Boundary conditions for nonlocal one-sided pseudo-differential operators and the associated stochastic processes II

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Abstract
We connect boundary conditions for one-sided pseudo-differential operators with the generators of modified one-sided Lévy processes. On one hand this allows modellers to use appropriate boundary conditions with confidence when restricting the modelling domain. On the other hand it allows for numerical techniques based on differential equation solvers to obtain fast approximations of densities or other statistical properties of restricted one-sided Lévy processes encountered, for example, in finance. In particular we identify a new nonlocal mass conserving boundary condition by showing it corresponds to fast-forwarding, i.e. removing the time the process spends outside the domain. We treat all combinations of killing, reflecting and fast-forwarding boundary conditions.

In Part I we show wellposedness of the backward and forward Cauchy problems with a one-sided pseudo-differential operator with boundary conditions as generator. We do so by showing convergence of Feller semigroups based on grid point approximations of the modified Lévy process.

In Part II we show that the limiting Feller semigroup is indeed the semigroup associated with the modified Lévy process by showing continuity of the modifications with respect to the Skorokhod topology.

Keywords: nonlocal operator, nonlocal differential equation, spectrally positive Lévy process, Feller process
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1. Introduction

The last three decades have seen a surge in the application and theoretical development of fractional \([1, 2, 3, 4, 5, 6]\) and nonlocal \([7, 8, 9, 10, 11, 12]\) integro-differential equations. An important reason is the discovery of clear probabilistic explanations for solutions and boundary conditions involving processes with jumps (hence the nonlocality) \([4]\), which is the foundation of particle tracking/Monte Carlo numerical methods, e.g. \([13]\). In contrast to the complete understanding of boundary conditions for one-dimensional diffusion (continuous) processes \([14, 15]\), one-dimensional Lévy jump (discontinuous) processes are yet to be fully classified with respect to their pseudo-differential operators/boundary conditions for the associated backward and forward Kolmogorov equations. Indeed this is an active field of research \([7, 16, 17, 18, 19, 20, 21]\). Identifying boundary conditions is especially complex when one wants to impose a mass conserving boundary condition modeling jumps across the boundary of the domain, as a variety of natural modifications of the trajectories appear. Four examples are: censoring and its maximal extension \([17]\), stochastic reflection \([16]\) and fast-forwarding \([22]\). A particularly well-understood case is the one of the spectrally positive \(\alpha\)-stable Lévy processes \(Y\), for \(\alpha \in (1, 2)\). In fact, the work \([23]\), which originates in \([19]\), gives a detailed description of the backward and forward equations identified by restricting trajectories of \(Y\) to \([-1, 1]\) either by killing, stochastically reflecting or fast-forwarding. In particular, fast-forwarding \(Y\) by removing the time \(Y\) spends outside \([-1, 1]\) results in the nonlocal (or “reinsertion in the interior”) boundary condition

\[
\frac{\partial}{\partial x}^{-\alpha} f(-1) = \int_0^2 f'(y - 1) \frac{y^{1-\alpha}}{\Gamma(2-\alpha)} \, dy = 0
\]
for the backward equation, where $\partial_+^{\alpha-1}$ is a (right) Caputo derivative of order $\alpha - 1 \in (0, 1)$ on $(-\infty, 1]$. The methods of \cite{23} are based on a finite difference/Gr"{u}nwald approximation of the Feller generator of $Y$, which provides a rigorous and yet intuitive explanation of (1). Briefly, the nonlocal boundary condition (1) describes the following conservation of mass: when $Y$ leaves $(-1, \infty)$ by a drift, its mass is redistributed inside the domain (hence the nonlocality) according to the location of $Y$ at its first jump back inside $(-1, \infty)$. This is entirely different from stochastic reflection, where exiting particles are forced to stay on $\{-1\}$, which results in the standard Neumann boundary condition $f'(1) = 0$.

The main purpose of this work is to extend the methods and results for stable processes in \cite{23} to recurrent one-sided Lévy processes without the aid of scaling properties. This means characterising the backward and forward Cauchy problems of the restrictions to an interval of $Y$ via a finite difference approximation. For a detailed discussion of these Cauchy problems we refer to the introduction of Part I \cite{24}.

We now introduce our one-sided process on $\mathbb{R}$ (before we restrict it to an interval) and then discuss the results presented in this article. Spectrally positive Lévy processes possess a rich and well developed theory \cite{25, 26, 27} along with several applications, for example in finance, hydrology, and queues \cite{28, 29, 30, 31, 8, 32, 33, 34, 35}. Importantly, their fluctuation theory features several explicit and semi-explicit formulae not available for most Lévy processes. Such formulae are expressed in terms of the scale function $k_\psi$, defined by its Laplace transform $1/\psi$, for $\psi$ being the Laplace exponent of the process \cite[Chapter VII]{25}. We follow this tradition obtaining a full description in terms of scale functions of backward and forward generators of our Feller processes on an interval. We denote by $Y$ any recurrent spectrally positive Lévy process with paths of unbounded variation and no diffusion component. Our main contribution is Theorem 5.3, which we reworded below for convenience.

**Theorem 1.1.** Restrict $Y$ to a process $Y^{LR}$ on $[-1, 1]$ by imposing two boundary conditions at $\{-1, 1\}$ for any combination of killing, reflecting and fast-forwarding. Then $Y^{LR}$ is a Feller process with backward and forward generators given by the Caputo/Riemann–Liouville type operators with boundary conditions in Table 1.

This is obtained by combining the $J_1$-Skorokhod continuity theory developed in Section 3 with the strong convergence of the approximating (backward) semigroups in \cite[Theorem 5.1]{24}. Let us give here a worded description of the processes $Y^{LR}$.

1. $Y^{DD}$: $Y$ is killed as soon as it leaves $(-1, 1)$.

| Process $Y^{LR}$ | Forward generator $D_+^\alpha f(Y)_t$ | Backward generator $D_-^\alpha f(Y)_t$ |
|------------------|---------------------------------------|---------------------------------------|
| $1. Y^{DD}$      | $D_+^\alpha f(Y)_t$                  | $D_-^\alpha f(Y)_t$                  |
| $2. Y^{DN}$      | $D_+^\alpha f(N^+(Y))_t$             | $D_-^\alpha f(N^+(Y))_t$             |
| $3. Y^{ND}$      | $D_+^\alpha f(N^-(Y))_t$             | $D_-^\alpha f(N^-(Y))_t$             |
| $4. Y^{NN}$      | $D_+^\alpha f(N^+(Y))_t$             | $D_-^\alpha f(N^+(Y))_t$             |
| $5. Y^{NP}$      | $D_+^\alpha f(N^{+1}(Y))_t$          | $D_-^\alpha f(N^{+1}(Y))_t$          |
| $6. Y^{NN}$      | $D_+^\alpha f(N^{+2}(Y))_t$          | $D_-^\alpha f(N^{+2}(Y))_t$          |

Table 1: This is the same as \cite[Table 1]{24}. It lists the restrictions of the spectrally positive process $Y$ to $[-1, 1]$ and the associated forward and backward generators of strongly continuous contraction semigroups on $X = L^1([-1, 1])$ and $X = C_0(\Omega)$, respectively. The maps to construct $Y^{LR}$ are defined in Section 3. The generators $(G, LR)$ are defined in \cite[Definition 2.8]{24} and the explicit representation for the domains of the generators $(G, LR)$ can be found in \cite[Table 2]{24}. 

2. $Y^\text{DN}$: $Y$ is killed if it drifts across the left boundary. If it jumps across the right boundary we make a time change deleting the time for which $Y$ is to the right of the right boundary. (By [36, Lemma 2], $Y^\text{DN}$ equals in law $Y^{\text{DN}^*}$, reflecting $Y$ at the right boundary and then killing it at the left boundary.)

3. $Y^\text{ND}$: we make a time change deleting the time for which $Y$ is to the left of the left boundary. This process is then killed if it jumps across the right boundary.

4. $Y^\text{NN}$: we make a time change deleting the time for which $Y$ is outside the domain $(-1, 1)$. (This process equals in law $Y^{\text{NN}^*}$, i.e. reflecting $Y$ at the right boundary and then fast-forwarding the paths at the left boundary.)

5. $Y^\text{N*D}$: $Y$ is reflected at the left boundary and then killed if it jumps across the right boundary.

6. $Y^\text{N*N}$: $Y$ is reflected at the left boundary and then fast-forwarded at the right boundary. (This process equals in law the two sided reflection of $Y$ as defined in [16], which we prove in Corollary 5.2.)

From the descriptions above we see that we treat all possible combinations of LR for a recurrent $Y$, and in Table 2 we list possible alternative definitions for this processes.

We obtain as a corollary of Table 1 several new and known results concerning exit problems and resolvent measures for one-sided Lévy processes. The new results are the representation of the resolvent measures for $Y^\text{NN}$ and $Y^{\text{ND}}$, the identification of Lebesgue measure as an invariant measure for $Y^\text{NN}$, and the solution for the exit problem for $Y^{\text{ND}}$. Note that this exit problem describes a natural quantity, namely the time spent by a spectrally positive Lévy process in an interval $[a, b]$ before its first jump above $b$. As for the known results, we provide a new proof of the representation of the resolvent measures for $Y^{\text{DN}}$, $Y^{\text{N*D}}$ and $Y^{\text{N*N}}$, first proved in [37] and [38]. Unfortunately, in the cases $Y^{\text{N*D}}$ and $Y^{\text{N*N}}$ our work provides a new proof only by introducing the regularity assumption [24, (H1)] on the Laplace exponent to tame the singularities arising from the interpolated schemes in [24, Section 4] (as discussed in [24, Section 1.3]). (We also recall that we treat only recurrent $Y$ with paths of unbounded variation and no diffusion component.) However, our proof is based on strong convergence of finite difference schemes and continuity of maps on Skorokhod spaces, which appears very different from the several existing proofs. This is because none of the existing proofs employ compound Poisson approximations of $Y$. See for example [39] which uses martingale arguments, [40] using Itô excursion theory and [41] using potential theory, and we refer to [27, 42, 43, 44] for further discussions of these known results and proofs. Also worth highlighting that several key steps of our proof are not tied to the one-sidedness of our processes. Thus we believe it can be extended to the study of other processes, such as the symmetric stable process, which is part of our current investigation.

The main technical challenge in this article is proving that the paths of $Y$ are points of continuity of the fast-forwarding map in the $J_1$ topology. In Theorem 3.11 (Corollary 3.14) we give simple conditions that characterise the points of continuity of fast-forwarding maps on half-lines (intervals). These conditions cover a wide range of processes with jumps and therefore the presentation in Section 4.2.1 is given independently of its application in the rest of this article. To illustrate the difficulties behind this result, consider fast-forwarding below 0, defined for a path $f$ as the composition $f \circ A_f^{-1}$, where $A_f^{-1}$ is the right inverse of the additive functional
\[ A_I(t) = \int_0^t 1_{\{f(z) > 0\}} \, dz. \]
First of all, note that one must have that the limit path does not spend (Lebesgue) positive time at 0 (see for example Remark 3.9-(ii)). Once this is assumed, the natural strategy would be to give conditions such that \( f_n \to f \) implies \((f_n, A_{f_n}^{-1}) \to (f, A_f^{-1})\) and then use known results on \( J_1 \) continuity of composition of functions (see, e.g., [45, Theorem 13.2.2]). However, to the best of our knowledge, existing results do not cover one-sided processes, essentially because, almost surely, if \( t \) is a point of discontinuity of \( A_f^{-1} \), then \( f \) is discontinuous at \( A_f^{-1}(t) \), as we illustrate in Remark 3.9-(iii). Another issue is that proving this joint convergence to \((f, A_f^{-1})\) is a large part of our (constructive) proof, which would not be significantly shortened even if we were to assume that \( f \) is continuous (to apply, say, [45, Theorem 13.2.2]).

Another important results is Theorem 4.10, where we identify the transition rate matrix of fast-forwarding the discrete Grönwall type approximating process. This allows us to identify the fast-forwarding boundary condition (in the limit) as discussed in Part I [24, Section 1.3]. Also note that this work gives the details for the Skorokhod continuity part of [23]. Let us finally remark that the articles [22, 36, 46] applied fast-forwarding to a one-sided Lévy process, and [22] is the only one that applied fast-forwarding to the boundary where the (stable) process exits by a drift.

This work structured as follows: Section 2 introduces basic results on stochastic processes and Skorokhod spaces; Section 3 studies the continuity properties of our maps on the Skorokhod spaces; Section 4 applies these maps to \( Y \) and its approximation and connects the latter to the approximating processes constructed in [24, Section 3.3]; Section 5 applies the continuity results to these processes to derive our main results.

2. Preliminaries

We introduce basic notation and results stochastic processes in Section 2.1 and Skorokhod spaces 2.2. We will use the standard notation \( \mathbb{R}, \mathbb{N}, \mathbb{Z}, C^\infty_0(\mathbb{R}), L^1(E), a \wedge b, a \vee b \) and \( 1_A \), which we defined in [24]. Also, we denote by \( \| \cdot \|_X \) the norm of a Banach space \( X \), and by \( C^\infty_0(E) \) the Banach space of real-valued continuous functions on \( E \) vanishing at infinity with the supremum norm for \( E \) a locally compact metric space [9, Page 1]. For a complex valued function \( f \) with domain containing \( E \), we write \( \| f \|_{E, \infty} = \sup\{|f(x)| : x \in E\} \). We recall the convention that \( \Omega \) refers to the interval
\[ \Omega \in \{(-1,1), [-1,1), (-1,1], [-1,1]\} \]
with an endpoint excluded if the problem has a Dirichlet boundary condition (D) there. The symbol \( \lambda \) is reserved for the Lebesgue measure (contrarily to Part I [24]).

2.1. Stochastic processes

If \( E \) is a metric space we denote by \( \mathcal{B}(E) \) its Borel \( \sigma \)-algebra, and if \( \{X_n\}_{n \in \mathbb{N}}, X \) are random variables on \( E \), we say that \( \{X_n\}_{n \in \mathbb{N}} \) converges weakly to \( X \), denoted by \( X_n \rightharpoonup X \) as \( n \to \infty \), if \( \mathbb{E}[g(X_n)] \to \mathbb{E}[g(X)] \) for all real-valued bounded continuous functions on \( E \). We recall some basic facts about spaces of càdlàg (right continuous with left limits) functions that can be found, for example, in [47, Chapter VI.1]. Let \( D([0,\infty),\mathbb{R}) \) be the space of càdlàg functions \( f : [0,\infty) \to \mathbb{R} \) for \( E \) being a locally compact separable metric space. We always equip \( D([0,\infty), E) \) with the \( J_1 \) Skorokhod topology and the induced Borel \( \sigma \)-algebra \( D_{J_1} \), and recall that \( D([0,\infty), E) \) is a Polish space. For each \( t \geq 0 \) we define the projection map \( \pi_t : D([0,\infty), E) \to E \).
by \( \pi_t(f) = f(t) \) and recall that \( \pi_t \) is measurable. By a càdlàg process (with state-space \( E \)) we mean a probability measure \( \mathbb{P} \) on \( D([0, \infty), E) \). So we construct the random variables \((X_t)_{t \geq 0}\) such that \( \mathbb{P}[X_t \in B] = \mathbb{P} \circ \pi_t^{-1}(B), B \in \mathcal{B}(E), t \geq 0 \). Then by [48, Problem 1, Chapter 2] and standard relabelling, we can identify any càdlàg process with a progressive process \( X : D([0, \infty), E) \times [0, \infty) \to E \) with respect to the \( \mathbb{P} \)-completion of the natural filtration of \( X \), in the terminology of [48, Chapter 2].

Vice versa, we identify any progressive process with state-space \( E \) and càdlàg paths with a càdlàg process on \( D([0, \infty), E) \), by Kolmogorov’s Extension Theorem. We use the standard terminology defined in [48, Section 1, Chapter 2] for stopping times, their filtrations and related properties. For any \( B \in \mathcal{B}(E) \) we define the first exit time from \( B \in \mathcal{B}(E) \) for \( X \) by

\[
\tau^X_B = \inf \left\{ t > 0 : X_t \notin B \right\},
\]

with the convention that \( \inf \emptyset = \infty \). Note that by [48, Theorem 1.6, Chapter 2], assuming that \( X \) starts at \( x \in B \) and it is progressive with respect to a complete right continuous filtration, if \( B \) open or closed, then \( \tau^X_B \) is a stopping time with respect to this filtration. We remark that in order to simplify notation within certain proofs, the symbols \( \tau \) and \( \tau_n \) might be defined differently in different proofs. By convention, for a bounded measurable function \( g : D([0, \infty), E) \to E \) and denoting by \( \mu \) the distribution of \( X_0 \), we write \( \mathbb{E}_\mu[g(X)] = \mathbb{E}[g(X)] \) and \( \mathbb{P}_\mu[X \in B] = \mathbb{P}[X \in B] \) for \( B \in \mathcal{D}_I \), and substitute \( \mu \) with \( x \in E \) if \( \mu \) is the Dirac delta measure at \( x \).

We say that \( X \) is a (time-homogeneous) Markov or strong Markov process (with respect to its natural filtration) according to the standard definitions in [48, pages 156 and 158], respectively. We say that \( X \) is a Feller process if \( X \) is a strong Markov process and for each \( g \in C_0(E) \), \( x \mapsto \mathbb{E}_x[g(X)] \in C_0(E) \) for every \( t > 0 \) and \( \| \mathbb{E}_x[g(X_t)] - g \|_{C_0(E)} \to 0 \) as \( t \downarrow 0 \). We recall that the completion of the natural filtration of a Feller process is right continuous [48, Theorem 2.7, Chapter 2], and that Feller processes with state space \( E \) are in one-to-one correspondence with Feller semigroups on \( C_0(E) \) as defined in [9, Chapter 1]. For any càdlàg process \( X \) (or a càdlàg path) and \( B \in \mathcal{B}(E) \) we define the continuous non-decreasing functional

\[
t \mapsto A^X_B(t) = \int_0^t 1_{(X_s, \in B)} \, ds
\]

and its right continuous inverse taking values in \([0, \infty]\)

\[
(A^X_B)^{-1}(t) = \inf \left\{ s > 0 : A^X_B(s) > t \right\}, \quad t \in [0, \infty).
\]

We will often ease notation by writing \( A^X_B = A^X \) and \( (A^X_B)^{-1} = (A^X)^{-1} \) when the dependence on \( B \) is clear. For any Markov process \( X \) in this work \( A^X_B \) will be a continuous additive functional for \( X \) in the sense of [49, Definition IV.1.15]. We say that \( X \) is a Lévy process if \( E = \mathbb{R} \) and \( X \) is a càdlàg process with independent and stationary increments, and we recall that if \( X \) has paths of unbounded variation, then, for any \( x \in (a, b) \),

\[
\tau^X_{(-\infty, b)}(x) = \tau^X_{(-\infty, b)} \quad \text{and} \quad \tau^X_{(a, \infty)} = \tau^X_{(a, \infty)}, \quad \mathbb{P}_x\text{-a.s.}
\]

which is a consequence of the strong Markov property and regularity of \((-\infty, 0)\) and \((0, \infty)\) for \([26, Theorem 6.5]\).

A continuous time Markov process \( X = (X_t)_{t \geq 0} \) for a collection of initial distributions \( X_0 \in S_0 \subset E \) is a discrete-space continuous-time Markov process if for each \( X_0 \) the range \( S \) of \( X \) is countable and its transition rate matrix \( Q = R(P - I) \) has bounded
rates, i.e. sup_{i\in S} R_i < \infty$, where $P = (P_{i,j})_{i,j\in S} \geq 0$ is such that $\sum_{j} P_{i,j} = 1$, $R = (R_i)_{i\in S}$ is a diagonal matrix with finite non-negative entries, and $I$ is the identity matrix. Then, for a given initial distribution any $X$ is characterised by its transition rate matrix $Q$ [50, Chapter 4.2] and we deduce from [50, Eq. (4.2.1)] that for all $i \in S_0$ and $j \in S$

$$e^{-tR_i} = P_i[J_j > t] = P_i[X_s = i \text{ for all } s \in [0,t]];$$

where $J_j$ is the time of the first jump of $X$, and

$$P_{i,j} = \lim_{t\downarrow 0} P_i[t < J_1, X_{J_1} = j],$$

and note that the above implies that $P_{i,i} = 0$ for all $i$. Recall that by the ergodic result in [50, Eq. (4.4.11)], the $j$-th entry of the vector $\lim_{i\uparrow 0} \beta (\beta I - Q^T)^{-1} \bar{e}_i$ is given by

$$\lim_{\beta \downarrow 0} \int_0^\infty e^{-\beta t} \mathbb{P}_i[X_t = j] \, dt = \mathbb{P}_i[\sigma^X_f < \infty], \quad \text{if } i \neq j \text{ and } R_i = 0,$$

where $\sigma^X_f := \inf\{t \geq J_1 : X_t = j\}$, $Q^T$ is the transpose of $Q$ and $\bar{e}_i$ is the vector with $i$-th entry equal to 1 and 0 otherwise.

2.2. Skorokhod spaces and the Continuous Mapping Theorem

For $T \in (0,\infty)$, let $D([0,T],\mathbb{R})$ and $D([0,\infty],\mathbb{R})$ be the spaces real-valued càdlàg functions on $[0,T]$ and $[0,\infty)$, respectively, and we equip the first space with the Skorokhod metric

$$d_{J_1,T}(f,g) := \inf_{\gamma \in \Gamma} \{ \|\gamma - I\|_{C[0,T]} \vee \|f(\gamma) - g\|_{[0,T],\infty} \},$$

where $\Gamma$ is the space of strictly increasing continuous bijections of $[0,T]$ to itself and $I \in \Gamma$ is the identity map, and the second space with the Skorokhod metric

$$d_{J_1}(f,g) := \sum_{m=1}^{\infty} 2^{-m} (1 \wedge d_{J_1,m}(f_m,g_m))$$

with $f_m$ and $g_m$ respectively being the smoothed restrictions to $D([0,m],\mathbb{R})$ (as defined in [25, Eq. (16.3)]) of $f$ and $g$. These metrics induce the Skorokhod $J_1$ topology on the respective spaces [25, Sections 12 and 16]. We will use the following simple property

$$d_{J_1}(f,cf) \leq |1-c|, \quad \text{for any } c \in \mathbb{R}, \ f \in D([0,\infty),\mathbb{R}),$$

which is follows from $d_{J_1,m}(f_m,(cf)|m) \leq |1-c|$. We recall that $f_n \to f$ as $n \to \infty$ in $D([0,\infty),\mathbb{R})$ if and only if there exists a sequence $\{T_m\}_{m\in \mathbb{N}}$ of continuity points of $f$ such that $T_m \to \infty$ and for each $m \in \mathbb{N}$ the restrictions of $f_n$ and $f$ to $[0,T_m]$ are such that $f_n \to f$ in $D([0,T_m],\mathbb{R})$ as $n \to \infty$ [45, Section 12.9]. We denote one dimensional open balls by $B_r(x) = \{ y \in \mathbb{R} : |y-x| < r \}$ for any $x \in \mathbb{R}$ and $r > 0$. We recall that the Borel $\sigma$-algebra on $D([0,T],\mathbb{R})$ (respectively $D([0,\infty),\mathbb{R})$) equals the smallest $\sigma$-algebra containing $\{ \pi_t^{-1}(B_r(x)) : x \in \mathbb{R}, r > 0, t \in T \}$ for $T$ a dense set in $[0,T]$ containing $T$ (respectively a dense set in $[0,\infty]$) [51, Theorems 12.5.iii and 16.6.iii]. Then, for any closed set $C \subset \mathbb{R}$ and reals $a < b$ the sets

$$\{ f : f(t) \in C \text{ for all } t \in [a,b] \} \quad \text{and} \quad \{ f : f(t) \in C^c \text{ for some } t \in [a,b] \}$$
are measurable in $D([0, \infty), \mathbb{R})$. Also, by the argument in [52, page 232], for any $B \in \mathcal{B}(\mathbb{R})$ the map from $D([0, \infty), \mathbb{R})$ to $[0, \infty]$

$$f \mapsto \lambda(t \in [0, \infty) : f(t) \in B),$$

is Borel measurable, and for completeness we give the proof in Appendix A. To ensure that the fast-forwarding maps of Section 4.2.1 are well-defined we sometimes restrict our analysis to paths that spend infinite amount of time in the interval $C$ (see Remark 3.9-(i)). Namely, for any $a, b$ such that $-\infty \leq a < b \leq \infty$ we define the sub-space of $D([0, \infty), \mathbb{R})$ by

$$D_{a,b}([0, \infty), \mathbb{R}) = D([0, \infty), \mathbb{R}) \cap S_{a,b},$$

where

$$S_{a,b} := \{ f \in D([0, \infty), \mathbb{R}) : \lambda(t \in [0, \infty) : f(t) \in (a, b)) = \infty \}.$$

We recall two classical theorems and a standard proposition. Below, for a function $g$ between two metric spaces, we denote by $\text{Disc}(g)$ the set of points of discontinuity of $g$.

**Theorem 2.1.** ([Continuous Mapping Theorem [45, Theorem 3.4.3]]) If $X_n \Rightarrow X$ on $E$ and $g : E \to E'$ is a measurable map between two separable metric spaces satisfying $\mathbb{P}[X \in \text{Disc}(g)] = 0$, then $g(X_n) \Rightarrow g(X)$ on $E'$.

**Theorem 2.2.** ([Skorokhod Representation Theorem [45, Theorem 3.2.2]]) If $X_n \Rightarrow X$ on a separable metric space $E$, then there exist $\tilde{X}_n, \tilde{X}$ on a common probability space such that $\tilde{X}_n = X_n$ and $\tilde{X} = X$ in law, and $\mathbb{P}[\lim_{n \to \infty} \tilde{X}_n = \tilde{X}] = 1$.

**Proposition 2.3.** If $E$ is a metric space and $\{X_n\}_{n \in \mathbb{N}}$, $X$ are random variables on the same probability space on $E$ satisfying $\mathbb{P}[[\lim_{n \to \infty} X_n = X]] = 1$, then $X_n \Rightarrow X$ as $n \to \infty$. Moreover, if $E = D([0, \infty), \mathbb{R})$ and if $\mathbb{P}[\pi_t(X) = \pi_{t-}(X)] = 1$ for some $t > 0$, then $\pi_t$ is continuous at $X$ $\mathbb{P}$-a.s. and $\pi_t(X_n) \Rightarrow \pi_t(X)$ as $n \to \infty$ on $\mathbb{R}$.

**Proof.** The first statement is standard. The second statement follows by [51, Theorem 16.6.i] and Theorem 2.1.}

The next corollary is a simple weakening of the Continuous Mapping Theorem (CMT), which we need when dealing with the killing maps in Section 5.1.

**Corollary 2.4.** Suppose that $X_n \Rightarrow X$ as $n \to \infty$ on $E$, for each $n \in \mathbb{N} \cup \{0\}$ $g_n : E \to E'$ satisfy $\mathbb{P}[X \in \text{Disc}(f)] = 0$. If it also holds for each $n \in \mathbb{N}$ that

$$g_n(X_n) = f(X_n) \text{ a.s. and } g_0(X) = f(X) \text{ a.s.},$$

then $f(X_n) \Rightarrow f(X)$ as $n \to \infty$ on $E'$.

**Proof.** The proof is a simple adaptation of the standard argument in [45, Theorem 3.2.2]. Namely, construct the pushforward measures of $f(X_n)$ and $f(X)$ using (7), then use Theorem 2.2 to obtain versions of $X_n$ and $X$ such that $\tilde{X}_n \Rightarrow \tilde{X}$ a.s. in $E$, so that $g_n(\tilde{X}_n) \Rightarrow g_0(\tilde{X})$ a.s. in $E'$ by the continuity assumption on $f$. Then (7) and Proposition 2.3 conclude the proof.

3. Reflecting, killing and fast-forwarding maps

In this section we study the properties of several maps on (subspaces of) $D([0, \infty), \mathbb{R})$ which impose either reflecting, killing or fast-forwarding boundary conditions. The key result is the characterisation of the continuity points of fast-forwarding maps (one and two-sided). These maps are interesting in their own right and therefore we present them with no reference to Part I [24].
3.1. Reflecting \((N^*)\)

**Definition 3.1.** For any \(a, b \in \mathbb{R}\) with \(a < b\) we define the reflecting maps on \([a, \infty)\), \((-\infty, b]\) and \([a, b]\) from \(D([0, \infty), \mathbb{R})\) to itself respectively as

\[
N^*_a(f)(t) = f(t) + \inf_{s \leq t} \{(f(s) - a) \wedge 0\},
\]

\[
N^*_b(f)(t) = f(t) + \sup_{s \leq t} \{(f(s) - b) \vee 0\},
\]

\[
[N^*_aN^*_b](f)(t) = f(t) + \eta_a(t) - \eta_b(t),
\]

for each \(t \in [0, \infty)\), where the non-decreasing functions \(\eta_a, \eta_b \in D([0, \infty), \mathbb{R})\) are such that \([N^*_aN^*_b](f)\) takes values in \([a, b]\), \(\eta_a(0) = \eta_b(0)\) and they satisfy the minimal pushing conditions

\[
\int_0^\infty 1_{[\eta_a(t) < \eta_b(t) \leq b]} \, d\eta_b(t) = 0, \quad \int_0^\infty 1_{[\eta_a(t) \geq \eta_b(t) > a]} \, d\eta_b(t) = 0.
\]

If \(a = -1\) and \(b = 1\) we may simplify notation by writing \(N^*_{-1} = N^*_a\) and \(N^*_1 = N^*_b\).

**Remark 3.2.** Recall that \(N^*_{-1}, N^*_1\) and \([N^*_aN^*_b]\) are all Lipschitz continuous on \(D([0, \infty), \mathbb{R})\) (see, e.g., [53, Corollary 1.6]), and thus they are all Lipschitz continuous as maps from \(D_{c,d}([0, \infty), \mathbb{R})\) to \(D([0, \infty), \mathbb{R})\) for any \(c < d\).

3.2. Killing (D)

We model killing/Dirichlet boundary conditions by absorbing the path at the exit barrier the first time it touches or crosses the barrier (exit from an open set). These maps are not continuous in \(D([0, \infty), \mathbb{R})\), and we believe they are also not measurable. To overcome these issues with little technicality, we identify simple assumptions to show the paths of our spectrally positive process are continuity points of these maps. And to obtain measurability (in order to apply Corollary 2.4), we show that the first exit from open intervals equals the first exit from certain closed intervals for our processes, as the first exit from closed intervals is measurable, thanks to (5).

**Definition 3.3.** We define the killing maps on \((-1, \infty)\), \((-\infty, 1)\) and \((-1, 1)\) from \(D([0, \infty), \mathbb{R})\) to itself respectively as

\[
D'(f)(t) = \begin{cases} 
  f(t), & t < \tau^f_{(-1, \infty)}, \\
  -1, & t \geq \tau^f_{(-1, \infty)},
\end{cases}
\]

\[
D''(f)(t) = \begin{cases} 
  f(t), & t < \tau^f_{(-\infty, 1)}, \\
  1, & t \geq \tau^f_{(-\infty, 1)},
\end{cases}
\]

for each \(t \in [0, \infty)\), and \(D'(D'')\), recalling that \(\tau^f_{(-1, \infty)} = \inf\{s > 0 : f(s) \leq -1\}\) and \(\tau^f_{(-\infty, 1)} = \inf\{s > 0 : f(s) \geq 1\}\).

Note that we have the commutative property \(D'(D'') = D''(D')\).

**Proposition 3.4.** For all \(h \geq 0\), the maps \(\overline{D}_h, \underline{D}_h\) and \(\overline{D}_h(\underline{D}_h)\) are measurable, where

\[
\overline{D}_h(f)(t) = \begin{cases} 
  f(t), & t < \tau^f_{([0,\infty), h]}, \\
  -1, & t \geq \tau^f_{([0,\infty), h]},
\end{cases}
\]

\[
\underline{D}_h(f)(t) = \begin{cases} 
  f(t), & t < \tau^f_{([-\infty, 1), h]}, \\
  1, & t \geq \tau^f_{([-\infty, 1), h]},
\end{cases}
\]
Proof. We show that $D_h^+$ is measurable, then so are $D_h^-$ and $D_h^+(D_h^-)$. Recall it is enough to check that $D_h^-(A)$ is a Borel set in $D([0, \infty), \mathbb{R})$ for any $A = \pi_t^{-1}(B_r(x))$ with $x \in \mathbb{R}, r > 0$ and $t > 0$. Suppose first that $h > 0$. If $B_r(x) \cap (-\infty, h-1) = \emptyset$, then $D_h^-(A) = A \cap \{ f \geq h - 1 \text{ on } [0, t] \}$, which is measurable by (5). Otherwise we have $B_r(x) \cap (-\infty, h-1) \neq \emptyset$. Then, if $B_r(x) \subset (-\infty, -1)$, then $D_h^-(A) = \emptyset$. If $B_r(x) \subset (-1, \infty)$, then write $B_r(x) = B_1 \cup B_2$ where $B_1 \subset (-1, h-1)$ and $B_2 \subset [h-1, \infty)$, so that
\[ D_h^-(A) = D_h^-(\pi_t^{-1}(B_2)), \]
which is measurable by the same argument as in the first part of this proof. The only remaining possibility is that $-1 \in B_r(x)$. In this latter case rewrite $B_r(x) = B_1 \cup \{-1\} \cup B_2$ where $B_1 \subset (-\infty, -1)$ and $B_2 \subset (-1, \infty)$. Then
\[ D_h^-(A) = D_h^-(\pi_t^{-1}(\{-1\})) \cup D_h^+(\pi_t^{-1}(B_2)), \]
and the second set in the right hand side is measurable by the first part of the proof, and the first set is measurable because of (5) and
\[ D_h^-(\pi_t^{-1}(\{-1\})) = \{ f : f(s) < h - 1 \text{ for some } s \leq t \}. \]

We now treat the case $h = 0$ in an analogous fashion. Suppose that $B_r(x) \cap (-\infty, -1) = \emptyset$, then $D_0^-(A) = A \cap \{ f \geq -1 \text{ on } [0, t] \}$, which is measurable by (5). Otherwise $B_r(x) \cap (-\infty, -1) \neq \emptyset$. In this case, if $B_r(x) \subset (-\infty, -1)$, then $D_0^-(A) = \emptyset$. The remaining case is that $-1 \in B_r(x)$ and thus we can write $B_r(x) = B_1 \cup \{-1\} \cup B_2$ where $B_1 \subset (-\infty, -1)$ and $B_2 \subset (-1, \infty)$. Then
\[ D_0^-(A) = D_0^-(\pi_t^{-1}(\{-1\})) \cup D_0^-(\pi_t^{-1}(B_2)), \]
and the second set in the right hand side is measurable by the first part of the proof, and the first set is measurable because of (5) and
\[ D_0^-(\pi_t^{-1}(\{-1\})) = \{ f : f(s) < -1 \text{ for some } s \leq t \} \cup \pi_t^{-1}(\{-1\}). \]

The proof of the following corollary is straightforward and we omit it.

Corollary 3.5. For any $h > 0$ we have the identities: $D_h^+(f_h) = D_h^-(f_h)$ if $f_h$ takes values in $-1 + h \mathbb{Z}$; $D_h^+(f_h) = D_h^-(f_h)$ if $f_h$ takes values in $1 + h \mathbb{Z}$; $D_0^+(f) = D_0^-(f)$ if $\tau_{[-1, \infty)} = \tau_{(-\infty, 1)}$; and $D_0^+(f) = D_0^-(f)$ if $\tau_{(-\infty, 1]} = \tau_{(-\infty, 1)}$.

Proposition 3.6. Suppose $f \in D([0, \infty), \mathbb{R})$ is such that $f(0) < 1$ and for each $\delta > 0$
\[ \sup_{0 \leq t < \tau - \delta} f(t) < 1, \]
and if $\tau := \tau_{(-\infty, 1]} < \infty$, then $f(\tau) > 1$. Then, $f$ is a continuity point for $D'$.

Proof. Recall it is enough to check continuity of the restriction of $f$ to $D([0, T_\alpha], \mathbb{R})$ for a sequence of continuity points $T_\alpha \to \infty$ of $f$. Continuity is clear if $\tau = \infty$, as the first condition guarantees that $D'(f) = f$ and the same holds for any $\tilde{f}$ close enough to $f$. Otherwise $\tau < \infty$, and suppose first that $f(\tau) = 1$. For any continuity
point $T > \tau$ of $f$ we run the following argument. For an arbitrary $\epsilon > 0$, denote by $
abla^* = f(\tau) - 1 > 0$, let $\delta > 0$ such that $\sup_{t \in [\tau - \delta, \tau]} |f(t) - 1| \leq \epsilon/2$ and let $\epsilon_* > 0$ such that 

$$
\sup_{0 \leq t < \tau - \delta} f(t) \leq 1 - \epsilon_.*
$$

Let $\tilde{f}$ such that 

$$
\inf_{\gamma \in \Gamma} \max \left\{ \|I - \gamma\|_{C[0,T]}, \|\tilde{f}(\gamma) - f\|_{[0,T], \infty} \right\} < (\epsilon \wedge \epsilon^* \wedge \epsilon_*)/2,
$$

and denote by $\gamma$ any time change satisfying the above inequality and define $\tilde{\gamma}_* = \inf \{ t : f(\gamma(t)) \geq 1 \}$. Then, directly from the definitions, $D^*(\tilde{f}(\gamma)) = D^*(\tilde{f}(\gamma))$ and 

$$
\|D^*(\tilde{f}(\gamma)) - D^*(f)\|_{[0,T], \infty} = \|\tilde{f}(\gamma) - f\|_{[0,\tilde{\gamma}_*, \tau \wedge \tau]} + \|D^*(\tilde{f}(\gamma)) - D^*(f)\|_{[\tilde{\gamma}_*, \tau \wedge \tau], \infty} \leq \frac{\epsilon}{2} + \|D^*(\tilde{f}(\gamma)) - D^*(f)\|_{[\tilde{\gamma}_*, \tau \wedge \tau], \infty}.
$$

Suppose that $\tau < \tilde{\gamma}$, but because $f(\tau) = \epsilon^* + 1$ and $\|\tilde{f}(\gamma) - f\|_{[0,T], \infty} \leq \epsilon^*/2$, we obtain that 

$$
\tilde{\gamma}(\gamma) = \tilde{f}(\gamma(\tau)) - f(\tau) + f(\tau) \geq -\frac{\epsilon^*}{2} + \epsilon^* + 1 = \frac{\epsilon^*}{2} + 1,
$$

which contradicts the definition of $\tilde{\gamma}$. Suppose instead that $\tau - \delta \geq \tilde{\gamma}$. Then, by right continuity of $\tilde{f}(\gamma)$ we have $\tilde{f}(\gamma(\tilde{\gamma})) \geq 1$, but $f(\tilde{\gamma}) \leq 1 - \epsilon_*$, so that 

$$
\tilde{f}(\gamma(\tilde{\gamma})) - f(\tilde{\gamma}) \geq \tilde{f}(\gamma(\tilde{\gamma})) + \epsilon_* - 1 \geq \epsilon_* \geq \epsilon_* / 2,
$$

which contradicts $\|\tilde{f}(\gamma) - f\|_{[0,T], \infty} \leq \epsilon^*/2$. Therefore the only possibility is that $\tau - \delta < \tilde{\gamma} \leq \tau$, and so 

$$
\|D^*(\tilde{f}(\gamma)) - D^*(f)\|_{[\tilde{\gamma}_*, \tau \wedge \tau], \infty} = \|D^*(\tilde{f}(\gamma)) - D^*(f)\|_{[\tilde{\gamma}_*, \tau], \infty} = \|1 - f\|_{[\tilde{\gamma}_*, \tau], \infty} \leq \|1 - f\|_{[\tau - \delta, \tau], \infty} \leq \epsilon/2.
$$

We omit the simpler proof for the case $f(\tau-) < 1$, as a simple contradiction argument above proves that $\tau = \tilde{\gamma}$ for $\tilde{f}$ close enough to $f$.

**Proposition 3.7.** Suppose $f \in D([0, \infty), \mathbb{R})$ is such that $f(0) > -1$ and for each $s \in [0, \infty)$, the function $t \mapsto \inf \{ f(z) : z \in [s, t] \}$ is continuous for $t \leq s$. Then $f$ is a continuity point for $D^f$.

**Proof.** The proof is similar to the one of Proposition 3.6 and is omitted.

**3.3. Fast-forwarding (N)**

We define the left fast-forwarding map for an arbitrary barrier $a \in \mathbb{R}$, prove its measurability and characterise its points of continuity. The corresponding results for the right version and fast-forwarding outside an interval are then easily derived at the end of this section.

**Definition 3.8.** For any $a \in \mathbb{R}$ and all $f \in D_{a, \infty}([0, \infty), \mathbb{R})$ we define the left fast-forwarding map at $a$ by $N^l_a : D_{a, \infty}([0, \infty), \mathbb{R}) \to D([0, \infty), \mathbb{R})$ by $N^l_a(f) = f(A_f^{-1})$ where 

$$
A_f^{-1}(t) = \inf \{ s > 0 : A_f(s) > t \}, \quad A_f(t) = \int_0^t 1_{\{f(z) > a\}} \, dz.
$$
Remark 3.9. We discuss several issues with studying the continuity of $N^l_t$. Also recall that if $N^l_t$ is continuous and non-decreasing, we recall that $A^{-1}_f(A_f(z)) \geq z$ for all $z \in [0, \infty)$. We also recall that if $z \in [0, \infty)$ is such that

$$A_f(z) < A_f(z + \epsilon) \text{ for all } \epsilon > 0,$$

then $A^{-1}_f(A_f(z)) = z$, and we say that $z$ is a point of increase for $A_f$.

**Remark 3.9.** We discuss several issues with studying the continuity of $N^l_t$.

(i) The map $N^l_t$ cannot be extended to a map from $D([0, \infty), \mathbb{R})$ to itself by setting $N^l_t(f) = \delta$ for all $t \geq \zeta := \lim_{t \to \infty} A_f(t)$, where $\mathbb{R}$ is the one-point compactification of $\mathbb{R}$. This is because some paths $f \in D([0, \infty), \mathbb{R})$ spend only a finite amount of time $\zeta$ above $\alpha$, and there is no left limit at $\zeta$ for $N^l_t(f)$. An example is $f(t) = \sum_{n=1}^{\infty} (2 + (-1)^n)I_{[n\leq t<n+1]}$ with $\alpha = 0$, then the life-time of $N^l_t(f)$ is $\zeta = \sum_{n=1}^{\infty} n^{-2}$, but $\lim_{t \to \zeta} N^l_t(f)(t)$ does not converge (oscillation between 3 and 1). A natural space $\mathcal{D}$ such that $N^l_t : \mathcal{D} \to \mathcal{D}$ is $\mathcal{D} = \bigcup_{T \in (0, \infty)} D_\delta([0, T], \mathbb{R})$, where $D_\delta([0, T], \mathbb{R}) := \{f \in D([0, T], \mathbb{R}) \text{ and } f = \delta \text{ on } [T, \infty)\}$. But we do not know if we can extend the $J_1$ topology in a natural way for our CMT, thus we avoid this technicality by simply working with $D_{a, \infty}([0, \infty), \mathbb{R})$.

(ii) The map $N^l_t$ is not continuous. Indeed let $\alpha = 0$, $f(t) = 1_{\{t \geq 1\}}$ and $f_n(t) = \frac{1}{n}1_{\{t<1\}} + 1_{\{t\geq 1\}}$. Then $f_n \to f$ uniformly (and thus in $D([0, \infty), \mathbb{R})$), but $N^l_0(f) = 1$ and $N^l_0(f_n) = n$ for all $n$ and of course $f_n \not\to 1$ in $D([0, \infty), \mathbb{R})$. The issue here is that $\lambda(t \in [0, \infty) : f(t) = 0) = 1$, implying that at times when $f \leq 0 < f_n$ the fast-forwarding map produces a large delay (note that it is easy to give a similar example for $f$ continuous). By assuming (9) in Theorem 3.11, we are able to make this delay arbitrarily small for all $f_n$ close enough to $f$ (see Step (ii) of the proof). A small delay allows a small tilting of the Skorokhod time change we construct for $N^l_0(f_n)$. However requiring (9) is not enough for continuity. For example, take

$$f(t) = (t - 1)1_{\{t<1\}} + 1_{\{t\leq 1\}} \quad \text{and} \quad f_n(t) = (t - 1 + 1/n)1_{\{t<1\}} + 1_{\{t\leq 1\}}.$$

Clearly $\lambda(t \in [0, \infty) : f(t) = 0) = 0$ and $f_n \to f$ uniformly. But $N^l_0(f) = 1$ and $N^l_0(f_n) = 1_{\{t<1/n\}} + 1_{\{t\leq 1\}}$, and so the Skorokhod distance must be bounded below by $1 - 1/n$. We rule out this behaviour with the following assumption

**NI** if there exists an $\epsilon > 0$ and $t_1 < t_2 < t_3 < t_4$ such that

$$f(t) \geq a + \epsilon \text{ for all } t \in [t_1, t_2) \cup [t_3, t_4) \text{ and } f(t) \leq a \text{ for all } t \in [t_2, t_3),$$

then $\sup\{f(t) : t \in [t_2, t_3]\} < a$.

This condition is of course satisfied by continuous processes. It is also satisfied by one-sided processes, as these processes jump only in one direction.

(iii) We believe that Theorem 3.11 cannot be obtained using existing results on composition of functions. For example, the standard result [45, Theorem 13.2.2] does not apply to our case as it requires either the limit path $f$ or its time change $A^{-1}_f$ to be continuous. The most general composition theorem we know of is [54, Theorem 1.2], but it is easy to see that assumption [54, A2] fails almost surely even for our spectrally positive process $Y$, as for any discontinuity point $t$ of $A^{-1}_Y$, we have $N^l_0(Y)(t) = Y(A^{-1}_Y(t)) > a = Y(A^{-1}_Y(t-))$. (This example
also rules out the possibility of working with $M$ topologies and applying [45, Thorem 13.2.4.]. Moreover, even if we wanted to prove continuity of $N^t_0$ at $f \in C[0,\infty]$ satisfying (9), most of our proof would still be needed to prove the joint convergence to $(f, A^{-1}_f)$ required by the composition theorems.

**Proposition 3.10.** The map $N^t_0$ is measurable.

**Proof.** It can be found in Appendix A. □

**Theorem 3.11.** Assume that $f(0) > a$. Then, the map $N^t_0$ is continuous at $f$ if and only if $f$ satisfies (NI) and

$$\lambda(t \in [0, \infty) : f(t) = a) = 0.$$ (9)

**Proof.** (Conventions: to simplify notation in this proof we write $|| \cdot ||_{E, \infty} = || \cdot ||_E$ and we drop the brackets when composing additive functionals, their inverses and Skorokhod time changes, for example by writing $A^{-1}_f(\gamma(A_0))(t) = A^{-1}_f \gamma A_0(t)$ for $f, g \in D_0([0, \infty), \mathbb{R}), t \in [0, \infty)$ and $\gamma \in \Gamma$.)

We first prove the ‘if’ direction. It is enough to prove sequential continuity, and so we take a sequence $\{f_m\}_{m \in \mathbb{N}} \subset D_0([0, \infty), \mathbb{R})$ such that $f_m \to f$ as $m \to \infty$. The proof is divided into five steps. Namely: (i) choose how close $f_m$ has to be to $f$; (ii) prove a key estimate stating that the time delay due to $f_m \leq a < f$ or $f \leq a < f_m$ is small; (iii) construct the time change $\gamma$ for $N^t_0(f_m)$; (iv) show the time change is close to the identity; (v) show $N^t_0(f_m)(\gamma)$ is close to $N^t_0(f)$.

**[Step (i)]** For an arbitrary $\epsilon \in (0, 1]$ satisfying $\epsilon < f(0) - a$, let $\bar{\tau}_0 = 0$ and define recursively $(\tau_n, \bar{\tau}_n) \in (0, \infty)^2$ for $n \in \mathbb{N}$ as

$$\tau_n = \inf \{ t > \bar{\tau}_{n-1} : f(t) \leq a + \epsilon/2 \}, \quad \bar{\tau}_n = \inf \{ t > \tau_n : f(t) \geq a + \epsilon \}.$$  

Note that we must have $\tau_n < \bar{\tau}_n < \tau_{n+1}$ for each $n \in \mathbb{N}$ such that $\tau_{n+1} < \infty$. We first assume that $\tau_n < \infty$ for each $n \in \mathbb{N}$, which implies that $\tau_n \to \infty$ as $n \to \infty$ because $f$ is càdlàg (the proof in the remaining cases is similar and explained at the end of the proof). Fix $N \in \mathbb{N}$ large and let $T$ be a point of continuity of $f$ such that $\bar{\tau}_N < T < \tau_{N+1}$. Then $T$ is a point of increase of $A_f$, which implies that $A^{-1}_f A_f(T) = T$, and so $A_f(T)$ is a point of continuity of $N^t_0(f)$. Thus, recalling Section 2.2, because as $T$ goes to infinity $A_f(T) \to \infty$, it is enough to prove that $N^t_0(f_m) \to N^t_0(f)$ as $m \to \infty$ in $D([0, A_f(T)], \mathbb{R})$ for the restrictions of $N^t_0(f_m)$ and $N^t_0(f)$ to $[0, A_f(T)]$.

We now select four positive constants less than 1, denoted by $\delta_1, \delta_2, \delta_3$ and $\delta_4$, and we set $\delta(\epsilon, T) = \delta = \min\{\delta_1, \delta_2, \delta_3, \delta_4\}$.

Let $\delta_1 = \delta_{1,1} \wedge \delta_{1,2}$, where, for each of the $n_j \leq N$ such that $f > a$ at some point in $[\tau_{n_j}, \bar{\tau}_{n_j})$ we fix a $t_{n_j} \in [\tau_{n_j}, \bar{\tau}_{n_j})$ such that $f(t_{n_j}) > a$ and then we set $\delta_{1,1} = \min_{i}(f(t_{n_j}) - a)/2$, then using right continuity of $f$ we can select $\delta_4(\delta_{1,1}) = \delta_4 \in (0, 1]$ such that for all $n_j$

$$t_{n_j} + \delta_4 < \bar{\tau}_{n_j} \quad \text{and} \quad \inf_{t \in [\tau_{n_j}, t_{n_j} + \delta_4]} f(t) > f(t_{n_j}) - \delta_{1,1} \quad (10)$$

(and observe it implies $A_f(t_{n_j} + \delta_4) - A_f(t_{n_j}) = \delta_4$ for all $n_j$); meanwhile for all the $n_i$ such that $f \leq a$ for all points in $[\tau_{n_i}, \bar{\tau}_{n_i})$ we use (NI) to find $\delta_{1,2} \in (0, 1]$ such that

$$\max_{t \in [\tau_{n_i}, \bar{\tau}_{n_i})} f(t) < a - \delta_{1,2}. \quad (11)$$
Let $\delta_2 \in (0, 1]$ be such that $T + \delta_2$ is a continuity point of $f$ and
\[
\|f - f(T)\|_{[T-\delta_2, T+\delta_2]} \leq \epsilon \wedge \frac{f(T) - a}{3}.
\] (12)
By (9), we can now let $\delta_3(\epsilon, T, \delta_2, \delta_4) = \delta_3 \in (0, 1]$ such that
\[
\lambda(t \in [0, T] : |f(t) - a| \leq \delta_3) \leq \frac{\epsilon \delta_2 \delta_4}{6}.
\] (13)
By [51, Lemma 1 Section 12] there exists $N' \in \mathbb{N}$ and $0 = t_1 < t_2 < \ldots < t_{N'+1} = T$ such that
\[
\max_{n \in \{1, \ldots, N'\}} \sup_{s, t \in [t_n, t_{n+1})} |f(s) - f(t)| \leq \frac{\delta}{6}.
\] (14)
By our definitions $f(t) - a \geq \epsilon/2$ for all $t \in \cup_{n=0}^{N'} [\tau_n, \tau_{n+1})$. Let $M \in \mathbb{N}$ such that for all $m \geq M$, $f_m$ is
\[
\frac{\delta^2 \epsilon}{6N'} \text{ close to } f \text{ in } D([0, T], \mathbb{R}) \text{ and }
\epsilon \wedge \frac{f(T) - a}{2} \text{ close to } f \text{ in } D([T, T + \delta_2], \mathbb{R}).
\] (15)
Denote by $g$ any such $f_m$, and let $\gamma$ be a Skorokhod time change on $[0, T]$ such that
\[
\|\gamma - I\|_{C([0, T])} \wedge \|g(\gamma) - f\|_{[0, T]} \leq \frac{\delta^2 \epsilon}{6N'}.
\] (16)
Then by (14) and (16)
\[
\max_{n \in \{1, \ldots, N'\}} \sup_{s, t \in [t_n, t_{n+1})} |g(\gamma(s)) - g(\gamma(t))|
\leq \max_{n \in \{1, \ldots, N'\}} \sup_{s, t \in [t_n, t_{n+1})} |g(\gamma(s)) - f(s)| + |f(s) - f(t)| + |f(t) - g(\gamma(t))|
\leq \frac{\delta}{2}.
\] (17)
In Step (iii) we construct a Skorokhod time change $\tilde{\gamma}$ on $[0, A_f(T)]$ to show that $g(A_f^{-1})$ is (up to a constant depending only on $T$) $\epsilon$-close to $f(A_f^{-1})$ in $D([0, A_f(T)], \mathbb{R})$.

[Step (ii)] But first we prove the key bound on time delay
\[
\sup_{t \in [0, T]} |A_g(\gamma(t)) - A_f(t)| \leq \epsilon \delta_2 \delta_4.
\] (18)
To prove (18), observe that for any $t \in [0, T]$, using (16) in the last inequality,
\[
|A_g(\gamma(t)) - A_f(t)|
\leq \|\gamma - I\|_{[0, T]} + \lambda(z \in [0, T] : g(\gamma(z)) \leq a < g(z) \text{ or } g(z) < a < g(\gamma(z)))
+ \lambda(z \in [0, T] : g(\gamma(z)) \leq a < f(z) \text{ or } f(z) \leq a < g(\gamma(z)))
\[
\leq \frac{\epsilon \delta_2 \delta_4}{5} + \lambda(z \in [0, T] : g(\gamma(z))) \leq a < g(z) \text{ or } g(z) \leq a < g(\gamma(z))
\]
\[
+ \lambda(z \in [0, T] : |f(z) - a| \leq \delta),
\]
and from (13) we know the last term is bounded by \(\epsilon \delta_2 \delta_4 / 5\). We now bound the second term by \(\epsilon \delta_2 \delta_4 / 3 / 5\) using (17). Denote by
\[
S(z) \text{ the condition } g(z) \leq a < g(\gamma(z)) \text{ or } g(\gamma(z)) \leq a < g(z),
\]
and observe that whenever \(z, \gamma^{-1}(z) \in [t_n, t_{n+1})\), \(n = 1, 2, ..., N',\) condition \(S(z)\) implies that \(|g(\gamma(z)) - a| \leq \delta / 2\), by (17). Also recall that \(\gamma\) and \(\gamma^{-1}\) are increasing, so that \(z \leq \gamma(z)\) if and only if \(\gamma^{-1}(z) \leq z\). Then
\[
\lambda(z \in [0, T] : S(z))
\]
\[
= \sum_{n=1}^{N'} \lambda(z \in [t_n, t_{n+1}) : S(z))
\]
\[
= \sum_{n=1}^{N'} \left( \lambda(z \in [t_n, t_{n+1}) : S(z)) \right) 1_{\{t_n \geq \gamma(t_n), \gamma(t_{n+1}) \geq t_{n+1}\}}
\]
\[
+ \lambda(z \in [t_n, t_{n+1}) : S(z)) 1_{\{t_n < \gamma(t_n), \gamma(t_{n+1}) \geq t_{n+1}\}}
\]
\[
+ \lambda(z \in [t_n, t_{n+1}) : S(z)) 1_{\{t_n \geq \gamma(t_n), \gamma(t_{n+1}) < t_{n+1}\}}
\]
\[
+ \lambda(z \in [t_n, t_{n+1}) : S(z)) 1_{\{t_n < \gamma(t_n), \gamma(t_{n+1}) < t_{n+1}\}}
\]
\[
\leq \sum_{n=1}^{N'} \left( \lambda(z \in [t_n, t_{n+1}) : |g(\gamma(z)) - a| \leq \delta / 2) 1_{\{t_n \geq \gamma(t_n), \gamma(t_{n+1}) \geq t_{n+1}\}}
\]
\[
+ \lambda([t_n, \gamma(t_n)) \cup \{z \in [\gamma(t_n), t_{n+1}) : S(z))\} 1_{\{t_n < \gamma(t_n), \gamma(t_{n+1}) \geq t_{n+1}\}}
\]
\[
+ \lambda([\gamma(t_{n+1}), t_{n+1}] \cup \{z \in [t_n, \gamma(t_{n+1})) : S(z))\} 1_{\{t_n \geq \gamma(t_n), \gamma(t_{n+1}) < t_{n+1}\}}
\]
\[
+ \lambda(z \in [\gamma(t_n), \gamma(t_{n+1})) \cup [\gamma(t_{n+1}), t_{n+1}]) 1_{\{t_n < \gamma(t_n), \gamma(t_{n+1}) < t_{n+1}\}}
\]
\[
\leq \sum_{n=1}^{N'} \left( \lambda(z \in [t_n, t_{n+1}) : |g(\gamma(z)) - a| \leq \delta / 2) 1_{\{t_n \geq \gamma(t_n), \gamma(t_{n+1}) \geq t_{n+1}\}}
\]
\[
+ \lambda(z \in [\gamma(t_n), t_{n+1}) : |g(\gamma(z)) - a| \leq \delta / 2) 1_{\{t_n < \gamma(t_n), \gamma(t_{n+1}) \geq t_{n+1}\}}
\]
\[
+ \lambda(z \in [t_n, \gamma(t_{n+1})) : |g(\gamma(z)) - a| \leq \delta / 2) 1_{\{t_n \geq \gamma(t_n), \gamma(t_{n+1}) < t_{n+1}\}}
\]
\[
+ \lambda(z \in [\gamma(t_n), \gamma(t_{n+1})) : |g(\gamma(z)) - a| \leq \delta / 2) 1_{\{t_n < \gamma(t_n), \gamma(t_{n+1}) < t_{n+1}\}}
\]
\[
+ 2N'\|\gamma - I\|_{C[0, T]}
\]
\[
\leq \lambda(z \in [0, T] : |g(\gamma(z)) - a| \leq \delta / 2) + 2N'\|\gamma - I\|_{C[0, T]}
\]
\[
\leq \lambda(z \in [0, T] : |f(z) - a| \leq \delta) + 2\frac{\delta_2 \delta_4}{5}
\]
\[
\leq \frac{\epsilon \delta_2 \delta_4}{5} + 2\frac{\epsilon \delta_2 \delta_4}{5}.
\]
[Step (iii)] We partition \([0, A_f(T)]\) as
\[
\left( \bigcup_{n=1}^{N} [A_f(\bar{\tau}_{n-1}), A_f(\tau_n)] \cup [A_f(\tau_n), A_f(\bar{\tau}_n)] \right) \cup [A_f(\bar{\tau}_N), A_f(T)],
\]
and we recall that \([A_f(\bar{\tau}_{n-1}), A_f(\tau_n)] \neq \emptyset\) for all \(n \leq N\) as well as \([A_f(\bar{\tau}_N), A_f(T)]\), meanwhile \([A_f(\tau_n), A_f(\bar{\tau}_n)]\) could be empty or not. Note that by the definition of \(\bar{\tau}_{n-1}\) and \(\tau_n\), for \(t \in [A_f(\bar{\tau}_{n-1}), A_f(\tau_n)]\), \(\bar{\tau}_{n-1} + s\) is a point of increase for \(A_f\), where \(s = t - A_f(\bar{\tau}_{n-1})\), because \(0 \leq s \leq A_f(\tau_n) - A_f(\bar{\tau}_{n-1}) = \tau_n - \bar{\tau}_{n-1}\) and thus
\[
t = A_f(\bar{\tau}_{n-1}) + s = A_f(\bar{\tau}_{n-1} + s).
\]
Then we just proved that for any \(n = 1, \ldots, N\) and \(t \in [A_f(\bar{\tau}_{n-1}), A_f(\tau_n)]\)
\[
A_f^{-1}(t) = \bar{\tau}_{n-1} + s, \quad \text{where} \quad s = t - A_f(\bar{\tau}_{n-1}),
\]
which immediately implies
\[
f(A_f^{-1}(t)) = f(\bar{\tau}_{n-1} + s), \quad \text{for} \quad t \in [A_f(\bar{\tau}_{n-1}), A_f(\tau_n)].
\]
Also for each \(n \in \mathbb{N}\) such that \([A_f(\tau_n), A_f(\bar{\tau}_n)] \neq \emptyset\),
\[
a \leq f(A_f^{-1}(t)) \leq a + \epsilon, \quad t \in [A_f(\tau_n), A_f(\bar{\tau}_n)].
\]
We now construct a Skorokhod time change \(\bar{\gamma}\) on \([0, A_f(T)]\) so that \(N^\delta_n(g)(\bar{\gamma}) = g(A_f^{-1}\bar{\gamma}(t)) = g(\gamma(\bar{\tau}_{n-1} + s))\) for \(t \in [A_f(\bar{\tau}_{n-1}), A_f(\tau_n)]\), to synchronize \(N^\delta_n(g)\) with (20) and apply (16); meanwhile we linearly rescale the time change on the remaining intervals exploiting bounds for \(N^\delta_n(g)(\gamma)\) similar to (21) (due to \(g(\gamma) \leq a + 2\epsilon\) on \([\tau_n, \bar{\tau}_n]\)). This linear rescaling is small thanks to the key estimate (18). For the very last interval we need an extra rescaling to get \(\bar{\gamma}A_f(T) = A_f(T)\), and there we use the \(\delta_2\) “control” from (12) and (15).
For \(n = 1, 2, \ldots, N\) we define
\[
\bar{\gamma}(t) = \begin{cases} 
A_g\gamma A_f^{-1}(t), & t \in [A_f(\bar{\tau}_{n-1}), A_f(\tau_n)], \\
c_n(t - A_f(\tau_n)) + A_g\gamma(\tau_n), & t \in [A_f(\tau_n), A_f(\bar{\tau}_n)], \\
A_gA_f^{-1}(t), & t \in [A_f(\bar{\tau}_N), A_f(T - \delta_2)], \\
c_{N+1}(t - A_f(T - \delta_2)) + A_g\gamma(T - \delta_2), & t \in [A_f(T - \delta_2), A_f(T)],
\end{cases}
\]
where
\[
c_n = \frac{A_g\gamma(\bar{\tau}_n) - A_g\gamma(\tau_n)}{A_f(\bar{\tau}_n) - A_f(\tau_n)},
\]
when \([A_f(\tau_n), A_f(\bar{\tau}_n)] \neq \emptyset\), and
\[
c_{N+1} = \frac{A_f(T) - A_g\gamma(T - \delta_2)}{A_f(T) - A_f(T - \delta_2)}
\]
It is immediate to verify that \(\bar{\gamma}(0) = \gamma(0) = 0\) (recalling that \(A_f^{-1}(0) = 0\) as \(f(0) > a\)) and \(\bar{\gamma}(A_f(T)) = A_f(T)\). Moreover, \(\bar{\gamma}\) is increasing and continuous as:
- \(A_f^{-1}\) is increasing and continuous on \([A_f(\bar{\tau}_{n-1}), A_f(\tau_n)]\), \(n = 1, \ldots, N + 1\);
- \(\gamma\) is increasing and continuous;
• $A_f$ is increasing and continuous on $[\gamma(1), \gamma(N+1)]$ by (16) and the definition of $\tau_{n-1}$ and $\tau_n$, $n = 1, \ldots, N+1$;

• if $[A_f(\tau_n), A_f(\tau_{n-1})] \neq \emptyset$ we are just linearly interpolating with a positive slope because by (10)

$$A_f(\tau_n) - A_f(\tau_{n-1}) > \delta_4 \quad \text{for all} \quad n = 1, \ldots, N,$$

and by the $\delta$-bound in (16) and (10)

$$A_g(\tau_n) - A_g(\tau_{n-1}) > \delta_4 \quad \text{for all} \quad n = 1, \ldots, N,$$

and similar observations yield $c_{N+1} > 0$;

• if $[A_f(\tau_n), A_f(\tau_{n-1})] = \emptyset$ we only need to check that $A_gA_f^{-1}A_f(\tau_n) = A_g(\tau_n)$ equals

$$\lim_{t \to A_f(\tau_n)} A_gA_f^{-1}(t) = \lim_{t \to \tau_n} A_gA_f^{-1}(t) = A_g(\tau_n),$$

where in the identities above we used that $\tau_n$ and every point in $[\tau_{n-1}, \tau_n)$ are points of increase for $A_f$, and in the last equality we used continuity of both $A_g$ and $\gamma$. To check this, observe that (11) holds, and so, by the $\delta$-bound in (16), $g(\gamma(t)) \leq a$ for all $t \in [\tau_n, \tau_{n+1})$ and thus $A_g(\tau_n) = A_g(\tau_{n+1})$.

So we proved that $\hat{\gamma}$ is a Skorokhod time change on $[0, A_f(T)]$.

[Step (iv)] We now verify that $\|\hat{\gamma} - I\|_{C[0, A_f(T)]} \leq C\epsilon$, where $C > 0$ depends only on $T$. First note that by (18) and (22)

$$|c_n - 1| \leq \frac{2\delta_4\epsilon}{\delta_4} = 2\epsilon,$$

$$|c_{N+1} - 1| \leq \frac{|A_g(\tau_n - \delta_2) - A_f(T - \delta_2)|}{\delta_2} \leq \epsilon. \quad (23)$$

If $t \in [A_f(\tau_n), A_f(\tau_{n-1})] \neq \emptyset$, $n = 1, \ldots, N$, then, using (23) and (18),

$$|\gamma(t) - t| = |c_n (t - A_f(\tau_n)) + A_g(\tau_{n-1}) - t| \leq (t - A_f(\tau_n))|c_n - 1| + |A_g(\tau_{n-1}) - A_f(\tau_n)| \leq A_f(T)2\epsilon + \epsilon,$$

and similarly for $t \in [A_f(T - \delta_2), A_f(T)]$. Meanwhile, if $t \in [A_f(\tau_{n-1}), A_f(\tau_n)]$, $n = 1, \ldots, N$, then, using (19) and (18),

$$|\gamma(t) - t| = |A_g(\tau_{n-1} + s) - A_f(\tau_{n-1} + s)| \leq \epsilon,$$

and similarly for $t \in [A_f(\tau_N), A_f(T - \delta_2)]$.

[Step (v)] It remains to prove that $\|g(A_g^{-1}\bar{\gamma}) - f(A_f^{-1})\|_{C[0, A_f(T)]} \leq C\epsilon$ for $C > 0$ only dependent on $T$. For $t \in [A_f(\tau_n), A_f(\bar{\tau}_n)] \neq \emptyset$, $n = 1, \ldots, N$, the increasing function $A_g^{-1}\bar{\gamma}(t) = A_g^{-1}(c_n (t - A_f(\tau_n)) + A_g(\tau_{n-1}))$ is such that

$$\gamma(\tau_n) \leq A_g^{-1}A_g(\tau_n) \leq A_g^{-1}\bar{\gamma}(t) < A_g^{-1}A_g(\bar{\tau}_n) = \gamma(\bar{\tau}_n),$$

so that by (16) and the definition of $\tau_n$ and $\bar{\tau}_n$

$$a \leq g(A_g^{-1}\bar{\gamma}(t)) \leq \sup_{z \in [\gamma(\tau_n), \gamma(\bar{\tau}_n)]} g(z) \leq a + 2\epsilon.$$
Thus, for \( n = 1, \ldots, N \), using (21),
\[
\|g(A^{-1}g) - f(A^{-1}f)\|_{|A_f(\tau_n), A_f(\tau_n)|} \\
\leq \|g(A^{-1}g) - a\|_{|A_f(\tau_n), A_f(\tau_n)|} + \|a - f(A^{-1}f)\|_{|A_f(\tau_n), A_f(\tau_n)|} \leq 3\varepsilon.
\]

For \( t \in [A_f(\tau_n), A_f(\tau_n)] \), \( n = 1, \ldots, N \), using (19),
\[
g(A^{-1}g(t)) = g(A^{-1}A_gA^{-1}A_f(\bar{\tau}_n - s)) = g(A^{-1}A_g\gamma(\bar{\tau}_n - s)),
\]
and by the \( \varepsilon/4 \)-bound from (16) we have that \( \gamma(\bar{\tau}_n - s) \in [\gamma(\bar{\tau}_n - 1), \gamma(\tau_n)) \) is a point of increase for \( A_g \), where we used
\[
\gamma(\bar{\tau}_n - 1 + A_f(\tau_n) - A_f(\bar{\tau}_n)) = \gamma(\bar{\tau}_n + \tau_n - \bar{\tau}_n - 1).
\]

So we proved that \( g(A^{-1}g(t)) = g(\gamma(\bar{\tau}_n - 1 + s)) \), and by (16) and (20)
\[
\|g(A^{-1}g) - f(A^{-1}f)\|_{|A_f(\tau_n), A_f(\tau_n)|} \leq \varepsilon.
\]

The same argument holds for \( [A_f(\bar{\tau}_N), A_f(T - \delta_2)] \). Finally, for \( t \in [A_f(T - \delta_2), A_f(T)] \),
\[
|g(A^{-1}g(t)) - f(A^{-1}f(t))| \leq |g(A^{-1}g(t)) - f(T)| + |f(T) - f(T - \delta_2 + s)|,
\]
for \( s = t - A_f(T - \delta_2) \) so that the second term is bounded by \( \varepsilon \) by (12), and thus it only remains to bound the first term uniformly for \( t \in [A_f(T - \delta_2), A_f(T)] \). Observe that
\[
A^{-1}A_g(T - \delta_2) = A^{-1}A_g\gamma(T - \delta_2) = \gamma(T - \delta_2),
\]
and
\[
A_g(T) = A^{-1}A_f(T) = A^{-1}(A_g(T + \Delta)) = A^{-1}A_g(T + \Delta) = T + \Delta,
\]
where \( |\Delta| = |A_f(T) - A_g(T)| \leq \delta_2 \) by (18) and we used that all points in \( [\gamma(T - \delta), T] \) are points of increase for \( A_g \), by the \( \varepsilon/4 \)-bound in (16), as well as all the points \( t \in [T, T + \delta_2] \), by
\[
g(t) - a = g(\tau(t)) - f(t) + (f(T) - a) = (f(t) - f(T)) \geq (f(T) - a)/3 > 0,
\]
using (12) and (15), with \( \tau \) being any Skorokhod time change satisfying (15). So it remains to show that \( g \) is close to \( f(T) \) on \( [\gamma(T - \delta_2), T + \delta_2] \). Using (12) and (15)
\[
\|g - f(T)\|_{[T, T + \delta_2]} = \|g(\tau(t)) - f(T)\|_{[T, T + \delta_2]} \\
\leq \|g(\tau(t)) - f(T)\|_{[T, T + \delta_2]} + \|f(T)\|_{[T, T + \delta_2]} \leq 2\varepsilon.
\]

and using (16) and (15)
\[
\|g - f(T)\|_{[\gamma(T - \delta_2), T]} = \|g(\gamma) - f(T)\|_{[\gamma(T - \delta_2), T]} \\
\leq \|g(\gamma) - f(T)\|_{[\gamma(T - \delta_2), T]} + \|f(T)\|_{[T - \delta_2, T]} \leq 2\varepsilon.
\]

To complete the proof of the ‘if’ direction, it remains to consider the case where \( \tau_n = \infty \) for some \( n \in \mathbb{N} \). Let \( m \) denote the smallest integer such that \( \tau_n = \infty \). If \( \tau_{n-1} < \tau_m = \infty \) the same proof above applies easily as the limit process path \( f \) is eventually bounded below by \( \varepsilon/2 > 0 \). Otherwise, \( \tau_{m-2} < \tau_{m-1} = \infty \), and then the same proof can be applied because for all \( M \in \mathbb{N} \) there exists a continuity point \( T > M \) of \( f \) such that \( f(T) > a \), as \( f \) spends an infinite amount of time above \( a \).

Then the last interval \( [\tau_{m-2}, T] \) is easy to treat as \( \|N^\gamma_a(f - a)\|_{[A_f(\tau_{m-2}), A_f(T)]} \leq \varepsilon \).
And so we proved the ‘if’ direction.

The ‘only if’ direction is simple and we only sketch the proof. If (Nl) fails, then let $f_n$ equal to $f$ on $\mathbb{R} \setminus [t_2, t_3)$ and equal to $f + 1/n$ on $[t_2, t_3)$, then $f_n \to f$ uniformly but for any $T$ large and any Skorokhod time change $\gamma$, either $\|N_n(f_n) - N_n(f)\|_{[0, T]} \geq \epsilon/2$ for all $n$ large or $\|\gamma - I\|_{C([0, T])} \geq (t_2 - t_1) \wedge (t_4 - t_3) > 0$. Alternatively, if $\lambda(t) : f(t) = a$, we can find $t_1 < t_2 < t_3 < t_4 \in [0, \infty)$ such that $\lambda(t) \in [t_1, t_3) : f(t) = a$, $t_2 - t_1$, $f \leq a$ on (the possibly empty interval) $[t_2, t_3)$ and $f > a$ on $(t_3, t_4)$. Define $f_n$ equal to $f$ everywhere but from $[t_1, t_2)$ where we let $f_n = f + 1/n$ and a similar argument as above concludes.

We immediately have the version of Theorem 3.11 for a right barrier $b \in \mathbb{R}$ and a two sided barrier.

**Definition 3.12.** Let $a, b \in \mathbb{R}$ with $a < b$. Define the right fast-forwarding map at $b \in \mathbb{R}$, $N_b^r : D_{-\infty, b}((0, \infty), \mathbb{R}) \to D((0, \infty), \mathbb{R})$, by $N_b^r(f) = f(A_f^{-1})$ where $A_f^{-1}(t) = \inf \{ s > 0 : A_f(s) > t \}$ and $A_f(t) = \int_0^t 1_{\{(f(z) < b)\}} dz$; and define the two-sided fast-forwarding map at $a, b$ as

$$N_b^l(N_b^r) : D_{a, b}((0, \infty), \mathbb{R}) \to D((0, \infty), \mathbb{R}).$$

Note that $N_b^r$ is clearly measurable by using the same argument as in Proposition 3.10 and so is $N_a^l(N_b^r)$.

**Corollary 3.13.** Assume that $f(0) < b \in \mathbb{R}$. Then, the map $N_b^r$ is continuous at $f$ if and only if $f$ satisfies $\lambda(t \in [0, \infty) : f(t) = b) = 0$ and

(Nr) : if there exists an $\epsilon > 0$ and $t_1 < t_2 < t_3 < t_4$ such that $f(t) \leq b - \epsilon$ for all $t \in [t_1, t_2) \cup [t_3, t_4]$ and $f(t) \geq b$ for all $t \in [t_2, t_3)$, then $\inf_{t \in \{t_2, t_3\}} f(t) > b$.

**Corollary 3.14.** Assume that $f(0) \in (a, b)$. Then, the map $N_b^l(N_b^r)$ is continuous at $f$ if and only if $f$ satisfies (Nl), (Nr) and $\lambda(t \in [0, \infty) : f(t) \in \{a, b\}) = 0$.

**Remark 3.15.** If $a = -1$ and $b = 1$ we write $N_1^l = N_l$ and $N_b^r = N_r$.

We conclude this section with a simple statement showing that $N_a^l$ and $N_b^r$ commute and that their composition equals the natural definition of deletion the time spent outside $(a, b)$. The proof can be found in Appendix A.

**Proposition 3.16.** For any $a < b$, $N_a^l(N_b^r) = N_b^r(N_a^l)$ on $D_{a, b}((0, \infty), \mathbb{R})$ and $N_a^l(N_b^r)$ can be equivalently defined as $f \mapsto f((A_f(a, b))^{-1})$ where $A_f(a, b)(t) = \int_0^t 1_{\{(f(z) \in (a, b))\}} dz$.

4. One-sided and Grünwald type processes

This section studies the pathwise construction of our Markov processes on $[-1, 1]$, obtained by applying the reflecting, killing and fast-forwarding maps introduced in Section 3. In Section 4.1 we study the restrictions of our spectrally positive process $Y$, meanwhile in Section 4.2 we study the restrictions of our discrete-valued Grünwald type approximation to $Y$. These last discrete-valued processes allow us to gain crucial insights on the boundary behaviour of the limit processes, along with providing a tool to derive the boundary conditions identified in Theorem 1.1.
4.1. One-sided processes and their boundary modifications

We denote by $Y = (Y_t)_{t \geq 0}$ the càdlàg modification of the spectrally positive Lévy process with Laplace exponent $\psi$, i.e. $\exp\{\psi(\xi)\} = \mathbb{E}_0[\exp\{-\xi Y_1\}]$ for $\Re \xi \geq 0$ [25, Page 188], with $\psi$ defined in (H0).

(H0) For a nonnegative Borel measure $\phi$ on $(0, \infty)$ such that

$$\int_{(0,\infty)} (y^2 \wedge y) \phi(dy) < \infty \quad \text{and} \quad \int_{(0,1)} y \phi(dy) = \infty,$$

we define $\psi(\xi) = \int_{(0,\infty)} \left(e^{-\xi y} - 1 + \xi y \right) \phi(dy)$, for $\Re \xi \geq 0$.

Note that $\psi(\xi) = \xi^2 \int_0^\infty e^{-\xi x} \Phi(x) \, dx$ where $\Phi(x) = \int_x^\infty \phi((y, \infty)) \, dy$, and by Dominated Convergence Theorem, as $|\xi| \downarrow 0$

$$\psi(\xi) \to 0 \quad \text{and} \quad \psi'(\xi) \to 0. \quad (24)$$

Remark 4.1. We collect some basic facts about the process $Y$. For examples we refer to [24, Example 2.4].

(i) The process $Y$ is a Feller process and its Feller semigroup on $C_0(\mathbb{R})$ is generated by the closure of $(D^\infty_{-\infty}, C_c(\mathbb{R}))$ (see proof of Proposition 4.15), where

$$D^\psi_{-\infty} g(x) := \frac{d^2}{dx^2} \int_0^\infty \Phi(y) g(x + y) \, dy.$$

(ii) Under assumption (H0), the Laplace exponent $\psi$ characterises spectrally positive, recurrent Lévy processes with paths of unbounded variation and no diffusion component [24, Remark 2.3.i]. In particular, $P_{Y_0}$-a.s., $\tau^Y_{(-1,1)} = \tau^Y_{[1,1]}$, $\tau^Y_{(-1,\infty)} = \tau^Y_{[-1,\infty)}$ and $\tau^Y_{(-\infty,1)} = \tau^Y_{(-\infty,-1)}$, all immediate consequences of (2).

(iii) By [26, Theorem 8.1.iii] we know that for all $\beta \geq 0$

$$x \mapsto \mathbb{E}_x \left[ e^{-\beta \tau^Y_{(-\infty,-1)}} 1_{\{\tau^Y_{(-\infty,-1)} < \tau^Y_{(-1,\infty)}\}} \right] \in C_0(-1,1)$$

and it equals 1 for $x = 1$, and

$$x \mapsto \mathbb{E}_x \left[ e^{-\beta \tau^Y_{(-1,\infty)}} 1_{\{\tau^Y_{(-1,\infty)} > \tau^Y_{(-1,-1)}\}} \right] \in C_0[-1,1]$$

and it equals 1 for $x = -1$. Moreover, for any $\beta > 0$ and $g \in C[-1,1]$

$$x \mapsto D^\beta_{-\infty} g(x) := \int_0^\infty e^{-\beta t} \mathbb{E}_x [g(Y_t) 1_{\{t < \tau^Y_{(-1,-1)}\}}] \, dt \in C_0(-1,1),$$

which is a consequence of [26, Theorem 8.7].

To conclude the argument of our main result, Theorem 1.1, we need the resolvent operator of each $Y_{LR}$ to map $C_0(\Omega)$ to itself, hence we prove this below.

Proposition 4.2. The càdlàg processes $Y_{LR}$ in Table 2 are strong Markov and the respective $\beta$-resolvent operators ($\beta > 0$) map $C_0(\Omega)$ to itself.
Table 2: The definition of each process $Y^{LR}$, recalling that $Y$ is defined in Section 4.1 and the maps are defined in Section 3.

| Definition of $Y^{LR}$ | Equals a.s. | Equals in law |
|-------------------------|-------------|---------------|
| 1. $Y^{DD} = D'(D'(Y))$ | $D'(D'(Y))$ | $D'(N^*(Y))$ |
| 2. $Y^{NN} = D'(N^r(Y))$ | $N^r(D'(Y))$ | $D'(N^*(Y))$ |
| 3. $Y^{ND} = D'(N^r(Y))$ | $N^r(D'(Y))$ | $D'(N^*(Y))$ |
| 4. $Y^{NN} = N'(N^r(Y))$ | $N'(N^r((Y)))$ | $N^r(N^*(Y))$ |
| 5. $Y^{N} = D'(N^*(Y))$ | | $[N^r(N^*(Y))](Y)$ |
| 6. $Y^{N} = N'(N^*(Y))$ | | |

Proof. Recall that $Y$ is a Feller process and so is $N^*(Y)$ by [25, Proposition VI.1]. Then by Theorem [55, Theorem 3.14, Chapter IV, page 104] they are standard (or Hunt) processes (see [55, Definition 3.23, Chapter IV, page 104]). Then, by [49, Exercise V.2.11, page 212] the processes $N'(N^r(Y))$, $N^r(Y)$, $N^*(Y)$ and $N^r(N^*(Y))$ are strong Markov càdlàg processes and (the completion of) their natural filtration is right continuous. We remark that for $N'(N^r(Y))$ we used the “two-sided fast-forwarding” representation from Proposition 3.16. The remaining processes are obtained by killing the processes $Y$, $N^r(Y)$, $N^*(Y)$ and $N^r(N^*(Y))$ at the stopping time defined as their first exit from the open (in the respective state spaces) sets $(-1,1)$, $[-1,1)$, $(-1,1]$ and $[-1,1)$, respectively. Then the resulting càdlàg processes are strong Markov with right continuous filtration by [55, Theorem 10.1 and Remarks 10.2,(1,3,4), Chapter X, page 301-302]).

We now simplify notation by writing $Z = Y^{LR}$. Then $\tau_{(-1,1)}^Z$ is a stopping time for each process $Z$ [48, Theorem 1.6, Chapter 2] and $\tau_{(-1,1)}^Z = \tau_{(-1,1)}^Y = \tau_{(-1,1)}^Y$ $P_x$-a.s. for any $x \in (-1,1)$ by Remark 4.1(ii). Now we claim that for any $x \in (-1,1)$ it holds $P_x$-a.s. that

$$Z_{\tau_{(-1,1)}^Z} = 1 \text{ on } \{\tau_{(-1,1)}^Z < \tau_{(-1,1)}^Z\} \text{ and } Z_{\tau_{(-1,1)}^Z} = -1 \text{ on } \{\tau_{(-1,1)}^Z > \tau_{(-1,1)}^Z\} \tag{25}$$

for all boundary conditions (recalling that for killing we conventionally absorb at 1 or $-1$ in the natural way). For any process and boundary that involves killing (D) and reflecting ($N^*$) the claim is clear. For the cases NR on $\{\tau_{(-1,1)}^Z > \tau_{(-1,1)}^Z\} = \{\tau_{(-1,1)}^Y > \tau_{(-1,1)}^Y\}$ we have

$$Z_{\tau_{(-1,1)}^Z} = Y_A^{-1}(\tau_{(-1,1)}^Y) = -1,$$

because $\tau_{(-1,1)}^Y = A_Y(\tau_{(-1,1)}^Y)$, $\tau_{(-1,1)}^Y$ is a point of increase for $A_Y$ (by the strong Markov property and $-1$ being regular for $(-1,\infty)$) and $Y_{\tau_{(-1,1)}^Y} = -1$ because $Y$ is spectrally positive. For the cases LN, on $\{\tau_{(-1,1)}^Z < \tau_{(-1,1)}^Z\} = \{\tau_{(-1,1)}^Y < \tau_{(-1,1)}^Y\}$ we have

$$Z_{\tau_{(-1,1)}^Z} = Y_A^{-1}(\tau_{(-1,1)}^Y) = 1,$$

as $A_Y^{-1}(\tau_{(-1,1)}^Y)$ is the time of first re-entry in $(-\infty,1)$ which must be at 1 because $Y$ is spectrally positive. We now can show that the resolvent maps $C_0(\Omega)$ to itself, i.e. $R^\beta_{LR} C_0(\Omega) \subset C_0(\Omega)$ where

$$R^\beta_{LR} g(x) = E_x \left[ \int_0^\infty e^{-\beta t} g(Z_t) \, dt \right], \quad \beta > 0, \ x \in [-1,1], \ g \in C_0(\Omega).$$
Indeed by the strong Markov property and (25)

\[
R^\text{LR}_\beta g(x) = \mathbb{E}_x \left[ \int_0^{\tau_{(-1,1)}} e^{-\beta t} g(Z_t) \, dt \right] + \mathbb{E}_x \left[ \int_{\tau_{(-1,1)}}^\infty e^{-\beta t} g(Z_t) \, dt \right]
\]

\[= R^\text{DD}_\beta g(x) + \mathbb{E}_x \left[ e^{-\beta \tau_{(-1,1)}} R^\text{LR}_\beta g(Z_{\tau_{(-1,1)}}) \right] \]

\[= R^\text{DD}_\beta g(x) + R^\text{LR}_\beta g(1) \mathbb{E}_x \left[ e^{-\beta \tau_{(-1,1)}} 1_{\{\tau_{(-1,1)} < \tau_{x} \}} \right] \]

\[+ R^\text{LR}_\beta g(-1) \mathbb{E}_x \left[ e^{-\beta \tau_{(-1,1)}} 1_{\{\tau_{(-1,1)} > \tau_{x} \}} \right] \]

\[= R^\text{DD}_\beta g(x) + R^\text{LR}_\beta g(1) \mathbb{E}_x \left[ e^{-\beta \tau_{(-1,1)}} 1_{\{\tau_{(-1,1)} < \tau_{y} \}} \right] + R^\text{LR}_\beta g(-1) \mathbb{E}_x \left[ e^{-\beta \tau_{(-1,1)}} 1_{\{\tau_{(-1,1)} > \tau_{y} \}} \right], \]

which defines a continuous function on $C_0(\Omega)$ by Remark 4.1-(iii).

\[ \Box \]

**Remark 4.3.** By Proposition 4.2, we can use the Markov property and right continuity of the paths of $Y^{\text{LR}}$ to conclude that $t \mapsto \mathbb{E}_x [g(Y^\text{LR}_t)]$ is right continuous for each $x \in [-1, 1]$ and $g \in C_0(\Omega)$.

The following proposition will allow us to apply the CMT in Section 5.1.

**Proposition 4.4.** Assume (H0) and let $x \in (-1, 1)$. Then, $\mathbb{P}_x$-a.s. the paths of $X$ satisfy the conditions on $f$ in the statement $S$, where $(X, S)$ can be:

\begin{align*}
(Y, \text{Proposition 3.6}), & \quad (D^\text{r}(Y), \text{Proposition 3.7}), \\
(N^\text{r}(Y), \text{Proposition 3.7}), & \quad (N^\text{r}(Y), \text{Proposition 3.6}), \\
(N^*\text{r}(Y), \text{Proposition 3.6}), & \quad (N^*\text{r}(Y), \text{Corollary 3.13 with } b = 1), \\
(Y, \text{Corollary 3.13 with } b = 1), & \quad (Y, \text{Corollary 3.14 with } \{a, b\} = \{-1, 1\}).
\end{align*}

**Proof.** Let $x < 1$ be the starting point of $Y$ and simplify notation by writing $\tau = \tau_{(-1,1)}$. Concerning (Y, Proposition 3.6), combining (2) with the fact that $Y$ cannot creep upward [26, Section 8.1], we obtain that $\mathbb{P}_x$-a.s. $Y_x > 1$. Now suppose there exists a $\delta > 0$ such that $\sup_{t \in [0, \tau - \delta]} Y(t) = 1$ with positive probability, then there exists $\tau' < \tau$ such that $f(\tau') = 1$ and $f(\tau') < 1$, but this implies that $Y$ has a negative jump, which is a contradiction. The claim from the other two cases concerning Proposition 3.6 follow similarly. Concerning Proposition 3.7 the result follows as $Y$ moves downward continuously and this property is preserved by $D^\text{r}$ and $N^\text{r}$. The remaining cases follow by Remark 4.7, the fact that the occupation measure of $Y$ allows a density ($Y$ moves downward by a drift and 0 is regular for $(-\infty, 0)$), so its range has positive Lebesgue measure on any compact time interval [56, Section 0]) which implies (9), and that (NI) and (Nr) are always satisfied by one-sided processes.

\[ \Box \]

4.2. Grünwald type processes and their boundary modifications

In this section we identify the pathwise description of the processes with generators $G^\text{LR}_h$ from [24, Lemma 3.14] when started on a gridpoint $x \in \text{Grid}_h$, where for any $n \in \mathbb{N}$, we define

$$\text{Grid}_h := \{-1 + hj : j = 0, 1, ..., 2/h\}, \quad \text{where} \quad h = 2/(n + 1).$$
We first recall from [24, Proposition 3.11] that $Y^h$ denotes the compound Poisson process generated by

$$\partial^\psi_{\ast,h} f(x) = \Pi^{-1}_h \Pi f(x) = \sum_{j=0}^{\infty} f(x + (j-1)h) G^\psi_{j,h},$$

with transition rate matrix

$$G_h = \begin{pmatrix}
G^\psi_{1,h} & G^\psi_{2,h} & \cdots & G^\psi_{n-1,h} & G^\psi_{n,h} \\
G^\psi_{0,h} & G^\psi_{1,h} & \cdots & G^\psi_{n-2,h} & G^\psi_{n-1,h} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & G^\psi_{0,h} & \cdots & G^\psi_{1,h} & G^\psi_{2,h} \\
0 & 0 & \cdots & G^\psi_{0,h} & G^\psi_{1,h}
\end{pmatrix},$$

where $\{G^\psi_{j,h} : j \in \mathbb{N}\}$ is the set of coefficients determined by $\psi((1-\xi)/h) = \sum_{j=0}^{\infty} G^\psi_{j,h} \xi^j$ for $|\xi| < 1$. We recall from [24, Lemma 3.3] that

$$-G^\psi_{1,h}, G^\psi_{0,h}, G^\psi_{j,h} > 0, \text{ for } j \geq 2, \text{ and } -G^\psi_{1,h} = \sum_{j \neq 1} G^\psi_{j,h},$$

and that we set $G^\psi_{j,h} = 0$ for $j \in \mathbb{N}$. We refer to [24, Sections 3.1 and 3.3] for an in depth discussion of these coefficients and their properties.

**Remark 4.5** (Convention). From now on, to simplify notation, we drop the $h$ subscript by writing $G^\psi_j = G^\psi_{j,h}$.

We will apply the maps of Section 3 to $Y^h$ with $Y^h_0 \in \text{Grid}_h$ and show that the transition matrix in each case is

$$G^\text{LR}_{n+2} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
d_0 & b_1 & b_2 & \cdots & b_{n-1} & b_n & d_{n+1} \\
0 & G_0^\psi & G_1^\psi & \cdots & G_{n-2}^\psi & b_{n-1}^r & d_n^r \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & G_0^\psi & b_2^r & d_3^r & \vdots \\
0 & 0 & 0 & 0 & G_0^\psi & b_1^r & d_2^r \\
0 & 0 & 0 & 0 & 0 & G_0^\psi & \vdots
\end{pmatrix}$$

with the coefficients \{b_j, b_j^r, b_n : j = 1, 2, \ldots, n - 1\} determined for each boundary condition in Table 3. Note that (28) equals [24, Eq. (40)] with the addition of absorbing Dirichlet boundary conditions at $-1$ and $1$.

We will use the next proposition and the remark afterward to guarantee the fast-forwarding maps are well-defined.

**Proposition 4.6.** Assuming (H0), the process $Y$ is recurrent, and so is $Y^h$ for every $h > 0$. 


Proof. By [57, Theorem 25.3, page 159] and (H0) the process $Y^h$ has finite first moment, and so does $Y^h$ (due to its finite Lévy measure). By [57, Theorem 36.7, page 248] it is enough to prove that $E_0[Y^1] = 0$ and $E_0[Y_n^h] = 0$. Recalling that the Laplace exponents of $Y$ and $Y^h$ are respectively $\psi(ix)$ and $e^{ixh}\psi((1-e^{-ixh})/h)$ (see [24, Remark 3.2.ii]), we obtain

$$E_0[Y^1] = \lim_{x \to 0} \frac{d}{dx} iE_0[e^{-ixY^1}] = \lim_{x \to 0} \frac{d}{dx} ie^{\psi(ix)} = \lim_{x \to 0} i^2\psi'(ix) e^{\psi(ix)} = 0,$$

by (24), and similarly

$$E_0[Y_n^h] = \lim_{x \to 0} \left(he^{ixh}\psi((1-e^{-ixh})/h) + \psi'((1-e^{-ixh})/h)\right)i^2E_0[e^{-ixY^1}] = 0.$$

\qed

Remark 4.7. Proposition 4.6 with [57, Theorem 35.4.iii, page 239] implies that for any $x \in \mathbb{R}$, $\mathbb{P}_x$-a.s., $Y$, $Y^h \in D_{a,b}([0,\infty), \mathbb{R})$ for any $-\infty \leq a < b \leq \infty$ (for all $h$ small), and using $\liminf Y_t = \liminf Y_t^h = -\infty \mathbb{P}_x$-a.s. by [57, Proposition 37.10.3, page 255], we obtain that, $\mathbb{P}_x$-a.s., $N^f(Y), N^f_{h-1}(Y^h) \in D_{-\infty,1}([0,\infty), \mathbb{R})$.

| Case | Rates $b^i, d^i$ |
|------|------------------|
| DR   | $b^i = G^i_0$, $d^i = 0$ |
| NR   | $b^i = -\sum_{j=0}^{i-1} G^i_j$, $d^i = 0$ |
| N*R  | $b^i = \sum_{j=0}^{i-1} G^i_j$, $b^i = G^i_0$, $i \geq 2$, $d^i = 0$ |
| LD   | $b^i = G^i_0$, $b_n = b^i_n$, $d^i_n = \sum_{j=0}^{n-1} G^i_j$ |
| LN   | $b^i = -\sum_{j=0}^{i-1} G^i_j$, $b_n = -\sum_{j=0}^{n-1} b^i_j$, $d^i_n = 0$ |

Table 3: Table of boundary weights used to build the transition rate matrix (28), assuming $b^i_0 = 0$ for NR and N*R, and with the exception that in the case ND we set $d^i_{n+1} = \sum_{j=n+1}^\infty b^i_j$.

With the help of Section 4.2.1 we prove below the desired pathwise representation on gridpoints $\text{Grid}_h$ from [24, Lemma 3.14]. In the second column we listed the associated transition rate matrix as defined in (28).

| Process $Y_{LR, h}$ | Rate matrix $G_{LR, h}$ |
|----------------------|-------------------------|
| 1. $Y_{DD, h}^D = D'(D(Y^h))$ | $G_{DD}^{LR, h}$ |
| 2. $Y_{DN, h}^D = D'(N(Y^h))$ | $G_{DN}^{LR, h}$ |
| 3. $Y_{ND, h}^N = D'(N(Y^h))$ | $G_{ND}^{LR, h}$ |
| 4. $Y_{NN, h}^N = N'(N(Y^h))$ | $G_{NN}^{LR, h}$ |
| 5. $Y_{DN, h}^D = D'(N^{h-1}(Y^h))$ | $G_{DN}^{LR, h}$ |
| 6. $Y_{NN, h}^N = N'(N^{h-1}(Y^h))$ | $G_{NN}^{LR, h}$ |

Table 4: Pathwise representation $Y_{LR, h}$ of the processes generated by the interpolated matrices $G_{LR, h}$ on gridpoints $\text{Grid}_h$ from [24, Lemma 3.14]. In the second column we listed the associated transition rate matrix as defined in (28).

Proposition 4.8. Assuming (H0), for any $n \in \mathbb{N}$ and letting $h = 2/(n + 1)$, the Markov process taking values in $\text{Grid}_h$ with transition rate matrix $G_{n+2}^{LR, h}$ is $Y_{LR, h}$ given in Table 4.

Proof. Recall that all processes are absorbed once they visit 1 or −1, which justifies the first and the last row. It is immediate to observe that for all cases the rate
of leaving a point in Grid_{h−1} \{−1, 1−1, −1, 1−h, 1\} remains −G^v_1, and this holds also for Y^{DR,h} leaving h−1 and Y^{LD,h} leaving 1−h. Also the rates of landing on Grid_{h−1} \{−1, 1−h, 1\} from Grid_{h−1} \{−1, 1−1, −1, 1−h, 1\} are clear. The rates for Y^{NR,h} and Y^{RN,h} leaving h−1 and Y^{LN,h} leaving 1−h are proved in Theorem 4.10 and Proposition 4.9. The first (last) column in G^{NR}_{n+2} is clear as D^I (D') allows to reach −1 (1) and does so by accumulating the intensities of all the jumps less (greater) or equal than −1 (1), meanwhile the other maps never allow to reach \{−1, 1\}. Note that the map N^r accumulates the intensities of jumps of Y^h to a state greater or equal to 1−h on 1−h, by Proposition 4.6. The coefficients for the second row of G^{NR}_{n+2} and the second last row of G^{RN}_{n+2} are obtained immediately from Theorem 4.10. The last n coefficients for the second row of G^{NR}_{n+2} are clear and we are done.

\[ \square \]

Proposition 4.9. For any h > 0, if (H0) holds and Y_0^h = 0, then the first jump of N_{n+1}^{a,l}(Y^h) and of N_{n+1}^{a,r}(Y^h) are exponentially distributed with rate −(G^v_1 + G^v_0) and G^v_0, respectively.

Proof. Denote by J_m the time of the m-th jump of Y^h with the convention J_0 = J_{−1} = 0 and compute using standard arguments and (27),

\[ \mathbb{P}_0[N_{s}^{a,r}(Y^h)] = 0 \text{ for all } s < t] = e^{G^v_1 t} \sum_{n=0}^{\infty} \left( -\frac{G^v_1 t}{n!} \right)^n \mathbb{P}_0[Y^h_m \leq t, Y^h_{m−1} > 0 \text{ for all } 0 \leq m \leq n] = e^{G^v_1 t} \sum_{n=0}^{\infty} \left( \frac{G^v_1 t}{n!} \right)^n = e^{G^v_1 t}. \]

A similar computation using \mathbb{P}_0[Y^h_m \leq t, Y^h_{m−1} > 0 \text{ for all } 0 \leq m \leq n] = (−G^v_0 / G^v_1)^n yields \mathbb{P}_0[N_{s}^{a,l}(Y^h)] = 0 \text{ for all } s < t] = e^{(G^v_1 + G^v_0)t}. \]

4.2.1. Details for fast-forwarding

We derive the waiting times and the distribution of the jumps of fast-forwarding Y^h at a boundary. We use the auxiliary process Y^{h, \text{stop}} which is defined as Y^h stopped the first time it visits hN. Then, for each h > 0, the transition rate matrix of Y^{h, \text{stop}} is

\[ G_{\text{stop}} = (g_{i,j})_{i,j \in \mathbb{Z}} \text{ with } g_{i,j} = \begin{cases} G^v_{j−i+1}, & i \leq 0, \\ 0, & i > 0. \end{cases} \quad (29) \]

Theorem 4.10. Assume (H0). Then the process N^d(Y^h) with Y_0^h = −1+h jumps the first time according to an exponential waiting time with rate G^v_0, and it lands on −1 + (1+j)h with probability −(\sum_{i=0}^{j} G^v_i)/G^v_0 for each j ≥ 1. The process N^r(Y^h) with Y_0^h = 1−h jumps the first time according to an exponential waiting time with rate G^v_0, and it lands on 1−2h.

Proof. Without loss of generality we look at the process N_{−h}^d(Y^h) with Y_0^h = 0. We first justify the identities for j ∈ \mathbb{N}

\[ \mathbb{P}_0 \left[ N_{−h}^d(Y^h)_{j1} = jh \right] = \mathbb{P}_0 \left[ Y^h_{\sigma_{jh}^h} = jh \right] = \mathbb{P}_0 \left[ \sigma_{jh, \text{stop}}^h < \infty \right] = \frac{1}{G^v_0} \sum_{i=0}^{j} G^v_i, \]

where J_1 is the first jump time of N_{−h}^d(Y^h), \sigma_{jh}^{Y^h} = \inf\{t > 0 : Y^h_t = jh\} and \sigma_{jh, \text{stop}}^{Y^h} = \inf\{t > 0 : Y^h_{t, \text{stop}} = hj\}. The first identity is a direct consequence of the pathwise
definition $N_{-h}^t$ (Definition 3.8) along with the piecewise constant paths of $Y^h$ (or simply put, $N_{-h}^t(Y^h)$ visits $jh$ the first time it moves if and only if $Y^h$ visits $jh$ the first time it enters $h$); the second identity is clear; and the third identity is due Corollary 4.14 and (3). Also by Proposition 4.11 we know that $J_1$ at 0 is an exponential waiting time with rate $G_1^0$, which concludes the proof for $N_{-h}^t(Y^h)$. For $N_{h}^t(Y^h) = 0$, the proof is immediate from Propositions 4.11 and 4.6.

**Proposition 4.11.** For any $h > 0$, assuming (H0) and $Y_0^h = 0$, the first jump of $N_{-h}^t(Y^h)$ and of $N_{h}^t(Y^h)$ for are both exponentially distributed with rate $G_0^0$.

**Proof.** Denote by $E_m^h$ the waiting time between the $(m-1)$-th and the $m$-th jump of $Y^h$ for $m \in \mathbb{N}$, and recall that $\{E_m^h : m \in \mathbb{N}\}$ is a collection of iid exponential random variables with rate $\mu = -G_1^0$ and $P[\sum_{m=1}^{\infty} E_m^h > t] = e^{-\mu t} \sum_{m=0}^{\infty} \frac{(\mu t)^m}{m!}$ (Erlang distribution). Let $J_m$ denote the $m$-th jump time of the stopped process $Y_{\text{stop}}^h$ ($Y^h$ stopped the first time it visits $h\mathbb{N}$) and define $\sigma(m)$ to be the $m$-th return time at 0 of $Y_{\text{stop}}^h$ with the convention that $\sigma(0) = 0$, and define the function $j(\sigma(m)) = \{\sigma(m) < \infty\} \rightarrow \mathbb{N} \cup \{0\}$ such that $\sigma(m) = J_{j(\sigma(m))}$. We now justify the following identities

\[
P_0[N_{-h}^t(Y^h)_s = 0 \text{ for all } s \leq t]
= \sum_{n=0}^{\infty} P_0 \left[ \sum_{m=0}^{n} \int_{j(\sigma(m))}^{\min(j(\sigma(m)) + 1, t)} 1_{\{Y^h_{\text{stop}} = 0\}} \, dz > t \right) \cap \{\sigma(n) < \infty, \sigma(n+1) = \infty\}
= \sum_{n=0}^{\infty} \left( e^{-\mu t} \sum_{m=0}^{n} \frac{(\mu t)^m}{m!} \right) P_0 \left[ \sigma(n) < \infty, \sigma(n+1) = \infty \right]
= e^{-\mu t} \sum_{m=0}^{\infty} \frac{(\mu t)^m}{m!} \sum_{n \geq m} P_0 \left[ \sigma(n) < \infty, \sigma(n+1) = \infty \right]
= e^{-\mu t} \sum_{m=0}^{\infty} \frac{(\mu t)^m}{m!} P_0 \left[ \sigma(m) < \infty \right]
= e^{-\mu t} \sum_{m=0}^{\infty} \frac{(\mu t)^m}{m!} P_0 \left[ \sigma(m) < \infty \right] P_0 \left[ \sigma(1) < \infty \right]
\]

The first identity is immediate from the definition of $N_{-h}^t(Y^h)$, as this process does not move for $t$-units of time if and only if the free process $Y^h$ spend at least $t$-units of time at 0 before visiting $h\mathbb{N}$; the second identity uses $\{\sigma(m) < \infty, \sigma(m+1) = \infty\} \cap \{\sigma(n) < \infty, \sigma(n+1) = \infty\} = \emptyset$ for all $n \neq m$; the third and fourth identities are clear observing that, on $\{\sigma(n) < \infty, \sigma(n+1) = \infty\}$, $Y_{\text{stop}}^h$ visits 0 only at times $\sigma(m)$ for $m \leq n$; the sixth identity is an application of Tonelli’s Theorem: the second last identity is clear: the last identity uses $P_0 \left[ \sigma(m) < \infty \right] = P_0 \left[ \sigma(1) < \infty \right]^m$ [50, page 96]; the fifth identity follows by observing that

\[
(J_{j(\sigma(m)) + 1} - J_{j(\sigma(m))}) = E_{j(\sigma(m))}^h \quad \text{on} \quad \{\sigma(m) < \infty\},
\]
and computing
\[
\mathbb{P}_0 \left( \left\{ \sum_{m=0}^n E_{p_j}^{Y_h} > t \right\} \cap \{ \sigma^{(n)} < \infty, \sigma^{(n+1)} = \infty \} \right)
\]
\[
= \sum_{p \in P_n} \mathbb{P}_0 \left( \left\{ \sum_{m=0}^n E_{p(m)}^{Y_h} > t \right\} \cap \{ \sigma^{(n)} < \infty, \sigma^{(n+1)} = \infty \} \cap \{ j(\sigma^{(m)}) = p(m) : m \leq n \} \right)
\]
\[
= \sum_{p \in P_n} \mathbb{P}_0 \left( \sum_{m=0}^n E_{p(m)}^{Y_h} > t \right) \mathbb{P}_0 \left( \{ \sigma^{(n)} < \infty, \sigma^{(n+1)} = \infty \} \cap \{ j(\sigma^{(m)}) = p(m) : m \leq n \} \right)
\]
\[
= \mathbb{P}_0 \left( \sum_{m=1}^{n+1} E_m^{Y_h} > t \right) \sum_{p \in P_n} \mathbb{P}_0 \left( \{ \sigma^{(n)} < \infty, \sigma^{(n+1)} = \infty \} \cap \{ j(\sigma^{(m)}) = p(m) : m \leq n \} \right),
\]
where \( P_n = \{ p : \{0,1,\ldots,n\} \rightarrow \mathbb{N}; p(m) < p(m+1) \} \) is countable as \( P_n = \bigcup_{m=0}^{n} \{ p : \{0,1,\ldots,n\} \rightarrow \{1,\ldots,l+1\}, p(m) < p(m+1) \} \), and note that we used independence of the event \( \{ \sigma^{(n)} < \infty, \sigma^{(n+1)} = \infty, j(\sigma^{(m)}) = p(m), m \leq n \} \) and the random variables \( \{ E_{p}^{Y_h} : t \geq 0 \} \) (the event depends only on the jump magnitudes of \( Y_h \), not on the length of the waiting times of \( Y_h \) [50, Page 81]).

Hence we showed that the waiting time is exponential, and it remains to compute \( \mathbb{P}_0 [\sigma^{(1)} < \infty] \). We directly compute
\[
\mathbb{P}_0 [\sigma^{(1)} < \infty] = \mathbb{P}_0 \left[ Y_{h,J_1}^{\text{stop}} = 1, s \mapsto Y_{h,J_1+s} \text{ hits } 0 \right]
\]
\[
= \mathbb{P}_0 \left[ Y_{h,J_1}^{\text{stop}} = 1 \right] \mathbb{P}_0 \left[ Y_{h,J_1}^{\text{stop}} \text{ hits } 0 \right]
\]
\[
= \mathbb{P}_0 \left[ Y_{h,J_1}^{\text{stop}} = 1 \right] \mathbb{P}_0 \left[ Y_{h,J_1}^{\text{stop}} \text{ hits } 0 \right]
\]
\[
= \frac{G_0^0 - \varrho_0^0 - \varrho_1^0}{-\varrho_1^0},
\]
where \( J_1^{Y_h} \) is the first jump time of \( Y_h \), and we used the strong Markov property in the second identity, the third identity is clear, and we used Corollary 4.14 with (3) in the last identity. To prove the statement for \( N_0^m(Y_h) \) observe that the waiting time is exponential by the same proof above, but using \( Y_{h}^{\text{stop}-} \) instead of \( Y_{h}^{\text{stop}} \), where \( Y_{h}^{\text{stop}-} \) is the process \( Y_h \) stopped on its first visit of \(-hN\). To calculate the average of the exponential time it remains to compute \( \mathbb{P}_0 [\sigma^{(1)-} < \infty] \), where \( \sigma^{(1)-} \) is the time of first return at 0 of \( Y_{h}^{\text{stop}-} \). Because \( Y_h \) moves to the left by single step and it is recurrent (Proposition 4.6) we easily obtain, using (27),
\[
\mathbb{P}_0 [\sigma^{(1)-} < \infty] = \mathbb{P}_0 \left[ Y_{J_1^{Y_h}} = 1 \right] = \sum_{m=2}^{\infty} \frac{G_m^0}{-\varrho_1^0} = \frac{-\varrho_0^0 - G_0^0}{-\varrho_1^0}.
\]

The following is an easy consequence of Propositions 4.9 and 4.11, and we omit the proof.

**Corollary 4.12.** For any \( h > 0 \), assuming \((H0)\) and \( Y_0^h \in \{1-jh : j \in \mathbb{N}\} \), it holds that \( N'(Y_h) = N_{1-h}^\infty(Y_h) \) in law.

In Proposition 4.11 we found the exponential waiting time of \( N'(Y_h) \) at \( h - 1 \). To find the distribution of the where \( N'(Y_h) \) moves the first time it jumps we prove the
generalised version of [23, Theorem 16], with the proof essentially unchanged. We denote by \( G_{\text{stop}}^T \) the transpose of \( G_{\text{stop}} \), so that

\[
(G_{\text{stop}}^T \vec{x})_n = \sum_{k \in \mathbb{Z}} g_{k,n} \vec{x}_k = \sum_{k \geq n+1} G_k^\psi \vec{x}_{n+1-k}.
\]

**Lemma 4.13.** Assume \((H0)\). Then the resolvent of \( G_{\text{stop}}^T \) for \( \beta > 0 \) evaluated at \( \vec{e}_0 \) is given by \( ((\beta I - G_{\text{stop}}^T)^{-1} \vec{e}_0) = \vec{y}/G_0^\psi \), where

\[
\vec{y}_n = \begin{cases} e^{h(n-1)\varphi^{-1}(\beta)}, & n \leq 0, \\ \frac{1}{\beta} \sum_{k=n}^{\infty} G_k^\psi e^{h(n-k)\varphi^{-1}(\beta)}, & n > 0,
\end{cases}
\]

and \( \varphi(\beta) := e^{h\beta}(1 - e^{-h\beta})/h = \sum_{k=0}^{\infty} G_k^\psi e^{h(1-k)\beta} \).

**Proof.** Note that \( \varphi \) is invertible on \([0, \infty)\), as its derivative is positive on \((0, \infty)\) and bounded below away from 0, which is immediate from \( \psi', \psi > 0 \) on \((0, \infty)\) and

\[
\varphi'(\beta) = he^{h\beta}(1 - e^{-h\beta})/h + \varphi'(1 - e^{-h\beta})/h > 0.
\]

Next we apply \( \beta I - G_{\text{stop}}^T \) to our candidate resolvent and show that the result is indeed \( \vec{e}_0 \). Recall that \( \vec{y} \) is given by (30). For \( n \leq -1 \),

\[
((\beta I - G_{\text{stop}}^T)\vec{y})_n = \beta e^{h(n-1)\varphi^{-1}(\beta)} - \sum_{k=0}^{\infty} G_k^\psi e^{h(n-k)\varphi^{-1}(\beta)}
\]

\[
= \beta e^{h(n-1)\varphi^{-1}(\beta)} - e^{h(n-1)\varphi^{-1}(\beta)} \sum_{k=0}^{\infty} G_k^\psi e^{h(1-k)\varphi^{-1}(\beta)}
\]

\[
= \beta e^{h(n-1)\varphi^{-1}(\beta)} - e^{h(n-1)\varphi^{-1}(\beta)} \varphi(\varphi^{-1}(\beta)) = 0.
\]

For \( n = 0 \),

\[
((\beta I - G_{\text{stop}}^T)\vec{y})_0 = \beta e^{-h\varphi^{-1}(\beta)} - \sum_{k=1}^{\infty} G_k^\psi e^{-hk\varphi^{-1}(\beta)}
\]

\[
= \beta e^{-h\varphi^{-1}(\beta)} - e^{-h\varphi^{-1}(\beta)} \sum_{k=0}^{\infty} G_k^\psi e^{h(1-k)\varphi^{-1}(\beta)} + G_0^\psi
\]

\[
= \beta e^{-h\varphi^{-1}(\beta)} - e^{-h\varphi^{-1}(\beta)} \varphi(\varphi^{-1}(\beta)) + G_0^\psi = G_0^\psi.
\]

For \( n \geq 1 \), \( ((\beta I - G_{\text{stop}}^T)\vec{y})_n = 0 \) by the definition of \( G_{\text{stop}}^T \) and \( \vec{y} \). Hence \( (\beta I - G_{\text{stop}}^T)\vec{y} = G_0^\psi \vec{e}_0 \) and therefore \( (\beta I - G_{\text{stop}}^T)^{-1} \vec{e}_0 = \vec{y}/G_0^\psi \). \( \square \)

**Corollary 4.14.** Assuming \((H0)\), we have that \( \lim_{\beta \downarrow 0} \beta(\beta I - G_{\text{stop}}^T)^{-1} \vec{e}_0 = \vec{z} \), where \( \vec{z}_j = 0 \) for \( j \leq 0 \) and \( \vec{z}_j = \frac{1}{\sqrt{\psi_0}} \sum_{k=0}^{j} G_k^\psi \) for \( j > 0 \).

**Proof.** As \( \beta \downarrow 0 \), \( \varphi^{-1}(\beta) \to 0 \) and hence

\[
(\beta(\beta I - G_{\text{stop}}^T)^{-1} \vec{e}_0)_n \to 0
\]

for all \( n \leq 0 \). For \( n > 0 \),

\[
(\beta(\beta I - G_{\text{stop}}^T)^{-1} \vec{e}_0)_n \to \frac{1}{G_0^\psi} \sum_{k=n}^{\infty} G_k^\psi,
\]

and we conclude with (27). \( \square \)
4.3. Convergence of the Grünwald type process

**Proposition 4.15.** Assume (H0) and let $-\infty \leq a < b \leq \infty$. Then, if $Y^h_0 \Rightarrow Y_0$ on $\mathbb{R}$ as $h \to 0$, then $Y^h \Rightarrow Y$ on $D_{c,d}([0, \infty), \mathbb{R})$ as $h \to 0$.

**Proof.** Recall that on $C^\infty_c(\mathbb{R})$, two integration by parts prove

$$D^-_{\infty}\psi g(x) = \int_0^\infty \Phi(y)g''(x+y)\,dy = \int_0^\infty (g(x+y)-g(x)-yg'(x))\,\phi(dy),$$

so that the closure of $(D^-_{\infty}\psi, C^\infty_c(\mathbb{R}))$ in $C_0(\mathbb{R})$ generates the Feller process $Y$ [9, Corollary 2.10]. Let $0 \neq g \in C^\infty_c(\mathbb{R})$ and $a, b \in \mathbb{R}$ such that the support of $g$ is contained in $[a, b]$. Let $\epsilon > 0$ be arbitrary. Then, for any $a'$ small enough with $a' < a$ and all $h > 0$ small,

$$|\partial_{-h}\psi g(x) - D_{-\infty}\psi g(x)| = \begin{cases} 0, & x \in (b, \infty), \\ |\partial_{-h}\psi g(x) - D_{-\infty}\psi g(x)|, & x \in [a', b], \\ |e_h(x)|, & x \in (-\infty, a'), \end{cases}$$

where

$$e_h(x) = \sum_{j=m(x,h)}^\infty G_j \psi g(x+(j-1)h) - D_{-\infty}\psi g(x),$$

with $m(x,h) \geq 2$ being the largest integer so that $(m(x,h)-1)h \leq a-x$. Thus we can choose $a'$ such that for all $x \leq a'$

$$|e_h(x)| \leq \sum_{j=m(x,h)}^\infty G_j \psi |g(x+(j-1)h)| + \epsilon \leq \|g\|_{C_0(\mathbb{R})} \left( \sum_{j=m(a',h)}^\infty G_j \psi \right) + \epsilon,$$

because $D_{-\infty}\psi g \in C_0(\mathbb{R})$ and for each $h > 0$, $m(a',h) \leq m(x,h)$ for all $x \leq a'$. By (27) and an easy adaptation of the proof of the third limit in [24, Eq. (32)],

$$\limsup_{h \to 0} \sum_{j=m(a',h)}^\infty G_j \psi = -\limsup_{h \to 0} \sum_{j=0}^{m(a',h)-1} G_j \psi \leq 3\epsilon \left( \phi([a-a', \infty)) + \phi((a-a', \infty)) \right),$$

and because $\lim_{y \to \infty} \phi((y, \infty)) = 0$, we can choose a possibly smaller $a'$ such that

$$3\epsilon \left( \phi([a-a', \infty)) + \phi((a-a', \infty)) \right) \leq \frac{\epsilon}{2\|g\|_{C_0(\mathbb{R})}},$$

so that for all small $h > 0$ we can use the bound

$$\sup_{x \in (-\infty, a']} |e_h(x)| \leq 2\epsilon.$$

On the other hand, by [24, Corollary 3.8], for all $h > 0$ small

$$\left\| \partial_{-h}\psi g - D_{-\infty}\psi g \right\|_{C_{a',b]} \leq \epsilon.$$

As we proved strong convergence of the generators on the core $C^\infty_c(\mathbb{R})$, by [58, Theorem 17.25] we obtain the weak convergence of the respective stochastic processes on $D([0, \infty), \mathbb{R})$. The convergence on $D_{c,d}([0, \infty), \mathbb{R})$ follows immediately from Remark 4.7. □
5. Convergence of processes and semigroups

We are finally ready to prove our main results, by combining the Skorokhod continuity results of Section 3 with the Trotter–Kato convergence proved in Part I in [24, Theorem 5.1]. To do so, we first show that we can find a sequence of grids from the approximation scheme of Part I so that \( Y_{t}^{LR,h} \Rightarrow Y_{t}^{LR} \) for almost every \( t > 0 \) and \( Y_{0}^{LR} = Y_{0}^{LR,h} = x \) in a dense subset of \((-1, 1)\). Then we combine this weak convergence with the Trotter–Kato convergence of the (interpolated) Grünwald type semigroups of Part I. As we showed that on gridpoints these semigroups are given by \( Y^{LR,h} \) (Proposition 4.8), we can characterise pathwise the limit semigroups of Part I, and the proof of Theorem 1.1 is complete. Finally, we apply our results to derive new resolvent measures for the processes involving a left fast-forwarding boundary condition.

5.1. Skorokhod convergence

**Lemma 5.1** (Skorokhod convergence). Assume (H0) and recall the definitions of \( Y, Y^{LR}, Y^{h} \) and \( Y^{LR,h} \) from Tables 2 and 4. For any \( n \in \mathbb{N} \), let \( h = 2/(n + 1) \) and \( Y_{0}^{h} = Y_{0} = x \in \text{Grid}_{h}\{−1, 1\} \). Let \( \{h_{j}(x) = h_{j} : j \in \mathbb{N}\} \) be a sequence such that \( h_{j} \to 0 \) as \( j \to \infty \) and \( x \in \text{Grid}_{h_{j}} \) for all \( j \in \mathbb{N} \). Then

\[
Y^{LR,h_{j}} \Rightarrow Y^{LR} \quad \text{as } j \to \infty \quad \text{on } D([0, \infty), \mathbb{R}),
\]

and in particular, for all \( t > 0 \) outside of a countable set and \( g \in C_{0}(\Omega) \),

\[
\mathbb{E}_{x}[g(Y^{LR,h_{j}})] \to \mathbb{E}_{x}[g(Y_{t}^{LR})].
\]

**Proof.** Note that we only need to prove the six weak convergences on \( D([0, \infty), \mathbb{R}) \), as then the last convergence follows immediately from Proposition 2.3, as \( \mathbb{P}_{x}[Y_{t}^{LR} \neq Y_{t}^{LR}] = 0 \) for all \( t > 0 \) outside of a countable set [51, Section 16, page 174]. We recall the definitions of the killing maps in Definition 3.3 and Proposition 3.4, of the (measurable) fast-forwarding maps in Definitions 3.8 and 3.12 and Remark 3.15, and the (continuous) reflecting maps in Definition 3.1. Also, in each case we use, without mention, Proposition 4.15 and every convergence is understood “as \( j \to \infty \)”.

1. **DD:** by Corollary 3.5 and Proposition (2) we have \( Y^{DD,h} = D'((D^{\alpha}(Y^{h})) = D'_{0}(D'_{h}(Y^{h})) \) and \( Y^{DD} = D'(D^{\alpha}(Y)) = D'_{0}(D'_{0}(Y)) \). Then the result follows by Corollary 2.4 and Propositions 3.4, 3.6, 3.7 and 4.4.

2. **DN:** By Propositions 4.4 and Corollary 3.5, \( N'(Y) \) is \( \mathbb{P}_{x}-\text{a.s. a continuity point of } D^{\alpha}(N'(Y)) = D'_{0}(D'_{0}(N'(Y))) \) and \( D'(N'(Y^{h})) = D'_{0}(D'_{0}(N'(Y^{h}))) \). Moreover, Corollary 3.13 and Proposition 4.4 prove that \( N'(Y^{h_{j}}) \Rightarrow N'(Y) \) on \( D([0, \infty), \mathbb{R}) \). Then we conclude with Corollary 2.4.

3. **ND:** Similar as the above and omitted.

4. **NN:** By Proposition 4.4, \( Y^{h} \) is \( \mathbb{P}_{x}-\text{a.s. a continuity point of } N'(N') \) and by Proposition 4.6 the paths of \( Y^{h} \) belong to \( D_{-1,1}([0, \infty), \mathbb{R}) \) for any \( h > 0 \). Then the result follows from Theorem 2.1 and Corollary 3.14.

5. **N*D:** If we show that

\[
N^{*,l}_{h_{j}^{-1}}(Y^{h_{j}}) \Rightarrow N^{*,l}(Y) \quad \text{on } D([0, \infty), \mathbb{R}), \tag{31}
\]

30
then the result follows from Proposition 4.4 and Corollaries 3.5 and 2.4. To do so, by Theorem 2.2 we denote by the same notation versions of $Y$ and $Y^{h_j}$ such that $Y^{h_j} \rightarrow Y$ in $D([0,\infty),\mathbb{R})$ ($\mathbb{P}_x$-a.s.). Then, by continuity of $N^{*,l}$, we known that $N^{*,l}(Y^{h_j}) \rightarrow N^{*,l}(Y)$ in $D([0,\infty),\mathbb{R})$. Thus, by the triangle inequality, if we to show that $d_{J_1}(N^{*,l}_{h_j-1}(Y^{h_j}), N^{*,l}(Y^{h_j})) \rightarrow 0$, then (31) follows by Proposition 2.3. And so, observing that $c_j N^{*,l}_{h_j-1}(Y^{h_j}) = N^{*,l}(c_j Y^{h_j})$ pointwise if $c_j = (1 - h_j)^{-1}$, then, using (4) in the second inequality,

$$d_{J_1}(N^{*,l}_{h_j-1}(Y^{h_j}), N^{*,l}(Y^{h_j})) \leq d_{J_1}(N^{*,l}_{h_j-1}(Y^{h_j}), N^{*,l}(Y^{h_j})) + d_{J_1}(N^{*,l}(c_j Y^{h_j}), N^{*,l}(Y^{h_j})) \leq (c_j - 1) + d_{J_1}(N^{*,l}(c_j Y^{h_j}), N^{*,l}(Y^{h_j})),$$

and the second term vanishes by continuity of $N^{*,l}$ and $c_j Y^{h_j} \rightarrow Y$ in $D([0,\infty),\mathbb{R})$ again by triangle inequality and (4).

6. $N^*$: by noting that the convergence (31) holds on $D_{-\infty,1}([0,\infty),\mathbb{R})$, this case follows by Theorem 2.1 and Proposition 4.4.

\[\square\]

**Corollary 5.2.** Assume (H0). Then the process $Y^{N^*N}$ equals the process $[N^{*,l}N^{*,r}](Y)$ in law.

**Proof.** Let $x \in (-1,1)$ and a corresponding sequence $\{h_j : j \in \mathbb{N}\}$ as in Lemma 5.1 and let $Y^{h_j} = x$ for all $j \in \mathbb{N}$ so that $Y^{N^*N,h_j} = [N^{*,l}_{h_j-1}N^{*,r}_{1-h_j}](Y^{h_j})_0 = x$. Then, using Propositions 4.9 and 4.11 with the piecewise constant paths of $Y^{h_j}$, $Y^{N^*N,h_j} = [N^{*,l}_{h_j-1}N^{*,r}_{1-h_j}](Y^{h_j})$ $\mathbb{P}_x$-a.s. and by same trick with multiplication by $c_j = (1 - h_j)^{-1}$ in step 5 of the proof of Lemma 5.1, we obtain that $[N^{*,l}_{h_j-1}N^{*,r}_{1-h_j}](Y^{h_j}) \Rightarrow [N^{*,l}N^{*,r}](Y)_t$, as $j \to \infty$ on $[-1,1]$ for all $t$ outside of a countable set. And so we obtain, by the Dominated Convergence Theorem,

$$\int_0^\infty e^{-\beta t} \mathbb{E}_x[g(Y^{N^*N})_t] dt = \int_0^\infty e^{-\beta t} \mathbb{E}_x[g([N^{*,l}N^{*,r}](Y))_t] dt$$

for and for any $\beta > 0$, $g \in C[-1,1]$ and $x$ in a dense subset of $(-1,1)$. Then, by Proposition 4.2 and [24, Remark 2.21], the resolvents agree everywhere on $[-1,1]$. As for each $x \in [-1,1]$, $t \mapsto \mathbb{E}_x[g(Y^{N^*N})]$ is right continuous (Remark 4.3) and $t \mapsto \mathbb{E}_x[g([N^{*,l}N^{*,r}](Y))_t]$, we obtain that the semigroups agree on $C[-1,1]$ by [55, Lemma 1.1, Chapter I] and we are done. \[\square\]

### 5.2. Pathwise characterisation of backward and forward equations

**Theorem 5.3.** [Pathwise characterisation] Under assumption (H0), the six processes in Table 1 induce Feller semigroups on $C_0(\Omega)$ with backward and forward generators given in Table 1.

**Proof.** By [24, Corollaries 5.2 and 5.4], the operators in the last column of Table 1 generate Feller semigroups on $C_0(\Omega)$ and the operators in the second column equal the restriction to $L^1[-1,1]$ of the respective dual operators on $C_0(\Omega)^*$, the dual space of

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Note that this remark derives its conclusion for the two-sided reflection $[N^{*,l}N^{*,r}](Y)$. 31
implies the following

the conclusion follows by \[24\], Remark 2.21] and Corollary 5.2. For case 3 or 4 in Table 1, denote by \(P\) the Feller semigroup on \(C_0(\Omega)\) generated by \((\partial^\psi, LR)\), and let \(P^h\) be the respective approximating semigroup as constructed in \[24\], Theorem 5.1]. Then, by \[24\], Theorem 5.1], \(P^h g(x) \rightarrow P_t g(x)\) as \(h \rightarrow 0\) for any \(g \in C_0(\Omega), x \in \Omega\) and \(t > 0\). By Proposition 4.8, \(P^h g(x) = E_x[g(Y^{LR,h}_t)]\) for any \(x \in \text{Grid}_h, t > 0\) and \(g \in C_0(\Omega).\) And so we choose \(\{h_j\}_{j \in \mathbb{N}}\) as in Lemma 5.1, and the same lemma implies that as \(j \rightarrow \infty\)

\[
E_x[g(Y^{LR,h}_t)] \rightarrow E_x[g(Y^{LR}_t)] \quad \text{for almost every } t > 0.
\]

Therefore, by the Dominated Convergence Theorem, we obtain that for any \(\beta > 0, \ g \in C_0(\Omega)\) and \(x\) in a dense subset of \(\Omega,\)

\[
\int_0^\infty e^{-\beta t} P_t g(x) \, dt = \int_0^\infty e^{-\beta t} E_x[g(Y^{LR}_t)] \, dt,
\]

and by Proposition 4.2 these resolvents agree for all \(x \in \Omega.\) Then, by Remark 4.3, continuity of \(t \mapsto P_t g(x)\) and \[55\], Lemma 1.1, Chapter I], the semigroups agree on \(C_0(\Omega).\)

In the following corollary \(\Omega_3\) denotes the one-point compactification of \(\Omega\) whenever \(\Omega\) is not compact.

**Corollary 5.4.** Under assumption \[24\], (H1)], if the initial conditions of the Feller processes constructed in \[24\], Lemma 3.14] converge weakly on \(\Omega_3\) to the initial condition of \(Y^{LR}\), then these processes converge weakly to \(Y^{LR}\) on \(D([0, \infty), \Omega_3)\). This is true under (H0) for the cases ND, NN and DN.

**Proof.** Recalling the canonical extension of a Feller process on \(\Omega\) to \(\Omega_3\) (see, e.g., [9], page 12), this is an immediate consequence of Theorem 5.3, \[24\], Theorem 5.1] and \[58\], Theorem 17.25].

Recall that if \(g \in \mathcal{D}\), then \(G g = 0\) if and only if \(P_t g = g\) for all \(t > 0\), where \(P\) is a strongly continuous semigroup with generator \((G, \mathcal{D}).\) Then, from the domain representation of \((\partial^\psi_N, NN)\) in \[24\], Table 4] we see that Theorem 5.3 implies the following corollary.

**Corollary 5.5.** Assume (H0). Then \(\mu(dx) = 2^{-1} dx\) on \([-1, 1]\) is an invariant probability measure for \(Y^{NN}\), i.e. \(P_t [Y^{NN}_t \in dx] = \mu(dx)\) for all \(t > 0.\)

Let us also observe that from the domain representation of \((D^\psi_N, N^*N)\) in \[24\], Table 4) (and recalling \[24\], Remark 2.15] we see that \(dW(y + 1)/W(2)\) is an invariant probability measure for \(Y^{N*N}\), which was proved in \[38\], Theorem 2.ii] (here \(W \in C(0, \infty)\) is the function with Laplace transform \(1/\psi)).

### 5.3. Resolvent measures for one-sided processes

We provide below a new representation for the resolvent measures and an exit problem that involve fast-forwarding. To ease the comparison with \[26\], Theorem 8.11], we state it for the case of fast-forwarding the spectrally negative Lévy process \(-Y.\) We assume that any function \(g \in C[0, a]\) is extended by 0 on \(\mathbb{R} \setminus [0, a]\) upon being convolved, we denote by \(*\) the convolution operator and we define \(I_g(x) = g * 1(x) = \int_0^x g(y) \, dy\)
\((x \geq 0)\) for the constant function \(1 \in C[0, a]\). Define for any \(q \geq 0\) the \textit{operator scale function} \(Z^{(q)}\) acting on \(g \in C[0, a]\) as

\[
Z^{(q)}[g](x) = g(x) + \sum_{n=1}^{\infty} q^n (W*)^n g(x), \quad x \in [0, a],
\]

and we also let \(Z^{(q)}(x) := Z^{(q)}[1](x)\), where \(W\) is the positive non-decreasing function with Laplace transform \(\int_0^\infty e^{-\xi x} W(x) \, dx = \psi(\xi)^{-1}, \Re \xi > 0\), and define

\[
W^{(q)}(x) = W(x) + \sum_{n=1}^{\infty} q^n (W*)^n W(x), \quad x > 0,
\]

extended by 0 on \((-\infty, 0)\) (cf. [27, Theorem 2.1]). Note that \(Z^{(q)}[g]\) defines an absolutely uniformly convergent series, in the sense that for all \(x \in [0, a]\)

\[
\left\| Z^{(q)}[g] \right\|_{[0, x], \infty} \leq \left\| Z^{(q)}[g] \right\|_{[0, x], \infty} \leq \|g\|_{C[0, x]} \left(1 + qW(x) \sum_{n=0}^{\infty} \frac{(qW(x)x)^n}{n!}\right).
\]

**Remark 5.6.** Denoting \(dZ^{(q)}/dx\) by \((Z^{(q)})'\), observe that

\[
qZ^{(q)}[W] = (Z^{(q)})' = qW^{(q)},
\]

and for \(1, g \in C[0, a]\),

\[
qZ^{(q)}[W \ast 1] = qIZ^{(q)}[W] = I(Z^{(q)})' = Z^{(q)} - 1,
\]

and \(IZ^{(q)}[g] = Z^{(q)}[g \ast 1] = g \ast Z^{(q)}\). Also, we recall that \(Z^{(q)} = E^+_{+q}\) and \(W = k_0^+ (-1)\) in the notation of Part I (see [24, Remark 2.15]).

**Corollary 5.7** (Fast-forwarding resolvents). Let \(-Y\) be any recurrent spectrally negative process with paths of unbounded variation and no diffusion component.

(i) Let and \(\tau_0 = \inf\{t > 0 : N^*_a(-Y), g \in (0, a]\}. Then, for any \(q \geq 0\), the \(q\)-resolvent measure on \((0, a]\) at \(x \in (0, a]\) \(N^*_a(-Y)\) killed at \(\tau_0\) is

\[
U_{q,x}^{\text{DN}}(dy) = \left(\frac{W^{(q)}(x)}{Z^{(q)}(a)}\right)Z^{(q)}(a-y) - W^{(q)}(x-y)\right)dy,
\]

and the corresponding exit problem allows the solution

\[
\mathbb{E}_x[e^{-q\tau_0}] = Z^{(q)}(x) - \frac{q}{Z^{(q)}(a)} \int_0^a Z^{(q)}(z) dz W^{(q)}(x).
\]

(ii) For any \(q > 0\), the \(q\)-resolvent measure on \([0, a]\) \(N^*_a(-Y)\) at \(x \in [0, a]\) is

\[
U_{q,x}^{\text{NN}}(dy) = \left(\frac{Z^{(q)}(x)}{q \int_0^a Z^{(q)}(z) dz}\right) Z^{(q)}(a-y) - W^{(q)}(x-y)\right)dy.
\]

**Proof.** Recall that \((H0)\) characterises \(Y\), and so the same strategy of Theorem 5.3 (cf. resolvent for \(D(\partial^\psi_{-X}, \text{ND})\) in [24, Table 4]) proves that for any \(g \in C_0(0, a]\)

\[
\mathbb{E}_x \left[ \int_0^{\tau_0} e^{-qy} g(N^*_a(-Y)_t) dt \right] = Z^{(q)} \left[ \frac{IZ^{(q)}[g](a)}{1 + qZ^{(q)}[W](a)} \right] W - W \ast g(x), \tag{32}
\]
and by the identities in Remark 5.6 the identity for \( U_{q,x}^{DN} \) is proved. The same proof holds for \( U_{q,N}^{NN} \) but using the resolvent for \( D(Q_N^{+}, NN) \) in [24, Table 4] (observing that in this table, for the coefficient \( d, 2 = I_d(\pm 1) \)). For the exit problem, note that for any \( g \in C_0(0, a) \) and \( q > 0, \) (32) holds. Then, by Monotone Convergence Theorem applied to \( 0 \leq g_n \uparrow 1 \) everywhere on \((0, a],\)

\[
\frac{1}{q} - \mathbb{E}_x[e^{-q_0\alpha}] = \frac{IZ_1[(a)(1)]}{1 + qZ_1[IW]\alpha]Z_1[W](x) - Z_1[W*1]|(x)} = \frac{IZ_1[(a)]Z_1[W](x)}{Z_1[(a)]W_1(x)} + 1 - \mathbb{E}_x[e^{-q_0\alpha}],
\]

Example 5.8. In the stable/fractional case \( \phi(dy) = y^{-1-\alpha}/\Gamma(-\alpha), \) \( \alpha \in (1, 2), \) we have \( W(x) = x^{\alpha-1}/\Gamma(\alpha) \) and \( Z_1[(a)](x) = E_{\alpha,1}(qx^{\alpha}), \) where \( E_{\gamma,\beta}(x) = \sum_{n=0}^{\infty} x^n/\Gamma(\gamma n + \beta) \) is the Mittag-Leffler function for two positive parameters \( \gamma \) and \( \beta \) [26, Exercises 8.2.ii and 8.2.iii]. Then Corollary 5.7-(i) implies that for all \( x \in (0, a], \)

\[
\mathbb{E}_x[e^{-\tau_0^{DN}}] = E_{\alpha,1}(qx^{\alpha}) - \frac{q \int_0^\alpha E_{\alpha,1}(qz^{\alpha}) dz}{E_{\alpha,1}(qa^{\alpha})} x^{\alpha-1} E_{\alpha,a}(q \alpha^{\alpha}),
\]

and

\[
\mathbb{E}_x[\tau_0^{DN}] = aW(x) - \int_0^x W(z) dz = a \frac{x^{\alpha-1}}{\Gamma(\alpha)} - \frac{x^{\alpha}}{\Gamma(\alpha + 1)},
\]

where \( \tau_0^{DN} = \tau_{(0,a)}^{N^*}(-Y). \) Instead the average time spent in \((0, a)\) by \( N_{a}^{\ast,-r}(-Y) \) before exiting \((0, a)\) is, by [26, Theorem 8.10.i], of the form

\[
\mathbb{E}_x[\tau_0^{DN*}] = \frac{W(a)}{W'(a)} W(x) - \int_0^x W(z) dz = \frac{a \alpha^{\alpha-1}}{\alpha - 1 \Gamma(\alpha)} - \frac{x^{\alpha}}{\Gamma(\alpha + 1)},
\]

where \( \tau_0^{DN*} = \tau_{(0,a)}^{N_{a}^{\ast,-r}}(-Y). \) Note that for each \( x \in (0, a], \)

\[
\mathbb{E}_x[\tau_0^{DN*}] \rightarrow \infty \text{ meanwhile } \mathbb{E}_x[\tau_0^{DN}] \rightarrow a - x \text{ as } \alpha \downarrow 1,
\]

suggesting that, in applications, modelling fast-forwarding boundary conditions (particles free to move in and out of the domain) incorrectly with Neumann/reflecting boundary conditions can lead to significant prediction errors. For completeness, let us note that by [26, Theorem 8.10.ii],

\[
\mathbb{E}_x[\tau_a^{ND}] = \int_0^a W(z) dz - \int_0^x W(z) dz = a \frac{\alpha^{\alpha}}{\Gamma(\alpha + 1)} - \frac{x^{\alpha}}{\Gamma(\alpha + 1)},
\]

where \( \tau_a^{ND} = \tau_{(0,a)}^{N_{a}^{\ast}}(-Y) \) equals \( \tau_{(0,a)}^{N^*} \) in law.

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Appendix

A. Additional proofs

Proof.[of Proposition 3.10] Recall that the cylinder sets \( \mathcal{A} = \{ \pi_t^{-1}(B_r(x)) : t, r > 0, x \in \mathbb{R} \} \) generate the \( \sigma \)-algebra of \( D([0, \infty), \mathbb{R}) \). Let \( A \in \mathcal{A} \). If \( B_r(x) \cap (a, \infty) = \emptyset \) then \( N_a^{-1}(A) = \emptyset \). Otherwise \( B_r(x) \cap (a, \infty) \neq \emptyset \) and we consider two sub-cases, namely \( a \notin B_r(x) \) and \( a \in B_r(x) \). For the first case we claim that \( N_a^{-1}(A) \) equals the intersection with \( S_{a,\infty} \) of

\[
\bigcup_{r \in \mathbb{Q}^+, \bar{x} \in \mathbb{Q}, n \in \mathbb{N}} \left( \bigcup_{t \leq s \in \mathbb{Q}} \pi_q^{-1}(B_r(\bar{x})) \cap \left\{ f : \lambda(s \in [0, q] : f(s) > a) \in (t, t + 1/n) \right\} \right),
\]

which is a measurable set in \( D_{a,\infty}([0, \infty), \mathbb{R}) \) by (6). To prove this first observe that for a function \( f \in D_{a,\infty}([0, \infty), \mathbb{R}) \) and any \( t > 0 \)

\[
N_a^l(f)(t) \in B_r(x) \\
\iff \exists \bar{t} \geq t : f(\bar{t}) \in B_r(x) \& \lambda(\{ s \in [0, \bar{t}] : f(s) > a \}) = t \\
\iff \exists \bar{x} \in B_r(x), \bar{t} \in \mathbb{Q}^+, \bar{x} \in \mathbb{Q} : \\
\forall n \in \mathbb{N} \exists q \geq t, q \in \mathbb{Q}^+ : f(q) \in B_r(\bar{x}) \& \lambda(\{ s \in [0, q] : f(s) > a \}) \in (t, t + 1/n).
\]

The first \( \iff \) is clear from the definition of \( N_a^l \) and \( a \notin B_r(x) \). To prove the second \( \Rightarrow \) observe that by right continuity there exists \( B_r(\bar{x}) \) and a sequence of rationals \( q_n \downarrow \bar{t} \) such that \( f(q_n) \in B_r(\bar{x}) \) for all \( n \) and then it has to hold that \( \lambda(\{ s \in [0, q_n] : f \geq a \}) \downarrow t \). To prove the second second \( \iff \) observe that we can select a sequence \( q_n \geq t \) such that \( f(q_n) \in B_r(\bar{x}) \) for all \( n \) and

\[
t < \lambda(s \in [0, q_n] : f(s) > a) < t + 1/n.
\]

As \( f \in S_{a,\infty} \) and \( q_n \geq t \), we know \( \{ q_n : n \in \mathbb{N} \} \) is contained in some compact interval \([t, b]\), so that there exists a \( t' \geq t \) such that \( q_{n_j} \to t \) for some subsequence. If there exists a \( q_{n_m} < t' \), then we get a contradiction because

\[
t < \lambda(s \in [0, q_{n_m}] : f(s) > a) \leq \lim_{j \to \infty} \lambda(s \in [0, q_{n_j}] : f(s) > a) = t.
\]

Then we can choose a further subsequence \( q_m \downarrow t' \) and by right continuity \( f(t') \in B_r(\bar{x}) \subset B_r(x) \), and we are done.

It remains to treat the case \( a \in B_r(x) \). In this case we can select \( \bar{r}, \bar{x} \) so that \( N_a^l(f) \in A \) if and only if either \( N_a^l(f)(t) \in B_r(\bar{x}) \subset (a, \infty) \) or \( N_a^l(f)(t) = a \). The set of functions \( f \) that satisfy the first condition are measurable by the same argument as above. We can now conclude if we prove that for each \( t > 0 \) the set \( N_a^{-1}(\pi_t^{-1}(\{a\})) \) equals the intersection of \( S_{a,\infty} \) with

\[
\bigcap_{n, m \in \mathbb{N}} \left( \bigcup_{t \leq q \in \mathbb{Q}} \pi_q^{-1}(B_{1/m}(a)) \right) \cap \left\{ f : \lambda(s \in [0, q] : f(s) > a) \in (t, t + 1/n) \right\}.
\]

So we prove that

\[
N_a^l(f)(t) = a
\]
\[ \Longleftrightarrow \exists t' \geq t : f(t') = a & \lambda(s \in [0, t']) : f(s) > a = t & \forall n \in \mathbb{N} \lambda(s \in [t', t' + 1/n]) : f(s) > a > 0 \]

\[ \Longleftrightarrow \forall n, m \in \mathbb{N} \exists q \in \mathbb{Q}^+: f(q) \in B_{1/m}(a) & \lambda(s \in [0, q]) : f(s) > a > (t, t + 1/n). \]

The first ‘\( \Longleftrightarrow \)’ is immediate due to

\[ \inf \left\{ \varepsilon : \int_0^{t'} 1_{\{f(z) > a\}}dz > \varepsilon \right\} = t' \]

\[ \Longleftrightarrow \int_0^{t'} 1_{\{f(z) > a\}}dz = t & \forall n \in \mathbb{N} \int_{t'}^{t'+1/n} 1_{\{f(z) > a\}}dz > 0. \]

For the second ‘\( \Rightarrow \)’, by right continuity of \( f \) we can take a sequence of rationals \( q_n \downarrow t' \) such that \( f(q_n) \in B_{1/m}(a) \), then, because \( \lim_{m \to \infty} \lambda(s \in [0, q_n]) : f(s) > a = t \), we can find \( n_m \geq m \) such that

\[ t = \lambda(s \in [0, t']) : f(s) > a < \lambda(s \in [0, q_{n_m}]) : f(s) > a < t + 1/n. \]

For the second ‘\( \Leftarrow \)’, select a sequence \( \{q_n : n \in \mathbb{N}\} \) such that \( f(q_n) \in B_{1/n}(a) \) and \( \lambda(s \in [0, q_n]) : f(s) > a \in (t, t + n^{-1}) \). Then, again by recurrence of \( f \), the \( q_n \)'s live in a compact set so take a subsequence \( \{q_{n_i}\} \) converging to some \( t' \). Note that again the existence of one \( q_{n_i} < t' \) leads to a contradiction because

\[ t < \lambda(s \in [0, q_{n_i}] : f(s) > a \leq \lim_{j \to \infty} \lambda(s \in [0, q_{n_j}] : f(s) > a) = t, \]

thus we can select a further subsequence \( q_i \downarrow t' \) so that, as \( i \to \infty \), \( f(q_i) \) converges to \( a \) but also to \( f(t') \) by right continuity, and clearly for any \( \epsilon > 0 \) there exists a \( q_i \) \( \epsilon \)-close to \( t' \) implying

\[ \lambda(s \in [t', t' + \epsilon] : f(s) > a \geq \lambda(s \in [t', t' + q_i] : f(s) > a) > 0. \]

**Proof.**[of Proposition 3.16] The two maps are clearly well-defined and measurability is immediate as they are the composition of measurable maps by Proposition 3.10. Now observe that for any \( x \in (a, b) \) there exists \( t > 0 \) with

\[ N_{\varepsilon}(N_x(f))(t) = f \left( (A_b^t)^{-1} \left( (A_b^{f((A_b^t)^{-1})})^{-1} (t) \right) \right) = x \]

if and only if there exist \( t, t' \geq 0 \) such that \( f(t') = x \) and

\[ (A_b^t)^{-1} \left( (A_b^{f((A_b^t)^{-1})})^{-1} (t) \right) = t', \quad (A.1) \]

where we simplified notation by writing \( (A_b^t)^{-1} \) for the right inverse of \( s \mapsto A_b^t(s) := \int_0^s 1_{\{f(z) > a\}}dz \) and \( (A_b^{f((A_b^t)^{-1})})^{-1} \) for the right inverse of \( s \mapsto \int_0^s 1_{\{f((A_b^t)^{-1}(z)) < b\}}dz \). Equation \( (A.1) \) holds if and only if there exist \( t, t' \geq 0 \) such that \( f(t') = x \) and

\[ (A_b^{f((A_b^t)^{-1})})^{-1} (t) = A_b^t(t') \quad \text{and} \quad \int_{t'}^{t'+1/n} 1_{\{f(z) > a\}}dz > 0 \quad \forall n \in \mathbb{N}, \]

which in turn holds if and only if there exist \( t, t' \geq 0 \) such that \( f(t') = x \) and

\[ t = \int_0^{A_b^t(t')} 1_{\{f((A_b^t)^{-1}(z)) < b\}}dz \quad \text{and} \quad \int_{A_b^t(t')}^{A_b^t(t') + 1/n} 1_{\{f((A_b^t)^{-1}(z)) < b\}}dz > 0 \quad \forall n \in \mathbb{N}, \quad (A.2) \]

36
Then the above function is measurable by standard results in measure theory. Let $(a,b)$ be an interval.

Finally, to show the paths agree at times when they equal $a$, observe that $N_b^n(a,b) = f((A_{(a,b)}^t)^{-1})$. Clearly the same argument above proves that $N_b^n(a,b) = f((A_{(a,b)}^t)^{-1})$ on $(a,b)$.}

Proof. The first identity in (A.2) equals $t = \int_0^t 1_{(a < f(z) < b)} \, dz$, meanwhile the inequality can be rewritten for all large $n$ as

$$0 < \int_{A_{(a,b)}^t(a')} \, 1_{(f((A_{(a,b)}^t)^{-1}) < b)} \, dz$$

$$= \int_{a'}^{t' + 1/n} \, 1_{(f((A_{(a,b)}^t)^{-1}) < b)} \, dz$$

$$= \int_{a'}^{t' + 1/n} \, 1_{(a < f(z) < b)} \, dz,$$

because for all small $\epsilon > 0$ we have $f(z) > a$ for all $z \in \{t', t' + \epsilon\}$ which implies $(A_{(a,b)}^t)^{-1}(A_{(a,b)}^t(t') + \epsilon) = (A_{(a,b)}^t)^{-1}(A_{(a,b)}^t(t' + \epsilon)) = t' + \epsilon$. And so we proved that on $(a,b)$

$$N_b^n(f) = f((A_{(a,b)}^t)^{-1}).$$

Clearly the same argument above proves that $N_b^n(N_b^n(f)) = f((A_{(a,b)}^t)^{-1})$ on $(a,b)$. Finally, to show the paths agree at times when they equal $a$ it is now enough to observe that $N_b^n(N_b^n(f))(t) = a$ implies that there exists a sequence of decreasing times $t_n \downarrow t$ such that $N_b^n(N_b^n(f))(t_n) = f((A_{(a,b)}^t)^{-1})(t_n) > a$ and we can conclude by right continuity of the fast-forwarded paths. The exact same argument holds for $b$ and we are done.

Proof. [Measurability of (6)] We first prove that $(f, t) \mapsto \pi_t(f)$ is $\sigma(D, B(\mathbb{R}^d)) \setminus B(\mathbb{R})$ measurable. To do we first show that for any $\epsilon > 0$ the function

$$(f, t) \mapsto h_\epsilon(f, t) = \frac{1}{\epsilon} \int_t^{t+\epsilon} f(s) \, ds,$$

is continuous. Let $(f_n, t_n) \to (f, t)$. Then

$$\epsilon |h_\epsilon(f_n, t_n) - h_\epsilon(f, t)| = \left| \int_{t_n}^{t_n+\epsilon} f_n(s) \, ds - \int_t^{t+\epsilon} f(s) \, ds \right|$$

$$= \left| \int_{t_n}^{t_n+\epsilon} f_n(s) \, ds - \int_t^{t+\epsilon} f_n(s) \, ds + \int_t^{t+\epsilon} f_n(s) - f(s) \, ds \right|$$

$$\leq \sup_n \|f_n\|_{[t-\epsilon, t+\epsilon]} 2|t_n - t| + \int_t^{t+\epsilon} |f_n(s) - f(s)| \, ds$$

which vanishes because convergence in $D([0, \infty), \mathbb{R})$ implies convergence almost everywhere and the sequence must be uniformly bounded. By the right continuity of $\epsilon |h_\epsilon(f, t) \to \pi_t(f)$ as $\epsilon \to 0$, and thus $(f, t) \mapsto \pi_t(f)$ is measurable. Now we write

$$f \mapsto \lambda(t \in [0, \infty) : f(t) \in B) = \int_0^\infty 1_{(\pi_t(f) \in B)} \, dt = \int_0^\infty \varphi(f, t) \, dt.$$

Then the above function is measurable by standard results in measure theory [59, Chapter 7] as $\varphi : D([0, \infty), \mathbb{R}) \times [0, \infty) \to \mathbb{R}$ is a non-negative bounded measurable function.
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