Schur function expansions of KP \(\tau\)-functions associated to algebraic curves

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Abstract. The Schur function expansion of Sato–Segal–Wilson KP \(\tau\)-functions is reviewed. The case of \(\tau\)-functions related to algebraic curves of arbitrary genus is studied in detail. Explicit expressions for the Plücker coordinate coefficients appearing in the expansion are obtained in terms of directional derivatives of the Riemann \(\theta\)-function or Klein \(\sigma\)-function along the KP flow directions. By using the fundamental bi-differential it is shown how the coefficients can be expressed as polynomials in terms of Klein’s higher-genus generalizations of Weierstrass’ \(\zeta\) and \(\wp\)-functions. The cases of genus-two hyperelliptic and genus-three trigonal curves are detailed as illustrations of the approach developed here.

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1. Introduction

In the mid-1970s Novikov and Dubrovin [1]–[3], Its and Matveev [4], [5] and others [6]–[9] applied the Lax pair (isospectral deformation) approach to Hill’s operator for periodic potentials with finite band spectrum and thereby determined finite gap periodic solutions to the KdV equation

\[ U_t = 6U U_x - U_{xxx}. \]  (1.1)

Its and Matveev discovered a remarkable formula that provides periodic and, more generally, quasi-periodic solutions of the KdV equation explicitly as a second logarithmic derivative of the Riemann $\theta$-function:

\[ U(x, t) = -\frac{\partial^2}{\partial x^2} \log \theta(Ux + Vt + W) + C \]  (1.2)

with $U, V, W = \text{const} \in \mathbb{C}^g$ and $C \in \mathbb{C}$. The $\theta$-function appearing here is determined by the period lattice of a hyperelliptic curve $X$ of arbitrary genus $g$, and the ‘winding vectors’ $U, V$ are periods of Abelian differentials of the second kind. This follows from a more general formula (3.7) for the associated Baker–Akhiezer function (see (2.61)), which is valid for the KP hierarchy and was first obtained for the KdV case by Its (appearing in the appendix to [10]).

This formed an important part of the general theory of algebro-geometric solutions of the KdV equation (see, for example, [10] for a review). Krichever [11] extended these considerations more generally to quasi-periodic solutions of the KP hierarchy. This led to a general method of integration of such partial differential equations determined by the specification of an algebraic curve and certain additional data on it. (See [12] for an overview of this approach and further applications.) These results had a great influence on subsequent developments in the theory of integrable non-linear hierarchies and their applications in various areas of mathematics and physics.

The phenomenon of algebro-geometric integrability has been considered from various viewpoints. In this paper we discuss the theory of $\tau$-functions as initiated by M. and Y. Sato [13]–[15] and developed in the works of M. Sato, Date, Jimbo, Kashiwara, Miwa, and others (see, for example, [16]). We also make use of the geometrical formulation of Segal and Wilson [17]. In this approach the $\tau$-function

\[ \tau = \tau_w(t), \quad \tau_w(0) \neq 0, \]  (1.3)

is understood to depend on two sets of variables, an infinite-dimensional vector $t = (t_1, t_2, \ldots) \in \mathbb{C}^\infty$ and an element $w \in \text{Gr}_{\mathcal{H}+}(\mathcal{H})$ of an infinite-dimensional Grassmannian consisting of subspaces of a polarized Hilbert space (the direct sum of two mutually orthogonal subspaces of essentially equal size)

\[ \mathcal{H} = \mathcal{H}+ + \mathcal{H}_- \]  (1.4)

that are ‘commensurable’ with the fixed subspace $\mathcal{H}_+$ (that is, orthogonal projection to $\mathcal{H}_+$ is ‘large’—a Fredholm operator—while projection to $\mathcal{H}_-$ is, for example, of Hilbert–Schmidt class). The Plücker relations, defining the embedding of $\text{Gr}_{\mathcal{H}+}(\mathcal{H})$ as a subvariety of the projectivized infinite exterior space $\mathbb{P}(\Lambda \mathcal{H})$,
are equivalent to an infinite set of bilinear differential relations in the $t$ variables, the Hirota equations, which in turn imply the equations of the KP hierarchy.

In the special case to be considered here, $w$ is determined by certain algebro-geometric data, the Dubrovin–Krichever–Novikov (DKN) data, consisting of an algebraic curve $X$ of genus $g$, a non-special positive divisor of degree $g$,

$$\mathcal{D} = \sum_{i=1}^{g} p_i, \quad p_i \in X,$$

(1.5)
a point $p_\infty$ identified as ‘infinity’, and a local parameter $\xi = 1/z$ with $\xi(p_\infty) = 0$.

We further make a choice of a homology basis $\{a_1, \ldots, a_g; b_1, \ldots, b_g\}$ for $X$ consisting of $a$ and $b$ cycles satisfying the intersection relations

$$a_i \circ a_j = 0, \quad b_i \circ b_j = 0, \quad a_i \circ b_j = \delta_{ij},$$

(1.6)
a normalized basis $\{u_1, \ldots, u_g\}$ of holomorphic Abelian differentials, and a canonical polygonization of $X$ obtained by cutting along the $a$ and $b$-cycles. After selecting an arbitrary base point $p_0$, the Abel map

$$\mathcal{A} : \mathcal{I}^g X \rightarrow \mathcal{J}(X) = \mathbb{C}^g/\Gamma$$

(1.7)
translated by the Riemann constant $K$ corresponding to the polygonization. We can then denote, in short, the corresponding $\tau$-function as $\tau(e, t)$.

The $\tau$-function can be represented as a Taylor series in $t$ and then re-expressed as an expansion in a basis consisting of Schur functions $s_\lambda(t)$ labelled by partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)})$ (where the $\lambda_i$ form a weakly decreasing sequence of non-negative integers $\lambda_1 \geq \lambda_2 \geq \cdots$ with the last non-zero term $\lambda_{\ell(\lambda)}$, where $\ell(\lambda)$ is the length of $\lambda$). The flow parameters $(t_1, t_2, \ldots)$ can be identified with monomial sums

$$t_i = \frac{1}{i} \sum_{a=1}^{N} x_a^i,$$

(1.9)
in terms of $N$ auxiliary variables for any $N \geq \ell(\lambda)$, by taking the (stable) limit as $N \rightarrow \infty$.

The Cauchy–Littlewood identity [18] (or equivalently, the Abelian group property of the KP flow) permits us to express this expansion in the form

$$\tau(e, t) = \sum_{\lambda} \left[ s_\lambda \left( \frac{\partial}{\partial t_1}, \ldots, \frac{1}{k} \frac{\partial}{\partial t_k}, \ldots \right) \right]_{t=0} s_\lambda(t),$$

(1.10)
where the summation runs over all partitions $\lambda$. The important point is that the coefficients

$$\pi_\lambda(w) := \left[ s_\lambda \left( \frac{\partial}{\partial t_1}, \ldots, \frac{1}{k} \frac{\partial}{\partial t_k}, \ldots \right) \right]_{t=0} \tau(e, t),$$

(1.11)
in this expansion are just the Plücker coordinates of the element \( w \in \text{Gr}_{\mathcal{H}_+}(\mathcal{H}) \) under the Plücker embedding

\[
\mathfrak{P} \colon \text{Gr}_{\mathcal{H}_+}(\mathcal{H}) \to \mathbf{P}(\mathcal{F})
\]

into the projectivization of the exterior space

\[
\mathcal{F} = \Lambda \mathcal{H},
\]

which is a completion of the space of sums over a basis consisting of semi-infinite wedge products of basis elements of \( \mathcal{H} \) (the Fermionic Fock space). In this setting, the Hirota bilinear relations of the KP hierarchy are simply equivalent to the Plücker relations satisfied by the coefficients \( \{ \pi_{\lambda}(w) \} \).

To each partition \( \lambda = (\lambda_1, \lambda_2, \ldots) \) we associate, as usual, a Young diagram with \( \lambda_1 \) boxes in the first row, \( \lambda_2 \) in the second, and so on. For example,

\[
\lambda = (3, 3, 1) \iff \begin{array}{|c|c|c|} \hline & & \hline \end{array}
\]

Partitions of the form \((k, 1, 1, \ldots, 1) \equiv (k, 1^j)\) are called hooks. Partitions can equivalently be expressed in the Frobenius notation as

\[
\lambda \sim (a_1, \ldots, a_k | b_1, \ldots, b_k),
\]

where \( k \) is the number of diagonal boxes in the Young diagram, called the rank of the partition, and the \( a_i \) and \( b_i \) are, respectively, the numbers of boxes to the right of and below the diagonal ones \([19]\). The Giambelli formula

\[
s_{(a_1, \ldots, a_k | b_1, \ldots, b_k)} = \det(s_{(a_i | b_j)})
\]

expresses the Schur function corresponding to an arbitrary partition as a determinant whose entries are the Schur functions corresponding to hook diagrams only. M. Sato also used Giambelli’s formula in the decomposition of the coefficients of the expansion (1.10), since the same determinantal relations are valid when expressed for the Plücker coordinates of the image of any element \( w \) of the Grassmannian (that is, for any completely decomposable element of \( \Lambda \mathcal{H} \)) in terms of those for hook partitions (see (2.23)). This amounts to an explicit solution of the Plücker relations that is valid on the affine neighborhood corresponding to the ‘big cell’. By (1.11), the resulting relations appear as partial differential equations for which the \( \tau \)-function plays the role of generating function.

Such expansions are valid for any \( \tau \)-function \( \tau_w(t) \), but here we will mainly consider \( \tau \)-functions \( \tau(e, t) \) associated to the DKN data on an algebraic curve \( X \) of genus \( g \), and compute the coefficients in the Schur function expansion for this case.

Let \( X \) be equipped with a canonical homology basis \((a_1, \ldots, a_g; b_1, \ldots, b_g)\) and corresponding polygonization, and choose a basis of holomorphic differentials \( u = (u_1, \ldots, u_g)^T \) ordered according to the degree of their vanishing at the Weierstrass point \( p_\infty \) at infinity, \( n_g + 1, \ldots, n_1 + 1 \), where \( \mathfrak{W} = (n_1, \ldots, n_g) \) is the Weierstrass gap sequence at \( p_\infty \) (see §3.2).
Denote by
\[ A = \left( \oint_{a_j} u_i \right)_{i,j=1,\ldots,g}, \quad B = \left( \oint_{b_j} u_i \right)_{i,j=1,\ldots,g} \tag{1.17} \]
the period matrices. The Jacobian of the curve is then \( \text{Jac}(X) = \mathbb{C}^g / A \oplus B. \)

We will refer to \((A, B)\) as the first period matrices or Riemann period matrices. The second period matrices, \((S, T)\), are similarly formed from the periods of meromorphic differentials \( r = (r_1, \ldots, r_g)^T \) with poles only at the Weierstrass point \( p_\infty \), of orders \( n_g + 1, \ldots, n_1 + 1 \), respectively. The matrices
\[ S = -\left( \oint_{a_j} r_i \right)_{i,j=1,\ldots,g}, \quad T = -\left( \oint_{b_j} r_i \right)_{i,j=1,\ldots,g} \tag{1.18} \]
are normalized by the condition (generalized Legendre equation)
\[ \begin{pmatrix} A & B \\ S & T \end{pmatrix} \begin{pmatrix} A & B \\ S & T \end{pmatrix}^T = -2i\pi J, \quad J = \begin{pmatrix} 0_g & -1_g \\ 1_g & 0_g \end{pmatrix}. \tag{1.19} \]
The matrix
\[ \kappa := SA^{-1} \tag{1.20} \]
is necessarily symmetric, \( \kappa^T = \kappa \), and
\[ G = \kappa A, \quad T = \kappa B - \frac{i\pi}{2} (A^{-1})^T. \tag{1.21} \]

**Remark 1.1.** In the basis \((u, r)\) the meromorphic differentials \( r \) are not uniquely defined, only up to the addition of a holomorphic differential. Therefore, the matrix \( \kappa \) is only defined up to the addition of an arbitrary symmetric matrix. Nevertheless, for our purposes it is sufficient to choose a specific representative of the class of differentials \( r \), and this will be specified in each case treated in detail in the examples.

Following Baker [20], we associate to the curve \( X \) the fundamental bi-differential \( \Omega(p, q) \) which is the unique symmetric meromorphic bi-differential on \( X \times X \) with a second-order pole on the diagonal \( p = q \) that is elsewhere holomorphic in each variable. Locally expressed, this has the form
\[ \Omega(p, q) = \frac{d\xi(p) d\xi(q)}{(\xi(p) - \xi(q))^2} + \sum_{i,j=0}^\infty \mu_{ij}(p_0) \xi(p)^i \xi(q)^j d\xi(p) d\xi(q), \tag{1.22} \]
where \( \xi(p) \) and \( \xi(q) \) are local coordinates in a neighbourhood of a base point \( p_0 \), \( \xi(p_0) = 0 \), and the coefficients \( \mu_{ij}(p_0) \) are symmetric with respect to the \( i, j \)-indices: \( \mu_{ij}(p_0) = \mu_{ji}(p_0) \). The bi-differential \( \Omega(p, q) \) is normalized by the conditions
\[ \oint_{a_j} \Omega(p, q) = 0, \quad j = 1, \ldots, g. \tag{1.23} \]

Usually \( \Omega(p, q) \) is realized as the second logarithmic derivative of the prime-form or \( \theta \)-function [21], [22]. But in our development we use an alternative representation
of $\Omega(p, q)$ in an algebraic form that goes back to Weierstrass and Klein, as described by Baker in [23]:

$$\Omega(p, q) = \frac{\mathcal{F}(p, q)}{P_y(p)P_w(q)(x - z)^2} \, dx \, dz + 2u(p)^T \kappa u(q), \quad (1.24)$$

where $p = (x, y)$, $q = (z, w)$, the function $\mathcal{F}(p, q) = \mathcal{F}((x, y), (z, w))$ is a polynomial in its arguments, with coefficients depending on the parameters defining the curve $X$ explicitly as a planar model given by the polynomial equation

$$P(x, y) = 0, \quad (1.25)$$

$u$ is the $g$-component vector whose entries are the holomorphic differentials, and $\kappa$ is a symmetric matrix (1.20), thereby providing the normalization (1.23) of $\Omega(p, q)$.

We will refer to the first term on the right-hand side of (1.24), which involves the polynomial $F(p, q)$, as the algebraic part, denoting it by $\Omega^{\text{alg}}(p, q)$. In a neighbourhood of the base point $p_0$, $\xi(p_0) = 0$, the form $\Omega^{\text{alg}}(p, q)$ is expanded in a power series as

$$\Omega^{\text{alg}}(p, q) = \frac{d\xi(p)}{(\xi(p) - \xi(q))^2} + \sum_{i, j=0}^{\infty} \mu^{\text{alg}}_{ij}(p_0)\xi(p)^i\xi(q)^j \, d\xi(p) \, d\xi(q), \quad (1.26)$$

where the quantities $\mu^{\text{alg}}_{ij}(p_0)$ are algebraic functions of $\xi(p_0)$ and the coefficients of the curve. The transcendental part $\Omega^{\text{trans}}(p, q)$ of $\Omega(p, q)$ is holomorphic, and its series expansion in a neighbourhood of the base point $p_0$ has the form

$$\Omega^{\text{trans}}(p, q) \equiv 2u(p)^T \kappa u(q) = \sum_{i, j=0}^{\infty} \mu^{\text{trans}}_{ij}(p_0)\xi(p)^i\xi(q)^j \, d\xi(p) \, d\xi(q). \quad (1.27)$$

Therefore,

$$\Omega(p, q) = \Omega^{\text{alg}}(p, q) + \Omega^{\text{trans}}(p, q) \quad (1.28)$$

and

$$\mu_{ij}(p_0) = \mu^{\text{alg}}_{ij}(p_0) + \mu^{\text{trans}}_{ij}(p_0) \quad (1.29)$$

for all $p_0$ and $i, j \in \mathbb{Z}$.

The algebraic representation of the fundamental bi-differential, as described above, lies behind the definition of the multivariable $\sigma$-function in terms of the multivariable (Riemann) $\theta$-functions $\theta$. It differs from $\theta$ by an exponential factor and a modular factor $C$:

$$\sigma(v) = C \exp \left\{ \frac{1}{2} v^T \kappa v \right\} \theta(\mathfrak{A}^{-1}v; \tau), \quad (1.30)$$

where $\kappa$ and $\mathfrak{A}$ are defined in (1.20) and (1.17). These modifications make $\sigma(v)$ invariant with respect to the action of the symplectic group, so that for any $\gamma \in \text{Sp}(2g, \mathbb{Z})$, we have

$$\sigma(v; \gamma \circ \mathfrak{M}) = \sigma(v; \mathfrak{M}), \quad (1.31)$$

\[1\]The factor 2 in the normalization makes the case $g = 1$ agree with the usual one in the Weierstrass theory of elliptic functions.

\[2\]The representation (1.24) was further developed by Buchstaber, Leykin, and one of the authors [24] and more recently by Nakayashiki [25].
where $\mathcal{M}$ is the set of periods of the curve $X$. The multivariable $\sigma$-function is the natural generalization of the Weierstrass $\sigma$-function to algebraic curves of higher genera.

Remark 1.2. In his lectures, [26], Weierstrass defined the $\sigma$-function in terms of a series with coefficients given recursively, a key point of the Weierstrass theory of elliptic functions. A generalization of this result to genus-two curves was begun by Baker [20] and recently completed by Buchstaber and Leykin [27], who obtained recurrence relations between the coefficients of the $\sigma$-function series in closed form. Buchstaber and Leykin also recently found an operator algebra that annihilates the $\sigma$-function of higher-genera $(n, s)$-curves [28]. Finding a suitable recursive definition of the higher-genera $\sigma$-functions along the lines of [28] remains a challenging problem.

In this paper we study relations between the multivariable $\sigma$-function and the Sato $\tau$-function [13]–[15] for the case of quasi-periodic solutions associated to DKN data on an algebraic curve. Thus, we chiefly consider this class of ‘algebro-geometric $\tau$-function’ solutions. These are essentially the same as those studied by Fay [29], [30] in terms of $\theta$-functions. In terms of $\sigma$-functions such $\tau$-functions can be expressed by (see Proposition 3.1)

$$\frac{\tau(e, t)}{\tau(e, 0)} = \frac{\sigma\left(\sum_{k=1}^{\infty} 2 A U_k(p_\infty) t_k + 2 A e\right)}{\sigma(2 A e)} \exp\left\{\frac{1}{2} \sum_{k,l=0}^{\infty} \mu_{kl}^{\text{alg}}(p_\infty) t_k t_l\right\}. \tag{1.32}$$

Here as above, $A$ is the period matrix of holomorphic differentials, $U_k(p_\infty), k = 1, 2, \ldots,$ are ‘winding vectors’, that is, $b$-periods of normalized differentials of the second kind with poles of order $k + 1$ at the point $p_\infty$, and $e$ is an arbitrary point of the Jacobian variety $\text{Jac}(X)$.

Remark 1.3. In the series of publications [32]–[34] Buchstaber and Shorina also discussed the relations between the $\theta$-function and $\sigma$-function solutions of the KdV-hierarchy.

The paper is organized as follows. In §2 we review the geometric formulation of the Sato–Segal–Wilson $\tau$-function in terms of Hilbert space Grassmannians. The interpretation of the coefficients of the Schur function expansion as Plücker coordinates is derived (Proposition 2.2), as well as their expression in terms of the affine coordinates on the big cell, and hook partitions (Corollary 2.3).

In §3 we restrict ourselves to the special case of $\tau$-functions associated to DKN data on an algebraic curve. The explicit formula for such $\tau$-functions in terms of Riemann $\theta$-functions is given in equation (3.37). The affine coordinates determining the Schur function expansion are computed explicitly (equation (3.31)) in terms of directional derivatives of the Riemann $\theta$-function along the flow directions. In §3.2 we review the Weierstrass gap theorem and introduce the normalized symmetric bi-differential $\Omega(p, q)$. This allows us to make explicit the splitting into ‘algebraic’ and ‘transcendental’ parts (Corollary 3.1) of the infinite quadratic form $Q$ appearing

\^Recently A. Nakayashiki [31] has independently suggested a similar expression for the algebro-geometric $\tau$-functions in terms of multivariable $\sigma$-functions and studied properties of the $\sigma$-function series.
in the exponential factor in equation (3.37) and in the formula (3.31) determining
the affine coordinates. The equivalent expression for the \( \tau \)-function in terms of the
\( \sigma \)-function, which displays more clearly its modular transformation properties, is
derived in Proposition 3.1, equation (3.77). From this, it is shown how to compute
the affine coordinate matrix elements explicitly as polynomials in the Kleinian
functions \( \{ \zeta_i, \wp_{ij} \} \) which generalize the Weierstrass \( \zeta \)- and \( \wp \)-functions to curves
of arbitrary genus.

In \( \S \, 4 \) we consider several examples for which the affine coordinates are explicitly
calculated and use the Plücker relations to derive identities relating Kleinian \( \zeta \)-
and \( \wp \)-functions of different degrees. The case of hyperelliptic curves is treated in
detail, with explicit formulae computed for the case of genus \( g = 2 \). In Proposition
4.1 we derive algebraic relations between the Kleinian \( \wp \)-functions of different orders
for this case. A further example based on a planar model of a class of trigonal
curves is considered, and explicit expressions for the lowest affine coordinate matrix
elements are computed in terms of the Kleinian \( \zeta \)- and \( \wp \)-functions for this case, as
well as identities relating \( \wp \)-functions of different orders that again follow from the
Plücker relations.

2. Background. Sato–Segal–Wilson \( \tau \)-function

The following is a brief summary of the Sato [15] and Segal and Wilson [17]
approach to KP \( \tau \)-functions. The first is essentially algebraic in nature, the second
functional analytic, but we combine elements of both. For more precise defini-
tions of the main ingredients (Hilbert space Grassmannians, infinite transforma-
tion groups, determinant line bundles, Fermionic Fock space, the Plücker map, and
so on), the reader is referred to these two sources, which agree in the geometric
framework but not in analytic details. The introductory summary given here is
intended to be as simple and self-contained as possible, and applicable to the cases
at hand. Functional analytic details of the general formulation are either omitted
or referred to [17], and the ingredients are made as much as possible to appear like
the finite-dimensional case.

2.1. Hilbert space Grassmannian and Plücker coordinates. Following [17],
we start with the Hilbert space

\[
\mathcal{H} := L^2(S^1) = \mathcal{H}_+ \oplus \mathcal{H}_- \tag{2.1}
\]

of complex-valued functions \( f \) on the unit circle \( \{ z = e^{i\phi} \} \) in the complex plane
which are square-integrable, and which can be split as a sum

\[
f = f_+ + f_-,
\]

where \( f_\pm \in \mathcal{H}_\pm \) are the positive and negative parts of the Fourier series.

Equivalently, \( \mathcal{H}_+ \) can be interpreted as the space of holomorphic functions on
the interior of the unit disc and \( \mathcal{H}_- \) as the space of holomorphic functions on the
exterior that vanish at \( z = \infty \), with orthonormal bases provided by the monomials
in \( z \):

\[
\mathcal{H}_+ = \text{span}\{ e_j := z^{-j-1} \}_{j=-1,-2,...}, \quad \mathcal{H}_- = \text{span}\{ e_j := z^{-j-1} \}_{j=0,1,2,...}. \tag{2.3}
\]
Remark 2.1. The convention of labelling the basis vectors $e_j$ so that $\mathcal{H}_+$ is the span of those having negative indexes $j$ is chosen so that under the Plücker map (see below) $\mathcal{H}_+$ is taken into $|0\rangle = e_{-1} \wedge e_{-2} \wedge \cdots$, which is the ‘Dirac sea’, in which all negative ‘energy’ states are occupied.

We denote by $\text{Gr}_{\mathcal{H}_+}(\mathcal{H})$ the Hilbert space Grassmannian whose points are closed subspaces $w \subset \mathcal{H}$ that are commensurable with $\mathcal{H}_+$ in the sense that orthogonal projection $\pi^+: w \to \mathcal{H}_+$ onto $\mathcal{H}_+$ along $\mathcal{H}_-$ is a Fredholm map with index zero, and orthogonal projection $\pi^-: w \to \mathcal{H}_-$ onto $\mathcal{H}_-$ along $\mathcal{H}_+$ is Hilbert–Schmidt. (In [17] this is called the zero virtual dimension component of the full Hilbert space Grassmannian.)

Let
\[
W = (W_0, W_1, \ldots) = (W_+, W_-) = \begin{pmatrix}
W_+ & \vdots & \vdots & \vdots & \vdots \\
& -1 & \vdots & \vdots & \vdots \\
& & 0 & \vdots & \vdots \\
& & & \ddots & \ddots \\
& & & & +\infty
\end{pmatrix},
\]
(2.5)

where the rows are labelled by the integers, increasing downward with the 0th row at the top of the $W_-$ block, and the columns are labelled by the non-negative integers, starting with 0 on the left and increasing to the right.

Remark 2.2. Note that the column labelling corresponds to the monomial degrees in $\mathcal{H}_+$ whereas the row labelling corresponds to the basis elements $\{e_j\}$.

Here $W_-$ can be viewed as representing a map $w_-: \mathcal{H}_+ \to \mathcal{H}_-$ and $W_+$ as representing a map $w_+: \mathcal{H}_+ \to \mathcal{H}_+$ such that $w$ is the image of their sum
\[
w = (w_- + w_+)(\mathcal{H}_+).
\]
(2.6)

The Grassmannian $\text{Gr}_{\mathcal{H}_+}(\mathcal{H})$ can be interpreted as the quotient of the bundle $\text{Fr}_{\mathcal{H}_+}(\mathcal{H})$ of admissible frames by the right action of the group $\text{Gl}(\mathcal{H}_+)$ of linear changes of basis. (See [17] for the precise definition, which requires elements of $\text{Gl}(\mathcal{H}_+)$ to have a non-vanishing, finite determinant.) Thus, $\text{Fr}_{\mathcal{H}_+}(\mathcal{H})$ can be viewed, as in the finite-dimensional case, as a principal $\text{Gl}(\mathcal{H}_+)$-bundle over $\text{Gr}_{\mathcal{H}_+}(\mathcal{H})$, to which we can associate, through the determinant representation, a line bundle $\text{Det} \to \text{Gr}_{\mathcal{H}_+}(\mathcal{H})$ and its dual $\text{Det}^* \to \text{Gr}_{\mathcal{H}_+}(\mathcal{H})$. A holomorphic section of the latter is determined by each partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell(\lambda) > 0)$, $\lambda_i \in \mathbb{N}$, where $\ell(\lambda)$ denotes the length, in a way that mimics the finite-dimensional
case. Extending the set of parts \( \{ \lambda_j \} \) in the usual way \([18]\) to an infinite sequence \( \{ \lambda_j \} \) by defining
\[
\lambda_j = 0 \quad \text{if } j > \ell(\lambda),
\]
the determinant \( \det(W_\lambda) \) of the submatrix \( W_\lambda \) of \( W \) consisting of the rows \( \{ \lambda_i - i \} \) defines a holomorphic section \( \sigma: \text{Gr}_+ (\mathcal{H}) \rightarrow \text{Det}^* \) of the bundle \( \text{Det}^* \rightarrow \text{Gr}_+ (\mathcal{H}) \), and these sections span the class of admissible sections. (Again, for the full analytic details required to define the class of admissible sections, see \([17]\).)

As in the finite-dimensional case, the space of such holomorphic sections can be identified with a certain subspace \( F_0 \subset F \) of the exterior space \( F := \Lambda \mathcal{H} \), the ‘zero charge sector’ of \( F \), with the latter interpreted as the full ‘Fermionic Fock space’. Let \( F_0 \) be the span of the exterior elements
\[
|\lambda \rangle := e_{l_1} \wedge e_{l_2} \wedge e_{l_3} \wedge \cdots, \tag{2.8}
\]
where for any partition \( \lambda = (\lambda_1, \lambda_2, \ldots) \) the \( l_i \) are the ‘particle coordinates’
\[
l_i = \lambda_i - i. \tag{2.9}
\]
These exterior elements form an orthonormal basis with respect to the inner product induced on \( F_0 \subset \Lambda \mathcal{H} \) by the one on \( \mathcal{H} \).

For each partition \( \lambda \) we can also define \( \mathcal{H}_\lambda \in \text{Gr}_+ (\mathcal{H}) \) as
\[
\mathcal{H}_\lambda = \text{span} \{ e_{l_i} \}_{i \in \mathbb{N}}. \tag{2.10}
\]
In particular, the element \( \mathcal{H}_0 \) corresponding to the trivial partition \( \lambda = 0 \) is \( \mathcal{H}_+ \).

We define the Plücker map \( \hat{\mathcal{P}}: \text{Fr}_+ (\mathcal{H}) \rightarrow F_0 \) in the natural way:
\[
\hat{\mathcal{P}}: \text{Fr}_+ (\mathcal{H}) \rightarrow F_0, \tag{2.11}
\]
\[
\hat{\mathcal{P}}: \{ w_0, w_1, \ldots \} \mapsto w_0 \wedge w_1 \wedge w_2 \wedge \cdots. \tag{2.12}
\]
Changing the frame \( \{ w_0, w_1, \ldots \} \) spanning the subspace \( w \subset \mathcal{H} \) by application (on the right) of an element \( g \in \text{GL}(\mathcal{H}_+) \) just changes the image under \( \hat{\mathcal{P}} \) by the non-zero multiplicative factor \( \det(g) \). Therefore, the Plücker map \( \hat{\mathcal{P}} \) on the frame bundle projects to a map embedding \( \text{Gr}_+ (\mathcal{H}) \) into the projectivization \( \mathcal{P}(F_0) \) of \( F_0 \)
\[
\mathcal{P}: \text{Gr}_+ (\mathcal{H}) \rightarrow \mathcal{P}(F_0),
\]
\[
\mathcal{P}: \{ w_0, w_1, \ldots \} \mapsto [w_0 \wedge w_1 \wedge w_2 \wedge \cdots], \tag{2.13}
\]
where \([|v\rangle]\) denotes the projective equivalence class of \(|v\rangle \in F_0 \). In particular,
\[
\mathcal{P}(\mathcal{H}_\lambda) = [|\lambda\rangle]. \tag{2.14}
\]
The image of \( \text{Gr}_+ (\mathcal{H}) \) under \( \mathcal{P} \) is the orbit of \( \mathcal{H}_+ \) under the identity component of the general linear group \( \text{GL}(\mathcal{H}) \) (again, suitably defined as in \([15]\) or \([17]\)).
The Plücker coordinates \( \{ \pi_{\lambda}(|v|) \} \) of an element \(|v| \in \mathcal{F}_0\) are just its components relative to the orthonormal basis \( \{|\lambda\rangle\} \):

\[
\pi_{\lambda}(|v|) = \langle \lambda | v \rangle, \\
|v\rangle = \sum_{\lambda} \pi_{\lambda} |\lambda\rangle.
\]  

(2.15)

Upon applying the Plücker map \( \hat{\mathfrak{P}} \) to an element \( \{w_0, w_1, \ldots \} \in \text{Fr}_{\mathcal{H}^+}(\mathcal{H}) \) spanning \( w \in \text{Gr}_{\mathcal{H}^+}(\mathcal{H}) \), the Plücker coordinates of its image are the homogeneous coordinates of \( \mathfrak{P}(w) \) under the Plücker map (2.13):

\[
\pi_{\lambda}(w) := \pi_{\lambda}(\mathfrak{P}(w)).
\]  

(2.16)

It follows from the above that

\[
\pi_{\lambda}(w) = \det(W_{\lambda}),
\]  

(2.17)

and hence the basis of the space of holomorphic sections \( H^0(\text{Gr}_{\mathcal{H}^+}(\mathcal{H}), \text{Det}^*) \) of \( \text{Det}^* \) defined by the partitions \( \lambda \) corresponds precisely to the Plücker coordinates.

The Plücker relations are the infinite set of quadratic relations that determine the image of \( \text{Gr}_{\mathcal{H}^+}(\mathcal{H}) \) under the Plücker map. They follow from the fact that for any \( w \in \text{Gr}_{\mathcal{H}^+}(\mathcal{H}) \) the image \( \mathfrak{P}(w) \) is a decomposable element of \( \mathcal{F}_0 \). The Plücker coordinates of \( \pi_{\lambda}(w) \) are therefore not independent; it is possible to express them on an open dense affine subvariety as finite determinants in terms of a much smaller subset consisting, for example, of those Plücker coordinates corresponding to hook partitions.

It is convenient to use the Frobenius notation \((a_1, a_2, \ldots, a_r \mid b_1, b_2, \ldots, b_r)\) for a partition [18], where \((a_i, b_i)\) are the numbers of elements to the right of and below the \( i \)th diagonal element of the Young diagram for \( i = 1, \ldots, r \). A hook partition \( \lambda = (a+1, 1^b) \) is one for which \( r = 1 \), and hence in the Frobenius notation is expressed as \((a \mid b)\).

To see how to express the Plücker coordinate \( \pi_{(a_1, a_2, \ldots, a_r \mid b_1, b_2, \ldots, b_r)}(w) \) corresponding to an arbitrary partition in terms of the coordinates for the hook partitions \( \{(a_i \mid b_j)\}_{1 \leq i, j \leq r} \), it is easiest to assume that \( w \) is in the ‘big cell’, in which the map \( w_+ : \mathcal{H}_+ \rightarrow \mathcal{H}_+ \) is invertible. The infinite matrix \( W_+ \) in (2.5) is therefore also invertible, and we can define affine coordinates as the matrix entries of

\[
A := W_- W_+^{-1}.
\]  

(2.18)

By convention, we will let the indices \((a, b)\) range over the non-negative integers, and therefore the componentwise interpretation of (2.18) is

\[
A_{ab} := (W_- W_+^{-1})_{ab}, \quad a, b \in \mathbb{N}.
\]  

(2.19)

The homogeneous coordinates in this basis have the form

\[
WW_+^{-1} = \begin{pmatrix} I \\ A \end{pmatrix},
\]  

(2.20)

where \( I \) is the semi-infinite identity matrix with \( I_{ij} = \delta_{-i-1,j} \), which, in view of the labelling convention, has \((i, j)\)th entry for \( i \leq -1, j \geq 0 \), and the labelling
convention for the matrix $A$ in the lower block is that the indices $(a, b)$ start with $(0, 0)$ in the upper left corner and increase downward and to the right. This allows us to express all the Plücker coordinates as finite determinants in terms of the affine coordinates on the big cell.

**Proposition 2.1.** The Plücker coordinate $\pi_{(a_1, a_2, \ldots, a_r \mid b_1, b_2, \ldots, b_r)}(w)$ corresponding to the partition $(a_1, a_2, \ldots, a_r \mid b_1, b_2, \ldots, b_r)$ is

$$\pi_{(a_1, a_2, \ldots, a_r \mid b_1, b_2, \ldots, b_r)}(w) = \det(\pi_{(a_i \mid b_j)}(w)|_{1 \leq i, j \leq r}) \det(W_+). \quad (2.21)$$

In particular, we can consider the case of hook partitions, which, according to (2.21), coincide, within a sign, with the components of the affine coordinate matrix $A$, allowing all other Plücker coordinates to be expressed as finite determinants in terms of these.

**Corollary 2.1.** The Plücker coordinate corresponding to a hook partition $(a \mid b)$ is, to within a sign, the $(a, b)$ affine coordinate,

$$\pi_{(a \mid b)}(w) = (-1)^b A_{ab}, \quad (2.22)$$

and hence

$$\pi_{(a_1, a_2, \ldots, a_r \mid b_1, b_2, \ldots, b_r)}(w) = \det\left(\pi_{(a_i \mid b_j)}(w)|_{1 \leq i, j \leq r}\right). \quad (2.23)$$

**2.2. Abelian flow group $\Gamma_+$ and the KP $\tau$-function.** We now introduce the Abelian subgroup $\Gamma_+ \subset \mathfrak{gl}(\mathcal{H})$ consisting of the non-zero elements of $\mathcal{H}_+$ normalized to equal 1 at the origin $z = 0$:

$$\Gamma_+ := \left\{ \gamma(t) := \exp\left(\sum_{i=1}^{\infty} t_i z^i\right) \right\}, \quad t := (t_1, t_2, \ldots), \quad (2.24)$$

acting on $\mathcal{H}$ by multiplication

$$\Gamma_+ \times \mathcal{H} \to \mathcal{H}, \quad (\gamma, f) \mapsto \gamma f. \quad (2.25)$$

This induces an action on the Grassmannian,

$$\Gamma_+ \times \text{Gr}_{\mathcal{H}_+}(\mathcal{H}) \to \text{Gr}_{\mathcal{H}_+}(\mathcal{H}), \quad (\gamma(t), w) \mapsto \gamma(t)w, \quad (2.26)$$

which lifts to one on the bundle $\text{Det}^* \to \text{Gr}_{\mathcal{H}_+}(\mathcal{H})$ and determines an action on the space of holomorphic sections $H^0(\text{Gr}_{\mathcal{H}_+}(\mathcal{H}), \text{Det}^*)$:

$$\Gamma_+ \times H^0(\text{Gr}_{\mathcal{H}_+}(\mathcal{H}), \text{Det}^*) \to H^0(\text{Gr}_{\mathcal{H}_+}(\mathcal{H}), \text{Det}^*), \quad (\gamma(t), \sigma) \mapsto \tilde{\gamma}(t)\sigma := \sigma \circ \gamma^{-1}(t), \quad (2.27)$$

$$\tilde{\gamma}(t)\sigma(w) := \tilde{\gamma}(t)\sigma(\gamma^{-1}(t)w).$$
The latter coincides, up to normalization, with the induced action on $\mathcal{P}_0 \subset \Lambda\mathcal{H}$. Let

$$w(t) = \gamma(t)(w)$$

be the image of $w \in \text{Gr}_{\mathcal{H}}(\mathcal{H})$ under the action of the group element $\gamma(t)$ and let $W(t)$ be its matrix of homogeneous coordinates relative to the standard basis of monomials $\{e_j\}_{j \in \mathbb{Z}}$. Then

$$\pi_\lambda(w(t)) = \det(W_\lambda(t))$$

is the Plücker coordinate of $w(t)$ corresponding to the partition $\lambda$. The KP $\tau$-function is defined to be the Plücker coordinate corresponding to the trivial partition $\lambda = 0$:

$$\tau_w(t) := \pi_0(w(t)) = \det(W_+(t)).$$

Since, as will be seen in the next subsection, all other Plücker coordinates of $w(t)$ can be determined from $\tau_w(t)$ by applying constant-coefficient differential operators with respect to the $t$ variables, defined in terms of the Schur functions, the Plücker relations for $\mathcal{P}(w(t))$ can be expressed as an infinite system of bilinear differential relations satisfied by $\tau_w(t)$, namely, the Hirota relations [15], which are equivalent to the hierarchy of KP flow equations.

### 2.3. Schur function expansions.

Recall that if we view the flow parameters $t = (t_1, t_2, \ldots)$ as power sums in terms of a set $\{x_1, \ldots, x_N\}$ of $N$ auxiliary variables,

$$t_i = \frac{1}{i} \sum_{a=1}^{N} x_a^i,$$

then the Schur function $s_\lambda$, which is the irreducible character of the tensor representation of $\mathfrak{sl}(N)$ with symmetry type corresponding to the partition $\lambda$, is given by the Jacobi–Trudi formula [18]

$$s_\lambda(t) = \det(h_{\lambda_i-i+j})_{1 \leq i, j \leq n}$$

for any $n \geq \ell(\lambda)$, where $\{h_j(t)\}_{j=1,\ldots,\infty}$ are the complete symmetric functions defined by the generating-function formula

$$\exp \left( \sum_{i=1}^{\infty} t_iz^i \right) = \sum_{j=0}^{\infty} h_j(t)z^j.$$

We have $h_0(t) = 1$ and it is understood in (2.32) that $h_j(t) := 0$ for $j < 0$.

Given any function $f(t)$ that admits a Taylor series expansion in the flow variables about the origin $0 := (0, 0, \ldots)$,

$$f(t) = \left. \left( \exp \left( \sum_{i=1}^{\infty} \tilde{t}_i \frac{\partial}{\partial \tilde{t}_i} \right) f(t) \right) \right|_{\tilde{t} = 0},$$

where $\tilde{t} = (\tilde{t}_1, \tilde{t}_2, \ldots)$, we can use the Schur functions as a basis and express the series as

$$f(t) = \sum_{\lambda} f_\lambda s_\lambda(t).$$
(This determines the ‘Bose–Fermi equivalence’, associating an element \( \sum \lambda f_\lambda |\lambda \rangle \) of the Fermi Fock space \( \mathcal{F}_0 \) to an element \( f \) of the Bose Fock space, with \( f \) viewed as a symmetric function of an underlying infinite series of parameters \( \{x_a\}_{a=1,...,\infty} \) related by \( (2.31) \), in the inductive limit, to the flow parameters \( t \).) Using the Cauchy–Littlewood identity \([18]\)

\[
\exp \left( \sum_{i=1}^{\infty} i t_i \dot{t}_i \right) = \sum_{\lambda} s_\lambda(t) s_\lambda(\dot{t})
\]

in the form

\[
\exp \left( \sum_{i=1}^{\infty} t_i \frac{\partial}{\partial t_i} \right) = \sum_{\lambda} s_\lambda(t) s_\lambda(\partial_t),
\]

where

\[
\partial_t := \left\{ \frac{1}{i} \frac{\partial}{\partial t_i} \right\}_{i=1,2,...},
\]

we obtain

\[
 f_\lambda = s_\lambda(\partial_t)(f(t)) \big|_{t=0}.
\]

For the \( \tau \)-function \( \tau_w(t) \), the coefficients in the expansion coincide with the Plücker coordinates \( \pi_\lambda(w) \) of the initial point \( w \in \text{Gr}_{\mathcal{H}_+}(\mathcal{H}) \).

**Proposition 2.2** (M. and Y. Sato \([15]\)). The Schur function expansion of \( \tau_w(t) \) is

\[
\tau_w(t) = \sum_{\lambda} \pi_\lambda(w) s_\lambda(t).
\]

The Plücker coordinates are therefore given by

\[
\pi_\lambda(w) = s_\lambda(\partial_t)(\tau_w(t)) \big|_{t=0}.
\]

**Proof.** By using formula \( (2.33) \) it is easy to see that the metric representation of the action \( (2.25) \) is given by

\[
W(t) = \begin{pmatrix} H_{++}(t) & H_{+-}(t) \\ 0 & H_{-+}(t) \end{pmatrix} \begin{pmatrix} W_+ \\ W_- \end{pmatrix},
\]

where

\[
H_{++}(t) := \begin{pmatrix} \ddots & \ddots & \ddots & \vdots \\ \ddots & 1 & h_1 & h_2 \\ \vdots & 0 & 1 & h_1 \\ \vdots & 0 & 0 & 1 \end{pmatrix},
\]

\[
H_{+-}(t) := \begin{pmatrix} \vdots & 1 & h_1 & h_2 \\ h_3 & \ddots & \ddots & \vdots \\ h_2 & h_3 & \ddots & \ddots \\ h_1 & h_2 & h_3 & \ddots \end{pmatrix}, \quad H_{-+}(t) := \begin{pmatrix} 1 & h_1 & h_2 & h_3 & \cdots \\ 0 & 1 & h_1 & h_2 & \cdots \\ 0 & 0 & 1 & h_1 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}.
\]
Letting
\[ H(t) := (H_{++}(t) \quad H_{+-}(t)), \tag{2.45} \]
we have
\[ \tau_w(t) = \det(H(t)W) = \sum_{\lambda} \det(H_{\lambda}(t)) \det(W_\lambda) = \sum_{\lambda} \pi_\lambda(H^T(t)) \pi_\lambda(w), \tag{2.46} \]
where the second equality is the Cauchy–Binet identity and
\[ \pi_\lambda(H(t)) := \det(h_{\lambda_i-i+j}(t))|_{1 \leq i, j \leq n} = s_\lambda(t) \tag{2.47} \]
by (2.32). We thus obtain the Schur function expansion (2.40) of the \( \tau \)-function.

On the ‘big cell’, each \( \pi_\lambda(w) \) is determined through (2.21) as a finite determinant in terms of the affine coordinates \( \{A_{ab}\} \) of \( w \), which by (2.22) coincide, within a sign, with the hook partition Plücker coordinates. Therefore, we need only apply (2.41) to obtain
\[ (-1)^b A_{ab} = \pi_{(a \mid b)}(w) = s_{(a \mid b)}(\partial_t)(\tau_w(t)) \big|_{t=0}. \tag{2.48} \]
Substituting the expression (2.21) for the Plücker coordinates \( \pi_{(a_1, \ldots, a_r \mid b_1, \ldots, b_r)} \) in (2.40), we obtain the following.

**Corollary 2.2.**
\[ \frac{\tau_w(t)}{\tau_w(0)} = \sum_{\lambda} (-1)^r \sum_{k=1}^{b-1} b_k \det(A_{a,b_k})|_{1 \leq i, j \leq r} s_\lambda(t). \tag{2.49} \]

A further simplification can be made using the following identity, for which a simple proof follows from the above definitions.

**Lemma 2.1.**
\[ s_{(a \mid b)}(t) = (-1)^b \sum_{j=1}^{b+1} h_{b-j+1}(-t) h_{a+j}(t). \tag{2.50} \]

**Proof.** By the Jacobi–Trudi formula (2.32),
\[ s_{(a \mid b)}(t) = \det \begin{pmatrix} h^T & h \\ H & k \end{pmatrix}, \tag{2.51} \]
where
\[ h^T = (h_{a+1}(t), h_{a+2}(t), \ldots, h_{a+b}(t)), \quad h := h_{a+b+1}(t), \tag{2.52} \]
\[ k := \begin{pmatrix} h_b(t) \\ h_{b-1}(t) \\ \vdots \\ h_1(t) \end{pmatrix}, \quad H := \begin{pmatrix} 1 & h_1(t) & h_2(t) & \cdots & h_{b-1}(t) \\ 0 & 1 & h_1(t) & h_2(t) & \cdots & h_{b-2}(t) \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & \cdots & \ddots & 1 \end{pmatrix}. \tag{2.53} \]
It follows from the generating function formula (2.33) that the inverse $H^{-1}$ is given by

\[
H^{-1} := \begin{pmatrix}
1 & h_1(-t) & h_2(-t) & \cdots & \cdots & h_{b-1}(-t) \\
0 & 1 & h_1(-t) & h_2(-t) & \cdots & h_{b-2}(-t) \\
0 & 0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & 1
\end{pmatrix}
\]  
(2.54)

and

\[
\sum_{j=-a}^{b} h_{a+j}(-t)h_{b-j}(t) = \delta_{ab}.
\]  
(2.55)

The matrix

\[
\begin{pmatrix}
0^T & H^{-1} \\
1 & -h^{-1}h^TH^{-1}
\end{pmatrix}
\]  
(2.56)

has determinant $(-1)^b$, and its product with the matrix in (2.51) is

\[
\begin{pmatrix}
h^T \\
H \\
k
\end{pmatrix}
\begin{pmatrix}
0^T & H^{-1} \\
1 & -h^{-1}h^TH^{-1}
\end{pmatrix}
= \begin{pmatrix}
h \\
k
I - h^{-1}kh^TH^{-1}
\end{pmatrix}.
\]  
(2.57)

Therefore,

\[
s_{(a\mid b)}(t) = (-1)^b h \det(I - h^{-1}kh^TH^{-1}) = (-1)^b(h - H^{-1}kh^T).
\]  
(2.58)

But it follows from (2.54) and (2.55) that

\[
H^{-1}k = -\begin{pmatrix}
h_b(-t) \\
h_{b-1}(-t) \\
\vdots \\
h_1(-t)
\end{pmatrix},
\]  
(2.59)

from which (2.50) follows, in view of the definition (2.52) of $h$. QED

Substituting the identity (2.50) into (2.48) thus gives the following.

**Corollary 2.3.**

\[
A_{ab} = -\sum_{j=0}^{b} h_{a+j-1}(-\partial_t)h_{b-j}(\partial_t)(\tau_w(t))\big|_{t=0}.
\]  
(2.60)

The relations (2.21), (2.41), (2.48) determining the Plücker coordinates of $w \in \text{Gr}_+ (H)$ on the ‘big cell’ are equivalent to the fact that the formal Baker–Akhiezer function [13], [14], [17] defined by the Sato formula [15]

\[
\psi_w(z, t) = \exp\left(\sum_{i=1}^{\infty} t_i z^i\right) \frac{\tau_w(t - [z^{-1}])}{\tau_w(t)},
\]  
(2.61)
where

\[ [z^{-1}] := \left( \frac{1}{z}, \frac{1}{2z^2}, \frac{1}{3z^3}, \ldots \right). \quad (2.62) \]

takes its values in \( w \in \text{Gr}_{H^+}(\mathcal{H}) \) for all values of \( t \).

Following Sato, we can also introduce the dual Baker function:

\[
\Psi^*_w(z, t) = -\frac{\tau_w(t + [z^{-1}])}{\tau(t)} \exp \left\{ -\sum_{i=1}^{\infty} t_i z^i \right\}. \quad (2.63)
\]

Then, as shown in [13]–[15], the KP hierarchy equations can also be expressed in the form of Hirota bilinear equations for the \( \tau \)-function.

**Theorem 2.1** (Hirota bilinear relation [13]–[15]).

\[
\text{Res}_{z=0} \Psi^*_w(z, t) \Psi^*_w(z, \tilde{t}) \equiv 0, \quad (2.64)
\]

where \( \text{Res}_{z=0} \) just means the coefficient of \( z^{-1} \) in the formal Laurent expansion about \( z = 0 \), and the relation is satisfied identically in the infinite set of KP flow variables \( t = (t_1, t_2, \ldots) \), \( \tilde{t} = (\tilde{t}_1, \tilde{t}_2, \ldots) \).

**Remark 2.3.** In view of (2.63), equation (2.64) can be written equivalently as

\[
\text{Res}_{z=0} \exp \left\{ \sum_{i=1}^{\infty} t_i z^i \right\} \exp \left\{ -\sum_{i=1}^{\infty} \tilde{t}_i z^i \right\} \tau(t - [z^{-1}]) \tau(\tilde{t} + [z^{-1}]) \equiv 0 \quad (2.65)
\]

identically in \( t \) and \( \tilde{t} \).

**Remark 2.4.** It is also shown in [13]–[15] that (2.61) is simply an expression of the infinite set of Plücker relations satisfied by the coefficients \( \pi_\lambda(w) \) appearing in the Schur function expansion (2.40).

### 3. Algebraic curves

**3.1. Baker–Akhiezer function and \( \tau \)-function for algebraic curves.** A particularly important class of \( \tau \)-functions consists of those associated to algebraic curves [11], [12], [16]. The relevant data needed to define these are: an algebraic curve \( X \) of genus \( g \), a positive non-special divisor of degree \( g \)

\[
D := \sum_{i=1}^{g} p_i, \quad p_i \in X, \quad (3.1)
\]

(or, equivalently, a degree-\( g \) positive line bundle \( \mathcal{L} \to X \) in general position satisfying suitable generic stability conditions), a point ‘at infinity’ \( p_\infty \in X \), and a local parameter \( \xi = 1/z \) defined on a disc

\[
D_\infty := \{ p(\zeta), \ |\zeta| \leq 1, \ p(0) = p_\infty \} \quad (3.2)
\]

centered at \( p_\infty \). The points \( p_i \) are assumed to lie in the complement \( D_0 := X - D_\infty \).

If we identify \( S^1 \) with \( \partial D_\infty \), then the associated element \( w := w(X, D, p_\infty, \zeta) \in \text{Gr}_{H^+}(\mathcal{H}) \) is the closure of the space of functions \( f \in L^2(S^1) \) admitting a meromorphic extension to \( D_0 \) with pole divisor subordinate to \( \mathcal{D} \).
To realize the construction we use canonical $\theta$-functions of $g$ variables, $g \geq 1$:

$$\theta(z) = \sum_{m \in \mathbb{Z}^g} \exp\{i\pi m^T T m + 2i\pi m^T z\}, \quad z \in \mathbb{C}^g,$$

where $T$ is a complex symmetric $g \times g$ matrix with positive-definite imaginary part. The space of such matrices (the Siegel upper half-space) will be denoted by $S^g$.

The $\theta$-function is holomorphic on $\mathbb{C}^g \times S^g$ and satisfies

$$\theta(z + n) = \theta(z), \quad \theta(z + Tn) = \exp\{-i\pi(n^T Tn + 2z^T n)\} \theta(z).$$

(3.4)

In the case of $\theta$-functions associated to an algebraic curve we have

$$T := A^{-1}B,$$

(3.5)

where $A$ and $B$ are the period matrices of a basis of holomorphic differentials.

As shown in [11], [12], [16], the corresponding Baker–Akhiezer function can be chosen as the restriction to $\partial D_+$ of a meromorphic function on $X - p_\infty$ with pole divisor $\mathcal{D}$ and with an essential singularity at $p_\infty$ of the form

$$\psi_w(p(\zeta), t) \sim \exp\left\{ \sum_{i=1}^{\infty} t_i z^i \right\} \left( 1 + o\left( \frac{1}{z} \right) \right).$$

(3.6)

The Riemann–Roch theorem implies that there is just a one-dimensional space of such functions. They can be expressed, up to a $t$-dependent normalization, as

$$\tilde{\psi}_w(p, t) = \exp\left\{ \int_{p_0}^{p} \Omega(t) \right\} \frac{\theta(\mathcal{A}(p) - \mathcal{A}(\mathcal{D}) + \sum_{i=1}^{\infty} U_i t_i - K)}{\theta(\mathcal{A}(p) - \mathcal{A}(\mathcal{D}) - K)},$$

(3.7)

where $\theta$ is the $\theta$-function with $T$ equal to the Riemann period matrix defined below, relative to a suitable polygonization obtained by cutting along a canonical homology basis $(a_1, \ldots, a_g; b_1, \ldots, b_g)$ with intersection matrix

$$a_i \circ a_j = b_i \circ b_j = 0, \quad a_i \circ b_j = \delta_{ij},$$

(3.8)

$p_0$ is an arbitrarily chosen base point,

$$\mathcal{A} : \mathcal{I}^g(X) \to \mathbb{C}^g,$$

$$\mathcal{A} : \sum_{j=1}^{g} p_j \mapsto \sum_{j=1}^{g} \int_{p_0}^{p_j} \omega, \quad \omega = \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_g \end{pmatrix},$$

(3.9)

is the Abel map with $i$th component

$$\mathcal{A}_i := \sum_{j=1}^{g} \int_{p_0}^{p_j} \omega_i, \quad i = 1, \ldots, g,$$

(3.10)

where $(\omega_1, \ldots, \omega_g)$ is a canonically normalized basis of the space $H^0(K)$ of holomorphic Abelian differentials,

$$\oint_a \omega_j = \delta_{ij}, \quad \oint_b \omega_j = T_{ij},$$

(3.11)
and $K \in \mathbb{C}^g$ is the Riemann constant, chosen so that $\theta(\mathcal{A}(p) - \mathcal{A}(D) - K)$ vanishes at the $g$ points $p = p_i$ in the divisor $D$.

We define the linear family of Abelian differentials of the second kind

$$\Omega(t) = \sum_{j=1}^{\infty} \Omega_j t_j,$$

(3.12)

where $\Omega_i$ is the unique normalized Abelian differential of the second kind with pole divisor of degree $j + 1$ at $p_\infty$ having the local form

$$\Omega_j \sim d(z^j) + \text{(holomorphic)}$$

(3.13)

near $p_\infty$, with vanishing $a$-cycles

$$\oint_{a_i} \Omega_j = 0, \quad i = 1, \ldots, g.$$ 

(3.14)

Then $2\pi i U_j \in \mathbb{C}^g$ is defined to be the vector of $b$-cycles with components

$$\oint_{b_k} \Omega_j = 2\pi i (U_j)_k, \quad k = 1, \ldots, g.$$ 

(3.15)

For comparison with the formal Baker function $\psi_a(z, t)$ appearing in the Sato formula (2.61), we must interpret $p = p(z)$ within the punctured disc $D_\infty - p_\infty$ and on its boundary, and normalize $\tilde{\psi}(p, t)$ in formula (3.7) so as to obtain the correct local expansion (3.6) near $\xi = 0$:

$$\psi_w(p(\xi), t) = \frac{\tilde{\psi}_w(p(\xi), t)}{a_0(t)},$$

(3.16)

where

$$\tilde{\psi}_w(p(\xi), t) \sim \exp \left\{ \int_{p_0}^{p(\xi)} \Omega(t) \right\} \left( a_0(t) + a_1(t)\xi + \cdots \right).$$

(3.17)

Since $\int_{p_0}^{p} \Omega_i$ has the local expansion

$$\int_{p_0}^{p} \Omega_i = \xi^{-i} + \sum_{j=1}^{\infty} \frac{1}{j} Q_{ij} \xi^j + q_j,$$

(3.18)

where

$$Q_{ij} = Q_{ji}, \quad 1 \leq i, j \leq \infty,$$

(3.19)

and $\mathcal{A}(p(z))$ has the expansion [12], [35]

$$\mathcal{A}(p(z)) = \mathcal{A}(p_\infty) - \sum_{j=1}^{\infty} \frac{1}{j} U_j z^{-j},$$

(3.20)
this gives the formula

\[
\psi_w(p(z), t) = \exp \left\{ \sum_{i=1}^{\infty} t_i \left( z^i + \sum_{j=1}^{\infty} \frac{1}{j} Q_{ji} z^{-j} \right) \right\} \\
\times \frac{\theta(e + \sum_{i=1}^{\infty} U_i (t_i - z^{-i}/i)) \theta(e)}{\theta(e - \sum_{i=1}^{\infty} U_i z^{-i}/i) \theta(e + \sum_{i=1}^{\infty} U_i t_i)} \tag{3.21}
\]

near \( z = \infty \)
where
\[
e := \mathcal{A}(p_{\infty}) - \mathcal{A}(\mathcal{D}) - \mathcal{K}. \tag{3.22}
\]

(Note that the assumptions behind formula (3.7) imply that \( e \) is not on the theta divisor; \( \theta(e) \neq 0 \).) The last ratio of \( \theta \)-function factors in (3.21),
\[
\frac{\theta(e)}{\theta(e + \sum_{i=1}^{\infty} U_i t_i)},
\]
does not depend on \( z \), and hence the space spanned by the values of \( \psi_w(p(z), t) \) is the same as that spanned by
\[
\tilde{\psi}_w(p(z), t) := \exp \left\{ \sum_{i=1}^{\infty} t_i \left( z^i + \sum_{j=1}^{\infty} \frac{1}{j} Q_{ji} z^{-j} \right) \right\} \frac{\theta(e + \sum_{i=1}^{\infty} U_i (t_i - z^{-i}/i))}{\theta(e - \sum_{i=1}^{\infty} U_i z^{-i}/i)}.
\tag{3.23}
\]
We can expand the remaining ratio of \( \theta \)-function terms as a power series in \( z^{-1} \):
\[
\frac{\theta(e + \sum_{i=1}^{\infty} U_i (t_i - z^{-i}/i))}{\theta(e - \sum_{i=1}^{\infty} U_i z^{-i}/i)} \left( e + \sum_{i=1}^{\infty} U_i t_i \right) = \sum_{j=0}^{\infty} z^{-j} h_j(-\nabla_U) \theta(e + \sum_{i=1}^{\infty} U_i t_i) \tag{3.24}
\]
and
\[
\frac{\theta(e - \sum_{i=1}^{\infty} \frac{1}{i} U_i z^{-i})}{\theta(e + \sum_{i=1}^{\infty} U_i z^{-i})} = \sum_{j=0}^{\infty} z^{-j} h_j(-\nabla_U) \theta(e), \tag{3.25}
\]
where
\[
\nabla_U := \left( \nabla_{U_1}, \frac{1}{2} \nabla_{U_2}, \frac{1}{3} \nabla_{U_3}, \ldots \right) \tag{3.26}
\]
and \( \nabla_{U_i} \) is the directional derivative in \( \mathbb{C}^g \) along \( U_i \).

We now define a basis \( \{ w_0, w_1, \ldots \} \) for \( w \) as
\[
w_0(z) := \tilde{\psi}(z, t)|_{t=0} = 1, \quad w_j(z) := \frac{\partial \tilde{\psi}(z, t)}{\partial t_j} \bigg|_{t=0}, \quad j \geq 1. \tag{3.27}
\]
Then
\[
w_j(z) = z^j + P_{0j} + \sum_{i=1}^{\infty} \left( \frac{1}{i} Q_{ij} + P_{ij} \right) z^{-i}, \quad j = 1, 2, \ldots \tag{3.28}
\]
where
\[
\sum_{i=0}^{\infty} P_{ij} z^{-i} := \sum_{i=0}^{\infty} M_{ij} z^{-i} \sum_{i=0}^{\infty} N_{i} z^{-i}
\] (3.29)
and
\[
M_{ij} := \nabla_{U_j} h_i (-\nabla_{U}) \theta(e), \quad N_i := h_i (-\nabla_{U}) \theta(e).
\] (3.30)
It follows that the affine coordinates \( A_{ij} \) of the element \( w \) are
\[
A_{ij} := \frac{1}{i+1} Q_{i+1,j} + P_{i+1,j}, \quad i = 0, 1, 2, \ldots, \quad j = 1, 2, \ldots,
A_{i0} = 0, \quad i = 0, 1, 2, \ldots.
\] (3.31)
By Corollary 2.1 this determines the Plücker coordinates for all hook partitions, and hence, by (2.23), for all partitions.

Comparing formula (3.21) for the Baker function with the Sato formula (2.61), we see that the \( \tau \)-function for \( w = w(X,D,p_{\infty},\zeta) \) is given by (cf. [35])
\[
\tau_{w}(t) = \exp \sum_{i=1}^{\infty} \lambda_i t_i \exp \left( -\frac{1}{2} \sum_{i,j=1}^{\infty} Q_{ij} t_i t_j \right) \theta \left( e + \sum_{i=1}^{\infty} t_i U_i \right),
\] (3.32)
where
\[
\lambda_i := \mu_i + \sum_{k=0}^{i-1} \frac{Q_{k,i-k}}{2k(i-k)}.
\] (3.33)
with the \( \mu_i \) determined by the formula
\[
\theta(e) \exp \left\{ \sum_{i=1}^{\infty} \frac{\mu_i}{iz^i} \right\} := \theta \left( e - \sum_{i=1}^{\infty} \frac{U_i}{iz^i} \right) = \sum_{i=0}^{\infty} N_i z^{-i}.
\] (3.34)
From the viewpoint of the KP hierarchy, however, the linear-exponential factor \( \exp \{ \sum_{i=1}^{\infty} \lambda_i t_i \} \) in (3.32) can be removed, since for the Baker-Akhiezer function this just corresponds to a gauge transformation that is constant with respect to the \( t \) variables:
\[
\psi_w(z, t) \rightarrow k(z) \psi_w(z, t), \quad k(z) := \exp \left\{ \sum_{i=1}^{\infty} \frac{\lambda_i}{iz^i} \right\},
\] (3.35)
which leaves the solutions to the KP flow equations invariant. Equivalently, this means replacing \( w \in \text{Gr}_{\mathcal{H}}(\mathcal{H}) \) by
\[
w_k := \text{span} \{ kv, v \in w \}.
\] (3.36)
Therefore, the \( \tau \)-function can be chosen in the simpler gauge equivalent form (cf. [16], [29], [35])
\[
\tau_{w}(t) = \exp \left\{ -\frac{1}{2} \sum_{i,j=1}^{\infty} Q_{ij} t_i t_j \right\} \theta \left( e + \sum_{i=1}^{\infty} t_i U_i \right).
\] (3.37)
Henceforth, we denote this \( \tau \)-function as \( \tau(e,t) = \tau_{w}(t) \).
Remark 3.1. Whereas this gauge factor has no effect on the solutions of the KP hierarchy, it plays an important role in the more general setting of non-isospectral deformations. It is related to the tensor weight of the \( \tau \)-function, viewed as a section of a line bundle over the moduli space. (See \([36]–[38]\).)

Applying formula (2.60) directly to \( \tau_w(t) \) as defined in (3.37) provides an alternative way to compute the affine coordinates \( A_{ab} \) that is equivalent, up to such a gauge transformation, to (3.31).

The Hirota bilinear relations (2.65) defining the KP hierarchy can in this case be written equivalently as

\[
\text{Res}_{\xi=0} \frac{1}{\xi^2} \left[ \exp \left\{ \sum_{n=1}^{\infty} t_n \xi^{-n} \right\} \exp \left\{ -\sum_{n=1}^{\infty} \frac{\xi^n}{n} \frac{\partial}{\partial t_n} \right\} \tau(e; t) \right. \\
\left. \times \exp \left\{ -\sum_{n=1}^{\infty} \tilde{t}_n \xi^{-n} \right\} \exp \left\{ \sum_{n=1}^{\infty} \frac{\xi^n}{n} \frac{\partial}{\partial \tilde{t}_n} \right\} \tau(e; \tilde{t}) \right] = 0,
\]

where \( \xi = \xi(q) \) is the local coordinate of the point \( q \) near \( p \), \( \xi(p) = 0 \).

3.2. Weierstrass gaps, bases, and the fundamental bi-differential. The Weierstrass gap theorem \([39]\) states the following.

**Theorem 3.1 (Lückensatz).** For any point \( p \in X \) on a non-singular algebraic curve \( X \) of genus \( g \) there exist precisely \( g \) distinct non-negative integers \( n_1, \ldots, n_g \), satisfying the inequalities

\[
n_1 = 1 < n_2 < \cdots < n_g < 2g
\]

and such that no meromorphic function on \( X \) can have pole divisors solely at \( p \) of degrees \((n_1, \ldots, n_g)\).

The set of integers \((n_1, \ldots, n_g)\) is called the Weierstrass gap numbers, and will be denoted by \( \mathcal{W}(p) \). For a point ‘in general position’ the gap numbers are \( 1, \ldots, g \). A point \( p \in X \) for which there is a meromorphic function with a pole of order smaller than \( g + 1 \) at \( p \) is called a Weierstrass point.

For the given point \( p_\infty \) and local parameter \( \xi(p) \), the \( 2g \)-dimensional space \( H_1^*(X, \mathbb{Z}) \) dual to the homology group \( H_1(X, \mathbb{Z}) \) can be identified with a space of meromorphic differentials having poles only at \( p_\infty \), with vanishing residues, the pairing being given by integration over cycles. A basis \( \{u_1, \ldots, u_g, \Omega_{n_1}, \ldots, \Omega_{n_g}\} \) for this space consists of the \( g \) elements \( \{u_1, \ldots, u_g\} \) providing a basis for the subspace \( H^0(X, K) \) of holomorphic differentials defined so that \( u_j \) vanishes to order \((n_j - 1)\) at \( p_\infty \):

\[
u_k = -\left( (\xi(p))^{n_k-1} + \text{(higher-order terms)} \right) d(\xi(p)), \quad k = 1, \ldots, g, \quad n_k \in \mathcal{W}(p),
\]

with higher-order terms \( \xi(p)^k \) consisting only of powers \( k \) for which \( k + 1 \notin \mathcal{W}(p) \).

The remaining basis elements \( \{\Omega_{n_1}, \ldots, \Omega_{n_g}\} \) are normalized differentials of the second kind, where the \( \Omega_k \) are as defined in (3.13), (3.14). These are mutually dual under the pairing

\[
\frac{1}{2\pi i} \int_X u_j \wedge \Omega_{n_k} = \text{Res}_{p=p_\infty} \left( \left( \int_{p_0}^p u_j \right) \Omega_{n_k}(p) \right) = \delta_{jk}.
\]
We denote the non-vanishing period matrices with respect to the a- and b-cycles by
\[
A_{ij}, \quad B_{ij},
\]
\[\Omega_{n_i} =: C_{ij} = (U_{n_i})_j, \quad i, j = 1, \ldots, g.\] (3.42)

The columns of the $g \times g$ matrix $C$ are thus given by the vectors $U_{n_i}$ corresponding to the Weierstrass gaps
\[C = (U_{n_1}, \ldots, U_{n_g}), \quad n_j \in \mathcal{W}(p_\infty).\] (3.44)

The Riemann bilinear relations, obtained by applying Stokes’ theorem to the 2-forms $u_j \wedge u_k$ and $u_j \wedge \Omega_k$ on the canonical polygonization of the curve, then imply
\[AB^T = BA^T, \quad AC^T = 1_g.\] (3.45)

The relation to the basis \{\omega_1, \ldots, \omega_1, \Omega_{n_1}, \ldots, \Omega_{n_g}\} of normalized differentials is thus
\[\omega_i = \sum_{j=1}^g A_{ij}^{-1} u_j = \sum_{j=1}^g C_{ji} u_j,\] (3.46)

and the normalized Riemann period matrix $T$ is
\[T := A^{-1}B = C^T B.\] (3.47)

Defining the vectors
\[R_j = A U_j, \quad j = 1, 2, \ldots,\] (3.48)

we get from (3.20) that the differentials $u_i$ have the local expansion
\[u_i(p(\xi)) = -\sum_{j=n_i}^\infty (R_j)_i \xi(p)^{j-1} d\xi(p).\] (3.49)

It is also convenient to introduce the normalized symmetric bi-differential $\Omega(p, q)$ on $X \times X$ (see [40], [41], [21]) defined by the following conditions:
• $\Omega(p, q)$ has a second-order pole on the diagonal $p = q$ where its local form, expressed in terms of the parameters $\xi(p)$ and $\xi(q)$, is
\[\Omega(p, q) = \left(\frac{1}{(\xi(q) - \xi(p))^2} + \sum_{i,j=0}^\infty \mu_{ij} \xi(p)^i \xi(q)^j\right) d\xi(p) d\xi(q),\] (3.50)

with
\[\mu_{ij} = \mu_{ji};\] (3.51)

• $\Omega(p, q)$ is holomorphic elsewhere with respect to both $p$ and $q$;
• the a-cycles all vanish,
\[\oint_{p \in a_j} \Omega(p, q) = \oint_{q \in a_j} \Omega(p, q) = 0.\] (3.52)
These conditions uniquely determine $\Omega(q,p)$, which can be given the explicit representation

$$\Omega(p,q) = d_p d_q \log \left( \omega(p) - \omega(q) + \delta \right),$$

where $\delta$ is a non-singular odd half-period.

The following further properties are also consequences of the definitions:

- The $b$-cycles are given by the normalized holomorphic differentials $\omega_j$, $j = 1, \ldots, g$,

$$\oint_{p \in b_j} \Omega(p,q) = 2\pi i \omega_j(q), \quad \oint_{q \in b_j} \Omega(p,q) = 2\pi i \omega_j(p).$$

- The residues at $p_\infty$ of the locally defined bi-differentials $\xi(p)^{-j} \Omega(p,q)$ and $\xi(q)^{-j} \Omega(p,q)$ are given by the normalized differentials $\Omega_j$ of the second kind,

$$\text{Res}_{p = p_\infty} \xi(p)^{-j} \Omega(p,q) = -\Omega_j(q), \quad \text{Res}_{q = p_\infty} \xi(q)^{-j} \Omega(p,q) = -\Omega_j(p).$$

- The coefficients $\mu_{ij}$ in the expansion (3.50) are related to the coefficients in the expansion (3.18) by

$$\mu_{ij} = -Q_{i+1,j+1}.$$  

3.3. Planar model of the curve. Henceforth, we assume that the algebraic curve $X$ of geometric genus $g \geq 1$ is given by the equation

$$X : \quad P(x,y) = 0, \quad x, y \in \mathbb{C},$$

with $P$ a polynomial in $x$ and $y$,

$$P(x,y) = y^n + a_1(x)y^{n-1} + \cdots + a_n(x),$$

where the $a_k(x)$, $k = 0, \ldots, n$, are polynomials in $x$ and $n > 1$. Suppose that the curve $X$ has a Weierstrass point at infinity $p_\infty$, where the coordinates $x$, $y$ are locally expressed as

$$x = \frac{1}{\xi^n} + \cdots, \quad y = \frac{1}{\xi^s} + \cdots,$$

and the order of any monomial term in $P(x,y)$ is the order of its pole at $p_\infty$. Now suppose that the curve $X$ can be written in the form

$$y^n - x^s + (\text{lower-order terms}) = 0,$$

where $n, s > 2$ are positive integers.

In what follows, the planar coordinates of a point $p$ are denoted by $(x(p), y(p))$. When considering local expansions near a reference point $p_\infty$ with local parameter $\xi(p)$, we also use $p(\xi)$ to denote the point, with $p(0) = p_\infty$. It is then possible to include the principal part of $\Omega$ in an explicit algebraic expression in terms of the coefficients of the curve $X$. 

Theorem 3.2 [23]. The fundamental bi-differential can be expressed in the form

\[
\Omega(p, q) = \frac{\mathcal{F}(p, q)}{(x(p) - x(q))^2} \frac{dx(p) \, dx(q)}{P_y(x(p), y(p))P_y(x(q), y(q))} + \sum_{i=1}^g \sum_{j=1}^g \omega_i(p) \gamma_{ij} \omega_j(q) \\
= \frac{\mathcal{F}(p, q)}{(x(p) - x(q))^2} \frac{dx(p) \, dx(q)}{P_y(x(p), y(p))P_y(x(q), y(q))} + \sum_{i=1}^g \sum_{j=1}^g \omega_i(p) \chi_{ij} \omega_j(q),
\]

(3.61)

(3.62)

where \(\mathcal{F}(p, q)\) is a polynomial function of the coordinates \((x(p), y(p), x(q), y(q))\) which is a linear form in the coefficients of the polynomials \(\{a_0(x), \ldots, a_n(x)\}\) defining the planar model of the curve (3.58), and the symmetric \(g \times g\) matrices \(\chi\) and \(\gamma\) have elements \(\chi_{ij}, \gamma_{ij}\) related by

\[
\gamma_{ij} = (\mathfrak{L}^T \chi \mathfrak{L})_{ij} = -\int_{p \in \alpha_i} \int_{q \in \alpha_j} \frac{\mathcal{F}(p, q)}{(x(p) - x(q))^2} \frac{dx(p) \, dx(q)}{P_y(x(p), y(p))P_y(x(q), y(q))}.
\]

(3.63)

Remark 3.2. The representation (3.62) of the fundamental bi-differential \(\Omega(p, q)\) in the hyperelliptic case is classical and can be found in the books [23] and [20]. It is summarized in [24] and extended to non-hyperelliptic curves in [42], [43].

Remark 3.3. As already mentioned above in Remark 1.1, the matrix \(\chi\) appearing in the definition of the multivariable \(\sigma\)-function is defined only up to the addition of an arbitrary symmetric matrix, say \(\chi\). The change \(\chi \to \chi + \chi\), however, does not affect the higher Klein formula nor its consequences, the algebraic and differential relations between the \(\varphi\)-functions. In the examples to follow, explicit expressions will be given for the polynomial \(\mathcal{F}(p, q)\) defining the fundamental bi-differential \(\Omega(p, q)\).

The following is a consequence of (3.62).

Corollary 3.1. The coefficients \(\mu_{ij}\) in the expansion (3.50) can be decomposed as a sum

\[
\mu_{ij} = -Q_{i+1, j+1} = \mu_{ij}^{alg} + \mu_{ij}^{trans},
\]

(3.64)

where \(\mu_{ij}^{alg}\) is determined by the expansion

\[
\Omega^{alg}(p, q) = \frac{\mathcal{F}(p, q)}{(x(p) - x(q))^2} \frac{dx(p) \, dx(q)}{P_y(x(p), y(p))P_y(x(q), y(q))} \\
= \left(\frac{1}{(\xi(q) - \xi(p))^2} + \sum_{i, j=0}^{\infty} \mu_{ij}^{alg} \xi(p)^i \xi(q)^j\right) d\xi(q) \, d\xi(p),
\]

(3.65)

\(\xi(p)\) and \(\xi(q)\) are local coordinates of the points \(p\) and \(q\), and \(\mu_{ij}^{trans}\) is given by

\[
\mu_{ij}^{trans} = \mathbf{R}_{i+1}^T \chi \mathbf{R}_{j+1}.
\]

(3.66)
Proof. The coefficients $\mu_{ij}$ in the expansion in (3.50) of the normalized symmetric bi-differential $\Omega(p, q)$ (projective connection) near the diagonal $p = q$ can be expressed as a sum $\mu_{ij} = \mu_{ij}^{\text{alg}} + \mu_{ij}^{\text{trans}}$. The first term is obtained by expansion of the left-hand side of (3.65) as a rational bi-differential on $X \times X$. The second term follows from the local expansion in powers of $\xi$ of the holomorphic differentials $u_i$ in the second term of the right-hand side of (3.62). It is transcendental and given by (3.66). The term $\mu_{ij}^{\text{alg}}$ defines the holomorphic part of the expansion of the rational bi-differential appearing in the first term in (3.61) and (3.62).

Remark 3.4. Note that the double integral in (3.63) can be decomposed into periods of basic holomorphic and meromorphic differentials of the second kind. Following the Baker construction [23], [44], we represent the integrand $\Omega^{\text{alg}}(p, q)$ in the form

$$\Omega^{\text{alg}}(p, q) = \frac{\mathcal{F}((x, y); (z, w))}{(x - z)^2} \frac{dx \, dz}{P_g(x, y)P_z(z, w)} = \frac{1}{2} \frac{d}{dz} \Pi_{(z, w)}^{(z', w')} (x, y) \, dz + u^T(x, y) \mathbf{r}(z, w).$$

(3.67)

Here $\Pi_{(z, w)}^{(z', w')} (x, y)$ is a differential of the third kind with first-order poles at the points $(z, w)$ and $(z', w')$ and corresponding residues $\pm 1$, $\mathbf{u} = (u_1, \ldots, u_g)^T$ is the vector of holomorphic differentials normalized as in equation (3.40), and $\mathbf{r} = (r_1, \ldots, r_g)^T$ is the vector of meromorphic differentials with poles of orders $n_1 + 1, \ldots, n_g + 1$ at $p_{\infty}$. (The pole location $(z', w')$ can be taken as arbitrary and does not affect the construction.) The differentials $\mathbf{r}$ are then chosen to satisfy the symmetry condition

$$\Omega^{\text{alg}}(p, q) = \Omega^{\text{alg}}(q, p).$$

(3.68)

The explicit algebraic construction of the differentials $\mathbf{r}$ is described in [23] and further developed in [24]. In particular, in the case of a hyperelliptic curve we have

$$P(x, y) = y^2 - \mathcal{P}_{2g+1}(x),$$

(3.69)

where $\mathcal{P}_{2g+1}(x)$ is a polynomial in $x$ of degree $2g + 1$, and

$$\Pi_{(z, w)}^{(z', w')} (x, y) = \frac{1}{2y} \left\{ \frac{y + w}{x - z} - \frac{y + w'}{x - z'} \right\} dx.$$

(3.70)

The second term in (3.70) can be taken as an arbitrary finite point of the curve.

The matrices of $\alpha$- and $\beta$-periods of the differentials $\mathbf{r}$ as given in the Introduction by equations (1.21) and (1.20) enter in the definition of the multivariable $\sigma$-function (3.72) below.

3.4. $\sigma$-functions and algebro-geometric formulae for $\pi_\lambda(w)$. We denote by $\mathcal{D} = p_1 + \cdots + p_g$ a positive non-special divisor of degree $g$ and let

$$\mathbf{v} := \sum_{i=1}^g \int_{p_{\infty}}^{q_i} \mathbf{u} + 2\mathbf{K} = -2\mathbf{e},$$

(3.71)
where $\mathfrak{A}$ is defined in (3.42), $K$ is the vector of Riemann constants with base point at $p_\infty$, and $\mathbf{u} = (u_1, \ldots, u_g)^T$.

The multivariable $\sigma$-function $\sigma(\mathbf{v})$ is defined by the formula

$$\sigma(\mathbf{v}) = C \theta(\mathbf{e}) \exp \left\{ \frac{1}{2} \mathbf{v}^T \mathbf{r} \mathbf{v} \right\}, \quad \mathbf{e} = -\mathfrak{A}^{-1} \mathbf{v}, \quad (3.72)$$

where $C$ is a constant depending on the moduli of the curve and its explicit form is not needed here. This definition is a natural generalization of the Weierstrass $\sigma$-function from the theory of elliptic functions to higher genera.

The multivariable $\zeta$-functions $\zeta = (\zeta_1, \ldots, \zeta_g)$ are defined as

$$\zeta_k(\mathbf{v}) = \frac{\partial}{\partial v_k} \log \sigma(\mathbf{v}), \quad k = 1, \ldots, g. \quad (3.73)$$

The Kleinian $\wp$-functions are the second logarithmic derivatives of the $\sigma$-function:

$$\wp_{ik}(\mathbf{v}) = -\frac{\partial^2}{\partial v_i \partial v_k} \log \sigma(\mathbf{v}), \quad i, k = 1, \ldots, g. \quad (3.74)$$

More generally, we denote higher-order logarithmic derivatives as

$$\wp_{i_1, \ldots, i_l, \ldots, k_1, \ldots, k_m}(\mathbf{v}) = -\frac{\partial^{m_1i_1 + \cdots + m_ki_k}}{\partial v_{i_1}^{m_1} \cdots \partial v_{i_k}^{m_k}} \log \sigma(\mathbf{v}), \quad i_1, \ldots, i_k \in \{1, \ldots, g\}. \quad (3.75)$$

In the classical theory, the following theorem provides the basic means for deriving algebraic and differential relations between multivariable $\wp$-functions. (See, for example, [23] and the more recent exposition [24].)

**Theorem 3.3** (Klein formula). Let the planar curve $X$ of genus $g$ be defined by the polynomial equation $P(x, y) = 0$. Choose a set of independent holomorphic differentials in the form

$$u_k = \frac{\phi_k(x, y)}{f_y(x, y)} \, dx, \quad k = 1, \ldots, g, \quad (3.75)$$

where the $\phi_k(x, y)$ are polynomials in $x$ and $y$. Let $p = (x, y)$ be an arbitrary point of $X$ and $\mathcal{D} = p_1 + p_2 + \cdots + p_g$ a positive non-special divisor on $X$, $p_k = (x_k, y_k)$. Let $\mathbf{v}$ be the shifted image of $\mathcal{D}$ under the Abel map as in (3.71). Then

$$\sum_{j,k=1}^g \wp_{jk} \left( \int_{p_0}^p \mathbf{u} - \mathbf{v} \right) \phi_k(x, y) \phi_j(x_r, y_r) = \frac{\mathcal{F}(p, p_r)}{(x - x_r)^2}, \quad r = 1, \ldots, g, \quad (3.76)$$

where the polynomial $\mathcal{F}(p, p_r) = \mathcal{F}(\langle x, y \rangle; (x_r, y_r))$ defines the fundamental bi-differential $\Omega(p, p_r)$.

**Remark 3.5.** This formula was first given for hyperelliptic curves in [40], [41]. But the proof, which is based on the Riemann vanishing theorem and the representation of the fundamental bi-differential in the form (3.62), can easily be extended to non-hyperelliptic curves.
The Weierstrass–Poincaré theorem (see, for example, [45]) says that any \( g + 1 \) Abelian functions on the Jacobian of an algebraic curve of genus \( g \) are algebraically dependent. In particular, it follows from this that, in the case of a hyperelliptic curve of genus \( g \), among the \( (g + 1)g/2 \) functions \( \wp_{ij} \) there are only \( g \) that are algebraically independent, so these functions must satisfy \( g(g - 1)/2 \) relations. (By using the Klein formula (3.76) it was shown in [24] that these relations are quartics that represent a Kummer variety. (See the example of a genus-two curve in § 4.1 below.))

Another set of relations that follow from the Klein formula describe the Jacobi variety of the curve \( X \) and integrable flows of KP type using \( \wp \)-functions as coordinates. In particular, in the case of hyperelliptic curves all products \( \wp_{ijk} \wp_{pqr} \) are cubic polynomials in \( \wp_{ij} \). (For more details see [24].) Here we will show that these results can also be obtained within \( \tau \)-function theory on the basis of the Sato formula, or equivalently, from the bilinear identity. To do so, we represent the Sato \( \tau \)-function in terms of the multivariable \( \sigma \)-function of Klein. These methods of derivation of integrable hierarchies of KP type for Jacobi and Kummer varieties are compared in [46].

**Proposition 3.1.** The normalized algebro-geometric \( \tau \)-function \( \frac{\tau(e, t)}{\tau(e, 0)} \) is expressible in terms of the multivariable \( \sigma \)-function as

\[
\frac{\tau(e, t)}{\tau(e, 0)} = \frac{\sigma\left(\sum_{k=1}^{\infty} R_k t_k + v\right)}{\sigma(v)} \exp\left\{ \sum_{k=1}^{\infty} \Lambda_k(v)t_k \right\} \exp\left\{ \frac{1}{2} \sum_{k,l=1}^{\infty} \mu^{\text{alg}}_{kl} t_k t_l \right\}. \tag{3.77}
\]

Here \( v = A e \) is the shifted Abelian image (3.71) of the positive non-special divisor \( D \), and \( \mu^{\text{alg}}_{ik} \) are the coefficients in the expansion of the algebraic part of the bi-differential \( \Omega(p, q) \) near the point \( p_\infty \). The coefficients \( \Lambda_k(v) \) are given by

\[
\Lambda_k(v) = R^T_k \cdot v, \quad k = 1, 2, \ldots. \tag{3.78}
\]

**Proof.** The algebro-geometric \( \tau \)-function in the gauge-transformed form (3.37) leads to the relation

\[
\frac{\tau(e, t)}{\tau(e, 0)} = \exp\left\{ \frac{1}{2} \sum_{i,j=0}^{\infty} \mu^{\text{alg}}_{ij} t_i t_j + \frac{1}{2} \sum_{i,j=0}^{\infty} \mu^{\text{trans}}_{ij} t_i t_j \right\} \frac{\theta(e + \sum_{i=1}^{\infty} U_i t_i)}{\theta(e)}, \tag{3.79}
\]

where the \( \lambda_i \) are given in (3.33) and the relation (3.56) was used.

On the other hand, from the definition of the \( \sigma \)-function we get

\[
\sigma\left(\sum_{k=1}^{\infty} R_k t_k + v\right) = C \theta\left(\sum_{k=1}^{\infty} U_k t_k + A^{-1} v\right) \exp\left\{ \frac{1}{2} \sum_{k,l=1}^{\infty} R^T_k \cdot R_l t_k t_l \right\}
\times \exp\left\{ \sum_{k=1}^{\infty} R^T_k \cdot v t_k \right\} \exp\left\{ \frac{1}{2} v^T \cdot v \right\}, \tag{3.80}
\]

where \( C \) is the constant given in the definition (3.72) of the \( \sigma \)-function. Taking equation (3.66) into account, we get (3.77) with the coefficients \( \Lambda_k \) given in (3.78).
Since the $\tau$-function is defined only up to a linear-exponential factor in $t$, we omit the linear-exponential factor in (3.77) to get the simpler formula

$$\frac{\tau(e, t)}{\tau(e, 0)} = \frac{\sigma\left(\sum_{k=1}^{\infty} R_k t_k + v\right)}{\sigma(v)} \exp\left\{\frac{1}{2} \sum_{k, l=1}^{\infty} \mu_{kl}^{\text{alg}} t_k t_l\right\}, \quad v = -A e. \quad (3.81)$$

**Remark 3.6.** A similar formula for the algebro-geometric $\tau$-functions in terms of the $\sigma$-function was given by Nakayashiki in [31], where the terms linear in $t_k$ in the exponent were also taken into account.

At first glance the use of the multivariable $\sigma$-function instead of the Riemann $\theta$-function in the expression for the algebro-geometric $\tau$-function seems a trivial change. But as a result, the quadratic form in the exponent has coefficients that are algebraically expressed in terms of coefficients of the curve. Moreover, these coefficients, $\mu^{\text{alg}}_{kl}$, are polynomials in the coefficients of the curve (see [47] for details).

As shown below, such a representation of the $\tau$-function results in solutions of the integrable KP hierarchy expressed as differential polynomials in the $\wp$-functions with polynomial coefficients determined directly in terms of the coefficients of the polynomials $P(x, y)$ defining the curve.

According to Propositions 2.1 and 2.2, for any curve $X$ of genus $g$ the associated algebro-geometric $\tau$-function admits the expansion

$$\frac{\tau(e, t)}{\tau(e, 0)} = \sum_{\lambda} \pi_\lambda(w) s_\lambda(t), \quad (3.82)$$

where

$$\pi_\lambda(w) = (-1)^{\sum_{k=1}^{\infty} b_k} \det(A_{a_i b_j})|_{1 \leq i, j \leq r}, \quad (3.83)$$

the sum being over all partitions $\lambda$, which in the Frobenius notation have the form $(a_1, \ldots, a_r | b_1, \ldots, b_r)$. Here $A_{ij}$ with $i, j = 0, \ldots, \infty$ are the elements of the affine coordinate matrix $A$ representing the Grassmannian element $w(X, \mathcal{G}, p_\infty, \zeta) \in \text{Gr}_H^+(\mathcal{H})$.

**Corollary 3.2.** The elements $A_{ij}$ are expressible as polynomials in the Kleinian symbols

$$\zeta_i(v), \varphi_{ij}(v), \ldots, \quad i, j \in \{1, \ldots, g\}, \quad (3.84)$$

and the coefficients of the polynomial $P(x, y)$.

**Proof.** The quantities $\pi_\lambda(w)$ in the $\tau$-expansion (3.82) are expressible in terms of the quotients $\sigma_i(v)/\sigma(v), \sigma_{ij}(v)/\sigma(v), \ldots$. But for $i, j, k, \ldots = 1, \ldots, g$ we get

$$\frac{\sigma_i(v)}{\sigma(v)} = \zeta_i(v), \quad \frac{\sigma_{ij}(v)}{\sigma(v)} = \zeta_i(v)\zeta_j(v) - \varphi_{ij}(v), \quad (3.85)$$

$$\frac{\sigma_{ijk}(v)}{\sigma(v)} = \zeta_i(v)\varphi_{jk}(v) + \zeta_j(v)\varphi_{ik}(v) + \zeta_k(v)\varphi_{ij}(v) - \zeta_i(v)\zeta_j(v)\zeta_k(v) + \varphi_{ijk}(v), \quad (3.86)$$

and so on...
To each symbol $\wp_{k_1\ldots k_g}$ we assign a weight

$$\wp_{k_1\ldots k_g} \iff \mathcal{W}_{k_1\ldots k_g} = \sum_{j=1}^{g} k_j n_j,$$  \hfill (3.87)

where $\{n_i\}_{i=1,\ldots,g}$ is the Weierstrass gap sequence at infinity.

To each coefficient $a_{kl}$ of a monomial term $a_{kl}x^ky^l$, $k < s$, $l < n$, in the polynomial $P(x,y)$ defining the curve (3.57) we assign the weight $\mathcal{W}$ that is the sum of the weights of the factors,

$$a_{kl} \iff \mathcal{W}_{kl} = ns - (nk + ls).$$  \hfill (3.88)

Finally, we assign to a monomial whose factors are $\wp$-symbols and coefficients $a_{k,j}$ the weight $\mathcal{W}$ that is the sum of the weights of the factors,

$$a_{ij} \cdots a_{kl} \wp_{i_1\ldots i_g} \cdots \wp_{k_1\ldots k_g} \iff \mathcal{W} = \mathcal{W}_{ij} + \cdots + \mathcal{W}_{kl} + \mathcal{W}_{i_1\ldots i_g} + \cdots + \mathcal{W}_{k_1\ldots k_g}.$$  \hfill (3.89)

Consider the set of (Giambelli-like) relations

$$\pi(a_1,\ldots,a_r | b_1,\ldots,b_r) = (-1)\sum_{i=1}^{r} b_i \det(A_{a_ib_j}),$$  \hfill (3.90)

corresponding to a partition $\lambda = (a_1,\ldots,a_r | b_1,\ldots,b_r)$ of weight $\mathcal{W}$. The above procedure reduces the relations (3.90) to corresponding homogeneous polynomial relations of weight $\mathcal{W}$ between the $\wp$-functions $\wp_{i_1\ldots i_k}$, with coefficients that are polynomials in the coefficients of the polynomial $P(x,y)$ defining the curve. Such relations describe KP-type hierarchies in terms of $\wp$-coordinates.

4. Examples and applications of Schur function expansions

4.1. $\tau$-function of a hyperelliptic curve. Let $X$ be a hyperelliptic curve of genus $g$ defined by the equation

$$X : \quad P(x,y) = 0,$$  \hfill (4.1)

with polynomial $P(x,y)$ given as

$$P(x,y) = y^2 - 4x^{2g+1} - \cdots - \alpha_0 = y^2 - 4 \prod_{j=1}^{2g+1} (x - a_j).$$  \hfill (4.2)

As above, denote by $p = (x,y)$ an arbitrary point of $X$ and let $p_\infty = (\infty, \infty)$. We choose a canonical basis of cycles $(a_1,\ldots,a_g; b_1,\ldots,b_g) \in H_1(X,\mathbb{Z})$ and fix the basic holomorphic differentials $u = (u_1,\ldots,u_g)^T$ as

$$u_i(p) = \frac{x^{i-1} dx}{y}, \quad i = 1,\ldots,g.$$  \hfill (4.3)

As above, we denote the period matrices by $\mathfrak{A}$ and $\mathfrak{B}$ and let $T = \mathfrak{A}^{-1}\mathfrak{B}$. 
Let \( \mathcal{D} = q_1 + \cdots + q_g \) be a non-special divisor of degree \( g \), and let \( \mathbf{v} \) be the shifted Abel map given in (3.71). In this case the vector of Riemann constants \( \mathbf{K} \) can be given as the image under the Abel map of the divisor \( \mathcal{D} = (p_1, \ldots, p_g) \), where \( p_k = (a_k, 0) \) are branch points whose Abelian images are odd half-periods (§VII.1.2):

\[
\mathbf{K} = - \sum_{k=1}^{g} \int_{p_k}^{p_{\infty}} \omega = - \mathfrak{A}^{-1} \sum_{k=1}^{g} \int_{p_k}^{p_{\infty}} \mathbf{u}.
\]

(4.4)

Therefore, we can write

\[
\mathbf{v} = \sum_{k=1}^{g} \int_{(a_k, 0)}^{q_k} \mathbf{u}.
\]

(4.5)

In the classical theory (see, for example, [23], [44]) it was shown that the quadratic bi-differential \( \Omega(p, q) \) can be chosen in the form

\[
\Omega(p, q) = \frac{F(x, z) + 2yz}{4(x-z)^2} \, dx \, dz + 2\mathbf{v}^T(p) \mathbf{\zeta}(q),
\]

(4.6)

where the polynomial \( F(x, z) \) is the Kleinian 2-polar

\[
F(x, z) = \sum_{m=0}^{g} x^m z^m (2\alpha_{2m} + (x+z)\alpha_{2m+1}),
\]

(4.7)

and the symmetric \( g \times g \) matrix \( \mathbf{\zeta} \) is given as \( \mathbf{\zeta} = \mathfrak{A}^{-1} \mathbf{\Theta} \), where \( \mathbf{\Theta} \) is the matrix of \( \alpha \)-periods of the meromorphic differentials

\[
r_j = \sum_{k=j}^{2g+1-j} (k+1-j)\alpha_{k+1+j} \frac{x^k \, dx}{4y}, \quad j = 1, \ldots, g.
\]

(4.8)

**Theorem 4.1** (Klein formula for a hyperelliptic curve). Let the planar curve \( X \) be defined by the polynomial equation (4.2). Let \( p = (x, y) \) be an arbitrary point of \( X \) and let \( \mathcal{D} = p_1 + \cdots + p_g \) be a non-special divisor on \( X \), \( p_k = (x_k, y_k) \). Let \( \mathbf{v} \) be the shifted Abel map of \( \mathcal{D} \) given in (4.5). Then

\[
\sum_{i,k=1}^{g} \phi_{ik} (\mathcal{A}(p) - \mathcal{A}(p_{\infty}) + \mathbf{v}) x^{k-1} x_{r}^{i-1} = \frac{F(x, x_r) - 2yy_r}{4(x-x_r)^2}, \quad r = 1, \ldots, g,
\]

(4.9)

where the polynomial

\[
\mathcal{F}(p, p_r) = \mathcal{F}( (x, y), (x_r, y_r) ) = F(x, x_r) - 2yy_r
\]

defines the fundamental bi-differential \( \Omega(p, p_r) \).
Remark 4.1. The differential relations between the \( \wp_{ij} \) describe all possible integrable equations associated with the given curve. Moreover, one can show that, for arbitrary genus \( g \), any even derivative \( \wp_{i_1 \ldots i_{2k}} \), \( k > 1 \), can be written as a polynomial in the \( \wp_{ik} \) with coefficients expressible in terms of the invariants of the curve [24]. A complete set of differential relations in the particular cases \( g = 2 \) and \( g = 3 \) can be found in [48], [20]; in [49] these relations were obtained in covariant form.

We now restrict ourselves to the case of a genus-two curve whose equation can be taken in the form

\[
y^2 = 4x^5 + \alpha_4 x^4 + \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0 = 4(x - a_1)(x - a_2)(x - a_3)(x - a_4)(x - a_5), \quad a_i \neq a_j.
\]  

(4.10) \hspace{1cm} (4.11)

The holomorphic differentials \( u = (u_1, u_2)^T \) are related to \( v = (v_1, v_2)^T \), where

\[
v_1 = \frac{x \, dx}{y}, \quad v_2 = \frac{dx}{y},
\]  

(4.12)

by the transformation

\[
\left( \begin{array}{cc}
1 & \alpha_4/8 \\
0 & 1
\end{array} \right).
\]  

(4.13)

According to the definition (3.65), the first few coefficients \( \mu_{ij}^{\text{alg}} \) are

\[
\begin{align*}
\mu_{ij}^{\text{alg}} &= 0 \quad \text{if } i \text{ or } j \text{ or both are even}, \\
\mu_{11}^{\text{alg}} &= -\frac{1}{16} \alpha_4, \\
\mu_{13}^{\text{alg}} &= \mu_{31} = -\frac{1}{16} \alpha_3 + \frac{3}{256} \alpha_2^2, \\
\mu_{15}^{\text{alg}} &= \mu_{51} = -\frac{1}{16} \alpha_2 + \frac{3}{128} \alpha_3 \alpha_4 - \frac{5}{2048} \alpha_4^3, \\
\mu_{33}^{\text{alg}} &= \frac{3}{16} \alpha_2 + \frac{1}{32} \alpha_3 \alpha_4 - \frac{3}{1024} \alpha_4^3, \\
\vdots
\end{align*}
\]  

(4.14)

We introduce the set of residue vectors

\[
\mathbf{R}_{2k} = 0, \quad k = 1, 2, \ldots,
\]

and

\[
\mathbf{R}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{R}_3 = \begin{pmatrix} -\alpha_4/8 \\ 1 \end{pmatrix}, \quad \mathbf{R}_5 = \begin{pmatrix} -\alpha_3/8 + 3\alpha_2^2/128 \\ -\alpha_4/8 \end{pmatrix}, \quad \ldots.
\]  

(4.15)
It follows from formulae (2.48) and (2.60) that the first few affine matrix components are

\[
A_{00}(v) = \zeta_1,
\]
\[
A_{01}(v) = \frac{1}{2} \zeta_1^2 - \frac{1}{2} \varphi_{11} - \frac{1}{16} \alpha_4,
\]
\[
A_{02}(v) = \frac{1}{6} \zeta_1^3 + \frac{1}{3} \zeta_2 - \left( \frac{1}{2} \varphi_{11} + \frac{5}{48} \alpha_4 \right) \zeta_1 - \frac{1}{6} \varphi_{111},
\]
\[
A_{03}(v) = \frac{1}{24} \zeta_1^4 + \frac{1}{3} \zeta_1 \zeta_2 - \left( \frac{7}{96} \alpha_4 + \frac{1}{4} \varphi_{11} \right) \zeta_1 - \frac{1}{6} \varphi_{111} \zeta_1
\]
\[
\quad - \frac{1}{24} \varphi_{1111} - \frac{1}{3} \varphi_{12} + \frac{1}{8} \varphi_{11}^2 + \frac{7}{96} \alpha_4 \varphi_{11} - \frac{1}{24} \alpha_3 + \frac{5}{512} \alpha_4^2,
\]
\[
A_{10}(v) = A_{01}(v),
\]
\[
A_{11}(v) = \frac{1}{3} \zeta_1^3 - \frac{1}{3} \zeta_2 - \left( \varphi_{11} + \frac{1}{12} \alpha_4 \right) \zeta_1 - \frac{1}{3} \varphi_{111},
\]
\[
A_{12}(v) = \frac{1}{8} \zeta_1^4 - \frac{1}{2} \zeta_1 \varphi_{111} + \frac{3}{8} \varphi_{11}^2 - \frac{1}{8} \varphi_{1111} - \left( \frac{3}{4} \varphi_{11} + \frac{3}{32} \alpha_4 \right) \zeta_1^2
\]
\[
\quad + \frac{3}{32} \alpha_4 \varphi_{11} + \frac{3}{512} \alpha_4^2.
\]
\[
A_{13}(v) = \frac{1}{30} \zeta_1^5 + \frac{1}{6} \zeta_1^2 \zeta_2 - \frac{1}{3} \varphi_{1111} \zeta_1^2
\]
\[
\quad - \left( \frac{1}{3} \varphi_{11} + \frac{1}{16} \alpha_4 \right) \zeta_1^3 - \left( \frac{1}{6} \varphi_{11} + \frac{1}{48} \alpha_4 \right) \zeta_2
\]
\[
\quad + \left( - \frac{1}{6} \varphi_{1111} - \frac{1}{3} \varphi_{12} + \frac{1}{2} \varphi_{11}^2 + \frac{3}{16} \alpha_4 \varphi_{11} - \frac{1}{24} \alpha_3 + \frac{7}{384} \alpha_4^2 \right) \zeta_1
\]
\[
\quad + \frac{1}{3} \varphi_{11} \varphi_{111} - \frac{1}{6} \varphi_{1111} - \frac{1}{30} \varphi_{11111} + \frac{1}{16} \alpha_4 \varphi_{111},
\]
\[
\vdots
\]

All the Kleinian symbols \( \zeta_k, \varphi_{kl}, \ldots \) in these formulae are evaluated at \( v \).

**Proposition 4.1.** The Plücker relations written for the partition \( \lambda = (2, 2) \) of weight \( 4 \) and the partitions \( \lambda = (3, 2) \) and \( \lambda = (2, 2, 1) \) of weight \( 5 \) are equivalent to the equation

\[
\varphi_{1111}(v) = 6 \varphi_{111}^2(v) + 4 \varphi_{12}(v) + \alpha_4 \varphi_{11}(v) + \frac{1}{2} \alpha_3.
\]  

**Proof.** The first non-trivial Plücker relation for the partition \( \lambda = (2, 2) \),

\[
\pi_{(1,0,1,0)} = \begin{vmatrix} A_{11} & A_{10} \\ A_{01} & A_{00} \end{vmatrix},
\]
is written in detailed form as

$$\tau(0, v) \left[ \frac{1}{12} \frac{\partial^4}{\partial t_1^4} + \frac{1}{4} \frac{\partial^2}{\partial t_2^2} - \frac{1}{3} \frac{\partial^2}{\partial t_1 \partial t_3} \right] \tau(t, v) \bigg|_{t=0} = 0$$

$$= \left[ \left( \frac{1}{3} \frac{\partial^3}{\partial t_1^3} - \frac{1}{3} \frac{\partial}{\partial t_3} \right) \tau(t, v) \bigg|_{t=0} \right] + \left[ \left( \frac{1}{2} \frac{\partial}{\partial t_2} + \frac{\partial^2}{\partial t_1^2} \right) \tau(t, v) \bigg|_{t=0} \right] = 0. \quad (4.19)$$

Substituting the expression \( (3.81) \) for \( \tau(t, e) \) into this relation and the expressions \( (4.14) \) for \( \mu_{ij} \) and computing directional derivatives, we obtain equation \( (4.17) \). Expressions for the hook diagram coefficients appearing in the determinant are given above, and

$$\pi(1,0|1,0) = -\frac{1}{12} \varphi_{1111} + \frac{1}{12} \zeta_1 - \left( \frac{1}{2} \varphi_{11} + \frac{1}{48} \alpha_4 \right) \zeta_1^2 - \frac{1}{3} \zeta_2 \zeta_1 - \frac{1}{3} \varphi_{111} \zeta_1 + \frac{1}{3} \varphi_{12} + \frac{1}{4} \varphi_{11}^2 + \frac{1}{48} \alpha_4 \varphi_{11} - \frac{1}{256} \alpha_4^2 + \frac{1}{24} \alpha_3. \quad (4.20)$$

The partitions \( \lambda = (3, 2) \) and \( \lambda = (2, 2, 1) \) of weight 5 give Plücker relations that imply the action of \( \zeta_1(v) + \partial/\partial v_1 \) on the above equation. These all yield the single equation

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\begin{align*}
\varphi_{1111}(v) &= 6\varphi_{11}^2(v) + 4\varphi_{12}(v) + \alpha_4 \varphi_{11}(v) + \frac{1}{2} \alpha_3. \quad (4.21)
\end{align*}

\begin{align*}
\varphi_{1122}(v) &= 6\varphi_{11}\varphi_{12}(v) - 2\varphi_{22}(v) + \alpha_4 \varphi_{12}(v), \quad (4.21)
\varphi_{1111}(v) &= 4\varphi_{11}^3(v) + \varphi_{22}(v) + 4\varphi_{12}(v)\varphi_{11}(v) + \alpha_4 \varphi_{11}^2(v) + \alpha_3 \varphi_{11}(v). \quad (4.22)
\end{align*}

Using the definition of the weight \( \mathcal{W} \) for the functions appearing in \( (4.21) \) and \( (4.22) \), we see that all these equations are homogeneous.
The process described here can be continued. It seems reasonable to conjecture that the whole set of differential relations between multi-index Kleinian symbols can be put into correspondence with the Young diagrams of the partitions of type $u = (2, 2, i_1, \ldots, i_n)$ in such the way that partitions of weights $2k$ and $2k + 1$ correspond to a set of equations of weight $\mathcal{W} = 2k$.

To complete the interpretation of the basic equations describing Abelian functions in terms of Plücker relations, we find that the Kummer surface arises from the Plücker relation corresponding to the $\lambda = (4, 4, 4, 4)$ diagram with weight 16:

$$
\pi_{(3,2,1,0|3,2,1,0)} = \begin{vmatrix}
A_{33} & A_{32} & A_{31} & A_{30} \\
A_{23} & A_{22} & A_{21} & A_{20} \\
A_{13} & A_{12} & A_{11} & A_{10} \\
A_{03} & A_{02} & A_{01} & A_{00}
\end{vmatrix}.
$$

(4.23)

The equation

$$
\pi_{(3,2,1,0|3,2,1,0)} = 0
$$

(4.24)

can be written in the form

$$
\begin{vmatrix}
\alpha_0 & \alpha_1/2 & -2\varphi_{22} & -2\varphi_{12} \\
\alpha_1/2 & \alpha_2 + 4\varphi_{22} & \alpha_3/2 + 2\varphi_{12} & -2\varphi_{11} \\
-2\varphi_{22} & \alpha_3/2 + 2\varphi_{12} & \alpha_4 + 4\varphi_{11} & 2 \\
-2\varphi_{12} & -2\varphi_{11} & 2 & 0
\end{vmatrix} = 0.
$$

(4.25)

This is the celebrated quartic Kummer surface, $\text{Kum}(X)$, defined as the surface in $\mathbb{C}^3$ with coordinates $x = \varphi_{11}$, $y = \varphi_{12}$, $z = \varphi_{22}$. $\text{Kum}(X)$ is the quotient of the Jacobi variety: $\text{Kum}(X) = \text{Jac}(X)/\langle u \rightarrow -u \rangle$.

Therefore, we conclude that

$$
\begin{array}{c|c}
\end{array}
\text{Kum}(X).
$$

Remark 4.2. The equation for the Kummer surface in this form was derived by Baker [20], and a generalization to higher genera was given in [24]. Also note that the equations written for all $2 \times 2$ minors of the matrix $(A_{ij})_{i,j=0,\ldots,3}$ in (4.23),

$$
\pi_{(i,k|j,l)} = \begin{vmatrix}
A_{ij} & A_{il} \\
A_{kj} & A_{kl}
\end{vmatrix}, \quad i \geq j, \quad k \geq l, \quad i, j, k, l \in \{0, 1, 2, 3\},
$$

(4.26)

give a complete set of algebraic equations describing the Jacobi variety $\text{Jac}(X)$ as an algebraic variety, and also the flows of KdV type on $\text{Jac}(X)$. This resembles the matrix realization of the Jacobi and Kummer varieties given by Baker [20], which was generalized to higher genera in [24].

The above considerations lead to the following result.

**Theorem 4.2.** Each column vector $A_k(v)$, $k = 0, 1, \ldots$, in the matrix of affine coordinates of the Grassmannian element whose components $A_{kl}$, $l = 1, 2, \ldots$, correspond to hook partitions $(k+1, l^1)$ is a polynomial in a finite set of Kleinian symbols $\zeta_i(v), \varphi_{ij}(v), \varphi_{ijk}(v)$. 
4.2. \( \tau \)-function of a trigonal curve. In this section we demonstrate how the above results appear in the case of a trigonal curve. The \( \sigma \)-function theory of trigonal Abelian functions was developed in [43]. Various results in this area were obtained in [27], [42], [50]–[53]. In order to emphasize the main idea and avoid cumbersome formulae, we restrict ourselves to the first non-trivial case of the cyclic family of trigonal curves \( X \) of genus 3 defined by

\[
P(x, y) = y^3 - (x^4 + \beta_3 x^3 + \beta_6 x^2 + \beta_9 x + \beta_{12}) = 0,
\]

and we fix a canonical basis of cycles \((a_1, a_2, a_3; b_1, b_2, b_3) \in H_1(X, \mathbb{Z})\) on \( X \).

An explicit calculation of the canonical holomorphic differentials and the meromorphic differentials conjugate to them is given in [50]. In particular, we have for \( p = (x, y) \)

\[
\begin{align*}
u_1(p) &= \frac{dx}{3y}, &\quad u_2(p) &= \frac{x dx}{3y^2}, &\quad u_3(p) &= \frac{dx}{3y^2}, \\
r_1(p) &= \frac{x^2 dx}{3y^2}, &\quad r_2(p) &= -\frac{2x dx}{3y}, &\quad r_3(p) &= -\frac{(5x^2 + 3\beta_3 x + \beta_6) dx}{3y}.
\end{align*}
\]

As above, denote the period matrices by \( \mathfrak{A} \) and \( \mathfrak{B} \) and let \( \mathbf{T} = \mathfrak{A}^{-1}\mathfrak{B} \) and \( \mathbf{z} = \mathfrak{A}^{-1}\mathfrak{G} \).

Let \( \mathcal{D} = p_1 + p_2 + p_3 \) be a non-special divisor of degree 3 and

\[
v = \sum_{i=1}^{3} \int_{p_{\infty}}^{p_i} u + \mathfrak{A} \mathbf{K},
\]

where \( \mathbf{K} \) is the vector of Riemann constants with base point at \( p_{\infty} \).

The polynomial \( \mathcal{F}((x, y), (z, w)) \) appearing in the fundamental bi-differential \( \Omega(p, q) \) is given by

\[
\mathcal{F}((x, y), (z, w)) = 3w^2y^2 + wT(x, z) + yT(z, x),
\]

where

\[
T(x, z) = 3\beta_{12} + (z + 2x)\beta_9 + x(x + 2z)\beta_6 + 3\beta_3 x^2 z + x^2 z^2 + 2x^3 z.
\]

Expanding about \( p_{\infty} \) gives the following expressions for the quantities \( \mu_{ij}^{\text{alg}} \):

\[
\begin{align*}
\mu_{00}^{\text{alg}} &= 0, \\
\mu_{01}^{\text{alg}} &= -\frac{2}{3} \beta_3, \\
\mu_{04}^{\text{alg}} &= -\frac{2}{3} \beta_6 + \frac{5}{9} \beta_3^2, \\
\mu_{13}^{\text{alg}} &= -\frac{2}{3} \beta_6 + \frac{4}{9} \beta_3^2, \\
\mu_{22}^{\text{alg}} &= 0, \\
\vdots
\end{align*}
\]
Remark 4.3. $\mu_{ij}^{alg} = 0$ unless $(i + j) + 2 \equiv 0 \mod 3$. This is a consequence of the cyclic symmetry of the curve.

In this case the Klein formula is

$$\sum_{i,k=1}^{3} \varphi\left( \int_{p_{\infty}}^{p} u - v \right) \phi_i(x, y) \phi_k(x_r, y_r) = \frac{\mathcal{F}(p, p_r)}{(x - x_r)^2}, \quad r = 1, 2, 3,$$

where $p = (x, y)$, $p_k = (x_k, y_k)$, and also $\phi_1(x, y) = y$, $\phi_2(x, y) = x$, $\phi_3(x, y) = 1$.

Expanding this relation in a neighbourhood of $p_{\infty}$ where the local coordinate $x = 1/\xi^3$ is introduced, and equating the principal parts at the poles, we obtain a set of equations involving the variables $x_x$, $y_k$ and $\varphi$-symbols. We now show that these relations can be obtained as consequences of the Plücker relations via Giambelli-type formulae.

The first Young diagram leading to a non-trivial Plücker relation, as in the hyperelliptic genus-two case, corresponds to the partition $\lambda = (2, 2)$. In this case we obtain, after simplification, the equation

$$\varphi_{1111} = 6\varphi_{11}^2 - 3\varphi_{22}. \quad (4.32)$$

Differentiating with respect to the coordinate $v_1$ gives the Boussinesq equation.

The derivation of (4.32) is based on the formula (4.18) for the trigonal curve (4.27). Namely, we have

$$\pi_{(1,0|1,0)} = \frac{1}{4} \varphi_{22} + \frac{1}{4} \zeta_2^2 - \frac{1}{12} \varphi_{1111} - \frac{1}{3} \varphi_{111} \zeta_1 + \frac{1}{4} \varphi_{11}^2 - \frac{1}{2} \varphi_{11} \zeta_1^2 + \frac{1}{12} \zeta_4^4 \quad (4.33)$$

and also

$$A_{00}(v) = \zeta_1(v),$$
$$A_{01}(v) = -\frac{1}{2} \varphi_{11}(v) + \frac{1}{2} \zeta_1^2(v) - \frac{1}{2} \zeta_2(v),$$
$$A_{10}(v) = \frac{1}{2} \zeta_2(v) - \frac{1}{2} \varphi_{11}(v) + \frac{1}{2} \zeta_1^2(v),$$
$$A_{11}(v) = -\frac{1}{3} \varphi_{111}(v) - \varphi_{11}(v) \zeta_1(v) + \frac{1}{3} \zeta_1^3(v). \quad (4.34)$$

In the trigonal case we no longer have the symmetry about the diagonal of the Young diagram that we have in the genus-two case, but we can restrict ourselves to equations of even degree by taking symmetric combinations of the two diagrams related by transposition. In the weight 5 case we have the symmetric combination

$$\begin{array}{ccc}
\begin{array}{ccc}
 & & \\
 & & \\
& & \\
 & & \\
& & \\
\end{array}
+ \begin{array}{ccc}
\begin{array}{ccc}
 & & \\
 & & \\
& & \\
& & \\
& & \\
\end{array}
\end{array},
$$

which gives the weight 5 trigonal partial differential equation

$$\varphi_{1112} = 6\varphi_{11} \varphi_{12} + 3\beta_3 \varphi_{11}.$$
For weight 6 we have the three sets of diagrams

\[
\begin{align*}
\begin{array}{c}
\text{Diagram A} \\
\text{Diagram B} \\
\text{Diagram C}
\end{array}
\end{align*}
\]

which lead to an overdetermined set of equations with the unique solution

\[
\begin{align*}
\varphi_{111}^2 &= 4\varphi_{11}^3 + \varphi_{12}^2 + 4\varphi_{13} - 4\varphi_{11}\varphi_{22}, \\
\varphi_{1122} &= 4\varphi_{13} + 4\varphi_{12}^2 + 2\varphi_{11}\varphi_{22} + 3\beta_3\varphi_{12} + 2\beta_6.
\end{align*}
\]

Continuing in this way, we recover the cyclic trigonal versions of the full set of equations given in [50].

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