Asymptotic Justification of Models of Plates Containing Inside Hard Thin Inclusions

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Abstract: An equilibrium problem of the Kirchhoff–Love plate containing a nonhomogeneous inclusion is considered. It is assumed that elastic properties of the inclusion depend on a small parameter characterizing the width of the inclusion ε as ε^N with N < 1. The passage to the limit as the parameter ε tends to zero is justified, and an asymptotic model of a plate containing a thin inhomogeneous hard inclusion is constructed. It is shown that there exists two types of thin inclusions: rigid inclusion (N < −1) and elastic inclusion (N = −1). The inhomogeneity disappears in the case of N ∈ (−1, 1).

Keywords: Kirchhoff-Love plate; composite material; thin inclusion; asymptotic analysis

1. Introduction

An equilibrium problem of a Kirchhoff–Love plate containing a nonhomogeneous inclusion is considered. It is assumed that the elastic properties of the inclusion depend on a small parameter characterizing the width of the inclusion ε as ε^N with N < 1. The problem is formulated as a variational one; namely, as a minimization problem of the energy functional over a set of admissible deflections in the Sobolev space H^2. This implies that the deflections function is a solution of a boundary value problem for bi-harmonic operator (pure bending, see, e.g., [1–4]).

The aim of the present work is to justify passing to the limit as ε → 0. To do this, we apply a method that was originally introduced in [5,6] for problems of gluing plates. The method is based on variational properties of the solution to the corresponding minimization problem and allows for finding a limit problem for any N < 1 simultaneously. It is shown that there exist two types of hard inclusions in dependence of N: thin rigid inclusion (N < −1) and thin elastic inclusion (N = −1). In case N ∈ (−1, 1), the influence of the inhomogeneity disappears in the limit. We get limit problems in a variational form, which is convenient, for example, for numerical analysis by the finite element method.

Let us give a short survey of works that are close to the present investigation. Note that there are not so many works devoted to study of models of thin inclusions in plates. We mention [7–9], in which thin elastic inclusions in plates were studied. Papers [10–13] are devoted investigations of thin rigid inclusions. We refer to [14–21] for asymptotic analyses for different models of bonded structures in Elasticity. We indicate also paper [22], where a geometry-dependent state problem for a heterogeneous medium with defects is investigated in framework of anti-plane elasticity.

Finally, we mention paper [23], where the mechanical behavior of an anisotropic nonhomogeneous linearly elastic three-layer plate with soft adhesive, including the inertia forces, was studied, and the various limiting models in the dependence of the size and the stiffness of the adhesive was derived. The problem under consideration in the present paper is different from the mentioned paper because we consider the hard inhomogeneity lying strictly inside the plate and derive limiting problem depending on the size and stiffness of the inclusion. Wherein, the plate size does not vary and remains constant.
2. Statement of Problem

Let us fix a small parameter $\varepsilon \in (0, 1)$ and consider an inhomogeneous rectangular plate $\Omega \subset \mathbb{R}^2$ with a thin rectangular inclusion $\Omega_{\text{inc}}^\varepsilon \subset \Omega$ of width $2d$, where $d$ is diameter of $\Omega$. Let us specify some notations:

$$\Omega = (-a_1, a_2) \times (-b_1, b_2), \quad a_\alpha, b_\alpha > 0, \quad \alpha = 1, 2,$$

$$\Omega_{\text{inc}}^\varepsilon = (-\varepsilon d, \varepsilon d) \times (-c_1, c_2), \quad 0 < c_\alpha < b_\alpha, \quad \alpha = 1, 2,$$

$$\Omega_{\pm} = \{(y_1, y_2) \in \Omega \mid \pm y_1 > 0\},$$

$$S = \partial \Omega_- \cap \partial \Omega_+,$$

$$S_{\text{inc}} = S \cap \Omega_{\text{inc}}^\varepsilon,$$

$$\Omega_{\text{mat}}^\varepsilon = \Omega \setminus \Omega_{\text{inc}}^\varepsilon, \quad \Omega_{\pm}^\varepsilon = \Omega_{\text{mat}}^\varepsilon \cap \Omega_{\pm}.$$

Note that, for all small enough $\varepsilon > 0$ a family of subdomains $\Omega_{\text{inc}}^\varepsilon$ lies strictly inside $\Omega$. Besides, let us define the following notations:

$$\Omega_{\text{mid}}^\varepsilon = \{(y_1, y_2) \in \Omega \mid -\varepsilon d < y_1 < \varepsilon d, \quad y_2 \in S\},$$

$$S_{\pm}^\varepsilon = \{(y_1, y_2) \in \Omega \mid y_1 = \pm \varepsilon d, \quad y_2 \in S\},$$

We assume that $S_{\text{inc}}$ is divided into three subsets $S_\alpha \subset S_{\text{inc}}$, where each $S_\alpha$ is an union of finite number of segments or empty set, $\alpha = 1, 2, 3$.

In our consideration, $\Omega$ is a composite plate, consisting of the elastic matrix $\Omega_{\text{mat}}^\varepsilon$ and the inhomogeneous inclusion $\Omega_{\text{inc}}^\varepsilon$, where

$$\Omega_{\alpha}^\varepsilon = \{(y_1, y_2) \in \mathbb{R}^2 \mid -\varepsilon d < y_1 < \varepsilon d, \quad y_2 \in S_\alpha\}, \quad \alpha = 1, 2, 3.$$

Moreover, in the sequel, we will use the following notations:

$$\Omega_0^\varepsilon = \Omega_{\text{mid}}^\varepsilon \setminus \bigcup_{\alpha=1}^3 \Omega_{\alpha}^\varepsilon,$$

$$S_0 = S \setminus S_{\text{inc}}.$$

Denote, by $E_0$, $E_\alpha$, and $k_0$, $k_\alpha$, Young’s modules and Poisson’s ratios of parts $\Omega_{\text{mat}}^\varepsilon$ and $\Omega_\alpha^\varepsilon$ of the composite plate $\Omega$, respectively, $\alpha = 1, 2, 3$. The compound character of the structure is expressed by the fact that $E_0$, $k_0$, and $k_\alpha$ are constants, while Young’s modulus $E_\alpha$ depends on $\varepsilon$, as follows:

$$E_\alpha = \varepsilon^{N_\alpha} E_\alpha^0 \quad \text{in} \quad \Omega_\alpha^\varepsilon, \quad \alpha = 1, 2, 3,$$

where $N_1$, $N_2$, $N_3$ are real numbers, such that

$$N_1 < -1, \quad N_2 = -1, \quad N_3 \in (-1, 1).$$

Parameters $N_1$ and $N_2$ correspond to hard inclusions in the plate $\Omega$ (see [6,24,25]). Moreover, put $N_0 = 0$.

Denote, by $w$, deflections of the composite plate $\Omega$. Then the bending moments are defined by formulae (see, e.g., [26,27])

$$m_{ij}(w) = d_{ijkl} w_{ijl}, \quad i, j = 1, 2, \quad w_{ijl} = \frac{\partial^2 w}{\partial y_i \partial y_j}.$$
where the positive definite and symmetric tensor \( \{d_{ijkl}\} \) is orthotropic with the following components:

\[
d_{iiii}(y) = D_\varepsilon(y),
\]

\[
d_{jjjj}(y) = D_\alpha(1 - k_\alpha(y))/2, \quad i \neq j,
\]

\[
d_{ijij}(y) = d_{ijji}(y) = D_\varepsilon(y)(1 - k_\varepsilon(y))/2,
\]

\[
D_\varepsilon(y) = \begin{cases} D_0 \text{ in } \Omega_{\text{mat}}, \\ \epsilon^\alpha D_\alpha \text{ in } \Omega_{\alpha}, \quad \alpha = 1, 2, 3, \\ \end{cases}
\]

\[
h = \text{a thickness of the plate } \Omega \text{ that is constant. Note paper } [28], \text{ where it was shown non-standard behaviour in the asymptotic two-dimensional reduction from three-dimensional elasticity, when the thickness and size of inclusions depend on the same parameter.}
\]

The potential energy functional of the plate has the following representation (see [27]):

\[
\Pi(w) = \frac{1}{2} \int_\Omega d_{ijkl} w_{kl} w_{ij} \, dy - \int_\Omega f w \, dy,
\]

where \( f \in L^2(\Omega) \) is a bulk force acting on the plate \( \Omega \). Subsequently, the equilibrium problem of nonhomogeneous plate clamped on the external boundary \( \partial \Omega \) can be formulated as the minimization problem: find a function \( w_\varepsilon \in H^2_0(\Omega) \) such that

\[
\Pi(w_\varepsilon) = \inf_{w \in H^2_0(\Omega)} \Pi(w).
\]

Problem (2) is known to have a unique solution \( w_\varepsilon \) (see, e.g., [26, 29]), which satisfies the variational equality:

\[
\int_\Omega d_{ijkl} w_{kl} w_{ij} \, dy = \int_\Omega f w \, dy \quad \forall w \in H^2_0(\Omega).
\]

Moreover, the function \( w_\varepsilon \) is a unique solution the following boundary value problem:

\[
(d_{ijkl} w_{kl}, ij) = f \text{ in } \Omega,
\]

\[
w_\varepsilon = \frac{\partial w_\varepsilon}{\partial \nu} = 0 \text{ on } \partial \Omega,
\]

where \( \nu \) is a unit normal vector \( \partial \Omega \).

3. Decomposition of the Problem and Coordinate Transformations

In the sequel, we will have deal with the problem (3). Let us rewrite it in an equivalent form. For this, we introduce the following set:

\[
K_\varepsilon = \{v = (v_-, v_+, v_m) \in H^2(\Omega_-) \times H^2(\Omega_+) \times H^2(\Omega_m) \mid \}
\]

\[
v_\pm = v_m, \quad v_{\pm,1} = v_{m,1} \text{ a.e. on } S_\pm, \quad v_\pm = \frac{\partial v_\pm}{\partial \nu} = 0 \text{ a.e. on } \partial \Omega_\pm \cap \partial \Omega \}.
\]
Taking into account the (1), problem (3) can be reformulated, as follows: find a triplet $(w_{ε-}, w_{ε+}, w_{εm}) \in K_{ε}$ satisfying a variational equality

\[ b_{ε-}(w_{ε-}, v_-) + b_{ε+}(w_{ε+}, v_+) + b_{εm}(w_{εm}, v_m) = l_-(v_-) + l_+(v_+) \forall (v_-, v_+, v_m) \in K_{ε}, \tag{4} \]

where

\[
\begin{align*}
  b_{ε±}(u, v) &= D_{ω_ε} \int_{ω_ε} (u_{11}v_{11} + u_{22}v_{22} + \kappa_0(u_{11}v_{22} + u_{22}v_{11}) + 2(1 - \kappa_0)u_{12}v_{12}) dy, \\
  b_{εm}(u, v) &= \sum_{n=0}^{3} D_{ω_n}^{ε} \int_{ω_n} (u_{11}v_{11} + u_{22}v_{22} + \kappa_n(u_{11}v_{22} + u_{22}v_{11}) + 2(1 - \kappa_n)u_{12}v_{12}) dy. \\
  l_±(u) &= \int_{ω_±} fu dy, \\
  l_{εm}(u) &= \int_{ω_m} fu dy.
\end{align*}
\]

From the Calculus of Variations, it follows that problem (4) has a unique solution $(w_{ε-}, w_{ε+}, w_{εm}) \in K_{ε}$ for all $ε > 0$ small enough (see, e.g., \cite{2,26}). Herewith, $w_{ε±}$ and $w_{εm}$ are restrictions of $w_{ε}$ on subdomains $Ω_{ε±}$ and $Ω_{εm}$, respectively.

Next, we introduce coordinate transformations that map domains $Ω_{ε±}$ and $Ω_{εm}$ onto domains independent of $ε$. For this, we consider two convex domains $ω_1$ and $ω_2$, such that

\[ S \subset ω_1, \bar{ω}_1 \subset ω_2, \partial ω_2 \cap \{y_1 = -a_1\} = ∅, \partial ω_2 \cap \{y_1 = a_2\} = ∅, \]

and a smooth cut-off function $θ$, such that

\[ θ = 1 \text{ in } ω_1, \ 0 < θ < 1 \text{ in } ω_2, \ θ = 0 \text{ in } R^2 \setminus ω_2. \]

Let us introduce the following notations:

\[ Ω_m = \{(z_1, z_2) \in R^2 \mid -d < z_1 < d, z_2 \in S\}, \]

\[ S_± = \{(z_1, z_2) \in R^2 \mid z_1 = ±d, z_2 \in S\}, \]

\[ Ω_n = \{(z_1, z_2) \in R^2 \mid -d < z_1 < d, z_2 \in S_n\}, \ a = 0, 1, 2, 3, \]

\[ S_n^± = \{(z_1, z_2) \in R^2 \mid z_1 = ±d, z_2 \in S_n\}, \ a = 0, 1, 2, 3. \]

and define coordinate transformations in the domains $Ω_±$ and $Ω_m$ as follows:

\[ y_1 = x_1 ± εdθ(x_1, x_2), \ y_2 = x_2, \ (x_1, x_2) \in Ω_±, \ (y_1, y_2) \in Ω_±^x, \]

\[ y_1 = εz_1, \ y_2 = z_2, \ (z_1, z_2) \in Ω_m, \ (y_1, y_2) \in Ω_m^x. \]

It is not difficult to show that for all sufficiently small coordinate transformations (5) and (6) map bijectively the domains $Ω_±$ and $Ω_m$ onto $Ω_±^x$ and $Ω_m^x$, respectively, (see, e.g., \cite{30,31}). Note that the subdomain $Ω_n^x$ is mapped into subdomains $Ω_n, a = 0, 1, 2, 3$.

Denote, by $Φ_ε^+(x)$ and $J_ε^x$, Jacobian matrices and Jacobians of transformations (5), respectively,

\[ Φ_ε^+(x_1, x_2) = \begin{pmatrix} 1 ± εdθ_1(x_1, x_2) & ±εdθ_2(x_1, x_2) \\ 0 & 1 \end{pmatrix}, \]

\[ J_ε^x(x_1, x_2) = \det Φ_ε^+(x_1, x_2) = 1 ± εdθ_1(x_1, x_2). \]
Coordinate transformations (5) and (6) establish one-to-one correspondences between spaces $H^2(\Omega_\pm)$, $H^2(\Omega_m)$ and $H^2(\Omega_\pm)$, $H^2(\Omega_m)$, respectively. Moreover, the set $K^e$ is transformed into a set $K^e$,

$$K^e = \{ v = (v_-, v_+, v_m) \in H^2(\Omega_-) \times H^2(\Omega_+) \times H^2(\Omega_m) \mid v_\pm|_S = v_m|_{S_\pm}, v_{\pm,1}|_S = \frac{1}{\varepsilon} v_{m,1}|_{S_\pm} \},$$

where

$$H^2(\Omega_\pm) = \{ v_\pm \in H^2(\Omega_\pm) \mid \frac{\partial v_\pm}{\partial \nu} = 0 \text{ a.e. on } \partial \Omega_\pm \cap \partial S \}. $$

Hereinafter, we assume that, for any functions $v_\pm(x), x \in \Omega_\pm$, and $v_m(z), z \in \Omega_m$, equality $v_\pm|_S = v_m|_{S_\pm}$ means that

$$v_\pm(0, x_2) = v_m(\pm d, z_2), x_2 = z_2 \in S.$$

Introduce the following notations:

$$w^e_\pm(x_1, x_2) = w_\pm(x_1 \pm \varepsilon \theta(x_1, x_2), x_2), (x_1, x_2) \in \Omega_\pm,$$

$$w^e_m(z_1, z_2) = w_m(\varepsilon z_1, z_2), (z_1, z_2) \in \Omega_m.$$

Because of the smoothness of coordinate transformations (5), we have asymptotic expansions for the transformations of the second-order derivatives for (5) (see, e.g., [30–33])

$$w^e_{\pm,ij} = w^e_{\pm,ij} + \varepsilon P^e_{ij}(\varepsilon, w^e_\pm),$$

(7)

with

$$|P^e_{ij}(\varepsilon, w^e_\pm)| \leq C(|w^e_{\pm,j}| + |w^e_{\pm,l}|), i, j, k, l = 1, 2.$$

Besides, we have for (6)

$$w_{im,11}(y_1, y_2) = \frac{w^e_{m,11}(z_1, z_2)}{\varepsilon^2}, w_{im,12}(y_1, y_2) = \frac{w^e_{m,12}(z_1, z_2)}{\varepsilon}, w_{im,22}(y_1, y_2) = \frac{w^e_{m,22}(z_1, z_2)}{\varepsilon}.$$

After applying coordinate transformations (5) and (6) to (4), we get that the triplet $(w^e_-, w^e_+, uw^e_m) \in K^e$ is a unique solution to the following variational equality:

$$b^e_-(w^e_-, v_-) + b^e_+(w^e_+, v_+) + b^e_m(w^e_m, v_m) = b^e_-(v_-) + b^e_+(v_+) + b^e_m(v_m) \forall (v_-, v_+, v_m) \in K^e,$$

(8)

where, taking into account (7) and (1),

$$b^e_-(u, v) = b_-(u, v) + r_-(\varepsilon, u, v),$$

$$b_\pm(u, v) = D_0 \int_{\Omega_\pm} \left( u_{11}v_{11} + u_{22}v_{22} + k_\pm(u_{11}v_{22} + u_{22}v_{11}) + 2(1 - k_\pm)u_{12}v_{12} \right) dx,$$

$$|r_\pm(\varepsilon, u, v)| \leq c_\pm(\varepsilon) \left( ||u||^2_{H^2(\Omega_\pm)} + ||v||^2_{H^2(\Omega_\pm)} \right), 0 \leq c_\pm(\varepsilon) = o(1) \text{ as } \varepsilon \to 0,$$

(9)
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Theorem 1. Let variational equality:

\[ b_m^+(u, v) = \int_{\Omega_m} \left( \frac{u_{11}v_{11}}{\varepsilon^2} + u_{22}v_{22} + \frac{k_m}{\varepsilon} (u_{11}v_{22} + v_{22}w_{11}) + \frac{2(1 - k_m)}{\varepsilon} u_{12}v_{12} \right) dz + \]

\[ + D_1 \int_{\Omega_m} \left( \frac{u_{11}v_{11}}{\varepsilon^2} + \frac{u_{22}v_{22}}{\varepsilon - \varepsilon_1} + \frac{k_m}{\varepsilon - \varepsilon_1} (u_{11}v_{22} + v_{22}w_{11}) + \frac{2(1 - k_m)}{\varepsilon - \varepsilon_1} u_{12}v_{12} \right) dz + \]

\[ + D_2 \int_{\Omega_m} \left( \frac{u_{11}v_{11}}{\varepsilon^4} + u_{22}v_{22} + \frac{k_m}{\varepsilon^2} (u_{11}v_{22} + v_{22}w_{11}) + \frac{2(1 - k_m)}{\varepsilon^2} u_{12}v_{12} \right) dz + \]

\[ + D_3 \int_{\Omega_m} \left( \frac{u_{11}v_{11}}{\varepsilon^3 - \varepsilon_3} + e^{N+1} u_{22}v_{22} + \frac{k_m}{\varepsilon - \varepsilon_3} (u_{11}v_{22} + v_{22}w_{11}) + \frac{2(1 - k_m)}{\varepsilon - \varepsilon_3} u_{12}v_{12} \right) dz, \]

\[ l_m^+(v) = \int_{\Omega_m} \frac{f(x_1 + \pm d\theta(x_1, x_2), x_2)(1 \mp d\theta, 1(x_1, x_2))}{v} dx, \]

\[ l_m^-(v) = \varepsilon \int_{\Omega_m} \frac{f(z_1, z_2) v}{dz}, \]

\[ |l_m^+(v)| \leq C\|v\|_{L^2(\Omega_m)}, \quad (10) \]

\[ |l_m^-(v)| \leq C\varepsilon\|v\|_{L^2(\Omega_m)}. \quad (11) \]

4. Limit Problem

To justify passing to the limit as \( \varepsilon \to 0 \), we need some auxiliary lemma proved in [5,6].

Lemma 1 (Poincare-type inequalities). For any triplet \((v_-, v_+, v_m) \in K_\varepsilon \) and \( \varepsilon \in (0, 1) \), the inequalities

\[ \|v_m\|_{L^2(\Omega_m)}^2 \leq C \left( \|v_{m,1}\|_{L^2(\Omega_m)}^2 + \|v_{\pm}\|_{L^2(\Omega_\pm)}^2 \right), \]

\[ \|v_{m,1}\|_{L^2(\Omega_m)}^2 \leq C \left( \|v_{m,1}\|_{L^2(\Omega_m)}^2 + \varepsilon^2 \|v_{\pm,1}\|_{L^2(S)}^2 \right) \]

hold, where a constant \( C > 0 \) does not depend on \((v_-, v_+, v_m)\) and \( \varepsilon > 0 \).

Our main result is the following theorem.

Theorem 1. Let \( w^\varepsilon = (w_-^\varepsilon, w_+^\varepsilon, w_m^\varepsilon) \) be a solution to (8); let \( w_0 \in K_0 \) be a solution to the following variational equality:

\[ b(w_0, w) + 4d(1 - k_2)D_m \int_{S_2} \frac{\partial(w_{0,1}\mid_{S_2})}{\partial z_2} \frac{\partial(w_{1,1}\mid_{S_2})}{\partial z_2} dz_2 = l(w) \forall w \in K_0, \quad (12) \]

where

\[ K_0 = \{ w \in H^2_0(\Omega) \mid w = \alpha x_2 + \beta a.e. \text{ on } S_1, \alpha, \beta \in \mathbb{R}; w_{1,1} \in H^1(S_2) \}. \]

Denote, by \( w_\pm \), a restriction of \( w \) to subdomain \( \Omega_\pm \) and, moreover, put

\[ w_m(z_1, z_2) = w_0(z_1, 0) \quad \text{for} \quad (z_1, z_2) \in \Omega_m. \]

Then, the following convergences

\[ w_-^\varepsilon \to w_- \quad \text{weakly in} \quad H^2(\Omega_\pm), \]

\[ w_m^\varepsilon \to w_m \quad \text{weakly in} \quad L^2(\Omega_m), \]
take place as $\varepsilon \to 0$.

**Proof.** Let us substitute $(w^\varepsilon, w^\varepsilon_+, w^\varepsilon_m)$ in (8) as a test function. Taking into account Lemma, (9)–(11), we obtain an estimate

$$
\|w^\varepsilon\|^2_{H^2(\Omega)} + \|w^\varepsilon_+\|^2_{H^2(\Omega)} + \|w^\varepsilon_m\|^2_{H^2(\Omega)} + \|\varepsilon w^\varepsilon_m\|_{L_2(\Omega)}^2 + \|\varepsilon w^\varepsilon_{m,1}\|_{L_2(\Omega)}^2 + \|\varepsilon w^\varepsilon_{m,12}\|_{L_2(\Omega)}^2 + \|\varepsilon w^\varepsilon_{m,22}\|_{L_2(\Omega)}^2 \leq C
$$

(13)

with a constant $C$ independent of $\varepsilon$. Here, by $w^\varepsilon_{m,\alpha}$ denote a restriction of $w^\varepsilon$ to $\Omega_\alpha$, $\alpha = 0, 1, 2, 3$. Moreover, from (13), Lemma, and definition of the set $K_\varepsilon$, we additionally have

$$
\|w^\varepsilon_m\|_{L_2(\Omega)} \leq C, \quad \|w^\varepsilon_{m,1}\|_{L_2(\Omega)} \leq C\varepsilon.
$$

(14)

Estimates (13) and (14) entail the existence of functions $w_\pm \in H^{2,0}(\Omega_\pm)$, $w_m \in L_2(\Omega_m)$, $p_\alpha, q_\alpha, r_\alpha \in L_2(\Omega_\alpha)$, $\alpha = 0, 1, 2, 3$, such that for some subsequence $\{\varepsilon_n\}_{n=1}^\infty$ still denoted by $\varepsilon$, the following convergences:

$$
\begin{align*}
\varepsilon^{-\frac{1}{2}} w^\varepsilon_{m,1} & \rightharpoonup p_\alpha \quad \text{weakly in } L_2(\Omega_\alpha), \\
\varepsilon^{-\frac{1}{2}} w^\varepsilon_{m,12} & \rightharpoonup q_\alpha \quad \text{weakly in } L_2(\Omega_\alpha), \\
\varepsilon^{-\frac{1}{2}} w^\varepsilon_{m,22} & \rightharpoonup r_\alpha \quad \text{weakly in } L_2(\Omega_\alpha)
\end{align*}
$$

(15)

hold as $\varepsilon \to 0$, with $r_2 = w^\varepsilon_{m,22}$. Moreover, from (13) and (14), it follows that

$$
\begin{align*}
w^\varepsilon_{m,1} & \to w_{m,1} = 0 \quad \text{strongly in } L_2(\Omega_m), \\
w^\varepsilon_{m,11} & \to w_{m,11} = 0 \quad \text{strongly in } L_2(\Omega_m), \\
w^\varepsilon_{m,22} & \to w_{m,22} = 0 \quad \text{strongly in } L_2(\Omega_1)
\end{align*}
$$

(16)-(18)

and there exists $u \in L_2(\Omega_{m_2})$ such that

$$
\frac{w^\varepsilon}{\varepsilon} \rightharpoonup u \quad \text{weakly in } L_2(\Omega_2).
$$

From definition of the set $K_\varepsilon$, after passing to the limit as $\varepsilon \to 0$, we obtain

$$
w_m|_{\Omega_\pm} = w_\pm|_{\Omega_\pm}.
$$

(19)
Because \( w_{m,1} = 0 \) in \( \Omega_m \) (see (16)), \( w_m \) does not depend on \( z_2 \). Therefore, taking into account (17), we conclude that there exists a function \( \beta(z_2) \in L_2(\Omega_m) \) such that

\[
w_m(z_1, z_2) = \beta(z_2), \quad (z_1, z_2) \in \Omega_m.
\]

Condition (18) means that the function \( w_m \) is affine in the domain \( \Omega_m \) with respect to \( z_2 \), i.e., there exists \( \delta, \gamma \in \mathbb{R} \), such that

\[
w_m(z_1, z_2) = \delta z_2 + \gamma \text{ in } \Omega_1.
\]

(20)

Because of (19), we have

\[
w_{-} \bigg|_{S} = - w_{+} \bigg|_{S}.
\]

(21)

Now, let us show that \( w_{\pm} \) satisfy the following equality:

\[
w_{+},1 = w_{-},1 \text{ on } S.
\]

(22)

Indeed, from the relation

\[
\int_{-d}^{d} w_{m,11}^{e}(z_1, z_2) dz_1 = w_{m,1}^{e}(d, z_2) - w_{m,1}^{e}(-d, z_2),
\]

it follows that

\[
\int_{-d}^{d} |w_{m,1}^{e}(d, z_2) - w_{m,1}^{e}(-d, z_2)|^2 dz_2 \leq 2d \|w_{m,11}^{e}\|^2_{L_2(\Omega_m)}.
\]

Due to estimate (13) and the equalities \( w_{m,1}^{e}(\pm d, z_2) = \varepsilon w_{m}^{e}(0, z_2) \) for \( z_2 \in (a, b) \) (see the definition of the set \( K_\varepsilon \)), we obtain

\[
\|w_{+,1}^{e} - w_{-,1}^{e}\|_{L_2(S)} \leq \frac{2d}{\varepsilon} \|w_{m,11}^{e}\|_{L_2(\Omega_m)} \to 0
\]

as \( \varepsilon \to 0 \). From (15) (the first line) and the compactness of trace operator, it follows

\[
w_{\pm,1}^{e} \to w_{\pm,1} \text{ strongly in } L_2(S)
\]

as \( \varepsilon \to 0 \), and (22) holds.

At last, using the same arguments as in [6], we can prove additionally that

\[
w_{\pm,1} \bigg|_{S_2} \in H^1(S_2)
\]

and, moreover,

\[
p_2 = -k_m w_{m,22} \text{ in } \Omega_2,
\]

\[
q_2 = \frac{\partial (w_{-1} \bigg|_{S_2})}{\partial z_2} \text{ in } \Omega_2,
\]

\[
u = w_{-1} \bigg|_{S_2} \text{ in } \Omega_2.
\]

Now, let us define a function

\[
w_0(x) = \begin{cases} w_{-}(x) & x \in \Omega_{-}, \\ w_{+}(x) & x \in \Omega_{+}. \end{cases}
\]

(24)

Conditions (19)–(23) imply that the function \( w_0 \) belongs to the set \( K_0 \).
In order to proceed with a problem defining the function \( w_0 \), we take arbitrary function \( v \in C^2(\Omega) \cap K_0 \) and define three functions \( v_-, v_+, v_m \) by

\[
v_-(z_1, z_2) = v(0, z_2), \quad v_+(z_1, z_2) = v(z_1, z_2), \quad v_m(z_1, z_2) = v(z_1, z_2) \in \Omega_m.
\]

Subsequently, for these functions, we consider a triplet \( (v_-, \epsilon \psi_-, \epsilon \psi_+ + v_m + \epsilon \psi_m) \in K_\epsilon \), where \( \psi_m(z_1, z_2) = \psi_z(0, z_2)z_1 \) for \( (z_1, z_2) \in \Omega_m \), and \( \psi_\pm \in H^{2,0}(\Omega_\pm) \) is arbitrary extensions of \( \psi_m \) in domains \( \Omega_\pm \), such that

\[
\psi_\pm|_S = \psi_m|_{S_\pm}, \quad \psi_\pm, 1 = 0 \text{ on } S,
\]

and substitute it in (8). Since \( v_{m,11} = 0 \) and \( \psi_{m,11} = 0 \) in \( \Omega_m \), weak convergences in (15) and Formulas (23) allows for us to pass to the limit as \( \epsilon \to 0 \) and obtain the following relation:

\[
b_-(w_-, v_-) + b_+(w_+, v_+) + 4d(1 - k_2)D_2 \int_{S_2} \frac{\partial(w_-|S_1)}{\partial z_2} \frac{\partial(v_-|S_1)}{\partial z_2} dz_2 = l_-(v_-) + l_+(v_+) \quad \forall v \in C^2(\Omega) \cap K_0.
\]

Taking into account (24) and the fact that \( C^2(\Omega) \cap K_0 \) is dense in \( K_0 \), we obtain (12).

Assuming that the solution \( w_0 \) to variational problem (12) has additional regularity, by applying the generalized Green formula (see, e.g., [2, 26]), we deduce differential equations and boundary conditions for the functions \( w_0 \):

\[
D_0 \Delta^2 w_0 = f \text{ in } \Omega \setminus (S_1 \cup S_2),
\]

\[
w_0 = \frac{\partial w_0}{\partial v} = 0 \text{ on } \partial \Omega,
\]

\[
w_0 = \delta_0 x_2 + \beta_0 \text{ on } S_1, \quad \delta_0, \beta_0 \in \mathbb{R},
\]

\[
\left[m^1(w_0)\right] = 0 \text{ on } S_1,
\]

\[
\int_{S_1} [t^1(w_0)] dx_2 = 0, \quad \int_{S_1} [\tilde{t}^1(w_0)] x_2 dx_2 = 0,
\]

\[
\left[t^2(w_0)\right] = 0 \text{ on } S_2,
\]

\[
p = w_{0,1} \text{ on } S_2,
\]

\[
4dD_2(1 - k_2)p_{2,2} = [m^2(w_0)] \text{ on } S_2,
\]

\[
p_{2,2} = 0 \text{ at } \partial S_2,
\]

where \( m^\alpha(w_0) \) and \( t^\alpha(w_0) \) are bending moments and transverse forces, respectively, defined by

\[
m^\alpha(w_0) = D_\alpha \left(k_\alpha \Delta w_0 + (1 - k_\alpha) \frac{\partial^2 w_0}{\partial v^2}\right),
\]

\[
t^\alpha(w_0) = D_\alpha \frac{\partial}{\partial v} \left(\Delta w_0 + (1 - k_\alpha) \frac{\partial^2 w_0}{\partial v^2}\right),
\]

\( \nu = (1, 0) \) and \( \tau = (-1, 0) \) are an unit normal vector and an unit tangent vector, respectively, \( \alpha = 1, 2 \).

The mechanical interpretation of boundary conditions can be found in [6], see also [10, 34, 35].
5. Concluding Remarks

We proposed a method of asymptotic derivation of plate models containing hard thin inclusions lying strictly inside the plate. The method is based on the variational properties of the solution of the equilibrium problem and allows for one to simultaneously construct all possible cases of hard thin inclusions. It is shown that there exist two type of thin inclusions in the Kirchhoff–Love plate, namely, the rigid inclusion $S_1$ for $N < -1$ and the elastic inclusion $S_2$ for $N = -1$. The inhomogeneity disappears in the case of $N \in (-1, 1)$. The last means that we have no any peculiarity along the set $S_3$.

In the conclusion, we note that the proposed method does not allow considering the case of the exponent $N \geq 1$ simultaneously with the case of the exponent $N < 1$, because, for the first case, we need to use other type of test functions (see [6]), which cannot be substituted in variational equality for the second case of the exponent.

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