GENERAL CONCAVITY OF MINIMAL $L^2$ INTEGRALS RELATED TO MULTIPLIER IDEAL SHEAVES

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Abstract. In this note, we present a general version of the concavity of the minimal $L^2$ integrals related to multiplier ideal sheaves.

1. Introduction

The multiplier ideal sheaves related to plurisubharmonic functions plays an important role in complex geometry and algebraic geometry (see e.g. [4, 31, 39, 9, 10, 6, 11, 28, 41, 42, 7]). We recall the definition of the multiplier ideal sheaves as follows.

Let $\varphi$ be a plurisubharmonic function (see [8, 37, 38]) on a complex manifold. It is known that the multiplier ideal sheaf $\mathcal{I}(\varphi)$ was defined as the sheaf of germs of holomorphic functions $f$ such that $|f|^2 e^{-\varphi}$ is locally integrable (see [7]).

In [6], Demailly posed the so-called strong openness conjecture on multiplier ideal sheaves (SOC for short) i.e. $\mathcal{I}(\varphi) = \mathcal{I}_+ (\varphi) := \cup_{\varepsilon > 0} \mathcal{I}((1 + \varepsilon) \varphi)$. When $\mathcal{I}(\varphi) = \mathcal{O}$, SOC degenerates to the openness conjecture (OC for short) posed by Demailly-Kollár [10].

The dimension two case of OC was proved by Favre-Jonsson [13], and the dimension two case of SOC was proved by Jonsson-Mustata [25]. OC was proved by Berndtsson [3]. SOC was proved by Guan-Zhou [20], see also [29] and [24].

In [1], Berndtsson establishes an effectiveness result of OC. Simulated by Berndtsson’s effectiveness result of OC, continuing the solution of SOC [20], Guan-Zhou [21] establish an effectiveness result of SOC.

Recently, we [16] establish a sharp version of the effectiveness result of SOC by considering a concavity property of the minimal $L^2$ integrals related to multiplier ideals.

In the present note, we obtain a general version of the above concavity property.

1.1. A general concavity property. Let $M$ be a $n-$dimensional Stein manifold, and let $K_M$ be the canonical (holomorphic) line bundle on $M$. Let $\psi < -T$ be a plurisubharmonic function on $M$, and let $\varphi$ be a Lebesgue measurable function on $M$, such that $\varphi + \psi$ is a plurisubharmonic function on $M$, where $T \in (-\infty, +\infty)$.

We call a positive smooth function $c$ on $(T, +\infty)$ in class $\mathcal{P}_T$ if the following three statements hold

1. $\int_T^{+\infty} c(t) e^{-t} dt < +\infty$;
2. $c(t) e^{-t}$ is decreasing with respect to $t$;

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(3) for any compact subset $K \subseteq M$, $e^{-\varphi} c(-\psi)$ has a positive lower bound on $K$. Especially, if $\varphi \equiv 0$, then (3) is equivalent to $\lim_{t \to +\infty} c(t) > 0$.

Let $Z_0$ be a subset of $\{ \psi = -\infty \}$ such that $Z_0 \cap \text{Supp}(\{ \mathcal{O}/ \mathcal{I}(\varphi + \psi) \}) \neq \emptyset$. Let $U \supseteq Z_0$ be an open subset of $M$ and let $f$ be a holomorphic $(n, 0)$ form on $U$. Let $\mathcal{F} \supseteq \mathcal{I}(\varphi + \psi)|_U$ be a coherent subsheaf of $\mathcal{O}$ on $U$.

Denote
\[
\inf \left\{ \int_{\{ \psi < -t \}} |\tilde{f}|^2 e^{-\varphi} c(-\psi) : \exists \text{ open set } U' \text{ s.t. } Z_0 \subset U' \subset U, \& \ (\tilde{f} - f) \in H^0(\{ \psi < -t \} \cap U', \mathcal{O}(K_M) \otimes \mathcal{F}) \right\},
\]
by $G(t; c)$ ($G(t)$ for short without misunderstanding), where $c \in \mathcal{P}_T$, and $|f|^2 := \sqrt{-1} a^2 f \wedge \bar{f}$ for any $(n, 0)$ form $f$.

If there is no holomorphic holomorphic $(n, 0)$ form $\tilde{f}$ on $\{ \psi < -t \}$ satisfying $(\tilde{f} - f) \in H^0(\{ \psi < -t \} \cap U, \mathcal{O}(K_M) \otimes \mathcal{F})$, then we set $G(t) = -\infty$.

In the present note, we obtain the following concavity of $G(t)$.

**Theorem 1.1.** $G(g^{-1}(r))$ is concave with respect to $r \in (0, \int_T^{+\infty} c(t)e^{-t}dt]$, where $g(t) = \int_t^{+\infty} c(t_1)e^{-t_1}dt_1$, $t \in [T, +\infty)$.

Especially, when $c(t) \equiv 1$ and $A = 0$, Theorem 1.1 degenerates to the concavity of the minimal $L^2$ integrals related to multiplier ideals in [16] (Proposition 4.1 in [16]).

Theorem 1.1 implies the following.

**Corollary 1.2.** For any $c \in \mathcal{P}_T$, the following three statements are equivalent
   (1) $G(g^{-1}(r))$ is linear with respect to $r \in (0, \int_T^{+\infty} c(t)e^{-t}dt]$, i.e.,
   \[
   G(t) = \frac{G(T)}{\int_T^{+\infty} c(t)e^{-t}dt} \int_t^{+\infty} c(t_1)e^{-t_1}dt_1,
   \]
   holds for any $t \in [T, +\infty)$;
   (2) $\frac{G(g^{-1}(r_0))}{r_0} \leq \frac{G(T)}{\int_T^{+\infty} c(t)e^{-t}dt}$ holds for some $r_0 \in (0, \int_T^{+\infty} c(t)e^{-t}dt]$, i.e.,
   \[
   \frac{G(t_0)}{\int_{t_0}^{+\infty} c(t_1)e^{-t_1}dt_1} \leq \frac{G(T)}{\int_T^{+\infty} c(t)e^{-t}dt},
   \]
   holds for some $t_0 \in (T, +\infty)$;
   (3) $\lim_{r \to 0^+} \frac{G(g^{-1}(r))}{r} \leq \frac{G(T)}{\int_T^{+\infty} c(t)e^{-t}dt}$ holds, i.e.,
   \[
   \lim_{t \to +\infty} \frac{G(t)}{\int_t^{+\infty} c(t_1)e^{-t_1}dt_1} \leq \frac{G(T)}{\int_T^{+\infty} c(t)e^{-t}dt},
   \]
   holds.

2. Proof of Theorem 1.1

In this section, we modify some techniques in [16] and prove Theorem 1.1.
2.1. \textbf{L}² \textbf{m}ethods \textbf{r}elated \textbf{t}o \textbf{L}² \textbf{e}xtension \textbf{t}heorem. Let \( c(t) \) be a positive function in \( C^\infty((T, +\infty)) \) satisfying \( \int_T^\infty c(t)e^{-t}dt < \infty \) and

\[
\left( \int_T^t c(t_1)e^{-t_1}dt_1 \right)^2 > c(t)e^{-t} \int_T^t \left( \int_T^{t_2} c(t_1)e^{-t_1}dt_1 \right)dt_2,
\]

for any \( t \in (T, +\infty) \), where \( T \in (-\infty, +\infty) \). This class of functions was denoted by \( C_T \). Especially, if \( c(t)e^{-t} \) is decreasing with respect to \( t \) and \( \int_T^\infty c(t)e^{-t}dt < \infty \), then inequality (2.1) holds (see [19]).

In this section, we present the following Lemma, whose various forms already appear in [18, 19, 16] etc.:

\textbf{Lemma 2.1.} Let \( B \in (0, +\infty) \) and \( t_0 \geq 0 \) be arbitrarily given. Let \( M \) be a \( n \)–dimensional Stein manifold. Let \( \psi < −T \) be a plurisubharmonic function on \( M \). Let \( \varphi \) be a plurisubharmonic function on \( M \). Let \( F \) be a holomorphic \((n,0)\) form on \( \{ \psi < -t_0 \} \), such that

\[
\int_{K \cap \{ \psi < -t_0 \}} |F|^2 < +\infty
\]

for any compact subset \( K \) of \( M \), and

\[
\int_M \frac{1}{B} \mathbb{I}_{\{ -t_0 - B < \psi < -t_0 \}} |F|^2 e^{-\varphi} d\lambda_n \leq C < +\infty.
\]

Then there exists a holomorphic \((n,0)\) form \( \tilde{F} \) on \( M \), such that,

\[
\int_M |\tilde{F} - (1 - b(\psi))F|^2 e^{-\varphi + v(\psi)} c(-v(\psi)) d\lambda_n \leq C \int_T^{t_0 + B} c(t)e^{-t}dt
\]

where \( b(t) = \int_{-\infty}^t \frac{1}{B} \mathbb{I}_{\{ -t_0 - B < s < -t_0 \}} ds \), \( v(t) = \int_0^t b(s)ds \), and \( c(t) \in C_T \).

It is clear that \( \mathbb{I}_{(-t_0, +\infty)} b(t) \leq \mathbb{I}_{(-t_0 - B, +\infty)} \) and \( \max\{t, -t_0 - B\} \leq v(t) \leq \max\{t, -t_0\} \).

2.2. \textbf{S}ome \textbf{p}roperties \textbf{of} \( G(t) \).

Following the notations and assumptions in Section 1.1, we present some properties related to \( G(t) \) in the present section.

The following lemma is a characterization of \( G(T) \neq 0 \).

\textbf{Lemma 2.2.} \( f \notin \mathcal{F}(U) \Leftrightarrow G(T) \neq 0 \) (maybe \(+\infty\)).

\textbf{Proof.} It is clear that \( f \in \mathcal{F}(U) \Rightarrow G(T) = 0 \).

In the following part, we prove that \( f \notin \mathcal{F}(U) \Rightarrow G(T) \neq 0 \) (maybe \(-\infty \) or \(+\infty\)). We prove it by contradiction: if not, then there exists holomorphic \((n,0)\) forms \( \{ \tilde{f}_j \}_{j \in \mathbb{N}^+} \) on \( M \) such that \( \lim_{j \to \infty} \int_M |\tilde{f}_j|^2 e^{-\varphi} c(-\psi) = 0 \) and \( (\tilde{f}_j|_{U} - f) \in H^0(U, \mathcal{O}(K_M) \otimes \mathcal{F}) \) for any \( j \). As \( e^{-\varphi} c(-\psi) \) has positive lower bound on any compact subset of \( M \), there exists a subsequence of \( \{ \tilde{f}_j \}_{j \in \mathbb{N}^+} \) denoted by \( \{ \tilde{f}_{j_k} \}_{k \in \mathbb{N}^+} \) compactly convergent to \( 0 \). It is clear that \( \tilde{f}_{j_k} - f \) is compactly convergent to \( 0 - f = f \) on \( U \). It follows from the closeness of the sections of coherent analytic sheaves under the topology of compact convergence (see [16]) that \( f \in H^0(U, \mathcal{O}(K_M) \otimes \mathcal{F}) \), which contradicts \( f \notin H^0(U, \mathcal{O}(K_M) \otimes \mathcal{F}) \). Then we obtain \( f \notin H^0(U, \mathcal{O}(K_M) \otimes \mathcal{F}) \Rightarrow G(T) > 0 \) (maybe \(+\infty\)). This proves Lemma 2.2. \( \square \)

The following lemma shows the uniqueness of the holomorphic \((n,0)\) form related to \( G(t) \).
Lemma 2.3. Assume that $G(t) < +\infty$ for some $t \in [T, +\infty)$. Then there exists a unique holomorphic $(n, 0)$ form $F_1$ on $\{\psi < -t\}$ satisfying $(F_1 - f) \in H^0(\{\psi < -t\} \cap U, \mathcal{O}(K_M) \otimes \mathcal{F})$ and $\int_{\{\psi < -t\}} |F_1|^2 e^{-\varphi} c(-\psi) = G(t)$. Furthermore, for any holomorphic $(n, 0)$ form $\hat{F}$ on $\{\psi < -t\}$ satisfying $(\hat{F} - f) \in H^0(\{\psi < -t\} \cap U, \mathcal{O}(K_M) \otimes \mathcal{F})$ and $\int_{\{\psi < -t\}} |\hat{F}|^2 e^{-\varphi} c(-\psi) < +\infty$, we have the following equality

$$
\int_{\{\psi < -t\}} |F_1|^2 e^{-\varphi} c(-\psi) + \int_{\{\psi < -t\}} |\hat{F} - F_1|^2 e^{-\varphi} c(-\psi) = \int_{\{\psi < -t\}} |\hat{F}|^2 e^{-\varphi} c(-\psi).
$$

(2.5)

Proof. Firstly, we prove the existence of $F_1$. As $G(t) < +\infty$ then there exists holomorphic $(n, 0)$ forms $\{f_j\} \in \mathcal{O}^+\{\psi < -t\}$ such that $\int_{\{\psi < -t\}} |f_j|^2 e^{-\varphi} c(-\psi) \to G(t)$, and $(f_j - f) \in H^0(\{\psi < -t\} \cap U, \mathcal{O}(K_M) \otimes \mathcal{F})$. Then there exists a subsequence of $\{f_j\}$ compact convergence to a holomorphic $(n, 0)$ form $F$ on $\{\psi < -t\}$ satisfying $\int_K |f|^2 e^{-\varphi} c(-\psi) \leq G(t)$ for any compact set $K \subset \{\psi < -t\}$, which implies $\int_{\{\psi < -t\}} |f|^2 e^{-\varphi} c(-\psi) \leq G(t)$ by Levi’s Theorem. As $e^{-\varphi} c(-\psi)$ has positive lower bound on any compact subset of $M$, it follows from the closedness of the sections of coherent analytic sheaves under the topology of compact convergence (see [15]) that $(F - f) \in H^0(\{\psi < -t\} \cap U, \mathcal{O}(K_M) \otimes \mathcal{F})$. Then we obtain the existence of $F_1 = F$.

Secondly, we prove the uniqueness of $F_1$ by contradiction: if not, there exist two different holomorphic $(n, 0)$ forms $f_1$ and $f_2$ on $\{\psi < -t\}$ satisfying $\int_{\{\psi < -t\}} |f_1|^2 e^{-\varphi} c(-\psi) = \int_{\{\psi < -t\}} |f_2|^2 = G(t)$, $(f_1 - f) \in H^0(\{\psi < -t\} \cap U, \mathcal{O}(K_M) \otimes \mathcal{F})$ and $(f_2 - f) \in H^0(\{\psi < -t\} \cap U, \mathcal{O}(K_M) \otimes \mathcal{F})$. Note that

$$
\int_{\{\psi < -t\}} \frac{|f_1|}{2} e^{-\varphi} c(-\psi) + \int_{\{\psi < -t\}} \frac{|f_2|}{2} e^{-\varphi} c(-\psi) = \frac{\int_{\{\psi < -t\}} |f_1 + f_2|^2}{2} e^{-\varphi} c(-\psi) = G(t),
$$

(2.6)

then we obtain that

$$
\int_{\{\psi < -t\}} \frac{|f_1|}{2} e^{-\varphi} c(-\psi) + \int_{\{\psi < -t\}} \frac{|f_2|}{2} e^{-\varphi} c(-\psi) < G(t),
$$

and $(f_1 + f_2 - f) \in H^0(\{\psi < -t\} \cap U, \mathcal{O}(K_M) \otimes \mathcal{F})$, which contradicts the definition of $G(t)$.

Finally, we prove equality 2.6. For any holomorphic $h$ on $\{\psi < -t\}$ satisfying $\int_{\{\psi < -t\}} |h|^2 e^{-\varphi} c(-\psi) < +\infty$ and $h \in H^0(\{\psi < -t\} \cap U, \mathcal{O}(K_M) \otimes \mathcal{F})$, it is clear that for any complex number $\alpha$, $F_1 + \alpha h$ satisfying $((F_1 + \alpha h) - f) \in H^0(\{\psi < -t\} \cap U, \mathcal{O}(K_M) \otimes \mathcal{F})$, and $\int_{\{\psi < -t\}} |F_1 + \alpha h|^2 e^{-\varphi} c(-\psi) \leq \int_{\{\psi < -t\}} |F_1|^2 e^{-\varphi} c(-\psi) < +\infty$. Note that

$$
\int_{\{\psi < -t\}} |F_1 + \alpha h|^2 e^{-\varphi} c(-\psi) - \int_{\{\psi < -t\}} |F_1|^2 e^{-\varphi} c(-\psi) \geq 0
$$

implies

$$
\Re \int_{\{\psi < -t\}} F_1 h e^{-\varphi} c(-\psi) = 0
$$
by considering \( \alpha \to 0 \), then
\[
\int_{\{\psi < -t\}} |F_t + h|^2 e^{-\varphi} c(-\psi) = \int_{\{\psi < -t\}} (|F_t|^2 + |h|^2)e^{-\varphi}c(-\psi).
\]
Choosing \( h = \hat{F} - F_t \), we obtain equality \( \square \).

The following function shows the lower semi-continuity property of \( G(t) \).

**Lemma 2.4.** Assume that \( G(T) < +\infty \). Then \( G(t) \) is decreasing with respect to \( t \in [T, +\infty) \), such that \( \lim_{t \to t_0^+} G(t) = G(t_0) \) \( (t_0 \in [T, +\infty)) \), \( \lim_{t \to t_0^-} G(t) \geq G(t_0) \) \( (t_0 \in (T, +\infty)) \), and \( \lim_{t \to +\infty} G(t) = 0 \), where \( t_0 \in [T, +\infty) \). Especially \( G(t) \) is lower semi-continuous on \([T, +\infty)\).

**Proof.** By the definition of \( G(t) \), it is clear that \( G(t) \) is decreasing on \([T, +\infty)\) and \( \lim_{t \to t_0^-} G(t) \geq G(t_0) \). It suffices to prove \( \lim_{t \to t_0^+} G(t) = G(t_0) \). We prove it by contradiction: if not, then \( \lim_{t \to t_0^+} G(t) < G(t_0) \).

By Lemma 2.3, there exists a unique holomorphic \((n,0)\) form \( F_t \) on \{\( \psi < -t \)\} satisfying \( (F_t - f) \in H^0(\{\psi < -t\} \cap U, \mathcal{O}(K_M) \otimes \mathcal{F}) \) and \( \int_{\{\psi < -t\}} |F_t|^2 e^{-\varphi} c(-\psi) = G(t) \). Note that \( G(t) \) is decreasing implies that \( \int_{\{\psi < -t\}} |F_t|^2 e^{-\varphi} c(-\psi) \leq \lim_{t \to t_0^+} G(t) \) for any \( t > t_0 \). As \( e^{-\varphi} c(-\psi) \) has positive lower bound on any compact subset of \( M \), for any compact subset \( K \) of \{\( \psi < -t_0 \)\}, there exists \( \{F_{t_j}\} \) \( (t_j \to t_0 + 0, j \to +\infty) \) uniformly convergent on \( K \). Then there exists a subsequence of \( \{F_{t_j}\} \) (also denoted by \( \{F_{t_j}\} \)) convergent on any compact subset of \{\( \psi < -t_0 \)\}.

Let \( \hat{F}_{t_0} := \lim_{j \to +\infty} F_{t_j} \), which is a holomorphic \((n,0)\) form on \{\( \psi < -t_0 \)\}. Then it follows from the decreasing property of \( G(t) \) that
\[
\int_K |\hat{F}_{t_0}|^2 e^{-\varphi} c(-\psi) \leq \lim_{j \to +\infty} \int_K |F_{t_j}|^2 e^{-\varphi} c(-\psi) \leq \lim_{j \to +\infty} G(t_j) \leq \lim_{t \to t_0^+} G(t)
\]
for any compact set \( K \subset \{\psi < -t_0\} \). It follows from Levi’s theorem that
\[
\int_M |\hat{F}_{t_0}|^2 e^{-\varphi} c(-\psi) \leq \lim_{t \to t_0^+} G(t).
\]
Then we obtain that \( G(t_0) \leq \int_M |\hat{F}_{t_0}|^2 e^{-\varphi} c(-\psi) \leq \lim_{t \to t_0^+} G(t) \), which contradicts \( \lim_{t \to t_0^+} G(t) < G(t_0) \). \( \square \)

We consider the derivatives of \( G(t) \) in the following lemma.

**Lemma 2.5.** Assume that \( G(T) < +\infty \). Then for any \( t_0 \in (T, +\infty) \), we have
\[
\frac{G(T) - G(t_0)}{\int_T^\infty c(t)e^{-t} dt} = \liminf_{B \to 0^+} \frac{G(t_0) - G(t_0 + B)}{B}.
\]

**Proof.** By Lemma 2.3, there exists a holomorphic \((n,0)\) form \( F_{t_0} \) on \{\( \psi < 0 \)\}, such that \( (F_{t_0} - f) \in H^0(\{\psi < t_0\} \cap U, \mathcal{O}(K_M) \otimes \mathcal{F}) \) and \( \int_{\{\psi < -t_0\}} |F_{t_0}|^2 e^{-\varphi} c(-\psi) = G(t_0) \).

It suffices to consider that \( \lim_{B \to 0^+} \frac{G(t_0) - G(t_0 + B)}{B} \in (-\infty, 0] \) because of the decreasing property of \( G(t) \). Then there exists \( B_j \to 0 + 0 \) \( (j \to +\infty) \) such that
\[
\lim_{j \to +\infty} \frac{G(t_0) - G(t_0 + B_j)}{B_j} = \liminf_{B \to 0^+} \frac{G(t_0) - G(t_0 + B)}{B}
\]
and \( \{\frac{G(t_0) - G(t_0 + B_j)}{B_j}\}_{j \in \mathbb{N}^+} \) is bounded.
As \( t \leq v(t) \), the decreasing property of \( c(t)e^{-t} \) shows that
\[
e(t)e^{-t} \geq c(-v(-t))e^{v(-t)}
\]
for any \( t \geq 0 \), which implies
\[
e^{-\psi+v(\psi)}c(-v(\psi)) \geq c(-\psi).
\]

Lemma 2.1 (\( \varphi \sim \varphi + \psi \) here \( \sim \) means the former replaced by the latter and the notation will be used throughout the paper) shows that for any \( B_j \), there exists holomorphic \((n, 0)\) form \( \tilde{F}_j \) on \( M \), such that \( (\tilde{F}_j - f_{t_0}) \in H^0(\{\psi < -t_0\} \cap U, \mathcal{O}(K_M) \otimes L(\varphi + \psi)) \subseteq H^0(\{\psi < -t_0\} \cap U, \mathcal{O}(K_M) \otimes \mathcal{F}) \) \( (\Rightarrow (\tilde{F}_j - f) \in H^0(\{\psi < -t_0\} \cap U, \mathcal{O}(K_M) \otimes \mathcal{F}) \) and
\[
\int_M |\tilde{F}_j - (1 - b_{t_0,B_j}(\psi))f_{t_0}|^2 e^{-\varphi} e^{-\psi+v(\psi)}c(-v(\psi))
\]
\[
\leq \int_M |\tilde{F}_j - (1 - b_{t_0,B_j}(\psi))f_{t_0}|^2 e^{-\varphi} c(-\psi)
\]
\[
\leq \int_0^{t_0+B_j} c(t)e^{-t} dt \int_M \frac{1}{B_j} (\mathbb{J}_{-t_0-B_j<\psi<-t_0})|f_{t_0}|^2 e^{-\varphi} c(-\psi)
\]
\[
\leq \frac{e^{t_0+B_j} \int_0^{t_0+B_j} c(t)e^{-t} dt}{\inf\{e(t_0,t_0+B_j) c(t)\}} \times \left( \int_M \frac{1}{B_j} (\mathbb{J}_{-t_0-B_j<\psi<-t_0})|f_{t_0}|^2 e^{-\varphi} c(-\psi) \right)
\]
\[
- \int_M \frac{1}{B_j} \mathbb{J}_{\psi<-t_0-B_j} |f_{t_0}|^2 e^{-\varphi} c(-\psi))
\]
\[
\leq \frac{e^{t_0+B_j} \int_0^{t_0+B_j} c(t)e^{-t} dt}{\inf\{e(t_0,t_0+B_j) c(t)\}} \times \frac{G(t_0) - G(t_0 + B_j)}{B_j}
\]

Firstly, we will prove that \( \int_M |\tilde{F}_j|^2 e^{-\varphi} c(-\psi) \) is bounded with respect to \( j \).

Note that
\[
(\int_M |\tilde{F}_j - (1 - b_{t_0,B_j}(\psi))f_{t_0}|^2 e^{-\varphi} c(-\psi))^{1/2}
\]
\[
\geq (\int_M |\tilde{F}_j|^2 e^{-\varphi} c(-\psi))^{1/2} - (\int_M |(1 - b_{t_0,B_j}(\psi))f_{t_0}|^2 e^{-\varphi} c(-\psi))^{1/2}
\]
then it follows from inequality 2.7 that
\[
(\int_M |\tilde{F}_j|^2 e^{-\varphi} c(-\psi))^{1/2}
\]
\[
\leq \frac{e^{t_0+B_j} \int_0^{t_0+B_j} c(t)e^{-t} dt}{\inf\{e(t_0,t_0+B_j) c(t)\}}^{1/2} \left( \frac{G(t_0) - G(t_0 + B_j)}{B_j} \right)^{1/2}
\]
\[
+ \left( \int_M |(1 - b_{t_0,B_j}(\psi))f_{t_0}|^2 e^{-\varphi} c(-\psi) \right)^{1/2}.
\]
Since \( \left\{ \frac{G(t_0 + B_j) - G(t_0)}{B_j} \right\} \) is bounded and \( 0 \leq b_{t_0,B_j}(\psi) \leq 1 \), then \( \int_M |\tilde{F}_j|^2 \) is bounded with respect to \( j \).

Secondly, we will prove the main result.
It follows from $b_{t_0}(\psi) = 1$ on $\{\psi \geq -t_0\}$ that

$$
\int_M |\tilde{F}_j - (1 - b_{t_0,B_j}(\psi))F_{t_0}|^2 e^{-\varphi}c(-\psi)
= \int_{\{\psi \geq -t_0\}} |\tilde{F}_j|^2 e^{-\varphi}c(-\psi) + \int_{\{\psi < -t_0\}} |\tilde{F}_j - (1 - b_{t_0,B_j}(\psi))F_{t_0}|^2 e^{-\varphi}c(-\psi)
$$

(2.10)

Denote that $|| \cdot ||_2 := (\int_{\{\psi < -t_0\}} |\cdot|^2 e^{-\varphi}c(-\psi))^{1/2}$. It is clear that

$$
||\tilde{F}_j - (1 - b_{t_0}(\psi))F_{t_0}||^2_2
\geq ||\tilde{F}_j - F_{t_0}||^2_2 - 2||\tilde{F}_j - F_{t_0}||^2 ||b_{t_0,B_j}(\psi)F_{t_0}||_2
\geq ||\tilde{F}_j - F_{t_0}||^2_2 - 2||\tilde{F}_j - F_{t_0}||_2 (\int_{\{-t_0 - B_j < \psi < -t_0\}} |F_{t_0}|^2 e^{-\varphi}c(-\psi))^{1/2},
$$

(2.11)

where the last inequality follows from $0 \leq b_{t_0,B_j}(\psi) \leq 1$ and $b_{t_0,B_j}(\psi) = 0$ on $\{\psi \leq -t_0 - B_0\}$.

Combining equality 2.10, inequality 2.11 and equality 2.10, we obtain that

$$
\int_M |\tilde{F}_j - (1 - b_{t_0,B_j}(\psi))F_{t_0}|^2 e^{-\varphi}c(-\psi)
= \int_{\{\psi \geq -t_0\}} |\tilde{F}_j|^2 e^{-\varphi}c(-\psi) + ||\tilde{F}_j - (1 - b_{t_0}(\psi))F_{t_0}||^2_2
\geq \int_{\{\psi \geq -t_0\}} |\tilde{F}_j|^2 e^{-\varphi}c(-\psi) + ||\tilde{F}_j - F_{t_0}||^2_2 - 2||\tilde{F}_j - F_{t_0}||_2 ||b_{t_0,B_j}(\psi)F_{t_0}||_2
\geq \int_{\{\psi \geq -t_0\}} |\tilde{F}_j|^2 e^{-\varphi}c(-\psi) + ||\tilde{F}_j||^2_2 - ||F_{t_0}||^2_2
- 2||\tilde{F}_j - F_{t_0}||_2 (\int_{\{-t_0 - B_j < \psi < -t_0\}} |F_{t_0}|^2 e^{-\varphi}c(-\psi))^{1/2}
= \int_M |\tilde{F}_j|^2 e^{-\varphi}c(-\psi) - ||F_{t_0}||^2_2
- 2||\tilde{F}_j - F_{t_0}||_2 (\int_{\{-t_0 - B_j < \psi < -t_0\}} |F_{t_0}|^2 e^{-\varphi}c(-\psi))^{1/2}.
$$

(2.12)

It follows from equality 2.10 that

$$
||\tilde{F}_j - F_{t_0}||_2 = (||\tilde{F}_j||^2_2 - ||F_{t_0}||^2_2)^{1/2} \leq ||\tilde{F}_j||_2 \leq (\int_M |\tilde{F}_j|^2 e^{-\varphi}c(-\psi))^{1/2}.
$$

(2.13)

Since $\int_M |\tilde{F}_j|^2 e^{-\varphi}c(-\psi)$ is bounded with respect to $j$, inequality 2.13 implies that $(\int_{\{\psi < -t_0\}} |\tilde{F}_j - F_{t_0}|^2 e^{-\varphi}c(-\psi))^{1/2}$ is bounded with respect to $j$. Using the dominated convergence theorem and $\int_{\{\psi < -t_0\}} |F_{t_0}|^2 e^{-\varphi}c(-\psi) = G(t_0) \leq G(0) < +\infty$, we obtain that $\lim_{j \to +\infty} \int_{\{-t_0 - B_j < \psi < -t_0\}} |F_{t_0}|^2 e^{-\varphi}c(-\psi) = 0$. Then

$$
\lim_{j \to +\infty} ||\tilde{F}_j - F_{t_0}||_2 (\int_{\{-t_0 - B_j < \psi < -t_0\}} |F_{t_0}|^2 e^{-\varphi}c(-\psi))^{1/2} = 0.
$$
Combining with inequality 2.12 we obtain
\[
\liminf_{j \to +\infty} \int_M |\tilde{F}_j - (1 - b_{t_0,B_j}(\psi))F_{t_0}|^2 e^{-\varphi}c(-\psi)
\]
(2.14)
\[
\geq \liminf_{j \to +\infty} \int_M |\tilde{F}_j|^2 e^{-\varphi}c(-\psi) - \|F_{t_0}\|^2_2.
\]
Using inequality 2.7 (3rd “≥”) and inequality 2.14 (4th “≥”), we obtain
\[
\frac{\int_{t_0}^{t} c(t)e^{-t}dt}{c(t_0)e^{-t_0}} \lim_{j \to +\infty} \frac{G(t_0) - G(t_0 + B_j)}{B_j}
\]
\[
= \lim_{j \to +\infty} \frac{\int_{t_0}^{t} c(t)e^{-t}dt}{\inf_{t \in (t_0,t_0+B_j)} c(t)} \int_M \frac{1}{B_j} |\{t \leq t_0 - B < \psi < t_0\}| |F_{t_0}|^2 e^{-\varphi}c(-\psi)
\]
(2.15)
\[
\geq \liminf_{j \to +\infty} \int_M |\tilde{F}_j - (1 - b_{t_0,B_j}(\psi))F_{t_0}|^2 e^{-\varphi}c(-\psi)
\]
\[
\geq \liminf_{j \to +\infty} \int_M |\tilde{F}_j|^2 e^{-\varphi}c(-\psi) - \|F_{t_0}\|^2_2
\]
\[
\geq G(T) - G(t_0).
\]
This proves Lemma 2.5. \(\square\)

Lemma 2.5 implies the following lemma.

Lemma 2.6. Assume that \(G(T) < +\infty\). Then for any \(t_0, t_1 \in [T, +\infty)\), we have
\[
\frac{G(t_1) - G(t_1 + t_0)}{\int_{t_1}^{t_1+t_0} c(t)e^{-t}dt} \leq \liminf_{B \to 0+0} \frac{G(t_1+t_1+B) - G(t_0+t_1+B)}{c(t_0)e^{-t_0}},
\]
i.e.
\[
\frac{G(t_1) - G(t_1 + t_0)}{\int_{t_1}^{+\infty} c(t)e^{-t}dt - \int_{t_1+t_0}^{+\infty} c(t)e^{-t}dt}
\]
(2.16)
\[
\leq \liminf_{B \to 0+0} \frac{G(t_0 + t_1) - G(t_0 + t_1 + B)}{\int_{t_1+t_0}^{+\infty} c(t)e^{-t}dt - \int_{t_1+t_0+B}^{+\infty} c(t)e^{-t}dt}
\]

2.3. Proof of Theorem 1.1 As \(G(\varphi^{-1}(r);c)\) is lower semicontinuous (Lemma 2.4), then it follows from the following well-known property of concave functions (Lemma 2.7) that Lemma 2.6 implies Theorem 1.1.

Lemma 2.7. Let \(a(r)\) be a lower semicontinuous function on \((0,R]\). Then \(a(r)\) is concave if and only if
\[
\frac{a(r_1) - a(r_2)}{r_1 - r_2} \leq \liminf_{r_3 \to r_2} \frac{a(r_3) - a(r_2)}{r_3 - r_2},
\]
holds for any \(0 < r_2 < r_1 \leq R\).

3. Appendix: Proof of Lemma 2.1

In this section, we prove Lemma 2.1.
3.1. Preparations. It follows from Lemma [3,5] that there exist smooth plurisubharmonic functions $\psi_n$ and $\varphi_m$ on $M$ decreasing convergent to $\psi$ and $\varphi$ respectively.

The following remark shows that it suffices to consider Lemma [2,1] for the case that $M$ is a relatively compact open Stein submanifold of a Stein manifold, and $F$ is a holomorphic $(n,0)$ form on \{ $\psi < -t_0$ \} such that $\int_{\{ \psi < -t_0 \}} |F|^2 < +\infty$, which implies that $\sup_m \sup_M \psi_m < -T$ and $\sup_m \sup_M \varphi_m < +\infty$ on $M$.

In the following remark, we recall some standard steps (see e.g. [39, 18, 19]) to illustrate it.

Remark 3.1. It is well-known that there exist open Stein submanifolds $D_1 \subset \cdots \subset D_j \subset \cdots \subset \cup_{j=1}^{\infty} D_j = M$.

If inequality [2,1] holds on any $D_j$ and inequality [2,3] holds on $M$, then for any $B > 0$, we obtain a sequence of holomorphic $(n,0)$ forms $\tilde{F}_j$ on $D_j$ such that

$$
\int_{D_j} |\tilde{F}_j - (1-b(\psi))F|^2 e^{-\varphi+\varphi(\psi)} c(-v(\psi))d\lambda_n \leq C \int_{D_j} B^{-1} B^{-1} d\lambda_n \leq C \int_{D_j} c(t)e^{-t} \ dt
$$

is bounded with respect to $j$. Note that for any given $j$, $e^{-\varphi+\varphi(\psi)} c(-v(\psi))$ has a positive lower bound, then it follows that for any any given $j$, $\int_{D_j} |\tilde{F}_j - (1-b(\psi))F|^2$ is bounded with respect to $j' \geq j$. Combining with

$$
\int_{D_j} (1-b(\psi))F|^2 \leq \int_{D_j \cap \{ \psi < -t_0 \}} |F|^2 < +\infty
$$

and inequality [2,1], one can obtain that $\int_{D_j} |\tilde{F}_j|^2$ is bounded with respect to $j' \geq j$.

By diagonal method, there exists a subsequence $F_{j'}$ uniformly convergent on any $M_j$ to a holomorphic $(n,0)$ form on $M$ denoted by $\tilde{F}$. Then it follows from inequality [3,2] and the dominated convergence theorem that

$$
\int_{D_j} |\tilde{F}_j - (1-b(\psi))F|^2 e^{-\max\{\varphi(-v(\psi)),\varphi(-N)\}} c(-v(\psi)) \leq C \int_{D_j} c(t)e^{-t} \ dt
$$

for any $N > 0$, which implies

$$
\int_{D_j} |\tilde{F}_j - (1-b(\psi))F|^2 e^{-(\varphi(\psi))} c(-v(\psi)) \leq C \int_{D_j} c(t)e^{-t} \ dt,
$$

then one can obtain Lemma [2,1] when $j$ goes to $+\infty$.

For the sake of completeness, we recall some lemmas on $L^2$ estimates for some $\bar{\partial}$ equations, and $\bar{\partial}^*$ means the Hilbert adjoint operator of $\bar{\partial}$.

In the following part of this subsection, we recall some lemmas on $L^2$ estimates for some $\bar{\partial}$ equations. Denote by $\bar{\partial}^*$ or $D''$ means the Hilbert adjoint operator of $\bar{\partial}$.

Lemma 3.2. (see [33] or [35]) Let $(X, \omega)$ be a Kähler manifold of dimension $n$ with a Kähler metric $\omega$. Let $(E, h)$ be a hermitian holomorphic vector bundle. Let $\eta, \varphi > 0$ be smooth functions on $X$. Then for every form $\alpha \in \Omega(X, \Lambda^{n,\varphi}T_X \otimes E)$, which is the space of smooth differential forms with values in $E$ with compact support, we have
functions on $\mathbb{R}$

Lemma 3.3. Let $\lambda \in \Lambda$ is positive definite everywhere on $\mathbb{R}$ for any $(u, v)$ have Lemma 3.4. Assume that there are smooth and bounded functions $\eta, g > 0$ on $X$ such that the (Hermitian) curvature operator

$$B := [\eta \sqrt{-1} \Theta_E - \sqrt{-1} \partial \bar{\partial} \eta - \sqrt{-1} g \partial \bar{\partial} \eta, \Lambda]$$

is positive definite everywhere on $\Lambda^{n,q} T_X^* \otimes E$, for some $q \geq 1$. Then for every form $\lambda \in L^2(X, \Lambda^{n,q} T_X^* \otimes E)$ such that $D'\lambda = 0$ and $\int_X (B^{-1} \lambda, \lambda) dV_\omega < \infty$, there exists $u \in L^2(X, \Lambda^{n,q-1} T_X^* \otimes E)$ such that $D'' u = \lambda$ and

$$\int_X (\eta + g^{-1})^{-1} |u|^2 dV_\omega \leq \int_X (B^{-1} \lambda, \lambda) dV_\omega.$$

The following Lemma belongs to Fornaess and Narasimhan on approximation property of plurisubharmonic functions of Stein manifolds.

Lemma 3.5. Let $X$ be a Stein manifold and $\varphi \in \text{PSH}(X)$. Then there exists a sequence $\{\varphi_n\}_{n=1,2,\ldots}$ of smooth strongly plurisubharmonic functions such that $\varphi_n \downarrow \varphi$.

3.2. Proof of Lemma 2.1. For the sake of completeness, let’s recall some steps in the proof in [16] (see also [18, 19, 21]) with some slight modifications in order to prove Lemma 2.1.

It follows from Remark 3.1 that it suffices to consider that $M$ is a Stein manifold, and $F$ is holomorphic $(n, 0)$ form on $U \cap \{\psi < -t_0\}$ and

$$\int_{\{\psi < -t_0\}} |F|^2 < +\infty,$$

and there exist smooth plurisubharmonic functions $\psi_m$ and $\varphi_m$ on $M$ decreasing convergent to $\psi$ and $\varphi$ respectively, satisfying $\sup_m \sup_M \psi_m < -T$ and $\sup_m \sup_M \varphi_m < +\infty$.

Step 1: recall some Notations

Let $\varepsilon \in (0, \frac{1}{4} B)$. Let $\{v_{\varepsilon}\}_{\varepsilon \in (0, \frac{1}{4} B)}$ be a family of smooth increasing convex functions on $\mathbb{R}$, which are continuous functions on $\mathbb{R} \cup \{-T\}$, such that:

1. $v_{\varepsilon}(t) = t$ for $t \geq -t_0 - \varepsilon$, $v_{\varepsilon}(t) = \text{constant}$ for $t < -t_0 - B + \varepsilon$;
2. $v''_{\varepsilon}(t)$ are pointwise convergent to $\frac{1}{B} 1_{(-t_0-B,-t_0)}$, when $\varepsilon \to 0$, and $0 \leq v''_{\varepsilon}(t) \leq \frac{1}{B} 1_{(-t_0-B+\varepsilon,-t_0-\varepsilon)}$ for any $t \in \mathbb{R}$;
3). \( v'(t) \) are pointwise convergent to \( b(t) \) which is a continuous function on \( \mathbb{R} \), when \( \varepsilon \to 0 \), and \( 0 \leq v'(t) \leq 1 \) for any \( t \in \mathbb{R} \).

One can construct the family \( \{v_{\varepsilon}\}_{\varepsilon \in (0,1/\varepsilon)} \) by the setting

\[
v_{\varepsilon}(t):=\int_{-\infty}^{t} \left( \int_{-\infty}^{t_{1}} \frac{1}{B-4\varepsilon} I_{(-t_{0}-B+2\varepsilon,-t_{0}-2\varepsilon)}*\rho_{\varepsilon}(s)ds \right) dt_{1} - \int_{-\infty}^{0} \left( \int_{-\infty}^{t_{1}} \frac{1}{B-4\varepsilon} I_{(-t_{0}-B+2\varepsilon,-t_{0}-2\varepsilon)}*\rho_{\varepsilon}(s)ds \right) dt_{1},
\]

where \( \rho_{\varepsilon} \) is the kernel of convolution satisfying \( \text{supp}(\rho_{\varepsilon}) \subset (-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}) \). Then it follows that

\[
v''(t) = \frac{1}{B-4\varepsilon} I_{(-t_{0}-B+2\varepsilon,-t_{0}-2\varepsilon)}*\rho_{\varepsilon}(t),
\]

and

\[
v'(t) = \int_{-\infty}^{t} \left( \int_{-\infty}^{t_{1}} \frac{1}{B-4\varepsilon} I_{(-t_{0}-B+2\varepsilon,-t_{0}-2\varepsilon)}*\rho_{\varepsilon}(s)ds \right) dt_{1}.
\]

It suffices to consider the case that

\[
\int_{M} \frac{1}{B} \mathbb{I}_{(-t_{0}-B<\psi<\alpha)} |F|^2 e^{-\psi} < +\infty.
\]

Let \( \eta = s(-v_{\varepsilon}(\psi_{m})) \) and \( \phi = u(-v_{\varepsilon}(\psi_{m})) \), where \( s \in C^{\infty}((T, +\infty)) \) satisfies \( s \geq 0 \), and \( u \in C^{\infty}((T, +\infty)) \), satisfies \( \lim_{t \to +\infty} u(t) \) exists, such that \( u''-s'' > 0 \), and \( s' - u's = 1 \). It follows from \( \sum_{m} \psi_{m} < -T \) that \( \phi = u(-v_{\varepsilon}(\psi_{m})) \) are uniformly bounded on \( M \) with respect to \( m \) and \( \varepsilon \), and \( u(-v_{\varepsilon}(\psi)) \) are uniformly bounded on \( M \) with respect to \( \varepsilon \). Let \( \Phi = \phi + \varphi_{m'} \), and let \( \tilde{h} = e^{-\phi} \).

**Step 2: Solving \( \bar{D} \)-equation with smooth polar function and smooth weight**

Now let \( \alpha \in D(M, \Lambda^{n-1}T^{*}_{M}) \) be a smooth \((n, 1)\) form with compact support on \( M \). Using Lemma 3.2 and Lemma 3.3, the inequality \( s \geq 0 \) and the fact that \( \varphi_{m} \) is plurisubharmonic on \( M \), we get

\[
\| (\eta + g^{-1})^{1/2} D''\alpha \|^{2}_{M, \tilde{h}} + \| \eta^{1/2} D''\alpha \|^{2}_{M, \tilde{h}} \geq \| [\eta^{1/2} D''\alpha] \|^{2}_{M, \tilde{h}}
\]

\[
\geq \| [\eta^{1/2} D''\alpha] \|^{2}_{M, \tilde{h}} \geq \| [\eta^{1/2} D''\alpha] \|^{2}_{M, \tilde{h}}.
\]

where \( g \) is a positive continuous function on \( M \). We need the following calculations to determine \( g \).

\[
\nabla\bar{D}\eta = s'(-v_{\varepsilon}(\psi_{m})) \nabla\bar{D}(v_{\varepsilon}(\psi_{m})) + s''(-v_{\varepsilon}(\psi_{m})) \nabla v_{\varepsilon}(\psi_{m}) \wedge \nabla v_{\varepsilon}(\psi_{m}).
\]

and

\[
\nabla\bar{D}\phi = -u'(-v_{\varepsilon}(\psi_{m})) \nabla\bar{D}(v_{\varepsilon}(\psi_{m})) + u''(-v_{\varepsilon}(\psi_{m})) \nabla v_{\varepsilon}(\psi_{m}) \wedge \nabla v_{\varepsilon}(\psi_{m}).
\]

Then we have

\[
\begin{align*}
\bar{D}\eta + \eta \nabla\bar{D}\phi - g(\partial\eta) \wedge \bar{D}

= (s' - su') \nabla\bar{D}(v_{\varepsilon}(\psi_{m}))(u''s - s'') - gs^{2} \nabla\bar{D}(v_{\varepsilon}(\psi_{m})) \nabla v_{\varepsilon}(\psi_{m}) \\
= (s' - su')(v_{\varepsilon}(\psi_{m})) \nabla v_{\varepsilon}(\psi_{m}) + v_{\varepsilon}(\psi_{m}) \nabla(\psi_{m}) \wedge \nabla (\psi_{m}) \\
+ ((u''s - s'') - gs^{2}) \nabla\bar{D}(v_{\varepsilon}(\psi_{m})) \wedge \nabla v_{\varepsilon}(\psi_{m}).
\end{align*}
\]
We omit composite item \(-v_c(\psi_m)\) after \(s' - su'\) and \((u''s - s'') - gs'^2\) in the above equalities.

As \(v_{\alpha, \varepsilon}'\geq 0\) and \(s' - su' = 1\), using Lemma 3.3, equality 3.10 and inequality 3.11, we obtain

\[
(B_\alpha, \alpha)_{\bar{h}} = \langle \eta \sqrt{-1} \Theta_{\bar{h}} - \sqrt{-1} \partial \bar{\partial} \eta - \sqrt{-1} g \partial \eta \wedge \bar{\partial} \eta, \Lambda_\omega \rangle_{\alpha, \alpha} \bar{h}
\]

(3.11)

\[
\geq \langle (v''_{\alpha, \varepsilon} \circ \psi_m) \sqrt{-1} \partial \bar{\partial} \psi_m \wedge \bar{\partial} \psi_m, \Lambda_\omega \rangle_{\alpha, \alpha} \bar{h}
\]

\[= \langle ((v''_{\alpha, \varepsilon} \circ \psi_m) \partial \psi_m \wedge (\Lambda_\omega (\partial \psi_m)^2), \alpha \rangle_{\bar{h}}.
\]

Using the definition of contraction, Cauchy-Schwarz inequality and the inequality 3.11, we have

\[
\|(v''_{\alpha, \varepsilon} \circ \psi_m) \partial \psi_m \wedge \gamma, \alpha \rangle_{\bar{h}}|^2 \leq \|(v''_{\alpha, \varepsilon} \circ \psi_m) \gamma, \alpha \lambda (\partial \psi_m)^2 \rangle_{\bar{h}}^2
\]

\[
\leq \|(v''_{\alpha, \varepsilon} \circ \psi_m) \gamma, \gamma \rangle_{\bar{h}} \langle (v''_{\alpha, \varepsilon} \circ \psi_m) \partial \psi_m \wedge (\Lambda_\omega (\partial \psi_m)^2), \alpha \rangle_{\bar{h}}
\]

(3.12)

for any \((n, 0)\) form \(\gamma\) and \((n, 1)\) form \(\alpha\).

As \(F\) is holomorphic on \(\{\psi < -t_0\} \supset Supp(v'_c(\psi_m))\), then \(\lambda := \tilde{\partial}[(1 - v'_c(\psi_m))F]\) is well-defined and smooth on \(M\).

Taking \(\gamma = F\), and \(\alpha = B^{-1} \tilde{\partial} \psi \wedge \bar{\partial} \), note that \(\bar{h} = e^{-\Phi}\), using inequality 3.12, we have

\[
\langle B^{-1} \lambda, \lambda \rangle_{\bar{h}} \leq (v''_{\alpha, \varepsilon} \circ \psi_m) |F|^2 e^{-\Phi}.
\]

Then it follows that

\[
\int_M \langle B^{-1} \lambda, \lambda \rangle_{\bar{h}} \leq \int_M (v''_{\alpha, \varepsilon} \circ \psi_m) |F|^2 e^{-\Phi}.
\]

Using Lemma 3.3, we have locally \(L^1\) function \(u_{m, m', \varepsilon}\) on \(M\) such that \(\bar{\partial} u_{m, m', \varepsilon} = \lambda\), and

\[
\int_M |u_{m, m', \varepsilon}|^2 (\eta + g^{-1})^{-1} e^{-\Phi} \leq \int_M \langle B^{-1} \lambda, \lambda \rangle_{\bar{h}} \leq \int_M (v''_c(\psi_m)) |F|^2 e^{-\Phi}.
\]

(3.13)

Let \(g = \frac{u''_{s \rightarrow s'} - u''_{s'}}{s' - s} (-v_c(\psi_m))\). It follows that \(\eta + g^{-1} = (s + \frac{s'^2}{s' - s}) (-v_c(\psi_m))\).

Let \(\mu := (\eta + g^{-1})^{-1}\).

Assume that we can choose \(\eta\) and \(\phi\) such that \(e^{v_c(\psi_m) e\phi c(-v_c \circ \psi_m)} = (\eta + g^{-1})^{-1}\).

Then inequality 3.13 becomes

\[
\int_M |u_{m, m', \varepsilon}|^2 e^{v_c(\psi_m) - \phi(\psi_m) e(-v_c \circ \psi_m)} \leq \int_M (v''_c(\psi_m)) |F|^2 e^{-\phi - \phi'}. \]

(3.14)

Let \(F_{m, m', \varepsilon} := -u_{m, m', \varepsilon} + (1 - v'_c(\psi_m)) F\). Then inequality 3.14 becomes

\[
\int_M |F_{m, m', \varepsilon}|^2 e^{v_c(\psi_m) - \phi(\psi_m) e(-v_c \circ \psi_m)} \leq \int_M (v''_c(\psi_m)) |F|^2 e^{-\phi - \phi'}. \]

(3.15)

Step 3: Singular polar function and smooth weight
As $\sup_{m,\varepsilon} |\phi| = \sup_{m,\varepsilon} |u(-v_\varepsilon(\psi_m))| < +\infty$ and $\varphi_m'$ is continuous on $\bar{M}$, then $\sup_{m,\varepsilon} e^{-\phi - \varphi_m'} < +\infty$. Note that

$$v''_\varepsilon(\psi_m) |F|^2 e^{-\phi - \varphi_m'} \leq \frac{2}{B} I_{\{\psi < -t_0\}} |F|^2 \sup_{m,\varepsilon} e^{-\phi - \varphi_m'}$$

on $M$, then it follows from inequality (3.4) and the dominated convergence theorem that

$$\lim_{m \to +\infty} \int_M v''_\varepsilon(\psi_m) |F|^2 e^{-\phi - \varphi_m'} = \int_M v''_\varepsilon(\psi) |F|^2 e^{-u(-v_\varepsilon(\psi)) - \varphi_m'}$$

(3.16)

Note that $\inf_m \inf_M e^{-v_\varepsilon(\psi_m) - \varphi_m'} c(-v_\varepsilon \circ \psi_m) > 0$, then it follows from inequality (3.15) and (3.16) that $\sup_m \int_M |F_{m,m',\varepsilon} - (1 - v'_\varepsilon(\psi_m)) F|^2 < +\infty$. Note that

$$\sup_m \int_M |F_{m,m',\varepsilon} - (1 - v'_\varepsilon(\psi_m)) F|^2 \leq (|1 - v'_\varepsilon(\psi_m)) F|^2) \leq 2(\int_M |F_{m,m',\varepsilon} - (1 - v'_\varepsilon(\psi_m)) F|^2 < +\infty$$

(3.17)

then it follows from inequality (3.16) that $\sup_m \int_M |F_{m,m',\varepsilon}|^2 < +\infty$, which implies that there exists a subsequence of $\{F_{m,m',\varepsilon}\}_m$ (also denoted by $F_{m,m',\varepsilon}$ compactly convergent to a holomorphic $F_{m',\varepsilon}$ on $M$.

Note that $v_\varepsilon(\psi_m) - \varphi_m'$ are uniformly bounded on $M$ with respect to $m$, then it follows from $|F_{m,m',\varepsilon} - (1 - v'_\varepsilon(\psi_m)) F|^2 \leq (|1 - v'_\varepsilon(\psi_m)) F|^2) \leq 2(\int_M |F_{m,m',\varepsilon}|^2 + \int_{\{\psi < -t_0\}} F|^2) \leq 2(\int_M |F_{m,m',\varepsilon}|^2 + \int_{\{\psi < -t_0\}} F|^2)$ and the dominated convergence theorem that

$$\lim_{m \to +\infty} \int_K |F_{m,m',\varepsilon} - (1 - v'_\varepsilon(\psi_m)) F|^2 e^{-v_\varepsilon(\psi) - \varphi_m'} c(-v_\varepsilon \circ \psi_m)$$

(3.18)

$$= \int_K |F_{m',\varepsilon} - (1 - v'_\varepsilon(\psi)) F|^2 e^{-v_\varepsilon(\psi) - \varphi_m'} c(-v_\varepsilon \circ \psi)$$

holds for any compact subset $K$ on $M$. Combining with inequality (3.15) and (3.19), one can obtain that

$$\int_K |F_{m',\varepsilon} - (1 - v'_\varepsilon(\psi)) F|^2 e^{-v_\varepsilon(\psi) - \varphi_m'} c(-v_\varepsilon \circ \psi)$$

(3.19)

$$\leq \int_M v''_\varepsilon(\psi) |F|^2 e^{-u(-v_\varepsilon(\psi)) - \varphi_m'},$$

which implies

$$\int_M |F_{m',\varepsilon} - (1 - v'_\varepsilon(\psi)) F|^2 e^{-v_\varepsilon(\psi) - \varphi_m'} c(-v_\varepsilon \circ \psi)$$

(3.20)

$$\leq \int_M v''_\varepsilon(\psi) |F|^2 e^{-u(-v_\varepsilon(\psi)) - \varphi_m'},$$

Step 4: Nonsmooth cut-off function

Note that $\sup_{\varepsilon} \sup_M e^{-u(-v_\varepsilon(\psi)) - \varphi_m'} < +\infty$, and

$$v''_\varepsilon(\psi) |F|^2 e^{-u(-v_\varepsilon(\psi)) - \varphi_m'} \leq \frac{2}{B} I_{\{|t_0 - \psi | < t_0\}} |F|^2 \sup_{\varepsilon} \sup_M e^{-u(-v_\varepsilon(\psi)) - \varphi_m'},$$
then it follows from inequality (3.21) and the dominated convergence theorem that
\[
\lim_{\varepsilon \to 0} \int_M v_\varepsilon'(\psi)|F|^2 e^{-u(-v_\varepsilon(\psi)) - \varphi_{m'}}
\]
(3.21)
\[
= \int_M \frac{1}{B} \mathbb{I}_{\{-t_0 - b < \psi < -t_0\}} |F|^2 e^{-u(-v(\psi)) - \varphi_{m'}}
\]
\[
\leq \sup_M e^{-u(-v(\psi))} \int_M \frac{1}{B} \mathbb{I}_{\{-t_0 - b < \psi < -t_0\}} |F|^2 e^{-\varphi_{m'}} < +\infty.
\]

Note that \(\inf \varepsilon \inf_M e^{v_\varepsilon(\psi)} - \varphi_{m'}(c(-v_\varepsilon \circ \psi)) > 0\), then it follows from inequality (3.20) and (3.21) that \(\sup \varepsilon \int_M |F_{m', \varepsilon} - (1 - v_\varepsilon'(\psi))F|^2 < +\infty\). Combining with
\[
\sup \varepsilon \int_M |(1 - v_\varepsilon'(\psi))F|^2 \leq \int_M \mathbb{I}_{\{\psi < -t_0\}} |F|^2 < +\infty,
\]
one can obtain that \(\sup \varepsilon \int_M |F_{m', \varepsilon}|^2 < +\infty\), which implies that there exists a subsequence of \(\{F_{m', \varepsilon}\}_{\varepsilon \to 0}\) (also denoted by \(\{F_{m', \varepsilon}\}_{\varepsilon \to 0}\)) compactly convergent to a holomorphic \((n, 0)\) form on \(M\) denoted by \(F_{m'}\).

Note that \(\sup \varepsilon \sup_M e^{v_\varepsilon(\psi)} - \varphi_{m'}(c(-v_\varepsilon \circ \psi)) < +\infty\) and \(|F_{m', \varepsilon} - (1 - v_\varepsilon'(\psi))F|^2 \leq 2(|F_{m', \varepsilon}|^2 + |\mathbb{I}_{\{\psi < -t_0\}} F|^2)\), then it follows from inequality (3.22) and the dominated convergence theorem on any given \(K \subset D\) (with dominant function
\[
2(\sup \varepsilon M \mathbb{I}_{\{F_{m', \varepsilon}\}}^2 + \mathbb{I}_{\{\psi < -t_0\}} |F|^2) \sup \varepsilon M e^{v_\varepsilon(\psi) - \varphi_{m'}(c(-v_\varepsilon \circ \psi))}
\]
that
\[
\lim_{\varepsilon \to 0} \int_K |F_{m', \varepsilon} - (1 - b(\psi))F|^2 e^{v(\psi) - \varphi_{m'}(c(-v \circ \psi))}
\]
(3.23)
\[
= \int_K |F_{m'} - (1 - b(\psi))F|^2 e^{v(\psi) - \varphi_{m'}(c(-v \circ \psi))}.
\]
Combining with inequality (3.21) and (3.20), one can obtain that
\[
\int_K |F_{m'} - (1 - b(\psi))F|^2 e^{v(\psi) - \varphi_{m'}(c(-v \circ \psi))}
\]
(3.24)
\[
\leq (\sup_M e^{-u(-v(\psi))}) \int_M \frac{1}{B} \mathbb{I}_{\{-t_0 - b < \psi < -t_0\}} |F|^2 e^{-\varphi_{m'}}
\]
which implies
\[
\int_M |F_{m'} - (1 - b(\psi))F|^2 e^{v(\psi) - \varphi_{m'}(c(-v \circ \psi))}
\]
(3.25)
\[
\leq (\sup_M e^{-u(-v(\psi))}) \int_M \frac{1}{B} \mathbb{I}_{\{-t_0 - b < \psi < -t_0\}} |F|^2 e^{-\varphi_{m'}}.
\]

Step 5: Singular weight

Note that
\[
\int_M \frac{1}{B} \mathbb{I}_{\{-t_0 - b < \psi < -t_0\}} |F|^2 e^{-\varphi_{m'}} \leq \int_M \frac{1}{B} \mathbb{I}_{\{-t_0 - b < \psi < -t_0\}} |F|^2 e^{-\varphi} < +\infty,
\]
and \(\sup_M e^{-u(-v(\psi))} < +\infty\), then it from (3.25) that
\[
\sup_M \int_M |F_{m'} - (1 - b(\psi))F|^2 e^{v(\psi) - \varphi_{m'}(c(-v \circ \psi))} < +\infty.
\]
Combining with $\inf_{m'} \inf_M e^{v(\psi) - \varphi_{m'}} e(-v(\psi)) > 0$, one can obtain that
\[
\sup_{m'} \int_M |F_{m'} - (1 - b(\psi)) F|^2 < +\infty.
\]
Note that
\[
\int_M |(1 - b(\psi)) F|^2 \leq \int_M |\Pi_{t < t_0} F|^2 < +\infty.
\]
Then $\sup_{m'} \int_M |F_{m'}|^2 < +\infty$, which implies that there exists a compactly convergent subsequence of $\{F_{m'}\}$ denoted by $\{F_{m''}\}$, which is convergent a holomorphic (n,0) form $F$ on $M$.

Note that $\sup_{m'} \sup_M e^{v(\psi) - \varphi_{m'}} e(-v \circ \psi) < +\infty$, then it follows from inequality (3.27) and the dominated convergence theorem on any given compact subset $K$ of $M$ (with dominant function $2[\sup_{m''} \sup_K (|F_{m''}|^2 + \Pi_{t < t_0} |F|^2) \sup_M e^{v(\psi) - \varphi_{m''}}]$) that
\[
\lim_{m'' \to +\infty} \int_K |F_{m''} - (1 - b(\psi)) F|^2 e^{v(\psi) - \varphi_{m''}} c(-v \circ \psi) = \int_K \tilde{F} - (1 - b(\psi)) F|^2 e^{v(\psi) - \varphi_{m''}} c(-v \circ \psi).
\]
Note that for any $m'' \geq m'$, $\varphi_{m''} \leq \varphi_{m'}$ holds, then it follows from inequality (3.25) and (3.26) that
\[
\lim_{m'' \to +\infty} \int_K |F_{m''} - (1 - b(\psi)) F|^2 e^{v(\psi) - \varphi_{m''}} c(-v \circ \psi) \leq \lim sup_{m'' \to +\infty} \int_K |F_{m''} - (1 - b(\psi)) F|^2 e^{v(\psi) - \varphi_{m''}} c(-v \circ \psi) \leq \lim sup_{m'' \to +\infty} \sup_M e^{-u(-v(\psi))} \int_M \frac{1}{B} \Pi_{t < t_0 - B < \psi < -t_0} |F|^2 e^{-\varphi_{m''}} \leq (\sup_M e^{-u(-v(\psi))}) C < +\infty.
\]
Combining with equality (3.28), one can obtain that
\[
\int_K \tilde{F} - (1 - b(\psi)) F|^2 e^{v(\psi) - \varphi_{m''}} c(-v \circ \psi) \leq (\sup_M e^{-u(-v(\psi))}) C,
\]
for any compact subset of $M$, which implies
\[
\int_M \tilde{F} - (1 - b(\psi)) F|^2 e^{v(\psi) - \varphi_{m''}} c(-v \circ \psi) \leq (\sup_M e^{-u(-v(\psi))}) C.
\]
When $m' \to +\infty$, it follows from Levi’s Theorem that
\[
(3.30) \quad \int_M |\tilde{F} - (1 - b(\psi)) F|^2 e^{v(\psi) - \varphi} c(-v(\psi)) \leq (\sup_M e^{-u(-v(\psi))}) C.
\]

**Step 6: ODE system**

It suffices to find $\eta$ and $\phi$ such that $(\eta + g^{-1}) = e^{-v_m} e^{-\phi} \frac{1}{e(-v_m(\psi_m))}$ on $M$. As $\eta = s(-v_m(\psi_m))$ and $\phi = u(-v_m(\psi_m))$, we have $(\eta + g^{-1}) e^{v_m(\psi_m)} e^\phi = (s + \frac{e^{-s} e^{u(\psi_m)}}{e^{-s} e^{u(\psi_m)}}) e^u \circ (-v_m(\psi_m)).$
Summarizing the above discussion about $s$ and $u$, we are naturally led to a system of ODEs (see [17–18,19,21]):

\begin{align}
(3.31) \quad & 1. \ (s + \frac{s'^2}{u''s - s''})e^{u-t} = \frac{1}{c(t)}, \\
& 2. \ s' - su' = 1,
\end{align}

where $t \in (T, +\infty)$.

It is not hard to solve the ODE system (3.31) and get $u(t) = -\log(\int_T^t c(t_1)e^{-t_1}dt_1)$ and $s(t) = \frac{\int_T^t c(t_1)e^{-t_1}dt_1}{\int_T^t c(t_1)e^{-t_1}dt_1}$ (see [19]). It follows that $s \in C^\infty((T, +\infty))$ satisfies $s \geq 1$, $\lim_{t \to +\infty} u(t) = -\log(\int_T^t c(t_1)e^{-t_1}dt_1)$ and $u \in C^\infty((T, +\infty))$ satisfies $u''s - s'' > 0$.

As $u(t) = -\log(\int_T^t c(t_1)e^{-t_1}dt_1)$ is decreasing with respect to $t$, then it follows from $-T \geq v(t) \geq \max\{t, -t_0 - B_0\} \geq -t_0 - B_0$ for any $t \leq 0$ that

\begin{align}
(3.32) \quad & \sup Me^{-u(v)} \leq \sup_{t \in [T,t_0+B]} e^{-u(t)} = \int_T^{t_0+B} c(t_1)e^{-t_1}dt_1,
\end{align}

therefore we are done. Thus we prove Lemma 2.1.

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