ALGEBRAIC DENSITY PROPERTY OF HOMOGENEOUS SPACES

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Abstract. Let $X$ be an affine algebraic variety with a transitive action of the algebraic automorphism group. Suppose that $X$ is equipped with several fixed point free non-degenerate $SL_2$-actions satisfying some mild additional assumption. Then we prove that the Lie algebra generated by completely integrable algebraic vector fields on $X$ coincides with the set of all algebraic vector fields. In particular, we show that apart from a few exceptions this fact is true for any homogeneous space of form $G/R$ where $G$ is a linear algebraic group and $R$ is its proper reductive subgroup.

1. Introduction

In this paper we develop further methods introduced by F. Kutzschebauch and the third author in [16] which they used to obtain new results in the Andersén-Lempert theory ([1], [2]). D. Varolin was the first one to realize the importance of the following notion in that theory ([27]).

1.1. Definition. A complex manifold $X$ has the density property if in the compact-open topology the Lie algebra $\text{Lie}_{\text{hol}}(X)$ generated by completely integrable holomorphic vector fields on $X$ is dense in the Lie algebra $\text{VF}_{\text{hol}}(X)$ of all holomorphic vector fields on $X$. An affine algebraic manifold $X$ has the algebraic density property if the Lie algebra $\text{Lie}_{\text{alg}}(X)$ generated by completely integrable algebraic vector fields on it coincides with the Lie algebra $\text{VF}_{\text{alg}}(X)$ of all algebraic vector fields on it (clearly, the algebraic density property implies the density property).

For any complex manifold with the density property the Andersén-Lempert theory is applicable and its effectiveness in complex analysis was demonstrated in several papers (e.g., see [12], [27], [28]). Furthermore, Kollár and Mangolte found a remarkable application of this theory to real algebraic geometry [18]. However until recently the class of manifolds for which this property was established was quite narrow (mostly Euclidean spaces and semi-simple Lie groups, and homogeneous spaces of semi-simple groups with trivial centers [25], [26]). In [15] this class was enlarged by hypersurfaces of form $uv = p(\bar{x})$ and in [16] by connected complex algebraic groups except for $\mathbb{C}^+, \mathbb{C}^*$ (for which the density property is not true) and the higher dimensional tori (for which the validity of this property is still unknown). Furthermore, it was shown in [15], [16] that these varieties have the algebraic density property.

In this paper we study a smooth complex affine algebraic variety $X$ with a transitive action of the algebraic automorphism group $\text{Aut} X$ (which is natural because complex
manifolds with the algebraic density property have transitive automorphism groups). Though the facts we prove about such objects are rather straightforward extension of [16], in combination with Lie group theory they lead to a much wider class of homogeneous spaces with the algebraic density property. Our new technique yields, in particular, to the following.

Theorem. Let $G$ be a linear algebraic group and $R$ be its proper reductive subgroup such that the homogeneous space $G/R$ is different from $\mathbb{C}_+$, a torus, or a $\mathbb{Q}$-homology plane $\mathcal{P}$ with a fundamental group $\mathbb{Z}_2$. Then $G/R$ has the algebraic density property.

Besides the criteria developed in [16] the main new ingredient of the proof is the Luna slice theorem. For convenience of readers we remind it in Section 2 together with basic facts about algebraic quotients and some crucial results from [16]. In Section 3 we prove our main theorem. As an application we obtain the Theorem before in section 4 using some technical fact from the Lie group theory presented in the Appendix.

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2. Preliminaries

Let us fix some notation first. In this paper $X$ will be always a complex affine algebraic variety and $G$ be an algebraic group acting on $X$, i.e. $X$ is a $G$-variety. The ring of regular functions on $X$ will be denoted by $\mathbb{C}[X]$ and its subring of $G$-invariant functions by $\mathbb{C}[X]^G$.

2.1. Algebraic (categorical) quotients. Recall that the algebraic quotient $X//G$ of $X$ with respect to the $G$-action is $\text{Spec}(\mathbb{C}[X]^G)$. By $\pi : X \to X//G$ we denote the natural quotient morphism generated by the embedding $\mathbb{C}[X]^G \hookrightarrow \mathbb{C}[X]$. The main (universal) property of algebraic quotients is that any morphism from $X$ constant on orbits of $G$ factors through $\pi$. In the case of a reductive $G$ several important facts (e.g., see [24], [23], [8], [13]) are collected in the following.

2.2. Proposition. Let $G$ be a reductive group.

(1) The quotient $X//G$ is an affine algebraic variety which is normal in the case of a normal $X$ and the quotient morphism $\pi : X \to X//G$ is surjective.

(2) The closure of every $G$-orbit contains a unique closed orbit and each fiber $\pi^{-1}(y)$ (where $y \in X//G$) contains also a unique closed orbit $O$. Furthermore, $\pi^{-1}(y)$ is the union of all those orbits whose closures contain $O$.

(3) In particular, if every orbit of the $G$-action on $X$ is closed then $X//G$ is isomorphic to the orbit space $X/G$.

(4) The image of a closed $G$-invariant subset under $\pi$ is closed.

If $X$ is a complex algebraic group, and $G$ is a closed subgroup acting on $X$ by multiplication, clearly all the orbits are closed. If $G$ is reductive, the previous proposition implies that the quotient $X/G$ is affine. The next proposition (Matsushima’s criterion) shows that the converse is also true.
2.3. **Proposition.** Let $G$ be a complex reductive group, and $H$ be a closed subgroup of $G$. Then the quotient space $G/H$ is affine if and only if $H$ is reductive.

Besides reductive groups actions in this paper, a crucial role will be played by $\mathbb{C}_+$-actions. In general algebraic quotients in this case are not affine but only quasi-affine [29]. However, we shall use later the fact that for the natural action of any $\mathbb{C}_+$-subgroup of $SL_2$ generated by multiplication one has $SL_2/\mathbb{C}_+ \cong \mathbb{C}^2$.

2.4. **Luna’s slice theorem** (e.g., see [8, 23]). Let us remind some terminology first. Suppose that $f : X \to Y$ is a $G$-equivariant morphism of affine algebraic $G$-varieties $X$ and $Y$. Then the induced morphism $f_G : X//G \to Y//G$ is well defined and the following diagram is commutative.

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
X//G & \xrightarrow{f_G} & Y//G
\end{array}
\]

2.5. **Definition.** A $G$-equivariant morphism $f$ is called strongly étale if

1. The induced morphism $f_G : X//G \to Y//G$ is étale
2. The quotient morphism $\pi_G : X \to X//G$ induces a $G$-isomorphism between $X$ and the fibred product $Y \times_{Y//G} (X//G)$.

From the properties of étale maps (8) it follows that $f$ is étale (in particular, quasi-finite).

Let $H$ be an algebraic subgroup of $G$, and $Z$ an affine $H$-variety. We denote $G \times_H Z$ the quotient of $G \times Z$ by the action of $H$ given by $h(g, z) = (gh^{-1}, hz)$. The left multiplication on $G$ generates a left action on $G \times_H Z$. The next lemma is an obvious consequence of 2.2.

2.6. **Lemma.** Let $X$ be an affine $G$-variety and $G$ be reductive. Then the $H$-orbits of $G \times X$ are all isomorphic to $H$. Therefore the fibers of the quotient morphism $G \times X \to G \times_H X$ coincide with the $H$-orbits.

The isotropy group of a point $x \in X$ will be denoted by $G_x$. Recall also that an open set $U$ of $X$ is called saturated if $\pi^{-1}_G(\pi_G(U)) = U$. We are ready to state the Luna slice theorem.

**Theorem 1.** Let $G$ be a reductive group acting on an affine algebraic variety $X$, and let $x \in X$ be a point in a closed $G$-orbit. Then there exists a locally closed affine algebraic subvariety $V$ (called a slice) of $X$ containing $x$ such that

1. $V$ is $G_x$-invariant;
2. the image of the $G$-morphism $\varphi : G \times_{G_x} V$ induced by the action is a saturated open set $U$ of $X$;
3. the restriction $\varphi : G \times_{G_x} V \to U$ is strongly étale.
Given a saturated open set $U$, we will denote $\pi_G(U)$ by $U//G$. It follows from 2.2 that $U//G$ is open. This theorem implies that the following diagram is commutative

\[
\begin{array}{ccc}
G \times_{G_x} V & \longrightarrow & U \\
\downarrow & & \downarrow \\
V//G_x & \longrightarrow & U//G
\end{array}
\]

and $G \times_{G_x} V \simeq U \times_{U//G} V//G_x$.

2.7. **The compatibility criterion.** This section presents the criteria for the algebraic density property, introduced in [16], that will be used to prove the main results of this paper.

2.8. **Definition.** Let $X$ be an affine algebraic manifold. An algebraic vector field $\sigma$ on $X$ is semi-simple if its phase flow is an algebraic $\mathbb{C}^*$-action on $X$. A vector field $\delta$ is locally nilpotent if its phase flow is an algebraic $\mathbb{C}_+$-action on $X$. In the last case $\delta$ can be viewed as a locally nilpotent derivation on $\mathbb{C}[X]$. That is, for every nonzero $f \in \mathbb{C}[X]$ there is the smallest $n = n(f)$ for which $\delta^n(f) = 0$. We set $\deg \delta(f) = n - 1$. In particular, elements from the kernel Ker $\delta$ have the zero degree with respect to $\delta$.

2.9. **Definition.** Let $\delta_1$ and $\delta_2$ be nontrivial algebraic vector fields on an affine algebraic manifold $X$ such that $\delta_1$ is a locally nilpotent derivation on $\mathbb{C}[X]$, and $\delta_2$ is either also locally nilpotent or semi-simple. That is, $\delta_i$ generates an algebraic action of $H_i$ on $X$ where $H_1 \simeq \mathbb{C}_+$ and $H_2$ is either $\mathbb{C}_+$ or $\mathbb{C}^*$. We say that $\delta_1$ and $\delta_2$ are semi-compatible if the vector space $\text{Span}(\text{Ker} \delta_1 \cdot \text{Ker} \delta_2)$ generated by elements from Ker $\delta_1 \cdot$ Ker $\delta_2$ contains a nonzero ideal in $\mathbb{C}[X]$.

A semi-compatible pair is called compatible if in addition one of the following conditions holds

1. when $H_2 \simeq \mathbb{C}^*$ there is an element $a \in \text{Ker} \delta_2$ such that $\deg \delta_1(a) = 1$, i.e. $\delta_1(a) \in \text{Ker} \delta_1 \setminus \{0\}$;
2. when $H_2 \simeq \mathbb{C}_+$ (i.e. both $\delta_1$ and $\delta_2$ are locally nilpotent) there is an element $a$ such that $\deg \delta_1(a) = 1$ and $\deg \delta_2(a) \leq 1$.

2.10. **Remark.** If $[\delta_1, \delta_2] = 0$ then condition (1) and condition (2) with $a \in \text{Ker} \delta_2$ hold automatically.

2.11. **Example.** Consider $SL_2$ (or even $PSL_2$) with two natural $\mathbb{C}_+$-subgroups: namely, the subgroup $H_1$ (resp. $H_2$) of the lower (resp. upper) triangular unipotent matrices. Denote by

\[
A = \begin{pmatrix}
a_1 & a_2 \\
b_1 & b_2
\end{pmatrix}
\]

an element of $SL_2$. Then the left multiplication generate actions of $H_1$ and $H_2$ on $SL_2$ with the following associated locally nilpotent derivations on $\mathbb{C}[SL_2]$

\[
\delta_1 = a_1 \frac{\partial}{\partial b_1} + a_2 \frac{\partial}{\partial b_2}
\]
\[ \delta_2 = b_1 \frac{\partial}{\partial a_1} + b_2 \frac{\partial}{\partial a_2}. \]

Clearly, \( \ker \delta_1 \) is generated by \( a_1 \) and \( a_2 \) while \( \ker \delta_2 \) is generated by \( b_1 \) and \( b_2 \). Hence \( \delta_1 \) and \( \delta_2 \) are semi-compatible. Furthermore, taking \( a = a_1 b_2 \) we see that condition (2) of Definition 2.9 holds, i.e. they are compatible.

It is worth mentioning the following geometrical reformulation of semi-compatibility which will be needed further.

2.12. **Proposition.** Suppose that \( H_1 \) and \( H_2 \) are as in Definition 2.9, \( X \) is a normal affine algebraic variety equipped with nontrivial algebraic \( H_i \)-actions where \( i = 1, 2 \) (in particular, each \( H_i \) generates an algebraic vector field \( \delta_i \) on \( X \)). Let \( X_i = X//H_i \) and \( \rho_i : X \to X_i \) the quotient morphisms. Set \( \rho = (\rho_1, \rho_2) : X \to Y := X_1 \times X_2 \) and \( Z \) equal to the closure of \( \rho(X) \) in \( Y \). Then \( \delta_1 \) and \( \delta_2 \) are semi-compatible iff \( \rho : X \to Z \) is a finite birational morphism.

2.13. **Definition.** A finite subset \( M \) of the tangent space \( T_x X \) at a point \( x \) of a complex algebraic manifold \( X \) is called a generating set if the image of \( M \) under the action of the isotropy group (of algebraic automorphisms) of \( x \) generates \( T_x X \).

It was shown in [16] that the existence of a pair of compatible derivations \( \delta_1 \) and \( \delta_2 \) from Definition 2.9 implies that \( \text{Lie}_{\text{alg}}(X) \) contains a \( \mathbb{C}[X] \)-submodule \( I\delta_2 \) where \( I \) is a nontrivial ideal in \( \mathbb{C}[X] \). This yields the central criterion for algebraic density property [16].

**Theorem 2.** Let \( X \) be a smooth homogeneous (with respect to \( \text{Aut} \, X \)) affine algebraic manifold with finitely many pairs of compatible vector fields \( \{\delta^k_i, \delta^k_2\}_{k=1}^m \) such that for some point \( x_0 \in X \) vectors \( \{\delta^k_2(x_0)\}_{k=1}^m \) form a generating set. Then \( \text{Lie}_{\text{alg}}(X) \) contains a nontrivial \( \mathbb{C}[X] \)-module and \( X \) has the algebraic density property.

As an application of this theorem we have the following.

2.14. **Proposition.** Let \( X_1 \) and \( X_2 \) be smooth homogeneous (with respect to algebraic automorphism groups) affine algebraic varieties such that each \( X_i \) admits a finite number of integrable algebraic vector fields \( \{\delta^k_i\}_{k=1}^m \) whose values at some point \( x_i \in X_i \) form a generating set and, furthermore, in the case of \( X_1 \) these vector fields are locally nilpotent. Then \( X_1 \times X_2 \) has the algebraic density property.

We shall need also two technical results (Lemmas 3.6 and 3.7 in [16]) that describe conditions under which quasi-finite morphisms preserve semi-compatibility.

2.15. **Lemma.** Let \( G = \text{SL}_2 \) and \( X, X' \) be normal affine algebraic varieties equipped with non-degenerate \( G \)-actions. Suppose that subgroups \( H_1 \) and \( H_2 \) of \( G \) are as in Example 2.11, i.e. they act naturally on \( X \) and \( X' \). Let \( \rho_i : X \to X_i := X//H_i \) and

\(^1\text{In the case of condition (2) in Definition 2.9 this fact was proven in [16] only for } \text{deg}_{\delta_2}(a) = 0 \text{ but the proof works for } \text{deg}_{\delta_2}(a) = 1 \text{ as well without any change.} \)
\[ \rho_i : X' \to X'_i := X'/H_i \text{ be the quotient morphisms and let } p : X \to X' \text{ be a finite } G\text{-equivariant morphism, i.e. we have commutative diagrams} \]
\[
\begin{array}{ccc}
X & \xrightarrow{\rho_i} & X_i \\
\downarrow p & & \downarrow q_i \\
X' & \xrightarrow{\rho'_i} & X'_i
\end{array}
\]
for \( i = 1, 2 \). Treat \( \mathbb{C}[X_i] \) (resp. \( \mathbb{C}[X'_i] \)) as a subalgebra of \( \mathbb{C}[X] \) (resp. \( \mathbb{C}[X'] \)). Let \( \text{Span}(\mathbb{C}[X_1]\cdot\mathbb{C}[X_2]) \) contain a nonzero ideal of \( \mathbb{C}[X] \). Then \( \text{Span}(\mathbb{C}[X_1]\cdot\mathbb{C}[X'_2]) \) contains a nonzero ideal of \( \mathbb{C}[X'] \).

The second result is presented here in a slightly different form but with a much simpler proof.

2.16. Lemma. Let the assumption of Lemma 2.15 hold with two exceptions: we do not assume that \( G\)-actions are non-degenerate and instead of the finiteness of \( p \) we suppose that there are a surjective étale morphism \( r : M \to M' \) of normal affine algebraic varieties equipped with trivial \( G\)-actions and a surjective \( G\)-equivariant morphism \( \tau' : X' \to M' \) such that \( X \) is isomorphic to fibred product \( X' \times_{M'} M \) with \( p : X \to X' \) being the natural projection (i.e. \( p \) is surjective étale). Then the conclusion of Lemma 2.15 remains valid.

Proof. By construction, \( X_i = X'_i \times_{M'} M \). Thus we have the following commutative diagram
\[
\begin{array}{ccc}
X & \xrightarrow{\rho} & (X'_1 \times X'_2) \times_{(M' \times M')} (M \times M) \\
\downarrow p & & \downarrow q \\
X' & \xrightarrow{\rho'} & X'_1 \times X'_2 \\
& & \overset{(\tau',\tau')}\longrightarrow M' \times M'.
\end{array}
\]
Set \( Z \) (resp. \( Z' \)) equal to the closure of \( \rho(X) \) in \( X_1 \times X_2 \) (resp. \( \rho'(X') \) in \( X'_1 \times X'_2 \)) and \( D \simeq M \) (resp. \( D' \simeq M' \)) be the diagonal subset in \( M \times M \) (resp. \( M' \times M' \)). Since \( X = X' \times_{M'} M \) we see that \( Z = Z' \times_{D'} D \). For any affine algebraic variety \( Y \) denote by \( Y_{\text{norm}} \) its normalization, i.e. \( Z_{\text{norm}} = Z'_{\text{norm}} \times_{D'} D \). By Lemma 2.12 \( \rho : X \to Z_{\text{norm}} \) is an isomorphism. Since \( r \) is surjective it can happen only when \( \rho' : X' \to Z' \) is an isomorphism. Hence the desired conclusion follows from Lemma 2.12.

The last result from [16] that we need allows us to switch from local to global compatibility.

2.17. Proposition. Let \( X \) be an \( SL_2 \)-variety with associated locally nilpotent derivations \( \delta_1 \) and \( \delta_2 \), \( Y \) be a normal affine algebraic variety equipped with a trivial \( SL_2 \)-action, and \( r : X \to Y \) be a surjective \( SL_2 \)-equivariant morphism. Suppose that for any \( y \in Y \) there exists an étale neighborhood \( g : W \to Y \) such that the vector fields induced by \( \delta_1 \) and \( \delta_2 \) on the fibred product \( X \times_Y W \) are semi-compatible. Then \( \delta_1 \) and \( \delta_2 \) are semi-compatible.
3. Algebraic density property and $SL_2$-actions

3.1. Notation. We suppose that $H_1, H_2, \delta_1$ and $\delta_2$ are as in Example 2.11. Note that if $SL_2$ acts algebraically on an affine algebraic variety $X$ then we have automatically the $C^*_+$-actions of $H_1$ and $H_2$ on $X$ that generate locally nilpotent vector fields on $X$, which by abuse of notation will be denoted by the same symbols $\delta_1$ and $\delta_2$. If $X$ admits several (say, $N$) $SL_2$-actions, we denote by $\{\delta_i^{k_1}, \delta_i^{k_2}\}_{k=1}^N$ the corresponding collection of pairs of locally nilpotent derivations on $\mathbb{C}[X]$.

Here is the first main result of this paper.

Theorem 3. Let $X$ be a smooth complex affine algebraic variety whose group of algebraic automorphisms is transitive. Suppose that $X$ is equipped with $N$ fixed point free non-degenerate actions of $SL_2$-groups $\Gamma_1, \ldots, \Gamma_N$. Let $\{\delta_i^{k_1}, \delta_i^{k_2}\}_{k=1}^N$ be the corresponding pairs of locally nilpotent vector fields. If $\{\delta_i^{k_2}(x_0)\}_{k=1}^N \subset T_{x_0}X$ is a generating set at some point $x_0 \in X$ then $X$ has the algebraic density property.

3.2. Remark. Note that we can choose any nilpotent element of the Lie algebra of $SL_2$ as $\delta_2$. Since the space of nilpotent elements generate the whole Lie algebra we can reformulate Theorem 3 as follows: a smooth complex affine algebraic variety $X$ with a transitive group of algebraic automorphisms has the algebraic density property provided it admits “sufficiently many” fixed point free non-degenerate $SL_2$-actions, where “sufficiently many” means that at some point $x_0 \in X$ the tangent spaces of the corresponding $SL_2$-orbits through $x_0$ generate the whole space $T_{x_0}X$.

By virtue of Theorem 2 the main result will be a consequence of the following.

Theorem 4. Let $X$ be a smooth complex affine algebraic variety equipped with a fixed point free non-degenerate $SL_2$-action that induces a pair of locally nilpotent vector fields $\{\delta_1, \delta_2\}$. Then these vector fields are compatible.

The proof of the last fact requires some preparations and until we finish this proof completely the assumption is that all $SL_2$-actions we consider are non-degenerate.

3.3. Lemma. Let the assumption of Theorem 4 hold and $x \in X$ be a point contained in a closed $SL_2$-orbit. Then the isotropy group of $x$ is either finite, or isomorphic to the diagonal $\mathbb{C}^*$-subgroup of $SL_2$, or to the normalizer of this $\mathbb{C}^*$-subgroup (which is the extension of $\mathbb{C}^*$ by $\mathbb{Z}_2$).

Proof. By Matsushima’s criterion (Proposition 2.3) the isotropy group must be reductive and it cannot be $SL_2$ itself since the action has no fixed points. The only two-dimensional reductive group is $\mathbb{C}^* \times \mathbb{C}^*$ (III) which is not contained in $SL_2$. Thus besides finite subgroups we are left to consider the one-dimensional reductive subgroups that include $\mathbb{C}^*$ (which can be considered to be the diagonal subgroup since all tori are conjugated) and its finite extensions. The normalizer of $\mathbb{C}^*$ which is its extension by $\mathbb{Z}_2$ generated by

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
is reductive. If we try to find an extension of $\mathbb{C}^*$ by another finite subgroup that contains an element $B$ not from the normalizer then $\mathbb{C}^*$ and $B\mathbb{C}^*B^{-1}$ meet at the identical matrix. In particular, the reductive subgroup must be at least two-dimensional, and we have to disregard this case. \hfill \Box

3.4. Proposition. Let $X, \delta_1, \delta_2$ be as in Theorem [4]. Then there exists a regular function $g \in \mathbb{C}[X]$ such that $\deg_{\delta_1}(g) = \deg_{\delta_2}(g) = 1$.

Proof. Let $x \in X$ be a point of a closed $SL_2$-orbit. Luna’s slice Theorem yields diagram (2) with $G = SL_2$ and $G_x$ being one of the subgroups described in Lemma 3.3. That is, we have the natural morphism $\varphi : SL_2 \times V \to U$ that factors through the étale morphism $SL_2 \times_{G_x} V \to U$ where $V$ is the slice at $x$. First, consider the case when $G_x$ is finite. Then $\varphi$ itself is étale. Furthermore, replacing $V$ by its Zariski open subset and $U$ by the corresponding Zariski open $SL_2$-invariant subset one can suppose that $\varphi$ is also finite. Set $f = a_1b_2$ where $a_i, b_i$ are as in Example 2.11. Note that each $\delta_i$ generates a natural locally nilpotent vector field $\delta_i$ on $SL_2 \times V$ such that $\mathbb{C}[V] \subset \text{Ker} \delta_i$ and $\varphi_*(\delta_i)$ coincides with the vector field induced by $\delta_i$ on $X$. Treating $f$ as an element of $\mathbb{C}[SL_2 \times V]$ we have $\deg_{\delta_i}(f) = 1, i = 1, 2$. For every $h \in \mathbb{C}[SL_2 \times V]$ we define a function $\hat{h} \in \mathbb{C}[U]$ by $\hat{h}(u) = \sum_{y \in \varphi^{-1}(u)} h(y)$. One can check that if $h \in \text{Ker} \delta_i$ then $\delta_i(\hat{h}) = 0$. Hence $\delta_i^2(\hat{f}) = 0$ but we also need $\delta_i(\hat{f}) \neq 0$ which is not necessarily true. Thus multiply $f$ by $\beta \in \mathbb{C}[V]$. Since $\beta \in \text{Ker} \delta_i$ we have $\delta_i(\hat{\beta f})(u) = \sum_{y \in \varphi^{-1}(u)} \beta(y) \delta_i(f)(y)$. Note that $\delta_i(f)(y_0)$ is not zero at a general $y_0 \in SL_2 \times V$ since $\delta_i(f) \neq 0$. By a standard application of the Nullstellensatz we can choose $\beta$ with prescribed values at the finite set $\varphi^{-1}(u_0)$ where $u_0 = \varphi(y_0)$. Hence we can assure that $\delta_i(\hat{\beta f})(u_0) \neq 0$, i.e. $\deg_{\delta_i}(\hat{\beta f}) = 1$. There is still one problem: $\hat{\beta f}$ is regular on $U$ but necessarily not on $X$. In order to fix it we set $g = \alpha \hat{\beta f}$ where $\alpha$ is a lift of a nonzero function on $X//G$ that vanishes with high multiplicity on $(X//G) \setminus (U//G)$. Since $\alpha \in \text{Ker} \delta_i$ we still have $\deg_{\delta_i}(g) = 1$ which concludes the proof in the case of a finite isotropy group.

For a one-dimensional isotropy group note that $f$ is $\mathbb{C}^*$-invariant with respect to the action of the diagonal subgroup of $SL_2$. That is, $f$ can be viewed as a function on $SL_2 \times_{\mathbb{C}^*} V$. Then we can replace morphism $\varphi$ with morphism $\psi : SL_2 \times_{\mathbb{C}^*} V \to U$ that factors through the étale morphism $SL_2 \times_{G_x} V \to U$. Now $\psi$ is also étale and the rest of the argument remains the same. \hfill \Box

In order to finish the proof of Theorem 4 we need to show semi-compatibility of vector fields $\delta_1$ and $\delta_2$ on $X$. Let $U$ be a saturated set as in diagram (2) with $G = SL_2$. Since $U$ is $SL_2$-invariant it is $H_i$-invariant (where $H_i$ is from Notation 3.1) and the restriction of $\delta_i$ to $U$ is a locally nilpotent vector field which we denote again by the same letter. Furthermore, the closure of any $SL_2$-orbit $O$ contains a closed orbit, i.e. $O$ is contained in an open set like $U$ and, therefore, $X$ can be covered by a finite collections of such open sets. Thus Proposition 2.11 implies the following.
3.5. **Lemma.** If for every \( U \) as before the locally nilpotent vector fields \( \delta_1 \) and \( \delta_2 \) are semi-compatible on \( U \) then they are semi-compatible on \( X \).

3.6. **Notation.** Suppose further that \( H_1 \) and \( H_2 \) act on \( SL_2 \times V \) by left multiplication on first factor. The locally nilpotent vector fields associated with these actions of \( H_1 \) and \( H_2 \) are, obviously, semi-compatible since they are compatible on \( SL_2 \) (see Example 2.11). Consider the \( SL_2 \)-equivariant morphism \( G \times V \rightarrow G \times_{G_x} V \) where \( V \), \( G = SL_2 \), and \( G_x \) are as in diagram (2). By definition \( G \times_{G_x} V \) is the quotient of \( G \times V \) with respect to the \( G_x \)-action whose restriction to the first factor is the multiplication from the right. Hence \( H_1 \) action commutes with \( G_x \)-action and, therefore, one has the induced \( H_1 \)-action on \( G \times_{G_x} V \). Following the pattern of Notation 3.1 we denote the associated locally nilpotent derivations on \( G \times_{G_x} V \) again by \( \delta_1 \) and \( \delta_2 \). That is, the \( SL_2 \)-equivariant étale morphism \( \varphi : G \times_{G_x} V \rightarrow U \) transforms vector field \( \delta_i \) on \( G \times_{G_x} V \) into vector field \( \delta_i \) on \( U \).

From Lemma 2.16 and Luna’s slice theorem we have immediately the following.

3.7. **Lemma.** (1) If the locally nilpotent vector fields \( \delta_1 \) and \( \delta_2 \) are semi-compatible on \( G \times_{G_x} V \) then they are semi-compatible on \( U \).

(2) Furthermore, if the isotropy group \( G_x \) is finite \( \delta_1 \) and \( \delta_2 \) are, indeed, semi-compatible on \( G \times_{G_x} V \).

Now we have to tackle semi-compatibility in the case of one-dimensional isotropy subgroup \( G_x \) using Proposition 2.12 as a main tool. We start with the case of \( G_x = \mathbb{C}^* \).

3.8. **Notation.** Consider the diagonal \( \mathbb{C}^* \)-subgroup of \( SL_2 \), i.e. elements of form

\[
\begin{pmatrix}
\lambda^{-1} & 0 \\
0 & \lambda
\end{pmatrix}.
\]

The action of \( s_\lambda \) on \( v \in V \) will be denoted by \( \lambda.v \). When we speak later about the \( \mathbb{C}^* \)-action on \( V \) we mean exactly this action. Set \( Y = SL_2 \times V \), \( Y' = SL_2 \times \mathbb{C}^* \), \( V \), \( Y_1 = Y'/H_1 \), \( Y_1' = Y'/H_1 \). Denote by \( \rho \) : \( Y \rightarrow Y_1 \) the quotient morphism of the \( H_1 \)-action and use the similar notation for \( Y' \), \( Y_1' \). Set \( \rho = (\rho_1, \rho_2) : Y \rightarrow Y_1 \times Y_2 \) and \( \rho' = (\rho_1', \rho_2') : Y' \rightarrow Y_1' \times Y_2' \).

Note that \( Y_i \simeq \mathbb{C}^2 \times V \) since \( SL_2/\!/\mathbb{C}^* \simeq \mathbb{C}^2 \). Furthermore, looking at the kernels of \( \delta_1 \) and \( \delta_2 \) from Example 2.11 we see for

\[
A = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \in SL_2
\]

the quotient maps \( SL_2 \rightarrow SL_2/\!/H_1 \simeq \mathbb{C}^2 \) and \( SL_2 \rightarrow SL_2/\!/H_2 \simeq \mathbb{C}^2 \) are given by \( A \mapsto (a_1, a_2) \) and \( A \mapsto (b_1, b_2) \) respectively. Hence morphism \( \rho : SL_2 \times V = Y \rightarrow Y_1 \times Y_2 \simeq \mathbb{C}^2 \times V \times V \) is given by

\[
(3) \quad \rho(a_1, a_2, b_1, b_2, v) = (a_1, a_2, b_1, b_2, v, v).
\]
As we mentioned before, to define $Y' = SL_2 \times_{C^*} V$ we let $C^*$ act on $SL_2$ via right multiplication. Since $H_1$ and $H_2$ act on $SL_2$ from the left, there are well-defined $C^*$-actions on $Y_1$ and on $Y_2$ and a torus $T$-action on $Y_1 \times Y_2$, where $T = C^* \times C^*$. Namely,

$$
(\lambda, \mu). (a_1, a_2, b_1, b_2, v, w) = (\lambda a_1, \lambda^{-1}a_2, \mu b_1, \mu^{-1}b_2, \lambda v, \mu w)
$$

for $(a_1, a_2, b_1, b_2, v, w) \in Y_1 \times Y_2$ and $(\lambda, \mu) \in T$.

Since the $C^*$-action on $Y$ and the action of $H_i$, $i = 1, 2$ are commutative, the following diagram is also commutative.

$$
\begin{array}{ccc}
Y & \xrightarrow{\rho} & Y_1 \times Y_2 \\
\downarrow p & & \downarrow q \\
Y' & \xrightarrow{\rho'} & Y'_1 \times Y'_2 
\end{array}
$$

where $q$ (resp. $p$) is the quotient map with respect to the $T$-action (resp. $C^*$-action). It is also worth mentioning that the $C^*$-action on $Y$ induces the action of the diagonal of $T$ on $\rho(Y)$, i.e. for every $y \in Y$ we have $\rho(\lambda y) = (\lambda, \lambda, \rho(y)$).

3.9. Lemma. Let $Z = \rho(Y)$ in diagram (5) and $Z'$ be the closure of $\rho'(Y')$.

(i) The map $\rho : Y \rightarrow Z$ is an isomorphism and $Z$ is the closed subvariety of $Y_1 \times Y_2 = C^4 \times V \times V$ that consists of points $(a_1, a_2, b_1, b_2, v, w) \in Y_1 \times Y_2$ satisfying the equations $a_1 b_2 - a_2 b_1 = 1$ and $v = w$.

(ii) Let $T$ be the $T$-orbit of $Z$ in $Y_1 \times Y_2$ and $\bar{T}$ be its closure. Then $T$ coincides with the $(C^* \times 1)$-orbit (resp. $(1 \times C^*)$-orbit) of $Z$. Furthermore, for each $(a_1, a_2, b_1, b_2, v, w) \in \bar{T}$ one has $\pi(v) = \pi(w)$ where $\pi : V \rightarrow V/\!/C^*$ is the quotient morphism.

(iii) The restriction of diagram (5) yields the following

$$
\begin{array}{ccc}
Y & \xrightarrow{\rho} & Z \subset \bar{T} \\
\downarrow p & & \downarrow q \\
Y' & \xrightarrow{\rho'} & q(Z) \subset Z'
\end{array}
$$

where $Y' = Y/\!/C^* = Y/\!/C^*$, $q$ is the quotient morphism of the $T$-action (i.e. $Z' = \bar{T}/\!/\mathbb{T}$), and $q(Z) = \rho'(Y')$.

Proof. The first statement is an immediate consequence of formula (3). The beginning of the second statement follows from the fact that the action of the diagonal $C^*$-subgroup of $T$ preserves $Z$. This implies that for every $t = (a_1, a_2, b_1, b_2, v, w) \in T$ points $v, w \in V$ belong to the same $C^*$-orbit and, in particular, $\pi(v) = \pi(w)$. This equality holds for each point in $\bar{T}$ by continuity.

In diagram (5) $Y' = Y/\!/C^* = Y/\!/C^*$ because of Proposition 2.2 (3) and Lemma 2.6 and the equality $q(Z) = \rho'(Y')$ is the consequence of the commutativity of that diagram. Note that $\bar{T}$ is $T$-invariant. Hence $q(\bar{T})$ coincides with $Z'$ by Proposition 2.2 (4). Being the restriction of the quotient morphism, $q|_{\bar{T}} : T \rightarrow Z'$ is a quotient morphism itself (e.g., see [S]) which concludes the proof.
3.10. **Lemma.** There is a rational \( T \)-quasi-invariant function \( f \) on \( \bar{T} \) such that for \( t = (a_1, a_2, b_1, b_2, w, v) \in T \) one has

1. \( \frac{1}{t_0} a_1 b_2 - f(t) a_2 b_1 = 1 \) and \( w = f(t).v \);
2. the set \( \bar{T} \setminus T \) is contained in \((f)_0 \cup (f)_\infty\); 
3. \( f \) generates a regular function on a normalization \( T_N \) of \( T \).

**Proof.** By Lemma 3.9 (ii) any point \( t = (a_1, a_2, b_1, b_2, w, v) \in T \) is of form \( t = (\lambda, 1).z_0 \) where \( z_0 \in Z \) and \( \lambda \in \mathbb{C}^* \). Hence formula (4) implies that \( w = \lambda.v \) and \( \lambda^{-1} a_1 b_2 - \lambda a_2 b_1 = 1 \). The last equality yields two possible values (one of which can be \( \infty \) or \( 0 \) if any of numbers \( a_1, a_2, b_1, \) or \( b_2 \) vanish)

\[
\lambda_\pm = \frac{-1 \pm \sqrt{1 + 4a_1 a_2 b_1 b_2}}{2a_2 b_1}
\]

and we assume that

\[
\lambda = \lambda_- = \frac{-1 - \sqrt{1 + 4a_1 a_2 b_1 b_2}}{2a_2 b_1},
\]

i.e. \( w = \lambda_- v \). Note that \( \lambda_+.v = w \) as well only when

\[
\tau = \frac{\lambda_+}{\lambda_-} = \frac{-1 + \sqrt{1 + 4a_1 a_2 b_1 b_2}}{-1 - \sqrt{1 + 4a_1 a_2 b_1 b_2}}
\]

is in the isotropy group of \( v \).

Consider the set of points \( t \in T \) such that \( v \) is not a fixed point of the \( \mathbb{C}^* \)-action on \( V \) and \( \tau.v = v \). Denote its closure by \( S \). Since \( S \) is a proper subvariety of \( T \), one has a well-defined branch \( \lambda_- \) of the two-valued function \( \lambda_\pm \) on the complement to \( S \). Its extension to \( \bar{T} \), which is denoted by \( f \), satisfies (1).

Let \( t_n \in T \) and \( t_n \to t \in \bar{T} \) as \( n \to \infty \). By Lemma 3.9 (ii) \( t_n \) is of form \( t_n = (f(t_n) a_1^n, \frac{1}{f(t_n)} a_2^n, b_1^n, b_2^n, f(t_n).v_n, v_n) \) where

\[
\begin{pmatrix}
a_1^n & a_2^n \\
b_1^n & b_2^n
\end{pmatrix} \in SL_2 \text{ and } v = \lim_{n \to \infty} v_n.
\]

If sequences \( \{f(t_n)\} \) and \( \{1/f(t_n)\} \) are bounded then switching to a subsequence one can suppose that \( f(t_n) \to f(t) \in \mathbb{C}^* \), \( w = f(t)v \), and \( t = (f(t)a_1', \frac{1}{f(t)} a_2', b_1', b_2', f(t).v, v) \) where

\[
\begin{pmatrix}
a_1' & a_2' \\
b_1' & b_2'
\end{pmatrix} \in SL_2,
\]

i.e. \( t \in T \). Hence \( \bar{T} \setminus T \) is contained in \((f)_0 \cup (f)_\infty\) which is (2).

Function \( f \) is regular on \( T \setminus S \) by construction. Consider \( t \in S \) with \( w \) and \( v \) in the same non-constant \( \mathbb{C}^* \)-orbit, i.e. \( w = \lambda.v \) for some \( \lambda \in \mathbb{C}^* \). Then \( w = \lambda'.v \) if and only if only \( \lambda' \) belongs to the coset \( \Gamma \) of the isotropy subgroup of \( v \) in \( \mathbb{C}^* \). For any sequence of points \( t_n \) convergent to \( t \) one can check that \( f(t_n) \to \lambda \in \Gamma \) by continuity, i.e. \( f \) is bounded in a neighborhood of \( t \). Let \( \nu : T_N \to T \) be a normalization morphism. Then function \( f \circ \nu \) extends regularly to \( \nu^{-1}(t) \) by the Riemann extension theorem. The set
of point of \( S \) for which \( v \) is a fixed point of the \( \mathbb{C}^* \)-action is of codimension at least 2 in \( T \). By the Hartogs’ theorem \( f \circ \nu \) extends regularly to \( T_N \) which concludes (3).

\[ \square \]

3.11. **Remark.** Consider the rational map \( \kappa : T \to Z \) given by \( t \mapsto (\frac{1}{f(t)}, 1).t \). It is regular on \( T \setminus S \) and if \( t \in T \setminus S \) and \( z \in Z \) are such that \( t = (\lambda, 1).z \) then \( \kappa(t) = z \). In particular \( \kappa \) sends \( T \)-orbits from \( T \) into \( \mathbb{C}^* \)-orbits of \( Z \). Furthermore, morphism \( \kappa_N = \kappa \circ \nu : T_N \to Z \) is regular by the same reason as function \( f \circ \nu \).

3.12. **Lemma.** Let \( E_i = \{ t = (a_1, a_2, b_1, b_2, w, v) \in T | b_i = 0 \} \) and \( \bar{T}^b \) coincide with \( T \cap (f(0) \setminus E_2) \cup (f(\infty) \setminus E_1) \). Suppose that \( \bar{T}^b \) is a normalization of \( \bar{T}^b \). Then there is a regular extension of \( \kappa_N : T_N \to Z \) to a morphism \( \bar{k}_N : \bar{T}^b \to Z \).

**Proof.** Since the set \( (f)_0 \cap (f)_{\infty} \) is of codimension 2 in \( \bar{T} \), the Hartogs’ theorem implies that it suffices to prove the regularity of \( k_N \) on the normalization of \( \bar{T} \setminus ((f)_0 \cap (f)_{\infty}) \). Furthermore, by the Riemann extension theorem it is enough to construct a continuous extension of \( \kappa \) from \( T \setminus S \) to \( T \setminus ((f)_0 \cap (f)_{\infty}) \)

By Lemma 3.10 (2) we need to consider this extension, say, at \( t = (a_1, a_2, b_1, b_2, w, v) \in (f)_0 \setminus (f)_{\infty} \). Let \( t_n \to t \) as \( n \to \infty \) where

\[
 t_n = (f(t_n)a_1^n, 1 - f(t_n)a_2^n, b_1^n, b_2^n, f(t_n).v_n, v_n) \in T
\]

with \( a_1^n b_2^n - a_2^n b_1^n = 1 \) and \( f(t_n) \to 0 \). Perturbing, if necessary, this sequence \( \{ t_n \} \) we can suppose every \( t_n \notin S \), i.e. \( \kappa(t_n) = (a_1^n, a_2^n, b_1^n, b_2^n, v_n, v_n) \). Note that \( \lim v_n = v, b_k = \lim b_k^n, k = 1, 2 \) and \( a_2^n \to 0 \) since \( a_2 \) is finite. Hence \( 1 = a_1^n b_2^n - a_2^n b_1^n \approx a_1^n b_2^n \) and \( a_1^n \to 1/b_2 \) as \( n \to \infty \). Now we get a continuous extension of \( \kappa \) by putting \( \kappa(t) = (1/b_2, 0, b_1, b_2, v, v) \). This yields the desired conclusion.

\[ \square \]

3.13. **Remark.** If we use the group \((1 \times \mathbb{C}^*)\) instead of the group \((\mathbb{C}^* \times 1)\) from Lemma 3.9 (ii) in our construction this would lead to the replacement of \( f \) by \( f^{-1} \). Furthermore for the variety \( \bar{T}^a = T \cup ((f)_0 \setminus \{ a_1 = 0 \}) \cup ((f)_{\infty} \setminus \{ a_2 = 0 \}) \) we obtain a morphism \( \bar{k}_N : \bar{T}^a_N \to Z \) similar to \( \bar{k}_N \).

The next fact is intuitively obvious but requires some work.

3.14. **Lemma.** The complement \( \bar{T}^0 \) of \( \bar{T}^a \cup \bar{T}^b \) in \( \bar{T} \) (which is \( \bar{T}^0 = (\bar{T} \setminus T) \cap \bigcup_{i \neq j} \{ a_i = b_j = 0 \} \) has codimension at least 2.

**Proof.** Let \( t_n \to t = (a_1, a_2, b_1, b_2, w, v) \) be as in the proof of Lemma 3.12. Since for a general point of the slice \( V \) the isotropy group is finite after perturbation we can suppose that each \( v_n \) is contained in a non-constant \( \mathbb{C}^* \)-orbit \( O_n \subset V \). Treat \( v_n \) and \( f(t_n).v_n \) as numbers in \( \mathbb{C}^b \cong O_n \) such that \( f(t_n).v_n = f(t_n)v_n \). Let \( |v_n| \) and \( |f(t_n).v_n| \) be their absolute values. Then one has the annulus \( A_n = \{ |f(t_n).v_n| < \zeta < |v_n| \} \subset O_n \), i.e. \( \zeta = \eta v_n \) where \( |f(t_n)| < |\eta| < 1 \) for each \( \zeta \in A_n \). By Lemma 3.9 (iii) \( \pi(v) = \pi(w) \) but by Lemma 3.10 (3) the \( \mathbb{C}^* \)-orbit \( O(v) \) and \( O(w) \) are different unless \( w = v \) is a
fixed point of the $\mathbb{C}^*$-action. In any case, by Proposition 2.2 (2) the closures of these orbits meet at a fixed point $\bar{v}$ of the $\mathbb{C}^*$-action.

Consider a compact neighborhood $W = \{ u \in V | \varphi(u) \leq 1 \}$ of $\bar{v}$ in $V$ where $\varphi$ is a plurisubharmonic function on $V$ that vanishes at $\bar{v}$ only. Note that the sequence $\{(\lambda, \mu).t_n\}$ is convergent to $(\lambda a_1, a_2/\lambda, \mu b_1, b_2/\mu, \lambda . w, \mu . v)$. In particular, replacing $\{t_n\}$ by $\{(\lambda, \mu).t_n\}$ with appropriate $\lambda$ and $\mu$ we can suppose that the boundary $\partial A_n$ of any annulus $A_n$ is contained in $W$ for sufficiently large $n$. By the maximum principle $A_n \subset W$. The limit $A = \lim_{n \to \infty} A_n$ is a compact subset of $W$ that contains both $v$ and $w$, and also all points $\eta.v$ with $0 < |\eta| < 1$ (since $|f(t_n)| \to 0$). Unless $O(v) = \bar{v}$ only one of the closures of sets $\{\eta.v| 0 < |\eta| < 1 \}$ or $\{\eta.v| |\eta| > 1 \}$ in $V$ is compact and contains the fixed point $\bar{v}$ (indeed, otherwise the closure of $O(v)$ is a complete curve in the affine variety $V$). The argument before shows that it is the first one.

That is, $\mu . v \to \bar{v}$ when $\mu \to 0$. Similarly, $\lambda . w \to \bar{v}$ when $\lambda \to \infty$. It is not difficult to check now that the dimension of the set of such pairs $(w, v)$ is at most $\dim V$.

Consider the set $(\bar{T} \setminus T) \cap \{a_1 = b_2 = 0\}$. It consists of points $t = (0, a_2, b_1, 0, w, v)$ and, therefore, its dimension, is at most $\dim V + 2$. Thus it has codimension at least 2 in $\bar{T}$ whose dimension is $\dim V + 4$. This yields the desired conclusion.

□

The next technical fact may be somewhere in the literature, but unfortunately we did not find a reference.

3.15. Proposition. Let a reductive group $G$ act on an affine algebraic variety $X$ and $\pi : X \to Q := X//G$ be the quotient morphism such that one of closed $G$-orbits $O$ is contained in the smooth part of $X$. Suppose that $\nu : X_N \to X$ and $\mu : Q_N \to Q$ are normalization morphisms, i.e. $\pi \circ \nu = \pi_N \circ \mu$ for some morphism $\pi_N : X_N \to Q_N$. Then $Q_N \simeq X_N//G$ for the induced $G$-action on $X_N$ and $\pi_N$ is the quotient morphism.

Proof. Let $\psi : X_N \to R$ be the quotient morphism. By the universal property of quotient morphisms $\pi_N = \varphi \circ \psi$ where $\varphi : R \to Q_N$ is a morphism. It suffices to show that $\varphi$ is an isomorphism. The points of $Q$ (resp. $R$) are nothing but the closed $G$-orbits in $X$ (resp. $X_N$) by Proposition 2.2 and above each closed orbit in $X$ we have only a finite number of closed orbits in $X_N$ because $\nu$ is finite. Hence $\mu \circ \varphi : R \to Q$ and, therefore, $\varphi : R \to Q_N$ are at least quasi-finite. There is only one closed orbit $O_N$ in $X_N$ above orbit $O \subset \text{reg} X$. Thus $\varphi$ is injective in a neighborhood of $\psi(O_N)$. That is, $\varphi$ is birational and by the Zariski Main theorem it is an embedding.

It remains to show that $\varphi$ is proper. Recall that $G$ is a complexification of a its compact subgroup $G^R$ and there is a so-called Kempf-Ness real algebraic subvariety $X_R$ of $X$ such that the restriction $\pi|_{X_R}$ is nothing but the standard quotient map $X^R \to X^R/G^R = Q$ which is automatically proper (e.g., see [21]). Set $X_R^N = \nu^{-1}(X^R)$. Then the restriction of $\pi \circ \nu$ to $X_R^N$ is proper being the composition of two proper maps. On the other hand the restriction of $\mu \circ \pi_N = \pi \circ \nu$ to $X_R^N$ is proper only when morphism $\varphi$, through which it factors, is proper which concludes the proof.

□
3.16. Proposition. Morphism $\rho' : Y' \to Z'$ from diagram $\Box$ is finite birational.

Proof. Morphism $\rho'$ factors through $\rho'_N : Y' \to Z'_N$ where $\mu : Z'_N \to Z'$ is a normalization of $Z'$ and the statement of the proposition is equivalent to the fact that $\rho'_N$ is an isomorphism. Set $Z'(b) = q(T^b)$ and $Z'(a) = q(T^a)$. Note that $Z' \setminus (Z'(a) \cup Z'(b))$ is in the $q$-image of the $T$-invariant set $T^0$ from Lemma 3.14. Hence $Z' \setminus (Z'(b) \cup Z'(a))$ is of codimension 2 in $Z'$ and by the Hartogs’ theorem it suffices to prove that $\rho'_N$ is invertible over $Z(b)'$ (resp. $Z'(a)$).

By Remark 3.11 $\tilde{\kappa}^b_N$ sends each orbit of the induced $T$-action on $\tilde{T}_N^b$ onto a $C^*$-orbit in $Z$. Thus the composition of $\tilde{\kappa}^b_N$ with $p : Z \simeq Y \to Y'$ is constant on $T$-orbits and by the universal property of quotient spaces it must factor through the quotient morphism $q^b_N : \tilde{T}_N^b \to Q$. By Proposition 3.15 $Q = Z'_N(b)$ where $Z'_N(b) = \mu^{-1}(Z'(b))$. That is, $p \circ \tilde{\kappa}^b_N = \tau^b \circ q^b_N$ where $\tau^b : Z'_N(b) \to Y'$. Our construction implies that $\tau^b$ is the inverse of $\rho'_N$ over $Z'_N(b)$. Hence $\rho'_N$ is invertible over $Z'_N(b)$ which concludes the proof. 

3.17. Proof of Theorems 4 and 3. Let $G = SL_2$ act algebraically on $X$ as in Theorem 4 and $V$ be the slice of this action at point $x \in X$ so that there is an étale morphism $G \times_{G_x} V \to U$ as in Theorem 4. By Lemmas 3.5 and 3.7 for validity of Theorem 4 it suffices to prove semi-compatibility of vector fields $\delta_1$ and $\delta_2$ on $\mathcal{Y} = G \times_{G_x} V$, which was already done in the case of a finite isotropy group $G_x$ (see Lemma 3.7 (2)). Consider the quotient morphisms $\varrho_i : \mathcal{Y} \to \mathcal{Y}_i := \mathcal{Y} / H_i$ where $H_i$, $i = 1, 2$ are as in Example 2.11. Set $\varrho = (\varrho_1, \varrho_2) : \mathcal{Y} \to \varrho(\mathcal{Y}) \subset \mathcal{Y}_1 \times \mathcal{Y}_2$. By Proposition 2.12 Theorem 4 is true if $\varrho$ is finite birational. If $G_x = C^*$ then $\varrho : \mathcal{Y} \to \mathcal{Y}_1 \times \mathcal{Y}_2$ is nothing but morphism $\rho' : Y' \to Y'_1 \times Y'_2$ from Proposition 3.16, i.e. we are done in this case as well. By Lemma 3.3 the only remaining case is when $G_x$ is an extension of $C^*$ by $\mathbb{Z}_2$. Then one has a $\mathbb{Z}_2$-action on $Y'$ such that it is commutative with $H_i$-actions on $Y'$ and $\mathcal{Y} = Y'/\mathbb{Z}_2$. Since vector fields $\delta_1$ and $\delta_2$ are semi-compatible on $Y'$ by Propositions 3.16 and 2.12 they generate also semi-compatible vector fields on $\mathcal{Y}$ by Lemma 2.15. This concludes Theorem 4 and, therefore, Theorem 3.

3.18. Remark. (1) Consider $\mathcal{Y} = G \times_{G_x} V$ in the case when $G = G_x = SL_2$, i.e. the $SL_2$-action has a fixed point. It is not difficult to show that morphism $\varrho = (\varrho_1, \varrho_2) : \mathcal{Y} \to \varrho(\mathcal{Y}) \subset \mathcal{Y}_1 \times \mathcal{Y}_2$ as in the proof before is not quasi-finite. In particular, $\delta_1$ and $\delta_2$ are not compatible. However, we do not know if the condition about the absence of fixed points is essential for Theorem 3. In examples we know the presence of fixed points is not an obstacle for the algebraic density property. Say, for $\mathbb{C}^n$ with $2 \leq n \leq 4$ any algebraic $SL_2$-action is a representation in a suitable polynomial coordinate system (see, [22]) and, therefore, has a fixed point; but the validity of the algebraic density property is a consequence the Andersén-Lempert work.
(2) The simplest case of a degenerate \( SL_2 \)-action is presented by the homogeneous space \( SL_2/\mathbb{C}^* \) where \( \mathbb{C}^* \) is the diagonal subgroup. Let
\[
A = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}
\]
be a general element of \( SL_2 \). Then the ring of invariants of the \( \mathbb{C}^* \)-action is generated by \( u = a_1 a_2, v = b_1 b_2 \), and \( z = a_2 b_1 + 1/2 \) (since \( a_1 b_2 = 1 + a_2 b_1 = 1/2 + z \)). Hence \( SL_2/\mathbb{C}^* \) is isomorphic to a hypersurface \( S \) in \( \mathbb{C}^3_{u,v,z} \) given by the equation \( uv = z^2 - 1/4 \). In particular, it has the algebraic density property by [15].

(3) However, the situation is more complicated if we consider the normalizer \( T \) of the diagonal \( \mathbb{C}^* \)-subgroup of \( SL_2 \) (i.e. \( T \) is an extension of \( \mathbb{C}^* \) by \( \mathbb{Z}_2 \)). Then \( \mathcal{P} = SL_2/T \) is isomorphic to \( S/\mathbb{Z}_2 \) where the \( \mathbb{Z}_2 \)-action is given by \( (u,v,z) \rightarrow (-u,-v,-z) \). It can be shown that this surface \( \mathcal{P} \) is the only \( \mathbb{Q} \)-homology plane which is simultaneously a Danilov-Gizatullin surface (i.e. it has a trivial Makar-Limanov invariant (see [10])), and its fundamental group is \( \mathbb{Z}_2 \). We doubt that \( \mathcal{P} \) has algebraic density property.

4. Applications

Theorem [3] is applicable to a wide class of homogeneous spaces. Let us start with the following observation: given a reductive subgroup \( R \) of a linear algebraic group \( G \) any \( SL_2 \)-subgroup \( \Gamma < G \) yields a natural \( \Gamma \)-action on \( G/R \). Furthermore, for each point \( aR \in G/R \) its isotropy subgroup under this action is isomorphic to \( \Gamma \cap aRa^{-1} \). In particular, the action has no fixed points if \( a^{-1} \Gamma a \) is not contained in \( R \) for any \( a \in G \) and it is non-degenerate if \( \Gamma a := a^{-1} \Gamma a \cap R \cong \Gamma \cap aRa^{-1} \) is finite for some \( a \in R \). Thus Theorem [3] implies the following.

4.1. Proposition. Let \( G \) be an algebraic group and \( \Gamma_1, \ldots, \Gamma_k \) be its \( SL_2 \)-subgroups such that at some \( x \in G \) the set \( \{ \delta_i^a(x) \} \) is a generating one (where \( \delta_i^a, \delta_2^a \) is the corresponding pair of locally nilpotent vector fields on \( G \) generated by the natural \( \Gamma_i \)-action). Suppose that for each \( i = 1, \ldots, k \) and any \( a \in G \) the group \( \Gamma_i^a := a^{-1} \Gamma_i a \) is not isomorphic to \( \Gamma_i \), and furthermore \( \Gamma_i^a \) is finite for some \( a \). Then \( G/R \) has the algebraic density property.

Note that for a simple Lie group \( G \) a generating set at any \( x \in G \) consists of one nonzero vector since the adjoint representation is irreducible. Therefore, in this case the algebraic density property is a consequence of the following.

Theorem 5. Let \( G \) be a simple Lie group with Lie algebra different from \( \mathfrak{sl}_2 \) and \( R \) be its proper reductive subgroup. Then there exists an \( SL_2 \)-subgroup \( \Gamma \) in \( G \) such that \( \Gamma^a \) is not isomorphic to \( \Gamma \) for any \( a \in G \), and, furthermore, \( \Gamma^a \) is finite for some \( a \in G \).

Surprisingly enough the proof of this Theorem (at least in our presentation) requires some serious facts from Lie group theory and we shall postpone it till the Appendix.

4.2. Corollary. Let \( X = G/R \) be an affine homogeneous space of a semi-simple Lie group \( G \). Suppose that \( X \) is different from a \( \mathbb{Q} \)-homology plane with \( \mathbb{Z}_2 \) as a fundamental
group. Then $X$ is equipped with $N$ pairs $\{\delta_k^1, \delta_k^2\}_{k=1}^N$ of compatible derivations such that the collection $\{\delta_k^1(x_0)\}_{k=1}^N \subset T_{x_0}X$ is a generating set at some point $x_0 \in X$. In particular, $X$ has the algebraic density property by Theorem 2.

Proof. Note that $R$ is reductive by Proposition 2.3 (Matsushima’s theorem). Then $X$ is isomorphic to a quotient of form $G/R$ where $G = G_1 \oplus \ldots \oplus G_N$, each $G_i$ is a simple Lie group and $R$ is not necessarily connected. However, we can suppose that $R$ is connected by virtue of Proposition 2.1. Consider the projection homomorphism $\pi_k : G \to G_k$ and $R_k = \pi_k(R)$ which is reductive being the image of a reductive group. If $N$ is minimal possible then $R_k \neq G_k$ for every $k$. Indeed, if say $R_N = G_N$ then $X = (G_1 \oplus \ldots \oplus G_{N-1})/R$ where $R = \text{Ker} \pi_N$ which contradicts minimality.

Assume first that none of $G_i$’s is isomorphic to $SL_2$. By Theorem 3 one can choose an $SL_2$-subgroup $\Gamma_k < G_k$ such that the natural $\Gamma_k$-action on $G_k/\pi_k(R)$ and, therefore, on $G/R$ is fixed point free. We can also assume that each $\Gamma_k$-action is non-degenerate. Denote by $\delta_1^k$ and $\delta_2^k$ the corresponding pair of locally nilpotent derivations for the $\Gamma_k$-action. Since the adjoint representation is irreducible for a simple Lie group, $\{\delta_2^k(e)\}_{k=1}^N$ is a generating set of the tangent space $T_{e}G$ at $e = e_1 \oplus \ldots \oplus e_N \in G$, where $e_k$ is a unit of $G_k$. Consider $X = G/R$ as the set of left cosets, i.e. $X$ is the quotient of $G$ with respect to the action generated by multiplication by elements of $R$ from the right. Hence this action commutes with multiplication by elements of $\Gamma_k$ from the left, and, therefore, it commutes with any field $\delta_1^k$. Pushing the actions of $\Gamma_k$’s to $X$ we get fixed point free non-degenerate $SL_2$-actions on $X$ and the desired conclusion in this case follows from by Theorem 3.

In the case when some of $G_k$’s are isomorphic to $SL_2$ we cannot assume that each $\Gamma_k$-action is non-degenerate, but now $N \geq 2$ and the $\Gamma_k$-actions are still fixed point free. Consider an isomorphism $\varphi_k : \Gamma_k \to \Gamma_1$. Then we have an $SL_2$-group $\Gamma^{\varphi_k} = \{(\varphi_k(\gamma), \gamma) | \gamma \in \Gamma_k\} < \Gamma_1 \times \Gamma_k$ acting naturally on $G_1 \times G_k$ and, therefore, on $G$. This isomorphism $\varphi_k$ can be chosen so that the $\Gamma^{\varphi_k}$-action is non-degenerate. Indeed, if, say, a $C^\infty$-subgroup $L < \Gamma_k$ acts on $G_k$ trivially choose $\varphi_k$ so that $\varphi_k(L)$ acts nontrivially on $G_1$ which makes the action non-degenerate. In particular, by Theorem 4 we get pairs of compatible locally nilpotent derivations $\tilde{\delta}_1^{\varphi_k}$ and $\tilde{\delta}_2^{\varphi_k}$ corresponding to such actions. Set $G' = G_2 \oplus \ldots \oplus G_N$ and $e' = e_2 \oplus \ldots \oplus e_N \in G'$. Since the adjoint representation is irreducible for a simple Lie group the orbit of the set $\{\tilde{\delta}_2^{\varphi_k}(e)\}_{k=2}^N$ under conjugations generates a subspace of $S$ of $T_{e}G$ such that the restriction of the natural projection $T_{e}G \to T_{e'}G'$ to $S$ is surjective. In order to enlarge $\{\tilde{\delta}_2^{\varphi_k}(e)\}_{k=2}^N$ to a generating subset of $T_{e}G$ consider an isomorphism $\psi_2 : \Gamma_2 \to \Gamma_1$ different from $\varphi_2$ and such that the $\Gamma^{\psi_2}$-action is non-degenerate. Denote the corresponding compatible locally nilpotent derivations by $\tilde{\delta}_1^{\psi_2}$ and $\tilde{\delta}_2^{\psi_2}$ on $G_1 \oplus G_2$ (and also by abusing notation on $G$). Note that the vectors $\tilde{\delta}_2^{\psi_2}(e_1 \oplus e_2)$ and $\tilde{\delta}_2^{\psi_2}(e_1 \oplus e_2)$ can be assumed different with an appropriate choice of $\psi_2$. Hence these two vectors form a generating subset of $T_{e_1 \oplus e_2}G_1 \oplus G_2$. Taking into consideration the remark about $S$ we see that $\{\tilde{\delta}_2^{\varphi_k}(e)\}_{k=2}^N \cup \{\tilde{\delta}_2^{\psi_2}(e)\}$ is
a generating subset of $T_xG$. Now pushing these $SL_2$-actions to $X$ we get the desired conclusion.

\[\square\]

**Theorem 6.** Let $G$ be a linear algebraic group and $R$ be its proper reductive subgroup such that the homogeneous space $G/R$ is different from $\mathbb{C}_+$, a torus, or the $\mathbb{Q}$-homology plane with fundamental group $\mathbb{Z}_2$ (i.e. the surface $\mathcal{P}$ from Remark 3.18 (3)). Then $G/R$ has the algebraic density property.

**Proof.** Since all components of $G/R$ are isomorphic as varieties we can suppose that $G$ is connected. Furthermore, by Corollary 4.2 and Remark 3.18 (2) we are done with a semi-simple $G$.

Let us consider first the case of a reductive but not semi-simple $G$. Then the center $Z \simeq (\mathbb{C}^*)^n$ of $G$ is nontrivial. Let $S$ be the semi-simple part of $G$. Assume for the time being that $G$ is isomorphic as group to the direct product $S \times Z$ and consider the natural projection $\tau : G \to Z$. Set $Z' = \tau(R) = R/R'$ where $R' = R \cap S$. Since we are going to work with compatible vector fields we can suppose that $R$ is connected by virtue of Lemma 2.15. Then $Z'$ is a subtorus of $Z$ and also $R'$ is reductive by Proposition 2.3.

Hence $G/R = (G/R')/Z'$ and $G/R' = S/R' \times Z$. Note that there is a subtorus $Z''$ of $Z$ such that $Z'' \simeq Z/Z'$ and $Z' \cdot Z'' = Z$. (Indeed, $Z' \simeq (\mathbb{C}^*)^k$ generates a sublattice $L \simeq \mathbb{Z}^k$ of homomorphisms from $\mathbb{C}^*$ into $Z'$ of the similar lattice $Z^n$ of $Z \simeq (\mathbb{C}^*)^n$ such that the quotient $Z^n/L$ has no torsion, i.e. it is isomorphic to $\mathbb{Z}^n$. Since any short exact sequence of free $\mathbb{Z}$-modules splits we have a $\mathbb{Z}$-submodule $K \simeq \mathbb{Z}^{n-k}$ in $\mathbb{Z}^n$ such that $K + L = \mathbb{Z}^n$. This lattice $K$ yields a desired subtorus $Z'' \simeq (\mathbb{C}^*)^{n-k}$.) Hence $G/R$ is isomorphic to $\varrho^{-1}(Z'') \simeq S/R' \times Z''$ where $\varrho : G/R' \to Z$ is the natural projection. Note that both factors are nontrivial since otherwise $G/R$ is either a torus or we are in the semi-simple case again. Thus $X$ has the algebraic density property by Proposition 2.14 with $S/R'$ playing the role of $X_1$ and $Z''$ of $X_2$. In particular, we have a finite set of pairs of compatible vector fields $\{\delta_1^k, \delta_2^k\}$ as in Theorem 2. Furthermore, one can suppose that the fields $\delta_1^k$ correspond to one parameter subgroups of $S$ isomorphic to $\mathbb{C}_+$ and $\delta_2^k$ to one parameter subgroups of $Z$ isomorphic to $\mathbb{C}^*$. In the general case $G/R$ is the factor of $X$ with respect to the natural action of a finite (central) normal subgroup $F < G$. Since $F$ is central the fields $\delta_1^k, \delta_2^k$ induce completely integrable vector fields $\tilde{\delta}_1^k, \tilde{\delta}_2^k$ on $G/R$ while $\tilde{\delta}_2^k(x_0)$ is a generating set for some $x_0 \in G/R$. By Lemma 2.15 the pairs $\{\tilde{\delta}_1^k, \tilde{\delta}_2^k\}$ are compatible and the density property for $G/R$ follows again from Theorem 2.

In the case of a general linear algebraic group $G$ different from a reductive group, $\mathbb{C}^n$, or a torus $(\mathbb{C}^*)^n$ consider the nontrivial unipotent radical $\mathcal{R}_u$ of $G$. It is automatically an algebraic subgroup of $G$ ([6], p. 183). By Mostow’s theorem 20 (see also [6], p. 181) $G$ contains a (Levi) maximal closed reductive algebraic subgroup $G_0$ such that $G$ is the semi-direct product of $G_0$ and $\mathcal{R}_u$, i.e. $G$ is isomorphic as affine variety to the product $\mathcal{R}_u \times G_0$. Furthermore, any other maximal reductive subgroup is conjugated to $G_0$. Hence, replacing $G_0$ by its conjugate, we can suppose that $\mathcal{R}_u$ is contained in $G_0$. [ALGEBRAIC DENSITY PROPERTY OF HOMOGENEOUS SPACES 17]
Therefore $G/R$ is isomorphic as an affine algebraic variety to the $G_0/R \times R_u$ and we are done now by Proposition 2.14 with $R_u$ playing the role of $X_1$ and $G_0/R$ of $X_2$. □

4.3. Remark. (1) The algebraic density property implies, in particular, that the Lie algebra generated by completely integrable algebraic (and, therefore, holomorphic) vector fields is infinite-dimensional, i.e. this is true for homogeneous spaces from Theorem 6. For Stein manifolds of dimension at least two that are homogeneous spaces of holomorphic actions of a connected complex Lie groups the infinite dimensionality of such algebras was also established by Huckleberry and Isaev [14].

(2) Note that as in [16] we proved actually a stronger fact for a homogeneous space $X = G/R$ from Theorem 6. Namely, it follows from the construction that the Lie algebra generated by vector fields of form $f\sigma$, where $\sigma$ is either locally nilpotent or semi-simple and $f \in \text{Ker}\sigma$ for semi-simple $\sigma$ and $\deg_{\sigma} f \leq 1$ in the locally nilpotent case, coincides with $\text{AVF}(X)$.

5. Appendix: The proof of Theorem 6

Let us start with the following technical fact.

5.1. Proposition. Let $R$ be a semi-simple subgroup of a semi-simple group $G$. Suppose that the number of orbits of nilpotent elements in the Lie algebra $\mathfrak{r}$ of $R$ under the adjoint action is less than the number of orbits of nilpotent elements in the Lie algebra of $G$ under the adjoint action. Then $G$ contains an $SL_2$-subgroup $\Gamma$ such that $\Gamma g := g^{-1}\Gamma g \cap R$ is different from $g^{-1}\Gamma g$ for any $g \in G$.

Proof. By the Jacobson-Morozov theorem ([5], Chap. 8.11.2, Proposition 2 and Corollary) for any semi-simple group $G$ there is a bijection between the set of $G$-conjugacy classes of $sl_2$-triples and the set of $G$-conjugacy classes of nonzero nilpotent elements from $G$ which implies the desired conclusion. □

In order to exploit Proposition 5.1 we need to remind some terminology and results from [5].

5.2. Definition. (1) Recall that a semi-simple element $h$ of a Lie algebra is regular, if the kernel of its adjoint action is a Cartan subalgebra. An $sl_2$-subalgebra of the Lie algebra $\mathfrak{g}$ of a semi-simple group $G$ is called principal if in its triple of standard generators the semi-simple element $h$ is regular and the adjoint action of $h$ has even eigenvalues (see Definition 3 in [5] Chapter 8.11.4). The subgroup generated by this subalgebra is called a principal $SL_2$-subgroup of $G$. As an example of such a principal subgroup one can consider an $SL_2$-subgroup of $SL_n$ that acts irreducibly on the natural representation space $\mathbb{C}^n$. In general, principal $sl_2$-subalgebras exist in any semi-simple Lie algebra $\mathfrak{g}$ (see Proposition 8 in [5] Chapter 8.11.4). Any two principal $SL_2$-subgroups are conjugated (see Proposition 6 in [5] Chapter 8.11.3 and Proposition 9 in [5] Chapter 8.11.4).
A connected closed subgroup $P$ of $G$ is called principal if it contains a principal $SL_2$-subgroup. A rank of $P$ is the rank of the maximal torus it contains. If this rank is 1 then $P$ coincides with its principal $SL_2$-subgroup (see Exercise 21 for Chapter 9.4).

5.3. Proposition. Let $R$ be a proper reductive subgroup of a simple group $G$ different from $SL_2$ or $PSL_2$. Then there exists an $SL_2$-subgroup $\Gamma$ of $G$ such that $\Gamma^g := g^{-1}\Gamma g \cap R$ is different from $g^{-1}\Gamma g$ for any $g \in G$.

Proof. If $R$ is not principal it cannot contain a principal $SL_2$-subgroup and we are done. Thus it suffices to consider the case of principal subgroup $R$ only.

Suppose first that $R$ is of rank 1. If $R$ contains $g^{-1}\Gamma g$ it must coincide with this subgroup by the dimension argument. Hence it suffices to choose non-principal $\Gamma$ to see the validity of the Proposition in this case.

Suppose now that $R$ is of rank at least 2. Then there are the following possibilities (Exercise 20c-e for Chapter 9.4):

1. $R$ is of type $B_2$ and $G$ is of type $A_3$ or $A_4$;
2. $R$ is of type $G_2$ and $G$ is of type $B_3$, $D_4$, or $A_6$;
3. $G$ is of type $A_{2l}$ with $l \geq 3$ and $R$ is of type $B_l$;
4. $G$ is of type $A_{2l-1}$ with $l \geq 3$ and $R$ is of type $C_l$;
5. $G$ is of type $D_l$ with $l \geq 4$ and $R$ is of type $B_{l-1}$;
6. $G$ is of type $E_6$ and $R$ is of type $F_4$.

In order to apply Proposition 5.1 to these cases we need the Dynkin classification of nilpotent orbits (with Elkington’s corrections) as described in the Bala-Carter paper page 6-7).

By this classification the number $a_n$ of such orbits in a simple Lie algebra of type $A_n$ coincides with the number of partitions $\lambda$ of $n+1$, i.e. $\lambda = (\lambda_1, \ldots, \lambda_k)$ with natural $\lambda_i$ such that $|\lambda| = \lambda_1 + \cdots + \lambda_k = n+1$.

For a simple Lie algebra of type $B_{2n}$ the number $b_n$ of nilpotent orbits coincides with number of partitions $\lambda$ and $\mu$ such that $2|\lambda| + |\mu| = 2m + 1$ where $\mu$ is a partition with distinct odd parts.

For a simple Lie algebra of type $C_m$ the number $c_n$ of nilpotent orbits coincides with number of partitions $\lambda$ and $\mu$ such that $|\lambda| + |\mu| = m$ where $\mu$ is a partition with distinct parts.

For a simple Lie algebra of type $D_m$ the number $d_n$ of nilpotent orbits coincides with number of partitions $\lambda$ and $\mu$ such that $2|\lambda| + |\mu| = 2m$ where $\mu$ is a partition with distinct odd parts.

The number of nilpotent orbits of algebras of type $G_2, F_4, E_6, E_7, E_8$ is 5,16, 21,45, and 70 respectively.

Now one has $a_4 > a_3 = 5 > b_2 = 4$ which settles case (1) by Proposition 5.1. Then $b_3, d_4, a_6 > 5$ which settles case (2). Similarly, $a_{2l} > b_l$ for $l \geq 3$, $a_{2l-1} > c_l$ for $l \geq 3$, $d_l > b_{l-1}$ for $l \geq 4$, and $21 > 16$ which settles cases (3)-(6) and concludes the proof.

\[2\text{A definition of a principal subgroup in [5] is different (see Exercise 18 for Chapter 9.4) but it coincides with this one in the case of a complex Lie group (see Exercise 21c for Chapter 9.4).}\]
5.4. Remark. In fact, the statement of the above Proposition is true for any proper maximal subgroup $R$ of $G$. This can be deduced from Dynkin’s classification of maximal subalgebras in semisimple Lie algebras. We outline the argument below.

Let us consider a maximal subalgebra $\mathfrak{r}$ in $\mathfrak{g}$, where $\mathfrak{g}$ is a simple Lie algebra. If $\mathfrak{r}$ is regular (i.e., if its normalizer contains some Cartan subalgebra in $\mathfrak{g}$), then $\mathfrak{r}$ does not contain any principal $\mathfrak{sl}_2$-triple $[21]$ Section 6.2.4. Thus we may assume that $\mathfrak{r}$ is non-regular.

If $\mathfrak{g}$ is exceptional, the list of such $\mathfrak{r}$ is given in $[21]$ Theorems 6.3.4, 6.3.5. All of them are semisimple, and we will only consider simple subalgebras (otherwise, $\mathfrak{r}$ once again does not contain any principal $\mathfrak{sl}_2$ ’s). The list of simple maximal non-regular subalgebras of rank $\geq 2$ in exceptional Lie algebras is short: $B_2$ in $E_8$, $A_2$ in $E_7$ and $A_2$, $G_2$, $C_4$, $F_4$ in $E_6$. In all these cases Proposition 5.1 applies.

It remains to consider non-regular maximal subalgebras $\mathfrak{r}$ of classical Lie algebras. Any such $\mathfrak{r}$ is simple, and an embedding of $\mathfrak{r}$ in $\mathfrak{g}$ is defined by a nontrivial linear irreducible representation $\varphi : \mathfrak{r} \to \mathfrak{sl}(V)$. Let $n = \dim V$ and $m = \left[\frac{n}{2}\right]$. If the module $V$ is not self-dual, $\mathfrak{r}$ is a maximal subalgebra in $\mathfrak{g} = A_{n-1}$. If $V$ is self-dual and endowed with a skew-symmetric invariant form, $\mathfrak{r}$ is a maximal subalgebra in $\mathfrak{g} = C_m$; and if $V$ is self-dual with a symmetric invariant form, $\mathfrak{r}$ is a maximal subalgebra in $\mathfrak{g} = B_m$ or $D_m$. Denote by $o(V)$ the number of nilpotent orbits in $\mathfrak{g}$, then

$$o(V) \geq o_n = \begin{cases} \min(a_{n-1}, b_m), & \text{if } n \text{ is odd} \\ \min(a_{n-1}, d_m, c_m), & \text{if } n \text{ is even} \end{cases}.$$ 

We want to check that for any irreducible $\mathfrak{r}$-module $V$ (except those corresponding to the trivial embedding $\mathfrak{r} = \mathfrak{g}$), the number $o(\mathfrak{r})$ of the nilpotent orbits in $\mathfrak{r}$ is less than $o(V)$. In what follows, representations $\varphi$ generating trivial embeddings of $\mathfrak{r}$ in $\mathfrak{g}$ are excluded. For exceptional $\mathfrak{r}$ of types $G_2$, $F_4$, $E_6$, $E_7$, $E_8$ the smallest irreducible representation has dimension $n = 7, 26, 27, 56, 248$ respectively. In all cases, the inequality $o(\mathfrak{r}) < o_n$ holds.

If $\mathfrak{r}$ is of type $A_k$, then either $V$ is not self-dual and $n > k + 1$ (in which case $o(\mathfrak{r}) = a_k < a_{n-1} = o(V))$ or $V$ is self-dual and $n > 2(k + 1)$. Then $a_k < o_n \leq o(V)$.

If $\mathfrak{r}$ is of type $B_k$ ($k \geq 2$), then all irreducible $V$ are self-dual and $n > 2k + 1$, hence $b_k < o_n$. If $\mathfrak{r}$ is of type $C_k$ ($k \geq 3$), then all irreducible $V$ are self-dual and $n > 4k$, hence $c_k < o_n$. If $\mathfrak{r}$ is of type $D_k$ ($k \geq 4$), then for any irreducible $V$, $n > 3k$ and $d_k < o_n$.

From this we conclude that Proposition 5.1 applies to any simple non-regular $\mathfrak{r}$ in $\mathfrak{g}$, where $\mathfrak{g}$ is a classical simple Lie algebra.

5.5. Lemma. Each orbit $O$ of a fixed point free degenerate $SL_2$-action on an affine algebraic variety $X$ is two-dimensional and closed, and the isotropy group of any point $x \in X$ is either $\mathbb{C}^*$ or $\mathbb{Z}_2$-extension of $\mathbb{C}^*$.
Proof. In the case of a fixed point free $SL_2$-action the isotropy group $I_x$ of a point from a closed orbit is either finite or $\mathbb{C}^*$ or $\mathbb{Z}_2$-extension of $\mathbb{C}^*$ by Lemma 5.5. Because the action is also degenerate $I_x$ cannot be finite and, therefore the closed orbit $SL_2/I_x$ is two-dimensional. By Proposition 2.2 (2) the closure of $O$ must contain a closed orbit. Since $O$ itself is at most two-dimensional it must coincide with this closed orbit.

□

Next we need two lemmas with the proof of the first one being straightforward.

5.6. Lemma. Let $G$ be a simple Lie group of dimension $N$ and rank $n$, $a$ be an element of $G$ and $C(a)$ be its centralizer. Suppose that $k$ is the dimension of $C(a)$. Then the dimension of the orbit $O$ of a under conjugations is $N - k$. In particular, when $a$ is a regular element (i.e. $\dim C(a) = n$) we have $\dim O = N - n$ coincides with the codimension of the centralizer of $a$.

5.7. Lemma. Let $G$ be a simple Lie group of dimension $N$ and rank $n$, $R$ be its proper reductive subgroup of dimension $M$ and rank $m$, $\Gamma$ be a $SL_2$-subgroup of $G$ such that its natural action on $G/R$ is fixed point free degenerate. Suppose that $a$ is a semi-simple non-identical element of $\Gamma$ and $k$ is the dimension of $C(a)$. Then $M \geq N - k - 1$. Furthermore, if $a$ is regular $M = N - n + m - 2$.

Proof. Since the $\Gamma$-action on $G/R$ is fixed point free and degenerate the isotropy group of any element $gR \in G/R$ is either $\mathbb{C}^*$ or a $\mathbb{Z}_2$ extension of $\mathbb{C}^*$ by Lemma 5.5. Recall that this isotropy group is $\Gamma \cap gRg^{-1}$ and therefore, $R$ contains a unique subgroup of form $g^{-1}L'g$ where $L'$ is a $\mathbb{C}^*$-subgroup of $\Gamma$. That is, $L' = \gamma_0^{-1}L\gamma_0$ for some $\gamma_0 \in \Gamma$ where $L$ is the $\mathbb{C}^*$-subgroup generated by $a$. Furthermore, this $\gamma_0$ is unique modulo a normalizer of $L$ in $\Gamma$ because otherwise $\Gamma g$ contain another $\mathbb{C}^*$-subgroup of $g^{-1}\Gamma g$ and, therefore, it would be at least two dimensional. The two-dimensional variety $W_{a,g} = \{(\gamma g)^{-1}a(\gamma g) | \gamma \in \Gamma\}$ meets $R$ exactly at two points $(\gamma_0 g)^{-1}a(\gamma_0 g)$ and $(\gamma_0 g)^{-1}a^{-1}(\gamma_0 g)$ (since the normalizer of $L$ has two components). Varying $g$ we can suppose that $W_{a,g}$ contains a general point of the $G$-orbit $O_a$ of $a$ under conjugations. Since it meets subvariety $R \cap O_a$ of $O_a$ at two points we see that $\dim R \cap O_a = \dim O_a - 2 = N - k - 2$ by Lemma 5.6. Thus, with $a$ running over $L$ we have $\dim R \geq N - k - 1$.

For the second statement note $b = g^{-1}ag \in R$ is a regular element in $G$. Hence the maximal torus in $G$ (and, therefore, in $R$) containing $b$ is determined uniquely. Assume that two elements $b_l = g_l^{-1}ag_l \in R$, $l = 1, 2$ are contained in the same maximal torus $T'$ of $R$ and, therefore, the same maximal torus $T$ of $G$. Then $g_2g_1^{-1}$ belongs to the normalizer of $T$, i.e. $b_2$ is of form $w^{-1}b_1w$ where $w$ is an element of the Weyl group of $T$. Thus $R \cap O_a$ meets each maximal torus $T'$ at a finite number of points. The space of maximal tori of $R$ is naturally isomorphic to $R/T'_{\text{norm}}$ where $T'_{\text{norm}}$ is the normalizer of $T'$. Hence $\dim R \cap O_a = \dim R - \dim T'_{\text{norm}} = M - m$. We showed already that the last dimension is also $N - n - 2$ which implies $M = N - n + m - 2$.

□
5.8. **Proposition.** Let the assumption of Lemma 5.7 hold. Then $a$ cannot be a regular element of $G$.

**Proof.** Assume the contrary, i.e. $a$ is regular. Let $\mathfrak{g}$ be the Lie algebra of $G$, $\mathfrak{h}$ be its Cartan subalgebra, $\mathfrak{r}_+$ (resp. $\mathfrak{r}_-$) be the linear space generated by positive (resp. negative) root spaces. Set $\mathfrak{s} = \mathfrak{r}_+ + \mathfrak{r}_-$ and suppose that $\mathfrak{g}', \mathfrak{h}', \mathfrak{s}'$ be the similar objects for $R$ with $\mathfrak{h}' \subset \mathfrak{h}$. Put $\mathfrak{r}'_\pm = \mathfrak{s}' \cap \mathfrak{r}_\pm$.

Each element of a root space $x'$ from $\mathfrak{s}'$ is of form $x' = h_0 + x_+ + x_-$ where $h_0 \in \mathfrak{h}$ and $x_\pm \in \mathfrak{r}_\pm$. Then there exists element $h' \in \mathfrak{h}'$ such that the Lie bracket $[x', h']$ is a nonzero multiple of $x'$ which implies that $h_0 = 0$, since $[h_0, h'] = 0$. Thus $\mathfrak{s}' \subset \mathfrak{s}$. By assumption $\mathfrak{h}'$ is a linear subspace of $\mathfrak{h}$ of codimension $n - m$. Hence Lemma 5.7 implies that $\mathfrak{s}'$ is of codimension 2 in $\mathfrak{s}$. We have two possibilities: (1) either, say, $\mathfrak{r}'_+ = \mathfrak{r}_+$ and $\mathfrak{r}'_-$ is of codimension 2 in $\mathfrak{r}_-$ or (2) $\mathfrak{r}'_\pm$ is of codimension 1 in $\mathfrak{r}_\pm$. In the first case each element of a root space $x \in \mathfrak{r}_+$, being in an eigenspace of $\mathfrak{h}' \subset \mathfrak{h}$, is also an element of a root space of $\mathfrak{g}'$. However for each root the negative of it is also contained in $\mathfrak{g}'$ which implies that $\mathfrak{r}'_- = \mathfrak{r}_-$. A contradiction.

In case (2) consider the generators $x_1, \ldots, x_l$ (resp. $y_1, \ldots, y_l$) of all root spaces in $\mathfrak{r}_+$ (resp. $\mathfrak{r}_-$) such that $h = [x_i, y_i]$ is a nonzero element of $\mathfrak{h}$. Their linear combination $\sum_{i=1}^l c_i^+ x_i$ is contained in $\mathfrak{r}'_+$ if and only if its coefficients satisfy a nontrivial linear equation $\sum_{i=1}^l c_i^+ = 0$. Similarly $\sum_{i=1}^l c_i^- y_i$ is contained in $\mathfrak{r}'_-$ if and only if its coefficients satisfy a nontrivial linear equation $\sum_{i=1}^l c_i^- = 0$. Note that $d_i^+ = 0$ if and only if $d_i^- = 0$ since otherwise one can find a root of $\mathfrak{g}'$ whose negative is not a root. Without loss of generality we suppose that the simple roots are presented by $x_1, \ldots, x_n$, i.e. $h_1, \ldots, h_n$ is a basis of $\mathfrak{h}$. Hence at least one coefficient $d_i^+ \neq 0$ for $i \leq n$. Indeed, otherwise $\mathfrak{r}'_+$ contains $x_1, \ldots, x_n$ which implies that $\mathfrak{r}'_+ = \mathfrak{r}_+$ contrary to our assumption. Note that $[a, x_i] = 2x_i$ for $i \leq n$ ([5], Chapter 8.11.4, Proposition 8).

Furthermore, since any $x_j$, $j \geq n + 1$ is a Lie bracket of simple elements one can check via the Jacobi identity that $[a, x_j] = sx_j$ where $s$ is an even number greater than 2. If we assume that $d_j^+ \neq 0$ then a linear combination $x_i + cx_j$, $c \neq 0$ is contained in $\mathfrak{r}'_+$ for some $x_i$, $i \leq n$. Taking Lie bracket with $a$ we see that $2x_i + scx_j \in \mathfrak{r}'_+$. Hence $x_j \in \mathfrak{r}'_+$, i.e. $d_j^+ = 0$ which is absurd. Thus $d_i^+ = 0$ only for $k \leq n$. We can suppose that $d_i^+ \neq 0$ for $i \leq l_0 \leq n$ and $d_j^+ = 0$ for any $j \geq l_0 + 1$. Note that $h_j \in \mathfrak{h}'$. If $l_0 \geq 3$ pick any three distinct numbers $i, j$, and $k \leq l_0$. Then up to nonzero coefficients $x_i + x_j \in \mathfrak{r}'_+$ and $y_i + y_k \in \mathfrak{r}'_-$. Hence $h_i = [x_i + x_j, y_i + y_k] \in \mathfrak{h}'$, i.e. $\mathfrak{h}' = \mathfrak{h}$. In this case we can find $h \in \mathfrak{h}'$ such that $[h, x_i] = s_i x_i$ and $[h, x_j] = s_j x_j$ with $s_i \neq s_j$. As before this implies that $x_i \in \mathfrak{r}'_+$ which is a contradiction. Thus we can suppose that at most $d_i^+$ and $d_j^+$ are different from zero.

If $l_0 \leq 2$ and $n \geq 3$ we can suppose that $[x_2, x_3]$ is a nonzero nilpotent element. The direct computation shows that up to nonzero coefficients $[[x_2, x_3], [y_2, y_3]]$ coincides with $h_2 - h_3$. Since $h_3 \in \mathfrak{h}'$, so is $h_2$. The same argument works for $h_1$, i.e. $\mathfrak{h}' = \mathfrak{h}$ again which leads to a contradiction as before. If $n = 2$ then the rank $m$ of $R$ is 1 (since we do
not want \( h' = h \), i.e. \( R \) is either \( \mathbb{C}^* \) or \( SL_2 \). In both cases \( \dim R < \dim G - n + m - 2 \) contrary to Lemma 5.7 which yields the desired conclusion.

\[ \square \]

Combining this result with Definition 5.2 and Proposition 5.3 we get the following.

5.9. **Corollary.** Let \( G \) be a simple Lie group and \( R \) be its reductive non-principal subgroup. Then for the principal \( SL_2 \)-subgroup \( \Gamma \subset G \) we have \( \Gamma^g \) is finite for some \( g \in G \) and \( \Gamma^g \) is different from \( g^{-1} \Gamma g \) for any \( g \in G \).

5.10. **Lemma.** Let \( R \) be a principal subgroup of \( G \). Then there exists an \( SL_2 \)-subgroup \( \Gamma \subset G \) such that \( \Gamma^g \) is finite for some \( g \in G \) and \( \Gamma^g \) is different from \( g^{-1} \Gamma g \) for any \( g \in G \).

**Proof.** Recall that the subregular nilpotent orbit is the unique nilpotent orbit of codimension \( \text{rank } g + 2 \) in \( g \) [7, Section 4.1]. It can be characterized as the unique open orbit in the boundary of the principal nilpotent orbit. The corresponding \( sl_2 \)-triple \( (X, H, Y) \) in \( g \) is also called subregular. The dimension of the centralizer of the semisimple subregular element \( H \) in this triple is \( \text{rank } g + 2 \) [7]. We denote the subregular \( SL_2 \) subgroup of \( G \) by \( \Gamma_{sr} \).

| \( G \)   | \( R \)   | \( \text{rank } G + 3 \) | \( \text{dim } G - \text{dim } R \) |
|----------|----------|--------------------------|-------------------------------|
| \( B_3 \) | \( G_2 \) | 6                        | 7                             |
| \( D_4 \) | \( G_2 \) | 7                        | 14                            |
| \( A_6 \) | \( G_2 \) | 9                        | 34                            |
| \( E_6 \) | \( F_4 \) | 9                        | 26                            |
| \( A_{2l-1} \) | \( C_l \) | \( 2l + 2 \)              | \( l(2l - 1) - 1 \)           |
| \( A_{2l} \) | \( B_l \) | \( 2l + 3 \)              | \( 2l^2 + 3l \)               |
| \( D_l \) | \( B_{l-1} \) | \( l + 3 \)              | \( 2l - 1 \)                  |

We will demonstrate that in the cases listed in the table above, no conjugates of \( \Gamma_{sr} \) can belong to \( R \). Then by Lemma 5.7 the statement of the current Lemma follows whenever \( \dim G - \dim R > \text{rank } G + 3 \). From the table above we see that it covers all the principal embeddings from the proof of Proposition 5.3 with the exceptions of the inclusions \( B_3 \subset D_4 \) and \( C_2 \subset A_3 \).

For \( G = A_r \), the subregular \( sl_2 \) corresponds to the partition \( (r, 1) \). If \( r \) is odd, this partition is not symplectic (since in symplectic partitions all odd entries occur with even multiplicity), and if \( r \) is even, this partition is not orthogonal (since in orthogonal partitions all even entries occur with even multiplicity). In other words, the subregular \( SL_2 \)-subgroup \( \Gamma_{sr} \) in \( A_{2l-1} \) (respectively, \( A_{2l} \)) does not preserve any nondegenerate symplectic (resp., orthogonal) form on \( \mathbb{C}^{2l} \) \( (\mathbb{C}^{2l+1}) \) and thus does not belong to \( R = C_l \) (resp., \( R = B_l \)). The same is true for any conjugate of \( \Gamma_{sr} \) in \( G \).

If \( G = D_l \), the embedding of \( R = SO_{2l-1} \subset SO_{2l} \) is defined by the choice of the non-isotropic vector \( v \in \mathbb{C}^{2l} \) which is fixed by \( R \). The subregular \( sl_2 \) in \( so_{2l} \) corresponds to
the partition \((2l - 3, 3)\). Thus we see that \(\Gamma_{\text{sr}} \subset SO_{2l}\) does not fix any one-dimensional subspace in \(C^{2l}\) (its invariant subspaces have dimensions \(2l - 3\) and \(3\)) and thus none of its conjugates can belong to \(R\). Moreover, we can choose \(v\) such that \(xv \neq v\) for \(x \in \Gamma_{\text{sr}}, x \neq 1\). Thus \(\Gamma_{\text{sr}} \cap SO_{2l-1} = \{e\}\). This establishes the desired conclusion for the embeddings \(SO_7 \subset SO_8\) (i.e. the case of \(B_3 \subset D_4\)) and \(SO_5 \subset SO_6\) (i.e. the case of \(B_2 \simeq C_2 \subset A_3 \simeq D_3\)), in which the dimension count of Lemma 5.7 by itself is not sufficient.

The alignments of \(\mathfrak{sl}_2\)-triples in exceptional cases were analyzed in [19]. In particular, it was observed there that any conjugacy class of \(\mathfrak{sl}_2\)-triples in \(f_4\) lifts uniquely to a conjugacy class of \(\mathfrak{sl}_2\)-triples in \(e_6\). Consulting the explicit correspondence given in [19, 2.2], we observe that the largest non-principal nilpotent orbit in \(e_6\) which has nonempty intersection with \(f_4\) has codimension 10. This implies that no \(\mathfrak{sl}_2\)-triple in \(f_4\) lifts to a subregular \(\mathfrak{sl}_2\) in \(e_6\). In other words, no conjugate of \(\Gamma_{\text{sr}} \subset E_6\) belongs to \(F_4\).

When \(R = G_2\), its embedding in \(SO_8\) is defined by the triality automorphism \(\tau: SO_8 \to SO_8\), with \(R\) being a fixed point group of this automorphism. Equivalently, \(R = SO_7 \cap \tau(SO_7)\). In particular, only those \(\mathfrak{sl}_2\)-triples in \(\mathfrak{so}_8\) which are fixed under the triality automorphism belong to \(\mathfrak{g}_2\). Observe that the subregular \((5, 3)\) \(\mathfrak{sl}_2\)-triple is not fixed by triality (cf. [19, Remark 2.6]). Similarly, the subregular \(\mathfrak{sl}_2\)-triple in \(\mathfrak{so}_7\) corresponds to the partition \((5, 1, 1)\). Since the \((5, 1, 1, 1)\) \(\mathfrak{sl}_2\)-triple in \(\mathfrak{so}_8\) is not invariant under triality, neither is subregular \(\mathfrak{sl}_2\) in \(\mathfrak{so}_7\). Thus no conjugates of \(\Gamma_{\text{sr}}\) in \(B_3\) or \(D_4\) are fixed by \(\tau\), and no conjugates of \(\Gamma_{\text{sr}}\) belong to \(G_2\).

Finally, when \(G = A_6\), the subregular triple in \(A_6\) does not belong to \(B_3\) (see above), and thus none of its conjugates lie in \(G_2 \subset B_3\).

Now Theorem 5 follows immediately from the combined statements of Corollary 5.9 and Lemma 5.10.

5.11. Remark. Note that we proved slightly more than required. Namely, the \(SL_2\)-subgroup \(\Gamma\) in Theorem 5 can be chosen either principal or subregular.

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