A RECTANGULAR INTERVAL OF A RECTANGULAR LATTICE
IS A RECTANGULAR LATTICE

G. GRÄTZER

Abstract. Let $L$ be a slim, planar, semimodular lattice (slim means that it does not contain $M_3$-sublattices). We call the interval $I = [o, i]$ of $L$ rectangular, if there are $u_l, u_r \in [o, i] - \{o, i\}$ such that $o = u_l \land u_r$ and $i = u_l \lor u_r$, where $u_l$ is to the left of $u_r$.

We prove that a rectangular interval of a rectangular lattice is a rectangular lattice.

As an application, we get a recent result of G. Czédli.

1. Introduction

We started studying planar, semimodular lattices in my papers with E. Knapp [4]–[8]. More than four dozen publications have been devoted to this topic since; see G. Czédli’s list http://www.math.u-szeged.hu/~czedli/m/listak/publ-psml.pdf

An SPS lattice $L$ is a planar semimodular lattice that is also slim, that is, it does not contain $M_3$-sublattices.

Following my paper with E. Knapp [7], a planar semimodular lattice $L$ is rectangular, if its left boundary chain has exactly one doubly-irreducible element (the left corner) and its right boundary chain has exactly one doubly-irreducible element (the right corner).

Rectangular lattices are easier to work with than planar semimodular lattices, because they have much more structure. Moreover, a planar semimodular lattice has a (congruence-preserving) extension to a rectangular lattice, so we can prove many result for planar semimodular lattices by verifying them for rectangular lattices (G. Grätzer and E. Knapp [7]).

There are many interesting and useful facts about SPS lattices. In this paper, we present a new one, closely related to the main result in G. Czédli [1]. Before we state it, we need a definition. Let $L$ be a planar lattice. We call the interval $I = [o, i]$ of $L$ rectangular, if there are $u_l, u_r \in [o, i] - \{o, i\}$ such that $o = u_l \land u_r$ and $i = u_l \lor u_r$, where $u_l$ is to the left of $u_r$.

Theorem 1. Let $L$ be an SPS lattice and let $I$ be a rectangular interval of $L$. Then the lattice $I$ is slim and rectangular.

We will apply this theorem to get a recent result of G. Czédli [1].

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Basic concepts and notation. We assume that the reader is familiar with the basic concepts as in CFL2 (cite [3]). You can find the complete Part I. A Brief Introduction to Lattices and Glossary of Notation of [3] at tinyurl.com/lattices101

2. Preliminaries

We discuss in Section 4.3 of CFL2 a result of G. Czédli and E. T. Schmidt [2]: for an SPS lattice \( L \) and covering square \( C \) in \( L \), we can insert a fork at \( C \) to obtain the lattice extension \( L[C] \), which is also an SPS lattice.

Sometimes, we can delete a fork, see G. Czédli and E. T. Schmidt [2].

Lemma 2. Let \( L \) be an SPS lattice and let \( S \) be a covering \( N_7 \) in \( L \), with middle element \( m \), left corner \( b_l \) and right corner \( b_r \). Let us assume that the top element \( t \) of \( S \) is minimal, that is, there is no \( S' \) a covering \( N_7 \) with top element \( t' \) satisfying that \( t' < t \). Then \( L \) has a sublattice \( L^- \) with 4-cell \( C = S - \{ m, b_l, b_r \} \) such that \( L = L^-[C] \).

![Figure 1. Deleting a fork.](image)

The structure of slim rectangular lattices is described as follows.

Theorem 3 (G. Czédli and E. T. Schmidt [2]). \( L \) is a slim rectangular lattice iff it can be obtained from a grid by inserting forks.

There is a slightly stronger version of this result, implicit in G. Czédli and E. T. Schmidt [2]. We present it with a short proof.

Theorem 4. For every slim rectangular lattice \( K \), there is a grid \( G \) and sequences \( G = K_1, K_2, \ldots, K_n = K \) of slim rectangular lattices and

\[
C_1 = \{ o_1, c_1, d_1, i_1 \}, C_2 = \{ o_2, c_2, d_2, i_2 \}, \ldots, C_{n-1} = \{ o_{n-1}, c_{n-1}, d_{n-1}, i_{n-1} \}
\]

of 4-cells in the appropriate lattices such that

\[
G = K_1, K_1[C_1] = K_2, \ldots, K_{n-1}[C_{n-1}] = K_n = K.
\]

Moreover, the principal ideals \( \downarrow c_{n-1} \) and \( \downarrow d_{n-1} \) are distributive.
Proof. We prove this result by induction on the number \( n \) of covering \( N_7 \)-s in \( K \). If \( n = 0 \), then \( K \) is distributive by G. Grätzer and E. Knapp [7], so the statement is trivial. Now let us assume that the statement holds for \( n-1 \). Let \( K \) be a slim rectangular lattice with \( n \) covering \( N_7 \)-s. As in Lemma \( 2 \) we take \( S \), a minimal covering \( N_7 \) in \( K \). Then we form the sublattice \( K^- \) by deleting the fork at \( S \). So we get a 4-cell \( C = C_{n-1} = \{ o_{n-1}, c_{n-1}, d_{n-1}, i_{n-1} \} \) of \( K^- \) such that \( K = K^-[C] \). Since \( K^- \) has \( n-1 \) covering \( N_7 \)-s, we get the sequence

\[
G = K_1, K_1[C_1] = K_2, \ldots, K_{n-2}[C_{n-2}] = K_{n-1} = K^-,
\]

which, along with \( K = K^-[C] \), prove the statement for \( K \).

By the minimality of \( S \), the principal ideals \( \downarrow c_{n-1} \) and \( \downarrow d_{n-1} \) are distributive. \( \square \)

3. Proving Theorem \( 1 \)

Theorem \( 1 \) obviously holds for grids.

Otherwise, we are given the slim rectangular lattice \( K \), the slim rectangular lattice \( K^- \) as defined in the proof of Theorem \( 3 \) and the covering square

\[
C_{n-1} = \{ o_{n-1}, c_{n-1}, d_{n-1}, i_{n-1} \}
\]

in \( K^- \), so that we obtain \( K \) from \( K^- \) by inserting a fork in \( C_{n-1} \), adding the elements \( m \) in the middle of \( C_{n-1} \), adding the sequences of elements \( x_1, \ldots \) on the left going down and \( y_1, \ldots \) on the right going down.

Let \( I = [o, i]_K \) be a rectangular interval in \( K \) with bounds \( o, i \) and corners \( u_l, u_r \).

We want to prove that \( I \) is a slim rectangular lattice. Of course, the lattice \( I \) is slim.

We induct on \( n \), the number of fork extension to get from \( G \) to \( K \), in equation \( 1 \).

There are three types of subcases.

Case 1. \( I \) has no element internal to \( \downarrow i_{n-1} \). For instance, \( I \cap \downarrow i_{n-1} = \emptyset \). Then \( [o, i]_{K^-} = I \). By induction, \( [o, i]_{K^-} \) is rectangular, therefore, so is \( I \).

Case 2. \( m \) is an internal element of \( I \). For instance, \( u_l \) is \( c_{n-1} \) or it is to the left of \( c_{n-1} \) and symmetrically. In this case, \( C \) is a covering square in \( [o, i]_{K^-} \) and we obtain \( [o, i]_K \) by adding a fork to \( C \) in \( [o, i]_{K^-} \). A fork extension of a slim rectangular lattice is also slim rectangular, so \( I \) is slim rectangular.

Case 3. \( m \) is not an internal element of \( I \) but some \( x_j \) (or \( y_j \)) is. For instance, \( x_2 \) is an internal element of \( I \). Then we obtain \( I \) from \( [o, i]_{K^-} \) by replacing a cover preserving \( C_m \times C_2 \) by \( C_m \times C_3 \), and so it is rectangular.

4. Applications

Corollary 5. Let \( L \) be an SPS lattice and let \( I \) be a rectangular interval of \( L \). Let \( (P) \) be any property of slim rectangular lattices. Then \( (P) \) holds for the lattice \( I \).

For instance, let \( (P) \) be the property: the intervals \( [o, u_l] \) and \( [o, u_r] \) are chains and all elements of the lower boundary of \( I \) except for \( u_l, u_r \) are meet-reducible. Then we get the main result of G. Czédli [11]:

Corollary 6. Let \( L \) be an SPS lattice and let \( I \) be a rectangular interval of \( L \). then \( [o, u_l] \) and \( [o, u_r] \) are chains and all elements of the lower boundary of \( I \) except for \( u_l, u_r \) are meet-reducible.

Another nice application is the following.
Corollary 7. Let $L$ be an SPS lattice and let $I$ be a rectangular interval of $L$ with corners $u_l, u_r$. Then for any $x \in I$, the following equation holds:

$$x = (x \land u_l) \lor (x \land u_r).$$

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Email address: gratzer@mac.com

URL: http://server.maths.umanitoba.ca/homepages/gratzer/