On a nonlinear sequential four-point fractional q-difference equation involving q-integral operators in boundary conditions along with stability criteria

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Abstract

In this paper, we consider a nonlinear sequential q-difference equation based on the Caputo fractional quantum derivatives with nonlocal boundary value conditions containing Riemann–Liouville fractional quantum integrals in four points. In this direction, we derive some criteria and conditions of the existence and uniqueness of solutions to a given Caputo fractional q-difference boundary value problem. Some pure techniques based on condensing operators and Sadovskii’s measure and the eigenvalue of an operator are employed to prove the main results. Also, the Ulam–Hyers stability and generalized Ulam–Hyers stability are investigated. We examine our results by providing two illustrative examples.

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1 Introduction

In several areas of sciences, such as biology, chemistry, economics, physics, and engineering, fractional calculus and its relevant differential equations and BVPs have been used extensively [1–3]. Indeed, fractional derivatives are not only a generalization of ordinary derivatives, but also they explain dynamical behavior of various physical processes specifically and effectively (real life phenomena) in contrast to integer order derivatives. References [4–18] are available for some improvements on the fractional differential equations theory.

By virtue of developments in fractional quantum calculus (q-FC), a number of scientists and researchers [19, 20] were attracted to a study of fractional q-difference equations, beginning in the nineteenth century, and wide interest lately [21–23].

In 2007, Atici et al. [24] studied some notions in relation to fractional q-calculus on time scales. Then in 2012, Annaby and Mansour presented their investigations by pub-
lishing a book on equations and BVPs in the context of fractional $q$-calculus [25]. Jarad et al. [26] turned to the stability notion on $q$-fractional non-autonomous systems and after that, Abdeljawad et al. [27] introduced Gronwall-type inequality in $q$-operator settings. By combining the two above notions, Butt et al. [28] investigated Ulam stability for a Caputo delay $q$-difference equation by means of $q$-Gronwall-type inequality. Also, some fascinating insights concerning IVPs and BVPs containing $q$-difference equations can be found in [29–35] and the references therein. Ahmad, Nieto, Alsaedi and Al-Hutami [36] turned to the stability notion on sequential four-point Caputo fractional $q$-difference equation by means of $q$-fractional derivatives in Caputo sense of orders $\theta$. In 2014, Ahmad et al. [37] studied the existence criteria of the following $q$-difference equation involving two nonlinear terms and four-point nonlocal boundary conditions:

\[
\begin{align*}
\mathcal{D}_q^\rho (\mathcal{D}_q^\beta + b)u(t) &= f(t, u(t)) \\
u(0) &= d_1 I_q^{\gamma - 1} u(\theta) = d_1 \int_0^\theta \frac{(\theta - s)^{(\gamma - 1)}}{\Gamma(\gamma)} u(s) \, ds, \\
u(1) &= d_2 I_q^{\gamma - 1} u(\pi) = d_2 \int_0^\pi \frac{(\pi - s)^{(\gamma - 1)}}{\Gamma(\gamma)} u(s) \, ds, \quad \gamma > 2, \theta > 0, \pi < 1,
\end{align*}
\]

where $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$, $\rho, \beta \in (0,1]$, $b, d_1, d_2 \in \mathbb{R}$ and $\mathcal{D}_q^\rho, \mathcal{D}_q^\beta$ denote the $q$-fractional derivatives in Caputo sense of orders $\rho$ and $\beta$.

In continuation to the investigation of the $q$-variant of fractional problems and inspired by the aforementioned work, we aim to examine this area from another angle. Several known methods of functional analysis are used to establish required results on the existence of solutions for a class of $q$-difference problems. More specifically, we consider the sequential four-point Caputo fractional $q$-difference boundary value problem ($q$-CFBVP) of the format

\[
\begin{align*}
\mathcal{D}_q^\mu (\mathcal{D}_q^\alpha + b)u(t) - g(t, u(t)) &= f(t, u(t)), \quad t \in J := [0, T], \\
a_1 u(0) + b_1 \mathcal{D}_q^\gamma u(0) &= \lambda_1 \int_0^{\eta_1} \frac{(\eta_1 - s)^{(\eta_1 - 1)}}{\Gamma(\eta_1)} u(s) \, ds, \quad \eta_1 \in (0, T), \sigma_1 > 0, \\
a_2 u(T) + b_2 \mathcal{D}_q^\gamma u(T) &= \lambda_2 \int_0^{\eta_2} \frac{(\eta_2 - s)^{(\eta_2 - 1)}}{\Gamma(\eta_2)} u(s) \, ds, \quad \eta_2 \in (0, T), \sigma_2 > 0,
\end{align*}
\]

where $\mathcal{D}_q^\mu$ is the $\mu$th $q$-difference derivative in the Caputo structure with $\mu \in \{\gamma, \beta, \alpha\}$ such that $0 < \alpha, \beta, \gamma \leq 1$, $\mathcal{D}_q^\gamma$ is the $\theta$th $q$-difference integral in the Riemann–Liouville structure with $\theta = 0$ subject to $\theta \in (\sigma_1, \sigma_2)$ and also $f, g : J \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions. $a_1, a_2, b_1, b_2, \lambda_1, \lambda_2$ are suitably chosen constants in $\mathbb{R}^*$. Regarding to the novelty of the paper, in comparison to above $q$-problems, our supposed sequential $q$-CFBVP is more general. Under the given boundary value conditions, we have used both Caputo and Riemann–Liouville $q$-fractional operators in four different
points of domain of the unknown solution function $u$ simultaneously, in which the linear combinations of the unknown function and its fractional derivative is corresponding to a multiple of $q$-Riemann–Liouville integral in two mid-points. In this paper, we have designed an extended form of Langevin equations by providing a nonlinear function $g$ in the left-side hand of the given boundary value problem (3). Also, to prove the existence of solutions for such an applied $q$-problem, we shall utilize some pure notions of functional analysis based on the measure of non-compactness, condensing operators and eigenvalue of the operator, which have been used in papers limited in this regard so far and this distinguishes our research from the work of others. Moreover, we here emphasize that this paper may have useful and effective applications in physics and quantum mechanics such as Langevin systems in the context of quantum operators.

The remaining part of this paper is organized as follows: Sect. 2 is devoted to the primitive notions of $q$-FC. At first, in Sect. 3, we give an auxiliary lemma which provides the solution of the supposed $q$-CFBVP (3) and then based on the obtained integral equation, by using fixed point theorems due to Sadovski, Krasnoselskii–Zabreiko and O’Regan, we establish the existence of solutions for the $q$-CFBVP (3) and also for its uniqueness, we utilize the famous Banach principle. In Sect. 4, the stability criteria of Ulam–Hyers type and its generalized type are checked. Additionally, in Sect. 5, we provide two examples which ensure the usability of the results presented in Sect. 3. The manuscript is ended by our conclusions in Sect. 6.

2 Preliminaries regarding $q$-operators

We collect some important basic notions of $q$-FC in this section. For details, we refer to [19, 21, 38, 39]. Let $q \in (0, 1)$. A $q$-real number is denoted by $[m]_q$ and is defined as

$$[m]_q = \frac{1 - q^m}{1 - q}, \quad m \in \mathbb{R}.$$ 

The $q$-power function $(m - n)^k$ with $m, n \in \mathbb{R}$ is

$$(m - n)^{(0)} = 1, \quad (m - n)^{(k)} = \prod_{j=0}^{k-1} (m - n q^j), \quad k \in \mathbb{N} \cup \{0\},$$

and, if $\beta \in \mathbb{R}$, then

$$(m - n)^{[\beta]} = m^{\beta} \prod_{i=0}^{\infty} \frac{m - n q^i}{m - n q^{\beta i}}.$$ 

On the other side, $[c(m - n)]^{[\beta]} = c^{[\beta]} (m - n)^{[\beta]}$ holds for $c \in \mathbb{R}$ and also notice that $m^{[0]} = m^0$ if $n = 0$. The $q$-Gamma function is given by

$$\Gamma_q(\alpha) = \frac{(1 - q)^{[\alpha - 1]}}{(1 - q)^{\alpha - 1}}, \quad \alpha \in \mathbb{R} \setminus \{0, -1, \ldots\},$$

and satisfies $\Gamma_q(\alpha + 1) = [\alpha]_q \Gamma_q(\alpha)$.

The 1st-$q$-derivative of an arbitrary mapping $\phi$ is defined by the following rule:

$$(D_q \phi)(x) = \frac{\phi(q x) - \phi(x)}{(1 - q)x}, \quad x \neq 0,$$
and for the higher orders, it becomes

\[ D_0^k \phi = \phi, \quad D_0^k \phi = D_q (D_0^{k-1} \phi), \quad k \in \mathbb{N} \cup \{0\}. \]

The 1st-\(q\)-integral of an arbitrary mapping \(\phi\) given on the interval \([0, n]\) is given by

\[ (I_q \phi)(x) = \int_0^x \phi(r) \, dq r = x(1 - q) \sum_{k=0}^{\infty} \phi(xq^k) q^k, \quad x \in [0, n]. \]

If \(m \in [0, n]\), then

\[ \int_m^n \phi(r) \, dq r = \int_0^n \phi(r) \, dq r - \int_0^m \phi(r) \, dq r. \]

Similarly, for the higher orders, it becomes

\[ I_0^k \phi = \phi, \quad I_0^k \phi = I_q (I_0^{k-1} \phi), \quad k \in \mathbb{N} \cup \{0\}. \]

For two first order \(q\)-operators \(D_q\) and \(I_q\), we have

\[ D_q I_q \phi(x) = \phi(x). \]

Here, we assemble some definitions about such \(q\)-operators from the fractional point of view.

**Definition 1** ([39]) Let \(\alpha \geq 0\). The \(\alpha\)th-\(q\)-integral of the Riemann–Liouville type for \(\phi\) defined on \([0, \infty)\) is given by \(I_0^\alpha \phi(t) = \phi(t)\) and

\[ I_q^\alpha \phi(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qr)^{(\alpha-1)} \phi(r) \, dq r, \quad \alpha > 0. \]

**Definition 2** ([19]) The Caputo \(\alpha\)th-\(q\)-derivative for an absolutely continuous mapping \(\phi\) is formulated by

\[ D_q^\alpha \phi(t) = I_0^{[\alpha]-\alpha} D_q^{[\alpha]} \phi(t), \]

where \([\alpha]\) denotes the integer part of \(\alpha\).

For more information on the fractional \(q\)-operators, we refer the reader to [38].

**Lemma 3** ([19]) Let \(\alpha, \beta \in \mathbb{R}_+. Then we have the following formulas:

1. \(I_q^\alpha I_q^\beta \phi(t) = I_q^{\alpha+\beta} \phi(t)\);
2. \(D_q^\alpha I_q^\beta \phi(t) = \phi(t)\).

**Lemma 4** ([40]) Let \(\alpha \in \mathbb{R}_+\) and \(\beta \in (-1, \infty)\). One has

\[ I_q^\alpha t^\beta = \frac{\Gamma_q(\beta + 1)}{\Gamma_q(\alpha + \beta + 1)} t^{\alpha+\beta}, \quad t > 0. \]
In particular, if $\phi \equiv 1$, then

$$I_q^\alpha 1(t) = \frac{1}{\Gamma_q(1 + \alpha)} t^{(\alpha)} \quad \text{for all } t > 0.$$ 

**Lemma 5** ([19]) Let $\alpha, \sigma > 0$. Then

$$I_q^\alpha D_q^\sigma \phi(t) = D_q^\sigma I_q^\alpha \phi(t) - \sum_{j=0}^{\sigma-1} \frac{t^{\sigma-\sigma+j}}{\Gamma_q(\sigma + j - \sigma + 1)} D_q^j \phi(0).$$

**Lemma 6** ([30, 40]) Let $k - 1 < \alpha < k$. Then

$$I_q^\alpha D_q^\alpha \phi(t) = \phi(t) - \sum_{j=0}^{k-1} \frac{t^j}{\Gamma_q(j + 1)} D_q^j \phi(0).$$

For the homogeneous $q$-difference equation $D_q^\alpha \phi(t) = 0$, the general series solution by Lemma 6 is given as $\phi(t) = \mu_0 + \mu_1 t + \mu_2 t^2 + \cdots + \mu_{k-1} t^{k-1}$ via $\mu_j \in \mathbb{R}$ and $k = [\alpha] + 1$ [19].

So, we have

$$(I_q^\alpha D_q^\alpha \phi)(t) = \phi(t) + \mu_0 + \mu_1 t + \mu_2 t^2 + \cdots + \mu_{k-1} t^{k-1}.$$ 

3 Results regarding existence property

In the present section, before moving to our fundamental results, we define $\| \cdot \|$ on $X = C(\mathbb{J}, \mathbb{R})$ as $\| u \| = \sup_{t \in \mathbb{J}} |u(t)|$, which in this phase, $X$ transforms into a Banach space. Now, in the first place, we provide the next auxiliary lemma.

**Lemma 7** Let $\psi \in C(\mathbb{J}, \mathbb{R})$, $\alpha, \beta, \gamma \in (0, 1)$, $\sigma_1, \sigma_2 > 0$, $a_1, a_2, b_1, b_2, \lambda_1, \lambda_2 \in \mathbb{R}^+$ and $g_u(t) = g(t, u(t))$. Then the solution of the linear sequential four-point $q$-CFBVP defined by

$$\begin{cases} 
D_q^\alpha (D_q^\beta u(t) - g_u(t)) = \psi(t), & t \in \mathbb{J} := [0, T], \\
a_1 u(0) + b_1 D_q^\gamma u(0) = \lambda_1 I_q^\sigma_1 u(\eta_1), & 0 < \eta_1 < T, \sigma_1 > 0, \\
a_2 u(T) + b_2 D_q^\gamma u(T) = \lambda_2 I_q^\sigma_2 u(\eta_2), & 0 < \eta_2 < T, \sigma_2 > 0, 
\end{cases}$$

is given by

$$u(t) = I_q^\alpha g_u(t) + I_q^{\alpha+\beta} \psi(t) + \mu_1 (I_q^{\alpha+\sigma_1} g_u(\eta_1) + I_q^{\alpha+\beta+\sigma_1} \psi(\eta_1)) + \mu_2 (I_q^{\alpha+\sigma_2} g_u(\eta_2) + I_q^{\alpha+\beta+\sigma_2} \psi(\eta_2)) - b_1 (I_q^{\alpha+\gamma} g_u(T) + I_q^{\alpha+\beta+\gamma} \psi(T)) - a_2 (I_q^{\alpha} g_u(T) + I_q^{\alpha+\beta} \psi(T)).$$
where

\[
\mu_1(t) = \lambda_1 \left( \frac{(\Lambda - \Lambda_2 \Lambda_3) \Gamma_q(\beta + 1) - t^\beta \Lambda_1 \Lambda_3}{\Lambda \Lambda_1 \Gamma_q(\beta + 1)} \right),
\]

\[
\mu_2(t) = \frac{\Lambda_1 t^\beta + \Lambda_2 \Gamma_q(\beta + 1)}{\Lambda \Gamma_q(\beta + 1)},
\]

\[
\Lambda_1 = a_1 - \frac{\lambda_1 \eta_1^\sigma_1}{\Gamma_q(\sigma_1 + 1)}, \quad \Lambda_2 = \frac{\lambda_1 \eta_1^\beta + \sigma_1}{\Gamma_q(\beta + \sigma_1 + 1)},
\]

\[
\Lambda_3 = a_2 - \frac{\lambda_2 \eta_2^\sigma_2}{\Gamma_q(\sigma_2 + 1)},
\]

\[
\Lambda_4 = \frac{a_2 T^\beta}{\Gamma_q(\beta + 1)} + \frac{b_2 T^\beta - \gamma}{\Gamma_q(\beta - \gamma + 1)} - \frac{\lambda_2 \eta_2^\beta + \sigma_2}{\Gamma_q(\beta + \sigma_2 + 1)},
\]

and \( \Lambda \) is given by

\[
\Lambda = \left( \frac{a_2 T^\beta}{\Gamma_q(\beta + 1)} + \frac{b_2 T^\beta - \gamma}{\Gamma_q(\beta - \gamma + 1)} - \frac{\lambda_2 \eta_2^\beta + \sigma_2}{\Gamma_q(\beta + \sigma_2 + 1)} \right) \left( a_1 - \frac{\lambda_1 \eta_1^\sigma_1}{\Gamma_q(\sigma_1 + 1)} \right)
\]

\[
+ \frac{\lambda_1 \eta_1^\beta + \sigma_1}{\Gamma_q(\beta + \sigma_1 + 1)} \left( a_2 - \frac{\lambda_2 \eta_2^\sigma_2}{\Gamma_q(\sigma_2 + 1)} \right) = \Lambda_4 \Lambda_1 + \Lambda_2 \Lambda_3 \neq 0.
\]

**Proof** By using Lemma 6, we obtain the integral equation corresponding to (4):

\[
u(t) = I_q^\alpha g_u(t) + f_q^{\alpha + \beta} \psi(t) + \frac{t^\beta}{\Gamma_q(\beta + 1)} k_0 + k_1, \quad k_0, k_1 \in \mathbb{R}.
\]

Using the given boundary conditions in (4), we may obtain

\[
\mathcal{I}_q^\alpha u(t) = I_q^{\alpha + \beta} g_u(t) + f_q^{\alpha + \beta + \alpha} \psi(t) + \frac{t^\beta}{\Gamma_q(\beta + \alpha + 1)} + \frac{t^\sigma_1}{\Gamma_q(\beta + \sigma_1 + 1)}
\]

for \( i = 1, 2 \) and

\[
\mathcal{D}_q^\gamma x(t) = I_q^{\beta - \gamma} g_u(t) + f_q^{\alpha + \beta - \gamma} \psi(t) + \frac{t^\beta}{\Gamma_q(\beta - \gamma + 1)}.
\]

By categorizing similar terms, we obtain the expressions

\[
\Lambda_1 k_1 - \Lambda_2 k_0 = \lambda_1 I_q^{\sigma_1 + \beta} g_u(\eta_1) + \lambda_1 I_q^{\beta + \sigma_1} \psi(\eta_1)
\]

and

\[
\Lambda_3 k_1 + \Lambda_4 k_0 = \lambda_2 f_q^{\beta + \sigma_2} g_u(\eta_2) + \lambda_2 f_q^{\beta + \sigma_2 + \alpha} \psi(\eta_2)
\]

\[
- a_2 \left( t^\beta g_u(T) + I_q^\beta \psi(T) \right) - b_2 \left( I_q^{\beta - \gamma} g_u(T) + I_q^{\beta + \alpha - \gamma} \psi(T) \right).
\]

Therefore, from (9) and (10), we get

\[
k_0 = - \frac{\Lambda_3}{\Lambda} \left( \lambda_1 I_q^{\beta + \sigma_1} g_u(\eta_1) + \lambda_1 I_q^{\beta + \sigma_1 + \alpha} \psi(\eta_1) \right)
\]
\[ + \frac{\Lambda_1}{\Lambda} \left[ \lambda_2 (I_{q}^{\rho} g_u(\eta_2) + I_{q}^{\rho+\alpha} \psi(\eta_2)) \ight. \\
\left. - a_2 (I_{q}^{\rho} g_u(T) + I_{q}^{\rho+\alpha} \psi(T)) - b_2 (I_{q}^{\rho-\gamma} g_u(T) + I_{q}^{\rho+\alpha-\gamma} \psi(T)) \right], \]

and, by inserting \( k_0 \) into (9), we obtain

\[
k_1 = \frac{\Lambda - \Lambda_2 \Lambda_3}{\Lambda_1 \Lambda} \left( \lambda_1 I_{q}^{\rho+\beta} g_u(\eta_1) + \lambda_1 I_{q}^{\rho+\beta+\alpha} \psi(\eta_1) \right) \\
+ \frac{\Lambda_2}{\Lambda} \left[ \lambda_2 (I_{q}^{\rho+\beta} g_u(\eta_2) + I_{q}^{\rho+\beta+\alpha} \psi(\eta_2)) - a_2 (I_{q}^{\rho} g_u(T) + I_{q}^{\rho+\alpha} \psi(T)) \ight. \\
- b_2 (I_{q}^{\rho+\gamma} g_u(T) + I_{q}^{\rho+\alpha+\gamma} \psi(T)). \]

Substituting the value of \( k_0 \), \( k_1 \) in (8), we get (5), which completes the proof. \( \square \)

Note that, for simplicity, we set \( g(t, u(t)) = g_u(t) \) and \( f(t, u(t)) = f_u(t) \) throughout the manuscript.

### 3.1 The first existence criterion

In this subsection, we prove an existence result for the sequential four-point \( q \)-CFBVP (3) by making use of Sadovskiĭ’s fixed-point theorem. Before moving towards it, we would like to recall several auxiliary facts which are our main tools. \( X \) is supposed as a Banach space.

**Definition 8** Consider a bounded subset \( M \) of \((X, d)\). The Kuratowski measure of non-compactness, denoted by \( \alpha(M) \), is defined by

\[
\alpha(M) : = \inf \left\{ \epsilon > 0 : \exists \text{ finitely many sets } M_i \text{ s.t. } M = \bigcup_{i=1}^{n} M_i \text{ and } D(M_i) \leq \epsilon \right\},
\]

where \( D(M_i) = \sup \{ |u - \bar{u}| : u, \bar{u} \in M_i \} \).

**Definition 9 ([41])** Consider a bounded and continuous function \( \Phi : \text{Dom}(\Phi) \subseteq X \rightarrow X \) on \( X \). For an arbitrary bounded set \( M \subseteq \text{Dom}(\Phi) \), the map \( \Phi \) is condensing if

\[
\alpha(\Phi(M)) < \alpha(M),
\]

in which \( \alpha \) is introduced above.

**Lemma 10 ([42])** Let \( \mathcal{K}_1, \mathcal{K}_2 : E \subseteq X \rightarrow X \). The operator \( \mathcal{K}_1 + \mathcal{K}_2 \) is condensing if

i. \( \mathcal{K}_1 \) is \( k \)-contraction; that is, \( \forall u, v \in E \) and \( \exists k \in (0, 1) \), so that

\[
\| \mathcal{K}_1 u - \mathcal{K}_1 v \| \leq k \| u - v \|;
\]

ii. \( \mathcal{K}_2 \) is compact.

**Theorem 11 ([43])** Consider the bounded, closed and convex subset \( B \) of \( X \) and the condensing mapping \( \Phi : B \rightarrow B \). Then \( \Phi \) has a fixed point.
From now on, we put
\[
\Theta_1 = \frac{T^{\alpha + \beta}}{\Gamma_q(\alpha + \beta + 1)} + \left| \mu_1(T) \right| \frac{\eta_1^{\alpha + \beta + \sigma_1}}{\Gamma_q(\alpha + \beta + \sigma_1 + 1)} \\
+ \left| \mu_2(T) \right| \left( \lambda_2 \frac{\eta_2^{\alpha + \beta + \sigma_2}}{\Gamma_q(\alpha + \beta + \sigma_2 + 1)} + b_2 \frac{T^{\alpha + \beta - \gamma}}{\Gamma_q(\alpha + \beta - \gamma + 1)} \\
+ a_2 \frac{T^{\alpha + \beta}}{\Gamma_q(\alpha + \beta + 1)} \right)
\]
(11)
and
\[
\Theta_2 = \frac{T^\beta}{\Gamma_q(\beta + 1)} + \left| \mu_1(T) \right| \frac{\eta_1^{\beta + \sigma_1}}{\Gamma_q(\beta + \sigma_1 + 1)} \\
+ \left| \mu_2(T) \right| \left( \lambda_2 \frac{\eta_2^{\beta + \sigma_2}}{\Gamma_q(\beta + \sigma_2 + 1)} + b_2 \frac{T^{\beta - \gamma}}{\Gamma_q(\beta - \gamma + 1)} + a_2 \frac{T^\beta}{\Gamma_q(\beta + 1)} \right).
\]
(12)

**Theorem 12** Consider the following assertions:

(B1) \( \exists L > 0 \) so that \( |f_u(t) - f_v(t)| \leq L|u(t) - v(t)|, \forall t \in I, u, v \in \mathbb{R} \);

(B2) \( |f_u(t)| \leq \sigma(t) \) and \( |g_u(t)| \leq \rho(t) \), where \( \sigma, \rho \in C(I, \mathbb{R}^+) \).

Then the sequential four-point \( q \)-CFBVP (3) has a solution on \( J \) if \( Q := L\Theta_1 < 1 \), by introducing \( \Theta_1 \) as (11).

**Proof** Consider a bounded, closed and convex subset \( B_r = \{ u \in X : \|u\| \leq r \} \) of \( X = C(I, \mathbb{R}) \) for a fixed constant \( r \). With regard to Lemma 7, define \( K : X \to X \) as follows:

\[
Ku(t) = I^\alpha_q g_u(t) + I^\alpha_q f_u(t) \\
+ \mu_1(t) \left[ I^\alpha_q g_u(\eta_1) + I^\alpha_q f_u(\eta_1) \right] \\
+ \mu_2(t) \left[ \lambda_2 (I^\alpha_q g_u(\eta_2) + I^\alpha_q f_u(\eta_2)) - b_2 (I^\alpha_{q^\gamma} g_u(T) + I^\alpha_q f_u(T)) - a_2 (I^\alpha_q g_u(T) + I^\alpha_q f_u(T)) \right].
\]
(13)

We split the operator \( K \) on the set \( B_r \) into \( K = K_1 + K_2 \), where

\[
K_1 u(t) = I^\alpha_q f_u(t) + \mu_1(t) \left( I^\alpha_q f_u(\eta_1) \right) \\
+ \mu_2(t) \left[ \lambda_2 (I^\alpha_q f_u(\eta_2)) - b_2 (I^\alpha_{q^\gamma} f_u(T)) - a_2 (I^\alpha_q f_u(T)) \right]
\]
(14)
and

\[
K_2 u(t) = I^\beta_q g_u(t) + \mu_1(t) \left( I^\beta_q g_u(\eta_1) \right) \\
+ \mu_2(t) \left[ \lambda_2 (I^\beta_q g_u(\eta_2)) - b_2 (I^\beta_{q^\gamma} g_u(T)) - a_2 (I^\beta_q g_u(T)) \right].
\]
(15)

We want to prove that the operators \( K_1 \) and \( K_2 \) follow all the assertions of Theorem 11. We proceed to implement the proof in four steps.
Step 1: \( \mathcal{KB}_r \subset B_r \)
Let us select \( r \) so that \( r \geq \|\sigma\|\Theta_1 + \|\rho\|\Theta_2 \), where \( \Theta_2, \Theta_1 \) are given by (11) and (12) and \( \|\sigma\| = \sup_{t \in J} |\sigma(t)| \) and \( \|\rho\| = \sup_{t \in J} |\rho(t)| \). For any \( u \in B_r \), we have

\[
| (\mathcal{K}u)(t) | \leq \sup_{t \in J} \left| I_q^\beta g_u(t) + I_q^{\alpha + \beta} f_u(t) \right|
+ \left| \mu_1(t) \left[ I_q^{\alpha + \sigma_1} g_u(\eta_1) + I_q^{\alpha + \sigma_1} f_u(\eta_1) \right] \right|
+ \left| \mu_2(t) \left[ I_q^{\sigma_2} g_u(\eta_2) \right] \right|
+ b_2 \left| I_q^{-\gamma} g_u(T) \right|
+ a_2 \left| I_q^{\beta - \gamma} f_u(T) \right|
\]

which implies that \( \mathcal{KB}_r \subset B_r \).

Step 2: \( \mathcal{K}_2 \) is compact
In view of Step 1, we observe that the operator \( \mathcal{K}_2 \) is uniformly bounded; indeed for any \( u \in B_r \):

\[
| (\mathcal{K}_2u)(t) | \leq I_q^\beta g_u(t) + | \mu_1(t) \left[ I_q^{\alpha + \sigma_1} g_u(\eta_1) \right] |
+ | \mu_2(t) \left[ I_q^{\sigma_2} g_u(\eta_2) \right] |
+ b_2 | I_q^{-\gamma} g_u(T) |
+ a_2 | I_q^{\beta - \gamma} f_u(T) |
\]

\[
\leq \|\rho\| \left[ \frac{T^\beta}{\Gamma_q(\beta + 1)} + \frac{\eta_1^{\alpha + \sigma_1}}{\Gamma_q(\beta + \sigma_1 + 1)} \right]
+ \frac{T^{-\gamma}}{\Gamma_q(\beta - \gamma + 1)} + a_2 \frac{T^\beta}{\Gamma_q(\beta + 1)} \]

Now, take \( t_1, t_2 \in J \) by assuming \( t_1 < t_2 \) and \( u \in B_r \). Hence we have

\[
| \mathcal{K}_2u(t_2) - \mathcal{K}_2u(t_1) |
\leq I_q^\beta | g_u(t_2) - g_u(t_1) | + | \mu_1(t_2) - \mu_1(t_1) | ( I_q^{\alpha + \sigma_1} | g_u(\eta_1) | )
+ | \mu_2(t_2) - \mu_2(t_1) | ( I_q^{\sigma_2} | g_u(\eta_2) | )
+ b_2 | I_q^{-\gamma} g_u(T) | + a_2 | I_q^{\beta - \gamma} f_u(T) |
\]

\[
\leq \frac{\|\rho\|}{\Gamma_q(\beta + 1)} \left[ T^{\beta_2} - t_1^\beta + 2(t_2 - t_1) + 2(t_2 - t_1)^\beta \right]
+ | \mu_1(t_2) - \mu_1(t_1) | ( I_q^{\alpha + \sigma_1} | \rho \| )
+ b_2 | I_q^{-\gamma} | \rho \| |
+ a_2 | I_q^{\beta - \gamma} | \rho \| |
\]  \hspace{1cm} (16)

The right-hand side of (16) tends to zero (not depending upon \( u \)) as \( t_2 \to t_1 \). This shows that \( \mathcal{K}_2 \) is equicontinuous. From the above reasons, it is clear that \( \mathcal{K}_2 \) is relatively compact on \( B_r \). Application of the Arzelà–Ascoli theorem proves the compactness of \( \mathcal{K}_2 \) on \( B_r \).

Step 3: \( \mathcal{K}_1 \) is \( Q \)-contractive.
From \((B_1)\) and \((B_2)\) and for each \( u, v \in B_r \), we have

\[
| \mathcal{K}_1u(t) - \mathcal{K}_1v(t) | \leq \sup_{t \in J} \left[ I_q^{\alpha + \beta} f_u - f_v \right] + | \mu_1(t) | ( I_q^{\alpha + \sigma_1} f_u - f_v(\eta_1) )
+ | \mu_2(t) | ( I_q^{\sigma_2} f_u - f_v(\eta_2) ) + b_2 | I_q^{-\gamma} f_u - f_v(T) |
+ a_2 | I_q^{\beta - \gamma} f_u - f_v(T) |
\]
\[ + a_2 \left( \frac{f_q}{q^{r+\beta}} |f_u - f_v(T)| \right) \right] \right] \right) \right) \right) \right) \leq L \Theta_1 \| u - v \|. \]

So, \( \| K_1 u - K_1 v \| \leq L \Theta_1 \| u - v \|. \) Thus \( K_1 \) is \( Q \)-contractive because of \( Q := L \Theta_1 < 1. \)

**Step 4:** \( K \) is condensing.

As \( K_1 \) and \( K_2 \) are continuous \( Q \)-contraction and compact, respectively, thus by Lemma 10, \( K : B_r \to B_r \) with \( K = K_1 + K_2 \) is a condensing map on \( B_r \). From the above arguments, by Theorem 11, we conclude that the map \( K \) has a fixed point, which leads to the existence of at least one solution for the sequential four-point \( q \)-CFBVP (3) in \( X \). \( \square \)

### 3.2 The second existence criterion

We now use another fixed point result due to Krasnoselskii–Zabreiko to demonstrate the following existence criterion for the sequential four-point \( q \)-CFBVP (3).

**Theorem 13** ([44]) Consider a completely continuous map \( K \) on a Banach space \( X \). If a bounded linear map \( L \) exists on \( X \) so that 1 is not an eigenvalue of it and

\[
\lim_{\| u \| \to \infty} \frac{\| K(u) - L(u) \|}{\| u \|} = 0,
\]

then \( K \) has a fixed point in \( X \).

**Theorem 14** Consider the following assertions:

(H1) \( f : J \times \mathbb{R} \to \mathbb{R} \) is continuous and for some \( t \in J, f(t,0) \neq 0 \) and

\[
\lim_{\| u \| \to \infty} \frac{f(t,u)}{u} = \lambda(t). \tag{17}
\]

(H2) The function \( g : J \times \mathbb{R} \to \mathbb{R} \) is continuous and \( \exists A \in \mathbb{R}^+ \) so that

\[ |g(t,u(t))| \leq A |u(t)|. \]

Then there exists at least one solution for the sequential four-point \( q \)-CFBVP (3) on \( J \) such that

\[
\lambda_{\text{max}} := \max_{t \in J} |\lambda(t)| < \frac{1 - A \Theta_2}{\Theta_1}, \tag{18}
\]

where \( \Theta_1 \) and \( \Theta_2 \) are, respectively, given by (11) and (12).

**Proof** Consider a sequence \( \{u_n\} \subset B_r \) which converges to \( u \). We know that \( f \) and \( g \) are continuous, so, by letting \( n \to \infty \), we get

\[ |f_{u_n} - f_u(t)| \to 0, \quad |g_{u_n} - g_u(t)| \to 0. \]

Thus, for \( t \in J \), we write

\[
| (K u_n)(t) - (K u)(t) | \leq I_q^{p+\beta} |g_{u_n} - g_u(t)| + I_q^{p+\beta} |f_{u_n} - f_u|(t)
\]
+ \mu_1(t) \left( I_q^{\beta+\alpha_1} [g_u] - g_u(\eta_1) + I_q^{\alpha+\beta+\alpha_1} [f_u] - f_u(\eta_1) \right) \\
+ \mu_2(t) \left( \lambda_2 \left( I_q^{\beta+\alpha_2} [g_u] - g_u(\eta_2) + I_q^{\alpha+\beta+\alpha_2} [f_u] - f_u(\eta_2) \right) \right) \\
+ b_2 \left( I_q^{\beta+\gamma} [g_u(T) + I_q^{\alpha+\beta+\gamma} [f_u(T)] \right) + a_2 \left( I_q^{\alpha+\beta} [g_u(T)] + I_q^{\alpha+\beta} [f_u(T)] \right) \\
+ a_2 \left( I_q^{\alpha+\beta} [g_u(T) + I_q^{\alpha+\beta} [f_u(T)] \right) \to 0. \quad (19)

Therefore the right-hand side of (19) tends to zero. Therefore, the continuity of $K$ is established. Now, for $r > 0$, we set $N = \{u \in C(J, \mathbb{R}); \|u\| \leq r\}$ and $\|f^*\| = \sup_{(t,\omega)\in J \times N} |f_u(t)|$. Thus,

$$
\|Ku(t)\| \leq I_q^\beta [g_u(t)] + I_q^{\alpha+\beta} [f_u(t)] \\
+ \|\mu_1(t) \left( I_q^{\beta+\alpha_1} [g_u(\eta_1)] + I_q^{\alpha+\beta+\alpha_1} [f_u(\eta_1)] \right) \|f^*\| \\
+ \|\mu_2(t) \left( \lambda_2 \left( I_q^{\beta+\alpha_2} [g_u(\eta_2)] + I_q^{\alpha+\beta+\alpha_2} [f_u(\eta_2)] \right) \right) \|f^*\| \\
+ b_2 \left( I_q^{\beta+\gamma} [g_u(T)] + I_q^{\alpha+\beta+\gamma} [f_u(T)] \right) + a_2 \left( I_q^{\alpha+\beta} [g_u(T)] + I_q^{\alpha+\beta} [f_u(T)] \right) \|f^*\| \\
\leq I_q^\beta (1 + \|\mu_1(t) \left[ I_q^{\beta+\alpha_1} (1(t)) \right] \|f^*\| + I_q^{\alpha+\beta} \|f_u(T)\| \\
+ \|\mu_2(t) \left[ \lambda_2 \left( I_q^{\beta+\alpha_2} (1(t)) \right) \right) \|f^*\| + I_q^{\alpha+\beta} \|f_u(T)\| \\
+ b_2 \left( I_q^{\beta+\gamma} (1(T)) + I_q^{\alpha+\beta+\gamma} \|f_u(T)\| \right) + a_2 \left( I_q^{\alpha+\beta} (1(T)) + I_q^{\alpha+\beta} \|f_u(T)\| \right) \|f^*\| \\
+ a_2 \left( I_q^{\alpha+\beta} (1(T)) \right) \|f^*\| + a_2 \left( I_q^{\alpha+\beta} (1(T)) \right) \|f^*\| \\
+ \|\mu_1(t) \left( I_q^{\beta+\alpha_1} (1(t)) \right) \|f^*\| + I_q^{\alpha+\beta} \|f_u(T)\| \|f^*\| \\
+ \|\mu_2(t) \left( \lambda_2 \left( I_q^{\beta+\alpha_2} (1(t)) \right) \right) \|f^*\| + I_q^{\alpha+\beta} \|f_u(T)\| \|f^*\| \\
+ b_2 \left( I_q^{\beta+\gamma} (1(T)) + I_q^{\alpha+\beta+\gamma} \|f_u(T)\| \right) \|f^*\| + a_2 \left( I_q^{\alpha+\beta} (1(T)) + I_q^{\alpha+\beta} \|f_u(T)\| \right) \|f^*\| \\
\leq \Theta_1 \|f^*\| + \Theta_2 r,
$$

which yields $\|Ku\| \leq \Theta_1 \|f^*\| + A \Theta_2 r$. This shows the uniformly boundedness of $K$. We now claim that $K$ is equicontinuous.

Let $t_1, t_2 \in J$ via $t_1 < t_2$. Then, by setting $\|f^*\| = \sup_{(t,\omega)\in J \times N} |f_u(t)|$, we obtain

$$
|Ku(t_2) - Ku(t_1)| \\
\leq I_q^\beta [g_u(t_2) - g_u(t_1)] + I_q^{\alpha+\beta} [f_u(t_2) - f_u(t_1)] \\
+ \|\mu_1(t_2) - \mu_1(t_1) \left[ I_q^{\beta+\alpha_1} [g_u(\eta_1)] + I_q^{\alpha+\beta+\alpha_1} [f_u(\eta_1)] \right) \|f^*\| \\
+ \|\mu_2(t_2) - \mu_2(t_1) \left[ \lambda_2 \left( I_q^{\beta+\alpha_2} [g_u(\eta_2)] + I_q^{\alpha+\beta+\alpha_2} [f_u(\eta_2)] \right) \right) \|f^*\| \\
+ b_2 \left( I_q^{\beta+\gamma} [g_u(T)] + I_q^{\alpha+\beta+\gamma} [f_u(T)] \right) + a_2 \left( I_q^{\alpha+\beta} [g_u(T)] + I_q^{\alpha+\beta} [f_u(T)] \right) \|f^*\| \\
\leq \frac{Ar}{\Gamma_q(\beta+1)} \left[ I_q^\beta (t_2 - t_1) + 2(t_2 - t_1)^{\alpha+\beta} \right] \\
+ \frac{\|f^*\|}{\Gamma_q(\alpha+\beta+1)} \left[ I_q^{\alpha+\beta} (t_2 - t_1)^{\alpha+\beta} + 2(t_2 - t_1)^{\alpha+\beta} \right] \\
+ \|\mu_1(t_2) - \mu_1(t_1) \left[ I_q^{\beta+\alpha_1} (1(t_1)) + I_q^{\alpha+\beta+\alpha_1} (1(t_1)) \right) \|f^*\| \\
+ \|\mu_2(t_2) - \mu_2(t_1) \left[ \lambda_2 \left( I_q^{\beta+\alpha_2} (1(t_1)) + I_q^{\alpha+\beta+\alpha_2} (1(t_1)) \right) \right) \|f^*\| \\
+ b_2 \left( I_q^{\beta+\gamma} (1(T)) + I_q^{\alpha+\beta+\gamma} (1(T)) \right) \|f^*\| + a_2 \left( I_q^{\alpha+\beta} (1(T)) + I_q^{\alpha+\beta} (1(T)) \right) \|f^*\| \\
\to 0 \quad \text{as} \quad t_2 \to t_1 \quad \text{independent of} \quad u. \quad \text{In consequence, from the above arguments,} \quad K \quad \text{is relatively compact on} \quad N. \quad \text{Application of the Arzelà–Ascoli theorem proves the compactness of} \quad K \quad \text{on} \quad N.
Now, by considering the sequential four-point \(q\)-CFBVP (3) to be linear by taking \(f_u(t) = f(t, u(t)) = \lambda(t)u(t)\), the operator \(\mathcal{L}\), by Lemma 7, is formulated by

\[
\mathcal{L}u(t) = I_q^{\beta} g_u(t) + I_q^{\alpha+\beta} \lambda(t)u(t) \\
+ \mu(t) \left[ I_q^{\alpha+\beta} \left[ f_u(t) - \lambda(t)u(t) \right] \right] \\
+ b_2 \left( I_q^{\alpha+\beta} \left[ \frac{f_u(T)}{u(T)} - \lambda(T) \right] \right) \\
+ a_2 \left( I_q^{\alpha+\beta} \left[ \frac{f_u(T)}{u(T)} - \lambda(T) \right] \right) \\
- a_2 \left( I_q^{\alpha+\beta} \left[ \frac{f_u(T)}{u(T)} - \lambda(T) \right] \right).
\]

Our next claim is that 1 is not an eigenvalue of \(\mathcal{L}\). If it is so, by (18), we estimate

\[
\|u\| = \sup_{t \in J} |(\mathcal{L}u)(t)| \\
\leq \sup_{t \in J} \left[ I_q^{\beta} g_u(t) + I_q^{\alpha+\beta} |\lambda(t)||u(t)| \right] \\
+ \left| \mu_1(t) \left[ I_q^{\alpha+\beta} \left[ f_u(t) - \lambda(t)u(t) \right] \right] \right| \\
+ \left| \mu_2(t) \left[ I_q^{\alpha+\beta} \left[ f_u(t) - \lambda(t)u(t) \right] \right] \right| \\
+ b_2 \left( I_q^{\alpha+\beta} \left[ \frac{f_u(T)}{u(T)} - \lambda(T) \right] \right) \\
+ a_2 \left( I_q^{\alpha+\beta} \left[ \frac{f_u(T)}{u(T)} - \lambda(T) \right] \right)
\]

\[
\leq (\lambda_{\max} \Theta_1 + A \Theta_2)\|u\| < \|u\|,
\]

which is not possible. Hence we established our claim.

Finally, we show that \(\|\mathcal{K}(u) - \mathcal{L}(u)\|\|u\|\) vanishes as \(\|u\| \to \infty\). For \(t \in J\), one may write

\[
\left| (\mathcal{K}u)(t) - (\mathcal{L}u)(t) \right| \\
\leq I_q^{\alpha+\beta} \left| f_u(t) - \lambda(t)u(t) \right| \\
+ \left| \mu_1(t) \left[ I_q^{\alpha+\beta} \left[ f_u(t) - \lambda(t)u(t) \right] \right] \right| \\
+ \left| \mu_2(t) \left[ I_q^{\alpha+\beta} \left[ f_u(t) - \lambda(t)u(t) \right] \right] \right| \\
+ b_2 \left( I_q^{\alpha+\beta} \left[ \frac{f_u(T)}{u(T)} - \lambda(T) \right] \right) \\
+ a_2 \left( I_q^{\alpha+\beta} \left[ \frac{f_u(T)}{u(T)} - \lambda(T) \right] \right)
\]

\[
\leq I_q^{\alpha+\beta} \left[ \frac{f_u(t)}{u(t)} - \lambda(t) \right] \|u(t)\|
\]

This means that

\[
\frac{\|\mathcal{K}u - \mathcal{L}u\|}{\|u\|} \leq I_q^{\alpha+\beta} \left[ \frac{f_u(t)}{u(t)} - \lambda(t) \right].
\]
By (17) and letting \( \|u\| \to \infty \), it is concluded that \( \frac{\|K(t) - \lambda(t)\|}{\|u\|} \to 0 \). Thus we obtain

\[
\lim_{\|u\| \to \infty} \frac{\|K(u) - L(u)\|}{\|u\|} = 0.
\]

Consequently, by Theorem 13, the supposed sequential four-point \( q \)-CFBVP (3) admits a solution in \( X \). The proof is ended.

### 3.3 The third existence criterion

We now present our last existence criterion based on the O’Regan theorem [45].

**Theorem 15** ([45]) Consider a closed and convex set \( E \neq \emptyset \) belonging to a Banach space \( X \) containing an open set \( O \). Define \( K : \bar{O} \to E \) as \( K = K_1 + K_2 \) subject to \( K(\bar{O}) \) being bounded. Moreover, \( K_1 : \bar{O} \to E \) is continuous and completely continuous, \( K_2 : \bar{O} \to E \) is nonlinear contraction (i.e., a nonnegative nondecreasing function \( \Upsilon : [0, \infty) \to [0, \infty) \) exists which satisfies \( \Upsilon(t) < t \) for \( t > 0 \), and \( \|K_2u - K_2u'\| \leq \Upsilon(\|u - u'\|), \forall u, u' \in O \)). Then either

1. \( C_1 \) \( K \) has a fixed point \( u \in \bar{O} \);

or

2. \( C_2 \) there exist \( u \in \partial O \) and \( \mu \in (0, 1) \) such that \( u = \mu K(u) \).

**Theorem 16** Let \( f, g \in C([-\pi, \pi] \times \mathbb{R}, \mathbb{R}) \) and assume that:

1. **(D1)** there exist a nonnegative mapping \( b \in C(J, [0, \infty)) \) and a nondecreasing function \( T : [0, \infty) \to (0, \infty) \) such that

\[
|f(t, u)| \leq b(t)T(\|u\|), \quad \forall (t, u) \in J \times \mathbb{R};
\]

2. **(D2)** there exist a continuous function \( \phi_1 : [0, \infty) \to [0, \infty) \) and \( \kappa > 0 \) such that

\[
|g(t, u) - g(t, v)| \leq \phi_1(\|u - v\|) \quad \text{and} \quad \phi_1(\|u\|) \leq \kappa |u|, \quad \forall t \in J, u, v \in \mathbb{R};
\]

3. **(D3)** there exists \( \varepsilon > 0 \) such that

\[
\sup_{t \in (0, \infty)} \left[ \frac{\varepsilon}{\Theta_1 \beta^\gamma (\varepsilon) + \mu \Theta_2} \right] > \frac{1}{1 - k \Theta_2},
\]

where \( \varepsilon > 0 \) so that

\[
\frac{\varepsilon}{\Theta_1 \beta^\gamma (\varepsilon) + \mu \Theta_2} > \frac{1}{1 - k \Theta_2},
\]

Then there exists a solution for the supposed sequential four-point \( q \)-CFBVP (3) on \( J \).

**Proof** We consider \( K : X \to X \) defined by (13) as

\[
Ku(t) = K_1u(t) + K_2u(t), \quad t \in J,
\]

where the operators \( K_1 \) and \( K_2 \) are, respectively, given in (14) and (15). By (D3), \( \exists \varepsilon > 0 \) so that

\[
\frac{\varepsilon}{\Theta_1 \beta^\gamma (\varepsilon) + \mu \Theta_2} > \frac{1}{1 - k \Theta_2},
\]
and take $B_\varepsilon = \{u \in X : \|u\| < \varepsilon\}$. We demonstrate the continuity and complete continuity of $K_1$. Before this, we prove the uniform boundedness of $K_1$. Taking any $u \in B_\varepsilon$, we have

$$
\|(K_1u)(t)\| \leq T_\varepsilon^{\alpha+\beta}|f_u(t)| + \left\{\mu_1(t)\left(|T_\varepsilon^{\alpha+\beta+1}|f_u(\eta_1)\right) + \left\{|\mu_2(t)|\left[2 T_\varepsilon^{\alpha+\beta+1}g_u(\eta_2)\right] + b_2(T_\varepsilon^{\alpha+\beta+1}|f_u(T)|)\right\} + a_2(T_\varepsilon^{\alpha+\beta}|f_u(T)|)\right\} \\
\leq \Theta_1 b^* \mathbb{T}(\varepsilon),
$$

in which $b^* = \sup_{t\in J} |b(t)|$. Thus $K_1$ is uniformly bounded. Let $t_1, t_2 \in J$ such that $t_1 < t_2$. Then

$$
\|(K_1u)(t_2) - (K_1u)(t_1)\| \\
\leq T_\varepsilon^{\alpha+\beta}|f_u(t_2) - f_u(t_1)| + \left\{|\mu_1(t_2) - \mu_1(t_1)|\left(|T_\varepsilon^{\alpha+\beta+1}|f_u(\eta_1)\right) + \left\{|\mu_2(t_2) - \mu_2(t_1)|\left[2 T_\varepsilon^{\alpha+\beta+1}g_u(\eta_2)\right] + b_2(T_\varepsilon^{\alpha+\beta+1}|f_u(T)|)\right\} + a_2(T_\varepsilon^{\alpha+\beta}|f_u(T)|)\right\} \\
\leq \frac{b^* \mathbb{T}(\varepsilon)}{\Gamma_\varepsilon(\alpha + \beta + 1)}\left[ T_\varepsilon^{\alpha+\beta} - T_\varepsilon^{\alpha+\beta+2} + 2(t_2 - t_1)T_\varepsilon^{\alpha+\beta} \right] \\
+ \left\{|\mu_1(t_2) - \mu_1(t_1)|\left(|T_\varepsilon^{\alpha+\beta+1}|f_u(\eta_1)\right) + \left\{|\mu_2(t_2) - \mu_2(t_1)|\left[2 T_\varepsilon^{\alpha+\beta+1}g_u(\eta_2)\right] + b_2(T_\varepsilon^{\alpha+\beta+1}|f_u(T)|)\right\} + a_2(T_\varepsilon^{\alpha+\beta}|f_u(T)|)\right\},
$$

which tends to zero as $t_2 \to t_1$ free of $u$. This gives the equicontinuity of $K_1$. Application of the Arzelà–Ascoli theorem proves the compactness of $K_1$ and consequently its complete continuity. Furthermore, the continuity of $K_1$ can be deduced from that of $f$ by the hypothesis.

We now show that $K_2$ is a nonlinear contraction. By (D2) and for $u, v \in B_\varepsilon$, we have

$$
\|(K_2u)(t) - (K_2v)(t)\| \\
\leq T_\varepsilon^{\alpha+\beta}|g_u - g_v|(t) + \left\{|\mu_1(t)|\left(|T_\varepsilon^{\alpha+\beta+1}|g_u - g_v(\eta_1)\right) + \left\{|\mu_2(t)|\left[\lambda_2 T_\varepsilon^{\alpha+\beta+1}|g_u - g_v(\eta_2)\right] + b_2 T_\varepsilon^{\alpha+\beta+1}|g_u - g_v(\eta_2)|\right\} + a_2(T_\varepsilon^{\alpha+\beta}|g_u - g_v(\eta_2)|)\right\} \\
\leq \Theta_2 \phi_\kappa \left(\|u(t) - v(t)\|\right) \\
\leq \Theta_2 \kappa \|u(t) - v(t)\|.
$$

By setting $\Upsilon(u) = \Theta_2 \kappa u$, note that $\Upsilon(0) = 0$ and $\Upsilon(u) = \Theta_2 \kappa u < u$ for $u > 0$ since $\kappa \Theta_2 < 1$. Thus

$$
\|K_2u - K_2v\| \leq \Upsilon(\|u - v\|).
$$

Hence $K_2$ is a nonlinear contraction. Now again, by (D2), for arbitrary $u \in B_\varepsilon$, we estimate

$$
|g_u(t)| = |g(t, u)| \leq |g(t, u) - g(t, 0)| + |g(t, 0)| \leq \phi_1(\|u\|) + |g(t, 0)| \leq \kappa \varepsilon + I.
$$
where \( l = \sup_{t \in J} |g(t, 0)| \). Hence, we get

\[
\|K_2u\| \leq \Theta_2 (\kappa \varepsilon + l),
\]

which confirms the boundedness of \( K_2 \). Thus, \( K = K_1 + K_2 \) is bounded.

In the final step, we prove that the assumption (C2) of Theorem 15 does not hold. To prove this, consider the existence of \( \mu \in (0, 1) \) and \( u \in \partial B_\varepsilon \) such that \( u = \mu K_1 u \). So \( \|u\| = \varepsilon \) and

\[
|u(t)| = \mu |(Ku)(t)| = \mu |K_1u(t) + K_2u(t)| \leq |K_1u(t)| + |K_2u(t)| \leq \Theta_1 b^* \mathbb{T}(\varepsilon) + \Theta_2 (\kappa \varepsilon + 1).
\]

Taking the supremum for all \( t \in J \) yields

\[
\|u\| \leq \Theta_1 b^* \mathbb{T}(\varepsilon) + (\kappa \varepsilon + l) \Theta_2.
\]

Hence, we get

\[
\frac{\varepsilon}{\Theta_1 b^* \mathbb{T}(\varepsilon) + l \Theta_2} \leq \frac{1}{1 - \kappa \Theta_2},
\]

which contradicts (D3). Thus \( K_1 \) and \( K_2 \) satisfy all the assertions of Theorem 15. Therefore, a fixed-point of \( K \) in \( B_\varepsilon \) exists, which is the same solution of the sequential four-point \( q \)-CFBVP (3). The proof is finished. \( \square \)

### 3.4 The uniqueness property

Finally, we investigate the uniqueness property for the solutions of the sequential four-point \( q \)-CFBVP (3) by referring to the Banach principle.

**Theorem 17** Let

\( (H_4) \) \( \exists \alpha > 0 \) satisfying

\[
|g_u(t) - g_v(t)| \leq \alpha |u(t) - v(t)|, \quad \forall t \in J, u, v \in \mathbb{R};
\]

\( (H_5) \) \( \exists \ell > 0 \) satisfying

\[
|f_u(t) - f_v(t)| \leq \ell |u(t) - v(t)|, \quad \forall t \in J, u, v \in \mathbb{R}.
\]

Then the sequential four-point \( q \)-CFBVP (3) has a unique solution on \( J \) if

\[
\ell \Theta_1 + a \Theta_2 < 1,
\]

where \( \Theta_1, \Theta_2 \) are given in (11) and (12), respectively.
Proof To prove the relevant result, define the ball $B_r = \{u \in X : \|u\| \leq r\}$ for some $r > 0$ satisfying
\[
r \geq \frac{\Theta_1 f_0^* + \Theta_2 g_0^*}{1 - \ell \Theta_1 - a \Theta_2},
\]
where $g_0^* = \sup_{t \in J} |g(t, 0)|$ and $f_0^* = \sup_{t \in J} |f(t, 0)|$ and $\Theta_1$ and $\Theta_2$ are, respectively, given by (11) and (12). Now, we prove $KB_r \subset B_r$ in which the operator $K : X \to X$ is illustrated as (13). Similar to Step 1 in Theorem 12, for $u \in B_r$, we get
\[
\left|(Ku)(t)\right| \leq I_q^\beta |g_u(t)| + I_q^{\alpha + \beta} |f_u(t)| + |\mu_1(t)| I_q^{\beta + \alpha_1} |g_u(\eta_1)| + b_2 \left(\frac{I_q^{\beta + \alpha}}{g_u(T)} + \frac{I_q^{\beta + \gamma}}{f_u(T)} + \frac{f_u(T)}{I_q^{\beta + \alpha}}\right) + a_2 \left(\frac{I_q^\beta |g_u(T)|}{f_u(T)} + \frac{f_u(T)}{I_q^{\beta + \alpha}}\right) \leq (\ell \|u\| + f_0^*) \sup_{t \in J} I_q^{\beta + \alpha_1} (t) + |\mu_1(t)| I_q^{\beta + \alpha_1} (1(\eta_1) - a_2 I_q^{\beta + \alpha_1} (T)) + a_2 I_q^{\alpha + \beta} (1(\eta_1) - a_2 I_q^{\beta + \alpha_1} (T)) \leq \Theta_1 (r + f_0^*) + \Theta_2 (ar + g_0^*) < r,
\]
which implies $\|K(u)\| \leq r$. Thus, $K$ maps $B_r$ into itself. Next, we prove that $K$ is a contraction. For $u,v \in X$, and applying (11) and (12), we have
\[
\left|(Ku)(t) - (Kv)(t)\right| \leq I_q^\beta |g_u - g_v|(t) + I_q^{\alpha + \beta} |f_u - f_v|(t) + |\mu_1(t)| I_q^{\beta + \alpha_1} |g_u - g_v(\eta_1)| + b_2 \left(\frac{I_q^{\beta + \alpha}}{g_u(T) - g_v(T)} + \frac{I_q^{\beta + \gamma}}{f_u(T) - f_v(T)} + \frac{f_u(T) - f_v(T)}{I_q^{\beta + \alpha}}\right) + a_2 \left(\frac{I_q^\beta |g_u(T) - g_v(T)|}{f_u(T) - f_v(T)} + \frac{f_u(T) - f_v(T)}{I_q^{\beta + \alpha}}\right) \leq (\ell \Theta_1 + a \Theta_2) \|u - v\|.
\]
Consequently, we get
\[
\|K(u)(t) - K(v)(t)\| \leq (\ell \Theta_1 + a \Theta_2) \|u - v\|.
\]
Since $\ell \Theta_1 + a \Theta_2 < 1$, the above inequality proves that $K$ is a contraction. Thus application of the Banach principle shows that $K$ has a unique fixed point, corresponding to unique solution of the sequential four-point $q$-CFBVP (3) on $J$. This ends the proof.

4 The criterion of Ulam–Hyers stability
Due to the importance of the notion of stability for possible solutions of different dynamical systems, in this section, we review two Ulam–Hyers and generalized Ulam–Hyers...
stabilities for solutions of the sequential four-point $q$-CFBVP (3). For more information, see [46–48].

**Definition 18** ([49]) The sequential four-point $q$-CFBVP (3) is Ulam–Hyers stable if $c^*$ exists in $\mathbb{R}^+$ such that for all $\epsilon > 0$ and all $u^*(t) \in C(J, \mathbb{R})$ as a solution function satisfying

$$
\left| D_q^\alpha (D_q^\beta u^*(t)) - g(t, u^*(t)) \right| < \epsilon,
$$

there exists a function $u(t) \in C(J, \mathbb{R})$ as the solution of the sequential four-point $q$-CFBVP (3) with

$$
\left| u^*(t) - u(t) \right| \leq c^* \epsilon, \quad t \in J.
$$

**Definition 19** ([49]) The sequential four-point $q$-CFBVP (3) is generalized Ulam–Hyers stable if $H \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $H(0) = 0$ such that for all $\epsilon > 0$ and all $u^*(t) \in C(J, \mathbb{R})$ as a solution of

$$
\left| D_q^\alpha (D_q^\beta u^*(t)) - g(t, u^*(t)) \right| < \epsilon,
$$

there exists a function $u(t) \in C(J, \mathbb{R})$ as the solution of the sequential four-point $q$-CFBVP (3) with

$$
\left| u^*(t) - u(t) \right| \leq H(\epsilon), \quad t \in J.
$$

**Remark 1** ([49]) It is evident that Def. 18 $\Rightarrow$ Def. 19.

**Remark 2** ([49]) It is notable that $u^*(t) \in C(J, \mathbb{R})$ is a solution for (21) iff $G \in C(J, \mathbb{R})$ depending on $u^*$ such that

1. $|G(t)| < \epsilon$, $t \in J$.
2. $D_q^\alpha (D_q^\beta u^*(t)) - g(t, u^*(t)) = f(t, u(t)) + G(t)$, $t \in J$.

Now, we can discuss the above stabilities for solutions to the sequential four-point $q$-CFBVP (3).

**Theorem 20** If $(H_4)$ and $(H_5)$ are fulfilled, then the sequential four-point $q$-CFBVP (3) is Ulam–Hyers stable on $J$ and accordingly is generalized Ulam–Hyers stable whenever

$$
\ell \mathcal{\Theta}_1 + a \mathcal{\Theta}_2 < 1,
$$

where $\mathcal{\Theta}_1$, $\mathcal{\Theta}_2$ are in the same forms given in (11) and (12), respectively.

**Proof** For each $\epsilon > 0$ and each function $u^*(t) \in C(J, \mathbb{R})$ as a solution of the inequality

$$
\left| D_q^\alpha (D_q^\beta u(t)) - g(t, u(t)) \right| < \epsilon,
$$

a function $G(t)$ exists which satisfies

$$
D_q^\beta (D_q^\beta u(t)) - g(t, u(t)) = f(t, u(t)) + G(t)
$$
with $|G(t)| \leq \varepsilon$. It gives

$$u^*(t) = I_q^\beta g_\alpha(t) + I_q^{\alpha + \beta} f_\alpha(t) + I_q^{\alpha + \beta} G(t)$$

$$+ \mu_1(t) [I_q^{\alpha + \beta} g_\alpha(\eta_1) + I_q^{\alpha + \beta} f_\alpha(\eta_1)]$$

$$+ \mu_2(t) [I_q^{\alpha + \beta} g_\alpha(\eta_2) + I_q^{\alpha + \beta} f_\alpha(\eta_2)]$$

$$- b_2 (I_q^{\alpha + \gamma} g_\alpha(T) + I_q^{\alpha + \beta - \gamma} f_\alpha(T)) - a_2 (I_q^\beta g_\alpha(T) + I_q^{\alpha + \beta} f_\alpha(T)).$$

On the other side, let a unique function $u(t) \in C(J, \mathbb{R})$ be the solution of (3). Then $u(t)$ is written by

$$u(t) = I_q^\beta g_\alpha(t) + I_q^{\alpha + \beta} f_\alpha(t)$$

$$+ \mu_1(t) [I_q^{\alpha + \beta} g_\alpha(\eta_1) + I_q^{\alpha + \beta} f_\alpha(\eta_1)]$$

$$+ \mu_2(t) [I_q^{\alpha + \beta} g_\alpha(\eta_2) + I_q^{\alpha + \beta} f_\alpha(\eta_2)]$$

$$- b_2 (I_q^{\alpha + \gamma} g_\alpha(T) + I_q^{\alpha + \beta - \gamma} f_\alpha(T)) - a_2 (I_q^\beta g_\alpha(T) + I_q^{\alpha + \beta} f_\alpha(T)).$$

We estimate

$$|u^*(t) - u(t)| \leq I_q^\beta |G(t)| + I_q^{\alpha + \beta} |g_\alpha - g_\alpha|(t) + I_q^{\alpha + \beta} |f_\alpha - f_\alpha|(t)$$

$$+ |\mu_1(t)| [I_q^{\alpha + \beta} |g_\alpha - g_\alpha|(\eta_1) + I_q^{\alpha + \beta} |f_\alpha - f_\alpha|(\eta_1)]$$

$$+ |\mu_2(t)| [I_q^{\alpha + \beta} |g_\alpha - g_\alpha|(\eta_2) + I_q^{\alpha + \beta} |f_\alpha - f_\alpha|(\eta_2)]$$

$$+ b_2 (I_q^{\alpha + \gamma} |g_\alpha - g_\alpha|(T) + I_q^{\alpha + \beta - \gamma} |f_\alpha - f_\alpha|(T))$$

$$+ a_2 (I_q^\beta |g_\alpha - g_\alpha|(T) + I_q^{\alpha + \beta} |f_\alpha - f_\alpha|(T))$$

$$\leq \frac{\varepsilon T_q^{\alpha + \beta}}{\Gamma_q(\alpha + \beta + 1)} + (\ell \Theta_1 + a \Theta_2) \|u^* - u\|.$$

Hence

$$\|u^* - u\| \leq \frac{\varepsilon T_q^{\alpha + \beta}}{\Gamma_q(\alpha + \beta + 1)} + (\ell \Theta_1 + a \Theta_2) \|u^* - u\|,$$

where $\Theta_1, \Theta_2$ are the same constants as represented in (11) and (12), respectively. In consequence,

$$\|u^* - u\| \leq \frac{\varepsilon T_q^{\alpha + \beta}}{\Gamma_q(\alpha + \beta + 1)[1 - (\ell \Theta_1 + a \Theta_2)]}.$$ 

By assuming $c^* = \frac{T_q^{\alpha + \beta}}{\Gamma_q(\alpha + \beta + 1)[1 - (\ell \Theta_1 + a \Theta_2)]}$, the Ulam–Hyers stability for $q$-system (3) is satisfied. Also, for

$$H(\varepsilon) = \frac{\varepsilon T_q^{\alpha + \beta}}{\Gamma_q(\alpha + \beta + 1)[1 - (\ell \Theta_1 + a \Theta_2)]}$$

with $H(0) = 0$, the condition of the generalized Ulam–Hyers stability is fulfilled for solutions of the $q$-system (3). This completes the proof.
5 Two examples

Here, we aim to present some examples to examine the obtained results.

Example 1 Let us consider the sequential four-point \( q \)-CFBVP with the following data:

\[
\begin{align*}
D_{1/4}^{1/3}[D_{1/4}^{1/2}u(t) - g(t, u(t))] &= f(t, u(t)), \quad t \in J = [0, 1], \\
u(0) + 2D_{1/4}^{1/3}u(0) &= 2/5I_{1/4}^{3/4}u(1/2), \\
2u(1) + D_{1/4}^{1/3}u(1) &= 3/7I_{1/4}^{3/4}u(3/4),
\end{align*}
\]

(22)

where \( \alpha = 1/3, \beta = 2/3, q = 1/4, T = 1, \gamma = 1/2, a_1 = b_2 = 1, a_2 = b_1 = 2, \sigma_1 = 3/4, \sigma_2 = 1/4, \lambda_1 = 2/5, \lambda_2 = 3/7, \eta_1 = 1/2, \eta_2 = 3/4 \) and \( g(t, u), f(t, u) \) are defined by

\[
f(t, u) = \frac{tu}{56(1 + t)^5} \left( \frac{|u| + 2}{|u| + 1} \right) + \frac{1}{2} \text{ and } g(t, u) = \frac{3t^2}{6} \left( \frac{|u|}{3(|u| + 1)} \right).
\]

The continuity of \( f \) is obvious and we reach \( f(t, 0) = \frac{1}{2} \) (see Fig. 1). Now, we divide \( f(t, u) \) by \( u \) and we get

\[
\frac{f(t, u)}{u} = \frac{t}{56(1 + t)^5} \left( 1 + \frac{1}{|u| + 1} \right) + \frac{1}{2u}.
\]

Hence

\[
\lim_{|u| \to \infty} \frac{f(t, u)}{u} = \frac{t}{56(1 + t)^5}.
\]

Setting \( \lambda(t) = \frac{t}{56(1 + t)^5} \), we get \( \lambda_{\max} = 0.0179 \). On the other side,

\[
|g(t, u)| \leq \frac{1}{6}|u|.
\]

Letting \( A = 1/6 \), we obtain \( \Theta_1 = 3.5597 \) and \( \Theta_2 = 4.8600 \). Since \( (1 - A\Theta_2)/\Theta_1 = 0.0548 > \lambda_{\max} \), where \( \Theta_1 \) and \( \Theta_2 \) are, respectively, given by Eqs. (11) and (12). therefore, by Theorem 14, the sequential four-point \( q \)-CFBVP (22) has a solution on \([0, 1]\).

Example 2 By considering \( \alpha = 1/3, \beta = 2/3, q = 1/4, T = 1, \gamma = 1/2, a_1 = b_2 = 1, a_2 = b_1 = 2, \sigma_1 = 3/4, \sigma_2 = 1/4, \lambda_1 = 2/5, \lambda_2 = 3/7, \eta_1 = 1/2, \eta_2 = 3/4 \) the sequential four-point \( q \)-CFBVP...
is then given by

\[
\begin{aligned}
D^{1/3}_{1/4}[D^{1/2}_{1/4}u(t) - g(t, u(t))] &= f(t, u(t)), \quad t \in J = [0, 1], \\
u(0) + 2D^{1/2}_{1/4}u(0) &= 2/5I^{1/4}_{1/4}u(1/2), \\
2u(1) + D^{1/2}_{1/4}u(1) &= 3/7I^{1/4}_{1/4}u(3/4),
\end{aligned}
\]  

(23)

where \( f(t, u) \) and \( g(t, u) \) are given by (see Fig. 2)

\[
f(t, u) = \frac{1}{3\sqrt{900 + t^2}}(\arctan u + e^{-t}) \quad \text{and} \quad g(t, u) = \frac{1}{100(1 + t^2)} \sin u + \frac{\cos t}{25}.
\]

By usual computations, we obtain \( \Theta_1 = 3.5597 \) and \( \Theta_2 = 4.8600 \). Taking \( a = 1/100 \) and \( \ell = 1/90 \), it is clear that \((H_1)\) and \((H_2)\) are verified. Moreover, \( \ell \Theta_1 + a \Theta_2 \approx 0.8775 < 1 \). Thus, Theorem 17 is fulfilled and hence based on it, one can find that a unique solution exists for the sequential four-point \( q \)-CFBVP (23) on \([0, 1]\). On the other side, as \( \ell \Theta_1 + a \Theta_2 < 1 \) is valid, so, by Theorem 20, the given sequential four-point \( q \)-CFBVP (23) is Ulam–Hyers and also generalized Ulam–Hyers stable on \( J \).

6 Conclusions

In the present research, we considered a new boundary problem in the context of the quantum fractional operators. In other words, we defined a sequential \( q \)-fractional system of \( q \)-difference equation in which boundary conditions are designed as a linear combination of an unknown function and its \( q \)-derivative corresponding to a multiple of \( q \)-integrals in four points. The main focus of this research is on the solution’s existence and its uniqueness with the help of some methods inspired by several pure concepts in functional analysis. We used three different fixed-point methods for this aim relying on the measure of non-compactness and condensing operators and compact operators. The existence of a unique solution is investigated based on the Banach criterion. The investigation of stability of the given \( q \)-CFBVP system in two formats based on Ulam–Hyers’ conditions is implemented. Lastly, two examples are provided to ensure the findings. It is evident that this structure is more general and has many special applied cases. By assuming \( g(t, u(t)) = -\mu \in \mathbb{R} \) and \( a_1 = b_1 = a_2 = b_2 = 1 \) and \( \sigma_1 = \sigma_2 = 1 \) and by letting \( q \to 1 \), our proposed sequential four-point \( q \)-CFBVP (3) is transformed into a fractional Langevin
equation with integral conditions

\[
\begin{align*}
D_q^\alpha (D^\beta_qu(t) + \mu)) &= f(t, u(t)), \quad t \in J := [0, T], \\
u(0) + D^\gamma u(0) &= \lambda_1 \int_0^{\eta_1} u(s) \, ds, \quad \eta_1 \in (0, T), \\
u(T) + D^\gamma u(T) &= \lambda_2 \int_0^{\eta_2} u(s) \, ds, \quad \eta_2 \in (0, T),
\end{align*}
\]

which is considered as one of the most important equations in mathematical physics. Therefore, one can observe that the research study presented in the manuscript is not only new in the existing structure, but will also lead to other various quantum fractional problems as special cases. In future studies, we can generalize our boundary conditions to multi-point ones and investigate similar results in the context of newly-defined fractional \((p, q)\)-operators in both cases of difference equations and inclusions.

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Authors’ contributions

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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