The boundary supersymmetric sine-Gordon model revisited

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Abstract

We argue that, contrary to previous claims, the supersymmetric sine-Gordon model with boundary has a two-parameter family of boundary interactions which preserves both integrability and supersymmetry. We also propose the corresponding boundary $S$ matrix for the first supermultiplet of breathers.
1 Introduction

Ghoshal and Zamolodchikov [1] formulated the general framework of integrable quantum field theory in the presence of a boundary. Among the examples which they investigated is the boundary version of the sine-Gordon (SG) model [2, 3]. They argued that this model has a two-parameter family of boundary interactions which preserves integrability, and they proposed the corresponding soliton boundary $S$ matrix.

Inami et al. [4] subsequently investigated a boundary version of the supersymmetric sine-Gordon (SSG) model [5, 6, 7]. They argued that the combined constraints of integrability and supersymmetry do not allow any free parameters in the boundary interaction. The boundary SSG and SSHG (supersymmetric sinh-Gordon) models have been studied further in [8, 9, 10, 11].

We argue here that, contrary to the claim in [4], the boundary SSG model has a two-parameter family of boundary interactions which preserves both integrability and supersymmetry. The key point is that the class of boundary interactions considered in [4] is too restrictive: one must instead introduce a Fermionic boundary degree of freedom, as was done by Ghoshal and Zamolodchikov [1] in order to describe the Ising model in a boundary magnetic field. The boundary SG model at the free Fermion point [12, 13] is also formulated with such Fermionic boundary degrees of freedom.

Our result resolves an interesting paradox which arose from our recent work [14] on the boundary supersymmetric Yang-Lee model, which is defined using the notion of perturbed conformal field theory [15]. Indeed, let us first recall that, just as the bulk Yang-Lee (YL) model [16] is a restriction [17] of the bulk SG model, the bulk supersymmetric Yang-Lee (SYL) model [18] is a restriction [19] of the bulk SSG model. The same is true for the corresponding boundary versions: the boundary YL [1] and boundary SYL [19] models are restrictions of the boundary SG and boundary SSG models, respectively. In [14] we found strong evidence (both from perturbed conformal field theory and the conjectured boundary $S$ matrix) that the boundary SYL model with one free boundary parameter has supersymmetry. This strongly suggests that its “parent”, the boundary SSG model, should also have at least a one-parameter family of boundary interactions which maintains both integrability and supersymmetry.

The outline of this Letter is as follows. In Section 2, we propose the boundary interaction in terms of two unknown functions $f(\phi)$ and $B(\phi)$. We determine these functions in Section 3 from the requirements of supersymmetry and integrability. In Section 4 we briefly discuss our results. In particular, we propose the boundary $S$ matrix for the first supermultiplet of breathers.

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1 Earlier related work includes [2, 3].
2 The boundary SSG model

In order to facilitate comparison with [9], which hereafter we refer to as I, we adopt the same notations. The Euclidean-space action of the boundary SSG model is given by

\[ S = \int_{-\infty}^{\infty} dy \int_{-\infty}^{0} dx \mathcal{L}_0 + \int_{-\infty}^{\infty} dy \mathcal{L}_b, \]

where the bulk Lagrangian density is given by

\[ \mathcal{L}_0 = 2 \partial_z \phi \partial_{\bar{z}} \phi - 2 \bar{\psi} \partial_z \psi + 2 \psi \partial_{\bar{z}} \psi - 4 \cos \phi - 4 \bar{\psi} \psi \cos \frac{\phi}{2}, \]

where \( \psi \) and \( \bar{\psi} \) are the two components of a Majorana Fermion field, and \( z = x + iy, \bar{z} = x - iy \). We propose the following boundary Lagrangian at \( x = 0 \)

\[ \mathcal{L}_b = \bar{\psi} \psi + ia \partial_y a - 2 f(\phi) a (\psi - \bar{\psi}) + B(\phi), \]

where \( a \) is a Hermitian Fermionic boundary degree of freedom which anticommutes with both \( \psi \) and \( \bar{\psi} \). As mentioned in the Introduction, such an approach was first considered by Ghoshal and Zamolodchikov [1] to describe the Ising model in a boundary magnetic field. Moreover, \( f(\phi) \) and \( B(\phi) \) are potentials (functions of the scalar field \( \phi \), but not of its derivatives), which are still to be determined.

Variation of the action gives the classical equations of motion. For the bulk, the equations are (I2.2), (I3.3)

\[ \partial_z \partial_{\bar{z}} \phi = \sin \phi + \frac{1}{2} \bar{\psi} \psi \sin \frac{\phi}{2}, \]
\[ \partial_{\bar{z}} \psi = -\bar{\psi} \cos \frac{\phi}{2}, \quad \partial_z \bar{\psi} = -\psi \cos \frac{\phi}{2}. \]

and the boundary conditions at \( x = 0 \) are

\[ \psi + \bar{\psi} = 2fa, \]
\[ i \partial_y a = f(\psi - \bar{\psi}), \]
\[ \partial_x \phi = -\frac{\partial B}{\partial \phi} + 2 \frac{\partial f}{\partial \phi} a (\psi - \bar{\psi}). \]

If we assume that \( f \) is nonzero, then we can eliminate \( a \) from Eqs. (8)-(7), and thereby obtain the following boundary conditions at \( x = 0 \):

\[ \partial_y (\psi + \bar{\psi}) = \frac{\partial \ln f}{\partial \phi} \partial_y \phi (\psi + \bar{\psi}) - 2if^2 (\psi - \bar{\psi}), \]
\[ \partial_x \phi = -\frac{\partial B}{\partial \phi} + 2 \frac{\partial \ln f}{\partial \phi} \bar{\psi} \psi. \]

\footnote{We have already performed, following I, the field rescalings \( \phi \to \phi/\beta, \psi \to \psi/\beta, \bar{\psi} \to \bar{\psi}/\beta, \) and we consider the classical limit \( \beta \to 0 \); moreover, we have set the mass \( m = 2 \) (cf. Eq. (I2.1)).}
3 Integrals of motion

We now proceed to determine the potentials $f(\phi)$ and $B(\phi)$ which appear in the boundary action by demanding both supersymmetry and integrability. We recall [9, 11] that the bulk SSG model has an infinite number of integrals of motion constructed from the densities $T_{s+1}, \Theta_{s-1}, \overline{\Theta}_{s-1}$, which obey

$$\partial_z T_{s+1} = \partial_{\overline{z}} \Theta_{s-1}, \quad \partial_{\overline{z}} T_{s+1} = \partial_z \overline{\Theta}_{s-1}.$$  \hspace{1cm} (10)

The $s = \frac{1}{2}$ and $s = 1$ densities are given by the supercurrent

$$T_{\frac{1}{2}} = \partial_z \phi \psi, \quad \overline{T}_{\frac{1}{2}} = \partial_{\overline{z}} \phi \overline{\psi},$$

$$\Theta_{-\frac{1}{2}} = -2\overline{\psi} \sin \frac{\phi}{2}, \quad \overline{\Theta}_{-\frac{1}{2}} = -2\psi \sin \frac{\phi}{2},$$  \hspace{1cm} (11)

and the energy-momentum tensor

$$T_2 = (\partial_z \phi)^2 - \partial_z \overline{\psi} \psi, \quad \overline{T}_2 = (\partial_{\overline{z}} \phi)^2 + \partial_{\overline{z}} \overline{\psi} \overline{\psi},$$

$$\Theta_0 = \overline{\Theta}_0 = -2 \cos \phi - \overline{\psi} \psi \cos \frac{\phi}{2},$$  \hspace{1cm} (12)

respectively. The $s = 3$ densities are given by [9, 11]

$$T_4 = (\partial_z \phi)^2 - \frac{1}{4} (\partial_z \phi)^4 + \frac{3}{4} (\partial_z \phi)^2 \partial_z \psi \overline{\psi} - \partial_z^2 \psi \partial_z \overline{\psi},$$

$$\overline{T}_4 = (\partial_{\overline{z}} \phi)^2 - \frac{1}{4} (\partial_{\overline{z}} \phi)^4 - \frac{3}{4} (\partial_{\overline{z}} \phi)^2 \partial_{\overline{z}} \overline{\psi} \overline{\psi} + \partial_{\overline{z}}^2 \overline{\psi} \partial_{\overline{z}} \overline{\psi},$$

$$\Theta_2 = (\partial_z \phi)^2 \cos \phi - \partial_z \overline{\psi} \psi \cos^2 \frac{\phi}{2} - \overline{\psi} \partial_z \psi \partial_z \cos \frac{\phi}{2} + \frac{1}{2} \overline{\psi} \psi (\partial_z \phi)^2 \cos \frac{\phi}{2},$$

$$\overline{\Theta}_2 = (\partial_{\overline{z}} \phi)^2 \cos \phi + \partial_{\overline{z}} \overline{\psi} \overline{\psi} \cos^2 \frac{\phi}{2} + \psi \partial_{\overline{z}} \overline{\psi} \partial_{\overline{z}} \cos \frac{\phi}{2} + \frac{1}{2} \overline{\psi} \psi (\partial_{\overline{z}} \phi)^2 \cos \frac{\phi}{2}. $$  \hspace{1cm} (13)

As observed by Ghoshal and Zamolodchikov [1], it follows from the continuity Eqs. [10] that the boundary model has the integral of motion

$$P_s = \int_{-\infty}^0 dx \left( T_{s+1} + \overline{T}_{s+1} + \Theta_{s-1} + \overline{\Theta}_{s-1} \right) - i \Sigma_s(y),$$  \hspace{1cm} (14)

provided that the following condition holds at $x = 0$

$$T_{s+1} - \overline{T}_{s+1} - \Theta_{s-1} + \overline{\Theta}_{s-1} = \partial_y \Sigma_s(y),$$  \hspace{1cm} (15)

where $\Sigma_s(y)$ is a (local) boundary term.

As we shall show below, the requirement that $P_3$ be an integral of motion implies that the potential $B(\phi)$ is the same as for the (Bosonic) boundary SG model [11]

$$B(\phi) = 2\alpha \cos \frac{1}{2} (\phi - \phi_0),$$  \hspace{1cm} (16)
where $\alpha$ and $\phi_0$ are arbitrary real parameters.

The requirement of on-shell supersymmetry is that $P_{\frac{1}{2}}$ also be an integral of motion. The constraint (15) for $s = \frac{1}{2}$ reads

$$
\left(\frac{1}{2} \partial_x \phi - 2 \sin \phi \right)(\psi - \bar{\psi}) - \frac{i}{2} \partial_y \phi (\psi + \bar{\psi}) = \partial_y \Sigma_{\frac{1}{2}}.
$$

We now substitute for $\partial_x \phi$ the boundary condition (9) with the potential $B(\phi)$ given by (16); and then we use (5) and (6) to substitute for $\psi + \bar{\psi}$ and $\psi - \bar{\psi}$, respectively. Moreover, we make the following Ansatz for the boundary term

$$
\Sigma_{\frac{1}{2}} = \frac{i}{2} g(\phi) a,
$$

where $a$ is the Fermionic boundary degree of freedom appearing in the boundary Lagrangian (3), and $g(\phi)$ is a function of $\phi$ which is yet to be determined. Eq. (17) implies that $g(\phi)$ must satisfy the pair of equations

$$
g = \frac{\alpha \sin \frac{1}{2} (\phi - \phi_0) - 4 \sin \phi}{f}, \quad \frac{\partial g}{\partial \phi} = -2 f.
$$

It is not difficult to solve these equations for $f(\phi)$ (and hence, also $g(\phi)$),

$$
f(\phi) = \left[ \alpha \sin \frac{1}{2} (\phi - \phi_0) - 4 \sin \phi \right] \left[ \frac{1}{8 \alpha \cos \frac{1}{2} (\phi - \phi_0) - 4 \cos \frac{\phi}{2} + C} \right]^\frac{1}{2},
$$

where $C$ is an integration constant.

We now turn to the requirement that $P_3$ be an integral of motion. The LHS of the constraint (15) for $s = 3$ is given by (I3.10)

$$
T_4 - \bar{T}_4 - \Theta_2 + \bar{\Theta}_2 =
-\frac{i}{2} (4 \phi_x^2 - \phi_y^2) (\bar{\psi}_y \bar{\psi} - \psi_y \bar{\psi}) + 3i \cos^2 \frac{\phi}{2} (\bar{\psi}_y \bar{\psi} - \psi_y \bar{\psi})
+ 3i \cos^2 \frac{\phi}{2} (\bar{\psi}_x \bar{\psi} - \psi_x \bar{\psi})
+ \frac{3}{8} \alpha \cos \frac{1}{2} (\phi - \phi_0) - 4 \cos \frac{\phi}{2} + C.
$$

Our task is to try to write the above expression as $\partial_y \Sigma_3$.

We begin by replacing $\phi_x$ by the boundary condition (9), and $\phi_{xy}$ by

$$
\phi_{xy} = -\frac{\partial^2 B}{\partial \phi^2} \phi_y + 2 \frac{\partial^2 \ln f}{\partial \phi^2} \phi_y \bar{\psi} \psi + 2 \frac{\partial \ln f}{\partial \phi} (\bar{\psi}_y \psi + \psi_y \bar{\psi}).
$$
Since the pure Bosonic terms in (21) (which evidently do not depend on the potential \( f \)) must by themselves form a total \( y \)-derivative, it is clear that \( \mathcal{B}(\phi) \) must be the same as for the pure Bosonic boundary SG model, as anticipated in Eq. (16) above. (See also I.)

The last two terms in (21) can be re-expressed as

\[
3\partial_y \cos \frac{\phi}{2}(\bar{\psi}_y \psi - \bar{\psi}_y y) + \partial_y \left[ \cos \frac{\phi}{2}(\bar{\psi}_y y - \bar{\psi}_y y) \right].
\]  

Let us now consider the Fermionic term in (21) with three derivatives, \( i(\bar{\psi}_{yy}\bar{\psi}_y - \psi_{yy}\psi_y) \). We can extract a total \( y \)-derivative from this term as follows: first, observe the identity

\[
\partial_y (\psi_y \bar{\psi}_y y - \bar{\psi}_y \psi_y y) = \psi_{yy} \bar{\psi}_y + \psi_y \bar{\psi}_{yy}.
\]  

Next, differentiate the boundary condition (8),

\[
\psi_{yy} + \bar{\psi}_{yy} = \left( \frac{\partial^2 \ln f}{\partial \phi^2} \phi_y^2 + \frac{\partial \ln f}{\partial \phi} \phi_{yy} \right) (\psi + \bar{\psi}) + \frac{\partial \ln f}{\partial \phi} \phi_y (\psi + \bar{\psi}) - 4i f^2 \frac{\partial \ln f}{\partial \phi} \phi_y (\psi - \bar{\psi}) - 2i f^2 (\psi_y - \bar{\psi}_y).
\]  

Solving (25) for \( \psi_{yy} \) and substituting into (24), and repeating for \( \bar{\psi}_{yy} \), we obtain the desired result

\[
i(\bar{\psi}_{yy}\bar{\psi}_y - \psi_{yy}\psi_y) = -i \partial_y (\psi_y \bar{\psi}_y) + i \left( \frac{\partial^2 \ln f}{\partial \phi^2} \phi_y^2 + \frac{\partial \ln f}{\partial \phi} \phi_{yy} \right) \left[ (\psi + \bar{\psi}) \bar{\psi}_y + \psi_y (\psi + \bar{\psi}) \right] + 4f^2 \frac{\partial \ln f}{\partial \phi} \phi_y \left[ (\psi - \bar{\psi}) \bar{\psi}_y + \psi_y (\psi - \bar{\psi}) \right] + 2i \frac{\partial \ln f}{\partial \phi} \phi_y \bar{\psi}_y \psi_y.
\]  

Our general strategy now is to use the boundary condition (8) to express \( \psi_y \bar{\psi}_y \) in terms of \( \bar{\psi}_y \psi \) (plus terms with \( \bar{\psi} \psi \)); and similarly to express \( \bar{\psi}_y \psi \) in terms of \( \psi_y \bar{\psi}_y \), and \( \psi_y \bar{\psi}_y \) in terms of \( \bar{\psi}_y \psi \) and \( \psi_y \bar{\psi}_y \). In this way, all the remaining Fermion bilinears have at most one \( y \)-derivative; and those bilinears with one derivative appear in only two combinations: \( (\bar{\psi}_y \bar{\psi} - \psi_y \psi) \) and \( (\bar{\psi}_y \bar{\psi} + \psi_y \psi) \). The former combination can be further reduced using the identity

\[
\bar{\psi}_y \bar{\psi} - \psi_y \psi = \partial_y (\bar{\psi} \psi) - 2 \frac{\partial \ln f}{\partial \phi} \phi_y \bar{\psi} \psi.
\]  

However, the latter combination cannot be so reduced, and thus, its coefficient must vanish. After some computation, we find that the total contribution of such terms in (21) is

\[
(\bar{\psi}_y \bar{\psi} + \psi_y \psi) \phi_y \left\{ -\frac{3}{8} \left[ \alpha \sin \frac{1}{2}(\phi - \phi_0) - 4 \sin \frac{\phi}{2} \right] + 12f^2 \frac{\partial \ln f}{\partial \phi} \right\}.
\]
Recalling the result (20) for $f(\phi)$, we find that the expression in (28) within braces does vanish, provided that the integration constant $C$ is given by

$$C = \sqrt{\alpha^2 - 8 \alpha \cos \frac{\phi_0}{2} + 16}. \quad (29)$$

The expression for $f(\phi)$ then takes the simplified form

$$f(\phi) = \frac{\sqrt{C}}{2} \sin \frac{1}{4}(\phi - D), \quad \text{where} \quad \tan \frac{D}{2} = \frac{\alpha \sin \frac{\phi_0}{2}}{\alpha \cos \frac{\phi_0}{2} - 4}. \quad (30)$$

After further computation, we find that (21) is given by

$$i \overline{\psi} \psi \left\{ \phi_y \left[ \frac{3}{2} \sin \phi - \frac{3}{4} \alpha \sin \frac{\phi_0}{2} - \frac{3}{32} \alpha^2 \sin(\phi - \phi_0) - 6f^2 \sin \frac{\phi}{2} ight. \\
+ \left( \frac{3}{8} \alpha^2 \sin^2 \frac{1}{2}(\phi - \phi_0) + 6 \cos \phi - 6 \cos^2 \frac{\phi}{2} - 24f^2 \right) \frac{\partial \ln f}{\partial \phi} \right] \\
+ 2\phi_y^3 \left[ \frac{\partial^3 \ln f}{\partial \phi^3} + 3 \frac{\partial \ln f}{\partial \phi} \frac{\partial^2 \ln f}{\partial \phi^2} + \frac{1}{16} \frac{\partial \ln f}{\partial \phi} + \left( \frac{\partial \ln f}{\partial \phi} \right)^3 \right] \\
+ 2\phi_y \phi_{yy} \left[ \frac{3}{16} + 3 \frac{\partial^2 \ln f}{\partial \phi^2} + 3 \left( \frac{\partial \ln f}{\partial \phi} \right)^2 \right] \right\}, \quad (31)$$

up to a total $y$-derivative. Remarkably, upon substituting the result (30) for $f(\phi)$, we find that the quantities in each of the square brackets vanish. We conclude that $P_3$ is indeed an integral of motion.

Finally, we remark that the boundary model also has the integral of motion

$$P_s' = \int_{-\infty}^{0} dx \ (T_{s+1} - T_{s+1} + \Theta_{s-1} - \Theta_{s-1}) - i \Sigma_s'(y), \quad (32)$$

provided that the constraint

$$T_{s+1} + T_{s+1} - \Theta_{s-1} - \Theta_{s-1} = \partial_y \Sigma_s(y) \quad (33)$$

is satisfied at $x = 0$. We expect that the boundary SSG model has (for $s = \frac{1}{2}$, at least) such integrals of motion if a different set of boundary conditions is imposed, corresponding to a boundary Lagrangian of the form

$$\mathcal{L}_b' = -\overline{\psi} \psi + ia \partial_y a - 2if(\phi)a(\psi + \overline{\psi}) + B(\phi). \quad (34)$$

However, we shall not pursue those calculations here.

\[3\text{There is a second (negative) root, which is ruled out by the requirement that } f \text{ be real.}\]
4 Discussion

We have shown that the boundary SSG model with boundary Lagrangian \((3)\), with \(\mathcal{B}(\phi)\) given by \((16)\) and \(f(\phi)\) given by \((30)\), has the integrals of motion \(P_s\) \((14)\) with \(s = \frac{1}{2}\) and \(s = 3\). The fact that \(P_{\frac{1}{2}}\) is an integral of motion means that the model has on-shell supersymmetry. The fact that \(P_3\) is an integral of motion is strong evidence that the model is integrable.

We emphasize that this boundary interaction, just like the one for the (Bosonic) boundary SSG model, has two continuous parameters, \(\alpha\) and \(\phi_0\). As mentioned in the Introduction, we were motivated to look for such boundary interactions by our earlier results \([19]\) on the boundary SYL model.

We conjecture that the boundary SSG model is integrable, and that the corresponding boundary \(S\) matrix \(S(\theta)\) for the \(n = 1\) breather supermultiplet is given by \([15]\)

\[
S(\theta) = S_{SG}(\theta ; \eta, \vartheta) \, S_{SUSY}^{(\varepsilon = -1)}(\theta ; \varphi),
\]

where \(S_{SG}(\theta ; \eta, \vartheta)\) is the \(n = 1\) breather boundary SG reflection factor \([20]\), and \(S_{SUSY}^{(\varepsilon)}(\theta ; \varphi)\) is a \(2 \times 2\) matrix given in \([13, 15]\). Indeed, \(S(\theta)\) satisfies the boundary Yang-Baxter equation, as well as unitarity and boundary cross-unitarity \([1]\). Moreover, \(S(\theta)\) commutes with a supersymmetry charge corresponding to the integral of motion \(P_{\frac{3}{2}}\) \([19]\). The proof is essentially the same as the one given in \([19]\) for the boundary SYL model. Note that the boundary term \(\Sigma_{\frac{1}{2}}\) \((18)\) corresponds (up to a proportionality factor) to \((-1)^F\), where \(F\) denotes Fermion number.

Although the \(S\) matrix \((33)\) involves three boundary parameters \(\eta, \vartheta, \varphi\), we expect that there is one relation among these parameters, so that only two parameters are independent, in concordance with the boundary action. Indeed, a parallel situation occurs for the boundary SYL model \([13]\), whose boundary \(S\) matrix nominally involves two boundary parameters, but only one of them is independent. For the boundary SSG model, the relation among the parameters can presumably be found in the same manner as for the boundary SYL model \([19]\): namely, by imposing the constraint \([1]\) that near a pole \(i\nu_0\) of the boundary \(S\) matrix associated with the excited boundary state \(|\alpha\rangle_B\) (which can be interpreted as a boundary bound state of particle \(A_a\) with the boundary ground state \(|0\rangle_B\)), the boundary \(S\) matrix must have the form

\[
\mathcal{S}_a^b(\theta) \simeq i \frac{g_{a_0}^\alpha g_{0_0}^{b_0}}{2 \, \theta - i\nu_0},
\]

where \(g_{a_0}^\alpha\) are boundary-particle couplings. It would be interesting to explicitly work out this relation for the boundary SSG model, and to find the precise relation of the two independent parameters of the boundary \(S\) matrix to the parameters \(\alpha, \phi_0\) of the boundary action. The latter problem has already been addressed for the case of the (Bosonic) boundary SG model \([27, 28]\), and we expect that similar techniques can be applied to the supersymmetric case. Moreover, analogous results should also hold for boundary \(S\) matrices for the SSG higher breathers and solitons.

\(^4\)The matrix \(S_{SUSY}^{(\varepsilon)}(\theta ; \varphi)\) with \(\varepsilon = -1\) commutes with a supersymmetry charge corresponding to \(P_s'\) \((22)\) with \(s = \frac{1}{2}\).
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Note added:

We have verified that the boundary SSG model with the boundary Lagrangian $L'_b$ (34) indeed has the integrals of motion $P'_1$ (32) and $P_3$ (14), with the same $B(\phi)$ (16), and $f(\phi) = \frac{\sqrt{C}}{2} \cos \frac{\phi}{4} (\phi - D)$ where $C$ and $D$ are given by (29) and (30) with $\alpha \to -\alpha$. In the conformal limit, the boundary Lagrangians (3) and (34) give Neveu-Schwarz and Ramond boundary conditions, respectively.

References

[1] S. Ghoshal and A.B. Zamolodchikov, Int. J. Mod. Phys. A9 (1994) 3841.
[2] I.V. Cherednik, Theor. Math. Phys. 61 (1984) 977.
[3] E.K. Sklyanin, J. Phys. A21 (1988) 2375.
[4] L. Mezincescu and R.I. Nepomechie, J. Phys. A25 (1992) 2533.
[5] H.J. de Vega and A. González-Ruiz, J. Phys. A26 (1993) L519.
[6] A. Fring and R. Köberle, Nucl. Phys. B421 (1994) 159.
[7] I.Ya. Aref’eva and V.E. Korepin, JETP Lett. 20 (1974) 312.
[8] A.B. Zamolodchikov and Al.B. Zamolodchikov, Ann. Phys. 120 (1979) 253.
[9] T. Inami, S. Odake and Y-Z Zhang, Phys. Lett. B359 (1995) 118.
[10] P. Di Vecchia and S. Ferrara, Nucl. Phys. B130 (1977) 93; J. Hruby, Nucl. Phys. B131 (1977) 275.
[11] S. Ferrara, L. Girardello and S. Sciuto, Phys. Lett. B76 (1978) 303; L. Girardello and S. Sciuto, Phys. Lett. B77 (1978) 267; R. Sasaki and I. Yamanaka, Prog. Theor. Phys. 79 (1988) 1167.
[12] R. Shankar and E. Witten, Phys. Rev. D17 (1978) 2134; C. Ahn, D. Bernard and A. LeClair, Nucl. Phys. B346 (1990) 409; C. Ahn, Nucl. Phys. B354 (1991) 57.
[13] C. Ahn and W.M. Koo, J. Phys. A29 (1996) 5845; Nucl. Phys. B482 (1996) 675.
[14] M. Moriconi and K. Schoutens, Nucl. Phys. B487 (1997) 756.
[15] C. Ahn and R.I. Nepomechie, Nucl. Phys. B586 (2000) 611.

[16] M. Ablikim and E. Corrigan, hep-th/0007214.

[17] M. Ameduri, R. Konik and A. LeClair, Phys. Lett. B354 (1995) 376.

[18] L. Mezincescu and R.I. Nepomechie, “Integrals of motion for the sine-Gordon model with boundary at the free Fermion point,” in Unified Symmetry in the Small and in the Large, eds. B.N. Kursunoglu, S. Mintz and A. Perlmutter (Plenum, 1995) p. 213; Int. J. Mod. Phys. A13 (1998) 2747.

[19] C. Ahn and R.I. Nepomechie, Nucl. Phys. B594 (2001) 660.

[20] A.B. Zamolodchikov, Advanced Studies in Pure Mathematics 19 (1989) 641; “Fractional-spin integrals of motion in perturbed conformal field theory,” in Fields, Strings and Quantum Gravity, eds. H. Guo, Z. Qiu and H. Tye, (Gordon and Breach, 1989).

[21] J.L. Cardy and G. Mussardo, Phys. Lett. 225 (1989) 275.

[22] F.A. Smirnov, Int. J. Mod. Phys. A4 (1989) 4213; Nucl. Phys. B337 (1990) 156.

[23] K. Schoutens, Nucl. Phys. B344 (1990) 665.

[24] C. Ahn, Nucl. Phys. B422 (1994) 449.

[25] P. Dorey, A. Pocklington, R. Tateo and G. Watts, Nucl. Phys. B525 (1998) 641.

[26] S. Ghoshal, Int. J. Mod. Phys. A9 (1994) 4801.

[27] Al.B. Zamolodchikov, unpublished work reported at the 1999 Bologna CFT workshop.

[28] A. Chenaghlou and E. Corrigan, Int. J. Mod. Phys. A15 (2000) 4417; E. Corrigan and A. Taormina, J. Phys. A33 (2000) 8739.