A Maximum Linear Arrangement Problem on Directed Graphs

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Abstract

We propose a new arrangement problem on directed graphs, Maximum Directed Linear Arrangement (MaxDLA). This is a directed variant of a similar problem for undirected graphs, in which however one seeks maximum and not minimum; this problem known as the Minimum Linear Arrangement Problem (MinLA) has been much studied in the literature. We establish a number of theorems illustrating the behavior and complexity of MaxDLA. First, we relate MaxDLA to Maximum Directed Cut (MaxDiCut) by proving that every simple digraph \( D \) on \( n \) vertices satisfies \( \frac{2}{3} \text{maxDiCut}(D) \leq \text{MaxDLA}(D) \leq (n-1)\text{maxDiCut}(D) \). Next, we prove that MaxDiCut is NP-Hard for planar digraphs (even with the added restriction of maximum degree 15); it follows from the above bounds that MaxDLA is also NP-Hard for planar digraphs. In contrast, Hadlock (1975) and Dorfman and Orlova (1972) showed that the undirected Maximum Cut problem is solvable in polynomial time on planar graphs.

On the positive side, we present a polynomial-time algorithm for solving MaxDLA on orientations of trees with degree bounded by a constant, which translates to a polynomial-time algorithm for solving MinLA on the complements of those trees. This pairs with results by Goldberg and Klipker (1976), Shiloach (1979) and Chung (1984) solving MinLA in polynomial time on trees. Finally, analogues of Harper’s famous isoperimetric inequality for the hypercube, in the setting of MaxDLA, are shown for tournaments, orientations of graphs with degree at most two, and transitive acyclic digraphs.

1 Introduction

Except where otherwise noted, we shall assume that all directed graphs (digraphs) are simple (i.e. do not have loops or parallel edges) but may have opposite edges (an edge \((u, v)\) is opposite \((v, u)\)). A digraph is symmetric if every edge has an opposite, and in this case we consider it an (undirected) graph.

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Let $D$ be an $n$-vertex digraph. A \emph{(linear) arrangement} $\pi$ of $D$ is a bijection from $V(D)$ to \{1, \ldots, n\}. We will write $\pi = (v_1, v_2, \ldots, v_n)$ to indicate that $\pi(v_i) = i$ for every $1 \leq i \leq n$. Given an arrangement $\pi$ of a digraph $D$, the \emph{value of an edge} $e = (u, v)$ of $D$ is defined to be $\text{val}_\pi(e) = \max\{0, \pi(v) - \pi(u)\}$. The \emph{value of $\pi$} is defined as $\text{val}_D(\pi) = \sum_{e \in E(D)} \text{val}_\pi(e)$. The Maximum Directed Linear Arrangement Problem can now be stated as follows.

**Maximum Directed Linear Arrangement (MaxDLA)**

\textbf{Input:} $n$-vertex digraph $D$, integer $k$

\textbf{Problem:} Is there an arrangement $\pi$ of $D$ with value at least $k$?

MaxDLA is related to two well-studied problems: Minimum Linear Arrangement, and Simple Maximum Directed Cut.

The Minimum Linear Arrangement problem (MinLA) asks, given an undirected graph $G$ and integer $k$, if there is an arrangement of $G$ with value at most $k$. MinLA is \text{NP}-complete in general [6, GT42], even for bipartite [14], and interval [3] graphs, but is polynomial-time solvable for many classes of graphs such as trees [7, 14, 2], and unit interval graphs [11, 13]. See [5, 12] for a survey.

The relation between MinLA and MaxDLA is straightforward. Given a digraph $D$, its \text{complement} $\overline{D}$ is the digraph with vertex set $V(D)$ and edge set $E(\overline{D})$. Observe that, for an arrangement $\pi$ of an $n$-vertex digraph $D$, $\text{val}_D(\pi)$ is maximum exactly when $\text{val}_{\overline{D}}(\pi)$ is minimum. Indeed, $\text{val}_D(\pi) + \text{val}_{\overline{D}}(\pi) = \binom{n-1}{2}$, which is the value of an arrangement of an $n$-vertex complete digraph. [6, p. 244]. Theorem 1.1 follows.

**Theorem 1.1.** MaxDLA is \text{NP}-Complete.

As stated previously, MinLA is solvable in polynomial time for trees. This was first shown for trees Goldberg and Klipker in 1976 with an $O(n^3)$ algorithm [7]. And improved by Shiloach to all trees in 1979, with an interesting $O(n^{2.2})$ algorithm [14]. Finally, in 1984, Chung improved Shiloach’s algorithm and, with a careful analysis of running time, gave a $O(n^{3})$ algorithm where $\lambda \approx 1.6$ [2] which is the current best. In Section 4 of this paper, we present a polynomial-time algorithm solving MinLA for the complements of trees with degree bounded by a constant. This was discovered first in the context of MaxDLA. The algorithm for MaxDLA, which is polynomial-time for orientations of trees with constant-bounded degree, occupies the bulk of the section. Hence, the main results of section 4 are the following Theorem and Corollary.

**Theorem 1.2.** Let $G$ be a forest on $n$ vertices with $\Delta(G) = d$ and let $D$ be an orientation of $G$. Then the MaxDLA of $D$ is solvable in time $O(n^{4d})$.

**Corollary 1.3.** Let $G$ be a graph on $n$ vertices whose complement is a forest of maximum degree $d$. Then MinLA of $G$ is solvable in time $O(n^{4d})$. 
The Simple Maximum Directed Cut problem (MaxDiCut) is also related to MaxDLA. Given a partition of the vertices of a digraph $D$ into two sets $S$ and $T$, the directed cut from $S$ to $T$, written $E(S,T)$, is the set of edges with tail in $S$ and head in $T$. MaxDiCut asks, given a digraph $D$ and integer $k$, if there is a directed cut of $D$ containing at least $k$ edges. MaxDiCut is NP-complete in general [6, pp. 244-246], and remains so for many symmetric graph classes such as chordal, tripartite, split [1], and unit-disk graphs [4]. It is solvable in polynomial time for symmetric cographs [1] and symmetric planar graphs [9]. The following theorem, proved at the beginning of Section 3, contrasts the last result.

**Theorem 1.4.** MaxDiCut is NP-complete even when restricted to planar multi-digraphs of maximum degree 15.

As a corollary we show that the same holds for MaxDLA.

**Corollary 1.5.** MaxDLA is NP-complete even when restricted to planar multi-digraphs of maximum degree 15.

We round out Section 3 by showing a dependence between MaxDLA and MaxDiCut, whose bounds are best possible.

**Theorem 1.6.** For every digraph $D$,

$$\frac{n}{2} \text{MaxDiCut}(D) \leq \text{MaxDLA}(D) \leq (n-1) \text{MaxDiCut}(D).$$

For any arrangement $(v_1, v_2, \ldots, v_n)$ of a directed graph $D$, there are $n-1$ associated directed cuts of the form $E(\{v_1, \ldots, v_k\}, \{v_{k+1}, \ldots, v_n\})$ where $1 \leq k \leq n-1$. We call these the cuts of the arrangement.

**Corollary 1.7.** The largest cut in a MaxDLA of digraph $D$ has size at least $\frac{1}{2} \text{MaxDiCut}(D)$.

The last section of this paper examines digraphs for which the maximum directed linear arrangement has the property that every cut of the arrangement $E(\{v_1, \ldots, v_k\}, \{v_{k+1}, \ldots, v_n\})$ is a largest directed cut in $G$ of the form $E(X,Y)$ where $|X| = k$. This is inspired by a famous edge isoperimetric inequality of Harper. In 1966, Harper gave an arrangement of the vertices of a hypercube so that every cut of the arrangement was minimum over all cuts separating the same size vertex sets. In contrast, this paper presents three classes of digraphs, all with an arrangement where every cut is maximum over all cuts separating the same size vertex sets. These digraphs are tournaments, orientations of graphs $G$ with $\Delta(G) \leq 2$, and transitive acyclic digraphs.
2 Properties of Directed Linear Arrangements

In this section we establish some basic properties of Directed Linear Arrangements that will be helpful in our investigations.

Fix an arrangement $\pi = (v_1, v_2, ..., v_n)$ of a digraph $D$ and define $S_i = \{v_j \in V(D) \mid j \leq i\}$ and $T_i = \{v_j \in V(D) \mid j > i\}$. So the cuts $C_1, \ldots, C_{n-1}$ of the arrangement $\pi$ are defined by the rule that $C_i$ is the directed cut from $S_i$ to $T_i$. We say that the arrangement $\pi$ contains each $C_i$ and we let $c_i = |C_i|$ for the purposes of this discussion.

Observe that the value of an arrangement $\pi$ can be calculated by summing its cuts

$$val(\pi) = \sum_{i=1}^{n-1} c_i.$$ (i)

The level of a vertex $v_i$ in an arrangement $\pi = (v_1, v_2, ..., v_n)$ can be thought of as its contribution to $c_i$, and is defined as follows

$$l(v_i) = c_i - c_{i-1},$$ (ii)

where we let $c_0 = c_n = 0$.

The value of a cut $c_i$ is therefore the sum of the levels to its left. And so there are three equivalent ways to calculate the value of an arrangement, based on edges, cuts, and levels respectively.

**Property 2.1.** The value of an arrangement $\pi = (v_1, v_2, ..., v_n)$ of a digraph $D = (V, E)$ is

$$val(\pi) = \sum_{e \in E(D)} val(e) = \sum_{i=1}^{n-1} c_i = \sum_{i=1}^{n-1} \sum_{j=1}^{i} l(v_i).$$

Levels lead to a nice abstraction of MaxDLA. Let $\pi = (v_1, v_2, ..., v_n)$ be an arrangement of digraph $D$. Given vertex $v_i$, its level in $\pi$ is the number of its in-neighbours to the its left subtracted from the number of its out-neighbours to its right $l(v_i) = |N^+(v_i) \cap T_i| - |N^-(v_i) \cap S_i|$. This is just a direct application of the definition of directed cut. From this, by adding and subtracting $|N^+(v_i) \cap S_i|$, it follows that the level of a vertex is

$$l(v_i) = d^+(v_i) - |N(v_i) \cap S_i|.$$ (iii)

Therefore the levels, and hence the value, of an arrangement $\pi$ can be calculated without knowing the direction of the edges of $D$, only the out-degree of each vertex. From this it follows that the value of $\pi$ is unchanged when a directed cycle of $D$ is reversed.
Property 2.2. Let $D$ be a digraph with directed cycle $C$, and let $D'$ be the digraph obtained from $D$ by reversing $C$. If $\pi$ is an arrangement of $D$, then $\text{val}_D(\pi) = \text{val}_{D'}(\pi)$.

Corollary 2.3. Let $G$ be a graph, and let $D$ and $D'$ be orientations of $G$ so that every vertex has the same outdegree in both $D$ and $D'$. If $\pi$ is an arrangement of $D$, then $\text{val}_D(\pi) = \text{val}_{D'}(\pi)$.

Corollary 2.4. Let $D$ be an Eulerian digraph, and $D'$ be $D$ with every edge reversed. If $\pi$ is an arrangement of $D$, then $\text{val}_D(\pi) = \text{val}_{D'}(\pi)$.

The signature $s$ of an arrangement $\pi = (v_1, v_2, ..., v_n)$ is the $(n - 1)$-tuple of its cut sizes

$$s(\pi) = (c_1, c_2, ..., c_{n-1}).$$

And we use the shorthand $s_i$ to mean the $i^{th}$ element of $s$.

For a digraph $D$, let $S$ be the set of signatures of $D$. That is, $s \in S$ whenever there is an arrangement $\pi$ of $D$ so that $s(\pi) = s$. For $s, s' \in S$ we write $s' \leq s$ whenever $s'_i \leq s_i$ for all $i$. This notation captures the idea of $s$ being no worse than $s'$ on every cut. It is easy to see that $S$ is a partial order under $\leq$. We say an arrangement $\pi$ is maximum (maximal) if $s(\pi)$ is maximum (maximal) in $S$.

Observation 2.5. Let $D$ be a disconnected digraph and let $H$ be one of its components. An arrangement $\pi$ is a maximal (maximum) arrangement of $D$ only if $\pi$ restricted to $H$ is a maximal (maximum) arrangement of $H$.

Having covered the more pressing definitions, we move on to properties of directed linear arrangements.

The Maximum Directed Linear Arrangement (MaxDLA) problem is contained in a problem on undirected graphs with vertex weights. We define the Weighted Maximum Linear Arrangement (W-MaxLA) problem, which asks, given an undirected graph $G$ with weight function $f : V(G) \to \mathbb{Z}$ and positive integer $k$, if there is a linear arrangement $\pi$ with value at least $k$.

Since the value of an arrangement is completely determined by the level of its vertices, the following property is immediate.

Property 2.6. MaxDLA($D, k$) is equivalent to W-MaxLA($G, f, k$) when $G$ is the underlying graph of $D$ and $f$ maps the vertices of $G$ to their outdegree in $D$.

Property 2.7. The vertices of a maximal linear arrangement are arranged by levels, in non-decreasing order.
Proof. Suppose, for a contradiction, \( \pi = (v_1, v_2, ..., v_n) \) is a maximal arrangement having vertices \( v_i, v_j \) so that \( i < j \) but \( l(v_i) < l(v_j) \). We may assume, without loss of generality, \( j = i + 1 \). But then interchanging \( v_i \) and \( v_j \) increases the value of \( c_i \), while leaving the other cuts unchanged. This contracts the maximality of \( \pi \).

\[ \square \]

3 Relation to Maximum Directed Cut

A maximum directed cut in \( D \) is a directed cut \( E(X, Y) \) with largest cardinality over all partitions \( \{X, Y\} \). We denote the number of edges in such a cut by \( \text{maxDiCut}(D) \).

A maximum linear arrangement of a digraph \( D \) may or may not contain a maximum directed cut of \( D \). However, \( \text{maxDiCut}(D) \) and \( \text{val}(D) \) are related by the following theorem.

**Theorem 3.1.** If \( D \) is a digraph with \( n \) vertices and \( \text{maxDiCut}(D) = t \), then \( \frac{1}{2}nt \leq \text{val}(D) \leq (n - 1)t \).

**Proof.** For the upper bound, observe that a linear arrangement of \( D \) contains exactly \( n - 1 \) cuts, none of which is larger than \( t \).

For the lower bound, let \( E(X, Y) \) be a maximum directed cut of \( D \), and construct \( D' \) from \( D \) by deleting all edges not in \( E(X, Y) \). By construction, \( D' \) is an orientation of a bipartite graph with all edges directed from \( X \) to \( Y \). We will now build an arrangement \( \pi \) of \( D' \) with value at least \( \frac{1}{2}nt \). Let \( \pi_x \) be an arrangement of \( X \) by non-increasing outdegree in \( D' \). Similarly, let \( \pi_y \) be an arrangement of \( Y \) by non-decreasing indegree in \( D' \). Then \( \pi = \pi_x, \pi_y \) is an arrangement of \( D \) satisfying the lower bound.

\[ \square \]

The following corollary is immediate.

**Corollary 3.2.** The largest cut in a MaxDLA of digraph \( D \) has size at least \( \frac{1}{2}\text{MaxDiCut}(D) \).

We now move on to an NP-Completeness result for MaxDLA on orientations of planar graphs. We do this through two reductions. First, we reduce Planar Max 2SAT to Planar MaxDiCut. And then reduce MaxDiCut to MaxDLA in a way that preserves planarity.

Max2SAT asks, given a 2-CNF \( \phi \) and positive integer \( k \), if there is a truth assignment to the variables of \( \phi \) so that at least \( k \) clauses in \( \phi \) are satisfied. Given a CNF \( \phi \), its variable graph is the graph with vertex set \( V \) so that each \( v \in V \) is associated with exactly one variable in \( \phi \), and edge set \( E \) so that \( \{u, v\} \in E \) if and only if the variables associated with \( u \) and \( v \) occur together in some clause of \( \phi \). Planar Max2SAT is an instance of Max2SAT whose 2-CNF has a variable graph that is planar. Planar Max2SAT is NP-complete [8, p. 254].
Theorem 3.3. Planar MaxDiCut is NP-complete.

Proof. The edges in a directed cut can be counted and compared to \( k \) in linear time, so Planar MaxDiCut is in NP.

We proceed by reduction from Planar Max2Sat. Given an instance of Planar Max2Sat with 2-CNF \( \phi \), construct a digraph \( D = (V, E) \) from the vertices of the variable graph of \( \phi \), by adding one gadget per clause \( C_i \) in \( \phi \) as follows.

1. If \( C_i \) consists of a single variable \( v \) in positive form, then add one new vertex \( x_i \) and edge \((x_i, v)\) with multiplicity two.

2. If \( C_i \) consists of a single negated variable \( v \), then add one new vertex \( x_i \) and edge \((v, x_i)\) with multiplicity two.

3. If \( C_i \) consists of two variables \( u \) and \( v \), both in positive form, then add one new vertex \( x_i \) and edges \((u, v)\), \((v, u)\), \((x_i, v)\), and \((x_i, u)\).

4. If \( C_i \) consists of two variables \( u \) and \( v \), both negated, then add one new vertex \( x_i \) and edges \((u, v)\), \((v, u)\), \((v, x_i)\), and \((u, x_i)\).

5. If \( C_i \) consists of two variables, \( u \) in negative, and \( v \) in positive form, then add one new vertex \( x_i \) and edges \((u, x_i)\) and \((x_i, v)\) both with multiplicity two.

Now we claim that there is a truth assignment to the variables of \( \phi \) so that at least \( k \) clauses are satisfied if and only if \( D \) has a directed cut of size at least \( 2k \).

Suppose \( D \) has a dicut \( E = (F, T) \) of size at least \( 2k \). Let \( \tau \) be a truth assignment to the variables of \( \phi \) so that variables associated with a vertex in \( F \) (\( T \)) are assigned false (true). The gadgets are edge-disjoint so each may be examined independently, and it is simple to observe that a gadget contributes exactly two edges to \( E \) when its clause is satisfied, and exactly zero edges otherwise. Hence \( \tau \) satisfies at least \( k \) clauses.

Conversely, suppose \( \phi \) has a truth assignment \( \tau \) so that at least \( k \) clauses are satisfied. Then construct in \( D \) a dicut \( E = (F, T) \) by placing vertices associated with variables true under \( \tau \) in \( T \) and false under \( \tau \) in \( F \). Place the vertices that were added with a gadget so the number of edges across \( E \) is maximized. Then by examining each gadget independently, it is clear that gadgets whose clause is true under \( \tau \) contribute exactly two edges to \( E \). Hence \( D \) has a dicut of size at least \( 2k \).

Corollary 3.4. Let \( D \) be a planar digraph with \( \Delta(G) \leq 15 \). Then the MaxDiCut of \( D \) is NP-Complete.
Proof. This is obtained by following existing well-known reductions.

We now reduce MaxDiCut to MaxDLA in a simple way that, among other things, preserves planarity and maximum degree. This is inspired by Garey, Johnson and Stockmeyer’s 1976 reduction of MaxCut to MinLA [6, pp. 244 - 246].

Theorem 3.5. $\text{MaxDiCut} \leq \text{MaxDLA}$

Proof. Let $D = (V, E)$ be a digraph on $n$ vertices, and let $k$ be a positive integer. We may assume $D$ has at least one edge. Suppose $D$ and $k$ are inputs to MaxDiCut.

Construct a new graph $D'$ by adding $n^3$ isolated vertices to $D$. Denote the additional isolated vertices by $S$. We claim that $D$ has a cut of size at least $k$ if and only if $G'$ has a linear arrangement of size at least $kn^3$.

Indeed, suppose $G$ has a cut $C = E(A, B)$ of size at least $k$. Then, arrange the vertices of $G'$ so that all vertices in $A$ come before all vertices in $S$, which in turn come before all vertices in $B$. This arrangement has value at least $kn^3$ because the large cut $C$ is repeated for every vertex in $S$.

Now, suppose $G$ has no cut of size at least $k$. Then $G'$ also has no cut of size at least $k$. Hence an arrangement $\pi$ of $G'$ can have value at most $(k - 1)(n^3 + n - 1)$ because every one of the $(n^3 + n - 1)$ cuts in $\pi$ has size at most $(k - 1)$. But $k < n^2$ as $k$ is limited by the number of edges in $G$. So that $n^3 > (k - 1)n$ and $\pi$ has value at most $(k - 1)(n^3 + n) < kn^3 + (k - 1)n - n^3 < kn^3$. \qed

Corollary 3.6. Let $S$ be a class of digraphs closed under the operation of adding an isolated vertex. If $\text{MaxDiCut}$ is $\text{NP}$-Complete for $S$, then $\text{MaxDLA}$ is also $\text{NP}$-Complete for $S$.

4 Algorithm for Oriented Trees

In this section, an algorithm solving MaxDLA on orientations of trees with degree bounded by a constant is described. This same algorithm with a slight modification, described at the end of this section, solves MinLA on complements of those graphs.

Fix a positive integer $d$. Let $G = (V, E)$ be a tree on $n$ vertices with $\Delta(G) \leq d$, and let $D$ be an orientation of $G$. Let $k$ be a positive integer. The following describes an algorithm solving MaxDLA for $D, k$ in time $O(n^{4d})$.

The first step is to convert the problem on digraph $D$ to one on its underlying graph $G$, with vertex weight $f(v) = d_D^+(v)$. We then solve $W - \text{MaxLA}(G, f, k)$ which is equivalent to our original problem by Property 2.4. Next, we generate all possible maximal signatures of arrangements of $G, f$, of which there are only polynomially many, using Algorithm 4.2. And finally we check in polynomial time if one has value at least $k$. 
Before proceeding, we make the following observation, on which we base Algorithm 4.2.

**Observation 4.1.** Let $G = (V, E)$ be an undirected graph, $e = (u, v) \in E(G)$, $f : V(G) \mapsto \mathbb{Z}$ be a weight function, and $\pi$ be an arrangement of $V(G)$ with $\pi(u) < \pi(v)$. Define $f_v$ as follows:

$$f_v(x) = \begin{cases} f(x) - 1 & \text{if } x = v \\ f(x) & \text{otherwise.} \end{cases}$$

Then the value of $\pi$ on $G, f$ is the same as the value of $\pi$ on $G - e, f_v$.

**Algorithm 4.2.** \texttt{FindMaximalSignatures}(G, f)

INPUT: A forest $G = (V, E)$ and a weight function from $V(G)$ to the positive integers.

OUTPUT: $S$, a set of maximal signatures of $G, f$.

Step 1: Divide $G$ into connected components. For each connected component $H_i$ do step 2.

Step 2: If $H_i$ is an isolated vertex $v$, set $S_i = \{(f(v))\}$. Otherwise go to Step 3.

Step 3: Find an edge $e = (u, v)$ that minimizes the size of the largest component when $e$ is deleted.

Step 4: Define two new weight functions $f_u$ and $f_v$ so that

$$f_u(x) = \begin{cases} f(x) - 1 & \text{if } x = u \\ f(x) & \text{otherwise} \end{cases} \quad \text{(vi)}$$

and

$$f_v(x) = \begin{cases} f(x) - 1 & \text{if } x = v \\ f(x) & \text{otherwise.} \end{cases} \quad \text{(vii)}$$

Step 5: Recursively calculate the maximal signatures of $H_i - e$ with both $f_u$ and $f_v$. Do this by setting $S_u = \text{FindMaximalSignatures}(H - e, f_u)$ and $S_v = \text{FindMaximalSignatures}(H - e, f_v)$.

Step 6: Define $S_i = S_u \cup S_v$ and delete non-maximal signatures from $S_i$.

Step 7: Return $S = \bigcup_i S_i$ with non-maximal signatures deleted.

The correctness of this algorithm follows from Property 2.4 and Observations 2.3 and 4.1. We must show that the running time is $O(n^{4d})$. 
In Step 3 of Algorithm 4.2, two graph components are generated and recursed upon. These components each have size at most $\frac{(n-1)(d-1)}{d}$ where $n = |V(G)|$. Because Step 1 immediately divides the connected components, this means that at each step of recursion the size of the problem is reduced by a factor of at least $\frac{d-1}{d}$.

Steps 6 and 7 take up the majority of computation time. But this remains polynomial because we are only keeping track of signatures. A signature is concerned only with the number of vertices at each level. There are $2^d + 1$ possible levels, so only $O(n^{2d})$ possible signatures. This means at Step 6, that $|S_u|, |S_v|$ are $O(n^{2d})$ and so can be compared in time $O(n^{4d})$. At Step 7, again, $|S_i|$ is $O(n^{2d})$, so this step can also be done in time $O(n^{4d})$. This leads to a recurrence of $T(n) = 4T\left(\frac{d-1}{d}n\right) + O(n^{4d})$, and running time $O(n^{12})$.

This algorithm may be applied to solve MinLA on graphs $G$ when $\overline{G}$ is a forest with degree bounded by a constant. We consider $G$ as a symmetric digraph, and proceed as written except for Step 4. At Step 4, the weight function is simply reduced by two instead of one.

5 Graphs with a Maximum Arrangement

Inspired by a theorem of Harper exhibiting cuts of the hypercube minimum for every cardinality [10], we examine the opposite. In this section, we examine three classes of digraphs having a maximum arrangement: tournaments, orientations of graphs $G$ with $\Delta G \leq 2$, and transitive acyclic digraphs. These types of graphs are interesting for MaxDLA because, by Observation 2.3, they are closed over disjoint union.

A tournament is an orientation of a complete graph. The MaxDLA of a tournament is obtained by arranging its vertices by non-increasing out-degree. Further, each of the cuts contained in that arrangement is maximum for its cardinality. This becomes clear when abstracting MaxDLA to W-MaxLA (Property 2.4). After which we are left arranging a weighted complete graph.

Orientations of graphs $G$ with $\Delta G \leq 2$ also have a maximum arrangement, which is seen by abstracting as was done for tournaments.

Finally, a transitive acyclic digraph also has a maximum arrangement. It is obtained by arranging the vertices $v$ by non-increasing $d^+(v) - d^-(v)$. This is only slightly more difficult to see.

**Theorem 5.1.** Let $D = (V,E)$ be a transitive acyclic digraph, and $\pi$ an arrangement of $V(D)$ by non-increasing $d^+(v) - d^-(v)$. Then $\pi$ is maximum.

**Proof.** First we show that a maximum arrangement of $D$ must be a topological sort. Indeed, suppose not, suppose there is an arrangement $\sigma$ which does better than a topological sort. Then $\sigma$ must have an arc pointing backwards. Choose the shortest such arc $e = (u,v)$. Then $u$ and $v$ must have
no neighbours in between them in the arrangement \( \sigma \). But this means reversing \( u \) and \( v \) strictly increases the value of that arrangement, a contradiction. Further, since the improvement was local, it follows that a cut maximum for its cardinality must have no backward arcs. Thus a maximum arrangement of \( D \) is a topological sort.

Finally, we observe that the level of each vertex \( v \) is \( d^+(v) - d^-(v) \). And since vertices must be arranged by non-increasing level, the theorem follows. \( \square \)

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