THE EFFECT OF NONLOCAL REACTION IN AN EPIDEMIC MODEL WITH NONLOCAL DIFFUSION AND FREE BOUNDARIES

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Abstract. In this paper, we examine an epidemic model which is described by a system of two equations with nonlocal diffusion on the equation for the infectious agents $u$, while no dispersal is assumed in the other equation for the infective humans $v$. The underlying spatial region $[g(t), h(t)]$ (i.e., the infected region) is assumed to change with time, governed by a set of free boundary conditions. In the recent work [33], such a model was considered where the growth rate of $u$ due to the contribution from $v$ is given by $cv$ for some positive constant $c$. Here this term is replaced by a nonlocal reaction function of $v$ in the form $c \int_{h(t)}^{g(t)} K(x-y)v(t,y)dy$ with a suitable kernel function $K$, to represent the nonlocal effect of $v$ on the growth of $u$. We first show that this problem has a unique solution for all $t > 0$, and then we show that its longtime behaviour is determined by a spreading-vanishing dichotomy, which indicates that the long-time dynamics of the model is not vastly altered by this change of the term $cv$. We also obtain sharp criteria for spreading and vanishing, which reveal that changes do occur in these criteria from the earlier model in [33] where the term $cv$ was used; in particular, small nonlocal dispersal rate of $u$ alone no longer guarantees successful spreading of the disease as in the model of [33].

1. Introduction. To describe the cholera epidemic which spread in the European Mediterranean regions in 1973, Capasso and Maddalena [6] proposed the following mathematical model:

\[
\begin{aligned}
  u_t &= d \Delta u - au + cv, & t > 0, & x \in \Omega, \\
  v_t &= -bv + G(u), & t > 0, & x \in \Omega, \\
  \frac{\partial u}{\partial n} + \alpha u &= 0, & t > 0, & x \in \partial \Omega, \\
  u(0, x) &= u_0(x), & v(0, x) &= v_0(x), & x \in \Omega,
\end{aligned}
\]

(1.1)

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where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, representing the epidemic region, $a$, $b$, $c$ are all positive constants, $u(t,x)$ and $v(t,x)$, respectively, stand for the average population concentration of the infectious agents and the infective humans in the infected area at time $t$ and location $x$. $1/a$ stands for the mean lifetime of the agents in the environment, $1/b$ represents the mean infectious period of the infective humans, $c$ is the multiplicative factor of the infectious agents due to the infective humans, and $G(u)$ is the infection rate of the human population due to the concentration of $u$ in the infected area. Two basic assumptions of the model (1.1) are: (i) The total susceptible human population is large enough compared to the infective population, and is assumed to be constant during the evolution of the epidemic; (ii) The infectious agents disperse randomly, and the mobility of the infective human population is small and thus neglected. The function $G$ is assumed to satisfy

\begin{align*}
&\textbf{(G1)} \quad G \in C^1([0, \infty)), \quad G(0) = 0, \quad G'(z) > 0 \text{ for } \forall z \geq 0; \\
&\textbf{(G2)} \quad \frac{G(z)}{z} \text{ is strictly decreasing}^1 \text{ and } \lim_{z \to +\infty} \frac{G(z)}{z} < \frac{ab}{c}.
\end{align*}

An example is $G(z) = \alpha \frac{z}{1 + z}$ with $\alpha \in (0, \frac{ab}{c})$. It was shown in [6] that the number

$$
\hat{R}_0 := \frac{cG'(0)}{(a + d\lambda_1)b}
$$

is a threshold value for the long-time dynamical behaviour of (1.1): The epidemic will eventually tend to extinction if $0 < \hat{R}_0 \leq 1$, and there is a globally asymptotically stable endemic state if $\hat{R}_0 > 1$, where $\lambda_1$ is the first eigenvalue of

$$
-\Delta \phi = \lambda \phi \text{ in } \Omega, \quad \frac{\partial \phi}{\partial n} + \alpha \phi = 0 \text{ on } \partial \Omega.
$$

In order to better describe the spreading of the epidemic front, Ahn et al. [1] investigated this problem via the following free boundary model:

\begin{align*}
&u_t = du_{xx} - au + cv, \quad t > 0, \quad x \in (g(t), h(t)), \\
v_t = -bv + G(u), \quad t > 0, \quad x \in (g(t), h(t)), \\
u(t,x) = v(t,x) = 0, \quad t > 0, \quad x = g(t) \text{ or } x = h(t), \\
g(0) = -h_0, \quad g'(t) = -\mu u_x(t, g(t)), \quad t > 0, \\
h(0) = h_0, \quad h'(t) = -\mu u_x(t, h(t)), \quad t > 0, \\
u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad x \in [-h_0, h_0].
\end{align*}

They proved a spreading-vanishing dichotomy for its long-time dynamical behaviour: the unique solution $(u, v, g, h)$ of (1.2) satisfies one of the following:

(i) Vanishing:

$$
\lim_{t \to \infty} [h(t) - g(t)] < \infty \text{ and } \lim_{t \to \infty} (\|u(t, \cdot)\|_{C([g(t), h(t)])} + \|v(t, \cdot)\|_{C([g(t), h(t)])}) = 0;
$$

(ii) Spreading (which is possible only if $R_0 > 1$):

$$
\lim_{t \to \infty} [h(t) - g(t)] = \infty \text{ and } \lim_{t \to \infty} (u, v) = (K_1, K_2) \text{ locally uniformly in } \mathbb{R},
$$

where

$$
R_0 := \frac{cG'(0)}{ab}.
$$

\footnote{1For most results to hold here, it suffices to assume that $G(z)/z$ is nonincreasing for $z > 0$, and is strictly decreasing in a small neighbourhood of the set $\{z > 0 : G(z)/z = ab/c\}$ when $G'(0) > ab/c$; see details at the end of the paper.}
and \((K_1, K_2)\) is uniquely determined by, when \(R_0 > 1\),
\[
G(K_1) = \frac{ab}{c}K_1, \quad K_2 = \frac{G(K_1)}{b}.
\]

Furthermore,
(i) if \(R_0 \leq 1\), then vanishing happens;
(ii) if \(R_0 \geq 1 + \frac{d}{a} \left(\frac{\pi h_0}{2} \right)^2\), then spreading happens;
(iii) if \(1 < R_0 < 1 + \frac{d}{a} \left(\frac{\pi h_0}{2} \right)^2\), then vanishing happens for small initial data \((u_0, v_0)\), and spreading happens for large initial data.

When spreading happens, the spreading speed of (1.2) was established in [32].

We note that a derivation of the free boundary conditions in (1.2) can be found in [3], and spreading-vanishing type dynamical behaviour for many related free boundary models has been established in recent years, after Du and Lin [10] where this type of free boundary condition was used in a logistic type local diffusion model, and a spreading-vanishing dichotomy was first established; we refer to [8, 11, 12, 14, 15, 16, 17, 18, 19, 20, 21, 22, 26, 27, 28, 29, 30, 31] and the references therein for some of the recent works in this direction.

Note that in (1.1) and (1.2), the dispersal of the infectious agents is assumed to follow the rules of random diffusion, which is not ideal in many practical situations. This kind of dispersal may be alternatively described by a nonlocal diffusion operator of the form
\[
d \int \mathbb{R} J(x-y)u(t,y)dy - du(t,x),
\]
which can capture short-range as well as long-range factors in the dispersal process by choosing the kernel function \(J\) properly [2, 24].

Recently, Cao et al. [4] proposed a nonlocal version of the free boundary model in [10], and extended many basic results of [10] to the corresponding nonlocal model. More recently, Du et al. [9] investigated the spreading speed of the nonlocal model in [4], and demonstrated that, depending on the choice of the kernel function \(J\), the spreading speed of the nonlocal model can be finite or infinite, contrasting sharply to the spreading speed determined by the local diffusion model, which is always finite.

Motivated by the work [4], some related models with nonlocal diffusion and free boundaries have been considered in several recent works (see, for example, [13, 23, 33]). In particular, Zhao et al. [33] studied a nonlocal version of (1.2), which has the form
\[
\begin{aligned}
&\begin{cases}
  u_t = d \int_{g(t)}^{h(t)} J(x-y)u(t,y)dy - du - au + cv, \\
  v_t = -bv + G(u), \\
  u(t,x) = v(t,x) = 0,
\end{cases} \\
&\begin{cases}
  g'(t) = -\int_{h(t)}^{g(t)} J(x-y)u(t,x)dydx, \\
  h'(t) = \int_{g(t)}^{h(t)} J(x-y)u(t,x)dydx,
\end{cases} \\
&-g(0) = h(0) = h_0, \\
u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad x \in [-h_0, h_0].
\end{aligned}
\]

Here the kernel function \(J : \mathbb{R} \to \mathbb{R}\) is assumed to satisfy
(J) $J \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $J$ is symmetric and nonnegative, $J(0) > 0$, $\int_{\mathbb{R}} J(x)dx = 1$.

The parameters $a$, $b$, $c$, $d$, $\mu$ and $h_0$ are positive constants. The initial functions $u_0(x), v_0(x)$ satisfy

$$u_0, v_0 \in C([-h_0, h_0]), \quad u_0(\pm h_0) = v_0(\pm h_0) = 0, \quad u_0, v_0 > 0 \text{ in } (-h_0, h_0). \quad (1.6)$$

As before, $G$ satisfies (G1)-(G2). They showed that, for (1.5), a spreading-vanishing dichotomy similar to that in [1] for the corresponding local model (1.2) still holds. But the criteria for spreading and vanishing is different from that in [1]; more precisely, it was shown in [33] that

(i) If $R_0 \leq 1$, then vanishing happens for all admissible initial data $(u_0, v_0)$.

(ii) If $R_0 \geq 1 + \frac{d}{a}$, then spreading happens for all admissible initial data $(u_0, v_0)$.  

(iii) If $1 < R_0 < 1 + \frac{d}{a}$, then there exists $t^* > 0$ (independent of the initial data) such that spreading happens when $2h_0 \geq t^*$, and if $2h_0 < t^*$, then there exists $\mu^* > 0$ depending on $(u_0, v_0)$ such that spreading happens if and only if $\mu > \mu^*$.

Since $R_0$ is independent of $d$, from (ii) above we immediately see that, when $R_0 > 1$, for all small $d > 0$, $R_0 \geq 1 + \frac{d}{a}$ and hence spreading happens regardless of the initial data $(u_0, v_0)$.

Note that in (1.1), (1.2) and (1.5), the infective agents $u$ at a spatial point $x$ is assumed to depend only on the infective humans $v$ at spatial location $x$. In reality, it should also depend on the infective humans $v$ at some spatial neighbourhood of $x$. Such a consideration was included in the model in [5], where instead of $cv$, the growth rate of infectious agents due to the infective humans is described by

$$\int_{\Omega} K(x, y)v(t, y)dy,$$

with $K(x, y)$ representing the transfer kernel of infectious agents produced by the infective humans at $y$ and made available at $x$.

In this paper, we would like to examine the effect of such a change on the dynamics of (1.5). We assume that $K(x, y)$ depends only on the distance between $x$ and $y$, and so we may write $K = K(x - y)$. The modified (1.5) has the following form:

$$\begin{cases}
    u_t = d \int_{g(t)}^{h(t)} J(x - y)u(t, y)dy - du - au + c \int_{g(t)}^{h(t)} K(x - y)v(t, y)dy, & t > 0, \quad x \in (g(t), h(t)), \\
    v_t = -bv + G(u), & t > 0, \quad x \in (g(t), h(t)), \\
    u(t, g(t)) = u(t, h(t)) = v(t, g(t)) = v(t, h(t)) = 0, & t > 0, \\
    g'(t) = -\mu \int_{g(t)}^{h(t)} J(x - y)u(t, x)dydx, & t > 0, \\
    h'(t) = \mu \int_{g(t)}^{h(t)} J(x - y)u(t, x)dydx, & t > 0, \\
    -g(0) = h(0) = h_0, \\
    u(0, x) = u_0(x), \quad v(0, x) = v_0(x), & x \in [-h_0, h_0],
\end{cases} \quad (1.7)$$

where the parameters $a$, $b$, $c$, $d$, $\mu$ and $h_0$ are positive constants, $J(x)$ and $K(x)$ satisfy (J), and the initial functions $u_0(x)$ and $v_0(x)$ satisfy (1.6).

The main results of this paper are the following theorems.

**Theorem 1.1** (Global existence and uniqueness). Suppose that $J(x)$ and $K(x)$ satisfy (J), and $G$ satisfies (G1) and (G2). Then for any given $h_0 > 0$ and $u_0(x)$,
\(v_0(x)\) satisfying (1.6), problem (1.7) admits a unique solution \((u(t, x), v(t, x), g(t), h(t))\) defined for all \(t > 0\).

**Theorem 1.2** (Spreading-vanishing dichotomy). Let the conditions of Theorem 1.1 hold and \((u, v, g, h)\) be the unique solution of (1.7). Then one of the following alternatives must happen:

(i) Spreading: \(\lim_{t \to \infty} [h(t) - g(t)] = \infty\) (and necessarily \(R_0 > 1\)),

\[ \lim_{t \to +\infty} (u(t, x), v(t, x)) = (K_1, K_2)\] locally uniformly in \(\mathbb{R}\),

where \((K_1, K_2)\) is given by (1.4).

(ii) Vanishing: \(\lim_{t \to \infty} (g(t), h(t)) = (g_\infty, h_\infty)\) is a finite interval, and

\[ \lim_{t \to \infty} \max_{x \in [g(t), h(t)]} u(t, x) = 0, \quad \lim_{t \to \infty} \max_{x \in [g(t), h(t)]} v(t, x) = 0. \]

Let us recall that \(R_0\) is given by (1.3).

**Theorem 1.3** (Spreading-vanishing criteria). In Theorem 1.2, the dichotomy can be determined as follows:

(i) If \(R_0 \leq 1\), then vanishing happens for all admissible initial data \((u_0, v_0)\).

(ii) If \(R_0 > 1\), then there exists \(l^* > 0\) independent of \((u_0, v_0)\) such that spreading happens when \(2h_0 \geq l^*\), and if \(2h_0 < l^*\), then there exists \(\mu^* > 0\) depending on \((u_0, v_0)\) such that spreading happens if and only if \(\mu > \mu^*\).

**Remark 1.** For the nonlocal diffusion model (1.5), we see from [33, Theorem 1.3] that if \(R_0 > 1\) then spreading happens to (1.5) for all small \(d > 0\), regardless of the initial data. But, due to the introduction of the nonlocal effect involving \(K\), we found from Theorem 1.3 that the epidemic modelled by (1.7) will not always spread for small \(d > 0\); instead, the initial data play a role through \(h_0\) or through \((u_0, v_0)\). Biologically, this means that the nonlocal effect may decrease the chance of epidemic disease spreading, compared with the case that there is no nonlocal effect. We should emphasise that \(l^*\) is determined by an eigenvalue problem involving \(d\). In [25], the local diffusion model (1.2) with the term \(cv\) replaced by \(\int_{g(t)}^{h(t)} \mathcal{K}(x-y)v(t,y)dy\) was investigated, and similar results to [1] was obtained.

The rest of this paper is devoted to the proof of Theorems 1.1, 1.2 and 1.3, which consists of the next section. The approach follows that of [33], but considerable changes are needed, since the nonlocal reaction term \(e^{\int_{g(t)}^{h(t)} \mathcal{K}(x-y)v(t,y)dy}\) causes a number of difficulties. For example, new techniques are required to treat the associated linearised eigenvalue problem of (1.7), as well as the associated auxiliary fixed boundary problems, as both are significantly different from the corresponding problems for (1.5).

**2. Proof of the main results.** Throughout this section, we always assume that \(J(x)\) and \(K(x)\) satisfy (J), and \(G\) satisfies (G1) and (G2). For \(h_0 > 0\), \((u_0, v_0)\) satisfying (1.6), and any given \(T > 0\), we introduce the following notations:

\[ A := \max \left\{ K_1, \|u_0\|_\infty, \frac{C}{a} \|v_0\|_\infty \right\} \quad \text{with} \quad K_1 = 0 \quad \text{if} \quad R_0 \leq 1, \]

\[ B := \max \left\{ \|v_0\|_\infty, \frac{G(A)}{b} \right\}, \quad G_T := \{ g \in C([0, T]) : -g \in \mathcal{H}_T^{h_0} \}, \]

\[ H_T := \left\{ h \in C([0, T]) : h(0) = h_0, \quad \inf_{0 \leq t_1 < t_2 \leq T} \frac{h(t_2) - h(t_1)}{t_2 - t_1} > 0 \right\}, \]
Now we prove \((2.1)\). Then \(u(t,x)\) and \(v(t,x)\) are continuous in \(D_{T}^{g,h}\) and satisfy

\[
\begin{cases}
  u_t \geq d \int_{g(t)}^{h(t)} J(x-y)u(t,y)dy - du + c_{11}u \\
  + c_{12} \int_{g(t)}^{h(t)} K(x-y)v(t,y)dy, & 0 < t \leq T, x \in (g(t),h(t)), \\
  v_t \geq c_{21}u + c_{22}v, & 0 < t \leq T, x \in (g(t),h(t)), \\
  u(t,x) \geq 0, v(t,x) \geq 0, & 0 < t \leq T, x \in [g(t),h(t)], \\
  u(0,x) \geq 0, v(0,x) \geq 0, & x \in [-h_{0},h_{0}].
\end{cases}
\]

(2.1)

Then \((u(t,x),v(t,x)) \geq (0,0)\) for all \(0 \leq t \leq T\) and \(g(t) \leq x \leq h(t)\). Moreover, if we assume additionally \(u(0,x) \neq 0\) in \([-h_{0},h_{0}]\), then \(u(t,x) > 0\) in \(D_{T}^{g,h}\).

Proof. Let
\[
w(t,x) = e^{kt}u(t,x) \text{ and } z(t,x) = e^{kt}v(t,x),
\]
where \(k\) is large enough such that \(k > d + \|c_{11}\|_{\infty} + \|c_{22}\|_{\infty}\), and then
\[
p(t,x) := k + c_{22}(t,x) + c_{21}(t,x) \geq k - \|c_{22}\|_{\infty} > 0 \text{ for all } (t,x) \in D_{T}^{g,h}.
\]

Then \(w(t,x)\) and \(z(t,x)\) satisfy

\[
\begin{cases}
  w_t \geq d \int_{g(t)}^{h(t)} J(x-y)w(t,y)dy + (k - d + c_{11})w \\
  + c_{12} \int_{g(t)}^{h(t)} K(x-y)z(t,y)dy, & 0 < t \leq T, x \in (g(t),h(t)), \\
  z_t \geq (k + c_{22})z + c_{21}w.
\end{cases}
\]

(2.2)

Denote
\[
p_{0} = \sup_{(t,x) \in D_{T}^{g,h}} p(t,x) \text{ and } T^{*} = \min \left\{ T, \frac{1}{4(k + \|c_{11}\|_{\infty} + \|c_{12}\|_{\infty})}, \frac{1}{4p_{0}} \right\}.
\]

Now we prove \(w \geq 0\) and \(z \geq 0\) in \(D_{T}^{g,h}\). Suppose that
\[
m := \min \left\{ \inf_{(t,x) \in D_{T}^{g,h}} w(t,x), \inf_{(t,x) \in D_{T}^{g,h}} z(t,x) \right\} < 0.
\]
By (2.1), \( w \geq 0 \) and \( z \geq 0 \) on the parabolic boundary of \( D^{g,h}_T \). Hence, there exists \((t^*, x^*) \in D^{g,h}_T\) such that \( \frac{m}{2} = w(t^*, x^*) < 0 \) or \( \frac{m}{2} = z(t^*, x^*) < 0 \). We now define

\[
t_0 = t_0(x^*) := \begin{cases} 
t_h^{\infty}, \quad x^* \in (g(t^*), -h_0) \text{ and } x^* = g(t_h^{\infty}), \\
0, \quad x^* \in [-h_0, h_0], \\
t_l^h, \quad x^* \in (h_0, h(t^*)) \text{ and } x^* = h(t_l^h).
\end{cases}
\]

Clearly, \( u(t_0, x^*) \geq 0 \) and \( v(t_0, x^*) \geq 0 \).

If \( \frac{m}{2} = w(t^*, x^*) < 0 \), then it follows from the first equation of (2.2) that

\[
w(t^*, x^*) - w(t_0, x^*) \\
\geq d \int_{t_0}^{t^*} \int_{g(t)}^{h(t)} J(x^* - y)w(t, y)dydt + c_{12} \int_{t_0}^{t^*} \int_{g(t)}^{h(t)} K(x^* - y)z(t, y)dydt \\
+ \int_{t_0}^{t^*} (k - d + c_{11})w(t, x^*)dt
\]

\[
\geq d \int_{t_0}^{t^*} \int_{g(t)}^{h(t)} J(x^* - y)m^2dydt + c_{12} \int_{t_0}^{t^*} \int_{g(t)}^{h(t)} K(x^* - y)m_2dydt \\
+ \int_{t_0}^{t^*} (k - d + c_{11})m^2dt
\]

\[
\geq m(k + \|c_{11}\|_\infty + \|c_{12}\|_\infty)(t^* - t_0).
\]

Since \( w(t_0, x^*) = e^{kt_0}u(t_0, x^*) \geq 0 \), we deduce

\[
\frac{m}{2} \geq m(k + \|c_{11}\|_\infty + \|c_{12}\|_\infty)(t^* - t_0) \geq m(k + \|c_{11}\|_\infty + \|c_{12}\|_\infty)T^* \geq \frac{m}{4},
\]

which is a contradiction to \( m < 0 \).

If \( \frac{m}{2} = z(t^*, x^*) < 0 \), then it follows from the second equation of (2.2) that

\[
z(t^*, x^*) - z(t_0, x^*) \geq \int_{t_0}^{t^*} [(k + c_{22})z(t, x^*) + c_{21}w(t, x^*)]dt
\]

\[
\geq \int_{t_0}^{t^*} [(k + c_{22})m + c_{21}m]dt \geq mp_0(t^* - t_0).
\]

Since \( z(t_0, x^*) = e^{kt_0}v(t_0, x^*) \geq 0 \), we deduce

\[
\frac{m}{2} \geq mp_0(t^* - t_0) \geq mp_0T^* \geq \frac{m}{4},
\]

which is a contradiction to \( m < 0 \).

If \( T^* = T \), then \((u(t, x), v(t, x)) \geq (0, 0)\) for all \( 0 \leq t \leq T \) and \( g(t) \leq x \leq h(t) \) follows directly; while if \( T^* < T \), we may repeat this process with \((u_0(x), v_0(x))\) replaced by \((u(T^*, x), v(T^*, x))\), and \((0, T)\) replaced by \((T^*, T)\). Clearly after repeating this process finitely many times, we will obtain \((u(t, x), v(t, x)) \geq (0, 0)\) for all \( 0 \leq t \leq T \) and \( g(t) \leq x \leq h(t) \).

If \( u(0, x) \neq 0 \) in \([-h_0, h_0]\), then it follows directly from the Lemma 2.2 of [4] that \( u(t, x) > 0 \) in \( D^{g,h}_T \). \( \Box \)

The existence and uniqueness of solutions to the problem (1.7) can be done in a similar fashion as in [33]. We only list the main steps in the proof.
The proof of Theorem 1.1. For any given $T > 0$, $(g^*, h^*) \in G_T \times H_T$ and $v^* \in X^0_T$, it follows from
$$
\int_{\mathbb{R}} K(x - y)dy = 1 \text{ and } v^*(t, x) \leq B \text{ for } (t, x) \in D_T^{g^*, h^*}
$$
that
$$
\int_{g^*(t)}^{h^*(t)} K(x - y)u^*(t, y)dy \leq B \text{ for } (t, x) \in D_T^{g^*, h^*}.
$$
Then we can apply the same argument in the proof of [33, Lemma 2.2] to obtain
$$
\int_{g^*(t)}^{h^*(t)} K(x - y)v^*(t, y)dy \leq B \text{ for } (t, x) \in D_T^{g^*, h^*}.
$$
Then we can define the mapping $F$ with
$$
F(u(t, g^*(t)), h^*(t)) = 0, \quad 0 < t \leq T,
$$
$$
u(0, x) = u_0(x), \quad x \in [-h_0, h_0]
$$
amits a unique solution $u^* \in C(D_T^{g^*, h^*})$. Moreover,
$$
0 < u^* \leq A \text{ for any } (t, x) \in D_T^{g^*, h^*}.
$$
For such $(u^*, g^*, h^*)$, we can define $\tilde{u}_0(x)$ as the zero extension of $v_0(x)$ to $x \in \mathbb{R} \setminus [-h_0, h_0]$ and then define $\tilde{t}_x$ as in Step 1 of the proof of [33, Lemma 2.2], but with $(g, h)$ replaced by $(g^*, h^*)$. To mark the difference, we denote $\tilde{t}_x$ by $\tilde{t}_x^*$. Now, for each $x \in (g^*(T), h^*(T))$, we consider the initial value problem
$$
\begin{cases}
  v_t = -bv + G(u^*), & t_x^* < t \leq T, \\
  v(\tilde{t}_x^*, x) = \tilde{u}_0(x).
\end{cases}
$$
By the Fundamental Theorem of ODEs and some simple comparison argument, it can be easily shown that (2.4) has a unique solution $\tilde{v}^*(t, x)$, and it is continuous and satisfies
$$
0 \leq \tilde{v}^*(t, x) \leq B \text{ for } t \in (0, T] \text{ and } x \in [g^*(t), h^*(t)].
$$
Therefore $\tilde{v}^* \in X_T^0$.

Next we define $(\tilde{g}^*, \tilde{h}^*)$ for $t \in [0, T]$ by
$$
\begin{cases}
  \tilde{g}^*(t) := -\tilde{h}_0 - \mu \int_0^t \int_{g^*(\tau)}^{h^*(\tau)} J(x - y)u^*(\tau, x)dydx\,d\tau, \\
  \tilde{h}^*(t) := \tilde{h}_0 + \mu \int_0^t \int_{g^*(\tau)}^{h^*(\tau)} J(x - y)u^*(\tau, x)dydx\,d\tau.
\end{cases}
$$
Then we can define the mapping $F(v^*, g^*, h^*) = (\tilde{v}^*, \tilde{g}^*, \tilde{h}^*)$. Let
$$
\Sigma_T := \left\{(v, g, h) \in X_T^0 \times G_T^{h_0} \times H_T^{h_0} : \sup_{0 \leq t_1 < t_2 \leq T} \frac{g(t_2) - g(t_1)}{t_2 - t_1} \leq -\mu \tilde{\sigma}_0, \right\}
$$
$$
\inf_{0 \leq t_1 < t_2 \leq T} \frac{h(t_2) - h(t_1)}{t_2 - t_1} \geq \mu \sigma_0, \quad h(t) - g(t) \leq 2h_0 + \frac{\epsilon_0}{4} \text{ for } t \in [0, T].
$$
Using the corresponding argument of [4], we can show that there exists some sufficiently small $T_0 = T_0(\mu, A, h_0, \epsilon_0, t_0, J) > 0$ such that $(\tilde{v}^*, \tilde{g}^*, \tilde{h}^*) \in \Sigma_T$ for any $T \in (0, T_0)$, which implies that $F(\Sigma_T) \subset \Sigma_T$ for $T \in (0, T_0)$. Let $V(t, x) =
\(v_1^*(t, x) - v_2^*(t, x)\). For \(T \in (0, T_0]\) and any given \((u_i^*, g_i^*, h_i^*) \in \Sigma_T\) \((i = 1, 2)\), it is easy to check that
\[
\left| \int_{g_i^*}^{h_i^*} K(x^* - y)v_1^*(t, y)dy - \int_{g_i^*}^{h_i^*} K(x^* - y)v_2^*(t, y)dy \right|
\]
\[
= \left| \int_{g_i^*}^{h_i^*} K(x^* - y)V(t, y)dy + \left( \int_{g_i^*}^{h_i^*} h_i^*(t) - \int_{g_i^*}^{h_i^*} g_i^*(t) \right)K(x^* - y)v_2^*(t, y)dy \right|
\]
\[
\leq \|V\|_{C([0, T] \times \mathbb{R})} + \left( \int_{g_i^*}^{h_i^*} g_i^*(t) + \int_{g_i^*}^{h_i^*} h_i^*(t) \right)K(x^* - y)v_2^*(t, y)dy
\]
\[
\leq \|V\|_{C([0, T] \times \mathbb{R})} + B\|K\|_{\infty} \left[ \|g_i^* - g_i^*\|_{C([0, T])} + \|h_i^* - h_i^*\|_{C([0, T])} \right].
\]

By this fact, we can follow the approach of the proof of [33, Theorem 1.1] to show \(\bar{F}\) is a contraction mapping for sufficiently small \(T \in (0, T_0]\), and hence for such \(t\), \(\bar{F}\) has a unique fixed point in \(\Sigma_T\), which clearly is a solution of (1.7) for \(t \in [0, T]\).

Similar to Steps 3 and 4 in the proof of [33, Theorem 1.1], we can show that this is the unique solution of (1.7) and it can be extended uniquely to all \(t > 0\).

**Lemma 2.2.** Let \(h_0, T > 0\) and \(\Omega_0 := [0, T] \times [-h_0, h_0]\). Suppose that \(\alpha \in L^\infty(\Omega_0)\) is nonnegative, and \((u(t, x), v(t, x))\) as well as \((u_i(t, x), v_i(t, x))\) are continuous in \(\Omega_0\) and satisfy
\[
\begin{align*}
  u_t &\geq d \int_{-h_0}^{h_0} J(x - y)u(t, y)dy - du - au, \\
  &\quad + c \int_{-h_0}^{h_0} K(x - y)v(t, y)dy, \quad t \in (0, T], \quad x \in [-h_0, h_0], \\
  v_t &\geq -bv + \alpha(t, x)u, \quad t \in (0, T], \quad x \in [-h_0, h_0], \\
  u(0, x) &\geq 0, \quad v(0, x) \geq a, \quad x \in [-h_0, h_0].
\end{align*}
\]

Then \(u(t, x), v(t, x) \geq 0\) for all \(0 \leq t \leq T\) and \(-h_0 \leq x \leq h_0\).

**Proof.** This follows from a simple variation of the argument in the proof of Lemma 2.1. We omit the details.

**Lemma 2.3.** For \(T \in (0, +\infty)\), suppose that \(\bar{\gamma}, \bar{h} \in C([0, T]), \bar{u}, \bar{v} \in C(\bar{\mathcal{T}}_T)\), \(\bar{u}, \bar{v} \geq 0\) in \(\bar{\mathcal{T}}_T\). If \((\bar{u}, \bar{v}, \bar{\gamma}, \bar{h})\) satisfies
\[
\begin{align*}
  \bar{u}_t &\geq d \int_{\mathcal{T}(t)} J(x - y)\bar{u}(t, y)dy - d\bar{u} - a\bar{u}, \\
  &\quad + c \int_{\mathcal{T}(t)} K(x - y)\bar{v}(t, y)dy, \quad 0 < t \leq T, \quad x \in (\bar{\gamma}(t), \bar{h}(t)), \\
  \bar{v}_t &\geq -b\bar{v} + G(\bar{u}), \quad 0 < t \leq T, \quad x \in (\bar{\gamma}(t), \bar{h}(t)), \\
  \bar{\gamma}'(t) &\leq -\mu \int_{\mathcal{T}(t)} J(x - y)\bar{u}(t, x)dydx, \quad 0 < t \leq T, \\
  \bar{h}'(t) &\geq \mu \int_{\mathcal{T}(t)} J(x - y)\bar{u}(t, x)dydx, \quad 0 < t \leq T, \\
  \bar{u}(0, x) &\geq u_0(x), \quad \bar{v}(0, x) \geq v_0(x), \quad x \in [-h_0, h_0], \\
  \bar{\gamma}(0) &\leq -h_0, \quad \bar{h}(0) \geq h_0,
\end{align*}
\]
then the unique solution \((u, v, g, h)\) of (1.7) satisfies
\[
\begin{align*}
  u(t, x) & \leq u(t, x), \quad v(t, x) \leq v(t, x), \\
  g(t) & \geq g(t), \quad h(t) \leq h(t) \text{ for } 0 < t \leq T, \quad g(t) \leq x \leq h(t).
\end{align*}
\]

**Proof.** By using Lemma 2.1, we can argue as in [33] to prove this lemma. We omit the details. \(\square\)

The following result is a direct consequence of the above comparison principle (Lemma 2.3), where to stress the dependence on the parameter \(\mu\), we use \((u_\mu, v_\mu, g_\mu, h_\mu)\) to denote the solution of problem (1.7).

**Corollary 1.** If \(\mu_1 \leq \mu_2\), we have \(u_{\mu_1}(t, x) \leq u_{\mu_2}(t, x)\), \(v_{\mu_1}(t, x) \leq v_{\mu_2}(t, x)\), \(g_{\mu_1}(t) \geq g_{\mu_2}(t)\), \(h_{\mu_1}(t) \leq h_{\mu_2}(t)\) for \(0 < t \leq T\) and \(u_{\mu_1}(t) \leq x \leq h_{\mu_1}(t)\).

It is easily seen that \(h(t)\) is monotonically increasing and \(g(t)\) is monotonically decreasing. Therefore
\[
\lim_{t \to \infty} h(t) = h_\infty \in (h_0, +\infty) \quad \text{and} \quad \lim_{t \to \infty} g(t) = g_\infty \in [-\infty, -h_0)
\]
are always well-defined.

Let
\[
\theta := \frac{cG'(0)}{b} - a.
\]
Clearly \(\theta \leq 0\) is equivalent to \(R_0 \leq 1\).

**Lemma 2.4.** If \(\theta \leq 0\), then vanishing happens.

**Proof.** We follow the ideas in [25, Theorem 4.4]. Direct calculations yield
\[
\begin{align*}
  & \frac{d}{dt} \int_{g(t)}^{h(t)} \left[ u(t, x) + \frac{c}{b} \int_{g(t)}^{h(t)} K(x-y)v(t,y)dy \right] dx \\
  = & \int_{g(t)}^{h(t)} \left[ u_t(t, x) + \frac{c}{b} \int_{g(t)}^{h(t)} K(x-y)v_t(t,y)dy + h'(t)K(x-h(t))v(t,h(t)) \\
  & \quad - g'(t)K(x-g(t))v(t,g(t)) \right] dx \\
  & + h'(t) \left[ u(t, h(t)) + \frac{c}{b} \int_{g(t)}^{h(t)} K(h(t)-y)v(t,y)dy \right] \\
  & - g'(t) \left[ u(t, g(t)) + \frac{c}{b} \int_{g(t)}^{h(t)} K(g(t)-y)v(t,y)dy \right] \\
  = & \int_{g(t)}^{h(t)} \left[ d \int_{g(t)}^{h(t)} J(x-y)u(t,y)dy - d \int_{\mathbb{R}} J(x-y)u(t,x)dy - au \\
  & + \frac{c}{b} \int_{g(t)}^{h(t)} K(x-y)G(u(t,y))dy \right] dx \\
  & + h'(t) \frac{c}{b} \int_{g(t)}^{h(t)} K(h(t)-y)v(t,y)dy - g'(t) \frac{c}{b} \int_{g(t)}^{h(t)} K(g(t)-y)v(t,y)dy \\
  = & - \frac{d}{\mu} (h'(t) - g'(t)) + \int_{g(t)}^{h(t)} \left[ -au + \frac{c}{b} \int_{g(t)}^{h(t)} K(x-y)G(u(t,y))dy \right] dx \\
  & + h'(t) \frac{c}{b} \int_{g(t)}^{h(t)} K(h(t)-y)v(t,y)dy - g'(t) \frac{c}{b} \int_{g(t)}^{h(t)} K(g(t)-y)v(t,y)dy.
\end{align*}
\]
Let \((\bar{u}(t), \bar{v}(t))\) be the solution of the ODE problem
\[
\begin{align*}
    u'(t) &= -au + cv, \quad t > 0, \\
    v'(t) &= -bv + G(u), \quad t > 0, \\
    u(0) &= \|u_0\|_\infty, \quad v(0) = \|v_0\|_\infty.
\end{align*}
\]
(2.6)
Since \(\theta = \frac{cG'(0)}{b} - a \leq 0\) and hence \(R_0 = \frac{cG'(0)}{ab} \leq 1\), the solution \((0, 0)\) of problem (2.6) is globally attractive and so \((\bar{u}(t), \bar{v}(t)) \to (0, 0)\) as \(t \to \infty\).

By the comparison principle (Lemma 2.1), we have
\[
    u(t, x) \leq \bar{u}(t), \quad v(t, x) \leq \bar{v}(t) \quad \text{for } t \geq 0 \text{ and } x \in [g(t), h(t)].
\]
Hence,
\[
    \lim_{t \to \infty} \max_{x \in [g(t), h(t)]} u(t, x) = \lim_{t \to \infty} \max_{x \in [g(t), h(t)]} v(t, x) = 0. \tag{2.7}
\]
For any given \(\varepsilon \in (0, \frac{h(t) - g(t)}{2\pi r})\), there exists \(T_0 = T_0(\varepsilon)\) such that \(v(t, x) < \varepsilon\) for \(t \geq T_0\) and \(x \in [g(t), h(t)]\). Then we have 
\[
    \frac{d}{dt} \int_{g(t)}^{h(t)} \left[ u(t, x) + \frac{c}{b} \int_{g(t)}^{h(t)} K(x-y)v(t,y)dy \right] dx 
    \leq \frac{d}{\mu} (h'(t)-g'(t)) + \int_{g(t)}^{h(t)} \left[ -au + \frac{c}{b} \int_{g(t)}^{h(t)} K(x-y)G(u(y))dy \right] dx + \frac{c\varepsilon}{b} (h'(t)-g'(t)).
\]
Integrating from \(T_0\) to \(t\) gives
\[
    \int_{g(t)}^{h(t)} \left[ u(T_0, x) + \frac{c}{b} \int_{g(T_0)}^{h(T_0)} K(x-y)v(T_0,y)dy \right] dx 
    \leq \int_{g(T_0)}^{h(T_0)} \left[ u(T_0, x) + \frac{c}{b} \int_{g(T_0)}^{h(T_0)} K(x-y)v(T_0,y)dy \right] dx 
    + \left( \frac{d}{\mu} - \frac{c\varepsilon}{b} \right) (h(T_0) - g(T_0)) - \left( \frac{d}{\mu} - \frac{c\varepsilon}{b} \right) (h(t) - g(t)) 
    + \int_{T_0}^{t} \int_{g(s)}^{h(s)} \left[ -au(s, x) + \frac{c}{b} \int_{g(s)}^{h(s)} K(x-y)G(u(s,y))dy \right] dx ds.
\]
For all \(T_0 \leq s \leq t\), by \(\frac{G(z)}{z} < G'(0)\) for \(z > 0\) and Fubini theorem, we have
\[
    \int_{g(s)}^{h(s)} \left[ -au(s, x) + \frac{c}{b} \int_{g(s)}^{h(s)} K(x-y)G(u(s,y))dy \right] dx 
    \leq \int_{g(s)}^{h(s)} \left[ -au(s, x) + \frac{cG'(0)}{b} \int_{g(s)}^{h(s)} K(x-y)u(s,y)dy \right] dx 
    = -a \int_{g(s)}^{h(s)} u(s, x)dx + \frac{cG'(0)}{b} \int_{g(s)}^{h(s)} \int_{g(s)}^{h(s)} K(x-y)dxu(s,y)dy 
    \leq \left( \frac{cG'(0)}{b} - a \right) \int_{g(s)}^{h(s)} u(s, x)dx \leq 0.
\]
Hence, we have
\[
    \left( \frac{d}{\mu} - \frac{c\varepsilon}{b} \right) (h(t) - g(t)) \leq \int_{g(T_0)}^{h(T_0)} \left[ u(T_0, x) + \frac{c}{b} \int_{g(T_0)}^{h(T_0)} K(x-y)v(T_0,y)dy \right] dx.
\[ + \left( \frac{d}{\mu} - \frac{c \varepsilon}{b} \right) (h(T_0) - g(T_0)), \]
then we can get that \( h_\infty - g_\infty < \infty \) by letting \( t \to \infty \). This fact and (2.7) implies that vanishing happens.

In the following, we consider the case \( \theta > 0 \), i.e., \( R_0 > 1 \). Denote
\[
\bar{J}(x) := \frac{b}{db + cG'(0)} \left[ dJ(x) + \frac{cG'(0)}{b} K(x) \right].
\]
Then we consider the operator \( \mathcal{L}_\Omega + \theta : C(\Omega) \to C(\Omega) \) defined by
\[
(\mathcal{L}_\Omega + \theta)[\phi](x) = \left( d + \frac{cG'(0)}{b} \right) \left[ \int_\Omega \bar{J}(x-y)\phi(y)dy - \phi(x) \right] + \theta \phi(x),
\]
where \( \Omega \) is an open interval in \( \mathbb{R} \), possibly unbounded. Define
\[
\lambda_p(\mathcal{L}_\Omega + \theta) = \inf \left\{ \lambda \in \mathbb{R} : (\mathcal{L}_\Omega + \theta)[\phi] \leq \lambda \phi \text{ for some } \phi \in C(\Omega), \phi > 0 \right\}.
\]

It is easy to check \( \bar{J}(x) \) satisfies the condition (J). Therefore, \( \lambda_p(\mathcal{L}_\Omega + \theta) \) is a principal eigenvalue, namely, \( \lambda_p(\mathcal{L}_\Omega + \theta) \) is an eigenvalue of the operator \( \mathcal{L}_\Omega + \theta \) with a continuous and positive eigenfunction. Let \( \Omega = (l_1, l_2) \), it follows from [4, Proposition 3.4] that
(i) \( \lambda_p(\mathcal{L}_{(l_1, l_2)} + \theta) \) is strictly increasing and continuous in \( l = l_2 - l_1 \), and also strictly increasing in \( \theta \).
(ii) \( \lim_{l_2 - l_1 \to +\infty} \lambda_p(\mathcal{L}_{(l_1, l_2)} + \theta) = \theta \).
(iii) \( \lim_{l_2 - l_1 \to 0} \lambda_p(\mathcal{L}_{(l_1, l_2)} + \theta) = -a - d \).

In the following, we first consider
\[
d \int_{l_1}^{l_2} \bar{J}(x-y)w(y)dy - dw - aw + \frac{c}{b} \int_{l_1}^{l_2} K(x-y)G(w(y))dy = 0, \quad x \in [l_1, l_2], \quad (2.8)
\]
and we can prove the following result.

**Lemma 2.5.** The following hold true:
(i) If \( \lambda_p(\mathcal{L}_{(l_1, l_2)} + \theta) > 0 \), then (2.8) admits a unique positive continuous solution.
(ii) If \( \lambda_p(\mathcal{L}_{(l_1, l_2)} + \theta) \leq 0 \), then any non-negative uniformly bounded solution of (2.8) is identically 0.

**Proof.** The idea comes from [7].
(i) We will prove this conclusion by the following two steps.
Step 1. We prove (2.8) admits a positive solution. Let \( \phi(x) \) be the corresponding normalized positive eigenfunction of \( \lambda_p(\mathcal{L}_{(l_1, l_2)} + \theta) \), namely, \( \|\phi\|_\infty = 1 \) and
\[
(\mathcal{L}_{(l_1, l_2)} + \theta)[\phi](x) = \lambda_p \phi(x), \quad x \in [l_1, l_2];
\]
then by definition of \( \mathcal{L}_{(l_1, l_2)} + \theta \) we have
\[
\left( d + \frac{cG'(0)}{b} \right) \int_{l_1}^{l_2} \bar{J}(x-y)\phi(y)dy - d\phi - a\phi = \lambda_p \phi(x). \quad (2.9)
\]
We first claim that there exists a positive constant \( c_0 \) such that
\[
\phi(x) > c_0 \text{ in } [l_1, l_2]. \quad (2.10)
\]
Let
\[
\alpha(x) := \int_{l_1}^{l_2} \bar{J}(x-y)\phi(y)dy,
\]
then it follows from (2.9) that
\[ \alpha(x) = \frac{b(\lambda_p + d + a)}{db + cG'(0)} \phi(x). \]

By construction, \( \alpha(x) \) is non-negative continuous function in \([l_1, l_2]\). Therefore, \( \alpha(x) \) achieves a non-negative minimum at some point \( x_0 \in [l_1, l_2] \). To prove (2.10), we only need to show \( \alpha(x_0) > 0 \). Otherwise, we have
\[ \alpha(x_0) = \int_{l_1}^{l_2} J(x_0 - y)\phi(y)dy = 0. \]

Hence, \( \phi(x) = 0 \) a.e. in \( \Omega_{x_0} := \{ y \in [l_1, l_2] \mid x_0 - y \in \text{supp}(\tilde{J}) \} \). Then we can find some \( x_1 \in \Omega_{x_0} \), but closing to \( \partial\Omega_{x_0} \), such that \( \phi(x_1) = 0 \), and then \( \alpha(x_1) = 0 \). Similarly, we can show that \( \phi(x) = 0 \) a.e. in \( \Omega_{x_1} \). By repeating this argument, we have \( \phi(x) = 0 \) a.e. in \([l_1, l_2]\). Since \( \phi(x) \) is continuous in \([l_1, l_2]\), we have \( \phi(x) \equiv 0 \) in \([l_1, l_2]\), which is a contradiction. As a consequence, (2.10) holds.

Define
\[ A[w](x) = \frac{1}{d + a} \left[ d \int_{l_1}^{l_2} J(x - y)w(y)dy + \frac{c}{b} \int_{l_1}^{l_2} K(x - y)G(w(y))dy \right]. \]

Then \( w \) solve (2.8) if and only if \( w \) is a fixed point of \( A \). We will use a basic iterative scheme to show \( A \) has a fixed point. Next we show that
\[ A[\epsilon \phi](x) \geq \epsilon \phi(x), \quad A[K_1](x) \leq K_1, \quad (2.11) \]
where \( K_1 \) is given by (1.4) and \( \epsilon > 0 \) will be determined later. Let us compute \( A[\epsilon \phi](x) \) and \( A[K_1](x) \). For \( x \in [l_1, l_2] \), we have
\[
A[\epsilon \phi](x) = \frac{1}{d + a} \left[ d \int_{l_1}^{l_2} J(x - y)\epsilon \phi(y)dy + \frac{c}{b} \int_{l_1}^{l_2} K(x - y)G(\epsilon \phi(y))dy \right]
= \epsilon \phi + \frac{1}{d + a} \left[ \lambda_p \epsilon \phi(x) - \frac{cG'(0)}{b} \int_{l_1}^{l_2} K(x - y)\epsilon \phi(y)dy \right]
+ \frac{c}{b} \int_{l_1}^{l_2} K(x - y)G(\epsilon \phi(y))dy
= \epsilon \phi + \frac{1}{d + a} \left[ \lambda_p \epsilon \phi(x) + \frac{c}{b} \int_{l_1}^{l_2} K(x - y)\left[G'(\eta_c(y)) - G'(0)\right]\epsilon \phi(y)dy \right]
= \epsilon \phi + \frac{\epsilon \phi(x)}{d + a} \left[ \lambda_p + \frac{c}{b\phi(x)} \int_{l_1}^{l_2} K(x - y)\left[G'(\eta_c(y)) - G'(0)\right]dy \right]
\geq \epsilon \phi + \frac{\epsilon \phi(x)}{d + a} \left[ \lambda_p - \frac{c}{b\phi(0)} \left\| G'(\eta_c(y)) - G'(0) \right\|_{C([l_1, l_2])} \int_{l_1}^{l_2} K(x - y)dy \right]
\geq \epsilon \phi + \frac{\epsilon \phi(x)}{d + a} \left[ \lambda_p - \frac{c}{bc_0} \left\| G'(\eta_c(y)) - G'(0) \right\|_{C([l_1, l_2])} \right],
\]
and
\[
A[K_1](x) = \frac{1}{d + a} \left[ dK_1 \int_{l_1}^{l_2} J(x - y)dy + \frac{cG(K_1)}{b} \int_{l_1}^{l_2} K(x - y)dy \right]
\leq \frac{1}{d + a} \left[ dK_1 + \frac{cG(K_1)}{b} \right] = \frac{1}{d + a} (dK_1 + aK_1) = K_1,
\]
where \(0 < \eta_r(y) < \epsilon \phi(y) < \epsilon\) for \(y \in (l_1, l_2)\). If we choose \(\epsilon\) small enough such that
\[
\lambda_p - \frac{1}{bc_0}\|G'(\eta(y)) - G'(0)\|_{C([l_1, l_2])} > 0,
\]
then (2.11) holds. We shrink \(\epsilon\) if necessary so that \(\epsilon \phi(x) \leq K_1\). We now define
\[
w_0(x) = \epsilon \phi(x), \ w_{n+1}(x) = A[w_n](x), \ n = 0, 1, 2, \cdots .
\]
The monotonicity of \(G\) implies that, if \(z_1 \leq z_2\), then \(A[z_1](x) \leq A[z_2](x)\). Using this fact and (2.11) we obtain
\[
w_0(x) \leq w_n(x) \leq w_{n+1}(x) \leq K_1, \quad \text{for } x \in [l_1, l_2] \text{ and } n = 1, 2, \cdots .
\]
Therefore, we can define
\[
w^*(x) = \lim_{n \to \infty} w_n(x).
\]
By the Lebesgue dominated convergence theorem, we deduce from \(w_{n+1}(x) = A[w_n](x)\) that
\[
w^*(x) = A[w^*](x),
\]
and hence \(w^*\) solve (2.8).

Step 2. We show that when a positive solution of (2.8) exists then it is unique. Let \(w_1^*(x)\) and \(w_2^*(x)\) be two solutions of (2.8). Then
\[
d \int_{l_1}^{l_2} J(x - y)w_i^*(y)dy + \frac{c}{b} \int_{l_1}^{l_2} K(x - y)G(w_i^*(y))dy = (d + a)w_i^*(x), \quad i = 1, 2.
\]
Arguing as the proof of (2.10), we have there exist two positive constants \(c_i^*\) and \(c_2^*\) such that \(w_1^*(x) \geq c_i^*\) in \([l_1, l_2], i = 1, 2\). Then we can define
\[
\gamma^* = \inf \{\gamma > 0 \mid \gamma w_1^*(x) \geq w_2^*(x) \text{ for } x \in [l_1, l_2]\}.
\]
Clearly
\[
\gamma^* w_1^*(x) \geq w_2^*(x) \text{ for } x \in [l_1, l_2], \quad \text{and} \quad \gamma^* w_1^*(x_0) = w_2^*(x_0) \text{ for some } x_0 \in [l_1, l_2].
\]
We claim that \(\gamma^* \leq 1\). Assume by contradiction that \(\gamma^* > 1\). Since \(G(z)\) is strictly decreasing in \(z\), we have \(G(\gamma^* w_1^*(y)) < \gamma^* G(w_1^*(y))\) for \(y \in [l_1, l_2]\). Then we have
\[
d \int_{l_1}^{l_2} J(x - y)\gamma^* w_1^*(y)dy + \frac{c}{b} \int_{l_1}^{l_2} K(x - y)G(\gamma^* w_1^*(y))dy < (d + a)\gamma^* w_1^*(x).
\]
Since \(G'(z) > 0\) and \(\gamma^* w_1^*(x) \geq w_2^*(x)\) for \(x \in [l_1, l_2]\), we have
\[
G(\gamma^* w_1^*(x)) \geq G(w_2^*(x)) \text{ for } x \in [l_1, l_2].
\]
Therefore
\[
0 < d \int_{l_1}^{l_2} J(x_0 - y)(\gamma^* w_1^* - w_2^*)(y)dy + \frac{c}{b} \int_{l_1}^{l_2} K(x_0 - y)(G(\gamma^* w_1^*) - G(w_2^*))dy
\]
\[
< (d + a)(\gamma^* w_1^* - w_2^*)(x_0) = 0.
\]
This contradiction implies \(\gamma^* \leq 1\). Hence, \(w_1^*(x) \geq w_2^*(x)\) in \([l_1, l_2]\). Similarly, we can show \(w_2^*(x) \geq w_1^*(x)\) in \([l_1, l_2]\). This proves the uniqueness of the solution.

(ii) Suppose that \(w\) is a non-negative, not identically zero and bounded solution of (2.8), then we have
\[
\int_{l_1}^{l_2} \left[dJ(x - y)w(y) + \frac{c}{b}K(x - y)G(w(y))\right]dy - dw - aw = 0.
\]
Arguing as the proof of (2.10), we can show that there exists a positive constant \(c_1\) such that \(w(x) > c_1\) in \([l_1, l_2]\).
Let
\[ M_{(I_1, I_2)}[\phi](x) := \left( d + \frac{cG'(0)}{b} \right) \int_{I_1}^{I_2} J(x-y)\phi(y)dy. \]
By \( K(0) > 0, w(x) > c_1 \) in \([I_1, I_2]\) and \( \frac{G(z)}{z} < G'(0) \) for \( z > 0 \), we have
\[ \frac{c}{b} \int_{I_1}^{I_2} K(x-y)G(w(y))dy < \frac{cG'(0)}{b} \int_{I_1}^{I_2} K(x-y)w(y)dy. \]
For \( x \in [I_1, I_2]\), it follows from this inequality that
\[ M_{(I_1, I_2)}[w](x) - dw - aw \]
\[ = d \int_{I_1}^{I_2} J(x-y)w(y)dy - dw - aw + \frac{cG'(0)}{b} \int_{I_1}^{I_2} K(x-y)w(y)dy \]
\[ > d \int_{I_1}^{I_2} J(x-y)w(y)dy - dw - aw + \frac{c}{b} \int_{I_1}^{I_2} K(x-y)G(w(y))dy = 0. \]
By definition of \( \lambda_p(\mathcal{L}_{(I_1, I_2)} + \theta) \), there exists \( \phi(x) > 0 \) for \( x \in [I_1, I_2]\) such that
\[ M_{(I_1, I_2)}[\phi](x) - d\phi - a\phi = (\mathcal{L}_{(I_1, I_2)} + \theta)[\phi](x) = \lambda_p\phi \leq 0. \]
Define
\[ \tau^* = \inf \{ \tau > 0 \mid w(x) \leq \tau\phi(x) \text{ for } x \in [I_1, I_2] \}. \]
We will show that \( w \equiv 0 \) by proving that \( \tau^* = 0 \). Assume that \( \tau^* > 0 \). By the definition of \( \tau^* \), \( w(x) \leq \tau^*\phi(x) \) for \( x \in [I_1, I_2]\), and there exists some \( x_0 \in [I_1, I_2]\) such that \( w(x_0) = \tau^*\phi(x_0) \).
At \( x_0 \), we have, by the positivity of the operator \( M_{(I_1, I_2)} \) and (2.12),
\[ 0 \leq M_{(I_1, I_2)}[\tau^*\phi - w](x_0) = M_{(I_1, I_2)}[\tau^*\phi - w](x_0) - (d + a)(\tau^*\phi - w)(x_0) < 0. \]
This contradiction implies that \( \tau^* = 0 \).

For the problem below,
\[ \begin{cases} 
  d \int_{I_1}^{I_2} J(x-y)w(y)dy - dw - aw + c \int_{I_1}^{I_2} K(x-y)z(y)dy = 0, & x \in [I_1, I_2], \\
  -bz + G(w) = 0, & x \in [I_1, I_2], \end{cases} \]
by Lemma 2.5, the following result holds:

**Corollary 2.** If \( \lambda_p(\mathcal{L}_{(I_1, I_2)} + \theta) > 0 \), then problem (2.13) admits a unique positive solution \((W(x), Z(x))\) with \( Z(x) = G(W(x))/b \). Moreover, if \( \lambda_p(\mathcal{L}_{(I_1, I_2)} + \theta) \leq 0 \), then any nonnegative uniformly bounded solution of (2.13) is identically zero.

Next, we consider the fixed boundary problem
\[ \begin{cases} 
  w_t = d \int_{I_1}^{I_2} J(x-y)w(t,y)dy - dw - aw \\
  + c \int_{I_1}^{I_2} K(x-y)z(t,y)dy, & t > 0, x \in [I_1, I_2], \\
  z_t = -bz + G(w), & t > 0, x \in [I_1, I_2], \\
  w(0, x) = w_0(x), z(0, x) = z_0(x), & x \in [I_1, I_2]. \end{cases} \]

**Lemma 2.6.** If the initial functions \( w_0, z_0 \in C([I_1, I_2]) \) are nonnegative, and \( w_0, z_0 \not\equiv 0 \), then (2.14) has a unique positive solution \((w(t, x), z(t, x))\) defined for all \( t > 0 \).
Lemma 2.7. For unique solution \((w, z)\) of (2.14):

(i) If \(\lambda_p(L_{(t, t_2)} + \theta) > 0\), then
\[
\lim_{t \to +\infty} (w(t, x), z(t, x)) = (W(x), Z(x)) \text{ uniformly in } [l_1, l_2].
\]

(ii) If \(\lambda_p(L_{(t, t_2)} + \theta) \leq 0\), then
\[
\lim_{t \to +\infty} (w(t, x), z(t, x)) = (0, 0) \text{ uniformly in } [l_1, l_2].
\]

Proof. We can prove this lemma by following the proof of [33, Lemma 3.8].

(i) Let \(\phi\) be the positive normalized eigenfunction corresponding to \(\lambda_p\), namely, \(||\phi\|_\infty = 1\) and
\[
d \int_{l_1}^{l_2} J(x-y) \phi(y) dy - d \phi(x) - a \phi + \frac{c G'(0)}{b} \int_{l_1}^{l_2} K(x-y) \phi(y) dy = \lambda_p \phi, x \in [l_1, l_2].
\]

Arguing as the proof of (2.10), we can have there exists some positive constant \(c_1\) such that \(\phi(x) \geq c_1\) in \([l_1, l_2]\). Define, for small \(\epsilon > 0\),
\[
\bar{w}^*(x) = \epsilon \phi(x), \quad \bar{z}^*(x) = \frac{1}{b} G(\epsilon \phi(x)) \text{ for } x \in [l_1, l_2].
\]

Direct computations yield
\[
d \int_{l_1}^{l_2} J(x-y) \bar{w}^*(y) dy - d \bar{w}^* - a \bar{w}^* + c \int_{l_1}^{l_2} K(x-y) \bar{z}^*(y) dy
\]
\[
= \epsilon \left[ d \int_{l_1}^{l_2} J(x-y) \phi(y) dy - d \phi(x) - a \phi + \frac{c G'(0)}{b} \int_{l_1}^{l_2} K(x-y) \phi(y) dy \right]
\]
\[
+ \frac{c}{b} \int_{l_1}^{l_2} K(x-y) G(\epsilon \phi(y)) dy - \frac{c}{b} \int_{l_1}^{l_2} K(x-y) G'(0) \epsilon \phi(y) dy
\]
\[
\geq \epsilon \phi \left[ \lambda_p - \frac{c}{b \epsilon} \|G'(\eta_\epsilon(y)) - G'(0)\|_{C([l_1, l_2])} \right]
\]
and \(-b \bar{z}^* + G(\bar{w}^*) = 0\), where \(0 < \eta_\epsilon(y) < \epsilon \phi(y) < \epsilon\). Hence we can choose \(\epsilon\) small enough such that \((\bar{w}^*(x), \bar{z}^*(x))\) is a lower solution of (2.13).

Since \(\gamma_{(K_1)} \gamma_{(z)} = \frac{a b}{c}\) and \(\gamma_{(z)}\) is decreasing, we can choose \(M_1\) and \(M_2\) such that
\[
M_1 > \max\{K_1, \|w_0\|_\infty\}, \quad M_2 > \|z_0\|_\infty, \quad G(M_1) < b M_2 < \frac{a b}{c} M_1.
\]

Then it is easy to check that \((\bar{w}^*(x), \bar{z}^*(x)) = (M_1, M_2)\) is an upper solution of (2.13). Moreover, we can further guarantee that
\[
(\bar{w}^*(x), \bar{z}^*(x)) \geq (w(1, x), z(1, x)) \geq (\bar{w}^*(x), \bar{z}^*(x)) \text{ componentwisely.}
\]
Let \((\tilde{w}(t, x), \tilde{z}(t, x))\) and \((w(t, x), z(t, x))\) be the solution of (2.14) with \((w_0(x), z_0(x))\) replaced by \((\tilde{w}(x), \tilde{z}(x))\) and \((w^*(x), z^*(x))\). By Lemma 2.2 and a simple comparison argument, we have, for \(t \geq 0\) and \(x \in [l_1, l_2],\)

\[
(w(t, x), z(t, x)) \leq (w(t + 1, x), z(t + 1, x)) \leq (\tilde{w}(t, x), \tilde{z}(t, x)).
\]

By using Corollary 2, we can apply the same arguments in (ii) of [33, Lemma 3.8] to obtain

\[
\lim_{t \to +\infty} (w(t, x), z(t, x)) = \lim_{t \to +\infty} (\tilde{w}(t, x), \tilde{z}(t, x)) = (W(x), Z(x)) \text{ uniformly in } [l_1, l_2].
\]

It follows from this fact and (2.15) that

\[
\lim_{t \to +\infty} (w(t, x), z(t, x)) = (W(x), Z(x)) \text{ uniformly in } [l_1, l_2].
\]

(ii) In this case we can construct \((\tilde{w}(t, x), \tilde{z}(t, x))\) as above, and obtain

\[
(0, 0) \leq (w(t, x), z(t, x)) \leq (\tilde{w}(t, x), \tilde{z}(t, x)) \text{ for } t \geq 0 \text{ and } x \in [l_1, l_2].
\]

By using Corollary 2, we can apply the same arguments in (ii) of [33, Lemma 3.8] to obtain

\[
\lim_{t \to +\infty} (\tilde{w}(t, x), \tilde{z}(t, x)) = (0, 0) \text{ uniformly in } [l_1, l_2].
\]

Hence, we have

\[
\lim_{t \to +\infty} (w(t, x), z(t, x)) = (0, 0) \text{ uniformly in } [l_1, l_2].
\]

The proof is complete. \(\Box\)

Next, we consider the solution \((W(x), Z(x))\) to (2.13) when \(-l_1, l_2 \to +\infty\). To stress the dependence of the solution on \((l_1, l_2)\), we use \((W_{(l_1, l_2)}(x), Z_{(l_1, l_2)}(x))\) and \((w_{(l_1, l_2)}(t, x), z_{(l_1, l_2)}(t, x))\) to denote the solution of (2.13) and (2.14) respectively.

**Lemma 2.8.** Assume \(\theta > 0\). Then there exists \(L > 0\) such that for every interval \((l_1, l_2)\) with length \(l_2 - l_1 > L\), we have \(\lambda_p(\mathcal{L}_{(l_1, l_2)} + \theta) > 0\) and then (2.14) has a unique positive steady state \((W_{(l_1, l_2)}(x), Z_{(l_1, l_2)}(x))\). Moreover,

\[
\lim_{-l_1, l_2 \to +\infty} (W_{(l_1, l_2)}(x), Z_{(l_1, l_2)}(x)) = (K_1, K_2) \text{ locally uniformly in } \mathbb{R}. \tag{2.16}
\]

**Proof.** This can be proven by the same arguments as [4, Proposition 3.6], and we give the details below for completeness.

Since \(\theta > 0\), it follows from [4, Proposition 3.4] that there exists \(L > 0\) such that \(\lambda_p(\mathcal{L}_{(l_1, l_2)} + \theta) > 0\) for \(l_2 - l_1 > L\). Applying Lemma 2.7 gives that (2.14) has a unique positive steady state \((W_{(l_1, l_2)}(x), Z_{(l_1, l_2)}(x))\).

Denote \(I = (l_1, l_2)\). Fix a positive function \(u_0 \in L^\infty(\mathbb{R}) \cap C(\mathbb{R})\). By Lemma 2.2 we can use a simple comparison argument to show that the unique solution \((w_I(t, x), z_I(t, x))\) of (2.14) has the property

\[
(w_I(t, x), z_I(t, x)) \leq (w_{l_2}(t, x), z_{l_2}(t, x)) \text{ for } t > 0 \text{ and } x \in I_1
\]

if \(I_1 := (l_1, l_2) \subset I_2 := (l_1^2, l_2^2)\). Letting \(t \to \infty\), it follows that, when \(|I_1| > L,\)

\[
(W_{l_1}(x), Z_{l_1}(x)) \leq (W_{l_2}(x), Z_{l_2}(x)) \text{ for } x \in I_1.
\]

This monotonicity property implies that to show (2.16), it suffices to prove it along any particular sequence \((l_1^n, l_2^n)\) with \(-l_1^n, l_2^n \to +\infty\) as \(n \to +\infty\).

Fix \(z_1, z_2 \in \mathbb{R}\) and consider \((z_i - n, z_i + n), i = 1, 2\). There exists \(N \in \mathbb{N}\) such that for \(n \geq N,\)

\[
\lambda_p(\mathcal{L}_{(z_i - n, z_i + n)} + \theta) > 0, \quad i = 1, 2.
\]
Let \((w_{i,n}(t, x), z_{i,n}(t, x))\) be the solution of the problem
\[
\begin{cases}
w_t = d \int_{z_i-n}^{z_i+n} J(x - y)w(t, y)dy - dw - aw \\
+ c \int_{z_i-n}^{z_i+n} K(x - y)z(t, y)dy, \\
z_t = -bz + G(w), \\
w(0, x) = w_0(x), \quad z(0, x) = z_0(x),
\end{cases}
\] for \(t > 0, \quad x \in [z_i - n, z_i + n], \quad (2.17)\)

Due to Lemma 2.7, one sees that for \(n \geq N, \quad (2.17)\) admits a unique positive steady state \((W_{i,n}, Z_{i,n}) \in [C([z_i - n, z_i + n])]^2,\)

\[
\lim_{t \to \infty} (w_{i,n}(t, x), z_{i,n}(t, x)) = (W_{i,n}(x), Z_{i,n}(x)) \text{ in } [C([z_i - n, z_i + n])]^2, \quad i = 1, 2.
\]

Moreover, if \(m > n,\) it follows from Lemma 2.2 that
\[
(w_{i,m}(t, x), z_{i,m}(t, x)) \geq (w_{i,n}(t, x), z_{i,n}(t, x)) \quad \text{for } t > 0, \quad x \in [z_i - n, z_i + n],
\]

and then
\[
(W_{i,m}(x), Z_{i,m}(x)) \geq (W_{i,n}(x), Z_{i,n}(x)) \quad \text{for } x \in [z_i - n, z_i + n]. \quad (2.18)
\]

Let \((\pi(t), \tau(t))\) be the solution of the ODE problem
\[
\begin{cases}
w'(t) = -aw + cz, \\
z'(t) = -bz + G(w), \quad t > 0, \\
w(0) = \|w_0\|_{\infty}, \quad z(0) = \|z_0\|_{\infty}.
\end{cases}
\]

Making use of Lemma 2.2, we deduce
\[
(w_{i,n}(t, x), z_{i,n}(t, x)) \leq (\pi(t), \tau(t)) \quad \text{and hence } (W_{i,n}(x), Z_{i,n}(x)) \leq (K_1, K_2).
\]

By this fact and (2.18), there exists \((W_i(x), Z_i(x)) \in L^\infty(\mathbb{R}) \times L^\infty(\mathbb{R})\) such that
\[
\lim_{n \to +\infty} (W_{i,n}(x), Z_{i,n}(x)) = (W_i(x), Z_i(x)) \quad \text{for every } x \in \mathbb{R}.
\]

We now show that
\[
(W_1(x), Z_1(x)) \equiv (W_2(x), Z_2(x)). \quad (2.19)
\]

Fix \(N_1 \in \mathbb{N}\) with \(N_1 > |z_1 - z_2|\). Then \([z_i - n - N_1, z_i + n + N_1] \supset [z_2 - n, z_2 + n]\) and hence, for \(n \geq N_1,\)
\[
(w_{1,n+N_1}(t, x), z_{1,n+N_1}(t, x)) \geq (w_{2,n}(t, x), z_{2,n}(t, x)),
\]

and so
\[
(W_{1,n+N_1}(x), Z_{1,n+N_1}(x)) \geq (W_{2,n}(x), Z_{2,n}(x)) \quad \text{for } x \in [z_2 - n, z_2 + n],
\]

which implies \((W_1(x), Z_1(x)) \geq (W_2(x), Z_2(x))\) in \(\mathbb{R}\) by letting \(n \to +\infty\). Similarly, we have \((W_1(x), Z_1(x)) \leq (W_2(x), Z_2(x))\) in \(\mathbb{R}\). Hence (2.19) holds.

From (2.19) it follows immediately that \((W_1(x), Z_1(x)) \equiv (c_1, c_2)\) is a constant solution in \(\mathbb{R} \times \mathbb{R}\) as \(z_1\) and \(z_2\) are arbitrary. It then follows that \(W_{i,n}(x)\) converges to \(c_i\) locally uniformly in \(\mathbb{R}\) as \(n \to +\infty\) thanks to Dini’s theorem. This in turn implies that \((c_1, c_2)\) is the solution of (2.14) with \((l_1, l_2)\) replaced by \((\infty, \infty)\), and then \((c_1, c_2)\) must be a positive zero of
\[
-aw + cz = 0, \quad -bz + G(w) = 0.
\]

Thus \((c_1, c_2) = (K_1, K_2),\) and \((W_{1,n}(x), Z_{1,n}(x))\) converges to \((K_1, K_2)\) locally uniformly in \(\mathbb{R} \times \mathbb{R}\) as \(n \to +\infty\), which implies (2.16).
By using Lemma 2.7, we can apply the same argument as [4, Theorem 3.7] (see also [33, Lemma 3.9]) to show

**Lemma 2.9.** Assume that \( \theta > 0 \). Let \((u, v, g, h)\) be the unique solution of (1.7). If \( h_\infty - g_\infty < \infty \), then

\[
\lim_{t \to \infty} \max_{g(t) \leq x \leq h(t)} u(t, x) = \lim_{t \to \infty} \max_{g(t) \leq x \leq h(t)} v(t, x) = 0,
\]

and

\[
\lambda_p(L_{(g_\infty, h_\infty)}) + \theta \leq 0. \tag{2.20}
\]

For \( \theta > 0 \), it follows from [4, Proposition 3.4] that there exists \( l^* \) such that

\[
\begin{cases}
\lambda_p(L_{(l_1, l_2)}) + \theta = 0 & \text{if } l_2 - l_1 = l^*, \\
(l_2 - l_1 - l^*)\lambda_p(L_{(l_1, l_2)}) > 0 & \text{if } l_2 - l_1 \in (0, +\infty) \setminus \{l^*\}.
\end{cases}
\]

**Lemma 2.10.** Assume that \( \theta > 0 \). Then the following hold:

(i) If \( h_\infty - g_\infty < \infty \), then \( h_\infty - g_\infty \leq l^* \).

(ii) If \( 2h_0 \geq l^* \), then \( h_\infty - g_\infty = \infty \).

(iii) If \( 2h_0 < l^* \), then there exist \( 0 < \mu_0 \leq \mu^0 \) such that \( h_\infty - g_\infty = \infty \) for \( \mu > \mu^0 \) and \( h_\infty - g_\infty < \infty \) for \( 0 < \mu < \mu_0 \).

**Proof.** (i) Arguing indirectly we assume that \( h_\infty - g_\infty > l^* \). Since \( \theta > 0 \), we have \( \lambda_p(L_{(h_\infty, g_\infty)}) + \theta > 0 \). This is a contradiction to (2.20).

(ii) This conclusion follows directly from (i).

(iii) By [13, Lemma 3.9], there exists \( \mu_0 \) such that \( h_\infty - g_\infty = \infty \) for \( \mu > \mu_0 \).

It remains to prove the conclusion for \( \mu_0 \). This can be done by same argument as [33, Lemma 3.11 (iii)]. Since the nonlocal reaction term appears, some considerable changes are needed and so we give the details below.

Since \( 2h_0 < l^* \), we have \( \lambda_p(L_{(-h_0, h_0)}) + \theta < 0 \). There exists some small \( \varepsilon > 0 \) such that \( h^* := h_0 (1 + \varepsilon) \) satisfies

\[
\lambda_p^* := \lambda_p(L_{(-h^*, h^*)}) + \theta < 0.
\]

Let \( \phi_1 \) be the positive normalized eigenfunction corresponding to \( \lambda_p^* \), namely, \( \phi_1 > 0 \), \( \|\phi_1\|_\infty = 1 \) and

\[
d \int_{-h^*}^{h^*} J(x-y)\phi_1(y)dy - a\phi_1 + \frac{cG'(0)}{b} \int_{-h^*}^{h^*} K(x-y)\phi_1(y)dy = \lambda_p^* \phi_1, \quad x \in [-h^*, h^*].
\]

Arguing as in the proof of (2.10), we can show there exists some positive constant \( c_0 \) such that \( \phi_1(x) > c_0 \) in \( [-h^*, h^*] \). Choose a positive constant \( M \) large enough such that

\[
M\phi_1(x) \geq u_0(x) \quad \text{and} \quad \left( \frac{G'(0)}{b} - \frac{\lambda_p^* c_0}{4c} \right) M\phi_1(x) \geq v_0(x) \quad \text{for } x \in [-h_0, h_0].
\]

Define

\[
\bar{\theta}(t) = h_0 \left[ 1 + \varepsilon \left( 1 - e^{-\delta t} \right) \right], \quad \overline{\theta}(t) = -\bar{\theta}(t), \quad t \geq 0,
\]

\[
\bar{\pi}(t, x) = M e^{-\delta t} \phi_1(x), \quad t \geq 0, \quad x \in [\overline{\theta}(t), \bar{\theta}(t)],
\]

\[
\overline{\pi}(t, x) = \left( \frac{G'(0)}{b} - \frac{\lambda_p^* c_0}{4c} \right) \bar{\pi}(t, x), \quad t \geq 0, \quad x \in [\overline{\theta}(t), \bar{\theta}(t)],
\]

where \( \delta > 0 \) will be determined later. Clearly \( h_0 \leq \bar{\theta}(t) \leq h^* \).
Direct calculations yield
\[
\begin{align*}
\pi(t) - d \int_{\mathcal{F}(t)} J(x-y) \pi(t,y) dy + a \pi - c \int_{\mathcal{F}(t)} K(x-y) \pi(t,y) dy \\
\geq M e^{-\delta t} \left[ - \delta \phi_1(x) - d \int_{-h}^{h} J(x-y) \phi_1(y) dy + d \phi_1 + a \phi_1 \\
- \frac{cG'(0)}{b} \int_{-h}^{h} K(x-y) \phi_1(y) dy + \frac{\lambda_p^* c_0}{4} \int_{-h}^{h} K(x-y) \phi_1(y) dy \right] \\
= M e^{-\delta t} \phi_1(x) \left[ - \delta - \lambda_p^* + \frac{\lambda_p^* c_0}{4 \phi_1(x)} \int_{-h}^{h} K(x-y) \phi_1(y) dy \right] \\
\geq M e^{-\delta t} \phi_1(x) \left[ - \delta - \frac{3 \lambda_p^*}{4} \right]
\end{align*}
\]
and
\[
\pi(t) + b \pi - G(\pi) > (b - \delta) \left( \frac{G'(0)}{b} - \frac{\lambda_p^* c_0}{4c} \right) \pi - G'(0) \pi = \left[ - \delta - \frac{G'(0)}{b} - (b - \delta) \frac{\lambda_p^* c_0}{4c} \right] \pi
\]
for \( t > 0 \) and \( x \in (\mathcal{G}(t), \overline{\mathcal{F}(t)}) \). Since \( \lambda_p^* < 0 \), we can choose \( \delta \) small enough such that
\[
- \delta - \frac{3 \lambda_p^*}{4} > 0 \quad \text{and} \quad - \delta - \frac{G'(0)}{b} - (b - \delta) \frac{\lambda_p^* c_0}{4c} > 0.
\]
Moreover, \( \mathcal{H}'(t) = h_{00} \delta e^{-\delta t} \) and
\[
\mu \int_{\mathcal{G}(t)} \mathcal{H}(t) \int_{\mathcal{H}(t)}^{+\infty} J(x-y) \pi(t,x) dy dx \leq 2 \mu M e^{-\delta t} h^*.
\]
If
\[
\mu \leq \frac{h_{00} \delta}{2 \lambda^*_p} := \mu_0,
\]
then we have
\[
\mathcal{H}(t) \geq \mu \int_{\mathcal{G}(t)} \mathcal{H}(t) \int_{\mathcal{H}(t)}^{+\infty} J(x-y) \pi(t,x) dy dx.
\]
Similarly, we can derive
\[
\mathcal{G}'(t) \leq - \mu \int_{\mathcal{G}(t)} \mathcal{G}(t) \int_{-\infty}^{-\mathcal{H}(t)} J(x-y) \pi(t,x) dy dx.
\]
We may now apply Lemma 2.3 to obtain
\[
u(t, x) \leq \pi(t, x), \quad v(t, x) \leq \pi(t, x),
\]
\[
g(t) \geq \mathcal{G}(t), \quad h(t) \leq \mathcal{H}(t) \quad \text{for} \ t > 0 \quad \text{and} \ x \in [g(t), h(t)].
\]
It follows that \( \lim_{t \to \infty} (h(t) - g(t)) \leq \lim_{t \to \infty} (\mathcal{H}(t) - \mathcal{G}(t)) \leq 2 h^* < \infty. \)

**Lemma 2.11.** Assume that \( \theta > 0 \). If \( 2 h_{00} < 1 \), then there exists \( \mu^* > 0 \) such that
\( h_{\infty} = g_{\infty} = \infty \) for \( \mu > \mu^* \) and \( h_{\infty} = g_{\infty} < \infty \) for \( 0 < \mu \leq \mu^* \).

**Proof.** By Corollary 1, the proof of this lemma follows from the same method in [4, Theorem 3.14] (see also [33, Lemma 3.12]).

**Lemma 2.12.** Assume that \( \theta > 0 \). Then \( h_{\infty} = +\infty \) if and only if \( g_{\infty} = -\infty \).

**Proof.** The proof of this lemma can be done by the same argument as in [4, Lemma 3.8] (see also [33, Lemma 3.13]).
Lemma 2.13. Suppose $\theta > 0$. If $h_{\infty} - g_{\infty} = +\infty$, then
\[
\lim_{t \to \infty} (u(t, x), v(t, x)) = (K_1, K_2) \text{ locally uniformly in } \mathbb{R}.
\]

Proof. By Lemmas 2.12, 2.7, 2.8, we can apply the same argument as [33, Lemma 3.14] to obtain this lemma. \qed

Clearly, Theorem 1.2 follows directly from Lemmas 2.4, 2.9 and 2.13. Theorem 1.3 follows directly from Lemmas 2.4, 2.10 and 2.11.

We end this paper with some details about footnote 1 in the introduction. So suppose (G2) is replaced by
\[
(G2)^{'} \quad G(z)/z \text{ is nonincreasing for } z > 0 \text{ and strictly decreasing in the neighbourhood of } \Sigma := \{ z > 0 : G(z)/z = ab/c \} \text{ in the case } G'(0) > ab/c.
\]

We now explain the changes we have to make to the results proved in this paper. Firstly we note that the strict monotonicity near $\Sigma$ is necessary to guarantee the uniqueness of $K_1$ determined by $G(K_1)/K_1 = ab/c$.

Secondly we note that under $(G2)^{'}$, in Lemma 2.5 part (i), the same arguments can be used to prove there are a maximal and a minimal positive solution, say $(\overline{W}, \overline{Z})$ and $(\underline{W}, \underline{Z})$, respectively, but the uniqueness result is lost in general. The nonexistence result in part (ii) still holds as the proof is not affected. Corollary 2 should be modified accordingly.

Thirdly we have to modify the conclusion in Lemma 2.7: part (i) should be changed to
\[
\liminf_{t \to \infty} (w(t, \cdot), z(t, \cdot)) \geq (\overline{W}, \overline{Z}), \quad \limsup_{t \to \infty} (w(t, \cdot), z(t, \cdot)) \leq (\overline{W}, \overline{Z})
\]
uniformly in $[l_1, l_2]$; part (ii) remains valid.

Fourth, the conclusions in Lemma 2.8 still hold, since the same argument can be applied to $(\overline{W}(l_1, l_2), \overline{Z}(l_1, l_2))$ and $(\underline{W}(l_1, l_2), \underline{Z}(l_1, l_2))$ to show that each of them converges to a constant positive solution of (2.13) over $\mathbb{R}$ as $-l_1, l_2 \to \infty$, which by $(G2)^{'}$ is necessarily $(K_1, K_2)$.

The rest of the paper is not affected by the changes from $(G2)$ to $(G2)^{'}$.

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