Numerical solutions to a class of scalar elliptic BVPs for anisotropic exponentially graded media

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Abstract. The Boundary Element Method (BEM) is used for obtaining solutions to another class of elliptic boundary value problems (BVPs) of anisotropic exponentially graded media. A technique of transforming the variable coefficient governing equation to a constant coefficient equation is utilized for deriving a boundary integral equation. Some particular problems are considered to illustrate the application of the BEM. The results show the convergence, consistency, and accuracy of the BEM solutions.

1. Introduction
The BEM has been successfully used for solving many types of problems of homogeneous media. Some works on homogeneous media problems have been recently done by Azis et. al [1, 2] and Haddade et. al [3] in which the authors considered pollutant transport problems governed by 2D diffusion-convection.

However, this is generally not the case for inhomogeneous media. There are two techniques generally used in dealing with inhomogeneous media. The first one uses a relevant Green function or fundamental solution to the inhomogeneous problem. Cheng in [4] had applied this technique. The second technique is by transforming the variable coefficient governing equation to a constant coefficient equation. Some progress on using the second technique has been made. For examples, Clements and Azis in [5]. For anisotropic inhomogeneous some works have been studied by Azis and Clements in [6] and Azis et. al in [7] for elasticity problems, Azis and Clements in [8, 9] for heat conduction problems. In addition to this, recently Salam et. al in [10] have been working on a class of elliptic problems for anisotropic inhomogeneous media. In this work a boundary integral equation was derived after a transformation of the variable governing equation to a constant coefficient equation. The governing equation considered by Salam et. al in [10] takes the form

$$\frac{\partial}{\partial x_i} \left[ \lambda_{ij} (x_1, x_2) \frac{\partial \phi (x_1, x_2)}{\partial x_j} \right] = 0 \quad (1)$$

where the coefficients $\lambda_{ij}$ depend on $x_1$ and $x_2$ and the repeated summation convention (summing from 1 to 2) is employed.

This paper is intended to extend the work by Salam et. al [10] for problems with governing equation (1) to for 2D boundary value problems governed by another type of (dimensionless) elliptic equation of the form

$$\frac{\partial}{\partial x_i} \left[ \lambda_{ij} (x_1, x_2) \frac{\partial \phi (x_1, x_2)}{\partial x_j} \right] + \beta (x_1, x_2) \phi (x_1, x_2) = 0 \quad (2)$$
A variety of problems of both isotropic and anisotropic inhomogeneous media are usually modelled with equation (2). Steady infiltration problems (when $\beta < 0$, see for examples [11, 12]), acoustic problems (when $\beta > 0$, see for examples [13, 14]), and antiplane strain in elastostatics and plane thermostatic problems (when $\beta = 0$) are the areas for which the governing equation is of the type (2).

The technique of transforming (2) to a constant coefficient equation will again be used for obtaining a boundary integral equation for the solution of (2). Some constraint will be placed on the class of coefficients $\lambda_{ij}$ and $\beta$ for which the solution obtained is valid. The analysis of this paper is purely formal; the main aim being to construct effective BEM for class of equations which falls within the type (2).

2. The boundary value problem

Referred to a Cartesian frame $Ox_1x_2$ a solution to (2) is sought which is valid in a region $\Omega$ in $\mathbb{R}^2$ with boundary $\partial \Omega$ which consists of a finite number of piecewise smooth closed curves. On $\partial \Omega_1$ the dependent variable $\phi (x)$ ($x = (x_1, x_2)$) is specified and on $\partial \Omega_2$ $P(x) = \lambda_{ij} (\partial \phi / \partial x_j) n_i$ (3)
is specified where $\partial \Omega = \partial \Omega_1 \cup \partial \Omega_2$ and $n = (n_1, n_2)$ denotes the outward pointing normal to $\partial \Omega$.

For all points in $\Omega$ the matrix of coefficients $[\lambda_{ij}]$ is a real symmetric positive definite matrix so that throughout $\Omega$ equation (2) is a second order elliptic partial differential equation. Therefore equation (2) may be written explicitly as

$$\frac{\partial}{\partial x_1} \left( \lambda_{11} \frac{\partial \phi}{\partial x_1} \right) + 2 \frac{\partial}{\partial x_1} \left( \lambda_{12} \frac{\partial \phi}{\partial x_2} \right) + \frac{\partial}{\partial x_2} \left( \lambda_{22} \frac{\partial \phi}{\partial x_2} \right) + \beta \phi = 0$$

Further, the coefficients $\lambda_{ij}$ and $\beta$ are required to be twice differentiable functions of the two independent variables $x_1$ and $x_2$.

The method of solution will be to obtain boundary integral equations from which numerical values of the dependent variables $\phi$ and $P$ may be obtained for all points in $\Omega$. The analysis here is specially relevant to an anisotropic medium but it equally applies to isotropic media. For isotropy, the coefficients in (2) take the form $\lambda_{11} = \lambda_{22}$ and $\lambda_{12} = 0$ and use of these equations in the following analysis immediately yields the corresponding results for an isotropic medium.

3. The boundary integral equation

The boundary integral equation is derived by transforming the variable coefficient equation (2) to a constant coefficient equation. The coefficients $\lambda_{ij}$ and $\beta$ are required to take the form

$$\lambda_{ij}(x) = \lambdabar_{ij} g(x) \quad (4)$$
$$\beta(x) = \betabar g(x) \quad (5)$$

where the $\lambdabar_{ij}$ and $\betabar$ are constants and $g$ is a differentiable function of $x$. Use of (4) and (5) and in (2) yields

$$\lambdabar_{ij} \frac{\partial}{\partial x_i} \left( g \frac{\partial \phi}{\partial x_j} \right) + \betabar g \phi = 0 \quad (6)$$

Let

$$\phi(x) = g^{-1/2}(x) \psi(x) \quad (7)$$
so that (6) may be written in the form

$$\lambda_{ij} \frac{\partial}{\partial x_i} \left[ g \frac{\partial (g^{-1/2}\psi)}{\partial x_j} \right] + \beta g^{1/2}\psi = 0$$

That is

$$\lambda_{ij} \left[ \left( \frac{1}{4} g^{-3/2} \frac{\partial g}{\partial x_i} \frac{\partial g}{\partial x_j} \right) - \frac{1}{2} g^{-1/2} \frac{\partial^2 g}{\partial x_i \partial x_j} \right) \psi + g^{1/2} \frac{\partial^2 \psi}{\partial x_i \partial x_j} \right] + \beta g^{1/2}\psi = 0 \quad (8)$$

Use of the identity

$$\frac{\partial^2 g^{1/2}}{\partial x_i \partial x_j} = -\frac{1}{4} g^{-3/2} \frac{\partial g}{\partial x_i} \frac{\partial g}{\partial x_j} + \frac{1}{2} g^{-1/2} \frac{\partial^2 g}{\partial x_i \partial x_j}$$

permits (8) to be written in the form

$$g^{1/2} \lambda_{ij} \frac{\partial^2 \psi}{\partial x_i \partial x_j} - \psi \lambda_{ij} \frac{\partial^2 g^{1/2}}{\partial x_i \partial x_j} + \beta g^{1/2}\psi = 0$$

It follows that if $g$ is such that

$$\lambda_{ij} \frac{\partial^2 g^{1/2}}{\partial x_i \partial x_j} - \beta g^{1/2} = 0 \quad (9)$$

then the transformation (7) carries the variable coefficients equation (6) to the constant coefficients equation

$$\lambda_{ij} \frac{\partial^2 \psi}{\partial x_i \partial x_j} = 0 \quad (10)$$

We will put a constraint for the function $g(x)$ satisfying (9) to take exponentially form

$$g(x) = [A \exp (\alpha_i x_i)]^2 \quad \beta = \lambda_{ij} \alpha_i \alpha_j \quad (11)$$

Also, substitution of (4) and (7) into (3) gives

$$P = -P_g \psi + P_\psi g^{1/2} \quad (12)$$

where

$$P_g (x) = \lambda_{ij} \frac{\partial g^{1/2}}{\partial x_j} n_i \quad P_\psi (x) = \lambda_{ij} \frac{\partial \psi}{\partial x_j} n_i$$

A boundary integral equation for the solution of (10) is given in the form

$$\eta (x_0) \psi (x_0) = \int_{\partial \Omega} [\Gamma (x, x_0) \psi (x) - \Phi (x, x_0) P_\psi (x)] ds (x) \quad (13)$$

where $x_0 = (a, b)$, $\eta = 0$ if $(a, b) \notin \Omega \cup \partial \Omega$, $\eta = 1$ if $(a, b) \in \Omega$, $\eta = 1/2$ if $(a, b) \in \partial \Omega$ and $\partial \Omega$ has a continuously turning tangent at $(a, b)$.

The so called fundamental solution $\Phi$ in (13) is any solution of the equation

$$\lambda_{ij} \frac{\partial^2 \Phi}{\partial x_i \partial x_j} = \delta (x - x_0)$$

and the $\Gamma$ is given by

$$\Gamma (x, x_0) = \lambda_{ij} \frac{\partial \Phi (x, x_0)}{\partial x_j} n_i$$
where δ is the Dirac delta function. Following Azis in [15], for two-dimensional problems Φ and Γ are given by

$$\Phi (x, x_0) = \frac{K}{2\pi} \ln R$$

$$\Gamma (x, x_0) = \frac{K}{2\pi R} \frac{\partial R}{\partial x_j} n_j$$

(14)

where

$$K = \frac{\dot{\tau}}{\zeta}$$

$$\zeta = \left[ \lambda_{11} + 2\lambda_{12} \dot{\tau} + \lambda_{22} (\dot{\tau}^2 + \ddot{\tau}^2) \right] / 2$$

$$R = \sqrt{(\dot{x}_1 - \dot{a})^2 + (\dot{x}_2 - \dot{b})^2}$$

$$\dot{x}_1 = x_1 + \dot{\tau} x_2$$

$$\dot{a} = a + \dot{\tau} b$$

$$\dot{x}_2 = \dot{\tau} x_2$$

$$\dot{b} = \ddot{\tau} b$$

where ι and ι̅ are respectively the real and the positive imaginary parts of the complex root τ of the quadratic

$$\lambda_{11} + 2\lambda_{12} \tau + \lambda_{22} \tau^2 = 0$$

The derivatives $\partial R/\partial x_j$ needed for the calculation of the Γ in (14) are given by

$$\frac{\partial R}{\partial x_1} = \frac{1}{R} (\dot{x}_1 - \dot{a})$$

$$\frac{\partial R}{\partial x_2} = \dot{\tau} \left[ \frac{1}{R} (\dot{x}_1 - \dot{a}) \right] + \ddot{\tau} \left[ \frac{1}{R} (\dot{x}_2 - \dot{b}) \right]$$

Use of (7) and (12) in (13) yields

$$\eta (x_0) g^{1/2} (x_0) \phi (x_0) = \int_{\partial \Omega} \left\{ \left[ g^{1/2} (x) \Gamma (x, x_0) - P_g (x) \Phi (x, x_0) \right] \phi (x) \right. \right.$$  

$$\left. - \left[ g^{-1/2} (x) \Phi (x, x_0) \right] P (x) \right\} ds (x)$$

(15)

This equation provides a boundary integral equation for determining φ and P at all points of Ω.

4. Numerical examples
Some particular boundary value problems will be solved numerically by employing the integral equation (15). The main aim is to show the validity of the analysis for deriving the boundary integral equation (15) and the appropriateness of the BEM in solving the problems through the derived boundary integral equation (15). Standard boundary element method is employed to obtain numerical results. The integrals in equation (15) are evaluated numerically using the Bode’s quadrature (see Abramowitz and Stegun in [16]). For all problems considered the function $g(x)$ is of the form (11).

4.1. Examples with analytical solutions
In order to see the convergence and accuracy of the BEM we will consider some examples of problems with analytical solutions. The parameters for the exponential inhomogeneity function $g(x)$ are $A = 3, \alpha_1 = 0.35, \alpha_2 = 0.65$. Plot of $g(x)$ is shown in Figure 1. The geometry of the
region Ω and the boundary conditions are as depicted in Figure 2. The values of the constant coefficients \( \lambda_{ij} \) for the governing equation (2) are

\[
\lambda_{11} = 0.75, \lambda_{12} = 0.75, \lambda_{22} = 1
\]

Therefore from (9)

\[
\beta = \lambda_{ij} \alpha_i \alpha_j = 0.855625
\]

The function \( g(x) \) satisfies (9). Therefore equation (10) has to be the corresponding constant coefficient equation for \( \psi(x) \). For test problems, we will take \( \psi(x) \) as a linear function satisfying (10)

\[
\psi(x) = B (\gamma_0 + \gamma_1 x_1 + \gamma_2 x_2)
\]

4.1.1. Problem 4.1.1 We choose analytical solution with parameters for \( \psi(x) \)

\[
B = 2, \gamma_0 = 1, \gamma_1 = 0.65, \gamma_2 = 0.45
\]
Table 1. BEM and analytical solutions for Problem 4.1.1

| $x_1, x_2$ | $\phi$ | $\partial \phi / \partial x_1$ | $\partial \phi / \partial x_2$ | $\phi$ | $\partial \phi / \partial x_1$ | $\partial \phi / \partial x_2$ |
|------------|--------|-------------------------------|-------------------------------|--------|-------------------------------|-------------------------------|
| (0.1,0.5)  | 0.6021 | 0.1077                        | -0.2025                       | 0.6010 | 0.0957                        | -0.1824                       |
| (0.3,0.5)  | 0.6172 | 0.0637                        | -0.2001                       | 0.6165 | 0.0640                        | -0.2011                       |
| (0.5,0.5)  | 0.6278 | 0.0442                        | -0.2190                       | 0.6271 | 0.0430                        | -0.2216                       |
| (0.7,0.5)  | 0.6346 | 0.0265                        | -0.2374                       | 0.6338 | 0.0246                        | -0.2391                       |
| (0.9,0.5)  | 0.6390 | 0.0879                        | -0.3143                       | 0.6371 | 0.0176                        | -0.2636                       |
| (0.5,0.1)  | 0.7145 | -0.0684                       | -0.1084                       | 0.7167 | 0.1017                        | -0.2358                       |
| (0.5,0.3)  | 0.6716 | 0.0634                        | -0.2187                       | 0.6717 | 0.0644                        | -0.2242                       |
| (0.5,0.7)  | 0.5841 | 0.0263                        | -0.2163                       | 0.5833 | 0.0252                        | -0.2158                       |
| (0.5,0.9)  | 0.5415 | 0.1523                        | -0.3055                       | 0.5408 | 0.0061                        | -0.2020                       |

| $\phi$ | $\partial \phi / \partial x_1$ | $\partial \phi / \partial x_2$ |
|------------|-------------------------------|-------------------------------|
| BEM 20 elements | BEM 40 elements |
| (0.5,0.8)  | 0.6004 | 0.0910                        | -0.1788                       | 0.6000 | 0.0923                        | -0.1807                       |
| (0.3,0.5)  | 0.6161 | 0.0654                        | -0.2031                       | 0.6158 | 0.0664                        | -0.2051                       |
| (0.5,0.5)  | 0.6269 | 0.0432                        | -0.2234                       | 0.6267 | 0.0435                        | -0.2254                       |
| (0.7,0.5)  | 0.6336 | 0.0239                        | -0.2407                       | 0.6334 | 0.0234                        | -0.2420                       |
| (0.9,0.5)  | 0.6368 | 0.0097                        | -0.2562                       | 0.6363 | 0.0058                        | -0.2554                       |
| (0.5,0.1)  | 0.7175 | 0.0090                        | -0.2266                       | 0.7185 | 0.0089                        | -0.2310                       |
| (0.5,0.3)  | 0.6720 | 0.0641                        | -0.2271                       | 0.6723 | 0.0640                        | -0.2298                       |
| (0.5,0.7)  | 0.5828 | 0.0267                        | -0.2174                       | 0.5823 | 0.0270                        | -0.2187                       |
| (0.5,0.9)  | 0.5400 | 0.0135                        | -0.2097                       | 0.5394 | 0.0139                        | -0.2103                       |

Table 2 shows the results of the analytical and BEM solutions with 20, 40 and 80 elements of equal length. The BEM solution converges to the analytical solution as the number of elements increases.

4.1.2. Problem 4.1.2 We choose analytical solution with parameters

$$B = 4, \gamma_0 = 1, \gamma_1 = 0.65, \gamma_2 = 0.45$$

so that

$$\phi(x) = 4 (1 + 0.65x_1 + 0.45x_2) / [3 \exp (0.35x_1 + 0.65x_2)]$$

Table 2 shows the results of the analytical and BEM solutions with 20, 40 and 80 elements of equal length. The BEM solution converges to the analytical solution as the number of elements increases.

4.1.3. Problem 4.1.3 We choose analytical solution with parameters

$$B = 1, \gamma_0 = 1, \gamma_1 = 0.65, \gamma_2 = 0.45$$

so that

$$\phi(x) = (1 + 0.65x_1 + 0.45x_2) / [3 \exp (0.35x_1 + 0.65x_2)]$$

Table 3 shows the results of the analytical and BEM solutions with 20, 40 and 80 elements of equal length. The BEM solution converges to the analytical solution as the number of elements increases.
Table 2. BEM and analytical solutions for Problem 4.1.2

\[(x_1, x_2) \quad \phi \quad \partial \phi / \partial x_1 \quad \partial \phi / \partial x_2 \quad \phi \quad \partial \phi / \partial x_1 \quad \partial \phi / \partial x_2\]

|               | BEM 20 elements | BEM 40 elements |
|---------------|-----------------|-----------------|
| \((0.1,0.5)\) | 1.2042 0.2154  -0.4051 | 1.2020 0.1913  -0.3648 |
| \((0.3,0.5)\) | 1.2345 0.1273  -0.4002 | 1.2330 0.1280  -0.4021 |
| \((0.5,0.5)\) | 1.2556 0.0884  -0.4380 | 1.2541 0.0860  -0.4433 |
| \((0.7,0.5)\) | 1.2692 0.0530  -0.4747 | 1.2676 0.0492  -0.4783 |
| \((0.9,0.5)\) | 1.2779 0.1759  -0.6285 | 1.2742 0.0352  -0.5273 |
| \((0.5,0.1)\) | 1.4290 -0.1368  -0.2167 | 1.4334 0.2034  -0.4717 |
| \((0.5,0.3)\) | 1.3433 0.1268  -0.4374 | 1.3434 0.1287  -0.4484 |
| \((0.5,0.7)\) | 1.1683 0.0527  -0.4325 | 1.1666 0.0505  -0.4315 |
| \((0.5,0.9)\) | 1.0831 0.3046  -0.6111 | 1.0816 0.0123  -0.4041 |

|               | BEM 80 elements | Analytical |
|---------------|-----------------|------------|
| \((0.1,0.5)\) | 1.2009 0.1821  -0.3576 | 1.2000 0.1847  -0.3614 |
| \((0.3,0.5)\) | 1.2322 0.1308  -0.4061 | 1.2316 0.1327  -0.4103 |
| \((0.5,0.5)\) | 1.2538 0.0863  -0.4468 | 1.2535 0.0869  -0.4509 |
| \((0.7,0.5)\) | 1.2671 0.0479  -0.4814 | 1.2668 0.0468  -0.4841 |
| \((0.9,0.5)\) | 1.2736 0.0194  -0.5124 | 1.2725 0.0116  -0.5108 |
| \((0.5,0.1)\) | 1.4349 0.1800  -0.4532 | 1.4369 0.1788  -0.4620 |
| \((0.5,0.3)\) | 1.3440 0.1282  -0.4542 | 1.3446 0.1280  -0.4596 |
| \((0.5,0.7)\) | 1.1656 0.0533  -0.4349 | 1.1646 0.0540  -0.4374 |
| \((0.5,0.9)\) | 1.0800 0.0269  -0.4195 | 1.0788 0.0277  -0.4206 |

Table 3. BEM and analytical solutions for Problem 4.1.3

\[(x_1, x_2) \quad \phi \quad \partial \phi / \partial x_1 \quad \partial \phi / \partial x_2 \quad \phi \quad \partial \phi / \partial x_1 \quad \partial \phi / \partial x_2\]

|               | BEM 20 elements | BEM 40 elements |
|---------------|-----------------|-----------------|
| \((0.1,0.5)\) | 0.3010 0.0539  -0.1013 | 0.3005 0.0478  -0.0912 |
| \((0.3,0.5)\) | 0.3086 0.0318  -0.1000 | 0.3082 0.0320  -0.1005 |
| \((0.5,0.5)\) | 0.3139 0.0221  -0.1095 | 0.3135 0.0213  -0.1108 |
| \((0.7,0.5)\) | 0.3173 0.0133  -0.1187 | 0.3169 0.0123  -0.1196 |
| \((0.9,0.5)\) | 0.3195 0.0440  -0.1571 | 0.3185 0.0088  -0.1318 |
| \((0.5,0.1)\) | 0.3572 -0.0342  -0.0542 | 0.3583 0.0508  -0.1179 |
| \((0.5,0.3)\) | 0.3358 0.0317  -0.1094 | 0.3359 0.0322  -0.1121 |
| \((0.5,0.7)\) | 0.2921 0.0132  -0.1081 | 0.2916 0.0126  -0.1079 |
| \((0.5,0.9)\) | 0.2708 0.0761  -0.1528 | 0.2704 0.0031  -0.1010 |

|               | BEM 80 elements | Analytical |
|---------------|-----------------|------------|
| \((0.1,0.5)\) | 0.3002 0.0455  -0.0894 | 0.3000 0.0462  -0.0903 |
| \((0.3,0.5)\) | 0.3081 0.0327  -0.1015 | 0.3079 0.0332  -0.1026 |
| \((0.5,0.5)\) | 0.3134 0.0216  -0.1117 | 0.3134 0.0217  -0.1127 |
| \((0.7,0.5)\) | 0.3168 0.0120  -0.1204 | 0.3167 0.0117  -0.1210 |
| \((0.9,0.5)\) | 0.3184 0.0048  -0.1281 | 0.3181 0.0029  -0.1277 |
| \((0.5,0.1)\) | 0.3587 0.0450  -0.1133 | 0.3592 0.0447  -0.1155 |
| \((0.5,0.3)\) | 0.3360 0.0321  -0.1135 | 0.3362 0.0320  -0.1149 |
| \((0.5,0.7)\) | 0.2914 0.0133  -0.1087 | 0.2912 0.0135  -0.1094 |
| \((0.5,0.9)\) | 0.2700 0.0067  -0.1049 | 0.2697 0.0069  -0.1051 |
4.2. Examples without analytical solutions

In this section we will consider some examples of problems without simple analytical solutions. We setup some problems for a homogeneous isotropic material by taking $\lambda_{11} = \lambda_{22} = 1, \lambda_{12} = 0$ and with symmetrical boundary conditions. The aim is to see the consistency of the BEM of whether it produces symmetrical solutions.

4.2.1. Problem 4.2.1 For this problem we take $g(x) = [A \exp(\alpha_i x_i)]^2 = 1$ with $A = 1, \alpha_1 = 0, \alpha_2 = 0$ so that $\beta = \lambda_{ij}\alpha_i\alpha_j = 0$. And the symmetrical boundary conditions are as shown in Figure 3. Table 4 shows the results of the BEM solution using 40, 80, 160 and 320 elements of equal length. As expected, the results converge as the number of elements increases and also they are symmetrical about the axes $x_2 = 0.5$.

4.2.2. Problem 4.2.2 We consider a problem with $g(x) = 4$. The boundary conditions are as shown in Figure 3. Table 4 shows the results of the BEM solution using 40, 80, 160 and 320 elements of equal length. As expected, the results converge as the number of elements increases and also they are symmetrical about the axes $x_2 = 0.5$.

4.2.3. Problem 4.2.3 Now we consider a problem with $g(x) = 4$ again and the symmetrical boundary conditions are as shown in Figure 4. Table 6 shows the results of the BEM solution using 40, 80, 160 and 320 elements of equal length. The results converge as the number of elements increases and also they are symmetrical about the axes $x_1 = 0.5$.

5. Conclusion

The scalar elliptic governing equation (2) is used for modelling physical problems such as steady infiltration problems (when $\beta < 0$), acoustic problems (when $\beta > 0$), and antiplane strain in elastostatics and plane thermostatic problems (when $\beta = 0$). The boundary integral equation (15) was derived from this governing equation (2) and straight from (15) a BEM was then constructed for calculation of numerical solutions to the problems for anisotropic exponentially graded media. The results show the convergence, consistency, and accuracy of the BEM solutions. Together with its ease in implementation, it may be concluded that BEM is a good numerical method for solving such kind of problems.

Figure 3. The geometry of Problem 4.2.1 and Problem 4.2.2
### Table 4. BEM solution for Problem 4.2.1

| $(x_1, x_2)$ | $\phi$ | $\partial \phi / \partial x_1$ | $\partial \phi / \partial x_2$ | $\phi$ | $\partial \phi / \partial x_1$ | $\partial \phi / \partial x_2$ |
|--------------|--------|-------------------------------|-------------------------------|--------|-------------------------------|-------------------------------|
| BEM 40 elements | BEM 80 elements |
| (0.1,0.5)    | 0.8924 | -0.9973 | 0.0000 | 0.8967 | -0.9987 | 0.0000 |
| (0.3,0.5)    | 0.6932 | -0.9952 | 0.0000 | 0.6971 | -0.9980 | 0.0000 |
| (0.5,0.5)    | 0.4942 | -0.9943 | 0.0000 | 0.4975 | -0.9977 | 0.0000 |
| (0.7,0.5)    | 0.2955 | -0.9932 | 0.0000 | 0.2980 | -0.9973 | 0.0000 |
| (0.9,0.5)    | 0.0970 | -0.9919 | 0.0000 | 0.0986 | -0.9968 | -0.0000 |
| (0.5,0.1)    | 0.4940 | -0.9975 | 0.0000 | 0.4974 | -0.9982 | 0.0000 |
| (0.5,0.3)    | 0.4942 | -0.9945 | 0.0007 | 0.4975 | -0.9978 | 0.0002 |
| (0.5,0.7)    | 0.4942 | -0.9945 | -0.0007 | 0.4975 | -0.9978 | -0.0002 |
| (0.5,0.9)    | 0.4940 | -0.9975 | -0.0005 | 0.4974 | -0.9982 | -0.0002 |

| BEM 160 elements | BEM 320 elements |
|------------------|------------------|
| (0.1,0.5)        | 0.8985 | -0.9994 | -0.0000 | 0.8993 | -0.9997 | 0.0000 |
| (0.3,0.5)        | 0.6987 | -0.9991 | -0.0000 | 0.6994 | -0.9996 | 0.0000 |
| (0.5,0.5)        | 0.4989 | -0.9990 | -0.0000 | 0.4995 | -0.9996 | 0.0000 |
| (0.7,0.5)        | 0.2991 | -0.9989 | -0.0000 | 0.2996 | -0.9995 | 0.0000 |
| (0.9,0.5)        | 0.0993 | -0.9987 | -0.0000 | 0.0997 | -0.9994 | 0.0000 |
| (0.5,0.1)        | 0.4988 | -0.9992 | 0.0001  | 0.4995 | -0.9997 | 0.0000 |
| (0.5,0.3)        | 0.4989 | -0.9991 | 0.0001  | 0.4995 | -0.9996 | 0.0000 |
| (0.5,0.7)        | 0.4989 | -0.9991 | -0.0001 | 0.4995 | -0.9996 | -0.0000 |
| (0.5,0.9)        | 0.4988 | -0.9992 | -0.0001 | 0.4995 | -0.9997 | -0.0000 |

### Table 5. BEM solution for Problem 4.2.2

| $(x_1, x_2)$ | $\phi$ | $\partial \phi / \partial x_1$ | $\partial \phi / \partial x_2$ | $\phi$ | $\partial \phi / \partial x_1$ | $\partial \phi / \partial x_2$ |
|--------------|--------|-------------------------------|-------------------------------|--------|-------------------------------|-------------------------------|
| BEM 40 elements | BEM 80 elements |
| (0.1,0.5)    | 0.2231 | -0.2493 | 0.0000 | 0.2242 | -0.2497 | -0.0000 |
| (0.3,0.5)    | 0.1733 | -0.2488 | 0.0000 | 0.1743 | -0.2495 | -0.0000 |
| (0.5,0.5)    | 0.1236 | -0.2486 | 0.0000 | 0.1244 | -0.2494 | -0.0000 |
| (0.7,0.5)    | 0.0739 | -0.2483 | 0.0000 | 0.0745 | -0.2493 | -0.0000 |
| (0.9,0.5)    | 0.0242 | -0.2480 | -0.0000 | 0.0246 | -0.2492 | -0.0000 |
| (0.5,0.1)    | 0.1235 | -0.2494 | 0.0001  | 0.1244 | -0.2496 | 0.0000 |
| (0.5,0.3)    | 0.1235 | -0.2486 | 0.0002  | 0.1244 | -0.2495 | 0.0001 |
| (0.5,0.7)    | 0.1235 | -0.2486 | -0.0002 | 0.1244 | -0.2495 | -0.0001 |
| (0.5,0.9)    | 0.1235 | -0.2494 | -0.0001 | 0.1244 | -0.2496 | -0.0000 |

| BEM 160 elements | BEM 320 elements |
|------------------|------------------|
| (0.1,0.5)        | 0.2246 | -0.2498 | -0.0000 | 0.2248 | -0.2499 | 0.0000 |
| (0.3,0.5)        | 0.1747 | -0.2498 | -0.0000 | 0.1748 | -0.2499 | 0.0000 |
| (0.5,0.5)        | 0.1247 | -0.2498 | -0.0000 | 0.1249 | -0.2499 | 0.0000 |
| (0.7,0.5)        | 0.0748 | -0.2497 | -0.0000 | 0.0749 | -0.2499 | 0.0000 |
| (0.9,0.5)        | 0.0248 | -0.2497 | -0.0000 | 0.0249 | -0.2499 | 0.0000 |
| (0.5,0.1)        | 0.1247 | -0.2498 | 0.0000  | 0.1249 | -0.2499 | 0.0000 |
| (0.5,0.3)        | 0.1247 | -0.2498 | 0.0000  | 0.1249 | -0.2499 | 0.0000 |
| (0.5,0.7)        | 0.1247 | -0.2498 | -0.0000 | 0.1249 | -0.2499 | -0.0000 |
| (0.5,0.9)        | 0.1247 | -0.2498 | -0.0000 | 0.1249 | -0.2499 | -0.0000 |
Figure 4. The geometry of Problem 4.2.3

Table 6. BEM solution for Problem 4.2.3

| (x₁, x₂) | φ   | ∂φ/∂x₁ | ∂φ/∂x₂ | φ   | ∂φ/∂x₁ | ∂φ/∂x₂ |
|---------|-----|---------|---------|-----|---------|---------|
|         |     |         |         |     |         |         |
| BEM 40 elements | BEM 80 elements |
| (0.1, 0.5) | 0.5185 | 0.1822 | 0.9382 | 0.5187 | 0.1821 | 0.9351 |
| (0.3, 0.5) | 0.5485 | 0.1090 | 0.8357 | 0.5487 | 0.1088 | 0.8339 |
| (0.5, 0.5) | 0.5597 | -0.0000 | 0.8001 | 0.5598 | -0.0000 | 0.7982 |
| (0.7, 0.5) | 0.5485 | -0.1090 | 0.8356 | 0.5487 | -0.1088 | 0.8339 |
| (0.9, 0.5) | 0.5185 | -0.1822 | 0.9382 | 0.5187 | -0.1819 | 0.9351 |
| (0.5, 0.1) | 0.3078 | 0.0000 | 0.4000 | 0.3087 | 0.0000 | 0.3979 |
| (0.5, 0.3) | 0.4138 | -0.0000 | 0.6451 | 0.4143 | -0.0000 | 0.6433 |
| (0.5, 0.7) | 0.7290 | -0.0000 | 0.8833 | 0.7287 | -0.0000 | 0.8811 |
| (0.5, 0.9) | 0.9099 | -0.0000 | 0.9190 | 0.9091 | -0.0000 | 0.9163 |
| BEM 160 elements | BEM 320 elements |
| (0.1, 0.5) | 0.5188 | 0.1820 | 0.9344 | 0.5188 | 0.1820 | 0.9341 |
| (0.3, 0.5) | 0.5488 | 0.1088 | 0.8329 | 0.5489 | 0.1088 | 0.8324 |
| (0.5, 0.5) | 0.5599 | -0.0000 | 0.7971 | 0.5600 | -0.0000 | 0.7966 |
| (0.7, 0.5) | 0.5488 | -0.1088 | 0.8329 | 0.5489 | -0.1088 | 0.8324 |
| (0.9, 0.5) | 0.5188 | -0.1820 | 0.9344 | 0.5188 | -0.1820 | 0.9341 |
| (0.5, 0.1) | 0.3092 | -0.0000 | 0.3971 | 0.3095 | 0.0000 | 0.3967 |
| (0.5, 0.3) | 0.4146 | -0.0000 | 0.6423 | 0.4147 | 0.0000 | 0.6418 |
| (0.5, 0.7) | 0.7286 | -0.0000 | 0.8799 | 0.7286 | -0.0000 | 0.8793 |
| (0.5, 0.9) | 0.9087 | -0.0000 | 0.9150 | 0.9086 | -0.0000 | 0.9143 |

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