ON THE STABILITY OF A MULTI-AGENT SYSTEM SATISFYING A GENERALIZED LIÉNARD EQUATION

CARL KOLON, CONSTANTINE MEDYNETS, AND IRINA POPOVICI

ABSTRACT. We study the stability of the system of \( n \)-coupled second-order differential equations
\[
\ddot{r}_k = (1 - |\dot{r}_k|^2)\dot{r}_k - (r_k - R), \quad r_k \in \mathbb{R}^2, \quad R = (r_1 + \cdots + r_n)/n.
\]
Previous numerical experiments have shown that for a large set of initial conditions the system converges to a rotating limit cycle with a fixed center of mass (the coordinates of the center of mass depend on the initial conditions), dubbed a ring state. We prove that a ring state (particles spinning in the same direction) is stable and we show that every solution that starts near a ring state asymptotically approaches a ring state. The proof uses singular perturbations ideas to improve the approximations of the flow on the center manifold in the presence of non-isolated fixed points. We also provide the full description of limit cycles and the stability analysis for the decoupled system \((R = \text{const})\).

1. INTRODUCTION

This note is devoted to the study of the second-order \( n \)-particle system (oscillators) on the plane governed by the equations:
\[
\ddot{r}_k = (1 - |\dot{r}_k|^2)\dot{r}_k - (r_k - R)
\]
\[
R = \frac{1}{n} \sum_{j=1}^{n} r_j
\]
Here, \( r_k \) represents the two-dimensional position vector of the \( k \)-th particle and \( \dot{r}_k \) stands for the time derivative \( \frac{d}{dt} \). Every agent accelerates based on a nonlinear function of its velocity, sometimes referred to as self-propulsion, and a linear attraction to the center of mass. This model has been studied before as a tool in collective/opinion dynamics with applications to pattern formation in robotics and biological systems [18, 20, 6].

Numerical simulations have captured a range of limit behaviors for this \( 4n \)-dimensional system, including configurations in which the agents rotate around a stationary center of mass, \( r_k(t) = e^{\pm i(t + \theta_k)} + R_0 \) (Figure 1), and perturbations of the linear motion with unit speed, \( r_k(t) = v_0 t + d, v_0, d \in \mathbb{R}^2 \) (Figure 2). Following [20], we call (i) the circular limit cycles ring states when all agents rotate in the same direction about the stationary center of mass and (ii) identical linear solutions translating states.

In this paper, we investigate the stability of ring states. We note that the collection of ring states contains an \((n - 2)\)-dimensional submanifold consisting of (non-isolated) fixed points when viewed from a special rotating frame of reference. The presence of non-isolated fixed points in dynamical systems renders ineffective the usual simplifications and dimension-reduction techniques such as normal forms.
Figure 1. Asymptotic behavior of (1) in the basin of attraction of a non-degenerate ring state for \( t \in [0, 18] \). The center of mass \( R(t) \) at \( t = 18 \) is indicated as a red dot. The dashed line represents the trajectory of \( R(t) \). The positions of the agents at \( t = 0 \) (at \( t = 18 \)) are represented as hollow (solid) dots. Agents eventually rotate around the circle of radius 1 centered at \( R(t) \) at a constant speed of 1, as the center of mass becomes stationary. In ring states, the polar angles \( \{ \theta_k \} \) of the agents about the center of mass satisfy \( \sum_{k=1}^{n} e^{i\theta_k} = 0 \).

or Taylor approximations of the flow on the center manifold. In our proof of stability, we overcome this limitation by introducing a new technique for approximating the center manifold of dynamical systems with non-isolated fixed points, Section 5.3 and Appendix. The following is the main result of the paper.

**Theorem 1.1.** Every ring state solution of Equation (1) is stable. Furthermore, every solution that starts near a ring state converges to a nearby ring state.

More specifically, we show that for any \( \epsilon > 0 \) there exists \( \delta > 0 \) such that given any initial center of mass location \( R_0 \) and polar angles \( \theta_{0,k} \) such that \( \sum_{j=1}^{n} e^{i\theta_{0,k}} = 0 \) if \( r_k(0) \) and \( \dot{r}_k(0) \) satisfy \( |r_k(0) - e^{i\theta_{0,k}}| < \delta \) and \( |\dot{r}_k(0) - ie^{i\theta_{0,k}}| < \delta \), then there exists \( R_\infty \) and \( \theta_{\infty,k} \) with \( \sum_{j=1}^{n} e^{i\theta_{\infty,k}} = 0 \) such that

\[
|r_k(t) - (R_0 + e^{i\theta_{0,k}} e^{it})| < \epsilon, \quad |\dot{r}_k(t) - ie^{i\theta_{0,k}} e^{it}| < \epsilon
\]

and

\[
\lim_{t \to \infty} \left[ r_k(t) - (R_\infty + e^{i\theta_{\infty,k}} e^{it}) \right] = 0, \quad \lim_{t \to \infty} \left[ \dot{r}_k(t) - ie^{i\theta_{\infty,k}} e^{it} \right] = 0.
\]

Figure 2. Translating state behavior of ten agents with random initial positions in the square \([-1, 1] \times [-1, 1]\) and random initial velocities drawn from the square \([0, 1] \times [0, 1]\). The curves represent the agents’ trajectories.
It is interesting to notice that by restricting the motion of \((1)\) to a line through the origin, that is by considering a 1D particle system, and by switching to the velocity-acceleration coordinates, we obtain a system of coupled van der Pol equations. Some classes of coupled van der Pol equations have been shown to admit nontrivial periodic cycles [13].

The dynamics of \((1)\) in the plane when the center of mass is stationary (referred to as the decoupled system) is described by the equation \(\ddot{r} = (1 - |\dot{r}|^2)\dot{r} - r\), which belongs to the class of generalized Liénard systems. If the motion of the decoupled system is restricted to the \(x\)-axis or to any other line through the origin, the stability properties follow from the classic Liénard theorem [16], [21, Theorem 4.6]. In this case, the system admits a unique nontrivial periodic cycle and every other non-zero solution approaches this cycle. In Theorem 2.4 we discuss the decoupled problem when the initial position \(r(0)\) and velocity \(\dot{r}(0)\) are not collinear. We construct a Lyapunov function for the system and show that every non-zero solution of the decoupled system approaches one of the cycles \(r(t) = e^{\pm it}\).

We would like to mention several papers in which systems similar to \((1)\) have been studied. An interesting discussion of the synchronization and boundedness of coupled dissipative systems can be found in [10]. However, the couplings studied in [10] do not agree with the one in Equation \((1)\). Some sufficient conditions for ultimate boundedness for coupled systems are presented in [3]. General second-order gradient-like systems \(\ddot{r} = f(\dot{r}) - \nabla U(r)\) were studied in [11] and [12, Ch. 7], where it was shown that under some conditions on \(f\) and the potential function \(U\), most notably \(f(\dot{r}) \cdot \dot{r} < 0\) if \(\dot{r} \neq 0\), every bounded solution converges to a configuration that solves \(\nabla U(r) = 0\). We note that the function \((1 - |\dot{r}_k|^2)\dot{r}_k \cdot \dot{r}_k\) in \((1)\) is not negative-definite and, thus, the results of [11] do not apply. We also note that the limit behavior of flocks and rings whose particles are equally distributed on a circle (i.e. mills) have been numerically analyzed extensively, using both individual-based and continuum models, see, for example, [5], [15], [1] for self-propelled particles, and [14] for the steady state pattern formation.

It is not immediately apparent why for all initial conditions the solutions of \((1)\) are (globally) defined for all \(t > 0\). The existence of global solutions follows from the classic existence theorems and the fact that the deviations of particles from the center of mass, \(r_k - R\), and their velocities \(\dot{r}_k\) are eventually bounded by a constant that depends on the number of particles [19]. Note also that \(|R|\) can only increase linearly in time [19].

A brief outline of the contents of the paper follows. In Section 2 we introduce main definitions, give an overview of various limit configurations for System \((1)\), and study stability properties of the decoupled system (Theorem 2.4).

The proof of Theorem 1.1 is divided into three parts which are presented in Sections 3, 4, and 5. In Section 3 we introduce a new coordinate system \(\{X_k, Y_k\}\) based on a rotating frame of reference, by setting \(r_k = e^{it}(X_k + iY_k)\), and establish that the set of equilibrium points in this new coordinate system corresponds to the set of ring states centered at the origin. In Section 4 we present the proof of Theorem 1.1 for the case of a non-degenerate ring state, that is, ring states for which the family of vectors \(r_k - R\), \(k = 1, \ldots, n\), is not collinear. In this case we conduct spectral analysis of the (rotating frame) Jacobian to show that the center manifold has dimension \(n\) for any non-degenerate ring state configuration. We show that the spectral gap decreases to zero as the initial configurations of the non-degenerate rings approach a degenerate one. Moreover, we explicitly show that the collection of all ring states in a neighborhood of a non-degenerate ring state forms a center manifold and we completely describe the dynamics of the system.
Viewed in isolation the dynamics near one non-degenerate ring state (i.e. near a non-degenerate fixed point in the rotated frame) are rather simple, yet piecing together the local phase portraits is complicated when the configurations approach a degenerate ring state \((n)\) must necessarily be even). In this case, as the particles get more concentrated near two polar opposite locations, the spectral gap of the non-degenerate fixed points decreases to zero. We will illustrate this phenomenon and challenge on a singular perturbation example in the plane that shares some of the features with our \(4n\) system.

Consider the homoclinic rose with six petals, described in cartesian, polar, or complex coordinates as

\[
\begin{align*}
\dot{x} &= -(x^4 - 6x^2y^2 + y^4) \\
\dot{y} &= -(4x^3y - 4xy^3) \\
\dot{r} &= -r^4 \cos(3\theta) \\
\dot{\theta} &= -r^3 \sin(3\theta)
\end{align*}
\]

or \(\dot{z} = -z^4\)

and the function \(f(x, y) = 4x^2 - x^4 - y^2\). For a positive perturbation parameter \(\epsilon\) consider the system

\[
\begin{align*}
\dot{x} &= (4x^2 - x^4 - y^2 + \epsilon^2)(-1)(x^4 - 6x^2y^2 + y^4) \\
\dot{y} &= (4x^2 - x^4 - y^2 + \epsilon^2)(-1)(4x^3y - 4xy^3).
\end{align*}
\]

The region \(4x^2 - x^4 - y^2 + \epsilon^2 > 0\) is a Jordan domain containing the origin, so the phase portrait of (2) is identical to that of the homoclinic rose (Figure (3)) within the ball of radius \(\epsilon\) from the origin, albeit having the homoclinic curves traveled at progressively slower rates.

Figure 3. The phase portrait for the system \(\dot{z} = -z^4\) is the homoclinic rose with six petals. Near the origin, System (3) has a similar phase portrait.

As \(\epsilon\) approaches zero, the system

\[
\begin{align*}
\dot{x} &= (4x^2 - x^4 - y^2)(-1)(x^4 - 6x^2y^2 + y^4) \\
\dot{y} &= (4x^2 - x^4 - y^2)(-1)(4x^3y - 4xy^3).
\end{align*}
\]

has fixed points that accumulate at the origin tangential to the lines \(|y| = 2|x|\).

Figure (4) illustrates the location of the fixed points for (3). Note that inside the two lobes the function \(4x^2 - x^4 - y^2\) is positive, so the flow of the rose is preserved. Outside the lobes the flow of (3) has opposite orientation from the original rose.

Figure (5) shows the phase portrait of (3) in the first quadrant near the origin with one of the fixed points labeled by \(P\). The center manifold through any given fixed point is 1-dimensional. As the fixed points approach the origin, the corresponding negative real eigenvalues increase to zero. The union of the all attracting manifolds and their limit points is the angle sector between \(\pi/3\) and \(2\pi/3\) and does not include a neighborhood of the origin. The phase portrait near the origin is
Figure 4. The curve $4x^2 - x^4 - y^2 = 0$ and its two tangent lines, $y = \pm 2x$.

Figure 5. The phase portrait for the singularly perturbed system (3), zoomed in close to the origin. Shown in red is the set of fixed points, which is indistinguishable from the tangential line $y = 2x$. The fixed point $P$ has the curve $4x^2 - x^4 - y^2 = 0$ for its center manifold. Its rate of decay along the stable manifold of $P$ and the angle between the stable and central manifolds decrease to zero as $P$ approaches the origin. Although all the fixed points in the first quadrant are stable, the origin is not.

Section 5 is devoted to the proof of Theorem 1.1 for the case of degenerate ring states, which necessitates that the number of particles $n$ is even and the particles in these ring states split into two equinumerous polar opposite groups. The presence of an extra symmetry in degenerate ring states leads to an increase in the dimension of the center manifold (now $n + 1$), and, as a result, to more complicated dynamics. An additional challenge with the degenerate case is that the center manifold cannot be found explicitly or satisfactorily approximated with Taylor polynomials. In this paper, one of our main contributions is that we provide a framework for producing an approximation of the center manifold that combines the singular perturbation approach for fast-slow dynamical systems [7, 9] and the contraction mapping principle as outlined in [2].

much richer outside that sector as it contains the homoclinic orbits of the first rose petal as illustrated in Figure 5.
More on this method can be found in the Appendix. In section 5 we construct a center manifold by using equilibrium points as error-free anchors for the center manifold and show that the flow on the center manifold is stable and that the limit dynamics is that of a ring state (either degenerate or non-degenerate depending on the initial conditions). We also include the associated rates of convergence, which can be loosely summarized as: for a perturbation of magnitude \( \epsilon \) from a degenerate configuration, the distance to the new limit configuration decreases as slow as \( \frac{\epsilon}{1 + \epsilon \sqrt{t}} \) or \( \epsilon e^{-\epsilon t} \) as \( t \) goes to infinity.

In Appendix, we present in more general terms the framework for approximating the center manifold and the flow on it near non-isolated equilibrium points. The reader may want to refer to it to provide broader context for the approximations stated in Lemma 5.7, and the definition of the point \( Q_Z \) in (36). Our method is effective where the traditional techniques such as Taylor approximations and normal forms fail. For dynamical systems with non-isolated equilibrium points, truncating the vector fields can destroy the degeneracies that make all the nullclines of the system intersect, thus leading to the elimination of the equilibrium points. Our approach for approximating the center manifold turns the complication of having a big set of equilibrium points accumulating at the origin into a computational advantage – we use the location of the equilibrium points to anchor the approximation of the center manifold. To the best of our knowledge, this approach has never been applied before to multidimensional coupled systems like (1).

Acknowledgement. We learned about System (1) and applications of swarm systems to robotics from Dr. Ira Schwartz, Dr. Jason Hindes, and Dr. Klementyna Szwaykowska of the Naval Research Laboratory, Washington, D.C., when they presented their research in the U.S. Naval Academy Applied Mathematics Seminar. C. Medynets and C. Kolon also visited Dr. Schwartz and his research group at the NRL in the summer of 2016 and 2017, respectively. We are thankful for their hospitality and fruitful discussions. C. M. and I.P. also acknowledge the support from the Office of Naval Research, Grant # N0001421WX00045.

2. Special Configurations

In this section we introduce the terminology, which we mainly borrow from [20], and discuss several special solutions of (1), including fixed points, solutions with stationary center of mass, circular solutions, and solutions when the agents move along straight lines. We begin by noticing that the set of solutions of (1) is

- translation invariant,
- rotation invariant,
- invariant with respect to the reflections about the coordinate axes, that is, with respect to the transformations \( y \mapsto -y \) and \( x \mapsto -x \).

Thus, if \( r(t) \) is a solution of the system, then \( e^{i\theta}r(t) \) and \( r(t) + c \) are also solutions for any \( \theta \in \mathbb{R} \) and any \( c \in \mathbb{R}^2 \).

**Definition 2.1.** Any solution of (1) of the form \( r_k = vt + d \), where \( v, d \in \mathbb{R}^2 \), is called a translating state. Necessarily, \( ||v|| = 1 \).

The fixed points of (1) correspond to the solutions with \( r_k = c, k = 1, \ldots, n \), for some \( c \in \mathbb{R}^2 \), that is, to the solutions when all agent occupy the same point in the plane. The fixed points form a two-dimensional manifold in \( \mathbb{R}^{4n} \). It is straightforward to check that the Jacobian at any fixed point has 1 as its eigenvalue, which yields the following result.

**Proposition 2.2.** The zero solution of (1) is unstable.
ON THE STABILITY OF A PARTICLE SYSTEM IN THE PLANE

Definition 2.3. (1) A **ring state** is any solution of (1) of the form $r_k(t) = e^{\epsilon t} e^{i\theta_k} + c$, where $\epsilon = \pm 1$ and has the same sign for every $k$, $c \in \mathbb{R}^2$ is a constant vector, and the polar angles $\{\theta_k\}$ satisfy: $\sum_{k=1}^{n} e^{i\theta_k} = 0$. (2) We call a ring state **degenerate** if $\theta_i = \theta^* + \phi_i$, where $\phi_i \in \{0, \pi\}$ and $\theta^* \in [0, 2\pi)$, that is, the particles split into two equinumerous polar-opposite groups that rotate about the stationary center of mass with a unit angular velocity. Degenerate ring states can only arise for an even number of particles.

The system decouples whenever the center of mass is stationary. For example, this can happen when the set of $n$ particles can be partitioned into subsets with rotational symmetries. It also happens when the system is in a ring state. We say that the particles $\{k_1, k_2, \ldots k_p\}$ have rotational symmetry about the point $R_0$ if

$$r_{k_m}(t) - R_0 = e^{(m-1)\frac{2\pi}{p}} (r_{k_1}(t) - R_0), \quad m = 2, \ldots, p, \quad t \geq 0.$$ 

Note that as a consequence of existence and uniqueness, if the set of particles partitions into subsets having initial positions and velocities that are rotationally symmetric about a common point $R_0$, then the rotational symmetries of the subsets continue for all times $t$.

Asymptotic decoupling, say, as in Figure 1, often occurs in simulations with random initial conditions and relatively small velocities. However, as with any multidimensional system of oscillators, the limit behavior can be very complex, see Figure 6.

**Figure 6.** Numerically simulated asymptotic behavior of (1) with $n = 3$. The initial conditions are $\hat{r}(0) = [(2.04, -1.36), (2.49, 0.05), (1, -0.02)]$, $\hat{r}(0) = [(0.34, 1.16), (0.77, -0.51), (-0.76, -0.34)]$. The positions of the agents at $t = 0$ (at $t = 50$) are represented as hollow (solid) dots. The center of mass, represented by the red dot, $R(t)$ moves along a nontrivial cycle (the dashed curve).

The trajectory of a particle in the decoupled system centered at the origin is described by the equation:

$$\ddot{r} = (1 - \dot{r} \cdot \dot{r}) \dot{r} - r \tag{4}$$

Consider a solution $\{r, \dot{r}\}$ of Equation (4). If the vectors $r$ and $\dot{r}$ are parallel at some instant, then they remain parallel and we can describe them as $r(t) = c(t) \hat{r}$ and $\dot{r}(t) = \dot{c}(t) \hat{r}$ for some unit vector $\hat{r}$ and a real-valued function $c$ in which case the function $c(t)$ must solve the equation $\ddot{c} = \dot{c}^3 - c$. Setting $x = \dot{c}$, $y = -c$, and $f(x) = x^3 - x$, we obtain the system $\dot{x} = y - f(x)$, $\dot{y} = -x$, which belongs to the class of Liénard systems. The function $f(x)$ satisfies the Liénard theorem [16], see also [17] for a more general discussion, which guarantees the existence of a unique nontrivial limit cycle and this limit cycle is stable. The zero solution of (4) is unstable.
The following result describes the limit behavior of (4) when the position vector and the velocity vector are not initially parallel.

**Theorem 2.4.** Let \( \{r, \dot{r}\} \) be a solution of (4) and \( \phi \) be the angle between the vectors \( r \) and \( \dot{r} \) when measured in the counter-clockwise direction starting from the vector \( r \). If \( 0 < \phi < \pi \) (\( \pi < \phi < 2\pi \)) at some instant, then the solution \( r \) converges to a limit cycle \( r_+(t) = ce^{it} \) (\( r_-(t) = ce^{i(t-\pi)} \)), where \( c \in \mathbb{C} \) has \( |c| = 1 \). The limit cycles \( r_+ \) and \( r_- \) are stable.

**Proof.** In the plane, we can use the following substitution:

\[
    r = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \dot{r} = \begin{pmatrix} u \\ v \end{pmatrix}
\]

which gives the 4-dimensional system:

\[
    \begin{align*}
    \dot{x} &= u \\
    \dot{y} &= v \\
    \dot{u} &= (1 - u^2 - v^2)u - x \\
    \dot{v} &= (1 - u^2 - v^2)v - y
    \end{align*}
\]

(5)

The derivative of the function \( vx - uy \) along trajectories of (5) is equal to \( (1 - u^2 - v^2)(vx - uy) \); this implies that if at an instant \( vx = uy \) then the vectors \( r \) and \( \dot{r} \) are parallel (or zero) at all times, and the system reduces to the one-dimensional case. We are therefore interested in the behavior of the system only in the region \( vx \neq uy \), which we call \( \Omega \).

\[
    \Omega = \{(u, v, x, y) \in \mathbb{R}^4 : vx \neq uy\}
\]

Note that the subsets \( \Omega_+ \) and \( \Omega_- \) of \( \Omega \) satisfying \( vx - uy > 0 \) and \( vx - uy < 0 \), respectively, are invariant.

We define the function \( L : \Omega \to \mathbb{R} \) as follows:

\[
    L = x^2 + y^2 + u^2 + v^2 - \log((vx - uy)^2)
\]

This function is defined throughout \( \Omega \), since \( vx \neq uy \) in \( \Omega \). Differentiating \( L \) along trajectories of (5) yields:

\[
    \dot{L} = -2(1 - u^2 - v^2)^2 \leq 0
\]

This indicates that \( L \) is decreasing along trajectories of system (5). We may note that

\[
    L = x^2 + y^2 + u^2 + v^2 - \log((vx - uy)^2) \\
    = x^2 + y^2 + u^2 + v^2 - \log((x^2 + y^2)(u^2 + v^2) - (xu + yv)^2) \\
    \geq x^2 + y^2 + u^2 + v^2 - \log((x^2 + y^2)(u^2 + v^2)) \\
    = (x^2 + y^2 - \log(x^2 + y^2)) + (u^2 + v^2 - \log(u^2 + v^2))
\]
Therefore, as \((x^2 + y^2) \to \infty\) or \((u^2 + v^2) \to \infty, L \to \infty\). This shows that \(L\) is radially unbounded. Since, \(t - \log(t) \geq 1\) for all \(t > 0\), we obtain that \(L \geq 2\). In fact, the minimum value of \(L \to 2\). The function \(L\) attains this minimum value on the set \(\Omega_{\min} = \{(u,v,x,y) \in \Omega: (u^2 + v^2 = 1), (x^2 + y^2 = 1), (cx - uy = \pm 1)\} \).

Thus, according to Lasalle’s Invariance Principle, every trajectory approaches the largest invariant set inside \(\Omega_{\min}\). The trajectories contained entirely in \(\Omega_{\min}\) correspond to the solutions of the form \(ce^{it}\) and \(ce^{-it}, c \in \mathbb{C}, |c| = 1\).

Thus, if \(0 < \phi < \pi\), then the \((x,y,u,v)\)-representation of the solution \(\{r, \dot{r}\}\) lies inside \(\Omega_+\) and this solution will have \(r_+\) as its limit cycle. Similarly, the solutions for which \(\pi < \phi < 2\pi\) will approach a limit cycle \(r_-\).

\[\Box\]

3. Ring States: Change of Coordinates

In the following three sections we prove the main result of the paper. In the present section, we introduce a change of coordinates in which ring states centered at the origin form fixed points. We also calculate the Jacobian of the system.

Next section focuses on non-degenerate fixed points. We show that (locally) ring states form a center manifold containing stable fixed points and stable cycles. The result will then follow from the reduction principle of center manifold theory.

Since the system is translation and reflection invariant, it suffices to establish our main result for ring states about the origin with all agents spinning counterclockwise. Fix a ring state \(r_k = e^{it}e^{i\theta_{0k}}\). Notice that \(\sum_{k=1}^n e^{i\theta_{0k}} = 0\).

Observe that the set of trajectories of (1) is rotation-invariant and the ring state \(r_k = e^{it}e^{i\theta_{0k}}\) when rotated by an angle \(\theta^*\) corresponds to the ring state \(r_k = e^{it}e^{i(\theta_{0k} + \theta^*)}\). It is important to note that rotated solutions have the same stability properties.

Note that, working with rotated ring states if necessary, we can always assume that the polar angles \(\{\theta_{0k}\}\) of the ring state additionally satisfy:

\[\sum_{k=1}^n \cos(\theta_{0k}) \sin(\theta_{0k}) = 0.\] (6)

That reduction is possible since for any choice of real numbers \(A\) and \(B\), the equation \(A\sin(2\theta^*) + B\cos(2\theta^*) = 0\) always has a solution in \(\theta^*\), we can find \(\theta^*\) such that

\[\sum_k \sin(\theta_{0k} + \theta^*) \cos(\theta_{0k} + \theta^*) = \left(\frac{1}{2} \sum_k \sin(2\theta_{0k})\right) \cos(2\theta^*) + \left(\frac{1}{2} \sum_k \cos(2\theta_{0k})\right) \sin(2\theta^*) = 0.\]

Consider the following change of coordinates

\[r_k = e^{it}e^{i\theta_{0k}}[(a_k + 1) + ib_k], \text{ where } a_k, b_k \in \mathbb{R}.\] (7)

Then

\[\dot{r}_k = e^{it}e^{i\theta_{0k}}[(\dot{a}_k - b_k) + i((a_k + 1) + \dot{b}_k)]\]
\[\ddot{r}_k = e^{it}e^{i\theta_{0k}}[(\ddot{a}_k - 2\dot{b}_k - a_k - 1) + i(\dot{b}_k + 2\dot{a}_k - b_k)].\] (8)

Note that

\[1 - |\dot{r}_k|^2 = 1 - (\dot{a}_k - b_k)^2 - (a_k + 1 + \dot{b}_k)^2 = -(\dot{a}_k - b_k)^2 - 2(a_k + \dot{b}_k) - (a_k + \dot{b}_k)^2.\]
Thus, substituting Equations (5) into System (1) and dividing both sides by $e^{i t} e^{i \theta_{0k}}$, we get

\[(\ddot{a}_k - 2 \dot{b}_k - a_k - 1) + i(\ddot{b}_k + 2 \dot{a}_k - b_k) = \left[1 - (\dot{a}_k - b_k)^2 - ((a_k + 1) + \dot{b}_k)^2\right](\dot{a}_k - b_k) + i((a_k + 1) + \dot{b}_k) - (a_k + 1 + ib_k) + \frac{1}{n} \sum_{m=1}^{n} e^{i(\theta_{0m} - \theta_{0k})}(a_m + 1 + ib_m).
\]

Note that $\sum_{m=1}^{n} e^{i(\theta_{0m} - \theta_{0k})} = 0$. Now separating the real part and imaginary part and setting $\dot{a}_k = \ddot{a}_k$ and $w_k = \ddot{b}_k$, we obtain the system

\[
\begin{align*}
\ddot{a}_k &= \dot{a}_k \\
\ddot{b}_k &= \dot{b}_k \\
\ddot{a}_k &= 2w_k - \left[(u_k - b_k)^2 + 2(a_k + w_k) + (a_k + w_k)^2\right](u_k - b_k) + \frac{1}{n} \sum_{m=1}^{n} \cos(\theta_{0m} - \theta_{0k})a_m - \frac{1}{n} \sum_{m=1}^{n} \sin(\theta_{0m} - \theta_{0k})b_m \\
\ddot{b}_k &= -2w_k - 2a_k - 2w_k - \left[(u_k - b_k)^2 + (a_k + w_k)^2\right] - \left[(u_k - b_k)^2 + 2(a_k + w_k) + (a_k + w_k)^2\right](a_k + w_k) + \frac{1}{n} \sum_{m=1}^{n} \sin(\theta_{0m} - \theta_{0k})a_m + \frac{1}{n} \sum_{m=1}^{n} \cos(\theta_{0m} - \theta_{0k})b_m.
\end{align*}
\]

We notice that the ring state solution $r_k = e^{i t} e^{i \theta_{0k}}$, $\dot{r}_k = i e^{i t} e^{i \theta_{0k}}$, $k = 1, \ldots, n$, corresponds to the fixed point $a_k = 0$, $b_k = 0$, $u_k = 0$, $w_k = 0$, $k = 1, \ldots, n$, in the new coordinates.

Thus, we have just introduced a coordinate system in which the given ring state becomes an equilibrium point. Our next objective is to describe the set of all equilibrium points in this coordinate system.

It is useful to think of the coordinate change of (7) as a two-step change: first introduce the auxiliary coordinate system \(\{X_k, Y_k\}\) that sets $r_k = e^{i t}(X_k + i Y_k)$, which we refer to as the rotating frame, then perform rotations and translations within the \(\{X_k, Y_k\}\) and \(\{\bar{X}_k, \bar{Y}_k\}\) coordinate planes to have the fixed points of interest located at the origin.

In Equation (1), switching to $X_k$ and $Y_k$, we obtain the system of equations

\[
\begin{align*}
\ddot{X}_k &= 2\dot{Y}_k + \left(1 - (X_k - Y_k)^2 - (X_k + Y_k)^2\right)(\dot{X}_k - Y_k) + \bar{X} \\
\ddot{Y}_k &= -2\dot{X}_k + \left(1 - (\dot{X}_k - Y_k)^2 - (X_k + Y_k)^2\right)(\dot{X}_k + Y_k) + \bar{Y}
\end{align*}
\]

where $\bar{X} = (X_1 + \cdots + X_n)/n$ and $\bar{Y} = (Y_1 + \cdots + Y_n)/n$. Our next results describes the set of equilibrium points for (10).

**Lemma 3.1.** A point \((X_k, Y_k, \bar{X}, \bar{Y})\), \(k = 1, \ldots, n\), is an equilibrium point of (10) if and only if (i) $\dot{X}_k = 0$, $\dot{Y}_k = 0$ for all $k$, (ii) $\bar{X} = \bar{Y} = 0$ and (iii) for all $k$, either $X_k = Y_k = 0$ or $X_k = \cos(\delta_k)$, $Y_k = \sin(\delta_k)$ for some $\delta_k$.

Note that the fixed points of (10), seen in the original coordinates, correspond to having a juxtaposition between $p$ particles placed at the origin (an unstable fixed point) and $N - p$ particles in a rotating ring state about the origin.

**Proof.** Equating the right-hand sides of (10) to zero, setting $\dot{X}_k = 0$ and $\dot{Y}_k = 0$, and solving for $\dot{X}$ and $\dot{Y}$, we obtain

\[
\begin{align*}
\dot{X} &= -(1 - X_k^2 - Y_k^2)Y_k \\
\dot{Y} &= (1 - X_k^2 - Y_k^2)X_k.
\end{align*}
\]
It follows that
\[(\bar{X})^2 + (\bar{Y})^2 = (1 - (X_k^2 + Y_k^2))(X_k^2 + Y_k^2), \quad k = 1, \ldots, n.\] (12)

Set \(s = (\bar{X})^2 + (\bar{Y})^2\). We claim that \(s = 0\). Indeed, assume towards contradiction that \(s > 0\). Consider the function \(f(t) = (1 - t)^2t\) for \(t \geq 0\). Then each value \(s_k = X_k^2 + Y_k^2\) is a solution of the equation \(f(s_k) = s\). Note that \(s_k \neq 1\) for otherwise \(s = f(s_k) = 0\). Solving (11) for \(X_k\) and \(Y_k\), we get that
\[X_k = \frac{\bar{Y}}{1 - s_k} \quad \text{and} \quad Y_k = -\frac{\bar{X}}{(1 - s_k)^2}.\]

It follows that
\[\bar{X} = \frac{1}{n} \sum_k \frac{\bar{Y}}{(1 - s_k)^2} \quad \text{and} \quad \bar{Y} = -\frac{1}{n} \sum_k \frac{\bar{X}}{(1 - s_k)^2}.\]

Multiplying both sides of the first equation by \(\bar{X}\) and the second equation by \(\bar{Y}\) and then adding them together, we obtain that \(s = (\bar{X})^2 + (\bar{Y})^2 = 0\), which is a contradiction.

Thus \(s = 0\). It follows from Equation (12) that either \(X_k, Y_k = 0\) or \(X_k^2 + Y_k^2 = 1\). Denote by \(I\) the set of indices \(k\) such that \(X_k^2 + Y_k^2 = 1\). For \(k \in I\), choose \(\delta_k \in [0, 2\pi]\) such that \((X_k, Y_k) = (\cos(\delta_k), \sin(\delta_k))\). Note that \(\sum_{k \in I} e^{i\delta_k} = n(X + iY) = 0\).

Conversely, we notice that any family \((X_k, Y_k)\) of constant functions such that \(X_k = Y_k = 0\) or \(X_k = \cos(\delta_k), Y_k = \sin(\delta_k)\) with \(X = Y = 0\) is an equilibrium solution for (10). The result now follows.

The rotating states about the origin of (1) correspond to the fixed points of (10) that are within the set \(\bigcap_{k=1}^n \{X_k^2 + Y_k^2 = 1\}\). We denote this subset of fixed point by \(\mathcal{F}\). The \(\epsilon, \delta\) formulation of Theorem (1.1) when \(R_0 = 0\) becomes the standard stability and limit cycle asymptotic for the fixed points of (10) in \(\mathcal{F}\).

We notice that the rotating frame coordinates \(\{X_k, Y_k\}\) relate to \(\{a_k, b_k\}\) as
\[(X_k + iY_k) = e^{i\theta_k}(a_k + 1 + ib_k),\]

via an affine transformation. Thus, there is a one-to-one correspondence between equilibrium points of these systems. Lemma 3.1 gives a full description of equilibrium points for the rotating frame. Thus, each point of the form \((X_k = \cos(\delta'_k), Y_k = \sin(\delta'_k), \bar{X} = 0, \bar{Y} = 0)\), corresponds to the point \((a_k = \cos(\delta'_k - \theta_{0k}), b_k = \sin(\delta'_k - \theta_{0k}), u_k = 0, w_k = 0)\). Thus, renaming \(\delta'_{k} - \theta_{0k}\) as \(\delta_k\), we obtain a complete description of the set of equilibrium points in the \(\{a, b, u, w\}\)-coordinates.

**Lemma 3.2.** Let
\[\Delta = \{ (\delta_1, \ldots, \delta_n) \in \mathbb{R}^n : \sum_{k=1}^n e^{i(\theta_{0k} + \delta_k)} = 0 \},\]

the set of all possible perturbations of the ring state \((\theta_{01}, \ldots, \theta_{0n})\) within the family of ring states. Suppose
\[\mathcal{R} = \{ a_k = \cos(\delta_k) - 1, b_k = \sin(\delta_k), u_k = 0, w_k = 0 : \{ \delta_k \} \in \Delta \} .\]

Then \(\mathcal{R}\) consists of all equilibrium solutions of (9) near the origin.

**Jacobian.** We finish this section by calculating the Jacobian \(J\) of (9) about the origin. The rows and the columns of the Jacobian are indexed by the variables \(\{a_k\}, \{b_k\}, \{u_k\}, \{w_k\}\) and their derivatives, respectively. Then
In the case of degenerate ring state, either one of these vectors is zero or they are collinear. We claim that \( \ker(S) \) of the Jacobian (seen as a function of the angles \( \theta \)) of the periodic orbits described explicitly in (16), and (ii) that the spectral gap (13), which we use it to establish (i) that the flow on the center manifold consists of the periodic orbits described explicitly in (16), and (ii) that the spectral gap approach a degenerate ring state (only possible if \( n \) is even). We note that for degenerate ring states the negative spectral gap is determined by a multiplicity-one negative eigenvalue.

We claim that \( \ker(S) = \ker(C) = \{ \bar{c}, \bar{s} \} \), the set of vectors \( \bar{c} \) and \( \bar{s} \) span a two-dimensional linear space. In the case of degenerate ring state, either one of these vectors is zero or they are collinear.

In this section we introduce a basis informed by the eigenvectors of the Jacobian (13), which we use it to establish (i) that the flow on the center manifold consists of the periodic orbits described explicitly in (16), and (ii) that the spectral gap of the Jacobian (seen as a function of the angles \( \theta_{k,0} \)) decreases to zero when the configurations \( \{ \theta_{k,0} \}_{k=1}^{n} \) approach a degenerate ring state (only possible if \( n \) is even). We note that for degenerate ring states the negative spectral gap is determined by a multiplicity-one negative eigenvalue.

Assume that the ring state \( r_k = e^{it}e^{i\theta_{0k}} \) is non-degenerate. Define the column-vectors

\[
\bar{c} = (\cos(\theta_{01}), \ldots, \cos(\theta_{0n}))^T \quad \text{and} \quad \bar{s} = (\sin(\theta_{01}), \ldots, \sin(\theta_{0n}))^T.
\]

Equation (6) ensures that the vectors \( \bar{c} \) and \( \bar{s} \) are orthogonal. We note that for non-degenerate ring states the vectors \( \bar{c} \) and \( \bar{s} \) span a two-dimensional linear space. In the case of degenerate ring state, either one of these vectors is zero or they are collinear.

The matrix \( S \) is the \( n \times n \) zero matrix, \( I_n \) is the \( n \times n \) identity matrix,

\[
S = \left\{ \frac{1}{n} \sin(\theta_{0m} - \theta_{0k}) \right\}_{k,m=1}^{n} \quad \text{and} \quad C = \left\{ \frac{1}{n} \cos(\theta_{0m} - \theta_{0k}) \right\}_{k,m=1}^{n}.
\]

4. Stability of Non-Degenerate Ring States

In this section we introduce a basis informed by the eigenvectors of the Jacobian (13), which we use it to establish (i) that the flow on the center manifold consists of the periodic orbits described explicitly in (16), and (ii) that the spectral gap of the Jacobian (seen as a function of the angles \( \theta_{k,0} \)) decreases to zero when the configurations \( \{ \theta_{k,0} \}_{k=1}^{n} \) approach a degenerate ring state (only possible if \( n \) is even). We note that for degenerate ring states the negative spectral gap is determined by a multiplicity-one negative eigenvalue.

We claim that \( \ker(S) = \ker(C) = \{ \bar{c}, \bar{s} \} \), the set of vectors \( x \in \mathbb{R}^{n} \) such that \( \bar{c} \cdot x = 0 \) and \( \bar{s} \cdot x = 0 \). Indeed, the \( k \)-th row of the vector \( n\bar{S}x \) is

\[
\sum_{m=1}^{n} \sin(\theta_{0m} - \theta_{0k})x_m = \cos(\theta_{0k}) \sum_{m=1}^{n} \sin(\theta_{0m})x_m - \sin(\theta_{0k}) \sum_{m=1}^{n} \cos(\theta_{0m})x_m
\]

Since the vectors \( \bar{c} \) and \( \bar{s} \) are linearly independent, we obtain that \( x \in \ker(S) \) if and only if \( x \) is orthogonal to both \( \bar{c} \) and \( \bar{s} \). Similarly, we can show that \( \ker(C) = \{ \bar{s}, \bar{c} \} \). It follows that \( \ker(S) = \ker(C) \) and that it is a \((n-2)\)-dimensional subspace in \( \mathbb{R}^{n} \).

Let \( V \) be a matrix whose columns form an orthonormal basis for \( \ker(C) = \ker(S) \). Note that the matrix \( V \) is \( n \times (n-2) \) dimensional. Consider the linear subspaces \( L_1 \) and \( L_2 \) of \( \mathbb{R}^{4n} \) spanned by the columns of the matrices \( B_1 \) and \( B_2 \), respectively, where

\[
B_1 = \begin{bmatrix}
\forall & O_{n,n-2} & O_{n,n-2} & O_{n,n-2} \\
O_{n,n-2} & \forall & O_{n,n-2} & O_{n,n-2} \\
O_{n,n-2} & O_{n,n-2} & \forall & O_{n,n-2} \\
O_{n,n-2} & O_{n,n-2} & O_{n,n-2} & \forall
\end{bmatrix}, \quad B_2 = \begin{bmatrix}
\bar{c} & \bar{s} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \bar{c} & \bar{s} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \bar{c} & \bar{s} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \bar{c} & \bar{s} & 0 & 0
\end{bmatrix}
\]

Here \( O_{n,n-2} \) is the \( n \times (n-2) \)-zero matrix. Note that \( \mathbb{R}^{4n} = L_1 \oplus L_2, L_1 \perp L_2, \)
\( \dim(L_1) = 4n - 8 \), and \( \dim(L_2) = 8 \). Then
ON THE STABILITY OF A PARTICLE SYSTEM IN THE PLANE

\[ JB_1 = \begin{bmatrix}
O_{n,n-2} & O_{n,n-2} & \vdots & O_{n,n-2} \\
O_{n,n-2} & O_{n,n-2} & \vdots & \vdots \\
O_{n,n-2} & O_{n,n-2} & \vdots & 2\lambda \\
-2\lambda & -2\lambda & \vdots & -2\lambda
\end{bmatrix} \quad (14) \]

Set \( f = \frac{1}{n} \sum_{k=1}^{n} \cos^2(\theta_k) \) and \( d = \frac{1}{n} \sum_{k=1}^{n} \sin^2(\theta_k) \). Then, using Equation (6), one can check that

\[ C\tilde{c} = f\tilde{c}, \quad S\tilde{c} = -f\tilde{s}, \]
\[ C\tilde{s} = d\tilde{s}, \quad S\tilde{s} = d\tilde{c}. \]

It follows that

\[ JB_2 = \begin{bmatrix}
0 & 0 & 0 & 0 & \tilde{c} & \tilde{s} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \tilde{c} & \tilde{s} \\
f\tilde{c} & d\tilde{s} & f\tilde{s} - d\tilde{c} & 0 & 0 & 2\tilde{c} & 2\tilde{s} \\
-2\tilde{c} - f\tilde{s} & d\tilde{s} - 2\tilde{s} & f\tilde{c} & d\tilde{s} & -2\tilde{c} & -2\tilde{c} & -2\tilde{s}
\end{bmatrix}. \quad (15) \]

Note that \( L_1 \) and \( L_2 \) are \( J \)-invariant subspaces. Thus, to describe the spectrum of the Jacobian it is enough to do so for the restrictions \( J|L_1 \) and \( J|L_2 \).

**Restriction of \( J \) onto \( L_1 \).** It follows from Equation (14) that the matrix of the restriction of \( J \) onto \( L_1 \) in the basis \( B_1 \) has the form

\[ J_1 = \begin{bmatrix}
O_{n-2} & O_{n-2} & I_{n-2} & O_{n-2} \\
O_{n-2} & O_{n-2} & O_{n-2} & I_{n-2} \\
O_{n-2} & O_{n-2} & O_{n-2} & 2I_{n-2} \\
-2I_{n-2} & O_{n-2} & -2I_{n-2} & -2I_{n-2}
\end{bmatrix} \]

Here \( O_{n-2} \) is the \( (n-2) \times (n-2) \) zero matrix and \( I_{n-2} \) is the identity matrix of the same dimension. Using the formula for the determinant for block matrices, we compute the characteristic polynomial

\[ \det(J_1 - \lambda I) = \lambda^{n-2}(\lambda^3 + 2\lambda^2 + 4\lambda + 4)^{n-2}. \]

The eigenvalues of \( J_1 \) are \( \lambda = 0, \lambda \approx -0.35 + 1.72i, \lambda \approx -0.35 - 1.72i, \lambda \approx -1.29 \), each of geometric multiplicity \( n-2 \).

**Restriction of \( J \) onto \( L_2 \).** Observe that \( d + f = 1 \). Set \( \omega \) such that \( f = 1/2 + \omega \) and \( d = 1/2 - \omega \). Then using Formula (15), we find the matrix of the restriction of \( J \) onto \( L_2 \) in the basis \( B_2 \):

\[ J_2 = \begin{pmatrix}
O_4 & I_4 \\
0 & 0 & 0 & 0 & 2 & 0 \\
0 & d & f & 0 & 0 & 0 & 2 \\
-2 & d & f & 0 & -2 & 0 & -2 \\
-f & -2 & 0 & d & 0 & -2 & 0
\end{pmatrix} \]

Then the characteristic polynomial of \( J_2 \) is equal to

\[ \det(J_2 - \lambda I) = (1 + \lambda^2)(1 - 4\omega^2 + 4\lambda + 12\lambda^2 + 14\lambda^3 + 9\lambda^4 + 4\lambda^5 + \lambda^6). \]

Consider the Hurwitz matrix \( H \) for the polynomial \( f_\rho(\lambda) = 1 - 4\omega^2 + 4\lambda + 12\lambda^2 + 14\lambda^3 + 9\lambda^4 + 4\lambda^5 + \lambda^6 \).
that the center manifold of the system must be zero eigenvalues, \(X\). Observe that the periodic orbits in (16) have simpler description in the rotating frame (next section).

The leading principal minors of \(H\) are \(\Delta_1 = 4, \Delta_2 = 22, \Delta_3 = 132, \Delta_4 = -352(\omega^2 - 3), \Delta_5 = -8(-333 - 636\omega^2 + 128\omega^4)\), and \(\Delta_6 = 8(-1 + 2\omega)(1 + 2\omega)(-333 - 636\omega^2 + 128\omega^4)\). Note that \(-1/2 < \omega < 1/2\). It is straightforward to check that the leading principal minors of \(H\) are strictly positive. Thus, by the Routh-Hurwitz criterion [8, Ch. XV, Section 6], the roots of the polynomial \(f(\lambda)\) are in the negative half-plane.

Moreover, for \(\omega = \pm 1/2\) the polynomial \(f_{\pm 1/2}\) has \(\lambda = 0\) as a simple root, with the remaining five bounded away from the imaginary line (in fact \(f_{\pm 1/2} = \lambda(\lambda + 1)^2(\lambda^2 + 2\lambda^2 + 4\lambda + 4)\), consistent with the characteristic polynomial in the next section).

**Center manifold.** We have established that the spectrum of \(J\) consists of \(n - 2\) zero eigenvalues, \(i, -i\), and \(3n\) eigenvalues in the negative half-plane. It implies that the center manifold of the system must be \(n\)-dimensional.

To understand the \(\pm i\) eigenspace, consider a nearby ring state of the form \(r_k(t) = R_0 + e^{i(\theta_k + \delta_k)}e^{it}, k = 1, \ldots, n, R_0 = |R_0|e^{i\rho}\). Using Equation (8) and the fact that \(ie^{i\delta_k} = (\bar{a}_k - \bar{b}_k) + i((a_k + 1) + b_k)\), we can write down \(r_k\) in the \(\{a, b, u, w\}\)-coordinates as

\[
\begin{bmatrix}
a_k \\
b_k \\
u_k \\
w_k
\end{bmatrix} = \begin{bmatrix}
\cos(\delta_k) - 1 \\
\sin(\delta_k) \\
0 \\
0
\end{bmatrix} + |R_0| \cos(\rho - t) \begin{bmatrix}
\cos(\theta_{0k}) \\
-\sin(\theta_{0k}) \\
-\sin(\theta_{0k}) \\
-\cos(\theta_{0k})
\end{bmatrix} + |R_0| \sin(\rho - t) \begin{bmatrix}
\sin(\theta_{0k}) \\
\cos(\theta_{0k}) \\
\cos(\theta_{0k}) \\
-\sin(\theta_{0k})
\end{bmatrix} + |R_0| \cos(t) \begin{bmatrix}
\bar{c} \\
-\bar{s} \\
-\bar{s} \\
\bar{c}
\end{bmatrix}
\]

Denote by \(\bar{a}\) the column-vector \((a_1, \ldots, a_n)^T\). The vectors \(\bar{b}, \bar{u}, \bar{w}\) are defined analogously. For a vector \(\delta = (\delta_1, \ldots, \delta_n)^T\) and a function \(f\), we will write \(f(\delta)\) to denote the vector \(f(\delta_1), \ldots, f(\delta_n))^T\). Then the \(\{a, b, q, p\}\)-coordinates of the ring state \(r_k = R_0 + e^{i(\theta_k + \delta_k)}e^{it}, k = 1, \ldots, n, R_0 = |R_0|e^{i\psi}\) can be compactly represented as

\[
\begin{bmatrix}
\bar{a} \\
\bar{b} \\
\bar{u} \\
\bar{w}
\end{bmatrix} = \begin{bmatrix}
\cos(\delta) - 1 \\
\sin(\delta) \\
0 \\
0
\end{bmatrix} + |R_0| \cos(\rho - t) \begin{bmatrix}
\bar{c} \\
-\bar{s} \\
-\bar{s} \\
\bar{c}
\end{bmatrix} + |R_0| \sin(\rho - t) \begin{bmatrix}
\bar{c} \\
\bar{s} \\
\bar{s} \\
-\bar{c}
\end{bmatrix} + |R_0| \cos(t) \begin{bmatrix}
\bar{c} \\
-\bar{s} \\
-\bar{s} \\
\bar{c}
\end{bmatrix}
\]

Observe that the periodic orbits in (16) have simpler description in the rotating frame coordinates as \(X_\delta(t) + iY_\delta(t) = e^{i(\theta_k + \delta_k)} + R_0e^{-it}\) and \(\dot{X}_\delta(t) + i\dot{Y}_\delta(t) = -R_0e^{-it}\). Necessarily, the point \(\{(\cos(\theta_{0k} + \delta_k))^n\}_{k=1}^n, \{(\sin(\theta_{0k} + \delta_k))^n\}_{k=1}^n, R_0, 0, 0\) is a fixed point from \(F\). Using the \(n\)-dimensional column vector

\[
1_n = (1, \ldots, 1)^T
\]

we have that using the rotating frame notations \(\text{col}[X + iY, \dot{X} + i\dot{Y}]\) the periodic orbits of (16) equal \(\text{col}[X_\delta + iY_\delta, 0_n + i0_n] + \text{col}[R_0e^{-it}1_n, -iR_0e^{-it}1_n]\), where
col]X_n + iY_n, 0_n + i0_n] ∈ F. We note that the main advantage of using the rotation frame of reference in comparison to the {a,b,u,w}-coordinates is that the coordinates \{X_k, Y_k\} are independent of the initial configuration angles \{θ_{0,k}\}.

Set
\[ \Delta = \{(δ_1, \ldots, δ_n) ∈ \mathbb{R}^n : \sum_{k=1}^n e^{i(θ_{0,k}+δ_k)} = 0 \}, \]
the set of all possible perturbations of the ring state \(\bar{θ} = (θ_1, \ldots, θ_n)\) within the family of ring states. Set \(R = \{(\cos(\bar{δ}) - 1, \sin(\bar{δ}), 0, 0)^T : \bar{δ} ∈ ∆\}\) and

\[ \mathcal{M} = R + \text{Span} \begin{pmatrix} \bar{c} & \bar{s} \\ -\bar{s} & \bar{c} \\ -\bar{c} & -\bar{s} \end{pmatrix}, \]

(17)

Notice that the set \(Δ\) consists of solutions to the equations
\[ \bar{c} \cdot \cos(\bar{δ}) - \bar{s} \cdot \sin(\bar{δ}) = 0 \quad \text{and} \quad \bar{c} \cdot \sin(\bar{δ}) + \bar{s} \cdot \cos(\bar{δ}) = 0. \]

Since the vectors \(\bar{c}\) and \(\bar{s}\) span a two-dimensional vector space, the Implicit Function Theorem implies that \(Δ\) is an \((n-2)\)-dimensional manifold in a neighborhood of the origin in \(\mathbb{R}^n\). Since, the function \(ξ ↦ (\cos(ξ) - 1, \sin(ξ))\) is an embedding for small values of \(ξ\), we get that \(R\) is an \((n-2)\)-dimensional manifold in a neighborhood of the origin in \(\mathbb{R}^{4n}\). It follows from (17) that in a neighborhood of the origin \(M\) is an \(n\)-dimensional submanifold of \(\mathbb{R}^{4n}\).

According to Lemma 3.2, the submanifold \(R\) consists of equilibrium solutions. Therefore, the set \(M\) contains full trajectories of (9) and, thus, \(M\) is flow-invariant. Since every center manifold contains all equilibrium solutions and all closed trajectories that remain within a predefined sufficiently small neighborhood of the origin, we obtain that \(M\) must be a submanifold of the center manifold. Since the center manifold is \(n\)-dimensional and \(\dim(M) = n\), we conclude that \(M\) is the center manifold of the system. One can also verify directly that (i) the linear space spanned by the vectors that determine the second summand of \(M\) are \(J\)-invariant and the matrix of \(J\) restricted to this subspace has eigenvalues \(i\) and \(-i\) and that (ii) the kernel of \(J\) is tangent to \(R\).

![Figure 8](image.png)

**Figure 8.** Graphical depiction of the manifold \(M\). The submanifold \(R\) (represented by the line) consists of fixed points for (9) that correspond to ring states about the origin. Through each point in \(R\) there goes a plane (the second summand in (17)) made up of closed cycles, concentric ellipses. The ellipses reflect the motion of the center of mass of (1) relative to the original ring state \(r_k = e^{it}e^{iθ_k}\), as the center of mass shifts away from the origin.

Each trajectory in \(M\) corresponds to a ring state with equilibrium states corresponding to ring states about the origin and elliptical trajectories correspond to ring states centered about other points, see Figure 8 for a graphical depiction of \(M\).
The flow on $\mathcal{M}$ is described by (16). The structure of the flow on $\mathcal{M}$ ensures that the zero solution is stable for the reduced flow. Thus, in view of center manifold theory [3, Theorem 2 in Section 2.4], we immediately establish the stability of the zero solution (ring state) of (9). Furthermore, any solution that starts in a small neighborhood of the origin approaches the center manifold. In other words, all ring states starting in a neighborhood of a non-degenerate ring state will converge to another ring state configuration.

5. Stability of Degenerate Ring States

In this section, we prove Theorem 1.1 for the case of degenerate ring states, or ring states of the form $r_k(t) = e^{it} e^{i\theta_k}$, where $\theta_k = 0, \pi$. Note this case is only possible for an even number of particles. Thus, in what follows the number of particles $n$ is assumed to be even. Set

$$N = \frac{n}{2}.$$ 

Start near the degenerate ring state rotating according to $(e^{it})_{k=1,\ldots,N}$ and $(e^{it}e^{i\pi})_{k=N+1,\ldots,n}$ and switch to the $\{a_k, b_k, u_k, w_k\}$-coordinates.

The proof of stability involves the following major steps.

1. We first change the coordinates to explicitly separate neutral and stable directions. We will denote the neutral variables associated with eigenvalue $\lambda = 0$ by $Z \in \mathbb{R}^{n-1}$ and with $\lambda = \pm i$ by $\delta_1, \delta_2 \in \mathbb{R}$. The stable variables will be denoted by $\beta, \beta_0, \beta_* \in \mathbb{R}^{n-1}$ and $\gamma_1, \gamma_2 \in \mathbb{R}$.

2. We decouple the motion of the first $4n - 2$ components from the variables $\delta_1$ and $\delta_2$. We accomplish it by observing that the neutral variables $\delta_1$ and $\delta_2$ have no affect on the dynamics of the remaining variables. We think of these two components as being a linear system in $\delta_1, \delta_2$ driven by the output of the decoupled $(Z, \beta, \beta_0, \beta_*, \gamma_1, \gamma_2)$-system.

3. In Theorem 5.10 we construct a manifold that approximates the center manifold for the reduced $(Z, \beta, \beta_0, \beta_*, \gamma_1, \gamma_2)$-system and find an approximation for the flow on the center manifold.

4. We establish stability for the reduced system.

5. Finally, we incorporate the driven variables $\delta_1$ and $\delta_2$ and establish stability for the full system.

The numbering and the content of the subsections below correspond to the items in the list above.

5.1. The change of basis from $(a, b, u, w)$ to $(Z, \beta, \beta_0, \beta_*, \gamma_1, \gamma_2, \delta_1, \delta_2)$. We proceed from the definition of $(a, b, u, w)$ in (7) applied to

$$\theta_0 = \ldots = \theta_{0,N} = 0 \text{ and } \theta_{0,N+1} = \ldots = \theta_{0,2N} = \pi.$$ 

Define

$$s_k = u_k - b_k \text{ and } \varepsilon_k = a_k + w_k.$$ 

Then the non-linear parts of $\dot{u}_k$ and $\dot{w}_k$ in (9) can be represented as

$$U_k = -(s_k^2 + 2\varepsilon_k + \varepsilon_k^2)s_k = -s_k ((\varepsilon_k + 1)^2 + s_k^2 - 1)$$

$$W_k = -(s_k^2 + \varepsilon_k^2) - (s_k^2 + 2\varepsilon_k + \varepsilon_k^2)\varepsilon_k = (-1)(\varepsilon_k + 1) ((\varepsilon_k + 1)^2 + s_k^2 - 1) + 2\varepsilon_k,$$

respectively. It follows that System (9) can be rewritten in the form

$$(\dot{a}_k, \dot{b}_k, \dot{u}_k, \dot{w}_k)^T = J(a_k, b_k, u_k, w_k)^T + (0, 0, U_k, W_k)^T,$$

where $J$ is the Jacobian of the $(a, b, u, w)$-system at the origin.
Denote by $X$ the $n$-dimensional column vector

$$X = (1, \ldots, 1, -1, \ldots, -1)^T.$$ 

Note that $X = \bar{x}$, where $\bar{x}$ was defined in Section 3 (§ is 0). Since the variables $c_k$ are used extensively throughout the remaining part of the paper, to avoid any confusion, we decided to rename $\bar{x}$ as $X$. It follows from (13) that

$$J = \begin{pmatrix}
\partial_v \\
\partial_u \\
\partial_w
\end{pmatrix}
\begin{pmatrix}
O_n & O_n & I_n & O_n \\
O_n & O_n & O_n & I_n \\
C & O_n & O_n & 2I_n
\end{pmatrix}$$

(20)

where $C$ is the $n \times n$ matrix

$$C = \frac{1}{n} \left[ X, \ldots, X, -X, \ldots, -X \right].$$

Consider the $N$-dimensional column vector

$$1_N = (1, \ldots, 1)^T$$

and choose a basis for the subspace \{ $v \in \mathbb{R}^N$ : $v \cdot 1_N = 0$ \}. Arrange these basis vectors in a matrix form and denote the resultant $N \times (N-1)$-matrix by $V$. Notice that the columns of the matrix $V$ are orthogonal to the vector $1_N$, which will be important later. Set

$$\mathcal{V} = \begin{pmatrix}
V & \mathbf{O}_{N,N-1} & 1_N \\
\mathbf{O}_{N,N-1} & V & 1_N
\end{pmatrix}$$

(21)

Notice that $\mathcal{V}$ is an $n \times (n-1)$-matrix and

$$C\mathcal{V} = \mathbf{O}_{n,n-1} \text{ since } 1_N^T V = 0_{N-1}.$$  

(22)

Remark 5.1. The columns of the matrix $\mathcal{V}$ are orthogonal to the vector $X$ and together they form a basis of $\mathbb{R}^n$.

We introduce the change of coordinates $Z, \beta, \beta_0, \beta_+ \in \mathbb{R}^{n-1}$, and $\gamma_1, \gamma_2, \delta_1, \delta_2 \in \mathbb{R}$ as follows:

$$(a, b, u, w)^T = P \cdot (Z, \beta, \beta_0, \beta_+, \gamma_1, \gamma_2, \delta_1, \delta_2)^T,$$

where the transformation matrix is

$$P = \begin{pmatrix}
O_{n,n-1} & \mathbf{O}_{n,n-1} & \mathbf{O}_{n,n-1} & -X & -X & \mathbf{O}_n & -X \\
\mathbf{O}_{n,n-1} & \mathbf{O}_{n,n-1} & \mathbf{O}_{n,n-1} & \gamma_1 & \gamma_2 & \mathbf{O}_n & \mathbf{O}_n \\
\mathbf{O}_{n,n-1} & \mathbf{O}_{n,n-1} & \mathbf{O}_{n,n-1} & \mathbf{O}_n & \mathbf{O}_n & \mathbf{O}_n & \mathbf{O}_n \\
\mathbf{O}_{n,n-1} & \mathbf{O}_{n,n-1} & \mathbf{O}_{n,n-1} & \mathbf{O}_n & \mathbf{O}_n & \mathbf{O}_n & \mathbf{O}_n
\end{pmatrix}$$

(23)

Note that

$$a = \mathbf{V} \beta - \gamma_1 X - \gamma_2 X - \delta_2 X$$

$$b = \mathbf{V} Z + \frac{1}{2} \mathbf{V} \beta_0 - \gamma_2 X + \delta_1 X$$

$$u = \mathbf{V} \beta_0 + \gamma_1 X + \delta_1 X$$

$$w = \mathbf{V} \beta_+ + \gamma_2 X + \delta_2 X$$

$$c = \mathbf{V} \beta + \mathbf{V} \beta_+ - \gamma_1 X$$

$$s = -\mathbf{V} Z + \frac{1}{2} \mathbf{V} \beta_0 + (\gamma_1 + \gamma_2) X.$$ 

(24)

With a slight abuse of notation, denote by $\tilde{Z}, \tilde{\beta}, \ldots, \tilde{\delta}_2$ the matrices containing the columns of the transformation matrix $P$ from (23) labeled by the corresponding symbols, such as $\tilde{Z}$ denoting the $4n \times (n-1)$ matrix col$[O_{n,n-1}, \mathbf{V}, O_{n,n-1}, O_{n,n-1}]$. 
It follows from (20), (22), and (23) that \( J\hat{Z} = 0, J\delta_1 = -\delta_2, J\delta_2 = \delta_1, J\gamma_1 = -\gamma_1, J\gamma_2 = \gamma_1 - \gamma_2 \) and

\[
J[\beta, \beta_0, \beta_*] = \begin{pmatrix}
O_{n,n-1} & \mathbb{V} & O_{n,n-1} \\
O_{n,n-1} & O_{n,n-1} & \mathbb{V} \\
O_{n,n-1} & O_{n,n-1} & 2\mathbb{V}
\end{pmatrix}
= [-2\beta_*, \beta - 2\beta_*, 2\beta_0 - 2\beta_*].
\]

The subspace spanned by the vectors \( \{\beta, \beta_0, \beta_*\} \) is \( J \)-invariant and as will be seen from the characteristic polynomial of \( J \), the eigenvalues of \( J \) restricted to this subspace lie in the negative half-plane. The columns \( \delta_1 \) and \( \delta_2 \) span the eigenspace of \( J \) corresponding to the eigenvalues \( \pm \iota \); the columns \( \hat{Z} \) represent the kernel of \( J \); the columns \( \gamma_1 \) and \( \gamma_2 \) span the generalized eigenspace of \(-1\). The Jacobian in the new coordinates becomes:

\[
J_{\text{deg}} = P^{-1}JP = \begin{pmatrix}
O_{n-1,n-1} & J_{\text{top}} \\
-1 & 1 \\
0 & -1 \\
-1 & 0
\end{pmatrix}
\]

where the blank spaces indicate zero entries and

\[
J_{\text{top}} = \begin{pmatrix}
O_{n-1} & I_{n-1} & O_{n-1} \\
O_{n-1} & O_{n-1} & 2I_{n-1} \\
-2I_{n-1} & -2I_{n-1} & -2I_{n-1}
\end{pmatrix}.
\]

The characteristic polynomial of \( J_{\text{deg}} \) is \((\lambda^3 + 2\lambda^2 + 4\lambda + 4)^{n-1}\lambda^{n-1}(\lambda + 1)^2(\lambda^2 + 1)\). The roots of the cubic polynomial are \( \lambda \approx -0.35 \pm 1.72i, \lambda \approx -1.29 \). We remark that \( J_{\text{top}} \) is the matrix of the restriction of \( J \) on \( \text{span}(\beta, \beta_0, \beta_*) \) in the basis \( \{\beta, \beta_0, \beta_*\} \).

Thus, the variables \( Z, \delta_1, \delta_2 \) represent the neutral directions of the system and the variables \( \beta, \beta_0, \beta_*, \gamma_1, \) and \( \gamma_2 \) capture the stable directions.

### 5.2. Decoupling the motion

Our goal is to show that the dynamics of the rest of the variables can be decoupled from \( \delta_1 \) and \( \delta_2 \) and to work with the reduced \( 4n - 2 \) system. For particles in a ring state \( R_x + iR_y + e^{i(\theta_0 + \tau)} \) near the degenerate state with angles \( 0 \) and \( \pi \), the components \( (\delta_1, \delta_2) \) are equal to the center of mass \((R_x, R_y)\). Thus, decoupling \( \delta_1 \) and \( \delta_2 \) suppresses the location of the center of rotation of the limit cycle (once we prove that said limit cycle exists).

We need to understand the non-linear parts in the \( \hat{Z}, \hat{\beta}, \hat{\beta}_0, \hat{\beta}_*, \hat{\gamma}_1, \hat{\gamma}_2, \hat{\delta}_1, \) and \( \hat{\delta}_2 \) equations.

Notice that \( X^T \mathbb{V} = 0_{n-1}^T \) and thus the matrix \([V, X]\) has full rank and is invertible. Let \( Q \) be the inverse of \([V, X]\). Note that \( Q \) is \( n \times n \)-dimensional. Denote by \( \mathbb{I} \) the matrix consisting of the top \( n - 1 \) rows of \( Q \). Similarly, the \( N \) by \( N \) matrix \([V, 1_N]\) has full rank; denote by \( Q \) its inverse and by \( T \) the top \( N - 1 \) rows of \( Q \). Then \( \mathbb{I} \) has \([T, 0_{N-1, N}]\) for its top \( N - 1 \) rows, followed by the \([0_{N-1, N}, T] \) and \( \mathbb{I}^T \) for its last row.

Using the block-structure of \([V, X]\), we can see that \( \mathbb{I}V = I_{n-1} \) and \( \mathbb{I}X = 0_{n-1} \). Then, additionally using the fact that \( X^T \mathbb{V} = 0_{n-1}^T \), we can verify that the inverse of the transformation matrix is
on the subspaces \( V \) are small. Compact region it equals \( Df(0) \), and so that within the neighborhood of the origin the nonlinearities \( \delta \) independent (decouples) from Lemma 5.2. The nonlinear terms of the \( (\cdot) \)-system. It follows from (18) that the nonlinear terms of (27) is tangent to \( V \), the global flow is also stable near the origin.

Consider a smooth vector field \( f: \mathbb{R}^m \rightarrow \mathbb{R}^m \) with \( f(0) = 0 \) and the corresponding flow \( \dot{x} = f(x) \). Let \( A = Df(0) \), and let \( V^s, V^u, V^c \) be the corresponding stable, unstable, center subspaces. Denote by \( \pi_s, \pi_u, \pi_c \) the projections on the subspaces \( V^s, V^u, V^c \). Then there is a \( C^2 \)-function \( \phi: V^c \rightarrow \mathbb{R}^m \) and a small neighborhood \( V \) of the origin in \( V^c \) such that the manifold \( M = \{ \phi(z) : z \in V \} \) is locally invariant and \( M \) is tangent to \( V^c \) at the origin. Now, if \( V^u = \emptyset \) and the flow on \( M \) is stable near the origin, in view of center manifold theory [3, Theorem 2], the global flow is also stable near the origin.

Now we will explain the construction of \( M \). To simplify the exposition, we will consider only the case most relevant to this paper when \( V^u = \emptyset \). Choose \( \eta > 0 \) and 

\[
\begin{pmatrix}
\dot{Z} \\
\dot{\beta} \\
\dot{\beta}_0 \\
\dot{\beta}_s \\
\dot{\gamma}_1 \\
\dot{\gamma}_2 \\
\end{pmatrix} = 
J_{top}

\begin{pmatrix}
O_{n-1,n-1} \\
\cdot \\
\cdot \\
\cdot \\
-1 \\
0 \\
\end{pmatrix}

\begin{pmatrix}
Z \\
\beta \\
\beta_0 \\
\beta_s \\
\gamma_1 \\
\gamma_2 \\
\end{pmatrix} +

\begin{pmatrix}
\frac{1}{\eta}U \\
0 \\
\frac{1}{\eta}U \\
\frac{1}{\eta}W \\
\frac{1}{\eta}X^T \cdot (U + W) \\
\frac{1}{\eta}X^T \cdot U \\
\end{pmatrix} 
\]  

where \( U = \{ U_k \}_{k=1}^n \) and \( W = \{ W_k \}_{k=1}^n \) are the vector-valued functions that capture the nonlinear terms of the \( (a, b, u, w) \)-system. It follows from (18) that the nonlinear terms \( U_k \) and \( W_k \) depend only on \( s_k = u_k - b_k \) and \( c_k = a_k + w_k \), which, in view of (24), are independent of \( \delta_1 \) and \( \delta_2 \). Thus, we get the following result.

**Lemma 5.2.** The RHS of the variables \( \{ Z, \beta, \beta_0, \beta_s, \gamma_1, \gamma_2 \} \) in Equation (28) is independent (decouples) from \( \delta_1 \) and \( \delta_2 \).

In what follows we study the stability of the reduced \( \{ Z, \beta, \beta_0, \beta_s, \gamma_1, \gamma_2 \} \) and then show that the dynamics incorporating the driven variables \( \delta_1 \) and \( \delta_2 \) is also stable about the origin.

5.3. Approximating the center manifold for the reduced system. In our construction of the center manifold, we will follow the approach based on the contraction mapping principle as outlined in [2]. Namely, the main result of [2] is the construction of the center manifold, we will follow the approach based on the contraction mapping principle as outlined in [2].
strictly less than the spectral gap of $A = Df(0)$, where $\dot{x} = f(x)$. Rewrite $\dot{x} = f(x)$ in the form $\dot{x} = Ax + g(x)$. Consider the space of “slow”-growing functions

$$F_\eta = \{ y : \mathbb{R} \to \mathbb{R}^m : \sup_{t \in \mathbb{R}} |y(t)e^{-\eta t}| < \infty \}$$

and for each $z : V^c$ and each function $y \in F_\eta$, define the operator

$$\Gamma_z(y)(t) = e^{At}z + \int_0^t e^{A(t-\tau)}\pi_z g(y(\tau))d\tau + \int_{-\infty}^t e^{A(t-\tau)}\pi_z g(y(\tau))d\tau. \quad (30)$$

In [2], the author shows that $\Gamma_z$ is a contraction mapping on $F_\eta$. We note that the norm $|| \cdot ||_\eta$ on $F_\eta$ is defined as

$$||y(t)||_\eta = \sup_{t \in \mathbb{R}} |y(t)e^{-\eta t}|. \quad (31)$$

Thus, for each $z \in V^c$, there is a unique fixed point $y_z \in F_\eta$ for the operator $\Gamma_z$. Define the function $\phi : V^c \to \mathbb{R}^m$ as $\phi(z) = y_z(0)$. Then the manifold $M = \{ \phi(z) \}$, where $z$ runs through a small neighborhood of the origin in $V^c$, satisfies all the relevant properties of the center manifold.

Given $Z \in \mathbb{R}^{n-1}$, note that the vector $(Z, 0_{(n+1)})$ belongs to the center subspace of the Jacobian $J_{deg}$. Let $\eta$ be the spectral gap of $J_{deg}$. Thus, applying (30) to System (28), we get

$$\Gamma_{\eta}(y)(t) = \begin{pmatrix}
Z + \int_0^t \frac{1}{2}I\mathcal{U}(y(t))d\tau \\
\int_0^t e^{J_{top}(t-\tau)} \begin{pmatrix}
0 \\
\mathcal{U}(y(t)) \\
\mathcal{W}(y(t))
\end{pmatrix} d\tau \\
\int_{-\infty}^t e^{J_{top}(t-\tau)} \begin{pmatrix}
-1 \\ 0 \\ -1
\end{pmatrix}^{(t-\tau)} \begin{pmatrix}
\frac{1}{n}X^T, \mathcal{U}(y(t)) \\
\frac{1}{n}X^T, \mathcal{W}(y(t))
\end{pmatrix} d\tau
\end{pmatrix} \quad (32)$$

We skip the proof of the following standard fact.

**Lemma 5.3.** Let $A$ be a matrix with eigenvalues in the negative half-plane. Then

$$\int_{-\infty}^t e^{A(t-\tau)}d\tau = -A^{-1}.$$ 

Notice that

$$\begin{pmatrix}
-1 \\ 0 \\ -1
\end{pmatrix}^{-1} = \begin{pmatrix}
1 \\ 0 \\ 1
\end{pmatrix}$$

and

$$-J_{top}^{-1} = -\begin{pmatrix}
O_{n-1} & I_{n-1} & O_{n-1} \\
O_{n-1} & O_{n-1} & 2I_{n-1} \\
-2I_{n-1} & -2I_{n-1} & -2I_{n-1}
\end{pmatrix}^{-1} = \begin{pmatrix}
I_{n-1} & \frac{1}{2}I_{n-1} & \frac{1}{2}I_{n-1} \\
-2I_{n-1} & -I_{n-1} & O_{n-1} \\
O_{n-1} & \frac{1}{2}I_{n-1} & O_{n-1}
\end{pmatrix}.$$

When the operator (32) is applied to a constant function $y \in F_\eta$, the functions $\mathcal{U}(y(\tau))$ and $\mathcal{W}(y(\tau))$ are independent of $\tau$ and, thus, can be taken out of the integral. Thus, using the matrix equations above, we can simplify the operator $\Gamma_{\eta}$, $Z \in \mathbb{R}^{n-1}$, as the operator $\Gamma_{Z}$ simplifies to

$$\Gamma_{Z}(y)(t) = \begin{pmatrix}
Z - \frac{1}{2}\mathcal{U}(y) \\
\frac{1}{2}(\mathcal{U}(y) + \mathcal{W}(y)) \\
0_{n-1} \\
-\frac{1}{2}\mathcal{U}(y) \\
\frac{1}{n}X^T(\mathcal{U}(y) + \mathcal{W}(y))
\end{pmatrix} \quad (33)$$
Since the center manifold locally contains all equilibrium points of the dynamical system, our next step is to describe a subset of equilibrium points of (28).

In the discussion that follows all our arguments are assumed to apply in a small enough neighborhood of the origin.

**Definition 5.4.**  
(1) For \( Z = (z_1, \ldots, z_{n-1})^T \in \mathbb{R}^{n-1} \), denote by \( Z_U \) (upper) and \( Z_L \) (lower) the \((N-1)\)-dimensional column-vectors:

\[
Z = \begin{pmatrix}
Z_U \\
Z_L \\
z_{n-1}
\end{pmatrix}.
\]

(2) For \( Z = (z_1, \ldots, z_{n-1}) \in \mathbb{R}^{n-1} \), denote by \(|Z| = \sum_{i=1}^{n-1} |z_i|\), the \(l^1\)-norm of \( Z \). Denote by \(|Z|_r\) the reduced norm of \( Z \) defined as \(|Z|_r = |Z_U| + |Z_L| = \sum_{i=1}^{n-2} |z_i|\).

(3) Given \( Z \in \mathbb{R}^{n-1} \), define the variables \( \{\theta_k\}_{k=1}^n \) implicitly as \( \sin(\theta_i) = (\forall Z)_i \).

Remark 5.5.  
(1) Notice that \( V_n \) and the columns of \( V \) form a basis in \( \mathbb{R}^n \) and the columns of \( V \) are orthogonal to the vector \( X \). Given small enough \( Z \in \mathbb{R}^{n-1} \), define the scalar \( E = E(Z) \) and vector \( \alpha = (\alpha_i(Z))_{i=1, \ldots, n-1} \) as the coefficients of \( \cos(\Theta) - 1_n \) in the basis \( \{V, X\} \).

Namely,

\[
\cos(\Theta) - 1_n = \forall \alpha + EX.
\]

(5) Denote by \( \sin(\Theta_U) \) and \( \sin(\Theta_L) \) the \( N \)-dimensional column-vectors

\[
\sin(\Theta) = \begin{pmatrix} \sin(\Theta_U) \\ \sin(\Theta_L) \end{pmatrix}.
\]

(6) Finally, we will need the following notation

\[
\mathcal{A}(Z) = \frac{1}{n} 1_n^T \cdot \cos(\Theta) = \frac{1}{n} \sum_{k=1}^n \cos(\theta_k).
\]

Remark 5.6.  
(1) Notice that

\[
E(Z) = \frac{1}{n} X^T (\cos(\Theta) - 1_n) = \frac{1}{n} X^T \cos(\Theta).
\]

(2) It follows from the structure of the matrix \( V \) that

\[
\sin(\Theta_U) = VZ_U + z_{n-1} 1_N \quad \text{and} \quad \sin(\Theta_L) = VZ_L + z_{n-1} 1_N.
\]

Since \( 1_N^T \cdot V = 0_{N-1}^T \), we get that

\[
\sum_{k=1}^N \sin \theta_k = \sum_{l=1+N}^{2N} \sin \theta_l = Nz_{n-1}.
\]

Remark 5.6. To illustrate the geometrical meaning of \( \Theta \) and \( z_{n-1} \), consider a perturbation \((e^{i\omega_k e^{i\theta_k}})_{k=1}^N \) and \((e^{i\pi e^{i\omega_k e^{i\theta_k}}})_{k=1}^N \) of the state \((e^{i\theta_k})_{k=1}^N \), \((e^{i\pi e^{i\theta_k}})_{k=1}^N \), that is itself a ring state about the origin. In the \((a, b, u, w)\)-coordinate system we obtain that \( a_k = \cos(\omega_k) - 1 \), \( b_k = \sin(\omega_k) \), and \( u_k = w_k = 0 \). Using the transformation matrix \( P^{-1} \) as described in [27], we obtain that \( Z = T \sin(\Omega) \) where
\(\sin(\Omega) = \{\sin(\omega_k)\}_{k=1}^{n}\). In particular, since the last row of the matrix \(T\) is \(1_n^T\), we have that \(z_{n-1} = \frac{1}{n} 1_n^T \sin(\Omega)\).

For a ring state centered about the origin, the polar angles \(\{\omega_k\}\) must satisfy

\[
\sum_{k=1}^{N} \left( e^{i\omega_k} + e^{i(\pi + \omega_k + \pi)} \right) = 0;
\]

we conclude that

\[
\frac{1}{N} \sum_{k=1}^{N} \sin \omega_k = \frac{1}{N} \sum_{k=1}^{N} \sin \omega_{k+N} = z_{n-1}.
\]

Thus, \(\sin(\Theta) = Z = VT \sin(\Omega)\). It can be checked that

\[
VT = I_n - \frac{1}{n} \left( \begin{array}{ccc}
I_N & -I_N \\
-I_N & I_N
\end{array} \right).
\]

Since \(\sum_{k=1}^{N} \sin(\omega_k) = \sum_{k=N+1}^{n} \sin(\omega_k)\), we obtain that \(\sin(\theta_k) = \sin(\omega_k)\). In other words, \(\theta_k = \omega_k\) captures the perturbation angles relative to the angles of the degenerate ring state in question. The geometric meanings of \(\theta\)'s and \(z_{n-1}\) are illustrated in Figure 9.

![Figure 9](image)

**Figure 9.** A perturbation of the degenerate ring state of six particles into a non-degenerate ring state. The particles naturally split into two groups, one appearing on the left and one appearing on the right of the picture. The center of mass for each group is depicted as a hollow circle on the slanted center (red solid) line. These centers are polar opposites of each other. The \(y\)-coordinate of the center of mass of the right group is given by the variable \(z_{n-1}\). The variables \(\theta\)'s introduced above measure the perturbation of the polar angles relative to the \(x\)-axis. The quantity \(D_L = |p_3 - c|^2 + |p_2 - c|^2 + |p_1 - c|^2\) captures the dispersion of the right group and will be used below as a Lyapunov function in the proof of stability.

In the following lemma, we describe some of the equilibrium points for (28). In what follows by “small enough” we mean vectors \(Z\) for which the functions \(E(Z)\) and \(\alpha(Z)\) are well-defined.

**Lemma 5.7.** Suppose \(Z_0 \in \mathbb{R}^{n-1}\) is small enough with \(E(Z_0) = 0\). Then the point \((Z_0, \alpha(Z_0), 0_{n-1}, 0_{n-1}, 0, 0)\) is a stationary point for the reduced flow (28).
Proof. Let \( P = (Z_0, \alpha(Z_0), 0_{n-1}, 0_{n-1}, 0, 0) \). Substituting \( Z = Z_0, \beta = \alpha(Z_0), \beta_0 = \beta_n = 0_{n-1}, \gamma_1 = \gamma_2 = 0 \) in \( c \) and \( s \) from (24), we get that
\[
\begin{align*}
c &= \nabla \alpha(Z_0) = \nabla \alpha(Z_0) + \mathcal{E}(Z_0)X = \cos(\Theta) - 1_n \quad \text{and} \quad s = -\nabla Z_0 = -\sin(\Theta).
\end{align*}
\]

Hence, \((x_k + 1)^2 + s_k^2 - 1 = (\cos \theta_k)^2 + (-\sin \theta_k)^2 - 1 = 0\).

It follows from (18) that \( U_k = 0 \) and \( W_k = 2\epsilon_k \) at the point \( P \). Note that at \( P \) we have that
\[
\nabla \mathcal{V} = \nabla (2\epsilon) = 2\nabla (\cos(\Theta) - 1_n) = 2\nabla \cdot \nabla \alpha(Z_0) = 2\alpha(Z_0),
\]
where the last equality follows from the fact that \( \nabla \mathcal{V} = 1_{n-1} \), see Section (5.2) for details. Similarly,
\[
X^T \mathcal{V} = X^T (2\epsilon) = 2X^T \nabla \alpha(Z_0) = 0.
\]

Thus, substituting the point \( P \) into the RHS of (28), we obtain
\[
\begin{bmatrix}
O_{n-1,n-1} \\
\alpha \circ \mathcal{J}_{top}
\end{bmatrix}
\begin{bmatrix}
Z_0 \\
\alpha(Z_0) \\
0_{n-1} \\
0_{n-1} \\
0 \\
\end{bmatrix}
+ \begin{bmatrix}
0_{n-1} \\
0_{n-1} \\
0_{n-1} \\
0 \end{bmatrix} = 0
\]

It follows from (28) that \( \mathcal{J}_{top} \cdot (\alpha(Z_0), 0_{n-1}, 0_{n-1})^T = (0_{n-1}, 0_{n-1}, -2\alpha(Z_0)) \). Thus, the RHS of (28) vanishes at the point \( P \), whereby establishing that \( P \) is an equilibrium point for the reduced system (28).

\[\square\]

Remark 5.8. (1) Since every centre manifold \( \mathcal{M} \) must contain all nearby stationary points of the flow, in view of Lemma 5.7, the points \((Z, \alpha(Z), 0_{n-1}, 0_{n-1}, 0, 0)\) with \( \mathcal{E}(Z) = 0 \) belong to the centre manifold.

(2) Additionally, since \( \mathcal{M} \) can be locally represented by the graph of a function \( h : \mathbb{R}^{n-1} \to \mathbb{R}^{3n+1} \), if \( Y = h(Z) \) and \( \mathcal{E}(Z) = 0 \), then \( Y = (\alpha(Z), 0_{n-1}, 0_{n-1}, 0, 0) \), that is, \((Z, Y)\) is a fixed point of the reduced flow. In other words, the set \( \{(Z, Y) \in \mathcal{M} : \mathcal{E}(Z) = 0\} \) consists of fixed points and, thus, is flow-invariant.

(3) Given a point on the centre manifold with initial value \( \mathcal{E}(Z) > 0 \), we have \( \mathcal{E}(Z(t)) > 0 \) for all \( t \) for which \( \mathcal{E}(Z(t)) \) is defined. A similar result holds for negative values of \( \mathcal{E} \).

(4) In view of Lemmas 3.1 and 5.7, points of the center manifold satisfying \( \mathcal{E}(Z) = 0 \) represent ring states of the original system (1).

In the following theorem we construct an approximation for the flow on the center manifold of the reduced system (28). In the proof, we extensively exploit the fact that the set of fixed points is “large” within the center manifold, as witnessed by Lemma 5.7. In what follows \( \mathcal{O}(f(x)) \) will stand for the standard big-O notation as \( x \to 0 \) and “\( \circ \)” will denote the Hadamard (component wise) product of vectors.

We refer the reader to Definition 5.4 for the concepts and notations pertaining to the statement and the proof of the following lemma and approximation theorem.

Lemma 5.9. For small \( Z \in \mathbb{R}^{n-1} \) we have
\[
\begin{align*}
(1) \sin(\Theta) &= \mathcal{O}(|Z|), \\
(2) \sin(\Theta) - z_{n-1}^2 1_n &= \mathcal{O}(|Z|_r), \\
(3) \sin^2(\Theta) - z_{n-1}^2 1_n &= \mathcal{O}(|Z|_r |Z|), \\
(4) \mathcal{E}(Z) &= \mathcal{O}(|Z|_r |Z|), \\
(5) \cos(\Theta) &= \mathcal{A}(Z) 1_n + \mathcal{O}(|Z|_r |Z|).
\end{align*}
\]
and \( (2) \) the projection of the flow on \( M \) for the reduced flow \((28)\). In a neighborhood of the origin, \( (1) \) the projection of the flow on \( M \) belongs to the space of slow-growing functions \( \mathcal{F}_n \).

Using the triangle inequality, we get that 
\[
|\sin(\theta_i) - \sin(\theta_j)| \leq |\sin(\theta_i) - z_{n-1}| + |\sin(\theta_j) - z_{n-1}| \leq |VZu| + |VZL| = O(|Z|) \text{.}
\]

Now we need the following fact: \( \frac{d}{dt} \sqrt{1 - t^2} = O(t) \). Applying the mean value theorem to the function \( \sqrt{1 - t^2} \), we obtain that
\[
|\cos(\theta_i) - \cos(\theta_j)| = |\sqrt{1 - \sin^2(\theta_i)} - \sqrt{1 - \sin^2(\theta_j)}| = O(|\sin(\theta_i)|)\sin(\theta_i) - \sin(\theta_j)| = O(|Z||Z|) \text{.}
\]

It follows that
\[
\cos(\theta_k) - A(Z) = \frac{1}{n} \sum_{m=1}^{n} (\cos(\theta_k) - \cos(\theta_m)) = O(|Z| |Z|) \text{.}
\]

Similarly,
\[
\mathcal{E}(Z) = \frac{1}{n} \sum_{m=1}^{N} (\cos(\theta_m) - \cos(\theta_{N+m})) = O(|Z| |Z|) \text{.}
\]

\[\Box\]

**Theorem 5.10** (The Approximation Theorem). Let \( M \) be the local center manifold for the reduced flow \( (28) \). In a neighborhood of the origin, \( (1) \) the projection of the flow on \( M \) onto the neutral directions satisfies
\[
\frac{dZ}{dt} = \mathcal{E}(Z)A(Z) \begin{bmatrix} Zu \\ ZL \\ 0 \end{bmatrix} + \mathcal{E}(Z)O(|Z| |Z|)1_{n-1}
\]

and \( (2) \) the projection of the flow on \( M \) onto the \( \gamma = (\gamma_1, \gamma_2) \) coordinates satisfies
\[
\frac{d\gamma}{dt} = \mathcal{E}(Z)O(|Z|).
\]

**Proof.** \( (1) \) For small enough \( Z \in \mathbb{R}^{n-1} \), define the functions \( f_1(Z), f_2(Z) \) as unique scalar-valued functions that solve the system of linear equations
\[
-(1 + f_1) \left( \frac{1}{n} \mathbf{1}_n^T \sin(2\Theta) \right) - (2 + f_2) \left( \frac{1}{n} \mathbf{1}_n^T \sin^2(\Theta) \right) = f_1
\]
\[
2(1 + f_1) \left( \frac{1}{n} \mathbf{1}_n^T \cos^2(\Theta) \right) + (2 + f_2) \left( \frac{1}{n} \mathbf{1}_n^T \sin(2\Theta) \right) = 2 + f_2
\]
\[(35)\]

Note that in the special case when \( Z = 0_{n-1} \) the linear system that defines \( f_1 \) and \( f_2 \) is nonsingular with solution \( f_1(0_{n-1}) = 0 \) and \( f_2(0_{n-1}) = 0 \), thus, the functions \( f_1 \) and \( f_2 \) are well-defined and small \( ( f_i(Z) = O(Z) ) \) for all small enough Z.

Given \( Z \in \mathbb{R}^{n-1} \), set
\[
Q_Z = [Z, \alpha(Z), 0_{n-1}, 0_{n-1}, \mathcal{E}(Z)f_1(Z), \mathcal{E}(Z)(2 - f_1(Z) + f_2(Z))]^T \in \mathbb{R}^{4n-2}. \quad (36)
\]

Our goal is to show that the center manifold \( M \) can be locally approximated by \( \{Q_Z : Z \in \mathbb{R}^{n-1}\} \) and then use the explicit form of \( Q_Z \) to approximate the RHS of \( (28) \) and obtain an approximation for the flow on \( M \). Observe that as a constant function \( Q_Z \) belongs to the space of slow-growing functions \( \mathcal{F}_n \). We also mention that all the functions appearing in this proof are analytic with respect to \( Z \).

**Lemma 5.11.** For small enough \( Z \in \mathbb{R}^{n-1} \), \( |\Gamma(Z) - Q_Z| = O(|Z|) \).
Thus, Equation (38) becomes

\[ c = \mathcal{V} \alpha(Z) - \mathcal{E}(Z) f_1(Z) X \text{ and } s = -\mathcal{V} Z + \mathcal{E}(Z)(2 + f_2(Z)). \]

To simplify the notation, we will omit the argument in the functions \( \mathcal{E} \) and \( f_1, f_2 \).

Using Definition (5.4), we get

\[ (c + 1_n)^2 + s^2 - 1_n = \mathcal{E}^2(1 + f_1)^2 1_n - 2\mathcal{E}(1 + f_1) \cos(\Theta) \oplus X + \mathcal{E}^2(2 + f_2)^2 1_n - 2\mathcal{E}(2 + f_2) \sin(\Theta) \oplus X. \]

Simplify

\[ (c + 1_n)^2 + s^2 - 1_n = \mathcal{E}^2(1 + f_1)^2 1_n - 2\mathcal{E}(1 + f_1) \cos(\Theta) \oplus X + \mathcal{E}^2(2 + f_2)^2 1_n - 2\mathcal{E}(2 + f_2) \sin(\Theta) \oplus X. \]

Recall that according to Lemma 5.9, we have that \( \sin(\Theta) = \mathcal{O}(|Z|) \) and \( \mathcal{E}(Z) = \mathcal{O}(|Z| \cdot |Z|). \) Also \( f_1(Z) = \mathcal{O}(|Z|) \). Therefore,

\[ (c + 1_n)^2 + s^2 - 1_n = \mathcal{E} \left[ -2(1 + f_1) \cos(\Theta) \oplus X - 2(2 + f_2) \sin(\Theta) \oplus X + \mathcal{O}(|Z| \cdot |Z|) \right] \]

and

\[ -s = \sin(\Theta) + \mathcal{O}(|Z| \cdot |Z|). \]

From (18), we get that

\[ U = -s ((c + 1)^2 + s^2 - 1) \]

\[ = \mathcal{E} \left[ -2(1 + f_1) \cos(\Theta) \oplus X - 2(2 + f_2) \sin^2(\Theta) \oplus X + \mathcal{O}(|Z| \cdot |Z|) \right] \]

\[ = \mathcal{E} \left[ -(1 + f_1) \sin(2\Theta) \oplus X - 2(2 + f_2) \sin^2(\Theta) \oplus X + \mathcal{O}(|Z| \cdot |Z|) \right]. \]

Therefore, using (35) we get that

\[ \frac{1}{n} X^T \cdot U = \mathcal{E} \left[ -\frac{1}{n} 1_n^T \cdot \sin(2\Theta)(1 + f_1) - \frac{2}{n} 1_n^T \cdot \sin^2(\Theta)(2 + f_2) + \mathcal{O}(|Z| \cdot |Z|) \right]. \]

From Lemma 5.9, \( \cos(\Theta) \sin(\Theta) = A(Z) \sin(\Theta) + \mathcal{O}(|Z| \cdot |Z|) \) and \( \sin^2(\Theta) = z_n^2 1_n + \mathcal{O}(|Z| \cdot |Z|) \).

Thus, Equation (38) becomes

\[ U = \mathcal{E} \left[ -2(1 + f_1) A(Z) \sin(\Theta) \oplus X - 2(2 + f_2) z_n^2 1_n + \mathcal{O}(|Z| \cdot |Z|) \right] \]

Recall that the matrix \( \mathcal{U} \) was defined as the top \( n - 1 \) rows of the matrix \( [\mathcal{V}, X]^{-1} \) and that we denoted by \( T \) the top \( N - 1 \) rows of the matrix \( [V, 1_N]^{-1} \). Using the block-structure of the matrix \( \mathcal{V} \) from (21), we can verify that

\[
\text{col}[\mathcal{V}, X]^{-1} = \begin{bmatrix}
T & O_{N-1,N} \\
\frac{1}{2N}1_N^T & T \\
\frac{1}{2N}1_N^T & \frac{1}{2N}1_N^T
\end{bmatrix}.
\]

Thus,

\[
\mathcal{U} = \begin{bmatrix}
T & O_{N-1,N} \\
\frac{1}{2N}1_N^T & T \\
\frac{1}{2N}1_N^T & \frac{1}{2N}1_N^T
\end{bmatrix}.
\]
Using the fact that $TV = 1_{N-1}$ and $T1_N = 0_{N-1}$, we get that
\[
\Gamma \cdot (\sin(\Theta) \odot X) = \begin{bmatrix} T_{O_{N-1,N}}^T & O_{N-1,N}^T \end{bmatrix} \cdot \begin{bmatrix} VZ_U + z_{n-1}1_N \\ -VZ_L - z_{n-1}1_N \end{bmatrix} = \begin{bmatrix} Z_U \\ -Z_L \end{bmatrix}.
\]

Note also that $\Gamma \cdot X = 0_n$. Thus, using (40), we obtain that
\[
\mathbb{T}U = 2\mathcal{E}(1 + f_1)A(Z) \begin{bmatrix} -Z_U \\ Z_L \\ 0 \end{bmatrix} + \mathcal{E}O(|Z|r|Z|)
\]
\[
= 2\mathcal{E}A(Z) \begin{bmatrix} -Z_U \\ Z_L \\ 0 \end{bmatrix} + \mathcal{E}O(|Z|r|Z|) + \mathcal{E}O(|Z|r|Z|)
\]
\[
= 2\mathcal{E}A(Z) \begin{bmatrix} -Z_U \\ Z_L \\ 0 \end{bmatrix} + \mathcal{E}O(|Z|r|Z|).
\]

We now pursue similar approximations for $W$. Recall that $W = -(c + 1_n)((c + 1_n)^2 + s^2 - 1_n) + 2c$.

The term $2c = 2\mathcal{V}a - 2\mathcal{E}f_1X$ contributes
\[
\mathbb{T}(2c) = 2\mathcal{V}a - 2\mathcal{E}f_1\mathbb{1}X = 2\alpha 1_{n-1} \text{ and } \frac{1}{n}X^T(2c) = -2\mathcal{E}f_1.
\]

Our goal is to show that $-(c + 1_n)((c + 1_n)^2 + s^2 - 1_n)$ gives $\mathcal{E}O(|Z|r|Z|)$ when multiplied by $\Gamma$, and that it gives $\mathcal{E}(2 + f_2 + \mathcal{E}O(|Z|r|Z|)$ when multiplied by $\frac{1}{n}X^T$.

Combining Equation (37) and the fact $-1(c + 1_n) = -\cos(\Theta) + (1 + f_1)\mathcal{E}X = -\cos(\Theta) + \mathcal{O}(|Z|r|Z|)$, we obtain that $W = 2\mathcal{E}f_1X + \mathcal{E}O(|Z|r|Z|)$.

Thus, from the second equation of the system (35) defining $f_1, f_2$:
\[
\frac{1}{n}X^T \cdot (W - 2c) = E \left[ 2(1 + f_1)\frac{1}{n}X^T \cos^2(\Theta) + \frac{1}{n}X^T(2 + f_2)\sin(\Theta) + \mathcal{O}(|Z|r|Z|) \right]
\]
\[
= E [2 + f_2 + \mathcal{O}(|Z|r|Z|)]
\]
and, using $\cos(\Theta) = A(Z) + \mathcal{O}(|Z|r|Z|)$,
\[
\Gamma(W - 2c) = \mathbb{T}E \left[ 2(1 + f_1)A(Z)X + 2(2 + f_2)A(Z)\sin(\Theta) \odot X + \mathcal{O}(|Z|r|Z|) \right]
\]
\[
= 0 + 2(2 + f_2)\mathcal{E}A(Z) \begin{bmatrix} Z_U \\ -Z_L \\ 0 \end{bmatrix} + \mathcal{E}O(|Z|r|Z|)
\]
\[
= \mathcal{E}O(|Z|r|Z|).
\]

Combining these equations with (42), we obtain that
\[
\frac{1}{n}X^T \cdot W = \mathcal{E}(2 - 2f_1 + f_2) + \mathcal{E}O(|Z|r|Z|)
\]
\[
\mathbb{T}W = 2\alpha(Z)1_{n-1} + \mathcal{E}O(|Z|r|Z|).
\]

It follows from (41) that $\mathbb{T}U = \mathcal{E}O(|Z|r|Z|)$. Using it along with Equations (39) and (43) in Equation (33), which defines the contraction operator $\Gamma_Z$, we obtain
Combining (46) with (41), we obtain that which completes the proof of the first part of the theorem.

\[
\Gamma_Z(Q_Z(t)) = \begin{pmatrix}
Z - \frac{1}{2} \mathcal{U}(Q_Z) \\
\frac{1}{2} \mathcal{U}(Q_Z) + \mathcal{V}(Q_Z) \\
0_{n-1} \\
-\frac{1}{2} \mathcal{U}(Q_Z) \\
\frac{1}{n} X^T \mathcal{U}(Q_Z)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
Z + \mathcal{E}\mathcal{O}(|Z|_r) \\
\alpha(Z) + \mathcal{E}\mathcal{O}(|Z|_r) \\
0_{n-1} \\
\mathcal{E}f_1 + \mathcal{E}\mathcal{O}(|Z|_r) \\
\mathcal{E}f_1 + \mathcal{E}O(|Z|_r)|Z|
\end{pmatrix}
\]

\[
= Q_Z + \mathcal{E}\mathcal{O}(|Z|_r),
\]

which completes the proof of the lemma.

In the space \( F_\eta \) of slow-growing functions, the operator \( \Gamma_Z \) contracts with a rate of 1/2, see [2, (3.7)]. It follows from the contraction mapping theorem that the fixed point function \( F_Z(t) \) and the constant function \( Q_Z \) satisfy

\[
||F_Z(t) - Q_Z||_{\eta} \leq \frac{1}{1 - \frac{1}{2}} ||\Gamma_Z(Q_Z) - Q_Z||_{\eta} = \mathcal{E}(Z)\mathcal{O}(|Z|_r),
\]

where the \( \eta \)-norm is defined in (31). We also have that

\[
||F_Z(0) - Q_Z|| \leq ||F_Z(t) - Q_Z||_{\eta} = \mathcal{E}(Z)\mathcal{O}(|Z|_r). \tag{45}
\]

The center manifold theorem [2] states that the set \( \mathcal{M} = \{F_Z(0)\} \), when \( Z \) runs over a small neighborhood of the origin in \( R^{n-1} \), is a center manifold of the system. The stable variables of \( \mathcal{M} \) are quadratic, \( \mathcal{O}(|Z|^2) \), in \( Z \). Thus, since \( 24 \), \( c = \mathcal{V}\beta + \mathcal{V}\beta + \gamma_1 X \) and \( s = -\mathcal{V}Z + \frac{1}{2} \mathcal{V}\beta_0 + (\gamma_1 + \gamma_2) \), we have that \( c = \mathcal{O}(|Z|^2) \) and \( s = \mathcal{O}(|Z|) \) when evaluated at \( F_Z(0) \in \mathcal{M} \). It follows from \( 45 \) that \( s = \mathcal{O}(|Z|) \) and \( c = \mathcal{O}(|Z|) \) along any line segment connecting \( F_Z(0) \) and \( Q_Z \).

Therefore, calculating the gradient of the function \( \mathcal{U}_k \), we obtain that \( |\nabla \mathcal{U}_k| \leq 2|s_k| + 2|s| + |\epsilon_k|^2 + 2|\epsilon_k| + 3|\epsilon_k| = \mathcal{O}(|Z|) \) along the line segment connecting \( F_Z(0) \) and \( Q_Z \). It follows from the mean value theorem that

\[
|\mathcal{U}(F_Z(0) - Q_Z)| \leq \max|\nabla \mathcal{U}| |F_Z(0) - Q_Z| = \mathcal{O}(|Z|)\mathcal{E}\mathcal{O}(|Z|_r) = \mathcal{E}Z\mathcal{O}(|Z|_r)|Z|. \tag{46}
\]

Combining (46) with (41), we obtain that

\[
\frac{dZ}{dt} |_{F_Z(0)} = \frac{1}{2} \mathcal{U}(F_Z(0)) = \mathcal{E}(Z)A(Z) \begin{pmatrix}
Z \\
-Z_L \\
0
\end{pmatrix} + \mathcal{E}(Z)\mathcal{O}(|Z|_r)|Z|,
\]

which completes the proof of the first part of the theorem.

(II) Our next goal is to obtain estimates for \((\hat{\gamma}_1, \hat{\gamma}_2)\) for the flow on the center manifold. Recall that

\[
\begin{pmatrix}
\hat{\gamma}_1 \\
\hat{\gamma}_2
\end{pmatrix} = \begin{pmatrix}
-1 & 1 \\
0 & -1
\end{pmatrix} \begin{pmatrix}
\gamma_1 \\
\gamma_2
\end{pmatrix} + \begin{pmatrix}
-\frac{1}{n} X^T \cdot W \\
\frac{1}{n} X^T \cdot (\mathcal{U} + \mathcal{W})
\end{pmatrix}. \tag{47}
\]

Fix \( Z \in R^{n-1} \) and consider \( Q_Z \) and the corresponding point \( F_Z(0) \) on the center manifold. Recall that \( \gamma_1|Q_Z = \mathcal{E}f_1 \) and \( \gamma_2|Q_Z = \mathcal{E}(2 - f_1 + f_2) \). It follows from (45) and the structure of \( Q_Z \) that

\[
\gamma_1|F_Z(0) = \mathcal{E}f_1 + \mathcal{E}\mathcal{O}(|Z|_r) \quad \text{and} \quad \gamma_2|F_Z(0) = 2\mathcal{E} - \mathcal{E}f_1 + \mathcal{E}f_2 + \mathcal{E}\mathcal{O}(|Z|_r). \tag{48}
\]
From (49), we get that 
\[ \frac{1}{n} X^T \cdot U(F_Z(0)) = \mathcal{E} f_1 + \mathcal{E} O(|Z| r) . \]
Combining it with (46), we obtain that
\[ \frac{1}{n} X^T \cdot U(F_Z(0)) = \mathcal{E} f_1 + \mathcal{E} O(|Z| r). \]  
(49)

Using arguments similar to those preceding Equation (46), we obtain that \[ |W(F_Z(0))| = \frac{1}{n} |\nabla W| \cdot |F_Z(0) - Q_Z| = \mathcal{O}(|Z|) \mathcal{E} \mathcal{O}(|Z| r) \]
\[ = \mathcal{E}(Z) \mathcal{O}(|Z| r)|Z|. \]
Combining this estimate with (43), we obtain that
\[ \frac{1}{n} X^T \cdot W(F_Z(0)) = 2\mathcal{E} - 2\mathcal{E} f_1 + \mathcal{E} f_2 + \mathcal{E} O(|Z| r). \]  
(50)

Substituting (48), (49), and (50) into (47), we obtain that
\[ \gamma_1|_{F_Z(0)} = \mathcal{E}(Z) \mathcal{O}(|Z| r) \text{ and } \gamma_2|_{F_Z(0)} = \mathcal{E}(Z) \mathcal{O}(|Z| r), \]
which completes the proof of the theorem.

5.4. Stability of the reduced flow. In view of center manifold theory, to establish the stability of the reduced \((Z, \beta, \beta_0, \beta_r, \gamma_1, \gamma_2)\)-system, it suffices to establish the stability of the flow on the center manifold. Furthermore, we will show that every solution that starts near the origin approaches an equilibrium point. Recall that in view of Lemma 3.1 and Remark 5.8, the set \(\mathcal{M} \cap \{ \mathcal{E} = 0 \} \) consists of equilibrium points of the flow, which in turn coincides with the set of ring state solutions centered at the origin.

**Theorem 5.12.** The flow on the center manifold \(\mathcal{M}\) of the reduced system \(28\) is stable at the origin. Furthermore, any trajectory that starts near the origin approaches an equilibrium point of the form \((Z_n, \alpha(Z_n), 0_{2n}), Z_n \in \mathbb{R}^{n-1}, \mathcal{E}(Z_n) = 0\).

**Proof.** In view of Theorem 5.10, the flow on the center manifold is governed by
\[ \frac{dZ}{dt} = \mathcal{E}(Z) A(Z) \begin{bmatrix} Z_U \\ -Z_L \\ 0 \end{bmatrix} + \mathcal{E}(Z) \mathcal{O}(|Z| r)|Z|. \]

Using Remark 5.8, we notice that the sets \(\mathcal{M}_0 = \mathcal{M} \cap \{ \mathcal{E} = 0 \}, \mathcal{M}_+ = \mathcal{M} \cap \{ \mathcal{E} > 0 \}, \) and \(\mathcal{M}_- = \mathcal{M} \cap \{ \mathcal{E} < 0 \} \) are flow-invariant and the set \(\mathcal{M}_0\) consists of equilibrium points. Thus, to establish stability of the flow on \(\mathcal{M}\), we will separately show that the flow on \(\mathcal{M}_+\) and \(\mathcal{M}_-\) is stable.

Furthermore, we will prove that the flow converges towards the set \(\mathcal{M}_0\), the set of ring states. Define the functions
\[ D_U(Z) = \frac{1}{N} (VZ_L)^T \cdot VZ_U \quad \text{and} \quad D_L(Z) = \frac{1}{N} (VZ_L)^T \cdot VZ_L. \]  
(51)

Using the definition of \(\sin(\theta)\), we notice that \(D_L(Z) = \sum_{k=1}^{N} \sin(\theta_k) - z_{n-1})^2\). Furthermore, in view of Remark 5.5, \(z_{n-1} = \frac{1}{N} \sum_{k=1}^{N} \sin(\theta_k)\). Thus, \(D_L(Z)\) can be interpreted as the dispersion/scattering of the right group of particles. Similarly, \(D_U(Z)\) can be viewed as the dispersion of the left group of particles.

We will prove that \(D_L\) and \(D_U\) are Lyapunov functions for the flow on \(\mathcal{M}_+\) and \(\mathcal{M}_-\), respectively, which will ensure the stability of the flow near the origin. Finally, applying LaSalle’s Invariance Principle, we will establish the convergence of solutions to the set \(\mathcal{M}_0\).
Notice that there is a universal constant $C > 0$ such that $\sqrt{\mathcal{D}_L(Z)} \leq C|Z|_r$ and $\sqrt{\mathcal{D}_U(Z)} \leq C|Z|_r$. In the following lemma we show that $\mathcal{D}_L(Z)$ is comparable to $|Z|_r$ in $\{\mathcal{E} > 0\}$ and that $\mathcal{D}_U(Z)$ is comparable to $|Z|_r$ in $\{\mathcal{E} < 0\}$ for all small enough $Z$.

**Lemma 5.13.** There exists a constant $C > 0$ such that for all $Z$ small enough: (i) $\frac{1}{C}|Z|_r \leq \sqrt{\mathcal{D}_L(Z)} \leq C|Z|_r$ in the region where $\{\mathcal{E} > 0\}$, (ii) $\frac{1}{C}|Z|_r \leq \sqrt{\mathcal{D}_U(Z)} \leq C|Z|_r$ in the region where $\{\mathcal{E} < 0\}$, and (iii) $\mathcal{D}_L(Z) \geq \frac{1}{2}\mathcal{D}_U(Z)$ in $\{\mathcal{E} > 0\}$.

**Proof.** By definition of $\mathcal{E}$, we have that

$$\mathcal{E}(Z) = \frac{1}{n} \sum_{k=1}^N \left( \cos(\theta_k) - \cos(\theta_{N+k}) \right) = \frac{2}{n} \sum_{k=1}^N \left[ \sin^2 \left( \frac{\theta_k + N}{2} \right) - \sin^2 \left( \frac{\theta_k}{2} \right) \right]$$

where $t_i = \sin(\theta_i) - z_{n-1}$ or, equivalently, the $i$-th component of the vector $VZ_U$ for $1 \leq i \leq N$ and of the vector $VZ_L$ if $N + 1 \leq i \leq n$.

Given a small $z \in \mathbb{R}$, consider the function $t \mapsto \sin^2 \left( \frac{1}{2} \sin^{-1}(t + z) \right)$ and using direct computations find its Taylor series expansion of order 2. Then,

$$\sin^2 \left( \frac{1}{2} \sin^{-1}(t + z) \right) = \sin^2 \left( \frac{1}{2} \sin z \right) + \frac{z}{2\sqrt{1 - z^2}} t + \frac{1}{4(1 - z^2)^2} t^2 + R_2(t).$$

Expressing the remainder $R_2(t)$ in the Lagrange form, we can directly verify that $R_2(t) = t^3 h(t, z)$, where $|h(t, z)| \leq K(|t| + |z|)$ for some constant $K > 0$. Note that the constant $K$ depends only on the radius of the neighborhood of the origin.

Substituting the Taylor series expansion with $t = t_i$ and $z = z_{n-1}$ in the equation for $\mathcal{E}$ above, we get that

$$\mathcal{E}(Z) = \frac{1}{N} \frac{z_{n-1}}{\sqrt{1 - z_{n-1}^2}} \left( \sum_{k=1}^N t_{N+k} - \sum_{k=1}^N t_k \right)$$

$$+ \frac{1}{4\sqrt{1 - z_{n-1}^2}} \left( \sum_{k=1}^N t_{N+k}^2 - \sum_{k=1}^N t_k^2 \right) + \frac{1}{N} \sum_{k=1}^N (R_2(t_{N+k}) - R_2(t_k)). \quad (52)$$

Note that $\sum_{k=1}^N t_k = \mathbf{1}_N^T VZ_U = 0$ as the columns of the matrix $V$ are orthogonal to the vector $\mathbf{1}_N$. Similarly, $\sum_{k=1}^N t_{N+k} = 0$. Note also that $\mathcal{D}_U = \frac{1}{N} \sum_{k=1}^N t_k^2$ and $\mathcal{D}_L = \frac{1}{N} \sum_{k=1}^N t_{N+k}^2$. Finally, observe that $h(t_k, z_{n-1}) = o(|Z|)$ and $t_k = o(|Z|) = o(\sqrt{\mathcal{D}_L + \mathcal{D}_U})$. Thus, continuing with Equation (52), we obtain that

$$\mathcal{E}(Z) = \frac{1}{4\sqrt{1 - z_{n-1}^2}} \left( \mathcal{D}_L(Z) - \mathcal{D}_U(Z) \right)$$

$$+ \frac{1}{N} \sum_{k=1}^N \left( t_{N+k}^2 h(t_{N+k}, z_{n-1}) - t_k^2 h(t_k, z_{n-1}) \right) \quad (53)$$

$$= \frac{1}{4\sqrt{1 - z_{n-1}^2}} \left( \mathcal{D}_L(Z) - \mathcal{D}_U(Z) \right) + (\mathcal{D}_L(Z) + \mathcal{D}_U(Z))^{3/2} H(Z),$$

where $H(Z) = o(|Z|)$. Set $f(Z) = 4\sqrt{1 - z_{n-1}^2}$. Then

$$\mathcal{D}_U(Z) - \mathcal{D}_L(Z) = -f(Z)\mathcal{E}(Z) + (\mathcal{D}_L(Z) + \mathcal{D}_U(Z))^{3/2} H(Z) f(Z).$$
Consider the case $E(Z) > 0$. Choose a small neighborhood of the origin in $\mathbb{R}^{n-1}$ in which $(3/2)^{3/2} D_U(Z)^{1/2} |H(Z)| f(Z) < \frac{1}{2}$. We claim that in this neighborhood $D_L(Z) \geq \frac{1}{2} D_U(Z)$. Indeed, assume towards contradiction that $D_L(Z) < \frac{1}{2} D_U(Z)$.

It follows that $D_L(Z) - D_U(Z) < -\frac{1}{2} D_U(Z)$. Hence,

$$\frac{1}{2} D_U(Z) < D_U(Z) - D_L(Z) = f(Z) E(Z) + (D_L(Z) + D_U(Z))^{3/2} H(Z) f(Z) \leq (D_L(Z) + D_U(Z))^{3/2} H(Z) f(Z).$$

The last inequality implies that $H(Z) > 0$. It follows that

$$\frac{1}{2} D_U(Z) < (D_L(Z) + D_U(Z))^{3/2} H(Z) f(Z) \leq \left( \frac{3}{2} D_U(Z) \right)^{3/2} H(Z) f(Z) \leq \frac{1}{2} D_U(Z),$$

which is a contradiction. Therefore, $|Z| = O(\sqrt{D_U + D_L}) = O(\sqrt{D_L})$ and the result follows. The proof in the case when $E(Z) < 0$ is similar.

**Lemma 5.14.** On the center manifold $M$, if $Z$ is near the origin, then

$$\frac{dD_L}{dt} = -2E(Z)A(Z)D_L(Z)(1 + O(|Z|)) \quad \text{whenever } Z \in M_+,$$

$$\frac{dD_U}{dt} = 2E(Z)A(Z)D_U(Z)(1 + O(|Z|)) \quad \text{whenever } Z \in M_-.$$ \hspace{1cm} (54)

**Proof.** Consider the case when $Z \in M_+$, that is, $E(Z) > 0$. In view of Lemma 5.13 $|Z| = O(\sqrt{D_U})$. By definition, $D_L = \frac{1}{N} (VZ_L)^T \cdot VZ_L$. Differentiating both sides of the equation along trajectories of the flow on the center manifold and applying Theorem 5.10 we obtain that

$$\frac{dD_L}{dt} = \frac{2}{N} (VZ_L)^T \cdot V \dot{Z} = -\frac{2}{N} E(Z) (VZ_L)^T \cdot [A(Z)VZ_L + O(|Z|^2|Z|)]$$

$$= -2E(Z) (A(Z)D_L(Z) + O(|Z|^2|Z|)) = -2E(Z) A(Z) D_L(Z) (1 + O(|Z|)).$$

The proof for the case $Z \in M_-$ is analogous and will be omitted.

Now we are ready to show that the flow on submanifolds $M_+$ and $M_-$ is stable. We will consider only $M_+$ as the proof for $M_-$ uses similar arguments and will be left to the reader. Recall that the submanifold $M_+$ is flow-invariant. Lemma 5.13 implies that $D_L(Z) > 0$ for all small enough $Z \in M_+$. Thus, it follows from Equation (54) that the function $t \mapsto D_L(Z(t))$ is decreasing along any trajectory $Z(t)$ that stays inside $M_+$.

Fix any trajectory $Z(t) \subset M_+$. Using Theorem 5.10 and the previous lemma, we differentiate the functions $\sqrt{D_L} \pm z_{n-1}$ along $Z(t)$:

$$\frac{d(\sqrt{D_L} \pm z_{n-1})}{dt} = -\frac{2E(Z)A(Z)D_L(Z)(1 + O(|Z|))}{2 \sqrt{D_L(Z)}} \pm E(Z) O(|Z|)$$

$$= -2E(Z) A(Z) \sqrt{D_L(Z)}(1 + O(|Z|)) \pm E(Z) \sqrt{D_L(Z)} O(|Z|)$$

$$= -2E(Z) \sqrt{D_L(Z)} A(Z) [1 + O(|Z|)],$$

which shows that for all $Z$ small enough the functions $\sqrt{D_L} \pm z_{n-1}$ are decreasing along trajectories in $M_+$. It follows from the definition of $D_L$ and Lemma 5.13 that there are constants $C > 0$ and $K > 0$ such that $\sqrt{D_L(Z)} \leq C|Z|$ and $|Z| \leq K\sqrt{D_L(Z)}$ for all small enough $Z$. Fix an arbitrary trajectory $Z(t) = (z_1(t), \ldots, z_{n-1}(t))$ in $M_+$ that starts in a small neighborhood of the origin. Since the functions $\sqrt{D_L}$ and $\sqrt{D_L} \pm z_{n-1}$ are decreasing along $Z(t)$, we obtain that for $t > 0$:

$$\sqrt{D_L(Z(t))} \leq \sqrt{D_L(Z(0))} \leq C|Z(0)|$$
Furthermore, each trajectory that starts near the origin approaches a fixed point of
Corollary 5.15. The reduced system converges to a fixed point.

at the origin), we conclude that every trajectory of the reduced system near the
manifold \(M\). Since trajectories in \(M\) approach fixed points (ring states centered
near the origin), the flow on the reduced center manifold is stable near

Finally, consider an initial condition \(Z_0\) small enough so that for all \(t > 0\) its
trajectory \(Z(t)\) is confined to the neighborhood where the center manifold exists.
Note that the existence of such a neighborhood follows from stability of the system
at the origin. If \(\mathcal{E}(Z_0) = 0\), then \(Z_0\) is a fixed point. If \(\mathcal{E}(Z_0) \neq 0\), without loss of
generality, assume \(\mathcal{E}(Z_0) > 0\), that is, \(Z(t) \subset M_+\). Note that any limit point \(Z_\omega\)
of \(Z(t)\) satisfies \(\mathcal{E}(Z_\omega) = 0\).

Applying LaSalle’s Invariance Principle to the negative semi-definite \((\mathcal{D}_L \leq 0)\)
function \(\mathcal{D}_L\), we obtain that \(Z_\omega\) satisfies \(\mathcal{D}_L(Z_\omega) = 0\), which according to Lemma
5.14 is equivalent to \(\mathcal{E}(Z_\omega) = 0\) or \(\mathcal{D}_L(Z_\omega) = 0\). If \(\mathcal{E}(Z_\omega) = 0\), then in view of
Remark 5.8 the limit point \((Z_\omega, \alpha(Z_\omega), 0_{\mathbb{Z}_n})\) is an equilibrium point of the flow and the
result follows. If \(\mathcal{E}(Z_\omega) > 0\), then \(\mathcal{D}_L(Z_\omega)\) must be equal to zero. Thus, Lemma
5.13 implies that \(|Z_\omega|_r = 0\) and \(\sin(\Theta) = z_{n-1}^* 1_{n-1}\), where \(\Theta = \Theta(Z_\omega)\). It follows that
\(\mathcal{E}(Z_\omega) = 0\), which is a contradiction.

Now Theorem 5.12 and center manifold theory immediately imply the stability of
the reduced system near the origin. According to Section 2.4 every trajectory that starts near
the origin approaches a trajectory in the center manifold \(M\). Since trajectories in \(M\) approach fixed points (ring states centered
at the origin), we conclude that every trajectory of the reduced system near the
origin converges to a fixed point.

Corollary 5.15. The reduced \((Z, \beta, \beta_0, \beta_2, \gamma_1, \gamma_2)\)-system is stable near the origin.
Furthermore, each trajectory that starts near the origin approaches a fixed point of
the system.

5.5. Stability of the full system. In the full system, the coordinates \((Z, \delta_1, \delta_2)\)
are neutral and the coordinates \((\beta, \beta_0, \beta_2, \gamma_1, \gamma_2)\) are stable. In this section, we
establish stability of the expanded system with the variables \((\delta_1, \delta_2)\) being driven by
the stable subsystem. Note that in the theorem below the clockwise rotation in the
\(\delta\) components corresponds to a translation of the center of mass of the system in the
original coordinates to the limit position \((\delta_{s,1}, \delta_{s,1})\). The equilibrium point \(Q_*\)
of the reduced system determines the angles of the limit configuration of the particles
that in the original coordinate system rotate with unit speed about \((\delta_{s,1}, \delta_{s,1})\).

Theorem 5.16. The \((Z, \beta, \beta_0, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2)\)-system governed by \([28]\) is stable
at the origin.

Moreover, there exists a neighborhood \(N\) of the origin in \(\mathbb{R}^{1n}\) such that the flow of
\([28]\) that starts in \(N\) asymptotically approaches the periodic “steady-state”
oscillatory solution of the form \((Q_*, \cos(t)\delta_{s,1} + \sin(t)\delta_{s,2}, -\sin(t)\delta_{s,1} + \cos(t)\delta_{s,2}))\),
where \(Q_*\) is an equilibrium point for the reduced system \([28]\) and \(\delta_{s,1}, \delta_{s,2} \in \mathbb{R}\).
Proof. Denote by \( \varphi_t \) the flow of the full system governed by (28). Denote by \( M_{n-1} \) the center manifold of the reduced system constructed in Theorem 5.12. Recall that \( M_{n-1} = \{ F_Z(0) : Z \in \mathbb{R}^{n-1}, |Z| < r_0 \} \) for some \( r_0 > 0 \). In view of Theorem 5.12, the flow \( \varphi_t \) on \( M_{n-1} \subseteq \mathbb{R}^{4n} \) is stable near the origin. Recall also that the first \( 4n - 2 \) components of (28) decouple from the last two \( (\delta_1, \delta_2) \). It follows that the manifold \( M = M_{n-1} \times \mathbb{R}^2 \) is (locally) forward invariant under (28) and tangent to the linear center subspace. Thus \( M \) is the center manifold for (28). In view of center manifold theory, the stability of the flow of the full system is equivalent to the stability of the flow on \( M \). Note the projection of the flow \( (M, \varphi_t) \) onto \( \mathbb{R}^{4n-2} \) is stable near the origin. Thus, to establish stability it suffices to show that for all initial conditions close enough to the origin, the projection of the flow \( (M, \varphi_t) \) onto \( (\delta_1, \delta_2) \)-coordinates remains close to the origin in \( \mathbb{R}^2 \) for small initial conditions from \( M \).

If \( Z \in \mathbb{R}^{n-1} \) is such that \( \mathcal{E}(Z) = 0 \), then in view of Remark 5.8, \( F_Z(0) \) is an equilibrium point on the center manifold. It follows from the proof of Lemma 5.7 that the functions \( X^T \cdot U = 0 \) and \( X^T \cdot W = 0 \) at \( F_Z(0) \). Therefore, using (28), we observe that whenever \( (F_Z(0), \delta_1, \delta_2) \in M \) satisfies \( \mathcal{E}(Z) = 0 \), the flow in \( (\delta_1, \delta_2) \) is governed by

\[
\begin{pmatrix}
\dot{\delta}_1 \\
\dot{\delta}_2
\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0
\end{pmatrix} \begin{pmatrix}
\delta_1 \\
\delta_2
\end{pmatrix}.
\]

The solutions of the flow have the form \( e^{-t} \delta_0 \), which shows that they remain near the origin for all \( t > 0 \). Now, it remains to establish stability in the (flow-invariant) regions \( M \cap \{ \mathcal{E} > 0 \} \) and \( M \cap \{ \mathcal{E} < 0 \} \). In what follows we present the proof only for \( M \cap \{ \mathcal{E} > 0 \} \) as the proof for \( \mathcal{E} < 0 \) is similar and will be left to the reader. Without mentioning it explicitly, we will be working in a small neighborhood of the origin where the computations make sense.

The proof we are about to present has the underlying argument that \( (\delta_1, \delta_2) \) are the solutions of a linear harmonic oscillator driven by functions that are non-oscillatory in nature. Computationally, it is easier to follow the changes in \( \gamma_1, \gamma_2 \) than the functions \( \frac{1}{n} X^T \cdot U \) and \( \frac{1}{n} X^T \cdot W \) forcing the system. Thus, we use (28) to express

\[
\begin{pmatrix}
\frac{1}{n} X^T \cdot W \\
\frac{1}{n} X^T \cdot (U + W)
\end{pmatrix} = \begin{pmatrix} \gamma_1 \\
\gamma_2 \end{pmatrix} - \frac{1}{n} \begin{pmatrix} 1 & 1 \\ 0 & -1
\end{pmatrix} \begin{pmatrix}
\gamma_1 \\
\gamma_2
\end{pmatrix}.
\]

Solving this system for \( -\frac{1}{n} X^T \cdot U \) and \( \frac{1}{n} X^T \cdot (U + W) \) and substituting the solutions into the \( (\delta_1, \delta_2) \)-components of (28), we obtain that

\[
\begin{pmatrix}
\dot{\delta}_1 \\
\dot{\delta}_2
\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0
\end{pmatrix} \begin{pmatrix}
\delta_1 \\
\delta_2
\end{pmatrix} + \frac{1}{n} \begin{pmatrix} X^T \cdot (U + W) \\
X^T \cdot U
\end{pmatrix}
\]

or, equivalently,

\[
\frac{d}{dt} \begin{pmatrix}
\delta \\
\gamma
\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -1
\end{pmatrix} \begin{pmatrix}
\gamma_1 \\
\gamma_2
\end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0
\end{pmatrix} \begin{pmatrix}
\gamma_1 \\
\gamma_2
\end{pmatrix},
\]

where \( \delta = (\delta_1, \delta_2) \) and \( \gamma = (\gamma_1, \gamma_2) \). Setting

\[
\nu = \delta - \begin{pmatrix} 0 \\
-1
\end{pmatrix} \gamma, \quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0
\end{pmatrix}, \quad \text{and} \quad B(t) = \begin{pmatrix} -1 & 0 \\ -1 & -1
\end{pmatrix} \gamma(t),
\]

the last equation can be rewritten as \( \dot{\nu} = A\nu + B(t) \). Using the variation of parameters, we obtain that

\[
\nu(t) = e^{At} \nu(0) + \int_0^t e^{A(t-s)} B(s) ds
\]
or, equivalently,

\[
\begin{bmatrix}
0 & 1 \\
-1 & -1
\end{bmatrix}
\delta(t) = \begin{bmatrix}
\cos t & \sin t \\
-\sin t & \cos t
\end{bmatrix}
\gamma(t) = \begin{bmatrix}
\cos t & \sin t \\
-\sin t & \cos t
\end{bmatrix}
\delta(0) - \begin{bmatrix}
0 & 1 \\
-1 & 1
\end{bmatrix}
\gamma(0)
\]

\[+ \begin{bmatrix}
\cos t & \sin t \\
-\sin t & \cos t
\end{bmatrix}
\int_0^t
\begin{bmatrix}
\cos s & -\sin s \\
\sin s & \cos s
\end{bmatrix}
\begin{bmatrix}
-\gamma_1(s) \\
-\gamma_1(s) - \gamma_2(s)
\end{bmatrix}
ds.
\]  

(55)

It follows from Theorem 5.12 that \(\gamma(t) \subseteq \mathcal{M}_{n-1}\) remains small and converges to \(O_2\) as \(t \to \infty\). The term

\[
\left|
\begin{bmatrix}
\cos t & \sin t \\
-\sin t & \cos t
\end{bmatrix}
\delta(0) - \begin{bmatrix}
0 & 1 \\
-1 & 1
\end{bmatrix}
\gamma(0)
\right|
\]

contributes a small clockwise rotation. Thus, in view of (55) to establish the stability of the flow on \(\mathcal{M}\) we only need to investigate the convergence and the bounds for the integral

\[
\int_0^t
\begin{bmatrix}
\cos s & -\sin s \\
\sin s & \cos s
\end{bmatrix}
\begin{bmatrix}
-\gamma_1(s) \\
-\gamma_1(s) - \gamma_2(s)
\end{bmatrix}
ds
\]

as \(t \to \infty\).

Theorem 5.10 implies that on the center manifold \(\mathcal{M}_{n-1}\) the projection onto \(\gamma\) satisfies

\[
\frac{d\gamma}{dt} = \mathcal{E} \mathcal{O}(\|Z\|) = \mathcal{E} \mathcal{O}(\sqrt{\mathcal{D}_L}).
\]

The last equality follows from Lemma 5.13.

Let \(C > 0\) be such that

\[
\left|\frac{d\gamma}{dt}\right| \leq C\mathcal{E}\sqrt{\mathcal{D}_L} \quad \text{for all sufficiently small } Z.
\]  

(56)

Introduce (auxiliary) functions

\[
B_j(t) = \sqrt{\mathcal{D}_L(t)} - \sqrt{\mathcal{D}_L,\infty} + \frac{1}{4C}\gamma_j(t),
\]

where \(\mathcal{D}_L,\infty = \lim_{t \to \infty} \mathcal{D}_L(t)\). Notice that the limit exists since the function \(\mathcal{D}_L\) is decreasing and bounded below. Then

\[
\int_0^t e^{is}\gamma_j(s)ds = 4C \int_0^t e^{is}B_j(s)ds - 4C \int_0^t e^{is}(\sqrt{\mathcal{D}_L(s)} - \sqrt{\mathcal{D}_L,\infty})ds
\]  

(57)

The function \(t \to (\sqrt{\mathcal{D}_L(t)} - \sqrt{\mathcal{D}_L,\infty})\) is decreasing to zero, therefore, by the Dirichlet test its oscillatory integral (the second term in the RHS of (57)) converges when \(t \to \infty\). Moreover, the range of \(\int_0^t e^{is}(\sqrt{\mathcal{D}_L(s)} - \sqrt{\mathcal{D}_L,\infty})ds\) is bounded by

\[
4 \left(\sqrt{\mathcal{D}_L(0)} - \sqrt{\mathcal{D}_L,\infty}\right).
\]

We claim that the functions \(B_j(t) = \sqrt{\mathcal{D}_L(t)} - \sqrt{\mathcal{D}_L,\infty} + \frac{1}{4C}\gamma_j(t)\) are decreasing on \([0,\infty)\) for all \(Z\) small enough. Indeed, differentiating \(B_j(t)\) and using (56) and Lemma 5.14 we get that

\[
\frac{dB_j}{dt} = -\frac{2}{2\sqrt{\mathcal{D}_L(t)}}\mathcal{E}(Z)\mathcal{A}(Z)\mathcal{D}_L(Z)(1 + \mathcal{O}(|Z|)) - \frac{1}{4C}\frac{d\gamma_j}{dt}
\]

\[
\leq -\mathcal{E}(Z)\sqrt{\mathcal{D}_L(Z)}\mathcal{A}(Z)(1 + \mathcal{O}(|Z|)) + \frac{1}{4C}\mathcal{E}(Z)\sqrt{\mathcal{D}_L(Z)}
\]

\[
= \mathcal{E}(Z)\sqrt{\mathcal{D}_L(Z)}\left(\frac{1}{4} + \mathcal{O}(|Z|) - \mathcal{A}(Z)\right).
\]  

(58)

Since, in view of Lemma 5.3 \(\mathcal{A}(Z) \to 1\) as \(|Z| \to 0\), we can always find a neighborhood of the origin in which \((\frac{1}{4} + \mathcal{O}(|Z|) - \mathcal{A}(Z)) < 0\), which shows that the functions \(B_j(t)\) are decreasing for all small enough \(Z\).
Thus, the functions $B_{j}(t)$ are decreasing. Note also that $B_{j}(t) \to 0$ as $t \to \infty$ as $\gamma(t) \to 0$ (Theorem 5.12). It follows form the Dirichlet test that the improper integral in (57) converges to some point in $\mathbb{R}^{2}$ as $t \to \infty$ and is bounded by

$$\left|\int_{0}^{t} e^{s} \gamma_{j}(s) ds\right| \leq 4C \left[ 8(\sqrt{D_{L}(0)} - \sqrt{D_{L,\infty}}) + \frac{1}{4C} \gamma_{j}(0) \right].$$

Apply the bounds obtained for the improper integrals to (55); conclude that for some constant $K > 0$, $\delta(t)$ remains within a distance of $K \max(|\delta(0)|, |\gamma(0)|, \sqrt{D_{L}(0)})$ from the origin. The stability of the full system near the origin follows.

We finish the paper with several remarks regarding the convergence

Remark 5.17. Recall that we have the correspondence between the ring states with the origin as their center of mass and the compact set $F$ of fixed points of the rotating frame system (see Section 3). Thus, given a ring state about the origin $P_{0} = \text{col}[X_{0} + iY_{0}, 0_{n} + i0_{n}] \in F$; and any $\epsilon > 0$, there exists $\delta = \delta(P_{0}, \epsilon) > 0$ such that if $P_{\epsilon} = \text{col}[X_{\epsilon} + iY_{\epsilon}, X_{\epsilon} + iY_{\epsilon}]$ is in the ball $B(P_{0}, \delta)$ of radius $\delta$ from $P_{0}$ then the solution $\phi_{t}(P_{\epsilon})$ of (10) with initial condition $P_{\epsilon}$ satisfies $|\phi_{t}(P_{\epsilon}) - P_{0}| < \epsilon$ for all $t > 0$.

Moreover, there exists $P_{\infty} \in F$ and $R_{\infty} \in \mathbb{C}$ such that

$$\lim_{t \to \infty} \left[ \phi_{t}(P_{\epsilon}) - (P_{\infty} + \text{col}[R_{0e^{-it}}1_{n}, -iR_{0e^{-it}}1_{n}]) \right] = 0.$$

Let $\epsilon > 0$. For each point $P$ in $F$ we can construct $\delta(P, \epsilon)$ as above and, by compactness of $F$, cover $F$ with finitely many balls $B(P_{0}, \delta(P_{0}, \epsilon))$. Let $\delta = \frac{1}{2} \min \delta(P_{0}, \epsilon)$. We get that for any $P_{0} \in F$, if $|P_{\epsilon} - P_{0}| < \delta$ then $|\phi_{t}(P_{\epsilon}) - P_{0}| < 2\epsilon$ for all $t > 0$. Recall that System (1) is translation invariant. Thus, similar estimates hold for ring states centered anywhere.

Finally, Remark 5.17 and the stability results of Sections 4 and 5 complete the proof of Theorem 1.1.

Remark 5.18. We finish the paper with several remarks regarding the convergence of the system to its limit configurations.

1) In this paper we established stability of ring states. The methods developed in this paper also allow us to obtain some estimates on the speed of convergence towards ring states. The functions $D_{L}$ and $D_{U}$ captured the dispersions of the right and left group of particles, respectively. These functions together with the variable $z_{n-1}$ (see Figure 8 for the geometrical meaning of $z_{n-1}$) ensured the stability of the degenerate ring state configuration. In a neighborhood of a degenerate ring state, the energy function $E$ decreases to zero, thus, ensuring that the system converges towards a ring state. In the region $\{E > 0\}$, the functions $D_{L}$ and $\sqrt{D_{L}} \pm z_{n-1}$ are
decreasing, but they do not have to converge to zero as the limit ring state need not be degenerate.

(2) Consider a situation when the left group particles \( r_1, \ldots, r_N \) start out with the same initial conditions and initial velocities, with initial conditions near those of a ring state. Due to symmetries of the system, the left particles will follow the same trajectory at all times, in which case, \( D_U = 0 \) at all times. For the limit cycle (assume its center is the origin): since the left particles’ unit vector positions sum up to magnitude \( N \), the limit cycle must have the right particles’ position vectors also sum up to a magnitude \( N \) vector. The triangle inequality implies that for the limit cycle the right particles have equal positions as well. In view of Lemma 5.13, the condition \( D_U = 0 \) places the system in the region \( \{ E > 0 \} \). It follows from (53) in the proof of Lemma 5.13 that

\[
E(Z) = \frac{1}{4\sqrt{1 - z_n^2}} (D_L(Z) - D_U(Z)) + O(|Z|_r^3).
\]

Since \( D_U = 0 \), we can show that \( E(Z) = \frac{1}{4} D_L(Z) + O(|Z|_r^3) \). Substituting this identity for \( E \) into \( D_L \) as presented in the statement of Lemma 5.14, we obtain that

\[
\frac{dD_L}{dt} = -\frac{1}{4} A(Z) D_L^2(Z) (1 + O(|Z|)) .
\]

Thus, for small enough \( Z \), the quantity \( \frac{dD_L}{dt} = -\frac{d}{dt} \left( \frac{1}{4} D_L \right) \) remains within a neighborhood of 1/4, say, within \([1/6, 1/3] \). It follows that

\[
\frac{D_L(0)}{1 + \frac{1}{4} D_L(0)t} \leq D_L(t) \leq \frac{D_L(0)}{1 + \frac{1}{3} D_L(0)t},
\]

which shows that mutual distances within the right group of particles are approaching zero, but rather slowly – at a rate comparable to \( \frac{1}{\sqrt{t}} \). Also this proves that the limit configuration is a degenerate ring state.

(3) Finally, we discuss the rate of convergence of the energy function \( E \). Recall that

\[
E = \frac{1}{n} \sum_{k=1}^{n} \varepsilon_k \cos(\theta_k) = \frac{1}{n} \sum_{k=1}^{n} \varepsilon_k \sqrt{1 - \sin^2 \theta_k},
\]

where \( \varepsilon_1 = \cdots = \varepsilon_N = 1 \) and \( \varepsilon_{N+1} = \cdots = \varepsilon_n = -1 \).

Therefore,

\[
\frac{dE}{dt} = -\frac{1}{n} \sum_{k=1}^{n} \varepsilon_k \left( \sin \theta_k \frac{d\sin \theta_k}{dt} \frac{1}{\cos \theta_k} \right).
\]

Recall \( \sin(\Theta) = \nabla Z \). Thus, using Theorem 5.10, we obtain that \( d(\sin(\Theta))/dt = \nabla \tilde{Z} = E \mathcal{O}(|Z_r|) \). It follows from Lemma 5.9 that \( \frac{1}{\cos(\Theta)} = \frac{1}{\mathcal{A}(Z)} + \mathcal{O}(|Z_r|) \).

Hence,

\[
\frac{dE}{dt} = -\frac{1}{n \mathcal{A}(Z)} \sum_{k=1}^{n} \varepsilon_k \left( \sin \theta_k \frac{d\sin \theta_k}{dt} \right) + E \mathcal{O}(|Z|_r^2|Z|_r^2).
\]

Recall that \( \sin(\Theta_U) = V Z_U + z_{n-1} \mathbf{1}_N, \sin(\Theta_L) = V Z_L + z_{n-1} \mathbf{1}_N \), and that the columns of the matrix \( V \) are orthogonal to \( \mathbf{1}_N \). Therefore,

\[
\frac{dE}{dt} = \frac{1}{n \mathcal{A}(Z)} \left[ -(V Z_U + z_{n-1} \mathbf{1}_N)^T \cdot (V \tilde{Z}_U + \dot{z}_{n-1} \mathbf{1}_N) 
\right.
\]

\[
+ (V Z_L + z_{n-1} \mathbf{1}_N)^T \cdot (V \tilde{Z}_L + \dot{z}_{n-1} \mathbf{1}_N) \left. \right] + E \mathcal{O}(|Z|_r^2|Z|_r^2)
\]

\[
= \frac{1}{n \mathcal{A}(Z)} \left[ -(V Z_U)^T \cdot (V \tilde{Z}_U) + (V Z_L)^T \cdot (V \tilde{Z}_L) \right] + E \mathcal{O}(|Z|_r^2|Z|_r^2)
\]
Using Theorem 5.10 and the definition of the functions \( D_U \) and \( D_L \), we get that
\[
\frac{dE}{dt} = -\frac{1}{2} E[D_U + D_L] + E O(|Z|^2 |Z|).
\]

Since \( D_U + D_L \geq \max (D_L, D_U) \geq O(|Z|^2) \), we get that
\[
\frac{dE}{dt} = -\frac{1}{2} E[D_U + D_L] (1 + O(|Z|)),
\]
which describes the rate of convergence of the energy function.

(4) Using the methods of Lemma 5.14 one can show that in the region \( \{ E > 0 \} \) the function \( D_U \) satisfies
\[
\frac{dD_U}{dt} = 2E(Z) A(Z) \left[ D_U + D_L O(|Z|) \right].
\]

Thus, if the initial dispersion of the left group of agents \( D_U(Z_0) \) exceeds \( K D_L(Z_0) |Z_0| \), the dispersion \( D_U \) will continue to increase in time. It means that the particles will rotate in a non-degenerate configuration approximating a non-degenerate ring state at a rate captured by
\[
\frac{dE}{dt} = \frac{1}{2} [D_U + D_L] (1 + O(|Z|)) \leq -\frac{1}{2} D_U(Z_0).
\]

It follows that
\[
E \leq E(Z_0) \exp \left( -\frac{1}{2} D_U(Z_0) t \right) \approx E(Z_0) \exp \left( -\frac{1}{2} |Z_0|^2 t \right).
\]

Finally, combine \((-1) \frac{dE}{dt} \leq \frac{1}{2} [D_U + D_L] \leq \frac{1}{2} D_U(\infty) \leq 3 D_U(0) \) and part (iii) of Lemma 5.13. Since \( D_U(\infty) + D_L(0) \leq 2 D_L(\infty) + D_L(0) \leq 3 D_L(0) \), we get
\[
(-1) \frac{dE}{dt} \leq \frac{3}{2} D_L(0)
\]
We conclude that for that for such initial conditions the decay of \( E \) is very slow:
\[
E \geq E(Z_0) \exp \left( -\frac{3}{2} |Z_{0,U}|^2 t \right).
\]

Appendix: An Approximation of Center Manifolds

In this section we provide a general description of the method that we developed to approximate the center manifold of \( \Sigma \). The method is best suited for determining the stability of dynamical systems whose sets of fixed points have dimension at least one (systems with non-isolated fixed points). In this section, we present the general framework for constructing center manifolds for such systems.

We consider systems with stable linearizations (eigenvalues \( \lambda \) satisfying \( \text{Re} \lambda \leq 0 \)) that have already been converted to a block-diagonal form, having the origin as a non-isolated equilibrium point:
\[
\begin{align*}
\dot{x} &= A_s x + f(x, y) \\
\dot{y} &= B y + g(x, y)
\end{align*}
\]

Here \( A_s \) is an \( s \times s \) matrix whose eigenvalues have negative real part, \( B \) is a \( c \times c \) real matrix whose eigenvalues are purely imaginary or zero; \( f \) and \( g \) are nonlinear \( C^r \) functions that equal zero and have zero gradients at the origin (\( r \geq 2 \)).

The Center Manifold Theorem guarantees the existence of a center manifold function \( h = h(y) \) defined for points \( y \) in a neighborhood of the origin in \( \mathbb{R}^c \) with values in \( \mathbb{R}^s \) with \( h(0) = 0 \), \( \nabla h(0) = 0 \) and such that its graph \( (h(y), y) \) is locally invariant under \( \Sigma \). Moreover, the stability of \( \Sigma \) near the origin is equivalent to the stability of the reduced-dimensional flow governed by the equation
\[
\dot{y} = B y + g(h(y), y)
\]

near the origin in \( \mathbb{R}^c \).
Our method of approximating the function $h$ references its classical construction as a family of fixed points for operators defined on the space of slow-growing functions. Let $\eta > 0$ be less than the spectral gap of $A_s$; let

$$X_\eta = \left\{ w : (-\infty, \infty) \to \mathbb{R}^{++}, w \text{ continuous}, \sup_{t \in (-\infty, \infty)} |w(t)e^{-\eta|t|}| < \infty \right\}.$$ 

Equipped with the norm $||w||_\eta = \sup_{t \in (-\infty, \infty)} |w(t)e^{-\eta|t|}|$, the space $X_\eta$ becomes a Banach space. Define $\Gamma : \mathbb{R}^{c} \times X_\eta \to X_\eta$ as follows. For each $y \in \mathbb{R}^{c}$, in a neighborhood of the origin, define the operator

$$\Gamma_y(w)(t) = \left[ \int_{-\infty}^{t} e^{A_s(t-\tau)} f(w(\tau))d\tau \right] e^{B_y t} + \int_{0}^{t} e^{B_y(t-\tau)} g(w(\tau))d\tau$$

(61)

Each operator $\Gamma_y$ is a contraction. By restricting $y$ to a small enough neighborhood $N$ of the origin, one can ensure that the operators $\Gamma_y$ contract by a uniform factor of $1/2$. Let $F_y$ denote the (unique) fixed point function for the operator $\Gamma_y$. Then the set $M = \{ F_y(0) : y \in N \}$ is the center manifold of the system (59). Define the function $h : \mathbb{R}^{c} \to \mathbb{R}^{a}$ as $(h(y), y) \in M$.

When studying specific systems such as (59), it is often the case that the exact equation for $h$ cannot be explicitly determined. The traditional workaround is to use the equation

$$ Dh(y)[By + g(h(y), y)] - A_s h(y) - f(h(y), y) = 0 $$

to compute the Taylor polynomial expansion for the components of $h(y)$ up to the desired degree of accuracy, provided that $f, g$ are smooth enough, and to substitute that approximate expression of $h(y)$ into (60) to determine the stability of the flow.

Unfortunately, for center manifolds of dimension two or higher, there is no guarantee that the dynamics of the truncated system mirrors that of (60) regardless of how high the degree of the Taylor approximation is.

The Taylor approximation approach is particularly ill-suited for dynamical systems with non-isolated equilibrium points since by truncating $h$ one may destroy the degeneracy that makes all the nullclines of (60) intersect, leading to the elimination of the equilibrium points away from the origin.

In this new approach for approximating the function $h$ we use the presence of a big set of equilibrium points as a computational advantage by using their locations (which is often known a priori) to anchor the approximation for $h$.

We illustrate our approach and explain the need for a new technique on a simple example of a 3-dimensional system with a 1-dimensional set of equilibrium points within a 2-dimensional center manifold whose center manifold function $h$ is not a polynomial. Consider

$$\dot{x} = x(y - x \sin x) $$
$$\dot{y} = -y^2(y - z) $$
$$\dot{z} = -z + x(y - x \sin x)(\sin x + x \cos x) + x \sin x.$$ 

(62)

The set of equilibrium points for this system is $\{(x, x \sin x, x \sin x), x \in \mathbb{R}\}$. The origin – which is one of the equilibrium points – has Jacobian matrix

$$\text{diag}\{0, 0, -1\}.$$ 

Note that from (62) we get $d(z - x \sin x)/dt = (-1)(z - x \sin x)$. Thus the surface parametrized by $(x, y, x \sin x)$ is invariant under the flow and is tangent to the

\[ \text{References} \]

2Technically the functions $f, g$ are modified outside of a neighborhood of the origin so that they have compact support and so that their support is also within a neighborhood of the origin where the nonlinearities are small.
\((x,y)\) plane at the origin. Therefore, this surface is the center manifold. The stable manifold is the \(z\)-axis.

Using the actual center manifold function \(z = h(x,y) = x \sin x\), the study of stability of (62) can be reduced to the study of

\[
\begin{align*}
\dot{x} &= x(y - x \sin x) \\
\dot{y} &= -y^2(y - x \sin x)
\end{align*}
\]

(63)

The points with \(y = x \sin x, \ x \in \mathbb{R}\), are fixed by System (63). Thus, the regions \(y < x \sin x\) and \(y > x \sin x\) are invariant under the flow. Moreover, the line \(y = 0\) and the lower half-plane \(y < 0\) are forward invariant. Using LaSalle’s invariance principle one can show that the origin is stable (but not asymptotically stable) and that the \(ω\)-limit points for (63) are precisely the equilibrium points \((x,x \sin x)\).

The flow on the center manifold is illustrated below:

If the classical Taylor polynomial technique is used to approximate \(h\), we get the truncated dynamical system

\[
\begin{align*}
\dot{x} &= x(y - x \sin x) \\
\dot{y} &= -y^2(y - xT_{2k-1}(x))
\end{align*}
\]

(64)

where \(T_{2k-1}(x)\) denotes the Taylor polynomial of degree \(2k - 1\) for \(\sin x\). Depending on whether \(k\) itself is even or odd, \(xT_{2k-1}(x)\) is an under or overestimate of \(h\).

We only focus on the case when \(k\) is even. In this case, \(xT(x) \leq x \sin x\) (we dropped the subscript \(2k - 1\) from the Taylor polynomial). One can show that the region where \(xT(x) \leq y \leq x \sin x, \ y < 1/10\) sketched below is forward invariant for (64) – on the isocline \(y = x \sin x\), labeled \(COA\) and on the segments \(DC\) and \(AB\) the value of \(\dot{y}\) is negative due to \(xT(x)\) underestimating \(x \sin x\), so the vector field points inside the region. On the isocline \(y = xT(x)\), labeled \(DOB\), we have \(\dot{x}^2 = 2x^4|xT(x) - x \sin x|\) so vector field points to reduce the magnitude of \(x\), therefore pointing inside the region. A similar argument shows that the region above \(y = xT(x)\) and below \(y = 1/6\) is also forward invariant.
Using the Lyapunov functions $L_1 = x^2 + y^2$ for orbits that cross into region $xT(x) \leq y \leq x \sin x$ and $L_2 = y - x \sin x$ for points in the region $y > x \sin x$ one can show that the origin is asymptotically stable for (64), a dynamical feature distinct from the original system (63). That misrepresentation persists regardless of the degree of the Taylor approximation.

We propose to approximate center manifold $z = h(x, y)$ using the solution $z = \Psi(x, y)$ from the equation of the nullcline $\dot{z} = 0$. In this case we can explicitly solve for $z$ to get

$$
\Psi(x, y) = x(y - x \sin x)(\sin x + x \cos x) + x \sin x.
$$

The error introduced by this approximation is

$$
||h(x, y) - \Psi(x, y)|| \leq C||[x(y - x \sin x), -y^2(y - \Psi(x, y))]| = |y - x \sin x|O(||(x, y)||).
$$

Using $h(x, y) = x(y - x \sin x)(\sin x + x \cos x) + x \sin x + |y - x \sin x|O(||(x, y)||)$ in (62), we get

$$
\dot{x} = x(y - x \sin x)
$$
$$
\dot{y} = -y^2(y - x \sin x)[1 + O(||(x, y)||)]
$$

The system (65) faithfully captures the equilibrium points at locations $(x, x \sin x)$ and the behavior of the trajectories as converging to the origin or to an off-origin equilibrium point, depending on whether the initial condition was in $\{(x_0, y_0) | x_0 = 0 \text{ or } y_0 < x_0 \sin x_0 \}$ or not.

Now we return to the general set up of (59). Denote by $E$ the projection of the set of equilibrium points from $\mathcal{N}$ of System (59) onto $0_s \times \mathbb{R}^c$. The set of equilibrium points in $\mathcal{N}$ is $\{(h(y), y) \mid y \in E\}$. Denote by $\mathcal{N}_e$ the projection of the neighborhood $\mathcal{N}$ onto the neutral directions.

The alternate approximation technique we introduce caters to systems for which

(A0) $E$ contains some non-trivial curve $\gamma_E$ through the origin;

(A1) There are no purely imaginary eigenvalues;

(A2) The geometric multiplicity of the zero eigenvalue equals its algebraic multiplicity.

Note that $A_s$ is non-singular. Using the Implicit Function Theorem we can solve for $x$ as a function of $y$ in the equation $A_s x + f(x, y) = 0$ (when $y$ is in the neighborhood of the origin in $\mathbb{R}^c$). Thus, there exists a $C^2$ function $\Psi : \mathcal{N}_e \to \mathbb{R}^s$ such that

$$
A\Psi(y) + f(\Psi(y), y) = 0 \quad \text{for all } y \in \mathcal{N}_e.
$$

Note that if $y_0 \in \mathcal{N}_e \cap E$ then $A\Psi(y_0) + f(\Psi(y_0), y_0) = 0$. Therefore due to the uniqueness of solutions produced by the Implicit Function Theorem, we get $h(y) = \Psi(y)$. In other words, our approximation is error-free on the set of equilibrium points.
and the function \( \Psi \) can be used as a first approximation to the center manifold function \( h \).

Let \( d(y) \) denote the distance from a point \( y \in \mathcal{N}_c \) to the set of equilibrium solutions \( E \). Note that in general \( d(y) \leq ||y|| \), but for points very close to \( E \) we could have \( d(y) \ll ||y|| \).

The following proposition provides an estimate on the accuracy of the approximation.

**Proposition 5.19.**

\[
|h(y) - \Psi(y)| = \mathcal{O}(||y||\text{dist}(y,E)).
\]

Moreover, the error of approximation equations governing the flow in (66) is

\[
||g(h(y),y) - g(\Psi(y),y)|| = \text{dist}(y,E)\mathcal{O}(||y||^2).
\]

**Proof.** Let \( y \in \mathcal{N}_c \). We begin by simplifying the operator \( \Gamma_y \) when applied to time-independent functions. Given a constant function \( w(t) = (w_{0s},w_{0c}) \), we get that

\[
\Gamma_y(w)(t) = \begin{bmatrix}
\left( \int_{-\infty}^{t} e^{A_s(t-\tau)} \, dt \right) f(w_{0s},w_{0c}) \\
e^{Bt}w_{0c} + \left( \int_{0}^{t} e^{B(t-\tau)} \, dt \right) g(w_{0s},w_{0c})
\end{bmatrix} = \begin{bmatrix}
-A_s^{-1}f(w_{0s},w_{0c}) \\
w_{0c} + tg(w_{0s},w_{0c})
\end{bmatrix}.
\]

(67)

It follows that

\[
\Gamma_y(w)(t) - w = \begin{bmatrix}
-A_s^{-1}(Aw_{0s} + f(w_{0s},w_{0c})) \\
tg(w_{0s},w_{0c})
\end{bmatrix}.
\]

(68)

Note that for \( w(t) = (w_{0s},w_{0c}) \), its norm in the Banach space \( X_\eta \) coincides with its norm in \( \mathbb{R}^{s+c} \) and the norm of the linear function \( tg(w_{0s},w_{0c}) \) is \( \frac{1}{\eta}||g(w_{0s},w_{0c})|| \)

since the supremum of \( te^{-\eta|t|} \) is \( \frac{1}{\eta} \).

Let \( y \in \mathcal{N}_c \) be fixed. Consider the time-invariant function \( w(t) = (\Psi(y),y) \) for all \( t \in \mathbb{R} \). Using (68) and (66) we obtain that

\[
\Gamma_y(\Psi(y),y)(t) - (\Psi(y),y) = \begin{bmatrix}
-A_s^{-1}(A\Psi(y) + f(\Psi(y),y)) \\
tg(\Psi(y),y)
\end{bmatrix} = \begin{bmatrix}
0 \\
tg(\Psi(y),y)
\end{bmatrix}.
\]

(69)

Therefore, \( ||\Gamma_y(\Psi(y),y)(t) - (\Psi(y),y)||_\eta \leq \frac{1}{\eta}||g(\Psi(y),y)|| \). This ensures that the function \( F_y \) that is the fixed point of the contraction operator satisfies

\[
||F_y - (\Psi(y),y)||_\eta \leq \frac{2}{\eta}||g(\Psi(y),y)||.
\]

It follows from (66) and \( \nabla f(0,0) = 0 \) that \( \nabla \Psi(0) = 0 \). Thus, \( \Psi(y) \leq C||y||^2 \).

Given a point \( y \), denote by \( e_y \) the point in \( E \) that is nearest to \( y \). Note that \( e_y \) lies in the closed ball \( B(y,2||y||) \) and satisfies \( g(\Psi(e_y),e_y) = 0 \). We have

\[
||g(\Psi(y),y)|| = ||g(\Psi(y),y) - g(\Psi(e_y),e_y)|| \leq C||y||||y - e_y||.
\]

Therefore,

\[
||h(y) - \Psi(y)|| = ||F_y(0) - \Psi(y)|| = \mathcal{O}(||F_y - (\Psi(y),y)||_\eta) = \mathcal{O}(||y||\text{dist}(y,E)).
\]

\[\square\]
Remark 5.20. (1) In some situations, one may be able to identify only an approximate (instead of explicit) solution \( \xi(y) \) to (66). In those situations, if
\[
A\xi(y) + f(\xi(y), y) = \text{dist}(y, E)O(||y||).
\]
Then
\[
||h(y) - \xi(y)|| = \text{dist}(y, E)O(||y||)
\]
and the error of approximation equations governing the flow in (60) is
\[
||g(h(y), y) - g(\xi(y), y)|| = \text{dist}(y, E)O(||y||^2).
\]
The only difference between using the explicit solution \( \Psi \) and the approximation \( \xi \) is in the top row of (60), where the component is not precisely zero, but \( \text{dist}(y, E)O(||y||^2) \).

(2) Our approximation for the center manifold of the reduced system introduced in the proof of Theorem 5.10 uses the point \( Q_Z \) from (36). The role of Lemma 5.11 was to establish that \( Q_Z \) played the role of the function \( \xi(y) \) with a slightly improved order of approximation. The error term was \( \mathcal{E}(Z)O(||Z||) \) and \( \mathcal{E}(Z) \approx \text{dist}(Z, E)||Z|| \).

References

[1] G. Albi, D. Balagué, J. A. Carrillo, and J. von Brecht. Stability analysis of flock and mill rings for second order models in swarming. SIAM Journal on Applied Mathematics, 74(3):794–818, 2014.

[2] Alberto Bressan. Tutorial on the center manifold theorem. In Hyperbolic systems of balance laws, volume 1911 of Lecture Notes in Math., pages 327–344. Springer, Berlin, 2007.

[3] Jack Carr. Applications of centre manifold theory, volume 35 of Applied Mathematical Sciences. Springer-Verlag, New York-Berlin, 1981.

[4] Rui J. P. de Figueiredo and Chieng-yi Chang. On the boundedness of solutions of classes of multidimensional nonlinear autonomous systems. SIAM J. Appl. Math., 17:672–680, 1969.

[5] Maria Dorsogna, Yao-Li Chuang, Andrea Bertozzi, and L Chayes. Self-propelled particles with soft-core interactions: Patterns, stability, and collapse. Physical review letters, 96:104302, 04 2006.

[6] W. Ebeling and F. Schweitzer. Swarms of particle agents with harmonic interactions. Theory in Biosciences, 120/3-4:207–224, 2001.

[7] Neil Fenichel. Geometric singular perturbation theory for ordinary differential equations. J. Differential Equations, 31(1):53–98, 1979.

[8] F. R. Gantmacher. The theory of matrices. Vols. 1, 2. Translated by K. A. Hirsch. Chelsea Publishing Co., New York, 1959.

[9] C. W. Gear, T. J. Kaper, I. G. Kevrekidis, and A. Zagaris. Projecting to a slow manifold: Singularity perturbed systems and legacy codes. SIAM Journal on Applied Dynamical Systems, 4(3):711–732, 2005 2005.

[10] Jack K. Hale. Diffusive coupling, dissipation, and synchronization. J. Dynam. Differential Equations, 9(1):1–52, 1997.

[11] A. Haraux and M. A. Jendoubi. Convergence of solutions of second-order gradient-like systems with analytic nonlinearities. Journal of Differential Equations, 144(DE973393):313–320, 1998.

[12] Alain Haraux and Mohamed Ali Jendoubi. The convergence problem for dissipative autonomous systems. SpringerBriefs in Mathematics. Springer, Cham; BCAM Basque Center for Applied Mathematics, Bilbao, 2015. Classical methods and recent advances, BCAM SpringerBriefs.

[13] Norimichi Hirano and Slawomir Rybicki. Existence of limit cycles for coupled van der Pol equations. J. Differential Equations, 195(1):194–209, 2003.

[14] Theodore Kolokolnikov, Hui Sun, David Uminsky, and Andrea L. Bertozzi. Stability of ring patterns arising from two-dimensional particle interactions. Phys. Rev. E, 84:015203, Jul 2011.

[15] Yao Li Chuang, Maria R. D’Orsogna, Daniel Marthaler, Andrea L. Bertozzi, and Lincoln S. Chayes. State transitions and the continuum limit for a 2d interacting, self-propelled particle system. Physica D: Nonlinear Phenomena, 232(1):33–47, 2007.

[16] A. Léviard. Étude des oscillations entretenues. Revue générale de l’électricité, 1928.

[17] A. Lins, W. de Melo, and C. C. Pugh. On lénard’s equation. In Lecture Notes in Mathematics, pages 335–357. Springer Berlin Heidelberg, 1977.
[18] Robert Mach and Frank Schweitzer. Modeling vortex swarming in daphnia. Bulletin of Mathematical Biology, 69(2):539–562, aug 2006.
[19] C. Medynets and I. Popovici. On the boundedness of a multi-particle system satisfying a generalized Liénard equation. preprint, 2021.
[20] Klementyna Szwaykowska, Ira B. Schwartz, Luis Mier y Teran Romero, Christoffer R. Heckman, Dan Mox, and M. Ani Hsieh. Collective motion patterns of swarms with delay coupling: Theory and experiment. Physical Review E, 93(3), mar 2016.
[21] Ferdinand Verhulst. Nonlinear differential equations and dynamical systems. Universitext. Springer-Verlag, Berlin, second edition, 1996. Translated from the 1985 Dutch original.

Email address: carl@kolon.org

United States Navy

Email address: medynets@usna.edu

Mathematics Department, United States Naval Academy, Annapolis, MD 21402

Email address: popovici@usna.edu (corresponding author)

Mathematics Department, United States Naval Academy, Annapolis, MD 21402