Measurement of the CKM angle $\gamma$ with $B^+ \to D^{(*)} [K^0, \pi^- \pi^+] K^{(*)+}$ decays in BABAR$^*$

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We report on the measurement of the Cabibbo-Kobayashi-Maskawa angle $\gamma$ through a Dalitz analysis of neutral $D$ decays to $K^0, \pi^-, \pi^+$ in the processes $B^+ \to D^{(*)} K^+$ and $B^+ \to D K^{*+}$, $D^* \to D \pi^0, D \gamma$, with the BABAR detector at the SLAC PEP-II $e^+e^-$ asymmetric-energy collider.

I. INTRODUCTION AND OVERVIEW

The angle $\gamma$ of the unitarity triangle is the phase of the Cabibbo-Kobayashi-Maskawa (CKM) matrix $\mathbb{V}$ defined as $\gamma \equiv \arg \left( -V^*_{ud} V_{ub}/V^*_{cd} V_{cb} \right)$, which corresponds to the phase of the element $V^*_{ub}$, i.e. $\Delta m = |V_{ub}| e^{-i\gamma}$, in the Wolfenstein parameterization $\mathbb{B}$. The precise measurement of the angle $\gamma$ is a crucial goal of the physics program at the B-factories, however, it is also one of the most difficult to achieve.

Among all methods proposed to extract $\gamma$, only those using $B^+ \to D^{(*)} K^+$ decays are theoretically clean because the main contributions to the amplitudes come from tree-level diagrams (the symbol $\mathbb{B}$). The sensitivity to $\gamma$ in the $B \to K^* \gamma$ state $\mathbb{D}$, introduces a relative phase $\gamma$ in the decay amplitude. The sensitivity to $\gamma$ depends on the magnitude of the ratio of the $B \to D$ amplitude with respect to the $B \to D$ one, $r_B$, which plays a key role on the ability to measure $\gamma$ at the B-factories. Theoretical expectations, consistent with current experimental limits, give $r_B \approx |V^*_{ub}| V^*_{ct}/V^*_{cb}$, with $c_F \sim 0.1$, where $c_F \sim 0.2$ is the color suppression factor.

When the $D^0$ is reconstructed in a 3-body final state like $K^0, \pi^-, \pi^+$, the interference between doubly-Cabibbo suppressed, Cabibbo allowed and $CP$-eigenstate amplitudes provides strong phases to ensure the sensitivity to $\gamma$ $\mathbb{C}$. The angle $\gamma$ can then be extracted through an analysis of the distribution of the events in the $D^0$ Dalitz plane.

Assuming negligible $D^0$ - $\bar{D}^0$ mixing and $CP$ asymmetries in $D$ decays, the $B^+ \to D^{(*)} K^+$, with $D^{(*)} \to D^0 \pi^0, D^{*0} \pi^0, D^0 \to K^0 \pi^- \pi^+$ decay chain rate $\Gamma (m^2, m^2_\pi)$ can be written as

$$\Gamma (m^2, m^2_\pi) \propto |A_{D^+}|^2 + r_B^2 |A_{D^+2}|^2 + 2 \epsilon \left\{ x \Re[A_{D^+2} A_{D^+2}^\ast] + y \Im[A_{D^+2} A_{D^+2}^\ast] \right\},$$ (1)

where $m^2$ and $m^2_\pi$ are the squared invariant masses of the $K^0, \pi^-$ and $K^0, \pi^+$ combinations, respectively, $A_{D^+} \equiv A_D (m^2, m^2_\pi)$, with $A_{D^+2} (A_{D^+2}^\ast)$ the amplitude of the $D^+ \to K^0 \pi^- \pi^+$ ($\bar{D}^0 \to K^0 \pi^- \pi^+$) decay. We introduce the CP (cartesian) parameters $\mathbb{C}$.

Equation (1) also applies to $B^+ \to D^{(*)} K^+$ decays, with the replacements $r_B \to r_s$, $\delta \to \delta_s$, $x \to x_s$, $y \to y_s$, with $r_s \approx 1$, and $\delta_s \approx 1$.

II. DATA SAMPLE AND EVENT SELECTION

The analysis for $B^+ \to D^{(*)} K^+$ - $B^- \to D^{(*)} K^-$ decays $\mathbb{C}$ is based on a sample of approximately 347 (227) million $B\bar{B}$ pairs collected by the BABAR detector $\mathbb{C}$ at the SLAC PEP-II $e^+e^-$ asymmetric-energy storage ring. For each signal channel we also reconstruct its own control sample, $B^- \to D^{(*)} \pi^-$ ($B^- \to D^{(*)} \pi^-$).

The reconstruction and selection criteria are described in detail elsewhere $\mathbb{C}$. $B$ meson candidates are characterized by using the energy difference $\Delta E$, the beam-energy substituted mass $m_{ES}$, and a Fisher discriminant $\mathbb{F}$ to separate $e^+e^- \to q\bar{q}$, $q = u, d, s, c$ (continuum) and $B\bar{B}$ events $\mathbb{C}$. If both $B^- \to D^{(*)} \pi^-$ - $D^0 \pi^0 K^-$ and $B^- \to D^{(*)} \pi^-$ - $D^0 \pi^0 K^-$ candidates are selected in the same event, only the $B^- \to D^{(*)} \pi^-$ - $D^0 \pi^0 K^-$ is kept. The cross-feed among the different samples is negligible except for $B^- \to D^{(*)} \pi^-$ - $D^0 \pi^0 K^-$, where the background from $B^- \to D^{(*)} \pi^-$ is about 5% of the signal yield.

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This contamination has a negligible effect on the CP parameters.

The reconstruction efficiencies are 15%, 7%, 9%, and 11%, for the $B^- \to \bar{D}^0 K^-$, $B^- \to \bar{D}^0[\bar{D}^0\pi^0]K^-$, $B^- \to \bar{D}^{\ast 0}[\bar{D}^{\ast 0}\gamma]K^-$, and $B^- \to \bar{D}^0 K^{\ast -}$ decay modes, respectively. Figure 1 shows the $m_{ES}$ distributions for all selection criteria, for $|\Delta E| < 30(25)$ MeV, for $B^- \to \bar{D}^{\ast 0} K^-(B^- \to \bar{D}^0 K^{\ast -})$. The largest background contribution comes from continuum events or $B\bar{B}$ decays where a fake or true $D^0$ is combined with a random track. Another source of background for $B^- \to \bar{D}^{\ast 0} K^-$ is given by $B^- \to \bar{D}^{\ast 0}\pi^-\pi^0$ decays where the prompt pion is misidentified as kaon. These decays are separated from the signal using their different $\Delta E$ distribution.

FIG. 1: Distributions of $m_{ES}$ for (a) $B^- \to \bar{D}^0 K^-$, (b) $B^- \to \bar{D}^{\ast 0}[\bar{D}^{\ast 0}\pi^0]K^-$, (c) $B^- \to \bar{D}^{\ast 0}[\bar{D}^{\ast 0}\gamma]K^-$, and (c) $B^- \to \bar{D}^{\ast 0} K^{\ast -}$. The curves superimposed represent the overall fit projections (solid black lines), the continuum contribution (dotted red lines), and the sum of all background components (dashed blue lines).

III. THE $D^0 \to K^0_{\pi\pi}\pi^+\pi^-$ DECAY MODEL

The $D^0 \to K^0_{\pi\pi}\pi^+\pi^-$ decay amplitude $A_D(m_{2},m_{1}^2)$ is determined from an unbinned maximum-likelihood fit to the Dalitz plot distribution of a high-purity (97.7%) tagged $D^0$ sample from 390328 $D^{+} \to D^0\pi^+$ decays reconstructed in 270 fb$^{-1}$ of data, shown in Fig. 2. Our phenomenological reference model to describe $A_D(m_{2},m_{1}^2)$ uses a sum of two-body amplitudes (subscript $r$) and a non-resonant (subscript NR) contribution,

$$A_D(m_{2},m_{1}^2) = \sum_{r} a_{r} e^{i\phi_{r}} A_r(m_{2}^2,m_{1}^2) + a_{NR} e^{i\phi_{NR}} , (2)$$

where the parameters $a_{r}$ ($\alpha_{NR}$) and $\phi_{r}$ ($\phi_{NR}$) are the magnitude and phase of the amplitude for component $r$ (NR). The function $A_r = F_r \times T_r \times W_r$ is the Lorentz-invariant expression that describes the dynamic properties of the $D^0$ meson decaying into $K^0_{\pi\pi}\pi^+\pi^-$ through an intermediate resonance $r$, as a function of position in the Dalitz plane. Here, $F_r$ is the Blatt-Weisskopf centrifugal barrier factor for the resonance decay vertex $[16]$ with radius $R = 1.5$ GeV$^{-1}$ (0.3 fm), $T_r$ is the resonance propagator, and $W_r$ describes the angular distribution in the decay. For $T_r$ we use a relativistic Breit-Wigner (BW) parameterization, except for $r = \rho(770)$ and $\rho(1450)$ where we use the functional form suggested in Ref. [17]. The angular dependence $W_r$ is described with the helicity formalism as shown in [18].$^1$. Mass and width values are taken from [19], with the exception of $K^0_{s}(1430)^{\pm}$ taken from [20]. The model consists of a total of 13 resonances leading to 16 two-body decay amplitudes and phases (see Table I), and accounts for efficiency variations across the Dalitz plane and the small background contribution. All the resonances considered in this model are well established except for the two scalar $\pi\pi$ resonances, $\sigma$ and $\sigma'$, whose masses and widths are obtained from our sample [21]. Their addition to the model is motivated by an improvement in the description of the data.

The possible absence of the $\sigma$ and $\sigma'$ resonances is considered in the evaluation of the systematic errors through the use of a K-matrix formalism [22] to parameterize the $\pi\pi$ S-wave states. The K-matrix method provides a direct way of imposing the unitarity constraint of the scattering matrix that is not guaranteed in the case of the BW model and is suited to the study of broad and overlapping resonances in multi-channel decays, avoiding the need to introduce the two $\sigma$ scalars,

$$A_D(m_{2}^2,m_{1}^2) = F_1(s) + \sum_{r \neq \pi\pi} a_{r} e^{i\phi_{r}} A_r(m_{2}^2,m_{1}^2) , (3)$$

where $F_1(s) = \sum_{j} [I - i K(s)\rho(s)]^{-1} P_j(s)$ is the contribution of $\pi\pi$ S-wave states. Here, $s = m_{\pi\pi}^2$, $I$ is the identity matrix, $K$ is the matrix describing the S-wave scattering process, $\rho$ is the phase-space matrix, and $P$ is the initial production vector [22]. The index $j$ represents the $j^{\text{th}}$ channel (1 = $\pi\pi$, 2 = $K K$, 3 = multi-meson [23], 4 = $\eta\eta$, 5 = $\eta\eta'$). The K-matrix parameters are obtained from a global fit to the available $\pi\pi$ scattering data below 1900 MeV/c$^2$ [24], while the initial production vector is obtained from our fit to the tagged $D^0 \to K^0_{s}\pi^-\pi^+$ data.

IV. CP FIT RESULTS AND SYSTEMATIC UNCERTAINTIES

Once the decay amplitude $A_D(m_{2}^2,m_{1}^2)$ is known it can be fed into Eq. (1). The extraction of the CP-

$^1$ The label A and B should be swapped in Eq. (6) of [18].
TABLE I: Complex amplitudes $a_r e^{i\theta_r}$ and fit fractions of the different components ($K_S\pi^0$, $K_S\pi^+$, and $\pi^+\pi^-$ resonances) obtained from the fit of the $D^0 \to K_S\pi^+\pi^-$ Dalitz distribution from $D^{+}\to D^{0}\pi^+\pi^-$ events. Errors are statistical only. The fit fraction is defined for the resonance terms as the integral of $a_r^2 |A_r (m_1^2, m_2^2)|^2$ over the Dalitz plane divided by the integral of $|A_D (m_1^2, m_2^2)|^2$. The sum of fit fractions is 119.5%. A value different from 100% is a consequence of the interference among the amplitudes.

| Component | $Re\{a_r e^{i\theta_r}\}$ | $Im\{a_r e^{i\theta_r}\}$ | Fraction (%) |
|-----------|-----------------------------|-----------------------------|--------------|
| $K^+(892)$ | -1.223 ± 0.011 | 1.346 ± 0.010 | 58.1 |
| $K^0_S(1430)$ | -1.698 ± 0.022 | -0.576 ± 0.024 | 6.7 |
| $K^0_S(1430)$ | -0.834 ± 0.021 | 0.931 ± 0.022 | 6.3 |
| $K^+(1410)$ | -0.25 ± 0.04 | -0.11 ± 0.03 | 0.1 |
| $K^+(1680)$ | -1.285 ± 0.014 | 0.205 ± 0.013 | 0.6 |
| $K^0_S(1430)^*$ | 0.100 ± 0.004 | -0.127 ± 0.003 | 0.5 |
| $K^0_S(1430)^+$ | -0.027 ± 0.016 | -0.076 ± 0.017 | 0.0 |
| $K^0_S(1430)^-$ | 0.019 ± 0.017 | 0.177 ± 0.018 | 0.1 |
| $\rho(1700)$ | 1 | 0 | 21.5 |
| $\omega(782)$ | -0.0219 ± 0.0010 | 0.0394 ± 0.0007 | 0.7 |
| $f_2(1270)$ | -0.699 ± 0.018 | 0.387 ± 0.018 | 2.1 |
| $\rho(1450)$ | 0.25 ± 0.04 | 0.04 ± 0.06 | 0.1 |
| Non-resonant | -0.99 ± 0.19 | 3.82 ± 0.13 | 3.5 |
| $f_0(980)$ | 0.447 ± 0.006 | 0.257 ± 0.008 | 6.4 |
| $f_0(1370)$ | 0.95 ± 0.11 | -1.619 ± 0.011 | 2.0 |
| $\sigma$ | 1.28 ± 0.02 | 0.273 ± 0.024 | 7.6 |
| $\sigma'$ | 0.260 ± 0.011 | -0.066 ± 0.010 | 0.9 |

The resulting values are consistent with what is found from Ref. [24] with parameters extracted from our fit. The uncertainty on the description of the $K\pi$ S-wave is estimated by floating in our flavor tagged $D^0$ sample the mass and width of the BW describing the $K^*(1430)$, and using an additional parameterization taken from Ref. [25] with parameters extracted from our fit. Since the $K\pi$ P-wave is dominated by the $K^*(892)$ in both Cabibbo allowed and doubly Cabibbo suppressed amplitude, the mass and the width of this resonance, taken from Ref. [19] in the reference model, are changed to the values obtained from our fit to the tagged $D^0$ sample. The resulting values are consistent with what is found in Ref. [17] for $J/\Psi K\pi$ decays selected in $B$ factories. For the $\pi\pi$ S-wave we use the K-matrix approach described in Ref. [26], while for the P-wave we change the mass and width describing the $\rho(770)$ within their quoted uncertainty [19].

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1%, −17%, −14%, −10%, and 2%, respectively. Figure 8 shows the one- and two-standard deviation confidence-level contours (including statistical and systematic uncertainties) in the $(x, y)$ plane for all the reconstructed modes, and separately for $B^−$ and $B^+$ decays. The separation of the $B^−$ and $B^+$ contours is an indication of direct CP violation.

The largest single contribution to the systematic uncertainties on the CP parameters comes from the choice of the Dalitz model used to describe the $D^0 \to K^0_S\pi^+\pi^-$ decay amplitude. We use a set of alternative models where some resonances are removed or the parameterization of the different amplitudes are changed. For the $\pi\pi$ S-wave we use the K-matrix approach described in Ref. [17], while for the P-wave we change the mass and width describing the $\rho(770)$ within their quoted uncertainty [19].
TABLE II: CP-violating parameters \( x_{\pi}^{(*)}, y_{\pi}^{(*)}, x_{s\pi}, \) and \( y_{s\pi} \), as obtained from the CP fit. The first error is statistical, the second is experimental systematic uncertainty and the third is the systematic uncertainty associated with the Dalitz model.

| CP parameter | \( B^{-} \to \bar{D}^{0}K^{-} \) | \( B^{-} \to \bar{D}^{0}s\bar{K}^{-} \) | \( B^{-} \to \bar{D}^{0}s\bar{K}^{+} \) |
|--------------|----------------|----------------|----------------|
| \( x_\pi \) \( / x_s \) | \( 0.041 \pm 0.059 \pm 0.018 \pm 0.011 \) | \( -0.106 \pm 0.091 \pm 0.020 \pm 0.009 \) | \( -0.20 \pm 0.20 \pm 0.11 \pm 0.03 \) |
| \( y_\pi / y_s \) | \( 0.056 \pm 0.071 \pm 0.007 \pm 0.023 \) | \( -0.019 \pm 0.096 \pm 0.022 \pm 0.016 \) | \( 0.26 \pm 0.30 \pm 0.16 \pm 0.03 \) |
| \( x_s / x_s \) | \( -0.072 \pm 0.056 \pm 0.014 \pm 0.029 \) | \( 0.084 \pm 0.088 \pm 0.015 \pm 0.018 \) | \( -0.07 \pm 0.23 \pm 0.13 \pm 0.03 \) |
| \( y_s / y_s \) | \( -0.033 \pm 0.066 \pm 0.007 \pm 0.018 \) | \( 0.096 \pm 0.111 \pm 0.032 \pm 0.017 \) | \( -0.01 \pm 0.32 \pm 0.18 \pm 0.05 \) |

![FIG. 3: Contours at 39.3% (dark) and 86.5% (light) confidence level (corresponding to two-dimensional one- and two-standard deviation regions), including statistical and systematic uncertainties, for (a) \((x_\pi, y_\pi)\), (b) \((x_s, y_s^*)\), and (c) \((x_{s\pi}, y_{s\pi})\) parameters, for \( B^- \) (thick and solid lines) and \( B^+ \) (thin and dotted lines) decays.](image)
TABLE III: Summary of the main contributions to the experimental systematic error on the CP parameters.

| Source | $x_-$ | $y_-$ | $x_+$ | $y_+$ | $x_-^*$ | $y_-^*$ | $x_+^*$ | $y_+^*$ |
|--------|-------|-------|-------|-------|---------|---------|---------|---------|
| $m_{ES}, \Delta E, \mathcal{F}$ shapes | 0.002 | 0.004 | 0.003 | 0.004 | 0.011 | 0.012 | 0.008 | 0.008 |
| Real $D^0$ fractions | 0.002 | 0.000 | 0.000 | 0.000 | 0.002 | 0.003 | 0.002 | 0.016 |
| Charge-$D^0$ flavor correlation | 0.008 | 0.002 | 0.002 | 0.002 | 0.005 | 0.005 | 0.001 | 0.022 |
| Efficiency in the Dalitz plot | 0.014 | 0.000 | 0.013 | 0.001 | 0.001 | 0.002 | 0.000 | 0.001 |
| Background Dalitz shape | 0.006 | 0.003 | 0.001 | 0.004 | 0.012 | 0.015 | 0.009 | 0.009 |
| Dalitz amplitudes and phases | 0.004 | 0.004 | 0.004 | 0.004 | 0.008 | 0.008 | 0.008 | 0.008 |
| $B^- \rightarrow D^{0}\bar{D}^0K^-$ cross-feed | 0.000 | 0.000 | 0.000 | 0.000 | 0.004 | 0.001 | 0.004 | 0.004 |
| CP violation in $D^+$ and $B^0$ bkg | 0.000 | 0.000 | 0.000 | 0.000 | 0.005 | 0.002 | 0.002 | 0.005 |
| Total experimental | 0.018 | 0.007 | 0.014 | 0.007 | 0.020 | 0.022 | 0.015 | 0.015 |

FIG. 4: Projections onto the (a) $(r_B, \gamma)$ and (b) $(r_B^*, \gamma)$ planes of the 3.7% (dark) and 45.1% (light) five-dimensional confidence level regions, corresponding to one- and two-standard deviation intervals, respectively, including statistical and systematic uncertainties.

addition of different $B$ (e.g. $B^+ \rightarrow D K^*\bar{K}$) and $D$ (e.g. $D^0 \rightarrow \pi^0\pi^-\pi^+; K^0\bar{K}^-K^+$) decay channels, and the combination with other methods \cite{4,5} will be helpful. Assuming $r_B = 0.1$ it will be possible to measure $\gamma$ with $\sim 10^5$ error with a 1 ab$^{-1}$ data sample, which is within the reach of the $\bar{B}a\bar{B}ar$ experiment.

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