A characterization of the Maass space on $O(2, m + 2)$ by symmetries

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In this paper, we define certain symmetries for automorphic forms on $O(2, m + 2)$ and show that the space of automorphic forms satisfying these symmetries coincides with the Maass space, the image of Saito-Kurokawa lifting.

Keywords: Automorphic forms; Theta lifting; Maass space

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1 Introduction

In [3], the first named author introduced certain symmetries for Siegel modular forms of even degree and showed that the space of Siegel modular forms of degree two satisfying the symmetries coincides with the so-called Maass Spezialschar, which is the space of Siegel modular forms whose Fourier coefficients satisfy Maass relations. Note that this space coincides with the image of Saito-Kurokawa lifting. Later Bringmann and Heim proved certain symmetries for Jacobi Eisenstein series of degree two ([1]).

On the other hand, Oda ([5]) and Rallis-Schiffmann ([6]) independently studied a theta lifting from elliptic modular forms of integral or half-integral weight to automorphic forms on the orthogonal group $O(2, m + 2)$. Note that, if $m = 1$, the theta lifting coincides with the Saito-Kurokawa lifting. Later Gritsenko ([2]) and Sugano ([7]) studied the theta lifting in terms of Jacobi forms of degree 1. It is known that the image of the theta lifting coincides with the space of holomorphic automorphic forms whose Fourier coefficients satisfy the Maass relation ([7]). Thus it is natural to ask whether certain symmetries characterizes the Maass space in the general orthogonal group case. The object of the paper is to give an affirmative answer to this question.

In this paper, we introduce symmetries of automorphic forms on $G = O(2, m + 2)$ arising from two embeddings of $SL_2$ into $G$, and show that the space of holomorphic automorphic forms on $G$ satisfying the symmetries coincides with the Maass space.

The paper is organized as follows. In Section 2, we first recall the definitions of automorphic forms on $G = O(2, m + 2)$ and the Maass space. After defining certain symmetries for automorphic forms on $G$, we state the main result of the paper (Theorem 2.2): The space of automorphic forms satisfying these symmetries coincides with the space of those satisfying Maass relations. As a direct consequence of the characterization of the Maass space by symmetries, we show that the restriction mapping induced by an embedding $G' = O(2, m + 1) \hookrightarrow G = O(2, m + 2)$ maps the Maass space on $G$ to that on $G'$. The proof of Theorem 2.2 is carried out in Section 3. By

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using some combinatorics, we prove an algebraic result (Proposition 3.2), from which Theorem 2.2 follows.

Notation

The upper half plane is denoted by \( \mathcal{H} = \{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \} \). For a real symmetric matrix \( R \) of degree \( n \), we put \( R(x, y) = x^t R y \) and \( R(x) = x^t R x \) for \( x, y \in \mathbb{C}^n \). For a condition \( P \), we put
\[
\delta(P) = \begin{cases} 
1 & \text{if } P \text{ holds}, \\
0 & \text{otherwise}.
\end{cases}
\]
Denote by \( \mathbb{N} \) the set of natural numbers. We put \( e[z] = e^{2\pi \sqrt{-1} z} \) for \( z \in \mathbb{C} \).

2 Main results

2.1 The orthogonal group \( G \)

Let \( S \) be a positive definite even integral symmetric matrix of degree \( m \). We put
\[
Q_1 = \begin{pmatrix} 1 & -S \cr -S & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 \\
0 & Q_1 \end{pmatrix}.
\]
In the following, we include the case of \( m = 0 \). Note that the signatures of \( Q_1 \) and \( Q_2 \) are \((1, m + 1)\) and \((2, m + 2)\), respectively.

Let\[
L_0 = Z^m, L_0^* = S^{-1} L_1, V_0 = L_0 \otimes \mathbb{Z} Q = \mathbb{Q}^m, \\
L_1 = Z^{m+2}, L_1^* = Q_1^{-1} L_1, V_1 = L_1 \otimes \mathbb{Z} Q = \mathbb{Q}^{m+2}, \\
L = Z^{m+4}, L^* = Q^{-1} L, V = L \otimes \mathbb{Z} Q = \mathbb{Q}^{m+4}.
\]

Let \( G = O(Q) \) be the orthogonal group of \( Q \) and \( G_\infty^+ \) the identity component of \( G_\infty = G(\mathbb{R}) \).

Let
\[
\mathcal{D} = \left\{ Z = \begin{pmatrix} \tau & w \\
w & z \end{pmatrix} \in \mathbb{C}^{m+2} \mid \tau, z \in \mathcal{H}, w \in \mathbb{C}^m, Q_1[\text{Im}(Z)] = \text{Im}(\tau)\text{Im}(z) - \frac{1}{2} S[\text{Im}(w)] > 0 \right\}.
\]
As is well-known, \( \mathcal{D} \) is a hermitian symmetric domain of type (IV). We often write \((\tau, w, z)\) for \( \left( \begin{array}{c} \tau \\
w \\
z \end{array} \right) \in \mathcal{D} \). We define an action of \( G_\infty^+ \) on \( \mathcal{D} \) and an automorphic factor \( J: G_\infty^+ \times \mathcal{D} \to \mathbb{C}^\times \) by \( g \widetilde{Z} = g(Z) J(g, Z) \) for \( g \in G_\infty^+ \) and \( Z \in \mathcal{D} \), where
\[
\widetilde{Z} = \begin{pmatrix} -2^{-1} Q_1[Z] \\
Z \\
1 \end{pmatrix} \in \mathbb{C}^{m+4}.
\]
Let $k$ be an integer and $F$ a function on $\mathcal{D}$. For $g \in G^+_{\infty}$, we define the Petersson slash operator by $(F|_k g)(Z) = J(g, Z)^{-k} F(g\langle Z \rangle)$.

### 2.2 Embeddings of $\text{SL}_2$ into $G$

Let $H = \text{SL}_2$. For $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H_{\infty}$ and $z \in \mathcal{S}$, let $h(z) = (az + b)(cz + d)^{-1}$ and $j(h, z) = cz + d$ as usual. We define two embeddings $\iota^\uparrow$ and $\iota^\downarrow$ of $H$ into $G$ by

$$
\iota^\uparrow(h) = \begin{pmatrix} a & -b \\ a & b \\ 1_m & d \\ c & d \end{pmatrix}, \\
\iota^\downarrow(h) = \begin{pmatrix} a & -b \\ -c & d \\ 1_m & a & b \\ c & d \end{pmatrix}
$$

for $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H$, respectively. It is easily verified that $\iota^\uparrow(h)$ and $\iota^\downarrow(h)$ commute each other and that $\iota^\uparrow(H_{\infty}), \iota^\downarrow(H_{\infty}) \subset G^+_{\infty}$. A straightforward calculation shows the following.

**Lemma 2.1.** For $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H_{\infty}$ and $Z = (\tau, w, z) \in \mathcal{D}$, we have

$$
\iota^\uparrow(h)\langle Z \rangle = \begin{pmatrix} h\langle \tau \rangle \\ j(h, \tau)^{-1}w \\ z - \frac{c}{2j(h, \tau)} S[w] \end{pmatrix}, \quad J(\iota^\uparrow(h), Z) = j(h, \tau)
$$

and

$$
\iota^\downarrow(h)\langle Z \rangle = \begin{pmatrix} \tau - \frac{c}{2j(h, z)} S[w] \\ j(h, z)^{-1}w \\ h\langle z \rangle \end{pmatrix}, \quad J(\iota^\downarrow(h), Z) = j(h, z).
$$

### 2.3 Automorphic forms

Let $\Gamma$ be a discrete subgroup of $G_{\infty}$ commensurable with $\Gamma(L) = \{ \gamma \in G^+_{\infty} \mid \gamma L = L \}$. We assume that

$$
(2.1) \quad \begin{pmatrix} 1 & -xQ_1 \\ 0 & 1_{m+2} \\ 0 & 0 & 1 \end{pmatrix}, \quad \iota^\uparrow(\gamma), \iota^\downarrow(\gamma') \in \Gamma \quad (x \in \mathbb{Z}^{m+2}, \gamma, \gamma' \in \text{SL}_2(\mathbb{Z})).
$$
Note that \( \Gamma(L) \) and \( \Gamma^*(L) = \{ \gamma \in \Gamma(L) \mid \gamma l \equiv l \pmod{L} \} \) for any \( l \in L^* \) satisfy this condition.

For a positive integer \( k \), let \( M_k(\Gamma) \) denote the space of holomorphic functions \( F \) on \( D \) satisfying the following two conditions:

\[(2.2)\quad F|_{k\gamma} = F \quad \text{for any} \quad \gamma \in \Gamma.\]
\[(2.3)\quad \text{If} \quad m = 0, \quad F \text{ is holomorphic at any cusp of} \quad \Gamma.\]

Let
\[(2.4)\quad \Lambda = \mathbb{Z} \times L_0^* \times \mathbb{Z},\]
\[(2.5)\quad \Lambda^+ = \{(a, \alpha, b) \in \Lambda \mid a, b, 2ab - S[\alpha] \geq 0\}.

An automorphic form \( F \in M_k(\Gamma) \) admits the Fourier expansion
\[F(\tau, w, z) = \sum_{(a, \alpha, b) \in \Lambda^+} A_F(a, \alpha, b) e[az - S(\alpha, w) + br].\]

We say that \( \lambda = (a, \alpha, b) \in \Lambda \) is primitive if \( (n^{-1}a, n^{-1}\alpha, n^{-1}b) \notin \Lambda \) for any \( n \in \mathbb{N}, n > 1 \). Denote by \( \Lambda_{\text{prm}} \) (respectively \( \Lambda_{\text{prm}}^+ \)) the set of primitive elements of \( \Lambda \) (respectively \( \Lambda^+ \)).

### 2.4 The Maass space and symmetries

We now define two subspaces of \( M_k(\Gamma) \).

Let \( M_k^M(\Gamma) \) be the space of \( F \in M_k(\Gamma) \) satisfying
\[(2.6)\quad A_F(la, l\alpha, lb) = \sum_{r|l} r^{k-1} A_F((r^{-1}l)^2ab, (r^{-1}l)\alpha, 1)\]
for any \( l \in \mathbb{N} \) and \( (a, \alpha, b) \in \Lambda_{\text{prm}}^+ \), where \( r \) runs over the positive divisors of \( l \). Note that
\[A_F(a, \alpha, b) = \sum_{d \in \mathbb{Z}_{>0}, d^{-1}(a, \alpha, b) \in \Lambda} d^{k-1} A_F \left( \frac{ab}{d^2}, \frac{\alpha}{d}, 1 \right)\]
for \( F \in M_k^M(\Gamma) \). When \( \Gamma = \Gamma^*(L) \), this space coincides with the Maass space introduced by Maass ([4]) when \( m = 1 \) and Sugano ([7]) when \( m > 1 \).

To define symmetries, let
\[T_n = \{ \xi \in M_2(\mathbb{Z}) \mid \det \xi = n \} = \bigcup_j SL_2(\mathbb{Z})\xi_j \quad \text{(a disjoint union)}\]
for \( n \in \mathbb{N} \). Define
\[F|_{kT_n^+} = n^{k/2-1} \sum_j F|_{k^j}(n^{-1/2}\xi_j),\]
\[F|_{kT_n^-} = n^{k/2-1} \sum_j F|_{k^j}(n^{-1/2}\xi_j)\]
for \( F \in M_k(\Gamma) \). Note that \( F|_k T^\uparrow_n \) and \( F|_k T^\downarrow_n \) are not in \( M_k(\Gamma) \) in general. We define the space \( M^S_k(\Gamma) \) to be the space of \( F \in M_k(\Gamma) \) satisfying \( F|_k T^\uparrow_n = F|_k T^\downarrow_n \) for any \( n \in \mathbb{N} \). It is easy to see that \( F \in M^S_k(\Gamma) \) if and only if \( F|_k T^\uparrow_p = F|_k T^\downarrow_p \) for any prime number \( p \). Observe that, for a prime number \( p \), we have

\[
(F|_k T^\uparrow_p)(\tau, w, z) = p^{k-1} F(p\tau, \sqrt{pw}, z) + p^{-1} \sum_{c=0}^{p-1} F(p^{-1}(\tau + c), \sqrt{p}^{-1}w, z)
\]

\[
(F|_k T^\downarrow_p)(\tau, w, z) = p^{k-1} F(\tau, \sqrt{pw}, pz) + p^{-1} \sum_{c=0}^{p-1} F(\tau, \sqrt{p}^{-1}w, p^{-1}(z + c))
\]

The main result of the paper is stated as follows.

**Theorem 2.2.** The Maass space \( M^M_k(\Gamma) \) coincides with \( M^S_k(\Gamma) \).

### 2.5 The compatibility with restrictions

Let \((L'_0, S')\) be a quadratic sub-lattice of \((L_0, S)\). Let \(Q'_1, Q', G'\) and \(D'\) be as in 2.1 corresponding to \(S'\). We assume that the inverse image \( \Gamma' \) of \( \Gamma \) by the embedding \( G' \subset G \) satisfies a condition similar to (2.1). Then the restriction of \( F \in M_k(\Gamma) \) to \( D' \) gives rise to a linear mapping \( j: M_k(\Gamma) \to M_k(\Gamma') \). Since the symmetry is compatible with \( j \), we have proved the following.

**Theorem 2.3.** We have \( j(M^S_k(\Gamma)) \subset M^S_k(\Gamma') \) and hence \( j(M^M_k(\Gamma)) \subset M^M_k(\Gamma') \).

### 3 Proof of Theorem 2.2

#### 3.1 Symmetries and Fourier expansion

**Lemma 3.1.** Let \( F \in M_k(\Gamma) \). Then \( F \in M^S_k(\Gamma) \) if and only if the following holds for any \((a, b, \alpha) \in \Lambda^+\) and any prime number \( p \):

\[
p^{k-1} A_F(a, p^{-1}a, p^{-1}b) - p^{k-1} A_F(p^{-1}a, p^{-1} \alpha, b) + A_F(a, \alpha, pb) - A_F(pa, \alpha, b) = 0.
\]

Here we make a convention that \( A_F(a, \alpha, b) = 0 \) if \((a, \alpha, b) \not\in \Lambda^+\).

**Proof.** We have

\[
(F|_k T^\uparrow_p)(\tau, \sqrt{pw}, z) = p^{k-1} F(p\tau, pw, z) + p^{-1} \sum_{c=0}^{p-1} F(p^{-1}(\tau + c), w, z)
\]

\[
= p^{k-1} \sum_{(a, \alpha, b) \in \Lambda^+} A_F(a, p^{-1}a, p^{-1}b) e[b\tau + az - S(\alpha, w)]
\]

\[
+ \sum_{(a, \alpha, b) \in \Lambda^+} A_F(a, \alpha, pb) e[b\tau + az - S(\alpha, w)]
\]

Theorem 2.2. The Maass space \( M^M_k(\Gamma) \) coincides with \( M^S_k(\Gamma) \).
(F|T_k)(τ, \sqrt{pw}, z) = p^{k-1} \sum_{(a,\alpha,b)\in\Lambda^+} A_F(p^{-1}a,p^{-1}\alpha,b)e[br + az - S(\alpha, w)]
+ \sum_{(a,\alpha,b)\in\Lambda^+} A_F(pa,\alpha,b)e[br + az - S(\alpha, w)],

from which the lemma immediately follows. \qed

3.2 The spaces $F^M$ and $F^S$

For a function $f$ on $V = \mathbb{Q} \times V_0 \times \mathbb{Q}$ and $r \in \mathbb{N}$, we put

$$M(r)f(a, \alpha, b) = f(r^2ab, r\alpha, 1),$$
$$N(r)f(a, \alpha, b) = f(ra, r\alpha, rb) \quad ((a, \alpha, b) \in V).$$

It is easy to see that

$$(3.1) \quad M(r)f(a, \alpha, b) = M(r)f(ma, \alpha, m^{-1}b),$$
$$M(mr)f(a, \alpha, b) = M(r)f(m^2a, m\alpha, b) = M(r)f(a, m\alpha, m^2b)$$

for $r, m \in \mathbb{N}$. Let $F$ be the space of functions on $V$ whose support is contained in $\Lambda$. For $f \in F$, a prime number $p$ and $(a, \alpha, b) \in V$, we set

$$I_p f(a, \alpha, b) = p^{k-1}f(a, p^{-1}\alpha, p^{-1}b) - p^{k-1}f(p^{-1}a, p^{-1}\alpha, b) + f(a, \alpha, pb) - f(pa, \alpha, b).$$

We define two subspaces of $F$ as follows:

$$F^M = \{ f \in F \mid N(l)f(X) = \sum_{r|l} r^{k-1}M(r^{-1}l)f(X) \text{ for any } l \in \mathbb{N} \text{ and } X \in \Lambda_{\text{prim}} \},$$
$$F^S = \{ f \in F \mid I_p f(X) = 0 \text{ for any prime number } p \text{ and } X \in \Lambda \}.$$

Let $F \in M_k(\Gamma)$ and consider $A_F$ as an element of $F$. In view of (2.6) and Lemma 3.1, we see that $F \in M^M_k(\Gamma)$ if and only if $A_F \in F^M$ and that $F \in M^S_k(\Gamma)$ if and only if $A_F \in F^S$. Thus the proof of Theorem 2.2 is now reduced to that of the following result.

**Proposition 3.2.** We have $F^M = F^S$.

3.3 Proof of $F^M \subset F^S$

In this subsection, we let $p$ be a prime number and $f \in F^M$, and show that $I_p f(X) = 0$ for any $X \in \Lambda$.

**Lemma 3.3.** Let $l = p^sn \in \mathbb{N}$ with $s \geq 0, n \in \mathbb{N}, p \nmid n$. For $X \in \Lambda_{\text{prim}}$, we have

$$N(pl)f(X) - p^{k-1}N(l)f(X) = \sum_{r|n} r^{k-1}M(pr^{-1}l)f(X).$$
Proof. The left-hand side of (3.3) is equal to
\[
\sum_{r \mid p^{s+1}n} M(r^{-1}p^{s+1}n)f(X) - p^{k-1} \sum_{r \mid p^sn} M(r^{-1}p^sn)f(X)
\]
\[
= \sum_{j=0}^{s+1} \sum_{r \mid n} (p^jr)^{k-1}M(p^sjr^{-1}n)f(X) - \sum_{j=0}^{s} \sum_{r \mid n} (p^jr)^{k-1}M(p^sjr^{-1}n)f(X)
\]
\[
= \sum_{r \mid n} r^{k-1}M(p^sr^{-1}n)f(X),
\]
which proves the lemma. \(\square\)

Let \(X \in \Lambda\). Then \(X = (lc, l\beta, ld)\) with \(l = p^sn \in \mathbb{N} (s \geq 0, n \in \mathbb{N}, p \nmid n)\) and \((c, \beta, d) \in \Lambda_{\text{prim}}\). To simplify the notation, we write \(I\) for \(I_p f(X)\). We have
\[
I = p^{k-1}N(l)f(c, p^{-1}\beta, p^{-1}d) - N(l)f(pc, \beta, d)
- p^{k-1}N(l)f(p^{-1}c, p^{-1}\beta, d) + N(l)f(c, \beta, pd).
\]

First consider the case where \(\beta \in L_0^* - pL_0^*\). Then \((pc, \beta, d), (c, \beta, pd) \in \Lambda_{\text{prim}}\) and
\[
M(r^{-1}l)f(pc, \beta, d) = M(r^{-1}l)f(c, \beta, pd)
\]
for \(r \mid l\). Suppose that \(s = 0\). Then we have
\[
N(l)f(c, p^{-1}\beta, p^{-1}d) = N(l)f(p^{-1}c, p^{-1}\beta, d) = 0,
\]
since \(p^{-1}l\beta \notin L_0^*\). It follows that
\[
I = -N(l)f(pc, \beta, d) + N(l)f(c, \beta, pd)
= - \sum_{r \mid l} r^{k-1}M(r^{-1}l)f(pc, \beta, d) + \sum_{r \mid l} r^{k-1}M(r^{-1}l)f(c, \beta, pd)
= 0.
\]

Next suppose that \(s > 0\). By Lemma 3.3 we have
\[
I = p^{k-1}N(p^{-1}l)f(pc, \beta, d) - N(l)(pc, \beta, d) - p^{k-1}N(p^{-1}l)(c, \beta, pd) + N(l)f(c, \beta, pd)
= - \sum_{r \mid n} r^{k-1}M(r^{-1}l)f(pc, \beta, d) + \sum_{r \mid n} r^{k-1}M(r^{-1}l)f(c, \beta, pd)
= 0.
\]

Next consider the case where \(\beta \in pL_0^*, p\mid c\) and \(p \nmid d\). Then \((pc, \beta, d), (p^{-1}c, p^{-1}\beta, d) \in \Lambda_{\text{prim}}\) and
\[
M(r^{-1}l)f(pc, \beta, d) = M(pr^{-1}l)f(p^{-1}c, p^{-1}\beta, d).
\]
First suppose that $s = 0$. Then $N(l)f(c, p^{-1}\beta, p^{-1}d) = 0$ since $p^{-1}ld \notin \mathbb{Z}$. By Lemma 3.3, we have

$$I = -N(l)f(pc, \beta, d) - p^{-1}N(l)f(pc, \beta, d) + N(pl)f(pc, \beta, d)$$

$$= -\sum_{r|l} r^{-1}M(r^{-1}l)f(pc, \beta, d) + \sum_{r|l} r^{-1}M(r^{-1}l)f(pc, \beta, d)$$

$$= 0.$$ 

If $s > 0$, we have

$$I = p^{-1}N(l)(pc, \beta, d) - N(l)f(pc, \beta, d)$$

$$- p^{-1}N(l)(pc, \beta, d) + N(pl)f(pc, \beta, d)$$

$$= -\sum_{r|n} r^{-1}M(r^{-1}l)f(pc, \beta, d) + \sum_{r|n} r^{-1}M(r^{-1}l)f(pc, \beta, d)$$

$$= 0$$

by Lemma 3.4.

We can show that $I = 0$ in the other cases in a similar way.

### 3.4 Proof of $\mathcal{F}^S \subset \mathcal{F}^M$

Let $f \in \mathcal{F}^S$. We will show that

$$(3.4) \quad N(l)f(X) = \sum_{r|l} r^{-1}M(r^{-1}l)f(X)$$

holds for $l \in \mathbb{N}$ and $X = (a, \alpha, b) \in \Lambda_{\text{prim}}$ by induction on $bl$. If $bl = 1$, both sides of (3.4) are equal to $f(X)$. Suppose that $bl > 1$ and let $p$ be a prime factor of $bl$. Then we have

$$N(l)f(X) = N(l)f(pa, \alpha, p^{-1}b) + p^{-1}N(l)f(a, \alpha, b) - p^{-1}N(l)f(a, p^{-1}\alpha, p^{-2}b).$$

First consider the case where $p \nmid l$. Then $b$ is divisible by $p$. Since $(p^{-1}la, p^{-1}l\alpha, p^{-1}lb) \notin \Lambda$, we have $N(p^{-1}l)f(a, \alpha, b) = 0$. Suppose that $\alpha \in L^*_0 - pL^*_0$. Then $(pa, \alpha, p^{-1}b) \in \Lambda_{\text{prim}}$ and $N(l)f(a, p^{-1}\alpha, p^{-2}b) = 0$. By induction, we have

$$N(l)f(X) = N(l)f(pa, \alpha, p^{-1}b) = \sum_{r|l} r^{-1}M(r^{-1}l)f(pa, \alpha, p^{-1}b) = \sum_{r|l} r^{-1}M(r^{-1}l)f(a, \alpha, b),$$

which proves the claim (3.4). Next suppose that $\alpha \in pL^*_0$ and $\text{ord}_p b = 1$. A similar argument as above shows that

$$N(l)f(X) = N(l)f(pa, \alpha, p^{-1}b) = \sum_{r|l} r^{-1}M(r^{-1}l)f(pa, \alpha, p^{-1}b) = \sum_{r|l} r^{-1}M(r^{-1}l)f(a, \alpha, b).$$
Suppose that $\alpha \in pL_0^*$ and $\text{ord}_p b \geq 2$. Then $p \nmid a$ and $(a, p^{-1} \alpha, p^{-2}b) \in \Lambda_{\text{prim}}$. By induction, we have

$$N(l)f(X) = N(pl)f(a, p^{-1} \alpha, p^{-2}b) - p^{k-1}N(l)f(a, p^{-1} \alpha, p^{-2}b)$$

$$= \sum_{r \mid l} r^{k-1}M(pr^{-1}l)f(a, p^{-1} \alpha, p^{-2}b) + \sum_{r \mid l} (pr)^{k-1}M(p(pr)^{-1}l)f(a, p^{-1} \alpha, p^{-2}b)$$

$$- p^{k-1} \sum_{r \mid l} r^{k-1}M(r^{-1}l)f(a, p^{-1} \alpha, p^{-2}b)$$

$$= \sum_{r \mid l} r^{k-1}M(pr^{-1}l)f(a, p^{-1} \alpha, p^{-2}b)$$

$$= \sum_{r \mid l} r^{k-1}M(r^{-1}l)f(a, \alpha, b).$$

Next consider the case where $p \mid l$. We let $l = p^s n$ with $s \geq 1, s \in \mathbb{N}$ and $p \nmid n$. First suppose that $p \nmid b$. Then $(p^2a, pa, b) \in \Lambda_{\text{prim}}$. By induction, $N(l)f(X)$ is equal to

$$N(p^{-1}l)f(p^2a, pa, b) + p^{k-1}N(p^{-1}l)f(a, \alpha, b) - p^{k-1}\delta(s \geq 2)N(p^{-2}l)f(p^2a, pa, b)$$

$$= \sum_{j=0}^{s-2} \sum_{r \mid n} (p^j r)^{k-1}M(p^{s-j-1}r^{-1}n)f(p^2a, pa, b) + \sum_{j=0}^{s-1} \sum_{r \mid n} (p^{j+1} r)^{k-1}M(p^{s-j-2}r^{-1}n)f(a, \alpha, b)$$

$$- \delta(s \geq 2) \sum_{j=0}^{s-2} \sum_{r \mid n} (p^j r)^{k-1}M(p^{s-j-2}r^{-1}n)f(p^2a, pa, b)$$

$$= \sum_{j=0}^{s-1} \sum_{r \mid n} (p^j r)^{k-1}M(p^{s-j}r^{-1}n)f(a, \alpha, b) + (p^s r)^{k-1}M(r^{-1}n)f(a, \alpha, b)$$

$$= \sum_{j=0}^{s} \sum_{r \mid n} (p^j r)^{k-1}M(p^{s-j}r^{-1}n)f(a, \alpha, b)$$

$$= \sum_{r \mid l} r^{k-1}M(r^{-1}l)f(a, \alpha, b).$$

Suppose that either $\text{ord}_p b = 1$ or “$\text{ord}_p b \geq 2$ and $\alpha \in L_0^* - pL_0^*$” holds. Then $(pa, \alpha, p^{-1}b) \in \Lambda_{\text{prim}}$. By induction, $N(l)f(X)$ is equal to

$$N(l)f(pa, \alpha, p^{-1}b) + p^{k-1}N(p^{-1}l)f(a, \alpha, b) - p^{k-1}N(p^{-1}l)f(pa, \alpha, p^{-1}b)$$

$$= \sum_{r \mid l} r^{k-1}M(r^{-1}l)f(pa, \alpha, p^{-1}b) + \sum_{r \mid p^{-1}l} (pr)^{k-1}M(p^{-1}r^{-1}l)f(a, \alpha, b)$$

$$- \sum_{r \mid p^{-1}l} (pr)^{k-1}M(p^{-1}r^{-1}l)f(pa, \alpha, p^{-1}b)$$

$$= \sum_{r \mid l} r^{k-1}M(r^{-1}l)f(a, \alpha, b).$$

Finally suppose that $\text{ord}_p b \geq 2$ and $\alpha \in pL_0^*$. Then $p \nmid a$ and $(a, p^{-1} \alpha, p^{-2}b) \in \Lambda_{\text{prim}}$. By
induction, $N(l)f(X)$ is equal to

\[
N(pl)f(a, p^{-1} \alpha, p^{-2}b) + p^{k-1}N(p^{-1}l)f(a, \alpha, b) - p^{k-1}N(l)f(a, p^{-1} \alpha, p^{-2}b) \\
= \sum_{r|pl} r^{k-1}M(pr^{-1}l)f(a, p^{-1} \alpha, p^{-2}b) + \sum_{r|p^{-1}l} (pr)^{k-1}M(p^{-1}r^{-1}l)f(a, \alpha, b) \\
- \sum_{r|l} (pr)^{k-1}M(r^{-1}l)f(a, p^{-1} \alpha, p^{-2}b) \\
= \sum_{j=0}^{s+1} \sum_{r|n} (p^j r)^{k-1}M(p^{s-j+1}r^{-1}n)f(a, p^{-1} \alpha, p^{-2}b) + \sum_{j=0}^{s-1} \sum_{r|n} (p^{j+1} r)^{k-1}M(p^{s-j-1}r^{-1}n)f(a, \alpha, b) \\
- \sum_{j=0}^{s} \sum_{r|n} (p^{j+1} r)^{k-1}M(p^{s-j-1}r^{-1}n)f(a, p^{-1} \alpha, p^{-2}b) \\
= \sum_{r|n} r^{k-1}M(p^{s+1}r^{-1}n)f(a, p^{-1} \alpha, p^{-2}b) + \sum_{j=0}^{s-1} \sum_{r|n} (p^{j+1} r)^{k-1}M(p^{s-j-1}r^{-1}n)f(a, \alpha, b) \\
= \sum_{j=0}^{s} \sum_{r|n} (p^{j} r)^{k-1}M(p^{s-j}r^{-1}n)f(a, \alpha, b) \\
= \sum_{r|l} r^{k-1}M(r^{-1}l)f(a, \alpha, b),
\]

which completes the proof of (3.4).

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