TREES, QUIVERS, BIGRAPHS: COMBINATORIAL BIALGEBRAS FROM MONOIDAL MÖBIUS CATEGORIES

ULRICH KRÄHMER AND LUCIA ROTHERAY

Abstract. The Hopf algebras associated by Gálvez-Carillo, Kock and Tonks to monoidal Möbius categories are shown to be distinct from those associated by Cibils and Rosso to Hopf quivers, except in the trivial case of a group algebra. Combinatorial left-sided Hopf algebras in the sense of Loday and Ronco do, however, provide examples, including planar rooted trees. As a new example, Milner’s bigraphs also define a bialgebra in this way.

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1. INTRODUCTION

Bigraphs [218] are a combinatorial structure originally developed in theoretical computer science to model mobile computation. Over the past four decades, various authors have identified natural constructions that express combinatorics in terms of Hopf algebras. The aim of the present paper is to apply these methods to bigraphs.

To this end, it is first necessary to clarify that these combinatorial Hopf-algebraic constructions are similar but distinct. For example, Cibils and Rosso [3] initiated the study of so-called Hopf quivers, i.e. quivers whose path coalgebra admits a Hopf algebra structure. The vertices of the quiver become the set of group-like elements and the arrows define a vector space basis of the space of twisted primitive elements. More generally, the study of Hopf algebra structures on the incidence coalgebras of suitable categories goes back to Joni and Rota [13]. In [9], this lead to the concept of a decomposition space, a generalised setting for incidence (co)algebras, and of which Möbius categories (as in [14, 15]) provide examples. The basic idea is to turn the free vector space on the set of morphisms of a small category into a Hopf algebra whose product is given by a monoidal structure on the category and whose coproduct is given by decomposition.

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It is our pleasure to thank Gabriella Böhm, Joachim Kock and Michele Sevegnani for discussions and suggestions.
Perhaps counterintuitively, the construction of Rosso and Cibils is not subsumed by the latter:

**Theorem 1.** The product in the Hopf algebra $kQ$ defined by a Hopf quiver $Q$ is a linear extension of a monoidal product on the free category $C_Q$ generated by $Q$ if and only if the quiver is empty (contains no arrows).

Motivated by the Connes-Kreimer Hopf algebra of rooted trees [1,5] and combinatorial Hopf algebras in the sense of Loday and Ronco [16], we also wanted to view these algebras in a common context alongside M"obius categories. In particular, we give a criterion for the Hopf algebra of a monoidal M"obius category to be left-sided combinatorial in the sense of Loday and Ronco, see Proposition 3 below.

It turned out that this is also the exact setting that is applicable to bigraphs - they yield a bialgebra which as an algebra is the free algebra on a set of generators given by connected bigraphs, and whose coproduct encodes the bigraph dynamics.

The structure of the paper is as follows:

In Section 2 we recall the definition of a M"obius category, and define the term combinatorial category and the associated bialgebra.

In Section 3 we present four examples of such combinatorial bialgebras. The first three of these are not new examples, but this is the first time (to our knowledge) that they have been studied in a common framework: incidence algebras of relations, the Connes-Kreimer Hopf algebra of rooted trees, and the Hopf quivers. The final example is the one which motivated the whole study: a bialgebra on bigraphs, defined using the categorical structure introduced by Milner in [18].

### 2. Bialgebras from monoidal categories

#### 2.1. M"obius categories

Throughout, all categories are assumed to be small, and all monoidal categories to be strict. The set of morphisms in a category $C$ will be denoted just by $C$. The set of morphisms $f: x \to y$ will be denoted $C(x,y)$, and we write $s(f) = x, t(f) = y$. The identity morphism of $x$ will be denoted $i_x$.

**Definition 1** (Decompositions and length). Given a morphism $f \in C$, define

$$N_n(f) := \{ (a_1, \ldots, a_n) \in C \times \cdots \times C \mid a_1 \circ \cdots \circ a_n = f \}$$

and let $\hat{N}_n(f)$ be the subset of nondegenerate decompositions, i.e. those for which none of the $a_i$ is an identity morphism. We further set

$$\hat{N}(f) := \bigcup_n \hat{N}_n(f), \quad \ell(f) := \sup\{ n \mid \hat{N}_n(f) \neq \emptyset \}, \quad C_n := \{ f \in C \mid \ell(f) \leq n \}.$$

We call $\ell(f)$ the length of $f$. A morphism is indecomposable if $\ell(f) = 1$.

Note that by definition, $N_n := \bigcup_{f \in C} N_n(f)$ is the set of $n$-simplices in the nerve of the category $C$. This leads to the more topological viewpoint of [9].

The following concept appears under various names in the literature; we follow the terminology of Joni and Rota [13] and Leroux [15], see also [14].

**Definition 2** (Locally finite category, M"obius category). A category $C$ is locally finite if the set $N_2(f)$ is finite for all $f \in C$. It is a M"obius category if all $\hat{N}(f)$ are finite.

**Lemma 1.** A category $C$ is locally finite if and only if for all $n \geq 0$ and $f \in C$, $N_n(f)$ is finite. It is M"obius if and only if it is locally finite and every $f$ has finite length.

Evidently, a M"obius category does not contain any nontrivial isomorphisms or idempotents. Note that the sequence of sets

$$C_n := \{ f \in C \mid \ell(f) \leq n \}$$
is a filtration of $C$ which is exhaustive precisely when $C$ is M"obius. The set $C_0$ contains only identity morphisms, and $C$ is called strongly one-way if all identity morphisms have length 0. This holds in particular if $C$ is M"obius.

2.2. Combinatorial categories. By definition, a functor preserves compositions. For the constructions below we must consider functors that also preserve decompositions. We use the following property, defined in [14, Lemma 4.3]:

**Definition 3 (Unique lifting of factorisations property).** A functor $F: D \to C$ has the unique lifting of factorisations (ULF) property if for all $f \in D$, the map
\[
N_2(f) \to N_2(F(f)), \quad (a, b) \mapsto (F(a), F(b))
\]
is bijective. If $F$ has the ULF property, we call it a ULF functor.

**Remark.** This notion is explored further in the work of Gálvez-Carillo, Kock and Tonks, see in particular [9, Section 1.5] and [10, Section 3].

Note that functors with this property also reflect identity morphisms:

**Lemma 2.** If $F$ is a ULF functor, then $F(f) = i_{F(x)}$ implies $f = i_x$. 

**Proof.** Follows directly from the injectivity of the map $N_2(i_x) \to N_2(i_{F(x)})$. 

We are in fact solely interested in the case in which $D = C \times C$ and $F$ is a monoidal structure $\cdot$ on $C$. So, explicitly, we require that for all $f, g \in C$, the map
\[
N_2(f) \times N_2(g) \to N_2(f \cdot g), \quad ((a, b), (\gamma, \delta)) \mapsto (a \cdot \gamma, b \cdot \delta)
\]
is bijective. This means that a tensor product of morphisms has exactly the decompositions that arise from decomposing the two tensor components, and all identity morphisms are monoidally indecomposable. So if $f \cdot g = i_x$, then both $f$ and $g$ must themselves be identity morphisms.

**Definition 4 (Combinatorial category).** A M"obius category which is monoidal and whose monoidal structure is decomposition-preserving will be called combinatorial.

2.3. Combinatorial bialgebras. Modulo details, the following result is known and contained in [9][13][14], but we wanted to present the precise correspondence between categorical and Hopf-algebraic concepts in one statement:

**Theorem 2.** Let $C$ be a category, $k$ a commutative ring, and $kC := \text{span}_k C$ the free $k$-module on $C$. We understand $\otimes$ to mean the tensor product $\otimes_k$.

1. If $C$ is locally finite, then $kC$ is a (counital coassociative) coalgebra with coproduct $\Delta: kC \to kC \otimes kC$ and counit $\varepsilon: kC \to k$ given by
\[
\Delta(f) := \sum_{(a, b) \in N_2(f)} a \otimes b, \quad \varepsilon(f) := \begin{cases} 1, & f = i_x \text{ for some } x, \\ 0, & \text{else.} \end{cases}
\]

2. If $C$ is M"obius and $k$ is a field, the coalgebra $kC$ is pointed.

3. If $C = (C, \cdot, 1_C)$ is monoidal, then $kC$ is a (unital associative) algebra with product induced by $\cdot$ and unit element given by $1_k$.

4. A functor between locally finite categories is a ULF functor if and only if it induces a homomorphism between the associated coalgebras. In particular, if $C$ is both locally finite and monoidal, then $(kC, \cdot, 1_k, \Delta, \varepsilon)$ is a bialgebra if and only if $\cdot$ is ULF.

5. If $C$ is a combinatorial category, then $kC$ is a Hopf algebra if and only if the monoid $(C_0, \cdot, 1_C)$ formed by the objects is a group.
Proof. 1., 3. and 4. are immediate. The local finiteness corresponds to the fact that the coproduct $\Delta$ lands in the algebraic tensor product $kC \otimes kC$.

2. By the remarks at the end of the preceding subsection, the family $\{kC_n\}_{n \geq 0}$ is a $k$-module filtration of $kC$ which is exhaustive if $C$ is Möbius. In this case it is also a coalgebra filtration, that is

$$\Delta(kC_n) \subset \sum_{i + j = n} kC_i \otimes kC_j.$$ 

If $k$ is a field, the fact that $kC$ is pointed now follows from [19, Proposition 4.1.2]. Note that the converse is not necessarily true: the path coalgebra of a non-Möbius category can be pointed.

5. follows now from [19, Lemma 7.6.2].

The following terminology is more general than the one used for example by Loday and Ronco [16]. We will explain the relation in Section 3.3 below.

Definition 5 (Combinatorial bialgebra). The path bialgebras of combinatorial categories will be called combinatorial.

3. Examples

3.1. Relations on monoids. We begin with a simple class of examples to illustrate the definitions.

Let $\preceq$ be a reflexive and transitive relation on a monoid $M$ for which

$$x \preceq y, z \preceq t \Rightarrow x \cdot z \preceq y \cdot t$$

holds. Then

$$C_M := \{(x, y) \in M \times M \mid x \preceq y\}$$

is the morphism set of a monoidal category with object set $M$ and $(x, y)$ the morphism $y \rightarrow x$. The operations $\circ, \cdot$ are given by

$$(x, y) \circ (y, z) := (x, z), \quad (x, y) \cdot (z, t) := (x \cdot z, y \cdot t).$$

The identity morphisms are given by $i_x = (x, x)$. By definition, we have:

Proposition 1. (1) The category $C_M$ is locally finite if and only if all intervals $[x, y] := \{z \in M \mid x \preceq z \preceq y\}$ are finite.

(2) The category $C_M$ is Möbius if and only if given $x \preceq y \in M$ there is $\ell(x, y) \in \mathbb{N}$ such that there is no chain $x = z_1 \preceq z_2 \preceq \ldots \preceq z_l = y$ with $z_i \neq z_{i+1}$ and $\ell(x, y)$.

(3) The monoidal structure is ULF if and only if it defines for all $x \preceq y, z \preceq t \in M$ a bijection $[x, y] \times [z, t] \rightarrow [x \cdot z, y \cdot t]$.

Thus if $C_M$ is Möbius, $\preceq$ is a partial ordering.

In particular, when $\preceq$ is an equivalence relation, then the interval $[x, y]$ is simply the equivalence class of $x$ (and of $y$), and [1] demands that the monoid structure descends to the quotient $M/\preceq$. The category $C_M$ is locally finite if and only if all equivalence classes are finite, but it is not Möbius unless $\preceq$ is $\approx$, as any $(x, y)$ with $x \preceq y$ but $x \neq y$ is a nontrivial isomorphism. As a coalgebra, we have

$$kC_M \cong \bigoplus_{x \in M/\preceq} M_{[x]}(k),$$

where $M_{[x]}(k)$ is the matrix coalgebra of size $n \times n$, so $kC_M$ is cosemisimple. The monoidal structure is ULF if and only if $M/\preceq \rightarrow (\mathbb{N}, \cdot), x \mapsto |x|$ is a map of monoids.
Example 1. As an example of a combinatorial bialgebra arising in this way, consider $M = \langle S \rangle$, the free monoid on $S = \{x, y\}$, in which two words are equivalent if they have the same length. Then $C_M = (S \times S)$ and $k(S \times S)$ is the free algebra on four generators $\alpha = (x, x), \beta = (x, y), \gamma = (y, x), \delta = (y, y)$ whose coproduct is given by
\[
\Delta(\alpha) = \alpha \otimes \alpha + \beta \otimes \gamma, \quad \Delta(\beta) = \alpha \otimes \beta + \beta \otimes \delta, \\
\Delta(\gamma) = \gamma \otimes \alpha + \delta \otimes \gamma, \quad \Delta(\delta) = \gamma \otimes \beta + \delta \otimes \delta.
\]
In particular, it is not a Hopf algebra.

3.2. Place graphs. We now consider path bialgebras that are examples of what Loday and Ronco call left-sided combinatorial Hopf algebras \[16\].

Proposition 2. Let $C = \langle \Gamma \rangle$ be a monoidal Möbius category which is a free monoid with respect to $\cdot$. Then the monoid of objects is isomorphic to the free monoid over $\Gamma_0 \subset \Gamma$, the set of identity morphisms inside $\Gamma$.

Proof. By identifying each object with its identity morphism, we view the monoid of objects as a submonoid of $C$, hence it is also free.

Assume now that $x$ is a monoidally indecomposable object. It remains to show that the identity $i_x$ is monoidally indecomposable as a morphism, i.e. is an element of $\Gamma$. So assume $i_x = f \cdot g$ with $f : y \to z, g : u \to v$. It follows that $i_x = (i_x, g \circ f_i)$.

Thus $i_x = i_z \cdot g = f \cdot i_u$, as the existence of non-trivial isomorphisms violates the Möbius condition. Repeating the argument with $f$ replaced by $i_z$ yields $i_x = i_z \cdot i_u$, contradicting the assumption that $x$ is monoidally indecomposable. □

Definition 6 (Left-sided combinatorial category). If $C = \langle \Gamma \rangle$ is a combinatorial category which is a free monoid with respect to $\cdot$ and for all $(a, b) \in N_2(\gamma), \gamma \in \Gamma$ we have $a \in \Gamma$, then we call $C$ left-sided.

The examples we will consider are of the following type:

Proposition 3. Let $C = \langle \Gamma \rangle$ be a free monoid. Furthermore, assume that $C(1, 1) = \{1\}, C(x, 1) = \emptyset$ for all objects $x \neq 1$, and that the codomains $t(\gamma)$ of all $\gamma \in \Gamma$ are in $\Gamma_0$. Then $C$ is left-sided.

Proof. If $\gamma = a \circ b$ and if $a = a_1 \cdot a_2$, then $t(a) = t(a_1) \cdot t(a_2) \in \Gamma_0$. Hence we either have $a_1 = i_1$ or $a_2 = i_1$, so $a$ is monoidally indecomposable. □

We will consider categories whose morphisms are types of planar rooted forests. The most general is the following.

Definition 7 (Place graph). A place graph consists of totally ordered finite disjoint sets $R, S, V$ and a non-decreasing map $\text{prnt} : V \cup S \to V \cup R$ such that $R \subseteq \text{im} \ \text{prnt}$ and for all $v \in V \cup S$, there is $n \in \mathbb{N}$ with $\text{prnt}^n(v) \in R$.

The elements of $R$ are called roots, those of $S$ sites, and those of $V$ vertices. The element $\text{prnt}(v)$ is the parent of $v$, and $\text{prnt}^{-1}(v)$ contains its children.

Definition 8 (Composition). Let $f_1 = (R_1, S_1, V_1, \text{prnt}_1), f_2 = (R_2, S_2, V_2, \text{prnt}_2)$ be place graphs with $R_1 = S_1$ and all other involved sets pairwise disjoint. Then one defines their composition $f_2 \circ f_1$ as $(R_2, S_1, V_1 \oplus V_2, \text{prnt})$, where $\text{prnt}(v) := \text{prnt}_1(v)$ for $v \in V_1, S_1$ except when $\text{prnt}_1(v) \in R_1 = S_2$, in which case $\text{prnt}(v) := \text{prnt}_2(\text{prnt}_1(v))$.

Here $\oplus$ denotes the concatenation (ordinal sum) of ordered sets. With this composition, place graphs are the morphisms in a monoidal category whose objects are finite ordered sets with concatenation as monoidal product.

Passing to isomorphism classes yields a small category $C_{PG}$ whose monoid of objects is $(\mathbb{N}, +)$ (that is, $C_{PG}$ is a non-symmetric PROP).
Example 2. Roots and sites are depicted in white, vertices in black. We draw place graphs with roots at the top.

\[ f = \begin{array}{c}
\text{root} \\
\text{site} \\
\text{vertex}
\end{array} \quad f \circ g = \begin{array}{c}
\text{root} \\
\text{site} \\
\text{vertex}
\end{array} \]

\[ g = \begin{array}{c}
\text{root} \\
\text{site} \\
\text{vertex}
\end{array} \quad g \circ (f \cdot f) = \begin{array}{c}
\text{root} \\
\text{site} \\
\text{vertex}
\end{array} \]

Remark 2. Observe that \( \nu(R, S, V, \text{prnt}) := |V| \) defines a grading, i.e. we have
\[
(2) \quad \nu(f_2 \circ f_1) = \nu(f_2) + \nu(f_1), \quad \nu(f_2 \cdot f_1) = \nu(f_2) + \nu(f_1).
\]
In general, this is different from the length filtration, though - for example, the place graph
\[
\begin{array}{c}
\text{root} \\
\text{site} \\
\text{vertex}
\end{array}
\]

is not an identity morphism but has \( \nu(f) = 0 \). Note also that in any place graph \( f \), we have
\[
(3) \quad |R| \leq |S| + \nu(f).
\]

Using these facts, we show:

Proposition 4. \( C_{PG} \) is a left-sided combinatorial category. The set \( \Gamma \) of generators is the set of place graphs with a single root.

Proof. That \( (C_{PG}, \cdot) \) is free on the set of place graphs with one root is immediate. Let now \( f = f_2 \circ f_1 \) be a decomposition of a place graph as in Remark 2. The latter shows that
\[
|R| = |R_2|, \quad |S| = |S_1|, \quad \nu(f) = \nu(f_1) + \nu(f_2)
\]

and
\[
|S_2| = |R_1| \leq \nu(f_1) + |S_1| \leq \nu(f) + |S|.
\]
The number of different place graphs whose sets of roots, sites and vertices are of a given size is finite. Thus \( N_2(f) \) is finite, that is, \( C_{PG} \) is locally finite.

Similarly, \( f \in C_{PG} \) has finite length: assume \( f = f_t \circ \ldots \circ f_1 \) with none of the \( f_j \) being an identity morphism. In view of (2), at most \( \nu(f) \) of the \( f_j \) have any vertex at all, and for these, (3) yields \( |S_{j+1}| = |R_j| \leq |S_j| + \nu(f_j) \). Composing with these increases the number of roots in total by at most \( \nu(f) \). For all other \( f_k \), we have \( \nu(f_k) = 0 \) and (since \( f_k \) is not an identity morphism) \( |S_{k+1}| = |R_k| \leq |S_k| - 1 \), and it follows that there can be at most \( |S| + \nu(f) \) of these \( f_k \). In combination, \( l \leq 2\nu(f) + |S| \), so \( C_{PG} \) is Möbius by Lemma 1.

To see that \( \cdot \) is ULF, assume \( f = g \cdot h = f_2 \circ f_1 \). Let \( f_2 = t_1 \cdots t_n \) and \( f_1 = b_1 \cdots b_m \), \( t_i, b_j \in \Gamma \). If \( t_i \) has \( d_i \) sites, set \( c_1 := b_1 \cdots b_{d_i}, c_2 := b_{d_i+1} \cdots b_{d_i+d_2} \) and analogously define \( c_1, \ldots, c_n \). Then \( f = (t_1 \circ c_1) \cdots (t_n \circ c_n) \), \( t_i \circ c_i \in \Gamma \), and \( N_2(f) \rightarrow N_2(g) \times N_2(h) \) given by
\[
(f_2, f_1) \mapsto ((t_1 \cdots t_w, c_1 \cdots c_w), (t_{w+1} \cdots t_n, c_{w+1} \cdots c_n))
\]
(where \( g \) has word length \( w \)) is an inverse to the map induced by \( \cdot \).

The left-sidedness of \( C_{PG} \) now follows easily from Proposition 3. \( \square \)
3.3. Rooted trees and the Connes-Kreimer Hopf algebra. In this section, we explain how the Connes-Kreimer Hopf algebra fits into the framework discussed in the present paper. To begin with, we consider a combinatorial subcategory \( \mathcal{C}_{RF} \) of \( \mathcal{C}_{PG} \). We follow the terminology of [1][11][20]:

**Definition 9** (Operadic planar rooted forest). An operadic planar rooted forest is a place graph with \( |\text{prnt}^{-1}(r)| = 1 \) for all roots \( r \in R \).

These rooted forests are in many respects easier to control than general place graphs. For example, we have:

**Proposition 5.** In \( \mathcal{C}_{RF} \) a morphism has length zero if and only if it is an identity. More generally, \( \ell(f) = \nu(f) \) for all \( f \in \mathcal{C}_{RF} \).

The bialgebras \( k\mathcal{C}_{PG} \), \( k\mathcal{C}_{RF} \) are not Hopf algebras, as in both cases there are group-like without inverses. However, the latter can be used to define the Connes-Kreimer Hopf algebra \( H_{CK} \) (which in itself is not a combinatorial bialgebra in the sense of Definition 5).

**Remark 3.** Consider the left coideal subalgebra \( H_{CK} \subset k\mathcal{C}_{RF} \) spanned by the forests with no sites, \( \bigcup_{n \in \mathbb{N}} \mathcal{C}_{RF}(0,n) \), and define a map \( \pi : k\mathcal{C}_{RF} \to H_{CK} \) which acts on a forest with \( \nu(f) > 0 \) by removing all sites and maps all others (which are identities) to \( \iota_0 \). Then \( \pi \) is a morphism of left \( k\mathcal{C}_{RF} \)-comodule algebras and hence \( \Delta := (\pi \otimes \text{id})\Delta \) defines a coassociative coproduct on \( H_{CK} \). The only degree zero morphism in \( H_{CK} \) is the empty graph, so by [19] Lemma 7.6.2, it is a Hopf algebra. This was called in [7] the planar Hopf algebra of rooted trees and is a non-commutative version of the Hopf algebra originally considered by Connes and Kreimer [5]. Analogously, Manchon [17] presents a bialgebra of Feynman graphs, and obtains the Connes-Kreimer Hopf algebra of Feynman graphs by removing all non-empty degree zero graphs. This example shows that in applications it is maybe not even the combinatorial bialgebra itself that matters, but certain right coideal subalgebras therein, which become Hopf algebras using a suitable projection.

3.4. Quiver bialgebras. We now turn to the case in which the category is free with respect to \( \circ \) rather than \( \cdot \): let \( Q = (Q_0,Q_1) \) be a quiver, \( k \) be a field, and \( \mathcal{C}_Q \) be the category of all paths in \( Q \), which is locally finite. By definition, the coalgebra \( k\mathcal{C}_Q \) is the path coalgebra \( kQ \) of the quiver.

Cibils and Rosso [3] (see also [4]) showed that a graded Hopf algebra structure on \( kQ \) endows \( Q_0 \) with a group and \( kQ_1 \) with a \( kQ_0 \)-Hopf bimodule structure. This in turn gives rise to what Cibils and Rosso call ramification data.

**Definition 10** (Ramification data). Let \( G \) be a group. By ramification data for \( G \), we mean a sum \( r = \sum_{\mathcal{C} \in \mathcal{E}} r_{\mathcal{C}} \mathcal{C} \) for \( \mathcal{E} \) the set of conjugacy classes of \( G \) and all \( r_{\mathcal{C}} \in \mathbb{N} \).

Alternatively, ramification data for \( G \) is a class function \( G \to \mathbb{N} \).

Conversely, groups with ramification data define Hopf algebras:

**Definition 11** (Hopf quiver). Let \( G \) be a group and \( r \) some ramification data for \( G \). The quiver with \( Q_0 = G \) and \( r_{\mathcal{C}} \) arrows from \( g \) to \( cg \) for each \( g, c \in G \), where \( C \) is the conjugacy class containing \( c \), is called the Hopf quiver determined by \((G,r)\).

**Theorem 3.** The path coalgebra \( kQ \) of a quiver \( Q \) admits graded Hopf algebra structures if and only if \( Q \) is a Hopf quiver.

To be more precise, for a Hopf quiver \( Q \), there is a canonical Hopf algebra structure on \( kQ \), with \( kQ_1 \cong \bigoplus_{\mathcal{C} \in \mathcal{E}} r_{\mathcal{C}} kC \otimes kQ_0 \) as vector space. The \( kQ_0 \)-bimodule structure is given by

\[
x(c \otimes g) = xcx^{-1} \otimes xgy
\]
and the $kQ_0$-bicomodule structure is given by
\[ c \otimes g \mapsto cq \otimes (c \otimes g) \otimes g. \]

The group structure on $kQ_0$ and the $kQ_0$-bimodule structure on $kQ_1$ define the algebra structure on $kQ$ in lowest degrees, which is extended universally to all of $kQ$, see [3, Theorem 3.8]. However, given a Hopf quiver, there are in general also Hopf algebra structures on $kQ$ compatible with $r$ that are different from the canonical one. The choice is in the $kQ_0$-bimodule structure of $kQ_1$ which amounts to a choice of an $n_C$-dimensional representation of the centralizer of an element $c \in C$. The extension of the product to paths is then unique.

Huang and Torecillas [12] proved that a quiver path coalgebra $kQ$ always admits graded bialgebra structures. The results of Green and Solberg [8] are also closely related, but different in that they study path algebras rather than path coalgebras. Here we focus on the Hopf algebra setting as considered in [3].

Our key aim is to stress that the product in a quiver Hopf algebra $kQ$ is not a linear extension of a monoidal product on $C_Q$ as in Theorem [2] unless $Q_1 = \emptyset$. To do this, we classify all monoidal structures on path categories of quivers whose vertices form a group under $\cdot$:

**Lemma 3.** Assume that the path category $C_Q$ of a quiver is monoidal such that $(Q_0, \cdot)$ forms a group. Then:

1. The monoidal product defines commuting left and right $Q_0$-actions on $Q_1$.
2. The path length $\ell$ is a grading with respect to both $\cdot$ and $\circ$.
3. Either $Q_1$ is empty, or there exists an element $z \in Z(Q_0)$ such that $Q_1$ contains for each $a \in Q_0$ exactly one arrow $f_a : a \to z \cdot a$.

**Proof.** If $Q_1 = \emptyset$ there is nothing to prove, so we assume $Q_1 \neq \emptyset$.

1. To begin with, we prove that for any identity morphism $i_a, a \in Q_0$, and any arrow $f : b \to c \in Q_1$, we have $i_a \cdot f, f \cdot i_a \in Q_1$. Indeed, we have
\[ i_a \cdot f = g_1 \circ \ldots \circ g_n \]
for unique arrows $g_i \in Q_1$, and then
\[ f = i_{a-1} \cdot (g_1 \circ \ldots \circ g_n) = (i_{a-1} \cdot g_1) \circ \ldots \circ (i_{a-1} \cdot g_n) \]
where we used that $i_a \cdot i_{a-1} = i_{a-1} = i_1$ and that $\cdot$ is a monoidal product. It is impossible that $i_{a-1} \cdot g_i = b_0$ for some $b_0 \in Q_0$, as we would then have $g_i = i_a \cdot i_b = i_{a+b}$.

2. For any arrow $f$, let $s(f), t(f)$ denote its source and target vertices respectively. As $\cdot$ is a monoidal product, we have for any two arrows $f, g$:
\[ f \cdot g = (i_{t(f)} \cdot g) \circ (f \cdot i_{s(g)}) = (f \cdot i_{t(g)}) \circ (i_{s(f)} \cdot g) \]
By what has been shown already, this is a path of length 2. Continuing inductively, one proves that $t(f \cdot g) = t(f) + t(g)$ for all paths $f, g \in C_Q$.

3. We also conclude from [1] that
\[ s(f) \cdot t(g) = t(f) \cdot s(g) \Rightarrow t(f) \cdot s(g)^{-1} = s(f)^{-1} \cdot t(f) = : z \in Q_0 \]
is constant (independent of $f$). So
\[ t(f) = s(f) \cdot z = z \cdot s(f) \]
and any arrow $f \in Q_1$ with source $a$ has the same target $z \cdot a = a \cdot z$.

Given any arrow $f : a \to a \cdot z$ and $b \in Q_0$, there exists $i_b \cdot f : b \cdot a \to z \cdot b \cdot a$. This means that the same number of arrows go out of each vertex and that $z$ is in the centre of $Q_0$. 

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Finally, assume there are two arrows \( f, g \) with source 1 (the unit element of \( Q_0 \)) and consider again (4).

\[
f \cdot g = (i_z \cdot g) \circ f = (f \cdot i_z) \circ g.
\]

We deduce that \( f = g. \)

If \( Q_1 = \emptyset \), then \( C_Q = \{i_a : a \in Q_0\} \) is just a group. The monoidal prouct is the group multiplication, and this is ULF. The Hopf algebra \( kC_Q \) is the group algebra of \( Q_0 \) and is combinatorial.

If \( Q_1 \neq \emptyset \), there are two sub-cases: \( z = 1 \) and \( z \neq 1 \). In the first case, each vertex \( a \in Q_0 \) has a unique arrow \( f_a : a \to a \). In the second case, each vertex \( a \) has one incoming arrow \( f_{z^{-1}a} : z^{-1}a \to a \) and one outgoing arrow \( f_a : a \to za \).

In both cases, any morphism in \( C_Q \) can be uniquely expressed as

\[
f_{z^n} \circ \ldots \circ f_a, \quad a \in Q_0, n \in \mathbb{N},
\]

to be interpreted as \( i_a \) when \( n = 0 \). The monoidal product is given by

\[
(f_{z^n} \circ \ldots \circ f_a) \cdot (f_{z^m} \circ \ldots \circ f_b) = f_{z^{n+m}} \circ \ldots \circ f_{a \cdot b}.
\]

This product is evidently not ULF, so we obtain:

**Theorem 4.** Given a Hopf quiver \( Q \), the quiver Hopf algebra \( kQ \) is a combinatorial Hopf algebra if and only if \( Q_1 = \emptyset \).

**Remark 4.** In [3], Crossley defines several Hopf algebra structures on \( k(S) \) for a set \( S \). In two of them, \( \Delta(w_1 \ldots w_n) = \sum_{i=1}^n w_1 \ldots w_i \otimes w_{i+1} \ldots w_n \), which is the coproduct we consider here. However, just like the quiver Hopf algebras of Cibils and Rosso, these two Hopf algebras are not combinatorial Hopf algebras in the sense of the present paper, as in both cases, \( m(\langle S \rangle \times \langle S \rangle) \not\subseteq \langle S \rangle \).

### 3.5. Bigraphs.

Bigraphs are a formal language developed by Milner [18] and his collaborators in order to model systems of communicating agents. They provide an intuitive graphical implementation of many process algebras (abstract programming languages employed in theoretical computer science). The notation used for bigraphs and the operations defined on them are immediately suggestive of an underlying monoidal category with bigraphs themselves as morphisms.

**Definition 12** (Link graph). A link graph consists of totally ordered finite disjoint sets \( P, X, Y \) and an equivalence relation \( \sim \) on \( P \cup X \cup Y \). Each equivalence class must contain at least two elements and cannot be entirely contained in \( X \) or \( Y \).

The equivalence classes are called hyperedges, and the elements of \( X \) and \( Y \) are called inner and outer names respectively. Like place graphs, link graphs form the morphisms of a combinatorial category:

**Definition 13** (Composition). If \( f_i = (P_i, X_i, Y_i, \sim_i) \), \( i = 1, 2 \), are link graphs with \( Y_1 = X_2 \), then

\[
f_2 \circ f_1 := (P_1 \oplus P_2, X_1, Y_2, \sim)
\]

where \( \sim \) is generated by \( \sim_1 \) and \( \sim_2 \) (including \( x \sim y \) if \( x \in X_1 \cup P_1, y \in P_2 \cup Y_2 \), and \( x \sim_1 z \sim_2 y \) for some \( z \in Y_1 = X_2 \)).

Our real focus is on the combination of place and link graphs into a single object:

**Definition 14** (Bigraph). A bigraph \( g = (g_p, g_l, \rho) \) consists of

1. A place graph \( g_p = (R, S, V, prnt) \).
2. A link graph \( g_l = (P, X, Y, \sim) \).
3. A map \( \rho : P \to V \).
We call \( \text{ctrl}(v) := |\rho^{-1}(v)| \) the arity of \( v \in V \), and the pairs \((S, X)\) and \((R, Y)\) the inner and outer interfaces of \( g \) respectively. We denote by \( g_0 \) the empty bigraph with inner and outer interfaces both \((\emptyset, \emptyset)\).

In general it is easier to specify a bigraph by drawing it:

**Example 3.** The bigraph \( g = (g_p, g_l, \rho) \) with

\[
\begin{align*}
    g_p &= (\{r_0\}, \{s_0, s_1\}, \{v\}, \text{prnt}), \\
    \text{prnt}(v) &= r_0 \quad \text{prnt}(s_0) = r_0, \quad \text{prnt}(s_1) = v \\
    g_l &= (\{p_0, p_1\}, \{x_0, x_1\}, \emptyset, \sim) \\
    p_0 &\sim x_0, \quad p_1 \sim x_1 \\
    \text{ctrl}(v) &= \{p_1, p_2\}
\end{align*}
\]

can be drawn as

In the sequel, when drawing bigraphs we will suppress the labels of \( V \) and \( P \) and identify \( X, Y, R, S \) with the sets \( \{0, \ldots, z - 1\} \) of the relevant cardinality.

**Remark 5.** In computer science terminology, we consider here only abstract bigraphs (no labels on \( V, P \)) with private names \((X, Y\) labelled as \([X], [Y])\) and without sharing. The construction of a combinatorial category can be extended to concrete bigraphs with sharing.

Adapting Milner’s definitions \([18]\), we define a combinatorial category of bigraphs and therefore an associated bialgebra.

**Definition 15** (Juxtaposition). The *juxtaposition of two bigraphs* \( g = (g_p, g_l, \rho_g) \) and \( f = (f_p, f_l, \rho_f) \) is given componentwise by concatenations:

\[
f g = (f_p \oplus g_p, f_l \oplus g_l, \rho_g \oplus \rho_f).
\]
Example 4.

\[ g = \begin{array}{c}
\bullet \\
\end{array}, \quad f = \begin{array}{c}
\bullet \\
\bullet \\
\end{array} \]

\[ gf = \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\end{array}, \quad fg = \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\end{array} \]

**Definition 16** (Composition). The composition of two bigraphs \( g = (g_p, g_l, \rho_g) \) and \( f = (f_p, f_l, \rho_f) \) is given componentwise by compositions:

\[ g \circ f = (g_p \circ f_p, g_l \circ f_l, \rho_g \oplus \rho_f) \]

if the inner interface of \( g \) equals the outer interface of \( f \).

Example 5.

We define a category \( C_B \) whose objects are pairs \((n, m)\) of natural numbers, with monoidal product \((n, m) \cdot (k, l) = (n + k, m + l)\). \( C_B ((n, m), (k, l)) \) consists of all bigraphs with inner and outer interfaces \((n, m)\) and \((k, l)\) respectively. The monoidal product and composition of morphisms are given by juxtaposition and composition respectively. The only invertible element is \( i_{0,0} \), the empty bigraph. \( C_0 \) consists of all bigraphs generated by \( i_{(0,1)} \) and \( i_{(1,0)} \), drawn respectively as

The combinatorial bialgebra \( kC_B \) has a grading given by number of vertices. This does not agree with the length filtration, as there are many non-identity bigraphs with no vertices (including those which are pure place graphs, as mentioned above).
Milner proposed in [18] that the category of bigraphs provides the right framework for the description of bigraph dynamics. Based on the success of Hopf algebraic treatments of Feynman graphs and other combinatorial objects (mentioned above and in wider literature), we feel that the combinatorial bialgebra associated to this category may prove useful in the study of bigraphs. The impact of such a linearisation has been demonstrated even in the most basic case of a group algebra, where the tools of linear algebra and ring theory lead to substantial progress in the study of the underlying combinatorial structure of the group itself. We thus hope that similar techniques can be applied to the problems arising in bigraph theory.

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