Microcanonical statistics of black holes and bootstrap condition

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Abstract

The microcanonical statistics of the Schwarzschild black holes as well as the Reissner-Nordström black holes are analyzed. In both cases we set up the inequalities in the microcanonical density of states. These are then used to show that the most probable configuration in the gases of black holes is that one black hole acquires all of the mass and all of the charge at high energy limit. Thus the black holes obey the statistical bootstrap condition and, in contrast to the other investigation, we see that U(1) charge does not break the bootstrap property.

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1. INTRODUCTION

The thermodynamics and statistical mechanics of black hole are the interesting areas of black hole physics [1]. As the black hole may radiate away completely the incoming pure states will totally evolve into the outcoming mixed states. Thus the quantum coherence is lost in the black hole decay and the unitary principle in the law of quantum mechanics is violated [2].

An attempt to resolve this problem is to take into account the effect of quantum hair [3,4]. The quantum hair can have dramatic and computable effects on the thermodynamical behavior of a black hole. It is hoped that black holes will carry a lot of quantum hairs which could generate enough effects to recover the quantum coherence.

Another approach, which is more fundamental, is to find a consistent theory of quantum gravity which can correctly describe the black hole radiation. The only candidate of the quantum gravity, to this day, is the string theory [5]. t Hooft, in a series of inspiring articles, had shown that black hole behaves like as a special case of string [6]. Thus the black holes may be thought of being made of string, in some senses. Since the strings carry a lot of massive excitations, thus a lot of quantum hairs therein may recover the quantum coherence. The microcanonical analysis [7] had found that strings obey the bootstrap condition [8,9]

$$\frac{\Omega(E)}{\rho(E)} \rightarrow 1, \quad as \quad E \rightarrow \infty. \quad (1.1)$$

where $\Omega(E)$ is the microcanonical density of states and $\rho(E)$ the degeneracy of string states. It is therefore interesting to see whether the black hole systems obey the bootstrap condition.

Historically, the bootstrap model was applied to the statistical model of the hadrons, in which the hadrons are assumed to be compounds of hadrons. The bootstrap model of hadron developed by Hagedron [8], Frautschi and others [9] in the early 70th can be used to explain the ever-increasing number resonances found in the higher energy. The statistical mechanics of black holes suggested by Harms and Leblanc [10] treated the black hole as composite objects which are made of other black holes, in the spirit of the old bootstrap model of hadron. Such a description of black hole might be of some interesting - in view of the present uncertainty concerning the black hole entropy problem [11]. In this paper we will reinvestigate the microcanonical density of states of the Schwarzschild as well as the Reissner-Nordström black holes, which had been studied by Harms and Leblanc[10].

Two kinds of the degeneracy of the states for Schwarzschild hole will be considered in section II. In the natural units ($\hbar = c = G = 1$), the first form is [4]

$$\rho(m) = C \exp(4\pi m^2), \quad (1.2)$$

where $m$ is the mass of the black hole and $C$ is a model-dependent constant. The second form is the hole with a quantized spectrum and [12-14]

$$\rho(n) = C \exp(4\pi n), \quad n = 1, 2, ... \quad (1.3)$$

In section III, two kinds of the degeneracy of the states for the Reissner-Nordström hole will be considered. The first form is [14]
\[ \rho(m, q) = C \exp[\pi (m + \sqrt{m^2 - q^2})^2], \]  

where \( m \) and \( q \) are the mass and charge of the hole, respectively. The second form is the hole with quantized spectrum and [15]

\[ \rho(n, k) = C \exp[\pi (\sqrt{n} + \sqrt{n - k})^2], \quad k, n = 1, 2, \ldots (k \leq n). \]  

Note that the hole mass is quantized in terms of the Plank mass \( \sqrt{c\hbar/G} \) while the electric charge is quantized in terms of the Plank charge \( e/\sqrt{\alpha} \).

In both cases we will set up the useful inequalities in the microcanonical density of states. These are then used to show that the most probable configuration in the gases of Schwarzschild is that one black hole acquires all of the mass in high energy limit. Thus the Schwarzschild black holes naturally obey the bootstrap property. In the Reissner-Nordström black holes system it shows that the most probable configuration is that one black hole acquires all of the mass and all of the charge at high energy limit. Thus charged black holes obey the statistical bootstrap condition and, in contrast to the other investigation [10], we see that U(1) charge does not break the bootstrap property.

**2. SCHWARZSCHILD BLACK HOLES**

**A. Black holes with continuous spectrum**

The microcanonical density of a gas of Schwarzschild black hole with continuous spectrum can be written as [9]

\[ \Omega(E, V) = \sum_{N=1}^{\infty} \Omega_N(E, V). \]  

The microcanonical density for the configuration with \( N \) black holes is [9,10]

\[ \Omega_N(E, V) = \frac{1}{N!} \left( \frac{V}{(2\pi)^3} \right)^N \prod_{i=1}^{N} \int_{m_0}^{\infty} dm_i \rho(m_i) \int_{-\infty}^{\infty} dp_i \delta(E - \sum_i E_i) \delta^3(\sum_i p_i), \]  

where \( m_0 \) is the lightest mass of the black hole, if it exists. The above equation for the density of state first presented by Frautschi (the Eq.(1.8) in reference 9) was used to investigate the statistical bootstrap model of hadrons. It was then adopted by Harms and Leblanc [10] to investigate the statistical mechanics of black hole, by regarding the black hole as the compound of black holes.

As in [10] we assume that the black holes obey the dispersion relation, \( m_i^2 = E_i^2 - p_i^2 \). Then from Eq.(1.2) we see that since \( \rho(m_i) = C \exp[4\pi(E_i^2 - p_i^2)] \) the high-momentum state in Eq.(2.2) will contribute negligibly to the momentum integration. Therefore we can neglect the momentum-conservation \( \delta \) function and Eq.(2.2) simply becomes \( N \) decoupled Gaussian integrals [9,10]

\[ \Omega_N(E, V) = \frac{1}{N!} \left( \frac{V}{(2\pi)^3} \right)^N \prod_{i=1}^{N} \int_{m_0}^{\infty} dE_i \rho(E_i) \delta(E - \sum_i E_i). \]
When \( N = 2 \) then
\[
\Omega_1(E, V) = \frac{CV}{(2\pi)^3} e^{4\pi E^2}.
\]  

(2.4)

When \( N = 2 \) then
\[
\Omega_2(E, V) = \frac{1}{2} \left[ \frac{CV}{(2\pi)^3} \right]^2 \int_{m_0}^\infty dE_1 \int_{m_0}^\infty dE_2 e^{4\pi (E_1^2 + E_2^2)} \delta(E - E_1 - E_2)
\]
\[
< \frac{1}{2} \left[ \frac{CV}{(2\pi)^3} \right]^2 \int_{m_0}^\infty dE_1 \int_{m_0}^\infty dE_2 e^{4\pi (E_1^2 + E_2^2)} \delta(E - E_1 - E_2)
\]
\[
= \frac{1}{2} \left[ \frac{CV}{(2\pi)^3} \right]^2 \int_0^E dx e^{4\pi [x^2 + (E - x)^2]}
\]
\[
= \frac{1}{2} \left[ \frac{CV}{(2\pi)^3} \right]^2 e^{4\pi E^2} F(E),
\]  

(2.5)

in which we define
\[
F(E) \equiv \int_0^E dx e^{8\pi (x^2 - Ex)} < \frac{1}{2\pi E} (1 - e^{-2\pi E^2}) < f_0 \approx 0.255,
\]  

(2.6)

as shown in the appendix A. Note that the function \( F(E) \) is increasing from zero, at \( E = 0 \), to the maximum value \( f_0 \), at \( E \approx 0.445 \), and then approaches to \( \frac{1}{2\pi E} \) at large \( E \).

To proceed, we see that Eq.(2.3) can be expressed as
\[
\Omega_N(E, V) < \frac{1}{N!} \left[ \frac{V}{(2\pi)^3} \right]^N \prod_{i=1}^N \int_0^\infty dE_i \rho(E_i) \delta(E - \sum_i E_i)
\]
\[
= \frac{1}{N!} \left[ \frac{CV}{(2\pi)^3} \right]^N e^{4\pi E^2} \int_0^E dx_1 e^{8\pi E^2 (x_1^2 - Ex_1)} \int_0^{E-x_1} dx_2 e^{8\pi E^2 [x_2^2 - (E-x_1)x_2]}
\]
\[
\times \cdots
\]
\[
\times \int_0^{E-x_1-x_2-\cdots-x_{N-3}} dx_{N-2} e^{8\pi E^2 [x_{N-2}^2 - (E-x_1-x_2-\cdots-x_{N-3})x_{N-2}]}
\]
\[
\times \int_0^{E-x_1-x_2-\cdots-x_{N-2}} dx_{N-1} e^{8\pi E^2 [x_{N-1}^2 - (E-x_1-x_2-\cdots-x_{N-2})x_{N-1}]}
\]
\[
= \frac{1}{N!} \left[ \frac{CV}{(2\pi)^3} \right]^N e^{4\pi E^2} \int_0^E dx_1 e^{8\pi E^2 (x_1^2 - Ex_1)} \int_0^{E-x_1} dx_2 e^{8\pi E^2 [x_2^2 - (E-x_1)x_2]}
\]
\[
\times \cdots
\]
\[
\times \int_0^{E-x_1-x_2-\cdots-x_{N-3}} dx_{N-2} e^{8\pi E^2 [x_{N-2}^2 - (E-x_1-x_2-\cdots-x_{N-3})x_{N-2}]}
\]
\[
\times F(E - x_1 - x_2 - \cdots - x_{N-2})
\]
\[
< \frac{1}{N!} \left[ \frac{CV}{(2\pi)^3} \right]^N (f_0)^{N-2} e^{4\pi E^2} F(E)
\]
\[
< \frac{1}{N!} \left[ \frac{CV}{(2\pi)^3} \right]^N (f_0)^{N-2} e^{4\pi E^2} \frac{1}{2\pi E} (1-e^{-2\pi E^2}),
\]
if \( N > 2 \).

Using this inequality we can obtain the relation
\[
\sum_{N=2}^{\infty} \Omega_N(E, V) < e^{4\pi E^2} \frac{1}{2\pi E} \frac{f_0^{-2}}{f_0} \exp\left[ \frac{CV}{(2\pi)^3} f_0 \right].
\]
Thus, at high energy limit
\[
E >> \frac{(2\pi)^2}{CV} f_0^{-2} \exp\left[ \frac{CV}{(2\pi)^3} f_0 \right],
\]
then
\[
\Omega_1(E, V) > \sum_{N=2}^{\infty} \Omega_N(E, V),
\]
and the microcanonical density of a gas of black holes can be approximated as
\[
\Omega(E, V) \approx \Omega_1(E, V) = \frac{CV}{(2\pi)^3} e^{4\pi E^2}.
\]
Thus the most probable configuration for a gas of Schwarzschild black holes with continuous spectrum will be that at \( N = 1 \). This implies that one hole acquires all of the mass and the bootstrap condition is obeyed.

**B. Black holes with discrete spectrum**

Next, we investigate the black holes system with discrete spectrum. The microcanonical density a gas of black holes is written as
\[
\Omega(E) = \sum_{N=1}^{\infty} \Omega(N, E).
\]
The density for the configuration with \( N \) black holes is
\[
\Omega(N, E) = \frac{1}{N!} \prod_{i=1}^{N} \sum_{l_i=1}^{\infty} p(l_i) \delta_{E, \sum_{i=1}^{N} E_{l_i}} = \frac{1}{N!} \prod_{i=1}^{N} \sum_{l_i=1}^{\infty} C g^{l_i} \delta_{E, \sum_{i=1}^{N} \sqrt{l_i}},
\]
in which \( g \equiv e^{4\pi} \) according to the Eq.(1.3). Note that the value of \( g \) (order 1) which may be model-dependent [12-14] does not affect the following analysis.

We now analyze the microcanonical density Eq.(2.13). When \( N=1 \) then
\[
\Omega(1, E) = C g^{E^2}.
\]
When \( N = 2 \) then
\[
\Omega(2, E) = \frac{C^2}{2} \sum_{l_1=1}^{\infty} \sum_{l_2=1}^{\infty} g^{l_1} g^{l_2} \delta_{E, \sqrt{l_1+l_2}}
\]
\[ \Omega(N, E) = \frac{C^2}{2} g^{E^2} \sum_{l=1}^{(E-1)^2} g^{2(l-E \sqrt{l})} \]

\[ \equiv \frac{C^2}{2} g^{E^2} K(E, g). \quad (2.15) \]

Now, through a simple calculation we can see that \( K(E, e^{4\pi}) \) is a rapidly decaying function with respect to the variable \( E \). For examples, \( K(2, e^{4\pi}) \approx 10^{-6}, K(5, e^{4\pi}) \approx 10^{-22}, \ldots, K(10, e^{4\pi}) \approx 10^{-49} \). Note that the hole mass is quantized in terms of the "Plank mass". Keep the property of the function \( K \) in mind we see that Eq.(2.13) can be expressed as

\[ \Omega(N, E) = \frac{C^n}{N!} g^{E^2} \sum_{l_1=1}^{(E-(N-1))^2} g^{2(l_1-E \sqrt{l_1})} \sum_{l_2=1}^{(E-(N-2)-\sqrt{l_2})^2} g^{2(l_2-\sqrt{l_2}(E-\sqrt{l_1}))} \]

\[ \times \cdots \]

\[ \times \sum_{l_{N-2}=1}^{(E-\sqrt{l_1}-\cdots-\sqrt{l_{N-3}})^2} g^{2(l_{N-2}-\sqrt{l_{N-2}(E-\sqrt{l_1}-\cdots-\sqrt{l_{N-3}}))} \]

\[ \times \sum_{l_{N-1}=1}^{(E-\sqrt{l_1}-\cdots-\sqrt{l_{N-2}})^2} g^{2(l_{N-1}-\sqrt{l_{N-1}(E-\sqrt{l_1}-\cdots-\sqrt{l_{N-2}}))} \]

\[ < \frac{C^n}{N!} g^{E^2} \sum_{l_1=1}^{(E-1)^2} g^{2(l_1-E \sqrt{l_1})} \sum_{l_2=1}^{(E-\sqrt{l_1})^2} g^{2(l_2-\sqrt{l_2}(E-\sqrt{l_1}))} \]

\[ \times \cdots \]

\[ \times \sum_{l_{N-2}=1}^{(E-\sqrt{l_1}-\cdots-\sqrt{l_{N-3}})^2} g^{2(l_{N-2}-\sqrt{l_{N-2}(E-\sqrt{l_1}-\cdots-\sqrt{l_{N-3}}))} \]

\[ \times \sum_{l_{N-1}=1}^{(E-\sqrt{l_1}-\cdots-\sqrt{l_{N-2}})^2} g^{2(l_{N-1}-\sqrt{l_{N-1}(E-\sqrt{l_1}-\cdots-\sqrt{l_{N-2}}))} \]

\[ < \frac{C^n}{N!} g^{E^2} \sum_{l_1=1}^{(E-1)^2} g^{2(l_1-E \sqrt{l_1})} K(1, g)^{N-1} \]

\[ < \frac{C^n}{N!} g^{E^2} K(E, g). \quad (2.16) \]

if \( N > 2 \).

Thus

\[ \sum_{n=2}^{\infty} \Omega(N, E) < g^{E^2} \left( \frac{C^2}{2!} + \frac{C^3}{3!} + \ldots \right) K(E, g) \]

\[ = g^{E^2} K(E, g)(e^C - 1). \quad (2.17) \]
Since the function $K(E, g)$ is a rapidly decaying function with respect to the variable $E$ we conclude that

$$\Omega(N, E) \approx \Omega(1, E) = CgE^2,$$  

(2.18)

if the energy of the system is sufficiently large. Thus the most probable configuration for a gas of Schwarzschild black holes with quantized spectrum will be that at $N = 1$. This means that one hole acquires all of the mass and the bootstrap condition is obeyed. Note that the term corrected to the Hawking’s temperature found in Ref.[16] in the canonical ensemble of black holes (for example, the eq.(26) in [16]) does not show in the microcanonical treatment of this paper.

3. REISSNER-NORDSTRÖM BLACK HOLES

The analysis of microcanonical density of a gas of the Reissner-Nordström black holes is very similar to that in the Schwarzschild black holes. To begin with, let us mention the main point in the section II. In there we first show that the two-hole density $\Omega_2(E, V)$ (or $\Omega(2, E)$) is less then the product of one-hole density $\Omega_1(E, V)$ (or $\Omega(1, E)$) by an energy-dependent function $F(E)$ (or $K(E, g)$). The crucial property is that this function is never larger then one and will approach to zero at high energy. This result can also be used to see that the $N$-hole density is always less then $(N - 1)$-hole density. Then, repeatedly using this property we thus show that the $N$-hole density is small then the one-hole density at high energy limit.

Therefore, the only work we now need to do is to show that the two-Reissner-Nordström-hole density is less then the product of one-Reissner-Nordström-hole density by an energy-dependent function, and this function is never larger then one and shall approach to zero at high energy.

A. Charged black holes with continuous spectrum

Let us first analyze the case with continuous spectrum. The microcanonical density for the configuration with $N$ charged black holes is [10]

$$\Omega_N(E, Q, V) = \frac{1}{N! (2\pi)^{3N}} \prod_{i=1}^{N} \int_{m_0}^{\infty} dm_i \int_{-m_i}^{m_i} dq_i \int_{-\infty}^{\infty} dp_i^3 \times \rho(m_i, q_i) \delta(E - \sum_i E_i) \delta(Q - \sum_i q_i) \delta^3(\sum_i p_i),$$  

(3.1)

where $m_0$ is the lightest mass of the black hole, if it exists. Once again, we assume that black holes obey the dispersion relation, $m_i^2 = E_i^2 - p_i^2$. Then, likes that in the neutral holes, the high-momentum state in Eq.(3.1) will contribute negligibly to the momentum integration and we can neglect the momentum-conservation $\delta$ function. Thus Eq.(3.1) becomes $N$ decoupled Gaussian integrals [10]

$$\Omega_N(E, Q, V) = \frac{1}{N! (2\pi)^{3N}} \prod_{i=1}^{N} \int_{m_0}^{\infty} dE_i \int_{-E_i}^{E_i} dq_i \rho(E_i, q_i) \delta(E - \sum_i E_i) \delta(Q - \sum_i q_i).$$  

(3.2)
We now analyze the microcanonical density Eq.(3.2). When \( N=1 \) then from Eq.(1.4) we have

\[
\Omega_1(E,Q,V) = C e^{\pi (E + \sqrt{E^2 - Q^2})^2}.
\]  

(3.3)

When \( N=2 \) then

\[
\Omega_2(E,Q,V) = \frac{1}{2} \left[ \frac{CV}{(2\pi)^3} \right]^2 \int_{j_0}^{E_1} dE_1 \int_{-E_2}^{E_2} dq_1 \int_{j_0}^{E_2} dE_2 \int_{-E_2}^{E_2} dq_2
\]
\[
\times e^{\pi (E_1 + \sqrt{E_1^2 - q_1^2})^2 + \pi (E_2 + \sqrt{E_2^2 - q_2^2})^2}
\]
\[
\times \delta(E - E_1 - E_2)\delta(Q - q_1 - q_2)
\]
\[
< \frac{1}{2} \left[ \frac{CV}{(2\pi)^3} \right]^2 \int_{0}^{E_1} dE_1 \int_{0}^{E_2} dE_1 \int_{0}^{E_2} dE_2 \int_{0}^{E_2} dq_2
\]
\[
\times e^{\pi (E_1 + \sqrt{E_1^2 - q_1^2})^2 + \pi (E_2 + \sqrt{E_2^2 - q_2^2})^2}
\]
\[
\times \delta(E - E_1 - E_2)\delta(Q - q_1 - q_2)
\]
\].

(3.4)

Since \( E = E_1 + E_2 \) we have the relation

\[
(E_1 + \sqrt{E_1^2 - q_1^2})^2 + (E_2 + \sqrt{E_2^2 - q_2^2})^2
\]
\[
= (E_1^2 + E_2^2) + (E_1^2 - q_1^2) + (E_2^2 - q_2^2) + 2E (\sqrt{E_1^2 - q_1^2} + \sqrt{E_2^2 - q_2^2})
\]
\[
- 2(E_2\sqrt{E_1^2 - q_1^2} + E_1\sqrt{E_2^2 - q_2^2})
\]
\[
< (E_1^2 + E_2^2) + (E_1^2 - q_1^2) + (E_2^2 - q_2^2) + 2E (\sqrt{E_1^2 - q_1^2} + \sqrt{E_2^2 - q_2^2}) - 2E_2\sqrt{E_1^2 - q_1^2}.
\]

(3.5)

To proceed, using the appendices B and C we see that

\[
(E_1^2 - q_1^2) + (E_2^2 - q_2^2) < E^2 - Q^2,
\]

(3.6)

\[
2E (\sqrt{E_1^2 - q_1^2} + \sqrt{E_2^2 - q_2^2}) < 2E\sqrt{E^2 - Q^2},
\]

(3.7)

\[
\int_{0}^{E_1} dq_1 \int_{0}^{E_2} dq_2 e^{\pi (E_1^2 + E_2^2)} \delta(E - E_1 - E_2) < e^{\pi E^2} G(E),
\]

(3.9)

in which we define

\[
G(E) \equiv \int_{0}^{E} dx e^{2\pi(x^2 - Ex)} < \frac{2}{\pi E} (1 - e^{-\pi E^2/2}) < g_0 \approx 0.509.
\]

(3.10)
Note that the function $G(E)$ is increasing from zero, at $E = 0$, to the maximum value $g_0$, at $E \approx 0.894$, and then approaches to $\frac{2}{\pi E}$ at large $E$.

Substituting the inequalities Eqs. (3.6)-(3.10) into Eq. (3.4) we thus have the inequality

$$\Omega_2(E, Q, V) < \frac{1}{2} \left[ \frac{CV}{(2\pi)^3} \right]^2 4h_0 G(E) \exp[\pi(E + \sqrt{E^2 - Q^2})^2].$$

(3.11)

Now we have shown that the two-hole density $\Omega_2(E, Q, V)$ is less than the product of one-hole density $\Omega_1(E, Q, V)$ by an energy-dependent function $G(E)$ which is never larger than one and will approach to zero at high energy. Then, as that in section II, after repeatedly using this property we can easily find that

$$\sum_{N=2}^{\infty} \Omega_N(E, Q, V) < \frac{\pi E^2}{2} (1 - e^{-\pi E^2/2}) (4h_0 f_0)^{-2} e^{\pi(E + \sqrt{E^2 - Q^2})^2} [CV(2\pi)^3(4h_0 f_0)].$$

(3.12)

Therefore at high energy limit

$$E >> \frac{(2\pi)^2}{CV}(4h_0 f_0)^{-2} e^{\pi(E + \sqrt{E^2 - Q^2})^2} [CV(2\pi)^3(4h_0 f_0)],$$

(3.13)

then

$$\Omega_1(E, Q, V) > \sum_{N=2}^{\infty} \Omega_N(E, Q, V),$$

(3.14)

and the microcanonical density of a gas of charged black holes can be approximated as

$$\Omega(E, Q, V) \approx \Omega_1(E, Q, V) = \frac{CV}{(2\pi)^3} \exp[\pi(E + \sqrt{E^2 - Q^2})^2].$$

(3.15)

Thus the most probable configuration for a gas of Reissner-Nordström black holes with continuous spectrum will be that at $N = 1$. This implies that one hole acquires all of the mass and all of the charge. Note that the bootstrap condition is obeyed in our analysis and thus the U(1) charge does not break the bootstrap property of the black hole, in contrast to the claim in [10].

**B. Charged black holes with discrete spectrum**

Next, we investigate the charged black holes system with discrete spectrum. The microcanonical density is written as

$$\Omega(E, Q) = \sum_{N=1}^{\infty} \Omega(N, E, Q).$$

(3.16)

The density for the configuration with $N$ charged black holes is

$$\Omega(N, E, Q) = \frac{1}{N!} \prod_{i=1}^{N} \sum_{n_i=1}^{\infty} \sum_{k_i=1}^{n_i} \rho(n_i, k_i) \delta_{E, \sum_{i=1}^{N} \sqrt{n_i} \delta_{Q, \sum_{i=1}^{N} \sqrt{k_i}}.$$
Using the similar prescription as that in deriving Eq.(2.15) we have the relation
\[ K_{\text{in}} \] in which, as mentioned in Eq.(2.15),
\[ E, Q \]

Using the inequalities Eqs.(3.21)-(3.24) we thus have the inequality
\[ \Omega(2, E, Q) = C \exp[\pi(E + \sqrt{E^2 - Q^2})^2] \].

When \( N = 2 \) then
\[ \Omega(1, E, Q) = C \exp[\pi(E + \sqrt{E^2 - Q^2})^2] \].

When \( N = 2 \) then
\[ \Omega(2, E, Q) = \frac{C^2}{2} \sum_{n_1=1}^{\infty} \sum_{k_1=1}^{n_1} \sum_{n_2=1}^{\infty} \sum_{k_2=1}^{n_2} \exp[\pi(n_1 + \sqrt{n_1 - k_1})^2] \exp[\pi(n_2 + \sqrt{n_2 - k_2})^2] \]
\[ \times \delta_{E, \sqrt{n_1} + \sqrt{n_2}} \delta_{Q, \sqrt{k_1} + \sqrt{k_2}}. \]

Since \( E = \sqrt{n_1} + \sqrt{n_2} \) we have the relation
\[ (\sqrt{n_1} + \sqrt{n_1 - k_1})^2 + (\sqrt{n_2} + \sqrt{n_2 - k_2})^2 \]
\[ = (n_1 + n_2) + (n_1 - k_1) + (n_2 - k_2) + 2E (\sqrt{n_1 \sqrt{n_1 - k_1} + \sqrt{n_2 \sqrt{n_2 - k_2}}}) \]
\[ - 2(\sqrt{n_2 \sqrt{n_1 - k_1} + \sqrt{n_1 \sqrt{n_2 - k_2}}}) \]
\[ < (n_1 + n_2) + (n_1 - k_1 + n_2 - k_2) + 2E (\sqrt{n_1 \sqrt{n_1 - k_1} + \sqrt{n_2 \sqrt{n_2 - k_2}}}) - 2\sqrt{n_2 \sqrt{n_1 - k_1}}. \]

Using the identities shown in the appendices B and C we have
\[ (n_1 - k_1 + n_2 - k_2) < E^2 - Q^2, \]
\[ 2E (\sqrt{n_1 - k_1} + \sqrt{n_2 - k_2}) < 2E \sqrt{E^2 - Q^2}, \]
\[ \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} \exp[-2\pi \sqrt{n_1 - k_1} \delta_{Q, \sqrt{n_1} + \sqrt{n_2}} < J(Q) \leq 1. \]

Using the similar prescription as that in deriving Eq.(2.15) we have the relation
\[ \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \exp[\pi(n_1 + n_2)] \delta_{E, \sqrt{n_1} + \sqrt{n_2}} < K(E, e^\pi) \exp(\pi E^2), \]

in which, as mentioned in Eq.(2.15), \( K(E, e^\pi) \) is a rapidly decaying function with respect to the variable \( E \). For examples, \( K(2, e^\pi) \approx 0.04, K(5, e^\pi) \approx 10^{-6}, \ldots, K(10, e^\pi) \approx 10^{-12} \). Using the inequalities Eqs.(3.21)-(3.24) we thus have the inequality
\[ \Omega_2(E, Q, V) < \frac{C^2}{2} K(E, e^\pi) \exp[\pi(E + \sqrt{E^2 - Q^2})^2]. \]

As we have shown that two-hole density \( \Omega(2, E, Q) \) is less then the product of one-hole density \( \Omega(1, E, Q) \) by an energy-dependent function \( K(E, e^\pi) \) which is never larger then one and approaches to zero at high energy. Then, as that in section II, repeatedly using this property we can easily find that
\[
\sum_{n=2}^{\infty} \Omega(N, E, Q) < K(E, e^\pi)(e^C - 1)\exp[\pi(E + \sqrt{E^2 - Q^2})^2].
\] (3.26)

Since the function \(K(E, e^\pi)\) is a rapidly decaying function with respect to the variable \(E\) we conclude that
\[
\Omega(N, E, Q) \approx \Omega(1, E, Q) = C\exp[\pi(E + \sqrt{E^2 - Q^2})^2],
\] (3.27)
if the energy of the system is sufficiently large. Thus the most probable configuration for a gas of Reissner-Nordström black holes with quantized spectrum will be that at \(N = 1\). This means that one hole acquires all of the mass and all of charge. Thus the bootstrap condition is still obeyed.

4. CONCLUSION

In conclusion, I have used the microcanonical treatment to study the statistical mechanics of a gas of Schwarzschild black holes or Reissner-Nordström black holes. The black holes may have the discrete spectrum or have the continuous spectrum. In these systems I have set up the inequalities in the microcanonical ensemble of \(N\) black holes. The central ideal in our treatment is first to show that the two-hole density is always less than the product of one-hole density by an energy-dependent function. This function is found to be never larger then one and will approach to zero at high energy. Once this relation is established then it can be adopted to show that the \(N\)-hole density is always less than the \((N-1)\)-hole density. Then, repeatedly using this property we thus finally show that the \(N\)-hole density is small then the one-hole density at high energy limit. Thus the most probable configuration is that with \(N = 1\), if the energy of the system is sufficiently large. This implies that the bootstrap condition is obeyed in the black holes system and \(U(1)\) charge does not break the bootstrap property.
APPENDIX A

From the definition
\[ F(E) \equiv \int_0^E dx e^{8\pi(x^2 - Ex)} = E \int_0^1 dy e^{8\pi E^2(y^2 - y)} \]
\[ = E e^{-2\pi E^2} \int_0^1 dy e^{8\pi E^2(y - 1/2)^2} = 2E e^{-2\pi E^2} \int_0^{1/2} dz e^{8\pi E^2 z^2} \]
\[ < 2E e^{-2\pi E^2} \int_0^{1/2} dz e^{4\pi E^2 z} = \frac{1}{2\pi E} (1 - e^{-2\pi E^2}), \quad (A1) \]
which is used in Eq.(2.5)

APPENDIX B

From the figure 1 we see that
\[ (E_1^2 - q_1^2) + (E_2^2 - q_2^2) \equiv \overline{ac}^2 - \overline{ab}^2 + \overline{cd}^2 - \overline{ce}^2 = \overline{cb}^2 + \overline{de}^2 \]
\[ = ef^2 + de^2 < (ef + de)^2 = ad^2 - abf^2 \]
\[ < (\overline{ac} + \overline{cd})^2 - abf^2 = E^2 - Q^2, \quad (B1) \]
which is used in Eq.(3.6).

From figure 1 we also see that
\[ \sqrt{E_1^2 - q_1^2} + \sqrt{E_2^2 - q_2^2} \equiv \sqrt{\overline{ac}^2 - \overline{ab}^2} + \sqrt{\overline{cd}^2 - \overline{ce}^2} = \overline{cb} + \overline{de} \]
\[ = ef + de = def = \sqrt{ad^2 - abf^2} \]
\[ < \sqrt{(\overline{ac} + \overline{cd})^2 - abf^2} = \sqrt{E^2 - Q^2}, \quad (B2) \]
which is used in Eq.(3.7).

Once letting \( n_1 = E_1^2, n_2 = E_2^2, k_1 = q_1^2 \) and \( k_2 = q_2^2 \), then the above inequalities also imply the relations used in Eqs.(3.21) and (3.22).

APPENDIX C

Since \( q_2 \leq E_2 \) we have
\[ \int_0^{E_1} dq_1 \int_0^{E_2} dq_2 e^{x(2\pi E_2 \sqrt{E_1^2 - q_1^2})} \delta(Q - q_1 - q_2) \]
\[ \leq \int_0^{E_1} dq_1 \int_0^{E_2} dq_2 e^{x(2\pi q_2 \sqrt{E_2^2 - q_1^2})} \delta(Q - q_1 - q_2) \]
\[ = \int_0^{E_1} dq_1 e^{x(2\pi (Q - q_1) \sqrt{E_1^2 - q_1^2})}. \quad (C1) \]

(i) When \( Q \geq E_1 \) (note that \( E = E_1 + E_2 \geq Q \)) then Eq.(C1) becomes
\[ \int_{0}^{E_1} dq_1 \exp[-2\pi (Q - q_1) \sqrt{E_1^2 - q_1^2}] \leq \int_{0}^{E_1} dq_1 \exp[-2\pi (E_1 - q_1) \sqrt{E_1^2 - q_1^2}] \equiv H(E_1). \]  

(C2)

After a simple calculation we see that the function H is increasing from zero, at \( E_1 = 0 \), to the maximum value \( h_0 \approx 0.275 \), at \( E_1 \approx 0.506 \), and then rapidly approaches to zero at large \( E_1 \).

(ii) When \( Q < E_1 \) then since \( 0 \leq q_2 = Q - q_1 \) we have to constrain the integration of \( q_1 \) in Eq.(C1) to be from zero to \( Q \), thus

\[ \int_{0}^{E_1} dq_1 \exp[-2\pi (Q - q_1) \sqrt{E_1^2 - q_1^2}] \leq \int_{0}^{Q} dq_1 \exp[-2\pi (Q - q_1) \sqrt{Q^2 - q_1^2}] \equiv H(Q). \]  

(C3)

As before, \( H(Q) < h_0 \approx 0.275 \). This establishes the inequality Eq.(3.8)

The prove the relation Eq.(3.23) we can let \( n_1 = E_1^2, n_2 = E_2^2, k_1 = q_1^2 \) and \( k_2 = q_2^2 \) in the above treatment. Then we have the relation

\[ \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} \exp[-2\pi \sqrt{n_2} \sqrt{n_1 - k_1}] \delta_{Q, \sqrt{k_1} + \sqrt{k_2}} \leq \sum_{k_1=1}^{Q^2} \exp[-2\pi (Q - \sqrt{k_1}) \sqrt{Q^2 - k_1}] = J(Q). \]  

(C4)

After a simple calculation we see that the function \( J(Q) \) is a rapidly decaying function with respect to the variable \( Q \). For examples, \( J(1) = 1, J(5) \approx 10^{-34}, \ldots, J(10) \approx 10^{-78} \).
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FIG. 1 Since $E_1 \geq q_1$ and $E_2 \geq q_2$ we can let $\bar{ac} = E_1$, $\bar{ab} = q_1$, $\bar{cd} = E_2$ and $\bar{ce} = q_2$. Note that $\angle abc = \angle ced = \pi/2$. 