TWO-GRID ECONOMICAL ALGORITHMS FOR PARABOLIC
INTEGRO-DIFFERENTIAL EQUATIONS WITH NONLINEAR
MEMORY

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Abstract. In this paper, several two-grid finite element algorithms for solving parabolic integro-differential equations (PIDEs) with nonlinear memory are presented. Analysis of these algorithms is given assuming a fully implicit time discretization. It is shown that these algorithms are as stable as the standard fully discrete finite element algorithm, and can achieve the same accuracy as the standard algorithm if the coarse grid size \( H \) and the fine grid size \( h \) satisfy \( H = O(h^{\frac{1}{r}}) \). Especially for PIDEs with nonlinear memory defined by a lower order nonlinear operator, our two-grid algorithm can save significant storage and computing time.

Key words. parabolic integro-differential equation, error estimate, finite element method, stability, two-grid method, backward Euler scheme

AMS subject classifications. 65M60, 65R20, 65L05

1. Introduction. The main purpose of this paper is to present some discretization techniques based on two finite element subspaces for solving parabolic integro-differential equations (PIDEs) with nonlinear memory:

\[
u_t + A u + \int_0^t K(t-s) B u(s) ds = f(x,t), \quad (x,t) \in \Omega \times (0,T], \tag{1.1}
\]

\[u(x,t) = 0, \quad (x,t) \in \partial \Omega \times (0,T], \tag{1.2}
\]

\[u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.3}
\]

where \( \Omega \subset \mathbb{R}^d (d \geq 1) \) is a bounded and polyhedral domain with a piecewise smooth boundary \( \partial \Omega \), \( K(t) \) is a smooth or nonsmooth memory kernel, and \( f \) is a known function. \( A \) is a symmetric positive definite second-order elliptic operator with smooth coefficients in \( x \) and \( t \), and \( B \) is a nonlinear operator of at most second order; that is,

\[B u = -\nabla \cdot (\alpha(x,u)\nabla u + \beta(x,u)) + \gamma(x,u) \cdot \nabla u + g(x,u). \tag{1.4}\]

For brevity, we will drop the dependence of variable \( x \) in \( \alpha(x,u), \beta(x,u), \gamma(x,u) \), and \( g(x,u) \) in the following exposition. We assume that the functions \( \alpha(u), \beta(u), \gamma(u), \) and \( g(u) \) (with the range \( \mathbb{R}^{d \times d}, \mathbb{R}^d, \mathbb{R}^d \), and \( \mathbb{R}^1 \), respectively) are smooth and bounded together with the Gateaux derivative. For the functions \( \beta(u) \) and \( g(u) \), we also assume that \( \beta(0) = 0 \) and \( g(0) = 0 \).

Equations of the above type, or linear versions thereof, can arise from many physical processes in which it is necessary to take into account the effects of memory due to the deficiency of the usual diffusion equations [20, 33, 39]. For approximating the solution \( u \) of PIDEs, both finite difference and finite element methods have been investigated extensively in the past for both the linear and nonlinear problem (see, for example, [8, 9, 29, 31, 37, 12, 54]). Recently, several new numerical methods such as mixed finite element method, finite volume element method, and discontinuous

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Galerkin method for space discretization or time discretization have been proposed to solve PIDEs (see, for example, [14, 22, 36, 36, 38, 39, 40, 41, 41, 42, 43, 44]).

The two-grid method based on two finite element spaces, one on a coarse grid and one on a fine grid, was first developed by Xu [47, 48, 49, 50] for nonsymmetric linear and nonlinear elliptic problems. Since then, the two-grid method for elliptic problems has been investigated further, e.g., Axelsson and Layton [3], Xu and Zhou [51], Li and Huang [28], and Bi and Ginting [4, 5]. In these works, theoretical study and numerical experiments show that the combined use of the numerical method such as finite element method and finite difference method, and the two-grid technique is computationally more efficient than the original method. Due to this better practical performance, the two-grid method has been widely applied to the study of eigenvalue problems [52, 53, 24], steady Navier Stokes equations [27, 21, 23, 15], the time-dependent Navier Stokes problem [26, 1, 2, 40, 44], the nonlinear parabolic problem [10, 32, 17, 10, 40, 14, 33, 58], and nonlinear hyperbolic equations [11]. Recently, Jin, Shu, and Xu [26] used this technique to solve decoupling systems of partial differential equations; Mu and Xu [34] and Cai, Mu, and Xu [7] employed it for the mixed Stokes-Darcy model. In [45], we proposed the two-grid algorithms based on the backward Euler scheme and finite element approximation for semi-linear PIDEs, and studied the long-time stability and error estimates of the two-grid algorithms.

In this paper, we present some two-grid algorithms for PIDEs with nonlinear memory and perform theoretical analysis that demonstrates our methods’ ability to match the accuracy of the classic finite element method by (1) solving a nonlinear problem on a coarse space $S_H$ and (2) solving a symmetric positive definite linear problem on the fine space $S_h$. Thus, solving PIDEs with nonlinear memory is not much more difficult than solving one linear problem, as $\dim S_H \ll \dim S_h$ and the work involved in solving the nonlinear problem on the coarse grid is relatively limited.

It is worth adding that when $\alpha \equiv 0$, our algorithm significantly reduces computational memory and storage requirements. A practical difficulty of numerical methods for PIDEs is that all previous values must be stored, as they all enter subsequent equations. In order to reduce memory requirements, some economical schemes have been proposed (for example, see, [43, 25]). However, these schemes either require more regularities on the solution $u$ [43], or they cannot be applied to nonlinear problem [25].

The remainder of this article is organized as follows: In Section 2, we present some conventions and notations that will be used throughout the article. In Section 3, the stability and error estimate of the classic fully discrete finite element method are discussed. The two-grid algorithms for PIDEs with nonlinear memory are presented and the stability and error estimates of these algorithms are discussed in Section 4. In Section 5, we offer some concluding remarks.

Throughout this paper, we use the letters $C$ and $c$ (with and without subscripts) to denote a generic positive constant that stand for different values depending on the context in different equations. When it is not important to keep track of these constants, we conceal the letter $C$ or $c$ in the notation $\lesssim$ or $\gtrsim$, such that $x \lesssim y$ means $x \leq Cy$ and $x \gtrsim y$ means $x \geq cy$.

2. Preliminaries. For any non-negative integer $r$ and number $p \geq 1$, let $W^{r,p}(\Omega)$ be the standard Sobolev space with a norm $\|\cdot\|_{r,p}$ given by $\|v\|_{r,p} = \sum_{|\kappa| \leq r} \|D^\kappa v\|_{L^p(\Omega)}$ (with the usual modification if $p = \infty$). This Sobolev space is also equipped with the seminorm $|v|_{r,p} = \sum_{|\kappa| = r} \|D^\kappa v\|_{L^p(\Omega)}$. For $p = 2$, we denote $H^r = W^{r,2}(\Omega)$ and take $H_0^1$ as the subspace of $H^1$ consisting of functions with a vanishing trace on $\partial \Omega$. For
simplicity, we also use notations $\| \cdot \|_r$, $\| \cdot \|_2$, and $\| \cdot \|_{\infty}$, and $\| \cdot \|$, such that $\| \cdot \|_r = \| \cdot \|_{r,2}$, $\| \cdot \| = \| \cdot \|_2$, and $\| \cdot \|_{\infty} = \| \cdot \|_{0,\infty}$, and $\| \cdot \| = \| \cdot \|_{r,2}$.

Let $\{S_h\}_{0<h\leq 1}$ be a family of finite-dimensional subspaces of $H^1_0$, with the following approximation properties:

$$
\inf_{\chi \in S_h} \{ \|v - \chi\| + h\|v - \chi\|_1 \} \lesssim h^r\|v\|_r, \quad v \in H^r \cap H^1_0, \quad r \geq 1 + \frac{d}{2}. \tag{2.1}
$$

We also assume that $\{S_h\}_{0<h\leq 1}$ satisfies the inverse hypothesis: there exists a constant $C > 0$ independent of $h$ such that

$$
\|\nabla \chi\|_{\infty} \leq C h^{-d/2} \|\chi\|, \quad \chi \in S_h. \tag{2.2}
$$

The weak formulation of the problem (1.1), (1.3) is: Find $u \in H^1_0(\Omega)$ such that

$$
(u_t, v) + A(u, v) + \int_0^t K(t - s)B(u(s), v)ds = (f, v), \quad v \in H^1_0, \tag{2.3}
$$

$$
u(0) = u_0, \tag{2.4}
$$

where $A(\cdot, \cdot)$ is the bilinear form associated with the operator $A$ on $H^1_0 \times H^1_0$ and $B(\cdot, \cdot)$ is defined by

$$
B(u, v) = (\alpha(u)\nabla u + \beta(u), \nabla v) + (\gamma(u) \cdot \nabla u + g(u), v), \quad u, v \in W^{1,\infty} \cap H^1_0.
$$

$(\cdot, \cdot)$ denotes the inner product in $L^2(\Omega)$. We always assume that $A$ is coercive and continuous with coercivity constant $\nu_0$ and continuity constant $\nu_1$. That is, we have

$$
A(v, v) \geq \nu_0 \|v\|^2_1 \quad \forall v \in H^1_0, \tag{2.5}
$$

$$
|A(u, v)| \leq \nu_1 \|u\|_1 \|v\|_1 \quad \forall u, v \in H^1_0. \tag{2.6}
$$

In view of the assumptions on the functions $\alpha(u)$, $\beta(u)$, $\gamma(u)$, and $g(u)$, it is easily verified that there exists a positive constant $\mu_0$ such that

$$
|B(u, v)| \leq \mu_0 \|u\|_1 \|v\|_1. \tag{2.7}
$$

For the time discretization of (1.1)-(1.3) we will consider the backward Euler scheme. To analyze the discretization on a time interval $(0, T)$, let $N$ be a positive integer, $\Delta t = T/N$, and let $t_n = n\Delta t$. As the truncation error of the backward Euler scheme is $O(\Delta t)$, we introduce a quadrature formula with a truncation error $O(\Delta t)$,

$$
\Delta t \sum_{i=1}^n \omega_{ni}g(t_i) = \int_0^{t_n} K(t_n - s)g(s)ds + O(\Delta t). \tag{2.8}
$$

Given our emphasis on two-grid discretization in space, we will not discuss how to obtain the numbers $\omega_{ni}$, but only assume that there exists a positive constant $K_1$ such that $|\omega_{ni}| \leq K_1$ for any $1 \leq n \leq N$, $1 \leq i \leq n$ and that $\omega_{nn} \neq 0$. Therefore, the problem considered in this paper must be discretized by a fully implicit scheme. Thus, the backward Euler fully discrete finite element approximation of problem (1.1), (1.3) is defined as a sequence $\{U^n\}_{n=0}^N$, such that

$$
(\partial_t U^n, v) + A(U^n, v) + \Delta t \sum_{i=1}^n \omega_{ni}B(U^i, v) = (f^n, v), \quad v \in S_h, \quad n \geq 1, \tag{2.9}
$$

$$
U^0 = u_0^h, \tag{2.10}
$$
where \( \bar{\partial} U^n = U^n - U^{n-1} \), \( u^n \) is an appropriate approximation of \( u_0 \) in \( S_h \), \( f^n = f(t_n) \). We know that (2.9) will result in a truncation error \( O(\Delta t) \) in time. But for nonlinear problems considered in this paper (\( \omega_{nn} \neq 0 \)), the solution of a nonlinear algebraic system is required at each time step. To decrease the amount of computational work, we propose using a two-grid technique to solve the PIDEs with nonlinear memory. With this technique, at each time step, solving a nonlinear problem on the fine space \( S_h \) is reduced by solving a nonlinear problem on the coarse space \( S_H \) and solving a linear SPD problem on the fine space \( S_h \).

For functions that vanish on the boundary, we recall Poincare’s inequality: there exists a constant \( \mathcal{P} \) such that \( \forall v \in H^1 \), \( \|v\| \leq \mathcal{P}|v|_1 \).

We make extensive use of the \( \epsilon \)-type inequality \( 2ab \leq \epsilon a^2 + b^2 \), \( \epsilon > 0 \), and of the inequality \( a^2 + b^2 \leq (|a| + |b|)^2 \). The results of this paper are based on the identity

\[
2(a^{n+1}, a^{n+1} - a^n) = |a^{n+1}|^2 - |a^n|^2 + |a^{n+1} - a^n|^2,
\]

(2.11)

and the following Gronwall lemma proved in [18].

**Lemma 2.1** (Discrete Gronwall lemma [18]). Let \( 0 \leq \lambda < 1 \), and \( a_n, b_n, c_n, \lambda_n \geq 0 \) with \( \{c_n\} \) being monotonically increasing. Then

\[
a_n + b_n \leq \sum_{j=\omega}^{n-1} \lambda_j a_j + \lambda c_n + c_n, \quad n = \omega, \omega + 1, \ldots
\]

(2.12)

implies for \( n = \omega, \omega + 1, \ldots \)

\[
a_n + b_n \leq \frac{c_n}{1 - \lambda} \prod_{j=\omega}^{n-1} \left(1 + \frac{\lambda_j}{1 - \lambda}\right) \leq \frac{c_n}{1 - \lambda} \exp \left(\frac{1}{1 - \lambda} \sum_{j=\omega}^{n-1} \lambda_j\right).
\]

### 3. Error estimate for the classic fully discrete finite element method.

In this section, we discuss the stability and error estimate of the standard fully discrete finite element method (2.9), (2.10). First, we prove the stability of the solution of (2.9) and (2.10).

**Theorem 3.1.** Let \( U^n \) be the solution obtained by (2.9) and (2.10). Then for all

\[
\Delta t \leq \min \left\{ \frac{1}{2}, \frac{7\nu_0^2}{8\mu_0^2 \mathcal{K}^2 T} \right\}.
\]

(3.1)

we have

\[
\|U^n\| + \left(\sum_{i=1}^{n} \|U^i - U^{i-1}\|^2\right)^{1/2} + \frac{\sqrt{\nu_0}}{2} \left(\sum_{i=1}^{n} \Delta t \|U^i\|_1^2\right)^{1/2}
\]

\[
\leq E_n^{1/2} \left(\|U^0\|^2 + \Delta t \sum_{i=1}^{n} \|f_i\|^2\right)^{1/2},
\]

(3.2)

where \( E_n = 6 \max\{e^{2\Delta t}, e^{(2\mu_0 K_{\text{ct}}/\nu_0)^2}\} \).
Proof. By taking \( v = 2\Delta tU^n \) in (2.9) and using (2.11), we obtain

\[
\|U^n\|^2 - \|U^{n-1}\|^2 + \|U^n - U^{n-1}\|^2 + 2\nu_0 \Delta t \|U^n\|^2 + 2(\Delta t)^2 \sum_{i=1}^{n} \omega_{ni} B(U^i, U^n) 
\leq 2\Delta t \|f^n\| \|U^n\|. \tag{3.3}
\]

Using (2.7), we have

\[
\|U^n\|^2 - \|U^{n-1}\|^2 + \|U^n - U^{n-1}\|^2 + 2\nu_0 \Delta t \|U^n\|^2 
\leq \mu_0 (\Delta t)^2 \sum_{i=1}^{n} |\omega_{ni}| (\frac{1}{\epsilon} \|U^i\|^2 + \epsilon \|U^n\|^2) + \Delta t (\|U^n\|^2 + \|f^n\|^2). \tag{3.4}
\]

Choose \( \epsilon = \nu_0/(\mu_0 K_1 t_n) \) to obtain

\[
\|U^n\|^2 + \|U^n - U^{n-1}\|^2 + \nu_0 \Delta t \|U^n\|^2 
\leq \|U^{n-1}\|^2 + \frac{\mu_0^2 K_1^2 t_n}{\nu_0} (\Delta t)^2 \sum_{i=1}^{n} \|U^i\|^2 + \Delta t \sum_{i=1}^{n} \|f^i\|^2. \tag{3.5}
\]

By summation, we have

\[
\|U^n\|^2 + \sum_{i=1}^{n} \|U^i - U^{i-1}\|^2 + \nu_0 \Delta t \sum_{i=1}^{n} \|U^i\|^2 
\leq \|U^0\|^2 + \Delta t \sum_{i=1}^{n} \|U^i\|^2 + (\Delta t)^2 \sum_{i=1}^{n} \frac{\mu_0^2 K_1^2 t_i}{\nu_0} \sum_{j=1}^{i} \|U^j\|^2 + \Delta t \sum_{i=1}^{n} \|f^i\|^2, \tag{3.6}
\]

which implies that

\[
(1 - \Delta t) \|U^n\|^2 + \sum_{i=1}^{n} \|U^i - U^{i-1}\|^2 + \left( \nu_0 - \frac{\mu_0^2 K_1^2 t_n}{\nu_0} \Delta t \right) \Delta t \sum_{i=1}^{n} \|U^i\|^2 
\leq \|U^0\|^2 + \Delta t \sum_{i=1}^{n-1} \max \left\{ 1, \frac{4\mu_0^2 K_1^2 t_i}{\nu_0^2} \right\} \left( \|U^i\|^2 + \frac{\nu_0}{4} (\Delta t \sum_{j=1}^{i} \|U^j\|^2) \right) + \Delta t \sum_{i=1}^{n} \|f^i\|^2. \tag{3.7}
\]

Since condition (3.1) implies that \( 1 - \Delta t \geq \frac{1}{2} \) and \( \nu_0 - \frac{\mu_0^2 K_1^2 t_n}{\nu_0} \Delta t \geq \frac{\nu_0}{8} \), with the aid of discrete Gronwall lemma 2.1, we obtain

\[
\|U^n\|^2 + 2 \sum_{i=1}^{n} \|U^i - U^{i-1}\|^2 + \frac{\Delta t \nu_0}{4} \sum_{i=1}^{n} \|U^i\|^2 
\leq E_n \left( \|U^0\|^2 + \Delta t \sum_{i=1}^{n} \|f^i\|^2 \right), \tag{3.8}
\]

which implies (3.2). Thus the proof is completed.

Remark. From (3.7), we find that for a given integral interval \((0, T)\) the stepsize \( \Delta t \) is determined by the ratio of \( \nu_0 \) to \( \mu_0 \) and increases as the value of coercivity constant \( \nu_0 \) increases.
Now let us take $v = 2\Delta t\bar{U}^n$ in (2.9) to obtain
\[
2\Delta t||\bar{U}^n||^2 + A(U^n, U^n) - A(U^{n-1}, U^{n-1}) + A(U^n - U^{n-1}, U^n - U^{n-1})
\]
\[+2(\Delta t)^2\sum_{i=1}^{n} \omega_{ni} B(U^i, \bar{U}^n)
\]
\[= 2\Delta t(f^n, \bar{U}^n). \tag{3.9}
\]

Since
\[
2\Delta t(f^n, \bar{U}^n) \leq \frac{1}{2}\Delta t||f^n||^2 + 2\Delta t||\bar{U}^n||^2 \tag{3.10}
\]
and
\[
2(\Delta t)^2\sum_{i=1}^{n} |\omega_{ni} B(U^i, \bar{U}^n)| \leq 2\Delta t\mu_0 \sum_{i=1}^{n} |\omega_{ni}|||U^i||_1||U^n - U^{n-1}||_1
\]
\[\leq \frac{t_n\mu_0^2 K_1^2}{\nu_0} \Delta t \sum_{i=1}^{n} ||U^i||_1^2 + \nu_0||U^n - U^{n-1}||_1^2, \tag{3.11}
\]
(3.9) becomes
\[
\nu_0||U^n||_1^2 \leq \frac{1}{2}\Delta t||f^n||^2 + \frac{t_n\mu_0^2 K_1^2}{\nu_0} \Delta t \sum_{i=1}^{n} ||U^i||_1^2 + \nu_1||U^n - U^{n-1}||_1^2. \tag{3.12}
\]
Then we have the following result.

**Theorem 3.2.** Let $U^n$ be the solution obtained by (2.9) and (2.10). Then for all $\Delta t \leq \frac{\nu_0}{2\mu_0^2 K_1^2 T}$,
\[
\Delta t \leq \frac{\nu_0^2}{2\mu_0^2 K_1^2 T}, \tag{3.13}
\]
we have
\[
||U^n||_1 \leq C \left( ||U^0||_1^2 + \Delta t \sum_{i=1}^{n} ||f^i||^2 \right)^{1/2}. \tag{3.14}
\]

**Proof.** It follows from (3.13) that $\nu_0 - \frac{t_n\mu_0^2 K_1^2}{\nu_0} \Delta t \geq \frac{\nu_0}{2}$. Then an application of discrete Gronwall lemma 2.1 to (3.12) leads to
\[
||U^n||_1 \leq C \left( ||U^0||_1^2 + \Delta t \sum_{i=1}^{n} ||f^i||^2 \right),
\]
which implies (3.14). This completes the proof.

To estimate the error of the fully discrete approximation (2.10), we define, for $w, u, v \in W^{1,\infty} \cap H^1_0(\Omega)$,
\[
B_1(w; u, v) = (\alpha(w) \nabla u, \nabla v) + (\gamma(w) \cdot \nabla u, v).
\]
Due to the assumptions on $\alpha(u)$ and $\gamma(u)$, there exist a constant $\sigma$ such that
\[
|B_1(w; u, v)| \leq \sigma ||u||_1 ||v||_1. \tag{3.15}
\]
As usual, we write the error \( e^n = u(t_n) - U^n \) as
\[
e^n = u(t_n) - U^n = (u(t_n) - V_h u(t_n)) + (V_h u(t_n) - U^n) = \rho^n + \theta^n,
\]
where \( V_h u \) is the Ritz-Volterra projection of the solution \( u \) and defined by [9]
\[
A(u - V_h u, v) + \int_0^t K(t - s)B_1(u(s); u(s) - V_h u(s), v)ds = 0, \; v \in S_h. \quad (3.16)
\]
For \( \rho(t) = u(t) - V_h u(t) \), following the line of Cannon and Lin [9], we show that there exists \( C_0 > 0 \), independent of \( h \) and \( t \), such that (see, also, [12, 30])
\[
\|\rho(t)\|_1 + h\|\rho(t)\|_1 \leq C_0 h^{r'} \|u(t)\|_r, \quad t \geq 0, \quad (3.17)
\]
\[
\|\rho(t)\|_1 \leq C_0 h^{r'} (\|u(t)\|_r + \|u_t(t)\|_r), \quad (3.18)
\]
\[
\|\rho(t)\|_{\infty} \leq C_0 h^{r'} \ln h^{\|u(t)\|_{r, \infty}}, \quad (3.19)
\]
where
\[
\|u(t)\|_r = \|u(t)\|_r + \int_0^t \|u(\tau)\|_r d\tau, \quad \|u(t)\|_{r, \infty} = \|u(t)\|_{r, \infty} + \int_0^t \|u(\tau)\|_{r, \infty} d\tau,
\]
and there exists a positive constant \( C = C(u) \), independent of \( h \), such that
\[
\|\nabla V_h u\|_{\infty} + \|\nabla (V_h u)_e\|_{\infty} \leq C. \quad (3.20)
\]
Now we need to estimate the error \( \theta^n = V_h u(t_n) - U^n \).

**Theorem 3.3.** Let \( u \) and \( U^n \) be the solutions of (2.3)-(2.4) and (2.9)-(2.10), respectively. If
\[
16\sigma^2 K^2 T < \nu_0^2, \quad (3.21)
\]
then, for sufficiently small \( \Delta t \), we have
\[
\|\theta^n\| + ||\theta^n - \theta^{n-1}\| + \sqrt{\nu_0 \Delta t} \|\theta^n\|_1 \lesssim h^r + \Delta t. \quad (3.22)
\]

**Proof.** Firstly, it follows from (2.23) and (2.24) that
\[
\begin{align*}
(u_t - \partial U^n, v) + A(u - U^n, v) + \int_0^{t_n} K(t - s)B(u(s), v)ds - \Delta t \sum_{i=1}^n \omega miB(U^i, v) & = 0, \quad v \in S_h.
\end{align*}
\]
Then we find that \( \theta^n \) satisfies
\[
\begin{align*}
(\partial \theta^n, v) + A(\theta^n, v) + A(\rho^n, v) + \int_0^{t_n} K(t - s)B_1(u(s); u(s) - V_h u(s), v)ds & + \int_0^{t_n} K(t - s)B_1(u(s); V_h u(s), v)ds \nonumber \\
+ \int_0^{t_n} K(t - s)[(\beta(u(s), \nabla v) + (g(u(s)), v)]ds - \Delta t \sum_{i=1}^n \omega miB(U^i, v) & = -\left(\frac{\rho^n - \rho^{n-1}}{\Delta t}, v\right) - \left(u_t - \frac{u(t_n) - u(t_{n-1})}{\Delta t}, v\right), \quad v \in S_h. \quad (3.23)
\end{align*}
\]
By virtue of the assumptions on $\alpha$ satisfy Lipschitz conditions with Lipschitz constant $C$.

Using (3.16), we have

\[ (\partial \theta^n, v) + A(\theta^n, v) + \int_0^{t_n} K(t - s) B_1(u(s); V_h u(s), v) \, ds \]

\[-\Delta t \sum_{i=1}^{n} \omega_{ni} B_1(u(t_i); V_h u(t_i), v) + \Delta t \sum_{i=1}^{n} \omega_{ni} B_1(u(t_i); \theta^i, v) \]

\[ + \Delta t \sum_{i=1}^{n} \omega_{ni} \left[ \left( (\alpha(u(t_i)) - \alpha(U^i)) \nabla U^i, \nabla v \right) + \left( (\gamma(u(t_i)) - \gamma(U^i)) \cdot \nabla U^i, v \right) \right] \]

\[ + \int_0^{t_n} K(t - s)[\beta(u(s)), \nabla v] + (g(u(s)), v) \, ds \]

\[-\Delta t \sum_{i=1}^{n} \omega_{ni} \left[ (\beta(u(t_i)), \nabla v) + (g(u(t_i)), v) \right] \]

\[ + \Delta t \sum_{i=1}^{n} \omega_{ni} \left[ (\beta(u(t_i))) - \beta(U^i), \nabla v \right) + (g(u(t_i)) - g(U^i), v) \]

\[ = - \left( \rho^n - \rho^{n-1} \frac{\Delta t}{\Delta t}, v \right) - \left( u_t - \frac{u(t_n) - u(t_{n-1})}{\Delta t}, v \right), \quad \forall v \in S_h. \quad (3.24) \]

Now, in view of (2.8), we have

\[ \left| \int_0^{t_n} K(t - s) B_1(u(s); V_h u(s), v) \, ds - \Delta t \sum_{i=1}^{n} \omega_{ni} B_1(u(t_i); V_h u(t_i), v) \right| \lesssim \Delta t \| v \|_1 (3.25) \]

and

\[ \left| \int_0^{t_n} K(t - s)[\beta(u(s)), \nabla v] + (g(u(s)), v) \, ds \right| - \Delta t \sum_{i=1}^{n} \omega_{ni} \left[ (\beta(u(t_i)), \nabla v) + (g(u(t_i)), v) \right] \lesssim \Delta t \| v \|_1. \quad (3.26) \]

Due to (3.15), the fifth term on the left-hand side in (3.24) can be bounded as

\[ \left| \Delta t \sum_{i=1}^{n} \omega_{ni} B_1(u(t_i); \theta^i, v) \right| \leq \Delta t K_1 \sum_{i=1}^{n} \sigma \| \theta^i \|_1 \| v \|_1. \quad (3.27) \]

By virtue of the assumptions on $\alpha(u)$, $\beta(u)$, $\gamma(u)$ and $g(u)$, we know $\alpha$, $\beta$, $\gamma$ and $g$ satisfy Lipschitz conditions with Lipschitz constant $C_L$, and thus the sixth and ninth
The first term on the right-hand side in (3.24) can be bounded as

\[ \Delta t \sum_{i=1}^{n} \omega_{ni} \left[ \left( \alpha(u(t_i)) - \alpha(U^{i}) \right) \nabla U^i, \nabla v \right] + \left( \gamma(u(t_i)) - \gamma(U^{i}) \right) \cdot \nabla U^i, v \]

\[ \leq \Delta t \sum_{i=1}^{n} \omega_{ni} \left[ \left( \alpha(u(t_i)) - \alpha(U^{i}) \right) \nabla \theta^i, \nabla v \right] + \left( \gamma(u(t_i)) - \gamma(U^{i}) \right) \cdot \nabla \theta^i, v \]

\[ \leq \Delta t \sum_{i=1}^{n} \omega_{ni} \left[ \left( \alpha(u(t_i)) - \alpha(U^{i}) \right) \nabla V_h u(t_i), \nabla v \right] + \left( \gamma(u(t_i)) - \gamma(U^{i}) \right) \cdot \nabla V_h u(t_i), v \]

\[ \leq \Delta t K_1 \sum_{i=1}^{n} \sigma \| \theta^i \|_1 \| v \|_1 + C_L \Delta t K_1 \sum_{i=1}^{n} \| u(t_i) - U^i \| \| \nabla V_h u(t_i) \|_\infty \| v \|_1 \]

\[ \leq \Delta t K_1 \sum_{i=1}^{n} \sigma \| \theta^i \|_1 \| v \|_1 + C C_L \Delta t K_1 \sum_{i=1}^{n} (\| \rho^i \| + \| \theta^i \|) \| v \|_1, \quad (3.28) \]

where the estimate (3.20) has been used, and

\[ \left| \Delta t \sum_{i=1}^{n} \omega_{ni} \left[ \left( \beta(u(t_i)) - \beta(U^{i}) \right), \nabla v \right] + \left( g(u(t_i)) - g(U^{i}) \right), v \right| \]

\[ \leq C_L \Delta t K_1 \sum_{i=1}^{n} \| u(t_i) - U^i \| \| v \|_1 \leq C_L \Delta t K_1 \sum_{i=1}^{n} (\| \rho^i \| + \| \theta^i \|) \| v \|_1. \quad (3.29) \]

The first term on the right-hand side in (3.24) can be bounded as

\[ \left| \left( \frac{\rho^n - \rho^{n-1}}{\Delta t}, v \right) \right| \leq \frac{1}{\Delta t} \| \rho^n - \rho^{n-1} \| \| v \| \leq \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} \| \rho_t(s) \| ds \| v \|

\leq \frac{C_0 h^r}{\Delta t} \int_{t_{n-1}}^{t_n} (\| u(s) \|_r + \| u_t(s) \|_r) ds \| v \|; \quad (3.30) \]

and the last term can be bounded as

\[ \left| \left( \frac{u(t_n) - u(t_{n-1})}{\Delta t}, v \right) \right| \leq \int_{t_{j-1}}^{t_j} \| u_{tt} \| ds \| v \|. \quad (3.31) \]

Taking \( v = 2\Delta t \theta^n \) and substituting all the above estimates (3.25)-(3.31) into
we obtain
\[ \|\theta^n\|^2 - \|\theta^{n-1}\|^2 + \|\theta^n - \theta^{n-1}\|^2 + 2\nu_0 \Delta t \|\theta^n\|_1^2 \]
\[ \leq C \Delta t^2 \|\theta^n\|_1 + C(\Delta t^2 + \Delta t \nu^*) \|\theta^n\| + 4\sigma \Delta t^2 K_1 \sum_{i=1}^n \|\theta_i\| \|\theta^n\|_1 \]
\[ + 4C C_1 \Delta t^2 K_1 \sum_{i=1}^n (\|\rho^n_i\| + \|\theta^n\|) \|\theta^n\|_1 \]
\[ \leq C \Delta t^2 + C \Delta t^2 \|\theta^n\|_1^2 + C(\Delta t + \nu^*)^2 + C \Delta t^2 \|\theta^n\|_1^2 + 2\sigma \Delta t^2 K_1 \sum_{i=1}^n \|\rho^n_i\|_1^2 + 2 \epsilon_1 \nu_0 \Delta t \|\theta^n\|_1^2 
+ 2 C C_1 \Delta t^2 K_1 \sum_{i=1}^n \|\rho^n_i\|_1^2 + 2 \epsilon_2 \nu_0 \Delta t K_1 \sum_{i=1}^n \|\rho^n_i\|_1^2 \]
\[ + \frac{2}{\epsilon_2} C C_1 \Delta t^2 K_1 \sum_{i=1}^n \|\theta^n\|_1^2 + 2 \epsilon_2 \nu_0 \Delta t K_1 \sum_{i=1}^n \|\theta^n\|_1^2, \quad (3.32) \]

where we have used the inequality
\[ \Delta t^2 \sum_{i=1}^n a_i b_n = C \sum_{i=1}^n \Delta t^{1/2} a_i \Delta t^{3/2} b_n \]
\[ \leq \frac{C}{2} \Delta t \sum_{i=1}^n a_i^2 + \frac{C}{2} \Delta t \sum_{i=1}^n b_n^2 \leq \frac{C}{2} \Delta t \sum_{i=1}^n a_i^2 + \frac{C}{2} \Delta t^2 \sum_{i=1}^n b_n^2. \]

Using the estimate (3.17) for \(\rho^n\), and taking \(\epsilon_1 = \frac{\nu_0}{4\sigma K_1} t_n\) and \(\epsilon_2 = \frac{\nu_0}{4C C_1 K_1} t_n\), we have
\[ \|\theta^n\|^2 - \|\theta^{n-1}\|^2 + \|\theta^n - \theta^{n-1}\|^2 + \nu_0 \Delta t \|\theta^n\|_1^2 \]
\[ \leq C(\Delta t + \nu^*)^2 + \left( C + \frac{8}{\nu_0} C^2 C_1^2 K_1^2 t_n \right) \Delta t^2 \|\theta^n\|_1^2 \]
\[ + \left( C + \frac{8}{\nu_0} \sigma C_1^2 K_1^2 t_n + 2 C C_1 K_1 t_n \right) \Delta t^2 \|\theta^n\|_1^2 + \frac{8}{\nu_0} \sigma^2 \Delta t^2 K_1^2 t_n \sum_{i=1}^{n-1} \|\theta_i\|_1^2 \]
\[ + \frac{8}{\nu_0} C^2 C_1^2 \Delta t^2 K_1^2 t_n \sum_{i=1}^{n-1} \|\theta_i\|^2, \quad (3.33) \]

Noting the condition (3.21) and taking sufficiently small \(\Delta t\) such that
\[ \left( C + \frac{8}{\nu_0} \sigma^2 K_1^2 t_n + 2 C C_1 K_1 t_n \right) \Delta t \leq \frac{\nu_0}{2} \quad \text{and} \quad \left( C + \frac{8}{\nu_0} C^2 C_1^2 K_1^2 t_n \right) \Delta t \leq \frac{1}{2}, \]
we obtain
\[ \|\theta^n\|^2 - \|\theta^n - \theta^{n-1}\|^2 + \nu_0 \Delta t \|\theta^n\|_1^2 \]
\[ \leq \|\theta^n\|^2 + \frac{1}{2} \Delta t^2 \sum_{i=1}^{n-1} \|\theta_i\|^2 + \frac{1}{2} \Delta t \|\theta^n\|_1^2 + C(\Delta t + \nu^*)^2 \]
\[ + \frac{\nu_0}{2} \Delta t \|\theta^n\|_1^2 + \frac{\nu_0}{2} \Delta t^2 \sum_{i=1}^{n-1} \|\theta_i\|_1^2. \quad (3.34) \]
Applying discrete Gronwall lemma 2.1 to the above inequality yields

\[ \|\theta^n\|^2 + \|\theta^n - \theta^{n-1}\|^2 + \nu_0 \Delta t \|\theta^n\|^2 \lesssim (h^r + \Delta t)^2. \]  

which implies (3.22). This proves the theorem.

Note that the condition (3.21), which implies that the equation (1.1) is diffusion-dominant, is appropriate, since the system may be blowup if the integral term is dominant. Under the condition (3.21), we can not study the long time behaviour of the numerical solution. Of course, if we assume that there exist positive constants \( \alpha_0, \alpha_1 > 0 \) such that

\[ \alpha_0 |\xi|^2 \leq \xi^T \alpha(u) \xi \leq \alpha_1 |\xi|^2, \quad \forall u \in \mathbb{R}, \quad \xi \in \mathbb{R}^d, \]  

then following the approach of [15], we can study the long time behavior of the exact solution and the numerical solution.

We now give the \( H^1 \) estimate of the error \( \theta^n \).

**Theorem 3.4.** Let \( u \) and \( U^n \) be the solutions of (2.3)-(2.4) and (2.9)-(2.10), respectively. Then, for all \( \Delta t \) satisfying

\[ \Delta t < \frac{\nu_0^2}{16 \sigma^2 K_1^2 h}, \]  

we have

\[ \|\theta^n\|_1 \lesssim h^r + \Delta t. \]  

**Proof.** Taking \( v = 2 \Delta t \partial \theta^n \) in (3.24), and estimating every terms in a way similar to Theorem 3.3, we get

\[ 2 \Delta t \|\partial \theta^n\|^2 + A(\theta^n, \theta^n) - A(\theta^{n-1}, \theta^{n-1}) + A(\theta^n - \theta^{n-1}, \theta^n - \theta^{n-1}) \]

\[ \leq C \Delta t^2 \|\partial \theta^n\|_1 + C(\Delta t^2 + \Delta t h^r) \|\partial \theta^n\| + 4 \sigma \Delta t^2 K_1 \sum_{i=1}^n \|\theta^i\|_1 \|\partial \theta^n\|_1 \]

\[ + 4 C C_L \Delta t^2 K_1 \sum_{i=1}^n (\|\theta^i\| + \|\theta^i\|_1) \|\partial \theta^n\|_1 \]

\[ \leq \frac{C \Delta t^2}{\nu_0^2} + \frac{\nu_0}{4} \|\theta^n - \theta^{n-1}\|^2 + \frac{C^2 \Delta t}{8} (\Delta t + h^r)^2 + 2 \Delta t \|\partial \theta^n\|^2 \]

\[ + \frac{16}{\nu_0} \sigma \Delta t K_1^2 t_n \sum_{i=1}^n \|\theta^i\|^2 + \frac{16}{\nu_0} \|\theta^n - \theta^{n-1}\|^2 + \frac{16}{\nu_0} C^2 C_L^2 \Delta t K_1^2 t_n \sum_{i=1}^n \|\theta^i\|^2 \]

\[ + \frac{\nu_0}{4} \|\theta^n - \theta^{n-1}\|^2 + \frac{16}{\nu_0} C^2 C_L^2 \Delta t K_1^2 t_n \sum_{i=1}^n \|\theta^i\|^2 + \frac{\nu_0}{4} \|\theta^n - \theta^{n-1}\|^2. \]  

(3.39)

Using (2.3), (2.4), (3.17) and (3.35) yields

\[ \nu_0 \|\theta^n\|^2 \leq \nu_1 \|\theta^{n-1}\|^2 + C (\Delta t + h^r)^2 + \frac{16}{\nu_0} \sigma \Delta t K_1^2 t_n \sum_{i=1}^n \|\theta^i\|^2. \]  

(3.40)

Then when \( \Delta t \) satisfies (3.37), an application of discrete Gronwall lemma 2.1 to the above inequality leads to (3.38). This completes the proof.
We observe that if (3.21) holds, then for any $\Delta t < 1$, the conclusion (3.22) is valid.

In the next theorem, we will establish the error estimate for the solution computed by the standard fully discrete finite element method (2.9)-(2.10).

**Theorem 3.5** (Error estimate for classic FEM). Let $u$ be the solution of (2.3)-(2.4) and $U^n$ be the solution of (2.9)-(2.10). Then, for sufficiently small $\Delta t$, we have, for all $n \geq 1$,

$$
\|U^n - u(t_n)\| \lesssim h^r + \Delta t, \quad \|U^n - u(t_n)\|_1 \lesssim h^{r-1} + \Delta t.
$$

**Proof.** The first inequality is a direct result of Theorem 3.3 and (3.17). From Theorem 3.4 and (3.17), we can prove the second inequality in (3.41).

4. Two-grid algorithms for PIDEs with nonlinear memory. In this section, we present three two-grid algorithms of the backward Euler finite element method for PIDEs with nonlinear memory. The basic mechanism in these algorithms is the construction of two regular triangulations of $\Omega$: a coarse triangulation $T_H$ with mesh size $H$ and a fine one $T_h$ with mesh size $h$ ($h \ll H$). For practical purposes, $T_h$ is a refinement of $T_H$. The corresponding finite element spaces are $S_H$ and $S_h$, which will be called coarse and fine space, respectively. To state the algorithms, we define, for $w, u, v \in W^{1,\infty} \cap H^1_0(\Omega)$,

$$
\tilde{B}(w; u, v) = (\alpha(w)\nabla u + \beta(w), \nabla v) + (\gamma(w) \cdot \nabla u + g(w), v).
$$

Due to the assumptions on $\alpha(u), \beta(u), \gamma(u)$, and $g(u)$, there exist two constants $\mu_1$ and $\mu_2$ such that

$$
|\tilde{B}(w; u, v)| \leq \mu_1 \|u\|_1 \|v\|_1 + \mu_2 \|w\| \|v\|_1.
$$

(4.1)

Let us now present our first two-grid algorithm.

**Algorithm 4.1.**

Step one (nonlinear problem on coarse grid $T_H$): Given $U_{H}^{n-1}$, find $U_{H}^{n} \in S_{H}$ such that

$$
\frac{1}{\Delta t} (U_{H}^{n} - U_{H}^{n-1}, v) + A(U_{H}^{n}, v) + \Delta t \sum_{i=1}^{n} \omega_{ni} \tilde{B}(U_{H}^{i}; U_{H}^{i}, v) = (f_{n}, v),
$$

$\quad v \in S_{H}, \quad n \geq 1,$

(4.2)

$$
U_{H}^{0} = u_{0}^{H}.
$$

(4.3)

Step two (linear problem on fine grid $T_h$): Given $U_{H}^{n}$, find $U_{h}^{n} \in S_{h}$ such that

$$
\frac{1}{\Delta t} (U_{h}^{n} - U_{h}^{n-1}, v) + A(U_{h}^{n}, v) + \Delta t \sum_{i=1}^{n} \omega_{ni} \tilde{B}(U_{h}^{i}; U_{h}^{i}, v) = (f_{n}, v),
$$

$\quad v \in S_{h}, \quad n \geq 1,$

(4.4)

$$
U_{h}^{0} = u_{0}^{h}.
$$

(4.5)

Firstly, we observe that for the solution of (4.4) and (4.5), our stability result is similar to the solution of (2.9) and (2.10).

**Theorem 4.1** (Stability of two-grid FEM Algorithm 4.1). Let $U_{h}^{n}$ be the solution obtained by Algorithm 4.1. Then when $\Delta t$ satisfies (3.7) and

$$
\Delta t \leq \min \left\{ \frac{1}{2}, \frac{3\nu^2}{2\mu^2 K^2 T} \right\},
$$

(4.6)
we have
\[ \sup_{1 \leq i \leq n} ||U_h^n|| + \left( \sum_{i=1}^{n} ||U_h^i - U_h^{i-1}||^2 \right)^{1/2} + \frac{\sqrt{\nu_0}}{2} \left( \sum_{i=1}^{n} \Delta t ||U_h^i||_1^2 \right)^{1/2} \]
\[ \leq C \left( ||U_h^n||^2 + ||U_h^0||^2 + \Delta t \sum_{i=1}^{n} ||f^i||^2 \right)^{1/2}. \] (4.7)

**Proof.** Similar to (3.3), using (1.1), we have
\[ ||U_h^n||^2 - ||U_h^{n-1}||^2 + ||U_h^n - U_h^{n-1}||^2 + 2\Delta t \nu_0 ||U_h^n||_1^2 \]
\[ \leq (\Delta t)^2 \sum_{i=1}^{n} |\omega_{ni}| \left( \frac{\mu_1}{4\epsilon_1} ||U_h^i||_1^2 + \mu_1 \epsilon_1 ||U_h^n||_1^2 + \frac{\mu_2}{4\epsilon_2} ||U_h^i||_1^2 + \mu_2 \epsilon_2 ||U_h^n||_1^2 \right) \]
\[ + \Delta t \left( ||U_h^n||^2 + ||f^n||^2 \right). \] (4.8)
After choosing \( \epsilon_1 = \frac{\mu_0}{2\mu_1 K_{1n}} \) and \( \epsilon_2 = \frac{\mu_0}{2\mu_2 K_{1n}} \), (4.8) becomes
\[ ||U_h^n||^2 + ||U_h^n - U_h^{n-1}||^2 + 2\Delta t \nu_0 ||U_h^n||_1^2 \]
\[ \leq ||U_h^{n-1}||^2 + (\Delta t)^2 \sum_{i=1}^{n} \left( \frac{\mu_2 K_{2tn}^2}{2\nu_0} ||U_h^i||_1^2 + \frac{\mu_2 K_{2tn}^2}{2\nu_0} ||U_h^i||_1^2 \right) + \Delta t ||f^n||^2 + \Delta t ||U_h^n||^2. \] (4.9)

With arguments similar to those in **Theorem 3.1** we obtain
\[ ||U_h^n||^2 + \sum_{i=1}^{n} ||U_h^i - U_h^{i-1}||^2 + \frac{\Delta t \nu_0}{4} \sum_{i=1}^{n} ||U_h^i||_1^2 \]
\[ \leq C \left( ||U_h^n||^2 + \sup_{1 \leq i \leq n} ||U_h^i||_1^2 + \Delta t \sum_{i=1}^{n} ||f^i||^2 \right). \] (4.10)

As \( U_h^n \) satisfies inequality (3.14), we can obtain (4.7).

To establish the error estimate for the solution computed by **Algorithm 4.1**, we need the following lemmas.

**Lemma 4.2.** Let \( U^n \) and \( U_h^n \) be the solutions obtained by (2.9) and (2.10) and **Algorithm 4.1**, respectively. If \( \Delta t \) satisfies condition
\[ \Delta t < \frac{\nu_0}{8\mu_1 K_{1n}^2 T}, \] (4.11)
then for any \( n \geq 1 \), we have
\[ \frac{2}{\sqrt{\nu_0 \Delta t}} ||W_h^n - W_h^{n-1}|| + ||W_h^n||_1 \lesssim H^r + h^{r-1} + \Delta t, \] (4.12)
where \( W_h^n = U_h^n - U^n \).

**Proof.** It follows from (2.9) and (4.9) that
\[ \frac{1}{\Delta t} (W_h^n - W_h^{n-1}, v) + A(W_h^n, v) + \Delta t \sum_{i=0}^{n} \omega_{ni} (\tilde{B}(U_h^i; U_h^i, v) - \tilde{B}(U^i; U^i, v)) = 0 \] (4.13)
Now let us bound \(|\tilde{B}(U_h^i; U_h^i, v) - \tilde{B}(U^i; U^i, v)|\). Firstly, we split \(\tilde{B}(U_h^i; U_h^i, v) - \tilde{B}(U^i; U^i, v)\) as follows:

\[
\tilde{B}(U_h^i; U_h^i, v) - \tilde{B}(U^i; U^i, v) = (\alpha(U_h^i)\nabla(U_h^i - U^i), \nabla v) + ((\alpha(U_h^i) - \alpha(U^i))\nabla U^i, \nabla v) + (\beta(U_h^i) - \beta(U^i), \nabla v) + (\gamma(U_h^i) - \gamma(U^i)) \cdot \nabla U^i, v) + (g(U_h^i) - g(U^i), v).
\]

(4.14)

It follows that

\[
\begin{align*}
&|((\alpha(U_h^i)\nabla(U_h^i - U^i), \nabla v)) + |(\beta(U_h^i) - \beta(U^i), \nabla v))| + |(\gamma(U_h^i) - \gamma(U^i)) \cdot \nabla U^i, v)| + |(\alpha(U_h^i) - \alpha(U^i))\nabla u(t_i), \nabla v)| \\
&\leq \mu_1\|W_h^i\|_1\|v\|_1 + C_L\|U_h^i - U^i\|\|v\|_1 \\
&\leq \mu_1\|W_h^i\|_1\|v\|_1 + C_L\|U_h^i - U^i\|\|u(t_i) - U^i\|\|v\|_1 \\
&\leq \mu_1\|W_h^i\|_1\|v\|_1 + C_L(H^r + h^r + \Delta t)\|v\|_1.
\end{align*}
\]

(4.15)

On the other hand, due to the assumption on \(\alpha\) and \(\gamma\), which implies that \(\alpha\) and \(\gamma\) are bounded and satisfy Lipschitz condition, we have

\[
\begin{align*}
&|((\alpha(U_h^i) - \alpha(U^i))\nabla U^i, \nabla v)| \\
&\leq |((\alpha(U_h^i) - \alpha(U^i))\nabla (U^i - u(t_i)), \nabla v)| + |((\alpha(U_h^i) - \alpha(U^i))\nabla u(t_i), \nabla v)| \\
&\leq C\|\nabla (U^i - u(t_i))\|\|\nabla v\| + C_L\|U_h^i - U^i\|\|\nabla u(t_i)\|\|\nabla v\| \\
&\leq C(u(H^r + h^r + \Delta t))\|\nabla v\|,
\end{align*}
\]

(4.16)

and

\[
\begin{align*}
&|((\gamma(U_h^i) - \gamma(U^i)) \cdot \nabla U^i, v)| \\
&\leq |((\gamma(U_h^i) - \gamma(U^i)) \cdot \nabla (U^i - u(t_i)), v)| + |((\gamma(U_h^i) - \gamma(U^i)) \cdot \nabla u(t_i), v)| \\
&\leq C\|\nabla (U^i - u(t_i))\|\|v\| + C_L\|U_h^i - U^i\|\|\nabla u(t_i)\|\|v\| \\
&\leq C(u)(H^r + h^r + \Delta t)\|v\|.
\end{align*}
\]

(4.17)

Take \(v = 2(W_h^n - W_h^{n-1})\) in (4.13), and combine (4.13), (4.14), (4.15), (4.16) and (4.17) to get

\[
\begin{align*}
2\Delta t \sum_{i=1}^n |\omega_{ni}| (\mu_1\|W_h^n\|_1 + C(H^r + h^r + \Delta t))\|W_h^n - W_h^{n-1}\|_1 + \nu_1\|W_h^{n-1}\|_1^2 \\
\leq 2\Delta t \sum_{i=1}^n |\omega_{ni}| (\mu_1\|W_h^n\|_1 + C(H^r + h^r + \Delta t))\|W_h^n - W_h^{n-1}\|_1 + \nu_1\|W_h^{n-1}\|_1^2.
\end{align*}
\]

(4.18)

The first term on the right-hand side of the above inequality can be bounded as

\[
\begin{align*}
2\Delta t \sum_{i=1}^n |\omega_{ni}| (\mu_1\|W_h^n\|_1 + C(H^r + h^r + \Delta t))\|W_h^n - W_h^{n-1}\|_1 \\
\leq \frac{4}{\nu_0}\Delta t K_1^2 \mu_1^2 \|W_h^n\|_1^2 + \frac{4}{\nu_0}C^2 K_1^2 \|W_h^n\|_1^2 (H^r + h^r + \Delta t)^2 \\
+ \nu_0\|W_h^n - W_h^{n-1}\|_1^2.
\end{align*}
\]

(4.19)
where we have used
\[
2\Delta t \sum_{i=1}^{n} a_i b \leq \frac{4t_n}{\nu_0} \Delta t \sum_{i=1}^{n} a_i^2 + \frac{\nu_0}{4t_n} \Delta t \sum_{i=1}^{n} b^2 = \frac{4t_n}{\nu_0} \Delta t \sum_{i=1}^{n} a_i^2 + \frac{\nu_0}{4} b^2.
\]

Substituting (4.19) into (4.18), we get
\[
\frac{2}{\Delta t} \| W_h^n - W_h^{n-1} \|^2 + \frac{\nu_0}{2} \| W_h^n \|^2 \leq \nu_0 \| W_h^{n-1} \|^2 + \frac{4\mu_1^2 K^2 t_n}{\nu_0} \Delta t \sum_{i=1}^{n} \| W_h^i \|_1 + C(H^r + h^{r-1} + \Delta t)^2. \tag{4.20}
\]

In view of (4.11), application of discrete Gronwall lemma 2.1 to the above inequality yields
\[
\frac{4}{\nu_0 \Delta t} \| W_h^n - W_h^{n-1} \|^2 + \| W_h^n \|^2 \leq C(H^r + h^{r-1} + \Delta t)^2. \tag{4.21}
\]

Then we arrive at (4.12).

Combining Theorem 3.5 and Lemma 4.2 immediately yields the following theorem.

**Theorem 4.3 (Error estimate for two-grid FEM Algorithm 4.1).** Let \( u \) be the solution of (2.3)-(2.4) and \( U^n_H \) be the solution of Algorithm 4.1. Then, for sufficiently small \( \Delta t \), we have, for all \( n \geq 1 \),
\[
\| U^n_H - u(t_n) \|_1 \lesssim H^r + h^{r-1} + \Delta t. \tag{4.22}
\]

**Proof.** Using the triangular inequality \( \| U^n_H - u(t_n) \|_1 \leq \| U^n - u(t_n) \|_1 + \| U^n_H - U^n \|_1 \), the second inequality in (3.41), and (4.12), we can obtain (4.22).

From (4.22), it is easy to find that when the mesh sizes satisfy \( H = O(h^{\frac{1}{r-1}}) \) the two-grid Algorithm 4.1 achieves the same approximation for PIDEs with nonlinear memory as the classic finite element method does.

Next we will present an algorithm that reduces a nonlinear problem to a symmetric positive definite (SPD) linear problem and a nonlinear system of smaller size.

**Algorithm 4.2.**

**Step one (nonlinear problem on coarse grid \( \mathcal{T}_H \)):** Given \( U^n_{H-1} \), find \( U^n_H \in \mathcal{S}_H \) such that
\[
\frac{1}{\Delta t} (U^n_H - U^{n-1}_H, v) + A(U^n_H, v) + \Delta t \sum_{i=1}^{n} \omega_n i \tilde{B}(U^n_H; U^n_H, v) = (f^n, v),
\]
\[
v \in \mathcal{S}_H, \quad n = 1, 2, \ldots, \tag{4.23}
\]
\[
U^0_H = u^0_H. \tag{4.24}
\]

**Step two (SPD linear problem on fine grid \( \mathcal{T}_h \)):** Given \( U^n_H \), find \( U^n_h \in \mathcal{S}_h \) such that
\[
\frac{1}{\Delta t} (U^n_h - U^{n-1}_h, v) + A(U^n_h, v) + \Delta t \sum_{i=1}^{n-1} \omega_n i \tilde{B}(U^n_h; U^n_h, v) + \Delta t \omega_{nn} \tilde{B}(U^n_H; U^n_H, v)
\]
\[
= (f^n, v), \quad v \in \mathcal{S}_h, \tag{4.25}
\]
\[
U^0_h = u^0_h, \quad n = 1, 2, \ldots. \tag{4.26}
\]
Obviously, this algorithm can also be applied to the nonsymmetric linear problem.

**Theorem 4.4** (Stability of two-grid FEM Algorithm 4.2). Let $U^n_h$ be the solution obtained by Algorithm 4.2. If $\Delta t$ satisfies (3.11), then we have

$$\|U^n_h\| + \left(\sum_{i=1}^{n} \|U^i_h - U^{i-1}_h\|^2\right)^{1/2} + \frac{\sqrt{\nu}}{2} \left(\sum_{i=1}^{n} \Delta t \|U^i_h\|_1^2\right)^{1/2} \leq C \left(\|U_0^n\|^2 + \Delta t \|U_H^0\|^2 + \Delta t \sum_{i=1}^{n} \|f^i\|^2\right)^{1/2}, \quad (4.27)$$

for any $n \geq 1$.

**Proof.** Similar to (3.4), using (4.1), we have

$$\|U^n_h\|^2 - \|U^{n-1}_h\|^2 + \|U^n_h - U^{n-1}_h\|^2 + 2\Delta t \nu ||U^n_h||^2_1 \leq (\Delta t)^2 \sum_{i=1}^{n} \|\omega_{ni}(\frac{\mu_0}{\nu} ||U^n_h||^2 + \mu_0 \epsilon ||U^n_h||^2_1) + (\Delta t)^2 ||\omega_{n}|| \left(\frac{\mu_0}{\nu} ||U^0_H||^2 + \mu_0 \epsilon ||U^n_H||^2_1\right) + \Delta t (||U^n_h||^2 + ||f^n||^2). \quad (4.28)$$

After choosing $\epsilon = \frac{\nu_0}{\mu_0 + \nu_0}$, the above inequality becomes

$$\|U^n_h\|^2 + \|U^n_h - U^{n-1}_h\|^2 + \Delta \nu \nu ||U^n_h||^2_1 \leq \|U^{n-1}_h\|^2 + (\Delta t)^2 \left(\sum_{i=1}^{n} \frac{\mu_0^2 K^2 t}{\nu_0} ||U^i_h||^2 + \frac{\mu_0^2 K^2 t_\nu}{\nu_0} ||U^n_H||^2_1\right) + \Delta t ||f^n||^2 + \Delta t ||U^n_H||^2. \quad (4.29)$$

With arguments similar to those in **Theorem 3.1**, we obtain

$$\|U^n_h\|^2 + \frac{n}{4} \sum_{i=1}^{n} \|U^i_h - U^{i-1}_h\|^2 + \Delta \nu \nu \sum_{i=1}^{n} ||U^i_h||^2_1 \leq C \left(\|U_0^n\|^2 + (\Delta t)^2 \sum_{i=1}^{n} ||U^i_h||^2 + \Delta t \sum_{i=1}^{n} ||f^i||^2\right),$$

in view of (3.14), therefore,

$$\|U^n_h\|^2 + \frac{n}{4} \sum_{i=1}^{n} ||U^i_h - U^{i-1}_h||^2 + \frac{\Delta \nu \nu}{4} \sum_{i=1}^{n} ||U^i_h||^2_1 \leq C \left(\|U_0^n\|^2 + \Delta t ||U^0_H||^2 + \Delta t \sum_{i=1}^{n} ||f^i||^2\right), \quad (4.30)$$

which implies (4.27). This completes the proof.

**Theorem 4.5** (Error estimate for two-grid FEM Algorithm 4.2). Let $U^n_h$ be the solution obtained by Algorithm 4.2. Then for sufficient small $\Delta t$, we have

$$\|U^n_h - u(t_n)\| \leq \sqrt{\Delta t H^{-1}} + h^r + \Delta t, \quad (4.31)$$

for any $n \geq 1$. 
Proof. As in Theorem 4.4, \(W^n_h = U^n_h - U^n\) satisfies the following error equation:

\[
\frac{1}{\Delta t} (W^n_h - W^{n-1}_h, v) + A(W^n_h, v) + \Delta t \sum_{i=0}^{n-1} \omega_{ni}(\tilde{B}(U^n_i; U^n_h, v) - \tilde{B}(U^n_i; U^n, v))
\]

\[
+ \Delta t \omega_{nn}(\tilde{B}(U^n_n; U^n_h, v) - \tilde{B}(U^n; U^n, v)) = 0. \tag{4.32}
\]

In view of the assumption on the coefficients of \(B\), there exists a constants \(\mu_B\) such that

\[
|\tilde{B}(u; u, v) - \tilde{B}(w; w, v)| \leq \mu_B \|u - w\|_1 \|v\|_1.
\]

Then we have

\[
|\Delta t \omega_{nn}(\tilde{B}(U^n_n; U^n_h, v) - \tilde{B}(U^n; U^n, v))| \leq \mu_B \Delta t \||U^n_h - U^n\|_1 \|v\|_1
\]

\[
\leq \mu_B K_1 \Delta t (\|U^n_h - u(t_n)\|_1 + \|u(t_n) - U^n\|_1) \|v\|_1
\]

\[
\leq \mu_B K_1 \Delta t (H^{-1} + h^{-1} + \Delta t) \|v\|_1. \tag{4.33}
\]

Take \(v = 2\Delta t W^n_h\) in (4.32) to obtain

\[
||W^n_h||^2 - ||W^{n-1}_h||^2 + ||W^n_h - W^{n-1}_h||^2 + 2\Delta t \nu_0||W^n_h||^2
\]

\[
\leq \mu_B (\Delta t)^2 \sum_{i=1}^{n-1} |\omega_{ni}| (\frac{1}{\epsilon} ||W^n_i||^2 + \epsilon ||W^n_i||^2)
\]

\[
+ \mu_B K_1 (\Delta t (\epsilon \Delta t ||W^n_h||_1^2 + \frac{1}{\epsilon} \Delta t (H^{-1} + \Delta t)^2).
\tag{4.34}
\]

By choosing \(\epsilon = \frac{\nu_0}{t_n \mu_B K_1}\), we get

\[
||W^n_h||^2 - ||W^{n-1}_h||^2 + ||W^n_h - W^{n-1}_h||^2 + \Delta t \nu_0||W^n_h||^2
\]

\[
\leq \frac{t_n \mu_B^2 K_1^2}{\nu_0} (\Delta t)^2 \sum_{i=1}^{n-1} ||W^n_i||^2 + \frac{t_n \mu_B^2 K_1^2}{\nu_0} (\Delta t)^2 (H^{-1} + \Delta t)^2. \tag{4.35}
\]

Sum from 1 up to n to obtain

\[
||W^n||^2 - ||W^0||^2 + \sum_{i=1}^{n} ||W^n_i - W^{i-1}_h||^2 + \Delta t \nu_0 \sum_{i=1}^{n} ||W^n_i||^2
\]

\[
\leq (\Delta t)^2 \sum_{i=1}^{n} \frac{t_i \mu_B^2 K_1^2}{\nu_0} \sum_{j=1}^{i-1} ||W^n_j||^2 + \sum_{i=1}^{n} \frac{t_i \mu_B^2 K_1^2}{\nu_0} (\Delta t)^2 (H^{-1} + \Delta t)^2
\]

\[
\leq (\Delta t)^2 \sum_{i=0}^{n-1} \frac{t_i+1 \mu_B^2 K_1^2}{\nu_0} \sum_{j=1}^{i} ||W^n_j||^2 + \sum_{i=1}^{n} \frac{t_i \mu_B^2 K_1^2}{\nu_0} (\Delta t)^2 (H^{-1} + \Delta t)^2
\]

\[
\leq (\Delta t)^2 \sum_{i=0}^{n} \frac{t_i+1 \mu_B^2 K_1^2}{\nu_0} \sum_{j=1}^{i} ||W^n_j||^2 + \frac{t_n \mu_B^2 K_1^2}{\nu_0} \Delta t (H^{-1} + \Delta t)^2. \tag{4.36}
\]

An application of discrete Gronwall Lemma 2.1 yields

\[
||W^n||^2 + \sum_{i=1}^{n} ||W^n_i - W^{i-1}_h||^2 + \Delta t \nu_0 \sum_{i=1}^{n} ||W^n_i||^2
\]

\[
\leq \frac{t_n \mu_B^2 K_1^2}{\nu_0} \Delta t (H^{-1} + \Delta t)^2 \exp \left( \frac{t_n \mu_B^2 K_1^2}{\nu_0} \right). \tag{4.37}
\]
Finally, (4.31) follows readily from this result when a triangular inequality is also applied.

Next we will present an algorithm that significantly reduces computational memory and storage requirements when $B$ gathers lower-order spatial derivatives and nonlinear terms. To state the algorithm, we define

$$\tilde{B}(w; u, v) = (\alpha(w) \nabla u, \nabla v),$$

and

$$N(w; u, v) = (\beta(w), \nabla v) + (\gamma(w) \cdot \nabla u + g(w), v).$$

In view of the assumptions on $\alpha(u)$, $\beta(u)$, $\gamma(u)$, and $g(u)$, we find that there exist two constants $\mu_3$ and $\mu_4$ such that

$$|\tilde{B}(w; u, v)| \leq \mu_3 \|u\|_1 \|v\|_1, \quad (4.38)$$

$$|N(w; u, v)| \leq \mu_4 \|u\| \|v\|_1. \quad (4.39)$$

Then the algorithm can be stated as follows.

**Algorithm 4.3.**

Step one (nonlinear problem on coarse grid $\mathcal{T}_H$): Given $U_{H}^{n-1}$, find $U_{H}^{n} \in S_H$ such that

$$\frac{1}{\Delta t} (U_{H}^{n} - U_{H}^{n-1}, v) + A(U_{H}^{n}, v) + \Delta t \sum_{i=1}^{n} \omega_{ni} (\tilde{B}(U_{H}^{n}; U_{H}^{n}, v) + N(U_{H}^{n}; U_{H}^{n}, v)) = (f^{n}, v), \quad v \in S_H, \quad n \geq 1,$$

$$U_{H}^{0} = u_{H}^{0}. \quad (4.40)$$

Step two (linear problem on fine grid $\mathcal{T}_h$): Given $U_{H}^{n}$, find $U_{h}^{n} \in S_h$ such that

$$\frac{1}{\Delta t} (U_{h}^{n} - U_{h}^{n-1}, v) + A(U_{h}^{n}, v) + \Delta t \sum_{i=1}^{n} \omega_{ni} (\tilde{B}(U_{H}^{i}; U_{h}^{n}, v) + N(U_{H}^{i}; U_{H}^{i}, v)) = (f^{n}, v), \quad v \in S_h, \quad n \geq 1,$$

$$U_{h}^{0} = u_{h}^{0}. \quad (4.42)$$

The stability of Algorithm 4.3 can be obtained by the same argument for Theorem 4.1.

**Theorem 4.6 (Stability of two-grid FEM Algorithm 4.3).** Let $U_{h}^{n}$ be the solution obtained by Algorithm 4.3. Then when

$$\Delta t \leq \min \left\{ \frac{1}{2}, \frac{3\nu_0^2}{2\mu_3^2 K_{TT}^2} \right\}, \quad (4.44)$$

we have

$$\sup_{1 \leq i \leq n} \left\| U_{h}^{i} \right\| + \left( \sum_{i=1}^{n} \left\| U_{h}^{i} - U_{h}^{i-1} \right\|^2 \right)^{1/2} + \frac{\sqrt{\nu_0}}{2} \left( \sum_{i=1}^{n} \Delta t \left\| U_{h}^{i} \right\|^2 \right)^{1/2} \leq C \left( \left\| U_{h}^{0} \right\|^2 + \left\| U_{H}^{0} \right\|^2 + \Delta t \sum_{i=1}^{n} \left\| f^{i} \right\|^2 \right)^{1/2}. \quad (4.45)$$
Proof. Similar to (4.3), using (4.38) and (4.39), we have
\[ \|U^n_h\|^2 - \|U^{n-1}_h\|^2 + \|U^n_h - U^{n-1}_h\|^2 + 2\Delta t\nu_0 \|U^n_h\|^2 \]
\[ \leq (\Delta t)^2 \sum_{i=1}^{n} |\omega_{ni}| \left( \frac{\mu_1}{4\epsilon_1} \|U^n_i\|^2 + \mu_3 \epsilon_1 \|U^n_i\|^2 + \frac{\mu_4}{4\epsilon_2} \|U^n_H\|^2 + \mu_4 \epsilon_2 \|U^n_i\|^2 \right) \]
\[ + \Delta t \left( \|U^n_h\|^2 + \|f^n\|^2 \right). \] (4.46)
After choosing \( \epsilon_1 = \frac{\nu_0}{2\mu_3 K_t t_n} \) and \( \epsilon_2 = \frac{\nu_0}{2\mu_4 K_t t_n} \), (4.46) becomes
\[ \|U^n_h\|^2 + \|U^n_h - U^{n-1}_h\|^2 + \Delta t\nu \|U^n_h\|^2 \]
\[ \leq \|U^{n-1}_h\|^2 + (\Delta t)^2 \sum_{i=1}^{n} \left( \frac{\mu_3^2 K^2 t_n}{2\nu_0} \|U^n_j\|^2 + \frac{\mu_4^2 K^2 t_n}{2\nu_0} \|U^n_i\|^2 \right) + \Delta t\|f^n\|^2 + \Delta t\|U^n_h\|^2. \] (4.47)
With arguments similar to those in THEOREM 3.1, we obtain
\[ \|U^n_h\|^2 + \sum_{i=1}^{n} \|U^n_i - U^{i-1}_h\|^2 + \frac{\Delta t\nu_0}{4} \sum_{i=1}^{n} \|U^n_i\|^2 \]
\[ \leq C \left( \|U^n_0\|^2 + \sup_{1 \leq i \leq n} \|U^n_H\|^2 + \Delta t \sum_{i=1}^{n} \|f^n\|^2 \right). \] (4.48)
As \( U^n_H \) satisfies inequality (3.2), we can obtain (4.45).

To get an idea of the accuracy of ALGORITHM 4.3, we have the following theorem.

THEOREM 4.7 (Error estimate for two-grid FEM Algorithm 4.3). Let \( U^n_h \) be the solutions obtained by ALGORITHM 4.3. Then for sufficient small \( \Delta t \), we have
\[ \|U^n_h - u(t_n)\| \lesssim \|U^n_h\| + \Delta t, \] (4.49)
for any \( n \geq 1 \).

Proof. Set \( W^n_h = U^n_h - U^n \) to get
\[ \frac{1}{\Delta t} (W^n_h - W^{n-1}_h, v) + A(W^n_h, v) + \Delta t \sum_{i=1}^{n} \omega_{ni}(\tilde{B}_s(U^n_H; U^n_H, v) - \tilde{B}_s(U^n_H; U^n_i, v)) \]
\[ + \Delta t \sum_{i=1}^{n} \omega_{ni}(N(U^n_H; U^n_H, v) - N(U^n_H; U^n_i, v)) = 0. \] (4.50)
Similar to the proof of LEMMA 4.2, we have
\[ N(U^n_H; U^n_H, v) - N(U^n_H; U^n_i, v) \]
\[ = (\beta(U^n_H) - \beta(U^n_i), \nabla v) + (\gamma(U^n_H) \cdot \nabla(U^n_H - U^n_i), v) \]
\[ + ((\gamma(U^n_H) - \gamma(U^n_i)) \cdot \nabla U^n, v) + (g(U^n_H) - g(U^n_i), \nabla v) \] (4.51)
and
\[ |(\gamma(U^n_H) \cdot \nabla(U^n_H - U^n_i), v)| \leq C \|U^n_H - U^n_i\| \|v\|_1 \]
\[ \leq C_L (\|u(t_i) - U^n_H\| + \|u(t_i) - U^n_i\|) \|v\|_1 \]
\[ \leq C_L (H^r + h^r + \Delta t) \|v\|_1. \] (4.52)
The desired estimate can then be obtained in a way similar to proofs of Theorem 4.3 and Lemma 4.2.

**Remark.** Observe that when $\alpha \equiv 0$, the approximation of the integral term on the fine grid is identical to the approximation of the integral term on the coarse grid. This means that when we solve $U_n^h$, all $U_i^h$ ($i < n - 1$) do not need to be stored on a fine grid. It also means that once the approximation of the integral term has been computed on the coarse grid it does not need to be computed on the fine grid. This significantly reduces computational memory and storage requirements. This result is novel and interesting even for linear problem.

5. **Concluding remarks.** We have presented and derived error estimates for several two-grid finite element algorithms for PIDEs with nonlinear memory. With the backward Euler scheme, the two-grid strategy consists of two steps: (1) discretizing the fully nonlinear problem in space on a coarse grid with mesh-size $H$ and time step-size $\Delta t$ and (2) discretizing the linearized problem in space on a fine grid with mesh-size $h$ and the same time step-size as in step (1). It is shown that these algorithms are as stable as the standard fully discrete finite element algorithm. We also present the error estimate at each time step. Compared with standard finite element methods, our algorithm not only keep good accuracy but also saves a lot of computational cost. As a byproduct of these results, we found that one of these algorithms significantly reduces computational memory and storage requirements if the nonlinear memory is defined by a first-order or zero-order nonlinear differential operator. Thus, the two-grid methods studied in this paper provide a new approach that takes advantage of some of the nice properties hidden in a complex problem.

The analysis herein was carried out for an implicit Euler discretization in time. However, the results could be extended to the second-order accuracy backward differentiation formula (BDF) scheme. Moreover, the analysis is valid for a state-dependent forcing term $f$ that satisfies certain conditions, e.g.,

$$|\frac{\partial}{\partial u} f(x, t, u)| + |\frac{\partial^2}{\partial u^2} f(x, t, u)| \leq M, \quad u \in \mathbb{R},$$

where $M$ is a positive constant.

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