Stability of determination of Riemann surface from its DN-map in terms of Teichmüller distance

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Abstract

As is known, the Dirichlet-to-Neumann operator $Λ$ of a Riemannian surface $(M, g)$ determines the surface up to conformal equivalence class $[(M, g)]$. Such classes constitute the Teichmüller space with the distance $d_T$. We show that the determination is continuous: $\|Λ - Λ'\|_{H^1(\partial M) \to L_2(\partial M)} \to 0$ implies $d_T([(M, g)], [(M', g')]) \to 0$.

Key words: electric impedance tomography of surfaces, holomorphic immersions, Dirichlet-to-Neumann map, stability of determination.

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Introduction

Two-dimensional EIT problem

- Let \((M, g)\) be a smooth compact two-dimensional Riemann manifold (a surface) with the smooth boundary \((\Gamma, dl)\), \(g\) the smooth metric tensor. For the sake of simplicity, we assume that \(\Gamma\) is diffeomorphic to a circle, and denote by \(dl\) and \(\nu\) the length element and unit normal on \(\Gamma\). Let \(\Delta_g\) be the Beltrami-Laplace operator on \(M\).

The Dirichlet-to-Neumann operator (DN map) of the surface \((M, g)\) acts on smooth functions on \(\Gamma\) by the rule \(\Lambda f := \partial_\nu u^f|_{\Gamma}\), where \(u^f\) satisfies

\[
\Delta_g u = 0 \quad \text{in} \ M \setminus \Gamma; \quad u = f \quad \text{on} \ \Gamma.
\]

The electric impedance tomography problem (EIT) is to determine the surface from its DN map. Here are some known facts and results.

- Suppose that \((M, g)\) and \((M', g')\) are the surfaces with common boundary \(\partial M = \partial M' = \Gamma\); let \(\Lambda\) and \(\Lambda'\) be their DN maps. We write \((M', g') \sim (M, g)\) if they are conformally equivalent via a diffeomorphism that does not move points of the boundary. By \([(M, g)]\) we denote the equivalence class of \((M, g)\).

In [11] M.Lassas and G.Uhlmann showed that \(\Lambda\) uniquely determines not the metric \(g\) on \(M\) but its conformal class, so that the correspondence \(\Lambda \leftrightarrow [(M, g)]\) is a bijection. Also, a procedure that recovers \(M\) by means of harmonic continuation from the boundary, is provided. In [2], the same fact is established by algebraic version of the boundary control method (BCM) [3]. Recently, it was extended to the case of non-orientable surfaces [4, 5] and surfaces with (unknown) internal holes [11].

The paper [9] by G.Henkin and V.Michel provides a constructive way to recover the surface from its DN map by the use of multidimensional complex analysis technique. Moreover, a characteristic description of the inverse data is given that is the necessary and sufficient conditions on an operator \(\Lambda\) to be the DN map of a surface. In [6] another simpler characterization in the framework of the algebraic BCM is provided.

- The question on stability of determination \(\Lambda \mapsto (M, g)\) can be posed as follows. Let the operators \(\Lambda\) and \(\Lambda'\) that correspond to the surfaces \((M, g)\)

\(^1\) throughout the paper, smooth means \(C^\infty\)-smooth

\(^2\) in the subsequent, the term ‘common boundary’ means that the surface metrics induce the same length element on the boundary.
and \((M', g')\), be close (in a relevant sense). Can one claim that the surfaces are also close (in appropriate sense)? An affirmative answer given in our paper \([7]\) is as follows.

First, as is clarified in \([10]\), to discuss a closeness of the surfaces is reasonable only under assumption that \(M\) and \(M'\) are diffeomorphic, i.e., have the same genus \(m \geq 0\). Accepting this for the rest of the given paper, we deal with the \textit{diffeomorphic} surfaces \((M, g)\) and \((M', g')\) with the common boundary \(\Gamma\) and the length element \(dl\) on it. Also, the surfaces are assumed orientable and oriented in accordance with a fixed orientation of \(\Gamma\).

Let \(E : M \mapsto \mathbb{C}^n\) be a holomorphic immersion of \((M, g)\); let us write \(M' \in \mathbb{M}_t\) if \(\|\Lambda' - \Lambda\|_{H^1(\partial M; \mathbb{R}) \to L^2(\partial M; \mathbb{R})} \leq t\) is fulfilled. Then the convergence

\[
\sup_{M' \in \mathbb{M}_t} \inf_{E'} d_H(\mathcal{E}'(M'), \mathcal{E}(M)) \longrightarrow 0 \quad t \to 0
\]

holds, where \(d_H\) is the Hausdorff distance in \(\mathbb{C}^n\) and the infimum is taken over all holomorphic immersions \(\mathcal{E}' : M' \mapsto \mathbb{C}^n\) \([7]\).

### Teichmüller metric

- The motivation to make the latter result more expressive and natural is as follows. As was noted above, the DN map \(\Lambda\) determines not the metric \(g\) on \(M\) but the conformal equivalence class of metrics \([\!(\!(M, g)\!\!)\!\!]\). Such classes constitute the Teichmüller space \(\mathfrak{T}_m\) \([13]\) (the subscript \(m\) indicates the genus of the surfaces). This space is endowed with the natural Teichmüller metric defined below. The set \(\mathfrak{D}_m\) of the DN maps of surfaces of genus \(m\) is contained in the normed space \(\mathfrak{B}_{10}\) of the bounded linear operators, which act from \(H^1(\Gamma; \mathbb{R})\) to \(H^0(\Gamma; \mathbb{R}) := L^2(\Gamma; \mathbb{R})\). Therefore, it would be most relevant to consider a continuity of the correspondence \(\Lambda \mapsto \![\!(\!(M, g)\!\!)\!\!]\) from \(\mathfrak{D}_m\) to \(\mathfrak{T}_m\). Such a continuity is the main result of our work. To present it, we begin with basic notions and known facts.

- Let \((M, g)\) and \((M', g')\) be the surfaces and let \(q : M \to M'\) be an orientation preserving diffeomorphism. In the holomorphic local coordinates \(z\) on \(M\) and \(z'\) on \(M'\), its differential is of the form

\[
dq(z) = \partial_z q(z) \left[ dz + \mu(z) d\bar{z} \right],
\]

where

\[
\mu(z) := \frac{\partial_{\bar{z}} q(z)}{\partial_z q(z)}
\]
is called the Beltrami quotient of $q$. The Jacobian of $q$ obeys $\text{Jac}(q) = |\partial_z q|^2 - |\partial_{\bar{z}} q|^2 > 0$, whence $|\mu(z)| < 1$. Also, $|\mu(z)|$ does not depend on the choice of holomorphic coordinates $z$ and $z'$. Thus, the function

$$k(x) := \frac{1 + |\mu(z)|}{1 - |\mu(z)|}$$

(where $z$ is a holomorphic coordinate of $x$) is continuous on $M$. Its value $k(x)$ is called a dilatation of the map $q$ at the point $x$. Note that $k(x)$ is equal to the square of the ratio between length of major and minor axis of the ellipse by pulling back along $q$ the unit circle in $T_q(x)M'$. The map $q$ is called $K$-quasi-conformal if its dilatation obeys $\sup_{x \in M} k(x) = K$; in this case the number $K$ is called the dilatation of $q$.

If $(M, g)$ and $(M', g')$ are conformally equivalent (belong to the same class $[(M, g)]$) via a diffeomorphism $q$ then one has $\partial_z q = 0$, $\mu(\cdot) = 0$, $k(\cdot) = 1$, and $K = 1$. Otherwise, the value $K \neq 1$ shows to what extent the map $q$ differs from being conformal. This motivates the basic definition of the Teichmüller distance: for $[(M, g)], [(M', g')] \in \mathfrak{T}_m$ one puts

$$d_T([(M, g)], [(M', g')]) := \frac{1}{2} \log \inf_q K(q),$$

(1)

where the infimum is taken over all orientation preserving diffeomorphisms $q : M \to M'$ and $K(q)$ denotes the dilatation of $q$. Note that $d_T([(M, g)], [(M', g')])$ does not depend on the choice of elements of conformal classes.

**Main result**

- Recall that $\mathfrak{T}_m$ is the set of conformal equivalence classes $[(M, g)]$ of the genus $m$ oriented surfaces $(M, g)$ with the common boundary $\Gamma$ and length element $dl$ on it. The surfaces $(M, g) \in [(M, g)]$ have the same DN map $\Lambda$. For a fixed class $[(M, g)]$, by $\mathfrak{M}_t \subset \mathfrak{T}_m$ we denote the set of the classes $[(M', g')]$ provided $\|\Lambda - \Lambda'\|_{B_10} \leq t$, where $\Lambda'$ is DN map of $(M', g')$. The result of the paper is the following.

**Theorem 0.1.** Let $[(M, g)] \in \mathfrak{T}_m$ and $\Lambda$ be the DN map of $[(M, g)]$. Then the relation

$$\sup_{[(M', g')] \in \mathfrak{M}_t} d_T([(M, g)], [(M', g')]) \xrightarrow{t \to 0} 0.$$  

(2)

holds and means that the correspondence $\Lambda \mapsto [(M, g)]$ is continuous from $\mathfrak{D}_m$ to $\mathfrak{T}_m$.
The remainder of the paper is devoted to the proof of Theorem 0.1. Its idea is to provide constructively a map \( \alpha : (M, g) \to (M', g') \), whose dilatation is close to 1 uniformly with respect to \((M', g')\), the closeness being derived from the closeness of \( \Lambda \) and \( \Lambda' \).

1 Preliminaries

In this section, we recall the basic notions and facts along with the results of [7] that will be used in the proof of Theorem 0.1. In the rest of paper, the objects associated with the surface \( M \) are designated by unprimed symbols, while objects associated with the surface \( M' \) are designated by primed symbols.

Holomorphic functions

- Since the the surface \((M, g)\) is orientable, there is the smooth family \( \Phi = \{ \Phi_x \}_{x \in M} \) of 'rotations' \( \Phi_x \in \text{End} T_x M \) such that
  \[
g(\Phi_x a, \Phi_x b) = g(a, b), \quad g(\Phi_x a, a) = 0, \quad \forall a, b \in T_x M, \quad x \in M.
\]

The boundary \( \Gamma \) of \( M \) is oriented by the tangent field \( \gamma := \Phi \nu \). In the subsequent, dealing with the set of surfaces \((M, g), (M', g'), \ldots\) with the common boundary \((\Gamma, dl)\), we agree that their orientations are consistent in such a way that \( \Phi \nu = \Phi' \nu' = \ldots = \gamma \).

A smooth function \( w : M \to \mathbb{C} \) is holomorphic if the Cauchy-Riemann condition \( \nabla_g \Im w = \Phi \nabla_g \Re w \) holds in \( M \); then its real and imaginary parts are harmonic i.e. \( \Delta_g \Re w = \Delta_g \Im w = 0 \) holds in \( M \). Denote the lineal of holomorphic smooth functions on \( M \) by \( \mathcal{H}(M) \).

- The one instrument in the proof of Theorem 0.1 is the generalized argument principle:
  \[
  \frac{1}{2\pi i} \int_{\Gamma} \tilde{w} \frac{dw}{w - z} = \sum_{x \in w^{-1}(\{z\})} \text{ord}_x(w - z) \tilde{w}(x)
  \]
  (3)
  for any \( w, \tilde{w} \in \mathcal{H}(M) \) and \( z \in \mathbb{C} \setminus \eta(\Gamma) \); here \( \text{ord}_x(w - z) \) is the order of zero \( x \) of the function \( w - z \). Formula (3) follows from Stokes theorem (see Theorem 3.16, [12]) and the residue theorem (see Lemma 3.12, [12]) for meromorphic 1-form \( \tilde{w} dw/(w - z) \). If \( \tilde{w} = 1 \), then the right-hand side of (3) is just a total
multiplicity \( \text{mul}(w - z) \) of zeroes of \( w - z \) and \( \boxed{3} \) becomes a usual argument principle.

- The trace operator \( \text{Tr} : w \mapsto w|_\Gamma \) is a bijection from \( \mathcal{H}(M) \) to its image \( \text{Tr}\mathcal{H}(M) \). Let us characterize the lineal \( \text{Tr}\mathcal{H}(M) \) in terms of DN-map \( \Lambda \). Denote the differentiation on the boundary \( \Gamma \) with respect to the tangent field \( \gamma \) by \( \partial_\gamma \). Also, denote the orthogonal complement to constants in \( L_2(\Gamma; \mathbb{R}) \) by \( \dot{L}_2(\Gamma; \mathbb{R}) \) and introduce the integration \( J = \partial_\gamma^{-1} \) on \( \dot{L}_2(\Gamma; \mathbb{R}) \).

As is known, the DN map \( \Lambda \) is a first-order pseudo-differential operator that acts continuously from \( H^m(\Gamma, \mathbb{R}) \) to \( H^{m-1}(\Gamma, \mathbb{R}) \), \( m = 1, 2, \ldots \), where \( H^m(\ldots) \) are Sobolev spaces. Its closure in \( L_2(\Gamma; \mathbb{R}) \) (still denoted by \( \Lambda \)) obeys \( \ker \Lambda = \{\text{const}\} \) and \( \text{ran} \Lambda = \dot{L}_2(\Gamma, \mathbb{R}) \). In particular, \( J \Lambda \) is well defined bounded operator in each \( H^m(\Gamma, \mathbb{R}) \), \( m = 1, 2, \ldots \). Similarly, \( \Lambda J \) is well defined bounded operator in each \( H^m(\Gamma, \mathbb{R}) \cap \dot{L}_2(\Gamma, \mathbb{R}), \ m = 0, 1, \ldots \).

Denote the projector in \( L_2(\Gamma, \mathbb{R}) \) onto the subspace \( \ker[I + (\Lambda J)^2]^* \oplus \mathbb{R} \) by \( P \) (here \( \mathbb{I} \) is the unit operator in \( L_2(\Gamma; \mathbb{R}) \)). As is shown in [2], the formula

\[
\text{Tr}\mathcal{H}(M) = \{ Pf + i[J\Lambda Pf + c] \mid f \in C^\infty(\Gamma, \mathbb{R}), \ c \in \mathbb{R} \} \tag{4}
\]

is valid. Also, the dimension of \( (I - P)L_2(\Gamma, \mathbb{R}) \) is finite and it is equal to 2 \( \text{gen}(M) \); hence, the projector \( P \) determines the topology of \( M \). In particular, if \( M \) is a disc in \( \mathbb{R}^2 \), then \( P = \mathbb{I} \) and \( \Lambda J \) coincides with the classical Hilbert transform that relates real and imaginary parts of traces of holomorphic functions.

- Let \( (M', g') \in \mathbb{M}_t \) be a surface with DN-map \( \Lambda' \). Due to \( \boxed{1} \), we can define a real linear map \( \hat{\beta}_{M'} : \text{Tr}\mathcal{H}(M) \rightarrow \text{Tr}\mathcal{H}(M') \) by the rule

\[
\hat{\beta}_{M'} \eta = P'\Re \eta + i[J\Lambda P' \Re \eta + \langle \Im \eta \rangle],
\]

where \( \langle f \rangle := \int_\Gamma f \, dl \), and \( P' \) is a projector in \( L_2(\Gamma, \mathbb{R}) \) onto the subspace \( \ker[I + (\Lambda J)^2]^* \oplus \mathbb{R} \). In [?], it is proved that, for sufficiently small \( t \in [0, t_0] \) and any \( M' \in \mathbb{M}_t \), the map \( \hat{\beta}_{M'} \) is bijective and the estimate

\[
\| \hat{\beta}_{M'} \eta - \eta \|_{C^m(\Gamma; \mathbb{C})} \leq c_m t \| \eta \|_{H^{m+1}(\Gamma; \mathbb{C})}, \quad \forall \eta \in \text{Tr}\mathcal{H}(M) \tag{5}
\]

is valid for \( m = 1, 2, \ldots \). Here and in the subseque, all constants in the estimates are assumed to be independent of \( t \) and \( M' \in \mathbb{M}_t \). We introduce the ‘canonical’ real linear map

\[
\beta_{M'} : \mathcal{H}(M) \rightarrow \mathcal{H}(M'), \quad \beta_{M'} = \text{Tr}^{-1} \circ \hat{\beta}_{M'} \circ \text{Tr}. \tag{6}
\]
Holomorphic embeddings

- Recall that an immersion is a differentiable map \( \alpha : M_1 \hookrightarrow M_2 \) between two differentiable manifolds \( M_1 \) and \( M_2 \), whose differential \( d\alpha_x : T_x M_1 \hookrightarrow T_{\alpha(x)} M_2 \) is injective for any \( x \in M_1 \). The immersion \( \alpha \) is an embedding if \( \alpha : M_1 \hookrightarrow \alpha(M_1) \) is a homeomorphism, (where the topology on \( \alpha(M_1) \) is induced by \( M_2 \)). We say that the embedding

\[
\mathcal{E} : M \to \mathbb{C}^n, \quad x \mapsto \{w_1(x), \ldots, w_n(x)\}
\]

is holomorphic if it is realised by holomorphic functions \( w_k \). In what follows, we denote \( \eta_k := \text{Tr} w_k \).

For \( \xi = \{\xi_1, \ldots, \xi_n\} \in \mathbb{C}^n \), introduce the coordinate projections \( \pi_i : \xi \mapsto \xi_i \). Let \( D \) be a domain in \( \mathbb{C} \), denote the cylinder \( \{\zeta \in \mathbb{C}^n \mid \pi_i \zeta \in D\} \) by \( \Pi_i[D] \). We say that \( \Pi_i[D] \) is \( \mathcal{E}(M) \)-projective if \( \pi_i : \Pi_i[D] \cap \mathcal{E}(M) \to \mathbb{C} \) is an embedding. The embedding \( \mathcal{E} \) is called projective if each point \( \xi \in \mathcal{E}(M) \) belongs to some \( \mathcal{E}(M) \)-projective cylinder. The existence of projective embeddings follows from the divisor theorem (see Theorem 7, [8]). In the parper, we deal with projective holomorphic embeddings only and, for short, call them just ‘embeddings’. The basic properties of such embeddings are presented below.

- The image of an embedding of a surface is determined only its DN-map and the choice of boundary traces \( \eta_k \). Indeed, let \((M, g)\) and \((M', g')\) be surfaces with the joint boundary (\(\Gamma, dl\)) and their DN-maps \(\Lambda\) and \(\Lambda'\) coincide. Then there exists a conformal map \( \beta : M \to M' \) that does not move the points of \(\Gamma\). Then the functions \( w_k \in \mathcal{H}(M), w'_k \in \mathcal{H}(M') \) with joint boundary traces \( \eta_k := \text{Tr} w_k = \text{Tr}' w'_k \) are connected by \( w_k = w'_k \circ \beta \). Hence, \( \mathcal{E}(M) = \mathcal{E}'(M) \) is valid for \( \mathcal{E} = \{w_1, \ldots, w_n\} \) and \( \mathcal{E}' = \{w'_1, \ldots, w'_n\} \).

- Let \( \mathcal{E} = \{w_1, \ldots, w_n\} \) be an embedding of \((M, g)\). The surface \( \mathcal{E}(M) \) is endowed with the metric \( \bar{g} \) induced by standard metric in \( \mathbb{C}^n \). With this choice of metric, the map \( \mathcal{E} : M \to \mathcal{E}(M) \) is conformal. Indeed, suppose that \( \phi : [0,1] \to M \) is a smooth curve with the beginning at \( x_0 \) and the tangent vector \( \theta \) at \( x_0 \). Let \( \tilde{\phi} = \mathcal{E} \circ \theta \) be a corresponding curve in \( \mathcal{E}(M) \) with the beginning at \( \xi_0 = \mathcal{E}(x_0) \) and the tangent vector \( \tilde{\theta} = d\mathcal{E}_{x_0} \theta \) at \( \xi \). Chose a projective cylinder \( \Pi_i[D] \) containing \( \xi \), then \( \mathcal{E}(M) \cap \Pi_i[D], \pi_i \) is a complex chart on \( \mathcal{E}(M) \) and \( \mathcal{E}^{-1}(\Pi_i[D]), w_i = \mathcal{E} \circ \pi_i \) is a complex chart on \( M \). In local coordinates \((x^1, x^2) = (\Re w_i(x), \Im w_i(x)) = (\Re \pi_i \xi, \Im \pi_i \xi)\), where
\[ \xi = \mathcal{E}(x), \] the components of vectors \( \theta \) and \( \tilde{\theta} \) satisfy
\[
\theta^1 + i\theta^2 = \frac{dw_i \circ \phi}{dt}(0) = \frac{d\pi_i \circ \mathcal{E} \circ \phi}{dt}(0) = \frac{d\pi_i \circ \tilde{\phi}}{dt}(0) = \tilde{\theta}^1 + i\tilde{\theta}^2.
\]

Since \( w_i \) is holomorphic, we have \( g_{kl} = \rho_1(x)\delta_{kl} \) in local coordinates \((x^1, x^2)\), where \( \rho_1 \) is a conformal factor. Hence, \( g(\theta, \theta) = \rho_1(x_0)|\theta^1 + i\theta^2|^2 \). Since the metric \( \tilde{g} \) on \( \mathcal{E}(M) \) is induced by \( \mathbb{C}^n \), we have
\[
\tilde{g}(\tilde{\theta}, \tilde{\theta}) = \sum_{k=1}^n \left| \frac{dw_k \circ w_i^{-1}}{dz} \frac{d\pi_i \circ \tilde{\phi}}{dt}(0) \right|^2 = \rho_2(x_0)|\tilde{\theta}^1 + i\tilde{\theta}^2|^2,
\]
where \( z = \pi_i \xi = w_i(x) \) and \( \rho_2(x) = \sum_{k=1}^n |\partial_k w_k \circ w_i^{-1}|^2 \). Hence, we have \( \tilde{g}(\tilde{\theta}, \tilde{\theta}) = \rho_2(x_0)\rho_1(x_0)^{-1}g(\theta, \theta) \) and, since \( \phi \) is arbitrary, \( \tilde{g} = \rho \mathcal{E}^*g \), where \( \rho \in C^\infty(\mathcal{E}(M); (0, +\infty)) \).

Note that the collection \( \{ U = \mathcal{E}^{-1}(\Pi_i[D]), w_i|_U \} \) (where \( \Pi_i[D] \)) is \( \mathcal{E}(M) \)-projective cylinder) provides a byholomorphic atlas on \( (M, g) \) whereas the collection \( \{ V = \mathcal{E}(M) \cap \Pi_i[D], \pi_i|_V \} \) provide a byholomorphic atlas on \( (\mathcal{E}(M), \tilde{g}) \) (both atlases are consistent with the metrics). With such atlases, the map \( \mathcal{E} : M \to \mathcal{E}(M) \) is byholomorphism.

- For the embedding \( \mathcal{E} = \{ w_1, \ldots, w_n \} \), application of the generalized argument principle \( (3) \) yields the following. Let \( \xi = \mathcal{E}(x) \) belongs to some projective cylinder \( \Pi_i[D] \) and \( z = \pi_i x \). Since the map \( \xi \to \pi_i(x) \) is an embedding, \( x \) is a unique and simple zero of the function \( w_i - z = \pi_i \circ \mathcal{E} - z \). Due to \( (3) \), the projection \( \pi_k \xi = w_k(x) \) can be found by
\[
\pi_k \xi = \frac{1}{2\pi i} \int_\Gamma \eta_k \frac{\partial_\gamma \eta_i}{\eta_i - z} dl. \tag{7}
\]

So, the conformal copy \( \mathcal{E}(M) \) of \( M \) can be reconstructed from \( \{ \eta_1, \ldots, \eta_n \} \) by using \( (7) \).

**Induced embeddings**

- Let \( \mathcal{E} = \{ w_1, \ldots, w_n \} \) be a (fixed) embedding of \((M, g)\). For a surface \((M', g') \in \mathbb{M}_t \), we say that its embedding \( \mathcal{E}' = \{ w'_1, \ldots, w'_n \} \) is induced by \( \mathcal{E} \) if the functions \( w'_k \) are connected with \( w_k \) by \( w'_k = \beta_{M'} w_k \), where the map \( \beta_{M'} \) is defined in \( (4) \). Then \( (4) \) yields
\[
\|\eta'_k - \eta_k\|_{C^m(\Gamma; \mathbb{C})} \leq c_m t \quad (m = 1, 2, \ldots) \tag{8}
\]
for boundary traces \( \eta_k = \text{Tr} w_k, \eta'_k = \text{Tr}' w'_k \).

In the subsequent, \( \mathcal{E}' \) always denotes the embedded of \( (M', g') \) induced by \( \mathcal{E} \). In [?], it is proved that, for sufficiently small \( t \in [0, t_0) \) and any \( (M', g') \in \mathcal{M}_t \), the induced embedding \( \mathcal{E}' \) is actually projective and

\[
\sup_{(M', g') \in \mathcal{M}_t} d_H(\mathcal{E}'(M'), \mathcal{E}(M)) \to 0,
\]

where \( d_H(K_1, K_2) \) is the Hausdorff distance between two compacts \( K_1 \) and \( K_2 \) in \( \mathbb{C}^n \) defined as the infimum of positive \( s \) such that \( s \)-neighbourhood of \( K_1 \) contains \( K_2 \) and \( s \)-neighbourhood of \( K_2 \) contains \( K_1 \).

## 2 The map \( \alpha \)

- Assume that \( (M, g) \) is a surface with boundary \( (\Gamma, dl) \) and \( \mathcal{E} \) is a fixed embedding of \( (M, g) \). Also, suppose that \( (M', g') \in \mathcal{M}_t \) with small \( t > 0 \) and \( \mathcal{E}' \) is an embedding of \( (M', g') \) induced by \( \mathcal{E} \). In this section, we construct the near-isometric diffeomorphism \( \alpha = \alpha_{M'} \) from \( \mathcal{E}(M) \) onto \( \mathcal{E}'(M') \). Informally speaking, the map \( \alpha \) is defined in the following way. If \( \xi \in \mathcal{E}(M) \) is separated from \( \mathcal{E}(\Gamma) \), then \( \alpha(\xi) \) is a point \( \xi' \) of \( \mathcal{E}'(M') \) closest to \( \xi \) in \( \mathbb{C}^n \). If \( \xi \in \mathcal{E}(M) \) is close to \( \mathcal{E}(\Gamma) \), then \( \alpha(\xi) \) is the point \( \xi' \in \mathcal{E}'(M') \) whose semi-geodesic coordinates \( (l', r') \) on \( \mathcal{E}'(M') \) coincides with the semi-geodesic coordinates \( (l, r) \) of \( \xi \).

- More precisely, \( \alpha(\xi) \) is defined as the point of minimum of a certain function \( \mathcal{D}(\xi, \cdot) : \mathcal{E}'(M') \to [0, +\infty) \) which is constructed in the following way. First, introduce the function

\[
\mathcal{D}_{\text{int}} : \mathcal{E}(M) \times \mathcal{E}'(M') \to [0, +\infty), \quad \mathcal{D}_{\text{int}}(\xi, \xi') := |\xi' - \xi|^2; \quad (9)
\]

then \( \mathcal{D}_{\text{int}}(\xi, \cdot) \) attain the minimum at a point \( \xi' \) of \( \mathcal{E}'(M') \) closest to \( \xi \).

For \( \xi \in \mathcal{E}(M) \), let \( r \) stands for distance between \( \xi \) and \( \mathcal{E}(\Gamma) \) in \( (\mathcal{E}(M), \tilde{g}) \). Also, let \( l := \mathcal{E}^{-1}(\xi) \), where \( \xi \) is a point of \( \mathcal{E}(\Gamma) \) closest to \( \xi \) in \( (\mathcal{E}(M), \tilde{g}) \). The pair \( (l, r) \) provides semi-geodesics coordinates on \( (\mathcal{E}(M), \tilde{g}) \), which are regular at least on a certain near-boundary strip \( r \leq c \). The semi-geodesics coordinates \( (l', r') \) on \( (\mathcal{E}'(M'), \tilde{g}') \) are defined in the same way. In what follows, we show that, for sufficiently small \( t \), the coordinates \( (l', r') \) are regular on the near-boundary strip \( r' \leq r_0 \), where \( r_0 > 0 \) does not depend on \( t \) and \( M' \). Introduce the function

\[
\mathcal{D}_\Gamma : \mathcal{E}(M) \times \mathcal{E}'(M') \to [0, +\infty),
\]

\[
\mathcal{D}_\Gamma(\xi, \xi') := (\text{dist}_\Gamma(l'(\xi'), l(\xi))^2 + (r'(\xi') - r(\xi))^2; \quad (10)
\]
then $\mathfrak{D}_\Gamma(\xi, \cdot) \left( r(\xi) < r_0 \right)$ attain the minimum at a point $\xi'$ with semi-geodesics coordinates $l' = l$, $r' = r$.

Let $\kappa$ be a non-negative smooth cut-off function on $\mathcal{E}(M)$ which is equal to zero for $r > 2r_0/3$ and to one for $r < r_0/3$. The function $\mathfrak{D} : \mathcal{E}(M) \times \mathcal{E}'(M') \to [0, +\infty)$ is defined by

$$\mathfrak{D} := (1 - \kappa)\mathfrak{D}_\text{int} + \kappa\mathfrak{D}_\Gamma.$$  \hfill (11)

- Along with $\mathfrak{D}$, we also consider the ‘unperturbed’ function

$$\mathfrak{D}_0 := (1 - \kappa)\mathfrak{D}_\text{int,0} + \kappa\mathfrak{D}_\Gamma,0$$

which coincides with $\mathfrak{D}$ in the case $\mathcal{E}'(M') = \mathcal{E}(M)$. Here the the functions $\mathfrak{D}_\text{int,0}$, $\mathfrak{D}_\Gamma,0$ are defined by (9) and (11), where $\mathcal{E}'(M')$ is replaced by $\mathcal{E}(M)$ and $l'$, $r'$ are replaced by $l$, $r$. Note that $\xi' = \xi$ is a unique and non-degenerated point of global minimum of the function $\xi' \mapsto \mathfrak{D}_0(\xi, \xi') \, (\xi \in \mathcal{E}(M))$.

2.1 Construction of $\alpha$ in a zone separated from $\mathcal{E}(\Gamma)$.

- Let $\Pi_i[D]$ be an $\mathcal{E}(M)$-projective cylinder whose closure does not intersects with $\mathcal{E}(\Gamma)$. Then $|\eta_i(l) - z| > c > 0$ and $\text{mul}(w_i - z) = 1$ for $z \in \overline{D}$, $l \in \Gamma$ and (3), (8) yields

$$\text{mul}(w' - z) - \text{mul}(w - z) = \left| \frac{1}{2\pi i} \int_{\Gamma} \frac{\partial_1 \eta_i'}{\eta_i' - z} dl - \frac{1}{2\pi i} \int_{\Gamma} \frac{\partial_1 \eta_i}{\eta_i - z} dl \right| \leq ct.$$  

Thus, for sufficiently small $t > 0$, we have $\text{mul}(w'_i - z) = 1$, whence the cylinder $\Pi_i[D]$ is also $\mathcal{E}'(M')$-projective. Similarly, differentiating (7) and applying estimate (8), we obtain

$$\|w'_j \circ w'^{-1}_i - w_j \circ w^{-1}_i\|_{C^m(D;\mathcal{E})} \leq c_m t, \quad (m = 1, 2, \ldots). \hfill (13)$$

Assuming that $\kappa = 0$ on $\mathcal{E}(M) \cap \Pi_i[D]$ and considering $\mathfrak{D} = \mathfrak{D}_\text{int}$ and $\mathfrak{D}_0 = \mathfrak{D}_\text{int,0}$ as functions of local coordinates $x^1 + ix^2 = z = \pi_i \xi$, $x^1 + ix^2 = z' = \pi_i \xi'$, we obtain

$$\mathfrak{D}(\xi, \xi') = \mathfrak{D}_\text{int}(\xi, \xi') = \sum_{k=1}^n |w'_j \circ w'^{-1}_i(z') - w_j \circ w^{-1}_i(z)|^2,$$

$$\mathfrak{D}_0(\xi, \xi') = \mathfrak{D}_\text{int,0}(\xi, \xi') = \sum_{k=1}^n |w_j \circ w^{-1}_i(z') - w_j \circ w^{-1}_i(z)|^2,$$
where \( z, z' \in D \). Then (13) implies
\[
\| \mathcal{D} - \mathcal{D}_0 \|_{C^m(D \times D; [0, +\infty))} \leq c_m t, \quad m = 1, 2, \ldots.
\]

Decreasing the diameter of \( D \), we can assume that \( \xi' = \xi \) is a unique point of local minimum of the function \( \xi' \mapsto \mathcal{D}_0(\xi, \xi') \) in \( E(M) \cap \Pi_i[D] \).

- Let \( \Pi_i[D] \) is a sub-cylinder of \( \Pi_i[D] \) such that \( \mathcal{D} \subset D \). To construct the map \( \xi \mapsto \alpha(\xi) \) for \( \xi \in E(M) \cap \Pi_i[D] \), we need the following

**Lemma 2.1.** Let \( X, \mathcal{X}, Y \) be domains with compact closures in \( \mathbb{R}^n \), \( \mathcal{X} \subset X \), and \( F \in C^2(\mathcal{X} \times Y; \mathbb{R}^n) \) satisfies i) the zero set of \( F \) is the graph of the function \( f \in C^1(X; Y) \cap C(\mathcal{X} \times Y) \) and ii) for any \( x \in X \) there exists \( (F'_y(x, f(x)))^{-1} \). Then there exists \( t_* > 0 \) such that, for any \( t \in (0, t_*) \) and any \( H \in C^1(X \times Y; \mathbb{R}^n) \) satisfying
\[
\| H - F \|_{C^1(X \times Y; \mathbb{R}^n)} \leq t,
\]
the zero set of \( H \) in \( X \times Y \) is the graph of the function \( h \in C^1(\mathcal{X}; Y) \) and
\[
\| h - f \|_{C^1(\mathcal{X}; Y)} \leq ct
\]
(the constant \( c \) does not depend on \( H \)). Also, if \( f \) is a diffeomorphism from \( X \) to \( f(X) \), then \( h \) is a a diffeomorphism from \( \mathcal{X} \) to \( g(\mathcal{X}) \).

**Lemma 2.1** is just a variant of the implicit functions theorem; for the convenience of the reader, its proof is contained in Appendix A.

In **Lemma 2.1** put \( X = Y = D, \mathcal{X} = D, F = (\partial_x, \partial_x D_0, \partial_x D_0), \) and \( H = (\partial_x, \partial_x D_0, \partial_x D_0) \). Then, in view of (14), we obtain that, for sufficiently small \( t > 0 \), any \( (M', g') \in \mathcal{M}_0 \), and any \( \xi \in E(M) \cap \Pi_i[D] \), there exists a unique point \( \xi' \in E'(M') \cap \Pi_i[D] \) which is a local minimum of the function \( \mathcal{D}(\xi, \cdot) \). Moreover, the map \( \alpha = \alpha_{\Pi_i[D]} : \xi \mapsto \xi' \) is a diffeomorphism from \( E(M) \cap \Pi_i[D] \) onto \( \tilde{\alpha}(E(M) \cap \Pi_i[D]) \subset E'(M') \cap \Pi_i[D] \) satisfying
\[
\| \pi_i \circ \alpha - \pi_i \|_{C^1(E(M) \cap \Pi_i[D]; C)} \leq ct.
\]
In particular, (13) and (16) imply
\[
|\alpha(\xi) - \xi| \leq ct, \quad \forall \xi \in E(M) \cap \Pi_i[D].
\]
Recall that \( \{ U = E(M) \cap \Pi_i[D], \phi = \pi_i|_U \} \) is a holomorphic chart on \( E(M) \) and \( \{ U' = E'(M') \cap \Pi_i[D], \phi' = \pi_i|_{U'} \} \) is holomorphic chart on \( E'(M') \). Formula (16) yields
\[
\| J - I \|_{C(\mathcal{D}; M_2)} \leq ct.
\]
where $J = \{ \partial x^k / \partial x^l \}_{k,l=1,2}$ is the Jacoby matrix of the map $\alpha$. Alternatively, the map $\tilde{\alpha} = \phi' \circ \alpha \circ \phi^{-1}$ obeys $\| \partial_{\tilde{\alpha}} - 1 \|_{C(D; \mathbb{C})} \leq ct$, $\| \partial_{\tilde{\alpha}} \|_{C(D; \mathbb{C})} \leq ct$. Hence, the Beltrami quotient $\mu_\alpha(\xi) = \| \partial_{\tilde{\alpha}}(z) / \partial_{\tilde{\alpha}}(z) \|_{z=\pi_{i}\xi}$ satisfies

$$\| \mu_\alpha \|_{C(U; \mathbb{C})} \leq ct. \quad (19)$$

- Let $\xi$ be a point of $U$ with the projection $z = \pi_{i}\xi$, and $\xi' = \alpha(\xi)$, $\xi' = \pi_{i}\xi'$. Let also $\theta \in T_\xi \mathcal{E}(M)$ be a tangent vector, and $\theta' = d\alpha_\xi(\theta)$. Denote by $\theta^k (\theta'^k)$ the components of $\theta (\theta')$ in local coordinates $x^k (x'^k)$ and put $\vartheta = \theta^1 + i\theta^2 (\vartheta' = \theta'^1 + i\theta'^2)$. Then $\| \vartheta' \vartheta^{-1} - 1 \| \leq ct$ in view of (18). By definition of the metrics on $\mathcal{E}(M)$ and $\mathcal{E}'(M')$, we have

$$\tilde{g}(\theta, \theta) = \rho(\theta)|\vartheta|^2, \quad \rho = \sum_{k=1}^{n} |\partial_{z}w_k \circ \vartheta^{-1}|^2, \quad \rho' = \sum_{k=1}^{n} |\partial_{z'}w'_k \circ \vartheta'^{-1}|^2. \quad (20)$$

In view of (13) and (16), we have $\| \rho' \rho^{-1} - 1 \|_{C(D; \mathbb{C})} \leq ct$. Therefore, we obtain the estimate

$$\left| \frac{\alpha^* \tilde{g}'(\theta, \theta)}{\tilde{g}(\theta, \theta)} - 1 \right| \leq ct, \quad \forall \theta \in T_\xi \mathcal{E}(M), \, \xi \in U, \quad (21)$$

which means that the map $\alpha$ is close to the isometry for small $t$.

### 2.2 Construction of $\alpha$ in a zone near $\mathcal{E}(\Gamma)$

- Now, let $\Pi_i[D]$ be an $\mathcal{E}(M)$-projective cylinder that intersects with $\mathcal{E}(\Gamma)$. Moreover, let its base $D$ be a disc with center at some point $z_0 = \eta_i(l_0)$. Denote $U = \mathcal{E}(M \setminus \Gamma) \cap \Pi_i[D]$. By decreasing the radius of $D$, we obtain the following: a) $\eta_i(\Gamma) \cap D$ is a smooth curve, $D \setminus \eta_i(\Gamma)$ has two connected components $D_0$ and $D_1 = \pi_{i}(U)$, and $\text{mul}(w_i - z) = p$ on $D_\rho$; b) the function

$$\tilde{l} : \eta_i^{-1}(D) \to \mathbb{R}, \quad \tilde{l}(l) := \Re \frac{\eta_i(l) - z_0}{\partial_l \eta_i(l_0)}$$

is invertible and the coordinates $z \mapsto \tilde{z} = (x^1, x^2)$,

$$x^1 = \Re \frac{z - z_0}{\partial_l \eta_i(l_0)}, \quad x^2 = \Im \frac{z - \eta_i \circ \tilde{l}^{-1}(x^1)}{\partial_l \eta_i(l_0)} \quad (22)$$
are regular on $D$. Note the ‘rectification property’ of coordinates \([22]\); in these coordinates, the curve $\eta_i(\Gamma) \cap D$ becomes a segment of the axis $x^2 = 0$ (see pic. ??).

Assume that $t$ is sufficiently small. Denote $U' = E'(M' \setminus \Gamma) \cap \Pi_i[D]$. Applying the argument principle \([3]\) and estimate \([5]\), we obtain that the properties a),b) are also valid for the curve $\eta'(\Gamma)$. Namely, we have a’) $\eta'_i(\Gamma) \cap D$ is a smooth curve, $D \setminus \eta'_i(\Gamma)$ has two connected components $D_0'$ and $D'_1 = \pi_i(U')$, and $\text{mul}(w_i - z) = p$ on $D'_0$; b’) the function

$$\bar{p} : \eta_i'^{-1}(D) \to \mathbb{R}, \quad \bar{p}(l) := \Re \frac{\eta'_i(l) - z_0}{\partial_{\eta'_i}(l_0)}$$

is invertible and the coordinates $z' \mapsto \tilde{z}' = (x'^1, x'^2)$,

$$x'^1 = \Re \frac{z' - z_0}{\partial_{\eta'_i}(l_0)}, \quad x'^2 = \Im \frac{z' - \eta'_i \circ \bar{p}^{-1}(x'^1)}{\partial_{\eta'_i}(l_0)}$$

are regular on $D$. In particular, the cylinder $\Pi_i[D]$ is also $E'(M')$-projective. Also, there exists a rectangle $\mathcal{R} = (-a, a) \times [0, b)$, on which both functions $\tilde{z}^{-1}, \tilde{z}'^{-1}$ are defined. As a corollary of \([3], [22], \) and \([24]\), we obtain

$$\|\tilde{z}'^{-1} - \tilde{z}^{-1}\|_{C^m(\mathcal{R})} + \|\bar{p}^{-1} - \bar{p}'^{-1}\|_{C^m([-a, a])} \leq c_m t, \quad m = 1, 2, \ldots$$

Now, let us prove that

$$\|w'_j \circ w_i^{-1} \circ \tilde{z}'^{-1} - w_j \circ w_i^{-1} \circ \tilde{z}^{-1}\|_{C^m(\mathcal{R}; \mathbb{C})} \xrightarrow{t \to 0} 0$$

uniformly with respect to $(M', g') \in \mathbb{M}_t$. Let $\xi \in U$, $\xi' \in U'$ and their projections $z = \pi_i \xi$, $z' = \pi_i \xi'$ are connected by $\tilde{z}(z) = \tilde{z}'(z') = (x'^1, x'^2) \in \mathcal{R}$. According to \([7]\), we have

$$\partial_{z'}^{m}[w'_j \circ w_i^{-1}](z') = \partial_{z'}^{m}(\pi_j \xi') = \frac{m!}{2\pi i} \int_{\Gamma} \frac{\eta'_j d\eta'_i}{(\eta'_i - z')^{m+1}}.$$  

Denote $\zeta_0 := \tilde{z}^{-1}((x^1, 0))$ and $\zeta'_0 := \tilde{z}'^{-1}((x^1, 0))$ (see pic. ??). Let $\zeta' \in \eta'_i(\Gamma)$ and $\tilde{z}'(\zeta') = (y^1, 0) \in \mathcal{R}$. According to \([24]\), we have

$$y^1 - x'^1 = \Re \frac{\zeta' - \zeta'_0}{\partial_{\eta'_i}(l_0)}.$$
From Taylor formula for $\eta_j' \circ \tilde{\eta}^{-1}$ in a neighbourhood of $x^1$, we obtain

$$|\eta_j' \circ \tilde{\eta}^{-1}(y^1) - \mathcal{P}_{m,x'^1}^{(k)}(\zeta')| \leq c|\zeta' - \zeta_0'|^{m+1} \tag{29}$$

($c$ is independent of $(M',g')$ due to (8) and (25)). Here $\mathcal{P}_{m,x'}^{(k)}$ is a $m$-th order polynomial in $\mathcal{P}'_{m,x'}^{(k)}(\zeta')$ and $\mathcal{P}'_{m,x'}^{(k)}(\zeta') = w_j' \circ \tilde{\eta}_i^{-1}(\zeta')$. Since $w_j' \circ \tilde{\eta}_i^{-1}$ is holomorphic on $D'$, the terms with $\zeta' - \zeta_0'$ are not included in the polynomial $\mathcal{P}'_{m,x'}^{(k)}$. Thus,

$$\mathcal{P}_{m,x'}^{(k)}(\zeta') = \sum_{k=0}^{m} \mathcal{P}_{m,x'}^{(k)}(\zeta' - \zeta_0')^k$$

and

$$\partial_z^m[w_j' \circ \tilde{\eta}_i^{-1}](z') = \frac{m!}{2\pi i} \int_{\Gamma_\delta} \frac{\eta_j' - \mathcal{P}_{m,x'}^{(k)}(\zeta')}{(\eta_i' - z')^{m+1}} \, d\eta_i'$$

$$+ \int_{\eta_i(d\Gamma_\delta)} \frac{\eta_j' \circ \eta_i^{-1} - \mathcal{P}_{m,x'}^{(k)}(\zeta')}{(\zeta' - z')^{m+1}} \, d\zeta' + m! \mathcal{P}_{m,x'}^{(m)}(\zeta') \tag{30}$$

where $\Gamma_\delta$ is a $\delta$-neighbourhood of the point $\eta_i^{-1}(\zeta_0')$. In view of (29),

$$\left|\eta_j' \circ \eta_i^{-1} - \mathcal{P}_{m,x'}^{(k)}(\zeta')\right| \leq c\left|\zeta' - \zeta_0'\right|^{m+1}.$$

Formulas (24), (28), and (25) lead to the estimate

$$|\zeta' - \zeta_0'| \leq c\|z^{-1}\|_{C^m(\mathcal{R})} |y^1 - x^1| \leq c|\zeta' - z'|.$$

Thus, the second integral in (30) tends to zero as $\delta \to 0$ uniformly with respect to $(M',g') \in \mathcal{M}_\delta$. Also, due to (8), the denominator in the first integral in (30) satisfies $|(\eta_i' - z')^{m+1}| \geq c_\delta > 0$ for any fixed $\delta$ and all $z' \in z'^{-1}(\mathcal{R})$. Of course, the facts above are also valid if we omit the primes in (27)-(30). Also, from (25) we have

$$|\mathcal{P}_{m,x'}^{(m)} - \mathcal{P}_{m,x'}^{(m)}| \leq ct$$

for the coefficients in the polynomials in (29) in primed and non-primed cases. Due to this and (8), we have

$$\left|\int_{\Gamma_\delta \setminus \Gamma_\delta} \frac{\eta_j' - \mathcal{P}_{m,x'}^{(k)}(\zeta')}{(\eta_i' - z')^{m+1}} \, d\eta_i' - \int_{\Gamma_\delta \setminus \Gamma_\delta} \frac{\eta_j - \mathcal{P}_{m,x'}^{(k)}(\zeta')}{(\eta_i - z')^{m+1}} \, d\eta_i\right| \leq c(\delta)t$$
for any fixed δ and all \((x^1, x^2) \in \mathcal{R}\). From these facts, formula (26) follows immediately.

- Recall the expression (20) for the metrics \(\tilde{g}, (\tilde{g}')\) induced by \(C^m\) on \(\mathcal{E}(M)\) \((\mathcal{E}'(M'))\) in coordinates \((\Re z, \Im z), z = \pi_j \xi\) (coordinates \((\Re z', \Im z'), z' = \pi_j \xi'\)). Due to this and formulas (25), (26), we obtain the estimate

\[
\|\tilde{g}_{ij}' - \tilde{g}_{ij}\|_{C^m(\mathcal{R})} \xrightarrow{t \to 0} 0
\]

for the components of metric tensors \(\tilde{g}\) and \(\tilde{g}'\) in local coordinates \(\tilde{z}\) and \(\tilde{z}'\).

Note that the convergence in (31) is uniform with respect to \((M', g') \in \mathcal{M}\).

Let \((l, r)\) and \((l', r')\) stand for the semi-geodesic coordinates of points \(\xi \in \mathcal{E}(M)\) and \(\xi' \in \mathcal{E}'(M')\), respectively. Denote

\[
\begin{align*}
V &= \{\xi \in \mathcal{E}(M) \mid \text{dist}_t(l, l_0) < q, \ r \in [0, q]\}, \\
V' &= \{\xi' \in \mathcal{E}'(M') \mid \text{dist}_t(l', l_0) < q, \ r' \in [0, q]\}.
\end{align*}
\]

We are going to prove that there exists a sufficiently small \(q > 0\) independent of \((M', g') \in \mathcal{M}\) and such that the coordinates \((l, r)\) and \((l', r')\) are regular on \(V\) and \(V'\), respectively. Also, we will prove that the map \(\alpha_V : V \to V'\), defined by the rule

\[
l'(\alpha_V(\xi)) = l(\xi), \quad r'(\alpha_V(\xi)) = r(\xi),
\]

is near-isometric for small \(t\). To this end, we need the following auxiliary fact.

Let \(\mathcal{B}\) be a neighbourhood of the origin in the half-plane \(\Re \times [0, +\infty)\) containing a segment \([-r_0, r_0] \times \{0\}\) and \(h\) be a metric tensor on \(\mathcal{B}\) with components \(h_{ij} \in C^2(\mathcal{B}; \Re)\). Denote the outward normal on \([-r_0, r_0] \times \{0\}\) corresponding to the metric \(h\) by \(\nu_h\). Introduce the bundle \(x_h\) of semi-geodesics (with respect to the metric \(h\)) curves \(r \mapsto x_h(r, \mu)\) in \(\mathcal{B}\), where \(\mu \in [-r_0, r_0], r > 0,\) and

\[
x_h(0, \mu) = (0, \mu), \quad \partial_r x_h(0, \mu) = -\nu_h(\mu).
\]

Put \(\mathcal{D}_r = (-r, r) \times [0, r)\).

**Lemma 2.2.** Suppose that \(x_g\) is a diffeomorphism from \(\mathcal{D}_{r_0}\) onto \(x_h(\mathcal{D}_{r_0}) \subset \mathcal{B}\). Then there exist \(s_0 > 0\) and the sub-rectangle \(\mathcal{D}_r \subset \mathcal{D}_{r_0}\) such that, for \(s \in (0, s_0)\) and any metrics \(h'\) on \(\mathcal{B}\), satisfying

\[
\|h_{ij}' - h_{ij}\|_{C^2(\mathcal{B}; \Re)} \leq s,
\]

\[
15
\]
the map \( x_{h'} \) is a diffeomorphism from \( \mathcal{D}_{r_1} \) onto \( x_{h'}(\mathcal{D}_{r_1}) \subset \mathcal{B} \) and

\[
\|x_{h'} - x_h\|_{C^1(\mathcal{D}_{r_1};\mathbb{R}^2)} \leq cs. \tag{34}
\]

In particular, the map \( \varpi := x_{h'} \circ x_h^{-1} \) obeys

\[
\|(\varpi h')_{ij} - h_{ij}\|_{C(\mathcal{D}_{r_1};\mathbb{R})} \leq cs. \tag{35}
\]

Lemma 2.2 is just a variant of the theorem on the local solvability of Cauchy problem; for the convenience of the reader, this proof is contained in Appendix B.

Put \( \mathcal{B} = \mathfrak{H} \) and \( h_{ij} = \tilde{g}_{ij}, \ h'_{ij} = \tilde{g}'_{ij} \). According to Lemma 2.2 and formula (31), the semi-geodesics coordinates \((t, r)\) and \((t', r')\) are regular on the neighbourhoods \( V \) and \( V' \) given by (32) for sufficiently small \( t > 0 \) and \( q = q(l_0) > 0 \). For the map \( \alpha_V \), formula (35) takes the form

\[
\|(\alpha_V^* \tilde{g}')_{ij} - \tilde{g}_{ij}\|_{C(\tilde{\xi}_0;\mathbb{R}^2)} \overset{t \to 0}{\longrightarrow} 0,
\]

where \( \tilde{g}_{ij} \) and \((\alpha_V^* \tilde{g}')_{ij}\) are the components of metric tensors \( \tilde{g} \) and \( \alpha_V^* \tilde{g}' \) in local coordinates \( \tilde{z} \). In view of (25), formula (34) yields

\[
\|\pi_i \circ \alpha_V - \pi_i\|_{C^1(V';\mathbb{C})} \overset{t \to 0}{\longrightarrow} 0. \tag{37}
\]

Chose the holomorphic charts \((\pi_i(V), \phi = \pi_i|_V)\) and \((\pi_i(V'), \phi' = \pi_i|_V')\) on \( \mathcal{E}(M) \) and \( \mathcal{E}'(M') \), respectively. Due to (37), the map \( \tilde{\alpha}_V = \phi' \circ \alpha_V \circ \phi^{-1} \) obeys \( \|\partial \tilde{\alpha}_V - 1\|_{C(\pi_i(V);\mathbb{C})} \leq ct \), \( \|\partial \tilde{\alpha}_V\|_{C(\pi_i(V);\mathbb{C})} \leq ct \). Hence, its Beltrami quotient \( \mu_{\alpha_V}(\xi) = \partial \tilde{\alpha}_V(z)/\partial \tilde{\alpha}_V(z)\big|_{z = \pi_i \xi} \) satisfies

\[
\|\mu_{\alpha_V}\|_{C(\pi_i(V);\mathbb{C})} \overset{t \to 0}{\longrightarrow} 0. \tag{38}
\]

Estimates (36), (37), and (38) are uniform with respect to \((M', g') \subset M_t\).

The neighbourhoods (32) with \( l_0 \in \Gamma \) and \( q = q(l_0) \) provide an open cover of the compact \( \mathcal{E}(\Gamma) \) in \( \mathcal{E}(M) \). Chose a finite sub-cover \( \{V_k\}_{k=1}^L \) and the positive \( r_0 > 0 \) such that the near-boundary strip \( r(\xi) \in [0, r_0) \) is contained in the union of \( V_k \). Chose the cut-off function \( \kappa \) in (11) in such a way that \( \kappa = 0 \) for \( r > 2r_0/3 \) and \( \kappa = 1 \) for \( r < r_0/3 \). By definition, the map \( \alpha \) satisfies

\[
l'(\alpha(\xi)) = l(\xi), \quad r'(\alpha(\xi)) = r(\xi)
\]
for any \( \xi \) in the strip \( r(\xi) < r_0/3 \) and, hence, \( \alpha(\xi) = \alpha_V(\xi) \) for each \( V \ni \xi \). Then estimate (36) yields

\[
\sup_{\theta \in T_\xi \mathcal{E}(M), \ r(\xi) \in [0,r_0)} \left| \frac{\alpha^* \tilde{g}'(\theta, \theta)}{\tilde{g}(\theta, \theta)} - 1 \right| \rightarrow 0.
\] (39)

Estimate (37) implies

\[
\sup_{r(\xi) \in [0,r_0)} |\alpha(\xi) - \xi| \rightarrow 0.
\] (40)

Note that both convergences (39) and (40) are uniform with respect to \( (M', g') \in \mathcal{M}_t \).

### 2.3 Construction of \( \alpha \) in an intermediate zone

- Let \( \Pi_i[D] \) be an \( \mathcal{E}(M) \)-projective cylinder whose intersection \( \mathcal{E}(M) \cap \Pi_i[D] \) is contained in the strip \( r \in (r_0/6, 5r_0/6) \). For \( \xi \in \mathcal{E}(M) \cap \Pi_i[D] \), we have \( \kappa(\xi) \in (0, 1) \) and the functions \( \mathfrak{D} \) and \( \mathfrak{D}_0 \) in (11) and (12) are of general type. Nonetheless, due to estimates (37) and (26), we still have the (uniform with respect to \( (M', g') \in \mathcal{M}_t \)) estimate

\[
\| \mathfrak{D} - \mathfrak{D}_0 \|_{C^1(\mathcal{E}(M) \cap \Pi_i[D] ; [0, +\infty))} \rightarrow 0.
\]

Also, recall that the function \( \xi' \mapsto \mathfrak{D}_0(\xi, \xi') \) has a unique and non-degenerated global minimum point \( \xi' = \xi \). So, by repeating of arguments of Subsection 2.1 (including the application of Lemma 2.1), we obtain that the map \( \alpha \) is well defined on \( U = \mathcal{E}(M) \cap \Pi_i[D] \), where \( \Pi_i[D] \) is a sub-cylinder of \( \Pi_i[D] \). Also,

\[
\sup_{\xi \in U} |\alpha(\xi) - \xi| \rightarrow 0,
\] (41)

\[
\sup_{\theta \in T_\xi \mathcal{E}(M), \ \xi \in U} \left| \frac{\alpha^* \tilde{g}'(\theta, \theta)}{\tilde{g}(\theta, \theta)} - 1 \right| \rightarrow 0
\] (42)

and the map \( \tilde{\alpha} = \phi' \circ \alpha \circ \phi^{-1} \) (where \( \phi = \pi_i |_{\mathcal{E}(M) \cap \Pi_i[D]} \) and \( \phi' = \pi_i |_{\mathcal{E}'(M) \cap \Pi_i[D]} \)) satisfies

\[
\| \mu_\alpha \|_{C(\pi_i(U); \mathbb{C})} \rightarrow 0,
\] (43)

where \( \mu_\alpha(\xi) = \partial_z \tilde{\alpha}(z)/\partial_z \tilde{\alpha}(z) |_{z=\pi_i(\xi)} \) is the Beltrami quotient of \( \tilde{\alpha} \). Note that all convergences (11)-(13) are uniform with respect to \( (M', g') \in \mathcal{M}_t \).
2.4 Construction of the global diffeomorphism $\alpha$

- Consider the open cover of the compact $\mathcal{E}(M)$ by the following sets: a) the near-boundary strip $U_0 = \{\xi \in \mathcal{E}(M) \mid r(\xi) \in [0, r_0/3]\}$, b) domains $U = \mathcal{E}(M) \cap \Pi_i[D]$, where $\Pi_i[D]$ is a sub-cylinder of $\mathcal{E}(M)$-projective cylinder $\Pi_i[D]$ which does not intersect the strip $r \in [0, r_0/6]$ in $\mathcal{E}(M)$. Fix some its finite sub-cover $\{U_0, U_1, \ldots, U_K\}$ and denote the cylinders corresponding to $U_k$, $k = 1, \ldots, K$ by $\Pi_0[D_k]$ and $\Pi_0[D]$. According to the Subsections 2.1, 2.3 the map $\alpha = \alpha_k$ is defined locally on each $U_k$.

Suppose that $\xi \in U_k \cap U_s$, where $k \neq 0$ and $s \neq k$. Put $\xi' = \alpha_k(\xi)$, $\xi'' = \alpha_s(\xi)$. Due to estimates (17), (10), (11), for any $\delta > 0$, there exists $t(\delta) > 0$ such that $|\xi'' - \xi'| < \delta$ for any $t \in (0, t(\delta))$. Hence, for sufficiently small $t$, the point $\xi''$ belongs to $\mathcal{E}'(M')\Pi_{ik}[D_k]$. Thus, $\xi'$ and $\xi''$ are both points of minimum of the function $D(\xi, \cdot)$ in $\mathcal{E}'(M')\Pi_{ik}[D_k]$. Due to results of Subsections 2.1 and 2.3 the function $D(\xi, \cdot)$ has a unique point of minimum in $\mathcal{E}'(M')\Pi_{ik}[D_k]$. Thus, $\xi'' = \xi'$. So, the map $\alpha$ is well-defined globally on $\mathcal{E}(M)$ for sufficiently small $t$. By definition, $\alpha \circ \mathcal{E}(l) = \mathcal{E}(l')$ for $l \in \Gamma$.

Now, suppose that $\alpha(\xi_1) = \alpha(\xi_2) = \xi'$, where $\xi_1 \neq \xi_2$. Since $\alpha|_{U_k}$ is a diffeomorphism of $U_k$ and $\alpha(U_k)$ for each $k$, there is no domains $U_k$ containing both $\xi_1$ and $\xi_2$. Hence, $|\xi_2 - \xi_1| \geq c > 0$. Otherwise, estimates (17), (10), (11) imply that $|\xi_2 - \xi_1| < c/2$ for sufficiently small $t$, a contradiction. Thus, the map $\alpha$ is an injection and, thus, a diffeomorphism from $\mathcal{E}(M)$ to $\alpha(\mathcal{E}(M))$. In particular, the set $\alpha(\mathcal{E}(M))$ is open in $\mathcal{E}(M)$. Let $\alpha(\xi_s) \to \xi'$ as $s \to \infty$. Since $\mathcal{E}(M)$ is compact, there is a subsequence of $\{\xi_s\}$ which converges to some point $\xi$ in $\mathcal{E}(M)$ (for simplicity, let this subsequence coincides with $\{\xi_s\}$). Since $\alpha$ is continuous, we have $\alpha(\xi_s) \to \alpha(\xi)$ as $s \to \infty$ and, hence, $\xi' = \alpha(\xi)$. Thus, the set $\alpha(\mathcal{E}(M))$ is also closed in $\mathcal{E}(M)$. So, we have $\alpha(\mathcal{E}(M)) = \mathcal{E}'(M')$ and the map $\alpha = \alpha_{M'}$ is a diffeomorphism from $\mathcal{E}(M)$ onto $\mathcal{E}'(M')$ for small $t$ and all $(M', g') \in \mathcal{M}_t$.

Estimates (21), (36), (42) imply that

$$\sup_{\theta \in T_\xi \mathcal{E}(M), \xi \in \mathcal{E}(M)} \left| \frac{\alpha^*g'(\theta, \theta)}{\tilde{g}(\theta, \theta)} - 1 \right| \to 0$$

uniformly with respect to $(M', g') \in \mathcal{M}_t$. So, the map $\alpha$ is near isometric for small $t$.

Introduce the diffeomorphism $\sigma_M = \mathcal{E}'^{-1} \circ \alpha_{M'} \circ \mathcal{E}$. In view of (19), (38), (43), the map $\sigma_M : (M, g) \to (M', g')$ is $K$-quasi-conformal with the
dilatation $K = K(\sigma_{M'})$ satisfying

$$\sup_{(M',g') \in \mathcal{M}_t} K(\sigma_{M'}) \to 0.$$  

In view of the definition (1) of Teichmüller distance between $[(M,g)]$ and $[(M',g')]$, we obtain

$$\sup_{(M',g') \in \mathcal{M}_t} d_T([(M,g)], [(M',g')]) \leq \frac{1}{2} \log \sup_{(M',g') \in \mathcal{M}_t} \inf K(\sigma_{M'}) \to 0.$$  

So, we have proved (2) and Theorem 0.1.

### A Proof of Lemma 2.1

Due to condition ii, the function $x \to |\det F'(x, f(x))|$ attains the positive minimum $m_0$ on the compact $\overline{X}$. Hence, in view of (15), we have $|\det F'(x, f(x))| > m_0/2$ for sufficiently small $t$ and any $x \in \overline{X}$. Introduce the functions

$$F(x, y) := y - (F_y'(x, f(x))^{-1} F(x, y),$$

$$H(x, y) := y - (H_y'(x, f(x))^{-1} H(x, y);$$

then $(x, y)$ is a zero of $F$ (or $H$) if and only if $y$ is a fixed point of the map $F(x, \cdot)$ (or $H(x, \cdot)$). Note that

$$F_y'(x, y) = I - (F_y'(x, f(x))^{-1} F_y'(x, y), \tag{44}$$

$$H_y'(x, y) = I - (H_y'(x, f(x))^{-1} H_y'(x, y) \tag{45}$$

and, in view of (15),

$$\|H_y'(x, \cdot) - F_y'(x, \cdot)\|_{C(X; Y)} \leq ct \tag{46}$$

(here and in the subsequent, the constants do not depend on $H$).

For $x \in X$, $y \in Y$, and $\varepsilon > 0$, denote the $\varepsilon$-neighbourhood of $x$ (or $y$) in $X$ (or $Y$) by $U_\varepsilon(x)$ (or $V_\varepsilon(y)$). Let $x_0 \in X$ and $y_0 = f(x_0)$, then $F(x_0, y_0) = 0$ and $F_y'(x_0, y_0) = 0$. Since $F \in C^1(X, Y)$, for any $\delta > 0$, there exists a sufficiently small $\varepsilon = \varepsilon(\delta) > 0$ such that $|F_y'(x, y)| \leq \delta$ for any $x \in U_\varepsilon(x_0)$ and $y \in V_\varepsilon(f(x))$. In view of (15), there exists $t_0 = t_0(\delta) > 0$ such that
\[|\mathcal{H}'_y(x, y)| \leq 2\delta \text{ for any } t \in (0, t_0) \text{ and any } x \in U_\varepsilon(x_0) \text{ and } y \in V_\varepsilon(f(x)).\]

Then, choosing sufficiently small \( \delta = \delta(x_0) \), we obtain
\[
|\mathcal{H}(x, y') - \mathcal{H}(x, y)| \leq c\delta|y' - y| < |y' - y|,
\]
\[
|\mathcal{H}(x, y) - \mathcal{H}(x, f(x))| \leq c\delta \varepsilon < \varepsilon
\]
for \( t \in (0, t_0), \ x \in U_\varepsilon(x_0), \) and \( y \in V_\varepsilon(f(x)) \). So, the map \( \mathcal{H}(x, \cdot) : V_\varepsilon(f(x)) \rightarrow V_\varepsilon(f(x)) \) is a contraction. Due to the Banach fixed-point theorem, for each \( x \in U_\varepsilon(x_0) \) and \( t \in (0, t_0) \), there exists a unique point \( y =: h(x) \in V_\varepsilon(f(x)) \) such that \( \mathcal{H}(x, y) = h(x) \) i.e. \( H(x, y) = 0 \). Also, the condition \( \|\mathcal{H}'_y(x, y)\| \leq 2\delta \) and formula (15) imply that \( H'_y(x, y)^{-1} \) exists for any \( t \in (0, t_0), \ x \in U_\varepsilon(x_0), \) and \( y \in V_\varepsilon(f(x)) \); in particular, there exists \( H'_y(x, g(x))^{-1} \). Therefore, since \( H \in C^1(X \times Y; \mathbb{R}^n) \), we obtain \( h \in C^1(U_\varepsilon(x_0); Y) \) from the implicit function theorem.

In view of (15) and the following equality
\[
0 = H(x, h(x)) - F(x, f(x)) = H(x, h(x)) - F(x, h(x)) + F(x, h(x)) - F(x, f(x)),
\]
we have
\[
|\int_0^t F'_y(x, f(x) + es)eds| = |F(x, h(x)) - F(x, f(x))| \leq ct,
\]
where \( \tau = |h(x) - f(x)| \) and \( e = (h(x) - f(x))/\tau \). Also, due to the condition \( \|F'_y(x, y)\| \leq \delta \) and formula (14), we have \( \|F'_y(x, y) - F'_y(x, f(x))\| \leq c\delta \) for \( t \in (0, t_0), \ x \in U_\varepsilon(x_0), \) and \( y \in V_\varepsilon(f(x)) \). Thus,
\[
|\int_0^t F_y(x, f(x) + es)eds| \geq \tau|F_y(x, f(x))| - c\delta \tau \geq c\tau,
\]
whence \( \tau \leq ct. \) So, we have proved that \( \|h - f\|_{C(U_\varepsilon(x_0); Y)} \leq ct. \) Next,
\[
0 = \frac{dF(x, f(x))}{dx} = F'_x(x, f(x)) + F'_y(x, f(x))f'(x),
\]
\[
0 = \frac{dH(x, h(x))}{dx} = H'_x(x, h(x)) + H'_y(x, h(x))h'(x),
\]
whence
\[
f'(x) - h'(x) = (H'_y(x, h(x)))^{-1}H'_x(x, h(x)) - (F'_y(x, f(x)))^{-1}F'_x(x, f(x)) =
\]
\[
= (H'_y(x, h(x)))^{-1}H'_x(x, h(x)) - (F'_y(x, h(x)))^{-1}F'_x(x, h(x)) +
\]
\[
+(F'_y(x, h(x)))^{-1}F'_x(x, h(x)) - (F'_y(x, f(x)))^{-1}F'_x(x, f(x)).
\]

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Thus, \( \|h - f\|_{C^1(U_e(x_0))} \leq ct \) since \( F \in C^2(\bar{X} \times \bar{Y}; \mathbb{R}^n) \) and the condition (15) holds.

The neighbourhoods \( U_{e(\delta)}(x_0) (x_0 \in \mathcal{X}, \delta = \delta(x_0)) \) provides an open cover of \( \mathcal{X} \). Choose a finite subcover \( U_{e(\delta)}(x_{0,k}) (k = 1, \ldots, K) \) and put \( t_* = \min\{t_0(\delta(x_{0,k}))\}_k \). Then, for \( t \in (0, t_*) \), the function \( h \) is defined globally on \( \mathcal{X} \) and obeys \( H(x, h(x)) = 0 \) for \( x \in \mathcal{X} \) and satisfies (15). Also, there exists neighbourhood \( N \) of the set \( \{(x, f(x)) | x \in \mathcal{X}\} \) in \( \mathcal{X} \times \mathcal{Y} \) such that all zeroes of \( H \) in \( N \) belong to the graph of \( h \). The set \( N' = (\mathcal{X} \times \mathcal{Y}) \setminus N \) is compact, so \( |F| \) attains positive minimum \( M \) on \( N' \). Due to (15), \( H \) has no zeroes on \( N' \) for sufficiently small \( t > 0 \).

Now, suppose that \( f \) is a diffeomorphism from \( X \) to \( f(X) \). Therefore, \( |\det f'(x)| > c_0 \) and \( |f(x') - f(x)| \geq c_1 \text{dist}_{\mathcal{X}}(x', x) \) \((c_0, c_1 > 0)\) for any \( x \in \mathcal{X} \).

Due to (15), \( |\det h'(x)| > c_0/2 > 0 \) for \( x \in \mathcal{X} \). Suppose that \( h(x) = h(x') \), then

\[
0 = \int_x^{x'} h'(s)ds = \int_x^{x'} f'(s)ds + \int_x^{x'} (h'(s) - f'(s))ds = f(x') - f(x) + \int_x^{x'} (h'(s) - f'(s))ds,
\]

where integral is taken over shortest curve in \( \mathcal{Y} \) connecting \( x \) and \( x' \). Thus,

\[
c_1 \text{dist}_{\mathcal{X}}(x', x) \leq |\int_x^{x'} (h'(s) - f'(s))ds| \leq ct \text{dist}_{\mathcal{X}}(x', x)
\]

due to (15). So, for sufficiently small \( t \), the equality \( h(x) = g(x') \) implies \( x' = x \). Therefore \( h \) is a a diffeomorphism from \( \mathcal{X} \) to \( h(\mathcal{X}) \).

**B Proof of Lemma 2.2**

Denote \( X_1 := x_h, X_2 := \partial_r x_h \) and \( Y_1 := x_{h'}, Y_2 := \partial_r x_{h'} \). The functions \( X = (X_1, X_2) \) and \( Y = (Y_1, Y_2) \) satisfy the following Cauchy problems for geodesic equations

\[
\frac{\partial X}{\partial r} = A(X), \quad X(0, \mu) = X_0(\mu), \tag{47}
\]
\[
\frac{\partial Y}{\partial r} = B(Y), \quad Y(0, \mu) = Y_0(\mu), \tag{48}
\]
where
\[ A(X) = (X^2, -\Gamma^i_{jk}(X^1)X^j_2X^k_2), \quad X_0(\mu) = ((0, \mu), -\nu_h(\mu)) \]
\[ B(Y) = (Y^2, -\tilde{\Gamma}^i_{jk}(Y^1)Y^j_2Y^k_2), \quad Y_0(\mu) = ((0, \mu), -\nu'_h(\mu)) \]
and \( \Gamma^i_{jk} \) and \( \tilde{\Gamma}^i_{jk} \) are Christoffel symbols corresponding to metrics \( h \) and \( h' \), respectively. Denote by \( V \) a bounded domain in \( \mathbb{R}^2 \) containing all \(-\nu_h(\mu)\) \((\mu \in (-r_0, r_0))\). Condition (33) implies
\[ \|A - B\|_{C^1(\mathbb{R} \times \mathcal{Y})} \leq cs, \quad \|Y_0 - X_0\|_{C^1([-r_0, r_0])} \leq cs. \tag{49} \]
The solutions of (47) and (48) are fixed points of the maps
\[ \mathcal{A}[X](r, \mu) = X_0(\mu) + \int_0^r A(X(r', \mu))dr', \]
\[ \mathcal{B}[Y](r, \mu) = Y_0(\mu) + \int_0^r B(Y(r', \mu))dr'. \]
Put \( a := \|A\|_{C^1(\mathbb{R} \times \mathcal{Y})} \). For \( r_1 > 0 \) and \( \epsilon > 0 \), denote the closed ball in \( C(\overline{\mathcal{B}_{r_1}}) \) of radius \( \epsilon \) with center at \( F \) by \( \mathcal{B}_{\epsilon, r_1}(F) \). According to (39), the inequalities
\[ \|B(Y_1) - A(Y_1)\|_{C(\overline{\mathcal{B}_{r_1}})} \leq c(1 + \epsilon)s, \]
\[ \|\mathcal{A}[X_1] - X_0\|_{C(\overline{\mathcal{B}_{r_1}})} \leq cr_1\epsilon, \]
\[ \|\mathcal{B}[Y_1] - Y_0\|_{C(\overline{\mathcal{B}_{r_1}})} \leq c(a + s)r_1\epsilon \]
and
\[ \|\mathcal{A}[X_1] - \mathcal{A}[X_2]\|_{C(\overline{\mathcal{B}_{r_1}})} \leq \epsilon\|X_1 - X_2\|_{C(\overline{\mathcal{B}_{r_1}})ar_1}, \]
\[ \|\mathcal{B}[Y_1] - \mathcal{B}[Y_2]\|_{C(\overline{\mathcal{B}_{r_1}})} \leq \epsilon\|Y_1 - Y_2\|_{C(\overline{\mathcal{B}_{r_1}})(a + s)r_1} \]
hold for any \( X_1, X_2 \in \mathcal{B}_{\epsilon, r_1}(X_0) \) and \( Y_1, Y_2 \in \mathcal{B}_{\epsilon, r_1}(Y_0) \). Thus, for sufficiently small \( \epsilon, r_1, s \), the maps \( \mathcal{A} \) (\( \mathcal{B} \)) are contractions on \( \mathcal{B}_{\epsilon, r_1}(X_0) \) and \( \mathcal{B}_{\epsilon/2, r_1}(Y_0) \subset \mathcal{B}_{\epsilon, r_1}(X_0) \), respectively, and their fixed points \( X, Y \) satisfy
\[ \|Y - X\|_{C(\overline{\mathcal{B}_{r_1}})} = \|\mathcal{B}[Y] - \mathcal{A}[X]\|_{C(\overline{\mathcal{B}_{r_1}})} \leq \|\mathcal{B}[Y] - \mathcal{A}[Y]\|_{C(\overline{\mathcal{B}_{r_1}})} + \|\mathcal{A}[Y] - \mathcal{A}[X]\|_{C(\overline{\mathcal{B}_{r_1}})} \leq cs + \|Y - X\|_{C(\overline{\mathcal{B}_{r_1}})/2}. \]

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Hence $\| Y - X \|_{C(Qr_1)} \leq cs$ and
\[
\| \partial_r Y - \partial_r X \|_{C(Qr_1)} = \| B[Y] - A[X] \|_{C(Qr_1)} \leq \\
\leq \| B[Y] - A[Y] \|_{C(Qr_1)} + \| A[Y] - A[X] \|_{C(Qr_1)} \leq \\
\leq c(s + \| Y - X \|_{C(Qr_1)}) \leq cs.
\]
Differentiation of (47) and (48) yields the following systems of linear equations
\[
\frac{\partial}{\partial r} \left( \frac{\partial X}{\partial \mu} \right) = A'_X(X) \frac{\partial X}{\partial \mu}, \quad \frac{\partial X}{\partial \mu}(0, \mu) = (X_0)'(\mu), \\
\frac{\partial}{\partial r} \left( \frac{\partial Y}{\partial \mu} \right) = B'_Y(Y) \frac{\partial Y}{\partial \mu}, \quad \frac{\partial Y}{\partial \mu}(0, \mu) = (Y_0)'(\mu).
\]
Thus, the estimates above lead to
\[
\| \frac{\partial Y}{\partial \mu} - \frac{\partial X}{\partial \mu} \|_{C(Qr_1)} \leq cs.
\]
and, hence, to (34). The remaining statements of Lemma 2.2 easily follows from (34).

**Statements and Declarations**

**Competing Interests.** On behalf of all authors, the corresponding author states that there is no conflict of interest.

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