Extensions of profinite duality groups

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Let $G$ be a profinite group and let $p$ be a prime number. By $\text{Mod}_p(G)$ we denote the category of discrete $p$-primary $G$-modules. For $A \in \text{Mod}_p(G)$ and $i \geq 0$, let

$$D_i(G, A) = \lim_{\rightarrow} H^i(U, A)^*,$$

where $^*$ is $\text{Hom}(-, \mathbb{Q}/p/\mathbb{Z}_p)$, the direct limit is taken over all open subgroups $U$ of $G$ and the transition maps are the duals of the corestriction maps. $D_i(G, A)$ is a discrete $G$-module in a natural way. Assume that $n = \text{cd}_p G$ is finite. Then the $G$-module

$$I(G) = \lim_{\nu \in \mathbb{N}} D_n(G, \mathbb{Z}/p^n\mathbb{Z})$$

is called the dualizing module of $G$ at $p$. Its importance lies in the functorial isomorphism

$$H^n(G, A)^* \cong \text{Hom}_G(A, I(G))$$

for all $A \in \text{Mod}_p(G)$. This isomorphism is induced by the cup-products $(V \subseteq U)$

$$H^n(G, A)^* \times _{p^n} A^U \longrightarrow H^n(V, \mathbb{Z}/p^n\mathbb{Z})^*, \ (\phi, a) \longmapsto (\alpha \mapsto \phi(\text{cor}_G^V(\alpha \cup a)))$$

by passing to the limit over $\nu$ and $V$, and then over $U$. The identity-map of $I(G)$ gives rise to the homomorphism

$$tr : H^n(G, I(G)) \longrightarrow \mathbb{Q}/p/\mathbb{Z}_p,$$

called the trace map.

The profinite group $G$ is called a duality group at $p$ of dimension $n$ if for all $i \in \mathbb{Z}$ and all finite $G$-modules $A \in \text{Mod}_p(G)$, the cup-product and the trace map
\begin{equation*}
H^i(G, \text{Hom}(A, I(G))) \times H^{n-i}(G, A) \xrightarrow{\cup} H^n(G, I(G)) \xrightarrow{\text{tr}} \mathbb{Q}_p / \mathbb{Z}_p
\end{equation*}

yield an isomorphism

\[ H^i(G, \text{Hom}(A, I(G))) \cong H^{n-i}(G, A)^*. \]

\textbf{Remark:} In \cite{Ve}, J.-L. Verdier used the name \textit{strict Cohen-Macaulay at } \( p \) for what we call a profinite duality group at \( p \) here. In \cite{Pl}, A. Pletch defined \( D^n_p \)-groups (and called them duality groups at \( p \) of dimension \( n \)). The \( D^n_p \)-groups of Pletch are exactly the duality groups at \( p \) (in our sense) which, in addition, satisfy the following finiteness condition:

\[ FC(G, p): \ H^i(G, A) \text{ is finite for all finite } A \in \text{Mod}_p(G) \text{ and for all } i \geq 0. \]

Since any finite, discrete \( G \)-module is trivialized by an open subgroup \( U \) of \( G \), condition \( FC(G, p) \) can also be rephrased in the form:

\[ FC(G, p): \ H^i(U, \mathbb{Z}/p\mathbb{Z}) \text{ is finite for all open subgroups } U \text{ of } G \text{ and all } i \geq 0. \]

By a duality theorem due to J. Tate, see \cite{Ta} Thm. 3 or \cite{Ve} Prop. 4.3 or \cite{NSW} (3.4.6), a profinite group \( G \) of cohomological \( p \)-dimension \( n \) is a duality group at \( p \) if and only if

\[ D_i(G, \mathbb{Z}/p\mathbb{Z}) = 0 \quad \text{for } 0 \leq i < n. \]

As a consequence we see that every open subgroup of a duality group at \( p \) is a duality group at \( p \) as well (of the same cohomological dimension), and if an open subgroup of \( G \) is a duality group at \( p \) and \( cd_p G < \infty \), then \( G \) is a duality group at \( p \) of the same cohomological dimension (use \cite{NSW} (3.3.5)(ii)). Furthermore, any profinite group of cohomological \( p \)-dimension 1 is a duality group at \( p \).

We call a profinite group \( G \) \textbf{virtually a duality group at } \( p \) of \textbf{(virtual) dimension } \( vcd_p G = n \) if an open subgroup \( U \) of \( G \) is a duality group at \( p \) of dimension \( n \).

The objective of this paper is to give a proof of Theorem \( \square \) below, which states that the class of duality groups is closed under group extensions \( 1 \rightarrow H \rightarrow G \rightarrow G/H \rightarrow 1 \) if the kernel satisfies \( FC(H, p) \). Weaker forms of Theorem \( \square \) were first proved by A. Pletch (for \( D^n_p \)-groups, see \cite{Pl}) and by the second author (for Poincaré groups, see \cite{Wi}).

\footnote{The proof given by Pletch in \cite{Pl} is only correct for pro-\( p \)-groups as the author assumes that finitely generated projective modules over the complete group ring \( \mathbb{Z}_p[G] \) are free.}
Theorem 1. Let

\[ 1 \rightarrow H \rightarrow G \rightarrow G/H \rightarrow 1 \]

be an exact sequence of profinite groups such that condition FC\((H, p)\) is satisfied. Then the following assertions hold:

(i) If \( G \) is a duality group at \( p \), then \( H \) is a duality group at \( p \) and \( G/H \) is virtually a duality group at \( p \).

(ii) If \( H \) and \( G/H \) are duality groups at \( p \), then \( G \) is a duality group at \( p \).

Moreover, in both cases we have:

\[ \text{cd}_p G = \text{cd}_p H + \text{vcd}_p G/H, \]

and there is a canonical \( G \)-isomorphism

\[ I(G)^\vee \cong I(H)^\vee \hat{\otimes}_{\mathbb{Z}_p} I(G/H)^\vee, \]

where \( \vee \) is the Pontryagin dual and \( \hat{\otimes}_{\mathbb{Z}_p} \) is the tensor-product in the category of compact \( \mathbb{Z}_p \)-modules.

Remark: The assumption FC\((H, p)\) is necessary, as the following examples show:

1. Let \( G \) be the free pro-\( p \)-group on two generators \( x, y \) and let \( H \subset G \) be the normal subgroup generated by \( x \). Then \( H \) is free of infinite rank, \( G/H \) is free of rank one and \( 1 \rightarrow H \rightarrow G \rightarrow G/H \rightarrow 1 \) is an exact sequence in which all three groups are duality groups of dimension one.

2. Let \( D \) be a duality group at \( p \) of dimension 2, \( F \) a duality group at \( p \) of dimension 1 and \( G = F * D \) their free product. The kernel of the projection \( G \rightarrow D \) has cohomological \( p \)-dimension 1, hence is a duality group at \( p \) of dimension 1. The group \( G \) has cohomological \( p \)-dimension 2 but is not a duality group at \( p \).

In the proof of Theorem 1 we make use of the following

Proposition 2. Let

\[ 1 \rightarrow H \rightarrow G \rightarrow G/H \rightarrow 1 \]

be an exact sequence of profinite groups. Assume that FC\((H, p)\) holds. Then there is a spectral sequence of homological type

\[ E^2_{ij} = D_i(G/H, \mathbb{Z}/p\mathbb{Z}) \otimes D_j(H, \mathbb{Z}/p\mathbb{Z}) 
\implies D_{i+j}(G, \mathbb{Z}/p\mathbb{Z}). \]
Proof. Let $g$ run through the open normal subgroups of $G$. Then $gH/H \cong g/g \cap H$ runs through the open normal subgroups of $G/H$. For a $G$-module $A \in \text{Mod}_p(G)$, we consider the Hochschild-Serre spectral sequence

\[ E(g, g \cap H, A) : E_{2}^{ij}(g, g \cap H, A) = H^{i}(g/g \cap H, H^{j}(g \cap H, A)) \Rightarrow H^{i+j}(g, A). \]

If $g' \subseteq g$ is another open normal subgroup of $G$, then the corestriction yields a morphism

\[ \text{cor} : E(g', g' \cap H, A) \to E(g, g \cap H, A) \]

of spectral sequences. The map

\[ E_{2}^{ij}(g', g' \cap H, A) \to E_{2}^{ij}(g, g \cap H, A) \]

is the composite of the maps

\[ H^{i}(g'/g' \cap H, H^{j}(g' \cap H, A)) \xrightarrow{\text{cor}^{g' \cap H}} H^{i}(g'/g' \cap H, H^{j}(g \cap H, A)) \]

\[ H^{i}(g/g \cap H, H^{j}(g \cap H, A)) \xrightarrow{\text{cor}^{g' /g' \cap H}} H^{i}(g/g \cap H, H^{j}(g \cap H, A)) \]

and the map between the limit terms is the corestriction

\[ \text{cor}^{g'} : H^{i+j}(g', A) \to H^{i+j}(g, A). \]

For $2 \leq r \leq \infty$ we set

\[ E_{ij}^{2} = D_{ij}^{2}(G, H, A) := \lim_{\to} E_{r}^{ij}(g, g \cap H, A)^{\ast}. \]

As taking duals and direct limits are exact operations, the terms $D_{ij}^{2}(G, H, A)$, $2 \leq r \leq \infty$, establish a homological spectral sequence which converges to $D_{h}(G, A)$. If $h$ runs through the open subgroups of $H$ which are normal in $G$, then the cohomology groups $H^{i}(h, A)$ are $G$-modules in a natural way. If $g$ is open in $G$ with $g \cap H \subseteq h$, then these groups are $g/g \cap H$-modules. We see that

\[ D_{ij}^{2}(G, H, A) = \lim_{h \cap H \leq h} \lim_{g \subseteq G} H^{i}(g/g \cap H, H^{j}(h, A))^{\ast}, \]

where for both limits the transition maps are (induced by) $\text{cor}^{\ast}$. In order to conclude the proof of the proposition, it remains to construct isomorphisms

\[ D_{ij}^{2}(G, H, \mathbb{Z}/p\mathbb{Z}) \cong D_{i}(G/H, \mathbb{Z}/p\mathbb{Z}) \otimes D_{j}(H, \mathbb{Z}/p\mathbb{Z}) \]

for $2 \leq r \leq \infty$. The proof then follows by induction on $r$. The details are left to the reader.
for all \(i\) and \(j\). To this end note that all occurring abelian groups are \(\mathbb{F}_p\)-vector spaces, so that \(*\) is \(\text{Hom}(\quad, \mathbb{F}_p)\). Further note that for vector spaces \(V, W\) over a field \(k\) the homomorphism

\[
V^* \otimes W^* \longrightarrow (V \otimes W)^*, \ \phi \otimes \psi \longmapsto (v \otimes w \mapsto \phi(v)\psi(w))
\]

is an isomorphism provided that \(V\) or \(W\) is finite-dimensional. Let \(h\) be an open subgroup of \(H\) which is normal in \(G\) and let \(g' \subseteq g\) be open subgroups of \(G\) such that \(g\) acts trivially on the finite group \(H^j(h, \mathbb{Z}/p\mathbb{Z})\). Then, by \(\text{[NSW]} \ (1.5.3)(iv)\), the diagram

\[
\begin{array}{ccc}
H^i(g'/g' \cap H, \mathbb{Z}/p\mathbb{Z}) \otimes H^j(h, \mathbb{Z}/p\mathbb{Z}) & \cup & H^i(g'/g' \cap H, H^j(h, \mathbb{Z}/p\mathbb{Z})) \\
\downarrow \text{cor} \otimes \text{id} & & \downarrow \text{cor} \\
H^i(g/g \cap H, \mathbb{Z}/p\mathbb{Z}) \otimes H^j(h, \mathbb{Z}/p\mathbb{Z}) & \cup & H^i(g/g \cap H, H^j(h, \mathbb{Z}/p\mathbb{Z}))
\end{array}
\]

commutes. For fixed \(h\), we therefore obtain isomorphisms

\[
D_i(G/H, \mathbb{Z}/p\mathbb{Z}) \otimes H^j(h, \mathbb{Z}/p\mathbb{Z})^* \cong \left( \lim_{\rightarrow g} H^i(g/g \cap H, \mathbb{Z}/p\mathbb{Z})^* \right) \otimes H^j(h, \mathbb{Z}/p\mathbb{Z})^*
\]

\[
\cong \lim_{\rightarrow g} H^i(g/g \cap H, \mathbb{Z}/p\mathbb{Z})^* \otimes H^j(h, \mathbb{Z}/p\mathbb{Z})^*
\]

\[
\cong \lim_{\rightarrow g} \left( H^i(g/g \cap H, \mathbb{Z}/p\mathbb{Z}) \otimes H^j(h, \mathbb{Z}/p\mathbb{Z}) \right)^*
\]

\[
\cong \lim_{\rightarrow g} H^i(g/g \cap H, H^j(h, \mathbb{Z}/p\mathbb{Z}))^*.
\]

Passing to the limit over \(h\), we obtain the required isomorphism

\[
D_i(G/H, \mathbb{Z}/p\mathbb{Z}) \otimes D_j(H, \mathbb{Z}/p\mathbb{Z}) \cong D^2_{ij}(G, H, \mathbb{Z}/p\mathbb{Z}).
\]

\[\square\]

**Corollary 3.** *Under the assumptions of Proposition \([2]\) let \(i_0\) and \(j_0\) be the smallest integers such that \(D_{i_0}(G/H, \mathbb{Z}/p\mathbb{Z}) \neq 0\) and \(D_{j_0}(H, \mathbb{Z}/p\mathbb{Z}) \neq 0\), respectively. Then \(D_{i_0+j_0}(G, \mathbb{Z}/p\mathbb{Z}) \neq 0\).*

**Proof.** The spectral sequence constructed in Proposition \([2]\) induces an isomorphism

\[
D_{i_0+j_0}(G, \mathbb{Z}/p\mathbb{Z}) \cong D_{i_0}(G/H, \mathbb{Z}/p\mathbb{Z}) \otimes D_{j_0}(H, \mathbb{Z}/p\mathbb{Z}) \neq 0.
\]

\[\square\]
Proof of Theorem 1. Assume that $G$ is a duality group at $p$ of dimension $d$. Let $\text{cd}_p H = m$ and $n = d - m$. Then there exists an open subgroup $H_1$ of $H$ such that $H^m(H_1, \mathbb{Z}/p\mathbb{Z}) \neq 0$. Let $G_1$ be an open subgroup of $G$ such that $H_1 = G_1 \cap H$. Then $G_1$ is a duality group at $p$ of dimension $d$, $\text{cd}_p H_1 = m$ and $G_1/H_1$ is an open subgroup of $G/H$. We consider the exact sequence

$$1 \longrightarrow H_1 \longrightarrow G_1 \longrightarrow G_1/H_1 \longrightarrow 1.$$ 

As $H^m(H_1, \mathbb{Z}/p\mathbb{Z})$ is finite and nonzero, we have $\text{vcd}_p G_1/H_1 = n$, see [NSW] (3.3.9). Furthermore, $D_i(G_1, \mathbb{Z}/p\mathbb{Z}) = 0$, $i < n + m$. Using Corollary 3 we see that $D_i(G_1/H_1, \mathbb{Z}/p\mathbb{Z}) = 0$ for all $i < n$ and $D_j(H_1, \mathbb{Z}/p\mathbb{Z}) = 0$ for all $j < m$. Thus $G_1/H_1$, hence $G/H$, is virtually a duality group at $p$ of dimension $n$, and $H_1$, and so $H$, is a duality group at $p$ of dimension $m$. This shows (i).

Assume now that $H$ and $G/H$ are duality groups at $p$ of dimension $m$ and $n$. Then, $\text{cd}_p G = n + m$ by [NSW] (3.3.8), and in the spectral sequence of Proposition 2 we have $E^2_{ij} = 0$ for $(i, j) \neq (n, m)$. Hence $D_r(G, \mathbb{Z}/p\mathbb{Z}) = 0$ for $r \neq n + m$ showing that $G$ is a duality group at $p$ of dimension $n + m$.

In order to prove the assertion about the dualizing modules, let $h$ run through all open subgroups of $H$ which are normal in $G$ and $g$ runs through the open subgroups of $G$. Since $m = \text{cd}_p H$, the Hochschild-Serre spectral sequence induces isomorphisms

$$H^{m+n}(g, \mathbb{Z}/p^r\mathbb{Z}) \cong H^n(g/g \cap H, H^m(g \cap H, \mathbb{Z}/p^r\mathbb{Z})), $$

and we obtain

$$I(G) \cong \lim_{\nu} \lim_{g} H^{m+n}(g, \mathbb{Z}/p^r\mathbb{Z})^*$$

$$\cong \lim_{\nu} \lim_{g} H^n(g/g \cap H, H^m(h, \mathbb{Z}/p^r\mathbb{Z}))^*$$

$$\cong \lim_{\nu} \lim_{g} \lim_{h} H^0(g/g \cap H, \text{Hom}(H^m(h, \mathbb{Z}/p^r\mathbb{Z}), I(G/H)))$$

$$\cong \lim_{\nu} \lim_{h} \text{Hom}(H^m(h, \mathbb{Z}/p^r\mathbb{Z}), I(G/H))$$

$$\cong \text{Hom}_{cts} \left( \lim_{\nu} \lim_{h} H^m(h, \mathbb{Z}/p^r\mathbb{Z}), I(G/H) \right)$$

$$\cong \text{Hom}_{cts} \left( \left( \lim_{\nu} \lim_{h} H^m(h, \mathbb{Z}/p^r\mathbb{Z})^* \right)^\vee, I(G/H) \right)$$

$$\cong \text{Hom}_{cts} (I(H)^\vee, I(G/H)) \cong (I(H)^\vee \otimes_{\mathbb{Z}_p} I(G/H)^\vee)^\vee$$

(see [NSW] (5.2.9) for the last isomorphism). This completes the proof of the theorem. \qed
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