Stringy orbifolds

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1 Introduction

During the last two years, there was an explosive growth of activities to study the "stringy" properties of orbifolds, motivated by the orbifold stringy theory from physics in the middle eighty. A year ago, this author wrote an article to survey some of these initial developments. In the May of 2000, a very successful conference on orbifold was held in Madison where more than 40 mathematician and physicists came together to talk about orbifolds. It was clear that the vast content of this new subject of mathematics is much beyond the author’s expectation. Hence, his initial survey was completely outdated. A new survey is needed to expose the new connections and new results the author leaned on and after conference to general audience. This is the main motivation and purpose of this survey article. Originally, the author planned to give a comprehensive survey to cover the most important results and bring the reader the frontier of research. Because of the limited time,

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the author has to scale down his plan to cover the only topics he is most familiar with. He will list other topics in the last sections.

The article consists of several relatively independent topics. Let me briefly sketch its content. On the Madison conference, the author learnt the interpretation of the orbifold in terms of groupoid by the work of Moerdijk and his collaborators [CM], [MP1], [MP2]. This is an important approach to orbifolds. This approach will be presented in the section 2. The groupoid approach is a categorical and more abstract approach. Although it is less concrete, but it does looks cleaner. Besides, Lupercio-Uribe’s twisted K-theory seems to be best described in this framework. Orbifold and groupoid provided two different points of view. The author feels that it is important to have diverse points of view on this subject. The work of Lupercio and Uribe [LU1], [LU2] on the twisted K-theory suggests that many of results in orbifold should be pushed to more general orbispace where the local model is the quotient of a smooth manifold by a Lie group. This kind of space is called Artin stack and also admitted a groupoid description. However, in many way, Satake’s original approach is more concrete and easier to work with in differential geometry. In the section 2, we present the version of orbispace and its morphism by W. Chen [C] which follows Satake’s original approach.

In the section 3, we will focus on the algebraic geometrical application of stringy orbifold. Here, we focus on the computation of the cohomological ring structure of the crepant resolution of orbifolds, a topic attracting many algebraic geometers in the past. The main example we use is the Hilbert scheme of points of the algebraic surfaces. In this section, we will present the beautiful calculation of the orbifold cohomology of the symmetry product of the complex manifolds by Vafa-Witten (vector space) [W], Fantechi-Gottsche-Uribe (ring structure) [FG2], [U]. Together with Lehn-Sorger’s computation of the cohomology of Hilbert scheme of points of $K3$ and $T^4$ [LS2]. They verified a conjecture of the author for the hyperkahler reslution of orbifolds. In the end, we will present the author’s conjecture for general crepant resolution and verify it for a few examples.

An important aspect of stringy orbifolds is the (twisted)-orbifold K-theory [AR]. Soon after the work of Adem-Ruan, the twisted orbifold K-theory and the twisted K-theory on smooth manifolds by [WIT2] were unified by Lupercio and Uribe [LU1] as the special cases of the twisted K-theory on groupoid by gerbes. We present their construction in the section 4. In the light of recent computation of equivariant twisted K-theory of a Lie group acting on itself by the conjugation by Freed-Hopkins-Teleman [FR], many of current theories should be pushed to the more general orbispace where an isotropy subgroup is not necessarily finite. There are many questions here. It seems to be a very exciting research direction in next few years. In this section, we will also present an interesting construction of Wang to twist the K-theory using spin representation (fermion) [W]. As Wang showed for the symmetric product, its Euler characteristic formula is much nicer than that of the twisted K-theory of Adem-Ruan. This suggests that more should be done regarding to Wang’s twisting.

One of attractions of the stringy orbifold is its unique orbifold feature, which does not exist on smooth manifolds. Orbifold cohomology is such an example. So far, the most of actions are on classical theories such as cohomology and K-theory. Now, we add an unique orbifold feature to its quantum theory. This is the integration of the theory of spin curves by Jarvis and others [J1] [J2] [JKV] [JKV2] into orbifolds. The theory of spin curves was motivated by the 2D-gravity coupled with the matter and has been around for ten years [WIT1]. In my view, it is a piece of mathematics before its time. There is no doubt that its natural home is in the stringy orbifold [AJ]. Furthermore, it also provides a missing link in the orbifold quantum cohomology. Recall that physically GW-invariants are correlation functions of 2D-gravities coupled with topological sigma model. The correct analogous of the topological sigma model in orbifold is not an ordinary map but a good map/groupoid morphism. It is amazing that the correct analogous of 2D-gravity in the
orbifold is not stable curves either. Instead, it is a spin curve or 2D-gravity coupled with the matter. From this point of view, it gives a satisfactory construction of the (spin) orbifold Gromov-Witten invariants over orbifolds. It is not clear at this moment for its implication in other topics of interest such as mirror symmetry. For example, what is the B-model correspondence of the spin orbifold quantum cohomology? In this section, we will sketch Jarvis-Ruan’s construction [JR] of the spin GW-invariants.

In last sections, we make some general remarks about other important topics in orbifolds. This author would like to thank all the participants to the Workshop on Mathematical Aspect of Orbifold Stringy Theory and their wonderful talks where the original motivation of this paper was conceived. In particular, he would like to thank his co-organizers A. Adem and J. Morava for the wonderful job they have done so that the current organizer doesn’t have to organize at all. He also would like to thank E. Lupercio for his unyielding effort to educate him for the theory of groupoid and stack.

2 Foundation

An important aspect of the foundation of orbifold came to my attention during the Madison conference is the interpretation of the orbifold as a groupoid through the earlier work of Moerdijk and his collaborators [CM1, MP1, MP2]. Compared to Satake’s approach, the groupoid is a categorical notion and more abstract. For my personal taste, I will probably prefer Satake’s ad hoc approach because it is more concrete and easy to work with in the differential geometry. But the more abstract approach of groupoid does have its advantage. First of all, it connects to the theory of the stack in the algebraic geometry. Secondly, many important constructions such as the twisted K-theory by gerbes [LU1] is best described using this framework. The author feels that it is useful to keep this dual approach of the orbifold. Therefore, we will present the groupoid approach of the orbifold in this section.

Compared to groupoid approach, Satake’s original approach looks like ad hoc. But it has the advantages to be more concrete and easier to work with in differential geometry. Even though its definition due to Satake was known since 1950’s, somehow it was very much under developed. Basically, it was viewed as a generalized smooth structure and has no distinct character of its own. When the author and W. Chen developed the Gromov-Witten theory over orbifolds, there was very little references we could use and we had to start from scratch. We discovered that Satake’s notion of an orbifold map is not good for pulling back bundles and we cooked out a new notion called good map. Recently, it was showed to be equivalent to the morphism of the groupoid (see [LU1]). However, it seems to be easier to do the computation for good maps. For example, in the orbifold quantum cohomology, one needs to compute the number of good maps/groupoid morphisms from the orbifold Riemann surface to a symplectic orbifold which realizes the same map on the underline topological space. For good maps, we have a concrete classification. It seems to be harder to do so for groupoid morphism.

Recently, Lupercio-Uribe unified the twisted orbifold K-theory of Adem-Ruan with the twisted K-theory on smooth manifold [LU1]. Moreover, they showed that their construction works for more general space of the Artin stack. It strongly suggests that the theory of stringy orbifolds can and should be push to this more general category. Groupoid approach automatically includes this case. It should be useful to extend Satake’s approach to this case as well. This was done by W. Chen [C], which we will present it here.
2.1 Orbifold and Groupoid

By the work of Moerdijk and his collaborators, an orbifold has a nice interpretation as a groupoid. We shall sketch this connection in this section. A groupoid can be thought of as a generalization of a group, a manifold and an equivalence relation. First as an equivalence relation, a groupoid has a set of relations $\mathcal{R}$ that we will think of as arrows. These elements arrows relate is a set $U$. Given an arrow $r \in \mathcal{R}$ it has a source $x = s(r) \in U$ and a target $y = t(r) \in U$. Then we say that $x \xrightarrow{r} y$, namely $x$ is related to $y$. We want to have an equivalence relation, for example we want transitivity and then we will need a way to compose arrows $x \xrightarrow{r} y \xrightarrow{s} z$. We also require $\mathcal{R}$ and $U$ to be more than mere sets. Sometimes we want them to be locally Hausdorff, paracompact, locally compact topological spaces, smooth manifolds.

**Definitions 2.1:** A groupoid is a pair of objects in a category $\mathcal{R}, U$ and morphisms

$$s, t : \mathcal{R} \to U$$

called respectively source and target, provided with an identity

$$e : U \to \mathcal{R}$$

a multiplication

$$m : \mathcal{R} \times_s \mathcal{R} \to \mathcal{R}$$

and an inverse

$$i : \mathcal{R} \to \mathcal{R}$$

satisfying the following properties:

1. The identity inverts both $s$ and $t$:

   $$\begin{array}{c}
   U \xrightarrow{e} \mathcal{R} \\
   \downarrow \text{id}_U \\
   U
   \end{array} \quad \text{and} \quad \begin{array}{c}
   U \xrightarrow{e} \mathcal{R} \\
   \downarrow \text{id}_U \\
   U
   \end{array}$$

2. Multiplication is compatible with both $s$ and $t$:

   $$\begin{array}{c}
   \mathcal{R} \xrightarrow{\pi_1} \mathcal{R} \\
   \downarrow m \\
   \mathcal{R}
   \end{array} \quad \text{and} \quad \begin{array}{c}
   \mathcal{R} \xrightarrow{\pi_2} \mathcal{R} \\
   \downarrow m \\
   \mathcal{R}
   \end{array}$$

3. Associativity:

   $$\begin{array}{c}
   \mathcal{R} \xrightarrow{\text{id}_R \otimes m} \mathcal{R} \xrightarrow{m} \mathcal{R} \\
   \downarrow \text{id}_R \otimes m \\
   \mathcal{R}
   \end{array}$$

4. Unit condition:

   $$\begin{array}{c}
   \mathcal{R} \xrightarrow{(e \circ s, \text{id}_R)} \mathcal{R} \xrightarrow{\text{id}_R} \mathcal{R} \\
   \downarrow m \\
   \mathcal{R}
   \end{array} \quad \text{and} \quad \begin{array}{c}
   \mathcal{R} \xrightarrow{(\text{id}_R \circ \text{cot})} \mathcal{R} \xrightarrow{\text{id}_R} \mathcal{R} \\
   \downarrow m \\
   \mathcal{R}
   \end{array}$$
5. **Inverse:**

\[ i \circ i = \text{id}_\mathcal{R} \]

\[ s \circ i = t \]

\[ t \circ i = s \]

with

\[
\begin{array}{ccc}
\mathcal{R} & \xrightarrow{(i, \text{id}_\mathcal{R})} & \mathcal{R}_t \times s \mathcal{R} \\
\downarrow s & & \downarrow m \\
U & \xrightarrow{e} & \mathcal{R}
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\mathcal{R} & \xrightarrow{(i, \text{id}_\mathcal{R})} & \mathcal{R}_t \times s \mathcal{R} \\
\downarrow t & & \downarrow m \\
U & \xrightarrow{e} & \mathcal{R}
\end{array}
\]

We denote the groupoid by \( \mathcal{R} \xrightarrow{\Psi} U := (\mathcal{R}, \mathcal{U}, s, t, e, m, i) \), and the groupoid is called \( \text{étale} \) if the base category is that of locally Hausdorff, paracompact, locally compact topological spaces and the maps \( s, t : \mathcal{R} \to \mathcal{U} \) are local homeomorphisms (diffeomorphisms). We will say that a groupoid is proper if \( s \) and \( t \) are proper maps. From now on we will assume that our groupoids are differentiable, \( \text{étale} \) and proper.

**Example 2.2:** For \( M \) a manifold and \( \{U_\alpha\} \) and open cover, let

\[
\mathcal{U} = \bigcup U_\alpha \quad \mathcal{R} = \bigcup_{(\alpha, \beta)} U_\alpha \cap U_\beta \quad (\alpha, \beta) \neq (\beta, \alpha)
\]

\[
s|_{U_{\alpha\beta}} : U_{\alpha\beta} \to U_\alpha, \quad t|_{U_{\alpha\beta}} : U_{\alpha\beta} \to U_\beta \quad e|_{U_\alpha} : U_\alpha \to U_\alpha
\]

\[
i|_{U_{\alpha\beta}} : U_{\alpha\beta} \to U_{\beta\alpha} \quad m|_{U_{\alpha\beta\gamma}} : U_{\alpha\beta\gamma} \to U_{\alpha\gamma}
\]

the natural maps. Note that in this example \( \mathcal{R}_t \times s \mathcal{R} \) coincides with the subset of \( \mathcal{R}_t \times s \mathcal{R} \) of pairs \((u, v)\) so that \( t(u) = s(v) \), namely the disjoint union of all possible triple intersections \( U_{\alpha\beta\gamma} \) of open sets in the open cover \( \{U_\alpha\} \).

**Example 2.3:** Let \( G \) be a group and \( U \) a set provided with a left \( G \) action \( G \times U \to U \)

\[
(g, u) \to gu
\]

we put \( \mathcal{U} = U \) and \( \mathcal{R} = G \times U \) with \( s(g, u) = u \) and \( t(g, u) = gu \). The domain of \( m \) is the same as \( G \times G \times U \) where \( m(g, h, u) = (gh, u) \), \( i(g, u) = (g^{-1}, gu) \) and \( e(u) = (\text{id}_G, u) \).

We will write \( G \times U \xrightarrow{\Psi} U \) (or sometimes \( X = [U/G] \)) to denote this groupoid. When \( U = \ast \) is a point, we denote the groupoid by \( \overline{G} \).

**Definition 2.4:** A morphism of groupoids \((\Psi, \psi) : (\mathcal{R}' \xrightarrow{\Psi} \mathcal{U}') \to (\mathcal{R} \xrightarrow{\psi} \mathcal{U})\) are the following commutative diagrams:

\[
\begin{array}{ccc}
\mathcal{R}' & \xrightarrow{\Psi} & \mathcal{R} \\
\downarrow s' & & \downarrow t \\
\mathcal{U}' & \xrightarrow{\psi} & \mathcal{U}
\end{array}
\quad \begin{array}{ccc}
\mathcal{R}' & \xrightarrow{\Psi} & \mathcal{R} \\
\downarrow e' & & \downarrow e \\
\mathcal{U}' & \xrightarrow{\psi} & \mathcal{U}
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{R}'_t \times s' \mathcal{R}' & \xrightarrow{\Psi} & \mathcal{R}_t \times s \mathcal{R} \\
\downarrow m' & & \downarrow m \\
\mathcal{R}' & \xrightarrow{\psi} & \mathcal{R}
\end{array}
\quad \begin{array}{ccc}
\mathcal{R}' & \xrightarrow{\Psi} & \mathcal{R} \\
\downarrow i' & & \downarrow i \\
\mathcal{R}' & \xrightarrow{\psi} & \mathcal{R}
\end{array}
\]

Now we need to say when two groupoids are “equivalent”

**Definition 2.5:** A morphism of groupoids \((\Psi, \psi)\) is called a Morita morphism whenever:
- The map $s \circ \pi_2 : \mathcal{U}_\psi \times \mathcal{R} \to \mathcal{U}$ is an étale surjection.

- The following square is a fibered product

\[
\begin{array}{ccc}
\mathcal{R}' & \xrightarrow{\Psi} & \mathcal{R} \\
(s', t') \downarrow \downarrow \downarrow & & \downarrow (s, t) \downarrow \\
\mathcal{U} \times \mathcal{U}' & \xrightarrow{\psi \times \psi} & \mathcal{U} \times \mathcal{U}
\end{array}
\]

Two groupoids $\mathcal{R}_1 \xrightarrow{\pi} \mathcal{U}_1$, $\mathcal{R}_2 \xrightarrow{\pi} \mathcal{U}_2$ are called Morita equivalent if there are Morita morphisms $(\Psi_i, \psi_i) : \mathcal{R}_i \xrightarrow{\pi} \mathcal{U}_i$ for $i = 1, 2$. This is an equivalence relation and in general we will consider the category of étale groupoids obtained by formally inverting the Morita equivalences (see [?] for details).

Let $X$ be an orbifold and $\{(V_p, G_p, \pi_p)\}_{p \in X}$ its orbifold structure, the groupoid $\mathcal{R} \xrightarrow{\pi} \mathcal{U}$ associated to $X$ will be defined as follows: $\mathcal{U} := \bigsqcup_{p \in X} V_p$ and an element $g : (v_1, V_1) \to (v_2, V_2)$ (an arrow) in $\mathcal{R}$ with $v_i \in V_i$, $i = 1, 2$, will be a equivalence class of triples $g = [\lambda_1, w, \lambda_2]$ where $w \in W$ for another uniformizing system $(W, H, \rho)$, and the $\lambda_i$’s are injections $(\lambda_i, \phi_i) : (W, H, \rho) \to (V_i, G_i, \pi_i)$ with $\lambda_i(w) = v_i$, $i = 1, 2$.

For another injection $\gamma, \psi : (W', H', \rho') \to (W, H, \rho)$ and $w' \in W'$ with $\gamma(w') = w$ then $[\lambda_1, w, \lambda_2] = [\lambda_1 \circ \gamma, w', \lambda_2 \circ \gamma]$. For another injection $\gamma, \psi : (W', H', \rho') \to (W, H, \rho)$ and $w' \in W'$ with $\gamma(w') = w$ then $[\lambda_1, w, \lambda_2] = [\lambda_1 \circ \gamma, w', \lambda_2 \circ \gamma]$. Now the maps $s, t, e, i, m$ are naturally described:

\[
s(\lambda_1, w, \lambda_2) = (\lambda_1(w), V_1), \quad t(\lambda_1, w, \lambda_2) = (\lambda_2(w), V_2) \quad e(x, V) = [id_V, x, id_V]
\]

\[
i(\lambda_1, w, \lambda_2) = [\lambda_2, w, \lambda_1] \quad m(\lambda_1, w, \lambda_2, [\mu_1, z, \mu_2]) = [\lambda_1 \circ \nu_1, y, \mu_2 \circ \nu_2]
\]

where $h = [\nu_1, y, \nu_2]$ is an arrow joining $w$ and $z$ (i.e. $\nu_1(y) = w \& \nu_2(y) = z$).

It can be given a topology to $\mathcal{R}$ so that $s, t$ will be étale maps, making it into a proper, étale, differentiable groupoid, and it is not hard to check that all the properties of groupoid are satisfied. This is a good place to note that an orbifold $X$ given by a groupoid $\mathcal{R} \to \mathcal{U}$ will be a smooth manifold if and only if the map $(s, t) : \mathcal{R} \to \mathcal{U} \times \mathcal{U}$ is one-to-one.

We want the morphism between orbifolds to be morphisms of groupoids, and this is precisely the case for the good maps given in [CR2].

**Proposition 2.6 [LU1]:** A morphism of groupoids induces a good map between the underlying orbifolds, and conversely, every good map arises in this way.

### 2.2 Orbispace

This section was taken from [□]. Let $U$ be a connected, locally connected topological space. A $G$-structure on $U$ is a triple $(\hat{U}, G_U, \pi_U)$ where $(\hat{U}, G_U)$ is a connected, locally connected $G$-space and $\pi_U : \hat{U} \to U$ is a continuous map inducing a homeomorphism between the orbit space $\hat{U}/G_U$ and $U$. An isomorphism between two $G$-structures on $U$, $(\hat{U}_i, G_{i, U}, \pi_{i, U})$ for $i = 1, 2$ is a pair $(\phi, \lambda)$ where $\lambda : G_{1, U} \to G_{2, U}$ is an isomorphism and $\phi : \hat{U}_1 \to \hat{U}_2$ is a $\lambda$-equivariant homeomorphism such that $\pi_{2, U} \circ \phi = \pi_{1, U}$. Note that each $g \in G_U$ induces an automorphism $(\phi_g, \lambda_g)$ on $(\hat{U}, G_U, \pi_U)$, defined by setting $\phi_g(x) = g \cdot x$, $\forall x \in \hat{U}$ and $\lambda_g(h) = ghg^{-1}$, $\forall h \in G_U$. However, it might not be true that every automorphism arises in this way. We shall only take into consideration the automorphisms $(\phi_g, \lambda_g)$, $g \in G_U$. More precisely, we define the automorphism group of the $G$-structure $(\hat{U}, G_U, \pi_U)$ to be $G_U$ via the induced isomorphisms $(\phi_g, \lambda_g)$ on it. We would like to point out that since the
action of $G_U$ on $\hat{U}$ is not required to be effective, two different automorphisms of the $G$-structure could have the same induced map.

Given a $G$-structure $(\hat{U}, G_U, \pi_U)$ on $U$, we consider the inverse image $\pi_U^{-1}(W)$ in $\hat{U}$, where $W$ is a connected open subset of $U$. Denote by $\hat{W}$ one of the connected components of $\pi_U^{-1}(W)$, by $G_W$ the subgroup of $G_U$ consisting of elements $g \in G_U$ such that $g \cdot \hat{W} = \hat{W}$, and let $\pi_W = (\pi_U)|_{\hat{W}}$.

**Lemma 2.7:** The triple $(\hat{W}, G_W, \pi_W)$ defines a $G$-structure on $W$. Moreover, the action of $G_U$ on $\hat{U}$ induces a transitive action on the set of all such $G$-structures on $W$ for which the following holds: Let $(\hat{W}_i, G_{W,i}, \pi_{W,i})$, $i = 1, 2$, be two such $G$-structures, and $\hat{W}_2 = g \cdot \hat{W}_1$ for some $g \in G_U$, then $G_{W,2} = gG_{W,1}g^{-1}$ in $G_U$. The stabilizer of the action at $G$-structure $(\hat{W}, G_W, \pi_W)$ is precisely the subgroup $G_W$ in $G_U$.

We will say that $(\hat{W}, G_W, \pi_W)$ is induced from the $G$-structure $(\hat{U}, G_U, \pi_U)$. Note that the subgroup $G_W$ is both closed and open in $G_U$. In fact, the space of cosets $G_U/G_W = \{gG_W | g \in G_U\}$ inherits a discrete topology from $G_U$.

**Transition maps.** Let $U_1$, $U_2$ be connected open subsets of a locally connected topological space with $U_1 \cap U_2 \neq \emptyset$. Suppose $U_1$ and $U_2$ are given with $G$-structures $(\hat{U}_1, G_{U_1}, \pi_{U_1})$ and $(\hat{U}_2, G_{U_2}, \pi_{U_2})$ respectively. Let $\{W_i | i \in I\}$ be the set of connected components of $U_1 \cap U_2$. We define $Iso_{W_i}(U_1, U_2)$ to be the set of isomorphisms from an element of $G_{U_1}(W_i)$ to an element of $G_{U_2}(W_i)$, and set

$$\text{(2.1)} \quad Iso(U_1, U_2) = \bigsqcup_{i \in I} Iso_{W_i}(U_1, U_2).$$

There is a canonical right action of $G_{U_1} \times G_{U_2}$ on $Iso(U_1, U_2)$ defined by

$$((\phi, \lambda), (g_1, g_2)) \mapsto (\phi, \lambda) \cdot (g_1, g_2) := (g_2^{-1} \circ \phi \circ g_1, Ad(g_2^{-1}) \circ \lambda \circ Ad(g_1)).$$

Note that $\text{Domain}(\lambda)$ is contained in the stabilizer at $(\phi, \lambda)$, through the identification to a subgroup of $G_{U_1} \times G_{U_2}$ by $g \mapsto (g, \lambda(g))$.

**Definition 2.8:** A set of transition maps from $U_1$ to $U_2$ is a topological space $\text{Tran}(U_1, U_2)$ together with a map

$$\text{(2.2)} \quad \chi_{U_1U_2} : \text{Tran}(U_1, U_2) \to Iso(U_1, U_2)$$

satisfying the following conditions:

(a) For any $i \in I$, $\text{Tran}_{W_i}(U_1, U_2) := \chi_{U_1U_2}^{-1}(Iso_{W_i}(U_1, U_2))$ is a non-empty open subset.

(b) There is a continuous right action of $G_{U_1} \times G_{U_2}$ on $\text{Tran}(U_1, U_2)$, written $(\xi, (g_1, g_2)) \mapsto \xi \cdot (g_1, g_2)$, which is transitive when restricted to each $\text{Tran}_{W_i}(U_1, U_2)$, such that the map $\chi_{U_1U_2}$ is $G_{U_1} \times G_{U_2}$-equivariant.

(c) The stabilizer at $\xi \in \text{Tran}(U_1, U_2)$, $\text{Stab}(\xi)$, is the image of the injective homomorphism $\text{Domain}(\lambda) \to G_{U_1} \times G_{U_2}$ defined by $g \mapsto (g, \lambda(g))$ where $\lambda$ is given in $\chi_{U_1U_2}(\xi) = (\phi, \lambda)$.

(d) For any $i \in I$ and $\xi \in \text{Tran}_{W_i}(U_1, U_2)$, the map $G_{U_1} \times G_{U_2} \to \text{Tran}_{W_i}(U_1, U_2)$ defined by $(g_1, g_2) \mapsto \xi \cdot (g_1, g_2)$ induces a homeomorphism between the space of left cosets $(G_{U_1} \times G_{U_2})/\text{Stab}(\xi)$ and $\text{Tran}_{W_i}(U_1, U_2)$.
Each element of $\text{Tran}(U_1, U_2)$ is called a transition map.

As an example, let us consider the case when $U_1 = U_2 = U$. Define a map $\chi_U : G_U \to \text{Iso}(U, U)$ by

\begin{equation}
\chi_U(g) = (g, \text{Ad}(g)),
\end{equation}

and a right action of $G_U \times G_U$ on $G_U$ by $g \cdot (g_1, g_2) = g_2^{-1}gg_1$. Then one can easily verify that $(G_U, \chi_U)$ satisfies Definition 2.8, hence defines a set of transition maps from $U$ to $U$.

The composition of transition maps is a map $\text{Tran}(U_1, U_2) \times \text{Tran}(U_2, U_3) \to \text{Tran}(U_1, U_3)$, written $(\xi_{12}, \xi_{23}) \mapsto \xi_{23} \circ \xi_{12}$, such that $\chi_{U_2U_3}(\xi_{23}) \circ \chi_{U_1U_2}(\xi_{12})$ equals $\chi_{U_1U_3}(\xi_{23} \circ \xi_{12})$ when restricted to the intersection of their domains. However, this could be problematic in general. First of all, in order for $\chi_{U_2U_3}(\xi_{23}) \circ \chi_{U_1U_2}(\xi_{12})$ to be defined, the following condition

\begin{equation}
\text{Range}(\chi_{U_1U_2}(\xi_{12})) \cap \text{Domain}(\chi_{U_2U_3}(\xi_{23})) \neq \emptyset
\end{equation}

is necessary, which does not hold in general. Secondly, the inverse image of the left-hand side of (2.4) under $\chi_{U_1U_2}(\xi_{12})$, which is a subset of $G_{U_1}(U_1 \cap U_2 \cap U_3)$, could be contained in more than one element of $G_{U_1}(U_1 \cap U_3)$. In this case, the composition map will pick up one of these elements of $G_{U_1}(U_1 \cap U_3)$ for the domain of $\chi_{U_1U_3}(\xi_{23} \circ \xi_{12})$, which seems quite unnatural.

Because of these reasons, we shall impose composition maps $\text{Tran}(U_1, U_2) \times \text{Tran}(U_2, U_3) \to \text{Tran}(U_1, U_3)$ only when the following condition for $U_1, U_2, U_3$ is met

\begin{equation}
\begin{align*}
(1) & \quad U_1 \subset U_2 \subset U_3 \quad \text{or} \\
(2) & \quad U_3 \subset U_2 \subset U_1 \quad \text{or} \\
(3) & \quad U_2 \subset U_1, U_2 \subset U_3.
\end{align*}
\end{equation}

It turns out that (2.6) is sufficiently rich so that the composition of transition maps in general can be derived from it, but on the other hand, (2.6) is also minimal that none of the cases in (2.6) may be removed.

With these preparations, we now introduce

**Definition 2.9:** Let $X$ be a locally connected topological space. An orbispace structure on $X$ is a set $\mathcal{U}$ of connected open subsets of $X$ satisfying the following conditions:

1. For any $U_1, U_2 \in \mathcal{U}$ with $U_1 \cap U_2 \neq \emptyset$, each connected component of $U_1 \cap U_2$ is also in $\mathcal{U}$.

2. Each element $U$ of $\mathcal{U}$ is assigned with a $G$-structure $(\hat{U}, G_U, \pi_U)$ satisfying the following conditions:

   a) Any ordered pair of $U_1, U_2 \in \mathcal{U}$ with $U_1 \cap U_2 \neq \emptyset$ is assigned with a set of transition maps $(\text{Tran}(U_1, U_2), \chi_{U_1U_2})$ as defined in Definition 2.8.

   b) For any $U_1, U_2, U_3 \in \mathcal{U}$ satisfying (2.6), there is a composition map $\text{Tran}(U_1, U_2) \times \text{Tran}(U_2, U_3) \to \text{Tran}(U_1, U_3)$, written $(\xi_{12}, \xi_{23}) \mapsto \xi_{23} \circ \xi_{12}$, such that $\chi_{U_2U_3}(\xi_{23}) \circ \chi_{U_1U_2}(\xi_{12})$ equals $\chi_{U_1U_3}(\xi_{23} \circ \xi_{12})$ when restricted to the domain of the latter. Moreover, the following condition is satisfied: for any $g_1 \in G_{U_1}$, $g_2, g_2' \in G_{U_2}$, and $g_3 \in G_{U_3}$,

   \begin{align*}
   (\xi_{23} \cdot (g_2, g_3)) \circ (\xi_{12} \cdot (g_1, g_2)) &= ((\xi_{23} \cdot (g_2g_2'^{-1}, 1_{G_{U_3}})) \circ \xi_{12}) \cdot (g_1, g_3) \\
   &= (\xi_{23} \circ (\xi_{12} \cdot (1_{G_{U_1}}, g_2(g_2'^{-1})))) \cdot (g_1, g_3).
   \end{align*}

   Combined with Definition 2.8 (d), this condition implies that the composition map is continuous.
Example 2.10: For any locally connected orbispace with a trivial orbispace structure, an orbispace $(X,\mathcal{U},G)$ and write for each trivial if $U$ is an orbispace structure, of which $x$ is isomorphic groups because of the existence of transition maps. An orbispace structure $G$ used. The reason for which we choose such a formulation lies in the following facts: First of all, the stabilizer $G_x$ is taken to be the set of all connected open subsets of the orbit $Y/G$. Each element $U$ of $\mathcal{U}$ is a fixed choice of the local group $G_x$ of $x$ in $G_U$ (i.e., $G_x = \{g \in G_U | g \cdot x = x\}$), and denote it by $G_p$. Clearly different choices of $x$ result in the same conjugacy class in $G_U$, and different choices of $U$ give rise to isomorphic groups because of the existence of transition maps. An orbispace structure $\mathcal{U}$ is called trivial if $G_U$ is trivial for each $U \in \mathcal{U}$. (Every locally connected topological space is canonically an orbispace with a trivial orbispace structure.) An orbispace $(X,\mathcal{U})$ is called orbifold if $G_U$ is finite for each $U \in \mathcal{U}$. As a notational convention, we very often only write $X$ for an orbispace $(X,\mathcal{U})$, and write $X_{top}$ for the underlying topological space $X$ for simplicity.

Example 2.10: For any locally connected $G$-space $(Y,G)$, the orbit space $Y/G$ canonically inherits an orbispace structure, of which $\mathcal{U}$ is taken to be the set of all connected open subsets of the orbit space $Y/G$. The $G$-structure assigned to each element of $\mathcal{U}$ is a fixed choice of the $G$-structures induced from the God-given $G$-structure $(Y,G)$ on the orbit space $Y/G$. Each set of transition maps is obtained by restricting $G$ to the corresponding induced $G$-structures. The verification of Definition 2.8 for this case is straightforward. An orbispace will be called global if it arises as the orbit space of a $G$-space equipped with the canonical orbispace structure as discussed in this example.

Remark 2.11: When a basic open set $U_\alpha$ is included in another one $U_\beta$, a transition map in $\text{Tran}(U_\alpha,U_\beta)$ is just an isomorphism from the $G$-structure of $U_\alpha$ onto one of the $G$-structures of $U_\beta$ induced from the $G$-structure of $U_\beta$. We will call a transition map in $\text{Tran}(U_\alpha,U_\beta)$ an injection when $U_\alpha \subset U_\beta$ holds.

Remark 2.12: In order to minimize the dependence of orbispace structure on the choice of the set $\mathcal{U}$ of basic open sets so that it becomes more intrinsic, it is appropriate to introduce an equivalence relation as follows: Two orbispace structures are said to be directly equivalent if one is contained in the other. The equivalence relation to be introduced is just a finite chain of direct equivalence.

Remark 2.13: The definition of good map/morphism can be modified from that of orbifold in an obvious fashion.

In a certain sense, the definition of orbispace structure in this paper is of a less intrinsic style than the definition of orbifold structure originally used in [3], where the language of germs was used. The reason for which we choose such a formulation lies in the following facts: First of all, an automorphism of a $G$-structure $(\hat{U},G_U,\pi_U)$, i.e., a pair $(\phi,\lambda)$ where $\lambda \in \text{Aut}(G_U)$ and $\phi$ is
a $\lambda$-equivariant homeomorphism of $\tilde{U}$ inducing identity map on $U$, might not be induced by an action of $G_U$ on $\tilde{U}$ (at least we have a burden of proving so). But we want to only consider those arising from the action of $G_U$. Secondly, suppose $W$ is a connected open subset of $U$, both of which are given with $G$-structures $(\tilde{W}, G_W, \pi_W)$ and $(\tilde{U}, G_U, \pi_U)$ respectively. Then an open embedding $\phi : \tilde{W} \to \tilde{U}$, which is $\lambda$-equivariant for some monomorphism $\lambda : G_W \to G_U$ and induces the inclusion $W \hookrightarrow U$, might not be an isomorphism onto one of the $G$-structures of $W$ induced from $(\tilde{U}, G_U, \pi_U)$. However, this kind of pathology can be ruled out if each $\tilde{U}$ is Hausdorff, $G_U$ acts on $\tilde{U}$ effectively.

3 Cohomology ring of crepant resolutions

Suppose that $X$ is an orbifold. In general, $K_X$ is an orbifold vector bundle or a $Q$-divisor only. When the $X$ is so called Gorenstein, $K_X$ is a bundle or a divisor. For the Gorenstein orbifold, a resolution $\pi : Y \to X$ is called a crepant resolution if $\pi^* K_X = K_Y$. Here, ”crepant” can be viewed as a minimality condition with respect to the canonical bundle. A crepant resolution always exists when dimension is two or three. A nice way to construct it is to use Hilbert scheme of points. However, the crepant resolution in dimension three is not unique. Different crepant resolutions are connected by flops. When the dimension is bigger than four, the crepant resolution does not always exist. It is an extremely interesting problem in algebraic geometry to find out when it does exist.

When the orbifold string theory was first constructed over the global quotient $[DHJVW]$, one of its first invariant is orbifold Euler characteristic. It was conjectured that the orbifold Euler characteristic is the same as the Euler characteristic of its crepant resolution. This fits well with McKay’s correspondence in algebro-geometry. It had been the main attraction before the current development. By the work of Batyrev and others $[HL, DL]$, this conjecture has been extended and solved for the orbifold Hodge number of Gorenstein global quotients. Very recently, it was solved in the complete generalities by Lupercio-Poddar $[LP]$ and Yasuda $[Y]$. Among all the examples, Hilbert scheme of points of algebraic surfaces are particularly interesting. Suppose that $M$ is an algebraic surface. We use $M^{[n]}$ to denote the Hilbert scheme of points of length $n$ of $M$. In his thesis $[G]$, Göttsche computed the generating function of the Euler number $\sum_{n=1}^{\infty} \chi(M^{[n]})q^n$ and showed that it has a surprising modularity. In 1994, in order to explain its modularity, Vafa-Witten $[VW]$ computed $\mathcal{H} = \oplus_n H^*(M^n/S_n, \mathbb{C})$. Motivated by the orbifold conformal field theory, they directly wrote $\mathcal{H}$ as a ”Fock space” or a representation of the Heisenberg algebra. Then, the generating function of the Euler characteristic is interpreted as the correlation function of an elliptic curve. Therefore, it should be invariant under the modular transformations of the elliptic curve. This shows that the space of cohomology itself has more structures. The orbifold string theory conjecture predicates that $\oplus_n H^*(M^{[n]}, \mathbb{C})$ should also admit a representation of the Heisenberg algebra. This conjecture was verified by a beautiful work of Nakajima $[N]$ and others.

The ring structure of $M^{[n]}$ is quite subtle and more interesting. Partial results have been obtained by Fantechi-Göttsche $[FG]$, Ellingsrud-Stromme $[ES1, ES2]$, Beauvill $[Bea]$, Mark $[Mar]$. Based on an important observation by Frenkel and Wang $[FW]$, Lehn-Sorger $[LS1]$ determined the cohomology ring of $(\mathbb{C}^2)^{[n]}$. At the same time, the author was computing orbifold cohomology $(\mathbb{C}^2)^{n}/S_n$ and the result from both calculations match perfectly. Based on the physical motivation and the strong evidence from $(\mathbb{C}^2)^{n}/S_n$, the author proposed $[R2]$ a conjecture in the case of a hyperkahler resolution.

**Cohomological Hyperkahler Resolution Conjecture:** Suppose that $\pi : Y \to X$ is a hyperkahler resolution. Then, the ordinary cohomology ring of $Y$ is isomorphic to the orbifold cohomology
ring of $X$.

In the case of the Hilbert scheme points of surfaces, the Cohomological Hyperkahler Resolution Conjecture (CHRC) implies that $K^3[n], (T^4)^n$ have an isomorphic cohomology ring as the orbifold cohomology rings of $K^3n/S_n, (T^4)^n/S_n$. The later was proved recently by the beautiful works of Lehn-Sorge $^{[LS2]}$, Fantechi-Göttsche $^{[FG2]}$ and Uribe $^{[U]}$. It should be mentioned that Fantechi-Göttsche-Uribe’s work computed the orbifold ring structure of $X^n/S_n$ for an arbitrary complex manifold $X$ which may or may not be $K^3, T^4$. There is a curious phenomenon that over rational number Lehn-Sorge, Fantanch-Göttsche and Uribe showed that one must modify the ring structure of the orbifold cohomology by a sign in order to match to the cohomology of Hilbert scheme. However, Qin-Wang observed that such a sign modification is unnecessary over complex number $^{[QW]}$. All the conjectures stated in this article (in fact any conjecture motivated by physics) are the statements over complex number.

In a different direction, the Hilbert scheme can be regarded as a moduli space. For a moduli space, there is a conjectural Mumford principal that the Künneth components of the Chern characters of the universal sheaf form a set of ring generators. Such a Mumford principal was well-known in the Donaldson theory. For the Hilbert scheme of points, the Mumford principal was proved by Li-Qin-Wang $^{[LQW]}$ which is instrumental in Lehn-Sorger’s calculation of $M^n$ for $M = K^3, T^4$ (see $^{[LQW]}$ for another set of generators). However, it is quite difficult to determine the relations for Li-Qin-Wang generators. The ring structure of $X^n$ for a general algebraic surface $X$ is still unknown.

It is easy to check that CHRC is false if we drop the hyperkahler condition. Motivated by physics and the work of the author with An-Min Li on the quantum cohomology and flop, the author proposed a conjecture for the arbitrary crepant resolution.

As mentioned previously, the crepant resolutions are not unique. The different crepant resolutions are connected by "K-equivalence" $^{[W]}$. Two smooth (or Gorenstein orbifolds) complex manifolds $X, Y$ are $K$-equivalent iff there is a common resolution $\phi, \psi : Z \to X, Y$ such that $\phi^* K_X = \psi^* K_Y$. Batyrev-Wang $^{[B], [W]}$ showed that two $K$-equivalent projective manifolds have the same betti number. It is natural to ask if they have the same ring structures. This question is obviously related to CHRC. Suppose that CHRC holds for non-hyperkahler resolutions. It implies that different resolutions (K-equivalent) have the same ring structures. Unfortunately, they usually have different ring structures, and hence CHRC fails in general. It is easy to check this in case of three dimensional flops. A key idea to remedy the situation is to include the quantum corrections. The author proposed $^{[R1]}$

**Quantum Minimal Model Conjecture:** Two $K$-equivalent projective manifolds have the same quantum cohomologies.

Li and the author proved Quantum Minimal Model conjecture in complex dimension three. In higher dimensions, it seems to be a difficult problem. In many ways, Quantum Minimal Model Conjecture unveils the deep relation between the quantum cohomology and the birational geometry $^{[R1]}$. However, it is a formidable task to master the quantum cohomology machinery for any non-experts. In $^{[R3]}$, the author proposed another conjecture focusing on the cohomology instead of the quantum cohomology. As mentioned before, the cohomology ring structures are not isomorphic for $K$-equivalent manifolds. Therefore, some quantum information must be included. The new conjecture requires a minimal set of quantum information involving the GW-invariants of the exceptional rational curve.
3.1 Orbifold cohomology of symmetry product

The orbifold cohomology of symmetry product is a beautiful subject. In this section, we first recall Vafa-Witten’s calculation of the orbifold cohomology group \( H = \oplus_n H^*_\text{orb}(X^n/S_n, \mathbb{R}) \) and directly write it as a "Fock space" or the representation of super Heisenberg algebra. Then, we sketch Fantechi-Göttsche [FG2] and Uribe [U]'s calculation of the ring structure.

Let \( X \) be an almost complex manifold and \( S_n \) be the symmetry group on \( n \)-letters. Then, \( X^n/S_n \) is an almost complex orbifold and its orbifold cohomology \( H^*_\text{orb}(X^n/S_n, \mathbb{R}) \) is well-defined.

**Theorem 3.1:** \( H = \oplus_n H^*_\text{orb}(X^n/S_n, \mathbb{R}) \) is an irreducible representation of the super Heisenberg algebra.

**Proof:** We first compute the cohomology of the nontwisted sector \( H^*(X^n/S_n, \mathbb{R}) \). It is easy to see that \( H^*(X^n/S_n, \mathbb{R}) = H^*(X^n, \mathbb{R})^{S_n} \). Pick a basis \( w^a \) of the cohomology of \( X \). Then, \( w^{a_1} \otimes w^{a_2} \otimes \cdots \otimes w^{a_n} \in H^*(X^n, \mathbb{R}) \). We obtain a class of \( H^*(X^n, \mathbb{R})^{S_n} \) by symmetrizing \( w^{a_1} \otimes w^{a_2} \otimes \cdots \otimes w^{a_n} \). We denote this class using symbol \( \alpha^a_{a_1} \cdots \alpha^a_{a_n} |0> \). This is a physical notation. \( 0 > \) is called the vacuum representing the cohomology of \( H^*(X^0/S_0, \mathbb{R}) \) and \( \alpha^a_{-1} \) is called a creation operator. We think formally that \( \alpha^a_{-1} \) acts on vacuum \( |0> \) to create an element of \( H^*_\text{orb}(X^n/S_n, \mathbb{R}) \) called 1-particle state. The creation operators satisfy commutation relation \( \alpha^a_{-1} \alpha^b_{-1} = (-1)^{deg(a)deg(b)} \alpha^a_{-1} \alpha^b_{-1} \). We will see this commutation relation again when we compute the cohomology of twisted sector.

The twisted sector is given by the connected components of \( X^n/C(g) \) for the conjugacy class \( g \). It is well-known that a conjugacy class is uniquely determined by the cycle decomposition \( g = i^{n_1} 2^{n_2} \cdots k^{n_k} \) where \( i^{n_i} \) means the \( n_i \)-many cycles of length \( i \). It is clear that \( n = \sum_i n_i \).

The fixed point loci \( X^n_g = X^{n_1} \times \cdots \times X^{n_k} \). The centralizer \( C(g) = S_{n_1} \times Z_{1} \cdots \times Z_{n_k} \). Hence, as a topological space, the twisted sector \( X^n_g/C(g) = X^{n_1}/S_{n_1} \cdots \times X^{n_k}/S_{n_k} \) although it has different orbifold structure given by the extra group \( Z_{1} \cdots Z_{n_k} \). The cohomology of the factor \( X^{n_i}/S_{n_i} \) is called the k-particle state. It can be written down by the same method of the nontwisted sector. We use a lower indices \( \alpha^a_{-i} \) to denote it. A cohomology class of a twisted sector can be written as

\[
\alpha^a_{-1} \cdots \alpha^a_{-1} \alpha^a_{-2} \cdots \alpha^a_{-k} |0>.
\]

When we consider the direct sum \( H = \oplus_n H^*_\text{orb}(X^n/S_n, \mathbb{R}) \), the constraint \( n = \sum_i n_i \) becomes a trivial condition. Hence, we conclude that the elements of the form (3.1) forms a basis of \( H \).

If you are a physicist, you stop here and claim \( H \) is a "Fock space" or an irreducible representation of super Heisenberg algebra. Since a common knowledge in physics may not be a common knowledge in mathematics, let me reconstruct the action of the Heisenberg algebra which is motivated by harmonic oscillator.

Before we reconstruct the action of super Heisenberg algebra, we compute the degree shifting number. Then, we will see that the theory is different when \( \dim X \) is even or odd. For each \( j \)-cycle, its action on \( (C^N)^j \) has eigenvalues \( N \)-copy of \( e^{2\pi i p/j} \), for \( p = 0, \cdots, j \). Therefore, its contribution to degree shifting number is \( \frac{1}{j}N \). Let \( deg(g) \) be the minimal number of transpositions to express \( g \) as the composition of transpositions. It is clear that \( deg(g) = \sum_i n_i (i-1) \) and degree shifting number \( \iota(g) = \frac{N}{j} deg(g) \). Notes that when \( N \) is even, \( \iota(g) \) is an integer. Otherwise, it is a fraction. In particular, when \( N = 2 \), \( \iota(g) = deg(g) \).

To construct the action of the full super Heisenberg algebra, we need to add so called annihilation operators. Let \( H \) be a superspace, i.e., \( H = H_{even} \oplus H_{odd} \). We further assume that \( H \) is equipped with an inner product such as Poincare paring. For any \( a \in H_{even} \), we define \( deg(a) = 0 \) and \( deg(b) = 1 \) if \( b \in H_{odd} \). A super Heisenberg algebra is the set of operators \( \alpha^a_l \) for \( a \in H, l \in \mathbb{Z} - \{0\} \).
We define super commutator

\begin{equation}
\{\alpha^a_l, \alpha^b_m\} = \alpha^a_l \alpha^b_m - (-1)^{\deg(a) \deg(b)} \alpha^b_m \alpha^a_l.
\end{equation}

Then, the operators $\alpha^a_l$ satisfies relation

\begin{equation}
\{\alpha^a_l, \alpha^b_m\} = -l \delta_{l+m,0} < a, b > Id.
\end{equation}

The operators $\alpha^a_l$ for $l > 0$ are called annihilation operators and $\alpha^a_l$ for $l < 0$ are called creation operators. The action of the annihilation operators on $H$ is determined by the commutation relation of (3.3) and

\begin{equation}
\alpha^a_l |0> = 0
\end{equation}

for $l > 0$. To recover the Euler characteristic, we define operator $L_0$ by the condition

\begin{equation}
L_0|0> = 1, \{L_0, \alpha^a_l\} = -l \alpha^a_l.
\end{equation}

Its character

\begin{equation}
trq^{L_0} = \sum q^n (\dim(n - \text{even eigenspace}) - \dim(n - \text{odd eigenspace})).
\end{equation}

From the general theory of Heisenberg representation, $trq^{L_0}$ is a modular form.

It is easy to check that

\begin{equation}
L_0(\alpha^a_{-l_1} \cdots \alpha^a_{-l_k}|0>) = \sum_i l_i \alpha^a_{-l_1} \cdots \alpha^a_{-l_k}|0>.
\end{equation}

Let $n = \sum_i$. Then, $\alpha^a_{-l_1} \cdots \alpha^a_{-l_k}|0> \in H^*_\text{orb}(X^n/S_n, \mathbb{R})$. Furthermore, when the dimension of $X$ is even, the degree shifting does not change the even/odd property of orbifold cohomology classes for its original degree. Hence,

\begin{equation}
q^{L_0} = \sum_n q^n \chi(H^*_\text{orb}(X^n/S_n, \mathbb{R})).
\end{equation}

An routine calculation shows that it equals to

\begin{equation}
\prod_n \frac{1}{(1 - q^n)\chi(X)}.
\end{equation}

Next, we calculate its ring structure. Although their goal is to compute the cohomology of the Hilbert scheme of points, the algebraic structure set up by Lehn-Sorger [LS2] is very convenient for the computation of the orbifold cohomology of a symmetry product. The actual computation of the orbifold product was carried out independently by Fantechi-Göttsche [FG2] and Uribe [U].

Before we carry out the computation, we take the advantage of the global quotient to reformulate the orbifold cohomology slightly.

Let $Y = X/G$ be an orbifold with $X$ a compact complex manifold. As before $X^g$ will denote the fixed point set of the action of $g$ on $X$.

The cohomology classes will be labeled by elements in $G$ and let the total ring $A(X, G)$ be

\begin{equation}
A(X, G) := \bigoplus_{g \in G} H^*(X^g; \mathbb{C}) \times \{g\}.
\end{equation}
Its group structure is the natural one and the ring structure that will be defined later will give us the orbifold cup product. The grading is the one in the orbifold cohomology, i.e.

\[ A^d(X, G) = \bigoplus_{g \in G} H^{d-2\iota(g)}(X^g; \mathbb{C}) \times \{g\} \]

For \( h \in G \) there is a natural map \( h : X^g \to X^{gh^{-1}} \) which can be extended to an action in \( A(X, G) \) inducing an isomorphism

\[ h : H^*(X^g; \mathbb{C}) \times \{g\} \to H^*(X^{gh^{-1}}; \mathbb{C}) \times \{hgh^{-1}\} \]

\( (\alpha, g) \mapsto ((h^{-1})^*\alpha, hgh^{-1}) \)

The invariant part under the action of \( G \) is isomorphic as a group to the orbifold cohomology,

\[ A(X, G)^G \cong \bigoplus_{(g)} H^*(X^g; \mathbb{C})^{C(g)} \cong H^*_{orb}(X/G; \mathbb{C}) \]

The construction of orbifold cup product use quotient \( X^g/C(g) \). We can obviously lift it to the fixed point set.

For \( X^{\langle h_1, h_2 \rangle} \), the fixed point set of \( \langle h_1, h_2 \rangle \), let

\[ f^{h_i,\langle h_1, h_2 \rangle} : H^*(X^{h_i}; \mathbb{C}) \to H^*(X^{\langle h_1, h_2 \rangle}; \mathbb{C}) \]

\[ \tilde{f}_{\langle h_1, h_2 \rangle, h_i} : H^*(X^{\langle h_1, h_2 \rangle}; \mathbb{C}) \to H^*(X^{h_i}; \mathbb{C}) \]

be the pull-back and the push-forward respectively of the diagonal inclusion map \( X^{\langle h_1, h_2 \rangle} \hookrightarrow X^{h_i} \) where \( i = 1, 2, 3 \) and \( h_3 = (h_1 h_2) \).

We need to make use of the obstruction bundle over \( X^{\langle h_1, h_2 \rangle} \); as \( Y_{\langle h_1, h_2 \rangle} = X^{\langle h_1, h_2 \rangle}/C(h_1, h_2) \) and taking the projection map \( \pi : X^{\langle h_1, h_2 \rangle} \to X^{\langle h_1, h_2 \rangle}/C(h_1, h_2) \) we will consider the Euler class of \( \pi^*(E_{(h)}) \).

Let the product \( A(X, G) \otimes A(X, G) \to A(X, G) \) be defined by

\[ (\alpha, h_1) \cdot (\beta, h_2) := f_{\langle h_1, h_2 \rangle, h_1 h_2} \left( f^{h_1,\langle h_1, h_2 \rangle}(\alpha) \wedge f^{h_2,\langle h_1, h_2 \rangle}(\beta) \wedge \pi^*c(E_{(h)}) \right) \]

whose three point function is

\[ <(\alpha, h_1), (\beta, h_2), (\gamma, (h_1 h_2)^{-1})> := \int_{X^{\langle h_1, h_2 \rangle}} f^{h_1,\langle h_1, h_2 \rangle}(\alpha) \wedge f^{h_2,\langle h_1, h_2 \rangle}(\beta) \wedge f^{(h_1 h_2)^{-1},\langle h_1, h_2 \rangle}(\gamma) \wedge \pi^*c(E_{(h)}) \]

The product \( A(X, G) \otimes A(X, G) \to A(X, G) \) previously defined is \( G \) equivariant.

This product induces a ring structure on the invariant group \( A(X, G)^G \) which will match with the orbifold cup product. Thus \( A(X, G)^G \) will inherit the properties of the orbifold cup product.

\( X \) will be an even dimensional complex manifold \( dim_{\mathbb{C}}X = 2N \), and the orbifold in mind will be \( X^n/S_n \) where the action of the symmetric group \( S_n \) on \( X^n \) is the natural one.

Now, we introduce more notations. For \( \sigma, \rho \in S_n \), let \( \Gamma \subset [n] := \{1, 2, \ldots, n\} \) be a set stable under the action of \( \sigma \); we will denote by \( O(\sigma; \Gamma) \) the set of orbits induced by the action of \( \sigma \) in \( \Gamma \). If \( \Gamma \) is \( \sigma \)-stable and \( \rho \)-stable, \( O(\sigma, \rho; \Gamma) \) will be the set of orbits induced by \( \langle \sigma, \rho \rangle \). When the set \( \Gamma \) is dropped from the expression, the set \( O(\sigma, [n]) \) will be denoted \( O(\sigma) \).
\(|\sigma|\) will denote the minimum number \(m\) of transpositions \(\tau_1, \ldots, \tau_m\) such that \(\sigma = \tau_1 \cdots \tau_m\); hence
\[
|\sigma| + |O(\sigma)| = n
\]

The set \(X^n\) will denote the fixed point set under the action of \(\sigma\) on \(X^n\). Superscripts on \(X\) will count the number of copies of itself on the cartesian product, and subscripts will be elements of the group and will determine fixed point sets.

For \(h_1, h_2 \in S_n\) the obstruction bundle \(E(h)\) over \(Y_{(h_1, h_2)}\) is defined by
\[
E_{(h)} = \left( H^1(\Sigma) \otimes e^*TY \right)^G
\]
where \(G = (h_1, h_2), Y = X^n/S_n\) and \(\Sigma\) is an orbifold Riemann surface provided with a \(G\) action such that \(\Sigma/G = (S^2, (x_1, x_2, x_3), (k_1, k_2, k_3))\) is an orbifold sphere with three marked points.

Because \(H^1(\Sigma)\) is a trivial bundle, the pullback of \(E(h)\) under \(\pi : X^n \to Y_{(h_1, h_2)}\) is
\[
E_{h_1, h_2} := \pi^*E_{(h)} = \left( H^1(\Sigma) \otimes \Delta^*TX^n \right)^G
\]
where \(\Delta : X^n_{h_1, h_2} \hookrightarrow X^n\) is the inclusion (if \(\rho : X^n \to Y\) is the quotient map, then \(\rho \circ \Delta = e \circ \pi\)).

Without loss of generality we can assume that \(|O(h_1, h_2)| = k\) and \(n_1 + \cdots + n_k = n\) a partition of such that
\[
\Gamma_i = \{ n_1 + \cdots + n_{i-1} + 1, \ldots, n_1 + \cdots + n_i \}
\]
and \(\{\Gamma_1, \Gamma_2, \ldots, \Gamma_k\} = O(h_1, h_2).\) We observe that the obstruction bundle \(E_{h_1, h_2}\) is the product of \(k\) bundles over \(X\) (i.e. \(E_{h_1, h_2} = \prod_i E_{h_1, h_2}^{i}\)), where each factor \(E_{h_1, h_2}^{i}\) corresponds to the orbit \(\Gamma_i\).

For \(\Delta_i : X \to X^{n_i}\) \(i = 1, \ldots, k\) the diagonal inclusions, the bundles \(\Delta_i^*TX^{n_i}\) become \(G\) bundles via the restriction of the action of \(G\) into the orbit \(\Gamma_i\) and
\[
\Delta^*TX^n \cong \Delta_{\Gamma_1}^*TX^{n_1} \times \cdots \times \Delta_{\Gamma_k}^*TX^{n_k}
\]
as \(G\) vector bundles. This comes from the fact that the orbits \(\Gamma_i\) are \(G\) stable, hence \(G\) induces an action on each \(X^{n_i}\). Hence, the obstruction bundle splits as
\[
E_{h_1, h_2} = \prod_{i=1}^k \left( H^1(\Sigma) \otimes \Delta_{\Gamma_i}^*TX^{n_i} \right)^G
\]

We can simplify the previous expression a bit further. Let \(G_i\) be the subgroup of \(S_{n_i}\) obtained from \(G\) when its action is restricted to the elements in \(\Gamma_i\); then we have a surjective homomorphism
\[
\lambda_i : G \to G_i
\]
where the action of \(G\) into \(\Delta_{\Gamma_i}^*TX^{n_i}\) factors through \(G_i\). So we have
\[
\left( H^1(\Sigma) \otimes \Delta_{\Gamma_i}^*TX^{n_i} \right)^G \cong \left( H^1(\Sigma)^{\ker(\lambda_i)} \otimes \Delta_{\Gamma_i}^*TX^{n_i} \right)^{G_i}.
\]
Now let \(\Sigma_i := \Sigma/\ker(\lambda_i)\), it is an orbifold Riemann surface with a \(G_i\) action so that \(\Sigma_i/G_i\) becomes an orbifold sphere with three marked points (the markings are with respect to the generators of \(G_i\): \(h_1, h_2, (h_1h_2)^{-1}\)). So, in the same way as in the definition of the obstruction bundle \(E(h)\) we get that
\[
E_{h_1, h_2} := \left( H^1(\Sigma_i) \otimes \Delta_{\Gamma_i}^*TX^{n_i} \right)^{G_i}.
\]
The obstruction bundle splits as
\[ E_{h_1,h_2} = \prod_{i=1}^{k} E_{h_1,h_2}^{i} \]

As the action of \( G_i \) in \( \Delta_i^* T X^{n_i} \) is independent on the structure of \( X \) (moreover, it depends only in the coordinates), hence

\[ \Delta_i^* T X^{n_i} \cong T X \otimes C^{n_i} \]

as \( G_i \)-vector bundles, where \( T X \) is the tangent bundle over \( X \) and \( G_i \subset S_{n_i} \) acts on \( C^{n_i} \) in the natural way. Then

\[ E_{h_1,h_2}^{i} \cong T X \otimes (H^1(\Sigma) \otimes C^{n_i})^{G_i} \]

Defining \( r(h_1,h_2)(i) := dim_C(H^1(\Sigma) \otimes C^{n_i})^{G_i} \) it follows that the Euler class of \( E_{h_1,h_2}^{i} \) equals the Euler class of \( X \) to some exponent \( c(E_{h_1,h_2}^{i}) = e(X)^{r(h_1,h_2)(i)} \). However, the underline space is one copy of \( X \) only. Hence,

\[
\begin{align*}
  c(E_{h_1,h_2}^{i}) = \begin{cases}
    1 & \text{if } r(h_1,h_2)(i) = 0 \\
    e(X) & \text{if } r(h_1,h_2)(i) = 1 \\
    0 & \text{if } r(h_1,h_2)(i) \geq 2
  \end{cases}
\end{align*}
\]

Therefore, we prove that

**Theorem 3.2:**

\[ c(E_{h_1,h_2}) = \prod_{i=1}^{k} c(E_{h_1,h_2}^{i}) \]

where \( c(E_{h_1,h_2}^{i}) \) is given by previous formula (3.8).

Theorem 3.2 matches the calculation of Lehn-Sorger for the cohomology of Hilbert scheme of points of \( K3, T^4 \) except a sign (see section 3.3).

### 3.2 Conjecture for general crepant resolution

Suppose that \( \pi : Y \to X \) is one crepant resolution of Gorenstein orbifold \( X \). Then, \( \pi \) is a Mori contraction and the homology classes of rational curves \( \pi \) contracted are generated by so called extremal rays. Let \( A_1, \ldots, A_k \) be an integral basis of extremal rays. We call \( \pi \) non-degenerate if \( A_1, \ldots, A_k \) are linearly independent. For example, the Hilbert-Chow map \( \pi : M^{[n]} \to M^n/S_n \) satisfies this hypothesis. Then, the homology class of any effective curve being contracted can be written as \( A = \sum_i a_i A_i \) for \( a_i \geq 0 \). For each \( A_i \), we assign a formal variable \( q_i \). Then, \( A \) corresponds to \( q_1^{a_1} \cdots q_k^{a_k} \). We define a 3-point function

\[
<\alpha, \beta, \gamma>_{qc} (q_1, \cdots q_k) = \sum_{a_1, \ldots, a_k} \Psi_{A}^{X}(\alpha, \beta, \gamma) q_1^{a_1} \cdots q_k^{a_k},
\]

where \( \Psi_{A}^{X}(\alpha, \beta, \gamma) \) is Gromov-Witten invariant and qc stands for the quantum correction. We view

\[ <\alpha, \beta, \gamma>_{qc} (q_1, \cdots q_k) \]

as analytic function of \( q_1, \cdots q_k \) and set \( q_i = -1 \) and let

\[
<\alpha, \beta, \gamma>_{qc} = <\alpha, \beta, \gamma>_{qc} (-1, \cdots, -1).
\]
We define a quantum corrected triple intersection
\[ <\alpha, \beta, \gamma>_\pi = <\alpha, \beta, \gamma> + <\alpha, \beta, \gamma>_{qc}, \]
where \(<\alpha, \beta, \gamma> = \int_X \alpha \cup \beta \cup \gamma\) is the ordinary triple intersection. Then we define the quantum corrected cup product \(\alpha \cup_\pi \beta\) by the equation
\[ <\alpha \cup_\pi \beta, \gamma>_\pi = <\alpha, \beta, \gamma>_\pi, \]
for arbitrary \(\gamma\). Another way to understand \(\alpha \cup_\pi \beta\) is as following. Define a product \(\alpha \ast_{qc} \beta\) by the equation
\[ <\alpha \ast_{qc} \beta, \gamma>_{qc} = <\alpha, \beta, \gamma>_{qc}, \]
for arbitrary \(\gamma \in H^*(Y, C)\). Then, the quantum corrected product is the ordinary cup product corrected by \(\alpha \ast_{qc} \beta\). Namely,
\[ (3.11) \quad \alpha \cup_\pi \beta = \alpha \cup \beta + \alpha \ast_{qc} \beta. \]
We denote the new quantum corrected cohomology ring as \(H^*_\pi(Y, C)\).

Cohomological Crepant Resolution Conjecture: Suppose that \(\pi\) is non-degenerate and hence \(H^*_\pi(Y, C)\) is well-defined. Then, \(H^*_\pi(Y, C)\) is the ring isomorphic to orbifold cohomology ring \(H^*_{orb}(X, C)\).

Recently, Li-Qin-Wang [LQW3] proved a striking theorem that the cohomology ring of \(M^{[n]}\) is universal in the sense that it depends only on homotopy type of \(M\) and \(K_X\). Combined with their result, CCRC yields

Conjecture 3.3: \(*\) product depends only on \(K_M\).

It suggests an interesting way to calculate \(*\) product by first finding a universal formula (depending only on \(K_M\)) and calculating a special example such as \(\mathbb{P}^2\) to determine the coefficient.

Next, we formulate a closely related conjecture for \(K\)-equivalent manifolds.

Suppose that \(X, X'\) are \(K\)-equivalent and \(\pi : X \to X'\) is the birational map. Again, exceptional rational curves makes sense. Suppose that \(\pi\) is nondegenerate. Then, we go through the previous construction to define ring \(H^*_\pi(X, C)\).

Cohomological Minimal Model Conjecture: Suppose that \(\pi, \pi^{-1}\) are nondegenerate. Then, \(H^*_\pi(X, C)\) is the ring isomorphic to \(H^*_{\pi^{-1}}(X', C)\)

When \(X, X'\) are the different crepant resolutions of the same orbifolds, Cohomological minimal model conjecture follows from Cohomological crepant resolution conjecture. However, it is well-known that most of \(K\)-equivalent manifolds are not crepant resolution of orbifolds. Cohomological minimal model conjecture can be generalized to orbifold provided that the quantum corrections are defined using orbifold Gromov-Witten invariants introduced by Chen-Ruan [CR2].

Remark 3.4: (1) The author does not know how to define quantum corrected product if \(\pi\) is not nondegenerate. (2) All the conjectures in this section should be understood as the conjectures up to certain slight modifications (see next section).

Example 3.5: Next, we use the work of Li-Qin [LQ] to verify Cohomological Crepant Resolution Conjecture for \(M^{[2]}\). To simplify the formula, we assume that \(M\) is simply connected.

It is easy to compute the orbifold cohomology \(H^*_{orb}(X, C)\) for \(X = M^2/\mathbb{Z}_2\). The nontwisted sector can be identified with invariant cohomology of \(M^2\). Let \(h_i \in H^2(M, C)\) be a basis and
\( H \in H^4(M, \mathbb{C}) \) be Poincare dual to a point. Then, the cohomology of the nontwisted sectors are generated by \( 1, 1 \otimes h_i + h_i \otimes 1, 1 \otimes H + H \otimes 1, h_i \otimes h_j + h_j \otimes h_i, h_i \otimes H + H \otimes h_i, H \otimes H \). The twisted sector is diffeomorphic to \( M \) with degree shifting number 1. We use \( \bar{1}, \bar{h}_i, \bar{H} \) to denote the generators. They are of degrees 2, 4, 6. By the definition, triple intersections

\[
< \text{twistedsector}, \text{nontwistedsector}, \text{nontwistedsector} >= 0,
\]

\[
< \text{twistedsector}, \text{twistedsector}, \text{twistedsector} >= 0.
\]

Following is the table of nonzero triple intersections involving classes from the twisted sector

\[
< \bar{1}, \bar{1}, 1 \otimes H + H \otimes 1 >= 1, < \bar{1}, \bar{1}, h_i \otimes h_j + h_j \otimes h_i >= h_i, h_j >,
\]

(3.12)

\[
< \bar{1}, \bar{h}_i, h_j \otimes 1 + h_j \otimes 1 >= h_i, h_j >.
\]

Next, we review the construction of \( Y = M^{[2]} \). Let \( \tilde{M}^2 \) be the blow-up of \( M^2 \) along the diagonal. Then, \( \mathbb{Z}_2 \) action extends to \( \tilde{M}^2 \). Then, \( Y = \tilde{M}^2 / \mathbb{Z}_2 \). It is clear that we should map the classes from nontwisted sector to its pull-back \( \pi : Y \to X \). We use the same notation to denote them. The exceptional divisor \( E \) of Hilbert-Chow map \( \pi : Y \to X \) is a \( P^1 \)-bundle over \( M \). Let \( \bar{1}, \bar{h}_i, \bar{H} \) be the Poincare dual to \( E \), \( p^{-1}(PD(h_i)) \) and fiber \( |C| \), where \( p : E \to M \) is the projection.

Notes that \( \bar{1}_{|E} = 2E \), where \( E \) is the tautological divisor of \( P^1 \)-bundle \( E \to M \). It is clear that \( E = P(N_{\Delta(X)|X^2}) \), where \( \Delta(X) \subset X^2 \) is the diagonal. Hence,

(3.13) \[
< \bar{1}, \bar{1}, \bar{h}_i > = 4E^2|_{p^{-1}(PD(h_i))} = 4C_1(N_{\Delta(X)|X^2})E|_{p^{-1}(PD(h_i))} = -4 < C_1(X), h_i >.
\]

(3.14) \[
< \bar{1}, \bar{1}, 1 \otimes H + H \otimes 1 >= 2E(1 \otimes H + H \otimes 1)(E) = 4E(C) = -4.
\]

(3.15) \[
< \bar{1}, \bar{h}_i, h_j \otimes 1 + h_j \otimes 1 > = -4 < h_i, h_j >.
\]

(3.16) \[
< \bar{1}, \bar{h}_i, 1 \otimes h_j + h_j \otimes 1 > = -4 < h_i, h_j >.
\]

Others are zero.

The quantum corrections have been computed by Li-Qin [LQ] (Proposition 3.021). The only nonzero terms are

(3.17) \[
< \bar{1}, \bar{1}, \bar{h}_i >_{qc} (q) = \sum_{d=1} \bar{1}(|C|)^2 \Psi_{d(C)}(\bar{h}) q^d
\]

Hence,

(3.18) \[
< \bar{1}, \bar{1}, \bar{h}_i >_{qc} = 4 < K_X, h_i > = 4 < C_1(X), h_i >
\]

cancels \( < \bar{1}, \bar{1}, \bar{h}_i > \).

It is clear that the map \( \bar{1} \to 2\bar{1}, \bar{h} \to 2\bar{h}, \bar{H} \to \bar{H} \) is a ring isomorphism. \( \square \)

Next, we give two examples to verify Cohomological Minimal Model Conjecture (CMMC).

**Example 3.6:** The first example is the flop in dimension three. This case has been worked out in great detail by Li-Ruan [LR]. For example, they proved a theorem that quantum cohomology
rings are isomorphic under the change of the variable $q \to \frac{1}{q}$. Notes that if we set $q = -1$, \( \frac{1}{q} = -1 \). We set other quantum variables zero. Then, the quantum product becomes the quantum corrected product $\alpha \cup_{\text{orb}} \beta$. Hence, CMMC follows from Li-Ruan’s theorem. However, it should be pointed out that one can directly verify CMMC without using Li-Ruan’s theorem. In fact, it is an much easier calculation.

**Example 3.7:** There is a beautiful four dimensional birational transformation called Mukai transform as follows. Let $P^2 \subset X^4$ with $N_{P^2|X^4} = T^*P^2$. Then, one can blow up $P^2$. The exceptional divisor of the blow up is a hypersurface of $P^2 \times P^2$ with the bidegree $(1,1)$. Then, one can blow down in another direction to obtain $X'$. $X, X'$ are $K$-equivalent. In his Ph.D thesis \cite{Z}, Wanchuan Zhang showed that the quantum corrections $<\alpha, \beta, \gamma >_{qc}$ are trivial, and cohomologies of $X, X'$ are isomorphic.

### 3.3 Miscellaneous issues

In the computation of orbifold cohomology of symmetry product and its relation to that of Hilbert scheme of points, there are two issues arisen. It was showed in the work of Lehn-Sorger, Fantechi-G"ottscbe and Uribe that one has to add a sign in the definition of orbifold product in order to match that of Hilbert scheme of points. This sign was described as follows.

Recall the definition of orbifold cup product

$$\alpha \cup_{\text{orb}} \beta = \sum_{(h_1, h_2) \in T_2, h_i \in (g_i)} (\alpha \cup_{\text{orb}} \beta)_{(h_1, h_2)},$$

where $(\alpha \cup_{\text{orb}} \beta)_{(h_1, h_2)} \in H^*(X(h_1h_2); \mathbb{C})$ is defined by the relation

$$<\alpha \cup_{\text{orb}} \beta, \gamma >_{\text{orb}} = \int_{X(h_1h_2)} \epsilon_1^* \alpha \wedge \epsilon_2^* \beta \wedge \epsilon_3^* \gamma \wedge e_A(E_{(g)}).$$

for $\gamma \in H^*_c(X((h_1h_2)-1); \mathbb{C})$. Then, we add a sign to each term.

$$\alpha \cup_{\text{orb}} \beta = \sum_{(h_1, h_2) \in T_2, h_i \in (g_i)} (-1)^{\epsilon(h_1, h_2)} (\alpha \cup_{\text{orb}} \beta)_{(h_1, h_2)},$$

where

$$\epsilon(h_1, h_2) = \frac{1}{2}(\iota(h_1) + \iota(h_2) - \iota(h_1h_2)).$$

Since

$$\epsilon(h_1, h_2) + \epsilon(h_1h_2, h_3) = \frac{1}{2}(\iota(h_1) + \iota(h_2) + \iota(h_3) - \iota(h_1h_2h_3)) = \epsilon(h_1, h_2h_3) + \epsilon(h_2, h_3),$$

such a sign modification does not affect the associativity of orbifold cohomology.

However, Qin-Wang \cite{QW} observed that the orbifold cohomology modified by such a sign is isomorphic to original orbifold cohomology over complex number by an explicit isomorphism

$$\alpha \to (-1)^{\iota(g) \over 2} \alpha$$

for $\alpha \in H^*(X(g); \mathbb{C})$. $\epsilon(h_1, h_2)$ is often an integer (for example symmetric product) while $\iota(g) \over 2$ is just a fraction. Hence, $(-1)^{\iota(g) \over 2}$ is a complex number only.
Another issue is the example of the crepant resolution of surface singularities $C^2/\Gamma$. As Fantechi-Göttsche [FG2] pointed out, the Poincare paring of $H^2_{orb}(C^2/\Gamma, \mathbb{C})$ is indefinite while the Poincare paring of its crepant resolution is negative definite. There is an easy way to fix this case (suggested to this author by Witten). We view the involution $I : H^*(X_g, \mathbb{C}) \to H^*(X_g, \mathbb{C})$ as a ”complex conjugation”. Then, we define a ”hermitian inner product”

$\langle \alpha, \beta \rangle = \langle \alpha, I^*(\beta) \rangle$.

If we use this ”hermitian” inner product, the intersection paring is positive definite again. The above process has its conformal theory origin (see [NW]). It is attempting to perform this modification on $H^*(X_g, \mathbb{C}) \oplus H^*(X_{g-1}, \mathbb{C})$ whenever $\iota(g) = \iota(g-1)$. The author does not know if it will affect the associativity of orbifold cohomology.

In a different direction, it is useful to modify orbifold cup product by a ”complex” sign for following reason. In ordinary cohomology, cup product is supercommutative

$\alpha \cup \beta = (-1)^{\deg(\alpha) \deg(\beta)} \beta \cup \alpha$.

For the orbifold cup product, we only have such supercommutativity when degree shifting number $\iota(g)$ is integer. In general case, degree shifting number is a rational number and we are talking about $(-1)^a$ for a fraction $a$. At the first, it may sounds odd since ”super” means odd/even property. However, there were other reasons to believe that it is important to consider a ”complex sign” $(-1)^a$ even for a fraction $a$. Here, we interpret $(-1)^a$ as a complex number $e^{\pi i a}$. The author does not know how to modify the orbifold product to achieve such a ”complex supercommutativity”.

### 4 Twisted orbifold K-theory

An important aspect of the stringy orbifold is the orbifold K-theory and its twisted version. Unfortunately, the twisted version developed by Adem-Ruan [AR] is less than satisfactory. As showed in [R3], one can perform the twisting of the orbifold cohomology for any inner local system. However, the twisted orbifold K-theory has only been constructed for the discrete torsion. The decomposition theorem by Chern homomorphism in the twisted case was only proved for the global quotient although the author expects it to be true in general. As Adem and the author was developing our twisted orbifold K-theory, they noted a parallel development of the twisted K-theory on smooth manifolds by Witten and others [W] for the purpose of describing the D-brane charge in physics. They were wondering if one can unify these two twisting. This was done beautifully by Lupercio-Uribe [LU1] using the notion of gerbes, which is an interesting concept of its own. Their construction will be presented in this section.

However, the corresponding decomposition theorem or the existence of corresponding Chern homomorphism is far less obvious. For example, we do not yet know the analogous of the orbifold cohomology for Witten’s twisted K-theory. Since Lupercio-Uribe’s construction works over more general orbit space or the Artin stack, it suggests that we should have a version of the orbifold cohomology for a general orbit space and its corresponding Chern homomorphism. There is a striking result recently by Freed-Hopkins-Teleman [FR] which shows that the twisted equivariant K-theory of Lie group $G$ acting on itself by the conjugation is isomorphic to the Verlinde algebra. However, their calculation did not explain the reason of the appearance of Verlinde algebra, i.e, establishing the explicit isomorphism. I am sure that the existence of the corresponding ”orbifold cohomology” and Chern homomorphism should clarify this issue. However, I would like to warn the reader that the straight forward generalization of Adem-Ruan’s decomposition theorem does
not work. In fact, there was an old article [H] to show that such a Chern homomorphism could not exist if an action has $S^1$-isotropy subgroup.

Another topic we will survey in this section is in different nature. Wang constructed the twisted K-theory using a Clifford algebra (Fermion) [W]. The generating function of its Euler characteristic has a better shape. It is an interesting construction. However, this author does not know how to incorporate it into other type of twisting.

4.1 Gerbe and twisted K-theory

Let $\mathcal{R} \to U$ be groupoid associated to an orbifold $X$.

**Definition 4.1:** A gerbe over an orbifold $\mathcal{R} \to U$, is a complex line bundle $L$ over $\mathcal{R}$ satisfying the following conditions

- $i^* L \cong L^{-1}$
- $\pi_1^* L \otimes \pi_2^* L \otimes m^* i^* L \cong 1$
- $\theta : \mathcal{R}_t \times_s \mathcal{R} \to U(1)$ is a 2-cocycle

where $\pi_1, \pi_2 : \mathcal{R}_t \times_s \mathcal{R} \to \mathcal{R}$ are the projections on the first and the second coordinates, and $\theta$ is a trivialization of the line bundle.

When the groupoid $G$ is given by the action of a finite group on a point $G \times \to \star$, a gerbe over $G$ is the same as an central extension

$$1 \to U(1) \to \tilde{G} \to G \to 1.$$ 

Gerbes over a discrete group $G$ are in 1-1 correspondance with the set of two cycles $Z(G, U(1))$.

The set of gerbes forms a group $Gb(\mathcal{R} \to U)$. Gerbes has a characteristic class $<L>$ given by $\theta : \mathcal{R}_t \times_s \mathcal{R} \to U(1)$. $<L>$ is a degree three cohomology in so called classifying space of groupoid. Moreover, the group $Gb(\mathcal{R} \to U)$ is independent of the Morita class of $\mathcal{R} \to U$.

**Example 4.2:** Consider an inclusion of (compact Lie) groups $K \subset G$ and consider the groupoid $G$ given by the action of $G$ in $G/K$,

$$G/K \times G \to G/K$$

Observe that the stabilizer of $[1]$ is $K$ and therefore we have that the following groupoid

$$[1] \times K \to [1]$$

is Morita equivalent to the one above.

From this we obtain

$$Gb(G) \cong H^3(K, Z)$$

For a smooth manifold $X$ we have that

$$Gb(X) = [X, BBC^*]$$

where $BBC^* = BU(H)$ for a Hilbert space $H$.

Let us write $PU(H)$ to denote the groupoid $\star \times PU(H) \to \star$. We have the following
Proposition 4.3: For an orbifold $X$ given by a groupoid $X$ we have
\[ Gb(X) = [X, \overline{\mathbf{PU}(H)}] \]
where $[X, \overline{\mathbf{PU}(H)}]$ represents the Morita equivalence classes of morphisms from $X$ to $\overline{\mathbf{PU}(H)}$.

Just as in the case of a gerbe over a smooth manifold, we can do differential geometry on gerbes over an orbifold groupoid $X = (\mathcal{R} \to \mathcal{U})$. Let us define a connection over a gerbe in this context.

Definition 4.4: A connection $(g, A, F, G)$ over a gerbe consists of a complex valued 0-form $g \in \Omega^0(\mathcal{R} \times_s \mathcal{R})$, a 1-form $A \in \Omega^1(\mathcal{R})$, a 2-form $F \in \Omega^2(\mathcal{U})$ and a 3-form $G \in \Omega^3(\mathcal{U})$ satisfying
- $G = dF$,
- $t^*F - s^*F = dA$ and
- $\pi_1^*A + \pi_2^*A + m^*i^*A = -\sqrt{-1}g^{-1}dg$

The 3-form $G$ is called the curvature of the connection. A connection is called flat if its curvature $G$ vanishes.

The 3-curvature $\frac{1}{2\pi \sqrt{-1}}G$ represents the integer characteristic class of the gerbe in cohomology with real coefficients, this is the Chern-Weil theory for a gerbe over an orbifold. Recall that for a line bundle over a manifold $X$ with a connection, the holonomy map can be considered as a map from loop space $LX$ to $U(1)$. For gerbes over $X$ with a connection, there is a notion of holonomy. But it produces a flat line bundle over $LX$. If we are given a gerbe $L$ with a flat connection over a groupoid $X$, using the holonomy we construct a "flat line bundle" $\Lambda$ over the loop groupoid $LX$ and hence obtain a flat line bundle over the fixed point set $\Lambda X = LXS^1$ of the natural circle action.

Theorem 4.5: The inertia orbifold $\Sigma_1X$ defined in [CR1] is represented by the groupoid $\Lambda X$. The holonomy line bundle $\Lambda$ over $\Lambda X$ is an inner local system as defined in [R3].

However, we do not know if every inner local system can be obtained in this way although it can recover the inner local system induced by discrete torsion (see the comments after Proposition 4.8).

Next, we use gerbes to construct twisted K-theory. We first present the construction of "twisted" vector bundle, a generalization of Adem-Ruan twisted orbifold vector bundle. But this construction only yields nontrivial K-theory when $< L >$ is a torsion class. For non-torsion class, we present a construction generalizing both twisted vector bundle and twisted K-theory in the smooth case.

Definition 4.6: An $n$-dimensional $L$-twisted bundle over $\mathcal{R} \to \mathcal{U}$ is a vector bundle $E \to \mathcal{U}$ together with a given isomorphism
\[ L \otimes t^*E \cong s^*E \]

Notice that we then have a canonical isomorphism
\[ m^*L \otimes \pi_2^*t^*E \cong \pi_1^*L \otimes \pi_2^*L \otimes \pi_2^*t^*E \cong \pi_1^*L \otimes \pi_2^*(L \otimes t^*E) \cong \pi_1^*L \otimes \pi_2^*s^*E \]

We can define the corresponding Whitney sum of $L$-twisted bundles similarly.

Definition 4.7: The Grothendieck group generated by the isomorphism classes of $L$ twisted bundles over the orbifold $X$ together with the addition operation just defined is called the $L$ twisted K-theory of $X$ and is denoted by $\mathcal{L}K_{\text{gpd}}(X)$. 

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Moerdijk and Pronk \[MP1\] proved that the isomorphism classes of orbifolds are in 1-1 correspondence with the classes of étale, proper groupoids up to Morita equivalence. As a direct consequence of the definitions, $\mathcal{L} K_{gpd}(X)$ is independent of the groupoid that is associated to $X$.

Using the group structure of $Gb(\mathcal{R} \to \mathcal{U})$ we can define a product between bundles twisted by different gerbes, so for $L_1$ and $L_2$ gerbes over $X$ 

$$L_1 K_{gpd}(X) \otimes L_2 K_{gpd}(X) \to L_1 \otimes L_2 K_{gpd}(X)$$

and we can define the total twisted orbifold $K$-theory of $X$ as 

$$TK_{gpd}(X) = \bigoplus_{\mathcal{L} \in Gb(\mathcal{R} \to \mathcal{U})} K_{gpd}(X)$$

This has a ring structure due to the following proposition.

**Proposition 4.8:** The twisted groups $\mathcal{L} K_{gpd}(G)$ satisfy the following properties:

1. If $\langle \mathcal{L} \rangle = 0$ then $\mathcal{L} K_{gpd}(G) = K_{gpd}(G)$.
2. $\mathcal{L} K_{gpd}(G)$ is a module over $\mathcal{L} K_{gpd}(G)$
3. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are two gerbes over $G$ then there is a homomorphism 

$$\mathcal{L}_1 K_{gpd}(G) \otimes \mathcal{L}_2 K_{gpd}(G) \to \mathcal{L}_1 \otimes \mathcal{L}_2 K_{gpd}(G)$$

4. If $\psi: \mathcal{G}_1 \to \mathcal{G}_2$ is a groupoid homomorphism then there is an induced homomorphism 

$$\mathcal{L} K_{gpd}(\mathcal{G}_2) \to \psi^* \mathcal{L} K_{gpd}(\mathcal{G}_1)$$

In the case when $Y$ is the orbifold universal cover of $X$ with orbifold fundamental group $\pi_1^{orb}(X) = H$, we can take a discrete torsion $\alpha \in H^2(H, U(1))$ and define the twisted $K$-theory of $X$ as in \[AR\]. The groupoid $\mathcal{R}_Y \times H \to \mathcal{U}_Y$ represents the orbifold $X$. We want to construct a gerbe $\mathcal{L}$ over $\mathcal{R}_Y \times H \to \mathcal{U}_Y$ so that the twisted $\mathcal{L} K_{gpd}(X)$ is the same as the twisted $\alpha K_{orb}(X)$ as defined by Adem-Ruan [?].

The discrete torsion $\alpha$ defines a central extension of $H$ 

$$1 \to U(1) \to \tilde{H} \to H \to 1$$

and doing the cartesian product with $\mathcal{R}_Y$ we get a line bundle 

$$\overline{U(1)} \to \mathcal{L}_\alpha = \mathcal{R}_Y \times \tilde{H} \downarrow \mathcal{R}_Y \times H$$

which ia a gerbe over $\mathcal{R}_Y \times H \to \mathcal{U}_Y$.

**Theorem 4.9:** $\mathcal{L}_\alpha K_{gpd}(X) \cong \alpha K_{orb}(X)$.

We should point out here that the theory so far described is essentially empty whenever the characteristic class $\langle \mathcal{L} \rangle$ is a non-torsion element in $H^3(M, \mathbb{Z})$. The following is true.
Proposition 4.10: If there is an \( n \)-dimensional \( L \)-twisted bundle over the groupoid \( \mathcal{G} \) then \( \langle L \rangle^n = 1 \).

We need to consider a more general definition when the class \( \langle L \rangle \) is a non-torsion class. With this in mind we would like to have a group model for the space \( \mathcal{F} \) of Fredholm operators. One possible candidate is the following.

Definition 4.11: For a given Hilbert space \( \mathcal{H} \) by a polarization of \( \mathcal{H} \) we mean a decomposition

\[
\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-
\]

where \( \mathcal{H}_+ \) is a complete infinite dimensional subspace of \( \mathcal{H} \) and \( \mathcal{H}_- \) is its orthogonal complement.

We define the group \( GL_{\text{res}} \) to be the subgroup of automorphisms of \( \mathcal{H} \) consisting of operators \( A \) that when written with respect to the polarization \( \mathcal{H}_+ \oplus \mathcal{H}_- \) look like

\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

where \( a: \mathcal{H}_+ \to \mathcal{H}_+ \) and \( d: \mathcal{H}_- \to \mathcal{H}_- \) are Fredholm operators, and \( b: \mathcal{H}_- \to \mathcal{H}_+ \) and \( c: \mathcal{H}_+ \to \mathcal{H}_- \) are Hilbert-Schmidt operators.

Notice that \( PU\mathcal{H}_+ \) acts by conjugation on \( GL_{\text{res}} \) for the map that sends \( g \in PU\mathcal{H}_+ \) to

\[
\iota(g) = \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}
\]

We have the following fact.

Proposition 4.12: The map \( GL_{\text{res}} \to \mathcal{F}: A \to a \) is a homotopy equivalence.

Consider a gerbe \( \mathcal{L} \) with characteristic class \( \alpha \) as a map \( X \to BBU(1) = BPU(\mathcal{H}_+) \), namely a Hilbert projective bundle \( Z_\alpha(M) \to M \). Then we form a \( GL_{\text{res}} \)-adjoint bundle over \( X \) by defining

\[
\mathcal{F}_\alpha(M) = Z_\alpha(M) \times_{PU\mathcal{H}} GL_{\text{res}}.
\]

Definition 4.13: We define the twisted K-theory

\[
\mathcal{L}K_{\text{gpd}}(X)
\]

as the homotopy class of the sections of \( \mathcal{F}_\alpha(X) \).

This definition works for a gerbe whose class is non-torsion and has the obvious naturality conditions. In particular it becomes Witten’s twisting if the groupoid represents a smooth manifold.

4.2 Twisting by fermion

It is known that \( H^2(S_n, U(1)) = \mathbb{Z}_2 \) for \( n \geq 4 \) and there is a twisted orbifold K-theory for symmetry product. The generating function of its Euler characteristic has been computed by Dijkgraaf [D], Wang [W] and Uribe [U1]. Unlike the generating function of Euler characteristic of ordinary orbifold K-theory, this twisted version does not have modularity. To remedy the situation, Wang [W] developed a super version of (equivariant) twisted K-theory for global quotient whose Euler characteristic again has nice property. It is a very interesting construction. Let me sketch his construction.
Let $\tilde{G}$ be a finite group and $d : \tilde{G} \to \mathbb{Z}_2$ be a group epimorphism understood as parity function. An element $a$ of $\tilde{G}$ is called odd/even if $d(a) = 1/0$. $(\tilde{G}, d)$ is called a finite supergroup. Recall that spin group is double cover of $SO(n)$. We also require $\tilde{G}$ is double cover of $G$ and the distinguished central element $\theta$ of order 2 is even. Next, we consider its representation theory. Again, a representation of a finite supergroup is the same as a module of its group superalgebra. Moreover, we only consider supermodule, i.e., $\mathbb{Z}_2$-graded. Given two supermodules $M = M_0 + M_1$ and $N = N_0 + N_1$ over a superalgebra $A = A_0 + A_1$, the linear map $f : M \to N$ between two $A$-supermodules is a homomorphism of degree $i$ if $f(M_j) \subset M_{i+j}$ and for any homogeneous element $a \in A$ and any homogeneous vector $m \in M$ we have

$$f(am) = (-1)^{d(f)d(a)}af(m).$$

The degree 0/1 part of a superspace is referred to as the even/odd part. We denote

$$Hom_A(M, N) = Hom_A(M, N)_0 \oplus Hom_A(M, N)_1,$$

where $Hom_A(M, N)_i$ consist of $A$-homomorphisms of degree $i$ from $M$ to $N$. The notions of submodules, tensor product, and irreducibility for supermodules are defined similarly. Given a finite supergroup $G$, a $\tilde{G}$-supermodule $V$ is called spin if the central element $\theta$ acts as $-1$, a reministic of spin representation. We will only consider spin supermodule. A spin supermodule can also be thought as a $\alpha$-twisted projective module for a nontrivial 2-cocycle $\alpha : G \times G \to \mathbb{Z}_2$.

There are two types of complex simple superalgebras $M(r|s)$ and $Q(n)$. $M(r|s)$ is the superalgebra consisting of the linear transformations of the superspace $C^{r+s} = C^r + C^s$. The superalgebra $Q(n)$ is the graded subalgebra of $M(n|n)$ consisting of matrices of the form

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix}$$

According to the classification of simple superalgebras above, the irreducible supermodules of a finite supergroup are divided into two types, type $M$ and type $Q$. We note that the endomorphism algebra of an irreducible supermodule $V$ is isomorphic to $C$ if $V$ is of type $M$ and isomorphic to the complex Clifford algebra $C_1$ in one variable if $V$ is of type $Q$.

For any conjugacy class $C$ of $G$, $\theta^{-1}(C)$ is either a conjugacy class of $\tilde{G}$ or it splits into two conjugacy classes of $\tilde{G}$. $C$ (and any element $g \in C$) is called split or non-split. It is easy to check that $g$ is split iff the twisted character $L^\alpha_g = \alpha(g,.)\alpha(g,.)^{-1}$ is a trivial character of the centralizer $C(g)$.

Now, we are ready to globalize above construction. Suppose that a finite group $G$ acts on a smooth manifold $X$ so that $X/G$ is a global quotient orbifold. Let $\tilde{G}$ be corresponding supergroup. A spin $\tilde{G}$-vector superbundle $E$ has property $E = E_0 + E_1$ (often denoted by $E_0|E_1$) over $X$ such that $\tilde{G}$ acts in a $\mathbb{Z}_2$-graded manner and $\theta$ acts as $-1$. Since $\tilde{G}$ always contains odd elements which interchange $E_0, E_1$. Hence, $\text{rank}E_0 = \text{rank}E_1$. The direct sum operation extends to spin $\tilde{G}$-bundle and we can define its K-theory $K_{\tilde{G}}$ $(X)$. For this super-twisted K-theory, Wang proved a decomposition theorem similar to that of Adem-Ruan [AR].

**Theorem 4.14**: Let $X, \tilde{G}$ be define as above. We have a natural $\mathbb{Z}_2$-graded isomorphism

$$K_{\tilde{G}}(X) \otimes C \cong \oplus_{[g]}(K(X^g) \otimes L^\alpha_g)^{C(g)},$$

where the summation runs over the even conjugacy classes in $G$. 

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If we drop super condition, we simply obtain twisted K-theory $\alpha K_G(X)$ in the sense of Adem-Ruan. The corresponding decomposition theorem is a summation over all the conjugacy classes.

**Example 4.15:** Recall that the symmetry group $S_n$ has nontrivial discrete torsion given by the fact that spin group is a double cover of $SO(n)$ and $S_n$ can be embedded into $SO(n)$ as a subgroup. This nontrivial discrete torsion can be described by exact sequence

$$1 \to \mathbb{Z}_2 \to \tilde{S}_n \to S_n,$$

where $\tilde{S}_n$ is generated by $\theta$ and $t_i, i = 1, \cdots, n - 1$ and subject to the relations

$$\theta^2 = 1, t_i^2 = (t_i t_{i+1})^3 = \theta, t_i t_j = \theta t_j t_i (i > j + 1), z t_i = t_i z.$$

The group $\tilde{S}_n$ carries a natural $\mathbb{Z}_2$ grading by letting $t_i$'s be odd and $\theta$ be even. Hence, $\tilde{S}_n$ is a finite supergroup. Wang studied $K^{-\tilde{S}_n}(X^n)$ in detail and shows that $\bigoplus_{n=0}^{\infty} K^{-\tilde{S}_n}(X^n)$ has many beautiful algebraic structure such as Hopf algebra, admitting an action of the twisted Heisenberg algebra. We should emphasis that the ordinary twisted K-theory by discrete torsion does not have these nice algebraic structure.

5 Spin Orbifold Quantum Cohomology

One of the attractions of the stringy orbifold is its unique orbifold feature which will not exist on smooth manifolds. In the on-going project of this author with T. Jarvis, such a unique orbifold feature is added to its quantum cohomology theory. This is the incorporation of the theory of spin curves to the orbifold quantum cohomology. It is even more remarkable that the theory of spin curves should be naturally an orbifold theory and was discovered ten years before its time. Now, it finally comes to its rightful home. This author should mention that an orbifoldish construction of the moduli space of spin curves has already been obtained by D. Abramovich and T. Jarvis [A]. Here, we go one more step to incorporate it into the Gromov-Witten theory.

The history of spin curves goes back ten years ago when Witten formulated a famous conjecture to relate intersection numbers on the moduli space of stable Riemann surfaces to the KdV-hierarchy. Witten’s conjecture was proved by Kontsevich [K], and motivated another well-known conjecture by Eguchi-Hori-Xiong and Katz in the quantum cohomology that the generating function of Gromov-Witten invariants satisfies a set of equations which form a Virasoro algebra. This conjecture is commonly known as the Virasoro conjecture. Witten’s conjecture was based on a matrix model description of so-called 2D-gravity. Around the same time, Witten also formulated a closely related but less known conjecture for so-called 2D-gravity coupled with topological matter. This conjecture is equivalent to saying that the generating functions of intersection numbers on the moduli space of higher spin curves satisfies constraints forming a $W$-algebra—a larger algebra than the Virasoro algebra. Since then, his conjecture has been given a rigorous mathematical formulation by Jarvis and his collaborators [J] [J1] [JKV] [JKV2].

An incorporation of spin curves into quantum cohomology yields an elegant theory of spin Gromov-Witten theory which is again unique for the orbifold. Furthermore, its generating function should satisfy a set of differential equations forming a $W$-algebra. This is again an orbifoldish generalization of the Virasoro conjecture. Recall that the Gromov-Witten theory corresponds to the topological sigma model coupled with 2-D gravity in physics. In [CR2], it was showed that the sigma model on orbifold is not the ordinary map but good map which is the basis for orbifold Gromov-Witten invariants. It is amazing that the counterpart of 2-D gravity in the orbifold theory is not 2-D gravity but 2-D gravity coupled with the topological matter. This realization should
has far reaching consequence to other area of interest such as mirror symmetry. For example, what is the B-model interpretation Spin orbifold quantum cohomology? A better understanding of this question should enrich the mirror symmetry as well.

To illustrate the idea of the present paper, recall that the quantum cohomology can be thought of as a theory to study the Cauchy-Riemann equation:

\[ \bar{\partial} f = 0. \]

for a map \( f : \Sigma \to X \), where \( \Sigma \) is a Riemann surface and \( X \) is a symplectic manifold. The orbifold quantum cohomology studies the same equation, where \( \Sigma \) and \( f \) are interpreted appropriately in an orbifold category. The spin orbifold quantum cohomology studies simultaneous solutions of the Cauchy-Riemann equation and another equation we call the Spin equation. The two equations are

\[ \bar{\partial} f = 0, \quad \bar{\partial}s + s^{r-1} = 0, \]

where \( s \in \Omega^0(L) \) for a line bundle \( L \) which is an \( r \)-th-root of the log canonical bundle (see Definition 5.7) of an orbifold Riemann surface. When \( \Sigma \) is smooth and \( r = 2 \), \( L \) corresponds a classical spin structure of \( \Sigma \). Hence, for arbitrary \( r \), \( L \) can be thought as a generalized spin structure.

An easy computation (Lemma 5.1) shows that the second equation has only zero as its solution. So its solution (moduli) space is closely related to that of the Cauchy-Riemann equation. However, the spin equation is non-transverse in general; hence, it gives rather different invariants.

Our spin orbifold quantum cohomology yields a new cohomology of orbifold, which we call \( r \)-spin orbifold cohomology or spin orbifold cohomology. We do not yet know how useful spin orbifold cohomology is.

### 5.1 Spin orbifold cohomology

One of remarkable consequences of ordinary orbifold quantum cohomology is the existence of a new cohomology ring of orbifold as the classical part of orbifold quantum cohomology. Our \( r \)-spin quantum cohomology also yields a new cohomology theory of orbifold, which is the main topic of another paper. In this section, we outline some of main ingredients.

Let \( X \) be an almost complex orbifold and \( B(\mathbb{Z}/r) \) be a point with the group action \( \mathbb{Z}/r \). Let \( X_{(g, l)} = X_{(g)} \times B(\mathbb{Z}/r)_{(l)} \) for \( 0 \leq l \leq r - 1 \).

**Definition 5.1.1:** We call \( X_{(g, l)} \) Neveu-Schwarz-sector for \( l \neq 0 \) and \( X_{(g, 0)} \) Ramond sector.

**Definition 5.1.2:** We define the \( r \)-spin orbifold cohomology \( H^*_{r, orb}(X, \mathcal{Q}) \) of \( X \) to be the vector space

\[ H^*_{r, orb}(X, \mathcal{Q}) := \bigoplus_{(g, l) > 0} H^*(X_{(g, l)}, \mathcal{Q}). \]

As in the "usual" (without \( r \)-spin structure) case, there is a shift in the grading of the cohomology, and a Poincare paring. Let

\[ \iota(g, l) := \iota(g) + \frac{l - 1}{r}, \]

where \( \iota(g) \) is the usual degree shifting number of \( g \). The \( r \)-spin orbifold degree \( \text{deg}_{r, orb}(\alpha) \) of \( \alpha \in H^j(X_{(g, l)}) \) is defined to be

\[ \text{deg}_{r, orb}(\alpha) = \text{deg}(\alpha) + 2\iota(g, l). \]
Definition 5.1.3: Let \( I : X_{(g,l)} \rightarrow X_{(g-1,r-l)} \) be the natural diffeomorphism. For all \( 0 \leq d \leq 2n + \frac{2(r-2)}{r} \) let

\[
\langle \rangle_{r,orb} : H^d_{r,orb}(X, Q) \oplus H^{d-2}(X_{(g,l)}, Q) \rightarrow \mathbb{Q}
\]

be defined as the direct sum of

\[
\langle (g,l) \rangle_{r,orb} : H^d(X_{(g,l)}, Q)[-2l(g,l)] \oplus H^{d-2}(X_{(g-1,r-l)}, Q)[-2l(g-1,r-l)] \rightarrow \mathbb{Q},
\]

where

\[
\langle \alpha, \beta \rangle_{r,orb} = \int_{X_{(g,l)}} \alpha \wedge I^*(\beta).
\]

Remark 5.1.4: The top degree of \( H^*_{r,orb}(X, Q) \) is \( 2n + \frac{2(r-2)}{r} \). Hence, it should be thought to have dimension \( 2n + \frac{2(r-2)}{r} \). In another words, \( \mathbb{B}(\mathbb{Z}/r) \) contributes a dimension \( \frac{2(r-2)}{r} \).

Proposition 5.1.5: \( \langle \rangle_{r,orb} \) is non-degenerate.

The proof is essentially identical to the usual (non-spin) orbifold case. \( H^*_{r,orb}(X, Q) \) has a ring structure as the degree zero part of r-spin orbifold quantum cohomology we will define in the later section.

Example 5.1.6: Let \( X \) be a point. However, r-spin cohomology \( H^*_{r,orb}(X, Q) \) can be described as follows. Let \( e_u \) for \( u \in \{1, \ldots, r-1\} \) be the constant function 1 on sector \( X_{(1,u)} \). The r-spin cohomology is generated by \( e_u \) with product \( e_u \ast e_v = e_{u+v-1} \) if \( u + v < r + 1 \) and \( e_u \ast e_v = 0 \) otherwise.

5.2 Spin equation

Recall that a pseudo-holomorphic map \( f : \Sigma \rightarrow X \) can be thought as a solution of the Cauchy-Riemann equation \( \partial f = 0 \). The solution space of \( \bar{\partial} f = 0 \) is the moduli space of pseudo-holomorphic maps. The moduli space of stable maps is a “nice” compactification of the moduli space of pseudo-holomorphic maps. A spin pseudo-holomorphic map is a solution of two nonlinear equations

\[
\bar{\partial} f = 0, \quad \bar{s} + \bar{\bar{s}}^{-1} = 0.
\]

The first equation is the usual one for pseudo-holomorphic maps, and the second equation uses a generalized spin structure on the Riemann surface as follows. Suppose that \( \Sigma \) is a marked Riemann surface with marked points \( z_1, \ldots, z_k \), and \( \pi : \bar{\Sigma} \rightarrow \Sigma \) is an orbifold Riemann surface with additional orbifold structure at marked point. If \( L \) is an orbifold line bundle on \( \bar{\Sigma} \), it can be uniquely lifted to an orbifold line bundle over \( \Sigma \). We shall use the same \( L \) to denote its lifting. Let \( K \) is the canonical bundle of \( \Sigma \).

Definition 5.2.1: Let

\[
K_{\log} := K \otimes \mathcal{O}(z_1) \otimes \cdots \otimes \mathcal{O}(z_k)
\]

be the log-canonical bundle. \( K_{\log} \) can be thought as canonical bundle of punctured Riemann surface \( \Sigma - \{z_1, \ldots, z_k\} \). Suppose that \( L \) is an orbifold line bundle on \( \bar{\Sigma} \) with an isomorphism \( \phi : L^r \rightarrow K_{\log} \),
where $K_{\log}$ is identified with its pull-back on $\tilde{\Sigma}$. The pair $(L, \phi)$ is called an \textit{rth-root}, or a generalized spin structure.

**Definition 5.2.2:** Suppose that the chart of $\tilde{\Sigma}$ at an orbifold point $z_i$ is $D/\mathbb{Z}/m$ with action $e^{2\pi i/m}(z) = e^{2\pi i}z$. Suppose that the local trivialization of $L$ is $D \times \mathbb{C}/\mathbb{Z}/m$ with the action

$$e^{2\pi i/m}(z, w) = (e^{2\pi i}z, e^{2\pi i}w).$$

When $d = 0$, we call $z_i$ a Ramond marked point. When $d > 0$, we call $z_i$ a Neveu-Schwarz marked point.

**Remark 5.2.3:** If $L$ is an orbifold line bundle on a smooth orbifold Riemann surface $\tilde{\Sigma}$, then the sheaf of local invariant sections of $L$ is locally free of rank one, and hence dual to a unique orbifold line bundle $|L|$ on $\Sigma$. $|L|$ corresponds to the desingularization of $L$ [CR] (Prop 4.1.2). It can be constructed as follows.

We keep the local trivialization at other places and change it at orbifold point $z_i$ by a $\mathbb{Z}/m$-equivariant map $\Psi : D - \{0\} \times \mathbb{C} \to D - \{0\} \times \mathbb{C}$ by

$$(z, w) \to (z^m, z^{-d}w),$$

where $\mathbb{Z}/m$ acts trivially on the second $D - \{0\} \times \mathbb{C}$. Then, we extend $L|_{D - \{0\} \times \mathbb{C}}$ to a smooth holomorphic line bundle over $\Sigma$ by the second trivialization. Since $\mathbb{Z}/m$ acts trivially, hence it is a line bundle over $\Sigma$, which we denote by $|L|$. Note that if $z_i$ is a Ramond marked point, $|L| = L$ locally. When $z_i$ is a Neveu-Schwarz marked point, $|L|$ will be different locally. In particular, it does not have a canonical fiber over $z_i$.

**Example 5.2.4:** An orbifold has a natural orbifold canonical bundle, defined as the top wedge product of its (orbifold) cotangent bundle. When $\tilde{\Sigma} = (\Sigma, z, m)$ is an orbifold Riemann surface, the desingularization

$$\pi_*(K_{\Sigma}) = K_{\Sigma} \otimes i \mathcal{O}(-(m_i - 1)z_i).$$

Next, we study the sections. Suppose that $s$ is a section of $|L|$ having local representative $g(u)$. Then, $(z, z^{d}g(z^{m}))$ is a local section of $L$. Therefore, we obtain a section $\pi(s) \in \Omega^0(L)$ which equals to $s$ outside of orbifold points under identification (5.1). It is clear that if $s$ is holomorphic, so it $\pi(s)$. If we start from an analytic section of $L$, we can reverse the above process to obtain a section of $|L|$. In particular, $L, |L|$ have the isomorphic space of holomorphic section. In the same way, there is a map $\pi : \Omega^{0,1}(|L|) \to \Omega^{0,1}(L)$, where $\Omega^{0,1}(L)$ is interpreted as orbifold $(0,1)$-form with value in $L$. Suppose that $g(u)du$ is a local representative of a section of $t \in \Omega^{0,1}(|L|)$. $\pi(t)$ has local representative $z^{k}g(z^{m})mz^{m-1}dz$. Moreover, $\pi$ induces an isomorphism from $H^1(|L|) \to H^1(L)$.

Suppose that $L^r \cong K_{\log}$. $rd = lm$ for some $l, d < m$ implies $l < r$. Therefore, $\frac{d}{m} = \frac{1}{r}$. Suppose that $s \in \Omega^0(|L|)$ with local representative $g(u)$. Then, $s^r$ has local representative $z^{rk}g(z^{d}) = z^{ml}g^r(z^{d}) = w^r g^r(u)$. Hence, $s^r \in \Omega^0(K_{\log} \otimes \mathcal{O}((-li)z_i)$. When $l_i > 0, s^r \in \Omega^0(K)$.

**Definition 5.2.5:** Suppose that all the marked points are Neveu-Schwarz. We have

$$s^{r-1} \in \Omega^0(K \otimes |L|^{-1}) \cong \Omega^{0,1}(|L|),$$

Hence, $\bar{s}s + s^{r-1}$ is an element of $\Omega^{0,1}(|L|)$. We define spin equation to be

$$\bar{s}s + s^{r-1} = 0.$$

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The following Lemma is due to Witten.

**Lemma 5.2.6:** Suppose that all the marked points are Neveu-Schwarz for $L$. Then, $\bar{\partial}s + s^{r-1} = 0$ iff $s = 0$.

**Proof:** Fix a Kähler metric on $\Sigma$ and an Hermitian metric on $L$.

$$\bar{\partial}s, s^{r-1} = \frac{1}{r} \int_\Sigma \partial(\bar{s}^r) = 0.$$ 

Hence,

$$(\bar{\partial}s + s^{r-1}, \bar{\partial}s + s^{r-1}) = (\bar{\partial}s, \bar{\partial}s) + (s^{r-1}, s^{r-1}) = 0$$

iff

$$\bar{\partial}s = s^{r-1} = 0,$$

and hence $s = 0$.

When $L$ possesses a Ramond marked point $z_i$, $s^{r-1} \in \Omega^0(\bar{K}_{\log} \otimes L) = \Omega^{0,1}(\Sigma - \{z_i\}, L) = \Omega^{0,1}(\Sigma - \{z_i\}, |L|)$. More generally, let $\Sigma$ be the puncture Riemann surface obtained by removing all the Ramond marked points.

**Definition 5.2.7:** For $s \in \Omega^0(\bar{\Sigma}, |L|)$,

$$\bar{\partial}s + s^{r-1} \in \Omega^{0,1}(\bar{\Sigma}, |L|),$$

and we define spin equation to be

$$\bar{\partial}s + s^{r-1} = 0.$$ 

To have an elliptic theory, we should perform a change of coordinate $z \to -\text{log} z$ around puncture point and view $\Sigma$ as a Riemann surface with cylindric ends. It is not clear if the solution of spin equation is zero in Ramond case.

We spend the rest of section to extend above discussion to nodal curve

**Definition 5.2.8:** A nodal curve with $k$ marked points is a pair $(\Sigma, z)$ of a connected topological space $\Sigma = \bigcup\pi_\nu(\Sigma_\nu)$, where $\Sigma_\nu$ is a smooth complex curve, and $\pi_\nu : \Sigma_\nu \to \Sigma$ is a continuous map, and $z = (z_1, \ldots, z_k)$ are distinct $k$ points in $\Sigma$ with the following properties:

- For each $z \in \Sigma_\nu$, there is a neighborhood of it such that the restriction of $\pi_\nu : \Sigma_\nu \to \Sigma$ to this neighborhood is a homeomorphism to its image.
- For each $z \in \Sigma$, we have $\sum_\nu \#\pi_\nu^{-1}(z) \leq 2$.
- $\sum_\nu \#\pi_\nu^{-1}(z_i) = 1$ for each $z_i \in z$.
- The number of complex curves $\Sigma_\nu$ is finite.
- The set of nodal points $\{z \mid \sum_\nu \#\pi_\nu^{-1}(z) = 2\}$ is finite.

A point $z \in \Sigma_\nu$ is called singular (or a node) if $\sum_\nu \#\pi_\nu^{-1}(\pi_\nu(z)) = 2$. A point $z \in \Sigma_\nu$ is said to be a marked point if $\pi_\nu(z) = z_i \in z$. Each $\Sigma_\nu$ is called a component of $\Sigma$. Let $k_\nu$ be the number of points on $\Sigma_\nu$ which are either singular or marked, and $g_\nu$ be the genus of $\Sigma_\nu$, a nodal curve $(\Sigma, z)$ is called stable if $k_\nu + 2g_\nu \geq 3$ holds for each component $\Sigma_\nu$ of $\Sigma$. 

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A map $\vartheta : \Sigma \to \Sigma'$ between two nodal curves is called an *isomorphism* if it is a homeomorphism, and if it can be lifted to biholomorphisms $\vartheta_{\nu} : \Sigma_{\nu} \to \Sigma'_{\nu}$ for each component $\Sigma_{\nu}$ of $\Sigma$. If $\Sigma, \Sigma'$ have marked points $z = (z_1, \cdots, z_k)$ and $z' = (z'_1, \cdots, z'_k)$, respectively, then we require $\vartheta(z_i) = z'_i$ for each $i$. Let $\text{aut}(\Sigma, z)$ be the group of automorphisms of $(\Sigma, z)$.

**Definition 5.2.9:** A nodal orbicurve is a nodal marked curve $(\Sigma, z)$ with an orbifold structure as follows:

- The set $z_{\nu}$ of orbifold points of each component $\Sigma_{\nu}$ is contained in the set of marked points and nodal points $z$.
- A neighborhood of a marked point is uniformized by a branched covering map $z \to z^{m_i}$ with $m_i \geq 1$.
- A neighborhood of a nodal point (viewed as a neighborhood of the origin of $\{xy = 0\} \subset \mathbb{C}^2$) is uniformized by a branched covering map $(x, y) \to (x^{n_j}, y^{n_j})$, with $n_j \geq 1$, and with group action $e^{2\pi i/n_j}(x, y) = (e^{2\pi i/n_j}x, e^{-2\pi i/n_j}y)$.

Here $m_i$ and $n_j$ are allowed to be equal to one, i.e., the corresponding orbifold structure is trivial there. We denote the corresponding nodal orbicurve by $(\Sigma, z, m, n)$ where $m = (m_1, \cdots, m_k)$ and $n = (n_j)$.

**Definition 5.2.10:** An isomorphism between two nodal orbicurves $\vartheta : (\Sigma, z, m, n) \to (\Sigma', z', m', n')$ is a collection of $C^\infty$ isomorphisms $\vartheta_{\nu}$ between orbicurves $\Sigma_{\nu}$ and $\Sigma'_{\nu}$ which induces an isomorphism $\vartheta : (\Sigma, z) \to (\Sigma', z')$. The group of automorphisms of a nodal orbicurve $(\Sigma, z, m, n)$ is denoted by $\text{aut}(\Sigma, z, m, n)$. It is easily seen that $\text{aut}(\Sigma, z, m, n)$ is a subgroup of $\text{aut}(\Sigma, z)$ of finite index.

**Definition 5.2.11:** A nodal r-spin orbicurve $\tilde{\Sigma} = (\Sigma, z, m)$ together with a pair $(L, \phi)$, where $L$ is an orbifold line bundle on $\tilde{\Sigma}$, and $\phi : L^{r} \to K^{1}_{\log}$ is an isomorphism. Suppose that the local chart of $L$ at nodal point is $D_0 \times \mathbb{C}/\mathbb{Z}/m$, where $D_0 = \{xy = 0, |x|, |y| < \epsilon\}$ and the action is $\lambda(x, y, w) = (e^{2\pi i/m}x, e^{-2\pi i/m}y, e^{2\pi i/w}).$ We call the nodal point a Ramond nodal point of $d = 0$. Otherwise, we call it a Neveu-Schwarz nodal point.

**Remark 5.2.12:** Near a Ramond nodal point, the local invariant sections are not necessarily locally free, but they do form a rank-one torsion-free sheaf (see [AD]). Hence, the desingularization $|L|$ is not a line bundle near a Ramond nodal point. However, we can desingularize the restriction of $L$ on each component and obtain a collection of line bundle over each component. It is clear that a NS-nodal point give rise a pair of NS-marked points on its components. Hence, the desingularization at this point has no canonical fiber. Therefore, there is no gluing condition at NS-nodal point. Actually, the situation is more subtle than this. For our purpose, it is enough to consider desingularization $|L|$ as a collection of line bundles over each component glued together at Ramond nodal points.

Next, we define spin equation. For simplicity, we assume $\Sigma = \Sigma_1 \cup \Sigma_2$ at nodal point $Q = \{p = q\}$. When $Q$ is NS,

$$\Omega^0(|L|) = \Omega^0(|L_1|) \times \Omega^0(|L_2|), \Omega^{0,1}(|L|) = \Omega^{0,1}(|L_1|) \times \Omega^{0,1}(|L_2|)$$

When $Q$ is Ramond,

$$\Omega^0(|L|) = \{s_1(p) = s_2(q); s_i\Omega^0(|L_i|)\}, \Omega^{0,1}(|L|) = \{\omega_1(p) = \omega_2(q); \omega_i \in \Omega^0(\Sigma_i, |L_i|)\}$$

In either case, there is a map

$$\pi : \Omega^0(|L|) \to \Omega^0(L), \Omega^{0,1}(|L|) \to \Omega^{0,1}(L)$$
inducing an isomorphism on the space of holomorphic sections (forms). To see this, recall that
\( \pi(s) \) for \( s \in \Omega^0(\mathcal|L|) \) is automatically zero at NS-marked points. Hence, \( \pi(s) \) can be glued (as zero) together at NS-nodal points even though \( s \) doesn’t. For the same reason, if \( \omega \in \Omega^0(\mathcal|L|) \), \( \pi(\omega) \) can be viewed as a 1-form with zero residue at nodal point and hence can be glued together.

Recall that for a nodal Riemann surface \( \Sigma \), its canonical bundle \( K_\Sigma \) restricts to \( K_{\text{log}} \) on each component. Therefore, spin equation is well-defined no matter the nodal point is Ramond or Neveu-Schwarz.

**Lemma 5.2.13:** Suppose that \( \Sigma \) is a nodal orbicurve and \( L \) is \( r \)-root of \( K_{\text{log}} \) such that all the marked points are Neveu-Schwarz. Then \( \bar{\partial}s + s^{r-1} = 0 \) iff \( s = 0 \).

**Proof:** Without the loss of generality, we assume that \( \Sigma = \Sigma_1 \vee \Sigma_2 \) has only two component and no other marked point. Suppose that \( s = (s_1, s_2) \). If the nodal point is NS, The lemma follows from Witten Lemma directly. Suppose that nodal point is Ramond,

\[
(\bar{\partial}s, s^{r-1}) = \sum \int_\Sigma \partial s_i \cdot \bar{s}_i^{r-1} = \sum \frac{1}{r} \int_\Sigma \partial(s^r) = \bar{s}^r(p) - s^r(q) = 0.
\]

Hence,

\[
(\bar{\partial}s + s^{r-1}, \bar{s} + s^{r-1}) = (\bar{\partial}s, \bar{s}) + (s^{r-1}, s^{r-1}) = 0
\]

iff

\[
\bar{\partial}s = s^{r-1} = 0,
\]

and hence \( s = 0 \).

### 5.3 The moduli space of spin orbifold stable maps

As we show in last section, the solution of equations (5.1) can be described using \( L \) only. In this section, we construct its moduli space.

**Definition 5.3.1:** Let \((X, J)\) be an almost complex orbifold. An \( r \)-spin orbifold stable map into \((X, J)\) is a quadruple \((f, (\Sigma, z, m, n), (L, \phi), \xi)\) described as follows:

1. \( f \) is a continuous map from a nodal orbicurve \((\Sigma, z)\) into \( X \) such that each \( f_\nu = f \circ \pi_\nu \) is a pseudo-holomorphic map from \( \Sigma_\nu \) into \( X \).
2. \( \xi \) is an isomorphism class of compatible structures (see definition in [CR2]).
3. Let \( k_\nu \) be the order of the set \( z_\nu \), namely the number of points on \( \Sigma_\nu \) which are special (i.e. nodal or marked ), if \( f_\nu \) is a constant map, then \( 2g_\nu - 2 + k_\nu > 0 \).
4. At any marked or nodal point \( p \) the induced homomorphism on the local group \( \rho_p : G_p \to U(1) \times G_{f(p)} \) is injective.

An isomorphism between \( r \)-spin orbifold stable maps

\[
(f, (\Sigma, z, m, n), (L, \phi), \xi) \to (f', (\Sigma', z', m', n'), (L', \phi'), \xi')
\]

is defined to be a pair \((\alpha, \beta)\) of an isomorphism \( \alpha : (f, (\Sigma, z, m, n), \xi) \to (f', (\Sigma', z', m', n'), \xi') \) of the underlying orbifold stable maps and an isomorphism \( \beta : f^*L' \to L \), which is compatible with \( \phi \) and \( f^*(\phi') \) in the obvious way.
Since the number of automorphisms of $L$ fixing $\phi$ is finite, and since the underlying map of an $r$-spin orbifold stable map is a stable map, its automorphism group is finite.

Any orbifold line bundle gives a natural representation over an orbifold Riemann surface $\Sigma$ can be modeled locally near a non-nodal point $x_0$ as $(D \times C)/(\mathbb{Z}/k)$, where $\mathbb{Z}/k$ acts on the disc $D$ by standard complex multiplication by $e^{2\pi i/k}$, and on $C$ by a representation $\mathbb{Z}/k \to U(1)$. Similarly, near a node $p$, $\Sigma$ can be modeled as $(D \times D')/(\mathbb{Z}/k)$ where $D \times D'$ is the union of two discs $D$ and $D'$ joined at 0. $\mathbb{Z}/k$ acts on $D$ via multiplication by $e^{2\pi i/k}$ and on $D'$ via multiplication by $e^{-2\pi i/k}$; and on $C$ by a representation $\mathbb{Z}/k \to U(1)$. A good orbifold map $f: \Sigma \to X$ induces a homomorphism $\mathbb{Z}/k \to U(1) \times G_{f(x_0)}$. We use $\rho_p$ to denote the product homomorphism $\mathbb{Z}/k \to U(1) \times G_{f(p)}$.

If $L$ is a $r$-spin structure, $\rho_p$ has image in $\mathbb{Z}/r \times G_{f(p)}$.

Given an $r$-spin orbifold stable map $(f, (\Sigma, z), (L, \phi), \xi)$, there is an associated homology class $f_*([\Sigma])$ in $H_2(X; \mathbb{Z})$ defined by $f_*([\Sigma]) = \sum_{\nu} f(\circ \pi_{\nu})_*[\Sigma_{\nu}]$. By previous argument, for each marked point $z$ on $\Sigma_{\nu}$, say $\pi_{\nu}(z) = z_i \in \mathbb{Z}$, $\xi_{\nu}$ determines, by the group homomorphism $\rho_{z_i}$, a conjugacy class $(g_i, l_i)$ where $g_i \in G_{f(z_i)}$ and $l_i \in \mathbb{Z}/r$. Let $\hat{X}$ be the disjoint union of $X_{(g_i,l_i)}$. We have a map $ev$ sending each (equivalence class of) stable maps into $\hat{X}^k$ by $(f, (\Sigma, z), \xi) \to ((f(z_1), (g_1,l_1)), \ldots, (f(z_k), (g_k,l_k)))$. $\hat{X}$ plays a role in r-spin orbifold theory analogous to the role played by inertial orbifold $X$ in “usual” (without spin structure) orbifold theory.

Similarly, we will call a type $\mathfrak{x} = \bigoplus_{i} X_{(g_i,l_i)}$ Ramond if any $l_i$ is zero, and Neveu-Schwarz otherwise.

**Remark 5.3.2:** Despite some superficial resemblance to the moduli of orbifold stable maps into $X \times \mathcal{B}(\mathbb{Z}/r)$, we will show in Section 5.3 that the moduli of $r$-spin orbifold stable maps has a completely different virtual fundamental class (even of a different dimension) and thus different Gromov-Witten invariants.

**Definition 5.3.3:** An $r$-spin stable map $(f, (\Sigma, z), (L, \phi), \xi)$ is said to be of type $\mathfrak{x}$ if $ev((f, (\Sigma, z), (L, \phi), \xi)) \in \mathfrak{x}$. Given a homology class $A \in H_2(X; \mathbb{Z})$, we let $\overline{\mathcal{M}}_{g,k,p}(X, J, A, \mathfrak{x})$ denote the moduli space of equivalence classes of $r$-spin orbifold stable maps of genus $g$, with $k$ marked points, homology class $A$, and type $\mathfrak{x}$, i.e.,

$$\overline{\mathcal{M}}_{g,k,p}(X, J, A, \mathfrak{x}) = \{ [(f, (\Sigma, z), (L, \phi), \xi)] | g_\Sigma = g, \# z = k, f_*([\Sigma]) = A, ev((f, (\Sigma, z), (L, p), \xi)) \in \mathfrak{x} \}.$$

**Remark 5.3.4:**

1. When $\overline{\mathcal{M}}_{g,k,p}(X, J, A, \mathfrak{x})$ is non-empty, the map forgetting $(L, \phi)$ makes $\overline{\mathcal{M}}_{g,k,p}(X, J, A, \mathfrak{x})$ into a finite branched cover of the moduli space of orbifold stable maps $\overline{\mathcal{M}}_{g,k}(X, J, A, \mathfrak{x})$. Here $\mathfrak{x}$ is the image of $\mathfrak{x}$ under the map $X^k \to \hat{X}^k$ forgetting the $l_i$.

2. When $X$ is a point with no orbifold structure, we obtain the usual moduli space of $r$-spin curves $\overline{\mathcal{M}}_{g,A}(J)$ (Mapping to a Point).

3. The orbifold line bundle $L$ has degree (or first Chern number)

$$\deg(L) = \deg(K_{\log})/r = \frac{2g - 2 + k}{r},$$

which must be congruent to $\sum \frac{l_i}{r} (\text{mod } \mathbb{Z})$, by [CR1, Prop 4.1.2]. Thus, $\overline{\mathcal{M}}_{g,k,p}(X, J, A, \mathfrak{x})$ is empty unless $2g - 2 + k - \sum l_i = 0 (\text{mod } r)$. 

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5.4 r-Spin orbifold Gromov-Witten invariants

Following the procedure in [CR2] for the moduli space of orbifold stable maps, we can define a natural stratification and topology for \( \overline{M}_{g,k,r}(X,J,\hat{x}) \).

**Proposition 5.4.1:** Suppose \( X \) is either a symplectic orbifold with a symplectic form \( \omega \) and an \( \omega \)-compatible almost-complex structure \( J \), or a projective orbifold with an integrable almost-complex structure \( J \). Then the moduli space \( \overline{M}_{g,k,r}(X,J,\hat{x}) \) is compact and metrizable.

By a slight modification of the construction in [CR2], one can obtain a virtual fundamental cycle for the moduli space of spin orbifold stable maps.

**Theorem 5.14:** If all \( l_i \) are non-zero, we can construct a Kuranishi structure of \( \overline{M}_{g,k,r}(X,A,J,\hat{x}) \) in the sense of [FO]. Hence, it defines a virtual fundamental cycle \( [\overline{M}_{g,k,r}(X,A,J,\hat{x})]_{\text{vir}} \) of degree

\[
2 \left[ c_1(A) + (n-3)(1-g) + k - \iota(\hat{x}) - \frac{1}{r} \left( (g-1)(r-2) + k - \sum_i l_i \right) \right].
\]

For any component \( \hat{x} = (X_{(g_1,l_1)}, \cdots, X_{(g_k,l_k)}) \), there are \( k \) evaluation maps

\[
e_i : \overline{M}_{g,k,r}(X,J,A,\hat{x}) \to X_{(g_i,l_i)}, \quad i = 1, \cdots, k.
\]

We also have a map

\[
p : \overline{M}_{g,k,r}(X,J,A,\hat{x}) \to \overline{M}_{g,k},
\]

where \( p \) contracts the unstable components of the domain to obtain a stable Riemann surface in \( \overline{M}_{g,k} \).

In addition to the forgetful morphism \( \overline{M}_{g,k,r} \to \overline{M}_{g,k} \), there is another important morphism of the moduli space of r-spin orbifold stable maps; namely, *forgetting tails*. When a nodal r-spin orbicurve \((\tilde{\Sigma},(L,\varphi)) = ((\Sigma,z,m),(L,\varphi))\) has a point \((z_i,m_i)\) such that the representation \(\mathbb{Z}/m_i \to \mathbb{Z}/r \to U(1)\), then \((L,\varphi)\) induces an r-spin structure \((\pi_*L,\pi_*\varphi)\) on the orbicurve \(\Sigma := (\Sigma, (z_1,\ldots,\hat{z}_i,\ldots,z_k), (m_1,\ldots,\hat{m}_i,\ldots,m_k))\) obtained by forgetting the \( i \)th marked point and its orbifold structure. Here \( \pi : \tilde{\Sigma} \to \Sigma \) is the obvious “forgetful” map.

Indeed, if \( \tilde{\Sigma} \) is the marked orbicurve

\[
\tilde{\Sigma} = (\Sigma, z, (m_1,\ldots,1,\ldots,m_k))
\]

obtained from \( \tilde{\Sigma} \) by forgetting the orbifold structure at \( z_i \), but keeping the marked point \( z_i \), we have a commuting diagram

\[
s \quad \pi \quad \downarrow \quad \pi
\]

By Remark ?? we have \( p_*\varphi : (p_*L)^{\otimes r} \to K_{\Sigma_{\log}} \otimes \mathcal{O}(z_i) \cong K_{\Sigma_{\log}} \), and of course \( p_*L = \pi_*L \), where we have simply forgotten the \( i \)th point \( z_i \).

Thus, if the type \( \hat{x} \) corresponds to points of \( \hat{X}^k \) with \( l_i = r-1 \), there exists a *forgetting tails* morphism

\[
\overline{M}_{g,k,r}(X,J,A,\hat{x}) \to \overline{M}_{g,k-1,r}(X,J,A,\hat{x}')
\]

where \( \hat{x}' \) is the connected component of \( \hat{X}^{k-1} \) obtained by mapping the component \( \hat{x} \) to \( \hat{X}^{k-1} \), via the map forgetting the \( i \)th component of \( \hat{X}^k \).
For any set of cohomology classes \( \gamma_i \in H^{* - 2(g_i)}(X_{(g_i, l_i)}; \mathbb{Q}) \subset H^*_\text{orb}(X; \mathbb{Q}), i = 1, \cdots, k \) and \( \sigma \in H^*(\overline{M}_{g, k}, \mathbb{Q}) \), the \( r \)-spin orbifold Gromov-Witten invariant is defined as the pairing

\[
\Psi_{(g, k, r, A, \hat{x})}^{X, J}(\sigma; \tau_{a_1}(\gamma_1), \cdots, \tau_{a_k}(\gamma_k)) = p^*\sigma(\prod_{i=1}^k \psi_i^{a_i})e^*(\sigma_i) \cap \overline{\mathcal{M}_{g, k, r}(X, J, A, \hat{x})}^\text{vir}.
\]

where \( \psi_i \) is the first Chern class of the line bundle generated by the cotangent space of the \( i \)-th marked point. When \( a_i < 0 \), we define it to be zero. The same argument as in the case of ordinary GW-invariants yields

**Proposition 5.4.2:**

1. \( \Psi_{(g, k, r, A, \hat{x})}^{X, J}(\sigma; \tau_{a_1}(\gamma_1), \cdots, \tau_{a_k}(\gamma_k)) = 0 \) unless \( \deg \sigma + \sum_i (\deg_{\text{orb}}(\gamma_i) + a_i) = 2c_1(A) + 2(n - 3)(1 - g) + 2k - 2(g - 1)(1 - \frac{2}{r}) - \frac{2k}{r} + 2 \sum l_i/r \), where \( \deg_{\text{orb}}(\gamma_i) \) is the orbifold degree of \( \gamma_i \) obtained after degree shifting.

2. \( \Psi_{(g, k, r, A, \hat{x})}^{X, J}(\sigma; \tau_{a_1}(\gamma_1), \cdots, \tau_{a_k}(\gamma_k)) \) is independent of the choice of \( J \) and hence is an invariant of symplectic structures.

Spin orbifold Gromov-Witten invariants are expected to satisfy a set of interesting axioms. Some of them are similar to the axiom of ordinary Gromov-Witten invariants. But some are new addition with the introduction of spin curves. We are in the process to verify these axioms.

### 6 Conclusion Remark

During the last year and half, this new subject is experiencing an explosion. This author often hears that a new idea would run through the physics like wild fire. We mathematician are used to be on the slower pace. However, it is fair to say that in such a short time, the basic idea of stringy orbifold is spreading like a wild fire in mathematics. This speed of new results is even beyond this author’s expectation. The excitement reminds him a lot the quantum cohomology era during 94-96. The diversity of papers in this volume is the evidence of such an explosion.

Because of the limited time and space, this author did not include several topics which he initially intends to. Recently, there is an algebraic construction of the orbifold quantum cohomology over Deligne-Mumford stack including an interesting integral version of the orbifold cohomology. Furthermore, the construction of the orbifold cohomology of \([CR]\) has been adapted to construct an orbifold Chow rings \([AV], [F]\). This exciting development is surveyed by their authors in this volume.

There are some very interesting constructions of the orbifold elliptic genus \([BL], [DLM]\). Physically, it belongs to a different type of string theory (heterotic string theory). Mathematically, they are also quite different from other orbifold theory such as orbifold cohomology. For examples, the twisted sectors in the orbifold cohomology is based on the commuting pair while the twisted sector in the elliptic genus is based on commuting triple.

There is very recent work of Borisov-Mavlyutov \([?, BM]\) who are considering the mirror symmetry from the vertex algebra point of view. In the process, they seems to suggest some conjectural solution for the orbifold cohomology product for certain toric varieties. It would be an interesting problem to check their answer.

There are also very recent papers by Lupercio-Poddar \([LP]\) and Yasuda \([Y]\), where a weak form of the orbifold string theory conjecture for the equivalence of orbifold the Hodge number and
Hodge number of the crepant resolution was solved in its complete generality. This is an exciting development.

We are living in a good time. This author expects more excitments to come!

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