ON SPECTRAL INCLUSION AND MAPPING THEOREMS
FOR SCALAR TYPE SPECTRAL OPERATORS
AND SEMIGROUPS

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Abstract. We establish spectral inclusion and mapping theorems for scalar type spectral operators, generalizing their counterparts for normal operators. Thereby, we extend a precise weak spectral mapping theorem, known to hold for $C_0$-semigroups of normal operators on complex Hilbert spaces, to the more general case of $C_0$-semigroups of scalar type spectral operators on complex Banach spaces. The finer spectrum structure is given itemized consideration.

I believe that mathematical reality lies outside us, that our function is to discover or observe it, and that the theorems which we prove, and which we describe grandiloquently as our “creations,” are simply the notes of our observations.

G.H. Hardy

1. Introduction

We establish spectral inclusion and mapping theorems for scalar type spectral operators, generalizing their counterparts for normal operators ([11, Theorem D.9] and [11, Theorem D.11]).

Thereby, we extend a precise weak spectral mapping theorem, known to hold for $C_0$-semigroups of normal operators on complex Hilbert spaces, to the more general case of $C_0$-semigroups of scalar type spectral operators on complex Banach spaces, the a priori requirement of eventual norm continuity remaining superfluous (cf. [15]).

The finer spectrum structure is given itemized consideration.

2. Preliminaries

Here, we concisely outline essential preliminaries.

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2.1. Resolvent Set and Spectrum.

For a closed linear operator $A$ in a complex Banach space $X$, the set

$$\rho(A) := \{ \lambda \in \mathbb{C} \mid \exists R(\lambda, A) := (A - \lambda I)^{-1} \in L(X) \}$$

($I$ is the identity operator on $X$, $L(X)$ is the space of bounded linear operators on $X$) and its complement $\sigma(A) := \mathbb{C} \setminus \rho(A)$ are called its resolvent set and spectrum, respectively.

The spectrum is partitioned into pairwise disjoint subsets, called the point, continuous, and residual spectrum of $A$, respectively, as follows:

$$\sigma_p(A) := \{ \lambda \in \mathbb{C} \mid A - \lambda I \text{ is not injective, i.e., } \lambda \text{ is an eigenvalue of } A \},$$

$$\sigma_c(A) := \{ \lambda \in \mathbb{C} \mid A - \lambda I \text{ is injective and } R(A - \lambda I) \neq X \},$$

$$\sigma_r(A) := \{ \lambda \in \mathbb{C} \mid A - \lambda I \text{ is injective and } R(A - \lambda I) = X \}$$

($R(\cdot)$ is the range of an operator and $\bar{\cdot}$ is the closure of a set) (see, e.g., [7,16]).

2.2. $C_0$-Semigroups.

A $C_0$-semigroup $\{T(t)\}_{t \geq 0}$ on a complex Banach space $(X, \| \cdot \|)$ with generator $A$ is said to be subject to a weak spectral mapping theorem if

$$(WSMT) \quad \sigma(T(t)) \setminus \{0\} = e^{t \sigma(A)} \setminus \{0\}, \quad t \geq 0.$$

An eventually norm-continuous $C_0$-semigroup $\{T(t)\}_{t \geq 0}$ on a complex Banach space, i.e., such that, for some $t_0 > 0$, the operator function

$$[t_0, \infty) \ni t \mapsto T(t) \in L(X),$$

is continuous relative to the operator norm $\| \cdot \|$ (here and henceforth, we use the same notation as for the norm on $X$), is subject to the following stronger version of (WSMT):

$$(SMT) \quad \sigma(T(t)) \setminus \{0\} = e^{t \sigma(A)}, \quad t \geq 0,$$

called a spectral mapping theorem (see [10, Proposition V.2.3] and [10, Theorem V.2.8]). The class of eventually norm-continuous $C_0$-semigroups encompasses $C_0$-semigroups with certain regularity properties, such as eventually compact and eventually differentiable, in particular analytic and uniformly continuous (see [10, Section II.5]).

A $C_0$-semigroup $\{T(t)\}_{t \geq 0}$ (of normal operators) on a complex Hilbert space generated by a normal operator $A$ is subject to the following precise version of weak spectral mapping theorem (WSMT):

$$(PWSMT) \quad \sigma(T(t)) = e^{t \sigma(A)}, \quad t \geq 0,$$

[10, Corollary V.2.12] without being a priori eventually norm-continuous.
2.3. Scalar Type Spectral Operators.

A scalar type spectral operator is a densely defined closed linear operator $A$ in a complex Banach space with strongly $\sigma$-additive spectral measure (the resolution of the identity) $E_A(\cdot)$, which assigns to the Borel sets of the complex plane projection operators on $X$ and has the operator’s spectrum $\sigma(A)$ as its support [5, 6, 9].

Associated with such an operator is the Borel operational calculus, assigning to each Borel measurable function $F : \sigma(A) \to \overline{\mathbb{C}}$ ($\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is the extended complex plane) with $E_A(\{\lambda \in \mathbb{C} \mid F(\lambda) = \infty\}) = 0$ a scalar type spectral operator

$$F(A) := \int_{\sigma(A)} F(\lambda) \, dE_A(\lambda)$$

in $X$, whose spectral measure is the image of $E_A(\cdot)$ under $F(\cdot)$, i.e.,

$$E_{F(A)}(\delta) = E_A(F^{-1}(\delta)), \quad \delta \in \mathcal{B}(\mathbb{C}),$$

$(\mathcal{B}(\mathbb{C})$ is the Borel $\sigma$-algebra on $\mathbb{C}$), with

$$A = \int_{\sigma(A)} \lambda \, dE_A(\lambda)$$

[2, 5, 6, 9].

On a complex finite-dimensional Banach space, scalar type spectral operators are those linear operators, which furnish an eigenbasis for the space, i.e., allow a diagonal matrix representation (see, e.g., [5, 6, 9]).

In a complex Hilbert space, scalar type spectral operators are those that are similar to normal operators [26] (see also [13, 14]), the latter being the scalar type spectral operators for which the corresponding spectral measure projections are orthogonal (see, e.g., [8, 23]).

Various examples of scalar type spectral operators, including differential operators arising in the study of linear systems of partial differential equations, in particular perturbed Laplacians, can be found in [9].

Due to its strong $\sigma$-additivity, the spectral measure is uniformly bounded, i.e.,

$$\exists M \geq 1 \, \forall \delta \in \mathcal{B}(\mathbb{C}) : \|E_A(\delta)\| \leq M$$

(see, e.g., [7]).

By [9, Theorem XVIII.2.11 (c)], for a Borel measurable function $F : \sigma(A) \to \overline{\mathbb{C}}$, the operator $F(A)$ is bounded iff $F(\cdot)$ is $E_A$-essentially bounded, i.e.,

$$E_A\text{-ess sup}_{\lambda \in \sigma(A)} |F(\lambda)| < \infty,$$

in which case

$$E_A\text{-ess sup}_{\lambda \in \sigma(A)} |F(\lambda)| \leq \|F(A)\| \leq 4M E_A\text{-ess sup}_{\lambda \in \sigma(A)} |F(\lambda)|,$$

where $M \geq 1$ is from (2.2).

For a scalar type spectral operator $A$, $\sigma(A) \neq \emptyset$,

$$\sigma_p(A) = \{\lambda \in \mathbb{C} \mid E_A(\{\lambda\}) \neq 0\},$$
with $E_A(\{\lambda\})X$ being the eigenspace associated with an eigenvalue $\lambda \in \sigma_p(A)$, i.e.,

$$E_A(\{\lambda\})X = \ker(A - \lambda I), \ \lambda \in \sigma_p(A),$$

moreover,

$$\sigma_c(A) = \{\lambda \in \sigma(A) \mid E_A(\{\lambda\}) = 0\}$$

and

$$\sigma_r(A) = \emptyset$$

[18, Corollary 3.1] (see also [19]).

A scalar type spectral $C_0$-semigroup $\{T(t)\}_{t \geq 0}$ (i.e., a $C_0$-semigroup of scalar type spectral operators) on a complex Banach space $X$ is generated by a scalar type spectral operator [4, 22], which is the case iff

$$s(A) < \infty$$

with

$$T(t) = e^{tA} := \int_{\sigma(A)} e^{t\lambda} dE_A(\lambda), \ t \geq 0,$$

[20, Proposition 3.1], the orbit maps of the semigroup

$$T(t)f = e^{tA}f, \ t \geq 0, f \in X,$$

being the weak solutions (also called the mild solutions) of the associated abstract evolution equation

$$y'(t) = Ay(t), \ t \geq 0,$$

[21] (see also [3, 10]).

3. Spectral Inclusion and Mapping Theorems

**Theorem 3.1** (Weak Spectral Inclusion and Mapping Theorem A.E.). Let $A$ be a scalar type spectral operator $A$ in a complex Banach space $(X, \| \cdot \|)$ with spectral measure $E_A(\cdot)$.

(1) If $F : \sigma(A) \to \mathbb{C}$ is a Borel measurable function, then

$$\sigma(F(A)) \subseteq F(\sigma(A) \setminus \sigma),$$

where $\sigma$ is an arbitrary Borel subset of $\sigma(A)$ for which $E_A(\sigma) = 0$.

(2) If furthermore the function $F : \sigma(A) \to \mathbb{C}$ is continuous on $\sigma(A) \setminus \sigma$, where $\sigma$ is a Borel subset of $\sigma(A)$ for which $E_A(\sigma) = 0$, then

$$\sigma(F(A)) = F(\sigma(A) \setminus \sigma).$$

**Proof.** Let $F : \sigma(A) \to \mathbb{C}$ be a Borel measurable function, $\sigma$ be an arbitrary $E_A$-null subset of $\sigma(A)$, i.e., $E_A(\sigma) = 0$, and

$$\lambda \in \mathbb{C} \setminus \overline{F(\sigma(A) \setminus \sigma)}$$

be arbitrary. Then

$$\text{dist}(\lambda, F(\sigma(A) \setminus \sigma)) := \inf_{\mu \in \sigma(A) \setminus \sigma} |F(\mu) - \lambda| > 0$$

(see, e.g., [16, 17]).
In view of $E_A(\sigma) = 0$, by the properties of the Borel operational calculus (see [9, Theorem XVIII.2.11 (e)]),

$$(F(A) - \lambda I)^{-1} = \int_{\sigma(A)} \frac{1}{F(\mu) - \lambda} dE_A(\mu) = \int_{\sigma(A)} \frac{1}{F(\mu) - \lambda} \chi_{\sigma(A) \setminus \sigma}(\mu) dE_A(\mu)
\quad =: \int_{\sigma(A) \setminus \sigma} \frac{1}{F(\mu) - \lambda} dE_A(\mu)$$

($\chi_\delta(\cdot)$ is the characteristic function of a set $\delta \subseteq \mathbb{C}$) is a bounded linear operator on $X$, for which, by (2.3), when applied to the function

$$\sigma(A) \ni \mu \mapsto \frac{1}{F(\mu) - \lambda} \chi_{\sigma(A) \setminus \sigma}(\mu) \in \mathbb{C},$$

we have:

$$\left\| (F(A) - \lambda I)^{-1} \right\| = \left\| \int_{\sigma(A)} \frac{1}{F(\mu) - \lambda} \chi_{\sigma(A) \setminus \sigma}(\mu) dE_A(\mu) \right\|
\leq 4M E_A \text{-ess sup}_{\mu \in \sigma(A)} \frac{1}{|F(\mu) - \lambda|} \chi_{\sigma(A) \setminus \sigma}(\mu)
= 4M E_A \text{-ess sup}_{\mu \in \sigma(A) \setminus \sigma} \frac{1}{|F(\mu) - \lambda|}
\leq 4M \sup_{\mu \in \sigma(A) \setminus \sigma} \frac{1}{|F(\mu) - \lambda|} = 4M \inf_{\mu \in \sigma(A) \setminus \sigma} \frac{1}{|F(\mu) - \lambda|}
= \frac{4M}{\text{dist}(\lambda, F(\sigma(A) \setminus \sigma))} < \infty,$$

where $M \geq 1$ is from (2.2).

Therefore,

$$\lambda \in \rho(F(A)).$$

Thus, we have the inclusion

$$\mathbb{C} \setminus \overline{F(\sigma(A) \setminus \sigma)} \subseteq \rho(F(A)),$$

or equivalently,

$$\sigma(F(A)) \subseteq \overline{F(\sigma(A) \setminus \sigma)},$$

which completes the proof of part (1).

Now, suppose that the function $F : \sigma(A) \to \mathbb{C}$ is also continuous on $\sigma(A) \setminus \sigma$, where $\sigma$ is a Borel subset of $\sigma(A)$ for which $E_A(\sigma) = 0$, and let

$$\lambda \in \rho(F(A))$$

be arbitrary.

By part (1), we have the inclusion

(3.3) $$\sigma(F(A)) \subseteq \overline{F(\sigma(A) \setminus \sigma)}.$$
In view of $E_A(\sigma) = 0$, by the properties of the Borel operational calculus (see [9, Theorem XVIII.2.11 (e)]),

$$
\int_{\sigma(A) \setminus \sigma} \frac{1}{F(\mu) - \lambda} dE_A(\mu) := \int_{\sigma(A)} \frac{1}{F(\mu) - \lambda} \chi_{\sigma(A) \setminus \sigma}(\mu) dE_A(\mu)
$$

$$
= \int_{\sigma(A)} \frac{1}{F(\mu) - \lambda} dE_A(\mu)
$$

$$
= (F(A) - \lambda I)^{-1} =: R(\lambda, F(A)) \in L(X).
$$

Hence, by (2.3), when applied to the function

$$
\sigma(A) \ni \mu \mapsto \frac{1}{F(\mu) - \lambda} \chi_{\sigma(A) \setminus \sigma}(\mu) \in \mathbb{C},
$$

we have:

$$
E_A \text{-ess sup}_{\mu \in \sigma(A) \setminus \sigma} \frac{1}{|F(\mu) - \lambda|} = E_A \text{-ess sup}_{\mu \in \sigma(A)} \frac{1}{|F(\mu) - \lambda|} \chi_{\sigma(A) \setminus \sigma}(\mu) \leq \|R(\lambda, A)\| < \infty,
$$

which, since

$$
E_A \text{-ess inf}_{\mu \in \sigma(A) \setminus \sigma} |F(\mu) - \lambda| > 0,
$$

Therefore, there exists a Borel set $\delta \subseteq \sigma(A) \setminus \sigma$ such that

$$
E_A(\delta) = 0
$$

(3.4)

and

$$
\text{dist} (\lambda, F((\sigma(A) \setminus \sigma) \setminus \delta)) := \inf_{\mu \in (\sigma(A) \setminus \sigma) \setminus \delta} |F(\mu) - \lambda| > 0,
$$

which implies that, for the open disk

$$
\Delta_r := \{ \mu \in \mathbb{C} \mid |\mu - \lambda| < r \}
$$

centered at $\lambda$ with radius

$$
r := \text{dist} (\lambda, F((\sigma(A) \setminus \sigma) \setminus \delta)) > 0,
$$

we have:

$$
(3.5) \quad \Delta_r \cap F((\sigma(A) \setminus \sigma) \setminus \delta) = \emptyset.
$$

Assume that

$$
(3.6) \quad \Delta_r \cap F(\delta) \neq \emptyset
$$

and let $\hat{F}(\cdot)$ be the restriction of $F(\cdot)$ to $\sigma(A) \setminus \sigma$.

Jointly, (3.5) and (3.6) imply that

$$
(3.7) \quad \emptyset \neq \hat{F}^{-1}(\Delta_r) \subseteq \delta.
$$
By the continuity of $\hat{F}(\cdot)$ on $\sigma(A) \setminus \sigma$, the set $\hat{F}^{-1}(\Delta_r)$ is open in $\sigma(A) \setminus \sigma$ (see, e.g., [17]), i.e.,

$$\hat{F}^{-1}(\Delta_r) = (\sigma(A) \setminus \sigma) \cap \theta = (\sigma(A) \cap \theta) \setminus \sigma,$$

where $\theta$ is a nonempty open set in $C$.

Whence, by the properties of spectral measure (see, e.g., [6,9]) and in view of $E_A(\sigma) = 0$, we infer that

$$E_A(\hat{F}^{-1}(\Delta_r)) = E_A(\sigma(A) \cap \theta) - E_A(\sigma(A) \cap \theta \cap \sigma)$$

$$= E_A(\sigma(A) \cap \theta) - E_A(\sigma(A) \cap \theta) E_A(\sigma)$$

$$= E_A(\sigma(A) \cap \theta) - E_A(\sigma(A) \cap \theta) 0 = E_A(\sigma(A) \cap \theta).$$

The latter, since $\sigma(A)$ is the support for the spectral measure $E_A(\cdot)$ (see Preliminaries) and $\theta$ is a nonempty open set in $C$, implies that

$$(3.8) \quad E_A(\hat{F}^{-1}(\Delta_r)) = E_A(\sigma(A) \cap \theta) \neq 0.$$

However, by the properties of spectral measure (see, e.g., [6,9]) and in view of (3.7) and (3.6),

$$E_A(\hat{F}^{-1}(\Delta_r)) = E_A(\hat{F}^{-1}(\Delta_r) \cap \delta)$$

$$= E_A(\hat{F}^{-1}(\Delta_r)) E_A(\delta) = E_A(\hat{F}^{-1}(\Delta_r)) 0 = 0.$$

The obtained contradiction with (3.8) proves that assumption (3.6) is false, and hence,

$$\Delta_r \cap F(\delta) = \emptyset.$$

Jointly, (3.5) and (3.9) imply

$$\Delta_r \cap F(\sigma(A) \setminus \sigma) = \emptyset,$$

which leads to the conclusion that

$$\lambda \in \mathbb{C} \setminus \overline{F(\sigma(A) \setminus \sigma)}.$$

Hence, we have the inclusion

$$\rho(F(A)) \subseteq \mathbb{C} \setminus \overline{F(\sigma(A) \setminus \sigma)},$$

or equivalently,

$$(3.10) \quad \overline{F(\sigma(A) \setminus \sigma)} \subseteq \sigma(F(A)).$$

Inclusions (3.3) and (3.10) jointly imply

$$\sigma(F(A)) = \overline{F(\sigma(A) \setminus \sigma)},$$

which completes the proof of part (2), and thus, of the entire statement. □

Remarks 3.1.

- Theorem 3.1 generalizes [11, Theorem D.11], which covers the case a normal operator $A$ in a complex Hilbert space and a Borel measurable function $F : \sigma(A) \to \mathbb{C}$ with a countable $E_A$-null set of discontinuities.
In [2, Theorem 5.2], equality (3.2) is stated for a *spectral operator* $A$ (not necessarily of scalar type) in a complex Banach space (see, e.g., [5,6,9]) and a function $F : \sigma(A) \to \mathbb{C}$ analytic on $\sigma(A)$ with the exception of a finite subset $\sigma$ of poles for which $E_A(\sigma) = 0$ and either analytic or having a pole at infinity (see also [9, Theorem XVIII.2.21]).

As the following examples demonstrate, for inclusion (3.1) and equality (3.2) to hold, the condition $E_A(\sigma) = 0$ is essential, and furthermore, without the requirement for the function $F : \sigma(A) \to \mathbb{C}$ to be continuous on $\sigma(A) \setminus \sigma$, inclusion (3.1) or even the inclusion

$$\sigma(F(A)) \subseteq F(\sigma(A) \setminus \sigma),$$

may be strict.

**Examples 3.1.**

1. For the *self-adjoint*, and hence, *scalar type spectral* (see Preliminaries), operator

$$l_2 \ni x := (x_n)_{n \in \mathbb{N}} \mapsto Ax := \left(0x_1, x_2, \frac{1}{2}x_3, \ldots\right) \in l_2$$

$(\mathbb{N} := \{1, 2, \ldots\})$ is the set of *natural numbers* of multiplication by the real-termed sequence

$$a_n := \begin{cases} 0, & n = 1, \\ 1/(n-1), & n \geq 2, \end{cases}$$

on the complex Hilbert space $l_2$ of square-summable sequences with

$$\sigma(A) = \sigma_p(A) = \{0\} \cup \{1/n\}_{n \in \mathbb{N}}$$

(see, e.g., [16]), the Borel measurable function

$$F(\lambda) := \begin{cases} 1/\lambda, & \lambda \neq 0, \\ 0, & \lambda = 0, \end{cases}$$

discontinuous at 0, and the Borel subset $\sigma := \{0\} \subseteq \sigma(A)$ with $E_A(\sigma) \neq 0$ (see (2.4)), $F(A)$ is the operator of multiplication by the sequence

$$F(a_n) := \begin{cases} 0, & n = 1, \\ n-1, & n \geq 2, \end{cases}$$

i.e.,

$$l_2 \supseteq D(F(A)) \ni x := (x_n)_{n \in \mathbb{N}} \mapsto F(A)x := (0x_1, x_2, 2x_3, \ldots) \in l_2,$$

where

$$D(F(A)) = \{(x_n)_{n \in \mathbb{N}} \in l_2 \mid (0x_1, x_2, 2x_3, \ldots) \in l_2\}$$

($D(\cdot)$ is the *domain* of an operator), with

$$\sigma(F(A)) = \sigma_p(F(A)) = \mathbb{Z}_+$$

($\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$ is the set of *nonnegative integers*).

However,

$$\sigma(F(A)) \nsubseteq F(\sigma(A) \setminus \sigma) = F(\sigma(A) \setminus \{0\}) = \mathbb{N}.$$
2. For the \textit{self-adjoint} operator

$$L_2(\mathbb{R}) \supseteq D(A) \ni f \mapsto [Af](x) := xf(x) \in L_2(\mathbb{R})$$

of multiplication by the independent variable in the complex Hilbert space

$$L_2(\mathbb{R}),$$

where

$$D(A) := \left\{ f \in L_2(\mathbb{R}) \left| \int_0^\infty |xf(x)|^2 \, dx < \infty \right. \right\}$$

($f(\cdot)$ is a representative of an equivalence class $f \in L_2(\mathbb{R})$), with

$$\sigma(A) = \sigma_c(A) = \mathbb{R}$$

(see, e.g., [1]), the Borel measurable function $\chi_{\{0\}}(\cdot)$ discontinuous at 0, and the Borel subset $\sigma := \emptyset \subseteq \sigma(A)$,

$$\chi_{\{0\}}(A) = E_A(\{0\}) = 0$$

(see (2.6)) (see, e.g., [8, 23]) with

$$\sigma (\chi_{\{0\}}(A)) = \sigma_p (\chi_{\{0\}}(A)) = \{0\}.$$

However,

$$\sigma (\chi_{\{0\}}(A)) \neq \chi_{\{0\}} (\sigma(A)) = \chi_{\{0\}} (\sigma(A)) = \{0, 1\}.$$

\textbf{Remark 3.2.} As is seen from the prior example, for any $\lambda \in \mathbb{R} \setminus \{0\}$, $\lambda \in \sigma_e(A)$, but $\chi_{\{0\}}(\lambda) = 0 \in \sigma_p (\chi_{\{0\}}(A))$ and $0 \in \sigma_e(A)$, but $\chi_{\{0\}}(0) = 1 \in \rho (\chi_{\{0\}}(A)).$

The subsequent statement, being the particular case of Theorem 3.1 for $\sigma := \emptyset$, generalizes [11, Theorem D.9], its counterpart for normal operators (cf. also [25, Lemma 4, Theorem 2]).

\textbf{Theorem 3.2 (Weak Spectral Inclusion and Mapping Theorem).} Let $A$ be a scalar type spectral operator $A$ in a complex Banach space.

(1) If $F : \sigma(A) \to \mathbb{C}$ is a Borel measurable function, then

\begin{equation}
\sigma(F(A)) \subseteq F(\sigma(A)).
\end{equation}

(2) If $F : \sigma(A) \to \mathbb{C}$ is a continuous function,

\begin{equation}
\sigma(F(A)) = F(\sigma(A)).
\end{equation}

\textbf{Remark 3.3.} As is seen from Examples 3.1, without the requirement of continuity for the Borel measurable function $F : \sigma(A) \to \mathbb{C}$, inclusion (3.11) may be strict.

\section*{4. Finer Spectrum Structure}

The finer spectrum structure is governed by the following statement.

\textbf{Theorem 4.1 (Finer Spectrum Structure).} Let $A$ be a scalar type spectral operator in a complex Banach space with spectral measure $E_A(\cdot)$.

(1) If $F : \sigma(A) \to \mathbb{C}$ is a Borel measurable function, then

\begin{equation}
\sigma_p(F(A)) \supseteq F(\sigma_p(A)) \quad \text{(Spectral Inclusion Theorem for Point Spectrum)}.
\end{equation}
(2) If the function $F : \sigma(A) \to \mathbb{C}$ is also injective, then

\begin{equation}
\sigma_p(F(A)) = F(\sigma_p(A)) \quad \text{(Spectral Mapping Theorem for Point Spectrum),}
\end{equation}

with the operators $A$ and $F(A)$ sharing the corresponding spectral measure projections, i.e.,

\begin{equation}
E_{F(A)}(\{F(\lambda)\}) = E_A(\{\lambda\}), \quad \lambda \in \sigma_p(A),
\end{equation}

and hence, the corresponding eigenspaces, i.e.,

\begin{equation}
\ker(F(A) - F(\lambda)I) = \ker(A - \lambda I), \quad \lambda \in \sigma_p(A),
\end{equation}

and

\begin{equation}
\rho(F(A)) \cup \sigma_c(F(A)) \supseteq F(\sigma_c(A)).
\end{equation}

(3) If furthermore the function $F : \sigma(A) \to \mathbb{C}$ is continuous and injective,

\begin{equation}
\sigma_c(F(A)) = F(\sigma_c(A)) \cup \left( \overline{F(\sigma(A)) \setminus F(\sigma(A))} \right).
\end{equation}

**Proof.** Let $\lambda \in \sigma_p(A)$ be arbitrary. Then

\[
E_A(\{\lambda\}) \neq 0
\]

(see (2.4)).

In view of the fact that the spectral measure $E_{F(A)}(\cdot)$ of $F(A)$ is defined by (2.1) (see [2, Theorem 3.3]) and by the properties of spectral measure (see, e.g., [6,9]),

\[
E_A(\{\lambda\})E_{F(A)}(\{F(\lambda)\}) = E_A(\{\lambda\})E_A(F^{-1}(\{F(\lambda)\})) = E_A(\{\lambda\} \cap F^{-1}(\{F(\lambda)\})) = E_A(\{\lambda\}) \neq 0.
\]

Whence, we conclude that

\[
E_{F(A)}(\{F(\lambda)\}) \neq 0,
\]

and hence,

\[
F(\lambda) \in \sigma_p(F(A))
\]

(see (2.4)).

Thus, inclusion (4.1) holds and the proof of part (1) is complete.

Now, suppose that the function $F : \sigma(A) \to \mathbb{C}$ is injective. Then there exists an inverse $F^{-1}(\cdot)$.

By (2.4) and (2.1),

\begin{equation}
\mu \in \sigma_p(F(A)) \iff E_{F(A)}(\{\mu\}) \neq 0 \iff E_A(F^{-1}(\{\mu\})) \neq 0 \\
\iff F^{-1}(\{\mu\}) \neq \emptyset \quad \text{and} \quad \lambda := F^{-1}(\mu) \in \sigma_p(A),
\end{equation}

which, by part (1), proves equality (4.2), with equalities (4.3) and (4.4) following by (2.1) and (2.5).

In view of the injectivity of the function $F : \sigma(A) \to \mathbb{C}$, equality (4.2) and the fact that

\begin{equation}
\sigma_r(F(A)) = \emptyset
\end{equation}

imply inclusion (4.5).

This completes the proof of part (2).
Finally, suppose that the function $F : \sigma(A) \to \mathbb{C}$ is continuous and injective.

By the Weak Spectral Inclusion and Mapping Theorem (Theorem 3.2), the continuity of the function $F : \sigma(A) \to \mathbb{C}$ implies that

$$(4.9) \quad \sigma(F(A)) = \overline{F(\sigma(A))},$$

or equivalently,

$$\rho(F(A)) = \mathbb{C} \setminus \overline{F(\sigma(A))}.$$  

By part (2), the injectivity of the function $F : \sigma(A) \to \mathbb{C}$ implies equality (4.2) and inclusion (4.5).

In view of equality (4.9), inclusion (4.5) turns into the inclusion

$$(4.10) \quad \sigma_c(F(A)) \supseteq F(\sigma_c(A)).$$

Further, by equalities (4.9), (4.2), and (4.8), we infer that

$$(4.11) \quad \sigma_c(F(A)) \supseteq \overline{F(\sigma(A))} \setminus F(\sigma(A)).$$

Thus, inclusions (4.10) and (4.11) imply the inclusion

$$(4.12) \quad \sigma_c(F(A)) \supseteq F(\sigma_c(A)) \cup \left( \overline{F(\sigma(A))} \setminus F(\sigma(A)) \right).$$

On the other hand, in view of equalities (4.9), (4.2), and (4.8), we also have the converse inclusion

$$(4.13) \quad \sigma_c(F(A)) \subseteq F(\sigma_c(A)) \cup \left( \overline{F(\sigma(A))} \setminus F(\sigma(A)) \right).$$

Inclusions (4.12) and (4.13) jointly imply equality (4.6).

This completes the proof of part (3), and thus, of the entire statement. \qed

Remarks 4.1.

- As is seen from Examples 3.1 (see Remark 3.2), without the requirement of injectivity for the Borel measurable function $F : \sigma(A) \to \mathbb{C}$, inclusion (4.1) may be strict and inclusion (4.5) need not hold.

- The following example shows that, with the requirement of injectivity but without the requirement of continuity for the Borel measurable function $F : \sigma(A) \to \mathbb{C}$, equality (4.6) need not hold.

Example 4.1. For the self-adjoint operator $A$ of multiplication by the independent variable in the complex Hilbert space $L_2(\mathbb{R})$ from Examples 3.1 and the real-valued Borel measurable function

$$F(\lambda) := \begin{cases} e^\lambda, & \lambda \neq 0, \\ -1, & \lambda = 0, \end{cases}$$

on $\sigma(A) = \sigma_c(A) = \mathbb{R}$, which is injective but discontinuous at 0 with $E_A(\{0\}) = 0$ (see (2.6)), $F(A)$ is the self-adjoint operator of multiplication by $F(\cdot)$.

By Theorem 4.1 (part (2)) and Theorem 3.1,

$$\sigma(F(A)) = \sigma_c(F(A)) = \overline{F(\sigma(A) \setminus \{0\})} = F(\mathbb{R} \setminus \{0\}) = [0, \infty).$$
Thus, for any \( \lambda \in \mathbb{R} \setminus \{0\} \), \( \lambda \in \sigma_c(A) \) and \( F(\lambda) = e^\lambda \in \sigma_c(F(A)) \) and, for \( 0 \in \sigma_c(A) \), \( F(0) = -1 \in \rho(F(A)) \).

**Remark 4.2.** As follows from Examples 3.1 and Example 4.1, generally, for a scalar type spectral operator \( A \) in a complex Banach space and a Borel measurable function \( F : \sigma(A) \to \mathbb{C} \),

\[
\rho(F(A)) \cup \sigma_c(F(A)) \cup \sigma_p(F(A)) \supseteq F(\sigma_c(A)).
\]

5. **Scalar Type Spectral \( C_0 \)-Semigroups**

Considering that, for a \( C_0 \)-semigroup \( \{T(t)\}_{t \geq 0} \) (of scalar type spectral operators) on a complex Banach space generated by a scalar type spectral operator \( A \) with spectral measure \( E_A(\cdot) \),

\[
T(t) = e^{tA} := \int_{\sigma(A)} e^{t\lambda} \, dE_A(\lambda), \quad t \geq 0,
\]

(see Preliminaries) and applying for each \( t \geq 0 \) the Weak Spectral Inclusion and Mapping Theorem (Theorem 3.2) relative to the continuous exponential function

\[
F_t(\lambda) := e^{t\lambda}, \quad \lambda \in \sigma(A),
\]

we obtain the following generalization of precise weak spectral mapping theorem (PWSMT), known to hold for \( C_0 \)-semigroups of normal operators (see Preliminaries).

**Theorem 5.1** (Precise Weak Spectral Mapping Theorem).

A \( C_0 \)-semigroup \( \{T(t)\}_{t \geq 0} \) (of scalar type spectral operators) on a complex Banach space generated by a scalar type spectral operator \( A \) is subject to precise weak spectral mapping theorem (PWSMT).

The finer spectrum structure for scalar type spectral \( C_0 \)-semigroups is governed by the following statement.

**Theorem 5.2** (Finer Spectrum Structure).

Let \( \{T(t)\}_{t \geq 0} \) be a \( C_0 \)-semigroup (of scalar type spectral operators) on a complex Banach space generated by a scalar type spectral operator \( A \). Then

\[
\begin{align*}
(5.1) \quad \sigma_p(T(t)) &= e^{t\sigma_p(A)}, \quad t \geq 0, \quad \text{(Spectral Mapping Theorem for Point Spectrum)}, \\
(5.2) \quad \sigma_r(T(t)) &= \emptyset, \quad t \geq 0, \\
(5.3) \quad \sigma_c(T(t)) &= e^{t\sigma(A)} \setminus e^{t\sigma_p(A)}, \quad t \geq 0.
\end{align*}
\]

If furthermore, for some \( t > 0 \), the restriction of the exponential function \( e^t \) to \( \sigma(A) \) is injective, then

\[
(5.4) \quad \sigma_c(T(t)) = e^{t\sigma_c(A)} \cup \left( e^{t\sigma(A)} \setminus e^{t\sigma(A)} \right).
\]

**Proof.** By [10, Theorem V.2.6],

\[
(5.5) \quad \sigma_p(T(t)) \setminus \{0\} = e^{t\sigma_p(A)}, \quad t \geq 0,
\]

even without the assumption of scalar type spectrality for the generator.
Since, in view of
\[ T(t) = e^{tA}, \quad t \geq 0, \]
(see Preliminaries), by the properties of the Borel operational calculus (see [9, Theorem XVIII.2.11 (h)]), for any \( t \geq 0 \), there exists the inverse
\[ T(t)^{-1} = (e^{tA})^{-1} = e^{-tA}, \]
we infer that
\[ (5.6) \quad 0 \not\in \sigma_p(T(t)), \quad t \geq 0. \]
From (5.5) and (5.6), we infer that equality (5.1) holds.
Equality (5.2) holds by (2.7).
Equality (5.3) follows by Theorem 5.1 from equalities (5.1) and (5.2).
If furthermore, for some \( t > 0 \), the restriction of the exponential function \( e^{t \cdot} \) to \( \sigma(A) \) is injective, then, by Theorem 4.1 (part (3)), we arrive at equality (5.4). □

6. The Case of Normal Operators and \( C_0 \)-Semigroups
For the case of normal operators and \( C_0 \)-semigroups (see Preliminaries), we arrive at the following corollaries of the corresponding statements.

**Corollary 6.1** (Weak Spectral Inclusion and Mapping Theorem A.E.).
Let \( A \) be a normal operator \( A \) in a complex Hilbert space with spectral measure \( E_A(\cdot) \).

(1) If \( F : \sigma(A) \rightarrow \mathbb{C} \) is a Borel measurable function, then
\[ \sigma(F(A)) \subseteq F(\sigma(A) \setminus \sigma), \]
where \( \sigma \) is an arbitrary Borel subset of \( \sigma(A) \) for which \( E_A(\sigma) = 0 \).

(2) If furthermore the function \( F : \sigma(A) \rightarrow \mathbb{C} \) is continuous on \( \sigma(A) \setminus \sigma \), where \( \sigma \) is a Borel subset of \( \sigma(A) \) for which \( E_A(\sigma) = 0 \), then
\[ \sigma(F(A)) = F(\sigma(A) \setminus \sigma). \]

Cf. [11, Theorem D.11] (see also [11, Theorem D.12]).

**Corollary 6.2** (Weak Spectral Inclusion and Mapping Theorem).
Let \( A \) be a normal operator \( A \) in a complex Hilbert space.

(1) If \( F : \sigma(A) \rightarrow \mathbb{C} \) is a Borel measurable function, then
\[ \sigma(F(A)) \subseteq F(\sigma(A)). \]

(2) If \( F : \sigma(A) \rightarrow \mathbb{C} \) is a continuous function,
\[ \sigma(F(A)) = F(\sigma(A)). \]

Cf. [11, Theorem D.9].
Corollary 6.3 (Finer Spectrum Structure).

Let $A$ be a normal operator in a complex Hilbert space with spectral measure $E_A(\cdot)$.

(1) If $F : \sigma(A) \to \mathbb{C}$ is a Borel measurable function, then

$$\sigma_p(F(A)) \supseteq F(\sigma_p(A)) \quad \text{(Spectral Inclusion Theorem for Point Spectrum)}.$$ 

(2) If the function $F : \sigma(A) \to \mathbb{C}$ is also injective, then

$$\sigma_p(F(A)) = F(\sigma_p(A)) \quad \text{(Spectral Mapping Theorem for Point Spectrum)},$$

with the operators $A$ and $F(A)$ sharing the corresponding spectral measure projections, i.e.,

$$E_{F(A)}(\{F(\lambda)\}) = E_A(\{\lambda\}), \quad \lambda \in \sigma_p(A),$$

and hence, the corresponding eigenspaces, i.e.,

$$\ker (F(A) - F(\lambda)I) = \ker (A - \lambda I), \quad \lambda \in \sigma_p(A),$$

and

$$\rho(F(A)) \cup \sigma_c(F(A)) \supseteq F(\sigma_c(A)).$$

(3) If furthermore the function $F : \sigma(A) \to \mathbb{C}$ is continuous and injective,

$$\sigma_c(F(A)) = F(\sigma_c(A)) \cup \left( F(\sigma(A)) \setminus F(\sigma(A)) \right).$$

Corollary 6.4 (Finer Spectrum Structure).

Let $\{T(t)\}_{t \geq 0}$ be a $C_0$-semigroup $\{T(t)\}_{t \geq 0}$ (of normal operators) on a complex Hilbert space generated by a normal operator $A$. Then

$$\sigma_p(T(t)) = e^{t\sigma_p(A)}, \quad t \geq 0, \quad \text{(Spectral Mapping Theorem for Point Spectrum)},$$

$$\sigma_r(T(t)) = \emptyset, \quad t \geq 0,$$

$$\sigma_c(T(t)) = e^{t\sigma_c(A)} \setminus e^{t\sigma_r(A)}, \quad t \geq 0.$$

If furthermore, for some $t > 0$, the restriction of the exponential function $e^t$ to $\sigma(A)$ is injective, then

$$\sigma_c(T(t)) = e^{t\sigma_c(A)} \cup \left( e^{t\sigma(A)} \setminus e^{t\sigma(A)} \right).$$

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