Renormalised iterated integrals of symbols with linear constraints

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Abstract

Given a holomorphic regularisation procedure (e.g. Riesz or dimensional regularisation) on classical symbols, we define renormalised multiple integrals of radial classical symbols with linear constraints. To do so, we first prove the existence of meromorphic extensions of multiple integrals of holomorphic perturbations of radial symbols with linear constraints and then implement either generalised evaluators or a Birkhoff factorisation. Renormalised multiple integrals are covariant and factorise over independent sets of constraints.

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Introduction

Regularisation methods are sufficient to handle ordinary integrals arising from one loop Feynman diagrams whereas renormalisation methods are required to handle multiple integrals arising from multiloop Feynman diagrams. Interesting algebraic constructions have been developed to disentangle the procedure used by physicists when computing such integrals [CK], [Kr]. Although they clarify the algebraic structure underlying the forest formula, these algebraic approaches based on the Hopf algebra structures on Feynman diagrams do not make explicit the corresponding manipulations on the multiple integrals. This paper aims at presenting analytic mechanisms underlying renormalisation procedures in physics on firm mathematical ground using the language pseudodifferential symbols in which locality in physics translates into a factorization property of integrals.

We consider integrals of symbols with linear constraints that reflect the conservation of momenta; properties of symbols clearly play a crucial role in the renormalisation procedure. When they converge we can write such integrals as follows:

$$\int_{\mathbb{R}^nL} (\tilde{\sigma} \circ B) (\xi_1, \cdots, \xi_L) \, d\xi_1 \cdots d\xi_L,$$

with $\tilde{\sigma} := \sigma_1 \otimes \cdots \otimes \sigma_I$ where the $\sigma_i$ are classical symbols on $\mathbb{R}^n$ and $B$ an $I \times L$ matrix of rank $L$. In the language of perturbative quantum field theory, $n$ stands for the dimension of space time so that $n = 4$, $L$ stands for the number of loops, $(\eta_1, \cdots, \eta_I) := B (\xi_1, \cdots, \xi_L)$ for the internal vertices and the matrix $B$ encodes the linear constraints they are submitted to as a result of the conservation of momenta. To illustrate this by an example take $I = 3, L = 2$, the symbols $\sigma_i, i = 1, 2, 3$ equal to $\sigma (\xi) = \frac{1}{(m^2 + |\xi|^2)^2}$ for some $m \in \mathbb{R}^*$ (which introduces a mass term) and the matrix $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$.

The corresponding integral for $n = 4$ reads

$$\int_{\mathbb{R}^4} \int_{\mathbb{R}^4} ((\sigma_1 \otimes \sigma_2 \otimes \sigma_3) \circ B) (\xi_1, \xi_2) \, d\xi_1 \, d\xi_2 = \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \frac{1}{(m^2 + |\xi_1|^2)^2} \frac{1}{(m^2 + |\xi_2|^2)^2} \frac{1}{(m^2 + |\xi_1 + \xi_2|^2)^2} \, d\xi_1 \, d\xi_2.$$

We wish to renormalise multiple integrals with linear constraints of this type when the integrand does not anymore lie in $L^1$ in such a way that

1. the renormalised integrals coincide with the usual integrals whenever the integrand lies in $L^1$,

2. they satisfy a Fubini type property, i.e. are invariant under permutations of the variables,

3. they factorise on disjoint sets of constraints, i.e. on products $(\sigma \circ B) \cdot (\sigma' \circ B') := (\sigma \otimes \sigma') \circ (B \oplus B')$ where $\oplus$ stands for the Whitney sum.

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1In the language of Feynman diagrams, we only deal with internal momenta namely we integrate on all the variables.
This last requirement, which would correspond in quantum field theory to the concatenation of Feynman diagrams, follows from the fundamental locality principle in physics.

Inspired by physicist’s computations of Feynman integrals, we present two renormalisation procedures, a first one which uses generalised evaluators and an alternative method using a Birkhoff factorisation procedure, both of which heavily rely on meromorphicity results and both of which lead to covariant expressions.

Let us briefly describe the structure of the paper.

Regularisation procedures for simple integrals of symbols are by now well known and provide a precise mathematical description for what physicists refer to as dimensional regularisation for one loop diagrams (see e.g. [C] from a physicist’s point of view and [P1] from a mathematician’s point of view for a review of some regularisation methods used in physics).

Regularisation techniques for simple integrals are reviewed in the first part of the paper. We describe in dimensional regularisation as an instance of more general holomorphic regularisations and compare it with cut-off regularisation in Theorem 1. Covariance, integration by parts and translation invariance properties are discussed in detail in section 3. In section 4, inspired by work by Lesch and Pflaum [LP] on strongly parametric symbols [2], we investigate parameter dependent integrals of symbols of the type that typically arises in the presence of external momenta in quantum field theory. The parameter dependence in the external parameters being affine in the context of Feynman diagrams, we study regularised integrals 

\[ \int_{\mathbb{R}^{nL}} (\sigma_1 \otimes \cdots \otimes \sigma_k) (\xi + \eta_1, \cdots, \xi + \eta_k) d\xi \]

which we actually define “modulo polynomials in the components of the external parameters” \( \eta_1, \cdots, \eta_k \). Since cut-off regularised integrals vanish on polynomials (see Proposition 1), the ambiguity arising from defining the integrals “up to polynomials” is not seen after implementing cut-off (or dimensional) regularisation in the remaining parameters (see Theorem 2 and Corollary 4).

The second part of the paper (sections 5-8) is dedicated to renormalisation techniques for multiple integrals with constraints. In section 5 we first renormalise multiple integrals without constraints (see Theorem 3) in the spirit of previous work with D. Manchon [MP]. The corner stone of renormalisation in our context is a meromorphicity result, which if fairly straightforward in the absence of constraints, becomes non trivial in the presence of linear constraints \[3\]. We show (see Theorem 4) that the map:

\[ (z_1, \cdots, z_I) \mapsto \int_{\mathbb{R}^{nL}} (\tilde{\sigma}(z) \circ B) (\xi_1, \cdots, \xi_L) d\xi_1 \cdots d\xi_L, \]

with \( \tilde{\sigma}(z) := \sigma_1(z_1) \otimes \cdots \otimes \sigma_I(z_I) \) obtained from a holomorphic perturbation \( \mathcal{R} : \sigma_i \mapsto \sigma_i(z) \) of radial symbols \( \sigma_i \) (which can e.g. arise from dimensional regularisation), has a meromorphic extension

\[ z \mapsto \int_{\mathbb{R}^{nL}} (\tilde{\sigma}(z) \circ B) (\xi_1, \cdots, \xi_L) d\xi_1 \cdots d\xi_L, \]

\[ \text{Although our setup is different from that of strongly parametric symbols, it turns out that the approach of [LP] can be partially adapted to our context.} \]

\[ \text{This issue is of course strongly related to the meromorphicity of Feynman integrals using dimensional regularisation previously investigated by various authors from different view points see e.g. [Sp], [CaM] [2], [CM], [BW1], [BW2].} \]
and we describe its pole structure. These meromorphicity results which are coherent with known results in the case of Feynman diagrams [SP], are to our knowledge new in such generality since they hold for any radial classical symbols and any holomorphic regularisation.

Since these meromorphic extensions coincide with ordinary multiple integrals on the domain of holomorphicity, by analytic continuation they factorise over disjoint sets of constraints i.e:

\[
\int_{\mathbb{R}^{nL}} (\tilde{\sigma}(z) \otimes \tilde{\sigma}'(z')) \circ (B \oplus B') = \left( \int_{\mathbb{R}^{nL}} \tilde{\sigma}(z) \circ B \right) \left( \int_{\mathbb{R}^{nL}} \tilde{\sigma}'(z') \circ B' \right),
\]

where we have set \( z := (z_1, \cdots, z_I) \) and \( z' := (z'_1, \cdots, z'_{I'}) \), \( \tilde{\sigma} := \sigma_1 \otimes \cdots \otimes \sigma_I \), \( \tilde{\sigma}' := \sigma'_1 \otimes \cdots \otimes \sigma'_{I'} \), \( B \) and \( B' \) being matrices of size \( I \times L \) and \( I' \times L' \) respectively.

We then describe two ways of extracting renormalised finite parts as \( z \to 0 \) while preserving this factorisation property:

1. Using generalised evaluators (see Theorem 5),
2. Using Birkhoff factorisation (see Theorem 6) after having identified \( z_i = z_4 \) and set the \( \sigma_i \)'s to be a fixed radial symbol \( \sigma \).

Just as in Connes and Kreimer’s pioneering work [CK], in this second approach the factorisation requirement translates to a character property on a certain Hopf algebra, the coproduct of which reflects the fact that one should in principle be able to perform iterated integrations “packetwise”, first integrating on any subset of variables and then on the remaining ones (see [BM] for comments on this point).

As well as being multiplicative (see (21)):

\[
\int_{\mathbb{R}^{(nL+L')}}^{\mathbb{R}_{\text{ren}}}(\tilde{\sigma} \otimes \tilde{\sigma}') \circ (B \oplus B') = \left( \int_{\mathbb{R}^{nL}}^{\mathbb{R}_{\text{ren}}} \tilde{\sigma} \circ B \right) \left( \int_{\mathbb{R}^{nL'}}^{\mathbb{R}_{\text{ren}}} \tilde{\sigma}' \circ B' \right),
\]

renormalised multiple integrals with constraints turn out to be covariant (see Theorem 7):

\[
\int_{\mathbb{R}^{nL}}^{\mathbb{R}_{\text{ren}}} (\tilde{\sigma} \circ B) \circ C = |\det C|^{-1} \int_{\mathbb{R}^{nL}}^{\mathbb{R}_{\text{ren}}} \tilde{\sigma} \circ B \quad \forall C \in GL_L(\mathbb{R}^n)
\]

and therefore obey a Fubini property (see (24)):

\[
\int_{\mathbb{R}^{nL}}^{\mathbb{R}_{\text{ren}}} \sigma \circ B(\xi_\rho(1), \cdots, \xi_\rho(L)) \, d \xi_1 \cdots d \xi_L = \int_{\mathbb{R}^{nL}}^{\mathbb{R}_{\text{ren}}} \sigma \circ B(\xi_1, \cdots, \xi_L) \, d \xi_1 \cdots d \xi_L \quad \forall \rho \in \Sigma_L.
\]

The above factorisation property (which reflects a locality principle in physics) does not fix the renormalised integrals uniquely; even when the holomorphic regularisation \( \mathcal{R} \) is fixed (e.g. dimensional regularisation), there still remains a freedom of choice left due the freedom of choice on the evaluator unless one imposes further constraints as one would do in quantum field theory.

This paper emphasises the analytic mechanisms underlying the renormalisation of multiple integrals of symbols with linear constraints, thereby raising further analytic questions which remain to be solved, namely

\footnote{Such an identification is natural in the context of dimensional regularisation by which the dimension \( n \) of the space is replaced by \( n - z \).}
1. Do these results which hold for radial symbols extend to all classical symbols? The meromorphicity established in Theorem 4 easily extends to polynomial symbols when using Riesz or dimensional regularisation due to the fact that such symbols can be obtained from derivatives of radial symbols \( (\xi_i = \frac{1}{2} \partial_i |\xi|^2) \) but it is not clear whether one can go beyond those classes of symbols.

2. How do these renormalisation procedures generalise to integrals of tensor products of symbols with affine constraints so as to allow for external momenta, one of the difficulties being how to control the symbolic behaviour of parameter dependent renormalised integrals in the external parameters?

3. How do the various renormalisation approaches described here compare? It follows from the pole structure of the meromorphic extensions described in the paper that the renormalised values obtained by different methods coincide for symbols \( \sigma_i \) whose orders have non integer partial sums since the renormalised values then correspond to ordinary evaluations of holomorphic maps at 0, but it is not clear what happens beyond this case.

4. It would be interesting to investigate all the coefficients of the Laurent expansion and to see when they can be recognized as motives.

Answering these questions can also be relevant for multiple discrete sums of symbols with constraints (see [P4]), multiple zeta functions being an important instance since they boil down to multiple discrete sums of symbols with conical constraints.

Table of contents

Part 1: Regularised integrals of symbols
1. Cut-off regularised integrals of log-polyhomogeneous symbols
2. Regularised integrals of log-polyhomogeneous symbols
3. Basic properties of integrals of holomorphic symbols
4. Regularised integrals with affine parameters

Part 2: Renormalised multiple integrals of symbols with linear constraints
5. Integrals of tensor products of symbols revisited
6. Linear constraints in terms of matrices
7. Multiple integrals of holomorphic families with constraints
8. Renormalised integrals with constraints

5see [BW2] and references therein for discussions along these lines
Part 1: Regularised integrals of symbols

In this first part we review and partially extend results of [MMP] and [MP]. Regularised integrals are defined using cut-off and holomorphic regularisations; dimensional regularisation is presented as an instance of holomorphic regularisations and then compared with cut-off regularisation.

1 Cut-off regularised integrals of log-polyhomogeneous symbols

We recall regularisation techniques for integrals of log-polyhomogeneous symbols which deal with ultraviolet divergences. Starting from cut-off regularisation we then turn to dimensional regularisation which we describe as an instance of more general holomorphic regularisation procedures. We discuss in how far such regularisation procedures also take care of infrared divergences. Such issues were previously discussed by many authors in the context of Feynman diagrams, starting with pioneering work of t’Hooft and Veltman [HV] on dimensional regularisation and later works of Smirnov [Sm1], [Sm2], [Sm3] and Speer [Sp] just to name a few later developments.

Since integrating classical symbols naturally leads to log-polyhomogeneous symbols, we describe regularisation procedures on the class of log-polyhomogeneous symbols.

1.1 From log-polyhomogeneous functions to symbols

We call a function \( f \in \mathcal{C}_\infty(\mathbb{R}^n - \{0\}) \) positively log-homogeneous of order \( a \) and log-degree \( k \) if

\[
 f(\xi) = \sum_{l=0}^{k} f_{a,l}(\xi) \log^l|\xi| \quad f_{a,l}(t \xi) = t^a f_{a,l}(\xi) \quad \forall \xi \in \mathbb{R}^n, \quad \forall t > 0.
\]

Following [L], given a positively log-homogeneous function of order \( a \) and log degree \( k \), let us set for any \( l \in \{1, \ldots, k\} \):

\[
 \text{res}_l(f) := \delta_{a+n} \int_{S^{n-1}} f_{a,l}(\xi) dS\xi
\]

where \( dS\xi \) is the volume form with respect to the standard metric on \( S^{n-1} \). Let us denote by \( \mathcal{P}^{a,k}_+(\mathbb{R}^n) \) the set of positively log-homogeneous functions on \( \mathbb{R}^n \) of order \( a \) and log degree \( k \).

**Example 1.** \( \xi \mapsto f(\xi) = \sum_{l=0}^{k} c_l|x|^a \log^l|\xi| \) with \( c_l \in \mathbb{R}, l = 0, \ldots, k \) belongs to \( \mathcal{P}^{a,k}_+(\mathbb{R}^n) \).

We call a function \( \sigma \in C_\infty(\mathbb{R}^n) \) a log-polyhomogeneous symbol of order \( a \) and log-type \( k \) with constant coefficients if

\[
 \sigma = \sum_{j=0}^{N-1} \chi_j \sigma_{a-j} + \sigma_{(N)}, \quad \tag{1}
\]

\( \chi \)

\( \chi \)We use a terminology which is slightly different from that of [L].
where $\chi$ is a smooth cut-off function which vanishes at 0 and equals to one outside the unit ball, where $\sigma_{a-j} \in \mathcal{P}^{n-j,k}(\mathbb{R}^n)$ and where $\sigma_{(N)}^X \in C^\infty(\mathbb{R}^n)$ satisfies the following requirement:

$$\exists C \in \mathbb{R}, \ \|\sigma \chi(N)(\xi)\| \leq C(\xi)^{\text{Re}(a)-N} \ \forall \xi \in \mathbb{R}^n$$

with $\langle \xi \rangle := (1 + |\xi|^2)^{\frac{1}{2}}$. Changing the cut-off function $\chi$ amounts to modifying the remainder term $\sigma_{(N)}^X$.

If the log-type $k$ vanishes then the symbol is called polyhomogeneous or classical.

We call a symbol $\sigma$ radial if $\sigma(\xi) = f(|\xi|)$ only depends on the radius.

**Remark 1.** For short one writes $\sigma \sim \sum_{j=0}^\infty \chi a_{a-j}$, the symbol $\sim$ controlling the asymptotics as $|\xi| \to \infty$ i.e. the ultraviolet behaviour.

**Example 2.** $\sigma(\xi) = \frac{1}{\langle \xi \rangle^{2\epsilon+1}}$ is a classical radial symbol of order $-2$ and

$$\sigma(\xi) \sim_{|\xi| \to \infty} \sum_{j=0}^\infty (-1)^k |\xi|^{-2-2k}.$$  

**Remark 2.** To deal with infrared divergences it can be useful to observe that a radial function

$$f(\xi) = \sum_{l=0}^{k} c_l |\xi|^a \log^l |\xi|$$

in $\mathcal{P}^{n,k}(\mathbb{R}^d)$ can be seen as a limit as $\epsilon \to 0$ of radial symbols

$$\sigma^\epsilon(\xi) = \sum_{l=0}^{k} c_l (|\xi|^2 + \epsilon^2)^{\frac{1}{2}} \log^l \left((|\xi|^2 + \epsilon^2)^{\frac{1}{2}}\right).$$

When $\epsilon \neq 0$ these are smooth functions on $\mathbb{R}^d$ which lie in $\mathcal{CS}^{a,k}(\mathbb{R}^d)$ and

$$\sigma^\epsilon(\xi) = (1 - \chi(\xi)) \sigma^\epsilon(\xi) + \chi(\xi) |\xi|^a \sum_{l=0}^{k} c_l \left(|\frac{\epsilon}{|\xi|}|^2 + 1\right)^{\frac{1}{2}} \left(-a \log |\xi| + \frac{1}{2} \log \left(\frac{|\xi|^2}{\epsilon^2} + 1\right)\right)^l$$

$$\sim \sum_{j=0}^{\infty} \sigma_{a-j}^\epsilon(\xi) \chi(\xi)$$

with $\sigma_{a-j}^\epsilon = e\sigma_{a-j}^1$. As before, $\chi$ is a smooth cut-off function which vanishes in a small neighborhood of 0 and is one outside the unit ball.

**Example 3.** Take $f(\xi) = |\xi|^{-2}$ which we write $f(\xi) = \lim_{\epsilon \to 0} (|\xi|^2 + \epsilon^2)^{-1}$. Then

$$\sigma^\epsilon(\xi) = \frac{1}{|\xi|^2 + \epsilon^2} \sim_{|\xi| \to \infty} \sum_{k=0}^{\infty} (-1)^k |\xi|^{-2-2k} \epsilon^{2k}.$$  

Let $\mathcal{CS}^{a,k}(\mathbb{R}^n)$ denote the set of log-polyhomogeneous symbols with constant coefficients of order $a$ and log-type $k$. It is convenient to introduce the following notation $\mathcal{CS}^{a,k}(\mathbb{R}^n) = \cup_{\epsilon \in \mathbb{R}} \mathcal{CS}^{a,k}(\mathbb{R}^n)$. The algebra

$$\mathcal{CS}(\mathbb{R}^n) := \cup_{a \in \mathbb{R}} \mathcal{CS}^{a,0}(\mathbb{R}^n)$$

7The following semi-norms labelled by multiindices $\gamma, \beta$ and integers $m \geq 0, p \in \{1, \cdots, k\}$,
generated by all log-polyhomogeneous symbols of log-type 0 is called the algebra of classical or polyhomogeneous symbols on $\mathbb{R}^n$ with constant coefficients. It contains the algebra $CS^{-\infty}(\mathbb{R}^n) := \bigcap_{a \in \mathbb{R}} CS^a(\mathbb{R}^n)$ of smoothing symbols. 

The algebra $CS^Z(\mathbb{R}^n) := \bigcup_{a \in \mathbb{Z}} CS^a(\mathbb{R}^n)$ of integer order log-polyhomogeneous symbols is strictly contained in the algebra generated by log-polyhomogeneous symbols of any order $CS^*(\mathbb{R}^n) := \langle \bigcup_{a \in \mathbb{C}} CS^a(\mathbb{R}^n) \rangle$.

### 1.2 Cut-off regularised integrals

We recall the construction of cut-off regularised integrals of log-polyhomogeneous symbols $[L]$ which generalises results previously established by Guillemin [G] and Wodzicki [W] in the case of classical symbols.

For any non negative integer $k$ and any log-polyhomogeneous symbol $\sigma \in CS^a,k(\mathbb{R}^n)$, the expression $\int_{B(0,R)} \sigma(\xi) d\bar{\xi}$ has an asymptotic expansion as $R$ tends to $\infty$ of the form $9$:

$$\int_{B(0,R)} \sigma(\xi) d\bar{\xi} \sim R \to \infty C(\sigma) + \sum_{j=0}^{\infty} \sum_{a-j+n \neq 0} \sum_{l=0}^{k} P_l(\sigma_{a-j,l})(\log R) R^{a-j+n} + \sum_{l=0}^{k} \frac{\text{res}_l(\sigma)}{l+1} \log^{l+1} R$$

where

$$\text{res}_l(\sigma) = \int_{S^{n-1}} \sigma_{-n,l}(\xi) dS\xi$$

is the higher $l$-th noncommutative residue, $P_l(\sigma_{a-j,l})(X)$ is a polynomial of degree $l$ with coefficients depending on $\sigma_{a-j,l}$ and $C(\sigma)$ is the constant term corresponding to the finite part called the cut-off regularised integral of $\sigma$:

$$\int_{\mathbb{R}^n} \sigma(\xi) d\xi := \int_{\mathbb{R}^n} \sigma(\chi_\mathbb{R}(\xi)) d\xi + \int_{B(0,1)} \chi(\xi)\sigma(\xi) d\xi$$

$$+ \sum_{j=0}^{N-1} \sum_{a-j+n \neq 0} \sum_{l=0}^{k} \frac{(-1)^{l+1}}{(a-j+n)^{l+1}} \int_{S^{n-1}} \sigma_{a-j,l}(\xi) dS\xi$$

$N$ give rise to a Fréchet topology on $CS^{a,k}(\mathbb{R}^n)$:

$$\sup_{\xi \in \mathbb{R}^n} (1 + |\xi|)^{-a+|\beta|} |D^\beta \sigma(\xi)|$$

$$\sup_{\xi \in \mathbb{R}^n} |\xi|^{-a+N+|\beta|} |D^\beta \sigma(\xi)|$$

$$\sup_{\xi \in \mathbb{R}^n} |\xi|^{-a-m+|\beta|} |D^\beta \sigma_{a-m}(\xi)|$$

8 $CS^Z(\mathbb{R}^n)$ is equipped with an inductive limit topology of Fréchet spaces

9 We have set $d\xi := (2\pi)^{-n/2} d\xi$ and $d\xi_i := (2\pi)^{-n/2} d\xi_i$. 

8
with the notations of \( \text{[MMP]} \).

It is independent of the choice of \( N \geq a + n - 1 \), as well as of the cut-off function \( \chi \). It is furthermore independent of the parametrisation \( R \) provided the higher noncommutative residue \( \operatorname{res}(\sigma) \) vanish for all integer \( 0 \leq l \leq k \) for we have:

\[
\mathcal{F}_{\mu} R_{\rightarrow \infty} \int_{B(0, R)} \sigma(\xi) d\xi = \mathcal{F}_{\mu} R_{\rightarrow \infty} \int_{B(0, R)} \sigma(\xi) d\xi + \sum_{l=0}^{k} \frac{\log^{l+1} \mu}{l+1} \cdot \operatorname{res}(\sigma)
\]

for any fixed \( \mu > 0 \).

If \( \sigma \) is a classical operator, setting \( k = 0 \) in the above formula yields

\[
\int_{\mathbb{R}^n} \sigma(\xi) d\xi := \int_{\mathbb{R}^n} \sigma(N)(\xi) d\xi + \sum_{j=0}^{N-1} \int_{B(0,1)} \chi(\xi) \sigma_{a-j}(\xi) d\xi
\]

\[
= \sum_{j=0, a-j+n \neq 0}^{N-1} \frac{1}{a-j+n} \int_{S^{n-1}} \sigma_{a-j}(\omega) d\omega.
\]

**Remark 3.** With the notations of Remark 2 we have the following Taylor expansion at \( \epsilon = 0 \):

\[
\int_{\mathbb{R}^d} \sigma^\epsilon(\xi) d\xi = \int_{\mathbb{R}^n} \sigma^\epsilon(N)(\xi) d\xi + \sum_{j=0}^{N-1} \int_{B(0,1)} \chi(\xi) \sigma^\epsilon_{a-j}(\xi) d\xi
\]

\[
+ \sum_{j=0, a-j+n \neq 0}^{N-1} \frac{(-1)^{l+1}!}{(a-j+n)^{l+1}} \int_{S^{n-1}} \sigma^\epsilon_{a-j,l}(\xi) d\omega
\]

\[
= \sum_{j=0}^{N-1} \epsilon^j \int_{B(0,1)} \chi(\xi) \sigma^1_{a-j}(\xi) d\xi
\]

\[
+ \sum_{j=0, a-j+n \neq 0}^{N-1} \epsilon^j \int_{B(0,1)} \sigma^1_{a-j,l}(\xi) d\omega
\]

\[
= O(\epsilon^N)
\]

since \( \sigma^\epsilon(N) = O(\epsilon^N) \) as a result of the fact that \( \sigma^\epsilon \sim \sum_{j=0}^{\infty} \epsilon^j \sigma^1_{a-j} \). It therefore turns out that the regularised cut-off integral which is built to deal with ultraviolet divergences also naturally takes care of infrared divergences in as far as it yields a Taylor expansion as \( \epsilon \to 0 \) of the map \( \epsilon \mapsto \int_{\mathbb{R}^n} \sum_{l=0}^{k} \epsilon^l \left( \frac{1}{(l+1)!} \log^l \left( \left( |\xi|^2 + \epsilon^2 \right) \right) \right) \).

An important property of cut-off regularised integrals already observed in \text{[MMP]} is that they vanish on polynomials.

**Proposition 1.** Let \( P(\xi_1, \ldots, \xi_k) = \sum_{a} c_a \xi_1^{a_1} \cdots \xi_k^{a_k} \) be a polynomial expression in the \( \xi_1, \cdots, \xi_n \) with complex coefficients \( c_a \), then

\[
\int_{\mathbb{R}^n} P(\xi_1, \cdots, \xi_n) d\xi = 0
\]
Proof: It suffices to prove that for any non negative integer $a$,
\[
\int_{\mathbb{R}^n} \xi_i^a d\xi = 0.
\]
Since for any $R > 0$
\[
\int_{B(0,R)} \xi_i^a d\xi = \left( \int_0^R r^{n+\alpha-1}dr \right) \int_{S^{n-1}} \xi_i^a d\xi = \frac{R^{n+\alpha}}{\alpha + n} \int_{S^{n-1}} \xi_i^a d\xi,
\]
we have
\[
\int_{\mathbb{R}^n} \xi_i^a d\xi = \lim_{R \to \infty} \frac{R^{n+\alpha}}{\alpha + n} \int_{S^{n-1}} \xi_i^a d\xi = 0.
\]
\[
\square
\]

2 Regularised integrals of log-polyhomogenous symbols

2.1 Cut-off regularised integrals of holomorphic families

Following [KV] (see [L] for the extension to log-polyhomogeneous symbols), we call a family $z \mapsto \sigma(z) \in CS^{\alpha,k}(\mathbb{R}^n)$ of log-polyhomogeneous symbols parametrised by $z \in \Omega \subset \mathbb{C}$ holomorphic if the following assumptions hold:

1. the order $\alpha(z)$ of $\sigma(z)$ is holomorphic in $z$,
2. for any $0 \leq l \leq k$, for any non negative integer $j$, the homogeneous components $\sigma_{\alpha(z)-j,l}(z)$ of the symbol $\sigma(z)$ yield holomorphic maps into $C^\infty(\mathbb{R}^n)$,
3. for any sufficiently large integer $N$, the map
\[
z \mapsto \int_{\mathbb{R}^n} e^{i\xi \cdot (z-y)} \left( \sigma(z)(\xi) - N \sum_{j=0}^{\infty} \chi(\xi) \sigma_{\alpha(z)-j}(z)(\xi) \right) d\xi
\]
yields a holomorphic map $z \mapsto K^{(\langle N)}$ into some $C^{\langle K(\langle N)}(\mathbb{R}^n \times \mathbb{R}^n)$ where $
\lim_{N \to \infty} K^{(\langle N)}(\langle N) = +\infty.
\]

We quote from [PS] the following theorem which extends results of [L] relating the noncommutative residue of holomorphic families of log-polyhomogeneous symbols with higher noncommutative residues. For simplicity, we restrict ourselves to holomorphic families with order $\alpha(z)$ given by an affine function of $z$, a case which covers natural applications.

**Proposition 2.** Let $k$ be a non negative integer. For any holomorphic family $z \mapsto \sigma(z) \in CS^{\alpha(z),k}(\mathbb{R}^n)$ of symbols parametrised by a domain $W \subset \mathbb{C}$ such



10
that $z \mapsto \alpha(z) = \alpha'(0) z + \alpha(0)$ is a non constant affine function, there is a Laurent expansion in a neighborhood of any $z_0 \in \mathbb{C}$

$$\int_{\mathbb{R}^n} \sigma(z)(\xi) d\xi = \text{fp}_{z=z_0} \int_{\mathbb{R}^n} \sigma(z)(\xi) d\xi + \sum_{j=1}^{k+1} \frac{r_j(\sigma)(z_0)(x)}{(z - z_0)^j}$$

+ $o\left((z - z_0)^{k}\right),$

where for $1 \leq j \leq k+1$, $r_j(\sigma)(z_0)(x)$ is explicitly determined by a local expression (see [L] for the case $\alpha'(0) = 1$)

$$r_j(\sigma)(z_0)(x) := \sum_{l=j-1}^{k} \frac{(-1)^{l+1}}{l!} \frac{l!}{(l+1-j)!} \text{res}\left((\sigma(l))^{(l+1-j)}\right)(z_0).$$

Here $\text{res}(\tau) = \int_{S^{n-1}} \tau_{n,0}(\xi) d\sigma_0 \xi$, $\sigma(l)(z)$ is the local symbol given by the coefficient of $\log |\xi|$ of $\sigma$ i.e. $\sigma(z) = \sum_{l=0}^{k} \sigma(l)(z) \log |\xi|$. On the other hand, the finite part $\text{fp}_{z=z_0} \int_{\mathbb{R}^n} \sigma(z)(\xi) d\xi$ consists of a global piece given by the cut-off regularised integral $\int_{\mathbb{R}^n} \sigma(0)(\xi) d\xi$ and a local piece expressed in terms of residues:

$$\text{fp}_{z=z_0} \int_{\mathbb{R}^n} \sigma(z)(\xi) d\xi = \int_{T_{n+U}} \sigma(z_0)(\xi) d\xi + \sum_{l=0}^{k} \frac{(-1)^{l+1}}{(\alpha'(z_0))^{l+1}} \frac{1}{l+1} \text{res}\left((\sigma(l))^{(l+1)}\right)(z_0).$$

As a consequence, the finite part $\text{fp}_{z=z_0} \int_{\mathbb{R}^n} \sigma(z)(\xi) d\xi$ is entirely determined by the derivative $\alpha'(z_0)$ of the order and by the derivatives of the symbol $\sigma(l)(z_0)$, $1 \leq k + 1$ via the cut-off integral and the noncommutative residue.

### 2.2 Regularised integrals

Let us briefly recall the notion of holomorphic regularisation taken from [KV] (see also [PS]) and adapted to physics applications in [PT]. It includes dimensional regularisation used in perturbative quantum field theories to cure singularities arising in loop diagrams see e.g. [HV], [Sm1], [Sm2], [Sm3].

**Definition 1.** A holomorphic regularisation procedure on a subset $S \subset CS^{*,k}(\mathbb{R}^n)$ is a map $\sigma \mapsto (z \mapsto \sigma(z))$ which sends $\sigma \in CS^{*,k}(\mathbb{R}^n)$ to a holomorphic family $\sigma \in CS^{*,k}(\mathbb{R}^n)$ such that

1. $\sigma(0) = \sigma,$
2. $\sigma(z)$ has holomorphic order $\alpha(z)$ (in particular, $\alpha(0)$ is equal to the order of $\sigma$) such that $\alpha'(0) \neq 0.$

We call a regularisation procedure $\mathcal{R}$ continuous whenever the map $\sigma \mapsto (z \mapsto \sigma(z))$ is continuous for the Fréchet topology on $CS^{*,k}(\mathbb{R}^n)$ (see previous footnote).
One often comes across holomorphic regularisations of the type:

\[
\mathcal{R}(\sigma)(z) = \sigma \cdot \tau(z)
\]

where \(\tau(z)\) is a holomorphic family of symbols in \(CS(\mathbb{R}^n)\) such that

1. \(\tau(0) = 1\),
2. \(\tau(z)\) has holomorphic order \(-q z\) with \(q > 0\).

Note that this implies that \(\sigma(z)\) has order \(\alpha(z) = \alpha(0) - q z\) with \(q \neq 0\).

This class of holomorphic regularisations contains known regularisation such as

- Riesz regularisation for which \(\tau(z)(\xi) := \chi(\xi) |\xi|^{-z}\), where \(\chi\) is some smooth cut-off function around 0 which is equal to 1 outside the unit ball.

- This is a particular instance of regularisations for which \(\tau(z)(\xi) = H(z) \cdot \chi(\xi) |\xi|^{-z}\) where \(H\) is a holomorphic function such that \(H(0) = 1\).

- In even dimensions, dimensional regularisation corresponds to the choice (see [P1]) \(H(z) := \frac{\pi - z}{\Gamma(p - \frac{z}{2})}\) which is a holomorphic function at \(z = 0\) such that \(H(0) = 1\).

The function \(H\) stands for the relative volume of the unit cotangent sphere in dimension \(n\) w.r.t. to its “volume” in dimension \(n(z) = n - z\).

**Example 4.** To illustrate this, let us consider integrals of a radial symbol \(\sigma(\xi) := f(|\xi|)\) following the physicists’ prescription for dimensional regularisation. Assuming that the symbol has order with real part \(< -n\), then

\[
\int_{\mathbb{R}^n} \sigma(\xi) d^n \xi = \text{Vol}(S^{n-1}) \int_{\mathbb{R}^n} f(r) r^{n-1} dr = \frac{2^{p} \pi^{p}}{\Gamma(p)} \int_{\mathbb{R}^n} f(r) r^{n-1} dr
\]

since the volume of the unit sphere \(S^{n-1}\) in even dimensions \(n = 2p\) is given by \(\text{Vol}(S^{n-1}) = \frac{\pi^{p}}{\Gamma(p)} = \frac{\pi \Gamma(p)}{\Gamma(p - 1)}\). Replacing \(n\) by \(n - z\) in the above expression yields a holomorphic map on the half plane \(\text{Re}(z) < \text{Re}(a) + n\):

\[
z \mapsto \frac{2^{p} \pi^{p-z}}{\Gamma(p - \frac{z}{2})} \int_{\mathbb{R}^n} f(r) r^{n-z-1} dr = \int_{\mathbb{R}^n} \sigma(z)(\xi) d\xi
\]

where we have set \(\sigma(z)(\xi) = H(z) \sigma(\xi) |\xi|^{-z}\) and \(H(z) := \frac{2^{p} \pi^{p-z} \Gamma(p)}{2 \pi \Gamma(p - \frac{z}{2})} = \frac{\pi^{p-z} \Gamma(p)}{\Gamma(p - \frac{z}{2})}\).

By the above constructions, we know that this extends to a meromorphic map \(z \mapsto \int_{\mathbb{R}^n} \sigma(z)(\xi) d\xi\) on the whole complex plane. Thus, dimensional regularisation on radial symbols boils down to holomorphic regularisation on the integrand.

These regularisation procedures are clearly continuous. They have in common that the order \(\alpha(z)\) of \(\sigma(z)\) is affine in \(z\):

\[
\alpha(z) = \alpha(0) - q z, \quad q \neq 0,
\]

which is why we restrict to this situation.
As a consequence of the results of the previous paragraph, given a holomorphic regularisation procedure \( \mathcal{R} : \sigma \mapsto \sigma(z) \) on \( CS^{*,k}(\mathbb{R}^n) \) and a symbol \( \sigma \in CS^{*,k}(\mathbb{R}^n) \), the map \( z \mapsto \int_{\mathbb{R}^n} \sigma(z)(\xi) \, d\xi \) is meromorphic with poles of order at most \( k+1 \) at points in \( \alpha^{-1}([-n, +\infty] \cap \mathbb{Z}) \) where \( \alpha(z) \) is the order of \( \sigma(z) \) so that we can define the finite part when \( z \to 0 \) as follows.

**Definition 2.** Given a holomorphic regularisation procedure \( \mathcal{R} : \sigma \mapsto \sigma(z) \) on \( CS^{*,k}(\mathbb{R}^n) \) and a symbol \( \sigma \in CS^{*,k}(\mathbb{R}^n) \), we define the regularised integral

\[
\int_{\mathbb{R}^n} \sigma(z) \, d\xi := \text{fp}_{z=0} \int_{\mathbb{R}^n} \sigma(z)(\xi) \, d\xi := \lim_{z \to 0} \left( \int_{\mathbb{R}^n} \sigma(z)(\xi) \, d\xi - \sum_{j=1}^{k+1} \frac{1}{z^j} \text{Res}_{z=0} \int_{\mathbb{R}^n} \sigma(z)(\xi) \, d\xi \right).
\]

In particular, in even dimensions we define the dimensional regularised integral of \( \sigma \) by

\[
\int_{\mathbb{R}^n} \sigma := \text{fp}_{z=0} \left( H(z) \int_{\mathbb{R}^n} \chi(\xi) \sigma(\xi) |\xi|^{-z} \, d\xi \right) + \int_{\mathbb{R}^n} (1 - \chi(\xi)) \sigma(\xi) \, d\xi \tag{6}
\]

which is independent of the choice of cut-off function \( \chi \).

**Example 5.** Simple computations show that Riesz and cut-off regularised integrals of symbols coincide.

**Theorem 1.** Dimensional regularised integrals of symbols in \( CS^{*,k}(\mathbb{R}^n) \) with \( n = 2p \) even differ from cut-off regularised integrals by a linear combination of the first \( k+1 \) derivatives of the function \( H(z) := \frac{\pi^{\frac{1}{2}} \Gamma(p)}{\Gamma(p - \frac{1}{2})} \) with coefficients involving the residues of derivatives of the symbol:

\[
\int_{\mathbb{R}^n} \sigma(\xi) d\xi = \int_{\mathbb{R}^n} \sigma(\xi) d\xi + \sum_{l=0}^{k} H^{(l+1)}(0) \sum_{j=1}^{k} \frac{1}{(j - l)!} \text{res} \left( (\sigma(j))^{(j-l)}(0) \right).
\]

When \( k = 0 \), \( \sigma \) is classical and:

\[
\int_{\mathbb{R}^n} \sigma(\xi) d\xi = \int_{\mathbb{R}^n} \sigma(\xi) d\xi + \frac{1}{2} \left( \sum_{j=1}^{p-1} \frac{1}{j} + \gamma \right) - \log \pi \right) \cdot \text{res}(\sigma).
\]

**Proof:** The fact that dimensional regularisation is obtained from Riesz regularisation \( \sigma \mapsto \sigma(z) \) by multiplying \( \sigma(z) \) by a function \( H(z) \) introduces extra terms involving complex residues:

\[
\text{fp}_{z=0} \left( H(z) \cdot \int_{\mathbb{R}^n} \sigma(z)(\xi) \, d\xi \right) = \int_{\mathbb{R}^n} \sigma(z)(\xi) \, d\xi + \sum_{l=0}^{k} \text{Res}^{l+1}(\sigma(z)) H^{(l+1)}(0)
\]

which, when combined with (6) yields:

\[
\text{fp}_{z=0} \left( H(z) \cdot \int_{\mathbb{R}^n} \sigma(z)(\xi) \, d\xi \right) = \int_{\mathbb{R}^n} \sigma(\xi) d\xi + \sum_{l=0}^{k} H^{(l+1)}(0) \sum_{j=1}^{k} \frac{1}{(j - l)!} \text{res} \left( (\sigma(j))^{(j-l)}(0) \right)
\]

13
since $\alpha(z) = \alpha(0) - z$. In particular, when $\sigma$ is classical (i.e. when $k = 0$) we have:

$$\text{fp}_{z=0} \left( H(z) \cdot \int_{\mathbb{R}^n} \sigma(z) d\xi \right) = \int_{\mathbb{R}^n} \sigma(d\xi) + \text{res}(\sigma) \cdot H'(0).$$

Since derivatives at $p \in \mathbb{N} - \{1\}$ of the Gamma function read: $\Gamma'(p) = \Gamma(k) \left( \sum_{j=1}^{p-1} \frac{1}{j} - \gamma \right)$ it follows that

$$H'(0) = \frac{1}{2} \left( -\log \pi + \frac{\Gamma'(p)}{\Gamma(p)} \right) = \frac{1}{2} \left( -\log \pi + \left( \sum_{j=1}^{p-1} \frac{1}{j} - \gamma \right) \right).$$

The result then follows. \( \square \)

**Remark 4.** Since we saw in Remark \( \text{Remark} \) that cut-off regularisation takes care of infrared divergences as well as ultraviolet ones, it follows that so does dimensional regularisation take care of infrared divergences. The additional residue terms only contribute by additional terms in the Taylor expansion at $\epsilon = 0$.

To close this paragraph, we observe that just as the cut-off regularised integral was, the map $\sigma \mapsto -\int_{\mathbb{R}^n} \sigma(\xi) d\xi$ is continuous for any continuous holomorphic regularisation procedure $\mathcal{R} : \sigma \mapsto \sigma(z)$ on $CS^{*,k}(\mathbb{R}^n)$, $k \in \mathbb{N}$.

## 3 Basic properties of integrals of holomorphic symbols

Cut-off regularisation turns out to have nice properties for non integer order symbols, such as a Stokes’ property, translation invariance and covariance. Consequently, computations involving dimensional regularisation can be carried out following the usual integration rules such as integration by parts, change of variables as long as this is done before taking finite parts. This in fact holds for any holomorphic regularisation procedure as is shown below, so in particular for dimensional regularisation, and provides a mathematical justification for the computations carried out by physicists when performing changes of variable and integrations by parts.

### 3.1 Integration by parts

An important property of cut-off regularised integrals is integration by parts, which is an instance of a more general Stokes’ property for symbol valued forms investigated in [MMP].

**Proposition 3.** Let $\sigma \in CS^{*,*}(\mathbb{R}^n)$ with order $\alpha \notin \mathbb{Z} \cap [-n, \infty[$. Then for any multiindex $\alpha$,

$$\int_{\mathbb{R}^n} \partial^\alpha \sigma(\xi) d\xi = 0.$$

**Remark 5.** This Stokes’ property actually characterises the cut-off regularised integral $\int_{\mathbb{R}^n}$ in as far as it is the only linear extension of the ordinary integral to non integer order symbols which vanishes on derivatives [P2].
Proof: We recall the general lines of the proof and refer to \cite{MMP} for further details. We only prove the result for classical symbols since the proof easily extends to log-polyhomogeneous symbols. It is clearly sufficient to prove the result for a multiindex $\alpha = i$ of length one.

- If $\sigma$ has order $< -n$ then we write:

$$\int_{\mathbb{R}^n} \partial_\xi \sigma(\xi) \, d\xi = \int_{\mathbb{R}^n} \partial_\xi \sigma(\xi) \, d\xi$$

$$= \lim_{R \to 0} \int_{B(0,R)} d\xi \partial_\xi \sigma(\xi)$$

$$= (-1)^{i-1} \lim_{R \to 0} \int_{B(0,R)} \sigma(\xi_1 \wedge \cdots \wedge \hat{\xi}_i \wedge \cdots \wedge \xi_n) \, d\xi$$

$$= (-1)^{i-1} \lim_{R \to 0} \int_{S(0,R)} \sigma(\xi) \, dS\xi$$

where we have set $dS\xi := d\xi_1 \wedge \cdots \wedge d\hat{\xi}_i \wedge \cdots \wedge d\xi_n$.

This limit vanishes. Indeed, $\sigma$ being a symbol of order $\alpha$, there is a positive constant $C$ such that

$$\left| \int_{S(0,R)} \sigma(\xi) \, dS\xi \right| \leq \int_{S(0,R)} |\sigma(\xi)| \, dS\xi$$

$$\leq C \int_{S(0,R)} (1 + |\xi|^2)^{\frac{\alpha}{2}} \, dS\xi$$

$$\leq C R^n (1 + R^2)^{\frac{\alpha}{2}} \operatorname{Vol}(S^{n-1}) .$$

Here $S(0, R) \subset B(0, R)$ is the sphere of radius $R$ centered at 0 in $\mathbb{R}^n$.

- If $\alpha \geq -n$, we write $\sigma = \sum_{j=0}^N \chi(\xi) \sigma_{\alpha-j}(\xi) + \sigma_{(N)}(\xi)$ where $\chi$ is a smooth cut-off function, $\sigma_{\alpha-j}(\xi)$ is positively homogeneous of degree $\alpha - j$ and $N$ is chosen large enough for $\sigma_{(N)}$ to be of order $< -n$. We have:

$$\int_{\mathbb{R}^n} \partial_\xi \sigma(\xi) \, d\xi = \sum_{j=0}^N \int_{\mathbb{R}^n} \chi(\xi) \partial_\xi \sigma_{\alpha-j}(\xi) \, d\xi + \int_{\mathbb{R}^n} \partial_\xi \sigma_{(N)}(\xi) \, d\xi .$$

It follows from the above computation that $\int_{\mathbb{R}^n} \partial_\xi \sigma_{(N)}(\xi) \, d\xi = 0$. On
the other hand, we have for large enough $R$ and any positive integer $i$:

$$
\int_{\mathbb{R}^n} \partial_{\xi_i} (\chi(\xi)\sigma_{\alpha-j}(\xi)) \, d\xi \\
= \lim_{R \to \infty} \left( \int_{B(0,R)} \partial_{\xi_i} (\chi(\xi)\sigma_{\alpha-j}(\xi)) \, d\xi \right) \\
= (-1)^{i-1} \lim_{R \to \infty} \left( \int_{S(0,R)} \chi(\xi)\sigma_{\alpha-j}(\xi) \, dS_{\xi} \right) \\
= \lim_{R \to \infty} \left( \int_{S(0,R)} \sigma_{\alpha-j}(\xi) \, dS_{\xi} \right)
$$

since $\chi|_{S(0,R)} = 1$ for large $R$

$$
= \lim_{R \to \infty} \left( R^{n-i+j+n+1} \int_{S^{n-1}} \sigma_{\alpha-j}(\xi) \, dS_{\xi} \right)
$$

which vanishes if $\alpha + n \notin \mathbb{N} \cup \{0\}$.

\[ \square \]

Let as before $\mathcal{R} : \sigma \mapsto \sigma(z)$ be a holomorphic regularisation on $CS(\mathbb{R}^n)$. The following result is a direct consequence of the above proposition.

**Corollary 1.** [MMPP] The following equality of meromorphic functions holds:

$$
\int_{\mathbb{R}^n} \partial^\alpha(\sigma(z))(\xi) \, d\xi = 0
$$

for any multiindex $\alpha$ and any $\sigma \in CS^{*,*}(\mathbb{R}^n)$.

**Proof:** The maps $z \mapsto \int_{\mathbb{R}^n} \partial^\alpha(\sigma(z))(\xi) \, d\xi$ are meromorphic as cut-off regularised integrals of a holomorphic family of symbols with poles in $\alpha^{-1}(\mathbb{Z} \cap [-n, +\infty])$. By Proposition 3 the expression $\int_{\mathbb{R}^n} \partial^\alpha(\sigma(z))(\xi) \, d\xi$ vanishes outside these poles so that the identity announced in the corollary holds as an equality of meromorphic maps. \[ \square \]

**Remark 6.** This does not imply that the same properties hold for $\int_\mathcal{R}$. Unless the total order of the symbols is non integer, one is in general to expect that

$$
\int_{\mathbb{R}^n} \partial^\alpha(\sigma)(\xi) \, d\xi \neq 0.
$$

### 3.2 Translation invariance

We let $\mathbb{R}^n$ act on $CS^{*,*}(\mathbb{R}^n)$ by translations as follows:

$$
\mathbb{R}^n \times CS^{*,*}(\mathbb{R}^n) \to CS^{*,*}(\mathbb{R}^n) \\
(\eta, \sigma) \mapsto \eta^* \sigma(\xi) := \sigma(\xi + \eta).
$$

$\eta$ can be seen as external momentum (usually denoted by $p$) whereas $\xi$ plays the role of internal momentum (usually denoted by $k$) in physics.
The map $\eta \mapsto t_0^*\sigma := \sigma(\cdot + \eta)$ has the following Taylor expansion at $\eta = 0$:

$$t_0^*\sigma(\xi) := \sum_{|\beta| \leq N} \partial^\beta \sigma(\xi) \frac{\eta^\beta}{|\beta|!} + \sum_{|\beta| = N+1} \frac{\eta^\beta}{|\beta|!} \int_0^1 (1-t)^N \partial^\gamma \sigma(\xi + t\eta) \, dt \quad \forall \xi \in \mathbb{R}^n. \tag{8}$$

Let us recall the following translation property for non integer order symbols.

**Proposition 4.** [MP] For any $\sigma \in CS^{a,*}(\mathbb{R}^n)$ and any $\eta \in \mathbb{R}^n$, the cut-off integral

$$\int_{\mathbb{R}^n} \sigma(\xi + \eta) \, d\xi := \lim_{R \to \infty} \int_{B(0,R)} \sigma(\xi + \eta) \, d\xi$$

is well defined. If $\sigma$ has order $a \notin \mathbb{Z} \cap [-n, +\infty[$ then:

$$\int_{\mathbb{R}^n} t_0^*\sigma(\xi) \, d\xi = \int_{\mathbb{R}^n} \sigma(\xi) \, d\xi.$$

**Remark 7.** Translation invariance actually characterises the cut-off regularised integral $\int_{\mathbb{R}^n}$ in as far as it is the only translation invariant linear extension of the ordinary integral to non integer order symbols [P2], [P3].

**Proof:** Let $\sigma \in CS^{a,*}(\mathbb{R}^n)$.

- If $a < -n$ then

$$\int_{\mathbb{R}^n} t_0^*\sigma(\xi) \, d\xi = \lim_{R \to \infty} \int_{B(0,R)} \sigma(\xi + \eta) \, d\xi = \int_{\mathbb{R}^n} \sigma(\xi) \, d\xi$$

is well defined. The second part of the statement then follows from translation invariance of the ordinary Lebesgue integral.

- Let us assume $\Re(a) \geq -n$. The derivatives $\partial^\gamma \sigma$ arising in the Taylor expansion lie in $CS(\mathbb{R}^n)$ so that their integrals over the ball $B(0,R)$ have asymptotic expansions when $R \to \infty$ in decreasing powers of $R$ with a finite number of powers of $\log R$. For $|\beta| = N + 1$ with $N$ chosen large enough, the asymptotic expansion converges as $R$ tends to infinity and has no logarithmic term. The integral $\int_{B(0,R)} \sigma(\xi + \eta) \, d\xi$ therefore has the same type of asymptotic expansion when $R \to \infty$ as $\int_{B(0,R)} \sigma(\xi) \, d\xi$ and the finite part:

$$\int_{\mathbb{R}^n} \sigma(\xi + \eta) \, d\xi := \lim_{R \to \infty} \int_{B(0,R)} \sigma(\xi + \eta) \, d\xi$$

$$= \sum_{|\beta| \leq N} \int_{\mathbb{R}^n} \partial^\beta \frac{\eta^\beta}{|\beta|!} + \sum_{|\beta| = N+1} \frac{\eta^\beta}{|\beta|!} \int_0^1 (1-t)^N \int_{\mathbb{R}^n} \partial^\gamma \sigma(\cdot + t\eta) \, dt$$

is well defined.

Mimicking the proof of Proposition 3 for $|\beta| > 0$ we write $\partial^\beta = \partial_{\xi_1} \circ \partial_{\eta_1}$.
equality of meromorphic functions holds:

\[ \text{Corollary 2.} \]

Following result is a direct consequence of the above proposition.

Let as before \( \text{meromorphic functions.} \)

\[ \text{for large enough} \quad N \]

\[ \text{by the remainder term} \quad \int \]

\[ \text{tegrals of holomorphic families of ordinary symbols and since the map} \]

\[ \text{given} \]

\[ \text{Proof:} \]

\[ \text{The Taylor expansion} \]

\[ \text{which tends to} \quad 0 \quad \text{as} \quad R \rightarrow \infty. \]

Hence the cut-off regularised integral of the remainder term vanishes.

If moreover \( a \notin \mathbb{Z} \) then by Proposition \[ \text{we have} \quad \int \]

\[ \text{for any} \quad \beta \neq 0 \quad \text{term remains in the Taylor expansion and the result follows.} \]

\[ \square \]

Let as before \( \mathcal{R} : \sigma \mapsto \sigma(z) \) be a holomorphic regularisation on \( CS(\mathbb{R}^n) \). The following result is a direct consequence of the above proposition.

\[ \text{Corollary 2.} \quad \text{MMP} \]

For any \( \sigma \in CS^*(\mathbb{R}^n) \) and any \( \eta \in \mathbb{R}^n \) the following equality of meromorphic functions holds:

\[ \int_{\mathbb{R}^n} \sigma(z)(\xi + \eta) d\xi = \int_{\mathbb{R}^n} \sigma(z)(\xi) d\xi. \]

\[ \text{Proof:} \]

\[ \text{The Taylor expansion} \]

\[ \int_{\mathbb{R}^n} \sigma(z)(\xi + \eta) d\xi \]

\[ = \sum_{|\beta| \leq N} \int_{\mathbb{R}^n} \partial^\beta \sigma(z) \frac{\eta^\beta}{\beta!} + \sum_{|\beta| = N+1} \frac{\eta^\beta}{\beta!} \int_0^1 (1-t)^N \int_{\mathbb{R}^n} \partial^\gamma \sigma(z)(\xi + t\eta) dt \]

provides meromorphy of the map \( z \mapsto \int_{\mathbb{R}^n} \sigma(z)(\xi + \eta) d\xi \) since we know that the maps \( z \mapsto \int_{\mathbb{R}^n} \partial^\beta \sigma(z)(\xi) \) are meromorphic as cut-off regularised integrals of holomorphic families of ordinary symbols and since the map given by the remainder term \( z \mapsto \sum_{|\beta| = N+1} \int_{\mathbb{R}^n} \partial^\beta \sigma(z)(\xi + \theta \eta) d\xi \) is holomorphic for large enough \( N \). Outside the set of poles we have by Proposition \[ \text{that} \quad \int_{\mathbb{R}^n} \sigma(z)(\xi + \eta) d\xi = \int_{\mathbb{R}^n} \sigma(z)(\xi) d\xi \] so that the equality holds as an equality of meromorphic functions. \( \square \)
Remark 8. This does not imply that translation invariance holds for \( \int_{\mathbb{R}^n} d\xi \sigma(\xi + \eta) \neq \int_{\mathbb{R}^n} d\xi \sigma(\xi) \).

3.3 Covariance

\( GL_n(\mathbb{R}^n) \) acts on \( CS^{*,*}(\mathbb{R}^n) \) as follows

\[
GL_n(\mathbb{R}^n) \times CS^{*,*}(\mathbb{R}^n) \rightarrow CS^{*,*}(\mathbb{R}^n)
(C, \sigma) \mapsto (\xi \mapsto \sigma(C\xi)) .
\]

We quote from [L] the following extension to log-polyhomogeneous symbols of a result proved in [KV] for classical symbols.

Proposition 5. Let \( \sigma \in CS^{a,*}(\mathbb{R}^n) \) with order \( a \notin \mathbb{Z} \cap [-n, \infty[ \). Then for any \( C \in GL_n(\mathbb{R}^n) \)

\[
|\det C| \int_{\mathbb{R}^n} \sigma(C\xi) d\xi = \int_{\mathbb{R}^n} \sigma(\xi) d\xi .
\]

Let as before \( \mathcal{R} : \sigma \mapsto \sigma(z) \) be a holomorphic regularisation on \( CS(\mathbb{R}^n) \).

The following result is a direct consequence of the above proposition.

Corollary 3. For any \( \sigma \in CS^{*,*}(\mathbb{R}^n) \) and for any \( C \in GL_n(\mathbb{R}^n) \) the following equality of meromorphic functions holds:

\[
|\det C| \int_{\mathbb{R}^n} \sigma(z)(C\xi) d\xi = \int_{\mathbb{R}^n} \sigma(z)(\xi) d\xi .
\]

Remark 9. This does not imply the covariance of the regularised integral \( \int_{\mathcal{R}} \).

Unless the order of the symbols is non integer, one is in general to expect that

\[
|\det C| \int_{\mathbb{R}^n} \sigma(C\xi) d\xi \neq \int_{\mathbb{R}^n} \sigma(\xi) d\xi .
\]

4 Regularised integrals with affine parameters

The aim of this section is to regularise and then investigate the dependence in the external parameters \( p_i \) of a priori divergent integrals of the type

\[
\int_{\mathbb{R}^n} \frac{P(k, p_1, \ldots, p_J)}{(L_i(k, p_1, \ldots, p_J))^2 + m^2)^s \cdots (L_I(k, p_1, \ldots, p_J))^2 + m^2)^s} dk , \quad (9)
\]

where \( P(k, p_1, \ldots, p_J) \) is a polynomial expression and \( L_i(k, p_1, \ldots, p_J), i = 1, \ldots, I \) are linear combinations of \( k \) and the \( p_j \)'s.

The definitions we adopt here are inspired by work of Lesch and Pflaum [LP] on traces of parametric pseudodifferential operators. Even though our symbols with affine parameters are not strongly parametric symbols as are the ones used in their work, their general approach can be adapted to our context, thereby offering an interpretation in terms of iterated integrals of symbols with linear
constraints of computations carried out by physicists to evaluate Feynman diagrams.
The affine parameters here play the role of external momenta in physics and we
describe two ways of regularising integrals of the type (9). We discuss the Taylor
truncation method implemented by physicists which gives rise to regularised
integrals defined modulo polynomials.
Following conventions used in the pseudodifferential literature, we choose to
denote by \( \eta \) the external parameters.

**Lemma 1.** For any \( \sigma_i \in CS^{a_{1}, \ldots, a_{n}}(\mathbb{R}^{n}) \) such that \( \sum_{i=1}^{k} \text{Re}(a_i) < -n \) where \( a_i \) is the order of \( \sigma_i \), then the integral with affine parameters \( \eta_i \in \mathbb{R}^{n}, i = 1, \ldots, k \)
\[
\int_{\mathbb{R}^{n}} (\sigma_1 \otimes \cdots \otimes \sigma_k) (\xi + \eta_1, \ldots, \xi + \eta_k) \, d\xi
\]
is well defined.

**Proof:** Since the \( \sigma_i \)'s are symbols we have \( |\sigma_i(\xi + \eta_i)| \leq C_i (\xi + \eta_i)^{a_i} \) for
some \( C_i \in \mathbb{R}_{+} \) and where we have set \( \langle \xi \rangle := \sqrt{1 + |\xi|^2} \). But the integral
\[
\int_{\mathbb{R}^{n}} \prod_{i=1}^{k} (\xi + \eta_i)^{a_i} \, d\xi
\]
is convergent whenever \( \sum_{i=1}^{k} \text{Re}(a_i) < -n \) hence the re-

**Remark 10.** A typical example of such an integral is:
\[
p \mapsto \int_{\mathbb{R}^{n}} \frac{P(k, p)}{(k^2 + m^2)^{s_2}((k - p)^2 + m^2)^{s_1}} \, dk
\]
where \( P \) is a polynomial expression and the \( s_i \) are complex numbers with real
part chosen large enough for the integral to converge. Here, we have adopted the
physicists notation \( k^2 \) for \( |k|^2 \). Writing the polynomial \( P(k, p) \) as a polynomial
\[
\sum_{a_{1}} a_{a_{1}}(p) k^{a_{1}}
\]
in \( k \) with coefficients depending polynomially on \( p \), we can rewrite the
integrand as a finite linear combination (with \( p \)-dependent coefficients) of
\[
k \mapsto (k^2 + m^2)^{s_1}((k - p)^2 + m^2)^{s_2}
\]
each of which reads \( k \mapsto (\tau_{a} \otimes \sigma_{1} \otimes \sigma_{2})(k, k, k - p) \)
where we have set \( \tau_{a}(k) := k^{a_{1}}, \sigma_{i}(k) = \frac{1}{(k^2 + m^2)^{s_{2}}}. \)

Let us now describe a procedure to regularise integrals with affine parameters
used by physicists to compute Feynman integrals. The idea is to truncate the
taylor series at the \( \eta_i \) (which correspond to external momenta in physics) about
the origin at a high enough order. With the notations of [57], we denote by \( \mathcal{M} \)
this truncation and define an alternative cut-off regularised integral \( \int_{\mathbb{R}^{n}} \)
\[
(\sigma_1 \otimes \cdots \otimes \sigma_k) (\xi + \eta_1, \ldots, \xi + \eta_k) \, d\xi = \int_{\mathbb{R}^{n}} (I - \mathcal{M}) (\sigma_1 \otimes \cdots \otimes \sigma_k) (\xi + \eta_1, \ldots, \xi + \eta_k) \, d\xi.
\]
In physics the order of truncation (given here by \( N \)) is chosen according to the
superficial degree of divergence of the diagram. In contrast, here we do not fix
the order of truncation as it will soon appear that it can be chosen arbitrarily
large.
It turns out that this regularised integral is defined “up to polynomials” in the
components of the external parameters and that it coincides with the previous-
ously defined cut-off regularised integral “up to polynomials” in these external
parameters.
Proposition 6. For any any \( \sigma_i \in C^{n,*}(R^n) \), modulo polynomials in the components of the parameters \( \eta_1 \in R^n, \ldots, \eta_k \in R^m \) the expression:

\[
\int_{R^n} (\sigma_1 \otimes \cdots \otimes \sigma_k) (\xi + \eta_1, \ldots, \xi + \eta_k) d\xi
\]

:= \int_{R^n} [ (\sigma_1 \otimes \cdots \otimes \sigma_k) (\xi + \eta_1, \ldots, \xi + \eta_k) \\
- \sum_{|\beta| = 0}^N \partial_1^{\beta_1} \cdots \partial_k^{\beta_k} (\sigma_1 \otimes \cdots \otimes \sigma_k) (\xi, \ldots, \xi) \frac{\eta_1^{\beta_1} \cdots \eta_k^{\beta_k}}{\beta_1! \cdots \beta_k!} ] d\xi
\]  

(10)
is well defined and coincides modulo polynomials in the components of the parameters \( \eta_1, \ldots, \eta_k \) with the ordinary integral whenever \( \sum_{i=1}^k \text{Re}(a_i) < -n \). Here \( \beta_1, \ldots, \beta_k \) are multiindices in \( N^n \) and we have set \( |\beta| := \sum_{i=1}^k |\beta_i| \).

Proof: A Taylor expansion of \((\sigma_1 \otimes \cdots \otimes \sigma_k)(\xi + \eta_1, \ldots, \xi + \eta_k)\) in \( \eta = (\eta_1, \ldots, \eta_k) \) at \( \eta = 0 \) yields

\[
(\sigma_1 \otimes \cdots \otimes \sigma_k)(\xi + \eta_1, \ldots, \xi + \eta_k) = \sum_{|\beta| = 0}^N \partial_1^{\beta_1} \cdots \partial_k^{\beta_k} (\sigma_1 \otimes \cdots \otimes \sigma_k)(\xi, \ldots, \xi) \frac{\eta_1^{\beta_1} \cdots \eta_k^{\beta_k}}{\beta_1! \cdots \beta_k!} + R_N(\xi, \eta_1, \ldots, \eta_k).
\]

Since the real part of the total order \( \sum_{i=1}^k a_i - |\beta| \) of \( \partial^\beta(\sigma_1 \otimes \cdots \otimes \sigma_k) \) decreases as \( |\beta| \) increases, the remainder term \( R_N(\xi, \eta_1, \ldots, \eta_k) \) lies in \( L^1(R^n) \) provided \( N \) is chosen large enough. The integral

\[
\int_{R^n} [ (\sigma_1 \otimes \cdots \otimes \sigma_k)(\xi + \eta_1, \ldots, \xi + \eta_k) \\
- \sum_{|\beta| = 0}^N \partial_1^{\beta_1} \cdots \partial_k^{\beta_k} (\sigma_1 \otimes \cdots \otimes \sigma_k)(\xi, \ldots, \xi) \frac{\eta_1^{\beta_1} \cdots \eta_k^{\beta_k}}{\beta_1! \cdots \beta_k!} ] d\xi
\]

therefore makes sense for large enough \( N \). A modification of \( N \) only modifies the expression by a polynomial in \( \eta_1, \ldots, \eta_k \) so that the expression is well-defined modulo polynomials.

When \( \sum_{i=1}^n \text{Re}(a_i) < -n \) the integral \( \int_{R^n} (\sigma_1 \otimes \cdots \otimes \sigma_k)(\xi + \eta_1, \ldots, \xi + \eta_k) d\xi \) converges by the above lemma and hence so does \( \xi \mapsto R_N(\xi, \eta_1, \ldots, \eta_k) \) lie in \( L^1(R^n) \). The Taylor expansion then yields \( \int_{R^n} \) with the cut-off regularised integral on the l.h.s. replaced by an ordinary integral. It follows that the cut-off regularised integral \( \int_{R^n} \) coincides (modulo polynomials in the components of \( \eta \)) with the usual integral whenever the integrand converges. \( \Box \)

The following result shows that derivations w.r. to the parameters commute with the regularised integral \( \int_{R^n} \).

Theorem 2. Modulo polynomials in the components of the parameters \( \eta_1, \ldots, \eta_k \) we have

\[
\partial^\gamma_\eta \int_{R^n} (\sigma_1 \otimes \cdots \otimes \sigma_k)(\xi + \eta_1, \ldots, \xi + \eta_k) d\xi
\]

= \int_{R^n} \partial^\gamma_\eta (\sigma_1 \otimes \cdots \otimes \sigma_k)(\xi + \eta_1, \ldots, \xi + \eta_k) d\xi, \quad \forall \gamma \in N^k
\]
where for the multiindex $\gamma = (\gamma_1, \cdots, \gamma_k)$ we have set $\partial_\gamma := \partial_{\gamma_1}^{\gamma_1} \cdots \partial_{\gamma_k}^{\gamma_k}$. Provided $|\gamma| = \gamma_1 + \cdots + \gamma_k$ is chosen large enough so that the integrand $\partial_{\gamma_1}^{\gamma_1} \cdots \partial_{\gamma_k}^{\gamma_k} (\sigma_1 \otimes \cdots \otimes \sigma_k) (\xi + \eta_1, \cdots, \xi + \eta_k)$ lies in $L^1(\mathbb{R}^n)$ we have:

$$
\partial_\gamma^n \int_{\mathbb{R}^n} (\sigma_1 \otimes \cdots \otimes \sigma_k) (\xi + \eta_1, \cdots, \xi + \eta_k) d\xi
= \int_{\mathbb{R}^n} \partial_\gamma^n (\sigma_1 \otimes \cdots \otimes \sigma_k) (\xi + \eta_1, \cdots, \xi + \eta_k) d\xi \quad \text{mod polynomials.}
$$

**Proof:** We prove the result for $\gamma = (0, \cdots, 0, \gamma_i, 0, \cdots, 0)$ from which the general statement then easily follows. Whenever $\sum_{i=1}^k \text{Re}(a_i) < -n$ we have

$$
\partial_{\gamma_i}^{\gamma_i} \int_{\mathbb{R}^n} (\sigma_1 \otimes \cdots \otimes \sigma_k) (\xi + \eta_1, \cdots, \xi + \eta_k) d\xi = \int_{\mathbb{R}^n} \partial_{\gamma_i}^{\gamma_i} (\sigma_1 \otimes \cdots \otimes \sigma_k) (\xi + \eta_1, \cdots, \xi + \eta_k) d\xi.
$$

Hence by (10), modulo polynomials in the $\eta_i$ we have

$$
\partial_{\gamma_i}^{\gamma_i} \int_{\mathbb{R}^n} (\sigma_1 \otimes \cdots \otimes \sigma_k) (\xi + \eta_1, \cdots, \xi + \eta_k) d\xi
= \partial_{\gamma_i}^{\gamma_i} \left[ \int_{\mathbb{R}^n} (\sigma_1 \otimes \cdots \otimes \sigma_k) (\xi + \eta_1, \cdots, \xi + \eta_k) d\xi - \sum_{|\beta|=0}^N \partial_{\gamma_i}^{\gamma_i} (\sigma_1 \otimes \cdots \otimes \sigma_k) (\xi, \cdots, \xi) \frac{\eta_1^{\beta_1} \cdots \eta_k^{\beta_k}}{\beta_1! \cdots \beta_k!} \right] d\xi
$$

$$
= \int_{\mathbb{R}^n} \partial_{\gamma_i}^{\gamma_i} \left[ (\sigma_1 \otimes \cdots \otimes \sigma_k) (\xi + \eta_1, \cdots, \xi + \eta_k) - \sum_{|\beta|=0}^N \partial_{\gamma_i}^{\gamma_i} (\sigma_1 \otimes \cdots \otimes \sigma_k) (\xi, \cdots, \xi) \frac{\eta_1^{\beta_1} \cdots \eta_k^{\beta_k}}{\beta_1! \cdots \beta_k!} \right] d\xi
$$

$$
= \int_{\mathbb{R}^n} \left[ \partial_{\gamma_i}^{\gamma_i} (\sigma_1 \otimes \cdots \otimes \sigma_k) (\xi + \eta_1, \cdots, \xi + \eta_k) - \sum_{|\beta|=0}^N \partial_{\gamma_i}^{\gamma_i} (\sigma_1 \otimes \cdots \otimes \sigma_k) (\xi, \cdots, \xi) \partial_{\gamma_i}^{\gamma_i} \frac{\eta_1^{\beta_1} \cdots \eta_k^{\beta_k}}{\beta_1! \cdots \beta_k!} \right] d\xi
$$

which ends the proof of the proposition. \(\Box\)
In certain situations the maps:
\[ \eta_i \mapsto \int_{\mathbb{R}^n} (\sigma_1 \otimes \cdots \otimes \sigma_k)(\xi + \eta_1, \cdots, \xi + \eta_k) \, d\xi \]
to define symbols (modulo polynomials in \( \eta_i \)), in which case one cannot expect the latter to be classical but rather log-polyhomogeneous. In that case, one can further integrate in the parameter \( \eta_i \) using cut-off integration.

The ambiguity that arises from having expressions defined “modulo polynomials” in the external parameters disappears after cut-off integration in these parameters as a result of the fact that the cut-off regularised integral \( \int_{\mathbb{R}^n} \) vanishes on polynomials. Consequently, the order of truncation at which the Taylor expansion was originally taken in the external parameters does not matter as long as it is chosen large enough: extra terms in the Taylor expansion are polynomials in the external parameters and hence vanish after cut-off integration in these parameters.

**Corollary 4.** Whenever \( \eta_i \mapsto \int_{\mathbb{R}^n} (\sigma_1 \otimes \cdots \otimes \sigma_k)(\xi + \eta_1, \cdots, \xi + \eta_k) \, d\xi \) lies in \( \text{CS}^{*,*}(\mathbb{R}^n) \) (modulo polynomials in \( \eta_i \)), the double cut-off regularised integral:
\[ \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} (\sigma_1 \otimes \cdots \otimes \sigma_k)(\xi + \eta_1, \cdots, \xi + \eta_k) \, d\xi \right) \, d\eta_i \]
is well defined modulo polynomials in the remaining \( \eta_j, j \neq i \).

If \( \eta_i \mapsto \int_{\mathbb{R}^n} (\sigma_1 \otimes \cdots \otimes \sigma_k)(\xi + \eta_1, \cdots, \xi + \eta_k) \, d\xi \) moreover has non integer order, then for large enough \( |\gamma_i| \) we have
\[ \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \partial_{\eta_i}^{\gamma_i} (\sigma_1 \otimes \cdots \otimes \sigma_k)(\xi + \eta_1, \cdots, \xi + \eta_k) \, d\xi \right) \, d\eta_i \]
where the cut-off integral \( \int_{\mathbb{R}^n} \) has now been replaced by an ordinary integral.

**Proof:** The first part of the statement follows from the fact that the cut-off regularised integral \( \int_{\mathbb{R}^n} \) vanishes on polynomials (see Proposition 1). The second part of the statement follows from integration by parts property (see Proposition 3) for the cut-off integral on non integer order symbols combined with (11). Indeed, we have:
\[ \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \partial_{\eta_i}^{\gamma_i} (\sigma_1 \otimes \cdots \otimes \sigma_k)(\xi + \eta_1, \cdots, \xi + \eta_k) \, d\xi \right) \, d\eta_i \]
by Proposition 3
\[ = (-1)^{|\gamma_i|} \int_{\mathbb{R}^n} \partial_{\gamma_i}^{\gamma_i} \left( \int_{\mathbb{R}^n} \partial_{\eta_i}^{\gamma_i} (\sigma_1 \otimes \cdots \otimes \sigma_k)(\xi + \eta_1, \cdots, \xi + \eta_k) \, d\xi \right) \, d\eta_i \]
by (11).
\[ \square \]
Part 2: Renormalised multiple integrals of symbols with linear constraints

The aim of this second part of the paper is to define renormalised multiple integrals with linear constraints. Instead of iterating regularised integrals as one might do for convergent integrals, we renormalise multiple integrals as a whole in the spirit of Connes and Kreimer’s approach to renormalisation of Feynman diagrams, keeping in mind that to such a diagram corresponds a multiple integral with affine constraints.

It is useful to first recall how multiple integrals of symbols without constraints can be renormalised using a Birkhoff factorisation.

5 Integrals of tensor products of symbols revisited

We report on and extend results of [MP] concerning integrals of tensor products of symbols, i.e. multiple integrals without constraints.

Following [MP], let us consider the tensor algebra of log-polyhomogeneous symbols:

\[ \mathcal{T} (CS(\mathbb{R}^n)) := \bigoplus_{k=0}^{\infty} \hat{\otimes}^k CS(\mathbb{R}^n) \]

built on the algebra \(CS(\mathbb{R}^n)\) of log-polyhomogeneous symbols on \(\mathbb{R}^n\). Here \(\hat{\otimes}\) denotes the Grothendieck completion.

The cut-off regularised integral being continuous on the subspace \(CS^n(\mathbb{R}^n)\) of classical symbols on \(\mathbb{R}^n\) with constant order \(a\) for any fixed \(a \in \mathbb{C}\), it can be extended by continuity and (multi-) linearity to the tensor algebra \(\mathcal{T}(CS(\mathbb{R}^n))\).

**Definition 3.** [MP] The cut-off regularised integral \(\int_{\mathbb{R}^n}\) defined on \(CS(\mathbb{R}^n)\) extends to a character:

\[ \mathcal{T}(CS(\mathbb{R}^n)) \to \mathbb{C} \]

\[ \sigma_1 \otimes \cdots \otimes \sigma_k \mapsto \int_{\mathbb{R}^n^k} \sigma_1 \otimes \cdots \otimes \sigma_k := \prod_{i=1}^{k} \int_{\mathbb{R}^n} \sigma_i. \]

As an immediate consequence of these definitions and the previous results on ordinary cut-off regularised integrals we have the following meromophicity result (This is a slight generalisation of results in [MP]).

**Lemma 2.** Given a continuous holomorphic regularisation procedure \(\mathcal{R}\) on \(CS(\mathbb{R}^n)\), for any \(\sigma_i \in CS(\mathbb{R}^n), i = 1, \cdots, k\) the map \(z \mapsto \int_{\mathbb{R}^k^n} \mathcal{R}(\sigma_1)(z_1) \otimes \cdots \otimes \mathcal{R}(\sigma_k)(z_k)\) is meromorphic with simple poles and we have the following factorisation property as an equality of meromorphic functions:

\[ \int_{\mathbb{R}^k^n} \mathcal{R}(\sigma_1)(z_1) \otimes \cdots \otimes \mathcal{R}(\sigma_k)(z_k) = \prod_{i=1}^{k} \int_{\mathbb{R}^n} \mathcal{R}(\sigma_i)(z_i). \]
Definition 4. Given a continuous holomorphic regularisation $\mathcal{R} : \sigma \mapsto \sigma(z)$ the regularised integral $\int^{\mathcal{R}}$ defined on $CS(\mathbb{R}^n)$ extends to a character:

$$\mathcal{I} (\mathcal{R})(\sigma_1 \otimes \cdots \otimes \sigma_k) := \prod_{i=1}^{k} \int_{\mathbb{R}^n} \sigma_i.$$

It coincides with the ordinary integral when the integrands $\sigma_i$ all lie in $L^1(\mathbb{R}^n)$:

$$\int_{\mathbb{R}^n} \sigma_1 \otimes \cdots \otimes \sigma_k = \int_{\mathbb{R}^n} \sigma_1 \otimes \cdots \otimes \sigma_k.$$

Remark 11. Unless the partial sums of the orders of the symbols $\sigma_i$ are non-integer valued or the integral converges, one is to expect that

$$\int_{\mathbb{R}^n} \sigma_1 \otimes \cdots \otimes \sigma_k \neq \lim_{z \to 0} \int_{\mathbb{R}^n} \mathcal{R}(\sigma_1)(z) \otimes \cdots \otimes \mathcal{R}(\sigma_k)(z)$$

since the finite part of a product of meromorphic functions, namely of the maps $z \mapsto \int_{\mathbb{R}^n} \mathcal{R}(\sigma_i)(z)$ with $i \in \{1, \cdots, I\}$, does not generally coincide with the product of the finite parts of these functions.

However, if one insists on setting $z_i = z$ for $i \in \{1, \cdots, I\}$ then one can implement a renormalisation procedure using Birkhoff factorisation to take care of the problem mentioned in the above remark. For this purpose we equip the tensor algebra $\mathcal{T} (CS(\mathbb{R}^n)) := \bigoplus_{k=0}^{\infty} \mathcal{T}_k (CS(\mathbb{R}^n))$ with the ordinary tensor product $\otimes$ and the deconcatanation coproduct:

$$\Delta : \mathcal{T} (CS(\mathbb{R}^n)) \to \bigoplus_{p+q=L} \left( \mathcal{T}^p (CS(\mathbb{R}^n)) \otimes \mathcal{T}^q (CS(\mathbb{R}^n)) \right)$$

$$\sigma_1 \otimes \cdots \otimes \sigma_k \mapsto \sum_{\{i_1, \cdots, i_k\} \subset \{1, \cdots, k\}} (\sigma_{i_1} \otimes \cdots \otimes \sigma_{i_{k'}}) \otimes (\sigma_{i_{k'+1}} \otimes \cdots \otimes \sigma_{i_k})$$

where $\{i_{k'+1}, \cdots, i_k\}$ is the complement in $\{1, \cdots, k\}$ of the set $\{i_1, \cdots, i_{k'}\}$. Let us recall the following well-known results (see e.g. [M]):

Lemma 3. $\mathcal{T}^0 := (\mathcal{T} (CS^{\star+}(\mathbb{R}^n)), \otimes, \Delta)$ is a graded (by the natural grading on tensor products) cocommutative connected Hopf algebra.

Remark 12. [M] This corresponds to the natural structure of cocommutative Hopf algebra on the tensor algebra of any vector space $V$ with the coproduct $\Delta$ given by the unique algebra morphism from $\mathcal{T}(V) \to \mathcal{T}(V) \otimes \mathcal{T}(V)$ such that $\Delta(1) = 1 \otimes 1$ and $\Delta(x) = x \otimes 1 + 1 \otimes x$.

Proof: We use Sweedler’s notations and write in a compact form

$$\Delta \sigma = \sum_{\{\sigma\}} \sigma_{(1)} \otimes \sigma_{(2)}.$$
• The coproduct $\Delta$ is clearly compatible with the filtration.
• The coproduct $\Delta$ is cocommutative for we have $\tau_{12} \circ \Delta = \Delta$ where $\tau_{ij}$ is the flip on the $i$-th and $j$-th entries:

$$\tau_{12} \circ \Delta(\sigma) = \tau_{12} \left( \sum_{(\sigma)} \sigma(1) \otimes \sigma(2) \right)$$

$$= \sum_{(\sigma)} \sigma(2) \otimes \sigma(1)$$

$$= \Delta(\sigma).$$

• The coproduct $\Delta$ is coassociative since

$$(\Delta \otimes 1) \circ \Delta(\sigma) = \sum_{(\sigma)} \left( \sigma(1,1) \otimes \sigma(1,2) \right) \otimes \sigma(2)$$

$$= \sum_{(\sigma)} \sigma(1) \otimes \left( \sigma(2,1) \otimes \sigma(2,2) \right)$$

$$= (1 \otimes \Delta) \circ \Delta(\sigma).$$

• The co-unit $\varepsilon$ defined by $\varepsilon(1) = 1$ is an algebra morphism.
• The coproduct $\Delta$ is compatible with the product $\otimes$.

$$\Delta \circ m.(\sigma \otimes \sigma') = \sum_{(\sigma \otimes \sigma')} \left( \sigma \otimes \sigma' \right)_{(1)} \bigotimes \left( \sigma \otimes \sigma' \right)_{(2)}$$

$$= (m \bigotimes m) \circ \tau_{23} \circ \left[ \left( \sigma(1) \otimes \sigma(2) \right) \bigotimes \left( \sigma'(1) \otimes \sigma'(2) \right) \right]$$

$$= (m \otimes m) \circ \tau_{23} \circ (\Delta \otimes \Delta) \left( \sigma \otimes \sigma' \right).$$

We derive the following meromorphicity result as an easy consequence of Lemma 2:

**Proposition 7.** Given a continuous holomorphic regularisation procedure $\mathcal{R}$ on $\mathcal{H}^0,$ the map

$$\Phi^\mathcal{R} : (\mathcal{H}^0, \otimes) \rightarrow \mathcal{M}(\mathbb{C})$$

$$\sigma_1 \otimes \cdots \otimes \sigma_k \mapsto \int_{\mathbb{R}^{nk}} \mathcal{R}(\sigma_1) \otimes \cdots \otimes \mathcal{R}(\sigma_k),$$

where $\mathcal{M}(\mathbb{C})$ denotes the algebra of meromorphic functions is well defined and induces an algebra morphism on $(\mathcal{H}^0, \otimes).$

A Birkhoff factorisation procedure then yields a complex valued character.
Theorem 3. A continuous holomorphic regularisation procedure $\mathcal{R}$ on $CS(\mathbb{R}^n)$ gives rise to a character on the Hopf algebra $(\mathcal{H}, \otimes)$:

$$\phi^\mathcal{R} : \mathcal{H} \to \mathbb{C},$$

$$\sigma_1 \otimes \cdots \otimes \sigma_k \mapsto \int_{\mathbb{R}^n}^{\mathcal{R}, \text{ren}} \sigma_1 \otimes \cdots \otimes \sigma_k$$

which therefore coincides with the extended regularised integral $\int_{\mathbb{R}^n}^{\mathcal{R}}$ on $\mathcal{T}(CS(\mathbb{R}^n))$.

In particular we have the following multiplicative property:

$$\int_{\mathbb{R}^n}^{\mathcal{R}, \text{ren}} (\sigma_1 \otimes \cdots \otimes \sigma_k) \otimes (\sigma'_1 \otimes \cdots \otimes \sigma'_{k'}) = (\int_{\mathbb{R}^n}^{\mathcal{R}, \text{ren}} \sigma_1 \otimes \cdots \otimes \sigma_k) \cdot (\int_{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}^{\mathcal{R}, \text{ren}} \sigma'_1 \otimes \cdots \otimes \sigma'_{k'}).$$

Proof: Birkhoff factorisation combined with a minimal substraction scheme yields the existence of a character on the connected filtered commutative Hopf algebra $\mathcal{H}$ [M] (Theorem II.5.1)

$$\Phi^\mathcal{R} : (\mathcal{H}, \otimes) \to \text{Hol}(\mathbb{C})$$

corresponding to the holomorphic part in the unique Birkhoff decomposition $\Phi^\mathcal{R} = (\Phi^\mathcal{R})^{-1} \ast \Phi^\mathcal{R}$ of $\Phi^\mathcal{R}$, $\ast$ being the convolution product on the Hopf algebra. Here Hol(\mathbb{C}) is the algebra of holomorphic functions. Its value $\phi^\mathcal{R} := \Phi^\mathcal{R}(0)$ at $z = 0$ yields in turn a character $\phi^\mathcal{R} : (\mathcal{H}, \otimes) \to \mathbb{C}$

$$\phi^\mathcal{R} (\sigma_1 \otimes \cdots \otimes \sigma_k) = \int_{\mathbb{R}^n}^{\mathcal{R}, \text{ren}} \sigma_1 \otimes \cdots \otimes \sigma_k$$

which extends the map given by the ordinary iterated integral. The multiplicativity of these renormalised integrals $\int_{\mathbb{R}^n}^{\mathcal{R}}$ w.r. to tensor products follows from the character property of $\phi^\mathcal{R}$. Since $\int_{\mathbb{R}^n}^{\mathcal{R}, \text{ren}}$ extends in a unique way to a character on $\mathcal{T}(CS(\mathbb{R}^n))$, the character $\int_{\mathbb{R}^n}^{\mathcal{R}, \text{ren}}$ coincides with the afore defined extension $\int_{\mathbb{R}^n}^{\mathcal{R}}$. □

6 Linear constraints in terms of matrices

Adding in linear constraints is carried out introducing matrices. To a matrix $B$ with real coefficients

$$B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1L} \\ \cdots & \cdots & \cdots & \cdots \\ b_{I1} & b_{I2} & \cdots & b_{IL} \end{pmatrix}$$

and symbols $\sigma_i \in CS(\mathbb{R}^n), i = 1, \cdots, I$ we associate the map

$$(\xi_1, \cdots, \xi_L) \mapsto (\sigma_1 \otimes \cdots \otimes \sigma_I) \circ B(\xi_1, \cdots, \xi_L) := \sigma_1 \left( \sum_{l=1}^{L} b_{1l} \xi_l \right) \cdots \sigma_I \left( \sum_{l=1}^{L} b_{Il} \xi_l \right)$$
and we want to investigate the corresponding multiple integral with linear constraints:
\[
\int_{\mathbb{R}^{nL}} (\sigma_1 \otimes \cdots \otimes \sigma_I) \circ B(\xi_1, \cdots, \xi_L) \, d\xi_1 \cdots d\xi_L.
\]

Remark 13. 1. A permutation \( \tau \in \Sigma_I \) on the lines of \( B \) amounts to relabelling the symbols \( \sigma_i \) in the tensor product.

2. A permutation \( \tau \in \Sigma_L \) on the columns of \( B \) amounts to relabelling the variables \( \xi_i \).

Example 6. Take \( I = 3, L = 2 \) and \( \sigma_i(\xi) = \frac{1}{m^2 + |\xi|^2} \) for all \( i = 1, 2, 3 \). Then
\[
\frac{1}{m^2 + |\xi_1|^2} \frac{1}{m^2 + |\xi_1 + \xi_2|^2} \frac{1}{m^2 + |\xi_2|^2} = ((\sigma_1 \otimes \sigma_2 \otimes \sigma_3) \circ B)(\xi_1, \xi_2, \xi_3)
\]
where \( B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \).

Feynman diagrams give rise to integrals with integrands of this type up to the fact that here we omit external momenta; allowing for external momenta would lead to affine constraints, a case which lies out of the scope of this article but which we hope to investigate in forthcoming work. Constraints on the momenta follow from the conservation of momentum as it flows through the diagram and \( L \) corresponds to the number of loops in the diagram.

Given a holomorphic regularisation \( \mathcal{R} : \sigma \mapsto \sigma(z) \), we extend it to \( \tilde{\sigma} \circ B \) with \( \tilde{\sigma} \in \mathcal{T}(CS(\mathbb{R}^n)) \) and \( B \) a matrix by:
\[
\tilde{\mathcal{R}}(\tilde{\sigma})(\tilde{z}) \circ B := (\mathcal{R}(\sigma_1)(z_1) \otimes \cdots \otimes \mathcal{R}(\sigma_k)(z_k)) \circ B \quad \forall \tilde{z} = (z_1, \cdots, z_k) \in \mathbb{C}^k
\]
which we also write \( \tilde{\sigma}(z) \circ B \) for short.

Proposition 8. Let \( \sigma_i \in CS(\mathbb{R}^n) \) of order \( a_i \). Let \( \mathcal{R} \) be a continuous holomorphic regularisation and let for \( i = 1, \cdots, I \), \( \alpha_i(z) \) denote the order of \( \sigma_i(z) \) which we assume is affine \( \alpha_i(z) = \alpha_i'(0)z + a_i \) with real coefficients and such that \( \alpha_i'(0) < 0 \).

If a matrix \( B = (b_{il}) \) of size \( I \times L \) and rank \( L \), the map
\[
\tilde{z} \mapsto \int_{\mathbb{R}^{nL}} \tilde{\mathcal{R}}(\tilde{\sigma}) \circ B(\tilde{z})
\]
is holomorphic on the domain \( D = \{ \tilde{z} \in \mathbb{C}^I, \quad \text{Re}(z_i) > -\frac{a_i}{\alpha_i'(0)} \} \).

Proof: The symbol property of each \( \sigma_i \) yields the existence of a constant \( C \) such that
\[
|\tilde{\sigma}(\tilde{z}) \circ B(\xi_1, \cdots, \xi_L)| \leq C \prod_{i=1}^I \left( \sum_{l=1}^L b_{il}\xi_l \right)^{\text{Re}(\alpha_i(z_i))}
\]
\[
\leq C \prod_{i=1}^I \left( \sum_{l=1}^L b_{il}\xi_l \right)^{\alpha_i'(0)\text{Re}(z_i) + a_i}
\]
where we have set $\langle \eta \rangle := \sqrt{1 + |\eta|^2}$.

We infer that for $\text{Re}(z_i) \geq \beta_i > 0$

$$|\tilde{\sigma}(z) \circ B(\xi_1, \cdots, \xi_L)| \leq \prod_{i=1}^{L} \left( \sum_{l=1}^{L} b_{il} \xi_l \right)^{\alpha_i(0) \beta_i + a_i}.$$  

We claim that the map $(\xi_1, \cdots, \xi_L) \mapsto \left( \sum_{l=1}^{L} b_{il} \xi_l \right)^{\alpha_i(0) \beta_i + a_i}$ lies in $L^1(\mathbb{R}^n_L)$ if $\beta_i > -\frac{a_i + b_i}{\alpha_i(0)}$. Indeed, the matrix $B$ being of rank $L$ by assumption, we can extract an invertible $L \times L$ matrix $D$. Assuming for simplicity (and without loss of generality, since this assumption holds up to permutation of the lines and columns) that it corresponds to the $L$ first lines of $B$ we write:

$$\prod_{i=1}^{L} \left( \sum_{l=1}^{L} b_{il} \xi_l \right)^{\alpha_i(0) \beta_i + a_i} = \prod_{i=1}^{L} \rho_i \circ B(\xi_1, \cdots, \xi_L) \leq \prod_{i=1}^{L} \rho_i \circ D(\xi_1, \cdots, \xi_L)$$

where we have set $\rho_i(\eta) := \langle \eta \rangle^{\alpha_i(0) \beta_i + a_i}$ and used the fact that $\rho_i(\eta) \geq 1$ and $\alpha_i(0) \beta_i + a_i < -n$.

But

$$\int_{\mathbb{R}^n_L} \otimes_{i=1}^{L} \rho_i \circ D = |\det D| \prod_{i=1}^{L} \int_{\mathbb{R}^n} \rho_i$$

converges as a product of integrals of symbols of order $< -n$ so that by dominated convergence, $\tilde{\mathcal{R}}(\tilde{\sigma})(z) \circ B$ lies in $L^1(\mathbb{R}^n_L)$ for any complex number $z \in D$.

On the other hand, the derivative in $z$ of holomorphic symbols have same order as the original symbols (see e.g. [PS]), the differentiation possibly introducing logarithmic terms. Replacing $\sigma_1(z_1), \cdots, \alpha_i(z_i)$ by $\partial_{z_1}^{\alpha_i} \sigma_1(z_1), \cdots, \partial_{z_i}^{\alpha_i} \sigma_1(z_i)$ in the above inequalities, we can infer by a similar procedure that for $\text{Re}(z_i) \geq \beta_i > -\frac{a_i + b_i}{\alpha_i(0)}$, the map $z \mapsto \tilde{\mathcal{R}}(\tilde{\sigma})(z) \circ B$ is uniformly bounded by an $L^1$ function.

The holomorphicity of $z \mapsto \int_{\mathbb{R}^n_L} \tilde{\mathcal{R}}(\tilde{\sigma})(z) \circ B$ then follows. □

As a straightforward consequence, we infer the existence of a meromorphic extension of the map $z \mapsto \int_{\mathbb{R}^n_L} \tilde{\mathcal{R}}(\tilde{\sigma})(z) \circ B$ to the whole plane when $L = 1$.

**Corollary 5.** Let $\sigma_i \in CS(\mathbb{R}^n)$ be of order $a_i$. Let $\mathcal{R}$ be a continuous holomorphic regularisation which sends $\sigma_i$ to $\sigma_i(z)$ of order $\alpha_i(z) = \alpha_i(0)z + a_i$ with real coefficients and such that $\alpha_i(0) < 0$.

Given an invertible matrix $B$ with $L$ columns, the map

$$z \mapsto \int_{\mathbb{R}^n_L} \tilde{\mathcal{R}}(\tilde{\sigma})(z) \circ B := |\det B|^{-1} \int_{\mathbb{R}^n_L} \tilde{\mathcal{R}}(\tilde{\sigma})(z)$$

yields a meromorphic extension to the whole complex plane of the holomorphic map $z \mapsto \int_{\mathbb{R}^n_L} \tilde{\mathcal{R}}(\tilde{\sigma})(z) \circ B$ defined on the domain $D = \{z \in \mathbb{C}^L, \text{Re}(z_i) > -\frac{a_i + b_i}{\alpha_i(0)}, \forall i \in \{1, \cdots, L\}\}$.

**Proof:** Let us set $\tilde{\sigma}(z) := \tilde{\mathcal{R}}(\tilde{\sigma}(z))$ as before. We know from the previous proposition that $z \mapsto \int_{\mathbb{R}^n_L} \tilde{\sigma}(z) \circ B$ defines a holomorphic map on $D = \{z \in \mathbb{C}^L, \text{Re}(z_i) > -\frac{a_i + b_i}{\alpha_i(0)}, \forall i \in \{1, \cdots, L\}\}$.  

29
\begin{equation}
\mathcal{C}, \text{Re}(z_i) = -\frac{\alpha_i + n}{\alpha_i(0)}, \quad \forall i \in \{1, \cdots, I\} \}
\end{equation}
By a change of variable it follows that in that region of the plane
\begin{equation}
\int_{\mathbb{R}^n} \tilde{\sigma}(z) \circ B := |\det B^{-1}| \int_{\mathbb{R}^n} \tilde{\sigma}(z).
\end{equation}
But by the results of the previous sections we know that \( \tilde{z} \mapsto \int_{\mathbb{R}^n} \tilde{\sigma}(\tilde{z}) \) extends to a meromorphic map on the whole complex plane given by a cut-off regularised integral of a tensor product of symbols \( \tilde{z} \mapsto \int_{\mathbb{R}^n} \tilde{\sigma}(\tilde{z}) \). Hence
\begin{equation}
\tilde{z} \mapsto \int_{\mathbb{R}^n} \tilde{\sigma}(\tilde{z}) \circ B := |\det B^{-1}| \int_{\mathbb{R}^n} \tilde{\sigma}(\tilde{z})
\end{equation}
provides a meromorphic extension of the l.h.s. \( \square \)

### 7 Multiple integrals of holomorphic families with linear constraints

Let us now show the existence of meromorphic extensions for integrals \( \tilde{z} \mapsto \int_{\mathbb{R}^n} \tilde{\mathcal{R}}(\tilde{\sigma})(\tilde{z}) \circ B \) built from more general matrices \( B \), where as before \( \mathcal{R} \) is a continuous holomorphic regularisation and \( \tilde{\sigma} := \sigma_1 \otimes \cdots \otimes \sigma_I \in T(CS(\mathbb{R}^n)) \).

The aim of this section is to prove the following result.

**Theorem 4.** Let \( \mathcal{R} : \sigma \mapsto \sigma(z) \) be a holomorphic regularisation procedure on \( CS(\mathbb{R}_+) \) and let \( \xi \mapsto \sigma_i(\xi) := \tau_i(|\xi|) \in CS(\mathbb{R}^n), i = 1, \cdots, I \) be radial polyhomogeneous symbols of order \( \alpha_i \), which are sent via \( \mathcal{R} \) to \( \xi \mapsto \sigma_i(z)(\xi) := \mathcal{R}(\tau_i)(z)(|\xi|) \) of non constant affine order \( \alpha_i(z) = -q z_i + a_i \), for some positive real number \( q \). For any matrix \( B \) of size \( I \times L \) and rank \( L \), the map
\begin{equation}
\tilde{z} \mapsto \int_{\mathbb{R}^n} \tilde{\mathcal{R}}(\tilde{\sigma})(\tilde{z}) \circ B
\end{equation}
which is well defined and holomorphic on the domain \( D = \{ \tilde{z} \in \mathbb{C}^I, \ \text{Re}(z_i) > -\frac{\alpha_i + n}{\alpha_i(0)}, \ \forall i \in \{1, \cdots, I\} \} \) extends to a meromorphic map on the whole complex plane with poles located on a countable set of affine hyperplanes
\begin{equation}
z_{\tau(1)} + \cdots + z_{\tau(i)} = -\frac{a_{\tau(1)} - \cdots - a_{\tau(i)} + \lambda_{\tau,i} + N_0}{q}, i \in \{1, \cdots, I\}, \quad \tau \in \Sigma_I,
\end{equation}
and where \( \lambda_{\tau,i} \in [-n i, 0][\mathbb{Z}] \) depends on the matrix \( B \).

**Remark 14.** An immediate but important consequence of this theorem is the fact that if none of the partial sums of the orders \( \alpha_i \) are integers then the hyperplanes of poles of the map \( \tilde{z} \mapsto \int_{\mathbb{R}^n} \tilde{\mathcal{R}}(\tilde{\sigma})(\tilde{z}) \circ B \) do not contain 0 so that the map is holomorphic in a neighborhood of 0.

Before going to the proof, let us illustrate this result by an example.

**Example 7.** If we choose \( I = 3, L = 2, \sigma_i, i = 1, 2, 3 \), \( \mathcal{R}(\sigma)(z)(\xi) = \sigma(\xi) \xi^{-z} \) (here \( q = 1 \)) with \( \xi := \sqrt{1 + |\xi|^2} \) and \( B \) as in Example 5, this yields back the known fact that the map \( (z_1, z_2, z_3) \mapsto \int_{\mathbb{R}^3} \frac{1}{(z_1 + z_2 + 1)^{1/2} (|z_1 + z_2| + 1)^{1/2} (|z_1| + 1)^{1/2} (|z_2| + 1)^{1/2}} \ d\xi_1 \ d\xi_2 \ d\xi_3 \).
has a meromorphic extension to the plane with poles on hyperplanes defined by equations involving partial sums of the $z_i$'s. Whenever $a_1, a_2, a_3, a_1 + a_2 + a_3, a_1 + a_3, a_1 + a_2 + a_3$ are not integers, the map is holomorphic in a neighborhood of 0.

Setting $z_i = z$ in the above theorem leads to the following result.

**Corollary 6.** Let $R : \sigma \mapsto \sigma(z)$ be a holomorphic regularisation procedure on $CS(\mathbb{R}_+)$ and let $\xi \mapsto \sigma_i(\xi) := \tau_i(|\xi|) \in CS(\mathbb{R}^n), i = 1, \cdots, I$ be radial polyhomogeneous symbols of order $a_i$ which are sent via $\mathcal{R}$ to $\xi \mapsto \sigma_i(z)(\xi) := R(\tau_i)(z)(\xi)$ of non constant affine order $\alpha_i(z) = -qz_i + a_i$, for some positive real number $q$. For any matrix $B$ of size $I \times L$ and rank $L$, the map

$$z \mapsto \int_{\mathbb{R}^{nL}} (R(\sigma_1)(z) \otimes \cdots \otimes R(\sigma_I)(z)) \circ B$$

which is well defined and holomorphic on the domain $D = \{z \in \mathbb{C}, \ \text{Re}(z) > \frac{-a_{\tau(1)} - \cdots - a_{\tau(I)} + \lambda_{\tau,i} + N_0}{q_i}, i \in \{1, \cdots, I\}, \ \tau \in \Sigma_I\}$, where as before $\lambda_{\tau,i} \in [-n_i, 0]$ is an integer depending on the matrix $B$.

In order to prove Theorem 4, we proceed in several steps, first reducing the problem to step matrices $B$, then to symbols of the type $\sigma_i : \xi \mapsto (\xi^2 + 1)^{a_i}$ and finally proving the meromorphicity for such symbols and matrices.

**Step 1: Reduction to step matrices**

We call an $I \times J$ matrix $B$ with real coefficients a step matrix if it fulfills the following condition

$$\exists i_1 < \cdots < i_L \ \text{in} \ \{1, \cdots, I\} \ \text{s.t} \ \ b_{i,i} = 0 \ \text{if} \ i < i_1 \ \text{and} \ b_{i,i} \neq 0. \ (12)$$

**Remark 15.** This condition actually says that the matrix has rank $\geq L$. It is $J = L$ then it has rank $L$.

**Proposition 9.** If Theorem 4 holds for step matrices then it holds for any matrix $B$.

**Proof:**

- Let us first observe that if the result holds for a matrix $B$ then it holds for any matrix $P B Q$ where $P$ and $Q$ are permutation matrices i.e. up to a relabelling of the symbols and the variables. Indeed, a permutation $\tau \in \Sigma_I$ on the lines induced by the matrix $P$ amounts to a relabelling of the symbols; since the statement should hold for all radial symbols, if it holds for $\tilde{\sigma} = \sigma_1 \otimes \cdots \otimes \sigma_I$ then it also holds for $\sigma_{\tau(1)} \otimes \cdots \otimes \sigma_{\tau(I)}$. Hence, if the statement of the theorem holds for a matrix $B$ it also holds for the matrix $P B$.

Assuming the statement of the theorem holds for a matrix $B$, then it also holds for the matrix $B Q$. Indeed, a permutation $\tau \in \Sigma_I$ on the columns
induced by the matrix $Q$ amounts to a relabelling of the variables $\xi_l$.

By Proposition 8 we know that if $B$ has rank $L$ then both the maps $\tilde{z} \mapsto \int_{\mathbb{R}^n} \tilde{R}(\tilde{\sigma})(\tilde{z}) \circ B$ and $\tilde{z} \mapsto \int_{\mathbb{R}^n} \tilde{R}(\tilde{\sigma})(\tilde{z}) \circ BQ$ are well defined and holomorphic on the domain $D = \{ \tilde{z} \in \mathbb{C}^I, \quad \text{Re}(z_i) > -\frac{a_i + n}{\alpha_i(0)}, \quad \forall i \in \{1, \cdots, I\} \}$. By the Fubini property we further have that

$$\int_{\mathbb{R}^n} \tilde{R}(\tilde{\sigma})(\tilde{z}) \circ B = |\det Q| \int_{\mathbb{R}^n} \tilde{R}(\tilde{\sigma})(\tilde{z}) \circ BQ \quad \forall \tilde{z} \in D.$$ 

If by assumption, the r.h.s has a meromorphic extension $\tilde{z} \mapsto \int_{\mathbb{R}^n} \tilde{R}(\tilde{\sigma})(\tilde{z}) \circ BQ$ then so does the l.h.s. have a meromorphic extension

$$\int_{\mathbb{R}^n} \tilde{R}(\tilde{\sigma})(\tilde{z}) \circ B := |\det Q| \int_{\mathbb{R}^n} \tilde{R}(\tilde{\sigma})(\tilde{z}) \circ BQ$$

which moreover has the same pole structure.

• Let $B$ be a non zero matrix. Then there is an invertible matrix $P$ and step matrix $T$ such that $P B^t = T$ where $B^t$ stands for the transpose of $B$. Hence the existence of an invertible matrix $Q = (P^t)^{-1}$ such that $B = T^t Q$. If $B$ has rank $L$ then so does the matrix $T^t$; along the same lines as above, one shows that if the statement of the theorem holds for $T^t$ then it holds for $B$. On the other hand, there are permutation matrices $P$ and $Q$ such that $S := P T^t Q$ is a step matrix for the transpose of a step matrix can be turned into a step matrix by iterated permutations on its lines and columns. If the theorem holds for step matrices then by the first part of the proof, it also holds for $T^t$ and hence for $B$.

\[\square\]

**Step 2: Reduction to symbols** $\sigma_i : \xi \mapsto (\xi^2 + 1)^{a_i}$

Let us first describe the asymptotic behaviour of classical radial symbols.

**Lemma 4.** Given a radial polyhomogeneous symbol $\sigma : \xi \mapsto \tau(|\xi|)$ on $\mathbb{R}^n$, $\tau \in \text{CS}(\mathbb{R}^+)$ of order $a$ there are real numbers $c_j, j \in \mathbb{N}_0$ such that

$$\sigma(\xi) \sim \sum_{j=0}^{\infty} c_j (\xi)^{a-j}$$

where $\sim$ stands for the equivalence of symbols modulo smoothing symbols. Here, as before we have set $|\xi| = \sqrt{1 + \xi^2}$.

**Proof:** A radial polyhomogeneous symbol $\sigma$ on $\mathbb{R}^n$ of order $a$ can be written

$$\sigma(\xi) = \sum_{j=0}^{N-1} \tau_{a-j}(|\xi|) \chi(|\xi|) + \tau^{(N)}(|\xi|)$$

where $N$ is a positive integer, $\tau^{(N)}$ is a polyhomogeneous symbol the order of which has real part no larger than $\text{Re}(a) - N$ and where $\tau_{a-j}$ are positively homogeneous functions of degree $a - j$. $\chi$ is a smooth cut-off function on $\mathbb{R}^+_0$,
which vanishes in a small neighborhood of 1 and is identically 1 outside the unit interval. Setting \( \gamma_{a-j} := \tau_{a-j}(1) \) we write
\[
\tau_{a-j}(\xi) = \gamma_{a-j}(\xi)^{a-j} \chi(|\xi|)
\]
\[
= \gamma_{a-j}(\xi)^2 - 1 \sum_{k_j=0}^{\infty} b_{k_j}(\xi)^{-2k_j}
\]
\[
\sim \gamma_{a-j}(\xi)^{a-j}\chi(\xi) \sum_{k_j=0}^{\infty} c_{k_j}(\xi)^{a-j-2k_j}
\]
where we have set \( c_{k_j} := \gamma_{a-j} b_{k_j} \) for some sequence \( b_{k_j}, k \in \mathbb{N}_0 \) of real numbers depending on \( a \) and \( j \) and used the fact that \( \chi \sim 1 \). Applying this to each \( \tau_{a-j} \) yields for any positive integer \( N \), the existence of a symbol \( \hat{\chi}^{(N)}(|\xi|) \) the order of which has real part no larger than Re(\( \alpha \)) and constants \( \tilde{c}_j \) such that
\[
\sigma(\xi) = \sum_{j=0}^{N-1} \tilde{c}_j \xi^{a-j} + \hat{\chi}^{(N)}(|\xi|)
\]
which ends the proof of the lemma. \( \square \)

Let \( \xi \mapsto \sigma_i(\xi) := \tau_i(\xi), \ldots, \xi \mapsto \sigma_f(\xi) := \tau_f(\xi) \) be radial polyhomogeneous symbol on \( \mathbb{R}^n \) of order \( a_1, \ldots, a_f \) respectively which we write
\[
\sigma_i(x_i) = \sum_{j_i=0}^{N_i-1} \tau_{a_i - j_i}(|x_i|) \gamma_i^{(N_i)}(|x_i|) \chi(|x_i|)
\]
\[
= \sum_{j_i=0}^{N_i-1} c_{j_i} x_i^{a_i-j_i} + \hat{\gamma}_i^{(N_i)}(|x_i|)
\]
where \( N_i, i = 1, \ldots, I \) are positive integers, \( \tau_{a_i - j_i}, i = 1, \ldots, I \) are homogeneous functions of degree \( a_i - j_i \), \( \gamma_i^{(N_i)}, i = 1, \ldots, I \) polyhomogeneous symbols of order with real part no larger than \( a_i - N_i \) and where we have set \( c_{j_i} := \tau_{a_i - j_i}(1), i = 1, \ldots, I \).

It follows that
\[
\prod_{i=1}^{I} \sigma_i(\xi_i) = \lim_{N \to \infty} \prod_{j_i=0}^{N_i-1} \sum_{j_i=0}^{N_i-1} c_{j_i}^{(N_i)} \langle \xi_i \rangle^{a_i-j_i} \cdot \hat{\gamma}_i^{(N_i)}(\xi_i)
\]
in the Fréchet topology on symbols of constant order \(^{12}\)

**Proposition 10.** If Theorem 4 holds for symbols \( \sigma_i : \xi \mapsto \xi^{a_i} \) then it holds for all classical radial symbols.

\(^{12}\)This Fréchet topology was described in a footnote in Section 1.
Proof: Let $B$ be an $L \times I$ matrix of rank $L$ and let $\sigma_1, \ldots, \sigma_I$ be radial polyhomogeneous symbols in $CS(\mathbb{R}^n)$ with orders $a_1, \ldots, a_I$ respectively. For each $j_i \in \mathbb{N}, i \in \{1, \ldots, I\}$ we set $\rho_i^j(\xi) := (\xi)^{a_i - j_i}$ and for all multiindices $(j_1, \ldots, j_I)$ we set $\tilde{\rho}^{j_1, \ldots, j_I} := \otimes_{i=1}^I \rho_i^{j_i}$.

Let us first observe that since $\text{Re}(a_i) - j_i \leq \text{Re}(a_i)$, the maps

$$z \mapsto \int_{\mathbb{R}^{nL}} \tilde{R}(\tilde{\rho}^{j_1, \ldots, j_I})(z) \circ B$$

are all well defined and holomorphic on the domain $D = \{z \in \mathbb{C}^I, \ \text{Re}(z_i) > -\frac{a_i + \tilde{\rho}^{j_1, \ldots, j_I}}{\alpha_i(0)}\} \quad \forall i \in \{1, \ldots, I\}$.

Let us assume that the theorem holds for this specific class of symbols. Then using again the fact that $\rho_i^j$ has order $a_i - j_i$ which differs from $a_i$ by a non negative integer, and replacing $a_i$ by $\alpha_i(z_i)$, it follows that these maps extend to meromorphic maps

$$z \mapsto \int_{\mathbb{R}^{nL}} \tilde{R}(\tilde{\rho}^{j_1, \ldots, j_I})(z) \circ B$$

on the whole complex plane with poles $z = (z_1, \ldots, z_I)$ such that

$$\alpha_{\tau(i)}(z_{\tau(1)}) + \cdots + \alpha_{\tau(i)}(z_{\tau(i)}) \in \lambda_{\tau,i} \cdot \mathbb{N}_0, \ \tau \in \Sigma_I,$$

or equivalently with poles located on a countable set of affine hyperplanes

$$z_{\tau(1)} + \cdots + z_{\tau(i)} \in -\frac{\alpha_{\tau(1)} - \cdots - \alpha_{\tau(i)} + \lambda_{\tau,i} \cdot \mathbb{N}_0}{q}, \quad \tau \in \Sigma_I,$$

with $\lambda_{\tau,i} \in [-n, 0]n\mathbb{Z}$ depending on the matrix $B$.

Then by (13) so does the map

$$z \mapsto \int_{\mathbb{R}^{nL}} \tilde{R}(\tilde{\sigma})(z) \circ B$$

extend to a meromorphic map on the complex plane:

$$z \mapsto \int_{\mathbb{R}^{nL}} \tilde{R}(\tilde{\rho}^{j_1, \ldots, j_I})(z) \circ B$$

$$:= \lim_{N \to \infty} \sum_{j_1=0}^{N-1} \cdots \sum_{j_I=0}^{N-1} \epsilon_{j_1}^1 \cdots \epsilon_{j_I}^I \int_{\mathbb{R}^{nL}} \prod_{i=1}^I \epsilon_{j_i}^I \cdot \left( \mathcal{R}(\rho_i^j(z_1)) \cdots \mathcal{R}(\rho_i^j(z_I)) \right) \circ B$$

with the same pole structure. Note that for large enough $j_i$’s, the hyperplanes of poles do not contain the origin and that their distance to the origin then increases as the $j_i$’s further increase. □

Step 3: The case of symbols $\sigma_i : \xi \mapsto (|\xi|^2 + 1)^{\alpha_i}$ and step matrices

We are therefore left to prove the statement of the theorem for an $I \times L$ matrix $B$ with real coefficients which fulfills condition (13) and symbols $\sigma_i : \xi \mapsto (|\xi|^2 + 1)^{\alpha_i}$. As previously observed, such a matrix has rank $L$. 34
Lemma 5. Under assumption (12) on \( B = (b_{i,l}) \) the matrix \( B^* B \) is positive definite. Note that with the notations of (12), we have \( i \geq l \).

Proof: For \( k \in \mathbb{R}^L \) in the kernel of \( B \), we have \( \sum_{l=1}^L b_{i,l} \xi_l = 0 \) for any \( i = 1, \cdots, I \), which applied to \( i = i_L \) yields \( \sum_{l=1}^L b_{i,l} \xi_l = 0 \). But since by assumption \( b_{i,l} = 0 \) for \( l < L \) only the term \( b_{i_L} \xi_l \) remains which shows that \( \xi_l = 0 \). Proceeding inductively yields the positivity of \( B^* B \). □

Proposition 11. Let \( B := (b_{i,l})_{i=1, \cdots, I; l=1, \cdots, L} \) be a matrix with property (12). The map

\[
(a_1, \cdots, a_I) \mapsto \int_{(\mathbb{R}^L)^I} \prod_{i=1}^I \left( \sum_{l=1}^L b_{i,l} \xi_l^{a_l} \right) d\xi_1 \cdots d\xi_L,
\]

which is holomorphic on the domain \( D := \{ a = (a_1, \cdots, a_I) \in \mathbb{C}^I, \text{Re}(a_i) < -n, \forall i \in \{1, \cdots, I\} \} \), has a meromorphic extension to the complex plane

\[
(a_1, \cdots, a_I) \mapsto \int_{(\mathbb{R}^L)^I} \prod_{i=1}^I \left( \sum_{l=1}^L b_{i,l} \xi_l^{a_l} \right) d\xi_1 \cdots d\xi_L
\]

\[
:= \frac{1}{\prod_{i=1}^I \Gamma(-a_i/2)} \sum_{\tau \in \Sigma_I} \prod_{i=1}^I \left( (a_{\tau(1)} + \cdots + a_{\tau(i)} + n s_{\tau,i}) \cdots (a_{\tau(1)} + \cdots + a_{\tau(i)} + n s_{\tau,i} - 2m_i) \right)
\]

for some holomorphic map \( H_{\tau, \underline{m}} \) on the domain \( \cap_{i=1}^I \{ \text{Re}(a_{\tau(1)} + \cdots + a_{\tau(i)}) + 2m_i < -ns_{\tau,i} \} \), with \( \tau \in \Sigma_I \) and \( \underline{m} := (m_1, \cdots, m_I) \) a multiindex of non negative integers. The \( s_{\tau,i} \leq i \)'s are positive integers which depend on the permutation \( \tau \), on the size \( L \times I \) and shape (i.e. on the \( i \)'s) of the matrix but not on the actual coefficients of the matrix.

The poles of this meromorphic extension lie on a countable set of affine hyperplanes \( a_{\tau(1)} + \cdots + a_{\tau(i)} \in a_{\tau,i} + \mathbb{N}_0 \) with \( \tau \in \Sigma_I, \ i \in \{1, \cdots, I\} \), \( a_{\tau,i} := -n s_{\tau,i} \in [-n, 0] \cap \mathbb{Z} \).

The proof, which is rather technical and lengthy is postponed to the Appendix. It closely follows Speer's proof [57] which uses iterated Mellin transforms and integrations by parts.

8 Renormalised multiple integrals with constraints

Let us consider the set

\[ A_I := \{ (\sigma_1 \otimes \cdots \otimes \sigma_I) \circ B, \ \sigma_i \in CS_{\text{rad}}(\mathbb{R}^n), \ B \in \mathcal{M}_{I,L}(\mathbb{R}), \ \text{rk} B = L, \ L \in \mathbb{N} \}, \]

where \( CS_{\text{rad}}(\mathbb{R}^n) \) stands for the algebra of classical radial symbols \( \xi \mapsto \tau(|\xi|) \) with \( \tau \in CS(\mathbb{R}^n) \), \( \mathcal{M}_{I,L}(\mathbb{R}) \) for the set of matrices of size \( I \times L \) with coefficients in \( \mathbb{R} \). The map

\[ A_I \times A_I' \rightarrow A_{I+I'}, \quad (\tilde{\sigma} \circ B) \times (\tilde{\sigma}' \circ B') \rightarrow (\tilde{\sigma} \circ B) \circ (\tilde{\sigma}' \circ B') := (\sigma \otimes \sigma') \circ (B \oplus B'), \]

where \( \oplus \) stands for the Whitney sum:

\[ B \oplus B' := \begin{pmatrix} B & 0 \\ 0 & B' \end{pmatrix} \]
induces a morphism of filtered algebras on $A := \bigcup_{I=1}^{\infty} A_I$.

Let us also introduce the set

$$B_I := \{ f : \mathbb{C}^I \to \mathbb{C}, \text{ s.t } \exists (m_1, \ldots, m_I) \in \mathbb{N}_0^I, \text{ the map}$$

$$\left( z_1, \ldots, z_I \right) \mapsto f(z_1, \ldots, z_I) \prod_{\tau \in \Sigma_I} \left( \prod_{i=1}^{I} (z_{\tau(i)} + \cdots + z_{\tau(i)})^{m_i} \right)$$

is holomorphic around $z = 0 \}$, \hspace{1cm} (16)

then $B := \bigcup_{I=1}^{\infty} B_I$ is a filtered algebra for the ordinary product of functions.

The following proposition is an easy consequence of Theorem 4.

**Proposition 12.** Let $R : \sigma \mapsto R(\sigma(z))$ be a holomorphic regularisation procedure on $CS(\mathbb{R}_+)$ which sends a symbol $\tau$ of order $a$ to $R(\tau)(z)$ of non constant affine order $-qz + a$, for some positive real number $q$. The map:

$$\Phi_R : A \to B$$

$$(\tilde{\sigma} \circ B) \mapsto \left( z \mapsto \int_{\mathbb{R}^n \times \mathbb{R}^I} R(\tilde{\sigma})(z) \circ B \right),$$

is a morphism of algebras.

**Proof:** It follows from Theorem 4 that if $\tilde{\sigma} \circ B$ lies in $A_I$ then the map $z \mapsto \int_{\mathbb{R}^n \times \mathbb{R}^I} R(\tilde{\sigma})(z) \circ B$ lies in $B_I$. The factorisation property w.r.to the product:

$$\Phi_R [ (\tilde{\sigma} \circ B) \ast (\tilde{\sigma}' \circ B') ] = \Phi_R (\tilde{\sigma} \circ B) \ast \Phi_R (\tilde{\sigma}' \circ B')$$

then follows by analytic continuation from the corresponding factorisation property on the domain of holomorphicity. \hfill \Box

### 8.1 Renormalisation via generalised evaluators

With the help of the morphism $\Phi_R$, we now build a character $\phi^R : A \to \mathbb{C}$ which boils down to building renormalised integrals which factorise on disjoint sets of constraints. Generalised evaluators (see e.g. [Sp]) at 0 provide an adequate procedure to extract "multiplicative" finite parts at 0 of meromorphic functions in a filtered algebra of the type $F = \bigcup_{I=1}^{\infty} F_I$ with:

$$F_I := \{ f : \mathbb{C}^k \to \mathbb{C}, \exists (m_1, \ldots, m_I) \in \mathbb{N}_0^I, \text{ s.t } \text{ the map}$$

$$\left( z_1, \ldots, z_I \right) \mapsto f(z_1, \ldots, z_I) \prod_{\tau \in \Sigma_I} \left( \prod_{i=1}^{I} (L^k_i(z_{\tau(i)})\cdots z_{(I)})^{m_i} \right)$$

is holomorphic around $z = 0 \}$ \hspace{1cm} (17)

where $L^k_1, \ldots, L^k_I$ are linear forms $L^k_i(z) = \sum_{j=1}^{I} a_{ij} z_j, i \in \{1, \ldots, I\}$ such that the matrix $(a_{ij}^I)$ can be embedded in the upper left corner of the matrix $(a_{ij}^{I+1})$.

**Example 8.** $B = \bigcup_{I \in \mathbb{N}} B_I$ with $B_I$ defined in (16) is such a filtered algebra with $L^k_i(z) = z_1 + \cdots + z_i$. 

36
Remark 16. Such filtered algebras are stable under holomorphic reparametterisations \(z \mapsto \kappa(z)\) such that \(\kappa(0) = 0\) and \(\kappa'(0) \neq 0\), i.e.
\[
f_I \in \mathcal{F}_I \Rightarrow f_I \circ \kappa^I \in \mathcal{F}_I.
\]
Indeed, the pole part \(\frac{1}{L(z_1, \ldots, z_I)}\) corresponding to a linear form \(L(z_1, \ldots, z_I) = \sum_{i=1}^I a_i z_i\) transforms to
\[
a_1 \kappa(z_1) + \cdots + a_I \kappa(z_I) = \frac{1}{\kappa'(0) \left( \sum_{i=1}^I (z_i^2 + o(z_i^2)) \right)} a_1 \left( z_1 + \frac{\kappa''(0)}{2\kappa'(0)} z_1^2 + o(z_1^2) \right) + \cdots + a_I \left( z_I + \frac{\kappa''(0)}{2\kappa'(0)} z_I^2 + o(z_I^2) \right)
\]
which is a meromorphic map with poles of the same type.

Definition 5. (see e.g. [Sp]) A generalised evaluator at 0 on the filtered algebra \(\mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{F}_I\) is a family of maps \(\mathcal{E} = \{E_I, I \in \mathbb{N}\}, E_I : \mathcal{F}_I \rightarrow \mathbb{C}\) such that
1. \(E\) is linear,
2. \(E\) coincides with the evaluation at 0 on analytic functions around 0,
3. \(E\) is continuous for the uniform convergence of analytic functions,
4. \(E\) is symmetric in the variables \(z_i\)’s,
5. \(E\) is compatible with the filtration on \(\mathcal{F}\),
6. \(E\) is multiplicative on tensor products:
\[
E_{I+I'}(f \otimes f') = E_I(f) E_{I'}(f')
\]
for any \(f \in \mathcal{F}_I\) depending only on the first \(I\) variables \(z_1, \ldots, z_I\), \(f' \in \mathcal{F}_{I'}\) on the remaining \(I'\) variables \(z_{I+1}, \ldots, z_{I+I'}\).

The map \(E\) on \(\mathcal{F}\) defined on \(\mathcal{F}_I\) by:
\[
E^0_I(f) := \frac{1}{I!} \sum_{\tau \in \Sigma_I} \text{fp}_{z_{\tau(I)}} = 0 \left( \cdots \left( \text{fp}_{z_{\tau(I)}} = 0 f(z) \right) \cdots \right)
\]
yields a generalized evaluator \(E^0\) at 0 on \(\mathcal{F}\).
A holomorphic reparametrisation \(z \mapsto \kappa(z)\) such that \(\kappa(0) = 0\) and \(\kappa'(0) \neq 0\) induces another evaluator \(E^\kappa\) defined on \(\mathcal{F}_k\) by
\[
E^\kappa_I(f) := E^0_I(f_I \circ \kappa^I)
\]
since \(f_k \in \mathcal{F}_I \Rightarrow f_I \circ \kappa^I \in \mathcal{F}_k\) by the above remark.
In general, \(E^\kappa \neq E^0\) as the following example shows.
**Example 9.** Note that

\[ F_1 = \{ f : \mathbb{C} \rightarrow \mathbb{C}, \ \exists m \in \mathbb{N}_0, \ \text{s.t.} \ z \mapsto f(z) z^m \ \text{is holomorphic around 0} \} \]

corresponds to functions in one variable meromorphic in a neighborhood of 0 with poles at \( z = 0 \). The evaluator \( E^0 \) on \( F_1 \) applied to \( f(z) = \sum_{i=1}^{m} i z^i + o(z) \) singles out the finite part \( E^0_1(f) = \lim_{z \to 0} \left( f(z) - \sum_{i=1}^{m} i z^i \right) \). When applied to \( f \circ \kappa(z) \) it picks up extra contributions since

\[
\frac{1}{(\kappa(z))^i} = \frac{1}{(\kappa'(0))^i z^i \left( 1 + \frac{\kappa''(0)}{2\kappa'(0)} z + o(z) \right)^i} = \frac{1}{(\kappa'(0))^i z^i} + \frac{1}{(\kappa'(0))^i} \sum_{j=1}^{i} \alpha_j \kappa^j \cdot \frac{z^j + o(z^j)}{(\kappa'(0))^j z^j} \quad \text{for some } (\alpha_1, \ldots, \alpha_i) \in \mathbb{C}^i
\]

which in turn implies that \( E^0_1(f \circ \kappa) = E^0_1(f) + \delta_{ij} \frac{\alpha_j}{(\kappa'(0))^j} \).

A change of variable \( T_k : (z_1, \ldots, z_k) \mapsto T_k(z_1, \ldots, z_k) \) with \( T = \{ T_k \in \text{GL}_k(\mathbb{C}), k \in \mathbb{N} \} \) a family of matrices nested in one another i.e. such that the matrix \( T_k \) can be embedded in the upper left corner of the matrix \( T_{k+1} \), gives rise to another evaluator \( E^T \) defined on \( F_k \) by

\[
E^T_k(f) := E^0_k(f \circ T_k).
\]

**Remark 17.** Clearly, \( E^T_k(f) := E^0_k((f_1 \circ T_1)^{(k)}) \) for any \( f \in F \) since the finite part at \( z = 0 \) of a meromorphic function \( z \mapsto f(z) \) around zero is insensitive to a linear transformation \( z \mapsto a z \) with \( a \neq 0 \).

However, in general, \( E^T \neq E^0 \) as the following example shows.

**Example 10.** The evaluator \( E^0 \) at 0 on \( B_2 \) applied to the map \( f : (z_1, z_2) \mapsto \frac{z_1 + z_2}{z_1 + z_2} \) yields:

\[
E^0_2(f) = \frac{\text{fp}_{z_2=0} \left( \text{fp}_{z_1=0} \left( 1 + \frac{z_2}{z_1} \right) \right) + \text{fp}_{z_2=0} \left( \text{fp}_{z_2=0} \left( 1 + \frac{z_2}{z_1} \right) \right)}{2} = 1.
\]

In contrast, the evaluator \( E^T \) corresponding to maps \( T_k(z_1, z_2, \ldots, z_k) = (z_1, z_2 - z_1, \ldots, z_k - z_{k-1}) \) yields

\[
E^T_2(f) = E^0_2((f \circ T_2)(f)) = \frac{\text{fp}_{u_2=0} \left( \text{fp}_{u_1=0} \frac{u_2}{u_1} \right) + \text{fp}_{u_1=0} \left( \text{fp}_{u_2=0} \frac{u_2}{u_1} \right)}{2} = 0.
\]

Combining the two types of transformations on evaluators, yields a family \( E^{\kappa \cdot T} \) of evaluators defined on \( F_k \) by:

\[
E^{\kappa \cdot T}_k(f) := E^0_k(f \circ T_k \circ \kappa).
\]
Remark 18. This raises the question whether such evaluators linearly span all evaluators.

Theorem 5. Let \( \mathcal{R} : \sigma \mapsto \sigma(z) \) be a holomorphic regularisation procedure on \( CS(\mathbb{R}_+^n) \) which sends a symbol \( \tau \) of order \( a \to \mathcal{R}(\tau)(z) \) of non constant affine order \(-aq_i + a\), for some positive real number \( q \) and let \( \mathcal{E} \) be a generalised evaluator at 0 on the algebra \( \mathcal{B} \) of meromorphic maps then the map \( \varphi^{\mathcal{R}, \mathcal{E}} : \mathcal{E} \circ \Phi^{\mathcal{R}} : \)

\[
\varphi^{\mathcal{R}, \mathcal{E}} : \mathcal{A} \to \mathbb{C}
\]

\[
\tilde{\sigma} \circ B \mapsto \mathcal{E} \circ \int_{\mathbb{R}^{nL}} \tilde{\sigma} \circ B := \mathcal{E} \circ \int_{\mathbb{R}^{nL}} \mathcal{R}(\tilde{\sigma}) \circ B,
\]

is a character. Whenever \( \tilde{\sigma} = \sigma_1 \otimes \cdots \otimes \sigma_l \) with \( \sigma_i \) of order \( a_i \) with real part \( < - n \) then \( \mathcal{E} \circ \int_{\mathbb{R}^{nL}} \mathcal{R}(\tilde{\sigma}) \circ B \) coincides with the ordinary integral \( \int_{\mathbb{R}^{nL}} \sigma \circ B \).

Proof: The multiplicativity easily follows from combining the multiplicative properties of the morphism \( \Phi^{\mathcal{R}} \) and the evaluator \( \mathcal{E} \). The fact that it coincides with the ordinary integral \( \int_{\mathbb{R}^{nL}} \tilde{\sigma} \circ B \) when \( \sigma_i \) has order \( a_i \) with real part \( < - n \), follows from the fact that the map \( \Phi^{\mathcal{R}} \) is then holomorphic around 0 combined with the fact that evaluators at \( \mathbb{Z}_n \) on holomorphic functions around a point \( \mathbb{Z}_n \) indeed boil down to evaluating the function at the point \( \mathbb{Z}_n \). □

8.2 Renormalisation via Birkhoff factorisation

We now give an alternative renormalisation procedure for multiple integrals of symbols with linear constraints in the case of equal symbols \( \sigma_1 = \sigma \) with \( \sigma \) some fixed classical radial symbol. The only freedom left is the choice of the matrix \( B \) corresponding to the linear constraints. Following Connes and Kreimer [CM], we carry out this renormalisation via a Birkhoff factorisation on a Hopf algebra (here a Hopf algebra of matrices plays the role of their Hopf algebra of Feynman diagrams) with the help of a morphism on this algebra with values in meromorphic maps.

We first introduce a Hopf algebra of matrices. Let \( \mathcal{H}_L := \{ B \in M_{L,L}(\mathbb{R}) \mid \text{rk}B = L, \quad I \in \mathbb{N} \} \) then \( \mathcal{H} = \bigcup_{L \in \mathbb{N}} \mathcal{H}_L \) is filtered by the rank of the matrix; if \( B \) has rank \( L \) and \( B' \) has rank \( L' \) then \( B \oplus B' \) has rank \( L + L' \).

Writing a matrix \( B = (b_{ij})_{i=1,\ldots,L;j=1,\ldots,L} \) in terms of its column vectors \( B = [C_1,\cdots,C_L] \), where \( C_i = (b_{ij})_{i=1,\ldots,L} \), we can equip \( \mathcal{H} \) with the following co-product which boils down to a deconcatenation coproduct on column vectors:

\[
\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}
\]

\[
[C_1,\cdots,C_L] \mapsto \sum_{\{l_1,\cdots,l_p\} \subset \{1,\cdots,L\}} [C_{l_1},\cdots,C_{l_p}] \otimes [C_{l_{p+1}},\cdots,C_{l_{p+q}}]
\]

where we have set \( L = p+q \) so that \( \{1,\cdots,L\} \) is the disjoint union of \( \{1,\cdots,l_p\} \) and \( \{l_{p+1},\cdots,l_{p+q}\} \).

Proposition 13. \( (\mathcal{H}, \oplus, \Delta) \) is a graded cocommutative Hopf algebra.

Proof: We use Sweedler’s notations and write in a compact form

\[
\Delta B = \sum_{(B)} B_{(1)} \otimes B_{(2)}.
\]
The coproduct $\Delta$ is compatible with the filtration since it sends $H_L$ to $\bigoplus_{p+q=L} H_p \otimes H_q$.

The product given by the Whitney sum $\oplus$ is not commutative since one does not expect $B \oplus B'$ to coincide with $B' \oplus B$ for any two matrices $B$ and $B'$.

The product is clearly associative $(B \oplus B') \oplus B'' = B \oplus (B' \oplus B'')$ for any three matrices $B, B', B''$.

The coproduct $\Delta$ is clearly cocommutative since $\tau_{12} \circ \Delta(B) = \Delta(B)$.

The coproduct $\Delta$ is coassociative since

$$(\Delta \otimes 1) \circ \Delta(B) = \sum (B_{(1:1)} \otimes B_{(1:2)}) \otimes B_{(2)}$$

$$= \sum (B_{(1)} \otimes (B_{(2:1)} \otimes B_{(2:2)})$$

$$= (1 \otimes \Delta) \circ \Delta(B).$$

The coproduct $\Delta$ is compatible with the Whitney sum.

$$\Delta \circ m(B \oplus B') = \Delta(B \oplus B')$$

$$= \sum_{(B \oplus B')_{(1)}} (B \oplus B')_{(2)}$$

$$= (m \otimes m) \circ \tau_{23} \left[(B_{(1)} \otimes B_{(2)}) \otimes (B'_{(1)} \otimes B'_{(2)})\right]$$

$$= (m \otimes m) \circ \tau_{23} \circ (\Delta \otimes \Delta)(B \oplus B').$$

\[\square\]

With the help of meromorphic extensions of integrals of holomorphic radial symbols with linear constraints built in the previous section, we build a morphism from $\mathcal{H}$ into the algebra of meromorphic functions. The following lemma follows from Corollary 6.

**Lemma 6.** Let $\mathcal{R} : \sigma \mapsto \sigma(z)$ be a holomorphic regularisation procedure on $CS(\mathbb{R}^+_1)$ and let $\sigma$ be a radial classical symbol of order $a$ which is sent via $\mathcal{R}$ to $\sigma(z)$ of non constant affine order $\alpha(z)$. The map

$$\Phi^R,\sigma_L : \mathcal{H}_L \mapsto \int_{\mathbb{R}^{nL}} \prod_{i=1}^J \sigma(z) \circ B(\xi_1, \ldots, \xi_L) d\xi_1 \cdots d\xi_L$$

yields a morphism of algebras

$$\Phi^R,\sigma : \mathcal{H}_L \mapsto \text{Mer}(\Phi)$$

$$B \mapsto \Phi^R,\sigma_L(B)$$

i.e.

$$\Phi^R,\sigma(B \oplus B') = \Phi^R,\sigma(B) \Phi^R,\sigma(B') \quad \forall (B, B') \in \mathcal{H}^2.$$
The following theorem then follows from Birkhoff factorisation combined with a minimal substraction scheme along the lines of a general procedure described in [M] (Theorem II.5.1).

**Theorem 6.** Let $R$ be a continuous holomorphic regularisation on $CS(\mathbb{R}^n)$ which sends a symbol of order $a$ to a symbol of order $\alpha(z) = -qz + a$ for some $q > 0$ and let $\sigma \in CS(\mathbb{R}^n)$ be a radial symbol. The map $\phi^{R,\sigma} := \Phi_R,\sigma(0)$ is a character

$$\phi^{R,\sigma} : \mathcal{H} \mapsto \mathbb{C}$$

$$B \mapsto \int_{\mathbb{R}^n,L} \sigma^\otimes I \circ B$$

where $\Phi_R,\sigma = (\Phi_R,\sigma)^{-1} \ast \Phi^+_R,\sigma$ is the unique Birkhoff decomposition of $\Phi^{R,\sigma}$, $\ast$ being the convolution product on the Hopf algebra.

When $\sigma$ has order with real part $< -n$, the renormalised integral $\int_{\mathbb{R}^n,L} \sigma^\otimes I \circ B$ coincides with the ordinary integral $\int_{\mathbb{R}^n} \sigma \otimes I \circ B$.

**Proof:** By the very construction of the birkhoff factorised morphism, the multiplicativity of $\phi^{R,Birk}$ follows from the multiplicative property of the morphism $\Phi^{R,\sigma}$. The fact that the resulting renormalised integral coincides with the ordinary integral $\int_{\mathbb{R}^n,L} \sigma^\otimes I \circ B$ when $\sigma$ has order with real part $< -n$ follows from the fact that the map $\Phi^{R,\sigma}$ is then holomorphic around 0 so that $\phi^+_R,\sigma = \phi^{R,\sigma}$. $\square$

### 8.3 Properties of renormalised multiple integrals with constraints

By construction, both renormalised multiple integrals of symbols with constraints given by some matrix $B \in M_{I,L}$, namely $\int_{\mathbb{R}^n,L} \sigma_1 \otimes \cdots \otimes \sigma_I \circ B$ obtained using evaluators, resp. $\int_{\mathbb{R}^n,L} \sigma^\otimes I \circ B$ obtained using Birkhoff factorisation

- factorise over disjoint sets of constraints:

  $$\int_{\mathbb{R}^n(L+L')}^{\mathcal{R},E} (\tilde{\sigma} \otimes \tilde{\sigma}') \circ (B \otimes B') = \left( \int_{\mathbb{R}^n,L}^{\mathcal{R},E} \tilde{\sigma} \circ B \right) \left( \int_{\mathbb{R}^n,L'}^{\mathcal{R},E} \tilde{\sigma} \circ B' \right)$$

  $$\int_{\mathbb{R}^n(L+L')}^{\mathcal{R},Birk} (\sigma^\otimes I \otimes (\sigma')^\otimes I') \circ (B \otimes B') = \left( \int_{\mathbb{R}^n,L}^{\mathcal{R},Birk} \sigma^\otimes I \circ B \right) \cdot \left( \int_{\mathbb{R}^n,L'}^{\mathcal{R},Birk} (\sigma')^\otimes I' \circ B' \right).$$

  Here, $B \in M_{I,L}(\mathbb{R})$ and $B' \in M_{I',L'}(\mathbb{R})$.

- coincide with the corresponding ordinary integrals with constraints when the integrands lie in $L^1$:

  $$\sigma_i \in L^1(\mathbb{R}^n) \forall i \in \{1, \cdots, I\} \Rightarrow \int_{\mathbb{R}^n,L}^{\mathcal{R},E} (\sigma_1 \otimes \cdots \otimes \sigma_I) \circ B = \int_{\mathbb{R}^n,L}^{\mathcal{R},E} (\sigma_1 \otimes \cdots \otimes \sigma_I) \circ B$$

  $$\int_{\mathbb{R}^n,L}^{\mathcal{R},Birk} \sigma^\otimes I \circ B = \int_{\mathbb{R}^n,L}^{\mathcal{R},Birk} \sigma^\otimes I \circ B$$  

  (22)
The following theorem shows that they moreover fulfill a covariance property and hence obey a Fubini property.

**Theorem 7.** Let \( R : \sigma \mapsto \sigma(z) \) be a holomorphic regularisation procedure on \( CS(I \mathbb{R}^+) \) which sends a symbol \( \tau \) of order \( a \) to \( R(\tau)(z) \) of non constant affine order \( -aqz + a \), for some positive real number \( q \) and let \( E \) be a generalised evaluator at \( 0 \) on the algebra \( B \) of meromorphic maps.

For any \( B \in M_{L,L}(\mathbb{R}) \) of rank \( L \), for any matrix \( C \in GL_L(\mathbb{R}) \) and any radial classical symbols \( \sigma_1, \ldots, \sigma_I \) on \( I \mathbb{R}^n \) we have

\[
\int_{\mathbb{R}^{nL}} R(\sigma_1 \otimes \cdots \otimes \sigma_I) \circ B \circ C = |\det C|^{-1} \int_{\mathbb{R}^{nL}} R(\sigma_1 \otimes \cdots \otimes \sigma_I) \circ B
\]

resp.

\[
\int_{\mathbb{R}^{nL}} R, \text{Birk} (\sigma^\otimes \circ B) \circ C = |\det C|^{-1} \int_{\mathbb{R}^{nL}} \sigma^\otimes \circ B.
\]

As a result, they obey a Fubini type property:

\[
\int_{\mathbb{R}^{nL}} R(\sigma_1 \otimes \cdots \otimes \sigma_I) \circ B(\xi_{\rho(1)}, \ldots, \xi_{\rho(L)}) = \int_{\mathbb{R}^{nL}} R, \text{Birk} (\sigma_1 \otimes \cdots \otimes \sigma_I) \circ B(\xi_1, \ldots, \xi_L)
\]

resp.

\[
\int_{\mathbb{R}^{nL}} R, \text{Birk} (\sigma_1 \otimes \cdots \otimes \sigma_I) (B(\xi_{\rho(1)}, \ldots, \xi_{\rho(L)})) = \int_{\mathbb{R}^{nL}} (\sigma_1 \otimes \cdots \otimes \sigma_I) (B(\xi_1, \ldots, \xi_L)) \quad \forall \rho \in \Sigma_L.
\]

**Proof:** The Fubini property follows from the covariance property choosing \( C \) to be a permutation matrix.

Covariance follows by analytic continuation from the usual covariance property of the ordinary integral; indeed this leads to the following equalities of meromorphic maps

\[
|\det C| \int_{\mathbb{R}^{nL}} (R(\sigma_1)(z_1) \otimes \cdots \otimes R(\sigma_I)(z_I)) \circ B \circ C = \int_{\mathbb{R}^{nL}} (R(\sigma_1)(z_1) \otimes \cdots \otimes R(\sigma_I)(z_I)) \circ B
\]

resp.

\[
|\det C| \int_{\mathbb{R}^{nL}} (R(\sigma)(z))^\otimes \circ B \circ C = \int_{\mathbb{R}^{nL}} (R(\sigma)(z))^\otimes \circ B.
\]

Applying a generalised evaluator \( E \) to either side of the first equality or implementing Birkhoff factorisation to the morphisms arising on either side of the second equality leads to the two identities of (23). \( \Box \)
Appendix : Proof of Proposition 11

To simplify notations, we set \( q_i(\xi) := \sum_{l=1}^{L} b_{il} \xi_l \) where \( \xi := (\xi_1, \ldots, \xi_L) \) and \( b_i = -a_i \). For \( \text{Re}(b_i) \) chosen sufficiently large, we write

\[
\frac{1}{\Gamma(b_1/2) \cdots \Gamma(b_L/2)} \int_0^\infty e^{\frac{1}{\xi_1} \cdot \cdots \cdot \frac{1}{\xi_L} - \frac{1}{\xi_1} \cdots \frac{1}{\xi_L}} e^{-\sum_{i=1}^{L} \epsilon_i (q_i(\xi))^2} d\xi_1 \cdots d\xi_L
\]

and

\[
\sum_{i=1}^{L} \epsilon_i q_i(\xi) = 0 \Rightarrow q_i(\xi) = 0 \quad \forall i \in \{1, \ldots, I\}
\]

where \( \xi_l \cdot \xi_m \) stands for the inner product in \( \mathbb{R}^n \) and where we have set \( \theta(\xi)_{lm} := \sum_{i=1}^{L} \epsilon_i b_{il} b_{im} \).

Since the \( \epsilon_i \) are positive \( \theta(\epsilon) \) is a non negative matrix, i.e. \( \theta(\epsilon)(\xi) \cdot \xi \geq 0 \). It is actually positive definite since

\[
\sum_{l,m=1}^{L} \theta(\xi)_{lm} \xi_l \cdot \xi_m = 0
\]

\[
\Rightarrow \sum_{i=1}^{L} \epsilon_i |q_i(\xi)|^2 = 0 \Rightarrow q_i(\xi) = 0 \quad \forall i \in \{1, \ldots, I\}
\]

\[
\Rightarrow \sum_{i=1}^{L} |q_i(\xi)|^2 = |B\xi|^2 = 0 \Rightarrow \xi = 0,
\]

using the fact that \( B^*B \) is positive definite. The map \( \xi \mapsto \sum_{l,m=1}^{L} \theta(\xi)_{lm} \xi_l \cdot \xi_m \) therefore defines a positive definite quadratic form of rank \( L \).

A Gaussian integration yields \( \int_{(\mathbb{R}^n)^L} e^{-\sum_{i=1}^{L} \epsilon_i |q_i(\xi)|^2} d\xi_1 \cdots d\xi_L = (\det(\theta(\xi)))^{-n/2}. \)

We want to perform the integration over \( \xi \):

\[
\frac{1}{\Gamma(b_1/2) \cdots \Gamma(b_L/2)} \int_0^\infty d\epsilon_1 \cdots d\epsilon_L e^{\frac{1}{\epsilon_1} \cdot \cdots \cdot \frac{1}{\epsilon_L} - \frac{1}{\epsilon_1} \cdots \frac{1}{\epsilon_L}} e^{-\sum_{i=1}^{L} \epsilon_i} \frac{1}{(\det(\theta(\xi)))^{n/2}} e^{-\sum_{i=1}^{L} \epsilon_i}.
\]

Let us decompose the space \( \mathbb{R}^k \) of parameters \( (\epsilon_1, \ldots, \epsilon_L) \) in regions \( D_\tau \) defined by \( \epsilon_{\tau(1)} \leq \cdots \leq \epsilon_{\tau(L)} \) for permutations \( \tau \in \Sigma_L \). This splits the integral \( \int_0^\infty d\epsilon_1 \cdots d\epsilon_L e^{\frac{1}{\epsilon_1} \cdot \cdots \cdot \frac{1}{\epsilon_L} - \frac{1}{\epsilon_1} \cdots \frac{1}{\epsilon_L}} (\det(\theta(\xi)))^{-\frac{n}{2}} e^{-\sum_{i=1}^{L} \epsilon_i} \) into a sum of integrals

\[
\int_{D_\tau} d\epsilon_1 \cdots d\epsilon_L e^{\frac{1}{\epsilon_1} \cdot \cdots \cdot \frac{1}{\epsilon_L} - \frac{1}{\epsilon_1} \cdots \frac{1}{\epsilon_L}} (\det(\theta(\xi)))^{-\frac{n}{2}} e^{-\sum_{i=1}^{L} \epsilon_i}.
\]

Let us focus on the integral over the domain \( D \) given by \( \epsilon_1 \leq \cdots \leq \epsilon_k \); the
results can then be transposed to other domains applying a permutation
$b_i \to a_{i(l)}$ on the $a_i$'s. We write the domain of integration as a union of cones
$0 \leq \epsilon_j \leq \cdots \leq \epsilon_f$. For simplicity, we consider the region $0 \leq \epsilon_1 \leq \cdots \leq \epsilon_f$
on which we introduce new variables $t_1, \cdots, t_I$ setting $\epsilon_i = t_I t_{I-1} \cdots t_i$. These
new variables vary in the domain $\Delta := \prod_{i=1}^{f-1} [0, 1] \times [0, \infty)$. Let us assume that
$b_{it} = 0$ for $i > i_I$, then the $l$-th line of $\theta$ reads
\[
\theta(it)_{im} = \sum_{i=1}^{I} t_I \cdots t_i b_{im} = \sum_{i=1}^{I} t_I \cdots t_i b_{im} = t_I \cdots t_i \left( b_{ii} + \sum_{i=1}^{I-1} t_{i-1} \cdots t_i b_{im} \right)
\]
or equivalently the $m$-the column of $\theta$ reads
\[
\theta(it)_{lm} = \sum_{i=1}^{I} t_I \cdots t_i b_{im} = \sum_{i=1}^{I} t_I \cdots t_i b_{im} = t_I \cdots t_i \left( b_{im} + \sum_{i=1}^{I-1} t_{i-1} \cdots t_i b_{im} \right).
\]
Factorising out $\sqrt{t_I \cdots t_i}$ from the $l$-th row and $\sqrt{t_I \cdots t_i}$ from the $m$-th column
for every $l, m \in \{1, I\}$ produces a symmetric matrix $\tilde{\theta}(l,m)$. In particular
we have factorised out $\tau_I \cdots \tau_i$ in the last expression. Let us as in [Sp]
choose $M = \max \{l, x_i \neq 0\}$; in particular $l > M \Rightarrow x_i = 0$. On the other hand,
since $l < M \Rightarrow i_t < i_M$ we have $l < M \Rightarrow b_{im} = 0$. Choosing $i = i_M$ in
reduces the sum to one term $b_{i_M M x_M}$ which would therefore vanish, leading to
a contradiction since neither $b_{i_M M}$ nor $x_M$ vanish by assumption.

\[\text{Note that a permutation } \tau \in \Sigma_f \text{ on the } a_i \text{'s (and hence the } b_i \text{'s) boils down to a permutation}
on the lines of the matrix } (a_i). \text{ Indeed, for any } \tau \in \Sigma_f
\begin{align*}
    \frac{1}{\Gamma(b_{e_1}) \cdots \Gamma(b_{e_f})} & \int_0^\infty \cdots \int_0^\infty \cdots \int_0^\infty \cdots \int_0^\infty e^{-\sum_{i=1}^k \tau_i(q_i)} e^{-(\sum_{i=1}^k \tau_i(q_i))^2} \\
    & = \frac{1}{\Gamma(b_1) \cdots \Gamma(b_f)} \int_0^\infty \cdots \int_0^\infty \cdots \int_0^\infty \cdots \int_0^\infty e^{-\sum_{i=1}^k \tau_i(q_i)} e^{-(\sum_{i=1}^k \tau_i(q_i))^2} \\
    & = \frac{1}{\Gamma(b_1) \cdots \Gamma(b_f)} \int_0^\infty \cdots \int_0^\infty \cdots \int_0^\infty \cdots \int_0^\infty e^{-\sum_{i=1}^k \tau_i(q_i)} e^{-(\sum_{i=1}^k \tau_i(q_i))^2} \\
    \end{align*}
so that the $q_i$'s which determine the lines of the matrix are permuted.
We thereby conclude that $\det\tilde{\theta}(\xi)$ does not vanish on the domain of integration. Performing the change of variable $(\epsilon_1, \ldots, \epsilon_L) \mapsto (t_1, \ldots, t_L)$ in the integral, which introduces a jacobian determinant $\prod_{i=1}^L t_i^{-1}$, we write the integral:

$$
\frac{1}{\Gamma(b_1/2) \cdots \Gamma(b_L/2)} \int_0^1 dt_1 \cdots \int_0^1 dt_{L-1} \int_0^\infty dt_L \prod_{i=1}^L (t_i \cdots t_L)^{-\frac{b}{2}} \cdot \prod_{i=1}^L (t_i) - e^{-\sum_{i=1}^L t_i} \left(\det\tilde{\theta}(t)\right)^{-n/2}
$$

$$
= \frac{1}{\Gamma(b_1/2) \cdots \Gamma(b_L/2)} \int_0^\infty dt_L \int_0^1 dt_1 \cdots \int_0^1 dt_{L-1} \prod_{i=1}^L (t_i \cdots t_{L-1})^{-\frac{b}{2}} (t_{L-1} \cdots t_{L-2})^{-n/2} \cdots (t_2 \cdots t_1)^{-\frac{b}{2}} h(\xi)
$$

$$
= \frac{1}{\Gamma(b_1/2) \cdots \Gamma(b_L/2)} \int_\Delta dt_L \cdots dt_1 \prod_{i=1}^L (t_i)^{-\frac{b}{2} - \frac{n}{2}} e^{-t_i} h(t)
$$

(27)

where the $s_i$'s are positive integers depending on the size and shape of the matrix $B$ (via the $t_i$'s) \[ \mathbb{N}_0 \] and where we have set

$$
h(\xi) := e^{-\sum_{i=1}^L t_i} \left(\det\tilde{\theta}(t)\right)^{-n/2} = \left(\det\tilde{\theta}(t)\right)^{-n/2} \prod_{i=1}^L e^{-t_i}.
$$

Since $\det\tilde{\theta}(t)$ is polynomial in the $t_i$'s, the convergence of the integral in $t_L$ at infinity is taken care of by the function $e^{-t_i}$ arising in $h$. On the other hand, $h$ is smooth on the domain of integration since it is clearly smooth outside the set of points for which $\det\tilde{\theta}(\xi)$ vanishes, which we saw is a void set. Thus, the various integrals converge at $t_i = 0$ for Re$(b_i)$ sufficiently large.

Integrating by parts with respect to each $t_i$, $\cdots$, $t_L$ introduces factors \[ \frac{1}{b_1 + \cdots + b_L - n s_i + 2m_i}, m_i \in \mathbb{N}_0 \] when taking primitives of $t_i^{-\frac{b}{2} - \frac{n}{2}}$ and differentiating $h(t)$.

We thereby build a meromorphic extension $\int_{\partial \mathbb{B}^L} \prod_{i=1}^L (\sum_{l=1}^{L+1} b_l \xi_l)^{s_i}$ to the whole complex plane as a sum over permutations $\tau \in \Sigma_L$ of expressions:

$$
\frac{1}{\Gamma(b_1) \cdots \Gamma(b_L)} \left( \prod_{i=1}^L \int_{t_i = 0}^{b_i} \left( (b_{\tau(1)} + \cdots + b_{\tau(i)} - n s_{\tau,i}) \cdots (b_{\tau(1)} + \cdots + b_{\tau(i)} - n s_{\tau,i} + 2m_i) \right) + \text{boundary terms} \right)
$$

where the boundary terms on the domain $\Delta$ are produced by the iterated $m_i$ integrations by parts in each variable $t_i$. Here $s_{\tau,i} \leq i$ is a positive integer depending on $\tau$ and the shape of the matrix and we have chosen the $m_i$ sufficiently large for the term $\int_{\partial \mathbb{B}^L} \prod_{i=1}^L t_i^{-\frac{b}{2} - \frac{n}{2}} e^{-t_i} h(t)^{s_i + \cdots + m_i} (\xi)$ to converge. The boundary terms are of the same type, namely they are proportional to

$$
\frac{1}{\Gamma(b_1) \cdots \Gamma(b_L)} \left( (b_{\bar{\tau}(1)} + \cdots + b_{\bar{\tau}(i)} - n s_{\tau,i}) \cdots (b_{\bar{\tau}(1)} + \cdots + b_{\bar{\tau}(i)} - n s_{\tau,i} + 2m_i) \right)
$$

\[ \text{boundary terms} \]

\[ \text{The integers $s_i$'s do not depend on the explicit coefficients of the matrix. We have $i \geq l$ so that $s_i \leq i$; in particular, Re$(a_i) < -n \Rightarrow \text{Re}(b_1) + \cdots + \text{Re}(b_i) - n s_i \geq \text{Re}(b_1) + \cdots + \text{Re}(b_i) - n i > 0$ so that as expected, the above integral converges.} \]
for some domain $\Delta' = \prod_{i=1}^{I'-1} [0,1] \times [0,\infty]$ for some $I' < I$ or $\Delta' = \prod_{i=1}^{I'-1} [0,1]$
for some $I' \leq I$ and some non negative integers $m'_i \leq m_i$ with at least one
$m'_{i_0} < m_{i_0}$.

This produces a meromorphic map which on the domain $\cap_{i=1}^{I} \{\Re(b(1) + \cdots + b_{(i)}) + 2m_i > ns_{\tau,i}\}$ reads

$$\frac{1}{\prod_{i=1}^{I} \Gamma(b_i)} \sum_{\tau \in \Sigma_I} \frac{H_{\tau,m}(b_1, \cdots, b_I)}{\prod_{i=1}^{I} ((b_{(1)} + \cdots + b_{(i)} - n s_{\tau,i}) \cdots (b_{(1)} + \cdots + b_{(i)} - n s_{\tau,i} + 2m_i))}$$

with $H_{\tau,m}$ holomorphic on that domain. It therefore extends to a meromorphic map on the whole complex space with simple simple poles on a countable set of affine hyperplanes $\{a_{\tau(1)} + \cdots + a_{\tau(i)} + n s_{\tau,i} \in \mathbb{C} \}$, where as before, the $s_{\tau,i}$'s are integers which depend on the permutation $\tau$ and on the size $L \times I$ shape (i.e. on the $l_i$'s) but not on the actual coefficients of the matrix.

Let us further observe that since $s_{\tau,i} \leq i$, if $\Re(a_i) < -n \Rightarrow \Re(b_i) > n$ for any $i \in \{1, \cdots, I\}$, then for any $\tau \in \Sigma_I$ we have $\Re(b_{(1)} + \cdots + b_{(i)} - n s_{\tau,i} > 0$ so that we recover the fact that the map $(a_1, \cdots, a_I) \mapsto f \in \mathbb{C} \prod_{i=1}^{I} \langle \sum_{i=1}^{L} b_{i l} \xi_i \rangle$ is holomorphic on the domain $D := \{a = (a_1, \cdots, a_I) \in \mathbb{C}^I, \Re(a_i) < -n, \ \forall i \in \{1, \cdots, I\}\}$. $\square$
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47
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48