Physical interpretation of Kundt spacetimes using geodesic deviation

J Podolský and R Švarc

1 Institute of Theoretical Physics, Charles University in Prague, Faculty of Mathematics and Physics, V Holešovičkách 2, 180 00 Praha 8, Czech Republic
2 Department of Physics, Faculty of Science, J. E. Purkinje University in Ústí nad Labem, České mládeže 8, 400 96 Ústí nad Labem, Czech Republic

E-mail: podolsky@mbox.troja.mff.cuni.cz and robert.svarc@mff.cuni.cz

Received 28 June 2013, in final form 6 September 2013
Published 1 October 2013
Online at stacks.iop.org/CQG/30/205016

Abstract
We investigate the fully general class of non-expanding, non-twisting and shear-free $D$-dimensional geometries using the invariant form of geodesic deviation equation which describes the relative motion of free test particles. We show that the local effect of such gravitational fields on the particles basically consists of isotropic motion caused by the cosmological constant $\Lambda$, Newtonian-type tidal deformations typical for spacetimes of algebraic type II, longitudinal motion characteristic for spacetimes of type III, and type N purely transverse effects of exact gravitational waves with $D(D-3)/2$ polarizations. We explicitly discuss the canonical forms of the geodesic deviation motion in all algebraically special subtypes of the Kundt family for which the optically privileged direction is a multiple Weyl aligned null direction, namely D(a), D(b), D(c), D(d), III(a), III(b), III(c), II(a), II(b), II(c) and II(d). We demonstrate that the key invariant quantities determining these algebraic types and subtypes also directly determine the specific local motion of test particles, and are thus measurable by gravitational detectors. As an example, we analyze an interesting class of type N or II gravitational waves which propagate on backgrounds of type O or D, including Minkowski, Bertotti–Robinson, Nariai and Plebański–Hacyan universes.

PACS numbers: 04.20.Jb, 04.50.−h, 04.30.−w

1. Introduction
Spacetimes of the Kundt class are defined by a purely geometric property, namely that they admit a geodesic null congruence which is non-expanding, non-twisting and shear-free. In the context of four-dimensional general relativity, such vacuum and pure radiation spacetimes of type N, III, or O were introduced and initially studied 50 years ago by Kundt [1, 2].

The whole Kundt class is, in fact, much wider. It admits a cosmological constant, electromagnetic field, other matter fields and supersymmetry. The solutions may be of various
algebraic types and can be extended to any number $D$ of dimensions. All Kundt spacetimes
(without assuming field equations) can be written as

$$ds^2 = g_{pq}(u, x) \, dx^p dx^q + 2 g_{up}(r, u, x) \, du \, dx^p - 2 \, du \, dr + g_{uu}(r, u, x) \, dr^2,$$

see [1–9]. In this metric, the coordinate $r$ is an affine parameter along the optically privileged
null congruence $\mathbf{k} = \partial_r$ (with vanishing expansion, twist and shear), $u =$ const. label null
(wave)surfaces, and $x \equiv (x^2, x^3, \ldots, x^{D-1})$ are $D - 2$ spatial coordinates in the transverse
Riemannian space. The spatial part $g_{pq}$ of the metric must be independent of $r$, all other metric
components $g_{up}$ and $g_{uu}$ can be functions of all the coordinates $(r, u, x)$.

The Kundt class of spacetimes is one of the most important families of exact solutions
in Einstein’s general relativity theory, see chapter 31 of the monograph [3] or chapter 18 of
[4] for reviews of the standard $D = 4$ case. It contains several famous subclasses, both in four and
higher number of dimensions, with interesting mathematical and physical properties. The
best-known of these are pp-waves (see [3–6, 10–14] and references therein) which admit a
covariantly constant null vector field. There are also VSI and CSI spacetimes [5, 6, 11–16] for
which all polynomial scalar invariants constructed from the Riemann tensor and its derivatives
vanish and are constant, respectively. Moreover, all the relativistic gyrotrons known so far
[17–24], representing the fields of localized spinning sources that propagate with the speed
of light, are also specific members of the Kundt class. Vacuum and conformally flat pure
radiation Kundt spacetimes provide an exceptional case for the invariant classification of exact
solutions [25–31], and all type D pure radiation solutions are also known [32, 33]. All vacuum
Kundt solutions of type D were found and classified a long time ago [34] and generalized to
electrovacuum and any value of the cosmological constant [35, 36]. These contain a subfamily
direct-product spacetimes, namely the Bertotti–Robinson, (anti-)Nariai and Plebański–
Hacyan spacetimes of type O and D (see chapter 7 of [4], [23] and [24] for higher-dimensional
generalizations) representing, for example, extreme limits and near-horizon geometries. With
Minkowski and (anti-)de Sitter spaces they form the natural backgrounds for non-expanding
gravitational waves of types N and II [37–44].

In our studies here we consider the fully general class of Kundt spacetimes of an arbitrary
dimension $D \geq 4$ (results for the standard general relativity are obtained by simply setting
$D = 4$). Taking the spacetime dimension as a free parameter $D$, we can investigate whether
the extension of the Kundt family to $D > 4$ exhibits some qualitatively different features and
unexpected properties. Our paper is thus also a contribution to the contemporary research
analyzing various aspects of Einstein’s gravity extended to higher dimensions. Explicit Kundt
solutions help us illustrate specific physical properties and general mathematical features of
such theories.

Specifically, we systematically investigate the complete $D \geq 4$ Kundt class of solutions
using geodesic deviation and discuss the corresponding effects on free test particles. In section 2
we summarize the equation of geodesic deviation, introduce invariant amplitudes of the
gravitational field, and we discuss them for the fully general Kundt family of geometries.
In section 3 we derive expressions for these amplitudes, and in section 4 we evaluate them
explicitly for all algebraically special Kundt spacetimes for which the optically privileged
congruence is generated by a multiple Weyl aligned null direction (WAND). The main results
are presented in section 5 where we discuss the specific structure of relative motion of
test particles for all possible algebraic types and subtypes of such Kundt geometries, see
subsections 5.1–5.12. In the final section 6 we present a particular example, namely an
interesting class of type II and N non-expanding gravitational waves on D and O backgrounds
of any dimension.
2. Geodesic deviation in the fully general Kundt spacetime

Relative motion of nearby free test particles (without charge and spin) is described by the equation of geodesic deviation [45, 46]. It has long been used as an important tool for studies of four-dimensional general relativity, in particular to analyze fields representing gravitational waves and black hole spacetimes (see [47–50] for more details and references). In our recent work [50] generalizing [41] we demonstrated that the equation of geodesic deviation in any $D$-dimensional spacetime can be expressed in the invariant form (using Einstein’s field equations $R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 8\pi T_{ab}$)

$$
\ddot{Z}^{(i)} = \frac{2\Lambda}{(D - 2)(D - 1)} Z^{(i)} + \Psi_{25} Z^{(i)} + \frac{1}{\sqrt{2}} (\Psi_{1T_i} - \Psi_{3T_i}) Z^{(i)}
$$

$$
+ \frac{8\pi}{D - 2} \left[ T_{(i)(j)} Z^{(i)} + T_{(1)(j)} Z^{(i)} - \left( T_{(0)(0)} + \frac{2T}{D - 1} \right) Z^{(i)} \right],
$$

$$
\ddot{Z}^{(i)} = \frac{2\Lambda}{(D - 2)(D - 1)} Z^{(i)} - \Psi_{2T^{(0)}} Z^{(i)} + \frac{1}{\sqrt{2}} (\Psi_{1T_i} - \Psi_{3T_i}) Z^{(i)} - \frac{1}{2} (\Psi_{0^i} + \Psi_{4^i}) Z^{(i)}
$$

$$
+ \frac{8\pi}{D - 2} \left[ T_{(i)(j)} Z^{(i)} + T_{(1)(j)} Z^{(i)} - \left( T_{(0)(0)} + \frac{2T}{D - 1} \right) Z^{(i)} \right].
$$

for $i, j = 2, \ldots, D - 1$. Here $Z^{(1)}, Z^{(2)}, \ldots, Z^{(D - 1)}$ are spatial components of $Z^{(i)} = e^{(i)} \cdot Z$ of the separation vector $Z$ between the two test particles in a natural interpretation orthonormal frame $\{e_a\}$, $e_\alpha$, $e_\eta$ is $\eta_{ab}$, where $e_{(0)} = u$ is the velocity vector of the fiducial test particle, and $\ddot{Z}^{(1)}, \ddot{Z}^{(2)}, \ldots, \ddot{Z}^{(D - 1)}$ are the corresponding relative physical accelerations $\ddot{Z}^{(i)} = e^{(i)} \cdot (D^2 Z/\mathrm{d}^2 t^2)$. The coefficients $T_{ab} = T(e_a, e_b)$ denote frame components of the energy–momentum tensor ($T$ is its trace), and the scalars

$$
\Psi_{0^i} = C_{abcd} k^a m^b k^c m^d,
$$

$$
\Psi_{1T_i} = C_{abcd} k^a b^b k^c m^d,
$$

$$
\Psi_{25} = C_{abcd} k^a b^b r^c m^d,
$$

$$
\Psi_{2T^{(0)}} = C_{abcd} k^a b^b r^c m^d,
$$

$$
\Psi_{3T_i} = C_{abcd} pl^a b^b r^c m^d,
$$

$$
\Psi_{3T^{(0)}} = C_{abcd} pl^a b^b r^c m^d,
$$

$$
\Psi_{4^i} = C_{abcd} pl^a b^b r^c m^d,
$$

with indices $i, j, k, l = 2, \ldots, D - 1$, are components of the Weyl tensor with respect to the null frame $\{k, l, m\}$ associated with $\{e_a\}$ via relations $k = \frac{1}{\sqrt{2}} (u + e_{(1)}), l = \frac{1}{\sqrt{2}} (u - e_{(1)}), m = e_{(0)}$, see figure 1.

The Weyl tensor components (4) are listed by their boost weight and directly generalize the standard Newman–Penrose complex scalars $\Psi_A$ known from the $D = 4$ case [50, 51]. In equations (2), (3), only the ‘electric part’ of the Weyl tensor with respect to $u$ represented by the scalars in the left column of (4) occurs. All these scalars exhibit the standard symmetries of the Weyl tensor, for example

$$
\Psi_{0i} = \Psi_{4i}, \quad \Psi_{0^i} = 0, \quad \Psi_{3^i} = \Psi_{3^i}.
$$

Moreover, there are relations between the left and right columns of (4), namely

$$
\Psi_{1T_i} = \Psi_{1T_i}, \quad \Psi_{3T_i} = \Psi_{3T_i},
$$

$$
\Psi_{25} = \Psi_{25}, \quad \Psi_{2T^{(0)}} = \Psi_{2T^{(0)}}, \quad \Psi_{2T^{(0)}} = \Psi_{2T^{(0)}}.
$$

Finally, let us remark that our notation which uses the symbols $\Psi_A$ in any dimension is related to the notations employed elsewhere, namely in [5, 12], [52, 53], and [6, 54]. Identifications
Figure 1. Evolution of the separation vector $Z$ that connects particles moving along geodesics $\gamma(\tau)$. $\tilde{\gamma}(\tau)$ is given by the equation of geodesic deviation (2) and (3). Its components are expressed in the orthonormal frame $\{e_a\}$ with $e_{(0)} = u$. The associated null frame $\{k, l, m\}$ is also indicated.

Table 1. Different equivalent notations used in the literature for those Weyl scalars that occur in the equation of geodesic deviation (2), (3).

| $\Psi_g$ | $\Psi_{1g}$ | $\Psi_{2g}$ | $\Psi_{2g(0)}$ | $\Psi_{3g}$ | $\Psi_{4g}$ |
|---------|-------------|-------------|----------------|-------------|-------------|
| $C_{0i0j}$ | $-C_{0i0j}$ | $-C_{0j0i}$ | $C_{1i0j}$ | $C_{1j0i}$ | Coley et al [5, 12] |
| $\Omega_{ij}$ | $-\Psi_j$ | $-\Phi_j$ | $\Psi_j'$ | $2\Psi_{ij}$ | Pravda et al [52, 53] |

Table 2. Other Weyl tensor components and their form in the GHP notation.

| $\Psi_{ijk}$ | $\Psi_{2ijk}$ | $\Psi_{2ijkl}$ | $\Psi_{2kl}$ | $\Psi_{gij}$ |
|-------------|--------------|----------------|-------------|-------------|
| $C_{ijkl}$ | $C_{ijkl}$ | $-C_{0ij0}$ | $-C_{0j0i}$ | Coley et al [5, 12] |
| $\Psi_{ijk}$ | $\Phi_{ijkl}$ | $-2\Phi_j^k$ | $-\Phi_j^s$ | $-\Psi_{ijk}$ | Pravda et al [52, 53] |

For the components present in the invariant form of the equation of geodesic deviation are summarized in table 1 (more details can be found in [50]). The remaining (independent) components of the Weyl tensor are listed in table 2.

For the most general Kundt spacetime (1), the null interpretation frame adapted to an arbitrary observer moving along the timelike geodesic $\gamma(\tau)$, whose velocity vector $u = 1/\sqrt{2} (\partial_r + \dot{u} \partial_u + \dot{x}^p \partial_p)$ (normalized as $u \cdot u = -1$, so that $\dot{u} \neq 0$), takes the form

$$k = \frac{1}{\sqrt{2} \dot{u}} \partial_r,$$

$$l = \left( \sqrt{2} \dot{r} - \frac{1}{\sqrt{2} \dot{u}} \right) \partial_r + \sqrt{2} \dot{u} \partial_u + \sqrt{2} \dot{x}^p \partial_p,$$

$$m_i = m^p_i \left( g_{pu} \dot{u} + g_{pq} \dot{x}^q \right) \partial_r + m_i^p \partial_p,$$

where $m_i^p$ satisfy $g_{pq} m_i^p m_j^q = \delta_{ij}$ to fulfil the normalization conditions $m_i \cdot m_j = \delta_{ij}$, $k \cdot l = -1$. Notice that the vector $k$ is oriented along the non-expanding, non-twisting and shear-free null congruence $k = \partial_r$ defining the Kundt family. Moreover, $u = 1/\sqrt{2} (k + l)$ and

3 In this paper, i, j, k, l are frame labels, whereas the indices p, q, m, n denote the spatial coordinate components. For example, $m_i^p$ stands for the $p$th spatial coordinate component of the vector $m_i$.  

4
\( e_{(1)} = \frac{1}{\sqrt{2}}(k - l) = \sqrt{3}k - u \). The spatial vector \( e_{(1)} \) is thus uniquely determined by the optically privileged null congruence of the Kundt family and the observer’s velocity \( u \). For this reason we call such a special direction \( e_{(1)} \), longitudinal, while we refer to the \( D - 2 \) directions \( e_{(i)} = m \), are transverse.

To evaluate the scalars (4) we have to project coordinate components of the Weyl tensor \( C_{abcd} \) of the generic Kundt spacetime, which can be found in appendix A of [9], onto the interpretation frame (7). Since \( C_{rreq} = 0 \), we immediately obtain \( \Psi_{qj} = 0 \), while the remaining Weyl scalars are in general non-zero. The relative motion of free test particles in any \( D \)-dimensional Kundt spacetime (1), determined by the equation of geodesic deviation (2), (3), can thus be naturally decomposed into the following components.

- **The presence of the cosmological constant \( \Lambda \) is encoded in the term**

  \[
  \begin{pmatrix}
  Z^{(1)} \\
  Z^{(0)}
  \end{pmatrix}
  = \frac{2\Lambda}{(D - 1)(D - 2)}
  \begin{pmatrix}
  1 & 0 \\
  0 & \delta_{ij}
  \end{pmatrix}
  \begin{pmatrix}
  Z^{(1)} \\
  Z^{(0)}
  \end{pmatrix},
  \tag{8}
  \]

  It causes isotropic motion of test particles, which is typical for (maximally symmetric) ‘background’ spacetimes of constant curvature, namely Minkowski space, de Sitter space and anti-de Sitter space.

- **The terms \( \Psi_{2S} \) and \( \Psi_{2T;j} \) are responsible for Newtonian-like tidal deformations** since the motion of test particles is given by

  \[
  \begin{pmatrix}
  Z^{(1)} \\
  Z^{(0)}
  \end{pmatrix}
  = \begin{pmatrix}
  \Psi_{2S} & 0 \\
  0 & -\Psi_{2T;j}
  \end{pmatrix}
  \begin{pmatrix}
  Z^{(1)} \\
  Z^{(0)}
  \end{pmatrix},
  \tag{9}
  \]

  where \( \Psi_{2S} = \Psi_{2T;j} \). These effects are typically present in type D spacetimes.

- **The scalars \( \Psi_{3T} \) and \( \Psi_{1T} \) represent longitudinal components of the gravitational field resulting in specific motion associated with the spatial direction \( +e_{(1)} \) and \( -e_{(1)} \), respectively. Such terms cause accelerations**

  \[
  \begin{pmatrix}
  Z^{(1)} \\
  Z^{(0)}
  \end{pmatrix}
  = -\frac{1}{\sqrt{2}}
  \begin{pmatrix}
  0 & \Psi_{3T} \\
  \Psi_{1T} & 0
  \end{pmatrix}
  \begin{pmatrix}
  Z^{(1)} \\
  Z^{(0)}
  \end{pmatrix},
  \tag{10}
  \]

  where \( \Psi_{3T} \), stand for either \( \Psi_{3T} \), or \( -\Psi_{1T} \), which are mutually equivalent under \( k \leftrightarrow l \), see (4). These scalars combine motion in the privileged longitudinal direction \( \pm e_{(1)} \) with motion in the transverse spatial directions \( e_{(i)} \). Such effects typically occur in spacetimes of type III, in particular IIIl.

- **The scalars \( \Psi_{qj} \), characteristic for type N spacetimes, can be interpreted as amplitudes of transverse gravitational waves propagating along the spatial direction \( +e_{(1)} \). These components of the field influence test particles as**

  \[
  \begin{pmatrix}
  Z^{(1)} \\
  Z^{(0)}
  \end{pmatrix}
  = -\frac{1}{2}
  \begin{pmatrix}
  0 & 0 \\
  0 & \Psi_{qj}
  \end{pmatrix}
  \begin{pmatrix}
  Z^{(1)} \\
  Z^{(0)}
  \end{pmatrix},
  \tag{11}
  \]

  They obviously cause purely transverse effects because there is no acceleration in the privileged longitudinal direction \( e_{(1)} \). The scalars \( \Psi_{qj} \) form a symmetric traceless matrix of dimension \( (D - 2) \times (D - 2) \), see (5). This matrix describing the amplitudes of gravitational waves has \( \frac{1}{2}D(D - 3) \) independent components corresponding to distinct polarization modes.

More details about these general effects of any gravitational field on test particles can be found in [30]. Their explicit discussion in the context of Kundt spacetimes will be presented in subsequent sections of this contribution.

There are also specific direct effects of matter in the equation of geodesic deviation (2), (3) which are determined by the frame components of the corresponding energy–momentum tensor \( T_{ab} \). As an explicit illustration, we present here two physically interesting examples:
For pure radiation (or ‘null dust’) aligned along the null direction \( k \), the energy–momentum tensor is \( T_{ab} = \rho k_a k_b \) where \( \rho \) represents radiation density. Its trace vanishes, \( T = 0 \), and the only non-vanishing components of \( T_{ab} \) in the equation of geodesic deviation (2), (3) reduce to

\[
\left( \frac{\dot{Z}^{(1)}}{\dot{Z}^{(0)}} \right) = -\frac{4 \pi \rho}{D - 2} \begin{pmatrix} 0 & 0 \\ 0 & \delta_{ij} \end{pmatrix} \left( \frac{Z^{(1)}}{Z^{(0)}} \right).
\]

(12)

There is no acceleration in the longitudinal spatial direction \( e_{(1)} \) and the effect in the transverse space is isotropic. Moreover, since \( \rho > 0 \), it causes a radial contraction which may eventually lead to exact focusing.

3. Explicit evaluation of the Weyl scalars \( \Psi_A \).

The invariant amplitudes \( \Psi_A \) of various gravitational field components (9)–(11) combine the local curvature of the Kundt spacetime with the kinematics of specific motion along an arbitrary timelike geodesic \( \gamma(\tau) \). These should be evaluated at any given event corresponding to the actual position of the observer along \( \gamma(\tau) \), with its actual velocity \( u = \dot{r} \partial_r + \dot{\varphi} \partial_{\varphi} + \dot{\rho} \partial_\rho \).

For an electromagnetic field aligned with the Kundt geometry (i.e., \( F_{ab} k^b = \mathcal{E} k_a \) where \( k = \partial_r \)) the most general form of the Maxwell tensor is

\[
F = \mathcal{E} \, dr \wedge du + \sigma_p \, du \wedge dx^p + \frac{1}{2} B_{pq} \, dx^p \wedge dx^q.
\]

(13)

Evaluating the energy–momentum tensor \( T_{ab} = \frac{1}{8\pi} \left( F_{ac} F_b^c - \frac{1}{4} g_{ab} F^{cd} F_{cd} \right) \) in the interpretation orthonormal frame, the corresponding effects on (uncharged) test particles, as given by the equation of geodesic deviation, take the form

\[
\left( \frac{\dot{Z}^{(1)}}{\dot{Z}^{(0)}} \right) = \begin{pmatrix} T & T_j \\ T_i & T_{ij} \end{pmatrix} \left( \frac{Z^{(1)}}{Z^{(0)}} \right),
\]

(14)

where

\[
T = -\frac{2}{D - 2} \left[ (\mathcal{E}^2 + B^2) + \frac{D - 4}{D - 1} (\mathcal{E}^2 - B^2) \right],
\]

\[
T_i = -\frac{2}{D - 2} m_p i \dot{x}^i (\mathcal{E}^2 g_{pq} + B_{pq}^2) + \dot{u} (\mathcal{E} \mathcal{E}_p - B_{pq} \mathcal{E}_m g^{pm}),
\]

\[
T_{ij} = \frac{2}{D - 2} \left[ m_p m_q B_{pq}^2 - \delta_{ij} \left( B^2 + \frac{D - 4}{D - 1} (\mathcal{E}^2 - B^2) + \dot{x}^i \dot{x}^j (\mathcal{E}^2 g_{pq} + B_{pq}^2) + 2 \dot{u} \dot{x}^i (\mathcal{E} \mathcal{E}_p - B_{pq} \mathcal{E}_m g^{pm}) \right) \right],
\]

(15)

with convenient auxiliary variables defined as

\[
\mathcal{E}_p \equiv \mathcal{E} g_{up} - \sigma_p, \quad B_{pq}^2 \equiv B_{pm} B_{qm} g^{pm}, \quad B^2 \equiv \frac{1}{2} B_{pq} B_{pq} g^{pq}. \]

(16)

The motion simplifies considerably if the magnetic field is absent \( (B_{pq} = 0) \):

\[
T = -\frac{2}{D - 2} \frac{2D - 5}{D - 1} \mathcal{E}^2,
\]

\[
T_i = -\frac{2}{D - 2} m_p i \dot{x}^i \mathcal{E}^2 + \dot{u} \mathcal{E}_p,
\]

(17)

\[
T_{ij} = -\frac{2}{D - 2} \delta_{ij} \left( \frac{D - 4}{D - 1} \mathcal{E}^2 + g_{pq} \dot{x}^p \dot{x}^q \mathcal{E}^2 + 2 \dot{u} \dot{x}^p \mathcal{E}_p + \dot{u}^2 \mathcal{E}_q \mathcal{E}^q \right),
\]

in particular when \( D = 4 \) and \( \sigma_p = 0 \) (in which case \( \mathcal{E}_p = \mathcal{E} g_{up} \)).
The scalars $\Psi_0$, $\Psi_{1/0}$, $\Psi_{1/1}$, $\Psi_{1/2}$, $\Psi_{1/3}$, and $\Psi_{0/0}$ (and $\Psi_{0/0}=0$) which enter the geodesic deviation equation (2), (3) can most conveniently be expressed explicitly if we employ the relation between the interpretation null frame (7) (adapted to the chosen geodesic observer) and the natural null frame for the Kundt geometry (1) which is
\begin{equation}
\begin{split}
k_{\text{nat}} &= \partial_x, \\
I_{\text{nat}} &= \frac{1}{2}g_{uu} \partial_u + \partial_u, \\
m_i &= m^p_i \left( g_{ap} \partial_a + \partial_p \right).
\end{split}
\end{equation}

The transition between the null frames (18) and (7) is a Lorentz transformation associated with the choice of different (timelike) observers, as explained in more detail in section 5 and appendix C of our work [50]. Specifically, the general interpretation frame is obtained from the natural one by combining a boost followed by a null rotation with fixed $k_{\text{nat}}$ (see equations (C3) and (C1) of [50]),
\begin{equation}
\begin{split}
k & = B k_{\text{nat}}, \\
I & = B^{-1} I_{\text{nat}} + \frac{1}{2} L I_{\text{nat}} + |L|^2 B k_{\text{nat}}, \\
m_i & = m_{i\text{nat}} + \frac{1}{2} L_i B k_{\text{nat}},
\end{split}
\end{equation}
where $|L|^2 = \delta^{ij} L_i L_j$ and
\begin{equation}
B = \frac{1}{\sqrt{2} \dot{u}}, \quad L_u = g_{pq} m_p^i \dot{x}^q.
\end{equation}

Conversely, the natural frame (18) is obtained from the interpretation frame (7) as a particular case when $\sqrt{2} \dot{u} = 1$, $\dot{x}^p = 0$ (and thus $\sqrt{2} \dot{r} = 1 = \frac{1}{2} g_{uu}$ due to the assumed normalization $u \cdot u = -1$), i.e., $B = 1$, $L_i = 0$. This corresponds to special observers with no motion in the transverse spatial directions ($\dot{x}^p = 0$ for all $p = 2, \ldots, D = 1$).

Under the Lorentz transformation (19), the Weyl scalars change as
\begin{equation}
\begin{split}
\Psi_0 & = 0, \quad \Psi_{1/0} = B \Psi_{1/0}^\text{nat}, \quad \Psi_{1/1} = B \Psi_{1/1}^\text{nat}, \\
\Psi_{2/0} & = \Psi_{2/0}^\text{nat} - 2 \sqrt{2} \Psi_{1/1}^\text{nat} B L^i, \\
\Psi_{2/1} & = \Psi_{2/1}^\text{nat} - 2 \sqrt{2} B L_i \Psi_{1/1}^\text{nat} - L_i \Psi_{1/0}^\text{nat}, \\
\Psi_{2/2} & = \Psi_{2/2}^\text{nat} + \sqrt{2} \Psi_{1/1}^\text{nat} B L^k - 2 \sqrt{2} \Psi_{1/1}^\text{nat} B L_j, \\
\Psi_{2/3} & = \Psi_{2/3}^\text{nat} + \sqrt{2} \Psi_{1/1}^\text{nat} B L^k - \sqrt{2} \Psi_{1/1}^\text{nat} B L_j, \\
\Psi_{3/0} & = B^{-1} \Psi_{3/0}^\text{nat} + \sqrt{2} \left( \Psi_{2/0}^\text{nat} L^j - \Psi_{2/1}^\text{nat} L_i + 2 L_i \Psi_{1/1}^\text{nat} \right) \\
& \quad + 4 \Psi_{1/1}^\text{nat} L_4 B L_4 - 2 \left( \Psi_{1/0}^\text{nat} L_4 + \Psi_{1/1}^\text{nat} L_4 - \Psi_{1/0}^\text{nat} L_4 \right) B L^i + \Psi_{1/0}^\text{nat} B |L|^2, \\
\Psi_{3/1} & = B^{-1} \Psi_{3/1}^\text{nat} + \sqrt{2} \Psi_{2/0}^\text{nat} L^j - \sqrt{2} \left( \Psi_{2/0}^\text{nat} L^k + \Psi_{2/3}^\text{nat} L_i \right) \\
& \quad + 2 \left( 2 \Psi_{1/1}^\text{nat} L_4 - \Psi_{1/1}^\text{nat} L^i L^j - \Psi_{1/1}^\text{nat} B |L|^2, \\
\Psi_{3/2} & = B^{-2} \Psi_{3/2}^\text{nat} + 2 \sqrt{2} B^{-1} \left( \Psi_{3/1}^\text{nat} L_j - \Psi_{3/0}^\text{nat} L^k \right) \\
& \quad + 2 \Psi_{2/0}^\text{nat} L^i L^j - 4 \Psi_{2/0}^\text{nat} L_i L_j L^k + 2 \Psi_{2/0}^\text{nat} |L|^2 - 2 \Psi_{2/2}^\text{nat} L_i L_j - 4 \Psi_{2/2}^\text{nat} L_i L_j L^k \\
& \quad - 2 \sqrt{2} B \left( 2 \Psi_{1/1}^\text{nat} L_j L^k L^l + \Psi_{1/1}^\text{nat} L^k L^l |L|^2 + \Psi_{1/1}^\text{nat} L_j L^k |L|^2 - 2 \Psi_{1/1}^\text{nat} L_j L^k L^l L_i \right),
\end{split}
\end{equation}

see expressions (C5) and (C7) of [50] in the particular case when $\Psi_{0/1} = 0$. The scalars $\Psi_{\text{nat}}^A$ represent the components (4) of the Weyl tensor in the natural null frame (18). Recall that the coordinate components of $C_{\text{abce}}$ were presented in appendix A of [9]. These scalars can also be used purely locally. For some purposes, it is not necessary to evaluate all the functions along $\gamma(\tau)$ and express them in terms of the proper time $\tau$ of the geodesic observer. For example, to determine the algebraic type of the spacetime at any given event, we only need to consider the values of the constants $\Psi_{\text{nat}}^A$ and their mutual relations. Moreover, they directly determine the actual acceleration of test particles in various spatial directions.
4. Algebraically special Kundt spacetimes

In our recent work [9], we analyzed the geometric and algebraic properties of all Kundt spacetimes for which the optically privileged (non-expanding, non-twisting, shear-free) congruence is generated by the null vector field \( k = \partial_t \), that is a multiple WAND.

Specifically, \( \Psi_{0/1}^{\text{nat}} = 0 \) immediately confirms the results of [6, 7, 55] that a generic Kundt geometry represented by the metric (1) is (at least) of algebraic type I (subtype I(b), in fact) and \( k = k^{\text{nat}} \) is WAND. In [9] we also demonstrated that the general Kundt spacetime of algebraic type II with respect to the double WAND \( k \) in any dimension \( D \) can be written in the form (1) with \( g_{\alpha\beta} = e_{\rho} + f_{\rho} r \) (at most) linear in \( r \),

\[
dx^2 = g_{\rho\sigma} \, dx^\rho \, dx^\sigma + 2 \left( e_{\rho} + f_{\rho} \right) \, dr \, dx^\rho - 2 \, du \, dr + g_{uu} (r, u, x) \, du^2,
\]

where \( g_{\rho\sigma}(u, x), \, e_{\rho}(u, x), \, f_{\rho}(u, x), \, p = 2, \ldots, D - 1 \), are functions independent of \( r \).

For such algebraically special Kundt geometries (22) with the multiple WAND \( k = k^{\text{nat}} \) there is \( \Psi_{1/\text{T/I}}^{\text{nat}} = 0 = \Psi_{1/\text{[I]}}^{\text{nat}} \). Moreover, in [9] we explicitly evaluated all the remaining Weyl scalars \( \Psi_{A}^{\text{nat}} \) of the boost weights \( 0, -1, -2 \). After lengthy calculations, we obtained the following surprisingly simple expressions, namely

\[
\Psi_{2S}^{\text{nat}} = \frac{D - 3}{D - 1} \left[ \frac{1}{2} g_{uu,rr} - \frac{1}{4} f^p f_p \right] + \frac{1}{D - 2} \left( \frac{1}{2} D - 3 + f \right),
\]

\[
\Psi_{2T/0}^{\text{nat}} = \frac{1}{D - 2} \left[ \frac{1}{2} g_{uu,rr} - \frac{1}{4} f^p f_p \right] + \frac{1}{2} (D - 4) \left( \frac{f_{pq}}{D - 2} - \frac{1}{D - 2} g_{pq} f \right),
\]

\[
\Psi_{2T/1}^{0/1} = \frac{1}{(D - 3)(D - 4)} \left( \delta_{ij} \delta_{kl} - \delta_{ik} \delta_{jl} + \frac{2}{D - 4} \left( \delta_{ik} \Psi_{2T/0}^{\text{nat}} - \delta_{ij} \Psi_{2T/1}^{\text{nat}} - \delta_{jk} \Psi_{2T/0}^{\text{nat}} + \delta_{kl} \Psi_{2T/1}^{\text{nat}} \right) \right),
\]

\[
\Psi_{2T/0}^{\text{nat}} = \frac{m_i m_j}{D - 2} \left[ \frac{1}{2} g_{uu,rr} - \frac{1}{4} f^p f_p \right] C_{mpnq},
\]

\[
\Psi_{2T/1}^{\text{nat}} = \frac{1}{2} \Psi_{2T/0}^{\text{nat}} = m_i^p f_{pq},
\]

\[
\Psi_{3T/1}^{\text{nat}} = -m_i^p \frac{D - 3}{D - 2} \left[ \frac{1}{2} \left( f_p g_{uu,rr} - g_{uu,cr} - f_p u \right) + e_p \left( \frac{1}{2} g_{uu,rr} - \frac{1}{4} f^p f_p \right) \right] + \frac{1}{4} f^p g_{pq} f_p - \frac{1}{2} f^p E_{jq} - \frac{1}{D - 3} X_p - \frac{1}{2} \left( f^p F_{jq} + \frac{1}{D - 3} Y_p \right),
\]

\[
\Psi_{3T/1}^{\text{nat}} = \Psi_{3T/1}^{\text{nat}} + \frac{1}{D - 3} \left( \delta_{ij} \Psi_{3T/1}^{\text{nat}} - \delta_{ik} \Psi_{3T/1}^{\text{nat}} \right),
\]

\[
\Psi_{3T/1}^{\text{nat}} = m_i^p m_j^{pq} \left[ \frac{1}{2} \left( g_{pq} X_{mq} - \frac{2}{D - 3} g_{pq} X_{mq} \right) + r \left( Y_{pq} - \frac{2}{D - 3} g_{pq} Y_{pq} \right) \right],
\]

\[
\Psi_{4T/1}^{\text{nat}} = \frac{1}{D - 2} g_{pq} W,
\]

\[
\Psi_{4T/1}^{\text{nat}} = \frac{1}{D - 2} g_{pq} W,
\]
in which $g_{pq} m_p^m m_q^q = \delta_{ij}$,

\[ X_{pqm} \equiv e_{(q|m)|p} + F_{qmn} e_p + F_{p|m} e_q + e_{p(m|f_q)} - f_{p|m} e_q + g_{p(m|u|q)}, \]  
\[ Y_{pqm} \equiv f_{(q|m)|p} + F_{qmn} f_p + F_{p|m} f_q, \]  
\[ W_{pq} \equiv -\frac{1}{2} (g_{au})_{(p|q)} + \frac{1}{2} g_{au}(f_{p|q}) + \frac{1}{2} g_{au,(p|q)} \]
\[-\frac{1}{2} g_{au}(r f_{p|q}) + e_{pq} - \frac{1}{2} g_{au,rr}(r f_{p|q} + 2 r f_{p|q} + e_{pq}) \]
\[ + \frac{1}{2} (f_{p|q} - g_{au,rr} r f_{p|q} + e_{pq} + (f_{p|q} - g_{au,rr}) r f_{p|q} + e_{pq}) \]
\[ + r^2 g^{rm} F_{rm} F_{pq} + r (f_{p,u|q}) - 2 g^{rn} E_{rn} F_{pq} + f_{mr} e_{pq} \]
\[ + e_{p(u|q)} - \frac{1}{2} g_{pq,uu} + g^{rn} E_{rn} F_{pq} + f_{mr} E_{rn} e_{pq} - e_{pq} F_{rn} f_{pq} \]
\[ + \frac{1}{2} (e_{pn} f_{pq} + f_{pn} e_{pq}) - \frac{1}{2} f_{pq} e_{pq}, \]

and their contractions are

\[ X_q \equiv g^{pq} X_{pqm}, \quad Y_q \equiv g^{pq} Y_{pqm}, \quad W \equiv g^{pq} W_{pq}. \] 

Note that $W_{pq} = W_{qp}$, while $X_{pq} = -X_{qpm}$ and $Y_{pq} = -Y_{qpm}$, so that $X_q$ and $Y_q$ are the only non-trivial contractions of $X_{pqm}$ and $Y_{pqm}$, respectively.

In these expressions we have introduced convenient geometric quantities

\[ f^p \equiv g^{pq} f_q, \] 
\[ f_{p|q} \equiv f_{p,q} - \Gamma^m_{pq} f_m, \] 
\[ f^p_{|p} \equiv g^{pq} f_{p|q}, \] 
\[ f_{pq} \equiv f_{(p|q)} + \frac{1}{2} f_{pq}, \] 
\[ f \equiv g^{pq} f_{pq} = f^p_{|p} + \frac{1}{2} f^p f_p, \] 
\[ F_{pq} \equiv f_{(p|q)} = f_{[p,q]}, \] 
\[ f_{m|q}|p \equiv f_{m|q,p} - \Gamma^p_{m|q} f_{[n,q]} - \Gamma^p_{m} f_{[m,n]}, \] 
\[ f_{p,u|q} \equiv f_{p,u|q} = f_{p,u} - f_{u,n} \Gamma^p_{pq}, \] 
\[ f_{(p,u|q)} = f_{(p,q),u} - f_{u,n} \Gamma^p_{pq}, \] 
\[ e^p \equiv g^{pq} e_q, \] 
\[ e_{pq} \equiv e_{(p|q)} = \frac{1}{2} g_{pq,uu}, \] 
\[ E_{pq} \equiv e_{(p|q)} + \frac{1}{2} g_{pq,uu}, \] 
\[ e_{(m|q)|p} \equiv e_{(m|q)|p} = e_{m|q,p} - \Gamma^p_{m} e_{[n,q]} - \Gamma^p_{m} e_{[m,n]}, \] 
\[ e_{m|q} \equiv e_{(m|q)} = e_{m|q,p} - e_{n,u} \Gamma^p_{pq}, \] 
\[ e_{(p,u)|q} \equiv e_{(p,u)|q} = e_{(p,u),u} - e_{n,u} \Gamma^p_{pq}, \] 
\[ g_{p|m|u|q} \equiv g_{p|m|u|q}, \] 
\[ (g_{uu})_{(p|q)} \equiv g_{uu,pq} - g_{uu,u} \Gamma^p_{pq}, \] 
\[ \Delta g_{uu} \equiv g^{pq} (g_{uu})_{|p|q}. \]
5. Geodesic deviation in Kundt spacetimes with a multiple WAND k

In the remaining parts of this paper we will discuss an important family of algebraically special Kundt spacetimes (22). As described in the previous section, Weyl scalars of the two highest boost weights vanish identically, \( \Psi_{g}^{\text{nat}} = 0, \Psi_{1T}^{\text{nat}} = 0 = \Psi_{1\alpha}^{\text{nat}} \). From (21) we then immediately obtain

\[
\Psi_{g} = 0, \quad \Psi_{1T} = 0, \quad \Psi_{1\alpha} = 0.
\]

The geodesic deviation equations (2), (3) (omitting the frame components of \( T_{ab} \) encoding the direct influence of matter, for example (12) or (14)) for the case of the Kundt class of (vacuum) spacetimes (22) thus reduce to

\[8\]
\[
\ddot{Z}^{(1)} = \frac{2\Lambda}{(D-2)(D-1)} Z^{(1)} + \frac{1}{\sqrt{2}} \Psi_{25} Z^{(1)},
\]
\[
\ddot{Z}^{(i)} = \frac{2\Lambda}{(D-2)(D-1)} Z^{(i)} - \frac{1}{\sqrt{2}} \Psi_{25,} Z^{(i)} - \frac{1}{2} \Psi_{4,} Z^{(i)}. \tag{58}
\]

The corresponding Weyl scalars in the interpretation null frame are given by expressions (21) with (20), which now simplify considerably due to (56):
\[
\Psi_{25} = \Psi_{25,}^{nat}, \quad \Psi_{25,0} = \Psi_{25,}^{nat},
\]
\[
\Psi_{3,} = \sqrt{2} \dot{u} \Psi_{3,}^{nat} + \sqrt{2} \dot{x}^p g_{pq} \left( (\Psi_{25}^{nat} - \Psi_{25,}^{nat}) m^q - \Psi_{25,}^{nat} m^q \right),
\]
\[
\Psi_{4,} = 2 \dot{u} \Psi_{4,}^{nat} + 4 \dot{u} \dot{x}^p g_{pq} \left( \Psi_{3,}^{nat} m^q - \Psi_{3,}^{nat} m^q \right) + 2 \dot{x}^p \dot{x}^q g_{pq} \left( \Psi_{25}^{nat} m^q - \Psi_{25,}^{nat} m^q \right),
\]
\[
-2 g_{pm} g_{qn} \left( \Psi_{25}^{nat} + \Psi_{25,0} m^m \right) m^{pn} m^{qn}, \tag{59}
\]

where \( m^{ij} \equiv \delta^{ij}m^i \) and both the frame indices \( i, j, k, l \) and the coordinate indices \( p, q, m, n \) take the ranges \( 2, 3, \ldots, D - 1 \). The coefficients \( \Psi_{x,}^{nat} \) are explicitly given by expressions (23)–(32). Notice that in (59), not only the ‘electric part’ but also the ‘magnetic part’ of the Weyl tensor with respect to \( u^{nat} \) occur due to the relative velocity \( x^p \) between \( u \) and \( u^{nat} = \frac{1}{\sqrt{2}}(k^{nat} + \dot{l}^{nat}) \). For completeness, the remaining ‘magnetic’ Weyl tensor components in the interpretation frame that do not enter directly the equations of geodesic deviation (57), (58) are
\[
\Psi_{2,} = \Psi_{2,}^{nat}, \quad \Psi_{4,} = \Psi_{4,}^{nat},
\]
\[
\Psi_{3,0} = \sqrt{2} \dot{u} \Psi_{3,}^{nat} + \sqrt{2} \dot{x}^p g_{pq} \left( \Psi_{25}^{nat} m^q - \Psi_{25,}^{nat} m^q \right) + 2 \dot{x}^p \dot{x}^q g_{pq} \left( \Psi_{25}^{nat} m^q - \Psi_{25,}^{nat} m^q \right). \tag{60}
\]

The specific relative motion of free test particles in any algebraically special Kundt spacetime (22) with a multiple WAND \( k \) thus consists of isotropic influence of the cosmological constant \( \Lambda \), Newtonian-like tidal deformations represented by \( \Psi_{25} \), \( \Psi_{25,0} \), longitudinal accelerations associated with the direction \( +e^{(1)} \) given by \( \Psi_{3,} \), and by transverse gravitational waves propagating along \( +e^{(1)} \) encoded in the symmetric traceless matrix \( \Psi_{4,} \). These components were described separately in (8)–(11). The invariant amplitudes (59) combine the curvature of the Kundt spacetime with the kinematics of the specific geodesic motion. In contrast to longitudinal and transverse effects, the Newtonian-like deformations caused by \( \Psi_{25} \) and \( \Psi_{25,0} \) are independent of the observer’s velocity components \( \dot{x}^p \) and \( \dot{u} \).

We will now describe systematically the canonical structure of the relative motion of free test particles in all possible algebraic types and subtypes of the Kundt family summarized in table 3.

### 5.1. Type O Kundt spacetimes

For type O Kundt spacetimes, the Weyl tensor vanishes identically, so that all the Weyl scalars \( \Psi_{x,}^{nat} \) given by (23)–(32) are zero. In view of (59), the geodesic deviation equations (57), (58) for the type O vacuum Kundt spacetimes reduce to
\[
\ddot{Z}^{(1)} = \frac{2\Lambda}{(D-2)(D-1)} Z^{(1)}, \quad \ddot{Z}^{(i)} = \frac{2\Lambda}{(D-2)(D-1)} Z^{(i)}. \tag{61}
\]

There is thus no distinction between the (generically privileged) longitudinal spatial direction \( e^{(1)} \) and the transverse spatial directions \( e^{(i)}, \ i = 2, \ldots, D - 1 \). The relative motion is isotropic and fully determined by the cosmological constant \( \Lambda \), see (8). This is in full agreement with
the well-known fact [3, 4] that the only type O vacuum spaces are just Minkowski space, de Sitter space or anti-de Sitter space.

For non-vacuum type O (conformally flat) Kundt spacetimes, it is necessary to add the terms representing direct influence of matter. For example, in the case of pure radiation (‘null dust’) aligned along \( \mathbf{k} \), the components (12) have to be superposed, and the equations of geodesic deviation become

\[
\dot{Z}^{(1)} = \frac{2\Lambda}{(D-2)(D-1)} Z^{(1)}, \quad \dot{Z}^{(i)} = \frac{2\Lambda}{(D-2)(D-1)} Z^{(i)} - \frac{4\pi \rho}{D-2} Z^{(i)}. \tag{62}
\]

Since \( \rho > 0 \), there is now an additional radial contraction in the transverse subspace.

For aligned electromagnetic field, the additional matter terms are given by (14).

5.2. Type N Kundt spacetimes

As shown in [9], for type N Kundt spacetimes (22) (with quadruple WAND \( \mathbf{k} = \hat{a}_k \)) the only non-trivial Weyl scalars are \( \Psi_{\Delta \nu}^{\text{nat}} \). Considering (59), the geodesic deviation equations (57), (58) for vacuum type N spacetimes thus take the form

\[
\dot{Z}^{(1)} = \frac{2\Lambda}{(D-2)(D-1)} Z^{(1)}, \tag{63}
\]

\[
\dot{Z}^{(i)} = \frac{2\Lambda}{(D-2)(D-1)} Z^{(i)} - \hat{u}^2 \Psi_{\Delta \nu}^{\text{nat}} Z^{(i)}, \tag{64}
\]

where, due to (32),

\[
\Psi_{\Delta \nu}^{\text{nat}} = m_i^2 m_j^2 \left( W_{pq} - \frac{1}{D-2} g_{pq} W \right) \tag{65}
\]

is a symmetric and traceless matrix fully determined by \( W_{pq} \). The symmetric matrix \( W_{pq} \) introduced in (35) simplifies, using all relevant conditions in table 3 (cf [9]), to

\[
W_{pq} = \frac{1}{2} a g_{pq} + U_{(p|\nu)} + U_{(\nu|p)}
\]

\[-\frac{1}{2}(c_{\nu} - c_{\nu} f_{pq})|_{\nu} + (c_{\nu} - c_{\nu} f_{pq})|_{p} - \frac{1}{2} b e_{pq} + (a - \frac{1}{4} f^{m} f_{m}) e_{qb} + Z_{(pq)}, \tag{66}
\]

in which

\[
U_{p} = \frac{1}{2} f_{p} - \frac{1}{4} f^{q} f_{q} e_{p} + \frac{1}{2} f^{q} e_{q} f_{p} - \frac{1}{2} f^{q} E_{q} f_{p} - \frac{1}{D-2} X_{p}, \tag{67}
\]

\[
Z_{pq} = e^{\nu} e_{\nu} f_{p} f_{q} + e_{p|\nu} f_{\nu} + \frac{1}{2} g_{pq} e_{\nu} - e^{\nu} E_{mp} f_{q} + g^{m} E_{mp} E_{aq} - \frac{1}{D-2} X_{p} e_{q}, \tag{68}
\]

and its trace is \( W = g^{\nu\nu} W_{pq} \). The matrix \( \Psi_{\Delta \nu}^{\text{nat}} \) represents the amplitudes of Kundt gravitational waves in any dimension \( D \). In general, their effect is superposed on the isotropic influence of the cosmological constant \( \Lambda \), as given by (61). In the case \( D = 4 \) this has been analyzed and described in our previous work [41].

Since the set of \((D-2) \times (D-2)\) scalars \( \Psi_{\Delta \nu}^{\text{nat}} \) forms a symmetric and traceless matrix, it has in general \( N \equiv \frac{1}{2} D(D-3) \) independent components corresponding to the polarization modes of the gravitational wave. The remaining freedom in the choice of the transverse vectors \( \mathbf{m}_{ij} \) of the interpretation frame (7) is given by the spatial rotations \( \mathbf{m}_{ij}' = \Phi_{ij}^{il} \mathbf{m}_{lj} \), where \( \Phi_{ij}^{il} \delta_{jl} = \delta_{ik} \). These rotations belong to the \( SO(D-2) \) group with \( N_{\text{rot}} \equiv \frac{1}{2} (D-2)(D-3) \) independent generators. Therefore, the number of physical degrees of freedom is

\[
N - N_{\text{rot}} = D - 3. \tag{69}
\]

This is exactly the number of independent eigenvalues of the matrix \( \Psi_{\Delta \nu}^{\text{nat}} \) which fully characterize the geodesic deviation deformation of a set of test particles. The sum of all
the eigenvalues must vanish (the traceless property), so that there is at least one positive and one negative eigenvalue. The number of distinct options of dividing the remaining eigenvalues into three groups with positive, null and negative signs is \( \binom{D-2}{2} \). Concerning the signs of the eigenvalues, we can thus distinguish \( \frac{1}{2}(D-2)(D-3) \) geometrically and physically distinct cases.

Diagonalizing \(-\Psi_4^{\text{nat}}\) and denoting its eigenvalues as \( A_2, A_3, \ldots \), we obtain

\[
-\Psi_4^{\text{nat}} = \text{diag}(A_2, A_3, \ldots) \quad \text{where} \quad \sum_{i=2}^{D-1} A_i = 0. \tag{70}
\]

In view of (64), the relative motion of (initially static) test particles is such that they recede in spatial directions with positive eigenvalues \( A_i > 0 \) , while they converge with negative eigenvalues \( A_i < 0 \). In the directions where \( A_i = 0 \) the particles stay fixed.

In the classic \( D = 4 \) case, there is just one possibility, namely \( A_3 = -A_2 \), and the diagonalized matrix of the gravitational wave amplitudes takes the form

\[
-\Psi_4^{\text{nat}} = \begin{pmatrix} A_2 & 0 \\ 0 & -A_2 \end{pmatrix}. \tag{71}
\]

In the transverse two-dimensional space perpendicular to the privileged propagation direction \( e_{(1)} \), we observe the standard gravitational wave effect, in which the set of test particles expands in the direction \( e_{(2)} \) when \( A_2 > 0 \) and simultaneously contracts by the same amount in the perpendicular direction \( e_{(3)} \) (or vice versa if \( A_2 < 0 \)), unless one has the trivial case \( A_2 = 0 \).

In higher dimensions, many more possibilities and new observable effects arise. For example, in the first non-trivial case \( D = 5 \), the corresponding transverse space spanned by \( (e_{(2)}, e_{(3)}, e_{(4)}) \) is three-dimensional. Concerning the deformation of a three-dimensional test sphere in this space, there are three physically distinct situations determined by \( A_2, A_3, A_4 \), namely:

- two eigenvalues are positive and one is negative, see figure 2(a),
- one eigenvalue is positive and two are negative, see figure 3(a),
- one eigenvalue is positive, one is zero and one is negative, see figure 4(a).

From the point of view of a gravitational wave detector located in our (1+3)-dimensional ‘real’ universe locally spanned by the vectors \( (u, e_{(1)}, e_{(2)}, e_{(3)}) \) where \( e_{(1)} \) is the propagation direction, \( (e_{(2)}, e_{(3)}) \) defines the plane of the detector, while \( e_{(4)} \) is the extra (directly unobservable) dimension, we would see the following ‘peculiar’ effects in which the usual traceless property in \( (e_{(2)}, e_{(3)}) \) is violated:

- both \( A_2, A_3 \neq 0 \) (either positive or negative), \( A_4 = -(A_2 + A_3) \) due to (70):

\[
-\Psi_4^{\text{nat}} = \begin{pmatrix} A_2 & 0 \\ 0 & A_3 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -A_3 & A_2 \\ 0 & -A_2 & -(A_2 + A_3) \end{pmatrix}. \tag{72}
\]

In the directly observable first sector of dimension \( 2 \times 2 \), the eigenvalues \( A_2 \) and \( A_3 \) can have arbitrary values now. Thus, the ring of test particles in these two directions (a section through the 3-sphere in the transverse space) may

- recede in both directions \( e_{(2)}, e_{(3)} \) \((A_2, A_3 > 0 \Rightarrow A_4 < 0)\) as in figure 2(b),
- converge in both directions \( e_{(2)}, e_{(3)} \) \((A_2, A_3 < 0 \Rightarrow A_4 > 0)\) as in figure 3(d),
- recede in one direction and converge in the other, but not by the same amount \((A_2, A_3)\) have opposite signs, \( |A_2| \neq |A_3| \Rightarrow A_4 \neq 0 \) as in figures 2(c), (d) or figures 3(b), (c),
Figure 2. Deformation of a sphere of test particles in the case when two eigenvalues of $-\Psi_{ij}^{nat}$ are positive and one is negative ($D = 5$, the wave propagates in the direction $e_{(1)}$, and the transverse 3-space shown is spanned by $e_{(2)}, e_{(3)}, e_{(4)}$). Plot (a) is a global view, (b), (c), (d) are views from the top, front, right, respectively.

Figure 3. Deformation of a sphere of test particles in the case when one eigenvalue of $-\Psi_{ij}^{nat}$ is positive and two are negative.

- behave as in the standard $D = 4$ general relativity ($A_3 = -A_2 \Rightarrow A_4 = 0$) as in figure 4(c), that is

\[
-\Psi_{ij}^{nat} = \begin{pmatrix}
A_2 & 0 & 0 \\
0 & -A_2 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]  

(73)
Figure 4. Deformation of a sphere of test particles in the case when one eigenvalue of $-\Psi_4^{nat}$ is positive, one is zero and one is negative.

- $A_3 = 0$ or $A_2 = 0$, so that

$$-\Psi_4^{nat} = \begin{pmatrix} A_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -A_2 \end{pmatrix}$$

or

$$-\Psi_4^{nat} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & A_3 & 0 \\ 0 & 0 & -A_3 \end{pmatrix}. \quad (74)$$

We can distinguish two subcases of this anomalous behavior, namely

- $A_2 > 0$ or $A_3 > 0$ as in figure 4(b),
- $A_2 < 0$ or $A_3 < 0$ as in figure 4(d).

Finally, in principle, it may also happen that the higher-dimensional gravitational wave would propagate in the extra spatial dimension $e^{(4)}$, say. Due to the formal swap $e^{(1)} \leftrightarrow e^{(4)}$, this would imply $A_4 = 0$ and $A_1 \neq 0$. In our real $(1+3)$-dimensional universe we would thus observe the complete three-dimensional transverse space with peculiar ‘longitudinal’ deformation of a sphere of test particles.

5.3. Type III Kundt spacetimes

For type III Kundt spacetimes (22) (which have a triple WAND $k = \tilde{a}_i$), all the Weyl tensor components of the boost weight 0 vanish, $\Psi_5^{nat} = \Psi_7^{nat} = \Psi_9^{nat} = \Psi_{2/\mathcal{M}}^{nat} = 0$. The equations of geodesic deviation (57), (58) thus become

$$\ddot{Z}^{(1)} = \frac{2\Lambda}{(D - 2)(D - 1)} Z^{(1)} - \dot{u} \Psi_3^{nat} Z^{(j)} - \frac{\dot{x}^\alpha g_{\beta\rho}}{\Psi_3^{nat} m_i} Z^{(j)}. \quad (75)$$

$$\ddot{Z}^{(i)} = \frac{2\Lambda}{(D - 2)(D - 1)} Z^{(i)} - \dot{u} \Psi_3^{nat} Z^{(j)} - \frac{\dot{x}^\alpha g_{\beta\rho}}{\Psi_3^{nat} m_i} Z^{(j)} - 2 \dot{u} \dot{x}^\beta \hat{g}_{\beta\rho} \psi_{3/\mathcal{M}} m_\lambda Z^{(j)}. \quad (76)$$
Using the conditions summarized in the first four rows of table 3, expressions (29)–(32) for the non-trivial Weyl scalars \( \Psi_{3Y}^{\text{nat}} \), \( \Psi_{3Y^i}^{\text{nat}} \), \( \Psi_{4Y^i}^{\text{nat}} \) reduce to

\[
\Psi_{3Y}^{\text{nat}} = -m_{Pj} \frac{D - 3}{D - 2} (a_{Pj} + f_{Pj} \alpha) r + \frac{1}{2} \left( b_{Pj} - f_{Pj,\alpha} - \frac{2}{D - 2} \left( \frac{S \Gamma}{D - 3} + f \right) + \frac{1}{2} f^q e_q f_P - f^q E_{qP} - \frac{2}{D - 3} X_P \right),
\]

\[
\Psi_{3Y^i}^{\text{nat}} = \tilde{\Psi}_{Y^i}^{\text{nat}} + \frac{1}{D - 3} (\delta_{ij} \Psi_{3Y^j}^{\text{nat}} - \delta_{ik} \Psi_{3Y^k}^{\text{nat}}),
\]

\[
\Psi_{4Y^i}^{\text{nat}} = m_i m_j f_0^2 \left( W_{pq} - \frac{1}{D - 2} g_{pq} W \right),
\]

where \( a = \frac{1}{4} f^q f_P - \frac{1}{D - 2} \left( \frac{S \Gamma}{D - 3} + f \right) \), \( X_q = g^{pq} X_{pq} \), \( W = g^{pq} W_{pq} \) and

\[
X_{pq} = e_{[q][p]} + e_{[q][\alpha]} f_{[\alpha]} - f_{[\alpha]} e_{[q]} + g^{pq} X_{pq},
\]

\[
W_{pq} = -r^2 \left[ \frac{1}{2} a_{[q]} + \frac{1}{2} f_{[q]} + \frac{1}{2} a_{[\alpha]} f_{[\alpha]} - \frac{1}{2} \left( \delta_{[q]} \Psi_{3Y^i}^{\text{nat}} - \delta_{[q]} X_{4Y^i} \right) \right]
\]

\[
= \frac{1}{2} e_{[q]} + \frac{1}{2} f_{[q]} - \frac{1}{2} \left( \Psi_{3Y}^{\text{nat}} - X_{4Y} \right) - \frac{1}{2} b_{[q]} - \frac{1}{2} \left( \Psi_{3Y^i}^{\text{nat}} - X_{4Y^i} \right) - \frac{1}{2} g_{pq,uu} - \frac{1}{2} f_{[q]} + \frac{1}{2} e_{[q]} + \frac{1}{2} f_{[q]} - \frac{1}{2} \left( \Psi_{3Y}^{\text{nat}} - X_{4Y} \right)
\]

\[
+ \frac{1}{2} \left( e_{[q]} + f_{[q]} + f_{[\alpha]} \right) - \frac{1}{2} \left( \Psi_{3Y^i}^{\text{nat}} - X_{4Y^i} \right).
\]

In addition to the isotropic influence of the cosmological constant \( \Lambda \) and the transverse effects of gravitational waves described by \( \Psi_{3Y}^{\text{nat}} \) (which are typical for type O and N spacetimes, respectively), type III Kundt spacetimes feature a longitudinal effect proportional to the scalars \( \Psi_{3Y}^{\text{nat}} \), see (75). Moreover, from (76) we conclude that there is also an additional kinematic effect for non-static observers — those with a non-vanishing velocity in the transverse space, \( \dot{x}^p \neq 0 \). Measuring relative motions between geodesic observers with non-trivial spatial velocities we can thus determine other components of the curvature tensor, namely the symmetric part of \( \Psi_{3Y}^{\text{nat}} \).

If, and only if, \( \Psi_{3Y}^{\text{nat}} = 0 \), the geometry is of algebraic type III, with respect to the triple \( \text{WAND} \ k^{\text{nat}} \) and \( \text{WAND} \ l^{\text{nat}} \).

### 5.4. Subtype III(a)

For the subtype III(a) of Kundt spacetimes, there is \( \Psi_{3Y}^{\text{nat}} = 0 \), see table 3 and [9]. The equations of geodesic deviation (75), (76) thus simplify to

\[
\dot{Z}^{(1)} = \frac{2\Lambda}{(D - 2)(D - 1)} Z^{(1)},
\]

\[
\dot{Z}^{(i)} = \frac{2\Lambda}{(D - 2)(D - 1)} Z^{(i)} - \dot{u}^2 \Psi_{4Y^i}^{\text{nat}} Z^{(i)} + 2 \dot{u} \dot{x}^p g_{pq} \Psi_{3Y}^{\text{nat}} m_{pq} Z^{(j)}. \tag{80}
\]

Apart from the cosmological background \( \Lambda \)-term, there are only transverse effects given by the scalars \( \Psi_{4Y^i}^{\text{nat}} \) and \( \Psi_{3Y}^{\text{nat}} \). The latter contribution is purely kinematical, i.e., it is absent for \( \dot{x}^p = 0 \). For such static observers, the geodesic deviation is the same as for type N spacetimes, cf (63), (64). The specific contributions of \( \Psi_{3Y}^{\text{nat}} \) can be identified by considering non-static observers with mutual velocities \( \dot{x}^p \neq 0 \).
5.5. Subtype III(b)

For the subtype III(b) of Kundt spacetimes, there is $\Psi_{3Y}^{\text{nat}} = 0$, see table 3 and [9]. In such a case, the equations (75), (76) become

$$\ddot{Z}^{(i)} = \frac{\Lambda}{(D-2)(D-1)} Z^{(i)} - \dot{u} \Psi_{3Y}^{\text{nat}} Z^{(j)} ,$$

$$\ddot{Z}^{(i)} = \frac{\Lambda}{(D-2)(D-1)} Z^{(i)} - \dot{u} \Psi_{3Y}^{\text{nat}} Z^{(1)} - \dot{u}^2 \Psi_{4Y}^{\text{nat}} Z^{(j)}$$

$$- \frac{2}{D-2} \frac{D-3}{3} \dot{u} \lambda^i g_{pq} \left( \Psi_{3Y}^{\text{nat}} m_p^q - \frac{\delta_i}{D-2} \Psi_{3Y}^{\text{nat}} m_{j} \right) Z^{(j)} .$$

The geodesic deviation is thus fully determined by the scalars $\Lambda$, $\Psi_{4Y}^{\text{nat}}$ and $\Psi_{3Y}^{\text{nat}}$ via the corresponding isotropic, transverse and longitudinal effects, respectively. For observers with non-vanishing spatial velocities $\dot{x}^0 \neq 0$, transverse motion is modified by the presence of $\Psi_{3Y}^{\text{nat}}$. This additional effect is traceless since $\dot{y}^0 \Psi_{3Y}^{\text{nat}} m^0 = \Psi_{3Y}^{\text{nat}} m^0$.

5.6. Type D Kundt spacetimes

For type D Kundt spacetimes (22) with a double WAND $k^{\text{nat}} = k = \partial_r$ and a double WAND $l^{\text{nat}} = \frac{1}{2} g_{ab} \partial_r + \partial_a$, all the Weyl scalars $\Psi_{4}^{\text{nat}}$ vanish, except for the boost weight 0. Therefore, the equations of geodesic deviation are (57), (58) with

$$\Psi_{2S} = \Psi_{2S}^{\text{nat}} , \quad \Psi_{2Y} = \Psi_{2Y}^{\text{nat}} ,$$

$$\Psi_{3Y} = \sqrt{2} \frac{\dot{x}^0}{g_{pq}} \left( \Psi_{2Y}^{\text{nat}} - \Psi_{2Y}^{\text{nat}} \right) m^q - \Psi_{2S}^{\text{nat}} m^q ,$$

$$\Psi_{4Y} = \frac{2}{\sqrt{2}} \sqrt{\frac{\dot{x}^0}{g_{pq}}} \left( g_{pq} \Psi_{2Y}^{\text{nat}} - g_{pm} g_{qn} \Psi_{2S}^{\text{nat}} m^p m^q + g_{pm} g_{qn} \Psi_{2Y}^{\text{nat}} m^p m^q - 2 \frac{\dot{x}^0}{g_{pq}} \Psi_{2Y}^{\text{nat}} \right) m^0 ,$$

where $\Psi_{2S}^{\text{nat}}$, $\Psi_{2Y}^{\text{nat}}$, $\Psi_{3Y}^{\text{nat}}$, $\Psi_{4Y}^{\text{nat}}$ are explicitly given by expressions (23)–(28).

For static observers that do not move in the transverse spatial directions ($\dot{x}^0 = 0$), we have $\Psi_{3Y} = 0 = \Psi_{4Y}$, so that the equations simplify considerably to

$$\ddot{Z}^{(1)} = \frac{2\Lambda}{(D-2)(D-1)} Z^{(1)} + \Psi_{2S}^{\text{nat}} Z^{(1)} ,$$

$$\ddot{Z}^{(i)} = \frac{2\Lambda}{(D-2)(D-1)} Z^{(i)} - \Psi_{2Y}^{\text{nat}} Z^{(j)} .$$

We can now explicitly discuss specific particle motion in various algebraic subtypes of the Kundt spacetimes of type D.

5.7. Subtype D(a)

The subtype D(a) is defined by the condition

$$\Psi_{2S}^{\text{nat}} = 0 .$$

This is equivalent to $g_{mn} = a r^2 + b r + c$ where $a = \frac{1}{2} f^0 f_0 - \frac{1}{D-2} (\frac{h_0}{D-2} + f)$, cf (23) and the first row in table 3. The geodesic deviation equations (84), (85) reduce to

$$\ddot{Z}^{(1)} = \frac{2\Lambda}{(D-2)(D-1)} Z^{(1)} ,$$

$$\ddot{Z}^{(i)} = \frac{2\Lambda}{(D-2)(D-1)} Z^{(i)} - \Psi_{2Y}^{\text{nat}} Z^{(j)} .$$
where, in view of (24), (86), we have $\Psi_{2T_0}^{nat} = 0$ which is explicitly expressed by (25). There is no longitudinal Newtonian motion, see (87), and the transverse Newtonian deformations (88) are traceless since $\delta^{ij} \Phi_{2T_0}^{nat} = 0$, see (6). Interestingly, in higher dimensions, the local behavior of test particles in subtype D(a) spacetimes, as given by expressions (87), (88), is very similar to the effect caused by type N gravitational waves (63), (64). Due to this close formal similarity, we can use figures 2–4 to illustrate particle motion in the $D = 5$ case. Such a situation does not appear in the $D = 4$ case since $\Psi_{2T_0}^{nat} = 0$, as we can see from (25).

For geodesics with spatial velocities $\dot{x}^i \neq 0$, there are additional terms $\Phi_{5T_0}^{nat}, \Phi_{4T_0}^{nat}$ given by (57), (58), (83). The scalars $\Psi_{2T_0}^{nat}, \Psi_{2T_0}^{nat}$ take the form (28) and (26), (27).

5.8. Subtype D(b)

The subtype D(b) occurs if, and only if, $\tilde{\Phi}_{2T_0}^{nat} = 0$. From (24) it thus follows that

$$\Psi_{2T_0}^{nat} = \frac{1}{D-2} \delta_{ij} \Psi_{2S}^{nat}.$$  \hspace{1cm} (89)

Due to (25), this is equivalent to $\tilde{\Phi}_{pq}^{nat} - \frac{1}{D-2} g_{pq} \tilde{\Phi} = -\frac{1}{2} (D-4) (f_{pq} - \frac{1}{D-2} g_{pq} f)$, see [9] and the second row in table 3. For such Kundt geometries, the equations of geodesic deviation (84), (85) take the form

$$\ddot{Z}^{(1)} = \frac{2\Lambda}{(D-2)(D-1)} \dot{Z}^{(1)} + \Psi_{2S}^{nat} Z^{(1)},$$  \hspace{1cm} (90)

$$\ddot{Z}^{(i)} = \frac{2\Lambda}{(D-2)(D-1)} \dot{Z}^{(i)} - \frac{\delta_{ij}}{D-2} \Psi_{2S}^{nat} Z^{(j)},$$  \hspace{1cm} (91)

We can see that the Newtonian part of the gravitational field is now fully determined by a single scalar $\Psi_{2S}^{nat}$ given by (23). Moreover, motion in the transverse spatial directions $i, j = 2, \ldots, D-1$ is isotropic (its sum is fully offset to zero by the longitudinal motion, $\delta^{ij} \Phi_{2T_0}^{nat} = 0$). A sphere of test particles, initially at rest, is thus deformed into a rotational ellipsoid with the axis $e^{(1)}$, see figure 5. Interestingly, this type of behavior enables us to determine experimentally the dimension $D$ of the spacetime. Subtracting the isotropic motion given by $\Lambda$, it is possible to measure the relative acceleration in the longitudinal direction

Figure 5. Deformation of a sphere of static test particles in the subtype D(b) when $D = 5$ for the cases (a) $\Psi_{2T_0}^{nat} < 0$, (b) $\Psi_{2T_0}^{nat} > 0$. Unlike in figures 2–4, here we show $e^{(1)}, e^{(2)}, e^{(3)}$, where $e^{(1)}$ is the longitudinal direction (oriented horizontally) while $e^{(2)}, e^{(3)}$ (plotted perpendicularly) are two directions of the transverse 3-space (the third equivalent transverse direction $e^{(4)}$ is suppressed).
\( e_{(1)} \) and compare it with the acceleration in any transverse direction \( e_{(2)} \), say, obtaining 
\[
\left( Z^{(1)} / Z^{(1)} \right) / \left( -Z^{(2)} / Z^{(2)} \right) = D - 2.
\]

For geodesics with \( i^p \neq 0 \), the additional terms \( \Psi_{3T} \) and \( \Psi_{4i} \) given by (57), (58), (83) have to be included.

5.9. Subtype D(ab)

Any Kundt spacetime that is both of the algebraic subtype D(a) and subtype D(b) must necessarily satisfy \( \Psi_{2i} = 0 \), see (86) and (89). Equations of geodesic deviation (57), (58) then reduce to
\[
\ddot{Z}^{(i)} = \frac{2\Lambda}{(D - 2)(D - 1)} Z^{(i)} - \frac{1}{\sqrt{2}} \Psi_{3T} Z^{(j)},
\]
and
\[
\ddot{Z}^{(1)} = \frac{2\Lambda}{(D - 2)(D - 1)} Z^{(1)} - \frac{1}{\sqrt{2}} \Psi_{3T} Z^{(1)} - \frac{1}{2} \Psi_{4i} Z^{(j)},
\]
where (83) becomes
\[
\Psi_{3T} = \frac{1}{2} \sqrt{2} m_i^p F_{p q} \dot{x}^q, \quad \Psi_{4i} = 2 m_i^p m_j^p (\tilde{C}_{mpnq} - 3 F_{pkm} g_{nq}) \dot{x}^p \dot{x}^q.
\]

Interestingly, for static observers (\( \dot{x}^p = 0 \)), we have \( \dot{x}^p = 0 = \Psi_{3T} \) and the equations contain only the cosmological constant \( \Lambda \)-term. The relative motion of such test particles is the same as in the type O spacetimes (61)—it is fully isotropic as in the background Minkowski, de Sitter or anti-de Sitter spaces.

Recall also that in the classic \( D = 4 \) case, the subtypes D(ab) and D(a) are identical because the condition for the subtype D(b) is always satisfied [9].

5.10. Subtype D(c)

The algebraic subtype D(c) is defined by the condition \( \Psi_{2i} = 0 \) which, using (27), is equivalent to \( \tilde{C}_{mpnq} = 0 \), cf the third row in table 3. Since \( \Psi_{3T} \) and \( \Psi_{4i} \) are generally non-vanishing, this subtype of the Kundt geometries cannot be distinguished by measuring the deviation (84), (85) of static geodesic observers. In principle, it can be detected in the relative motion of non-static particles with \( \dot{x}^p \neq 0 \) as the \( \Psi_{2i} \) component in the amplitude \( \Psi_{3T} \) determined by (83) is absent.

Moreover, as discussed in [9], the condition for subtype D(c) is identically satisfied in the cases \( D = 4 \) and \( D = 5 \).

5.11. Subtype D(d)

The subtype D(d) occurs if, and only if, \( \Psi_{2i} = 0 \). In view of (28), this is equivalent to \( F_{pq} = 0 \) (see table 3). As in the subcase D(c), this is not directly observable in the geodesic deviation (84), (85) of static observers, but it is implied by the absence of the \( \Psi_{2i} \) component entering the scalars \( \Psi_{3T}, \Psi_{4i} \) via (83). It is detectable by observers with \( \dot{x}^p \neq 0 \) for which the equations of geodesic deviation take the form (57), (58).

5.12. Type II Kundt spacetimes

The general form of the geodesic deviation equations for any Kundt spacetime (22) of algebraic type II (or more special) with (at least) a double WAND \( \kappa = \kappa_{\mathrm{nat}} = \partial r \) is
\[ \ddot{Z}^{(1)} = \frac{2\Lambda}{(D-2)(D-1)} Z^{(i)} + \Psi_{2S}^{\text{nat}} Z^{(i)} - \left[ \dot{u} \Psi_{2Y}^{\text{nat}} + \dot{x}^{\rho} g_{p\rho} \left( \frac{3}{2} \Psi_{2Y}^{\text{nat}} m_i^q - \Psi_{2Y}^{\text{nat}} m_i^q - \frac{D-1}{D-2} \Psi_{2S}^{\text{nat}} m_i^q \right) \right] Z^{(i)}. \] (95)

\[ \ddot{Z}^{(i)} = \frac{2\Lambda}{(D-2)(D-1)} Z^{(i)} - \left( \dot{u} \Psi_{2Y}^{\text{nat}} + \frac{\delta_{ij}}{D-2} \Psi_{2S}^{\text{nat}} \right) Z^{(j)} - u^2 \Psi_{4Y}^{\text{nat}} Z^{(j)} - \left[ \dot{u} \Psi_{2Y}^{\text{nat}} + \dot{x}^{\rho} g_{p\rho} \left( \frac{3}{2} \Psi_{2Y}^{\text{nat}} m_i^q - \Psi_{2Y}^{\text{nat}} m_i^q - \frac{D-1}{D-2} \Psi_{2S}^{\text{nat}} m_i^q \right) \right] Z^{(j)} - \left[ 2 \dot{x}^{\rho} g_{p\rho} \left( \frac{D-2}{D-3} \Psi_{2Y}^{\text{nat}} m_i^q - \frac{\delta_{ij}}{D-3} \Psi_{3Y}^{\text{nat}} m_i^q - \Psi_{3Y}^{\text{nat}} m_i^q \right) \right] Z^{(j)} + \dot{x}^{\rho} x^{\nu} \left[ \frac{\delta_{ik} \Psi_{2Y}^{\text{nat}} m_i^k - 3 \Psi_{2Y}^{\text{nat}} m_i^k - 2 \frac{D-2}{D-3} \Psi_{2Y}^{\text{nat}} m_i^k + \Psi_{2Y}^{\text{nat}} m_i^k \right] + \frac{\delta_{ij}}{D-3} \frac{D-1}{D-2} \Psi_{2Y}^{\text{nat}} m_i^q \right] Z^{(j)}. \] (96)

The behavior of test particles in the subtypes II(a), II(b), II(c) and II(d) is easily obtained by setting \( \Psi_{2S}^{\text{nat}} = 0, \Psi_{2Y}^{\text{nat}} = 0, \Psi_{2S}^{\text{nat}} = 0 \) and \( \Psi_{2Y}^{\text{nat}} = 0 \), respectively.

When all these Weyl scalars of the boost weight 0 vanish, we obtain the type III Kundt geometries with a quadruple WAND \( k \) and recover the results of sections 5.3–5.5. If, in addition, \( \Psi_{3Y}^{\text{nat}} = 0 = \Psi_{2Y}^{\text{nat}} \), the spacetimes are of type N with a quadruple W AND \( \Psi_{2Y}^{\text{nat}} = 0 \), see 5.1. Alternatively, if only \( \Psi_{2Y}^{\text{nat}} = 0 \), given by (32), (35), (36), the spacetime is of algebraic type II, with respect to a double WAND \( k^{\text{nat}} = \bar{\delta}_4 \) and WAND \( f^{\text{nat}} = \frac{1}{2} \Delta_{\text{nat}} \bar{\delta}_4 + \bar{\delta}_4 \). When only the scalars \( \Psi_{2Y}^{\text{nat}}, \Psi_{3Y}^{\text{nat}} \) are non-trivial, the geometry is of algebraic type III, with respect to a triple WAND \( k^{\text{nat}} \) and WAND \( f^{\text{nat}} \).

Type D Kundt geometries of section 5.6 arise by setting \( \Psi_{3Y}^{\text{nat}} = 0 = \Psi_{2Y}^{\text{nat}} \) and \( \Psi_{2Y}^{\text{nat}} = 0 \), in which case the expressions (95), (96) correspond to (83). The subtypes D(a), D(b), D(c) and D(d) are obtained when \( \Psi_{2S}^{\text{nat}} = 0, \Psi_{2Y}^{\text{nat}} = 0, \Psi_{2S}^{\text{nat}} = 0 \) and \( \Psi_{2Y}^{\text{nat}} = 0 \), respectively, reducing the results to those discussed in sections 5.7–5.11.

6. Example: type II and N gravitational waves on D and O backgrounds

As an interesting illustration, we can consider a line element of the form

\[ ds^2 = g_{pq} dx^p dx^q - 2 du dr + (a r^2 + c) du^2, \] (97)

where \( g_{pq} = g_{pq}(x) \), \( a = \text{const.} \) and \( c = c(u, x) \).

The possible algebraic structure of such Kundt geometries is summarized in table 4. The relative motion of free test particles in these spacetimes is described by equations (57), (58) where the scalars (59) take the form

\[ \Psi_{2S} = \frac{D-3}{D-1} \left( a + \frac{\delta R}{2(D-2)(D-3)} \right), \]

\[ \Psi_{2Y}^{(0)} = \frac{m_i^q}{D-2} \left( s R_{pq} - \frac{g_{pq}}{D-2} \delta R \right) + \frac{\delta_{ij}}{D-2} \frac{D-3}{D-1} \left( a + \frac{\delta R}{2(D-2)(D-3)} \right), \]

\[ \Psi_{3Y} = -\frac{\sqrt{2}}{D-2} \dot{x}^{\rho} m_i^q (\delta R_{pq} + (D-3) a g_{pq}), \]
Table 4. The structure of all Kundt geometries (97) with respect to a multiple WAND $k^\mu = \partial_\nu$ and (possibly double) WAND $l^\mu = \frac{1}{2}(ar^2 + c)\partial_\nu + \partial_{\nu'}$.

| Type   | Necessary and sufficient conditions |
|--------|-------------------------------------|
| II(a)  | $a = -\frac{D-2}{D-3}R$            |
| II(b)  | $R_{pq} = \frac{1}{D-2}g_{pq}R$   |
| II(c)  | $\mathcal{C}_{mpqn} = 0$          |
| II(d)  | Always                             |
| N      | II(abcd)                           |
| O      | N with $c_{[p|q]} = \frac{1}{D-2}g_{pq}\Delta c$ |
| D      | $c_{[p|q]} = \frac{1}{D-2}g_{pq}\Delta c$ |
| D(a)   | D with II(a)                       |
| D(b)   | D with II(b)                       |
| D(c)   | D with II(c)                       |
| D(d)   | D with II(d)                       |

$\Psi^{(i)} = -i^2 m^i m^j \left( c_{[p|q]} - \frac{g_{pq}}{D-2}\Delta c \right)$

$$
\begin{align*}
\dot{Z}^{(1)} & = \frac{2\Lambda}{(D-2)(D-1)}Z^{(1)}, \\
\dot{Z}^{(i)} & = \frac{2\Lambda}{(D-2)(D-1)}Z^{(i)} + \frac{1}{2}i^2 m^i m^j \left( c_{[p|q]} - \frac{g_{pq}}{D-2}\Delta c \right) Z^{(j)} - x^p \delta^i_m m^p m^j \mathcal{C}_{mpqn} Z^{(j)}. 
\end{align*}
$$

Notice that for the subtype II(ab)≡II(abd), this simplifies considerably to

When, in addition, $\mathcal{C}_{mpqn} = 0$, this becomes type II(abcd)≡N.

If, and only if, $c_{[p|q]} = \frac{1}{D-2}g_{pq}\Delta c$, the spacetimes are of type D or type O. When $c = 0$, these belong to the important family of direct-product spacetimes, see section 11 of [9], for which the first term of the metric (97) is a $(D-2)$-dimensional Riemannian space with metric $g_{pq}(x)$, while the second part is a two-dimensional Lorentzian spacetime of constant Gaussian curvature $a$. In general, $g_{pq}(x)$ need not be of constant curvature, but for the subtype D(a), $a$ is uniquely related to the constant Ricci scalar $\mathcal{R}$ of the transverse $(D-2)$-dimensional space. Such metrics represent natural higher-dimensional generalizations of the (anti-)Nariai, Plebański–Hacyan, Bertotti–Robinson and Minkowski spacetimes of types D or O, see [4].

For a non-trivial $c$, the spacetimes (97) are of type II or of type N. These can be naturally interpreted as the class of exact Kundt gravitational waves with the profile $c(a,x)$ propagating in various direct-product background universes of algebraic types D or O mentioned above (and listed in table 6 of [9]; see also [23, 24, 43]).

The class of metrics (97) clearly contains pp-waves (without gyratonic sources) propagating in flat space when $a = 0$. These are of type N if, and only if, $g_{pq} = \delta_{pq}$ (in which case they belong to the class of VSI spacetimes, see [9]).
Finally, let us observe that in the classic $D = 4$ case, the scalars (99) read
\[
\Psi_{25} = \frac{1}{3} \left( a + \frac{1}{2} \mathcal{R} \right), \quad \Psi_{27^{(0)}} = \frac{1}{6} \delta_{ij} \left( a + \frac{1}{2} \mathcal{R} \right),
\]
\[
\Psi_{37} = -\frac{\sqrt{\frac{3}{2}}}{2} \dot{x}^m \dot{x}^n \left( \mathcal{R}_{pq} + a g_{pq} \right),
\]
\[
\Psi_{47} = -\dot{u}^2 m_i^m m_j^n \left( c_{[i]} g_{[q]} - \frac{1}{2} g_{pq} \Delta c \right)
\]
\[
+ \dot{x}^m \dot{x}^n m_i^m m_j^n \left( a + \frac{1}{2} \mathcal{R} \right) (g_{pq} g_{mn} - 2 g_{pm} g_{qn}).
\]
The corresponding Kundt geometries (97) are thus generally of type II $\equiv II(bcd)$. They are of type $N \equiv II(abcd)$ if, and only if, $a = -\frac{1}{2} \mathcal{R}$ with the only non-vanishing Weyl scalar $\Psi_{47} = -\dot{u}^2 m_i^m m_j^n \left( c_{[i]} g_{[q]} - \frac{1}{2} g_{pq} \Delta c \right)$. In fact, this is the subfamily $\alpha = \beta, \epsilon = 1, C = 0$ of spacetimes discussed in [43] and in sections 18.6–18.7 of [4] (with the identification $\alpha = \beta = 1, \mathcal{R}, D = a$ and $H = -c$) which was interpreted as exact Kundt gravitational waves of type II propagating on type D backgrounds, and type N waves propagating on conformally flat type O backgrounds, respectively. These background universes with the geometry of a direct product of two constant curvature 2-spaces involve the standard Minkowski, Bertotti–Robinson, (anti-)Nariai and Plebański–Hacyan spacetimes, cf [23, 56, 57].

7. Conclusions

We have systematically analyzed the general class of Kundt geometries in an arbitrary dimension $D \geq 4$ using the geodesic deviation in Einstein’s theory. We have explicitly determined the specific motion of free test particles for all possible algebraically special spacetimes, including the corresponding subtypes, and demonstrated that the invariant quantities determining these (sub)types are measurable by detectors via characteristic relative accelerations. For example, the dimension of the spacetime can be measured directly by Newtonian-type tidal deformations of the algebraic subtype $D(b)$. The purely transverse type N effects represent exact gravitational waves with $D(D - 3)/2$ polarizations, which exhibit new and peculiar observable effects in higher dimensions $D > 4$. We have given an example of such geometric and physical interpretation of the Kundt family by analyzing the class of type N or II gravitational waves propagating on backgrounds of type O or D.

Acknowledgments

This work was supported by the grant GAČR P203/12/0118.

References

[1] Kundt W 1961 The plane-fronted gravitational waves Z. Phys. 163 77–86
[2] Kundt W 1962 Exact solutions of the field equations: twist-free pure radiation fields Proc. R. Soc. A 270 328–34
[3] Stephani H, Kramer D, MacCallum M, Hoenselaers C and Herlt E 2003 Exact Solutions of Einstein’s Field Equations (Cambridge: Cambridge University Press)
[4] Griffiths J and Podolský J 2009 Exact Space-Times in Einstein’s General Relativity (Cambridge: Cambridge University Press)
[5] Coley A 2008 Classification of the Weyl tensor in higher dimensions and applications Class. Quantum Grav. 25 033001
[6] Ortaggio M, Pravda V and Pravdová A 2013 Algebraic classification of higher dimensional spacetimes based on null alignment Class. Quantum Grav. 30 013001

22
[7] Podolský J and Žofka M 2009 General Kundt spacetimes in higher dimensions Class. Quantum Grav. 26 105008
[8] Coley A, Hervik S, Papadopoulos G and Pelavas N 2009 Kundt spacetimes Class. Quantum Grav. 26 105016
[9] Podolský J and Švarc R 2013 Explicit algebraic classification of Kundt geometries in any dimension Class. Quantum Grav. 30 125007
[10] Brinkmann H W 1925 Einstein spaces which are mapped conformally on each other Math. Ann. 94 119–45
[11] Coley A, Milson R, Pelavas N, Pravda V, Pravdová A and Zalaletdinov 2003 Generalizations of pp-wave spacetimes in higher dimensions Phys. Rev. D 67 104020
[12] Coley A, Milson R, Pravda V and Pravdová A 2004 Classification of the Weyl tensor in higher dimensions Class. Quantum Grav. 21 L35–41
[13] Coley A, Milson R, Pravda V and Pravdová A 2004 Vanishing scalar invariant spacetimes in higher dimensions Class. Quantum Grav. 21 5519–42
[14] Coley A, Fuster A, Hervik S and Pelavas N 2006 Higher dimensional VSI spacetimes Class. Quantum Grav. 23 7431–44
[15] Coley A, Hervik S and Pelavas N 2006 On spacetimes with constant scalar invariants Class. Quantum Grav. 23 3053–74
[16] Coley A, Hervik S and Pelavas N 2009 Spacetimes characterized by their scalar curvature invariants Class. Quantum Grav. 26 025013
[17] Bonnor W B 1970 Spinning null fluid in general relativity Int. J. Theor. Phys. 3 257–66
[18] Frolov V P and Fursaev D V 2005 Gravitational field of a spinning radiation beam pulse in higher dimensions Phys. Rev. D 71 104034
[19] Frolov V P, Israel W and Zelnikov A 2005 Gravitational field of relativistic gyratons Phys. Rev. D 72 084031
[20] Frolov V P and Zelnikov A 2005 Relativistic gyratons in asymptotically AdS spacetime Phys. Rev. D 72 104005
[21] Frolov V P and Zelnikov A 2006 Gravitational field of charged gyratons Class. Quantum Grav. 23 2119–28
[22] Caldarelli M M, Klemm D and Zoran E 2007 Supersymmetric gyratons in five dimensions Class. Quantum Grav. 24 1341–57
[23] Kadlecová H, Zelnikov A, Krtouš P and Podolský J 2009 Gyratons on direct-product spacetimes Phys. Rev. D 80 024004
[24] Krtouš P, Podolský J, Zelnikov A and Kadlecová H 2012 Higher-dimensional Kundt waves and gyratons Phys. Rev. D 86 044039
[25] McNutt D, Milson R and Coley A 2013 Vacuum Kundt waves Class. Quantum Grav. 30 055010
[26] Wils P 1989 Homogeneous and conformally Ricci flat pure radiation fields Class. Quantum Grav. 6 1243–51
[27] Koutras A and McIntosh C 1996 A metric with no symmetries or invariants Class. Quantum Grav. 13 L47–49
[28] Edgar S B and Ludwig G 1997 All conformally flat pure radiation metrics Class. Quantum Grav. 14 L65–68
[29] Skea J E F 1997 The invariant classification of conformally flat pure radiation spacetimes Class. Quantum Grav. 14 2393–404
[30] Griffiths J B and Podolský J 1998 Interpreting a conformally flat pure radiation space-time Class. Quantum Grav. 15 3863–71
[31] Barnes A 2001 On the symmetries of the Edgar–Ludwig metric Class. Quantum Grav. 18 5287–91
[32] Wils P and Van den Bergh N 1990 Petrov type D pure radiation fields of Kundt’s class J. Math. Phys. 51 102501
[33] De Groote L, Van den Bergh N and Wylléman L 2010 Petrov type D pure radiation fields of Kundt’s class J. Math. Phys. 51 102501
[34] Kinnersley W 1969 Type D vacuum metrics J. Math. Phys. 10 1195–203
[35] Plebański J F and Demiański M 1976 Rotating charged and uniformly accelerating mass in general relativity Ann. Phys. 98 98–127
[36] Griffiths J B and Podolský J 2006 A new look at the Plebański–Demiański family of solutions Int. J. Mod. Phys. D 15 335–69
[37] Ożváth I, Robinson I and Rügza K 1985 Plane-fronted gravitational and electromagnetic waves in spaces with cosmological constant J. Math. Phys. 26 1755–61
[38] Siklos S T C 1985 Lobatchevski plane gravitational waves Galaxies, Axissymmetric Systems and Relativity ed M A H MacCallum (Cambridge: Cambridge University Press) pp 247–74
[39] Podolský J 1998 Interpretation of the Siklos solutions as exact gravitational waves in the anti-de Sitter universe Class. Quantum Grav. 15 719–33
[40] Bičák J and Podolský J 1999 Gravitational waves in vacuum spacetimes with cosmological constant. I. Classification and geometrical properties of non-twisting type N solutions J. Math. Phys. 40 4495–505
[41] Bičák J and Podolský J 1999 Gravitational waves in vacuum spacetimes with cosmological constant. II. Deviation of geodesics and interpretation of non-twisting type N solutions J. Math. Phys. 40 4506–17
[42] Griffiths J B, Docherty P and Podolský J 2004 Generalized Kundt waves and their physical interpretation Class. Quantum Grav. 21 207–22
[43] Podolský J and Ortaggio M 2003 Explicit Kundt type II and N solutions as gravitational waves in various type D and O universes Class. Quantum Grav. 20 1685–701
[44] Podolský J and Belá M 2004 Geodesic motion in Kundt spacetimes and the character of the envelope singularity Class. Quantum Grav. 21 2811–29
[45] Levi-Civita T 1926 Sur l’écart géodésique Math. Ann. 97 291–320
[46] Synge J L 1934 On the deviation of geodesics and null-geodesics, particularly in relation to the properties of spaces of constant curvature and indefinite line-element Ann. Math. 35 705–13
[47] Synge J L 2009 Gen. Rel. Grav. 41 1205–14 (reprinted)
[48] Pirani F A E 1956 On the physical significance of the Riemann tensor Acta Phys. Pol. 15 389–405
[49] Pirani F A E 2009 Gen. Rel. Grav. 41 1215–32 (reprinted)
[50] Szekeres P 1965 The gravitational compass J. Math. Phys. 6 1387–91
[51] de Felice F and Bini D 2010 Classical Measurements in Curved Space-Times (Cambridge: Cambridge University Press)