such as an E_n-algebra in M as input and produces an object of M as output. We would like to explain the observation. Let k → A be a map of connective commutative ring spectra and let B be an E_n-algebra over A. Given a framed n-manifold M, we have the factorization homology \( \int_M B/A \), that is an A-module (spectrum). Now let us regard B as an E_n-algebra over k through the restriction along k → A. It gives rise to the factorization homology \( \int_M B/k \), that is a k-module. The point is that \( \int_M B/k \) is promoted to a module spectrum over \( A \otimes_k M \) and there is a canonical equivalence

\[
(\int_M B/k) \otimes_{A \otimes_k M} A \simeq \int_M B/A.
\]

The symbol A \( \otimes_k M \) means the tensor of A by the underlying topological space of M as a commutative algebra over k. Since A \( \otimes_k M \) is the ring of functions of the mapping stack Map(M, Spec A) in the sense of derived geometry, we may interpret \( \int_M B/k \) as an extension/deformation of \( \int_M B/A \) on S = Spec A to Map(M, S). Let us consider the special case when M = S^1 and B is an associative algebra (E_1-algebra) over A. In this case, \( \int_{S^1} B/A \) is the Hochschild homology spectrum \( \mathcal{H}_\bullet(B/A) \). If we replace B with an A-linear stable ∞-category C, the observation can naturally be extended to C. The work of Ben-Zvi and Nadler [4] and Preygel [21] relates equivariant complexes on the free loop space LS = Map(S^1, S) and 2-periodic D-modules in characteristic zero. Using this bridge, we associate a D-module structure on the periodic cyclic homology/complex \( \mathcal{H}_\bullet(C/A) \).
This approach is simple and easy. The extension to the free loop space may be understood as a simple case of the elementary observation about factorization homology. Moreover, using the extended object defined on the free loop space and the Koszul duality we can study the resulting $D$-module, but we defer the study to subsequent papers.

(II) Hochschild pair and moduli theory. Suppose that $k$ is a field of characteristic zero and $A$ is a smooth algebra over $k$. Let $C$ be an $A$-linear stable $\infty$-category. The second approach uses the Hochschild pair, that is, the pair of the Hochschild cochain complex $HH^\bullet(C/A)$ and the Hochschild chain complex $\mathcal{H}C_\bullet(C/A)$. Besides, the construction makes use of the local moduli (deformation) theory related to the Hochschild pair [15] and the theory of correspondence between pointed formal stacks and dg Lie algebras developed in [8], [12], [20, X]. The interplay of deformation functors, formal stacks, and dg Lie algebras plays a central role. In [14] for an $A$-linear stable $\infty$-category we give a conceptual construction of an algebra structure on the Hochschild pair $(HH^\bullet(C/A), HH_\bullet(C/A))$, encoded by means of so-called Kontsevich-Soibelman operad $\mathbf{KS}$. Moreover, in [15] we give a moduli-theoretic interpretation of $(HH^\bullet(C/A), HH_\bullet(C/A))$. Using them we extend $HH_\bullet(C/A)$ to an $S^1$-equivariant (Ind-coherent) complex on the loop space $LS$ (so that as above we can apply the relation of loop spaces and $D$-modules to it). In order to achieve this, we also construct a Kodaira-Spencer morphism for $C$ (where $C$ is regarded as a family of stable $\infty$-categories over Spec $A$). Our construction reveals how $(HH^\bullet(C/A), HH_\bullet(C/A))$ together with the Kodaira-Spencer morphism yields a promotion of the Hochschild homology $HH_\bullet(C/A)$ to the free loop space $LS$. Consider $LS$ to be a pointed formal stack $S \to LS \to S$ over $S = \text{Spec } A$, where $S \to LS$ is induced by constant loops. According to the correspondence between pointed formal stacks and dg Lie algebras, there is an essentially unique dg Lie algebra $L$ corresponding to $LS$. Moreover, a Lie algebra module over $L$ amounts to an Ind-coherent complex on $LS$. At a first glimpse, it seems like a good idea to have a Lie algebra action of $L$ on $HH_\bullet(C/A)$. However, $LS \to S$ does not commute with the natural $S^1$-action on $LS$. As a result, there is no way to describe an $S^1$-equivariant Ind-coherent complex on $LS$ as a Lie algebra module over $L$. Instead of $LS$, we consider a pointed formal stack $S \to S \times LS \xrightarrow{\ell} S$ which is a completion along the graph of $S \to LS$ and admits an $S^1$-action induced by that of $LS$ (actually, there are two versions of completions $S \times LS$ and $(S \times LS)^\wedge$, but in this introduction we do not distinguish them). We use a method of constructing complexes on the formal stacks in the diagram $S \xrightarrow{\ell} S \times S \to S \times LS$ in a compatible way.

The associated complex on $LS$ has a direct link to the structure arising from $(HH^\bullet(C/A), HH_\bullet(C/A))$ which are described as Lie algebra actions.

We introduced two different approaches. Both have their own features. The question naturally arises: what is the relationship between the two approaches? In a subsequent paper, we will prove that the associated two $D$-modules coincide. Actually, we will show that two promotions of $HH_\bullet(C/A)$ to the free loop spaces coincide.

Related works. Getzler’s method on connections on periodic cyclic homology uses Cartan homotopy/magic formula arising from the algebraic structure on the Hochschild cochain complex and the Hochschild chain complex. Our second approach also relies on Cartan homotopy/magic formula. In that sense, one may say shortly that the second approach is a generalization of Getzler’s original approach. On the other hand, closer inspection reveals that the strategy of the second approach differs quite fundamentally from Getzler’s one. Notwithstanding, both possess the advantage of having access to the algebraic structure on the Hochschild pair. Our construction uses methods which Getzler’s one does not use. The algebra $(HH^\bullet(C/A), HH_\bullet(C/A))$ over $\mathbf{KS}$ which is constructed in [14] has Morita invariance in the sense that an equivalence $C \simeq C'$ induces an equivalence of Hochschild pairs with algebraic structures. As mentioned above, we apply the moduli-theoretic description of $(HH^\bullet(C/A), HH_\bullet(C/A))$ established in [15] and the relation between free loop spaces and $D$-modules. These machineries allow us to obtain the bridge between $(HH^\bullet(C/A), HH_\bullet(C/A))$ and the geometry of the free loop space, and the $D$-module of the periodic cyclic homology/complex.

Recently, Hoyois, Safronov, Scherotzke and Sibilla develop the theory of categorified Chern character and categorified Grothendieck-Riemann-Roch theorem and apply it to produce a $D$-module structure on the periodic cyclic homology/complex [24, Section 6]. While they also use the free loop spaces, their work is based on different ideas and techniques so that it would be interesting to compare it with ours.
Organization. Let us give some instructions to the reader. Section 2 collects conventions and some of the notations that we will use. In Section 3, we will review some of background material relevant to this paper. In Section 4, we carry out the first construction (I): we make an observation about factorizations homology. The results are presented in Theorem 4.2, Theorem 4.8 and Theorem 4.12. Most of Sections 5–7 is devoted to the second construction (II). Section 5 contains the construction of Kodaira-Spencer morphisms. Section 6 contains a main procedure in (II). The main conclusion in Section 7 is Theorem 7.14. Section 4 and Sections 5–7 are logically independent from one another. The reader can read a preferred one. In Section 8, we apply the relation between free loop spaces and 2-periodic $D$-modules to show how to obtain $D$-module of periodic cyclic homology from equivariant complexes constructed in Section 4 and Section 5–7 (see Theorem 8.4).

Acknowledgements. The author would like to thank Takuo Matsuoka for valuable comments and advice on Section 4 and many discussions since the summer of 2017. The topic of this paper is included in the contents of the graduate course offered in fall term 2020 at Tohoku University. He also thanks students for their feedbacks. This work is supported by JSPS KAKENHI grant.

2. Notation and Convention

$(\infty, 1)$-categories. Throughout this paper we use the language of $(\infty, 1)$-categories. We use the theory of quasi-categories as a model of $(\infty, 1)$-categories. We assume that the reader is familiar with this theory. We will use the notation similar to that used in [14], [15]. A quasi-category is a simplicial set which satisfies the weak Kan condition of Boardman-Vogt. Following [18], we shall refer to quasi-categories as $\infty$-categories. Our main references are [18] and [19]. To an ordinary category, we can assign an $\infty$-category by taking its nerve, and therefore when we treat ordinary categories we often omit the nerve $N(\cdot)$ and directly regard them as $\infty$-categories.

Here is a list of (some) of the conventions and notation that we will use:

- $\Delta^n$: the standard $n$-simplex
- $N$: the simplicial nerve functor (cf. [18, 1.1.5])
- $S$: $\infty$-category of small spaces/$\infty$-groupoids. We denote by $\hat{S}$ the $\infty$-category of large spaces (cf. [18, 1.2.16]).
- $C^\geq$: the largest Kan subcomplex of an $\infty$-category $C$. Namely, $C^\geq$ is the largest $\infty$-groupoid contained in $C$.
- $C^{\mathsf{op}}$: the opposite $\infty$-category of an $\infty$-category. We also use the superscript “$\mathsf{op}$” to indicate the opposite category for ordinary categories and enriched categories.
- $S$: the stable $\infty$-category of spectra.
- $\mathsf{Fun}(A, B)$: the function complex for simplicial sets $A$ and $B$. If $A$ and $B$ are $\infty$-categories, we regard $\mathsf{Fun}(A, B)$ as the functor category.
- $\mathsf{Map}_C(C, C')$: the mapping space from an object $C \in C$ to $C' \in C$ where $C$ is an $\infty$-category. We usually view it as an object in $S$ (cf. [18, 1.2.2]).

Operads and Algebras. We will use operads. We employ the theory of $\infty$-operads which is thoroughly developed in [19]. The notion of $\infty$-operads gives one of the models of colored operads. Here is a list of (some) of the notation about $\infty$-operads and algebras over them that we will use:

- $\mathsf{Fin}_*$: the category of pointed finite sets $(0), (1), \ldots, (n), \ldots$ where $(n) = \{*, 1, \ldots, n\}$ with the base point $*$. We write $\Gamma$ for $N(\mathsf{Fin}_*)$. $(n)^0 = (n) \setminus \{\}$. Notice that the (nerve of) Segal’s gamma category is the opposite category of our $\Gamma$.
- Let $\mathcal{M}^\otimes \to O^\otimes$ be a fibration of $\infty$-operads. We denote by $\mathsf{Alg}_{/O^\otimes}(\mathcal{M}^\otimes)$ the $\infty$-category of algebra objects (cf. [19, 2.1.3.1]). We often write $\mathsf{Alg}_{/O^\otimes}(\mathcal{M}^\otimes)$.
- $\mathsf{CAlg}(\mathcal{M}^\otimes)$: $\infty$-category of commutative algebra objects in a symmetric monoidal $\infty$-category $\mathcal{M}^\otimes \to N(\mathsf{Fin}_*) = \Gamma$. When the symmetric monoidal structure is clear, we usually write $\mathsf{CAlg}(\mathcal{M}^\otimes)$ for $\mathsf{CAlg}(\mathcal{M}^\otimes)$.
- $\mathsf{Mod}_R^\otimes(\mathcal{M}^\otimes)$: the symmetric monoidal $\infty$-category of $R$-module objects where $\mathcal{M}^\otimes$ is a symmetric monoidal $\infty$-category. Here $R$ belongs to $\mathsf{CAlg}(\mathcal{M}^\otimes)$ cf. [19, 3.3.3, 4.5.2]. We write $\mathsf{Mod}_R(\mathcal{M}^\otimes)$ for the underlying $\infty$-category.
• Mod$_R$: Suppose that $R$ is a commutative ring spectrum, i.e., a commutative algebra object in Sp. Unless stated otherwise, we write Mod$_R^\infty$ for Mod$_R^\infty(Sp)$, that is, the symmetric monoidal ∞-category of $R$-module spectra. By Mod$_R$ we mean the underlying ∞-category. When $R$ is the Eilenberg-MacLane spectrum $HC$ of an ordinary commutative ring $C$, we write Mod$_C$ for Mod$_R$ (thus Mod$_C$ is not the category of usual $C$-modules). If $D^\otimes(C)$ denotes the symmetric monoidal stable ∞-category obtained from the category of (possibly unbounded) chain complexes of $C$-modules by inverting quasi-isomorphisms, there is a canonical equivalence Mod$_C^\infty \simeq D^\otimes(C)$ of symmetric monoidal ∞-categories, see [19, 7.1.2.7.1.2.13] for more details. Let $A$ be an object of CAlg(Mod$_R$) (see below), and let $A' \in$ CAlg(Sp) be the image of $A$ under the forgetful functor Mod$_R \to$ Sp. Then the induced functor Mod$_A$ (Mod$_R$) → Mod$_A$. (Sp) = Mod$_A$ is an equivalence. By abuse of notation, we usually write Mod$_A$ for Mod$_A$ (Mod$_R$).

• CAlg$_n$: the ∞-category CAlg(Mod$_R^\infty$) of commutative algebra objects in the symmetric monoidal ∞-category Mod$_R^\infty$ where $R$ is a commutative ring spectrum. We write CAlg$_R^+$ for (CAlg$_R^\infty$) $\simeq$ CAlg(Sp)$_R/R$. When $R$ is the Eilenberg-MacLane spectrum $HC$ with a commutative ring $C$, then we write CAlg$_C$ for CAlg$_HC$. If $C$ is an ordinary commutative ring over a field $k$ of characteristic zero, the ∞-category CAlg$_C = CAlg(Mod_C^\infty)$ is equivalent to the ∞-category obtained from the model category of commutative differential graded $C$-algebras by inverting quasi-isomorphisms (cf. [19, 7.1.4.11]). Therefore, we often regard an object of CAlg$_C$ as a commutative differential graded (dg) algebra over $C$. We shall refer to an object of CAlg$_C$ as a commutative dg algebra over $C$.

Suppose that $A \in$ CAlg$_k$ is a connective commutative dg algebra over a field $k$ of characteristic zero (a connective commutative dg algebra is a commutative dg algebra $A$ such that $H^n(A) = \pi_n(A) = 0$ for $i > 0$). Let Mod$_k^\infty$ be the full subcategory of Mod$_k$ spanned by connective objects (those objects $M$ such that $H^n(M) = 0$ for $i > 0$). We regard $A$ as an object of CAlg(Mod$_k^\infty$) and set CAlg$_A^\infty = CAlg(Mod_A(Mod_k^\infty))$. The symmetric monoidal fully faithful functor Mod$_k^\infty \to$ Mod$_k$ exhibits CAlg$_A^\infty = CAlg(Mod_A(Mod_k^\infty)))$ as a full subcategory of CAlg(Mod$_A$) = CAlg$_A$.

• $E^n_m$: the ∞-operad of little $n$-cubes (cf. [19, 5.1]). For a symmetric monoidal ∞-category $C^\otimes$, we write Alg$_n(C)$ or Alg$_{E^n_m}(C)$ for the ∞-category of algebra objects over $E^n_m$ in $C^\otimes$. We refer to an object of Alg$_n(C)$ as an $E_n$-algebra in $C$. If we denote by Ass$^\otimes$ the associative operad ([19, 4.1.1]), there is the standard equivalence Ass$^\otimes \simeq E^n_{1}$ of ∞-operads. We usually identify Alg$_1(C)$ with the ∞-category Alg$_{Ass}(C)$, that is, the ∞-category of associative algebras in $C$. We write Alg$_n^+(C)$ for the ∞-category Alg$_n(C)/_{1C}$ of augmented objects where $1_C$ denotes the unit algebra.

• $LM^\otimes$: the ∞-operad defined in [19, 4.2.1.7]. Roughly, an algebra over $LM^\otimes$ is a pair $(A, M)$ such that $A$ is an unital associative algebra and $M$ is a left $A$-module. For a symmetric monoidal ∞-category $C^\otimes \to \Gamma$, we write LMod($C^\otimes$) or LMod($C$) for Alg$_{LM^\otimes}(C^\otimes)$. There is the natural inclusion of ∞-operads Ass$^\otimes \to LM^\otimes$. This inclusion determines LMod($C$) → Alg$_{Ass}(C)$ $\simeq$ Alg$_1(C)$ which sends $(A, M)$ to $A$. For $A \in$ Alg$_1(C)$, we define LMod$_A(C)$ to be the fiber of LMod($C$) → Alg$_1(C)$ over $A$ in Cat$^\otimes$.

• $RM^\otimes$: these ∞-operads are variants of $LM^\otimes$ which are used to define structures of right modules over associative algebras [19, 4.2.1.36]. Informally, an algebra over $RM^\otimes$ is a pair $(A, M)$ such that $A$ is an unital associative algebra and $M$ is a right $A$-module. For a symmetric monoidal ∞-category $C^\otimes \to \Gamma$, we write RMod($C^\otimes$) (or simply RMod($C$)) for Alg$_{RM^\otimes}(C^\otimes)$.

• Unless otherwise stated, $k$ is a base field of characteristic zero.

Group actions. Let $G$ be a group object in $S$ (see e.g. [18, 7.2.2.1] for the notion of group objects). The main example in this paper is the circle $S^1$. Let $\mathcal{C}$ be an ∞-category. For an object $C \in \mathcal{C}$, a $G$-action on $C$ means a lift of $C \in \mathcal{C}$ to Fun$(BG, \mathcal{C})$, where $BG$ is the classifying space of $G$. A $G$-equivariant morphism means a morphism in Fun$(BG, \mathcal{C})$. We often identify Fun$(BG, \mathcal{C})$ as the limit (“$G$-invariants”/homotopy fixed points) of the trivial $G$-action on $C$ and write $C^G$ for Fun$(BG, \mathcal{C})$ (e.g., Mod$_A^{S^1} = \text{Fun}(BS^1, \text{Mod}_A)$). We remark that when we regard $G$ as a group object, $C^G$ is not the cotensor with the space $G$.
In the text, $G_{S}^{A}$ and $T_{A}^{B}[-1]$ are cotensors by $S^{A}$ in $\text{Lie}_{A}$, while $\text{End}^{R}(\mathcal{H}_{\bullet}(C/A))^{S}$ means the homotopy fixed points. We hope that these symbols may not be confusing.

**Stable $\infty$-categories.** We recall some formulations of $\infty$-categories of stable $\infty$-categoriess, see [15, Section 3.1], [5, Section 3], [19, 4.8] for details. Let $\text{St}$ be the $\infty$-category of small stable idempotent-complete $\infty$-categories whose morphisms are exact functors. This $\infty$-category is compactly generated. Let $\mathcal{C}$ be a small stable idempotent-complete $\infty$-category and let $\text{Ind}(\mathcal{C})$ denote the $\infty$-category of Ind-objects [18, 5.3.5]. Then $\text{Ind}(\mathcal{C})$ is a compactly generated stable $\infty$-category. The inclusion $\mathcal{C} \to \text{Ind}(\mathcal{C})$ identifies the essential image with the full subcategory $\text{Ind}(\mathcal{C})^{\circ}$ spanned by compact objects in $\text{Ind}(\mathcal{C})$.

We let $\text{Pr}^{L}$ denote the $\infty$-category of presentable $\infty$-categories such that mapping spaces are spaces of functors which preserve small colimits (i.e., left adjoint functors) [18, 5.5.3]. It has a closed symmetric monoidal structure whose internal Hom/mapping objects are given by the functor category $\text{Fun}^{L}(-,-)$ of left adjoint functors, see [19, 4.8.1.15]. The $\infty$-category $\text{St}$ also admits a closed symmetric monoidal structure whose internal Hom/mapping objects are given by the functor category $\text{Fun}^{\infty}(-,-)$ of exact functors. The stable $\infty$-category of compact spectra is a unit object in $\text{St}$. We set $\text{Pr}^{L}_{\text{St}} = \text{Mod}_{\text{Sp}}^{\infty}(\text{Pr}^{L})$, which can be regarded as the full subcategory of $\text{Pr}^{L}$ that consists of stable presentable $\infty$-categories. The Ind-construction $\mathcal{C} \mapsto \text{Ind}(\mathcal{C})$ determines a symmetric monoidal functor

$$\text{St} \longrightarrow \text{Pr}^{L}_{\text{St}}.$$  

Let $A \in \text{CAlg}(\text{Sp})$. Let $\text{Mod}^{\infty}_{A} \in \text{CAlg}(\text{Pr}^{L}_{\text{St}})$ be the symmetric monoidal $\infty$-category of $A$-modules in $\text{Sp}$. Let $\text{Perf}^{\infty}_{A} \in \text{CAlg}(\text{St})$ be the symmetric monoidal $\infty$-category of compact $A$-modules in $\text{Sp}$. Let $\text{Mod}^{\infty}_{\text{Perf}}(\text{St})$ be the symmetric monoidal $\infty$-category of $\text{Perf}^{\infty}_{A}$-modules in $\text{St}$. This symmetric monoidal $\infty$-category is presentable, and the tensor product functor preserves small colimits separately in each variable. We refer to an object of the underlying $\infty$-category $\text{Mod}^{\infty}_{\text{Perf}}(\text{St})$ as an $A$-linear small stable $\infty$-category. For ease of notation, put $\textbf{St}^{\infty}_{A} = \text{Mod}^{\infty}_{\text{Perf}}(\text{St})$ and $\textbf{St}_{A} := \text{Mod}^{\infty}_{\text{Perf}}(\text{St})$. We refer the reader to [14] for the description of $\textbf{St}^{\infty}_{A}$ by means of spectral categories. We write $\text{Pr}^{L}_{A}^{\infty}$ for $\text{Mod}^{\infty}_{\text{Mod}^{\infty}_{A}}(\text{Pr}^{L}_{\text{St}})$. We refer to an object of the underlying $\infty$-category $\text{Pr}^{L}_{A}$ as an $A$-linear stable presentable $\infty$-categories.

For $B \in \text{Alg}_{3}(\text{Mod}_{A})$, we denote by $\text{LMod}_{B}(\text{Mod}_{A})$ (resp. $\text{RMod}_{A}(\text{Mod}_{A})$) (or simply $\text{LMod}_{B}$ and (resp. $\text{RMod}_{A}$)) the $\infty$-category of left $B$-modules (resp. right $B$-module spectra) (there is a canonical equivalence $\text{LMod}_{B}(\text{Mod}_{A}) \simeq \text{LMod}_{B}(\text{Sp})$ induced by the forgetful functor $\text{Mod}_{A} \to \text{Sp}$).

### 3. Preliminaries

In this section, we review some of the theories we will use from the next section. While the prerequisite for Section 4.1 is Section 3.1 and 3.2, Section 5–7 needs also Section 3.3 and Section 3.4.

#### 3.1. Hochschild homology and Hochschild cohomology

Let $\text{St}_{A}$ denote $\text{Mod}^{\infty}_{\text{Perf}}(\text{St})$. Let

$$\mathcal{H}_{\bullet}(-/A) : \text{St}_{A} \longrightarrow \text{Mod}^{\infty}_{A} = \text{Fun}(BS^{1}, \text{Mod}_{A})$$

be the symmetric monoidal functor which carries $C \in \text{St}_{A}$ to the Hochschild homology $A$-module spectrum $\mathcal{H}_{\bullet}(C/A)$ over $A$. We refer the reader to [14, Section 6, 6.14] for the construction of $\mathcal{H}_{\bullet}(-/A)$ (in loc. cit. we use the symbol $\mathcal{H}_{\bullet}(C)$ instead of $\mathcal{H}_{\bullet}(C/A)$). Let $B \in \text{Alg}_{3}(\text{Mod}_{A})$. We denote by $\text{LMod}_{B} = \text{LMod}_{B}(\text{Mod}_{A})$ the stable $\infty$-category of left $B$-modules in $\text{Mod}_{A}$. Let $\text{Perf}_{B}$ be the smallest stable idempotent-complete subcategory of $\text{LMod}_{B}(\text{Mod}_{A})$ which contains $B$. We will write $\mathcal{H}_{\bullet}(B/A)$ for $\mathcal{H}_{\bullet}(\text{Perf}_{B}/A)$. If $B$ is an $E_{2}$-algebra in $\text{Mod}_{A}$, then $\text{LMod}_{B}$ has an associative monoidal structure. More precisely, $\text{LMod}_{B}$ lies in $\text{Alg}_{3}(\text{Pr}^{L}_{A})$. Moreover, $\text{Perf}_{B}$ is promoted to an object of $\text{Alg}_{3}(\text{St}_{A})$. Thus, the symmetric monoidal functor $\mathcal{H}_{\bullet}(-/A)$ gives $\mathcal{H}_{\bullet}(B/A)$ which belongs to $\text{Alg}_{3}(\text{Mod}^{S^{1}}_{A})$.

We review the Hochschild cohomology of $C \in \text{St}_{A}$. Let $\text{Ind}(\mathcal{C})$ be the Ind-category that belongs to $\text{Pr}^{L}_{A}$. Moreover, it is compactly generated. We denote by $\theta_{A} : \text{Alg}_{2}(\text{Mod}_{A}) \to \text{Alg}_{3}(\text{Pr}^{L}_{A})$ the functor informally given by $B \mapsto \text{LMod}_{B}^{\infty}$. By definition, the endomorphism algebra object $\text{End}_{A}(\text{Ind}(\mathcal{C})) \in \text{Alg}_{3}^{0}(\text{Pr}^{L}_{A})$ endowed with a tautological action on $\text{Ind}(\mathcal{C})$ is a final object of $\text{LMod}(\text{Pr}^{L}_{A}) \times_{\text{Pr}^{L}_{A}} \{\text{Ind}(\mathcal{C})\}$. To see this, note first
that there exists a right adjoint $\text{Alg}_1(\text{Pr}_3^A) \to \text{Alg}_2(\text{Mod}_A)$ of $\theta_A$, see [19, 4.8.5.11, 4.8.5.16]. Namely, there exists a adjoint pair

$$\theta_A : \text{Alg}_2(\text{Mod}_A) \xrightarrow{\text{affine}} \text{Alg}_1(\text{Pr}_3^A) : E_A$$

such that $E_A$ sends $\mathcal{M}^\otimes$ to the endomorphism algebra of the unit object $1_{\mathcal{A}}$. Then its final object is given by $E_A(\text{End}_{\mathcal{A}}(\text{Ind}(\mathcal{C}))) \in \text{Alg}_2(\text{Mod}_A)$ with the left module action of $\text{LMod}_{E_A(\text{End}_{\mathcal{A}}(\text{Ind}(\mathcal{C})))}$ on $\text{Ind}(\mathcal{C})$ determined by the counit map $\theta_A(E_A(\text{End}_{\mathcal{A}}(\text{Ind}(\mathcal{C})))) \to \text{End}_{\mathcal{A}}(\text{Ind}(\mathcal{C}))$. The Hochschild cohomology $\mathcal{H}\mathcal{H}^\bullet(\mathcal{C}/A)$ is defined to be $E_A(\text{End}_{\mathcal{A}}(\text{Ind}(\mathcal{C})))$. We refer to $\mathcal{H}\mathcal{H}^\bullet(\mathcal{C}/A)$ as the Hochschild cohomology of $\mathcal{C}$ (over $A$).

Though we use the word “homology” and “cohomology”, $\mathcal{H}\mathcal{H}^\bullet(\mathcal{C}/A)$ and $\mathcal{H}\mathcal{H}^\bullet(\mathcal{C}/A)$ are not graded modules obtained by passing to (co)homology but spectra (or chain complexes) with algebraic structures.

### 3.2. Derived schemes.

Let $A$ be a connective commutative dg algebra over $k$. We fix our convention on derived schemes. The references are [25], [8], [20]. We set $\text{Aff}_A = (\text{CAlg}_A^{\leq 0})^{\text{op}}$. We refer to an object of the functor category $\text{Fun}(\text{CAlg}_A^{\leq 0}, \hat{S})$ as a derived prestack or (simply) a functor. The $\infty$-category $\mathcal{C}_A$ of derived stacks is defined to be the full subcategory of $\text{Fun}(\text{CAlg}_A^{\leq 0}, \hat{S})$ which consists of derived prestacks satisfying the étale descent property. By the étale descent we mean the descent condition with respect to Cech nerves of étale coverings [8, I, Ch. 2, 2.3]. See e.g., [25] or [8, I, Ch. 2, 2.1] for flat, étale morphisms, open immersions, and related properties. By abuse of notation, we write $\text{Aff}_A$ also for the essential image of the Yoneda embedding $\text{Aff}_A \to \text{Fun}(\text{CAlg}_A^{\leq 0}, \hat{S})$. We denote by $\text{Spec} B$ the image of $B \in (\text{CAlg}_A^{\leq 0})^{\text{op}}$. Any representable prestack $\text{Spec} B$ satisfies the descent condition so that the essential image $\text{Aff}_A$ is contained in $\mathcal{C}_A$. If a stack $F$ is representable, we call $F$ a derived affine scheme. A derived scheme over $A$ is a stack $F$ which satisfies the following conditions:

- The diagonal morphism $F \to F \times F$ is affine schematic. That is, for any morphism $\text{Spec} B \to F \times F$, the fiber product $\text{Spec} B \times_{F \times F} F$ is represented by a derived affine scheme,

- There exists a set of derived affine schemes $\{T_i\}_{i \in I}$ and morphisms $\{\phi_i : T_i \to F\}_{i \in I}$ such that (i) each $\phi_i$ is an open immersion (i.e., for any Spec $B \to F$, the base change $T_i \times_F \text{Spec} B \to \text{Spec} B$ is an open immersion of derived affine schemes), and (ii) for any Spec $B \to F$ the set of base changes $\{T_i \times_F \text{Spec} B \to \text{Spec} B\}_{i \in I}$ is a family of open immersions which covers Spec $B$.

More precisely, this definition gives what we call derived schemes with affine diagonal.

**Mapping stacks.** Let $X$ be a derived scheme over $k$. Let $S$ be a space, that is, an object of $\mathcal{S}$. For simplicity, we suppose that $S$ is a pointed space; it has a map $\ast \to S$ from the contractible space $\ast$. We define the mapping stack $\text{Map}(S, X)$ (also called the mapping scheme). Let $J$ be the category (poset) of affine open sets in $X$. Let $\mathcal{O} : J^{\text{op}} \to \text{CAlg}(\text{Mod}_k)$ be the functor which is informally defined by the formula $\mathcal{O} J \ni U = \text{Spec} R \mapsto \Gamma(U, \mathcal{O}_X) = R \in \text{CAlg}(\text{Mod}_k)$ where $\mathcal{O}_X$ is the structure sheaf of $X$. Let $\otimes S : \text{CAlg}(\text{Mod}_k) \to \text{CAlg}(\text{Mod}_k)$ be the functor given by the tensor with $S \in \mathcal{S}$. Let $\mathcal{O} \otimes S : J \to \text{CAlg}(\text{Mod}_k)$ be the composite of $\mathcal{O}$ and $\otimes S$. The map $\ast \to S$ defines a natural transformation $\mathcal{O} \to \mathcal{O} \otimes S$. This natural transformation determines an object of $\text{CAlg}(\text{Fun}(J^{\text{op}}, \text{Mod}_k)) \otimes_{\mathcal{O}} \simeq \text{CAlg}(\text{Mod}_k(\text{Fun}(J^{\text{op}}, \text{Mod}_k)))$ (see [19, 3.4.1.7] for this equivalence). Let $\text{Mod}_k^{\text{op}(-)} : \text{CAlg}(\text{Mod}_k) \to \text{CAlg}(\text{Pr}_k)$ be the functor informally given by $R \mapsto \text{Mod}_k^{\text{op}(-)}$ (a morphism $R \to R'$ maps to a symmetric monoidal functor $\otimes R' : \text{Mod}_k \to \text{Mod}_k^{\text{op}(-)}$). The symmetric monoidal $\infty$-category of quasi-coherent complexes $\text{QC}(X)$ is defined to be a limit of $J^{\text{op}} \to \text{CAlg}(\text{Mod}_k) \xrightarrow{\text{op}(-)} \text{CAlg}(\text{Pr}_k)$. The symmetric monoidal $\infty$-category $\text{QC}(X)$ can be identified with a full subcategory of $\text{Mod}(\text{Fun}(J^{\text{op}}, \text{Mod}_k))$ that consists of those objects $M : J^{\text{op}} \to \text{Mod}_k$ such that for any morphism $\text{Spec} R' = V \to U = \text{Spec} R$ in $J$, the evaluation of $M$ at $V$ determines an equivalence $M(U) \otimes R' \to M(V)$. Let $\text{QC}(X)^{\leq 0} \subset \text{QC}(X)$ denote the full subcategory spanned by connective quasi-coherent complexes, that is, those objects $M$ having the property that for any affine open set $U = \text{Spec} R$ the restriction of $M$ to $U$ is connective in $\text{Mod}_R$. Observe that $\mathcal{O} \otimes S$ gives an object of $\text{CAlg}(\text{QC}(X)^{\leq 0})$. The opposite category $\text{CAlg}(\text{QC}(X)^{\leq 0})^{\text{op}}$ is equivalent to the full subcategory of $\text{Mod}(\text{CAlg}_k^{\leq 0}(\hat{S})_{/X}$, which consists of affine morphisms $Y \to X$ (cf. [20, VIII, the discussion before 3.2.7]). We define $\text{Map}(S, X) \to X$ to be an affine morphism determined by $\mathcal{O} \otimes S$ ($\text{Map}(S, X)$ over $X$, which
we refer to as the morphism induced by constant maps. We define the (free) loop space \( LX \) of \( X \) as \( \text{Map}(S^1, X) \). By definition, it is easy to see that \( \text{Map}(S^1, X) \simeq X \times_{X \times X} X \).

### 3.3. Formal stacks

Let \( A \) be a small \( \infty \)-category. Let \( \mathcal{P}(A) \) denote the functor category \( \text{Fun}(A^{op}, S) \), where \( S \) is the \( \infty \)-category of spaces/\( \infty \)-groupoids. There is the Yoneda embedding \( \mathfrak{h}_A : A \to \mathcal{P}(A) \). Let \( \mathcal{P}_\Sigma(A) \subset \mathcal{P}(A) \) be the full subcategory spanned by those functors \( A^{op} \to S \) which preserve finite products [18, 5.5.8.8]. The \( \infty \)-category \( \mathcal{P}_\Sigma(A) \) is compactly generated, and \( \mathcal{P}_\Sigma(A) \subset \mathcal{P}(A) \) is characterized as the smallest full subcategory which contains the essential image of the Yoneda embedding and is closed under sifted colimits.

Suppose that \( A \) admits finite coproducts and a zero object \( 0 \). We recall the \( \infty \)-category \( \mathcal{P}_\Sigma^lf(A) \) associated to \( A \) (cf. [12, 15]). Consider the set of morphisms \( S = \{ \mathfrak{h}_A(0) \sqcup \mathfrak{h}_A(M) \to \mathfrak{h}_A(0 \sqcup M) \} \) where \( M \in A \) such that the pushout \( 0 \sqcup M \) exists in \( A \). Let \( \mathcal{P}_\Sigma^lf(A) \) be the presentable \( \infty \)-category obtained from \( \mathcal{P}_\Sigma(A) \) by inverting morphisms in \( S \) (see e.g. [18, 5.5.4] for the localization). The \( \infty \)-category \( \mathcal{P}_\Sigma^lf(A) \) can be regarded as the full subcategory of \( \mathcal{P}_\Sigma(A) \) spanned by \( S \)-local objects. In other words, \( \mathcal{P}_\Sigma^lf(A) \) is the full subcategory which consists of functors \( F \) such that the canonical morphism \( F(0 \sqcup M) \to * \times F(M) \) is an equivalence for any \( M \in A \) such that \( 0 \sqcup M \) exists in \( A \). Any object of the essential image of the Yoneda embedding \( A \to \mathcal{P}_\Sigma(A) \) is \( S \)-local.

The presentable \( \infty \)-category \( \mathcal{P}_\Sigma^lf(A) \) can be characterized by a universal property. Let \( D \) be a presentable \( \infty \)-category. Let \( \text{Fun}^r(\mathcal{P}_\Sigma^lf(A), D) \) be the full subcategory of \( \text{Fun}(\mathcal{P}_\Sigma^lf(A), D) \) which consists of colimit-preserving functors (i.e., left adjoint functors). Let \( \text{Fun}^{lf}(A, D) \) be the full subcategory of \( \text{Fun}(A, D) \) spanned by those functors \( f \) which preserve finite coproducts and carry pushouts of the form \( 0 \sqcup M = \{ M \in A \} \) to \( f(0) \sqcup f(M) \). Taking into account the universal properties of \( \mathcal{P}_\Sigma \) and the localization [19, 5.5.8.15, 5.5.4.20], we see that the composition with the fully faithful functor \( A \mapsto \mathcal{P}_\Sigma^lf(A) \) induced by the Yoneda embedding determines an equivalence

\[
\text{Fun}^l(\mathcal{P}_\Sigma^lf(A), D) \sim \text{Fun}^{lf}(A, D).
\]

### 3.4. Formal stacks

Let \( A \) be a connective commutative dg algebra \( A \) over a field \( k \) of characteristic zero. We use a correspondence between pointed formal stacks and dg Lie algebras over \( A \), which is proved by Hennion [12] which generalizes the correspondence between dg Lie algebras and formal moduli problems over a field of characteristic zero (see Lurie [20, X]).

Let \( \text{Lie}_A \) be the \( \infty \)-category of dg Lie algebras. The \( \infty \)-category \( \text{Lie}_A \) is obtained from the model category of dg Lie algebras (whose fibrations are termwise surjective maps) by inverting quasi-isomorphisms (another equivalent approach is to define it as the \( \infty \)-category of algebras over the Lie operad \( \text{Lie} \)). Let \( \text{Art}^\text{tsz} \) be the full subcategory of \( \text{CAlg}^0_A//A := (\text{CAlg}^0_A)_{/A} \), which is spanned by trivial square zero extensions \( A = A \oplus 0 \to A \oplus M \xrightarrow{p} A \) such that \( M \) is a connective \( A \)-module of the form \( \oplus_1 \leq n A[di] \) (\( v_i \geq 0, d_i \geq 0 \)). By abuse of notation, we often write \( R \) for an object \( A \to R \to A \) of \( \text{CAlg}^0_A//A \). Similarly, we often omit the augmentations from the notation. Let \( \text{TSZ}_A \) denote the opposite category of \( \text{Art}^\text{tsz} \). We define the \( \infty \)-category \( \text{St}_A \) of pointed formal stacks over \( A \) to be \( \mathcal{P}_\Sigma^lf(\text{TSZ}_A) \). We often regard \( \text{TSZ}_A \) as a full subcategory of \( \text{St}_A \). We refer to an object of \( \text{Fun}(\text{Art}^\text{tsz}_A, S) \) as a pointed formal stack (or simply a formal stack). By definition, \( \text{St}_A \) is the full subcategory of \( \text{Fun}(\text{Art}^\text{tsz}_A, S) \) so that we usually think of a pointed formal stack as a functor \( \text{Art}^\text{tsz}_A \to S \). Unfolding the definition, a pointed formal stack over \( A \) is nothing but a functor \( F : \text{Art}^\text{tsz}_A \to S \) which satisfies the following conditions

- it preserves finite products,
- \( F(A \times_X A) \simeq * \times F(R) \ast \) for any \( R \in \text{Art}^\text{tsz}_A \) such that \( A \times_X A \in \text{Art}^\text{tsz}_A \).

Let \( \text{Free}_{\text{Lie}} : \text{Mod}_A \to \text{Lie}_A \) be the free Lie algebra functor which is a left adjoint to the forgetful functor \( \text{Lie}_A \to \text{Mod}_A \). Let \( \text{Mod}^f_A \subset \text{Mod}_A \) be the full subcategory that consists of objects of the form \( \oplus_1 \leq s A[di] \) (\( d_i \leq -1 \)). Let \( \text{Lie}^f_A \) be the full subcategory of \( \text{Lie}_A \), which is the essential image of the restriction of the free Lie algebra functor \( \text{Mod}^f_A \to \text{Lie}_A \). According to [12, 1.2.2], the inclusions \( \text{Mod}^f_A \to \text{Mod}_A \) and \( \text{Lie}^f_A \to \text{Lie}_A \) are extended to equivalences \( \mathcal{P}_\Sigma^lf(\text{Mod}^f_A) \sim \text{Mod}_A \) and \( \mathcal{P}_\Sigma^lf(\text{Lie}^f_A) \sim \text{Lie}_A \) in an essentially unique way (cf. Section 3.3). Let

\[
\text{Ch}^\bullet : \text{Lie}_A \longrightarrow (\text{CAlg}^0_A//A)^{op} : D_\infty
\]
be the adjoint pair where the left adjoint $Ch^*$ is the “Chevalley-Eilenberg cochain functor” which carries $L \in \text{Lie}_A$ to the Chevalley-Eilenberg cochain complex $Ch^*(L)$ (i.e., the $A$-linear dual of Chevalley-Eilenberg chain complex), see e.g. [12, 1.4], [20, X, 2.2]. Thanks to [12, 1.5.6], this adjoint pair induces an adjoint pair

$$\mathcal{F} : \text{Lie}_A \rightleftarrows \widehat{\text{St}}_A : \mathcal{L}.$$ 

Moreover, if $A$ is noetherian, both $\mathcal{F}$ and $\mathcal{L}$ are categorical equivalences. Recall that $A$ is noetherian if $H^n(A)$ is an ordinary noetherian ring, and $H^n(A)$ is trivial when $n$ is big enough and of finite type over $H^0(A)$ for any $n$. The left adjoint $\mathcal{F}$ is defined as follows. The restriction of the functor $Ch^*$ to $\text{Lie}_A$ induces $\text{Lie}^f_A \to \text{TSZ} \subset (\text{CAlg}_{A//})^{\text{op}}$ such that $\text{Lie}^f_A$ is the Chevalley-Eilenberg dual of $\text{TSZ}$. $\text{TSZ} \to \widehat{\text{St}}_A$ belongs to $\text{Fun}(\text{Lie}^f_A, \widehat{\text{St}}_A)$.

There exists an essentially unique left adjoint functor $\mathcal{F} : \text{Lie}_A \simeq \mathcal{P}_{\mathcal{E}}(\text{Lie}^f_A) \to \widehat{\text{St}}_A$ which extends $\text{Lie}^f_A \to \widehat{\text{St}}_A$ (cf. Section 3.3). The right adjoint $\mathcal{L}$ is given by the restriction of $\text{Fun}(\text{Art}^{\text{tass}}_A, \mathcal{S}) \to \text{Fun}(\text{Lie}^f_A)^{\text{op}}, \mathcal{S}$) determined by the composition with $\text{Lie}^f_A \to \text{TSZ}$. If $A$ is noetherian, $\mathcal{F}$ and $\mathcal{L}$ are reduced to a pair of mutually inverse functors $Ch^* : \text{Lie}^f_A \simeq \text{TSZ} : \mathcal{D}_\infty$. For $L \in \text{Lie}_A$, we usually write $\mathcal{F}_L$ for the associated formal stack $\mathcal{F}(L) \in \widehat{\text{St}}_A$.

**Completions.** Let $F : \text{CAlg}_{k^0} \to \mathcal{S}$ be a functor, that is, a derived prestack over $k$. Let $\text{Spec} A \hookrightarrow F \to \text{Spec} A$ be a lift to an object of $\text{Fun}(\text{CAlg}_{k^0}, \mathcal{S})_{\text{Spec} A//\text{Spec} A}$. We think of it as a derived prestack over Spec $A$ with a section from Spec $A$. It is equivalent to giving a functor $F' : (\text{CAlg}_{A//})^{\text{op}} \to \mathcal{S}$ such that $F'(A \hookrightarrow A)$ is a contractible space. We briefly recall the formal completion of $F$ (see [12, for a general treatment]). Since there exists a canonical fully faithful functor $\text{Art}^{\text{tass}}_A \subset (\text{CAlg}_{A//})^{\text{op}}$, the composite functor $\text{Art}^{\text{tass}}_A \to (\text{CAlg}_{A//})^{\text{op}} \to \mathcal{S}$ gives rise to an object $\widehat{F}$ of $\text{Fun}(\text{Art}^{\text{tass}}_A, \mathcal{S})$ such that $\widehat{F}(A) = \widehat{F}(A \hookrightarrow A)$ is a contractible space. We refer to $\widehat{F}$ as the formal completion of $F$ (along $i : \text{Spec} A \to F$). Suppose that $F$ is a derived scheme (or, more generally, a derived Artin stack [25, 1.3.3], [8, I, Ch. 2, Section 4]). Then the formal completion $\widehat{F}$ belongs to $\widehat{\text{St}}_A$. Namely, $\widehat{F}$ is a pointed formal stack over $A$ (see [12, Lemma 2.2]). In this case, there exists an essentially unique dg Lie algebra $L$ such that $\mathcal{F}_L \simeq \widehat{F}$. Suppose furthermore that $F$ is locally of finite presentation over $A$: this condition implies that the cotangent complex $L_{F//A}$ is a perfect complex. The underlying complex of $L$ is equivalent to the pullback $i^*(\mathcal{T}_{F//A}[-1]) \in \text{Mod}_A$ where $\mathcal{T}_{F//A}$ is the tangent complex that is a dual of $L_{F//A}$ (see [12, 1.5.6, 2.2.9]).

### 4. Extensions of factorization homology to Mapping stacks

**4.1.** We briefly recall the definition of factorization homology. We refer the reader to [2], [3], [13] for the theory of factorization homology we need (in [19], the theory is developed under the name of topological chiral homology). Let $(\text{Mfld}_n^\text{fr})^\otimes$ be the symmetric monoidal ∞-category of framed smooth $n$-manifolds such that the mapping spaces are the space of smooth embeddings endowed the data of compatibility of framings (see e.g. [2, 2.1], [3] or [13] for the detailed account). The symmetric monoidal structure is given by disjoint union. We write $\text{Mfld}_n$ for the underlying ∞-category. Let $\text{Disk}_n^\text{fr}$ be the full subcategory of $\text{Mfld}_n^\text{fr}$ spanned by those manifolds which is diffeomorphic to a (possibly empty) finite disjoint union of $\mathbb{R}^n$. Let $p : (\text{Disk}_n^\text{fr})^\otimes \to (\text{Mfld}_n^\text{fr})^\otimes$ denote the fully faithful symmetric monoidal functor.

By default, in this paper $\text{Mod}_R$ means $\text{Mod}_R(\text{Sp})$. However, results in this Section 4.1 holds in a more general setting. Let $\mathcal{P}^\otimes$ be a symmetric monoidal presentable ∞-category whose tensor product functor $\otimes : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ preserves small colimits separately in each variable. Let $R$ be a commutative algebra object in $\mathcal{P}$, that is, $R \in \text{CAlg}(\mathcal{P})$. In this Section 4.1, we write $\text{Mod}_R^\otimes$ for the symmetric monoidal ∞-category $\text{Mod}_R(\mathcal{P}^\otimes)$. As usual, $\text{Mod}_R$ means the underlying ∞-category.

Let $\text{Fun}^\otimes((\text{Disk}_n^\text{fr})^\otimes, \text{Mod}_R^\otimes) \simeq \text{Alg}_n(\text{Mod}_R)$ (obtained by identifying $(\text{Disk}_n^\text{fr})^\otimes$ with a symmetric monoidal envelope of the operad $\mathbf{E}_n$, see [19, Section 2.2.4] for monoidal envelopes). Consider the following adjoint pair

$$p^* : \text{Fun}^\otimes((\text{Disk}_n^\text{fr})^\otimes, \text{Mod}_R^\otimes) \rightleftarrows \text{Fun}^\otimes((\text{Mfld}_n^\text{fr})^\otimes, \text{Mod}_R^\otimes) : p^*$$
where $p^*$ is determined by composition with $p$. The left adjoint $p_!$ sends $\beta : (\text{Disk}^\text{fr}_n)^{\otimes} \to \text{Mod}_R^{\otimes}$ to a symmetric monoidal (operadic) left Kan extension $\text{Mfld}^\text{fr}_n \otimes \text{Mod}_R^{\otimes}$ of $\beta$ along $p$. Let us regard $\beta$ as an $\mathbb{E}_n$-algebra $B$ in $\text{Mod}_R$. For any $M \in \text{Mfld}^\text{fr}_n$, we write $\int_M B/R$ for the image of $M$ under $p(\beta)$. We refer to $\int_M B/R$ as the factorization homology of $M$ in coefficients in $B$ over $\text{Mod}_R$. The factorization homology $\int_M B/R$ can naturally be identified with a colimit of $(\text{Disk}^\text{fr}_n)_M \to \text{Disk}^\text{fr}_n \to \text{Mod}_R$ where the second functor is the underlying functor of the symmetric monoidal functor $\beta : (\text{Disk}^\text{fr}_n)^{\otimes} \to \text{Mod}_R^{\otimes}$. The functor $p_! : \text{Alg}_n(\text{Mod}_R) \simeq \text{Fun}^\circ((\text{Disk}^\text{fr}_n)^{\otimes}, \text{Mod}_R^{\otimes}) \to \text{Fun}^\circ((\text{Mfld}^\text{fr}_n)^{\otimes}, \text{Mod}_R^{\otimes})$ given by $B \mapsto \int_M B/R$ is extended to a symmetric monoidal functor. To see this, consider the symmetric monoidal structure on $\text{Fun}^\circ((\text{Mfld}^\text{fr}_n)^{\otimes}, \text{Mod}_R^{\otimes})$ which is induced by that of $\text{Mod}_R$. Namely, given two symmetric monoidal functors $F, G : (\text{Mfld}^\text{fr}_n)^{\otimes} \to \text{Mod}_R^{\otimes}$, the tensor product $F \otimes G$ is informally defined by $(F \otimes G)(M) = F(M) \otimes_R G(M)$ for any $M \in \text{Mfld}^\text{fr}_n$. For the precise construction of this symmetric monoidal structure, we refer to [14, Construction 7.9]. The $\infty$-category $\text{Fun}^\circ((\text{Disk}^\text{fr}_n)^{\otimes}, \text{Mod}_R^{\otimes}) \simeq \text{Alg}_n(\text{Mod}_R)$ is promoted to a symmetric monoidal $\infty$-category in a similar way, and the restriction functor $p^*$ is extended to a symmetric monoidal functor. We observe:

**Lemma 4.1.** The functor $p_!$ is a symmetric monoidal functor.

**Proof.** By the relative adjoint functor theorem [19, 7.3.2.11], the left adjoint $p_!$ of the symmetric monoidal functor $p^*$ is a lax symmetric monoidal functor. Let $F, G : (\text{Disk}^\text{fr}_n)^{\otimes} \to \text{Mod}_R^{\otimes}$ be two symmetric monoidal functors. It will suffice to prove that the canonical morphism $p_!(F \otimes G) \to p_!(F) \otimes p_!(G)$ is an equivalence. To this end, it is enough to show that the evaluation $t : p_!(F \otimes G)(M) \to p_!(F)(M) \otimes p_!(G)(M)$ is an equivalence in $\text{Mod}_R$ for any $M \in \text{Mfld}^\text{fr}_n$. The evaluation $p_!(F \otimes G)(M)$ can be identified with a colimit of

$$\phi : (\text{Disk}^\text{fr}_n)_M \to \text{Dis}^\text{fr}_n \overset{F \otimes G}{\to} \text{Mod}_R,$$

while $p_!(F)(M) \otimes p_!(G)(M)$ is a colimit of

$$\psi : (\text{Disk}^\text{fr}_n)_M \times (\text{Disk}^\text{fr}_n)_M \to \text{Dis}^\text{fr}_n \times \text{Dis}^\text{fr}_n \overset{F \times G}{\to} \text{Mod}_R \times \text{Mod}_R \simeq \text{Mod}_R.$$ 

The functor $F \otimes G : \text{Disk}_n \to \text{Mod}_R$ is naturally equivalent to the composite $\text{Disk}^\text{fr}_n \overset{\text{diagonal}}{\to} \text{Dis}^\text{fr}_n \times \text{Dis}^\text{fr}_n \overset{F \times G}{\to} \text{Mod}_R$ so that the composite of $\psi$ and the diagonal map $\delta : (\text{Disk}^\text{fr}_n)_M \to (\text{Disk}^\text{fr}_n)_M \times (\text{Disk}^\text{fr}_n)_M$ is equivalent to $\phi$. The map $t$ is induced by the universality of colimits. Thus, it is enough to show that $\delta$ is cofinal. According to [2, 3.22], $(\text{Disk}^\text{fr}_n)_M$ is sifted so that $\delta$ is cofinal. \hfill $\square$

Let us consider the composite of symmetric monoidal functors

$$\int_M (-) : \text{Alg}_n(\text{Mod}_R) \simeq \text{Fun}^\circ((\text{Disk}^\text{fr}_n)^{\otimes}, \text{Mod}_R^{\otimes}) \overset{p_!}{\to} \text{Fun}^\circ((\text{Mfld}^\text{fr}_n)^{\otimes}, \text{Mod}_R^{\otimes}) \overset{\text{ev}_M}{\to} \text{Mod}_R^{\otimes}$$

where $\text{ev}_M$ is a symmetric monoidal functor given by the evaluation at $M \in \text{Mfld}^\text{fr}_n$.

Let $k$ and $A$ be objects of $\text{CAlg}(\mathcal{P}^{\otimes})$ and let $k \to A$ be a morphism in $\text{CAlg}(\mathcal{P}^{\otimes})$. We note that $\text{Mod}_A := \text{Mod}_A(\mathcal{P}^{\otimes})$. We can think of $A$ as an object of $\text{CAlg}(\text{Alg}_n(\text{Mod}_A))$ which is equivalent to $\text{Alg}_n(\text{Mod}_A)$ where the equivalence is induced by the equivalence $\Gamma \otimes E_n \simeq \Gamma$ of $\infty$-operads coming from Dunn additivity theorem [19, 5.1.2.2] (we can also observe this from the facts that $\text{CAlg}(\text{Mod}_A)$ is a coCartesian symmetric monoidal $\infty$-category, and $E_n$ is a unital $\infty$-operad, see [19, 2.4.3.9]). The restriction functor $\text{Mod}_A \to \text{Mod}_k$ along $k \to A$ gives an object of $\text{CAlg}(\text{Alg}_n(\text{Mod}_k)) \simeq \text{CAlg}(\text{Mod}_k)$ which we denote by $A$. Since $\int_M (-) / k : \text{Alg}_n(\text{Mod}_k) \to \text{Mod}_k$ is symmetric monoidal, $\int_M A/k$ is a commutative algebra object in $\text{Mod}_k$. It gives rise to

$$\int_M (-) / k : \text{Mod}_A(\text{Alg}_n(\text{Mod}_k)) \to \text{Mod}_{\int_M A/k}(\text{Mod}_k)$$

and

$$\int_M (-) / A : \text{Alg}_n(\text{Mod}_A) \to \text{Mod}_A(\text{Mod}_k) \simeq \text{Mod}_A.$$


Theorem 4.2. There exists a canonical equivalence

$$\int_M B/k \otimes_{\int_M A/k} A \cong \int_M B/A$$

in $\text{Mod}_A$. Moreover, the equivalence is natural with respect to $B \in \text{Alg}_n(\text{Mod}_A)$ and $M \in \text{Mfd}^\text{fr}_n$ (see Construction 4.3 for the formulation).

Construction 4.3. We will construct $\int_M B/k \otimes_{\int_M A/k} A \to \int_M B/A$. We first consider the diagram

$$\begin{array}{ccc}
\text{Alg}_n(\text{Mod}_A) & \cong & \text{Fun}^\circ((\text{Disk}^\text{fr}_n)^\circ, \text{Mod}_A^\circ) \\
\downarrow{p^*} & & \downarrow{q^*} \\
\text{Alg}_n(\text{Mod}_k) & \cong & \text{Fun}^\circ((\text{Disk}^\text{fr}_n)^\circ, \text{Mod}_k^\circ)
\end{array}$$

where $a$ and $b$ are symmetric functors induced by compositions with $A \otimes_k : \text{Mod}_k^\circ \to \text{Mod}_A^\circ$, and $p_!$ is a left adjoint functor of the restriction functor $p^* : \text{Fun}^\circ((\text{Disk}^\text{fr}_n)^\circ, \text{Mod}_A^\circ) \to \text{Fun}^\circ((\text{Disk}^\text{fr}_n)^\circ, \text{Mod}_k^\circ)$. Likewise, $q_!$ is defined to be a left adjoint of the restriction functor $q^* : \text{Fun}^\circ((\text{Disk}^\text{fr}_n)^\circ, \text{Mod}_k^\circ) \to \text{Fun}^\circ((\text{Disk}^\text{fr}_n)^\circ, \text{Mod}_A^\circ)$. We denote by $r : \text{Alg}_n(\text{Mod}_A) \to \text{Alg}_n(\text{Mod}_k)$ the restriction functor (i.e., a right adjoint to $a$). The adjoint pairs $(a, r)$, $(p_!, p^*)$, and $(q_!, q^*)$ are relative adjoint pairs over $\Gamma$. The diagram commutes up to canonical homotopy. (To see this, note that $a \circ q^* \simeq p^* \circ b$ and $id \simeq q^* \circ q_!$ imply $a \simeq p^* \circ b \circ q_!$, that determines a natural transformation $\xi : p_! \circ a \to b \circ q_!$. Since $A \otimes_k : \text{Mod}_k \to \text{Mod}_A$ preserves small colimits, we see that $\xi$ is a natural equivalence.) It follows that $p_! \circ a \simeq b \circ q_!$ determines $t : b \circ q_! \circ r \simeq p_! \circ a \circ r \to p_!$. Recall $\text{Mod}(\mathcal{M}^\circ)$ for a symmetric monoidal $\infty$-category $\mathcal{M}^\circ$. Let $\Gamma^+ \to \Gamma$ denote the subcategory $\Gamma^+ = \Gamma(\{1\})$ with the forgetful functor to $\Gamma$ (i.e., the (nerve of) pointed finite sets, see Section 2). We say that a map in $\Gamma^+$ is inert if the image in $\Gamma$ is inert (see [19] for inert maps). Note that there exists a section $\Gamma \simeq \Gamma(\{0\}) \to \Gamma^+$ induced by $(1) \to (0)$. Given a symmetric monoidal $\infty$-category $u : \mathcal{M}^\circ \to \Gamma$, $\text{Mod}(\mathcal{M}^\circ)$ is the full subcategory of $\text{Fun}(\Gamma^+, \mathcal{M}^\circ)$ spanned by maps $\Gamma^+ \to \mathcal{M}^\circ$ which carries inert maps to $u$-coCartesian morphisms. Composition with the section $\Gamma \to \Gamma^+$ determines the forgetful functor $\text{Mod}(\mathcal{M}^\circ) \to \text{CAlg}(\mathcal{M}^\circ)$. By applying $\text{Mod}(-)$ to the unit $\alpha \circ r \to id$ of the relative adjoint pair, it is promoted to a natural transformation between functors $\text{Mod}(\text{Alg}_n(\text{Mod}_A)) \to \text{Mod}(\text{Alg}_n(\text{Mod}_A))$. Thus, $\xi$ induces $t^! : \text{Mod}(\text{Mod}(\text{Alg}_n(\text{Mod}_A)), \text{Mod}(\text{Fun}^\circ((\text{Disk}^\text{fr}_n)^\circ, \text{Mod}_k^\circ)))$. We have

$$\Delta^1 \times \text{Alg}_n(\text{Mod}_A) \xleftarrow{\Delta^1 \times \text{Mod}_A(\text{Alg}_n(\text{Mod}_A))} \Delta^1 \times \text{Mod}(\text{Alg}_n(\text{Mod}_A)) \xrightarrow{T^!} \text{Mod}(\text{Fun}^\circ((\text{Disk}^\text{fr}_n)^\circ, \text{Mod}_k^\circ)))$$

where $T^!$ corresponds to $t^!$, and $A$ in $\text{Mod}_A(\text{Alg}_n(\text{Mod}_A))$ indicates the unit algebra. Consider the induced map $\Delta^1 \times \{A\} \to \text{Mod}(\text{Fun}^\circ((\text{Disk}^\text{fr}_n)^\circ, \text{Mod}_k^\circ))$. The image of $\{1\} \times \{A\}$ is the constant functor $const(A)$ having the image $A$, which is regarded as a module over the unit algebra $\text{const}(A) \in \text{CAlg}(\text{Fun}^\circ((\text{Disk}^\text{fr}_n)^\circ, \text{Mod}_k^\circ))$. If we write $\int_M A/k$ for $q_!(r(A))$, the image of $\{0\} \times \{A\}$ is

$$b(q_!(r(A))) \cong A \otimes_k \left(\int_{\{0\}} A/k\right).$$

which is defined by formula $M \to A \otimes_k (\int_M A/k)$. Note that $A \otimes_k (\int_M A/k)$ is a colimit of $\text{Disk}^\text{fr}_n/_{\text{Mod}_A}$ where $a \circ r(A) = A \otimes_k A$. The counit $A \otimes_k (\int M A/k) \to A$ induces

$$A \otimes_k (\int M A/k) \to A$$

which corresponds to the composition of $e_{\text{Mod}} : \text{CAlg}(\text{Fun}^\circ((\text{Disk}^\text{fr}_n)^\circ, \text{Mod}_k^\circ)) \to \text{CAlg}(\text{Mod}_A)$ and

$$\theta : \Delta^1 \simeq \Delta^1 \times \{A\} \to \text{Mod}(\text{Fun}^\circ((\text{Disk}^\text{fr}_n)^\circ, \text{Mod}_k^\circ)) \to \text{CAlg}(\text{Fun}^\circ((\text{Disk}^\text{fr}_n)^\circ, \text{Mod}_k^\circ))).$$

Consider the induced functor

$$f : \text{Alg}_n(\text{Mod}_A) \to A := \text{Fun}(\Delta^1, \text{Mod}(\text{Fun}^\circ((\text{Disk}^\text{fr}_n)^\circ, \text{Mod}_k^\circ))) \times \text{Fun}(\Delta^1, \text{CAlg}(\text{Fun}^\circ((\text{Disk}^\text{fr}_n)^\circ, \text{Mod}_k^\circ))) \{\theta\}.$$
Let $\text{Mod}_{\text{const}(A)}(\Fun^\otimes((\text{Mfld}^\fr_n)^\otimes, \text{Mod}_A))$ denote the fiber over $\text{const}(A) \in \text{CAlg}(\Fun^\otimes((\text{Mfld}^\fr_n)^\otimes, \text{Mod}_A))$. Since $\Fun^\otimes((\text{Mfld}^\fr_n)^\otimes, \text{Mod}_A)$ has sifted colimits which commute with those in $\text{Mod}_A$, it follows from [19, 4.5.3.6, 4.5.1.4] that $\text{Mod}(\Fun^\otimes((\text{Mfld}^\fr_n)^\otimes, \text{Mod}_A)) \rightarrow \text{CAlg}(\Fun^\otimes((\text{Mfld}^\fr_n)^\otimes, \text{Mod}_A))$ is a coCartesian fibration. Thus the evident inclusion

$$\text{Mod}_{\text{const}(A)}(\Fun^\otimes((\text{Mfld}^\fr_n)^\otimes, \text{Mod}_A)) \rightarrow \mathcal{B} := \text{Mod}(\Fun^\otimes((\text{Mfld}^\fr_n)^\otimes, \text{Mod}_A)) \times_{\text{CAlg}(\Fun^\otimes((\text{Mfld}^\fr_n)^\otimes, \text{Mod}_A))} \text{CAlg}(\Fun^\otimes((\text{Mfld}^\fr_n)^\otimes, \text{Mod}_A))) / \text{const}(A)$$

admits a left adjoint. The composition of the left adjoint, the evident functor $A \rightarrow \Fun(\Delta^1, B)$, and $f$ gives

$$\chi : \text{Alg}_n(\text{Mod}_A) \rightarrow \Fun(\Delta^1, \text{Mod}_{\text{const}(A)}(\Fun^\otimes((\text{Mfld}^\fr_n)^\otimes, \text{Mod}_A))) \simeq \Fun(\Delta^1, \Fun^\otimes((\text{Mfld}^\fr_n)^\otimes, \text{Mod}_A)).$$

Since $\text{Mod}_A$ has (sifted) colimits whose tensor product functor $\text{Mod}_A \times \text{Mod}_A \rightarrow \text{Mod}_A$ preserves (sifted) colimits in each variable, it follows from [19, 3.2.3.1 (1), (2)] that $\Fun(\text{Alg}^\fr_n(\otimes,\otimes), \text{Mod}_A)$ has sifted colimits determined by the evaluation at $M$ preserves sifted colimits. Consequently, the base change functor $\text{Mod}_{A \otimes k(\int_{A/k})}(\Fun((\text{Mfld}^\fr_n)^\otimes, \text{Mod}_A)) \rightarrow \text{Mod}_{\text{const}(A)}(\Fun((\text{Mfld}^\fr_n)^\otimes, \text{Mod}_A))$ commutes with the base change functor $\text{Mod}_{A \otimes k(\int_{A/k})}(\text{Mod}_A) \rightarrow \text{Mod}_{A}(\text{Mod}_A) \simeq \text{Mod}_{A}$ under the evaluation functors, up to canonical homotopy. It follows that $\chi$ carries $B \in \text{Alg}_n(\text{Mod}_A)$ to a morphism in $\Fun^\otimes((\text{Mfld}^\fr_n)^\otimes, \text{Mod}_A)$ such that for any $M \in \text{Mfld}^\fr_n$, the evaluation at $M$ is

$$\chi_M(B) : A \otimes k \left( \int_M B/k \right) \otimes_{A \otimes k(\int_{A/k})} A \rightarrow \int_M B/A.$$

in $\text{Mod}_A$. The left-hand side is naturally equivalent to $\int_M B/k \otimes_{\int_{A/k}} A$.

**Proof of Theorem 4.2.** Recall that $\int_M B/A$ is a (sifted) colimit of $(\text{Disk}^\fr_n)^\otimes(M) \rightarrow \text{Disk}^\fr_n$ where $B$ denotes the underlying functor, that is, $\text{Disk}^\fr_n \rightarrow \text{Mod}_A$ carries $(\text{R}^n)^{\text{lim}}$ (i.e., $n$-disks with $m$ connected components) to the $m$-fold tensor product $B \otimes_A \cdots \otimes_A B$ in $\text{Mod}_A$. On the other hand, $\text{Mod}_A$ such that for any objects by

$$(\text{R}^n)^{\text{lim}} \rightarrow (A \otimes_k B) \otimes_A \cdots \otimes_A (A \otimes_k B) \otimes_{(A \otimes_k A) \otimes_A \cdots A(A \otimes_k A)) A \simeq (B \otimes_k \cdots \otimes_k B) \otimes_{(A \otimes_k \cdots \otimes_k A)} A$$

where $A \otimes_k B$ and $A \otimes_k A$ are image of $B \in \text{Mod}_A$ and $A \in \text{Alg}_1(\text{Mod}_A)$ under the base change $A \otimes_k$, respectively ($\bullet \otimes_A \cdots \otimes_A (\bullet)$ means the $m$-fold tensor product). By construction, the natural transformation $(A \otimes_k B) \otimes_{A \otimes_k A} A \rightarrow B$ between functors $\text{Disk}^\fr_n \rightarrow \text{Mod}_A$ induces $\chi_M(B) : A \otimes_k (\int_M B/k) \otimes_{A \otimes_k(\int_{A/k})} A \rightarrow \int_M B/A$. Thus, it is enough to show that for any $m \geq 0$, the canonical map

$$(A \otimes_k B) \otimes_A \cdots \otimes_A (A \otimes_k B) \otimes_{(A \otimes_k A) \otimes_A \cdots A(A \otimes_k A)) A \rightarrow B \otimes_A \cdots \otimes_A B$$

is an equivalence. This morphism is obviously an equivalence.

\[\boxdot\]

**Remark 4.4.** We outline a generalization of Theorem 4.2 without proof\footnote{We learn it from Takuo Matsuoka with multiple proofs.}. Let $A_1 \leftarrow A_0 \rightarrow A_2$ be a diagram in $\text{CAlg}(\mathcal{P})$. Let $B_0$ be an object of $\text{Alg}_{n+1}(\text{Mod}_{A_0})$. For $i = 1, 2$, let $B_i$ be an object of $\text{Alg}_{1}(\text{LMod}_{B_0 \otimes A_0 A_0})$. Let $M$ be a framed $n$-manifold. Consider $B_0 \in \text{Alg}_{n+1}(\text{Mod}_{A_0})$ as a symmetric monoidal functor $(\text{Disk}^\fr_n)^\otimes \rightarrow \text{Alg}(\text{Mod}_{A_0})$. We define $\int_M B_0/A_0$ to be a colimit of $(\text{Disk}^\fr_n)^\otimes \rightarrow \text{Alg}(\text{Mod}_{A_0})$. Then there exists an equivalence

$$\int_M B_1/A_1 \otimes_{\int_M B_0/A_0} \int_M B_2/A_2 \simeq \int_M (B_1 \otimes_{B_0} B_2)/(A_1 \otimes_{A_0} A_2)$$

in $\text{Mod}_{A_1 \otimes A_0 A_2}$. 

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\[\text{ON D-MODULES OF CATEGORIES I 11}\]
Consider the case that $\mathcal{P}^\circ = \text{Sp}^\circ$. We describe Theorem 4.2 in geometric terms. We write $A \otimes_k M$ for the tensor of $A$ with the underlying space of $M$ in $\text{CAlg}(\text{Mod}_k)$. By Remark 4.9 below, we see that $\int_M A/k$ can be promoted to an object of $\text{CAlg}(\text{Mod}_k)$ which is equivalent to $A \otimes_k M$. The canonical map $\int_M A/k \to A$ can be identified with $A \otimes_k M \to A \otimes_k A \ast \simeq A$ induced by the map $M \to \ast$ into the contractible space. Suppose that $A$ is connective. The affine scheme $\text{Spec}(A \otimes_k M)$ is the mapping stack $\text{Map}(M, \text{Spec} A)$ (see Section 3.2). We abuse notation by writing $\text{QC}((\text{Map}(M, \text{Spec} A))$ for $\text{Mod}_{A \otimes_k M}$.

**Corollary 4.5.** Let $B$ be an $E_n$-algebra in $\text{Mod}_A$. Let $M$ be a framed smooth $n$-manifold. Then $\int_M B/A \in \text{Mod}_A = \text{QC}((\text{Spec} A)$ is naturally promoted to $\int_M B/k \in \text{QC}((\text{Map}(M, \text{Spec} A))$ endowed with

$$\iota^* \int_M B/k \simeq \int_M B/A$$

where $\iota^*: \text{QC}((\text{Map}(M, \text{Spec} A)) \to \text{QC}((\text{Spec} A) = \text{Mod}_A$ is the pullback along the morphism $\iota: \text{Spec} A \to \text{Map}(M, \text{Spec} A)$ induced by constant maps. Moreover, if $\text{End}(M)$ is the monoid space ($E_n$-algebra in $\mathcal{S}$) of endomorphisms of $M$ in $\text{Mfld}^{fr}_n$, then $\int_M B/A \in \text{QC}((\text{Spec} A)^{\text{End}(M)}$ is naturally promoted to an object $\int_M B/k$ of $\text{QC}((\text{Map}(M, \text{Spec} A))^{\text{End}(M)}$.

We prove a variant of Theorem 4.2. Let $I$ be a small $\infty$-category and let $\text{Fun}(I, \text{Mod}_k)$ be the functor category. Note that $\text{Mod}_k$ has a symmetric monoidal structure, which is encoded by a coCartesian fibration $\text{Mod}^\circ_k \to \Gamma$. It follows that $\text{Fun}(I, \text{Mod}_k)$ admits a symmetric monoidal structure arising from that of $\text{Mod}_k$, which is defined by the coCartesian fibration

$$\text{Fun}(I, \text{Mod}^\circ_k) \times_{\text{Fun}(I, \Gamma)} \Gamma \to \Gamma.$$ 

Let $A_I$ be an object of $\text{CAlg}(\text{Fun}(I, \text{Mod}_k)) \simeq \text{Fun}(I, \text{CAlg}(\text{Mod}_k))$. We think of $\text{Fun}(I, \text{Mod}_k)$ and $A_I \in \text{CAlg}(\text{Fun}(I, \text{Mod}_k))$ as generalizations of $\text{Mod}_k$ and $A \in \text{CAlg}(\text{Mod}_k)$ in Theorem 4.2, respectively. Namely, we will discuss a diagram version.

We regard $A_I$ as a functor $a_I: I \to \text{CAlg}(\text{Mod}_k)$. Let $m_I$ be a section $I \to \text{Mod}_{A_I}(\text{Mod}_k)$ of the second projection $\text{Mod}_{A_I}(\text{Mod}_k) := \text{Mod}(\text{Mod}_k) \times_{\text{CAlg}(\text{Mod}_k)} I \to I$. If $m_I$ sends every morphism in $I$ to a coCartesian morphism in $\text{Mod}(\text{Mod}_k)$ (over $\text{CAlg}(\text{Mod}_k)$), we say that $m_I$ is coCartesian. Let $\text{Mod}_{A_I}(\text{Fun}(I, \text{Mod}_k))$ be the fiber of $\text{Mod}(\text{Fun}(I, \text{Mod}_k)) \to \text{CAlg}(\text{Fun}(I, \text{Mod}_k))$ over $A_I$. Unwinding the definition, we observe that a section $m_I$ is equivalent to giving an object of $\text{Mod}_{A_I}(\text{Fun}(I, \text{Mod}_k))$. If an object $M_I$ of $\text{Mod}_{A_I}(\text{Fun}(I, \text{Mod}_k))$ corresponds to a coCartesian section $I \to \text{Mod}_{A_I}(\text{Mod}_k)$, we say that $M_I$ is coCartesian. Let $\text{Mod}_{A_I}(\text{Fun}(I, \text{Mod}_k))^{\text{coCart}}$ denote the full subcategory of coCartesian objects. For any morphism $A \to A'$ in $\text{CAlg}(\text{Mod}_k)$, the base change functor $\otimes_A A' : \text{Mod}_A \to \text{Mod}_A$ is a symmetric monoidal functor which preserves small colimits. It follows that $\text{Mod}_{A_I}(\text{Fun}(I, \text{Mod}_k))^{\text{coCart}}$ is closed under tensor products and small colimits in $\text{Mod}_{A_I}(\text{Fun}(I, \text{Mod}_k))$. Let $\text{Mod}_{A_I}^{\circ}(\text{Fun}(I, \text{Mod}_k))$ denote $\text{Mod}_{A_I}(\text{Fun}(I, \text{Mod}_k))$ endowed with the standard symmetric monoidal structure. We denote by $\text{Mod}_{A_I}^{\circ}(\text{Fun}(I, \text{Mod}_k))^{\text{coCart}}$ the symmetric monoidal $\infty$-category whose symmetric monoidal structure is induced by $\text{Mod}_{A_I}(\text{Fun}(I, \text{Mod}_k))$. Let $B_I$ be an object of $\text{Alg}_n(\text{Mod}_{A_I}(\text{Fun}(I, \text{Mod}_k)))$. We will think that $B_I$ is a generalization of $B$ in Theorem 4.2. We say that $B_I$ is coCartesian if the underlying object of $B_I$ belongs to $\text{Mod}_{A_I}(\text{Fun}(I, \text{Mod}_k))^{\text{coCart}}$.

We consider the diagram of factorization homology associated to $A_I$ and $B_I$. As in the above case, there exists an adjoint pair

$$p^* : \text{Alg}_n(\text{Mod}_{A_I}(\text{Fun}(I, \text{Mod}_k))) \rightleftarrows \text{Fun}^{\circ}(\text{(Mfld}^n) \otimes, \text{Mod}_{A_I}^\circ(\text{Fun}(I, \text{Mod}_k))) : p^*$$

where the right-hand side is the $\infty$-category of symmetric monoidal functors. For any $M \in \text{Mfld}^n$, we write $\int_M B_I/A_I \in \text{Mod}_{A_I}(\text{Fun}(I, \text{Mod}_k))$ for the image of $M$ under $p(B_I)$. We refer to $\int_M B_I/A_I$ as the factorization homology of $M$ with coefficients in $B_I$ over $A_I$. Similarly, there exists an adjoint pair

$$q^* : \text{Alg}_n(\text{Fun}(I, \text{Mod}_k)) \rightleftarrows \text{Fun}^{\circ}(\text{(Mfld}^n) \otimes, \text{Fun}(I, \text{Mod}_k)) : q^*$$

Given $C_I \in \text{Alg}_n(\text{Fun}(I, \text{Mod}_k))$ we write $\int_M C_I/k \in \text{Fun}(I, \text{Mod}_k)$ for the image of $M$ under $q(C_I)$.

**Remark 4.6.** The restriction of $p_I$ induces

$$\text{Alg}_n(\text{Mod}_{A_I}(\text{Fun}(I, \text{Mod}_k))^{\text{coCart}} ) \to \text{Fun}^{\circ}(\text{(Mfld}^n) \otimes, \text{Mod}_{A_I}^\circ(\text{Fun}(I, \text{Mod}_k))^{\text{coCart}}).$$
To see this, it is enough to show that $\int_M B/I/A_I$ lies in $\text{Mod}_{A_I}(\text{Fun}(I, \text{Mod}_k))^{\text{coCart}}$ whenever $B/I$ is coCartesian. We note that $\int_M B/I/A_I$ is a colimit of $(\text{Disk}_n^r)/M \rightarrow \text{Disk}_n^r \rightarrow \text{Mod}_{A_I}(\text{Fun}(I, \text{Mod}_k))$ where the second functor is the underlying functor of $\beta_I : (\text{Disk}_n^r)^\otimes \rightarrow \text{Mod}_{A_I}(\text{Fun}(I, \text{Mod}_k))$ corresponding to $B/I$. Since the essential image of $\beta_I$ is contained in $\text{Mod}_{A_I}(\text{Fun}(I, \text{Mod}_k))^{\text{coCart}}$ and $\text{Mod}_{A_I}(\text{Fun}(I, \text{Mod}_k))^{\text{coCart}}$ is closed under small colimits in $\text{Mod}_{A_I}(\text{Fun}(I, \text{Mod}_k))$, it follows that the colimit $\int_M B/I/A_I$ lies in $\text{Mod}_{A_I}(\text{Fun}(I, \text{Mod}_k))^{\text{coCart}}$.

**Remark 4.7.** Consider the symmetric monoidal functor $e_i : \text{Fun}(I, \text{Mod}_k) \rightarrow \text{Mod}_k$ induced by the evaluation at an object $i \in I$. Let $A_I$ denote the image of $A_I$ in $\text{CAlg}(\text{Mod}_k)$ under the evaluation functor. There exists a diagram

$$
\text{Alg}_n(\text{Mod}_{A_I}(\text{Fun}(I, \text{Mod}_k))) \xrightarrow{p_i^*} \text{Fun}^\otimes((\text{Mfd}_n^r)^\otimes, \text{Mod}_{A_I}^\otimes(\text{Fun}(I, \text{Mod}_k)))
$$

where the vertical arrows are induced by $e_i : \text{Fun}(I, \text{Mod}_k) \rightarrow \text{Mod}_k$, and $((p_i)_*, (p_i)^*)$ is an adjoint pair such that the right adjoint $(p_i)^*$ is induced by composition with $(\text{Disk}_n^r)^\otimes \rightarrow (\text{Mfd}_n^r)^\otimes$. There is a natural transformation $(p_i)_! \circ e_i \rightarrow e_i' \circ p_i$ obtained from $e_i \simeq (p_i)^* \circ e_i' \circ p_i$. Remember that $\int_M B/I/A_I$ is a colimit of a diagram of the form $(\text{Disk}_n^r)/M \rightarrow \text{Mod}_{A_I}(\text{Fun}(I, \text{Mod}_k))$, and $\int_M B/I/A_I$ is a colimit of a diagram of the form $(\text{Disk}_n^r)/M \rightarrow \text{Mod}_{A_I}(\text{Mod}_k)$ where $B_i \in \text{Alg}_n(\text{Mod}_{A_I}(\text{Mod}_k))$ is the image of $B/I$ under the evaluation functor. We easily deduce an equivalence of $((p_i)_! \circ e_i \rightarrow e_i' \circ p_i)$ from the presentation of factorization homology in terms of colimits and the fact that forgetful functor $\text{Mod}_{A_I}(\text{Fun}(I, \text{Mod}_k)) \rightarrow \text{Mod}_{A_I}(\text{Fun}(I, \text{Mod}_k))$ preserves small colimits. This means that $p_i(B/I)$ is informally described as

$$
\text{Mfd}_n^r \ni M \mapsto [i \mapsto \int_M B_i/A_i] \in \text{Mod}_{A_I}(\text{Fun}(I, \text{Mod}_k)).
$$

**Theorem 4.8.** The factorization homology $\int_M B/I/A_I$ in $\text{Mod}_{A_I}(\text{Fun}(I, \text{Mod}_k))$ is promoted to an object $\int_M B/I/A_I/k$ of $\text{Mod}_{f_{A_I}}(\text{Fun}(I, \text{Mod}_k))$ with an equivalence

$$
\int_M B/I/k \otimes \int_M A/I \simeq \int_M B/I/A_I
$$

in $\text{Mod}_{A_I}(\text{Fun}(I, \text{Mod}_k))$. The equivalence is natural in $M \in \text{Mfd}_n^r$. (More precisely, there exists an equivalence $b \circ q \circ r(B/I) \otimes_{\text{Box}(\text{Mod}_k)} p(A/I) \simeq p_i(B/I)$ which we will describe in the proof. See the proof also for the notation.) Moreover, $B/I$ is coCartesian, then $\int_M B/I/k$ is coCartesian in $\text{Mod}_{f_{A_I}}(\text{Fun}(I, \text{Mod}_k))$.

**Remark 4.9.** The functor $\int_M A/I/k : I \rightarrow \text{CAlg}(\text{Mod}_k)$ can be identified with the functor given by $I \xrightarrow{\alpha} \text{CAlg}(\text{Mod}_k) \otimes_{\text{Mod}_k} \text{CAlg}(\text{Mod}_k)$ where the second functor is induced by the tensor with the underlying space of $M$ in $\mathcal{S}$. We write $\mathcal{M}^\otimes$ for $\text{Fun}(I, \text{Mod}_k)$ equipped with the symmetric monoidal structure. From [2, Prop. 5.1], there exists a commutative diagram of symmetric monoidal $\infty$-categories

$$
\begin{array}{c}
\text{CaAlg}(\mathcal{M}^\otimes) \\
\text{forget} \downarrow \\
\text{Alg}_n(\mathcal{M}^\otimes)
\end{array}
\xrightarrow{\xi} \begin{array}{c}
\text{Fun}^\otimes((\text{Mfd}_n^r)^\otimes, \mathcal{M}^\otimes) \\
\phi
\end{array}
$$

in $\text{CaAlg}(\text{Cat}_{\infty})$. The arrow $\xi$ is the composition of the functor $\text{CaAlg}(\mathcal{M}^\otimes) \rightarrow \text{Fun}^\otimes((\text{Mfd}_n^r)^\otimes, \text{CaAlg}(\mathcal{M}^\otimes))$ given by the tensor with the underlying spaces of manifolds, and the symmetric monoidal forgetful functor $\text{CaAlg}(\mathcal{M}^\otimes) \rightarrow \mathcal{M}^\otimes$. Apply $\text{CaAlg}(\text{Cat}_{\infty})$ to this diagram, that is, take the $\infty$-categories of commutative algebra objects. Since the tensor products $\Gamma \otimes \Gamma$ and $\Gamma \otimes \mathbb{E}_n$ of $\infty$-operads are naturally equivalent to the commutative $\infty$-operad $\Gamma$ (cf. [19, 3.2.4.5], Dunn additivity theorem [19, 5.1.2.2], [19, 5.1.1.5]), the
induced vertical functor $\text{CAlg}(\text{CAlg}(\mathcal{M}^\otimes)) \to \text{CAlg}(\text{Alg}_n(\mathcal{M}^\otimes))$ is equivalent to the identity functor $\text{CAlg}(\mathcal{M}^\otimes) \to \text{CAlg}(\mathcal{M}^\otimes)$. Thus the induced functor $\text{CAlg}(q_!) : \text{CAlg}(\mathcal{M}^\otimes) \simeq \text{CAlg}(\text{Alg}_n(\mathcal{M}^\otimes)) \to \text{CAlg}(\text{Fun}^\otimes((\text{Mfld}_n^\otimes)\otimes, \mathcal{M}^\otimes)) \simeq \text{Fun}^\otimes((\text{Mfld}_n^\otimes)\otimes, \text{CAlg}(\mathcal{M}^\otimes))$ is equivalent to $\text{CAlg}(\mathcal{M}^\otimes)$.

To observe that $\text{CAlg}(\mathcal{M}^\otimes) \simeq \text{CAlg}(\text{Fun}^\otimes((\text{Mfld}_n^\otimes)\otimes, \mathcal{M}^\otimes))$ induced by $\xi$. Note that $\int_M A_I/k$ is equivalent to the evaluation of $\text{CAlg}(q_!)(A_I)$ at $M$. To observe that $\int_M A_I/k \simeq A_I \otimes M$, it is enough to show that $\text{CAlg}(\mathcal{M}^\otimes) \to \text{Fun}^\otimes((\text{Mfld}_n^\otimes)\otimes, \text{CAlg}(\mathcal{M}^\otimes))$ is equivalent to the functor given by the tensor with the underlying spaces of manifolds. By $\Gamma \otimes \Gamma \simeq \Gamma$, there exist equivalences $\text{CAlg}(\mathcal{M}^\otimes) \simeq \text{CAlg}(\text{CAlg}(\mathcal{M}^\otimes)) \simeq \text{CAlg}(\mathcal{M}^\otimes)$ where the first left arrow $\alpha$ forgets the outer commutative algebra structure, and the second right arrow $\beta$ forgets the inner commutative algebra structure, such that the $\beta \circ \alpha^{-1}$ is equivalent to the identity functor. Using equivalences $\alpha$ and $\beta$, we easily deduce that $\text{CAlg}(\mathcal{M}^\otimes)$ is equivalent to the functor given by the tensor with the underlying spaces of manifolds.

Proof of Theorem 4.8. We first consider the unit algebra lying in $\text{CAlg}(\text{Alg}_n(\text{Mod}_A(I, \text{Fun}(I, \text{Mod}_k)))) \simeq \text{CAlg}(\text{Alg}_A(I, \text{Fun}(I, \text{Mod}_k)))$, which we denote by $A_I$. Replace $\text{Mod}_k$ and $\text{Mod}_A$ with $\text{Fun}(I, \text{Mod}_k)$ and $\text{Mod}_A(I, \text{Mod}_k)$, respectively in Construction 4.3. Then we have the diagram

\[ \begin{array}{ccc} \text{Alg}_n(\text{Mod}_A(I, \text{Fun}(I, \text{Mod}_k))) & \xrightarrow{p_n} & \text{Fun}^\otimes((\text{Mfld}_n^\otimes)\otimes, \text{Mod}_A^\otimes(I, \text{Fun}(I, \text{Mod}_k))) \\
\text{Alg}_n(\text{Fun}(I, \text{Mod}_k)) & \xrightarrow{q} & \text{Fun}^\otimes((\text{Mfld}_n^\otimes)\otimes, \text{Fun}(I, \text{Mod}_k)) \end{array} \]

where $a$ and $b$ are induced by the base change functor $\text{Fun}(I, \text{Mod}_k) \to \text{Mod}_A(I, \text{Fun}(I, \text{Mod}_k))$. Since $q_!$ induces $\text{Mod}_{\text{boq-or}}(\text{Alg}_A(I, \text{Fun}(I, \text{Mod}_k))) \to \text{Mod}_{\text{boq-or}}(\text{Mod}_n^\otimes((\text{Mfld}_n^\otimes)\otimes, \text{Mod}_A^\otimes(I, \text{Fun}(I, \text{Mod}_k))))$,

\[ q(r(B_I)) \text{ is promoted to an object of } \text{Mod}_{\text{boq-or}}(\text{Alg}_A(I, \text{Fun}(I, \text{Mod}_k))) \text{ which we denote by } \int_{(-)} B_I/k. \]

We write $\int_{(-)} A_I/k$ for $q(r(A_I))$. By repeating the construction of $\chi$ in Construction 4.3, we have

\[ \chi : \text{Alg}_n(\text{Mod}_A(I, \text{Fun}(I, \text{Mod}_k))) \to \text{Fun}(\Delta^1, \text{Fun}^\otimes((\text{Mfld}_n^\otimes)\otimes, \text{Mod}_A^\otimes(I, \text{Fun}(I, \text{Mod}_k)))) \]

This functor carries $B_I$ to $b \circ q_! \circ r(B_I) \otimes_{\text{boq-or}(A_I)} p_n(A_I) \to p_n(B_I)$ induced by $b \circ q_! \circ r \to p$, which we will denote by $\chi(B_I)$ (here we abuse the notation). Note that $p_n(A_I)$ is the unit (commutative) algebra, that is, the constant functor $\text{const}(A_I) : (\text{Mfld}_n^\otimes)\otimes \to \text{Mod}_A^\otimes(I, \text{Fun}(I, \text{Mod}_k))$ having value $A_I$ with the canonical module structure over $A_I$. We denote by $b \circ q_! \circ r(B_I) \otimes_{\text{boq-or}(A_I)} p_n(A_I)$ the image of $b \circ q_! \circ r(B_I)$ under the base change functor

\[ \text{Mod}_{\text{boq-or}(A_I)}(\text{Fun}^\otimes((\text{Mfld}_n^\otimes)\otimes, \text{Mod}_A^\otimes(I, \text{Fun}(I, \text{Mod}_k)))) \to \text{Mod}_{p_n(A_I)}(\text{Fun}^\otimes((\text{Mfld}_n^\otimes)\otimes, \text{Mod}_A^\otimes(I, \text{Fun}(I, \text{Mod}_k)))) \]

We will prove that the morphism

\[ h : b \circ q_! \circ r(B_I) \otimes_{\text{boq-or}(A_I)} p_n(A_I) \to p_n(B_I) \]

determined by $\chi(B_I)$ is an equivalence. We write $\int_M B_I/k$ and $\int_M A_I/k$ for the evaluations of $q_! \circ r(B_I)$ and $q_! \circ r(A_I)$ at $M$, respectively. As in Theorem 4.2, it is enough to show that for any $M \in \text{Mfld}_n^\otimes$, the evaluation of $h$ at $M$

\[ h_M : A_I \otimes_k \left( \int_M B_I/k \otimes_{A_I \otimes_k (\int_M A_I/k)} A_I \right) \to \int_M B_I/A_I \]

is an equivalence in $\text{Fun}(I, \text{Mod}_k)$ where $\otimes_k$ indicates the tensor product in $\text{Fun}(I, \text{Mod}_k)$. (The left-hand side is nothing but equivalent to $\int_M B_I/k \otimes_{\text{Alg}_A(I, \text{Mod}_k)} A_I$ in the statement.) For any object $i \in I$, if we write $A_i$ for the image of $A_I$ under $\text{CAlg}(\text{Fun}(I, \text{Mod}_k)) \to \text{CAlg}(\text{Alg}_k)$, then the symmetric monoidal functor $\text{Mod}_{A_i}(\text{Fun}(I, \text{Mod}_k)) \to \text{Mod}_{A_i}(\text{Mod}_k)$ induced by the evaluation $i$ preserves small colimits. Also, the forgetful functor $\text{Mod}_{A_i}(\text{Mod}_k) \to \text{Mod}_k$ preserves small colimits. Therefore, the evaluation functor (at $i \in I$) preserves the relative tensor products (given by geometric realizations). By Remark 4.7, the evaluation of $p_n(B_I)$ at $M$ and $i$ is naturally equivalent to $\int_M B_I/A_i$. Similarly, the
evaluation of \( q(r(B_I)) \) at \( M \) and \( i \) is equivalent to \( \int_M B_i/k \) in \( \text{Mod}_{\int_M A_i/k}(\text{Mod}_k) \). By the argument similar to Remark 4.7, we see that the evaluation of \( h_M \) at \( i \) is equivalent to the morphism

\[
A_i \otimes_k \left( \int_M B_i/k \right) \otimes_{A_i \otimes_k (\int_M A_i/k)} A_i \to \int_M B_i/A_i
\]

which appears just before the proof of Theorem 4.2 (with no subscript). The proof of Theorem 4.2 shows that this morphism is an equivalence. It follows that \( \chi(B_I) \) gives rise to an equivalence in \( \text{Fun}_\otimes((\text{Mfd}_n^\text{fr})^\otimes, \text{Mod}_A^\otimes_{\otimes}(\text{Fun}(I, \text{Mod}_k))) \).

Since the functor \( \int_M (\_)/k : \text{Alg}_n(\text{Mod}_k) \to \text{Mod}_k \) induced by factorization homology of \( M \) of \( \text{E}_n \)-algebras over \( \text{Mod}_k \) is a symmetric monoidal functor which preserves geometric realizations, it follows that any equivalence \( B_i \otimes_{A_i} A_j \simeq B_j \) induces

\[
\int_M B_i/k \otimes_{\int_M A_i/k} \int_M A_j/k \simeq \int_M (B_i \otimes_{A_i} A_j)/k \simeq \int_M B_j/k.
\]

Thus, if \( B_I \) is coCartesian, then \( \int_M B_I/k \) is coCartesian in \( \text{Mod}_{\int_M A_I/k}(\text{Fun}(I, \text{Mod}_k)) \).

\[\square\]

**Example 4.10.** Let \( X \) be a derived scheme over a base field \( k \) (more generally, \( k \) is allowed to be a connective commutative ring spectrum). Let \( J \) be the category (poset) of affine open sets in \( X \). Set \( I = J^{op} \). Let \( A_I : I \to \text{CAlg}(\text{Mod}_k) \) be the functor defined by \( I \ni U \mapsto \text{Spec} A_i \to A_i \in \text{CAlg}(\text{Mod}_k) \), where \( \mathcal{P}_\otimes = \text{Sp}_\otimes \). Consider \( \text{Mod}_{A_I}(\text{Fun}(I, \text{Mod}_k))^{\text{coCart}} \simeq (\text{Fun}(I, \text{Mod}(\text{Mod}_k)) \times_{\text{Fun}(I, \text{CAlg}(\text{Mod}_k))} \{A_I\})^{\text{coCart}} \) where the superscript in the latter indicates the full subcategory spanned by \( \text{coCart} \) sections. By [18, 3.3.3.2], the canonical symmetric monoidal functor

\[
\text{Mod}_{A_I}(\text{Fun}(I, \text{Mod}_k))^{\text{coCart}} \to \lim_{U_i \in I} \text{Mod}_{A_i}(\text{Mod}_k)
\]

is an equivalence. By the descent theory, the right-hand side is naturally equivalent to the symmetric monoidal stable \( \infty \)-category \( \text{QC}_\otimes(X) \) of quasi-coherent complexes on \( X \) (the reader may adopt \( \text{Mod}_{A_I}(\text{Fun}(I, \text{Mod}_k))^{\text{coCart}} \) as a definition of the category of quasi-coherent complexes/sheaves on \( X \)). We consider

\[
p_i : \text{Alg}_n(\text{QC}(X)) \simeq \text{Alg}_n(\text{Mod}_{A_I}(\text{Fun}(I, \text{Mod}_k))^{\text{coCart}}) \to \text{Fun}_\otimes((\text{Mfd}_n^\text{fr})^\otimes, \text{Mod}_{A_I}(\text{Fun}(I, \text{Mod}_k))^{\text{coCart}}) \]

\[
= \text{Fun}_\otimes((\text{Mfd}_n^\text{fr})^\otimes, \text{QC}(X)^\otimes)
\]

which carries \( \mathcal{B} \in \text{Alg}_n(\text{QC}(X)) \) to the symmetric monoidal functor \( p_i(\mathcal{B}) \) informally given by \( M \mapsto \int_M \mathcal{B}/X \) where \( \int_M \mathcal{B}/X \) should be regarded as the factorization homology of \( M \) with coefficients in \( \mathcal{B} \) over \( \text{QC}(X) \). Fix \( M \in \text{Mfd}_n^\text{fr} \). If we write \( \int_M (\_)/X \) for the composite \( \text{Alg}_n(\text{QC}(X)) \xrightarrow{\text{ev}_M} \text{Fun}_\otimes((\text{Mfd}_n^\text{fr})^\otimes, \text{QC}(X)^\otimes) \xrightarrow{\text{ev}_X} \text{QC}(X) \) with the evaluation at \( M \), Theorem 4.8 says that there exists a commutative diagram

\[
\begin{array}{ccc}
\text{Alg}_n(\text{QC}(X)) & \xrightarrow{\int_M (\_)/X} & \text{QC}(X) \\
\downarrow & & \downarrow \\
\text{Mod}_{\int_M A_I/k}(\text{Fun}(I, \text{Mod}_k))^{\text{coCart}} & \xrightarrow{\sim} & \text{QC}(\text{Map}(M, X))
\end{array}
\]

in \( \text{Cat}_\infty \). Here we use the notation in the proof of Theorem 4.8. Note that \( \int_M A_I/k \) is the composite \( I \xrightarrow{\text{ev}_I} \text{CAlg}(\text{Mod}_k) \otimes_M \text{CAlg}(\text{Mod}_k) \) which belongs to \( \text{CAlg}(\text{Fun}(I, \text{Mod}_k)) \simeq \text{Fun}(I, \text{CAlg}(\text{Mod}_k)) \) (cf. Remark 4.9). The vertical functor is induced by the base change along \( \int_M A_I/k \to A_I \) determined by the trivial map \( M \to * \). Suppose that the image of \( M \) in \( \mathcal{S} \) is connected. The mapping stack \( \text{Map}(M, X) \) is derived scheme obtained by gluing affine schemes \( \text{Spec}(A_i \otimes M) \) (cf. Section 3.2). Moreover, \( \text{Mod}_{\int_M A_I/k}(\text{Fun}(I, \text{Mod}_k))^{\text{coCart}} \) can be identified with \( \text{QC}(\text{Map}(M, X)) \), and the vertical functor is equivalent to \(*\)-pullback functor along the morphism \( X \to \text{Map}(M, X) \) induced by the constant maps. In particular, this means \( \int_M \mathcal{B}/X \in \text{QC}(X) \) is “extended to” an object of \( \text{QC}(\text{Map}(M, X)) \).
4.2. Let $k$ be a commutative ring spectrum and let $A \in \text{CAlg}(\text{Mod}_k(\text{Sp}))$. A typical situation is the case when $A$ be a commutative dg algebra over a field of characteristic zero $k$. Let $C$ be an $A$-linear small stable $\infty$-category.

Consider $\mathcal{HH}_\bullet(-/k) : \text{St}_k \rightarrow \text{Mod}^{St}_k$ be the Hochschild homology functor from $\text{St}_k$ (cf. Section 3.1). We interpret $\mathcal{HH}_\bullet(-/k)$ as a version of $\int_{S^1}(-)/k$ for stable $\infty$-categories in Section 4.1. In this Section, we prove a version of Theorem 4.2 in this setting. The fundamental idea of the proof is the same as that of Theorem 4.2. We note that $\text{St}_A \simeq \text{Mod}_{\text{Perf}_A}(\text{St}_k)$. Since $\mathcal{HH}_\bullet(-/k)$ is a symmetric monoidal functor, it induces

$$\mathcal{HH}_\bullet(-/k) : \text{St}_A \simeq \text{Mod}_{\text{Perf}_A}(\text{St}_k) \rightarrow \text{Mod}_{\mathcal{HH}_\bullet(\text{Perf}_A/k)(\text{Mod}^{St}_k)}.$$ 

Here we regard $\mathcal{HH}_\bullet(\text{Perf}_A/k)$ as an object of $\text{CAlg}_k^{St} = \text{Fun}(BS^1, \text{CAlg}_k)$. There exists an equivalence $\mathcal{HH}_\bullet(\text{Perf}_A/k) \simeq A \otimes_k S^1$ in $\text{CAlg}_k^{St}$, where $A \otimes_k S^1$ denote the tensor of $A$ by $S^1$ in $\text{CAlg}_k$ (see [15, Construction 4.3, Remark 4.4]).

Now consider $\mathcal{HH}_\bullet(C/k) \otimes_{\mathcal{HH}_\bullet(A/k)} A \simeq \mathcal{HH}_\bullet(C/A) \otimes_{A \otimes_k S^1} A$. Here $A \otimes_k S^1 \rightarrow A \simeq A \otimes_k \ast$ is determined by the $S^1$-equivariant map $S^1 \rightarrow \ast$ to the contractible space $\ast$.

**Lemma 4.11.** There exists a canonical equivalence $\mathcal{HH}_\bullet(C/k) \otimes_{\mathcal{HH}_\bullet(A/k)} A \simeq \mathcal{HH}_\bullet(C/A)$ in $\text{Mod}^{St}_A$.

**Proof.** The proof is a “many objects version” of the proof of Theorem 4.2 in the case of $M = S^1$. We achieve it by means of cyclic objects associated to spectral categories in Construction 4.13.

We set $S = \text{Spec} A$ and let $L_S$ denote the free loop space of $S$.

**Theorem 4.12.** We have a canonically defined object $\mathcal{HH}_\bullet(C/k) \in \text{QC}(LS)^{St}$ such that the pullback $i^* \mathcal{HH}_\bullet(C/k)$ along $i : S \rightarrow LS$ is naturally equivalent to $\mathcal{HH}_\bullet(C/A) \in \text{QC}(S)^{St}$.

**Construction 4.13.** We use the theory of symmetric spectra. The readers who do not know symmetric spectra are invited to skip the construction on the first reading. We use the notation and terminology in [14, Section 6]. We refer the reader to loc. cit. for details.

We denote by $\text{Sp}^{\Sigma}$ the symmetric monoidal category of symmetric spectra endowed with the stable $S$-model structure (see [23, Theorem 2.4, Proposition 2.5]). If $(\text{Sp}^{\Sigma})^c(W^{-1})$ denotes the $\infty$-category obtained from $(\text{Sp}^{\Sigma})^c$ by inverting weak equivalences (stable equivalences), there exists a symmetric monoidal equivalence $(\text{Sp}^{\Sigma})^c(W^{-1}) \simeq \text{Sp}$. Here $(-)^c$ indicates the full subcategory of cofibrant objects. See [19, 1.3.4, 4.1.7, 4.1.8] for localizations with respect to weak equivalences. Let $\text{CAlg}(\text{Sp}^{\Sigma})$ denote the category of commutative symmetric ring spectra, endowed with the model structure such that a morphism is a stable equivalence (resp. a fibration with respect to stable equivalences), there exists a canonical equivalence $\text{CAlg}(\text{Sp}^{\Sigma})^c(W^{-1}) \simeq \text{CAlg}(\text{Sp})$. Let $K$ be a cofibrant object in $\text{CAlg}(\text{Sp}^{\Sigma}_K)$, which represents $k \in \text{CAlg}(\text{Sp})$. Let $\text{Sp}^{\Sigma}_K$ be the category of $K$-module objects in $\text{Sp}^{\Sigma}_K$, which is endowed with the natural symmetric monoidal structure induced by the structure on $\text{Sp}^{\Sigma}_K$.

In virtue of [23, Theorem 2.6], there is a combinatorial symmetric monoidal projective model structure on $\text{Sp}^{\Sigma}_K$ satisfying the monoid axiom, in which a morphism is a weak equivalence (resp. a fibration) if the underlying morphism in $\text{Sp}^{\Sigma}_K$ is a stable equivalence (resp. a fibration with respect to stable $S$-model structure). We refer to this model structure as the stable $\Sigma$-category obtained from $\text{Sp}^{\Sigma}_K$ by inverting weak equivalences (stable equivalences), there exists a symmetric monoidal equivalence $(\text{Sp}^{\Sigma}_K)^c(W^{-1}) \simeq \text{Mod}_K$. Let $\text{CAlg}(\text{Sp}^{\Sigma}_K)$ be the category of commutative algebra objects in $\text{Sp}^{\Sigma}_K$, endowed with the model structure such that a morphism is a stable equivalence (resp. a fibration with respect to stable equivalences) if it is a stable equivalence (resp. a fibration) in $\text{Sp}^{\Sigma}_K$.

Let $\text{Cat}_{\mathcal{A}}$ (resp. $\text{Cat}_K$) denote the category of $\mathcal{A}$-spectral categories (resp. $K$-spectral categories), that is, categories enriched over the symmetric monoidal category $\text{Sp}^{\Sigma}_K$ (resp. $\text{Sp}^{\Sigma}_K$). Let $\text{Cat}_{\mathcal{A}}^c$ be the full subcategory of $\text{Cat}_{\mathcal{A}}$ that consists of pointwise cofibrant $\mathcal{A}$-spectral categories, that is, spectral categories whose Hom objects are cofibrant objects in $\text{Sp}^{\Sigma}_K$. We define $\text{Cat}_K^c$ in a similar way. The image of the forgetful functor $\text{Cat}_{\mathcal{A}}^c \rightarrow \text{Cat}_K$ is contained in $\text{Cat}_K^c$. To see this, it is enough to check...
that the forgetful functor $\text{Sp}^\Sigma(A) \to \text{Sp}^\Sigma(K)$ carries cofibrant objects to cofibrant objects in $\text{Sp}^\Sigma(K)$.

For this purpose, we recall that cofibrations in $\text{Sp}^\Sigma$ with respect to the stable $S$-model structure is the smallest weakly saturated class [18, A.1.2.2] of morphisms that contains $\{ S \otimes i \}_{i \in \text{Mon}}$ where $\text{Mon}$ is the class of monomorphisms of symmetric sequences, and $S \otimes i$ denotes the morphism of symmetric spectrum induced by $i$, namely, $S \otimes (-)$ is the left adjoint of the forgetful functor from $\text{Sp}^\Sigma$ to the category of symmetric sequences, see [23]. The class of cofibrations in $\text{Sp}^\Sigma(A)$ with respect to the stable $A$-model structure is the smallest weakly saturated class of morphisms containing $\{ A \otimes i = A \wedge (S \otimes i) \}_{i \in \text{Mon}}$.

Note that we assume that $A$ is a cofibrant object in $\text{CAlg}(\text{Sp}^\Sigma(K))$ so that $A$ is cofibrant in $\text{Sp}^\Sigma(K)$ (see [23, Proposition 4.1]). It follows that morphisms $A \otimes_K (K \otimes i)$ are cofibrations in $\text{Sp}^\Sigma(K)$. Since $\text{Sp}^\Sigma(A) \to \text{Sp}^\Sigma(K)$ preserves colimits, $\text{Sp}^\Sigma(A) \to \text{Sp}^\Sigma(K)$ preserves cofibrations.

Next, we briefly review the Hochschild homology functor (see [14]). Recall the definition of cyclic objects which determines Hochschild homology. Let $\mathbb{R}$ be a commutative ring spectrum and let $A$ be a pointwise cofibrant $\mathbb{R}$-spectral category. Let $A$ be the cyclic category (cf. [17]). The cyclic object $\Lambda^\text{op} \to \text{Sp}^\Sigma(\mathbb{R})$ is defined by the formula

$$\mathcal{H}(A/\mathbb{R})_p := \bigoplus_{(x_0, \ldots, x_p)} A(X_{p-1}, X_p) \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} A(X_0, X_1) \otimes_{\mathbb{R}} A(X_p, X_0).$$

The coproduct is taken over the set of sequences $(x_0, \ldots, x_p)$ of objects of $A$. See e.g. [14] for the definition of maps $\mathcal{H}(A/\mathbb{R})_p \to \mathcal{H}(A/\mathbb{R})_q$. Here we use the symbols $\otimes$ and $\oplus$ instead of the smash product $\wedge$ and the wedge sum $\vee$. The assignment $A \mapsto \mathcal{H}(A/\mathbb{R})_*$ determines a symmetric monoidal functor

$$\mathcal{H}(\_/: \mathbb{R})_* : \text{Cat}^c_{\text{hc}} \to \text{Fun}(\Lambda^\text{op}, \text{Sp}^\Sigma(\mathbb{R})^c).$$

We consider symmetric monoidal functors

$$\text{Cat}^c_{\text{hc}} \xrightarrow{\mathcal{H}(\text{\_/: } \mathbb{R})_*} \text{Fun}(\Lambda^\text{op}, \text{Sp}^\Sigma(\mathbb{K})^c) \to \text{Fun}(\Lambda^\text{op}, \text{Sp}^\Sigma(\mathbb{K})^c[W^{-1}]) \xrightarrow{L} \text{Fun}(\text{BS}^1, \text{Sp}^\Sigma(\mathbb{K})^c[W^{-1}]) \simeq \text{Mod}^S_{\mathbb{K}}.$$

There is a canonical functor $\text{Sp}^\Sigma(\mathbb{A})^c \to \text{Sp}^\Sigma(\mathbb{A})^c[W^{-1}]$ that induces the second functor. The third functor is the symmetric monoidal functor determined by left Kan extensions along the groupoid completion $\Lambda^\text{op} \to \text{BS}^1$. The composite carries Morita equivalences in $\text{Cat}^c_{\text{hc}}$ to equivalences in $\text{Mod}^S_{\mathbb{K}}$ (cf. [14, Lemma 6.11]). The composite induces a symmetric monoidal functor $\mathcal{H}(\_/: K) : \text{Cat}^c_{\text{hc}}[M^{-1}] \simeq \text{St}_k \to \text{Mod}^S_{\mathbb{K}}$ where $\text{Cat}^c_{\text{hc}}[M^{-1}]$ is the infinite category obtained by inverting Morita equivalences, and $\text{Cat}^c_{\text{hc}}[M^{-1}] \simeq \text{St}_k$ is a symmetric monoidal equivalence (see [14, Proposition 6.7]). We define $\mathcal{H}(\_/: A) : \text{Cat}^c_{\text{hc}}[M^{-1}] \simeq \text{St}_k \to \text{Mod}^S_{\mathbb{K}}$ in a similar way.

The forgetful functors $\text{Cat}^c_{\text{hc}} \to \text{Cat}^c_{\text{hc}}$ and $\text{Sp}^\Sigma(\mathbb{A})^c \to \text{Sp}^\Sigma(\mathbb{K})^c$ induces the diagram

$$\begin{array}{ccc}
\text{Cat}^c_{\text{hc}} & \xrightarrow{\mathcal{H}(\text{\_/: } \mathbb{K})_*} & \text{Fun}(\Lambda^\text{op}, \text{Sp}^\Sigma(\mathbb{A})^c) \\
| \pi_m & & | \\
\text{Cat}^c_{\text{hc}} & \xrightarrow{\mathcal{H}(\text{\_/: } \mathbb{K})_*} & \text{Fun}(\Lambda^\text{op}, \text{Sp}^\Sigma(\mathbb{K})^c) \\
\end{array}
\quad
\begin{array}{ccc}
\text{Fun}(\Lambda^\text{op}, \text{Sp}^\Sigma(\mathbb{A})^c[W^{-1}]) & \to & \text{Fun}(\text{BS}^1, \text{Mod}^S_{\mathbb{K}}) \\
| \pi_m & & | \\
\text{Fun}(\Lambda^\text{op}, \text{Sp}^\Sigma(\mathbb{K})^c[W^{-1}]) & \to & \text{Fun}(\text{BS}^1, \text{Mod}^S_{\mathbb{K}}).
\end{array}$$

The middle square commutes up to canonical homotopy. By [14, Lemma 6.9 (ii)] and the fact that $\text{Mod}^S_{\mathbb{A}} \to \text{Mod}^S_{\mathbb{K}}$ preserves colimits, we see that the right square also commutes. Let $B$ be a pointwise cofibrant $A$-spectral category, that we will think of as a model of an $A$-linear small stable $\infty$-category $C$. There exists a natural transformation $\theta : \mathcal{H}(\_/: \mathbb{K})_* \circ \pi_c \to \pi_m \circ \mathcal{H}(\_/: \mathbb{K})_*$ defined by the canonical maps

$$B(X_{p-1}, X_p) \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} B(X_0, X_1) \otimes_{\mathbb{K}} B(X_p, X_0) \to B(X_{p-1}, X_p) \otimes_{\mathbb{A}} \cdots \otimes_{\mathbb{A}} B(X_0, X_1) \otimes_{\mathbb{A}} B(X_p, X_0).$$

Let $B\mathbb{A}$ be the $A$-spectral category having one object $*$ such that the Hom spectrum is defined by $B\mathbb{A}(*, *) = A$. Namely, it is a unit object of $\text{Cat}_A$. Then $\mathcal{H}(B\mathbb{A}/A)_*$ is the constant cyclic object with value $A$. The induced map $\mathcal{H}(\pi_c(B\mathbb{A}/\mathbb{K})_* \to \pi_m(\mathcal{H}(B\mathbb{A}/A)_*)$ is a map in $\text{Fun}(\Lambda^\text{op}, \text{CAlg}(\text{Sp}^\Sigma(\mathbb{K})^c))$ because $B\mathbb{A}$ is a commutative algebra object. The functor $\pi_m \circ \mathcal{H}(\_/: \mathbb{K})_* \circ \pi_c$ is naturally promoted to $\alpha : \text{Cat}^c_{\text{hc}} \to \text{Mod}_{\text{const}(A)}(\text{Fun}(\Lambda^\text{op}, \text{Sp}^\Sigma(\mathbb{K})^c))$ where we write $\text{const}(A)$ for $\text{const}(\mathbb{A})$ and $\mathcal{H}(\text{\_/: } \mathbb{K})_* \circ \pi_c \circ \mathcal{H}(\_/: \mathbb{K})_* \circ \pi_c$. Likewise, $\mathcal{H}(\_/: \mathbb{K})_* \circ \pi_c$ is naturally promoted to $\beta : \text{Cat}^c_{\text{hc}} \to \text{Mod}_{\text{const}(A)}(\text{Fun}(\Lambda^\text{op}, \text{Sp}^\Sigma(\mathbb{K})^c))$. If $r$ denotes the restriction functor $\text{Mod}_{\text{const}(A)}(\text{Fun}(\Lambda^\text{op}, \text{Sp}^\Sigma(\mathbb{K})^c)) \to \text{Mod}_{\text{const}(A)}(\text{Fun}(\Lambda^\text{op}, \text{Sp}^\Sigma(\mathbb{K})^c)),$
there is a natural transformation $\theta' : \beta \to r \circ \alpha$ which extends $\theta$. We note that the image of $\mathcal{H}(\pi_c(B\mathbb{A}/K)_\bullet) \to \pi_m(\mathcal{H}(B\mathbb{A}/K)_\bullet)$ in $\text{Mod}_A^{S^1}$ is $A \otimes K S^1 \simeq \mathcal{H}(A/k) \to \mathcal{H}(A/A) \simeq A$ (induced by $S^1 \to *$) (see [15, Construction 4.3, Remark 4.4]). From these observations, passing to $\infty$-categories we have

$$
\begin{array}{c}
\xymatrix{
S_{t,A} \ar[d]^{\Pi_c} \ar[r]^{\mathcal{H}(/-/A)} & \text{Mod}_A^{S^1} \ar[d]^{\text{res}} \\
S_{t,k} \ar[r]^{\mathcal{H}(/-/k)} & \text{Mod}_k^{S^1}
}
\end{array}
$$

with $t : \mathcal{H}(/-/k) \circ \Pi_c \to \text{res} \circ \mathcal{H}(/-/A)$. Moreover, $t$ is promoted to a natural transformation between functors $S_{t,A} \to \text{Mod}_A^{\mathcal{H}(A/k)}(\text{Mod}_A^{S^1})$. The base change along $\mathcal{H}(A/k) \to A$ gives rise to a natural transformation between functors $S_{t,A} \to \text{Mod}_A^{S^1}$, which induces

$$
\phi : \mathcal{H}_*(B/k) \otimes \mathcal{H}_*(A/k) A \to \mathcal{H}_*(B/A)
$$

for each $B$, where we abuse notation by writing $B$ for the images of $B$ in $S_{t,A}$ and $S_{t,k}$.

Now we prove that $\phi$ is an equivalence. It is enough to prove an equivalence at the level of $\text{Fun}(\Lambda^\text{op}, \text{Sp}^\lozenge(K)^c[K^{-1}]) \simeq \text{Fun}(\Lambda^\text{op}, \text{Mod}_k)$. Write $C \to \text{const}_\infty(A)$ for the image of $\mathcal{H}(\pi_c(B\mathbb{A}/K)_\bullet) \to \pi_m(\mathcal{H}(B\mathbb{A}/K)_\bullet)$ in $\text{Fun}(\Lambda^\text{op}, \text{Mod}_k)$, where $\text{const}_\infty(A) : \Lambda^\text{op} \to \text{Mod}_k$ is the constant functor with value $A$. We write $P \to Q$ for the image of $\mathcal{H}(\pi_c(B)/K)_\bullet) \to \pi_m(\mathcal{H}(B/k)_\bullet)$ in $\text{Fun}(\Lambda^\text{op}, \text{Mod}_k)$. It suffices to show that $P \otimes C \text{const}_\infty(A) \to Q$ is an equivalence in $\text{Fun}(\Lambda^\text{op}, \text{Mod}_k)$. To this end, it is enough to prove that for any $p \geq 0$, the morphism of the $p$-th terms $(P \otimes C \text{const}_\infty(A))_p \to Q_p$ is an equivalence in $\text{Mod}_k$. Taking into account the definitions of the cyclic objects $\mathcal{H}(B\mathbb{A}/K)_\bullet$ and $\mathcal{H}(/-/k)_\bullet$, we are reduced to proving that

$$
\begin{array}{c}
\xymatrix{
(B(X_{p-1}, X_p) \otimes_k \cdots \otimes_k B(X_0, X_1) \otimes_k B(X_p, X_0)) \otimes (A \otimes_k \cdots \otimes_k A) A \\
B(X_{p-1}, X_p) \otimes A \cdots \otimes A B(X_0, X_1) \otimes A B(X_p, X_0)
}
\end{array}
$$

is an equivalence where $B(X_i, X_j)$ means the image of $B(X_i, X_j)$ in $\text{Mod}_k$, and $(A \otimes_k \cdots \otimes_k A) = A^{\otimes (p+1)} \to A$ is determined by the multiplication. This morphism is obviously an equivalence.

**Remark 4.14.** A slightly more effort allows $S$ to be a derived scheme (by working with a sheaf of commutative algebras $A$). We leave it to an interested reader.

**Remark 4.15.** From Construction 4.13, $\mathcal{H}_*(-/A) : S_{t,A} \to \text{Mod}_A^{S^1}$ factors as

$$
S_{t,A} \xrightarrow{\mathcal{H}_*(-/k)} \text{Mod}_A \otimes_k S^1(\text{Mod}_k^{S^1}) \xrightarrow{\text{res}_A \otimes_k S^1} \text{Mod}_A(\text{Mod}_k^{S^1}) \simeq \text{Mod}_A^{S^1}.
$$

Let $C_1 \to C_2 \to C_3$ be an exact sequence of $A$-linear small stable $\infty$-categories: it means that passing to $\text{Ind}$-categories, (i) $\text{Ind}(C_1) \to \text{Ind}(C_2)$ is fully faithful, (ii) the composite $\text{Ind}(C_1) \to \text{Ind}(C_2) \to \text{Ind}(C_3)$ is trivial, and (iii) $\text{Ind}(C_2)/\text{Ind}(C_1) \to \text{Ind}(C_3)$ is an equivalence where $\text{Ind}(C_2)/\text{Ind}(C_1)$ is a quotient $\text{Ind}(C_2)_{/\text{Ind}(C_1)} 0$ as a presentable stable $\infty$-category (cf. [5, Section 5]). In this situation, the localization theorem says that the induced sequence

$$
\begin{array}{c}
\xymatrix{
\mathcal{H}_*(C_1/k) \ar[r] & \mathcal{H}_*(C_2/k) \\
0 \ar[r] & \mathcal{H}_*(C_3/k)
}
\end{array}
$$

is a cofiber/fiber sequence in $\text{Mod}_A \otimes_k S^1(\text{Mod}_k^{S^1})$ (see [6, Theorem 7.1] for the localization theorem: in loc cit, spectral categories are treated, but the proof is applicable to $K$-spectral categories). This implies that the associated sequence $\Omega^p(C_1) \to \Omega^p(C_2) \to \Omega^p(C_3)$ of $D$-modules (constructed in Section 8) is also a cofiber/fiber sequence.
5. **Kodaira-Spencer morphism for a family of stable ∞-categories**

Let $A$ be a connective commutative dg algebra over $k$ and let $S = \text{Spec } A$ be a derived affine scheme over $k$. Suppose that $S$ is locally of finite presentation over $k$. Assume that $A$ is noetherian.

Let $\mathcal{C} = \mathcal{C}_A$ be an $A$-linear small stable ∞-category (cf. Section 2). Informally, we think of $\mathcal{C}_A$ as a family of stable ∞-categories over $\text{Spec } A$. In this section, we will construct a Kodaira-Spencer morphism for $\mathcal{C}$ over $A$. It is defined as a morphism of dg Lie algebras over $A$

$$T_{A/k}[-1] \rightarrow \mathcal{H}^\ast(\mathcal{C}/A)[1],$$

where $\mathcal{H}^\ast(D_{A/A})[1]$ is the shifted Hochschild cochain complex with a suitable Lie algebra structure, and $T_{A/k}$ is the tangent complex of $A$ over $k$, that is, the dual object of the cotangent complex $L_{A/k}$ in $\text{Mod}_A$.

5.1. Consider

$$S \xrightarrow{\Delta} S \times_k S \xrightarrow{\text{pr}_1} S$$

Let $	ilde{S} \times_k S$ denote the formal completion of $S \times_k S$ along $\Delta : S \to S \times_k S$ (see Section 3.4). The formal stack $\tilde{S} \times_k S : \text{Art}_{A}^{\text{tss}} \to S$ is informally given by

$$\text{Art}_{A}^{\text{tss}} \ni [A \to R \to A] \mapsto \text{Map}_{(\text{Art}_{k})_{S/S}}(\text{Spec } R, S \times_k S) \in S.$$}

Recall the equivalence $S_A^{\ast} \simeq \text{Lie}_A$ ([12, 1.5.6], Section 3.4). Then $S \times_k S$ corresponds to an object of $\text{Lie}_A$ whose underlying object is equivalent to the tangent complex $\Delta^!(\overline{T_{S \times_k S/S}[-1]})$. Since $\Delta^!(\overline{T_{S \times_k S/S}}) \simeq \Delta^!(\overline{\text{pr}_2^*(L_{S/k})}) \simeq L_{S/k}$ in $\text{Mod}_A$, it follows that $\Delta^!(\overline{T_{S \times_k S/S}[-1]}) \simeq T_{A/k}[-1]$. Here $\overline{\text{pr}}$ indicates the cotangent complex.

By abuse of notation we write $T_{A/k}[-1]$ for the corresponding object in $\text{Lie}_A$.

5.2. We review the functor associated to deformations of $\mathcal{C}_A = \mathcal{C} \subset \text{St}_A$. For the detailed construction, we refer to the reader to [15, Section 4.2]. Let $\text{Alg}_2(\text{Mod}_A)$ be the ∞-category of $E_2$-algebras and let $\text{Alg}_2^+ (\text{Mod}_A) = \text{Alg}_2(\text{Mod}_A)_{/A}$ be the ∞-category of augmented $E_2$-algebras. We often omit augmentations from the notation. Let $\text{RMod}_{\text{Perf}_{B}}(\text{St}_A)$ denote the ∞-category of $\text{Perf}_{B}$-module objects in $\text{St}_A$ (note that $B$ is an $E_2$-algebra so that $\text{Perf}_{B}$ belongs to $\text{Alg}_1(\text{St}_A)$). For a morphism $B \to B'$ in $\text{Alg}_2(\text{Mod}_A)$, there is the base change functor $\otimes_{\text{Perf}_{B}} \text{Perf}_{B'} : \text{RMod}_{\text{Perf}_{B}}(\text{St}_A) \to \text{RMod}_{\text{Perf}_{B'}}(\text{St}_A)$ which is a left adjoint to the functor $\text{RMod}_{\text{Perf}_{B'}}(\text{St}_A) \to \text{RMod}_{\text{Perf}_{B}}(\text{St}_A)$ induced by the restriction along $\otimes_{B} B' : \text{Perf}_{B} \to \text{Perf}_{B'}$. For $B \to A$ in $\text{Alg}_2^+ (\text{Mod}_A)$ we consider the ∞-category

$$\text{Def}_{E_2}(\mathcal{C})(B) : = \text{RMod}_{\text{Perf}_{B}}(\text{St}_A)^{\times}_{\text{St}_A} \{\mathcal{C}\}$$

such that $\text{RMod}_{\text{Perf}_{B}}(\text{St}_A) \to \text{St}_A \simeq \text{RMod}_{\text{Perf}_{B}}(\text{St}_A)$ is given by the base change along $B \to A$. An object of $\text{Def}_{E_2}(\mathcal{C})(B)$ can be regarded as a $B$-linear small stable ∞-category $C_B$ endowed with an equivalence $C_B \otimes_{\text{Perf}_{B}} \text{Perf}_{B'} \simeq \mathcal{C}$ in $\text{St}_A$. We think of it as the ∞-category of deformations of $\mathcal{C}$ to $B$. We write $\text{Def}_{E_2}(\mathcal{C})(B)$ for the largest ∞-groupoid (the largest Kan complex) contained in $\text{Def}_{E_2}(\mathcal{C})(B)$. Let

$$\text{Def}_{E_2}(\mathcal{C}) : \text{Alg}_2^+ (\text{Mod}_A) \to \tilde{S}$$

be the functor which is informally given by $[B \to A] \mapsto \text{Def}_{E_2}(\mathcal{C})(B)$. We call it the $E_2$-deformation functor of $\mathcal{C}$.

Let $\text{Def}(\mathcal{C}) : \text{Art}_A^{\text{tss}} \to \tilde{S}$ be the functor obtained from $\text{Def}_{E_2}(\mathcal{C})$ by the composition with the forgetful functor $\text{Art}_A^{\text{tss}} \to (C\text{Alg}_2^{\text{tss}})_{/A} \simeq (C\text{Alg}_2^{\text{tss}})_{/A/\text{A}} \to \text{Alg}_2^+ (\text{Mod}_A)$.

5.3. We define a morphism $\tilde{S} \times_k S \to \text{Def}(\mathcal{C})$ in $\text{Fun}(\text{Art}_A^{\text{tss}}, \mathcal{S})$. Let $X : \text{Art}_A^{\text{tss}} \to \tilde{S}$ be a formal prestack, i.e., a functor. We define the ∞-category of small stable ∞-categories over $X$ to be $\text{lim}_{\text{Spec } R \to X} \text{St}_R$ where the limit is taken over $(TS\mathcal{A})_{/\times}$. Set $\text{St}_X := \text{lim}_{\text{Spec } R \to X} \text{St}_R$. There is a canonical “augmentation” $\text{St}_X \to \text{St}_A$. The formal prestack $X$ is a colimit of $(TS\mathcal{A})_{/\times} \to TS\mathcal{A} \to \text{Fun}(\text{Art}_A^{\text{tss}}, \mathcal{S})$. It follows that there exists a natural equivalence

$$\text{Map}_{\text{Fun}(\text{Art}_A^{\text{tss}}, \mathcal{S})}(X, \text{Def}(\mathcal{C})) \simeq \text{St}_X^{\times} \times_{\text{St}_A^{\times}} \{\mathcal{C}\}.$$
by the base change \( pr_2^*(C) = C \otimes_{\text{Perf}_A} \text{Perf}_A \otimes_k A \in \text{St}_A \otimes_k A \) along the second projection \( pr_2 : S \times_k S = \text{Spec} A \otimes_k A \to \text{Spec} A = S \) in the natural way.

**Remark 5.1.** Informally, \( K_S \mathcal{C} \) on objects can be described as follows: Let \( [A \to R \to A] \in \text{Art}_{A}^{\text{tsz}} \) and let \( \alpha \times \beta : \text{Spec} R \to S \times_k S \) be an object of \( (\text{Aff}_k)_S / S \) which is regarded as an object of \( S \times_k S (R) \).

Then \( K_S \mathcal{C} \) sends \( \alpha \times \beta \) to the base change \( C \otimes_{\text{Perf}_A} \text{Perf}_R \) along \( \beta^* : \text{Perf}_A \to \text{Perf}_R \) with the equivalence \( (C \otimes_{\text{Perf}_A} \text{Perf}_R) \otimes_{\text{Perf}_R} \text{Perf}_A \simeq C \), which is a deformation of \( C \) to \( R \).

**5.4.** Let \( D_n = \text{Alg}_{E_n}^+ (\text{Mod}_A)^{\text{op}} \to \text{Alg}_{E_n}^+ (\text{Mod}_A) \) be the \( E_n \)-Koszul duality functor which sends an augmented \( E_n \)-algebra \( p : B \to A \) in \( \text{Mod}_A \) to the \( E_n \)-Koszul dual \( D_n(B) \to A \). The \( E_n \)-Koszul dual \( q : D_n(B) \to A \) has the following characterization: it represents the functor \( \text{Alg}_n (\text{Mod}_A)^{\text{op}} \to \mathcal{S} \) informally given by

\[
[q : C \to A] \mapsto \text{Map}_{\text{Alg}_n (\text{Mod}_A)} (B \otimes A C, A) \times \text{Map}_{\text{Alg}_n (\text{Mod}_A)} (B, A) \times \text{Map}_{\text{Alg}_n (\text{Mod}_A)} (C, A) \{ (p, q) \}.
\]

In particular, there is a tautological morphism \( B \otimes_A D_n(B) \to A \) in \( \text{Alg}_n (\text{Mod}_A) \) which extends \( p \) and \( \hat{p} \) up to homotopy. For Koszul duality of \( E_n \)-algebras we refer the reader to [19] or [20, X 4.4] (see also [15, Section 3] for the brief review).

Let \( \mathcal{F}_A \mathcal{E}_2^+: \text{Alg}_{E_2}^+ (\text{Mod}_A) \to \mathcal{S} \) be a functor informally given by

\[
[B \to A] \mapsto \text{Map}_{\text{Alg}_n (\text{Mod}_A)} (B \otimes A C, A) = \text{Map}_{\text{Alg}_n (\text{Mod}_A)} (D_2(B), \mathcal{H}^*(C/A)).
\]

In other words, \( \mathcal{F}_A \mathcal{E}_2^+(C/A) \) is the composite of \( D_2 \) and the functor \( \text{Alg}_{E_2}^+ (\text{Mod}_A)^{\text{op}} \to \mathcal{S} \) represented by \( \text{pr}_2 : A \otimes \mathcal{H}^*(C/A) \to A \). Let \( \mathcal{F}_A \mathcal{E}_2^+(C/A) : \text{Art}_{A}^{\text{tsz}} \to \mathcal{S} \) be the functor obtained from \( \mathcal{F}_A \mathcal{E}_2^+(C/A) \) by the composition with \( \text{Art}_{A}^{\text{tsz}} \to \text{Alg}^{\text{tsz}}_n (\text{Mod}_A)^{\text{op}} \to \text{Alg}_{E_2}^+ (\text{Mod}_A) \). See [15, Section 5.2, Lemma 5.7] for these functors.

There is a (canonical) morphism

\[
J_C^E : \text{Def}_{E_2} (\mathcal{C}) \to \mathcal{F}_A \mathcal{E}_2^+(C/A)
\]

in \( \text{Fun}(\text{Alg}_{E_2}^+ (\text{Mod}_A), \mathcal{S}) \) (see [15, Section 5.2] for the precise construction). The composition with \( \text{Art}_{A}^{\text{tsz}} \to \text{Alg}_{E_2}^+ (\text{Mod}_A) \) induces a morphism

\[
J_C : \text{Def}(\mathcal{C}) \to \mathcal{F}_A \mathcal{E}_2^+(C/A)
\]

in \( \text{Fun}(\text{Art}_{A}^{\text{tsz}}, \mathcal{S}) \).

**Remark 5.2.** Roughly speaking, \( J_C^E \) is universal among morphisms into functors arising from augmented \( E_2 \)-algebras (see [20, X, 5.3.16] for a statement: we will not use the universal property, but the argument in *loc. cit.* can be applied to our situation). Informally, \( J_C^E \) can be described as follows. Let \( B \to A \) be in \( \text{Alg}_{E_2}^+ (\text{Mod}_A) \) and let \( \beta_B = (C_B, \mathcal{C}_B \otimes_{\text{Perf}_A} \text{Perf}_A \simeq C) \) be a deformation of \( C \) to \( B \). The tautological morphism \( B \otimes_A D_2(B) \to A \) induces a morphism \( \text{Perf}_B \otimes \text{Perf}_{D_2(B)} \to \text{Perf}_A \) in \( \text{Alg}_1 (\text{St}_A) \), so that \( \beta_B \) is a left \( \text{Perf}_B \otimes \text{Perf}_{D_2(B)} \)-module. Thus, we may regard \( \text{Perf}_A \) as a \( \text{Perf}_B \)-\( \text{Perf}_{D_2(B)} \)-bimodule where \( \text{Perf}_{D_2(B)}^{\text{op}} \) means the “opposite algebra” of \( \text{Perf}_{D_2(B)} \). The relative tensor product \( \mathcal{C}_B \otimes_{\text{Perf}_B} \text{Perf}_A \) is a right \( \text{Perf}_{D_2(B)}^{\text{op}} \)-module, that is, a left \( \text{Perf}_{D_2(B)} \)-module. In particular, \( C \) is promoted to a left \( \text{Perf}_{D_2(B)} \)-module. By the Ind-construction it gives rise to a left \( \text{LM}_{\text{Mod}_{D_2(B)}} \)-module structure on \( D := \text{Ind}(C) \in \text{Pr}_A^L \). A left \( \text{LM}_{\text{Mod}_{D_2(B)}} \)-module structure on \( D \) amounts to a morphism \( \alpha_{\mathcal{C}_B} : \text{LM}_{\text{Mod}_{D_2(B)}} \to \text{End}_A(D) \) in \( \text{Alg}_1 (\text{Pr}_A^L) \) (see Section 3.1 for \( \text{End}_A(D) \)). By the definition of Hochschild cohomology \( \mathcal{H}^*(C/A) \), there is a canonical equivalence \( \text{Map}_{\text{Alg}_n (\text{Mod}_A)} (D_2(B), \mathcal{H}^*(C/A)) \simeq \text{Map}_{\text{Alg}_n (\text{Mod}_A)} (\text{LM}_{\text{Mod}_{D_2(B)}}, \text{End}_A(D)) \). Let \( \beta_{\mathcal{C}_B} : D_2(B) \to \mathcal{H}^*(C/A) \) be a morphism corresponding to \( \alpha_{\mathcal{C}_B} \) through the equivalence. Then \( J_C^E \) carries \( \beta_{\mathcal{C}_B} \) to \( \beta_{\mathcal{C}_B} \).

**5.5.** Let

\[
U_2 : \text{Lie}_A \to \text{Alg}_{E_2}^+ (\text{Mod}_A) : \text{res}_{E_2 / \text{Lie}}
\]

be the adjoint pair where the left adjoint \( U_2 \) is the universal enveloping \( E_2 \)-algebra functor, that is an “\( E_2 \)-generalization” of the universal enveloping functor (cf. [15, Section 3.5]). The composite \( \text{Alg}_{E_2}^+ (\text{Mod}_A) \to \text{Lie}_A \) \( \text{Lie}_A \to \text{Mod}_A \) is equivalent to the shift functor given by \( B \mapsto B[1] \).
Set $G_C = \text{res}_{E_2/Lie}(A \oplus \mathcal{H}^*(C/A) \to A)$. The underlying object of $G_C$ is equivalent to $\mathcal{H}^*(C/A)[1]$. Let $F_{G_C}$ be the formal stack associated to $G_C$. As proved in [15, Lemma 5.7], there exists a canonical equivalence

$$F_{A \oplus \mathcal{H}^*(C/A)} \simeq F_{G_C}.$$

**Construction 5.3.** We consider the sequence of functors

$$F_{\mathcal{T}_{A/k}[-1]} \simeq S \times_k S \overset{\mathcal{KS}}{\longrightarrow} \text{Def}(C) \overset{J_C}{\longrightarrow} F_{A \oplus \mathcal{H}^*(C/A)} \simeq F_{G_C}.$$

Since $A$ is noetherian, there is the equivalence $\hat{S}^*_A \simeq \text{Lie}_A$. The composite functor $F_{\mathcal{T}_{A/k}[-1]} \to F_{G_C}$ is a morphism in $\hat{S}^*_A$. Passing to $\text{Lie}_A$ we have a morphism in $\text{Lie}_A$

$$\text{KS}_C : \mathcal{T}_{A/k}[-1] \longrightarrow G_C.$$

We refer to $\text{KS}_C$ as the Kodaira-Spencer morphism for $C$.

**Remark 5.4.** To define $\text{KS}_C$, the noetherian condition is not necessary because we need only the functor $L : \hat{S}^*_A \to \text{Lie}_A$. To understand $\text{KS}_C$, let us consider the simple situation. Suppose that $R = A \oplus A[n] \in \text{Art}_{A[k]}$ is the trivial square extension of $A$ by $A[n]$ ($n \geq 0$). For $L \in \text{Lie}_A$, there are equivalences $F_L(R) \simeq \text{Map}_{\text{Lie}_A}(\text{Free}_{\text{Lie}_A}(A[-n-1]), L) \simeq \Omega^\infty(L[n+1])$ in $S$. We will consider $F_{\mathcal{T}_{A/k}[-1]}(R) \simeq \Omega^\infty(\mathcal{H}^*(C/A)[n+2]) \simeq G_{C}$. The space $F_{\mathcal{T}_{A/k}[-1]}(R)$ can be identified with $\text{Map}_{(\text{CAlg}_{k}^S)/A}(id : A \to A, R \to A)$. Let $p \in \pi_0(\Omega^\infty(\mathcal{T}_{A/k}[n]))$ and suppose that $f_p : A \to R$ in $(\text{CAlg}_{k}^S)_{/A}$ corresponds to $p$. Then we have a deformation $f_p^*C \subset C \otimes \text{Perf}_{/A} \text{Perf}_{/A}$ of $C$ to $R$. The image of $f_p^*C$ under $J_C$ determines a class/element $c(f_p^*C)$ of $\pi_0(\Omega^\infty(\mathcal{H}^*(C/A))[n+2]) \simeq H^n(\mathcal{H}^*(C/A) = H^n(\mathcal{H}^*(C/A)).$

By construction, $\pi_0(\Omega^\infty(\mathcal{T}_{A/k}[n])) = H^n(\mathcal{T}_{A/k}) \to \pi_0(\Omega^\infty(\mathcal{H}^*(C/A))[n+2]) \simeq H^n(\mathcal{H}^*(C/A))$ sends $p$ to $c(f_p^*C)$.

6. The Hochschild pair and Kodaira-Spencer morphism

6.1. The Lie algebra action and the extended Lie algebra action on Hochschild Homology

We assume that $A$ is a commutative noetherian dg algebra over $k$. Suppose that $C$ is an $A$-linear small stable $\infty$-category. We first recall an algebraic structure on the pair $(\mathcal{H}^*(C/A), \mathcal{H}^*(C/A))$. The pair $(\mathcal{H}^*(C/A), \mathcal{H}^*(C/A))$ is an algebra over the so-called Kontsevich-Soibelman operad $\text{KS}$. We use the construction in our previous work [14]. We will not recall the (colored) operad $\text{KS}$ in this paper, but we will explain equivalent algebraic data. According to [14, Theorem 1.2], an algebra in $\text{Mod}_A$ over $\text{KS}$ amounts to a triple of the following data

1. an $S^1$-action on $\mathcal{H}^*(C/A)$ in $\text{Mod}_A$. That is, $\mathcal{H}^*(C/A)$ is an object of $\mathcal{H}^*(C/A) \in \text{Mod}_A^{S^1} = \text{Fun}(BS^1, \text{Mod}_A)$.
2. an $E_2$-algebra structure on $\mathcal{H}^*(C/A)$,
3. an $S^1$-equivariant left module action of $\mathcal{H}^*(C/A)$ on $\mathcal{H}^*(C/A)$.

(Note that $\mathcal{H}^*(\mathcal{H}^*(C/A)) \in \text{Alg}_1(\text{Mod}_A^{S^1})$ and $\mathcal{H}^*(C/A) \in \text{Alg}_1(\text{Mod}_A^{S^1})$.)

More precisely, an algebra in $\text{Mod}_A$ over $\text{KS}$ is equivalent to giving an object of

$$\text{Alg}_2(\text{Mod}_A) \times_{\text{Alg}_1(\text{Mod}_A^{S^1})} \text{LMod}(\text{Mod}_A^{S^1})$$

where $\text{Alg}_2(\text{Mod}_A) \to \text{Alg}_1(\text{Mod}_A^{S^1})$ is induced by $\mathcal{H}^*(-/A)$.

Next, we recall Lie algebra actions on $\mathcal{H}^*(C/A)$ arising from the pair $(\mathcal{H}^*(C/A), \mathcal{H}^*(C/A))$. For details we refer to [15, Section 6]. Let $\text{End}(\mathcal{H}^*(C/A))$ denote the endomorphism associative algebra object in $\text{Mod}_A^{S^1}$ (cf. [19, 4.7.1]). The above datum (3) is equivalent to giving a morphism in $\text{Alg}_1(\text{Mod}_A^{S^1})$

$$\mathcal{H}^*(\mathcal{H}^*(C/A)) \longrightarrow \text{End}(\mathcal{H}^*(C/A)).$$

For $L \in \text{Lie}_A$ we let $L^{S^1}$ denote the cotensor of $L$ by $S^1$. More explicitly, $L^{S^1}$ is given by the fiber product $L \times_{S^1} L$ in $\text{Lie}_A$. (However, there is one exception to this: $\text{End}^L(\mathcal{H}^*(C/A))^{S^1}$ appeared below indicates the homotopy fixed points of an $S^1$-action.) Let $U_1 : \text{Lie}_A \to \text{Alg}_1(\text{Mod}_A)$ denote the universal enveloping algebra functor. Let $U_2(\mathcal{G}_C) \to \mathcal{H}^*(C/A)$ be the counit map arising from the
adjoint pair $U_2 : \text{Lie}_A \rightleftharpoons \text{Alg}_A(\text{Mod}_A)$ (we here adopt the unaugmented version). It gives rise to a sequence of morphisms

$$U_1(G^S_1) \simeq \mathcal{H}_\bullet(U_2(G_c)/A) \rightarrow \mathcal{H}_\bullet(\mathcal{H}_\bullet^*(C)/A) \rightarrow \text{End}(\mathcal{H}_\bullet(C/A))$$

in $\text{Alg}_A(\text{Mod}_A^S)$, where the first equivalence is proved in [15, Proposition 6.2], and the second morphism is induced by $U_2(G_c) \rightarrow \mathcal{H}_\bullet^*(C/A)$. Let $i : G_c \rightarrow G^S_1$ be the morphism in $\text{Lie}_A$ induced by $S^1 \rightarrow \ast$. Since $S \rightarrow \ast$ is $S^1$-equivariant, $G_c \rightarrow G^S_1$ is promoted to a morphism in $\text{Lie}_A^S = \text{Fun}(BS^1, \text{Lie}_A)$ where the $S^1$-action on $G_c$ is trivial. Using the composite of the above sequence we have a sequence of morphisms

$$U_1(G_c) \rightarrow U_1(G^S_1) \rightarrow \text{End}(\mathcal{H}_\bullet(C/A)).$$

in $\text{Alg}_A(\text{Mod}_A^S)$. Let $\text{End}^L(\mathcal{H}_\bullet(C/A)) \in \text{Lie}_A^S$ denote the dg Lie algebra with $S^1$-action associated to $\text{End}(\mathcal{H}_\bullet(C/A))$. Then these morphisms give rise to morphisms in $\text{Lie}_A^S$:

$$G_c \overset{i}{\rightarrow} G^S_1 \overset{\hat{A}_L}{\rightarrow} \text{End}^L(\mathcal{H}_\bullet(C/A)).$$

We write $A_L^C$ for the composite $\hat{A}_L \circ i : G_c \rightarrow \text{End}^L(\mathcal{H}_\bullet(C/A))$. (This symbol $A_L^C$ is different from $A_L^C$ in [15] which means the induced morphism $G_c \rightarrow \text{End}^L(\mathcal{H}_\bullet(C/A))^S_1$ in $\text{Lie}_A$.)

**Definition 6.1.** We call $A_L^C$ the canonical Lie algebra action of $G_c$ on $\mathcal{H}_\bullet(C/A)$. We call $\hat{A}_L^C$ the canonical extended Lie algebra action of $G^S_1$ on $\mathcal{H}_\bullet(C/A)$.

**6.2. Lie algebra actions of $T_{A/k}[-1]$ and $T_{A/k}[-1]^{S^1}$.**

**Construction 6.2.** Recall that $T_{A/k}[-1]$ is the dg Lie algebra that corresponds to $S \otimes_k S$. Let $KS_C : T_{A/k}[-1] \rightarrow G_c$ be the Kodaira-Spencer morphism (see Section 5). Recall the sequence $G_c \overset{\iota}{\rightarrow} G^S_1 \overset{\hat{A}_L}{\rightarrow} \text{End}^L(\mathcal{H}_\bullet(C/A))$ (see Section 6.1). We obtain the commutative diagram in $\text{Lie}(\text{Mod}_A^S)$:

$$\begin{array}{ccc}
T_{A/k}[-1] & \rightarrow & T_{A/k}[-1]^{S^1} \\
KS_C \downarrow & & \downarrow KS_C \\
G_c & \overset{\iota}{\rightarrow} & G^S_1 \overset{\hat{A}_L}{\rightarrow} \text{End}^L(\mathcal{H}_\bullet(C/A))
\end{array}$$

where $T_{A/k}[-1]^{S^1}$ is the cotensor of $T_{A/k}[-1]$ by $S^1$, and $T_{A/k}[-1] \rightarrow T_{A/k}[-1]^{S^1}$ is induced by $S^1 \rightarrow \ast$ ($\ast$ is the contractible space). The vertical morphism $KS_C^S : T_{A/k}[-1]^{S^1} \rightarrow G^S_1$ is induced by $KS_C$. The dg Lie algebras $T_{A/k}[-1]$ is endowed with the trivial $S^1$-action. It gives rise to an $S^1$-equivariant Lie algebra action of $T_{A/k}[-1]$ on $\mathcal{H}_\bullet(C/A)$, that is, $T_{A/k}[-1] \rightarrow \text{End}^L(\mathcal{H}_\bullet(C/A))$ in $\text{Lie}_A^S$. Moreover, this action factors through $T_{A/k}[-1] \rightarrow T_{A/k}[-1]^{S^1}$. The $S^1$-action on $T_{A/k}[-1]^{S^1}$ comes from the action of $S^1$ on $S^1$ defined by the multiplication $S^1 \times S^1 \rightarrow S^1$.

We give immediate consequences of Construction 6.2. Let $L$ be a dg Lie algebra over $A$. Let $\text{Rep}(L)(\text{Mod}_A)$ be the stable category of representations of $L$. We define $\text{Rep}(L)(\text{Mod}_A)$ to be $\text{LMod}_{U_1(L)}(\text{Mod}_A)$ where $U_1$ is the universal enveloping algebra functor $\text{Lie}_A \rightarrow \text{Alg}_A(\text{Mod}_A)$ (see e.g. [15, Section 3.5]). By the diagram in Construction 6.2, $\mathcal{H}_\bullet(C/A)$ is promoted to an object of $\text{Rep}(T_{A/k}[-1])(\text{Mod}_A)$. Note that the action of $T_{A/k}[-1]$ on $\mathcal{H}_\bullet(C/A)$ is extended to an action of $T_{A/k}[-1]^{S^1}$. We note that the diagram in Construction 6.2 is a diagram in $\text{Fun}(BS^1, \text{Mod}_A)$. For $L \in \text{Fun}(BS^1, \text{Lie}_A)$, we set $\text{Rep}(L)(\text{Mod}_A^S) = \text{LMod}_{U_1(L)}(\text{Mod}_A^S)$. Then $\mathcal{H}_\bullet(C/A)$ is promoted to an object of $\text{Rep}(T_{A/k}[-1]^{S^1})(\text{Mod}_A^S)$.

**Definition 6.3.** We shall refer to an object of $\text{Rep}(T_{A/k}[-1])(\text{Mod}_A^S)$ determined by $A_L^C \circ KS_C : T_{A/k}[-1] \rightarrow \text{End}^L(\mathcal{H}_\bullet(C/A))$ as the canonical $T_{A/k}[-1]$-module $\mathcal{H}_\bullet(C/A)$. We shall refer to an object of $\text{Rep}(T_{A/k}[-1]^{S^1})(\text{Mod}_A^S)$ determined by $A_L^C \circ KS_C^S : T_{A/k}[-1]^{S^1} \rightarrow \text{End}^L(\mathcal{H}_\bullet(C/A))$ as the canonical $T_{A/k}[-1]^{S^1}$-module $\mathcal{H}_\bullet(C/A)$.
Let us describe \( T_{A/k}[−1] \rightarrow T_{A/k}[−1]S^1 \) in terms of formal stacks. For this purpose, we first recall the (free) loop space of a derived scheme from the viewpoint of functors (cf. Section 3.2). Let \( X \) be a derived scheme over \( k \). Let \( L_XX \) be the free loop space of \( X \) which is defined as \( L_XX \cong X \times_X X \). If we regard it as a functor, \( L_XX : \text{CAlg}_k^{op} \rightarrow S \) is given by \( R \mapsto \text{Map}_{S}(S^1, X(R)) = X(R) \times X(R) \times X(R) \times X(R) \times X(R) \times X(R) \) where \( X(R) \) is the space \( \text{Map}_{\text{Fun}((\text{CAlg}_k^{op}, S), \text{Spec} R, X)} \) of \( R \)-valued points. The derived scheme \( L_XX \) has the \( S^1 \)-action induced by the canonical action on the domain in \( \text{Map}_{S}(S^1, X(R)) \). The evident map \( S^1 \rightarrow \) to the contractible space \( * \) induces \( X(R) = \text{Map}_{\mathcal{C}}(*, X(R)) = \text{Map}_{\mathcal{C}}(S^1, X(R)) \), so that there is a \( S^1 \)-equivariant canonical morphism \( \iota : X \rightarrow L_XX \) where the \( S^1 \)-action on \( X \) is trivial. This may be regarded as the morphism induced by constant loops.

Let \( S \times_k LS \) be the formal stack defined as the formal completion of \( S \times_k LS \) along the graph map \( S \rightarrow S \times_k LS \) induced by \( \iota : S \rightarrow LS \). Since \( S \times_k LS \) can be obtained from \( S \times_k S \) by the cotensor by \( S^1 \in \hat{\mathcal{S}}_A^s \), \( S \times_k LS \) corresponds to the dg Lie algebra \( T_{A/k}[−1]S^1 \) (obtained from \( T_{A/k}[−1] \) by the cotensor by \( S^1 \)). The morphism \( T_{A/k}[−1] \rightarrow T_{A/k}[−1]S^1 \) induced by \( S^1 \rightarrow * \) corresponds to

\[
S \times_k S \rightarrow S \times_k LS
\]

induced by \( S \rightarrow LS \).

### 6.3. Complexes/sheaves on formal stacks

Let \( X : \text{Art}^{tsz}_A \rightarrow S \) be a functor, that is, a formal prestack. We define the stable \( \infty \)-category \( QC_H(X) \) of quasi-coherent complexes on \( X \) (these are also called quasi-coherent sheaves in the literature, but we prefer to call them complexes). Let \( \text{Art}^{tsz}_A \rightarrow \hat{\text{Cat}}_\infty \) be the functor which carries \( R \) to \( \text{Mod}_R \rightarrow \text{Mod}_{R'} \). A morphism \( R \rightarrow R' \) maps to the base change functor \( (R) \otimes_R R' : \text{Mod}_R \rightarrow \text{Mod}_{R'} \). The functor corresponds to the base change of the coCartesian fibration \( \text{Mod}((\text{Mod}_A)) \rightarrow \text{CAlg}((\text{Mod}_A)) \) along \( \text{Art}^{tsz}_A \rightarrow \text{CAlg}((\text{Mod}_A)) \). Let \( \text{Art}^{tsz}_A \rightarrow \text{Fun}((\text{CAlg}_A^{op}, S)) \) be the formally faithful functor induced by the Yoneda embedding. Since \( \text{CAlg}_A^{op} \) has small limits, it follows that there exists a right Kan extension

\[
QC_H : \text{Fun}(\text{Art}^{tsz}_A, S)^{op} \rightarrow \hat{\text{Cat}}_\infty
\]

where \( QC_H(X) \) is equivalent to \( \lim_{\text{Spec} R \rightarrow X} \text{Mod}_{R} \) where the limit is taken over \( (TSZ_A)_X \) (\( TSZ_A \) denotes the opposite category of \( \text{Art}^{tsz}_A \)). If \( f : X \rightarrow Y \) is a morphism of formal presheaves, we denote by \( f^* : QC_H(Y) \rightarrow QC_H(X) \) the functor induced by \( QC_H \). We refer to \( f^* \) as the pullback functor. Since \( \text{Art}^{tsz}_A \) is a full subcategory of \( \text{Cat}(\text{CAlg}_A^{op}, A/)) \) and \( QC_H(A) = \text{Mod}_A \), the functor \( QC_H \) can be extended to \( \text{Fun}(\text{Art}^{tsz}_A, S)^{op} \rightarrow (\hat{\text{Cat}}_\infty)_{\text{Mod}_{A^{op}}}^{\text{op}} / \text{Mod}_{A} \).

Next, we define the stable \( \infty \)-category \( \text{Rep}_H(X) \). Let \( \text{Alg}_{1}(\text{Mod}_{A})^{op} \rightarrow \hat{\text{Cat}}_\infty \) be the functor informally given by \( B \mapsto \text{LMod}_B := \text{LMod}_{B(\text{Mod}_A)} \), which corresponds to the Cartesian fibration \( \text{LMod}_{B(\text{Mod}_A)} \rightarrow \text{Alg}_{1}(\text{Mod}_{A}) \). Consider the composite

\[
u : \text{Art}^{tsz}_A \rightarrow \text{Alg}_{1}(\text{Mod}_{A}) \xrightarrow{\mathbb{D}_1} \text{Alg}_{1}(\text{Mod}_{A})^{op} \rightarrow \hat{\text{Cat}}_\infty,
\]

where \( \mathbb{D}_1 \) is \( \mathbb{E}_1 \)-Koszul duality functor (cf. e.g. [20, X, 4.4], Section 5.4, [15]). We define

\[
\text{Rep}_H : \text{Fun}(\text{Art}^{tsz}_A, S)^{op} \rightarrow \hat{\text{Cat}}_\infty
\]

to be a right Kan extension of \( \nu \) along \( \text{Art}^{tsz}_A \rightarrow \text{Fun}(\text{Art}^{tsz}_A, S)^{op} \). By definition, \( \text{Rep}_H(X) \) is the limit \( \lim_{\text{Spec} R \rightarrow X} \text{LMod}_{\mathbb{D}_1(R)(\text{Mod}_{A})} \) where the limit is taken over \( (TSZ_A)_X \). The functor \( \text{Rep}_H \) has an extension \( \text{Fun}(\text{Art}^{tsz}_A, S)^{op} \rightarrow (\hat{\text{Cat}}_\infty)_{\text{Mod}_{A^{op}}}^{\text{op}} / \text{Mod}_{A} \) for the same reason as above. Since \( \mathbb{D}_1(R) \simeq U_1(\mathbb{D}_\infty(\mathbb{R})) \) for \( R \in \text{Art}^{tsz}_A \) (see e.g. [15, Proposition 3.3]), there exist categorical equivalences \( u(R) \simeq \text{LMod}_{\mathbb{D}_1(\mathbb{R})(\text{Mod}_{A})} \simeq \text{LMod}_{U_1(\mathbb{D}_\infty(\mathbb{R}))}(\text{Mod}_{A}) = \text{Rep}(\mathbb{D}_1(\mathbb{R}))(\text{Mod}_{A}) \).

**Lemma 6.4.** Let \( L \in \text{Lie}_{A} \). Let \( \mathcal{F}_L = X \) be the formal stack associated to \( L \). Then there exists a canonical equivalence \( \text{Rep}_H(X) \simeq \text{Rep}(L)(\text{Mod}_{A}) \).

**Proof.** Let \( \text{Lie}^{f}_A \) be the full subcategory of \( \text{Lie}_{A} \) spanned by free dg Lie algebras of the form \( \text{Free}_{\text{Lie}}(\mathbb{E}_1 \in \mathbb{E}_{n}^{\mathbb{R}}[p]) \) \( (p_1 \leq -1) \) (see Section 3.4). The equivalence \( \mathbb{D}_\infty : TSZ_A \simeq \text{Lie}^{f}_A \) induces \( \text{Fun}(\text{Art}^{tsz}_A, S) \simeq \text{Fun}(\text{Lie}^{f}_A, S) \). The fully faithful right adjoint of the composite of localization
functors $\text{Fun}((\text{Lie}_A^f)^{op}, S) \rightarrow \mathcal{P}_\Sigma(\text{Lie}_A^f) \rightarrow \mathcal{P}_\Sigma^g(\text{Lie}_A^f) \simeq \text{Lie}_A$ is given by the Yoneda embedding followed by the restriction $\text{Lie}_A \rightarrow \text{Fun}(\text{Lie}_A, S) \rightarrow \text{Fun}((\text{Lie}_A^f)^{op}, S)$ (see [12, 1.2.2], Section 3.3 for $\mathcal{P}_\Sigma^e(\text{Lie}_A^f) \simeq \text{Lie}_A$). It follows that $\text{Rep}_H(F_L)$ is a colimit of $(\text{Lie}_A^f)/L \rightarrow \text{Lie}_A^f \xrightarrow{\text{wCh}^*} \mathbb{C}^\infty_{\text{op}}$. The functor $u \circ \text{Ch}^*$ carries $P \in \text{Lie}_A^f$ to $\text{Rep}(P)(\text{Mod}_A) = \text{LMod}_{\text{Uni}(P)}$. By Claim 6.4.1 below, the $\infty$-category $(\text{TSZ}_A)/X \simeq (\text{Lie}_A^f)/L$ is sifted. By [20, X, 2.4.32, 2.4.33], a sifted colimit $\text{colim}_i \text{Rep}(L_i) = L$ of dg Lie algebras gives rise to a canonical equivalence $\text{Rep}(L)(\text{Mod}_A) \simeq \lim_{i \in I} \text{Rep}(L_i)(\text{Mod}_A)$. Thus, we obtain $\text{Rep}(L)(\text{Mod}_A) \simeq \lim_{[P \rightarrow L] \in (\text{Lie}_A^f)/L} \text{Rep}(P)(\text{Mod}_A) = \text{Rep}_H(F_L)$. \hfill \square

Claim 6.4.1. Let $X$ be a formal stack over $A$. Then $(\text{TSZ}_A)/X$ is sifted.

Proof. Put $D = (\text{TSZ}_A)/X$. It is enough to prove that the diagonal functor $D \rightarrow D \times D$ is a right adjoint (in particular, it is cofinal). We define $D \times D \rightarrow D$ by $\text{Spec} A \oplus M \rightarrow X, \text{Spec} A \oplus M' \rightarrow X) \mapsto \text{Spec } A \oplus (M \oplus M') \rightarrow X$ (keep in mind that there is the canonical equivalence $X(A \oplus (M \times M')) \simeq X(A \oplus M) \times X(A \oplus M')$ since $X$ is a formal stack). This functor is a left adjoint to the diagonal functor. \hfill \square

Remark 6.5. If $\phi : L \rightarrow L'$ is a morphism of dg Lie algebras, it induces a morphism $F_\phi : F_L \rightarrow F_{L'}$ of formal stacks. The contravariant functor $\text{Rep}_L$ gives rise to the pullback functor $F_\phi^* : \text{Rep}_H(F_{L'}) \rightarrow \text{Rep}_H(F_L)$. The pullback $F_\phi^*$ can naturally be identified with the restriction functor $\text{Rep}(L')(\text{Mod}_A) \rightarrow \text{Rep}(L)(\text{Mod}_A)$ determined by $\phi$. Indeed, there is the following diagram of restriction functors

$$
\begin{array}{ccc}
\text{Rep}(L')(\text{Mod}_A) & \rightarrow & \text{Rep}(L)(\text{Mod}_A) \\
\downarrow \simeq & & \downarrow \simeq \\
\text{lim}_{[P \rightarrow L'] \in (\text{Lie}_A^f)/L'} \text{Rep}(P)(\text{Mod}_A) & \rightarrow & \text{lim}_{[P \rightarrow L] \in (\text{Lie}_A^f)/L} \text{Rep}(P)(\text{Mod}_A)
\end{array}
$$

which commutes up to canonical homotopy.

Let $X$ be a functor $\text{Art}_{\text{TSZ}}^X \rightarrow S$. We will construct a fully faithful functor $QC_H(X) \rightarrow \text{Rep}_H(X)$. For $R \in \text{Art}_{\text{TSZ}}^X$, the functor $\text{Mod}_R(\text{Mod}_A) \rightarrow \text{LMod}_{\text{Uni}(R)}(\text{Mod}_A)$ which sends $P$ to $P \otimes_R A$ sends $1$. More precisely, $\text{Mod}_R(\text{Mod}_A) \rightarrow \text{LMod}_{\text{Uni}(R)}(\text{Mod}_A)$ is given by $\text{Rep}_L = \text{LMod}_{\text{Uni}(R)}(\text{Mod}_A) \simeq \text{Rep}(\text{Mod}_R(\text{Mod}_A))$ given the formula $P \mapsto P \otimes_R A$. By [12, 2.3.6], this functor is fully faithful. By the construction in [15, Remark 5.5], $I_R$ is functorial in $R \in \text{Art}_{\text{TSZ}}^X$. That is, there is a natural transformation $\text{Mod}_R(\text{Mod}_A) \rightarrow \text{LMod}_{\text{Uni}(R)}(\text{Mod}_A)$ between functors $(\text{Art}_{\text{TSZ}}^{X'})^{op} = \text{TSZ}_A \rightarrow \mathbb{C}^\infty_{\text{op}}$, such that the evaluation at each $R \in \text{Art}_{\text{TSZ}}^X$ is equivalent to $I_R$. By the definition of $QC_H$ and $\text{Rep}_H$ (as right Kan extensions), it gives rise to a morphism $\gamma : QC_H \rightarrow \text{Rep}_H$ in $\text{Fun}(\text{Art}_{\text{TSZ}}^X, S^{op}, \mathbb{C}^\infty_{\text{op}})$. For a functor $X : \text{Art}_{\text{TSZ}}^X \rightarrow S$ we have a canonical fully faithful functor

$$
\gamma_X : QC_H(X) \rightarrow \lim_{\text{Spec } R \rightarrow X} \text{LMod}_R(\text{Mod}_A) \rightarrow \lim_{\text{Spec } R \rightarrow X} \text{LMod}_{\text{Uni}(R)}(\text{Mod}_A) = \text{Rep}_H(X).
$$

Taking into account $\gamma_X$, we often regard $QC_H(X)$ as a stable subcategory of $\text{Rep}_H(X)$.

6.4. Let $N$ be an $A$-module spectrum, that is, an object of $\text{Mod}_A$. Let $\text{Def}(N) : \text{Art}_{\text{TSZ}}^X \rightarrow S$ be the deformation functor which assigns $[R \rightarrow A]$ to the space of deformations of $N$ to $\text{Mod}_R$. Namely, $\text{Def}(N)$ is given by $[R \rightarrow A] \rightarrow \text{Mod}_R^X \times \text{Mod}_A^\Sigma \{H\}$ (see [15, Section 4.3] or [20, X, 5.2] for a precise formulation). As with $\text{Def}(N)$, for any formal prestack $X : \text{Art}_{\text{TSZ}}^X \rightarrow S$, there is a canonical equivalence $\text{Map}_{\text{Fun}(\text{Art}_{\text{TSZ}}^X, S)}(X, \text{Def}(N)) \simeq QC_H(X) = \times \text{Mod}_A^\Sigma \{N\}$. Let $\text{End}(N) \in \text{Alg}_1(\text{Mod}_A)$ be the endomorphism algebra of $N$. We write $\text{End}_D^A(N)$ for the dg Lie algebra associated to $\text{End}_D^A(N)$. Let $F_{\text{End}_D^A(N)}$ denote the formal stack associated to $\text{End}_D^A(N)$ (through the equivalence $\text{Lie}_A \simeq \text{Spec}_A$).

There exists a canonical morphism $J_N : \text{Def}(N) \rightarrow F_{\text{End}_D^A(N)}$ in $\text{Fun}(\text{Art}_{\text{TSZ}}^X, S)$ such that $\text{Def}(N)(R) \rightarrow F_{\text{End}_D^A(N)}(R)$ is a fully faithful functor between $\infty$-groupoids/spaces for any $[R \rightarrow A] \in \text{Art}_{\text{TSZ}}^X$ (cf. [15,
Section 5.3, Proposition 4.11 and its proof, [20]). Here is a quick review of $J_N$. For any $L \in \text{Lie}_A$, we have
\[
\text{Map}_{\text{Fun}(\text{Art}_{\text{sz}}^A,S)}(\mathcal{F}_L, \mathcal{F}_{\text{End}^S(N)}) \simeq \text{Map}_{\text{Lie}_A}(L, \text{End}^S(N)) \simeq \text{Rep}(L)(\text{Mod}^A_X) \times_{\text{Mod}^A_{\mathbb{Z}}} \{N\}.
\]
Through this equivalence, the identity $\mathcal{F}_{\text{End}^S(N)} \to \mathcal{F}_{\text{End}^S(N)}$ corresponds to the tautological action of $\text{End}^S(N)$ on $N$. If $X$ is a functor $\text{Art}_{\text{sz}}^A \to S$, it follows from the definition of $\text{Rep}_H(X)$ that
\[
\text{Map}_{\text{Fun}(\text{Art}_{\text{sz}}^A,S)}(X, \mathcal{F}_{\text{End}^S(N)}) \simeq \text{Rep}_H(X)^{\times} \times_{\text{Mod}^A_{\mathbb{Z}}} \{N\}.
\]
That is, $\mathcal{F}_{\text{End}^S(N)}$ represents the functor $\text{Rep}_H(-)^{\times} \times_{\text{Mod}^A_{\mathbb{Z}}} \{N\} : \text{Fun}(\text{Art}_{\text{sz}}^A,S)^{\text{op}} \to \hat{S}$ given by $X \mapsto \text{Rep}_H(X)^{\times} \times_{\text{Mod}^A_{\mathbb{Z}}} \{N\}$. On the other hand, $\text{Def}(N)$ represents the functor $\text{QC}_H(-)^{\times} \times_{\text{Mod}^A_{\mathbb{Z}}} \{N\} : \text{Fun}(\text{Art}_{\text{sz}}^A,S)^{\text{op}} \to \hat{S}$ given by $X \mapsto \text{QC}_H(X)^{\times} \times_{\text{Mod}^A_{\mathbb{Z}}} \{N\}$. The morphism $\text{QC}_H \to \text{Rep}_H$ in $\text{Fun}(\text{Fun}(\text{Art}_{\text{sz}}^A,S)^{\text{op}}, \text{Cat}_{\infty})$ (cf. Section 6.3) induces $\text{QC}_H(-)^{\times} \times_{\text{Mod}^A_{\mathbb{Z}}} \{N\} \to \text{Rep}_H(-)^{\times} \times_{\text{Mod}^A_{\mathbb{Z}}} \{N\}$. By Yoneda lemma, it gives rise $J_N : \text{Def}(N) \to \mathcal{F}_{\text{End}^S(N)}$.

As we have constructed in [15, Construction 4.2, Section 5.4], there is a functor
\[
M^\text{plain}_C : \text{Def}(C) \to \text{Def}(\mathcal{H}_\bullet(C/A))
\]
which carries a deformation $(C_R, C_R \otimes_{\text{Perf}_A} \text{Perf}_A \simeq C)$ of $C$ to the deformation $((\mathcal{H}_\bullet(C_R/R), \alpha : (\mathcal{H}_\bullet(C_R/R) \otimes_R A \simeq \mathcal{H}_\bullet(C/A))$ of $\mathcal{H}_\bullet(C/A)$, where $\mathcal{H}_\bullet(C_R/R)$ is the relative Hochschild homology. We consider the diagram in $\text{Fun}(\text{Art}_{\text{sz}}^A,S)$:
\[
\begin{array}{ccc}
\mathcal{F}_{T_{A/k}[-1]} & \xrightarrow{S \times_k S} & \text{Def}(C) \\
& \downarrow{J_C} & \downarrow{M^\text{plain}_C} \\
\mathcal{F}_{\mathcal{G}_C} & \xrightarrow{A^\text{plain}_C} & \mathcal{F}_{\text{End}^S(\mathcal{H}_\bullet(C/A))} \\
\end{array}
\]
The morphism $\mathcal{F}_{T_{A/k}[-1]} \to \mathcal{F}_{\mathcal{G}_C}$ is induced by the Kodaira-Spencer morphism $KS_C$. The morphism $A^\text{plain}_C$ is induced by $A^\nu_C : \mathcal{G}_C \to \text{End}^S(\mathcal{H}_\bullet(C/A))$. According to [15, Theorem 1.1], the square commutes up to canonical homotopy.

Next, we explain a refinement of the above diagram that comes from the $S^1$-action on $\mathcal{H}_\bullet(C/A)$:
\[
\begin{array}{ccc}
\mathcal{F}_{T_{A/k}[-1]} & \xrightarrow{S \times_k S} & \text{Def}(C) \\
& \downarrow{J_C} & \downarrow{M^\nu_C} \\
\mathcal{F}_{\mathcal{G}_C} & \xrightarrow{A^\nu_C} & \mathcal{F}_{\text{End}^S(\mathcal{H}_\bullet(C/A))} \\
\end{array}
\]
whose squares and triangle commute up to canonical homotopy (see [15, Theorem 7.1] for details). Let us briefly explain this diagram. The functor $\text{Def}(\mathcal{H}_\bullet(C/A))^{S^1} : \text{Art}_{\text{sz}}^A \to S$ is given by $[R \to A] \mapsto (\text{Mod}_R^{S^1})^{\times} \times_{(\text{Mod}_A^{S^1})^{\times}} \{\mathcal{H}_\bullet(C/A)\}$. The map $* = \{\mathcal{H}_\bullet(C/A)\} \to \text{Mod}_A^{S^1} = \text{Fun}(BS^{1}, \text{Mod}_A)$ is determined by the Hochschild homology with the $S^1$-action. Informally, $M_C : \text{Def}(C) \to \text{Def}(\mathcal{H}_\bullet(C/A))^{S^1}$ can be described as follows: for any $R \in \text{Art}_{\text{sz}}^A$, $\text{Def}(C)(R) \to \text{Def}(\mathcal{H}_\bullet(C/A))^{S^1}(R) \to \text{Def}(\mathcal{H}_\bullet(C/A))(R)$ is naturally equivalent to
\[
S^1_R \times_{S^1_A} \{C\} \to (\text{Mod}_R^{S^1})^{\times} \times_{(\text{Mod}_A^{S^1})^{\times}} \{\mathcal{H}_\bullet(C/A)\}
\]
where the first arrow is induced by $\mathcal{H}_\bullet(-/R)$ and $\mathcal{H}_\bullet(-/A)$. Note that $A^\nu_C : \mathcal{G}_C \to \text{End}^S(\mathcal{H}_\bullet(C/A))$ is a morphism in $\text{Lie}_{\mathcal{A}}$, where the $S^1$-action on $\mathcal{G}_C$ is trivial. It gives rise to $\mathcal{G}_C \to \text{End}^S(\mathcal{H}_\bullet(C/A))^{S^1} \to \text{End}^S(\mathcal{H}_\bullet(C/A))$

in $\text{Lie}_A$ where $\text{End}^S(\mathcal{H}_\bullet(C/A))^{S^1}$ denotes the homotopy fixed points of the $S^1$-action on $\text{End}^S(\mathcal{H}_\bullet(C/A))$ (it is not the cotensor by $S^1$!!). Passing to formal stacks, we have $\mathcal{F}_{\mathcal{G}_C} \to \mathcal{F}_{\text{End}^S(\mathcal{H}_\bullet(C/A))^{S^1} \to}$
$F_{\text{End}^L(\mathcal{H}\mathcal{H}_*\mathcal{C}/A)}$. As in the case of $\mathcal{H}\mathcal{H}_*\mathcal{C}/A$,

$$J_{\mathcal{H}\mathcal{H}_*\mathcal{C}/A}(R) = \text{Def}(\mathcal{H}\mathcal{H}_*\mathcal{C}/A)^{S^1}(R) \rightarrow \mathcal{F}_{\text{End}^L(\mathcal{H}\mathcal{H}_*\mathcal{C}/A)^{S^1}(R)}$$

is fully faithful for any $[R \rightarrow A] \in \text{Art}_k^{\mathcal{T}}$ (indeed, $J_{\mathcal{H}\mathcal{H}_*\mathcal{C}/A}^{S^1}$ can be obtained as the limit ($S^1$-invariant) of $S^1$-equivariant morphism $J_{\mathcal{H}\mathcal{H}_*\mathcal{C}/A}$, cf. [15, Section 4.8]).

**Construction 6.6.** Applying $\text{QC}_H$ to the above diagram we obtain the following diagram

$$
\text{QC}_H(\mathcal{F}_{\text{End}^L(\mathcal{H}\mathcal{H}_*\mathcal{C}/A)}) \rightarrow \text{QC}_H(\mathcal{F}_{\mathcal{U}_c}) \rightarrow \text{QC}_H(\mathcal{F}_{\mathcal{T}_A/k[-1]}) \simeq \text{QC}_H(\mathcal{S} \times_k \mathcal{S})
$$

where each morphism is a pullback functor. If we replace $\text{QC}_H$ with $\text{Rep}_H$, then we have a similar diagram.

By Lemma 6.4 and Remark 6.5, the functors

$$\text{QC}_H(\mathcal{F}_{\text{End}^L(\mathcal{H}\mathcal{H}_*\mathcal{C}/A)}) \rightarrow \text{QC}_H(\mathcal{F}_{\mathcal{U}_c}) \rightarrow \text{QC}_H(\mathcal{F}_{\mathcal{T}_A/k[-1]}) \simeq \text{QC}_H(\mathcal{S} \times_k \mathcal{S})$$

is extended to

$$\text{Rep}(\text{End}^L(\mathcal{H}\mathcal{H}_*\mathcal{C}/A))(\text{Mod}_A) \rightarrow \text{Rep}(\mathcal{G}_c)(\text{Mod}_A) \rightarrow \text{Rep}(\mathcal{T}_A/k[-1])(\text{Mod}_A)$$

where functors are induced by the restriction of $\mathcal{T}_A/k[-1] \rightarrow \mathcal{G}_c \rightarrow \text{End}^L(\mathcal{H}\mathcal{H}_*\mathcal{C}/A)$ (cf. Construction 6.2). We note that the composition $\mathcal{T}_A/k[-1] \rightarrow \mathcal{G}_c \rightarrow \text{End}^L(\mathcal{H}\mathcal{H}_*\mathcal{C}/A)$ factors as the sequence of $S^1$-equivariant morphisms $\mathcal{T}_A/k[-1] \rightarrow \mathcal{T}_A/k[-1]^S \rightarrow \text{End}^L(\mathcal{H}\mathcal{H}_*\mathcal{C}/A)$. Moreover, $\mathcal{S} \times_k \mathcal{S}$ corresponds to $\mathcal{T}_A/k[-1]^S$ (cf. Section 6.2). Thus, we have pullback functors

$$\text{QC}_H(\mathcal{F}_{\text{End}^L(\mathcal{H}\mathcal{H}_*\mathcal{C}/A)}) \rightarrow \text{QC}_H(\mathcal{S} \times_k \mathcal{S}) \rightarrow \text{QC}_H(\mathcal{S} \times_k \mathcal{S}),$$

and the restriction functors

$$\text{Rep}(\text{End}^L(\mathcal{H}\mathcal{H}_*\mathcal{C}/A))(\text{Mod}_A) \rightarrow \text{Rep}(\mathcal{T}_A/k[-1]^S)(\text{Mod}_A) \rightarrow \text{Rep}(\mathcal{T}_A/k[-1])(\text{Mod}_A)$$

where the first sequence can be contained in the second sequence.

Let $\mathcal{U}$ be the universal deformation of $\mathcal{H}\mathcal{H}_*\mathcal{C}/A$. That is, $\mathcal{U}$ is the tautological object in

$$\text{QC}_H(\text{Def}(\mathcal{H}\mathcal{H}_*\mathcal{C}/A)) \times_{\text{Mod}_A} \{\mathcal{H}\mathcal{H}_*\mathcal{C}/A\}$$

which corresponds to the identity map $\text{Def}(\mathcal{H}\mathcal{H}_*\mathcal{C}/A) \rightarrow \text{Def}(\mathcal{H}\mathcal{H}_*\mathcal{C}/A)$ through the equivalence $\text{Map}_{\text{Fun}(\text{Art}_k^{\mathcal{T}}\mathcal{C})}(\text{Def}(\mathcal{H}\mathcal{H}_*\mathcal{C}/A), \text{Def}(\mathcal{H}\mathcal{H}_*\mathcal{C}/A)) \simeq \text{QC}_H(\text{Def}(\mathcal{H}\mathcal{H}_*\mathcal{C}/A)) \times_{\text{Mod}_A} \{\mathcal{H}\mathcal{H}_*\mathcal{C}/A\}$. (Informally, the image of $\mathcal{U}$ in $\text{QC}_H(\text{Def}(\mathcal{H}\mathcal{H}_*\mathcal{C}/A))$ is an object which associates to each $f : \text{Spec} R \rightarrow \text{Def}(\mathcal{H}\mathcal{H}_*\mathcal{C}/A)$ the $R$-module corresponding to $f$.)

Let $\mathcal{V}$ denote the image of $\mathcal{U}$ under $\text{QC}_H(\text{Def}(\mathcal{H}\mathcal{H}_*\mathcal{C}/A))) \rightarrow \text{QC}_H(\mathcal{S} \times_k \mathcal{S})$.

By the definition of $\mathcal{KH}_c$ classified by $\text{pr}_2^c(\mathcal{C})$ over $\mathcal{S} \times_k \mathcal{S}$ and the construction of $\mathcal{M}^{\text{plain}}_c$ induced by the relative Hochschild homology functor, we see:

**Lemma 6.7.** The object $\mathcal{V} \in \text{QC}_H(\mathcal{S} \times_k \mathcal{S})$ obtained from the base change $\mathcal{H}\mathcal{H}_*\mathcal{C}/A \otimes_A (A \otimes_k A) = \text{pr}_2^c(\mathcal{H}\mathcal{H}_*\mathcal{C}/A) \in \text{Mod}_A \otimes A =: \text{QC}(\mathcal{S} \times_k \mathcal{S})$ along the second projection $\text{pr}_2 : \mathcal{S} \times_k \mathcal{S} \rightarrow \mathcal{S}$. That is, $\mathcal{V}$ is the image of $\text{pr}_2^c(\mathcal{H}\mathcal{H}_*\mathcal{C}/A)$ through the canonical map $\text{QC}(\mathcal{S} \times_k \mathcal{S}) \rightarrow \text{lim}_{\text{Spec} R \rightarrow \mathcal{S} \times_k \mathcal{S}} \text{Mod}_R = \text{QC}_H(\mathcal{S} \times_k \mathcal{S})$.

Next, we observe:

**Proposition 6.8.** The object $\mathcal{V} \in \text{QC}_H(\mathcal{S} \times_k \mathcal{S})$ is promoted to an object $\mathcal{V}_L$ of $\text{Rep}_H(\mathcal{S} \times_k \mathcal{L})$. Moreover, $\mathcal{V}_L$ is promoted to $\text{Rep}_H(\mathcal{S} \times_k \mathcal{L})^{S^1}$. Here the $S^1$-action $\text{Rep}_H(\mathcal{S} \times_k \mathcal{L})$ comes from the action on $\mathcal{L}$. 

Lemma 6.9. The universal deformation \( U \in QC_H(\text{Def}(\mathcal{H}_H^\bullet(C/A)) \times_{\text{Mod}_A} \{\mathcal{H}_H^\bullet(C/A)\}) \) is naturally equivalent to the image of the \( \text{End}(\mathcal{H}_H^\bullet(C/A)) \)-module \( \mathcal{H}_H^\bullet(C/A) \) (defined by the tautological action) under the pullback along \( J_{\mathcal{H}_H^\bullet(C/A)} \)

\[
\text{Rep}(\text{End}(\mathcal{H}_H^\bullet(C/A)) \times_{\text{Mod}_A} \{\mathcal{H}_H^\bullet(C/A)\}) \to \text{Rep}(\text{Def}(\mathcal{H}_H^\bullet(C/A))) \times_{\text{Mod}_A} \{\mathcal{H}_H^\bullet(C/A)\}.
\]

Proof of Lemma 6.9. If \( X \) is a functor \( \text{Art}^\text{uns}_A \to S \),

\[
\text{Map}_{\text{Fun}(\text{Art}^\text{uns}_A,S)}(X, F_{\text{End}^L(\mathcal{H}_H^\bullet(C/A)))}) \simeq \text{Rep}_H(X)^\simeq \times_{\text{Mod}_A} \{\mathcal{H}_H^\bullet(C/A)\}.
\]

By the definition of \( J_{\mathcal{H}_H^\bullet(C/A)} \) (see the second paragraph of Section 6.4), the canonical morphism \( J_{\mathcal{H}_H^\bullet(C/A)} : \text{Def}(\mathcal{H}_H^\bullet(C/A)) \to F_{\text{End}^L(\mathcal{H}_H^\bullet(C/A)))} \) is classified by the universal deformation

\[
U \in QC_H(\text{Def}(\mathcal{H}_H^\bullet(C/A))) \times_{\text{Mod}_A} \{\mathcal{H}_H^\bullet(C/A)\} \subset \text{Rep}_H(\text{Def}(\mathcal{H}_H^\bullet(C/A))) \times_{\text{Mod}_A} \{\mathcal{H}_H^\bullet(C/A)\}.
\]

Thus, our claim follows. \( \Box \)

Proof of Proposition 6.8. Note that \( T_{A/k}[-1] \to \text{End}^L(\mathcal{H}_H^\bullet(C/A)) \) factors as \( T_{A/k}[-1] \to T_{A/k}[-1]^{S^1} \to \text{End}^L(\mathcal{H}_H^\bullet(C/A)) \). Thus, \( \mathcal{H}_H^\bullet(C/A) \) endowed with the tautological action of \( \text{End}(\mathcal{H}_H^\bullet(C/A)) \) gives rise to a \( T_{A/k}[-1]^{S^1} \)-module \( \mathcal{H}_H^\bullet(C/A) \), that is, an object \( \mathcal{V}_L \) of \( \text{Rep}(T_{A/k}[-1]^{S^1})(\text{Mod}_A) \) whose underlying module is \( \mathcal{H}_H^\bullet(C/A) \). We have the diagram

\[
\begin{array}{ccc}
F_{T_{A/k}[-1]} & \simeq & S \times_k S \\
\downarrow & & \downarrow \\
F_{T_{A/k}[-1]^{S^1}} & \simeq & S \times_k LS
\end{array}
\]

Lemma 6.9 shows that the image of \( \mathcal{V}_L \) in \( \text{Rep}_H(S \times_k LS) \) is naturally equivalent to \( \mathcal{V} \) because \( S \times_k S \to F_{\text{End}^L(\mathcal{H}_H^\bullet(C/A))} \) factors through \( \text{Def}(\mathcal{H}_H^\bullet(C/A)) \to F_{\text{End}^L(\mathcal{H}_H^\bullet(C/A))} \). Remember that \( T_{A/k}[-1]^{S^1} \to \text{End}^L(\mathcal{H}_H^\bullet(C/A)) \) is promoted to an \( S^1 \)-equivariant morphism of dg Lie algebras. Thus, \( \mathcal{V}_L \) is promoted to \( (\text{Rep}(T_{A/k}[-1]^{S^1})(\text{Mod}_A))^{S^1} \simeq \text{LMod}_{L^1(T_{A/k}[-1]^{S^1})(\text{Mod}_{S^1})} \) where \( T_{A/k}[-1]^{S^1} \) is regarded as an object of \( \text{Lie}_{S^1}^A = \text{Fun}(BS^1, \text{Lie}_A) \). Note that the \( S^1 \)-action on \( S \times_k LS \) amounts to the canonical \( S^1 \)-action on \( T_{A/k}[-1]^{S^1} \) since both \( S \times_k LS \) and \( T_{A/k}[-1]^{S^1} \) are obtained by the cotensor by \( S^1 \). Hence we see that \( \mathcal{V}_L \) belongs to \( \text{Rep}_H(S \times_k LS)^{S^1} \).

Next, we will take into account \( S^1 \)-actions on sheaves.

Construction 6.10. Let us recall from Construction 6.6 and Proposition 6.8 that the object \( \mathcal{V}_L \in \text{Rep}_H(S \times_k LS) \) and \( \mathcal{V} \in QC_H(S \times_k S) \subset \text{Rep}(T_{A/k}[-1])(\text{Mod}_A) \) are obtained from the \( S^1 \)-equivariant morphisms of dg Lie algebras

\[
T_{A/k}[-1] \to T_{A/k}[-1]^{S^1} \to \text{End}^L(\mathcal{H}_H^\bullet(C/A))
\]

and the tautological action \( \text{End}^L(\mathcal{H}_H^\bullet(C/A)) \) on \( \mathcal{H}_H^\bullet(C/A) \) (note that this tautological action defines an object of \( \text{Rep}(\text{End}^L(\mathcal{H}_H^\bullet(C/A)))(\text{Mod}_{S^1}) \)). Consequently, \( \mathcal{V}_L \) and \( \mathcal{V} \) are promoted to objects in

\[
\mathcal{V}'_L \in \text{Rep}_H(S \times_k LS)^{S^1} \quad \text{and} \quad \mathcal{V}' \in QC_H(S \times_k S)^{S^1} \subset \text{Rep}_H(S \times_k S)^{S^1}
\]

such that \( \mathcal{V}'_L \) maps to \( \mathcal{V}' \). Namely, \( \mathcal{V}'_L \) and \( \mathcal{V}' \) correspond to the canonical \( T_{A/k}[-1]^{S^1} \)-module and the canonical \( T_{A/k}[-1] \)-module, respectively (cf. Definition 6.3).

Note that \( \mathcal{V}' \in \text{Rep}(T_{A/k}[-1])(\text{Mod}_{S^1}) \) \( \simeq \text{Fun}(BS^1, \text{Rep}(T_{A/k}[-1])(\text{Mod}_A)) \) is obtained from the tautological action of \( \text{End}(\mathcal{H}_H^\bullet(C/A)) \) on \( \mathcal{H}_H^\bullet(C/A) \) and the \( S^1 \)-equivariant morphism of dg Lie algebras

\[
T_{A/k}[-1] \to \text{End}^L(\mathcal{H}_H^\bullet(C/A))^{S^1} \to \text{End}^L(\mathcal{H}_H^\bullet(C/A))
\]

where the actions on the left and middle ones are trivial. Set

\[
QC_H(\text{Def}(\mathcal{H}_H^\bullet(C/A))^{S^1})^{S^1} := \text{Fun}(BS^1, QC_H(\text{Def}(\mathcal{H}_H^\bullet(C/A))^{S^1})).
\]
Let \( \mathcal{U}' \in QC_H(Def(\mathcal{H}_\bullet(C/A))^S) \times_{Mod^{S_1}} \{ \mathcal{H}_\bullet(C/A) \} \) be the universal \( S^1 \)-equivariant deformation of \( \mathcal{H}_\bullet(C/A) \) (using the canonical action for any \( T \) tological action of \( End(s)^S \)). Consequently, \( \mathcal{U}' \) (this is a simple generalization of Lemma 6.9 to the \( S^1 \)-equivariant situation, see Remark 6.11). Thus, we see that the universal deformation \( \mathcal{U} \) along \( \mathcal{U}' \) is equivalent to the image of \( \mathcal{U}' \) under \( QC_H(Def(\mathcal{H}_\bullet(C/A))^S) \rightarrow QC_H(S) \). As in Lemma 6.7, we see that \( \mathcal{U}' \) is naturally equivalent to the object \( pr_2(\mathcal{H}_\bullet(C/A)) \) obtained from the pullback \( pr_2(\mathcal{H}_\bullet(C/A)) \in QC(S \times_k S) \) of \( \mathcal{H}_\bullet(C/A) \in QC(S) \). We regard \( \mathcal{U}' \) as an object of \( Rep_H(S) \times_k S ) \). The data \( \mathcal{V}' \rightarrow \mathcal{V} \) together with the equivalence between \( \mathcal{V}' \) and \( pr_2(\mathcal{H}_\bullet(C/A)) \) determines an object \( \mathcal{V} \) of

\[
QC(S) \times_{Rep_H(S)} Rep_H(S \times_k S) \times_{Rep_H(S)} \mathcal{V}.
\]

**Remark 6.11.** Consider the adjoint pair \( triv: Mod_A = Fun(*, Mod_A) \rightrightarrows Mod_{S_1} = Fun(BS^1, Mod_A) \) where the left adjoint induced by the composition with \( BS^1 \rightarrow * \) (so that the right adjoint is given by \( (-)^S \)). By this adjoint pair and the universal property of \( End(\mathcal{H}_\bullet(C/A)) \in \text{Alg}_{E_1}(Mod_{S_1}) \), we have

\[
\text{Map}_{Lie_A}(L, End^L(\mathcal{H}_\bullet(C/A))^S) \simeq \text{Map}_{\text{Alg}_{E_1}(Mod_{S_1})}(U_1(L), End(\mathcal{H}_\bullet(C/A))^S)
\]

\[
\simeq \text{Map}_{\text{Alg}_{E_1}(Mod_{S_1})}(triv(U_1(L)), End(\mathcal{H}_\bullet(C/A)))
\]

\[
\simeq L\text{Mod}_{triv(U_1(L))}(Mod_{S_1}^S) \times_{Mod_{S_1}^S} \{ \mathcal{H}_\bullet(C/A) \}
\]

\[
\simeq (\text{Rep}(L)(Mod_{S_1}^S)) \times_{Mod_{S_1}^S} \{ \mathcal{H}_\bullet(C/A) \}
\]

for any \( L \in Lie_A \). The identity map of \( End^L(\mathcal{H}_\bullet(C/A))^S \) maps to the tautological action of \( End^L(\mathcal{H}_\bullet(C/A))^S \) on \( \mathcal{H}_\bullet(C/A) \). As with \( F_{End(\mathcal{H}_\bullet(C/A))} \), there is a canonical equivalence

\[
\text{Map}_{\text{Fun}(\text{Art}_{E_1}^S, S)}(X, F_{End(\mathcal{H}_\bullet(C/A))}^S) \simeq (\text{Rep}_H(X)^S) \times_{Mod_{S_1}^S} \{ \mathcal{H}_\bullet(C/A) \}
\]

for any \( X \in \text{Fun}(\text{Art}_{E_1}^S, S) \). The morphism \( \text{Def}(\mathcal{H}_\bullet(C/A))^S \rightarrow F_{End(\mathcal{H}_\bullet(C/A))}^S \) is determined by

\[
QC_H((-)^S) \times_{Mod_{S_1}^S} \{ \mathcal{H}_\bullet(C/A) \} \rightarrow \text{Rep}_H((-)^S) \times_{Mod_{S_1}^S} \{ \mathcal{H}_\bullet(C/A) \}.
\]

Thus, we see that the universal deformation \( \mathcal{U}' \) can naturally be identified with the pullback of the tautological action of \( End^L(\mathcal{H}_\bullet(C/A))^S \) on \( \mathcal{H}_\bullet(C/A) \) along \( \text{Def}(\mathcal{H}_\bullet(C/A))^S \rightarrow F_{End(\mathcal{H}_\bullet(C/A))}^S \).

**Remark 6.12.** The canonical action of \( \mathcal{T}_{A/k}(-)^{S_1} \) on \( \mathcal{H}_\bullet(C/A) \) (Definition 6.3) can be described in terms of cyclic deformations of \( \mathcal{H}_\bullet(C/A) \) in the sense of [15]. The functor \( (-)^S \): \( Lie_A \rightarrow Lie_A \) determined by cotensor by \( S^1 \) preserves sifted colimits (cf. the proof of Proposition 6.2 in [15]). Therefore, \( \mathcal{T}_{A/k}(-)^{S_1} \) is a sifted colimit of the composite \( (Lie_A)^{S_1} \rightarrow Lie_A \rightarrow Lie_A \). It follows that there is an equivalence

\[
\text{Rep}(\mathcal{T}_{A/k}(-)^{S_1})(Mod_{S_1}^S) \times_{Mod_{S_1}^S} \{ \mathcal{H}_\bullet(C/A) \} \simeq \lim_{f:L \rightarrow \mathcal{T}_{A/k}(-)^{S_1}} \text{Rep}(L^{S_1})(Mod_{S_1}^S) \times_{Mod_{S_1}^S} \{ \mathcal{H}_\bullet(C/A) \}
\]
there exist a canonical equivalence $F \overset{\simeq}{\rightarrow} M$ which is functorial in the following composite of maps:

$$f : L \rightarrow T_{A/k}[-1] \mapsto J_{\overline{\mathcal{H}_*}(C/A)}(\mathcal{H}_*(C/A)) \in \text{Rep}(L_{S^1})(\text{Mod}^{S^1}_A) \times_{\text{Mod}^{S^1}_A} \{\mathcal{H}_*(C/A)\}$$

where $C_f$ is the base change of $\text{pr}_2^*\mathcal{C}$ along a morphism $\text{Spec} R \rightarrow S \times S$ corresponding to $f : L \rightarrow T_{A/k}[-1]$. Here if we regard $C_f$ as an $A$-linear small stable $\infty$-category, $\mathcal{H}_*(C/A)$ is a cyclic deformation of $\mathcal{H}_*(C/A)$, that is, a lift of $\mathcal{H}_*(C/A)$ to $\text{Mod}_{R \otimes A} S^1(\text{Mod}^{S^1}_A)$. The object $J_{\overline{\mathcal{H}_*}(C/A)}(\mathcal{H}_*(C/A))$ is determined by the image of $\mathcal{H}_*(C_f/A)$ under the Koszul duality functor $\text{Mod}_{R \otimes A} S^1(\text{Mod}^{S^1}_A) \rightarrow \text{LMod}_{R \otimes A} S^1(\text{Mod}^{S^1}_A) \simeq \text{Rep}(L_{S^1})(\text{Mod}^{S^1}_A)$ (see [15, Proposition 5.10] for $\mathbb{D}_1(R \otimes A S^1) \simeq U_1(L_{S^1})$).

We denote this object by $E^C_{\mathcal{C}}$. To be precise, $E^C_{\mathcal{C}}$ is constructed as follows. We use the notation in [15]. In particular, for $F_{A \otimes \text{End}(\mathcal{H}_*(C/A))}$ we refer to [15, Section 4.7]. By [15, Remark 6.6], for $M \in \text{Lie}_A$, there exist a canonical equivalence

$$\text{Map}_{\text{Fun}(\text{Art}_A^*), S}(F_M, F_{A \otimes \text{End}(\mathcal{H}_*(C/A))}) \simeq \text{Rep}((M^{S^1}_A) \times_{(\text{Mod}^{S^1}_A)} \{\mathcal{H}_*(C/A)\})$$

which is functorial in $M \in \text{Lie}_A$. By this equivalence, $E^C_{\mathcal{C}}$ is defined to be an object classified by the following composite of $KSC$ and maps in Corollary 7.2 in [15]:

$$u : F_{T_{A/k}}[-1] \simeq S \times_k S \rightarrow \text{Def}(C) M^{\mathcal{C}}_{\mathcal{C}} \rightarrow \text{Def}(\mathcal{H}_*(C/A)) J_{\overline{\mathcal{H}_*}(C/A)} F_{A \otimes \text{End}(\mathcal{H}_*(C/A))}.$$ 

Our claim follows from [15, Corollary 7.2], which says that $u$ is equivalent to

$$v : F_{T_{A/k}}[-1] \simeq S \times_k S \rightarrow \text{Def}(\mathcal{H}_*(C/A)) J_{\overline{\mathcal{H}_*}(C/A)} F_{\mathcal{C}}.$$ 

The final arrow is defined in [15, Construction 6.5] (in loc. cit., we denote it by $a : F_{A \otimes \mathcal{H}_*(C/A)} \rightarrow H_{\mathcal{C}} \otimes_{\mathcal{C}} \mathcal{H}_*(C/A)$). It corresponds to an object of

$$\text{Rep}(\mathcal{G}^{S^1}_{\mathcal{C}})((\text{Mod}^{S^1}_A) \times_{(\text{Mod}^{S^1}_A)} \{\mathcal{H}_*(C/A)\})$$

determined by $\mathbb{A}^{\mathcal{C}}_k : \mathbb{G}^{S^1}_{\mathcal{C}} \rightarrow \text{End}^x(\mathcal{H}_*(C/A))$. By the definition of Kodaira-Spencer morphism, $F_{T_{A/k}}[-1] \simeq S \times_k S \rightarrow \text{Def}(\mathcal{C}) F_{\mathcal{C}}$ is induced by the Kodaira-Spencer morphism $KSC$. We deduce that $v$ is classified by an object of $\text{Rep}(T_{A/k})[-1]^{S^1}((\text{Mod}^{S^1}_A) \times_{(\text{Mod}^{S^1}_A)} \{\mathcal{H}_*(C/A)\})$ determined by the composite $T_{A/k}[-1]^{S^1} \rightarrow \mathbb{G}^{S^1}_{\mathcal{C}} \rightarrow \text{End}^x(\mathcal{H}_*(C/A))$, that is, the canonical action of $T_{A/k}[-1]^{S^1}$ on $\mathcal{H}_*(C/A)$ (cf. Definition 6.3).

7. Extensions via Lie algebra actions

7.1. We use a theory of formal stacks in the formulation of Ind-coherent complexes/sheaves, which is extensively developed in Gaitsgory and Rozenblyum [8, Vol. II]. We give a minimal review of definitions which we will use. Suppose that $A$ is almost of finite type over $k$: $H^0(A)$ is a usual commutative algebra of finite type over $k$, and each $H^i(A)$ is a finitely generated $H^0(A)$-module. (By derived Hilbert basis theorem [19, 7.2.4.31], the condition of almost of finite type over $k$ is equivalent to the condition of almost of finite presentation over $k$ in [19, Definition 7.2.4.26].) Furthermore, assume that $A$ is eventually coconnective, that is, there is a nonpositive integer $n$ such that $H^i(A) = 0$ for $i < n$.

1-Formal stacks. Let $\text{CA}_{\leq 0, \leq 0}^k$ be the full subcategory of $\text{CA}_{0}^k$ spanned by those objects which are almost of finite type over $k$. Let $\text{CA}_{\leq 0, \leq 0}^k$ be the full subcategory of $\text{CA}_{\leq 0}^k$ spanned by those objects which are eventually coconnective and almost of finite type over $k$. For $R \in \text{CA}_{\leq 0}^k$, we let $R_{\text{red}}$ denote the reduced ring $H^0(R)_{\text{red}}$. Let $\text{Art}_{\leq 0}^k$ be the full subcategory of $(\text{CA}_{\leq 0, \leq 0}^k)_{A/\text{A}}$ that consists of objects of the form $A \twoheadrightarrow R \rightarrow A$ such that the associated ring homomorphisms of usual reduced rings $u_{\text{red}} : A_{\text{red}} \rightarrow R_{\text{red}}$ and $v_{\text{red}} : R_{\text{red}} \rightarrow A_{\text{red}}$ are isomorphisms. We let $\text{Nil}_A$ or $\text{Nil}_S$ denote the opposite category of $\text{Art}_{\leq 0}^k$. Since $A$ is eventually coconnective, there are natural fully faithful inclusions $\text{Art}^{\text{ss}}_A \subset \text{Art}_{\leq 0}^k$ and $\text{TSZ}_A \subset \text{Nil}_A$. We define the $\infty$-category $\text{St}_A^{1}$ to be the full subcategory
of $\text{Fun}(\text{Art}^\text{nil}_A, S)$ which consists of those functors $F : \text{Art}^\text{nil}_A \rightarrow S$ satisfying the following conditions (see [8, Vol. II, Chap.5, 1.5] for the detailed account):

- $F(A)$ is a contractible space,
- If $T_1 \sqcup_T T'$ is a pushout in $\text{Nil}_S$ such that $T \rightarrow T'$ is a square zero extension, then the canonical morphism $F(T_1 \sqcup_T T') \rightarrow F(T_1) \times_{F(T)} F(T')$ is an equivalence.

We will dub an object of $\text{St}_A^1$ as a pointed $!$-formal stack over $A$ (or $S = \text{Spec} A$) (in [8], objects of $\text{St}_A^1$ are referred to as pointed formal moduli problems). The Yoneda embedding $\text{Nil}_A \rightarrow \text{St}_A^1$ exhibits $\text{Nil}_A$ as a full subcategory.

**Remark 7.1.** The $\infty$-category $\text{St}_A^1$ is one of the formulations of the $\infty$-category of pointed formal moduli problems over $\text{Spec} A$ in [8]. See [8, Ch.5, 1.2.2, 1.4.2, 1.5.2] for several formulations.

**Ind-coherent complexes.** We review some definitions concerning Ind-coherent complexes/sheaves. The following is based on [7], [8]. Suppose that $B \in \text{CAlg}_k^0$ is almost of finite type $k$. Let $\text{Coh}(B)$ be the full subcategory of $\text{Mod}_B = \text{QC}(B)$ spanned by bounded complexes with coherent cohomologies. We define the stable $\infty$-category of Ind-coherent complexes over $\text{Spec} B$ to be $\text{QC}_1(B) := \text{Ind}(\text{Coh}(B))$. The stable presentable $\infty$-categories $\text{QC}_1(B)$ have a functoriality given by $!$-pullbacks: a morphism $f : \text{Spec} B' \rightarrow \text{Spec} B$ induces a colimit-preserving functor $f_1 : \text{QC}_1(\text{Spec} B') = \text{QC}_1(B) \rightarrow \text{QC}_1(B') = \text{QC}_1(\text{Spec} B')$. Let $f_* : \text{Coh}(B') \rightarrow \text{QC}(B)$ be the pushforward determined by the restriction along $B \rightarrow B'$. Here $\text{QC}(B) \subset \text{QC}(B')$ is the full subcategory spanned by left-bounded objects with respect to the standard $!$-structure. Since $\text{QC}(B) \simeq \text{Ind}(\text{Coh}(B))^+ \subset \text{Coh}(B)$, we have $\text{Coh}(B') \rightarrow \text{QC}(B') \simeq \text{Ind}(\text{Coh}(B))^+$. Passing to the $\infty$-category, it gives rise to a colimit-preserving functor $f_*^{\text{IndCoh}} : \text{QC}_1(B') = \text{Ind}(\text{Coh}(B')) \rightarrow \text{QC}_1(B) = \text{Ind}(\text{Coh}(B))$ which extends $\text{Coh}(B') \rightarrow \text{Ind}(\text{Coh}(B))$ in an essentially unique way. When $f$ is proper (the induced morphism of classical schemes is proper), $f_1$ is defined to be a right adjoint of $f_*^{\text{IndCoh}}$. When $f$ is an open immersion, $f_1$ is defined to be a left adjoint of $f_*^{\text{IndCoh}}$. If $f$ is arbitrary, we decompose $f$ as $j \circ \omega$ such that $j$ is an open immersion and $p$ is proper, and set $f_1^1 = p_1^1 \circ j_1$. It is necessary to prove that the definition is independent of the choice of decomposition. It is a difficult task to give a functorial and canonical definition of $\text{QC}_1(B)$. This was achieved in [8]. In particular, from [8] we have a functor

$$\text{QC}_1 : \text{CAlg}_k^{0,\infty} \rightarrow \text{Pr}_k^L$$

which carries $B$ to $\text{QC}_1(B)$, and carries $f$ to $f_1^1$ (in loc. cit., the symbol $\text{IndCoh}_{\text{Sch}^\lfloor 0,\infty \rceil }$ is used). There is a functor $\Upsilon_B : \text{Mod}_B = \text{QC}(B) \rightarrow \text{QC}_1(B)$ which sends $M \rightarrow M \otimes_B \omega_B$ where $\omega_B$ a dualizing object given by $\omega_B = p_1^1(k)$ where $p : \text{Spec} B \rightarrow \text{Spec} k$ is the structure morphism, see [8, Vol. I], [7, Section 5]. When $B$ is eventually cocoonnective, $\Upsilon_B$ is fully faithful. When $B$ is smooth over $k$, $\Upsilon_B$ is an equivalence. The functor $\Upsilon_B$ is functorial with respect to $B$: if $\Upsilon_B : \text{CAlg}_k^{0,\infty} \rightarrow \text{Pr}_k^L$ denotes the functor given by $B \mapsto \text{Mod}_B = \text{QC}(B)$, there is $\Upsilon : \text{QC} \rightarrow \text{QC}_1$ between functors $\text{CAlg}_k^{0,\infty} \rightarrow \text{Pr}_k^L$. In particular, $f : \text{Spec} B' \rightarrow \text{Spec} B$ induces the commutative diagram in $\text{Pr}_k^L$:

$$\begin{array}{ccc}
\text{QC}(B) & \xrightarrow{\Upsilon_B} & \text{QC}_1(B) \\
\downarrow f & & \downarrow f_1 \\
\text{QC}(B') & \xrightarrow{\Upsilon_B'} & \text{QC}_1(B').
\end{array}$$

The stable $\infty$-category $\text{QC}_1(B)$ admits a symmetric monoidal structure, and $\Upsilon_B$ is promoted to a symmetric monoidal functor. Moreover, $\Upsilon_B : \text{CAlg}_k^{0,\infty} \rightarrow \text{Pr}_k^L$ is promoted to $\text{CAlg}_k^{0,\infty} \rightarrow \text{CAlg}(\text{Pr}_k^L)$.

Let $\text{Fun}(\text{CAlg}_k^{0,\infty}, S)_{\text{lfk}} \subset \text{Fun}(\text{CAlg}_k^{0,\infty}, S)$ be the full subcategory of functors/prestacks locally almost of finite type over $k$, see [8, Vol.1, Chap.2, 1.7]. Here we do not recall the definition of the condition of locally almost of finite type, but the restriction functor along $\text{CAlg}_k^{0,\infty} \rightarrow \text{CAlg}_k^{0,\infty}$ induces an equivalence $\text{Fun}(\text{CAlg}_k^{0,\infty}, S)_{\text{lfk}} \simeq \text{Fun}(\text{CAlg}_k^{0,\infty}, S)$. For example, a derived affine scheme almost of finite type is an example of a prestack locally almost of finite type over $k$. Thus, $\text{Spec} B$ such that $B$
is almost of finite type, is completely determined by the functor \( \text{CAlg}_{k}^{\leq,0,\square} \rightarrow S \) defined by the formula \( R \mapsto \text{Map}_{\text{CAlg}_{k}^{\leq,0,\square}}(R, R) \). We define \( \text{Fun}(\text{CAlg}_{k}^{\leq,0,\square}, S)^{op} \rightarrow \text{Pr}_{k}^{L} \) as a right Kan extension of

\[
\text{CAlg}_{k}^{\leq,0,\square} \rightarrow \text{CAlg}_{k}^{\leq,0,\square \square} \xrightarrow{QC} \text{Pr}_{k}^{L}
\]
along the Yoneda embedding \( \text{CAlg}_{k}^{\leq,0,\square} \rightarrow \text{Fun}(\text{CAlg}_{k}^{\leq,0,\square}, S)^{op} \). We abuse notation by writing \( QC \) for the resulting functor \( \text{Fun}(\text{CAlg}_{k}^{\leq,0,\square}, S)^{op} \rightarrow \text{Pr}_{k}^{L} \). We refer to \( QC(F) \) as the stable \( \infty \)-category of Ind-coherent complexes on \( F \).

According to [8, Vol.II, Chap.7, 3.1.4, 3.3.2], there is a categorical equivalence

\[
\text{Lie}_{A}^{1} \simeq \hat{S}_{A}^{1}
\]
where \( \text{Lie}_{A}^{1} = \text{Lie}(QC_{\ast}(A)) \). The symmetric monoidal functor \( \Upsilon_{A} \) induces \( \text{Lie}_{A} \rightarrow \text{Lie}_{A}^{1} \). Since we assume that \( A \) is eventually coconnective, \( \Upsilon_{A} \) is a fully faithful left adjoint functor, see [7, Section 9.6]. Hence \( \text{Lie}_{A} \rightarrow \text{Lie}_{A}^{1} \) is also a fully faithful left adjoint functor.

7.2.

Construction 7.2. We will construct \( \Theta_{A} : \hat{S}_{A}^{\ast} \rightarrow \hat{S}_{A}^{1} \) that extends the fully faithful embedding \( \text{TSZ}_{A} \hookrightarrow \text{Nil}_{A} \subset \hat{S}_{A}^{1} \) induced by the Yoneda embedding. Recall the definition \( \hat{S}_{A}^{\ast} = \mathcal{P}_{\Sigma}^{\ast}((\text{TSZ}_{A})) \) and the universal property of \( \mathcal{P}_{\Sigma}^{\ast}((\text{TSZ}_{A})) \) (cf. Section 3.3 and Section 3.4). To construct \( \hat{S}_{A}^{\ast} \rightarrow \hat{S}_{A}^{1} \), it will suffice to construct \( \text{TSZ}_{A} \rightarrow \hat{S}_{A}^{1} \) which preserve finite coproducts and carry each morphism in \( T = \{ \text{Spec} A \in \text{Spec}(A \oplus M) \hookrightarrow \text{Spec} A \rightarrow \text{Spec}(A \oplus M[-1]) \} \) to an equivalence. Consider the Yoneda embedding \( \text{TSZ}_{A} \subset \text{Nil}_{A} \hookrightarrow \hat{S}_{A}^{1} \). This functor preserves finite coproducts (by the definition [8, Ch. 5, 1.5.2 (b)]). Moreover, it carries each morphism in \( T \) to an equivalence since (pointed) objects lying in \( \hat{S}_{A}^{1} \) admit deformation theory in the sense of [8, Ch. 1, 7.1.2] (in particular, objects in \( \hat{S}_{A}^{1} \) have cotangent complexes). Consequently, \( \text{TSZ}_{A} \subset \text{Nil}_{A} \hookrightarrow \hat{S}_{A}^{1} \) is (uniquely) extended to a colimit-preserving functor \( \hat{S}_{A}^{\ast} \rightarrow \hat{S}_{A}^{1} \), that we will denote by \( \Theta_{A} \).

Lemma 7.3. The functor \( \Theta_{A} : \hat{S}_{A}^{\ast} \rightarrow \hat{S}_{A}^{1} \) is fully faithful. The composite \( \text{Lie}_{A} \simeq \hat{S}_{A}^{\ast} \rightarrow \hat{S}_{A}^{1} \simeq \text{Lie}_{A}^{1} \) carries \( \text{Free}_{\text{Lie}_{A}}(N) \in \text{Lie}_{A}^{1} \rightarrow \Upsilon_{A}(\text{Free}_{\text{Lie}_{A}}(N)) \). See Section 3.4 for \( \text{Lie}_{A}^{1} \).

Proof. We first prove the second assertion. We note that the restriction of \( \hat{S}_{A}^{\ast} \rightarrow \hat{S}_{A}^{1} \simeq \text{Lie}_{A}^{1} \) determines an equivalence \( \text{TSZ}_{A} \simeq \text{Lie}_{A}^{1} \). If \( M \) is a sufficient-object \( A \)-module of the form \( M = \bigoplus_{0 \leq i} \text{Spec} A \oplus_{\text{Nil}_{A}} M \), the equivalence \( \text{TSZ}_{A} \simeq \text{Lie}_{A}^{1} \) sends \( \text{Spec}(A \oplus M) \to \text{Free}_{\text{Lie}_{A}}(N) \) where \( N = M^{\vee}[-1] \) (\( M^{\vee} \) is the dual object of \( M \) in the symmetric monoidal \( \infty \)-category \( \text{Mod}_{A} \)). According to [8, Vol. II, Chap.7, 3.7.1 (3.13, 3.7.10)], the equivalence \( \hat{S}_{A}^{1} \simeq \text{Lie}_{A}^{1} \) sends \( \text{Spec}(A \oplus M) \to \text{Free}_{\text{Lie}_{A}}(N) \) where \( \text{Free}_{\text{Lie}_{A}}(N) \in \text{Lie}_{A}^{1} \rightarrow \Upsilon_{A}(\text{Free}_{\text{Lie}_{A}}(N)) \) where \( \Upsilon_{A} : \text{QC}(A) \rightarrow \text{Lie}_{A}^{1} \) is the free functor, that is, a left adjoint to the forgetful functor. Thus the second claim follows.

Next, we prove the first assertion. To this end, it is enough to show that the composite \( F : \text{Lie}_{A} \simeq \hat{S}_{A}^{\ast} \rightarrow \hat{S}_{A}^{1} \simeq \text{Lie}_{A}^{1} \) is fully faithful. Observe that \( \text{TSZ}_{A} \rightarrow \hat{S}_{A}^{\ast} \rightarrow \hat{S}_{A}^{1} \) determines an equivalence to its essential image which we denote by \( \Theta_{A}(\text{TSZ}_{A}) \). Through the equivalence \( \hat{S}_{A}^{1} \simeq \text{Lie}_{A}^{1} \), \( \Theta_{A}(\text{TSZ}_{A}) \) corresponds to the full subcategory of \( \text{Lie}_{A}^{1} \) which consists of objects of the form \( \Upsilon_{A}(\text{Free}_{\text{Lie}_{A}}(N)) \) such that \( \text{Free}_{\text{Lie}_{A}}(N) \in \text{Lie}_{A}^{1} \). We write \( \text{Lie}_{A}^{1} \) for this full subcategory. The composite \( F : \text{Lie}_{A} \rightarrow \text{Lie}_{A}^{1} \) induces an equivalence \( \theta : \text{Lie}_{A} \rightarrow \text{Lie}_{A}^{1} \). By [12, 1.2.2], \( \text{Lie}_{A} \hookrightarrow \text{Lie}_{A}^{1} \) determines an equivalence \( \mathcal{P}_{\Sigma}^{\ast}(\text{Lie}_{A}^{1}) \simeq \text{Lie}_{A}^{1} \) where \( \mathcal{P}_{\Sigma}^{\ast}(\text{Lie}_{A}^{1}) \rightarrow \text{Lie}_{A}^{1} \) is induced by the universal property of \( \mathcal{P}_{\Sigma}^{\ast}(\text{Lie}_{A}^{1}) \) (see Section 3.3). Recall that \( \Upsilon_{A} : \text{Lie}_{A} \rightarrow \text{Lie}_{A}^{1} \) is fully faithful. It can be identified with \( \mathcal{P}_{\Sigma}^{\ast}(\text{Lie}_{A}^{1}) \rightarrow \mathcal{P}_{\Sigma}^{\ast}(\text{Lie}_{A}^{1}) \rightarrow \text{Lie}_{A}^{1} \) obtained from \( \text{Lie}_{A}^{1} \simeq \text{Lie}_{A}^{1} \hookrightarrow \text{Lie}_{A}^{1} \). The composite \( F : \text{Lie}_{A} \rightarrow \text{Lie}_{A}^{1} \) is a colimit-preserving functor that extends the restriction \( f : \text{Lie}_{A}^{1} \subset \text{Lie}_{A} \rightarrow \text{Lie}_{A}^{1} \) in an essentially unique way. In particular, \( F \) is given by \( \mathcal{P}_{\Sigma}^{\ast}(\text{Lie}_{A}^{1}) \rightarrow \mathcal{P}_{\Sigma}^{\ast}(\text{Lie}_{A}^{1}) \rightarrow \text{Lie}_{A}^{1} \) so that \( F \) is fully faithful. This proves the first assertion. □
Remark 7.4. When $A$ is smooth usual algebra over $k$, the proof of Lemma 7.3 shows that $\Theta_A : \hat{\mathcal{S}}_A \to \hat{\mathcal{S}}_A^1$ is an equivalence because the inclusion $\text{Lie}_A^1 \subset \text{Lie}_A^r$ induces an equivalence $\mathcal{P}_S^r(\text{Lie}_A^1) \simeq \text{Lie}_A^r$.

Let $X$ and $S$ be functors $\text{CAlg}^\leq_{k_0} \to S$ and let $i : S \to X$ be a morphism in $\text{Fun}(\text{CAlg}^\leq_{k_0}, S)$. We define $X^\wedge$ (or simply $X_S^\wedge$) which belongs to $\text{Fun}(\text{CAlg}^\leq_{k_0}, S)$ (see [8, Vol.II, Chap. 4]). The functor $X^\wedge$ may be regarded as the formal completion of $X$ along $X$ along $i$. As we shall observe, it is closely related to $X$ in Section 3.4 in a suitable situation. Let $\text{red} : \text{CAlg}^\leq_{k_0} \to \text{CAlg}^\leq_{k_0}$ be the functor given by $R \mapsto R_{\text{red}}$, that is a left adjoint of the inclusion $\text{CAlg}^\text{red}_{k_0} \hookrightarrow \text{CAlg}^\leq_{k_0}$ where $\text{CAlg}^\text{red}_{k_0}$ is the ordinary category of reduced commutative $k$-algebras. Let $X_{\text{dr}} : \text{CAlg}_{k_0} \to S$ be the composite functor $X \circ \text{red}$: it is given by $R \mapsto X(R_{\text{red}})$. We refer to $X_{\text{dr}}$ as the de Rham prestack of $X$. There is a natural transformation (a unit map of the adjoint pair) from the identity functor to $\text{red}$. It gives rise to a canonical morphism $X \to X_{\text{dr}}$. We define $X^\wedge : \text{CAlg}^\leq_{k_0} \to S$ to be $X \times_{X_{\text{dr}}} S_{\text{dr}}$. Suppose that $S = \text{Spec} A$ and $S \to X \to S$ is a pointed derived scheme locally almost of finite type over $S$. (cf. [8, Vol. I] for the condition of locally almost of finite type). We regard $S \to X \times_{X_{\text{dr}}} S_{\text{dr}} = X_S^\wedge \to S$ as an object of $\text{Fun}(\text{CAlg}^\leq_{k_0}, S)_S/S$ (by restricting to $\text{CAlg}^\leq_{k_0}$). The fully faithful functor $\text{Nil}_A \hookrightarrow \text{Fun}(\text{CAlg}^\leq_{k_0}, S)_S/S$ induced by Yoneda embedding and the functor $H^\bullet_\text{dR}X_S^\wedge \to S$ represented by $S \to X_S^\wedge \to S$ define a pointed $\mathfrak{l}$-formal stack $\text{Art}^\mathfrak{l}_A \hookrightarrow (\text{Fun}(\text{CAlg}^\leq_{k_0}, S)_S/S)^{\text{op}}$ $\xrightarrow{H^\bullet_\text{dR}X_S^\wedge \to S} S$. By abuse of notation, we will write $X_S^\wedge$ also for this pointed $\mathfrak{l}$-formal stack over $A$.

The formal completion of $S \to X \to S$ in the sense of Section 3.4 also gives rise to the pointed formal stack $\hat{X} \in \hat{\mathcal{S}}_A$. We will compare $\hat{X}$ and $X_S^\wedge$. We continue to assume that $S = \text{Spec} A \in \text{Fun}(\text{CAlg}^\leq_{k_0}, S)$. Let $S \to F \to S$ be an object of $\text{Fun}(\text{CAlg}^\leq_{k_0}, S)_S/S$. We remark that the third equivalence follows from the fact that $\text{Map}(S_{\text{red}}, X)$ is a discrete space because $X$ is a derived scheme (or algebraic space). We have a canonical equivalence

$\text{Map}_{\text{Fun}(\text{CAlg}^\leq_{k_0}, S)_S/S}(S, X_S^\wedge) \simeq \text{Map}_{\text{Fun}(\text{CAlg}^\leq_{k_0}, S)_S/S}(S, X)$

which is natural in $S_M$. Thus Lemma follows.

Corollary 7.6. There exists an equivalence between

$$(\text{TSZ}_A/X_S^\wedge) \subset (\hat{\mathcal{S}}_A^1/X_S^\wedge) \quad \text{and} \quad (\text{TSZ}_A/X) \subset \text{Fun}(\text{CAlg}^\leq_{k_0}, S)_S/S.$$ 

Proposition 7.7. Suppose that $X$ is a derived scheme locally of finite presentation over $S$, which is endowed with a section morphism $i : S \to X$. Moreover, suppose that $X$ is eventually coconnective. Then the functor $\Theta_A : \hat{\mathcal{S}}_A^1 \to \hat{\mathcal{S}}_A$ carries $\hat{X}$ to $X_S^\wedge$. More precisely, there is an equivalence $\Theta_A(\hat{X}) \to X_S^\wedge$ which is constructed in the proof.
Remark 7.8. If $S$ is smooth over $k$, the condition of eventually coconnectiveness on $X$ is not necessary, see the proof below.

Proof. For any $S_M = \text{Spec}(A \oplus M) \in \text{TSZ}_A$, by the definition of $\hat{X}$ there is an equivalence $\text{Map}_{\text{St}_A^e}(S_M, \hat{X}) \xrightarrow{\sim} \text{Map}_{\text{Fun}(\text{Alg}^{\leq 0, \square})}(S_M, X)$. Thus, by Corollary 7.6 we have canonical equivalences

$$\text{Map}_{\text{Fun}(\text{Alg}^{\leq 0, \square})}(S_M, X^\wedge_{S}) \simeq \text{Map}_{\text{Fun}(\text{Alg}^{\leq 0, \square})}(S_M, X) \simeq \text{Map}_{\text{St}_A^e}(S_M, \hat{X}).$$

Consequently, $(\text{TSZ}_A)^{\wedge}_{S} \rightarrow \text{St}_A^e \rightarrow \hat{X}$ is equivalent to $(\text{TSZ}_A)^{\wedge}_{S} \rightarrow \hat{X}$ (by abuse of notation, we here regard $\text{TSZ}_A$ as full subcategories of $\text{St}_A^e$ and $\hat{X}$; keep in mind that $\text{TSZ}_A \subset \text{St}_A^e \rightarrow \hat{X}$ is naturally equivalent to the Yoneda embedding.) Note that by definition $\Theta_A(\hat{X})$ is a colimit of $(\text{TSZ}_A)^{\wedge}_{S} \rightarrow \hat{X} \rightarrow \hat{X}$. We may and will assume that $\Theta_A(\hat{X})$ is a colimit of $(\text{TSZ}_A)^{\wedge}_{S} \rightarrow \hat{X}$. The universal property of the colimit determines a morphism $\Theta_A(\hat{X}) \rightarrow \hat{X}$. We prove that the morphism is an equivalence. Note that the restriction of the equivalence $\hat{X} \rightarrow \text{Lie}_A^1$ induces $\text{TSZ}_A \simeq \text{Lie}_A^1$. Let $L$ be an object of $\text{Lie}_A^1$ which corresponds to $X^\wedge_{S}$. It is enough to prove that $L$ is a colimit of $(\text{Lie}_A^1)^{\wedge}_{L} \rightarrow \text{Lie}_A^1$. Now assume for the moment that $L$ belongs to the essential image of $\text{Lie}_A^1 \rightarrow \text{Lie}_A^1$. Since its essential image coincides with the essential image of $\text{Y}_A : \text{Lie}_A \rightarrow \text{Lie}_A^1$ (see the construction and the proof of Lemma 7.3) and $\text{Y}_A : \text{Mod}_A \rightarrow \text{QC}_1(A)$ is fully faithful (because we assume that $A$ is eventually coconnective), $L$ satisfies the assumption if and only if the underlying object $L$ in $\text{QC}_1(A)$ belongs to the essential image of $\text{Y}_A$. In $\text{QC}_1(A)$, it follows from [8, Vol. II, Chap. 7, 3.7] (see also [12, 2.2.9]) that $L$ is obtained from the Serre dual of $\mathbb{L}_{X/S}$ (that lies in $\text{QC}_1(X)$) by the $!$-pullback along $i : S \rightarrow X$, where $\mathbb{L}_{X/S}$ is the cotangent complex that is a perfect complex with bounded coherent cohomologies because $X$ is a derived scheme locally of finite presentation over $S$ and is eventually coconnective. Thus, by [7, 9.6.5] it is equivalent to $\text{Y}_A(i^!(\mathbb{L}_{X/S}[-1])) = i^!(\mathbb{L}_{X/S} \otimes \omega_A[-1])$, where $\mathbb{L}_{X/S}$ is the dual object in $\text{Mod}_A$. Hence $L$ satisfies the assumption. Finally, we remark that if $S$ is smooth over $k$, any object of $\text{Lie}_A^1$ is a colimit of a sifted diagram of objects in $(\text{Lie}_A^1)$ (see Remark 7.4), so that the final part of the proof is not necessary. \hfill $\Box$

7.3. In Section 7, in what follows, $A$ is a commutative (ordinary) algebra which is smooth over the base field $k$. If $\overline{R}$ is the image of $R \in \text{Art}^{\text{tsz}}_A$ under the forgetful functor $\text{Art}^{\text{tsz}}_A \rightarrow \text{CAlg}_A$, then $\text{Map}_{\text{CAlg}_A}(\overline{R}, A)$ is a contractible space. Thus, the forgetful functor $\text{Art}^{\text{tsz}}_A \rightarrow \text{CAlg}_A$ is a fully faithful functor into the essential image. We often identify $\text{Art}^{\text{tsz}}_A$ with the essential image in $\text{CAlg}_A$.

Let $\text{QC}_1|_{\text{Art}^{\text{tsz}}_A} : \text{Art}^{\text{tsz}}_A \rightarrow \text{Cat}_\infty$ be the functor given by $R \mapsto \text{QC}_1(R)$ with $!$-pullback functors. See Construction 7.10 for a more precise formulation. Let $\text{Rep}_H|_{\text{Art}^{\text{tsz}}_A} : \text{Art}^{\text{tsz}}_A \rightarrow \tilde{\text{Cat}}_\infty$ be the restriction of $\text{Rep}_H : \text{Fun}(\text{Art}^{\text{tsz}}_A, S)^{\text{op}} \rightarrow \tilde{\text{Cat}}_\infty$ to $\text{Art}^{\text{tsz}}_A$, that is, the functor given by $R \mapsto L\text{Mod}_{\text{D}(R)}$ with respect to restriction functors.

Proposition 7.9. There exists a natural equivalence $\delta_A : \text{QC}_1|_{\text{Art}^{\text{tsz}}_A} \xrightarrow{\sim} \text{Rep}_H|_{\text{Art}^{\text{tsz}}_A}$.

We construct $\delta_A$ in the following Construction.

Construction 7.10. We first describe $\text{QC}_1|_{\text{Art}^{\text{tsz}}_A}$. Consider $\text{TSZ}_A = (\text{Art}^{\text{tsz}}_A)^{\text{op}} \rightarrow \tilde{\text{Cat}}_\infty$ which is given by $\text{Spec} R \mapsto \text{QC}_1(R)$. For $f : \text{Spec} R \rightarrow \text{Spec} R'$, it sends $f$ to $f_{\text{IndCoh}} : \text{QC}_1(R) \rightarrow \text{QC}_1(R')$. Note that any morphism $f : \text{Spec} R \rightarrow \text{Spec} R'$ that comes from $\text{TSZ}_A$ is proper. It follows that the restriction induces $f_* : \text{Coh}(R) \rightarrow \text{Coh}(R')$ which gives rise to $f_{\text{IndCoh}} : \text{Ind}(\text{Coh}(R)) \rightarrow \text{Ind}(\text{Coh}(R'))$ by passing to Ind-categories. Let $\rho : \text{Mod}^{\text{Coh}}(\text{Mod}_A) \rightarrow \text{Art}^{\text{tsz}}_A$ be the Cartesian fibration obtained from the Cartesian fibration $\text{Mod}(\text{Mod}_A) \rightarrow \text{CAlg}_A$ by the restriction to those pairs $(R, M)$ such that
If $R \in \text{Art}_{A}^{\text{tsz}}$ and $M \in \text{Coh}(R) \subset \text{Mod}(R)$ (see [19, 4.5.1] for $\text{Mod}(\text{Mod}_{A}) \to C\text{Alg}_{k}(A)$). The functor $F_{\rho} : \text{TSZ}_{A} \to \text{Cat}_{\infty}$ corresponding to $\rho$ is informally given on objects by $R \mapsto \text{Coh}(R)$ with respect to $f_{*} : (\text{Coh}(R) \to \text{Coh}(R'))$ for $f : \text{Spec} R \to \text{Spec} R'$. Note that Grothendieck/coherent duality gives an equivalence $\text{Coh}(R) \simeq \text{Coh}(R)^{op}$. There are canonical equivalences $\text{Coh}(R) \simeq \text{Coh}(R)^{op}$ which are functorial in $R \in \text{Art}_{A}^{\text{tsz}}$ with respect to restrictions [8, Vol.I, Chap.6, 4.2.4]. This gives a natural equivalence between $F_{\rho} : \text{TSZ}_{A} \to \text{Cat}_{\infty}$ and the functor $F_{\rho'} : \text{TSZ}_{A} \to \text{Cat}_{\infty}$ informally defined by $R \mapsto \text{Coh}(R)^{op}$ and functors $\text{Coh}(R)^{op} \to \text{Coh}(R')^{op}$ induced by restrictions $R' \to R$. More precisely, $F_{\rho'}$ is the composite functor $\text{TSZ}_{A} \xrightarrow{F_{\rho}} \text{Cat}_{\infty} \xrightarrow{\text{Cat}_{\infty}} \text{Cat}_{\infty}$ where the second functor is given by passing to opposite categories $\mathcal{X} \mapsto \mathcal{X}^{op}$. We let $\rho'$ denote a Cartesian fibration corresponding to $F_{\rho'}$. Applying the construction $D^{\text{lex}}$ in [20, X, 3.4.6–3.4.10] to $\rho'$, we have a coCartesian fibration $D^{\text{lex}}(\rho') \to \text{Art}_{A}^{\text{tsz}}$. The coCartesian fibration $D^{\text{lex}}(\rho') \to \text{Art}_{A}^{\text{tsz}}$ corresponds to the functor $\text{Art}_{A}^{\text{tsz}} \to \text{Cat}_{\infty}$ obtained from $\text{TSZ}_{A} \to \text{Pr}_{1}$ given by $R \mapsto \text{Ind}(\text{Coh}(R))$ (with the pushforwards $f_{*}^{\text{IndCoh}}$ for $f : \text{Spec} R \to \text{Spec} R'$) passing to right adjoint functors (cf. [18, 5.5.3.4]). For $f : \text{Spec} R \to \text{Spec} R'$, it gives rise to $f^{\dagger} : \text{Ind}(\text{Coh}(R')) = \text{Fun}^{\text{lex}}(\text{Coh}^{\text{op}}(R'), S) = \text{Fun}^{\text{lex}}(\text{Coh}(R)^{op}, S) = \text{Ind}(\text{Coh}(R))$ which is given by composition with $(f_{*})^{op} : \text{Coh}(R)^{op} \to \text{Coh}(R')^{op}$. Here the superscript in $\text{Fun}^{\text{lex}}(\text{Coh}(R), S)$ indicates the full subcategory spanned by left exact functors. We define $Q_{C_{1}|\text{Art}_{A}^{\text{tsz}} : \text{Art}_{A}^{\text{tsz}} \to \text{Cat}_{\infty}$ to be a functor corresponding to $D^{\text{lex}}(\rho') \to \text{Art}_{A}^{\text{tsz}}$. For ease of notation, we set $G = Q_{C_{1}|\text{Art}_{A}^{\text{tsz}}$. 

Next, we describe $\text{Rep}_{H}|_{\text{Art}_{A}^{\text{tsz}}}$. Consider the coCartesian fibration $p : \text{LMod}(\text{Mod}(A)) \to \text{Alg}_{k}(A)$ (cf. Section 2), which corresponds to the functor $\text{Alg}_{1}(\text{Mod}_{A}) \to \text{Cat}_{\infty}$ informally given by $B \mapsto \text{LMod}_{B}(\text{Mod}_{A})$, endowed with the base change functoriality. Let $\text{LMod}^{\text{perf}} (\text{Mod}_{A})$ be the full subcategory spanned by those pairs $(B, M)$ such that $M$ is a compact object in $\text{LMod}_{B}(\text{Mod}_{A}) \simeq \text{LMod}_{B}$. The restriction $p_{\text{perf}} : \text{LMod}^{\text{perf}}(\text{Mod}_{A}) \to \text{Alg}_{1}(\text{Mod}_{A})$ is also a coCartesian fibration. We apply $D^{\text{lex}}$ to the Cartesian fibration $(p_{\text{perf}})^{op} : \text{LMod}^{\text{perf}}(\text{Mod}_{A})^{op} \to \text{Alg}_{1}(\text{Mod}_{A})^{op}$. Then we obtain a coCartesian fibration $D^{\text{lex}}(p_{\text{perf}})^{op} \to \text{Alg}_{1}(\text{Mod}_{A})^{op}$. This coCartesian fibration is equivalent to $\text{LMod}(\text{Mod}_{A})^{op} \to \text{Alg}_{1}(\text{Mod}_{A})^{op}$ whose “opposite” is the Cartesian fibration $\text{LMod}(\text{Mod}_{A}) \to \text{Alg}_{1}(\text{Mod}_{A})$ (see [20, X, 3.4.10]). The functor $\text{Rep}_{H}|_{\text{Art}_{A}^{\text{tsz}}}$ corresponds to the Cartesian fibration $\text{LMod}(\text{Mod}_{A}) \times \text{Alg}_{1}(\text{Mod}_{A}) \to (\text{Art}_{A}^{\text{tsz})^{op} \to (\text{Art}_{A}^{\text{tsz})^{op} that is the base change of $p$ along $D_{1} : (\text{Art}_{A}^{\text{tsz})^{op} \to \text{Alg}_{1}(\text{Mod}_{A})$ (we omit the forgetful functor $\text{Art}_{A}^{\text{tsz}} \to \text{Alg}_{1}(\text{Mod}_{A})$ from the notation). We set $H = \text{Rep}_{H}|_{\text{Art}_{A}^{\text{tsz}}$. 

We will construct a natural equivalence $G \simeq H$. For this purpose, it is enough to prove that the base change $\text{LMod}^{\text{perf}}(\text{Mod}_{A})^{op} \times \text{Alg}_{1}(\text{Mod}_{A})^{op} \to \text{Art}_{A}^{\text{tsz}}$ is equivalent to $\rho'$ since the construction $D^{\text{lex}}$ gives $G \simeq H$. By the equivalence $\rho \simeq \rho'$, it will suffice to construct an equivalence between $\text{LMod}^{\text{perf}}(\text{Mod}_{A})^{op} \times \text{Alg}_{1}(\text{Mod}_{A})^{op} \to \text{Art}_{A}^{\text{tsz}}$ and $\rho$. The required construction is a straightforward generalization of [20, X, 3.5.3–3.5.6]. In loc. cit., the case of $A = k$ was carried out. Note that $R$ is of the form $A \otimes_{k} R_{0}$ such that $R_{0} \in \text{Art}_{k}^{\text{tsz}}$. It follows from [7, 4.6.2] that $\text{Coh}(R) \simeq \text{Coh}(A) \otimes_{k} \text{Coh}(R_{0}) \simeq \text{Perf}_{A} \otimes_{k} \text{Coh}(R_{0})$. Thus, $\text{Coh}(R)$ is generated by the essential image of $\text{Perf}_{A} \simeq \text{Coh}(A) \to \text{Coh}(R)$ by the restriction along $R \to A$. We see that $\text{Coh}(R)$ is the smallest stable subcategory of $\text{Mod}_{B}$, which contains $A$ and is closed under retracts (we use the assumption that $A$ is smooth over $k$). If we use this description of $\text{Coh}(R)$ for $R \in \text{Art}_{A}^{\text{tsz}}$, the construction in [20] works in our situation: it gives rise to a pullback square of ∞-categories

\[
\begin{array}{ccc}
\text{Mod}^{\text{Coh}}(\text{Mod}_{A}) & \xrightarrow{\rho} & \text{LMod}^{\text{perf}}(\text{Mod}_{A})^{op} \\
\downarrow & & \downarrow (p_{\text{perf}})^{op} \\
\text{Art}_{A}^{\text{tsz}} & \xrightarrow{\beta_{1}} & \text{Alg}_{1}(\text{Mod}_{A})^{op}.
\end{array}
\]

Consequently, we obtain $G \simeq H$. Unfolding the construction, when $R \in \text{Art}_{A}^{\text{tsz}}$ is $A$, $\delta_{A}$ induces the equivalence $Q_{C_{1}}(A) \xrightarrow{\sim} Q_{C}(A)$ determined by the tensor with the dual $\omega_{A}^{\vee}$ of the dualizing complex (that is, it is given by the formula $M \mapsto M \otimes_{A} \omega_{A}^{\vee}$).

7.4. Let $F : \text{Art}_{A}^{\text{nil}} \to S$ be a pointed !-formal stack over $S$. To $F$ we associate $F^{\prime} : C\text{Alg}_{k}^{\leq 0, \square} \to S$ equipped with $F^{\prime} \to \text{Spec} A = S$ and its section $S \to F^{\prime}$. We then define $Q_{C_{1}}(F)$ as $Q_{C_{1}}(F^{\prime})$. Let $F^{\ast} : (C\text{Alg}_{k}^{\leq 0, \square})_{A/} \to S$ be a left Kan extension of $F$ along $\text{Art}_{A}^{\text{nil}} \to (C\text{Alg}_{k}^{\leq 0, \square})_{A/}$. That is, $F^{\ast}$ is the image
of $F$ under the left adjoint of the forgetful functor $\text{Fun}((\text{CAlg}_k^{\leq 0, \square})_{/A}, S) \to \text{Fun}(\text{Art}_A^{nil}, S)$. Through the canonical equivalence $\text{Fun}((\text{CAlg}_k^{\leq 0, \square})_{/A}, S) \simeq \text{Fun}(\text{CAlg}_k^{\leq 0, \square}, S)/S$, $F'$ corresponds to $F'' \to S$ in $\text{Fun}((\text{CAlg}_k^{\leq 0, \square})_{/A}, S)/S$. Specifically, $F'' : \text{CAlg}_k^{\leq 0, \square} \to S$ can be defined as follows: if $Q_{F'} : (\text{CAlg}_k^{\leq 0, \square})_{/A} \to S$ is a left fibration classified by $F'$, then $F'' : \text{CAlg}_k^{\leq 0, \square} \to S$ corresponds to the composite of left fibrations $Q_{F'} \to (\text{CAlg}_k^{\leq 0, \square})_{/A} \to \text{CAlg}_k^{\leq 0, \square}$. If $O$ denotes the both initial and final pointed $d$-formal stack, then $O'$ is a final object in $\text{Fun}((\text{CAlg}_k^{\leq 0, \square})_{/A}, S)$. Thus, $F'' \to S$ admits a section $S \to F''$. We think of $F'' : \text{CAlg}_k^{\leq 0, \square} \to S$ as the underlying prestack locally almost of finite type over $k$. We define the stable $\infty$-category $QC_{\Gamma}(F)$ of Ind-coherent complexes/sheaves to be the stable $\infty$-category $QC_{\Gamma}(F'')$ of the Ind-coherent complexes on the underlying prestack $F'' : \text{CAlg}_k^{\leq 0, \square} \to S$. Recall that by definition $QC_{\Gamma}(F'')$ is informally given by the limit

$$\lim_{\text{Spec } B \to F'' \in ((\text{CAlg}_k^{\leq 0, \square})_{/S}), F'' \to S} QC_{\Gamma}(B)$$

of $\infty$-categories, where the diagram is induced by $\delta$-pullback functors. Suppose that colim$_{t \in I} F_t = F$ is a colimit of a sifted diagram $I \to \hat{\mathcal{S}}^1_{/A}$. By the compatibility of $QC_{\Gamma}$ with sifted colimits in $\hat{\mathcal{S}}^1_{/A}$ ([8, Vol. I, Chap. 7, 5.3]), the diagram of $\delta$-pullback functors yields an equivalence of symmetric monoidal $\infty$-categories

$$QC_{\Gamma}(F) \xrightarrow{\sim} \lim_{t \in I} QC_{\Gamma}(F_t).$$

In particular, if each $F_t$ is representable by Spec $B_t$, we have $QC_{\Gamma}(F) \simeq \lim_{t \in I} QC_{\Gamma}(B_t)$. For $Y \in \hat{\mathcal{S}}^1_{/A}$, we have

$$\text{Rep}_H(Y) = \lim_{\text{Spec } (A \oplus M) \in (TSZ_A)_{/Y}} \text{Rep}_H(\text{Spec } (A \oplus M)) \to \lim_{\text{Spec } (A \oplus M) \in (TSZ_A)_{/\theta_A(Y)}} \text{Rep}_H(\text{Spec } (A \oplus M))$$

$$\simeq \lim_{\text{Spec } (A \oplus M) \in (TSZ_A)_{/\theta_A(Y)}} QC_{\Gamma}(\text{Spec } (A \oplus M))$$

$$\simeq QC_{\Gamma}(\theta_A(Y))$$

where the second equivalence is induced by $\delta$ in Proposition 7.9 and Construction 7.10. By Lemma 7.3, the first functor is an equivalence. Since $\theta_A(Y)$ is a sifted colimit of $(TSZ_A)_{/\theta_A(Y)} \to \hat{\mathcal{S}}^1_{/A}$, the final equivalence follows from the compatibility of $QC_{\Gamma}$ with sifted colimits in $\hat{\mathcal{S}}^1_{/A}$. We write

$$\Gamma_Y : \text{Rep}_H(Y) \xrightarrow{\sim} QC_{\Gamma}(\theta_A(Y))$$

for the induced equivalence. Suppose that $Y$ is a formal completion $\hat{X}$ along $S \to X \to S = \text{Spec } A$ in the sense of Section 3.4 where $X$ is a derived scheme locally of finite presentation over $S$ and is eventually coconnective. According to Lemma 7.3, Proposition 7.7 and its proof, $\hat{\mathcal{S}}^1_{/A} \to \hat{\mathcal{S}}^1_\delta$ is a fully faithful left adjoint, and a sifted colimit of $(TSZ_A)_{/\theta_A(Y)} \simeq (TSZ_A)/X^\delta_\delta \to \hat{\mathcal{S}}^1_{/A}$ is canonically equivalent to $X^\delta_\delta$. In summary, in this situation we see:

**Proposition 7.11.** There exists an equivalence $\Gamma_{\hat{X}} : \text{Rep}_H(\hat{X}) \xrightarrow{\sim} QC_{\Gamma}(X^\delta_\delta)$.

7.5. Let $LS = S \times_{S_k} S$ be the free loop space of derived scheme $S = \text{Spec } A$ over $k$. Let $\iota : S \to LS$ be the morphism induced by constant loops, so that $\iota$ is an $S^1$-equivariant morphism. Let $(S \times_k S)^\delta_\delta$ denote $(S \times_k S) \times (S \times_k S)_{\text{dir}} S_{\text{dir}}$, that is determined by the diagonal $\Delta : S \to S \times_k S$. Let $(S \times_k LS)^\delta_\delta$ denote $(S \times_k LS) \times (S \times_k LS)_{\text{dir}} S_{\text{dir}}$, determined by $\iota \times \iota : S \to S \times_k LS$. Let $LS^\delta_\delta$ be the formal completion $LS \times (LS)_{\text{dir}} S_{\text{dir}}$ along the morphism $\iota : S \to LS$. Here we consider $(S \times_k S)^\delta_\delta$, $(S \times_k LS)^\delta_\delta$ and $LS^\delta_\delta$ as functors $\text{CAlg}_k^{\leq 0, \square} \to S$ (i.e., prestacks locally almost of finite type). Observe that $S_{\text{dir}} \to (LS)_{\text{dir}}$ is an equivalence so that the canonical morphism $LS^\delta_\delta \to LS$ is an equivalence. Indeed, by definition, for $R \in \text{CAlg}_k^{\leq 0}$ the mapping space $\text{Map}_{\text{Fun}}(\text{CAlg}_k^{\leq 0, \square}, S)(\text{Spec } R, (LS)_{\text{dir}})$ is equivalent to

$$\text{Map}_{\text{Fun}}(\text{CAlg}_k^{\leq 0, \square}, S)(\text{Spec } R_{\text{red}}, \text{Map}(S^1, S)) \simeq \text{Map}_{S}(S^1, S(R_{\text{red}})) \simeq S(R_{\text{red}}).$$

Here the final equivalence follows from the facts that $S^1$ is connected and $S(R_{\text{red}})$ is a discrete space because $S$ is a derived scheme (or algebraic space). Thus, $S_{\text{dir}} \simeq (LS)_{\text{dir}}$. The derived (affine) schemes $S \times_k S$, $S \times_k LS$ and $LS$ are almost of finite type over $k$. (By [19, 7.2.4.31], the condition of almost of
finite type over \( k \) is equivalent to the condition of almost of finite presentation over \( k \) in [19, Definition 7.2.4.26].) Indeed, since we assume that \( A \) is almost of finite type over \( k \), it follows from \([8, \text{Vol. I, Chap.2, 1.6.6, 1.7.10}]\) that the finite limits \( LS = S \times_{S \times k S} S, S \times_k S \) and \( S \times_k (S \times_{S \times k S} S) \) are also almost of finite type over \( k \).

Consider the square diagram in \( \text{Fun}(\text{CAlg}_{k}^{\leq 0, \underline{\Omega}}, S) \):

\[
\begin{array}{ccc}
(S \times_k S)_{S}^{\wedge} & \xrightarrow{id \times 1} & (S \times_k LS)_{S}^{\wedge} \\
\downarrow & & \downarrow \\
S & \xrightarrow{\iota} & LS_{S}^{\wedge}
\end{array}
\]

where the vertical morphisms are second projections. The functor \( QC_{1} \) gives rise to

\[
G : QC_{1}(LS) \simeq QC_{1}(LS_{S}^{\wedge}) \rightarrow QC_{1}(S) \times_{QC_{1}((S \times_{k S})_{S}^{\wedge})} QC_{1}((S \times_k LS)_{S}^{\wedge})
\]

**Proposition 7.12.** The functor \( G \) is an equivalence of \( \infty \)-categories. Moreover, this functor is promoted to an \( S^{1} \)-equivariant functor.

**Proof.** Consider the trivial action of \( S^{1} \) on \( (S \times_k S)_{S}^{\wedge} \) and \( S \). The actions of \( S^{1} \) on \( LS \simeq LS_{S}^{\wedge} \) and \( (S \times_k LS)_{S}^{\wedge} \) are induced by the canonical \( S^{1} \)-action. The above square is promoted to an \( S^{1} \)-equivariant diagram. Thus, \( G \) is also promoted to an \( S^{1} \)-equivariant diagram. It will suffice to show that the underlying functor is an equivalence. For this purpose, we first note that the square diagram can be regarded as a diagram in \( \mathcal{S}_{A}^{1} \). The morphisms from \( S \) is given by canonical maps \( id : S \rightarrow S, \Delta_{S} : S \rightarrow (S \times_k S)_{S}^{\wedge}, id \times \iota : S \rightarrow (S \times_k LS)_{S}^{\wedge} \) and \( \iota : S \rightarrow LS \simeq LS_{S}^{\wedge} \), respectively. The morphisms to \( S \) is given by \( id : S \rightarrow S \), the second projection \( (S \times_k S)_{S}^{\wedge} \rightarrow S \), \( (S \times_k LS)_{S}^{\wedge} \rightarrow LS \rightarrow S \) and \( LS \simeq LS_{S}^{\wedge} \rightarrow S \), respectively. Then the square diagram in \( \mathcal{S}_{A}^{1} \) induces the square diagram in \( \text{Lie}_{A}^{1} \):

\[
\begin{array}{ccc}
L((S \times_{k} S)_{S}^{\wedge}) & \xrightarrow{\iota_{S} \times \iota_{S}} & L((S \times_{k} LS)_{S}^{\wedge}) \\
\downarrow & & \downarrow \\
0 & \xrightarrow{0} & L_{LS_{S}^{\wedge}}
\end{array}
\]

For ease of notation, we put \( L_{1} = L((S \times_{k} S)_{S}^{\wedge}), L_{2} = L((S \times_k LS)_{S}^{\wedge}), L_{3} = L_{LS_{S}^{\wedge}} \). Thanks to \([8, \text{Vol. II, Chap.7, 5.1.2, 5.2}]\), \( G \) can be identified with the morphism induced by the restriction functors

\[
G_{\text{rep}} : \text{Rep}(L_{3})(QC_{1}(A)) \rightarrow QC_{1}(A) \times_{\text{Rep}(L_{1})(QC_{1}(A))} \text{Rep}(L_{2})(QC_{1}(A)).
\]

According to \([20, \text{Lemma 2.4.32, 2.4.33}]\) (one can apply the argument to our situation by replacing \( \text{Mod}_{k} \) with \( QC_{1}(A) \) in loc. cit.), if the morphism from the pushout \( 0 \cup_{L_{1}} L_{2} \rightarrow L_{3} \) in \( \text{Lie}_{A}^{1} \) is an equivalence (or \( A \rightrightarrows U_{L_{1} \cup L_{2}} U_{L_{2}} \simeq U_{L_{3}} \) in \( \text{Alg}_{A}(\text{Mod}_{A}) \)), then \( G_{\text{rep}} \) is an equivalence. Since \( ((S \times_k S)_{S}^{\wedge}) \times_{S} (LS_{S}^{\wedge}) \simeq (S \times_k LS)_{S}^{\wedge} \) in \( \mathcal{S}_{A}^{1} \), the sequence \( L_{1} \rightarrow L_{2} \rightarrow L_{3} = L_{1} \times \{0\} \rightarrow L_{1} \times L_{3} \simeq L_{3} \). Thus, \( 0 \cup_{L_{1}} L_{2} \simeq L_{2}/L_{1} \simeq L_{3} \). In other words, passing to universal enveloping algebras \( U_{L_{1}}(0) \cup_{U_{L_{1}}(L_{1})} U_{L_{1}}(L_{1} \times L_{3}) \simeq A \rightrightarrows U_{L_{1}(L_{1})} (U_{L_{1}}(L_{1}) \otimes_{A} U_{L_{1}(L_{3})}) \simeq A \rightrightarrows U_{L_{1}(L_{3})} ((A \oplus U_{L_{1}(L_{1})}) \otimes_{A} U_{L_{1}(L_{3})}) \simeq U_{L_{1}(L_{3})} \). Therefore, our assertion follows.

**Construction 7.13.** We first observe that \( \Theta_{A}(S \times_k LS) \simeq (S \times_k LS)_{S}^{\wedge} \). \((A \text{ is assumed to be a commutative ordinary algebra smooth over } k \) \). \( LS = \text{Spec } B \) is also locally of finite presentation over \( S \) since the compactness of the cotangent complex \( L_{A/k} \simeq \Omega_{A/k}^{1} \) implies that the free commutative algebra object \( \text{Sym}_{A}^{\ast}(L_{A/k}[1]) \simeq B \) is compact in \( \text{CAlg}_{A} \) where Hochshild-Kostant-Rosenberg theorem gives us an equivalence \( \text{Sym}_{A}^{\ast}(L_{A/k}[1]) \simeq B \). If \( p : LS \rightarrow S \) is the projection induced by a point \( * \rightarrow S^{1} \) on the circle, there is an exact sequence in \( \text{QC}(LS) \) i.e., a distinguished triangle in the homotopy triangulated category

\[
p^{\ast}L_{S/k} \rightarrow L_{LS/k} \rightarrow L_{LS/S} \rightarrow p^{\ast}L_{S/k}[1].
\]

The characterization of locally of finite presentation in term of the compactness (=perfect complex) of cotangent complexes \([19, 7.4.3.18] \) shows that \( p^{\ast}L_{S/k} \) and \( L_{LS/S} \) are compact in \( \text{QC}(LS) = \text{Mod}_{B} \). It follows immediately that \( L_{LS/k} \) is compact in \( \text{QC}(LS) \). Since both \( LS \) and \( S \) have the same classical
We consider the diagram

\[
\begin{array}{ccc}
\text{QC}(S) & \xrightarrow{pr_2} & \text{Rep}_H(S \times_k S) \leftarrow \text{Rep}_H(S \times_k LS) \\
\tau_A & \simeq & \tau_{S \times K S} \\
\text{QC}_1(S) & \xrightarrow{pr_2} & \text{QC}_1((S \times_k S)^\wedge_S) \leftarrow \text{QC}_1((S \times_k LS)^\wedge_S)
\end{array}
\]

which commutes up to canonical homotopy, where \( pr^* \) and \( pr_2^* \) are induced by the \(*\)-pullback and the \(!\)-pullback functor, respectively. The functor \( pr_2^* : \text{QC}(S) \to \text{Rep}_H(S \times_k S) \) is the composite \( \text{QC}(S) \to \text{QC}_H(S \times_k S) \to \text{Rep}_H(S \times_k S) \) where the first functor is induced by the \(*\)-pullback along the second projection \( S \times S \to S \). The functor \( pr_2^* : \text{QC}(S) \to \text{QC}_1((S \times_k S)^\wedge_S) \) can also be identified with \( \text{QC}_1(A) \to \text{QC}_1(S) = \text{QC}(\text{Spec} R) \) induced by \(!\)-pullbacks determined by the second projection \((S \times_k S)^\wedge_S \to S\). This diagram induces

\[
\text{QC}(S) \times_{\text{Rep}_H(S \times_k S)} \text{Rep}_H(S \times_k LS) \sim \text{QC}_1(S) \times \text{QC}_1((S \times_k S)^\wedge_S) \text{QC}_1((S \times_k LS)^\wedge_S) \simeq \text{QC}_1(LS)
\]

where the equivalence follows from Proposition 7.12. Since the above diagram consists of \( S^1 \)-equivariant functors (the actions on left and middle ones are trivial), we obtain the commutative diagram of \( \infty \)-categories

\[
\begin{array}{ccc}
\text{QC}(S)^{S^1} & \xrightarrow{pr_2} & \text{Rep}_H(S \times_k S)^{S^1} \leftarrow \text{Rep}_H(S \times_k LS)^{S^1} \\
\tau_A & \simeq & \tau_{S \times K S} \\
\text{QC}_1(S)^{S^1} & \xrightarrow{pr_2} & \text{QC}_1((S \times_k S)^\wedge_S)^{S^1} \leftarrow \text{QC}_1((S \times_k LS)^\wedge_S)^{S^1}
\end{array}
\]

where \((-)^{S^1}\) means the limit of an \( S^1 \)-action (i.e., \( S^1 \)-invariants) in \( \text{Cat}_{\infty} \) (or equivalently in the \( \infty \)-category of stable \( \infty \)-categories). It gives rise to an equivalence

\[
\text{QC}(S)^{S^1} \times_{\text{Rep}_H(S \times_k S)^{S^1}} \text{Rep}_H(S \times_k LS)^{S^1} \sim \text{QC}_1(S)^{S^1} \times_{\text{QC}_1((S \times_k S)^\wedge_S)^{S^1}} \text{QC}_1((S \times_k LS)^\wedge_S)^{S^1} \simeq \text{QC}_1(LS)^{S^1}.
\]

To obtain an object of \( \text{QC}_1(LS)^{S^1} \), it will suffice to construct an object of

\[
\text{QC}_1(S)^{S^1} \times_{\text{QC}_1((S \times_k S)^\wedge_S)^{S^1}} \text{QC}_1((S \times_k LS)^\wedge_S)^{S^1}.
\]

In Construction 6.10, we have constructed \( V_1 \in \text{QC}(S)^{S^1} \times_{\text{Rep}_H(S \times_k S)^{S^1}} \text{Rep}_H(S \times_k LS)^{S^1} \). We let \( H_\wedge(C) \) denote the image of \( V_1 \) in \( \text{QC}_1(LS)^{S^1} \).

We deduce the following result from the above Construction.

**Theorem 7.14.** Let \( C \) be an \( A \)-linear small stable \( \infty \)-category. We have constructed an object \( H_\wedge(C) \) of \( \text{QC}_1(LS)^{S^1} \) having the following properties:

(i) the image in \( \text{QC}_1(A)^{S^1} \) obtained by the \(!\)-pullback along \( \iota : S \to LS \) is naturally equivalent to \( H_\wedge*(C/A) \otimes_A \wedge_A \) (so that it maps to \( H_\wedge*(C/A) \in \text{Mod}_A = \text{QC}(S)^{S^1} \) through the equivalence \( \text{QC}(S)^{S^1} \simeq \text{QC}_1(S)^{S^1} \)),

(ii) the image in \( \text{QC}_1((S \times_k LS)^\wedge_S)^{S^1} \) obtained by the \(!\)-pullback along \( (S \times_k LS)^\wedge_S \to LS \) is naturally equivalent to the object corresponding to the canonical \( T_{A/k}[-1] \)-module \( H_\wedge*(C/A) \) (see Definition 6.3, Remark 6.12) through equivalences

\[
\text{QC}_1((S \times_k LS)^\wedge_S)^{S^1} \simeq \text{Rep}_H(S \times_k LS)^{S^1} \simeq \text{Rep}(T_{A/k}[-1])^{S^1} \text{(Mod}_A^{S^1} \). \]

(iii) the image in \( \text{QC}_1((S \times_k S)^\wedge_S)^{S^1} \) is naturally equivalent to the object corresponding to the canonical \( T_{A/k}[-1] \)-module \( H_\wedge*(C/A) \) (see Definition 6.3) through equivalences

\[
\text{QC}_1((S \times_k S)^\wedge)^{S^1} \simeq \text{Rep}_H(S \times_k S)^{S^1} \simeq \text{Rep}(T_{A/k}[-1])(\text{Mod}_A^{S^1} \). \]
8. D-modules

Let $C$ be an $A$-linear small stable $\infty$-category over a base field of characteristic zero $k$. In this section, we apply Theorem 4.12 and Theorem 7.14 to prove that the periodic cyclic homology/complex (that computes complex computing periodic cyclic homology) admits a $D$-module structure.

8.1. Let $S$ be a derived scheme locally almost of finite type over $k$. We first consider

$$
\begin{array}{ccc}
S & \xrightarrow{i} & LS \\
\downarrow{\pi} & & \downarrow{\rho} \\
S_{dR} & \xrightarrow{\sim} & (LS)_{dR}
\end{array}
$$

where the vertical functors are the canonical functors. As observed in Section 7.5, $\iota_{dR}$ is an equivalence.

We denote by $\pi$ the composite $LS \rightarrow (LS)_{dR} \xrightarrow{\iota_{dR}^{-1}} S_{dR}$. Let $\text{Coh}(LS) \subset \text{QC}_1(LS)$ be the full subcategory of coherent complexes. Suppose further that $LS$ is quasi-compact and separated over $k$. From [21, 4.4.3], $\text{QC}_1(LS) = \text{Ind}(\text{Coh}(LS))$ so that $\text{Coh}(LS)$ coincides with the full subcategory of compact objects. We write $\text{Coh}(LS)^{S^1} \otimes_{k[t]} k[t, t^{-1}]$ for the Tate construction $\text{Coh}(LS)^{S^1} \otimes_{\text{Perf}_{k[t]}} \text{Perf}_{k[t, t^{-1}]}$ where $\text{Perf}_{k[t]}$ is the full subcategory of $\text{Mod}_{k[t]}$ spanned by compact objects, and $k[t]$ is the free commutative dg algebra generated by one element $t$ of cohomological degree two (see e.g. [4], [21] for the Tate construction).

Let $\text{Ind}(\text{Coh}(LS)^{S^1}) \otimes_{k[t]} k[t, t^{-1}]$ denote $\text{Ind}(\text{Coh}(LS)^{S^1}) \otimes_{\text{Ind}(\text{Perf}_{k[t]})} \text{Ind}(\text{Perf}_{k[t, t^{-1}]}) = \text{Ind}(\text{Coh}(LS)^{S^1}) \otimes_{\text{Mod}_{k[t]}} \text{Mod}_{k[t, t^{-1}]}$. Let $\text{Coh}(S_{dR}) \otimes_k k[t]$ be $\text{Coh}(S_{dR}) \otimes_k \text{Perf}_{k[t]}$ which can be identified with the limit of the trivial $S^1$-action (homotopy fixed points) on $\text{Coh}(S_{dR})$ (the full subcategory $\text{Coh}(S_{dR})$ may and will be identified with the full subcategory of the compactly generated $\infty$-category $\text{QC}_1(S_{dR})$, which consists of compact objects, see [8] or [21, 4.4.4]). By [21, 4.4.4, 4.5.4], there exists the standard equivalence $\text{Coh}(S_{dR})^{S^1} \simeq \text{Coh}(S_{dR}) \otimes_k \text{Perf}_{k[t]}$ that comes from the trivial $S^1$-action. Similarly, we let $\text{Coh}(S_{dR}) \otimes_k k[t, t^{-1}]$ denote $\text{Coh}(S_{dR}) \otimes_k \text{Perf}_{k[t, t^{-1}]}$. Now we use the relation between loop spaces and $D$-modules (cf. [4] and [21, Theorem 1.3.5]). We use the result presented in [21]: the pushforward functor $(\pi_*)^{S^1} : \text{Coh}(LS)^{S^1} \rightarrow \text{Coh}(S_{dR})^{S^1} \simeq \text{Coh}(S_{dR}) \otimes_k \text{Perf}_{k[t]}$ with the base change to $k[t, t^{-1}]$ gives an equivalence

$$(\pi_*)^{S^1} \otimes_{k[t]} k[t, t^{-1}] : \text{Coh}(LS)^{S^1} \otimes_{k[t]} k[t, t^{-1}] \xrightarrow{\sim} \text{Coh}(S_{dR}) \otimes_k k[t, t^{-1}]$$

of $k[t, t^{-1}]$-linear small stable $\infty$-categories. Passing to $\text{Ind}$-categories, it gives rise to

$$L \text{D}_{S} : \text{Ind}(\text{Coh}(LS)^{S^1}) \otimes_{k[t]} k[t, t^{-1}] \xrightarrow{\sim} \text{Ind}(\text{Coh}(S_{dR})) \otimes_k k[t, t^{-1}] = \text{Ind}(\text{Coh}(S_{dR})) \otimes_k \text{Mod}_{k[t, t^{-1}]}.$$  

We define the $\infty$-category of crystals (right $D$-modules) on $S$ to be $\text{Ind}(\text{Coh}(S_{dR}))$. Set $\text{Crys}(S) := \text{Ind}(\text{Coh}(S_{dR}))$. We then regard an object of $\text{Crys}(S) \otimes_k k[t, t^{-1}]$ as a $\mathbb{Z}/2\mathbb{Z}$-periodic crystal/right $D$-module. Therefore, this equivalence brings us a relation between $\text{Ind}(\text{Coh}(LS)^{S^1})$ and $D$-modules up to Tate construction.

Construction 8.1. We apply the construction of $\text{Ind}$-categories to the exact functor $\text{Coh}(LS)^{S^1} \hookrightarrow \text{QC}_1(LS)^{S^1}$ to obtain a colimit-preserving functor $\Phi_{LS} : \text{Ind}(\text{Coh}(LS)^{S^1}) \rightarrow \text{QC}_1(LS)^{S^1}$. By adjoint functor theorem [18], there is a right adjoint functor

$$\Phi_{LS} : \text{QC}_1(LS)^{S^1} \rightarrow \text{Ind}(\text{Coh}(LS)^{S^1}).$$

Consider the diagram

$$
\begin{array}{ccc}
\text{Ind}(\text{Coh}(LS)^{S^1}) & \xrightarrow{\text{Ind}(\text{Coh}(S_{dR})) \otimes_k \text{Mod}_{k[t]}} & \text{Ind}(\text{Coh}(S_{dR})) \otimes_k \text{Mod}_{k[t, t^{-1}]}
\end{array}
$$

and the map $L \text{D}_{S} : \text{Ind}(\text{Coh}(LS)^{S^1}) \otimes_{k[t]} k[t, t^{-1}] \rightarrow \text{Ind}(\text{Coh}(S_{dR})) \otimes_k k[t, t^{-1}]$.
Let $\mathcal{H}_\omega(C)$ be the image of $\mathcal{H}(C)$ (see Construction 7.13) under $\Phi_{LS} : \text{QC}((LS)^{S_1}) \to \text{Ind}(\text{Coh}(LS)^{S_1})$.

Let $\mathcal{H}_+\mathcal{H}_\omega(C/k) \in \text{QC}(LS)^{S_1}$ under $\text{QC}(LS)^{S_1} \cong \text{QC}_1((LS)^{S_1}) \to \text{Ind}(\text{Coh}(LS)^{S_1})$.

Now we use the sequence of functors

$$\text{QC}_1((LS)^{S_1}) \to \text{Ind}(\text{Coh}(LS)^{S_1}) \to \text{Ind}(\text{Coh}(S_{\text{dR}})) \otimes_k \text{Mod}_{k[t]} \to \text{Ind}(\text{Coh}(S_{\text{dR}})) \otimes_k \text{Mod}_{k[t,t^{-1}]}.$$  

Let $\Omega_{\text{crys}}(C)$ and $\Omega_{\text{crys}}^i(C)$ be images of $H_\omega(C)$ and $H_\omega^i(C)$, respectively. Let $\Omega_{\text{crys}}(C)$ and $\Omega_{\text{crys}}^i(C)$ be images of $H_\omega(C)$ and $H_\omega^i(C)$, respectively.

**Remark 8.2.** In a subsequent paper, we will prove that $H_\omega^i(C) \simeq H_\omega(C)$. Consequently, in Theorem 8.4 below we have $\Omega_{\text{crys}}^i(C) \simeq \Omega_{\text{crys}}^i(C)$.

**8.2.** We briefly overview the definition of the periodic cyclic homology/complex. In what follows, we assume that $A = \text{Spec} \, A$ is smooth over $k$. We apply the construction of $\text{Ind}$-categories to the fully faithful functor $\text{Perf}_A^S \to \text{Mod}_A^S$ to obtain a colimit-preserving functor

$$\psi_S : \text{Ind}(\text{Perf}_A^S) \cong \text{Ind}(\text{Perf}_A \otimes_k \text{Perf}_k(t)) \to \text{Mod}_A^S \cong \text{Mod}_A \otimes_k \text{Mod}_k^S \cong \text{Mod}_A \otimes_k \text{Mod}_{k[t]}$$

where $k[t]$ is the free commutative dg algebra generated by one element $t$ of homological degree one. (The composite $\psi_S$ can be identified with a functor induced by $\text{Mod}_{k[t]} \cong \text{Ind}(\text{Perf}_k(t)) \cong \text{Ind}(\text{Perf}_k^S) \to \text{Mod}_k^S \cong \text{Mod}_{k[t]}$.) Here, if we write $A[t]$ for $A \otimes_k k[t]$ we use an equivalence $\text{Perf}_k^S \cong \text{Coh}(A)^{S_1} \cong \text{Coh}(A) \otimes_k \text{Perf}_k(t) \cong \text{Perf}_{A[t]}$ which carries $M$ to the homotopy fixed points $M^{S_1}$ with the canonical module structure over $A^{S_1} = A[t]$, where we use the smoothness of $A$ which implies $\text{Perf}_A \cong \text{Coh}(A)$. Then $\text{Perf}_A^S \cong \text{Perf}_{A[t]}$ gives rise to

$$\text{Ind}(\text{Perf}_A^S) \cong \text{Ind}(\text{Perf}_A(t)) \cong \text{Mod}_{A[t]} \cong \text{Mod}_A \otimes_k \text{Mod}_{k[t]}.$$

We note that the inverse equivalence $\text{Perf}_{A[t]} \to \text{Perf}_A^S$ is given by $A \otimes_k A[t] \to N \to A \otimes_k A[t] N$ with the module structure over the endomorphism algebra $\text{End}_{\text{Mod}_{A[t]}}(A) \cong A[t]$.

Thus, we can regard

$$\text{Ind}(\text{Perf}_A^S) \to \text{Mod}_A^S \cong \text{Mod}_{A[t]} \to \text{Mod}_A[t]$$

given by $N \to A \otimes_k A[t] N$. The right adjoint functor of $\text{Mod}_{A[t]} \to \text{Mod}_A[t]$ is given by $(-)^{S_1} : \text{Mod}_A \to \text{Mod}_{A[t]}$; $N \to N^{S_1}$ (where $N^{S_1}$ can be represented by the (derived) Hom complex $\text{Hom}_{\text{Mod}_{A[t]}}(A, N)$). Namely, there is an adjoint pair

$A \otimes_k A[t] \to N \to A \otimes_k A[t] N$.

Consequently, the right adjoint $\phi_S : \text{Mod}_A \otimes_k \text{Mod}_{k[t]} \to \text{Mod}_A \otimes_k \text{Mod}_{k[t]}$ of $\psi_S$ is equivalent to the functor $(-)^{S_1}$. This functor sends $N \in \text{Mod}_A$ to the $S^1$-invariants $N^{S_1} \in \text{Mod}_{A[t]}$. Thus, it sends $\mathcal{H}_+\mathcal{H}_\omega(C/A)$ to $\mathcal{H}_+\mathcal{H}_\omega(C/A)^{S_1}$. We define the negative cyclic homology/complex $\mathcal{H}_-\mathcal{H}_\omega(C/A)$ to be $\mathcal{H}_-\mathcal{H}_\omega(C/A)^{S_1} \cong \text{Mod}_{A \otimes_k \text{Mod}_{k[t]}}$. The periodic cyclic homology/complex $\mathcal{H}_\mathcal{P}_\omega(C/A)$ is defined as the image of $\mathcal{H}_-\mathcal{H}_\omega(C/A)$ under the canonical functor $\text{Mod}_A \otimes_k \text{Mod}_{k[t]} \to \text{Mod}_A \otimes_k \text{Mod}_{k[t,t^{-1}]}$. As the Hochschild homology, we refer to the complex $\mathcal{H}_\mathcal{P}_\omega(C/A)$ as the periodic cyclic homology.

**8.3.**

**Lemma 8.3.** Let $H \in \text{QC}_1((LS)^{S_1})$ and suppose that $(i^!)^S(H) \in \text{QC}_1((S)^{S_1})$ is equivalent to $\mathcal{H}_A(\mathcal{H}_-\mathcal{H}_\omega(C/A))$ with the (standard) $S^1$-action. (By Theorem 4.12 and Theorem 7.14, $\mathcal{H}_\omega(C)$ and $\mathcal{H}_\omega(C)$ satisfy this condition.) Let $X$ denote the image of $H$ in $\text{QC}_1((\text{S}_{dR}) \otimes_k \text{Mod}_{k[t,t^{-1}]}$. The forgetful functor $u : \text{QC}_1((\text{S}_{dR}) \otimes_k \text{Mod}_{k[t,t^{-1}]} \to \text{QC}_1((S) \otimes_k \text{Mod}_{k[t,t^{-1}]}$ sends $X$ to $\mathcal{H}_A(\mathcal{H}_-\mathcal{H}_\omega(C/A)^{S_1}) \otimes_k [k[t,t^{-1}]]$ in $\text{QC}_1((S) \otimes_k \text{Mod}_{k[t,t^{-1}]}$ (i.e., the image of the periodic cyclic homology).

**Proof.** By the construction of the forgetful functor $u : \text{Ind}(\text{Perf}_{S_{\text{dR}}}) \otimes_k \text{Mod}_{k[t,t^{-1}]} \to \text{QC}_1((S) \otimes_k \text{Mod}_{k[t,t^{-1}]}$ of objects of Coh$(S)$ to compact objects of QC$(S_{\text{dR}})$. Therefore, we can regard $i_\ast$ as a functor obtained from the restriction Coh$(S) \to Coh(S_{\text{dR}})$ by the Ind-construction. Note that $\left(i_\ast \right)^S\text{Coh}(S)^{S_1} : \text{Coh}(S)^{S_1} \to Coh(S_{\text{dR}})^{S_1}$ can naturally be identified with $i_\ast \otimes \text{Perf}_{k[t]} : \text{Coh}(S) \otimes_k \text{Perf}_{k[t]} \to Coh(S_{\text{dR}}) \otimes_k \text{Perf}_{k[t]}$. It follows that
filtration on the crystal/D-module $\Omega^1$. The objects $\Omega^•_{\text{crystal}}$ are canonically defined objects (in particular, it is fully faithful), we are reduced to showing that $\Phi_1 : \Omega^1 \to \Omega^1$ is a right adjoint of $\Phi_2 : \Omega^1 \to \Omega^1$, which is induced by the $t$-pullback right adjoint functor $i^*_t : \Omega^1 \to \Omega^1$ of $i^*_{\text{crystal}} : \Omega^1 \to \Omega^1$. By construction, we have a commutative diagram

$$
\begin{array}{ccc}
\Phi_2 & \text{Ind}(\text{Coh}(S))^S & \Phi_1 \\
| & \text{Ind}(\text{Coh}(S))^S & |\\
\downarrow & r_* & \downarrow r_* \\
\Omega^1 & \text{Ind}(\text{Coh}(S))^S & k[t,t^{-1}]
\end{array}
$$

Let $\text{Ind}(\text{Coh}(S))^S \to \text{Ind}(\text{Coh}(LS))^S$ be a colimit-preserving functor which is the Ind-extension of the pushforward $(i^*_{\text{crystal}})^S : \text{Coh}(S)^S \to \text{Coh}(LS)^S$. Let $r_* : \text{Ind}(\text{Coh}(LS))^S \to \text{Ind}(\text{Coh}(S))^S$ be its right adjoint. Let $(i^*)_S : \text{Coh}(LS)^S \to \text{Coh}(S)^S$ be a right adjoint functor of $(i^*_{\text{crystal}})^S$, which is induced by the $t$-pullback right adjoint functor $i^*_t : \text{Coh}(LS) \to \text{Coh}(S)$ of $i^*_{\text{crystal}} : \text{Coh}(S) \to \text{Coh}(LS)$. By construction, we have a commutative diagram

$$
\begin{array}{ccc}
\Phi_2 & \text{Ind}(\text{Coh}(S))^S & \Phi_1 \\
| & \text{Ind}(\text{Coh}(S))^S & |\\
\downarrow & r_* & \downarrow r_* \\
\Omega^1 & \text{Ind}(\text{Coh}(S))^S & k[t,t^{-1}]
\end{array}
$$

Let $\text{Ind}(\text{Coh}(S))^S \to \text{Ind}(\text{Coh}(LS))^S$ be a right adjoint of $\text{Ind}(\text{Coh}(LS))^S$, and $k[t,t^{-1}]$ is a right adjoint of $\text{Ind}(\text{Coh}(LS))^S$. By construction, we have a commutative diagram

$$
\begin{array}{ccc}
\Phi_2 & \text{Ind}(\text{Coh}(S))^S & \Phi_1 \\
| & \text{Ind}(\text{Coh}(S))^S & |\\
\downarrow & r_* & \downarrow r_* \\
\Omega^1 & \text{Ind}(\text{Coh}(S))^S & k[t,t^{-1}]
\end{array}
$$

By Proposition 8.1 and the above Lemma, we have:

**Theorem 8.4.** Suppose that $A$ is smooth over $k$. We have constructed two right $\mathbb{Z}/2\mathbb{Z}$-periodic crystals (D-modules) as objects $\Omega^\bullet(C), \Omega^\bullet(C) \in \text{Ind}(\text{Coh}(S^\text{dR})) \otimes_k \text{Mod}_{k[t,t^{-1}]}$ which lies in $\text{H}^\bullet(A)$ and $\text{H}^\bullet(C)$. More precisely, there exist two canonically defined objects $\Omega^\bullet(C), \Omega^\bullet(C)$ of $\text{Crys}(S) \otimes_k \text{Mod}_{k[t,t^{-1}]}$ whose image in $\text{Crys}(S) \otimes_k \text{Mod}_{k[t,t^{-1}]}$ under $u_\text{SS}$ is equivalent to $\text{Ind}(\text{Coh}(S)^{\text{crystal}})$ and $\text{Ind}(\text{Coh}(S)^{\text{crystal}})$. Moreover, $\Omega^\bullet(C)$ and $\Omega^\bullet(C)$ are obtained from $\Omega^{\text{crystal}}(C)$ and $\Omega^{\text{crystal}}(C)$ respectively.

**Remark 8.5.** The objects $\Omega^{\text{crystal}}(C)$ and $\Omega^{\text{crystal}}(C)$ can be thought of as a kind of a (semi-infinite) Hodge filtration on the crystal/D-module $\Omega^\bullet(C)$ and $\Omega^\bullet(C)$, respectively.

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