TWO CHARACTERIZATIONS OF TOPOLOGICAL SPACES
WITH NO INFINITE DISCRETE SUBSPACE

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Abstract. We give two characteristic properties of topological spaces with
no infinite discrete subspaces. The first one was obtained recently by the
first author. The full result extends well-known characterizations of posets
with no infinite antichain.

A topological space \( T := (E, \mathcal{F}) \), where \( \mathcal{F} \) is the set of closed subsets, is
noetherian if every descending sequence of closed subspaces is stationary. A
subset \( X \) of \( E \) is discrete if every subset of \( X \) is closed with respect to the
induced topology. A closed subset is irreducible if it is non-empty and not the
union of two proper closed subsets.

Theorem 1. The following properties are equivalent for a topological space
\( T := (E, \mathcal{F}) \).

(i) No infinite subset of \( E \) is discrete;
(ii) Every closed set is a finite union of irreducible closed subsets;
(iii) Every closed set contains a dense subset on which the induced topology is
noetherian.

The equivalence between (i) and (ii) was proved recently by the first author
[9]. We give a proof of the equivalence of (iii) with (i) and (ii) in Section 1.
That also gives an alternative proof of the equivalence between (i) and (ii),
and will stress the importance of the notion of infinite separating chain of
closed sets, inspired from [3, 2]. We make additional remarks in Section 2.

1. The proof

We mimic the proof of a similar result for posets, fairly well-known, which we
give at the end of the paper. The significant part is the implication \((i) \Rightarrow (iii)\).

We recall that a closure system is a pair \((E, \varphi)\) where \( \varphi \) (the closure) is a
map from the power set of \( E \) into itself which is extensive, order-preserving
and idempotent. A subset \( C \subseteq E \) is closed if \( \varphi(C) = C \); it is independent if
\( x \notin \varphi(C \setminus \{x\}) \) for every \( x \in C \); it is generating if \( \varphi(C) = E \). The closure \( \varphi_{|E'} \),
induced by \( \varphi \) on a subset \( E' \) of \( E \) is defined by \( \varphi_{|E'}(X) := \varphi(X) \cap E' \) for every

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Research Agency ANR (project BRAVAS).
We recall that \( \varphi \) defines a topology if and only if it preserves finite unions. In this case, we say that \( \varphi \) is topological, also independent sets are called discrete, generating sets are called dense.

We will use the following notion and result, adapted from \[3 \text{ Section } 3, \text{ p.7}], see also \[2 \text{ p.25}] \). A non-empty descending chain \( \mathcal{I} \) of closed sets of \((E, \varphi)\) is separating if for every \( I \in \mathcal{I} \setminus \{ \bigcup \mathcal{I} \} \) and every finite set \( F \subseteq \bigcup \mathcal{I} \setminus I \), there is a set \( J \in \mathcal{I} \) necessarily included in \( I \)—such that \( I \notin \varphi(F \cup J) \).

**Lemma 1.** A closure system \((E, \varphi)\) contains an infinite independent set if and only if \( E \) contains a subset \( E' \) for which the induced closure contains an infinite separating chain of closed sets.

**Proof.** Any infinite independent set contains a countably infinite independent subset, so we may as well assume a given infinite independent set \( X \) of the form \{ \( x_n : n < \omega \) \}, where \( x_m \neq x_n \) for all \( m \neq n \). Set \( E' := X \). Then the chain \( \mathcal{I} = \{ I_n : n < \omega \} \), where \( I_n := \varphi_{|E'}(X \setminus \{ x_i : i < n \}) \), is separating in \( E' \). Indeed, first \( \bigcup I = \emptyset = E' = X \), next every \( I \in \mathcal{I} \setminus \{ \bigcup \mathcal{I} \} \) is an \( I_n \) for some \( n \geq 1 \). For every finite set \( F \) of points of \( I_0 \setminus I_n \), define \( J \) as \( I_{n+1} \). Since \( x_n \) is different from every \( x_i \), \( i < n \), \( x_n \) is in \( X \setminus \{ x_i : i < n \} \) hence in \( I_n \). It follows that \( x_n \) is not in \( F \). It is not in \( J \) either, because \( J \notin \varphi(X \setminus \{ x_i : i < n+1 \}) \subseteq \varphi(X \setminus \{ x_n \}) \), and \( X \) is independent. Therefore \( x_n \) is not in \( F \cup J \). We rewrite that as \( F \cup J \subseteq X \setminus \{ x_n \} \), and conclude that \( \varphi(F \cup J) \subseteq \varphi(X \setminus \{ x_n \}) \). Since \( X \) is independent again, \( x_n \) cannot be in \( \varphi(F \cup J) \). However, \( x_n \) is in \( I_n \), so \( I_n \notin \varphi(F \cup J) \).

Conversely, let \( E' \) be a subset of \( E \) such that the induced closure \( \varphi' := \varphi_{|E'} \) on \( E' \) contains an infinite separating chain \( \mathcal{I} \) of closed sets. We construct an infinite independent subset for the induced closure \( \varphi' \). It will be independent for the closure \( \varphi \). For that, define inductively an infinite sequence \( x_0, I_0, \ldots, x_n, I_n, \ldots \) such that \( I_0 \in \mathcal{I} \setminus \{ \bigcup \mathcal{I} \} \), \( x_0 \in \bigcup \mathcal{I} \setminus I_0 \) and such that, for every \( n \geq 1 \):

\[
\begin{align*}
(a_n) & \ i_n \in \mathcal{I}; \\
(b_n) & \ i_n \subset I_{n-1}; \\
(c_n) & \ x_n \in I_{n-1} \setminus \varphi'\left(\{x_0, \ldots, x_{n-1}\} \cup I_n\right).
\end{align*}
\]

Since \( \mathcal{I} \) is infinite, \( \mathcal{I} \setminus \{ \bigcup \mathcal{I} \} \neq \emptyset \). Choose arbitrarily \( I_0 \in \mathcal{I} \setminus \{ \bigcup \mathcal{I} \} \) and \( x_0 \in \bigcup \mathcal{I} \setminus I_0 \). Let \( n \geq 1 \). Suppose \( x_k, I_k \) are defined and satisfy \((a_k), (b_k), (c_k)\) for every \( k \leq n-1 \). Set \( I := I_{n-1} \) and \( F := \{ x_0, \ldots, x_{n-1} \} \). Since \( I \in \mathcal{I} \) and \( F \subseteq \bigcup \mathcal{I} \setminus I \), there is some \( J \in \mathcal{I} \) such that \( I \notin \varphi'(F \cup J) \). Let \( z \in I \setminus \varphi'(F \cup J) \). Set \( x_n := z, I_n := J \).

It remains to check that the set \( X := \{ x_n : n < \omega \} \) is independent. For every \( x \in X \), say \( x = x_n \), we know that \( x_n \) is in \( I_{n-1} \) and not in the closed set \( C := \varphi'\left(\{x_0, \ldots, x_{n-1}\} \cup I_n\right) \), by \((c_n)\). The set \( C \) contains \( x_0, \ldots, x_{n-1} \). For every \( k > n+1 \), \( I_k \subset I_n \) by \((b_{k-1}), \ldots, (b_{n+1})\), and \( x_k \in I_k \) by \((c_k)\), so \( I_n \), hence also \( C \), contains every \( x_k \) with \( k > n \). It follows that \( C \) contains every element of \( X \) except \( x = x_n \). Therefore \( C \supseteq \varphi'(X \setminus \{ x \}) \), from which we conclude that \( x \), not being in \( C \), is not in \( \varphi'(X \setminus \{ x \}) \) either. \( \square \)
Remark 1. Assuming \( \varphi \) topological, we can take \( E' = E \) in Lemma 1. Indeed, let \( X \) be an independent subset of the form \( \{ x_n : n < \omega \} \). Let \( I_n := \varphi(X \setminus \{ x_i : i < n \}) \). The chain \( \mathcal{I} = \{ I_n : n < \omega \} \) is separating, as we now check. First \( \bigcup \mathcal{I} = I_0 = \varphi(X) \). Every \( I \in \mathcal{I} \setminus \{ \bigcup \mathcal{I} \} \) is an \( I_n \) for some \( n \geq 1 \). For every finite set \( F \) of points of \( \varphi(X) \setminus I_n \), we define \( J \) as \( I_{n+1} \) and we check that \( I_n \not\subseteq \varphi(F \cup J) \) by showing that \( x_n \), which is in \( I_n \), is not in \( \varphi(F \cup I_{n+1}) \). Since \( \varphi \) is topological, it suffices to show that \( x_n \) is neither in \( \varphi(F) \) nor in \( \varphi(I_{n+1}) = I_{n+1} \). The latter—that \( x_n \) is not in \( I_{n+1} \) is clear. As for the former, we note that \( I_n \cup \varphi(X \setminus \{ x_n \}) = \varphi((X \setminus \{ x_i : i < n \}) \cup (X \setminus \{ x_n \})) = \varphi(X) \), using the fact that \( \varphi \) is topological. That implies \( \varphi(X) \setminus I_n \not\subseteq \varphi(X \setminus \{ x_n \}) \), so \( F \subseteq \varphi(X \setminus \{ x_n \}) \), whence \( \varphi(F) \subseteq \varphi(X \setminus \{ x_n \}) \). If \( x_n \) were in \( \varphi(F) \), it would then be in \( \varphi(X \setminus \{ x_n \}) \), and that is impossible since \( X \) is independent.

Remark 2. In general, e.g. if \( \varphi \) is not topological, we cannot take \( E = E' \).

Consider \( E := \mathcal{P}(\mathbb{N}) \), \( \varphi(A) := \mathcal{P}(\bigcup A) \). Note that the closed subsets of \( E \) are exactly the sets of the form \( \mathcal{P}(A) \) with \( A \subseteq \mathbb{N} \). There is an infinite independent set, say \( X := \{ \{ n \} \mid n \in \mathbb{N} \} \). However, no subset of \( E \) has any separating chain of closed sets \( \mathcal{I} \). Indeed, assume one existed. Pick \( I \in \mathcal{I} \setminus \{ \bigcup \mathcal{I} \} \). Since \( I \) is closed, \( I = \mathcal{P}(A) \) for some \( A \subseteq \mathbb{N} \). Take \( F := \{ A \} : \) for every \( J \in \mathcal{I} \), \( \varphi(F \cup J) \supseteq \varphi(F) = \mathcal{P}(A) = I \), so \( \mathcal{I} \) cannot be separating.

In a quasi-ordered set \( E \), for every \( A \subseteq E \), write \( \downarrow A \) for \( \{ x \in E : \exists y \in A, x \leq y \} \). A subset \( A \) is cofinal in \( E \) if and only if \( E = \downarrow A \). An initial segment is a subset \( I \) of \( E \) such that \( I = \downarrow I \). The finitely generated initial segments are the sets of the form \( \downarrow A \), \( A \) finite. We write \( \downarrow x \) for \( \downarrow \{ x \} \).

Every closure system \( E \) is quasi-ordered by \( x \leq y \) if and only if \( x \in \varphi(\{ y \}) \). For every \( x \in X \), \( \downarrow x = \varphi(\{ x \}) \). Every closed set is an initial segment. If \( \varphi \) is topological, then every finitely generated initial segment \( \downarrow A \) is equal to \( \varphi(A) \), hence is closed.

We say that a quasi-ordered set is well-founded if and only if it has no infinite strictly descending sequence \( x_0 > x_1 > \cdots > x_n > \cdots \), where \( x < y \) if and only if \( x \leq y \) and \( y \not\subseteq x \). That extends the same notion on posets.

Lemma 2. If \( (E, \varphi) \) is a closure system then \( E \) contains a generating subset \( D \) on which the collection of finitely generated initial segments of the quasi-ordered set \( (D, \subseteq_{\downarrow D}) \) is well-founded.

Proof. According to a result of Birkhoff (1937), the poset \( \mathbf{I}_{\omega}(P) \) of finitely generated initial segments of a poset \( P \) is well-founded if \( P \) is well-founded [1, Theorem 2, p.182]. This property holds for initial segments of a quasi-ordered set too, since initial segments of a quasi-ordered set are inverse images of initial segments of the order quotient. For the reader’s convenience, we give a proof of Birkhoff’s result. Let \( \downarrow A_1 \supset \downarrow A_2 \supset \cdots \supset \downarrow A_n \supset \) be an infinite descending sequence where each \( A_i \) is finite. We may assume that each \( A_i \) is an antichain. Construct a tree \( T \) whose vertices are finite chains \( \{ x_1, x_2, \ldots, x_n \} \) where each \( x_i \in A_i \) and \( x_1 \geq x_2 \geq \cdots \geq x_n \). The cardinality of such a set is \( n \), or some lower number (if some element appears several times), and is the depth of the vertex.
in $T$. The unique parent of a depth $n$ set, $n \geq 1$, is obtained by removing its least element ($x_n$, but also $x_{n-1}$ if that happens to be equal to $x_n$, and so on). The empty chain is the root. Every element of each $A_n$ is the least element of at least one such chain, of depth at most $n$. Hence every set $\downarrow A_n$ appears as the initial segment generated by a finite set of vertices of $T$. Since there are infinitely many sets $\downarrow A_n$, the tree $T$ is infinite. Since $T$ is finitely branching, by König’s Lemma it has an infinite branch, and that is an infinite descending sequence of elements: contradiction.

Thus, in order to prove the lemma, it suffices to prove that $E$ contains a generating subset on which the quasi-order $\preceq$ is well-founded.

A result due to Hausdorff (see [3] Chapter 2, p.57] states that every poset contains a well-founded cofinal subset. That is also valid for quasi-ordered sets such as $E$. Indeed, consider the family $\mathcal{W}$ of all well-founded subsets of the set $E$, and order it by prefix: $A \subseteq B$ if and only if $B \cap \downarrow A = A$. By Zorn’s Lemma, it has a maximal element $A$. If $A$ were not cofinal, there would be a point $x$ that is not in $\downarrow A$. Then $B = A \cup \{x\}$ would be a strictly larger well-founded subset of $E$, contradicting maximality.

Hence let $D$ be a well-founded cofinal subset of $E$. For every $x \in E$, there is a point $y \in D$ such that $x \preceq y$. In other words, $x$ is in $\varphi(\{y\})$, hence in the larger set $\varphi(D)$. Therefore $\varphi(D) = E$ and $D$ is generating. $\square$

**Proof of the implication (i) $\Rightarrow$ (iii).** Let $\varphi$ be the closure associated with $\mathcal{F}$, and remember that it is topological. Let $C$ be a closed set. We define a dense subset of $C$ on which the induced topology is noetherian. This will be $D$, as given by Lemma [2] Note that $D$ is dense in $C$.

Let $\varphi_{\uparrow D}$ be the closure induced on $D$, namely $\varphi_{\uparrow D}(X) := \varphi(X) \cap D$ for every $X \subseteq D$. This is also a topological closure, and we claim that it is noetherian.

By (i), $E$ contains no infinite discrete set, so $D$ does not contain any infinite discrete set either. Imagine that $D$ contained an infinite strictly descending sequence $I_0 \triangleright I_1 \triangleright \cdots \triangleright I_n \triangleright \cdots$ of closed subsets. By Lemma [1] that chain must fail to be separating: there must be an index $m_1 \geq 1$ and a finite set $F_1$ of points of $I_0 \setminus I_{m_1}$ such that, for every $n < \omega$ (in particular, for every $n > m_1$), $I_{m_1} \subseteq \varphi_{\uparrow D}(F_1 \cup I_n) = \varphi_{\uparrow D}(F_1) \cup I_n$. The last equality is because $\varphi_{\uparrow D}$ is topological, and $I_n$ is closed. Hence $m_1 \geq 1$, $F_1 \subseteq I_0 \setminus I_{m_1}$, and for every $n > m_1$, $I_{m_1} \setminus I_n \subseteq \varphi_{\uparrow D}(F_1)$. We do the same with the infinite subsequence starting at $I_{m_1}$: there is an index $m_2 \geq m_1 + 1$ and a finite set $F_2 \subseteq I_{m_1} \setminus I_{m_2}$ such that for every $n > m_2$, $I_{m_2} \setminus I_n \subseteq \varphi_{\uparrow D}(F_2)$. Proceeding this way, we obtain indices $m_{k+1} \geq m_k + 1$ and finite sets $F_{k+1} \subseteq I_{m_k} \setminus I_{m_{k+1}}$ such that for every $n > m_{k+1}$, $I_{m_{k+1}} \setminus I_n \subseteq \varphi_{\uparrow D}(F_{k+1})$, for every $k \geq 1$.

Note that $F_{k+1} \subseteq I_{m_k} \setminus I_{m_{k+1}} \subseteq \varphi_{\uparrow D}(F_k)$, so $\varphi_{\uparrow D}(F_{k+1}) \subseteq \varphi_{\uparrow D}(F_k)$. It follows that the sequence $(\varphi_{\uparrow D}(F_k))_{k \geq 1}$ is descending. Since $D$ was obtained from Lemma [2] that sequence must be finite. Pick $k \geq 2$ such that $\varphi_{\uparrow D}(F_k) = \varphi_{\uparrow D}(F_{k+1})$. $F_k$ cannot be empty, since $\varphi_{\uparrow D}(F_k)$ contains $I_{m_k} \setminus I_{m_{k+1}}$, which is non-empty. Pick $x \in F_k$. In particular, $x$ is in $I_{m_{k-1}} \setminus I_{m_k}$, hence is not in $I_{m_k}$. 

However, \( x \) is also in \( \varphi(D)(F_k) = \varphi(D)(F_{k+1}) \), and since \( F_{k+1} \subseteq I_{m_k} \setminus I_{m_{k+1}} \subseteq I_{m_k} \), \( x \) is also in \( \varphi(D)(I_{m_k}) = I_{m_k} \): contradiction.

**Proof of the implication (iii) \( \Rightarrow \) (ii).** Let \( \varphi \) be the closure on \( E \). Let \( C \) be a closed set and \( D \) be a dense subset of \( C \) on which the closure \( \varphi(D) \) is well-founded.

On \( D \) every closed set \( D' \) is a finite union of irreducible closed sets. This fact goes back to Noether, see [11, Chapter VIII, Corollary, p.181]. Indeed, if \( D \) is not such, then, since the collection of closed sets on \( D \) is well-founded, there is a minimal member \( C' \) which is not a finite union of irreducible members. In particular, \( C' \) is non-empty. If \( C' \) is the union of two proper closed subsets, by minimality those closed subsets must be finite unions of irreducible subsets of \( D \), hence so must be \( C' \). It follows that \( C' \) is irreducible: contradiction.

Now \( D \) is itself closed in \( D \), so we can write \( D \) as a finite union of irreducible closed subsets \( C_i \) of \( D \), \( 1 \leq i \leq n \). For each \( C_i \), \( \varphi(C_i) \) is irreducible in \( E \), as one easily checks [8, Lemma 8.4.10]. By density and the fact that \( \varphi \) is topological, \( C = \varphi(D) = \bigcup_{i=1}^{n} \varphi(C_i) \).

**Proof of the implication (ii) \( \Rightarrow \) (i).** Let \( \varphi \) be the closure on \( E \) again, and let \( X \) be a discrete subspace. Write \( \varphi(X) \) as a finite union of irreducible closed sets \( I_1, \ldots, I_n \).

For each \( x \in X \), \( x \) is in some \( I_k \). We claim that \( I_k = \varphi(\{x\}) \). To that end, we note that \( I_k \subseteq \varphi(X) = \varphi(\{x\}) \cup \varphi(X \setminus \{x\}) \), since \( \varphi \) is topological. Therefore \( I_k \) is equal to the union of the two closed sets \( \varphi(\{x\}) \cap I_k \) and \( \varphi(X \setminus \{x\}) \cap I_k \).

Since \( X \) is discrete, hence independent, \( x \) is not in \( \varphi(X \setminus \{x\}) \), and as \( x \in I_k \), \( \varphi(X \setminus \{x\}) \cap I_k \) cannot be a proper closed subset of \( I_k \). Because \( I_k \) is irreducible, \( \varphi(\{x\}) \cap I_k \) cannot be a proper closed subset of \( I_k \), so \( \varphi(\{x\}) \cap I_k = I_k \). This means that \( I_k \subseteq \varphi(\{x\}) \), and the converse inclusion follows from \( x \in I_k \).

It follows that for any two distinct points \( x, y \in X \), \( x \) and \( y \) cannot be in the same \( I_k \). Otherwise \( \varphi(\{x\}) = \varphi(\{y\}) \), but since \( X \) is independent, \( x \) is not in \( \varphi(X \setminus \{x\}) \), hence not in the smaller set \( \varphi(\{y\}) \). That is impossible since \( \varphi(\{x\}) = \varphi(\{y\}) \) contains \( x \).

Since each \( I_k \) can contain at most one point from \( X \), \( X \) is finite.

2. Remarks and comments

2.1. Other characterizations. A.H. Stone [14, Theorem 2] has shown that (i) is equivalent to two further properties: (ii) every open cover of every subspace \( X \) of \( T \) has a finite subfamily whose union is dense in \( X \), and (iv) every continuous real-valued function on every subspace of \( T \) is bounded.

2.2. Noetherian topological spaces. Noetherian topological spaces have been studied for their own sake by A.H. Stone [14]. They are an important basic notion in algebraic geometry, since the spectrum of any noetherian ring in a noetherian topological space, with the so-called Zariski topology. They have also found applications in verification, the domain of computer science concerned with finding algorithms that prove properties of other computer
systems, automatically \[\text{[7]}.\] One can consult Section 9.7 of \[\text{[8]},\] which is devoted to noetherian topological spaces.

2.3. **A related result.** The implication \((i) \Rightarrow (ii)\) follows from the following result about closure systems. We recall that an *up-directed* subset of a poset \(P\) is a non-empty subset \(A\) of \(P\) such that any two elements of \(A\) have an upper bound in \(A\), and that an *ideal* is an up-directed initial segment. We always order powersets by inclusion.

**Theorem 2.** [9] If a closure system \((E, \varphi)\) contains no infinite independent set then: \((*)\) there are finitely many pairwise disjoint subsets \(A_i\) of \(E\) and, for each \(A_i\), a proper ideal \(N_i\) of \(\varphi(A_i)\) such that for every \(X \subseteq \bigcup_{i \in I} A_i\), the set \(X\) generates \(E\) if and only if \(A_i \cap X \notin N_i\), for each \(i \in I\).

As it will become apparent in Proposition \[\text{[3]},\] this result specialized to topological closures is just implication \((i) \Rightarrow (ii)\). Decompositions of topological closures were considered in \[\text{[6]},\] but this consequence was totally missed.

**Remark 3.** In Theorem \[\text{[3]},\] we may suppose that:

\[A_i \cap \varphi(\bigcup_{j \in I \setminus \{i\}} A_j) = \emptyset \text{ for each } i \in I.\]

This is Remark 1 of \[\text{[6]}.\] We repeat the argument. If \([1]\) does not hold, set \(A'_i = A_i \setminus \varphi(\bigcup_{j \neq i} A_j)\) and \(N'_i = N_i \cap \varphi(A'_i)\). Let us check that the \(A'_i\) and \(N'_i\) satisfy the condition in Theorem \[\text{[2]},\] First, each \(N'_i\) is a proper ideal (if \(N'_i\) is not proper, that is \(A'_i \in N'_i\), then \(A'_i \in N_i\). Set \(X := A'_i \cup \bigcup_{j \neq i} A_j\); since \(A_i \subseteq A'_i \cup \varphi(X)\), \(X\) generates \(E\), but violates the condition of the theorem). Next, let \(X \subseteq \bigcup \{A'_i : i \in I\}\). Suppose that \(X\) generates \(E\). Since \(X \subseteq \bigcup \{A_i : i \in I\}\), \(A_i \cap X \notin N_i\), for each \(i \in I\). Hence \(A'_i \cap X = A_i \cap X \notin N'_i\). Conversely, if \(A'_i \cap X \notin N'_i\), for each \(i \in I\), then \(A'_i \notin N_i\), hence \(X\) generates \(E\).

**Proposition 3.** Let \(\varphi\) be a topological closure operator on a set \(E\). Then \(E\) is a finite union of irreducible closed sets iff \(E\) has a decomposition satisfying the condition on generating sets in Theorem \[\text{[2]},\]

**Proof.** The result is a consequence of the following two claims.

**Claim 4.** Let \(A_i (i \in I)\) be a finite decomposition satisfying Condition \((*)\) on generating sets of Theorem \[\text{[2]},\] According to Remark \[\text{[3]},\] we may assume that it satisfies Condition \([1]\). Then \(Y \in N_i\) iff \(Y \subseteq A_i\) and \(\varphi(Y) \notin A_i\). In particular, \(X_i := \varphi(A_i)\) is irreducible.

**Proof of Claim 4** First note that \(E = \varphi(\bigcup_{i \in I} A_j)\). Indeed, it suffices to apply \((*)\) to \(X := \bigcup_{i \in I} A_j\), and to realize that \(A_i \cap X = A_i\) cannot be in \(N_i\), since \(N_i\) is a proper ideal of \(\varphi(A_i)\).

Suppose that \(Y \in N_i\). Since \(N_i\) is an initial segment, \(A_i \cap Y\) is in \(N_i\). For \(X := Y \cup \bigcup_{i \neq \{i\}} A_j, A_i \cap X = A_i \cap Y\) by Condition \([1]\), so \(A_i \cap X\) is in \(N_i\). By \((*)\), \(X\) does not generate \(E\). Since \(\varphi\) is a closure operator, if \(A_i \subseteq \varphi(Y)\) then \(E = \varphi(\bigcup_{i \in I} A_j) \subseteq \varphi(Y \cup \bigcup_{i \neq \{i\}} A_j) = \varphi(X)\), which is impossible. Hence \(A_i \notin \varphi(Y)\).
Conversely, since $\varphi(Y) \notin A_i$, there is a point $x$ in $A_i$—hence not in $\varphi(\bigcup_{j \in I \setminus \{i\}} A_j)$ by Condition 1—which is not in $\varphi(Y)$, hence not in $\varphi(Y) \cup \varphi(\bigcup_{j \in I \setminus \{i\}} A_j)$. The latter is equal to $\varphi(X)$, where $X := Y \cup \bigcup_{j \in I \setminus \{i\}} A_j$, since $\varphi$ is topological, so $X$ does not generate $E$. Using $(*)$, $A_j \cap X$ is in $N_j$ for some $j \in I$. If $j \neq i$, then $A_j \cap X \supseteq A_j$, and $A_j \cap X \in N_j$ would imply $A_j \in N_j$, contradicting the fact that $N_j$ is proper. Therefore $j = i$. This means that $A_i \cap X$, which is equal to $A_i \cap Y$ by Condition 1, hence to $Y$ since $Y \subseteq A_i$, is in $N_i$.

We finally show that $X_i$ is irreducible. Since $N_i$ is proper, $A_i$ is non-empty, hence $X_i$ is non-empty. Assume that $X_i$ is the union of two proper closed subsets $C_1$ and $C_2$. Consider $Y := C_1 \cap A_i$ (resp., $Y := C_2 \cap A_i$). Then $Y \subseteq A_i$, and $\varphi(Y) \subseteq C_1$ (resp., $C_2$) cannot contain $X_i = \varphi(A_i)$, hence cannot contain $A_i$. It follows that $Y$ is in $N_i$. In other words, both $C_1 \cap A_i$ and $C_2 \cap A_i$ are in $N_i$. Since $N_i$ is an ideal, $(C_1 \cap A_i) \cup (C_2 \cap A_i) = (C_1 \cup C_2) \cap A_i = X_i \cap A_i = A_i$ is in $N_i$, which is impossible since $N_i$ is proper.

Claim 5. If $E$ is a finite union of irreducible closed sets, let $(X_i)_{i \in I}$ be a family of such sets with $|I|$ minimum. Set $A_i := X_i \setminus \bigcup_{j \notin i} X_j$ and $N_i := \{A' \subseteq A_i : \varphi(A') \neq X_i\}$. This decomposition satisfies Condition $(*)$ of Theorem 2.

Proof of Claim 5. We check that $N_i$ is an ideal. Given $A', B' \in N_i$, $\varphi(A' \cup B') = \varphi(A') \cup \varphi(B')$, since $\varphi$ is topological. If that were equal to the whole of $X_i$, and since $\varphi(A')$ and $\varphi(B')$ are both proper closed subsets of $X_i$, $X_i$ would fail to be irreducible. Hence $\varphi(A' \cup B') \neq X_i$, so that $A' \cup B'$ is in $N_i$.

Then we check that $N_i$ is proper, namely that $A_i$ is not in $N_i$. By definition of $A_i$, $X_i \subseteq A_i \cup (\bigcup_{j \notin i} X_j) \subseteq \varphi(A_i) \cup (\bigcup_{j \notin i} X_j)$, so $X_i = (X_i \cap \varphi(A_i)) \cup (X_i \cap (\bigcup_{j \notin i} X_j))$, a union of two closed subsets. The second one, $X_i \cap (\bigcup_{j \notin i} X_j)$, is a proper subset of $X_i$ since we have chosen a family of least cardinality. Since $X_i$ is irreducible, the other one cannot be a proper subset. Therefore $\varphi(A_i) = X_i$. It follows that $A_i$ is not in $N_i$.

Finally, let $X \subseteq \bigcup_{i \in I} A_i$.

If $A_i \cap X$ belongs to $N_i$ for no $i \in I$, then by definition of $N_i$, $\varphi(A_i \cap X) = X_i$, hence $\varphi(X) = (\bigcup_{i \in I} A_i \cap X) = \bigcup_{i \in I} \varphi(A_i \cap X) = \bigcup_{i \in I} X_i = E$.

Conversely, assume that $\varphi(X) = E$. Since $X \subseteq \bigcup_{j \in I} A_j$, $X = \bigcup_{j \in I} A_j \cap X$, and since $\varphi$ is topological, $E = \varphi(X) = \bigcup_{i \in I} \varphi(A_j \cap X)$. Recall that $E = \bigcup_{i \in I} X_i$, so for every $i \in I$, $X_i \subseteq \bigcup_{j \in I} \varphi(A_j \cap X)$, and since $X_i$ is irreducible, there is a $j \in I$ such that $X_i \subseteq \varphi(A_j \cap X)$. If $j \neq i$, then $X_i \subseteq \varphi(A_j) \subseteq \varphi(X_j) = X_j$, which is impossible since we have chosen $(X_i)_{i \in I}$ of least cardinality. Hence $j = i$, meaning that for every $i \in I$, $X_i \subseteq \varphi(A_i \cap X)$. Since $\varphi(A_i \cap X) \subseteq \varphi(A_i) \subseteq \varphi(X_i) = X_i$, $X_i = \varphi(A_i \cap X)$, and that shows that $A_i \cap X$ is not in $N_i$.

2.4. Topological properties versus lattice properties. Item (ii) in Theorem 4 is a property about the lattice of closed sets of a topological space: if two topological spaces have the same lattice of closed sets, they both satisfy (ii) or none satisfies it. The same is true for item (i). Indeed, as it is well-known, the existence of an infinite discrete subspace (or more generally of
an infinite independent subset for a closure system) amounts to the existence of an embedding of \( P(\mathbb{N}) \), the collection of subsets of \( \mathbb{N} \) ordered by inclusion, into the lattice of closed sets. It not clear that this is the case for item (iii) without having a proof of Theorem \[1\] For example, the notion of density is not a lattice property: the smallest cardinality of dense subsets of \( \mathbb{N} \) with the cofinite topology (whose closed sets are the finite subsets of \( \mathbb{N} \) plus \( \mathbb{N} \) itself) is infinite, but the sobrification \( \mathbb{N}^s \) of this space obtained by adding a new point \( \infty \) to \( \mathbb{N} \), and whose closed sets are the finite subsets of \( \mathbb{N} \) plus \( \mathbb{N}^s \), has a one-point dense subset, \( \{\infty\} \), but an isomorphic lattice of closed sets.

2.5. The case of posets. Theorem \[1\] has a well-known predecessor in the theory of poset. It is worth to recall it.

Let \( P \) be a poset, and \( A \) be a subset of \( P \). An upper bound of \( A \) is any \( z \in P \) such that \( x \leq z \) for every \( x \in A \). The set \( A \) is \up-independent if no pair of distinct members of \( A \) have a common upper bound; it is consistent (or compatible) if every pair of members of \( A \) has an upper bound.

The final segments of \( P \) are the initial segments of \( P^d \), the opposite order; we denote by \( \uparrow A \), resp. \( \uparrow a \), the final segment of \( P \) generated by \( A \subseteq P \), resp. \( a \in P \).

The set \( I(P) \) of initial segments of \( P \) is the set of closed sets of a topology, the Alexandroff topology. In this setting, a subset \( A \) is discrete if and only if it is an antichain, \( A \) is dense if and only if it is cofinal, and \( A \) is irreducible if and only if it is an ideal.

A poset \( P \) is well-quasi-ordered (w.q.o. for short) if it is well-founded and contains no infinite antichain. According to Higman [11], \( P \) is w.q.o. iff \( I(P) \) is well-founded.

We recall the following result (see [5, Chapter 4]):

\textbf{Theorem 6.} The following properties are equivalent for a poset \( P \):

\begin{enumerate}[(a)]
\item \( P \) contains no infinite antichain;
\item every initial segment of \( P \) is a finite union of ideals;
\item every initial segment of \( P \) contains a cofinal subset which is well-quasi-ordered.
\end{enumerate}

\textit{Proof.} This is just Theorem \[1\] applied to \( P \) with the Alexandroff topology, provided one notes that a poset is well-quasi-ordered if and only if it is noetherian in its Alexandroff topology. But the proof simplifies.

\( (a) \Rightarrow (c) \). Let \( P' \) be an initial segment. By an already cited result of Hausdorff, \( P' \) contains a well-founded cofinal subset \( A \). Since \( P \) has no infinite antichain, \( P' \) has no infinite antichain; being well-founded it is w.q.o.

\( (c) \Rightarrow (b) \). Let \( P'' \) be an initial segment and \( A \) be a cofinal subset of \( P' \) which is w.q.o. Being w.q.o., \( A \) is a finite union of ideals \( I_1, \ldots, I_q \). This is a basic result of the theory of w.q.o. [1] Chapter VIII, Corollary, p.181]. Indeed, as in the proof of implication \((iii) \Rightarrow (ii)\) of Theorem \[1\] replacing “closed” by “initial segment” and “irreducible closed” by “ideal”, if \( A \) is not such, then,
since $I(A)$ is well-founded, there is a minimal member $A' \in I(A)$ which is not a finite union of ideals. This $A'$ is irreducible, hence is an ideal: contradiction.

Now $P' = \downarrow A = \downarrow I_1 \cup \ldots \downarrow I_q$ and the set $\downarrow I_i$ are ideals of $P'$.

(b) $\Rightarrow$ (a). Let $A$ be an antichain of $P$. An ideal $I$ of $\downarrow A$ cannot contain more than one element of $A$. Since $\downarrow A$ is a finite union of ideals, $A$ is finite. □

Remark 4. A direct proof of (a) $\Rightarrow$ (b) can be obtained from a special case of a result of Erdős and Tarski [4]. This special case is a prototypal Min-Max result which has been overlooked. For the reader’s convenience, we recall it and give a proof.

**Proposition 7.** If a poset $P$ contains no infinite up-independent set then there is a finite upper bound on the size of up-independent sets. In this case, the maximum size of up-independent sets, the least number of ideals whose union is $P$ and the least number of consistent sets whose union is $P$ are equal.

**Proof.** Let $P$ be a poset. Let $Q := \{x \in P : \uparrow x$ is up-directed$\}$, let $U := \downarrow Q$ and $R := P \setminus U$.

If $R$ is non-empty, define recursively a map $f$ from $2^{\omega}$, the set of finite 0-1 sequences, into $R$, by picking some arbitrary element of $R$ for $f(\epsilon)$ (where $\epsilon$ is the empty sequence), and choosing two incompatible elements of $\uparrow f(s)$ for $f(s.0)$ and $f(s.1)$ (here $s.0$ and $s.1$ are the two sequences obtained from $s$ by adding 0 and 1 to the right of the sequence $s$, respectively). Ordered by extension of sequences, $2^{\omega}$ is the dyadic tree $T_2$; the map $f$ from $T_2$ preserves the incompatibility relation. Since $T_2$ contains infinite up-independent sets, so does $R$. Hence $R$ is empty.

Among the subsets of $Q$ which are up-independent (in $Q$) let $L$ be a maximal member with respect to inclusion and $\kappa := |L|$. Then $\uparrow L$ is cofinal in $Q$ hence in $U$. Since for each $x \in L$, the set $I_x := \downarrow F_x$, where $F_x := \uparrow x$, is an ideal, $U$ is covered by at most $\kappa$ ideals. Since $R$ is empty, the poset $P$ is covered by at most $\kappa$, and in fact exactly $\kappa$, ideals. It is immediate that $\kappa$ is the common value to the parameters defined in the above proposition. □

Now, assuming (a), (b) can be proved as follows. Let $I$ be an initial segment of $P$. Since $P$ contains no infinite antichain, $I$ does not contain an infinite antichain either. In the subposet $I$, the up-independent subsets are all finite, since every up-independent subset is an antichain. Their maximal size is then the least number of ideals whose union is $I$, by the Erdős-Tarski theorem.

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