Intrinsic Resistivity via Quantum Nucleation of Phase Slips in a
One-Dimensional Josephson Junction Array

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Abstract

The resistivity of a one-dimensional Josephson junction array at zero temperature is calculated by estimating the nucleation rate of quantum phase slips. We choose a certain collective coordinate which describes the nucleation process and estimate the corresponding effective mass. In the strong coupling regime, where Josephson coupling energy exceeds the charging energy, we calculate the nucleation rate by means of WKB method. Our estimation is in good agreement with recent experimental data. The superconductor-insulator transition point is also discussed.

74.50.+r, 73.40.Gk, 64.60.Qb, 64.60.My
Quantum fluctuations have much influence on the properties of one-dimensional systems at low temperatures. For example, in extremely thin superconducting wires which can be regarded as nearly one-dimensional, the quantum fluctuations of the phase of the order parameter nucleate phase slips, which result in finite resistance at zero temperature [1]. Similar effects are expected for a one-dimensional Josephson junction (1D JJ) array. In a 1D JJ array, the phase difference and charge difference across a junction are canonically conjugate variables, so the charging effect is the origin of the quantum fluctuation of phase. When the system size becomes small and the charging energy large enough, nucleation occurs frequently and at some critical point, the quantum phase transition from a phase-coherent (superconducting) state to a charge-ordered (insulating) state occurs. Theoretically, this Superconductor-Insulator (S-I) transition in a 1D JJ array can be understood as some version of Kosterlitz-Thouless transition in a (1+1) dimensional classical spin model, extra dimension being imaginary time [2,3].

Recently, such S-I transition was experimentally investigated in detail with 1D JJ arrays having different numbers of superconducting grains [4]. In that work, the S-I transition was actually observed for each \( N \) near the point predicted by Kosterlitz-Thouless theory, and what is also remarkable, it was found that arrays have finite resistance at zero temperature even in the superconducting side. It is probable that these intrinsic resistance in the superconducting side are caused by the quantum nucleation of phase slips, as in the case of thin superconducting wires. In order to confirm this interpretation, we study the quantum nucleation process of a phase slip and calculate the nucleation rate in this Letter. The S-I transition point is also discussed.

To begin with, we consider a linear array of \( N \gg 1 \) superconducting grains embedded in an insulator at zero temperature. The grains are assumed to be small compared with bulk coherence length, so that the state of \( i \)th grain can be described by a single phase \( \theta_i \) and number \( n_i \) of Cooper pairs on the \( i \)th grain. The Euclidean action and Hamiltonian are

\[
S_E[\{n_i, \theta_i\}] = \int d\tau \left\{ \sum_{i}^N \hbar n_i \frac{\partial \theta_i}{\partial \tau} + H[\{n_i, \theta_i\}] \right\} ,
\]

(1)
\[ H[n_i, \theta_i] = \sum_{i=1}^{N-1} \frac{1}{2} E_C \left( \frac{n_{i+1} - n_i}{2} \right)^2 + \sum_{i=1}^{N} \frac{1}{2} E_G (n_i - \bar{n})^2 - \sum_{i=1}^{N-1} E_J \cos(\theta_{i+1} - \theta_i). \tag{2} \]

In Eq. (2), we take into account two kinds of charging energies, \( E_G \) and \( E_C \), the former is between a grain and the ground and the latter between nearest neighbor grains. \( E_J \) is Josephson coupling energy and \( \bar{n} \) is the average number of Cooper pairs on a grain. Although the normal component is not explicitly mentioned here, the property of the system largely depends on the value of normal tunnel resistivity \( R_T \) [3-7]. In this Letter, we restrict our attention to the superconducting regime, so \( R_T \) is assumed to be smaller than the quantum resistance \( R_Q = \frac{h}{4e^2} \sim 6.45 \text{k\Omega} \).

A supercurrent state of the above Hamiltonian is metastable, even at zero temperature. Such a state corresponds to a local minimum of the energy functional of \( n_i \) and \( \theta_i \), and definitely decay to lower local minima, due to quantum tunneling. When the effective inertial mass of the phase slip is not greatly dependent on the tunneling path, the most probable tunneling path will be through the valley, passing through two local minima and one saddle point in the functional space. In this situation, the wave function which describes the phase slip process is determined so as to minimize the energy functional at each stage of the tunneling process. Of many tunneling paths, we adopt only paths which describe slips of \( 2\pi \). Other paths of \( 4\pi, 6\pi, \ldots \) can be omitted because of their much higher tunnel barriers.

To determine the shape of the path, we temporarily fix the phase slip center between the \( i \)th and \((i + 1)\)th grain and restrict the average value of supercurrent, so add the following two terms in Hamiltonian Eq. (2)

\[ \lambda_1 (\theta_{j+1} - \theta_{j} - q) + \lambda_2 \left\{ \sum_{i=1}^{N-1} \left( \theta_{i+1} - \theta_i - \frac{2\pi n}{N-1} \right) \right\}, \]

where \( q \) is the "phase jump" at the phase slip center satisfying \(-\pi < q < \pi\). The "winding number of the phase" \( n \) is a real number, but not necessarily an integer. The extremal solution of the energy functional with additional terms is
\[ n_i \equiv \bar{n}, \quad \theta_{i+1} - \theta_i = \kappa = \frac{2\pi n - q}{N - 2} \quad (i \neq j), \]
\[ n_j \equiv \bar{n}, \quad \theta_{j+1} - \theta_j = q. \quad (3) \]

When we take the phase jump \( q \) as the collective coordinate describing each stage of the tunneling process, Eq. (3) is the energy minimum configuration with given \( q \) and \( n \). When \( q = \kappa \), Eq. (3) gives the local minimum solution \( \{n_{m,i}, \theta_{m,i}\}: \)
\[ n_{m,i} \equiv \bar{n}, \quad \theta_{m,i+1} - \theta_{m,i} \equiv \kappa_n = \frac{2\pi n}{N - 1}, \quad (4) \]
and when \( q = \pi - \kappa \), the saddle point solution \( \{n_{s,i}, \theta_{s,i}\}: \)
\[ n_{s,i} \equiv \bar{n}, \quad \theta_{s,i+1} - \theta_{s,i} = \kappa = (2n - 1)\frac{\pi}{N - 3} \quad (i \neq j), \]
\[ n_{s,j} \equiv \bar{n}, \quad \theta_{s,j+1} - \theta_{s,j} = \pi - \kappa. \quad (5) \]

Now the wave function is constructed which interpolate two local minimum points in the functional space.

Hereafter we adopt \( \kappa \), rather than \( q \), as the collective coordinate, because using \( \kappa \) allows us to consider the whole of the tunneling process while \( q \) doesn’t. The potential energy is now a function of the collective coordinate \( \kappa \),
\[ V(\kappa) = -E_J \left[ (N - 2) \cos \kappa + \cos \left\{ 2\pi n - \kappa (N - 2) \right\} \right]. \quad (6) \]

Next, we discuss the zero-point fluctuation of the collective coordinate \( \kappa \) around the local minimum. As already mentioned, \( n_i \) and \( \theta_i \) are canonically conjugate variables, therefore the field necessarily fluctuates around the local minimum configuration Eq. (4) and a certain mode of this fluctuations corresponds to the zero-point fluctuation of \( \kappa \) which nucleates phase slips. In order to estimate frequencies of collective modes, we add the fluctuation terms \( \{\delta n_i, \delta \theta_i\} \) to the local minimum configuration Eq. (4), substitute it into the Euclidean action, Eq. (1), and retain the terms up to the second order of \( \delta n_i \) and \( \delta \theta_i \), then we obtain
\[ \delta S_E = \sum_{i=1}^{N} \int d\tau \left( \begin{array}{c} \delta n_i \\ \delta \theta_i \end{array} \right) G^{-1} \left( \begin{array}{c} \delta n_i \\ \delta \theta_i \end{array} \right). \quad (7) \]
Here, $G$ is the Green’s function for $\delta n_i$ and $\delta \theta_i$, and Fourier component of $G^{-1}$ is given by

$$G^{-1} = \begin{pmatrix} \frac{1}{2}E_G + \frac{1}{2}E_C \sin^2 \frac{ka}{2} & \frac{\hbar}{2} \omega_n \\ -\frac{\hbar}{2} \omega_n & 2E_J \cos \left( \frac{2\pi n}{N-1} \right) \sin^2 \frac{ka}{2} \end{pmatrix},$$

(8)

where $a$ is the distance between nearest neighbor grains. The poles of $G$ give collective modes, whose dispersion relation is

$$\hbar \omega_k = 2 \sqrt{\left\{ E_G + E_C \sin^2 \frac{ka}{2} \right\} E_J \cos \left( \frac{2\pi n}{N-1} \right) \sin^2 \frac{ka}{2}}.$$ \hspace{1cm} (9)

Fig. 1 suggests that a phase slip has a local character, thus as the effective frequency of the zero-point oscillation of $\kappa$, we can extract from Eq. (9) the mode whose wave length is $a$:

$$\hbar \omega_{eff}(\kappa_n) = \hbar \omega_{k=\frac{\pi}{a}} = 2 \sqrt{(E_G + E_C) E_J \cos \left( \frac{2\pi n}{N-1} \right)}.$$ \hspace{1cm} (10)

We have already determined the potential energy, Eq. (6), so the effective inertial mass of $\kappa$ can be estimated from the frequency of the collective mode and the curvature of the potential energy at the local minimum. In this procedure, we assume that the inertial mass is not greatly dependent on the collective coordinate, and approximate it to the value at local minimum. From Eq. (10) the potential curvature at the local minimum $\kappa \sim \kappa_n$ is

$$V''(\kappa_n) = E_J (N-2) (N-1) \cos \left( \frac{2\pi n}{N-1} \right),$$ \hspace{1cm} (11)

where the prime denotes differentiation with respect to $\kappa$. We can derive the effective inertial mass for $\kappa$ from Eq. (10) and Eq. (11),

$$M_\kappa (n) = \frac{V''(\kappa_n)}{\omega_{eff}^2 (n)} = \frac{\hbar^2 (N-2) (N-1)}{4 \left( E_G + E_C \right)}.$$ \hspace{1cm} (12)

We now consider the tunneling problem from the local minimum $\kappa_n$ to the next lower local minimum $\kappa_{n-1}$. According to WKB formula, the tunneling rate $\Gamma(\kappa_n \rightarrow \kappa_{n-1})$ is given by

$$\Gamma(\kappa_n \rightarrow \kappa_{n-1}) = \frac{\omega_{eff}(\kappa_n)}{2\pi} \exp \left\{ -\frac{2}{\hbar} S(\kappa_n) \right\},$$

$$S(\kappa_n) = \int_{\kappa_n}^{\kappa_{n-1}} \sqrt{2M_{\kappa} (\kappa_n) \left[ V(\kappa) - \frac{1}{2} \hbar \omega_{k=\frac{\pi}{a}} \right]},$$ \hspace{1cm} (13)
where $\kappa_n^-$ and $\kappa_n^+$ are the classical turning points, located between $\kappa_{n-1}$ and $\kappa_n$. Here $\kappa_n^+$ doesn’t coincide with $\kappa_n$ because we adopt not the instanton method, but WKB method which explicitly includes the zero point energy in the action. This point will be discussed later. Besides, we implicitly assume the existence of the environments, so that the energy is conserved before and after the tunneling.

In the limit of large $N$ and small bias current, we can reduce the action in Eq. (13) up to the first order of a small parameter $\frac{n}{N}$, to obtain

$$S(\kappa_n) \sim 2\hbar J \times \left\{ 2 \left( E \left( \frac{\pi}{2}, \sqrt{1 - \frac{1}{2J}} \right) - \frac{1}{2} F \left( \frac{\pi}{2}, \sqrt{1 - \frac{1}{2J}} J \right) \right) - \pi^2 \frac{n}{N} F \left( \frac{\pi}{2}, \sqrt{1 - \frac{1}{2J}} \right) \right\}, \quad (14)$$

where $J = \sqrt{\frac{E_1}{E_G+E_C}}$ is the coupling parameter representing the strength of the phase coherence. $F$ and $E$ denotes the first and second kind of elliptic integrals respectively.

From the tunneling rate, we can derive an expression for the resistivity. Combining Josephson’s relation

$$\frac{d\theta_{tot}}{dt} = 2e \frac{\hbar}{V} V \quad \text{ (15)}$$

and the equation of motion of the phase

$$\frac{d\theta_{tot}}{dt} = 2\pi \Gamma(\kappa_n \rightarrow \kappa_{n-1}), \quad \text{ (16)}$$

we conclude that the voltage at metastable state $\kappa \sim \kappa_n$ is

$$V(\kappa_n) = \frac{\hbar\pi}{e} \Gamma(\kappa_n \rightarrow \kappa_{n-1}), \quad \text{ (17)}$$

where $\theta_{tot}$ denotes the total difference of the phase throughout the array. The zero bias resistivity $R$ is defined as follows:

$$R = \left. \frac{\partial V(\kappa_n)}{\partial I(\kappa_n)} \right|_{I \rightarrow 0}, \quad \text{ (18)}$$

where $I(\kappa_n)$ is the applied current equal to $E_J \sin \kappa_n$. 
In the limit \( N \to \infty \), we can actually calculate the zero bias resistivity in the analytical form. From Eq. (13), (14), (17) and (18), \( R \) is obtained in the following form:

\[
R = 2R_Q \times F \left( \frac{\pi}{2}, \sqrt{1 - \frac{1}{2J}} \right) \times \\
\exp \left[ -8E \left( \frac{\pi}{2}, \sqrt{1 - \frac{1}{2J}} \right) J + 4F \left( \frac{\pi}{2}, \sqrt{1 - \frac{1}{2J}} \right) \right].
\]

(19)

Note that \( J \) contains both \( E_G \) and \( E_C \). Many theoretical works have treated only the limiting case in which either \( E_G \) or \( E_C \) is absent, while our derivation includes the case in which the two charging energies are comparable. Incidentally, it is reported that the resistivity is nonlinearly dependent on the applied current in the case of superconducting wires [8], but that might be caused by applying, instead of WKB method, the instanton method [9] which doesn’t explicitly take into account the zero point energy in the action. This is the reason why we adopt WKB method, instead of the instanton method.

The resistivity Eq. (19) is shown in Fig. 2 compared with the experimental result. Because the experiment is in the limiting case where \( E_G \gg E_C \), the data are plotted against the reduced parameter \( J_0 = \sqrt{\frac{E_J}{E_G}} \). Without any fitting parameter, our theory and the experiment show good agreement asymptotically in the superconducting side \((J_0 \gtrsim 0.5)\).

However WKB approximation, on which our theory is based, is expected to be valid in the strong coupling limit \((J_0 \gg 1)\), so more experimental data in the stronger coupling regime \((J_0 > 0.8)\) are necessary to confirm our theory. Further, the data of arrays in which \( E_G \) and \( E_C \) are comparable are also hoped for.

We can also estimate the S-I transition points for various number of grains \( N \). In a strict sense, there is no critical point for a finite \( N \) because there is no sharp phase transition in a finite size system. But instead we can consider some critical region within which the crossover from the superconducting state to the insulating state takes place. In fact, in the experiment [4], the behaviors of resistance for finite \( N \)'s (63,127,255) change around \( J_0 \sim 0.5 \), so we can state that the transitions occur around there. The positions of critical regions are slightly dependent on \( N \). To be more specific, with increasing \( N \), the crossover is seen to
occur at larger $J$. This seems to correspond to the ascent of the transition temperature due to the size effect in the case of a 2D $XY$ model, which can be detected experimentally in superfluid $^4$He in mesoporous media [10,11].

In our theoretical derivation, when the zero point fluctuation of $\kappa$ and its potential barrier are comparable, the phase cannot be localized and the phase coherence is destroyed. So we can define this configuration as the boundary between superconducting and insulating states. The critical value $J_C$ is estimated from Eq. (6) and Eq. (10),

$$J_C \sim \left\{ (N - 1) - (N - 3) \cos \frac{\pi}{N - 3} \right\}^{-1}.$$  \hspace{1cm} (20)

For large $N$, Eq. (20) reduces to

$$J_C \sim \frac{1}{2} \left\{ 1 - \frac{1}{4} \left( \frac{\pi}{N} \right)^2 \right\}.$$  \hspace{1cm} (21)

Eq. (21) qualitatively explains the size dependence of the critical region in the experiment, although the quantitative agreement is rather poor. This quantitative discrepancy may result from the harmonic approximation we have employed for estimating zero-point fluctuation of $\kappa$.

Now, we give some comments to our argument up to this point. There might be an assertion that we should take into account the translational symmetry about the position of a phase slip center and therefore multiply the tunneling rate by the number of virtual centers of a phase slip formation, say, the number of junctions. But the mode we have adopted as the zero point fluctuation of $\kappa$ is created by the collective motion of the field and not localized at some junction, so the degree of freedom about the position of a phase slip center is already summed up. Besides, it is worth considering additional tunneling paths slightly shifted from the valley paths which we have adopted. This supplement will be crucial when the effective inertial mass of a phase slip is greatly dependent on the tunneling path and the correction by these paths probably increase the tunneling rate. On the contrary, it is widely known that the coupling of superconducting component with the environments generally suppresses the tunneling rate [12], which we have ignored in this Letter. To evaluate the contribution from this dissipation effect is also left for the future work.
In conclusion, we have studied the quantum nucleation process of a phase slip in a 1D JJ array using the collective coordinate method. The theoretical value of the zero bias resistivity calculated by means of WKB method in the case of large $N$ limit is in good agreement with the experimental data asymptotically in the superconducting side. We also have estimated the S-I transition point for an array with a general number of junctions, which roughly agrees with the experiment. All of our results suggest that the intrinsic resistivity in the superconducting side in a 1D JJ array at zero temperature can be understood with the concept of quantum nucleation of phase slips and it’s value can be actually estimated.

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FIG. 1. The phase of the interpolation function Eq. (3) for the case $N = 63$, $n = 2$ and $q = 3$ is plotted (solid line). The dotted line depicts the local minimum solution Eq. (4).
FIG. 2. The zero bias resistivity plotted against the coupling parameter $J_0$. In the superconducting side ($J_0 \gtrsim 0.5$), our theoretical value is in good agreement with the experimental data, especially with the ones for $N = 127$ and $N = 255$. 

$J_0 = \left(\frac{E_J}{E_G}\right)^{1/2}$