TWO-DIMENSIONAL SUSY–PSEUDO-HERMITICITY WITHOUT SEPARATION OF VARIABLES

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We study SUSY–intertwining for non-Hermitian Hamiltonians with special emphasis to the two-dimensional generalized Morse potential, which does not allow for separation of variables. The complexified methods of SUSY–separation of variables and two-dimensional shape invariance are used to construct particular solutions - both for complex conjugated energy pairs and for non-paired complex energies.

PACS: 03.65.-w; 03.65.Ge; 03.65.Fd; 11.30.

Keywords: Supersymmetry; Complex two-dimensional Hamiltonians; SUSY-pseudo-Hermiticity; Quasi-exactly-solvable models; Partial solvability

1. Introduction

Recently $PT$–invariance of one-dimensional models in Quantum Mechanics has been investigated by C. Bender and collaborators [1]-[3] (see also [4]) with special emphasis on the spectrum of the associated Hamiltonians. Since in many cases the spectrum was found to be...
real, $PT$–invariance was proposed as a generalization of standard Hermiticity. However it soon became clear that there are simple $PT$–symmetric examples, for which the spectrum is not real and therefore alternative criteria for reality of the spectrum were explored.

The most systematic investigation has been performed by A.Mostafazadeh [5]-[7] (see also [8], [9]) elaborating on the so-called pseudo-Hermiticity:

$$\eta H \eta^{-1} = H^\dagger$$

with $\eta$ a Hermitian invertible operator, expressed in terms of a biorthogonal basis. This type of approach requires a complete solution of the spectral problem.

For non-solvable problems, it is convenient to use the intertwining relations of SUSY Quantum Mechanics (SUSY QM) [10],[11] to relate pairs of Hamiltonians. One Hermitian and one non-Hermitian Hamiltonian may be intertwined [12], [13], [14] or, in other cases, non-Hermitian Hamiltonians may be intertwined [15]-[16] ($SUSY$–pseudo-Hermiticity) with their Hermitian conjugates. Both these constructions might lead to complex models with real spectra. We would like to remark that $SUSY$–pseudo-Hermiticity differs from the pseudo-supersymmetry of [17].

While one dimensional models of this kind have been investigated in many different frameworks like SUSY QM [12], [13], [18], [19], $PT$ symmetry [1]-[3], [20]-[23], [19], for two dimensions the advance is really at the start. To our knowledge there are only the papers [24] and [25], where some complex two-dimensional potentials are studied numerically, which are $PT$–symmetric and therefore are two dimensional extensions of the $x^2 + igx^3$ potential [26].

Within SUSY QM a class of non-trivial two-dimensional models (not allowing for separation of variables) was constructed in [27]-[30]. One model of this class (generalized singular Morse potential) was investigated [31] in detail by two novel methods: $SUSY$–separation of variables and two-dimensional shape invariance. The model is partially solvable (see, for example [32]) or, in alternative terminology, quasi-exactly-solvable [33],[34], this means that only part of eigenvalues and eigenfunctions can be found.

In Section 2 we introduce $SUSY$–pseudo-Hermiticity with supercharges of first and
second order in derivatives in one and two dimensions. In Section 3 the complexification of two-dimensional model of [31] is implemented in the context of \(SUSY\)-pseudo-Hermiticity with special attention to the appearance of levels in complex conjugated pairs. In particular, Subsection 3.1 contains the \(SUSY\)-separation of variables method, and 3.2 - the complex form of the two-dimensional shape invariance method.

2. \textbf{SUSY QM and SUSY—pseudo-Hermiticity.}

For the case of Hermitian Hamiltonians the main algebraic relation of SUSY Quantum Mechanics [10],[11], in all possible formulations and generalizations (for example, [35]-[39] ) is given by intertwining relations:

\[
\tilde{H}Q^+ = Q^+H; \tag{2}
\]
\[
Q^-\tilde{H} = HQ^-; \quad Q^- = (Q^+)^\dagger \tag{3}
\]

for a pair of Schrödinger operators (superpartners):

\[
\tilde{H} = -\Delta + \tilde{V}; \quad H = -\Delta + V.
\]

These relations connect eigenfunctions with the same eigenvalues (up to zero modes of \(Q^\pm\)):

\[
H\phi_{E_n} = E_n\phi_{E_n}; \quad \tilde{H}\tilde{\phi}_{E_n} = E_n\tilde{\phi}_{E_n};
\]
\[
\phi_{E_n} = Q^-\tilde{\phi}_{E_n}; \quad \tilde{\phi}_{E_n} = Q^+\phi_{E_n}.
\]

For compactness we do not introduce explicitly an index associated to possible degeneracy.

If \(H\) and \(\tilde{H}\) are non-Hermitian, the two intertwining relations (2) and (3) may become independent, and the supercharges \(Q^\pm\) not necessarily Hermitian conjugate. A particular case, considered earlier [12], \(\tilde{H}\) - Hermitian, and \(H\) - not, leads to the reality of the spectrum of \(H\).
Another possibility is to examine non-Hermitian partner Hamiltonians related by what we call \textit{SUSY-pseudo-Hermiticity}:

\begin{align}
Q^+ H &= H^\dagger Q^+; \tag{4} \\
HQ^- &= Q^- H^\dagger. \tag{5}
\end{align}

Since we deal with scalar potentials, Hermitian and complex conjugations are equivalent for our purposes (see also [39]). The eigenstates of $H^\dagger$ with eigenvalues $E_n$ will be denoted by $(\phi_{E_n}^\star)^*$ with eigenvalues $E_n$. They are related to those of $H$ not only by the intertwining but also by direct conjugation. This has been established by using the formalism of the biorthogonal expansion [5]-[7]. The operator $Q^+$ has in general zero modes becoming non-invertible.

As a consequence of Eq.(4), one can obtain a relation which can be useful for the classification of the spectrum:

\begin{equation}
(E_n - E_m^\star) < \phi_{E_m} | Q^+ \phi_{E_n} >= 0. \tag{6}
\end{equation}

First, let us notice that diagonal matrix elements in the subspace of zero modes of $Q^+$ are trivially zero. So, in this case (6) does not provide any restriction on the energy $E_n$, which in particular can be complex having no complex conjugate partner. Clearly, a non-zero value of the matrix element in (6) for $n = m$ outside the subspace, considered above, implies that the energy $E_n$ is real, while off-diagonal non-vanishing matrix element signals that complex energies appear in complex conjugated pairs $E_n = E_m^\star$.

For introducing \textit{SUSY-pseudo-Hermiticity} we provide an explicit exhaustive construction in the framework of complex SUSY QM [12] in one-dimension with first order supercharges [15],[16]. In this case without loss of generality:

\begin{align*}
H &= Q^- Q^+ + Const; \quad H^\dagger = Q^+ Q^- + Const^\star; \\
Q^+ &= -\partial + ig(x); \quad Q^- = +\partial + ig(x) = -(Q^+)^*; \quad \partial \equiv \frac{d}{dx}
\end{align*}

with $Const$ an arbitrary complex number and $g(x)$ an arbitrary real function, leading to the potential:

\begin{equation}
V(x) = -g^2(x) + ig'(x) + Const.
\end{equation}
In this case the equation for zero modes of $Q^+$ has no normalizable solution, so that $SUSY-$pseudo-Hermiticity is effectively equivalent to pseudo-Hermiticity (1). Connection with usual $PT-$symmetry can be established by choosing $g(x)$ to be of even parity. Second order generalizations (HSUSY) of one-dimensional supercharges [12] for $SUSY-$pseudo-Hermiticity were recently discussed [15],[16].

For two-dimensional SUSY QM models solutions of the intertwining relations (2) for scalar Schrödinger Hamiltonians only exist with second order supercharges [27]-[31] and only particular solutions have been found by suitable ansatzes. For the supercharges with Lorentz metrics [31]

$$Q^+ = (\partial_1^2 - \partial_2^2) + C_k \partial_k + B = 4\partial_+ \partial_- + C_+ \partial_+ + C_- \partial_+ + B, \quad (7)$$

a solution of (2) can be provided [27]-[30] by solving the system:

$$\partial_- (C_- F) = -\partial_+ (C_+ F); \quad (8)$$

$$\partial_+^2 F = \partial_-^2 F; \quad (9)$$

where $x_\pm \equiv x_1 \pm x_2$, $\partial_\pm \equiv \partial / \partial x_\pm$ and $C_\pm$ depend only on $x_\pm$, respectively:

$$C_+ \equiv C_1 - C_2 \equiv C_+(x_+); \quad C_- \equiv C_1 + C_2 \equiv C_-(x_-).$$

The function $F$, solution of (9), is represented as

$$F = F_1 (x_+ + x_-) + F_2 (x_+ - x_-).$$

The potentials $\tilde{V}(\vec{x}), V(\vec{x})$ and the function $B(\vec{x})$ are expressed in terms of $F_1(2x_1), F_2(2x_2)$ and $C_\pm(x_\pm)$, solutions of the system (8), (9):

$$\tilde{V} = \frac{1}{2} (C'_+ + C'_-) + \frac{1}{8} (C^2_+ + C^2_-) + \frac{1}{4} \left( F_2(x_+ - x_-) - F_1(x_+ + x_-) \right),$$
$$V = \frac{1}{2} (C'_+ + C'_-) + \frac{1}{8} (C^2_+ + C^2_-) + \frac{1}{4} \left( F_2(x_+ - x_-) - F_1(x_+ + x_-) \right),$$
$$B = \frac{1}{4} \left( C_+ C_- + F_1(x_+ + x_-) + F_2(x_+ - x_-) \right).$$

(10)
The linear character of Eq.(8) in $C_{\pm}$ allows to multiply $C_{\pm}$ by the imaginary unit keeping $F_{1,2}$ real. This sort of complexification renders $\tilde{V}$ of Eq.(10) complex conjugate to $V$. Thus the intertwining relations (2) in this case lead automatically to $\textit{SUSY}$—pseudo-Hermiticity. For the two-dimensional models the existence of zero modes of $Q^\pm$ can not be avoided: actually in the class of models studied in [31] the equation for zero modes allows separation of variables ($\textit{SUSY}$—separation of variables). Thus these zero modes can be constructed from normalizable solutions of two one-dimensional equations of second order (see Section 3 of [31]). In fact, the similarity relation, which eliminates first order derivatives from the supercharges, is now unitary.

The next Section will consider the complexification of the partially solvable (quasi-exactly-solvable) two-dimensional model studied in [31] (generalized singular Morse potential). The spectral problem was partially solved by two methods, one based on the $\textit{SUSY}$—separation of variables and the second - on the shape invariance. This model is a natural candidate to elucidate $\textit{SUSY}$—pseudo-Hermiticity in two dimensions because it is not amenable to separation of variables. Furthermore in this model $\textit{SUSY}$—pseudo-Hermiticity is not equivalent to pseudo-Hermiticity due to the existence of zero modes of $Q^+$ and also because the spectral problem is not exactly solvable.

We stress that in the class of models (10) the partner Hamiltonians are not [27]-[31] factorizable in terms of supercharges $Q^\pm$. But there are symmetry operators of fourth order in derivatives which can be factorized:

$$R = Q^- Q^+; \quad \tilde{R} = Q^+ Q^-.$$  \hspace{1cm} (12)

3. Complex two-dimensional generalized (singular) Morse potential.

The model is defined [31] in terms of a specific choice for $C_{\pm}$ and $F_{1,2}$ in Eqs.(10) and (11):

$$C_+(ia) = 4i\alpha \equiv \hat{C}_+(a); \quad C_-(ia) = 4i\alpha \cdot \coth \frac{\alpha x}{2} \equiv \hat{C}_-(a);$$

6
\[ f_1(x_1) \equiv \frac{1}{4} F_1(2x_1) = -A \left( \exp(-2\alpha x_1) - 2 \exp(-\alpha x_1) \right); \]
\[ f_2(x_2) \equiv \frac{1}{4} F_2(2x_2) = +A \left( \exp(-2\alpha x_2) - 2 \exp(-\alpha x_2) \right); \]
\[ \hat{V}^*(\vec{x}; a) \equiv V^*(\vec{x}; ia) = -\alpha^2 a(2a + i) \sinh^{-2} \left( \frac{\alpha x -}{2} \right) + 
+ A \left[ \exp(-2\alpha x_1) - 2 \exp(-\alpha x_1) + \exp(-2\alpha x_2) - 2 \exp(-\alpha x_2) \right]; \]
\[ \hat{V}(\vec{x}; a) \equiv V(\vec{x}; ia) = -\alpha^2 a(2a - i) \sinh^{-2} \left( \frac{\alpha x -}{2} \right) + 
+ A \left[ \exp(-2\alpha x_1) - 2 \exp(-\alpha x_1) + \exp(-2\alpha x_2) - 2 \exp(-\alpha x_2) \right], \tag{13} \]

where \( A \) is an arbitrary positive constant, and \( a \) is a real parameter. Below we will use for all operators and functions the "hat" notation following the definitions above. We stress that only for real values of the parameter \( a \) the model described above satisfies SUSY—pseudo-Hermiticity.

Within this complexification the supercharges \( \hat{Q}^+(a) \) are Hermitian because \( \hat{C}_\pm = \hat{C}_\pm(x_\pm) \) in Eq.(7) commute with \( \partial_\mp \). In contrast, the supercharges \( Q^-(a) \) for \( a \in \mathbb{R} \) are Hermitian conjugate to \( Q^+(a) \), but after the complexification \( a \to ia \) they are related by complex conjugation: \( \hat{Q}^-(a) = (\hat{Q}^+(a))^* \):

\[ \hat{Q}^\pm(a) = 4 \partial_+ \partial_- \pm 4i \alpha a \partial_- \pm 4i \alpha \coth(\frac{\alpha x -}{2}) \partial_+ + \hat{B}(a). \tag{14} \]

The Hamiltonian has no definite \( PT \)—symmetry, but has a \( x_- \)—reflection symmetry \( x_1 \leftrightarrow x_2 \) in coordinate space (permutation symmetry). The supercharges (14) are odd. Therefore this model has vanishing diagonal matrix elements in (6).

In addition, the Hamiltonian has a discrete symmetry (involution):

\[ \hat{V}(\vec{x}; a) = \hat{V}(\vec{x}; -a + \frac{i}{2}); \tag{15} \]
\[ \hat{V}(\vec{x}; a) = \hat{V}^*(\vec{x}; -a). \tag{16} \]
3.1. The method of SUSY—separation of variables.

In order to apply the method [31], one has to separate variables in \( \hat{Q}^\pm \) Eq.(14). This can be achieved by the transformation, which is unitary for \( b \in \mathbb{R} \):

\[
\hat{U}(\vec{x}; b) \equiv \exp \left( -ib\alpha(x_+ + \int \coth\left( \frac{\alpha x_-}{2} \right)d x_- \right) = \left( \frac{\alpha}{\sqrt{A}} \cdot \frac{\xi_1 \xi_2}{|\xi_2 - \xi_1|} \right)^{2ib} ;
\]

\[
\hat{Q}^-(0) \equiv \hat{U}(\vec{x}; b)\hat{Q}^-(b)\hat{U}^{-1}(\vec{x}; b) = \partial_1^2 - \partial_2^2 + \frac{1}{4}(F_1(2x_1) + F_2(2x_2)),
\]

where

\[
\xi_i \equiv \frac{2\sqrt{A}}{\alpha} \exp(-\alpha x_i); \quad i = 1, 2.
\]

The zero modes of \( \hat{Q}^+ \) can be parametrized as:

\[
\hat{\Omega}_n(\vec{x}; a) = \hat{U}(\vec{x}; a)\hat{\omega}_n(\vec{x}; a);
\]

\[
\hat{\omega}_n(\vec{x}) = \exp\left( -\frac{\xi_1 + \xi_2}{2} \right)(\xi_1 \xi_2)^{s_n}F(-(n, 2s_n + 1; \xi_1)F(-n, 2s_n + 1; \xi_2),
\]

where \( F(-(n, 2s_n + 1; \xi) \) is the standard degenerate (confluent) hypergeometric function, reducing to a polynomial for integer \( n \), and

\[
s_n = \frac{\sqrt{A}}{\alpha} - n - \frac{1}{2} > 0.
\]

Normalizable eigenfunctions \( \hat{\Psi}_{E_k}(\vec{x}; a) \) of the Hamiltonian \( \hat{H}(\vec{x}; a) \) can be obtained by linear superposition of zero modes (19) according to [31], and their eigenvalues (after complexification \( a \to ia \)) read:

\[
\hat{E}_k(a) = -4ia\alpha^2 s_k + 4a^2 \alpha^2 + 2\epsilon_k; \quad \epsilon_k \equiv -A \left[ 1 - \frac{\alpha}{\sqrt{A}}(k + \frac{1}{2}) \right]^2 < 0.
\]

Since the operator \( \hat{U}(\vec{x}; a) \) in (19) is unitary, the condition for normalizability of eigenfunctions \( \hat{\Psi}_{E_k}(\vec{x}; a) \) now does not depend on the parameter \( a \) and is expressed by the inequality (21): \( s_n > 0 \). The number of normalizable zero modes \( \hat{\Psi}_{E_k} \) is also determined by this inequality.

\*As a consequence of (18), one can derive \( \hat{Q}^+(0) = (\hat{Q}^-(0))^* = \hat{U}^*(b)\hat{Q}^+(b)(\hat{U}^{-1}(b))^* \).
Apparently, the energies (22) have nonzero imaginary part but we remind that (6) is trivially satisfied by the vanishing of the matrix elements \( \langle \hat{\Psi}_{E_m} | \hat{Q}^+ \hat{\Psi}_{E_n} \rangle \), since we deal with zero modes of \( \hat{Q}^+ \).

In order to find examples of complex conjugate energies we have to explore states outside the linear space of zero modes of \( \hat{Q}^+ \). Following the procedure of [31], we construct three eigenfunctions

\[
\hat{\Phi}^{(i)}(\vec{x}; a) \equiv \hat{\Omega}_0(\vec{x}; a) \cdot \hat{\Theta}^{(i)}(\vec{x}; a),
\]

with energies \( \hat{E}^{(i)}(a) \) given in (22):

\[
\hat{E}^{(i)}(a) = \hat{E}_0(a) + \hat{\gamma}^{(i)}(a),
\]

and

\[
\hat{\Theta}^{(1)}(\vec{x}; a) = |z_2|^{(4ia+1)}; \quad \hat{\gamma}^{(1)}(a) = \alpha^2(2s_0-1)(4ia+1); \quad (25)
\]

\[
\hat{\Theta}^{(2)}(\vec{x}; a) = |z_2|^{(4ia+1)} \left( z_1 + \frac{2}{4ia - 2s_0 + 3} \right); \quad \hat{\gamma}^{(2)}(a) = 4\alpha^2(s_0-1)(2ia+1); \quad (26)
\]

\[
\hat{\Theta}^{(3)}(\vec{x}; a) = z_1 - \frac{2}{4ia + 2s_0 - 1}; \quad \hat{\gamma}^{(3)}(a) = \alpha^2 \left( 4ia + 2s_0 - 1 \right). \quad (27)
\]

Here

\[
z_1 = \frac{1}{\xi_1} + \frac{1}{\xi_2}; \quad z_2 = \frac{1}{\xi_1} - \frac{1}{\xi_2}.
\]

In contrast to the case \( a \in \mathbb{R} \), where only \( \Phi^{(3)}(\vec{x}; a) \) is normalizable, all three eigenfunctions (25) - (27) become normalizable after \( a \to ia \) if

\[
s_0 > 2. \quad (28)
\]

One can argue from Eq.(1) that \( \hat{Q}^+ \hat{\Phi}^{(i)} \) are eigenfunctions of \( \hat{H}^I \) with the eigenvalues \( \hat{E}^{(i)}(a) \). As explained after (1), this means that \( \hat{H}(a) \), in addition to eigenvalues (24), has also the complex conjugate eigenvalues:

\[
\hat{H}(a)(\hat{Q}^+ \hat{\Phi}^{(i)}(\vec{x}; a))^* = (\hat{E}^{(i)}(a))^*(\hat{Q}^+ \hat{\Phi}^{(i)}(\vec{x}; a))^*.
\]

The condition of normalizability of all three wave functions in (29) coincides with (28). Orthogonality of all these wave functions can not be secured in general due to non-Hermiticity
of the Hamiltonian\(^f\), but a pseudo-orthogonality can be derived in agreement with the formalism [5]-[7] of the biorthogonal expansion with particular components \(\hat{\Phi}^{(i)}\); \(\hat{Q}^{-}(\hat{\Phi}^{(i)})^{*}\) and \((\hat{\Phi}^{(i)})^{*}\); \(\hat{Q}^{+}(\hat{\Phi}^{(i)})\)

\[
\begin{align*}
< (\hat{\Phi}^{(i)})^{*}|\hat{\Phi}^{(j)} > & = 0 \quad i \neq j \quad (30) \\
< (\hat{\Phi}^{(i)})^{*}|\hat{Q}^{-}(\hat{\Phi}^{(j)})^{*} > & = 0 \quad i, j = 1, 2, 3 \quad (31) \\
< \hat{Q}^{+}\hat{\Phi}^{(i)}|\hat{Q}^{-}\hat{\Phi}^{(j)} > & = 0 \quad i \neq j \quad (32)
\end{align*}
\]

Equations (32) can be derived by taking into account [31] that wave functions \(\hat{\Phi}^{(i)}\), \(\hat{\Phi}^{(j)}\) are eigenfunctions of the symmetry operator (12) with different eigenvalues. Equation (30) is proportional to (32). Eq.(31) follows from \(x_{-}\)—reflection symmetry considerations (\(\hat{\Phi}^{(i)}\) are even and \(\hat{Q}^{\pm}\) are odd).

Using the symmetry property (15) one can generate a new series of levels of \(\hat{H}(a) = \hat{H}(-a + \frac{i}{2})\) by considering each eigenfunction (and corresponding eigenvalues) from (22), (23), (29) with parameter shift \(a \rightarrow -a + \frac{i}{2}\). The partner eigenfunctions with complex conjugated energies can also be constructed along the same line as (29).

### 3.2. The shape invariance method.

Starting from a Schrödinger equation with potential of (13)

\[
\hat{H}(a)\hat{\phi}_{E_{n}}(\vec{x}; a) = \hat{E}_{n}(a)\hat{\phi}_{E_{n}}(\vec{x}; a)
\]

with \(\hat{\phi}_{E_{n}}(\vec{x}; a)\) - arbitrary eigenfunction, taking into account that \(\hat{H}(a + \frac{i}{2}) = \hat{H}^{\dagger}(a)\), we get:

\[
\hat{H}^{\dagger}(a)\hat{\phi}_{E_{n}}(\vec{x}; a + \frac{i}{2}) = \hat{E}_{n}(a + \frac{i}{2})\hat{\phi}_{E_{n}}(\vec{x}; a + \frac{i}{2}).
\]

\(\hat{Q}^{-}(a)\hat{\phi}_{E_{n}}(\vec{x}; a)\) from the intertwining relation:

\[
\hat{H}(a)\hat{Q}^{-}(a) = \hat{Q}^{-}(a)\hat{H}^{\dagger}(a),
\]

\(\hat{Q}^{+}\hat{\phi}_{E_{n}}(\vec{x}; a)\)

\(^f\)Due to \(x_{-}\)—reflection considerations one can however easily conclude that \(< \hat{Q}^{-}(\hat{\Phi}^{(i)})^{*}|\hat{\Phi}^{(j)} > = 0.\)
we obtain
\[
\hat{H}(a)\left(\hat{Q}^{-}(a)\hat{\phi}_{E_n}(\vec{x}; a + \frac{i}{2})\right) = \hat{E}_n(a + \frac{i}{2})\left(\hat{Q}^{-}(a)\hat{\phi}_{E_n}(\vec{x}; a + \frac{i}{2})\right)
\]

Thereby, we are at the first step [40] of a “shape invariance chain” of wave functions and eigenvalues for a complex value of the parameter. Notice that the definition of potentials now differs from that in [31] by a constant shift \(4\alpha^2a^2\). This leads to a vanishing of \(\mathcal{R}(a)\). In addition, we remark that our construction will contain complex values for the parameters in wave functions, in operators etc only in intermediate steps, but the parameter \(a\) will always be kept real.

We thus can construct an additional class of levels starting from \(\hat{\phi}_{E_n}\), an eigenstate of the kind \(\hat{\Psi}_{E_k}; \hat{\Phi}^{(i)}; \hat{Q}^{-}(\hat{\Phi}^{(i)})^*\) etc (see (22), (23), (29)). These states and their complex conjugated are additional (particular) components of the biorthogonal basis and will fulfill equations similar to (30), (31), (32).

Iterating this procedure, one generates the shape invariance chain:
\[
\Sigma^k_n(\vec{x}; a) \equiv \left[\hat{Q}^{-}(a)\hat{Q}^{-}(a + \frac{i}{2})\hat{Q}^{-}(a + i)...\hat{Q}^{-}(a + \frac{i(k-1)}{2})\hat{\phi}_{E_n}(\vec{x}; a + \frac{ik}{2})\right],
\]
associated to the energy
\[
\hat{E}^k_n(a) \equiv \hat{E}_n(a + \frac{ik}{2}).
\]

In particular, for \(\hat{\phi}_{E_n} - \) linear combination of zero modes of \(\hat{Q}^+\) (see (22))
\[
\hat{\phi}_{E_n}(\vec{x}; a + \frac{ik}{2}) = \hat{\Psi}_{E_n}(\vec{x}; a + \frac{ik}{2})
\]
the eigenvalues are:
\[
\hat{E}^k_n(a) = -\alpha^2(4ia(s_n - k) + (s_n - k)^2 + s_n^2 - 4a^2).
\]

In order to investigate the normalizability of these eigenfunctions, it is crucial to study their behaviour in \((\xi_1, \xi_2)\) plane following different paths. As already discussed in Section 4.3. of [31] for the case \(a \in \mathbb{R}\), the relevant singularities should occur for the origin, for \(\xi_2 \to 0\) and
for $\xi_1 \to \xi_2$. By using (19), (20) and (18) for supercharges and wave functions, one can study the suitable critical limits in the following representation for the norm of (33):

$$\|\hat{\Sigma}_n^k(a)\| \sim \|\hat{U}^{-1}(a) \hat{Q}^{-}(0)\hat{U}(-\frac{i}{2})\hat{Q}^{-}(0)\hat{U}(-\frac{i}{2})...\hat{Q}^{-}(0)\hat{U}(-\frac{i}{2}) \hat{U}(2a + ik)\hat{\omega}_n\|,$$

(35)

where the $\vec{x}$ dependence has been dropped for conciseness. Normalizability can be established for

$$s_n \equiv \sqrt{\frac{A}{a}} - \frac{1}{2} - n > k,$$

(36)

which is just the normalizability condition for (34). In other words the repeated application of $\hat{Q}^-$ does not restrict the relevant region of normalizability.

From the same condition one can estimate the number of normalizable states generated by successive applications of $\hat{Q}^-$ operators in (33). It depends only on the value of the integer part: $N \equiv \lfloor \sqrt{\frac{A}{a}} - \frac{1}{2} \rfloor$ and does not depend on $a$. For example, in the case of non-integer $\sqrt{\frac{A}{a}} - \frac{1}{2}$ the total number of states can be estimated to be $N(N + 1)/2$, in other words typically $A/(2\alpha^2)$ for large values of $N$.

As a final remark, we would like to mention that analogous results for shape invariance chains and their normalizability can be obtained for the model [31] (before complexification), though they were not explicitly discussed. In that case $Q^\pm$ are interrelated by Hermitian conjugation, and the calculation of the norm of the chain can be performed by an explicit introduction of the symmetry operator $\hat{R} = Q^+Q^-$, provided the arguments match. After this it is clear that results equivalent to (36) hold.

Our main results can be summarized as follows. In the context of the notion of $SUSY$—pseudo-Hermiticity two methods (already studied [31] for Hermitian models) - $SUSY$—separation of variables and two-dimensional shape invariance - were used to build explicitly a set of eigenvalues and eigenfunctions for the complex two-dimensional singular Morse potential. This part of the spectrum includes complex conjugated energy pairs and in addition non-paired complex energies for states - linear superpositions of zero modes of the intertwining operators. In contradistinction to one-dimensional models, pseudo-Hermiticity and $SUSY$-pseudo-Hermiticity are not equivalent for two-dimensional models just due to
the nontrivial role of the zero modes of the supercharges. For two-dimensional scalar models the intertwining relations can only be solved for second order supercharges, which a priori may have infinite number of zero modes. Therefore in the two-dimensional case the intertwined Hamiltonians are not anymore factorizable (compare with the factorizability in pseudo-supersymmetry of [17]), but factorizable symmetry operators (12) of fourth order in derivatives exist.

In the class of models considered in [31] and here these symmetry operators cannot be expressed in terms of the Hamiltonians and in general signal their integrability and possible degeneracy of their spectra, though no direct evidence of such degeneracy was found in the solved part of the spectrum. We remind that in [31] the selected energy eigenfunctions which were explored were simultaneously eigenfunctions of the symmetry operator, so there was no indication for degeneracy.

Acknowledgements

M.V.I. and D.N.N. are indebted to INFN, the University of Bologna for the support and hospitality. This work was partially supported by the Russian Foundation for Fundamental Research (Grant No.02-01-00499).

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