EXISTENCE AND ASYMPTOTIC BEHAVIOUR OF SOLUTIONS
FOR A QUASILINEAR SCHRÖDINGER-POISSON SYSTEM
UNDER A CRITICAL NONLINEARITY

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Abstract. In this paper we consider the following quasilinear Schrödinger-Poisson system
\[
\begin{aligned}
-\Delta u + u + \phi u &= \lambda f(x, u) + |u|^{2^*-2}u & \text{ in } \mathbb{R}^3 \\
-\Delta \phi - \varepsilon^4 \Delta_4 \phi &= u^2 & \text{ in } \mathbb{R}^3,
\end{aligned}
\]
depending on the two parameters \(\lambda, \varepsilon > 0\).

We first prove that, for \(\lambda\) larger than a certain \(\lambda^* > 0\), there exists a solution for every \(\varepsilon > 0\).
Later, we study the asymptotic behaviour of these solutions whenever \(\varepsilon\) tends to zero, and we prove
that they converge to the solution of the Schrödinger-Poisson system associated.

1. Introduction

In [6, 12] Kavian, Benmilh, Illner and Lange have attracted the attention on a new kind of elliptic
system which, to the best of our knowledge, was never been considered before in the mathematical
literature, although the problem was known among the physicists. It seems it has been named quasi-
linear Schrödinger-Poisson system and indeed is a generalization of the well-known Schrödinger-Poisson
system.

This new system appears by studying a quantum mechanical model of extremely small devices in semi-
conductor nanostructures taking into account quantum structure and the longitudinal field oscillations
during the beam propagation. Indeed the intensity-dependent dielectric permittivity has the form
\[c_{\text{died}}(\nabla \phi) = 1 + \varepsilon^4 |\nabla \phi|^2, \quad \varepsilon > 0 \text{ and constant}\]
that is, it depends on the field itself. We are considering the simplified case of constant coefficient in
\(c_{\text{died}}\) which corresponds to homogeneous medium. Here \(\phi\) is the electric field and \(\varepsilon\) appears to the power
4 just for convenience. It seems this physical model and the corresponding equation of propagation
has appeared for the first time in [1] where the authors proposed and discussed this new model (see
also [17]).

The main novelty with respect to the huge existing literature describing beam propagation whenever
\(c_{\text{died}}\) does not depend on the field, is that, from a mathematical point of view, the equation of the
electrostatic potential is not linear, that is it is not the usual Poisson equation (or Gauss law in physical
terms) given by \(-\Delta \phi = u^2\). Without entering in physical details here, the system one arrives by looking
for standing waves solutions is something of type
\[
\begin{aligned}
-\Delta u + \omega u + (\phi + \phi^*) u &= 0 \\
-\Delta \phi - \varepsilon^4 \Delta_4 \phi &= u^2 - n^*
\end{aligned}
\]
where \(u, \phi\) are the unknown functions (here \(u\) represents the modulus of the wave function and \(\phi\) the
electrostatic potential) and \(n^*, \phi^*: \mathbb{R}^3 \to \mathbb{R}\) are given data of the problem: they represent, respectively,
the dopant density and the effective external potential. The operator \(\Delta_4\) is the 4-Laplacian, defined
as \(\Delta_4 u := \text{div}(|\nabla u|^2 \nabla u)\). Indeed this is exactly the system introduced in the mathematical literature,
as we said before, in the papers [6, 12].
Under minimal summability conditions on the data \( n^* \) and \( \tilde{\phi} \) the authors in [6], by means of minimization techniques, proves the existence of ground state solutions and study its behaviour whenever \( \varepsilon \to 0^+ \). Indeed they converges to the ground state solution of the Schrödinger-Poisson system associated (that is, whenever \( \varepsilon = 0 \) in (1.1)).

A similar problem with periodicity conditions is studied in [12] where the existence of infinitely many solutions normalized in \( L^2 \) by means of the Krasnoselkii genus is proved.

Observe that the Schrödinger equation in the above system (i.e. the first equation) is linear in \( u \).

We point out that few other papers are known to treat this type of systems: we revise now them here.

In the recent paper [9], Ding, Li, Meng and Zhuang deal with an asymptotically linear nonlinearity in the Schrödinger equation and study the existence and the behaviour of the ground state solution as \( \varepsilon \to 0^+ \). Again the solutions converge to the solution of the “limit” problem with \( \varepsilon = 0 \).

Illner, Lange, Toomire and Zweifel in [13] consider the quasilinear Schrödinger-Poisson system in the unitary cube under periodic boundary conditions and by using Galerkin scheme, they prove global existence and uniqueness of solutions.

In [16] Li and Yang study the existence and uniqueness of a global mild solution to the initial boundary value problem in the one dimensional case.

Finally in the paper [8] of d’Avenia and Pisani, the Born-Infeld Lagrangian density interacting with the Klein-Gordon equation is considered. They find infinitely many radial solutions in the subcritical case via the Symmetric Mountain Pass Theorem. We cite this paper because the use of the Born-Infeld Lagrangian density for the electromagnetic field (in place of the classical Maxwell Lagrangian density) gives rise to the quasilinear equation for the electrostatic field. Indeed this will be our approach to derive the system in the next Section.

It is clear that in theoretical analysis, numerical studies the most frequently used model for beam propagation assumes \( c_{\text{dielectric}}(\nabla \phi) = 1 \) which gives rise to the Poisson equation \(-\Delta \phi = u^2 \) in the system. The advantage of working with the Poisson equation is that the solution is explicitly given by the convolution \( \phi^{\text{Poiss}}(u) = |\cdot|^{-1} * u^2 \) (up to a multiplicative factor) so that many good properties of the solution are known; in particular the homogeneity \( \phi^{\text{Poiss}}(tu) = t^2 \phi^{\text{Poiss}}(u) \), \( t \in \mathbb{R} \). For the Schrödinger-Poisson system, the existing literature is so huge that is almost impossible to give a satisfactory list of papers. As a matter of fact, the main difficult dealing with the quasilinear Poisson equation of type

\[-\Delta \phi - \Delta_4 \phi = u^2\]

is due exactly to the lack of good properties for the solution.

Coming back to the present paper, our aim is to study a system similar to (1.1) where the Schrödinger equation has a critical nonlinearity; more specifically, we are concerning here with the following system

\[
(P_{\lambda, \varepsilon}) \quad \begin{cases} 
-\Delta u + u + \phi u = \lambda f(x, u) + |u|^{2^* - 2} u & \text{in } \mathbb{R}^3, \\
-\Delta \phi - \varepsilon^4 \Delta_4 \phi = u^2 & \text{in } \mathbb{R}^3,
\end{cases}
\]

where
- \( \lambda > 0 \) and \( \varepsilon > 0 \) are parameters,
- \( 2^* = 6 \) is the critical Sobolev exponent in dimension 3,
- \( f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R} \) is a continuous function that satisfies the following assumptions

(0) \( f(x, t) = 0 \) for \( t \leq 0 \),

(1) \( \lim_{t \to 0} \frac{f(x, t)}{t} = 0 \), uniformly on \( x \in \mathbb{R}^3 \),

(2) there exists \( q \in (2, 2^*) \) verifying \( \lim_{t \to +\infty} \frac{f(x, t)}{t^{q-1}} = 0 \) uniformly on \( x \in \mathbb{R}^3 \).
(f3) there exists \( \theta \in (4, 2^*) \) such that

\[
0 < \theta F(x, t) = \theta \int_0^t f(x, s)ds \leq tf(x, t), \quad \text{for all } x \in \mathbb{R}^3 \text{ and } t > 0.
\]

A typical example of function satisfying the above conditions is

\[
f(x, t) = \sum_{i=1}^{k} C_i(x) t_i \frac{1}{i - 1}
\]

with \( k \in \mathbb{N}, \ 4 < q_i < 2^*, \ C_i \) bounded and positive functions and \( t_+ = \max\{t, 0\} \).

Before introducing the notion of solution, we establish few basic standard notations.

For \( p \in [1, +\infty], \ L^p(\mathbb{R}^3) \) is the usual Lebesgue space with norm \( ||u||_p \).

We denote with \( H^1(\mathbb{R}^3) \) the usual Sobolev space endowed with scalar product and norm given by

\[
\langle u, v \rangle_{H^1} := \int_{\mathbb{R}^3} \nabla u \nabla v + \int_{\mathbb{R}^3} uv, \quad ||u||_{H^1} := \langle u, u \rangle^{1/2}.
\]

For \( p \geq 2, D^{1,p}(\mathbb{R}^3) \) is the Banach space defined as the completion of the test functions \( C_c^\infty(\mathbb{R}^3) \) with respect to the \( L^p \)– norm of the gradient. We define

\[
X := D^{1,2}(\mathbb{R}^3) \cap D^{1,4}(\mathbb{R}^3)
\]

which is a Banach space under the norm

\[
||\phi||_X := ||\nabla \phi||_2 + ||\nabla \phi||_4.
\]

As a final convention, whenever we are understanding the Lebesgue measure \( dx \) in integrals, it will be always omitted; otherwise we will write explicitly the measure.

The natural functional spaces in which find the solutions of \((P_{\lambda,\varepsilon})\) are:

\[
u \in H^1(\mathbb{R}^3), \quad \phi \in X.
\]

By a solution of \((P_{\lambda,\varepsilon})\) we mean a pair \((u_{\lambda,\varepsilon}, \phi_{\lambda,\varepsilon}) \in H^1(\mathbb{R}^3) \times X\) such that

\[
(1.2) \quad \forall v \in H^1(\mathbb{R}^3) : \quad \int_{\mathbb{R}^3} \nabla u_{\lambda,\varepsilon} \nabla v + \int_{\mathbb{R}^3} u_{\lambda,\varepsilon} v + \int_{\mathbb{R}^3} \phi_{\lambda,\varepsilon} u_{\lambda,\varepsilon} v = \lambda \int_{\mathbb{R}^3} f(x, u_{\lambda,\varepsilon}) v + \int_{\mathbb{R}^3} |u_{\lambda,\varepsilon}|^{2^* - 2} u_{\lambda,\varepsilon} v
\]

\[
(1.3) \quad \forall \xi \in X : \quad \int_{\mathbb{R}^3} \nabla \phi_{\lambda,\varepsilon} \nabla \xi + \varepsilon \int_{\mathbb{R}^3} |\nabla \phi_{\lambda,\varepsilon}|^2 \nabla \phi_{\lambda,\varepsilon} \nabla \xi = \int_{\mathbb{R}^3} \xi u^2.
\]

The main results of this paper are the following.

**Theorem 1.** Assume that conditions (f0)-(f3) hold. Then, there exists \( \lambda^* > 0, \) such that

\[\forall \lambda \geq \lambda^*, \varepsilon > 0 : \text{problem } (P_{\lambda,\varepsilon}) \text{ admit a solution } (u_{\lambda,\varepsilon}, \phi_{\lambda,\varepsilon}) \in H^1(\mathbb{R}^3) \times X.\]

Moreover \( \phi_{\lambda,\varepsilon}, u_{\lambda,\varepsilon} \) are nonnegative, of Mountain Pass type and for every fixed \( \varepsilon > 0: \)

1. \( \lim_{\lambda \to +\infty} ||u_{\lambda,\varepsilon}||_{H^1} = 0, \)
2. \( \lim_{\lambda \to +\infty} ||\phi_{\lambda,\varepsilon}||_X = 0, \)
3. \( \lim_{\lambda \to +\infty} ||\phi_{\lambda,\varepsilon}||_\infty = 0. \)

Actually, except for the limit in 3., the Theorem also holds for \( \varepsilon = 0 \) by replacing \( X \) with \( D^{1,2}(\mathbb{R}^3) \).

We study also the behaviour with respect to \( \varepsilon \) of the solutions given in Theorem 1, indeed we prove they converge to the solution of the Schrödinger-Poisson system.

**Theorem 2.** Assume that conditions (f0)-(f3) hold. Let \( \lambda^* > 0 \) be the one given in Theorem 1 and \( \bar{X} \geq \lambda^* \) be fixed. Let \( \{(u_{\bar{X},\varepsilon}, \phi_{\bar{X},\varepsilon})\}_{\varepsilon > 0} \) be the solutions given above in correspondence of such fixed \( \bar{X}. \) Then

1. \( \lim_{\varepsilon \to 0^+} u_{\bar{X},\varepsilon} = u_{\bar{X},0} \text{ in } H^1(\mathbb{R}^3), \)
2. \( \lim_{\varepsilon \to 0^+} \phi_{\bar{X},\varepsilon} = \phi_{\bar{X},0} \text{ in } D^{1,2}(\mathbb{R}^3), \)
where \((u_{\lambda,0}, \phi_{\lambda,0}) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)\) is a positive solution, of Mountain Pass type of the Schrödinger-Poisson system

\[
\begin{cases}
-\Delta u + u + \phi u = \overline{\lambda} f(x, u) + |u|^{2^*-2} u & \text{in } \mathbb{R}^3, \\
-\Delta \phi = u^2 & \text{in } \mathbb{R}^3.
\end{cases}
\]

(1.4)

The important point of Theorem 1 is the vanishing of the solutions whenever \(\lambda\) is larger and larger. Moreover, thanks to a Moser iteration scheme, we get \(u_{\lambda,\varepsilon}, \phi_{\lambda} \in L^\infty(\mathbb{R}^3)\) This allow us to treat also the supercritical case, hence a problem of type

\[
\begin{cases}
-\Delta u + u + \phi u = \lambda f(x, u) + |u|^{p-2} u & \text{in } \mathbb{R}^3, p > 2^*, \\
-\Delta \phi - \varepsilon^4 \Delta_4 \phi = u^2 & \text{in } \mathbb{R}^3,
\end{cases}
\]

(1.5)

under the same assumptions on \(f\). More explicitly, as a consequence of Theorem 1 we have the following

Theorem 3. Theorem 1 and Theorem 2 hold also for problem (1.5).

Our approach in proving Theorem 1 is variational. Indeed a suitable functional can be defined whose critical points are exactly the solutions of \((P_{\lambda,\varepsilon})\). Hence the meaning of “Mountain Pass solution” will be clear.

In proving our results, we have to manage with various difficulties. Firstly, the fact that the problem is in the whole \(\mathbb{R}^3\) and no symmetry conditions on the solutions and on the datum \(f\) are imposed (as e.g. in [8]); even more we are in the critical case, then there is a clear lack of compactness. We are able to overcome this difficulty thanks to the Concentration Compactness of Lions (see [15]) and taking advantage of the parameter \(\lambda\).

Secondly, we have to face with the fact that the solution in the second equation of \((P_{\lambda,\varepsilon})\), which is quasilinear, has not an explicit formula, neither has homogeneity properties. To circumvent this last difficulty, a suitable truncation (already introduced in [14]) is used in front of the “bad” part of the functional. This type of truncation is also used in [3] to treat the classical Schrödinger-Poisson problem under a general nonlinearity of Berestycki-Lions type.

Our contribution in this paper is then to give a better understanding of this intriguing problem, especially to see as the truncation argument, which appears for the first time for a quasilinear Schrödinger-Poisson system, is useful to deal with a piece of the functional which has not good properties.

Note that when \(\varepsilon = 0\), that is in the case of the Schrödinger-Poisson system, Theorem 1 gives the result of Zhao and Zhao [18], which indeed concerns with a slightly different nonlinearity of type \(g(x, u) = \mu Q(x)|u|^{2^*-2} u + K(x)|u|^{2^*-2} u\). However with slight changes, our theorems also hold if in front of the critical nonlinearity there is coefficient \(K(x)\) as in [18].

Moreover, as a byproduct of our Theorem 3 we deduce the existence of a positive solution (for \(\lambda\) large) for the Schrödinger-Poisson system even in presence of a supercritical nonlinearity, fact that we were not able to find in the literature.

The paper is organized as follows.

In Section 2 we deduce the set of equations we are going to study. Indeed, differently from the paper of Benmilh and Kavian [6], we deduce the equations under study in the framework of the Abelian Gauge Theories by considering the interaction of the Schrödinger equation with the Maxwell equation described by the Born-Infeld Lagrangian which is the second order approximation of the classical Maxwell Lagrangian.

In Section 3 the variational framework of the problem is introduced. We study some properties of the second equation in the system and define the functional \(J_{\lambda,\varepsilon}\) whose critical points will be the solution of the system.

In Section 4 we introduce the truncation in the original functional \(J_{\lambda,\varepsilon}\). This will help to deal with the lack of properties of the solution of the second equation, in contrast to the case of the Schrödinger-Poisson system.

In Sections 5, 6 and 7 the proof of Theorem 1, 2 and 3, respectively, is given.
2. Derivation of the system

Let us spend few words in this section on the physical derivation of system \((P_{\lambda,\varepsilon})\) in the framework of Abelian Gauge Theory.

Our starting point is the Lagrangian of the nonlinear Schrödinger equation. Indeed it is well known that the Euler Lagrange equation of the Lagrangian density

\[ L_S(\psi) = ih\psi\partial_t\psi - \frac{\hbar^2}{2m}|\nabla\psi|^2 + G(x,|\psi|) \]

is exactly the nonlinear Schrödinger equation. Here \(G(x,|\psi|)\) is a suitable nonlinearity depending on the physical model.

The interaction of the wave function \(\psi\) with the electromagnetic field generated by its motion, is described by means of the covariant derivative in the framework of the Abelian Gauge Theory. In Physics this is known also as minimal coupling rule and, practically, consists in substituting the ordinary derivative in \(L_S\) with the new operators (the covariant, or Wayl derivatives):

\[ \partial_i \rightarrow \partial_i + \frac{iq}{\hbar}\phi, \quad \nabla \rightarrow \nabla - \frac{iq}{\hbar}A \]

where \(\phi\) and \(A\) are the gauge potentials of the electromagnetic field, that is

\[ E = -\nabla\phi - \partial_tA, \quad B = \nabla \times A, \]

\(q\) is the electric charge and \(\hbar\) the normalized Plank constant. In this way one obtains from \(L_S\) the Lagrangian density of the interaction

\[ L_{\text{int}}(\psi, \phi, A) = ih\psi\partial_t\psi - q\phi|\psi|^2 - \frac{\hbar^2}{2m} \left| \nabla\psi - \frac{iq}{\hbar}A \psi \right|^2 + G(x,|\psi|). \]

It is convenient to write the wave function in polar form, i.e. \(\psi(x, t) = u(x, t)e^{iS(x, t)/\hbar}\) with \(u, S : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}\). Then the Lagrangian density of the interaction takes the form

\[ L_{\text{int}}(u, S, \phi, A) = ihu\partial_tu - \frac{\hbar^2}{2m} |\nabla u|^2 - \left( \partial_tS + q\phi + \frac{1}{2m} |\nabla S - qA|^2 \right) u^2 + G(x, u). \]

However this is not the total Lagrangian density of the system since the e.m. field (and then \(\phi\) and \(A\)) is an unknown, hence also the Lagrangian density of the e.m. has to be considered.

The existing literature concerning the Schrödinger-Maxwell system, mainly consider the usual classical Lagrangian density of Maxwell:

\[ L_M(\phi, A) = \frac{1}{8\pi} (|E|^2 - |B|^2) = \frac{1}{8\pi} (|\nabla \phi + \partial_tA|^2 - |\nabla \times A|^2). \]

Here we use the Lagrangian density of the Born-Infeld theory, that is

\[ L_{\text{BI}} = \frac{1}{4\pi} \left[ \frac{1}{2} (|E|^2 - |B|^2) + \frac{\beta}{4} (|E|^2 - |B|^2)^2 \right], \quad \beta > 0. \]

In this way the total Lagrangian density, which describes the dynamic of the motion of the matter field \(\psi\) and the e.m. field \((E, B)\), is given by

\[ L_{\text{tot}}(u, S, \phi, A) = L_{\text{int}}(u, S, \phi, A) + L_{\text{BI}}(\phi, A) \]

\[ = ihu\partial_tu - \frac{\hbar^2}{2m} |\nabla u|^2 - \left( \partial_tS + q\phi + \frac{1}{2m} |\nabla S - qA|^2 \right) u^2 + G(x, u) \]

\[ + \frac{1}{4\pi} \left[ \frac{1}{2} (|E|^2 - |B|^2) + \frac{\beta}{4} (|E|^2 - |B|^2)^2 \right]. \]

The Euler Lagrange equations of this Lagrangian (that is, by making the variations with respect to \(u, S, \phi, A\) are easily computed and are

\[ -\frac{\hbar^2}{2m} \Delta u + \left( \partial_tS + q\phi + \frac{1}{2m} |\nabla S - qA|^2 \right) u = g(x, u). \]
\[
\partial_t u^2 + \frac{1}{m} \nabla \cdot \left[ (\nabla S - qA) u^2 \right] = 0
\]
\[
- \nabla \cdot \left( Z_{A,\phi} (\nabla \phi + \partial_t A) \right) = 4\pi qu^2
\]
\[
\partial_t \left( Z_{A,\phi} (\partial_t A + \nabla \phi) \right) + \nabla \times \left( Z_{A,\phi} \nabla \times A \right) = 4\pi q(\nabla S - qA)u^2
\]
where we have set, for brevity,
\[
Z_{A,\phi} := 1 + \beta |A_t + \nabla \phi|^2 - \beta |\nabla \times A|^2
\]
and \(g(x,s) = \partial_s G(x,s)\).

An interesting physical situation is that of standing waves in the purely electrostatic case which appears when we look for solutions of type
\[
u(x,t) = \nu(x), S(x,t) = \omega \hbar t, \phi(x,t) = \phi(x), A(x,t) = 0
\]
which gives rise to wave functions of type \(\psi(x,t) = \nu(x)e^{i\omega t}\). In this case the above set of equations is reduced to
\[
\begin{align*}
\hbar^2 \frac{\Delta u}{2m} + \omega u + q\phi u &= g(x,u) \quad \text{in } \mathbb{R}^3, \\
-\nabla \cdot (\nabla \phi + \beta |\nabla \phi|^2 \nabla \phi) &= 4\pi qu^2 \quad \text{in } \mathbb{R}^3.
\end{align*}
\]
Observing that up to change \(\phi\) with \(-\phi\), we can assume without lost of generality that \(q > 0\). By “normalizing” the constants
\[
\frac{\hbar^2}{2m} = \omega = q = 4\pi = 1,
\]
and setting
\[
\beta = \varepsilon \quad \text{and} \quad g(x,u) = \lambda f(x,u) + |u|^{2^* - 2} u
\]
problem (2.1) becomes exactly problem \((P_{\lambda,\varepsilon})\).

3. The variational framework

We begin by saying that that the single equation
\[
-\Delta \phi - \beta \Delta_4 \phi = \rho \quad (\beta > 0)
\]
has been very studied in the mathematical literature, since it falls down into the class of equations involving the \(p \& q\) Laplacian. In particular in [11], where the authors study the case in which the distribution \(\rho\) is a Dirac delta or an \(L^1\) function, it is shown that there is the continuous embedding
\[
X \hookrightarrow L^\infty(\mathbb{R}^3)
\]
(see [11, Proposition 8]). As a consequence, the solutions \(\phi_{\lambda,\varepsilon}\) given in Theorem 1 will be automatically in \(L^\infty(\mathbb{R}^3)\); moreover once we prove that \(\lim_{\lambda \to +\infty} ||\phi_{\lambda,\varepsilon}||_X = 0\), then we have for free that \(\lim_{\lambda \to +\infty} ||\phi_{\lambda,\varepsilon}||_\infty = 0\).

We have now a first variational principle; indeed, it is easy to see that the critical points of the \(C^2\) functional
\[
\mathcal{J}_{\lambda,\varepsilon}(u, \phi) = \frac{1}{2} ||u||_{H^1}^2 + \frac{1}{2} \int_{\mathbb{R}^3} \phi u^2 - \lambda \int_{\mathbb{R}^3} F(x, u) - \frac{1}{2^*} \int_{\mathbb{R}^3} |u|^{2^*} - \frac{1}{4} \int_{\mathbb{R}^3} |\nabla \phi|^2 - \frac{\varepsilon^4}{8} \int_{\mathbb{R}^3} |\nabla \phi|^4
\]
on \(H^1(\mathbb{R}^N) \times X\) are exactly the weak solutions of \((P_{\lambda,\varepsilon})\), according to (1.2) and (1.3). However since this functional \(\mathcal{J}_{\lambda,\varepsilon}\) is strongly indefinite, we adopt a reduction procedure which is successfully used with the “classical” Schrödinger-Poisson system.
3.1. Study of the quasilinear Schrödinger-Poisson equation. Let us consider for convenience the following general problem

\begin{equation}
- \Delta \phi - \Delta_4 \phi = g \in X'.
\end{equation}

This problem has a unique solution \( \phi_g \). This follows by the fact that the \( C^1 \) functional

\[
\phi \in X \mapsto \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \phi|^2 + \frac{1}{4} \int_{\mathbb{R}^3} |\nabla \phi|^4 - g[\phi] \in \mathbb{R}
\]

is strictly convex, coercive and weakly lower semicontinuous; hence possess a unique critical point, denoted with \( \phi(g) \), which is a minimum and a solution of (3.2). Alternatively, the existence of a unique solution \( \phi(g) \) can be deduced by using the Minty-Browder’s Theorem [7, Teorema V. 15], since the operator \( T : X \to X' \) defined by duality by

\[
\langle T(\phi), \xi \rangle := \int_{\mathbb{R}^3} \nabla \phi \nabla \xi + \int_{\mathbb{R}^3} |\nabla \phi|^2 \nabla \phi \nabla \xi,
\]

is continuous, strictly monotone and coercive. Anyway, from unicity result, it is well defined the solution operator

\[
\Phi : X' \to X, \quad \Phi(g) := \phi(g)
\]

associated to equation (3.2).

In the next result, we show that the solution operator \( \Phi \) is continuous.

**Lemma 1.** Let \( g_n \to g \) in \( X' \). Then, we have

\[
\int_{\mathbb{R}^3} |\nabla \phi(g_n)|^2 \to \int_{\mathbb{R}^3} |\nabla \phi(g)|^2, \quad \int_{\mathbb{R}^3} |\nabla \phi(g_n)|^4 \to \int_{\mathbb{R}^3} |\nabla \phi(g)|^4
\]

and consequently

\[
\phi(g_n) \to \phi(g) \quad \text{in} \quad L^\infty(\mathbb{R}^3).
\]

In particular the operator \( \Phi \) is continuous.

**Proof.** By assumptions for every \( w \in X \),

\[
\int_{\mathbb{R}^3} \nabla \phi(g_n) \nabla w + \int_{\mathbb{R}^3} |\nabla \phi(g_n)|^2 \nabla \phi(g_n) \nabla w = g_n[w]
\]

and

\[
\int_{\mathbb{R}^3} \nabla \phi(g) \nabla w + \int_{\mathbb{R}^3} |\nabla \phi(g)|^2 \nabla \phi(g) \nabla w = g[w].
\]

We conclude that

\[
\int_{\mathbb{R}^3} \nabla \phi(g_n) \nabla w + \int_{\mathbb{R}^3} |\nabla \phi(g_n)|^2 \nabla \phi(g_n) \nabla w - \int_{\mathbb{R}^3} \nabla \phi(g) \nabla w - \int_{\mathbb{R}^3} |\nabla \phi(g)|^2 \nabla \phi(g) \nabla w = o_n(1).
\]

Considering \( w = \phi(g_n) - \phi(g) \), we derive

\[
\int_{\mathbb{R}^3} |\nabla \phi(g_n) - \nabla \phi(g)|^2 + \int_{\mathbb{R}^3} (|\nabla \phi(g_n)|^2 \nabla \phi(g_n) - |\nabla \phi(g)|^2 \nabla \phi(g)) \quad (\nabla \phi(g_n) - \nabla \phi(g)) = o_n(1)
\]

and then by the Simon inequality there exists \( C > 0 \) such that

\[
\int_{\mathbb{R}^3} |\nabla \phi(g_n) - \nabla \phi(g)|^2 + C \int_{\mathbb{R}^3} |\nabla \phi(g_n) - \nabla \phi(g)|^4 \leq \int_{\mathbb{R}^3} |\nabla \phi(g_n) - \nabla \phi(g)|^2 + \int_{\mathbb{R}^3} (|\nabla \phi(g_n)|^2 \nabla \phi(g_n) - |\nabla \phi(g)|^2 \nabla \phi(g)) \quad (\nabla \phi(g_n) - \nabla \phi(g)) = o_n(1).
\]

which concludes the proof. \( \square \)

Of course all that we have seen here also holds for the problem

\[
- \Delta \phi - \varepsilon^4 \Delta_4 \phi = g \in X',
\]

by considering the map \( \Phi_\varepsilon \), for every \( \varepsilon > 0 \).
3.2. The reduction argument. Let us consider now a particular case of the previous subsection. Let \( u \in H^1(\mathbb{R}^3) \) and note that \( u^2 \in X' \) in the sense that the map
\[
g_{u^2} : \phi \in X \mapsto \int_{\mathbb{R}^3} \phi u^2 \in \mathbb{R}
\]
is linear and continuous. Then for every \( u \in H^1(\mathbb{R}^3) \) fixed, there exists a unique element in \( X \), that we denote with \( \phi_\varepsilon(u) \), such that
\[
-\Delta \phi_\varepsilon(u) - \varepsilon^4 \Delta \phi_\varepsilon(u) = u^2 \quad \text{in} \ \mathbb{R}^3.
\]
In the remaining of the paper, \( \phi_\varepsilon(u) \) will always denote the unique solution of (3.3), which, en passant, satisfies
\[
\int_{\mathbb{R}^3} |\nabla \phi_\varepsilon(u)|^2 + \varepsilon^4 \int_{\mathbb{R}^3} |\nabla \phi_\varepsilon(u)|^4 = \int_{\mathbb{R}^3} \phi_\varepsilon(u) u^2.
\]
In particular we have the following.

Lemma 2. If \( \{u_n\} \) converges to \( u \) in \( L^{12/5}(\mathbb{R}^3) \), then, for every fixed \( \varepsilon > 0 \):

(a) \( \lim_{n \to +\infty} \int_{\mathbb{R}^3} |\nabla \phi_\varepsilon(u_n)|^2 = \int_{\mathbb{R}^3} |\nabla \phi_\varepsilon(u)|^2 \),

(b) \( \lim_{n \to +\infty} \int_{\mathbb{R}^3} |\nabla \phi_\varepsilon(u_n)|^4 = \int_{\mathbb{R}^3} |\nabla \phi_\varepsilon(u)|^4 \),

(c) \( \lim_{n \to +\infty} \int_{\mathbb{R}^3} \phi_\varepsilon(u_n) u_n^2 = \int_{\mathbb{R}^3} \phi_\varepsilon(u) u^2 \),

(d) \( \lim_{n \to +\infty} \phi_\varepsilon(u_n) = \phi_\varepsilon(u) \) in \( L^\infty(\mathbb{R}^3) \).

Proof. Under our assumptions we have,
\[
\|g_{u_n^2} - g_{u^2}\| = \sup_{\|\phi\| = 1} \left| \int_{\mathbb{R}^3} \phi(u_n^2 - u^2) \right| \leq \|\phi\|_2 \|u_n^2 - u^2\|_{2'} \leq C \|u_n^2 - u^2\|_{6/5} \to 0.
\]
Then we can apply Lemma 1 and conclude the proof. \( \square \)

We introduce the map
\[
\Phi_0 : u \in H^1(\mathbb{R}^3) \mapsto \phi_0(u) \in D^{1,2}(\mathbb{R}^3).
\]
Many properties of this map are well known, in particular (a) and (c) of Lemma 2.

The next result is a consequence of the fact that \( J_{\lambda,\varepsilon} \) is \( C^2 \) and the Implicit Function Theorem. The arguments used to prove Lemma 3 and Lemma 4 are exactly the same as in [4] for the Schrödinger-Poisson system (that is for the map \( \Phi_0 \) defined above), or [5] for the Klein-Gordon-Maxwell system.

Lemma 3. For all \( \varepsilon > 0 \), let \( G_{\Phi_\varepsilon} \) be the graph of the map \( \Phi_\varepsilon : u \in H^1(\mathbb{R}^3) \mapsto \phi_\varepsilon(u) \in X \). Then
\[
G_{\Phi_\varepsilon} = \{ (u, \phi) \in H^1(\mathbb{R}^3) \times X : \partial_\phi J_{\lambda,\varepsilon}(u, \phi) = 0 \}.
\]
Moreover
\[
\Phi_\varepsilon \in C^1(H^1(\mathbb{R}^3); X).
\]
In view of this, the functional (recall (3.4))
\[
J_{\lambda,\varepsilon}(u) := J_{\lambda,\varepsilon}(u, \phi_\varepsilon(u)) \quad \text{is of class} \ C^1 \ \text{and in particular we have}
\]
\[
J_{\lambda,\varepsilon}'(u)[v] = \partial_u J_{\lambda,\varepsilon}(u, \phi_\varepsilon(u))[v] + \partial_\phi J_{\lambda,\varepsilon}(u, \phi_\varepsilon(u)) \circ \Phi_\varepsilon(u)[v]
\]
\[
= \partial_u J_{\lambda,\varepsilon}(u, \phi_\varepsilon(u))[v].
\]
Then by (3.1) we have
\[ J_{\lambda,\varepsilon}'(u)[v] = \int_{\mathbb{R}^3} \nabla u \nabla v + \int_{\mathbb{R}^3} uv + \int_{\mathbb{R}^3} \phi_\varepsilon(u)uv - \lambda \int_{\mathbb{R}^3} f(x,u)v - \int_{\mathbb{R}^3} |u|^{2^* - 2}uv \]
from which a second variational principle holds:

**Lemma 4.** Let \( \lambda, \varepsilon > 0 \) be fixed. The following statements are equivalent:

(i) the pair \((u_{\lambda,\varepsilon}, \phi_{\lambda,\varepsilon})\) \(\in H^1(\mathbb{R}^3) \times X\) is a critical point of \(J_{\lambda,\varepsilon}\) (i.e. \((u_{\lambda,\varepsilon}, \phi_{\lambda,\varepsilon})\) is a solution of \((P_{\lambda,\varepsilon})\)),

(ii) \(u_{\lambda,\varepsilon}\) is a critical point of \(J_{\lambda,\varepsilon}\) and \(\phi_{\lambda,\varepsilon} = \phi_\varepsilon(u_\lambda)\).

The functional \(J_{\lambda,\varepsilon}\) of the unique variable \(u\) obtained by \(J_{\lambda,\varepsilon}\) is usually called the reduced functional.

In view of Lemma 4, the critical points of \(J_{\lambda,\varepsilon}\) satisfy the equation
\[ -\Delta u + u + \phi_\varepsilon(u)u = \lambda f(x,u) + |u|^{2^* - 2}u \quad \text{in} \quad \mathbb{R}^3, \]
which is the equation we are going to consider in the following.

It will be convenient to introduce the functional
\[ I_\varepsilon : u \in H^1(\mathbb{R}^3) \mapsto \frac{1}{4} \int_{\mathbb{R}^3} |\nabla \phi_\varepsilon(u)|^2 + \frac{3\varepsilon^4}{8} \int_{\mathbb{R}^3} |\nabla \phi_\varepsilon(u)|^4 \in \mathbb{R} \]
in such a way that we can write
\[ J_{\lambda,\varepsilon}(u) = \frac{1}{2} \|u\|_{H^1}^2 + I_\varepsilon(u) - \lambda \int_{\mathbb{R}^3} F(x,u) - \frac{1}{2^*} \int_{\mathbb{R}^3} |u|^{2^*}. \]
With this notation it is
\[ I_0(u) = \frac{1}{4} \int_{\mathbb{R}^3} |\nabla \phi_0(u)|^2, \]
where of course \(-\Delta \phi_0(u) = u^2 \in \mathbb{R}^3\).

**Remark 1.** We observe that in [6] it has been proved by hand that \(J_{\lambda,\varepsilon}\) is \(C^1\) and that its critical points are solutions of (3.5). Of course the nontrivial part there was to show that \(I_\varepsilon\) is \(C^1\) with Fréchet derivative \(I_\varepsilon'(u)\) given by
\[ \forall v \in H^1(\mathbb{R}^3) : I_\varepsilon'(u)[v] = \int_{\mathbb{R}^3} \phi_\varepsilon(u)uv. \]
In other words, it gives rise exactly to the nonlocal term \(\phi_\varepsilon(u)u\) in the equation (3.5). See [6, Proposition 4.1].

**Remark 2.** A useful consequence of the differentiability of \(I_\varepsilon\) is that, if \(v \in H^1(\mathbb{R}^3)\) is fixed, then the function \(t \in (0, \infty) \mapsto I_\varepsilon(tv)\) is \(C^1\) with
\[ \frac{d}{dt} I_\varepsilon(tv) = I_\varepsilon'(tv)[v] = t \int_{\mathbb{R}^3} \phi_\varepsilon(tv)\psi^2. \]
In view of the above arguments, we are reduced to find a solution \(u_{\lambda,\varepsilon}\) of equation (3.5), that is a critical point of the functional \(J_{\lambda,\varepsilon}\).

### 4. The Truncated Functional

In order to overcome the lack of compactness and the “growth” of order 4 in \(I_\varepsilon\), let us define a truncation for the functional \(J_{\lambda,\varepsilon}\) in the following way. Consider a smooth cut-off function \(\psi : [0, +\infty) \to \mathbb{R}_+\) such that
\[
\begin{aligned}
\psi(t) &= 1, \quad t \in [0, 1], \\
0 &\leq \psi(t) \leq 1, \quad t \in (1, 2), \\
\psi(t) &= 0, \quad t \in [2, \infty), \\
|\psi'|_\infty &\leq 2.
\end{aligned}
\]
For each $T > 0$ we define $h_T(u) := \psi \left( \|u\|^2_{H^1}/T^2 \right)$ and the truncated functional $J^T_{\lambda, \varepsilon} : H^1(\mathbb{R}^3) \to \mathbb{R}$ given by

$$J^T_{\lambda, \varepsilon}(u) := \frac{1}{2} \|u\|^2_{H^1} + h_T(u) \left[ \frac{1}{4} \int_{\mathbb{R}^3} |\nabla \phi_\varepsilon(u)|^2 + \frac{3\varepsilon^2}{8} \int_{\mathbb{R}^3} |\nabla \phi_\varepsilon(u)|^4 \right] - \lambda \int_{\mathbb{R}^3} F(x, u) - \frac{1}{2^*} \int_{\mathbb{R}^3} |u|^{2^*}.$$ 

The functional $J^T_{\lambda, \varepsilon}$ is $C^1$ with differential given, for all $u, v \in H^1(\mathbb{R}^3)$, by

$$\langle J^T_{\lambda, \varepsilon}', (u, v) \rangle \triangleq \int_{\mathbb{R}^3} \phi_\varepsilon(u) u v - \lambda \int_{\mathbb{R}^3} f(x, u) v - \int |u|^{2^* - 2} v.$$ 

Then $u_{\lambda, \varepsilon} \in H^1(\mathbb{R}^3)$ is a critical point of $J^T_{\lambda, \varepsilon}$, if and only if the pair $(u_{\lambda, \varepsilon}, \phi_\varepsilon(u_{\lambda, \varepsilon})) \in H^1(\mathbb{R}^3) \times X$ is a weak solution of

$$\begin{cases}
(\Delta u + u) \left( 1 + \frac{2}{T^2} \psi' \left( \frac{\|u\|^2_{H^1}}{T^2} \right) \right) I_\varepsilon(u) + h_T(u) \phi_u = \lambda f(x, u) + |u|^{2^* - 2} u & \text{in } \mathbb{R}^3, \\
-\Delta \phi - \varepsilon^4 \Delta^4 \phi = u^2 & \text{in } \mathbb{R}^3.
\end{cases}$$

Let us observe the following

**Lemma 5.** Let $T, \lambda, \varepsilon > 0$ be fixed. For every $v \in H^1(\mathbb{R}^3) \setminus \{0\}$, the function

$$t \in [0, +\infty) \mapsto J^T_{\lambda, \varepsilon}(tv) \in \mathbb{R}$$

has a global maximum point which does not depend on $\varepsilon$ and is strictly positive. It will be denoted hereafter with $t^T_{\lambda}(v)$. Moreover,

$$\forall T > 0 : \lim_{\lambda \to +\infty} t^T_{\lambda}(v) = 0.$$ 

**Proof.** First of all let us see the existence of such $t^T_{\lambda}(v)$ for every $T, \lambda, \varepsilon > 0$.

It follows from (f1) and (f2) that, for each $\eta > 0$, there exists a positive constant $C(\eta)$ such that

$$F(x, t) \leq \eta \frac{1}{2} |t|^2 + \frac{1}{q} C(\eta) |t|^q.$$ 

Then, fixed $v \neq 0$,

$$J^T_{\lambda, \varepsilon}(tv) = \frac{t^2}{2} \|v\|^2_{H^1} + h_T(tv) I_\varepsilon(tv) - \lambda \int_{\mathbb{R}^3} F(x, tv) - \frac{t^{2^*}}{2^*} \int_{\mathbb{R}^3} |v|^{2^*} \geq \frac{t^2}{2} \|v\|^2_{H^1} - \lambda \int_{\mathbb{R}^3} F(x, tv) - \frac{t^{2^*}}{2^*} \int_{\mathbb{R}^3} |v|^{2^*}$$

and choosing $\eta$ sufficiently small and using the Sobolev embeddings, we have

$$J^T_{\lambda, \varepsilon}(tv) \geq \frac{t^2}{2} \left( 1 - C_1 \eta \lambda \|v\|^2_{H^1} - t^s C_2 \|v\|^q_{H^1} - t^{2^*} C_3 \|v\|^2_{H^1} \right) > 0$$

for small $t$. Here $C_i, i = 1, 2, 3$ are the embedding constants of $H^1(\mathbb{R}^3)$, respectively, into $L^2(\mathbb{R}^3), L^q(\mathbb{R}^3)$ and $L^{2^*}(\mathbb{R}^3)$. In the previous inequalities the dependence on $\varepsilon$ disappeared since the term involving $\varepsilon$ was thrown away being positive.

On the other hand it is easily seen that

$$\lim_{t \to +\infty} J^T_{\lambda, \varepsilon}(tv) = \lim_{t \to +\infty} \left( \frac{t^2}{2} \|v\|^2_{H^1} + h_T(tv) I_\varepsilon(tv) - \lambda \int_{\mathbb{R}^3} F(x, tv) - \frac{t^{2^*}}{2^*} \int_{\mathbb{R}^3} |v|^{2^*} \right) = -\infty$$

and again we avoid the dependence on $\varepsilon$ since for $s$ large $h_T(s) = 0$. Then the existence of $t^T_{\lambda}(v) > 0$, which does not depend on $\varepsilon$, is guaranteed.
Now let us prove the limit. We set for brevity $t_\lambda := t_\lambda^T(v)$. Such a $t_\lambda$ satisfies (recall Remark 2):

\begin{equation}
(4.3) \quad t_\lambda^2 + \frac{2T^2}{T^2} \left( \frac{t^2}{T^2} \right)^2 f_\lambda(t_\lambda v) + t_\lambda^2 \int_{\mathbb{R}^3} \phi_\varepsilon(t_\lambda v) v^2 = \lambda \int_{\mathbb{R}^3} f(x, t_\lambda v) t_\lambda v + t_\lambda^2 \int_{\mathbb{R}^3} |v|^2^*.
\end{equation}

and then, by (33),

$$t_\lambda^2 + t_\lambda^2 \left( \frac{t^2}{T^2} \right)^2 \int_{\mathbb{R}^3} \phi_\varepsilon(t_\lambda v) v^2 \geq t_\lambda^2 \int_{\mathbb{R}^3} |v|^2^*.$$ 

If it were $\lim_{\lambda \to +\infty} t_\lambda = +\infty$, then, there exists $\bar{\lambda} > 0$ such that for every $\lambda \geq \bar{\lambda}$ it is $t_\lambda > \sqrt{2T}$ and then the previous inequality becomes

$$t_\lambda^2 \geq t_\lambda^2 \int_{\mathbb{R}^3} |v|^2^*.$$ 

But this is impossible for $\lambda$ large. Thus, $\lim_{\lambda \to +\infty} t_\lambda = \beta > 0$. Of course we need to show that $\beta = 0$.

If $\beta > 0$, coming back to (4.3), and recalling that by Lemma 2 it is $\phi_\varepsilon(t_\lambda v) \to \phi_\varepsilon(\beta v)$ in $L^\infty(\mathbb{R}^3)$ as $\lambda \to +\infty$, we deduce

$$+\infty \leftarrow \lambda \int_{\mathbb{R}^3} f(x, t_\lambda v) t_\lambda v + t_\lambda^2 \int_{\mathbb{R}^3} |v|^2^* \leq \beta^2 + \beta^2 |\varepsilon|_{\infty} M + o_n(1) \quad \text{as} \quad \lambda \to +\infty.$$ 

which is an absurd. Thus we conclude that $\beta = 0$. This means that for every fixed $T > 0$ it is $\lim_{\lambda \to +\infty} t_\lambda^T(v) = 0$, completing the proof. 

\section{The Mountain Pass Geometry for $J_{T,\varepsilon}^x$.} In the sequel, we prove that the functional $J_{T,\varepsilon}^x$ has the Mountain Pass Geometry with some kind of uniformity with respect to the parameters. Observe that even $\varepsilon = 0$ is allowed, up to change $\phi_\varepsilon$ into $\phi_0$, the solution of the Poisson equation, and the function space $X$ into $D^{1,2}(\mathbb{R}^3)$.

\begin{lem}
Assume that conditions (f1) and (f2) hold. Then, for every $\lambda > 0$ there exists numbers $\rho_\lambda, \alpha_\lambda > 0$ such that,

$$\forall T > 0, \forall \varepsilon \geq 0 : \quad J_{T,\varepsilon}^x(u) \geq \alpha_\lambda, \quad \text{whenever} \quad \|u\|_{H^1} = \rho_\lambda.$$ 

We observe explicitly that indeed $\rho_\lambda, \alpha_\lambda$ does not depend on $T$ neither on $\varepsilon$; indeed in the proof we will simply throw away the term $h_T(u) I_\varepsilon(u)$, being positive.

\begin{proof}
Let $\lambda > 0$ be fixed. It follows from (f1) and (f2) that, for each $\eta > 0$, there exists a positive constant $C(\eta)$ such that

\begin{equation}
(4.4) \quad F(x, t) \leq \eta \frac{1}{2} |t|^2 + \frac{1}{q} C(\eta) |t|^q.
\end{equation}

By (4.4) we have

$$J_{T,\varepsilon}^x(u) \geq \frac{1}{2} \|u\|_{H^1}^2 - \frac{\eta}{2} \int_{\mathbb{R}^3} |u|^2 - \frac{1}{q} C(\eta) \lambda \int_{\mathbb{R}^3} |u|^q - \frac{1}{2^*} \int_{\mathbb{R}^3} |u|^{2^*}.$$ 

So, using the Sobolev Embedding Theorem, there is a positive constant $C > 0$ such that

$$J_{T,\varepsilon}^x(u) \geq C \|u\|_{H^1}^2 - \lambda C \|u\|_{H^1}^q - C \|u\|_{H^1}^{2^*}.$$ 

Since $2 < q < 2^*$, the result follows by choosing $\rho_\lambda > 0$ small enough.
\end{proof}

\begin{lem}
Assume that conditions (f1)-(f3) hold. Then for every $T > 0$, there exists $e_T \in H^1(\mathbb{R}^N)$ such that

$$\forall \lambda > 0, \forall \varepsilon \geq 0 : \quad J_{T,\varepsilon}^x(e_T) < 0 \quad \text{and} \quad \|e_T\|_{H^1} > \rho_\lambda,$$

where $\rho_\lambda$ is given in Lemma 6.
\end{lem}

Here the fact that $e_T$ does not depends on $\varepsilon$ is a consequence of the fact that the with the truncation we kill the term $I_\varepsilon$. 

Proof. Let $T > 0$ be fixed. Let now $v \in C_0^\infty(\mathbb{R}^N)$, positive, with $\|v\|_{H^1} = 1$. Using (3) and considering $t > 2T$, we get

$$J_{\lambda,\varepsilon}^T(tv) \leq \frac{1}{2} t^2 - \lambda t^2 \int_{\mathbb{R}^3} v^2 - \frac{t^2}{\varepsilon} \int_{\mathbb{R}^3} |v|^2 < \frac{1}{2} t^2 - \frac{t^2}{\varepsilon} \int_{\mathbb{R}^3} |v|^2.$$ 

Since $2 < \theta$, the result follows by choosing some $t_* > 2T$ large enough and setting $e_T := t_* v$. \qed

We recall that a sequence $\{u_n\} \subset H^1(\mathbb{R}^3)$ is a Palais-Smale sequence for the functional $J_{\lambda,\varepsilon}^T$ at the level $d \in \mathbb{R}$ if

$$J_{\lambda,\varepsilon}^T(u_n) \to d \quad \text{and} \quad (J_{\lambda,\varepsilon}^T)'(u_n) \to 0 \quad \text{in} \quad H^{-1}(\mathbb{R}^3).$$

If every Palais-Smale sequence of $J_{\lambda,\varepsilon}^T$ has a strong convergent subsequence, then one says that $J_{\lambda,\varepsilon}^T$ satisfies the Palais-Smale condition, or (PS) for short.

Then, since the functional $J_{\lambda,\varepsilon}^T$ satisfies the geometric assumptions of Mountain Pass Theorem (see [2]), we know that (see [19, p.12]), for every $T, \lambda > 0, \varepsilon \geq 0$ there exists a sequence $\{u_n\} \subset H^1(\mathbb{R}^3)$ satisfying

$$J_{\lambda,\varepsilon}^T(u_n) \to c_{\lambda,\varepsilon}^T > 0 \quad \text{and} \quad (J_{\lambda,\varepsilon}^T)'(u_n) \to 0,$$

where

$$c_{\lambda,\varepsilon}^T := \inf_{\gamma \in \Gamma_{\lambda,\varepsilon}} \max_{t \in [0,1]} J_{\lambda,\varepsilon}^T(\gamma(t)) > 0.$$

and

$$\Gamma_{\lambda,\varepsilon} := \{ \gamma \in C([0,1], H^1(\mathbb{R}^3)) : \gamma(0) = 0, \ J_{\lambda,\varepsilon}^T(\gamma(1)) < 0 \}.$$ 

It is clear that this sequence should depend also on $T, \lambda, \varepsilon$ but we omit this for simplicity. In other words, $\{u_n\}$ is a (PS) sequence at level $c_{\lambda,\varepsilon}^T$ for the functional $J_{\lambda,\varepsilon}^T$.

Observe that, since $e_T$ found in Lemma 7 does not depends on $\lambda$, neither on $\varepsilon$, by defining the path

$$\gamma_* : t \in [0,1] \mapsto t e_T \in H^1(\mathbb{R}^3)$$

we get $\gamma_* \in \bigcap_{\lambda > 0, \varepsilon \geq 0} \Gamma_{\lambda,\varepsilon}^T$.

Remark 3. Observe that the Mountain Pass structure of $J_{\lambda,\varepsilon}^T$ does not depend on $\varepsilon \geq 0$.

4.2. Estimates of $c_{\lambda,\varepsilon}^T$. Here we study the behaviour of the Mountain Pass levels $c_{\lambda,\varepsilon}^T$ with respect to $\lambda$, whenever $T, \varepsilon$ are fixed. This will be fundamental in order to prove that the (PS) sequences at level $c_{\lambda,\varepsilon}^T$ are bounded for $\lambda$ large.

Actually the next result is again independent on $\varepsilon \geq 0$, which is not surprising in view of Remark 3.

Lemma 8. If the conditions (ii)-(iii) hold, then

$$\forall T > 0 : \lim_{\lambda \to +\infty} \sup_{\varepsilon \geq 0} c_{\lambda,\varepsilon}^T = 0.$$ 

Proof. Let $T > 0$ be fixed. We prove that for every $\eta > 0$ there exists $\tilde{\lambda} > 0$ such that

$$\forall \lambda > \tilde{\lambda} : \quad 0 < \max_{t \in [0,1]} J_{\lambda,\varepsilon}^T(\gamma_*(t)) < \eta, \quad \forall \varepsilon \geq 0.$$ 

This of course will give the conclusion.

Then let us fix $\eta > 0$. Let $v \in C_0^\infty(\mathbb{R}^N), v \geq 0$ with $\|v\| = 1$ be the same function fixed in the proof of Lemma 7. By Lemma 5 there exists $t_\lambda^T = t_\lambda^T(v) > 0$ verifying $J_{\lambda,\varepsilon}^T(t_\lambda^T v) = \max_{t \geq 0} J_{\lambda,\varepsilon}^T(t v)$ and $\lim_{\lambda \to +\infty} \sup_{\varepsilon \geq 0} t_\lambda^T = 0$. Hence, due to the continuity of the maps $h_T$ and $I_\varepsilon$ we get the uniform limits in $\varepsilon \geq 0$:

$$\lim_{\lambda \to +\infty} h_T(t_\lambda^T v) = 1, \quad \lim_{\lambda \to +\infty} I_\varepsilon(t_\lambda^T v) = 0.$$ 

Then there exists $\tilde{\lambda} > 0$ such that

$$\forall \lambda > \tilde{\lambda} : \quad \frac{1}{2} (t_\lambda^T)^2 + h_T(t_\lambda^T v) I_\varepsilon(t_\lambda^T v) < \eta, \quad \forall \varepsilon \geq 0.$$
Since, as we know, for $\lambda > \bar{\lambda}$ it is (see (4.5)) $\gamma_s \in \cap_{\varepsilon \geq 0} \Gamma_{\lambda, \varepsilon}^T$, we get the following estimate:

$$0 < \max_{t \in [0, 1]} J_{\lambda, \varepsilon}^T(\gamma_s(t)) = J_{\lambda, \varepsilon}^T(t_{\lambda}^T v)$$

$$\leq \frac{1}{2}(t_{\lambda}^T)^2 + h_{T}(t_{\lambda}^T v) \left[ \frac{1}{4} \int_{\mathbb{R}^3} |\nabla \phi_{\varepsilon}(t_{\lambda}^T v)|^2 + \frac{3\varepsilon^4}{8} \int_{\mathbb{R}^3} |\nabla \phi_{\varepsilon}(t_{\lambda}^T v)|^4 \right]$$

$$= \frac{1}{2}(t_{\lambda}^T)^2 + h_{T}(t_{\lambda}^T v) \lambda_{\varepsilon} \varepsilon^2 (t_{\lambda}^T v)$$

$$< \eta$$

concluding the proof. \qed

We remark explicitly the important fact that the limit in Lemma 8 is uniform in $\varepsilon \geq 0$.

Thanks to the previous Lemma we have the following important result. Recall that $\theta \in (4, 2^*)$ is the constant given in the Ambrosetti-Rabinowitz condition (3).

Lemma 9. Let $T > 0$ be fixed and let $\lambda$ sufficiently large, let us say $\lambda \geq \lambda(T)$, such that

$$\sup_{\varepsilon \geq 0} c_{\lambda, \varepsilon}^T < \frac{\theta - 2}{2\theta} T^2,$$

(this is possible, in view of the previous Lemma 8).

Then, given $\varepsilon \geq 0$, any $(PS)$ sequence $\{u_n\}$ (we do not write the dependence on $T, \lambda, \varepsilon$) at level $c_{\lambda, \varepsilon}^T$ for the functional $J_{\lambda, \varepsilon}$ is bounded; more precisely it satisfies, up to subsequence, $\|u_n\|_{H^1} \leq T$. In particular, for all $\lambda \geq \lambda(T), \varepsilon \geq 0$, the sequence $\{u_n\}$ is such that

$$J_{\lambda, \varepsilon}(u_n) \to c_{\lambda, \varepsilon}, \quad J_{\lambda, \varepsilon}'(u_n) \to 0,$$

that is, it is a $(PS)$ sequence at the Mountain Pass level $c_{\lambda, \varepsilon}$ for the untruncated functional $J_{\lambda, \varepsilon}$.

Proof. Given $\varepsilon \geq 0$, let us first show that $\{u_n\}$ is bounded by $2T^2$. Assume by contradiction that there exists a subsequence of $\{u_n\}$, still denoted with $\{u_n\}$, such that $\|u_n\|_{H^1}^2 > 2T^2$. Taking into account (4.1) and (3), it follows that

$$c_{\lambda, \varepsilon}^T = J_{\lambda, \varepsilon}^T(u_n) - \frac{1}{\theta} (J_{\lambda, \varepsilon}^T)'(u_n)[u_n] + o_n(1)$$

$$\geq \frac{\theta - 2}{\theta} \|u_n\|_{H^1}^2 + \psi \left( \frac{\|u_n\|_{H^1}^2}{T^2} \right) \left[ I_{\varepsilon}(u_n) - \frac{1}{\theta} \int_{\mathbb{R}^3} \phi_{\varepsilon}(u_n) u_n^2 \right]$$

$$- \frac{2}{\theta T^2} \psi' \left( \frac{\|u_n\|_{H^1}^2}{T^2} \right) \|u_n\|_{H^1}^2 I_{\varepsilon}(u_n) + o_n(1)$$

$$\geq \frac{\theta - 2}{\theta} T^2 + o_n(1)$$

being $\psi' \leq 0$. This is a contradiction and proves that $\|u_n\|_{H^1}^2 \leq 2T^2$. We can prove now the Lemma. Assume by contradiction that $T^2 < \|u_n\|_{H^1}^2 \leq 2T^2$. We have, with similar computations as before and using that $\psi$ is decreasing, that

$$c_{\lambda, \varepsilon}^T = J_{\lambda, \varepsilon}^T(u_n) - \frac{1}{\theta} (J_{\lambda, \varepsilon}^T)'(u_n)[u_n] + o_n(1)$$

$$\geq \frac{\theta - 2}{2\theta} T^2 + \psi(2) \left[ I_{\varepsilon}(u_n) - \frac{1}{\theta} \int_{\mathbb{R}^3} \phi_{\varepsilon}(u_n) u_n^2 \right] + o_n(1)$$

$$= \frac{\theta - 2}{2\theta} T^2 + o_n(1)$$

which contrasts with the assumption and conclude the proof. \qed
5. Proof of Theorem 1

Here we prove Theorem 1 hence $T$ and $\varepsilon$ have to be considered fixed. From Lemma 8 there exists $\lambda'(T) > 0$ (actually which does not depend on $\varepsilon$ being the limit in Lemma 8 uniform in $\varepsilon$) such that

$$\forall \lambda \geq \lambda'(T) : \frac{2^* - \theta}{2^*} S^{3/2} \varepsilon \geq 0. \tag{5.1}$$

Here, $S$ is the best constant for the embedding $H^1(\mathbb{R}^3) \hookrightarrow L^{2^*}(\mathbb{R}^3)$.

In the remaining of this Section, the fact that inequality (5.1) is independent on $\varepsilon$ will not be used. However it will be important in the final Section 7.

Now, fix $\lambda \geq \max \{ \lambda'(T), \lambda(T) \}$ where $\lambda(T)$ is given in Lemma 9. Let us show that the truncated functional $J_{\lambda, \varepsilon}^T$ admits a critical point with norm less then $T$; then this will be a critical point of $J_{\lambda, \varepsilon}$ and hence a solution of our problem ($P_{\lambda, \varepsilon}$).

From Lemmas 6, 7 and 9 there exists a bounded ($PS$) sequence $\{u_n\} \subset H^1(\mathbb{R}^3)$ at level $c_{\lambda, \varepsilon}$ for the functional $J_{\lambda, \varepsilon}^T$. Since the sequence $\{u_n\}$ verifies also $\|u_n\|_{H^1} \leq T$, then it is actually a ($PS$) sequence for the functional $J_{\lambda, \varepsilon}$ at level $c_{\lambda, \varepsilon} = c_{\lambda, \varepsilon}^T$ and we can assume that there exists $u_{\lambda, \varepsilon}^T \in H^1(\mathbb{R}^3)$ such that $u_n \rightharpoonup u_{\lambda, \varepsilon}^T$ in $H^1(\mathbb{R}^3)$ and $\|u_{\lambda, \varepsilon}^T\|_{H^1} \leq T$.

We show now the following

Claim: $\|u_n\|_{H^1} \to \|u_{\lambda, \varepsilon}^T\|_{H^1}$ as $n \to \infty$.

It order to prove the Claim we suppose, up to a subsequence, that

$$|\nabla u_n|^2 \to |\nabla u_{\lambda, \varepsilon}^T|^2 + \mu \quad \text{and} \quad |u_n|^2 \to |u_{\lambda, \varepsilon}^T|^2 + \nu \quad \text{(weak$^*$sense of measures).} \tag{5.2}$$

Using the Concentration Compactness Principle due to Lions (see [15, Lemma 2.1]), we get the existence of a set, at most countable $\Lambda$, sequences $\{x_i\}_{i \in \Lambda} \subset \mathbb{R}^3$, $\{\nu_i\}_{i \in \Lambda}$, $\{\mu_i\}_{i \in \Lambda} \subset [0, \infty)$, such that

$$\nu = \sum_{i \in \Lambda} \nu_i \delta_{x_i}, \quad \mu \geq \sum_{i \in \Lambda} \mu_i \delta_{x_i}, \quad \text{and} \quad S\nu_i^{2/2} \leq \mu_i \quad \forall i \in \Lambda, \tag{5.3}$$

where $\delta_{x_i}$ is the Dirac mass centered in $x_i \in \mathbb{R}^3$.

Note that, if it were $\nu_i \geq S^{3/2}$ for some $i \in \Lambda$, since $\{u_n\}$ is a $(PS)$ sequence for $J_{\lambda, \varepsilon}$ at level $c_{\lambda, \varepsilon}$, we have

$$c_{\lambda, \varepsilon} = J_{\lambda, \varepsilon}(u_n) - \frac{1}{\theta} \lambda J_{\lambda, \varepsilon}(u_n)[u_n] + o_n(1)$$

$$= \frac{\theta - 2}{\theta} \frac{\|u_n\|^2}{H^1} + \frac{\theta - 4}{4\theta} \int_{\mathbb{R}^3} |\nabla \phi(x, u_n)|^2 + \frac{8 - 3\theta}{8\theta} \int_{\mathbb{R}^3} |\nabla \phi(x, u_n)|^4$$

$$+ \lambda \int_{\mathbb{R}^3} \left( \frac{1}{\theta} f(x, u_n)u_n - F(x, u_n) \right) + \frac{2^* - \theta}{2^*} \int_{\mathbb{R}^3} |u_n|^2$$

$$\geq \frac{2^* - \theta}{2^*} \int_{\mathbb{R}^3} |u_n|^2 \psi_{\rho}$$

Then, passing to the limit in $n$,

$$c_{\lambda, \varepsilon} \geq \frac{2^* - \theta}{2^*} \int_{\mathbb{R}^3} |u_{\lambda, \varepsilon}^T|^2 + \int_{\mathbb{R}^3} \sum_{i \in \Lambda} \delta_{x_i} \psi_{r_i} \geq \frac{2^* - \theta}{2^*} \nu_i \geq \frac{2^* - \theta}{2^*} S^{3/2}$$

which is absurd for our choice of $\lambda$. Thus it has necessarily to be

$$\forall i \in \Lambda : \nu_i < S^{3/2}. \tag{5.4}$$

On the other hand, fix $i \in \Lambda$. Consider $\psi \in C_0^\infty(\mathbb{R}^3, [0, 1])$ such that $\psi \equiv 1$ on $B_1(0)$, $\psi \equiv 0$ on $\mathbb{R}^3 \setminus B_2(0)$ and $|\nabla \psi|_\infty \leq 2$. Defining $\psi_r(x) := \psi((x - x_i)/r)$ where $r > 0$, we have that $\{\psi_r u_n\}$ is in $H^1(\mathbb{R}^3)$. Since $\|u_n\| \leq T$, it holds $J_{\lambda, \varepsilon}^T(u_n)[\psi_r u_n] \to 0$, explicitly,

$$\int_{\mathbb{R}^3} u_n \nabla u_n \nabla \psi_r + \int_{\mathbb{R}^3} |\nabla u_n|^2 \psi_r + \int_{\mathbb{R}^3} \phi u_n u_n^2 \psi_r - \lambda \int_{\mathbb{R}^3} f(x, u_n)u_n \psi_r - \int_{\mathbb{R}^3} |u_n|^2 \psi_r = o_n(1) \tag{5.5}$$
Let us pass to the limit, first as \( n \to \infty \) and then as \( r \to 0 \), in (5.5). We first note that
\[
\left| \int_{\mathbb{R}^3} u_n \nabla u_n \nabla \psi_r \right| \leq \int_{B_{2r}(x_i)} |\nabla u_n| |u_n \nabla \psi_r| \leq C \left( \int_{B_{2r}(x_i)} |u_n \nabla \psi_r|^2 \right)^{1/2},
\]
and then
\[
\lim_{n \to +\infty} \sup_{r \to 0} \left| \int_{\mathbb{R}^3} u_n \nabla u_n \nabla \psi_r \right| \leq C \left( \int_{B_{2r}(x_i)} |u \nabla \psi_r|^2 \right)^{1/2}
\]
from which
\[
\lim_{n \to +\infty} \left( \lim_{r \to 0} \sup_{r \to 0} \left| \int_{\mathbb{R}^3} u_n \nabla u_n \nabla \psi_r \right| \right) = 0.
\]
Analogously it is easy to see that
\[
\lim_{n \to +\infty} \left( \lim_{r \to 0} \sup_{r \to 0} \int_{\mathbb{R}^3} \phi_\varepsilon(u_n) u_n^2 \psi_r \right) = \lim_{n \to +\infty} \left( \lim_{r \to 0} \sup_{r \to 0} \int_{\mathbb{R}^3} f(x, u_n) u_n \psi_r \right) = 0
\]
Moreover we have
\[
\lim_{n \to +\infty} \int_{\mathbb{R}^3} |\nabla u_n|^2 \psi_r \geq \int_{\mathbb{R}^3} \psi_r d\mu
\]
\[
\lim_{n \to +\infty} \int_{\mathbb{R}^3} |u_n|^2 \psi_r \geq \int_{\mathbb{R}^3} \psi_r d\nu
\]
in (5.5), taking into account (5.2), (5.6) and (5.7), by (5.5) we deduce
\[
\int_{\mathbb{R}^3} \psi_r d\nu \geq \int_{\mathbb{R}^3} \psi_r d\mu + o_r(1).
\]
But then passing to the limit as \( r \to 0 \) we get \( \nu_i \geq \mu_i \) and by (5.3), we infer that
\[
\nu_i \geq S^{3/2}.
\]
This of course contrasts with (5.4) and gives that \( \Lambda = 0 \).

As a consequence of this, \( u_n \to u_{\lambda, \varepsilon}^T \) in \( L^2(\mathbb{R}^3) \), from which we deduce in a standard way that that
\[
\|u_n\|_{H^1} \to \|u_{\lambda, \varepsilon}^T\|_{H^1},
\]
proving the Claim.

Then \( u_n \to u_{\lambda, \varepsilon}^T \) in \( H^1(\mathbb{R}^3) \) and hence since the functional \( J_{\lambda, \varepsilon} \) is \( C^1 \) and \( \|u_n\|_{H^1} \leq T \):
\[
J_{\lambda, \varepsilon}(u_n) = J_{\lambda, \varepsilon}^T(u_n) \to J_{\lambda, \varepsilon}(u_{\lambda, \varepsilon}^T) = c_{\lambda, \varepsilon}^T = c_{\lambda, \varepsilon} \quad \text{and} \quad J'_{\lambda, \varepsilon}(u_n) = (J_{\lambda, \varepsilon}^T)'(u_n) \to (J_{\lambda, \varepsilon}^T)'(u_{\lambda, \varepsilon}^T) = 0
\]
i.e.
\[
J_{\lambda, \varepsilon}(u_{\lambda, \varepsilon}^T) = c_{\lambda, \varepsilon} > 0 \quad \text{and} \quad J'_{\lambda, \varepsilon}(u_{\lambda, \varepsilon}^T) = 0,
\]
showing that \( u_{\lambda, \varepsilon}^T \) is the solution of (3.5) we were looking for.

The first part of Theorem 1 is proved, with
\[
\lambda^* := \max\{\lambda'(T), \lambda(T)\}, \quad u_{\lambda, \varepsilon} := u_{\lambda, \varepsilon}^T, \quad \Phi_{\lambda, \varepsilon} := \Phi_{\varepsilon}(u_{\lambda, \varepsilon}) = \phi_{\varepsilon}(u_{\lambda, \varepsilon}).
\]

For what concerns the positivity of the solutions, we observe that for every \( u \in H^1(\mathbb{R}^3) \), the solution \( \phi_{\varepsilon}(u) \) of the second equation in \( (P_{\lambda, \varepsilon}) \) is nonnegative; indeed this is easily seen by multiplying the second equation by \( \phi_{\varepsilon}(u)^- := \max\{-\phi_{\varepsilon}(u), 0\} \) and integrating; we arrive at
\[
\int_{\mathbb{R}^3} |\nabla \phi_{\varepsilon}(u)|^2 + \int_{\mathbb{R}^3} |\nabla \phi_{\varepsilon}(u)^-|^4 \leq 0
\]
and the conclusion follows. Then, having \( \phi_{\varepsilon}(u) \geq 0 \) we see analogously that the solution \( u_{\lambda, \varepsilon} \) of
\[
-\Delta u + u + \phi_{\varepsilon}(u) u = \lambda f(x, u) + |u|^{2^* - 2} u
\]
found above has to be nonnegative, being \( f(x, t) = 0 \) for \( t \leq 0 \).
Finally by similar computations as in the proof of Lemma 9 we have that, fixed $\varepsilon > 0$:

$$0 = \lim_{\lambda \to +\infty} c_{\lambda, \varepsilon} = J_{\lambda, \varepsilon}(u_{\lambda, \varepsilon}) - \frac{1}{\theta} J'_{\lambda, \varepsilon}(u_{\lambda, \varepsilon})|u_{\lambda, \varepsilon}| \geq \frac{\theta - 2}{\theta} \|u_{\lambda, \varepsilon}\|^2_{H^1}.$$ 

Then $\lim_{\lambda \to +\infty} u_{\lambda, \varepsilon} = 0$ in $H^1(\mathbb{R}^3)$ and by the continuity of the map $\Phi_{\varepsilon}$ defined in Lemma 3, we get also $\lim_{\lambda \to +\infty} \|\phi_{\lambda, \varepsilon}\|_X = 0$. As we have already said, the fact that $\lim_{\lambda \to +\infty} |\phi_{\lambda, \varepsilon}| = 0$ follows by the continuous embedding of the space $X$ into $L^\infty(\mathbb{R}^3)$.

Theorem 1 is completely proved.

**Remark 4.** Actually we have proved an additional property on the solution of (3.5). Indeed our method shows that, for every $T > 0$ there exists a $\lambda^* = \lambda^*(T)$ such that for all $\lambda > \lambda^*(T)$ and $\varepsilon \geq 0$ there exists a solution $u_{\lambda, \varepsilon}$ of (3.5) with has norm less then $T$.

6. Proof of Theorem 2

From now on we fix the parameter $\lambda$ greater then $\lambda^* = \max\{\lambda'(T), \lambda(T)\}$. Our aim now is to show the behaviour of the solutions $u_{\lambda, \varepsilon}$ with respect to $\varepsilon$.

Let us begin to show that $\{u_{\lambda, \varepsilon}\}_{\varepsilon \geq 0}$ is bounded. We know that

$$J_{\lambda, \varepsilon}(u_{\lambda, \varepsilon}) = c_{\lambda, \varepsilon}, \quad J'_{\lambda, \varepsilon}(u_{\lambda, \varepsilon}) = 0.$$ 

Moreover, for this fixed $\lambda > 0$ we can invoke (5.1) and obtain that

$$\forall \varepsilon > 0: \quad 0 < c_{\lambda, \varepsilon} \leq \frac{2^* - \theta}{2^* \theta} S^{3/2}.$$ 

Then if we assume that $\lim_{\varepsilon \to 0^+} \|u_{\lambda, \varepsilon}\|_{H^1} = +\infty$, exactly as in the proof of Lemma 9 (where we can replace $J'_{\lambda, \varepsilon}$ with $J_{\lambda, \varepsilon}$), by (6.1) we have:

$$c_{\lambda, \varepsilon} = J_{\lambda, \varepsilon}(u_{\lambda, \varepsilon}) = \frac{1}{\theta}(J_{\lambda, \varepsilon})'(u_{\lambda, \varepsilon})|u_{\lambda, \varepsilon}| + o_\varepsilon(1) \geq \frac{1}{2} \|u_{\lambda, \varepsilon}\|^2_{H^1} + o_\varepsilon(1)$$

which contrasts with (6.2). Then, there exists $u_{\lambda, 0} \in H^1(\mathbb{R}^3)$ such that up to subsequence,

$$\lim_{\varepsilon \to 0^+} u_{\lambda, \varepsilon} = u_{\lambda, 0} \quad \text{in} \quad H^1(\mathbb{R}^3) \quad \text{as} \quad \varepsilon \to 0^+.$$ 

The fact that this convergence is strong, is done exactly in a straightforward way as in Section 5. This is based on the fact that the inequality in (5.1) is true for every $\varepsilon$. Then the proof follows as before by replacing the limits in $n$ with limits with respect to $\varepsilon$. In this way we first obtain the strong convergence into $L^2(\mathbb{R}^3)$, and then

$$\lim_{\varepsilon \to 0^+} u_{\lambda, \varepsilon} = u_{\lambda, 0} \quad \text{in} \quad H^1(\mathbb{R}^3).$$ 

In particular we have that $u_{\lambda, \varepsilon} \to u_{\lambda, 0}$ in $L^{6/5}(\mathbb{R}^3)$.

At this point we recall the following result which we rewrite adapted to our notations.

**Lemma 10.** (See [6, Lemma 3.2]) Let $f \in L^{6/5}(\mathbb{R}^3)$, $\{f_\varepsilon\}_{\varepsilon > 0} \subset L^{6/5}(\mathbb{R}^3)$ and assume $\lim_{\varepsilon \to 0^+} f_\varepsilon = f$ in $L^{6/5}(\mathbb{R}^3)$. Then

$$\lim_{\varepsilon \to 0^+} \phi_\varepsilon(f_\varepsilon) = \phi_0(f) \quad \text{in} \quad D^{1,2}(\mathbb{R}^3),$$

$$\lim_{\varepsilon \to 0^+} \varepsilon \phi_\varepsilon(f_\varepsilon) = 0 \quad \text{in} \quad D^{1,4}(\mathbb{R}^3).$$

In our case we have then

$$\phi_\varepsilon(u_{\lambda, \varepsilon}) \to \phi_0(u_{\lambda, 0}) \quad \text{in} \quad D^{1,2}(\mathbb{R}^3), \quad \varepsilon \phi_\varepsilon(u_{\lambda, \varepsilon}) \to 0 \quad \text{in} \quad D^{1,4}(\mathbb{R}^3).$$

To conclude the proof of Theorem 2, let $v \in C_\infty^\infty(\mathbb{R}^3)$ with supp$(v) \subset K$. We know that

$$(u_{\lambda, \varepsilon}, v)_{H^1} + \int_K \phi_\varepsilon(u_{\lambda, \varepsilon}) u_{\lambda, \varepsilon} v = \int_K \phi_\varepsilon(u_{\lambda, \varepsilon}) u_{\lambda, \varepsilon} v - \int_K |u_{\lambda, \varepsilon}|^2 - 2 u_{\lambda, \varepsilon} v.$$
We want to pass to the limit as \( \varepsilon \to 0^+ \) in the above identity. Let us see every term.

Of course
\[
(u_{\lambda,0}^\varepsilon, v)^{H^1} \to (u_{\lambda,0}^0, v)^{H^1}.
\]

Since \( \phi_{\varepsilon}(u_{\lambda,0}^\varepsilon) \to \phi_0(u_{\lambda,0}^0) \) in \( L^6(\mathbb{R}^3) \), \( u_{\lambda,0}^\varepsilon \to u_{\lambda,0}^0 \) in \( L^{12/5}(K) \) and \( v \in L^{12/5}(K) \) we easily find
\[
\int_K \phi_{\varepsilon}(u_{\lambda,0}^\varepsilon) u_{\lambda,0}^\varepsilon v \to \int_K \phi_0(u_{\lambda,0}^0) u_{\lambda,0}^0 v.
\]

Moreover in a standard way we have also
\[
\int_K f(x, u_{\lambda,0}^\varepsilon) v \to \int_K f(x, u_{\lambda,0}^0) v.
\]

and
\[
\int_K |u_{\lambda,0}^\varepsilon|^{2^* - 2} u_{\lambda,0}^\varepsilon v \to \int_K |u_{\lambda,0}^0|^{2^* - 2} u_{\lambda,0}^0 v.
\]

By (6.5)-(6.9) we deduce that
\[
(u_{\lambda,0}^\varepsilon, v) + \int_K \phi_0(u_{\lambda,0}^0) u_{\lambda,0}^0 v = \int_K \lambda f(x, u_{\lambda,0}^0) v - \int_K |u_{\lambda,0}^0|^{2^* - 2} u_{\lambda,0}^0 v
\]
and this says that \( u_{\lambda,0}^\varepsilon \) gives rise to a solution \( (u_{\lambda,0}^\varepsilon, \phi_0(u_{\lambda,0}^0)) \) of the Schrödinger-Poisson system (1.4).

Then by setting
\[
\phi_{\lambda,0} := \phi_{\varepsilon}(u_{\lambda,0}^\varepsilon), \quad \phi_{\lambda,0} := \phi_0(u_{\lambda,0}^0),
\]
the proof of Theorem 2 follows by (6.3), (6.4) and (6.10).

**Remark 5.** As a byproduct we get \( I_\varepsilon(u_{\varepsilon}) \to I_0(u_0) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \phi_0(u_{\lambda,0}^0)|^2 \) and consequently we have the convergence of the mountain pass energy levels, \( c_{\lambda,\varepsilon} \to c_{\lambda,0} \).

### 7. Proof of Theorem 3

In this section we study the supercritical case, that is the problem
\[
\begin{cases}
-\Delta u + u + |u|^{p-2} u = \lambda f(x, u) & \text{in } \mathbb{R}^3, p > 2^*, \\
-\Delta \phi - \varepsilon^4 \Delta_4 \phi = u^2 & \text{in } \mathbb{R}^3,
\end{cases}
\]

under the same assumptions on \( f \). We already know it is equivalent to consider the equation
\[
-\Delta u + u + \phi_u(x, u) u = \lambda f(x, u) + \lambda |u|^{p-2} u \quad \text{in } \mathbb{R}^3, \quad p > 2^*.
\]

To deal with this case we consider a new nonlinearity \( g_K \), for \( K > 0 \), given by
\[
g_K(x, t) = \begin{cases}
\lambda f(x, t) + |t|^{p-2} t & \text{if } |t| \leq K \\
\lambda f(x, t) + K^{p-2} t^2 - K & \text{if } |t| > K.
\end{cases}
\]

Once that
\[
|g_K(x, t)| \leq \lambda f(x, t) + K^{p-2} |t|^{2^*-2} t \quad \text{if } t \in \mathbb{R},
\]
we are in a position to apply Theorem 1 to the equation
\[
-\Delta u + u + \phi_u(x, u) u = g_K(x, u) \quad \text{in } \mathbb{R}^3,
\]
and then there exists a solution \( u_{\lambda,\varepsilon,K} \) of (7.3). It is sufficient now to show that there exists \( C > 0 \) independent on \( \lambda \) and \( K \) such that
\[
|u_{\lambda,\varepsilon,K}|_\infty \leq C|u_{\lambda,\varepsilon,K}|_{H^1}.
\]

Indeed, since we know that \( \|u_{\lambda,\varepsilon,K}\|_{H^1} \to 0 \) as \( \lambda \to \infty \), then, there is a \( \lambda^* > 0 \) such that, for all \( \lambda \geq \lambda^* \), \( u_{\lambda,\varepsilon,K} \) is indeed a solution of (7.2).

However the proof of (7.4) can be obtained by repeating the arguments in the proof of [10, Theorem 1.1, pages 10-13], taking into account the positivity of the solution of \( -\Delta \phi - \varepsilon^4 \Delta_4 \phi = u^2 \).
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