Abstract. Fix a pair of relatively prime integers $n > k \geq 1$, and a point $(\eta|\tau) \in \mathbb{C} \times \mathbb{H}$, where $\mathbb{H}$ denotes the upper-half complex plane, and let $\left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \text{SL}(2, \mathbb{Z})$. We show that Feigin and Odesskii’s elliptic algebras $Q_{n,k}(\eta|\tau)$ have the property $Q_{n,k}(\eta|\tau) \cong Q_{n,k}(\eta|\tau)$. As a consequence, given a pair $(E, \xi)$ consisting of a complex elliptic curve $E$ and a point $\xi \in E$, one may unambiguously define $Q_{n,k}(E, \xi) := Q_{n,k}(\eta|\tau)$ where $\tau \in \mathbb{H}$ is any point such that $\mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau \cong E$ and $\eta \in \mathbb{C}$ is any point whose image in $E$ is $\xi$. This justifies Feigin and Odesskii’s notation $Q_{n,k}(E, \xi)$ for their algebras.

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References

1. Introduction

We will use the notation $e(z) := e^{2\pi iz}$.
Always, $\tau$ denotes a point in the upper-half complex plane, $\mathbb{H} := \{x + iy \mid x, y \in \mathbb{R}, y > 0\}$.

1.1. **The main result.** Let

$$\vartheta(z \mid \tau) := \sum_{n \in \mathbb{Z}} e\left(nz + \frac{1}{2}n^2\tau\right).$$

In 1828, C.G.J. Jacobi [Jac28] proved the remarkable identity

$$\vartheta\left(\frac{z}{\tau} - \frac{1}{\tau}\right) = \sqrt{-i\tau} e\left(\frac{z^2}{2\tau}\right) \vartheta(z \mid \tau),$$

where $\sqrt{-i\tau}$ is the square root of $-i\tau$ having non-negative real part.\(^1\) Jacobi’s identity is a special case of the following functional equation: if

$$\vartheta\left(\frac{az+b}{c\tau+d}\right) = \zeta^{c\tau+d} e\left(\frac{cz^2}{2(c\tau+d)}\right) \vartheta(z \mid \tau).$$

The precise value of $\zeta$, which is not important in this paper, can be found at [Mum07, Thm. 7.1, p. 32], for example.

In 1989, Feigin and Odesskii defined a remarkable family of graded associative $\mathbb{C}$-algebras which we denote here by $Q_{n,k}(\eta \mid \tau)$. These are the elliptic algebras of the title. They depend on a pair of relatively prime integers $n > k \geq 1$ and a point $(\eta \mid \tau) \in \mathbb{C} \times \mathbb{H}$. We define $Q_{n,k}(\eta \mid \tau)$ in §1.2.

The main result in this paper is the following.

**Theorem 1.1 (Theorem 4.5).** If (1-3) holds, then there is an isomorphism of graded $\mathbb{C}$-algebras

$$Q_{n,k}\left(\frac{\eta}{\tau} \mid \frac{a\tau+b}{c\tau+d}\right) \cong Q_{n,k}(\eta \mid \tau).$$

In §4.5 we explain how this result can be interpreted as saying that $Q_{n,k}(\cdot \mid \cdot)$ is a “weakly modular function of weight $-1$ taking values in the category of graded $\mathbb{C}$-algebras”.

1.2. **The definition of $Q_{n,k}(\eta \mid \tau)$.** Fix the data $(n, k, \eta, \tau)$ as above; i.e., $n > k \geq 1$ is a pair of relatively prime positive integers and $(\eta \mid \tau)$ is a point in $\mathbb{C} \times \mathbb{H}$.

Write $\Lambda_{\tau} := \mathbb{Z} \oplus \mathbb{Z}\tau$ and $E_{\tau} := \mathbb{C}/\Lambda_{\tau}$.

Fix a complex vector space $V$ with basis $x_0, \ldots, x_{n-1}$ indexed by $\mathbb{Z}_n$, the cyclic group of order $n$. In [CKS21a, Prop. 2.6], we defined a particular basis $\theta_0, \ldots, \theta_{n-1}$, also indexed by $\mathbb{Z}_n$, for the vector space $\Theta_n(\Lambda_{\tau})$ which, by definition, consists of all holomorphic functions $f : \mathbb{C} \to \mathbb{C}$ such that

$$f(z + 1) = f(z) \quad \text{and} \quad f(z + \tau) = -e(-nz)f(z)$$

for all $z \in \mathbb{C}$. Copying Feigin and Odesskii, we then defined for each $\eta \in \mathbb{C} - \frac{1}{n}\Lambda_{\tau}$ a family of holomorphic linear operators

$$R_{n,k}(z, \eta \mid \tau) : V \otimes V \longrightarrow V \otimes V, \quad z \in \mathbb{C},$$

given by the formula

$$R_{n,k}(z, \eta \mid \tau)(x_i \otimes x_j) := \frac{\theta_0(-z)\cdots\theta_{n-1}(-z)}{\theta_1(0)\cdots\theta_{n-1}(0)} \sum_{r \in \mathbb{Z}_n} \frac{\theta_{j-i+r(k-1)}(-z + \eta)}{\theta_{j-i-r}(-z)\theta_{kr}(\eta)} x_{j-r} \otimes x_{i+r}$$

\(^1\)Jacobi did not use this notation, nor did he provide a proof in [Jac28]. Jacobi’s identity is equivalent to a similar identity for the function $\vartheta_1(z \mid \tau)$ that appears in Chapter XXI of Whittaker-Watson [WW62, pp. 475–476]. On the other hand, (1-2) is equivalent to a similar identity for Whittaker and Watson’s function $\vartheta_3(z \mid \tau)$ since $\vartheta(z \mid \tau) = \vartheta_3(\pi z \mid \tau)$ and, as Whittaker and Watson indicate, their identities for $\vartheta_1(z \mid \tau)$ and $\vartheta_3(z \mid \tau)$ are equivalent.
for all \((i, j) \in \mathbb{Z}_n^2\). Although this formula does not make sense when \(\eta \in \frac{1}{n} \Lambda\), because in that case \(\theta_{kr}(\eta) = 0\) for some \(r \in \mathbb{Z}_n\), the function \(R_{n,k}(\eta, \eta | \tau) : \mathbb{C} - \frac{1}{n} \Lambda, \tau \rightarrow \text{End}_\mathbb{C}(V^{\otimes 2})\) extends in a unique way to a holomorphic function on the entire complex plane \([\text{CKS21a}, \text{Lem. 3.13}]\)—we will write \(R_{n,k}(\eta, \eta | \tau)\) for the extension. This allows us to define, for all \(\eta \in \mathbb{C}\), the algebra

\[
Q_{n,k}(\eta | \tau) := \frac{TV}{(\text{the image of } R_{n,k}(\eta, \eta | \tau))}
\]

where \(TV\) denotes the tensor algebra on \(V\).

Although the formula for \(R_{n,k}(z, \eta | \tau)\) is not very illuminating, by building on results and ideas of Richey and Tracy \([\text{RT86}]\) we were able to show in \([\text{CKS20}]\) that \(R(z) = R_{n,k}(z, \eta | \tau)\) satisfies the quantum Yang-Baxter equation: for all \(u, v \in \mathbb{C}\),

\[
R(u)_{12}R(u + v)_{23}R(v)_{12} = R(v)_{23}R(u + v)_{12}R(u)_{23}
\]
as operators on \(V^{\otimes 3}\), where \(R(z)_{12} := R(z) \otimes I, R(z)_{23} := I \otimes R(z),\) and \(I\) is the identity operator on \(V\). This identity plays a key role in the proof of several results in \([\text{CKS20}]\) which showed, for suitable \(\eta\), that \(Q_{n,k}(\eta | \tau)\) has many of the good properties enjoyed by the polynomial ring on \(n\) variables; for example, when the image of \(\eta\) in \(E_\tau\) is not a torsion point, \(Q_{n,k}(\eta | \tau)\) has the same Hilbert series as that polynomial ring.

The work of Feigin and Odesskii in \([\text{FO89}, \text{OF95}, \text{Ode02}]\), and our later work in \([\text{CKS21a, CKS19, CKS21b, CKS20}]\), shows that \(Q_{n,k}(\eta | \tau)\) has many beautiful properties. Roughly speaking, the algebras \(Q_{n,k}(\eta | \tau)\) are a step beyond enveloping algebras and quantized enveloping algebras in the same way as elliptic and theta functions are a step beyond rational and trigonometric functions.

### 1.3. Feigin and Odesskii’s elliptic algebras \(Q_{n,k}(E, \xi)\)

Let \(X\) denote a compact Riemann surface of genus one. A pair \((X, p)\) consisting of a compact Riemann surface \(X\) of genus one and a point \(p\) on it will be called a complex elliptic curve. We usually omit the adjective “complex”. There is a unique way to make \(X\) an algebraic (or Lie) group with \(p\) as its identity—see, e.g., \([\text{Hai11, Cor. 1.11}]\). We will always view \((X, p)\) as a group in this way.

Given an elliptic curve \(E = (X, p)\) and a point \(\xi \in E\), i.e., a second point \(\xi \in X\), which is allowed to be \(p\), Feigin and Odesskii defined an algebra they denoted by \(Q_{n,k}(E, \xi)\)—see \([\text{OF89}]\), and \([\text{CKS21a}]\) for a definition when \(\xi\) is one of the points that Feigin and Odesskii disallow.\(^2\)

At \([\text{OF89, §1.2}]\), Feigin and Odesskii define \(Q_{n,k}(E, \xi)\) by the formula on the right-hand side of \((1-7)\) where \((\eta | \tau) \in \mathbb{C} \times \mathbb{H}\) is such that \(E\) is isomorphic to \(E_\tau\) and \(\eta\) is a point whose image in \(\mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau\) is \(\xi\).

There are many points \((\eta | \tau) \in \mathbb{C} \times \mathbb{H}\) with the property in the previous sentence. It is easy to see that \(Q_{n,k}(\eta | \tau) = Q_{n,k}(\eta' | \tau)\) if \(\eta\) and \(\eta'\) have the same image in \(E_\tau\) so Feigin and Odesskii’s definition of \(Q_{n,k}(E, \xi)\) does not depend on the choice of \(\eta\). However, Feigin and Odesskii do not show that the algebra they denote by \(Q_{n,k}(E, \xi)\) stays the same when \(\tau\) is replaced by another point \(\tau' \in \mathbb{H}\) for which \(E\) is isomorphic to \(E_{\tau'}\). We also failed to address this issue in our earlier papers \([\text{CKS21a, CKS19, CKS21b, CKS20}]\). Theorem 1.1 remedies this oversight and, as we will now explain, allows us to conclude that \(Q_{n,k}(E, \xi)\) is well-defined up to isomorphism when we define it as

\[
Q_{n,k}(E, \xi) := Q_{n,k}(\eta | \tau)
\]

where \(\tau \in \mathbb{H}\) is any point such that \(E_\tau \cong E\) and the image of \(\eta\) under this isomorphism is \(\xi\).

There is a notion of isomorphism between such pairs \((E, \xi) = (X, p, \xi)\), and it follows from Theorem 1.1 that \(Q_{n,k}(E, \xi)\) is isomorphic to \(Q_{n,k}(E', \xi')\) when \((E, \xi)\) is isomorphic to \((E', \xi')\). More precisely:

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\(^2\)In truth, Feigin and Odesskii use the notation \(Q_{n,k}(E, \tau)\) where \(\tau\) denotes a point on \(E\)—see §§1.3.1 and 1.3.2 below.
Theorem 1.2 (Corollary 4.8). If \((E, \xi) = (E_\tau, \eta + \Lambda_\tau)\) and \((E', \xi') = (E_\tau', \eta' + \Lambda_\tau')\) and \(\mu : E \to E'\) is an isomorphism of algebraic groups such that the diagram

\[
\begin{array}{ccc}
E & \xrightarrow{\mu} & E' \\
\downarrow{x \to x + \xi} & & \downarrow{x' \to x' + \xi'} \\
E & \xrightarrow{\mu} & E'
\end{array}
\]

commutes, then

\[
Q_{n,k}(\eta \mid \tau) \cong Q_{n,k}(\eta' \mid \tau').
\]

All the results about \(Q_{n,k}(E, \xi)\) in Feigin and Odesskii’s papers and in our earlier papers are, in fact, results about \(Q_{n,k}(\eta \mid \tau)\).

1.3.1. Changing the roles of the symbols \(\tau\) and \(\eta\). In our papers [CKS21a, CKS19, CKS21b, CKS20] we used \(\eta\) to denote a point in \(\mathbb{H}\) and \(\tau\) to denote a point in \(\mathbb{C}\). In this paper we switch that notation in order to agree with the common convention in the literature on elliptic functions, theta functions, and modular forms, that \(\tau\) denotes a point in \(\mathbb{H}\).

1.3.2. Warning. In [FO89, OF89], Feigin and Odesskii use the symbol \(\tau\) for both a point in \(\mathbb{C}\) and a point on \(E\)—in [OF89, §1.1] the symbol \(\tau\) denotes a point on \(E\) but in [OF89, §1.2] the notation \(\theta_0(\tau)\) only makes sense when \(\tau \in \mathbb{C}\).

1.4. The case when \(k = 1\). When \(k = 1\), Theorem 1.2 was proved by Tate and Van den Bergh [TVdB96]. In fact, they proved more because they work with elliptic curves over arbitrary fields.

In [TVdB96, §4.1], Tate and Van den Bergh define a graded \(\mathbb{F}\)-algebra \(A(E, \sigma, \mathcal{L})\) when \(E\) is an elliptic curve over an arbitrary field \(\mathbb{F}\), \(\sigma\) is a translation automorphism of \(E\), and \(\mathcal{L}\) is an invertible \(\mathcal{O}_E\)-module of degree \(n \geq 3\). They then prove in Proposition 4.1.1 the following result: if \(\mu : E' \to E\) is an isomorphism between elliptic curves, then there is a canonical isomorphism \(A(E, \sigma, \mathcal{L}) \to A(E', \mu^{-1}\sigma\mu, \mu^*\mathcal{L})\) sending \(x \in A(E, \sigma, \mathcal{L}) = H^0(E, \mathcal{L})\) to \(\mu^*x \in H^0(E', \mu^*\mathcal{L})\).

When \(E\) is a complex elliptic curve, \(\sigma\) is translation by \(\xi\), and \(\deg \mathcal{L} = n\), then

\[
A(E, \sigma, \mathcal{L}) \cong Q_{n,1}(E, \xi) = Q_{n,1}(\eta \mid \tau)
\]

where \(\tau \in \mathbb{H}\) is any point such that \(E \cong E_\tau\) and \(\xi\) is the image of \(\eta\) under an isomorphism \(E \to E_\tau\).3 Thus, [TVdB96, Prop. 4.1.1] proves Theorem 1.2 when \(k = 1\).

1.5. Remarks on the proof of the main theorem. The main theorem is an immediate consequence of the “functional equation” (1-8) below. Let us explain.

Let \(M = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \text{SL}(2, \mathbb{Z})\) and let \((\eta' \mid \tau') = M \triangleright (\eta \mid \tau) := (\begin{smallmatrix} \eta/d + \tau/a \\ c\tau + d \end{smallmatrix})\). Since the space of defining relations for \(Q_{n,k}(\eta \mid \tau)\) is the image of \(R_{n,k}(\eta, \eta' \mid \tau) : V^{\otimes 2} \to V^{\otimes 2}\), to show that \(Q_{n,k}(\eta' \mid \tau')\) is isomorphic to \(Q_{n,k}(\eta \mid \tau)\) we must show there is an automorphism \(\psi \in \text{GL}(V)\), depending on \(M\), such that \(\psi^{\otimes 2}\) sends the image of \(R_{n,k}(\eta, \eta' \mid \tau)\) to the image of \(R_{n,k}(\eta', \eta' \mid \tau')\); i.e., such that

\[
(\psi^{\otimes 2}) \circ R_{n,k}(\eta, \eta' \mid \tau) = R_{n,k}(\eta', \eta' \mid \tau') \circ \phi
\]

for some \(\phi \in \text{GL}(V^{\otimes 2})\); (1-8) is a stronger and more precise result.

To state (1-8) we extend the action of \(\text{SL}(2, \mathbb{Z})\) on \((\eta \mid \tau) \in \mathbb{C} \times \mathbb{H}\) to an action on \((z, \eta \mid \tau) \in \mathbb{C} \times \mathbb{C} \times \mathbb{H}\),

\[
M \triangleright (z, \eta | \tau) := \left(\begin{array}{c} z \\ (c\tau + d) \end{array} \right) = \left(\begin{array}{c} \eta/d + \tau/a \\ c\tau + d \end{array} \right).
\]

Theorem 4.4 shows there is a \(\psi \in \text{GL}(V)\) and a nowhere-vanishing holomorphic function \(f(z)\), both depending on \(M \in \text{SL}(2, \mathbb{Z})\), such that

(1-8) \[ R_{n,k}(M \triangleright (z, \eta | \tau)) = f(z) (\psi^{\otimes 2}) \circ R_{n,k}(z, \eta | \tau) \circ (\psi^{\otimes 2})^{-1}. \]

3A proof that \(A(E, \sigma, \mathcal{L})\) is isomorphic to \(Q_{n,1}(\eta \mid \tau)\) is given in [CKS21a, §3.2.6].
16. The contents of this paper. The equality (1-8) and the main theorem are proved in section 4. Section 2, which concerns certain normalizations \( w_{(u,v)}(z, \eta \mid \tau) \) of theta functions with characteristics, and section 3, which is about a finite Heisenberg group \( H_n \) of order \( n^3 \) and its extension \( \tilde{H}_n \) when \( n \) is even, are essential preliminaries.

The key to proving (1-8) is to use a presentation of the \( R \)-matrix defined in (1-6) that has more structure. This is done in several steps.

In section 3, we make \( V \), the degree-one component of \( Q_{n,k}(\eta \mid \tau) \), into an irreducible representation of \( H_n \); as a consequence there is a surjective homomorphism \( \rho : CH_n \to \text{End}_C(V) \) so we can, and do, choose a basis for \( \text{End}_C(V) \) consisting of images of certain elements \( J_{(u,v)} \in H_n \)—this works as stated when \( n \) is odd but needs a small adjustment when \( n \) is even; we then write \( R_{n,k}(z, \eta \mid \tau) \) in terms of this new basis and the functions \( w_{(u,v)}(z, \eta \mid \tau) \)—the essential part of this rewriting involves the operator \( T_k(z, \eta \mid \tau) : V^{\otimes 2} \to V^{\otimes 2} \) which is defined in (4-2)—its relation to \( R_{n,k}(z, \eta \mid \tau) \) is given in (4-3).

The “matrix coefficients” for \( T_k(z, \eta \mid \tau) \) involve the functions \( w_{(u,v)}(z, \eta \mid \tau) \) indexed by \((u,v) \in \mathbb{Z}^2 \), which are introduced in section 2. The main result in section 2 is that for each \( M \in \text{SL}(2, \mathbb{Z}) \) there is a nowhere-vanishing holomorphic function \( g(z) \), depending on \( M \) but not on \((u,v) \), such that

\[
w_{(u,v)}(M \triangleright (z, \eta \mid \tau)) = g(z) w_{(u,v)}(z, \eta \mid \tau).
\]

Thus, as far as \( w_{(u,v)}(z, \eta \mid \tau) \) is concerned, the modular action of \( \text{SL}(2, \mathbb{Z}) \) on \( \mathbb{C} \times \mathbb{C} \times \mathbb{H} \) can be replaced by the natural right action of \( \text{SL}(2, \mathbb{Z}) \) on \( \mathbb{Z}^2 \) or \( \mathbb{Z}^2_n \).

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2. Theta functions with characteristics and the functions \( w_{(u,v)}(z) \)

We will use the notation \( \omega := e(\frac{1}{n}) \).

When \( \tau \in \mathbb{H} \) we will write \( \Lambda_\tau := \mathbb{Z} + \mathbb{Z}\tau \) and \( E_\tau := \mathbb{C}/\Lambda_\tau \).

We write \( M^t \) for the transpose of a matrix \( M \) and \( M^{-t} \) for the inverse of \( M^t \) when it exists.

2.1. Definition of \( \theta_{u,v} \). As in [CKS20, §2.5], we will use theta functions with characteristics, namely the functions

\[
\theta_{u,v}(z \mid \tau) = \vartheta \begin{bmatrix} u \\ v \end{bmatrix}(z \mid \tau) := e\left( u(z + v) + \frac{1}{2}u^2\tau \right) \vartheta(z + u\tau + v \mid \tau)
\]

where \( \vartheta(z \mid \tau) \) is the function defined in (1-1). In particular, \( \theta_{0,0} = \vartheta \).

Proposition 2.1. Let \( s, t \in \mathbb{Z} \).

1. \( \theta_{u,v}(z + s \mid \tau) = \theta_{u,v+s}(z \mid \tau) \).
2. \( \theta_{u,v}(z + s\tau + t \mid \tau) = e\left( -s(z + v) - \frac{s^2\tau}{2} + tu \right) \theta_{u,v}(z \mid \tau) \).
3. \( \theta_{u,v}(z \mid \tau) = 0 \) if and only if \( z \in \frac{1}{2}(\tau + 1) - (u\tau + v) + \Lambda_\tau \).
4. Each of these zeros is a simple zero.

Proof. Parts (1), (2), and (3) were stated in [CKS20, the introductory remarks in §2.5, and Prop. 2.8], with the roles of \( \eta \) and \( \tau \) interchanged. For (3), see also [RT86, p. 315, (iv)] and the discussion before Proposition 2.8 in [CKS20]. By [CKS20, Lem. 2.5], \( \theta_{u,v}(z \mid \tau) \) has a single zero in each fundamental parallelogram for \( \Lambda_\tau \) located at \( \frac{1}{2}(1 + \tau) - (u\tau + v) \) modulo \( \Lambda_\tau \).
2.2. **Definition of** \( w_{(u,v)} \). As in [CKS20, §3], we define

\[
(2-1) \quad w_{(u,v)}(z) := \frac{\theta_{a \tau + b}^{a \tau + b}(z + \zeta \mid \tau)}{\theta_{c \tau + d}^{c \tau + d}(\zeta \mid \tau)},
\]

where

\[
(2-2) \quad \zeta := \eta + \frac{1}{2}(\tau + 1).
\]

We denoted \( \eta + \frac{1}{2}(\tau + 1) \) by \( \xi \) in [CKS20], but in this paper \( \xi \) always denotes an arbitrary point on \( E \).

The next result follows immediately from Proposition 2.1.

**Proposition 2.2.** The functions \( w_{(u,v)}(z) \) have the following properties:

1. \( w_{(u,v)}(0) = 1 \);
2. if \( s, t \in \mathbb{Z} \), then
   \[
   \frac{w_{(u,v)}(z + s\tau + t)}{w_{(u,v)}(z)} = e\left(-s(z + \eta) - \frac{s}{2}(s\tau + \tau + 1) + \frac{1}{2}(tu - sv)\right);
   \]
3. \( w_{(u,v)}(z) = 0 \) if and only if \( z = -\eta - \frac{1}{4}(u\tau + v) \) modulo \( \Lambda_{\tau} \);
4. each of these zeros is a simple zero.

In particular, \( w_{(u,v)}(z + 1) = e(\frac{a}{n})w_{(u,v)}(z) \) and \( w_{(u,v)}(z + \tau) = e(-z - \eta - \tau - \frac{1}{2} - \frac{1}{n}v)w_{(u,v)}(z) \).

2.2.1. The notation \( \theta_{a \tau + b}^{a \tau + b} \) and \( w_{(u,v)} \) makes sense for arbitrary complex numbers \( u \) and \( v \). As explained in the discussion preceding Theorem 3.1 in [CKS20], when \( u \) and \( v \) are integers \( w_{(u,v)}(z) \) depends only on their images in \( \mathbb{Z}_n \), so, in such a case, we will think of the subscript in \( w_{(u,v)}(z) \) as an element of \( \mathbb{Z}_n^2 \).

2.3. **The action of** \( \text{SL}(2, \mathbb{Z}) \) **on** \( \mathbb{C} \times \mathbb{C} \times \mathbb{H} \). If \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathbb{R}) \) and \( \tau \in \mathbb{H} \) we define

\[
M \triangleright \tau := \frac{a\tau + b}{c\tau + d}.
\]

If \( \Im(z) \) denotes the imaginary part of a complex number \( z \), \( \Im(M \triangleright \tau) = \det(M) |c\tau + d|^{-2} \Im(\tau) \). Hence the group \( \text{GL}^+(2, \mathbb{R}) \) of \( 2 \times 2 \) real matrices with positive determinant acts on \( \mathbb{H} \). That action extends to actions on \( \mathbb{C} \times \mathbb{H} \), given by the formula

\[
(2-3) \quad \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \triangleright (z \mid \tau) := \left( \begin{array}{c} z \\ c\tau + d \end{array} \right) \begin{pmatrix} a\tau + b \\ c\tau + d \end{pmatrix},
\]

and on \( \mathbb{C} \times \mathbb{C} \times \mathbb{H} \), given by the formula

\[
(2-4) \quad \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \triangleright (z, \eta \mid \tau) := \left( \begin{array}{c} z \\ c\tau + d \\ c\tau + d \end{array} \right) \begin{pmatrix} a\tau + b \\ \eta \\ c\tau + d \end{pmatrix}.
\]

In particular, the modular group \( \text{SL}(2, \mathbb{Z}) \) acts on \( \mathbb{C} \times \mathbb{H} \) and \( \mathbb{C} \times \mathbb{C} \times \mathbb{H} \). The action on \( \mathbb{C} \times \mathbb{H} \) is the same as the action in [RT86, (2.12)], which is a source we often refer to, both here and in [CKS20].

2.4. **Equivariance properties of** \( w_{(u,v)}(z, \eta \mid \tau) \). To state the next two results, we make explicit the dependence of \( w_{(u,v)} \) on all parameters by writing

\[
(2-5) \quad w_{(u,v)}(z, \eta \mid \tau) := w_{(u,v)}(z).
\]

**Proposition 2.3.** Let \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) \) act on the triples \( (z, \eta \mid \tau) \) via (2-4). The functions

\[
(2-6) \quad z \mapsto w_{(u,v)}(z, \eta \mid \tau)
\]

and

\[
(2-7) \quad z \mapsto w_{(u,v)}(M \triangleright (z, \eta \mid \tau))
\]

have the same zeros with the same multiplicities.
Proof. Let
\[(z', \eta' \mid \tau') := M \triangleright (z, \eta \mid \tau) = \begin{pmatrix} z & \eta \\ c\tau + d & c\tau + d \end{pmatrix} \begin{pmatrix} a\tau + b \\ c\tau + d \end{pmatrix} \).
By Proposition 2.2(3), \(w_{(u,v)}M(z, \eta \mid \tau) = 0\) if and only if
\[z + \eta + \frac{1}{n}(au + cv)\tau + (bu + dv)) \in \Lambda = \mathbb{Z}(a\tau + b) + \mathbb{Z}(c\tau + d),\]
where the equality uses the fact that \(M \in \text{SL}(2, \mathbb{Z})\). After dividing by \(c\tau + d\), this condition becomes
\[z' + \eta' + \frac{1}{n}(u(a\tau + b) + v(c\tau + d)) \in \mathbb{Z}\tau' + \mathbb{Z} = \Lambda'.\]
Therefore \(w_{(u,v)}M(z, \eta \mid \tau) = 0\) if and only if
\[z' + \eta' + \frac{1}{n}(u\tau' + v) \in \Lambda'.\]
By Proposition 2.2(3), this happens if and only if \(w_{(u,v)}(z', \eta' \mid \tau') = 0\); i.e., if and only if \(w_{(u,v)}(M \triangleright (z, \eta \mid \tau)) = 0\). Thus the zeros of the functions are the same. By Proposition 2.2(4) all the zeros are simple zeros so their multiplicities are the same. \(\square\)

We will now improve Proposition 2.3. First, we recall some standard terminology.

2.4.1. **Quasi-periodicity.** Suppose that \(\{\omega_1, \omega_2\}\) is an \(\mathbb{R}\)-basis for \(\mathbb{C}\), and let \(\Lambda := \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2\). We say that a function \(f : \mathbb{C} \to \mathbb{C}\) is **quasi-periodic with respect to** \(\Lambda\) if there are elements \(a, b, c, d \in \mathbb{C}\) such that
\[f(z + \omega_1) = e^{2\pi i (az + b)} f(z) \quad \text{and} \quad f(z + \omega_2) = e^{2\pi i (cz + d)} f(z)\]
for all \(z \in \mathbb{C}\).

2.4.2. **Factors of automorphy.** A **factor of automorphy** for \(\Lambda\) is a function
\[\Lambda \times \mathbb{C} \longrightarrow \mathbb{C}^\times, \quad (\lambda, z) \mapsto e_\lambda(z),\]
such that \(z \mapsto e_\lambda(z)\) is a nowhere-vanishing holomorphic function on \(\mathbb{C}\) and
\[e_{\lambda + \mu}(z) = e_\lambda(z + \mu)e_\mu(z)\]
for all \(\lambda, \mu \in \Lambda\); i.e., the function \(\lambda \mapsto e_\lambda(-)\) is a \(1\)-cocycle on \(\Lambda\) taking values in the group of nowhere-vanishing holomorphic functions on \(\mathbb{C}\), where the latter is a representation of \(\Lambda\) via the action
\[(\lambda \cdot f)(z) := f(z + \lambda)\]
If \(f(z \mid \tau)\) is quasi-periodic in \(z\) with respect to \(\mathbb{Z} + \mathbb{Z}\tau\), then its quasi-periodicity properties imply that the function
\[(s\tau + t, z) \mapsto e_{s\tau + t}(z) := \frac{f(z + s\tau + t \mid \tau)}{f(z \mid \tau)} \quad (s, t \in \mathbb{Z})\]
is a factor of automorphy for \(\mathbb{Z} + \mathbb{Z}\tau\). We call it the **factor of automorphy** for \(f(z)\). For example, Proposition 2.2(2) says that the factor of automorphy for \(w_{(u,v)}(z, \eta \mid \tau)\) is
\[e \left( -s(z + \eta) - \frac{s}{2}(s\tau + \tau + 1) + \frac{1}{n}(u, v) \left( \frac{t}{s} \right) \right)\].

2.4.3. The following fact will be used in the next proof: if \(f(z \mid \tau)\) and \(g(z \mid \tau)\) are quasi-periodic with respect to \(\mathbb{Z} + \mathbb{Z}\tau\) having the same factors of automorphy and the same zeros (counted with multiplicity), then there is a constant \(c\) such that \(f(z \mid \tau) = cg(z \mid \tau)\) for all \(z \in \mathbb{C}\). The reason is that \(f(z \mid \tau)/g(z \mid \tau)\) is an elliptic function with respect to \(\Lambda\) having no zeros and no poles, and hence constant.
2.4.4. Vague remark. The terminology “factor of automorphy” is also used in the context of line bundles on the elliptic curve $E_\tau := \mathbb{C}/\Lambda_\tau$. Theta functions with respect to $\Lambda_\tau$ are essentially the same things as sections of line bundles on $E_\tau$ (see, e.g., [BL04, p. 24 and Appendix B]).

**Theorem 2.4.** Let $M \in \text{SL}(2, \mathbb{Z})$. There is a nowhere-vanishing holomorphic function $f(z) = f_M(z)$, that does not depend on $(u, v)$, such that

$$w_{(u,v)M}(z, \eta | \tau) = f(z) w_{(u,v)}(M \triangleright (z, \eta | \tau)).$$

**Proof.** If the theorem holds for $M$ and $N$, then it holds for $MN$ with $f_{MN}(z) = f_M(z)f_N(z)$ and for $M^{-1}$ with $f_{M^{-1}}(z) = f_M(z)^{-1}$. It therefore suffices to prove the theorem for each of the two generators ([Ser80, §1.5.3])

$$X := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad Y := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

of $\text{SL}(2, \mathbb{Z})$. To prove the theorem it suffices to show that $w_{(u,v)M}(z, \eta | \tau)$ and $w_{(u,v)}(M \triangleright (z, \eta | \tau))$ have the same quasi-periodicity properties with respect to $\Lambda_\tau$, and the same zeros. We already showed in Proposition 2.3 that they have the same zeros, so it remains to show that they have the same factors of automorphy with respect to $\Lambda_\tau$.

Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$. As in the previous proof, we adopt the notation

$$(z', \eta' | \tau') := M \triangleright (z, \eta | \tau) = \begin{pmatrix} z \\ c\tau + d \end{pmatrix}, \quad \eta' := \frac{a\tau + b}{c\tau + d} \quad \text{and} \quad \tau' := \frac{1}{c\tau + d}.$$

Claim. There are unique integers $s', t'$ such that $(z' + s'\tau' + t', \eta' | \tau') = M \triangleright (z + s\tau + t, \eta | \tau)$, namely $s' := ds - ct$ and $t' := at - bs$, i.e., $(\tau') = M(\tau)$. 

**Proof.** By definition,

$$M \triangleright (z + s\tau + t, \eta | \tau) = \begin{pmatrix} z + s\tau + t \\ c\tau + d \end{pmatrix}, \quad \eta' := \frac{a\tau + b}{c\tau + d} \quad \text{and} \quad \tau' := \frac{1}{c\tau + d}.$$

Since $z' = \frac{z}{c\tau + d}$ it suffices to show there are unique integers $s', t'$ such that $s'\tau' + t' = \frac{z + s\tau + t}{c\tau + d}$. Certainly, if such an $(s', t') \in \mathbb{Z}^2$ exists it is unique because $\{1, \tau'\}$ is linearly independent over $\mathbb{Z}$—even over $\mathbb{R}$. Since $\tau' = \frac{ct + d}{c\tau + d}$, a simple calculation shows that $(ds - ct)\tau' + (at - bs) = \frac{ct + d}{c\tau + d}$: just multiply both sides by $c\tau + d$ and use the fact that $ad - bc = 1$.\footnote{Alternatively, if $(s', t') \in \mathbb{R}^2$ is such that $(c\tau + d)(s', t')(\tau') = (s, t)(\tau)$, then $(s, t)(\tau) = (s', t')\frac{(ct + d)(\tau')}{c\tau + d} = (s', t')M(\tau)$ and, since $\{1, \tau\}$ is linearly independent over $\mathbb{R}$, $(s', t') = (s, t)M^{-1} = (s, t)(\frac{d}{c} - \frac{b}{a}) = (ds - ct, at - bs)$.}

By Proposition 2.2(2), the factor of automorphy for $w_{(u,v)M}(z \mid \tau)$ is

$$e \left(-s(z + \eta) - \frac{s}{2}(s\tau + \tau + 1) + \frac{1}{n}(u, v)M(\tau) \right).$$

By Proposition 2.2(2), the factor of automorphy for $w_{(u,v)}(M \triangleright (z, \eta | \tau))$, i.e., the ratio

$$\frac{w_{(u,v)}(M \triangleright (z + s\tau + t, \eta | \tau))}{w_{(u,v)}(M \triangleright (z, \eta | \tau))} = \frac{w_{(u,v)}(z' + s'\tau' + t', \eta' | \tau')}{w_{(u,v)}(z', \eta' | \tau')},$$

is

$$e \left(-s'(z' + \eta') - \frac{s'}{2}(s'\tau' + \tau' + 1) + \frac{1}{n}(u, v)\left(\frac{t'}{s'}\right)\right).$$

But the terms involving $(u, v)$ in (2-8) and (2-9) are equal, so

$$\frac{(2-8)}{(2-9)} = e \left(-s(z + \eta) - \frac{s}{2}(s\tau + \tau + 1)\right) e \left(-s'(z' + \eta') - \frac{s'}{2}(s'\tau' + \tau' + 1)\right).$$
(1) (Proof for $M = Y$.) In this case, $(z', \eta' | \tau') = (z, \eta | \tau + 1)$ and $(t', -s') = (t, -s)Y^t = (t - s, -s)$. Thus, when $M = Y$, the denominator in (2-10) is
\[ e \left( -s(z + \eta) - \frac{s}{2}(s\tau + s + \tau + 1) \right), \]
which is equal to the numerator in (2-10) because $e \left( -\frac{s}{2}(s + 1) \right) = 1$. Hence (2-8) = (2-9) when $M = Y$ so, by the remark in §2.4.3, $w_{(u, u+v)}(z, \eta | \tau) = c w_{(u, v)}(z, \eta | \tau + 1)$ for some constant $c$. But $w_{(u, u+v)}(0, \eta | \tau) = 1 = w_{(u, v)}(0, \eta | \tau + 1)$ so $c = 1$, and we conclude that the theorem holds for $Y$ with $f_Y(z) = 1$ for all $z$.

(2) (Proof for $M = X$.) In this case, $(u, v)M = (v, -u)$, $(z', \eta' | \tau') = (z/\tau, \eta/\tau | -1/\tau)$, and $(s', t') = (-t, s)$. Thus, when $M = X$,
\[ \frac{(2-8)}{(2-9)} = e \left( -s(z + \eta) - \frac{s}{2}(s\tau + \tau + 1) - \frac{t}{2}(\frac{t}{\tau} - \frac{1}{\tau} + 1) - \frac{s}{2}(z + \eta) \right). \]
If we replaced the term $w_{(u, v)}M(z, \eta | \tau)$ by $g(z) w_{(u, v)}M(z, \eta | \tau)$ where $g(z) = e(Az^2 + Bz)$, then (2-11) would be multiplied by
\[ \frac{g(z + s\tau + t)}{g(z)} = e(A(z + s\tau + t)^2 + B(z + s\tau + t)) = e(A(2z + s\tau + t)(s\tau + t) + B(s\tau + t)). \]
Hence if $A = \frac{1}{2\tau}$ and $B = \frac{1}{2\tau} - \frac{s}{2\tau} + \frac{t}{\tau}$, then (2-11) would be multiplied by
\[ e \left( \frac{1}{2\tau}(2z + s\tau + t)(s\tau + t) + \left( \frac{1}{2} - \frac{1}{2\tau} + \frac{t}{\tau} \right)(s\tau + t) \right) \]
which equals
\[ e \left( sz + \frac{s}{2} + \frac{t}{2\tau}(s^2\tau^2 + 2st\tau + t^2) + \frac{s}{2} + s\eta + \frac{t}{2} - \frac{t}{\tau} + \frac{t}{\tau} \right). \]
A straightforward calculation shows that the product of this with (2-11) equals 1, so we conclude that $g(z) w_{(u, v)}X(z, \eta | \tau)$ and $w_{(u, v)}(X \triangleright (z, \eta | \tau))$ have the same factors of automorphy with respect to $\Lambda_\tau$. Since $g(z) \neq 0$ for all $z$, they also have the same zeros by Proposition 2.3, and both take the value 1 when $z = 0$, so we conclude that
\[ w_{(u, v)}X(z, \eta | \tau) = e \left( -\frac{1}{2\tau}z^2 + \left( \frac{1}{2\tau} - \frac{1}{2} - \frac{2}{\tau} \right)z \right) w_{(u, v)}(X \triangleright (z, \eta | \tau)). \]
The proof is complete. \hfill \Box

3. An extension of the finite Heisenberg group

Heisenberg groups play an important organizational role in the theory of theta functions and abelian varieties—see, for example, [BL04, Ch. 6], [Mum07, Ch. I, §3], [Mum91], and [Pol03]. In the present setting, the relevant Heisenberg group is an extension $1 \to \mu_n \to H_n \to \mathbb{Z}_n^2 \to 0$ where $\mu_n$ is the group of complex $n^{th}$ roots of unity. Below we will make $V$, the degree-one component of $Q_{n,k}(\eta | \tau)$, an irreducible representation for $H_n$. As Feigin and Odesskii first noticed, that action lifts to an action of $H_n$ as automorphisms of $Q_{n,k}(\eta | \tau)$ (see [CKS21a, §§2.3 and 3.5] for details).

Proposition 3.1 shows that when $n$ is odd the natural action of each $M \in \text{SL}(2, \mathbb{Z})$ on $\mathbb{Z}_n^2 = H_n / \langle \epsilon \rangle$ lifts to an action of $M$ as an automorphism of $H_n$. (But this does not yield a homomorphism $\text{SL}(2, \mathbb{Z}) \to \text{Aut}(H_n)$.) Such a lifting does not exist when $n$ is even. For this reason we introduce an index-two extension $\tilde{H}_n$ of $H_n$ for which such a lifting does exist. The utility of $H_n$ and $\tilde{H}_n$ becomes apparent in section 4 when we introduce a new presentation for the linear operator $R_{n,k}(z, \eta | \tau) : V \otimes^2 \to V \otimes^2$ in terms of a new linear operator $T_k(z, \eta | \tau) : V \otimes^2 \to V \otimes^2$ that transforms nicely with respect to the actions of, first, $\text{SL}(2, \mathbb{Z})$ on $(z, \eta | \tau)$ and the subscripts $(u, v) \in \mathbb{Z}_n^2$ in $w_{(u, v)}(z)$, and, second, $\tilde{H}_n$ on $V \otimes^2$, and, third, an action of $\text{SL}(2, \mathbb{Z})$ on $\tilde{H}_n$. 

3.1. The Heisenberg group $H_n$. The Heisenberg group of order $n^3$ is

$$H_n := \langle S, T, \epsilon \mid S^n = T^n = \epsilon^n = 1, [S, \epsilon] = [T, \epsilon] = 1, [S, T] = \epsilon \rangle.$$ 

The following equalities are useful when computing in $H_n$:

$$S^a T^b = T^b S^a \epsilon^{ab},$$

$$S^a T^b S^a T^b S^a T^b \cdots = T^B S^A \epsilon^C,$$

where $A = a_1 + a_2 + \cdots$, $B = b_1 + b_2 + \cdots$, $C = a_1 b_1 + (a_1 + a_2) b_2 + (a_1 + a_2 + a_3) b_3 + \cdots$,

$$(S^a T^b)^m = T^{bm} S^{am} \epsilon^X,$$

where $X = \frac{1}{2} m (m + 1) ab,$

$$(T^b S^a)^m = T^{bm} S^{am} \epsilon^Y,$$

where $Y = \frac{1}{2} m (m - 1) ab$.

Proposition 3.1. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathbb{Z})$. If $n$ is odd, there is an automorphism $\Psi'_M : H_n \to H_n$ such that

(3-1) \quad $\Psi'_M : \quad T \mapsto T^a S^c, \quad S \mapsto T^b S^d, \quad \epsilon \mapsto \epsilon^{\text{det} M}.$

Proof. The expressions $T^a S^c$ and $T^b S^d$ make sense when $a, b, c, d \in \mathbb{Z}$ because $T$ and $S$ have order $n$. The result follows from the fact that $T^n S^c$ and $T^b S^d$ have order $n$, and

$$(T^b S^d) (T^a S^c) (T^b S^d)^{-1} (T^a S^c)^{-1} = \epsilon^{ad-bc}.$$ 

The calculations contain no surprises. \qed

The map $\text{GL}(2, \mathbb{Z}) \to \text{Aut}(H_n)$, $M \mapsto \Psi'_M$, is not a group homomorphism. For example, if $M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then $\Psi'_M(T) = S$, $\Psi'_M(S) = T^{-1} S$, $\Psi'_M(S^2) = S^2 T^{-1} S = \epsilon T^{-1}$, but $\Psi'_M(S^2) = T^{-1}$. Similarly, if $M = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ and $N = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, then $\Psi'_M \Psi'_N(T) = \Psi'_M(T S) = S T \neq T S = \Psi'_M N(T)$. Because $M \mapsto \Psi'_M$ is not a group homomorphism we will make no use of $\Psi'_M$.

When $n$ is even, an additional problem arises: the map $\Psi'_M$ in (3-1) does not extend to an automorphism of $H_n$ because $T^n S^c$ and $T^b S^d$ need not have order $n$. For example, $TS$ has order $2n$. In order to address this problem we introduce a group when $n$ is even, the group $\tilde{H}_n$ defined below.

3.2. The extension $\tilde{H}_n$ of $H_n$. When $n$ is even we define

$$\tilde{H}_n := \langle S, T, \epsilon^{1/2} \mid S^n = T^n = \epsilon^n = 1, [S, \epsilon^{1/2}] = [T, \epsilon^{1/2}] = 1, [S, T] = \epsilon = (\epsilon^{1/2})^2 \rangle.$$ 

Here $\epsilon^{1/2}$ is a symbol that behaves like a square root of $\epsilon$. We note that $\langle S, T, \epsilon \rangle$ is a normal subgroup of index two in $\tilde{H}_n$ and is isomorphic to $H_n$.

3.2.1. The notation $\tilde{H}_n$ when $n$ is odd. For convenience, we define $\tilde{H}_n := H_n$ when $n$ is odd, with the convention that $\epsilon^{1/2} := \epsilon^{(n+1)/2}$. In that case, $(\epsilon^{(n+1)/2})^2 = \epsilon$, so $\langle \epsilon^{(n+1)/2} \rangle = \langle \epsilon \rangle$, and $(\epsilon^{1/2})^{-1} = \epsilon^{-(n-1)/2}$.

3.2.2. The action of $\tilde{H}_n$ as automorphisms of $Q_{n,k}(\eta | \tau)$. Feigin and Odesskii observed that $H_n$ acts as automorphisms of $Q_{n,k}(\eta | \tau)$ [Ode02, p. 1143] via the actions

$$S \cdot x_i = \omega^i x_i, \quad T \cdot x_i = x_{i+1}, \quad \epsilon \cdot x_i = \epsilon^{\left(\frac{1}{n}\right)} x_i.$$ 

A proof of this can be found at [CKS21a, Prop. 3.23]. It is easy to see that this extends to an action of $\tilde{H}_n$ as automorphisms of $Q_{n,k}(\eta | \tau)$ when $n$ is even.

Proposition 3.2. The group $\tilde{H}_n$ acts as degree-preserving $C$-algebra automorphisms of $Q_{n,k}(\eta | \tau)$ by

(3-2) \quad $S \cdot x_i = \omega^i x_i, \quad T \cdot x_i = x_{i+1}, \quad \epsilon^{1/2} \cdot x_i = -\epsilon^{\left(\frac{1}{2n}\right)} x_i, \quad \epsilon \cdot x_i = \epsilon^{\left(\frac{n+1}{2n}\right)} x_i.$

3.2.3. The reason for choosing $\epsilon^{1/2} \cdot x_i = -\epsilon^{\left(\frac{1}{2n}\right)} x_i$ rather than $\epsilon^{1/2} \cdot x_i = \epsilon^{\left(\frac{1}{2n}\right)} x_i$ in (3-2) is so the formula works for both even and odd $n$; i.e., the choice we have made for $\epsilon^{1/2} \cdot x_i$ is compatible with the convention in §3.2.1 that $\epsilon^{1/2} = \epsilon^{(n+1)/2}$ when $n$ is odd, but the other choice is incompatible.

\footnote{If $b$ and $c$ are even residues modulo $n$, however, both $T^n S^c$ and $T^b S^d$ have order $n$.}
3.3. The action of $\text{SL}(2, \mathbb{Z})$ as automorphisms of $\tilde{H}_n$. There are many actions of $\text{SL}(2, \mathbb{Z})$ as automorphisms of $\tilde{H}_n$, but we single out one particular action in Proposition 3.4 that has the virtue of simplicity.

**Lemma 3.3.** Let $\nu = \epsilon^{1/2}$ when $n$ is even and let $\nu = \epsilon^{(n+1)/2}$ when $n$ is odd. If $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathbb{Z})$, then there is an automorphism $\Psi_M$ of $\tilde{H}_n$ such that

$$\Psi_M : T \mapsto T^a S^c \nu^{ac}, \quad S \mapsto T^b S^d \nu^{bd}, \quad \nu \mapsto \nu^{ad-bc},$$

where $Z = acm^2 + bdr^2 + 2bcmr$.

**Proof.** Since

$$\Psi_M(S)\Psi_M(T) = T^b S^d \nu^{bd} T^a S^c \nu^{ac} = T^{a+b} S^{c+d} \epsilon^{ad+bc},$$

we have $\Psi_M(S)\Psi_M(T) = \Psi_M(T)\Psi_M(S)\epsilon^{ad-bc} = \Psi_M(T)\Psi_M(S)\Psi_M(\epsilon)$.

Furthermore,

$$\Psi_M(T)^n = (T^a S^c \nu^{ac})^n = T^{an} S^{cn} \epsilon^{n\nu}$$

where $D = \frac{1}{2}n(n-1)ac$. If $n$ is odd, then $D$ is an integer multiple of $n$ so $\epsilon^n = 1$, and $\nu^{acn} = \epsilon^{acn(n+1)/2} = 1$ so $\Psi_M(T)^n = 1$. If $n$ is even, then $D = m(n-1)ac$ where $m = n/2$ and $\nu^{acn} = \epsilon^{acm}$, whence $\epsilon^n \nu^{acn} = \epsilon^{nmac} = 1$; therefore $\Psi_M(T)^n = 1$ when $n$ is even. In conclusion, $\Psi_M(T)^n = 1$ when $n$ is odd and when $n$ is even. Similar calculations show that $\Psi_M(S)^n = 1$ when $n$ is odd and when $n$ is even.

Hence $\Psi_M$ is a group homomorphism.

Clearly, $\ker(\Psi_M) \cap \langle \nu \rangle = \{1\}$. However, every non-trivial normal subgroup of $\tilde{H}_n$ has non-trivial intersection with $\langle \nu \rangle$ so we conclude that $\ker(\Psi_M) = \{1\}$. Hence $\Psi_M$ is an automorphism of $\tilde{H}_n$.

We have

$$\Psi_M(T^m S^r) = (T^a S^c \nu^{ac})^m (T^b S^d \nu^{bd})^r = T^{am} S^{cm} \epsilon^X T^{br} S^{dr} \epsilon^Y \nu^{acm+bdr} = T^{am+br} S^{cm+dr} \epsilon^{X+Y+bcmr} \nu^{acm+bdr}$$

where $X = \frac{1}{2}m(m-1)ac$ and $Y = \frac{1}{2}r(r-1)bd$. Since $\epsilon = \nu^2$, the last part of the lemma holds with $Z = m(m-1)ac + r(r-1)bd + 2bcmr + acm + bdr$. Hence the result. $\Box$

**Proposition 3.4.** Let $\nu = \epsilon^{1/2}$ when $n$ is even and let $\nu = \epsilon^{(n+1)/2}$ when $n$ is odd. The map $\Psi : \text{SL}(2, \mathbb{Z}) \to \text{Aut}(\tilde{H}_n)$ that sends $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$ to the automorphism

$$\Psi_M : T \mapsto T^a S^c \nu^{ac}, \quad S \mapsto T^b S^d \nu^{bd}, \quad \nu \mapsto \nu,$$

is a group homomorphism.

**Proof.** Let $N = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$. Then

$$(\Psi_N \Psi_M)(T) = (T^a S^c \nu^{ac})^a (T^b S^d \nu^{bd}) \epsilon \nu^{ac} = T^{a+da'} S^{c+dc'} \epsilon^{a+da'} N^{b+db'} \epsilon^{d+dc'}$$

and

$$\Psi_{NM}(T) = T^{a+a'} S^{c+c'} \epsilon^{a+a'} N^{b+b'} \epsilon^{c+c'}$$

where $A = \frac{1}{2}a(a-1)a'c'$ and $B = \frac{1}{2}c(c-1)b'd'$. We have

$$\nu(t^{a+a'}N^{b+b'}(a+a'c'(c+c'))^b \nu = \nu^D$$
where
\[
D = a'c'a^2 + a'd'ac + b'c'ac + b'd'c^2 - a'c'a - b'd'c - ac \\
= ac(a'd' + b'c' - 1) + a'd'(a^2 - a) + b'd'(c^2 - c) \\
= 2acb'c + a(a - 1)a'c' + c(c - 1)b'd' \\
= 2(acb'c + A + B).
\]
Since \(\nu^2 = \epsilon, \nu^D = \epsilon^{acb'c + A + B}\). Hence \((\Psi N \Psi M)(T) = \Psi_{NM}(T)\). A similar calculation shows that \((\Psi N \Psi M)(S) = \Psi_{NM}(S)\). Hence \(\Psi N \Psi M = \Psi_{NM}\), so \(\Psi\) is a group homomorphism. \(\square\)

3.3.1. Remark. There is another way to obtain the homomorphism \(\Psi : \text{SL}(2, \mathbb{Z}) \to \text{Aut}(\tilde{H}_n)\).

By [Ser80, §1.5.3] or [CCS12, Prop. 2.1], for example, \(\text{SL}(2, \mathbb{Z})\) is the amalgamated free product \(\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6\) with \(\mathbb{Z}_4\) and \(\mathbb{Z}_6\) generated by
\[
(3-3) \quad X := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad Y := \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix},
\]
respectively. Independently of Lemma 3.3, it is easy to check that the formulas
\[
\Psi_X : \quad T \mapsto S^{-1}, \quad S \mapsto T, \quad \nu \mapsto \nu, \\
\Psi_Y : \quad T \mapsto TS^{-1}\nu^{-1}, \quad S \mapsto T, \quad \nu \mapsto \nu,
\]
(where \(\nu = \epsilon^{1/2}\)) extend to automorphisms of \(\tilde{H}_n\). Since
\[
\Psi_X^2 = \Psi_Y^3 : \quad T \mapsto T^{-1}, \quad S \mapsto S^{-1}, \quad \nu \mapsto \nu,
\]
\(\Psi_X^4 = \Psi_Y^6 = \text{id}\). Hence there is a unique group homomorphism \(\Psi : \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6 = \langle X \rangle *_{\mathbb{Z}_2} \langle Y \rangle \to \text{Aut} H_n\) with the property that \(\Psi(X) = \Psi_X\) and \(\Psi(Y) = \Psi_Y\).

The drawback to this alternative approach is that some additional calculation is needed to produce a formula for \(\Psi(M)\) for all \(M \in \text{SL}(2, \mathbb{Z})\).

3.3.2. Remark. In Proposition 3.4, the automorphism \(\Psi_M\) only depends on the image of \(M\) in \(\text{SL}(2, \mathbb{Z}_n)\) because \(T^n\) and \(S^n\) only depend on \(m\) modulo \(n\).

3.3.3. Notation. If \(M \in \text{SL}(2, \mathbb{Z})\) and \(x \in \tilde{H}_n\) we will use the notation
\[
M \triangleright x := \Psi_M(x).
\]

3.3.4. A nice feature of the action of \(\text{SL}(2, \mathbb{Z})\) on \(\tilde{H}_n\) in Proposition 3.4 is the following: if we define \(J_{(u, v)} := T^u S^v \in \tilde{H}_n\), then in \(\tilde{H}_n / (\epsilon^{1/2}) \cong \mathbb{Z}_2^n\) we have \(M \triangleright J_{(u, v)} = J_{(u, v)M^\ast}\).

3.4. Extending representations of \(H_n\) to representations of \(\tilde{H}_n\). It is well-known that the \(n\)-dimensional irreducible representations of \(H_n\) are determined up to isomorphism by their central characters; see, for example, [GH01, §2].

For a moment, let \(\rho : H_n \to \text{GL}(n, \mathbb{C})\) be any irreducible representation of \(H_n\) on \(\mathbb{C}^n\). Because \(\epsilon\) has order \(n\) it acts on \(\mathbb{C}^n\) as multiplication by an \(n\)th root of unity, \(\zeta\) say.

When \(n\) is even \(\rho\) can be extended to a representation \(\tilde{\rho}\) of \(\tilde{H}_n\) by having \(\epsilon^{1/2}\) act by a chosen square root of \(\zeta\). By [FH91, Prop. 5.1], for example, there are exactly two irreducible \(\tilde{H}_n\)-representations \(\tilde{\rho}\) such that \(\tilde{\rho}|_{H_n} \cong \rho\), namely \(\tilde{\rho}\) and \(\tilde{\rho} \otimes \epsilon\), where \(\epsilon\) is the non-trivial 1-dimensional character of \(\tilde{H}_n\) that sends \(H_n\) to 1. In particular, such a representation of \(\tilde{H}_n\) is also determined by its central character, i.e., the scalar by which the generator \(\epsilon^{1/2}\) of the center acts. The next statement follows.

Proposition 3.5. Let \(\rho : H_n \to \text{GL}(n, \mathbb{C})\) be any \(n\)-dimensional irreducible representation of \(H_n\) on which \(\epsilon\) acts as multiplication by \(\zeta\). Fix a square root, \(\zeta^{1/2}\), of \(\zeta\). Extend \(\rho\) to a group homomorphism \(\tilde{\rho} : \tilde{H}_n \to \text{GL}(n, \mathbb{C})\) by declaring that
\[
\tilde{\rho}(T^n S^b \epsilon^{b/2}) := \rho(T)^a \rho(S)^b (\zeta^{1/2})^c.
\]
If \( \phi \in \text{Aut}(\tilde{H}_n) \) acts as the identity on \( \epsilon^{1/2} \), then the representations \( \tilde{\rho}\phi \) and \( \tilde{\rho} \) are isomorphic.

**Corollary 3.6.** Let \( \rho : H_n \to \text{GL}(n, \mathbb{C}) \) be any \( n \)-dimensional irreducible representation of \( H_n \) and \( \tilde{\rho} : H_n \to \text{GL}(n, \mathbb{C}) \) an extension of it to \( \tilde{H}_n \). For \( M \in \text{SL}(2, \mathbb{Z}) \), let \( \Psi_M \in \text{Aut}(\tilde{H}_n) \) be the automorphism in Proposition 3.4. There is a set map—not a group homomorphism, in general—

\[
\psi : \text{SL}(2, \mathbb{Z}) \to \text{GL}(V)
\]

such that

\[
\tilde{\rho}(\Psi_M(x)) = \psi(M) \tilde{\rho}(x) \psi(M)^{-1}
\]

for all \( x \in \tilde{H}_n \).

**Proof.** This follows from the previous proposition because \( \Psi_M(\epsilon^{1/2}) = \epsilon^{1/2} \).

3.4.1. The notation \( \rho \). From now on we will use the notation \( \rho : \tilde{H}_n \to \text{GL}(V) \) for the representation of \( \tilde{H}_n \) in Proposition 3.2, and also for its linear extension \( \mathbb{C}\tilde{H}_n \to \text{End}_\mathbb{C}(V) \) to the group algebra. We will therefore write (3-5) as

\[
(3-6) \quad \rho(M \triangleright x) = \psi(M) \rho(x) \psi(M)^{-1}.
\]

3.4.2. Remark. In analogy with the remark in §3.3.2, the automorphism \( \psi(M) : V \to V \) depends only on the image of \( M \) in \( \text{SL}(2, \mathbb{Z}_n) \). For example, if the image of \( M \) in \( \text{SL}(2, \mathbb{Z}_n) \) is the identity, then \( \Psi_M = \text{id}_{\tilde{H}_n} \) so, in Corollary 3.6, we can take \( \psi(M) = \text{id}_V \).

3.5. The algebra \( A_n \) and the action of \( \text{SL}(2, \mathbb{Z}) \) on it. Let

\[
A_n := \mathbb{C}\tilde{H}_n \otimes_{\mathbb{C}(\epsilon^{1/2})} \mathbb{C}\tilde{H}_n,
\]

the tensor square of the group algebra \( \mathbb{C}\tilde{H}_n \) over the group algebra of its center, \( (\epsilon^{1/2}) \). The action of \( \text{SL}(2, \mathbb{Z}) \) on \( \tilde{H}_n \) extends to an action of \( \text{SL}(2, \mathbb{Z}) \) on \( \mathbb{C}\tilde{H}_n \) and, because elements of \( \text{SL}(2, \mathbb{Z}) \) act as the identity on the center of \( \tilde{H}_n \), there is an induced action of \( \text{SL}(2, \mathbb{Z}) \) on \( A_n \), namely \( M \triangleright (x \otimes y) = (M \triangleright x) \otimes (M \triangleright y) \).

4. The main results

4.1. Introduction. Let \( g, h \in \text{GL}(V) \) be the linear operators

\[
g \cdot x_i := \omega^i x_i \quad \quad h \cdot x_i := x_{i-1}
\]

where \( \omega := e^{\frac{1}{n}} \). The map \( S \leftrightarrow g, T \leftrightarrow h \), extends to an irreducible representation of \( H_n \) on \( V \) (essentially because if \( m \) is an integer, then \( g^m \) and \( h^m \) depend only on \( m \) modulo \( n \)) in which \( \epsilon \) acts as multiplication by \( \omega^{-1} \). For \( (a, b) \in \mathbb{Z}_n^2 \), we define the linear operator

\[
I_{a,b} := h^a g^b : V \to V,
\]

and its “universal” version

\[
J_{a,b} := J_{(a,b)} := T^a S^b \in \mathbb{C}\tilde{H}_n.
\]

4.1.1. We note that \( M \triangleright J_{(a,b)} = c J_{(a,b)M^t} \) where \( c \) is some element in the center of \( \tilde{H}_n \). Thus, in \( A_n \) we have the useful equality

\[
(4-1) \quad M \triangleright (J_{(a,b)} \otimes J_{(a,b)}^{-1}) = J_{(a,b)M^t} \otimes J_{(a,b)M^t}^{-1}.
\]

This is the reason we treat \( J_{(a,b)} \otimes J_{(a,b)}^{-1} \) as an element in \( A_n \) rather than \( \mathbb{C}\tilde{H}_n \otimes \mathbb{C}\tilde{H}_n \) when we define \( L_k(z, \eta \mid \tau) \) in (4-4) below.
4.1.2. Let \( T_k(z, \eta \,|\, \tau) : V^\otimes 2 \to V^\otimes 2 \) be the linear operator
\[
T_k(z, \eta \,|\, \tau) := \sum_{(u,v) \in \mathbb{Z}_+^2} w_{(u,v)}(-nz, \eta \,|\, \tau) I_{-k' u, v} \otimes I_{-k' u, v}^{-1}
\]
where \( k' := \) the unique integer such that \( n > k' \geq 1 \) and \( kk' = 1 \mod n \).

4.1.3. Let \( P : V \otimes V \to V \otimes V \) be the linear map \( P(x \otimes y) = y \otimes x \).

**Theorem 4.1.** [CKS20, Prop. 3.5] The images of \( R_{n,k}(z, \eta \,|\, \tau) \) and \( PT_k(z, \eta \,|\, \tau) \) are the same because
\[
R_{n,k}(z, \eta \,|\, \tau) = \frac{1}{n} e^{(-\frac{1}{2}n(n+1)z)} P \, T_k(z, \eta \,|\, \tau).
\]
Hence
\[
Q_{n,k}(\eta \,|\, \tau) = \frac{TV}{(\text{the image of } PT_k(\eta, \eta \,|\, \tau))}.
\]

The only difference between the operator \( R_{n,k}(z, \eta \,|\, \tau) \) defined in (1-6) and \( \frac{1}{n} e^{(-\frac{1}{2}n(n+1)z)} P \, T_k(z, \eta \,|\, \tau) \)
defined via (4-2) is that they are the matrix representations of the same linear operator with respect
to different bases for \( \text{End}_C(V) \). The advantage of \( T_k(z, \eta \,|\, \tau) \) over \( R_{n,k}(z, \eta \,|\, \tau) \) is that we can exploit
the way in which the actions of \( \tilde{H}_n \) and \( A_n \) on \( V^\otimes 2 \) interact with the actions of \( \text{SL}(2, \mathbb{Z}) \) on \( \tilde{H}_n \) and
the functions \( w_{(u,v)}(z, \eta \,|\, \tau) \).\(^6\) To determine how \( Q_{n,k}(\eta \,|\, \tau) \) transforms when \( \text{SL}(2, \mathbb{Z}) \) acts on \( (\eta \,|\, \tau) \) we will
determine how \( T_k(z, \eta \,|\, \tau) \) transforms under the action of \( \text{SL}(2, \mathbb{Z}) \) on \( (z, \eta \,|\, \tau) \in \mathbb{C} \times \mathbb{C} \times \mathbb{H} \).

4.2. \( \text{SL}(2, \mathbb{Z}) \)-equivariance properties of \( T_k(z, \eta \,|\, \tau) \) and \( L_k(z, \eta \,|\, \tau) \). Following [CKS20, Thm. 3.2]
and the discussion before it, we will determine how \( T_k(z, \eta \,|\, \tau) \) transforms under the action of \( \text{SL}(2, \mathbb{Z}) \)
by examining the action of \( \text{SL}(2, \mathbb{Z}) \) on the “universal” version of \( T_k(z, \eta \,|\, \tau) \) which belongs to the algebra \( A_n \). That universal version is
\[
L_k(z, \eta \,|\, \tau) := \sum_{(u,v) \in \mathbb{Z}_+^2} w_{(u,v)}(-nz, \eta \,|\, \tau) J_{-k' u, v} \otimes J_{-k' u, v}^{-1} \in A_n.
\]

The next “result” formalizes the statement that \( L_k(z, \eta \,|\, \tau) \) is the universal version of \( T_k(z, \eta \,|\, \tau) \).

**Lemma 4.2.** If \( \rho : \mathbb{C} \tilde{H}_n \to \text{End}(V) \) is the representation in **Proposition 3.2**, then
\[
T_k(z, \eta \,|\, \tau) = (\rho \otimes \rho)(L_k(z, \eta \,|\, \tau)).
\]

The main result in this section, **Theorem 4.3**, determines the “equivariance” properties of the map
\[
L_k : \mathbb{C} \times \mathbb{C} \times \mathbb{H} \to A_n, \quad (z, \eta \,|\, \tau) \mapsto L_k(z, \eta \,|\, \tau)
\]
with respect to the actions of \( \text{SL}(2, \mathbb{Z}) \) on its domain and co-domain. To state it we introduce the matrix
\[
D := \begin{pmatrix} -k & 0 \\ 0 & 1 \end{pmatrix} \in \text{GL}(2, \mathbb{R})
\]
and, for \( M \in \text{SL}(2, \mathbb{Z}) \), define
\[
M' := D^{-1} M' D.
\]

Note that \( D^{-t} = D^{-1} \) and \( M'' = (M')' = M \).

**Theorem 4.3.** Fix the data \((n, k, \eta, \tau)\) and \( M \in \text{SL}(2, \mathbb{Z}) \). There is a nowhere-vanishing holomorphic
function \( f(z) \) such that
\[
L_k(M \triangleright (z, \eta \,|\, \tau)) = f(z) M' \triangleright L_k(z, \eta \,|\, \tau),
\]
where \( M' \triangleright \) denotes the action of \( M' \in \text{SL}(2, \mathbb{Z}) \) on \( A_n \).

\(^6\)We learned this from Richey and Tracy’s paper [RT86].
Proof. Note that \((-k'u, v) = (u, v)D^{-1}\) for all \((u, v) \in \mathbb{Z}_n^2\).

Since \(M'' = M\), (4.7) is equivalent to the statement that there is a nowhere-vanishing holomorphic function \(f(z)\) such that
\[
L_k\left(M' \triangleright (z, \eta \mid \tau)\right) = f(z) \left(\rho \otimes \rho\right)\left(L_k(z, \eta \mid \tau)\right).
\]
That is what we will prove.

By the definition of \(L_k(z, \eta \mid \tau)\) in (4-4),
\[
M \triangleright L_k(z, \eta \mid \tau) = \sum_{u,v} w_{(u,v)}(-nz, \eta \mid \tau) J_{(u,v)D^{-1}M'} \otimes J_{(u,v)D^{-1}M'}^{-1}
\]
\[
= \sum_{u,v} w_{(u,v)M^{-1}D}(-nz, \eta \mid \tau) J_{u,v} \otimes J_{u,v}^{-1}
\]
\[
= \sum_{u,v} w_{(u,v)D(D^{-1}M^{-1}D)}(-nz, \eta \mid \tau) J_{u,v} \otimes J_{u,v}^{-1}.
\]

(The first equality in the previous computation uses the fact that \(L_k(z, \eta \mid \tau)\) is, by definition, an element of \(A_n\), not an element of \(\mathcal{C}\mathcal{H}_n \otimes \mathbb{C} \mathcal{H}_n\).) By Theorem 2.4, there is a nowhere-vanishing holomorphic function \(g(z)\) such that this equals
\[
g(z) \sum_{u,v} w_{(u,v)}D \left(D^{-1}M^{-1}D \triangleright (-nz, \eta \mid \tau)\right) J_{u,v} \otimes J_{u,v}^{-1},
\]
which is, as we see after re-indexing the summation over \((u, v)\), the same thing as
\[
g(z) \sum_{u,v} w_{(u,v)} \left(M' \triangleright (-nz, \eta \mid \tau)\right) J_{-k'u,v} \otimes J_{-k'u,v}^{-1} = g(z) L \left(M' \triangleright (z, \eta \mid \tau)\right).
\]
This finishes the proof.

\[\square\]

If \(M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})\), then the left-hand side of (4.7) is
\[
L_k\left(\frac{z}{ct+d}, \frac{\eta}{ct+d} \mid \frac{at+b}{ct+d}\right).
\]

4.3. \textbf{R-matrix and elliptic-algebra equivariance, and proof of the main theorem.} We now translate (4.7), which is a statement about how \(L_k(z, \eta \mid \tau)\) transforms under the action of \(\text{SL}(2, \mathbb{Z})\), into a statement about how \(R_{n,k}(z, \eta \mid \tau)\) transforms under the action of \(\text{SL}(2, \mathbb{Z})\). To do that we make use of the set map \(\psi : \text{SL}(2, \mathbb{Z}) \to GL(V)\) in Corollary 3.6.

\textbf{Theorem 4.4.} Fix the data \((n, k, \eta, \tau)\). Let \(M \in \text{SL}(2, \mathbb{Z})\) and let \(\psi(M')\) be the automorphism of \(V\) identified in Corollary 3.6. There is a nowhere-vanishing holomorphic function \(f(z)\) such that
\[
R_{n,k}(M \triangleright (z, \eta \mid \tau)) = f(z) \psi(M') \otimes \psi(M')^{-1} \cdot R_{n,k}(z, \eta \mid \tau) \cdot \left(\psi(M') \otimes \psi(M')^{-1}\right) - 1.
\]

\textbf{Proof.} Let \(\rho : \mathcal{C}\mathcal{H}_n \to \text{End}_{\mathbb{C}}(V)\) denote the representation in Proposition 3.2.

Since there is a nowhere-vanishing holomorphic function \(f(z)\) such that
\[
R_{n,k}(z, \eta \mid \tau) = f(z) PT_k(z, \eta \mid \tau) = f(z) P(\rho \otimes \rho)(L_k(z, \eta \mid \tau)),
\]
it suffices to show that
\[
(\rho \otimes \rho)(L_k(M \triangleright (z, \eta \mid \tau))) = g(z) \psi(M') \otimes \psi(M')^{-1} \cdot L_k(z, \eta \mid \tau) \cdot \left(\psi(M') \otimes \psi(M')^{-1}\right) - 1
\]
for some nowhere-vanishing holomorphic function \(g(z)\). However, it follows from Theorem 4.3 that there is a nowhere-vanishing holomorphic function \(g(z)\) such that
\[
(\rho \otimes \rho)(L_k(M \triangleright (z, \eta \mid \tau))) = g(z) (\rho \otimes \rho)(M' \triangleright L_k(z, \eta \mid \tau)),
\]
which is equal to
\[
g(z) \psi(M') \otimes \psi(M')^{-1} \cdot (\rho \otimes \rho)(L_k(z, \eta \mid \tau)) \cdot \left(\psi(M') \otimes \psi(M')^{-1}\right) - 1
\]
because, by (3-6), $\rho(M' x) = \psi(M')\rho(x)\psi(M')^{-1}$ for all $x \in \tilde{H}_n$, and hence for all $x \in \mathbb{C}\tilde{H}_n$. The proof is complete. \hfill \Box

**Theorem 4.5 (Theorem 1.1).** Let $M \in \text{SL}(2, \mathbb{Z})$. The linear isomorphism $\psi(M') : V \to V$ identified in Corollary 3.6 extends to a graded $\mathbb{C}$-algebra isomorphism

$$
\psi(M') : Q_{n,k}(\eta | \tau) \xrightarrow{\sim} Q_{n,k}(M \triangleright (\eta | \tau)).
$$

In particular, if $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$, then

$$
Q_{n,k} \left( \begin{array}{c} \eta \\ c\tau + d \end{array} \right) \begin{pmatrix} a\tau + b \\ c\tau + d \end{pmatrix} \cong Q_{n,k}(\eta | \tau).
$$

**Proof.** By definition, $Q_{n,k}(M \triangleright (\eta | \tau)) = TV/I$ where $I$ is the ideal generated by the image of

$$
R_{n,k}(M \triangleright (\eta, \eta | \tau)) := \lim_{z \to \eta} R_{n,k}(M \triangleright (z, \eta | \tau))
$$

By Theorem 4.4, this is the same as the image of $\psi(M') \otimes^2 \cdot R_{n,k}(\eta, \eta | \tau)$. The conclusion follows. \hfill \Box

4.3.1. $\psi(M')$ is $\tilde{H}_n$-equivariant if and only if $M$ is the identity. If $\psi : R \to R'$ is an isomorphism of rings there is an induced isomorphism of automorphism groups $\tilde{\psi} : \text{Aut}(R) \to \text{Aut}(R')$, $\alpha \mapsto \psi\alpha\psi^{-1}$.

Since $\tilde{H}_n$ acts as automorphisms of $Q_{n,k}(\eta | \tau)$ and $Q_{n,k}(\eta' | \tau')$ it is natural to ask whether $\tilde{\psi}(M')$ sends the copy of $\tilde{H}_n$ in $\text{Aut}(Q_{n,k}(\eta | \tau))$ to the copy of $\tilde{H}_n$ in $\text{Aut}(Q_{n,k}(\eta' | \tau'))$. It does: if $x \in \tilde{H}_n$, then $\tilde{\psi}(M') \rho(x)\tilde{\psi}(M')^{-1} = \rho(M' \triangleright x)$; i.e., $\tilde{\psi}(M')$ restricts to an isomorphism between the two copies of $\tilde{H}_n$ and that isomorphism is $x \mapsto M' \triangleright x$. In particular, if $M$ is not the identity, then $\tilde{\psi}(M')$ is not $\tilde{H}_n$-equivariant.

4.3.2. The next result involves an equality, not just an isomorphism.

**Corollary 4.6.** If $m \in \mathbb{Z}$, then $Q_{n,k} \left( \frac{\eta}{mn\tau + 1} \middle| \frac{\tau}{mn\tau + 1} \right) = Q_{n,k}(\eta | \tau)$.

**Proof.** As we remarked in §§3.3.2 and 3.4.2, the automorphisms $\Psi_M$ of $\tilde{H}_n$ and $\psi(M)$ of $V$ defined for $M \in \text{SL}(2, \mathbb{Z})$ only depend on the image of $M$ in $\text{SL}(2, \mathbb{Z})$; in particular, if the image of $M'$ in $\text{SL}(2, \mathbb{Z})$ is the identity, then we can take $\psi(M') = \text{id}_V$. If $M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then the image of $M'$ in $\text{SL}(2, \mathbb{Z})$ is the identity so we can take $\psi(M') = \text{id}_V$, whence Theorem 4.5 gives the result. \hfill \Box

4.4. The proof of Theorem 1.2. That result is a consequence of the following lemma.

**Lemma 4.7.** Let $\Lambda_1$ and $\Lambda_2$ be lattices in $\mathbb{C}$ and $f : \mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2$ a morphism of algebraic groups.

1. There is $u \in \mathbb{C}^\times$ such that $f(z + \Lambda_1) = uz + \Lambda_2$ for all $z \in \mathbb{C}$.

2. If $f$ is an isomorphism of algebraic groups, then the $u$ in (1) is unique.

3. Assume $\Lambda_1 = \Lambda_{\tau_1}$ and $\Lambda_2 = \Lambda_{\tau_2}$ for some $\tau_1, \tau_2 \in \mathbb{H}$. If $f$ is an isomorphism of algebraic groups, then there is a unique

$$
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})
$$

such that $f(z + \Lambda_1) = \frac{z}{c\tau_1 + d} + \Lambda_2$ for all $z \in \mathbb{C}$ and $\tau_2 = \frac{a\tau_1 + b}{c\tau_1 + d}$.

**Proof.** (1) This is [Hai11, Lem. 1.9].

(2) If $u_1z + \Lambda_2 = u_2z + \Lambda_2$ for all $z \in \mathbb{C}$, then $(u_1 - u_2)z \in \Lambda_2$ for all $z \in \mathbb{C}$, which implies that $u_1 - u_2 = 0$. Hence the $u$ in (1) is unique.

(3) (Existence.) Since $f(z + \Lambda_1) = uz + \Lambda_2$ for all $z \in \mathbb{C}$, $u\Lambda_1 = \Lambda_2$. Hence $\tau_2 = u(a\tau_1 + b)$ and $1 = u(c\tau_1 + d)$ for some $a, b, c, d \in \mathbb{Z}$. Since $u\Lambda_1 = \Lambda_2$, $\{a\tau_1 + b, c\tau_1 + d\}$ is a $\mathbb{Z}$-basis for $\Lambda_{\tau_1}$. Hence $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathbb{Z})$. Since $\tau_2 = \frac{a\tau_1 + b}{c\tau_1 + d}$, the imaginary part of $\tau_2$ is

$$
\Im(\tau_2) = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} |c\tau_1 + d|^{-2}\Im(\tau_1).
$$
Since the imaginary parts of $\tau_1$ and $\tau_2$ are positive, $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} > 0$. Hence $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in $\text{SL}(2, \mathbb{Z})$.

(Uniqueness.) Since $\tau_1 \notin \mathbb{Q}$, the integers $c$ and $d$ in the equality $1 = u(c\tau_1 + d)$ are unique. Likewise, since $\tau_1 \notin \mathbb{Q}$, the integers $a$ and $b$ in the equality $a\tau_1 + b = u^{-1}\tau_2$ are unique. $\square$

**Corollary 4.8 (Theorem 1.2).** If $f : E_\tau \to E_{\tau'}$ is an isomorphism of algebraic groups such that $f(\eta + \Lambda_\tau) = \eta' + \Lambda_{\tau'}$, then

$$Q_{n,k}(\eta \mid \tau) \cong Q_{n,k}(\eta' \mid \tau').$$

**Proof.** By Lemma 4.7(3), there is $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$ such that $f(z + \Lambda_\tau) = \frac{az + b}{cz + d} + \Lambda_{\tau'}$ for all $z \in \mathbb{C}$ and $\tau' = \frac{az + b}{cz + d}$. In particular, $\eta' + \Lambda_{\tau'} = \frac{\eta}{cz + d} + \Lambda_{\tau'}$. By Theorem 4.5,

$$Q_{n,k}(\eta \mid \tau) \cong Q_{n,k}(\frac{\eta}{cz + d} \mid \frac{az + b}{cz + d}) = Q_{n,k}(\eta' \mid \tau')$$

where the last equality holds since $\frac{\eta}{cz + d}$ and $\eta'$ have the same image in $E_{\tau'}$. $\square$

### 4.5. Modularity

The action of the modular group $\text{SL}(2, \mathbb{Z})$ on $\mathbb{C} \times \mathbb{H}$ is the starting point for the theory of modular forms and modular functions. For example, (see, e.g., [DS05, Defn. 1.1.1] and [Sil94, §I.3]) a meromorphic function $f : \mathbb{H} \to \mathbb{C}$ is called a **weakly modular function of weight $k$** if

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)$$

for all $\tau \in \mathbb{H}$ and all

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}).$$

We will now restate (4-10) in a way that resembles the isomorphism in (4-9).

First, we associate to a function $f : \mathbb{H} \to \mathbb{C}$ its (backward) graph

$$\Gamma_f := \{(f(\tau) \mid \tau) \mid \tau \in \mathbb{H}\} \subseteq \mathbb{C} \times \mathbb{H}.$$ 

Subsets of a given set can be viewed as $\{0,1\}$-valued functions on that set. Here such a procedure produces the function

$$F : \mathbb{C} \times \mathbb{H} \to \{0,1\}, \quad F(z \mid \tau) := \begin{cases} 1 & \text{if } f(\tau) = z, \\
0 & \text{if } f(\tau) \neq z. \end{cases}$$

With this notation, $f$ satisfies (4-10) if and only if

$$F\left((c\tau + d)^k z \mid \frac{a\tau + b}{c\tau + d}\right) = F(z \mid \tau).$$

In particular, $f(\tau)$ is a weakly modular function of weight $-1$ if and only if the $\{0,1\}$-valued function $F(z \mid \tau)$ satisfies the identity

$$F\left(\frac{z}{c\tau + d} \mid \frac{a\tau + b}{c\tau + d}\right) = F(z \mid \tau),$$

which resembles the isomorphism in (4-9).

If we express this identity by saying that $F$ is a “weakly modular function of weight $-1$ taking values in $\{0,1\}$”, and extend this terminology in the obvious way, then Theorem 4.5 says that $Q_{n,k}$ is a “weakly modular function of weight $-1$ taking values in the category of graded $\mathbb{C}$-algebras”.

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