SHI ARRANGEMENTS RESTRICTED TO WEYL CONES

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Abstract. We consider the restrictions of Shi arrangements to Weyl cones, their relations to antichains in the root poset, and their intersection posets. For any Weyl cone, we provide bijections between regions, flats intersecting the cone, and antichains of a naturally-defined subposet of the root poset. This gives a refinement of the parking function numbers via the Poincaré polynomials of the intersection posets of all Weyl cones. Finally, we interpret these Poincaré polynomials as the Hilbert series of two isomorphic graded rings, one arising from the Varchenko-Gel’fand ring and another, which we call the order ring since it turns out to be naturally associated to the order polytope.

1. Introduction and main results

Shi arrangements are well-studied hyperplane arrangements associated to an irreducible finite Weyl group. These were initially defined by Shi to study Kazhdan-Lusztig cells of affine Weyl groups [18] and since then have appeared in many algebraic, geometric, and combinatorial contexts. We refer to [2, §5.1.4] and [14] for their history in the context of finite Weyl groups. Among many, two classical results state that

- the number of regions is the parking function number \((h + 1)^\ell\) where \(h\) is the Coxeter number and \(\ell\) is the rank, and
- the number of dominant regions is the Catalan number \(\prod_{i=1}^\ell (d_i + h)/d_i\) where \(h\) and \(\ell\) are as above and \(d_1, \ldots, d_\ell\) are the invariant degrees.

In this paper, we study interpretations and refinements of these two (and of related) formulas in terms of the intersection poset of the Shi arrangement. In the final section, we discuss two ring constructions, one for arrangements and one for posets, which turn out to be isomorphic in the present context.

1.1. Background. We start with briefly fixing the necessary notation. We refer to [15] for further background on root systems and to [14] for a more detailed introduction to Shi arrangements. In Section 1.3, we illustrate the main results with a detailed example in type \(B_2\).

Let \(\Delta \subseteq \Phi^+ \subseteq \Phi \subset V\) be an irreducible crystallographic root system with a given choice of simple roots \(\Delta\), positive roots \(\Phi^+\) and negative roots \(\Phi^- = -\Phi^+\) inside a Euclidean space \(V\) of dimension \(\ell\) with inner product \(\langle \cdot, \cdot \rangle\). Then the Shi arrangement is

\[
\text{Shi}(\Phi^+) = \text{Shi}_0(\Phi^+) \cup \text{Shi}_1(\Phi^+) ,
\]

where

\[
\text{Shi}_i(\Phi^+) = \{ H_{\beta,i} = \{ v \in V \mid \langle v, \beta \rangle = i \} \mid \beta \in \Phi^+ \} .
\]
We observe that \( \text{Shi}_0(\Phi^+) = \mathcal{A}(W) \) is the reflection arrangement of the corresponding Weyl group \( W = W(\Phi) \). It in particular does not depend on the choice of simple and positive roots. The dominant cone in \( \mathcal{A}(W) \) is the (open) cone of \( V \) defined by
\[
C = \{ v \in V \mid \langle v, \beta \rangle > 0 \text{ for all } \beta \in \Phi^+ \}
\]
and for any \( w \in W \), the map \( w \mapsto wC \) is a bijection between elements in \( W \) and cones of \( \mathcal{A}(W) \), that is, the (open) connected components of the complement \( V \setminus \bigcup_{H \in \mathcal{A}(W)} H \). These cones have the explicit description
\[
wC = \bigcap_{\beta \in \text{Inv}(w)} H_{\beta,0}^- \cap \bigcap_{\beta \in \Phi^+ \setminus \text{Inv}(w)} H_{\beta,0}^+,
\]
where \( \text{Inv}(w) = \Phi^+ \cap w\Phi^- \) is the inversion set of \( w \in W \) and
\[
H_{\beta,i}^+ = \{ v \in V \mid \langle v, \beta \rangle > i \} \quad \text{and} \quad H_{\beta,i}^- = \{ v \in V \mid \langle v, \beta \rangle < i \}
\]
are the open halfspaces defined by the hyperplane \( H_{\beta,i} \). Each cone \( wC \) contains (possibly multiple) Shi regions. This suggests the following refinement
\[
(h + 1)^\ell = \# R(\text{Shi}(\Phi^+)) = \sum_{w \in W} \# R(wC),
\]
where \( R(\text{Shi}(\Phi^+)) \) is the set of Shi regions and
\[
R_w = \{ R \in R(\text{Shi}(\Phi^+)) \mid R \subset wC \}
\]
is the set of Shi regions inside the cone \( wC \). The Shi regions \( R_e \) for the identity element \( e \in W \) inside the dominant cone \( C \) are the dominant regions and are counted by the Catalan numbers. These are in bijection with certain subsets of positive roots, namely antichains in the root poset which is the poset on positive roots \( \Phi^+ \) given by the cover relations \( \beta < \gamma \) for \( \gamma - \beta \in \Delta \) [5].

1.2. Main results. Our first result generalizes this bijective correspondence to Shi regions inside any cone \( wC \). The ceiling set \( \text{ceil}(R) \subseteq \Phi^+ \) of a Shi region \( R \) is the set of positive roots \( \beta \) such that \( H_{\beta,1} \) is a facet-defining hyperplane of \( R \) with \( R \subset H_{\beta,-1}^- \). The following gives a bijection between \( R_w \) and the set \( A_w \) of antichains in the subposet \( \Phi^+ \setminus \text{Inv}(w^{-1}) \) of the root poset. To an antichain \( A \in A_w \), we associate the order ideal of \( \Phi^+ \setminus \text{Inv}(w^{-1}) \) generated by \( A \),
\[
\mathcal{I}(A) = \{ \gamma \in \Phi^+ \setminus \text{Inv}(w^{-1}) \mid \gamma < \beta \text{ for some } \beta \in A \}.
\]

**Theorem 1.1.** The map \( \varphi_w : R_w \to A_w \) given by
\[
R \mapsto w^{-1}(\text{ceil}(R))
\]
is a bijection with inverse \( \varphi_w^{-1} : A_w \to R_w \) given by
\[
A \mapsto \left\{ v \in V \left| \begin{array}{l}
0 \prec \langle v, \gamma \rangle < 1 \text{ for } \gamma \in w(\mathcal{I}(A)) \\
1 \prec \langle v, \gamma \rangle \text{ for } \gamma \notin w(\mathcal{I}(A))
\end{array} \right. \right\}.
\]

This theorem solves a problem originally proposed by Athanasiadis-Linusson in [6, Section 4], but we will see that it follows directly from previous work of Athanasiadis [5] and Armstrong-Reiner-Rhoades [3]. Indeed, without making it explicit, this correspondence was already implicitly considered in [3], see Lemma 2.15 below.

The main result of this paper is the following, which gives a bijection between antichains in \( A_w \) (and thus also of regions in \( R_w \)) and
\[
\mathcal{L}_w = \{ X \in \mathcal{L}(\text{Shi}(\Phi^+)) \mid X \cap wC \neq \emptyset \}.
\]
where \( \mathcal{L}(\text{Shi}(\Phi^+)) \) is the set of (non-empty) intersections of hyperplanes of \( \text{Shi}(\Phi^+) \). While having a similar flavor as Theorem 1.1, this result is—to the best of our knowledge—an interpretation of antichains of the root poset that has not appeared in the literature to the best of our knowledge.

**Theorem 1.2.** The map \( \psi_w : \mathcal{L}_w \rightarrow \mathcal{A}_w \) given by
\[
X \mapsto \{ w^{-1}(\beta) \mid X \subseteq H_{\beta,1} \}
\]
is a bijection with inverse \( \psi_w^{-1} : \mathcal{A}_w \rightarrow \mathcal{L}_w \) given by
\[
A \mapsto \bigcap_{\beta \in wA} H_{\beta,1}.
\]

**Remark 1.3** (Connection to nonnesting partitions). Antichains in the root poset (that is, elements of \( \mathcal{A}_e \) for the identity element \( e \in W \)) are called nonnesting partitions. Theorem 1.2 shows that the map \( A \mapsto \bigcap_{\beta \in A} H_{\beta,1} \) is injective. Athanasiadis and Reiner showed injectivity of the map \( A \mapsto \bigcap_{\beta \in A} H_{\beta,0} \), see [7, Corollary 6.2]. Their result establishes the embedding of the nonnesting flats into the intersection lattice \( \mathcal{L}(\mathcal{A}(W)) \). Thus the above result may be seen as a lift of the nonnesting flats into the affine intersection lattice \( \mathcal{L}(\text{Shi}_1(\Phi^+)) \) whose image are exactly the flats cutting through the dominant cone \( C \) (that is, flats in \( \mathcal{L}_e \)). Our proof of injectivity does not rely on their argument, nor can we deduce their result from ours without reproducing their argument.

**Remark 1.4** (Similarities to a poset of Biane and Josuat-Vergès). In [8, Definition 4.1], Biane and Josuat-Vergès consider the partial order on noncrossing partitions, which depends on a choice of a Coxeter element, given by the intersection of the absolute order and the Bruhat order. The rank sizes of their posets are also given by the Narayana numbers and all lower intervals are also Boolean lattices, compare Theorem 1.5. In type \( A_4 \), there is no choice of Coxeter element for which that poset is isomorphic to the intersection poset \( \mathcal{L}_e \).

Our third result is proved as part of the proof of Theorem 1.2 and concerns the structure of \( \mathcal{L}_w \) as a poset ordered by reverse inclusion. This poset is an order ideal of of the intersection poset \( \mathcal{L}(\text{Shi}(\Phi^+)) \) of the full arrangement and thus a meet-semilattice ranked by codimension, for which every lower interval \( [V, X] \subseteq \mathcal{L}_w \) is a geometric lattice. The following theorem strengthens this property and shows that the Möbius function values of lower intervals are all equal to \( \pm 1 \), implying that \( \mathcal{L}_w \) is Eulerian.

**Theorem 1.5.** All lower intervals of the poset \( \mathcal{L}_w \) are Boolean. Thus, \( \mu(V, X) = (-1)^{\text{codim}(X)} \) for all \( X \in \mathcal{L}_w \) and in particular
\[
\# \mathcal{L}_w = \# \mathcal{R}_w.
\]

The next results follow immediately from the above. We state them here, as they interpret and generalize previously-studied combinatorial properties of Shi arrangements. The **Poincaré polynomial** of the cone \( wC \) is
\[
\text{Poin}(wC, t) = \sum_{X \in \mathcal{L}_w} |\mu(V, X)| t^{\text{codim}(X)} = \sum_{k \geq 0} c_k(wC) t^k,
\]
and the coefficients \( c_k(wC) \) are its **Whitney numbers**. Thus Theorem 1.5 gives the following corollary.

**Corollary 1.6.** The \( k \)-th Whitney number of \( wC \) is
\[
c_k(wC) = \# \{ A \in \mathcal{A}_w \mid \# A = k \}.
\]
Summing up these Whitney numbers yields the following refinement of the parking function numbers, which was studied by Armstrong-Rhoades in type $A$ [4, Theorem 4.1] and then extended to type $C$ by Mészáros [17, Theorem 6].

**Corollary 1.7.** We have

$$\#\left\{ R \in R_w \mid \text{ceil}(R) = k \right\} = \sum_{w \in W} c_k(wC) .$$

Summing over all $k$ yields

$$\sum_{w \in W} \#A_w = R(\text{Shi}(\Phi^+)) = (h + 1)^\ell .$$

The *Narayana numbers* are a refinement of the Catalan numbers and one way to define these is by counting antichains in the root poset by cardinality, i.e.,

$$\text{Nar}^k(\Phi^+) = \#\left\{ A \in A_e \mid \#A = k \right\} .$$

Corollary 1.6 thus gives another interpretation of the Narayana numbers in terms of the Whitney numbers of the dominant cone.

**Corollary 1.8.** We have $\text{Nar}^k(\Phi^+) = c_k(C) .$

**Remark 1.9** (There is no $m$-eralization). The Shi arrangement has a well-known $m$-eralization given by

$$\text{Shi}^{(m)}(\Phi^+) = \bigcup_{-m < k \leq m} \text{Shi}_k(\Phi^+) .$$

Its region count is the *Fuss-parking function number* $(mh + 1)^\ell$ and its dominant region count is the *Fuss-Catalan number* $\prod_{i=1}^\ell (d_i + mh)/d_i$. There is also an $m$-eralization of antichains by Athanasiadis [5]. Unfortunately, the theory presented in this paper does not extend beyond $m = 1$. In type $A_2$ with $m = 2$, the dominant cone $C$ of $\text{Shi}^{(2)}(\Phi^+)$ is

The two shaded regions have the same single-element ceiling set, $H_{\alpha+\beta,2}$ (dashed). Moreover, the three hyperplanes

$$H_{\alpha,1} \cap H_{\beta,1} \cap H_{\alpha+\beta,2} = \{ \bullet \}$$

meet at a zero-dimensional flat with absolute Möbius function value $|\mu(V, \bullet)| = 2$. Thus we can have neither a bijection between regions in the dominant cone and certain antichain generalizations defined via ceiling sets (indeed, the bijection in [5] uses more information than only ceilings), nor can we have a bijection between flats cutting through $C$ and these dominant regions. In particular, the above flat is the unique flat with absolute Möbius function value not equal to 1 and the number of flats inside $C$ is 11 while there are 12 dominant regions.

Although we do not define them here, the Fuss-Catalan numbers are refined by Fuss-Narayana numbers and one can check that the Whitney number distribution matches
Fuss-Narayana distribution in rank 2 (for all $m$). This pattern fails in higher ranks as, for example, the the Poincaré polynomial of the dominant cone of in type $A_3$ with $m = 2$ is $1 + 12t + 29t^2 + 13t^3$ while the corresponding Fuss-Narayana polynomial is $1 + 12t + 28t^2 + 14t^3$.

After a detailed example in Section 1.3, the remainder of this paper is organized as follows. In Section 2, we extend and prove all above main results for the dominant cone $C$ and certain subarrangements of the Shi arrangement, namely those containing the reflection arrangement. We then deduce the main results from that setup in Section 2.2. In Section 3, we finally interpret the above Poincaré polynomials as the Hilbert series of two isomorphic graded rings, one arising from the Varchenko-Gel’fand ring and another, which we call the order ring of a finite poset since it turns out to be naturally associated to the order polytope.

1.3. The results in action. We close this introduction with a detailed illustration of the main results in the example of type $B_2$. The simple and positive roots are

$$\Delta = \{\alpha, \beta\} \subseteq \Phi^+ = \{\alpha, \beta, \alpha + \beta, 2\alpha + \beta\}.$$  
We write $s = s_\alpha$ and $t = s_\beta$ to be the reflections orthogonal to $\alpha$ and to $\beta$, respectively. One may realize this root system in the standard basis $\{e_1, e_2\}$ of $\mathbb{R}^2$ by $\alpha = e_2, \beta = e_1 - e_2$. In this realization, the arrangements $\mathcal{A}(W)$ (left) and Shi($\Phi^+$) (right) are

where we shaded the fundamental cone $C$ and the cone $stC$ and labelled all cones $wC$ of the reflection arrangement by the elements $w \in W$. The root poset $\Phi^+$ (left) and its subposet on $E = \{\alpha, 2\alpha + \beta\}$ (right) are

We observe that $E = \Phi^+ \setminus \text{Inv}(ts)$ by checking that $\text{Inv}(ts) = \{H_\beta, H_{\alpha+\beta}\}$ is given by the two hyperplanes separating $tsC$ from $C$. Next we illustrate the main results in these two settings, i.e., for the elements $e, st = (ts)^{-1} \in W$. We have

$$\mathcal{A}_e = \{\emptyset, \{\alpha\}, \{\beta\}, \{\alpha, \beta\}, \{\alpha + \beta\}, \{2\alpha + \beta\}\}, \quad \mathcal{A}_{st} = \{\emptyset, \{\beta\}, \{2\alpha + \beta\}\}. $$

Theorem 1.1 now yields that the ceilings of the chambers of the regions inside the cones $C$ and $stC$ are given by $e\mathcal{A}_e = \mathcal{A}_e$ and $st\mathcal{A}_{st} = \{\emptyset, \{\beta\}, \{\alpha + \beta\}\}$, respectively. The cone $C$
and \( stC \) are shaded above and one can check that the regions inside these cones have ceiling sets contained in \( A_e \) and \( stA_{st} \), respectively. Next, Theorem 1.2 gives a bijection between \( A_w \) and \( L_w \). The intersection posets \( L_e \) and \( L_{st} \) are drawn below, on the left and right respectively.

\[
\begin{align*}
H_{\alpha,1} \cap H_{\beta,1} & \\
H_{\alpha,1} & \quad H_{\beta,1} & \quad H_{\alpha+\beta,1} & \quad H_{2\alpha+\beta,1} & \quad H_{\beta,1} & \quad H_{\alpha+\beta,1} \\
V & \quad V
\end{align*}
\]

The six elements of \( L_e \) correspond to the antichains of \( \Phi^+ = \Phi^+ \setminus \text{Inv}(e) \), and the three elements of \( L_{st} \) correspond, after applying the element \( st \in W \), to the antichains of \( \Phi^+ \setminus \text{Inv}(ts) \). Their Poincaré polynomials are

\[
Poin(C, t) = 1 + 4t + t^2, \quad Poin(stC, t) = 1 + 2t.
\]

In particular, the Whitney numbers of \( C \)—that is, the coefficients of \( Poin(C, t) \)—are precisely the type \( B_2 \) Narayana numbers \((1, 4, 1)\). We sum the Poincaré polynomials of all chambers to obtain

\[
(1 + 4t + t^2) + (1 + 3t) + (1 + 2t) + (1 + t) + 1 + (1 + t) + (1 + 2t) + (1 + 3t)
= 8 + 16t + t^2.
\]

Evaluating at \( t = 1 \) gives the total number \( 25 = 5^2 \) of Shi regions. Note that this is not the Poincaré polynomial of the full arrangement, which is \( 1 + 8t + 16t^2 \). In general, the constant term of the sum of Poincaré polynomials is the cardinality of the group, whereas the Poincaré polynomial of the arrangement always has constant term 1.

## 2. Proofs of main results

In this section, we further generalize the main results to certain subarrangements of the Shi arrangement and consider the situation inside the dominant cone. We then show in Section 2.2 how these imply the results for other cones.

### 2.1. Generalized results for Shi deletions

Let \( E \subseteq \Phi^+ \) be a subposet and define the \( E \)-subarrangement

\[\text{Shi}(E) = \text{Shi}_0(\Phi^+) \cup \{H_{\beta,1} \mid \beta \in E\} \subseteq \text{Shi}(\Phi^+)\] .

Shi deletions were considered by Armstrong and Rhoades under the name graphical Shi arrangements [4]. Throughout this section let

- \( R_E \) be the regions of \( \text{Shi}(E) \) which lie inside the dominant cone \( C \), i.e., \( \{R \in R(\text{Shi}(E)) \mid R \subseteq C\} \);
- \( L_E \) the intersections of \( \text{Shi}(E) \) with nonempty intersection with \( C \), i.e., \( \{X \in L(\text{Shi}(E)) \mid X \cap C \neq \emptyset\} \), and
- \( A_E \) the set of antichains of roots in \( E \subseteq \Phi^+ \) (in root poset order).

Next we cast the main theorems into this framework.
Theorem 2.1. The map $\varphi_E : R_E \to A_E$ given by

$$R \mapsto \text{ceil}(R)$$

is a bijection with inverse $\varphi_E^{-1} : A_E \to R_E$ given by

$$A \mapsto \left\{ v \in V \mid \begin{array}{l} 0 < \langle v, \gamma \rangle < 1 \text{ for } \gamma \in I(A) \\ 1 < \langle v, \gamma \rangle \text{ for } \gamma \notin I(A) \end{array} \right\},$$

where $I(A)$ is the order ideal generated by $A$ inside the root poset restricted to $E$.

First observe that for $E = \Phi^+$, this map is the well-studied bijection of Shi [18, Theorem 1.4] which was reframed and generalized by Athanasiadis [5, Theorem 3.6]. We record this in the following proposition.

Proposition 2.2 (Shi [18]). Theorem 2.1 holds for $E = \Phi^+$.

The more general case above can be deduced from this known case as follows.

Claim 2.3. The map $\varphi_E : R_E \to A_E$ is well-defined.

Proof. Let $R \in R_E$. We first observe that ceil$(R)$ is a subset of $E$ by construction. Thus we only need to show that ceil$(R)$ is an antichain in the root poset. To this end, let $\beta \in \text{ceil}(R)$ and let $\gamma \in E$ such that $\gamma \prec \beta$. Then $\beta - \gamma = \sum_{\alpha \in \Delta} c_{\alpha} \alpha$ where $c_{\alpha} \geq 0$ are integers and not all $c_{\alpha} = 0$. Then for all $v \in R \cap C$, we have

$$\langle \gamma, v \rangle = \langle \gamma - \beta + \beta, v \rangle = \langle \beta, v \rangle - \langle \beta - \gamma, v \rangle \leq 1 - \sum_{\alpha \in \Delta} c_{\alpha} \langle \alpha, v \rangle < 1,$$

where the final inequality uses the fact that $\langle \alpha, v \rangle > 0$ for all $\alpha \in \Delta$, since $v \in R \cap C \subseteq C$. Thus the hyperplane $H_{\gamma,1}$ cannot be a ceiling of $R$. \qed

Claim 2.4. The map $\varphi_E : R_E \to A_E$ is injective.

Proof. Let $R, R' \in R_E$ be two different regions. Since $R, R' \subseteq C$, we have that they are separated by a hyperplane of the form $H_{\beta,1}$ for $\beta \in E$. Let $\beta$ index such a hyperplane and assume that $\beta$ is maximal with this property (in root poset order). Up to possibly swapping $R$ and $R'$, we may assume that $\langle \beta, v \rangle < 1$ for all $v \in R$ while $\langle v, \beta \rangle > 1$ for all $v \in R'$. Since $\langle \gamma, v \rangle < 1$ for all $\gamma \in E$ with $\beta \prec \gamma$, we see that $\beta$ is a ceiling of $R$ which is not a ceiling of $R'$. \qed

Claim 2.5. The map $\varphi_E^{-1} : A_E \to R_E$ is well-defined.

Proof. The definition of $\varphi_E^{-1}$ specifies, for each hyperplane, on which side of that hyperplane we lie. Thus, if $\varphi_E^{-1}(A)$ is nonempty, then it must be a region of the arrangement. From Proposition 2.2 we have $\varphi_E^{-1}(A) \neq \emptyset$, since $\varphi_E^{-1}(A)$ contains $\varphi_{\Phi^+}^{-1}(A)$ (as subsets of $V$). \qed

Claim 2.6. The map $\varphi_E^{-1} : A_E \to R_E$ is injective.

Proof. Two antichains $A, A' \in A_E$ are sent to the same region if and only if they define the same order ideal, which only happens if they are the same antichain. \qed

Proof of Theorem 2.1. We have already shown that $\varphi_E$ and $\varphi_E^{-1}$ are both well-defined and injective. They are thus both bijective. The discussion in the proof of Claim 2.3 also shows that they are inverses of each other. \qed
Theorem 2.7. The map \( \psi_E : \mathcal{L}_E \to \mathcal{A}_E \) given by
\[
X \mapsto \{ \beta \mid X \subseteq H_{\beta,1} \}
\]
is a bijection with inverse \( \psi_E^{-1} : \mathcal{A}_E \to \mathcal{L}_E \) given by
\[
A \mapsto \bigcap_{\beta \in A} H_{\beta,1}.
\]

Before proving this theorem, we provide (without proof) the following well-known property for later reference.

Proposition 2.8. Every lower interval in \( \mathcal{L}_E \) is a geometric lattice. In particular, \( \mathcal{L}_E \) is atomic in the sense that every element \( X \in \mathcal{L}_E \) is the join of the atoms in \( \mathcal{L}_E \) below \( X \).

We also recall the following result of Sommers.

Proposition 2.9 ([19]). Let \( A \subseteq \Phi^+ \) be an antichain. Then there exists an element \( w \in W \) in the Weyl group such that \( w(A) \subseteq \Delta \). In particular, \( A \) is linearly independent.

Claim 2.10. The map \( \psi_E : \mathcal{L}_E \to \mathcal{A}_E \) is well-defined.

Proof. Assume that \( B \subseteq E \) is not an antichain. Then there exist \( \gamma \prec \beta \) in \( B \) and therefore
\[
X = \bigcap_{\delta \in B} H_{\delta,1} \subset H_{\gamma,1} \cap H_{\beta,1}.
\]
Since \( \langle \gamma, v \rangle = \langle \beta, v \rangle = 1 \) for \( v \in H_{\gamma,1} \cap H_{\beta,1} \), we also have \( \langle \beta - \gamma, v \rangle = 0 \). Because \( \beta - \gamma = \sum_{\alpha \in \Delta} c_\alpha \alpha \) with \( c_\alpha \geq 0 \) and not all \( c_\alpha = 0 \), we obtain that
\[
\langle \beta - \gamma, v \rangle > 0
\]
and thus \( v \notin H_{\gamma,1} \cap H_{\beta,1} \). Equation \( \ast \) implies \( X \notin \mathcal{L}_E \), as desired. \( \square \)

Claim 2.11. The map \( \psi_E : \mathcal{L}_E \to \mathcal{A}_E \) is injective.

Proof. It follows from Proposition 2.8 that \( \psi_E(X) \) can be identified with the set of atoms below \( X \) in \( \mathcal{L}_E \). This yields the injectivity. \( \square \)

Claim 2.12. We have that the interval \([V, X]\) is Boolean for all \( X \in \mathcal{L}_E \). In particular, \( \mu(V, X) = (-1)^{\text{codim}(X)} \) and
\[
\# \mathcal{L}_E = \# \mathcal{A}_E.
\]

Proof. Certainly \( X = \bigcap_{\beta \in A} H_{\beta,1} \) for \( A \subseteq E \) and we have seen in the proof of Claim 2.10 that \( A \) is an antichain. It thus follows from Proposition 2.9 that \( A \) is linearly independent and \([V, X]\) is a Boolean lattice. This implies \( \mu(V, X) = (-1)^{\text{codim}(X)} \) and Zaslavsky’s theorem [23, Example A] gives
\[
\# \mathcal{R}_E = \sum_{X \in \mathcal{L}_E} |\mu(V, X)| = \# \mathcal{L}_E.
\]

\( \square \)

Note 2.13. Since each lower interval \([V, X]\) is Boolean, the previous proof also shows that
\[
\mu(X, Y) = (-1)^{\text{codim}(X) + \text{codim}(Y)}
\]
for all \( X, Y \in \mathcal{L}_E \) with \( Y \subseteq X \). In particular, this means that \( \mathcal{L}_E \) is Eulerian.

Claim 2.12 has the following immediate consequence.
Corollary 2.14. The Poincaré polynomial of the dominant cone $C$ in the Shi deletion $\text{Shi}(E)$ is given by

$$\text{Poin}_E(C,t) := \sum_{X \in \mathcal{L}_E} t^{\text{codim}(X)} = \sum_{k \geq 0} c_{E,k}(C) t^k$$

where the $k$th Whitney number of $\mathcal{L}_E$ is given by

$$c_{E,k}(C) = \#\{A \in A_E \mid \#A = k\}.$$

Proof of Theorem 2.7. The preceding sequence of claims shows that $\psi_E : \mathcal{L}_E \to A_E$ is well-defined and injective. In Theorem 2.1, we saw that $\#A_E = \#R_E$. With Claim 2.12, we obtain $\#\mathcal{L}_E = \#A_E$ and thus, $\psi_E$ is a bijection.

It remains to show that $\psi_E^{-1}$ is well-defined and the inverse of $\psi_E$. We prove both simultaneously. Let $A \in A_E$. Since $\psi_E$ is a bijection, there is a unique $X \in \mathcal{L}_E$ with $\psi_E(X) = A$. This means that $X = \bigcap_{\beta \in A} H_{\beta,1}$. This implies that

$$X = \psi_E^{-1}(A) = \psi_E^{-1} \circ \psi_E(X).$$

Since this holds for any $A \in A_E$, we have that $\psi_E^{-1}$ is well-defined and the inverse of $\psi_E$. \hfill $\Box$

2.2. Deduction of main results. In this section we deduce the main results of this paper from the results in the previous section. The main ingredient is the following lemma of Armstrong-Reiner-Rhoades (we include the proof for completeness, as the proof is short and we use a slightly reframed version of their statement). Recall that the inversion set of an element $w \in W$ is

$$\text{Inv}(w) = \Phi^+ \cap w\Phi^- = \{\beta \in \Phi^+ \mid w^{-1}(\beta) \in \Phi^-\}.$$

Lemma 2.15 ([3, Lemma 10.2]). Let $w \in W$ and $\beta \in \Phi^+$. Then $H_{\beta,1}$ cuts through $wC$ if and only if $\beta$ is not an inversion of $w$. In symbols,

$$H_{\beta,1} \cap wC = \emptyset \iff \beta \in \text{Inv}(w).$$

Proof. We have

$$H_{\beta,1} \cap wC \neq \emptyset \iff \text{there exists } v \in C \text{ such that } \langle \beta, w(v) \rangle = 1 \iff \text{there exists } v \in C \text{ such that } \langle w^{-1}(\beta), v \rangle = 1 \iff w^{-1}(\beta) \in \Phi^+ \iff \beta \notin \text{Inv}(w).$$

Here, the first and last equivalences are the respective definitions, the second uses the $W$-invariance of the inner product, and the third uses the fact that the dominant cone can be described by $C = \{v \in V \mid \langle \gamma, v \rangle > 0 \text{ for all } \gamma \in \Phi^+\}$. \hfill $\Box$

Lemma 2.16. For $w \in W$, we have that $\Phi^+ \setminus \text{Inv}(w) = w\left(\Phi^+ \setminus \text{Inv}(w^{-1})\right)$.

Proof. The following is a straightforward calculation.

$$w^{-1}(\Phi^+ \setminus \text{Inv}(w)) = w^{-1}(\Phi^+) \setminus w^{-1}(\Phi^+ \cap w\Phi^-) = w^{-1}(\Phi^+) \setminus (w^{-1}\Phi^+ \cap \Phi^-) = w^{-1}\Phi^+ \cap \Phi^+ = \Phi^+ \setminus \text{Inv}(w^{-1}).$$

Applying now $w$ to both sides yields the statement. \hfill $\Box$
Proof of Theorems 1.1, 1.2 and 1.5. These follow from the respective theorems in Section 2 applied to the set
\[ E = \Phi^+ \setminus \text{Inv}(w^{-1}). \]
To this end, we first observe that \( A_w = A_E \). Second, we have \( R_w = w(R_E) \). To see this, recall that \( R_w \) consists of the Shi regions inside \( wC \) and that Lemma 2.15 yields that \( R_w = w(F) \) for \( F = w^{-1}(\Phi^+ \setminus \text{Inv}(w)) \). Lemma 2.16 finally gives that \( F = E \). By the very same argument and the fact that the intersection poset is invariant under linear isomorphisms, we obtain the poset isomorphism \( L_w \cong w(L_E) \).

3. Algebraic Interpretation

In this final section we present two equivalent viewpoints on an algebraic interpretations of the Poincaré polynomial
\[ \text{Poin}_E(C, t) = \sum_{k \geq 0} c_{E, k}(C) \]
from Corollary 2.14 as the Hilbert series of a graded ring, where we fix a subset \( E \subseteq \Phi^+ \) throughout.

3.1. Two equivalent ring constructions. We present the following two constructions of \( \mathbb{Z} \)-modules associated to the subset \( E \subseteq \Phi^+ \). The first can be associated to any cone in an arrangement [11, 22], while the second can be associated to any finite poset [9]. We specialize the definitions to our current context. Both rings are free \( \mathbb{Z} \)-modules with componentwise multiplication. We thus emphasize that it is not that abstract ring construction that is interesting but the presentation using Heaviside functions that describes them as quotients of a polynomial ring as given in Corollary 3.4.

The Varchenko-Gel’fand ring of the dominant cone \( C \) of the Shi deletion \( \text{Shi}(E) \) is defined as
\[ \text{VG}(E, C) = \{ f : R_E \rightarrow \mathbb{Z} \} \]
with pointwise addition and multiplication. It is linearly generated by the indicator functions
\[ \{ \delta_R : R_E \rightarrow \mathbb{Z} \mid R \in R_E \}. \]
It is not hard to see (and is carefully discussed in [22] and in [11, Section 2.3]) that \( \text{VG}(E, C) \) is also generated (as a ring) by the Heaviside functions \( \{ x_\beta : R_E \rightarrow \mathbb{Z} \mid \beta \in E \} \) with
\[ R \mapsto \begin{cases} 1 & \text{if } R \subseteq H_{\beta, 1}^- \\ 0 & \text{else.} \end{cases} \]
To be explicit, we have
\[ x_\beta = \sum_{R \subseteq H_{\beta, 1}^-} \delta_R, \quad \delta_R = \prod_{R \subseteq H_{\beta, 1}^-} x_\beta \prod_{R \subseteq H_{\beta, 1}^+} (1 - x_\beta). \]
In the proof of Theorem 2.1, we saw that, for a region \( R \in R_E \) and a root \( \beta \in E \),
\[ x_\beta(R) = 1 \iff R \subseteq H_{\beta, 1}^- \iff \beta \in \mathcal{I}(\varphi_E(R)). \]
This means that the Varchenko-Gel’fand ring is in the present case isomorphic to the following ring construction for any finite poset.
Following a construction by Chapoton [9], we define the order ring of the sub-poset $E \subseteq \Phi^+$. To this end, let $\mathcal{I}_E = \{ \mathcal{I}(A) \mid A \in \mathcal{A}_E \}$ be its sets of order ideals inside $E \subseteq \Phi^+$ and define

$$\text{OR}(E) = \{ f : \mathcal{I}_E \to \mathbb{Z} \}$$

with pointwise addition and multiplication. It is generated linearly by the indicator functions

$$\{ \delta_I : \mathcal{I}_E \to \mathbb{Z} \mid I \in \mathcal{I}_E \}$$

and it is also generated (as a ring) by the Heaviside functions $\{ y_\beta : \mathcal{I}_E \to \mathbb{Z} \mid \beta \in E \}$ given by

$$I \mapsto \left\{ \begin{array}{ll}
1 & \text{if } \beta \in I \\
0 & \text{else.}
\end{array} \right.$$  

Concretely, we have

$$y_\beta = \sum_{I \in \mathcal{I}_E : \beta \in I} \delta_I, \quad \delta_I = \prod_{\beta \in I} y_\beta \cdot \prod_{\beta \in E \setminus I} (1 - y_\beta).$$

Proposition 3.1. Sending $\delta_R$ to $\delta_I$ for $I = \mathcal{I}(\varphi_E(R))$ or, equivalently, sending $x_\beta$ to $y_\beta$ is a ring isomorphism between $\text{VG}(E, C)$ and $\text{OR}(E)$.

Proof. This is immediate from Equation (⋆⋆). □

We will come back to the present setup and these two equivalent ring constructions in Corollary 3.5, after we give a general motivation for the order ring of any finite poset.

3.2. The order ring of a finite poset. Although our initial interest in order rings came from studying subposets of the root poset, the definition of the order ring—as well as its description via Heaviside functions—extends verbatim to any (finite) poset $(P, \preceq)$. We illustrate this construction in a detailed example in Example 3.7.

In this section, we motivate our choice of name by noting a simple connection between the order ring of an arbitrary finite poset $P$ and its order polytope introduced in [21] as

$$\mathcal{O}(P) = \{ a \in [0, 1]^P \mid a_p \leq a_q \text{ for all } p \preceq q \} \subset \mathbb{R}^P.$$ 

The vertex set is given by the indicator vectors of order filters, i.e.,

$$V_P = \{ a \in \{0, 1\}^P \mid a_p = 1 \Rightarrow a_q = 1 \text{ for all } p \preceq q \} \subset \{0, 1\}^P.$$ 

Since $V_P$ is a finite set of points, it is an algebraic variety with vanishing ideal

$$I(V_P) = \langle f \mid f(a) = 0 \text{ for all } a \in V_P \rangle \subseteq \mathbb{C}[z_p \mid p \in P].$$

The following theorem shows that this vanishing ideal has the same generators as the quotienting ideal of the order ring. Thus the order ring can be viewed as a $\mathbb{Z}$-analogue of the coordinate ring of the coordinate ring $\mathbb{C}[z_p \mid p \in P] / I(V_P)$ of $V_P$.

Theorem 3.2. Let $\mathcal{G} = \{ z_p(1 - z_q) \mid p \preceq q \}$. Then $I(V_P) = \langle \mathcal{G} \rangle$ and in particular, the coordinate ring of $V_P$ is $\mathbb{C}[z_p \mid p \in P] / \langle \mathcal{G} \rangle$. We moreover have a $\mathbb{Z}$-algebra isomorphism

$$\text{OR}(P) \cong \mathbb{Z}[z_p \mid p \in P] / \langle \mathcal{G} \rangle.$$

Remark 3.3. We introduced $\text{OR}(P)$ as a $\mathbb{Z}$-analogue of the coordinate ring of $V_P$, but it is also a $\mathbb{Z}$-analogue of the Möbius algebra (see [20, Definition 3.9.1]) of the Birkhoff lattice of order ideals. From this perspective, Theorem 3.2 says that the Möbius algebra of a finite distributive lattice is the coordinate ring of an algebraic variety (via the
Fundamental Theorem of Finite Distributive Lattices, see [20, Theorem 3.4.1]), and gives explicit presentation for the rings and a description of the variety.

For the proof, we applying a standard argument from Gröbner basis theory (adapted in [11, Lemma 8] to \( \mathbb{Z} \)-algebras). For the relevant background on term orders and initial ideals, see [1, §1.4] or [10, §2.3].

**Proof.** We prove the second statement first. Consider the \( \mathbb{Z} \)-algebra map

\[
\rho : \mathbb{Z}[z_p \mid p \in P] \to \text{OR}(P)
\]

defined by \( z_p \mapsto y_p \). Since the Heaviside functions \( y_p \) generate \( \text{OR}(P) \), this map is surjective. It moreover follows from \( y_p y_q = y_p \) for \( p \preceq q \) that

\[
G = \{ z_p(1 - z_q) \mid p \preceq q \} \subseteq \ker(\rho).
\]

For any term order, the initial terms of \( G \) are \( \{ z_p z_q \mid p \preceq q \} \), meaning that the standard monomials with respect to \( \{ z_p z_q \mid p \preceq q \} \) are monomials of the form \( \prod_{a \in A} z_a \) where \( A \) is an antichain of \( P \). Since the rank of \( \text{OR}(P) \) as a \( \mathbb{Z} \)-module is the number of order ideals (or of antichains), applying [11, Lemma 8] gives the desired \( \mathbb{Z} \)-algebra isomorphism.

Now we show that \( I(V_P) = \langle G \rangle \). Since \( I(V_P) \) is the vanishing ideal of an algebraic variety, the coordinate ring is a free \( \mathbb{C} \)-module of dimension \( \#V_P \) given by number of antichains of \( P \). The argument from the first part of this proof (now performed over \( \mathbb{C} \)) tells us that \( \mathbb{C}[z_p \mid p \in P]/\langle G \rangle \) is a free \( \mathbb{C} \)-algebra of dimension \( \#V_P \). Since the polynomial \( z_p(1 - z_q) \) vanishes on every point in \( V_P \), we have \( \langle G \rangle \subseteq I(V_P) \). This gives the surjection

\[
\mathbb{C}[z_p \mid p \in P]/\langle G \rangle \twoheadrightarrow \mathbb{C}[z_p \mid p \in P]/I(V_P),
\]

between two free \( \mathbb{C} \)-modules of the same dimension. Thus this is the desired isomorphism, see for example [16, Propositions 1.7, 1.12]. \( \square \)

We obtain the following as a corollary of our proof.

**Corollary 3.4.** For any term order, we have a \( \mathbb{Z} \)-algebra isomorphisms

\[
\text{gr}(\text{OR}(P)) \cong \mathbb{Z}[z_p \mid p \in P]/\langle z_p z_q \mid p \preceq q \rangle
\]

where \( \text{gr}(\text{OR}(P)) \) is the associated graded ring with respect to the degree filtration. Moreover, the \( k \)th graded component is linearly generated by the squarefree monomials

\[
\left\{ \prod_{p \in A} z_p \mid A \in A_P \text{ with } |A| = k \right\}.
\]

We specify this corollary to the setting above for the poset \( E \subseteq \Phi^+ \) as it connects the order ring to the Poincaré polynomial of the arrangement.

**Corollary 3.5.** We have

\[
\text{Hilb} \left( \text{gr}(\text{OR}(E)), t \right) = \sum_{A \in A_E} t^{|A|} = \text{Poin}_E(C, t).
\]

\footnote{This is a standard fact, which can be obtained obtained from Hilbert’s Nullstellensatz (see [12, Theorem 1.6], for example) and a general theorem about ideals of an algebraically-closed field (see [10, Proposition 1.3.7(iii)]), for example.}
Remark 3.6 (A recursion on standard monomials). We have seen in Corollary 3.4 that the standard monomials for $\text{OR}(P)$ with respect to any term order are identified with the set of antichains in $P$. Since $V_P \subset \{0, 1\}^P$, we may on the other hand revisit [13, Lemma 3.1], where a recursive description of the standard monomials are given for the lexicographic term order induced by a total ordering of the variables $\{z_p \; | \; p \in P\}$. These two viewpoints match as we describe now. For ease of notation, we assume that that $P$ is naturally labelled by $P = \{1, \ldots, k\}$, meaning that if $i \leq j$ in $P$, then $i \leq j$. Now let $P_1 = P \setminus \{k\}$ be the subposet of $P$ obtained by deleting the element $k$ (which is necessarily maximal in $P$) and let $P^0 = P \setminus I(k)$ be obtained by removing the principal order ideal generated by $k$. The recursive description in [13, Lemma 3.1] for the lexicographic term order on $\mathbb{Z}[z_1, \ldots, z_1]$ induced by $z_1 > z_2 > \cdots > z_k$ then becomes the well-known recursive description of order ideals of $P$ given by

$$A_P = A_{P_1} \cup \left\{ A \cup \{k\} \mid A \in A_{P^0} \right\}.$$ 

This equality says that an antichain in $P$ does either not contain $k$ and is thus an antichain in $P_1$, or it does contain $k$ and it thus is the union of antichain in $P^0$ with $\{k\}$. Given that $P^0 \subseteq P_1$ and thus $A_{P^0} \subseteq A_{P_1}$, we may rewrite this as

$$A_P = A_{P_1} \cup A_{P^0} \cup \left\{ A \cup \{k\} \mid A \in A_{P^0} \cap A_{P_1} \right\},$$

which is the recursive description given in [13, Lemma 3.1].

Example 3.7. We close this section with an example, for the poset

$$\begin{array}{ccc}
4 & 5 \\
3 & 2 \\
1 &
\end{array}$$

The vertices of its order polytope $\mathcal{O}(P)$ are

$$V_P = \{00000, 10000, 01000, 11100, 11110, 11101, 11000, 11111\}.$$

Its order ring has presentation $\text{OR}(P) \cong \mathbb{Z}[z_1, z_2, z_3, z_4, z_5]/\langle \mathcal{G} \rangle$, where $\mathcal{G}$ consists of $z_i - z_i^2$ for $i \in [5]$ as well as

$$z_1(1-z_3), z_1(1-z_4), z_1(1-z_5), z_2(1-z_3), z_2(1-z_4), z_2(1-z_5),$$

From Corollary 3.4, the associated graded (with respect to the degree filtration) of the order ring of $P$ is

$$\text{gr} \text{OR}(P) \cong \mathbb{Z} \cdot \{1\} \oplus \mathbb{Z} \cdot \{z_1, z_2, z_3, z_4, z_5\} \oplus \mathbb{Z} \cdot \{z_1 \cdot z_2, z_4 \cdot z_5\}$$

and its Hilbert series is $\text{Hilb}(\text{gr} \text{OR}(P), t) = 1 + 5t + 2t^2$. Note that the basis of the order ring is indexed by the antichains of $P$:

$$A_P = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1, 2\}, \{4, 5\}\}.$$

These antichains can be separated into antichains containing 5 and antichains not containing 5, thus giving

$$A_{P^0} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}\} \quad \text{and} \quad A_{P_1} = \{\emptyset, \{4\}\}$$

for the posets $P_1$ and $P^0$ obtained by deleting the vertex 5 and, respectively, deleting is vertex and everything below.
References

[1] William W. Adams and Philippe Loustaunau. *An introduction to Gröbner bases*, volume 3 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 1994.

[2] Drew Armstrong. Generalized noncrossing partitions and combinatorics of Coxeter groups. *Mem. Amer. Math. Soc.*, 202(949):x+159, 2009.

[3] Drew Armstrong, Victor Reiner, and Brendon Rhoades. Parking spaces. *Adv. Math.*, 269:647–706, 2015.

[4] Drew Armstrong and Brendon Rhoades. The Shi arrangement and the Ish arrangement. *Trans. Amer. Math. Soc.*, 364(3):1509–1528, 2012.

[5] Christos A. Athanasiadis. Generalized Catalan numbers, Weyl groups and arrangements of hyperplanes. *Bull. London Math. Soc.*, 36(3):294–302, 2004.

[6] Christos A. Athanasiadis and Svante Linusson. A simple bijection for the regions of the Shi arrangement of hyperplanes. *Discrete Math.*, 204(1-3):27–39, 1999.

[7] Christos A. Athanasiadis and Victor Reiner. Noncrossing partitions for the group $D_n$. *SIAM J. Discrete Math.*, 18(2):397–417, 2004.

[8] Philippe Biane and Matthieu Josuat-Vergès. Noncrossing partitions, Bruhat order and the cluster complex. *Ann. Inst. Fourier (Grenoble)*, 69(5):2241–2289, 2019.

[9] Frédéric Chapoton. Antichains of positive roots and Heaviside functions. *arXiv 0303220*, pages 1–7, 2003.

[10] David A. Cox, John Little, and Donal O'Shea. *Ideals, Varieties, and Algorithms*. Undergraduate Texts in Mathematics. Springer, fourth edition, 2015. An introduction to computational algebraic geometry and commutative algebra.

[11] Galen Dorpalen-Barry. The Varchenko-Gel’fand ring of a cone. *arXiv 2104.02740*, pages 1–16, 2021.

[12] David Eisenbud. *Commutative Algebra*, volume 150 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1995. With a view toward algebraic geometry.

[13] Alexander Engström, Raman Sanyal, and Christian Stump. Standard complexes of matroids and lattice paths. *Vietnam Journal of Mathematics*, 2022.

[14] Susanna Fishel. A survey of the Shi arrangement. 16:75–113, 2019.

[15] James E. Humphreys. *Reflection Groups and Coxeter Groups*, volume 29 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1990.

[16] T. Y. Lam. *Lectures on modules and rings*, volume 189 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1999.

[17] Karola Mészáros. Labeling the regions of the type $C_n$ Shi arrangement. *Electron. J. Combin.*, 20(2):Paper 31, 12, 2013.

[18] Jian-Yi Shi. The number of $\oplus$-sign types. *Quart. J. Math. Oxford Ser. (2)*, 48(189):93–105, 1997.

[19] Eric N. Sommers. B-stable ideals in the nilradical of a Borel subalgebra. *Canadian Mathematical Bulletin*, 48(3):460–472, 2005.

[20] Richard Stanley. *Enumerative Combinatorics*. Cambridge University Press, New York, NY, USA, 2 edition, 2012.

[21] Richard P. Stanley. Two poset polytopes. *Discrete Comput. Geom.*, 1(1):9–23, 1986.

[22] Alexander Varchenko and Izrail Moiseevich Gel’fand. Heaviside functions of a configuration of hyperplanes. *Funktional. Anal. i Prilozhen.,* 21(4):1–18, 96, 1987.

[23] Thomas Zaslavsky. A combinatorial analysis of topological dissections. *Advances in Math.*, 25(3):267–285, 1977.

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