Numerical modeling of compressible fluid flow through elastic porous medium

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Abstract. A numerical technique for solving an initial-boundary value problem of the single compressible fluid flow through elastic porous medium is presented. The model of compressible fluid flow in elastic porous medium is derived within the theory of hyperbolic thermodynamically compatible systems of conservation laws that provides a formulation of mathematically and physically well-posed governing equations of physical processes. Numerical method is based on the WENO–Runge–Kutta algorithm. For the boundary flux evaluation the characteristic decomposition and linearized boundary conditions are used. A two-dimensional problem of the fluid inflow into the saturated porous rectangular domain from its left boundary is studied numerically.

1. Introduction
Mathematical and numerical modeling of a fluid flow in a porous medium is of interest in many scientific areas and industrial applications. Despite of many research efforts there is still no a conventional approach allowing one to formulate a continuum model of multiphase flow in deformable porous media. For slow flows in the filtration regime there is a well established Darcy approach which gives the resulting governing equations of parabolic type for many kinds of filtration flows. This approach is not applicable if one need to model processes with finite speed of the fluid flow or flow in porous deformable elasto-plastic medium. There are some studies in the development of such models which are more general than the Darcy models, see for example [1], but they are still far from being generally accepted. A present research continues the development of mathematical and computational model for a compressible fluid flow in a porous medium [2, 3, 4, 5]. The model is derived in the framework of the theory of hyperbolic thermodynamically compatible systems of conservation laws [6] that gives a way to formulate mathematically and physically well-posed governing equations of physical processes in different areas of continuum mechanics (see, for example [7] and references therein). In this paper we apply the WENO–Runge–Kutta numerical method for the solution of an initial-boundary value problem of the single compressible fluid flow through elastic porous medium and present some numerical results showing the applicability of the model to real problems.

2. Formulation of the model
The governing differential equations for modeling a fluid flow through elastic porous medium are derived within the theory of thermodynamically compatible systems of conservation laws [6].
These equations can be found in [2]. Formulation of the governing system consists of several key steps:

(i) Introduce a set of physical variables characterizing the medium.

(ii) Define a set of thermodynamically compatible conservation laws written in terms of generating potential and variables, which are used to construct the system of governing equations.

(iii) Establish a relationship between the generating potential and variables with the physical variables and thermodynamic potential (internal energy).

(iv) Introduce source terms responsible for the dissipation and phase interaction in the system of governing equations.

In accordance with (i), we introduce a set of physical variables which characterize the elastic porous medium saturated by the compressible fluid. The fluid flow in the porous medium can be considered as the mixture of two phases, in which each phase (fluid and elastic porous) can be characterized by its own parameters of state. Note that we consider processes in which the temperature variations are negligibly small and consider the isentropic approximation of the model. We characterize the element of the mixture with the following state parameters:

- $\alpha_1$ — volume fraction of the elastic medium;
- $\rho$ — mass density of the mixture;
- $c_1$ — mass fraction of the elastic medium;
- $u^i$ — velocity of the mixture;
- $w^i$ — relative velocity of motion of the elastic porous medium with respect to the fluid;
- $F_{ij}$ — elastic deformation gradient of the mixture.

Volume and mass fractions of the liquid can be calculated as

$$\alpha = \frac{\rho_1}{\rho}, \quad c = \frac{\rho_2}{\rho}, \quad u = u^1 + u^2, \quad w = u^1 - u^2,$$

where $\rho_1, \rho_2$ — mass densities of the elastic porous medium and liquid; $u^1, u^2$ — their velocities.

The governing equations of the fluid flow in porous medium presented in [2] are formulated first in terms of the generating potential and variables in a conservative form and are hyperbolic if the generating potential is a convex function of generating variables. It turns out that for solution of specific problems it is more convenient to use individual phase parameters and the so-called generalized internal energy $E$, which is a function of $\alpha_1, \rho, c_1, w^k, F_{ij}$. We call $E$ the equation of state of the mixture and define it as a mass average of the phase internal energies $e_1(\rho, F), e_2(\rho)$, supplemented with the kinematic energy of the relative motion

$$E(\alpha, \rho, c_1, w^k, F) = c_1 e_1(\rho, F) + c_2 e_2(\rho_2) + \frac{1}{2} c_1 c_2 w^j w^i.$$

Thus, the thermodynamically compatible system of conservation laws for the fluid flow through elastic porous medium presented in [2] reads as

$$\begin{align*}
\frac{\partial \rho_1}{\partial t} + \frac{\partial \rho_1 u^k}{\partial x_k} = \lambda (p_1 - p_2), \\
\frac{\partial \rho}{\partial t} + \frac{\rho u^k}{\partial x_k} &= 0, \\
\frac{\partial (u^k - u^s)}{\partial t} + \frac{\partial (u^l u^l u^k/2 - u^s u^s/2 + e_1 + p_1/(\rho_1 - e_2 - p_2/\rho_2)}{\partial x_k} &= c_{klj} u^j \omega_j - \chi E_{w^k}, \\
\frac{\partial (\alpha_1 p_1 u^k_1 + \alpha_2 p_2 u^k_2)}{\partial t} + \frac{\partial (\alpha_1 p_1 u^k_1 + \alpha_2 p_2 u^k_2 + (\alpha_1 p_1 + \alpha_2 p_2) \delta_{ik} - \alpha_1 \sigma_{ik})}{\partial x_k} &= 0, \\
\frac{\partial F_{ij}}{\partial t} + \frac{\partial (\rho F_{ij} w^k - \rho F_{kj} w^i)}{\partial x_k} &= 0,
\end{align*}$$

(1)
where \( p_1 = (\rho_1)^2 \partial e_1 / \partial \rho_1, \) \( p_2 = (\rho_2)^2 \partial e_2 / \partial \rho_2 \) are pressures of the elastic porous medium and fluid, respectively, \( \bar{\sigma}_{ik} \) are shear stresses of the mixture; \( \chi \) is an interfacial friction coefficient, \( \lambda \) is a coefficient characterizing the rate of phase pressures relaxation to the common value; \( e_{klj} \) is the Levi–Civita symbol, \( \delta_{ik} \) is the Kronecker symbol; \( \omega_j \) is the artificial variable introduced to write the equation for the relative velocity in a divergence form

\[
\frac{\partial w^k}{\partial x_i} - \frac{\partial w^i}{\partial x_k} = -e_{k\alpha\omega\alpha}.
\]

Now we choose a particular form of the phase internal energies. The internal energy for the fluid is defined by the stiffened gas equation of state (isentropic case)

\[
e_2(\rho_2) = \frac{(C_2)^2}{\gamma_2(\gamma_2 - 1)} \left( \rho_2 \rho_2^\gamma - \frac{(C_2)^2 \rho_2 \rho_2^\gamma - \gamma_2 \rho_2^\gamma}{\gamma_2 \rho_2^\gamma} \right),
\]

where \( C_2 \) is the sound speed, \( \rho_{20} \) is the reference density, \( p_{20} \) is the reference pressure, and \( \gamma_2 \) is the polytropic coefficient.

The internal energy for the porous medium is defined in the following way:

\[
e_1(\rho_1, F) = e_1(\rho_2) + e_s(F),
\]

where

\[
e_1(\rho_1) = \frac{(C_1)^2}{\gamma_1(\gamma_1 - 1)} \left( \rho_1 \rho_1^\gamma - \frac{(C_1)^2 \rho_1 \rho_1^\gamma - \gamma_1 \rho_1^\gamma}{\gamma_1 \rho_1^\gamma} \right),
\]

and \( C_1, \rho_{10}, \rho_{10}, \gamma_1 \) have the same meaning as for the internal energy \( e_2 \) of the fluid. \( e_s(F) \) is defined according to [8]

\[
e_s(F) = \frac{\mu}{8\rho_{10}}(J_2 - 1),
\]

where \( \mu \) is the Lame coefficient and

\[
J_2 = \text{tr}(g^2), \quad g = \frac{G}{(\det G)^{1/3}}, \quad G = B^{-1}, \quad B = FF^T.
\]

Thus, using the above phase internal energies we obtain the thermodynamically compatible system of conservation laws which describes compressible fluid flow through the elastic porous medium.

### 3. Numerical method

We apply the presented model to the solution of two-dimensional test problems for the purpose of verification of the model. Since the governing equations form a hyperbolic system of conservation laws, the standard WENO–Runge–Kutta numerical method [9] can be directly applied for solving two-dimensional version of equations. The 2D version of system (1) consists of 11 equations, which formally can be written as

\[
\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x_1} + \frac{\partial \mathbf{G}(\mathbf{U})}{\partial x_2} = \mathbf{Q}(\mathbf{U}),
\]

where \( \mathbf{U} \) is a vector of conservative variables, \( \mathbf{F}(\mathbf{U}), \mathbf{G}(\mathbf{U}) \) are fluxes and \( \mathbf{Q}(\mathbf{U}) \) is a vector of source terms. Application of the WENO–Runge–Kutta method of the third order in space and fifth order in space for the solution of the initial value problem for the presented model can
be found in [3]. Below we apply the same method for the initial-boundary value problem. For the flux evaluation at intercells we use the Lax–Wendroff flux whereas for the boundary flux evaluation we use characteristic decomposition and linearized boundary conditions. Note that here we use the instantaneous pressure relaxation assuming the characteristic scale of pore space is small. The numerical treatment of the instantaneous pressure relaxation consists in correction of the phase volume fraction \( C \) and densities \( \rho_1, \rho_2 \) by solution of algebraic equation \( p_1 = p_2 \) in the end of each time step at each mesh cell.

### 3.1. Boundary value problem

Let us consider a two-dimensional problem of the fluid inflow into the saturated porous rectangular domain from its left boundary. To construct necessary left boundary condition we rewrite system of equations (2) in quasilinear form (in terms of physical variables)

\[
A_0(\mathbf{V}) \frac{\partial \mathbf{V}}{\partial t} + A_1(\mathbf{V}) \frac{\partial \mathbf{V}}{\partial x_1} + A_2(\mathbf{V}) \frac{\partial \mathbf{V}}{\partial x_2} = \mathbf{P}(\mathbf{V}),
\]

where \( \mathbf{V} \) is a vector of physical variables. Then we suppose that solution on the previous time step \((\mathbf{V}^n)\) is known. For the characteristic decomposition applied to the boundary conditions we linearize system (3) about the constant value of \( \mathbf{V}^n \) (the solution is sought in the following form \( \mathbf{V} = \mathbf{V}^n + \mathbf{v} \)). We also neglect the source terms and arrive at the following system of linear equations at the left boundary:

\[
\tilde{A}_0(\mathbf{V}^n) \frac{\partial \mathbf{v}}{\partial t} + \tilde{A}_1(\mathbf{V}^n) \frac{\partial \mathbf{v}}{\partial x_1} = 0.
\]

The latter can be rewritten as

\[
\frac{\partial \mathbf{v}}{\partial t} + \tilde{A}_1(\mathbf{V}^n) \frac{\partial \mathbf{v}}{\partial x_1} = 0,
\]

where \( \tilde{A}_1(\mathbf{V}^n) = \tilde{A}_0^{-1}(\mathbf{V}^n) \tilde{A}_1(\mathbf{V}^n) \). Assuming that initially both phases are at rest and the medium moves from the left to the right, we have 3 incoming and 8 outgoing characteristics. Three incoming characteristics correspond to two longitudinal velocities \((C_1, C_2)\) and one transverse velocity \((C_s)\). Among the eight outgoing characteristics there are three of them corresponding to \( C_1, C_2, C_s \) (sound characteristics) and five contact characteristics (equal to the velocity of the mixture \( u_1 \) in \( x_1 \) direction). Thus there are three Riemann invariants that are transported from border cell to the left boundary: \( r^1, r^2, r^3 \). To set the left boundary condition correctly we need 8 more conditions. We set them in the following manner: define phases velocities and set shear stress in \( x_2 \)-direction equal to zero (this condition should be linearized)

\[
\begin{align*}
  u_{1}^1 &= u_{10}^1, \\
  u_{2}^2 &= u_{20}^2, \\
  \Sigma_{12} &= \sigma_{12} = 0,
\end{align*}
\]

For setting the rest values the structure of \( R^{-1} \) must be considered, where \( R \) is the matrix of right eigenvectors of \( \tilde{A}_1(\mathbf{V}^n) \). We can only set the conditions that are linearly independent of previously set ones \((r^1(\mathbf{v}) = r^1(\mathbf{V}^n), r^2(\mathbf{v}) = r^2(\mathbf{V}^n), r^3(\mathbf{v}) = r^3(\mathbf{V}^n) \) and (5)). Those can be the following conditions

\[
\alpha_1^{n+1} = \alpha_1^n, \quad F_{11}^{n+1} = F_{11}^n, \quad F_{12}^{n+1} = F_{12}^n, \quad F_{22}^{n+1} = F_{22}^n, \quad (u_{1}^1)^{n+1} = (u_{1}^1)^n.
\]

We suppose that these values varies so slowly that they can be taken from the previous time step.
Thus we have non-degenerate system of linear algebraic equations for boundary values of the parameters of state. The solution of this system gives vector of physical variables $V^{n+1}$ on the left boundary on the new time step.

On the right boundary non-reflecting conditions are set, on the top and bottom boundaries we set periodical boundary conditions.

4. Examples

In this section we consider a two-dimensional problem of the fluid inflow into the saturated porous rectangular 3 cm long domain from its left boundary. We take sandstone-like material as the elastic porous medium with material parameters

$$C_1 = 3800 \text{ m/s}, \quad \rho_{10} = 2650 \text{ kg/m}^3, \quad p_{20} = 10^5 \text{ Pa}, \quad \gamma_1 = 2.8, \quad \mu = 21.3 \text{ GPa}.$$  

Fluid is a water with constants

$$C_2 = 1500 \text{ m/s}, \quad \rho_{20} = 1000 \text{ kg/m}^3, \quad p_{20} = 10^5 \text{ Pa}, \quad \gamma_2 = 2.4.$$  

Constant $\chi$ is equal to $10^5 \text{ s}^{-1}$. Initially both phases are at rest:

$$p_1 = p_2 = 10^5 \text{ Pa},$$

$$\rho_1 = 2650 \text{ kg/m}^3, \quad \rho_2 = 1000 \text{ kg/m}^3,$$

$$\alpha_1 = 0.5,$$

$$u_1^1 = u_2^1 = u_1^2 = u_2^2 = 0,$$

$$F_{11} = F_{22} = 1, \quad F_{12} = F_{21} = 0.$$  

We consider three different test problems with the following left boundary conditions:

- (Test 1) Liquid velocity is equal to elastic porous medium velocity: $u_1^1 = u_1^2 = 1 \text{ cm/s}.$
- (Test 2) Liquid flows into fixed porous medium: $u_1^1 = 0, \ u_2^1 = 1 \text{ cm/s}.$
- (Test 3) Liquid and elastic porous medium flow with different velocities:

  $u_1^1 = 1 \text{ cm/s}, \ u_2^1 = 2 \text{ cm/s}.$

Below all the data are displayed at four time moments: $t_1 = 0.1776315 \cdot 10^{-5} \text{ s},$  

$t_2 = 0.4144735 \cdot 10^{-5} \text{ s},$  

$t_3 = 0.6513155 \cdot 10^{-5} \text{ s},$  

$t_4 = 0.8881575 \cdot 10^{-5} \text{ s}.$  In the figures the pressure and stress are measured in bars (1 bar = $10^5$ Pa) whereas the velocities are measured in centimetres per second.

One can see the correct propagating wave structure for all test problems. The difference in left boundary velocities results in the difference in wave fields which depends also on the interfacial friction coefficient. The results of computations look physically reasonable. The model and numerical method require further verification and comparison with available experiments.
4.1. Test 1

Figure 1. Test 1. Pressure and stress in the skeleton ($p - \tilde{\sigma}_{11}$).

Figure 2. Test 1. Shear stress in $x_2$-direction and velocities in $x_1$-direction.
4.2. Test 2

Figure 3. Test 2. Pressure and stress in the skeleton $(p - \tilde{\sigma}_{11})$.

$\tilde{\sigma}_{22}$

Figure 4. Test 2. Shear stress in $x_2$-direction and velocities in $x_1$-direction.
4.3. Test 3

Figure 5. Test 3. Pressure and stress in the skeleton ($p - \tilde{\sigma}_{11}$).

Figure 6. Test 3. Shear stress in $x_2$-direction and velocities in $x_1$-direction.
5. Conclusion

The numerical method for initial-boundary value problems for the governing equations of the model of compressible fluid flow through elastic porous medium is developed. The model is based on the theory of thermodynamically compatible systems of conservation laws and governing equations of the model form a hyperbolic system of divergence form equations. The WENO–Runge–Kutta method with characteristic decomposition at the boundary is applied for the numerical solution of the test problems of fluid inflow from the boundary into the saturated porous medium. The numerical computations give physically reasonable results and further research will include the improvement of the numerical method and verification of the model.

Acknowledgments

Work of ER was supported by the Russian Foundation for Basic Research (Grants No 16-29-15131, 15-05-01310). Work of AAY and AKK was supported by the Grant of Government of the Russian Federation No 14.W03.31.0002.

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