A SIMPLE PROOF OF THE TREE-WIDTH DUALITY THEOREM

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Abstract. We give a simple proof of the “tree-width duality theorem” of Seymour and Thomas that the tree-width of a finite graph is exactly one less than the largest order of its brambles.

1. Introduction

A tree-decomposition $\mathcal{T} = (T, l)$ of a graph $G = (V, E)$ is tree whose nodes are labelled in such a way that
i. $V = \bigcup_{t \in V(T)} l(t)$;
ii. every $e \in E$ is contained in at least one $l(t)$;
iii. for every vertex $v \in V$, the nodes of $T$ whose bags contain $v$ induce a connected subtree of $T$.

The label of a node is its bag. The width of $\mathcal{T}$ is $\max\{|l(t)| ; t \in V(T)\} - 1$, and the tree-width $\text{tw}(G)$ of $G$ is the least width of any of its tree-decomposition.

Two subsets $X$ and $Y$ of $V$ touch if they meet or if there exists an edge linking them. A set $B$ of mutually touching connected vertex sets in $G$ is a bramble. A cover of $B$ is a set of vertices which meets all its elements, and the order of $B$ is the least size of one of its covers.

In this note, we give a new proof of the following theorem of Seymour and Thomas which Reed [Ree97] calls the “tree-width duality theorem”.

Theorem 1 ([ST93]). Let $k \geq 0$ be an integer. A graph has tree-width $\geq k$ if and only if it contains a bramble of order $> k$.

Although our proof is quite short, our goal is not to give a shorter proof. The proof in [Die05] is already short enough. Instead, we claim that our proof is much simpler than previous ones. Indeed, the proofs in [ST93, Die05] rely on a reverse induction on the size of a bramble which is not very enlightening. A new conceptually much simpler proof appeared in [LMT10] but this proof is a much more general result on sets of partitions which through a translation process unifies all known duality theorem of this kind such as the branch-width/tangle or the path-width blockade Theorems. We turn this more general proof back into a specific proof for tree-width which we believe is interesting both as an introduction to the framework of [AMNT09, LMT10], and to a reader which does not want to dwell into this framework but still want to have a better understanding of the tree-width duality Theorem.

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2. The proof

So let $G = (V, E)$ be a graph and let $k$ be a fixed integer. A bag of a tree-decomposition of $G$ is small if it has size $\leq k$ and is big otherwise. A partial ($< k$)-decomposition is a tree-decomposition $T$ with no big internal bag and with at least one small bag. Obviously, if all its bags are small, then $T$ is a tree-decomposition of width $< k$. If not, it contains a big leaf bag and the neighbouring bag $l(u)$ of any such big leaf bag $l(t)$ is small. The nonempty set $l(t) - l(u)$ is a $k$-flap of $T$.

Now suppose that $X$ and $Y$ are respectively $k$-flaps of some partial ($< k$)-decompositions $(T_X, l_X)$ and $(T_Y, l_Y)$, and that $S = N(X) \subseteq N(Y)$. Then by identifying the leaves of the two decompositions which respectively contains $X$ and $Y$ and relabelling this node $S$, then we obtain a new “better” partial ($< k$)-decomposition.

This gluing process is quite powerful. Indeed let $S \subseteq V$ have size $\leq k$ and let $C_1, \ldots, C_p$ be the components of $G - S$. The star whose centre $u$ is labelled $l(u) = S$ and whose $p$ leaves $v_1, \ldots, v_p$ are labelled by $l(v_i) = C_i \cup N(C_i)$ is a partial ($< k$)-decomposition which we call the star decomposition from $S$. It can be shown that if $\text{tw}(G) < k$, then an optimal tree-decomposition can always be obtained by repeatedly applying this gluing process from star decompositions from sets of size $\leq k$. But this process is not powerful enough for our purpose. We need the following lemma.

**Lemma 1.** Let $X$ and $Y$ be respectively $k$-flaps of some partial ($< k$)-decompositions $(T_X, l_X)$ and $(T_Y, l_Y)$ of some graph $G = (V, E)$. If $X$ and $Y$ do not touch, then there exists a partial ($< k$)-decomposition $(T, l)$ whose $k$-flaps are subsets of $k$-flaps of $(T_X, l_X)$ and $(T_Y, l_Y)$ other than $X$ and $Y$.

**Proof.** Since, $X$ and $Y$ no not touch, there exists $S \subseteq V$ such that no component of $G - S$ meet both $X$ and $Y$ (for example $N(X)$). Choose such an $S$ with $|S|$ minimal. Note that $|S| \leq |N(X)| \leq k$. Let $A$ contain $S$ and all the components of $G - S$ which meet $X$, and let $B = (V - A) \cup S$.

**Claim 1.** There exists a partial ($< k$)-decomposition of $G[B]$ with $S$ as a leaf and whose $k$-flaps are subsets of the $k$-flaps of $(T_X, l_X)$ other than $X$.

Let $x$ be the leaf of $T_X$ whose bag contains $X$. Since $|S|$ is minimum, there exists $|S|$ vertex disjoint paths $P_s$ from $X$ to $S$ ($s \in S$). Note that $P_s$ only meets $B$ in $s$. For each $s \in S$, pick a node $t_s$ in $T_X$ with $s \in l_X(t_s)$, and let $l'_X(t) = (l_X(t) \cap B) \cup \{s \mid t \in \text{path from } x \text{ to } t_s\}$ for all $t \in T$. Then $(T_X, l'_X)$ is the tree-decomposition of $G[B]$. Indeed, since we removed only vertices not in $B$, every vertex and every edge of $G[B]$ is contained in some bag $l'_X(t)$. Moreover, for any $v \notin S$, $l'_X(t)$ contains $v$ if and only if $l_X(t)$ does. And $l'_X(t)$ contains $s \in S$ if $l_X(t)$ does or if $t$ is on the path from $x$ to $t_s$. In other cases, the vertices $t \in V(T_X)$ whose bag $l'_X(t)$ contain a given vertex induce a subtree of $T_X$.

Now the size of a bag $l'_X(t)$ is at most $|l_X(t)|$. Indeed, since $P_s$ is a connected subgraph of $G$, it induces a connected subtree of $T_X$, and this subtree contains the path from $x$ to $t_s$. So for every vertex $s \in l'_X(t) \setminus l_X(t)$, there exists at least one other vertex of $P_s$ which as been removed. The decomposition $(T_X, l'_X)$ is thus indeed a partial ($< k$)-decomposition of $G[B]$. It remains to prove that the $k$-flaps of $(T_X, l'_X)$ are contained in the $k$-flaps of $(T_X, l_X)$ other than $X$. But by construction, the only leaf whose bag received new vertices is $x$ and $l'_X(x) = S$ which is small. This finishes the proof of the claim.
Let \((T_Y,l'_Y)\) be obtained in the same way for \(G[A]\). By identifying the leaves \(x\) and \(y\) of \(T_X\) and \(T_Y\), we obtain a partial \((< k)\)-decomposition which satisfies the conditions of the lemma. \(\Box\)

We are now ready to prove the tree-width duality Theorem.

Proof. For the backward implication, let \(\mathcal{B}\) be a bramble of order \(> k\) in a graph \(G\). We show that every tree-decomposition \((T,l)\) of \(G\) has a part that covers \(\mathcal{B}\), and thus \(T\) has width \(\geq k\).

We start by orienting the edges \(t_1t_2\) of \(T\). Let \(T_i\) be the component of \(T \setminus t_1t_2\) which contains \(t_i\) and let \(V_i = \cup_{t \in V(T_i)} l(t)\). If \(X := l(t_1) \cap l(t_2)\) covers \(\mathcal{B}\), we are done. If not, then because they are connected, each \(B \in \mathcal{B}\) disjoint from \(X\) in \(G\) contained is some \(B \subseteq V_i\). This \(i\) is the same for all such \(B\), because they touch. We now orient the edge \(t_1t_2\) towards \(t_i\). If every edge of \(T\) is oriented in this way and \(t\) is the last vertex of a maximal directed path in \(T\), then \(l(t)\) covers \(\mathcal{B}\).

To prove the forward direction, we now assume that \(G\) has tree-width \(\geq k\), then any partial \((< k)\)-decomposition contains a \(k\)-flap. There thus exists a set \(\mathcal{B}\) of \(k\)-flaps such that

(i) \(\mathcal{B}\) contains a flap of every partial \((< k)\)-decomposition;
(ii) \(\mathcal{B}\) is upward closed, that is if \(C \in \mathcal{B}\) and \(D \supseteq C\) is a \(k\)-flap, then \(D \in \mathcal{B}\).

So far, the set of all \(k\)-flaps satisfies (i) and (ii).

(iii) Subject to (i) and (ii), \(\mathcal{B}\) is inclusion-wise minimal.

The set \(\mathcal{B}\) may not be a bramble because it may contain non-connected elements but we claim that the set \(\mathcal{B}'\) which contains the connected elements of \(\mathcal{B}\) is a bramble of order \(\geq k\). Obviously, its elements are connected. To see that its order is \(> k\), let \(S \subseteq V\) have size \(\leq k\). Then \(\mathcal{B}'\) contains a \(k\)-flap of the star-decomposition from \(S\), and \(S\) is thus not a covering of \(\mathcal{B}'\).

We now prove that the elements of \(\mathcal{B}\) pairwise touch, which finishes the proof that \(\mathcal{B}'\) is a bramble. Suppose not, then let \(X\) and \(Y\) \(\in \mathcal{B}\) witness this. Obviously, no subsets of \(X\) and \(Y\) can touch so let us suppose that they are inclusion-wise minimal in \(\mathcal{B}\). The set \(X\) being minimal, \(\mathcal{B} \setminus \{X\}\) is still upward closed and is a strict subset of \(\mathcal{B}\). There thus exists at least one partial \((< k)\)-decomposition \((T_X,l_X)\) whose only flap in \(\mathcal{B}\) is \(X\). Likewise, let \((T_Y,l_Y)\) have only \(Y\) as a flap in \(\mathcal{B}\). Let \((T,l)\) be the partial \((< k)\)-decomposition satisfying the conditions of Lemma[H] Since \(\mathcal{B}\) is upward closed and contains no \(k\)-flap of \((T_X,l_X)\) and \((T_Y,l_Y)\) other than \(X\) and \(Y\), it contains no \(k\)-flap of \((T,l)\), a contradiction. \(\Box\)

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