PCT, SPIN AND STATISTICS, AND ANALYTIC WAVE FRONT SET

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Abstract

A new, more general derivation of the spin-statistics and PCT theorems is presented. It uses the notion of the analytic wave front set of (ultra)distributions and, in contrast to the usual approach, covers nonlocal quantum fields. The fields are defined as generalized functions with test functions of compact support in momentum space. The vacuum expectation values are thereby admitted to be arbitrarily singular in their space-time dependence. The local commutativity condition is replaced by an asymptotic commutativity condition, which develops generalizations of the microcausality axiom previously proposed.

Contents

1. Introduction 2
2. Carrier cones of analytic functionals: Basic theorems 3
3. Analytic wave front set and quasi-support 5
4. Asymptotic commutativity 7
5. Lorentz–invariant regularization 10
6. The role of Jost points in nonlocal field theory 12
7. Generalization of the spin-statistics theorem 15
8. Generalization of the PCT theorem 19
9. Concluding remarks 21

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1. Introduction

In this paper, we examine how essential the locality of interaction and the microcausality axiom are for deriving the two fundamental observed consequences of the general theory of quantum fields: the spin-statistics relation and the PCT symmetry. We show that both of them hold if local commutativity is replaced with a condition which is closer related to macrocausality and could be called asymptotic commutativity. Intuitively, it implies that the commutators of observable fields decrease for large spacelike separations of the arguments no slower than exponentially with order one and with maximum type. The precise definition of this condition given below is a refinement of that proposed in [1], where it was compared with generalizations of local commutativity suggested by other authors. Abandoning microcausality, we naturally eliminate the bounds on the high-energy (ultraviolet) behavior of the off-mass-shell amplitudes that follow from this axiom (see Sec. 9.1.D in [2] and Sec. VII.4 in [3]). In our setting, this behavior is arbitrary, and so the space-time dependence of fields can be arbitrarily singular. The usual derivation [2]–[4] of the spin-statistics relation and the PCT symmetry, which is based on the analyticity properties of vacuum expectation values in coordinate space, fails in this case, and an alternative construction of some envelopes of holomorphy in momentum space was suggested instead in [5]–[7]. In [8], it was observed that the problem can be solved using the notion of the analytic wave front set of distributions.

Here, we present all details of the new derivation of the spin-statistics and PCT theorems based on this approach. Aiming for maximal generality, we choose the Gelfand-Shilov spaces $S_0^\alpha$ as the functional domain of definition of fields. These spaces provide an enlarged framework as compared to the space $S_0^0$ used in [5]–[9], which is just the Fourier transform of Schwartz’s space $D$ of infinitely differentiable functions of compact support. In other words, the fields under study are treated as ultradistributions rather than distributions in the momentum-space variables. Some recent results [10]–[13] in the theory of analytic functionals are essential for our derivation, in particular, that every functional of the class $S_0^\alpha$ has a unique minimal carrier cone. Structure theorems concerning the properties of such a quasi-support allow handling highly singular generalized functions as easily as the standard Schwartz distributions.

The quantum theory of highly singular interactions is perhaps the most advanced branch of nonlocal field theory (see [14]–[17] for a review). A consistent theory of asymptotic states and particles was constructed within this framework, including the derivation of high-energy bounds on the scattering amplitudes [14], [18]. Using highly singular nonlocal form-factors proved efficient for the phenomenological description of strong interactions [16]. A possible interplay between this branch of field theory and string theory is of particular interest (see [1], [19]).

The work is organized as follows. Section 2 contains the above-mentioned structure theorems. In Sec. 3, a relationship between the analytic wave front set of ultradistributions and the carrier cones of their Fourier transforms is established. This relationship shows the incompatibility of certain support properties in the coordinate and momentum spaces and is a key point in our approach. In Sec. 4, the condition of asymptotic commutativity is formulated as a restriction on the matrix elements of the field commutators and is then rewritten in terms of the vacuum expectation values. In Sec. 5, the

\[^1\]From here on, the continuous dual space of a topological vector space is denoted by the same symbol with a prime.
features of the invariant ultraviolet regularization of the Lorentz-covariant functionals of class $S'_0$ are investigated. In Sec. 6, this regularization is used to demonstrate the role of the Jost points in nonlocal field theory. In Sec. 7, analogues of all main steps of the classical proof [2–4] of the spin-statistics theorem are derived for a field theory subject to the asymptotic commutativity condition, starting from the Dell’Antonio lemma and finishing with the Araki theorem, which establishes the existence of the Klein transformation that reduces fields to the normal commutation relations. In Sec. 8, a condition of weak asymptotic commutativity is defined and its equivalence to the PCT invariance is proved. We give some concluding remarks in Sec. 9.

2. Carrier cones of analytic functionals: Basic theorems

In the standard axiomatic approach [2–4], quantum fields are assumed to be tempered operator-valued distributions defined on the test functions belonging to the Schwartz space $S$ because the singularities occurring in the perturbation theory are always of finite order. However, when we seek exact solutions or consider general consequences of QFT, this assumption proves too restrictive (see [20]). To generalize the setting, we use the Gelfand-Shilov spaces $S_{\alpha}^\beta$ [21]. We recall that $S_{\alpha}^\beta(\mathbb{R}^n)$ is the union (more precisely, the inductive limit) of the family of spaces $S_{\alpha,a}^{\beta,b}(a, b > 0)$ consisting of those infinitely differentiable functions on $\mathbb{R}^n$ for which the norm

$$\|f\|_{a,b} = \sup_{x \in \mathbb{R}^n} \sup_{k,q \in \mathbb{Z}_+^n} \frac{|x^k \partial^q f(x)|}{a^{|k|} b^{|q|} k^\alpha q^\beta}$$

is finite. Clearly, the index $\alpha$ characterizes the growth rate of the functionals belonging to the dual space $S_{\alpha}^{\beta}$, and $\beta$ characterizes their singularity: in the QFT context, these indices control the respective infrared and ultraviolet behavior of fields. In accordance with the introduction, we set $\beta = 0$ in what follows. The space $S_{\alpha}^0$ is nontrivial only if $\alpha > 1$, which is assumed throughout this paper. The Fourier transformed space $\mathcal{F}(S_{\alpha}^0) = S_{\alpha}^0$ consists of functions with compact support. Therefore, the local properties of the generalized functions belonging to $S_{\alpha}^{0,\alpha}$, which are called ultradistributions of the Roumieu class $\{k^{\alpha k}\}$, are the same as those of the Schwartz tempered distributions, and their supports are defined in the standard way through a partition of unity. In contrast, the space $S_{\alpha}^0$ itself consists of entire analytic functions, i.e., in the $x$-representation, the elements of its dual space are analytic functionals for which the notion of a support is useless. Nevertheless, the functionals of this class retain a kind of angular localizability, which can be described as in [10]. To each open cone $U \subset \mathbb{R}^n$, we assign a space $S_{\alpha,a}^{0,b}(U)$, which is defined similarly to $S_{\alpha}^0$ with the only difference that in the analogue of formula (1), the supremum must be taken over $x \in U$. Every function $f \in S_{\alpha,a}^{0,b}(U)$ allows an analytic continuation to the whole space $\mathbb{C}^n$ and satisfies the estimate

$$|f(x + iy)| \leq C \exp\{-|x/a'|^{1/\alpha} + b'd(x, U) + b'|y|\},$$

where $d(\cdot, U)$ is the distance from the point to the cone $U$ and where $a'$ and $b'$ differ from $a$ and $b$ by a number factor that depends on the choice of the norm$^2$ in $\mathbb{R}^n$. This difference is unessential for the union taken over all the indices $a$ and $b$, and estimate (2) can therefore be used as a starting point for the definition of $S_{\alpha}^{0}(U)$ as well. The spaces

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$^2$As a rule this norm is assumed to be Euclidean in what follows.
over cones possess the same properties convenient from the standpoint of functional analysis as those of the original $S^0_\alpha$. Namely, these spaces are complete, barrelled, reflexive and Montel (see [17]). The space $S^0_\alpha$ is evidently a linear subspace of every $S^0_\alpha(U)$, and $S^0_\alpha(U)$ is identified with a linear subspace of $S^0_\alpha$ by the following theorem.

**Theorem 1 (density theorem).** The space $S^0_\alpha$ is sequentially dense in $S^0_\alpha(U)$ for each open cone $U \subset \mathbb{R}^n$.

We call a closed cone $K \subset \mathbb{R}^n$ a carrier cone of an analytic functional $v \in S^0_\alpha$ or say that this functional is carried by $K$ if $v$ admits a continuous extension to every space $S^0_\alpha(U)$, where $U \supset K \setminus \{0\}$. This continuity property is equivalent to the belonging $v \in S^0_\alpha(K)$, where $S^0_\alpha(K)$ is by definition the union of the spaces $S^0_\alpha(U)$ and is equipped with the inductive limit topology. The space $S^0_\alpha(\{0\})$ that corresponds to the degenerate cone $\{0\}$ consisting of the origin needs special consideration. This is the space of all entire functions satisfying the inequality $|f(z)| \leq C \exp(b|z|)$, where $z = x + iy$ and the constants $C$ and $b$ depend on $f$. From the above estimate of the test function behavior, the property of $v$ to be carried by a cone $K$ may be thought of as a faster than exponential decrease of this functional in the complement of $K$.

**Theorem 2 (quasi-localizability theorem).** If a functional $v \in S^0_\alpha$ is carried by each of the closed cones $K_1$ and $K_2$, then it is carried by their intersection.

The standard compactness arguments then imply that there is a smallest closed cone $K$ such that $v \in S^0_\alpha(K)$. This cone is called the quasi-support of $v$.

**Theorem 3 (decomposition theorem).** Every functional $v$ belonging to $S^0_\alpha$ and carried by the union of two closed cones $K_1$, $K_2$ allows a decomposition of the form $v = v_1 + v_2$, where $v_j \in S^0_\alpha(K_j)$, $j = 1, 2$.

When dealing with classes of analytic functionals possessing the properties from Theorems 1–3, it is natural to call them quasi-distributions.

If a cone $K$ is properly convex, i.e., its dual cone $K^* = \{\eta : \eta x \geq 0, \ \forall x \in K\}$ has a nonempty interior, then $e^{i\xi x} \in S^0_\alpha(K)$ for all $\Im \xi$ in this interior and the Laplace transform

$$\tilde{v}(\xi) = (2\pi)^{-n/2} (v, e^{i\xi x})$$

is well defined for each $v \in S^0_\alpha(K)$.

**Theorem 4 (Paley-Wiener-Schwartz-type theorem).** Let a functional $v \in S^0_\alpha$ be carried by a closed properly convex cone $K$. Its Laplace transform $\tilde{v}$ is analytic in the tubular domain $\mathbb{R}^n + iV$, where $V$ is the interior of $K^*$, and satisfies the estimate

$$|\tilde{v}(\xi)| \leq C_{\varepsilon,R}(V') \exp(\varepsilon |\Im \xi|^{-(\alpha-1)}) \quad (\Im \xi \in V', \ |\xi| \leq R)$$

(3)

for all $\varepsilon, R > 0$ and for each cone $V'$ such that $V' \setminus \{0\} \subset V$. If $\Im \xi \to 0$ inside a fixed $V'$, then $\tilde{v}(\xi)$ tends to the Fourier transform $\check{v}$ of the functional $v$ in the topology of $S^0_\alpha$. Furthermore, if the cone $K$ is properly convex, then the Laplace transformation $v \to \tilde{v}$ is a linear topological isomorphism of $S^0_\alpha(K)$ onto the space of functions that are analytic in $\mathbb{R}^n + iV$ and satisfy estimate (3).
Theorem 1 was proved in [12], Theorems 2 and 3 were derived in [10], and Theorem 4 was established in [11], [12] using the Hörmander $L^2$-estimates both for solutions of the system of nonhomogeneous Cauchy-Riemann equations and for solutions of the dual equation of this system. Analogous theorems hold for the functional classes $S''_{\alpha \beta}$, $0 < \beta < 1$, and the proof is even simpler in this case (see [1], [17]), but the spaces $S'_\alpha$ are of particular importance because they are universal in the sense that no restrictions are imposed on the ultraviolet behavior of fields when we use these spaces. It is worth noting that we can use the terms carrier cone and quai-support without specifying the index $\alpha$ (which is quite natural because this index characterizes the behavior at infinity, not the local properties). In fact, the condition $v \in S'_{\alpha}(K)$ straightforwardly implies that $v \in S'_{\alpha'}(K)$ for all $\alpha' < \alpha$. On the other hand, the following theorem was proved in [13].

**Theorem 5.** If $v \in S'_{\alpha} \cap S'_{\alpha'}(K)$, where $\alpha' < \alpha$, then $v \in S'_{\alpha}(K)$.

An analogous statement for $S'_{\alpha \beta}$ with $\beta \neq 0$ follows immediately from Theorem 3 in [1], while for $\beta = 0$, we must again use the Hörmander estimates. Another simple fact, which is a special case of Theorem 2 in [1], is also useful in what follows.

**Lemma 1 (convolution lemma).** For every $v \in S'_{\alpha}(K)$ and for each test function $f \in S_0'$, the convolution $(v * f)(x) = (v, f(x - \cdot))$ belongs to the space $S_0'(C)$, where $C$ is any open cone such that $C \setminus \{0\} \subset \mathbb{C} K$. Moreover, the mapping $S_{\alpha} \rightarrow S_{\alpha}(C): f \rightarrow v * f$ is continuous.

This is almost evident because shifting a test function into the interior of the complementary cone $\mathbb{C} K$ means removing it from the interior of the carrier cone $K$.

3. Analytic wave front set and quasi-support

It is well known (see, e.g., Sec. I.2.b in [22]) that the ultradistributions are embedded in the space of hyperfunctions with preservation of their supports. Therefore, we can use the general facts [23] from hyperfunction theory to analyze the analytic wave front set $WF_A(u)$ of an ultradistribution $u \in S'_{\alpha}(\mathbb{R}^n)$. We recall that $WF_A(u)$ is a closed subset of the product $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$, which is conic relative to the second variable. Its projection on the first space in the product is the smallest closed subset of $\mathbb{R}^n$ outside of which $u$ is analytic. This subset is denoted by sing supp$_A u$ and the cone in $\mathbb{R}^n \setminus \{0\}$ associated with a point $p \in$ sing supp$_A u$ is formed by those directions of a “bad” behavior of the Fourier transform of $u$ at infinity that are responsible for the nonanalyticity at this point.

**Lemma 2.** If $u \in S'_{\alpha}(\mathbb{R}^n)$ and if a closed cone $K \subset \mathbb{R}^n$ is a carrier of the Fourier transform $\hat{u}$, then

$$WF_A(u) \subset \mathbb{R}^n \times (K \setminus \{0\}).$$

The operator $u \rightarrow \hat{u}$ is the transpose of the test function transformation

$$f(x) \rightarrow (2\pi)^{-n/2} \int e^{-i p x} f(x) \, dx,$$

where the minus sign in the exponent is opposite to the sign for the (inverse) operator $v \rightarrow \hat{v}$ in Theorem 4.
Proof. We cover \( K \) with a finite number of closed properly convex cones \( K_j \) and decompose
\[
\hat{u} = \sum \hat{u}_j, \quad \hat{u}_j \in S^{0}_{\alpha}(K_j),
\]
using Theorem 3. By Theorem 4, decomposition (5) induces a representation of \( u \) in the form of a sum of boundary values of functions \( u_j(\zeta) \) analytic in the tubular domains \( \mathbb{R}^n + iV_j \), where \( V_j \) is the interior of the cone \( K_j^* \) that is the dual of \( K_j \). These boundary values coincide with the boundary values in the sense of hyperfunctions (see, e.g., Theorem 11.5 in [24]). By Theorem 9.3.4 in [23], we have the inclusion
\[
WF_{A}(u) \subset \mathbb{R}^n \times (\cup K_j^{**} \setminus \{0\}),
\]
where \( K_j^{**} = K_j \) because the cones \( K_j \) are closed and convex. By making a refinement of the shrinking it to \( K \), we obtain inclusion (4).

Remark 1. Lemma 2 improves Lemma 8.4.17 in [23], which states that for each tempered distribution \( u \in \mathbb{S}' \), the inclusion \( WF_{A}(u) \subset \mathbb{R}^n \times (L \setminus \{0\}) \) holds, where \( L \) is the limit cone of the set \( \text{supp} \hat{u} \) at infinity.

The cone \( L \) is certainly a carrier of the restriction \( \hat{u}|_S \). In fact, this cone by definition consists of the limits of various sequences \( t_\nu x_\nu \), where \( x_\nu \in \text{supp} \hat{u} \) and \( 0 < t_\nu \to 0 \). If \( L = \{0\} \), then the support of \( \hat{u} \) is a bounded set and \( \hat{u}, f_\nu) \to 0 \) for any sequence \( f_\nu \) that converges to zero in \( S_0^0(\{0\}) \) because this convergence implies the uniform convergence of all derivatives of \( f_\nu \) on compact sets. If \( L \neq \{0\} \), then the set \( \text{supp} \hat{u} \setminus U \) is compact for every open cone \( U \supset L \setminus \{0\} \). The convergence of a sequence \( f_\nu \) to zero in \( S_0^0(U) \) implies the convergence to zero in each of the norms \( \sup_{x \in U \cup B} |x^k \partial^q f(x)| \), where \( B \) is any bounded neighborhood of \( \text{supp} \hat{u} \setminus U \), and we again conclude that \( \hat{u}, f_\nu) \to 0 \).

Theorem 6. Let \( u \in S_0^0(\mathbb{R}^n) \) be a nontrivial ultradistribution whose support is contained in a properly convex cone \( V \). Then only the whole \( \mathbb{R}^n \) is a carrier cone of its Fourier transform \( \hat{u} \).

Proof. At first we assume that \( 0 \in \text{supp} u \). Then every vector belonging to the cone \( -V^* \setminus \{0\} \) is an external normal to the support at the point 0. By the Kashiwara theorem (see Theorem 9.6.6 and Corollary 9.6.8 in [23]), all nonzero elements of the linear span of the set of external normals belong to \( WF_{A}(u)_{p=0} \). The interior of \( V^* \) is not empty because the cone \( V \) is properly convex, and therefore this linear span covers \( \mathbb{R}^n \). By Lemma 2, each carrier cone of \( \hat{u} \) must then coincide with \( \mathbb{R}^n \).

We now suppose that \( 0 \notin \text{supp} u \) and that there exists an open cone \( U \) such that \( U \neq \mathbb{R}^n \) and \( \hat{u} \in S_0^0(U) \). We then consider the series of “contracted” ultradistributions
\[
\sum_{\nu=1}^{\infty} c_\nu u_\nu,
\]
where
\[
(u_\nu, g(p)) \overset{\text{def}}{=} \nu^{-n}(u, g(p/\nu)).
\]
Let \( \| \cdot \|_{U, a, b} \) be the norm of the Banach space that is dual of \( S_{a,a}^0(U) \). If
\[
0 < c_\nu < (\nu^2 \|\hat{u}_\nu\|_{U, \nu, a}^{-1}),
\]
then...
then the series \( \sum c_\nu \hat{u}_\nu \) is convergent in every space \( S_{0,0}^{\alpha,a}(U) \), and series \( [7] \) certainly converges in \( S_{0,0}^{\alpha} \). We let \( \hat{u} \) denote its sum. The coefficients \( c_\nu \) can be chosen so that the support of \( \hat{u} \) contains the point 0. In fact, let \( p_0 \) be the point of \( \text{supp} \ u \) nearest to 0. We can assume without loss of generality that \( |p_0| = 1 \). For each \( \mu = 1, 2, \ldots \), there exists a function \( g_\mu \in S_{0,0}^{\alpha} \) with support in the ball \( |p - p_0/\mu| < (1/2) (1/\mu - 1/(\mu + 1)) \) such that \( (u_\mu, g_\mu) = 1 \). Note that \( \text{supp} \ u_\nu \cap \text{supp} \ g_\mu = \emptyset \) for \( \nu < \mu \). We sequentially specify the coefficients \( c_\nu \) by imposing the conditions
\[
c_\nu |(u_\nu, g_\mu)| \leq a_\mu/2^\nu \quad \text{for} \quad \mu < \nu,
\]
in addition to (8). Then
\[
(u_\nu, g_\mu) = a_\mu + \sum_{\nu > \mu} c_\nu (u_\nu, g_\mu) \neq 0
\]
for each \( \mu \). Therefore \( 0 \in \text{supp} \ \hat{u} \), and we have the situation examined above, which completes the proof.

Theorem 6 can be reformulated as a uniqueness theorem, which is used below. Namely, if we known that an ultradistribution \( u \in S_{0,0}^{\alpha}(\mathbb{R}^n) \) has support in a properly convex cone and \( \hat{u} \) is carried by a cone different from \( \mathbb{R}^n \), then \( u \equiv 0 \).

4. Asymptotic commutativity

We now consider a finite family of fields \( \{\phi_\ell\} \), \( \ell = 1, \ldots, I \), that are operator-valued generalized functions defined on the space \( S_{0,0}^{\alpha}(\mathbb{R}^4) \), \( \alpha > 1 \), and transform according to irreducible representations of the proper Lorentz group \( L_+ \) or its covering group \( SL(2, \mathbb{C}) \). We adopt all the standard assumptions of the Wightman axiomatic approach \([2]–[4]\), except local commutativity, which cannot be formulated in terms of the test functions belonging to \( S_{0,0}^{\alpha} \) because they are entire analytic in coordinate space. As usual, we let \( D_0 \) denote the minimal common invariant domain, which is assumed to be dense, of the field operators in the Hilbert space \( \mathcal{H} \) of states, i.e., the vector subspace that is spanned by the vacuum state \( \Psi_0 \) and by various vectors of the form
\[
\phi_{\ell_1 \ell_1}(f_1) \cdots \phi_{\ell_n \ell_n}(f_n)\Psi_0, \quad n = 1, 2, \ldots,
\]
where \( f_k \in S_{0,0}^{\alpha}(\mathbb{R}^4) \) and \( \ell_k \) are the Lorentzian indices.

**Definition 1.** The field components \( \phi_{i \ell} \) and \( \phi_{i' \ell'} \) commute (anticommute) asymptotically for large spacelike separation of their arguments if the analytic functional
\[
\langle \Phi, [\phi_{i \ell}(x), \phi_{i' \ell'}(x')] \rangle_{(+)} - \Psi
\]
is carried by the closed cone \( \hat{V} = \{(x, x') \in \mathbb{R}^8 : (x - x')^2 \geq 0\} \) for any \( \Phi, \Psi \in D_0 \).

The matrix element \([9]\) can be treated as a generalized function on \( S_{0,0}^{\alpha}(\mathbb{R}^8) \) because (see \([12]\)) this space coincides with the \( \tau_\ell \)-completed tensor product \( S_{0,0}^{\alpha}(\mathbb{R}^4) \otimes_1 S_{0,0}^{\alpha}(\mathbb{R}^4) \) whose dual space, by definition of the inductive topology \( \tau_\ell \), is canonically isomorphic to the space of bilinear separately continuous forms on \( S_{0,0}^{\alpha}(\mathbb{R}^4) \times S_{0,0}^{\alpha}(\mathbb{R}^4) \). Such a coincidence also obtains in the case of the tensor product of several spaces; therefore the \( n \)-point vacuum expectation values of fields uniquely determine Wightman generalized
functions \( W_{\alpha_{1}, \ldots, \alpha_{n}} \in S^{0}_{\alpha}(\mathbb{R}^{4n}) \). We identify these objects in what follows, just as in the standard scheme [2]–[4]. Next we routinely define the expressions

\[
\int \phi_{1,\ell_{1}}(x_{1}) \cdots \phi_{n,\ell_{n}}(x_{n}) f(x_{1}, \ldots, x_{n}) \, dx_{1} \cdots dx_{n} \Psi_{0} \quad (n = 1, 2, \ldots),
\]

where \( f \in S^{0}_{\alpha}(\mathbb{R}^{4n}) \). Namely, vector \( \Psi \) is the limit of the sequence

\[
\Psi_{N} = \sum_{\nu=1}^{N} \phi_{1,\ell_{1}}(f_{1}^{\nu}) \cdots \phi_{n,\ell_{n}}(f_{n}^{\nu}) \Psi_{0},
\]

where

\[
\sum_{\nu=1}^{N} f_{1}^{\nu}(x_{1}) \cdots f_{n}^{\nu}(x_{n}) = f_{N} \in S^{0}_{\alpha}(\mathbb{R}^{4})^\otimes n
\]

and \( f_{N} \to f \) in \( S^{0}_{\alpha}(\mathbb{R}^{4n}) \). The set of vectors of form \( \Psi \) spans a subspace \( D_{1} \supset D_{0} \) to which every operator \( \phi_{i}(f) \) can be extended by continuity. The field operators comprise an irreducible system, which can be established in a manner analogous to that used in the standard Wightman framework [2]. Let \( M(f) = \phi_{1,\ell_{1}}(f_{1}) \cdots \phi_{n,\ell_{n}}(f_{n}) \) be a monomial in the field components and the operator \( T \) implement the space-time translations. If a bounded operator \( B \) acting in \( \mathcal{H} \) commutes weakly with all the field operators, then

\[
\langle M(f)^* \Psi_{0}, B \Psi_{0} \rangle = \langle \Psi_{0}, B M(f) \Psi_{0} \rangle.
\]

Replacing \( M(f) \) here with the monomial \( T(r)M(f)T^{-1}(r) = M(f(\cdot - r)) \) and taking the translation invariance of the vacuum into account, we obtain

\[
\langle M(f)^* \Psi_{0}, T^{-1}(r)B \Psi_{0} \rangle = \langle B^* \Psi_{0}, T(r)M(f) \Psi_{0} \rangle.
\]

This function of \( r \) is smooth and bounded by the constant \( \|B^* \Psi_{0}\| \cdot \|M(f) \Psi_{0}\| \). Hence, its Fourier transform is a tempered distribution. By the spectral condition, the Fourier transform of the left-hand side of \( 11 \) is supported in the closed forward light cone \( \mathbb{V}_{+} \), while that of the right-hand side is supported in \( \mathbb{V}_{-} \). Therefore, the support is the one-point set \( \{0\} \), and function \( 11 \) is a bounded polynomial, i.e., a constant. The fact that the left-hand side of \( 11 \) is independent of \( r \) implies the invariance of the vector \( B \Psi_{0} \) with respect to the space-time translations if we take into account that the monomial \( M(f) \) is arbitrary and \( D_{0} \) is dense in \( \mathcal{H} \). Because the vacuum is assumed to be the only invariant state, it follows that \( B \Psi_{0} = \lambda \Psi_{0} \). Writing

\[
\langle M(f)^* \Phi, B \Psi_{0} \rangle = \langle \Phi, BM(f) \Psi_{0} \rangle = \lambda \langle \Phi, M(f) \Psi_{0} \rangle,
\]

where \( \Phi \in D_{0} \), and again using the denseness of \( D_{0} \) in \( \mathcal{H} \), we obtain \( B = \lambda I \), as was to be proved.

We replace the local commutativity axiom with the asymptotic commutativity condition, which means that any two field components either commute or anticommute asymptotically. This condition is evidently weaker than local commutativity in the sense that it is certainly fulfilled for the restrictions of local fields to \( S^{0}_{\alpha}(\mathbb{R}^{4}) \). The standard considerations of Lorentz covariance imply that the type of commutation relations depends only on the type of the participating fields, not on their Lorentzian indices. Because of this we drop these indices in what follows.
We now consider the generalized function determined by the vacuum expectation value
\[
\langle \Psi_0, \phi_{\ell_1}(x_1) \ldots \phi_{\ell_{k-1}}(x_{k-1})|\phi_{\ell_k}(x_k), \phi_{\ell_{k+1}}(x_{k+1})] - \phi_{\ell_{k+2}}(x_{k+2}) \ldots \phi_{\ell_n}(x_n)\rangle_0, \tag{12}
\]
where the sign $-$ or $+$ corresponds to the type of commutation relation between the fields $\phi_{\ell_k}$ and $\phi_{\ell_{k+1}}$.

**Lemma 3.** The asymptotic commutativity condition implies that the functional defined by (12) on $S_0^0(\mathbb{R}^{4n})$ extends continuously to the space $S_0^0(\mathbb{R}^{4(k-1)} \times U \times \mathbb{R}^{4(n-k-1)})$, where $U$ is any open cone in $\mathbb{R}^8$ such that $\mathcal{V} \setminus \{0\} \subset U$. Hence, this functional is carried by the closed cone $\mathcal{V}_{n,k} = \mathbb{R}^{4(k-1)} \times \mathcal{V} \times \mathbb{R}^{4(n-k-1)}$.

**Proof.** First, we examine the simplest nontrivial case where $n = 3$ and $k = 1$. We then have a bilinear separately continuous form on $S_0^0(\mathbb{R}^8) \times S_0^0(\mathbb{R}^4)$. The asymptotic commutativity condition implies that it remains separately continuous after giving $S_0^0(\mathbb{R}^8)$ the topology induced by that of $S_0^0(U)$. It is not immediately clear that the extension by continuity to the latter space in the first argument, which is possible when the second argument is held fixed, yields a bilinear separately continuous form again. However, the space $S_0^0(\mathbb{R}^4)$, being the inductive limit of Fréchet spaces, is barrelled, and we can therefore use standard facts on the extension of bilinear mappings. By Theorem III.5.2 in [25], the form under consideration is $\mathcal{B}$-hypocontinuous, where $\mathcal{B}$ is the family of all bounded subsets of the space $S_0^0(\mathbb{R}^8)$ under the topology induced by that of $S_0^0(U)$. By Theorem 1, the space $S_0^0(\mathbb{R}^8)$ is sequentially dense in $S_0^0(U)$, and hence the family of closures of the specified bounded subsets in $S_0^0(U)$ covers $S_0^0(U)$. We now can apply Theorem III.5.4 in [25], which shows that the extension in question is indeed bilinear and separately continuous. It can therefore be identified with a continuous linear functional on $S_0^0(U \times \mathbb{R}^4) = S_0^0(U) \hat{\otimes}_1 S_0^0(\mathbb{R}^4)$, as was to be proved.

For $n = 4$ and $k = 1$, we consider expression (12) as a bilinear form on $S_0^0(\mathbb{R}^{12}) \times S_0^0(\mathbb{R}^4)$. The previous considerations show that it is separately continuous under the topology induced on $S_0^0(\mathbb{R}^{12})$ by that of $S_0^0(U \times \mathbb{R}^4)$. Repeating this reasoning and this time exploiting the denseness of $S_0^0(\mathbb{R}^{12})$ in $S_0^0(U \times \mathbb{R}^4)$, we obtain a unique continuous extension to the space $S_0^0(U \times \mathbb{R}^8)$. In the general case, Lemma 3 can be proved by induction on $n$.

**Corollary.** If the asymptotic commutativity condition is satisfied for the domain $D_0$, then it holds for the larger domain $D_1$ composed of various finite linear combinations of the vacuum and vectors of form (11).

As an application of Theorem 6, we note that under the asymptotic commutativity condition, the closure of the subspace $L \subset \mathcal{H}$ consisting of vectors of form (10), with the fixed indices $\ell_1, \ldots, \ell_n$ coincides with the closure of any subspace spanned by vectors of the same type but with a different order of the field operators. We let $\Psi(f)$ denote vector (10), $\pi$ denote a permutation of the indices $1, \ldots, n$, and $\Psi_{\pi}(f)$ denote the vector that corresponds to the new arrangement of the operators. The orthogonality of a vector $\Phi$ to the subspace $L$ means that the functional $\langle \Phi, \Psi(f) \rangle$ is identically zero. But then the functional $\langle \Phi, \Psi_{\pi}(f) \rangle$ is carried by a cone different from $\mathbb{R}^{4n}$. Hence, it is also equal to zero because, according to the spectral condition, its Fourier transform is supported in a properly convex cone.
5. Lorentz–invariant regularization

Let $u$ be a Lorentz-covariant ultradistribution defined on a test function space $S_0^\alpha(\mathbb{R}^4)$, $\alpha > 1$, and taking values in a finite-dimensional vector space $\mathcal{E}$ on which the group $SL(2, \mathbb{C})$ acts via a representation $T$. We regularize the asymptotic behavior of $u$ at infinity by multiplying it with an invariant function of the form

$$\omega(p/\mu) = \omega_0((p \cdot p)/\mu^2),$$

(13)

where $p \cdot p$ is the Lorentz square of the vector $p$ and $\omega_0 \in S_0^{\alpha'}(\mathbb{R})$, $1 < \alpha' < \alpha$, $\text{supp} \omega_0 \subset (-1, 1)$, and $\omega_0(t) = 1$ for $|t| \leq 1/2$.

Because the mapping $p \to p \cdot p$ is analytic, the function $\omega$ belongs to the Gevrey class $C^k$ with $L = (n + 1)^{\alpha'}$ (see Proposition 8.4.1 in [23]). This means that $\omega$ satisfies the estimate

$$\sup_{p \in B} |\partial^k \omega(p)| \leq C_B h_B^{k/2}$$

(14)

for every compact set $B \subset \mathbb{R}^4$. The constant $h_B$ grows with increasing $B$ and because of this, the parameter $\alpha'$ must be taken different from $\alpha$ and as close to unity as possible (see below). Using (14), we can easily verify that $\omega$ is a multiplier for $S_0^\alpha(\mathbb{R}^4)$. Evidently,

$$u_{\text{reg}} \overset{\text{def}}{=} u \omega(p/\mu) \to u$$

in the topology of $S_0^{\alpha}(\mathbb{R}^4, \mathcal{E})$ as $\mu \to \infty$.

**Theorem 7.** Let $u \in S_0^{\alpha'}(\mathbb{R}^4, \mathcal{E})$ be a Lorentz-covariant ultradistribution and $\omega$ be defined by (13). Then the regularized functional $u_{\text{reg}}$ (more precisely, the restriction $u_{\text{reg}}|S_0^{\alpha'}$) has a continuous extension to the space $S_0^{\alpha'}$. In particular, if $\alpha > 2$ and $\alpha' < \alpha - 1$, then the Fourier transform of $u_{\text{reg}}$ is strictly localizable (because $S_0^{\alpha' - \alpha}$ then contains functions of compact support).

**Proof.** First of all we note that $S_0^{\alpha'}$ is dense in $S_0^{\alpha}$ as well as in $S_0^{\alpha' - \alpha'}$. Therefore, the desired extension is unique. We first consider the simplest case where $u$ is Lorentz invariant. We can set $\mu = 1$ without loss of generality. As shown in [20], the possibility of the extension depends on the asymptotic behavior of the smoothed functional, i.e., the convolution $(u_{\text{reg}} \ast g)(q) = (u_{\text{reg}}, g(q - \cdot))$, $g \in S_0^{\alpha'}$. For simplicity, we assume that $\text{supp} g$ is contained in the ball $|p| < 1$, set $q^2 = q^3 = 0$ and use the light-cone variables $q^\pm = (q^0 \pm q^1)/\sqrt{2}$. We also set $|q^-| \leq q^+$, $q^+ > 1$ and let $\Lambda$ denote the boost $p^+ \to p^+/q^+$, $p^- \to q^+p^-$ in the plane $(p^0, p^3)$. Because $u$ and $\omega$ are Lorentz invariant, we have

$$(u_{\text{reg}} \ast g)(q) = (u, g_q), \quad g_q(p) \overset{\text{def}}{=} \omega(p)g(q - \Lambda^{-1}p).$$

(15)

We now estimate the values of a functional $u \in S_0^{\alpha}$ on test functions with support in the ball $|p| \leq B$:

$$|(u, g_q)| \leq \|u\|_{\alpha, A, B}\|g_q\|_{\alpha, A, B},$$

(16)

where

$$\|g\|_{\alpha, A, B} = \sup_{|p| \leq B} \sup_{k \in \mathbb{Z}^4} \frac{|\partial^k g(p)|}{A^k k! k_{\alpha k}},$$

(17)

by the definition of the topology of $S_0^{\alpha}$. The points of $\text{supp} g_q$ satisfy the inequalities $|p \cdot p| < 1$ and $(p^2)^2 + (p^3)^2 < 1$ by construction, and hence $|p^+ p^-| < 1$. Furthermore, $|q^+ - q^+ p^+| < 1$ and, as a consequence, $|p^-| < 1/(1 - 1/q^+)$.

Therefore, $\text{supp} g_q$ is
contained in the ball of radius 2 if $q^+$ is large enough. In estimating the derivatives we take into account that the norm $\| \cdot \|_{\alpha',a,1}$ with a suitable $a$ is finite for the function $g$ and that the transformation $\Lambda^{-1}$ contracts the graph of the function in the variable $p^+$ in $q^+$ times. Because the change to the light-cone variables, as well as any linear transformation of coordinates, is an automorphism of the Gelfand-Shilov spaces, we obtain

$$|\partial^k g(q - \Lambda^{-1}p)| \leq \sup_{p \in \mathbb{R}^4} |\partial^k g(\Lambda^{-1}p)| \leq \|g\|_{\alpha',a,1}(a'q^+)^{|k|}k^{\alpha'k}, \quad (18)$$

where $a'$ differs from $a$ by a number factor. (It is easy to verify that $a' \leq 2^{\alpha'+1}a$, but this refinement is unessential in what follows.) Inequalities (14) and (18) combined with the Leibnitz formula give

$$|\partial^k g_q(p)| \leq C\|g\|_{\alpha',a,1}(a'q^+ + h)^{|k|}k^{\alpha'k},$$

where $h$ corresponds to the compact set $|p| \leq 2$. Therefore,

$$|(u_{\text{reg}} * g)(q)| \leq C\|u\|_{\alpha,A,2} \|g\|_{\alpha',a,1} \sup_k \frac{(a'q^+ + h)^{|k|}}{A^{k|\alpha' - \alpha|k}} \leq C\|g\|_{\alpha',a,1} \exp \left\{ (a''|q|/A)^{1/(\alpha' - \alpha)} \right\}. \quad (19)$$

In other quadrants of the plane $q^2 = q^3 = 0$, the asymptotic behavior of the convolution can be similarly estimated using $|q^+|$ or $1/|q^-|$ as a boost parameter. For $q$ ranging a bounded set, the use of a boost is unnecessary and the convolution is obviously majorized by the constant $C''\|g\|_{\alpha',a,1}$. Further, any vector $q \in \mathbb{R}^4$ can be carried to a point $\tilde{q}$ of this plane by a suitable space rotation $R$ and $(u_{\text{reg}} * g)(q) = (u_{\text{reg}} * g_R)(\tilde{q})$, where $g_R(\cdot) = g(R^{-1}(\cdot))$. The correspondence $g \rightarrow g_R$ is a continuous mapping from $S_{0,a}^{\alpha}$ to $S_{0,a}^{\alpha'}$ (where $\tilde{a} \leq 3a$ as is easy to see). Hence, (19) is satisfied in the whole space $\mathbb{R}^4$, possibly with some new constants instead of $C'$ and $a''$. Because $A$ can be taken arbitrarily large, estimate (19) implies that $u_{\text{reg}}$ increases no faster than exponentially of order $1/(\alpha - \alpha')$ and type zero. By Proposition 1 in [26], it follows that $u_{\text{reg}}$ has an extension to the space $S_{\alpha - \alpha'}^{\alpha'}$, which can be determined by the formula

$$(u_{\text{reg}}, g) = \int (u_{\text{reg}}, \chi_0(q - \cdot)g(\cdot)) \, dq, \quad g \in S_{\alpha - \alpha'}^{\alpha'}, \quad (20)$$

where $\chi_0$ is an element of $S_0^{\alpha'}(\mathbb{R}^4)$ with support in the unit ball and such that $\int \chi_0(p) \, dp = 1$. Inequality (19) implies that the integral in the right-hand side of (20) does exist (see [26] for more details). In the general case of a Lorentz-covariant generalized function $u_t$, the only complication is due to the $q$ dependence of the matrix elements $T_{\ell'}(\Lambda^{-1})$ of the representation under which $u_t$ transforms, because the analogue of formula (15) involves these matrix elements. However, this dependence is polinomially bounded and has no effect on the exponential estimates. Theorem 7 is thus proved.

We also need the following result for the case of several variables $(p_1, \ldots, p_n) \in \mathbb{R}^{4n}$.

**Theorem 8.** Let $u$ be a Lorentz-covariant ultradistribution taking values in $\mathcal{E}$ and defined on $S_0^{\alpha}(\mathbb{R}^{4n})$. If supp $u \subset \overline{V}_+ \times \cdots \times \overline{V}_+ = \overline{V}_+$ and the regularizing multiplier has the form $\omega(p) = \omega_0(P \cdot p)$, where $P = \sum_{i=1}^n p_i$ and $\omega_0 \in S_0^{\alpha}(\mathbb{R})$, $1 < \alpha' < \alpha$, then the conclusion of Theorem 7 holds for $u_{\text{reg}} = u \omega(p/\mu)$.
Proof. We again consider the convolution \((u_{\text{reg}} * g)(q), g \in S^o_0\), assuming now that \(\text{supp} \, g\) is contained in the set \(\{p \in \mathbb{R}^{4n} : |p| < 1/n\}\). Letting \(Q = \sum q_i\), setting \(Q^2 = Q^+ = 0\) and assuming that \(|Q^-| < Q^+, \, Q^+ > 1\), we use the transformation \(\Lambda : p_i^+ \rightarrow p_i^+/Q^+, \, p_i^- \rightarrow Q^+ p_i^-\), \(i = 1, \ldots, n\). For the points at which the function \(g_\Lambda = \omega(p)g(q - \Lambda^{-1}p)\) differs from zero, the inequalities \(|P \cdot P| < 1, \,(P^2)^2 + (P^3)^2 < 1,\) and \(|Q^+ - Q^+P^+| < 1\) hold. Therefore, the previous reasoning shows that \(\text{supp} \, g_\Lambda\) is contained in the set \(|P| < 2\) if \(Q^+\) is large enough. We fix a neighborhood \(U\) of \(\text{supp} \, u\) by taking the union of a neighborhood of the origin with the product \(U^n\), where \(U\) is an open properly convex cone in \(\mathbb{R}^4\) containing \(\mathbb{Q}_+ \setminus \{0\}\). For the points of \(U^n\), the inequality \(|p| < \theta|P|\) holds with some constant \(\theta > 0\) because otherwise we could find a sequence of points \(p_{\nu} \in U^n\) such that \(|p_{\nu}| = 1\) and \(|P_{\nu}| < 1/\nu\). Then we could choose a convergent subsequence whose limit \(\bar{p}\) is a nonzero vector in \(U^n\) such that \(|\bar{P}| = 0\), contradicting the assumption that the cone \(U\) is properly convex. Therefore, the set \(\text{supp} \, g_\Lambda \cap U\) lies in the ball of radius \(2\theta\). Let \(\chi\) be a multiplier for \(S^o_0\) equal to unity in a neighborhood of \(\text{supp} \, u\) and zero outside \(U\). Then \((u, g_\Lambda) = (u, \chi g_\Lambda)\). It remains to estimate the norm \(\|\chi g_\Lambda\|_{a,A,2\theta}\). This reduces to a minor modification of the arguments used in the derivation of inequality (19) and yields the same result, which completes the proof.

6. The role of Jost points in nonlocal field theory

In local field theory [2]–[4], the real points of the extended domain of analyticity of the Wightman functions \(\mathcal{W}(x_1, \ldots, x_n)\) are referred to as Jost points. The Bargman-Hall-Wightman theorem shows that this extension is obtainable by applying various complex Lorentz transformations to the primitive domain of analyticity determined by the spectral condition. In terms of the difference variables \(\xi_k = x_k - x_{k+1}, k = 1, \ldots, n-1\), on which the Wightman functions actually depend because of the translation invariance, the set of Jost points is written as

\[
\mathbb{J}_{n-1} = \left\{ \xi \in \mathbb{R}^{4(n-1)} : \left( \sum \lambda_k \xi_k \right)^2 < 0 \quad \forall \lambda_k \geq 0, \sum \lambda_k \neq 0 \right\}. \tag{21}
\]

We let \(\mathcal{J}_n\) denote the inverse image of this open cone in \(\mathbb{R}^{4n}\). If the vacuum expectation values grow faster than exponentially of order one in momentum space, then the domain of analyticity in coordinate space is empty because such growth is incompatible with the Laplace transformation. However, Lücke observed [7] that the Jost points still play an important part in this essentially nonlocal case. In this section, we prove that for arbitrary high-energy behavior, the complement of the Jost cone is a carrier for some combinations of vacuum expectation values arising when deriving the spin-statistics relation and PCT symmetry.

We first pass to the difference variables, which requires a slightly more involved argument than in the standard theory [2]–[4] of tempered fields.

Lemma 4. For every translation-invariant functional \(\mathcal{W} \in S^o_{0}(\mathbb{R}^{4n})\), there exists a functional \(W \in S^o_{0}(\mathbb{R}^{4(n-1)})\) such that

\[
(\mathcal{W}, f) = \left( W, \int f_t(\xi) \, d\xi_n \right), \quad \text{where} \quad f_t(\xi) = f(\xi_1 + \cdots + \xi_n, \xi_2 + \cdots + \xi_n, \ldots, \xi_n). \tag{22}
\]

The condition \(W \in S^o_{0}(U)\), where \(U\) is an open cone in \(\mathbb{R}^{4(n-1)}\), amounts to the condition \(\mathcal{W} \in S^o_{0}(U)\), where \(U = \{ x \in \mathbb{R}^{4n} : (x_1 - x_2, \ldots, x_{n-1} - x_n) \in U \}\).

12
\[ t : (x_1, \ldots, x_n) \to (\xi_1 = x_1 - x_2, \ldots, \xi_{n-1} = x_{n-1} - x_n, \xi_n = x_n), \quad (23) \]

that takes each test function \( f(\xi) \) to \( f_t(\xi) = f(t^{-1}\xi) \), is an automorphism of \( S_\alpha^0(\mathbb{R}^{4n}) \), and the integration over \( \xi_n \) maps this space continuously onto \( S_\alpha^0(\mathbb{R}^{4(n-1)}) \). Hence, assigning a functional \( \mathcal{W} \) to each \( W \in S_\alpha^0(\mathbb{R}^{4(n-1)}) \) by formula (22), we obtain an injective mapping \( S_\alpha^0(\mathbb{R}^{4(n-1)}) \to S_\alpha^0(\mathbb{R}^{4n}) \), which is evidently continuous under the weak topologies of the dual spaces. Lemma 4 asserts that every translation-invariant functional \( \mathcal{W} \) belongs to the range of this mapping. In fact, its regularization through the convolution by a \( \delta \)-function-like sequence of test functions yields a sequence \( \mathcal{W}_\nu \) for which representation (22) is obviously valid with smooth functions \( W_\nu \). The sequence \( W_\nu \) is weakly fundamental, and because \( S_\alpha^0 \) is a Montel space,\(^4\) it converges to a functional \( W \) whose image is \( \mathcal{W} \).

The second conclusion of Lemma 4 is also evident because transformation (23) converts the subspace of functionals invariant with respect to simultaneous translation in all variables into the subspace of functionals invariant with respect to translation in the last variable and because the indicator function in the definition of \( S_\alpha^0(U) \) can be taken in the multiplicative form, i.e., as the product of a function depending on \( \xi_n \) and a function of the remaining variables \( \xi_k \) (see Sec. 3 in [12]), which completes the proof.

Let \( K \) be a carrier cone of \( W \). It follows from Lemma 4 that its inverse image \( K \) in \( \mathbb{R}^{4n} \) is a carrier of \( \mathcal{W} \). In fact, if an open cone \( U \) contains the cone \( K \setminus \{0\} \) and shrinks to it, then \( U \) is contained in any given conic neighborhood of \( K \).

**Theorem 9.** Let \( \phi \) be a field defined on the space \( S_\alpha^0(\mathbb{R}^4) \), \( \alpha > 2 \), and transforming according to an irreducible representation of the group \( SL(2, \mathbb{C}) \). Let \( \mathcal{W}(x_1, x_2) \) denote the generalized Wightman function determined by the vacuum expectation value \( \langle \Psi_0, \phi(x_1)\phi^*(x_2)\Psi_0 \rangle \). If \( \phi \) has an integer spin, then, as a consequence of the Poincaré covariance and the spectral condition, the difference

\[ \mathcal{W}(x_1, x_2) - \mathcal{W}(x_2, x_1) \]

is carried by the cone \( \mathcal{V} = \mathcal{C} J_2 \). In the case of half-integer spin, this cone is a carrier of the sum

\[ \mathcal{W}(x_1, x_2) + \mathcal{W}(x_2, x_1). \]

**Proof.** Lemma 4 reduces the problem to the derivation of the corresponding properties for the functional \( W \in S_\alpha^0(\mathbb{R}^4) \). We apply the ultraviolet regularization described above and use the notation \( \tilde{W}_\mu(p) = \tilde{W}(p) \omega(p/\mu) \), where \( \omega \) is chosen as in Sec. 5. Because of the spectral condition, \( \text{supp} \tilde{W}_\mu \) lies in the cone \( \mathcal{V}_+ \) and by Theorem 7, the functional \( \tilde{W}_\mu \) is defined on the space \( S_{\alpha'-\alpha}^0 \), where \( \alpha - \alpha' > 1 \), i.e., the growth of \( \tilde{W}_\mu \) at infinity is no worse than exponential of an order less than one. Therefore, \( \tilde{W}_\mu \) has an (inverse) Laplace transform \( \mathcal{W}_\mu(\zeta) \) holomorphic in the usual tubular domain \( \mathcal{T}_+ = \mathbb{R}^4 - i\mathcal{V}_+ \), whose boundary value is \( W_\mu \in S_{\alpha'-\alpha}^0 \) (see, e.g., Theorem 4 in [20], where details of the extension of the Paley-Wiener-Schwartz theorem to the generalized functions of this class were set forth). Because the regularization preserves

\(^4\)Instead of “Montel space,” which is currently conventional, the term “perfect space” was used in [21].
the Lorentz covariance, we can apply the Bargman-Hall-Wightman theorem \[2 \text{-} 4\] to $W_{\mu}(\zeta)$, which shows that this function allows an analytic continuation to the extended domain $T_+^{\text{ext}}$ and the continued function is covariant under the complex Lorentz group $L_+(\mathbb{C})$. For the field combination $\phi \phi^*$ in question, the transformation properties of the analytic Wightman function under the space-time reflection $PT \in L_+(\mathbb{C})$ are

$$W_{\mu}(\zeta) = \pm W_{\mu}(-\zeta),$$

where (from here on) the upper and lower signs correspond to the respective fields with integer and half-integer spins. This transformation law is the basic point, just as in the classical proof of the spin-statistics theorem \[2 \text{-} 4\]. Since $T_+^{\text{ext}}$ contains all spacelike points, relation \[26\] implies that the generalized function $F_{\mu} \overset{\text{def}}{=} W_{\mu}(\xi) \mp W_{\mu}(-\xi)$ has support in the closed light cone $\bar{\mathbb{V}}$ and therefore allows a continuous extension to the space $S^{0}_{\alpha'}(\mathbb{V})$. In fact, this extension can be defined by the formula $(\tilde{F}_{\mu}, f) = (F_{\mu}, \chi f)$, where $\chi$ is a multiplier for $S^{\alpha - \alpha'}_{\alpha'}$, which is identically equal to unity in an $\epsilon$-neighborhood of $\bar{\mathbb{V}}$ and vanishes outside the $2\epsilon$-neighborhood. Such a multiplier satisfies the estimate

$$|\partial^\alpha \chi(x)| \leq Ch^{|\alpha|} q^{(\alpha - \alpha')|q|},$$

while the derivatives of any function $f \in S^{0}_{\alpha'}(\mathbb{V})$ satisfies the inequalities

$$|\partial^\alpha f(x)| \leq C_\epsilon \|f\|_{a,b} b^{|\alpha|} \exp\{-|x/a|^{1/\alpha'}\}$$

on its support, as is easy to verify using the Taylor formula. Hence, the multiplication by $\chi$ continuously maps $S^{0}_{\alpha'}(\mathbb{V})$ into $S^{\alpha - \alpha'}_{\alpha'}$. It is important that the extensions $\tilde{F}_{\mu}$ are compatible with each other if $\mu$ and $\mu'$ are large enough compared to $b$, namely,

$$\tilde{F}_{\mu}|_{S^{0,b}_{\alpha',a}(\mathbb{V})} = \tilde{F}_{\mu'}|_{S^{0,b}_{\alpha',a}(\mathbb{V})}.$$ \[28\]

To prove this claim, we note that $(W_{\mu}, f) = (W, f)$ for $f \in S_{\alpha',a'}^{0, \mu/4}$ at arbitrary $a'$. In fact, for such a function, we have

$$|f(z)| \leq C \exp\left\{-|x/a'|^{1/\alpha'} + (\mu/4) \sum |y_i|\right\}$$

and this estimate implies (again by the Paley-Wiener-Schwartz theorem, this time in its simplest version dealing with functions in $\mathcal{D}$; see, e.g., Theorem 7.3.1 in \[23\]) that supp $\tilde{f}$ is contained in the ball $|p| \leq \mu/2$, where $\omega(p/\mu) = 1$ by the construction in Sec. 5. The formalized formulation of Theorem 1 given in \[12\] shows that there is a constant $c$ such that for $b' > cb$ and $a' > ca$, the space $S^{0,b'}_{\alpha',a'}$ is dense in $S^{0,b}_{\alpha',a'}(\mathbb{V})$ in the topology of $S^{0,b'}_{\alpha',a'}(\mathbb{V})$. Hence, equality \[28\] is satisfied for $\mu, \mu' > 4cb$. Therefore, the nonregularized functional $W(\xi) \mp W(-\xi)$ also has a continuous extension to $S^{0}_{\alpha'}(\mathbb{V})$. Applying Theorem 5 and Lemma 4, we complete the proof.

Theorem 9 is a special case of the following more general statement.

**Theorem 10.** Let $\{\phi_i\}$ be a family of fields that are defined on the test function space $S^{0}_{\alpha}(\mathbb{R}^4)$, $\alpha > 2$, and transform according to irreducible representations $(j_1, k_1)$ of the group $\text{SL}(2, \mathbb{C})$. Let $\mathcal{W}_{i_1 \ldots i_n}$ be the Wightman function determined by the $n$-point vacuum expectation value $\langle \Psi_0, \phi_{i_1}(x_1) \ldots \phi_{i_n}(x_n) \Psi_0 \rangle$. The cone $\mathcal{C}_J$ complementary to the Jost cone is a carrier of the generalized function

$$\mathcal{W}_{i_1 \ldots i_n}(x_1, \ldots, x_n) - (-1)^J \mathcal{W}_{i_1 \ldots i_n}(-x_1, \ldots, -x_n),$$

where $J = j_{i_1} + \cdots + j_{i_n}$. 

14
The proof is completely analogous to that given above, with the only difference that we now use a cutoff function of the form $\omega(P/\mu)$, where $P = p_1 + \cdots + p_{n-1}$ and $p_k$ are conjugate with $\xi_k = x_k - x_{k+1}$, then appeal to Theorem 8 instead of Theorem 7, and apply the familiar transformation law of the $n$-point analytic Wightman function of irreducible fields with respect to the space-time reflection.

7. Generalization of the spin-statistics theorem

We begin by deriving an analogue of the Dell’Antonio lemma, which shows that each pair of nonzero local fields $\phi, \psi$ has the same type of commutation relations as the pair $\phi, \psi^*$.

**Theorem 11.** Let the fields $\phi, \psi$, and their Hermitian adjoints be defined on the test function space $S^0_\alpha(\mathbb{R}^4)$, $\alpha > 1$. If $\phi$ has different asymptotic commutation relations with $\psi$ and $\psi^*$, then either $\phi(x)\Psi_0 = 0$, or $\psi(x)\Psi_0 = 0$.

**Proof.** For definiteness, we assume that $\phi$ commutes asymptotically with $\psi$ and anticommutes asymptotically with $\psi^*$ for large spacelike separations of the arguments. We consider the following sum of vacuum expectation values:

\[ \langle \Psi_0, \phi^*(x_1)\phi(x_2)\psi^*(y_1)\psi(y_2)\Psi_0 \rangle + \langle \Psi_0, \phi^*(x_1)\psi^*(y_1)\psi(y_2)\phi(x_2)\Psi_0 \rangle = \\
= \langle \Psi_0, \phi^*(x_1)[\phi(x_2), \psi^*(y_1)] + \psi(y_2) \rangle + \langle \Psi_0, \phi^*(x_1)\psi^*(y_1)[\psi(y_2), \phi(x_2)] \rangle \Psi_0 \rangle. \tag{30} \]

By Lemma 3, this functional is carried by the union of the cones $\{(x, y): (x_2 - y_1)^2 \geq 0\}$ and $\{(x, y): (x_2 - y_2)^2 \geq 0\}$. We average it with a test function of the form

\[ \bar{f}(x_1)f(x_2)\bar{g}(y_1 - \lambda r)g(y_2 - \lambda r), \]

where $r$ is a fixed spacelike vector and $\lambda > 0$. The result of averaging is a convolution considered on the ray $x_1 = x_2 = 0$, $y_1 = y_2 = \lambda r$, and by Lemma 1, it decreases as $\lambda \to \infty$ because this ray does not belong to the carrier cone. On the other hand, just as in the original reasoning of Dell’Antonio, it can be written in the form

\[ \langle \Psi_0, \phi(f)^*\phi(f)T(\lambda r)\psi(g)^*\psi(g)\Psi_0 \rangle + \|\psi(g)\|_{T}^{\lambda(\lambda r)\phi(f)\Psi_0}^{2}, \]

where $T(\lambda r)$ implements space-time translations. As $\lambda \to \infty$, the first term of the sum tends to

\[ \|\phi(f)\|_{T}^{2}\|\psi(g)\|_{T}^{2} \]

by the cluster decomposition property, which can be derived from the Wightman axioms without using locality (see [2], [3]). Therefore, if $\psi(g)\Psi_0 \neq 0$ for at least one test function $g$, then $\phi(f)\Psi_0 = 0$ for all $f \in S^0_\alpha(\mathbb{R}^4)$, which completes the proof.

**Theorem 12.** Let $\phi$ be a field that is defined on the space $S^0_\alpha(\mathbb{R}^4)$ with the index $\alpha > 2$ and transforms according to an irreducible representation of the group $L(2, \mathbb{C})$. The anomalous asymptotic commutation relation between $\phi$ and its adjoint $\phi^*$ (that is, anticommutativity in the case of integer spin and commutativity in the case of half-integer spin) implies that $\phi(f)\Psi_0 = \phi^*(f)\Psi_0 = 0$ for all $f \in S^0_\alpha(\mathbb{R}^4)$.
Proof. Suppose $\phi$ is an integer spin field. The anomalous commutation relation would imply, in particular, that the sum
\[ \langle \Psi_0, \phi^*(x_1)\phi(x_2)\Psi_0 \rangle + \langle \Psi_0, \phi(x_2)\phi^*(x_1)\Psi_0 \rangle \] (31)
is carried by the cone $\vec{\mathcal{V}}$. Then Theorem 9 shows that this cone is also a carrier of the sum
\[ \langle \Psi_0, \phi^*(x_1)\phi(x_2)\Psi_0 \rangle + \langle \Psi_0, \phi(x_1)\phi^*(x_2)\Psi_0 \rangle. \] (32)
In momentum space, both of the vacuum expectation values in (32) have support in the properly convex cone \( \{p \in \mathbb{R}^8: p_1 + p_2 = 0, \quad p_1 \in \mathbb{V}_+ \} \). Therefore, generalized function (32) is equal to zero by Theorem 6. Averaging it with $f(x_1)f(x_2)$, we get
\[ \|\phi(f)\Psi_0\|^2 + \|\phi^*(f)\Psi_0\|^2 = 0. \]
In the case of half-integer spin, the reasoning is the same with a proper change of the signs in the formulas. Theorem 12 is proved.

**Corollary.** In any field theory satisfying the asymptotic commutativity condition with test functions in $\mathcal{S}_0^0(\mathbb{R}^4)$, $\alpha > 2$, the equality
\[ \langle \Psi_0, \phi^*(x_1)\phi(x_2)\Psi_0 \rangle = \langle \Psi_0, \phi(x_1)\phi^*(x_2)\Psi_0 \rangle. \] (33)
holds.

**Proof.** The difference of these vacuum expectation values can be written as
\[ \langle \Psi_0, [\phi^*(x_1)\phi(x_2)]_\mp \Psi_0 \rangle \pm \langle \Psi_0, \phi(x_2)\phi^*(x_1)\Psi_0 \rangle - \langle \Psi_0, \phi(x_1)\phi^*(x_2)\Psi_0 \rangle. \] (34)
By Theorem 9, expression (34) is carried by the cone $\vec{\mathcal{V}}$ for both integer and half-integer spin cases, and Theorem 6 shows that this property is compatible with the spectral condition only if (33) is satisfied.

**Theorem 13.** If in a field theory satisfying the asymptotic commutativity condition with test functions in $\mathcal{S}_0^0(\mathbb{R}^4)$, $\alpha > 2$, we have $\phi(f)\Psi_0 = 0$ for all test functions, then the field $\phi$ vanishes.

**Proof.** It follows from the assumptions of the theorem that all vacuum expectation values involving at least one operator $\phi$ vanish. For instance, if $\phi$ stands in the next to the last position, then the vacuum expectation value
\[ \langle \Psi_0, \phi_{i_1}, \ldots, \phi_{i_{n-1}}\phi \phi_{i_n}\Psi_0 \rangle \]
coincides with the generalized function
\[ \langle \Psi_0, \phi_{i_1}, \ldots, \phi_{i_{n-1}}[\phi, \phi_{i_n}]_\mp \Psi_0 \rangle \]
carried by the cone $\mathbb{R}^{4(n-1)} \times \vec{\mathcal{V}}$, while the support of its Fourier transform lies in the properly convex cone determined by the spectral condition. Hence, it vanishes by Theorem 6. Next, we use the induction argument. Taking cyclicity of the vacuum into account, we obtain $\langle \Phi, \phi(f)\Psi \rangle = 0$ for all $\Phi \in \mathcal{H}$, $\Psi \in \mathcal{D}_0$, and $f \in \mathcal{S}_0^0(\mathbb{R}^4)$. The operator $\phi(f)$ is closable because its adjoint is densely defined. Hence $\phi(f) = 0$, which completes the proof.

The reasoning above remains valid if $\phi$ is replaced with any monomial $M$ in the field components. Therefore, $M\Psi_0 = 0$ implies $M = 0$, which equally follows from Theorem 13 and the cluster decomposition property.

We also need the following simple assertion whose proof is similar to the proof of Theorem 11.
Theorem 14. Let \( \{ \phi_\iota \} \) be a family of fields defined on the space \( S_0^0(\mathbb{R}^4) \), \( \alpha > 1 \). If two monomials \( M = \phi_{\iota_1}(x_1) \ldots \phi_{\iota_n}(x_n) \) and \( N = \phi_{\iota'_1}(y_1) \ldots \phi_{\iota'_n}(y_n) \) anticommute asymptotically for large spacelike separations of the set of points \( (x_1, \ldots, x_n) \) from the set of points \( (y_1, \ldots, y_n) \), then either \( \langle \Psi_0, M \Psi_0 \rangle = 0 \) or \( \langle \Psi_0, N \Psi_0 \rangle = 0 \).

Proof. In this case, we have the generalized function

\[
\langle \Psi_0, M(x_1, \ldots, x_n)N(y_1, \ldots, y_n)\Psi_0 \rangle + \langle \Psi_0, N(y_1, \ldots, y_n)M(x_1, \ldots, x_n)\Psi_0 \rangle
\]

carried by the cone

\[
\bigcup_{k=1}^{m+n} \{ (x, y) \in \mathbb{R}^{4(m+n)} : (x_k - y_l)^2 \geq 0 \}.
\]

Averaging it with a test function of the form \( f(x_1, \ldots, x_n)g(y_1 - \lambda r, \ldots, y_n - \lambda r) \), as before, we obtain a smooth function of the parameter \( \lambda \), which decreases rapidly as \( \lambda \to \infty \) by Lemma 1. On the other hand, this function can be written as

\[
\langle \Psi_0, M(f)T(\lambda r)N(g)\Psi_0 \rangle + \langle \Psi_0, N(g)T^{-1}(\lambda r)M(f)\Psi_0 \rangle,
\]

where both terms in sum tend to \( \langle \Psi_0, M(f)\Psi_0 \rangle \langle \Psi_0, N(g)\Psi_0 \rangle \) by the cluster property. Therefore, this product of vacuum expectation values is equal to zero for all \( f \in S_0^0(\mathbb{R}^{4m}) \) and \( g \in S_0^0(\mathbb{R}^{4n}) \). Theorem 14 is thus proved.

We can now derive an analogue of the Araki theorem on the reduction of commutation relations to the normal form. We follow the usual stipulation \([2]\) that the family under consideration does not include fields that vanish identically and that for every non-Hermitian field \( \phi_\iota \), its adjoint \( \phi_\iota^* \) enters in this family with some index \( \iota \neq \iota' \).

Theorem 15. In any theory of Wightman fields \( \{ \phi_\iota \}, \iota = 1, \ldots, I \), satisfying the asymptotic commutativity condition with test functions in \( S_0^0(\mathbb{R}^4) \), \( \alpha > 2 \), there exists a Klein transformation \( \phi_\iota \Rightarrow \phi'_\iota \), which reduces the commutation relations to the normal form, that is, the fields \( \phi'_\iota \) of integer spin commute asymptotically with any field in the new family for large spacelike separations of the arguments, and the transformed fields of half-integer spin anticommute asymptotically with each other. Furthermore, the fields \( \phi'_\iota \) satisfy all the other Wightman axioms, and the Hermitian conjugation condition \( (\phi'_\iota^*) = \phi'_\iota \) holds.

Proof. Theorems 11–14 reduce the proof of this statement to an almost literal repetition of the classical derivation \([2, 4]\) of the Araki theorem.

Let \( F_\iota \) denote the spinorial number of \( \phi_\iota \),

\[
F_\iota = \begin{cases} 
0, & \text{if } j_\iota + k_\iota \text{ is integer}, \\
1, & \text{if } j_\iota + k_\iota \text{ is half-integer}.
\end{cases}
\]

The asymptotic commutativity condition states that for each pair of fields \( \phi_\iota, \phi'_\iota \) belonging to the family under consideration, the matrix elements of the combination

\[
\phi_\iota(x)\phi'_\iota(y) - (-1)^{F_\iota F'_\iota + \omega_{\iota \iota'}}\phi'_\iota(y)\phi_\iota(x),
\]

are carried by the cone \( \bar{\mathcal{V}} \). Here, \( \omega_{\iota \iota'} = 0 \) if the commutation relation is normal and \( \omega_{\iota \iota'} = 1 \) otherwise. The matrix \( \omega \) is symmetric and possesses the properties

\[
\omega_{\iota \iota'} = \omega_{\iota' \iota} = \omega_{\iota', \iota}, \quad \omega_{\iota \iota} = 0
\]
by Theorems 11 and 12. It determines the sign in the commutation relation for any two monomials in fields. This sign depends only on how many fields of each type occur in the monomials, more precisely, on the parity of these numbers. This characteristic of a monomial $M$ can be written as a row $m = (m^1, \ldots, m^I)$, where $m^I = 0$ if $M$ contains an even number of $\phi_i$ and $m^I = 1$ if this number is odd. The set $\mathfrak{M}$ of such rows whose components are 0 or 1 has the structure of a vector space over the field $\mathbb{Z}_2$ and the mapping $M \rightarrow m$ is consistent with this structure in the sense that

$$m(M_1M_2) = m(M_1) + m(M_2).$$  \hfill (37)

The system of spinor numbers generates the linear form $F(m) = \sum F_i m^i$ and the matrix $\omega_i$ generates the bilinear form $(m_1, m_2) = \sum \omega_{ij} m_1^i m_2^j$ on $\mathfrak{M}$. In this notation, the sign factor in the commutation relation for monomials $M_1$ and $M_2$ becomes

$$(-1)^{F(m_1)F(m_2) + (m_1, m_2)},$$

and properties (36) can be written as

$$(e_i, m) = (e_i, m), \quad (m, m) = 0,$$ \hfill (38)

where $e_i = m(\phi_i)$. The second formula from (38) implies that the form $(\cdot, \cdot)$ induces a symplectic structure on $\mathfrak{M}$. In particular, it implies the identity $(m_1, m_2) + (m_2, m_1) = 0$, which is equivalent to $(m_1, m_2) = (m_2, m_1)$ in the case of the field $\mathbb{Z}_2$. We let $\mathfrak{A}$ denote the set of those vectors in $\mathfrak{M}$ that correspond to the monomials whose vacuum expectation values are not identically zero. From (37) and the cluster property, it follows that $\mathfrak{A}$ is a linear subspace of $\mathfrak{M}$. The restriction of the form $(\cdot, \cdot)$ to $\mathfrak{A}$ is zero because such monomials contain an even number of half-integer spin fields and the relation $(m_1, m_2) = 1$ would mean that $M_1$ and $M_2$ anticommute for large spacelike separation of the arguments, which contradicts Theorem 14. Every basis $(a_1, \ldots, a_q)$ in $\mathfrak{A}$ can be completed to a symplectic basis $(a_1, \ldots, a_r; b_1, \ldots, b_r; e_1, \ldots, e_s)$, $r \geq q$, $2r + s = I$, in the whole $\mathfrak{M}$. In this basis,

$$(a_j, b_j) = 1, \quad j = 1, \ldots, r,$$

while all other pairings vanish. In particular,

$$ (a_j, m) = 0 \quad \text{for all} \quad j = 1, \ldots, r, \quad m \in \mathfrak{A}. \hfill (39)$$

To each $a_j$, we can assign an operator $\theta_j$ acting in the Hilbert space $H$. Namely, we set

$$\theta_j \Psi_0 = \Psi_0, \quad \theta_j M(f) \Psi_0 = (-1)^{(a_j, m)} M(f) \Psi_0.$$\hfill

These operators are well defined. In fact, if $M_1(f) \Psi_0 = M_2(f) \Psi_0 \neq 0$, then $\langle \Psi_0, M_1^* M_2 \Psi_0 \rangle \neq 0$, i.e., $m(M_1^* M_2) \in \mathfrak{A}$ and relations (37)–(39) imply $(a_j, m_1) + (a_j, m_2) = 0$, which amounts to $(a_j, m_1) = (a_j, m_2)$. Further, the definition is consistent with the linear operations in $H$ because the relation $M(f) \Psi_0 = M_1(f) \Psi_0 + M_2(f) \Psi_0$, where all vectors are assumed nonzero, implies $(a_j, m) = (a_j, m_1) = (a_j, m_2)$. For example, if $(a_j, m_1)$ differs from the other two scalar products, then again using (37)–(39), we obtain a contradiction because at least one of the vacuum expectation values $\langle \Psi_0, M_1^* M \Psi_0 \rangle$ and $\langle \Psi_0, M_2^* M \Psi_0 \rangle$ does not vanish. Hence, the $\theta_j$’s can be extended to $D_0$ by linearity. Relations (37)–(39) imply that $\langle \theta_j \Phi, \theta_j \Psi \rangle = \langle \Phi, \Psi \rangle$ for all $\Phi, \Psi \in D_0$.  

18
Therefore, every operator $\theta_j$ can be uniquely extended to a unitary involution defined on the whole of $\mathcal{H}$. We set

$$U_i = \prod_{j=1}^{r} \theta_j^{(\varepsilon, b_j)}, \quad i = 1, \ldots, I.$$ (40)

The operators $U_i$ commute with each other, and their commutation relations with the fields are

$$U_i \phi_{i'} = (-1)^{\sigma_{i, i'}} \phi_{i'} U_i, \quad \text{where} \quad \sigma_{i, i'} = \sum_{j=1}^{r} (\varepsilon, b_j)(a_j, c_{i'}).$$ (41)

The Klein transformation is defined by the formula

$$\phi_i \Rightarrow \phi_i' = i^{\sigma_{i, i'}} U_i \phi_i.$$ (42)

Using (41) and the equality $\omega_{i, i'} = \sigma_{i, i'} + \sigma_{i', i}$, which holds by the definition of the symplectic basis, we deduce that for any $\Phi$,

$$\Theta \Phi \Theta^{-1} = (-1)^{2j_i} i^{F_i} \Phi_i^* \Theta_i = (-1)^{2j_i} i^{F_i} \Phi_i. \quad \text{Proof. It is well known that in terms of the Wightman functions, a necessary and sufficient condition for the operator $\Theta$ to exist is}$$ (43)

$$\Theta \phi_i(x) \Theta^{-1} = (-1)^{2j_i} i^{F_i} \phi_i^*(-x),$$ (43)

where $F_i$ is the spinorial number of the field.

8. Generalization of the PCT theorem

**Theorem 16.** In the field theory satisfying the asymptotic commutativity condition with test functions in $S^0_\alpha(\mathbb{R}^4)$, $\alpha > 2$, and with the normal spin-statistics relation, there exists an antiunitary PCT-symmetry operator $\Theta$. This operator leaves the vacuum state invariant, and if $\phi_i$ transforms according to the $(j_i, k_i)$ representation of $SL(2, \mathbb{C})$, then its transformation law under $\Theta$ is

$$\Theta \phi_i(x) \Theta^{-1} = (-1)^{2j_i} i^{F_i} \phi_i^*(-x),$$ (43)

where $F_i$ is the spinorial number of the field.

**Proof.** It is well known that in terms of the Wightman functions, a necessary and sufficient condition for the operator $\Theta$ to exist is

$$W_{i_1 \ldots i_n}(x_1, \ldots, x_n) = (-1)^{2J} i^F W_{i_1 \ldots i_n}(-x_n, \ldots, -x_1),$$ (44)

where $J = j_{i_1} + \cdots + j_{i_n}$ and $F = F_{i_1} + \cdots + F_{i_n}$ is the number of half-integer spin fields in the set $\phi_{i_1}, \ldots, \phi_{i_n}$. We now write the difference of the right- and left-hand sides of (44) in the form

$$[W_{i_1 \ldots i_n}(x_1, \ldots, x_n) - (-1)^{2J} W_{i_1 \ldots i_n}(-x_1, \ldots, -x_n)] +$$

$$+ (-1)^{2j_1} [W_{i_1 \ldots i_n}(-x_1, \ldots, -x_n) - i^F W_{i_1 \ldots i_n}(-x_n, \ldots, -x_1)].$$ (45)
The asymptotic commutativity condition implies that the expression in the first square brackets is carried by the cone $\bigcup_{k<l} \{x \in \mathbb{R}^{4n} : (x_k - x_l)^2 \geq 0\}$, which is contained in the complement of the Jost cone because the Jost points are totally spacelike. By Theorem 10, the cone $\mathbb{C}J_n$ is a carrier of the functional in the second square brackets as well. In momentum space, both the generalized functions involved in (44) have support in the properly convex cone $\mathbb{C}J_n$, whose $p$ component is carried by the cone $\mathbb{C}J_n$ also.

Therefore, equality (44) holds identically by Theorem 6. We can now construct the operator $\Theta$ in the standard way. First, we define it on those vectors that are obtained by applying monomials in fields to the vacuum:

$$\Theta \Psi_0 = \Psi_0, \quad \Theta \phi_{\iota_1} (f_1) \ldots \phi_{\iota_n} (f_n) \Psi_0 = (-1)^{2J} i^F \phi_{\iota_1}^* (\tilde{f}_1) \ldots \phi_{\iota_n}^* (\tilde{f}_n) \Psi_0,$$

where $\tilde{f}(x) = \tilde{f}(-x)$. It is easy to verify that $\Theta$ is well defined. In fact, taking into account that $\phi^*_\iota$ transforms according to the conjugate representation $(\iota_k, \iota_l)$, we see that (44) implies the relation $\langle \Theta \Phi, \Theta \Psi \rangle = \langle \Phi, \Psi \rangle$ for vectors of this special form.

Therefore, if a vector $\Psi$ is generated by different monomials $M_1(f_1)$ and $M_2(f_2)$, then the scalar product $\langle \Theta M_1 \Psi_0, \Theta M_2 \Psi_0 \rangle$ is equal to the squared length of either of the two vectors $\Theta M_1 \Psi_0$ and $\Theta M_2 \Psi_0$, i.e., these vectors coincide. Analogously, if $\Psi = \Psi_1 + \Psi_2$, where all vectors are obtained by applying monomials to $\Psi_0$, then $\Theta \Psi = \Theta \Psi_1 + \Theta \Psi_2$.

Therefore, $\Theta$ can be extended to $D_0$ by antilinearity. A further extension by continuity yields an antiunitary operator defined on the whole of $\mathcal{H}$.

A stronger formulation of Theorem 16 uses an analogue of the Jost-Dyson weak local commutativity condition.

**Definition 2.** We say that the fields $\{\phi_\iota\}$ defined on $S^0_\alpha(\mathbb{R}^4)$ satisfy the weak asymptotic commutativity condition if for each system of indices $\iota_1, \ldots, \iota_n$, the functional

$$\langle \Psi_0, \phi_{\iota_1} (x_1) \ldots \phi_{\iota_n} (x_n) \Psi_0 \rangle - i^F \langle \Psi_0, \phi_{\iota_n} (x_n) \ldots \phi_{\iota_1} (x_1) \Psi_0 \rangle$$

is carried by the cone $\mathbb{C}J_n$ complementary to the Jost cone.

The above consideration shows that this condition is equivalent to relation (44), i.e., the following statement is valid.

**Theorem 17.** A field theory satisfying all Wightman axioms with test functions in $S^0_\alpha(\mathbb{R}^4)$, $\alpha > 2$, but with the possible exception of local or asymptotic commutativity, has PCT symmetry if and only if the weak asymptotic commutativity condition is satisfied.

Moreover, by Theorem 6, the field theory has PCT symmetry even if difference (46) is carried by the complement of a cone generated by an arbitrarily small real neighborhood of a Jost point. Our consideration also shows that an analogue of the theorem on global nature of local commutativity is valid for the weak local commutativity. The most refined version of this theorem is due to Borchers and Pohlmeyer [27], who considered the theory of a scalar tempered field and established that a bound of the form

$$|\langle \Psi_0, [\phi(x_1), \phi(x_2)] \phi(x_3) \ldots \phi(x_n) \Psi_0 \rangle| \leq C_n \exp \{- \gamma ||(x_1 - x_2)^2||^{p/2}\} \quad (p > 1) \quad (47)$$
on the behavior of the commutators at those points \((x_1, x_2, x_3, \ldots, x_n)\) that belong to the cone \(J_n\) together with \((x_2, x_1, x_3, \ldots, x_n)\) results in the strict local commutativity. Analogously, if all fields \(\phi_i\) are defined on Schwartz’s space \(S\) and functional (46) decreases in the cone \(J_n\) by an exponential law of type (47), then it actually vanishes everywhere in this cone. In fact, using a partition of unity and taking the remark in Sec. 3 into account, we see that the restriction of functional (46) to each space \(S^0(\mathbb{R}^{4n})\) is carried by the cone \(\mathcal{C}J_n\) in this case. Therefore, the theory has PCT as a symmetry and must satisfy the weak local commutativity condition by the usual PCT theorem [2]–[4].

9. Concluding remarks

The main result of this work is a rigorous proof that PCT symmetry and the standard spin–statistics relation are preserved under replacing the microcausality axiom with the condition of a fast decrease of the (anti)commutators at large spacelike separations of the arguments, correctly formulated in terms of the theory of analytic functionals. The absence of any experimental indications of violation of these fundamental properties of quantum physics is customary believed to be evidence for the locality of interaction. On the contrary, we see that these properties have deeper roots in the mathematical structure of quantum field theory and are essentially asymptotic in character. The established generalization of the spin-statistics and PCT theorems is of greatest possible accuracy because, as already noted by Pauli [28], allowing an exponential decrease of the field (anti)commutators with order 1 and a finite type implies the possibility of quantizing the scalar field with an abnormal relation \([\phi(x), \phi(y)]_+ = \Delta^{(1)}(x - y)\), where \(\Delta^{(1)}\) is the even solution to the Klein-Gordon equation, which behaves like \(\exp(-m|x - y|)\) at spatial infinity.

In relation to the above remark about the theorem on the global nature of local commutativity, it should be emphasized that the asymptotic commutativity condition, being applied to the tempered fields (i.e., to their restrictions to \(S^0(\mathbb{R}^{4})\)), does not amount to naive bound (17) and does not imply local commutativity. Strictly speaking, this condition means a fast decrease not of the field commutator itself but of the result of smoothing it by convolution with appropriate test functions (see [1] for more details), which seems reasonable from the physical standpoint.

The theorems on carrier cones of analytic functionals presented in Sec. 2 were established in [10]–[13] for applications to the covariant quantization of gauge models in which singularities are of infrared origin and for which the spaces \(S^0(\mathbb{R}^{4})\) are natural functional domains of definition of fields in the momentum representation when the models are treated in a generic covariant gauge (see [29]). In particular, this formalism gives a simple, general method of constructing Wick–ordered entire functions of the indefinite metric free fields in the Hilbert-Fock-Krein spaces [30]. In the present paper, the efficiency of the developed technique has been demonstrated by an example of solving classical problems in nonlocal quantum field theory. Actually, this formalism accomplishes the extension of the Wightman axiomatic approach to nonlocal quantum fields with arbitrary high-energy behavior.

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