Stability and Hopf bifurcation analysis of a two state delay
differential equation modeling the human respiratory system

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Abstract

We study the two state model which describes the balance equation for carbon dioxide and oxygen. These are nonlinear parameter dependent and because of the transport delay in the respiratory control system, they are modeled with delay differential equation. So, the dynamics of a two state one delay model are investigated. By choosing the delay as a parameter, the stability and Hopf bifurcation conditions are obtained. We notice that as the delay passes through its critical value, the positive equilibrium loses its stability and Hopf bifurcation occurs. The stable region of the system with delay against the other parameters and bifurcation diagrams are also plotted. The three dimensional stability chart of the two state model is constructed. We find that the delay parameter has effect on the stability but not on the equilibrium state. The explicit derivation of the direction of Hopf bifurcation and the stability of the bifurcation periodic solutions are determined with the help of normal form theory and center manifold theorem to delay differential equations. Finally, some numerical example and simulations are carried out to confirm the analytical findings. The numerical simulations verify the theoretical results.

1 Introduction

In human respiratory system, the goal is to exchange the unwanted byproduct such as carbon dioxide for oxygen. The carbon dioxide is exchanged for oxygen by passive diffusion. Alveoli is the tiny air sacs in the lungs where the exchange of oxygen and carbon dioxide takes place. The respiratory control system changes the rate of ventilation in response to the levels of oxygen and carbon dioxide in the body. The time delay is due to the physical distance where carbon dioxide and oxygen level information is transported to the sensory control system before the ventilatory response can be adapted.

Understanding the human respiratory system is important for many medical conditions. The human respiratory and its control mechanics have been studied for more than hundred years. This system has important medical implications some of which are listed below [6, 28, 23, 15].

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• Periodic Breathing: Periodic Breathing is defined as three or more episodes of central apnea lasting at least 3 seconds, separated by no more than 20 seconds of normal breathing.

• Sleep Apnea: Sleep Apnea is a disorder in which breathing repeatedly stops and starts again. It is classified in two ways.
  
  – Central Sleep Apnea is a disorder which is characterized by a lack of drive to breathe.
  – Obstructive Sleep Apnea is a disorder which is characterized by episodes of partial or complete physical obstruction of the airflow.

• Cheyne-Stokes Respiration: Cheyne-Stokes Respiration is a disorder which is characterized by gradual increase in breathing followed by decrease or absence of breathing.

These disorders have been associated with a number of medical conditions such as hypertension, heart failures, diabetes and others [27, 21, 1].

1.1 Components of the Model

The carbon dioxide and oxygen levels are monitored at two respiratory centers in the body. These are called central and peripheral chemoreceptors.

1.1.1 Central chemoreceptors

Central chemoreceptors are located at the ventral surface of the medulla in the brain. These respond to the changes in the partial pressure of carbon dioxide in the brain.

1.1.2 Peripheral chemoreceptors

Peripheral chemoreceptors are located in the carotid bodies at the junction of the common carotid arteries and also at the aortic bodies. These respond to the changes in the partial pressure of both carbon dioxide and oxygen in arterial blood.

Since these respiratory centers are located at a distance from the lungs where the levels of the carbon dioxide and oxygen are regulated, there will be some delay (two transport delays) in the process. This regulation is modeled with the ventilation function.

1.1.3 Ventilation function

We assume some conditions for the ventilation functions $V(x, y)$ to be biologically realistic model.

• $V(x, y) \geq 0$ and $V(0, 0) = 0$
• $V(x, y)$ is differentiable
• $V(x, y)$ is an increasing function in both $x$ and $y$
• $\frac{\partial V(x, y)}{\partial x} > 0$ and $\frac{\partial V(x, y)}{\partial y} > 0$

1.2 Model equation

Although a five-state model involving three compartments and two control loops with multiple delays is investigated in [23], here we will study a two state model with one time delay discussed in [12] and [24]. A block diagram of the respiratory system is shown in Figure 1. The controller adjusts to inputs from the state (i.e., sleep, wakefulness) and the chemoreceptors which respond to the change in carbon dioxide and oxygen concentration.
We consider the following two state model for studying stability and bifurcation of a human respiratory system.

\[
\begin{align*}
\frac{d\tilde{x}}{dt} &= p - \alpha W(\tilde{x}(t - \tau), \tilde{y}(t - \tau)) (\tilde{x}(t) - x_I) \\
\frac{d\tilde{y}}{dt} &= -\sigma + \beta W(\tilde{x}(t - \tau), \tilde{y}(t - \tau)) (y_I - \tilde{y}(t)) \\
\end{align*}
\] (1)

where

- $\tilde{x}(\cdot), \tilde{y}(\cdot)$ represent the arterial carbon dioxide and oxygen concentration
- $W(\cdot, \cdot)$ is the ventilation function which represents the volume of gas moved by the respiratory system
- $\tau$ is the transport delay ($\tau > 0$ and $\tau = T_{D_1}$ in Figure 1)
- $x_I, y_I$ are inspired carbon dioxide and oxygen concentration
- $p$ is the carbon dioxide production rate
- $\sigma$ is the oxygen consumption rate
- $\alpha, \beta$ are positive constants associated with the diffusibility of carbon dioxide and oxygen respectively

For studying the stability analysis with a more convenient system, we convert the system (1) using

\[
\begin{align*}
x(t) &= a(\tilde{x}(t) - x_I) \\
y(t) &= b(y_I - \tilde{y}(t)) \\
\end{align*}
\] (2)

Solving for $\tilde{x}(t - \tau)$ and $\tilde{y}(t - \tau)$, we get,
\[ \dot{x}(t - \tau) = x_I + \frac{1}{a} x(t - \tau) \]
\[ \dot{y}(t - \tau) = y_I - \frac{1}{b} y(t - \tau) \]

Using Equation (3) in Equation (1), we obtain
\[ \frac{dx}{dt} = a \frac{dx}{dt} = a p - a \alpha W \left( x_I + \frac{1}{a} x(t - \tau), y_I - \frac{1}{b} y(t - \tau) \right) \frac{x(t)}{a} \]
\[ \frac{dy}{dt} = -b \frac{dy}{dt} = b \sigma - b \beta W \left( x_I + \frac{1}{a} x(t - \tau), y_I - \frac{1}{b} y(t - \tau) \right) \frac{y(t)}{b} \]

Setting \( a = 1/p \) and \( b = 1/\sigma \), we obtain the equations
\[ \frac{dx}{dt} = 1 - \alpha V(x(t - \tau), y(t - \tau)) x(t) \]
\[ \frac{dy}{dt} = 1 - \beta V(x(t - \tau), y(t - \tau)) y(t) \]

where the ventilation function is given by
\[ V(x(t - \tau), y(t - \tau)) = W(\tilde{x}(t - \tau), \tilde{y}(t - \tau)) \]

We will study the system (5) with
\[ V(x(t - \tau), y(t - \tau)) = 0.14 e^{-0.05(100-y(t-\tau))} x(t-\tau) \]

The state variables are concentrations in our model.

2 Stability and Hopf Bifurcation

In recent years, a lot of delay differential equations modeling various chemical, biological, ecological systems have been studied \[8, 14, 16, 38, 50, 39\]. With the outbreak of COVID-19 pandemic, many mathematical models using delay differential equations have been proposed \[35, 38, 40, 37, 13, 19, 33\].

Li and Zhang \[30\], Bilazero˘ glu \[9\], studied the dynamic analysis and Hopf bifurcation of a Lengyel-Epstein system with two delays. Li \[32\] studied a class of delay differential equations with two delays. Kumar et al \[25\] proposed a multiple delayed innovation diffusion model with Holling II functional response. Delayed predator-prey system have been investigated by many researchers \[42, 51, 2, 41, 22, 52, 47, 44, 11, 10, 45, 53\].

Ghosh et al \[17\] studied the rumor spread mechanism and the influential factors using epidemic like model. Several researchers have analyzed the U¸car prototype system \[46, 7, 29, 31\]. Wei \[48\] discussed the dynamics of a scalar delay differential equation. Gilsinn \[18\] estimates the bifurcation parameter of delay differential equation with application to machine tool chatter. There has been a focus on studying stability and Hopf bifurcation by choosing the delay as a parameter of the system with the linear stability methods.

The complex system modeling the human respiratory control system have been studied for several decades. Mackey and Glass \[34\], Khoo et al \[24\], Batzel et al \[4, 5\] have investigated stability analysis.

2.1 Equilibrium point

**Lemma 2.1.** There is a unique positive equilibrium point \( E_*(x_*, y_*) \) of system (5).

**Proof.** The equilibrium point \( E_*(x_*, y_*) \) is obtained by solving
\[ 1 - \alpha 0.14 e^{-0.05(100-y_*)} x_*^2 = 0 \]
\[ 1 - \beta 0.14 e^{-0.05(100-y_*)} y_* = 0 \]

We also notice that
\[ x_* \neq 0, y_* \neq 0, \text{ and } x_* = \frac{\beta}{\alpha} y_* \]
Then, rewriting as exact fractions and solving the equation

\[ 1 - \alpha \left( \frac{14}{100} \right) e^{-\frac{\beta y^2}{\alpha^2}} = 0 \]  

we get,

\[ y_* = 40 W \left( \frac{e^{5/2} \sqrt{\frac{\beta}{\alpha}}}{4\sqrt{14}} \right) \]  

and

\[ x_* = 40 \left( \frac{\beta}{\alpha} \right) W \left( \frac{e^{5/2} \sqrt{\frac{\alpha}{\beta^2}}}{4\sqrt{14}} \right) \]

where \( W \) represents the Lambert \( W \)-function.

Since \( V(0,0) = 0 \) and

\[ V \left( \frac{\beta y_*}{\alpha}, y_* \right) = \left( \frac{14}{100} \right) e^{-\frac{\beta y_*^2}{\alpha^2}} \]  

is increasing in \( y_* \), there is a unique positive solution \( y_* \).

For the default values of \( \alpha = 0.5 \) and \( \beta = 0.8 \), we get \((x_*, y_*) \approx (29.1842, 18.2401)\).

We plot the Equations \( 10 \) and \( 11 \) as a function of \( \alpha \) and \( \beta \) in Figure 2 and 3.

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**Figure 2**: Equilibrium points as a function of \( \alpha \) with \( \beta = 0.8 \).

**Figure 3**: Equilibrium points as a function of \( \beta \) with \( \alpha = 0.5 \).
2.2 Stability of the Equilibrium point

Let \( u(t) = x(t) - x_* \), \( v(t) = y(t) - y_* \). Then, the linearized system of (15) is given as follows:

\[
\begin{align*}
\frac{du(t)}{dt} &= -\alpha V(x_*, y_*) u(t) - \alpha x_* V_x(x_*, y_*) u(t - \tau) - \alpha x_* V_y(x_*, y_*) v(t - \tau) \\
\frac{dv(t)}{dt} &= -\beta V(x_*, y_*) v(t) - \beta y_* V_x(x_*, y_*) u(t - \tau) - \beta y_* V_y(x_*, y_*) v(t - \tau)
\end{align*}
\]  

(12)

This could be written in the form as

\[
\frac{d}{dt} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} + A_1 \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} + B_1 \begin{pmatrix} u(t - \tau) \\ v(t - \tau) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]  

(13)

where

\[
A_1 = \begin{pmatrix} \alpha V(x_*, y_*) & 0 \\ 0 & \beta V(x_*, y_*) \end{pmatrix} = \begin{pmatrix} \frac{7}{50} \alpha x_* e^{\frac{7}{20}(y_* - 100)} & 0 \\ 0 & \frac{7}{50} \beta x_* e^{\frac{7}{20}(y_* - 100)} \end{pmatrix}
\]

and

\[
B_1 = \begin{pmatrix} \alpha x_* V_x(x_*, y_*) & \alpha x_* V_y(x_*, y_*) \\ \beta y_* V_x(x_*, y_*) & \beta y_* V_y(x_*, y_*) \end{pmatrix} = \begin{pmatrix} \frac{7}{50} \alpha x_* e^{\frac{7}{20}(y_* - 100)} & \frac{700}{1000} \alpha x_* e^{\frac{7}{20}(y_* - 100)} \\ \frac{7}{50} \beta y_* e^{\frac{7}{20}(y_* - 100)} & \frac{700}{1000} \beta y_* e^{\frac{7}{20}(y_* - 100)} \end{pmatrix}
\]

The associated characteristic equation of the linear system (13) is

\[
\Delta(\lambda, \tau) = \det(\lambda I + A_1 + B_1 e^{-\tau \lambda}) = 0
\]  

(14)

That is,

\[
\lambda^2 + \frac{7\alpha x_* e^{-\frac{7\lambda}{20}}}{1000} \left( 20(\alpha + \beta) e^{\lambda \tau} + 20\alpha + \beta y_* \right) + \frac{49\alpha \beta x_* e^{-\frac{7\lambda}{20}}}{50000} \left( 20 e^{\lambda \tau} + y_* + 20 \right) = 0
\]  

(15)

This could be also written as

\[
\Delta(\lambda, \tau) = \lambda^2 + (A + B e^{-\lambda \tau}) \lambda + (C + D e^{-\lambda \tau}) = 0
\]  

(16)

where

\[
\begin{align*}
A &= \frac{7}{50} \alpha x_* e^{\frac{7}{20}} - 5 \\
B &= \frac{7}{50} \alpha x_* e^{\frac{7}{20}} - 5 + \frac{7}{50} \beta y_* e^{\frac{7}{20}} - 5 y_* \\
C &= \frac{49\alpha \beta x_* e^{\frac{7}{20}}}{2500} - 10 \\
D &= \frac{49\alpha \beta x_* e^{\frac{7}{20}}}{2500} - 10 + \frac{49\alpha \beta x_* e^{\frac{7}{20}}}{50000} y_*
\end{align*}
\]  

(17)

- For \( \tau = 0 \)

The characteristic equation (15) simplifies to

\[
\lambda^2 + \frac{7\lambda x_* e^{\frac{7\lambda}{20}} - 5}{1000} \left( 40\alpha + \beta (y_* + 20) \right) + \frac{49\alpha \beta x_* e^{\frac{7\lambda}{20}} - 10}{50000} (y_* + 40) = 0
\]  

(18)

The coefficients in this equation are positive, and therefore the roots have negative real parts.

- For \( \tau > 0 \)
The change in stability of eigenvalue $\lambda$ can occur if $\text{Re}(\lambda) = 0$. Let $\lambda = i\omega$ and characteristic equation (16) takes the form

$$-\omega^2 + i\omega (A + Be^{-i\tau}) + (C + De^{-i\tau}) = 0$$

(19)

Solving for the real and imaginary parts of both sides, we get

$$C - \omega^2 = -B\omega \sin(\tau\omega) - D\cos(\tau\omega)$$
$$A\omega = D\sin(\tau\omega) - B\omega \cos(\tau\omega)$$

(20)

Squaring and adding (20), we get the relation

$$f(\omega) = -\omega^4 + (-A^2 + B^2 + 2C)\omega^2 + (-C^2 + D^2) = 0$$

(21)

Let

$$M = -A^2 + B^2 + 2C = \frac{49\beta x_1^2 e^{\frac{2\pi}{\sqrt{5}} - 10} (-400\beta + 40\alpha y_* + \beta y_*)}{1000000}$$

(22)

Then $M$ is positive if $40\alpha y_* + \beta y_*^2 > 400\beta$ and let

$$N = -C^2 + D^2 = \frac{2401\alpha^2 \beta^2 x_1^4 e^{\frac{2\pi}{\sqrt{5}} - 200} y_* (y_* + 40)}{2500000000} > 0$$

(23)

Assuming that $Y = \omega^2$, we can write (21) as

$$\Phi(Y) = -Y^2 + MY + N = 0$$

(24)

This means that the Equation (24) has one positive root. Solving for $\tau$ from (20), we have the critical curves given by

$$\tau_*(n) =$$

$$\pm \frac{2000\sqrt{3e^5} \pi n}{7 \sqrt{N_3 + N_4}}$$
$$1000i\sqrt{2e^5} \log \left( \frac{e^{-\frac{\pi}{\sqrt{5}}} (\beta N_2 x_1^2 e^{\frac{2\pi}{\sqrt{5}}} (40\alpha + \beta N_1)) - 20i\sqrt{2} \sqrt{N_3 + N_4} x_* e^{\frac{\pi}{\sqrt{5}} (\alpha + \beta) + \sqrt{N_3}}}{x_* \left( 40\alpha \beta N_1 x_* e^{\frac{\pi}{\sqrt{5}} + i\sqrt{2} \sqrt{N_3 + N_4} (20\alpha + \beta y_*)} \right)} \right)$$
$$7\sqrt{N_3 + N_4}$$

(25)

where

$$N_1 = (y_* + 20)$$
$$N_2 = (y_* - 20)$$
$$N_3 = \beta^2 x_1^4 e^{\frac{\pi}{\sqrt{5}}} (y_* + 20) (3200\alpha^2 y_* + 80\alpha \beta (y_* - 20) y_* + \beta^2 (y_* - 20)^2 (y_* + 20))$$
$$N_4 = \beta x_1^2 e^{\frac{\pi}{\sqrt{5}}} (40\alpha y_* + \beta (y_*^2 - 400))$$

(26)

Consequently, we can count that the stability region are restricted between the set of two curves

$$\tau_1(n) =$$

$$\pm \frac{2000\sqrt{3e^5} \pi n}{7 \sqrt{N_3 + N_4}}$$
$$1000i\sqrt{2e^5} \log \left( \frac{e^{-\frac{\pi}{\sqrt{5}}} (\beta N_2 x_1^2 e^{\frac{2\pi}{\sqrt{5}}} (40\alpha + \beta N_1)) - 20i\sqrt{2} \sqrt{N_3 + N_4} x_* e^{\frac{\pi}{\sqrt{5}} (\alpha + \beta) + \sqrt{N_3}}}{x_* \left( 40\alpha \beta N_1 x_* e^{\frac{\pi}{\sqrt{5}} + i\sqrt{2} \sqrt{N_3 + N_4} (20\alpha + \beta y_*)} \right)} \right)$$
$$7\sqrt{N_3 + N_4}$$

(27)

($n = 0, 1, 2, ...$)
\[ \tau_2(n) = \frac{2000\sqrt{2}e^5\pi n}{7\sqrt{N_3 + N_4}} \]

\[ -1000i\sqrt{2}e^{5\log} \left( \frac{e^{\frac{\pi i}{4} (\beta N_2 x^2 + e^{\frac{\pi i}{4} (40\alpha + \beta N_1)} - 20\sqrt{2}\sqrt{N_3 + N_4} x e^{\frac{\pi i}{4} (\alpha + \beta)} + \sqrt{N_2})}}{40\alpha \beta N_1 x e^{\frac{\pi i}{4} (400\alpha + \beta y_0)}} \right) \]

\[ \right) \left( \frac{7\sqrt{N_3 + N_4}}{7\sqrt{N_3 + N_4}} \right) \]

\[ (n = 1, 2, \ldots) \] (28)

Notice that Equation (27) starts with \( n = 0, 1, 2, \ldots \) and Equation (28) starts with \( n = 1, 2, \ldots \) for the pair of curves to have positive values of \( \tau \). If Re \( (d\lambda/d\tau) \) have different sign on any two consecutive critical curves, then the stability region is confined between these two curves in the \((\tau, \alpha, \beta)\) parameter space.

Now we look to verify the transversality condition:

\[ \text{Re} \left( \frac{d\lambda}{d\tau} \right) > 0 \] (29)

at \( \tau = \tau_* \) with \( n = 0 \).

Differentiating characteristic equation (16) with respect to \( \tau \), we get

\[ 2\frac{d\lambda}{d\tau} + (A + Be^{-\lambda \tau}) \frac{d\lambda}{d\tau} + Be^{-\lambda \tau} \lambda \left( -\lambda - \frac{d\lambda}{d\tau} \right) + De^{-\lambda \tau} \left( -\lambda - \tau \frac{d\lambda}{d\tau} \right) = 0 \] (30)

Solving for \( [d\lambda/d\tau]^{-1} \), we get

\[ \left[ \frac{d\lambda}{d\tau} \right]^{-1} = \frac{(A + 2\lambda)e^{\lambda \tau}}{\lambda(B\lambda + D)} - \frac{B\lambda - B + D\tau}{\lambda(B\lambda + D)} \]

\[ = \frac{AD - BC + B\lambda^2 + 2D\lambda}{\lambda(A\lambda + C + \lambda^2)(B\lambda + D)} - \frac{\tau}{\lambda} \] (31)

On critical curves i.e with \( \tau = \tau_* \) and \( \lambda = i\omega \), and solving for the real part, we have

\[ \text{Re} \left( \left[ \frac{d\lambda}{d\tau} \right]^{-1} \right) = \frac{A^2 - 2C + 2\omega^2}{A^2\omega^2 + (C - \omega^2)^2} - \frac{B^2}{B^2\omega^2 + D^2} \]

\[ = \frac{A^2 - 2C + 2\omega^2}{B^2\omega^2 + D^2} - \frac{B^2}{B^2\omega^2 + D^2} \]

\[ = \frac{A^2 - B^2 - 2C + 2\omega^2}{B^2\omega^2 + D^2} \]

\[ = \frac{-C^2 + D^2 + \omega^4}{\omega^2(B^2\omega^2 + D^2)} \]

\[ = \frac{N + \omega^4}{\omega^2(B^2\omega^2 + D^2)} > 0, \text{ since } N > 0 \] (32)

Since Re \( (d\lambda/d\tau) \) > 0 for all the critical curves (27) and (28), the corresponding slopes have positive values on all the stability determining critical curves. Thus, there are no eigenvalues with negative real part across the critical curves. Further, we know that for \( \tau = 0 \) the equilibrium points \((x_*, y_*)\) are stable. Therefore, there can be only one stable region in the \((\tau, \alpha)\) or \((\tau, \beta)\) plane enclosed between the line \( \tau = 0 \) and the curve \( \tau_1(0) \).

The critical curves for various \( n \) as a function of \( \alpha \) and \( \beta \) are shown in Figures 3 and 5.
Figure 4: Critical curves for equilibrium \((x_*, y_*)\) with \(\beta = 0.8\). The solid curves represent \(\tau_1\) for \(n = 0, +1, +2\) and dashed curves represent \(\tau_2\) for \(n = +1, +2\). The region enclosed (shaded region) between the line \(\tau = 0\) and the curve \(\tau = \tau_1(0)\) is the only stable region.

Figure 5: Critical curves for equilibrium \((x_*, y_*)\) with \(\alpha = 0.5\). The solid curves represents \(\tau_1\) for \(n = 0, +1, +2\) and dashed curves represent \(\tau_2\) for \(n = +1, +2\). The region enclosed (shaded region) between the line \(\tau = 0\) and the curve \(\tau = \tau_1(0)\) is the only stable region.

The critical surfaces in the parameter space \((\alpha, \beta, \tau)\) that encompass the stable region is shown in Figure 6.
Figure 6: Critical surfaces of the two state human respiratory system.

Figure 7: Stability chart of the two state human respiratory system.
The 3 dimensional stability chart of the two state model of a human respiratory system in the parameter space \((\alpha, \beta, \tau)\) is shown in Figure 7.

For a more general case, the following theorem was proved in [12].

**Theorem 2.2.** Let 
\[
V^* = V(x^*, y^*), \quad V_x = V_x(x^*, y^*) \quad \text{and} \quad V_y = V_y(x^*, y^*).
\]

1. If 
   \[
   V^* \geq x^* V_x^* + y^* V_y^*,
   \]
   then the equilibrium \((x^*, y^*)\) is asymptotically stable for all delay \(\tau \geq 0\).

2. If 
   \[
   V^* < x^* V_x^* + y^* V_y^*,
   \]
   then there exists \(\tau^* > 0\) such that the equilibrium \((x^*, y^*)\) is asymptotically stable if \(0 \leq \tau < \tau^*\) and unstable if \(\tau > \tau^*\).

Therefore, from the above discussions, the following results can be directly deduced for our case.

Let \(\tau^*\) be defined by (25). Then,

(i) The positive equilibrium \(E^*_s(x^*, y^*)\) of system (5) is asymptotically stable for \(0 \leq \tau < \tau^*\).

(ii) The positive equilibrium \(E^*_s(x^*, y^*)\) of system (5) is unstable for \(\tau > \tau^*\).

(iii) System (5) undergoes Hopf bifurcation at the positive equilibrium \(E^*_s(x^*, y^*)\) for \(\tau = \tau^*\).

### 2.3 Critical Delay and Bifurcation

If we consider \(\tau\) as a parameter, then as \(\tau\) passes through its critical value \(\tau^*\), the positive equilibrium \(E_s(x^*, y^*)\) loses its stability. The maximum value of the real part of the characteristic equation is computed for several values of \(\tau\). This is shown in figure 8. For the default values of \(\alpha = 0.5, \beta = 0.8\), the equilibrium is \(E^*_s(x^*, y^*) \approx (29.1842, 18.2401)\) and the critical delay is \(\tau^* \approx 30.8017\).

![Figure 8: The maximum value of the real part of the eigenvalue with \(\alpha = 0.5, \beta = 0.8\).](image)

Table 1 lists the largest real part of the characteristic root computed for several values of \(\tau\) near the critical delay with \(\alpha = 0.5\) and \(\beta = 0.8\).
Table 1: The largest real part of the eigenvalues with $\alpha = 0.5$ and $\beta = 0.8$

| $\tau$ | Max(Re$[\lambda]$) |
|--------|---------------------|
| 25     | -0.00386067         |
| 26     | -0.0029966          |
| 27     | -0.00222993         |
| 28     | -0.00154774         |
| 29     | -0.000939186        |
| 30     | -0.000395051        |
| 31     | 0.0000925033        |
| 32     | 0.000530187         |
| 33     | 0.000923769         |
| 34     | 0.00127823          |
| 35     | 0.00159789          |

Figure 9 shows the maximum value of the real part of the characteristic equation for varying the parameters $\alpha$ and $\beta$.

The following tables list out the largest part of the eigenvalues with various combination of $\alpha$ and $\beta$ near the critical delay.

Table 2: The largest real part of the eigenvalues with $\alpha = 0.3$

| $\beta$ | $\tau$ | Max(Re$[\lambda]$) | $\beta$ | $\tau$ | Max(Re$[\lambda]$) | $\beta$ | $\tau$ | Max(Re$[\lambda]$) |
|--------|--------|---------------------|--------|--------|---------------------|--------|--------|---------------------|
| 0.3    | 20     | -0.00686887         | 0.6    | 30     | -0.00237351         | 0.9    | 45     | -0.00076911         |
| 0.3    | 21     | -0.00526380         | 0.6    | 31     | -0.00181421         | 0.9    | 46     | -0.00057352         |
| 0.3    | 22     | -0.00387139         | 0.6    | 32     | -0.00130903         | 0.9    | 47     | -0.00039104         |
| 0.3    | 23     | -0.00265816         | 0.6    | 33     | -0.00085181         | 0.9    | 48     | -0.00022063         |
| 0.3    | 24     | -0.00159690         | 0.6    | 34     | -0.00043725         | 0.9    | 49     | -0.00006136         |
| 0.3    | 25     | -0.00066533         | 0.6    | 35     | -0.00006074         | 0.9    | 50     | 0.00008762           |
| 0.3    | 26     | 0.00015497          | 0.6    | 36     | 0.00028174          | 0.9    | 51     | 0.00022707           |
| 0.3    | 27     | 0.00087928          | 0.6    | 37     | 0.00059370          | 0.9    | 52     | 0.00035769           |
| 0.3    | 28     | 0.00152043          | 0.6    | 38     | 0.00087822          | 0.9    | 53     | 0.00048012           |
| 0.3    | 29     | 0.00208920          | 0.6    | 39     | 0.0013802           | 0.9    | 54     | 0.00059495           |
| 0.3    | 30     | 0.00259473          | 0.6    | 40     | 0.00137550          | 0.9    | 55     | 0.00070271           |
Table 3: The largest real part of the eigenvalues with $\alpha = 0.6$

| $\beta$ | $\tau$ | Max(Re[$\lambda$]) | $\beta$ | $\tau$ | Max(Re[$\lambda$]) | $\beta$ | $\tau$ | Max(Re[$\lambda$]) |
|---------|--------|---------------------|---------|--------|---------------------|---------|--------|---------------------|
| 0.3     | 15     | -0.01148690         | 0.6     | 20     | -0.00560191         | 0.9     | 25     | -0.00312474         |
| 0.3     | 16     | -0.0053185          | 0.6     | 21     | -0.00412443         | 0.9     | 26     | -0.00229615         |
| 0.3     | 17     | -0.00070124         | 0.6     | 22     | -0.00284708         | 0.9     | 27     | -0.00156234         |
| 0.3     | 18     | -0.000040648        | 0.6     | 23     | -0.00173803         | 0.9     | 28     | -0.00091068         |
| 0.3     | 19     | -0.00226226         | 0.6     | 24     | -0.00077144         | 0.9     | 29     | -0.00033055         |
| 0.3     | 20     | -0.00078019         | 0.6     | 25     | -0.00007382         | 0.9     | 30     | 0.00018705          |
| 0.3     | 21     | 0.00048556          | 0.6     | 26     | -0.00081518         | 0.9     | 31     | 0.00064978          |
| 0.3     | 22     | 0.00157136          | 0.6     | 27     | -0.00146711         | 0.9     | 32     | 0.00106420          |
| 0.3     | 23     | 0.00250636          | 0.6     | 28     | 0.00204171          | 0.9     | 33     | 0.00143594          |
| 0.3     | 24     | 0.00331419          | 0.6     | 29     | 0.00254916          | 0.9     | 34     | 0.00176985          |
| 0.3     | 25     | 0.00401411          | 0.6     | 30     | 0.00299806          | 0.9     | 35     | 0.00207014          |

Table 4: The largest real part of the eigenvalues with $\alpha = 0.9$

| $\beta$ | $\tau$ | Max(Re[$\lambda$]) | $\beta$ | $\tau$ | Max(Re[$\lambda$]) | $\beta$ | $\tau$ | Max(Re[$\lambda$]) |
|---------|--------|---------------------|---------|--------|---------------------|---------|--------|---------------------|
| 0.3     | 13     | -0.01396730         | 0.6     | 16     | -0.00957134         | 0.9     | 20     | -0.00478992         |
| 0.3     | 14     | -0.01012840         | 0.6     | 17     | -0.00709898         | 0.9     | 21     | -0.00339803         |
| 0.3     | 15     | -0.00701248         | 0.6     | 18     | -0.00502401         | 0.9     | 22     | -0.00219773         |
| 0.3     | 16     | -0.00445748         | 0.6     | 19     | -0.00327084         | 0.9     | 23     | -0.00115831         |
| 0.3     | 17     | -0.00234397         | 0.6     | 20     | -0.00178089         | 0.9     | 24     | -0.00025488         |
| 0.3     | 18     | -0.00058240         | 0.6     | 21     | -0.00050815         | 0.9     | 25     | 0.00053292          |
| 0.3     | 19     | 0.00089540          | 0.6     | 22     | 0.00058388          | 0.9     | 26     | 0.00122183          |
| 0.3     | 20     | 0.00214212          | 0.6     | 23     | 0.00152449          | 0.9     | 27     | 0.00182576          |
| 0.3     | 21     | 0.00319896          | 0.6     | 24     | 0.00233739          | 0.9     | 28     | 0.00235633          |
| 0.3     | 22     | 0.00409852          | 0.6     | 25     | 0.00304190          | 0.9     | 29     | 0.00282329          |
| 0.3     | 23     | 0.00486685          | 0.6     | 26     | 0.00365396          | 0.9     | 30     | 0.00323487          |

We now list the equilibrium point $(x_*, y_*)$ and critical delay $\tau_*$ for these combinations of $\alpha$ and $\beta$.

Table 5: The equilibrium point and critical delay for various values of $\alpha$ and $\beta$

| $\alpha$ | $\beta$ | $(x_*, y_*)$     | $\tau_*$ |
|---------|---------|------------------|----------|
| 0.3     | 0.3     | (28.8782, 28.8782) | 25.8012  |
| 0.3     | 0.6     | (37.2949, 18.6474) | 35.1706  |
| 0.3     | 0.9     | (41.9183, 13.9728) | 49.4039  |
| 0.6     | 0.3     | (17.5118, 35.0237) | 20.5978  |
| 0.6     | 0.6     | (23.4108, 23.4108) | 24.9072  |
| 0.6     | 0.9     | (26.8631, 17.9087) | 29.6255  |
| 0.9     | 0.3     | (12.9723, 38.9169) | 18.3737  |
| 0.9     | 0.6     | (17.6832, 26.5248) | 21.4466  |
| 0.9     | 0.9     | (20.5381, 20.5381) | 24.3089  |

In figure 10, the characteristic roots of the smallest modulus are shown for the default values of $\alpha = 0.5$, and $\beta = 0.8$ while varying the parameter $\tau$ near the critical delay.
Figure 10: The characteristic roots of the smallest modulus with various values of $\alpha$, $\beta$, and $\tau$ near the critical delay.

Figure 11 shows the characteristic roots of the smallest modulus for various parameters of $\alpha$, $\beta$ and $\tau$.

Figure 11: The characteristic roots of the smallest modulus with various values of $\alpha$, $\beta$, and $\tau$.

Thus, we deduced that the (5) is asymptotically stable for $0 \leq \tau < \tau_*$ and unstable for $\tau > \tau_*$, and undergoes a Hopf bifurcation at the positive equilibrium $E_*(x_*, y_*)$ for $\tau = \tau_*(n)$ for $n = 0, 1, 2, \ldots$

2.4 Bifurcation Diagram

We vary the time delay $\tau$ with $\alpha = 0.5$ and $\beta = 0.8$. When $\tau \geq 30.8017$, the fixed point loses its stability through Hopf bifurcation as discussed in the previous section. The system (5) is run for a time span of $[0, 20,000]$ and we consider only the last one-fourth part to eliminate the possible transient responses. The bifurcation diagram is shown in Figure 12.
We also plot the bifurcation diagram of the long term values of the system with the parameters $\alpha$ and $\beta$ listed in Table 5. Since these diagrams are plotted by a different method, this provides us another opportunity to verify the values we found in that table. This is shown in Figures 13 - 15.
Figure 13: Bifurcation diagram with $\alpha = 0.3$ and various $\beta$. 
Figure 14: Bifurcation diagram with $\alpha = 0.6$ and various $\beta$. 
3 Direction and Stability of the Hopf Bifurcation

In this section, we apply the normal form theory and the center manifold theorem of Hassard et al [20] to get some properties of the Hopf bifurcation. In order to determine the direction and the stability of the Hopf bifurcation, we consider the following system whose equilibrium is shifted to the origin. Let $\tau = \tau_0 + \mu$, then $\mu = 0$ is the Hopf bifurcation value of system (5) at the positive equilibrium $E(x_*, y_*)$ and $\pm i\omega_*$ is the corresponding purely imaginary roots of the characteristic equation.

For the convenience of discussion, let $x_1 = x - x_*$, $x_2 = y - y_*$. The system (5) can be regarded as FDE in $C = C([-1, 0], \mathbb{R}^2)$ as

$$\dot{\mathbf{x}} = L_\mu(x_t) + f(\mu, x_t)$$

where $x(t) = (x_1, x_2)^T \in \mathbb{R}^2$, and $L_\mu : C \to \mathbb{R}$, $f : \mathbb{R} \times C \to \mathbb{R}$ are given respectively by
are eigenvalues of 

\[ L_{\mu}(\phi) = (\tau + \mu) \begin{pmatrix} \frac{1}{25} (-7) \beta e^{\frac{2\pi}{7} - 5 \gamma} & -\frac{7 \beta^2 e^{\frac{2\pi}{7} - 5 \gamma^2}}{1000 \alpha} \\ \frac{1}{50} (-7) \beta e^{\frac{2\pi}{7} - 5 \gamma} & -\frac{7 \beta^2 e^{\frac{2\pi}{7} - 5 \gamma^2}}{1000 \alpha} \end{pmatrix} \begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix} \]

\[ + (\tau + \mu) \begin{pmatrix} \frac{1}{50} (-7) \alpha e^{\frac{2\pi}{7} - 5 x_s} \frac{1}{50} (-7) \beta e^{\frac{2\pi}{7} - 5 y_s} & \frac{1}{50} (-7) \beta e^{\frac{2\pi}{7} - 5 y_s} \end{pmatrix} \begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \end{pmatrix} \]

and

\[ f(\mu, \phi) = (\tau + \mu) \begin{pmatrix} -\frac{7 \alpha}{50} e^{\frac{2\pi}{7}} \phi_1(0) \\ -\frac{7 \alpha}{50} e^{\frac{2\pi}{7}} \phi_1(-1) \end{pmatrix} \]

According to the Riesz representation theorem, there exists a matrix function \( \eta(\vartheta, \mu) \), \( \vartheta \in [-1, 0] \) with bounded variation components such that

\[ L_{\mu}(\phi) = \int_{-1}^0 d\eta(\vartheta, 0)\phi(\vartheta) \quad \text{for} \quad \phi \in C \]

Actually we can take

\[ \eta(\vartheta, \mu) = (\tau + \mu) \begin{pmatrix} \frac{1}{25} (-7) \beta e^{\frac{2\pi}{7} - 5 \gamma} & -\frac{7 \beta^2 e^{\frac{2\pi}{7} - 5 \gamma^2}}{1000 \alpha} \\ \frac{1}{50} (-7) \beta e^{\frac{2\pi}{7} - 5 \gamma} & -\frac{7 \beta^2 e^{\frac{2\pi}{7} - 5 \gamma^2}}{1000 \alpha} \end{pmatrix} \delta(\vartheta) \]

\[ - (\tau + \mu) \begin{pmatrix} \frac{1}{50} (-7) \alpha e^{\frac{2\pi}{7} - 5 x_s} \frac{1}{50} (-7) \beta e^{\frac{2\pi}{7} - 5 y_s} & \frac{1}{50} (-7) \beta e^{\frac{2\pi}{7} - 5 y_s} \end{pmatrix} \delta(\vartheta + 1) \]

where \( \delta \) is the Dirac Delta function. For \( \phi \in C^1([-1, 0], \mathbb{R}^2) \), define

\[ A(\mu)\phi = \left\{ \frac{d\vartheta(\vartheta)}{d\vartheta}, \int_0^\vartheta d\eta(\mu, s)\phi(s) \right\} \quad \vartheta \in [-1, 0), \quad \vartheta = 0, \]

and

\[ R(\mu)\phi = \left\{ 0, \int f(\mu, \vartheta) d\vartheta \right\} \quad \vartheta \in [-1, 0), \quad \vartheta = 0, \]

The system (33) can be represented as

\[ \dot{x} = A_\mu(x_t) + R(\mu)x_t \]

where \( x_t(\vartheta) = x(t + \vartheta) \) for \( \vartheta \in [-1, 0] \).

For \( \psi \in C^1([0, 1], (\mathbb{R}^2)^*) \), define

\[ A^* \psi(s) = \left\{ \frac{d\psi(x)}{dx}, \int_0^s d\nu(t, 0)\psi(-t) \right\} \quad s \in (0, 1], \quad s = 0, \]

Furthermore, for \( \phi \in C^1([-1, 0], (\mathbb{R}^2)^*) \), and \( \psi \in C^1([0, 1], (\mathbb{R}^2)^*) \), we give the bilinear inner product as

\[ \langle \psi(s), \phi(\vartheta) \rangle = \psi(0)\phi(0) - \int_{-1}^0 \int_0^{\vartheta} \psi(\xi - \vartheta) d\eta(\vartheta) \phi(\xi) d\xi \]

where \( \eta(\vartheta) = \eta(\vartheta, 0) \). Then \( A(0) \) and \( A^* \) are adjoint operators. From previous section, we have that \( \pm i\omega_s \tau_0 \) are eigenvalues of \( A(0) \). It is evident that they are also the eigenvalues of the linear operator \( A^* \). We need to compute the eigenvectors of \( A(0) \) and \( A^* \) corresponding to \( i\omega_s \tau_0 \) and \(-i\omega_s \tau_0 \).

Assume that

\[ q(\vartheta) = \left( \frac{1}{e} \right) e^{i\omega_s \tau_0 \vartheta} \]
is the eigenvector of $A(0)$ corresponding to $i\omega_0\tau_s$ and when $\vartheta = 0$, we have

$$q(0) = \begin{pmatrix} 1 \\ c \end{pmatrix}$$

(44)

Then

$$A(0)q(\vartheta) = i\omega_0\tau_s q(\vartheta)$$

(45)

From the definition of $A(0)$ and (33), (36) and (37), we get

$$A(0)q(0) = \begin{pmatrix} i\omega_0\tau_s \\ (i\omega_0\tau_s)c \end{pmatrix}$$

(46)

or,

$$\tau_s \begin{pmatrix} \frac{7}{25}\beta e^{-5\gamma} + \frac{7}{20}\alpha x e^{-i\tau\omega - 5} + i\omega & \frac{7\beta^2 e^{-5\gamma} + 7\gamma e^{-i\tau\omega} - 5}{1000(\gamma + 20)} + \frac{\beta e^{-5\gamma}}{1000} + i\omega \\ \frac{7}{20}\beta y e^{-5\gamma} + \frac{7}{20}\beta y e^{-i\tau\omega - 5} & \frac{7\beta^2 e^{-5\gamma} + 7\gamma e^{-i\tau\omega} - 5}{1000(\gamma + 20)} + \frac{7\gamma e^{-5\gamma}}{1000} + i\omega \end{pmatrix} \begin{pmatrix} 1 \\ c \end{pmatrix}$$

(47)

Thus we obtain,

$$c = -\frac{140\alpha\beta e^{\gamma/20}(y + \gamma e^{i\tau\omega})}{1000i\omega e^{5+i\tau\omega} + 7\beta^2 e^{5+i\tau\omega} + 140\beta^2 e^{5+i\tau\omega} + 7\gamma e^{5+i\tau\omega} + 7\gamma e^{20\gamma/20}y}$$

(48)

Similarly, we can get the eigenvector

$$q^*(s) = D \begin{pmatrix} 1 \\ c^* \end{pmatrix} e^{i\omega_0\tau_s s}$$

(49)

of $A^*$ corresponding to $-i\omega_0\tau_s$, where

$$c^* = \frac{50i\omega e^{5+i\tau\omega} - 7(\alpha x + 2\beta y e^{i\tau\omega})}{7\beta (y + \gamma e^{i\tau\omega})}$$

(50)

Now we evaluate the value of $D$ such that $\langle q^*(s), q(\vartheta) \rangle = 1$. From the bilinear inner product of (32), it follows that

$$1 = \langle q^*(s), q(\vartheta) \rangle$$

$$= \bar{q}^*(0)q(0) - \int_{-1}^{0} \int_{\xi = 0}^{\vartheta} \bar{q}^*(\xi - \vartheta) d\eta(\vartheta) q(\xi) d\xi$$

$$= \bar{D}(1, \bar{c}^*)(1, c)^T - \int_{-1}^{0} \int_{\xi = 0}^{\vartheta} \bar{D}(1, \bar{c}^*) e^{-i\omega_0\tau_s(\xi - \vartheta)} d\eta(\vartheta)(1, c)^T e^{i\omega_0\tau_s \xi} d\xi$$

$$= \bar{D}(1, \bar{c}^*)(1, c)^T - \bar{D}(1, \bar{c}^*) \int_{-1}^{0} d\eta(\vartheta)(1, c)^T e^{i\omega_0\tau_s \vartheta} d\xi$$

$$= \bar{D}(1, \bar{c}^*)(1, c)^T - \bar{D}(1, \bar{c}^*) \int_{-1}^{0} \vartheta e^{i\omega_0\tau_s \vartheta} d\eta(\vartheta)(1, c)^T$$

$$= \bar{D} \left\{ 1 + \bar{c}^* c^* \tau_s \left( -\frac{7\beta e^{-5\gamma}(20\alpha + \beta \gamma e^{i\tau\omega})(\beta c^* y_s + \alpha x_s)}{1000\alpha} \right) e^{-i\omega_0\tau_s} \right\}$$

(51)

Thus, we have

$$\bar{D} = \frac{1}{1 + \bar{c}^* c^* \tau_s \left( -\frac{7\beta e^{-5\gamma}(20\alpha + \beta \gamma e^{i\tau\omega})(\beta c^* y_s + \alpha x_s)}{1000\alpha} \right) e^{-i\omega_0\tau_s}}$$

(52)
In addition, from \( \langle \psi, A\phi \rangle = \langle A^*\psi, \phi \rangle \) and \( Aq(\vartheta) = -i\omega_\tau \gamma\vartheta(\vartheta) \), we can obtain

\[
-i\omega_\tau \langle q^*, \bar{q} \rangle = \langle q^*, A\bar{q} \rangle = \langle A^*q^*, \bar{q} \rangle = i\omega_\tau \langle q^*, \bar{q} \rangle
\]

Hence \( \langle q^*(s), \bar{q}(\vartheta) \rangle = 0 \).

In rest of the section, we calculate the coordinates to describe the center manifold \( C_0 \) at \( \mu = 0 \) by the method used in Hassard paper. Let \( x_t \) be the solution of equation (41) and define \( z(t) = (q^*, x_t) \); then

\[
\dot{z}(t) = \langle q^*, \dot{x}_t \rangle = \langle q^*, A(0)\dot{x}_t + R(0)x_t \rangle = \langle q^*, A(0)\dot{x}_t \rangle + \langle q^*, R(0)x_t \rangle = \langle A^*(0)q^*, \dot{x}_t \rangle + \bar{q}^*(0)f_0(z, \bar{z}) = i\omega_\tau x_\tau z + g(z, \bar{z})
\]

where

\[
g(z, \bar{z}) = \bar{q}^*(0)f_0(z, \bar{z}) = g_{20} \frac{z^2}{2} + g_{11} z\bar{z} + g_{20} \frac{\bar{z}^2}{2} + g_{21} \frac{z^2\bar{z}}{2} + \ldots
\]

Let

\[
W(t, \vartheta) = x_t(\vartheta) - z(t)q(\vartheta) - \bar{z}(t)\bar{q}(\vartheta) = x_t(\vartheta) - 2\Re \{z(t)q(\vartheta)\}
\]

On the center manifold \( C_0 \), we have

\[
W(t, \vartheta) = W(z(t), \bar{z}(t), \vartheta),
\]

where

\[
W(z, \bar{z}, \vartheta) = W_{20}(\vartheta) \frac{z^2}{2} + W_{11}(\vartheta)z\bar{z} + W_{02}(\vartheta) \frac{\bar{z}^2}{2} + W_{30}(\vartheta) \frac{z^3}{6} + \ldots
\]

\( z \) and \( \bar{z} \) are local coordinates for center manifold \( C_0 \) in the direction of \( q^* \) and \( \bar{q}^* \). Note that \( W \) is real if \( x_t \) is real. We only consider real solutions.

It follows from (34) and (38) that

\[
x_t(\vartheta) = (x_{1t}(\vartheta), x_{2t}(\vartheta))^T = W(t, \vartheta) + 2\Re \{z(t)q(\vartheta)\} = W(t, \vartheta) + z(t)q(\vartheta) + \bar{z}(t)\bar{q}(\vartheta) = W_{20}(\vartheta) \frac{z^2}{2} + W_{11}(\vartheta)z\bar{z} + W_{02}(\vartheta) \frac{\bar{z}^2}{2} + W_{30}(\vartheta) \frac{z^3}{6} + \ldots
\]

From (35), it follows that

\[
q(z, \bar{z}) = \bar{q}^*(0)f(0, x_t) = \tau_\vartheta \bar{D}(1, \bar{c}) \left( -\frac{79}{500\vartheta^2}x_{1t}(0) \right)
\]

and with

\[
x_{1t}(0) = z + \bar{z} + W_{20}(0) \frac{z^2}{2} + W_{11}(0)z\bar{z} + W_{02}(0) \frac{\bar{z}^2}{2} + \ldots
\]

\[
x_{1t}(-1) = z e^{-i\tau_\vartheta} \vartheta + \bar{z} e^{i\tau_\vartheta} \vartheta + W_{20}(0) \frac{z^2}{2} + W_{11}(0)z\bar{z} + W_{02}(0) \frac{\bar{z}^2}{2} + \ldots
\]

\[
x_{2t}(-1) = z e^{-i\tau_\vartheta} \vartheta + \bar{z} e^{i\tau_\vartheta} \vartheta + W_{20}(0) \frac{z^2}{2} + W_{11}(0)z\bar{z} + W_{02}(0) \frac{\bar{z}^2}{2} + \ldots
\]
we get,

\[ q(z, \bar{z}) = \tau_* \tilde{D} \left\{ -\frac{7\alpha}{50e^5} \left( z + \bar{z} + W_{20}^1(0) \frac{z^2}{2} + W_{11}^1(0) z \bar{z} + W_{02}^1(0) \bar{z}^2 + \ldots \right)^2 \right. \\
\left. - \frac{7\beta e^5}{50e^5} \left( z \bar{e}^{-i\tau_* \omega_*} + \bar{z} \bar{e}^{i\tau_* \omega_*} + W_{20}^1(0) \frac{z^2}{2} + W_{11}^1(0) z \bar{z} + W_{02}^1(0) \bar{z}^2 + \ldots \right) \right\} \]  

(62)

Comparing the coefficients with (55), we have

\[ g_{20} = 2\tau_* \tilde{D} \left\{ -\frac{7\alpha}{50e^5} - \frac{7\beta e^5}{50e^5} \right\} \]

\[ g_{11} = 2\tau_* \tilde{D} \left\{ -\frac{7\alpha}{25e^5} - \frac{7\beta e^5}{50e^5} \right\} \]

\[ g_{02} = 2\tau_* \tilde{D} \left\{ -\frac{7\alpha}{50e^5} - \frac{7\beta e^5}{50e^5} \right\} \]

\[ g_{21} = \tau_* \tilde{D} \left\{ \frac{7\alpha}{50e^5} \frac{4W_{11}^1(0) + 2W_{20}^1(0)}{50e^5} - \frac{7\beta e^5}{50e^5} e^{-5+2i\tau_* \omega_*} \times \right. \]

\[ (e^{2i\tau_* \omega_*} (W_{20}^1(-1) \bar{c} + W_{20}^2(-1)) + 2cW_{11}^1(-1) + 2W_{11}^2(-1)) \right\} \]

(63)

Since we have \( W_{20}(\vartheta) \) and \( W_{11}(\vartheta) \) in \( g_{21} \), we still need to calculate these terms. From (60) and (57), we get

\[ \bar{W} = \bar{x} - z - \bar{z} \bar{q} = \left\{ \begin{array}{l} AW - 2\text{Re} \{ q^*(0)f_{03}(\vartheta) \}, \quad \vartheta \in [-1, 0), \\
AW - 2\text{Re} \{ q^*(0)f_{03}(\vartheta) \} + f_0, \quad \vartheta = 0, \end{array} \right. \]

\[ \text{def} \quad AW + H(z, \bar{z}, \vartheta), \]

where

\[ H(z, \bar{z}, \vartheta) = H_{20}(\vartheta) \frac{z^2}{2} + H_{11}(\vartheta) z \bar{z} + H_{02}(\vartheta) \frac{\bar{z}^2}{2} + \ldots \]

(65)

Thus, we have

\[ AW(t, \vartheta) - \bar{W} = -H(z, \bar{z}, \vartheta) = -H_{20}(\vartheta) \frac{z^2}{2} - H_{11}(\vartheta) z \bar{z} - H_{02}(\vartheta) \frac{\bar{z}^2}{2} + \ldots \]

(66)

From (58), we obtain

\[ AW(t, \vartheta) = AW_{20}(\vartheta) \frac{z^2}{2} + AW_{11}(\vartheta) z \bar{z} + AW_{02}(\vartheta) \frac{\bar{z}^2}{2} + AW_{30}(\vartheta) \frac{z^3}{6} + \ldots \]

\[ \bar{W} = W_z \bar{z} + W_{\bar{z}} \bar{z} = W_{20}(\vartheta) z \bar{z} + W_{11}(\vartheta)(\bar{z} \bar{z} + z \bar{z}) + \ldots \]

(67)

Thus we have,

\[ (A - 2i\omega_* \tau_*) W_{20}(\vartheta) = -H_{20}(\vartheta), \]

\[ AW_{11}(\vartheta) = -H_{11}(\vartheta) \]

(68)

For \( \vartheta \in [-1, 0) \),

\[ H(z, \bar{z}, \vartheta) = -q^*(0)f_{03}(\vartheta) - q^*(0)f_0(\vartheta) = -g(z, \bar{z})q(\vartheta) - \bar{g}(z, \bar{z})\bar{q}(\vartheta) \]

(69)

Comparing the coefficients with (55), we get

\[ H_{20}(\vartheta) = -g_{20}\vartheta q(\vartheta) - \bar{g}_{02}\bar{q}(\vartheta) \]

\[ H_{11}(\vartheta) = -g_{11}q(\vartheta) - \bar{g}_{11}\bar{q}(\vartheta) \]

(70)
From \( (68) \) and \( (70) \) and the definition of \( A \), it follows that
\[
W_{20}(\vartheta) = 2i\omega_s^n T_s W_{20}(\vartheta) + g_{20q}(\vartheta) + \bar{g}_{02} \bar{q}(\vartheta) \tag{71}
\]
Notice that
\[
q(\vartheta) = \left( \frac{1}{c} \right) e^{i\omega_s^n T_s \vartheta}, \tag{72}
\]
so
\[
W_{20}(\vartheta) = \frac{i g_{20}}{\omega_s^n T_s} q(0) e^{i\omega_s^n T_s \vartheta} + \frac{i \bar{g}_{02}}{3\omega_s^n T_s} \bar{q}(0) e^{-i\omega_s^n T_s \vartheta} + E_1 e^{2i\omega_s^n T_s \vartheta} \tag{73}
\]
where \( E_1 = (E_{1}^{(1)}, E_{1}^{(2)}) \in \mathbb{R}^2 \) is a two-dimensional constant vector.

Similarly from \( (68) \) and \( (70) \), we obtain
\[
W_{11}(\vartheta) = -\frac{i g_{11}}{\omega_s^n T_s} q(0) e^{i\omega_s^n T_s \vartheta} + \frac{i \bar{g}_{11}}{\omega_s^n T_s} \bar{q}(0) e^{-i\omega_s^n T_s \vartheta} + E_2 \tag{74}
\]
where \( E_2 = (E_{2}^{(1)}, E_{2}^{(2)}) \in \mathbb{R}^2 \) is a two-dimensional constant vector.

Next, we need to compute \( E_1 \) and \( E_2 \). From the definition of \( A \) and \( (68) \), we obtain
\[
\int_{-1}^{0} d\eta(\vartheta) W_{20}(\vartheta) = 2i\omega_s^n T_s W_{20}(0) - H_{20}(0)
\]
and
\[
\int_{-1}^{0} d\eta(\vartheta) W_{11}(\vartheta) = -H_{11}(0) \tag{75}
\]
where \( \eta(\vartheta) = \eta(0, \vartheta) \).

By \( (71) \), we have
\[
H_{20}(0) = -g_{20} q(0) - \bar{g}_{02} \bar{q}(0) + 2\tau_s \left( -\frac{7\alpha}{50\epsilon e^{50}} \right) \tag{76}
\]
and
\[
H_{11}(0) = -g_{11} q(0) - \bar{g}_{11} \bar{q}(0) + 2\tau_s \left( -\frac{7\alpha}{25\epsilon} \right) \tag{77}
\]
Substituting \( (73) \) and \( (74) \) into \( (75) \), and noticing that
\[
\left( i\omega_s^n T_s I - \int_{-1}^{0} e^{i\omega_s^n T_s \vartheta} d\eta(\vartheta) \right) q(0) = 0 \tag{78}
\]
and
\[
\left( -i\omega_s^n T_s I - \int_{-1}^{0} e^{-i\omega_s^n T_s \vartheta} d\eta(\vartheta) \right) \bar{q}(0) = 0 \tag{79}
\]
we obtain,
\[
\left( 2i\omega_s^n T_s I - \int_{-1}^{0} e^{2i\omega_s^n T_s \vartheta} d\eta(\vartheta) \right) E_1 = 2\tau_s \left( -\frac{7\alpha}{50\epsilon e^{50}} \right) \tag{80}
\]
This leads to
\[
\left( \frac{7}{25} \beta e^{50} + \frac{7}{50} \alpha x e^{50} - 2i\tau_s \omega_s - 5 + 2i\omega_s \right) - \frac{7}{25} \beta e^{50} - 5\gamma + \frac{7}{50} \beta y e^{50} - 2i\tau_s \omega_s - 5 \right) e^{2i\omega_s^n T_s \vartheta} + \frac{7}{25} \gamma e^{50} - 5\gamma + \frac{7 \beta \gamma e^{50} - 2i\tau_s \omega_s - 5}{1000\alpha} + \frac{7 \beta \gamma e^{50} - 2i\tau_s \omega_s - 5}{1000\alpha} + \frac{2i\omega_s}{1000\alpha} \right) \times E_1 = 2 \left( -\frac{7\alpha}{50\epsilon e^{50}} \right) \tag{81}
\]
It follows that

$$E_1^{(1)} = \frac{2}{A} \left| -\frac{7\alpha}{(50 \epsilon^5)} - \frac{7\beta e^{-2i\tau_\epsilon} - 20}{1000a} \right| (82)$$

and

$$E_1^{(2)} = \frac{2}{A} \left| \frac{7\beta e^{-2i\tau_\epsilon} - 20}{1000a} + \frac{7\beta e^{-2i\tau_\epsilon} - 20}{1000a} \right| (83)$$

where $A =$

$$\left| \frac{7\beta e^{-2i\tau_\epsilon} - 20}{1000a} + \frac{7\beta e^{-2i\tau_\epsilon} - 20}{1000a} \right| (84)$$

Similarly, substituting (74) and (77) into (75), we get

$$\left( \frac{7\beta e^{-2i\tau_\epsilon} - 20}{1000a} + \frac{7\beta e^{-2i\tau_\epsilon} - 20}{1000a} \right) \times E_2 = 2 \left( -\frac{7\alpha}{(50 \epsilon^5)} - \frac{7\beta e^{-2i\tau_\epsilon} - 20}{1000a} \right)$$

It follows that

$$E_2^{(1)} = \frac{2}{B} \left| -\frac{7\alpha}{(50 \epsilon^5)} - \frac{7\beta e^{-2i\tau_\epsilon} - 20}{1000a} \right| (85)$$

and

$$E_2^{(2)} = \frac{2}{B} \left| -\frac{7\beta e^{-2i\tau_\epsilon} - 20}{1000a} + \frac{7\beta e^{-2i\tau_\epsilon} - 20}{1000a} \right| (86)$$

where

$$B = \left| \frac{7\beta e^{-2i\tau_\epsilon} - 20}{1000a} + \frac{7\beta e^{-2i\tau_\epsilon} - 20}{1000a} \right| (87)$$

Thus, we can determine $W_{20}(\theta)$ and $W_{11}(\theta)$ from (73) and (74). Furthermore, $g_{21}$ in (63) can be expressed by the parameters and delay. Thus, we can compute the following values:

$$c_1(0) = \frac{i}{2\omega_\epsilon \tau_\epsilon} \left( g_{20}g_{11} - 2 |g_{11}|^2 - |g_{21}|^2 \right) / 3 + g_{21}$$

$$\mu_2 = -\frac{\text{Re} \{ c_1(0) \}}{\text{Re} \{ \lambda(\tau_\epsilon) \}}$$

$$\beta_2 = 2\text{Re} \{ c_1(0) \},$$

$$T_2 = -\frac{\text{Im} \{ c_1(0) \} + \mu_2 \text{Im} \{ \lambda(\tau_\epsilon) \}}{\omega_\epsilon \tau_\epsilon}$$

which determines the qualities of bifurcation periodic solution in the center manifold at the critical value $\tau_\epsilon$.

Here $\mu_2$ determines the direction of the Hopf bifurcation. If $\mu_2 > 0$, then the bifurcation is supercritical and the bifurcation periodic solutions exist for $\tau > \tau_\epsilon$. $\beta_2$ determines the stability of the bifurcation periodic solutions: it is asymptotically stable if $\beta_2 < 0$. $T_2$ determines the period of the bifurcation periodic solutions; the period increases if $T_2 > 0$. 

24
4 Numerical Simulations

In section 2, we derived that the positive equilibrium $E_*(x_*, y_*)$ is asymptotically stable for $0 \leq \tau < \tau_*$ and unstable for $\tau > \tau_*$ and the system (5) undergoes a Hopf bifurcation when $\tau = \tau_*$. Here we will give the dynamic behaviors of the system with different values of the parameters $\alpha$ and $\beta$ with different time delay $\tau$. The simulation results of system (5) are plotted using the software Mathematica Version 12.1 [49].

4.1 The dynamic behavior with $\alpha = 0.5$ and $\beta = 0.8$

Here we study the dynamic behavior of system (5) by changing the time delay $\tau$ with $\alpha = 0.5$ and $\beta = 0.8$. All the simulations have initial conditions of $x(t) = 35.5$ and $y(t) = 26.5$. This has a unique positive equilibrium at $(x_*, y_*) = (29.1842, 18.2401)$. We observe that the equilibrium is stable for $\tau < 30.8017$ and unstable for $\tau > 30.8017$. At a Hopf bifurcation, no new equilibrium arise. A periodic solution emerges at the equilibrium point as $\tau$ passes through the bifurcation value.
Figure 16: The time series plots, phase plots and the ventilation plot with $\alpha = 0.5$, $\beta = 0.8$ and $\tau = 15$. 
Figure 17: The time series plots, phase plots and the ventilation plot with $\alpha = 0.5$, $\beta = 0.8$ and $\tau = 25$. 
Figure 18: The time series plots, phase plots and the ventilation plot with $\alpha = 0.5$, $\beta = 0.8$ and $\tau = 30.81$. 
Figure 19: The time series plots, phase plots and the ventilation plot with $\alpha = 0.5$, $\beta = 0.8$ and $\tau = 35$. 
Figure 20: The time series plots, phase plots and the ventilation plot with $\alpha = 0.5$, $\beta = 0.8$ and $\tau = 55$. 
4.2 The dynamic behavior with $\alpha = 0.6$ and $\beta = 0.6$

Here we study the dynamic behavior of system (5) by changing the time delay $\tau$ with $\alpha = 0.6$ and $\beta = 0.6$. Again the simulations have initial conditions of $x(t) = 35.5$ and $y(t) = 26.5$. This has a unique positive equilibrium at $(x_*, y_*) = (23.4108, 23.4108)$ as listed in [5]. We observe that the equilibrium is stable for $\tau < 24.9072$ and unstable for $\tau > 24.9072$. At a Hopf bifurcation, no new equilibrium arise. A periodic solution emerges at the equilibrium point as $\tau$ passes through the bifurcation value.
Figure 21: The time series plots, phase plots and the ventilation plot with $\alpha = 0.6$, $\beta = 0.6$ and $\tau = 15$. 
Figure 22: The time series plots, phase plots and the ventilation plot with $\alpha = 0.6$, $\beta = 0.6$ and $\tau = 24.9072$. 
Figure 23: The time series plots, phase plots and the ventilation plot with $\alpha = 0.6$, $\beta = 0.6$ and $\tau = 35$. 
5 Conclusion

We applied nonlinear delay differential equation in modeling the human respiratory system. The two state model which describes the balance equation for carbon dioxide and oxygen was studied. This model has three parameters $\alpha$, $\beta$ and $\tau$. The parameters $\alpha$ and $\beta$ affect the unique positive equilibrium $(x^*, y_*)$ of the model (see Figures 2 and 3) and the time delay $\tau$ affects the stability of the system (see Figures 4, 5, 6 and 8). The critical curves (Equation 25) were used in studying the stability of our model. The three dimensional stability chart is constructed as shown in Figure 7. There is a region enclosed by $\tau = 0$ and the curve $\tau = \tau_1(0)$ in the $(\tau, \alpha)$ and $(\tau, \beta)$ plane where the equilibrium $(x^*, y_*)$ is stable. We have derived analytical expression for equilibrium point and critical delay as a function of $\alpha$ and $\beta$ and we list some of these numerical values in Table 5. These values are also verified by plotting bifurcation diagrams. By picking the delay $\tau$ as the bifurcation parameter, the stability of the positive equilibrium $E^*(x_*, y_*)$ and the existence of Hopf bifurcation are derived. The equilibrium is asymptotically stable for $0 \leq \tau \leq \tau^*$, unstable for $\tau > \tau^*$. The system shows a supercritical Hopf bifurcation giving rise to stable periodic oscillations. These periodic oscillations may be related to the medical condition we refer as periodic breathing. It is to be noted that the delay parameter has effect on the stability but not on the equilibrium state. Additionally, the explicit derivation of the direction of Hopf bifurcation and the stability of the bifurcation periodic solutions are determined with the help of normal form theory and center manifold theorem to delay differential equations.

Finally, some numerical example and simulations are carried out to confirm the analytical findings. The numerical simulations verify the theoretical results.

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