MEASURES OF FINITE PLURICOMPLEX ENERGY

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Abstract. In this note we study a complex Monge-Ampère type equation of the form

\[(dd^c u)^n = \frac{ke^{-u}dV}{\int e^{-u}dV} \]

February 17, 2022

1. Introduction

This paper is a revised version of [C]. Throughout this paper let \( \Omega \subseteq \mathbb{C}^n, n \geq 1 \), be a bounded, connected, open, and hyperconvex set. Let \( \mathcal{E}_0, \mathcal{F}_1, \) and \( \mathcal{F} \) be the energy classes introduced in [C1, C2] (see section 2 for details). In [C1], the author proved the following theorem:

**Theorem 3.1.** Let \( \mu \) be a non-negative Radon measure. Then the following conditions are equivalent:

1. there exists a function \( u \in \mathcal{E}_1 \) such that \((dd^c u)^n = \mu,\)
2. there exists a constant \( B > 0 \), such that

\[ \int_{\Omega} (-\varphi) \, d\mu \leq B \left( \int_{\Omega} (-\varphi)(dd^c \varphi)^n \right)^{\frac{1}{n+1}} \]

for all \( \varphi \in \mathcal{E}_0, \) where \( (dd^c \cdot)^n \) is the complex Monge-Ampère operator.

This theorem gives a complete characterization of measures for which there exist a solution of the Dirichlet problem for the complex Monge-Ampère operator in the class \( \mathcal{E}_1. \)

In section 3 we give a direct proof of Theorem 3.1 without use of the Rainwater lemma (see [R]). The solutions to the Dirichlet problem in Theorem 3.1 are always unique. We are not going to discuss that point in this article.

In section 4, we prove

**Theorem 4.6:** For every \( k < (2n)^n \) there is a function \( u \in \mathcal{E}_0 \cap C \) with

\[(dd^c u)^n = \frac{ke^{-u}dV}{\int e^{-u}dV},\]

where \( dV \) is the normalized Lebesgue measure on \( \Omega. \)

2000 Mathematics Subject Classification. Primary 32U20; Secondary 31C15.

Key words and phrases. Complex Monge-Ampère operator, energy classes, Dirichlet problem, plurisubharmonic function.
The proof make use of Theorem 3.1. In section 5, we give an alternative proof when \( k=1 \), where we use variational methods together with the following theorem:

**Theorem 4.4** To every \( b > 1/(2n)^n \) there exists a constant \( B > 0 \), such that

\[
\int_{\Omega} \exp(-u) \, dV \leq B \exp\left(b \int_{\Omega} (-u)(dd^c u)^n\right) \quad \text{for all } u \in \mathcal{E}_1.
\]

**Remark:** A stronger version of Theorem 4.4 is proved in [BB].

2. Preliminaries

By \( \mathcal{E}_0 \) we denote the family of all bounded plurisubharmonic functions \( \varphi \) defined on \( \Omega \) such that

\[
\lim_{z \to \xi} \varphi(z) = 0 \quad \text{for every } \xi \in \partial \Omega, \quad \text{and} \quad \int_{\Omega} (dd^c \varphi)^n < \infty,
\]

where \((dd^c \cdot)^n\) is the complex Monge-Ampère operator, normalized so that \( dd^c = \frac{i \pi}{2} \partial \bar{\partial} \). Assume now \( u \) that is a function such that there exists a decreasing sequence \( \{u_j\} \), \( u_j \in \mathcal{E}_0 \), that converges pointwise to \( u \) on \( \Omega \), as \( j \) tends to \( +\infty \). For \( p > 0 \), we say that

- \( u \in \mathcal{F}_p \), if
  \[
  \sup_{j \geq 1} \int_{\Omega} ((-u_j)^p + 1)(dd^c u_j)^n < \infty,
  \]

- \( u \in \mathcal{E}_p \), if
  \[
  \sup_{j \geq 1} \int_{\Omega} (-u_j)^p (dd^c u_j)^n < \infty,
  \]

- \( u \in \mathcal{F} \), if
  \[
  \sup_{j \geq 1} \int_{\Omega} (dd^c u_j)^n < \infty.
  \]

The complex Monge-Ampère operator is well-defined on these classes. See e.g., [C1, C2, CZ, K] for more information about the energy classes.

**Theorem 2.1.** Let \( p \geq 1 \), and \( n \geq 2 \). Then there exists a constant \( D(n, p) \geq 1 \), depending only on \( n \) and \( p \), such that for any \( u_0, u_1, \ldots, u_n \in \mathcal{E}_p \) it holds that

\[
\int_{\Omega} (-u_0)^p dd^c u_1 \wedge \cdots \wedge dd^c u_n
\leq D(n, p) \left( \int_{\Omega} (-u_0)^p (dd^c u_0)^n \right)^{1/(n+p)} \cdots \left( \int_{\Omega} (-u_n)^p (dd^c u_n)^n \right)^{1/(n+p)}.
\]

Furthermore, \( D(n, 1) = 1 \) and \( D(n, p) \geq 1 \) for \( p \neq 1 \).

**Proof.** This is Theorem 3.4 in [P] (see also [AC1, ACP, C1, CP2]). \( \square \)

It was proved in [AC1] (see also [AC2]) that for \( p \neq 1 \) the constant \( D(n, p) \) in Theorem 2.1 is strictly great than 1.

The following variant is proved in [C2].
**Theorem 2.2.** Let $n \geq 2$. For any $u_0, u_1, \ldots, u_n \in \mathcal{F}$ it holds that
\[
\int_{\Omega} (-u_0) dd^c u_1 \wedge \cdots \wedge dd^c u_n \leq \left( \int_{\Omega} (-u_0)(dd^c u_1)^n \right)^{1/n} \cdots \left( \int_{\Omega} (-u_0)(dd^c u_n)^n \right)^{1/n}
\]

3. **Dirichlet’s problem in $\mathcal{E}_1$**

**Theorem 3.1.** Let $\mu$ be a non-negative Radon measure. Then the following conditions are equivalent:

1. There exists a function $u \in \mathcal{E}_1$ such that $(dd^c u)^n = \mu$,
2. There exists a constant $B > 0$, such that
   \[
   \int_{\Omega} (-\varphi) d\mu \leq B \left( \int_{\Omega} (-\varphi)(dd^c \varphi)^n \right)^{\frac{1}{n+1}}
   \]
   for all $\varphi \in \mathcal{E}_0$.

This theorem gives a complete characterization of measures for which there exist a solution of the Dirichlet problem for the complex Monge-Ampère operator in the class $\mathcal{E}_1$. Originally, the theorem was proved by the author in [C1]. The approximation theorem below is the main result in this section. It gives a direct proof of the Dirichlet problem without use of the Rainwater lemma. The solutions to the Dirichlet problem at hand are always unique. We are not going to discuss this here.

We say that a non-negative Radon measure $\mu$ belongs to $\mathcal{M}_1$ if there exists constant $A$ such that
\[
\int_{\Omega} (-u) d\mu \leq A \left( \int_{\Omega} (-u)(dd^c u)^n \right)^{\frac{1}{n+1}},
\]
holds for all $u \in \mathcal{E}_0$.

The setup: It is no loss of generality to assume that $\mu$ has compact support. So let $\mu \in \mathcal{M}_1$ with compact support. Let $\varphi$ be a usual regularization kernel and put $\mu_j = \varphi_j * \mu$ which is a well-defined non-negative compactly supported smooth function. Solve, using [BT], $(dd^c u_j)^n = \mu_j$ for $u_j \in \mathcal{E}_0$. We show that this sequence converges to the solution of the Dirichlet problem.

**Theorem 3.2.** (Approximation theorem). With notations as above, $u_j$ converges as distributions and in $L^1(\mu)$ to a function $u \in \mathcal{F}_1$ and $(dd^c u)^n = \mu$.

**Proof.** We claim that
\[
u = \lim_{j \to +\infty} \left( \sup_{k \geq j} u_k \right)^* \in \mathcal{F}_1 \quad \text{and} \quad (dd^c u)^n = \mu.
\]

For choose a weak*- convergent subsequence, again denoted by $u_j$ converging weak* to $u$. Then by the construction of $u_j$ we have that
\[
\int_{\Omega} -u_j(dd^c u_j)^n \leq \int_{\Omega} -u_j \mu \leq A \left( \int_{\Omega} (-u_j)(dd^c u_j)^n \right)^{\frac{1}{n+1}}
\]
so it follows from integration by parts
\[
\int_{\Omega} - (\sup_{k \geq j} u_k)^*(dd^c(\sup_{k \geq j} u_k))^* \leq \int_{\Omega} -u_j(dd^c u_j)^n \leq A^{(n+1)/n}.
\]
Note that integration by parts also gives the inequality

$$\int v\mu \leq \int v(dd^c u)^n$$

for all negative plurisubharmonic functions $v$ (see [C3]). Theorem 2.1 gives

$$\int -u\mu = \lim \int -u(dd^c u_j)^n \leq \left(\int -u(dd^c u)^n\right)^{1/(n+1)} \lim \left(\int -u_j(dd^c u_j)^n\right)^{n/n+1}$$

and we will show that

$$\lim \int -u_j(dd^c u_j)^n \leq \int -u\mu$$

so it follows that $\int u\mu = \int u(dd^c u)^n$. Let $j \leq k$:

$$\int -u_j(dd^c u_j)^n \leq \int -u_j(dd^c u_k)^n \leq \left(\int -u_j(dd^c u_j)^n\right)^{1/(n+1)} \left(\int -u_k(dd^c u_k)^n\right)^{n/n+1}$$

so $\int -u_j(dd^c u_j)^n$ is monotonically increasing to some $c < +\infty$. Theorem 2.2 gives

$$\int -u_j(dd^c u_j)^n \leq \int -u_k dd^c u_j \wedge (dd^c u_k)^{n-1} \leq \left(\int -u_k(dd^c u_j)^n\right)^{1/n} \left(\int -u_k(dd^c u_k)^n\right)^{(n-1)/n}.$$}

Hence,

$$\int -u_j(dd^c u_j)^n \leq \left(\int -u_k(dd^c u_j)^n\right)^{1/n} c^{(n-1)/n}.$$

Now

$$\lim_{k \to \infty} \int -u_k(dd^c u_j)^n = \int \varphi_j * u\mu = \int -u(dd^c u_j)^n$$

since $\varphi_j * u_k$ tends uniformly to $\varphi_j * u, j \to \infty$ on the support of $\mu$ and we have that $\int u\mu = \int u(dd^c u)^n$ since $\lim \int -u(dd^c u_j)^n = \int -u\mu$ by construction. Let now $v \in \mathcal{E}_0$ be given. We have for $t>0$

$$\int -(u + tv)\mu = \lim \int -(u + tv)(dd^c u_j)^n \leq \left(\int -(u + tv)(dd^c u + tv)^n\right)^{1/(n+1)} \lim \left(\int -u_j(dd^c u_j)^n\right)^{n/n+1} =$$

$$\left(\int -(u + tv)(dd^c u + tv)^n\right)^{1/(n+1)} \left(\int -u(dd^c u)^n\right)^{n/n+1} \leq$$

$$\int -(u + tv)(dd^c u + tv)^n \left(\int -u(dd^c u)^n\right)^{n/n+1}.$$
determined so the original sequence was already weak*-convergent to $u$. This completes the proof.

**Corollary 3.3.** Let $\mu$ be a positive measure with $\mu(\Omega) < +\infty$ and $\mu(P) = 0$ for every pluripolar set. Then there is a uniquely determined function $u \in \mathcal{F}$ with $(dd^c u)^n = \mu$.

It follows from Rainwater’s lemma that $\mu = f(dd^c v)^n$ for a $v \in \mathcal{F}_1$ so the solutions $u_j$ to $(dd^c u_j)^n = \min(f, j)(dd^c v)^n$ decreases to $u, j \to +\infty$.

4. Compact and convex sets in $\mathcal{E}_1$

We consider $\mathcal{F}(\Omega)$ as a convex cone in $L^1(\Omega, dV)$. The Theorems 2.1 and 2.2 gives on

$$\mathcal{F} : \left( \left( (dd^c (u + v))^n \right)^{\frac{1}{n}} \leq \left( (dd^c u)^n \right)^{\frac{1}{n}} + \left( (dd^c v)^n \right)^{\frac{1}{n}} \right)$$

and on

$$\mathcal{E}_1 : \left( \left( -(u + v)(dd^c (u + v))^n \right)^{\frac{1}{n+1}} \leq \left( -(dd^c u)^n \right)^{\frac{1}{n+1}} + \left( -(dd^c v)^n \right)^{\frac{1}{n+1}} \right).$$

Therefore,

$$\left\{ u \in \mathcal{F} : (dd^c u)^n \leq C \right\} \quad \text{and} \quad \left\{ u \in \mathcal{E}_1 : -(dd^c u)^n \leq C \right\}$$

are convex in $\mathcal{F}$ and $\mathcal{E}_1$, resp. Both are also compact.

**Lemma 4.1.** Assume $u_j, u \in \mathcal{E}_1$ and $\sup \int -u_j(dd^c u_j)^n < +\infty$. If $u_j \to u, j \to +\infty$ as distributions, then $u_j \to u, j \to +\infty$ in $L^1((dd^c w)^n)$ for every $w \in \mathcal{E}_1$.

Note: It may happen that $\int -u_j(dd^c u_j)^n = 1$ but $u_j \to 0, j \to +\infty$ as distributions.

**Proof.** Let $m > 0$. Then

$$|u - u_j| \leq |u - \max(u, mw)| + \max(u, mw) - \max(u_j, mw) + \max(u_j, mw) - u_j$$

$$\leq \max(u, mw) - u + |\max(u, mw) - \max(u_j, w)| + \max(u_j, mw) - u_j$$

so

$$\int |u - u_j|(dd^c w)^n \leq \int (\max(u, mw) - u)(dd^c w)^n$$

$$+ \int |\max(u, mw) - \max(u_j, w)|(dd^c w)^n + \int \max(u_j, mw) - u_j)(dd^c w)^n.$$ 

At the right hand side, when $m \to +\infty$, the first integral tends to 0 by monotone convergence and that the second tends to 0 when $j \to 0$ follows from Lemma 1.4 in [CK]. We use Theorems 3.1 and 2.1 to estimate the third term:

$$\int (\max(u_j, mw) - u_j)(dd^c w)^n \leq \int_{\{u_j < mw\}} -u_j(dd^c w)^n$$

$$= \int -u_j \chi_{\{u_j < mw\}}(dd^c w)^n \leq \left( \left( -u_j(dd^c u_j)^n \right)^{\frac{1}{n+1}} \left( \left( -w \frac{u_j}{mw}(dd^c w)^n \right)^{\frac{1}{n+1}} \right) \right) \leq \frac{c}{m^{n+1}} \to 0, \quad \text{as} \quad m \to +\infty.$$
We need the following two theorems.

**Theorem 4.2.** [CP1] There exist a constant $C$ such that
\[
\sup |u - v| \leq C ||g - h||^\frac{1}{2}
\]
where $u, v \in F$ and $(dd^c u)^n = gdV, (dd^c v)^n = hdV$ and $f, g \in L^2(dV)$.

**Theorem 4.3.** [ACKPZ], Theorem B. There exist a uniform constant $a_n > 0$, depending only on $n$, such that for any positive number $0 \leq \mu < n$ and any $u \in F(\Omega)$ such that $\int_{\Omega} (dd^c u)^n \leq \mu^n$, we have that
\[
\int_{\Omega} e^{-2u}dV \leq \left( \pi^n + a_n \frac{\mu}{(n - \mu)^n} \right) \delta_{\Omega}^{2n},
\]
where $V$ is the $2n$-dimensional Lebesgue measure on $C^n$ and $\delta_{\Omega}$ is the diameter of $\Omega$.

We prove

**Theorem 4.4.** To every $b > 1/(2n)^n$ there exists a constant $B > 0$, such that
\[
\int_{\Omega} \exp(-u) dV \leq B \exp \left( b \int_{\Omega} (-u)(dd^c u)^n \right)
\]
for all $u \in E_1$,

*Proof.* Set
\[
a = \int_{\Omega} (-u)(dd^c u)^n.
\]
Then
\[
(dd^c u)^n = \chi_{\{u > -ab\}} (dd^c u)^n + \chi_{\{u \leq -ab\}} (dd^c u)^n
\]
\[
= \chi_{\{u > -ab\}} (dd^c \max(u, -ab))^n + \chi_{\{u \leq -ab\}} (dd^c u)^n
\]
\[
\leq (dd^c \max(u, -ab))^n + \chi_{\{u \leq -ab\}} (dd^c u)^n.
\]
Solve $(dd^c w)^n = \chi_{\{u \leq -ab\}} (dd^c u)^n$ for $w \in F$. By Theorem 4.5 in [CP]
\[
u \geq \max(u, -ab) + w
\]
and since
\[
\int_{\Omega} (dd^c w)^n \leq a/ab < (2n)^n
\]
it follows from Theorem 4.3 that
\[
\int_{\Omega} \exp(-u)dV \leq D \exp(ab)
\]
and the proof is complete. \hfill \square

**Theorem 4.5.** (Consequence of Schauder’s fixed point theorem) Suppose $A$ is a convex and compact subset of $E_1$. If $T : A \to A$ is a continuous map then there is $u \in A$ with $u = T(u)$.

**Theorem 4.6.** For every $k < (2n)^n$ there is a function $u \in E_0 \cap C$ with
\[
(dd^c u)^n = \frac{ke^{-u}dV}{\int e^{-u}dV},
\]
where $dV$ is the normalized Lebesgue measure on $\Omega$. 

\hfill \square
Proof. We wish to use the fixed point theorem and define \( B = \{ u \in \mathcal{F}, (dd^c u)^n \leq k \} \). Using Theorem 4.3 and Corollary 3.3 we can consider the map \( u : B \to T(u) \in B \) where \( T(u) \) is the unique function in \( \mathcal{F} \) with

\[
(dd^c T(u))^n = \frac{ke^{-u}dV}{\int e^{-u}dV}.
\]

Choose \( m \) such that \( \left( \frac{m}{m-1} \right)^n k < (2n)^n \). By Theorem 4.3, there is a constant \( c \) such that

\[
\int -T(u)(dd^c T(u))^n = k \int -T(u)e^{-u}dV / \int e^{-u}dV \leq k\left( \int (T(u))^{m} dV \right)^{\frac{1}{m}} \leq (m!)^{\frac{1}{m}} k c
\]

for all \( u \in B \).

Hence it follows that \( T(u) \in \mathcal{F} \) and Theorem 4.3 and Theorem 4.2 now gives that \( T(u) \in \mathcal{E}_0 \cap C \).

We restrict \( T \) to the convex and compact set

\[
A = \left\{ u \in \mathcal{F}, (dd^c u)^n \leq k, -u(dd^c u)^n \leq (m!)^{\frac{1}{m}} k c \right\}.
\]

Then \( A \to T(u) \in A \) and it remains to show that \( T \) is continuous on \( A \). It is then enough to prove that if \( u_j, u \in A \) and \( u_j \to u, j \to +\infty \) as distributions, then

\[
\frac{e^{-uj}dV}{\int e^{-u}dV} \to \frac{e^{-u}dV}{\int e^{-u}dV} \quad \text{as } j \to +\infty \text{ in } L^2(dV).
\]

Choose \( t > 1 \) so that \( kt^n < n^a \) and \( p, \frac{1}{p} + \frac{1}{p} = 1 \) and define \( w_j = \sup_{k \geq j} u_k^* \). Now,

\[
\int |e^{-uj} - e^{-u}|^2 dV \leq 2 \int |e^{-uj} - e^{-w_j}|^2 dV + 2 \int |e^{-u} - e^{-w_j}|^2 dV
\]

and

\[
\int |e^{-uj} - e^{-w_j}|^2 dV = \int e^{-2uj} |1 - e^{u_j - w_j}|^2 dV
\]

\[
\leq \left( \int e^{-2u_j} \left( \int |1 - e^{u_j - w_j}| dV \right)^{\frac{1}{p}} \right)^{p} \leq q \left( \int (u_j - w_j) dV \right)^{\frac{1}{p}} \to 0 \quad \text{as } j \to +\infty
\]

by Lemma 4.1. The constant \( q \) can be estimated using Theorem 4.4.

\[ \square \]

5. The variational method

Let \( u, v \in \mathcal{E}_1 \), and assume that \( v \) is continuous. For \( t < 0 \), put \( P(u + tv) = \sup \{ w \in \mathcal{E}_1 : w \leq u + tv \} \). Then \( P(u + tv) \in \mathcal{E}_1 \) (see [ACC]). Write

\[
e(u) = \int -u(dd^c u)^n
\]

and let \( J : A \to R \) be a continuous functional on \( \mathcal{E}_1 \). Put \( F(u) = \frac{1}{n+1} e(u) + J(u) \). If

\[
\liminf_{t \to 0^+} \frac{J(P(u + tv)) - J(u)}{t} \geq \liminf_{t \to 0^+} \frac{J((u + tv) - J(u))}{t}
\]

for all \( v \in \mathcal{E}_0 \cap C \) and if \( u_m \) is a minimum point of \( F \), then

\[
\int -v(dd^c u_m)^n + J'(u_m + tv) |_{t=0} = 0 \quad \text{for all } v \in \mathcal{E}_0 \cap C.
\]
Example 5.1. Take $J(u) = - \log \int e^{-u}dV$. Then $J$ is defined on $\mathcal{E}$ and Theorem 4.4 shows that we only have to minimize over a compact convex subset. A calculation shows that

$$\lim_{t \to 0^-} \frac{J(P(u + tv)) - J(u)}{t} \geq \lim_{t \to 0^+} \frac{J(u + tv) - J(u)}{t} = \int \frac{ve^{-u}dV}{\int e^{-u}dV}$$

for all $v \in \mathcal{E} \cap C$ so

$$-(dd^cu)^n + \frac{e^{-u}dV}{\int e^{-u}dV} = 0.$$  

For let $v \in \mathcal{E} \cap C$. Then, for $t < 0$ we have that

$$\frac{J(P(u + tv)) - J(u)}{t} = \log \left( 1 + \frac{\int e^{-P(u + tv) + u}dV}{\int e^{-u}dV} \right) \geq \log \left( 1 + \frac{\int e^{-u}dV}{\int e^{-u}dV} \right) - t \geq \log \left( 1 - \frac{\int e^{-u}tvdV}{\int e^{-u}dV} \right) \geq \frac{\int e^{-u}vdV}{\int e^{-u}dV} \quad \text{as } t \to 0.$$  

□

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