Random Matrix approach to the crossover from Wigner to Poisson statistics of energy levels

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Abstract

We analyze a class of parametrized Random Matrix models, introduced by Rosenzweig and Porter, which is expected to describe the energy level statistics of quantum systems whose classical dynamics varies from regular to chaotic as a function of a parameter. We compute the generating function for the correlations of energy levels, in the limit of infinite matrix size. Our computations show that for a certain range of values of the parameter, the energy-level statistics is given by that of the Wigner-Dyson ensemble. For another range of parameter values, one obtains the Poisson statistics of uncorrelated energy levels. However, between these two ranges, new statistics emerge, which is neither Poissonian nor Wigner. The crossover is measured by a renormalized coupling constant. The model is exactly solved in the sense that, in the limit of infinite matrix size, the energy-level correlation functions and their generating function are given in terms of a finite set of integrals.
1 Introduction

Random Matrix Theory (RMT) [1], originally introduced by Wigner, to characterize the statistical behaviour of the energy levels of nuclei, has found many successful applications in various fields of physics in recent years. Originally, it was thought that RMT was applicable only to complex systems with many degrees of freedom. Hence, it came as a surprise when it was found that it could equally well describe simple quantum systems, with very few degrees of freedom, as long as their classical dynamics were chaotic. The first evidence of this fact was provided in the seminal paper by Bohigas et al [2] in which the energy level fluctuations of the quantum Sinai billiard were analysed and shown to be consistent with the predictions of the Gaussian Orthogonal ensemble of RMT. Since this pioneering work, it has been checked numerically, on a wide variety of systems, that the local statistical properties of a quantum system, whose classical counterpart is chaotic, are well-described by RMT. In particular, the nearest neighbour spacing distribution was found to be in excellent agreement with the spacing distribution between adjacent eigenvalues of random matrices.

In contrast, Berry and Tabor [1] had given strong arguments to justify that, for integrable systems with more than one degree of freedom, the nearest neighbour spacing distribution of the quantum energy levels should have a Poisson distribution, characteristic of uncorrelated levels. This has been confirmed by many numerical studies. There now exist excellent reviews on this topic [see [3] and [7]].

However, it is well-known in classical mechanics that purely integrable or purely chaotic systems are rare (at least for systems with a few degrees of freedom). For most systems, the phase space is partitioned into regular and chaotic regions and hence these systems can be referred to as mixed systems.

An important physical system illustrating these different behaviours, is the hydrogen atom in a magnetic field. The classical system is essentially integrable (chaotic) at weak (strong) fields but appears to be mixed at intermediate values of the field. This classical behaviour has its counterpart in the energy level statistics of the corresponding quantum system, which exhibits a crossover from Poisson to Wigner type, when the magnetic field is increased [5]. It is, therefore, important to find models of random matrices which could describe the statistics of the energy levels of such mixed systems. Qualitatively, such a model should be governed by a Hamiltonian matrix which is essentially a sum of two parts, one describing the chaotic part of phase space and hence belonging to the Wigner-Dyson ensemble of the relevant symmetry, and the other corresponding to the regular part of phase space. A number of authors have studied models in which block-diagonal GOE matrices are weakly coupled by matrix elements also belonging to a GOE ensemble [12, 9]. In this paper, we consider another class of models, first introduced by Rosenzweig and Porter [15] to describe the observed deviations from the Wigner- and Poisson statistics in the spectra of some transition metal atoms.
This class of models is governed by an ensemble of $N \times N$ matrices of the form

$$H = A + \frac{\lambda}{N^\alpha} G$$  \hspace{1cm} (1.1)$$

where $A$ and $G$ are either real symmetric matrices (in the orthogonal case) or complex self-adjoint ones (in the unitary case). The matrix elements of $G$ are chosen to be independent random variables with a Gaussian distribution of unit variance. In the context of quantum chaos, $G$ is supposed to correspond to the chaotic region of the classical phase space and hence its eigenvalue distribution should obey the Wigner-Dyson statistics. In contrast, $A$ should correspond to the classically regular region and hence its eigenvalues should exhibit Poisson statistics. It is easily seen that the statistics of the energy levels of $H$ depend only on the eigenvalues of $A$. Hence, without loss of generality, the matrix $A$ can be chosen to be diagonal. The simplest type of model which can be considered is, therefore, the following one: $A$ is a diagonal matrix, whose elements are independent random variables with a probability distribution $\nu(\cdot)$. Different behaviours can be expected by varying the exponent $\alpha$ in (1.1). The case $\alpha = 1/2$, corresponds to a perturbed Wigner-Dyson ensemble. It was recently analyzed by Brezin and Hikami [4]. They considered the case in which $G$ belongs to the Gaussian Unitary ensemble (GUE) and $A$ is a fixed diagonal matrix. They showed that the energy level statistics for such a matrix ensemble was the same as that of $G$, i.e., the statistics relevant to the GUE. If $\alpha > 1$, the energy level statistics is expected to be Poissonian. The value $\alpha = 1$ corresponds to the crossover regime and for it one expects new statistics. In fact, by making a numerical study of this model, Rosenzweig and Porter showed that if one chooses the exponent $\alpha$ to be unity, then one obtains energy level statistics which is intermediate between Wigner- and Poisson statistics.

Analytical studies of the model for $\alpha = 1$, has been done only in the unitary case. These studies made use of certain special features of unitary matrices. However, the case of the GOE, which one encounters more often, and is technically more challenging, had remained virtually unsolved thus far. The only results for this case were perturbative ones in the small $\lambda$ limit [4].

In this article, we develop a technique which can be used to study the spectral correlations for the case in which $G$ belongs to the GOE as well as to the one in which it belongs to the GUE. We compute the generating function for the average value of the product of traces of advanced Green’s functions, and the mixed product of traces of advanced and retarded Green’s functions. All the correlation functions of energy levels can be obtained from it, in the limit where $N$ goes to infinity.

In the case $\alpha = 1/2$ we show that the statistics for correlations between energy levels, on the scale of the mean level spacing, is the same as that of the Wigner-Dyson Ensemble. In the unitary case, we hence recover the result of Brezin and Hikami [4]. This matching of our result with theirs is not apriori obvious, because the mean level spacing depends on $A$ and we average over the distribution of $A$, whereas they take it to be fixed.
In the case $\alpha = 1$, our result for the generating function (in the infinite $N$ limit) is in the form of a finite set of ordinary integrals. Quite generally, we show that the density of states at an energy $e$ is given by $\nu(e)$, and that all the correlation functions are universal functions depending only on the “renormalized” coupling constant $\Lambda = \lambda \nu(e)$. This suggests that in order to make a comparison of the results of the model with a given quantum system, it might be plausible to take $\Lambda$ to be the ratio of $\rho(e)$ and $\rho_{\text{reg}}(e)$, where $\rho(e)$ is the classical Louiville measure of the energy surface and $\rho_{\text{reg}}(e)$ is the measure corresponding to the regular part of the phase space.

In order to obtain more concrete results, one needs to evaluate the integrals appearing in the expression for the generating functions. This explicit computation, which turns out to be a rather lengthy one, has been done in this paper for the generating function for the two-point correlation function, in the unitary case. From it we can recover the two-point correlation function itself, in the form of certain integrals over modified Bessel functions. In the orthogonal case, the generating function, even for the two-point correlation function, appears in the form of integrals over elliptic functions and we postpone the study of it to a future paper.

It may be worth giving some hint about the technique used in this paper. We basically use integrals over auxiliary Grassmannian variables to compute the average over the distribution of the Hamiltonian. However, finally, we evaluate these Grassmannian integrals so as to arrive at a representation in terms of ordinary integrals, in the large $N$ limit. Although similar in spirit to the familiar supersymmetric approach, introduced by Efetov [6] in this kind of problems, our technique is different in that we never compute supersymmetric integrals (the only case in which we could find supersymmetry useful is when $\alpha = 1/2$).

2 Generating Function

We want to calculate the correlation functions $\rho^{(n)}(e_1, \ldots, e_n)$ of the eigenvalues $\lambda_j$ of an $N \times N$ self–adjoint matrix $H$. They are defined as

$$\rho^{(n)}(e_1, \ldots, e_n) = \langle \prod_{\alpha=1}^{n} \hat{\rho}(e_\alpha) \rangle,$$

(2.1)

where

$$\hat{\rho}(e) = \frac{1}{N} \sum_j \delta(e - \lambda_j),$$

(2.2)

is the local density of eigenvalues at the energy $e$. The angular brackets will henceforth indicate an average over the probability distribution of $H$.

If $G^s(e)$ denotes the advanced ($s = +1$) and retarded ($s = -1$) Green’s function

$$G^s(e) = \frac{1}{e - H - is\varepsilon},$$

(2.3)

then

$$\hat{\rho}(e) = \lim_{\varepsilon \uparrow 0^+} \left\{ \frac{1}{2\pi iN} \text{tr} \left[ G^+(e) - G^-(e) \right] \right\}.$$

(2.4)
We will use the following identity to compute \( \text{tr} \ G^s(e)/N \).

\[
\frac{\partial}{\partial \varepsilon} \left. \frac{\det \left[ (\varepsilon - \varepsilon_0) I_N + is (e I_N - H) \right]}{\det \left[ (\varepsilon_0 + \varepsilon_0) I_N + is (e I_N - H) \right]} \right|_{\varepsilon_0 = \varepsilon} = \frac{1}{isN} \text{tr} \ G^s(e). \tag{2.5}
\]

The symbol \( I_N \) is used to denote the \( N \times N \) identity matrix. Hence, it is evident that the correlation functions of energy levels can be obtained from the generating function

\[
J^S_n = \left\langle \prod_{\alpha = 1}^{n} \frac{\det [\varepsilon_-(\alpha)/N + is_\alpha (e_\alpha - H)]}{\det [\varepsilon_+ (\alpha)/N + is_\alpha (e_\alpha - H)]} \right\rangle, \tag{2.6}
\]

where \( S = \{s_\alpha\}_{0 = 1}^{n} \), \( s_\alpha \in \{1, -1\} \) and \( \varepsilon_\pm(\alpha) > 0 \), by taking suitable derivatives of it with respect to the variables \( \varepsilon_-(\alpha) \) or \( \varepsilon_+ (\alpha) \).

In particular, the density of states is given by

\[
\rho(e) = \rho^{(1)}(e) = \lim_{\varepsilon_- \uparrow 0^+} \Re \left\{ \frac{1}{\pi} \frac{\partial}{\partial \varepsilon_-} J^+_1 \bigg|_{\varepsilon_- = \varepsilon_+} \right\} \tag{2.7}
\]

and the two-point correlation function by

\[
\rho^{(2)}(e_1, e_2) = \lim_{\varepsilon_- \uparrow 0^+} \Re \left\{ \frac{1}{2\pi^2} \frac{\partial^2 [J^+_2 - J^{-+}_2]}{\partial \varepsilon_- (1) \partial \varepsilon_- (2)} \bigg|_{\varepsilon_- = \varepsilon_+} \right\}. \tag{2.8}
\]

In this paper, we will consider Hamiltonians \( H \) of the form

\[
H = A + \frac{\lambda}{N} G, \tag{2.9}
\]

where \( G \) is an \( N \times N \) matrix whose elements are independent random variables with a Gaussian distribution of unit variance and zero mean. When the matrix elements of \( G \) are real (complex), the matrix \( G \) belongs to the Gaussian orthogonal (unitary) ensemble of standard Random Matrix Theory. Since the probability distribution of \( G \) is independent of the basis, the correlations of the energy levels of \( H \) depend only on the eigenvalues \( \{a_j\} \) of \( A \). Hence, without loss of generality, we can choose \( A \) to be a diagonal matrix whose elements are independent random variables with a probability distribution \( \nu(a_j) \).

2.1 The orthogonal case for finite matrix size

Consider a mixed system governed by a Hamiltonian matrix of the form (2.9), with the matrix \( G \) belonging to the Gaussian Orthogonal Ensemble (GOE). In this case, it is convenient to express the generating function \( J^S \) as the ratio of the square roots of the determinants of an antisymmetric matrix and a symmetric one. This is because one can cast such a ratio as a product of integrals over real and Grassmannian variables, by making use of the integral identities (A.1) and (A.2). [Note that here and henceforth, we suppress the subscript \( n \) of the generating function]
We can achieve this as follows: We start from the original expression (2.6) of the generating function, which can be rewritten in the form
\[ J^S = (\det S)^N \left( \frac{\det C_-}{\det C_+} \right), \] (2.10)
with \( C_+ \) and \( C_- \) being \( nN \times nN \) matrices given by
\[ C_- = z_- \otimes I_N - i I_n \otimes H, \] (2.11)
\[ C_+ = z_+ \otimes I_N - i S \otimes H. \] (2.12)
Here \( z_+, z_- \) and \( S \) are \( n \times n \) matrices with elements
\[ z_- (\alpha \alpha') = \delta_{\alpha \alpha'} \left[ \frac{\varepsilon_-(\alpha)}{N} s_{\alpha} + ie_{\alpha} \right] \] (2.13)
\[ z_+ (\alpha \alpha') = \delta_{\alpha \alpha'} \left[ \frac{\varepsilon_+(\alpha)}{N} + ie_{\alpha} s_{\alpha} \right] \] (2.14)
\[ S (\alpha \alpha') = \delta_{\alpha \alpha'} s_{\alpha}. \] (2.15)
The RHS of (2.10) can be cast in the form
\[ (\det S)^{N/2} \left( \frac{\det M_+^{1/2}}{\det M_-^{1/2}} \right), \] (2.16)
where \( S = I_2 \otimes S \), and \( M_+ \) and \( M_- \) are, respectively, the \( 2nN \times 2nN \) symmetric and antisymmetric matrices given by
\[ M_+ = I_2 \otimes C_+ ; \quad M_- = \Upsilon [I_2 \otimes C_-], \] (2.17)
with \( \Upsilon \) being the \( 2n \times 2n \) antisymmetric matrix:
\[ \Upsilon = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}. \]
This leads to the expression
\[ J^S = (\det S)^{N/2} \left( \frac{\det (\Upsilon Z_- \otimes I_N - i \Upsilon \otimes H)}{\det (Z_+ \otimes I_N - i S \otimes H)} \right)^{1/2}, \] (2.18)
where the matrices, \( Z_\pm = I_2 \otimes z_\pm \), have the matrix elements
\[ Z_\pm (p \alpha | p' \alpha') = \delta_{\alpha \alpha'} \delta_{pp'} z_\pm (\alpha \alpha'), \] (2.19)
with \( p = 1, 2 \). Moreover,
\[ S (p \alpha | p' \alpha') = \delta_{\alpha \alpha'} \delta_{pp'} s_{\alpha}, \] (2.20)
\[ \Upsilon (p \alpha | p' \alpha') = \delta_{\alpha \alpha'} \gamma (pp'), \] (2.21)
where
\[ \gamma(pp) = 0 ; \quad \gamma(12) = 1; \quad \gamma(21) = -1. \] (2.22)

We now use the standard trick of expressing the square root of the determinant appearing in the numerator of the expression (2.18) as a Gaussian integral over Grassmannian variables, and for the one appearing in the denominator, a usual Gaussian integral over real variables. [See Appendix A for a summary of some useful identities]. In this way the generating function can be written as a superintegral.

\[ J^S = (\det S)^{N/2} \int D \Phi \langle e^{-\Phi (\Gamma Z \otimes I_N - \Gamma S \otimes H) \Phi} \rangle, \] (2.23)

where \( \Gamma, Z \) and \( S \) are \( 4n \times 4n \) matrices defined as follows:

For \( \sigma = +, - \):
\[ Z = Z_\sigma \delta_{\sigma\sigma'}; \quad S = S_\sigma \delta_{\sigma\sigma'}; \quad \Gamma = \Upsilon_\sigma \delta_{\sigma\sigma'}, \]

with
\[ \Upsilon_+ = \mathds{I}_{2n}, \quad \Upsilon_- = \Upsilon, \quad S_+ = S, \quad S_- = \mathds{I}_{2n}; \] (2.24)

and \( \Phi \) denotes a supervector
\[ \Phi = \begin{pmatrix} \varphi_+ \\ \varphi_- \end{pmatrix}, \]
i.e., \( \varphi_+ \) and \( \varphi_- \) are, respectively, vectors with real and Grassmannian elements, labeled by
\[ \varphi_{\pm i}(p\alpha) \quad \text{with} \quad i \in \{1, \ldots, N\}, \quad p \in \{1, 2\}, \quad \alpha \in \{1, \ldots, n\}. \]

At this point, it is important to note a basic difference between the sector of real variables corresponding to \( \varphi_+ \), and the Grassmannian sector corresponding to \( \varphi_- \). If in (2.13) and (2.14), all the energies \( e_\alpha \) are the same and \( \varepsilon_\alpha = 0 \) for all \( \alpha \in \{1 \ldots n\} \), then the integrand in (2.23) as well as the measure \( D\Phi \) remain invariant under the following change of variables:

\[ \varphi_{+i}(p\alpha) = \sum_{p', \alpha'} \tau(p\alpha|p'\alpha') \varphi_{+i}'(p'\alpha'), \] (2.25)

where \( \tau \) is a \( 2n \times 2n \) matrix satisfying the relation
\[ \tau^t S \tau = S; \] (2.26)

and
\[ \varphi_{-i}(p\alpha) = \sum_{p', \alpha'} U(p\alpha|p'\alpha') \varphi_{-i}'(p'\alpha'), \] (2.27)

where \( U \) is a \( 2n \times 2n \) matrix such that
\[ U^t \Upsilon U = \Upsilon. \] (2.28)
The matrices $\tau$ form a group. If there are $p$ variables $s_\alpha$ taking the value $+1$ and $q$ variables taking the value $-1$, then this group is $O(p, q)$. Unless $q = 0$ or $p = 0$, this is a non-compact hyperbolic group. Since the sign of $s_\alpha$ corresponds to the type of the Green’s function (i.e., advanced or retarded), it is evident that this hyperbolic symmetry comes into play when we compute the average of a product of $p$ advanced and $q$ retarded Green’s functions. In contrast, the matrices $U$ belong to the $2n$ symplectic group. We have chosen to extract the factors of $s_\alpha$ in the numerator of the expression (2.18), of the generating function, in order to emphasize the difference between these two sectors.

Using the expression, (2.9), of the mixed Hamiltonian $H$, we obtain

$$J^S = \left( \det S \right)^{N/2} \int D\Phi \int D\nu(A) \exp \left[ -\left( \Phi, (\Gamma Z \otimes \mathbb{I}_N - \Gamma S \otimes A)\Phi \right) \right]$$

$$\times \langle \exp(\Phi, (i\frac{\lambda}{N} S \otimes G)\Phi) \rangle,$$

where

$$D\nu(A) := \prod_{j=1}^{N} da_j \nu(a_j)$$

In (2.29) the angular brackets denote an averaging over the Gaussian Orthogonal ensemble.

By integrating over the Gaussian probability distribution of the matrix elements of $G$, we see that

$$\langle \exp(\Phi, (i\frac{\lambda}{N} S \otimes G)\Phi) \rangle = \exp \left( -\frac{\lambda^2}{2N^2} \text{tr} Y^2 \right),$$

where $Y$ is the $2n \times 2n$ matrix defined as follows:

$$Y = \sum_{\sigma=+,-} Y_\sigma,$$

with

$$Y_\sigma(ij) = Y_\sigma(ji) = \sum_{p,p'=1}^{n} \sum_{\alpha=1}^{n} \varphi_{\alpha}(p) \varphi_{\alpha}(p') \Upsilon_\sigma(\alpha)pS_{\sigma}(\alpha).$$

Hence,

$$J^S = \left( \det S \right)^{N/2} \int D\varphi_+ \int D\varphi_- \int D\nu(A) \exp \left[ -\left( \varphi_+, (Z_+ \otimes \mathbb{I}_N - iS \otimes A)\varphi_+ \right) \right]$$

$$\exp \left[ -\left( \varphi_-, (\Upsilon_- \otimes \mathbb{I}_N - i\Upsilon_- \otimes -A)\varphi_- \right) \right] e^{-\frac{\lambda^2}{2N^2} \text{tr} Y^2}$$

Since $\lambda$ appears on the RHS of (2.31) only in the form of $\lambda^2$, it follows that we can always choose $\lambda$ to be positive. We shall make this choice for the rest of the paper.

We can now decompose $\text{tr} Y^2$ in the following way:

$$\text{tr} Y^2 = \text{tr} (L_+ S)^2 + \text{tr} (L_- \Upsilon_-)^2 + 2 (\varphi_-, (\Upsilon_- \otimes T)\varphi_-),$$
where the $2n \times 2n$ matrices $L_\sigma$ are given by

$$L_\sigma(p\alpha|p'\alpha') = \sum_i \varphi_{\sigma i}(p\alpha) \varphi_{\sigma i}(p'\alpha') \tag{2.36}$$

and the $N \times N$ matrix $T$ is given by

$$T_{ij} = \sum_{p\alpha} \varphi_{+i}(p\alpha) \varphi_{+j}(p\alpha) s_{\alpha}. \tag{2.37}$$

We will consider correlations between energy levels around some energy $e$, on the scale of the mean level spacing, i.e., $1/(N\rho(e))$. For this purpose we decompose the energies $e_{\alpha}$ into

$$e_{\alpha} = e + \frac{r_{\alpha}}{N} \tag{2.38}$$

so that if we define the matrices $\rho_+$ and $\rho_-$ of elements

$$\rho_+(p\alpha|p'\alpha') = \delta_{\alpha\alpha'} \delta_{pp'} [\varepsilon_+(\alpha) + ir_{\alpha} s_{\alpha}], \tag{2.39}$$

$$\rho_-(p\alpha|p'\alpha') = \delta_{\alpha\alpha'} \delta_{pp'} [\varepsilon_-(\alpha) s_{\alpha} + ir_{\alpha}], \tag{2.40}$$

then the generating function can be written as

$$J_S = \left( \det S \right)^{N/2} e^{\frac{-N}{2} \text{tr} \left[ \rho_+ S + \rho_- S \right]} \int D\nu(a) \int D\varphi_+ \int D\varphi_- e^{-\frac{N^2}{2} \text{tr} \left[ L^t_\sigma S + \rho_+ S \right]^2}$$

$$\times \frac{N^2}{2} \text{tr} \left[ L_\sigma^t \varphi_+ \left( iS \otimes \frac{e1_{N-A}}{N} \right) \varphi_- \right] e^{-\frac{1}{N^2} \text{tr} \left[ Q_+ (iS \otimes \frac{e1_{N-A}}{N}) \varphi_+ \right]} e^{-\frac{1}{N^2} \text{tr} \left[ Q_- (iS \otimes \frac{e1_{N-A}}{N}) \varphi_- \right]} \tag{2.41}$$

2.2 Pseudo–Gaussian transformations

With the help of Gaussian integrations over auxiliary matrices $Q_+$ and $Q_-$, the exponents in the integrands [on the RHS of (2.41)] can be reduced to quadratic forms in $\varphi_+$ and $\varphi_-$. This procedure, which is analogous to the Hubbard Stratonovich transformation, yields the following identities:

$$e^{-\frac{N^2}{2} \text{tr} \left( L^t_\sigma S + \frac{1}{N} S \rho_+ \right)^2} = \frac{1}{d_+} \int DQ_+ e^{-\frac{1}{2} \text{tr} Q^2_+} e^{-\frac{1}{N^2} \text{tr} \left[ Q_+ (L^t_\sigma S + \frac{1}{N} S \rho_+) \right]}, \tag{2.42}$$

and

$$e^{-\frac{N^2}{2} \text{tr} \left( L^t_\sigma \varphi_- \frac{1}{N} \rho_- \right)^2} = \frac{1}{d_-} \int DQ_- e^{-\frac{1}{2} \text{tr} Q^2_-} e^{-\frac{1}{N^2} \text{tr} \left[ Q_- (L^t_\sigma \varphi_- - \frac{1}{N} \rho_-) \right]}, \tag{2.43}$$

with

$$d_\pm = \int DQ_\pm e^{-\frac{1}{2} \text{tr} Q^2_\pm}. \tag{2.44}$$

Our aim is to substitute the above identities on the RHS of (2.41) and change the order of integration over $\varphi_+, \varphi_-$ and $Q_+, Q_-$. This is because the resulting integrals
over $\varphi_+$ and $\varphi_-$ turn out to be Gaussian integrals, whose values are obtained by making use of the integral identities (A.1) and (A.2). However, the change in the order of the integrations imposes a restriction on the form of the matrix $Q_+$. It is required to be of the following form [as introduced in [16]]:

\[
Q_+ = \begin{pmatrix}
Q_{11} - i\sqrt{\delta^2 + Q_{12} Q_{21}} & Q_{12} \\
Q_{21} & Q_{22} + i\sqrt{\delta^2 + Q_{21} Q_{12}}
\end{pmatrix},
\]

(2.45)

The block structure refers to the decomposition of the diagonal matrix $S$ [2.13] into $p$ elements equal to +1 and $q$ elements equal to −1, so that $Q_{11}$ is a $2np \times 2np$ real symmetric matrix, $Q_{22}$ a $2nq \times 2nq$ real symmetric matrix, $Q_{12}$ is a $2nq \times 2np$ real matrix and the $2np \times 2nq$ matrix $Q_{21}$ satisfies

\[
Q_{21}^t = Q_{12},
\]

(2.46)

where the superscript $t$ denotes the transpose of the matrix. The variable $\delta$ appearing on the RHS of (2.45) denotes an arbitrary positive number. This structure of the matrix $Q_+$ is dictated by the invariance of the integral over $\varphi_+$ (on the RHS of (2.41)) under the pseudo-orthogonal group $O(p,q)$, where $p$ denotes the number of advanced Green’s functions and $q$ denotes the number of retarded Green’s functions.

We consider $Q_-$ to be Hermitian: $Q_{-}^\dagger = Q_{-}$, so for $n = 1$ it reduces to a real number. In addition, we require $Q_-$ to satisfy the following relation:

\[
(Q_- \Upsilon_-)^t = -(Q_- \Upsilon_-)
\]

(2.47)

This constraint is imposed so that the Gaussian integration over the Grassmannian variables $\{\varphi_{-j}\}$ can be expressed in terms of the determinant of a matrix, as in (A.2) [see (2.50) – (2.52) below].

The constraint (2.47) on the matrix $Q_-$ is imposed so as to ensure the validity of the identity (2.43). Indeed the latter identity, involving such a matrix $Q_-$, uses the fact that the linear term in $Q_-$, appearing in the exponential, is of the form $\text{tr} \left( Q_- E \Upsilon_- \right)$, with $E$ being an antisymmetric matrix. We can therefore write the generating function in the form

\[
J^S = \frac{(\det S)^{N/2}}{d_+ d_-} \int DQ_+ \int DQ_- \exp \left( -\frac{1}{2} \text{tr} \left[ Q_+ + iS \rho_+ \right] \right)^2 \times \exp \left( -\frac{1}{2} \text{tr} \left[ Q_- - \frac{\rho_-}{\lambda} \right] \right)^2 \tilde{K}(Q_+, Q_-)
\]

(2.48)

with

\[
\tilde{K}(Q_+, Q_-) = \int D\nu(a) \int D\varphi_+ e^{-\left( \varphi_+ \left( iS \otimes (e_1 I_N - A) \right) \varphi_+ \right)} e^{-\left( \varphi_+ \left( iA \otimes S Q_+ \otimes I_N \right) \varphi_+ \right)} \times \mathbb{I}_-, \quad (2.49)
\]
where $\mathbb{I}_-$ is the integral over the Grassmannian variables $\{\varphi_j\}$ and is given by
\[
\mathbb{I}_- = \int D\varphi_- \exp \left[ - (\varphi_-, R\varphi_-) \right],
\]
(2.50)
with
\[
R := \frac{\lambda}{N} (\Upsilon_- Q_- \otimes \mathbb{I}_N + \Upsilon_\otimes \left( i(e\mathbb{I}_N - A) + \frac{\lambda^2}{N^2} T \right)).
\]
(2.51)
The matrix $T$ is defined by (2.37). In the following section we evaluate the integral $\mathbb{I}_-$ and show that $\hat{K}(Q_+, Q_-)$ depends only on the eigenvalues of the matrices $Q_+$ and $Q_-$.  

2.3 Evaluation of the integral $\mathbb{I}_-$

Note that the matrix $R$, defined by (2.51), is antisymmetric. This follows from our choice [(2.47)] of $(Q_- \Upsilon_-)$ to be an antisymmetric matrix. Hence, using the Gaussian identity, (A.2), we obtain
\[
\mathbb{I}_- = \sqrt{\det R}.
\]
(2.52)
In fact, the constraint (2.47) on the matrix $Q_-$ was imposed so as to obtain the above result.

Moreover, we can show, as follows, that the matrix $R$ depends only on the eigenvalues of the matrix $Q_-$. Defining an $N \times N$ matrix $A_1$:
\[
A_1 := i(e\mathbb{I}_N - A) + \frac{\lambda^2}{N^2} T,
\]
(2.53)
we write
\[
\det R = \det (\Upsilon_- \otimes \mathbb{I}_N) \det \left[ \frac{\lambda}{N} Q_- \otimes \mathbb{I}_N + \mathbb{I}_{2n} \otimes A_1 \right],
\]
\[
= \det (\Upsilon_- \otimes \mathbb{I}_N) \det \left[ \frac{\lambda}{N} q_- \otimes \mathbb{I}_N + \mathbb{I}_{2n} \otimes A_1 \right]
\]
\[
=: \det C,
\]
(2.54)
where $q_-$ is the diagonal matrix whose diagonal elements are the eigenvalues of $Q_-$.  

The elements of the matrix $C$, (2.54), are given by
\[
C(p\alpha, j|j\alpha', j') = \Upsilon_- (pp') \left[ \frac{\lambda}{N} q_-(\alpha)\delta_{\alpha\alpha'} \delta_{jj'} + \delta_{\alpha\alpha'} A_1(jj') \right],
\]
(2.55)
where $q_-(\alpha) \equiv q_-(p\alpha)$. This follows from the fact that the eigenvalues of $Q_-$ are doubly degenerate (as shown in Appendix B). Hence, the matrix $C$ is antisymmetric in the label $p$ and has the form
\[
C = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}
\]
where $D$ is an $nN \times nN$ matrix defined as follows:

$$D := \frac{\lambda}{N} \tilde{q}_- \otimes \mathbb{I}_N + \mathbb{I}_n \otimes A_1. \tag{2.56}$$

In (2.56), $\tilde{q}_-$ denotes an $n \times n$ diagonal matrix with diagonal elements $\tilde{q}_-\alpha = q_-\alpha$. This leads to the result

$$\mathbb{I}_- = \det D \tag{2.57}$$

From (2.53) and (2.56) it follows that

$$\det D = \prod_{\alpha=1}^{n} \det \left( \tilde{q}_-\alpha \frac{\lambda}{N} N + i(e \mathbb{I}_N - A) + \frac{\lambda^2}{N^2} T \right). \tag{2.58}$$

For each $\alpha \in \{1, \ldots, n\}$, let us define an $N \times N$ diagonal matrix

$$b_\alpha := \tilde{q}_-\alpha \frac{\lambda}{N} N + i(e \mathbb{I}_N - A) \tag{2.59}$$

In terms of this matrix, we can write

$$\mathbb{I}_- \equiv \det D = \prod_{\alpha} \det b_\alpha \det \left( \mathbb{I}_N + \frac{\lambda^2}{N^2} b^{-1}_\alpha T \right) \tag{2.60}$$

For each $\alpha$, let $F_\alpha$ denote a $2n \times 2n$ matrix whose elements are given by

$$F_\alpha(p\gamma|p'\gamma') := \sum_{j=1}^{N} \left( b^{-1}_{\alpha} \right) j \varphi_{+j}(p\gamma) \varphi_{+j}(p'\gamma') \tag{2.61}$$

It is easy to see that

$$\text{tr} \left( b^{-1}_\alpha T \right)^j = \text{tr} \left( SF_\alpha \right)^j \tag{2.62}$$

for any arbitrary integer $j$. Hence by (A.7) we have the identity

$$\det \left( \mathbb{I}_N + \frac{\lambda^2}{N^2} b^{-1}_\alpha T \right) = \det \left( \mathbb{I}_{2n} + \frac{\lambda^2}{N^2} SF_\alpha \right) \tag{2.63}$$

Hence from (2.60), (2.59) and (2.63) it follows that

$$\mathbb{I}_- = \prod_{j=1}^{N} \det \left( \frac{\lambda}{N} \tilde{q}_- + i(e - a_j) \mathbb{I}_n \right) \prod_{\alpha=1}^{n} \det \left( \mathbb{I}_{2n} + \frac{\lambda^2}{N^2} SF_\alpha \right) \tag{2.64}$$

where $a_j$ denotes a diagonal element of the diagonal matrix $A$. Using the Grassmannian integral representation (A.4) for a determinant we can write

$$\prod_{\alpha} \det \left( \mathbb{I}_{2n} + \frac{\lambda^2}{N^2} SF_\alpha \right) = \int D\overline{\Psi} D\Psi e^{-\langle \overline{\Psi},\Psi \rangle} e^{-\langle \overline{\Psi},B\Psi \rangle}, \tag{2.65}$$
where \( \Psi(\Psi) \) is a column (row) vector of length \( 2n^2 \), and
\[
(\overline{\Psi}, B\Psi) = \sum_{\alpha=1}^{n} \sum_{\beta,\beta'=1}^{2n} \overline{\Psi}_\alpha(\beta) B_\alpha(\beta\beta') \Psi_\alpha(\beta'),
\] (2.66)
where \( \beta \) now refers to the double index \((p\gamma)\) and \( B_\alpha \) denotes a \( 2n \times 2n \) matrix whose elements are given by
\[
B_\alpha(\beta\beta') = \frac{\lambda^2}{N^2} S(\beta) F_\alpha(\beta\beta'),
\] (2.67)
with \( S(\beta) = s_\gamma \). The relations (2.64) and (2.65) yield the following expression for the integral \( \mathbb{I}_- \):
\[
\mathbb{I}_- = \int D\overline{\Psi} D\Psi e^{-\overline{\Psi}\Psi} \prod_{j=1}^{N} \det \left( \frac{\overline{\lambda}}{N} \tilde{q}_- + i(e - a_j) \mathbb{I}_n \right) \exp \left( -\left[ \frac{\lambda^2}{N^2} \sum_{j=1}^{N} \sum_{\beta,\beta'=1}^{2n} \varphi_{+j}(\beta) \varphi_{+j}(\beta') S_\beta \sum_{\alpha=1}^{n} \left[ b^{-1}_\alpha \right]_j \overline{\Psi}_\alpha(\beta) \Psi_\alpha(\beta') \right] \right).
\] (2.68)

We can now insert the representation of \( \mathbb{I}_- \), given by (2.68), in the expression (2.49) for \( \hat{K} \), and perform the integration over \( \varphi_+ \). This yields
\[
\hat{K}(Q_+, Q_-) = \int D\overline{\Psi} D\Psi e^{-\overline{\Psi}\Psi} \int D\nu(a) \left[ \prod_{j=1}^{N} \det \left( \frac{\overline{\lambda}}{N} \tilde{q}_- + i(e - a_j) \mathbb{I}_n \right) \right] \prod_{j=1}^{N} \det \left[ S \left( i(e - a_j) \mathbb{I}_{2n} + i \frac{\lambda}{N} Q_+ + \frac{\lambda^2}{N^2} R_j \right) \right]^{-\frac{1}{2}},
\] (2.69)
where each \( R_j \) is a \( 2n \times 2n \) matrix with elements
\[
R_j(\beta\beta') = \sum_{\alpha=1}^{n} \left[ b^{-1}_\alpha \right]_j \overline{\Psi}_\alpha(\beta) \Psi_\alpha(\beta')
\[
= \sum_{\alpha=1}^{n} \left( \frac{\lambda}{N} \tilde{q}_{-\alpha} \mathbb{I}_N + i(e \mathbb{I}_N - A) \right)^{-1} \left[ b^{-1}_\alpha \right]_j \overline{\Psi}_\alpha(\beta) \Psi_\alpha(\beta')
\] (2.70)
From the definition (2.30) of the measure \( D\nu(a) \), it follows that the expression for \( \hat{K}(Q_+, Q_-) \) involves a product of \( N \) identical integrals and hence can be written in the form:
\[
\hat{K}(Q_+, Q_-) = \left( \det S \right)^{-N/2} \int D\overline{\Psi} D\Psi e^{-\overline{\Psi}\Psi} Z^N,
\] (2.71)
where, writing \( a \) for \( a_j \) and \( \mathcal{R} \) for \( R_j \), we define
\[
Z := \int da \nu(a) \det \left( \frac{\lambda}{N} \tilde{q}_- + i(e - a) \mathbb{I}_n \right) \det \left[ S \left( i(e - a) \mathbb{I}_{2n} + i \frac{\lambda}{N} Q_+ + \frac{\lambda^2}{N^2} \mathcal{R} \right) \right]^{-\frac{1}{2}}.
\] (2.72)
That this expression depends only on the eigenvalues $q_+(p\alpha)$ of $Q_+$ can be easily seen by making the following change of variables in the Grassmannian variables

$$\chi_\alpha(\beta) = \sum_{\beta''} G^{-1}(\beta\beta'') \Psi_\alpha(\beta'') \quad (2.73)$$

$$\chi_\alpha(\beta') = \sum_{\beta} G^{t}(\beta'\beta) \Psi_\alpha(\beta), \quad (2.74)$$

where $G$ is the matrix which diagonalizes $Q_+$, i.e.,

$$Q_+ = G q_+ G^{-1}, \quad (2.75)$$

and $q_+$ is a $2n \times 2n$ diagonal matrix with diagonal elements $q_+(p\alpha)$. Hence $\tilde{K}$ takes the form

$$\tilde{K} = (\det S)^{-N/2} \int D\chi D\chi e^{-(\chi,\chi)} P_N \quad (2.76)$$

where

$$P := \int da \nu(a) \det \left( \frac{\lambda}{N} \tilde{q} + i(e-a)I_n \right) \det \left[ i(e-a)I_{2n} + \frac{\lambda}{N} q_+ + \frac{\lambda^2}{N^2} R \right]^{-1/2}, \quad (2.77)$$

with

$$R(\beta\beta') = \sum_{\alpha=1}^n \left( \frac{\lambda}{N} q_{-\alpha} + i(e-a) \right)^{-1} \chi_\alpha(\beta) \chi_\alpha(\beta'). \quad (2.78)$$

We will cast $P$ in a slightly simpler form by making use of the degeneracy of the eigenvalues of $Q_-$. Let $X_\pm$ be the $2n \times 2n$ diagonal matrices of diagonal elements

$$x_+(p\gamma) := i(e-a) + \frac{\lambda}{N} q_+(p\gamma); \quad (2.79)$$

and

$$x_-(p\gamma) \equiv x_-(\gamma) := i(e-a) + \frac{\lambda}{N} q_-(\gamma). \quad (2.80)$$

We can rewrite $P \quad (2.77)$ as

$$P = \int da \nu(a) \left[ \det X_+ \right]^{1/2} \det \left( I_{2n} + \frac{\lambda^2}{N^2} X_+^{-1} R \right)^{-1/2}. \quad (2.81)$$

The matrix elements of $(X_+^{-1} R)$ are

$$(X_+^{-1} R)(p\gamma|p'\gamma') = \frac{1}{x_+(p\gamma)} \sum_{\alpha=1}^n \frac{\chi_\alpha(p\gamma) \chi_\alpha(p\gamma)}{x_-(\alpha)} \quad (2.82)$$

It is easy to see that for any arbitrary integer $j$

$$\text{tr} \left( X_+^{-1} M \right)^j = -\text{tr} \left( X_-^{-1} M \right)^j, \quad (2.83)$$
where $M$ is the $n \times n$ matrix of elements

$$M_{\alpha_1 \alpha_2}(q) = \sum_{p=1}^{2} \sum_{\gamma=1}^{n} \frac{X_{\alpha_1}(p, \gamma) X_{\alpha_2}(p, \gamma)}{X_+(p, \gamma)}, \quad (2.84)$$

and $x_-$ is a diagonal $n \times n$ matrix of elements $x_-(\gamma)$. We can therefore write $P$ as

$$P = \int da \nu(a) \left[ \frac{\text{det} X_-}{\text{det} X_+} \right]^{1/2} \text{det} \left( \mathbb{I}_n + \frac{\lambda^2}{N^2} M(X_+) \right)^{1/2} \cdot \quad (2.85)$$

Hence, the multiple integral $\tilde{K}(Q_+, Q_-)$, defined through (2.49), is seen to depend only on the eigenvalues of the matrices $Q_+$ and $Q_-$. It is given by (2.70), with $P$ being given by (2.85), above.

Hence, the generating function is given by

$$J^S = (\text{det} S)^{N/2} \int DQ_+ \int DQ_- \exp \left( -\frac{1}{2} \text{tr} \left[ Q_+ + i \frac{S \rho_+}{\lambda} \right] \right)^2 \times \exp \left( -\frac{1}{2} \text{tr} \left[ Q_- - \frac{\rho_-}{\lambda} \right] \right)^2 \tilde{K}(Q_+, Q_-), \quad (2.86)$$

where

$$\tilde{K} = (\text{det} S)^{-N/2} \int D\chi D\bar{\chi} e^{-\langle \chi, \chi \rangle} P^N, \quad (2.87)$$

and $P$ is given by (2.85).

**2.4 The unitary case for finite matrix size**

When the Hamiltonian matrix (2.9),

$$H = A + \frac{\lambda}{N} G,$$

is such that $G$ belongs to the Gaussian Unitary ensemble (GUE), we arrive at an analogous expression for the generating function.

In this case, we once again proceed from the expression (2.10), where the angular brackets now denote an averaging over the GUE. We directly cast this expression into integrals over complex and Grassmannian variables by making use of the integral identities (A.3) and (A.4). This yields

$$J^S = (\det S)^N \int D\varphi_+ D\varphi_- \langle e^{-(\varphi_+ C_+ \varphi_+)} e^{-(\varphi_- C_- \varphi_-)} \rangle, \quad (2.88)$$

where the matrices $C_+, C_-$ and $S$ are defined through (2.11) – (2.15).

The symmetry of the matrices $G$ with respect to unitary transformations allow us to use complex variables. There is no necessity to double the dimension of the matrices so as to accommodate real variables, as we did in the case of the GOE. Hence the label $p$ which appeared in the relations for the GOE, and resulted from
this doubling of the dimension, do not appear in the corresponding relations for the GOE. All $2n \times 2n$ matrices appearing in the case of the GOE are replaced by analogous $n \times n$ matrices labeled by a single index $\alpha$.

Evaluating the average over the GUE in (2.88), we obtain an expression similar to (2.41), with the following changes
\[(\det S)^{N/2} \rightarrow (\det S)^N\]
\[\Upsilon_- \rightarrow I_n\] (2.89)
(2.90)

We notice that in this case the symmetry in the Grassmannian sector is simply the unitary one. This implies that the matrix $Q_-$ is now simply an $n \times n$ self-adjoint matrix. The $n \times n$ matrix $Q_+$ has the same block structure as before but now $Q_{ii}$, with $i = 1, 2$, are self-adjoint matrices and
\[Q_{12} = Q_{21}^\dagger.\] (2.91)

By proceeding exactly as before, we finally arrive at the following expression for the generating function:
\[J^S = \frac{1}{d_+ d_-} \int DQ_+ \int DQ_- \exp \left( -\frac{1}{2} \text{tr} (Q_+ + i\frac{\rho_+ S}{A})^2 \right) \exp \left( -\frac{1}{2} \text{tr} (Q_- - \frac{\rho_-}{A})^2 \right) \widetilde{K}(q_+, q_-)\] (2.92)

where
\[\widetilde{K} = \int D\overline{\chi} D\chi e^{-\langle \overline{\chi}, \chi \rangle} P^N.\] (2.93)

Here $\overline{\chi}, \chi$ are Grassmannian vectors of length $n$, and $P$ is given by
\[P := \int da \nu(a) \left[ \frac{\det X_-}{\det X_+} \right] \det \left( I_n + \frac{\lambda^2}{N^2} M(X_+) \right),\] (2.94)

where
\[M_{\alpha_1 \alpha_2} = \sum_{\gamma=1}^{n} \frac{\chi_{\alpha_1}(\gamma) \overline{\chi}_{\alpha_2}(\gamma')}{X_+(\gamma)};\] (2.95)

and $X_\pm$ are diagonal $n \times n$ matrices with diagonal elements
\[x_+(\gamma) := i(e - a) + i\frac{\lambda}{N} q_+(\gamma)\] (2.96)
\[x_-(\gamma) := i(e - a) + \frac{\lambda}{N} q_-(\gamma)\] (2.97)

respectively. Equivalently, we could also write $P$ in a form analogous to (2.77),
\[P := \int da \nu(a) \det \left( \frac{\lambda}{N} q_- + i(e - a) I_n \right) \det \left[ i(e - a) I_n + i\frac{\lambda}{N} q_+ + \frac{\lambda^2}{N^2} R \right]^{-1},\] (2.98)
with the matrix $\mathcal{R}$ being defined by (2.78) as before, but the label $\beta$ taking the values 1 to $n$ only.

The above relations, (2.92) – (2.94), for the GUE, are found to be very similar in form to the corresponding relations (2.48) – (2.85) for the GOE. The only difference lies in the fact that the expression for $P$ involves square roots of determinants for the GOE [see (2.85)], whereas there is no square root appearing in the corresponding relation (2.94), in the case of the GUE.

2.5 Supersymmetric formulation for the orthogonal and unitary case

Up to now we have used Grassmannian variables only as auxiliary variables which are to be finally eliminated by integration. We can, however, alternatively, cast the expressions for the generating function in a supersymmetric form, where Grassmannian and ordinary variables are put on the same footing. If we introduce the usual parameter $\beta$ taking the value 1 for the orthogonal case and 2 for the unitary one, then the generating function in both cases can be expressed in the following elegant form:

$$J^S_{\beta} = \frac{\int DQ^\beta \exp \left(-\frac{1}{2} \text{Str}(Q^\beta - p^\beta / \lambda)^2\right) F^\beta(Q^\beta)^N}{\int DQ^\beta \exp \left(-\frac{1}{2} \text{Str}(Q^\beta)^2\right)}$$  \hspace{1cm} \text{(2.99)}

where

$$F^\beta(Q^\beta) = \int da \nu(a) \text{Sdet} \left( (e^{-a})_{4n/\beta} + \frac{\lambda}{N} Q^\beta \right)^{-\beta/2},$$  \hspace{1cm} \text{(2.100)}

the $4n/\beta \times 4n/\beta$ supermatrix $Q^\beta$ being given by

$$Q^\beta = \begin{pmatrix} Q^\beta_+ & Q^\beta_- \\ Q^\beta_- & -iQ^\beta_+ \end{pmatrix},$$  \hspace{1cm} \text{(2.101)}

and $p^\beta$ denoting the diagonal supermatrix

$$p^\beta = \begin{pmatrix} -i\rho_+ S^\beta & 0 \\ 0 & -i\rho_- \end{pmatrix},$$  \hspace{1cm} \text{(2.102)}

Here

$$S^\beta = \mathbb{I}_2 \otimes S \quad \text{for} \quad \beta = 1$$

$$= S \quad \text{for} \quad \beta = 2.$$  \hspace{1cm} \text{(2.103)} \hspace{1cm} \text{(2.104) \hspace{1cm} (2.105)}

The measure of the superintegral is

$$DQ^\beta := DQ^\beta_+ DQ^\beta_- DQ^\beta_+ DQ^\beta_+.$$
Let us show this in the orthogonal case. We start from the expression (2.71) for $\hat{K}$ and we note that $Z$ [(2.72)] can be written as

$$
(detrS)^{-\frac{1}{2}} \int da \nu(a) C^{-\frac{1}{2}} \quad (2.106)
$$

where $C$ can be written as

$$
C = \text{det} \left[ (e-a)I_{2n} - i\frac{\lambda}{N} Q^- \right]^{-1} \\
\times \text{det} \left[ (e-a)I_{2n} - \frac{\lambda}{N} Q_+ - \Lambda U^{-1} \left( (e-a)I_{2n} - i\frac{\lambda}{N} Q_- \right)^{-1} U \right],
$$

(2.107)

where

$$
\Lambda(\beta|p\alpha) = \frac{1}{\sqrt{2}} \Psi^\alpha(\beta) \quad (2.108)
$$

$$
\Lambda(p\alpha|\beta) = \frac{1}{\sqrt{2}} \Psi_\alpha(\beta), \quad (2.109)
$$

for $\alpha = 1 \ldots n$, and $U$ is the matrix which diagonalizes $Q_-$. In this form we see that $C$ is a superdeterminant:

$$
C = \text{Sdet} \left( (e-a)I_{4n} + \frac{\lambda}{N} Q \right), \quad (2.110)
$$

where

$$
Q = \begin{pmatrix} Q_+ & Q_{+-} \\ Q_{-+} & -iQ_- \end{pmatrix}, \quad (2.111)
$$

and

$$
Q_{+-} = \Lambda U^{-1} \quad Q_{-+} = U \Lambda. \quad (2.112)
$$

Noting that $\text{tr} (Q_{+-} Q_{-+}) = (\Psi, \Psi)$, we can easily see that the exponential term in $J^S$, (2.92), is indeed of the form

$$
-\frac{1}{2} \text{Str} (Q - \frac{P}{\lambda})^2.
$$

The unitary case is treated in the same way.

We would like to stress that here the supersymmetric formalism gives us only an elegant notation. The formalism itself is only useful when we want to mix Grassmannian and real or complex variables. Such a mixing arises in the standard case in which the matrix $A = 0$. This case has been treated by Efetov [3], by using the supersymmetric technique. In the large-$N$ limit a saddle point is constructed which mixes up the Grassmannian and ordinary variables. The supersymmetric technique is also useful in the case where $\lambda = \sqrt{N}$ and the matrix $A$ is non-zero, where one
can show that the energy level correlations agree with the predictions of standard Random Matrix theory. However, in the model that we are mainly concerned with in this paper, namely the one in which $\lambda$ remains fixed for large $N$, the supersymmetric formulation does not give us any additional advantage.

### 2.6 The orthogonal and unitary case in the limit of infinite matrix size

We will now evaluate the generating function in the limit $N \to \infty$, from which the correlation functions can be obtained by taking various derivatives with respect to the variables $\varepsilon_-(\alpha)$. It can be easily shown that there is no problem in the interchange of the limit $N \to \infty$, with the derivation with respect to $\varepsilon_-$, and the subsequent limit $\varepsilon_- = \varepsilon_+ \to 0$.

The result for the generating function both in the orthogonal ($\beta = 1$) and the unitary case ($\beta = 2$) can be put in the following form:

$$J^S = \exp\left(\frac{i\omega_\beta}{2} \text{tr} \left(\varepsilon_- - \varepsilon_+\right) S\right) \int DQ_+ Dq_- D\chi D\bar{\chi} \exp\left(-\frac{1}{2} \text{tr} \left[Q_+ + \frac{i\rho_+ S}{\lambda} - \frac{\beta \lambda p_+^2}{2}\right]\right)$$

$$\times \exp\left(-\frac{1}{2} \text{tr} \left[Q_- - \frac{\rho_-}{\lambda} - \frac{i\beta \lambda p_-^2}{2}\right]\right) - (\bar{\chi}, \chi)$$

$$\times \exp \left(\lambda \nu(e) \left[\frac{\beta A_\alpha}{2} + A_1^\beta + A_2^\beta\right]\right).$$

(2.113)

Here $\varepsilon_\pm$ are $n \times n$ matrices with diagonal elements $\varepsilon_\pm(\alpha)$, and

$$p = p(e) = \mathbb{P} \int da \frac{\nu(a)}{a - e},$$

(2.114)

with the symbol $\mathbb{P}$ denoting the principal value of the integral. The matrix $A_0$ is given by

$$A_0 = i\pi \text{tr} (q_+ + iq_-) \sigma(q_+),$$

(2.115)

where $\sigma(q_+)$ is the diagonal matrix whose elements are the signs of the imaginary parts of the eigenvalues $q_+(p\gamma)$ of the matrix $Q_+$.

$$A_1^\beta = \int_{-\infty}^{+\infty} dt \left\{ \left(\frac{\det(t - i q_-)}{\det(t + q_+)}\right)^\frac{\beta}{2} - \left[1 - \frac{\beta}{2} \text{tr} \frac{q_+ + iq_-}{t + q_+}\right]\right\}$$

(2.116)

and

$$A_2^\beta = \int_{-\infty}^{+\infty} dt \left(\frac{\det(t - i q_-)}{\det(t + q_+)}\right)^\frac{\beta}{2} \left[\det(1 - R_\alpha^\beta)^\frac{\beta}{2} - 1\right].$$

(2.117)

where

$$R_{\alpha\alpha'}^\lambda = \frac{1}{t - iq_-(\alpha)} \sum_{p=1}^{2} \sum_{\gamma=1}^{n} \chi_{\alpha}(p\gamma) \chi_{\alpha'}(p\gamma) \frac{\chi_{\alpha}(p\gamma)}{t + q_+(p\gamma)}$$

(2.118)
and

\[ R_{\alpha\alpha'}^2 = \frac{1}{t - iq_-(\alpha)} \sum_{\gamma=1}^{n} \frac{\bar{\chi}_\alpha(\gamma)\chi_{\alpha'}(\gamma)}{t + q_+(\gamma)} \]  

(2.119)

This is the main result of this paper. We have expressed the generating function for the correlation functions in terms of a finite set of integrals. Hence we have reduced the problem of the computation of the generating function, in the limit of infinite matrix size, to that of the evaluation of a finite set of integrals. This was our main purpose, since, starting from this explicit expression, we can proceed to evaluate the physically relevant correlation functions. However, as we shall see, the task of evaluating these integrals is non-trivial. Nevertheless, a general conclusion can be drawn from this expression by noting that the generating function has the following structure:

\[ J^S = \exp\left(\frac{ip\beta}{2} \text{tr} \left( (\varepsilon_- - \varepsilon_+)S \right) \right) K^S \left( v\varepsilon_-, v\varepsilon_+; \{ v\alpha + \frac{\lambda^2\beta\nu p}{2}, \Lambda \} \right) , \]  

(2.120)

where \( \Lambda = \lambda \nu(e) \), can be called the renormalized coupling constant.

Since the correlation functions can be computed from the generating function by using the formula

\[ \left( \frac{1}{2\pi} \right)^n \prod_{\alpha=1}^{n} \frac{\partial}{\partial \varepsilon_- (\alpha)} J^{(I_{n},-I_{n})} \bigg|_{\varepsilon_- = \varepsilon_+ = 0} = \rho^{(n)}(r_1, \ldots, r_n), \]  

(2.121)

where

\[ J^{(I_{n},-I_{n})} = \left( \prod_{\alpha=1}^{n} \frac{\det [\varepsilon_-^2(\alpha) + (e_\alpha - H)^2]}{\det [\varepsilon_+^2(\alpha) + (e_\alpha - H)^2]} \right), \]  

(2.122)

is positive, it follows that

\[ J^{(I_{n},-I_{n})} = |K^{(I_{n},-I_{n})}|, \]

and therefore \( \rho^{(n)} \) has the structure

\[ \rho^{(n)}(r_1, \ldots, r_n) = v^n \ f^{(n)}_{\beta}(r_1\nu + a, r_2\nu + a, \ldots, r_n\nu + a; \Lambda), \]  

(2.123)

with \( a = \lambda^2\beta/2 \). However, since the correlation functions are translation invariant, the RHS of (2.123) does not depend on \( a \).

We shall prove that the density of states \( \rho(e) \) is equal to \( \nu(e) \). We can therefore conclude that, on the scale of energy where the mean level spacing is equal to unity, the correlation functions are universal, i.e., they depend only on \( \beta \) and \( \Lambda \). More precisely,

\[ \left[ \frac{1}{\rho(e)} \right]^n \rho^{(n)}(r_1, \ldots, r_n) = f^{(n)}_{\beta}(r_1, r_2, \ldots, r_n; \Lambda). \]  

(2.124)

Let us now derive eqn. (2.113). We decompose \( P \), as given by

\[ P = \int da \nu(a) \left( \frac{\det X_+}{\det X_-} \right)^{\frac{a}{2}} \left( \frac{\det \left[ 1 + \frac{\lambda^2M(X_+)}{N^2X_-} \right]}{\det X_-} \right)^{\frac{a}{2}}, \]  

(2.125)
into three terms, i.e., \( P = P_0 + P_1 + P_2 \), where

\[
P_0 := 1 + \frac{\beta}{2} \int da \nu(a) \text{tr} \frac{X_- - X_+}{X_+},
\]

(2.126)

\[
P_1 := \int da \nu(a) \left\{ \left( \frac{\det X_-}{\det X_+} \right)^{\beta/2} - \left[ 1 + \frac{\beta}{2} \text{tr} \frac{X_- - X_+}{X_+} \right] \right\}
\]

(2.127)

and

\[
P_2 := \int da \nu(a) \left\{ \left( \frac{\det X_-}{\det X_+} \right)^{\beta/2} \right\} \left\{ \left[ \det \left( 1 + \frac{\lambda^2 M(X_+)}{N^2 X_-} \right) \right]^{\frac{\beta}{2}} - 1 \right\}.
\]

(2.128)

Let us first evaluate \( P_0 \) in the large-\( N \) limit.

\[
\int da \nu(a) \text{tr} \frac{X_- - X_+}{X_+} = -\frac{\lambda}{N} \int da \nu(a) \text{tr} \left[ (q_+ + iq_-) \left( (e - a) I_{2n} + \frac{\lambda}{N} q_+ \right)^{-1} \right].
\]

(2.129)

Since \( \text{Im} q_+(\gamma) \neq 0 \), the integral on the RHS of (2.129) tends to the expression

\[-p(e) - i\pi \nu(e) \sigma_+(p\gamma)\]

as \( N \to \infty \), where \( \sigma_+(p\gamma) \) denotes the sign of the imaginary part of the eigenvalue \( q_+(p\gamma) \).

Hence, for large \( N \),

\[
P_0 = 1 + \frac{\lambda \beta}{N} \left[ \text{ptr} (q_+ + i q_-) + i \pi \nu \text{tr} (q_+ + i q_-) \sigma(q_+) \right],
\]

(2.130)

where \( \sigma(q_+) \) is the diagonal matrix with elements \( \sigma_+(p\gamma) \). In the second term, \( P_1 \), we make the change of variables \( e - a = t\lambda/N \), so that it reads

\[
P_1 = \frac{\lambda}{N} \int_{-\infty}^{+\infty} dt \nu(e - t\lambda/N) \left\{ \left( \frac{\det(t - iq_-)}{\det(t + q_+)} \right)^{\frac{\beta}{2}} - \left[ 1 - \frac{\beta}{2} \text{tr} \frac{q_+ + iq_-}{t + q_+} \right] \right\}.
\]

(2.131)

The term in the paranthesis is bounded in \( t \) and decays like \( 1/t^2 \) when \( t \) is large, since \( \text{Im} q_+(\gamma) \neq 0 \). Hence, we can use the dominated convergence theorem \([14]\) to show that if

\[
\sup_t \nu(t) < \infty,
\]

and \( \nu(t) \) is continuous, then for large \( N \),

\[
P_1 = \frac{\lambda \nu(e)}{N} A_1^\beta.
\]

(2.132)
The term \( P_2 \) is treated in exactly the same way as \( P_1 \), so that asymptotically,

\[
P_2 = \frac{\lambda \nu(e)}{N} A^\beta_2. \tag{2.133}
\]

Finally,

\[
\lim_{N \to \infty} P^N = \exp \left( \frac{\lambda \beta p}{2} \text{tr} (Q_+ + iQ_-) + \lambda \nu \left( \frac{\beta}{2} A_0 + A^\beta_1 + A^\beta_2 \right) \right). \tag{2.134}
\]

Here and henceforth, we write \( \nu \) for \( \nu(e) \). The expression given by eqn. (2.113) for \( J^S \) is obtained by completing the square in \( Q_+ \) and \( Q_- \).  

2.7 The density of states and the average of the product of traces of advanced Green’s functions

The only computation which is easy in the general case, is that of the generating function for traces of advanced Green’s functions. This corresponds to the choice \( s_\alpha = 1 \) for all \( \alpha \in \{1 \ldots n \} \).

Let \( q_\pm(j) \) denote the eigenvalues of the matrices \( Q_\pm \). For the above-mentioned choice of the matrix \( S \), we know that \( q_+(j) = q'_+(j) - i\delta \), where \( q'_+(j) \) is real and \( \delta \) is positive. Since \( q_-(j) \) is also real (and doubly degenerate in the \( \beta = 1 \) case) we see that the integrands in the expressions for \( A^\beta_1 \) and \( A^\beta_2 \) are analytic in the variable \( t \) in the lower half-plane, and decay like \( 1/t^2 \). We can therefore apply Cauchy’s theorem to simply conclude that \( A^\beta_1 = A^\beta_2 = 0 \). In contrast, if we computed these quantities for the case of a mixed product of advanced and retarded Green’s functions \( (s_\alpha = 1, \alpha = 1 \ldots p, s_\alpha = -1, \alpha = p + 1 \ldots n) \), there would be \( n - p \) singularities in the lower half-plane, and, therefore, \( A^\beta_1 \) and \( A^\beta_2 \) would be non-zero.

Hence, it follows easily from (2.113) that the generating function factorizes as follows:

\[
J^I_n = \exp \left( i(p + i\pi \nu) \sum_{\alpha=1}^{n} [\varepsilon_-(\alpha) - \varepsilon_+(\alpha)] \right) J_+ J_- \tag{2.135}
\]

where

\[
J_+ = \frac{1}{d_+} \int DQ_+ \exp \left( -\frac{1}{2} \text{tr} \left[ Q_+ + \frac{i\rho_+ S}{\lambda} - \frac{\beta \lambda}{2} (p + i\pi \nu) \right]^2 \right), \tag{2.136}
\]

and

\[
J_- = \frac{1}{d_-} \int DQ_- \exp \left( -\frac{1}{2} \text{tr} \left[ Q_- - \frac{\rho_-}{\lambda} - \frac{i\beta \lambda}{2} (p + i\pi \nu) \right]^2 \right). \tag{2.137}
\]

Hence, we see from the definitions of \( d_\pm \) [(2.44)] that \( J_\pm = 1 \) and

\[
J^I_n = \exp \left( i(p + i\pi \nu) \sum_{\alpha=1}^{n} [\varepsilon_-(\alpha) - \varepsilon_+(\alpha)] \right). \tag{2.138}
\]

This implies that

\[
\lim_{N \to \infty} \left( \prod_{\alpha=1}^{n} \frac{1}{N} \text{tr} G^+_{e_\alpha} \right) = [-p - i\pi \nu]^n, \tag{2.139}
\]
which in turn shows that average of a product of the traces of advanced (or retarded) Green’s functions factorize. In particular, we see that the density of states, \( \rho(e) \), is simply given by

\[
\rho(e) = \nu(e).
\] (2.140)

3 The perturbed Wigner-Dyson ensembles

We can also look at the situation considered by Brezin and Hikami in the unitary case, that is, the situation in which the coupling constant \( \lambda \) is of the order of \( \sqrt{N} \). Up to a trivial rescaling, we can simply take \( \lambda = \sqrt{N} \). We will in this case start from the supersymmetric formula (2.99), which we repeat here for convenience.

\[
J^S_\beta = \frac{\int DQ^\beta \exp \left( -\frac{1}{2} \text{Str}(Q^\beta - p^\beta / \lambda)^2 \right) F_\beta(Q^\beta)^N}{\int DQ^\beta \exp \left( -\frac{1}{2} \text{Str}(Q^\beta)^2 \right)}
\] (3.1)

where

\[
F_\beta(Q^\beta) = \int da \nu(a) \text{Sdet} \left( (e - a) \mathbb{1}_{4n/\beta} + \frac{\lambda}{N} Q^\beta \right)^{-\beta/2},
\] (3.2)

the \( 4n/\beta \times 4n/\beta \) supermatrix \( Q^\beta \) being given by

\[
Q^\beta = \left( \begin{array}{cc} Q^\beta_+ & Q^\beta_- \\ Q^\beta_- & Q^\beta_+ \end{array} \right),
\] (3.3)

and \( p^\beta \) denoting the diagonal supermatrix

\[
p^\beta = \left( \begin{array}{cc} -i\rho_+ S_\beta & 0 \\ 0 & -i\rho_- \end{array} \right).
\] (3.4)

The measure of the superintegral is

\[
DQ^\beta := DQ^\beta_+ DQ^\beta_- DQ^\beta_+ DQ^\beta_-.
\]

Following Efetov, [3], let us diagonalize the supermatrix \( Q \), i.e., we define

\[
Q = V q V^{-1},
\] (3.5)

where \( q \) is a diagonal supermatrix. Here and henceforth, we suppress the superscript \( \beta \), unless explicitly required. The measure \( DQ \) hence factorizes as follows

\[
DQ = d\mu(\nu) \, m(q) \, dq.
\] (3.6)

The generating function will therefore be of the form

\[
J_\beta = \frac{1}{d} \int d\mu(\nu) m(q) dq \exp \left( -\frac{1}{2} \text{Str} \left( q - \frac{V^{-1} p V}{\sqrt{N}} \right)^2 \right) \langle \text{Sdet} \left( (e - a) \mathbb{1}_m + \frac{q}{\sqrt{N}} \right)^{-\frac{\beta}{2}} \rangle^N,
\] (3.7)
where the brackets $\langle \cdot \rangle$ denotes an average with respect to the measure $\nu(a)$ on $a$ and $m = 4n/\beta$.

Keeping $V$ fixed, we look for a saddle point $\hat{q}$, of order $\sqrt{N}$, of the integral over $q$. We take

$$\hat{q}_+ = \hat{q}_- = \sqrt{N} (b \mathbb{I}_{m/2} - ic \beta),$$  

(3.8)

where $c$ is positive and $b$ is real. Since $\text{Str} \, \hat{q}^2 = 0$ and $\text{Sdet}[(e - a) \mathbb{I}_m + \hat{q}] = 1$, we see that the condition for $\hat{q}$ to be a saddle point is

$$\frac{\hat{q}}{\sqrt{N}} = -\frac{\beta}{2} \langle r \rangle,$$

(3.9)

where

$$r = \left((e - a) \mathbb{I}_m + \sqrt{N} \hat{q}\right)^{-1}.$$  

(3.10)

Note that for a function $f(a)$, the symbol $\langle f(a) \rangle$ denotes the average

$$\langle f(a) \rangle = \int da \, \nu(a) \, f(a).$$  

(3.11)

The condition (3.9) is satisfied if $z = b - ic$ is a solution of the equation (first obtained by Pastur [13]):

$$z + \frac{\beta}{2} \langle (e - a + z)^{-1} \rangle = 0.$$  

(3.12)

Taking into account the contribution of the fluctuations around this saddle-point, we see that, if $S$ is the supermatrix

$$S = \mathbb{I}_2 \otimes S,$$

then, to order $1$ in $1/(\sqrt{N})$, we have

$$J = \frac{1}{d} \exp \left( b \text{Str}(p) \right) \int d\mu(\nu) \, m(\hat{q}) \, F \exp \left( -ic \text{Str}(VSV^{-1}p) \right),$$  

(3.13)

where

$$F = \int d(\delta q) \exp(g),$$  

(3.14)

with

$$g = -\frac{1}{2} \text{Str}(\delta q)^2 + \frac{\beta}{4} \text{Str} \langle (r \delta q)^2 \rangle + \frac{\beta^2}{8} \left[ \langle (\text{Str}(r \delta q))^2 \rangle - \langle \text{Str}(r \delta q) \rangle^2 \right].$$  

(3.15)

However, we see that $F$ and $m(\hat{q})$ depend only on $b$ and $c$, and hence the generating function $J^S$ has the structure

$$J^S = \exp \left( ibtr \left[ (\varepsilon_- - \varepsilon_+) S \right] \right) C(b, c) K(c \varepsilon_+, c \varepsilon_-, cr),$$  

(3.16)
K being the same function as the one in the case where the matrix A is zero, i.e., the standard case of Random Matrix Theory. It is easy to see that \( c = \pi \rho(e) \), where \( \rho(e) \) is the density of states. Therefore, proceeding as before, we conclude that, on the scale of energy in which the mean level spacing is unity, the correlation functions are the same as those of Random Matrix Theory. More precisely,

\[
\left( \frac{1}{\rho(e)} \right)^n \rho^{(n)}(\frac{r_1}{\rho(e)}, \frac{r_2}{\rho(e)}, \ldots, \frac{r_n}{\rho(e)}) = \rho^{(n)}_\beta(r_1, \ldots, r_n),
\]

(3.17)

\( \rho^{(n)}_\beta \) being \( n \)-point correlation function of the standard Random Matrix Theory ensemble, characterised only by \( \beta \).

In the unitary case (\( \beta = 1 \)), this conclusion agrees with that of Brezin and Hikami, [4], who considered an ensemble with a fixed matrix A. Even though their result appears formally to be valid for any set of values of \( a_j \) (\( a_j \) being the diagonal elements of A), it is clear that at best it is true with probability one with respect to some probability distribution on the \( a_j \)’s. In contrast, our result concerns correlation functions which are averaged over the \( a_j \)’s.

In the case \( \alpha > 1 \), we take \( \lambda = 1/(N^{\alpha-1}) \) in the above expressions, and it is easily seen that when \( N \) is large, the dominant term in the generating function is the generating function for the matrix A. In other words, the statistics is Poissonian in this case.

In order to proceed further, we will now look at the simplest case, that of the two-point correlation functions.

4 Unitary case: The two-point generating function

There is one major simplification in the unitary case. The integral defining \( A_2 \) (2.117) can be explicitly evaluated and gives a meromorphic function of the eigenvalues \( \{q_j^-\} \) and \( \{q_j^+\} \) of the matrices \( Q^- \) and \( Q^+ \), respectively. We will, however, only consider the case of the two-point generating function \( (n = 2) \), with \( s_1 = +1 \) and \( s_2 = -1 \). The result of the integration over \( t \) can be expressed as:

\[
A_2 = \frac{\pi i}{2w^2}[(x_2 + iy_1)C_{22}^1 + (x_2 + iy_2)C_{11}^1 - (x_1 + iy_1)C_{22}^2 - (x_1 + iy_2)C_{11}^2]
- \frac{\pi i}{4w^3}[\text{det}C^1 + \text{det}C^2 - Z^2],
\]

(4.1)

where

\[
x_j = q_{+j} \quad ; \quad y_j = q_{-j}
\]

(4.2)

and

\[
w = \frac{x_1 - x_2}{2}
\]

(4.3)

We choose \( \text{Im} \ x_1 < 0 \) and \( \text{Im} \ x_2 > 0 \) so that \( \text{Im} \ w < 0 \); The matrix \( C^\alpha \), with \( \alpha \in \{1, 2\} \), has for elements

\[
C_{ij}^\alpha = \chi_i^\alpha \chi_j^\alpha.
\]

(4.4)
while
\[ Z = C_{11}^1 C_{22}^2 - C_{12}^1 C_{21}^2 + C_{11}^2 C_{22}^1 - C_{12}^2 C_{21}^1. \]  
(4.5)

We first integrate over the Grassmannian variables \( \{ \chi^\alpha_j, \bar{\chi}^\alpha_j \} \) so that if we define
\[
I := \int D\bar{\chi} D\chi \exp(-\sum_{\alpha=1}^2 \text{tr} C^\alpha + \lambda \nu A_2),
\]  
(4.6)

we find that
\[
I = F(y_1)F(y_2)g(y_1)g(y_2) + 3d^2 + d[F(y_1) - g(y_1)][F(y_2) - g(y_2)],
\]  
(4.7)

with
\[
F(y) = 1 - \pi i \lambda \nu \frac{w}{2} (x_2 + iy),
\]  
(4.8)
\[
g(y) = 1 + \pi i \lambda \nu \frac{w}{2} (x_1 + iy),
\]  
(4.9)
and
\[
d = -\pi i \lambda \nu \frac{w}{2}. \]  
(4.11)

Grouping these results we can express the generating function as:
\[
J^{+-} = \frac{1}{d_+ d_-} \int dQ_+ \int dQ_- \exp \left[ -\frac{1}{2} \text{tr} (Q_+ + i \rho_+ S)^2 - \frac{1}{2} \text{tr} (Q_- - \rho_-)^2 \right]
\times I \exp \left( -\lambda \alpha [x_1 + x_2 + i(y_1 + y_2)] + i \pi \lambda \nu [-2w + i(y_2 - y_1)] \right)
\times \exp \left( -\pi i \lambda \nu \frac{w}{2} (x_1 + iy_1)(x_2 + iy_2) \right)
\]  
(4.12)

We now integrate over \( Q_- \). The integral has the structure
\[
A = \int dQ_- F(y_1, y_2) \exp \left( -\frac{1}{2} \text{tr} Q_-^2 + \frac{1}{\lambda} \text{tr} (\rho_- Q_-) \right),
\]  
(4.13)

where \( F(y_1, y_2) \) is some symmetric function of \( y_1 \) and \( y_2 \). If we define
\[
z_1 := \frac{y_1 - y_2}{2},
\]  
(4.14)
\[
z_2 := \frac{y_1 + y_2}{2},
\]  
(4.15)
we can write
\[
Q_- = \begin{pmatrix}
z_2 + z_1 \cos \varphi & z_1 \sin \varphi e^{i\psi} \\
z_1 \sin \varphi e^{-i\psi} & z_2 - z_1 \cos \varphi
\end{pmatrix}
\]
with \( \varphi \in [0, \pi] \) and \( \psi \in [0, 2\pi] \). In terms of the variables \( z_1 \) and \( z_2 \), we find that

\[
A = \frac{2\pi \lambda}{\text{tr}(\rho - s)} \sum_{\sigma \in \{-1, +1\}} \sigma \int_{-\infty}^{\infty} dz_1 \int_{-\infty}^{\infty} dz_2 \, z_1 F(z_1 + z_2, z_2 - z_1) \\
\times \exp\left(-[z_1^2 + z_2^2] + \frac{z_2}{\lambda} \text{tr} \rho_- + \frac{\sigma z_1}{\lambda} \text{tr}(\rho_- S)\right)
\]

and the generating function is given by the expression

\[
J^{+-} = \frac{2\pi \lambda}{d_+ d_- \text{tr}(\rho_- S)} \exp\left(\frac{1}{2\lambda^2} \left[\text{tr}(\rho_+ S)^2 - \text{tr} \rho_-^2\right]\right) \\
\times \sum_{\sigma} \sigma \int dQ_+ \exp\left[-\frac{1}{2} \text{tr} Q_+^2 - \frac{i}{\lambda} \text{tr} Q_+ S\right] G_\sigma(z_3, w),
\]

where

\[
z_3 = \frac{x_1 + x_2}{2}
\]

and

\[
G_\sigma(z_3, w) = \int dz_1 \int dz_2 z_1 I \exp\left[-z_2^2 (1 + \frac{i\pi \lambda \nu}{w}) - z_1^2 (1 - \frac{i\pi \lambda \nu}{w})\right] \\
\exp(-\pi i \lambda \nu w) \exp\left(\frac{2\pi \lambda \nu}{w} z_2 z_1 - 2 \lambda \alpha [z_3 + iz_2]\right).
\]

This last integral converges if \(|\text{Im} \ w| > \pi \lambda \nu\), a condition that we can impose by choosing the free parameter \( \delta \) appearing in \( Q_+ \).

Now the matrix \( Q_+ \) has the form

\[
Q_+ = \begin{pmatrix}
z_3 + q - i \sqrt{\delta^2 + h} & \sqrt{\delta^2 + h} \\
\sqrt{\delta^2 + h} e^{-i\theta} & z_3 - q + i \sqrt{\delta^2 + h}
\end{pmatrix}
\]

where \( \theta \in [0, 2\pi] \), \( h \geq 0 \), \( q \in \mathbb{R} \) and \( dQ_+ = 2dz_3 \, dq \, dh \, d\theta \). Since

\[
\frac{1}{2} \text{tr} Q_+^2 = w^2 + z_3^2 \quad \text{with} \quad w = \sqrt{(q - i \sqrt{\delta^2 + h})^2 + h},
\]

we have that

\[
\text{tr} (Q_+ \rho_+ h) = z_3 \text{tr}(\rho_+ h) + (q - i \sqrt{\delta^2 + h}) \text{tr} \rho_+.
\]

Hence, we need to compute an integral of the form

\[
B = \int_{-\infty}^{\infty} dq \int_0^\infty dh \ \exp\left(-w^2 - i \text{tr} \rho_+(q - i \sqrt{\delta^2 + h})\right) G_\sigma(z_3, w).
\]

It can be shown that since \( G_\sigma(z_3, w) \) is analytic and bounded in the domain \( \text{Im} \ w < -\pi \lambda \nu \), we can change the integral over \( q \) into an integral over \( w \) along the
path C := Im w = −b, with b > πλν, by choosing δ > πλν, so that the integral over h can be evaluated. Finally, we obtain

\[ B = \frac{2i\lambda}{\text{tr}\rho_+} \int_C dw w \exp\left(-[w^2 + \frac{iw}{\lambda} \text{tr}\rho_+]\right) G_\sigma(z_3, w). \] (4.23)

Hence,

\[ J^+ = 8\pi^2i\lambda^2 \frac{\exp\left(\frac{\lambda^2}{2w^3} - \frac{\lambda^2}{2w^3}\right)}{(\text{tr}\rho_+)(\text{tr}\rho_-)d_+d_-} \sum_\sigma \int_C dw w \exp\left(-[w^2 + i\pi\lambda\nu w + \frac{iw}{\lambda} \text{tr}\rho_+]\right) K_\sigma(w), \] (4.24)

where

\[ K_\sigma(w) = \int dz_1 dz_2 dz_3 \exp\left(-\sum_j z_j^2 - \frac{\pi i\lambda\nu}{w} [(z_3 + iz_2)^2 + z_1^2] - 2\lambda\alpha(z_3 + iz_2)\right) \]
\[ \times \exp\left(\frac{\sigma}{\lambda} z_1 \text{tr}(\rho_-) + \frac{z_2}{\lambda} \text{tr}(\rho_-) - \frac{i\lambda}{\lambda} \text{tr}(\rho_+ s)\right), \] (4.25)

and

\[ I = A_1 + (z_3 + iz_2)^2 A_2 + z_1^2 A_3 + [(z_3 + iz_2)^2 + z_1^2] A_4, \] (4.26)

with

\[ A_1 = \left(1 + \frac{i\Lambda}{2w}\right)^4 - \frac{3\Lambda^2}{4w^6} \]
\[ A_2 = -\frac{\Lambda^2}{2w^4} \left[-\frac{i\Lambda}{2w^3} - (1 + \frac{i\Lambda}{2w})^2\right] \]
\[ A_3 = -\frac{\Lambda^2}{2w^4} \left[-\frac{i\Lambda}{2w^3} + (1 + \frac{i\Lambda}{2w})^2\right] \]
\[ A_4 = \frac{1}{16} \frac{\Lambda^4}{w^8}, \] (4.27)

where

\[ \Lambda = \pi\lambda\nu. \] (4.28)

We now define

\[ M = \int dz_1 dz_2 dz_3 \exp\left[-\sum_{j=1}^3 z_j^2 - \frac{c\Lambda}{w} [(z_3 + iz_2)^2 + z_1^2]\right] \]
\[ \times \exp\left[\sigma t z_1 - y(z_3 + iz_2) - u(z_3 - iz_2)\right], \] (4.29)

with

\[ u = \frac{1}{2i\lambda} \text{tr}(\rho_- - \rho_+) \]
\[ y = 2\lambda\alpha - \frac{1}{2i\lambda} \left[\text{tr}\rho_- + \text{tr}(\rho_+ s)\right] \]
\[ t = \frac{1}{\lambda} \text{tr}(\rho_- s), \] (4.30) (4.31) (4.32)
so that
\[ \sigma K_\sigma = \frac{\partial}{\partial t} \left[ A_1 + A_2 \frac{\partial^2}{\partial y^2} + A_3 \frac{\partial^2}{\partial t^2} + A_4 \left( \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial t^2} \right)^2 \right] M, \] (4.33)

and the Gaussian integral \( M \) is given by
\[ M = \frac{\pi^{3/2}}{(1 + \frac{i\Lambda}{w})^{3/2}} \times \exp \left[ \frac{t^2}{4(1 + \frac{i\Lambda}{w})} + uy - u^2 \frac{i\Lambda}{w} \right]. \] (4.34)

Since \( d_\pm = 2\pi^2 \), we can express the generating function as
\[ J^+ = \frac{4i}{\pi vt} \exp \left[ uy + \frac{1}{2\Lambda^2} \text{tr} \left( \rho_+^2 - \rho_-^2 \right) \right] \partial_t R, \] (4.35)
where
\[ R = \langle w A_1 \rangle + u^2 \langle w [A_2 - A_3] \rangle + \hat{L} \langle w A_3 \rangle + \hat{L}^2 \langle w A_4 \rangle; \] (4.36)

If \( F(w) \) is a function of \( w \), we define
\[ \langle F(w) \rangle = \int_C dw F(w) \exp \left[ -w^2 - iw(v + \Lambda) - \frac{i\Lambda u^2}{w} \right] \times \int_{-\infty}^{+\infty} dz \exp \left[ -z^2(1 + \frac{i\Lambda}{w}) - tz \right], \] (4.37)
and \( \hat{L} \) is the operator
\[ \hat{L} = u^2 + \frac{\partial^2}{\partial y^2}. \] (4.38)

We can express \( R \) as a linear combination of the functions
\[ B_n = \langle w^{-n} \rangle, \] (4.39)
if we note that
\[ \hat{L} B_n = -\frac{i}{\Lambda} (n - 2) B_{n-1} - \frac{2i}{\Lambda} B_{n-3} + (v + \Lambda) B_{n-2}, \] (4.40)
so that
\[ R = \sum_{j=1}^{3} \alpha_j B_j, \] (4.41)
with
\[ \alpha_{-1} = 1, \quad \alpha_0 = 2i\Lambda, \quad \alpha_1 = -\frac{13}{4} \Lambda^2 - \frac{v\Lambda}{2}, \]
\[ \alpha_2 = -\frac{3}{2} i\Lambda^3 + \frac{i\Lambda}{2} - \frac{3}{4} uv^2, \quad \alpha_3 = \frac{\Lambda^4}{4} - \frac{7}{8} \Lambda^2 + \frac{v^2}{4} + \frac{v^2}{16} \Lambda^2 + u^2 \Lambda^2, \]
\[ \alpha_4 = iu^2 \Lambda^3 - \frac{3}{8} i\Lambda^3, \quad \alpha_5 = -\frac{u^2}{4} \Lambda^4. \] (4.42)
One can also use the recursion formula
\[ \partial_t B_n = \frac{t}{2} B_n - i\Lambda \partial_t B_{n+1} \] (4.43)
to express the generating function in terms of the \( B_n \), for \( n \in \{-1, 4\} \) and \( \partial_t B_5 \).

One can also show that \( B_n \) can be represented as an integral over the modified Bessel functions. However, we shall not use this representation now, since we do not need to compute the generating function itself, but only its derivatives.

### 4.1 Unitary case: The two-point correlation function

In order to compute the correlation function, we start from the expression (4.35) for the generating function and note that when \( \varepsilon_+ = \varepsilon_- \) and \( u = 0 \). Hence, since \( J^{+-}(\varepsilon_+ = \varepsilon_-) = 1 \), we have that
\[
\partial_t R \bigg|_{u=0} = \frac{vt\pi}{4i}.
\] (4.44)

Moreover,
\[
\frac{\partial J^{+-}}{\partial \varepsilon_+(1)}_{\varepsilon_- = \varepsilon_+} = -\pi \nu + i\alpha,
\]
\[
\frac{\partial J^{+-}}{\partial \varepsilon_+(2)}_{\varepsilon_- = \varepsilon_+} = -\pi \nu - i\alpha,
\] (4.45)
since these derivatives give the average value of
\[
\langle -\frac{1}{N} \text{tr} \left( \frac{\varepsilon_+(j)}{N} + i(e_j - H) \right)^{-1} \rangle
\]
in the limit \( N \to \infty \). Using these relations, one can show that
\[
\frac{\partial^2 J^{+-}}{\partial \varepsilon_+(1) \partial \varepsilon_+(2)}_{\varepsilon_- = \varepsilon_+ = 0} = \alpha^2 + \frac{1}{\chi^2} \left[ \frac{3}{2} - \frac{v^2}{4} - iv + \frac{2\Lambda}{v} + M \right],
\] (4.46)
with
\[
M = \frac{4i}{\pi v^2} \left[ \partial_t \left( \frac{1}{2} \frac{\partial^2 R}{\partial u^2} + \frac{\partial^2 R}{\partial v^2} \right) \right]_{u=0,v=t=r_1-r_2}.
\] (4.47)

Since the two-point correlation function \( \rho_2(r_1, r_2) \) is given by
\[
2\pi^2 \rho_2(r_1, r_2) = \text{Re} \left[ \frac{\partial^2}{\partial \varepsilon_+(1) \partial \varepsilon_+(2)}(J^{+-} + J^{++}) \right]_{\varepsilon_- = \varepsilon_+ = 0},
\] (4.48)
we see that the unfolded cluster function
\[
Y(r) = \frac{1}{\nu^2} \left[ \nu^2 - \rho_2 \left( \frac{r_1}{\nu}, \frac{r_2}{\nu} \right) \right], \text{ with } r = r_1 - r_2,
\] (4.49)
is given by
\[ Y(r) = \frac{1}{2\Lambda^2} \text{Re} \left( \frac{3}{2} - \frac{v^2}{4} - \Lambda v - \Lambda^2 + M \right), \quad (4.50) \]
where \( v = \pi ir/\Lambda \).

It remains, therefore, to compute \( M \). This is a lengthy computation, which is simplified by making use of the recursion formula (4.43) and the following relations:
\[ \partial_u B_n = -i\Lambda B_{n+1}, \quad \partial_v B_n = -iB_{n-1}. \quad (4.51) \]
In this way we get
\[ M = \frac{2i}{\pi v} \sum_{j=-3}^{4} \beta_j B_j^0 + \frac{4i}{\pi v^2} \beta_5 (\partial_4 B_5)^0. \quad (4.52) \]
The \( \beta_j \) are some polynomials of second degree in \( v \), and the superscript 0 indicates that the quantities are computed with \( u = 0 \) and \( v = t = i\pi r/\Lambda \).

It remains to compute the \( B_n \). From eqns. (4.37) and (4.39), we see that if we made the change of variables \( z = s - iw \), when \( u = 0 \) and \( v = t \), then we could write \( B_n \) in the following form
\[ B_n = \int_C dw \, w^{-n} \int_{-\infty}^{+\infty} ds \exp \left( -s^2 - s(v + 2\Lambda) - i\Lambda^2 + 2iws \right). \quad (4.53) \]
When \( n \geq 1 \), we can interchange the two integrals and replace the \( w \) integral by a contour integral around the origin when \( s \geq 0 \) (because \( \text{Im} w < -\Lambda \)), whereas, if \( s < 0 \), the \( w \)-integral vanishes. In this way one finds that, when \( n \geq 1 \)
\[ B_n = 2\pi i \int_0^{\infty} ds \, f(s)(2is)^{n-1} F_{n-1}(2\Lambda s^3) \quad (4.54) \]
where \( f(s) = \exp(-s^2 - s(v + 2\Lambda)) \) and
\[ F_n(x) = \sum_{j=0}^{\infty} \frac{x^j}{j!(n+j)!} = x^{-\frac{2}{n+2}} I_n(2\sqrt{x}), \quad (4.55) \]
\( I_n \) being the modified Bessel function. In order to get this form, we have simply expanded \( \exp(-i\Lambda s^2/w) \) in powers of \( w^{-1} \) in the \( w \) integral.

When \( n \leq 0 \), we have to do the interchange of the \( s \)-integral and the \( w \)-integral more carefully, and one finds that
\[ B_n = B_n' + B_n'', \quad (4.56) \]
where
\[ B_n' = 2\pi i \int_0^{\infty} ds \, f(s)(-i\Lambda s^2)^{n+1} F_{n+1}(2\Lambda s^3) \quad (4.57) \]
and $B''_n$ is some polynomial of degree $n$ in $v$. Similarly, when $n \geq 1$, we have that
\[
\partial_t B_n = iB_{n-1} - R_n
\]  
where
\[
R_n = \pi \int_0^\infty ds f(s)(2is)^n F_{n-1}(2\Lambda s^3).
\]  
We have now at our disposal all the quantities appearing in $M$, and, therefore, from (4.50), the cluster function $Y$. The final result for $Y$ is given by the following expression;
\[
Y = \int_0^\infty ds \exp \left( -s^2 - 2\Lambda s \right) \left[ \sum_{j=0}^4 \alpha_j F_j(2\Lambda s^3) \right] \cos \left( \frac{\pi rs}{\Lambda} \right)
+ \int_0^\infty ds \exp \left( -s^2 - 2\Lambda s \right) \left[ \sum_{j=0}^4 \beta_j F_j(2\Lambda s^3) \right] \sin \left( \frac{\pi rs}{\Lambda} \right)
\]  
where
\[
\begin{align*}
\alpha_0 &= -\frac{(s + \Lambda)}{4} \\
\alpha_1 &= \frac{s^2 \Lambda}{4} \\
\alpha_2 &= s^4 \Lambda - 3\Lambda s^2 \\
\alpha_3 &= s^3 \Lambda^2 \left[ 2\Lambda^2 - 3 - \frac{8\Lambda^2}{\pi^2 r^2} \right] \\
\alpha_4 &= s^5 \Lambda^4 \left[ \frac{16\Lambda^2}{\pi^2 r^2} - 2 - 4\Lambda^2 \right]
\end{align*}
\]  
and
\[
\begin{align*}
\beta_0 &= \frac{7\Lambda}{4\pi r} \\
\beta_1 &= \frac{s^2 \Lambda^3}{2\pi r} + s^4 \left[ -\frac{5\Lambda^3}{2\pi r} + \pi r \right] \\
\beta_2 &= -\frac{5s^4 \Lambda^3}{2\pi r} + s^2 \left[ \frac{\Lambda^3 - 7\Lambda}{\pi r} + \frac{\pi r \Lambda}{2} \right] \\
\beta_3 &= \frac{4s^6 \Lambda^3}{\pi r} + s^3 \Lambda^2 \left[ \frac{1 + 8\Lambda^2}{\pi r} + \frac{\pi r}{2} (1 + 2\Lambda^2) \right] \\
\beta_4 &= -\frac{2s^8 \Lambda^3}{\pi r} - s^5 \frac{16\Lambda^6}{\pi r}.
\end{align*}
\]  
This expression appears to be more complicated than the one given in [8], but it can probably be transformed into it by using various recursion formula for the functions $F_n$, like
\[
n \geq 1 \quad x F_{n+1} = F_{n-1} - n F_n \quad \text{and} \quad \frac{dF_n}{dx} = F_{n+1}.
\]
5 Orthogonal case: The two-point generating function

We start from the equation

$$
\rho^{(2)}(r_1, r_2) = \frac{1}{2\pi^2} \text{Re} \left\{ \frac{\partial^2 [J_{2}^{+} - J_{2}^{-}] }{\partial \varepsilon_{-}(1) \partial \varepsilon_{-}(2)} \right|_{\varepsilon_{-}=\varepsilon_{+}=0} \right\},
$$

(5.1)

in order to compute the correlation function $\rho^{(2)}(e_1, e_2)$. Using the fact (established in Section 2.7) that

$$
\frac{\partial J^{+}}{\partial \varepsilon_{-}(1)} = \pi \nu + ip \quad (5.2)
$$

$$
\frac{\partial J^{+}}{\partial \varepsilon_{-}(2)} = \pi \nu - ip \quad (5.3)
$$

$$
\frac{\partial^2 J^{++}}{\partial \varepsilon_{-}(1) \partial \varepsilon_{-}(2)} \right|_{\varepsilon_{-}=\varepsilon_{+}=0} = (\pi \nu + ip)^2, \quad (5.4)
$$

we can express $\rho^{(2)}(r_1, r_2)$ as

$$
\rho^{(2)}(r_1, r_2) = -\frac{1}{\lambda^2 \pi^2} p(r_1 + r_2) - \frac{2}{\lambda^2 \pi^2} r_1 r_2 + \frac{1}{2} \nu^2 - \frac{p^2}{2\pi^2}
$$

$$
- \frac{1}{2d_+d_- \lambda^2 \pi^2} \int DQ_+ DQ_- \prod_{\alpha=1}^{2} \left[ Q_{-(1\alpha)}(1\alpha) + Q_{-(2\alpha)}(2\alpha) \right]
$$

$$
\times \exp\left( -\frac{1}{2} \text{tr} \left[ (Q_- - \frac{\nu}{\lambda} - i\beta p \lambda)^2 + (Q_+ + \frac{i\rho + S}{\lambda} - \frac{\beta p \lambda}{2})^2 \right] \right) \times K(q_+, q_-), \quad (5.5)
$$

(5.7)

where

$$
K(q_+, q_-) = \int D\chi \chi \exp\left( -\langle\chi, \chi\rangle \right) \exp\left( \lambda \nu(e)[\frac{1}{2}A_0 + A_1 + A_2] \right); \quad (5.8)
$$

with

$$
A_0 = i\pi \text{tr} \left[ ((q_+ + iq_-)\sigma(q_+)) \right], \quad (5.9)
$$

$$
A_1 = \int_{-\infty}^{+\infty} dt \left[ \Delta(t) \right]^{\frac{1}{2}} \left[ 1 - \frac{1}{2} \sum_{\rho=1}^{2} \sum_{\gamma=1}^{2} \frac{q_+(p\gamma) + iq_-(p\gamma)}{t + q_+(p\gamma)} \right], \quad (5.10)
$$

$$
A_2 = \int_{-\infty}^{+\infty} dt \left[ \Delta(t) \right]^{\frac{1}{2}} \left[ \det(\mathbb{I}_2 - R) \right]^{\frac{1}{2}} - 1, \quad (5.11)
$$

and

$$
\Delta(t) = \frac{(t - iq_-(1))^{2} (t - iq_-(2))^{2}}{\prod_{\rho=1}^{2} \prod_{\gamma=1}^{2} (t + q_+(p\gamma))}. \quad (5.12)
$$
The $2 \times 2$ matrix $R$ is given by

$$R_{\alpha \alpha'} = \frac{1}{t - iq_-(\alpha)} \sum_{\gamma=1}^{2} \sum_{\gamma=1}^{2} \frac{x_\alpha(p\gamma)x_{\alpha'}(p\gamma)}{t + q_+(p\gamma)}.$$  

(5.13)

We recall that $q_+(p\gamma)$ and $q_-(\gamma)$ are the eigenvalues of $Q_+$ and $Q_-$. Two eigenvalues of $Q_+$ have a positive imaginary part. The other two have a negative imaginary part.

$A_1$ can be expressed in terms of elliptic integrals. The same is true for $A_2$, once we note that

$$\det(1 - R) = 1 - X \quad \text{with} \quad X = \text{tr} R - \det R,$$  

(5.14)

so that

$$\det(1 - R)^{1/2} - 1 = \sum_{n=1}^{8} X^n c_n,$$  

(5.15)

since the $X$ are Grassmannian variables [$c_n$ are constants]. One can then integrate over the Grassmannian variables, to compute $K$. We will not reproduce this computation here, since we have not analysed the resulting expression for the correlation function. It can finally be noted that for higher order correlation functions, hyper-elliptic integrals appear.

**Appendix A: Mathematical Preliminaries**

In this appendix we list certain mathematical relations that we make use of later in the paper.

The following Gaussian integrals play a central role in our technique for studying mixed systems:

For a $2n \times 2n$, antisymmetric matrix $M_-$, we have the relation

$$\int D \varphi_- e^{-(\varphi_, M_- \varphi_-)} = \sqrt{\det M_-},$$  

(A.1)

where $\varphi_-$ denotes a vector whose elements $\varphi_{-j}$ are Grassmannian variables, and

$$D \varphi_- := \prod_j \frac{d \varphi_{-j}}{\sqrt{2}}.$$  

We use the convention that

$$\int d \varphi_{-j} \varphi_{-j} = 1.$$  

For a symmetric matrix $M_+$, with a positive definite real part, we have the relation

$$\int D \varphi_+ e^{-(\varphi_+, M_+ \varphi_+)} = [\det M_+]^{-1/2},$$  

(A.2)
where \( \varphi_+ \) denotes a vector whose elements \( \varphi_{+j} \) are real variables and
\[
D\varphi_+ := \prod_j \left( \frac{1}{\sqrt{\pi}} \right) d\varphi_{+j}.
\]
A similar identity for integrals over pairs of complex variables \( \varphi_{+j}, \overline{\varphi}_{+j} \), which is valid for any matrix \( M \) such that \((M + M^\dagger)\) is positive definite, is
\[
\int D\varphi_+ e^{-\langle \varphi_+, M \varphi_+ \rangle} = [\det M]^{-1}, \tag{A.3}
\]
where
\[
D\varphi_+ = \prod_j d\varphi_{+j} d\varphi_{+j} 2\pi i.
\]

The determinant of any matrix \( M \) can be expressed as an integral over Grassmannian variables as follows
\[
\int D\varphi_- e^{-\langle \varphi_-, M \varphi_- \rangle} = \det M_- \tag{A.4}
\]
where \( \varphi_- \) and \( \varphi_- \) denote vectors whose elements, denoted by \( \varphi_{-j} \) and \( \varphi_{-j} \), are Grassmannian variables; the measure is given by
\[
D\varphi_- = D\varphi_- D\varphi_- = \prod_j d\varphi_{-j} d\varphi_{-j}.
\]

The following identities involving determinants and traces of matrices are also used frequently: Let \( A \) be an \((n \times n)\) matrix and \( B \) be an \((m \times m)\) matrix. Then
\[
\bullet \quad \det(A \otimes B) = (\det A)^m (\det B)^n \tag{A.5}
\]
\[
\bullet \quad \det(A \otimes I_m + I_m \otimes B) = \prod_{\alpha=1}^n \prod_{j=1}^m (a_{\alpha} + b_j) = \prod_{\alpha=1}^n \det (a_{\alpha} I_m + B) = \prod_{j=1}^m \det (A + b_j I_n), \tag{A.6}
\]
where \( a_{\alpha} \) and \( b_j \) denote the eigenvalues of the matrices \( A \) and \( B \) respectively. The symbol \( I_j \) is used to denote the \( j \times j \) identity matrix.

\[
\bullet \quad \text{If } tr A^j = tr B^j \text{ for any arbitrary integer } j, \text{ then} \quad \det (I_n + A) = \det (I_m + B). \tag{A.7}
\]

**Appendix B: Properties of the matrix \( Q_- \)**

The \( 2n \times 2n \) matrix \( Q_- \) is self-adjoint and satisfies the property
\[
Q_-^t = -\Upsilon_- Q_- \Upsilon_- \tag{B.1}
\]
where
\[
\Upsilon = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}
\]
We will show that such matrices have doubly degenerate eigenvalues and can be diagonalised by unitary matrices, which also belong to the symplectic group. More precisely

\[ Q_- = U q_- U^\dagger, \quad \text{(B.2)} \]

where

\[ U U^\dagger = U^\dagger U = 1 \quad \text{(B.3)} \]
\[ U^\dagger \Upsilon_- U = \Upsilon_- \quad \text{(B.4)} \]

and \( q_- \) is the diagonal matrix of eigenvalues of \( Q_- \). Consider the spectral decomposition of \( Q_- \):

\[ Q_- = \sum_{j=1}^{n} \lambda_j P_j. \quad \text{(B.5)} \]

Then the projectors \( P_j \) also have the property (B.1), which implies that

\[ P_j(1\alpha|1\beta) = P_j(2\beta|2\alpha). \]

hence,

\[ \text{tr} P_j = 2 \sum_{\alpha=1}^{n} P_j(1\alpha|1\alpha), \quad \text{(B.6)} \]

which implies that for almost all \( Q_- \) (with respect to the Lebesgue measure), \( \text{Tr} P_j = 2 \), and, therefore, the eigenvalues of \( Q_- \) are doubly degenerate.

On the other hand, (B.1) and (B.2) can be expressed as

\[ M q_- = q_- M, \quad \text{(B.7)} \]

where

\[ M = U^\dagger \Upsilon_- (U^\dagger)^t. \quad \text{(B.8)} \]

However,

\[ M^t = -M \quad \text{and} \quad M^t M = I_{2n}, \quad \text{(B.9)} \]

and from (B.7) it follows that

\[ M(p\alpha|q\beta) = \delta_{\alpha,\beta} e^{i\psi_\alpha} [\delta_{p,1}\delta_{q,2} - \delta_{p,2}\delta_{q,1}] \quad \text{(B.10)} \]

Noting, however, that \( U \) can be replaced by \( U e^{i\varphi} \) in (B.2), where \( e^{i\varphi} \) is a diagonal matrix, we see from (B.8) that we can choose \( \psi_\alpha = 0 \) in (B.10). In other words

\[ M = \Upsilon_- \]

which gives the desired property (B.4) for \( U \).

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