PHANTOM PROBABILITY

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Abstract. Classical probability theory supports probability measures, assigning a fixed positive real value to each event, these measures are far from satisfactory in formulating real-life occurrences. The main innovation of this paper is the introduction of a new probability measure, enabling varying probabilities that are recorded by ring elements to be assigned to events; this measure still provides a Bayesian model, resembling the classical probability model.

By introducing two principles for the possible variation of a probability (also known as uncertainty, ambiguity, or imprecise probability), together with the “correct” algebraic structure allowing the framing of these principles, we present the foundations for the theory of phantom probability, generalizing classical probability theory in a natural way. This generalization preserves many of the well-known properties, as well as familiar distribution functions, of classical probability theory: moments, covariance, moment generating functions, the law of large numbers, and the central limit theorem are just a few of the instances demonstrating the concept of phantom probability theory.

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Introduction

Over the years much effort has been invested in trying to understand aspects of probability in which the evaluations of occurrences, as well as their likelihoods of happening, are uncertain. Although the terminology for this type of phenomena is varied (uncertainty for physicists, ambiguity for economists, imprecise probability for mathematicians, and phantom for us), fundamentally, the absence of theory enabling the formulation of such phenomena is a common problem for many fields of study. In this paper we introduce a new approach, supported by a novel probability measure, allowing a natural mathematical framing of this type of problems.

Two main principles underlie our approach to treating probability measures associated with varied evaluations:

- For each event, the sum of its probability and its possible distortion lies in the real interval \([0,1]\);
- The overall distortions always sum up to 0.

Having the right algebraic structure, termed here the ring of phantom numbers that naturally records probabilities and their oriented variations, these principles lead to the introduction of our new phantom probability measure, on which much of the theory of classical probability can be generalized. This generalization captures both the uncertainty of outcomes and ambiguous likelihoods, and it is still Bayesian.

The ring \(\mathbb{C}_8\) of phantom numbers consists of elements of the form \(z = a + \varphi b\), each of which is a compound of the real term \(a\) and the phantom term \(b\) (notated, like the complex numbers, by \(\varphi\) instead of \(i\)), and whose operations, addition and multiplication respectively, are

\[
(a_1 + \varphi b_1) \oplus (a_2 + \varphi b_2) := (a_1 + a_2) + \varphi (b_1 + b_2),
\]

\[
(a_1 + \varphi b_1) \otimes (a_2 + \varphi b_2) := a_1a_2 + \varphi (a_1b_2 + b_1a_2 + b_1b_2).
\]

This arithmetic makes \(\mathbb{C}_8\) suitable for the purpose of carrying a theory of probability. In many ways this ring resembles the field of complex numbers, but its arithmetic is different; here \(\varphi\) is idempotent, i.e. \(\varphi^2 = \varphi\), while \(i^2 = -1\) for the complexes. Similar structures, though sometimes using different terminology, have been studied in the literature, mainly from the abstract point of view of algebra; the innovation of this paper is the utilization in probability theory, which requires some special setting like phantom conjugate, reduced elements, absolute value, and norm. With these notions suitably defined, the way toward the development of a phantom probability theory is prepared.

One of the main advantages of phantom functions \(f : \mathbb{C}_8 \rightarrow \mathbb{C}_8\), mainly polynomial-like functions, is that they can be rewritten as

\[
f = f_{re} + \varphi (\hat{f} - f_{re}),
\]

where \(f_{re}\) and \(\hat{f}\) are real functions \(\mathbb{R} \rightarrow \mathbb{R}\). We call this property, which plays a main role in our exposition, the realization property of phantoms functions.

With this realization property satisfied, most of the phantom calculations are reduced simply to the real familiar calculations. Moreover, for \(z = a + \varphi b\), the real term \(a\) of \(z\) is the only argument involved in \(f_{re}\); this shows that when \(G\) is a pantomization of a real function \(g : \mathbb{R} \rightarrow \mathbb{R}\), the real component \(G_{re}\) of \(G\) is just \(g\). Surprisingly, the pantomizations of all classical probability functions (moments, variances, covariances, etc.) admit the realization property.

Using the phantom ring structure, together with our measure principles, we keep track of the evolution of the classical theory of probability. The leading motif throughout our exposition is that restricting the theory to the real terms of all the arguments involved always leaves ones with the well-known classical theory. Given this foundation, as well as the appropriate definitions, the probability insights are much clearer and their proofs become more transparent.

The main topics covered by this paper include:
• Conditional probability, independence, and Bayes’ rule;
• Random variables (discrete, continuous, and multiple);
• Attributes of random variables: moments, variances, covariances, moment generating functions;
• Inequalities (appropriately defined);
• Limit theorems.

Along our exposition we also provide many examples demonstrating how classical results naturally carry over to the phantom framework, further results and applications will be appear in our future papers.

The fact that the phantom probability space provides a Bayesian probability model paves the way for developing a theory of phantom stochastic processes and phantom Markov chains [9] with a view towards applications in dynamical systems.

We use the notion of imprecise probability as a generic term to cover all mathematical models which measure chance or uncertainty without sharp numerical probabilities [10]. The known results of past efforts to find a theory that frames imprecise probability give only partial or complicated answers. For example, fuzzy probability [17] only treats uncertain outcomes but not varying probabilities; conversely, complex probability provides a partial answer for deformed probabilities but only for fixed outcomes [11][18]. On the other hand, the operator measure theory [14] is very complicated and not intuitive, while the min-max model [7] is not Bayesian and [13] sometimes becomes non-additive.

These probability theories have a tremendous range of applications, like quantum mechanics, statistics, stochastic processes, dynamical systems, game theory, economics, mathematical finance, or decision making theory; to name just a few. Our development, together with the attendant examples, which smoothly extend the known theories that have already proven to be significant, lead one to believe that phantom theory could contribute to these applications, and make for a better understanding of phenomenons that arise in the real life.

1. The phantom ground ring

1.1. Ground ring structure. The central idea of our new approach is a generalization of the field \((\mathbb{R}, +, \cdot)\) of real numbers to a ring structure whose binary operations are induced by the familiar addition and multiplication of \(\mathbb{R}\). Focusing on application to probability theory, to make our exposition clearer, we give the explicit description for the certain extension of \(\mathbb{R}\) of order 1, which is suitable enough for the scope of this paper. For the sake of completeness, and in an effort to attract audiences from various fields of study, we recall some of the standard algebraic definitions (see [13]) and present the full proofs related to the basics of the algebraic structure for the extension of order 1. The more general phantom framework is outlined in the next subsection.

Set theoretically, our ground ring \(\mathbb{P}H_{(1)}(\mathbb{R})\), called a ring with phantoms or phantom ring, for short, is the Cartesian product \(\mathbb{R} \times \mathbb{R}\); for simplicity, we write \(a + \varphi b\) for a pair \((a, b) \in \mathbb{P}H_{(1)}(\mathbb{R})\). We say that \(\mathbb{P}H_{(1)}(\mathbb{R})\) is a phantom ring of order 1 over the reals, and denote it as \(\mathbb{P}H\), for short. (The general case of order \(> 1\) is spelled out later in Subsection 1.2) The elements of \(\mathbb{P}H\) are called phantom numbers, usually denoted \(x, y, z\).

In what follows we use the generic notation that \(a, b \in \mathbb{R}\) for reals and write \(z := a + \varphi b\) for a phantom number \(z\); we call \(a\) the real term of \(z\) while \(b\) is termed the phantom term of \(z\). We use the notation

\[
\text{re}(z) := a \quad \text{and} \quad \text{ph}(z) := b
\]

for the real term and the phantom term of \(z = a + \varphi b\), respectively. (The reason for calling the second argument “phantom” arises from the meaning assigned to this value in the extension of the probability measure, as explained in Section 2).

The set \(\mathbb{P}H\) is then equipped with the two binary operations, addition and multiplication, respectively,

\[
(a_1 + \varphi b_1) \oplus (a_2 + \varphi b_2) := (a_1 + a_2) + \varphi (b_1 + b_2),
\]

\[
(a_1 + \varphi b_1) \otimes (a_2 + \varphi b_2) := a_1 a_2 + \varphi (a_1 b_2 + b_1 a_2 + b_1 b_2),
\]

to establish the phantom ring (to be proved next), \((\mathbb{P}H, \oplus, \otimes)\), with unit \(1 := 1 + \varphi 0\) and zero \(0 := 0 + \varphi 0\). We write \(\mathbb{P}H^\times\) for \(\mathbb{P}H \setminus \{0\}\), \(\varphi\) for \(\varphi 1\), and \(a - \varphi b\) for \(a + \varphi (-b)\).

Remark 1.1. In general, similar algebraic structures (with different terminologies) are known in the literature, mainly for graded algebras or \(k\)-algebras in semiring theory, usually applied to a tensor \(M \otimes_k k\), where \(M\) is a module over semiring \(k\), c.f., [10]. However, as will be seen immediately, in this paper we push the algebraic
theory much further for the special case where \( M = k \) is a field; then, in this case, \( k \otimes_k k \) has a much richer structure. Moreover, some of our definitions are unique with the aim of serving applications in probability and measure theory.

Although the multiplication of \((\mathbb{PH}, \oplus, \otimes)\) is somehow reminiscent of the multiplication of the complex numbers \( \mathbb{C} \), it is different: for the phantoms \( \wp = \wp^2 \) is multiplicative idempotent, while for the complexes \( i^2 = -1 \) is not idempotent.

**Proposition 1.2.** \((\mathbb{PH}^x, \otimes)\) is an Abelian semigroup.

*Proof.* Given \( z_i = a_i + \wp b_i \), where \( i = 1, 2, 3 \), we have

\[
(z_1 \otimes z_2) \otimes z_3 = (a_1 a_2 + \wp (a_1 b_2 + b_1 a_2 + b_1 b_2)) \otimes (a_3 + \wp b_3)
\]

\[
= (a_1 a_2) a_3 + \wp ((a_1 a_2) b_3 + (a_1 b_2 + b_1 a_2 + b_1 b_2) a_3 + (a_1 b_2 + b_1 a_2 + b_1 b_2) b_3)
\]

\[
= a_1 (a_2 a_3) + \wp (a_2 b_3) + a_1 (b_2 a_3) + b_1 (a_2 a_3) + b_1 (b_2 b_3) + (a_1 b_2 + b_1 a_2 + b_1 b_2) b_3
\]

\[
= a_1 (a_2 a_3) + \wp (a_1 a_2 b_3 + b_2 a_3 + b_2 b_3) + b_1 (a_2 a_3) + b_1 (b_2 a_3 + b_2 b_3 + b_2 b_3)
\]

\[
= (a_1 + \wp b_1) \otimes ((a_2 a_3) + \wp (a_2 b_3 + b_2 a_3 + b_2 b_3))
\]

\[
= z_1 \otimes (z_2 \otimes z_3),
\]

which proves associativity. Commutativity is obtained by

\[
(z_1 \otimes z_2) \otimes z_3 = a_1 a_2 + \wp (a_1 b_2 + b_1 a_2 + b_1 b_2) = a_2 a_1 + \wp (a_2 b_1 + b_2 a_1 + b_2 b_1) = z_2 \otimes z_1.
\]

This shows that \((\mathbb{PH}, \otimes)\) is a (multiplicative) Abelian semigroup.

**Theorem 1.3.** \((\mathbb{PH}, \oplus, \otimes)\) is a commutative ring.

*Proof.* Since \( \oplus \) is defined coordinate-wise, and \((\mathbb{R}, +)\) is an (additive) commutative group, it is clear that \((\mathbb{PH}, \oplus)\) is also a commutative group. The unique additive inverse \(-z\) of \( z = a + \wp b \) is

\[
-z := (-a) + \wp (-b).
\]

The pair \((\mathbb{PH}, \otimes)\) is a (multiplicative) Abelian semigroup, by Proposition 1.2 so we need to prove the distributivity of \( \otimes \) over \( \oplus \):

\[
z_1 \otimes (z_2 \oplus z_3) = (a_1 + \wp b_1) \otimes (a_2 + \wp b_2 + a_3 + \wp b_3)
\]

\[
= (a_1 + \wp b_1) \otimes ((a_2 + a_3) + \wp (b_2 + b_3))
\]

\[
= a_1 (a_2 + a_3) + \wp (a_1 b_2 + b_2 a_3) + b_1 (a_2 + a_3) + b_1 (b_2 + b_3)
\]

\[
= a_1 a_2 + a_1 a_3 + \wp (a_1 b_2 + b_1 a_2 + b_1 b_2) + \wp (a_1 b_3 + b_1 a_3 + b_1 b_3)
\]

\[
= (z_1 \otimes z_2) \oplus (z_1 \otimes z_3).
\]

All together we have proved that \((\mathbb{PH}, \oplus, \otimes)\) has the structure of a commutative ring.

Note that \((\mathbb{PH}^x, \otimes)\) is not a group, and thus \((\mathbb{PH}, \oplus, \otimes)\) is not a field, since there are non-zero numbers \( z \in \mathbb{PH}^x \) without an inverse; for example \( z = 0 + \wp b \).

Recalling that a nonzero ring element \( z_1 \) is a zero divisor if there exists a nonzero element \( z_2 \) such that \( z_1 \otimes z_2 = 0 \), one observes that the phantom ring \((\mathbb{PH}, \oplus, \otimes)\) is not an integral domain, i.e. it has zero divisors; for example

\[
(0 + \wp b) \otimes (-b + \wp b) = 0 + \wp (0 b + b (-b) + b b) = 0
\]

and thus \( 0 + \wp b \) and \(-b + \wp b\) are zero divisors.

**Proposition 1.4.** All the zero divisors of \((\mathbb{PH}, \oplus, \otimes)\) are of the form

\[
z = 0 + \wp a \quad \text{or} \quad z = (-a) + \wp a,
\]

for some \( a \in \mathbb{R} \).

*Proof.* Assume \( z_1 = a_1 + \wp b_1 \), and \( z_2 = a_2 + \wp b_2 \) are nonzero elements such that \( z_1 \otimes z_2 = 0 \), that is

\[
(a_1 + \wp b_1) \otimes (a_1 + \wp b_1) = a_1 a_2 + \wp (a_1 b_2 + a_2 b_1 + b_1 b_2) = 0.
\]

Suppose \( a_1 \neq 0 \), then by the real term of the product \( a_2 = 0 \). So, by the phantom term, we should have \( a_1 b_2 + b_1 b_2 = (a_1 + b_1) b_2 = 0 \). But \( b_2 \neq 0 \), since \( z_2 \neq 0 \), and thus \( b_1 = -a_1 \). This means that \( z_1 = a_1 - \wp a_1 \) and \( z_2 = 0 + \wp b_2 m \) as required.
A nonzero element \( z \in \mathbb{P}^\times \) which is not of the form \( (a, 0) \) is called a **nonzero divisor**: the collection of all zero divisors in \( (\mathbb{P}^\times, \oplus, \otimes) \) is denoted

\[
Z_{\text{div}}(\mathbb{P}^\times) = \{ z \in \mathbb{P}^\times \mid z \text{ is zero divisor} \}.
\]

We sometimes write \( Z_{\text{div}}^0 \) for the union \( Z_{\text{div}} \cup \{0\} \).

**Definition 1.5.** The **phantom conjugate** \( \tilde{z} \) of \( z = a + \varphi b \) is defined to be

\[
\tilde{z} := (a + b) - \varphi b.
\]

The real number

\[
\hat{z} := a + b
\]

is called the **(real) reduction** of \( z \).

Having the notion of (real) reduction, we can write the product of two phantom numbers as:

\[
z_1 \otimes z_2 = a_1 a_2 + \varphi (\tilde{z}_1 \tilde{z}_2 - a_1 a_2).
\]

**Remark 1.6.** By Proposition 1.4, one sees that \( \hat{z} = 0 \) if and only if both \( z = 0 \) or \( \tilde{z} = 0 \); when both of them are zero then \( z = 0 \). Moreover, in this view, given a suitable topology on \( \mathbb{P}^\times \), the complement of \( Z_{\text{div}}^0 \) in \( \mathbb{P}^\times \) is dense, so we can omit the zero divisor without detracting form the abstract theory.

One can easily verify the following properties for phantom conjugates and (real) reductions:

**Properties 1.7.** For any \( z = a + \varphi b \) the following properties are satisfied:

1. \( z = a + \varphi (\tilde{z} - a) = (\hat{z} - b) + \varphi b \),
2. \( \tilde{z}_1 \oplus \tilde{z}_2 = \tilde{z}_1 \oplus \tilde{z}_2 \),
3. \( (\hat{z}) = -\hat{z} \),
4. \( \tilde{z}_1 \otimes \tilde{z}_2 = \tilde{z}_1 \otimes \tilde{z}_2 \),
5. \( \text{ph}(z \oplus \tilde{z}) = \text{ph}(z \otimes \tilde{z}) = 0 \),
6. \( \tilde{z} = \hat{z} - \varphi b \),
7. \( \tilde{z}_1 \oplus \tilde{z}_2 = \tilde{z}_1 + \tilde{z}_2 \),
8. \( (\hat{z}) = -\hat{z} \),
9. \( \tilde{z}_1 \otimes \tilde{z}_2 = \tilde{z}_1 \cdot \tilde{z}_2 \).

**Remark 1.8.** The elements of \( \mathbb{P}^\times \) can be understood as intervals in \( \mathbb{R} \). This means that an element \( z = a + \varphi b \) stands for the interval that starts at \( a \) and ends at \( a + b \), i.e. the reduction of \( z \). Thus, \( \mathbb{P}^\times \) can be realized as a ring of intervals, given by:

\[
\begin{align*}
[a_1, a_1 + b_1] \oplus [a_2, a_2 + b_2] &= [a_1 + a_2, a_1 + a_2 + b_1 + b_2]; \\
[a_1, a_1 + b_1] \otimes [a_2, a_2 + b_2] &= [a_1 a_2, a_1 b_2 + a_2 b_1 + a_2 b_2].
\end{align*}
\]

In order to get a canonical interval representation, \( z = a + \varphi b \) is assigned to the half-open interval \( [a, \tilde{z}) \).

In this view, zero divisors are intervals with 0 as one of their endpoints. This view also provides the motivation for the definition of the conjugate: \( z \) and \( \tilde{z} \) represent the same interval but with switched endpoints.

In fact, \( (\mathbb{P}^\times, \oplus, \otimes) \) has a much richer structure than a standard ring; the division is well defined for all nonzero divisors in \( \mathbb{P}^\times \), and each has an inverse. Given a nonzero divisor \( z \in \mathbb{P}^\times \), we define the multiplicative inverse of \( z \) to be

\[
z^{-1} := \frac{1}{a} + \varphi \frac{(-b)}{a (a + b)} = \frac{1}{a} + \varphi \frac{(-b)}{a \hat{z}} = \frac{1}{a} + \varphi \left( \frac{1}{\hat{z}} - \frac{1}{a} \right);
\]

indeed \( z^{-1} \) is an inverse of \( z \),

\[
z \otimes z^{-1} = (a + \varphi b) \otimes \left( \frac{1}{a} + \varphi \frac{(-b)}{a (a + b)} \right) = a \frac{1}{a} + \varphi \left( a \frac{(-b)}{a (a + b)} + \frac{b}{a} + b \frac{(-b)}{a (a + b)} \right) = 1.
\]
One can easily verify that $z^{-1}$ is unique, and that the reduction of an inverse number has the form:

$$z^{-1} = \frac{1}{a+b} = \frac{1}{z}.
$$

Since the multiplicative inverse, defined only for all $z \in \text{PH} \setminus Z^0_{\text{div}}(\text{PH})$, and the additive inverse are unique, we define the division and the substraction, respectively, for $(\text{PH}, \oplus, \otimes)$ as

$$z_1 \otimes z_2 := z_1 \otimes z_2^{-1} \quad \text{and} \quad z_1 \ominus z_2 := z_1 \ominus (-z_2),$$

where $\otimes$ is defined only for a nonzero divisor $z_2 \not\in Z^0_{\text{div}}$. Accordingly, we write

$$(1.5) \quad z_1 \otimes z_2 = \frac{a_1}{a_2} + \varphi \frac{b_1 a_2 - a_1 b_2}{a_2 (a_2 + b_2)},$$

which leads to the following useful form:

$$(1.6) \quad z_1 \otimes z_2 = \frac{a_1}{a_2} + \varphi \frac{\zeta_1 a_2 - \zeta_1 \zeta_2}{a_2 \zeta_2} = \frac{a_1}{a_2} + \varphi \left( \frac{\zeta_1}{\zeta_2} - \frac{a_1}{a_2} \right).$$

**Definition 1.9.** A phantom number $z = a + \varphi b$ is said to be **positive** if $a > 0$ and $b > 0$. When $a > 0$ and $\hat{z} > 0$ we say that $z$ is **pseudo positive**. If $a < 0$ and $b < 0$, then $z$ is said to be **negative** and when $a < 0$ and $\hat{z} < 0$ we say that $z$ is **pseudo negative**. When $z$ is pseudo positive or 0 it is termed **pseudo nonnegative**, and if is pseudo negative or 0 is called **pseudo nonpositive**.

Clearly, any positive (negative) phantom number is also pseudo positive (negative). In particular, if $z$ is pseudo positive, or pseudo negative, then $z \not\in Z^0_{\text{div}}$, cf. Remark [1.6] and is multiplicatively invertible.

**Lemma 1.10.** Given two pseudo nonnegatives $z_1, z_2 \in \text{PH}$ then:

(i) Their sum is pseudo nonnegative,

(ii) Their product is pseudo nonnegative,

(iii) $z^2$ is pseudo nonnegative for each $z \in \text{PH},$

(iv) When $z_2$ is pseudo positive, the fraction $z_1 \otimes z_2$ is pseudo nonnegative.

Proof.

(i) Write $z_1 + z_2 = (a_1 + a_2) + \varphi (b_1 + b_2)$, since $a_1 + a_2$ is positive, and $b_1 \geq -a_1$ and $b_2 \geq -a_2$, the proof is clear.

(ii) By Equation (1.2), $z_1 z_2 = a_1 a_2 + \varphi (\hat{z}_1 \hat{z}_2 - a_1 a_2)$. Then, by the hypothesis, $a_1 a_2 \geq 0$ and $\hat{z}_1 \hat{z}_2 = a_1 a_2 + \hat{z}_1 \hat{z}_2 - a_1 a_2 = \hat{z}_1 \hat{z}_2 \geq 0$.

(iii) Use (ii) with $z = z_1 = z_2$, or write directly $z^2 = a^2 + \varphi (z^2 - a^2)$, so $a^2 \geq 0$ and then $(z^2) = a^2 + (a + b)^2 - a^2 = (a + b)^2 \geq 0$.

(iv) Writing $z_1 \otimes z_2$ as in Equation (1.4), $\frac{\zeta_1}{\zeta_2}$ and $\frac{a_1}{a_2}$ are (real) positives, and thus

$$\frac{\zeta_1}{\zeta_2} - \frac{a_1}{a_2} = \frac{a_1 + b_1 a_2 - a_1 (a_2 + b_2)}{\zeta_2 a_2} = \frac{b_1 a_2 - a_1 b_2}{\zeta_2 a_2} \geq -a_1 a_2 - a_1 b_2 = -a_1 = a_1.$$

Next, we outline the view of our structure in the category of rings. Categorically, we have the trivial embedding

$$\varphi : (\mathbb{R}, +, \cdot) \rightarrow (\text{PH}, \oplus, \otimes),$$

given by sending $\varphi : a \mapsto a + \varphi 0$. On the other hand, we also have the onto projection

$$\pi : (\text{PH}, \oplus, \otimes) \rightarrow (\mathbb{R}, +, \cdot),$$

given by sending $\pi : a + \varphi b \mapsto a a + \beta b$ for some real numbers $\alpha$ and $\beta$. If $\beta = 0$, the projection is phantom forgetful, i.e. $\pi : z \mapsto \alpha (\text{re}(z))$, while $\pi$ is real forgetful when $\alpha = 0$.

**Remark 1.11.** Viewing $(\text{PH}, \oplus, \otimes)$ as an $\mathbb{R}$-module, we define the scalar multiplication $\mathbb{R} \times \text{PH} \rightarrow \text{PH}$ as

$$r(a + \varphi b) := \varphi(r) \otimes (a + \varphi b),$$

for any $r \in \mathbb{R}$, which is written as $r(a + \varphi b) = (ra) + \varphi (rb)$, for simplicity. Similarly, we write $\frac{a}{\varphi} + \varphi \frac{b}{\varphi}$ for $\varphi(\frac{1}{a}) \otimes (a + \varphi b)$. 


It is easy to check that the set of real numbers forms a subfield in the ring of phantom numbers \( (\mathbb{PH}, \oplus, \otimes) \), and the phantom numbers whose real term is zero establish an ideal in \( (\mathbb{PH}, \oplus, \otimes) \).

1.2. Generalization. In the previous subsection we described the extension of order 1 of the field of real numbers. For completeness, we present the general definition of a phantom ring of arbitrary order.

Given a field \( K \) of characteristic \( \neq 2 \), usually the field \( \mathbb{R} \) of real numbers, the **phantom ring** \( \mathbb{PH}(n)(K) \) of order \( n \), or **n-phantom ring**, for short, is built over the product \( K \times \cdots \times K \) of \( n + 1 \) copies of \( K \) indexed \( 0, 1, \ldots, n \). Accordingly, the elements of \( \mathbb{PH}(n)(K) \) are just \( (n + 1) \)-tuples \( (x_0, x_1, \ldots, x_n) \) and \( (y_0, y_1, \ldots, y_n) \) denoted, respectively, as \( x \) and \( y \). \( \mathbb{PH}(n)(K) \) is then equipped with the following binary operations, addition and multiplication, respectively:

\[
\begin{align*}
\bar{x} &:= (x_0 + y_0, x_1 + y_1, \ldots, x_n + y_n), \\
\bar{y} &:= (x_0 y_0, x_1 y_1, \ldots, x_n y_n) \\
\bar{z} &:= x_0 + y_0 x_1 + y_1 x_2 + \cdots + y_n x_{n-1} + x_n y_0.
\end{align*}
\]

(1.7)

where \( \bar{x} = \sum_{j=0}^{i} x_j \) and \( \bar{y} = \sum_{j=0}^{i} y_j \).

(Not that the notation here is different from that used in the previous subsection, in particular the \( x_i \) and the \( y_i \).)

In the sequel, for simplicity, we apply our development only to \( \mathbb{PH}(1)(\mathbb{R}) \), which as we have said is denoted \( \mathbb{PH} \), though we note that extends smoothly to any \( \mathbb{PH}(n)(K) \), with \( n > 1 \), defined over a suitable field \( K \). Generalizing the future definitions suitably to \( n \), the \( n \)-phantom ring \( \mathbb{PH}(n)(K) \) carries also the same properties as \( \mathbb{PH}(1)(\mathbb{R}) \), to be described in the next sections.

**Notations:** For the rest of this paper, assuming that the reader is familiar with the arithmetical nuances, we write \( z_1 + z_2 \) for \( z_1 \oplus z_2 \), \( z_1 - z_2 \) for \( z_1 \ominus z_2 \), \( z_1 z_2 \) for the product \( z_1 \otimes z_2 \), \( \frac{z_1}{z_2} \) for the division \( z_1 \oslash z_2 \), and \( z^n \) for \( z \otimes \cdots \otimes z \) repeated \( n \) times. The phantom ring \( (\mathbb{PH}, \oplus, \otimes) \) is denoted \( \mathbb{PH} \), for short.

1.3. Relations and orders. In the sequel, mainly for the development of phantom probability theory, we need some relations that help to utilize the structure of \( \mathbb{PH} \).

To make our paper reasonably self-contained, let us recall the property of a binary relation on a set for being an order:

**Definition 1.12.** A binary relation \( \preceq_{\mathbb{PH}} \) is a **weak order** on a set \( S \) if the following properties hold:

(i) Reflexivity: \( s \preceq_{\mathbb{PH}} s \) for all \( s \in S \);

(ii) Transitivity: \( s_1 \preceq_{\mathbb{PH}} s_2 \) and \( s_2 \preceq_{\mathbb{PH}} s_3 \) implies \( s_1 \preceq_{\mathbb{PH}} s_3 \);

(iii) Comparability (trichotomy law): for any \( s_1, s_2 \in S \), either \( s_1 \preceq_{\mathbb{PH}} s_2 \) or \( s_2 \preceq_{\mathbb{PH}} s_1 \).

(When \( s_1 \preceq_{\mathbb{PH}} s_2 \) and \( s_2 \preceq_{\mathbb{PH}} s_1 \) we write \( s_1 \sim_{\mathbb{PH}} s_2 \).) A **weakly ordered set** is a pair \( (S, \preceq_{\mathbb{PH}}) \) where \( S \) is a set and \( \preceq_{\mathbb{PH}} \) is a weak order on \( S \). When \( S \) consists of phantom numbers we say that \( (S, \preceq_{\mathbb{PH}}) \) is a **phantom weakly ordered set**.

Adding the extra axiom:

(iii) antisymmetry: \( s_1 \preceq_{\mathbb{PH}} s_2 \) and \( s_2 \preceq_{\mathbb{PH}} s_1 \) implies \( s_1 = s_2 \);

the order \( \preceq_{\mathbb{PH}} \) is then a **total order**, or **order**, for short, and is denoted as \( \leq \).

Clearly, \( \sim_{\mathbb{PH}} \) induces an equivalent relation on \( \mathbb{PH} \), the classes of which are \( \mathbb{PH}/\sim_{\mathbb{PH}} \), and when \( \sim_{\mathbb{PH}} \) is a total order \( \sim_{\mathbb{PH}} \) is replaced by full equality =. We use the notation \( \sqsubseteq_{\mathbb{PH}} \) to distinguish this order, mainly when writing \( \preceq_{\mathbb{PH}} \), from the other orders used in the sequel. Therefore, the symbol \( < \) and \( \leq \) always denote the usual order of the real numbers.

Although, in general, many relations may serve as weak order on \( \mathbb{PH} \), in this paper we require the weak order \( \preceq_{\mathbb{PH}} \) to have the following properties:

**Properties 1.13.**
(i) **Compatibility with the standard order of the reals**, that is
\[ z_1 \preceq_{\text{wk}} z_2 \iff a_1 \leq a_2, \]
for any \( z_1 = a_1 + \wp 0 \) and \( z_2 = a_2 + \wp 0 \).

(ii) **Compatibility with the arithmetic operations of \( \mathbb{PH} \):**
(a) if \( z_1 \preceq_{\text{wk}} z_2 \) then \( z_1 + z_3 \preceq_{\text{wk}} z_2 + z_3 \), for any \( z_3 \in \mathbb{PH} \);
(b) if \( z_1 \preceq_{\text{wk}} z_2 \) then \( z_1 z_3 \preceq_{\text{wk}} z_2 z_3 \), for any pseudo positive \( z_3 \in \mathbb{PH} \);
(c) if \( z_1, z_3 \preceq_{\text{wk}} z_2 \) then \( z_1 \preceq_{\text{wk}} z_2 \), for any pseudo positive \( z_3 \in \mathbb{PH} \).

Viewing \( \mathbb{PH} \) as an Euclidean space, we usually assume that all the elements that are \( \sim_{\text{wk}} \) form a connectable set.

**Example 1.14.** Viewing \( \mathbb{PH} \) as \( \mathbb{R} \times \mathbb{R} \), our main example for a total order on \( \mathbb{PH} \) is the **lexicographic order** \( \preceq_{\text{lex}} \) defined as
\[ a_1 + \wp b_1 \preceq_{\text{lex}} a_2 + \wp b_2 \iff \begin{cases} a_1 < a_2, & a_1 \neq a_2; \\ b_1 \leq b_2, & a_1 = a_2; \end{cases} \]
which is a total order satisfying the above conditions.

In the continuation, when writing \( \preceq_{\text{wk}} \), we assume the weak order \( \preceq_{\text{wk}} \) is provided with the set structure. The reader should keep in mind that one interpretation for an order which is also total is the lexicographic order \( \preceq_{\text{lex}} \).

**Remark 1.15.** Note that our definition of pseudo positivity, cf. Definition 1.8, is independent of the given order \( \preceq_{\text{wk}} \) on \( \mathbb{PH} \).

In the sequel, mainly for probability theory, we also use the notation \( \preceq_{\text{re}} \) for the (real) relation
\[ z_1 \preceq_{\text{re}} z_2 \iff \text{re}(z_1) < \text{re}(z_2), \]
the other real relations \( =_{\text{re}}, \leq_{\text{re}}, >_{\text{re}}, \) and \( \geq_{\text{re}} \) are defined similarly.

We define the real-valued function \([ ]_\alpha : \mathbb{PH} \to \mathbb{R} \), with a real positive parameter \( \alpha \in \mathbb{R} \), given by
\[ [ ]_\alpha \colon a + \wp b \mapsto a + \frac{b}{\alpha}, \]
and write \([z]_\alpha \) for the image of \( z \in \mathbb{PH} \) in \( \mathbb{R} \). Then, \([ ]_\alpha \) determines the equivalence relation on \( \mathbb{PH} \) given by
\[ z_1 \sim_\alpha z_2 \iff [z_1]_\alpha = [z_2]_\alpha, \]
and written \( z_1 =_\alpha z_2 \). (Note that in the special case when \( \alpha = 2 \), by this definition, we always have \( z =_\alpha \bar{z} \).)

The quotient ring of \( \mathbb{PH} \), taken with respect to \([ ]_\alpha \), is denoted as \( \mathbb{PH}/\alpha \); clearly \( \mathbb{PH}/\alpha \cong \mathbb{R} \).

In the same way, \([ ]_\alpha \) induces a weak order on \( \mathbb{PH} \), provided as
\[ z_1 \preceq_\alpha z_2 \iff [z_1]_\alpha \leq [z_2]_\alpha; \]
and satisfying Properties 1.14, the relations \( \sim_\alpha, >_\alpha, \) and \( \geq_\alpha \) are determined similarly.

These relations are very important for advanced topics in phantom probability theory and their applications, mainly discussed in the sequel papers [8, 9].

### 1.4. Powers and exponents

Writing \( z^n \), with \( n \in \mathbb{N} \), for the product \( z \cdots z \) with \( z \) repeated \( n \) times, for any \( z = a + \wp b \) we have
\[ z^n = a^n + \wp \sum_{i=1}^{n} \binom{n}{i} a^{n-i} b^i; \]
as usual, \( z^0 \) is identified with the unit \( 1 \). This form leads to the following friendly formula:
\[ z^n = a^n + \wp ((a + b)^n - a^n) = a^n + \wp (z^n - a^n). \]
Following accepted standards, we write \( z^{-n} \) for \( \frac{1}{z^n} \), and therefore get the extension to integral powers of phantom numbers.

Equation (1.12) plays a main role throughout our development and, together with Equation (1.12), leads to the next important formula, which is used frequently in the sequel:
\[ z_1^n z_2^m = a_1^n a_2^m + \wp (z_1^n z_2^m - a_1^n a_2^m). \]
Properties 1.16. Given a phantom number \( z \in \Phi \), then:

1. \( z^i z^j = z^{i+j} \),
2. \( \frac{z^i}{z^j} = z^{i-j} \),
3. \( (z^i)^j = (z^j)^i = z^{ij} \),

for any \( i, j \in \mathbb{Z} \).

Of course, one can take an arbitrary finite number of multiplicands, \( z_1, z_2, \ldots, z_n \), and get recursively

\[
\sqrt[n]{z_1 z_2 \cdots z_n} = \sqrt[n]{a_1} \sqrt[n]{a_2} \cdots \sqrt[n]{a_n} + \varphi \left( \hat{z}_1 \hat{z}_2 \cdots \hat{z}_n - a_1 a_2 \cdots a_n \right),
\]

for any \( i_1, i_2, \ldots, i_n \in \mathbb{Z} \).

Definition 1.17. When a phantom equation \( Q \) can be written in terms of two real equations, \( Q_{re} \) and \( \hat{Q} \), as

\[
Q = Q_{re} + \varphi (\hat{Q} - Q_{re}),
\]

we say that \( Q \) has a realization form, or equivalently, that it admits the realization property.

For that matter an equation might be an arithmetic expression or a function, where \( Q \), \( Q_{re} \) and \( \hat{Q} \) stand respectively for the real and the reduction of each argument involved in \( Q \).

Equations (1.12), (1.13), and (1.14) above admit the realization property. In the sequel, we will see that many other familiar equations admit this nice property. Having this property, as spelled out later for probability theory, phantom results are induced by known results for reals, which makes the development much easier.

Since the realization property is satisfied for each \( z \in \Phi \) and any natural power \( n \in \mathbb{N} \), cf. Equation (1.12), it easy is to determine the \( n \)th root, if it exists, of a phantom number \( z = a + \varphi b \) as:

\[
\sqrt[n]{z} = \sqrt[n]{a} + \varphi (\sqrt[n]{z} - \sqrt[n]{a}),
\]

where \( \sqrt[n]{a} \) and \( \sqrt[n]{z} \) are, respectively, the real \( n \)th roots of \( a \) and \( z \), and \( n \in \mathbb{N} \) is a real positive number. Clearly, when \( n \) is even, both \( a \) and \( z \) must be nonnegative.

In the usual way, we sometimes write \( z^{\frac{1}{n}} \) for \( \sqrt[n]{z} \), and have the properties:

Properties 1.18. Given pseudo nonnegative phantom numbers \( z, z_1, \) and \( z_2 \) then:

1. \( \sqrt[n]{z_1} \sqrt[n]{z_2} = \sqrt[n]{z_1 z_2} \),
2. \( \sqrt[n]{z_1} \sqrt[n]{z_2} = \sqrt[n]{z_1 z_2} \), for pseudo positive \( z_2 \),
3. \( \sqrt[n]{z^m} = (\sqrt[n]{z})^m = z^{\frac{m}{n}} \),

for any positive \( m, n \in \mathbb{N} \).

In the specific case when \( n = 2 \), clearly, each pseudo nonnegative phantom number \( z = a + \varphi b \in \Phi \) has a square root

\[
\sqrt{a + \varphi b} = \sqrt{a} + \varphi (\sqrt{a + b} - \sqrt{a})
\]

\[
= \sqrt{a} + \varphi (\sqrt{\varphi} - \sqrt{a}).
\]

Actually, \( \sqrt{a} \) in the equation stands for \( \pm \sqrt{a} \); therefore there always exists a nonnegative square root of \( a + \varphi b \), i.e. a root whose real and phantom terms are both nonnegative.

In the standard way, we define the exponent of an element \( z \in \Phi \) to be the infinite phantom sum

\[
e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots
\]

Proposition 1.19. Given \( z, z_1 \) and \( z_2 \) in \( \Phi \) then:

1. \( e^0 = 1 \),
2. \( e^z = e^a + \varphi (e^{a+b} - e^a) \),
3. \( e^{z_1} e^{z_2} = e^{z_1 + z_2} \).
(4) \( e^{z_1} / e^{z_1} = e^{z_1-z_2} \).

**Proof.** (1) is by definition. (2) Expand \( e^z \) and use Equation (1.12), i.e.

\[
e^{a+b} = 1 + (a + \varphi b)^2 + \frac{(a+\varphi b)^3}{3!} + \ldots
\]

\( = 1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + \ldots \)

\[= \varphi \left( 1 + (a + b)^2 + \frac{(a+b)^3}{3!} + \ldots \right) - \left( 1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + \ldots \right) \]

\[= e^a + \varphi \left( e^{a+b} - e^a \right). \]

(3) Using the identity in (2), write

\[
e^{z_1} e^{z_2} = \left( e^{a_1} + \varphi \left( e^{a_1+b_1} - e^{a_1} \right) \right) \left( e^{a_2} + \varphi \left( e^{a_2+b_2} - e^{a_2} \right) \right)
\]

\[= e^{a_1} e^{a_2} + \varphi \left( e^{a_1} \left( e^{a_2+b_2} - e^{a_2} \right) + e^{a_2} \left( e^{a_1+b_1} - e^{a_1} \right) + \left( e^{a_1+b_1} - e^{a_1} \right) \left( e^{a_2+b_2} - e^{a_2} \right) \right)
\]

\[= e^{(a_1+a_2) + \varphi (b_1+b_2)}
\]

\[= e^{z_1+z_2}. \]

(4) Straightforward from (3) by taking \( e^{z_1+z_2} = e^{z_1+(-z_2)} \). \( \square \)

Proposition (1.19) (2) yields the following convenient form, i.e. the realization form, for the phantom exponent:

\[e^z = e^a + \varphi \left( e^{a+b} - e^a \right) = e^a + \varphi \left( e^z - e^a \right), \]

often used in phantom probability theory.

Analogously to classical theory, for any pseudo positive \( z \in \mathbb{P} \) we define the **logarithm** as

\[\log(z) = (z - 1) - \frac{(z - 1)^2}{2} + \frac{(z - 1)^3}{3} - \frac{(z - 1)^4}{4} + \ldots, \]

where \( 1 = 1 + \varphi 1 \), and prove that

\[\log(z) = \log(a) + \varphi \left( \log(a + b) - \log(a) \right)
\]

\[= \log(a) + \varphi \left( \log(z) - \log(a) \right), \]

that is, the realization property for phantom logarithm.

1.5. **Phantom spaces.** Modules over the phantom ring, called **phantom modules**, are just like standard modules over rings \( \mathbb{P} \). For the reader’s convenience we state this explicitly:

**Definition 1.20.** A phantom \( \mathbb{P} \)-**module** \( V \) is an additive group \( (V, \oplus, 0_V) \) together with a scalar multiplication \( \mathbb{P} \times V \to V \) satisfying the following properties for all \( z \in \mathbb{P} \) and \( v, w \in V \):

(i) \( z(v + w) = zv + zw; \)

(ii) \( (z_1 \oplus z_2)v = z_1v + z_2v; \)

(iii) \( (z_1 z_2)v = z_1(z_2v); \)

(iv) \( 1 v = v; \)

(v) \( 0 v = 0_V = z 0_V. \)

The direct sum \( \bigoplus_{J \in \mathcal{J}} \mathbb{P} \) of copies (indexed by \( \mathcal{J} \)) of the phantom ring \( \mathbb{P} \) is denoted as \( \mathbb{P}^{(\mathcal{J})} \), with zero element \( 0 = (0, \ldots, 0) \), and is called the **phantom space.** When \( \mathcal{J} = \{1, \ldots, n\} \), then the phantom space \( \mathbb{P}^{(\mathcal{J})} \) is denoted as \( \mathbb{P}^{(n)} \) and we say that \( \mathbb{P}^{(n)} \) is an \( n \)-phantom space. As element of \( \mathbb{P}^{(n)} \) is just an \( n \)-tuple \( (z_1, \ldots, z_n) \) and is denoted as \( z \).

Denoting the nonnegative real numbers as \( \mathbb{R}_+ \), we recall the standard definition of a norm, formulated for the \( n \)-phantom space:

**Definition 1.21.** A **norm** on \( \mathbb{P}^{(n)} \) is a real-valued function \( \| \cdot \| : \mathbb{P}^{(n)} \to \mathbb{R}_+ \) that satisfies:

(i) \( 0 \leq \| z \| \in \mathbb{R} \) and \( \| z \| = 0 \) iff \( z = 0, \)

(ii) \( \| rz \| = |r| \| z \| \) for each \( r \in \mathbb{R}, \)

(iii) \( \| z' \oplus z'' \| \leq \| z' \| + \| z'' \|, \)

for any \( z, z', z'' \in \mathbb{P}^{(n)} \).
In what follows we use the absolute value, also called a modulus, $|\cdot| : \mathbb{PH} \rightarrow \mathbb{R}_+$ given by

\[(1.18) \quad |a + \wp b| = \sqrt{\left(a + \frac{b}{2}\right)^2 + \left(b - \frac{b}{2}\right)^2}.
\]

(When $z$ is only a real term, i.e. $z = a + \wp 0$. This definition coincides with the familiar absolute value of the reals.)

**Proposition 1.22.** The absolute value $|\cdot|$ as defined in Equation (1.18) is a norm on $\mathbb{PH}$.

**Proof.** (i) and (ii) are immediate by definitions. To prove (iii), we show that $|z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2$. Expanding both sides of this form, and letting $\alpha_i = a_i + \frac{b_i}{2}$, $\beta_i = \frac{b_i}{2}$ for $i = 1, 2$, we have

\[ (\alpha_1 + \alpha_2)^2 + (\beta_1 + \beta_2)^2 \leq \alpha_1^2 + \beta_1^2 + 2|z_1| |z_2| + \alpha_2^2 + \beta_2^2. \]

Discard similar components on both sides and write $|z_1|$ and $|z_2|$ explicitly to get

\[ 2\alpha_1 \alpha_2 + 2\beta_1 \beta_2 \leq 2\alpha_1 \beta_1 \sqrt{\alpha_2^2 + \beta_2^2}. \]

Canceling the common multipliers and taking squares, we have $(\alpha_1 \alpha_2 + \beta_1 \beta_2)^2 \leq (\alpha_1^2 + \beta_1^2) (\alpha_2^2 + \beta_2^2)$, and thus

\[ 2\alpha_1 \alpha_2 \beta_1 \beta_2 \leq \alpha_1^2 \beta_2^2 + \beta_1^2 \alpha_2^2, \]

which implies $0 \leq \alpha_1^2 \beta_2^2 - 2\alpha_1 \alpha_2 \beta_1 \beta_2 + \beta_1^2 \alpha_2^2 = (\alpha_1 \beta_2 - \beta_1 \alpha_2)^2$. This proves property (iii) of Definition 1.21.

\[ \square \]

Using the reduced form of phantom numbers, and Properties 1.13 (1), $|z|^2$ can be written also as

\[ |z|^2 = \left(a + \frac{\bar{z} - a}{2}\right)^2 + \left(\frac{\bar{z} - a}{2}\right)^2 = \frac{a^2 + \bar{z}^2}{2}. \]

Having a weak order satisfying Properties 1.13 (i), one also has $|z| \gtrsim wk 0$ for any $z \in \mathbb{PH}$.

**Remark 1.23.** There are several main reasons for defining the absolute value on $\mathbb{PH}$ as it has been defined in Equation (1.18):

(i) $|z| = |\bar{z}|$, for each $z \in \mathbb{PH}$; indeed, to verify this identity, we have

\[ |\bar{z}| = |(a + b) - \wp b| = \sqrt{(a + b - \frac{b}{2})^2 + \left(\frac{b}{2}\right)^2} = \sqrt{(a + \frac{b}{2})^2 + \left(-\frac{b}{2}\right)^2} = |z|. \]
(ii) Considering $\mathbb{P} \mathbb{H}$ as $\mathbb{R} \times \mathbb{R}$, the two point compactification of each copy of $\mathbb{R}$ is viewed as a parallelogram with vertices $(0, 0)$, $(0, 1)$, $(1, 0)$, and $(1, -1)$ together with the following correspondences:

$(-\infty, -\infty) \mapsto (0, 0)$, \quad $(-\infty, \infty) \mapsto (0, 1)$, \quad $(\infty, -\infty) \mapsto (1, -1)$, \quad $(\infty, \infty) \mapsto (1, 0)$.

(See Figure 1 (a).) Accordingly, a $0 + \varphi 0$ is the unique point having absolute value 0, and $1 + \varphi 0$ is the unique point having absolute value 1. The significance of this property become apparent later in the discussion on the phantom probability theory.

(iii) The same view of (ii), applied for $[\cdot]_\alpha$ with $\alpha = 2$, cf. Equation (1.10), shows that the classes $[0]_\alpha$ and $[1]_\alpha$ in $\mathbb{P} \mathbb{H}/_\alpha$ are singletons. (See Figure 1 (b).)

These properties are very important for applications in phantom probability theory.

Having the norm $\| \|$ on $\mathbb{P} \mathbb{H}$, we equipped $\mathbb{P} \mathbb{H}$ with the following relation:

\[(1.19) \quad z_1 \precsim | z_2 | \iff | z_1 | < | z_2 | .\]

Accordingly, 0 is the unique minimal element in $\mathbb{P} \mathbb{H}$. (Clearly, this relation is also a weak order on $\mathbb{P} \mathbb{H}$; however, since $\precsim$ ignores signs, it does not satisfy Properties 1.13.)

Basing on Equation (1.18) we defined the norm on the $n$-phantom space $\mathbb{P} \mathbb{H}^{(n)}$ as:

\[\| z \| = \sqrt{| z_1 |^2 + \cdots + | z_n |^2}\]

where $z = (z_1, \ldots, z_n)$.

**Proposition 1.24.** $\| \|$ is norm.

**Proof.** Straightforward from $\| \|$ being a norm.

We use to $\| \|$ to define the map $d : \mathbb{P} \mathbb{H}^{(n)} \times \mathbb{P} \mathbb{H}^{(n)} \rightarrow \mathbb{R}_+$ given by

\[(1.20) \quad d : z_1 \times z_2 \mapsto \| z_1 - z_2 \| ,\]

where the substraction is taken coordinate-wise. We write $d(z_1, z_2)$ for the image of $z_1 \times z_2$ under $d$.

**Proposition 1.25.** $d$ is a metric on $\mathbb{P} \mathbb{H}^{(n)}$.

**Proof.** By Proposition 1.24 we have $d(z_1, z_2) \geq 0$, for any $z_1$ and $z_2$, and equals 0 iff $z_1 = z_2$. Symmetry is clear. $d(z_1, z_3) \leq d(z_1, z_2) + d(z_2, z_3)$ is derived from the triangular law satisfied by $\| \|$. Note that using the metric (1.20) we always have

\[d(z, \bar{z}) = | a - (a + b) + \varphi (b - (-b)) | = | -b + \varphi 2b | = b ,\]

for every $z = a + \varphi b$ in $\mathbb{P} \mathbb{H}$.

**Remark 1.26.** The fact that $\mathbb{P} \mathbb{H}^{(n)}$ is metric space allows us to define a Borel $\sigma$-algebra over $\mathbb{P} \mathbb{H}^{(n)}$ in the usual way.

1.6. **Polynomials.** Polynomials over the phantom ring, called **phantom polynomials**, are defined just as formal polynomials over rings [12]. As usual polynomials, say in $n$ phantom variables $\lambda_1, \ldots, \lambda_n$, form a ring which is denoted as $\mathbb{P} \mathbb{H}[\lambda_1, \ldots, \lambda_n]$; these polynomials can also be viewed as sums of polynomials in $2n$ real variables.

**Remark 1.27.** Given a polynomial $f = \sum_i \alpha_i \lambda_i^i \cdots \lambda_i^n$ in $\mathbb{P} \mathbb{H}[\lambda_1, \ldots, \lambda_n]$, it can be written as $f = f_{re} + \varphi f_{ph}$, where $f_{re}$ and $f_{ph}$ are real polynomials.

Suppose $\lambda_i = u_i + \varphi v_i$, $i = 1, \ldots, n$, is a sum of two variables $u$ and $v$ that take real values, and let

\[f_{re}(u_1, \ldots, u_n) = \sum_i \text{re}(\alpha_i) u_1^i \cdots u_i^n \quad \text{and} \quad f(\lambda_1, \ldots, \lambda_n) = \sum_i \alpha_i \lambda_1^i \cdots \lambda_i^n\]

be two polynomials over the reals. (Note that, since $\hat{\lambda}_i = u_i + v_i$, $\hat{f}$ is considered as a real polynomial in $2n$ variables.) Then, by Equation (1.13), $f$ is written as

\[f(\lambda_1, \ldots, \lambda_n) = f_{re}(u_1, \ldots, u_n) + \varphi \left( f(\hat{\lambda}_1, \ldots, \hat{\lambda}_n) - f_{re}(u_1, \ldots, u_n) \right) .\]

Therefore, phantom polynomials also admit the realization property, in this case in the sense of functions.
The conjugate polynomial $\bar{f}$ of $f = \sum_i \alpha_i \lambda_i^1 \cdots \lambda_i^n$ is defined as
$$\bar{f} = \sum_i \bar{\alpha}_i \lambda_i^1 \cdots \lambda_i^n.$$

**Proposition 1.28.** $\bar{f}(z_1, \ldots, z_n) = \bar{f}(\bar{z}_1, \ldots, \bar{z}_n)$ for any $f \in \mathbb{P}[\lambda_1, \ldots, \lambda_n]$ and each $(z_1, \ldots, z_n) \in \mathbb{P}H(n)$.

**Proof.** Straightforward by Properties [1.7] \qed

1.7. **Basic analysis.** Finally, we provide the necessary notions for basic analysis over the phantoms; we present only the general tools needed for our exposition. Most of these notions are the phantom analogues to those in complex analysis; in general we adopt the philosophy of analysis over the complexes.

**Definition 1.29.** Let $z_1, z_2, \ldots$ be an infinite sequence of phantom numbers, and let $z$ be another phantom number. We say that the sequence $z_n$ converges to $z$, written $\lim_{n \to \infty} z_n = z$, if for every real $\epsilon > 0$ there exists some $n_0$ such that $|z_n - z| < \epsilon$, for all $n > n_0$.

**Lemma 1.30.** A sequence $z_i = a_i + \wp b_i$, $i = 1, 2, \ldots$ converges to $z = a + \wp b$ if $\lim_{n \to \infty} a_n = a$ and $\lim_{n \to \infty} b_n = b$ as real sequences.

**Proof.** ($\Leftarrow$) Clear by definition, cf. Equation [1.18].

($\Rightarrow$) Write $\lim_{n \to \infty} \sqrt{(a_n - a + \frac{b_n - b}{2})^2 + (\frac{b_n - b}{2})^2} = 0$. Each, $(\frac{b_n - b}{2})^2$ and $(a_n - a + \frac{b_n - b}{2})^2$ is positive and converges to $0$. Thus, by definition, $\lim_{n \to \infty} a_n = a$. Then, by the first component, $a_n \to a$.

A function $f : D \to \mathbb{P}H$, whose domain is a subset $D \subset \mathbb{P}H(n)$, is termed a **phantom function**, while a function $g : \mathbb{R}^n \to \mathbb{R}$ is called a **real function**. We say that a function is a phantom-valued function if its range lies in $\mathbb{P}H$; similarly a function whose range lies in $\mathbb{R}$ is called real-valued.

**Definition 1.31.** Given a phantom function $f : D \to \mathbb{P}H$, we say that $w_0 \in \mathbb{P}H$ is the **limit** of $f$ when $z \to z_0$ in $D$ if for any real $\epsilon > 0$ there exists a real $\delta > 0$ such that for any $z$ with $|z - z_0| < \delta$ we have $|f(z) - w_0| < \epsilon$. In such a case we write $\lim_{z \to z_0} f(z) = w_0$.

A function $f$ is **continuous** at $z_0 \in D$ if $\lim_{z \to z_0} f(z) = f(z_0)$, and is said to be continuous on $D$ if it is continuous at each $z_0 \in D$.

Suppose $f : D \to \mathbb{P}H$ is a phantom function, where $D \subset \mathbb{P}H$ is a set, and $z_0$ is an interior point of $D$. The **derivative** of $f$ at $z_0$ is defined as
$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$
provided this limit exists (depending also on $z - z_0$ being a nonzero divisor). In this case, $f$ is called **differentiable** at $z_0$. If $f$ is differentiable for all points in an open disk centered at $z_0$ then $f$ is called **analytic** at $z_0$. The phantom function $f$ is analytic on the open set $D \subset \mathbb{P}H$ if it is differentiable (and hence analytic) at every point in $D$. (The latter properties of derivation are also satisfied for phantom derivation.)

**Example 1.32.** The derivative of a polynomial $f = \sum_i \alpha_i \lambda_i^1 \cdots \lambda_i^n$ at $z_0$, where $\lambda = u + \wp v$, written as $f(\lambda) = f_{\text{re}}(u) + \wp (\bar{f}(\bar{\lambda}) - f_{\text{re}}(u))$ by Remark [1.27] is provided by using Equation [1.3] as:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f_{\text{re}}'(a_0) + \wp \lim_{z \to z_0} \frac{(\bar{f}(\bar{z}_0)) - f_{\text{re}}'(a_0)(z - z_0)}{(z - z_0)} = f_{\text{re}}'(a_0) + \wp \left( f'(\bar{z}_0) - f_{\text{re}}'(a_0) \right).$$

When a phantom function has the realization property, i.e. it is of the form $f_{\text{re}}(t) + \wp f_{\text{ph}}(t)$, where both $f_{\text{re}}$ and $f_{\text{ph}}$ are real functions with $t \in \mathbb{R}$, the derivative of $f$ is given as
(1.21)$$f' = f_{\text{re}}'(t) + \wp f_{\text{ph}}'(t).$$

Indeed, write
$$f' = \lim_{t \to t_0} \frac{(f_{\text{re}}(t) + \wp f_{\text{ph}}(t)) - (f_{\text{re}}(t_0) + \wp f_{\text{ph}}(t_0))}{t - t_0}$$

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which by Equation (1.6) is
\[ f' = \lim_{t \to t_0} \frac{f_{re}(t) - f_{re}(t_0)}{t - t_0} + \varphi \lim_{t \to t_0} \frac{f_{ph}(t) - f_{ph}(t_0)}{(t - t_0)}. \]

Phantom integration is not really anything different from real integration over paths. For a continuous phantom-valued function \( \phi(t) : [a, b] \to \mathbb{P}, \) where \( \phi = \phi_{re} + \varphi \phi_{ph}, \) we define
\begin{equation}
(1.22)
\int_a^b \phi(t)dt = \int_a^b \phi_{re}(t)dt + \varphi \int_a^b \phi_{ph}(t)dt.
\end{equation}

For a function which takes phantom numbers as arguments, we integrate over a path \( \gamma \) (instead of a real interval) in \( \mathbb{P} \) realized as \( \mathbb{R} \times \mathbb{R} \). If one thinks about the substitution rule for real integrals, the following definition, which is based on Equation (1.22) should come as no surprise.

**Definition 1.33.** Suppose \( \gamma \) is a smooth path parameterized by \( \gamma(t) : [a, b] \to \mathbb{P}, \) \( a \leq t \leq b, \) where \( t, a, b \in \mathbb{R} \), and \( f : \mathbb{P} \to \mathbb{P} \) is a phantom function which is continuous on \( \gamma \). Then we define the integral of \( f \) on \( \gamma \) as
\begin{equation}
(1.23)
\int_{\gamma} f(z)dz = \int_{\gamma} f(\gamma(t))\gamma'(t)dt.
\end{equation}

This is simply the path integral of \( f \) along the path \( \gamma \). This integral can be defined analogously to the Riemann integral as the limit of sums of the form \( \sum (f \circ \gamma)(\tau_k)(t_k - t_{k-1}) \), so is the Riemann-Stieltjes integral of \( f \circ \gamma \) with respect to \( \tau \). Using this definition, the integral can be extended to rectifiable paths, i.e. ones for which \( \gamma \) is only of bounded variation.

**Properties 1.34.** Suppose \( \gamma \) is a smooth path, \( f \) and \( g \) are phantom functions which are continuous on \( \gamma \), and \( w \in \mathbb{P} \) is constant.

1. \( \int_{\gamma} (f + wg)dz = \int_{\gamma} fdz + w \int_{\gamma} gdz. \)

2. If \( \gamma \) is parameterized by \( \gamma(t) \), \( a \leq t \leq b \), define the path \( -\gamma \) through \( -\gamma(t) = (a + b - t) \), \( a \leq t \leq b. \) Then \( \int_{\gamma} fdz = -\int_{\gamma} f(z)dz. \)

3. If \( \gamma_1 \) and \( \gamma_2 \) are paths so that \( \gamma_2 \) starts where \( \gamma_1 \) ends then define the curve \( \gamma_1 \gamma_2 \) by following \( \gamma_1 \) to its end, and then continuing on to \( \gamma_2 \) to its end. Then \( \int_{\gamma_1 \gamma_2} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz. \)

Assume \( f \) is given as \( f_{re} + \varphi f_{ph} \), and is defined along a smooth path \( \gamma \), given in a parametric form
\[ \gamma = \{z = z(t) : \alpha \leq t \leq \beta, \ t \in \mathbb{R}\}, \]
for some real \( \alpha, \beta \), for \( z = z_{re} + \varphi z_{ph} \); we also write \( z_{re} = a(t) \) and \( z_{ph} = b(t) \). Then, using the familiar line integral from calculus, Equation (1.23) is written in the following useful form:
\begin{equation}
(1.24)
\int_{\gamma} f(z)dz = \int_{\alpha}^{\beta} f_{re}(t)a'(t)dt + \varphi \int_{\alpha}^{\beta} f_{ph}(t)(a'(t) + b'(t))dt,
\end{equation}
where \( f_{re}(t) \) and \( f_{ph}(t) \) stand respectively for \( f_{re}(a(t), b(t)) \) and \( f_{ph}(a(t), b(t)). \)

2. Phantom probability spaces

2.1. Phantom probability laws. We first recall the necessary basics of standard measure theory, then we further extend these basics to obtain the phantom setting that generalizes the familiar classical probability framework. We use [3, 5, 6] as general references for classical probability theory.

A measure space is a triple \( (\Omega, \Sigma, \mu) \), where \( \Sigma \) is a \( \sigma \)-algebra of subsets over a set \( \Omega \) and \( \mu : \Sigma \to [0, \infty] \) is a real-valued function, called a measure, that satisfies the properties:

(i) \( \mu(\emptyset) = 0; \)

(ii) \( \mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i) \) for any countable sequence \( A_1, A_2, A_3, \ldots \) of pairwise disjoint sets in \( \Sigma. \)

A measure \( \mu \) is monotonic if \( \mu(A_1) \leq \mu(A_2) \) for each \( A_1 \subseteq A_2. \)

A probability measure is a measure with total measure one (i.e. \( \mu(\Omega) = 1 \)), cf. [2]; a probability space \( (\Omega, \Sigma, P) \) is a measure space with a probability measure \( \mu := P \) that satisfies the additional probability axiom
\[ P(A) \geq 0, \quad \forall A \in \Sigma. \]

When \( (\Omega, \Sigma, P) \) is a probability space, \( P(A) \) is said to be the probability of \( A \), \( \Omega \) is called the sample space and its elements are called outcomes, usually denoted as \( \omega_1, \omega_2, \ldots. \) A collection of possible outcomes is
called an event. In the sequel, mainly in the examples, we use the letter $P$ to denote a standard probability measure, i.e. $P : \Sigma \to [0,1] \subset \mathbb{R}$.

**Terminology:** In what follows, when using the term “standard”, or “real standard”, we refer to the known classical results, based on the above (real) probability measure, appearing in the literature on probability theory, [3, 4].

Roughly speaking, our aim is to generalize the probability measure $P : \Sigma \to \mathbb{R}$ to a phantom-valued function $P : \Sigma \to \mathbb{P}$, whose real component is a standard probability measure while its phantom component satisfies an extra axiom. One way to realize this extra axiom, enforced only on the phantom component, is to understand the phantom as a signed distortion (either positive or negative) assigned to each evaluation of the probability measure. Therefore, given a fixed event, its probability together with an arbitrary distortion should still be positive (in the standard sense) and should not exceed 1.

**Remark 2.1.** In the continuation the sample space $\Omega$ need not be a standard sample space, and is also generalized to a **phantom sample space** – a sample space consisting of phantom elements, called **phantom outcomes**. In what follows, the notation $\Omega$ is also used for a phantom sample space, and we use the standard terminology of outcomes and events, respectively, for elements and subsets of $\Omega$.

Recall that $\mathbb{P}$ is assumed to be equipped with a weak order $\preceq_{wk}$, coinciding with the standard order on $\mathbb{R}$, usually a total order. However, to ensure that our formalism is abstract enough, we formulate our setting in terms of a general phantom weak order $\preceq_{wk}$ on $\mathbb{P}$.

A phantom-valued function

$$P : \Sigma \to \mathbb{P},$$

is called a **phantom probability measure** if it satisfies the following axioms. We denote the real component and the phantom component of $P$ as $P_{re}$ and $P_{ph}$, respectively, each being a real-valued function, and write $P = P_{re} + \phi P_{ph}$:

**Axiom 2.2 (Phantom probability measure).**

(i) **Nonnegativity:** $0 \leq P_{re}(A) \leq 1$ for each $A \in \Sigma$,

(ii) **Normalization:** $P(\Omega) = 1$,

(iii) **Additivity:** $P(A \cup B) = P(A) + P(B)$ for any pair of disjoint events $A$ and $B$ in $\Sigma$,

(iv) **Phantomization:** $-P_{re}(A) \leq P_{ph}(A) \leq 1 - P_{re}(A)$ for each $A \in \Sigma$.

(The order $\leq$ is the standard order of the real numbers.)

As one can see, conditions (i)-(iii) are none other than the well known classical probability axioms, referring to real component of $P$ (condition (iii) is also imposed on $P_{ph}$), while the extra axiom (iv) is enforced on the phantom component. (This axiom can be equivalently written as $0 \leq P_{re}(A) + P_{ph}(A) \leq 1$.) Therefore, the real component $P_{re}$ of any phantom probability measure $P$ is always a standard (real) probability measure. These axioms properly frame our earlier probability principles.

Let $\Lambda \subset \mathbb{P}$ be the set

$$(2.1) \quad \Lambda = \{ z \in \mathbb{P} \mid a \in [0,1], \ -a \leq b \leq 1 - a \},$$

each of whose points has a real term belonging to real interval $\Lambda_{re} = [0,1] \subset \mathbb{R}$ and a phantom term limited to the interval $[-a,1-a]$, conditional on the real term of the points. The set $\Lambda$ is called the **phantom probability zone**, all of whose elements are pseudo positive.

**Remark 2.3.** In order to define our probability theory appropriately, we need to enforce the following requirement on the weak order provided with $\mathbb{P}$:

$$(2.2) \quad 0 \prec_{wk} z \prec_{wk} 1, \quad \text{for each } z \in \Lambda.$$

For example, $\preceq_{\alpha}$ with $\alpha > 1$ (cf. Equation (1.11)), or $\preceq_{\mathbb{L}}$ (cf. Equation (1.19)), are weak orders that satisfy this condition. The total order $\leq_{\text{lex}}$ (cf. Equation (1.3)) also admits this property.

For any phantom probability measure $P$, one sees that always

$$P : \Sigma \to \Lambda,$$
compactification of \(C_8/C_0\). Therefore, the target of a phantom probability measure always lies in \(\bar{\Lambda}\). Given a fixed event \(A\), we use the Gothic letter \(p\) to denote the image \(\mathcal{P}(A) = p + \varphi q\) of \(A\), indicating that the corresponding phantom number belongs to \(\bar{\Lambda}\), and thus stands for a phantom probability value.

Note that a phantom probability measure is notated by a calligraphic letter, while a standard measure is notated by a capital letter.

To avoid nonzero annihilators, in the sequel exposition, we usually restrict the target of the phantom probability measure to the set

\[ \Lambda = \{ p \in \bar{\Lambda} : p \text{ is not a zero divisor} \}, \]

which we call the \textbf{restricted phantom probability zone}. (Note that \(0 \in \Lambda\) and the \(\bar{\Lambda}\) is the topological closure of \(\Lambda\).) In the remainder, unless otherwise specified, we always assume the probability values are in \(\Lambda\), i.e. we exclude all the possible zero divisors in \(\bar{\Lambda}\). (In view of Remark [2.3] (ii), \(\bar{\Lambda}\) is realized as the compactification of \(\mathbb{P}\mathbb{H}\), while the elements of \(\Lambda\) are all pseudo nonnegative, cf. Definition [1.9])

\textbf{Lemma 2.4.} \textit{Suppose} \(p, p' \in \Lambda\), \textit{then:}

(i) \((1 - p) \in \bar{\Lambda}\),

(ii) \(pp' \in \bar{\Lambda}\).

\textit{Proof.} (i) \textbf{Let} \(p = p + \varphi q\), \textbf{then} \(1 - p = (1 - p) + \varphi (-q)\). Clearly, as \(p \in [0, 1]\), the real term \((1 - p) \in [0, 1]\). The phantom term should satisfy

\[-(1 - p) \leq -q \leq 1 - (1 - p) \quad (= p),\]

\(\text{i.e.} \quad -p \leq q \leq (1 - p), \text{ but is given by the assumption that } p \in \Lambda.\)

(ii) \textbf{Let} \(p = p + \varphi q\) and \(p' = p' + \varphi q'; \textbf{then} pp' = pp' + \varphi (pq' + qp' + qq')\), \textbf{clearly} \(pp' \in \bar{\Lambda}_\text{re}. \) Using Axiom [2.2] (iv), write

\[p(-p') + (-p)p' + (-p)(-p') \leq pq' + qp' + qq'\]

\[\leq p(1 - p') + (1 - p)p' + (1 - p)(1 - p'),\]

and expand to get

\[p(-p') \leq pq' + qp' + qq' \leq 1 - pp',\]

as desired. \(\square\)

Given an element \(p \in \bar{\Lambda}\), the element \((1 - p)\), also in \(\bar{\Lambda}\), is regarded as the \textbf{phantom complement} of \(p\) in \(\bar{\Lambda}\).

\textbf{Lemma 2.5.} \textit{The image of a phantom probability measure} \(\mathcal{P}\) \textit{is well defined for phantom addition and multiplication, that is} \(\mathcal{P}(A) + \mathcal{P}(B), \mathcal{P}(A)\mathcal{P}(B), \text{ and } \mathcal{P}(A^c)\) \textit{are in} \(\Lambda\) \textit{for any} \(A, B \in \Sigma\).

\textit{Proof.} The addition is axiomatic, since \(A \cup B \subseteq \Sigma\), implies \(\mathcal{P}(A) + \mathcal{P}(B) \in \Lambda\); cf. Axiom [2.2] (iv).

For the multiplication, take \(p = \mathcal{P}(A)\) and \(q = \mathcal{P}(B)\) and apply Lemma [2.4] (ii). Since \(\mathcal{P}(A) \in \Lambda\), by Lemma [2.2] (i), \(\mathcal{P}(A^c) = 1 - p\) is in \(\Lambda\). \(\square\)

\textbf{Definition 2.6.} A triple \((\Omega, \Sigma, \mathcal{P})\), where \(\Sigma\) is a \(\sigma\)-algebra of subsets of \(\Omega\) and \(\mathcal{P}\) is a phantom probability measure, is called a \textbf{phantom probability space}. Given an event \(A \in \Sigma\), \(\mathcal{P}(A)\) is said to be the \textbf{phantom probability} of \(A\).

As mentioned earlier, in the context of phantom probability spaces, the phantom term should be realized as a signed bounded distortion, with respect to each event, dispersed non-uniformly over the probability space and it has total sum 0. Accordingly, the phantom probability measure can be understand as a family of real probability measures \(P_t : \Sigma \rightarrow [0, 1]\), each satisfying

\[(2.3)\quad P_t(A) \in [\mathcal{P}_{\text{re}}(A), \mathcal{P}_{\text{re}}(A) + \mathcal{P}_{\text{ph}}(A)], \quad \text{for any } A \in \Sigma,\]

(or \(P_t(A) \in [\mathcal{P}_{\text{re}}(A) + \mathcal{P}_{\text{ph}}(A), \mathcal{P}_{\text{re}}(A)], \text{ when } \mathcal{P}_{\text{ph}}(A)\) is negative).

We say that a phantom probability measure \(\mathcal{P} : \Sigma \rightarrow \mathbb{P}\mathbb{H}\), agrees with \(\mathcal{P} : \Sigma \rightarrow \mathbb{P}\mathbb{H}\) if it satisfies,

\[|P_t(A), \mathcal{P}_{\text{re}}(A) + \mathcal{P}_{\text{ph}}(A)] \leq [\mathcal{P}_{\text{re}}(A), \mathcal{P}_{\text{re}}(A) + \mathcal{P}_{\text{ph}}(A)], \quad \text{for any } A \in \Sigma.\]

A real phantom probability measure \(P : \Sigma \rightarrow \mathbb{R}\), is said to agrees with \(\mathcal{P} : \Sigma \rightarrow \mathbb{P}\mathbb{H}\) if it satisfies Equation [2.3].
This argument provides the basis for Axiom 2.2 (iv): the sum of a probability and a distortion can not exceed the probability of a whole $\Omega$ and is never negative, since otherwise it would violate the standard laws of probability.

**Remark 2.7.** In light of the previous paragraph, the former notions obtain the following special meaning:

1. A probability $P(A) = p + \varphi q$ and its conjugate $(p - q) - \varphi q$ resemble the same likelihoods (in the usual sense) lying between $p$ and $p + q$. This is one of the reasons for specifying a norm in which $|P(A)| = |P(\Omega)|$.

   When one wants to dismiss this similarity and to have a unique canonical representative, he can use the correspondence $q \rightarrow [0, q)$, and therefore $p + \varphi q$ is understood as $\{P \in [p, p + q]\}$. (The same setting can be used for sample spaces as well.)

2. Zero divisors in the image of $P$, if they exist, correspond to events whose likelihood might be equal 0, i.e. $P(A) = 0 + \varphi q$ or $P(A) = p - \varphi p$, with $p, q \geq 0$; cf. Proposition 1.3.

   An exclusive case is when nothing is known about the likelihood of an event $A$; this scenario is recorded by $P(A) = 0 + \varphi 1$.

3. Fixing $P_{ph} := 0$ for the phantom component of $P$, one gets the standard probability model.

4. Cases in which two probabilities $P(A)$ and $P(B)$ are both phantom numbers but their sum is real might happen and mean that the probability of $A \cup B$ is fixed, but the probability of the interior subdivision is uncertain.

**Remark 2.8.** Any phantom probability measure $P : \Sigma \rightarrow PH$ is associated neutrally with a (real) reduced probability measure

$$\hat{P} : \Sigma \longrightarrow [0, 1],$$

given by sending each $A \in \Sigma$ to $\hat{P}(A)$. Since $P(A^c) = 1 - P(A)$ for each $A \in \Sigma$ and $P(\Omega) = 1$, it is easy to verify that $\hat{P}$ is a proper standard real probability measure.

We recall that the real component, $P_{re} : \Sigma \rightarrow [0, 1]$, of the phantom measure $P$ is a proper standard real probability measure as well.

As will be seen in the sequel, this reduced probability measure plays a crucial role in our future development.

**2.2. Digression.** In view of Subsection 1.2 the phantom probability measure $P : \Sigma \rightarrow PH$, is a certain case of a phantom measure with probability zone $\Lambda \subset PH_{(1)}(R)$ of order 1. The general case is given with the phantom probability measure

$$P : \Sigma \longrightarrow PH_{(n)}(R),$$

of order $n$, and the following generalization of Axiom 2.2 (iv):

(iv) $-P_{re}(A) \leq \sum_{i=1}^{n} P_{\ell}(A) \leq 1 - P_{re}(A)$ for each $A \in \Sigma$ and $i = 1, \ldots, n$,

where $P_{\ell}$ denotes the phantom component of $P$ of level $\ell$.

Generalizing our definitions appropriately, most of the following theory extends smoothly to phantom measures of order $n$.

**2.3. Elementary properties of phantom probability.**

**Proposition 2.9** (Basic properties of phantom probability I). Given a phantom probability measure $P$, the following properties are satisfied for each $A$ and $B$ in $\Sigma$:

1. $P_{ph}(\Omega) = 0$,
2. $P(\emptyset) = 0$, i.e. $P_{ph}(\emptyset) = P_{re}(\emptyset) = 0$,
3. $-1 \leq P_{ph}(A) \leq 1$ for each $A \in \Sigma$,
4. $0 \leq |P(A)| \leq 1$,
5. $P(A^c) = 1 - P(A)$,
6. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$,
7. $P_{re}(A) \leq P_{re}(B)$ if $A \subseteq B \subseteq \Omega$,
8. $P_{re}(A \cup B) \leq P_{re}(A) + P_{re}(B)$.

**Proof.**
(1) By definition, cf. Axiom 2.2 (ii).

(2) By Axiom 2.2 (iii), \( \mathcal{P}(\Omega \cup \emptyset) = \mathcal{P}(\Omega) + \mathcal{P}(\emptyset) \). Thus, by Axiom 2.2 (i), \( \mathcal{P}_{re}(\Omega) + \mathcal{P}_{re}(\emptyset) = 1 + \mathcal{P}_{re}(\emptyset) \leq 1 \), namely \( \mathcal{P}_{re}(\emptyset) = 0 \). On the other hand, by property (1), \( \mathcal{P}_{ph}(\Omega) + \mathcal{P}_{ph}(\emptyset) = 0 + \mathcal{P}_{ph}(\emptyset) = 0 \), so \( \mathcal{P}_{ph}(\emptyset) = 0 \).

(3) Immediate by Axiom 2.2 (i) and Axiom 2.2 (iii).

(4) Let \( \mathcal{P}(A) = p + \varphi q \), then \( |p + \varphi q|^2 = p^2 + pq + \frac{q^2}{2} \). Thus, since \( q \leq 1 - p \) and \( 0 \leq p \leq 1 \),

\[
p^2 + pq + \frac{q^2}{2} \leq p^2 + p(1 - p) + \frac{(1 - p)^2}{2} = 1 + \frac{p^2}{2} \leq \frac{1 + 1}{2}.
\]

On the other hand, since \( q \geq -p \), \( p^2 + pq + \frac{q^2}{2} \geq p^2 + p(-p) + \frac{(-p)^2}{2} = \frac{p^2}{2} \geq 0 \).

(5) Straightforward from property (1).

(6) Write \( A \cup B = A \cup (A^c \cap B) \) and \( B = (A \cap B) \cup (A^c \cap B) \). The additivity axiom yields

\[
\mathcal{P}(A \cup B) = \mathcal{P}(A) + \mathcal{P}(A^c \cap B) \quad \text{and} \quad \mathcal{P}(B) = \mathcal{P}(A \cap B) + \mathcal{P}(A^c \cap B).
\]

Subtracting the second equality from the first and rearranging terms, we obtain the required.

(7) and (8) are precisely the well known relations for the real probability measure \( \mathcal{P}_{re} \).

One of the reasons for defining the absolute value as in Equation (1.15) is that \( 0 \in \Lambda \) is the unique element with \( |z| = 0 \) and 1 is the unique element with \( |z| = 1 \). The equiv-norm elements in \( \Lambda \) are a restriction of ellipses centered around the origin, to the first quadrant (see Figure 1).

The same reason also led to defining the relation \( <_\alpha \) as in Equation (1.11), since then for each \( z \in \Lambda \) we have \( 0 \leq |z|_\alpha \leq 1 \), where 0 and 1 are obtained uniquely, i.e. \( [1]_\alpha = 1 \) and \( [0]_\alpha = 0 \). Thus, their equivalent classes in \( \Lambda \) are singletons, and they are all the singletons in \( \Lambda/\alpha \). All the equivalent classes are parallel line segments having a slope \( = -\alpha \).

Next, we plug in the given weak order \( \preceq_{wk} \) on \( \mathbb{P} \); note that this weak order assumes satisfying the condition of Remark 2.3.

**Proposition 2.10 (Elementary properties of phantom probability II).** For any phantom probability measure \( \mathcal{P} \), the following properties are satisfied for each \( A \) and \( B \) in \( \Sigma \):

1. \( 0 \preceq_{wk} \mathcal{P}(A) \preceq_{wk} 1 \) for each \( A \in \Sigma \),
2. \( \mathcal{P}(A) \preceq_{wk} \mathcal{P}(B) \) whenever \( A \subseteq B \subseteq \Omega \),
3. \( \mathcal{P}(A) \preceq_{wk} \mathcal{P}(A \cup B) \), for any pair of disjoint events \( A \) and \( B \) in \( \Sigma \),
4. \( \mathcal{P}(A \cup B) \preceq_{wk} \mathcal{P}(A) + \mathcal{P}(B) \).

**Proof.**

1. \( \mathcal{P}(A) \in \bar{\Lambda} \) in which \( 0 \preceq_{wk} z \preceq_{wk} 1 \) for each \( z \in \bar{\Lambda} \), cf. Remark 2.3.

2. Write \( B = A \cup C \). So, by Axiom 2.2 (iii), \( \mathcal{P}(A \cup C) = \mathcal{P}(A) + \mathcal{P}(C) \). But \( \mathcal{P}(C) \preceq_{wk} 0 \) by property (1), and hence \( \mathcal{P}(A) + \mathcal{P}(C) \preceq_{wk} \mathcal{P}(A) \).

3. Immediate by property (2).

4. Combine property (1) and Proposition 2.9 (6).

**Proposition 2.11 (Compound phantom probability measure).** Suppose \( \mathcal{P}_i, \ i = 1, \ldots, m, \) are phantom probability measures and \( z_i \in \bar{\Lambda} \) are phantom numbers \( z_i = a_i + \varphi b_i \) such that \( \sum_i z_i = 1 \). Then \( \mathcal{P} = \sum_i z_i \mathcal{P}_i \) is also a legitimate phantom probability measure.

**Proof.** We need to verify the axioms of being a phantom probability measure; cf. Axiom 2.2:

1. For a fixed \( A \in \Sigma \), let \( \mathcal{P}_{min, re}(A) = \min \{ \mathcal{P}_{i, re}(A) \} \) and let \( \mathcal{P}_{max, re}(A) = \max \{ \mathcal{P}_{i, re}(A) \} \). Then,

\[
0 \leq \mathcal{P}_{min, re}(A) \sum_i a_i \leq \sum_i a_i \mathcal{P}_{i, re}(A) \leq \mathcal{P}_{max, re}(A) \sum_i a_i \leq 1.
\]

The fact that each \( z_i \) is in \( \Lambda \) insures that \( 0 \leq a_i \leq 1 \), and thus \( a_i \mathcal{P}_i(A) \in [0, 1] \).
(ii) \( P(\Omega) = \sum_i z_i P_i(\Omega) = \sum_i z_i = 1 \).

(iii) By the additivity of each \( P_i \), since \( A \) and \( B \) are assumed to be disjoint, write \( P(A \cup B) = \sum_i z_i\bar{P}_i(A \cup B) = \sum_i z_i\bar{P}_i(A) + \sum_i z_i\bar{P}_i(B) = P(A) + P(B) \).

(iv) To prove that \( P_{\text{ph}}(A) \leq 1 - P_{\text{re}}(A) \), we expand

\[
P_{\text{ph}}(A) = \sum_i (a_i + b_i)P_{i,\text{ph}}(A) + b_iP_{i,\text{re}}(A) \\
\leq \sum_i ((a_i + b_i)(1 - P_{i,\text{re}}(A)) + b_iP_{i,\text{re}}(A)) \\
= \sum_i (a_i + b_i) - \sum_i a_iP_{i,\text{re}}(A) \\
= 1 - P_{\text{re}}(A).
\]

The same argument shows that \( -P_{\text{re}}(A) \leq P_{\text{ph}}(A) \).

\[
\square
\]

**Corollary 2.12.** The space of phantom probability measures on a \( \sigma \)-algebra \( \Sigma \) is closed under an action of probability measures. That is, given a family of phantom probability measures \( P_i : \Sigma \to \Lambda \), \( i = 1, \ldots, m \), and a phantom probability measure \( Q : \{\{1\}, \ldots, \{m\}\} \to \Lambda \), then \( P \), defined as

\[
P = \sum_i Q(i)P_i,
\]

is also a phantom probability measure.

2.4. Initial examples. The following examples are presented mainly to demonstrate how nonstandard problems are formulated naturally using phantom probability models. Later we show the phantom analogues to well-known probability distributions.

We start with an example whose sample space is also a sample space in the usual sense.

**Example 2.13.** Consider an unfair coin whose probability \( P_t(H) \) to get head (in a single experiment) is unfixed, but belongs to the interval \([0.4, 0.6]\). Accordingly, for any possibility of \( P_t(H) \), the probability \( P_t(T) \) to get tail must satisfy \( P_t(H) + P_t(T) = 1 \), and thus is also restricted to the interval \([0.4, 0.6]\).

This situation is formulated phantomly by letting

\[
P(H) = 0.4 + \psi 0.2 \quad \text{and} \quad P(T) = 0.6 - \psi 0.2.
\]

In this view, the real term of \( P(H) + P(T) \) is constantly 1, while the distortion, which is at most 0.2, is encoded in the phantom terms of \( P(H) \) and \( P(T) \).

In classical probability theory the uniform probability is defined by assigning an identical probability to each event \( A \) in \( \Sigma \), which recall is formulated as \( P(A_i) = \frac{1}{k} \) for a discrete model with \( \Sigma = \{A_1, \ldots, A_k\} \). This trivial formulation becomes meaningless in the phantom framework, since by Axiom 2.2(iv) the phantom term must be identically 0 for each \( P(A_i) \). But, in view of Remark 2.7(i), one can alternate between phantom probabilities and their conjugates, unless \( k \) is even, to have the sum of phantom terms equal 0.

Next we consider an example with a phantom probability space.

**Example 2.14.** Assume a financial investment with an expected profit of \( 5M \$ \) up to \( 10M \$ \) in the case of success, which is estimated to have 40%–60% likelihood and 0 profit otherwise. Using a phantom probability space we formulate this investment with \( \Sigma = \{\{5M + \psi 5M\}, \{0\}\} \) where the phantom probability measure \( P : \Sigma \to \mathbb{P}_H \) is given by

\[
P : \{5M + \psi 5M\} \mapsto 0.4 + \psi 0.2, \quad P : \{0\} \mapsto 0.6 - \psi 0.2.
\]

An exclusive case, very difficult to formulate using classical probability theory, is the following:

**Example 2.15.** Assume a gambler who knows nothing about the chances of winning in a new roulette game. Denoting the event of winning and losing respectively by \( W \) and \( L \), we define the phantom probability measure:

\[
P : W \mapsto 0 + \psi 1, \quad P : V \mapsto 1 - \psi 1.
\]

Recall that these two phantom numbers are zero divisors in \( \mathbb{P}_H \).
2.5. Conditional phantom probability and Bayes’ rule. Analogously to classical theory, the conditional phantom probability of \( A \), given a fixed conditioning event \( B \), is denoted \( \mathcal{P}(A|B) \) and defined as

\[
\mathcal{P}(A|B) = \frac{\mathcal{P}(A \cap B)}{\mathcal{P}(B)},
\]

where \( \mathcal{P}(B) \) assumed nonzero and a nonzero divisor. As a consequence, given two disjoint events \( A \) and \( B \), where \( \mathcal{P}(B) \notin \mathbb{Z}_0^{\div} \), we have the equality:

\[
(2.4) \quad \mathcal{P}(A|B) \mathcal{P}(B) = \mathcal{P}(A \cap B).
\]

It can be verified that for a fixed event \( A \), the conditional phantom probability forms a legitimate phantom probability law that satisfies Axiom 2.2. The fact that the phantom term of \( \mathcal{P}(A|A) \) is 0 can be seen using Equation 1.5, that is

\[
\mathcal{P}(A|A) = \frac{p}{p} + \varphi \frac{pq - qp}{p(p + q)} = 1 + \varphi 0,
\]

for \( \mathcal{P}(A) = p + \varphi q \).

Assuming all the conditioning events have probabilities that are nonzero divisors and are \( \neq 0 \), applying Equation 2.4 recursively we have

\[
\mathcal{P}(\cap_{i=1}^n A_i) = \mathcal{P}(A_1) \mathcal{P}(A_2|A_1) \mathcal{P}(A_3|A_1 \cap A_2) \cdots \mathcal{P}(A_n|\cap_{i=1}^{n-1} A_i).
\]

A sequence of events \( A_1, \ldots, A_n \in \Sigma \) is said to be a partition of \( \Omega \) if each possible outcome is included in one and only one of the events \( A_1, \ldots, A_n \). That is, the sample space \( \Omega \) is the disjoint union of the events \( A_1, \ldots, A_n \).

**Theorem 2.16 (Total probability theorem).** Let \( A_1, \ldots, A_n \) be disjoint events that form a partition of the sample space \( \Omega \) and assume that \( \mathcal{P}(A_i) \neq 0 \) is not a zero divisor, for all \( i = 1, \ldots, n \). Then, for any event \( B \), we have

\[
\mathcal{P}(B) = \sum_i \mathcal{P}(B \cap A_i) = \sum_i \mathcal{P}(A_i) \mathcal{P}(B|A_i).
\]

**Proof.** The events \( A_1, \ldots, A_n \) form a partition of the sample space \( \Omega \), so the event \( B \) can be decomposed into the disjoint union of its intersections \( A_i \cap B \) with the sets \( A_i \). Using the additivity axiom, Axiom 2.2 (iii), it follows that \( \mathcal{P}(B) = \sum_i \mathcal{P}(B \cap A_i) \). The proof is completed by the definition of conditional probability, i.e. \( \mathcal{P}(B \cap A_i) = \mathcal{P}(A_i) \mathcal{P}(B|A_i) \).

**Theorem 2.17 (Bayes’ rule).** Let \( A_1, \ldots, A_n \) be disjoint events that form a partition of the sample space \( \Omega \) and assume that \( \mathcal{P}(A_i) \neq 0 \) is not a zero divisor, for all \( i = 1, \ldots, n \). Then, for any event \( B \), we have

\[
\mathcal{P}(A_i|B) = \frac{\mathcal{P}(A_i) \mathcal{P}(B|A_i)}{\mathcal{P}(B)} = \frac{\mathcal{P}(A_i) \mathcal{P}(B|A_i)}{\mathcal{P}(A_1) \mathcal{P}(B|A_1) + \cdots + \mathcal{P}(A_n) \mathcal{P}(B|A_n)}.
\]

**Proof.** To verify Bayes’ rule, note that \( \mathcal{P}(A_i) \mathcal{P}(B|A_i) \) and \( \mathcal{P}(A_i|B) \mathcal{P}(B) \) are equal, because they are both equal to \( \mathcal{P}(A_i \cap B) \). This yields the first equality. The second equality follows from the first by using the total probability theorem to rewrite \( \mathcal{P}(B) \).

**Remark 2.18.** In light of Theorems 2.16 and 2.17 the phantom probability space, as introduced in Definition 2.4, provides a Bayesian probability model. Being a Bayesian probability model is a crucial property of a theory of stochastic processes and Markov chains. The absence of this property is one of the main deficiencies in some alternative models that has been suggested in the past, cf. 111-113.

2.6. Independence. When the equality

\[
\mathcal{P}(A|B) = \mathcal{P}(A)
\]

holds, we say that the event \( A \) is (phantomly) independent of the event \( B \). Note that by the definition \( \mathcal{P}(A|B) = \mathcal{P}(A \cap B)/\mathcal{P}(B) \), this is equivalent to

\[
\mathcal{P}(A \cap B) = \mathcal{P}(A) \mathcal{P}(B).
\]

We adopt this latter relation as the definition of independence because it can be used even if \( \mathcal{P}(B) \) is a zero divisor or 0, in which case \( \mathcal{P}(A|B) \) is undefined.

The symmetry of this relation also implies that independence is a symmetric property: that is, if \( A \) is independent of \( B \), then \( B \) is independent of \( A \), and we can unambiguously say that \( A \) and \( B \) are independent events.
We noted earlier that the conditional phantom probabilities of events, conditioned on a particular event, form a legitimate probability law. Thus, we can talk about phantom independence of various events with respect to this conditional law. In particular, given an event \( C \), the events \( A \) and \( B \) are called **conditionally independent** if

\[
\mathcal{P}(A \cap B | C) = \mathcal{P}(A | C) \mathcal{P}(B | C).
\]

(2.5)

The definition of conditional probability and the multiplication rule yield

\[
\mathcal{P}(A \cap B | C) = \frac{\mathcal{P}(A \cap B \cap C)}{\mathcal{P}(C)} = \frac{\mathcal{P}(B \cap C) \mathcal{P}(A | B \cap C)}{\mathcal{P}(C)} = \frac{\mathcal{P}(C) \mathcal{P}(B | C) \mathcal{P}(A | B \cap C)}{\mathcal{P}(C)} = \mathcal{P}(B | C) \mathcal{P}(A | B \cap C),
\]

and thus, using Equation (2.5)

\[
\mathcal{P}(B | C) \mathcal{P}(A | B \cap C) = \mathcal{P}(A | C) \mathcal{P}(B | C).
\]

After canceling the factor \( \mathcal{P}(B | C) \), assumed nonzero divisor and \( \neq 0 \), we see that conditional independence is the same as the condition

\[
\mathcal{P}(A | B \cap C) = \mathcal{P}(A | C).
\]

In other words, this relation states that if \( C \) is known to have occurred, the additional knowledge that \( B \) also occurred does not change the probability of \( A \), even though it is a phantom probability (understood as a varied probability in the classical sense). Interestingly, like in the classical theory, independence of two events \( A \) and \( B \) with respect to the unconditional probability law, does not imply conditional independence, and vice versa.

We generalize the definition of phantom independence to finitely many events, and say that the events \( A_1, A_2, \ldots, A_n \) are independent if

\[
\mathcal{P}(\bigcap_{i \in S} A_i) = \prod_{i \in S} \mathcal{P}(A_i),
\]

for every subset \( S \) of \( \{1, 2, \ldots, n\} \).

3. Phantom random variables

In many probabilistic models, a random variable is a real-valued function \( X : \Omega \rightarrow \mathbb{R} \) of the outcomes of an experiment, which means that each outcome is assigned with a fixed single (real) numerical value. In real life this is far from being satisfactory; for example consider that these numerical values correspond to instrument readings or stock prices. Our module allows the assignment of a varied numerical value, recorded as a phantom number, to each outcome.

Consider a random experiment with a sample space \( \Omega \). A **phantom random variable**, written \( \text{p. r. v.} \) for short,

\[
X : \Omega \rightarrow \mathbb{P}\mathbb{H}
\]

is a single-phantom-valued function of the form

\[
X : \omega \mapsto x(t) = a_x(t) + \varphi b_x(t), \quad t \in \mathbb{R},
\]

that assigns a phantom number \( x = X(\omega) \), called the **value** of \( X \), to each sample element \( \omega \in \Omega \). We write \( X_{\text{re}} \) and \( X_{\text{ph}} \), respectively, for the real and the phantom components of \( X \). In this realization, \( X \) is parameterized by real numbers, denoted by \( t \); later we shall see that this parametrization is either discrete or continuous.

Note that the terminology which used here is the traditional terminology of probability theory, and for this reason we use the letter \( x \), which stands for \( a_x + \varphi b_x \), to denote the phantom evaluation of \( X \) at \( \omega \), while \( z \) stands for an arbitrary element in \( \mathbb{P}\mathbb{H} \). (Clearly a p. r. v. is not a variable at all in the usual sense, but a function.)

The sample space \( \Omega \) is called the **domain** of the p. r. v. \( X \) and is denoted \( \text{D}_X \). The collection of all phantom values of \( X(\omega) \), where \( \omega \in \Omega \), is termed the **phantom range**, or just **range**, for short, of the p. r. v. \( X \) and is denoted by \( \text{R}_X \). Thus, the range \( \text{R}_X \) of p. r. v. \( X(\Omega) \) is a certain subset of the set of all phantom numbers, usually assumed without zero divisors.

Note that two or more different sample elements might give the same value of \( X(\omega) \), but two different numbers in the range cannot be assigned to the same sample point.
Remark 3.1. When the real parametrization in Map \( \ref{map:1} \) is one-to-one, i.e. \( x(t_1) \neq x(t_2) \) for any \( t_1 \neq t_2 \), then the parametrization induces a total order on \( R_X \). We denote this order as \( \leq \).

Clearly, any function \( g : R_X \rightarrow \mathbb{P} \mathbb{H} \) of a p.r.v., i.e. a function whose domain contains the range of \( X \), defines another p.r.v.

To any p.r.v. \( X \) we associate the **reduced phantom random variable**, written r.p.r.v. for short,

\[
\hat{X} : \Omega \longrightarrow \mathbb{R}
\]

given by

\[
(3.2) \quad \hat{X} : \omega \longmapsto X_{\text{re}}(\omega) + X_{\text{ph}}(\omega),
\]

which in view of Map \( \ref{map:1} \) is \( \hat{X}(\omega) = a_x(t) + b_x(t) \), with \( t \in \mathbb{R} \).

Remark 3.2. Along our next development we use the weak order \( \preceq_{\text{wk}} \) on \( \mathbb{P} \mathbb{H} \), assumed to satisfy condition \( \ref{condition:2} \). Accordingly the range of a p.r.v. is well ordered. (Recall that the main examples are \( \preceq_{1} \) and \( \preceq_{||} \) for a weak order, cf. Remark \( \ref{remark:2} \) and the lexicographic order, cf. Equation \( \ref{equation:1.8} \) for a total order.)

We also remark that the range \( R_X \) of any given p.r.v. \( X \) can be embedded in \( \mathbb{R} \), cf. Map \( \ref{map:1} \), and therefore, as pointed out earlier, can be parameterized by the real numbers.

If \( X \) is a p.r.v. and \( z \in \mathbb{P} \mathbb{H} \) is a fixed phantom number, not necessarily in \( R_X \), we define the event \( (X = z) \) as the preimage of \( z \), i.e.

\[
(X = z) = \{ \omega \in \Omega : X(\omega) = z \},
\]

which has probability \( \mathcal{P}(X = z) \). Note that when \( z \notin R_X \), we set \( \mathcal{P}(X = z) = 0 \).

Similarly, for fixed numbers \( z, z_1, \) and \( z_2 \) in \( \mathbb{P} \mathbb{H} \), we define the following events:

\[
(X \sim_{\text{wk}} z) = \{ \omega \in \Omega : X(\omega) \sim_{\text{wk}} z \},
\]

\[
(X \preceq_{\text{wk}} z) = \{ \omega \in \Omega : X(\omega) \preceq_{\text{wk}} z \},
\]

\[
(X \succeq_{\text{wk}} z) = \{ \omega \in \Omega : X(\omega) \succeq_{\text{wk}} z \},
\]

\[
(z_1 \prec_{\text{wk}} X \preceq_{\text{wk}} z_2) = \{ \omega \in \Omega : z_1 \prec_{\text{wk}} X(\omega) \preceq_{\text{wk}} z_2 \},
\]

which have respectively the phantom probabilities \( \mathcal{P}(X \sim_{\text{wk}} z) \), \( \mathcal{P}(X \preceq_{\text{wk}} z) \), \( \mathcal{P}(X \succeq_{\text{wk}} z) \), and \( \mathcal{P}(z_1 \prec_{\text{wk}} X \preceq_{\text{wk}} z_2) \). (We emphasize that these values need not be real numbers.)

Note that when \( z \notin R_X \), we can still have \( \mathcal{P}(X \sim_{\text{wk}} z) \neq 0 \). Of course this can only happen for a weak order; for a total order \( \mathcal{P}(X = z) = 0 \) for each \( z \notin R_X \).

Given an arbitrary phantom number \( z \) and a p.r.v. \( X \), with one-to-one real parameterizations \( t \), we define the function \( \xi_X : \mathbb{P} \mathbb{H} \rightarrow R_X \) by

\[
(3.3) \quad \xi_X(z) := \max\{\max_{\leq t} \{x \in R_X : x \preceq_{\text{wk}} z \} \},
\]

and sometimes write \( x^z \) for \( \xi_X(z) \in R_X \). This function is well defined unless \( R_X = \emptyset \). Note that the interior max provides a set of elements in \( X \) which are in the same equivalence class, determined by \( \sim_{\text{wk}} \), while the exterior pick the maximal \( t \)-element in this class.

In the same way we define the function \( \xi_X : \mathbb{P} \mathbb{H} \rightarrow R_X \) as

\[
(3.4) \quad \xi_X(z) := \min\{\min_{\leq t} \{x \in R_X : x \succeq_{\text{wk}} z \} \},
\]

and write \( x_z \) for \( \xi_X(z) \in R_X \). As before, we set \( \xi_X(x) = x \) for each \( x \in R_X \). We emphasize that the interior “min” and the “max” above are taken with respect to \( \preceq_{\text{wk}} \) – the weak order on \( \mathbb{P} \mathbb{H} \), and the exteriors are taken with respect to \( \leq \) – the total order on \( R_X \). The use of these functions is mainly for continuous p.r.v. as will be seen later.

3.1. Discrete random variables.

**Definition 3.3.** A p.r.v. \( X \) is called **discrete** if its range \( R_X \), i.e. the set of values that it can take, is finite or at most countably infinite.
When \( X \) is discrete, we sometimes denote the values of \( X \) as \( x_1, x_2, \ldots, x_k, \ldots \), indicating that it is parameterized by \( t \in \mathbb{N} \).

The most important way to characterize a random variable is through the (phantom) probabilities of the values that it can take. For a discrete random variable \( X \), these values are captured by the **probability mass function** of \( X \), written \( p. m. f. \) for short, and denoted \( p_x \). In particular, if \( x \) is any possible value of \( X \), the **probability mass** of \( x \), denoted \( p_x(x) \), is the phantom probability of the event \( \{X \sim wk x\} \) consisting of all outcomes that give rise to a value of \( X \) equal to \( x \). That is

\[
p_x(x) = \mathcal{P}(X = x),
\]

and therefore \( 0 \leq wk p_x(x) \leq wk \), cf. Proposition 2.11(1). The mass function of \( X \) is extended to the whole \( \mathbb{PH} \) by setting \( p_x(z) := 0 \) for each \( z \notin R_x \), and thus

\[
p_x(z) = \begin{cases} \mathcal{P}(X = z), & z \in R_x; \\ 0, & z \notin R_x. \end{cases}
\]

By the axioms of the phantom probability measure, we therefore have

\[
\sum_{x \in X} p_x(x) = 1,
\]

where in the summation above, \( x \) ranges over all the possible numerical phantom values of \( X \). This follows from the additivity and normalization axioms, because the events \( \{X = x\} \) are disjoint and form a partition of the sample space \( \Omega \), as \( x \) ranges over all possible values of \( X \). By a similar argument, for any set \( S \) of phantom numbers, we also have

\[
\mathcal{P}(X \in S) = \sum_{x \in S} p_x(x),
\]

where \( X \in S \) means the values of \( X \) which are contained in \( S \). (This notation it is a bit misleading, but is the traditional notation.)

Let us summarize the properties of phantom mass functions:

**Properties 3.4. Properties of a p. m. f. \( p_x \):**

1. \( 0 \leq wk p_x(z) \leq wk 1 \),
2. \( p_x(z) \in \Lambda \), and thus \( 0 \leq |p_x(z)| \leq 1 \),
3. \( p_x(z) = 0 \) if \( z \neq x_1, x_2, \ldots \), for all \( x_i \in R_x \),
4. \( \sum_k p_x(x_k) = 1 \).

In view of Remark 2.8 with any \( p. m. f. \ p_x \) we associate the **reduced probability mass function**, written \( r. p. m. f. \) for short,

\[
\hat{\mathcal{P}}(X \in S) = \sum_{x \in S} \hat{p}_x(x).
\]

Given a phantom number, in particular a phantom probability \( p \in \Lambda \), by Definition 1.18 we always have

\[
\hat{\mathcal{P}}^\Lambda - p = 1 - \hat{p}.
\]

Note that for each phantom number \( p \in \Lambda \) we have the inclusions \( \hat{p} \in [0, 1] \) and \( 1 - \hat{p} \in [0, 1] \) in the real interval \([0, 1]\). We use this property in the forthcoming examples.

The following examples demonstrate how well-known probability mass functions are generalized naturally in the phantom framework. Moreover, these examples show that most of these probabilities have the realization property. We keep the traditional notation and denote the parameter as \( \lambda \), though, here it takes phantom values.

**Example 3.5 (The binomial p. r. v.).** A biased coin with ambiguous probability is tossed \( n \) times. At each toss, the coin comes up a head with phantom probability \( p = p + \varphi q \), and a tail with phantom probability \( q = 1 - p \), independently of prior tosses.

Let \( X \) be the number of heads in the \( n \)-toss sequence. We refer to \( X \) as a binomial p. r. v. with parameters \( n \in \mathbb{N} \) and \( p \in \mathbb{PH} \). The p. m. f. of \( X \) consists of the binomial probabilities:

\[
p_x(k) = \mathcal{P}(X = k) = \binom{n}{k} p^k q^{n-k}, \quad k = 0, 1, \ldots, n.
\]
Recalling that
\[ \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1. \]

To see that this property is satisfied, apply Equation (1.13) for \( p^k (1-p)^{n-k} \), that is
\[ p^k (1-p)^{n-k} = \varphi (\hat{p}^k (1-\hat{p})^{n-k} - p^k (1-p)^{n-k}). \]

Recalling that \( \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1 \) for any real \( p \in [0,1] \), if \( \hat{p} \in [0,1] \), we take the sum
\[ \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} + \varphi \sum_{k=0}^{n} \binom{n}{k} (\hat{p}^k (1-\hat{p})^{n-k} - p^k (1-p)^{n-k}) = 1 + \varphi (1-1) \]
to obtain the desired.

**Example 3.6 (The geometric p. r. v.)** The geometric p. r. v. is the number \( X \) of trials, each with phantom probability \( p \), needed for success the first time. Its p. m. f. is given by
\[ p_X(k) = (1-p)^{k-1} p, \quad k = 1, 2, \ldots. \]
This is a legitimate p. m. f. Indeed, use Equation (1.13) to write
\[ p_X(k) = ((1-p) - \varphi q)^{k-1}(p + \varphi q) = (1-p)^{k-1} p + \varphi ((1-\hat{p})^{k-1}(\hat{p}) - (1-p)^{k-1} p). \]
Recalling that \( \sum_{k=1}^{\infty} (1-p)^{k-1} p = 1 \) for any real \( p \in [0,1] \), we take the sum
\[ \sum_{k=1}^{\infty} p_X(k) = \sum_{k=1}^{\infty} (1-p)^{k-1} p + \varphi \sum_{k=1}^{\infty} ((1-\hat{p})^{k-1}(\hat{p}) - (1-p)^{k-1} p) = 1 + \varphi (1-1), \]
to get the required.

**Example 3.7 (The Poisson p. r. v.)** A Poisson p. r. v. takes nonnegative integer values. Its p. m. f. is given by
\[ p_X(k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots, \]
where \( \lambda \) is a pseudo positive phantom parameter characterizing the p. m. f. It is a legitimate p. m. f. because
\[ \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \left( 1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \ldots \right) = e^{-\lambda} e^\lambda = 1. \]
The latter equality is by Proposition 1.19 (3).

### 3.2. Continuous random variables.

The case when phantom random variables are continuous is much delicate than the discrete case, especially since paths are involved and their parametrization needs to be included carefully in our formulation.

**Definition 3.8.** A p. r. v. \( X \) is called continuous if its probability law can be described in terms of a piecewise continuous phantom function \( f_X : \mathbb{P} \rightarrow \mathbb{P} \), called the **probability density function** of \( X \), written p. d. f. for short, whose real component is nonnegative and which satisfies
\[
\mathbb{P}(X \in S) = \int_S f_X(x) dx,
\]
for every subset \( S \) of
\[
\gamma_X = \{ X = a_x(t) + \varphi b_x(t) \mid t \in \mathbb{R} \},
\]
where \( a_x(t) \) and \( b_x(t) \) are real piecewise differentiable functions.
In fact, we care only about \( f_X \) restricted to \( \gamma_X \), which is the range of \( X \), on which \( f_X \) is piecewise continuous. The set \( \gamma_X \) is realized as a path interval in \( \mathbb{P}^{0,1} = \mathbb{R} \times \mathbb{R} \), isomorphic to an interval in \( \mathbb{R} \), and it plays a main role in our exposition.

Note that \( S \) does not need not be continuous. In such a case, assuming \( S \) compounds of countably many continuous subsets \( S_i \), the integral is translated to the sum of integrals over the \( S_i \), i.e.

\[
\int_S f_X(x)dx = \sum_{S_i} \int_{S_i} f_X(x)dx
\]

where each \( S_i \) is continuous and the \( S_i \)'s are pairwise disjoint. In the sequel, for simplicity, we assume \( S \) is a continuous subset of \( X \), otherwise we apply the same consideration as (3.8).

**Remark 3.9.**

(i) Since \( \gamma_X \) is a parameterized path in \( \mathbb{P}^{0,1} \), i.e. \( x(t) = a_x(t) + \varphi b_x(t) \) for each \( x \in \gamma_X \), the map \( \mathbb{R} \rightarrow \gamma_X \) is not necessarily one-to-one, and several reals may have the same image. In other words \( \gamma_X \) might have self-intersection points, see for example Figure 2(a). These cases require special treatment that is beyond the scope of this paper. Therefore, in the rest of this paper, when dealing with paths, they are always assumed to have no self-intersections.

(ii) Having this assumption, i.e. \( \gamma_X \) has no self-intersection points, given a point \( x = a_x(t) + \varphi b_x(t) \) of \( \gamma_X \), for notional convenience, we write \( \tau(x) \) for the real \( t \)-value, determined by the parametrization of \( \gamma_X \), whose image is \( x \). Moreover, as mentioned before (cf. Remark 3.7), the given parametrization also determines an order on \( \gamma_X \), which we denote as \( \leq_t \), and write \( x_1 \leq_t x_2 \) when \( \tau(x_1) \leq \tau(x_2) \).

(iii) In view of Remark 3.7 (ii), in most applications the image of a path \( \gamma_X \) in the compactification of \( \mathbb{P}^{0,1} \) has the endpoints \((0,0)\) and \((1,0)\), as illustrated by Figure 2(b), and usually does not have points with the same real term. However, we do not limit ourself to this type of path.

(iv) Abusing the notation, as in the complex convention, in order to address the situation that a continuous p. r. v. \( X \) is provided as a path \( \gamma_X \), parameterized by \( t \in \mathbb{R} \), we sometime write \( \gamma_X(t) \) for \( x(t) \) and \( \gamma_X'(t) \) for the derivative \( x'(t) \) of \( x(t) \) with respect to \( t \), which by Equation (1.21) is just \( a'(t) + \varphi b'(t) \).

Since the integration of p. d. f.'s is performed along paths whose order does not need to be compatible with the weak order on \( \mathbb{P}^{0,1} \), we distinguish between cases in which the integration scope is determined by points that belong to the path and cases when these points are arbitrary phantom points. We start with the former case, then we extend it the latter case.

First, since we are dealing with an integral along a parameterized path, cf. Equation (3.6), this integral can be written as

\[
\mathcal{P}(X \in S) = \int_S f_X(x)dx = \int_S f_X(\gamma_X(t))\gamma_X'(t)dt,
\]
for any $S \subset \gamma_X$, assumed to be continuous. In particular, the probability that the value $x$ of $X$ falls within a path interval of $\gamma_X$, whose endpoints are $x_1$ and $x_2$, is

$$\mathcal{P}(x_1 \leq t X \leq x_2) = \int_{\tau(x_1)}^{\tau(x_2)} f_X(\gamma_X(t)) \gamma_X'(t) dt, \quad \text{for } x_1, x_2 \text{ are on } \gamma_X.$$  

Recall that by Equation (1.23), this integral decomposes into two real integrals, one for the real component and the second for the phantom component. The evaluation of each component can be interpreted as the areas confined between $\gamma_X$ and the graphs of the corresponding function.

To simplify the notation, we define

$$\hat{f}_{X, \gamma}(t) := f_X(\gamma_X(t)) \gamma_X'(t)$$

and rewrite Equation (3.9) as

$$\mathcal{P}(x_1 \leq t X \leq x_2) = \int_{\tau(x_1)}^{\tau(x_2)} \hat{f}_{X, \gamma}(t) dt.$$  

(In fact $\gamma_X$ is determined by $X$, but we use this notation to indicate that $\hat{f}_{X, \gamma}(t)$ depends on the path $\gamma_X$ in $\mathcal{PH}$.)

As in the discreet case, a p.d.f. $\tilde{f}_{X, \gamma}$ is associated with the reduced probability density function, written r.p.d.f. for short, defined as

$$\tilde{\mathcal{P}}(x_1 \leq t X \leq x_2) = \int_{\tau(x_1)}^{\tau(x_2)} \tilde{f}_{X, \gamma}(t) dt,$$

by taking the integral over the sum of the real component and the phantom component of $\hat{f}_{X, \gamma}$.

As in the standard theory, for any single value $x$ we have

$$\mathcal{P}(X = x) = \int_{\tau(x)}^{\tau(x)} \hat{f}_{X, \gamma}(t) dt = 0.$$  

Therefore, including or excluding the endpoints of an interval in $\gamma_X$ has no effect on its probability:

$$\mathcal{P}(x_1 \leq t X \leq x_2) = \mathcal{P}(x_1 < t X \leq x_2) = \mathcal{P}(x_1 \leq t X < x_2) = \mathcal{P}(x_1 < t X < x_2),$$

for any $x_1, x_2 \in \gamma_X$.

As usual, to be qualified as a p.d.f., $f_X$ must satisfy the normalization property

$$\int_{\gamma_X} f_X dx = \int_{-\infty}^{\infty} \tilde{f}_{X, \gamma}(t) dt = 1,$$

where its real component must take only nonnegative values, i.e., $\text{re}(f_X(x)) \geq 0$ for every $x \in \gamma_X$. Accordingly, the function $\tilde{f}_{X, \gamma}$ satisfies

$$\int_{-\infty}^{\infty} \text{re}(\tilde{f}_{X, \gamma}(t)) dt = 1 \quad \text{and} \quad \int_{-\infty}^{\infty} \text{ph}(\tilde{f}_{X, \gamma}(t)) dt = 0;$$

that is, normalization and vanishing of phantoms, respectively.

Now, we turn to the cases in which the scopes of random variables are determined by arbitrary phantom values. To extend $f_X$ to the whole $\mathcal{PH}$, we fix $f_X(z) := 0$ for each $z \notin \gamma_X$ and define

$$\mathcal{P}(z_1 \leq_{\text{wk}} X \leq_{\text{wk}} z_2) = \int_{S} \tilde{f}_{X, \gamma}(t) dt, \quad S = \{ x \in \gamma_X : z_1 \leq_{\text{wk}} x \leq_{\text{wk}} z_2 \}$$

for any $z_1, z_2 \in \mathcal{PH}$. In the case that $S$ is not continuous, the integral is decomposed into the sum of countably many integrals along the path intervals as in (3.8), taken with respect to $\leq_t$ in the positive direction.

When $S$ is continuous, Equation (3.13) has the form

$$\mathcal{P}(z_1 \leq_{\text{wk}} X \leq_{\text{wk}} z_2) = \int_{\tau(\overline{\xi}_X(z_1))}^{\tau(\overline{\xi}_X(z_2))} \tilde{f}_{X, \gamma}(t) dt,$$

which is just a line integral along a path interval of $\gamma_X$. Recall that $\overline{\xi}_X(z_1)$ and $\underline{\xi}_X(z_1)$, cf. Equations (3.3) and (3.4), provide the “top” point and the “bottom” point on $\gamma_X$ closest to $z_2$ and $z_1$, respectively; these points
are unique since ≤_t is a total order applied on the weak order ∼_Wk. In this sense, we capture all the elements in R_X which are less or ∼_Wk-equivalent to z_2 and greater or ∼_Wk-equivalent to z.

Accordingly, we also have,

\[ P(X \sim_{Wk} z) = \int_{\gamma_{(X)(z)}} f_x(t)dt, \]

which is the integration along all elements in R_X, assumed continuous, that are ∼_Wk-equivalent to z. In this view it is easy to see that we might have P(X \sim_{Wk} z) ≠ 0.

Let us outline some basic properties of probability density functions:

**Properties 3.10. Properties of p. d. f. f_x:**

1. \( 0 \leq_{Wk} f_X(z) \leq_{Wk} 1 \), for each z ∈ P_H,
2. \( f_X(z) \in \Lambda \), and thus \( 0 \leq |f_X(z)| \leq 1 \),
3. \( f_X(z) = 0 \) if \( z \notin \gamma_X \).

In most applications, the random variable is either discrete or continuous, but if the p. d. f. of a p. r. v. X possesses features of both discrete and continuous random variable’s, then the random variable X is called a mixed phantom random variable.

### 3.3. Cumulative phantom distribution function.

The cumulative phantom distribution function, written c. p. d. f. for short, F_X of a p. r. v. X provides the probability \( P(X \preceq_{Wk} z) \), i.e.

\[ F_X(z) = P(X \preceq_{Wk} z) = \left\{ \begin{array}{ll} \sum_{x_k \preceq_{Wk} z} p_X(x_k), & X \text{ discrete;} \\ \int_S \tilde{f}_x \gamma(t)dt, & X \text{ continuous.} \end{array} \right. \]

for every z in P_H. Here, S is defined as \{x ∈ \gamma_X : x \preceq_{Wk} z\}, cf. Equation (3.13) with z_1 tending to \(-\infty\), where \(-\infty\) stands for \(-\infty(1 + \varphi)\).

Loosely speaking, the c. p. d. f. \( F_X(z) \) “accumulates” phantom probability up to the phantom value z. As in classical theory, most of the information about a random experiment described by the p. r. v. X is recorded by the behavior of \( F_X(z) \).

Using Equation (3.13), if X is a continuous random variable, then

\[ P(z_1 \preceq_{Wk} X \preceq_{Wk} z_2) = \int_{S} \tilde{f}_x \gamma(t)dt = F_X(z_2) - F_X(z_1). \]

(The discrete analogue is obvious.) To emphasize, although \( F_X \) gets an argument that is a phantom number, it also depends on the parametrization of X.

**Properties 3.11. Writing z → \infty, for z = a + \varphi b with a → \infty and b → \infty, we have the following properties:**

1. \( 0 \leq_{Wk} F_X(z) \leq_{Wk} 1 \),
2. \( 0 \leq |F_X(z)| \leq 1 \),
3. \( F_X(z_1) \leq_{Wk} F_X(z_2) \) if \( z_1 \leq_{Wk} z_2 \),
4. \( \lim_{z \to \infty} F_X(z) = F_X(\infty) = 1, \)
5. \( \lim_{x \to -\infty} F_X(z) = F_X(-\infty) = 0, \)
6. \( \lim_{z \to z_0^+} F_X(z) = F_X(z_0^+), \) where \( z_0^+ = \lim_{0<|\varepsilon| \to 0} z_0^+ + \varepsilon. \)

(The verification of these properties is straightforward.)

Accordingly, having Properties 3.11, one can also compute other probabilities, such as

\[ P(X \succ_{Wk} z_1) = 1 - F_X(z_1), \]
\[ P(X \prec_{Wk} z_2) = F_X(z_2), \] where \( z_2^- = \lim_{0 \leq_{Wk} \varepsilon \to 0} (z_2 - \varepsilon). \]
3.4. Moments and variance. In the sequel, we use the notation \( \mathbb{E} [\cdot] \), \( \text{Var} [\cdot] \), and \( \text{Cov} [\cdot] \) respectively for moments, variance, and covariance. These notations are used for both the phantom sense and the standard sense (applied for real numbers) of the respective probability functions, where the meaning is understood from the context. We also point out that for these functions, and others, the standard form is always captured in the real component of the function. As will be seen, this attribute is provided for free by the phantom structure, and one should keep in mind that the generalization to the phantom framework is performed only through the phantom terms of the arguments. This is the leading idea of our forthcoming exposition.

**Definition 3.12.** The \( n \)’th moment of a p. r. v. \( X \) is defined by

\[
\mathbb{E}[X^n] = \begin{cases} 
\sum_x x^n p_X(x), & X \text{ discrete;} \\
\int_{\mathbb{R}} (\gamma_X(t))^{n} \hat{f}_{X,\gamma}(t) dt, & X \text{ continuous.}
\end{cases}
\]

(3.18)

The first moment \( \mathbb{E}[X^1] \) is called the mean, or the expected value, of \( X \) and is denoted by \( \mu_X \).

The \( n \)’th reduced moment \( \mathbb{E}([\hat{X}]^n) \) of a p. r. v. \( X \) is the standard moment for reals, applied to \( \hat{X} \) with \( \hat{\gamma}_X \), or \( \hat{f}_{\hat{X},\gamma} \), for a discrete or a continuous \( X \), respectively. In the same way we define the \( n \)’th conjugate moment \( \mathbb{E}([\tilde{X}]^n) \) of \( X \), by taking \( \tilde{\gamma}_X \), computed with respect to \( \tilde{\gamma}_X \) or \( \tilde{f}_{\tilde{X},\gamma} \); clearly this is a phantom function.

We write \( \mathbb{E}_{\text{re}}[X^n] \) and \( \mathbb{E}_{\text{ph}}[X^n] \), respectively, for the real term and the phantom term of \( \mathbb{E}[X^n] \) and therefore have

\[
\mathbb{E}_{\text{re}}[X^n] = \mathbb{E}[X_{\text{re}}^n],
\]

where \( \mathbb{E}[X_{\text{re}}^n] \) is taken with respect to the real component of \( p_X \) or \( \hat{f}_{\hat{X},\gamma} \). (This relation is not satisfied for the phantom component, i.e. \( \mathbb{E}_{\text{ph}}[X^n] \neq \mathbb{E}[X_{\text{ph}}^n] \), since it also involves the real term of the arguments.)

When \( X \) is discrete, using Equation (1.13), we write

\[
\mathbb{E}[X^n] = \sum_x x^n p_X(x) = \sum_x a^n_x p_x + \varphi((a_x + b_x)^n(p_x + q_x) - a^n_x p_x),
\]

\[
= \sum_x a^n_x p_x + \varphi(\hat{x}^n p_x - a^n_x p_x),
\]

for \( x = a_x + \varphi b_x \) and \( p_X(x) = p_x + \varphi q_x \); a similar form is also obtained for a continuous \( X \). Accordingly, the phantom moment satisfies the realization property, and Equation (3.18) gets the following friendly form:

\[
\mathbb{E}[X^n] = \mathbb{E}[X_{\text{re}}^n] + \varphi(\mathbb{E}[\hat{X}^n] - \mathbb{E}[X_{\text{re}}^n]),
\]

where \( \mathbb{E}[X_{\text{re}}^n] \) stands for \( \sum p_x a^n_x \) and \( \mathbb{E}[\hat{X}^n] \) stands for \( \sum \hat{p} \hat{x}^n \).

**Proposition 3.13.** \( \mathbb{E}[X^n] = \mathbb{E}[\tilde{X}^n] \), where \( \mathbb{E}[\tilde{X}^n] \) is computed with respect to the conjugates of \( X \) and the probability measure.

**Proof.** Straightforward by the additivity and the multiplicativity of the phantom conjugate, cf. Properties 1.7. \( \square \)

The variance of a p. r. v. \( X \), denoted by \( \sigma_X^2 \) or \( \text{Var}[X] \), is defined as

\[
\sigma_X^2 = \mathbb{E}[(X - \mu_X)^2]
\]

(3.20)

and thus

\[
\sigma_X^2 = \begin{cases} 
\sum_x (x - \mu_X)^2 p_X(x), & X \text{ discrete;} \\
\int_{\mathbb{R}} (\gamma_X(t) - \mu_X)^2 \hat{f}_{X,\gamma}(t) dt, & X \text{ continuous.}
\end{cases}
\]

(3.21)

As usual, \( \text{Var}_{\text{re}}[X] \) and \( \text{Var}_{\text{ph}}[X] \) denote respectively the real term and the phantom term of \( \text{Var}[X] \), where \( \text{Var}_{\text{re}}[X] = \text{Var}[X_{\text{re}}] \) are taken with respect to the real component of the probability measure. Thus, we always have \( \text{Var}_{\text{re}}[X] \geq 0 \), since it is just a standard (real) variance. Moreover, we have the following property:

**Proposition 3.14.** The variance \( \text{Var}[X] \) is pseudo nonnegative, for any p. r. v. \( X \).

**Proof.** In view of Equation (5.21), since the square of a phantom number and probabilities are pseudo nonnegative, then the proof is completed by Lemma 1.10 applied for the sum of the products. \( \square \)
The **reduced variance**, i.e. a real-valued function, is defined as

\[
\text{Var}\{\hat{X}\} = \text{E}[(\hat{X} - \text{E}[\hat{X}])^2].
\]

Since this is a standard (real) variance, then we always have \(\text{Var}\{\hat{X}\} \geq 0\). The **conjugate variance** is defined similarly as \(\text{Var}(X) = \text{E}[(\overline{X} - \text{E}[\overline{X}])^2]\), i.e. taken with respect to the conjugates of \(X\) and the probability measure.

Expanding the right-hand side of Equation (3.20), we obtain the following relation:

\[
\text{Var}\{X\} = \text{E}[X^2] - \text{E}[X]^2,
\]

which is a useful formula for determining the variance. Plugging Equation (3.19) into this form, and simplifying, one obtains the realization property for variance:

\[
\text{Var}\{X\} = \text{Var}\{X_{re}\} + \varphi \left(\text{Var}\{\hat{X}\} - \text{Var}\{X_{re}\}\right).
\]

We recall that the notation \(\text{Var}\{X\}\) is used for both the phantom and the standard variance, where the meaning is understood from the context.

**Remark 3.15.** Both \(E[X^n]\) and \(\text{Var}\{X\}\) are phantom functions whose real components satisfy the familiar properties of \(n\)’th moment and variance.

Assuming \(g(X)\) is a phantom function of a p. r. v. \(X\), the expected value of the p. r. v. \(g(X)\) is given by

\[
\text{E}[g(X)] = \begin{cases} \sum_x g(x)p_X(x), & X \text{ discrete;} \\
\int \gamma_X g(\gamma_X(t))\overline{f}_{X,\gamma}(t)dt, & X \text{ continuous.} \end{cases}
\]

It is straightforward to verify that when \(g\) is a linear phantom function, say \(g(X) = \alpha X + \beta\), with \(\alpha, \beta \in \mathbb{P}_H\), then

\[
\text{E}[g(X)] = \alpha \text{E}[X] + \beta \quad \text{and} \quad \text{Var}[g(X)] = \alpha^2 \text{Var}[X],
\]

the latter formula is obtained by (3.21).

**Proposition 3.16.** \(\text{Var}\{X\} = \text{Var}\{\overline{X}\}\), where \(\text{Var}\{ \overline{X}^n \}\) is computed with respect to the conjugates of \(X\) and the probability measure.

**Proof.** Use Equation (3.22) and Properties (1.7) to write \(\text{Var}\{X\} = \text{E}[X^2] - \text{E}[X]^2\), which by Proposition 3.13 is \(\text{E}[\overline{X}^2] - \text{E}[\overline{X}]^2\) and again by Properties (1.7) \(\text{E}[\overline{X}^2] - \text{E}[\overline{X}]^2\), that is \(\text{Var}\{\overline{X}\}\). \(\square\)

Having this property of Proposition 3.14 we attained the following additional phantom analog:

**Definition 3.17.** The **standard phantom deviation** \(\sigma_X\) of a p. r. v. \(X\), is defined to be the maximal nonnegative phantom square root of \(\text{Var}\{X\}\), cf. Equation (1.16), i.e. the root with the nonnegative real term and the maximal nonnegative phantom term.

Since each phantom number has a nonnegative square root, cf. Equation (1.16), and \(\text{Var}\{X\}\) is pseudo nonnegative, then the standard phantom deviation is well defined for any p. r. v.. Using Equation (1.16) it is easy to see that the standard phantom deviation also admits the realization property, i.e.

\[
\sigma_X = \sigma_{X_{re}} + \varphi (\sigma_X - \sigma_{X_{re}}).
\]

3.5. **Special examples.** The following examples show how the

- classical mean and variance carry naturally on to the phantom framework. When a p. r. v. is discrete, we can retain the exact classical setting, while the continuous cases require a modification of definitions which involves the parametrization of \(X\), i.e. that of \(\gamma_X\). Yet, the standard (real) distributions are received as the private cases for the phantom ones.

**Example 3.18 (Mean and variance of the Bernoulli).** Consider the experiment of tossing a biased coin, which comes up a head with phantom probability \(p\) and a tail with probability \(1 - p\), and the Bernoulli p. r. v. \(X\) with p. m. f.

\[
p_X(k) = \begin{cases} p, & k = 1; \\
1 - p, & k = 0. \end{cases}
\]

Then \(\text{E}[X] = 1p + 0(1 - p) = p\), \(\text{E}[X^2] = 1^2p + 0^2(1 - p) = p\) and thus \(\text{Var}[X] = \text{E}[X^2] - \text{E}[X]^2 = p - p^2\).
Example 3.19 (The mean of the Poisson). The mean of the Poisson p. m. f. with pseudo positive parameter \( \lambda \in \mathbb{P}\mathbb{H} \)

\[
p_X(k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots ,
\]

can be calculated as follows:

\[
E[X] = \sum_{k=0}^{\infty} ke^{-\lambda} \frac{\lambda^k}{k!} = \sum_{k=1}^{\infty} ke^{-\lambda} \frac{\lambda^{k-1}}{(k-1)!} \lambda = \sum_{k=0}^{\infty} ke^{-\lambda} \frac{\lambda^k}{k!} = \lambda.
\]

(* the component indexed \( k = 0 \) is zero.) A similar calculation shows that the phantom variance of a Poisson random variable is also \( \lambda \).

Example 3.20 (The phantom exponential p. r. v.). Let \( x(t) = a(t) + \varphi b(t) \). We write \( a := a(t), b := b(t), \) and \( x := x(t) \), for short. The notation \( x' \) stands for the derivative of \( x \) with respect to \( t \), and thus \( x' = a' + \varphi b' \).

A phantom exponential p. r. v. has a p. d. f. with the form

\[
f_X(x) = \left\{ \begin{array}{ll}
\frac{\lambda}{x} e^{-\lambda x}, & x \text{ is pseudo positive;} \\
0, & \text{otherwise},
\end{array} \right.
\]

where \( \lambda \in \mathbb{P}\mathbb{H} \) is a pseudo positive parameter. In particular \( \lambda \neq 0 \) and is not a zero divisor in \( \mathbb{P}\mathbb{H} \).

One observes that when \( X \) is a real random variable, i.e. \( x(t) := t \), then \( x' = 1 \) and the phantom exponential \( f_X \) collapses to the known exponential random variable.

Using Equation (1.17) and Equation (1.21), respectively, we have

\[
x\lambda = a\lambda_{re} + \varphi (\hat{x}\lambda - a\lambda_{re}), \quad \text{and} \quad \frac{\lambda}{x'} = \frac{\lambda_{re}}{a'} + \varphi \left( \frac{\hat{\lambda}}{x'} - \frac{\lambda_{re}}{a'} \right).
\]

Then, Equation (1.17) yields

\[
f_X = \frac{\lambda}{x} e^{-\lambda x} = \left( \frac{\lambda_{re}}{a'} + \varphi \left( \frac{\hat{\lambda}}{x'} - \frac{\lambda_{re}}{a'} \right) \right) e^{-a\lambda_{re} - \varphi (\hat{x}\lambda - a\lambda_{re})}
\]

\[
= \left( \frac{\lambda_{re}}{a'} + \varphi \left( \frac{\hat{\lambda}}{x'} - \frac{\lambda_{re}}{a'} \right) \right) \left( e^{-a\lambda_{re} + \varphi (\hat{x}\lambda - e^{-a\lambda_{re}})} \right)
\]

\[
= \frac{\lambda}{a'} e^{-a\lambda_{re}} + \varphi \left( \frac{\lambda}{x'} e^{-\hat{x}\lambda} - \frac{\lambda_{re}}{a'} e^{-a\lambda_{re}} \right).
\]

Now, for \( \tilde{f}_{X,\gamma} = f_X \gamma_X \) we get

\[
\tilde{f}_{X,\gamma} = \frac{\lambda}{x} e^{-\lambda x} x' = \lambda_{re} e^{-a\lambda_{re}} + \varphi \left( \hat{\lambda} e^{-\hat{x}\lambda} - \lambda_{re} e^{-a\lambda_{re}} \right),
\]

which shows that \( \tilde{f}_{X,\gamma} \) satisfies the normalization property. This because each component is by itself a real exponential, and thus the real component is 1 and the phantom component is summed up to 0.

A similar computation as before shows that

\[
x \tilde{f}_{X,\gamma} = a\lambda_{re} e^{-a\lambda_{re}} + \varphi \left( \hat{x}\lambda e^{-\hat{x}\lambda} - a\lambda_{re} e^{-a\lambda_{re}} \right).
\]

Recalling that for a real exponential random variable, \( E[X] = 1/\lambda \) and \( \text{Var}[X] = 1/\lambda^2 \), cf. [3], taking the integral

\[
E(X) = \int_0^{\infty} x(t) \tilde{f}_{X,\gamma}(t) dt = \frac{1}{\lambda_{re}} + \varphi \left( \frac{1}{\lambda} - \frac{1}{\lambda_{re}} \right)
\]

we get the mean of exponential p. r. v. in terms of the phantom parameter \( \lambda \) as

\[
E(X) = \frac{1}{\lambda_{re}} + \varphi \left( \frac{1}{\lambda} - \frac{1}{\lambda_{re}} \right) = \frac{1}{\lambda_{re}} + \varphi \left( \frac{-\lambda_{ph}}{\lambda_{re}(\lambda_{re} + \lambda_{ph})} \right) = \frac{1}{\lambda}.
\]

cf. Equation (1.4).

In fact we could also have obtained this relation in a shorter way by using Equation (3.19), but, for the matter of validation, we have presented the detailed computation.
3.6. Normal random variables. A continuous p. r. v. \( X \) is said to be **phantom normal**, or **phantom Gaussian**, if it has a p. d. f. of the form

\[
f_X(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2},
\]

where \( \mu \) and \( \sigma \) are two phantom scalar parameters characterizing the p. d. f., with \( \sigma \) assumed pseudo positive. For simplicity, we also assume \( x' \) differentiable, and write \( x' \) for the derivative of \( x := a(t) + \phi b(t) \) with respect to \( t \).

Note that in comparison to the classical case the normal p. r. v. includes the extra argument \( x' \) in the denominator. Yet, as we had for the exponential p. r. v., when \( X \) is assumed to take only real values, the phantom normal density function collapses to the classical normal density function.

**Proposition 3.21.** \( f_X(x) \) satisfies the the normalization property

\[
\frac{1}{\sigma \sqrt{2\pi}} \int \frac{1}{x'} e^{-(x-\mu)^2/2\sigma^2} \, dx = 1,
\]

where \( \gamma_X \) is parameterized by \( t \in \mathbb{R} \), assumed differentiable.

**Proof.** Let \( w = x - \mu \), and therefore \( w' = x' \). Then, \( (x - \mu)^2/\sigma^2 = w^2/\sigma^2 \), and using simple computation one can verify that

\[
w^2/\sigma^2 = w_{re}^2/\sigma_{re}^2 + \varphi \left( \hat{w}^2/\sigma^2 - w_{re}^2/\sigma_{re}^2 \right).
\]

Plugging this into \( e^{-(x-\mu)^2/2\sigma^2} \) and using Equation (3.24), we have

\[
e^{-(x-\mu)^2/2\sigma^2} = e^{-w^2/2\sigma^2} = e^{-w_{re}^2/2\sigma_{re}^2} + \varphi \left( e^{-\hat{w}^2/2\sigma^2} - e^{-w_{re}^2/2\sigma_{re}^2} \right).
\]

Thus,

\[
\frac{1}{x'} e^{-w^2/2\sigma^2} = \frac{1}{\sigma \sigma_{re} \sqrt{2\pi}} \frac{1}{x'} e^{-w_{re}^2/2\sigma_{re}^2} + \varphi \left( \frac{1}{\sigma \sigma_{re} \sqrt{2\pi}} \frac{1}{x'} e^{-\hat{w}^2/2\sigma^2} - \frac{1}{\sigma \sigma_{re} \sqrt{2\pi}} \frac{1}{x'} e^{-w_{re}^2/2\sigma_{re}^2} \right).
\]

But then, \( \frac{1}{x'} e^{-w^2/2\sigma^2} \gamma_X' \) is just \( e^{-w_{re}^2/2\sigma_{re}^2} + \varphi \left( e^{-\hat{w}^2/2\sigma^2} - e^{-w_{re}^2/2\sigma_{re}^2} \right) \).

Recalling that \( \frac{1}{\sigma} = \frac{1}{\sigma_{re}^2} - \varphi \sigma_{re} \sigma = \frac{1}{\sigma_{re}^2} + \varphi \left( \frac{1}{\sigma} - \frac{1}{\sigma_{re}} \right) \) and integrating this in Equation (3.24), given in parametric form, we get

\[
\frac{1}{\sigma_{re} \sqrt{2\pi}} \int e^{-w_{re}^2/2\sigma_{re}^2} \, dt + \varphi \left( \frac{1}{\sigma \sqrt{2\pi}} \int e^{-\hat{w}^2/2\sigma^2} \, dt - \frac{1}{\sigma_{re} \sqrt{2\pi}} \int e^{-w_{re}^2/2\sigma_{re}^2} \, dt \right) = 1 + \varphi (1 - 1),
\]

since the phantom component is the sum of two standard normal distributions, each equal to 1.

Equation (3.24) shows that the phantom normal p. d. f. admits the realization property.

**Proposition 3.22.** The mean and the variance of a normal p. r. v. \( X \) with phantom parameters \( \mu \) and \( \sigma \) are

\[
\mathbb{E}[X] = \mu \quad \text{and} \quad \text{Var}[X] = \sigma^2.
\]

**Proof.** Consider the realization property of Equation (3.20) combined respectively, with Equation (3.19) and Equation (3.24).

**Theorem 3.23.** Normality is preserved under linear transformations. If \( X \) is a normal p. r. v. with mean \( \mu \) and variance \( \sigma^2 \), and if \( \alpha, \beta \in \mathbb{R} \) are phantom scalars, then the p. r. v. \( Y = \alpha X + \beta \) is also normal, with mean and variance

\[
\mathbb{E}[Y] = \alpha \mu + \beta, \quad \text{Var}[Y] = \alpha^2 \sigma^2.
\]

**Proof.** Immediate by Proposition 3.22 and Equation (3.24).

A normal random variable \( Y \) with zero mean and unit variance is said to be a **standard phantom normal**. Its c. p. d. f., denoted as \( \Phi \), is given by

\[
\Phi(z) = \mathbb{P}(Y \leq z) = \frac{1}{\sqrt{2\pi}} \int_S \frac{1}{y} e^{-y^2/2} \, dy,
\]

where \( S = \{ y \in \gamma_Y : y \leq z \} \), assumed continuous and differentiable. Clearly, this integral can also be written in the parametric form as given in Equation (3.14).
Let \( X \) be a normal p. r. v. with mean \( \mu_X \) and variance \( \sigma_X^2 \). We “standardize” \( X \) by defining a new random variable \( Y \) given by
\[
Y = \frac{X - \mu_X}{\sigma_X}.
\]
Since \( Y \) is a linear transformation of \( X \), it is normal. Furthermore,
\[
E[Y] = \frac{E[X] - \mu_X}{\sigma_X} = 0, \quad \text{Var}[Y] = \frac{\text{Var}[X]}{\sigma_X} = 1.
\]
Thus, \( Y \) is a standard normal p. r. v. This fact allows us to calculate the probability of any event defined in terms of \( X \): we redefine the event in terms of \( Y \), and then use the standard normal p. r. v. in a variety of engineering, physical, and statistical contexts. As we have shown the normal p. r. v. preserves this property and generalizes the classical one in a natural way.

The (classical) normal random variable plays an important role in a broad range of probabilistic models. The main reason is that, generally speaking, it models well the additive effect of many independent factors, regardless of the c. p. d. f. of the individual random variables. This property is captured in the celebrated central limit theorem, extended to the phantom framework, which will be discussed in Section 6.

4. Multiple random variable

Consider a random experience having the sample space \( \Omega \). A multiple phantom random variable, written \( \text{m. p. r. v.} \) for short, is a multiple-phantom-valued function
\[
(X_1, \ldots, X_n) : \Omega \longrightarrow \mathbb{P}^{(n)},
\]
given by
\[
(X_1, \ldots, X_n) : \omega \longmapsto (X_1(\omega), \ldots, X_n(\omega)),
\]
with each \( X_i \) a p. r. v. on \( \Omega \) as in Equation (4.1). The phantom range of the m. p. r. v. \((X_1, \ldots, X_n)\) is denoted by \( R_{x_1, \ldots, x_n} \), and defined by
\[
R_{x_1, \ldots, x_n} = \{ (x_1, \ldots, x_n) : \omega \in \Omega, \ x_1 = X_1(\omega), \ldots, x_n = X_n(\omega) \}.
\]

If the \( X_i \)'s are each, by themselves, discrete p. r. v.'s, then \((X_1, \ldots, X_n)\) is called a discrete m. p. r. v.. Similarly, if the \( X_i \)'s are each, by themselves, continuous p. r. v.'s, then \((X_1, \ldots, X_n)\) is called a continuous m. p. r. v.. When \( n = 2 \) we write \((X, Y)\) for \((X_1, X_2)\) and call it a bivariate phantom random variable, written \( \text{b. p. r. v.} \) for short. In the remainder of this section, to make the exposition clearer, we present the case of b. p. r. v.; the extension to m. p. r. v. is straightforward.

Consider two discrete p. r. v.'s \( X \) and \( Y \) associated with the same experiment. The joint phantom mass function of \( X \) and \( Y \) is defined by
\[
p_{X,Y}(x, y) = P(X = x, Y = y)
\]
for all pairs of phantom numerical values \((x, y)\) that \( X \) and \( Y \) can take; otherwise it equals zero. (Here and elsewhere, we will use the abbreviated notation \( P(X = x, Y = y) \) instead of the more precise notation \( P((X = x) \cap (Y = y)) \).)

The joint p. m. f. determines the probability of any event that can be specified in terms of the p. r. v.'s \( X \) and \( Y \). For example, if \( A \) is the set of all pairs \((x, y)\) that have a certain property, then
\[
P((x, y) \in A) = \sum_{(x, y) \in A} p_{X,Y}(x, y).
\]

In fact, as in classical theory, we can calculate the p. m. f.'s of \( X \) and \( Y \) by using the formulas
\[
p_X(x) = \sum_y p_{X,Y}(x, y), \quad p_Y(y) = \sum_x p_{X,Y}(x, y),
\]
where \( x \) and \( y \) range respectively over all the phantom values of \( X \) and \( Y \).

**Definition 4.1.** Two discrete p. r. v.'s \( X \) and \( Y \) are said to be independent if
\[
p_{X,Y}(x, y) = p_X(x) p_Y(y), \quad \text{for all } x, y.
\]
We say that two continuous p. r. v.’s associated with a common experiment are **jointly continuous**, and can be described in terms of a joint p. d. f. $f_{X,Y}$, if $f_{X,Y}$ is a continuous function whose real component is nonnegative and that satisfies

$$
P((X, Y) \in B) = \int_B f_{X,Y}(x,y) dx dy
$$

for every subset $B$ of

$$
\gamma_X \times \gamma_Y = \{(x, y) : x \in \gamma_X, y \in \gamma_Y\},
$$

where $\gamma_X$ and $\gamma_Y$ are as in Equation (3.7).

Accordingly, given $z_1, z_2, w_1, w_2 \in \mathcal{P}H$, we define

$$
S_X = \{x \in \gamma_X : z_1 \preceq_{wk} x \preceq_{wk} z_2\}, \quad S_Y = \{y \in \gamma_X : w_1 \preceq_{wk} y \preceq_{wk} w_2\},
$$

and have

$$
P(z_1 \preceq_{wk} X \preceq_{wk} z_2, w_1 \preceq_{wk} Y \preceq_{wk} w_2) = \int_{S_Y} \int_{S_X} f_{X,Y}(x,y) dx dy.
$$

(Note that when $S_X$ or $S_Y$ are not connected, the integral decomposes into a sum of integrals, assumed finitely many.)

Furthermore, by letting $B$ in Equation (4.2) be the entire set $\gamma_X \times \gamma_Y$, we obtain the normalization property

$$
P((X, Y) \in B) = \int_{\gamma_Y} \int_{\gamma_X} f_{X,Y}(x,y) dx dy = 1.
$$

As before, when $S_X$ and $S_Y$ are subpaths of $\gamma_X$ and $\gamma_Y$, respectively, we can use the orders $\leq_t$ and $\leq_s$ induced on $\gamma_X$ and $\gamma_Y$ by their parametrization, respectively, together with the integral form (1.23), and write:

$$
P(x_1 \leq_t X \leq_t x_2, y_1 \leq_s Y \leq_s y_2) = \int_{\tau(y_2)}^{\tau(y_1)} \int_{\tau(x_1)}^{\tau(x_2)} f_{X,Y}(\gamma_X(t),\gamma_Y(s)) \gamma_X'(t)\gamma_Y'(s) dt ds.
$$

for $x_1, x_2 \in \gamma_X, y_1, y_2 \in \gamma_Y$.

The **marginal** p. d. f.’s $f_X$ and $f_Y$ of $X$ and $Y$, respectively, are given by:

$$
f_X(x) = \int_{\gamma_Y} f_{X,Y}(x,y) dy, \quad \text{and} \quad f_Y(y) = \int_{\gamma_X} f_{X,Y}(x,y) dx.
$$

In full analogy with the discrete case, we say that two continuous p. r. v.’s $X$ and $Y$ are **independent** if their joint p. d. f. is the product of their marginal p. d. f.’s:

$$
f_{X,Y}(x,y) = f_X(x)f_Y(y), \quad \text{for all } x,y.
$$

By simple computation one can verify that:

**Properties 4.2.** If $X$ and $Y$ are independent p. r. v.’s then:

1. The p. r. v.’s $g(X)$ and $h(Y)$ are independent, for any functions $g$ and $h$,
2. $E[z_1 X + z_2 Y + z_3] = z_1 E[X] + z_2 E[Y] + z_3$, for $z_i \in \mathcal{P}H$,
3. $E[XY] = E[X]E[Y]$, and more generally $E[g(X)h(Y)] = E[g(X)]E[h(Y)]$,
4. $\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y]$.

These properties can be verified easily by direct computation or by using the relaxation property admitted by $E[\cdot]$ and $\text{Var}[\cdot]$ and validation of these property for the standard (real) cases.

### 4.1. Covariance and correlation.

The **covariance**, denoted by $\text{Cov}[X,Y]$, of two p. r. v.’s $X$ and $Y$ is defined as

$$
\text{Cov}[X,Y] = E\left[ (X - E[X])(Y - E[Y]) \right].
$$

The p. r. v.’s $X$ and $Y$ are said to be **uncorrelated** if $\text{Cov}[X,Y] = 0$. When $\text{Cov}_{re}[X,Y] = 0$ we say that $X$ and $Y$ are **real uncorrelated**, and if $\text{Cov}_{ph}[X,Y] = 0$ we say that $X$ and $Y$ are **phantomly uncorrelated**.

We let

$$
\text{Cov}[\tilde{X}, \tilde{Y}] = E\left[ (\tilde{X} - E[\tilde{X}])(\tilde{Y} - E[\tilde{Y}]) \right],
$$

and call it the (real) **reduced covariance** of $X$ and $Y$, where expectations are computed with respect to the reductions of $X$ and the probability measure.
Let \( U = X - E[X] \) and \( V = Y - E[Y] \), then
\[
\text{Cov}[X, Y] = E[U_{re}V_{re} + \varphi(U\tilde{V} - U_{re}V_{re})],
\]
cf. Equation (1.13), which by Equation (3.19) is
\[
\text{Cov}[X, Y] = E[U_{re}V_{re}] + \varphi(E[U\tilde{V}] - E[U_{re}V_{re}]).
\]
In other words,
\[
(4.4) \quad \text{Cov}[X, Y] = \text{Cov}[X_{re}, Y_{re}] + \varphi(\text{Cov}[\tilde{X}, \tilde{Y}] - \text{Cov}[X_{re}, Y_{re}]),
\]
that is the phantom covariance admits the realization property.

Roughly speaking, positive or negative parts (cf. Definition 1.9) of covariance indicate that the values of \( X - E[X] \) and \( Y - E[Y] \) obtained in a single experiment tend to have the same or the opposite sign, respectively. Thus, the signs of the real and the phantom term of the covariance provide an important qualitative indicator of the relation between the real components and the phantom components of \( X \) and \( Y \). If \( X \) and \( Y \) are independent, then
\[
(4.5) \quad \rho(X, Y) = \frac{\text{Cov}[X, Y]}{\sqrt{\text{Var}[X] \text{Var}[Y]}}.
\]
This maybe viewed as a normalized version of the phantom covariance \( \text{Cov}[X, Y] \), and as the computation below shows, the real term of \( \rho(X, Y) \) ranges from \(-1\) to \(1\).

Using Equation (3.23) and Equation (1.14), together with the realization properties of the square root (1.16), Equation (4.5) receives the familiar form:
\[
\rho(X, Y) = \frac{\text{Cov}[X_{re}, Y_{re}] + \varphi(\text{Cov}[\tilde{X}, \tilde{Y}] - \text{Cov}[X_{re}, Y_{re}])}{\sqrt{(\text{Var}[X_{re}] + \varphi(\text{Var}[X] - \text{Var}[X_{re}]))(\text{Var}[Y_{re}] + \varphi(\text{Var}[Y] - \text{Var}[Y_{re}]))}}.
\]
Therefore,
\[
(4.6) \quad \rho(X, Y) = \rho(X_{re}, Y_{re}) + \varphi\left(\rho(\tilde{X}, \tilde{Y}) - \rho(X_{re}, Y_{re})\right),
\]
which is the realization property for phantom covariance.

Let \( \tilde{\Lambda} \) be the set
\[
(4.7) \quad \tilde{\Lambda} = \{ z \in \mathbb{H} | a \in [-1, 1], -(1 + a) \leq b \leq 1 - a \},
\]
i.e. it is the pointwise product \( 2\Lambda - 1 \). We write \( \tilde{\Lambda}_{(+, +)} \) for the subset of \( \tilde{\Lambda} \) consisting of all phantom points whose real and phantom terms are positive; \( \tilde{\Lambda}_{(+, -)} \), \( \tilde{\Lambda}_{(-, +)} \), and \( \tilde{\Lambda}_{(-, -)} \) are defined respectively according to the positivity signs of the real term and the phantom term of their points.

**Proposition 4.3.** Given any two p. r. v.’s \( X \) and \( Y \), then \( \rho(X, Y) \in \tilde{\Lambda} \) as defined in Equation (4.7).

**Proof.** Using classical theory, both \( \rho(\tilde{X}, \tilde{Y}) \) and \( \rho(X_{re}, Y_{re}) \) range from \(-1\) to \(1\); the proof is completed by Equation (4.6). \(\square\)

If \( \rho \in \tilde{\Lambda}_{(+, +)} \) (or \( \rho \in \tilde{\Lambda}_{(-, -)} \)), then the real and the phantom values of \( x - E[X] \) and \( y - E[Y] \) “tend” to have the same (or opposite, respectively) sign, and the size of \(|\rho|\) provides a normalized measure of the extent to which this is true. In fact, always assuming that \( X \) and \( Y \) have positive variances, it can be shown that
\( \rho = 1 \) (or \( \rho = -1 \)) if and only if there exists a constant positive phantom number \( \alpha \), or negative, respectively, such that
\[
y - E[Y] = \alpha (x - E[X]), \quad \text{for all possible numerical values } (x, y).
\]
When \( \rho \in \tilde{A}_{(+,-)} \), or \( \rho \in \tilde{A}_{(-,+)} \), then the real terms of \( x - E[X] \) and \( y - E[Y] \) “tend” to have the same (or opposite, respectively) sign opposite to that of their phantom terms.

5. Moment generating functions

The moment generating function, written \( \text{m.g.f.} \) for short, of the distribution function of a p.r.v. \( X \) (also referred to as the transform of \( X \)) is a phantom function \( M_X(\zeta) \) of a free phantom parameter \( \zeta \in \mathbb{PH} \), defined by
\[
M_X(\zeta) = E[e^{\zeta X}].
\]
In more detail, the corresponding transform of \( X \) is given by:
\[
M_X(\zeta) = \begin{cases} 
\sum_x e^{\zeta x} p_X(x), & \text{X discrete;} \\
\int_{\gamma_X} e^{\zeta x} f_X(x) dx, & \text{X continuous.}
\end{cases}
\]
Let \( \zeta = \zeta_{re} + \varphi \zeta_{ph} \) and use Equation \( (1.17) \) to write
\[
E[e^{\zeta X}] = E[e^{\zeta_{re} X_{re}} + \varphi (\tilde{X} - \zeta_{re} X_{re})] = E[e^{\zeta_{re} X_{re}} + \varphi \left( e^{\tilde{X}} - e^{\zeta_{re} X_{re}} \right)].
\]
Then, by Equation \( (3.10) \), one has the realization property for m.g.f.
\[
E[e^{\zeta X}] = E[e^{\zeta_{re} X_{re}}] + \varphi \left( E[e^{\tilde{X}}] - E[e^{\zeta_{re} X_{re}}] \right),
\]
and thus
\[
M_X(\zeta) = M_X(\zeta_{re}) + \varphi (M_X(\zeta) - M_X(\zeta_{re})),
\]
where \( M_X(\zeta_{re}) \) and \( M_X(\zeta) \) are standard (real) moment generating functions.

**Theorem 5.1 (Inversion property).** The m.g.f. \( M_X(\zeta) \) completely determines the probability law of the random variable \( X \). In particular, if \( M_X(\zeta) = M_Y(\zeta) \) for all \( \zeta \), then the random variables \( X \) and \( Y \) have the same probability law.

This property is a rather profound mathematical fact that is used frequently in classical probability theory. In light of Equation \( (5.2) \), i.e., the realization property of \( M_X(\zeta) \), this phantom property is derived directly from the known result for the standard (real) m.g.f., applied to each comportment, in classical probability theory \( [3] \).

Transform methods are particularly convenient when dealing with a sum of p.r.v.’s, since it covers addition of independent p.r.v. to multiplication of transforms, as we now show. Let \( X \) and \( Y \) be independent p.r.v.’s, and let \( W = X + Y \). The transform associated with \( W \) is, by definition,
\[
M_W(\zeta) = E[e^{\zeta W}] = E[e^{\zeta (X+Y)}] = E[e^{\zeta X} e^{\zeta Y}],
\]
the last equality is due to Equation \( (1.17) \).

Consider a fixed value of the parameter \( \zeta \in \mathbb{PH} \). Since \( X \) and \( Y \) are independent, \( e^{\zeta X} \) and \( e^{\zeta Y} \) are also independent p.r.v.'s. Hence, the expectation of their product is the product of the expectations, and thus
\[
M_W(\zeta) = M_X(\zeta) M_Y(\zeta).
\]
By the same argument, if \( X_1, \ldots, X_n \) is a collection of independent p.r.v.’s, and \( W = X_1 + \cdots + X_n \), then
\[
M_W(\zeta) = M_{X_1}(\zeta) \cdots M_{X_n}(\zeta).
\]
5.1. Examples of moment generating functions.

Example 5.2 (The transform of a linear function of a random variable). Let $M_x(\zeta)$ be the transform associated with a p. r. v. $X$. Consider a new p. r. v. $Y = uX + v$ for some $u, v \in \mathbb{P}$. We then have

$$M_Y(\zeta) = E[e^{\zeta(uX+v)}] = e^{\zeta v} E[e^{\zeta uX}] = e^{\zeta v} M_x(\zeta u).$$

Example 5.3 (The transform of the binomial). Let $X_1, \ldots, X_n$ be independent Bernoulli p. r. v.’s, cf. Example 3.18, with a common parameter $p$, assigned to probability. Then,

$$M_{X_i}(\zeta) = (1-p)e^{\zeta c} + pe^{\zeta c} = 1 - p + pe^{\zeta c}, \quad \text{for all } i.$$ 

The p. r. v. $Y = X_1 + \cdots + X_n$ is phantom binomial with parameters $n \in \mathbb{N}$ and $p \in \mathbb{P}$. Its transform is given by

$$M_Y(\zeta) = (1-p + pe^{\zeta})^n.$$

Example 5.4 (The sum of independent Poisson random variables is Poisson). Let $X$ and $Y$ be independent Poisson p. r. v.’s with means $\mu_x$ and $\mu_y$, respectively, and let $W = X + Y$. Then,

$$M_x(\zeta) = e^{\mu_x(e^{\zeta} - 1)}, \quad M_y(\zeta) = e^{\mu_y(e^{\zeta} - 1)},$$

and

$$M_w(\zeta) = M_x(\zeta)M_y(\zeta) = e^{\mu_x(e^{\zeta} - 1) + \mu_y(e^{\zeta} - 1)} = e^{(\mu_x + \mu_y)(e^{\zeta} - 1)}.$$ 

Thus, $W$ has the same transform as a Poisson p. r. v. with mean $\mu_x + \mu_y$. By the uniqueness property of transforms, $W$ is Poisson with mean $\mu_x + \mu_y$.

6. LIMIT THEOREMS

6.1. Some useful inequalities. Before getting to probability inequalities, we need further results about the weak order $\prec_{\text{wk}}$ on $\mathbb{P}$, including its relations with the phantom absolute value as defined in Equation (1.13). We recall that $\prec_{\text{wk}}$ assumed satisfying Properties (1.13).

Remark 6.1. The classical relation $z^2 = |z|^2$ does not always hold phantomly; we might have $z^2 \prec_{\text{wk}} |z|^2$ or $z^2 \succ_{\text{wk}} |z|^2$. Note that $|z|^2$ is real while $z^2$ is phantom. It is easy to verify that $z^2 = |z|^2$ holds iff $b = -2a$.

Moreover, from a metric point of view, there are numbers that are “close” in the sense of the weak order $\prec_{\text{wk}}$, but very far in the sense of $| |$: for example assuming $\prec_{\text{wk}}$ is the lexicographic order, a small increasing of $e$ makes $z_1 = e + \varphi \epsilon$ greater than $z_2 = e + \varphi b$, but still $|z_2| > |z_1|$.

The mismatch between $\prec_{\text{wk}}$ and the $| |$, as addressed in Remark 6.1, yields different versions for phantom Markov inequalities, aiming to provide later a phantom version of the Chebyshev inequality.

Proposition 6.2 (Markov phantom inequalities). Given a p. r. v. $X$ that takes only values $\prec_{\text{wk}} 0$. Then

(i) $P(X \prec_{\text{wk}} z) \leq \frac{E[X]}{z}$, for any pseudo positive $z \in \mathbb{P}$,

(ii) $P(|X| \geq |z|) \prec_{\text{wk}} \frac{E[|X|]}{|z|}$, for any $z \in \mathbb{P}$,

(iii) $|P(|X| \geq |z|)| \leq \frac{E[|X|]}{|z|}$, for any $z \in \mathbb{P}$.

Proof. (i) Fix a pseudo positive $z \in \mathbb{P}$ and consider the random variable $Y_z$ defined by

$$Y_z = \begin{cases} 0, & X \prec_{\text{wk}} z; \\ z, & X \succ_{\text{wk}} z. \end{cases}$$

It is seen that the relation $Y_z \prec_{\text{wk}} X$ always holds and therefore, using Properties (1.13) (ii) for sums and products, $E[Y_z] \prec_{\text{wk}} E[X]$. On the other hand, $E[Y_z] = zP(Y_z \prec_{\text{wk}} z) = zP(X \prec_{\text{wk}} z)$, from which we obtain $zP(X \prec_{\text{wk}} z) \prec_{\text{wk}} E[X]$. The proof is then completed by Property (1.13) (ii) for division.

(ii) Apply part (i) to $|X|$ and $|z|$, since both are positive.

(iii) We need to prove that $|E[|Y_z|]| \leq |E[|X|]|$, or equivalently that $0 \leq |E[|X|]|^2 - |E[|Y_z|]|^2$; then the required inequality is obtained by part (ii). We prove the assertion for a discrete p. r. v.; the continuous version is received similarly.

Let $y_z \in \{0, |z|\}$ for the value of $Y_z$ apply to $x \in X$; accordingly $p_x(x) = p_y(y_z)$ for each $x \in X$. We write $x$ and $y_z$ for $|x|$ and $|y_z|$, respectively, assuming both are real nonnegatives. Then, denoting
$p_{X, re}$ and $p_{X, ph}$ the real and the phantom component of $p_X$, respectively, 

$$|E[Y_z]|^2 = E_{re}[|Y_z|^2] + E_{ph}[|Y_z|^2] + |E_{ph}[|Y_z|^2]|/2$$

$$= \sum_{x', x''} y'x' y''x'' p_{X, re}(x') p_{X, ph}(x'') + \sum_{x', x''} y'x' y''x'' p_{X, re}(x') p_{X, ph}(x'')$$

$$+ \sum_{x', x''} y'x' y''x'' p_{X, ph}(x') p_{X, ph}(x'')/2$$

Letting $g_x(x', x'') = p_{X, re}(x') p_{X, ph}(x'') + p_{X, re}(x') p_{X, ph}(x'') + p_{X, ph}(x') p_{X, ph}(x'')/2$, as it is derived from the absolute value, one observes that $g_x(x', x'') \geq 0$.

Putting all together, and considering the difference, we have

$$|E[|X|]|^2 - |E[Y_z]|^2 = \sum_{x', x''} (x'' - y'x' y''x'') g_x(x', x''),$$

in which all components are $\geq 0$. Since $x' \geq y'x'$ and $x'' \geq y''x''$ then $x'x'' - y'x' y''x'' \geq 0$, and thus the sum is $\geq 0$ as desired.

We write $\mu_{|X|}$ and $\sigma^2_{|X|}$ for $E||X||$ and $\text{Var}(||X||)$, respectively, then have the phantom analogously to the Chebyshev inequality.

**Proposition 6.3 (Chebyshev phantom inequality).** If $X$ is a random variable with mean $\mu_{|X|}$ and variance $\sigma^2_{|X|}$, then

$$|\Pr (||X|| - \mu_{|X|} \geq |z|)\} \leq \frac{\sigma^2_{|X|}}{|z|^2}, \text{ for all } z \neq 0.$$

**Proof.** Consider the nonnegative random variable $(||X|| - \mu_{|X|})^2$ and apply the Markov inequality (iii) with $z = |w|^2$ to obtain

$$\Pr ((|X| - \mu_{|X|})^2 \geq |w|^2) \leq \frac{E[(|X| - \mu_{|X|})^2]}{|w|^2}.$$

Since, $(|X| - \mu_{|X|})^2$ is a real nonnegative number, $|(|X| - \mu_{|X|})^2| = (|X| - \mu_{|X|})^2$, and thus

$$\Pr ((|X| - \mu_{|X|})^2 \geq |w|^2) \leq \frac{E[(|X| - \mu_{|X|})^2]}{|w|^2} = \frac{\sigma^2_{|X|}}{|w|^2}.$$

The derivation is completed by observing that the event $(|X| - \mu_{|X|})^2 \geq |w|^2$ is identical to the event $||X|| - \mu_{|X|} \geq |w|$ and

$$\Pr (||X|| - \mu_{|X|} \geq |w|) = \Pr ((|X| - \mu_{|X|})^2 \geq |w|^2) \leq \frac{\sigma^2_{|X|}}{|w|^2}.$$ 

An alternative form of the Chebyshev inequality is obtained by letting $|w| = c\sigma_{|X|}$, where $c$ is a real positive, which yields

$$\Pr (||X|| - \mu_{|X|} \geq c\sigma_{|X|}) = \frac{\sigma^2_{|X|}}{(c\sigma_{|X|})^2} = \frac{1}{c^2}.$$ 

Thus, the probability that a random variable $|X|$ takes a value more than $c$ times the standard deviations away from the mean $\mu_{|X|}$ is at most $1/c^2$.

The Chebyshev inequality is generally more powerful than the Markov inequality (the bounds that it provides are more accurate), because it also makes use of information on the variance of $X$. Still, as usual, the mean and the variance of a random variable are only a rough summary of the properties of its distribution, and we cannot expect the bounds to be close approximations of the exact probabilities.
6.2. The weak law of large numbers. Consider a sequence $X_1, X_2, \ldots$ of independent identically distributed p. r. v.’s, each with mean $\mu$ and variance $\sigma^2$. Let

$$S_n = X_1 + \cdots + X_n$$

be the sum of the first $n$ of them. As in classical theory, phantom limit theorems are mostly concerned with the properties of $S_n$ and related p. r. v.’s, as $n$ becomes very large. In fact, the realization property of phantoms provides the phantom analogues to these theorems in a trivial way.

Because of the independence of $X_i$’s, we have

$$\text{Var}[S_n] = \text{Var}[X_1] + \cdots + \text{Var}[X_n] = n\sigma^2.$$ 

Thus, the distribution of $S_n$ spreads out as $n$ increases, and does not have a meaningful limit. The situation is different if we consider the sample mean

$$M_n = \frac{X_1 + \cdots + X_n}{n} = \frac{S_n}{n},$$

which can also be written as

$$(6.1) \quad M_n = M_{n,\text{re}} + \varphi(M_n - M_{n,\text{re}}).$$

A quick calculation, together with the independence, shows that

$$\text{E}[M_n] = \mu, \quad \text{Var}[M_n] = \frac{\sigma^2}{n}.$$ 

We apply Chebyshev inequality and obtain

$$(6.2) \quad |\mathcal{P}(|M_n - \mu_{|M_n|}| \geq \epsilon)| \leq \frac{\sigma^2}{n \epsilon^2} \quad \text{for any real } \epsilon > 0.$$ 

We observe that for any real fixed $\epsilon > 0$, the right-hand side of this inequality goes to zero as $n$ increases. This form gives one way to approach phantom limit theorems. However, in the sequel, we focus on the way established by the realization property. This means that we consider phantom probability for abstract events, or random variables.

Next we consider the phantom weak law of large numbers, stated below. It turns out that this law remains true even if the $X_i$ have infinite variance, but a much more elaborate argument is needed, which we omit. The only assumption needed is that $\text{E}[X_i]$ is well-defined and finite.

**Theorem 6.4 (The weak law of large numbers (WLLN)).** Let $X_1, X_2, \ldots$ be independent identically distributed p. r. v.’s with mean $\mu$. For every real $\epsilon \geq 0$, we have

$$\mathcal{P}(|M_n - \mu| \geq \epsilon) \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

or equivalently

$$\mathcal{P}(|M_n - \mu| < \epsilon) \rightarrow 1, \quad \text{as } n \rightarrow \infty.$$ 

**Proof.** Recall that $|M_n - \mu| \rightarrow 0$ iff both, $\text{re}(M_n - \mu) \rightarrow 0$ and $\text{ph}(M_n - \mu) \rightarrow 0$, cf. Lemma 1.30, and that $\mathcal{P}_{\text{re}}$ is a standard (real) probability measure for any phantom probability measure $\mathcal{P} = \mathcal{P}_{\text{re}} + \varphi \mathcal{P}_{\text{ph}}$. Then, since $|M_n - \mu| < \epsilon$ is an inequality of real random variables, by the known WLLN for real probabilities, $\mathcal{P}_{\text{re}}(|M_n - \mu| < \epsilon) \rightarrow 1$ as $n \rightarrow \infty$, which means $\mathcal{P}_{\text{ph}}(|M_n - \mu| < \epsilon) \rightarrow 0$, since $\mathcal{P}$ is a phantom probability measure.

As in classical theory, the phantom WLLN states that for a large $n$, the “bulk” of the distribution of $M_n$ is concentrated near $\mu$. That is, if we consider a neighborhood around $\mu$, which here is 2-dimensional, then there is a high probability that $M_n$ will fall in that neighborhood; as $n \rightarrow \infty$, this probability converges to 1. Of course, if $\epsilon$ is very small, we may have to wait longer (i.e., need a larger value of $n$) before we can assert that $M_n$ is highly likely to fall in that neighborhood.

**Corollary 6.5.** Let $X_1, X_2, \ldots$ be independent identically distributed p. r. v.’s with mean $\mu$. For every real $\epsilon \geq 0$, we have $|\mathcal{P}(|M_n - \mu| < \epsilon)| \rightarrow 1$, as $n \rightarrow \infty$. 

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6.3. The central limit theorem. We can interpret the WLLN as stating that $M_n$ converges to $\mu$. However, since $M_1, M_2, \ldots$ is a sequence of phantom random variables, not a sequence of phantom numbers, the meaning of convergence, in the phantom sense, has to be precise. A particular definition is provided below. To facilitate the comparison with the ordinary notion of convergence, we also include the definition of the latter.

**Definition 6.6.** Let $X_1, X_2, \ldots$ be a sequence of p.r.v.’s (not necessarily independent), and let $z$ be a phantom number. We say that the sequence $X_n$ converges to $z$ in probability, if for every real $\epsilon > 0$, we have

$$\lim_{n \to \infty} P( |X_n - z| \geq \epsilon ) = 0,$$

or equivalently, for every real $\delta > 0$ and for every real $\epsilon > 0$, there exists some $n_0$ such that

$$|P( |X_n - z| \geq \epsilon )| \leq \delta,$$

for all $n > n_0$.

According to the weak law of large numbers, the distribution of the sample mean $M_n$ is increasingly concentrated in the near vicinity of the true mean $\mu$. In particular, its variance tends to zero. On the other hand, the variance of the sum $S_n = X_1 + \cdots + X_n = nM_n$ is unbounded, and the distribution of $S_n$ cannot be said to converge to anything meaningful.

An intermediate view is obtained by considering the deviation $S_n - n\mu$ of $S_n$ from its mean $n\mu$, and scaling it by a (real) factor proportional to $1/\sqrt{n}$. What is special about this particular scaling is that it keeps the variance, even though it is phantom, at a constant level. The central limit theorem asserts that the distribution of this scaled phantom random variable approaches a normal phantom distribution.

More specifically, let $X_1, X_2, \ldots$ be a sequence of independent identically distributed p.r.v.’s with mean $\mu$ and variance $\sigma^2$. We define

$$W_n = \frac{S_n - n\mu}{\sqrt{n}} = \frac{X_1 + \cdots + X_n - n\mu}{\sqrt{n}}. \tag{6.3}$$

An easy calculation yields:

$$E[W_n] = \frac{E[X_1 + \cdots + X_n] - n\mu}{\sqrt{n}} = 0,$$

and

$$\text{Var}[W_n] = \frac{\text{Var}[X_1 + \cdots + X_n]}{\sqrt{n}} = \frac{\sigma^2}{\sqrt{n}} = 1.$$  

**Theorem 6.7 (The phantom central limit theorem).** Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed p.r.v.’s with common mean $\mu$ and a finite variance $\sigma^2$, and let $W_n$ be defined as in Equation (6.3). Then, the c.p.d.f. of $W_n$ converges to the standard normal c.p.d.f.; that is, for a given phantom value $z \in \mathcal{PH}$,

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_S \frac{1}{w} e^{-w^2/2} dw,$$

with $S = \{ w \in \gamma_{nW_n} : w \preceq_{wk} z \}$ assumed piecewise continuous and differentiable, in the sense that

$$\lim_{n \to \infty} P(W_n \preceq_{wk} z) = \Phi(z), \quad \text{for every } z \in \mathcal{PH}.$$  

*Proof.* The proof is established on the standard central limit theorem, known for real distributions, cf. [3] [4].

We also use the fact that if $z_{re} \to z_{0, re}$ and $z_{ph} \to z_{0, ph}$ as reals, then $z \to z_0$, and the properties of the standard phantom normal distribution are as addressed in Proposition 6.2.

By phantom computations, that are already familiar to the reader, we have

$$W_n = \frac{S_{n, re} - n\mu_{re}}{\sigma_{re}\sqrt{n}} + \varphi \left( \frac{\hat{S}_{n, re} - n\hat{\mu}}{\sigma_{re}\sqrt{n}} - \frac{S_{n, re} - n\mu_{re}}{\sigma_{re}\sqrt{n}} \right). \tag{6.4}$$

Suppose $z \in W_n$, then by the classical central limit theorem, each component converges to the standard normal cumulative distribution function, and thus using the realization property of Equation 6.20, we get the desired.

When $z \notin W_n$, apply the same argument to $\xi_{W_n}(z)$, cf. Equation 3.30, for which $\Phi(z) = \Phi(\xi_{W_n}(z))$ by definition.  

\qed
The central phantom limit theorem is surprisingly general, maybe even more general than the known classical one, which is a private case of the phantom theorem. (Note that here the integration is performed along a path.) Besides independence, and the implicit assumption that the mean and variance are well-defined and finite, it places no other requirement on the distribution of the \( X_i \), even though they are phantoms, which could be discrete, continuous, or mixed random variables.

This is of tremendous importance for several reasons, both conceptual and practical. On the conceptual side, it indicates that the sum of a large number of independent p. r. v.’s is approximately phantom normal. As such, it applies to many situations in which a random effect is the sum of a large number of small but independent random factors. Noise in many natural or engineered systems has this property.

In a wide array of contexts, it has been found empirically that the statistics of noise are well-described by (real) normal distributions, and the central limit theorem provides a convincing explanation of this phenomenon. Here, we add another argument, recorded by the phantom term which might provide more information about the behavior of the noise.

On the practical side, the phantom central limit theorem eliminates the need for detailed probabilistic models and for tedious manipulations of p. m. f.'s and p. d. f.'s. Rather, it reduces all the computations to a real familiar framework, and allows the calculation of specific probabilities by simply referring to the table of the standard normal distribution. Furthermore, these calculations only require knowledge about the phantom means and phantom variances.

6.4. The strong law of large numbers.

**Theorem 6.8 (The strong law of large numbers (SLLN)).** Let \( X_1, X_2, \ldots \) be a sequence of independent identically distributed p. r. v.’s with mean \( \mu \). Then, the sequence of sample means \( M_n = (X_1 + \cdots + X_n)/n \) converges to \( \mu \), in the sense that

\[
P \left( \lim_{n \to \infty} \frac{X_1 + \cdots + X_n}{n} = \mu \right) = 1.
\]

**Proof.** Using the same argument as in the proof of Theorem 6.4, \( P_{re} \) is a standard probability measure. By the classical SLLN, \( P_{re} \left( \lim_{n \to \infty} \frac{X_1 + \cdots + X_n}{n} = \mu \right) = 1 \), and thus, since \( P \) is a phantom probability measure, \( P_{ph} \left( \lim_{n \to \infty} \frac{X_1 + \cdots + X_n}{n} = \mu \right) = 0 \). (Note that, for this purpose, the fact that the random variable may take phantom values does not play a role; equivalently, the \( X_i \) can be viewed as random variables that take values in \( \mathbb{R}^2 \).) \( \square \)

Consider a sequence of p. r. v.’s, \( X_1, X_2, \ldots \), (not necessarily independent) associated with the same probability model. Let \( \omega \) be a phantom number. We say that \( X_n \) converges to \( \omega_0 \) with probability 1 (or almost surely) if

\[
P \left( \lim_{n \to \infty} X_n = \omega_0 \right) = 1.
\]

In order to interpret the SSLN, one needs to use probabilistic phantom models in terms of sample spaces. The contemplated experiment is infinitely long and generates experimental values for each one of the p. r. v.’s in the sequence \( X_1, X_2, \ldots \). Thus, one should rather think of the sample space \( \Omega \) as a set of infinite sequences \( \omega = (x_1, x_2, \ldots) \) of phantom numbers: any such sequence is a possible outcome of the experiment. Let us now define the subset \( A \) of \( \Omega \) consisting of those sequences \( (x_1, x_2, \ldots) \) whose long-term average is \( \mu \), i.e.,

\[
(x_1, x_2, \ldots) \in A \iff \lim_{n \to \infty} \frac{x_1 + \cdots + x_n}{n} = \mu.
\]

The SSLN states that most of the phantom probabilities are concentrated on this particular subset of \( \Omega \). Equivalently, the collection of outcomes that do not belong to \( A \) (infinitesimal sequences whose long-term average \( \neq \mu \)) has probability zero.

This means that the initial distortions of the probabilities become meaningless as \( n \to \infty \), as well as their phantom terms. (The latter have a special meaning when dealing with Markov chains and stochastic processes, cf. [9].) Moreover, in the long term, the contribution of the phantom term lessens and tends to zero.

The difference between the weak and the strong law is subtle and deserves close scrutiny. The weak law states that the probability \( P(|M_n - \mu| \geq \epsilon) \) of a significant deviation of \( M_n \) from \( \mu \) goes to zero as \( n \to \infty \). Still, for any finite \( n \), this probability can be positive and it is conceivable that once in a while, even if infrequently, \( M_n \) deviates significantly from \( \mu \). The weak law provides no conclusive information on the number of such
deviations, but the strong law does. According to the strong law, and with probability 1, \( M_n \) converges to \( \mu \). This implies that for any given \( \epsilon > 0 \), the difference \( |M_n - \mu| \) will exceed \( \epsilon \) only a finite number of times.

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