SATURATED SETS FOR NONUNIFORMLY HYPERBOLIC SYSTEMS

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Abstract. In this paper we prove that for an ergodic hyperbolic measure \( \omega \) of a \( C^{1+\alpha} \) diffeomorphism \( f \) on a Riemannian manifold \( M \), there is an \( \omega \)-full measured set \( \tilde{\Lambda} \) such that for every invariant probability \( \mu \in \mathcal{M}_{inv}(\tilde{\Lambda}, f) \), the metric entropy of \( \mu \) is equal to the topological entropy of saturated set \( G_\mu \) consisting of generic points of \( \mu \):

\[
h_\mu(f) = h_{top}(f, G_\mu).
\]

Moreover, for every nonempty, compact and connected subset \( K \) of \( \mathcal{M}_{inv}(\tilde{\Lambda}, f) \) with the same hyperbolic rate, we compute the topological entropy of saturated set \( G_K \) of \( K \) by the following equality:

\[
\inf\{h_\mu(f) \mid \mu \in K\} = h_{top}(f, G_K).
\]

In particular these results can be applied (i) to the nonuniform hyperbolic diffeomorphisms described by Katok, (ii) to the robustly transitive partially hyperbolic diffeomorphisms described by Mañé, (iii) to the robustly transitive non-partially hyperbolic diffeomorphisms described by Bonatti-Viana. In all these cases \( \mathcal{M}_{inv}(\tilde{\Lambda}, f) \) contains an open subset of \( \mathcal{M}_{erg}(M, f) \).

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1. Introduction

Let \((M, d)\) be a compact metric space and \(f : M \to M\) be a continuous map. Given an invariant subset \(\Gamma \subset M\), denote by \(\mathcal{M}(\Gamma)\) the set consisting of all Borel probability measures, by \(\mathcal{M}_{inv}(\Gamma, f)\) the subset consisting of \(f\)-invariant probability measures and, by \(\mathcal{M}_{erg}(\Gamma, f)\) the subset consisting of \(f\)-invariant ergodic probability measures. Clearly, if \(\Gamma\) is compact then \(\mathcal{M}(\Gamma)\) and \(\mathcal{M}_{inv}(\Gamma, f)\) are both compact spaces in the weak*-topology of measures. Given \(x \in M\), define the \(n\)-ordered empirical measure of \(x\) by

\[
\mathcal{E}_n(x) = \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)},
\]

where \(\delta_y\) is the Dirac mass at \(y \in M\). A subset \(W \subset M\) is called saturated if \(x \in W\) and the sequence \(\{\mathcal{E}_n(y)\}\) has the same limit points set as \(\{\mathcal{E}_n(x)\}\) then \(y \in W\). The limit point set \(V(x)\) of \(\{\mathcal{E}_n(x)\}\) is always a compact connected subset of \(\mathcal{M}_{inv}(M, f)\). Given \(\mu \in \mathcal{M}_{inv}(M, f)\), we collect the saturated set \(G_\mu\) of \(\mu\) by those generic points \(x\) satisfying \(V(x) = \{\mu\}\). More generically, for a compact connected subset \(K \subset \mathcal{M}_{inv}(M, f)\), we denote by \(G_K\) the saturated set consisting of points \(x\) with \(V(x) = K\). By Birkhoff Ergodic Theorem, \(\mu(G_\mu) = 1\) when \(\mu\) is ergodic. However, this is somewhat a special case. For non-ergodic \(\mu\), by Ergodic Decomposition Theorem, \(G_\mu\) has measure 0 and thus is “thin” in view of measure. In addition, when \(f\) is uniformly hyperbolic \((\text{20})\) or non uniformly hyperbolic \((\text{19})\), \(G_\mu\) is of first category hence “thin” in view of topology. Exactly, one can get this topological fact of first category as follows. Denote by \(C^0(M)\) the set of continuous real-valued functions on \(M\) provided with the sup norm. For non uniformly hyperbolic systems \((f, \mu)\), there is \(x \in M\) such that

\[
\overline{\text{orb}(x, f)} \subset \text{supp}(\mu) \quad \text{and} \quad \mathcal{E}_n(x) \text{ does not converge},
\]

where the support of a measure \(\nu\), denoted by \(\text{supp}(\nu)\), is the minimal closed set with \(\nu\)-total measure, see \(\text{30} \, \text{19}\). We can take \(0 < a_1 < a_2\) and \(\varphi \in C(M)\) such that

\[
\liminf_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) < a_1 < a_2 < \limsup_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)).
\]

Let

\[
R = \cap_N \cup_{n \geq N} \left\{ x \mid \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) < a_1 \right\} \cap \cap_N \cup_{n \geq N} \left\{ x \mid \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) > a_2 \right\}.
\]

Then

\[
R \cap \overline{\text{orb}(x, f)} \subset \overline{(\text{orb}(x, f) \setminus G_\mu)} \quad \text{and} \quad R \cap \overline{\text{orb}(x, f)} \text{ is a } G_\delta \text{ subset of } \overline{\text{orb}(x, f)}.
\]

Combining with \(x \in \text{orb}(x, f)\), we can see that \(\text{orb}(x, f) \setminus G_\mu\) is a residual set of \(\text{orb}(x, f)\). Hence, \(G_\mu\) is of first category in the subspace \(\text{orb}(x, f)\).
For a conservative system $(f, M, \text{Leb})$ preserving the normalized volume measure \text{Leb}, if $f$ is ergodic, then by ergodic theorem,
\[ E_n(x) \rightarrow \text{Leb}, \quad \text{as} \quad n \rightarrow +\infty, \]
for \text{Leb-}a.e. $x \in M$. In the general dissipative case where, a priori, there is no distinguished invariant probability measures, it is much more subtle what one should mean by describing the behavior of almost orbits in the physically observable sense. In this content, an invariant measure $\mu$ is called physical measure (or Sinai-Ruelle-Bowen measure) if the saturated set $G_{\mu}$ is of positive Lebesgue measure. SRB measures are used to measure the “thickness” of saturated sets in view of Leb-measure.

Motivated by the definition of saturated sets, it is reasonable to think that $G_{\mu}$ should put together all information of $\mu$. If $\mu$ is ergodic, Bowen\cite{Bowen} has succeeded this motivation to prove that
\[ h_{\text{top}}(f, G_{\mu}) = h_{\mu}(f). \]
When $f$ is mixing and uniformly hyperbolic (which implies uniform specification property), applying \cite{Bowen} it also holds that
\[ h_{\text{top}}(f, G_{\mu}) = h_{\mu}(f). \]
This implies that $G_{\mu}$ is “thick” in view of topological entropy. Indeed, the information of invariant measure can be well approximated by nearby measures \cite{Bowen, LI, LI1, LI2}. For non uniformly hyperbolic systems, in \cite{Liang} Liang, Sun and Tian proved $G_{\mu} \neq \emptyset$. Our goal in the present paper is to show the “thickness” of $G_{\mu}$ in view of entropy.

Now we start to introduce our results precisely. Let $M$ be a compact connected boundary-less Riemannian $d$-dimensional manifold and $f : M \rightarrow M$ a $C^{1+\alpha}$ diffeomorphism. We use $Df_x$ to denote the tangent map of $f$ at $x \in M$. We say that $x \in M$ is a regular point of $f$ if there exist numbers $\lambda_1(x) > \lambda_2(x) > \cdots > \lambda_{\phi(x)}(x)$ and a decomposition on the tangent space
\[ T_x M = E_1(x) \oplus \cdots \oplus E_{\phi(x)}(x) \]
such that
\[ \lim_{n \rightarrow +\infty} \frac{1}{n} \log \| (Df^n)_x u \| = \lambda_j(x) \]
for every $0 \neq u \in E_j(x)$ and every $1 \leq j \leq \phi(x)$. The number $\lambda_i(x)$ and the space $E_i(x)$ are called the Lyapunov exponents and the eigenspaces of $f$ at the regular point $x$, respectively. Oseledets theorem \cite{Oseledets} states that all regular points of a diffeomorphism $f : M \rightarrow M$ forms a Borel set with total measure. For a regular point $x \in M$ we define
\[ \lambda^+(x) = \max \{ 0, \min \{ \lambda_i(x) \mid \lambda_i(x) > 0, \ 1 \leq i \leq \phi(x) \} \} \]
and
\[ \lambda^-(x) = \max \{ 0, \min \{ -\lambda_i(x) \mid \lambda_i(x) < 0, \ 1 \leq i \leq \phi(x) \} \}. \]
We appoint $\min \emptyset = \max \emptyset = 0$. Taking an ergodic invariant measure $\mu$, by the ergodicity for $\mu$-almost all $x \in M$ we can obtain uniform exponents $\lambda_i(x) = \lambda_i(\mu)$ for $1 \leq i \leq \phi(\mu)$. In this content we denote $\lambda^+(\mu) = \lambda^+(x)$ and $\lambda^-(\mu) = \lambda^-(x)$. We say an ergodic measure $\mu$ is hyperbolic if $\lambda^+(\mu)$ and $\lambda^-(\mu)$ are both non-zero.
Definition 1.1. Given $\beta_1, \beta_2 \gg \epsilon > 0$, and for all $k \in \mathbb{Z}^+$, the hyperbolic block $\Lambda_k = \Lambda_k(\beta_1, \beta_2; \epsilon)$ consists of all points $x \in M$ for which there is a splitting $T_x M = E^s_x \oplus E^u_x$ with the invariance property $Df^t(E^s_x) = E^s_{f^t x}$ and $Df^t(E^u_x) = E^u_{f^t x}$, and satisfying:

(a) $\|Df^n|E^s_{f^t x}\| \leq e^{\epsilon k} e^{-(\beta_1 - \epsilon)n} e^{\epsilon |t|}, \forall t \in \mathbb{Z}, n \geq 1$;
(b) $\|Df^{-n}|E^u_{f^t x}\| \leq e^{\epsilon k} e^{-(\beta_2 - \epsilon)n} e^{\epsilon |t|}, \forall t \in \mathbb{Z}, n \geq 1$; and
(c) $\tan(Angle(E^s_{f^t x}, E^u_{f^t x})) \geq e^{-\epsilon k} e^{-\epsilon |t|}, \forall t \in \mathbb{Z}$.

Definition 1.2. $\Lambda(\beta_1, \beta_2; \epsilon) = \bigcup_{k=1}^{+\infty} \Lambda_k(\beta_1, \beta_2; \epsilon)$ is a Pesin set.

It is verified that $\Lambda(\beta_1, \beta_2; \epsilon)$ is an $f$-invariant set but usually not compact. Although the definition of Pesin set is adopted in a topology sense, it is indeed related to invariant measures. Actually, given an ergodic hyperbolic measure $\omega$ for $f$ if $\lambda^-(\omega) \geq \beta_1$ and $\lambda^+(\mu) \geq \beta_2$ then $\omega \in \mathcal{M}_{inv}(\Lambda(\beta_1, \beta_2; \epsilon), f)$. From now on we fix such a measure $\omega$ and denote by $\omega|_{\Lambda_i}$ the conditional measure of $\omega$ on $\Lambda_i$. Set $\bar{\Lambda}_i = \text{supp}(\omega|_{\Lambda_i})$ and $\bar{\Lambda} = \bigcup_{i \geq 1} \bar{\Lambda}_i$. Clearly, $f^\pm(\bar{\Lambda}_i) \subset \bar{\Lambda}_{i+1}$. and the sub-bundles $E^s_x, E^u_x$ depend continuously on $x \in \bar{\Lambda}_i$. Moreover, $\bar{\Lambda}$ is also $f$-invariant with $\omega$-full measure.

Theorem 1.3. For every $\mu \in \mathcal{M}_{inv}(\bar{\Lambda}, f)$, we have

$$h_\mu(f) = h_{top}(f, G_\mu).$$

Let $\{\eta_i\}_{i=1}^{\infty}$ be a decreasing sequence which approaches zero. As in [24] we say a probability measure $\mu \in \mathcal{M}_{inv}(M, f)$ has hyperbolic rate $\{\eta_i\}$ with respect to the Pesin set $\bar{\Lambda} = \bigcup_{i \geq 1} \bar{\Lambda}_i$ if $\mu(\bar{\Lambda}_i) \geq 1 - \eta_i$ for all $i \geq 1$.

Theorem 1.4. Let $\eta = \{\eta_i\}$ be a sequence decreasing to zero and $\mathcal{M}(\bar{\Lambda}, \eta) \subset \mathcal{M}_{inv}(M, f)$ be the set of measures with hyperbolic rate $\eta$. Given any nonempty compact connect set $K \subset \mathcal{M}(\bar{\Lambda}, \eta)$, we have

$$\inf\{h_\mu(f) \mid \mu \in K\} = h_{top}(f, G_K).$$

2. Dynamics of non uniformly hyperbolic systems

We start with some notions and results of Pesin theory [2] [16] [28].

2.1. Lyapunov metric. Assume $\Lambda(\beta_1, \beta_2; \epsilon) = \bigcup_{k \geq 1} \Lambda_k(\beta_1, \beta_2; \epsilon)$ is a nonempty Pesin set. Let $\beta'_1 = \beta_1 - 2\epsilon$, $\beta'_2 = \beta_2 - 2\epsilon$. Note that $\epsilon \ll \beta_1, \beta_2$, then $\beta'_1 > 0, \beta'_2 > 0$.

For $x \in \Lambda(\beta_1, \beta_2; \epsilon)$, we define

$$\|v_s\|_s = \sum_{n=1}^{+\infty} e^{\beta'_1 n} \|D_x f^n(v_s)\|, \forall v_s \in E^s_x,$$

$$\|v_u\|_u = \sum_{n=1}^{+\infty} e^{\beta'_2 n} \|D_x f^{-n}(v_u)\|, \forall v_u \in E^u_x,$$

$$\|v\|_r = \max(\|v_s\|_s, \|v_u\|_u)$$

where $v = v_s + v_u$.

1Here $\bar{\Lambda}$ is obtained by taking support for each hyperbolic block $\Lambda_i$ so even if an ergodic measure with Lyapunov exponents away from $[-\beta_1, \beta_2]$ it is not necessary of positive measure for $\bar{\Lambda}$. We will give more discussions on $\bar{\Lambda}$ in section 6.
We call the norm $\| \cdot \|^\prime$ a Lyapunov metric. This metric is in general not equivalent to the Riemannian metric. With the Lyapunov metric $f : \Lambda \to \Lambda$ is uniformly hyperbolic. The following estimates are known:

(i) $\| Df \|_{E_x^s} \|' \leq e^{-\beta_1'}$, $\| Df^{-1} \|_{E_x^u} \|' \leq e^{-\beta_2'}$;

(ii) $\frac{1}{2} \| v \| \leq \| v \|' \leq \frac{2}{1-e^{-\|v\|}} \| v \|', \quad \forall v \in T_x M, \; x \in \Lambda_k$.

**Definition 2.1.** In the local coordinate chart, a coordinate change $C_x : M \to GL(m, \mathbb{R})$ is called a Lyapunov change of coordinates if for each regular point $x \in M$ and $u, v \in T_x M$, it satisfies

$$< u, v >_x = < C_x u, C_x v >'_x.$$

By any Lyapunov change of coordinates $C_x$ sends the orthogonal decomposition $\mathbb{R}^{\dim E^s_x} \oplus \mathbb{R}^{\dim E^u_x}$ to the decomposition $E^s_x \oplus E^u_x$ of $T_x M$. Additionally, denote $A_x(x) = C_x(f(x))^{-1} Df_x C_x(x)$. Then

$$A_x(x) = \begin{pmatrix} A_x^s(x) & 0 \\ 0 & A_x^u(x) \end{pmatrix},$$

$$\| A_x^s(x) \| \leq e^{-\beta_1'}, \quad \| A_x^u(x) \|^{-1} \leq e^{-\beta_2'}.$$

2.2. **Lyapunov neighborhood.** Fix a point $x \in \Lambda(\beta_1, \beta_2, \epsilon)$. By taking charts about $x$, $f(x)$ we can assume without loss of generality that $x \in \mathbb{R}^d, f(x) \in \mathbb{R}^d$. For a sufficiently small neighborhood $U$ of $x$, we can trivialize the tangent bundle over $U$ by identifying $T_U M \equiv U \times \mathbb{R}^d$. For any point $y \in U$ and tangent vector $v \in T_y M$ we can then use the identification $T_U M = U \times \mathbb{R}^d$ to translate the vector $v$ to a corresponding vector $\bar{v} \in T_x M$. We then define $\|v\|'' = \|\bar{v}\|'$, where $\| \cdot \|'$ indicates the Lyapunov metric. This defines a new norm $\| \cdot \|''$ (which agrees with $\| \cdot \|'$ on the fiber $T_x M$). Similarly, we can define $\| \cdot \|''$ on $T_z M$ (for any $z$ in a sufficiently small neighborhood of $fx$ or $f^{-1}x$). We write $\bar{v}$ as $v$ whenever there is no confusion. We can define a new splitting $T_y M = E'^s_y \oplus E'^u_y, y \in U$ by translating the splitting $T_x M = E^s_x \oplus E^u_x$ (and similarly for $T_z M = E^s_z \oplus E^u_z$).

There exist $\beta'_1 = \beta_1 - 3\epsilon > 0, \beta'_2 = \beta_2 - 3\epsilon > 0$ and $\epsilon_0 > 0$ such that if we set $\epsilon_k = \epsilon_0 e^{-\epsilon k}$ then for any $y \in B(x, \epsilon_k)$ in an $\epsilon_k$ neighborhood of $x \in \Lambda_k$. We have a splitting $T_y M = E'^s_y \oplus E'^u_y$ with hyperbolic behavior:

(i) $\| D_y f(v) \|''_y \leq e^{-\beta'_1'} \| v \|''$ for every $v \in E'^s_y$;

(ii) $\| D_y f^{-1}(w) \|''_{f^{-1}y} \leq e^{-\beta'_2'} \| w \|''$ for every $w \in E'^u_y$.

The constant $\epsilon_0$ here and afterwards depends on various global properties of $f$, e.g., the Hölder constants, the size of the local trivialization, see p.73 in [28].

**Definition 2.2.** We define the Lyapunov neighborhood $\Pi = \Pi(x, a \epsilon_k)$ of $x$ in $\Lambda_k$ (with size $a \epsilon_k, 0 < a < 1$) to be the neighborhood of $x$ in $M$ which is the exponential projection onto $M$ of the tangent rectangle $(-a \epsilon_k, a \epsilon_k) E^s_x \oplus (-a \epsilon_k, a \epsilon_k) E^u_x$.

In the Lyapunov neighborhoods, $Df$ displays uniformly hyperbolic in the Lyapunov metric. More precisely, one can extend the definition of $C_x$ to the Lyapunov neighborhood $\Pi(x, a \epsilon_k)$ such that for any $y \in \Pi(x, a \epsilon_k)$,

$$A_x(y) := C_x(f(y))^{-1} Df_y C_x(y) = \begin{pmatrix} A_x^s(y) & 0 \\ 0 & A_x^u(y) \end{pmatrix},$$

$$\| A_x^s(y) \| \leq e^{-\beta'_1'}, \quad \| A_x^u(y)^{-1} \| \leq e^{-\beta'_2'}.$$
Let $\Psi_x = \exp_x \circ C_x(x)$. Given $x \in \Lambda_k$, we say that the set $H^u \subset \Pi(x, a\epsilon_k)$ is an admissible $(u, \gamma_0, k)$-manifold near $x$ if $H^u = \Psi_x(\text{graph } \psi)$ for some $\gamma_0$-Lipschitz function $\psi : (a\epsilon_k, a\epsilon_k)E^u_x \rightarrow (-a\epsilon_k, a\epsilon_k)E^u_x$ with $\|\psi\| \leq \alpha_k/4$. Similarly, we can also define $(s, \gamma_0, k)$-manifold near $x$. Through each point $y \in \Pi(x, a\epsilon_k)$ we can take $(u, \gamma_0, k)$-admissible manifold $H^u(y) \subset \Pi(x, a\epsilon_k)$ and $(s, \gamma_0, k)$-admissible manifold $H^s(y) \subset \Pi(x, a\epsilon_k)$. Fixing $\gamma_0$ small enough, we can assume that
\[
(i) \|D_z f(v)||^m_z \leq e^{-\beta_1^m + \epsilon ||v||^m} \text{ for every } v \in T_z H^s(y), z \in H^s(y);
\]
\[
(ii) \|D_z f^{-1}(v)||^m_{z^{-1}} \leq e^{-\beta_2^m + \epsilon ||v||^m} \text{ for every } w \in T_z H^u(y), z \in H^u(y).
\]

For any regular point $x \in \Lambda$, define $k(x) = \min \{i \in \mathbb{Z} \mid x \in \Lambda_i \}$. Using the local hyperbolicity above, we can see that each connected component of $f(H^u(y)) \cap \Pi(fz, a\epsilon_k(fz))$ is an admissible $(u, \gamma_0, k(fz))$-manifold; each connected component of $f^{-1}(H^s(y)) \cap \Pi(f^{-1}z, a\epsilon_k(f^{-1}z))$ is an admissible $(s, \gamma_0, k(f^{-1}z))$-manifold.

2.3. Weak shadowing lemma. In this section, we state a weak shadowing property for $C^{1+\alpha}$ non-uniformly hyperbolic systems, which is needful in our proofs.

Let $(\delta_k)_{k=1}^\infty$ be a sequence of positive real numbers. Let $(x_n)_{n=-\infty}^\infty$ be a sequence of points in $\Lambda = \Lambda(\beta_1, \beta_2, \epsilon)$ for which there exists a sequence $(s_n)_{n=-\infty}^\infty$ of positive integers satisfying:

(a) $x_n \in \Lambda_{s_n}$, $\forall n \in \mathbb{Z}$;

(b) $|s_n - s_{n-1}| \leq 1, \forall n \in \mathbb{Z}$;

(c) $d(f(x_n), x_{n+1}) \leq \delta_{s_n}, \forall n \in \mathbb{Z}$,

then we call $(x_n)_{n=-\infty}^\infty$ a $(\delta_k)_{k=1}^\infty$ pseudo-orbit. Given $c > 0$, a point $x \in M$ is an $c$-shadowing point for the $(\delta_k)_{k=1}^\infty$ pseudo-orbit if $d(f^n(x), x_n) \leq c\epsilon_{s_n}, \forall n \in \mathbb{Z}$, where $\epsilon_k = \epsilon_0 e^{-\epsilon k}$ are given by the definition of Lyapunov neighborhoods.

**Theorem 2.3. (Weak shadowing lemma)** Let $f : M \rightarrow M$ be a $C^{1+\alpha}$ diffeomorphism, with a non-empty Pesin set $\Lambda = \Lambda(\beta_1, \beta_2; \epsilon)$ and fixed parameters, $\beta_1, \beta_2 \gg \epsilon > 0$. For $c > 0$ there exists a sequence $(\delta_k)_{k=1}^\infty$ such that for any $(\delta_k)_{k=1}^\infty$ pseudo-orbit there exists a unique $c$-shadowing point.

3. Entropy for non compact spaces

In our settings the saturated sets are often non compact. In [7] Bowen gave the definition of topological entropy for non compact spaces. We state the definition in a slightly different way and they are in fact equivalent. Let $E \subset M$ and $C_n(E, \epsilon)$ be the set of all finite or countable covers of $E$ by the sets of form $B_m(x, \epsilon)$ with $m \geq n$. Denote

$$\mathcal{Y}(E; t, n, \epsilon) = \inf \{ \sum_{B_m(x, \epsilon) \in A} e^{-tm} \mid A \in C_n(E, \epsilon) \},$$

$$\mathcal{Y}(E; t, \epsilon) = \lim_{n \rightarrow \infty} \gamma(E; t, n, \epsilon).$$

Define

$$h_{top}(E; \epsilon) = \inf \{ t \mid \mathcal{Y}(E; t, \epsilon) = 0 \} = \sup \{ t \mid \mathcal{Y}(E; t, \epsilon) = \infty \}$$

and the topological entropy of $E$ is

$$h_{top}(E, f) = \lim_{\epsilon \rightarrow 0} h_{top}(E; \epsilon).$$
The following formulas from \[27\] (Theorem 4.1(3)) are subcases of Bowen's variational principle and true for general topological setting.

**Proposition 3.1.** Let \( K \subset M_{inv}(M, f) \) be non-empty, compact and connected. Then

\[
h_{\text{top}}(f, G_K) \leq \inf \{ h_\mu(f) | \mu \in K \}.
\]

In particular, taking \( K = \{ \mu \} \) one has

\[
h_{\text{top}}(f, G_\mu) \leq h_\mu(f).
\]

By the above proposition, to prove Theorem 1.3 and Theorem 1.4, it suffices to show the following theorems.

**Theorem 3.2.** For every \( \mu \in M_{inv}(\tilde{\Lambda}, f) \), we have

\[
h_{\text{top}}(f, G_\mu) \geq h_\mu(f).
\]

**Theorem 3.3.** Let \( \eta = \{ \eta_n \} \) be a sequence decreasing to zero and \( M(\tilde{\Lambda}, \eta) \subset M_{inv}(\tilde{\Lambda}, f) \) be the set of measures with hyperbolic rate \( \eta \). Given any nonempty compactly connected set \( K \subset M(\tilde{\Lambda}, \eta) \), we have

\[
h_{\text{top}}(f, G_K) \geq \inf \{ h_\mu(f) | \mu \in K \}.
\]

**Remark 3.4.** Let \( \mu \in M_{inv}(M, f) \) and \( K \subset M_{inv}(M, f) \) be a nonempty compactly connected set. In \[27\], C. E. Pfister and W. G. Sullivan proved that

(1) with almost product property (for detailed definition, see \[27\]), it holds that

\[
h_{\text{top}}(f, G_\mu) = h_\mu(f);
\]

(2) with almost product property plus uniform separation (for detailed definition, see \[27\]), it holds that

\[
h_{\text{top}}(f, G_K) = \inf \{ h_\mu(f) | \mu \in K \}.
\]

However, for nonuniformly hyperbolic systems, the shadowing and separation are inherent from the weak hyperbolicity of Lyapunov neighborhoods which varies in the index \( k \) of Pesin blocks \( \Lambda_k \), hence in general almost product property and uniform separation both fail to be true.

### 4. Proofs of Theorem 1.3 and Theorem 3.2

In this section, we will verify Theorem 3.2 and thus complete the proof of Theorem 1.3 by Proposition 3.1.

For each ergodic measure \( \nu \), we use Katok’s definition of metric entropy (see \[17\]). For \( x, y \in M \) and \( n \in \mathbb{N} \), let

\[
d^n(x, y) = \max_{0 \leq i \leq n-1} d(f^i(x), f^i(y)).
\]

For \( \varepsilon, \delta > 0 \), let \( N_n(\varepsilon, \delta) \) be the minimal number of \( \varepsilon \)- Bowen balls \( B_n(x, \varepsilon) \) in the \( d^n \)-metric, which cover a set of \( \nu \)-measure at least \( 1 - \delta \). We define

\[
h_{\nu}^{Kat}(f, \varepsilon | \delta) = \limsup_{n \to \infty} \frac{\log N_n(\varepsilon, \delta)}{n}.
\]

It follows by Theorem 1.1 of \[17\] that

\[
h_\nu(f) = \lim_{\varepsilon \to 0} h_{\nu}^{Kat}(f, \varepsilon | \delta).
\]
Recall that $\mathcal{M}_{\text{erg}}(M,f)$ denote the set of all ergodic $f$–invariant measures supported on $M$. Assume $\mu = \int_{\mathcal{M}_{\text{erg}}(M,f)} d\tau(\nu)$ is the ergodic decomposition of $\mu$ then by Jacobs Theorem

$$h_\mu(f) = \int_{\mathcal{M}_{\text{erg}}(M,f)} h_\nu(f) d\tau(\nu).$$

Define

$$h_\mu^{\text{Kat}}(f, \varepsilon \mid \delta) \triangleq \int_{\mathcal{M}_{\text{erg}}(M,f)} h_\nu^{\text{Kat}}(f, \varepsilon \mid \delta) d\tau(\nu).$$

By Monotone Convergence Theorem, we have

$$h_\mu(f) = \lim_{\varepsilon \to 0} h_\mu^{\text{Kat}}(f, \varepsilon \mid \delta) = \lim_{\varepsilon \to 0} h_\mu^{\text{Kat}}(f, \varepsilon \mid \delta).$$

**Proof of Theorem 3.2** Assume $\{\varphi_i\}_{i=1}^\infty$ is the dense subset of $C(M)$ giving the weak* topology, that is,

$$D(\mu, \nu) = \sum_{i=1}^\infty \left| \int \varphi_i d\mu - \int \varphi_i d\nu \right| \frac{1}{2i+1} \|\varphi_i\|$$

for $\mu, \nu \in \mathcal{M}(M)$. It is easy to check the affine property of $D$, i.e., for any $\mu, m_1, m_2 \in \mathcal{M}(M)$ and $0 \leq \theta \leq 1$,

$$D(\mu, \theta m_1 + (1 - \theta)m_2) \leq \theta D(\mu, m_1) + (1 - \theta)D(\mu, m_2).$$

In addition, $D(\mu, \nu) \leq 1$ for any $\mu, \nu \in \mathcal{M}(M)$. For any integer $k \geq 1$ and $\varphi_1, \cdots, \varphi_k$, there exists $b_k > 0$ such that

(1) $d(\varphi_j(x), \varphi_j(y)) < \frac{1}{k}\|\varphi_j\|$ for any $d(x,y) < b_k$, $1 \leq j \leq k$.

Now fix $\varepsilon, \delta > 0$.

**Lemma 4.1.** For any integer $k \geq 1$ and invariant measure $\mu$, we can take a finite convex combination of ergodic probability measures with rational coefficients,

$$\mu_k = \sum_{j=1}^{p_k} a_{k,j} m_{k,j}$$

such that

(2) $D(\mu, \mu_k) < \frac{1}{k^2} m_{k,j}(\tilde{\Lambda}) = 1$ and $|h_\mu^{\text{Kat}}(f, \varepsilon \mid \delta) - h_\mu^{\text{Kat}}(f, \varepsilon \mid \delta)| < \frac{1}{k}$. 

**Proof.** From the ergodic decomposition, we get

$$\int_{\tilde{\Lambda}} \varphi_i d\mu = \int_{\mathcal{M}_{\text{erg}}(\tilde{\Lambda},f)} \int_{\tilde{\Lambda}} \varphi_i dm d\tau(m), \quad 1 \leq i \leq k.$$

Use the definition of Lebesgue integral, we get the following steps. First, we denote

$$A_+ := \max_{1 \leq i \leq k} \int_{\tilde{\Lambda}} \varphi_i dm + 1, \quad A_- := \min_{1 \leq i \leq k} \int_{\tilde{\Lambda}} \varphi_i dm - 1, \quad F_+ := \sup_{\mathcal{M}_{\text{erg}}(\tilde{\Lambda},f)} h_\mu^{\text{Kat}}(f, \varepsilon \mid \delta) + 1, \quad F_- := \inf_{\mathcal{M}_{\text{erg}}(\tilde{\Lambda},f)} h_\mu^{\text{Kat}}(f, \varepsilon \mid \delta) - 1.$$

It is easy to see that:

$$-\infty < A_- < A_+ < +\infty, \quad -\infty < F_- < F_+ < +\infty.$$
For any integer $n > 0$, let

$$y_0 = A_-, y_j - y_{j-1} = \frac{A_+ - A_-}{n}, y_n = A_+.$$  

$$F_0 = F_-, F_j - F_{j-1} = \frac{F_+ - F_-}{n}, F_n = F_+.$$  

We can take $E_{i,j}, F_s$ to be measurable partitions of $\mathcal{M}_{\text{erg}}(\Lambda, f)$ as follows:

$$E_{i,j} = \{\mu \in \mathcal{M}_{\text{erg}}(\Lambda, f) \mid y_j \leq \int_{\Lambda} \varphi_i dm \leq y_{j+1}\},$$

$$F_n = \{\mu \in \mathcal{M}_{\text{erg}}(\Lambda, f) \mid F_j \leq h^K(f, \varepsilon | \delta) \leq F_{j+1}\}.$$  

Noticing the fact that $\bigcup_j E_{i,j} = \mathcal{M}_{\text{erg}}(\Lambda, f)$ and $\bigcup_j F_n = \mathcal{M}_{\text{erg}}(\Lambda, f)$, we can choose a new partition $\xi$ defined as:

$$\xi = \bigcap_{i,j} E_{i,j} \bigcap F_s,$$

where $\varsigma \bigcap \varsigma$ is given by $\{A \cap B \mid A \in \varsigma, B \in \varsigma\}$. For convenience, denote $\xi = \{\xi_{k,1}, \xi_{k,2}, \ldots, \xi_{k,p_k}\}$. To finish the proof of Lemma 4.1, we can let $n$ large enough such that any combination

$$\mu_k = \sum_{j=1}^{p_k} a_{k,j} m_{k,j}$$

where $m_{k,j} \in \xi_{k,j}$, rational numbers $a_{k,j} > 0$ with $|a_{k,j} - \tau(\xi_{k,j})| < \frac{1}{2k}$, satisfies:

$$D(\mu, \mu_k) < \frac{1}{k} m_{k,j} (\Lambda) = 1 \quad \text{and} \quad |h^K(f, \varepsilon | \delta) - h^K_{\mu_k}(f, \varepsilon | \delta)| < \frac{1}{k}.$$  

For each $k$, we can find $l_k$ such that $m_{k,j}(\Lambda_{l_k}) > 1 - \delta$ for all $1 \leq j \leq p_k$. Recalling that $c_\varepsilon$ is the scale of Lyapunov neighborhoods associated with the Pesin block $\Lambda_{l_k}$. For any $x \in \Lambda_{l_k}$, $Df$ exhibits uniform hyperbolicity in $B(x, c_{\varepsilon})$. For $c = \frac{\epsilon_\varepsilon}{2c_\varepsilon}$, by Theorem 2.3 there is a sequence of numbers $(\delta_k)_{k=1}^\infty$. Let $\xi_k$ be a finite partition of $M$ with $\text{diam} \xi_k < \min\{\delta_k, \frac{1}{2k}, \epsilon_{l_k}, \delta_{l_k}\}$ and $\xi_k = \{\Lambda_{l_k}, M \setminus \Lambda_{l_k}\}$. Given $t \in \mathbb{N}$, consider the set

$$\Lambda'(m_{k,j}) = \{x \in \Lambda_{l_k} \mid f^n(x) \in \xi_k(x) \text{ for some } q \in [t, [(1 + \frac{1}{k})t]] \}$$

and $D(\mathcal{E}_n(x), m_{k,j}) < \frac{1}{k}$ for all $n \geq t$},

where $\xi_k(x)$ denotes the element in the partition $\xi_k$ which contains the point $x$. Before going on the proof, we give the following claim.

**Claim**

$m_{k,j}(\Lambda'(m_{k,j})) \to m_{k,j}(\Lambda_{l_k})$ as $t \to +\infty$.

**Proof.** By ergodicity of $m_{k,j}$ and Birkhoff Ergodic Theorem, we know that for $m_{k,j} \text{ a.e.} x \in \Lambda_{l_k}$, it holds

$$\lim_{n \to \infty} \mathcal{E}_n(x) = m_{k,j}.$$
So we only need prove that the set 
\[ \Lambda_t^1(m_{k,j}) = \{ x \in \bar{A}_{t_k} \mid f^q(x) \in \xi_k(x) \text{ for some } q \in [t, [(1 + \frac{1}{k})t]] \} \]
satisfying the property 
\[ m_{k,j}(\Lambda_t^1(m_{k,j})) \rightarrow m_{k,j}(\bar{A}_{t_k}) \text{ as } t \rightarrow +\infty. \]

We next need the quantitative Poincaré’s Recurrence Theorem (see Lemma 3.12 in [3] for more detail) as following.

**Lemma 4.2.** Let \( f \) be a \( C^1 \) diffeomorphism preserving an invariant measure \( \mu \) supported on \( M \). Let \( \Gamma \subset M \) be a measurable set with \( \mu(\Gamma) > 0 \) and let
\[ \Omega = \cup_{n \in \mathbb{Z}} f^n(\Gamma). \]

Take \( \gamma > 0 \). Then there exists a measurable function \( N_0 : \Omega \rightarrow \mathbb{N} \) such that for a.e.\( x \in \Omega \), every \( n \geq N_0(x) \) and every \( t \in [0, 1] \) there is some \( l \in \{0, 1, \ldots, n\} \) such that \( f^l(x) \in \Gamma \) and \( |(l/n) - t| < \gamma \).

**Remark 4.3.** A slight modify (More precisely, replacing the interval \((n(t - \gamma), n(t + \gamma))\) by \((n(t, n(t + \gamma)))\), one can require that \((l/n) - t < \gamma \) in the above lemma. Hence we have \( l \in [n, n(t + \gamma)] \).

Take an element \( \xi_k^l \) of the partition \( \xi_k \). Let \( \Gamma = \xi_k^l, \gamma = \frac{1}{k} \). Applying Lemma 4.2 and its remark, we can deduce that for a.e.\( x \in \xi_k^l \), there exists a measurable function \( \mathcal{N}_0 \) such that for every \( t \geq N_0(x) \) there is some \( q \in \{0, 1, \cdots, n\} \) such that \( f^q(x) \in \xi_k^l = \xi_k(x) \) and \( q \in [t, t(1 + \frac{1}{k})] \). That is to say, \( t \geq N_0(x) \) implies \( x \in \Lambda_t^1(m_{k,j}) \). And this property holds for a.e.\( x \in \xi_k^l \). Hence it is true for a.e.\( x \in \xi_k^l \). This completes the proof of the claim.

Now we continue our proof of Theorem 3.2. By above claim, we can take \( t_k \) such that 
\[ m_{k,j}(\Lambda_t^1(m_{k,j})) > 1 - \delta \]
for all \( t \geq t_k \) and \( 1 \leq j \leq p_k \).

Let \( E_t(k, j) \subset \Lambda_t^1(m_{k,j}) \) be a \((t, \epsilon)\)-separated set of maximal cardinality. Then \( \Lambda_t^1(m_{k,j}) \subset \bigcup_{x \in E_t(k, j)} B_t(x, \epsilon) \), and by the definition of Katok’s entropy there exist infinitely many \( t \) satisfying 
\[ \# E_t(k, j) \geq e^{t(h_{\mu}^{\text{Kat}}(f, \epsilon) - \frac{\delta}{k})}. \]

For each \( q \in [t, [(1 + \frac{1}{k})t]], \) let 
\[ V_q = \{ x \in E_t(k, j) \mid f^q(x) \in \xi_k(x) \} \]
and let \( n = n(k, j) \) be the value of \( q \) which maximizes \( \# V_q \). Obviously, 
\[ t \geq \frac{n}{1 + \frac{1}{k}} \geq n(1 - \frac{1}{k}). \]

Since \( e^{\frac{\delta}{k}} > \frac{1}{k} \), we deduce that 
\[ \# V_0 \geq \frac{\# E_t(k, j)}{\epsilon} \geq e^{t(h_{\mu}^{\text{Kat}}(f, \epsilon) - \frac{\delta}{k})}. \]
Consider the element \( A_n(m_{k,j}) \in \xi_k \) for which \( \xi(V_n \cap A_n(m_{k,j})) \) is maximal. It follows that

\[
\xi(V_n \cap A_n(m_{k,j})) \geq \frac{1}{\| \xi \|} \xi V_n \geq \frac{1}{\| \xi \|} e^{t(h^{Kat}_{m,j}(f_\varepsilon|\delta) - \frac{1}{\delta})}.
\]

Thus taking \( t \) large enough so that \( e^t > \frac{1}{\| \xi \|} \), we have by inequality (3) that

\[
(4) \quad \xi(V_n \cap A_n(m_{k,j})) \geq e^{t(h^{Kat}_{m,j}(f_\varepsilon|\delta) - \frac{1}{\delta})} \geq e^{n(1 - \frac{1}{\delta})(h^{Kat}_{m,j}(f_\varepsilon|\delta) - \frac{1}{\delta})}.
\]

Notice that \( A_{n(k,j)}(m_{k,j}) \) is contained in an open subset \( U(k,j) \) of some Lyapunov neighborhood with \( \text{diam}(U(k,j)) < 2\text{diam}(\xi_k) \). By the ergodicity of \( \omega \), for any two measures \( m_{k_1,j_1}, m_{k_2,j_2} \) we can find \( y = y(m_{k_1,j_1}, m_{k_2,j_2}) \in U(k_1, j_1) \cap \Lambda_{k_1} \) satisfying that for some \( s = s(m_{k_1,j_1}, m_{k_2,j_2}) \) one has

\[ f^s(y) \in U(k_2, j_2) \cap \Lambda_{k_2}. \]

Letting \( C_{k,j} = \frac{a_k}{n(k,j)} \), we can choose an integer \( N_k \) larger enough so that \( N_kC_{k,j} \) are integers and

\[ \sum_{1 \leq r_1, r_2 \leq k+1, 1 \leq j_i, p_i, i = 1, 2} s(m_{r_1,j_1}, m_{r_2,j_2}) \geq N_k. \]

Arbitrarily take \( x(k,j) \in A_n(m_{k,j}) \cap V_n(k,j) \). Denote sequences

\[
X_k = \sum_{j=1}^{p_k-1} s(m_{k,j}, m_{k,j+1}) + s(m_{k,p_k}, m_{k,1})
\]

\[
Y_k = \sum_{j=1}^{p_k} N_k n(k,j) C_{k,j} + X_k = N_k + X_k.
\]

So,

\[
(5) \quad \frac{N_k}{Y_k} \geq \frac{1}{1 + \frac{1}{k}} \geq 1 - \frac{1}{k}.
\]

We further choose a strictly increasing sequence \( \{T_k\} \) with \( T_k \in \mathbb{N} \),

\[
(6) \quad Y_{k+1} \leq \frac{1}{k+1} \sum_{r=1}^{k} Y_r T_r,
\]

\[
(7) \quad \sum_{r=1}^{k} (Y_r T_r + s(m_{r_1,r+1,1})) \leq \frac{1}{k+1} Y_{k+1} T_{k+1}.
\]

In order to obtain shadowing points \( z \) with our desired property \( E_n(z) \to \mu \) as \( n \to +\infty \), we first construct pseudo-orbits with satisfactory property in the measure theoretic sense. For simplicity of the statement, for \( x \in M \) define segments of orbits

\[
L_{k,j}(x) \triangleq (x, f(x), \ldots, f^{n(k,j)-1}(x)), \quad 1 \leq j \leq p_k,
\]

\[
\hat{L}_{k_1,j_1,k_2,j_2}(x) \triangleq (x, f(x), \ldots, f^{s(m_{k_1,j_1}, m_{k_2,j_2})^{-1}}(x)), \quad 1 \leq j_i \leq p_{k_i}, \quad i = 1, 2.
\]
Consider now the pseudo-orbit

\[ O = O(x(1,1;1,1), \ldots, x(1,1,1,1), \ldots, x(1,p_1;1,1), \ldots, x(1,p_1;1,1,1,1); x(1,1;T_1,1), \ldots, x(1,1;T_1,1,1), \ldots, x(1,1;T_1,1,1,1); \ldots; x(k,1;1,1), \ldots, x(k,1;1,1,1), \ldots, x(k,p_k;1,1,1), \ldots, x(k,p_k;1,1,1,1); \ldots; x(k,1;T_k,1), \ldots, x(K,1;T_k,1,1), \ldots, x(k,p_k;T_k,1), \ldots, x(k,p_k;T_k,1,1); \ldots; x(k,1;T_k,1,1), \ldots, x(K,1;T_k,1,1,1), \ldots, x(k,p_k;T_k,1,1), \ldots, x(k,p_k;T_k,1,1,1); \ldots; x(k,1;T_k,1,1), \ldots, x(K,1;T_k,1,1,1); \ldots; \ldots; \)
with the precise form as follows

\[
\begin{align*}
\{ & L_{1,1}(x(1,1;1,1)), \ldots, L_{1,1}(x(1,1;1,N_1C_{1,1})), \hat{L}_{1,1,1,2}(y(m_{1,1},m_{1,2})); \\
& L_{1,2}(x(1,2;1,1)), \ldots, L_{1,2}(x(1,2;1,N_1C_{1,2})), \hat{L}_{1,2,1,3}(y(m_{1,1},m_{1,2})); \\
& L_{1,p_1}(x(1,p_1;1,1)), \ldots, L_{1,p_1}(x(1,p_1;1,N_1C_{1,p_1})), \hat{L}_{1,p_1,1,1}(y(m_{1,p_1},m_{1,1})); \\
& \ldots \\
& L_{1,1}(x(1,1;T_1,1)), \ldots, L_{1,1}(x(1,1;1,N_1C_{1,1})), \hat{L}_{1,1,1,2}(y(m_{1,1},m_{1,2})); \\
& L_{1,2}(x(1,2;T_1,1)), \ldots, L_{1,2}(x(1,2;1,N_1C_{1,2})), \hat{L}_{1,2,1,3}(y(m_{1,1},m_{1,2})); \\
& L_{1,p_1}(x(1,p_1;T_1,1)), \ldots, L_{1,p_1}(x(1,p_1;1,N_1C_{1,p_1})), \hat{L}_{1,p_1,1,1}(y(m_{1,p_1},m_{1,1})); \\
& \hat{L}(y(m_{1,1},m_{2,1})); \\
& : \\
& [L_{k,1}(x(k,1;1,1)), \ldots, L_{k,1}(x(k,1;1,N_kC_{k,1})), \hat{L}_{k,1,k,2}(y(m_{k,1},m_{k,2})); \\
& L_{k,2}(x(k,2;1,1)), \ldots, L_{k,2}(x(k,2;1,N_kC_{k,2})), \hat{L}_{k,2,k,3}(y(m_{k,1},m_{k,2})); \\
& L_{k,p_k}(x(k,p_k;1,1)), \ldots, L_{k,p_k}(x(k,p_k;1,N_kC_{k,p_k})), \hat{L}_{k,p_k,k,1}(y(m_{k,p_k},m_{k,1})); \\
& \hat{L}(y(m_{k,1},m_{k+1,1})); \\
& \ldots \\
& L_{k,1}(x(k,1;T_k,1)), \ldots, L_{k,1}(x(k,1;1,N_kC_{k,1})), \hat{L}_{k,1,k,2}(y(m_{k,1},m_{k,2})); \\
& L_{k,2}(x(k,2;T_k,1)), \ldots, L_{k,2}(x(k,2;1,N_kC_{k,2})), \hat{L}_{k,2,k,3}(y(m_{k,1},m_{k,2})); \\
& L_{k,p_k}(x(k,p_k;T_k,1)), \ldots, L_{k,p_k}(x(k,p_k;1,N_kC_{k,p_k})), \hat{L}_{k,p_k,k,1}(y(m_{k,p_k},m_{k,1})); \\
& \hat{L}(y(m_{k,1},m_{k+1,1})); \\
& : \\
& \ldots \},
\end{align*}
\]

where \(x(k,j;i,t) \in V_{n(k,j)} \cap A_{n(k,j)}(m_{k,j})\).
For \( k \geq 1, 1 \leq i \leq T_k, 1 \leq j \leq p_k, t \geq 1 \), let \( M_1 = 0 \),

\[
M_k = M_{k,1} = \sum_{r=1}^{k-1} (T_r Y_r + s(m_{r,1}, m_{r+1,1})), \\
M_{k,i} = M_{k,i,1} = M_k + (i-1)Y_k, \\
M_{k,i,j} = M_{k,i,j,1} = M_{k,i} + \sum_{q=1}^{j-1} (N_k n(k, q) C_{k,q} + s(m_{k,q}, m_{k,q+1})), \\
M_{k,i,j,t} = M_{k,i,j} + (t-1)n(k,j).
\]

By Theorem 2.3, there exists a shadowing point \( z \) of \( O \) such that

\[
d(f^{M_{k,i,j,t+q}}(z), f^q(x(k,j;i,t))) < \epsilon \epsilon_0 e^{-\epsilon_\text{l}k} < \frac{\epsilon}{4\epsilon_0} e^{-\epsilon_\text{l}k} \leq \frac{\epsilon}{4},
\]

for \( 0 \leq q \leq n(k,j) - 1, 1 \leq i \leq T_k, 1 \leq t \leq N_k C_{k,j}, 1 \leq j \leq p_k \). To be precise, \( z \) can be considered as a map with variables \( x(k,j;i,t) \):

\[
z = z(x(1,1;1,1), \ldots, x(1,1;1,1), N_1C_{1,1}, \ldots, x(1,1;1,1), \ldots, x(1,1;1,1), 1, \ldots, x(1,1;1,1), 1, 1), \ldots, x(1,1;1,1), 1, \ldots, x(1,1;1,1), 1, \ldots, x(1,1;1,1), N_1C_{1,1}; \\
\ldots; \\
x(1,1;T_1,1), \ldots, x(1,1;T_1,1), N_1C_{1,1}, \ldots, x(1,1;T_1,1), \ldots, x(1,1;T_1,1), N_1C_{1,1}; \\
\ldots; \\
x(k,1;1,1), \ldots, x(k,1;1,1), N_kC_{k,1}, \ldots, x(k,1;1,1), \ldots, x(k,1;1,1), N_kC_{k,1}; \\
\ldots; \\
x(k,1;T_k,1), \ldots, x(k,1;T_k,1), N_kC_{k,1}, \ldots, x(k,1;T_k,1), \ldots, x(k,1;T_k,1), N_kC_{k,1}; \\
\ldots; \\
\ldots)
\]

We denote by \( \mathcal{J} \) the set of all shadowing points \( z \) obtained in above procedure.

**Lemma 4.4.** \( \mathcal{J} \subset G_\mu \).

**Proof.** First we prove that for any \( z \in \mathcal{J} \),

\[
\lim_{k \to +\infty} \mathcal{E}_{M_k}(z) = \mu.
\]

We begin by estimating \( d(f^{M_{k,i,j,t+q}}(z), f^q(x(k,j;i,t))) \) for \( 0 \leq q \leq n(k,j) - 1 \). Recalling that in the procedure of finding the shadowing point \( z \), all the constructions are done in the Lyapunov neighborhoods \( \Pi(x(k,j;i,t), a\epsilon_k) \). Moreover, notice that we have required \( \text{diam}\, \xi_k < \frac{b_k(1-\epsilon^{-\infty})}{4\epsilon_0 2^{k+1}} \) which implies that for every two adjacent orbit segments \( x(k,j;i_1,t_1) \) and \( x(k,j;i_2,t_2) \), the ending point of the front orbit segment and the beginning point of the segment following are \( \frac{b_k(1-\epsilon^{-\infty})}{4\epsilon_0 2^{k+1}} \) close to each other. Let \( y \) be the unique intersection point of admissible manifolds \( H^s(z) \) and \( H^u(x) \). In what follows, define \( d'' \) to be the distance induced by \( || \cdot ||'' \) in
the local Lyapunov neighborhoods. By the hyperbolicity of $Df$ in the Lyapunov coordinates\(^2\) we obtain

\[
d(f^{M_{k,i,j,t}+q}(z), f^q(x(k,j;i,t))) \\
\leq d(f^{M_{k,i,j,t}+q}(z), f^q(y)) + d(f^q(y), f^q(x(k,j;i,t))) \\
\leq \sqrt{2}d''(f^{M_{k,i,j,t}+q}(z), f^q(y)) + \sqrt{2}d''(f^q(y), f^q(x(k,j;i,t))) \\
\leq \sqrt{2}\max\{e^{-(\beta''_1-c)q}, e^{-(\beta''_2-c)(n(k,j)-q)}\}d''(f^{M_{k,i,j,t}}(z), y) \\
+ d''(f^{n(k,j)}(y), f^{n(k,j)}(x(k,j;i,t))) \\
\leq \frac{2\sqrt{2}\epsilon^{(k+1)}}{1-e^{-\epsilon}}(d(f^{M_{k,i,j,t}}(z), y) + d(f^{n(k,j)}(y), f^{n(k,j)}(x(k,j;i,t)))) \\
\leq \frac{2\sqrt{2}\epsilon^{(k+1)}}{1-e^{-\epsilon}} - 2\diam(\xi_k) \\
< b_k
\]

for $0 \leq q \leq n(k,j) - 1$. Now we can deduce that

\[
|\varphi_p(f^{M_{k,i,j,t}+q}(z) - \varphi_p(f^q(x(k,j;i,t))))| < \frac{1}{k}\|\varphi_p\|, \quad 1 \leq p \leq k,
\]

which implies that

\[
(8) \quad D(\mathcal{E}_{n(k,j)}(f^{M_{k,i,j,t}}(z)), \mathcal{E}_{n(k,j)}(x(k,j;i,t))) < \frac{1}{k} + \frac{1}{2k-1} < \frac{2}{k},
\]

for sufficiently large $k$. By the triangle inequality, we have

\[
D(\mathcal{E}_k(f^{M_{k,i}}(z)), \mu) \leq D(\mathcal{E}_k(f^{M_{k,i}}(z)), \mu_k) + \frac{1}{k} \\
\leq D(\mathcal{E}_k(f^{M_{k,i}}(z)), \frac{1}{Y_k - X_k} \sum_{j=1}^{pk} N_kC_{k,j}n(k,j)\mathcal{E}_{n(k,j)}(f^{M_{k,i,j}}(z))) \\
+ D(\frac{1}{Y_k - X_k} \sum_{j=1}^{pk} N_kC_{k,j}n(k,j)\mathcal{E}_{n(k,j)}(f^{M_{k,i,j}}(z)), \mu_k) + \frac{1}{k}
\]

\(^2\)This hyperbolic property is crucial in the estimation of distance along adjacent segments, so the weak shadowing lemma\(^2\) (which is actually stated in topological way) does not suffice to conclude Theorem 1.3 and the following Theorem 1.4.
Note that for any \( \varphi \in C^0(M) \), it holds
\[
\| \int \varphi d\mathcal{E}_{Y_k}(f^{M_{k,i}}(z)) - \int \varphi d\frac{1}{Y_k - X_k} \sum_{j=1}^{p_k} N_k C_{k,j} n(k,j) \mathcal{E}_{n(k,j)}(f^{M_{k,i,j}}(z)) \| \\
= \| \frac{1}{Y_k} \sum_{q=1}^{Y_k-1} \varphi(f^{M_{k,i}+q}(z)) - \frac{1}{Y_k} \sum_{q=1}^{p_k} N_k C_{k,j} n(k,j) \varphi(f^{M_{k,i,j}+q}(z)) \| \\
\leq \| \frac{1}{Y_k} \sum_{j=1}^{p_k} N_k C_{k,j} \sum_{q=1}^{n(k,j)-1} \varphi(f^{M_{k,i,j}+q}(z)) - \frac{1}{Y_k} \sum_{q=1}^{p_k} N_k C_{k,j} \sum_{q=1}^{n(k,j)-1} \varphi(f^{M_{k,i,j}+q}(z)) \| \\
+ \| \frac{1}{Y_k} \sum_{j=1}^{p_k-1} \sum_{q=1}^{n(k,j)-1} \varphi(f^{M_{k,i,j}-s(m_{k,j}, m_{k,j+1})+q}(z)) \| \\
+ \| \sum_{q=1}^{s(m_{k,p_k+1}, m_{k,1})-1} \varphi(f^{M_{k,i,j}-s(m_{k,p_k}, m_{k,1})+q}(z)) \| \\
\leq \| \left( \frac{1}{Y_k} - \frac{1}{Y_k - X_k} \right) (Y_k - X_k) \| + \frac{X_k}{Y_k} \| \varphi \|. 
\]

Then by the definition of \( D \), the above inequality implies that
\[
D(\mathcal{E}_{Y_k}(f^{M_{k,i}}(z)), \frac{1}{Y_k - X_k} \sum_{j=1}^{p_k} N_k C_{k,j} n(k,j) \mathcal{E}_{n(k,j)}(f^{M_{k,i,j}}(z))) \\
\leq \| \frac{1}{Y_k} - \frac{1}{Y_k - X_k} \| (Y_k - X_k) \| + \frac{X_k}{Y_k} . 
\]

Thus, by the affine property of \( D \), together with the property \( a_{k,j} = n(k,j)C_{k,j} \) and \( N_k = Y_k - X_k \), we have
\[
D(\mathcal{E}_{Y_k}(f^{M_{k,i}}(z)), \mu) \leq D(\frac{1}{Y_k - X_k} \sum_{j=1}^{p_k} N_k C_{k,j} n(k,j) \mathcal{E}_{n(k,j)}(f^{M_{k,i,j}}(z)), \sum_{j=1}^{p_k} a_{k,j} m_{k,j}) \\
+ \| \left( \frac{1}{Y_k} - \frac{1}{Y_k - X_k} \right) (Y_k - X_k) \| + \frac{X_k}{Y_k} + \frac{1}{k} \\
\leq \frac{N_k}{Y_k - X_k} \sum_{j=1}^{p_k} a_{k,j} D(\mathcal{E}_{n(k,j)}(f^{M_{k,i,j}}(z), m_{k,j})) + \frac{2X_k}{Y_k} + \frac{1}{k} \\
= \sum_{j=1}^{p_k} a_{k,j} D(\mathcal{E}_{n(k,j)}(f^{M_{k,i,j}}(z), m_{k,j})) + \frac{2X_k}{Y_k} + \frac{1}{k} .
\]
Noting that
\[ \sum_{j=1}^{p_k} a_{k,j} D(\mathcal{E}_{n(k,j)}(f^{M_{k+1}}(z), m_{k,j}) \leq \sum_{j=1}^{p_k} a_{k,j} D(\mathcal{E}_{n(k,j)}(f^{M_{k+1}}(z)), \mathcal{E}_{n(k,j)}(x(k,j))) + \sum_{j=1}^{p_k} a_{k,j} D(\mathcal{E}_{n(k,j)}(x(k,j)), m_{k,j}) \]
and by the definition of \( \Lambda^f(m_{k,j}) \) which all \( x(k,j) \) belong to and by (8), we can further deduce that
\[ D(\mathcal{E}_k(f^{M_{k+1}}(z)), \mu) \leq \sum_{j=1}^{p_k} a_{k,j} D(\mathcal{E}_{n(k,j)}(f^{M_{k+1}}(z), \mathcal{E}_{n(k,j)}(x(k,j))) + \frac{1}{k} + 2 \frac{X_k}{Y_k} + \frac{1}{k} \]
\[ \leq \frac{2}{k} + \frac{1}{k} + 2 \frac{X_k}{Y_k} + \frac{1}{k} \]
\[ \leq \frac{6}{k} \quad \text{(by (5)).} \]
Hence, by affine property and inequalities (8) and (7) and \( D(\cdot, \cdot) \leq 1 \), we obtain that
\[ D(\mathcal{E}_{M_{k+1}}(z), \mu) \leq \frac{\sum_{r=1}^{k-1} (T_r Y_r + s(m_r, m_{r+1})) + s(m_1, m_{k+1})}{T_k Y_k + \sum_{r=1}^{k-1} (T_r Y_r + s(m_r, m_{r+1})) + s(m_1, m_{k+1})} \]
\[ \leq \frac{\sum_{r=1}^{k-1} (T_r Y_r + s(m_r, m_{r+1})) + s(m_1, m_{k+1})}{T_k Y_k + \sum_{r=1}^{k-1} (T_r Y_r + s(m_r, m_{r+1})) + s(m_1, m_{k+1})} \]
\[ \leq \frac{6}{k} \]
Thus,
\[ \lim_{k \to +\infty} \mathcal{E}_{M_k}(z) = \mu. \]
For \( M_{k,i} \leq n \leq M_{k,i+1} \) (here we appoint \( M_{k,p_{k+1}} = M_{k+1,1} \)), it follows that
\[ D(\mathcal{E}_n(z), \mu) \leq \frac{M_k}{n} D(\mathcal{E}_{M_k}(z), \mu) + \frac{1}{n} \sum_{i=1}^{n-1} D(\mathcal{E}_k(f^{M_{k,i-1}}(z)), \mu) \quad \text{(by affine property)} \]
\[ + \frac{n - M_{k,i}}{n} D(\mathcal{E}_{n-M_{k,i}}(f^{M_{k,i}}(z)), \mu) \]
\[ \leq \frac{M_k}{n} \left( 8 + \frac{(i - 1) Y_k}{n} + \frac{Y_k + s(m_1, m_{k+1})}{n} \right) \]
\[ \leq \frac{15}{k} \quad \text{(by (8) and (9)).} \]
Let $n \to +\infty$, then $k \to +\infty$ and $\mathcal{E}_n(z) \to \mu$. That is $\mathcal{J} \in G_\mu$. For any $z' \in \mathcal{J}$, we take $z_t \in \mathcal{J}$ with $\lim_n z_t = z'$. Observing that for $M_{k,i} \leq n \leq M_{k,i+1}$, $D(\mathcal{E}_n(z_t), \mu) \leq 15/k$ by continuity it also holds that $D(\mathcal{E}_n(z'), \mu) \leq 15/k$. This completes the proof of the Lemma 4.4.

To finish the proof of Theorem 4.2 we need to compute the entropy of $\mathcal{J} \subset G_\mu$. Notice that the choices of the position labeled by $x(k, j; i, t)$ in (10) has at least
\[ e^{n(k,j)(1 - \frac{1}{k})(h^\text{Kat}_{m_{k,j}}(f, \varepsilon|\delta) - \frac{1}{4})} \]
by (1). Moreover, fixing the position indexed $k, j, t$, for distinct $x(k, j; i, t), x'(k, j; i, t) \in V_{n(k,j)} \cap A_{n(k,j)}(m_{k,j})$, the corresponding shadowing points $z, z'$ satisfying
\[
d(f^{M_{k,i,j,t}}(z), f^{M_{k,i,j,t}}(z')) \
\geq d(f^q(x(k, j; i, t)), f^q(x'(k, j; i, t))) - d(f^q(x(k, j; i, t))) \
\geq d(f^q(x(k, j; i, t)), f^q(x'(k, j; t))) - \varepsilon/2.
\]
Since $x(k, j; t), x'(k, j; i, t)$ are $(n(k, j), \varepsilon)$-separated, $f^{M_{k,i,j,t}}(z), f^{M_{k,i,j,t}}(z')$ are $(n(k, j), \varepsilon/2)$-separated. Denote sets concerning the choice of quasi-orbits in $M_{k,i}$
\[
H_{k,i} = \{ (x(k, j; i, 1), \cdots, x(k, j; i, N_k C_k), \cdots, x(k, p_k; i, 1), \cdots, x(1, p_k; i, N_k C_k, p_k) \mid x(k, j; i, t) \in V_{n(k,j)} \cap A_{n(k,j)} \}.
\]
Then
\[
\sharp H_{k,i} \geq e^{\sum_{j=1}^{p_k} N_k C_k n(k,j)(1 - \frac{1}{k})(h^\text{Kat}_{m_{k,j}}(f, \varepsilon|\delta) - \frac{1}{4})}.
\]
Hence,
\[
\frac{1}{Y_k} \log \sharp H_{k,i} \geq \frac{Y_k - X_k}{Y_k} \sum_{j=1}^{p_k} a_{k,j} (1 - \frac{1}{k})(h^\text{Kat}_{m_{k,j}}(f, \varepsilon|\delta) - \frac{4}{k})
\]
\[
\geq (1 - \frac{1}{k}) \sum_{j=1}^{p_k} a_{k,j} (1 - \frac{1}{k})(h^\text{Kat}_{m_{k,j}}(f, \varepsilon|\delta) - \frac{4}{k})
\]
\[
= (1 - \frac{1}{k})^2 h^\text{Kat}_{m_k}(f, \varepsilon|\delta) - \frac{4}{k}(1 - \frac{1}{k})^2
\]
\[
\geq (1 - \frac{1}{k})^2 h^\text{Kat}_{m}(f, \varepsilon|\delta) - \frac{4}{k}(1 - \frac{1}{k})^2.
\]
Since $\mathcal{J}$ is compact we can take only finite covers $C(\mathcal{J}, \varepsilon/2)$ of $\mathcal{J}$ in the calculation of topological entropy $h_\text{top}(\mathcal{J}, \varepsilon/2)$. Let $r < h^\text{Kat}_{m_k}(f, \varepsilon|\delta)$. For each $A \in C(\mathcal{J}, \varepsilon/2)$ we define a new cover $\mathcal{A}'$ in which for $M_{k,i} \leq m \leq M_{k,i+1}$, $B_m(z, \varepsilon/2)$ is replaced by $B_{M_{k,i}}(z, \varepsilon/2)$, where we suppose $M_{k,0} = M_{k-1,p_k-1}$, $M_{k,p_k+1} = M_{k+1,1}$. Therefore,
\[
\mathcal{Y}(\mathcal{J}; r, n, \varepsilon/2) = \inf_{A \in C(\mathcal{J}, \varepsilon/2)} \sum_{B_m, (z, \varepsilon/2) \in A} e^{-r m} \geq \inf_{A \in C(\mathcal{J}, \varepsilon/2)} \sum_{B_{M_{k,i}}, (z, \varepsilon/2) \in A'} e^{-r M_{k,i+1}}.
\]
Denote
\[
b = b(\mathcal{A}') = \max\{M_{k,i} \mid B_m(z, \frac{\varepsilon}{2}) \in \mathcal{A}' \text{ and } M_{k,i} \leq m < M_{k,i+1}\}.
\]
Noticing that $A'$ is a cover of $J$ each point of $J$ belongs to some $B_{M_{k,i}}(x, \frac{\epsilon}{2})$ with $M_{k,i} \leq b$. Moreover, if $z, z' \in J$ with some position $x(k, j; i, t) \neq x'(k, j; i, t)$ then $z, z'$ can't stay in the same $B_{M_{k,i}}(x, \frac{\epsilon}{2})$. Define

$$W_{k,i} = \{B_{M_{k,i}}(z, \frac{\epsilon}{2}) \in A' \}.$$ 

It follows that

$$\sum_{M_{k,i} \leq b} \#W_{k,i} \leq b \#W_{k,i} \sum_{M_{k,i} \leq b} \#H_{k',i'} \geq 1.$$ 

So,

$$\sum_{M_{k,i} \leq b} \#W_{k,i} \left(\sum_{M_{k,i} \leq b} \#H_{k',i'} \right)^{-1} \geq 1.$$ 

From (9) it is easily seen that

$$\lim_{k \to \infty} \sum_{M_{k,i} \leq b} \#W_{k,i} \Pi_{M_{k',i'}} \leq b \#H_{k',i'} \exp(h_{Kat}(f, \varepsilon | \delta) M_{k,i}) \geq 1.$$ 

Since $r < h_{Kat}(f, \varepsilon | \delta)$ and $\lim_{k \to \infty} \frac{M_{k,i+1}}{M_{k,i}} = 1$, we can take $k$ large enough so that

$$\frac{M_{k,i+1}}{M_{k,i}} \leq \frac{h_{Kat}(f, \varepsilon | \delta)}{r}.$$ 

Thus there is some constant $c_0 > 0$ for large $k$

$$\sum_{M_{k,i} \leq b} e^{-rM_{k,i}} e^{-rM_{k,i+1}} = \sum_{M_{k,i} \leq b} \#W_{k,i} e^{-rM_{k,i+1}}$$

$$\geq \sum_{M_{k,i} \leq b} \#W_{k,i} \exp(-h_{Kat}(f, \varepsilon | \delta) M_{k,i})$$

$$\geq c_0 \sum_{M_{k,i} \leq b} \#W_{k,i} \left(\Pi_{M_{k',i'}} \leq b \#H_{k',i'} \right)^{-1}$$

$$\geq c_0,$$ 

which together with the arbitrariness of $r$ gives rise to the required inequality

$$h_{top}(J, \frac{\epsilon}{2}) \geq h_{Kat}(f, \varepsilon | \delta).$$

Finally, the arbitrariness of $\varepsilon$ yields:

$$h_{top}(f, G_{\mu}) \geq h_{\mu}(f).$$

\[ \square \]

5. Proofs of Theorem 1.4 and Theorem 3.3

We start this section by recalling the notion of entropy introduced by Newhouse [24]. Given $\mu \in M_{inv}(M, f)$, let $F \subset M$ be a measurable set. Define
(1) \( H(n, \rho \mid x, F, \varepsilon) = \log \max \{ zE \mid E \text{ is a } (d^n, \rho) - \text{separated set in } F \cap B_n(x, \varepsilon) \}; \)

(2) \( H(n, \rho \mid F, \varepsilon) = \sup_{x \in F} H(n, \rho \mid x, F, \varepsilon); \)

(3) \( h(\rho \mid F, \varepsilon) = \lim_{n \to +\infty} \frac{1}{n} H(n, \rho \mid F, \varepsilon); \)

(4) \( h(F, \varepsilon) = \lim_{\rho \to 0} h(\rho \mid F, \varepsilon); \)

(5) \( h_{\text{loc}}^{\text{New}}(\mu, \varepsilon) = \lim_{\sigma \to 1} \inf \{ h(F, \varepsilon) \mid \mu(F) > \sigma \}; \)

(6) \( h_{\text{loc}}^{\text{New}}(\mu, \varepsilon) = h_{\mu}(f) - h_{\text{loc}}^{\text{New}}(\mu, \varepsilon) \)

Let \( \{\theta_k\}_{k=1}^{\infty} \) be a decreasing sequence which approaches zero. One can verify that \( (h^{\text{New}}_{\mu}(\mu, \theta_k) \mid \mu \in \mathcal{M}_{\text{inv}}(M, f)) \) is in fact an increasing sequence of functions defined on \( \mathcal{M}_{\text{inv}}(M, f). \) Further more,

\[
\lim_{\theta_k \to 0} h^{\text{New}}_{\mu}(\mu, \theta_k) = h_{\mu}(f) \quad \text{for any } \mu \in \mathcal{M}(f).
\]

Let \( \mathcal{H} = (h_k) \) and \( \mathcal{H}' = (h'_k) \) be two increasing sequences of functions on a compact domain \( D. \) We say \( \mathcal{H}' \) uniformly dominates \( \mathcal{H}, \) denoted by \( \mathcal{H}' \geq \mathcal{H}, \) if for every index \( k \) and every \( \gamma > 0 \) there exists an index \( k' \) such that

\[
h'_{k'} \geq h_k - \gamma.
\]

We say that \( \mathcal{H} \) and \( \mathcal{H}' \) are uniformly equivalent if both \( \mathcal{H} \geq \mathcal{H}' \) and \( \mathcal{H}' \geq \mathcal{H}. \) Obviously, uniform equivalence is an equivalence relation.

Next we give some elements from the theory of entropy structures as developed by Boyle-Downarowicz [8]. An increasing sequence \( \alpha_1 \leq \alpha_2 \leq \cdots \) of partitions of \( M \) is called essential (for \( f \)) if

(1) \( \text{diam}(\alpha_k) \to 0 \) as \( k \to +\infty, \)

(2) \( \mu(\partial \alpha_k) = 0 \) for every \( \mu \in \mathcal{M}_{\text{inv}}(M, f). \)

Here \( \partial \alpha_k \) denotes the union of the boundaries of elements in the partition \( \alpha_k. \) Note that essential sequences of partitions may not exist (e.g., for the identity map on the unit interval). However, for any finite entropy system \( (f, M) \) it follows from the work of Lindenstrauss and Weiss [21,22] that the product \( f \times R \) with \( R \) an irrational rotation has essential sequences of partitions. Noting that the rotation doesn’t contribute entropy for every invariant measure, we can always assume \( (f, M) \) has an essential sequence. By an entropy structure of a finite topological entropy dynamical system \( (f, M) \) we mean an increasing sequence \( \mathcal{H} = (h_k) \) of functions defined on \( \mathcal{M}_{\text{inv}}(M, f) \) which is uniformly equivalent to \( (h_{\mu}(f, \alpha_k) \mid \mu \in \mathcal{M}_{\text{inv}}(M, f)). \) Combining with Katok’s definition of entropy, we consider an increasing sequence of functions on \( \mathcal{M}_{\text{inv}}(M, f) \) given by \( (h^{\text{Kat}}_{\mu}(f, \epsilon_k \mid \delta) \mid \mu \in \mathcal{M}_{\text{inv}}(f)). \)

**Theorem 5.1.** Both \( (h^{\text{Kat}}_{\mu}(f, \epsilon_k \mid \delta) \mid \mu \in \mathcal{M}_{\text{inv}}(M, f)) \) and \( (h^{\text{New}}_{\mu}(\mu, \theta_k) \mid \mu \in \mathcal{M}_{\text{inv}}(M, f)) \) are entropy structures hence they are uniformly equivalent.

**Proof.** This theorem is a part of Theorem 7.0.1 in [11].
Remark 5.2. The entropy structure in fact reflects the uniform convergence of entropy. It is well known that there are various notions of entropy. However, not all of them can form entropy structure, for example, the classic definition of entropy by partitions (see Theorem 8.0.1 in [11]).

Let \( \eta = \{\eta_n\}_{n=1}^{\infty} \) be a sequence decreasing to zero. \( \mathcal{M}(\Lambda, \eta) \) is the subset of \( \mathcal{M}_{inv}(M, f) \) with respect to the hyperbolic rate \( \eta \).

For \( \delta, \varepsilon > 0 \) and any \( \Upsilon \subset \mathcal{M}(M) \), define
\[
h_{Kat, loc}^{\Upsilon}(f, \varepsilon | \delta) = \max_{\mu \in \Upsilon} \{ h_{\mu}(f) - h_{Kat}^{\Upsilon}(f, \varepsilon | \delta) \}.
\]

Lemma 5.3. \( \lim_{\theta_k \to 0} h_{Kat}^{\mathcal{M}(\Lambda, \eta)}(f, \theta_k | \delta) = 0. \)

Proof. First we need a proposition contained in Page 226 of [24], which reads as
\[
\lim_{\varepsilon \to 0} \sup_{\mu \in \mathcal{M}(\Lambda, \eta)} h_{New}^{\mu}(\varepsilon) = 0.
\]

By Theorem 5.1
\[
(h_{New}(\mu, \theta_k) | \mu \in \mathcal{M}_{inv}(M, f)) \leq (h_{Kat}^{\mu}(f, \theta_k | \delta) | \mu \in \mathcal{M}_{inv}(M, f)).
\]
So, for any \( k \in \mathbb{N} \), there exists \( k' > k \) such that
\[
h_{Kat}^{\mu}(f, \theta_{k'} | \delta) \geq h_{New}^{\mu}(\mu, \theta_k) - \frac{1}{k'},
\]
for all \( \mu \in \mathcal{M}(\Lambda, \eta) \). It follows that
\[
h_{\mu}(f) - h_{Kat}^{\mu}(f, \theta_{k'} | \delta) \leq h_{\mu}(f) - (h_{New}^{\mu}(\mu, \theta_k) - \frac{1}{k'}) = h_{loc}^{New}(\mu, \theta_k) + \frac{1}{k'},
\]
for all \( \mu \in \mathcal{M}(\Lambda, \eta) \). Taking supremum on \( \mathcal{M}(\Lambda, \eta) \) and letting \( k \to +\infty \), we conclude that
\[
\lim_{\theta_{k'} \to 0} h_{Kat}^{\mathcal{M}(\Lambda, \eta), loc}(f, \theta_{k'} | \delta) = 0.
\]

Remark 5.4. In [24], Lemma 5.3 was used to prove upper semi-continuity of metric entropy on \( \mathcal{M}(\Lambda, \eta) \). However, the upper semi-continuity is broadly not true even if the underlying system is non uniformly hyperbolic. For example, in [25], T. Downarowicz and S. E. Newhouse established surface diffeomorphisms whose local entropy of arbitrary pre-assigned scale is always larger than a positive constant. Exactly, they constructed a compact subset \( E \) of \( \mathcal{M}_{inv}(\Lambda, f) \) such that there exist a periodic measure in \( E \) and a positive real number \( \rho_0 \) such that for each \( \mu \in E \) and each \( k > 0 \),
\[
\limsup_{\nu \in E, \nu \to \mu} h_{\nu}(f) - h_{k}(\nu) > \rho_0,
\]
which implies infinity of symbolic extension entropy and also the absence of upper semi-continuity of metric entropy and thus no uniform separation in [27].
Now we begin to prove Theorem 3.3 and hence complete the proof of Theorem 1.4 by Proposition 3.1. Throughout this section, for simplicity, we adopt the symbols used in the proof of Theorem 1.3. Except specially mentioned, the relative quantitative relation of symbols share the same meaning.

**Proof of Theorem 3.3**

For any nonempty closed connected set \( K \subset M(\tilde{\Lambda}, \eta) \), there exists a sequence of closed balls \( U_n \) in \( M_{inv}(M, f) \) with radius \( \zeta_n \) in the metric \( D \) with the weak* topology such that the following holds:

\[
\begin{align*}
(i) & \quad U_n \cap U_{n+1} \cap K \neq \emptyset; \\
(ii) & \quad \cap_{N \geq 1} \bigcup_{n \geq N} U_n = K; \\
(iii) & \quad \lim_{n \to +\infty} \zeta_n = 0.
\end{align*}
\]

By (1), we take \( \nu_k \in U_k \cap K \). Given \( \gamma > 0 \), using Lemma 5.3, we can find an \( \varepsilon > 0 \) such that

\[ h_{M(\tilde{\Lambda}, \eta), loc}^{K_{at}}(f, \varepsilon \mid \delta) < \gamma. \]

For each \( \nu_k \), we then can choose a finite convex combination of ergodic probability measures with rational coefficients,

\[ \mu_k = \sum_{j=1}^{p_k} a_{k,j} m_{k,j} \]

satisfying the following properties:

\[ D(\nu_k, \mu_k) < \frac{1}{k}, \quad m_{k,j}(\Lambda) = 1 \quad \text{and} \quad |h_{m_{k,j}}^{K_{at}}(f, \varepsilon \mid \delta) - h_{\nu_k}^{K_{at}}(f, \varepsilon \mid \delta)| < \frac{1}{k}. \]

For each \( k \), we can find \( l_k \) such that \( m_{k,j}(\Lambda_{l_k}) > 1 - \delta \) for all \( 1 \leq j \leq p_k \). For \( \epsilon = 8\varepsilon \), by Theorem 2.2 there is a sequence of numbers \( (\delta_k)_{k=1}^{\infty} \). Let \( \xi_k \) be a finite partition of \( M \) with \( \text{diam} \xi_k < \min\{\frac{b_k(1-e^{-\epsilon})}{4\sqrt{2}\epsilon^{(k+1)n}}, \epsilon_{l_k}, \delta_{l_k}\} \) and \( \xi_k \supset \{\Lambda_{l_k}, M \setminus \Lambda_{l_k}\} \).

For each \( m_{k,j} \), following the proof of Theorem 3.2, we can obtain an integer \( n(k, j) \) and an \((n(k, j), \varepsilon)\)-separated set \( \mathcal{W}_n \) contained in an open subset \( U(k, j) \) of some Lyapunov neighborhood with \( \text{diam}(U(k, j)) < 2\text{diam}(\xi_k) \) and satisfying that

\[ \# \mathcal{W}_{n(k,j)} \geq e^{n(k,j)(1-\frac{\epsilon}{2})(h_{m_{k,j}}^{K_{at}}(f,\varepsilon|\delta) - \frac{1}{2})}. \]

Then likewise, for \( k_1, k_2, j_1, j_2 \) one can find \( y = y(m_{k_1,j_1}, m_{k_2,j_2}) \in U(k_1, j_1) \) satisfying that for some \( s = s(m_{k_1,j_1}, m_{k_2,j_2}) \in \mathbb{N}, \)

\[ f^s(y) \in U(k_2, j_2). \]
In the same manner, we consider the following pseudo-orbit

\[ O = O(x(1, 1; 1, 1), \cdots, x(1, 1; 1, N_1 C_1), \cdots, x(1, p_1; 1, 1), \cdots, x(1, p_1; 1, N_1 C_1, p_1); \]

\[ \cdots; \]

\[ x(1, 1; T_1, 1), \cdots, x(1, 1; T_1, N_1 C_1), \cdots, x(1, p_1; T_1), \cdots, x(1, p_1; T_1, N_1 C_1, p_1); \]

\[ \cdots; \]

\[ x(k, 1; 1, 1), \cdots, x(k, 1; 1, N_k C_k), \cdots, x(k, p_k; 1, 1), \cdots, x(k, p_k; 1, N_k C_k, p_k); \]

\[ \cdots; \]

\[ x(k, 1; T_k, 1), \cdots, x(K, 1; T_k, N_k C_k), \cdots, x(k, p_k; T_k), \cdots, x(k, p_k; T_k, N_k C_k, p_k); \]

\[ \cdots; \]

with the precise type as \( \mathfrak{S} \), where \( x(k, j; i, t) \in W_{n(k,j)} \). Then Theorem 2.3 applies to give rise to a shadowing point \( z \) of \( O \) such that

\[ d(f^{M_{k,i}+q(z)} f^q(x(k, j; i, t))) < c \epsilon_0 e^{-\epsilon t_k} < \frac{\epsilon}{4 \epsilon_0}, \]

for \( 0 \leq q \leq n(k, j) - 1, 1 \leq i \leq T_k, 1 \leq t \leq N_k C_{k,j}, 1 \leq j \leq p_k \). By the construction of \( N_k \) and \( Y_k \), it is verified that

\[ D(\mathcal{E}_{Y_k}(f^{M_{k,i}}(z)), \nu_k) \leq \frac{6}{k}. \]

For sufficiently large \( M_{k,i} \leq n \leq M_{k,i+1} \), by affine property, we have that

\[
D(\mathcal{E}_n(z), \nu_k) \leq \frac{M_{k,i}-2}{n} D(\mathcal{E}_{M_{k,i}-2}(z), \nu_k) + \frac{Y_{k-1}}{n} \sum_{r=1}^{T_{k,i}} D(\mathcal{E}_{Y_k}(f^{M_{k,i}-r-1}(z)), \nu_k)
\]

\[
+ \frac{s_{M_{k,i}-1,r,1}}{n} D(\mathcal{E}_{s_{M_{k,i}-1,r,1}}(f^{M_{k,i}-r-1}(z)), \nu_k)
\]

\[
+ \frac{Y_k}{n} \sum_{r=1}^{i-1} D(\mathcal{E}_{Y_k}(f^{M_{k,i}-r-1}(z)), \nu_k)
\]

\[
+ \frac{n-M_{k,i}}{n} D(\mathcal{E}_{n-M_{k,i}}(f^{M_{k,i}}(z)), \nu_k).
\]

Noting that

\[ D(\mathcal{E}_{Y_k}(f^{M_{k,i}-r-1}(z)), \nu_k) \leq D(\mathcal{E}_{Y_k}(f^{M_{k,i}-r-1}(z)), \nu_{k-1}) + D(\nu_k, \nu_{k-1}) \]

and using the fact that \( D(\nu_k, \nu_{k-1}) \leq 2 \zeta_k + 2 \zeta_{k-1} \) and inequalities \( \mathfrak{S} \) and \( \mathfrak{T} \), one can deduce that

\[ D(\mathcal{E}_n(z), \nu_k) \leq \frac{1}{k} + \left( \frac{6}{k-1} + 2 \zeta_k + 2 \zeta_{k-1} \right) + \frac{1}{k} + \frac{6}{k} + \frac{1}{k}. \]

Letting \( n \to +\infty \), we get \( V(z) \subset K \). On the other hand, noting that

\[
\bigcap_{N \geq 1} \bigcup_{n \geq N} U_n = K,
\]
so $E_n(z)$ can enter any neighborhood of each $\nu \in K$ in infinitely times, which implies the converse side $K \subset V(x)$. Consequently, $V(z) = K$.

Next we show the inequality concerning entropy. Fixing $k, j, i, t$, the corresponding shadowing points of distinct $x(k, j, i)$ are $(n(k, j))$, $\frac{1}{2}$-separated. Let

$$H_{ki} = \{ x(k, j; i, 1), \cdots, x(k, j; i, N_kC_{k,j}), \cdots, x(k, p_k; i, 1), \cdots, x(1, p_k; i, N_kC_{k,p_k}) \}

| x(k, j; i, t) \in V_{n(k,j)} \cap A_{n(k,j)} \}.$$

Then

$$\# H_{ki} \geq e^{\sum_{j=1}^{p_k} N_kC_{k,j}n(k,j)(1-\frac{1}{4})(h_{Kat}^{inf}(f, \varepsilon | \delta) - \frac{1}{4})}.$$

So,

$$\frac{1}{Y_k} \log H_{ki} \geq \frac{Y_k - X_k}{Y_k} \sum_{j=1}^{p_k} a_{k,j}(1 - \frac{1}{k})(h_{Kat}^{inf}(f, \varepsilon | \delta) - \frac{4}{k})

\geq (1 - \frac{1}{k}) \sum_{j=1}^{p_k} a_{k,j}(1 - \frac{1}{k})(h_{Kat}^{inf}(f, \varepsilon | \delta) - \frac{4}{k})

= (1 - \frac{1}{k})^2 h_{Kat}^{inf}(f, \varepsilon | \delta) - \frac{4}{k}(1 - \frac{1}{k})^2

\geq (1 - \frac{1}{k})^2 (h_{Kat}^{inf}(f, \varepsilon | \delta) - \frac{1}{k}) - \frac{4}{k}(1 - \frac{1}{k})^2

\geq (1 - \frac{1}{k})^2 (h_{Kat}^{inf}(f) - \gamma - \frac{1}{k}) - \frac{4}{k}(1 - \frac{1}{k})^2.$$

In sequel by the analogous arguments in section 4, we obtain that

$$h_{top}(f, G_K) \geq \inf \{ h_{\mu}(f) | \mu \in K \} - \gamma.$$

The arbitrariness of $\gamma$ concludes the desired inequality:

$$h_{top}(f, G_K) \geq \inf \{ h_{\mu}(f) | \mu \in K \}.$$

\[ \square \]

6. On the Structure of Pesin set $\tilde{\Lambda}$

The construction of $\tilde{\Lambda}$ asks for many techniques that yields fruitful properties of Pesin set but meanwhile leads difficulty to check which measures support on $\tilde{\Lambda}$. Sometimes $M_{inv}(\tilde{\Lambda}, f)$ contains only the measure $\omega$ itself, for instance $\omega$ is atomic. In what follows, we will show that for several classes of diffeomorphisms derived from Anosov systems $M_{inv}(\tilde{\Lambda}, f)$ enjoys many members.

6.1. Symbolic dynamics of Anosov diffeomorphisms. Let $f_0$ be an Anosov diffeomorphism on a Riemannian manifold $M$. For $x \in M$, $\varepsilon_0 > 0$, we have the stable manifold $W_{\varepsilon_0}^s(x)$ and the unstable manifold $W_{\varepsilon_0}^u(x)$ defined by

$$W_{\varepsilon_0}^s(x) = \{ y \in M | d(f_0^n(x), f_0^n(y)) \leq \varepsilon_0, \text{ for all } n \geq 0 \}$$

$$W_{\varepsilon_0}^u(x) = \{ y \in M | d(f_0^{-n}(x), F^{-n}(y)) \leq \varepsilon_0, \text{ for all } n \geq 0 \}.$$
Fixing small $\varepsilon_0 > 0$ there exists a $\delta_0 > 0$ so that $W_{\varepsilon_0}^s(x) \cap W_{\varepsilon_0}^u(y)$ contains a single point $[x,y]$ whenever $d(x,y) < \delta_0$. Furthermore, the function $\pi: \{(x,y) \in M \times M \mid d(x,y) < \delta_0\} \to M$ is continuous. A rectangle $R$ is understood by a subset of $M$ with small diameter and $[x,y] \in R$ whenever $x,y \in R$. For $x \in R$ let

$$W^s(x,R) = W_{\varepsilon_0}^s(x) \cap R \quad \text{and} \quad W^u(x,R) = W_{\varepsilon_0}^u(x) \cap R.$$ 

For Anosov diffeomorphism $f_0$ one can obtain the following structure known as a Markov partition $\mathcal{R} = \{R_1, R_2, \ldots, R_l\}$ of $M$ with properties:

1. $\text{int } R_i \cap \text{int } R_j = \emptyset$ for $i \neq j$;
2. $f_0 W^u(x,R_i) \supset W^u(f_0 x,R_j)$ and $f_0 W^s(x,R_i) \subset W^s(f_0 x,R_j)$ when $x \in \text{int } R_i$, $f x \in \text{int } R_j$.

Using the Markov Partition $\mathcal{R}$ we can define the transition matrix $B = B(\mathcal{R})$ by

$$B_{i,j} = \begin{cases} 1 & \text{if } \text{int } R_i \cap f_0^{-1} (\text{int } R_j) \neq \emptyset; \\ 0 & \text{otherwise}. \end{cases}$$

The subshift $(\Sigma_B, \sigma)$ associated with $B$ is given by

$$\Sigma_B = \{q \in \Sigma_l \mid B_{q_{i,q_{i+1}}} = 1 \ \forall i \in \mathbb{Z}\}.$$ 

For each $q \in \Sigma_B$ by the hyperbolic property the set $\cap_{i \in \mathbb{Z}} f_0^{-i} R_{q_i}$ contains a single point, denoted by $\pi_0(q)$. We denote $\Sigma_B(i) = \{q \in \Sigma_B \mid q_0 = i\}$.

The following properties hold for the map $\pi_0$ (see Sinai [31] and Bowen [5, 6]).

**Proposition 6.1.**

1. The map $\pi_0: \Sigma_B \to M$ is a continuous surjection satisfying $\pi_0 \circ \sigma = f_0 \circ \pi_0$;
2. $\pi_0(\Sigma_B(i)) = R_i$, $1 \leq i \leq l$;
3. $h_{\text{top}}(\sigma, \Sigma_B) = h_{\text{top}}(f_0, M)$.

Since $B$ is $(0,1)$-matrix, using Perron Frobenius Theorem the maximal eigenvalue $\lambda$ of $B$ is positive and simple. $\lambda$ has the row eigenvector $u = (u_1, \ldots, u_l)$, $u_i > 0$, and the column eigenvector $v = (v_1, \ldots, v_l)^T$, $v_i > 0$. We assume $\sum_{i=1}^l u_i v_i = 1$ and denote $(p_1, \ldots, p_l) = (u_1 v_1, \ldots, u_l v_l)$. Define a new matrix

$$P = (p_{ij})_{l \times l}, \quad \text{where } \ p_{ij} = \frac{B_{ij} v_i}{\lambda v_i}.$$ 

Then $P$ can define a Markov chain with probability $\mu_0$ satisfying

$$\mu_0([a_0 a_1 \cdots a_l]) = p_{a_0} p_{a_0 a_1} \cdots p_{a_{l-1} a_l}.$$ 

Then $\mu_0$ is $\sigma$-invariant and Gurevich [12, 13] proved that $\mu_0$ is the unique maximal measure of $(\Sigma_B, \sigma)$, that is,

$$h_{\text{top}}(\sigma, \Sigma_B) = h_{\mu_0}(\sigma, \Sigma_B) = \log \lambda.$$ 

In addition, Bowen [5] proved that $\pi_0_{\ast}(\mu_0)(\partial \mathcal{R}) = 0$, where $\partial \mathcal{R}$ consists of all boundaries of $R_i$, $1 \leq i \leq l$. 

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Denote $\mu_1 = \pi_0(\mu_0)$. Then $\mu_1(\pi_0\Sigma_B(i)) = p_i$ for $1 \leq i \leq l$. For $0 < \gamma < 1$, $N \in \mathbb{N}$ define

$$\Gamma_N(i, \gamma) = \{x \in M \mid \sharp\{n \leq j \leq n + k - 1 \mid f_0^j(x) \in R_i\} \leq N + k(p_i + \gamma) + |n|\gamma, \forall k \geq 1, \forall n \in \mathbb{Z}\}.$$ 

Then $f_0^n(\Gamma_N(i, \gamma)) \subset \Gamma_{N+1}(i, \gamma)$. Let $\Gamma(i, \gamma) = \cup_{N \geq 1} \Gamma_N(i, \gamma)$.

**Lemma 6.2.** For any $m \in M_{\text{inv}}(M, f_0)$, if $m(R_i) < p_i + \gamma/2$ then $m(\Gamma(i, \gamma)) = 1$.

**Proof.** Since $m(R_i) < p_i + \gamma/2$, for $m$ almost all $x$ one can find $N(x) > 0$ such that

$$n(m(R_i) - \frac{\gamma}{2}) \leq \sharp\{0 \leq j \leq n - 1 \mid f_0^j(x) \in R_i\} \leq n(m(R_i) + \frac{\gamma}{2}), \forall n \geq N(x);$$

$$n(m(R_i) - \frac{\gamma}{2}) \leq \sharp\{0 \leq j \leq n - 1 \mid f_0^{-j}(x) \in R_i\} \leq n(m(R_i) + \frac{\gamma}{2}), \forall n \geq N(x).$$

Take $N_0(x)$ to be the smallest number such that for every $n \geq 1$,

$$-N_0(x) + n(m(R_i) - \frac{\gamma}{2}) \leq \sharp\{0 \leq j \leq n - 1 \mid f_0^j(x) \in R_i\} \leq N_0(x) + n(m(R_i) + \frac{\gamma}{2});$$

$$-N_0(x) + n(m(R_i) - \frac{\gamma}{2}) \leq \sharp\{0 \leq j \leq n - 1 \mid f_0^{-j}(x) \in R_i\} \leq N_0(x) + n(m(R_i) + \frac{\gamma}{2}).$$

Then for any $k \geq 1$,

$$\sharp\{n \leq j \leq n + k - 1 \mid f_0^j(x) \in R_i\} = \sharp\{0 \leq j \leq n + k - 1 \mid f_0^j(x) \in R_i\} - \sharp\{0 \leq j \leq n - 1 \mid f_0^j(x) \in R_i\} \leq N_0(x) + (n + k)(m(R_i) + \frac{\gamma}{2}) - (-N_0(x) + n(m(R_i) - \frac{\gamma}{2}))$$

$$= 2N_0(x) + k(m(R_i) + \frac{\gamma}{2}) + n\gamma.$$ 

In this manner we can also show

$$\sharp\{n \leq j \leq n + k - 1 \mid f_0^{-j}(x) \in R_i\} \leq 2N_0(x) + k(m(R_i) + \frac{\gamma}{2}) + n\gamma.$$ 

Thus, $x \in \Gamma_{N_0(x)}(i, \gamma)$.

By Lemma 6.2 $\mu_1(\Gamma(i, \gamma)) = 1$. We further define

$$\tilde{\Gamma}_N(i, \gamma) = \supp(\mu_1 \mid \Gamma_N(i, \gamma)) \quad \text{and} \quad \tilde{\Gamma}(i, \gamma) = \cup_{N \geq 1} \tilde{\Gamma}_N(i, \gamma).$$

It holds that $\tilde{\Gamma}(i, \gamma)$ is $f$-invariant and $\mu_1(\tilde{\Gamma}(i, \gamma)) = 1$.

**Proposition 6.3.** There is a neighborhood $U$ of $\mu_1$ in $M_{\text{inv}}(M, f_0)$ such that for any ergodic measure $m \in U$ we have $m \in M_{\text{inv}}(\tilde{\Gamma}(i, \gamma), f_0)$.

**Proof.** Observing that $\mu_1(\partial R_i) = 0$, for $\gamma > 0$ there exists a neighborhood $U$ of $\mu_1$ in $M_{\text{inv}}(M, F)$ such that for any $m \in U$ one has

$$m(R_i) < p_i + \frac{\gamma}{2}.$$ 

**Claim:** We can find an ergodic measure $m_0 \in M_{\text{inv}}(\Sigma_B, \sigma)$ satisfying $\pi_0m_0 = m$. 

Proof of Claim. Denote the basin of $m$ by
$$Q_m(M, f_0) = \{ x \in M \mid \lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f_j^i x) = \lim_{n \to -\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f_j^i x) = \int_M \varphi dm, \forall \varphi \in C^0(M) \}.$$Take and fix a point $x \in Q_m(M, f_0)$ and choose $q \in \Sigma_B$ with $\pi_0(q) = x$. Define a sequence of measures $\nu_n$ on $\Sigma_B$ by
$$\int \psi d\nu_n := \frac{1}{n} \sum_{i=0}^{n-1} \psi(\sigma^i(q)), \forall \psi \in C^0(\Sigma_B).$$By taking a subsequence when necessary we can assume that $\nu_n \to \nu_0$. It is standard to verify that $\nu_0$ is a $\sigma$-invariant measure and $\nu_\sigma$ covers $m$, i.e., $\pi_0(\nu_0) = m$. Set
$$Q'(\sigma) := \bigcup_{\nu \in \mathcal{M}_{\text{erg}}(\Sigma_B, \sigma)} Q_\nu(\Sigma_B, \sigma).$$Then $Q(\sigma)$ is a $\sigma$-invariant total measure subset in $\Sigma_B$. We have
$$m(Q_m(M, F) \cap \pi_0 Q(\sigma)) \geq \nu_0(\pi_0^{-1} Q_m(M, f_0) \cap Q(\sigma)) = 0.$$Then the set
$$A_0 := \left\{ \nu \in \mathcal{M}_{\text{erg}}(\Sigma_B, \sigma) \mid \exists q \in Q(\sigma), \pi_0(q) \in Q_m(M, f_0), \text{s.t.} \right.$$
$$\text{lim}_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \psi(\sigma^i(q)) = \text{lim}_{n \to -\infty} \frac{1}{n} \sum_{i=0}^{n-1} \psi(\sigma^i(q))$$
$$= \int_{\Sigma_B} \psi d\nu, \forall \psi \in C^0(\Sigma_B) \}$$is non-empty. It is clear that $\nu$ covers $m$, $\pi_0(\nu) = m$, for all $\nu \in A_0$.

We continue the proof of Proposition 6.3. Since $\pi_0(m_0) = m$ so $m_0(\pi_0^{-1}(R_i)) = m(R_i) < p_i + \gamma/2$ which together with $\Sigma_B(i) \subset \pi_0^{-1}(R_i)$ implies that
$$m_0(\Sigma_B(i)) < p_i + \frac{\gamma}{2}.$$In particular, $m_0(\Sigma_B(i)) < p_i + \frac{\gamma}{2}$. For $0 < \gamma < 1$, $N \in \mathbb{N}$ define
$$\Upsilon_N(i, \gamma) = \{ \bar{q} \in \Sigma_B \mid q \leq j \leq n + k - 1 \mid q_j = i \leq N + k(p_i + \gamma) + |n| \gamma, \quad \bar{q} \leq n \leq n + k - 1 \mid q_{-j} = i \leq N + k(p_i + \gamma) + |n| \gamma \quad \forall k \geq 1 \forall n \in \mathbb{Z} \}.$$Let $\Upsilon(i, \gamma) = \bigcup_{N \geq 1} \Upsilon_N(i, \gamma)$. Then $\mu_0(\Upsilon(i, \gamma)) = 1$. Further define
$$\Upsilon_N(i, \gamma) = \text{supp} \mu_0 \mid \Upsilon_N(i, \gamma) \} \quad \text{and} \quad \Upsilon(i, \gamma) = \bigcup_{N \geq 1} \Upsilon_N(i, \gamma).$$It also holds that $\Upsilon(i, \gamma)$ is $\sigma$-invariant and $\mu_0(\Upsilon(i, \gamma)) = 1$.

Lemma 6.4. Given $m_0 \in \mathcal{M}_{\text{erg}}(\Sigma_B, \sigma)$, if $m_0(\Sigma_B(i)) < p_i + \gamma/2$ then $m_0 \in \mathcal{M}_{\text{inv}}(\Upsilon(i, \gamma), \sigma)$. 

Proof of Lemma. Since $m_0(\Sigma_B(i)) < p_i + \gamma/2$ we obtain $m_0(\cup_{N \in \mathbb{N}} \Sigma_N(i, \gamma)) = 1$.
We can take $N_0$ so large that $m_0(\Sigma_N(i, \gamma)) > 0$ and $\mu_0(\Sigma_N(i, \gamma)) > 0$. Define
$$\Sigma(i, j) = \{q \in \Sigma_N(i, \gamma) \mid q_0 = j\}.$$ Then there exists $j \in [1, l]$ such that $\mu_0(\Sigma(i, j)) > 0$.

Noting that $(\Sigma_B, \sigma)$ is mixing, there is $L_0 > 0$ such that for each pair $j_1, j_2$ one can choose an sequence $L(j_1, j_2) = (q_1 \cdots q_L)$ satisfying $q_1 = j_1$, $q_L = j_2$ and $2 \leq L(j_1, j_2) \leq L_0$.

Arbitrarily taking $q \in \Sigma_N(i, \gamma)$, $z \in \Sigma(i, j)$, $n \in \mathbb{N}$, define
$$y(q, z, n) = (\cdot \cdot \cdot z_{-3}z_{-2}z_{-1}L(q_0, q_{-n})q_{-n+1} \cdots q_{-1}; q_0 q_1 \cdots q_{n-1}L(q_n, z_0)z_1 z_2 z_3 \cdots).$$ Denote $N_1 = 2L_0 + 2N_0 + 1$. For any $\theta > 0$ we can take large $n$ satisfying $n > N_1$ and $d(y(q, z, n), q) < \theta$. Define a new subset of $\Sigma_B$:
$$Y(q, n) = \{y(q, z, n) \in \Sigma_B \mid z \in \Sigma(i, j)\}.$$ Consider the positive and negative constitutions of $\Sigma(i, j)$ as follows
$$\Sigma^+(i, j) = \{w \in \Sigma_B \mid w_k = z_k, i \geq 0, \text{ for some } z \in \Sigma(i, j)\}$$
$$\Sigma^-(i, j) = \{w \in \Sigma_B \mid w_k = z_k, i \leq 0, \text{ for some } z \in \Sigma(i, j)\}.$$ Clearly $\Sigma^+(i, j) \supset \Sigma(i, j)$, $\Sigma^-(i, j) \supset \Sigma(i, j)$. Then by the Markov property of $\mu_0$ it holds that
$$\mu_0(Y(q, n)) \geq \mu_0(\Sigma^-(i, j))p_{jq_{-n}p_{q_{-n}q_{-n+1}} \cdots p_{q_{n-1}q_n}q_{n,j}) \mu_0(\Sigma^+(i, j)) > 0.$$ Moreover, for any $q \in Y(q, n)$ and $k \geq 1$, $s \in \mathbb{Z}$ we have
Case 1: $-n - \frac{s}{2}L \leq s \leq n + \frac{s}{2}L$, $s + k - 1 \leq n + \frac{s}{2}L$ it follows that
$$\sharp\{s \leq t \leq s + k - 1 \mid y_t = i\} \leq 2L_0 + \sharp\{s \leq t \leq s + k - 1 \mid q_t = i\} \leq 2L_0 + N_0 + k(p_i + \gamma) + |s|\gamma.$$ Case 2: $-n - L \leq s \leq n + L$, $s + k - 1 > n + L$ it follows that
$$\sharp\{s \leq t \leq s + k - 1 \mid y_t = i\} \leq L_0 + N_0 + (n + L - s)(p_i + \gamma) + |s|\gamma + N_0 + (s + k - 1 - n - L)(p_i + \gamma) \leq L_0 + 2N_0 + k(p_i + \gamma) + |s|\gamma.$$ Case 3: $s > n + L$ it follows that
$$\sharp\{s \leq t \leq s + k - 1 \mid y_t = i\} \leq N_0 + k(p_i + \gamma) + |s|\gamma.$$ Case 4: $s < -n - L$ it follows that
$$\sharp\{s \leq t \leq s + k - 1 \mid y_t = i\} \leq 2L_0 + 2N_0 + k(p_i + \gamma) + |s|\gamma.$$ The situation of $\sharp\{s \leq t \leq s + k - 1 \mid y_{-t} = i\}$ is similar. Altogether, since $N_1 = 2L_0 + 2N_0 + 1$, $Y(q, n) \subset \Sigma_N(i, \gamma)$. The arbitrariness of $\theta$ gives rise to that
$$q \in \text{supp}(\mu_0 \mid \Sigma_N(i, \gamma)).$$ That is, $\Sigma_N(i, \gamma) \subset \Sigma_N(i, \gamma)$. Since $m_0(\Sigma_N(i, \gamma)) > 0$ so $m_0(\Sigma_N(i, \gamma)) > 0$ which by the ergodicity of $m_0$ implies $m_0(\Sigma_N(i, \gamma)) = 1.$
Noting that \( \pi_0(\Gamma_N(i, \gamma)) \subset \Gamma_N(i, \gamma) \), by Lemma 6.4 we obtain
\[
m(\Gamma(i, \gamma)) = m_0(\pi_0^{-1}(\Gamma(i, \gamma))) \geq m_0(\Gamma(i, \gamma)) = 1
\]
which concludes Proposition 6.3.

\[\square\]

6.2. **Nonuniformly hyperbolic systems.** We shall verify \( \tilde{A} \) for an example due to Katok [15] (see also [1, 2]) of a diffeomorphism on the 2-torus \( \mathbb{T}^2 \) with nonzero Lyapunov exponents, which is not an Anosov map. Let \( f_0 \) be a hyperbolic linear automorphism given by the matrix
\[
A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}
\]
Let \( \mathcal{R} = \{R_1, R_2, \cdots, R_l\} \) be the Markov partition of \( f_0 \) and \( B = B(\mathcal{R}) \) be the associated transition matrix. \( f_0 \) has a maximal measure \( \mu_1 \). Without loss of generality, at most taking an iteration of \( f_0 \) we suppose there is a fixed point \( O \in \text{int} \, R_1 \). Consider the disk \( D_r \) centered at \( O \) of radius \( r \). Let \((s_1, s_2)\) be the coordinates in \( D_r \) obtained from the eigendirections of \( A \). The map \( A \) is the time-1 map of the local flow in \( D_r \) generated by the system of ordinary differential equations:
\[
\frac{ds_1}{dt} = s_1 \log \lambda, \quad \frac{ds_2}{dt} = -s_2 \log \lambda.
\]
We obtain the desired map by slowing down \( A \) near the origin.

Fix small \( r_1 < r_0 \) and consider the time-1 map \( g \) generated by the system of ordinary differential equations in \( D_{r_1} :\)
\[
\frac{ds_1}{dt} = s_1 \psi(s_1^2 + s_2^2) \log \lambda, \quad \frac{ds_2}{dt} = -s_2 \psi(s_1^2 + s_2^2) \log \lambda
\]
where \( \psi \) is a real-valued function on \([0, 1]\) satisfying:

(1) \( \psi \) is a \( C^\infty \) function except for the origin \( O \);

(2) \( \psi(0) = 0 \) and \( \psi(u) = 1 \) for \( u \geq r_0 \) where \( 0 < r_0 < 1 \);

(3) \( \psi(u) > 0 \) for every \( 0 < u < r_0 \);

(4) \( \int_0^1 \frac{du}{\psi(u)} < \infty \).

The map \( f \), given as \( f(x) = g(x) \) if \( x \in D_{r_1} \) and \( f(x) = A(x) \) otherwise, defines a homeomorphism of the torus, which is a \( C^\infty \) diffeomorphism everywhere except for the origin \( O \). To provide the differentiability of the map \( f \), the function \( \psi \) must satisfy some extra conditions. Namely, near \( O \) the integral \( \int_0^1 \frac{du}{\psi(u)} \) must converge “very slowly”. We refer the smoothness to [15]. Here \( f \) is contained in the \( C^0 \) closure of Anosov diffeomorphisms and even more there is a homeomorphism \( \pi : \mathbb{T}^2 \to \mathbb{T}^2 \) such that \( \pi \circ f_0 = f \circ \pi \) and \( \pi(O) = O \). By the constructions, there is a continuous decomposition on the tangent space \( T\mathbb{T}^2 = E^1 \oplus E^2 \) such that for any neighborhood \( V \) of \( O \), there exists \( \lambda_V > 1 \) such that

(1) \( \|Df_x|_{E^1(x)}\| \geq \lambda_V, \quad \|Df_x|_{E^2(x)}\| \leq \lambda_V^{-1}, \quad x \in \mathbb{T}^2 \setminus V \);

(2) \( \|Df_x|_{E^1(x)}\| \geq 1, \quad \|Df_x|_{E^2(x)}\| \leq 1, \quad x \in V \).
Let $H_i = \pi(R_i)$, $\nu_0 = \pi_* \mu_1$ and $p_i = \nu_0(H_i)$. Then $H_i$ is a closed subset of $\mathbb{T}^2$ with nonempty interior. Let

$$p_0 = \frac{1}{2} \min \{1 - p_i \mid 1 \leq i \leq l\},$$

$$\beta = (1 - p_1 - p_0 - \gamma) \log \lambda_V.$$

**Theorem 6.5.** There exists a neighborhood $U$ of $\nu_0$ in $\mathcal{M}_{\text{inv}}(\mathbb{T}^2, f)$ such that for any ergodic $\nu \in U$ it holds that $\nu \in \mathcal{M}_{\text{inv}}(\Lambda(\beta, \beta, \epsilon))$ for any $0 \leq \epsilon \ll \beta$.

**Proof.** Take a small neighborhood $V \subset H_1$ of $O$. Denote

$$\Phi_N(i, \gamma) = \{x \in M \mid \# \{n \leq j \leq n + k - 1 \mid f^j(x) \in H_i\} \leq N + k(p_i + \gamma) + |n|\gamma,$$

$$\# \{n \leq j \leq n + k - 1 \mid f^{-j}(x) \in H_i\} \leq N + k(p_i + \gamma) + |n|\gamma$$

$$\forall k \geq 1, \forall n \in \mathbb{Z}\}.$$

Define

$$\tilde{\Phi}_N(1, \gamma) = \text{supp}(\nu_0 | \Phi_N(1, \gamma)).$$

Then for some large $N$ we have $\nu_0(\tilde{\Phi}_N(1, \gamma)) > 0$. Noting that $\mu_1(\partial R_1) = 0$, by Proposition 6.3 and the conjugation $\pi$ there exists a neighborhood $U$ of $\nu_0$ in $\mathcal{M}(\mathbb{T}^2, f)$ such that for any ergodic $\nu \in U$,

$$\nu(\tilde{\Phi}_N(1, \gamma)) > 0.$$

For any $x \in \Phi_N(1, \gamma)$ and $k \geq 1, n \in \mathbb{Z}$ we have

Case 1: $k(p_1 + \gamma + p_0) \leq N + k(p_1 + \gamma) + |n|\gamma$, then

$$k \leq \frac{N + |n|\gamma}{p_0}.$$

So,

$$\|Df^{-k} |_{E^1(f^{n}x)}\| \leq e^{-k \beta} \exp\left(\frac{\beta}{p_0}\right) \left(N + |n|\gamma\right),$$

$$\|Df_{x}^{k} |_{E^2(f^{n}x)}\| \leq e^{-k \beta} \exp\left(\frac{\beta}{p_0}\right) \left(N + |n|\gamma\right).$$

Case 2: $k(p_1 + \gamma + p_0) > N + k(p_1 + \gamma) + |n|\gamma$, then

$$\|Df^{-k} |_{E^1(f^{n}x)}\| \leq \lambda_V^{-(1-p_1-p_0-\gamma)k} = e^{-\beta k},$$

$$\|Df_{x}^{k} |_{E^2(f^{n}x)}\| \leq \lambda_V^{-(1-p_1-p_0-\gamma)k} = e^{-\beta k}.$$

Let $N_2 = \lfloor \frac{\beta N}{p_0} \rfloor + 1$. Then

$$\Phi_N(1, \gamma) \subset \Lambda_{N_2}(\beta, \beta, \gamma) \text{ and } \tilde{\Phi}_N(1, \gamma) \subset \tilde{\Lambda}_{N_2}(\beta, \beta, \gamma).$$

Therefore,

$$\nu(\tilde{\Lambda}_{N_2}(\beta, \beta, \gamma)) > 0$$

which completes the proof of Theorem 6.5. \(\square\)
6.3. Robustly transitive partially hyperbolic systems. In [23] R. Mañé constructed a class of robustly transitive diffeomorphisms which is not hyperbolic. Firstly we recall the description of Mañé’s example. Let $T^n$, $n \geq 3$, be the torus $n$-dimentional and $f_0 : T^n \to T^n$ be a (linear) Anosov diffeomorphism. Assume that the tangent bundle of $T^n$ admits the $Df_0$-invariant splitting $T_T^n = E^{ss} \oplus E^u \oplus E^{uu}$, with $\dim E^u = 1$ and

$$
\lambda_s := \|Df|_{E^{ss}}\|, \quad \lambda_u := \|Df|_{E^u}\|, \quad \lambda_{uu} := \|Df|_{E^{uu}}\|
$$

satisfying the relation

$$
\lambda_s < 1 < \lambda_u < \lambda_{uu}.
$$

The following Lemma is proved in [29].

**Lemma 6.6.** Let $f_0 : T^n \to T^n$ be a linear Anosov map. Then there exists $C > 0$ such that for any small $r$ and any $f : T^n \to T^n$ with $\text{dist}_{C^0}(f, g) < r$ there exists $\pi : T^n \to T^n$ continuous and onto, $\text{dist}_{C^0}(\pi, \id) < Cr$, and

$$
f_0 \circ \pi = \pi \circ f.
$$

Let $R = \{R_1, R_2, \ldots, R_l\}$ be the Markov partition of $f_0$ and $B = B(R)$ be the associated transition matrix. Let $\mu_1$ be the maximal measure of $(T^n, f_0)$ and $p_i = \mu_1(R_i)$ for $1 \leq i \leq l$. Suppose there is a fixed point $O \in \text{int} R_1$. Take small $r$ satisfying the ball $B(O, Cr) \subset R_1$ and $d(B(O, Cr), \partial R_1) > Cr$. Then deform the Anosov diffeomorphim $f$ inside $B(p, r)$ passing through a flip bifurcation along the central unstable foliation $F_u(p)$ and then we obtain three fixed points, two of them with stability index equal to $\dim E^u$ and the other one with stability index equal to $\dim E^s + 1$. Moreover take positive numbers $\delta, \gamma \ll \min\{\lambda_s, \lambda_u\}$. Let $f$ satisfy the following $C^1$ open conditions:

1. $\|Df|_{E^{ss}}\| < e^\delta \lambda_s$, $\|Df|_{E^{uu}}\| > e^{-\delta} \lambda_{uu}$;
2. $e^{-\delta} \lambda_u < \|Df|_{E^u(x)}\| < e^\delta \lambda_u$, for $x \in T^n \setminus B(O, r)$;
3. $e^{-\delta} < \|Df|_{E^u(x)}\| < e^\delta \lambda_u$, for $x \in B(O, r)$.

As shown in [29] for the obtained $f$ there exists a unique maximal measure $\nu_0$ of $f$ with $\pi_* \nu_0 = \mu_1$. Let $H_i = \pi(R_i), p_i = \pi_\mu H_i$ and

$$
p_0 = \frac{1}{2} \min\{1 - p_i \mid 1 \leq i \leq l\},
$$

$$
\beta = (1 - p_1 - p_0 - \gamma) \min\{-\log \lambda_s - \delta, \log \lambda_u - \delta\}.
$$

We can see $E^{uu}$ is uniformly contracted by at least $e^{-\beta}$. 

\[\text{Figure 2.}\]
Theorem 6.7. There exist $0 < \epsilon \ll 1 < \beta$ and a neighborhood $U$ of $\nu_0$ in $\mathcal{M}_{inv}(\mathbb{T}^n, f)$ such that for any ergodic $\nu \in U$ it holds that $\nu \in \mathcal{M}_{inv}(\Lambda(\beta, \beta, \epsilon), f)$.

Proof. By Proposition 6.3 we can take a neighborhood $U_1$ of $\mu_1$ in $\mathcal{M}_{inv}(\mathbb{T}^n, f_0)$ such that every ergodic $\mu \in U_1$ also belongs to $\tilde{\Gamma}(i, \gamma)$, where $\tilde{\Gamma}(i, \gamma)$ is given by Proposition 6.3. Since $\pi$ is continuous, there is a neighborhood $U$ of $\nu_0$ in $\mathcal{M}_{inv}(\mathbb{T}^n, f)$ such that $\pi(U) \subset U_1$. For $N \in \mathbb{N}, \gamma > 0$, define

$$T_N(i, \gamma) = \{ x \in M \mid \sharp\{ n \leq j \leq n + k - 1 \mid f_j(x) \in B(O, r) \} \leq N + k(p_1 + \gamma) + |n|\gamma, \\text{\forall } k \geq 1, \forall n \in \mathbb{Z} \}. $$

For large $N$ we have $\nu_0(T_N(i, \gamma)) > 0$ and let

$$\tilde{T}_N(i, \gamma) = \text{supp}(\nu_0 \mid T_N(i, \gamma)).$$

For any $z \in T_N(i, \gamma)$, $n \in \mathbb{Z}$, $k \geq 1$ we have

Case 1: $k(p_1 + \gamma + p_0) \leq N + k(p_1 + \gamma) + |n|\gamma$, then

$$k \leq \frac{N + |n|\gamma}{p_0}. $$

So,

$$\|Df^{-k} \|_{E^n(x) \oplus E^n(f^n x)} \| \leq e^{-k\beta} \exp\left(\frac{\beta}{p_0}(N + |n|\gamma)\right).$$

Case 2: $k(p_1 + \gamma + p_0) > N + k(p_1 + \gamma) + |n|\gamma$, then

$$\|Df^{-k} \|_{E^n(x) \oplus E^n(f^n x)} \| \leq (\lambda u e^{\delta_k(1-p_1-p_0-\gamma)k} e^{\delta k(p_1+\gamma+p_0)} \leq e^{-\beta k} e^{\delta k(p_1+\gamma+p_0)}. $$

Let $N_2 = [\frac{\beta N}{p_0}] + 1, \epsilon = \max\{\delta(p_1 + \gamma + p_0), \gamma\}$. Then

$$T_N(1, \gamma) \subset \Lambda_{N_2}(\beta, \beta, \epsilon) \text{ and } \tilde{T}_N(1, \gamma) \subset \tilde{\Lambda}_{N_2}(\beta, \beta, \epsilon).$$

For any $x \in \Gamma_N(1, \gamma)$, $z \in \pi^{-1}(x)$, it holds that

$$d(f^i(z), f^i_0(x)) = d(f^i(z), \pi(f^i(x))) < Cr $$

which implies that if $f^i_0(x) \notin R_1$ then $f^i(z) \notin B(O, r)$ because $d(B(O, r), \partial R_1) > Cr$. Thus

$$\pi^{-1}(\Gamma_N(1, \gamma)) \subset T_N(1, \gamma) \subset \Lambda_{N_2}(\beta, \beta, \epsilon)$$

which yields that

$$\pi^{-1}(\tilde{\Gamma}_N(1, \gamma)) \subset \tilde{\Lambda}_{N_2}(\beta, \beta, \epsilon).$$

For any ergodic $\nu \in U$, $\pi_\ast \nu \in U_1$. So $\pi_\ast \nu(\tilde{\Gamma}_N(1, \gamma)) > 0$. We obtain

$$\nu(\tilde{\Lambda}_{N_2}(\beta, \beta, \epsilon)) \geq \nu(\pi^{-1}(\Gamma_N(1, \gamma))) = \pi_\ast \nu(\Gamma_N(1, \gamma)) > 0.$$ 

The ergodicity of $\nu$ concludes $\nu(\tilde{\Lambda}(\beta, \beta, \epsilon)) = 1.$

□
6.4. Robustly transitive systems which is not partially hyperbolic. In this subsection we will apply the structure of $\Lambda$ to a class of diffeomorphisms introduced by Bonatti-Viana. For our requirements we need do some additional assumptions on their constants. The class $\mathcal{V} \subset \text{Diff}(T^n)$ under consideration consists of diffeomorphisms which are also deformations of an Anosov diffeomorphism. To define $\mathcal{V}$, let $f_0$ be a linear Anosov diffeomorphism of the $n$-dimensional torus $T^n$. Let $\mathcal{R} = \{R_1, R_2, \cdots, R_l\}$ be the Markov partition of $f_0$ and $B = B(\mathcal{R})$ be the associated transition matrix. Let $\mu_1$ be the maximal measure of $(T^n, f_0)$ and $p_i = \mu_1(R_i)$ for $1 \leq i \leq l$. Suppose there is a fixed point $O \in \text{int} R_1$. Take small $r$ satisfying the ball $B(O, Cr) \subset R_1$ and $d(B(O, Cr), \partial R_1) > Cr$, where $C$ is given by Lemma 6.6.

Denote by $TM = E^s_0 \oplus E^u_0$ the hyperbolic splitting for $f_0$ and let

$$\lambda_s := \|Df|_{E^s_0}\|, \quad \lambda_u := \|Df|_{E^u_0}\|.$$  

We suppose that $f_0$ has at least one fixed point outside $V$. Fix positive numbers $\delta, \gamma \ll \lambda := \min\{\lambda_s, \lambda_u\}$. Let

$$p_0 = \frac{1}{2} \min\{1 - p_i \mid 1 \leq i \leq l\},$$

$$\beta = (1 - p_1 - p_0 - \gamma) \log \lambda - \delta.$$  

By definition $f \in \mathcal{V}$ if it satisfies the following $C^1$ open conditions:

1. There exist small continuous cone fields $C^{cu}$ and $C^{cs}$ invariant for $Df$ and $Df^{-1}$ containing respectively $E^s_0$ and $E^u_0$.

2. $f$ is $C^1$ close to $f_0$ in the complement of $B(O, r)$, so that for $x \in T^n \setminus B(O, r)$:

$$\|(Df|T_x D^{cu})\| > e^{-\delta} \lambda \quad \text{and} \quad \|(Df|T_x D^{cs})\| < e^{\delta} \lambda^{-1}.$$  

3. For $x \in B(O, r)$:

$$\|(Df|T_x D^{cu})\| > e^{-\delta} \quad \text{and} \quad \|(Df|T_x D^{cs})\| < e^{\delta},$$

where $D^{cu}$ and $D^{cs}$ are disks tangent to $C^{cu}$ and $C^{cs}$.

Immediately by the cone property, we can get a dominated splitting $T\mathbb{T}^n = E \oplus F$ with $E \subset D^{cs}$ and $F \subset D^{cu}$. We use Lemma 6.6 there exists $\pi : \mathbb{T}^n \to \mathbb{T}^n$ continuous and onto, $\text{dist}_{C^0}(\pi, \text{id}) < Cr$, and

$$f_0 \circ \pi = \pi \circ f.$$  

In [7] for the obtained $f$, Buzzi and Fisher proved that there exists a unique maximal measure $\nu_0$ of $f$ with $\pi_*\nu_0 = \mu_1$. This measure $\nu_0$ conforms good structure of Pesin set $\Lambda$ by the following Theorem.

**Theorem 6.8.** There exist $0 < \epsilon \ll 1 < \beta$ and a neighborhood $U$ of $\nu_0$ in $\mathcal{M}_{inv}(\mathbb{T}^n, f)$ such that for any ergodic $\nu \in U$ it holds that $\nu \in \mathcal{M}_{inv}(\Lambda(\beta, \beta, \epsilon)).$

**Proof.** The arguments are analogous of Theorem 6.7. Choose a neighborhood $U_1$ of $\mu_1$ in $\mathcal{M}(\mathbb{T}^n, f_0)$ such that every ergodic $\mu \in U_1$ is contained in $\Gamma(i, \gamma)$, where $\Gamma(i, \gamma)$ defined as Proposition 6.3. The continuity of $\pi$ give rise to a neighborhood $U$ of $\nu_0$ in $\mathcal{M}(\mathbb{T}^n, f)$ such that $\pi_* U \subset U_1$. For $N \in \mathbb{N}, \gamma > 0$, define

$$T_N(i, \gamma) = \{x \in M \mid \sharp\{n \leq j \leq n + k - 1 \mid f^j(x) \in B(O, r)\} \leq N + k(p_i + \gamma) + |n|\gamma,$$

$$\forall k \geq 1, \forall n \in \mathbb{Z}\}.$$
For large $N$ we have $\nu_0(T_N(i, \gamma)) > 0$ and let

$$\tilde{T}_N(i, \gamma) = \text{supp}(\nu_0 \mid T_N(i, \gamma)).$$

Let $N_2 = \lceil \frac{\beta N}{\gamma p} \rceil + 1$, $\varepsilon = \max\{\delta(p_1 + \gamma + p_0), \gamma\}$. Then

$$T_N(1, \gamma) \subset \Lambda_{N_2}(\beta, \beta, \varepsilon) \quad \text{and} \quad T_N(1, \gamma) \subset \tilde{\Lambda}_{N_2}(\beta, \beta, \varepsilon).$$

For any $x \in \Gamma_N(1, \gamma)$, $z \in \pi^{-1}(x)$, it holds that

$$d(f^i(z), f^i_0(x)) = d(f^i(z), \pi(f^i(x))) < Cr$$

which implies that if $f_0^i(x) \notin R_1$ then $f^i(z) \notin B(O, r)$ because $d(B(O, r), \partial R_1) > Cr$. Thus

$$\pi^{-1}(\Gamma_N(1, \gamma)) \subset T_N(1, \gamma) \subset \Lambda_{N_2}(\beta, \beta, \varepsilon)$$

which yields that

$$\pi^{-1}(\tilde{T}_N(1, \gamma)) \subset \tilde{\Lambda}_{N_2}(\beta, \beta, \varepsilon).$$

For any ergodic $\nu \in U$, $\pi_* \nu \in U_1$. So $\pi_* \nu(\tilde{T}_N(1, \gamma)) > 0$. We obtain

$$\nu(\tilde{\Lambda}_{N_2}(\beta, \beta, \varepsilon)) \geq \nu(\pi^{-1}(\tilde{T}_N(1, \gamma))) = \pi_* \nu(\tilde{T}_N(1, \gamma)) > 0.$$

Once more, the ergodicity of $\nu_0$ concludes $\nu(\tilde{\Lambda}(\lambda_1, \lambda_1, \varepsilon)) = 1$.

$\square$

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