Anti-self-dual metrics on Lie groups

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Abstract. The aim of the paper is to determine left-invariant, anti-self-dual, non conformally flat, Riemannian metrics on four-dimensional Lie groups.

Introduction

Let $G$ be a Lie group of dimension $n$ equipped with a left-invariant Riemannian metric $g$. The curvature tensor $R = R(g)$ is completely determined by its value at the identity element of $G$. In this way, $R$ is an element of the vector space

$$R = S^2(Λ^2 g^*) ⊕ Λ^4 g^*,$$

where $g$ is the Lie algebra of $G$ and $A ⊕ B$ denotes the orthogonal complement of $B$ in $A$. Moreover, there is an $O(n)$-invariant decomposition

$$R = V ⊕ W,$$

where $V ≃ S^2 g^*$. The component $\text{Ric}$ of $R$ in $V$ is the Ricci tensor, and the component $W$ in $W$ is the Weyl tensor. If $W = 0$, then $M$ is conformally flat in the sense that there exist local coordinates $x^i$ for which $g$ is a scalar function times $\sum_{i=1}^n dx^i ⊗ dx^i$.

For example, the compact Lie group $U(2)$ (double covered by $S^1 × S^3$) acquires such coordinates from its description as a discrete quotient of $\mathbb{R}^4 \setminus \{0\}$.

If $G$ is 4-dimensional and oriented, the Hodge involution $*: Λ^2 g^* → Λ^2 g^*$ gives rise to a decomposition

$$W = W^+ + W^-,$$

where $W^± ∈ S^2_0(Λ^± g^*)$ and $Λ^± g^*$ is the ±-eigenspace of $*$. If $W^+ = 0$ the metric $g$ is said to be anti-self-dual, or ‘ASD’ for short. In this paper, we classify left-invariant metrics on 4-dimensional Lie groups with $W^+ = 0$ and $W^- ≠ 0$. Such a group is necessarily solvable, and in §3 we describe the general form of its Lie algebra. We prove that the hypotheses eliminate all but a 1-parameter family of Lie algebras, and that there are essentially only two distinct left-invariant metrics that are anti-self-dual.

The proof of the main theorem (1.6 below) is accomplished by first eliminating Lie algebras that admit an orientation-reversing automorphism. A key feature of the method is that the Gram-Schmidt process is compatible with the triangular nature
of the nilpotent Lie algebra structure constants. This enables one to compute the structure constants for an orthonormal basis relative to an arbitrary inner product. Once freedom in the choice of basis has been removed, the equation $W^+ = 0$ is solved by means of a specially adapted Maple program reproduced in the Appendix. No doubt further effort would eliminate the need for a computer analysis, but this aspect of the work may have independent interest.

Acknowledgments. This paper was begun during a visit by the first author in Oxford in 1997, in parallel with the work [1] that establishes similar results independently. The second author is grateful for the encouragement received at the Tokyo conference to finally present the material in written form.

1. Summary of results

Let $V$ denote a real 4-dimensional vector space, with basis $(f_1, f_2, f_3, f_4)$. For $\lambda \in \mathbb{R}$, let $g_\lambda$ denote the Lie algebra defined on $V$ by the relations

\begin{align*}
[f_1, f_2] &= f_2 - \lambda f_3, \\
[f_1, f_3] &= \lambda f_2 + f_3, \\
[f_1, f_4] &= 2f_4, \\
[f_2, f_3] &= -f_4.
\end{align*}

(1.1)

The Jacobi identity is easily verified. The Lie algebra $g_\lambda$ is solvable and $g'_\lambda = [g_\lambda, g_\lambda]$ is isomorphic to the 3-dimensional Heisenberg algebra. Changing the signs of $f_3, f_4$ is equivalent to replacing $\lambda$ by $-\lambda$, though we shall see that $g_\lambda, g'_\lambda$ are not isomorphic if $|\lambda| \neq |\lambda'|$.

Later on, it will be convenient to express the relations (1.1) in terms of differential forms. To this end, let $(f^i)$ denote the dual basis of $g_\lambda^*$. With the convention that $d f^i(u, v) = -f^i[u, v]$, (1.1) asserts that

\begin{align*}
d f^1 &= 0, \\
d f^2 &= -f^{12} - \lambda f^{13}, \\
d f^3 &= \lambda f^{12} - f^{13}, \\
d f^4 &= -2f^{14} + f^{23}.
\end{align*}

(1.2)

Let $G_\lambda$ denote the simply-connected Lie group corresponding to $g_\lambda$. We shall not identify the group structure, but (1.2) can be integrated to yield coordinates $t, x, y, u$ on $G_\lambda$ relative to which

\begin{align*}
f^1 &= dt, \\
f^2 &= e^{-t}(dx + \lambda y dt), \\
f^3 &= e^{-t}(dy - \lambda x dt), \\
f^4 &= e^{-2t}(du + \frac{1}{2}(xdy - y dx) - \frac{1}{2}\lambda(x^2 + y^2) dt).
\end{align*}

(1.3)
For $k > 0$, let $\langle ., . \rangle_k$ denote the inner product on $V$ defined by

$$
\langle f_i, f_j \rangle =
\begin{cases} 
  k^2, & i = j = 1, \\
  1, & 2 \leq i = j \leq 4, \\
  0, & i \neq j.
\end{cases}
$$

(1.4)

This inner product induces a left-invariant metric

$$
g_k = k^2 f_1 \otimes f_1 + f_2 \otimes f_2 + f_3 \otimes f_3 + f_4 \otimes f_4
$$
on $G_\lambda$. We shall show that the corresponding metrics are all isometric for fixed $k$. The associated curvature tensor does not therefore depend upon $\lambda$. Using the program in the Appendix, it is easy to verify

1.5 Proposition. Relative to the orthonormal basis $(\frac{1}{k}f_1, f_2, f_3, f_4)$ of $g_\lambda$, the Ricci and Weyl tensors of $g_k$ have the diagonal forms

$$
\text{Ric} = \begin{pmatrix}
\frac{6}{k^2} & 0 & 0 & 0 \\
0 & \frac{4}{k^2 + \frac{1}{2}} & 0 & 0 \\
0 & 0 & \frac{4}{k^2 + \frac{1}{2}} & 0 \\
0 & 0 & 0 & \frac{8}{k^2 - \frac{1}{2}}
\end{pmatrix},
$$

$$
\text{W}^\pm = \frac{k^2 \mp 3k + 2}{3k^2} \begin{pmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 2
\end{pmatrix}.
$$

Observe that $\text{Ric}$ is degenerate if $k = 4$, and $\text{W}^+ = 0$ if $k \in \{1, 2\}$. We shall prove that, conformally speaking, there are no other non-flat ASD examples:

1.6 Theorem. Let $G$ be an oriented four-dimensional Lie group admitting a left-invariant Riemannian metric $g$ such that $W^+ = 0$ and $W^- \neq 0$. Then the Lie algebra of $G$ is isomorphic to $\mathfrak{g}_\lambda$ for some $\lambda \geq 0$, and $g$ is locally homothetic to either $g_1$ or $g_2$.

The distinction between $W^+, W^-$ depends upon a choice of orientation, and in this sense inclusion of the prefix ‘anti’ in the title is purely a matter of taste. However, an orientation is distinguished by the presence of a complex structure. The set of positively-oriented almost complex structures on $G$ compatible with a metric is isomorphic to $SO(4)/U(2) \cong S^2$, and the equation $W^+ = 0$ is precisely the integrability condition for a tautological almost complex structure on $G \times S^2$ [3]. A special situation in which $W^+ = 0$ occurs when each element in $S^2$ represents an integrable complex structure on $G$, which is then called hypercomplex.

Four-dimensional Lie groups with a left-invariant hypercomplex structure were classified by Barberis [4], and the above theorem can therefore be viewed as a generalization of this work. Indeed, [4] asserts that $G_\lambda$ admits a left-invariant hypercomplex structure if and only if $\lambda = 0$, and that $(G_0, g_1)$ is hyperhermitian. On the other hand,
$(G_0,g_2)$ is isometric to the complex hyperbolic plane $\mathbb{C}H^2 = SU(2,1)/S(U(2) \times U(1))$ with its symmetric metric, which is well known to satisfy $W^- = 0$ (with respect to the orientation relative to the natural complex structure). This metric appeared in Jensen’s classification of Einstein metrics on Lie groups [3], which provides a starting point for an alternative approach to classifying self-dual metrics and other structures on 4-dimensional Lie groups [1].

The fact that there are just two values of the parameter $k$ that solve our problem is reminiscent of the existence of two Einstein metrics on certain sphere bundles. For example, it is well known that $\mathbb{C}P^3$ (as a bundle over $S^4$) has its standard Kähler-Einstein metric as well as a nearly-Kähler metric with weak holonomy $U(3)$.

2. Building Lie algebras

Let $a_k$ denote the abelian Lie algebra whose underlying vector space is $\mathbb{R}^k$. Let $h_3$ denote the Lie algebra of the Heisenberg group; thus $h_3$ has a basis $(f_1, f_2, f_3)$ satisfying $f_1 = [f_2, f_3] = -[f_3, f_2]$ and all other brackets zero.

Given a Lie algebra $h$, let

$$\text{der } h = \{ D: h \to h : D[x,y] = [Dx,y] + [x,Dy] \}$$

denote the set of derivations of $h$. Suppose that $\rho: b \to \text{der } h$ is a homomorphism of Lie algebras. The extension of $h$ by $\rho$ is the Lie algebra $g$ with underlying vector space $b \oplus h$, in which $b$ and $h$ are both subalgebras and $[x,h] = \rho(x)(h)$ for all $x \in b$ and $h \in h$. We write

$$g = b \oplus_{\rho} h,$$

and if $b$ is abelian, we say that $g$ is an abelian extension of $h$.

2.1 Proposition. If $g$ is a 4-dimensional Lie algebra with zero centre then $g$ is isomorphic to an abelian extension of $a_2$, $a_3$ or $h_3$.

Proof. If $g$ is not solvable, its radical $r$ is a proper ideal. The Levi decomposition gives $g = s \oplus_{\rho} r$ where $s$ is a semi-simple algebra and $\rho: s \to \text{der } r$. It follows that $\dim s = 3$ and $\dim r = 1$. But 1-dimensional representations of a semi-simple Lie algebra are trivial, so the centre $z$ of $g$ equals $r$. We may therefore assume that $g$ is solvable.

A Lie algebra $g$ is solvable if and only if the ideal $g' = [g,g]$ is nilpotent. The only non-abelian nilpotent Lie algebra of dimension less than 4 is $h_3$, so $g'$ is one of $0, a_1, a_2, a_3, h_3$.

If $g' = 0$ then $z = g \neq 0$.

If $g' \cong a_1$ then $g$ has a basis $(f_1, f_2, f_3, f_4)$ with $g' = \langle f_1 \rangle$. Unless $f_1 \in z$, we may suppose that $f_2, f_3, f_4$ are chosen such that $[f_1, f_2] = f_1$ and $[f_1, f_i] = 0$ for $i = 3, 4$. We may further modify $f_3$ in order that $[f_2, f_3] = 0$. But then the Jacobi identity

$$[[f_2, f_3], f_4] + [[f_3, f_4], f_2] + [[f_4, f_2], f_3] = 0$$
implies that \([f_3, f_4] = 0\) and \(f_3 \in \mathfrak{z}\).

If \(g' \cong a_2\) then there is an isomorphism \(g \cong \mathbb{R}^2 \oplus g'\) of vector spaces, and the bracket determines a linear mapping \(\rho: \mathbb{R}^2 \to \text{der}(g')\). If \(\ker \rho\) is non-zero then it contains a 1-dimensional subspace that when added to \(g'\) yields an abelian algebra \(a_3\) of which \(g\) is an abelian extension. If \(\rho\) is injective then \(g'_C\) has a basis of eigenvectors for the commuting elements in \(\text{Im} \rho\). It follows that there exists a real basis \((f_1, f_2, f_3, f_4)\) of \(g\) such that \(g' = \langle f_3, f_4 \rangle\) and \(\text{ad} f_i\) restricts to the identity on \(g'\). Then \(f_2\) may be modified so that \([f_1, f_4] = 0\), and \(g = \mathbb{R}^2 \oplus \rho g'\) is an abelian extension.

If \(g' \cong a_3\) or \(g' \cong h_3\) then \(g\) is immediately an abelian extension of \(g'\).

Four-dimensional solvable Lie algebras can be broadly divided into 7 classes according to the triple \((d', d'', d''')\) of dimensions of \(g', g'' = [g', g'], g''' = [g'', g'']\). In [1], curvature computations are carried out for each of these classes in turn. For our purposes, the following observation helps to restrict the range of algebras that need to be considered.

2.2 Lemma. Let \(G\) be a simply-connected Lie group with a left-invariant metric \(g\). There exists an orientation-reversing isometry of \((G, g)\) in either of the two cases:

(i) the Lie algebra \(g\) has non-zero centre, or

(ii) \(g\) is an abelian extension of \(a_3\).

Proof. We shall exhibit an orthogonal transformation of \((g, \langle \ldots \rangle)\) which reverses the orientation of \(g\) in each case. This automorphism will induce the desired isometry of \((G, g)\).

(i) If \(\mathfrak{z} \neq 0\), choose an orthonormal basis \((f_1, f_2, f_3, f_4)\) of \(g\) with \(f_1 \in \mathfrak{z}\). It is then immediate that the linear mapping \(\phi: g \to g\) defined by

\[
\phi(f_i) = \begin{cases} 
-f_i & i = 1, \\
f_i & i \neq 1,
\end{cases}
\]

is a Lie algebra automorphism.

(ii) If \(g = \mathbb{R} \oplus \rho a_3\), choose an orthonormal basis \((f_1, \ldots, f_4)\) with \(f_2, f_3, f_4 \in a_3\). This time,

\[
\psi(f_i) = \begin{cases} 
f_i & i = 4, \\
-f_i & i \geq 2
\end{cases}
\]

is the required automorphism. \(\square\)

Combined with Proposition 2.1, this yields

2.3 Corollary. Let \(G\) be an oriented Lie group with a left-invariant Riemannian metric that satisfies \(W^+ = 0\) and \(W^- \neq 0\). Then \(g\) is an abelian extension of \(a_2\) or \(h_3\).

2.4 Proposition. The Lie algebras \(g_\lambda, g_{\lambda'}\) are isomorphic if and only if \(\lambda = \pm \lambda'\).
Proof. The derived algebra $g' = [g_\lambda, g_\lambda]$ in (1.1) has a basis $(f_2, f_3, f_4)$. With respect to this basis, $\text{ad} f_1$ acts on $g'_\lambda$ as

$$
\begin{pmatrix}
1 & \lambda & 0 \\
-\lambda & 1 & 0 \\
0 & 0 & 2
\end{pmatrix}.
$$

It follows that if $f$ is any element of $g_\lambda \setminus g'_\lambda$ then

$$\frac{\det(\text{ad}(f)|_{g'_\lambda})}{\text{tr}(\text{ad}(f)|_{g'_\lambda})} = \frac{1}{2}(1 + \lambda^2),$$

so $g_\lambda, g'_\lambda$ cannot be isomorphic unless $\lambda = \pm \lambda'$. We observed in §1 that they are isomorphic if $\lambda = -\lambda'$. \qed

2.5 Proposition. Let $\lambda \in \mathbb{R}$ and $k, k' > 0$. The Riemannian manifolds $(G_\lambda, g_k)$ and $(G_{\lambda'}, g_{k'})$ are isometric if and only if $k = k'$.

Proof. The fact that $(G_\lambda, g_k)$ is not isometric to $(G_{\lambda'}, g_{k'})$ if $k \neq k'$ follows immediately from 1.5, the computations of which can be carried out by starting from (3.7) below. It therefore suffices to show that $(G_\lambda, g_k)$ is isometric to $(G_0, g_k)$ for fixed $k > 0$.

Define $u$ as the abelian extension

$$u = \langle f_0 \rangle \oplus \rho g_0,$$

where $\rho$ is given by

$$
\begin{align*}
[f_0, f_2] &= -f_3, & [f_0, f_1] &= 0 \\
[f_0, f_3] &= f_2, & [f_0, f_4] &= 0.
\end{align*}
$$

(2.6)

Let $U$ denote the simply connected Lie group with Lie algebra $u$.

Comparing (2.6) with (1.1), we see that the mapping $g_\lambda \to u$ defined by

$$
\begin{align*}
f_1 &\mapsto \lambda f_0 + f_1, \\
f_i &\mapsto f_i, \quad 2 \leq i \leq 4,
\end{align*}
$$

is a Lie algebra homomorphism. We denote the induced immersion $G_\lambda \to U$ by $i_\lambda$.

The 1-parameter subgroup $\mathbb{R}$ of $U$ generated by $f_0$ acts on the coordinates $x, y$ by

$$
\begin{align*}
x &\mapsto x \cos s + y \sin s, \\
y &\mapsto -x \sin s + y \cos s.
\end{align*}
$$

(2.7)

This gives rise to a submersion $\pi : U \to U/\mathbb{R}$ onto a homogeneous manifold for which $\pi \circ i_\lambda$ is a diffeomorphism for each $\lambda$. By construction,

$$(\pi \circ i_\lambda)_* f_i = (\pi \circ i_0)_* f_i, \quad 1 \leq i \leq 4,$$
and $\phi = (\pi \circ \iota_{\lambda})^{-1} \circ (\pi \circ \iota_0)$ is the required isometry $(G_0, g_k) \to (G_{\lambda}, g_k)$.

More explicitly, we may identify $U$ with $\mathbb{R} \times G_0$ by means of coordinates $s, t, x, y, u$ relative to which $f^0 = ds$ and (1.3) holds with $\lambda = 0$. Then (2.7) and the fact that $i^*_\lambda(ds) = \lambda dt$ imply that

$$
\phi^*(dx^2 + dy^2) = (dx + \lambda y dt)^2 + (dy - \lambda x dt)^2,
\phi^*(du + \frac{1}{2}(xdy - ydx)) = du + \frac{1}{2}(xdy - ydx) - \frac{1}{2}\lambda(x^2 + y^2)dt,
$$

whence $\phi^*g_k = g_k$. □

3. Proof of the theorem

Let $G$ be a 4-dimensional Lie group with a chosen orientation, and let $g$ be a left-invariant Riemannian metric. The latter corresponds to an inner product $\langle \cdot, \cdot \rangle$ on the Lie algebra $g$ of $G$. The curvature tensor of $g$ and its components $W^\pm$ are completely determined by the structure constants of the Lie algebra $g$ and its inner product, so the problem is purely algebraic.

Because we are only interested in metrics that are not conformally flat, the Corollary tells us that $g$ must be an abelian extension of $a_2$ or $h_3$. This information alone is sufficient to proceed with the calculations, though these are simplified by appealing to a classification of 4-dimensional solvable Lie algebras that appears for example in [12]. The algebras we need to consider are listed in the following table, with the structure constants encapsulated in the exterior derivatives of a basis $(f^1, f^2, f^3, f^4)$ of $g^\ast$. In each case, $df^1 = 0$. The real parameters $\alpha, \beta$ satisfy $0 \leq \alpha \leq 2$ and $\beta \geq 0$.

| type          | $(df^2, df^3, df^4)$                                      | extension of |
|---------------|---------------------------------------------------------|--------------|
| $g_2 \oplus g_2$ | $(0, f^{13}, f^{24})$                                | $a_2$        |
| $g_{4,1}$     | $(0, f^{13} + f^{14} + f^{23})$                        | $a_2, h_3$   |
| $g_{4,2}$     | $(0, f^{13} - f^{42}, f^{14} - f^{23})$                | $a_2$        |
| $g_{4,9}(\alpha)$ | $((1 - \alpha)f^{12}, -f^{13}, -\alpha f^{14} - f^{23})$ | $h_3$        |
| $g_{4,10}$    | $(f^{12}, f^{12} + f^{13}, f^{23} + 2f^{14})$         | $h_3$        |
| $g_{4,11}(\beta)$ | $(\beta f^{12} + f^{13}, -f^{12} + \beta f^{13}, -f^{23} + 2\beta f^{14})$ | $h_3$        |

Let $g$ be one of $g_{4,1}$, $g_{4,9}(\alpha)$, $g_{4,10}$, $g_{4,11}(\beta)$. In each case, the above basis satisfies

$$
\begin{align*}
&df^1 = 0, \\
&df^2 \in \langle f^{12}, f^{13} \rangle, \\
&df^3 \in \langle f^{12}, f^{13} \rangle, \\
&df^4 \in \langle f^{23}, f^{14} \rangle.
\end{align*}
$$

(3.1)
Apply the Gram-Schmidt process to obtain an orthonormal basis \((e^1, e^2, e^3, e^4)\) satisfying

\[ e^j = \sum_{i=1}^{j} a^j_i e^i, \quad a^j_j \neq 0. \]  

(3.2)

The relations (3.1) then become

\[
\begin{align*}
    d e^1 &= 0, \\
    d e^2 &= c_{12}^2 e^{12} + c_{13}^3 e^{13}, \\
    d e^3 &= c_{12}^3 e^{12} + c_{13}^3 e^{13}, \\
    d e^4 &= c_{12}^4 e^{12} + c_{13}^4 e^{13} + c_{14}^4 e^{14} + c_{23}^4 e^{23},
\end{align*}
\]

(3.3)

for certain real constants \(c_{ij}^k\).

In the new scheme of things, each line on the right-hand side must include all terms from the previous line. Even if the isomorphism class of \(g\) is fixed, the structure constants \(c_{ij}^k\) are allowed to vary to reflect all possible choices of inner product. However, we must first ensure that (3.3) still describes a Lie algebra.

3.4 Lemma. Further to (3.3), we may suppose that (i) \(c_{23}^4 = 1\), (ii) \(c_{14}^4 = c_{12}^2 + c_{13}^3\), and (iii) \(c_{12}^1 = 0\).

Proof. The Jacobi identity is equivalent to the set of equations \(\{d(de^i) = 0 : 1 \leq i \leq 4\}\).

In our case, only the equation \(d(de^4) = 0\) is in doubt, and this yields

\[ 0 = c_{23}^4 (c_{12}^2 + c_{13}^3 - c_{14}^4). \]

For the Lie algebras under consideration, \(f^{23}\) appears with a non-zero coefficient in \(df^4\). It follows from (3.2) that \(c_{23}^1 \neq 0\) whence (ii).

Given the similar nature of \(de^2, de^3\), the form of (3.3) is preserved by a rotation of the form

\[
\begin{align*}
    e^2 &\mapsto (\cos \theta)e^2 + (\sin \theta)e^3, \\
    e^3 &\mapsto -(\sin \theta)e^2 + (\cos \theta)e^3.
\end{align*}
\]

(3.5)

Applying this substitution,

\[ d e^4 = (c_{12}^4 \cos \theta - c_{13}^4 \sin \theta) e^{12} + \ldots \]

and we can choose \(\theta\) so that (iii) is satisfied relative to the new orthonormal basis.

Finally, performing the overall scaling \(e^i \mapsto c_{23}^k e^i\) gives (i). \(\square\)

The Weyl tensor can now be computed in terms of the \(c_{ij}^k\). The orientation of \((e^i)\) can be reversed by changing the sign of \(e^1\). This will alter the signs of the \(c_{1ij}\), but will not interfere with the above assumptions. It therefore suffices to compute \(W^+\), whose vanishing is not affected by the rescaling that accompanied condition (i).

We now proceed to solve \(W^+ = 0\), that represents \(\dim S_0^2(\wedge^+ g^*) = 5\) quadratic equations in the coefficients of (3.3). There remain 5 unknowns, namely

\[ c_{12}^2, c_{13}^2, c_{12}^3, c_{13}^3, c_{13}^4. \]
3.6 Proposition. Given the above assumptions, there are only two real solutions of $W^+ = 0$, namely

(i) $a^3_{13} = -1$, $a^4_{13} = 0$, $a^3_{12} = -a^2_{13}$, $a^2_{12} = -1$;

(ii) $a^3_{13} = -\frac{1}{2}$, $a^4_{13} = 0$, $a^3_{12} = -a^2_{13}$, $a^2_{12} = -\frac{1}{2}$.

Proof. Involves an application of the Appendix. 

Setting $a^2_{13} = \lambda/k$ with $k = 1$ or 2 gives

$$d e^1 = 0,$$
$$k d e^2 = -e^{12} - \lambda e^{13},$$
$$k d e^3 = \lambda e^{12} - e^{13},$$
$$k d e^4 = -2e^{14} + k e^{23}. \tag{3.7}$$

Let $(e_i)$ denote the dual basis. Then $(f_1, f_2, f_3, f_4) = (k e_1, e_2, e_3, e_4)$ satisfies (1.2) and (1.4) (the new $f$'s are unrelated to those in (3.2)). In terms of the table, it is easy to verify that $g_{\lambda} \simeq \begin{cases} g_{4.9}(2), & \text{if } \lambda = 0, \\ g_{4.11}(\frac{1}{\lambda}), & \text{if } \lambda > 0. \end{cases}$

To complete the proof of the theorem, similar arguments are required for $g_2 \oplus g_2$ and $g_{4.2}$. In the first case, the analogue of (3.3) is

$$d e^1 = 0,$$
$$d e^2 = 0,$$
$$d e^3 = c^3_{12} e^{12} + c^3_{13} e^{13},$$
$$d e^4 = c^4_{12} e^{12} + c^4_{13} e^{13} + c^4_{14} e^{14} + c^4_{23} e^{23} + c^4_{24} e^{24}.$$

The Jacobi identity implies that

$$a^4_{23} (a^3_{13} - a^4_{14}) + a^4_{13} a^4_{24} = 0.$$

An overall scaling can be used to set $a^4_{24} = 1$ and this leaves 5 unknowns. There are no solutions to $W^+ = 0$.

The equations for $g_{4.2}$ are initially more complicated. Gram-Schmidt yields

$$d e^1 = 0,$$
$$d e^2 = 0,$$
$$d e^3 = c^3_{12} e^{12} + c^3_{13} e^{13} + c^3_{14} e^{14} + c^3_{23} e^{23} + c^3_{24} e^{24},$$
$$d e^4 = c^4_{12} e^{12} + c^4_{13} e^{13} + c^4_{14} e^{14} + c^4_{23} e^{23} + c^4_{24} e^{24}.$$

A rotation in the plane generated by $e^1, e^2$, in analogy to (3.5), enables us to arrange that $c^3_{14} = 0$ and $c^3_{24} \neq 0$. After an overall scaling of the basis we may assume that $c^3_{24} = 1$. The Jacobi identity then implies that

$$c^4_{13} = 0, \quad c^4_{13} = c^3_{13}.$$
This leaves 6 unknowns, sufficiently few for the program to reveal that once again there are no solutions with \( W^+ = 0 \).

4. Further properties

**Invariant complex structures.** To check whether the manifold \((G_\lambda, g_k)\) is hyper-hermitian, we take a standard basis
\[
\begin{align*}
\omega_1 &= e^{12} + e^{34}, \\
\omega_2 &= e^{13} + e^{42}, \\
\omega_3 &= e^{14} + e^{23}
\end{align*}
\]
of self-dual 2-forms. Let \( I_1, I_2, I_3 \) denote the corresponding left-invariant almost complex structures, and \( \theta_1, \theta_2, \theta_3 \) the Lee forms defined by \( d\omega_i = \omega_i \wedge \theta_i \). An easy consequence of (3.7) is that
\[
\begin{align*}
\theta_1 &= -\frac{3}{k} e^1 - \lambda e^4 = \theta_2, \\
\theta_3 &= -(1 + \frac{2}{k}) e^1.
\end{align*}
\]
The equality between \( \theta_1 \) and \( \theta_2 \) implies (via some standard theory \[11\]) that \( I_3 \) is always integrable. On the other hand, \( \theta_1 \) and \( \theta_3 \) are only equal if \( \lambda = 0 \) and \( k = 1 \).

4.1 Corollary. The Lie group \( G_\lambda \) admits an invariant complex structure \( I_3 \) for any \( \lambda \), but an invariant hypercomplex structure \((I_1, I_2, I_3)\) only for \( \lambda = 0 \). This hypercomplex structure is compatible with \( g_1 \).

**Properties of \( g_1 \).** Suppose that \( \lambda = 0 \) and \( k = 1 \). Using the coordinates \( t, x, y, u \) as in (1.3) and the substitution \( v = e^t \), take
\[
\begin{align*}
e^1 &= dt, \\
e^2 &= \frac{1}{v} dx, \\
e^3 &= \frac{1}{v} dy, \\
e^4 &= \frac{1}{v^2} (du + \frac{1}{2} (xdy - ydx)).
\end{align*}
\]
If we further set
\[
\begin{align*}
\tilde{\omega}_1^+ &= vd\nu \wedge dx \pm (dy \wedge du + \frac{1}{2} ydy \wedge dx) \\
\tilde{\omega}_2^+ &= vd\nu \wedge dy \pm (du \wedge dx + \frac{1}{2} xdx \wedge dy) \\
\tilde{\omega}_3^+ &= dv \wedge du - \frac{1}{2} (xdy \wedge dv - ydx \wedge dv) \pm vdx dy,
\end{align*}
\]
then \( \tilde{\omega}_i^+ = v^3 \omega_i \). The following is immediate.
4.2 Lemma. Five of the six 2-forms $\tilde{\omega}^\pm_i$ are closed, all except $\tilde{\omega}^-_3$.

The fact that $d\tilde{\omega}^+_i = 0$ for all $i$ implies that the almost-complex structures $I_1, I_2, I_3$ are all integrable, and $v^3g_1$ is a hyperkähler metric. This observation is contained in [5]; in fact

$$v^3g_1 = v(dx^2 + dy^2 + dv^2) + \frac{1}{v}(du + \frac{1}{2}(xdy - ydx))^2$$

(4.3)

is in the form of the Gibbons-Hawking ansatz for a hyperkähler metric with $S^1$ action. On the other hand, if $\sigma = (\cos \theta)\omega^-_1 + (\sin \theta)\omega^-_2$ then (for any $\theta$) $(v^3g_1, \sigma)$ is an almost-Kähler Einstein structure which is not Kähler. There is a family of similar local counterexamples to the Goldberg conjecture [2].

Links with other geometries. Let $z = x + iv$, and consider the upper half plane $H = \{z \in \mathbb{C} : v > 0\}$. Then $v^3g_1$ coincides with the natural metric on the cotangent bundle $T^*H$ induced from a ‘special Kähler’ metric on $H$. This aspect of the above example is explained in [11]. Here, we merely remark that special Kähler metrics are determined locally by a ‘holomorphic prepotential’ $F$ that here we take to be $\frac{1}{6}z^3$ [5]. This allows one to introduce a dual or ‘conjugate’ holomorphic coordinate

$$w = \frac{1}{2}z^2 = \frac{1}{2}(x^2 - v^2) - xv i,$$

whose real part we denote by $\tilde{x}$. The (incomplete) special Kähler metric $v(dx^2 + dv^2)$ on $H$ has associated 2-form

$$\omega = dx \wedge d\tilde{x} = \frac{v}{2i}dz \wedge d\bar{z}.$$  

Decreeing the 1-forms $dx, d\tilde{x}$ to be parallel defines a flat symplectic connection on $H$ which can be used to construct the hyperkähler metric (4.3) in a functorial manner.

The Lie algebra $\mathfrak{g}_0$ also features in [7], as one example in a study of Einstein metrics constructed from left-invariant metrics on nilpotent groups. Higher dimensional examples give rise to Einstein metrics on solvable extensions, and related metrics with exceptional holonomy.

Quadratic components of $W$. In four dimensions, the space $W$ of Weyl tensors has symmetric square

$$S^2W \cong S^2W^+ \oplus (W^+ \otimes W^-) \oplus S^2W^-,$$

(4.4)

and there exist $SO(4)$-irreducible 9-dimensional representations $\mathcal{V}^\pm$ for which

$$S^2\mathcal{W}^\pm \cong \mathcal{V}^\pm \oplus \mathcal{W}^\pm \oplus \mathbb{R}$$

(see for example [10]). Thus, the vanishing of $W^+$ reduces the number of non-zero components of $W \otimes W$ in (4.4) from 7 to 3. Moreover, if $W^+$ has repeated eigenvalues
as in 1.5 then the $\mathcal{W}^+$-component of $W^+ \otimes W^+$ is proportional to $W^+$ (this also implies that certain components of $W^+ \otimes W^+ \otimes W^+$ vanish in $S^3\mathcal{W}$).

On a higher dimensional manifold $M$, it is known that the Weyl tensor $W$ is irreducible. Appropriate analogues of anti-self-duality can in theory be defined in terms of components of $S^2\mathcal{W}$. Of special interest are likely to be conditions imposed on $W$ by the existence of one or more orthogonal complex structures on $M$.

5. Appendix

The essential part of the program used to solve the equation $W^+ = 0$ is reproduced below. It is set up to mimic the assumptions 3.3 and 3.4, through the remaining cases of $\mathfrak{g}_2 \oplus \mathfrak{g}_2$ and $\mathfrak{g}_{4,2}$ are similar.

Once the solutions are found it is an easy matter to substitute back using for example assign(so[1]) and determine $W^-$ by setting $z:=-1$.

```plaintext
# Define the structure constants
for i to 4 do defform(e[i]=1) od:
for i to 5 do defform(a[i]=const) od:
defform(d(e[1])=0):
defform(d(e[2])=a[1]*e[1]&^e[2]+a[2]*e[1]&^e[3]):
defform(d(e[3])=a[3]*e[1]&^e[2]+a[4]*e[1]&^e[3]):
defform(d(e[4])=a[5]*e[1]&^e[3]+(a[1]+a[4])*e[1]&^e[4]+e[2]&^e[3]):

# Define the set of connection coefficients
gset:={}:
for k to 4 do g[k]:=array(antisymmetric,1..4,1..4) od:
for k to 4 do for j to 4 do for i to j-1 do
defform(g[k][i,j]=const); gset:=gset union {g[k][i,j]} od od od:

# Define and solve the Cartan equations
fset:={}:
for k to 4 do
d[e[k]]:=sum(sum(g[m][k,n]*d(e[m]),m=1..4),n=1..4)
od:
c:=(k,i,j)->coeff(d[e[k]]-d(e[k]),e[i]&^e[j]):
for k to 4 do for j to 4 do for i to j-1 do
fset:=fset union {c(k,i,j)=c(k,j,i)}
od od od:
sol:=solve(fset,gset):
assign(sol):

# Define the Riemann curvature tensor
r1:=(i,j)->sum(g[p][i,j]*d(e[p]),p=1..4):
```
\[
\begin{align*}
  r2 & := (i,j) \rightarrow \sum \left( \sum \left( \sum (g[u][i,r] \ast g[v][r,j] \ast e[u] \&^e[v], u=1..4, v=1..4), r=1..4) \right) \\
  r & := (i,j,x,y) \rightarrow \text{coeff}(\text{simpform}(r1(i,j) - r2(i,j)), e[x] \&^e[y]) \\
  \text{for} \ i \ \text{to} \ 4 \ \text{do} \ \text{for} \ j \ \text{to} \ 4 \ \text{do} \\
  R[i,j] & := \text{array}(\text{antisymmetric}, 1..4, 1..4) \\
  \text{for} \ y \ \text{to} \ 4 \ \text{do} \\
  \quad \text{for} \ x \ \text{to} \ y-1 \ \text{do} \ R[i,j][x,y] := r(i,j,x,y) - r(i,j,y,x) \ \text{od} \ \\
  \text{od} \ \text{od} \\
  \# \ \text{Define the Ricci, scalar and Weyl+ curvature} \\
  \text{ric} & := \text{array}(\text{symmetric}, 1..4, 1..4) \\
  \text{for} \ j \ \text{to} \ 4 \ \text{do} \ \text{for} \ i \ \text{to} \ j \ \text{do} \\
  \quad \text{ric}[i,j] := \sum (R[i,p][j,p], p=1..4) \\
  \text{od} \ \text{od} \\
  \text{sc} & := \sum (\text{ric}[q,q], q=1..4) \\
  \text{W} & := \text{array}(\text{symmetric}, 1..3, 1..3): \ z:=1: \\
  \quad W[1,1] := R[1,2][1,2] + R[3,4][3,4] + 2z*\text{R}[1,2][3,4] - \text{sc}/6: \\
  \quad W[2,2] := R[1,3][1,3] + R[4,2][4,2] + 2z*\text{R}[1,3][4,2] - \text{sc}/6: \\
  \quad W[3,3] := R[1,4][1,4] + R[2,3][2,3] + 2z*\text{R}[1,4][2,3] - \text{sc}/6: \\
  \quad W[1,2] := R[1,2][1,3] + z*\text{R}[1,2][4,2] + z*\text{R}[3,4][1,3] + \text{R}[3,4][4,2]: \\
  \quad W[1,3] := R[1,2][1,4] + z*\text{R}[1,2][2,3] + z*\text{R}[3,4][1,4] + \text{R}[3,4][2,3]: \\
  \quad W[2,3] := R[1,3][1,4] + z*\text{R}[1,3][2,3] + z*\text{R}[4,2][1,4] + \text{R}[4,2][2,3]: \\
  \# \ \text{Solve Weyl+=0} \\
  \text{hset} & := \{W[1,1]=0, W[1,2]=0, W[1,3]=0, W[2,2]=0, W[2,3]=0\}: \\
  \text{so} & := \text{solve}(\text{hset}): \\
\end{align*}
\]

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