Ground State Degeneracy of Potts Antiferromagnets: Cases with Noncompact 
$W$ Boundaries Having Multiple Points at $1/q = 0$

Robert Shrock*, Shan-Ho Tsai†
Institute for Theoretical Physics
State University of New York
Stony Brook, N. Y. 11794-3840

We present exact calculations of the zero-temperature partition function, $Z(G, q, T = 0)$, and ground-state degeneracy (per site), $W\{G\}, q$, for the $q$-state Potts antiferromagnet on a number of families of graphs $\{G\}$ for which the boundary $\mathcal{B}$ of regions of analyticity of $W$ in the complex $q$ plane is noncompact and has the properties that (i) in the $z = 1/q$ plane, the point $z = 0$ is a multiple point on $\mathcal{B}$ and (ii) $\mathcal{B}$ includes support for $\text{Re}(q) < 0$. These families are generated by the method of homeomorphic expansion. Our results give further insight into the conditions for the validity of large-$q$ series expansions for the reduced function $W_{\text{red.}} = q^{-1}W$.

05.20.-y, 64.60.C, 75.10.H

*email: shrock@insti.physics.sunysb.edu
†current address: Department of Physics and Astronomy, University of Georgia, Athens, GA 30602; email: tsai@hal.physast.uga.edu
I. INTRODUCTION

This paper continues our study of nonzero ground state entropy, $S_0\{G, q\} \neq 0$, i.e., ground state degeneracy (per site) $W\{\{G\}, q\} > 1$, where $S_0 = k_B \ln W$, in $q$-state Potts antiferromagnets on various lattices and, more generally, families of graphs $\{G\}$. There is an interesting connection with graph theory here, since the zero-temperature partition function of the above-mentioned $q$-state Potts antiferromagnet on a graph $G$ satisfies $Z(G, q, T = 0)_{PAF} = P(G, q)$, where $P(G, q)$ is the chromatic polynomial expressing the number of ways of coloring the vertices of the graph $G$ with $q$ colors such that no two adjacent vertices have the same color. Thus,

$$W(\lim_{n \to \infty} G, q) = \lim_{n \to \infty} P(G, q)^{1/n}$$

where $n = v(G)$ is the number of vertices of $G$. An example of a real substance exhibiting nonzero ground state entropy is ice. Just as complex analysis provides deeper insights into real analysis in mathematics, the generalization from $q \in \mathbb{Z}_+$, to $q \in \mathbb{C}$ yields a deeper understanding of the behavior of $W\{\{G\}, q\}$ for physical (positive integral) $q$. In general, $W\{\{G\}, q\}$ is an analytic function in the $q$ plane except along a certain continuous locus of points, which we denote $B$. In the limit as $n \to \infty$, the locus $B$ forms by means of a coalescence of a subset of the zeros of $P(G, q)$ (called chromatic zeros of $G$). In a series of papers we have calculated and analyzed $W\{\{G\}, q\}$ for a variety of families of graphs, both for physical values of $q$ (via rigorous upper and lower bounds, large-$q$ series calculations, and Monte Carlo measurements) and for the generalization to complex values of $q$.

A basic question that one can ask about the locus $B$ for (the $n \to \infty$ limit of a) family of graphs $\{G\}$ is whether it extends only a finite distance from the origin of the $q$ plane or, instead, extends an infinite distance away from this origin (i.e., passes through the point $z = 0$ in the $z = 1/q$ plane), and hence is noncompact. The importance of this question for the statistical mechanics of $q$-state Potts antiferromagnets is that large-$q$ Taylor series provide a powerful means of obtaining approximate values of the ground state degeneracy even for moderate values of $q$, but these exist if and only if the reduced function

$$W_{red}\{\{G\}, q\} = q^{-1}W\{\{G\}, q\}$$

is analytic at $1/q = 0$, which, in turn, is true if and only if the nonanalytic boundary $B$ does not pass through $z = 0$. Large-$q$ series calculations of $W_{red}$ have been derived for regular lattices, but we noted that for the $r \to \infty$ limit of the graph $K_2 + C_r$, the locus $B$, which is noncompact, no such large-$q$ series exists. Here, $\overline{K}_r$ denotes the complement of the complete graph $K_r$. It is clearly important to understand better the differences between the behavior of Potts antiferromagnets on families of graphs that yield $W_{red}\{\{G\}, q\}$ functions analytic at $z = 0$ and those that do not, i.e. those that yield noncompact loci $B$ passing through $z = 0$. In the present paper, we carry out such a study for families of graphs which are homeomorphic expansions of the following type: we start with families of the form $(K_r)_b + G_r$, where $G_r$ is a given graph family with $r$ vertices, $b$ signifies the removal of certain bonds in the

1 This reduced function is a natural object to define since an obvious upper bound on $P(G, q)$ describing the coloring of an $n$-vertex graph with $q$ colors is $P(G, q) \leq q^n$, and hence $W\{\{G\}, q\} \leq q$; this guarantees that $\lim_{q \to \infty} W_{red}\{\{G\}, q\}$ is a finite quantity (even if it is sometimes nonanalytic at this point). The large-$q$ series take their simplest form as series in the expansion variable $y = 1/(q - 1)$ for the equivalent reduced function $W\{\{G\}, q\} = q^{-1}(1 - q^{-1})^{-\Delta/2}W\{\{G\}, q\}$, where $\Delta$ is the coordination number of the lattice.

2 The complete graph $K_n$ on $n$ vertices is defined as the graph where each of these vertices is connected to all of the other $n - 1$ vertices by bonds. The complement $\overline{K}_n$ of $K_n$ is the graph with $n$ vertices and no bonds.
We then perform homeomorphic expansion of the bonds connecting the $K_p$ and $G_r$ subgraphs. This family is categorized as being of type HEC. An interesting feature of the families analyzed here is that, in contrast to those that we have previously studied, they yield loci $B$ (which are boundaries of regions of analyticity of $W$) that have support for $\text{Re}(q) < 0$ and for which the point $z = 0$ is a multiple point on the algebraic curve $B$ in the technical terminology of algebraic geometry [20], i.e., a point where several branches of this curve cross each other. For basic definitions on homeomorphic classes of graphs, see Ref. [6]; some recent work on homeomorphism classes in a different direction than ours is Ref. [22].

A general form for the chromatic polynomial of an $n$-vertex graph $G$ is

$$P(G, q) = c_0(q) + \sum_{j=1}^{N_n} c_j(q)a_j(q)^{t_jn}$$

where $c_j(q)$ and $a_j(q)$ are certain functions of $q$. Here the $a_j(q)$ and $c_{j\neq 0}(q)$ are independent of $n$, while $c_0(q)$ may contain $n$-dependent terms, such as $(-1)^n$, but does not grow with $n$ like $(\text{const.})^n$. A term $a_\ell(q)$ is defined as “leading” if it dominates the $n \to \infty$ limit of $P(G, q)$; in particular, if $N_a \geq 2$, then it satisfies $|a_\ell(q)| \geq 1$ and $|a_\ell(q)| > |a_j(q)|$ for $j \neq \ell$, so that $|W| = |a_\ell|^{t_\ell}$. The locus $B$ occurs where there is a nonanalytic change in $W$ as the leading terms $a_\ell$ in eq. (1.2) changes.

Since for the families of graphs studied here, the boundary $B$ is noncompact in the $q$ plane, it is often more convenient to describe the boundary in the complex $z$ or $y$ planes, where, as above, $z = 1/q$, and

$$y = \frac{1}{q-1} = \frac{z}{1-z}$$

The variable $y$ is commonly used in large-$q$ series expansions. For the families considered here $B$ is actually compact in the $y$ and $z$ planes. We define polar coordinates as

$$z = \zeta e^{i\theta}$$

and

$$y = \rho e^{i\beta}$$

and the function

$$D_k(q) = \frac{P(C_k, q)}{q(q-1)} = a^{k-2} \sum_{j=0}^{k-2} (-a)^{-j} = \sum_{s=0}^{k-2} (-1)^s \binom{k-1}{s} q^{k-2-s}$$

where $a = q - 1 = 1/y$ and $P(C_k, q)$ is the chromatic polynomial for the circuit (cyclic) graph $C_k$ with $k$ vertices,

$$P(C_k, q) = a^k + (-1)^k a$$

---

3 The “join” of two graphs $G$ and $H$ is obtained by adding bonds connecting each of the vertices of $G$ to those of $H$. This was denoted $G \times H$ in Ref. [13]; here we use the more common notation for this object in the mathematical literature, viz., $G + H$. 

2
We shall also use a standard notation from combinatorics,
\[ q^{(p)} = p! \left( \frac{q}{q - p} \right) = \prod_{s=0}^{p-1} (q-s) \tag{1.8} \]

The organization of the paper is as follows. In section 2 we construct several general multiparameter homeomorphic classes of families of graphs that, in a certain limit, yield noncompact boundaries \( B(q) \). In sections 3-5 we give exact calculations of the respective boundaries \( B \) for three such families. These exhibit features going beyond those that we found in our earlier work in three main ways: (i) the point \( z=0 \) can be a multiple point; (ii) \( B \) includes support for \( \text{Re}(q) < 0 \), and (iii) it is no longer true in general that \( \text{Re}(q) = 0 \iff q = 0 \) for \( q \in B \). A general discussion of our findings is given in section 6, and our conclusions are presented in the final section.

II. HOMEOMORPHIC CLASSES OF HEC TYPE

We can construct a large variety of families of graphs with noncompact boundaries \( B(q) \) of regions where the respective \( W \) functions are analytic by performing homeomorphic expansions of the basic family constructed and analyzed before \[13\],
\[(K_p)_b + G_r , \tag{2.1}\]
where, as before, \( b \) denotes the removal of \( b \) bonds connecting a vertex \( v \) in the \( K_p \) subgraph to the other vertices of \( K_p \). Since each vertex \( v \in K_p \) has degree \( \Delta = p - 1 \), \( b \) is bounded above according to \( b \leq p - 1 \).

We have shown earlier that (i) the locus \( B \) for \( \lim_{r \to \infty} [(K_p)_b + G_r] \) is noncompact in the \( q \) plane, passing through \( z = 1/q = 0 \) \[13\] and (ii) the analogous locus for the \( r \to \infty \) limit of homeomorphic expansions of this family has the same noncompactness property. To construct specific homeomorphic expansions, select a vertex \( v_1 \) in the \( K_p \) subgraph and perform homeomorphic expansion on each of the bonds connecting this vertex with the vertices in \( G_r \) by inserting \( k_2 \) additional degree-2 vertices on each of these bonds, where \( k_2 \geq 3 \). Then continue this process with a second vertex \( v_2 \in K_p \), inserting \( k_2-2 \) additional degree-2 vertices on each of bonds connecting \( v_2 \) with the vertices of \( G_r \), and so forth for the other vertices in \( K_p \) (including the vertex \( v \) from which the \( b \) bonds were removed, if there are any bonds left linking with this vertex). We denote the resultant homeomorphic expansion as
\[ HEC_{k_1-2,k_2-2,...,k_p-2}[(K_p)_b(v) + G_r] \tag{2.2}\]

The labelling is chosen so that, counting the two vertices on the original bond connecting to \( v_j \in K_p \) together with the \( k_j - 2 \) inserted vertices, there is a total of \( k_j \) vertices in all along what was originally this single bond. Some illustrative examples of this and other types of families to be discussed are shown in Fig. \[3\].
Another starting family with a noncompact $B$ in the $r \to \infty$ limit is

$$(K_p)_{\{b\};\{v\}} + G_r$$ (2.3)

where the subscript $\{b\}$ refers to the removal of multiple bonds from a set of non-adjacent vertices $\{v\}$ of the $K_p$ subgraph. Performing the HEC-type homeomorphic expansion as above leads to the family

$$HEC_{k_1-2,k_2-2,...,k_p-2}(\{b\};\{v\} + G_r)$$ (2.4)

A special case of eq.(2.4) applies if in the starting family one removes all of the bonds connecting vertices in the $K_p$ subgraph to each other, so that this starting family is

$$\overline{K}_p + G_r$$ (2.5)

The HEC-type homeomorphic expansion of this family is thus

$$HEC_{k_1-2,k_2-2,...,k_p-2}(\overline{K}_p + G_r)$$ (2.6)

In this case one clearly obtains the same homeomorphically expanded family if one permutes the choices of $k_j$:

$$HEC_{k_1-2,k_2-2,...,k_p-2}(\overline{K}_p + G_r) = HEC_{\pi(k_1)-2,\pi(k_2)-2,...,\pi(k_p)-2}(\overline{K}_p + G_r)$$ (2.7)

where $\pi$ is an element of the permutation group on $p$ objects, $S_p$. The number of vertices in the homeomorphic classes of families (2.2), (2.4), and (2.6) is the same, namely

$$v\left(HEC_{k_1-2,k_2-2,...,k_p-2}(K_p)_{\{b\};\{v\}} + G_r\right) = v\left(HEC_{k_1-2,k_2-2,...,k_p-2}(K_p)_{\{b\};\{v\}} + G_r\right) = v\left(HEC_{k_1-2,k_2-2,...,k_p-2}(\overline{K}_p + G_r)\right)$$
\[ p + r(1 - 2p + \sum_{j=1}^{p} k_j) \] (2.8)

Note that for \( p = 2 \), since one can only remove \( b = 1 \) bond in the \( K_2 \) subgraph, thereby obtaining \((K_2)_{b=1} = \overline{K}_2\), it follows also that

\[ \text{HEC}_{k_1-2,k_2-2}(K_2)_{b=1} + G_r] = \text{HEC}_{k_1-2,k_2-2}(\overline{K}_2 + G_r) \] (2.9)

For the case \( G_r = \overline{K}_r \), the chromatic number is

\[ \chi(\text{HEC}_{k_1-2,k_2-2}, \ldots, k_p-2, [K_p + \overline{K}_r]) = 2 \] (2.10)

For \( G_r = T_r \), for the cases \( r \geq 2 \) of interest here,

\[ \chi(\text{HEC}_{k_1-2,k_2-2}, \ldots, k_p-2, [K_p + T_r]) = 3 \] (2.11)

so that for \( r \geq 2 \),

\[ P(\text{HEC}_{k_1-2,k_2-2}, \ldots, k_p-2, [K_p + T_r], q = 2) = 0 \] (2.12)

For the \( r = 1 \) case, this family degenerates to a tree graph,

\[ \text{HEC}_{k_1-2,k_2-2}, \ldots, k_p-2, [K_p + T_r] = T_{v_t} \quad \text{where} \quad v_t = 1 + \sum_{j=1}^{p} (k_j - 1) \] (2.13)

with chromatic polynomial \( P(T_{v_t}, q) = q(q - 1)^{v_t - 1} \) and chromatic number \( \chi = 2 \).

A general result for the chromatic polynomial in the case \( r = 2, G_r = T_r \) is

\[ P(\text{HEC}_{k_1-2,k_2-2}, \ldots, k_p-2, [K_p + T_2], q) = q(q - 1) \prod_{j=1}^{p} D_{2k_j - 1} \] (2.14)

Since the respective indices \( 2k_j - 1 \) of each of the \( D_{2k_j - 1} \) in eq. (2.14) are odd and since

\[ D_{k\text{ odd}} = (q - 2) \times \text{pol}(q) \] (2.15)

where \( \text{pol}(q) \) is a polynomial in \( q \) of degree \( k - 3 \), it follows that

\[ P(\text{HEC}_{k_1-2,k_2-2}, \ldots, k_p-2, [K_p + T_2], q) = q(q - 1)(q - 2)^p \times \text{pol}'(q) \] (2.16)

where \( \text{pol}'(q) \) is a polynomial of degree \( 2 \sum_{j=1}^{p} (k_j - 2) \).

Since the number of vertices \( n = v \) is a linear function of \( p, r, \) and \( k_1, \ldots, k_p \), there are several ways of producing the limit \( n \to \infty \) (\( L \) denotes limit):

\[ L_p : p \to \infty \quad \text{with} \quad r, k_1, \ldots, k_p \quad \text{fixed} \] (2.17)

\[ L_r : r \to \infty \quad \text{with} \quad p, k_1, \ldots, k_p \quad \text{fixed} \] (2.18)

\[ L_{k_j} : k_j \to \infty \quad \text{with} \quad p, k_1, \ldots, k_{j-1}, k_{j+1}, \ldots k_p \quad \text{fixed} \] (2.19)
and, for the case where all of the \( k_j \)'s are the same, i.e., \( k_j = k_{cb} \forall j \) (where the subscript \( cb \) denotes “connecting bond”),

\[
L_k : k_{cb} \to \infty \quad \text{with} \quad p, r \quad \text{fixed}
\]  

(2.20)

As discussed before \[13\], the limit \( L_p \) is not of much interest from the viewpoint of either statistical mechanics or graph theory, since for any given graph \( G_r \) and for any given (finite) value of \( q \in \mathbb{Z}_+ \), as \( p \) becomes sufficiently large, one will not be able to color the graph \( (K_p)_b + G_r \) or homeomorphic expansions thereof, and the chromatic polynomial will vanish. The limits \( L_k \) and \( L_k \) are also not of primary interest here since they generically yield compact boundaries \( B \), as will be illustrated below. We shall therefore concentrate on the limit \( L_r \). The lowest choice of \( p \) that yields a noncompact boundary \( B(q) \) is \( p = 2 \).

For general \( p \) and \( G_r \), we remark on two special types of HEC homeomorphic expansions. A minimal case is one in which the above homeomorphic expansion is carried out only on the bonds connecting a single vertex, taken to be \( v_1 \) with no loss of generality, in \( K_p \) to vertices in \( G_r \), so \( k_2 = k_3 = \ldots = k_p = 2: \)

\[
HEC_{k_1-2,0,...,0(K_p + G_r)}
\]  

(2.21)

A particularly symmetric case is the one in which the homeomorphic expansions are the same on all of the connecting bonds:

\[
HEC_{k_1-2,k_2-2,...,k_p-2(K_p + G_r)} \quad \text{with} \quad k_1 = k_2 = \ldots = k_p = k_{cb}
\]  

(2.22)

In this case, the right-hand side of eq. (2.8) reduces to \( v = p + r + pr(k_{cb} - 2) \). We proceed to give detailed analyses of some special families of homeomorphic expansions.

III. CLASS OF FAMILIES \( HEC_{k_1-2,k_2-2,K_2 + K_r} \)

The family \( HEC_{k_1-2,k_2-2,K_2 + K_r} \) has the special property that the graphs of this family only depend on the sum of \( k_1 \) and \( k_2 \), not on each of these parameters individually:

\[
HEC_{k_1-2,k_2-2(K_2 + K_r)} = HEC_{k_1'-2,k_2'-2(K_2 + K_r)} \quad \iff \quad k_1 + k_2 = k_1' + k_2'
\]  

(3.1)

To incorporate this symmetry, we define

\[
H_{k,r} = HEC_{k-3(K_2 + K_r)} = HEC_{k_1-2,k_2-2(K_2 + K_r)} \quad \text{where} \quad k = k_1 + k_2 - 1
\]  

(3.2)

As will be evident in our explicit results to be given below, the fact that this family has a noncompact \( W \) boundary \( B(q) \) in the \( L_r \) limit can be understood to follow from its construction as a homeomorphic expansion of the starting family \( (K_2)_b + G_r = K_2 + G_r \) which (as was shown previously \[13\]) has a noncompact \( B \) (here, \( G_r = K_r \)). The noncompactness of the boundary \( B \) obviously implies that in the \( L_r \) limit, the chromatic zeros have unbounded magnitudes (in the \( q \) plane), since \( B \) arises via the coalescence of these chromatic zeros in this limit \[13\]. We note that the case \( k_1 = k_2 = 2 \), i.e., \( k = 3 \) is a special case of

---

\[4\] For a family of graphs \( G_{k,r} \), rather than analyzing the continuous locus \( B \) resulting from the \( L_r \) limit in eq. (2.18) or the \( L_k \) in eq. (2.20), one may formulate a different problem: let \( \{q_0\}_{k,r} \) denote the set of chromatic zeros of \( G_{k,r} \), and consider the union of this set, summed over both \( r \) and \( k \), denoted \( \{q_0\}_{y,k} = \sum_{r=1}^{\infty} \sum_{k=2}^{\infty} \{q_0\}_{k,r} \). This problem has been considered for the family \( G_{k,r} = H_{k,r} \) by A. Sokal, who finds (private communication) that the image of this union is dense in the vicinity of the origin, \( y = z = 0 \).
the graphs with noncompact \( B(q) \) that we have constructed and studied before \([3]\), so we concentrate on the homeomorphic expansions \( k \geq 3 \) here. The number of vertices is given by the \( p = 2 \) special case of eq. (3.2), viz., \( v(H_{k,r}) = (k - 2)r + 2 \). For arbitrary \( k \geq 2 \) and \( r \geq 1 \), \( H_{k,r} \) is bipartite, i.e.,

\[
\chi(H_{k,r}) = 2
\]

(3.3)

For \( r \geq 2 \), the girth is \( g(H_{k,r}) = 2k - 2 \). Indeed, all (non-self-intersecting) closed paths are of this length. For \( r = 1 \) and \( r = 2 \), the family degenerates according to

\[
H_{k,1} = T_k
\]

(3.4)

and

\[
H_{k,2} = C_{2k-2}
\]

(3.5)

where, as defined above, \( T_n \) and \( C_n \) are, respectively, the tree and circuit graphs with \( n \) vertices. Hence, we shall take \( r \geq 3 \).

By use of the deletion-contraction theorem \([4] - [6]\) we obtain the recursion relation

\[
P(H_{k,r}, q) = D_k P(H_{k,r-1}, q) + (-1)^{k-1} q[(q - 1)D_{k-1}]^{r-1}
\]

(3.6)

which we solve to get the chromatic polynomial

\[
P(H_{k,r}, q) = q(q - 1) \left[ D_{2k-2}(D_k)^{r-2} - (q - 1)(D_{k-1})^2 \left( (D_k)^{r-2} - \{ (q - 1)D_{k-1} \}^{r-2} \right) \right]
\]

(3.7)

This has the form of eq. (1.2) with \( N_a = 2 \),

\[
a_1 = D_k
\]

(3.8)

and

\[
a_2 = (q - 1)D_{k-1}
\]

(3.9)

In the limit \( L_r \) of eq. (2.18), the nonanalytic boundary locus \( B \) is determined as the solution of the degeneracy of magnitudes of the leading terms

\[
|a_1| = |a_2|
\]

(3.10)

To determine \( B \), we multiply eq. (3.10) by \( |q(q - 1)| \) to obtain \( |P(C_k, q)| = |(q - 1)P(C_{k-1}, q)| \) (the resultant spurious solutions thereby introduced at \( q = 0, 1 \), i.e., \( y = -1, \infty \), are understood to be ignored in the following discussion). Dividing by \( |a|^k \) yields

---

5 We recall the statement of the addition-contraction theorem \([1, 3]\): let \( G \) be a graph, and let \( v \) and \( v' \) be two non-adjacent vertices in \( G \). Form (i) the graph \( G_{\text{add}} \) in which one adds a bond connecting \( v \) and \( v' \), and (ii) the graph \( G_{\text{contr}} \) in which one identifies \( v \) and \( v' \). Then the chromatic polynomial for coloring \( G \) with \( q \) colors, \( P(G, q) \), satisfies \( P(G, q) = P(G_{\text{add}}, q) + P(G_{\text{contr}}, q) \). The reverse process of bond deletion, leading to the same equation, is known as the deletion-contraction theorem.
\[ |1 + (-1)^ky^{k-1}| = |1 + (-1)^ky^{k-2}| \]  
\[ (3.11) \]

Since \( y = 0 \) is a solution of eq. (3.11), \( B \) is noncompact in the \( q \) plane, passing through \( z = 1/q = 0 \), as noted above (equivalently, eq. (3.10) satisfies the condition given in the theorem of section IV of Ref. [13] that guarantees that the solution locus \( B \) is noncompact in the \( q \) plane). (In contrast, for fixed \( k \) \( B \) is compact in the \( z \) or \( y \) plane.)

Furthermore, we find that (i) 2\((k-2)\) curves on \( B \), consisting of \((k-2)\) complex-conjugate (c.c.) pairs, intersect at \( z = y = 0 \); (ii) these curves approach the point \( z = y = 0 \) at the angles

\[ \phi_j = \beta_j = \pm \frac{(2j+1)\pi}{2(k-2)}, \quad j = 0, ..., k-3 \]  
\[ (3.12) \]

Thus, if and only if \( k \) is odd, these angles include \( \pm \pi/2 \), i.e., a branch of \( B \) crosses the point \( z = y = 0 \) vertically. To prove these results, we write (3.11) in terms of polar coordinates using eq. (1.5):

\[ \rho^{k-2}\left[\rho^{k-2}(\rho^2 - 1) + 2(-1)^k\left\{\rho \cos[(k-1)\beta] + \cos[(k-2)\beta]\right\}\right] = 0 \]  
\[ (3.13) \]

As \( \rho \to 0 \), so that \( z \to y \) (whence \( \beta \to \phi \)), the solution is given by \( \cos[(k-2)\beta] = 0 \), which yields the results (i) and (ii). In the terminology of algebraic geometry [20], the point \( z = y = 0 \) is a singular (multiple) point on the algebraic curve \( B \) of index \( k-2 \).

As a further result, we find that (iii) if \( k \) is odd, \( B \) crosses the real \( q \) axis once, at a value \( q_c \) that increases monotonically from \( 3/2 \) for \( k = 3 \), with \( \lim_{k \to \infty} q_c = 2 \); (iv) if \( k \) is even, \( B \) never crosses the real \( q \) axis, so \( q_c \) is not defined. To show these results, we first set \( \beta = 0 \) in eq. (3.13), which becomes

\[ (\rho + 1)^2\left[\rho^{k-2} + 2(-1)^k \sum_{s=0}^{k-3} (-\rho)^s\right] = 0 \]  
\[ (3.14) \]

For odd \( k \), this has a single real, positive solution for \( \rho \), which decreases monotonically from \( \rho = 2 \) for \( k = 3 \), approaching \( \rho = 1 \) as \( k \to \infty \). For even \( k \), eq. (3.14) has no (real) solution for \( \rho \). For \( \beta = \pi \), eq. (3.13) yields

\[ (\rho - 1)^2\left[\rho^{k-2} + 2 \sum_{s=0}^{k-3} \rho^s\right] = 0 \]  
\[ (3.15) \]

Aside from the root \( \rho = 1 \), i.e., \( y = -1 \), which is spurious as noted above, eq. (3.15) has no real roots. The result for \( q \) follows directly.
FIG. 2. Boundary $B$ in the $z = 1/q$ plane for $\lim_{r \to \infty} H_{k,r}$ with $k = (a) 4 (b) 5 (c) 6$. Chromatic zeros for $H_{k,r}$ with $(k,r) = (a) (4,30) (b) (5,18) (c) (6,10)$ are shown for comparison.
FIG. 3. Boundary $B$ in the $y = 1/(q - 1)$ plane for $\lim_{r \to \infty} H_{k,r}$ with $k = 6$. Chromatic zeros for $H_{k,r}$ with $r = 10$ are shown for comparison.

The curves comprising $B$ divide the $z$ (equivalently, $y$ or $q$) plane into $k - 1$ regions. For even $k = 2\ell$ and odd $k = 2\ell + 1$, these include $(\ell - 1)$ complex conjugate pairs of regions. In the $q$ plane, $B$ consists of $k - 2$ disjoint curves (a line for $k = 3$); each curve extends inward from complex infinity, turns around and heads back out to complex infinity along a different direction. For $k = 3$, $B$ is a circle in both the $y$ and $z$ planes, given by the equation $|y - 1| = 1$, i.e., $|z - 1/3| = 1/3$. For $k = 4, 5, 6$ we show our calculations of $B$ in the $z$ planes in Fig. 2. The corresponding plots in the $y$ plane look more like (slightly distorted) cloverleafs, reflecting the simplicity of the degeneracy equation (3.11) in the $y$ variable; as an example, we show the case $k = 6$ in Fig. 3 (Additional figures in the $y$ plane are given in Ref. [21]). As is clear from Fig. 2, for $k \geq 4$, $B$ has support for $\text{Re}(q) < 0$ (equivalent to $\text{Re}(z) < 0$). Note that $\text{Re}(z) = 0$ does not, in general, imply that $z = 0$, since for $k \geq 4$, $B$ intersects the imaginary $z$ axis away from the origin, e.g., at $z = \pm z_i$, where

$$z_i = i\sqrt{2/5} = 0.632456i \quad \text{for} \quad k = 4$$

$$z_i = i\sqrt{8/7} = 1.06904i \quad \text{for} \quad k = 5$$

$$z_i = (i/3)[10 + \sqrt{82}]^{1/2} = 0.323971i, 1.45508i \quad \text{for} \quad k = 6$$

In the $y$ plane, the intercepts $y = \pm i\rho_i$ of $B$ with the imaginary $y$ axis (aside from $y = 0$) are given, for even $k = 2\ell$, by
\[ \rho_i^{2(\ell - 1)}(\rho_i^2 - 1) - 2(-1)^\ell = 0 \]  

(3.19)

and, for odd \( k = 2\ell + 1 \), by eq. (3.13) multiplied by \( \rho \). Solving these equations for general \( k \), we find that (with \( k > 3 \)) (i) for \( k = (0 \text{ or } 1) \mod 4 \), \( \rho_i \) is nonzero, decreasing monotonically from \( \sqrt{\rho} \) for \( k = 4, 5 \) toward 1 as \( k \to \infty \); (ii) for \( k = (2 \text{ or } 3) \mod 4 \), there is no nonzero intercept \( \rho_i \). Although \( \mathcal{B} \) extends to \( \text{Re}(z) < 0 \) (equivalently, \( \text{Re}(q) < 0 \)), it never includes support for negative real \( z \) or \( q \).

In Fig. 2 we also show illustrative chromatic zeros for (a) \( k = 4, r = 30 \) (\( n = 62 \)), (b) \( k = 5, r = 18 \) (\( n = 56 \)), and (c) \( k = 6, r = 10 \) (\( n = 42 \)). Aside from the always-present zeros at \( q = 0, 1 \) (\( z = \infty, 1 \)), the chromatic zeros generally lie close to the asymptotic curves comprising \( \mathcal{B} \) onto which they coalesce as \( k \to \infty \). For a given set of (finite) \( k \) and \( r \), the moduli of the zeros are bounded, and hence they avoid the points \( z = y = 0 \), approaching these only as \( r \to \infty \) for a given \( k \).

As before [12][13], we let \( R_1 \) denote the region including the positive real \( q \) axis for \( q > q_\text{c} \). For \( k \) even, this includes the entire real \( q \) axis. For odd \( k \), we denote the region containing the rest of the real \( q \) axis as \( R_{2,k \text{ odd}} \); from our statement above, it follows that there is no \( R_2 \) phase for \( k \) even. For \( q \in R_1 \), we calculate

\[
W(\lim_{r \to \infty} H_{k,r}, q) = \begin{cases} 
\left| a_2(q) \right|^{1/(k-2)} & \text{if } k \text{ is odd} \\
\left| a_1(q) \right|^{1/(k-2)} & \text{if } k \text{ is even}
\end{cases} \quad (3.20)
\]

As discussed in Ref. [12], in regions other than \( R_1 \), there is no canonical choice of phase in taking the \( 1/n \)'th root in eq. (3.1), so that one can only determine \( |W(G, q)| \). We find

\[
|W(\lim_{r \to \infty} H_{k,r}, q)| = \left| a_1(q) \right|^{1/(k-2)}, \quad \text{for } k \text{ odd and } q \in R_{2,k \text{ odd}} \quad (3.21)
\]

\[
|W(\lim_{r \to \infty} H_{k,r}, q)| = \left| a_2(q) \right|^{1/(k-2)}, \quad \text{for } k \text{ odd and } q \notin R_1, R_{2,k \text{ odd}} \quad (3.22)
\]

and for \( k \) even and \( q \notin R_1 \)

On \( \mathcal{B} \), \( |W| \) is continuous but nonanalytic.

**IV. CLASS OF FAMILIES** \( HEC_{k_{1-2,0,0}(R_3 + R_r)} \)

When \( p \geq 3 \) in the homeomorphic class of families (2.2) or (2.6) with \( G_r = R_r \), the graphs and their chromatic polynomials depend on the individual values of the \( k_j \), not just on the sum, as in eq. (3.1). A simple homeomorphic class of families of this type is the special case of eq. (2.21) for \( p = 3 \) and \( G_r = R_r \), namely

\[
\Theta_{k,r} = HEC_{k_{1-2,0,0}(R_3 + R_r)}, \quad k_1 \equiv k \quad (4.1)
\]

We have \( v(\Theta_{k,r}) = r(k - 1) + 3 \) and

\[
\chi(\Theta_{k,r}) = 2 \quad (4.2)
\]

For our present purposes, it will suffice to study the family \( \Theta_{3,r} \). We calculate the chromatic polynomial

\[
P(\Theta_{3,r}, q) = (q - 2)^{2r-4}P(\Theta_{3,2}, q) - q(q - 1)(D_4)^2 \left[ (q - 2)^{2r-4} - (D_4)^{-2} \right] - q(q - 1)^2(q - 2)^r \left[ (q - 2)^{r-2} - (q - 1)^{r-2} \right]
\]

12
\[-q(q - 1)(q - 2)(q^2 - 4q + 5)^2 \left[ (q - 2)^{2r-4} - (q^2 - 4q + 5)^{r-2} \right] \] (4.3)

where
\[ P(\Theta_{3,2}, q) = q(q - 1) \left[ (q - 1)D_6 - (q - 2)D_5 \right] \] (4.4)

This has the form of eq. (1.2) with \( N_a = 4 \) and
\[ a_1 = D_4 = q^2 - 3q + 3 \] (4.5)
\[ a_2 = q^2 - 4q + 5 \] (4.6)
\[ a_3 = (q - 1)(q - 2) \] (4.7)
and
\[ a_4 = (q - 2)^2 \] (4.8)

FIG. 4. Boundary \( B \) in the \( z = 1/q \) plane for \( \lim_{r \to \infty} \Theta_{3,r} \). Chromatic zeros for \( \Theta_{3,r} \) with \( r = 16 \) are shown for comparison.
The nonanalytic boundary $\mathcal{B}$ for the limit as $r \to \infty$ is shown in Fig. 1. As with the $H_{k,r}$ family discussed above, and the other families to be discussed below, one sees the important feature that $\mathcal{B}$ includes support for $Re(z) < 0$ or equivalently $Re(q) < 0$. The boundary divides the $z$ (or equivalently $y$ or $q$) plane into six different regions: (i) $R_1$, including the interval $0 < z < 1/2$ on the positive real $z$ axis; (ii) $R_2$, including the union of the intervals $z > 1/2$ and $z < 0$; (iii) a complex-conjugate pair of regions $R_3$ and $R_3^*$ lying just above and below $R_1$ in the $z$ plane; and (iv) a second pair of complex-conjugate regions $R_4$ and $R_4^*$ located such that $R_4$ lies above and adjacent to $R_3$ (hence between $R_3$ and $R_2$) and $R_4^*$ lies below $R_3^*$. In the regions $R_1$ and $R_2$, $a_1$ and $a_2$ are dominant, respectively, while $a_j$ is dominant in regions $R_j$ and $R_j^*$ for $j = 3$ and $j = 4$, respectively. We have

$$W(\lim_{r \to \infty} \Theta_{k,r}, q) = (a_1)^{1/2} \quad \text{for} \quad q \in R_1$$

$$|W(\lim_{r \to \infty} \Theta_{k,r}, q)| = |a_2|^{1/2} \quad \text{for} \quad q \in R_2$$

and

$$|W(\lim_{r \to \infty} \Theta_{k,r}, q)| = |a_j|^{1/2} \quad \text{for} \quad q \in R_j, R_j^*, j = 3, 4$$

The region boundaries between regions $R_i$ and $R_j$ are the solutions of the respective degeneracy equation $|a_i| = |a_j|$ where $a_i$ and $a_j$ are leading terms. Dividing these equations by $|q|^2$ and re-expressing them in terms of $z$ to get the corresponding degeneracy equations in the $z$ plane, one has $|\tilde{a}_i| = |\tilde{a}_j|$ with $\tilde{a}_1 = 1 - 3z + 3z^2$, $\tilde{a}_2 = 1 - 4z + 5z^2$, $\tilde{a}_3 = (1 - z)(1 - 2z)$, and $\tilde{a}_4 = (1 - 2z)^2$. Clearly, each $(i, j)$ pair of the degeneracy equations $|\tilde{a}_i| = |\tilde{a}_j|$ in the $z$ plane has a solution at $z = 0$, which shows that, in accordance with the condition of Ref. [13], $\mathcal{B}$ is noncompact in the $q$ plane, passing through the point $z = 0$. Thus, as is evident in Fig. 1, all of the regions are contiguous at $z = 0$, where six curves (i.e., three branches) of $\mathcal{B}$ meet at a multiple point. For small $\zeta = |z|$, the degeneracy equation for the part of $\mathcal{B}$ separating $R_1$ and $R_3$ has the form $2\zeta^2(1 - 2\cos^2 \theta) + O(\zeta^3) = 0$, so that this boundary crosses the point $z = 0$ at the angles given by $\cos \theta = 1/\sqrt{2}$, i.e.,

$$\theta_{R_1, R_3}, \theta_{R_1, R_3^*} = \pm \frac{\pi}{4}$$

(The other solution, $\cos \theta = -1/\sqrt{2}$ is not relevant because in this region, $a_1$ and $a_3$ are not leading terms.) Similarly, for small $\zeta$, the degeneracy equation for the part of $\mathcal{B}$ separating the regions $R_3$ and $R_4$ is $\zeta(3\zeta - 2\cos \theta) = 0$ so that as $\zeta \to 0$, this boundary crosses the point $z = 0$ at the angles given by $\cos \theta = 0$, i.e.,

$$\theta_{R_3, R_4}, \theta_{R_3^*, R_4^*} = \pm \frac{\pi}{2}$$

The same type of reasoning applied to the $R_2, R_4$ and $R_2, R_4^*$ boundaries shows that they cross $z = 0$ at the angles

$$\theta_{R_2, R_4}, \theta_{R_2, R_4^*} = \pm \frac{3\pi}{4}$$

Concerning the boundary separating regions $R_1$ and $R_2$, we observe that the relevant degeneracy equation

$$B(R_1 - R_2) : \quad |1 - 3z + 3z^2| = |1 - 4z + 5z^2|$$

(4.15)
has the solution \( z = 1/2 \) as its only solution other than \( z = 0 \) where \( a_1 \) and \( a_2 \) are leading terms. Hence, \( q_c = 2 \) for this family. Concerning the two complex-conjugate (c.c.) pairs of multiple points, we note that the multiple point \( z_{2,3,4} \) where regions \( R_2 \), \( R_3 \), and \( R_4 \) are contiguous, and its c.c., are

\[
z_{2,3,4}, z_{2,3,4}^* = 3^{-1/2}e^{\pm i\pi/6} = \frac{1}{2} \pm \frac{i}{2\sqrt{3}} \tag{4.16}
\]

Hence this c.c. pair lies on the unit circle \( |y| = 1 \) or equivalently \( |q - 1| = 1 \) in the respective \( y \) and \( q \) planes:

\[
y_{2,3,4}, y_{2,3,4}^* = e^{\pm i\pi/3} \tag{4.17}
\]

\[
q_{2,3,4}, q_{2,3,4}^* = 3^{1/2}e^{\pm i\pi/6} \tag{4.18}
\]

A corresponding analysis can be given for the multiple point \( z_{1,2,3} \) where regions \( R_1 \), \( R_2 \), and \( R_3 \) are contiguous.

V. CLASS OF FAMILIES \( HEC_{k_1-2,k_2-2}(\overline{K}_2 + T_r) \)

Illustrative graphs of the family

\[
HEC_{k_1-2,k_2-2}(\overline{K}_2 + T_r) \tag{5.1}
\]

with \( (k_1, k_2) = (3, 2) \) and \( (k_1, k_2) = (3, 3) \) are shown in Fig. 2(c,d), respectively. From the general equation \( (2.8) \) with \( G_r = T_r \) and \( p = 2 \), we have

\[
v(HEC_{k_1-2,k_2-2}(\overline{K}_2 + T_r)) = 2 + r(k_1 + k_2 - 3) \tag{5.2}
\]

The special case \( k_1 = k_2 = 2 \) is the family \( \overline{K}_2 + T_r = (K_2)_{b=1} + T_r \) which we analyzed in detail earlier \[13\] so we concentrate on the homeomorphic expansions of this starting family here, i.e., \( k_1 \geq 3 \) and/or \( k_2 \geq 3 \). In order to calculate the chromatic polynomial for the graphs in this homeomorphic class of families, we shall utilize a generating function method as we did in Ref. [16] for infinitely long, finite-width strip graphs of various lattices. The generating function \( \Gamma \) is a function of a symbolic variable \( x \) and yields the chromatic polynomials \( P(HEC_{k_1-2,k_2-2}(\overline{K}_2 + T_r), q) \) via the Taylor series expansion around \( x = 0 \) according to

\[
\Gamma(HEC_{k_1-2,k_2-2}(\overline{K}_2 + T_r), q, x) = \sum_{r=1}^{\infty} P(HEC_{k_1-2,k_2-2}(\overline{K}_2 + T_r), q) x^r \tag{5.3}
\]

The summation starts at \( r = 1 \) since this is the minimum value of \( r \) in this family. As in our earlier work \[16\], we find that \( \Gamma \) is a rational function of \( x \) and, separately, of \( q \), of the form

\[
\Gamma(HEC_{k_1-2,k_2-2}(\overline{K}_2 + T_r), q, x) = \frac{N(HEC_{k_1-2,k_2-2}(\overline{K}_2 + T_r), q, x)}{D(HEC_{k_1-2,k_2-2}(\overline{K}_2 + T_r), q, x)} \tag{5.4}
\]

with

\[
N(HEC_{k_1-2,k_2-2}(\overline{K}_2 + T_r), q, x) = x \sum_{j=0}^{j_{max}} A_j(q)x^j \tag{5.5}
\]
(so that \( \deg_x(N) = j_{\text{max}} + 1 \); the prefactor of \( x \) reflects the fact that the minimum value of \( r \) is 1) and

\[
D\left( HEC_{k_1-2,k_2-2}(K_2 + T_r), q, x \right) = 1 + \sum_{j'=1}^{j_{\text{max}}'} b_{j'}(q)x^{j'}
\]

(5.6)

where \( j_{\text{max}} \) and \( j_{\text{max}}' \) depend on the specific family, the \( A_j \) and \( b_{j'} \) are polynomials in \( q \), and their dependence on \( k_1 \) and \( k_2 \) is left implicit. (This notation follows that of Ref. [16] except that in eq. (5.5), we use \( A \) rather than \( a \) to avoid confusion with the \( a_j \) functions in eq. (1.2).) A general formula for \( A_0 \) is

\[
A_0 = P(T_{k_1+k_2-1}, q) = q(q-1)^{k_1+k_2-2} = q(q-1)^{k_2}
\]

(5.7)

The other \( A_j \)'s and \( b_{j'} \) polynomials will be given below for specific families.

For the starting family, \( HEC_{0,0}(K_2 + T_r) = K_2 + T_r = (K_2)_{b=1} + T_r \), and indeed its generalization \( (K_p)_b + T_r \), one has

\[
P((K_p)_b + T_r, q) = q^{(p+1)} \left[ (q-p-1)^{r-1} + b(q-p)^{r-2} \right]
\]

(5.8)

(recall the notation in eq. (1.8)). The equivalent representation in terms of a generating function is obtained by noting that eq. (5.8) has the form of eq. (1.2) with \( c_0 = 0 \) and \( N_a = 2 \), i.e.,

\[
P((K_p)_b + T_r, q) = c_1(a_1)^r + c_2(a_2)^r
\]

(5.9)

with

\[
a_1 = q - p - 1, \quad a_2 = q - p
\]

(5.10)

\[
c_1 = \frac{q^{(p+1)}}{q-p-1}
\]

(5.11)

and

\[
c_2 = \frac{b q^{(p+1)}}{(q-p)^2}
\]

(5.12)

We find that the chromatic polynomial \( P((K_p)_b + T_r, q) \) is given as the coefficient of \( x^r \) in the Taylor series expansion about \( x = 0 \) of the generating function

\[
\Gamma((K_p)_b + T_r, q; x) = \frac{x(A_0 + A_1x)}{(1 - \lambda_1 x)(1 - \lambda_2 x)}
\]

(5.13)

where

\[
\lambda_j = a_j, \quad j = 1, 2
\]

(5.14)

\[
A_0 = a_1c_1 + a_2c_2
\]

(5.15)

and

\[
A_1 = -a_1a_2(c_1 + c_2)
\]

(5.16)

For a general graph \( G \), this type of relation between the form for the chromatic polynomial (5.9) and the generating function (5.13) has a straightforward generalization to the case where \( N_a \geq 3 \). In the present case \( (K_2)_{b=1} + T_r = HEC_{0,0}(K_2 + T_r) \), we find, for the upper limits on the sums in \( N(x) \) and \( D(x) \) the values \( j_{\text{max}} = 1 \) and \( j_{\text{max}}' = 2 \).
A. Families $HEC_{k_1-2,0}(\overline{K}_2 + T_r)$

We first consider the simplest homeomorphic expansion of type $HEC$ on the starting family $\overline{K}_2 + T_r$, in which one performs this expansion only on the bonds connecting one of the two vertices in the $\overline{K}_2$ subgraph with the vertices of the $T_r$ subgraph. With no loss of generality, we choose this vertex in $\overline{K}_2$ to be $v_1$, so that $k_2 - 2 = 0$ and $k_1 - 2 \geq 1$. For brevity of notation, define

$$O_{k,r} = HEC_{k_1-2,0}(\overline{K}_2 + T_r), \quad k_1 \equiv k$$

with $v(O_{k,r}) = 2 + r(k - 1)$. When referring to the collection of the graphs of this family for various $r$, we denote it as $\{O_k\}$. The case $k = 2$ is just the original starting family, $\overline{K}_2 + T_r$. For the family $O_{k,r}$ with $k \geq 3$ we find that

$$j_{\text{max}} = 2, \quad j'_{\text{max}} = 3$$

For general $k \geq 3$, the denominator of the generating function can be written as

$$D(\{O_k\}, q, x) = \left[1 - (q - 2)D_k x\right] \left[1 - (q - 3)D_k x - (q - 1)(q - 2)D_{k-1}D_k x^2\right]$$

$$= (1 - \lambda_2 x)(1 - \lambda_{1p} x)(1 - \lambda_{1m} x)$$

where

$$\lambda_{1p,m} = \frac{1}{2} \left[(q - 3)D_k \pm \left\{ (q - 3)D_k \right\}^2 + 4(q - 1)(q - 2)D_{k-1}D_k \right]^{1/2}$$

and

$$\lambda_2 = (q - 2)D_k$$

The coefficient functions in $N$ are $A_0$, given by the special case of eq. (5.17), viz., $A_0 = q(q - 1)^k$, and

$$A_1 = -q(q - 1)D_k \left[2(q - 5)(q - 1)^{k-1} - (q - 2)\left[qD_k - 2(-1)^k\right]\right]$$

$$A_2 = -q^2(q - 1)^2(q - 2)D_{k-1}D_k^2$$

(In deriving the expression for $A_1$, we have used the identity $D_{2k-1} = [qD_k - 2(-1)^k]D_k$.) We calculate

$$W(\{O_k\}, q) = (\lambda_{\text{max}})^{1/(k-1)} \quad \text{for} \quad q \in R_1$$

and

$$|W(\{O_k\}, q)| = |\lambda_{\text{max}}|^{1/(k-1)} \quad \text{for} \quad q \in R_j \neq R_1$$

where $\lambda_{\text{max}}(q)$ denotes the leading $\lambda$ in eq. (5.19) in the respective regions of the $q$ plane. For a general family $\{G\}$ of graphs with a generating function

$$\Gamma(\{G\}, q, x) = \frac{N(x)}{D(x)} = \sum_{j=j_0}^{\infty} P(G_r, q)x^r$$

(5.26)
with

\[ D(x) = \prod_{j' = 1}^{j'_{\text{max}}} (1 - \lambda_{j'} x) \]  

(5.27)

the boundary \( \mathcal{B} \) in the limit \( r \to \infty \) is given (see eq. (4.16) of Ref. [16]) as solution locus of the equation expressing the degeneracy in magnitude of two leading \( \lambda \) terms in eq. (5.27), where \( W \) switches form nonanalytically,

\[ |\lambda_{\text{max}}(q)| = |\lambda_{\text{max}}'(q)| \]  

(5.28)

As an illustration, we consider the lowest homeomorphic expansion, \( k = 3 \), for which eqs. (5.20) and (5.21) read

\[ \lambda_{1p,m} = \frac{1}{2} (q - 2) \left[ (q - 3) \pm \left( q^2 - 2q + 5 \right)^{1/2} \right] \]  

(5.29)

\[ \lambda_2 = (q - 2)^2 \]  

(5.30)

Let us now write \( \lambda_{1p}, \lambda_{1m} \) and \( \lambda_2 \) in terms of the variable \( z = 1/q \) in polar coordinates, as given by eq. (1.4) and Taylor-expand the resulting expressions for small \( \zeta \). We obtain for the magnitudes squared of the \( \lambda \)'s

\[ |\lambda_{1p}|^2 = 1 - 4\zeta \cos \vartheta + 2\zeta^2 (\cos(2\vartheta) + 2) + O(\zeta^3) \]  

(5.31)

\[ |\lambda_{1m}|^2 = \zeta^2 + O(\zeta^3) \]  

(5.32)

\[ |\lambda_2|^2 = 1 - 4\zeta \cos \vartheta + 4\zeta^2 + O(\zeta^3) \]  

(5.33)

Thus, for small values of \( |z| \), the boundary \( \mathcal{B} \) is given by the equation \( |\lambda_{1p}| = |\lambda_2| \), which yields \( \zeta \cos(2\vartheta) + O(\zeta^3) = 0 \) as \( \zeta \to 0 \). Hence, \( \mathcal{B} \) is noncompact in the \( q \) plane and crosses the origin of the \( z \) plane at angles such that \( \cos(2\vartheta) = 0 \), i.e.,

\[ \vartheta = \frac{\pi}{4} + \frac{n \pi}{2}, \quad 0 \leq n \leq 3 \]  

(5.34)

This is evident in Fig. 5. From eq. (1.3), it follows that \( y \to z \) as \( z \to 0 \), so that these angles are the same in the \( y \) plane. The boundary \( \mathcal{B} \) divides the \( z \) plane into four regions: (i) \( R_1 \), including the interval \( 0 < z < 1/3 \) on the real \( z \) axis; (ii) \( R_2 \), including the intervals \( -\infty < z < 0 \) and \( 1/3 < z < \infty \) on the real \( z \) axis; and (iii) the complex-conjugate pair of regions \( R_3 \) and \( R_3^* \) lying roughly above and below \( z = 0 \).

We find

\[ W(\lim_{r \to \infty} O_{3,r}, q) = (\lambda_{1p})^{1/2} \quad \text{for} \quad q \in R_1 \]  

(5.35)

\[ |W(\lim_{r \to \infty} O_{3,r}, q)| = |\lambda_{1p}|^{1/2} \quad \text{for} \quad q \in R_2 \]  

(5.36)

and
\[ |W(\lim_{r \to \infty} O_{3,r}, q)| = |\lambda_2|^{1/2} \quad \text{for} \quad q \in R_3, R^*_3 \]  

(5.37)

A portion of \( B \) crosses the positive real axis at \( z = z_c = 1/3 \) so that \( q_c = 3 \) for \( \lim_{r \to \infty} O_{3,r} \). Along this portion of the boundary \( |\lambda_{1m}| \) becomes degenerate with \( |\lambda_{1p}| \), although the former never dominates in magnitude over the latter. This portion of the boundary ends in two c.c. multiple points. Note that

\[ W(\lim_{r \to \infty} O_{3,r}, q) = 0 \quad \text{at} \quad q = 2 \]  

(5.38)

consistent with eqs. (2.11) and (2.12)

---

**FIG. 5.** Boundary \( B \) in the \( z = 1/q \) plane for \( \lim_{r \to \infty} O_{k,r} \) with \( k = 3 \). Chromatic zeros for \( O_{k,r} (k, r) = (3, 29) \) are shown for comparison.

---

**B. Families \( HEC_{k_1-2,k_2-2}(K_2 + T_r) \) with \( k_1 = k_2 \)**

We next consider the case of symmetric homeomorphic expansion of the bonds from the two vertices of the \( K_2 \) subgraph, i.e., \( k_1 = k_2 \equiv k \). For brevity of notation, we denote

\[ U_{k,r} = HEC_{k_1-2,k_2-2}(K_2 + T_r), \quad k_1 = k_2 \equiv k \]  

(5.39)

with \( v(U_{k,r}) = 2 + r(2k - 3) \). When referring to the collection of the graphs of this family for various \( r \), we denote it as \( \{ U_k \} \).

For the family \( U_{k,r} \) we find that
The denominator of the generating function can be written as

\[
D(\{U_k\}, q, x) = \left[1 - (q - 2)D_k x - (q - 1)^3 D_{k-1}^2 x^2\right] \times \left[1 - D_k(D_{k+1} - D_k)x - (q - 1)^2 D_{k-1}^2 x^2\right]
\]

(5.41)

Observe that each factor of \(x\) in \(D(\{U_k\}, q, x)\) occurs with at least one accompanying factor of \(D_k\). We have

\[
D(\{U_k\}, q, x) = \left(1 - \lambda_{1p,x}(1 - \lambda_{1m,x})(1 - \lambda_{2p,x})(1 - \lambda_{2m,x})\right)
\]

(5.42)

where

\[
\lambda_{1p,m} = \frac{D_k}{2} \left[(q - 2)D_k \pm \left[(q - 2)D_k^2 + 4(q - 1)^3(D_{k-1})^2\right]^{1/2}\right]
\]

(5.43)

and

\[
\lambda_{2p,m} = \frac{D_k}{2} \left[(D_{k+1} - D_k) \pm \left[(D_{k+1} - D_k)^2 + 4(q - 1)^2 D_{k-1}D_k\right]^{1/2}\right]
\]

(5.44)

For the polynomials in the numerator \(N(x)\), eq. (5.7) gives

\[
A_0 = q(q - 1)^{2k-2}
\]

and we find

\[
A_1 = q(q - 1) \left[(D_{2k-1})^2 - (q - 1)^{2k-3}[(q - 3)D_k + D_{k+1}]D_k\right]
\]

(5.45)

\[
A_2 = -q(q - 1)^2 D_{k-1}D_k^3 \left[q^2 D_k^2 - q(q + 2)(-1)^kD_k + 2q + q(q - 2)(q - 1)^{k-2}D_k + (q - 2)(-1)^k(q - 1)^{k-2} + 1\right]
\]

\[
A_3 = -q^3(q - 1)^4 D_{k-1}^3 D_k^3
\]

(5.46)

(5.47)

We calculate

\[
W(\{U_k\}, q) = [\lambda_{\max}(q)]^{1/(2k-3)} \text{ for } q \in R_1
\]

(5.48)

and

\[
|W(\{U_k\}, q)| = [\lambda_{\max}(q)]^{1/(2k-3)} \text{ for } q \in R_j \neq R_1
\]

(5.49)

where \(\lambda_{\max}(q)\) refers to the respective dominant \(\lambda\) in the given region. The boundary \(B\) is given, as before, by the solution locus of eq. (5.28). We show this boundary in Fig. (a), together with illustrative chromatic zeros for the case \(r = 19\). From the property (2.15), it follows that the \(\lambda\)'s are degenerate at zero when \(q = 2, \lambda_{1p,m}(q = 2) = \lambda_{2p,m}(q = 2) = 0\) for \(k\) odd

(5.50)

so that \(B\) passes through the point \(q = 2\) if \(k\) is odd, and, furthermore,
Thus, in the vicinity of $z = 0$, the boundary $B_k$ is given by the equation $|q|^3 = 0$ for $\zeta = 0$. Hence, six curves on $B$ (forming three branches) cross the point $z = 0$ (and hence also the point $y = 0$), at angles

$$\theta = \frac{\pi}{6} + n \frac{\pi}{3}, \quad 0 \leq n \leq 5$$

(5.58)

The boundary $B$ divides the $z$ plane into six regions. As one traverses a circle around the origin, $z = 0$, starting with a small positive real $z$ value, first $\lambda_{1p;k=3}$ is dominant, and then the dominant $\lambda$’s alternate between $\lambda_{1p;k=3}$ and $\lambda_{2p;k=3}$. The resultant $W$ functions are given by eqs. (5.48) and (5.49).

For the family $U_{k=4}$, eqs. (5.43) and (5.44) yield

$$\lambda_{1p,m;k=4} = \frac{1}{2} (q-2)(q^2-3q+3) \left[ (q^2-3q+3) \pm \left( q^4 - 3q^3 + 3q^2 - 6q + 5 \right)^{1/2} \right]$$

(5.59)

$$\lambda_{2p,m;k=4} = \frac{1}{2} (q-2)(q^2-3q+3) \left[ (q^3 - 5q^2 + 9q - 7) \pm \left( q^6 - 6q^5 + 15q^4 - 24q^3 + 35q^2 - 42q + 25 \right)^{1/2} \right]$$

(5.60)

As before, the boundary $B$ is determined by eq. (5.28). This boundary is plotted in Fig. (b). Let us write $\lambda_{1p,m}$ and $\lambda_{2p,m}$ in terms of the variable $z = 1/q$ in polar coordinates, as given by eq. (1.4), and
Taylor expand the resulting expressions for small $\zeta$. We find that the two $\lambda$’s that are dominant near $z = 0$, namely $\lambda_1 p$ and $\lambda_2 p$, have squared magnitudes that coincide through $O(\zeta^4)$ and differ in order $O(\zeta^5)$ in the coefficient of $\zeta^5 \cos(5\theta)$. (The other two $\lambda$’s are subdominant near $z = 0$; indeed, they vanish there, as $|\lambda_{1m}|^2 \sim |\lambda_{2m}|^2 = 4\zeta^2 + O(\zeta^3).$) It follows that as $\zeta \to 0$, the boundary $B$ is given by the equation $|\lambda_1 p| = |\lambda_2 p|$, which yields $\cos(5\theta) = 0$ for $\zeta \neq 0$. Hence, ten curves, forming five branches of $B$, cross the point $z = 0$ (and also $y = 0$) at the angles

$$\theta = \frac{\pi}{10} + \frac{n\pi}{5}, \quad 0 \leq n \leq 9$$

(5.61)

Here, the boundary $B$ divides the $z$ plane into eight regions. As one traverses a circle around the origin starting with a small positive real value of $z$, first $\lambda_{1p,k=4}$ is dominant, and then the dominant $\lambda$’s alternate between $\lambda_{1p,k=4}$ and $\lambda_{2p,k=4}$, as before with $k = 3$. The resultant $W$ functions again follow from eqs. (5.48) and (5.49). Thus, comparing our results for $U_{k,r}$ with $k = 3$ and $k = 4$, we observe that the number of curves on $B$ passing through $z = 0$, and the number of regions, increase as $k$ is increased. One should also remark that certain properties of the region depend on whether $k$ is even or odd, such as the fact that $B$ passes through $z = 1/2$ for odd $k \geq 3$. 

22
FIG. 6. Boundary $\mathcal{B}$ in the $z = 1/q$ plane for $\lim_{r \to \infty} U_{k,r}$ with $k = (a) 3 \ (b) 4$. Chromatic zeros for $U_{k,r}$ with $(k, r) = (a) (3,19) \ (b) (4,15)$ are shown for comparison.

VI. DISCUSSION

In this section we discuss general features of the Potts antiferromagnet $W$ functions on all of the families of graphs that we have constructed and studied in this paper and our earlier work, with loci $\mathcal{B}$ that are noncompact in the $q$ plane. In (the theorem of section IV of) Ref. [13] we gave the general algebraic condition that for a particular family of graphs, the infinite–vertex limit yields a locus $\mathcal{B}$ that passes through $z = 0$. From our calculations we have observed a geometrical regularity in the families of graphs that have this property, viz., that in this infinite-vertex limit they all contain an infinite number of different, non-overlapping (and non-self-intersecting) circuits, each of which passes through at least two fixed, nonadjacent vertices. This immediately implies that these aforementioned nonadjacent vertices have degrees $\Delta$ that go to infinity in this limit. One thus sees at the graphical level why the derivation of the large–$q$ series for the reduced function $W_{\text{red}}(\{G\}, q)$ or equivalently $\overline{W}(\{G\}, q)$ on regular lattices with free or some type of periodic boundary conditions works; in the thermodynamic limit, such lattices do not have the two or more nonadjacent vertices with degrees $\Delta \to \infty$ as in the criterion stated above for noncompact $\mathcal{B}$.

We remark that in studies of the Potts model on regular lattices, it has been useful to utilize certain boundary conditions that do involve vertices which, in the thermodynamic limit, have infinite degree $\Delta$, since these make it possible to preserve exact duality on finite lattices [23–25]. On the square lattice, the duality-preserving boundary conditions (DBC’s) of types DBC-1 and DBC-2, in the notation of Ref. [25] involve one such vertex with $\Delta \to \infty$ while the DBC-3 type involves two such vertices; however, in the latter
case, these are adjacent. Hence, none of these duality-preserving boundary conditions invalidates a large–q expansion of $W_{\text{red}}(\{G\}, q)$ for this lattice.

An elementary topological property should be noted: for an arbitrary family of graphs, the continuous locus of points $B$ where $W$ is nonanalytic in the infinite–vertex limit does not necessarily enclose regions in the $q$ plane. Indeed, in Ref. [16] we carried out exact calculations of a variety of infinitely long, finite-width strips of different lattices and found loci $B$ that consisted of arcs (and line segments) that did not enclose regions. (We have also performed similar calculations for other types of strips with specific types of end-graphs that yield loci $B$ that do enclose regions.) None of these loci were noncompact in the $q$ plane. In contrast, if the infinite-vertex limit of a family of graphs yields a locus $B$ that passes through $z = 0$, then, given that it is not a semi-infinite line segment on the positive or negative $z$ plane, it necessarily separates the $z$ and equivalently the $q$ or $y$ planes into at least two different regions where $W$ is an analytic function. In Ref. [16], we observed that the endpoints of the arcs (or line segments on the real $q$ axis) that comprised the nonanalytic locus $B$ were determined by the branch points of certain algebraic expressions occurring in the $\lambda$’s that entered in degeneracy equations of the form of eq. (5.28). Since, for example, the $\lambda$’s for the family $\{U_k\}$, eqs. (5.43) and (5.44), also contain square roots, one may wonder what role the branch points of these roots play with regard to the boundary $B$. We have investigated this and have found that the analogues of the arcs extending between branch points of the square roots, which comprised the various boundaries in Ref. [16], do not yield endpoint singularities on $B$ here. The reason for this is that either (i) these arcs involve degeneracy of $\lambda$’s that are nonleading in a given region, or (ii) although they coincide on part of their length with the actual boundary, the portion of the arcs containing the endpoints lies off this boundary, in a region where condition (i) holds. This is illustrated in Fig. 7 for the $\{U_k\}$ family with $k = 3$, where we show the actual boundary $B$, as in Fig. 6(a), together with the arcs (drawn in a thicker black) formed by the degeneracy conditions

$$|\lambda_{2p}| = |\lambda_{2m}|$$

and

$$|\lambda_{1p}| = |\lambda_{1m}|$$

(6.1)
FIG. 7. Boundary $B$ in the $z = 1/q$ plane for $\lim_{r \to \infty} U_{3,r}$, shown together with the locus of solutions of the degeneracy equations (6.1) and (6.2). See text for discussion.

The complex-conjugate pair of arcs which constitute the solution locus to eq. (6.1), with endpoints at $z_{2a}$, $z_{2a}^* = 0.04106 \pm 0.43627i$ and $z_{2b}$, $z_{2b}^* = 0.54717 \pm 0.08333i$, lie in the interior of the regions $R_4$ and $R_4^*$. But in these regions, neither $\lambda_{2p}$ nor $\lambda_{2m}$ is dominant; rather, as we discussed in the section on the $U_{k,r}$ family, it is $\lambda_{1p}$ that is dominant in these regions. Hence, this complex-conjugate pair of arcs is not relevant to the actual boundary. As shown in Fig. 3, the locus of solutions for the degeneracy equation (6.2) forms a complex-conjugate pair of arcs that cross each other and the real $z$ axis at $z = 1/2$ and have endpoints at $z_{1a}$, $z_{1a}^* = 0.10169 \pm 0.37529i$ and $z_{1b}$, $z_{1b}^* = 0.73164 \pm 0.12612i$. The portion of these arcs that lie to the right of $\text{Re}(z) = 1/2$ are not relevant for determining the boundary because in this region, denoted $R_2$ in our section above on the $\{U_k\}$ family, the dominant $\lambda$ is $\lambda_{2p}$. The portion of the complex-conjugate arcs lying between the multiple (crossing) point $z = 1/2$ on the right and the multiple points forming T-intersections on the left at $z_T$, $z_T^* = 0.3204 \pm 0.2110i$ does coincide with the boundary, since on this portion the boundary is determined by the degeneracy of magnitudes of leading eigenvalues $\lambda_{1p}$ and $\lambda_{1m}$, i.e., eq. (6.2). However, to the left of these T-intersection points, the arcs forming the solution locus of eq. (6.2) again deviate from the actual boundary, which is determined by the degeneracy of leading magnitudes $|\lambda_{2p}| = |\lambda_{1p}|$. Hence, once again, the left endpoints of these arcs are not relevant for $B$. Thus, the boundaries $B$ for the families considered here, even aside from the fact that they automatically enclose regions owing to their noncompactness, in contrast to the arcs comprising the loci $B$ for the strip graphs that we studied in Ref. [16], also differ qualitatively in that they do not contain any arc endpoints. We have explained why this is so even though some of the families do have $\lambda$’s containing branch points singularities. This emphasizes that the existence of the arc-like structure of the loci $B$ that we found in Ref. [16] depended
not just on the presence of branch point singularities in the \( \lambda \)'s entering via equations analogous to eq. \([5.43]\) in the denominators \( D \) of the generating functions for various strip graphs, but also on the fact that these branch point singularities occurred at endpoints of arcs in regions where these arcs were the solution loci of magnitude degeneracy equations for leading \( \lambda \)'s.

From our studies of many homeomorphic expansions, we have found the general feature that for a given homeomorphic class parametrized by some set of homeomorphic expansion indices \( \{k_j\} \), the number of regions separated by the locus \( B \) in the \( r \to \infty \) limit is a nondecreasing function of the above expansion indices. This is in accord with the constraints from algebraic geometry, in particular, the Harnack theorem \([20]\). The application of this theorem is simplest in the case where \( B \) is a nonsingular algebraic curve (so that the number of regions \( N_{\text{comp.}} = N_{\text{comp.}} + 1 \) where \( N_{\text{comp.}} \) denotes the number of connected components of \( B \)), as is the case for the \( r \to \infty \) limit of the families \( T_{k,r} = HEG_{k-2}(\overline{K}_2 + T_r) \) and \( S_{k,r} = HEG_{k-2}(\overline{K}_3 + T_r) \).

In these cases (where \( B \) has no multiple points) the Harnack theorem states that the number of regions is bounded above by \( g + 1 \), where \( g \) is the genus of the algebraic equation in the variables \( Re(q) \) and \( Im(q) \) whose solution set is \( B \), viz., \( g = (d-1)(d-2)/2 \) where \( d \) is the homogeneous degree of this equation. Since, as our exact solutions show, this degree and the resultant genus increase as a function of the homeomorphic expansion indices, the Harnack upper bound also increases. It should be noted, however, that the number of regions may remain the same as one performs a homeomorphic expansion (e.g., \( N_{\text{reg.}} = 2 \) for the \( r \to \infty \) limit of \( HEG_{k-2}(\overline{K}_2 + T_r) \) for \( k = 2, 3, \) and \( 4, \) then \( N_{\text{reg.}} = 3 \) for \( k = 5, 6, \) and \( 7, \) etc.)

For algebraic curves \( B \) with singularities such as multiple points, one no longer has the simple relation \( N_{\text{reg.}} = N_{\text{comp.}} + 1 \). Indeed, in the families constructed and analyzed in this paper, \( B \) consists of a single connected component, i.e., \( N_{\text{comp.}} = 1 \), while the number of regions \( N_{\text{reg.}} \) and the index of the multiple point at \( z = 0 \) increase as a function of the indices \( \{k_j\} \) describing the homeomorphic expansions. The origin of this increase is the increase in the degree of the polynomials in \( z \) that occur in the degeneracy equations relevant for the boundary \( B \) in the neighborhood of \( z = 0 \) (e.g., eqs. \([3.11]\) and the resultant \([3.12]\) for \( H_{k,r} \), etc.).

The boundaries \( B \) found for the families of graphs in the present paper exhibit some interesting differences with respect to the (similarly noncompact) boundaries that we have previously studied \([13]\): the latter shared three general properties: (i) a single curve on \( B \) passes through \( z = 0 \), so that this point is a regular point on \( B \) as an algebraic curve; (ii) if \( q \in B \), then \( Re(q) \geq 0 \), i.e. \( B \) includes support only for \( Re(q) \geq 0 \), or equivalently, for \( Re(z) \geq 0 \); and (iii) if \( z \in B \) and \( Re(z) = 0 \), then \( z = 0 \). In contrast, for the families studied in the present paper, the homeomorphic expansions, which are all of the type \( HEC \), yield, in the \( r \to \infty \) limit, respective boundaries \( B \) that differ in each of these aspects: (i) several branches of the algebraic curve \( B \) pass through the point \( z = 0 \), which is thus a singular (multiple) point on this curve; (ii) \( B \) includes support for \( Re(q) < 0 \) or equivalently \( Re(z) < 0 \), and (iii) \( B \) includes points with \( Re(z) = 0 \) other than \( z = 0 \) itself. Some insight into this can be gained by recalling the differences in the types of homeomorphic expansions of starting graphs. Since in the original starting families such as \( (K_p)_b + G_r \) and \( \overline{K}_p + G_r \), it was the connecting bonds linking the vertices of the \( K_p \) subgraph with the vertices of the \( G_r \) subgraph that gave rise to the noncompactness of the respective boundaries \( B \), it makes sense that homeomorphic expansions of these bonds would alter the nature of the boundaries at \( z = 0 \). Since this change involves multiple branches of \( B \) passing through this point at various angles, as in eqs. \([3.12]\), \([3.34]\), \([5.58]\) and \([6.61]\), this also shows why this type of homeomorphic expansion leads to boundaries that include support for \( Re(q) < 0 \), but \( q \neq 0, \infty \), i.e., \( Re(z) < 0 \) but \( z \neq \infty, 0 \). In contrast, since the bonds within the \( G_r \) and \( K_p \) subgraphs, by themselves, are not directly responsible for the noncompactness of \( B \), it is plausible that homeomorphic
expansions of these bonds would not change the nature of \( B \) at \( z = 0 \).

VII. BOUNDARY FOR \( L_k \) LIMIT

Finally, we briefly discuss the boundary \( B \) that results when one takes the limit \( L_k \) of eq. (2.20), i.e., \( k \to \infty \) with \( r \) and \( p \) fixed, for the families studied in this paper. We consider the nontrivial range \( r \geq 2 \) since for \( r = 1 \), families typically degenerate into tree graphs with \( B = \emptyset \). For the families studied here for which we have obtained the chromatic polynomials for general \( k \), including \( H_{k,r} \), \( O_{k,r} \), and \( U_{k,r} \), we find that this boundary is simply the unit circle \( |q - 1| = 1 \). This is easily understood, since one can reexpress our formulas for chromatic polynomials in terms of polynomials in \( a = q - 1 \) using eqs. (1.6) and (1.7). From eq. (1.7), it follows that the chromatic polynomials are of the form of eq. (1.2) with \( a \) as the quantity entering in terms raised to respective powers proportional to \( k \), together with possible \( c_0 \) terms involving \((-1)^k\), and hence, as discussed in our earlier work [12], \( B \) is determined by the condition \( |a| = 1 \). Hence, \( q_c = 2 \) and \( B \) has no support for \( \text{Re}(q) < 0 \). Since the locus \( |q - 1| = 1 \) is compact in the \( q \) plane and since our focus here is on the situation where \( B(q) \) is noncompact in the \( q \) plane, passing through \( z = 0 \), and the connection with the existence of large-\( q \) series for \( W_{\text{red}} \), the \( L_k \) limit is not of primary interest here. We mention, however, that \( B \) divides the \( q \) plane into two regions, \(( R_1)_L \) and \(( R_2)_L \), which are the exterior and interior of the unit circle \( |q - 1| = 1 \) (the subscripts \( L \) are appended to distinguish these regions from those for the \( L_r \) limit). It is straightforward to calculate the \( W \) function in these two regions in the \( L_k \) limit of the various families. For example, for the \( H_{k,r} \) family (for fixed \( r \geq 2 \)),

\[
W([\lim_{k \to \infty} H_{k,r}], q) = q - 1 \quad \text{for} \quad q \in (R_1)_L
\]

and

\[
|W([\lim_{k \to \infty} H_{k,r}], q)| = 1 \quad \text{for} \quad q \in (R_2)_L
\]

VIII. CONCLUSIONS

In this paper we have presented exact calculations of the zero-temperature partition function, \( Z(G, q, T = 0) \), and ground-state degeneracy (per site), \( W(\{G\}, q) \), for the \( q \)-state Potts antiferromagnet on a number of families of graphs \( \{G\} \) for which the boundary \( B \) of regions of analyticity of \( W \) in the complex \( q \) plane is noncompact and has the properties that (i) in the \( z = 1/q \) plane, the point \( z = 0 \) is a multiple point on \( B \); (ii) \( B \) includes support for \( \text{Re}(z) < 0 \); and (iii) \( B \) crosses the imaginary \( z \) axis away from \( z = 0 \). Our results yield insight into the conditions which preclude the existence of large-\( q \) Taylor series expansions for the reduced function \( W_{\text{red}} = q^{-1}W \). This insight is valuable since large-\( q \) expansions, where they exist, are of great utility in obtaining approximate values of the exponent of the ground state entropy, \( W \).

This research was supported in part by the NSF grant PHY-93-09888.
[1] Potts, R. B. 1952 Proc. Camb. Phil. Soc. 48, 106.
[2] Wu, F. Y. 1982 Rev. Mod. Phys. 54, 235; errata, ibid. 55, 315 (1983).
[3] Birkhoff, G. D. 1912 Ann. of Math. 14, 42.
[4] Read, R. C. 1968 J. Combin. Theory 4, 52; Read, R. C. and Tutte, W. T. 1988 “Chromatic Polynomials”, in Selected Topics in Graph Theory, 3, eds. Beineke, L. W. and Wilson, R. J. (Academic Press, New York)
[5] Tutte, W. T. 1984 Graph Theory, vol. 21 of Encyclopedia of Mathematics and its Applications, ed. Rota, G. C. (Addison-Wesley, New York).
[6] Harary, F. 1969 Graph Theory (Addison-Wesley, Reading); Biggs, N. L. 1993 Algebraic Graph Theory, (Cambridge, U.K., Cambridge Univ. Press).
[7] Pauling, L. 1960 The Nature of the Chemical Bond (Cornell Univ. Press, Ithaca), p. 466.
[8] Lieb, E. H. and Wu, F. Y. 1972 in Domb, C. and Green, M. S., eds., Phase Transitions and Critical Phenomena (Academic Press, New York) v. 1, p. 331.
[9] Some related early work is in Beraha, S., Kahane, J., and Weiss, N. 1980 J. Combin. Theory B 28, 52, Baxter, R. J. 1987 J. Phys. A 20, 5241, and Ref. [10].
[10] Read, R. C. and Royle, G. F. 1991 in Graph Theory, Combinatorics, and Applications (Wiley, New York), vol. 2, p. 1009.
[11] Shrock, R. and Tsai, S.-H. 1997 J. Phys. A 30, 495.
[12] Shrock, R. and Tsai, S.-H. 1997 Phys. Rev. E55, 5165; E56, 1342.
[13] Shrock, R. and Tsai, S.-H. 1997 Phys. Rev. E56, 3935.
[14] Shrock, R. and Tsai, S.-H. 1997 Phys. Rev. E56, 6791; E56, 2733; Tsai, S.-H. 1998 Phys. Rev. E57, 2686.
[15] Shrock, R. and Tsai, S.-H. 1997 Phys. Rev. E56, 4111.
[16] Roček, M., Shrock, R., and Tsai, S.-H. 1998 Physica A252, 505; Physica A259, 367; Shrock, R. and Tsai, S.-H. 1998 Physica A259, 315; Phys. Rev. E58, 4332.
[17] Nagle, J. F. 1971 J. Combin. Theory B 10, 42); Nagle, J. F 1968 J. Math. Phys. 9, 1007.
[18] Baker, G. A., Jr. 1971 J. Combin. Theory 10, 217.
[19] Kim, D. and Enting, I. G. 1979 J. Combin. Theory B 26, 327.
[20] Lefschetz, S. 1953 Algebraic Geometry (Princeton Univ. Press, Princeton); Hartshorne, R. 1977 Algebraic Geometry (Springer, New York).
[21] Tsai, S.-H. 1998 Ph.D. thesis, State University of New York.
[22] Read, R. C. and Whitehead, E. G. Discrete Math., in press. We thank Professor Read for kindly giving us a copy of this preprint.
[23] Martin, P. P. 1991 Potts Models and Related Problems in Statistical Mechanics (World Scientific, Singapore)
[24] Chen, C. N., Hu, C. K., and Wu, F. Y. 1996 Phys. Rev. Lett. 76, 169; Wu, F. Y. et al. 1996 Phys. Rev. Lett. 76, 173.
[25] Matveev, V. and Shrock, R. 1996 Phys. Rev. E54, 6174.