Instantons and Fixed Point Actions in $SU(2)$ Gauge Theory

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June 2021

Abstract

We describe the properties of instantons in lattice gauge theory when the action is a fixed point action of some renormalization group transformation. We present a theoretically consistent method for measuring topological charge using an inverse renormalization group transformation. We show that, using a fixed point action, the action of smooth configurations with non-zero topological charge is greater than or equal to its continuum value $8\pi^2/g^2$.

$^1$Work supported in part by NSF Grant PHY-9023257 and U. S. Department of Energy grant DE–AC02–91ER–40672
1 INTRODUCTION

The dynamics of asymptotically free gauge theories are strongly influenced by topological effects. In QCD instantons may be responsible for breaking axial symmetry and resolving the $U(1)$ problem\cite{1}. In the large-$N_c$ limit a combination of the masses of the $\eta, \eta'$, and $K$ mesons is related to the topological susceptibility through the Witten-Veneziano formula\cite{2}.

The topological susceptibility $\chi$ is defined as the infinite volume limit of

$$\chi_t = \frac{\langle Q^2 \rangle}{V}$$

where $Q$ is the topological charge and $V$ the space time volume. In QCD $\chi_t$ is a dimension-4 object with no weak coupling expansion. As such, a calculation of $\chi_t$ in physical units in the continuum requires nonperturbative techniques.

Over the years there have been many attempts to compute topological properties of QCD using lattice Monte Carlo simulations. A serious problem in lattice studies of topology is the presence of lattice artifacts. They can arise both from the form of the lattice action and from the choice of lattice operator to define and measure topological charge. A lattice action is, in general, not scale invariant, i.e. the action of a smooth continuum instanton can depend on its size. This leads to lattice artifacts, called “dislocations,” \cite{3}, that are non-zero charged configurations whose contribution to the topological charge comes entirely from small localized regions. If the minimal action of a dislocation is smaller than $6/11$ (for SU(2)) times the continuum value of a one-instanton configuration, then dislocations will dominate the path integral and spoil the scaling of $\chi_t$ \cite{4, 5}. Difficulties also arise because the topological charge is not conserved on the lattice. When the size of an instanton becomes small compared to the lattice spacing it can ”fall through” the lattice and its charge disappears. Topological charge operators can fail to identify this process.

These problems are circumvented by the use of fixed point actions. If one can construct a lattice action and lattice operators which live on the renormalized trajectory (RT) of some renormalization group transformation (RGT), then one’s predictions do not depend on the lattice spacing. A recent series of papers\cite{6, 7, 8, 9} have shown how to find a fixed point (FP) action for asymptotically free theories, with explicit examples for spin and gauge models. FP actions share the scaling properties of the RT (through one-loop quantum corrections) and as such may be taken as a first approximation to a RT. FP actions have scale-invariant instanton solutions with an action value of exactly $8\pi^2/g^2$ and, as we will show, one can define a topological charge using RG techniques which has no lattice artifacts.
In principle, the questions we are asking and the methods by which we answer them are similar to the problems associated with topology in two-dimensional spin models\[10, 11, 12\]. Because we deal with four dimensional gauge theories we face additional obstacles in numerical tests due to computer speed and memory limitations.

In Section 2 we describe the formal properties of instantons under RG transformations and review the argument that if the action is a FP action, an instanton solution is scale invariant. We then describe a consistent method for measuring topological charge which exploits the invariance of the instanton solution under RG transformations at the FP. In Section 3 we describe the construction of trial instanton solutions on the lattice and in Section 4 we review the construction of FP actions and display a FP action for SU(2) gauge theory for a particular RG transformation. Finally in Section 5 we show that configurations with nonzero topological charge have action greater than or equal to the classical value $8\pi^2/g^2$ while configurations with action less than the classical value have zero topological charge. In this paper we do not address the calculation of the topological susceptibility via Monte Carlo simulation.

# 2 Formal Considerations for Ideal Instantons

We consider an SU(N) pure gauge theory on the lattice and the RG transformation

$$e^{-\beta' S'(V)} = \int DU \exp\{-\beta (S(U) + T(U, V))\}, \tag{2}$$

where $U$ is the original link variable, $V$ is the blocked link variable and $T(U, V)$ is the blocking kernel that defines the transformation. At $\beta = \infty$ the transformation becomes a steepest descent relation with a fixed point solution

$$S^{FP}(V) = \min_{\{U\}} (S^{FP}(U) + T(U, V)), \tag{3}$$

where

$$T(U, V) = -\frac{\kappa}{N} \sum_{n_B, \mu} \left[ \text{ReTr} \left( V_{\mu}(n_B) Q^\dagger_{\mu}(n_B) \right) \right] - \max_{W} \left\{ \text{ReTr}(W Q^\dagger) \right\}. \tag{4}$$

In Eqn. (4) $W \in SU(N)$ and the $N \times N$ complex matrix $Q_{\mu}(n_B)$ is the block average. The block transformation which we consider here is a scale-two Swendsen type transformation, that is referred to as “type I” RGT and $Q$ is defined

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2 The actual numerical analysis and simulations in this paper were done for SU(2). The equations are written for general N if not indicated otherwise.
in Fig. 5 in [7]. For further details we refer the reader to Ref. [7]. Eqn. 3 transforms a coarse lattice \{V\} that has structure on the scale of one coarse lattice spacing to a fine lattice \{U\} with half the lattice spacing of \{V\}. We will refer to this procedure as “inverse blocking.”

In the continuum the action is conventionally defined as

\[
S = -\frac{1}{4} \int d^4x F_{\mu\nu} F^{\mu\nu}
\]

and the continuum instanton action is \(S = 8\pi^2/g^2\). In lattice conventions \(S = \beta S\) with \(\beta = 2N/g^2\) for \(SU(N)\), and the equivalent result for the instanton action is \(S_I = 4\pi^2/N\). In all tables and figures in the paper which deal with actions of instanton-like configurations we will present results in units of \(S_I\) so that the reader will not be forced to deal with the particular conventions we use for \(S\) (such as sums over space-time indices, etc.). In order to avoid domination of the functional integral by dislocations, the action of a configuration with nonzero topological charge must satisfy the constraint

\[
S > \frac{48\pi^2}{11N^2},
\]

or \(S/S_I > 6/11\) for \(SU(2)\).

The continuum instanton action is scale invariant. In general this is not the case on the lattice, as is well known, for example, for the plaquette Wilson action. On the other hand FP actions are scale invariant. The proof of that statement was given in Ref. [6] for the \(d = 2\) \(\sigma\)-model and easily generalizes for asymptotically free gauge models. If \(\{V\}\) is a solution of the classical equations of motion, so that \(\delta S_{FP}/\delta V = 0\), then

\[
\frac{\delta S_{FP}}{\delta V} = \frac{\delta T}{\delta V} + \frac{\delta U}{\delta V} \frac{\delta}{\delta U} (S_{FP} + T) = 0.
\]

Eqn. 3 implies that for the minimizing configuration \(\{U(V)\}\) the second term in Eqn. 4 vanishes. Then it follows that \(\delta T/\delta V = 0\), too, or that \(T(U,V)\) also takes its minimum value, zero. Thus

\[
\frac{\delta S_{FP}(U)}{\delta U} \bigg|_{U=U(V)} = 0,
\]

and \(S_{FP}(V) = S_{FP}(U(V))\). According to this result the FP action has exact scale invariant instanton solutions with action equal to \(8\pi^2/g^2\) as in the continuum theory.

It also follows that the inverse blocking transformation does not change the topological charge of a configuration. According to Eqn. 3 the action on the
fine lattice is always equal to or smaller than the action on the coarse lattice, because the blocking kernel $T(U, V)$ is positive-definite. Thus a coarse configuration that has zero topological charge will inverse block into a configuration with lower action, and after many levels of inverse blocking the action on the finest lattice will go to zero. In contrast, the action of a continuum instanton configuration will not change under blocking. A coarse configuration consisting of an instanton plus fluctuations, whose action is greater than $S_I$, will inverse block to a configuration whose action is closer to $S_I$, and under repeated inverse blocking will approach $S_I$ arbitrarily closely. The size of the instanton grows by a factor of two under the inverse blocking, so the physical picture of the many-times inverse blocked lattice configuration is a large smooth instanton with its appropriate action.

There are two types of definitions of topological charge in the literature. One is the “algebraic” or “field theoretic” definition where some lattice discretization of the continuum operator $F\tilde{F}$ is measured. The geometric definition reconstructs a fiber bundle from the lattice gauge field and identifies the second Chern number of this bundle with the topological charge. Both definitions are correct and produce equivalent results on smooth configurations. On a coarse lattice with a sufficiently rough gauge configuration both definitions of the topological charge break down. The field theoretical definition does not yield integer values of the charge and to obtain a continuum charge the lattice charge is multiplicatively renormalized by a coupling-dependent (and action-dependent) factor $Z(\beta)$. The algebraic definition could be improved by using a FP instanton charge as described by Ref. [12]—if it could be done non-perturbatively. We elect not to pursue the algebraic method in this paper due to the complication of the $Z(\beta)$ factor. The geometric definition also fails as it often identifies the remnants of an instanton that has fallen through the lattice as a non-zero charge object thus predicting an artificially large topological susceptibility.

Eqn. 3 provides a way to smooth rough configurations in a way which preserves their topological properties. We define the topological charge of a configuration by first inverse blocking it to a sufficiently smooth configuration and then measuring the charge on the fine lattice. If the initial configuration is sufficiently rough, one might need to inverse block more than once to get a sufficiently smooth configuration to reliably measure $Q$. Since the geometric definition works on rougher configurations than the field theoretic one, we elected to use the geometric method to measure $Q$. Practical considerations of computer memory and speed restrict us to a single step of inverse blocking.

Notice that the above definition of topological charge requires the specification of some renormalization group transformation and its associated FP
action. Inverse blocking with an arbitrary action is not guaranteed to preserve
topological charge from one level of blocking to the next.

Either the algebraic or geometric definition of topological charge could be—
and has been—combined with some other smoothing algorithm. One popular
smoothing algorithm is “cooling.” The idea behind cooling is that one takes a
gauge configuration and performs some local minimization of the action; this
minimization is supposed to eliminate short-distance fluctuations while preserv-
ing long-distance structure. Unfortunately, cooling is not a trustworthy indi-
cator of topological structure for rough gauge configurations. The topological
charge of a cooled configuration depends on the particular action whose value
is to be minimized \( \mathcal{L} \), and if the action is not constrained to be \( S_I \) for all
\( Q \neq 0 \) configurations, the act of cooling could change \( Q \). As a practical prob-
lem, the particular algorithm used to cool may be so efficient that it does not
find local minima of its action but instead global minima: the lowest global
minimum is some gauge transform of the identity. While it may be possible to
find a cooling algorithm which reproduces the results of inverse blocking, we
elect in this work to use only the theoretically reliable inverse blocking method
for measuring topological charge.

3 Practical Considerations for Ideal Instantons

We want to demonstrate that an FP action is indeed scale invariant by showing
that the action of an instanton that is the solution of the equations of motion
is independent of its size. We will parameterize these solutions by a radius or
scale factor \( \rho \) and measure the topological charge \( Q(\rho) \) and the action \( S(\rho) \). We
want to show that when \( Q = 1 \) \( S(\rho) \geq S_I \) and when \( S(\rho) \leq S_I \), \( Q = 0 \). We will
also illustrate that our definition of the topological charge is free of dislocations,
at least for smooth instantons.

We must first deal with a couple of technical problems.

3.1 Finite volume effects

We preface this section by writing down a few formulas for the gauge potential
for a single continuum instanton:

\[
A_\mu(x) = -if(x)g\partial_\mu g^\dagger
\]

(9)
where \( x \) is a Euclidean four-vector,

\[
f(x) = \frac{x^2}{x^2 + \rho^2}
\]

is the shape factor, and

\[
g = \frac{1}{x}(x_0 + i\vec{x} \cdot \vec{\sigma}).
\]

On a periodic lattice a single instanton is not a solution of the classical equations of motion; it does not obey periodicity. If we go ahead and lay down an instanton-like solution on a periodic torus of size \( L \), we find that its action diverges linearly:

\[
S(L) = S_0 + O(L).
\]

due to the discontinuity of the field configurations at \( x_0 = \pm L/2 \).

To ameliorate this problem we follow Pugh and Teper and consider instead trial solutions made of an instanton and a superimposed dislocation. (The dislocation can be regarded as the remnant of a small anti-instanton that fell through the lattice.) Take the solution defined by Eqn. 9 and perform a singular gauge transformation on it

\[
V_\mu(x) = g^\dagger(x)U_\mu(x)g(x + \hat{\mu})
\]

where \( g \) is defined in Eqn. 11. It is easy to show that on a periodic lattice the finite-volume correction to the action of this configuration is

\[
S(L) = S_0 + O(1/L^3)
\]

As we study discretized instantons, we will do computations on many different lattice sizes \( L \) and extrapolate results for the action to infinite volume using

\[
S(L) = S_0 + a_1/L^3 + a_2/L^5.
\]

3.2 Short distance effects

The discretization of the instanton solution on the lattice will also show lattice artifacts when the size of the instanton solution (parameterized by \( \rho \)) is of the order of the lattice spacing. A true instanton solution of a FP action has no such artifacts.

To place an instanton-like solution on a lattice we begin with a vector potential \( A_\mu(x) \) defined via Eqn. 9 with \( x \) measured at any continuum point with
respect to an origin \( x_0 \). Define link variables on a lattice on sites labeled by an integer \( n \) by approximately constructing the path-ordered exponential between neighboring sites. That is, define

\[
U_\mu(na) = \prod_j U_\mu(x = na + j\Delta x\hat{\mu})
\] (16)

where

\[
U_\mu(x = na + j\Delta x\hat{\mu}) = \exp(i\Delta x A_\mu(x = na + j\Delta x\hat{\mu}))
\] (17)

and \( A_\mu(x) \) is given by Eqn. 9. We typically broke the lattice spacing up into 20 intervals (\( \Delta x = a/20 \)). The interpolation corresponds to smoothing the classical solution along the lines of the lattice. Finally, we perform the singular gauge transformation of Eqn. 13 on the link variable of Eqn. 17. So far this definition follows Pugh and Teper[4].

This procedure does not guarantee that we have constructed a solution for the equations of motion of our FP action. Short distance roughness in the solution, especially for small instantons, is still present. In order to remove this roughness we block the instanton configuration (perhaps repeatedly) using the RGT of Eqn. 8. Blocking at \( \beta \to \infty \) moves the configuration towards the FP and transforms the instanton to a solution of the equations of motion. Blocking does not preserve the topological charge. With each blocking step the size of the instanton is halved and it might disappear from the lattice. Both the value of the action and the topological charge operator should signal when that happens.

We consider trial solutions made after one level of blocking (\( 40^4 \to 20^4 \), \( 32^4 \to 16^4 \), \( 20^4 \to 10^4 \) and \( 16^4 \to 8^4 \)) and after two levels of blocking (\( 40^4 \to 10^4 \), \( 32^4 \to 8^4 \), \( 16^4 \to 4^4 \)). In addition we consider two kinds of instanton-like solutions, which we call \( c_1 \) and \( c_2 \) instantons, whose centers are located at \( x_0 = (L/2 + 1/2, L/2 + 1/2, L/2 + 1/2) \) and \( x_0 = (L/2, L/2, L/2 + 1/2, L/2 + 1/2) \) respectively on an \( L^4 \) lattice.

### 4 Construction of a FP action

Determining the properties of instantons using a FP action requires knowing the action. The construction of a FP action quadratic in the gauge fields, valid for smooth gauge configurations, was described in [7]. However, instanton solutions with small \( \rho \) are not smooth and we need a parametrization of the FP action valid on rough configurations as well.

Our parametrization of the action is based on powers of the traces of the
loop products $V_C = \Pi_C V_\mu(n)$ (for SU(N)) where $C$ is an arbitrary closed path

\[
S^{FP}(V) = \frac{1}{N} \sum_C (c_1(C)(N - \text{ReTr}(V_C)) + c_2(C)(N - \text{ReTrTr}(V_C))^2 + \ldots \quad (18)
\]

Assume we know the value of the fixed point action on a set of gauge configurations $\{V\}$. As we can measure the traces of the loop products in Eqn. 18, we obtain a set of linear equations, one for each $\{V\}$ configuration, for the coefficients $c_i(C)$. Consequently the parameters of the fixed point action on a given set of $\{V\}$ configurations can be obtained from a linear fit.

All that is left is to calculate the value of the FP action on a given configuration. The inverse blocking procedure of Eqn. 3 can be used for that. For any coarse configuration $\{V\}$ we can find a fine configuration $\{U^{FP}\}$ that minimizes the right hand side of Eqn. 3 and calculate the FP action as

\[
S^{FP}(V) = S^{FP}(U^{FP}) + T(U^{FP}, V). \quad (19)
\]

For this procedure we need the FP action on the fine configurations only. As the $\{U\}$ configurations are very smooth, the FP action is well approximated by its analytically derived quadratic form $S^{FP}_q$ on the fine configurations. $S^{FP}_q$ is still too complicated for the numerical minimization. Fortunately it is possible to replace $S^{FP}_q$ with a simpler action $S_0$ in the minimization if its corresponding fine configuration $\{U_0\}$ is close to $\{U^{FP}\}$ and the difference $S^{FP}_q - S_0$ can be corrected perturbatively. If

\[
S'(V) = \min_U (S_0(U) + T(U, V)) = S_0(U_0) + T(U_0, V), \quad (20)
\]

then

\[
S^{FP}(V) = S^{FP}_q(U_0) + T(U_0, V) = S^{FP}(U_0) + S'(V) - S_0(U_0), \quad (21)
\]

assuming that $S^{FP}_q(U_0) - S_0(U_0)$ is small. We took for $S_0$ the single plaquette action

\[
S_0(U) = 0.9113(2 - \text{Tr}U_p) + 0.4564(2 - \text{Tr}U_p)^2 - 0.7723(2 - \text{Tr}U_p)^3 + 0.3007(2 - \text{Tr}U_p)^4 \quad (22)
\]

and found that the perturbative corrections are about 0.5% as illustrated in Fig. 1 for a set of sample $\{V\}$ configurations.

FP actions, even the most local ones, can be fairly complicated. At very large correlation lengths the FP action is well described by its quadratic form, given
by the $c_1(C)$ coefficients of Eqn. 18. As the correlation length decreases the higher representation terms become important and it is necessary to determine the $c_i(C), i > 1$ coefficients as well. At even smaller correlation lengths it is possible that new type of operators, which were not present in the quadratic limit (products of disconnected loops, for example), will become important.

We parametrized the fixed point action with the 12 loops present in the quadratic limit, each in 4 different representations, i.e. using the first 4 powers in the series of Eqn. 18. We found that a 48 parameter fit describes the fixed point action well if the lattice spacing of the coarse configurations is $a \leq 1/3T_c$.

The couplings of a 48 parameter fit are given in Table 1. In this fit we used about 500 configurations generated with the Wilson action, all with $a \geq 1/3T_c$, and about 20 smooth instanton configurations with $\rho > 1.0$ that we created with the procedure described in Sect. 3. The FP action value of the instanton configurations were fixed by requiring that on infinite volume their action extrapolates to the theoretical value $S = S_I$.

Now we turn our attention to the scale invariance of the full FP action.

Table 1: Couplings of the Type I RGT fixed point action given in terms of loops defined in Table 3 of Ref. 7. The quadratic couplings are labeled by $c_1$, and the coefficients of higher powers of $(2 - \text{Tr}U(C))$ from the fit described in Sec. 4 are also shown.

| loop | $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|------|-------|-------|-------|-------|
| 1    | .6744 | -.00716 | .0934 | -.0190 |
| 2    | -.02  | -.170  | .117  | -.0202 |
| 3    | .012  | -.136  | .0864 | -.0141 |
| 4    | .005  | -.101  | .0729 | -.0112 |
| 5    | -.0031| .0600  | -.0380| .00708 |
| 8    | -.0035| .0281  | -.0252| .00426 |
| 9    | .0027 | -.112  | .0697 | -.0103 |
| 15   | -.0024| .0542  | -.0346| .00531 |
| 16   | .0013 | .0348  | -.0242| .00416 |
| 20   | .003  | .106   | -.0695| .0122  |
| 23   | .0032 | -.0550 | .0385 | -.0060 |
| 25   | .0024 | -.0343 | .0077 | .000053 |

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5 The action and topological charge of instanton configurations

We measure both the topological charge and the value of the fixed point action using the inverse block transformation. For each trial configuration \( \{V\} \) we create a fine configuration \( \{U\} \) according to Eqn. 3. The fine configurations are sufficient enough that the topological charge can be measured by the geometric method

\[
Q(V) = Q_{\text{geom}}(U),
\]

and the minimization gives the value of the fixed point action as well

\[
S^{FP}(V) = S^{FP}(U) + T(U, V).
\]

Measuring the action using Eqn. 24 rather than by direct evaluation on the coarse lattice exploits the fact that we know the FP action better on the fine configuration since the gauge field is smoother there. For the minimization procedure we need the FP action on the fine configurations only.

We found that for the instanton configurations a Symanzik type action

\[
S_0 = \frac{5}{3}(N - TrU_{1\times1}) - \frac{1}{12}(N - TrU_{1\times2})
\]

was efficient in the minimization. The difference \( S^{FP}(U_0) - S_0(U_0) \) was less than a percent with this choice.

We have four approximations to a smooth instanton solution: \( c_1 \) or \( c_2 \) origins, single or double blocked. When the four approximations do not give identical results for the action or topological charge, we know that the configurations do not satisfy the equations of motion. They could be instanton configurations with small scale fluctuations or configurations where the instanton fell through the lattice. In the former case we expect \( Q = 1 \) and \( S > S_I \), in the latter \( Q = 0 \) with arbitrary action.

Table 2 gives the value of \( S_0(U_0) \), \( T(U_0, V) \), \( S_{\text{fine}} = S^{FP}(U_0) \), \( S^{FP}(V) = S^{FP}(U_0) + T(U_0, V) \), \( Q_{\text{geom}}(U_0) \) and \( Q_{\text{geom}}(V) \) for a series of \( c_1 \) configurations blocked once from \( 16^4 \) to \( 8^4 \) lattices. The instanton radius is measured on the coarse \( (8^4) \) lattice. We inverse block the \( 8^4 \) configuration to \( 16^4 \) and measure its action and charge there. At large \( \rho \) the topological charge is \( Q = 1 \) and \( T(U_0, V) \) is small, as one would expect for a smooth instanton. The action \( S^{FP}(V) \) is slightly larger than \( S_I \) due to the boundary effects. As \( \rho \) drops below 0.88, \( T \) increases about 2 orders of magnitude and \( S^{FP}(V) \) falls below \( S_I \). The instanton has fallen through the lattice. The topological charge measured on
the fine configuration correctly describes the situation. The charge measured on the coarse lattice is still $Q(V)_{geom} = 1$. The geometric definition incorrectly identifies a dislocation with a $Q = 1$ instanton.

Table 2: Action and charge of a set of instanton configurations blocked from $16^4$ to $8^4$ lattices.

| $\rho$ | $T/S_I$ | $S_{fine}/S_I$ | $(T + S_{fine})/S_I$ | $Q_{geom}(U)$ | $Q_{geom}(V)$ |
|-------|--------|----------------|----------------------|----------------|---------------|
| 0.5   | 0.119102 | 0.194042       | 0.313144             | 0              | 0             |
| 0.8   | 0.153455 | 0.698940       | 0.852395             | 0              | 1             |
| 0.85  | 0.145686 | 0.789170       | 0.934856             | 0              | 1             |
| 0.86  | 0.138044 | 0.816921       | 0.954965             | 0              | 1             |
| 0.88  | 0.001922 | 1.004702       | 1.00662              | 1              | 1             |
| 0.89  | 0.001359 | 1.004853       | 1.00621              | 1              | 1             |
| 0.9   | 0.001006 | 1.005316       | 1.00632              | 1              | 1             |
| 0.91  | 0.001010 | 1.005891       | 1.0069               | 1              | 1             |
| 0.92  | 0.000895 | 1.006427       | 1.00732              | 1              | 1             |
| 0.95  | 0.000838 | 1.007948       | 1.00879              | 1              | 1             |
| 1.0   | 0.000909 | 1.010088       | 1.011                | 1              | 1             |
| 1.2   | 0.001611 | 1.017157       | 1.01877              | 1              | 1             |
| 1.3   | 0.002159 | 1.021950       | 1.02411              | 1              | 1             |
| 1.5   | 0.003644 | 1.036813       | 1.04046              | 1              | 1             |

We cannot specify the FP action or the action of an instanton candidate to arbitrary accuracy. There are several sources for the uncertainty of the action. For larger $\rho$ the infinite volume extrapolation introduces error. For smaller instantons the main source of uncertainty is the specific parametrization of the FP action. Table I gives a 48 parameter form obtained from a fit to a data set which included both instantons and a collection of non-instanton “typical gauge field” configurations with $a \geq 1/3T_c$. The small core instantons are rougher than these configurations. If we use a different set of configurations with the same qualitative properties we obtain a slightly different parametrization. The new parametrization gives the same action for the configurations with $a > 1/3T_c$ but up to 10% different value for the small radius instantons (using Eqn. 24). We illustrate those differences by presenting a profile of action vs. $\rho$ for a set of once-blocked $c_1$ instantons in Fig. 2. The uncertainty in the action is represented as an error bar on the value of the action, obtained by taking two equally good parameterizations of the FP action and computing the action of the test configuration using Eqn. 24. The dotted lines show the extrapolation to infinite volume using Eqn. 17 and the couplings of Table I. The value of the topological charge (measured on the inverse blocked configuration) is overlaid on the figure. Notice that the action is consistent with the continuum value $S_I$ while $Q = 1$ and falls below that number when $Q = 0$. Notice also that while the difference in the action is larger for configurations which do not carry topological charge, the
two parameterizations agree closely for the value of the action for an instanton configuration.

The corresponding plots for once blocked $c_2$ and twice blocked $c_1$ instantons show the same qualitative features (Figs. 3 and 4) but the quantitative differences indicate that one or two blocking steps do not get rid of all the short distance fluctuations of small radius instantons.

6 Instantons and the Wilson Action

Our theoretically consistent definition of the topological charge is closely related to a FP action. In principle the same configuration can have different topological charge and action depending on the FP action we use to measure it, though in practice that happens only for small core instantons. The Wilson plaquette action is not a FP action of any RG transformation. It is not possible to define the topological charge of a configuration with respect to the Wilson action.

Most of the topological calculations so far used the Wilson action and the charge was measured directly either on the original lattice or on its cooled version. The recent overimproved cooling technique relies on actions improved according to their properties on smooth instanton configurations, whose topological charge is identified by the direct geometric definition [16, 17].

We feel it is important to demonstrate the (non-) scale invariant properties of the Wilson action and the importance of inverse blocking in defining the topological charge. In Fig. 5-6 we show the action of our once blocked $c_1$ and $c_2$ instantons measured with the Wilson action. The dotted line indicates the $Q = 0 \to Q = 1$ boundary on the original coarse lattice while the solid line is the boundary on the inverse blocked configuration. The inverse blocking was done with the FP action and RG transformation described in Sect. 2 and is not necessarily correct for the Wilson action. Nevertheless, the difference between the two definitions is obvious. For the $c_2$ once blocked instantons, for example, the boundary moves from $\rho = 0.54$ to $\rho = 0.70$ – enough to overcome the entropy problem discussed in Sect. 2!

We want to emphasize one more time that our definition of the topological charge has to be used with the corresponding FP action and it cannot be automatically applied for the plaquette Wilson action.
7 Conclusions

In this paper we have presented a theoretically consistent definition of the topological charge based on a renormalization group transformation and studied the scale invariant instanton solutions of the corresponding FP action.

We have demonstrated that the FP action shows the desired scale invariance, i.e., that instanton solutions with non-zero topological charge have action \( S(\rho) \geq \frac{8\pi^2}{g^2} \). For small radius smooth instanton configurations we showed that the RG topological charge is different from the direct geometrical definition and only the former is consistent with the scale invariance of the FP action, as the direct geometric method identifies dislocations with \( S(\rho) < \frac{8\pi^2}{g^2} \) with \( Q = 1 \) instantons.

Our definition of the topological charge works only with the FP action of a given RG transformation and cannot be used with arbitrary, non-FP actions. In the present work we have studied only one renormalization group transformation but the method generalizes easily and the qualitative features should be the same for any RGT.

The extension of the techniques we have described to SU(3) gauge theory is straightforward. The only complication will be the necessity to use the SU(3) analog of the geometric definition for topological charge [18].

A measurement of the topological susceptibility using an eight parameter approximate FP action valid for small correlation lengths is the subject of another paper [19].

8 Acknowledgements

We would like to thank M. Müller-Preussker for providing us with a copy of the DESY program for measuring topological charge. We very much want to thank P. Hasenfratz and F. Niedermayer for useful conversations. We would like to thank A. Barker, M. Horanyi and the Colorado high energy experimental group for allowing us to use their work stations. This work was supported by the U.S. Department of Energy and by the National Science Foundation.

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Figure 1: The perturbative corrections $\delta S_{PT} = S^{FP}(U_0) - S_0(U_0)$ for a set of configurations using the type I RGT, for the quadratic limit of the FP action.
Figure 2: Action of once blocked $c_1$ instanton configurations computed using Eqn. 3. The error bar represents the uncertainty in the action due to choice of different 48 parameter actions and reflects our inability to completely reconstruct the FP action. The vertical line corresponds to the $Q = 0 \rightarrow Q = 1$ boundary.
Figure 3: Action of once blocked $c_2$ instanton configurations presented as in Fig. 2.
Figure 4: Action of twice blocked $c_1$ instanton configurations presented as in Fig. 2.
Figure 5: Action of single blocked $c_1$ instanton configurations computed using the Wilson action. The dotted line indicates the $Q = 0 \to Q = 1$ boundary on the original coarse lattice while the solid line is the boundary on the inverse blocked configuration.
Figure 6: Action of single blocked $c_2$ instanton configurations computed using the Wilson action, displayed as in Fig. 5.