On Time-Dependent Queue-Size Distribution in a Model With Finite Buffer Capacity and Deterministic Multiple Vacations With Applications to LTE DRX Mechanism Modeling

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ABSTRACT A finite-capacity queueing model with the arrival stream of messages governed by the compound Poisson process and generally-distributed processing times is investigated. Whenever the system becomes empty (the server becomes idle), a number of deterministic independent vacations of equal length are initialized as far as at least one message is detected in the accumulating buffer at the completion epoch of one of them. During vacations the service process is suspended completely, while after finishing the last vacation it restarts normally. Identifying Markov moments in the evolution of the system and using the Korolyuk’s potential method, the compact-form representation for the transient queue-size distribution conditioned by the initial buffer state is found in terms of its Laplace transform. The considered model has potential applications in modeling the energy saving LTE DRX mechanism. A detailed simulation and numerical study is attached.

INDEX TERMS Discontinuous reception (DRX), multiple vacation policy, queue-size distribution, stochastic simulation, transient state.

I. INTRODUCTION

Global challenges related to the need to save energy are a permanent motivation to look for new solutions and techniques that could be used in more effective energy management and monitoring the phenomenon of its consumption. Energy saving is of particular importance in the operation of wireless networks (Wi-Fi, LTE, wireless sensor networks etc.) where nodes/mobile stations are usually powered by batteries (e.g. sensors in wireless sensor networks). In order to ensure the longest functioning of the relevant devices, various mechanisms are implemented to reduce energy consumption. One of the mechanisms implemented in LTE wireless data transmission standard is discontinuous reception (DRX). In DRX the state of the buffer accumulating incoming messages is monitored at regular intervals. If there are no messages, the buffer state is checked (see e.g. [7]). Evidently, queueing systems with different-type vacation policies can be effectively used in modeling the functioning of networks with the mechanism of energy saving. In particular, the construction of LTE DRX scheme suggests using the multiple vacation policy with exhaustive service and constant (deterministic) length of successive vacations. In this policy, at the moment at which the system empties, the server takes on a vacation during which the processing is blocked completely. After finishing the vacation the queue size is monitored. If the buffer is still empty, another independent vacation is taking on and so on.

Levy and Yechiali were the first who introduced and considered the concept of server’s vacation [19]. Since the publication of this work, a large number of articles have been published on analytical results and the application of queueing models with different-type of vacation policies. A detailed study on various vacation queueing models can be found e.g. in an excellent book of N. Tian and Z. G. Zhang [25] or in survey papers [9] and [24]. In [5] a queueing model with server vacations and batch Poisson arrivals is analyzed. A model with batch input flow and
gated service with multiple vacation policy is studied in [2]. A kind of a modification of the classical multiple vacation policy is proposed in [6] where after each vacation or service completion the server takes sequence of vacations until a batch of new messages enters. In [27] an infinite-capacity model with working vacations is considered in which, every time when the system becomes empty, the server takes a working vacation during which processing is provided with a lower rate. If at least one message is present at a service completion instant, the vacation is interrupted and the server initializes a normal working period (with a given probability) or continues the vacation. A discrete-time queueing model with multiple vacations is investigated in [11] by using the supplementary variables’ technique and the idea of imbedded Markov chain. The queue-size distributions at different time moments (pre-arrival, arbitrary, etc.) are obtained. In [29] the optimal service policy in the $M/G/1$-type queue with multiple vacation policy is analyzed. The optimal strategy for the Markovian queue with batch arrivals operated with multiple vacations and threshold-type $N$-policy is derived in [14]. Queueing models with server vacations and auto-correlated input flows are investigated in [3] and [4], where MAP and BMAP arrival processes are assumed, respectively. In [12] a Markovian queueing model with differentiated vacations and vacations interruptions is investigated.

As one can note, most results obtained for different-type models with vacation policies relate to the stationary state of the system. Transient results are much rarer, however, time-dependent analysis of the system is sometimes very important. For example, if we analyze the system just after its restarting after a repair period, or in the case of the application of a new control mechanism, transient analysis is especially desired. Moreover, according to network applications, in some wireless sensor networks the intensity of the input/output traffic for a network node is very low so the steady state does not reflect the behavior of the system.

One can find transient analysis of the $M/M/1$-type queue with multiple vacation policy combined with the $N$-policy in [28]. In [13] an $M/M/1$ model subjected to multiple differentiated vacations, customer impatience and a waiting server is investigated. A finite-buffer model with Poisson arrival stream and generally-distributed service times is studied in [16] under multiple vacation policy. Compact-form representation for the Laplace transform of the transient queueing delay is obtained there. In [15] new results for a general-type $GI/G/1$ model with batch arrivals of messages and exponentially-distributed server vacations are derived. In particular, it is obtained the Laplace transform of the joint distribution of the first busy period, the first vacation period and the number of messages served during the first busy period.

In [22] a new method is proposed for computing mean queueing delay in LTE DRX. A variant of the $M/G/1$-type queue with a modified service time and multiple vacations is proposed for the analysis of DRX mechanism in [8]. Different performance measures are obtained there. In [10] the DRX scheme is studied analytically. The problem of selecting optimal parameters of DRX scheme, while keeping Quality of Service (QoS) delay requirements is considered. In [23] a finite-buffer queueing model with a number of $c$ service stations is proposed for the control of Internet of Healthcare Things medical monitoring system.

In the paper, we study a finite-buffer queueing model with batch Poisson input flow, generally distributed service times operating under multiple vacation policy with deterministic (constant) durations of successive vacations. Applying analytical approach based on the idea of embedded Markov chain, continuous version of the formula of total probability and Korolyuk’s potential method, a compact-form representation for the Laplace transform of the queue-size distribution conditioned by the initial buffer state is obtained. The considered model has potential applications in modeling LTE DRX mechanism. A detailed simulation and numerical study is provided.

The remaining part of the article is organized as follows. In the next short Section 2, we give a precise mathematical description of the considered queueing model. In Section 3, we build a system of integral equations for the transient queue-size distribution where the condition is the buffer state at time $t = 0$. In Section 4, a corresponding system written for Laplace transforms is found and written in a specific form. Section 5 contains the main analytical result, namely the representation for the Laplace transform of the conditional transient queue-size distribution. The solution is obtained via Korolyuk’s potential approach. Section 6 is devoted to simulations and numerical examples. Numerical results obtained from analytical formulae are compared with simulations there. The last Section 7 contained conclusions and remarks on future works.

II. MODEL DESCRIPTION

In the paper, we deal with the $M^X/G/1/B$-type queueing model in which messages arrive according to a compound Poisson process with rate $\lambda > 0$. So, a group of $k$ messages enters simultaneously with a general discrete probability distribution $p_k$, where $\sum_{k=1}^{\infty} p_k = 1$. The processing is organized individually, according to the FIFO service discipline, with a general-type cumulative distribution function (CDF) $F(\cdot)$ of the processing time, with the Laplace-Stieltjes transform (LST) $f(\cdot)$. The capacity of the accumulating buffer is finite and equals $B - 1$, so the maximal system state (the number of messages simultaneously present in the system) equals $B$. It is assumed that if the size of the arriving group of messages exceeds the current available capacity of the accumulating buffer, the whole group is being lost (the so-called Whole Batch Acceptance Strategy).

Whenever the service station becomes idle, a multiple vacation is initialized during which the processing is suspended completely. More precisely, successive deterministic (constant) vacations of duration $T > 0$ are taken repeatedly. The queue size is monitored at completion epochs of successive vacations. If at least one message accumulated
in the buffer is detected at the end of one of them, the processing restarts immediately at this moment.

**III. TIME-DEPENDENT EQUATIONS FOR CONDITIONAL QUEUE-SIZE DISTRIBUTION**

In this section, applying the paradigm of embedded Markov chain and using the continuous version of total probability law, we establish the system of integral equations for the transient queue-size distribution conditioned by the initial state (number of messages present) of the considered queuing model.

Let us denote by \( X(t) \) the number of messages present in the system at time \( t \), including the one being processed at this moment, if any.

Introduce the time-dependent queue-size distribution conditioned by the system state at the initial moment \( t = 0 \) in the following way:

\[
Q_n(t, m) \overset{\text{def}}{=} P( X(t) = m \mid X(0) = n ),
\]

where \( t > 0, m, n \in \{0, \ldots, B\} \).

Let us begin with the analysis of the system being empty at the initial moment, namely \( X(0) = 0 \). Since no message is detected at this time, the server takes a multiple vacation period (we treat this moment as a regeneration time). It is easy to note that, for fixed time moment \( t \), the following three mutually exclusive random events may happen:

- Case 1: the first message arrives before time \( t \) and the first multiple vacation period also completes before \( t \);
- Case 2: the first message arrives before time \( t \) but the first multiple vacation period completes after \( t \);
- Case 3: the first message arrives after time \( t \).

Investigate firstly Case 1. Denoting by \( I[\mathcal{E}] \) the indicator of the random event \( \mathcal{E} \), we have the following representation:

\[
P( X(t) = m ) \cap \text{Case 1 } | X(0) = 0)
= \sum_{k=0}^{\infty} \int_{0}^{t} \lambda e^{-\lambda y} I[kT \leq y < (k + 1)T < t]
\times \left[ \sum_{i=1}^{B-1} p_i \left( \sum_{r=0}^{B-i} \sum_{j=0}^{r} \frac{[\lambda [(k + 1)T - y]] !}{j!} e^{-\lambda [(k + 1)T - y]} \right) p_r^j \right] dy,
\]

(2)

where

\[
p_{0}^{0} = 1, \quad p_{i}^{0} = 0, \quad p_{i}^{j} = \sum_{r=1}^{j} p_{i-r}^{(j-1),*} p_{r}
\]

for \( i, j \geq 1 \).

For Case 2 we obtain, similarly,
\[= \int_0^t \left[ \sum_{i=0}^{r-B-n-1} \int_{j=0}^{i} \frac{(\lambda y)^j}{j!} e^{-\lambda y} p_i^j Q_{n+i-1}(t-y, m) \right] dt + Q_{B-1}(t-y, m) \sum_{i=0}^{i} \int_{j=0}^{i} \frac{(\lambda y)^j}{j!} e^{-\lambda y} p_i^j \right] dF(y) + [1 - F(t)] e^{-\lambda t} \left( I[n < m \leq B - 1] \sum_{i=0}^{m-n} p_i^j \right) \frac{(\lambda t)^j}{j!} + I[m = B] \sum_{i=0}^{i} \int_{j=0}^{i} \frac{(\lambda t)^j}{j!} \right] \]

where \( n \in \{1, \ldots, B\} \) and we take the agreement that \( \sum_{i=0}^{n-1} = 0 \).

Indeed, the first summand on the right side of (8) relates to the situation in which the buffer does not overflow before the completion of the first service. The second summand describes the case in which before the first departure epoch the buffer becomes saturated so at time \( y \) the number of messages reduces to \( B - 1 \). The last summand on the right side of (8) relates to the situation in which the first message leaves the system after time \( t \).

IV. CORRESPONDING SYSTEM FOR LAPLACE TRANSFORMS

In this section, we transform the original system of equations (7)–(8) to the corresponding one written for Laplace transforms and next write it in a specific form.

To shorten the notation, let us introduce for \( Re(s) > 0 \) the following functions and functional sequences:

\[ q_0(s, m) = \int_0^\infty e^{-st} Q_{n}(t, m) dt; \]
\[ a_r(s) = \int_0^\infty e^{-s(t+\lambda s)} \sum_{j=0}^{r} \frac{(\lambda t)^j}{j!} p_i^j dF(t); \]
\[ \bar{a}_r(s) = \int_0^\infty e^{-s(t+\lambda s)} \sum_{j=0}^{r} \frac{(\lambda t)^j}{j!} p_i^j [1 - F(t)] dt; \]
\[ b_r(s, m) = I[r \leq m \leq B - 1] \sum_{i=0}^{\infty} \bar{a}_r(s); \]
\[ \alpha_r(s) = \frac{e^{-s(\lambda s + T)}}{1 - e^{-s(\lambda s + T)}} \sum_{j=0}^{r} \frac{(\lambda T)^{j+1}}{(j+1)!} p_i^j; \]
\[ \beta(s) = e^{-s(\lambda s + T)} \sum_{i=0}^{B-1} p_i \sum_{r=0}^{B-i} \sum_{j=0}^{r} \frac{(\lambda T)^{j+1}}{(j+1)!} \]
\[ + (e^{\lambda T} - 1) \sum_{i=0}^{B} p_i; \]
\[ \gamma(s, m) = \sum_{i=1}^{B-1} p_i \left( I[i \leq m \leq B - 1] \sum_{j=0}^{m-i} \frac{p_i^j}{j!} \right). \]

and

\[ \eta_r(s) = \frac{\lambda^{r+1}}{(1 - e^{-s(\lambda s + T)})^{r+2}} \times \left( 1 - e^{-s(\lambda s + T)} \sum_{i=0}^{r+1} \frac{[(\lambda s + T)^i]}{i!} \right). \]

Observe that now the Eqs. (7)–(8) can be written in the form

\[ q_0(s, m) = \sum_{i=0}^{m} p_i \sum_{r=0}^{B-i} a_r(s) q_i(r, s, m) + q_B(s, m) \beta(s) + \gamma(s, m), \]
\[ q_r(s, m) = \sum_{i=0}^{m} a_i(s) q_{n+i-1}(s, m) + q_{B-1}(s, m) \sum_{i=0}^{m} a_i(s) + b_n(s, m), \]

where \( n \in \{1, \ldots, B\} \).

By using the substitution

\[ x_n(s, m) \equiv q_{B-n}(s, m), \]

where \( n \in \{0, \ldots, B\} \), we can transform the last system (17)–(18) in the following way:

\[ \sum_{k=0}^{B-n} a_{k+1}(s) x_{n-k}(s, m) - x_n(s, m) = \theta_n(s, m), \]

where \( n \in \{0, \ldots, B - 1\} \), and

\[ x_0(s, m) = \sum_{i=0}^{r} p_i \sum_{r=0}^{B-i} a_{B-i-r}(s) x_r(s, m) + x_0(s, m) \beta(s) + \gamma(s, m), \]

where

\[ \theta_n(s, m) \equiv a_{n+1}(s) x_0(s, m) - x_1(s, m) \sum_{k=0}^{B-n} a_k(s) - b_{B-n}(s, m). \]

V. REPRESENTATION FOR SOLUTION

In this section, using the idea of Korolyuk’s potential, we obtain the compact-form representation for the solution of the system (20)–(21). The idea of Korolyuk’s potential was introduced in [18] as an alternative tool for the analysis of random walks based on compound Poisson processes. In particular, it is proved in [18] (see also [17]) that each
solution of the system of infinite number of equations of the form (compare (20))
\[
\sum_{k=1}^{n} a_{k+1} x_{n-k} - x_{n} = \theta_{n},
\]
where \( n \geq 0 \), can be written as
\[
x_{n} = CR_{n+1} + \sum_{k=0}^{n} R_{n-k} \theta_{k}, \quad n \geq 0,
\]
where \( C \in \mathbb{R} \) and the sequence \( (R_{k}) \) is defined as follows.

Define the following generating functions:
\[
r(z) \overset{def}{=} \sum_{k=0}^{\infty} z^{k} R_{k}
\]
and
\[
a(z) \overset{def}{=} \sum_{k=0}^{\infty} z^{k} a_{k},
\]
where \(|z| < 1\).

It can be proved ([17], [18]) that then we have
\[
r(z) = \frac{z}{a(z) - z}
\]
and hence, using the Maclaurin expansion and comparing coefficients at successive powers \( z^{k} \), we obtain
\[
R_{k} = \frac{1}{k!} \frac{d^{k}}{dz^{k}} \left( \frac{z}{a(z) - z} \right) \bigg|_{z=0}, \quad k \geq 1.
\]

Because in (20) we have dependence on \( s \) and on \((s, m)\), we get
\[
x_{n}(s, m) = C(s, m)R_{n+1}(s) + \sum_{k=0}^{n} R_{n-k}(s) \theta_{k}(s, m), \quad n \geq 0,
\]
where
\[
R_{k}(s) = \frac{1}{k!} \frac{d^{k}}{dz^{k}} \left( \frac{z}{a(s, z) - z} \right) \bigg|_{z=0},
\]
where now
\[
a(s, z) \overset{def}{=} \sum_{k=0}^{\infty} z^{k} a_{k}(s), \quad Re(s) > 0, \quad |z| < 1.
\]

Alternatively (see [17], [18]) the functional sequence \((R_{k}(s))\) can be defined recursively as follows:
\[
R_{0}(s) = 0, \quad R_{1}(s) = (a_{0}(s))^{-1},
\]
\[
R_{k}(s) = R_{1}(s) \left( R_{k-1}(s) - \sum_{j=0}^{k-1} a_{j+1}(s) R_{k-1-j}(s) \right), \quad k \geq 2.
\]

As it turns out, it is possible to express the unknown function \( C(s, m) \) explicitly, using the Eq. (21) as a boundary condition.

Substituting \( n = 0 \) into (29), we find the following simple relationship:
\[
C(s, m) = \frac{x_{0}(s, m)}{R_{1}(s)} = x_{0}(s, m)a_{0}(s).
\]

In fact, instead of finding \( C(s, m) \), it will be more convenient for us to find the closed-form formula for \( x_{0}(s, m) \).

Similarly, let us find the representation for \( x_{1}(s, m) \) in terms of \( x_{0}(s, m) \). Substituting \( n = 0 \) into (22) and utilizing the fact that
\[
\sum_{k=0}^{\infty} a_{k}(s) = f(s),
\]
we obtain
\[
\theta_{0}(s, m) = a_{1}(s)x_{0}(s, m) - x_{1}(s, m) (f(s) - a_{0}(s))
\]
\[
- b_{B}(s, m).
\]

From the other side, taking \( n = 0 \) in (20), we get
\[
\theta_{0}(s, m) = a_{0}(s)x_{1}(s, m) + a_{1}(s)x_{0}(s, m) - x_{0}(s, m).
\]

Comparing the right sides of (35) and (36) leads to
\[
x_{1}(s, m) = \frac{x_{0}(s, m) - b_{B}(s, m)}{f(s)}.
\]

To find the formula for \( x_{0}(s, m) \) we must use the “boundary” equation (21) written for \( n = B \). Introducing (29) and (33) into (21), we obtain
\[
x_{B}(s, m)
\]
\[
= \sum_{i=1}^{B-1} \sum_{r=1}^{B-i} a_{B-i-r} \left[ x_{0}(s, m)a_{0}(s)R_{r+1}(s)
\right.
\]
\[
+ \sum_{k=0}^{r} R_{r-k}(s) \left[ a_{k+1}(s)x_{0}(s, m) - (f(s))^{-1} x_{0}(s, m)
\right.
\]
\[
- b_{B}(s, m) \right] \sum_{j=k+1}^{\infty} a_{j}(s) - b_{B-k}(s, m) \right]
\]
\[
+ x_{0}(s, m) \beta(s) + \gamma(s, m).
\]

Similarly, writing (29) for \( n = B \) and applying (22), (33) and (37), we have
\[
x_{B}(s, m) = x_{0}(s, m)a_{0}(s)R_{B+1}(s) + \sum_{k=0}^{B} R_{B-k}(s)
\]
\[
\times \left[ a_{k+1}(s)x_{0}(s, m) - (f(s))^{-1} x_{0}(s, m)
\right.
\]
\[
- b_{B}(s, m) \right] \sum_{i=k+1}^{\infty} a_{i}(s) - b_{B-i}(s, m) \right].
\]

Now, comparing the right sides of (38) and (39), we obtain the following quotient-form compact representation for \( x_{0}(s, m) \):
\[
x_{0}(s, m) = \frac{g(s, m)}{h(s)},
\]
where

\[
g(s, m) \overset{\text{def}}{=} \sum_{i=1}^{B-1} \left[ \sum_{r=1}^{B-i} \sum_{k=0}^{r} R_{r-k}(s) \right] a_{i}(s) - b_{B-i-r}(s, m) + \sum_{k=0}^{B} R_{B-k}(s) \left( \sum_{j=k+1}^{\infty} a_{j}(s) - b_{B-j}(s, m) \right)
\]

and

\[
h(s) \overset{\text{def}}{=} a_{0}(s)R_{B+1}(s) + \sum_{k=0}^{B} R_{B-k}(s) \left[ a_{k+1}(s) - \left( f(s) \right)^{-1} \sum_{i=k+1}^{\infty} a_{i}(s) \right] - \beta(s).
\]

The consequence of the representations (19), (22), (29), (33) and (40) is the following main result:

**Theorem 1:** The formula for the Laplace transform of the transient conditional queue-size distribution in the considered M\(^X\)/G/1/B-type system with multiple vacation policy is following:

\[
q_{n}(s, m) = \left( a_{0}(s)R_{B-n+1}(s) + \sum_{k=0}^{B-n} R_{B-n-k}(s) \left[ a_{k+1}(s) - \left( f(s) \right)_{\beta(s)} \sum_{i=k+1}^{\infty} a_{i}(s) \right] \right)
\]

\[
+ \left( f(s) \right)^{-1} \sum_{i=k+1}^{\infty} a_{i}(s) \right) \frac{g(s, m)}{h(s)} + \sum_{k=0}^{B-n} R_{B-n-k}(s)
\]

\[
\times \left( \left( f(s) \right)_{\beta(s)}^{-1} b_{B}(s, m) \sum_{i=k+1}^{\infty} a_{i}(s) - b_{B-k}(s, m) \right).
\]

(43)

where \( T > 0 \) denotes duration of a single deterministic vacation, \( n \in \{0, \ldots, B\} \) indicates the initial system state and the representations for \( a_{i}(s), b_{i}(s, m), R_{i}(s), g(s, m) \) and \( h(s) \) are found in (10), (12), (30), (41) and (42), respectively.

From Theorem 1 the following corollary follows that states the formula for the Laplace transform of the loss (blocking) probability that the arriving group will be lost due to the buffer overflow:

**Corollary 1:** The Laplace transform \( q_{n}^{(\text{loss})}(s) \) of the probability that the arriving group will be completely lost (blocked) due to the buffer overflow, on condition that the system starts

\[
\sum_{i=0}^{B-1} \sum_{k=0}^{B-i} R_{r-k}(s) \left( f(s) \right)^{-1} b_{B-n-k}(s, m) - \beta(s)
\]

\[
+ \left( f(s) \right)^{-1} \sum_{i=k+1}^{\infty} a_{i}(s) \right) \frac{g(s, m)}{h(s)} + \sum_{k=0}^{B-n} R_{B-n-k}(s)
\]

\[
\times \left( \left( f(s) \right)_{\beta(s)}^{-1} b_{B}(s, m) \sum_{i=k+1}^{\infty} a_{i}(s) - b_{B-k}(s, m) \right).
\]

(44)

where \( n \in \{0, \ldots, B\} \) and the representation for \( q_{n}(s, m) \) is found in (43).

**VI. NUMERICAL AND SIMULATION STUDY**

To illustrate the operation of the described queueing model, we performed a numerical and simulation study. We applied Eq. 43 for two system setups. Numerical results prepared in Wolfram Mathematica 11.3 [20] were compared with the results of 100,000 simulation runs designed in OMNeT++ Discrete Event Simulator 5.4.1 [26].

In the first case, hereinafter referred to as Setup 1, the maximal system state \( B \) was set to 10 messages. Probabilities of arriving of \( m \) messages, where \( m \in \{1, 2, \ldots, 10\} \), were calculated from gamma distribution with a shape parameter of 30 and a scale parameter of 1 in the following way:

\[
p_{m} = \sum_{x=10 \cdot (m-1)+1}^{10 \cdot m} \text{PDF}_{\Gamma_{\text{30,1}}}(x),
\]

(45)

where PDF\(_{\Gamma_{\text{30,1}}}(x)\) is a probability density function of gamma distribution with parameters 30 and 1 in a point \( x > 0 \). A bar chart of calculated probabilities is presented in Fig. 1. The highest value is for \( p_{3} \) and it is equal to about 0.47, slightly more than \( p_{4} \approx 0.46 \). The mean time between arriving of jobs is 0.4 seconds while the vacations time is equal to 1.28 seconds. Messages are processed according to Erlang distribution with a shape parameter of 2 and a rate parameter of 20. For the numerical inversion, Abate-Choudhury-Whitt algorithm was used [1] which we described in detail in Appendix. Its parameters were changed compared to the Authors proposition due to the complexity of the inverted equations. Greater values lead to oscillations and the distortion of results.

This scenario is prepared in such a way that it can be used in LTE DRX mechanism modeling for downlink transmissions.
A DRX cycle consists of one subframe during which the device is active and data transmission is possible. This subframe corresponds to the processing period. During the rest of the cycle time the device is sleeping which in our model is represented by the vacation period \( T \). The value of \( T = 1.28 \) has been chosen according to the specification of an exemplary module which supports DRX cycles in Cat M1 and Cat NB1 transmission modes [21].

We also prepared calculations for the simpler model, referred to as Setup 2. In this case \( B \) is set to 3. The probability of arrival of 1 message is equal to the probability for 2 jobs, i.e. \( p_1 = p_2 = 0.5 \). The service occurs according to Erlang distribution with a shape parameter of 2 and a rate parameter of 10. The average job processing time is equal to 0.2 seconds. Parameters of Abate-Choudhury-Whitt algorithm were set as suggested by the Authors of [1]. The detailed comparison of two experiment setups is presented in Table 1.

In the case of Setup 1, 400 elements of \( p_{ij}^* \) were calculated, according to Eq. 3, where \( i, j \in \{0, 1, ..., 19\} \) while in the case of Setup 2 there were 169 elements designated, \( i, j \in \{0, 1, ..., 12\} \). Furthermore, in both presented cases for Eqs. (12) and (15) at most 3 components of infinite sums were designated.

Fig. 3 presents a comparison between numerical and simulation results for Setup 1 while detailed values of errors are shown in Table 2. In the case of \( X(t) \in \{0, 1, \ldots, 6\} \) results are averaged over 50 points with a time interval 0.1 second from 0.1 to 5 seconds after running the system. Due to problems with numerical inversion calculations start from 0.5 second for \( X(t) = 7 \) (46 points in total) and from 1 second for \( X(t) \in \{8, 9, 10\} \) (41 points in total).
TABLE 3: Mean absolute errors (MAEs) with standard deviation calculated between numerical and simulation results for Setup 2. Results are averaged over 30 points with a time interval 0.1 second from 0.1 to 3 seconds after running the system (30 points in total).

| X(t) | 0   | 1   | 2   | 3   |
|------|-----|-----|-----|-----|
| MAE  | 0.0012 | 0.0018 | 0.0276 | 0.0250 |
| std. dev. | 0.0006 | 0.0014 | 0.0299 | 0.0181 |

In the second tested scenario, called Setup 2, the tested system has smaller buffer size. Furthermore, messages are incoming less frequently. The single vacation duration has been extended to $T = 5$ seconds. For $X(t) \in \{0, 1\}$ curves derived from numerical equations fit almost perfectly to the simulation results. More significant differences are visible in the case of $X(t) = 2$ and 3. Still, for more than 60% of grid points, absolute errors are not greater than 0.01, which is presented in Fig. 2b. A detailed comparison of numerical and simulation results is shown in Fig. 4. Values of MAEs comparing numerical and simulation results are dependent from the $X(t)$ but even in the worst case they do not exceed 0.03.
VII. CONCLUSION AND FUTURE WORKS
In this work, we derived a compact-form representation for the Laplace transform of conditional queue-size distribution in the $M^X/G/1/B$-type system with multiple vacation policy based on deterministic vacation time $T > 0$ and batch arrivals. Thanks to numerical inversion of this formula with using Abate-Choudhury-Whitt algorithm it is possible to assess the number of messages in the queue at given time $t > 0$. This may facilitate the adjustment of system parameters like the vacation time duration $T$ to provide a compromise between energy consumption and the number of lost messages. The presented model can be useful in practical applications like LTE DRX. In the future, the model may be expanded with an intermediate state between vacations and the processing period. After ending of the idle period the system can “listen” for the arriving messages. In current solution, if at the end of the multiple vacation period there is no message buffered in the queue, the system starts a new vacation. It is possible to add a new state which prevents from new vacation for some time after the previous ones.

APPENDIX

ABATE-CHOUHDURY-WHITT ALGORITHM
This algorithm described in [1] is based on the Bromwich integral, which makes it possible to find the value of the function $\psi(\cdot)$ at fixed $t > 0$ from its Laplace transform $\hat{\psi}(\cdot)$, namely

$$\psi(t) = \frac{1}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} e^{-st} \hat{\psi}(s) ds, \quad (46)$$

where $\epsilon \in \mathbb{R}$ is located on the right side of all the singularities of $\hat{\psi}(\cdot)$. Next, by the use of the trapezoidal rule on Eq. (46) with a step length $\Delta$, we obtain the following approximation of $\psi(t)$:

$$\psi_{\Delta}(t) = \frac{\Delta e^{\epsilon t}}{2\pi} \hat{\psi}(\epsilon) + \frac{\Delta e^{\epsilon t}}{\pi} \sum_{k=1}^{\infty} \text{Re}[e^{ik\Delta t} \hat{\psi}(\epsilon + ik\Delta t)]. \quad (47)$$

Substituting $\Delta = \frac{\pi}{L}$ and $\epsilon = \frac{A}{2L}$, we obtain the following series representation of (47):

$$\psi_{\Delta}(t) = \psi_{A,L}(t) = \sum_{k=0}^{\infty} (-1)^k \varrho_k(t) \quad (48)$$

where

$$\varrho_k(t) = \frac{e^{A/2L}}{2Lt} \omega_k(t), \quad k \geq 0, \quad (49)$$

$$\omega_0(t) = \hat{\psi}(\frac{A}{2L}) + 2 \sum_{j=1}^{L} \text{Re} \left[ \hat{\psi} \left( \frac{A}{2L} + \frac{ij\pi}{L} \right) e^{ij\pi/L} \right], \quad (50)$$

and

$$\omega_k(t) = 2 \sum_{j=1}^{L} \text{Re} \left[ \hat{\psi} \left( \frac{A}{2L} + \frac{ij\pi}{L} + \frac{ik\pi}{t} \right) e^{ij\pi/L} \right], \quad (51)$$

where $k \geq 1$.

Finally, after the application of the Euler formula

$$\sum_{k=0}^{\infty} (-1)^k \mu_k \approx \sum_{k=0}^{m} \binom{m}{k} \frac{1}{2m} \sum_{j=0}^{n+k} (-1)^j \mu_j, \quad (52)$$
we obtain the final representation of inverse Laplace transform approximation in the following form:

\[
\psi(t) \approx \sum_{k=0}^{m} \left( \frac{1}{2m} \sum_{j=0}^{n+k} (-1)^j G_j(t) \right). \quad (53)
\]

To use the Abate-Choudhury-Whitt algorithm, it is necessary to choose values of \( L, A, m \) and \( n \). An exemplary Authors' proposition is \( L = 1, A = 19, m = 11 \) and \( n = 38 \).

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