Large Josephson current in Weyl nodal loop semimetals due to odd-frequency superconductivity

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The closed nodal line loop Fermi surface in a Weyl nodal loop material (WNL) results in protected zero-energy drumhead surface states; surface bands with flat energy dispersion. The drumhead states are also fully spin-polarized in WNLs breaking time-reversal symmetry. The large density of states of the drumhead states makes WNLs exceedingly prone to electronic ordering. At the same time, naively the spin-polarization would entirely prevent conventional superconductivity due to its spin-singlet nature. Here we show the complete opposite: WNLs are extremely promising materials for superconducting Josephson junctions due to odd-frequency superconductivity. By sandwiching a WNL between two conventional superconductors we theoretically demonstrate the presence of very large Josephson currents, even up to orders of magnitude larger than for normal metals. The large currents are generated both by an efficient transformation of spin-singlet pairs into odd-frequency spin-triplet pairing by the Weyl dispersion and the drumhead states ensuring exceptionally strong proximity effect.

WNLs are three-dimensional (3D) semimetals where the nodal loop Fermi surface position them in-between a conventional metal with its 2D Fermi surface and a Weyl semimetal which only has 0D Fermi points [1, 2]. Multiple candidate WNLs have recently been both proposed [3–7] and experimentally observed in compounds such as PbTaSe_2, ZrSiS, Ca_3P_2, CaAgAs [8–11]. Away from the nodal loop Fermi surface the dispersion is Weyl-like, completely locking the electron orbital and spin degrees of freedom together.

The nodal loop Fermi surface results in drumhead surface states at zero energy, whose area is set by the projection of the nodal loop on the surface plane [12–14]. The origin of this surface state is similar to that of the Weyl semimetals, but there only 1D surface arcs are formed due to the lower dimensionality of the bulk Fermi surface. The flat band dispersion of the WNL drumhead states results in a van-Hove like singularity in the surface density of states (DOS). As a result, the surface becomes extremely prone to electronic ordering. Full spin-polarization of the surface has already been established in the common WNLs breaking time-reversal symmetry considered here [15], while superconducting order has been suggested for other systems with surface flat bands [16–18]. Superconductivity has also been studied in bulk WNLs, with 3D chiral superconducting states proposed based on both symmetry analysis [19] and renormalization group calculations [20]. Moreover, it has been shown that the bulk chiral $p$-wave superconductivity develops $p+ip$ surface states [15]. However, in terms of surface superconductivity, the complete surface spin-polarization has been assumed to prohibit any spin-singlet superconductivity, including proximity effect from conventional superconductors (SCs) [15].

The incompatibility of spin-polarization and conventional (spin-singlet, $s$-wave) superconductivity has actually been remedied in a few cases by generating an exotic state of odd-frequency superconductivity [21–23]. Odd-frequency Cooper pairs are odd under the exchange of the relative time coordinate between the two electrons forming the pair, in contrast to the conventional equal-time pairing. As a consequence, odd-frequency pairing allows the common $s$-wave superconducting state to have spin-triplet symmetry and still satisfy the necessary fermionic nature of superconductivity. In this way, odd-frequency spin-triplet pairing has been evoked to explain the long-ranged superconducting proximity effect measured in superconducting-ferromagnet junctions [24], but odd-frequency superconductivity has also been found in nonmagnetic superconducting junctions [25], as well as in bulk multiband SCs [26] and driven systems [27]. As equal-time expectation values vanish for odd-frequency superconductivity, it becomes easily a hidden order and direct detection is notoriously hard. Still, odd-frequency superconductivity has been shown to impact physical properties ranging from the Meissner [28–33] and Kerr effects [34, 35], to the existence of a finite supercurrent in half-metal (HM) Josephson junctions [36–39]. In the last case, the full spin-polarization of the HM prohibits spin-singlet superconductivity, but Josephson effect has still been shown to be present in HMs with spin-active interfaces due to creating odd-frequency equal-spin triplet pairing [39].

In this work, we study a Josephson junction constructed by sandwiching a WNL between two conventional SCs, as shown schematically in Fig. 1(a). Despite the complete spin-polarization of the surfaces, which naively prohibits any proximity-induced superconductivity and thus Josephson effect, we find a huge Josephson current, even orders of magnitude larger than in normal metal (NM) and HM junctions. We first show how the spin-orbital Weyl interaction in WNLs results in a very efficient creation of equal-spin triplet Cooper pairs, mimicking the spin arrangement in the normal state. It is these equal-spin pairs that carry the Josephson current, which is further dramatically enhanced thanks to the zero-energy drumhead surface states generating excellent interfaces with the SCs. The combined effect is that WNLs produce what can be classified as optimal odd-frequency Josephson junctions.
We start by establishing the characteristic bulk and surface properties of WNLs. For this we use a prototype model that captures all important physical details and is described by the Hamiltonian [15, 19]:

$$H_{WNL} = t_w \sigma_x \left( 6 - \alpha_1 - 2 \cos(k_x a) - 2 \cos(k_y a) - 2 \cos(k_z a) \right) + 2 \alpha_2 t_w \sigma_z \sin(k_z a) - \mu.$$  

(1)

Here, $\mathbf{k}$ is the electron wave vector and $\sigma$ are the Pauli matrices in spin space, as we assume that the WNL breaks time-reversal symmetry. Moreover, $t_w$ is the overall hopping amplitude, $\mu$ the chemical potential, and $a$ the lattice constant. For simplicity we measure energy and length in units of $t_w$ and $a$, respectively. The Fermi surface is a nodal line loop at zero doping $\mu = 0$, while it forms a thin torus for nonzero $\mu$ with its shape tuned by the two parameters, $\alpha_1$, $\alpha_2$. We primarily use $\alpha_1 = \alpha_2 = 1$, which results in an essentially circular Fermi surface, but our results are not sensitive to this particular choice, see supplementary information (SI). The energy dispersion away from zero energy takes the Weyl-like $k \cdot \sigma$ form, giving the material class its name [10, 20].

To study a finite WNL and its surface states, we use the continuum Hamiltonian Eq. (1) in the $(k_x, k_y)$ directions, while we discretize the Hamiltonian in the $z$-direction. We assume a lattice with $n_w = 21$ layers along the $z$-direction, as indicated by black solid circles in Fig. 1(a). This creates stable bulk conditions in the middle of the slab and well-formed surface states, see Figs. 1(b,c). In Fig. 1(b) we plot the energy dispersion along the $k_x$ direction. At zero doping the band dispersion is electron-hole symmetric and two bands, indicated in red, have a vanishing electron group velocity and form zero-energy flat bands in a large region around the $\Gamma$ point. These two bands reside on the two slab surfaces, as clearly seen in Fig. 1(c), where we display the spin-polarized DOS for several different layers. The two surface layers have a very tall DOS peak at zero energy, which are fully spin-polarized but in opposite directions, as expected from WNL drumhead surface states [1]. The bulk on the other hand has only a small constant DOS at zero energy due to the nodal line Fermi surface and there is no net spin-polarization in the middle of the slab. It is the spin-orbit-like interaction term $\alpha_2 t_w \sigma_z \sin(k_z a)$ in the WNL Hamiltonian Eq. (1) that causes the characteristic spin rotation throughout a WNL material; from spin-down polarization at zero-energy for the left surface (layer 1) to spin-up polarization for the right surface (layer $n_w$), which we schematically illustrate with arrows in Fig. 1(a). It is only at very large energies that the first ($n_{th}$) surface hosts any up- (down-)spin polarization. At finite doping the drumhead surface states remain unchanged with a full spin-polarization, but now located at an energy $\mu$ below the Fermi level. At the same time, the DOS at the Fermi level in the bulk increases due to the torus-shaped Fermi surface at finite doping.

### Odd-Frequency Pairing

Next, we place two conventional spin-singlet $s$-wave SCs of the same superconducting material in proximity to the two surfaces of the WNL slab, see Fig. 1(a). The superconducting order parameter amplitude in the SCs is set by $\Delta$, but we allow for different phases, $\varphi_{L,R}$ such that a Josephson current can be generated across the superconducting heterostructure. We couple the WNL and the SCs using a generic spin-independent tunneling amplitude $t_{sc-w}$.

To study proximity-induced superconductivity in the WNL we extract the superconducting pair amplitudes in the WNL by calculating the anomalous Green’s function $F$ of the full heterostructure (see methods). Using $s$-wave SCs and requiring stability against disorder, allow us to primary focus on isotropic, or equivalently on-site, $s$-wave pairing. In the SI we also report all nearest neighbor extended $s$-wave pair amplitudes, but they are significantly smaller and thus less important. In terms of spins, the Weyl spectrum rotates the spin and can thus allow for both equal- and mixed-spin triplet pairs. We therefore study all possible spin configurations for the superconducting pairing.

In Fig. 2 we plot the real (upper panels, a-d) and imaginary (lower panels, e-h) parts of the anomalous Green’s func-
Figure 2. Pair amplitudes in WNL Josephson junctions and their frequency dependence. Real (upper panels, a-d) and imaginary (bottom panels, e-h) parts of the anomalous Green’s function $F$ as a function of frequency $\omega$, capturing the pair amplitudes divided into the equal-spin (↑↑ and ↓↓), mixed-spin (↑↓ + ↓↑) triplet, and spin-singlet (↑↓ − ↓↑) components. Left to right panels shows results for the $n = 1, 2, 3$ and middle $n = (n_w + 1)/2 = 11$ layers, respectively. Here $\mu_w = 0, t_{sc-w} = 0.5, \Delta = 0.01, t_{sc} = 1, \mu_{sc} = 2$, and $\varphi_L = \varphi_R = 0$.

The anomalous Green’s function $F$ as a function of frequency $\omega$, divided into all possible spin configurations. Each column represent a different layer in the WNL; layers 1, 2, 3, and $n = (n_w + 1)/2$. We note directly that all different spin-triplet components appear throughout the WNL and that they are always odd functions of frequency, as required by the Fermi-Dirac statistics of the Cooper pairs. In the first surface layer there is still notable spin-singlet pairing. This is to be expected since the first layer is directly coupled to the SC and therefore necessarily harbors superconducting pairs of same symmetry as in the SC. However, the spin-singlet amplitude decay extremely quickly into the WNL, such that it has essentially disappeared already in the second layer. This behavior is not surprising when considering that the drumhead states of the WNL are fully spin-polarized and thus the tunneling of opposite spins is energetically extremely costly. Similar immediate destruction of spin-singlet amplitudes have previously also been reported for HM junctions [38]. Despite the complete lack of proximity effect for spin-singlet superconductivity beyond the first surface layer, there is still significant pairing induced in the WNL. It is instead equal-spin triplet pairing with spins aligned with the spin-polarization of the drumhead state that grows and heavily dominates in the subsurface layers. Thus, the WNL essentially becomes an odd-frequency superconductor beyond the very first surface layer.

In the middle of the sample, Fig. 2(d,h), all pair amplitudes are suppressed due to the distance from the SC interface, but notably, the two equal spin pairing terms have exactly the same magnitude, just mirrored in $\omega = 0$. Plotting the pair amplitudes also for the right half of the WNL, we find exactly the same results as for the left part shown in Fig. 2, only with spin-up and spin-down interchanged. The behavior of the equal-spin pairing is the superconducting equivalent of the spin-polarization in the normal state twisting from full spin-down polarization in the left surface layer to full spin-up polarization in the right surface layer. Thus the appearance of large odd-frequency equal-spin triplet components in WNLs is guaranteed by the intrinsic Weyl spin-orbital structure of the WNL normal state.

To better probe the propagation of Cooper pairs inside the WNL, we plot in Figs. 3(a-c) the absolute value of the different pair amplitudes as a function of the layer index $n$ for the whole left side of the WNL. The pair amplitudes in the right part is obtained from the left layers by just interchanging spin-up and spin-down. We display the result for three different chemical potentials, $\mu = 0, 0.05, 0.1$, respectively, to capture both the nodal loop and torus-shaped Fermi surface WNLs. Further, we set $\omega = 0.5\Delta$, but keep all other parameters as in Fig. 2. Other choices of $\omega$ can be found in the SI, showing no change of trends compared to Fig. 3. We find the
Figure 3. Pair amplitude decay in WNL Josephson junctions compared to NM and HM junctions. Evolution of the absolute value of equal-spin (↑↑ and ↓↓), mixed-spin (↑↓ + ↓↑) triplet, and spin-singlet (↑↓ − ↓↑) pair amplitudes into the middle of the WNL (a,b,c) with chemical potential set to \( \mu_w = 0, 0.05, 0.1 \), respectively, and compared to similar junctions with NM (d) and HM without (e) and with (f) an spin-active interface. Pair amplitudes are extracted for \( \omega = 0.5\Delta \). Same parameters as in Fig. 2 for the WNL, while the NM and HM are the spin-independent part of the WNL with the HM having an additional \( m_z \sigma_z \) with \( m_z = 0.5 \) magnetization in the bulk and an interface \( m_z \sigma_z \) with \( m_z = 1 \) term modeling the spin-active interface.

same extremely fast suppression of the spin-singlet amplitude for all doping levels. The mixed-spin triplet state experiences the same decay, due to the same unfavorable spin alignment as the spin-singlet pairing. Instead, it is spin-down triplet pairing that is clearly dominating, also well into the WNL and for all \( \mu \). The finite spin-up triplet pair amplitude is due to probing the pair amplitudes at finite energies, where also bulk states give a finite contribution. Increasing the doping level thus show no significant changes in the relative importance of the different pairing channels. However, the magnitude of the pair amplitudes increases due to the increased bulk DOS in finite doped WNL. Overall this shows that the almost exclusively odd-frequency pairing state in the WNL is not sensitive to the tuning of the doping level.

To demonstrate the remarkable pairing in WNL Josephson junctions we compare the results with the behavior of similar Josephson junctions made of NM (d) and HMs without (e) and with (f) a spin-active interface region. In order to create systems with directly comparable properties we model the NM by removing all spin-dependence from the WNL Hamiltonian in Eq. (1), such that \( H_{NM} = t_w (6 - 2\cos(k_x) - 2\cos(k_y) - 2\cos(k_z)) - \mu \). This creates a prototype parabolically dispersive metal, where we set \( \mu = 0.1 \) to reach a finite bulk DOS.

For the NM, no term breaks the spin degeneracy, and thus only spin-singlet pairing is present in the NM. This spin-singlet amplitude experiences a regular slow decay, set by the conventional proximity effect.

For the HM junctions, we add the term \( m_z \sigma_z \), with \( m_z = 0.5 \), to the NM Hamiltonian \( H_{NM} \), such that only spin-down electrons are present at zero energy. This strong magnetization causes the same dramatic suppression of the spin-singlet amplitude as in the WNL. It also allows for spin-rotation into the mixed-spin triplet state. However, this odd-frequency spin-triplet pairing state is always small and fast-decaying, as the occurrence of spin-up components is not energetically favored by the magnetization [40]. To achieve a spin-down triplet component, an additional spin quantization axis has to be present in the HM junction. This is often achieved by introducing a spin-active region at the interface between the SC and HM, see e.g. [38, 39]. In this case the Cooper pair spin quantization axis rotates between the interface and the HM bulk, with the consequence that spin-equal pairing is induced beyond the interface. In Fig. 3(f) we therefore add a term \( m_x \sigma_x \), with \( m_x = 1 \), to the two surface layers of the HM to model a strongly spin-active interface. As a result, spin-equal triplet pairing is generated, of which the spin-down component survives throughout the HM region. The spin-up triplet state is also initially generated at the interface, but it is ener-
getically unfavorable and decays very quickly in the HM.

Comparing the WNL with the NM and HM junctions, we see that the WNL junction closest resembles that of the HM with an active spin interface, since they both experience a strong proximity effect consisting of odd-frequency equal-spin triplet pairing. However, WNL Josephson junctions are fundamentally different from HM Josephson junctions as they do not need any additional spin-active interface region added during manufacturing in order to generate equal-spin odd-frequency pairing. Moreover, the intrinsic Weyl spin-orbit coupling in the WNL causes the initial spin-down pairing in the left surface to rotate into spin-up pairing in the right surface. As a consequence, there is a clear decay of the spin-up triplet component into the middle of the WNL, which is not present in the HM. It is thus not the distance from the SC that causes the main decay of the equal-spin triplet components in the WNL, but mainly the continuous rotation of the spin orientation of the Cooper pairs. We also note that the size of the pair amplitudes in the WNL at zero doping is actually of the same order of magnitude as in the NM junctions. This is particularly surprising since the nodal line bulk state has a much smaller DOS at low energies compared to the NM (see SI for a detailed account on the zero-energy DOS). We attribute the large pair amplitudes in undoped WNLs to the singular zero-energy DOS of the drumhead surface states, which creates a naturally pair amplitudes in undoped WNLs to the singular zero-energy (see SI). We also note that for large µ the drumhead surface states are located far from the Fermi level and are thus less active in electric transport. Thus, the large current in heavily doped WNL junctions is instead primarily a manifestation of how powerful of the Weyl spin-orbit interaction is in generating odd-frequency equal-spin pairing to carry the current.

**EXOTIC JOSEPHSON CURRENT**

Having shown how odd-frequency spin-triplet pairing generates large proximity-induced superconductivity in a WNL, despite the spin-polarized surface states and the spin-singlet SCs, we now turn to the possibility of measuring a finite Josephson current in the WNL junction. For comparison we also calculate the current in NM and HM junctions. While we limited the investigation in Figs 2-3 to isotropic s-wave pairing, the contributions from all pairing channels are automatically included when calculating the current. In Fig. 4(a) we show in a log-plot the maximum Josephson current \( J \) between two conventional spin-singlet SCs as a function of the chemical potential \( \mu \) in the WNL and also directly compare with the results of NM and HM junctions. The results are obtained for \( \varphi_L = \pi/2, \varphi_R = 0 \), which to a very good approximation gives the maximum current as \( J \sim \sin(\varphi_L - \varphi_R) \), see SI.

We find that in all junctions the currents increases with increasing \( \mu \), while also displaying some overlaid oscillatory behavior. This is expected since the low-energy DOS increases with \( \mu \) for all junctions with some smaller oscillations due to finite size effects. Most notably, we find that the WNL junction carry the most current over a wide range of low to moderate doping levels, often even orders of magnitude larger than the NM and HM junctions. This is really quite remarkable considering both the fully spin-polarized surface states, which makes propagation of conventional pairing impossible, and the low bulk DOS of the nodal line/thin torus Fermi surface in the bulk. In fact, the WNL bulk DOS is for the full doping range smaller than that of the bulk NM, for \( \mu \leq 0.4 \) it is even several times smaller, see SI. We therefore must accredit the large Josephson current in WNL junctions to the existence of large odd-frequency spin-triplet pairing and the singular DOS of the drumhead surface states. WNL Josephson effect thus forms a key example on the importance of odd-frequency pairing in inhomogeneous superconducting systems. If it were not for the odd-frequency correlations, there would be no proximity-induced superconductivity or measurable Josephson effect in WNL junctions. Note that while there exists some spin-singlet pairing in finite doped WNLs due to higher energy states, the amplitude is negligible compared to the amplitude in NM junctions, and thus the wast amount of the current in doped WNL is still carried by the equal-spin triplet pairs.

Comparing with the HM junctions that also host odd-frequency spin-triplet pairs, we find that the HM without a spin-active interface only carry a minute Josephson current. Introducing a spin-active interface layer, we find a significantly increased current in the HM due to the perseverance of odd-frequency equal-spin triplet pairs. Yet, it is only at extremely large \( \mu \) that the HM system carries a similarly sized current to the WNL junction. Here we note that for large \( \mu \) the drumhead surface states are located far from the Fermi level and are thus less active in electric transport. Thus, the large current in heavily doped WNL junctions is instead primarily a manifestation of how powerful of the Weyl spin-orbit interaction is in generating odd-frequency equal-spin pairing to carry the current.

In Figs. 4(b,c) we further explore the parameter dependencies of the Josephson current. Fig. 4(b) shows the variation of the Josephson current with respect to the superconducting gap parameter \( \Delta \), keeping the chemical potential of NM and HM at \( \mu = 0.1 \), while the WNL is undoped. Thus, both the NM and HM have a large metallic DOS at low energies, while the WNL has only a 1D Fermi nodal loop. Usually, this parameter is lower than the hopping inside the junction and in the SCs, limiting it to \( t_{SC-SC} < 1 \), and in practice it is tuned by modifying the interfaces. Larger tunneling clearly enhances the Josephson current in all junctions, but we find that the WNL junctions carries the largest Josephson current for all values of \( t_{SC-SC} \). We provide additional plots for other choices of chemical potential in NM and HM in the SI, verifying that the large Josephson current through the WNL in Fig. 4 is generic and not restricted to a narrow range of physical parameters.
The physics of the WNL Josephson junction can be understood from the behavior of HM junctions with spin-active interfaces. The continuous rotation of the spin polarization makes the WNL Josephson junctions more similar to HM junctions with a helical magnet configuration instead of just a spin-active interface. Such helical magnets have recently become more concreteness we use $n_{a_n} = 21$ layers in the z-direction, but our results are not sensitive to the number of layers, granted that the surface states are spatially well separated and the junction still carry a supercurrent for a NM. By calculating the retarded Green’s function $G = (\omega + i0^+ - H)^{-1}$ we obtain the spin-resolved DOS in each WNL layer $n$ as $\rho_{n,\sigma} = -\frac{i}{\pi} \sum_{k} G_{n,\sigma}(\mathbf{k}||\omega + i0^+)$. To investigate Josephson junctions we attach a conventional SC to each of the two WNL slab surfaces with the Hamiltonian given in Nambu space as

$$\mathcal{H}_j^{\sigma} = \begin{pmatrix} \hbar e \sigma_0 & \Delta e^{i\varphi_0}/\sigma_y \\ \Delta e^{-i\varphi_0}/\sigma_y & -\hbar e \sigma_0 \end{pmatrix},$$

where $j = L, R$ indicates the left and right SC, respectively. Here the normal state Hamiltonian in $(k_x, k_y)$ takes the form $\hbar e = t_{sc}(2 - \cos(k_x a) - \cos(k_y a)) - \mu_{sc}$. While in the z-direction we assume a tunneling $t_{sc}$ between different layers of the SCs. Spin-singlet s-wave superconductivity is implemented as usual by an on-site order parameter $\Delta e^{i\varphi_0}$. We use $n_{a_n} = 20$ layers for each SC and set $t_{sc} = 1$ and $\mu_{sc} = 2$ to create SCs with large low-energy DOS. The WNL and SCs are connected by a generic spin-independent tunneling $t_{sc-w}$. The

**DISCUSSION**

To summarize we establish a large and exotic Josephson effect in WNL superconducting junctions, driven by the spin-polarized surface drumhead surface states and the bulk Weyl spin-orbit interaction. This is in spite of the spin-polarized drumhead states prohibiting propagation of the spin-singlet pairs in the conventional SC contacts. Instead, the Weyl spin-orbit interaction enables a very effective transformation of the spin-singlet pairing into an odd-frequency equal-spin triplet pairing state, which then carries the Josephson current. Furthermore, the singular DOS of the drumhead surface states make carrier transport between the SCs and the WNL very effective, which significantly enhances the current. In fact, we find that the WNL Josephson current can easily be orders of magnitude larger than the current in NM Josephson junctions, despite a much lower bulk DOS in the WNL due to its nodal line Fermi surface.

The physics of the WNL Josephson junction can be understood from the behavior of HM junctions with spin-active interfaces, as both junctions have dominating odd-frequency pairing due to their equal-spin triplet pairing carrying the Josephson current. Furthermore, the singular DOS of the drumhead surface states make carrier transport between the SCs and the WNL very effective, which significantly enhances the current. In fact, we find that the WNL Josephson current can easily be orders of magnitude larger than the current in NM Josephson junctions, despite a much lower bulk DOS in the WNL due to its nodal line Fermi surface.

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results are not qualitatively sensitive to $n_{sc}$ or other physical parameters in the SCs, as shown in the SI. Using the same Green’s function technique but now the whole system WNL + SCs, we extract the anomalous electron-hole part, which is proportional to $(c_{nr}^\dagger c_{nr})$, with $c_{nr}^\dagger$ the electron creation operator in layer $n$ with spin $\sigma$, and thus gives the pair correlations in all SC and WNL layers. By integrating over $k_0$ we obtain the $x$-wave contribution, which we report individually for each pair spin configuration. Note that in order to achieve the correct pair amplitudes as a function of $\omega$, we need to use the advanced (retarded) Green’s function for negative (positive) frequencies [23, 44].

To calculate the Josephson current between two SCs, we use the continuity equation $\nabla \cdot J + (\partial \rho/\partial t) = 0$, where $J$ is the current density vector and $\rho = \sum_{\sigma} c_{nr}^\dagger c_{nr}$ is the density operator [45, 46]. Since we are working only with layers in the $z$-direction, we sum over all $k_0$. Moreover, $\langle \ldots \rangle$ indicates the expectation value taken over whole system, which we obtain by summing over all occupied energy eigenstates. Finally, the derivative of the density operator $\rho_{\sigma}$ can be obtained from $(\partial \rho_{\sigma}/\partial t) = \{[H, \rho_{\sigma}]\}$. Here the right-hand side generates terms of the type, $c_{n+1}\sigma c_{n-1}$ and $c_{n}^\dagger c_{n-1}$, which are intuitively proportional to the in- and out-going currents ($J_{in}$, $J_{out}$) in each layer. Writing $\nabla \cdot J = (J_{out} - J_{in})/a^2$, we can obtain the Josephson current $J = e(J_{out} - J_{in})/h$.

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In this supplementary information (SI) we provide additional results to support the findings in the main text.

**Effects of anisotropy of the Fermi nodal loop**

As discussed in the main text, a WNL can be modeled with different choices of \( \alpha_1 \) and \( \alpha_2 \). For \( \alpha_1 = \alpha_2 \) the Fermi surface at zero doping is almost a circle (a perfect circle if the dispersion is \( E \sim k^2 \)), which we use in the main text. Here we show that our results are not dependent on this particular choice of \( \alpha_1 \) and \( \alpha_2 \) and thus the anisotropy of the Fermi nodal loop is not an important factor.

In Fig. S5(a-c), we set \( \alpha_1 = 1 \) and present the propagation of different pair amplitudes through the WNL for \( \alpha_2 = 0.5, 1, 2 \), respectively, such that (b) is the same as Fig. 3(a) in the main text. Although the strength of the pair amplitudes changes as we tune \( \alpha_2 \), we find rapidly decaying spin-singlet and more persistent spin-down triplet pairings throughout the parameter regime. In Fig. S5(d-f) we then plot the Josephson current as a function of the same parameters as in Fig. 4 in the main text, i.e. as a function of the chemical potential (d), order parameter \( \Delta \) (e), and tunneling between the SC and WNL \( t_{\text{sc-w}} \) (f), for the same choices of \( \alpha_2 = 0.5, 1, 2 \) and also compare the results with a NM junction. As seen in these figures, the large Josephson current in the WNL junction is preserved independently of the anisotropy of the Fermi nodal loop, and the current is always larger or comparable to that of a NM junction, even in the large doping regime. At the lowest doping levels larger \( \alpha_2 \) creates larger currents, as this term partially governs the hopping in the current direction. However, this is not the only parameter controlling the current and the behavior also changes at higher doping levels. We also calculate the Josephson current for several other choices of \( \alpha_{1,2} \) and find that as long as there exist fully spin-polarized drumhead surface states, the Josephson current is very large in the WNL junction. This further support our claim that the drumhead surface states are crucial for the Josephson effect in WNL junctions.

**Extended s-wave pair amplitudes**

In the main text, we focus on isotropic and \( k \)-independent pairing, represented by on-site amplitudes, but other pair amplitudes are also present in a WNL Josephson junction. Considering that disorder is always present in a real system we can limit our considerations to different \( s \)-wave amplitudes, as all other amplitudes average to zero or near zero for a reasonably symmetrically \( \Gamma \)-centered Fermi surface. In Fig. S6, we present the frequency dependence of all possible \( s \)-wave amplitudes residing on nearest neighbor sites in the same layer. These represent the simplest possible extended-\( s \)-wave sym-
Figure 5. (Top panels, a-c): Evolution of the absolute value of equal-spin (↑↑ and ↓↓), mixed-spin (↑↓ + ↓↑) triplet, and spin-singlet (↑↓−↓↑) pair amplitudes into the middle of a WNL for $\alpha_2=0.5, 1, 2$ with $\alpha_1=1$. All other parameters are the same as in Fig. 3(a) in the main text. (Bottom panels, d-f): Josephson current as a function of chemical potential (d), order parameter amplitude $\Delta$ (e), and tunneling between the SC and junction material $t_{sc-w}$ (f). Here we also compare the result of the WNL for $\alpha_2=0.5, 1, 2$, $\alpha_1=1$ with a NM junction. All other parameters are the same as in Fig. 4 in the main text.

Frequency dependence of the pair propagation

In Fig. 3 in the main text we present the absolute value of different pair amplitudes as a function of layer in the WNL sampled at $\omega = 0.5\Delta$. Based on this we conclude that the spin-singlet pairing decay extremely rapidly, while the spin-down amplitude survives well in the WNL, effects we attribute to the spin-down spin-polarized drumhead surface states. In Fig. S7 we present complementary results for two larger frequencies, $\omega = \Delta$ (a) and $\omega = 2\Delta$ (b). The results show that the discussion in the main text focused on $\omega = 0.5\Delta$ still holds for larger frequencies, even at energies far above the superconducting gap. In fact, we find that the spin-down triplet amplitudes become even more prominent at larger frequencies, which further underscores the importance of odd-frequency pairing in the WNL.

Comparison between normal-state DOS in WNL and NM

The large Josephson current in WNL junctions is particularly remarkable considering the nodal loop or thin torus Fermi surface of the WNL in comparison to a NM. In Fig. S8 we quantify the statements about the DOS in the WNL in comparison to a NM. Panel (a) shows the DOS in the WNL as a function of the chemical potential $\mu$. The spin-down polarized surface state is heavily dominating at low $\mu$, while at finite doping, the bulk achieves a comparable DOS. The equivalent plot for the NM is shown in (b). Here the DOS is spin-independent in all layers, has no significant surface contributions, and no singularities at low energies, as expected for the prototypical parabolic dispersion in the NM. The oscillations found in both the WNL and NM densities are due to the finite

metry and can be extracted by taking the summation

$$F(r, \omega) = \sum_{k_{||}} e^{ik_{||}r} F(k_{||}, \omega),$$

over all $k_{||} = (k_x, k_y)$, where $r$ points to the nearest neighbor sites in the square lattice. Fig. S6 is completely analogous to Fig. 2 in the main text, and we see directly that the pair amplitudes are significantly reduced in magnitude, they are roughly half as large for the extended-$s$-wave symmetry compared to the isotropic $s$-wave state. The reduction is actually largest for the spin-singlet pairing, which further emphasizes the importance of the odd-frequency spin-triplet correlations.
Figure 6. Real (top panels, a-d) and imaginary (bottom panels, e-f) parts of the anomalous Green’s function $F$ for in-layer extended-$s$-wave symmetry as a function of frequency $\omega$, capturing the pair amplitudes divided into the equal-spin ($\uparrow\uparrow$ and $\downarrow\downarrow$), mixed-spin ($\uparrow\downarrow + \downarrow\uparrow$) triplet, and spin-singlet ($\uparrow\downarrow - \downarrow\uparrow$) components. Left to right figures shows results for the $n = 1, 2, 3$ and middle $n = (n_w + 1)/2$ layers, respectively. Same parameters as Fig. 2 in the main text, which present the equivalent plot for the isotropic $s$-wave amplitudes.

Figure 7. Evolution of the absolute value of equal-spin ($\uparrow\uparrow$ and $\downarrow\downarrow$), mixed-spin ($\uparrow\downarrow + \downarrow\uparrow$) triplet and spin-singlet ($\uparrow\downarrow - \downarrow\uparrow$) pair amplitudes into the middle of the WNL extracted at frequencies $\omega = \Delta$ (a) and $\omega = 2\Delta$ (b). Same parameters as in Fig. 3(a) in the main text, which show the equivalent plot for $\omega = 0.5\Delta$.

In Fig. 8(c) we directly compare the WNL and NM by plotting the fraction of $\rho_{NM}/\rho_{WNL}$ as a function of $\mu$, divided into spin-polarized surface and bulk contributions. We see directly that the DOS in the bulk is always higher in the NM compared to the WNL. Thus a NM Josephson junction has more bulk carriers to carry the supercurrent. However, in the entire range $\mu \lesssim 0.5$ it is actually the WNL that carries the larger Josephson current, see Fig. 4(a) in the main text. Having chosen the exact same parameters for the SC and the tunneling into the junction for the WNL and NM junctions, it is thus not the bulk DOS that governs the supercurrent. The large Josephson effect in lightly doped WNL is instead explained by the spin-polarized drumhead surface states. If we also include this DOS in the comparison by summing the DOS in the full slab to arrive at the total DOS, we find that for very small $\mu$ the WNL indeed has the larger total DOS. Still, for $0.1 \lesssim \mu \lesssim 0.5$ the total DOS is actually larger in the NM than in the WNL.
Thus it is not just the total slab DOS that is important, but also the location of the drumhead states which generates a superior coupling between the external SC contacts and the WNL.

Current-phase relationship

In an ideal Josephson junction the current-phase relationship is given by $J = J_{\text{max}} \sin(\phi - \phi_R)$. Thus the maximum current is found at $\phi = \phi_L - \phi_R = \pm \pi/2$, which is also the value we use in the main text to extract the maximum Josephson current. However, the simple sinusoidal behavior can be modified in real materials and in Fig. S9 we explore the full current-phase relationship for WNL Josephson junctions. We plot the normalized current $J(\phi)/J_{\text{max}}$ for different SC properties, by both varying the order parameter amplitude $\Delta$ and the tunneling between the SCs and WNL $t_{sc-w}$. As seen, $\phi = \pi/2$ is an extremely good approximation for generating the largest current for all different parameters, thus supporting this choice in the main text.

![Figure 9. Normalized Josephson current through a WNL junction as a function of phase difference between the SCs $\phi = \phi_L - \phi_R$ and compared to the ideal $\sin(\phi)$ relationship. Different curves represent different order parameter amplitudes $\Delta$, with the tunneling between the SCs and WNL set to $t_{sc-w} = 0.5$ (a) and $t_{sc-w} = 1$ (b). Insets show zoom-in around $\phi = \pi/2$. Same parameters as in Fig. 2 in the main text, except $\phi_L, \phi_R$.](image)

Currents at other doping levels in NM and HM

In Fig. 4 in the main text we present the Josephson current for junctions consisting of WNL, NM, and HMs with and without spin-active interface region materials. In Figs. 4(b,c) in the main text we compare the results for different order parameter amplitudes $\Delta$ and tunneling from SCs to the junction material $t_{sc-w}$, respectively, and there have to select a particular doping level. In the main text we use a finite $\mu = 0.1$ for the NM and HM junctions in order to get a reasonable but not too large bulk DOS, while for the WNL we use the extreme limit of $\mu_w = 0$. To show that this particular choice of $\mu$ is not misleading, we plot in Fig. S10 the equivalent results for $\mu = 0.2$ (a,b) and $\mu = 0.4$ (c,d) for the NM and HM, while we keep $\mu_w = 0$ in the WNL as a comparison. For a large range of parameters we find that the WNL Josephson current is larger or of similar magnitude as in the NM junction. It is only for large $\mu$ we find that the NM current surpasses that of the WNL, but not by much and mainly for larger $\Delta$ values. This is driven by an increasing DOS in the NM when $\mu$ is increased. For example at $\mu = 0.4$ the bulk DOS of the NM is almost 5 times larger than that of the bulk in the WNL at zero doping. Thus it is still highly remarkable how well the WNL carries a Josephson current compared to the NM.

Currents with modified SC properties

We use prototype conventional spin-singlet $s$-wave SCs as the external contacts in our Josephson junctions. To appropriately capture interface effects we model the SCs with a finite number of layers to make sure bulk conditions are met in the middle of the SCs. In Fig. S11 we show the Josephson current through the WNL as a function of the thickness of the SCs $n_{sc}$, and also compare it to NM and HM junctions. As seen, there are some oscillatory behavior for small $n_{sc}$ due to finite size effects. However, for $n_{sc} \gtrsim 20$ we approach a nearly constant behavior in all junctions. Thus the choice of $n_{sc} = 20$ in the main text is a very good compromise between studying a
small system for computational purposes and reaching good bulk conditions in the SC contacts. Moreover, we also test our main results for different values of the chemical potential $\mu_{SC}$ in the SC and the interface tunneling $t_{sc}$. We find that for all larger values of the chemical potential, i.e. modeling a good metallic normal state as expected for a conventional SC, the results do not vary significantly and our choice of $\mu_{sc} = 2$ in the main text is very representative. Choosing different tunneling $t_{sc}$ also do not qualitatively change the result and our main conclusions holds for different models of the SC.

Figure 11. Josephson current in WNL, NM, and HM junctions as a function of the thickness of the SC contacts, $n_{sc}$. Same parameters as in Figs. 3(a,d,e,f) in the main text.