Families of cubic Thue equations with effective bounds for the solutions

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Abstract. To each non totally real cubic extension \( K \) of \( \mathbb{Q} \) and to each generator \( \alpha \) of the cubic field \( K \), we attach a family of cubic Thue equations, indexed by the units of \( K \), and we prove that this family of cubic Thue equations has only a finite number of integer solutions, by giving an effective upper bound for these solutions.

1 Statements

Let us consider an irreducible binary cubic form having rational integers coefficients
\[
F(X,Y) = a_0X^3 + a_1X^2Y + a_2XY^2 + a_3Y^3 \in \mathbb{Z}[X,Y]
\]
with the property that the polynomial \( F(X,1) \) has exactly one real root \( \alpha \) and two complex imaginary roots, namely \( \alpha' \) and \( \overline{\alpha'} \). Hence \( \alpha \notin \mathbb{Q}, \alpha' \neq \overline{\alpha'} \) and
\[
F(X,Y) = a_0(X - \alpha Y)(X - \alpha' Y)(X - \overline{\alpha'} Y).
\]

Let \( K \) be the cubic number field \( \mathbb{Q}(\alpha) \) which we view as a subfield of \( \mathbb{R} \). Define \( \sigma : K \to \mathbb{C} \) to be one of the two complex embeddings, the other one being the conjugate \( \overline{\sigma} \). Hence \( \alpha' = \sigma(\alpha) \) and \( \overline{\alpha'} = \overline{\sigma(\alpha)} \). If \( \tau \) is defined to be the complex conjugation, we have \( \overline{\sigma} = \tau \circ \sigma \) and \( \sigma \circ \tau = \sigma \).

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Let $\varepsilon$ be a unit $> 1$ of the ring $\mathbb{Z}_K$ of algebraic integers of $K$ and let $\varepsilon' = \sigma(\varepsilon)$ and $\varepsilon'' = \sigma(\varepsilon)$ be the two other algebraic conjugates of $\varepsilon$. We have

$$|\varepsilon'| = |\varepsilon''| = \frac{1}{\sqrt{\varepsilon}} < 1.$$

For $n \in \mathbb{Z}$, define

$$F_n(X, Y) = a_0(X - \varepsilon^n \alpha Y)(X - \varepsilon'^n \alpha' Y)(X - \varepsilon''^n \alpha'' Y).$$

Let $k \in \mathbb{N}$, where $\mathbb{N} = \{1, 2, \ldots\}$. We plan to study the family of Thue inequations

$$0 < |F_n(x, y)| \leq k,$$

where the unknowns $n, x, y$ take values in $\mathbb{Z}$.

**Theorem 1.** There exist effectively computable positive constants $\kappa_1$ and $\kappa_2$, depending only on $F$, such that, for all $k \in \mathbb{Z}$ with $k \geq 1$ and for all $(n, x, y) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ satisfying $\varepsilon^n \alpha \notin \mathbb{Q}$, $xy \neq 0$ and $|F_n(x, y)| \leq k$, we have

$$\max \{ |\varepsilon^n|, |x|, |y| \} \leq \kappa_1 \kappa_2^2.$$

From this theorem, we deduce the following corollary.

**Corollary 1.** For $k \in \mathbb{Z}$, $k > 0$, the set

$$\{(n, x, y) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \mid \varepsilon^n \alpha \notin \mathbb{Q}; xy \neq 0; |F_n(x, y)| \leq k\}$$

is finite.

This corollary is a particular case of the main result of [2], but the proof in [2] is based on the Schmidt subspace theorem which does not allow to give an effective upper bound for the solutions $(n, x, y)$.

**Example.** Let $D \in \mathbb{Z}$, $D \neq -1$. Let $\varepsilon := (\sqrt[3]{D^3 + 1} - D)^{-1}$. There exist two positive effectively computable absolute constants $\kappa_3$ and $\kappa_4$ with the following property. Define a sequence $(F_n)_{n \in \mathbb{Z}}$ of cubic forms in $\mathbb{Z}[X, Y]$ by

$$F_n(X, Y) = X^3 + a_n X^2 Y + b_n X Y^2 - Y^3,$$

where $(a_n)_{n \in \mathbb{Z}}$ is defined by the recurrence relation

$$a_{n+3} = 3D a_{n+2} + 3D^2 a_{n+1} + a_n$$

with the initial conditions $a_0 = 3D^2$, $a_{-1} = 3$ and $a_{-2} = -3D$, and where $(b_n)_{n \in \mathbb{Z}}$ is defined by $b_n = -a_{n-2}$. Then, for $x, y, n$ rational integers with $xy \neq 0$ and $n \neq -1$, we have

$$|F_n(x, y)| \geq \kappa_3 \max \{ |x|, |y|, |\varepsilon^n| \} \kappa_4.$$
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This result follows from Theorem 1 with \( \alpha = \varepsilon \) and

\[
F(X, Y) = X^3 - 3DX^2Y - 3D^2XY^2 - Y^3.
\]

Indeed, the irreducible polynomial of \( \varepsilon^{-1} = \sqrt{D^3 + 1} - D \) is

\[
F_{-2}(X, 1) = (X + D)^3 - D^3 - 1 = X^3 + 3DX^2 + 3D^2X - 1,
\]

the irreducible polynomial of \( \alpha = \varepsilon \) is

\[
F(X, 1) = F_0(X, 1) = F_{-2}(1, X) = X^3 - 3D^2X^2 - 3DX - 1,
\]

while

\[
F_{-1}(X, Y) = (X - Y)^3 = X^3 - 3X^2Y + 3XY^2 - Y^3.
\]

For \( n \in \mathbb{Z}, n \neq -1 \), \( F_n(X, 1) \) is the irreducible polynomial of \( \alpha \varepsilon^n = \varepsilon^{n+1} \), while for any \( n \in \mathbb{Z}, F_n(X, Y) = N_{\mathbb{Q}(\varepsilon)/\mathbb{Q}}(X - \varepsilon^{n+1}Y) \).

The recurrence relation for \( a_n = \varepsilon^{n+1} + \varepsilon^{n}+1 + \varepsilon^n \) follows from

\[
\varepsilon^{n+3} = 3D\varepsilon^{n+2} + 3D^2\varepsilon^{n+1} + \varepsilon^n
\]

and for \( b_n \), from \( F_{-n}(X, Y) = -F_{n-2}(Y, X) \).

2 Elementary estimates

For a given integer \( k > 0 \), we consider a solution \((n, x, y)\) in \( \mathbb{Z}^3 \) of the Thue inequality with \( \varepsilon^n \alpha \) irrational and \( xy \neq 0 \). We will use \( \kappa_5, \kappa_6, \ldots, \kappa_{55} \) to designate some constants depending only on \( \alpha \).

Let us firstly explain that in order to prove Theorem 1 we can assume \( n \geq 0 \) by eventually permuting \( x \) and \( y \). Let us suppose that \( n < 0 \) and write

\[
F(X, Y) = a_3(Y - \alpha^{-1}X)(Y - \alpha^{t-1}X)(Y - \overline{\alpha}^{-1}X).
\]

Then

\[
F_n(X, Y) = a_3(Y - e^{\lfloor n \rfloor} \alpha^{-1}X)(Y - e^{\lfloor n \rfloor} \alpha^{t-1}X)(Y - \overline{\alpha}^{-\lfloor n \rfloor} \alpha^{-1}X).
\]

Now it is simply a matter of using the result for \( \lfloor n \rfloor \) for the polynomial \( G(X, Y) = F(Y, X) \).

Let us now check that, in order to prove the statements of 1, there is no restriction in assuming that \( \alpha \) is an algebraic integer and that \( a_0 = 1 \). To achieve this goal, we define
\[ F(T, Y) = T^3 + a_1 T^2 Y + a_0 a_2 T Y^2 + a_0^2 a_3 Y^3 \in \mathbb{Z}[T, Y], \]

so that \( a_0^2 F(X, Y) = \tilde{F}(a_0 X, Y) \). If we define \( \tilde{\alpha} = a_0 \alpha \) and \( \tilde{\alpha}' = a_0 \alpha' \), then \( \tilde{\alpha} \) is a nonzero algebraic integer, and we have

\[ \tilde{F}(T, Y) = (T - \tilde{\alpha} Y)(T - \tilde{\alpha}' Y)(T - \overline{\tilde{\alpha}} Y). \]

For \( n \in \mathbb{Z} \), the binary form

\[ \tilde{F}_n(T, Y) = (T - \varepsilon^n \tilde{\alpha} Y)(T - \varepsilon^n \tilde{\alpha}' Y)(T - \overline{\varepsilon^n \tilde{\alpha}} Y) \]

satisfies

\[ a_0^2 \tilde{F}_n(X, Y) = \tilde{F}_n(a_0 X, Y). \]

The condition (1) implies \( 0 < |\tilde{F}_n(a_0 x, y)| \leq a_0^2 k \). Therefore it suffices to prove the statements for \( \tilde{F}_n \) instead of \( F_n \), with \( \alpha \) and \( \alpha' \) replaced by \( \tilde{\alpha} \) and \( \tilde{\alpha}' \). This allows us, from now on, to suppose \( \alpha \in \mathbb{Z}_K \) and \( a_0 = 1 \).

As already explained, we can assume \( n \geq 0 \). There is no restriction in supposing \( k \geq 2 \); (if we prove the result for a value of \( k \geq 2 \), we deduce it right away for smaller values of \( k \), since we consider Thue inequations and not Thue equations). If \( k \) were assumed to be \( \geq 2 \), we would not need \( \kappa_1 \), as is easily seen, and the conclusion would read

\[ \max\{n, |\alpha|, |\beta|\} \leq k. \]

Without loss of generality we can assume that \( n \) is sufficiently large. As a matter of fact, if \( n \) is bounded, we are led to some given Thue equations, and Theorem 1 follows from Theorem 5.1 of [3].

Let us recall that for an algebraic number \( \gamma \), the house of \( \gamma \), denoted \( \text{h} \), is by definition the maximum of the absolute values of the conjugates of \( \gamma \). Moreover, \( d \) is the degree of the algebraic number field \( K \) (namely \( d = 3 \) here) and \( R \) is the regulator of \( K \) (viz. \( R = \log \varepsilon \)), where, from now on, \( \varepsilon \) is the fundamental unit \( > 1 \) of the non totally real cubic field \( K \). The next statement is Lemma A.6 of [3].

**Lemma 1** Let \( \gamma \) be a nonzero element of \( \mathbb{Z}_K \) of norm \( \leq M \). There exists a unit \( \eta \in \mathbb{Z}_K \) such that the house \( \text{h} \) is bounded by an effectively computable constant which depends only on \( d \), \( R \) and \( M \).

We need to make explicit the dependence upon \( M \), and for this, it suffices to apply Lemma A.15 of [3], which we want to state, under the assumption that the \( d \) embeddings of the algebraic number field \( K \) in \( \mathbb{C} \) are noted \( \sigma_1, \ldots, \sigma_d \).

**Lemma 2** Let \( K \) be an algebraic number field of degree \( d \) and let \( \gamma \) be a nonzero element of \( \mathbb{Z}_K \) whose absolute value of the norm is \( m \). Then there exists a unit \( \eta \in \mathbb{Z}_K^\times \) such that

\[
\frac{1}{R} \max_{1 \leq j \leq d} \left| \log(m^{-1/d}|\sigma_j(\eta \gamma)|) \right|
\]

is bounded by an effectively computable constant which depends only on \( d \).
Since \( d = 3 \), \( K = \mathbb{Q}(\alpha) \) and the regulator \( R \) of \( K \) is an effectively computable constant (see for instance \( \text{(1)} \), §6.5), the conclusion of Lemma 2 is

\[
-\kappa_5 \leq \log(|\sigma_j(\eta \gamma)|/\sqrt{m}) \leq \kappa_5,
\]

which can also be written as

\[
\kappa_6 \sqrt{m} \leq |\sigma_j(\eta \gamma)| \leq \kappa_7 \sqrt{m},
\]

with two effectively computable positive constants \( \kappa_6 \) and \( \kappa_7 \). We will use only the upper bound\(^1\) under the hypotheses of Lemma 1 with \( d = 3 \), when \( \gamma \) is a nonzero element of \( \mathbb{Z}_K \) of norm \( \leq M \), there exists a unit \( \eta \) of \( \mathbb{Z}_K^* \) such that

\[
|\eta \gamma| \leq \kappa_7 \sqrt{M}.
\]

Our strategy is to prove that \(|\ell|\) is bounded by a constant times \( \log k \), and that \(|n|\) is also bounded by a constant times \( \log k \); then we will show that \(|y|\) is bounded by a constant power of \( k \) and deduce that \(|x|\) is also bounded by a constant power of \( k \).

Let us eliminate \( x \) in (2) and (4) to obtain

\[
y = -\frac{\varepsilon^n \alpha y - \varepsilon^m \alpha' y}{\varepsilon^n \alpha - \varepsilon^m \alpha'},
\]

since we supposed \( \varepsilon^n \alpha \) irrational, we did not divide by 0. The complex conjugate of (2) is written as

\[
x - \frac{\varepsilon^m \alpha y}{\varepsilon^n \alpha - \varepsilon^m \alpha'} = \frac{\varepsilon^m \alpha'}{\varepsilon^n \alpha - \varepsilon^m \alpha'}.
\]

\(^1\) The lower bound follows from looking at the norm!
We eliminate $x$ and $y$ in the three equations (2), (4) and (6) to obtain a unit equation à la Siegel:

$$
\varepsilon \ell \xi_1 (\alpha' \varepsilon^n - \alpha \varepsilon'^n) + \varepsilon' \ell \xi_1 (\alpha' \varepsilon'^n - \alpha \varepsilon^n) + \varepsilon' \ell \xi_1 (\alpha \varepsilon^n - \alpha' \varepsilon'^n) = 0. 
\tag{7}
$$

In the remaining part of this section 2, we suppose $\varepsilon^n |\alpha| \geq 2 |\varepsilon' n|$. \tag{8}

Note that if this inequality is not satisfied, then we have $\varepsilon_3 n / 2 < 2 |\alpha'| |\alpha|$ and this leads to the inequality (18), and to the rest of the proof of Theorem 1 by using the argument following the inequality (18).

For $\ell > 0$, the absolute value of the numerator $\varepsilon \ell \xi_1 - \varepsilon' \ell \xi'_1$ in (5) is increasing like $\varepsilon \ell$ and for $\ell < 0$ it is increasing like $\varepsilon'^2 / 2$; for $n > 0$, the absolute value of the denominator $\varepsilon^n \alpha - \varepsilon' n \alpha'$ is increasing like $\varepsilon^n$ and for $n < 0$ it is increasing like $\varepsilon'^n / 2$. In order to extract some information from the equation (5), we write it in the form

$$y = \pm \frac{A - a}{B - b}$$

with $B = \varepsilon^n \alpha, \ b = \varepsilon' n \alpha'$,

$$\{A, a\} = \left\{ \varepsilon \ell \xi_1, \varepsilon' \ell \xi'_1 \right\},$$

the choice of $A$ and $a$ being dictated by

$$|A| = \max \{ |\varepsilon \ell \xi_1|, |\varepsilon' \ell \xi'_1| \}, \quad |a| = \min \{ |\varepsilon \ell \xi_1|, |\varepsilon' \ell \xi'_1| \}.$$

Since $|A - a| \leq 2 |A|$ and since $|b| \leq |B| / 2$ because of (8), we have $|B - b| \geq |B| / 2$, so we get

$$|y| \leq 4 \frac{|A|}{|B|}.$$

We will consider the two cases corresponding to the possible signs of $\ell$, (remember that $n$ is positive).

**First case.** Let $\ell \leq 0$. We have

$$|A| \leq \kappa_{11} \varepsilon'^n / 2 \kappa_9 .$$

We deduce from (5)

$$1 \leq |y| \leq 4 \frac{\varepsilon \ell}{\alpha} \varepsilon'^{n / 2} \leq \kappa_{12} \varepsilon'^{n / 2} \kappa_9 \tag{9}$$

Hence there exists $\kappa_{13}$ such that
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\[ 0 \leq \log |y| \leq \left( \frac{|\ell|}{2} - n \right) \log \epsilon + \frac{13}{15} \log k, \]

from which we deduce the inequality

\[ n \leq \frac{|\ell|}{2} + \kappa_{14} \log k, \tag{10} \]

which will prove useful: \( n \) is roughly bounded by \(|\ell|\). From (4) we deduce the existence of a constant \( \kappa_{15} \) such that

\[ |x| \leq \epsilon^{-n/2}|\alpha'|y| + \kappa_{15} k^{\epsilon^{-1/2}}. \tag{11} \]

**Second case.** Let \( \ell > 0 \). We have

\[ |A| \leq \kappa_{16} \epsilon^{\ell/2}. \tag{12} \]

We deduce from (5) the upper bound

\[ 1 \leq |y| \leq 4 \left( \frac{\xi_1}{\alpha} \right) \epsilon^{\ell-n} \leq \kappa_{17} k^{\epsilon^{\ell-n}}; \tag{12} \]

hence there exists \( \kappa_{18} \) such that

\[ 0 \leq \log |y| \leq (\ell - n) \log \epsilon + \kappa_{18} \log k. \]

Consequently,

\[ n \leq \ell + \kappa_{19} \log k. \tag{13} \]

From the relation (4) we deduce the existence of a constant \( \kappa_{20} \) such that

\[ 1 \leq |x| \leq \epsilon^{-n/2}|\alpha'|y| + \kappa_{20} k^{\epsilon^{-1/2}}. \tag{14} \]

By taking into account the inequalities (9), (10) and (11) in the case \( \ell \leq 0 \), and the inequalities (12), (13) and (14) in the case \( \ell > 0 \), let us show that the existence of a constant \( \kappa_{21} \) satisfying \(|\ell| \leq \kappa_{21} \log k \) allows to conclude the proof of Theorem 1. As a matter of fact, suppose

\[ |\ell| \leq \kappa_{21} \log k. \tag{15} \]

Then (10) and (13) imply \( n \leq \kappa_{22} \log k \), whereupon \(|\ell|\) and \( n \) are effectively bounded by a constant times \( \log k \). This implies that the elements \( e^t \), with \( t \) being \((|\ell|/2) - n, \ell - n, -n/2, |\ell|/2 \) or \(-\ell/2\), appearing in (9), (12), (11) and (14) are bounded from above by \( k^{\kappa_{23}} \) for some constant \( \kappa_{23} \). Therefore the upper bound of \(|y|\) in the conclusion of Theorem 1 follows from (9) and (12) and the upper bound of \(|x|\) is a consequence of (11) and (14). Our goal is to show that sooner or later, we end up with the inequality (15).

In the case \( \ell > 0 \), the lower bound \(|x| \geq 1\) provides an extra piece of information. If the term \( \epsilon^{\ell/2} \xi_1 \) on the right hand side of (4) does not have an absolute value \( < 1/2 \),
then the upper bound (15) holds true and this suffices to claim the proof of Theorem 1. Suppose now $|\epsilon^{\ell}x_1'| < 1/2$. Since the relation (12) implies

$$e^{-n/2} |\alpha' y| \leq 4 \left| \frac{\xi_1 \alpha'}{\alpha} \right| e^{\ell - (3n/2)},$$

we have

$$1 \leq |x| \leq 4 \left| \frac{\xi_1 \alpha'}{\alpha} \right| e^{\ell - (3n/2)} + \frac{1}{2}$$

and

$$1 \leq 8 \left| \frac{\xi_1 \alpha'}{\alpha} \right| e^{\ell - (3n/2)}.$$

We deduce

$$\frac{3}{2} n \leq \ell + \kappa_{24} \log k. \quad (16)$$

The upper bound in (16) is sharper than the one in (13), but, amazingly, we used (13) to establish (16).

When $\ell < 0$, we have $|\ell - n| = n + |\ell| \geq |\ell|$, while in the case $\ell \geq 0$ we have

$$|\ell - n| \geq \frac{1}{3} \ell + \frac{2}{3} \ell - n \geq \frac{1}{3} |\ell| - \kappa_{24} \log k,$$

because of (16). Therefore, if $\ell$ is positive (recall (16)), zero or negative (recall (10)), we always have

$$n \leq \frac{2}{3} |\ell| + \kappa_{25} \log k \quad \text{and} \quad |\ell - n| \geq \frac{1}{3} |\ell| - \kappa_{24} \log k \quad (17)$$

with $\kappa_{24} > 0$ and $\kappa_{25} > 0$.

3 Diophantine tool

Let us remind what we mean by the absolute logarithmic height $h(\alpha)$ of an algebraic number $\alpha$ (cf. [4], Chap. 3). For $L$ a number field and for $\alpha \in L$, we define

$$h(\alpha) = \frac{1}{[L : Q]} \log H_L(\alpha),$$

with

$$H_L(\alpha) = \prod_{v} \max \{1, |\alpha|_v \}^{d_v}$$

where $v$ runs over the set of places of $L$, with $d_v$ being the local degree of the place $v$ if $v$ is ultrametric, $d_v = 1$ if $v$ is real, $d_v = 2$ if $v$ is complex. When $f(X) \in \mathbb{Z}[X]$ is the minimal polynomial of $\alpha$ and $f(X) = a_0 \prod_{1 \leq j \leq d} (X - \alpha_j)$, with $\alpha_1 = \alpha$, it
happens that
\[ h(\alpha) = \frac{1}{d} \log M(f) \quad \text{with} \quad M(f) = \left| a_0 \right| \prod_{1 \leq j \leq d} \max \{ 1, |\alpha_j| \}. \]

We will use two particular cases of Theorem 9.1 of [4]. The first one is a lower bound for the linear form of logarithms \( b_0 \lambda_0 + b_1 \lambda_1 + b_2 \lambda_2 \), and the second one is a lower bound for \( \gamma_1^2 \gamma_2^2 - 1 \). Here is the first one.

**Proposition 1.** There exists an explicit absolute constant \( c_0 > 0 \) with the following property. Let \( \gamma_0, \gamma_1, \gamma_2 \) be three rational integers such that \( \Lambda = b_0 \lambda_0 + b_1 \lambda_1 + b_2 \lambda_2 \) be nonzero. Write
\[ \gamma_0 = e^{\lambda_0}, \quad \gamma_1 = e^{\lambda_1}, \quad \gamma_2 = e^{\lambda_2} \quad \text{and} \quad D = \left[ \mathbb{Q}(\gamma_0, \gamma_1, \gamma_2) : \mathbb{Q} \right]. \]

Let \( A_0, A_1, A_2 \) and \( B \) be real positive numbers satisfying
\[ \log A_i \geq \max \left\{ h(\gamma_i), \left| \frac{\lambda_i}{D} \right| \right\} \quad (i = 0, 1, 2) \]
and
\[ B \geq \max \left\{ e, D, \frac{|b_2|}{D \log A_0}, \frac{|b_0|}{D \log A_2}, \frac{|b_2|}{D \log A_1}, \frac{|b_1|}{D \log A_2} \right\}. \]
Then
\[ |\Lambda| \geq \exp \left\{ -c_0 D^5 (\log D)(\log A_0)(\log A_1)(\log A_2)(\log B) \right\}. \]

The second particular case of Theorem 9.1 in [4] that we will use is the next Proposition 2. It also follows from Corollary 9.22 of [4]. We could as well deduce it from Proposition 1.

**Proposition 2.** Let \( D \) be a positive integer. There exists an explicit constant \( c_1 > 0 \), depending only on \( D \) with the following property. Let \( K \) be a number field of degree \( \leq D \). Let \( \gamma_1, \gamma_2 \) be nonzero elements in \( K \) and let \( b_1, b_2 \) be rational integers. Assume \( \gamma_1^2 \gamma_2^2 \neq 1 \). Set
\[ B = \max \{ 2, |b_1|, |b_2| \} \quad \text{and, for} \quad i = 1, 2, \quad A_i = \exp \left( \max \{ e, h(\gamma_i) \} \right). \]
Then
\[ |\gamma_1^2 \gamma_2^2 - 1| \geq \exp \left\{ -c_1 (\log B)(\log A_1)(\log A_2) \right\}. \]

Proposition 2 will come into play via its following consequence.

**Corollary 2** Let \( \delta_1 \) and \( \delta_2 \) be two real numbers in the interval \( [0, 2\pi) \). Suppose that the numbers \( e^{i \delta_1} \) and \( e^{i \delta_2} \) are algebraic. There exists an explicit constant \( c_2 > 0 \), depending only upon \( \delta_1 \) and \( \delta_2 \), with the following property: for each \( n \in \mathbb{Z} \) such that \( \delta_1 + n \delta_2 \notin \mathbb{Z} \pi \), we have
\[ |\sin(\delta_1 + n \delta_2)| \geq (|n| + 2)^{-c_2}. \]
Proof. Write $\gamma_1 = e^{i\delta_1}$ and $\gamma_2 = e^{i\delta_2}$. By hypothesis, $\gamma_1$ and $\gamma_2$ are algebraic with $\gamma_1\gamma_2 \neq 1$. Let us use Proposition 2 with $b_1 = 1$, $b_2 = n$. The parameters $A_1$ and $A_2$ depend only upon $\delta_1$ and $\delta_2$ and the number $B = \max\{2, |n|\}$ is bounded from above by $|n| + 2$. Hence

$$|\gamma_1\gamma_2 - 1| \geq (|n| + 2)^{-c_3}$$

where $c_3$ depends only upon $\delta_1$ and $\delta_2$. Let $\ell$ be the nearest integer to $(\delta_1 + n\delta_2)/\pi$ (take the floor if there are two possible values) and let $t = \delta_1 + n\delta_2 - \ell\pi$. This real number $t$ is in the interval $(-\pi/2, \pi/2]$. Now

$$|e^{it} + 1| = |1 + \cos(t) + i\sin(t)| = \sqrt{2(1 + \cos(t))} \geq \sqrt{2}.$$ 

Since $e^{it} = (-1)^{\ell}\gamma_1\gamma_2$, we deduce

$$|\sin(\delta_1 + n\delta_2)| = |\sin(t)| = \frac{1}{2} \left| (-1)^{2\ell} e^{2it} - 1 \right| = \frac{1}{2} \left| (-1)^{t} e^{it} + 1 \right| \left| (-1)^{t} e^{it} - 1 \right| \geq \frac{\sqrt{2}}{2} |\gamma_1\gamma_2 - 1|.$$ 

This secures the proof of Corollary 2.

The following elementary lemma makes clear that $e^{t} \sim 1$ for $t \to 0$. The first (resp. second) part follows from Exercise 1.1.a (resp. 1.1.b or 1.1.c) of [4]. We will use only the second part; the first one shows that the number $t$ in the proof of Corollary 2 is close to 0, but we did not need it.

**Lemma 3**  

(a) For $t \in \mathbb{C}$, we have

$$|e^{t} - 1| \leq |t| \max\{1, |e^{t}|\}.$$ 

(b) If a complex number $z$ satisfies $|z - 1| < 1/2$, then there exists $t \in \mathbb{C}$ such that $e^{t} = z$ and $|t| \leq 2|z - 1|$. This $t$ is unique and is the principal determination of the logarithm of $z$:

$$|\log z| \leq 2|z - 1|.$$ 

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4 Proof of Theorem [1]

Let us define some real numbers $\theta$, $\delta$ and $\nu$ in the interval $[0, 2\pi)$ by

$$e^{t} = \frac{1}{e^{1/2}} e^{i\theta}, \quad \alpha' = |\alpha'| e^{i\delta}, \quad \xi_1' = |\xi_1'| e^{i\nu}.$$ 

By ordering the terms of (7), we can write this relation as

$$T_1 + T_2 + T_3 = 0.$$
and the three terms involved are
\[
\begin{align*}
T_1 : &= e^\ell \xi_1 (\alpha' e^n - \overline{\alpha'} e^{\ell n}) = 2i\xi_1 |\alpha'| e^{\ell-n/2} \sin(\delta + n\theta), \\
T_2 : &= \alpha e^n (\overline{\xi}_1' - e^{\ell} \xi_1') = -2i\xi_1' |\alpha| e^{n-\ell/2} \sin(\nu + \ell \theta), \\
T_3 : &= \xi_1' e^{\ell n} \overline{\alpha'} e^{\ell n} - \overline{\xi}_1 e^\ell \alpha' e^n = 2i\xi_1' \alpha' e^{-(n+\ell)/2} \sin(\nu - \delta + (\ell - n)\theta).
\end{align*}
\]

It turns out that these three terms are purely imaginary. We write this zero sum as
\[
a + b + c = 0 \quad \text{with} \quad |a| \geq |b| \geq |c|,
\]
and we use the fact that this implies that $|a| \leq 2|b|$. Thanks to $(17)$, Corollary $2$ shows that a lower bound of the sinus terms is $|\ell|^{-\kappa_{26}}$ (and an obvious upper bound is $1$). Moreover,

\begin{itemize}
  \item The $T_1$ term contains a constant factor and the factors:
    \begin{itemize}
      \item $|\xi_1|$ with $k^{-2} \leq |\xi_1| \leq k^2$.
      \item $e^{\ell-n/2}$ (which is the main term).
      \item a sinus with a parameter $n$ (a lower bound of the absolute value of that sinus being $n^{-\kappa_{36}}$).
    \end{itemize}
  \item Similarly, $T_2$ contains a constant factor and the factors:
    \begin{itemize}
      \item $|\xi_1'|$ with $k^{-2} \leq |\xi_1'| \leq k^2$.
      \item $e^{n-\ell/2}$ (which the main term).
      \item a sinus with a parameter $\ell$ (a lower bound of the absolute value of that sinus being $|\ell|^{-\kappa_{26}}$).
    \end{itemize}
  \item Similarly, $T_3$ contains a constant factor and the factors:
    \begin{itemize}
      \item $|\xi_1'|$ with $k^{-2} \leq |\xi_1'| \leq k^2$.
      \item $e^{-(n+\ell)/2}$ (which the main term).
      \item a sinus with a parameter $\ell - n$ (a lower bound of the absolute value of that sinus being $|\ell - n|^{-\kappa_{36}}$).
    \end{itemize}
\end{itemize}

We will consider three cases, and we will use the inequalities $(3)$ and $(17)$. This will eventually allow us to conclude that there is an upper bound for $|\ell|$ and $n$ by an effective constant times $\log k$.

**First case.** If the two terms $a$ and $b$ with the largest absolute values are $T_1$ and $T_2$, from the inequalities $|T_1| \leq 2|T_2|$ and $|T_2| \leq 2|T_1|$ (which come from $|b| \leq |a| \leq 2|b|$), we deduce (thanks to $(17)$)
\[
k^{-\kappa_{30}} |\ell|^{-\kappa_{31}} \leq e^{\frac{3}{2} (\ell - n)} \leq k^{\kappa_{32}} |\ell|^{\kappa_{33}},
\]
whereupon, thanks again to $(17)$, we have
\[
-\kappa_{34} \log k + \frac{|\ell|}{3} \leq |\ell - n| \leq \kappa_{35} \log |\ell| + \kappa_{36} \log k,
\]
which leads to $|\ell| \leq \kappa_{37} (\log k + \log |\ell|)$. This secures the upper bound $(15)$, and ends the proof of Theorem $1$.
Second case. Suppose that the two terms $a$ and $b$ with the largest absolute values are $T_1$ and $T_3$. By writing $|T_1| \leq 2|T_3|$ and $|T_3| \leq 2|T_1|$, we obtain (thanks to (17))

$$k^{-1/3}|\ell|^{-\kappa_{38}} \leq \varepsilon^{3/2} \leq k^{1/3}|\ell|^{\kappa_{39}},$$

hence

$$|\ell| \leq \kappa_{40}(\log k + \log |\ell|).$$

Once more, we have $\varepsilon|\ell| \leq k\kappa_{41}$, and we saw that the upper bound (17) allows to draw the conclusion.

Third case. Let us consider the remaining case, namely, the two terms $a$ and $b$ with the largest absolute values being $T_2$ and $T_3$. Consequently, in the relation $T_1 + T_2 + T_3 = 0$, written in the form $a + b + c = 0$ with $|a| \geq |b| \geq |c|$, we have $c = T_1$. Writing $|T_2| \leq 2|T_3|$ and $|T_3| \leq 2|T_2|$, we obtain

$$k^{-1/3}|\ell|^{-\kappa_{42}} \leq \varepsilon^{3n/2} \leq k^{1/3}|\ell|^{\kappa_{43}}.$$

From the second of these inequalities, we deduce the existence of $\kappa_{44}$ such that

$$n \leq \kappa_{44}(\log k + \log |\ell|).$$

Remark. The upper bound (18) allows to proceed as in the usual proof of the Thue theorem where $n$ is fixed.

From the upper bound $|T_1| \leq |T_2|$, one deduces $n > \ell - \kappa_{45}\log k$, so that (18) leads right away to the conclusion if $\ell$ is positive.

Let us suppose now that $\ell$ is negative. Let us consider again the equation (7) that we write in the form

$$\rho_n \varepsilon^\ell + \mu_n \varepsilon'\ell - \overline{\mu_n} \varepsilon'^\ell = 0$$

(19)

with

$$\rho_n = \zeta_1(\alpha' \varepsilon'^n - \overline{\alpha} \varepsilon^n) \quad \text{and} \quad \mu_n = \zeta_1(\overline{\alpha'} \varepsilon'^n - \alpha \varepsilon^n).$$

We check (cf. Property 3.3 of [4])

$$h(\mu_n) \leq \kappa_{46}(n + \log k).$$

Let us divide each side of (19) by $-\mu_n \varepsilon'^\ell$:

$$\frac{\overline{\mu_n} \varepsilon'^\ell}{\mu_n \varepsilon'^\ell} - 1 = \frac{\rho_n \varepsilon^\ell}{\mu_n \varepsilon'^\ell}.$$

We have

$$|\alpha' \varepsilon'^n - \overline{\alpha} \varepsilon^n| \leq |\alpha' \varepsilon'^n| + |\overline{\alpha} \varepsilon^n| = 2 |\varepsilon'^n \alpha'|$$

and, using (8),

$$|\overline{\alpha'} \varepsilon'^n - \alpha \varepsilon^n| \geq \frac{1}{2} |\alpha| \varepsilon^n.$$
Since \( |\xi| \leq k^{1/9} \) and \( |\xi'| > k^{-1/9} \) by (3), we come up with
\[
|\rho_n| \leq \kappa_4 k^{5/9} \kappa_9 \epsilon_n / 2, \quad |\mu_n| \geq \kappa_4 k^{5/9} \epsilon_n.
\]
Therefore, since \( |\epsilon'|^{-1} = \epsilon^{1/2} \), we have
\[
\left| \frac{\mu_n \epsilon'}{\mu_n \epsilon'} - 1 \right| = \left| \frac{\rho_n \epsilon'}{\mu_n \epsilon'} \right| \leq \kappa_{49} \epsilon^{-(n+3|\ell|)/2} k^{5/9}.
\]
We denote by \( \log \) the principal value of the logarithm and we set
\[
\lambda_1 = \log \left( \frac{\epsilon'}{\epsilon} \right), \quad \lambda_2 = \log \left( \frac{\mu_n}{\mu_n} \right) \quad \text{and} \quad \Lambda = \log \left( \frac{\mu_n \epsilon'}{\mu_n \epsilon'} \right).
\]
We have
\[
\lambda_1 = 2i\pi \nu \quad \lambda_2 = 2i\pi \theta_n,
\]
where \( \nu \) and \( \theta_n \) are the real numbers in the interval \([0, 1)\) defined by
\[
\frac{\epsilon'}{\epsilon'} = e^{2i\pi \nu} \quad \text{and} \quad \frac{\mu_n}{\mu_n} = e^{2i\pi \theta_n}.
\]
From \( e^\Lambda = e^{\lambda_1 + \lambda_2} \) we deduce \( \Lambda - \ell \lambda_1 - \lambda_2 = 2i\pi h \) with \( h \in \mathbb{Z} \). From Lemma 3 we deduce \( |\Lambda| \leq 2 |e^\Lambda - 1| \). Using \( |\Lambda| < 2\pi \) and writing
\[
2i\pi h = \Lambda - 2i\pi \ell \nu - 2i\pi \theta_n,
\]
we deduce \( |h| \leq |\ell| + 2 \).

In Proposition 1 let us take
\[
b_0 = h, \quad b_1 = \ell, \quad b_2 = 1, \quad \gamma_0 = 1, \quad \lambda_0 = 2i\pi, \quad \gamma_1 = \frac{\epsilon'}{\epsilon'}, \quad \gamma_2 = \frac{\mu_n}{\mu_n},
\]
\[
A_0 = A_1 = \kappa_3, \quad A_2 = (k \epsilon^n)^{\kappa_1}, \quad B = e + \frac{|\ell|}{\log A_2}.
\]
Notice that the degree \( D \) of the field \( \mathbb{Q}(\gamma_0, \gamma_1, \gamma_2) \) is \( \leq 6 \). Then we obtain
\[
\left| \frac{\mu_n \left( \frac{\epsilon'}{\epsilon'} \right) ^\ell}{\mu_n \epsilon'} - 1 \right| = |e^\Lambda - 1| \geq \frac{1}{2} |\Lambda| \geq \exp \left\{ -\kappa_{32} (\log A_2) (\log B) \right\}.
\]
By combining this estimate with (20), we deduce
\[
|\ell| \leq \kappa_{33} (n + \log k) \log B,
\]
which can also be written as $B \leq \kappa_4 \log B$, hence $B$ is bounded. This allows to obtain

$$|\ell| \leq \kappa_5 (n + \log k).$$

We use (18) to deduce $e^{\ell} \leq k^{41}$ and we saw that the upper bound (15) leads to the conclusion of the main Theorem.

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