GLOBAL EXISTENCE AND BLOW-UP OF SOLUTIONS FOR SEMILINEAR HEAT EQUATION WITH NONLINEAR NONLOCAL BOUNDARY CONDITION

ALEXANDER GLADKOV AND TATIANA KAVITOVA

Abstract. In this paper we consider a semilinear parabolic equation with nonlinear and nonlocal boundary condition and nonnegative initial datum. We prove some global existence results. Criteria on this problem which determine whether the solutions blow up in finite time for large or for all nontrivial initial data are also given.

1. Introduction

In this paper we consider the initial boundary value problem for the following semilinear parabolic equation

\[ u_t = \Delta u + c(x,t)u^p, \quad x \in \Omega, \quad t > 0, \quad (1.1) \]

with nonlinear nonlocal boundary condition

\[ \frac{\partial u(x,t)}{\partial \nu} = \int_{\Omega} k(x,y,t)u^l(y,t) \, dy, \quad x \in \partial \Omega, \quad t > 0, \quad (1.2) \]

and initial datum

\[ u(x,0) = u_0(x), \quad x \in \Omega, \quad (1.3) \]

where \( p > 0, \ l > 0 \), \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) for \( n \geq 1 \) with smooth boundary \( \partial \Omega \), \( \nu \) is unit outward normal on \( \partial \Omega \).

Throughout this paper we suppose that the functions \( c(x,t) \), \( k(x,y,t) \) and \( u_0(x) \) satisfy the following conditions:

\[ c(x,t) \in C^\alpha_{\text{loc}}(\Omega \times [0, +\infty)), \quad 0 < \alpha < 1, \quad c(x,t) \geq 0; \]

\[ k(x,y,t) \in C(\partial \Omega \times \Omega \times [0, +\infty)), \quad k(x,y,t) \geq 0; \]

\[ u_0(x) \in C^1(\bar{\Omega}), \quad u_0(x) \geq 0 \text{ in } \Omega, \quad \frac{\partial u_0(x)}{\partial \nu} = \int_{\Omega} k(x,y,0)u^l_0(y) \, dy \text{ on } \partial \Omega. \]

Many authors have studied blow-up problem for parabolic equations and systems with nonlocal boundary conditions (see, for example, [11, 12, 8, 7, 8, 10, 13, 16, 17, 18, 19, 20, 21] and the references therein). In particular, the initial boundary value problem for equation (1.1) with nonlinear nonlocal boundary condition

\[ u(x,t) = \int_{\Omega} k(x,y,t)u^l(y,t) \, dy, \quad x \in \partial \Omega, \quad t > 0, \]

was considered for \( c(x,t) \leq 0 \) and \( c(x,t) \geq 0 \) in [8] and [10] respectively.

Local existence theorem, comparison principle, the uniqueness and nonuniqueness of solution for problem (1.1) have been considered in [9].

1991 Mathematics Subject Classification. Primary 35B44, 35K58, 35K61.
Key words and phrases. semilinear heat equation, nonlocal boundary condition, blow-up.
In this paper we obtain necessary and sufficient conditions for the existence of global solutions as well as for a blow-up of solutions in finite time for problem (1.1)–(1.3). Our global existence and blow-up results depend on the behavior of the functions \( c(x, t) \) and \( k(x, y, t) \) as \( t \to \infty \).

This paper is organized as follows. In the next section we show that all nonnegative solutions are global for \( \max(p, l) \leq 1 \). In Section 3 we prove blow-up of solutions for large and for all nontrivial initial data as well as global existence of solutions for small initial data. Finally, in Section 4 we establish that if \( p \leq 1 \) and \( l > 1 \) blow-up can occur only on the boundary.

2. Global existence

Let \( Q_T = \Omega \times (0, T) \), \( S_T = \partial \Omega \times (0, T) \), \( \Gamma_T = S_T \cup \Omega \times \{0\} \), \( T > 0 \).

**Definition 2.1.** We say that a nonnegative function \( u(x, t) \in C^{2,1}(Q_T) \cap C^{1,0}(Q_T \cup \Gamma_T) \) is a supersolution of (1.1)–(1.3) in \( Q_T \) if
\[
\begin{align*}
&\frac{\partial u(x, t)}{\partial t} \geq \Delta u + c(x, t) u^p, \ (x, t) \in Q_T, \quad (2.1) \\
&\frac{\partial u(x, t)}{\partial \nu} \geq \int_{\Omega} k(x, y, t) u^l(y, t) \, dy, \ (x, t) \in S_T, \quad (2.2) \\
&u(x, 0) \geq u_0(x), \ x \in \Omega, \quad (2.3)
\end{align*}
\]
and \( u(x, t) \in C^{2,1}(Q_T) \cap C^{1,0}(Q_T \cup \Gamma_T) \) is a subsolution of (1.1)–(1.3) in \( Q_T \) if \( u \geq 0 \) and it satisfies (2.1)–(2.3) in the reverse order. We say that \( u(x, t) \) is a solution of problem (1.1)–(1.3) in \( Q_T \) if \( u(x, t) \) is both a subsolution and a supersolution of (1.1)–(1.3) in \( Q_T \).

To prove the main results we use the positiveness of solution and the comparison principle which have been proved in [9].

**Theorem 2.2.** Suppose that \( u_0 \not= 0 \) in \( \Omega \) and \( u(x, t) \) is a solution of (1.1)–(1.3) in \( Q_T \). Then \( u(x, t) > 0 \) in \( Q_T \cup S_T \).

**Theorem 2.3.** Let \( u(x, t) \) and \( v(x, t) \) be a supersolution and a subsolution of problem (1.1)–(1.3) in \( Q_T \), respectively. Suppose that \( u(x, t) > 0 \) or \( v(x, t) > 0 \) in \( Q_T \cup \Gamma_T \) if \( \min(p, l) < 1 \). Then \( u(x, t) \geq v(x, t) \) in \( Q_T \cup \Gamma_T \).

The proof of the following statement relies on the continuation principle and the construction of a supersolution.

**Theorem 2.4.** Let \( \max(p, l) \leq 1 \). Then problem (1.1)–(1.3) has global solution for any initial datum.

**Proof.** In order to prove global existence of solution for (1.1)–(1.3) we construct a suitable explicit supersolution of (1.1)–(1.3) in \( Q_T \) for any positive \( T \). Since \( c(x, t) \) and \( k(x, y, t) \) are continuous functions there exists a constant \( M > 0 \) such that \( c(x, t) \leq M \) and \( k(x, y, t) \leq M \) in \( Q_T \) and \( \partial \Omega \times Q_T \), respectively. Let \( \lambda_1 \) be the first eigenvalue of the following problem
\[
\begin{cases}
\Delta \varphi + \lambda \varphi = 0, & x \in \Omega, \\
\varphi(x) = 0, & x \in \partial \Omega,
\end{cases}
\]
and \( \varphi(x) \) be the corresponding eigenfunction with \( \sup_{\Omega} \varphi(x) = 1 \). It is well known \( \varphi(x) \geq 0 \) in \( \Omega \) and \( \max_{\partial \Omega} \partial_x \varphi(x)/\partial \nu < 0 \).
Now we show that \( u = d \exp[bt - a\varphi(x)] \) is a supersolution of (1.1)–(1.3) in \( Q_T \), where constants \( a, b \) and \( d \) are satisfied the following inequalities:

\[
b \geq a^2 \sup_{\Omega} |\nabla \varphi|^2 + a\lambda_1 + M \max\{1, d^{p-1} \exp[a(1-p)]\},
\]

\[
a \geq M|\Omega|d^{l-1} \max_{\partial\Omega} \left(-\frac{\partial \varphi}{\partial \nu}\right)^{-1}, \quad d \geq \exp[a] \sup_{\Omega} u_0(x).
\]

Here \( |\Omega| \) is the Lebesgue measure of \( \Omega \). It is easy to check that

\[
\frac{\partial \varphi}{\partial \nu} - \int_{\Omega} k(x,y,t)u_l(y,t) \, dy = d \exp[bt] \left(a \exp[-a\varphi(x)] \left(-\frac{\partial \varphi}{\partial \nu}\right) - d^{l-1} \exp[b(l-1)t] \int_{\Omega} k(x,y,t) \exp[-al\varphi(y)] \, dy\right) \geq 0 \text{ on } S_T
\]

and

\[
\varphi(x,0) \geq u_0(x) \text{ in } \Omega.
\]

\[
\square \quad \square
\]

3. Blow-up and global existence for \( \max(p,l) > 1 \)

We set

\[
w(t) = \int_{\Omega} u(x,t) \, dx, \quad c_0(t) = |\Omega|^{1-p} \inf_{\Omega} c(x,t), \quad k_0(t) = |\Omega|^{1-l} \inf_{\partial\Omega} \int_{\partial\Omega} k(x,y,t) \, dS_x
\]

and consider the Cauchy problem given by one of the following equations:

\[
w'(t) = c_0(t)w^p, \quad p > 1, \quad l < 1, \quad (3.1)
\]

\[
w'(t) = k_0(t)w^l, \quad p < 1, \quad l > 1, \quad (3.2)
\]

\[
w'(t) = c_0(t)w^p + k_0(t)w^l, \quad p \geq 1, \quad l \geq 1, \quad (3.3)
\]

with initial datum

\[
w(0) = \int_{\Omega} u_0(x) \, dx. \quad (3.4)
\]

**Theorem 3.1.** Let \( \max(p,l) > 1 \) and the Cauchy problem (3.1), (3.2), (3.3) or (3.1), (3.4) does not have a global solution. Then the solution of (1.1)–(1.3) blows up in finite time.

**Proof.** Suppose \( p > 1 \) and \( l < 1 \). Integrating (1.1) over \( \Omega \) and using Green’s identity, we have

\[
w'(t) = \int_{\Omega} c(x,t)w^p(x,t) \, dx + \int_{\partial\Omega} \int_{\Omega} k(x,y,t)u_l(y,t) \, dy \, dS_x
\]

\[
\geq \inf_{\Omega} c(x,t) \int_{\Omega} w^p(x,t) \, dx + \inf_{\partial\Omega} \int_{\Omega} k(x,y,t) \, dS_x \int_{\Omega} u_l(y,t) \, dy. \quad (3.5)
\]

By Jensen’s inequality \( w'(t) \geq c_0(t)w^p \). Now the comparison principle for ordinary differential equation implies the claim. The proof for other cases is similar. \( \square \quad \square \)
Remark 3.2. Solving the Cauchy problem for equations (3.1)–(3.3) with initial datum (3.4), we obtain that (1.1)–(1.3) does not have global solutions in the following cases:

- \( p > 1 \) and \( \int_{\Omega} u_0(x) \, dx > \left( (p-1) \int_{0}^{\infty} c_0(t) \, dt \right)^{-1/(p-1)} \),
- \( l > 1 \) and \( \int_{\Omega} u_0(x) \, dx > \left( (l-1) \int_{0}^{\infty} k_0(t) \, dt \right)^{-1/(l-1)} \),

\( p = 1, l > 1 \) and \( \int_{\Omega} u_0(x) \, dx > \left( (l-1) \int_{0}^{\infty} k_0(t) \exp[(l-1) \int_{0}^{t} c_0(s) \, ds] \, dt \right)^{-1/(l-1)} \),

\( l = 1, p > 1 \) and \( \int_{\Omega} u_0(x) \, dx > \left( (p-1) \int_{0}^{\infty} c_0(t) \exp[(p-1) \int_{0}^{t} k_0(s) \, ds] \, dt \right)^{-1/(p-1)} \).

In particular, there are not nontrivial global solutions of (1.1)–(1.3) if \( p > 1 \) and

\[
\int_{0}^{\infty} c_0(t) \, dt = \infty, \tag{3.6}
\]

\( l > 1 \) and

\[
\int_{0}^{\infty} k_0(t) \, dt = \infty, \tag{3.7}
\]

\( p = 1, l > 1 \) and

\[
\int_{0}^{\infty} k_0(t) \exp[(l-1) \int_{0}^{t} c_0(s) \, ds] \, dt = \infty, \tag{3.8}
\]

\( p > 1, l = 1 \) and

\[
\int_{0}^{\infty} c_0(t) \exp[(p-1) \int_{0}^{t} k_0(s) \, ds] \, dt = \infty. \tag{3.9}
\]

Now we obtain sufficient conditions for blow-up of all nontrivial solutions of (1.1)–(1.3) for \( p > 1 \) if (3.6) is not fulfilled as well as for \( l > 1 \) if (3.7) is not fulfilled. Let us introduce the following auxiliary functions

\[
\tau(t) = \int c(x, t) \, dx, \quad k(t) = \int_{\partial \Omega} \int k(x, y, t) \, dy \, dS_x.
\]

Theorem 3.3. Problem (1.1)–(1.5) does not have nontrivial global solutions if \( p > 1 \) and

\[
\int_{0}^{\infty} c_0(t) \, dt < \infty \quad \text{and} \quad \lim_{t \to \infty} \int_{0}^{t} \tau(\tau) \, d\tau \left( \int_{\tau}^{\infty} c_0(\tau) \, d\tau \right)^{1/(p-1)} = \infty, \tag{3.10}
\]

or \( l > 1 \) and

\[
\int_{0}^{\infty} k_0(t) \, dt < \infty \quad \text{and} \quad \lim_{t \to \infty} \int_{0}^{t} \tau(\tau) \, d\tau \left( \int_{\tau}^{\infty} k_0(\tau) \, d\tau \right)^{1/(l-1)} = \infty.
\]

Proof. Suppose \( p > 1 \) and (3.10) holds. Let \( u(x, t) \) be nontrivial solution of (1.1)–(1.3) in \( Q_T \). Then by Theorem 2.2 we conclude \( u(x, \varepsilon) \geq \alpha \) for any \( x \in \Omega \) and some \( \varepsilon > 0 \) and \( \alpha > 0 \). It is easy to see that \( u(\alpha, x) = \alpha \) is a subsolution of the following problem

\[
\begin{aligned}
v_t &= \Delta v + c(x, t)v^p, \quad x \in \Omega, \quad \varepsilon < t < T, \\
\partial_{\nu}(x, t) &= \int_{\Omega} k(x, y, t)v^q(y, t) \, dy, \quad x \in \partial \Omega, \quad \varepsilon < t < T, \\
v(x, \varepsilon) &= u(\alpha, x), \quad x \in \Omega.
\end{aligned}
\]
Then by comparison principle $u(x, t) \geq \alpha$ for $t \in [\varepsilon, T)$. Using (3.11) and Jensen’s inequality, we get
\[ w'(t) \geq c_0(t)w^p + \alpha l(t) \quad \text{for} \quad \varepsilon \leq t < T. \quad (3.11) \]
From (3.11) we have
\[ w'(t) \geq c_0(t)w^p \quad (3.12) \]
and
\[ w'(t) \geq \alpha l(t). \quad (3.13) \]
By (3.10) we can choose a constant $t_0 > 0$ such that
\[ \left( \alpha l(p-1)^{1/(p-1)} \right)^{-1} < \int_{\varepsilon}^{t_0} k(\tau) d\tau \left( \int_{t_0}^{\infty} c_0(\tau) d\tau \right)^{1/(p-1)}. \quad (3.14) \]
Obviously, we need consider the case $t_0 < T$. For $\varepsilon \leq t_0 < t < T$ from (3.12), (3.13) we conclude
\[ w(t) \geq \left( w^{-p+1}(t_0) - (p-1) \int_{t_0}^{t} c_0(\tau) d\tau \right)^{1/(p-1)} \quad (3.15) \]
and
\[ w(t_0) \geq \alpha \int_{\varepsilon}^{t_0} k(\tau) d\tau. \quad (3.16) \]
By virtue of (3.14), (3.16) we have
\[ w^{-(p-1)}(t_0) < (p-1) \int_{t_0}^{\infty} c_0(\tau) d\tau. \quad (3.17) \]
From (3.15), (3.17) we can see that a solution of (1.1)–(1.3) blows up in finite time. The proof of the second part of the theorem is similar. □ □

To obtain sufficient conditions for the existence of bounded solutions for (1.1)–(1.3) we consider the following auxiliary problem
\[
\begin{cases}
  v_t = \Delta v, & x \in \Omega, \ t > 0 \\
  \frac{\partial v(x,t)}{\partial \nu} = g(t), & x \in \partial \Omega, \ t > 0, \\
  v(x,0) = v_0(x), & x \in \Omega.
\end{cases}
\quad (3.18)
\]
With respect to the data of (3.18) we suppose:
\[ g(t) \in C([0, +\infty)), \ g(t) \geq 0, \quad (3.19) \]
\[ v_0(x) \in C^1(\overline{\Omega}), \ v_0(x) \geq 0 \text{ in } \Omega, \ \frac{\partial v_0(x)}{\partial \nu} = g(0) \text{ on } \partial \Omega. \quad (3.20) \]

**Lemma 3.4.** Let (3.19), (3.20) hold. Then a solution of (3.18) is bounded if and only if
\[ \int_{0}^{\infty} g(t) dt < \infty \quad (3.21) \]
and there exist positive constants $\alpha$, $t_0$ and $c$ such that $\alpha > t_0$ and
\[ \int_{t-t_0}^{t} \frac{g(\tau)}{\sqrt{t-\tau}} d\tau \leq c \text{ for any } t \geq \alpha. \quad (3.22) \]
Proof. Let $G_N(x, y; t - \tau)$ be the Green function of the heat equation with homogeneous Neumann boundary condition. We note that $G_N(x, y; t - \tau)$ has the following properties (see, for example, [13]):

\begin{align}
G_N(x, y; t - \tau) & \geq 0, \ x, y \in \Omega, \ 0 \leq \tau < t < T, \\
\int_{\Omega} G_N(x, y; t - \tau) \, dy &= 1, \ x \in \Omega, \ 0 \leq \tau < t < T.
\end{align}

(3.23)

(3.24)

Moreover, similarly as in [12] and [13] we can show

\begin{align}
|G_N(x, y; t - \tau) - 1/|\Omega|| & \leq c_1 \exp[-c_2(t - \tau)], \ x, y \in \Omega, \ t - \tau \geq \varepsilon, \\
\frac{c_3}{\sqrt{t - \tau}} & \leq \int_{\partial \Omega} G_N(x, y; t - \tau) \, dS_y \leq \frac{c_4}{\sqrt{t - \tau}}, \ x \in \partial \Omega, \ 0 < t - \tau \leq \varepsilon,
\end{align}

(3.25)

(3.26)

for some small $\varepsilon > 0$ and

\begin{align}
\int_{\partial \Omega} G_N(x, \xi; t - \tau) \, dS_{\xi} & \geq c_5, \ x \in \overline{\Omega}, \ 0 \leq \tau < t < T.
\end{align}

(3.27)

Here and subsequently by $c_i (i \in \mathbb{N})$ we denote positive constants. Note that the upper bound in (3.26) is true for any $x \in \overline{\Omega}$. It is well known that problem (3.18) is equivalent to the equation

$$
v(x, t) = \int_{\Omega} G_N(x, y; t) v_0(y) \, dy + \int_0^t g(\tau) \int_{\partial \Omega} G_N(x, y; t - \tau) \, dS_y \, d\tau, \ x \in \overline{\Omega}, \ t > 0.
$$

(3.28)

Using (3.21)–(3.26), (3.28) for some $\varepsilon > 0$ we get

\begin{align}
v(x, t) & \leq \sup_{\Omega} v_0(x) + |\partial \Omega| \int_{t - \varepsilon}^t (c_1 \exp[-c_2(t - \tau)] + 1/|\Omega|) g(\tau) \, d\tau \\
& \quad + c_4 \int_{t - \varepsilon}^t \frac{g(\tau)}{\sqrt{t - \tau}} \, d\tau \leq c_6, \ x \in \overline{\Omega}, \ t \geq \alpha.
\end{align}

Hence, $v(x, t)$ is a bounded solution of (3.18). Necessity of (3.21), (3.22) for boundedness of a solution of (3.18) is proved similarly. \(\square\)

Remark 3.5. The function $g(t)$ satisfies (3.22) if there exist positive constants $\alpha$, $t_0$ and $c$ such that $\alpha > t_0$ and for some $q > 2$ the inequality

\begin{align}
\int_{t - t_0}^t g^q(\tau) \, d\tau \leq c
\end{align}

(3.29)

holds for any $t \geq \alpha$. Indeed, applying Hölder’s inequality, we obtain

\begin{align}
\int_{t - t_0}^t \frac{g(\tau)}{\sqrt{t - \tau}} \, d\tau & \leq \left( \int_{t - t_0}^t g^q(\tau) \, d\tau \right)^{1/q} \left( \int_{t - t_0}^t (t - \tau)^{-m/2} \, d\tau \right)^{1/m} \\
& \leq c(t_0),
\end{align}

where $1/q + 1/m = 1$, and hence $1 < m < 2$.

Now we construct a function which demonstrates that (3.19), (3.21) and (3.29) with $q = 2$ do not guarantee (3.22). Denote $O_n = [n - 1/n^3, n], \ n = 2, 3, \ldots,$ and consider the following function

\begin{align}
g(t) = \left\{ \begin{array}{ll}
g_n(t), & t \in O_n, \ n = 2, 3, \ldots, \\
f(t), & t \in [0, +\infty) \setminus \bigcup_{n=2}^{\infty} O_n,
\end{array} \right.
\end{align}

where $g_n(t)$ and $f(t)$ are suitable functions.
where
\[ g_{n}(t) = \frac{1}{\sqrt{n + 1/n^6 - t}} \ln(n + 1/n^6 - t)^{\alpha}, \alpha \in (1/2, 1), n = 2, 3, \ldots, \]

\(f(t)\) is a continuous function such that \(g(t)\) satisfies (3.19) and
\[ \int_{[0, +\infty) \setminus \bigcup_{n=2}^{\infty} O_{n}} (f^2(t) + f(t)) \, dt < \infty. \]

A straightforward computations show that \(g(t)\) satisfies (3.21), (3.29) with \(q = 2\) and
\[ \int_{n^{-1/n^3}}^{n} \frac{g(\tau)}{\sqrt{n - \tau}} \, d\tau \to \infty \text{ as } n \to \infty. \]

Put \(c_1(t) = \sup_{\Omega} c(x, t)\) and \(k_1(t) = \sup_{\partial \Omega \times \Omega} k(x, y, t)\). Suppose that \(c_1(t)\) and \(k_1(t)\) satisfy the following conditions:
\[ \int_{0}^{\infty} (c_1(t) + k_1(t)) \, dt < \infty \tag{3.30} \]
and there exist positive constants \(\alpha, t_0\) and \(K\) such that \(\alpha > t_0\) and
\[ \int_{t-t_0}^{t} \frac{k_1(\tau)}{\sqrt{t - \tau}} \, d\tau \leq K \text{ for any } t \geq \alpha. \tag{3.31} \]

**Theorem 3.6.** Let \(\min(p, l) > 1\) and (3.30), (3.31) hold. Then problem (1.1)–(1.3) has bounded global solutions for small initial data.

**Proof.** Let \(v(x, t)\) be a solution of (3.18) with boundary condition \(g(t) = k_1(t)\) and positive initial datum. According to Lemma 3.4 there exists positive constant \(V\) such that \(v(x, t) \leq V\) for any \(x \in \Omega\) and \(t \geq 0\).

To prove the theorem we construct a supersolution of (1.1)–(1.3) in the following form \(\overline{u}(x, t) = af(t)v(x, t)\), where
\[ f(t) = \left( A - (p-1)a^{p-1}V^{p-1} \int_{0}^{t} c_1(\tau) \, d\tau \right)^{-1/(p-1)}, \]
with
\[ A = 1 + (p-1)a^{p-1}V^{p-1} \int_{0}^{\infty} c_1(\tau) \, dt \text{ and } a \text{ is some positive constant.} \]

After simple computations it follows that
\[ \overline{u}_t - \Delta \overline{u} - c(x, t)\overline{u}^p = af'(t)v + afv_t - af p\Delta v - a^p c(x, t) f^p v^p \geq av(f'(t) - a^{p-1}V^{p-1}c_1(t)f^p) = 0, \quad x \in \Omega, \quad t > 0, \]
and
\[ \frac{\partial \overline{u}}{\partial \nu} - \int_{\partial \Omega} k(x, y, t)\overline{u}(y, t) \, dy \geq af(t)k_1(t)(1 - a^{l-1}f^{l-1}V^{l}(|\Omega|) \geq 0, \quad x \in \partial \Omega, \quad t > 0, \]
for small values of \(a\). Hence, \(\overline{u}(x, t)\) is a supersolution of problem (1.1)–(1.3) for an initial datum \(u_{0}(x) \leq aA^{-1/(p-1)}v(x, 0)\).

**Remark 3.7.** By Remark 3.2 and Theorem 3.6 condition (3.30) is optimal for global existence of solutions for (1.1)–(1.3) with \(c(x, t) = c(t)\) and \(k(x, y, t) = k(t)\). Arguing in the same way as in the proof of Lemma 3.4 it is easy to show that (3.31) is optimal for the existence of bounded global solutions for (1.1)–(1.3) with \(k(x, y, t) = k(t)\).
Remark 3.8. Assume that \( \min(p,l) > 1 \), (3.30) holds and there exist positive constants \( \alpha, t_0 \) and \( K \) such that \( \alpha > t_0 \) and for some \( q > 2 \) the inequality
\[
\int_{t-t_0}^t k_l^q(\tau) \, d\tau \leq K
\]
holds for any \( t \geq \alpha \). Then by Remark 3.5 and Theorem 3.6 problem (1.1)--(1.3) has bounded global solutions for small initial data.

3.1. The case \( p = 1 \) and \( l > 1 \). Suppose that for some \( K \geq 0 \) and \( \varepsilon > 0 \) the functions \( k(x,y,t) \) and \( c_1(t) \) satisfy
\[
\int_{\Omega} k(x,y,t) \, dy \leq K \exp \left[ -(l-1) \left\{ \int_0^t c_1(\tau) \, d\tau + \varepsilon t \right\} \right]
\]
for any \( x \in \partial \Omega \) and \( t \geq 0 \).

**Theorem 3.9.** Let \( p = 1 \), \( l > 1 \) and (3.32) hold. Then problem (1.1)--(1.3) has global solutions for small initial data.

**Proof.** Let \( \psi(x) \) be a solution of the following problem
\[
\Delta \psi = 1, \quad x \in \Omega, \quad \frac{\partial \psi(x)}{\partial \nu} = \gamma, \quad x \in \partial \Omega,
\]
where \( \gamma = |\Omega|/|\partial \Omega| \). To prove the theorem we construct a supersolution of (1.1)--(1.3) in such a form that \( v(x,t) = b \exp[f(t)]\psi(x) \), where \( f(t) = \int_0^t c_1(\tau) \, d\tau + \varepsilon t \) and \( b \) is some positive constant. Indeed, we have
\[
v_t - \Delta v - c(x,t)v = (c_1(t) + \varepsilon)v - \frac{v}{\psi(x)} - c(x,t)v \geq \left( \varepsilon - \frac{1}{\psi(x)} \right) v \geq 0, \quad x \in \Omega, \quad t > 0,
\]
for large values of \( \inf_{\Omega} \psi(x) \) and
\[
\frac{\partial v}{\partial \nu} = \gamma b \exp[f(t)] - b' \exp[f(t)] \sup_{\partial \Omega} \int_{\Omega} k(x,y,t)\psi(y) \, dy
\]
\[
\geq b \exp[f(t)] \left( \gamma - Kb^{-1} \sup_{\Omega} \psi(x) \right) \geq 0, \quad x \in \partial \Omega, \quad t > 0,
\]
for small values of \( b \). Consequently, \( v(x,t) \) is a supersolution of (1.1)--(1.3) for an initial datum \( u_0(x) \leq b\psi(x) \). \( \square \)

**Remark 3.10.** It is easy to see from Remark 3.2 that Theorem 3.9 does not hold for \( \varepsilon = 0 \).

Suppose that \( k_1(t) \) and \( c_1(t) \) satisfy
\[
\int_0^\infty k_1(t) \exp \left[ (l-1) \int_0^t c_1(\tau) \, d\tau \right] \, dt < \infty \tag{3.33}
\]
and there exist positive constants \( \alpha, t_0 \) and \( K \) such that \( \alpha > t_0 \) and
\[
\int_{t-t_0}^t k_1(\tau) \exp \left[ (l-1) \int_0^\tau c_1(s) \, ds \right] \, d\tau \leq K \text{ for any } t \geq \alpha. \tag{3.34}
\]
Theorem 3.11. Let $p = 1$, $l > 1$ and (3.33), (3.34) hold. Then problem (1.1)–(1.3) has bounded global solutions for small initial data.

Proof. Let $v(x, t)$ be a solution of (3.18) with boundary condition $g(t) = k_1(t) \exp[(l-1) \int_0^t c_1(\tau) d\tau]$ and positive initial datum. According to Lemma 3.4 there exists positive constant $V$ such that $v(x, t) \leq V$ for any $x \in \Omega$ and $t \geq 0$.

To prove the theorem we construct a supersolution of (1.1)–(1.3) in the following form $\bar{v}(x, t) = af(t)v(x, t)$, where

$$f(t) = \left(A - (p-1)a^{p-1}V^{p-1} \int_0^t c_1(t) dt\right)^{-1/(p-1)},$$

$A = 1 + (p-1)a^{p-1}V^{p-1} \int_0^\infty c_1(t) dt$ and $a$ is some positive constant. It is easy to check that

$$\bar{v}_t - \Delta \bar{v} - a(x, t)v = ac_1(t)v \exp\left[\int_0^t c_1(\tau) d\tau\right] + a \exp\left[\int_0^t c_1(\tau) d\tau\right] v_t$$

$$- a \exp\left[\int_0^t c_1(\tau) d\tau\right] \Delta v - ac(x, t) \exp\left[\int_0^t c_1(\tau) d\tau\right] v \geq 0, \quad x \in \Omega, \quad t > 0,$$

and

$$\frac{\partial \bar{v}}{\partial v} - \int_\Omega k(x, y, t) \bar{v}(y, t) dy \geq ak_1(t) \exp\left[l \int_0^t c(\tau) d\tau\right] (1 - a^{l-1}V^{l-1}|\Omega|) \geq 0$$

for $x \in \partial\Omega$, $t > 0$ and some small values of $a$. Hence, $\bar{v}(x, t)$ is a supersolution of problem (1.1)–(1.3) for an initial datum $v_0(x) \leq av(x, 0)$.

Remark 3.12. By Remark 3.2 and Theorem 3.11 condition (3.33) is optimal for global existence of solutions for (1.1)–(1.3) with $c(x, t) = c(t)$ and $k(x, y, t) = k(t)$.

Arguing in the same way as in the proof of Lemma 3.4 it is easy to show that (3.34) is optimal for the existence of bounded global solutions for (1.1)–(1.3) with $k(x, y, t) = k(t)$.

Remark 3.13. Assume $p = 1$, $l > 1$, (3.33) holds and there exist positive constants $\alpha$, $t_0$ and $K$ such that $\alpha > t_0$ and for some $q > 2$ the inequality

$$\int_{t_0}^t k_1^q(\tau) \exp\left[q(l-1) \int_0^\tau c_1(s) ds\right] d\tau \leq K$$

holds for any $t \geq \alpha$. Then by Remark 3.2 and Theorem 3.11 problem (1.1)–(1.3) has bounded global solutions for small initial data.

3.2. The case $l = 1$ and $p > 1$. Consider the auxiliary problem

$$\begin{cases}
\Delta h(x) = ah(x), \quad x \in \Omega, \\
\frac{\partial h(x)}{\partial \nu} = g(x) \int_\Omega h(y) dy, \quad x \in \partial\Omega.
\end{cases}$$

(3.35)

With respect to the data of (3.35) we suppose

$$g(x) \in C(\partial\Omega), \quad g(x) \geq 0, \quad a = \int_{\partial\Omega} g(x) dS > 0.$$

Lemma 3.14. Problem (3.35) has infinitely many nonnegative solutions.
Proof. Let \( \int_\Omega h(y) \, dy \neq 0 \). Set \( v(x) = h(x)/\int_\Omega h(y) \, dy \). It is easy to verify that (3.35) is reduced to the following problem

\[
\begin{aligned}
\Delta v(x) &= av(x), \quad x \in \Omega, \\
\frac{\partial v(x)}{\partial \nu} &= g(x), \quad x \in \partial \Omega,
\end{aligned}
\]

provided

\[
\int_\Omega v(y) \, dy = 1.
\] (3.37)

By (15) problem (3.36) has unique solution. Obviously, this solution satisfies (3.37). Now we show nonnegativity of \( v(x) \) cannot attain a negative minimum in \( \Omega \). Suppose there exists a point \( x_0 \in \partial \Omega \) such that \( v(x_0) = \min_{\Omega} v(x) < 0 \). Then \( \partial v(x_0)/\partial \nu < 0 \) (see [6]) which contradicts the boundary condition. Obviously, \( h(x) = \alpha v(x) \) is nonnegative solution of (3.35) for any \( \alpha > 0 \).

Suppose that \( k(x, y, t) \) and \( c_1(t) \) satisfy the following conditions:

\[
k(x, y, t) \leq k_2(x), \quad x \in \partial \Omega, \quad y \in \Omega, \quad t > 0,
\]

(3.38)

and

\[
\int_0^\infty c_1(t) \exp \left[ (p-1)t \int_{\partial \Omega} k_2(x) \, dS \right] \, dt < \infty,
\]

(3.39)

where \( k_2(x) \) is some nonnegative continuous on \( \partial \Omega \) function.

**Theorem 3.15.** Let \( l = 1 \), \( p > 1 \) and (3.38), (3.39) hold. Then problem (1.1)–(1.3) has global solutions for small initial data.

**Proof.** Let \( h(x) \) be some nonnegative solution of (3.35) with \( a = \int_{\partial \Omega} k_2(x) \, dS \) and \( g(x) = k_2(x) \). To prove the theorem we construct a supersolution of (1.1)–(1.3) in such a form that \( \overline{u}(x, t) = f(t)h(x) \), where

\[
f(t) = \exp[at] \left( A - (p-1) \sup_{\Omega} h^{p-1}(x) \int_0^t c_1(\tau) \exp[(p-1)a\tau] \, d\tau \right)^{-1/(p-1)},
\]

\[
A = 1 + (p-1) \sup_{\Omega} h^{p-1}(x) \int_0^\infty c_1(t) \exp[(p-1)at] \, dt.
\]

Indeed, we have

\[
\overline{u}_t - \Delta \overline{u} - c(x, t) \overline{u} = f'(t)h - afh - c(x, t)fph \]

\[
\geq h(f'(t) - af - \sup_{\Omega} h^{p-1}(x)c_1(t)f) = 0, \quad x \in \Omega, \quad t > 0,
\]

and

\[
\frac{\partial \overline{u}}{\partial \nu} - \int_{\Omega} k(x, y, t) \overline{u}(y, t) \, dy = f(t) \left( k_2(x) \int_{\Omega} h(y) \, dy - \int_{\Omega} k(x, y, t)h(y) \, dy \right) \geq 0
\]

for \( x \in \partial \Omega, \quad t > 0 \). Hence, \( \overline{u}(x, t) \) is a supersolution of (1.1)–(1.3) for an initial datum \( u_0(x) \leq A^{-1/(p-1)}h(x) \).

**Remark 3.16.** It is easy to see that (3.39) and (3.9) are optimal conditions for global existence and blow-up of solutions for (1.1)–(1.3) if, for example, \( c(x, t) = c(t) \) and \( k(x, y, t) = k(x) \).
4. Blow-up on the boundary

In this section we show that for problem (1.1)–(1.3) in the case \( l > 1 \) and \( p \leq 1 \) blow-up cannot occur at the interior domain. We introduce the following notation

\[
J(t) = \int_0^t \int_\Omega u'(x, \tau) \, dx \, d\tau.
\]

**Lemma 4.1.** Let \( l > 1 \), \( \inf_{\partial \Omega \times Q_T} k(x, y, t) > 0 \) and the solution \( u(x, t) \) of (1.1)–(1.3) blows up in \( t = T \). Then for \( t \in \left[0, T \right) \)

\[
J(t) \leq s (T - t)^{-1/(l-1)} , \quad s > 0.
\]

**Proof.** It is well known that \( u(x, t) \) is the solution of (1.1)–(1.3) in \( Q_T \) if and only if

\[
u(x, t) = \int_\Omega G_N(x, y; t) u_0(y) \, dy + \int_0^t \int_\Omega G_N(x, y; t - \tau) c(y, \tau) u^p(y, \tau) \, dy \, d\tau
\]

\[
+ \int_0^t \int_{\partial \Omega} G_N(x, \xi; t - \tau) \int_0^1 k(\xi, y, \tau) u^l(y, \tau) \, dy \, dS_\xi \, d\tau.
\]

By virtue of (3.23), (3.27), (4.1), (4.3) and Jensen’s inequality we have

\[
J'(t) = \int_\Omega u'(x, t) \, dx \geq k^l \left( \int \int_{\partial \Omega} G_N(x, \xi; t - \tau) \int_0^1 u^l(y, \tau) \, dy \, dS_\xi \, d\tau \right) \]

\[
\geq (c_3 k)^l |\Omega| J'(t),
\]

where \( k = \inf_{\partial \Omega \times Q_T} k(x, y, t) \). Thus,

\[
J'(t) \geq c_7 J'(t).
\]

Integrating (4.4) over \( (t; T) \), we obtain (4.2). \( \square \)

**Theorem 4.2.** Let \( p \leq 1 \) and the conditions of Lemma 4.1 hold. Then blow-up can occur only on the boundary.

**Proof.** In the proof we shall use some arguments of [4], [11]. Let \( u(x, t) = \exp[ct]v(x, t) \), where \( c = \sup c(x, t) \). It is easy to check that \( v(x, t) \) is a solution of

\[
\begin{cases}
\partial_\nu v + \Delta v + (1 - p)ct |c(x, t)u^p| - cv, \quad (x, t) \in Q_T,
\partial_\nu v = \exp[(l - 1)ct] \int_\Omega k(x, y, t) v^l(y, t) \, dy, \quad (x, t) \in S_T,
v(x, 0) = u_0(x), \quad x \in \Omega.
\end{cases}
\]

Then \( v(x, t) \) satisfies the following equation

\[
v(x, t) = \int_\Omega G_N(x, y; t) u_0(y) \, dy
\]

\[
+ \int_0^t \int_\Omega G_N(x, y; t - \tau) \exp[\exp[-(1 - p)ct] c(y, \tau) u^p(y, \tau) - cv(y, \tau)] \, dy \, d\tau
\]

\[
+ \int_0^t \int_{\partial \Omega} G_N(x, \xi; t - \tau) \int_0^1 \exp[(l - 1)ct] k(\xi, y, \tau) u^l(y, \tau) \, dy \, dS_\xi \, d\tau.
\]
for \((x,t) \in Q_T\). We now take an arbitrary \(\Omega' \subset \subset \Omega\) with \(\partial \Omega' \in C^2\) such that \(\text{dist}(\partial \Omega, \partial \Omega') = \varepsilon > 0\). It is known (see, for example, [12]) that
\[
0 \leq G_N(x, y; t - \tau) \leq c_\varepsilon, \quad x \in \Omega', \quad y \in \partial \Omega, \quad 0 < \tau < t < T,
\]
where \(c_\varepsilon\) is a positive constant depending on \(\varepsilon\). By (3.23), (3.24), (4.2), (4.5) and (4.6) we have
\[
\sup_{\Omega'} v(x,t) \leq \sup_{\Omega} u_0(x) + c \int_0^t \int_{\Omega} G_N(x, y; t - \tau) \, dy \, d\tau + c_\varepsilon \sup_{\partial \Omega \times Q_T} k(x,y,t) |\partial \Omega| \exp[(l-1)cT] \int_0^t \int_{\Omega} u^l(y, \tau) \, dy \, d\tau
\]
\[
\leq c_7 + c_8 J(t) \leq c_9 (T-t)^{-1/(l-1)}.
\]
Hence,
\[
\sup_{\Omega'} u(x,t) \leq c_{10} (T-t)^{-1/(l-1)}.
\]
As it is shown in [11], there exists a function \(f(x) \in C^2(\Omega')\) satisfying
\[
\Delta f - \frac{l-1}{l} \frac{|\nabla f|^2}{f} \geq -c_{11} \text{ in } \Omega', \quad f(x) > 0 \text{ in } \Omega', \quad f(x) = 0 \text{ on } \partial \Omega'.
\]  
(4.7)

We introduce the auxiliary function
\[
w(x,t) = c_{12} \exp[\mu t] \left( f(x) + c_{11}(T-t) \right)^{-1/(l-1)},
\]
where the positive constants \(\mu\) and \(c_{12}\) will be defined below. By (4.7) for \(x \in \Omega'\) and \(t \in [0,T)\) we get
\[
w_t - \Delta w - c(x,t) w^p = \mu w - c(x,t) w^p
\]
\[
+ \frac{w}{(l-1)[f(x) + c_{11}(T-t)]} \left( c_{11} + \Delta f - \frac{l|\nabla f|^2}{(l-1)[f(x) + c_{11}(T-t)]} \right) \geq 0
\] provided that

\[
\mu \geq c \left( \frac{[\sup_{\Omega'} f(x) + c_{11}T]^{1/(l-1)}}{c_{12}} \right)^{1-p}.
\]
Choosing \(c_{12}\) such that \(c_{12} > c_{11}^{1/(l-1)} c_{10}\) and \(w(x,0) \geq u(x,0)\) for \(x \in \Omega'\), by comparison principle we conclude
\[
u(x,t) \leq w(x,t) \text{ in } \Omega' \times [0,T).
\]
Hence, \(u(x,t)\) cannot blow up in \(\Omega' \times [0,T]\). Since \(\Omega'\) is an arbitrary subset of \(\Omega\), the proof is completed. 

\[\Box\]

References

[1] Z. Cui, Z. Yang, Roles of weight functions to a nonlinear porous medium equation with nonlocal source and nonlocal boundary condition, J. Math. Anal. Appl. 342 (2008) 559–570.
[2] Z. Cui, Z. Yang, R. Zhang, Blow-up of solutions for nonlinear parabolic equation with nonlocal source and nonlocal boundary condition, Appl. Math. Comput. 224 (2013) 1–8.
[3] K. Deng, Z. Dong, Blow-up for the equation with a general memory boundary condition, Comm. Pure Appl. Anal. 11 (2012) 2147–2156.
[4] K. Deng, C.L. Zhao, Blow-up for a parabolic system coupled in an equation and a boundary condition, Proc. Royal Soc. Edinb. 131A (2001) 1345–1355.
[5] Z.B. Fang, J. Zhang, Global existence and blow-up of solutions for p-Laplacian evolution equation with nonlinear memory term and nonlocal boundary condition, Boundary Value Problems 2014 (2014) 1–17.
[6] A. Friedman, Partial differential equations of parabolic type, Prentice-Hall, 1964.
[7] Y. Gao, W. Gao, Existence and blow-up of solutions for a porous medium equation with nonlocal boundary condition, Appl. Anal. 90 (2011) 799–809.
[8] A. Gladkov, M. Guedda, Blow-up problem for semilinear heat equation with absorption and a nonlocal boundary condition, Nonlinear Anal. 74 (2011) 4573–4580.
[9] A. Gladkov, T. Kavitova, Initial boundary value problem for a semilinear parabolic equation with nonlinear nonlocal boundary conditions, http://arxiv.org/abs/1412.5021.
[10] A. Gladkov, K. Ik Kim, Blow-up of solutions for semilinear heat equation with nonlinear nonlocal boundary condition, J. Math. Anal. Appl. 338 (2008) 264–273.
[11] B. Hu, H.M. Yin, The profile near blowup time for solution of the heat equation with a nonlinear boundary condition, Trans. Amer. Math. Soc. 346 (1994) 117–135.
[12] B. Hu, H.M. Yin, Critical exponents for a system of heat equations coupled in a non-linear boundary condition, Math. Meth. Appl. Sci. 19 (1996) 1099–1120.
[13] C.S. Kahane, On the asymptotic behavior of solutions of parabolic equations, Czechoslovak Math. J. 33 (1983) 262–285.
[14] D. Liu, C. Mu, Blowup properties for a semilinear reaction-diffusion system with nonlinear nonlocal boundary conditions, Abstr. Appl. Anal. 2010 (2010) 1–17.
[15] C. Miranda, Equazioni alle derivate pazziali di tipo ellittico, Springer-Verlag, 1955.
[16] C.V. Pao, Asymptotic behavior of solutions of reaction-diffusion equations with nonlocal boundary conditions, J. Comput. Appl. Math. 88 (1998) 225–238.
[17] Y. Wang, C. Mu, Z. Xiang, Blowup of solutions to a porous medium equation with nonlocal boundary condition, Appl. Math. Comput. 192 (2007) 579–585.
[18] L. Yang, C. Fan, Global existence and blow-up of solutions to a degenerate parabolic system with nonlocal sources and nonlocal boundaries, Monatshefte für Mathematik. 174 (2014) 493–510.
[19] Z. Ye, X. Xu, Global existence and blow-up for a porous medium system with nonlocal boundary conditions and nonlocal sources, Nonlinear Anal. 82 (2013) 115–126.
[20] H.M. Yin, On a class of parabolic equations with nonlocal boundary conditions, J. Math. Anal. Appl. 294 (2004) 712–728.
[21] S. Zheng, I. Kong, Roles of weight functions in a nonlinear nonlocal parabolic system, Nonlinear Anal. 68 (2008) 2406–2416.

Alexander Gladkov, Department of Mechanics and Mathematics, Belarusian State University, Nezavisimosti Avenue 4, 220030 Minsk, Belarus
E-mail address: gladkoval@mail.ru

Tatiana Kavitova, Department of Mathematics, Vitebsk State University, Moskovskii pr. 33, 210038 Vitebsk, Belarus
E-mail address: kavitovatv@tut.by