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To cite this version:
Nemanja Draganić, François Dross, Jacob Fox, António Girão, Frédéric Havet, et al.. Powers of paths in tournaments. Combinatorics, Probability and Computing, 2021, pp.1-5. 10.1017/S0963548321000067. hal-03269230

HAL Id: hal-03269230
https://inria.hal.science/hal-03269230
Submitted on 10 Nov 2021

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Powers of paths in tournaments

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Abstract

In this short note we prove that every tournament contains the $k$-th power of a directed path of linear length. This improves upon recent results of Yuster and of Girão. We also give a complete solution for this problem when $k = 2$, showing that there is always a square of a directed path of length $\lceil 2n/3 \rceil - 1$, which is best possible.

1 Introduction

One of the main themes in extremal graph theory is the study of embedding long paths and cycles in graphs. Some of the classical examples include the Erdős–Gallai theorem [?] that every $n$-vertex graph with average degree $d$ contains a path of length $d$, and Dirac’s theorem [?] that every graph with minimum degree $n/2$ contains a Hamilton cycle. A famous generalization of this, conjectured by Pósa and Seymour, and proved for large $n$ by Komlós, Sárközy and Szemerédi [?], asserts that if the minimum degree is at least $kn/(k + 1)$, then the graph contains the $k$-th power of a Hamilton cycle.

In this note, we are interested in embedding directed graphs in a tournament. A tournament is an oriented complete graph. The $k$-th power of the directed path $\vec{P}_\ell = v_0 \ldots v_\ell$ of length $\ell$ is the graph $\vec{P}_k^\ell$ on the same vertex set containing a directed edge $v_iv_j$ if and only if $i < j \leq i + k$. The $k$-th power of a directed cycle is defined analogously. An old result of Bollobás and Häggkvist [?] says that, for large $n$, every $n$-vertex tournament with all indegrees and outdegrees at least $(1/4 + \varepsilon)n$ contains the $k$-th power of a Hamilton cycle.

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(the constant $1/4$ is optimal). However, we cannot expect to find powers of directed cycles in general, as the transitive tournament contains no cycles at all.

What about powers of directed paths? A classical result, which appears in every graph theory book (see, e.g., [?]), says that every tournament contains a directed Hamilton path. On the other hand, Yuster [?] recently observed that some tournaments are quite far from containing the square of a Hamilton path. In particular, there is an $n$-vertex tournament that does not even contain the square of $\vec{P}_{2n/3}$, and more generally, for every $k \geq 2$, there are tournaments with $n$ vertices and no $k$-th power of a path with more than $nk/2^{k/2}$ vertices. In the other direction, Yuster proved that every tournament with $n$ vertices contains the square of a path of length $n^{0.295}$. This was improved very recently by Girão [?], who showed that for fixed $k$, every tournament on $n$ vertices contains the $k$-th power of a path of length $n^{1-o(1)}$.

Both papers noted that no sublinear upper bound is known. Our main result shows that the maximum length is in fact linear in $n$. Theorem 1. For $n \geq 2$, every $n$-vertex tournament contains the $k$-th power of a directed path of length $n/2^{4k+6k}$.

The proof of this theorem combines Kővári–Sós–Turán style arguments, used for the bipartite Turán problem, and median orderings of tournaments. A median ordering is a vertex ordering that maximizes the number of forward edges. ?? and Yuster’s construction show that an optimal bound on the length has the form $n/2^{\Theta(k)}$. It would be interesting to find the exact value of the constant factor in the exponent. Optimizing our proof can yield a lower bound of $n/2^{c_k+o(k)}$ with $c \approx 3.9$, but is unlikely to give the correct bound.

We also improve the exponential constant in the upper bound from $1/2$ to $1$.

Theorem 2. Let $k \geq 5$ and $n \geq k(k+1)2^k$. There is an $n$-vertex tournament that does not contain the $k$-th power of a directed path of length $k(k+1)n/2^k$.

Note that this theorem also holds trivially for $k \leq 4$, when $k(k+1)n/2^k > n$.

Finally, we can solve the problem completely in the special case of $k = 2$. Once again, the proof uses certain properties of median orderings.

Theorem 3. For $n \geq 1$, every $n$-vertex tournament contains the square of a directed path of length $\ell = [2n/3] - 1$, but not necessarily of length $\ell + 1$.

Theorems ??, ?? and ?? are proved in Sections ??, ?? and ??, respectively.

2 Lower bound

We will need the following Kővári–Sós–Turán style lemma.

Lemma 4. Let $G$ be a directed graph with disjoint vertex subsets $A$ and $B$ with $|A| = 2k + 1$, $|B| \geq 2^{4k+4}k$, and every vertex in $A$ has at least $(1 - \frac{1}{2k+1})|B|/2$ outneighbours in $B$. Then $A$ contains a subset $A'$ of size $k$ that has at least $(2k+1)2^{2k}$ common outneighbours in $B$. 
Proof. Suppose there is no such set $A'$. Then every $k$-subset of $A$ appears in the inneighbourhood of less than $(2k+1)2^{2k}$ vertices in $B$. So if $d^-(v)$ denotes the number of inneighbours a vertex $v \in B$ has in $A$, then we have

$$\binom{2k+1}{k} \cdot (2k+1)2^{2k} = \left(\frac{|A|}{k}\right) \cdot (2k+1)2^{2k} > \sum_{v \in B} \binom{d^-(v)}{k}. \tag{1}$$

On the other hand, $\sum_{v \in B} d^-(v) \geq |A|(1 - \frac{1}{2k+1})|B|/2 = k|B|$. By Jensen’s inequality, $\sum_{v \in B} \binom{d^-(v)}{k} \geq |B| \cdot \left(\frac{\sum_{v \in B} d^-(v)/|B|}{k}\right) = |B| \geq 2^{4k+4}k$. This contradicts (1).

One more ingredient we need for the proof of ?? is the folklore fact that every tournament on $2^m$ vertices contains a transitive subtournament of size $m + 1$. This is easily seen by taking a vertex of outdegree at least $2^{m-1}$ as the first vertex of the subtournament, and then recursing on the outneighbourhood.

**Proof of ??**. Order the vertices as $0, 1, \ldots, n - 1$ to maximize the number of forward edges, i.e., the number of edges $ij$ such that $i < j$. As was mentioned in the introduction, we will refer to such a sequence as a median ordering of the vertices. We denote an “interval” of vertices with respect to this ordering by $[i, j] = \{i, \ldots, j - 1\}$, where $0 \leq i < j \leq n$.

We will embed $P_\ell^k$ inductively using the following claim.

**Claim.** Let $t = 2^{4k+4}k$ and $t \leq i \leq n - (2k+1)t$. For every subset $A^* \subseteq [i - t, i)$ of size $2^k$, there is an index $i + t \leq j \leq i + (2k+1)t$ and a set $A' \subseteq A^*$ of size $k$ such that $A'$ induces a transitive tournament and its vertices have at least $2^{2k}$ common outneighbours in $[j - t, j)$.

**Proof.** There is a subset $A \subseteq A^*$ of size $2k + 1$ that induces a transitive tournament. Let $B = [i, i + (2k+1)t]$. Then every vertex $v \in A$ has at least $kt = \left(1 - \frac{1}{2k+1}\right) |B|/2$ outneighbours in $B$. Indeed, otherwise $v$ would have more than $(k + 1)t$ inneighbours in the interval $B$, so moving $v$ to the end of this interval would increase the number of forward edges in the ordering, contradicting our choice of the vertex ordering.

We can thus apply ?? to find a $k$-subset $A' \subseteq A$ with least $2^{2k + 1}2^{2k}$ common outneighbours in $B$. Partition $B$ into $2k + 1$ intervals of size $t$, and we can choose $j$ accordingly so that $A'$ has at least $2^{2k}$ common outneighbours in the interval $[j - t, j)$.

The theorem trivially holds for $n < 2^{2k}$, so assume $n \geq 2^{2k}$. Let $i_0 = 2^{2k}$ and $A_0 = [0, 2^{2k})$, and apply the Claim with $i = i_0$ and $A^* = A_0$. We get a set $A' \subseteq A_0$ of size $k$ that induces a transitive tournament, i.e., the $k$-th power of some path $v_0 \ldots v_{k-1}$. Moreover, this $A'$ has at least $2^{2k}$ common outneighbours in some interval $[j - t, j]$ with $i_0 + t \leq j \leq i_0 + (2k + 1)t$. Let us define $i_1 = j$, and choose $A_1$ to be any $2^{2k}$ of the common outneighbours.

At step $s$, we apply the Claim again with $i = i_s$ and $A^* = A_s$ to find the $k$-th power of some path $v_{sk} \ldots v_{(s+1)k-1}$ in $A_s$ with $2^{2k}$ common outneighbours in some $[i_{s+1} - t, i_{s+1})$ with $i_s + t \leq i_{s+1} \leq i_s + (2k + 1)t$, and repeat this process until some step $\ell$ with $i_\ell > n - (2k + 1)t$. Note that intervals $[i_s - t, i_s)$ and $[i_{s+1} - t, i_{s+1})$ are always disjoint. Finally, $A_\ell$ must also contain a transitive tournament of size $2k + 1$. Call these vertices $v_{i_\ell k}, \ldots, v_{(\ell + 2)k}$. Observe that $n - (2k + 1)t < i_\ell \leq 2^{2k + 1}(2k + 1)t$, so $n < (\ell + 2)(2k + 1)t$. 


Then \(v_0 \ldots v_{(\ell+2)k}\) is a directed path of length \((\ell+2)k \geq kn/(2k+1)t \geq n/(2^{4k+6}k)\) whose \(k\)-th power is contained in the tournament. In fact, we proved a bit more: the tournament contains all edges of the form \(v_av_b\) with \(a < b\) and \(\lfloor a/k \rfloor + 1 \geq \lfloor b/k \rfloor\).

\[\square\]

### 3 Upper bound

Let \(\ell_k(n)\) denote the smallest integer \(\ell\) such that there is an \(n\)-vertex tournament that does not contain \(\vec{P}_\ell^k\), or in other words, the largest integer such that every \(n\)-vertex tournament contains the \(k\)-th power of a directed path on \(\ell\) vertices.

To prove \(\ell_k(n)\), we first note that \(\ell_k(n)\) is subadditive.

**Lemma 5.** For any \(k, n, m \geq 1\), we have \(\ell_k(n + m) \leq \ell_k(n) + \ell_k(m)\).

**Proof.** Let \(T_1\) and \(T_2\) be extremal tournaments on \(n\) and \(m\) vertices, respectively, not containing the \(k\)-th power of any directed path of length \(\ell_k(n)\) and \(\ell_k(m)\). Let \(T\) be the tournament on \(n + m\) vertices, obtained from the disjoint union of \(T_1\) and \(T_2\) by adding all remaining edges directed from \(T_1\) to \(T_2\). Then any \(k\)-th power of a path in \(T\) must be the concatenation of the \(k\)-th power of a path in \(T_1\) and the \(k\)-th power of a path in \(T_2\), and hence it must have length at most \((\ell_k(n) - 1) + (\ell_k(m) - 1) + 1 < \ell_k(n) + \ell_k(m)\).

\[\square\]

Our improved upper bound is based on the following construction.

**Lemma 6.** For every \(k \geq 5\), we have \(\ell_k(2^{k-1}) < \frac{k(k+1)}{2}\).

**Proof.** Let \(n = 2^{k-1}\) and \(\ell = \frac{k(k+1)}{2}\), and note that \(\vec{P}_{\ell-1}^k\) has \(k\ell - \ell\) edges.

Let \(T\) be a random \(n\)-vertex tournament obtained by orienting the edges of \(K_n\) independently and uniformly at random. The probability that a fixed sequence of \(\ell\) vertices \(v_0 \ldots v_{\ell-1}\) forms a copy of \(\vec{P}_{\ell-1}^k\) is \(2^{-(k-1)\ell}\). There are \(\binom{n}{\ell}\ell!\) such sequences, so the probability that \(T\) contains the \(k\)-th power of a path of length \(\ell - 1\) is at most \(\binom{n}{\ell}\ell!2^{-(k-1)\ell} < n^\ell\cdot 2^{-(k-1)\ell} = 1\).

So with positive probability \(T\) does not contain \(\vec{P}_{\ell-1}^k\), therefore \(\ell_k(2^{k-1}) \leq \ell - 1\).

\[\square\]

Combining \(\ell_k(n)\) and using the monotonicity of \(\ell_k(n)\), we get \(\ell_k(n) \leq \left[\frac{n}{2^{k-1}}\right] \cdot \ell_k(2^{k-1}) \leq \left(\frac{n}{2^{k-1}} + 1\right)\left(\frac{k(k+1)}{2} - 1\right) \leq \frac{k(k+1)n}{2^k}\)

for \(n \geq k(k+1)2^k\), establishing \(\ell_k(n)\).

### 4 The square of a path

**Proof of \(\ell_2(n)\).** Recall that \(\ell_2(n)\) is the largest integer such that every \(n\)-vertex tournament contains the square of a path on \(\ell\) vertices. Proving \(\ell_2(n)\) is therefore equivalent to showing \(\ell_2(n) = \lceil 2n/3 \rceil\) for every \(n \geq 1\).

It is easy to check that \(\ell_2(1) = 1\) and \(\ell_2(2) = \ell_2(3) = 2\), so \(\ell_2(n) \leq \lceil 2n/3 \rceil\) follows from \(\ell_2(2)\) by induction, as \(\ell_2(n) \leq \ell_2(n-3) + \ell_2(3) = \ell_2(n-3) + 2\) holds for every \(n > 3\). For the lower bound we need to take a closer look at median orderings.
Claim. Every median ordering $x_1, \ldots, x_n$ of a tournament has the following properties:

(a) All edges of the form $x_ix_{i+1}$ are in the tournament.

(b) If $x_ix_{i-2}$ is an edge of the tournament, then “rotating” $x_{i-2}x_{i-1}x_i$ gives two other median orderings $x_1, \ldots, x_{i-3}, x_{i-1}, x_i, x_{i-2}, x_{i+1}, \ldots, x_n$ and $x_1, \ldots, x_{i-3}, x_i, x_{i-2}, x_{i-1}, x_{i+1}, \ldots, x_n$.

(c) If $x_ix_{i-2}$ is an edge of the tournament, then each of $x_{i-2}, x_{i-1}, x_i$ is an inneighbour of $x_{i+1}$, and at most one of them is an outneighbour of $x_{i+1}$.

Proof. Property (??) holds, as otherwise we could swap $x_i$ and $x_{i+1}$ to get an ordering with more forward edges, contradicting our assumption. Property (??) holds because rotating $x_{i-2}x_{i-1}x_i$ has no effect on the number of forward edges.

Indeed, if $x_ix_{i-2}$ is an edge, then each of $x_{i-2}, x_{i-1}, x_i$ is an inneighbour of $x_{i+1}$. This means that none of $i, i+1, i+2$ is a bad index in this new ordering, and hence the largest bad index is smaller than $i$. This is a contradiction.

Let us now say that $i$ is a bad index in a median ordering $x_1, \ldots, x_n$ if $x_ix_{i-2}$ is an edge, and at least one of $x_{i+2}x_i$ and $x_{i+2}x_{i-1}$ is also an edge.

Lemma 7. Every tournament has a median ordering without any bad indices.

Proof. Suppose this fails to hold for some tournament, and take a median ordering $x_1, \ldots, x_n$ that minimizes the largest bad index $i$. As $i$ is a bad index, $x_ix_{i-2}$ is an edge, and $x_i$ or $x_{i-1}$ is an outneighbour of $x_{i+2}$. By (??), $x_{i-2}x_{i-1}x_i$ can be rotated so that $x_{i+2}x_{i-2}$ is an edge in the new median ordering $x_1, \ldots, x_{i-3}, x_{i-2}, x_{i-1}, x_i, x_{i+1}, \ldots, x_n$. Then neither $x_{i+2}x_i$ nor $x_{i+2}x_{i-1}$ is an edge, since by (??), only one of $x_{i-2}, x_{i-1}, x_i$ is an outneighbour of $x_{i+2}$. Also by (??), $x_{i-1}x_{i+1}$ and $x_{i-1}x_{i+1}$ are edges, so both of $x_{i+1}$ and $x_{i+2}$ are outneighbours of $x_{i-1}$ and $x_i$. This means that none of $i, i+1, i+2$ is a bad index in this new ordering, and hence the largest bad index is smaller than $i$. This is a contradiction.

Now we are ready to prove $\ell_2(n) \geq \lceil 2n/3 \rceil$. Take an $n$-vertex tournament with median ordering $x_1, \ldots, x_n$ as in ??, and let $I = \{i_1 < i_2 < \cdots < i_k\}$ be the set of indices $i$ such that $x_i x_{i-2}$ is not an edge (in particular, $i_1 = 1$ and $i_2 = 2$). We claim that $x_{i_1}, \ldots, x_{i_k}$ is a directed path on $k \geq \lceil 2n/3 \rceil$ vertices whose square is contained in the tournament.

To see this, first observe that if the index $i+2$ is not in $I$, then both $i$ and $i+1$ are in $I$. Indeed, if $x_{i+2}x_i$ is an edge, then $x_{i+1}x_{i-1}$ cannot be one because of (??), and $x_ix_{i-2}$ cannot be one because $i$ is not a bad index. This immediately implies $k \geq \lceil 2n/3 \rceil$.

It remains to check that $x_{i-j-2}x_{ij}$ and $x_{i-1}x_{ij}$ are all edges in the tournament. By the above observation, we know that $i_j - 3 \leq i_j - 2 < i_j - 1 < i_j$. Here $x_{i_j-1}x_{i_j}$ is an edge by (??), and $x_{i_j-2}x_{i_j}$ is an edge by the definition of $I$. So the only case left is to show that $x_{i-2}x_{ij}$ is an edge when $i_j - 2 = i_j - 3$.

In this case there is an index $i_j - 3 < i < i_j$ that is not in $I$, i.e., $x_ix_{i-2}$ is an edge in the tournament. But then if $i = i_j - 1$, then $x_{i_j-1}x_{ij}$ is an edge because of (??), while otherwise $i = i_j - 2$, and $x_{i_j-2}x_{ij}$ is an edge because $i$ is not a bad index. This concludes our proof. \□
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