SEIBERG-WITTEN THEORY AND INTEGRABLE SYSTEMS *

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Abstract

We summarize recent results on the resolution of two intimately related problems, one physical, the other mathematical. The first deals with the resolution of the non-perturbative low energy dynamics of certain \( \mathcal{N} = 2 \) supersymmetric Yang-Mills theories. We concentrate on the theories with one massive hypermultiplet in the adjoint representation of an arbitrary gauge algebra \( \mathcal{G} \). The second deals with the construction of Lax pairs with spectral parameter for certain classical mechanics “Calogero-Moser” integrable systems associated with an arbitrary Lie algebra \( \mathcal{G} \). We review the solution to both of these problems as well as their interrelation.

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I. INTRODUCTION

Some of the most important physical problems of contemporary theoretical physics concern the behavior of gauge theories and string theory at strong coupling. For gauge theories, these include the problems of confinement of color, of dynamical chiral symmetry breaking, of the strong coupling behavior of chiral gauge theories, of the dynamical breaking of supersymmetry. In each of these areas, major advances have been achieved over the past few years, and a useful resolution of some of these difficult problems appears to be within sight. For string theory, these include the problems of dynamical compactification of the 10-dimensional theory to string vacua with 4 dimensions and of supersymmetry breaking at low energies. Already, it has become clear that, at strong coupling, the string spectrum is radically altered and effectively derives from the unique 11-dimensional M-theory.

This rapid progress was driven in large part by the Seiberg-Witten solution of $N = 2$ supersymmetric Yang-Mills theory for $SU(2)$ gauge group [1] and by the discovery of D-branes in string theory. Some of the key ingredients underlying these developments are

1. Restriction to solving for the low energy behavior of the non-perturbative dynamics, summarized by the low energy effective action of the theory.
2. High degrees of supersymmetry. This has the effect of imposing certain holomorphicity constraints on parts of the low energy effective action, and thus of restricting its form considerably. For gauge theories in 4-dimensions, we distinguish the following degrees of supersymmetry.
   - $N = 1$ supersymmetry supports chiral fermions and is the starting point for the Minimal Supersymmetric Standard Model, the simplest extension of the Standard Model to include supersymmetric partners.
   - $N = 2$ supersymmetry only supports non-chiral fermions and is thus less realistic as a particle physics model, but appears better “solvable”. This is where the Seiberg-Witten solution was constructed.
   - $N = 4$ is the maximal amount of supersymmetry, and a special case of $N = 2$ supersymmetry with only non-chiral fermions and vanishing renormalization group $\beta$-function. Dynamically, the latter theory is the simplest amongst 4-dimensional gauge theories, and offers the best hopes for admitting an exact solution.
3. Electric-magnetic and Montonen-Olive duality. The free Maxwell equations are invariant under electric- magnetic duality when $\vec{E} \rightarrow \vec{B}$ and $\vec{B} \rightarrow -\vec{E}$. In the presence of matter, duality will require the presence of both electric charge $e$ and magnetic...
monopole charge $g$ whose magnitude is related by Dirac quantization $e \cdot g \sim \hbar$. Thus, weak electric coupling is related to large magnetic coupling. Conversely, problems of large electric coupling (such as confinement of the color electric charge of quarks) are mapped by duality into problems of weak magnetic charge. It was conjectured by Montonen and Olive that the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory for any gauge algebra $\mathcal{G}$ is mapped under the interchange of electric and magnetic charges, i.e. under $e \leftrightarrow 1/e$ into the theory with dual gauge algebra $\mathcal{G}^\vee$. When combined with the shift-invariance of the instanton angle $\theta$ this symmetry is augmented to the duality group $SL(2,\mathbb{Z})$, or a subgroup thereof.

Finally, the Maldacena equivalence between Type IIB superstring theory on $AdS_5 \times S^5$ and 4-dimensional $\mathcal{N} = 4$ superconformal Yang-Mills theory is conjectured to hold at strong coupling. The AdS/SCFT correspondence thus establishes a link between certain non-perturbative phenomena in string theory and in gauge theory.

Of central interest to many of these exciting developments is the 4-dimensional supersymmetric Yang-Mills theory with maximal supersymmetry, $\mathcal{N} = 4$, and with arbitrary gauge algebra $\mathcal{G}$. Here, we shall consider a generalization of this theory, in which a mass term is added for part of the $\mathcal{N} = 4$ gauge multiplet, softly breaking the $\mathcal{N} = 4$ symmetry to $\mathcal{N} = 2$. As an $\mathcal{N} = 2$ supersymmetric theory, the theory has a $\mathcal{G}$-gauge multiplet, and a hypermultiplet in the adjoint representation of $\mathcal{G}$ with mass $m$.

This generalized theory enjoys many of the same properties as the $\mathcal{N} = 4$ theory: it has the same field contents; it is ultra-violet finite; it has vanishing renormalization group $\beta$-function, and it is expected to have Montonen-Olive duality symmetry. For vanishing hypermultiplet mass $m = 0$, the $\mathcal{N} = 4$ theory is recovered. For $m \to \infty$, it is possible to choose dependences of the gauge coupling and of the gauge scalar expectation values so that the limiting theory is one of many interesting $\mathcal{N} = 2$ supersymmetric Yang-Mills theories. Amongst these possibilities for $\mathcal{G} = SU(N)$ for example, are the theories with any number of hypermultiplets in the fundamental representation of $SU(N)$, or with product gauge algebras $SU(N_1) \times SU(N_2) \times \cdots \times SU(N_p)$, and hypermultiplets in fundamental and bi-fundamental representations of these product algebras.

Remarkably, the Seiberg-Witten theory for $\mathcal{N} = 2$ supersymmetric Yang-Mills theory for arbitrary gauge algebra $\mathcal{G}$ appears to be intimately related with the existence of certain classical mechanics integrable systems. This relation was first suspected on the basis of the similarity between the Seiberg-Witten curves and the spectral curves of certain integrable models [2]. Then, arguments were developed that Seiberg-Witten theory naturally produces integrable structures [3]. But a connection derived from first principles between
Seiberg-Witten theory and integrable models still seems to be lacking.

For the $\mathcal{N} = 2$ supersymmetric Yang-Mills theory with massive hypermultiplet, the relevant integrable system appears to be the *elliptic Calogero-Moser system*. For $SU(N)$ gauge group, Donagi and Witten [3] proposed that the spectral curves of the $SU(N)$ Hitchin system should play the role of the Seiberg-Witten curves. Krichever (in unpublished work), Gorsky and Nekrasov, and Martinec [4] recognized that the $SU(N)$ Hitchin system spectral curves are identical to those of the $SU(N)$ elliptic Calogero-Moser integrable system. That the $SU(N)$ elliptic Calogero-Moser curves (and associated Seiberg-Witten differential) do indeed provide the Seiberg-Witten solution for the $\mathcal{N} = 2$ theory with one massive hypermultiplet was fully established by the authors in [5], where it was shown that

1. the resulting effective prepotential $F$ (and thus the low energy effective action) reproduces correctly the logarithmic singularities predicted by perturbation theory;
2. $F$ satisfies a renormalization group type equation which determines explicitly and efficiently instanton contributions to any order;
3. the prepotential in the limit of large hypermultiplet mass $m$ (as well as large gauge scalar expectation value and small gauge coupling) correctly reproduces the prepotentials for $\mathcal{N} = 2$ super Yang Mills theory with any number of hypermultiplets in the fundamental representation of the gauge group.

The fundamental problem in Seiberg-Witten theory is to determine the Seiberg-Witten curves and differentials, corresponding to an $\mathcal{N} = 2$ supersymmetric gauge theory with arbitrary gauge algebra $\mathcal{G}$, and a massive hypermultiplet in an arbitrary representation $R$ of $\mathcal{G}$, subject to the constraint of asymptotic freedom or conformal invariance. With the correspondence between Seiberg-Witten curves and the spectral curves of classical mechanics integrable systems [3], this problem is equivalent to determining a general integrable system, associated with the Lie algebra $\mathcal{G}$ and the representation $R$.

The $\mathcal{N} = 2$ theory for arbitrary gauge algebra $\mathcal{G}$ and with one massive hypermultiplet in the adjoint representation was one such outstanding case when $\mathcal{G} \neq SU(N)$. Actually, as discussed previously, upon taking suitable limits, this theory contains a very large number of models with smaller hypermultiplet representations $R$, and in this sense has a universal aspect. It appeared difficult to generalize directly the Donagi-Witten construction of Hitchin systems to arbitrary $\mathcal{G}$, and it was thus natural to seek this generalization directly amongst the elliptic Calogero-Moser integrable systems. It has been known now for a long time, thanks to the work of Olshanetsky and Perelomov [6], that Calogero-Moser systems can be defined for any simple Lie algebra. Olshanetsky and Perelomov also showed that the
Calogero-Moser systems for *classical* Lie algebras were integrable, although the existence of a spectral curve (or Lax pair with spectral parameter) as well as the case of exceptional Lie algebras remained open. Thus several immediate questions are:

- Does the elliptic Calogero-Moser system for general Lie algebra \( G \) admit a Lax pair with spectral parameter?
- Does it correspond to the \( \mathcal{N} = 2 \) supersymmetric gauge theory with gauge algebra \( G \) and a hypermultiplet in the adjoint representation?
- Can this correspondence be verified in the limiting cases when the mass \( m \) tends to 0 with the theory acquiring \( \mathcal{N} = 4 \) supersymmetry and when \( m \to \infty \), with the hypermultiplet decoupling in part to smaller representations of \( G \)?

The purpose of this paper is to review the solution to these questions, which were obtained in [7], [8] and [9]. In summary, the answers can be stated succinctly as follows.

- The elliptic Calogero-Moser systems defined by an arbitrary simple Lie algebra \( G \) do admit Lax pairs with spectral parameters.
- The correspondence between elliptic \( G \) Calogero-Moser systems and \( \mathcal{N} = 2 \) supersymmetric \( G \) gauge theories with matter in the adjoint representation holds directly when the Lie algebra \( G \) is simply-laced. When \( G \) is not simply-laced, the correspondence is with new integrable models, *the twisted elliptic Calogero-Moser systems* introduced in [7,8].
- The new twisted elliptic Calogero-Moser systems also admit a Lax pair with spectral parameter [7].
- In the scaling limit \( m = M q^{-\frac{2}{2} \delta} \to \infty \), \( M \) fixed, the twisted (respectively untwisted) elliptic \( G \) Calogero-Moser systems tend to the Toda system for \( (G^{(1)})^\vee \) (respectively \( G^{(1)} \)) for \( \delta = \frac{1}{h_G} \) (respectively \( \delta = \frac{1}{h_G^\vee} \)). Here \( h_G \) and \( h_G^\vee \) are the Coxeter and the dual Coxeter numbers of \( G \) [8].

The remainder of this paper is organized as follows. In §II, we briefly review supersymmetric gauge theories, the set-up and basic constructions of Seiberg-Witten theory. In §III, we discuss the elliptic Calogero-Moser systems introduced by Olshanetsky and Perelomov long ago, and present the new twisted elliptic Calogero-Moser systems introduced in [7,8]. In §IV, we show how these systems tend to the Toda systems in certain limits and discuss their integrability properties and Lax pairs with spectral parameter in §V. Finally, in §VI and §VII, we discuss the Seiberg-Witten solution for the \( \mathcal{N} = 2 \) supersymmetric Yang-Mills theories and a massive hypermultiplet in the adjoint representation of the gauge algebra for \( G = SU(N) \) and for arbitrary \( G \) respectively.
II. SEIBERG-WITTEN THEORY

Supersymmetric Yang-Mills theories are ordinary field theories of scalar, spin 1/2 fermions and gauge fields, with field contents fitting into representations of the supersymmetry algebra and with certain special relations between the gauge, Yukawa and Higgs self-couplings. For each of $\mathcal{N} = 1, 2, 4$, there is a gauge multiplet $(g)$ in the adjoint representation of the gauge algebra $\mathcal{G}$ and for $\mathcal{N} = 1, 2$ there are matter multiplets $(m)$ in an arbitrary representation $R$ of $\mathcal{G}$.

a) Supersymmetry multiplets

(1) For $\mathcal{N} = 1$, we have
   - $(g)$ the gauge multiplet $(A_\mu, \lambda)$ containing a gauge field $A_\mu$ and a Majorana fermion $\lambda$;
   - $(m)$ the chiral multiplet $(\varphi, \psi)$ containing a complex scalar $\varphi$ and a chiral fermion $\psi$.

(2) For $\mathcal{N} = 2$, we have
   - $(g)$ the gauge multiplet $(A_\mu, \lambda_\pm, \phi)$ containing a gauge field $A_\mu$, a Dirac fermion $\lambda_\pm$ and a complex scalar $\phi$, which we shall often refer to as the gauge scalar. Under an $\mathcal{N} = 1$ supersymmetry subalgebra, the $\mathcal{N} = 2$ gauge multiplet is the direct sum of the $\mathcal{N} = 1$ gauge multiplet and an $\mathcal{N} = 1$ chiral multiplet in the adjoint representation of $\mathcal{G}$.
   - $(m)$ the hypermultiplet $(\psi_\pm, H_\pm)$ contains a Dirac fermion $\psi_\pm$ and two complex scalars $H_\pm$. Under an $\mathcal{N} = 1$ subalgebra, this multiplet is the sum of one left and one right $\mathcal{N} = 1$ chiral multiplets.

(3) For $\mathcal{N} = 4$, we have
   - $(g)$ the gauge multiplet $(A_\mu, \lambda_\alpha, \phi_I)$, containing a gauge field $A_\mu$, four Majorana spinors $\lambda_\alpha$, $\alpha = 1, \cdots, 4$ and six real scalars $\phi_I$, $I = 1, \cdots, 6$. Under an $\mathcal{N} = 2$ subalgebra, the multiplet is the sum of an $\mathcal{N} = 2$ gauge multiplet and an $\mathcal{N} = 2$ hypermultiplet in the adjoint representation of the gauge algebra $\mathcal{G}$.
   - $(m)$ there is no matter multiplet for $\mathcal{N} = 4$.

b) Supersymmetric Lagrangians

For the study of Seiberg-Witten theory, we shall need both the $\mathcal{N} = 2$ supersymmetric microscopic (renormalizable) Lagrangian as well as $\mathcal{N} = 2$ supersymmetric effective Lagrangians. Both types may be viewed as general Lagrangians involving the multiplets given above, but with the restriction that only terms are retained with at most two derivatives on any term involving boson fields, and one derivative on any term involving fermion fields. This is the usual approximation made when dealing with effective low energy theories, and
also happens to be one of the criteria for renormalizability. These effective Lagrangians are always polynomial in the gauge and fermion fields, but depend upon the various scalar fields through possibly general functions. Supersymmetry imposes certain holomorphicity conditions on some of these functions, a property fundamental in the Seiberg-Witten analysis. Henceforth, we restrict to considering only such Lagrangians.

For $\mathcal{N} = 1$ supersymmetric theories with gauge multiplet $(A^a_\mu, \lambda^a)$, $a = 1, \ldots, \dim \mathcal{G}$ and chiral multiplets $(\varphi^i, \psi^i)$, $i = 1, \ldots, N_f$, the key parts of the most general Lagrangian are given by the kinetic terms type and potential terms for the fields (all other terms such as Yukawa couplings are omitted, as we shall not need their form)

$$
\mathcal{L} = - g_{ij} \left[ D_\mu \varphi^i D^\mu \varphi^j + i \bar{\psi}^j \sigma^\mu D_\mu \psi^i \right] - \frac{1}{2} g^{ij} \frac{\partial W(\varphi)}{\partial \varphi^i} \frac{\partial \bar{W}(\bar{\varphi})}{\partial \bar{\varphi}^j} - \frac{1}{2} \tau_{ab}(\varphi) \left[ \frac{i}{16} F^a_{\mu\nu} F^{\mu\nu a} - \frac{1}{16} \tilde{F}^a_{\mu\nu} \tilde{F}^{\mu\nu a} + \bar{\lambda}^b \sigma^\mu D_\mu \lambda^a \right] + c.c. + \cdots
$$

Here $g_{ij} = \partial_i \partial_j K(\varphi, \bar{\varphi})$ is the Kähler metric on the scalar fields, $D_\mu$ are suitable covariant derivatives with respect to the gauge field (and the Kähler connection for $D_\mu$ on fermions), and $F_{\mu\nu}$ is the field strength of $A_\mu$. The superpotential $W(\varphi)$ and the gauge coupling field $\tau_{ab}(\varphi)$ are constrained by $\mathcal{N} = 1$ supersymmetry to be complex analytic functions of $\varphi$. For $\mathcal{N} = 1$ supersymmetric theories, it is very convenient to derive the above results from an $\mathcal{N} = 1$ superfield formulation, in which the complex analyticity of $W$ and $\tau_{ab}$ emerges from the fact that these functions arise in $F$-terms, while the Kähler potential comes from a $D$-term. In $F$-terms, only superfields of one chirality enter; since a chiral fermion is in the same multiplet as the complex scalar field $\varphi$, but not $\bar{\varphi}$, all $\varphi$-dependence emerging from $F$-terms is inherently complex analytic. With a generalization to $\mathcal{N} = 2$ and $\mathcal{N} = 4$ in mind, where no convenient off-shell superfield formulation is available, we prefer here to use component language throughout.

For $\mathcal{N} = 2$ supersymmetric theories, the gauge multiplet consists of an $\mathcal{N} = 1$ gauge multiplet and an $\mathcal{N} = 1$ chiral multiplet in the adjoint representation of the gauge algebra. Thus, part of the components of the chiral field $(\varphi^i, \psi^i)$ are in the adjoint representation, and we shall denote that part by $(\phi^a, \psi^a)$, with the index $a$ running through the adjoint representation. (The remaining components make up hypermultiplets.) Since the adjoint representation is real, there is no distinction between $a$ and $\bar{a}$. We shall concentrate on that part of the Lagrangian (2.1) which involves only the $\mathcal{N} = 2$ vector multiplet fields.

Enforcing $\mathcal{N} = 2$ supersymmetry on the $\mathcal{N} = 1$ Lagrangian (2.1) for the vector multiplet is not so easy. However, it is straightforward to enforce some necessary conditions.
The $\mathcal{N} = 2$ supersymmetry algebra is invariant under an $SU(2)_R$ group which rotates the two independent supercharges into one another, and thus rotates the two spinors in the $\mathcal{N} = 2$ gauge multiplet into one another as well. In the $\mathcal{N} = 1$ language used in (2.1), these two spinors are $\psi^a$ and $\lambda^a$. $\mathcal{N} = 2$ supersymmetry requires invariance under $SU(2)_R$, and thus invariance of the Lagrangian under this symmetry. Invariance of the kinetic terms for $\lambda$ and $\psi$ in (2.1) immediately yields a relation between the Kähler metric and the gauge coupling function

$$\frac{\partial^2 K(\phi, \bar{\phi})}{\partial \phi^a \partial \bar{\phi}^b} = \text{Im} \tau_{ab}(\phi)$$

(2.2)

Since $\tau_{ab}(\phi)$ is a complex analytic function of $\phi$, the partial derivative of (2.2) with respect to $\phi^a$ is complex analytic, and thus

$$\frac{\partial^2 K(\phi, \bar{\phi})}{\partial \phi^a \partial \bar{\phi}^c} = T_{abc}(\phi) \bar{\phi}^b$$

(2.3)

for some complex analytic function $T_{abc}$ of $\phi$. The most general solution to (2.3) is very easily obtained by integrating up twice, and may be expressed in terms of a single complex analytic function $F$, called the superpotential. In terms of $F$, the quantities $\tau$ and $K$ are given by

$$\tau_{ab}(\phi) = \frac{\partial^2 F(\phi)}{\partial \phi^a \partial \phi^b}$$

(2.4a)

$$K(\phi, \bar{\phi}) = -\frac{i}{2} \bar{\phi}^b \partial F \bar{\phi}^b + \frac{i}{2} \phi^c \partial \bar{F} \phi^c$$

(2.4b)

This restricted form of the $\mathcal{N} = 2$ effective action is closely related with special geometry.

Imposing $SU(2)_R$-symmetry on the Yukawa couplings requires that the superpotential for the gauge scalars be similarly restricted. The resulting expressions are rather complicated, and we shall give below only the special cases needed for our analysis. Analogous conditions are required upon inclusion of hypermultiplets, but we shall not give those here. Once these necessary conditions arising from $SU(2)_R$ invariance have been imposed, it may in fact be shown that the Lagrangian obtained in this way is indeed $\mathcal{N} = 2$ supersymmetric [10].

c) The Set-Up for Seiberg-Witten Theory

The starting point for Seiberg-Witten theory is an $\mathcal{N} = 2$ supersymmetric Yang-Mills theory with gauge algebra $G$ and hypermultiplets in a representation $R$ of $G$ with masses
The microscopic Lagrangian is completely fixed by $\mathcal{N} = 2$ supersymmetry in terms of the gauge coupling $g$ and the instanton angle $\theta$, and is given by

$$\mathcal{L} = \frac{1}{4g^2} F^a_{\mu\nu} F^{a\mu\nu} + \frac{\theta}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{a\mu\nu} + D_\mu \bar{\phi} D^{\mu} \phi + \text{tr}[\bar{\phi}, \phi]^2 + \cdots$$

(2.5)

where we have neglected hypermultiplet and fermion terms.

The low energy effective theory corresponding to this model can be analyzed by studying first the structure of the vacuum. $\mathcal{N} = 2$ supersymmetric vacuum states can occur whenever the vacuum energy is exactly zero. Since the energy is always positive in a supersymmetric theory, we are guaranteed that any zero energy solution is a vacuum. This is the case here for vanishing gauge fields and constant gauge scalar fields $\phi$ for which the potential energy term also vanishes. The potential energy vanishes if and only if $[\bar{\phi}, \phi] = 0$, a condition equivalent to the vacuum expectation value of $\phi$ being a linear combination of the Cartan generators of the gauge algebra $\mathcal{G}$,

$$<\phi> = \sum_{j=1}^{n} a_j h_j \quad n = \text{rank } \mathcal{G}$$

(2.6)

Here, the complex parameters $a_j$ are usually referred to as the quantum moduli, or also as the quantum order parameters of the $\mathcal{N} = 2$ vacua.

For generic values of the parameters $a_j$, the $\mathcal{G}$-gauge symmetry will be broken down to $U(1)^n$/Weyl($\mathcal{G}$), and the low energy theory is that of $n$ different Coulomb fields, up to global identifications by Weyl($\mathcal{G}$). Since $\mathcal{N} = 2$ supersymmetry is unbroken in any of these vacua, the low energy effective Lagrangian will have to be invariant under $\mathcal{N} = 2$ supersymmetry. But, we have already given a description of all such effective actions before, in terms of a complex analytic superpotential $\mathcal{F}(\phi)$. In the case of $n$ different $U(1)$ gauge fields, this effective Lagrangian is particularly simple, and we have

$$\mathcal{L}_{\text{effective}} = \frac{1}{4} \text{Im}(\tau_{ij}) F^{i\mu\nu} F^{j\mu\nu} + \frac{1}{4} \text{Re}(\tau_{ij}) F^{i\mu\nu} \tilde{F}^{j\mu\nu} + \partial_\mu \bar{\phi}^j \partial^{\mu} \phi_{Dj} + \text{fermions}$$

(2.7)

Here, the dual gauge scalar $\phi_{Dj}$ and the gauge coupling function $\tau_{ij}$ are both given in terms of the prepotential $\mathcal{F}$

$$\phi_{Dj} = \frac{\partial \mathcal{F}(\phi)}{\partial \phi_j} \quad \tau_{ij} = \frac{\partial^2 \mathcal{F}(\phi)}{\partial \phi_i \partial \phi_j}$$

(2.8)

The form of the effective Lagrangian (2.7) is the same for any of the values of the complex moduli of $\mathcal{N} = 2$ vacua, with the understanding that the fields $\phi_j$ take on the expectation value $<\phi_j> = a_j$. Since the prepotential $\mathcal{F}(\phi)$ is a function of the fields $\phi$ only, but
not of derivatives of $\phi$, the prepotential will be completely determined by its values on the vacuum expectation values of the field, namely by its values on the quantum order parameters $a_j$.

d) The Seiberg-Witten Solution

The object of Seiberg-Witten theory is the determination of the prepotential $F(a_j)$, from which the entire low energy effective action will be known. This is achieved by exploiting the physical conditions satisfied by $F$.

(1) $F(a_j)$ is complex analytic in $a_j$ in view of $\mathcal{N} = 2$ supersymmetry, as shown in b) above.

(2) The matrix $\text{Im} \tau_{ij} = \text{Im} \partial_i \partial_j F$ is positive definite, since by (2.7), it coincides with the metric on the kinetic terms for the gauge fields $A_j$.

(3) The large $a_j$ behavior is known from perturbative quantum field theory calculations and asymptotic freedom, and is given by $F(a) \sim (a_i - a_j)^2 \ln(a_i - a_j)^2$.

More precisely, for gauge algebra $G$ and hypermultiplets in the representation $R$ of $G$, $F(a)$ is of the form

$$F(a) = F^{\text{class}}(a) + \sum_{d=1}^{\infty} F_d(a) \Lambda^{(2h_G^\vee - I(R))d}$$

$$- \frac{1}{8 \pi i} \left[ \sum_{\alpha \in \mathcal{R}(G)} (\alpha \cdot a)^2 \ln \left( \frac{(\alpha \cdot a)^2}{\Lambda^2} \right) - \sum_{\lambda \in W(R)} (\lambda \cdot a + m)^2 \ln \left( \frac{(\lambda \cdot a + m)^2}{\Lambda^2} \right) \right].$$

(2.9)

Here $\Lambda$ is a dynamically generated scale introduced by renormalization, $h_G^\vee$ is the quadratic Casimir of $G$ (equal to the dual Coxeter number), $I(R)$ is the Dynkin index of the representation $R$, and $\mathcal{R}(G)$ and $W(R)$ denote respectively the roots of $G$ and the weights of the representation $R$. The terms on the right hand side of (2.9) represent respectively the classical prepotential, the one-loop perturbative corrections (higher loops do not contribute in view of non-renormalization theorems), and the instanton corrections $F^{(d)} = F_d \Lambda^{(2h_G^\vee - I(R))d}$ of all orders $d$. In general, it is prohibitively difficult to determine the coefficients $F_d$ from field theory methods. For conformally invariant theories, the expansion (2.9) is replaced by a similar one where the dynamical scale $\Lambda$ is replaced by a modular invariant $q = e^{2\pi i \tau}$ (see e.g. (6.2) and (6.7) below).

As a result of the requirements (1) and (2), it follows immediately that $F$ cannot be a single-valued function of the $a_j$. For if it were, $\text{Im} \tau_{ij}$ would be both harmonic and bounded from below, which would imply that it must be independent of $a_j$. But, from (3), we know
that $\tau_{ij}$ is not constant at large $a_j$. And indeed, from (3) again, it is clear that neither $F$ nor $\tau_{ij}$ are single valued functions of the $a_j$.

As is clear from the large $a_j$ behavior $\tau_{ij}(a) \sim \ln(a_i - a_j)$, one of the ways in which $\tau_{ij}(a)$ is multiple valued is by shifts of any of the matrix elements by an integer. This ambiguity does not affect the physics of the low energy effective action (2.7), because the constant shifts in Re$(\tau_{ij})$ are like the shifts of the instanton angle $\theta$ by $2\pi$ times an integer and not observable. A more complicated multiple-valuedness consists in taking $\tau \to -\tau^{-1}$, and corresponds to electric-magnetic duality, as shown by Seiberg and Witten [1]. The combination of these two types of transformations produces the full duality group $SL(2n, \mathbb{Z})$ of monodromies of $\tau$.

A natural setting in which the above monodromy problem may be solved is provided by families of Riemann surfaces, called the Seiberg Witten curves, denoted by $\Gamma$. Indeed, letting the quantum moduli $a_j$ correspond to moduli of the Riemann surfaces, there is automatically a complex analytic period matrix, whose imaginary part is positive definite, and whose monodromy group corresponds to the modular group of the surface. For $G = SU(2)$ gauge group and no hypermultiplets for example, the Seiberg-Witten curve is a of genus 1, and may be represented as a double sheeted cover of the complex plane, $\Gamma(u) = \{(x, y); \ y^2 = (x - \Lambda)(x + \Lambda)(x - u)\}$. Here $u$ is an auxiliary parameter, which will be related to the quantum modulus $a$, and $\Lambda$ is the renormalization scale. We shall choose the branch cut between the points $x = \pm \Lambda$. The quantum modulus and prepotential are then given by

$$a(u) = \frac{1}{2\pi i} \oint_A (x - u) \frac{dx}{y}, \quad a_D(u) = \frac{\partial F(a)}{\partial a} = \frac{1}{2\pi i} \oint_B (x - u) \frac{dx}{y}$$  \hspace{1cm} (2.10)$$

where the $A$-cycle may be chosen around the branch cut between $\pm \Lambda$ and the $B$-cycle between the branch points $\pm \Lambda$ and $u$. As $u \to \pm \Lambda$, the elliptic curve produces a singularity which physically is interpreted as caused by the vanishing of the mass of a magnetic monopole or dyon.

Starting from the Seiberg-Witten solution for gauge group $G = SU(2)$, one may abstract the general set-up of the Seiberg-Witten solution, expected for arbitrary gauge algebra $G$ with rank $n$ and general hypermultiplet representation. The ingredients are

1. The Seiberg-Witten curve is a family of Riemann surfaces $\Gamma(u_1, \cdots, u_n)$ dependent on $n$ auxiliary complex parameters $u_j$, which are related to the quantum moduli $a_j$. The Seiberg-Witten curve will also depend upon the gauge coupling $g$ and $\theta$-angle and on the hypermultiplet masses $m_k$. 
(2) The Seiberg-Witten meromorphic differential 1-form \( d\lambda \) on \( \Gamma \), whose residues are linear in the hypermultiplet masses \( m_k \). Since the hypermultiplet masses receive no quantum corrections as \( a_j \) varies, the derivatives \( \partial(d\lambda)/\partial a_j \) are holomorphic 1-forms.

![Image](image)

(3) The quantum moduli and the prepotential are given by

\[
a_j = \frac{1}{2\pi i} \oint_{A_j} d\lambda \\
a_{Dj} = \frac{\partial F}{\partial a_j} = \frac{1}{2\pi i} \oint_{B_j} d\lambda
\] (2.11)

Shortly after the initial work of Seiberg and Witten, the curves and differentials for general \( SU(N) \), with and without hypermultiplets in the fundamental representation were proposed, as well as generalizations to the gauge groups \( SO(N) \) and \( Sp(N) \) [11]. See also [12-14]. Use was made of the \( R \)-charge assignments of the fields, the singularity structure of the degenerations of the Seiberg-Witten curve, and much educated guess work.

III. TWISTED AND UNTWISTED CALOGERO-MOSER SYSTEMS

a) The \( SU(N) \) Elliptic Calogero-Moser System

The original elliptic Calogero-Moser system is the system defined by the Hamiltonian

\[
H(x, p) = \frac{1}{2} \sum_{i=1}^{N} p_i^2 - \frac{1}{2} m^2 \sum_{i \neq j} \wp(x_i - x_j) \] (3.1)

Here \( m \) is a mass parameter, and \( \wp(x) \) is the Weierstrass \( \wp \)-function, defined on a torus \( \mathbb{C}/(2\omega_1 \mathbb{Z} + 2\omega_2 \mathbb{Z}) \). As usual, we denote by \( \tau = \omega_2/\omega_1 \) the moduli of the torus, and set \( q = e^{2\pi i \tau} \). The well-known trigonometric and rational limits with respective potentials

\[
-\frac{1}{2} m^2 \sum_{i \neq j} \frac{1}{4 \sinh^2 \left( \frac{x_i - x_j}{2} \right)} \\
-\frac{1}{2} m^2 \sum_{i \neq j} \frac{1}{(x_i - x_j)^2}
\]

arise in the limits \( \omega_1 = -i \pi, \omega_2 \to \infty \) and \( \omega_1, \omega_2 \to \infty \). All these systems have been shown to be completely integrable in the sense of Liouville, i.e. they all admit a complete set of integrals of motion which are in involution [15-17]. For a recent review of some applications of these models see [18].

Our considerations require however a notion of integrability which is in some sense more stringent, namely the existence of a Lax pair \( L(z), M(z) \) with spectral parameter \( z \). Such a Lax pair was obtained by Krichever [19] in 1980. He showed that the Hamiltonian system (3.1) is equivalent to the Lax equation \( \dot{L}(z) = [L(z), M(z)] \), with \( L(z) \) and \( M(z) \) given by the following \( N \times N \) matrices

\[
L_{ij}(z) = p_i \delta_{ij} - m(1 - \delta_{ij}) \Phi(x_i - x_j, z) \\
M_{ij}(z) = m \delta_{ij} \sum_{k \neq i} \wp(x_i - x_k) - m(1 - \delta_{ij}) \Phi'(x_i - x_j, z). \] (3.2)
The function $\Phi(x, z)$ is defined by

$$\Phi(x, z) = \frac{\sigma(z-x)}{\sigma(z)} e^{x\zeta(z)},$$

(3.3)

where $\sigma(z)$, $\zeta(z)$ are the usual Weierstrass $\sigma$ and $\zeta$ functions on the torus $\mathbb{C}/(2\omega_1 \mathbb{Z} + 2\omega_2 \mathbb{Z})$. The function $\Phi(x, z)$ satisfies the key functional equation

$$\Phi(x, z)\Phi'(y, z) - \Phi(y, z)\Phi'(x, z) = (\wp(x) - \wp(y))\Phi(x+y, z).$$

(3.4)

It is well-known that functional equations of this form are required for the Hamiltonian equations of motion to be equivalent to the Lax equation $\dot{L}(z) = [L(z), M(z)]$ with a Lax pair of the form (3.2). Often, solutions had been obtained under additional parity assumptions in $x$ (and $y$), which prevent the existence of a spectral parameter. The solution $\Phi(x, z)$ with spectral parameter $z$ is obtained by dropping such parity assumptions for general $z$. It is a relatively recent result of Braden and Buchstaber [20] that, conversely, general functional equations of the form (3.4) essentially determine $\Phi(x, z)$.

b) Calogero-Moser Systems defined by Lie Algebras

As Olshanetsky and Perelomov [6] realized very early on, the Hamiltonian system (3.1) is only one example of a whole series of Hamiltonian systems associated with each simple Lie algebra. More precisely, given any simple Lie algebra $\mathcal{G}$, Olshanetsky and Perelomov [6] introduced the system with Hamiltonian

$$H(x, p) = \frac{1}{2} \sum_{i=1}^{r} p_i^2 - \frac{1}{2} \sum_{\alpha \in \mathcal{R}(\mathcal{G})} m_{|\alpha|}^2 \wp(\alpha \cdot x),$$

(3.5)

where $r$ is the rank of $\mathcal{G}$, $\mathcal{R}(\mathcal{G})$ denotes the set of roots of $\mathcal{G}$, and the $m_{|\alpha|}$ are mass parameters. To preserve the invariance of the Hamiltonian (3.5) under the Weyl group, the parameters $m_{|\alpha|}$ depend only on the orbit $|\alpha|$ of the root $\alpha$, and not on the root $\alpha$ itself. In the case of $A_{N-1} = SU(N)$, it is common practice to use $N$ pairs of dynamical variables $(x_i, p_i)$, since the roots of $A_{N-1}$ lie conveniently on a hyperplane in $\mathbb{C}^N$. The dynamics of the system are unaffected if we shift all $x_i$ by a constant, and the number of degrees of freedom is effectively $N - 1 = r$. Now the roots of $SU(N)$ are given by $\alpha = e_i - e_j$, $1 \leq i, j \leq N$, $i \neq j$. Thus we recognize the original elliptic Calogero-Moser system as the special case of (3.5) corresponding to $A_{N-1}$. As in the original case, the elliptic systems (3.5) admit rational and trigonometric limits. Olshanetsky and Perelomov succeeded in constructing a Lax pair for all these systems in the case of classical Lie algebras, albeit without spectral parameter [6].
c) Twisted Calogero-Moser Systems defined by Lie Algebras

It turns out that the Hamiltonian systems (3.5) are not the only natural extensions of the basic elliptic Calogero-Moser system. A subtlety arises for simple Lie algebras \( \mathcal{G} \) which are not simply-laced, i.e., algebras which admit roots of uneven length. This is the case for the algebras \( B_n, C_n, G_2, \) and \( F_4 \) in Cartan’s classification. For these algebras, the following twisted elliptic Calogero-Moser systems were introduced by the authors in [7,8]

\[
H^\text{twisted}_\mathcal{G} = \frac{1}{2} \sum_{i=1}^{r} p_i^2 - \frac{1}{2} \sum_{\alpha \in \mathcal{R}(\mathcal{G})} m_{\mathcal{G} | \alpha} \nu(\alpha) (\alpha \cdot x).
\]

Here the function \( \nu(\alpha) \) depends only on the length of the root \( \alpha \). If \( \mathcal{G} \) is simply-laced, we set \( \nu(\alpha) = 1 \) identically. Otherwise, for \( \mathcal{G} \) non simply-laced, we set \( \nu(\alpha) = 1 \) when \( \alpha \) is a long root, \( \nu(\alpha) = 2 \) when \( \alpha \) is a short root and \( \mathcal{G} \) is one of the algebras \( B_n, C_n, \) or \( F_4 \), and \( \nu(\alpha) = 3 \) when \( \alpha \) is a short root and \( \mathcal{G} = G_2 \). The twisted Weierstrass function \( \wp_\nu(z) \) is defined by

\[
\wp_\nu(z) = \sum_{\sigma=0}^{\nu-1} \wp(z + 2\omega_a \sigma / \nu),
\]

where \( \omega_a \) is any of the half-periods \( \omega_1, \omega_2, \) or \( \omega_1 + \omega_2 \). Thus the twisted and untwisted Calogero-Moser systems coincide for \( \mathcal{G} \) simply laced. The original motivation for twisted Calogero-Moser systems was based on their scaling limits (which will be discussed in the next section) [7,8]. Another motivation based on the symmetries of Dynkin diagrams was proposed subsequently by Bordner, Sasaki, and Takasaki [21].

IV. SCALING LIMITS OF CALOGERO-MOSER SYSTEMS

a) Results of Inozemtsev for \( A_{N-1} \)

For the standard elliptic Calogero-Moser systems corresponding to \( A_{N-1} \), Inozemtsev [22] has shown in the 1980’s that in the scaling limit

\[
m = M q^{-\frac{1}{N}}, \quad q \to 0
\]

\[
x_i = X_i - 2\omega_2 \frac{i}{N}, \quad 1 \leq i \leq N
\]

where \( M \) is kept fixed, the elliptic \( A_{N-1} \) Calogero-Moser Hamiltonian tends to the following Hamiltonian

\[
H_{\text{Toda}} = \frac{1}{2} \sum_{i=1}^{N} p_i^2 - \frac{1}{2} \left( \sum_{i=1}^{N-1} e^{X_{i+1} - X_i} + e^{X_1 - X_N} \right)
\]
The roots $e_i - e_{i+1}, 1 \leq i \leq N - 1$, and $e_N - e_1$ can be recognized as the simple roots of the affine algebra $A^{(1)}_{N-1}$. (For basic facts on affine algebras, we refer to [23]). Thus (4.3) can be recognized as the Hamiltonian of the Toda system defined by $A^{(1)}_{N-1}$.

b) Scaling Limits based on the Coxeter Number

The key feature of the above scaling limit is the collapse of the sum over the entire root lattice of $A_{N-1}$ in the Calogero-Moser Hamiltonian to the sum over only simple roots in the Toda Hamiltonian for the Kac-Moody algebra $A^{(1)}_{N-1}$. Our task is to extend this mechanism to general Lie algebras. For this, we consider the following generalization of the preceding scaling limit

$$m = M q^{-\frac{1}{2}} \delta,$$

$$x = X - 2 \omega_2 \delta \rho^\vee,$$

Here $x = (x_i), X = (X_i)$ and $\rho^\vee$ are r-dimensional vectors. The vector $x$ is the dynamical variable of the Calogero-Moser system. The parameters $\delta$ and $\rho^\vee$ depend on the algebra $G$ and are yet to be chosen. As for $M$ and $X$, they have the same interpretation as earlier, namely as respectively the mass parameter and the dynamical variables of the limiting system. Setting $\omega_1 = -i \pi$, the contribution of each root $\alpha$ to the Calogero-Moser potential can be expressed as

$$m^2 \wp(\alpha \cdot x) = \frac{1}{2} M^2 \sum_{n=-\infty}^{\infty} \frac{e^{2 \delta \omega_2}}{\text{ch}(\alpha \cdot x - 2n \omega_2) - 1}.$$  

It suffices to consider positive roots $\alpha$. We shall also assume that $0 \leq \delta \alpha \cdot \rho^\vee \leq 1$. The contributions of the $n = 0$ and $n = -1$ summands in (4.6) are proportional to $e^{2 \omega_2 (\delta - \delta \alpha \cdot \rho^\vee)}$ and $e^{2 \omega_2 (\delta - 1 + \delta \alpha \cdot \rho^\vee)}$ respectively. Thus the existence of a finite scaling limit requires that

$$\delta \leq \delta \alpha \cdot \rho^\vee \leq 1 - \delta.$$  

Let $\alpha_i, 1 \leq i \leq r$ be a basis of simple roots for $G$. If we want all simple roots $\alpha_i$ to survive in the limit, we must require that

$$\alpha_i \cdot \rho^\vee = 1, \ 1 \leq i \leq r.$$  

This condition characterizes the vector $\rho^\vee$ as the level vector. Next, the second condition in (3.7) can be rewritten as $\delta \{1 + \max_\alpha (\alpha \cdot \rho^\vee)\} \leq 1$. But

$$h_G = 1 + \max_\alpha (\alpha \cdot \rho^\vee)$$

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is precisely the Coxeter number of \(G\), and we must have \(\delta \leq \frac{1}{h_{\mathcal{G}}}\). Thus when \(\delta < \frac{1}{h_{\mathcal{G}}}\), the contributions of all the roots except for the simple roots of \(G\) tend to 0. On the other hand, when \(\delta = \frac{1}{h_{\mathcal{G}}}\), the highest root \(\alpha_0\) realizing the maximum over \(\alpha\) in (4.9) survives. Since \(-\alpha_0\) is the additional simple root for the affine Lie algebra \(G^{(1)}\), we arrive in this way at the following theorem, which was proved in [8]

**Theorem 1.** Under the limit (4.4-4.5), with \(\delta = \frac{1}{h_{\mathcal{G}}}\), and \(\rho^\vee\) given by the level vector, the Hamiltonian of the elliptic Calogero-Moser system for the simple Lie algebra \(G\) tends to the Hamiltonian of the Toda system for the affine Lie algebra \(G^{(1)}\).

(c) **Scaling Limit based on the Dual Coxeter Number**

If the Seiberg-Witten spectral curve of the \(\mathcal{N} = 2\) supersymmetric gauge theory with a hypermultiplet in the adjoint representation is to be realized as the spectral curve for a Calogero-Moser system, the parameter \(m\) in the Calogero-Moser system should correspond to the mass of the hypermultiplet. In the gauge theory, the dependence of the coupling constant on the mass \(m\) is given by

\[
\tau = \frac{i}{2\pi} h_{\mathcal{G}}^\vee \ln \frac{m^2}{M^2} \iff m = M q^{-\frac{1}{2h_{\mathcal{G}}}}
\]  

(4.10)

where \(h_{\mathcal{G}}^\vee\) is the quadratic Casimir of the Lie algebra \(\mathcal{G}\). This shows that the correct physical limit, expressing the decoupling of the hypermultiplet as it becomes infinitely massive, is given by (3.4), but with \(\delta = \frac{1}{h_{\mathcal{G}}}\). To establish a closer parallel with our preceding discussion, we recall that the quadratic Casimir \(h_{\mathcal{G}}^\vee\) coincides with the dual Coxeter number of \(\mathcal{G}\), defined by

\[
h_{\mathcal{G}}^\vee = 1 + \max_\alpha (\alpha^\vee \cdot \rho),
\]  

(4.11)

where \(\alpha^\vee = \frac{2\alpha}{\alpha^2}\) is the coroot associated to \(\alpha\), and \(\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha\) is the well-known Weyl vector.

For simply laced Lie algebras \(\mathcal{G}\) (ADE algebras), we have \(h_{\mathcal{G}} = h_{\mathcal{G}}^\vee\), and the preceding scaling limits apply. However, for non simply-laced algebras \((B_n, C_n, G_2, F_4)\), we have \(h_{\mathcal{G}} > h_{\mathcal{G}}^\vee\), and our earlier considerations show that the untwisted elliptic Calogero-Moser Hamiltonians do not tend to a finite limit under (3.10), \(q \to 0\), \(M\) is kept fixed. This is why the twisted Hamiltonian systems (3.6) have to be introduced. The twisting produces precisely to an improvement in the asymptotic behavior of the potential which allows a finite, non-trivial limit. More precisely, we can write

\[
m^2 \varphi_\nu(x) = \frac{\nu^2}{2} \sum_{n=-\infty}^{\infty} \frac{m^2}{\text{ch} \nu(x - 2n\omega_2) - 1}.
\]  

(4.12)
Setting \(x = X - 2\omega_2 \delta^\vee \rho\), we obtain the following asymptotics

\[
m^2 \varphi_\nu(x) = \nu^2 M^2 \left\{ \begin{array}{ll}
 e^{-2\omega_2(\delta^\vee \alpha^\vee \cdot \rho - \delta^\vee)} - \alpha^\vee \cdot X + e^{-2\omega_2(1-\delta^\vee \alpha^\vee \cdot \rho - \delta^\vee)} + \alpha^\vee \cdot X, & \text{if } \alpha \text{ is long;}
 e^{-2\omega_2(\delta^\vee \alpha^\vee \cdot \rho - \delta^\vee)} - \alpha^\vee \cdot X, & \text{if } \alpha \text{ is short.}
\end{array} \right.
\]

This leads to the following theorem [8]

**Theorem 2.** Under the limit \(x = X + 2\omega_2 \frac{1}{h_\nu^\vee} \rho\), \(m = Mq^{-\frac{1}{2h_\nu^\vee}}\), with \(\rho\) the Weyl vector and \(q \to 0\), the Hamiltonian of the twisted elliptic Calogero-Moser system for the simple Lie algebra \(G\) tends to the Hamiltonian of the Toda system for the affine Lie algebra \((G^{(1)})^\vee\).

So far we have discussed only the scaling limits of the Hamiltonians. However, similar arguments show that the Lax pairs constructed below also have finite, non-trivial scaling limits whenever this is the case for the Hamiltonians. The spectral parameter \(z\) should scale as \(e^z = Zq^{\frac{1}{4}}\), with \(Z\) fixed. The parameter \(Z\) can be identified with the loop group parameter for the resulting affine Toda system.

**V. LAX PAIRS FOR CALOGERO-MOSER SYSTEMS**

**a) The General Ansatz**

Let the rank of \(G\) be \(n\), and \(d\) be its dimension. Let \(\Lambda\) be a representation of \(G\) of dimension \(N\), of weights \(\lambda_I\), \(1 \leq I \leq N\). Let \(u_I \in \mathbb{C}^N\) be the weights of the fundamental representation of \(GL(N, \mathbb{C})\). Project orthogonally the \(u_I\)’s onto the \(\lambda_I\)’s as

\[
su_I = \lambda_I + u_I, \quad \lambda_I \perp v_J.
\]

It is easily verified that \(s^2\) is the second Dynkin index. Then

\[
\alpha_{IJ} = \lambda_I - \lambda_J
\]

is a weight of \(\Lambda \otimes \Lambda^*\) associated to the root \(u_I - u_J\) of \(GL(N, \mathbb{C})\). The Lax pairs for both untwisted and twisted Calogero-Moser systems will be of the form

\[
L = P + X, \quad M = D + X,
\]

where the matrices \(P, X, D\), and \(Y\) are given by

\[
X = \sum_{I \neq J} C_{IJ} \Phi_{IJ}(\alpha_{IJ}, z) E_{IJ}, \quad Y = \sum_{I \neq J} C_{IJ} \Phi'_{IJ}(\alpha_{IJ}, z) E_{IJ}
\]
and by
\[
P = p \cdot h, \quad D = d \cdot (h \oplus \tilde{h}) + \Delta. \tag{5.5}
\]

Here \(h\) is in a Cartan subalgebra \(H_G\) for \(G\), \(\tilde{h}\) is in the Cartan-Killing orthogonal complement of \(H_G\) inside a Cartan subalgebra \(H\) for \(GL(N, \mathbb{C})\), and \(\Delta\) is in the centralizer of \(H_G\) in \(GL(N, \mathbb{C})\). The functions \(\Phi_{IJ}(x, z)\) and the coefficients \(C_{IJ}\) are yet to be determined.

We begin by stating the necessary and sufficient conditions for the pair \(L(z), M(z)\) of (4.1) to be a Lax pair for the (twisted or untwisted) Calogero-Moser systems. For this, it is convenient to introduce the following notation
\[
\Phi_{IJ} = \Phi_{IJ}(\alpha_{IJ} \cdot x)
\]
\[
\varphi_{IJ}' = \Phi_{IJ}(\alpha_{IJ} \cdot x, z)\Phi_{IJ}'(\alpha_{IJ} \cdot x, z) - \Phi_{IJ}(\alpha_{IJ} \cdot x, z)\Phi_{IJ}'(\alpha_{IJ} \cdot x, z). \tag{5.6}
\]

Then the Lax equation \(\dot{L}(z) = [L(z), M(z)]\) implies the Calogero-Moser system if and only if the following three identities are satisfied
\[
\sum_{I \neq J} C_{IJ} C_{JI} \varphi_{IJ}' \alpha_{IJ} = s^2 \sum_{\alpha \in R(\mathcal{G})} m_{|\alpha|}^2 \varphi_{\nu(\alpha)}(\alpha \cdot x) \tag{5.7}
\]
\[
\sum_{I \neq J} C_{IJ} C_{JI} (\varphi_{IJ} - \varphi_{JI}) = 0 \tag{5.8}
\]
\[
\sum_{K \neq I, J} C_{IK} C_{KJ} (\Phi_{IK} \Phi_{KJ}' - \Phi_{IK}' \Phi_{KJ}) = sC_{IJ} d \cdot (v_I - v_J) + \sum_{K \neq I, J} \Delta_{IJ} C_{KJ} \Phi_{KJ} - \sum_{K \neq I, J} C_{IK} \Phi_{IK} \Delta_{KJ} \tag{5.9}
\]

The following theorem was established in [7]:

**Theorem 3.** A representation \(\Lambda\), functions \(\Phi_{IJ}\), and coefficients \(C_{IJ}\) with a spectral parameter \(z\) satisfying (4.7-4.9) can be found for all twisted and untwisted elliptic Calogero-Moser systems associated with a simple Lie algebra \(\mathcal{G}\), except possibly in the case of twisted \(G_2\). In the case of \(E_8\), we have to assume the existence of a \(\pm 1\) cocycle.
b) Lax Pairs for Untwisted Calogero-Moser Systems

We now describe some important features of the Lax pairs we obtain in this manner.

- In the case of the untwisted Calogero-Moser systems, we can choose $\Phi_{IJ}(x, z) = \Phi(x, z), \wp_{IJ}(x) = \wp(x)$ for all $G$.

- $\Delta = 0$ for all $G$, except for $E_8$.

- For $A_n$, the Lax pair (3.2-3.3) corresponds to the choice of the fundamental representation for $\Lambda$. A different Lax pair can be found by taking $\Lambda$ to be the antisymmetric representation.

- For the $BC_n$ system, the Lax pair is obtained by imbedding $B_n$ in $GL(N, \mathbb{C})$ with $N = 2n + 1$. When $z = \omega_n$ (half-period), the Lax pair obtained this way reduces to the Lax pair obtained by Olshanetsky and Perelomov [6].

- For the $B_n$ and $D_n$ systems, additional Lax pairs with spectral parameter can be found by taking $\Lambda$ to be the spinor representation.

- For $G_2$, a first Lax pair with spectral parameter can be obtained by the above construction with $\Lambda$ chosen to be the 7 of $G_2$. A second Lax pair with spectral parameter can be obtained by restricting the 8 of $B_3$ to the 7 $\oplus$ 1 of $G_2$.

- For $F_4$, a Lax pair can be obtained by taking $\Lambda$ to be the 26 $\oplus$ 1 of $F_4$, viewed as the restriction of the 27 of $E_6$ to its $F_4$ subalgebra.

- For $E_6$, $\Lambda$ is the 27 representation.

- For $E_7$, $\Lambda$ is the 56 representation.

- For $E_8$, a Lax pair with spectral parameter can be constructed with $\Lambda$ given by the 248 representation, if coefficients $c_{IJ} = \pm 1$ exist with the following cocycle conditions

\[
\begin{align*}
    c(\lambda, \lambda - \delta)c(\lambda - \delta, \mu) &= c(\lambda, \mu + \delta)c(\mu + \delta, \mu) \\
    &\text{when } \delta \cdot \lambda = -\delta \cdot \mu = 1, \ \lambda \cdot \mu = 0 \\
    c(\lambda, \mu)c(\lambda - \delta, \mu) &= c(\lambda, \lambda - \delta) \\
    &\text{when } \delta \cdot \lambda = \lambda \cdot \mu = 1, \ \delta \cdot \mu = 0 \\
    c(\lambda, \mu)c(\lambda, \lambda - \mu) &= -c(\lambda - \mu, -\mu) \\
    &\text{when } \lambda \cdot \mu = 1.
\end{align*}
\]

(5.10)

The matrix $\Delta$ in the Lax pair is then the $8 \times 8$ matrix given by
\[\Delta_{ab} = \sum_{\delta \cdot \beta_a = 1} m_2 \left( c(\beta_a, \delta) c(\delta, \beta_b) + c(\beta_a, \beta_a - \delta) c(\beta_a - \delta, \beta_b) \right) \varphi(\delta \cdot x) \]
\[- \sum_{\delta \cdot \beta_a = 1} m_2 \left( c(\beta_a, \delta) c(\delta, \beta_b) + c(\beta_a, \beta_a - \delta) c(\beta_a - \delta, \beta_b) \right) \varphi(\delta \cdot x)\]
\[\Delta_{aa} = \sum_{\beta_a, \delta = 1} m_2 \varphi(\delta \cdot x) + 2m_2 \varphi(\beta_a \cdot x), \quad (5.11)\]

where \(\beta_a, 1 \leq a \leq 8,\) is a maximal set of 8 mutually orthogonal roots.

- Explicit expressions for the constants \(C_{IJ}\) and the functions \(d(x),\) and thus for the Lax pair are particularly simple when the representation \(\Lambda\) consists of only a single Weyl orbit of weights. This is the case when \(\Lambda\) is either
  (1) the defining representation of \(A_n, C_n\) or \(D_n;\)
  (2) any rank \(p\) totally anti-symmetric representation of \(A_n;\)
  (3) an irreducible fundamental spinor representation of \(B_n\) or \(D_n;\)
  (4) the 27 of \(E_6;\) the 56 of \(E_7.\)

The weights \(\lambda\) and \(\mu\) of \(\Lambda\) provide unique labels instead of \(I\) and \(J,\) and the values of \(C_{IJ} = C_{\lambda\mu}\) are given by a simple formula

\[C_{\lambda\mu} = \begin{cases} \sqrt{\frac{\alpha^2}{2} m_{|\alpha|}} & \text{when } \alpha = \lambda - \mu \text{ is a root} \\ 0 & \text{otherwise} \end{cases} \]

The expression for the vector \(d\) may be summarized by

\[sd \cdot u_\lambda = \sum_{\lambda \cdot \delta = 1; \, \delta^2 = 2} m_{|\delta|} \varphi(\delta \cdot x)\]

(For \(C_n,\) the last equation has an additional term, as given in [7].) In each case, the number of independent couplings \(m_{|\alpha|}\) equals the number of different root lengths.

c) Lax Pairs for Twisted Calogero-Moser Systems

Recall that the twisted and untwisted Calogero-Moser systems differ only for non-simply laced Lie algebras, namely \(B_n, C_n, G_2\) and \(F_4.\) These are the only algebras we discuss in this paragraph. The construction (4.3-4.9) gives then Lax pairs for all of them, with the possible exception of twisted \(G_2.\) Unlike the case of untwisted Lie algebras however, the functions \(\Phi_{IJ}\) have to be chosen with care, and differ for each algebra. More specifically,
• For $B_n$, the Lax pair is of dimension $N = 2n$, admits two independent couplings $m_1$ and $m_2$, and
\[
\Phi_{IJ}(x, z) = \begin{cases} 
\Phi(x, z), & \text{if } I - J \neq 0, \pm n; \\
\Phi_2(\frac{1}{2}x, z), & \text{if } I - J = \pm n.
\end{cases} \tag{5.12}
\]
Here a new function $\Phi_2(x, z)$ is defined by
\[
\Phi_2\left(\frac{1}{2}x, z\right) = \frac{\Phi\left(\frac{1}{2}x, z\right)\Phi\left(\frac{1}{2}x + \omega_1, z\right)}{\Phi(\omega_1, z)} \tag{5.13}
\]

• For $C_n$, the Lax pair is of dimension $N = 2n + 2$, admits one independent coupling $m_2$, and
\[
\Phi_{IJ}(x, z) = \Phi_2(x + \omega_{IJ}, z),
\]
where $\omega_{IJ}$ are given by
\[
\omega_{IJ} = \begin{cases} 
0, & \text{if } I \neq J = 1, 2, \ldots, 2n + 1; \\
\omega_2, & \text{if } 1 \leq I \leq 2n, J = 2n + 2; \\
-\omega_2, & \text{if } 1 \leq J \leq 2n, I = 2n + 2.
\end{cases} \tag{5.14}
\]

• For $F_4$, the Lax pair is of dimension $N = 24$, two independent couplings $m_1$ and $m_2$,
\[
\Phi_{\lambda \mu}(x, z) = \begin{cases} 
\Phi(x, z), & \text{if } \lambda \cdot \mu = 0; \\
\Phi_1(x, z), & \text{if } \lambda \cdot \mu = \frac{1}{2}; \\
\Phi_2\left(\frac{1}{2}x, z\right), & \text{if } \lambda \cdot \mu = -1.
\end{cases} \tag{5.15}
\]
where the function $\Phi_1(x, z)$ is defined by
\[
\Phi_1(x, z) = \Phi(x, z) - e^{\pi i \zeta(z) + \eta_1 z} \Phi(x + \omega_1, z) \tag{5.16}
\]
Here it is more convenient to label the entries of the Lax pair directly by the weights $\lambda = \lambda_I$ and $\mu = \lambda_J$ instead of $I$ and $J$.

• For $G_2$, candidate Lax pairs can be defined in the 6 and 8 representations of $G_2$, but it is still unknown whether elliptic functions $\Phi_{IJ}(x, z)$ exist which satisfy the required identities.

We note that recently Lax pairs of root type have been considered [21] which correspond, in the above Ansatz (5.3-5), to $\Lambda$ equal to the adjoint representation of $G$ and the coefficients $C_{IJ}$ vanishing for $I$ or $J$ associated with zero weights. This choice yields another Lax pair for the case of $E_8$. 

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VI. CALOGERO-MOSER AND SEIBERG-WITTEN THEORY FOR SU(N)

The correspondence between Seiberg-Witten theory for $\mathcal{N} = 2$ super-Yang-Mills theory with one hypermultiplet in the adjoint representation of the gauge algebra, and the elliptic Calogero-Moser systems was first established in [5], for the gauge algebra $\mathcal{G} = SU(N)$. We describe it here in some detail.

All that we shall need here of the elliptic Calogero-Moser system is its Lax operator $L(z)$, whose $N \times N$ matrix elements are given by

$$L_{ij}(z) = p_i \delta_{ij} - m(1 - \delta_{ij})\Phi(x_i - x_j, z) \quad (6.1)$$

Notice that the Hamiltonian is simply given in terms of $L$ by

$$H(x, p) = \frac{1}{2} \text{tr} L(z)^2 + C\varphi(z)$$

with $C = -\frac{1}{2} m^2 N(N - 1)$.

a) Correspondence of Data

The correspondence between the data of the elliptic Calogero-Moser system and those of the Seiberg-Witten theory is as follows.

1. The parameter $m$ in (6.1) is the hypermultiplet mass;
2. The gauge coupling $g$ and the $\theta$-angle are related to the modulus of the torus $\Sigma = \mathbb{C}/(2\omega_1 \mathbb{Z} + 2\omega_2 \mathbb{Z})$ by

$$\tau = \frac{\omega_2}{\omega_1} = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}; \quad (6.2)$$

3. The Seiberg-Witten curve $\Gamma$ is the spectral curve of the elliptic Calogero-Moser model, defined by

$$\Gamma = \{(k, z) \in \mathbb{C} \times \Sigma, \det (kI - L(z)) = 0\} \quad (6.3)$$

and the Seiberg-Witten 1-form is $d\lambda = k \, dz$. $\Gamma$ is invariant under the Weyl group of $SU(N)$.

4. Using the Lax equation $\dot{L} = [L, M]$, it is clear that the spectral curve is independent of time, and can be dependent only upon the constants of motion of the Calogero-Moser system, of which there are only $N$. These integrals of motion may be viewed as parametrized by the quantum moduli of the Seiberg-Witten system.

5. Finally, $d\lambda = kdz$ is meromorphic, with a simple pole on each of the $N$ sheets above the point $z = 0$ on the base torus. The residue at each of these poles is proportional to $m$, as required by the general set-up of Seiberg-Witten theory, explained in §II.
b) Four Fundamental Theorems

While the above mappings of the Seiberg-Witten data onto the Calogero-Moser data is certainly natural, there is no direct proof of it, and it is important to check that the results inferred from it agree with known facts from quantum field theory. To establish this, as well as a series of further predictions from the correspondence, we give four theorems (the proofs may be found in [5] for the first three theorems, and in [24] for the last one).

**Theorem 4.** The spectral curve equation \( \det(k I - L(z)) = 0 \) is equivalent to

\[
\vartheta_1 \left( \frac{1}{2\omega_1} (z - m \frac{\partial}{\partial k}) \bigg| \tau \right) H(k) = 0 \quad (6.4)
\]

where \( H(k) \) is a monic polynomial in \( k \) of degree \( N \), whose zeros (or equivalently whose coefficients) correspond to the moduli of the gauge theory. If \( H(k) = \prod_{i=1}^{N} (k - k_i) \), then

\[
\lim_{q \to 0} \frac{1}{2\pi i} \oint_{A_i} k dz = k_i - \frac{1}{2} m.
\]

Here, \( \vartheta_1 \) is the Jacobi \( \vartheta \)-function, which admits a simple series expansion in powers of the instanton factor \( q = e^{2\pi i \tau} \), so that the curve equation may also be rewritten as a series expansion

\[
\sum_{n \in \mathbb{Z}} (-)^n q^{\frac{1}{2}n(n-1)} e^{n z} H(k - n \cdot m) = 0 \quad (6.5)
\]

where we have set \( \omega_1 = -i\pi \) without loss of generality. The series expansion (6.5) is superconvergent and sparse in the sense that it receives contributions only at integers that grow like \( n^2 \).

**Theorem 5.** The prepotential of the Seiberg-Witten theory obeys a renormalization group-type equation that simply relates \( F \) to the Calogero-Moser Hamiltonian, expressed in terms of the quantum order parameters \( a_j \)

\[
a_j = \frac{1}{2\pi i} \oint_{A_j} d\lambda \quad \frac{\partial F}{\partial \tau} \bigg|_{a_j} = H(x, p) = \frac{1}{2} \text{tr} L(z)^2 + C \varphi(z) \quad (6.6)
\]

Furthermore, in an expansion in powers of the instanton factor \( q = e^{2\pi i \tau} \), the quantum order parameters \( a_j \) may be computed by residue methods in terms of the zeros of \( H(k) \).

The proof of (6.6) requires Riemann surface deformation theory [5]. The fact that the quantum order parameters may be evaluated by residue methods arises from the fact that
$A_j$-cycles may be chosen on the spectral curve $\Gamma$ in such a way that they will shrink to zero as $q \to 0$. As a result, contour integrals around full-fledged branch cuts $A_j$ reduce to contour integrals around poles at single points, which may be calculated by residue methods only. These methods were originally developed in [25,26]. Knowing the quantum order parameters in terms of the zeros $k_j$ of $H(k) = 0$ is a relation that may be inverted and used in (6.6) to obtain a differential relation for all order instanton corrections. It is now only necessary to evaluate explicitly the $\tau$-independent contribution to $F$, which in field theory arises from perturbation theory. This may be done easily by retaining only the $n = 0$ and $n = 1$ terms in the expansion of the curve (6.5), so that $z = \ln H(k) - \ln H(k-m)$. The results of the calculations to two instanton order may be summarized in the following theorem [5].

**Theorem 6.** The prepotential, to 2 instanton order is given by $F = F^{(\text{pert})} + F^{(1)} + F^{(2)}$. The perturbative contribution is given by

$$F^{(\text{pert})} = \frac{\tau}{2} \sum_i a_i^2 - \frac{1}{8\pi i} \sum_{i,j} \left[ (a_i-a_j)^2 \ln(a_i-a_j)^2 - (a_i-a_j-m)^2 \ln(a_i-a_j-m)^2 \right]$$

(6.7a)

while all instanton corrections are expressed in terms of a single function

$$S_i(a) = \frac{\prod_{j=1}^N [(a_i-a_j)^2 - m^2]}{\prod_{j\neq i} (a-a_j)^2}$$

(6.7b)

as follows

$$F^{(1)} = \frac{q}{2\pi i} \sum_i S_i(a_i)$$

$$F^{(2)} = \frac{q^2}{8\pi i} \left[ \sum_i S_i(a_i) \partial_i^2 S_i(a_i) + 4 \sum_{i\neq j} \frac{S_i(a_i) S_j(a_j)}{(a_i-a_j)^2} - \frac{S_i(a_i) S_j(a_j)}{(a_i-a_j-m)^2} \right]$$

(6.7c)

The perturbative corrections to the prepotential of (6.7a) indeed precisely agree with the predictions of asymptotic freedom. The formulas (6.7c) for the instanton corrections $F^{(1)}$ and $F^{(2)}$ are new, as they have not yet been computed by direct field theory methods. Perturbative expansions of the prepotential in powers of $m$ have also been obtained in [27].

The moduli $k_i$, $1 \leq i \leq N$, of the gauge theory are evidently integrals of motion of the system. To identify these integrals of motion, denote by $S$ be any subset of $\{1, \cdots, N\}$, and let $S^* = \{1, \cdots, N\} \setminus S$, $\varphi(S) = \varphi(x_i - x_j)$ when $S = \{i, j\}$. Let also $p_S$ denote the subset of momenta $p_i$ with $i \in S$. We have [24]
Theorem 7. For any $K$, $0 \leq K \leq N$, let $\sigma_K(k_1, \cdots, k_N) = \sigma_K(k)$ be the $K$-th symmetric polynomial of $(k_1, \cdots, k_N)$, defined by $H(u) = \sum_{K=0}^{N} (-)^K \sigma_K(k) u^{N-K}$. Then

$$\sigma_K(k) = \sigma_K(p) + \sum_{l=1}^{[K/2]} m^{2l} \sum_{|S_i \cap S_j| = 2s_i j} \sigma_{K-2l}(p_{(\cup_{l=1}^s S_i)\cup}) \prod_{i=1}^{l}[\varphi(S_i) + \frac{m_i}{\omega_1}]$$

(6.8)

c) Partial Decoupling of the Hypermultiplet and Product Gauge Groups

The spectral curves of certain gauge theories can be easily derived from the Calogero-Moser curves by a partial decoupling of the hypermultiplet. Indeed,

- the masses of the gauge multiplet and hypermultiplet are $|a_i - a_j|$ and $|a_i - a_j + m|$. In suitable limits, some of these masses become $\infty$, and states with infinite mass decouple. The remaining gauge group is a subgroup of $SU(N)$.

- When the effective coupling of a gauge subgroup is 0, the dynamics freeze and the gauge states become non-interacting.

Non-trivial decoupling limits arise when $\tau \to \infty$ and $m \to \infty$. When all $a_i$ are finite, we obtain the pure Yang-Mills theory. When some hypermultiplets masses remain finite, the $U(1)$ factors freeze, the gauge group $SU(N)$ is broken down to $SU(N_1) \times \cdots \times SU(N_p)$, and the remaining hypermultiplets are in e.g. fundamental or bifundamental representations. For example, let $N = 2N_1$ be even, and set

$$k_i = v_1 + x_i, \quad k_{N_1+j} = v_2 + y_j, \quad 1 \leq i, j \leq N_1,$$

with $\sum_{i=1}^{N_1} x_i = \sum_{j=1}^{N_1} y_j = 0$. (The term $v = v_1 - v_2$ is associated to the $U(1)$ factor of the gauge group). In the limit $m \to \infty$, $q \to 0$, with $x_i$, $y_j$, $\mu = v - m$ and $\Lambda = mq \frac{N_1}{2}$ kept fixed, the theory reduces to a $SU(N_1) \times SU(N_1)$ gauge theory, with a hypermultiplet in the bifundamental $(N_1, \bar{N}_1) \oplus (\bar{N}_1, N_1)$, and spectral curve

$$A(x) - t(-)^{N_1} B(x) - 2^{N_1} \Lambda^{N_1} \left( \frac{1}{t} - t^2 \right) = 0,$$

(6.9)

where $A(x) = \prod_{i=1}^{N_1} (x - x_i)$, $B(x) = \prod_{j=1}^{N_1} (x + \mu - y_j)$, $t = e^z$. This agrees with the curve found by Witten [28] using M Theory, and by Katz, Mayr, and Vafa [29] using geometric engineering.

The prepotential of the $SU(N_1) \times SU(N_1)$ theory can be also read off the Calogero-Moser prepotential. It is convenient to introduce $x^{(I)}_i$, $I = 1, 2$, by $x^{(1)}_i = x_i$, $x^{(2)}_i = y_i$, $1 \leq i \leq N_1$. Set

$$A^I_i = \prod_{j \in I} (x - x^{(I)}_j), \quad B^I(x) = \prod_{j \in I} (\mu \pm (x - x^{(I)}_j)), \quad S^I_i(x) = \frac{B^I(x)}{A^I_i(x)^2},$$
where the ± sign in \( B^I(x) \) is the same as the sign of \( J - I \). Then the the first two orders of instanton corrections to the prepotential for the \( SU(N_1) \times SU(N_1) \) theory are given by

\[
\mathcal{F}^{(1)}_{SU(N_1) \times SU(N_1)} = \frac{(-2\Lambda)^{N_1}}{2\pi i} \sum_{I=1,2} \sum_{i \in I} S^I_i(x^{(I)}_i)
\]

\[
\mathcal{F}^{(2)}_{SU(N_1) \times SU(N_1)} = \frac{(-2\Lambda)^{2N_1}}{8\pi i} \sum_{I=1,2} \sum_{i \in I} S^I_i(x^{(I)}_i) \frac{\partial^2 S^I_i(x^{(I)}_i)}{\partial x^{(I)}_i^2} + \sum_{i \neq j, i, j \in I} \frac{S^I_i(x^{(I)}_i)S^I_j(x^{(I)}_j)}{(x^{(I)}_i - x^{(I)}_j)^2}.
\]

(6.10)

We note that an alternative derivation of (6.4) was recently presented in [30].

VII. CALOGERO-MOSER AND SEIBERG-WITTEN THEORY
FOR GENERAL \( \mathcal{G} \)

We consider now the \( \mathcal{N} = 2 \) supersymmetric gauge theory for a general simple gauge algebra \( \mathcal{G} \) and a hypermultiplet of mass \( m \) in the adjoint representation. Then [9]

- the Seiberg-Witten curve of the theory is given by the spectral curve \( \Gamma = \{(k, z) \in \mathbb{C} \times \Sigma; \det(kI - L(z)) = 0\} \) of the twisted elliptic Calogero-Moser system associated to the Lie algebra \( \mathcal{G} \). The Seiberg-Witten differential \( d\lambda \) is given by \( d\lambda = kdz \).

- The function \( R(k, z) = \det(kI - L(z)) \) is polynomial in \( k \) and meromorphic in \( z \). The spectral curve \( \Gamma \) is invariant under the Weyl group of \( \mathcal{G} \). It depends on \( n \) complex moduli, which can be thought of as independent integrals of motion of the Calogero-Moser system.

- The differential \( d\lambda = kdz \) is meromorphic on \( \Gamma \), with simple poles. The position and residues of the poles are independent of the moduli. The residues are linear in the hypermultiplet mass \( m \). (Unlike the case of \( SU(N) \), their exact values are difficult to determine for general \( \mathcal{G} \)).

- In the \( m \to 0 \) limit, the Calogero-Moser system reduces to a free system, the spectral curve \( \Gamma \) is just the product of several unglued copies of the base torus \( \Sigma \), indexed by the constant eigenvalues of \( L(z) = p \cdot h \). Let \( k_i, 1 \leq i \leq n \), be \( n \) independent eigenvalues, and \( A_i, B_i \) be the \( A \) and \( B \) cycles lifted to the corresponding sheets. For each \( i \), we readily obtain

\[
a_i = \frac{1}{2\pi i} \oint_{A_i} d\lambda = \frac{k_i}{2\pi i} \oint_A dz = \frac{2\omega_1}{2\pi i} k_i
\]

\[
a_{Di} = \frac{1}{2\pi i} \oint_{B_i} d\lambda = \frac{k_i}{2\pi i} \oint_B dz = \frac{2\omega_1}{2\pi i} \tau k_i
\]

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Thus the prepotential $F$ is given by $F = \frac{\tau}{2} \sum_{i=1}^{n} a_{i}^{2}$. This is the classical prepotential and hence the correct answer, since in the $m \to 0$ limit, the theory acquires an $\mathcal{N} = 4$ supersymmetry, and receives no quantum corrections.

- The $m \to \infty$ limit is the crucial consistency check, which motivated the introduction of the twisted Calogero-Moser systems in the first place [7,8]. In view of Theorem 2 and subsequent comments, in the limit $m \to \infty$, $q \to 0$, with $x = X + 2\omega_{2} \frac{1}{h_{\mathcal{G}}} \rho$, $m = Mq^{-\frac{1}{h_{\mathcal{G}}}}$ with $X$ and $M$ kept fixed, the Hamiltonian and spectral curve for the twisted elliptic Calogero-Moser system with Lie algebra $\mathcal{G}$ reduce to the Hamiltonian and spectral curve for the Toda system for the affine Lie algebra $(\mathcal{G}^{(1)})^{\vee}$. This is the correct answer. Indeed, in this limit, the gauge theory with adjoint hypermultiplet reduces to the pure Yang-Mills theory, and the Seiberg-Witten spectral curves for pure Yang-Mills with gauge algebra $\mathcal{G}$ have been shown by Martinec and Warner [31] to be the spectral curves of the Toda system for $(\mathcal{G}^{(1)})^{\vee}$.

- The effective prepotential can be evaluated explicitly in the case of $\mathcal{G} = D_{n}$ for $n \leq 5$. Its logarithmic singularity does reproduce the logarithmic singularities expected from field theory considerations.

- As in the known correspondences between Seiberg-Witten theory and integrable models [5,25], we expect the following equation to hold

$$\frac{\partial F}{\partial \tau} = H^{\text{twisted}}_{\mathcal{G}}(x,p), \quad (6.11)$$

Note that the left hand side can be interpreted in the gauge theory as a renormalization group equation.

- For simple laced $\mathcal{G}$, the curves $R(k, z) = 0$ are modular invariant. Physically, the gauge theories for these Lie algebras are self-dual. For non simply-laced $\mathcal{G}$, the modular group is broken to the congruence subgroup $\Gamma_{0}(2)$ for $\mathcal{G} = B_{n}, C_{n}, F_{4}$, and to $\Gamma_{0}(3)$ for $G_{2}$. The Hamiltonians of the twisted Calogero-Moser systems for non-simply laced $\mathcal{G}$ are also transformed under Landen transformations into the Hamiltonians of the twisted Calogero-Moser system for the dual algebra $\mathcal{G}^{\vee}$. It would be interesting to determine whether such transformations exist for the spectral curves or the corresponding gauge theories themselves.

Spectral curves for certain gauge theories with classical gauge algebras and matter in the adjoint representation have also been proposed in [32] and [33], based on branes and M-theory. Relations with integrable systems were discussed in [34].
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