TORSIONAL OSCILLATIONS OF NONBARE STRANGE STARS

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ABSTRACT

Strange stars are one of the possible compact stellar objects that can form after a supernova collapse. We consider a model of a strange star having an inner core in the color-flavor locked phase surmounted by a crystalline color superconducting (CCSC) layer. These two phases constitute the quarksphere, which we assume to be the largest and heaviest part of the strange star. The next layer consists of standard nuclear matter forming an ionic crust, hovering on the top of the quarksphere and prevented from falling by a strong dipolar electric field. The dipolar electric field arises because quark matter is confined in the quarksphere by the strong interaction, but electrons can leak outside forming an electron layer a few hundred fermi thick separating the ionic crust from the underlying quark matter. The ionic matter and the CCSC matter constitute two electromagnetically coupled crust layers. We study the torsional oscillations of these two layers. Remarkably, we find that if a fraction larger than \(10^{-4}\) of the energy of a Vela-like glitch is conveyed to a torsional oscillation, the ionic crust will likely break. The reason is that the very rigid and heavy CCSC crust layer will absorb only a small fraction of the glitch energy, leading to a large-amplitude torsional oscillation of the ionic crust. The maximum stress generated by the torsional oscillation is located inside the ionic crust and is very close to the star’s surface. This peculiar behavior leads to a much easier crust cracking than in standard neutron stars.

Key words: stars: neutron – stars: oscillations

1. INTRODUCTION

The properties of hadronic matter at densities larger than the nuclear saturation density are mostly unknown. Even though much effort has been devoted to the study of very dense hadronic matter, we still do not know the actual ground state of matter as a function of the baryonic density. In particular, we are still unable to prove or falsify the hypothesis advanced by Bodmer and Witten (Bodmer 1971; Witten 1984) that standard nuclei are not the ground state of matter. According to this hypothesis the energetically favored ground state of baryonic matter could be a collapsed state; a hadronic configuration corresponding to a (not yet experimentally observed) short-range free-energy minimum of the strong interaction. For small baryonic numbers, the collapsed state can be thought of as a droplet of quarks and gluons having a size smaller than standard nuclei. The droplet has a surface tension given by the bag pressure, which keeps the matter density inside the droplet at an almost constant value larger than the saturation density of standard nuclear matter. Therefore, with increasing baryonic number, \(A\), the size of the droplet grows as \(r \sim A^{1/3}\), as is characteristic of self-bound objects. In the following we will assume that for any baryonic density the quark droplet corresponds to the free-energy minimum of the system, meaning that it is not possible to minimize the free energy by fission processes. By contrast, standard nuclear matter can only form small clumps corresponding to fission-stable nuclei.

At vanishing temperature and considering that the only effect of the strong interaction is a bag pressure, the collapsed state can be thought of as consisting of almost free quarks filling the pertinent energy levels up to the corresponding Fermi energy. When the light-quark chemical potential exceeds the strange-quark mass, the strange-quark states start to be populated by means of weak decay processes. In this case, the collapsed state corresponds to “catalyzed” \(u, d, s\) quark matter (Witten 1984), the so-called strange matter. A very big clump of strange matter is called a strange star (Alcock et al. 1986; Haensel et al. 1986). Strange stars and neutron stars can be viewed as two different end products of supernova explosions and are classified as compact stellar objects (CSOs), which are observed as stars having a radius of about 10 km and a mass of about a solar mass, \(M_\odot\). Actually, assuming that strange matter is self-bound implies that the size of a strange star has no lower bound, meaning that strange stars much smaller than standard neutron stars can exist. Gravity plays a role only for very massive objects, restricting the mass to about 2\(M_\odot\) (for a sufficiently stiff equation of state (EOS); see for example Mannarelli et al. 2014).

Assuming that the ground state of hadronic matter consists of strange matter still does not clarify unambiguously the properties of the system. The largest value of the quark chemical potential that can be reached in massive strange stars is of the order of 400–500 MeV. Even considering this extreme case, the strong interaction is still nonperturbative and quantum chromodynamics is not under quantitative control. Therefore, approximation schemes must be used. Analyses using various models indicate that, at the densities relevant for strange stars, deconfined and cold quark matter is likely in a color superconducting (CSC) phase (see Rajagopal & Wilczek 2000; Alford et al. 2008; Anglani et al. 2014 for reviews), in which quarks form Cooper pairs, breaking the \(SU(3)\) color gauge symmetry. The reason is that the estimated critical temperature of color superconductors is at least of the order of a few MeV, much larger than the tens of keV temperature of a few seconds old CSOs. Thus, if quark matter is present in CSOs, it should be in a CSC phase.

The CSC phase is actually a collection of phases, and pinning down the favored quark pairing is not trivial. Using models based on one-gluon exchanges or on instanton exchanges, it can be shown that at asymptotic densities the color-flavor locked (CFL) phase (Alford et al. 1999) is
energetically favored. In the CFL phase, $u$, $d$, $s$ quarks of all colors pair coherently, maximizing the free-energy gain. Although this phase is very robust, it might be that strange stars do not completely consist of CFL matter. The reason is that when the effective strange quark mass has a value comparable to the quark chemical potential, a considerable free-energy penalty results for pairing the strange quarks. If the free-energy penalty is larger than the free-energy gain associated with the CFL pairing—exceeding the corresponding Chandrasekhar–Clogston limit (Chandrasekhar 1962; Clogston 1962; Anglani et al. 2014)—a different phase is favored. In the present paper we assume that the CFL phase is realized in the central and denser part of the strange star and the next favored phase down in density is the crystalline color superconducting (CCSC) phase (Alford et al. 2001a; Anglani et al. 2014), which is then realized in the outer and less dense part of the strange star (Rupak & Jaikumar 2012; Mannarelli et al. 2014). The CCSC phase is characterized by a periodic modulation of the diquark pairing. For our purposes, the most important property of the CCSC phase is that this periodic modulation is mechanically rigid, with an extremely large shear modulus (Mannarelli et al. 2007; Anglani et al. 2014). In our model the CFL phase and the CCSC phase form the quark sphere comprising most of the mass of the strange star. It is known from various model calculations (see for example Anglani et al. 2014) that the CCSC phase is the favored phase for sufficiently mismatched quark Fermi spheres, corresponding to a quark chemical potential in a certain range of values. The actual matter density at which the CCSC is favored depends on the detailed dependence of the strange quark mass on the quark chemical potential and the corresponding value of the CFL gap parameter. Since these quantities cannot be precisely computed, we will assume that at a certain radius, $R_{\text{CFL}}$, there exists a boundary between the CFL and the CCSC phases.

An important aspect of dense quark matter in CSOs is that it must be electrically neutral. The typical combined effect of charge neutrality, $\beta$ decays, and nonvanishing strange-quark mass, $M_s$, is to populate the electron states (see for example Alford et al. 2001b). Qualitatively, the reason is that a large strange-quark mass disfavors the appearance of strange quarks by light-quark $\beta$ decays, thus the electrical neutrality condition is satisfied by populating electron states. A notable exception is the CFL phase, in which the symmetric pairing induced by the strong interaction forces there to be an equal number of $u$, $d$, and $s$ quarks, thus no electrons are present. On the other hand, in the CCSC phase, the quark Fermi momenta are mismatched and electrons are needed to maintain the electrical charge neutrality. Therefore, our model of a strange star consists of a central electron-free region of CFL matter in contact with an electron-rich region of CCSC matter. The presence of electrons in the CCSC phase leads to an interesting phenomenon happening at the surface of the strange star where the strong interaction confines quarks beneath the CCSC surface. Electrons are bound only by the electromagnetic force and will typically spread outside the quark sphere for a length-scale of the order of the Debye screening length, of hundreds of fermi. A CSO made by quark matter surrounded by an electrosphere is called a bare strange star (see Alcock et al. 1986 for more details). In Mannarelli et al. (2014) we studied the torsional oscillations of bare strange stars, in particular we estimated the emitted power considering emission by an oscillating magnetic dipole, finding that it can be quite large, of the order of $10^{45}$ erg s$^{-1}$.

Besides the CCSC crust, it is possible that additional crust layers are present in a strange star. If the surface tension of quark matter is sufficiently small, a crust of charged strangelets could form close to the strange star surface (see Alford & Eby 2008). Unfortunately, the coexistence of strangelets within the CCSC crust has never been studied so far, thus we postpone the analysis of this system to future work. A second possibility is that a standard ionic crust hovers on the top of the CCSC crust. The reason is that in the formation process, or by accretion, the strange star attracts hadronic matter. Neutrons are absorbed by the quarksphere, but ions are repelled by the positively charged quark surface. Thus, ions hover on the top of the strange star as long as the repulsive electric force and the gravitational force balance. If the accreted material is sufficient, it will form an ionic crust separated from the deconfined quark matter surface by a thin electron layer. This CSO is called a nonbare strange star (Alcock et al. 1986).

In the present paper, we consider a model of a nonbare strange star having two crust layers: an inner crust layer consisting of CCSC matter and an outer crust layer consisting of standard ionic matter. For the first time we analyze the torsional oscillations of the two coupled crust layers. We focus on an analysis on $\ell = 1$ modes, corresponding to oscillatory twists of the crust. These modes do not conserve angular momentum, therefore we assume that they are triggered by events that transfer angular momentum to the crust of the strange star. A typical event of this sort is a stellar glitch. For definitiveness, we will assume that an energy of the order of that released in a Vela-like glitch is conveyed to the $\ell = 1$ modes; we will show how our results can be appropriately rescaled if a different energy scale is used. One of the most interesting results that we obtain is that the shear strain has a radial dependence, with a maximum much closer to the star surface than in standard neutron stars. Moreover, in standard neutron stars the energy of the torsional oscillations is spread across the entire crust, which is more than a kilometer thick. In the considered model of nonbare strange stars, for a sufficiently thin CCSC crust, all the energy of the torsional oscillation is conveyed in a layer a few hundred meters thick, corresponding to the ionic crust at densities below neutron drip. Therefore, the ionic crust layer will likely crack even if a small fraction, of order $10^{-4}$, of the Vela-like glitch is conveyed to the ionic crust. In other words, we show that, with respect to a standard neutron star, the probability of having a crustal crack in a nonbare strange star greatly increases thanks to the very different mechanical properties of the two crusts. Therefore, in our model the minimum energy needed for crust cracking is smaller than in standard neutron stars. We will discuss this topic in Section 4, considering various values of the parameters characterizing the ionic crust layer.

Among the possible astrophysical phenomena that might be considered in relation to crust cracking and torsional oscillations are the quasiperiodic oscillations (QPOs) observed in the tail of giant flares (see Israel et al. 2005; Strohmayer & Watts 2005, 2006). Since in our model the two crusts have very different shear moduli and densities, a nontrivial dynamical process can happen, leading to oscillation frequencies different from those obtained in previous studies of torsional oscillations in compact stars. QPOs at high frequency, in the range of kHz, turn out to be consistent with our results. However, in our
model we have neglected the magnetic field; including the magnetic field may completely change the dynamics. Note that, even in standard neutron stars, the magnetic field has a relevant role in explaining the frequencies of the QPOs (see Watts 2011; Turolla et al. 2015).

The present paper is organized as follows. In Section 2 we present a general description of the model of a nonbare strange star. In Section 3 we determine the stellar structure by solving the equations of hydrostatic equilibrium. In Section 4 we discuss the torsional oscillations of the stellar crust. In Section 5 we draw our conclusions. In the Appendix we discuss a simple toy model of two rigid slabs, determining the corresponding torsional eigenfrequencies.

2. GENERAL DESCRIPTION

The considered nonbare strange star model is depicted in Figure 1. The $u$, $d$, $s$ quark matter is radially confined by the strong interaction in the quarkphere, with radius $R_q$. This part of the star can be considered as a big hadron with an extremely large baryonic number. Since the quark chemical potential is large and the temperature is sufficiently low, deconfined quark matter is assumed to be in a CSC phase. The inner part of the quarksphere, up to the radius $R_{\text{CFL}}$, is in the CFL phase, and we treat

$$a = \frac{R_{\text{CFL}}}{R_q},$$

as a parameter. The CFL phase is surmounted by the CCSC phase, extending between $R_{\text{CFL}}$ and $R_q$. The outer part of the nonbare strange star consists of a thin electron layer, extending between $R_q$ and $R_c$. This region is only a few hundreds of fermi thick (Alcock et al. 1986), thus when considering the hydrostatic equilibrium configuration we will not distinguish between $R_q$ and $R_c$.

The next layer on the top of the electrosphere is the ionic crust, which is composed by ions and electrons. This layer extends between $R_c$ and $R$, reaching at most the neutron drip density (Alcock et al. 1986). Densities above the neutron drip point cannot be attained because the ionic crust is kept from collapsing on the underlying strange star surface by the electrostatic surface field. If the density of the crust reaches the neutron drip point, neutrons are liberated, falling on the underlying strange star surface by gravitational attraction, and are thus absorbed in the quarksphere. This process puts a limit on the maximum mass of the ionic crust (Alcock et al. 1986). The ionic crust layer of nonbare strange stars has the same mechanical properties as the outer crust of standard neutron stars, meaning that we will consider the known EOS of nuclear matter below neutron drip and the estimated values of the elastic moduli of nuclear matter. Moreover, as we will discuss below, the ionic crust is in part liquid forming the so-called ocean.

An important aspect to clarify is that while the background configuration is determined by the general relativistic hydrostatic equilibrium, the oscillations are described considering an elastic response to the mechanical stress in the Newtonian approximation. Although the descriptions of the equilibrium and of the mechanically excited state of the star seem in contradiction, they can be justified as follows. As is well known, general relativistic effects are relevant for the determination of the equilibrium configuration of compact stars, thus the background configuration is determined by solving the appropriate Tolman–Oppenheimer–Volkov (TOV) equations. These equations describe the hydrostatic equilibrium of a spherical liquid in general relativity. The liquid description is a reasonable approximation for the equilibrium state of the strange star. Indeed, for a matter density above $10^5$ g cm$^{-3}$ the pressure in the ionic crust is mainly due to degenerate electrons that can be treated as a fluid. The remaining part of the ionic crust, having lower density, has a very small mass that can be safely neglected in the solution of the hydrostatic equilibrium equations. Regarding quark matter, the CFL phase is a fluid and thus a liquid description is appropriate. The CCSC phase is rigid because of the rigidity of the periodic pattern of the gap parameter, but the equilibrium pressure of quark matter does depend weakly on the presence of the condensate. We shall assume that such a contribution can be absorbed in a liquid-like description of quark matter.

Regarding the mechanical response, we will only include the effect of the shear modulus, because we focus on torsional oscillations, which only have a tangential stress component. In this case no radial displacement occurs and a Newtonian description is applicable; however, see Schumaker & Thorne (1983) and Andersson et al. (2002) for a general relativistic discussion. Obviously, the oscillations are confined to the crusts of the nonbare strange star, because a fluid cannot support shear waves. The various interfaces between the different layers will be described by appropriate boundary conditions (BCs). Assuming that a small shear perturbation acts on the system, we expand the displacement field, the pressure,
and the matter density respectively as follows

\[ U = U_0 + u, \quad P_{\text{th}} = P_0 \delta_k - \Pi_{\text{th}}, \quad \rho = \rho_0 + \delta \rho, \quad (2) \]

where the quantities with a subscript 0 correspond to the time-independent background and the remaining quantities represent the linear perturbations.

3. BACKGROUND CONFIGURATION

We set as the equilibrium configuration the one with \( U_0 = 0 \) and assume negligible background magnetic field. Therefore, our analysis is strictly valid for slowly rotating nonmagnetar stars; however, we will briefly comment on the effect of a nonvanishing magnetic field. In the stationary state, electrons and quarks are confined in regions with a net charge and are in thermal equilibrium. This is possible because non-neutral systems in appropriate geometries can be confined and in thermodynamic equilibrium; a typical example are non-neutral plasmas (see Dubin & O’Neil 1999 for a review), which, unlike quasineutral plasmas, can be brought to equilibrium and confined.

The equilibrium is determined by the balance between the hydrostatic pressure and the gravitational attraction, which is appropriately described by the TOV equation

\[ \frac{\partial p}{\partial r} = -\frac{G(p + \rho)(m + 4\pi \rho r^3)}{r(r - 2G m)}, \quad (3) \]

where \( m(r) = \int_0^r d\rho' \rho'(r') \) and \( G \) is the gravitational constant. The gravitational potential \( \Phi \) can be determined by

\[ \frac{\partial \Phi}{\partial r} = \frac{G(m + 4\pi \rho r^3)}{r(r - 2G m)}, \quad (4) \]

once Equation (3) has been solved.

In our model two different EOSs must be used: \( p_{\text{QM}}(\rho) \) for quark matter in the quarkosphere and \( p_{\text{NM}}(\rho) \) for nuclear matter in the ionic crust. In both cases we assume that the temperature is so low that the background distribution can be approximated by a Fermi liquid at zero temperature. Quarks are strongly interacting and a first-principles calculation of the EOS is unfeasible. However, if the leading contribution of the quark interaction is quark pairing, we expect corrections of the order \( \Delta \mu^2 / \mu^2 \gtrsim 10\% \) to the EOS of the free Fermi gas. To effectively take into account the strong interaction we use the general parameterization of the quark matter EOS given in Alford et al. (2005):

\[ \Omega_{\text{QM}} = -\frac{3}{4\pi^2} a_4 \mu^4 + \frac{3}{4\pi^2} a_2 \mu^2 + B_{\text{eff}}, \quad (5) \]

where \( a_4, a_2, \) and \( B_{\text{eff}} \) are independent of the average quark chemical potential \( \mu \). We use the set of parameters \( a_4 = 0.7, a_2 = (200 \text{ MeV})^2, \) and \( B_{\text{eff}} = (165 \text{ MeV})^4 \) (similar results hold for the two sets of parameters discussed in Mannarelli et al. 2014).

For the ionic crust EOS we assume that it consists of a Coulomb crystal embedded in a degenerate electron gas. We only need an expression at a density below the neutron drip point, thus we use the data reported in Haensel & Pichon (1994); see Datta et al. (1995) for various nuclear matter EOSs. As in Glendenning & Weber (1992) we assume that the highest density of nuclear matter corresponds to the neutron drip point. This assumption corresponds to the maximization of the ionic crust radius. More refined studies show that properly taking into account the electronic pressure results in a reduction of the ionic crust mass and thickness (Martemyanov 1994; Huang & Lu 1997; Zdunik 2002). However, the detailed extent of the ionic crust is not relevant for our purposes; we are only interested in order-of-magnitude estimates of the effect of an ionic crust. Therefore, in our approach the radius, \( R_{\text{q}} \) corresponding to the boundary between the quarkosphere and the ionic crust, is determined by the condition that the pressure of quark matter equals the pressure of nuclear matter at the neutron drip point, i.e., \( p_{\text{QM}}(R_{\text{d}}) = p_{\text{NM}}(R_{\text{d}}) = p_{\text{ND}} \approx 7.8 \times 10^{-9} \text{ dyn cm}^{-2} \). The star’s radius, \( R \), is determined by the BC on the pressure \( p_{\text{ND}}(R) = 0 \), meaning that the pressure at the surface of the ionic crust vanishes. Note that the effect of the magnetic field on the matter distribution is small even when considering the extreme values characteristic of magnetars; see for example the discussion in Frieben & Rezzolla (2012).

In Figure 2 we report the matter density profile for the considered strange star model obtained by solving the TOV equation for a strange star having a mass of about 1.4 \( M_{\odot} \). The obtained star radius is \( R \approx 9.18 \text{ km} \), corresponding to the outer surface of the ionic crust layer. The matter density in the quarksphere is roughly constant, changing by less than a factor 2. There is a jump at \( R = R_{\text{q}} \approx 8.98 \text{ km} \) corresponding to the transition between quark matter and the ionic crust. The nuclear density changes by about 10 orders of magnitude across the ionic crust layer, from the iron-like density of \( \rho_{\text{Fe}} \approx 7.8 \text{ g cm}^{-3} \) to the neutron drip density \( \rho_{\text{ND}} \approx 4.3 \times 10^{11} \text{ g cm}^{-3} \). This is the same radial dependence found in the outer crust of standard neutron stars.

4. TORSIONAL OSCILLATIONS

Various modes of crust oscillations in strange stars have been studied in previous works: Chugunov (2006) analyzed the \( p \)-modes confined in the ionic crust of a nonbare strange star; Lin (2013) studied the gravitational wave emission of the \( \ell = 2 \) torsional oscillations of bare strange stars with a CCSC crust; Rupak & Jaikumar (2012) studied the \( r \)-mode instability of strange stars with a CCSC crust; Watts & Reddy (2007) studied
the torsional oscillations of strange stars without a CCSC crust, finding that the observed frequencies of QPOs pose a challenge to the considered models of strange stars. In the present paper we focus on a novel oscillation mode corresponding to torsional oscillations coupling the CCSC crust and the ionic crust.

In the Newtonian approximation and assuming a fluid description, the oscillations obey Euler’s equation

$$\rho \frac{d\mathbf{u}}{dt} + \partial_i \Pi_{ik} = 0,$$

(6)

and the continuity equation

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0.$$

(7)

For simplicity we have not considered dissipative processes and have assumed that the matter density of the various species does not change; in other words, the system is in “chemical” equilibrium. We have also omitted the collision term, meaning that we assume that collisions are so fast that local equilibrium is reached in a very short timescale as compared with that of the external forces. All the perturbations oscillate at the same frequency $\sigma$; in particular we focus on the eigenmodes

$$\mathbf{u} = e^{i \sigma t} \mathbf{\xi},$$

(8)

restricting the analysis to torsional oscillations, corresponding to transverse oscillations with no radial displacement

$$\nabla \cdot \mathbf{u} = 0, \quad u_r = 0.$$

(9)

Upon plugging the above expressions into Equation (7) it can be easily shown that no matter density fluctuation is produced.

For nonrotating and nonmagnetic stellar models the torsional oscillations do not couple with any other star oscillation, thus an eigenmodes analysis is possible. In accordance with McDermott et al. (1988) we will indicate with $\delta u$ the torsional mode having harmonic index $\ell$ and $n$ modes. The $t$-mode displacement field can be decomposed as follows:

$$\mathbf{\xi}_t = \frac{W}{\sin \theta} \frac{\partial Y_{lm}}{\partial \phi}, \quad \mathbf{\xi}_\phi = -W \frac{\partial Y_{lm}}{\partial \theta},$$

(10)

and Euler’s equation in spherical coordinates reads

$$\sigma^2 W_{t} + \frac{\nu_i}{r} \left[ -\frac{d}{dr} \left( \frac{W_{t}}r \right) - \frac{W_{t}}r \right] - \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{W_{t}}r \right) + \frac{\ell (\ell + 1)}{r^2} W_{t} = 0,$$

(11)

where $\nu_i$ is the shear modulus and $\nu_i = \sqrt{\nu_i/\rho_i}$ is the shear velocity. The index $i = 1, 2$ refers to the CCSC crust layer and to the ionic crust layer, respectively. The differential equations describing the oscillations of the two crusts are decoupled, because no long-range forces are present. However, a coupling between the two oscillations is produced by the BC, effectively describing the effect of short-range electromagnetic forces between the two crusts. The set of BCs that we use for solving the two Euler equations is the following:

$$\left( \frac{dW_{i}}{dr} - \frac{W_{i}}r \right)_{R_{CCSC}} = 0,$$

(12)

where $R_{CCSC} \approx 9.15 \text{ km}$ corresponds to the boundary between the solid crust and the ocean (at $\rho \approx 10^7 \text{ g cm}^{-3}$). Equations (12), (13), (15) describe the no-traction BCs and Equation (14) the no-slip BC. The no-traction BCs simply mean that no net force acts on any boundary between two adjacent layers. The no-slip BC means that the displacement of the two crust layers at the interface is the same. This is a reasonable assumption since there exists a very large electrostatic field between the two crusts, of the order of $10^{17} \text{ V cm}^{-1}$ (see Alcock et al. 1986). Thus, we are assuming that the static friction at this interface is so strong that at any time the two crust oscillations have the same amplitude and frequency, $\sigma$. Note that the assumption that the oscillations in the two crusts have the same phase and frequency was implicitly made in Equation (11).

At the CCSC–CFL boundary we only impose the no-traction BC, Equation (12), but the two materials have free slip. The reason is that the CFL phase is a neutral superfluid, therefore it should not react with a force to the oscillation of the CCSC internal surface. In this case we are assuming that no vortices are created by the oscillation of the CCSC boundary and that the temperature is so low that thermal fluctuations in the CFL superfluid can be neglected. A similar reasoning applies to the ionic crust–ocean boundary, which is described by the no-traction BC, Equation (14), and free slip.

The shear moduli and densities of the two crust layers are fundamental ingredients for determining the solutions of Equation (11). We already determined the matter density profile in Section 3. Regarding the shear modulus of the CCSC phase, it can be obtained from the low-energy Lagrangian description in terms of phonon-like excitations (see Casalbuoni et al. 2001, 2002a, 2002b; Mannarelli et al. 2007). In particular, the coefficients of the quadratic terms of the low-energy Lagrangian determine the linear response of the crystal to excitations. Unfortunately, various approximations have to be used to extract the shear modulus: the procedure used in Mannarelli et al. (2007) relies on a Ginzburg–Landau (GL) expansion of the Nambu–Jona-Lasinio model used to mimic the properties of quark matter at the relevant baryonic densities. It is known that this procedure is not under quantitative control (see Mannarelli et al. 2006), moreover the recent analyses of two flavor systems (Cao et al. 2015) seem to indicate that many terms in the GL expansion must be included to have a controlled approximation scheme. Using the GL expansion up to order $\Delta^2$, it was found in Mannarelli et al. (2007) that

$$\nu_{CCSC} \approx \nu_0 \left( \frac{\Delta (\mu) \text{ MeV}}{10 \text{ MeV}} \right)^2 \left( \frac{\mu}{400 \text{ MeV}} \right)^2,$$

(16)

where

$$\nu_0 = 2.47 \text{ MeV fm}^3,$$

(17)
will be our reference value. We will assume that the shear modulus is constant within the CCSC crust and hereafter we will identify the CCSC shear modulus with the reference value. It is certainly true that \( \nu_{\text{CCSC}} \) depends on the quark chemical potential \( \mu \). However, in the interior of compact stars the quark chemical potential is almost constant. Moreover, as we will see below, the most relevant aspect is that \( \nu_{\text{CCSC}} \) is much larger than the shear modulus of the iron crust. The quark matter density does not strongly depend on the radial coordinate. Therefore, we will consider the constant value \( \rho_{\text{QM}} = 10^{15} \text{ g cm}^{-3} \), a typical quark matter value, see Figure 2.

The shear modulus of the ionic crust depends on the particular crystalline structure considered and on the plane of application of the shear stress. Early calculations of the shear modulus of monovalent crystals were performed by Fuchs (1936). In compact stars the orientation of the crystals is unknown, but it is possible to define an effective shear modulus, \( \nu_{\text{eff}} \), averaging over directions, as shown in Strohmayer et al. (1991), which should give an excellent approximate result if the crust has a polycrystalline structure. From dimensional analysis and considering that the rigidity is due to the electromagnetic interaction between ions, it is clear that \( \nu_{\text{eff}}(r) \propto (Z(r)e^2n_N(r)/a_N(r))^2/3 \), where \( Z(r) \) is the radially dependent proton number and \( n_N(r) \) is the number density of nuclei. The order-of-magnitude estimate of the shear modulus gives a very large value, about \( 10^{30} \text{ dyn cm}^{-2} \) for the inner crust, as already noted in Smoluchowski (1970). Using Monte Carlo simulations (see Strohmayer et al. 1991) or molecular dynamics methods (see Hoffman & Heyl 2012), eventually including quantum fluctuations as in Baiko (2011), one can estimate the proportionality factor, finding that

\[
\nu_{\text{eff}}(r) = \frac{c n_N(r)Ze^2}{a_N(r)},
\]

(18)

where \( a_N(r) = (3/(4\pi n_N(r)))^{1/3} \) is the average inter-ion spacing and \( c \approx 0.1 \) gives an approximate result at any density larger than \( 10^6 \text{ g cm}^{-3} \), where electrons form a degenerate Fermi gas and the Coulomb crystal model can be applied. At lower density, the Coulomb crystal model cannot be applied and Equation (18) does not apply. As an example, extrapolating Equation (18) to the star’s surface does not reproduce the known shear modulus of iron.

Summarizing, the matter densities and shear moduli of the nonbare strange star are approximately given by

\[
\nu = \begin{cases} 
0 & \text{for } r < R_{\text{CFL}} \\
\nu_0 & \text{for } R_{\text{CFL}} < r < R_q \\
\nu_{\text{eff}}(r) & \text{for } R_q < r < R_{\text{ocean}}
\end{cases}
\]

(19)

and

\[
\rho = \begin{cases} 
\rho_{\text{QM}} & \text{for } R_{\text{CFL}} < r < R_q \\
\rho_{\text{NM}}(r) & \text{for } R_q < r < R_{\text{ocean}}
\end{cases}
\]

(20)

with \( \rho_{\text{QM}} = 10^{15} \text{ g cm}^{-3} \) and the CCSC shear modulus given in Equation (17). For a realistic description of amplitude and frequency of the torsional oscillations, the radial dependence of the nuclear matter density and of the effective shear modulus must be appropriately taken into account in Equation (11). However, to disentangle the various aspects of the problem in Section 4.1 we will approximate the ionic crust as a homogeneous system. We turn to a discussion of the inhomogeneous ionic crust layer in Section 4.2.

### 4.1. System of Two Homogeneous Crusts

Let us approximate the nonbare strange star crust as consisting of two homogeneous crust layers. We consider two different parameter sets for characterizing the properties of the ionic crust. Parameter set A corresponds to the nuclear matter density and the effective shear modulus at the inner surface of the ionic crust, respectively given by \( \rho_{\text{NM}}(R_q) = \rho_{\text{ND}} \) and \( \nu_{\text{eff}}(R_q) \approx 3.4 \times 10^{-5} \text{ MeV fm}^{-3} \). Parameter set B corresponds to the nuclear matter density and the effective shear modulus near the ocean surface, respectively given by \( \rho(R_{\text{ocean}}) = 2.8 \times 10^9 \text{ g cm}^{-3} \) and \( \nu_{\text{eff}}(R_{\text{ocean}}) = 2.1 \times 10^{-6} \text{ MeV fm}^{-3} \).

Since both crust layers are homogeneous, Equation (11) can be cast in the form of two Bessel equations, and we can determine an analytic expression for the oscillations for any \( \ell \). In the following we focus on \( \ell = 1 \), but similar results hold for different values of \( \ell \). Imposing the BCs given in Equations (12) and (15), we find that

\[
W_1(r) = \frac{1}{\sqrt{\ell}} C_1 (J_{3/2}(r\sigma/v_1) - K_1 Y_{3/2}(r\sigma/v_1)),
\]

(21)

where \( J_n \) and \( Y_n \) are Bessel functions of the first and second kinds, respectively, and

\[
K_1 = \frac{J_{5/2}(R_{\text{CFL}}\sigma_1/v_1)}{Y_{5/2}(R_{\text{CFL}}\sigma_1/v_1)},
\]

(22)

and

\[
K_2 = \frac{J_{5/2}(R_{\text{ocean}}\sigma_2/v_2)}{Y_{5/2}(R_{\text{ocean}}\sigma_2/v_2)}.
\]

(23)

From the two remaining BCs, Equations (13) and (14), we can determine the ratio \( C_1/C_2 \) and the eigenmode frequency.

In Figure 3 we report the dependence of the frequency of the first three eigenmodes on \( \sigma \), see Equation (1). The solid blue line corresponds to the \( \ell_1 \) mode, the dashed red line corresponds to the \( \ell_2 \) mode, and the dotted green line corresponds to the \( \ell_3 \) mode. The results reported in the upper panel are obtained with parameter set A, while the results reported in the bottom panel are obtained with parameter set B.

Let us focus on the fundamental \( \ell_1 \) mode obtained with parameter set A. For \( a \lesssim 0.75 \) the frequency of this mode increases with increasing \( a \). In this range of values of \( a \) the frequency and the amplitude of the oscillations are very similar to the results obtained in Mannarelli et al. (2014) for a bare strange star and are weakly dependent on the mechanical properties of the ionic crust layer. For \( a \gtrsim 0.75 \) the frequency becomes almost independent of \( a \) and the opposite behavior happens, with torsional oscillation segregated in the ionic crust layer with a frequency determined by the mechanical properties of the ionic lattice. Indeed, this frequency corresponds to \( \omega_2 \approx (\ell_2/2) \times v_2/(R - R_q) \) as described in the Appendix using a simpler model with planar geometry. The transition point between the two regimes corresponds to the crossing between the frequency of the \( n = 1 \) mode and that of the \( n = 2 \) mode and is approximately given by \( a^* = 1 - (2\nu_1/R_q) \times (R - R_q)/\nu_2 \approx 0.75 \), as determined in the planar approximation. The transition between the two...
behaviors corresponds to a node at the interface between the two crust layers.

The modes with more nodes have a similar behavior. For the $t_2$ mode there are two crossing points, one with the mode with $n = 3$ and one with the mode with $n = 1$. As can be seen in the upper panel of Figure 3 there exist two regions in which the frequency of the $t_2$ mode is almost independent of $a$ and two regions in which it is strongly dependent on $a$. The $t_3$ mode also has a similar behavior. We have checked that, for any considered mode, the range of values for which the frequency is almost constant corresponds to torsional oscillation basically confined to the ionic crust layer. Note that for any value of $a$ there is at least one of the three modes that is weakly dependent on $a$, meaning that there is at least one mode that is segregated to the ionic crust. For parameter set B, the frequencies of the three considered modes are independent of $a$ for any value of $a$, see the bottom panel of Figure 3. Therefore, for parameter set B the low-lying $t$-mode frequencies are completely determined by the mechanical properties of the ionic crust layer.

To better understand the difference between the two regions of frequencies, respectively dependent on and constant in $a$, let us discuss in detail the amplitude of the $t_1$ mode. In Figure 4 we show the amplitude of the $t_1$ mode obtained for parameter set A, considering $a = 0.4$ (upper panel) and $a = 0.8$ (bottom panel). To determine the absolute values of this amplitude we assume that all the energy of a Vela-like glitch, $E_{\text{Vela}} \sim 5 \times 10^{42}$ erg, is conveyed to the $t_1$ mode. If a fraction $\alpha$ of the Vela-like glitch is considered, then the amplitude has to be scaled by $\sqrt{\alpha}$. For $a = 0.4$, we obtain that the CCSC crust and the ionic crust oscillate with a similar amplitude that is order of tens of centimeters. For $a = 0.8$ the oscillation of the CCSC crust is negligible and the mode is segregated to the ionic crust layer. The energy of the glitch conveyed in the thin ionic layer generates an amplitude of the order of tens of meters. This result confirms the expectation that $t$-modes having an $a$-independent frequency are basically ionic crust oscillations (ICOs). We can therefore classify the nonbare strange star oscillations as coupled crust oscillations (CCOs) and ICOs. CCOs are $a$-dependent torsional oscillations with comparable amplitude in both crusts. ICOs are $a$-independent oscillations.
angular coordinates, but for simplicity we called the shear traction considering the deformation of the crust. Let us understand the physical behavior of the system that reported in the bottom panel of Figure 4. Therefore, for modes is negligible and the corresponding amplitude is always extremely large and can reach values of kilometers at the surface. Note that the absolute value of the amplitude in this case is

\[ a_1 = 0.4. \]

In both cases the strains are obtained assuming that the energy of a Vela-like glitch is conveyed into a \( t_1 \) mode oscillation.

Having negligible amplitude in the CCSC crust and a large amplitude in the ionic crust.

For parameter set B the dependence on \( a \) of the first three modes is negligible and the corresponding amplitude is always confined in the ionic crust layer with a behavior very similar to that reported in the bottom panel of Figure 4. Therefore, for parameter set B the considered torsional oscillations are ICOs. Note that the absolute value of the amplitude in this case is extremely large and can reach values of kilometers at the star’s surface. Let us understand the physical behavior of the system considering the deformation of the crust.

A measure of a solid deformation is the shear strain (also called the shear traction). The shear strain depends on the angular coordinates, but for simplicity we define

\[ [s] = \left| \frac{dW}{dr} - \frac{W}{r} \right|. \]

which only depends on the radial coordinate; the shear strain at any angle can be obtained multiplying \( [s] \) by the appropriate angular function, see Equation (10). We focus on the \( t_1 \) mode and in Figure 5 we report the corresponding shear strain across the ionic crust (the deformation of the CCSC crust layer is always much smaller). The upper panel of Figure 5 corresponds to the ionic crust strain induced by the \( t_1 \) ICO obtained with the parameter set A and \( a = 0.8 \), that is, the reported strain corresponds to the amplitude shown in the lower panel of Figure 4, showing the extreme values for the deformation discussed above. The bottom panel of Figure 5 corresponds to the ionic crust strain induced by the \( t_1 \) ICO obtained with the parameter set B and \( a = 0.4 \). Note that in both panels of Figure 5 the strain has a maximum at the boundary between the CCSC crust layer and the ionic-crust layer. The shear strain is a monotonic decreasing function of the radial coordinate because the ionic crust layer is homogenous and because the shear strain at the surface has to vanish because of the no-traction BC. As we will see in the next section, considering an inhomogenous matter density results in a nonmonotonic shear strain. For all considered cases the deformation of the ionic crust layer is so large that the crust very likely cracks before reaching such a large deformation. We will discuss crust cracking in Section 5; we now turn to a more realistic description of the ionic crust layer.

4.2. Inhomogeneous Ionic Crust

We now include the radial dependence of the mechanical properties of the ionic crust matter, while keeping both the matter density of the quark matter and the shear modulus of the CCSC crust layer constant. In particular, we numerically solve Equation (11) including the radial dependence of the ionic crust shear velocity, \( v_2(r) \), and shear modulus, \( \nu_2(r) \). The frequencies of the first three torsional eigenmodes are reported in Figure 6. This figure is rather similar to the one reported in the upper panel of Figure 3, showing that parameter set A is the one that gives a reasonably good approximation of the ionic crust, as far as \( t \)-mode oscillations are concerned.

Let us focus on the \( t_1 \) mode. For small values of \( a \) the frequency strongly depends on \( a \) and the oscillation amplitude is very large in both crusts; see the upper panel of Figure 7 for a representative behavior obtained with \( a = 0.4 \). Thus, this oscillation can be classified as a CCO, as defined in the previous section. This CCO amplitude is very similar to the one reported in the upper panel of Figure 4; the reason is that the

![Figure 5](image-url)

**Figure 5.** Absolute values of the shear strain, defined in Equation (24), across the ionic crust layer for two different ICOs. Top: results obtained with parameter set A and for \( a = 0.8 \). Bottom: results obtained with parameter set B and for \( a = 0.4 \). In both cases the strains are obtained assuming that the energy of a Vela-like glitch is conveyed into a \( t_1 \) mode oscillation.

![Figure 6](image-url)

**Figure 6.** Frequency of the \( t \)-modes with \( \ell = 1 \) as a function of \( a \) for different numbers of nodes: \( n = 1 \) mode, solid blue; \( n = 2 \) mode, dashed red; \( n = 3 \) mode, dotted green. These results have been obtained by considering the CCSC crust layer to be homogenous and a radial dependence of the shear modulus and of the matter density of the ionic crust layer.
same values of the CCSC crust density and shear modulus have been used. Increasing the value of $a$, one reaches a critical value around $a^\ast \sim 0.7$ for which there is a crossing between the $n = 1$ and the $n = 2$ modes. For $a > a^\ast$ the amplitude of the oscillations is mainly confined in the ionic crust layer and the oscillation frequency is almost independent of $a$, corresponding to an ICO as defined in the previous section. In the bottom panel of Figure 7 is reported the amplitude of the $\iota_1$ ICO obtained for $a = 0.8$. In this case, the qualitative behavior of the oscillation amplitude is very similar to the one reported in the bottom panel of Figure 4, but the amplitude of the oscillation is now larger by about an order of magnitude. The reason is that in this case the light matter at the top of the ionic crust is easily displaced by the torsional oscillation. This effect is akin to the seismic site effect leading to the amplification of earthquake seismic waves in the presence of certain geological conditions. In other words, by considering a constant density one underestimates the matter displacement.

As in the case of a homogenous ionic crust considered in the previous section, the amplitude of the oscillations is very large and indicates that a crust cracking may occur.

The shear strain corresponding to the ICO is reported in the upper panel of Figure 8 as a function of the radius. Comparing Figure 5 and the upper panel of Figure 8, one can see that a new feature has appeared. The shear strain does not have a maximum at the CCSC crust–ionic crust boundary. The reason is that the shear modulus of the ionic crust decreases on moving toward the surface and it is therefore easily displaced by a torsional oscillation. However, the shear strain at the surface of the star must vanish, because it corresponds to the no-traction BC, thus a maximum is produced close to the star’s surface.

This characteristic behavior of the shear strain is present also for the CCOs and the maximum is located where the inhomogeneous term in Equation (11) is maximum. In the bottom panel of Figure 8 we show the maximum value obtained for the strain as a function of the parameter $a$. From this picture it is clear that for any value of $a$ the shear strain is large, larger than $10^{-4}$, possibly leading to crust cracking. We will discuss this issue in more detail in Section 5.

### 4.3. Temperature and Magnetic Field Effects

So far we have neglected the effects of the temperature and of the magnetic field of the compact star. In this section we discuss the range of validity of the presented analysis; in particular we estimate the range of temperature and magnetic field for which the presented analysis is approximately applicable.

Regarding the temperature effects, it is known that the shear modulus decreases with increasing temperature, vanishing at the melting temperature. The surface temperature of a typical
compact star is about \(10^5 \sim 10^6\) K, but a larger temperature is attained in the interior (see for example Shapiro & Teukolsky 1983). In any case, the typical energy scale of the CCSC crust is tens of MeV, thus the temperature effect in the CCSC crust layer can be neglected. On the other hand, a fraction of the ionic crust is believed to be strongly affected by the temperature. We will estimate the temperature at which the present analysis is valid by considering the amount of ionic crust that is liquid, forming the ocean of the compact star. The existence of the ocean is due to the fact that the surface temperature of compact stars is higher than the melting temperature of some chemical element of the ionic crust. The transition from solid to liquid can be semiquantitatively determined by the ratio between the typical Coulomb energy and the thermal energy:

\[
\Gamma = \frac{Z(r)^2 e^2}{a_Y(r) T(r)}.
\]  

(25)

with the liquid/solid transition taking place at \(\Gamma = 175\). Therefore, the radial position of the liquid/solid transition point in the ionic crust depends on the local value of the temperature. Assuming a constant temperature in the ionic crust (which should be a very good thermal conductor), we can easily estimate the radial point corresponding to the solid/liquid phase transition in the ionic crust, see the upper panel of Figure 9. For \(T = 3 \times 10^7\) K, corresponding to the dashed red line in the upper panel of Figure 9, only a small fraction of the crust is liquid. The value of the stellar density at which the transition takes place is \(\rho \sim 10^7 \text{g cm}^{-3}\). Therefore, in the present analysis we have assumed that the crust has at most a temperature of the order of \(10^7\) K. For higher temperatures, indeed, say for \(T = 3 \times 10^8\) K corresponding to the solid blue line in the upper panel of Figure 9, most of the ionic crust is liquid.

Regarding the effect of the magnetic field, it is very complicated to properly take it into account (see Turolla et al. 2015). We focus on just two possible effects: a local effect on the net charge distribution located at \(r \sim R_q\) and a magnetohydrodynamic (MHD) effect in the bulk. The local effect corresponds to a deformation of the charge distribution at \(R_q\), and can be properly absorbed in the BCs. Thus, it leads to a shift of the quantized frequencies. Unless one considers magnetic fields \(B > 10^{14}\) G this shift is very small because the electric field energy density in the electron layer is enormous (Alcock et al. 1986). The bulk magnetic field produces a Lorentz restoring force that competes with the shear stress. If the Lorentz restoring force is bigger than the shear stress the associated oscillations are Alfvén waves. In particular, if the Alfvén velocity is much larger than the shear velocity, then MHD effects are dominant. In the bottom panel of Figure 9 we compare the shear velocity with the Alfvén velocity for three different values of the magnetic field. For \(B \leq 10^{13}\) G the shear velocity dominates and the presented analysis applies. Note that Alfvén waves are present even if the crust melts.

5. CONCLUSIONS

We have considered a model of a nonbare strange star comprising CSC quark matter surmounted by a standard nuclear matter crust. The internal part of the quarksphere is in the CFL phase; the external part of the quarksphere is in the CCSC phase and is separated from the ionic crust by an electron layer a few hundred fermi thick. We have determined the background configuration by solving the pertinent TOV equation considering a simple parameterization of the EOS of quark matter and a realistic EOS for the description of the ionic crust. We have considered one of the possible stellar structures in the sequence of configurations that solve the TOV equations. Similar results can be obtained if considering strange stars with different masses and radii.

Both the CCSC and the ionic crust are rigid and we have studied the torsional oscillations supported by these two electromagnetically coupled crusts. We have classified the torsional oscillations as CCOs, oscillations of the two crusts with comparable amplitude, and ICOs, oscillations confined in the ionic crust. CCOs are completely negligible if the CCSC crust is thin, say less than \(\approx 2\) km, and only ICOs are relevant. We have considered both a simplified model in which the two crusts are homogeneous and a more realistic model in which the ionic crust density and shear modulus are radially dependent. In any case we have assumed that the CCSC crust is homogeneous, because the density in the quarksphere changes by less than a factor 2. We have obtained that the oscillation frequencies of the \(\ell = 1\) modes are of the order of \(10\) kHz, and are not very sensitive to the extent of the CCSC

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**Figure 9.** Top: Coulomb parameter for two different values of the temperature. The horizontal line corresponds to \(\Gamma = 175\), which is the value that separates the liquid and solid phases. Bottom: shear and Alfvén velocities. The \(v_{s,1}\) curve is obtained for \(B = 10^{13}\) G, the \(v_{a,2}\) curve for \(B = 10^{14}\) G, and the \(v_{a,3}\) curve for \(B = 10^{15}\) G.
crust. The $\ell = 1$ modes correspond to oscillatory twists of the crust and do not conserve angular momentum, therefore these modes are activated by events that transfer angular momentum to the strange star’s crust. A typical event of this sort is a pulsar glitch associated with vortex drift from the CFL core to the CCSC crust. For that reason, we have assumed that the energy conveyed to the $t$-mode oscillation is of the order of that of a Vela-like glitch. For definiteness we have assumed that all the energy of a Vela-like glitch triggers one single mode. The obtained deformation of the ionic crust is very large even considering CCOS. If a fraction $\alpha$ of the Vela-like glitch energy is conveyed to the considered $t$-mode, then the amplitude and the shear deformation must be scaled by $\sqrt{\alpha}$. If $\alpha \sim 1$, the strain is such that it will possibly break the ionic crust layer. If the CCSC crust is sufficiently thin to segregate most of the oscillation in the ionic crust, a much smaller energy suffices to produce a large deformation of the ionic crust: even considering $\alpha \sim 10^{-3}$, the shear strain on the ionic crust is of order 0.1. Crust cracking happens if the shear strain is larger than the breaking strain, $s_{\text{max}}$, see Kittel (1976) for a discussion of the breaking strain in standard materials. The breaking strain of the ionic crust is highly uncertain, indeed it is not known which is the microscopic mechanism responsible for the nonelastic response. The widely used range of values is between $s_{\text{max}} = 10^{-4}$ and $10^{-2}$, but values of $10^{-1}$ could be appropriate for perfect crystals without defects (Kittel 1976), as also shown by recent results obtained by molecular dynamics simulations of Coulomb crystals (Horowitz & Kadau 2009). In order to properly assess where and when the crust breaks, it would be important to compute the breaking strain as a function of the radial distance. The result of our computation is, indeed, that the maximum strain produced by a $t_1$ oscillation is located a few tens of meters below the surface of the ionic crust. Therefore, this is the part of the ionic crust that will likely break during a glitch. We are not aware of any observable that might be related to the breaking of the ionic crust at a specific radial distance. Possibly, giant gamma-ray bursts (see Thompson & Duncan 2001) and QPOs (see Israel et al. 2005; Strohmayer & Watts 2005, 2006; Glampedakis et al. 2006; Watts & Reddy 2007; Watts 2011) might be related to such a phenomenon. However, our model cannot reproduce the low-frequency QPOs observed in giant flares. We postpone to future work the study of magnetoelastic oscillations including a very high magnetic field, possibly leading to lower oscillation frequencies as in standard magnetar models.

It would be interesting if some other specific observable could help to distinguish the crust-breaking process of our strange star model from standard crustal breaks. Note that the maximum of the shear strain in the present model is much closer to the star’s surface than in standard neutron stars (see for example McDermott et al. 1988). Moreover, in standard neutron stars the energy of the $t$-modes is spread across the entire crust, extending for more than a kilometer below the star’s surface. By contrast, in the considered model of nonbare strange stars, all the energy of the $t$-modes is conveyed in a layer few hundred meters thick, corresponding to the ionic crust at densities below neutron drip. For this reason, in our model it is easier to produce the crust cracking than in standard neutron stars, meaning that a smaller amount of energy is needed to crack the ionic crust than in a standard neutron star scenario. It is important to realize that in our discussion we have considered the less favorable model of the ionic crust for crust cracking: since the effect of a reduced crust layer is to maximize the deformation, in our model we have taken the largest possible extent of the ionic crust layer, underestimating the shear strain.

As a final remark, note that if the $t$-oscillations are not triggered by angular momentum transfer to the strange star crust, then $\ell = 1$ modes cannot be excited. Then, the low-lying oscillating mode is the $2\ell_0$ mode. We find that the oscillation frequency of this mode is of the order of 5 kHz and it is weakly dependent on $a$. Similar results were reported in Lin (2013), where a nonbare strange star model with a CCSC core was considered. Assuming that a Vela-like glitch energy is conveyed to this mode, we find that the corresponding shear strain is suppressed with respect to the $\ell = 1$ mode by about three orders of magnitude. We expect that similar results hold for higher values of $\ell$, meaning that—assuming that an equal amount of kinetic energy is deposited to the modes with different value of $\ell$—the $\ell = 1$ mode will produce the larger shear strain.

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APPENDIX

OSCILLATIONS OF TWO TIGHTLY BOUND RIGID SLABS

Consider the system depicted in Figure 10 consisting of two slabs with a common contact surface. Suppose that a torque is applied to the structure. In standard conditions the two slabs would easily slide along the common surface with some kinetic friction. If the applied force is below a threshold value, the static friction prevents the relative motion and the whole system oscillates transversely around the equilibrium configuration. In common materials, the static friction is due to electrostatic Van der Waals forces and is extremely small. Suppose, however, that a strong electrostatic field is present along the contact surface, in such a way that the two surfaces

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure10}
\caption{Torque acting on a system of two homogenous slabs in contact. We assume that the static friction between the two slabs is so large that the shear oscillations obey the no-slip boundary condition.}
\end{figure}
are tightly bound. Let us consider the fictitious extreme case in which the binding force between the two surfaces is of the same order as or larger than the force between the atoms in the two slabs. Then, the force necessary to make the two surfaces slide is larger than the force needed to break the bonds in the bulk of the two slabs. The system of two slabs can be considered as one single inhomogeneous body with a step-like matter density and shear modulus. In these conditions, the transverse displacement field, \( u \), is a continuous function across the surface (no-slip BC):

\[
 u_1(0^-) = u_2(0^+),
\]  

(26)

where we assumed that \( z = 0 \) corresponds to the contact surface. The no-slip BC leads to

\[
 \nu_1 \partial_z u_1(0^-) = \nu_2 \partial_z u_2(0^+),
\]  

(27)

meaning that for unequal shear moduli, \( \nu_1 \), the derivative of the transverse displacement field is not continuous at the interface (a cuspid point). Note that the matter densities of the two slabs do not explicitly appear in this equation; indeed the shear moduli have the dimensions of a pressure. The matter density dependence is hidden in \( \nu_1 \) which in general is a function of the molecular binding force and density. The above no-slip BC basically states that the transverse pressure on the contact surface must vanish.

The frequencies of the quantized oscillations are obtained and supplement Equations (26) and (27) with no-slip BCs for the free surfaces, which for the considered planar geometry can be expressed as the Neumann BCs \( \partial_z u_1(-D_1) = 0 = \partial_z u_2(D_2) \). In the linear approximation, a straightforward calculation gives

\[
 (\nu_1 v_2 + \nu_2 v_1) \sin \left( \frac{D_1}{v_1} + \frac{D_2}{v_2} \right) = (\nu_2 v_1 - \nu_1 v_2) \sin \left( \frac{D_1}{v_2} - \frac{D_1}{v_1} \right),
\]  

(28)

where \( v_1 = \sqrt{\nu_1/\rho_1} \) are the shear velocities. For \( \nu_1 v_2 > \nu_2 v_1 \) we obtain the approximate solution

\[
 2 \sin \left( \frac{D_1}{v_1} \right) \cos \left( \frac{D_2}{v_2} \right) = 0,
\]  

(29)

showing that in this case there are two distinct frequencies of oscillations, one determined by the mechanical property of slab 1 and one by the property of slab 2. Note that the frequency quantization

\[
 \omega_n = \frac{n \pi v_1}{D_1},
\]  

(30)

is the one that is obtained by imposing Dirichlet BCs for a single slab with width \( D_1 \) and shear velocity \( v_1 \). On the other hand, the frequency quantization

\[
 \omega_n = \frac{(2n-1) \pi v_2}{2D_2},
\]  

(31)

is the one that is obtained by imposing Neumann BCs considering a single slab with width \( D_2 \) and shear velocity \( v_2 \). Note that we can recast the condition \( \nu_1 v_2 \geq \nu_2 v_1 \) as

\[
 \sqrt{\frac{\nu_1}{\nu_2}} \gg \sqrt{\frac{\rho_2}{\rho_1}},
\]  

(32)

and assuming that the slab 1 describes the CCSC crust and that the slab 2 describes the ionic crust, this inequality is certainly satisfied. Therefore, even in the planar approximation, the oscillation can be divided into ICOs with eigenfrequencies given in Equation (30) and CCOs with eigenfrequencies given in Equation (31). As we have seen in Section 4.1, this classification remains approximately valid in the spherically symmetric case.

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