Spacetime foam

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Abstract

Spacetime foam is analyzed within the simplistic model of a set of scalar fields on a flat background. We suggest the formula for the path integral which allows to account for all possible topologies of spacetime. We show that the proper path integral defines a cutoff for the field theory. The form of the cutoff is fixed by the field theory itself and has no free additional parameters. New features of the Feynman diagram technic are outlined and possible applications in quantum gravity are discussed.
1 Introduction

Spacetime foam is commonly believed to cure divergencies in particle physics and therefore it will eventually allow to remove the unnatural and non-physical (and extremely restrictive) principle of the renormalizability of physical field theories. However so far such a property has not been explicitly established yet.

It seems that the basic difficulty here stems from the problem of classifying topologies in 4-dimensions. Indeed, the adequate description of spacetime foam effects is reached in the euclidean quantum gravity advocated primary by S. W. Hawking and developed by many authors (e.g., see Refs. [3]-[7]). The euclidean path integral for the expectation value of an observable $B$ is

$$\langle B \rangle = \sum B e^{-S} \sum e^{-S}$$

where $S$ is the euclidean action and sum is taken over all field configurations and all topologies of the euclidean spacetime. The path integral is usually supposed to be taken in the two steps. First, one integrates over all field configurations keeping a specific topology fixed and then sums over different topologies, so that the partition function can be presented as

$$Z = \sum e^{-S} = \sum_{\text{topologies}} e^{-S_{\text{eff}}}$$

where $S_{\text{eff}}$ is an independent effective action for each topology. Now one may use the semiclassical approximation (instantons) to evaluate contributions of different topological classes etc. and this is the way on which the further development of euclidean quantum gravity is going on (e.g., see Refs. [8] and references therein). We leave aside the loop quantum gravity [9], for essentials remain the same (as far as topologies is concerned).

It is clear however that results obtained on this way are rather restrictive in nature. Save the absence of an appropriate classification of different topologies, one can never justify that terms (topological classes) omitted give small effects. Even if such terms have smaller actions $S_{\text{eff}}$ the number of such additional terms is enormous. One may think that the semiclassical approximation in (2) (though useful in investigating particular features) is not suitable. In the present Letter we suggest absolutely equivalent formula for the path integral which allows to account for the all possible topologies of spacetime (even those which are apparently not smooth). As we shall see the proper path integral automatically defines a Lorentz invariant cutoff for the field theory as it was to be expected. The formula suggested follows quite naturally from the three well-established fundamental facts. 1) Any 4-dimensional manifold can be continued to the whole Euclidean space by adding non-physical regions of the space. Such a continuation is not unique however. In particular, the existence of a universal
covering is the well-known mathematical fact. However in the general case the universal covering requires considering a curved space, while at high energies (at least at laboratory scales) the space looks to be flat. Our claim is that there always exists a continuation when the space remains to be flat (e.g., see Ref. [10]). 2) The discrepancy between the actual Green functions and those for the euclidean space is described by a topological bias of sources (i.e., the topology or the proper boundary conditions for the actual Green functions can be accounted for by additional sources). 3) The topological bias of sources has an equivalent description in terms of multi-valued fields.

The first two facts represent the well known classical results. E.g., the universal covering (which is not more than the astrophysical way of the extrapolation of the laboratory coordinate system) and the concept of the topological bias were described in detail in Refs. [10, 11]. In particular, in astrophysics when we look at the sky we always have to deal with the universal covering and this allows to give the most natural explanation for the all the variety of the observed dark matter phenomena (see the above papers and Ref. [12] where theoretical rotation curves for spiral galaxies were shown to be in a very good agreement with observations). The bias of sources and the "standard" continuation (i.e., without introducing a non-flat metric) is the standard tool for solving different electrostatic problems in classical electrodynamics (e.g., see the image method in Ref. [13]).

The last fact (the multivalued nature of fields) being transparent is however less known. The basic construction was suggested in Ref. [14] and developed in Refs. [15, 16]. In fact it represent the most natural tool to describe the so-called coda waves and seismic noise [17]. Indeed, due to multiple scattering on topology (or in porous systems on boundaries) plane waves are not solutions to linear field equations (for a particular topology the homogeneity of space is broken). Thus if we consider any wave packet $\phi$, it, due to multiple scattering, transforms to $\phi = \sum \phi_j$. When the topology is random, the scattering randomizes phases and such a field acquires the diffuse nature $\langle \phi^2 \rangle = \sum \langle \phi_j^2 \rangle$, i.e., each term can be considered as an independent field. Thus although on the micro scale the field equations remain unchanged the intensities follow a diffusion equation. The diffuse nature of seismic fields has intensively been studied (e.g., see Refs. [18] and references therein). We point out that the physical field (which is measured in experiments) represents only the sum of terms $\phi = \sum \phi_j$ and it is defined only in the physically admissible region of space. Every term however becomes an "independent field" upon a continuation to the whole space. In quantum theory particles which are described by the diffuse fields obey the generalized statistics (in particular, the violation of the Pauli principle in such fields has a rather clear physical sense; the violation occurs due to the existence of "mirror" particles in non-physical regions of space, while upon restriction to the fundamental domain the statistics restores) [16].

For the sake of simplicity and to make the basic ideas clear (and to avoid usual technical problems in quantum gravity) we, in the present Letter, consider
the most simple example of a set of scalar fields in $R^4$. The metric is supposed to be everywhere flat, while the topology is described by some gluing procedure along some multi-connected hypersurfaces. We point out that in general when considering the universal covering such gluing leads to $\delta$-like singularities in the scalar curvature which rigorously speaking require to account for the gravitational action. To avoid such problem we shall suppose that every hypersurface is approximated by piecewise flat surfaces. Then the $\delta$-like terms in the curvature are concentrated on vertexes and ribs which have zero measure and do not contribute to the geodesic flow. Moreover such terms possess both signs (depending on the induced curvature on the hypersurfaces) and for sufficiently complex topologies the vanishing of the mean curvature is actually not restrictive. In considering the standard continuation (by the image method) the metric remains everywhere flat, while the scattering on the topology is completely described by the bias of sources and we need not to add the gravitational action.

2 The universal covering and the topological bias

The universal covering for an arbitrary non-trivial topology of space can be constructed as follows. We take a point $O$ in our space $\mathcal{M}$ and issue geodesics (straight lines) from $O$ in every direction. Then points in $\mathcal{M}$ can be labeled by the distance from $O$ and by the direction of the corresponding geodesic. In other words, for an observer at $O$ the space $\mathcal{M}$ will always look as $R^4$. However if we take a point $P \in \mathcal{M}$, there may exist many homotopically non-equivalent geodesics connecting $O$ and $P$. Thus, any source at the point $P$ will have many images in $R^4$. The topology of $\mathcal{M}$ can be determined by noticing that in the observed space $R^4$ there is a fundamental domain $\mathcal{D}$ such that every point in $\mathcal{D}$ has a number of copies outside $\mathcal{D}$. The actual manifold $\mathcal{M}$ is then obtained by identifying the copies. In this way, we may describe the topology of space $\mathcal{M}$ by indicating for each point $r \in R^4$ the set of its copies $E(r)$, i.e. the set of points that are images of the same point in $\mathcal{M}$.

Consider now the actual Green function for a scalar wave equation in the physically admissible region $\mathcal{D}$

\[ (-\Box_x + m^2) G(x, y) = 4\pi \delta(x - y) , \]

where $x, y \in \mathcal{D}$. Upon continuing to the universal covering $R^4$ this equation transforms as follows

\[ (-\Box_x + m^2) G(x, y) = 4\pi N(x, y) , \] (3)

where coordinates $x, y$ are extended to the whole space $R^4$ and

\[ N(x, y) = \delta(x - y) + \sum \delta(x - f_i(y)) \] (4)

(the sum is here taken over all images of the point $y$, i.e., over all $f_i(y) \in E(y)$). The two point function $N(x, y)$ was called the topological bias in Refs.
which describes the discrepancy between the actual physical space (the fundamental domain $D$) and the universal covering (the simple topology space) $R^4$. We point out that the topology is completely (one-to-one) defined by the specifying the bias $N(x,y)$.

The structure of the bias (4) on the universal covering has one important feature which allows it to mimic dark matter phenomena, i.e.,

$$\int_V N(x,y) d^4x = N(V) \geq 1$$

where $V$ is some volume around the point $y$. The number $N(V) - 1 = 0, 1, 2, ...$ gives the number of points $f_i(y)$ which get into the coordinate volume $V$. Roughly, this number characterizes how many times the volume $V$ covers the fundamental domain (or the physically admissible region) $D$.

Let us return to the path integral (2). Consider a particular virtual topology of space. It is clear that the action in (1)-(2) has the same value for all physical spaces which can be obtained by rotations and transitions of the coordinate system in $R^4$. Thus, upon averaging out over possible orientations and transitions the bias acquires always the structure $N(x,y) = N(|x-y|)$ and for the Green function we find

$$G(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{N(k)}{k^2 + m^2} \exp\{i k (x-y)\},$$

where $N(k)$ is the Fourier transform for the bias. The above Green function plays the most important role in particle theory and its UV (ultra-violet) behavior (actually that of the bias $N(k)$) defines whether the resulting quantum theory is finite or not. What we expect that the proper definition of the path integral over virtual topologies should fix the specific form of the bias $N(k)$.

We also point out that the universal covering is what we actually use in astrophysics when extrapolating our laboratory coordinate system to extremely large distances. Therefore, in expressions (3) (5) the coordinates $x$, $y$ have the direct physical (observational) status in applying to cosmological problems (DM and dark energy phenomena, origin of density perturbations etc.). In particular, we can never say (without additional subtle effects) if two points $x_1$ and $x_2$ are close or not (at least there are no external safe rulers to measure the distances). On the contrary, in high energy physics we use an extrapolation to very small scales (by means of our "safe" laboratory rulers). Again we cannot say if two points $x_1$ and $x_2$ are close or not. However we still can assign specific distances extrapolated from the laboratory coordinate system and this is exactly the coordinate system we use in particle physics. As we shall see the extrapolation in particle physics leads to the same expressions (3), (5) however the bias (4) acquires somewhat different features. By other words the Universe looks somewhat different when we look at small or large distances.

As it was shown in Ref. [19] in this case the bias $N(x,y)$ represents a projection operator onto physically admissible states. This means that $(\hat{N})^2 = \hat{N}$ and in the basis of eigenvectors it takes the form $N(x,y) = \sum N_k f_k^*(x) f_k(y)$ with eigenvalues $N_k = 0, 1$. While on the universal covering possible eigenvalues $N_k = 0, 1, 2, ...$. 

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[11] [19]
3 Topological bias in particle physics

In the present section we consider the bias which originates from a single wormhole. Such a bias was constructed first in Ref. [10] for the massless field in 3-dimensions, while the generalization to the euclidean 4-space is straightforward. We point out that a wormhole describes a virtual baby universe which may branch off and joint onto our mother Universe [3]-[7].

A single wormhole can be viewed as a couple of conjugated spheres $S_{\pm}$ of the radius $a$ and with a distance $d = |\vec{R}_+ - \vec{R}_-|$ between centers of spheres. The interior of the spheres is removed and surfaces are glued together. For the sake of simplicity we consider the massless case i.e., the Green function $\Delta G(x, y) = 4\pi \delta(x - y)$ for such a topology. In Ref. [10] we have shown that the proper boundary conditions (the actual topology) can be accounted for by adding the bias of the source $\delta(x - y) \to \delta(x - y) + b(x, y)$ where in the approximation $a/d \ll 1$ the bias in $R^3$ takes the form

$$b(x) \approx a \left( \frac{1}{R_-} - \frac{1}{R_+} \right) \left[ \delta(\vec{x} - \vec{R}_+) - \delta(\vec{x} - \vec{R}_-) \right]$$

(6)

where we set $y = 0$ and neglect the throat size, i.e., all additional sources (ghost images) are placed in the centers of spheres. The generalization to the space $R^4$ is trivial and gives

$$b(x) = a^2 \left( \frac{1}{R_+^2} - \frac{1}{R_-^2} \right) \left[ \delta(\vec{x} - \vec{R}_+) - \delta(\vec{x} - \vec{R}_-) \right].$$

(7)

We see that unlike [10] the function $b(x)$ has the property $\int b(x) d^4x = 0$ which gives $\int N(x) d^4x \equiv 1$ and for any volume $V$ we get $N(V) \leq 1$.

Let us introduce the probability distribution for parameters of the wormhole $P(R_{\pm}, a)$ which is defined by the action in (1). It is clear that due to homogeneity an isotropy of $R^4$ this function may depend only on $d = |\vec{R}_+ - \vec{R}_-|$ and we find for the mean bias

$$\overline{b}(r) = 2 \int \left( \frac{1}{R_+^2} - \frac{1}{r^2} \right) f \left( |\vec{R} - \vec{r}| \right) d^4\vec{R},$$

(8)

where $f(d) = \int a^2 P(d, a) da$. For the Fourier transforms $b(k) = (2\pi)^{-2} \int b(r) e^{-ikr} d^3r$ this expression takes the simplest form

$$\overline{b}(k) = \frac{8\pi (f(0) - f(k))}{k^2}.$$  

(9)

In the so-called long-wave approximation (the low energy physics) we can completely neglect the throat size $a \to 0$. In this limit the action for the wormhole does not depend on the separation distance $d = |\vec{R}_+ - \vec{R}_-|$ at all, i.e.,
\( P (d, a) = P (a) \), and the mean bias reduces merely to \( \bar{b} (x) = b \delta (x) \) (e.g., see Ref. [10]). Therefore, the effect of wormholes reduces merely to a renormalization of physical constants (e.g., of charge values) which is in the complete agreement with the previous results of Refs. [4, 5, 7]. Moreover, the value \( b < 0 \) [10] which means that virtual wormholes always diminish charge values as it was first pointed out in Ref. [7].

In conclusion of this section we point out that the multiplier \( 4 \pi / k^2 \) in (9) and \( 1 / R_+^2 \) in (5) is the standard Green function for \( R^4 \). In the case of massive particles it should be replaced with \( 4 \pi / (k^2 + m^2) \) and \( e^{-mR_+} / R_+^2 \) respectively. Thus, we see that in particle physics the structure of the Green functions (3), (5) remains the same, while the property of the bias for the universal covering \( N (V) \geq 1 \) changes drastically to \( N (V) \leq 1 \).

4 Multi-valued fields and the action

The structure of the bias [11] suggests the analogous decomposition of the true Green function

\[
G (x, y) = G_0 (x - y) + \sum G_0 (x - f_i (y)) \quad (10)
\]

where \( G_0 (x - y) = 1 / (x - y)^2 \) is the standard Green function for the euclidean space \( R^4 \). If we present it in the form of the path integral for a scalar particle in \( D \), i.e., \( G (x, y) = \sum_{x(s) \in D} \exp \left( - \int_y^x ds \right) \) then every term in (10) corresponds to the restriction of trajectories \( x (s) \) to a particular homotopic class [11]. When we continue such terms to the whole space \( R^4 \) they acquire the character of independent fields that is to say that such particles has to be described by a scalar field \( \phi \) which acquires the multi-valued (diffused) nature. An equivalent representation for such a field can be achieved in terms of the generalized statistics (e.g., see for details Ref. [16]). In the case of homogeneous and isotropic topological structure the multi-valued (diffuse) character of the scalar field is more convenient to describe in the Fourier representation \( (\phi = \frac{1}{(2 \pi)^{D/2}} \int d^4 k \phi_k e^{ikx} ) \) that is to replace the single-valued field \( \phi_k \) with a set of fields \( \phi_k^j \) where \( j = 1, 2, ..., N (k) \), while the bias \( N (k) \) has the meaning of the number of such fields (from the phenomenological standpoint such fields were introduced first in Ref. [14] and for the relation to the generalized statistics see Refs. [16]).

Consider now the euclidean action for the scalar field (we use the Planckian units in which \( M_{pl} = 1 \))

\[
S = \frac{1}{2} \int \left[ (\partial_\mu \phi)^2 + m^2 \phi^2 + V (\phi) \right] d^4 x. \quad (11)
\]

Rigorously speaking the integral here should run only over the fundamental domain \( D \). However to describe different possible topologies on an equal footing we should continue this expression on the whole space \( R^4 \). In what follows we shall use the Fourier transform for the field, while the actual topology will be
encoded by specifying $N(k)$ (we assume that the integration over transitions and orientations in $N$ is already carried out and therefore $N(k)$ defines a whole class of topologies, while $S$ is the modified action). Then the linear part of the action takes the structure:

$$S_0 = \frac{L^4}{2} \int \sum_{j=1}^{N(k)} (k^2 + m^2) \left| \phi^j_k \right|^2 \frac{d^4k}{(2\pi)^4},$$

(12)

while the non-linear term $S_{int}(\phi)$ should be accounted for by perturbations. We recall that in this expression the values of the number of fields $N(k)$ depend on scales under consideration and, therefore, the result for the cutoff function depends on the choice of the continuation used. As it was explained previously in astrophysical problems we use the universal covering and the number of fields takes values $N(k) = 0, 1, 2, ...$, while in particle physics the number of fields can take only two possible values $N(k) = 0, 1$.

The physical sense has the sum of fields, and therefore the generating functional should be taken as

$$\tilde{Z} [J] = \exp \left\{ -S_{int} \left( \frac{\delta}{\delta J} \right) \right\} \int D[\phi] \exp \left\{ -S_0(\phi) + L^4 \int J(-k) \tilde{\phi}_k d^4k \right\}$$

$$= \tilde{Z} [0] \exp \left\{ -S_{int} \left( \frac{\delta}{\delta J} \right) \right\} \exp \left\{ \frac{L^4}{2} \int \frac{|J(k)|^2}{k^2 + m^2} N(k) \frac{d^4k}{(2\pi)^4} \right\},$$

(13)

where $\tilde{\phi}_k = \sum_{j=1}^{N(k)} \phi^j_k$, while for $\tilde{Z} [0]$ we find

$$\tilde{Z} [0] = \exp \left\{ - \frac{L^4}{2} \int N(k) \frac{d^4k}{(2\pi)^4} \ln \frac{k^2 + m^2}{\pi} \right\}.$$  

(14)

In particular, we can write $\tilde{Z} [0] = \exp (-L^4 < \rho >_{eff})$, where $< \rho >_{eff}$ is the zero-point vacuum energy density which for a particular topology $N(k)$ is

$$< \rho >_{eff} = \frac{1}{2} \int N(k) \frac{d^4k}{(2\pi)^4} \ln \frac{k^2 + m^2}{\pi}.$$  

(15)

Thus we see whether the cosmological constant is finite or not depends on the topological structure of the actual space. Now to account for all possible virtual topologies (spacetime foam) and get the final expression for the generating function $Z [J]$ we have to sum over topologies, i.e., possible values of $N(k)$ in accordance to (2). For sure we may expect that all topologies which give infinite values of $< \rho >_{eff}$ should be suppressed.

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4We point out that such a simple form for the linear part of the action is reached only for isotropic and homogeneous class of topologies, while for a particular topology the bias has the structure $N = N(k, k')$ and the action diagonalizes in a specific (for given topology) basis e.g., see discussions in Refs. [10] [19].
5 Cutoff function in particle physics

While the topology is fixed, \( N(k) \) is an ordinary fixed function. Now we are ready to evaluate the cutoff for the particle physics in the case when topology may fluctuate. In this case possible values of \( N(k) \) are 0 and 1. The partition function \( \tilde{Z}[0] \) has the structure

\[
\tilde{Z}[0] = \prod_k Z_k^{N(k)}
\]

where \( Z_k \) is given by the standard single-field expression \( Z_k = \sqrt{\pi/(k^2 + m^2)} \) and the sum over possible values \( N(k) \) gives

\[
Z = \sum_{\text{topologies}} \tilde{Z}[0] = \prod_k \left( \sum_{N=0,1} Z_k^{N(k)} \right) = \prod_k (1 + Z_k), \tag{16}
\]

while for the mean cutoff we find from (11)

\[
\mathcal{N}(k) = \frac{Z_k}{(1 + Z_k)}. \tag{17}
\]

This expression straightforwardly generalizes on a multiplet of scalar fields or a set of bosonic fields of an arbitrary spin which gives

\[
\ln Z_k = \frac{1}{2} \sum_\alpha \ln \left( \frac{\pi}{(k^2 + m_\alpha^2)} \right), \tag{18}
\]

where the sum is taken over all fields and helicity states. The bias and the cutoff for Fermi fields require a separate consideration and we present it elsewhere.

The remarkable property of the cutoff function is the explicit Lorentz invariance (i.e., the function \( N(k) \) depends on the momenta via the Lorentz invariant expression \( k^2 \)). On the mass-shell \( Z_k \to \infty \) and it reduces to \( N(k) \to 1 \) which reflects the fact that on the mass shell the space looks as \( \mathbb{R}^4 \), while at very small (planckian) scales \( Z_k \ll 1 \) it has the behavior \( N(k) \sim 1/k^3 \to 0 \) as \( k \to \infty \), (where \( g \) is the total number of degrees of freedom). Thus, as it was expected for sufficiently big number of fields \( g \), \( N(k) \) provides indeed a Lorentz invariant cutoff which we discuss in the next section.

6 Finiteness of Feynman diagrams

The generating functional \( \tilde{Z}[J] \) leads to the standard perturbation scheme (e.g., see the standard textbooks [20]). A new features however appear. As we can see from (5) and (13) the integration measure for every closed loop takes the form \( \mathcal{N}(k) d^4k/(2\pi)^4 \) and, therefore, every diagram will include the factor

\[
\mathcal{N}(k) = \frac{Z_k}{(1 + Z_k)}.
\]

Actually \( N(k) \) defines the whole topological class, while a specific topology is fixed by a function \( N(k,k') \).
\[ \langle N(k_1) N(k_2) \ldots N(k_n) \rangle \text{ which in the first only approximation by topology fluctuations can be replaced with the product } \overline{N}(k_1) \overline{N}(k_2) \ldots \overline{N}(k_n) \text{ where } \overline{N}(k) \text{ gives the cutoff which is defined by (17). Thus every Feynman diagram acquires an additional decomposition onto a series by topology fluctuations of the cutoff function.} \]

The contribution in the cutoff function \( \overline{N}(k) \) comes from all physical fundamental fields \([18]\) and it is clear that all UV divergencies are automatically regularized (e.g., if we account only for gravitational \( h_{\mu\nu} \), electromagnetic \( A_\nu \), and weak \( Z_\nu, W^{\pm}_\nu \) interactions, the number of degrees of freedom is 10 and it defines the UV behavior \( N(k) \sim 1/k^{10} \) as \( k \to \infty \) which is already sufficient to regularize all divergent diagrams\(^6\). In (17) the characteristic UV scale of the cutoff has the planckian order \( Z_k \sim 1 \), which means that \( Z_k \) includes contribution of all fields with masses less than planckian mass \( m_{pl} \). This is not convenient for practical computations; for the actual cutoff occurs for much lower energies. To see this let us introduce the characteristic scale \( k \sim \mu \) which has the sense of the laboratory scale from which we extrapolate our laboratory coordinate system to very small distances (i.e. the actual scale of the cutoff). From the analogy with the statistical physics such a scale can be viewed as a specific chemical potential which corresponds to the additional cosmological constant term to the action\(^7\), i.e., the redefinition of (15) as

\[ \langle \rho \rangle_{eff} = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \ln \frac{k^2 + m^2}{\mu^2}. \quad (19) \]

Then \( Z_k \) modifies as \( Z_k \to Z_k/Z_\mu \) and the cutoff function (17) modifies as

\[ \overline{N}(k) = \frac{Z_k}{Z_\mu + Z_k}. \quad (20) \]

In such a form we may retain in \( Z_k \) only the necessary (smallest) number of fields with masses \( m_\alpha < \mu \), while all more massive particles give only a constant contribution to \( Z_k \sim \mu/m \) and lead merely to a renormalization of the scale \( \mu \) itself. By other words we may suppose that the contribution of the most heavy particles is already encoded in \( \mu \) (at least this allows also to account phenomenologically for all possible new particles and fields which may be found in the future at extremely high energies).

Thus the cutoff function acquires the structure

\[ \overline{N}(k) = \frac{\mu^g}{(\mu^g + k^{2\alpha_0}(k^2 + m_1^2)^{\alpha_1} \cdots (k^2 + m_n^2)^{\alpha_n})} \quad (21) \]

where \( m_\alpha < \mu \) and \( g = \sum 2\alpha_\alpha \) is the total number of fields we have to retain.

\(^6\)We point out that gauge fields have more components whose contribution to \( Z_k \) depends on the choice of the gauge fixing. Therefore the exponent in \( N(k) \sim 1/k^{10} \) may be even more than ten.

\(^7\)We recall that when we consider interactions all constants acquire a dependence on scales [20].
The most divergent expressions in quantum field theory come from terms of the type \( \langle (\partial \phi)^2 \rangle \), which in the momentum space have UV behavior \( \sim p^4 \). We point out that \( p^4 \) gives also the highest rate of divergence in quantum gravity as well, e.g. see Ref. [21]. As an example of such a term we consider the cosmological constant \((19)\). Since all fields which we retain in \((21)\) give some contribution to the cosmological constant \( \langle \rho \rangle_{\text{eff}} \) we sum \((19)\) over all fields which gives (upon simple transformations)

\[
\langle \rho \rangle_{\text{eff}} = \frac{\mu^4}{(16\pi^2)} F(\alpha, \tilde{m}),
\]

where

\[
F(\alpha, \tilde{m}) = \int_0^\infty \frac{\ln \left( x^{\alpha_0} (x + \tilde{m}_1)^{\alpha_1} \cdots (x + \tilde{m}_n)^{\alpha_n} \right) dx}{(1 + x^{\alpha_0} (x + \tilde{m}_1)^{\alpha_1} \cdots (x + \tilde{m}_n)^{\alpha_n})^4}
\]

and \( \tilde{m}_i = m_i^2/\mu^2 \). This expression is finite for \( \sum 2\alpha_n > 4 \) (i.e., we have to retain at least five field degrees of freedom). In the case when \( \tilde{m}_i = 0 \) it gives

\[
F(\alpha, 0) = -\frac{\pi^2 \cos(2\pi/\alpha)}{\alpha_0 \sin^2(2\pi/\alpha_0)}.
\]

Next "dangerous" terms are given by \( \langle \phi^2 \rangle \) which define the renormalization of the mass. We evaluate it for \( \lambda \phi^4 \) [20] which in the first order by \( \lambda \) gives the correction to the mass (the so-called "tadpole" diagram)

\[
\delta m^2 = \Sigma(p) = \frac{\lambda}{2} \int N(k) \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + m^2}
\]

which gives

\[
\Sigma(p) = \Sigma(0) = \frac{\lambda}{32\pi^2} \mu^4 G(\alpha, \tilde{m}),
\]

where

\[
G(\alpha, \tilde{m}) = \int_0^\infty \frac{xdx}{(x + \tilde{m})(1 + x^{\alpha_0} (x + \tilde{m}_1)^{\alpha_1} \cdots (x + \tilde{m}_n)^{\alpha_n})}
\]

which is already finite for \( \sum 2\alpha_n > 2 \). In the massless case it gives

\[
G(\alpha, 0) = \frac{1}{\alpha_0} \Gamma(1/\alpha_0) \Gamma(1 - 1/\alpha_0).
\]

In this manner we see that all divergencies in Feynman diagrams disappear when the contribution of a proper number of fields in the cutoff function is taken into account. It is quite clear that this result is valid almost in all theories (whose

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8 Actually the most divergent behavior will be given by \( \sim p^8 \), when fluctuations in the cutoff function itself are taken into account, since the Gaussian character of the distribution over \( N(k) \) gives \( \Delta N^2 \sim N \). However such terms should be treated in the complete analogy with the subsequent analysis.
dynamical equations do not include too high derivatives of fields which in general lead to \( p^n \) divergencies) and it seems to remain true in general relativity (GR) as well [21]. However unlike gauge fields (which are proved to be renormalizable) GR represents formally non-renormalizable theory\(^9\) and therefore it requires the more complete and rigorous proof which we leave for the future research.

7 Cutoff function on the universal covering

In observational cosmology when we look at the sky we always use the coordinate system which corresponds to the universal covering. Therefore, in solving astrophysical problems (quantum origin of density perturbations, quantum cosmology, etc.) we have to use the representation in which the actual space is described by the universal covering. We recall that in general the universal covering requires the introducing of a curved background and, therefore, the results of the present section have only a preliminary character.

In the present section we evaluate the astrophysical cutoff function as well. In this case the number of fields takes the values \( N(k) = 0, 1, 2, \ldots \) and (16) becomes

\[
Z = \sum_{\text{topologies}} \tilde{Z}[0] = \prod_k \left( \sum_{N=0}^{\infty} \frac{Z^N(k)}{N(k)!} \right) = \exp \left( L^4 \int \frac{d^4k}{(2\pi)^4} \right),
\]

where we have accounted for the fact that permutations of fields at the same \( k \) gives the same quantum state (i.e., the identity of fields which gives the factor \( 1/N! \)). Then for the mean cutoff we find from (1)

\[
\overline{N}(k) = Z_k.
\]

Thus (17) and (25) define the relation between the bias (cutoffs) in the two different representations for the same physical space.

The analogy with the statistical physics shows that (24) (25) correspond to the classical (or the Boltzmann) statistics. As it was discussed in the introduction such statistics corresponds to the so-called diffused fields [18]. However quantum topology should introduce some additional statistics between fields [14] which corresponds to third quantization and which we consider in what follows [14].

Consider first the density of fields in the configuration space (i.e., the space of fields)

\[
N[k, \phi] = \sum_j \delta \left( \phi - \phi^j_k \right)
\]

\(^9\)There is only a small chance that due to the entanglement in complex diagrams divergencies may remain.

\(^{10}\)Such correlations may be important in investigating corrections to the mean values of the type \( \langle N(k_1) N(k_2) \ldots N(k_n) \rangle \) which appear in Feynman diagrams.
so that the number of fields is merely

\[ N(k) = \int N[k, \phi] \, d\phi. \]

Then the action \((12)\) can be rewritten as

\[ S = \frac{L^4}{2} \int N[k, \phi] \left( k^2 + m^2 \right) |\phi|^2 \, D\phi \frac{d^4k}{(2\pi)^4} \]

which represents the functional of \(N[k, \phi]\). Thus, the partition function can be presented as

\[ Z = \sum_{N[k, \phi]} \exp \{-S(N[k, \phi])\}. \]

Here the sum over \(N[k, \phi]\) includes, in fact, both the sum over topologies and configuration variables. The further depends on the statistics of fields assumed (which is not the same as the statistics of particles, e.g., see Refs. [14, 16]). If we accept the Fermi statistics (i.e., numbers \(N[k, \phi] = 0, 1\) then such scalar particles will obey the so-called para-Bose statistics [16]. The choice should be made from experiment (though there may be some theoretical reasoning for a particular choice). In both cases we find for the mean density

\[ \langle N[k, \phi] \rangle = \left[ \exp \left( \frac{1}{2} (k^2 + m^2) |\phi|^2 \right) \mp 1 \right]^{-1} \]

and for the cutoff function we find the same expression (25) with an additional multiplier

\[ \overline{N}(k) = C_{\pm} Z^g_k \]

where the multiplier is given by \((g\) is the number of components of the scalar field \(\phi)\)

\[ C_{\pm} = \frac{1}{\pi^{g/2}} \int \frac{d^g\phi}{\exp \left( \frac{1}{2} |\phi|^2 \right) \mp 1}. \]

8 Conclusions

In conclusion we briefly repeat basic results. First of all we have explicitly demonstrated that spacetime foam provides quantum fields with a cutoff. The form of the cutoff is fixed by the field theory itself and it does not introduce additional parameters. It depends only on the standard set of naked parameters related to fields. It does also depend on the representation of the physical space used. We have to used the two types of different representations depending on the problem under consideration. In particle physics we extrapolate the laboratory coordinate system to extremely small scales and, therefore, we should use the so-called standard representation (the image method) which gives (21) for the cutoff. In astrophysics however we always have deal with the universal
covering and the cutoff becomes (25). Since we considered quantum topology fluctuations around the flat space, the cutoff has the Lorentz invariant form. This is always justified for particle physics, while in the astrophysical picture our results carry rather a preliminary character; for rigorous consideration requires a curved background. In the present Letter our consideration has a simplified character, i.e., a set of scalar fields. However it is clear that all the results can be straightforwardly extended to any non-linear field theory. In particular, the cutoff suggested automatically regularizes divergencies in quantum fields and, therefore, we can expect that general relativity represents in fact a renormalizable theory.

We also demonstrated that every Feynman diagram acquires an additional decomposition onto a series by topology fluctuations in the cutoff function which may lead to some new phenomena.

The cutoff function has the meaning of the topological bias of point sources which displays the discrepancy between the visual and the actual spaces. In astrophysics such a discrepancy is observed as the Dark Matter phenomenon [11, 19]. Analogous phenomena are widely known in particle physics which represent "Dark Charges" of all sorts. Those are not more than the standard (phenomenological) Higgs fields [22]. Therefore, we expect that quantum gravity provides the unique tool to fix all constants of nature (the lambda term, mass spectrum, charge values, etc.). However the self-consistent evaluation of such parameters requires considering the complete theory which is to be developed.

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