THE MASTER EQUATION OF
2D STRING THEORY

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Abstract

A general method is presented for deriving on-shell Ward-identities in
(2D) string theory. It is shown that all tree-level Ward identities can be
summarized in a single quadratic differential equation for the generating
function of all amplitudes. This result is extended to loop amplitudes
and leads to a master equation à la Batalin-Vilkovisky for the complete
partition function.

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1. Introduction and summary

A important feature of 2D string theory is that in addition to the usual physical operators at ghost level $n_{gh} = 2$ there are other BRST-invariant operators at different ghost levels $[1, 2, 3]$. The importance of this fact was emphasized by Witten, who showed that the $n_{gh} = 0$ states constitute a ring, called the ground ring, and that the operators with $n_{gh} = 1$ are associated with symmetries $[4]$. These symmetries were discovered previously in the context of matrix models $[5]$, and were independently considered by Klebanov and Polyakov $[6]$.

In the current literature most attention is given to the states at ghost level 0, 1 and 2, but there are equally many physical states at other ghost levels. In fact, for every physical state $\phi^I$ with $n_{gh} = n_I$ there is an associated ‘anti-state’ at $n_{gh} = 5 - n_I$ given by

$$|\tilde{\phi}^I\rangle = b_0^I |\phi^I_c\rangle$$

where $b_0^I = (b_0 - \bar{b}_0)$ and $\phi^I_c$ is the BRST-invariant state that is conjugate to $\phi^I_I$. In this paper we will use these anti-states to give a systematic derivation of the Ward-identities and discuss the generalization to loop amplitudes. We find that the information about all Ward-identities (and more) can be summarized in a single equation for the generating function $F = -\lambda^2 \log Z$ of all string amplitudes. It takes the form

$$- \lambda^2 \sum_I \frac{\partial^2 F}{\partial g^I \partial \bar{g}_I} + \sum_I \frac{\partial F}{\partial g^I} \frac{\partial F}{\partial \bar{g}_I} = 0$$

where $g^I$ and $\bar{g}_I$ are the couplings for the states and anti-states respectively and $\lambda$ is the string coupling constant. This equation is known in the Batalin-Vilkovisky formalism as the the (quantum) master equation $[7]$.

In section 2 we review the spectrum of physical states of 2D string theory and discuss the role of the corresponding charges in relation with the symmetries and the perturbations of the BRST-charge. The anti-states are introduced in section 3. We find that the structure constants of the symmetry algebra of all charges are given by the $n$-point amplitudes. The Ward-identities are derived in section 4. Finally, in section 5 we write these identities in the form of the master equation, and we discuss some aspects of the Batalin-Vilkovisky formalism.
2. Physical states and (un-)broken symmetries

In aim of this paper is to derive Ward identities associated with (unbroken) symmetries of string theory. Our motivation comes from the recent studies of 2D string theory, which showed that 2D strings have many unbroken symmetries. The structure of the physical spectrum and symmetries of 2D string theory in a flat background has been significantly clarified in the recent work of Witten and Zwiebach [8]. In the first two subsections we will summarize some of their results, in particular the construction of the symmetry charges∗. In section 2.3 we use this to study the perturbations of the BRST-charge.

2.1. The spectrum of 2D string theory

First we summarize the spectrum for the uncompactified string and we restrict our attention to the positive branch of Liouville momenta \( p_L \geq Q_L \) (\( Q_L \) is the background charge). It is known for a long time that the spectrum of physical states contains an infinite set of discrete states at special values of the momenta [9]. These discrete states, which also appear in matrix models [10], have been suggested to have a topological origin [11]. More recently it was discovered that each of the discrete states is accompanied by a state \( O \) at ghost level \( n_{gh} = 0 \) and a pair of states \( J^L \) and \( J^R \) at \( n_{gh} = 1 \) [1]–[4]. Together with the tachyon vertex operators \( V_p \), this gives the following set of physical states

\[
\begin{align*}
 n_{gh} = 0 : & \quad O_{lm} \\
 n_{gh} = 1 : & \quad J_{lm} \\
 n_{gh} = 2 : & \quad V_{lm}, V_p,
\end{align*}
\]  

(2.1)

where \( p \in R \), and \( l \) and \( m \) are the usual \( SU(2) \) isospin labels. These states are obtained by simply taking the product of the left- and right-moving BRST-invariant states. For example for \( l = 0 \), we have

\[
O_{0,0} = 1, \quad J^L_{0,0} = c\partial X, \quad J^R_{0,0} = \overline{c}\partial X, \quad V_{0,0} = c\overline{c}\partial X\overline{\partial}X,
\]  

(2.2)

*I thank E. Witten and B. Zwiebach for explaining these results to me before finishing their manuscript.
where $X$ is the string coordinate. In fact, (2.1) only represents half of the physical states for $p_L \geq Q_L$. By using the Liouville field $\varphi_L$ one can construct new states by

$$W = [Q, \varphi_L V]_{\pm}. \quad (2.3)$$

These states are clearly BRST-invariant, but not BRST-exact since $\varphi_L$ itself is not a good operator. We obtain in this way physical states $W_p$ and $W_{lm}$ at ghost level 3, for example

$$W_{0,0} = (c\partial c\overline{c} + \overline{c}\partial c\overline{c})\partial X\overline{\partial}X. \quad (2.4)$$

In the same way one can construct new states at ghost level 1 and 2 from $O_{lm}$ and $J_{lm}^{L,R}$. The precise details of this construction will not be important for our discussion.

A similar list of operators can be made for the compactified string models: the tachyons carry discrete momenta $(p_L, p_R)$ and there are additional discrete states with quantum numbers. While the tachyon spectrum varies continuously with $R$, one finds that the spectrum of discrete states behaves in a rather chaotic way. Only at the $SU(2)$-point the discrete states are labelled by the $SU(2)_L \times SU(2)_R$ quantum numbers $(l, m_L, m_R)$. When we perturb the radius $R$ away from its self-dual value $R_{SD}$, all discrete states, except those at $m_L = m_R = 0$ disappear, but at rational values of $R/R_{sd}$ some of the discrete states at non-zero $m_L$ and $m_R$ will re-appear. One of the aims of this section is to obtain a better understanding of this phenomenon.

### 2.2. Conserved charges and symmetries

Our following discussion is valid for any background of the 2D string theory, and so we will use a more general notation. For all known backgrounds there are BRST-invariant states at $n_{gh} = 0, 1, 2, 3$ which we denote by $O_\alpha, J_a, V_i$, and $W_A$, respectively. In the subsequent sections we write them collectively as

$$\phi_I = O_\alpha, J_a, V_i, W_A. \quad (2.5)$$

These operators commute when their ghost-number is even, and anti-commute when it is odd. We note that not all operators are factorizable in left and right movers, nor

\(^1\)Here and in the following $[,]_{\pm}$ is an anti-commutator when both operators have odd ghost-number and a commutator otherwise

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do they always have the same left and right ghost number. In this and other respects
2D string theory has similar features as topological field theory, and it is convenient
to borrow some of the formalism of the latter. In particular, one can apply the
descent equations to construct from a dimension zero field $\phi$ with $[Q, \phi]_{\pm} = 0$ an
associated current (=1-forms) $\phi^{(1)}$ and two-form operator $\phi^{(2)}$ through

$$d\phi^{(0)} = [Q, \phi^{(1)}]_{\pm},$$
$$d\phi^{(1)} = [Q, \phi^{(2)}]_{\pm}.$$  (2.6)

Note that this defines $\phi^{(1)}$ and $\phi^{(2)}$ up to BRST-commutators.

There are several ways in which one can derive relations among the different
states. One possibility is to consider the ground ring formed by the zero forms $O_\alpha$
of the ghost number 0 [4, 12]. A second approach, which is the one we will study in
this paper, is to use the charges constructed from the one-forms $\phi^{(1)}$. In this section
we will examine these charges associated with $J_a$ and $V_i$. The contour-integrals of
$J_a^{(1)}$ describe BRST-invariant observables with ghost-number zero

$$Q_a = \frac{1}{2\pi i} \oint J_a^{(1)}. \hspace{1cm} (2.7)$$

It can be shown that these charges commute with $b_0$, and so they generate symme-
tries between physical states [8]

$$Q_a |V_i\rangle = f_{ai}^j |V_j\rangle. \hspace{1cm} (2.8)$$

This equation holds at the level of BRST-cohomology classes, so we ignore any
BRST-trivial contributions. The symmetry charges form a closed algebra

$$[Q_a, Q_b] = f_{abc} Q_c. \hspace{1cm} (2.9)$$

For example for the flat uncompactified background the symmetry algebra is related
to area-preserving diffeomorphisms and takes the form [4, 3]

$$[Q_{lm}, Q_{l'm'}] = ((l + 1)m' - (l' + 1)m)Q_{l+l', m+m'}. \hspace{1cm} (2.10)$$

These symmetries represent the unbroken gauge-symmetries of the 2D string, as will
become clear below.
2.3. The perturbed BRST-charge.

Similarly as for the currents $J_a$ we can also use the 1-forms of the operators $V_i$ to construct charges

$$Q_i = \oint V_i^{(1)}.$$  \hfill (2.11)

These anti-commuting charges have ghost number 1 and map the usual physical operators $V_i$ on to the states at $n_{gh} = 3$

$$Q_i |V_j\rangle = C_{ij}^A |W_A\rangle.$$  \hfill (2.12)

The coefficients $C_{ij}^A$ are directly related to the quadratic $\beta$-function, which indicates that the charges $Q_i$ play an important role in studying the perturbations of the model. As usual the physical perturbations of the action are represented the integrated two-forms $V_i^{(2)}$

$$S' = S + \delta t^i \int V_i^{(2)}.$$  \hfill (2.13)

The perturbed BRST-charge is determined by a simple application of the Noether procedure as follows. Let us perform a BRST-transformation with a coordinate-dependent parameter $\epsilon$ in (2.13). Using the descent equation (2.6) we find

$$\delta_{\text{BRST}} S' = \int \epsilon \, d(j_{\text{BRST}} + \delta t^i V_i^{(1)}).$$  \hfill (2.14)

From this we read off the perturbation of the BRST-current $j_{\text{BRST}}$, from which we construct the perturbed BRST-charge. We find

$$Q' = Q + \delta t^i Q_i.$$  \hfill (2.15)

For example, for a vertex operator of the usual form $V = \bar{c} \Psi$, with $\Psi$ a dimension $(1,1)$ (matter) primary field, the variation of the BRST-charge is

$$\oint (d\bar{c}\Psi + d\bar{c}\Psi).$$

What happens to the symmetries under these perturbations. When the perturbation $\delta t^i V_i$ does not commute with one or more of the charges $Q_a$ the symmetry will be, at least partly, broken. As will be explained in the following section, we have the relation

$$Q' |J_a\rangle = \delta t^i Q_a |V_i\rangle.$$  \hfill (2.16)
The state on the r.h.s. can be identified with the Goldstone mode. When it is non-vanishing we conclude from (2.16) that after the perturbation the symmetry current $J_a$ is no longer physical. At the same time equation (2.16) tells that the Goldstone mode becomes a longitudinal mode, (=BRST-exact). This is the true signature of a spontaneously broken gauge-symmetry. In fact, this is the phenomenon that is responsible for the disappearance and re-appearance of the discrete states.

3. Anti-states and more symmetries

In this paper we will work with vanishing cosmological constant, and so, to get non-vanishing amplitudes, we have to include the states on the negative branch of Liouville momenta. On this branch the non-vanishing BRST-cohomology is at ghost-levels $n_{gh} = 5, 4, 3, 2$. We denote them by

$$\tilde{\phi}^I = \tilde{O}^\alpha, \tilde{J}^a, \tilde{V}^i, \tilde{W}^A. \quad (3.1)$$

They are related to the the states $\phi_I$ given in (2.5) by

$$\tilde{\phi}^I = b_0^{-1} \phi^I_c. \quad (3.2)$$

where $\langle \phi^I_c | \phi_I \rangle = \delta_I^J$. We will refer to $\tilde{\phi}^I$ as the ‘anti-state’ or ‘anti-field’ corresponding to $\phi_I$ because as will become clear they describe the on-shell modes of the space-time anti-fields [13]. In the subsequent sections all fields $\phi_I$ and anti-fields $\tilde{\phi}^I$ will be treated on equal footing.

Again we use the descent equations (2.6) to construct BRST-invariant charges from the fields $\phi_I$

$$Q_I = \frac{1}{2\pi i} \oint \phi^{(1)}_I, \quad (3.3)$$

which except for $Q_a$ have ghost number different from zero. This means that they not really generate physical symmetries, but they do lead to relations among the

* Strictly speaking the space-time anti-fields correspond to states with $n_{gh} \geq 3$, and therefore it may be more appropriate to associate $W^A$ with an anti-field and $\tilde{W}^A$ with a field. Fortunately this distinction will not be essential for our purposes.
as can be seen from (2.6) the charges $Q_I$ are conserved only up to BRST commutators. This fact will become important later on. In a similar way one can construct charges $\tilde{Q}^I$ for the anti-states, but these will not be considered in this section.

Both the states as their anti-states represent ‘good’ operators in the sense that they are annihilated by $b_0$ \[\ddagger\]. Due to this fact all (genus zero) two-point functions vanish, but the three-point functions

\[
\langle \phi_I \phi_J \tilde{\phi}^K \rangle = f_{IJ}^K \tag{3.4}
\]
\[
\langle \phi_I \tilde{\phi}^J \tilde{\phi}^K \rangle = \tilde{f}^{JK} \tag{3.4}
\]

are in general non-vanishing provided the total ghost-number adds up to 6. These three-point functions describe the action of the charges $Q_I$ on states

\[
Q_I |\phi_J \rangle = f_{IJ}^K |\phi_K \rangle, \quad Q_I |\tilde{\phi}^J \rangle = f_{IJ}^K |\tilde{\phi}^J \rangle + \tilde{f}^{JK} |\phi_J \rangle, \tag{3.5}
\]

as well as the (anti-)commutation relations among the charges

\[
[Q_I, Q_J]_\pm = f_{IJ}^K Q_K. \tag{3.6}
\]

Both these relations are valid modulo BRST-commutators. They are derived by some straightforward manipulations involving the $b$-fields, that at the same time show the consistency with the $b_0$-constraint \[\ddagger\]. Notice that the coefficients $f_{IJ}^K$ are (anti-)symmetric in $I$ and $J$. We already made use of that in (2.10).

Requiring that the states $\phi_I$ form a representation of the algebra gives the relations

\[
0 = f_{IJ}^M f_{MK}^L + \text{cycl. in } I, J, K, \tag{3.7}
\]
\[
f_{IJ}^M \tilde{f}_{MK}^L = \tilde{f}_{IJ}^{KM} f_{MK}^L + (\text{anti-})\text{symm. in } I, J \& K, L. \tag{3.8}
\]

An alternative way of arriving at the same relations is to consider Ward-identities. For example the Jacobi identity (3.7) follows from

\[
\langle \int d\phi_I^{(1)} \phi_J \phi_K \tilde{\phi}^L \rangle = 0, \tag{3.9}
\]

\[\ddagger\]I thank R. Dijkgraaf for helpful discussions on this point.
which leads to

\[
\langle (\oint \phi_I^{(1)} \phi_J) \phi_K \bar{\phi}^L \rangle + \langle \phi_J (\oint \phi_I^{(1)} \phi_K) \bar{\phi}^L \rangle + \langle \phi_I \phi_K (\oint \phi_J \bar{\phi}^L) \rangle = 0. \tag{3.10}
\]

Combined with (3.4) and (3.5) this indeed gives (3.7). Notice that it is irrelevant which operator actually played the role of the current.

In a similar way one can study more general Ward-identities with more operator insertions. This was done recently in refs. \cite{13} and \cite{8}, where it was noted that the symmetries can act non-linearly on the vertex operators. The origin of this fact is that the currents \( \phi_I^{(1)} \) are conserved only up to a BRST-derivative, which after ‘partial integration’ produces higher order contact terms. Thus, a Ward-identity is actually a statement about the decoupling of a total BRST-derivative in which none of the operators plays a special role; this is the point of view we will take in the next section.

4. Ward identities

String amplitudes are usually written as integrals of correlation functions of two-forms \( \phi^{(2)} \) on a genus \( g \) surface \( \Sigma \). A more symmetric and for our purpose more convenient way to represent the amplitudes is to use only zero-forms and extra \( b \)-field insertions. In this way we treat the integrations over the positions of the operators on equal footing with the other moduli of surface \( \Sigma \). Thus we write the \( g \)-loop \((r+s)\)-point amplitude as follows

\[
\langle \phi_{I_1} \cdots \phi_{I_r} \bar{\phi}^{J_1} \cdots \bar{\phi}^{J_s} \rangle_g = \int_{\mathcal{M}_{g,r+s}} \left\langle \prod_{i \in I} \phi_{I_i} \prod_{j \in J} \bar{\phi}^{J_j} \prod_{\alpha=1}^{3g-3+r+s} \left( \mu_\alpha, b \right) \left( \varpi_\alpha, \overline{b} \right) \right\rangle, \tag{4.1}
\]

where \( I = \{1, \ldots, r\} \) and \( J = \{1, \ldots, s\} \). The integral is over the moduli space \( \mathcal{M}_{g,r+s} \) of genus \( g \) surfaces with \( r+s \) (marked) punctures, and \( \mu_\alpha \) represent the Beltrami-differentials, which are integrated over the surface together with the \( b \)-fields. The double brackets on the r.h.s. indicate the correlation function in the 2d field theory of the ghost and matter fields. By ghost counting the combined
ghost-number of the states and anti-states must add up to \(2r+2s\) to get a non-zero result.

The requirement that unphysical modes in string theory decouple is usually stated as the condition that amplitudes containing BRST-trivial states vanish, i.e.

\[
\langle [Q, \Lambda] \pm \prod_i \phi_i \prod_j \tilde{\phi}^j \rangle = 0. \tag{4.2}
\]

Here the amplitude is defined as in (1.1). In the following we assume that (4.2) holds for all dimension zero operators \(\Lambda\) that satisfy \(b_0^\pm |\Lambda\rangle = 0\).

The naive reasoning for why (4.2) is true is as follows. One ‘partially integrates’ the BRST-derivative and because \(Q\) commutes with all \(\phi_I\) and \(\tilde{\phi}^J\) it only acts on the \(b\)-insertions. This produces an insertion of the stress-tensor \(T = [Q, b]_+\). Thus by a standard argument, the result is a total derivative on moduli space and therefore naively vanishes. In this naive argument we ignored the possible contributions that come from the boundary components of moduli space. It is a non-trivial fact that these contributions cancel, and this is what leads to the Ward-identities. We will analyse the above manipulations for a particularly convenient choice for \(\Lambda\) namely \(\Lambda = 1\). For this choice (4.2) is clearly true because \([Q, 1] = 0\), and secondly, as we will see the different contributions at the boundary only involve physical states.

First we consider the boundary components of \(\mathcal{M}_{g,r+s}\) where the surface splits into two surfaces \(\Sigma_1\) and \(\Sigma_2\) with genera \(0 \leq g_1, g_2 \leq g\), \(g_1 + g_2 = g\), and the operators are distributed over \(\Sigma_1\) and \(\Sigma_2\) so that \(r_1 + r_2 = r\) and \(s_1 + s_2 = s\); in fig. 1 we have drawn this situation for a one-loop 4-point amplitude. In going to the boundary of \(\mathcal{M}_{g,r+s}\) we take the length of one of the cycles to zero and consequently only states with scaling dimension zero give a finite contribution in the factorization expansion. The remaining moduli can be recognized as the moduli of \(\Sigma_1\) and \(\Sigma_2\), except for one ‘angular’ parameter being the ‘twist’ associated with the pinched cycle. The integration over this modulus, together with the corresponding \(b\)-ghost insertion, yields the BRST-invariant operator

\[
\Pi = (b_0 - \bar{b}_0) \delta(L_0 - \bar{L}_0). \tag{4.3}
\]

There exists a basis of the Hilbert space \(\mathcal{H}\) so that the basis elements that are not annihilated by \(Q\) are orthogonal to the physical states and conjugate to the
BRST-trivial states. We can use this fact to write the operator $\Pi$ as

$$\Pi = \sum_{K} |\tilde{\phi}^K\rangle\langle \phi^K| + \sum_{K} |\phi^K\rangle\langle \tilde{\phi}^K| + \text{ a } Q\text{-commutator.}$$

(4.4)

where the sum is only over the physical states. The $Q$-commutator term in (4.4) can be dropped because of the assumption that (4.2) holds for all proper zero-forms $\Lambda$. We conclude therefore that only physical states give a contribution to the Ward-identity.

For genus zero the boundary components of $\mathcal{M}_{0,r+s}$ are all of the type described above and are labeled by the way the operators are distributed over the two surfaces $\Sigma_1$ and $\Sigma_2$. Summing the different contributions leads to the following tree-level Ward identities

$$\sum_{K} \sum_{I_1 \cup I_2 = I, J_1 \cup J_2 = J} \left< \prod_{i \in I_1} \phi_{I_i} \prod_{j \in J_1} \tilde{\phi}_{J_j} \phi_{K} \right>_{0} \left< \prod_{i' \in I_2} \phi_{I_i'} \prod_{j' \in J_2} \tilde{\phi}_{J_j'} \right>_{0} = 0,$n \hspace{1cm} (4.5)

where the sum is over all subdivisions $I_1, I_2$ and $J_1, J_2$ of the sets $\mathcal{I}$ and $\mathcal{J}$. As a special case this contains the Jacobi identity (3.7) and the relation (3.8). Another interesting case is when we have one 'current' $J_a$ and for the rest operators with $n_{gh} = 2$. When the symmetry is linearly realized, only the three-point amplitudes of the current $J_a$ are non-vanishing. It is easily seen that in this case (4.5) reduces to the more familiar form of a linear Ward-identity.

For higher genus surfaces there is one other type of boundary component $\mathcal{M}_{g,r+s}$, namely when one of the handles is pinched (see fig. 2). Following a similar argument as sketched above we find that in this case the contribution to the Ward-identity can be expressed as a (graded) trace of the operator $\Pi$ times a BRST invariant operator that represents the rest of the surface. Again by using (4.4) we can, at least formally, reduce this trace to a sum over physical states. This leads to the following extension of (4.5) to loop amplitudes

$$0 = \sum_{K} \left< \phi_{K} \tilde{\phi}^{K} \prod_{i \in \mathcal{I}} \phi_{I_i} \prod_{j \in \mathcal{J}} \tilde{\phi}_{J_j} \right>_{g-1} +$$

$$+ \sum_{K} \sum_{g_1 + g_2 = g} \left< \prod_{i \in \mathcal{I}_1} \phi_{I_i} \prod_{j \in \mathcal{J}_1} \tilde{\phi}_{J_j} \phi_{K} \right>_{g_1} \left< \prod_{i' \in \mathcal{I}_2} \phi_{I_{i'}} \prod_{j' \in \mathcal{J}_2} \tilde{\phi}_{J_{j'}} \right>_{g_2},$$

(4.6)

*After we finished our calculations we noticed that this relation for the special case $s = 1$ also appeared in [8], where a similar derivation was given and it was noted that it defines a so-called homotopy lie-algebra.
where the first term comes from the degeneration of a handle.

How much can we learn from these Ward-identities? Are they sufficient, for example, to compute all string amplitude starting from, say, the genus zero three and four-point functions? The answer to this question of course depends on how large the symmetry group is, and whether it acts non-trivially on all the physical operators. For the uncompactified flat background the recent results of Klebanov [13], who reproduced all the tachyon amplitudes using Ward-identities, are an indication that this might be true. However, as will be explained in the next section, the Ward-identities (4.5) and (4.6) have a large gauge-invariance, and therefore to get a unique solution one in general needs more information.

An interesting special case of (4.6) is

$$f_{IJ}^K \langle \phi_K \rangle_1 = - \sum_K \langle \phi_I \phi_J \phi_K \tilde{\phi}^K \rangle_0.$$  \hspace{1cm} (4.7)

which shows that the one-loop tadpoles do not have to be invariant under the symmetry when the gauge-group acts non-linearly at tree-level. It also indicates that the classical symmetries are modified by the quantum corrections.

5. The Master Equation

In this section we will recast the Ward-identities in a more compact form as constraints on the partition function. Let us introduce coupling constants $g^I$ and $\tilde{g}_I$ for the states and anti-states and consider the perturbed action

$$S' = S + \sum_I g^I \int \phi_I + \sum_I \tilde{g}_I \int \tilde{\phi}^I.$$  \hspace{1cm} (5.1)

Depending on whether the ghost number of $\phi_I$ is even or odd the couplings $g^I$ are commuting or anti-commuting variables. The couplings $\tilde{g}_I$ have the opposite ‘statistics’ from their anti-partners $g^I$. More explicitly,

$$g^I = s^\alpha, c^a, t^i, d^A$$

$$\tilde{g}_I = \tilde{s}_\alpha, \tilde{c}_a, \tilde{t}_i, \tilde{d}_A$$  \hspace{1cm} (5.2)
represent the coupling for the states (2.5) and anti-states (3.1) respectively. Here $s^a, t^i, \bar{c}_a$, and $\bar{d}_A$ are even and $c^a, d^A, \bar{s}_a$ and $\bar{t}_i$ are odd variables. We note the couplings $c^a$ are in one to one correspondence with the (gauge-)symmetries generated by $J_a$ and can be identified with the on-shell modes of the space-time ghosts. The other couplings have similar space-time interpretations, for example $s^a$ are the ghost for ghosts, etc. In fact, one can define a space-time anti-ghost number $n_{ag}$ by

$$n_{ag}(g^I) = n_{gh}(\phi_I) - 2,$$
$$n_{ag}(\bar{g}_I) = n_{gh}((\bar{\phi})^I) - 2,$$ (5.3)

so that the couplings $t^i$ and $\bar{d}_A$ have $n_{ag} = 0$.

Our aim is to write the collection of all Ward-identities (4.5) and (4.6) as a differential equation for the free energy $F$ as a function of the couplings $g^I$ and $\bar{g}_I$. The free energy $F$ is the generating function for all connected string amplitudes

$$\langle \phi_{I_1} \ldots \phi_{I_r} \bar{\phi}_{J_1} \ldots \bar{\phi}_{J_s} \rangle = - \frac{\partial^{r+s} F(g^I, \bar{g}_J)}{\partial g^{I_1} \ldots \partial \bar{g}_{J_s}} |_{g^I = \bar{g}_J = 0}$$ (5.4)

and has a genus-expansion

$$F = \sum_g \lambda^{2g} F_g$$ (5.5)

in terms of the string coupling constant $\lambda$. It easily seen that $F$ is an even function and has total antighost number $n_{ag} = 0$. In this section $\langle \cdots \rangle$ denotes the full connected amplitude including the loop contributions, unless otherwise indicated.

5.1. The classical master action

When we turn on the couplings ghost-number is no longer conserved, and the operators $\phi_I$ and $\bar{\phi}^J$ can acquire non-vanishing one-point functions on the sphere. The tree-level Ward-identities (4.5) can be integrated several times to obtain an expression in terms of these one-point functions

$$\sum_K \langle \phi_K \rangle_0 \langle \bar{\phi}^K \rangle_0 = 0,$$ (5.6)

which holds for arbitrary values of couplings. In fact, this single equation contains the same information as the infinite set of identities (1.5), which simply follow by
differentiating \((5.6)\) \(r\) times with respect to the couplings \(g^I\), and \(s\) times with respect to \(\tilde{g}_J\). The ‘statistics’ of the couplings is important to get the correct signs for the different terms.

To write \((5.6)\) in terms of the tree level free energy \(F_0\) we introduce the so-called anti-bracket. It is defined by

\[
\{A, B\} = \sum_I \left( \frac{\partial A}{\partial \tilde{g}^I} \frac{\partial B}{\partial g^I} \pm \frac{\partial B}{\partial \tilde{g}^I} \frac{\partial A}{\partial g^I} \right),
\]

where the plus-sign occurs only when both \(A\) and \(B\) are commuting. Equivalently we may define the anti-bracket through

\[
\{\tilde{g}_J, g^I\} = \delta^I_J, \quad \{g^I, g^J\} = \{\tilde{g}_I, \tilde{g}_J\} = 0,
\]

supplemented with the rule that the anti-bracket acts as a derivation. So in the anti-bracket the fields \(g^I\) are conjugate to the anti-fields \(\tilde{g}_I\), but in contrast with the Poisson-bracket conjugate variables have opposite ‘statistics’.

Using the anti-bracket the relation \((5.6)\) can be compactly written as

\[
\{F_0, F_0\} = 0,
\]

which summarizes all tree level Ward identities. Equation \((5.9)\) is familiar from the Batalin-Vilkovisky formalism, where it is known as the classical master equation. This is no coincidence: the BV-formalism is developed precisely to give a general framework for writing Ward-identities in arbitrary gauge theories. The free energy \(F_0\) plays the role of the ‘classical’ master action for all fields and anti-fields and at the same time it is the generator of BRST-transformations. Thus, equation \((5.9)\) expresses the invariance of the ‘action’ \(F_0\) under the BRST-transformations

\[
\delta g^I = \epsilon \{F_0, g^I\} = \epsilon \frac{\partial F_0}{\partial \tilde{g}^I},
\]

\[
\delta \tilde{g}_J = \epsilon \{F_0, \tilde{g}_J\} = \pm \epsilon \frac{\partial F_0}{\partial g^J},
\]

as well as the nilpotency of the BRST charge. To make this interpretation a bit more apparent we note that, in lowest order, \(F_0\) is given by

\[
F_0 = c^a t^i f_{ai} \tilde{t}_j + \frac{1}{2} c^a \tilde{c}^b f_{ab} \tilde{c} c + \ldots - \frac{1}{2} t^i \tilde{t}^j C_{ij} A \tilde{d}_A + \ldots
\]

\[(5.11)\]
In the first two terms we recognize the conventional BRST-charge: the $c^a$ are the modes of the space-time ghost fields and that the coefficients $f_{ab}^j$ and $f_{abc}$ are the generators and structure constants of the physical symmetries. For symmetries that are linearly realized are the only terms involving the ghosts $c^a$, but in general there will be more terms.

Due to the analogy with the BV-formalism it is natural to think of the tree-level free energy $F_0$ as being (part of) the classical space-time action of the 2d string. Indeed, its equation of motion

$$\frac{\partial F_0}{\partial \tilde{d}_A} = -\frac{1}{2} t^i t^j C_{ij} A \ldots = 0$$

(5.12)

coincides in first order with the quadratic $\beta$-function condition, and one may conjecture that the higher order terms reproduce the full $\beta$-functions. This suggest that $F_0$ is closely related to Zamolodchikov’s $c$-function.

5.2. The quantum master equation

The Ward-identities (4.6) for the loop amplitudes have a similar structure, but contain the contributions due the pinching of a handle. In fact, this has a very natural interpretation within the BV-formalism, namely it leads to what is called the quantum master equation. It involves the quadratic differential operator

$$\Delta = \sum_I \frac{\partial^2}{\partial g^I \partial \tilde{g}^I}.$$  

(5.13)

Notice that $\Delta$ is nilpotent, i.e. it satisfies $\Delta^2 = 0$. The quantum master equation reads in terms of complete free energy $F$

$$- \lambda^2 \Delta F + \frac{1}{2} \{ F, F \} = 0.$$  

(5.14)

The second term describes the splitting of the surface in two component while the first term originates from the pinched handle. This becomes more evident if we rewrite it again in terms of the amplitudes. The master equation then becomes

$$\lambda^2 \sum_I \langle \tilde{\phi}_I \phi^I \rangle + \sum_I \langle \tilde{\phi}_I \rangle \langle \phi^I \rangle = 0,$$

(5.15)
which again is valid for arbitrary values of the couplings. The first term in (5.15) can be non-zero even when we put all couplings to zero. In this case also the one-point functions must be non-zero, and consequently the loop corrections to the ‘$\beta$-function’ (5.12) will modify the classical solutions. This phenomenon is known as the ‘Fischler-Susskind mechanism’ [16].

We can even further simplify the master equation by expressing it in the full partition function
\[ \mathcal{Z}(g^I, \tilde{g}_I) = e^{-\frac{1}{\lambda^2} F(g^I, \tilde{g}_I)}. \] (5.16)

It then takes the form
\[ \Delta \mathcal{Z} = 0, \] (5.17)
which looks misleadingly simple but as is clear from our derivation, it contains all the information about the Ward-identities (4.6). The fact that $\Delta^2 = 0$ is a direct consequence of the nilpotency of the BRST-charge $Q$ on the world sheet. We observe that (5.17) is invariant under
\[ \mathcal{Z} \rightarrow \mathcal{Z} + \Delta Y, \] (5.18)
where $Y$ an arbitrary odd function of the couplings with $n_{ag} = 1$. This gauge-invariance is related to the fact that we can change the physical fields $\phi_I$ by total BRST-derivatives of ‘bad’ operators, which do not decouple. At tree level this redundancy corresponds to reparametrizations in the space of couplings $(g^I, \tilde{g}_I)$ that preserve the anti-bracket. Therefore we believe that two partition functions that are related by a gauge transformation of the form (5.18) must be considered to be equivalent.

Although naively the master equation (5.17) does not depend on the background we emphasize that its derivation used a particular fixed background. The partition function is only formally defined as a perturbative expansion in the couplings around $g^I = \tilde{g}_I = 0$. In this point the first and second derivatives of $F$ vanish, but of course this is not true for other values of the couplings. This makes clear that if we compute the partition function perturbatively for some other background it will not produce the same answer. It is possible that the answers are the same after a gauge transformation (5.18), but we do not know whether this is true in general.

Finally, we note that equation (5.17) is quite reminiscent of the Virasoro and $W$-constraints for $c \leq 1$ gravity [17], and also its derivation is analogous to the way the Virasoro constraints were found in topological gravity [18]. An important difference is that the master equation gives only one constraint, and is in itself not sufficient to
fix $Z$, even up to equivalence. It seems natural to expect that the master equation also holds non-perturbatively; in this case it may even tell us something about the nature of the non-perturbative effects [19].

6. Concluding remarks

In this paper we have described a general framework for studying the symmetries of 2D string theory. We realize that our discussion has been rather formal, for example, we implicitly assumed that all amplitudes and all the sums over states are well defined and finite. In practice all kinds of divergences can arise, but we believe that the master equation itself can help in dealing with these divergences. It is useful, therefore, to apply and verify these ideas more explicitly in concrete situations.

An important open problem is how to relate different backgrounds. The partition function $Z$ was defined through its perturbative expansion in the coupling constants for the physical states. The main obstacle in defining $Z$ as a function of the background is the fact that the spectrum of physical states varies rather discontinuously. This brings us back to our discussion of section 2.3 where we studied the perturbations of the BRST-charge. We found that the perturbed BRST-charge can be expressed in terms of the symmetry charges as

$$Q' = Q + \sum_I g^I Q_I + \ldots.$$  (6.1)

The non-linear symmetry that is generated by these charges $Q_I$ is encoded in the master equation. This makes clear that by studying the master equation we should also learn something about the perturbations of the physical spectrum.

To make progress in this direction one could try to enlarge the space of coupling constants, by including states that become physical at other values of the couplings. In this way one might be forced to go to the full closed string field theory, but hopefully there is a more tractable intermediate formulation. The idea that such a intermediate formulation may exists has also been put forward in [8]. In this respect it is interesting to note that a BV-equation already exists for closed string
field theory \[20\]. The master equation that we derived in this paper presumably corresponds to the on-shell truncation of the full master equation of string field theory.

We have put our discussion in the context of 2D string theory, because at present it is the only non-trivial example of a string theory with a large symmetry algebra. One can try to apply the same techniques to \( c \leq 1 \) models, or maybe even to \( c \geq 1 \). Unfortunately, the representation theory of the Virasoro algebra with \( c \geq 1 \) indicates that it is not possible to have many discrete states in higher dimensional models. We note, however, that conformal invariance played only a minor role in our derivations. It is quite possible that more general backgrounds exist that are not necessarily based on conformal field theories. Such models can have large symmetry algebras, and then our results would still be applicable.

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FIGURE CAPTIONS:

Figure 1: A torus with 4 marked punctures that degenerates by pinching off a sphere with 2 punctures.

Figure 2: A torus with 4 marked punctures that degenerates by pinching a handle.