Dynamics of dipolar Atom-Molecular BEC in a double well potential: Effect of atom-molecular coherent coupling

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Abstract

In the present work we have studied the dynamics of dipolar atom-molecular Bose Einstein Condensates coupled via Feshbach Resonance in a double well potential. We have numerically solved four coupled GP like equations, two for left well and two for right well for this atom molecular coupled system. Our numerical results show that both the long-range dipole-dipole interaction (chosen to be positive) and the coherent coupling interaction (which is positive for bosons) facilitate the transmission of atoms and molecules from left well to the right well when the population in the right well dominates over that in the left well and is trapped for a period of time. Whereas in absence of any one of these interactions probability of transient transmission decreases. However in absence of both the interactions (dipole-dipole and coherent coupling) i.e. when only the repulsive contact interaction is present, it leads to self trapping in the left well for a period of time. It is also shown that the signature of coherent coupling between atoms and molecules on the density distribution of atoms in the double well potential is present both in absence and presence of dipole-dipole interaction.

1 Introduction

The experimental observation of Bose-Einstein condensation in a dilute trapped gas of alkali-metal atoms \cite{1} has stimulated a number of theoretical and experimental studies on the properties of these condensates. One of these fields is the study of steady state and dynamical properties of these condensates while trapped in a double well or multi well configurations \cite{2,3}. It explored the possibility of realising Josephson junction in ultra-cold atoms or condensates. Josephson effect (JE) in ultra-cold atoms was first predicted by Javanainen \cite{4} and theoretically realized in a double well potential occupied by a macroscopic number of bosons \cite{5}. It has also been shown that coherent quantum tunneling between two condensates in two traps gives rise to oscillation in atomic numbers...
The Josephson oscillation (JO) in atomic condensate in a double well potential was studied including the many body hard sphere interaction and discussed the validity of two-mode model for JO [7]. Besides the Josephson oscillations in atomic condensates in double well potential a new phenomena i.e. macroscopic quantum self-trapping (MQST) can be obtained which has no analog for superconducting Josephson junctions. MQST was first predicted by Smerzi and collaborators [8] and subsequently both the Josephson oscillation (JO) and MQST were observed in single bosonic Josephson junction (BJJ) [9, 10]. JE was also observed in an array of BJJ/periodic potential [11, 12] and theoretical investigation was carried out to provide an explanation of the observations in periodic potential [13, 14]. JO and MQST has been studied also in triple well [15]. Ananikian and Bergeman [16] have studied validity of two-mode model for BECs in double well potential and by using improved two-mode model reproduced the experimentally observed [9] tunneling oscillation frequency. Julia-Diaz et al [17] and Mele-Messeguer et al [18] studied JO in binary BECs in double well potential and explored the dependence of JO on different spin collision channels. Study on self trapping of BEC in DW shows by using periodically modulated double well potential MQST can be systematically controlled. Li-Hua Lu and You-Quan Li [19] has shown that in partially coherent BEC in DW both degree of coherence and initial phase difference affects the final particle distribution in two wells. Ottaviani et al [20] studied transport and self-trapping of BEC in a double well by controlling the energy bias between two wells as well as nonlinearity of the system. Shchesnovich and Trippenbach [21] used Fock space method to study JO and MQST in BEC in a double well potential and suggested that their model can also be applied for atom-molecular coherence and nonlinear tunneling of BEC in optical lattices. Nonlinear Josephson dynamics of BECs in a double-well potential has been studied including coupling to quasiparticle states and it has been suggested that JO and Rabi oscillation can be damped due to quasiparticle inelastic collisions [22]. Effect of decoherence on the Josephson dynamics of BEC in DW potential has been studied by analyzing the coupling of the condensate with the environment and have shown that damping of JO due to decoherence can enhance the self-trapping [23].

Time-dependent self-trapping of BECs in a DW potential has been discussed by phase space analysis within the mean field approximation and have shown that the population imbalance of BECs can be controlled by varying the atom-atom interaction and the driving time dependent magnetic field [24]. Gillet et al. [25] have studied Josephson dynamics in BECs in a symmetric one dimensional double well potential within the four mode approximation and have shown that the dissipation due to the presence of two excited modes affects the rich Josephson dynamics in BECs. Huang et al [26] have used quantum Fisher information to study evolution of JO to MQST and the parameter sensitivity of the stability of the condensate system. L. Fu and J. Liu [27] discussed the influence of many-body quantum fluctuations on the transition to self-trapping in a BEC in a symmetric double well. They showed manifestation of quantum entanglement of the transition which reaches to maximum at the transition point. Levy et al. [28] studied dc and ac Josephson effects in two weakly linked Bose-Einstein condensates considering a time-dependent barrier, moving adiabatically across the trapping potential and have shown that there is a critical velocity of the barrier at which sharp transition from dc to ac JE occurs. The transport of Bose-Einstein condensate (BEC) in a double-well trap can be controlled by con-
sidering the time-dependent (Gaussian) coupling between the wells and their relative detuning within the mean field approximation both for repulsive and attractive interactions [29]. This group has also explored the inverse population transfer of the repulsive Bose-Einstein condensate (BEC) in a weakly bound double-well trap within the 3D time-dependent Gross-Pitaevskii equation considering a time dependent barrier shift separating the two wells.

Josephson effects such as JO and MQST have also been studied in bosonic and fermionic mixtures in double well potential [30]. Luick et al [31] measured the frequency of Josephson oscillations as a function of the phase difference across the junction and found excellent agreement with the sinusoidal current phase relation in a strongly correlated 2D fermi gas. In our group we consider bosons or fermions trapped in a quasi-1D DW potential interacting via a 3D finite range two-body interaction potential with large scattering length $a_s$. Under tight-binding or two mode approximation, we describe in detail the effects of the range of interaction on the quantum dynamics and number-phase uncertainty in the strongly interacting or unitarity regime. We defined the standard quantum limit (SQL) for phase and number fluctuations and described two-mode squeezing for number and phase variables for this system [32].

With the observation of BECs of dipolar atoms with large magnetic moments e.g. $^{52}$Cr, $^{168}$Er, $^{164}$Dy etc [33] stimulated a large number of theoretical and experimental investigations on the effect of long range dipole-dipole interaction on the properties of BECs. The presence of long range, anisotropic and nonlocal dipole-dipole interactions the manifestation of JE in dipolar BECs becomes different than that in case of short range contact interactions. The dipolar interaction may drastically change the properties of quasi-1D dipolar condensates, even for situations in which the dipolar interaction would be completely overwhelmed by the short-range interactions in a 3D environment [34]. With the experimental advent in trapping and cooling of dipolar molecular gases of both fermionic and bosonic nature has stimulated the study of the properties of dipolar BECs e.g. structure, collective excitations, chemical reaction between dipolar molecules, vortex in spin-orbit coupled rotating dipolar molecules, dipolar BEC and superfluid BCS pairing transition etc [35]. The stability of dipolar BECs of $^{52}$Cr has been experimentally investigated which depends on the scattering length and strongly on the trap geometry [36]. First realization of heteronuclear dipolar quantum mixtures of highly magnetic erbium and dysprosium atoms has been reported and experimentally demonstrated binary Bose-Einstein condensation in five different Er-Dy isotope combinations, as well as one Er-Dy Bose-Fermi mixture [37].

Mixing, demixing, and structure formation in a binary dipolar Bose-Einstein condensate [38] and the miscibility in coupled dipolar and non-dipolar Bose-Einstein condensates have been studied. Stability of three-dimensional vortex lattice structures in purely dipolar Bose-Einstein condensate (BEC) has been experimentally and theoretically studied by varying the aspect ratio and the strength of dipole-dipole interaction [39]. Vortex lattice formation in dipolar BECs via rotation of the polarisation and the orientation of dipoles has also been explored [40, 41]. Anisotropic superfluid flow has been observed in dipolar BEC with strong magnetic dipole-dipole interaction by moving an attractive laser beam through the condensate [42]. Formation of spin-vortex pair, solitons, vortex-solitons and its stability in dipolar spinor BECs have been examined [43]. Within the Hamiltonian mean-field (HMF) model it has been shown that the
stationary solutions of generalized Gross-Pitaevskii equation (GGPE) can give rise to a tower of solitons in presence of long range dipolar interactions in a BEC [44]. To study structure and dynamics of trapped dipolar gases different theoretical models e.g. two mode/multimode, mean field approximation, numerical integration of Gross-Pitaevskii/modified Gross-Pitaevskii equation etc have been used. However we have derived analytical solutions for Gross-Pitaevskii and modified Gross-Pitaevskii equations for non-dipolar atomic condensate trapped in harmonic potential using modified Thomas-Fermi approximation [45] and Eberlien et al [46] first derived exact solution of the Thomas-Fermi equation for a trapped Bose-Einstein condensate with dipole-dipole interactions.

One of the interesting features in dipolar BECs is the appearance of Roton minimum in the spectrum of elementary excitations [47] which occurs when the dipole-dipole long range interaction dominates over short range contact interaction. Roton minimum in dipolar ultra cold gases has been observed and theoretical analysed [48] [49]. Instabilities in rotonization of 1D dipolar gas in 1D lattices and also in 2D dipolar gases have been discussed [50]. Softening of Roton mode has also been studied by considering finite range interaction and laser induced dipole-dipole interactions in single and multi-component BECs. Effect of tilting of the dipoles on the Roton mode excitation and post-roton instability in a dipolar BEC confined in pancake-shaped trap has been studied [51]. In the region of post-roton instability in dipolar BECs it is demonstrated both theoretically and experimentally stable droplets can be formed and it is stabilised by the quantum fluctuation/three-body interaction [52]. Moreover quantum fluctuations in dipolar condensates stabilize a new supersolid phase giving rise to high modulational contrast in density(cristalization) and near-perfect superfluidity [53] [54]. Supersolidity of dipolar BEC has been extensively studied in recent experiments and theoretical explanation was also provided [55]. The advances of theoretical and experimental studies in dipolar quantum gases has been reviewed and the physical insight has been presented by Lahaye et al [3].

In general in the study of JE in non-dipolar BECs in DW or multi well potentials nonlinear short range contact interactions are considered which controls the transition from JO to MQST. However for dipolar BECs in DW/multi well potentials interplay between short range contact interaction and the long-range dipole-dipole interaction leading to switch over from JO to MQST and vice versa has been studied by varying the scattering length and the number of atoms in dipolar BEC trapped in double well/optical lattice. It is shown that the system can translate from MQST to JO in presence of dipolar interaction [56]. Coherent tunneling phenomena leading to JO and MQST can occur in an effective ring-shaped double well potential induced by the dipolar interaction [58] [59]. Dependence of JE on alignment of dipoles/ magnetisation direction and the stability of the dipolar BEC in double well have been theoretically demonstrated [60]. Effect of excitation to higher excited state on the dynamical stabilisation of dipolar BEC in triple well configuration has been discussed [61]. It has also been shown introduction of asymmetric basis function can improve the results from two-mode model [62]. Spin dynamics of ultra cold magnetic atoms located in two wells has been experimentally studied and the effect of long-range dipolar interactions on the dynamics has been also explored theoretically [63] [64]. Recently our group has studied the effect of trap confinement and the atom-atom nonlinear interaction on JO and MQST in cold atoms in a
DW potential. We considered three types of interaction potentials (i) contact, (ii) long range dipolar and (iii) finite range potentials. We found that by varying the aspect ratio of the trap (i) transition from JO to MQSt occurs for small atom-atom interactions (ii) transition from Rabi to JO and JO to MQST occurs for long range dipolar interaction and transition from JO to MQST occurs even if scattering length is relatively large in the region of narrow Feshbach resonance due to finite range effects [65].

Besides Josephson effect, formation of vortices/vortex solitons in dipolar BECs confined in double well/optical lattice has been explored [66]. Formation of hidden vortices within the barrier and vortex lattice has been demonstrated in dipolar BEC in a double well potential [67].

With the advent of experimental realization of state selective molecules from atomic BECs/ultracold atoms by photo association (PA) [68, 69], by magnetic Feshbach resonance (FR) [70] and by magneto-photoassociation applying Feshbach resonance (FR) technique [71], many theoretical attempts were made to understand and set the guidelines for the realization of molecular BECs by the process of PA and magnetic FR [72]. In our group an attempt was made [73] to explain the experimental observation [70] on the dynamics of atom-molecular coherence in BEC of $^{85}$Rb atoms. Bose-Einstein condensation of $^{6}$Li$_2$ molecules from fermionic Lithium atoms in an optical trap was reported [74]. Dynamics of coupled atom-molecular condensates in presence of PA in a double well potential was studied to explore the effect of tunneling and PA laser intensity on the number of molecules formed [75]. Stability and JE of atom molecular coupled BECs in a double well potential has also been studied [76] by considering the four mode model.

In this paper we have studied dynamics of dipolar atomic condensates coupled with molecular condensate via magnetic Feshbach resonance confined in a double well potential. We explored the effect of atom-molecular coherence and the effect of long-range dipole-dipole interactions on the transport of atoms and molecules from one well to the other. In this theoretical approach we solved four time dependent coupled Gross-Pitaevskii equations (two for each well) to investigate the temporal evolution of atomic and molecular numbers in each well. To our knowledge the effect of atom-molecular coherence on the dynamics of dipolar atomic condensates in double well potential, coupled with molecules via magnetic Feshbach resonance has not been studied yet.

2 Theory

In the present work we have studied numerically the dynamics of coupled dipolar atom-molecular BECs from left to right well and vice versa in a double well potential in the axial direction. Atomic and molecular BECs are coupled via magnetic Feshbach resonance [70, 73] in both the wells. Four coupled time-dependent GP like equations for atomic and molecular BECs two for left and two for right well have been derived from the first principle considering the mean field energy of the coupled system. Four time-dependent three dimensional (3D) coupled GP like equations were then reduced to single dimension (1D) to study the dynamics of dipolar coupled BECs trapped in a double well potential in the axial direction.
2.1 Derivation of four coupled equations:

Mean field energy density for dipolar atom-molecular coupled condensate in double well is given as:

\[
E[\psi_{al}, \psi_{ar}, \psi_{ml}, \psi_{mr}] = \psi_{al}^* \left[-\frac{\hbar^2 \nabla^2}{2m} + v_a(r) + \lambda_a \psi_{al}\psi_{al}\right] + \frac{d_a}{2} \int V(r - r') \psi_{al}^* \psi_{al} dr' \psi_{al} + \psi_{al}^* \left[-\frac{\hbar^2 \nabla^2}{2m} + v_a(r) + \lambda_a \psi_{ar}\psi_{ar}\right] + \frac{d_a}{2} \int V(r - r') \psi_{ar}^* \psi_{ar} dr' \psi_{ar} + \psi_{ar}^* \left[-\frac{\hbar^2 \nabla^2}{4m} + v_m(r) + \lambda_m \psi_{ml}\psi_{ml}\right] + \frac{d_m}{2} \int V(r - r') \psi_{ml}^* \psi_{ml} dr' \psi_{ml} \]

\[
\psi_{mr}^* \left[-\frac{\hbar^2 \nabla^2}{4m} + v_m(r) + \lambda_m \psi_{mr}\psi_{mr}\right] + \frac{d_m}{2} \int V(r - r') \psi_{mr}^* \psi_{mr} dr' \psi_{mr} + \psi_{mr}^* \psi_{ar} \psi_{ar}^* \psi_{ml}^* \psi_{mr}^* + \frac{\chi}{2} \left[\psi_{mr}^* \psi_{al} + \psi_{ml}^* \psi_{al} + \psi_{ar}^* \psi_{al} + \psi_{mr}^* \psi_{ar} \psi_{al}\right] + \frac{\chi}{2} \left[\psi_{al}^* \psi_{al} + \psi_{ar}^* \psi_{al} + \psi_{mr}^* \psi_{al} + \psi_{mr}^* \psi_{ar} \psi_{al}\right] \tag{1}
\]

Here \(\psi_{al}, \psi_{ar}\) are the macroscopic wavefunctions for atomic condensate in left and right wells respectively and \(\psi_{ml}, \psi_{mr}\) are the same for the molecular condensate in the double well respectively. All the wavefunctions mentioned above are the function of \(r\) and \(t\). \(v_a(r)\) and \(v_m(r)\) are the double well potential for atomic and molecular traps respectively, which are given as:

\[
v_a(r) = \frac{1}{2} m \omega_r^2 (r^2 + z^2) + Ae^{-\kappa z^2}
\]

and

\[
v_m(r) = m \omega_z^2 (r^2 + z^2) + Ae^{-\kappa z^2}
\]

where \(\omega_r, \omega_z\) are the radial and axial trap frequencies and the trap asymmetry parameter \(\kappa = \frac{\omega_z}{\omega_r}\). The constants \(A\) and \(\kappa\) are the height and the width of the Gaussian barrier which gives rise to double well feature in the axial direction when added to the harmonic potential. \(\lambda_a, \lambda_m\) and \(\lambda_{am}\) are the s-wave contact interactions between atom-atom, molecule-molecule and atom-molecule respectively. \(\lambda_a = \frac{4 \pi \hbar^2 a}{m}\), where \(a\) is the s-wave scattering length and \(m\) is the mass of the atom. In the present work we considered these three interactions are the same. Atom-molecular coupling strength via magnetic Feshbach resonance

\[
\chi = \sqrt{4 \pi \hbar^2 a_{bg}(\Delta \mu)(\Delta B)/m}, \text{ where } a_{bg} \text{ is the background scattering length, } \Delta \mu \text{ is the difference between magnetic moments of two free atoms and the bound molecule and } \Delta B \text{ is the width of the resonance, } \epsilon \text{ is the detuning from the resonance. The dipole-dipole interaction coefficients between two atoms and between two molecules are given as } d_a = \mu_0 \mu_d^2 / 4 \pi \text{ and } d_m = \mu_0 \mu_d^2 / 4 \pi, \mu_0 \text{ is the permeability of free space, } \mu_d \text{ and } \mu_d' \text{ are the magnetic dipole moments of atoms and molecules respectively. The dipole-dipole interaction term}
\]
\[ V(\mathbf{r} - \mathbf{r}') = \frac{1 - 3 \cos^2 \theta}{(\mathbf{r} - \mathbf{r}')^2} \] where \( \theta \) is the angle between the vector \((\mathbf{r} - \mathbf{r}')\) and the polarization direction taken along the axial symmetry axis \(z\).

Four coupled time-dependent GP like equations for dipolar atom-molecular system coupled via magnetic Feshbach resonance and trapped in a double well potential can be obtained by varying mean field energy \(E\) with respect to \(\psi_{al}^*, \psi_{ar}^*, \psi_{ml}^*\) and \(\psi_{mr}^*\) and equating to the time derivative of the corresponding wave functions respectively as follows:

\[
\begin{align*}
    i\hbar \frac{\partial \psi_{al}(\mathbf{r}, t)}{\partial t} &= \frac{\delta E}{\delta \psi_{al}^*}; \\
    i\hbar \frac{\partial \psi_{ar}(\mathbf{r}, t)}{\partial t} &= \frac{\delta E}{\delta \psi_{ar}^*}; \\
    i\hbar \frac{\partial \psi_{ml}(\mathbf{r}, t)}{\partial t} &= \frac{\delta E}{\delta \psi_{ml}^*}; \\
    i\hbar \frac{\partial \psi_{mr}(\mathbf{r}, t)}{\partial t} &= \frac{\delta E}{\delta \psi_{mr}^*}.
\end{align*}
\]

Hence four time-dependent coupled equations are written as:

\[
\begin{align*}
    (3) \quad &\delta_{al} \int V(\mathbf{r} - \mathbf{r}')|\psi_{al}(\mathbf{r}', t)|^2 d\mathbf{r}' \psi_{al}(\mathbf{r}, t) + \chi \psi_{ml}(\mathbf{r}, t) \psi_{al}(\mathbf{r}, t) \\
    (4) \quad &\delta_{ar} \int V(\mathbf{r} - \mathbf{r}')|\psi_{ar}(\mathbf{r}', t)|^2 d\mathbf{r}' \psi_{ar}(\mathbf{r}, t) + \chi \psi_{mr}(\mathbf{r}, t) \psi_{ar}(\mathbf{r}, t) \\
    (5) \quad &\delta_{ml} \int V(\mathbf{r} - \mathbf{r}')|\psi_{ml}(\mathbf{r}', t)|^2 d\mathbf{r}' \psi_{ml}(\mathbf{r}, t) + \frac{\chi}{2} \psi_{al}(\mathbf{r}, t)^2 \\
    (6) \quad &\delta_{mr} \int V(\mathbf{r} - \mathbf{r}')|\psi_{mr}(\mathbf{r}', t)|^2 d\mathbf{r}' \psi_{mr}(\mathbf{r}, t) + \frac{\chi}{2} \psi_{ar}(\mathbf{r}, t)^2
\end{align*}
\]

Equations (3)-(6) can be written in dimensionless form \([73]\) by expressing energy, length, density and time in units of oscillator energy \(\hbar \omega_{z}\), oscillator length \(l_z = \sqrt{\frac{\hbar}{m \omega_{z}}}\), \(l_z^{-3}\) and \(\omega_{z}^{-1}\) respectively:

\[
\begin{align*}
    (7) \quad &\delta_{al} U_{al}(\mathbf{r}, t) \psi_{al}(\mathbf{r}, t) + \chi \psi_{ml}(\mathbf{r}, t) \psi_{al}(\mathbf{r}, t) \\
    &\delta_{ar} U_{ar}(\mathbf{r}, t) \psi_{ar}(\mathbf{r}, t) + \chi \psi_{mr}(\mathbf{r}, t) \psi_{ar}(\mathbf{r}, t) \\
    &\delta_{ml} U_{ml}(\mathbf{r}, t) \psi_{ml}(\mathbf{r}, t) + \frac{\chi}{2} \psi_{al}(\mathbf{r}, t)^2 \\
    &\delta_{mr} U_{mr}(\mathbf{r}, t) \psi_{mr}(\mathbf{r}, t) + \frac{\chi}{2} \psi_{ar}(\mathbf{r}, t)^2
\end{align*}
\]
\[
\frac{i}{\hbar} \frac{\partial \psi_{ar}(r,t)}{\partial t} = \left[ -\nabla^2 + v'_a(r) + \lambda'_a |\psi_{ar}(r,t)|^2 + \lambda'_{am} |\psi_{mr}(r,t)|^2 + \right. \\
\left. d'_a U_{ar}(r,t) |\psi_{ar}(r,t)|^2 + \chi' |\psi_{mr}(r,t)|^2 \psi_{ar}^*(r,t) \right]
\]

\[
\frac{i}{\hbar} \frac{\partial \psi_{ml}(r,t)}{\partial t} = \left[ -\nabla^2 + v'_m(r) + \lambda'_m |\psi_{ml}(r,t)|^2 + \lambda'_{am} |\psi_{al}(r,t)|^2 + \right. \\
\left. d'_m U_{ml}(r,t) |\psi_{ml}(r,t)|^2 + \chi' |\psi_{al}(r,t)|^2 \psi_{ml}^*(r,t) \right]
\]

\[
\frac{i}{\hbar} \frac{\partial \psi_{mr}(r,t)}{\partial t} = \left[ -\nabla^2 + v'_m(r) + \lambda'_m |\psi_{mr}(r,t)|^2 + \lambda'_{am} |\psi_{ar}(r,t)|^2 + \right. \\
\left. d'_m U_{mr}(r,t) |\psi_{mr}(r,t)|^2 + \chi' |\psi_{ar}(r,t)|^2 \psi_{mr}^*(r,t) \right]
\]

Here

\[
U_{al}(r,t) = \int V(r-r')|\psi_{al}(r',t)|^2 dr' \\
U_{ar}(r,t) = \int V(r-r')|\psi_{ar}(r',t)|^2 dr' \\
U_{ml}(r,t) = \int V(r-r')|\psi_{ml}(r',t)|^2 dr' \\
U_{mr}(r,t) = \int V(r-r')|\psi_{mr}(r',t)|^2 dr'
\]

\[
v'_a(r) = \frac{1}{2} \left( \frac{\rho^2}{\lambda^2} + z^2 \right) + Ae^{-\kappa z^2}; \quad v'_m(r) = \left( \frac{\rho^2}{\lambda^2} + z^2 \right) + Ae^{-\kappa z^2}; \\
d'_a = \frac{d_a}{l^2 \hbar \omega_z}; \\
d'_m = \frac{d_m}{l^2 \hbar \omega_z}; \quad \lambda'_a = \frac{\lambda_a}{\hbar \omega_z}; \quad \lambda'_m = \frac{\lambda_m}{\hbar \omega_z}; \quad \chi'_a = \frac{\chi_a}{\hbar \omega_z}; \quad \chi'_m = \frac{\chi_m}{\hbar \omega_z}; \quad \epsilon' = \frac{\epsilon}{\hbar \omega_z}
\]

2.2 Derivation of one dimensional coupled equations to study the dynamics of dipolar atom-molecular BECs in double well:

To derive the coupled equations in the axial direction (\(z\)) from the 3D equations (7)-(10), we consider strong confinement in the radial direction i.e \(\omega_z << \omega_\rho\) and hence the asymmetry parameter \(\lambda = \frac{\omega_z}{\omega_\rho} << 1\). Due to the strong confinement it is assumed that the BECs in the radial trap are confined in the ground state and the corresponding wavefunctions are given as:

\[
\psi_{a0}(\rho) = \frac{1}{\sqrt{\pi \lambda}} e^{-\frac{\rho^2}{2 \lambda}}; \quad \psi_{m0}(\rho) = \frac{1}{\sqrt{\pi \lambda}} e^{-\frac{\rho^2}{2 \lambda}}
\]
respectively. Hence the total wavefunction for atoms and molecules in the left and right wells are given as:

\[ \psi_{al}(r, t) = \psi_{a0}(\rho)\psi_{al}(z, t); \psi_{ar}(r, t) = \psi_{a0}(\rho)\psi_{ar}(z, t) \] (14)

\[ \psi_{ml}(r, t) = \psi_{m0}(\rho)\psi_{ml}(z, t); \psi_{mr}(r, t) = \psi_{m0}(\rho)\psi_{mr}(z, t) \] (15)

Hence by substituting equations (14) and (15) in equations (7)-(10), multiply-

\[ \frac{i}{\hbar}\frac{\partial}{\partial t}\psi_{al}(z, t) = \frac{1}{2}\frac{\partial^2}{\partial z^2}\psi_{al}(z, t) + v_a(z)\psi_{al}(z, t) + \lambda''_m\psi_{ml}(z, t)\psi_{al}(z, t) + \lambda''_a|\psi_{al}(z, t)|^2 + \frac{\lambda''_m|\psi_{ml}(z, t)|^2}{\hbar^2} \] (16)

\[ \frac{i}{\hbar}\frac{\partial}{\partial t}\psi_{ar}(z, t) = \frac{1}{2}\frac{\partial^2}{\partial z^2}\psi_{ar}(z, t) + v_a(z)\psi_{ar}(z, t) + \lambda''_m\psi_{mr}(z, t)\psi_{ar}(z, t) + \lambda''_a|\psi_{ar}(z, t)|^2 + \frac{\lambda''_m|\psi_{mr}(z, t)|^2}{\hbar^2} \] (17)

\[ \frac{i}{\hbar}\frac{\partial}{\partial t}\psi_{ml}(z, t) = \frac{1}{4}\frac{\partial^2}{\partial z^2}\psi_{ml}(z, t) + v_m(z)\psi_{ml}(z, t) + \lambda''_m\psi_{al}(z, t)\psi_{ml}(z, t) + \lambda''_m|\psi_{al}(z, t)|^2 + \frac{\lambda''_m|\psi_{al}(z, t)|^2}{\hbar^2} \] (18)

\[ \frac{i}{\hbar}\frac{\partial}{\partial t}\psi_{mr}(z, t) = \frac{1}{4}\frac{\partial^2}{\partial z^2}\psi_{mr}(z, t) + v_m(z)\psi_{mr}(z, t) + \lambda''_m\psi_{ar}(z, t)\psi_{mr}(z, t) + \lambda''_m|\psi_{ar}(z, t)|^2 + \frac{\lambda''_m|\psi_{ar}(z, t)|^2}{\hbar^2} \] (19)

where \( v_a(z) = \frac{1}{2}z^2 + Ae^{-\kappa x}; v_m(z) = z^2 + Ae^{-\kappa x}; \lambda''_a = \frac{\lambda_a}{2\pi}; \lambda''_m = \lambda''_a = \lambda''_m; \)

\[ \lambda'' = \frac{\lambda''_m}{\sqrt{2\pi}} \]

\[ U_{al}(z, t) = \int V(|z - z'|)|\psi_{al}(z', t)|^2dz' \] (20)

\[ U_{ar}(z, t) = \int V(|z - z'|)|\psi_{ar}(z', t)|^2dz' \]

\[ U_{ml}(z, t) = \int V(|z - z'|)|\psi_{ml}(z', t)|^2dz' \]

\[ U_{mr}(z, t) = \int V(|z - z'|)|\psi_{mr}(z', t)|^2dz' \] (21)
2.3 Derivation of dipole-dipole interaction potential in one dimension (z)

The dipole-dipole interaction potential in axial direction can be obtained by taking the Fourier transforms of 3D potentials given in equations (11) and (12) as follows: Fourier transform of dipole-dipole interaction is given as \[ 77 \]

\[ F[V(r - r')] = \frac{4\pi}{3} \left[ \frac{3k_z^2}{k^2 + k_z^2} - 1 \right] \] (22)

Fourier transforms of densities are given as:

\[ n_{a0}(k) = \int_0^\infty e^{ik\rho} n_{a0}(\rho) d\rho = e^{-\frac{\lambda^2}{4}} \] (23)

\[ n_{m0}(k) = \int_0^\infty e^{ik\rho} n_{m0}(\rho) d\rho = e^{-\frac{\lambda^2}{4}} \] (24)

\[ n_{ai}(k_z, t) = \int_{-\infty}^{\infty} dz e^{ik_z z} n_{ai}(z, t) \] (25)

\[ n_{mi}(k_z, t) = \int_{-\infty}^{\infty} dz e^{ik_z z} n_{mi}(z, t) \] (26)

Here the densities in coordinate space are given as \[ n_{j0}(\rho) = |\psi_{j0}(\rho)|^2 \]; \[ n_{ai}(z, t) = |\psi_{ai}(z, t)|^2 \] and \[ n_{mi}(z, t) = |\psi_{mi}(z, t)|^2 \] where the suffix 'i' is for suffix 'l' and 'r' corresponding to left and right well and the suffix 'j' corresponds to atoms and molecules. Hence using equations (22)-(26) the dipole-dipole interaction potentials in momentum space as a function of \( (k_{\rho}, k_z, t) \) is obtained. Then performing the integration over \( k_{\rho} \) and taking the inverse Fourier transform over \( k_z \) one can get the interactions in one dimension (z) as follows:

\[ U_{ai}(z, t) = \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} e^{-ik_z z} U_{ai}(k_z) n_{ai}(k_z, t) \] (27)

\[ U_{mi}(z, t) = \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} e^{-ik_z z} U_{mi}(k_z) n_{mi}(k_z, t) \] (28)

where

\[ U_{ai}(k_z) = \frac{4\pi}{3} \frac{1}{(2\pi)^2} \int_0^\infty \frac{dk_\rho}{k_\rho^2 + k_z^2} \left[ \frac{3k_\rho^2}{k_\rho^2 + k_z^2} - 1 \right] |n_{a0}(k_\rho)|^2 \]

\[ = \frac{4\pi}{3} \frac{1}{(2\pi)^2} \int_0^\infty \frac{dk_\rho}{k_\rho^2 + k_z^2} \left[ \frac{3k_\rho^2}{k_\rho^2 + k_z^2} - 1 \right] e^{-\frac{\lambda^2}{4}} \]

\[ = \frac{4\pi}{3} \frac{1}{2\pi\lambda} \int_0^\infty dv \left[ \frac{3\xi^2}{v + \xi^2} - 1 \right] e^{-v} \]

\[ = \frac{4\pi}{3} \frac{1}{2\pi\lambda} S_a(\xi) \] (29)
where \( S_a(\xi) = \left\lfloor \frac{3\xi^2}{v+\xi^2} \right\rfloor - 1 \) \( e^{-v} \); \( \xi = k_1 \sqrt{\frac{\lambda}{2}} \) and \( v = k_\rho^2 \frac{\lambda}{2} \)

\[
U_{m_1}(k_z) = \frac{4\pi}{3} \int_0^\infty dk_p \left[ \frac{3k_p^2}{k_p^2 + k_z^2} - 1 \right] |\eta_{m_0}k_p|^2
\]

\[
= \frac{4\pi}{3} \frac{1}{(2\pi)^2} \int_0^\infty dk_p \left[ \frac{3k_p^2}{k_p^2 + k_z^2} - 1 \right] e^{-\frac{v^2}{2}}
\]

\[
= \frac{4\pi}{3} \frac{2}{2\pi} \int_0^\infty du \left[ \frac{3\zeta^2}{u + \zeta^2} - 1 \right] e^{-u}
\]

\[
= \frac{4\pi}{3} \frac{2}{2\pi} \lambda S_m(\zeta)
\]

Therefore by equating equations (20) and (27) and substituting equation (29) one can write down the dipole-dipole interaction term for atoms in left and right wells in equations (16) and (17) as:

\[
\int V(|z-z'|)\psi_{\eta_1}(z', t)|^2 dz' = \int_{-\infty}^{\infty} dk_z \frac{1}{2\pi} e^{-ik_z z} U_{\eta_1}(k_z) \eta_{\eta_1}(k_z, t)
\]

\[
= \frac{4\pi}{3} \frac{1}{(2\pi)^2} \int_0^\infty dk_z \frac{1}{2\pi} e^{-ik_z z} S_a(\xi) \eta_{\eta_1}(k_z, t)
\]

Similarly by equating equations (21) and (28) and substituting equation (30) the dipole-dipole interaction term for molecules in left and right wells in equations (18) and (19) can be given as:

\[
\int V(|z-z'|)\psi_{\eta_m}(z', t)|^2 dz' = \int_{-\infty}^{\infty} dk_z \frac{1}{2\pi} e^{-ik_z z} U_{\eta_m}(k_z) \eta_{\eta_m}(k_z, t)
\]

\[
= \frac{4\pi}{3} \frac{2}{(2\pi)^2} \int_0^\infty dk_z \frac{1}{2\pi} e^{-ik_z z} S_m(\zeta) \eta_{\eta_m}(k_z, t)
\]

The normalization condition used is \( N = N_\eta(t) + 2N_m(t) \) where \( N \) is the initial total number of atoms, \( N_\eta(t) \) and \( N_m(t) \) are the total number of atoms and molecules respectively at each instant of time; \( N_\eta(t) = N_{\eta_1}(t) + N_{\eta_2}(t) + N_{\eta_3}(t) \); \( N_m(t) = N_{m_1}(t) + N_{m_2}(t) + N_{m_3}(t) \). \( N_{\eta_1}(t) \) and \( N_{\eta_2}(t) \) are the number of atoms in left and right well respectively at time \( t \) and are defined as: \( N_{\eta_1}(t) = \int_{-\infty}^{\infty} dz n_{\eta_1}(z, t) \); and \( N_{\eta_2}(t) = \int_{-\infty}^{\infty} dz n_{\eta_2}(z, t) \). Similarly the number of molecules in left and right well at time \( t \) are given as: \( N_{m_1}(t) = \int_{-\infty}^{\infty} dz n_{m_1}(z, t) \); and \( N_{m_2}(t) = \int_{-\infty}^{\infty} dz n_{m_2}(z, t) \) respectively. Population imbalance is defined as \( z_{\eta}(t) = (N_{\eta_1}(t) - N_{\eta_2}(t))/N_{\eta} \) for atoms and for molecules \( z_m(t) = (N_{m_1}(t) - N_{m_2}(t))/N_m \) where \( N_\eta \) and \( N_m \) are the initial total number of atoms and molecules respectively and \( N = N_\eta + 2N_m \).

### 3 Numerical Calculation

To study the dynamical behaviour of coupled atom-molecular BEC system with long range dipole-dipole interaction and short range contact interactions between particles we solved four one dimensional (in axial z direction) coupled equations (16)-(19) numerically using Crank-Nicholson scheme to discretize the
equations by using a space step \( h = 0.005 \) and time step \( \delta t = 5 \times 10^{-8} \). For initial input wavefunctions we have solved the time-independent atom-molecular coupled (two) equations by imaginary time method without considering the dipole-dipole interaction. To obtain initial guess wavefunctions in left and right wells we have divided the total wavefunctions for atom and molecules \( \psi_{al}(z) \) and \( \psi_{mr}(z) \) respectively in the range \(-\infty < z < \infty\) into two parts. The initial guess wavefunctions for atoms in left and right wells are defined as: \( \psi_{al}(z) = \psi_{ag}(z) \) for the values of \( z \) from \(-\infty \) to \( 0 \) and \( \psi_{ar}(z) = \psi_{ag}(z) \) for the values of \( z \) from \( 0 \) to \( \infty \) respectively. Similarly initial guess wavefunctions for molecules in left and right wells are defined as: \( \psi_{ml}(z) = \psi_{mg}(z) \) for the values of \( z \) from \(-\infty \) to \( 0 \) and \( \psi_{mr}(z) = \psi_{mg}(z) \) for the values of \( z \) from \( 0 \) to \( \infty \) respectively.

Initial total number of atoms is taken to be 5000 which is distributed unequally in left and right wells. The number of atoms in left and right wells is taken to be 1500 and 1000 respectively. Initial number of molecules has been chosen to be half of the number of atoms. Hence the number of molecules in the left and right wells are 750 and 500 respectively. Hence \( N_{al}(0) = 1500, N_{ar}(0) = 1000, N_{ml}(0) = 750, N_{mr}(0) = 500 \) and \( z_{a}(0) = z_{m}(0) = 0.2 \) for this calculation. To ensure specific initial number of atoms and molecules in left and right wells we have normalized \( \psi_{al}(z), \psi_{ar}(z), \psi_{ml}(z) \) and \( \psi_{mr}(z) \) accordingly. This type of unequal population distribution can be reached by considering the asymmetric double well potential [56, 9, 10]. The relative phase between atomic and molecular condensates is considered to be \( \pi \) in both the wells. However the atomic condensates in the left and right well are chosen to be in phase and similarly the relative phase is zero for molecular condensates. The convergence of the wavefunctions in this work has been checked to be \( \leq 10^{-8} \).

For simplicity we have assumed atom-atom, atom-molecule and molecule-molecule interactions are the same i.e. \( \lambda''_{a} = \lambda''_{m} = \lambda''_{am} \); and for this bosonic system the short range contact interaction is repulsive in nature. After simplification the coefficient for atom-atom contact interaction in equations (16) and (17) can be reduced to \( \lambda''_{a} = \frac{2a}{\lambda} \). In the present work we have chosen that the dipole-dipole interactions between molecules in left and right wells are the same as those of atoms i.e. \( d_{m}U_{ml}(k_{z}) = d_{m}U_{al}(k_{z}) \) where \( U_{al}(k_{z}) \) and \( U_{ml}(k_{z}) \) are given in equations (29) and (30) respectively. Here we have considered that the dipoles are aligned along the polarization axis [57], such that the dipole-dipole interaction is attractive in nature. The calculation of dipole-dipole interaction potential for the atoms and molecules given in equations (27) and (28) respectively has been done in two steps: (i) we have considered the functions \( S_{a}(\xi) \) and \( S_{m}(\xi) \) are equal and \( S_{a}(\xi) \) has been expressed as Exponential functions which has been multiplied by the Fourier transforms of atomic density \( n_{al}(k_{z}, t) \) and molecular density \( n_{ml}(k_{z}, t) \) for atoms and molecules respectively. Then (ii) the inverse Fourier transforms of the products has been done to obtain \( U_{al}(z, t) \) and \( U_{ml}(z, t) \) as given in equations (27) and (28). The Fourier transforms of densities and the inverse Fourier transforms have been done numerically. If a dipolar length scale is defined as \( \lambda_{dd} = \frac{\mu_{a}r_{a}^{2}m}{12\pi\hbar} \) then the coefficient of dipole-dipole interaction for atoms become \( d''_{a} = \frac{1}{2\pi\hbar} \frac{3\mu_{a}a_{0}^{2}}{2\hbar} \).

In this model calculation we have chosen different parameters as: (i) the coefficient of dipole-dipole interaction \( = 0.75 \times \) atom-atom contact interaction, (ii) the s-wave scattering length of atoms \( 'a' \) has been chosen to be \( a = 20a_{0} \) and the mass of atoms \( m \) is taken as that of Cr atoms, (iii) the angular frequency
\( \omega_z = 2\pi \times 194 Hz \) and the asymmetry parameter \( \lambda = 0.11 \), (iv) the double well parameters \( A = 15 \) and \( \kappa = 10 \) respectively, (v) the value of the atom-molecular coupling strength in dimensionless form is taken as \( \chi'' = 0.23012 \) \cite{73} and (vi) the detuning \( \epsilon = 8 \times 10^4 Hz \).

4 Results and Discussions

To study the dynamical behaviour of atomic and molecular coupled BECs in a double well potential in presence of dipole-dipole long range interaction as well as short range contact interaction four time-dependent coupled GP like equations have been solved by applying the steepest descent method in the Crank-Nicholson discretization scheme. Details of Crank-Nicholson scheme has been discussed previously \cite{73, 78}. The effect of dipole-dipole long range interaction and atom-molecular coherence through Feshbach resonance on the transmit/oscillation/trapping of atomic and molecular population between two wells has been studied. The total number of atoms and molecules in left well \( (N_{al}(t), N_{ml}(t)) \) and right well \( (N_{ar}(t), N_{mr}(t)) \) has been calculated at each instant of time in presence and absence of dipole-dipole interaction as well as in presence and absence of coherent coupling between atoms and molecules and plotted as a function of time. Corresponding population imbalance for atoms \( z_a(t) \) and molecules \( z_m(t) \) as a function of time has also been presented. To demonstrate these effects on the density profile of atoms, atomic density as a function of \( z \) has been plotted at times in the range of 76 to 120. The value of \( t \) mentioned here is in units of \( \omega_z^{-1} \).

4.1 Effect of coherent coupling on the dynamics of population in the presence of dipole-dipole interaction:

Evolution of total number of atoms and molecules in left and right wells is shown in Fig.1. It shows the dynamical behaviour of atoms and molecules in two wells considering the long range dipole-dipole interaction in presence and absence of coherent coupling \( (\chi) \) between atoms and molecules. In these figures (Fig.1(a) and Fig.1(b)) upper and lower panel show dynamics of atoms and molecules respectively. Both in Fig.1(a) and Fig.1(b) atomic and molecular populations are found to oscillate initially. But in Fig.1(a) (in presence of coherent coupling \( \chi \)) transmission of atoms from left to right well starts after time \( t \approx 55 \), when the population in right well becomes much higher than that in the left well and this transmission persists up to the time \( t \approx 125 \) with some oscillations where populations become equal or very close to each other. However this transient transmission effect is washed out in Fig.1(b) (in absence of \( \chi \)) where atoms are self trapped in left well for the period of time \( t=75 \) to 175 approximately with some oscillations where populations become equal or close to each other. These features are also demonstrated in Fig.2(a) and Fig.2(b) where the population imbalance \( z_a(t) \) of atoms corresponding to Figs.1(a) and 1(b) has been plotted. Fig.2(a) shows partial transmission for a time duration of \( t=55 \) to 125 approximately when most of the time \( z_a(t) \) remains negative except at some points where it touches zero. Whereas \( z_m(t) \) in Fig.2(b) is positive most of the time in the period \( t=75 \) to 175 with some oscillations touching zero. Therefore the presence of coherent coupling between atoms and molecules with dipole-dipole
long range interaction prefers transient transmission of atoms dominantly to the right well in contrary to transient self trapping in left well in absence of it. This type of oscillation with time in the atomic population imbalance for non-dipolar atom-molecular coupled BECs has been obtained in case of initial unequal population distribution of atoms and molecules in two wells and for moderately strong tunnelling effect \[76\].

Dynamical behaviour of molecules (lower panel) in Fig.1(a) shows initial oscillations and the transfer of population to right well starts at \( t=110 \) approximately and remains higher than that in the left well up to the time \( t=185 \) approximately. The feature of transient transmission of molecular population exists in presence of coherent coupling as in the case of atomic transmission, but it takes double the time that for atoms to start. In absence of coherent coupling transfer of molecular population (lower panel of Fig.1(b)) occurs at larger time \( t=130 \) and the prominence of transfer persists for a shorter time than that in case of Fig.1(a) (lower panel).

Molecular tunneling effect in non-dipolar atomic-molecular coupled (via photo
Population imbalance $z(t)$ for molecules

Time

Figure 3: Population imbalance for molecules for (a) $\chi \neq 0$ and $a_{dd} = 0.75a$ and (b) $\chi = 0$ and $a_{dd} = 0.75a$

association) BECs has been discussed previously choosing certain parameters. It has been shown that the molecular number in the right well dominates over that in the left well and oscillates with time (similar to the results shown here) for small atom-molecular coupling.

The transmission effect on molecules as shown in figures 1(a) and 1(b) has been supported by the plot of population imbalance of molecules $z_m(t)$ in Fig.3(a) and Fig.3(b) respectively. Therefore as in the case of atoms coherent coupling between atoms and molecules leads to transient transmission of molecules in the right well and the population in right well becomes larger than that in the left well for a period of time mentioned above. However this transient transmission time is shortened in absence of coherent coupling.

Therefore from these results it is found that transmission of both the atoms and molecules for a period of time is facilitated in presence of atom molecular coherence in a dipolar atom-molecular coupled BEC system. Whereas in absence of it this transient transmission effect is damped.

4.2 Effect of coherent coupling on the dynamics of population in the absence of dipole-dipole interaction:

To investigate the effect of coherent coupling on the dynamics of atoms and molecules in absence of dipole-dipole interaction we repeated the calculation with $a_{dd} = 0$ and plotted the total number of atoms and molecules (upper and lower panel respectively) as a function of time in Fig.4(a) and 4(b) in presence and absence of coherent coupling ($\chi$) respectively. It is found that for $\chi = 0$ and $a_{dd} = 0$, self trapping of atomic population (Fig.4(b)) is present for a long period of time $t=30$ to 140 approximately except at $t \approx 118$ where a small population exchange occurs. This feature of self trapping is also evident in the plot of corresponding population imbalance $z_a(t)$ in Fig.5(b). The self-trapping effect has been obtained previously in our group for non-dipolar cold atoms trapped in a double well in case of strong confinement i.e. the asymmetry parameter $\lambda << 1$.

It is shown here that in presence of coherent coupling this self trapping is disturbed and transmission of atomic population is prominent for a shorter period of time (65 to 95) in Fig.4(a) than that in Fig.1(a). However a small transfer of
Atomic and Molecular Number in left and right well

Figure 4: Atomic (upper panel) and Molecular (lower panel) population distribution as a function of time. Total number of atoms and molecules in left (black, violet) and right (red, green) wells are given for (a) $\chi \neq 0$ and $a_{dd} = 0$ and (b) $\chi = 0$ and $a_{dd} = 0$ respectively.

Figure 5: Population imbalance for atoms for (a) $\chi \neq 0$ and $a_{dd} = 0$ and (b) for $\chi = 0$ and $a_{dd} = 0$

population is present during the time 115 to 150 with small oscillations. This can be seen in the plot of corresponding population imbalance $z_a(t)$ in Fig.5(a). Therefore from these figures it is found that the transient self trapping in the left well and the transient transmission of population to be trapped in the right well can be induced in absence and presence of coherent coupling respectively, even when the long range dipole-dipole interaction is considered to be zero.

However for molecules the (lower panels of fig.4(b)) the transient effect of self trapping in the left well in absence of coherent coupling ($a_{dd} = 0$ and $\chi = 0$) is not so prominent as that in atoms. The molecular population in left well dominates over right well only for short period of time $t=125$ to 140 approximately. Otherwise it oscillates between two wells with the population in the left well greater than that in the right well in most of the time after $t = 140$. This feature is demonstrated in the plot of $z_m(t)$ in Fig.6(b). Similarly the effect of coherent coupling (for $a_{dd} = 0$ and $\chi \neq 0$) leading to transient transmission to the right well of the molecules (in the lower panel of Fig.4(a)) is not so prominent as in the case of atoms. Here molecular population in right well dominates over that in the left well only in the short period of times, the
longest of which is t=100 to 125 approximately at a stretch. This feature is evident from the corresponding plot of $z_m(t)$ in Fig.6(a).

Therefore from these figures it is found that the effect of coherent coupling in absence of dipole-dipole interaction on molecular dynamics is not so prominent as that in the case of atoms. The dipole-dipole long range interaction and the coherent coupling between atoms and molecules both are positive in this calculation and both lead to the transient transmission of population from the left well to the right well. The degree of this transient transmission effect will depend on the strength of these interactions which depends also on the number of particles taking part in the dynamics. The dipole-dipole interaction is proportional to the number of particles but the coherent coupling term is proportional to the square root of the number of particles. Hence in the absence of dipole-dipole interaction the effective strength of interaction leading to transient transmission is reduced. Moreover the number of molecules taking part in the dynamics is much less than the number of atoms and hence the transient transmission effect is much less effective in case of molecules than that in case of atoms in absence of dipole-dipole interaction.

**4.3 Effect of dipole-dipole interaction on the dynamics of population in presence and absence of coherent coupling:**

The dipole-dipole interaction which is a long range interaction, is expected to facilitate the transmission of atomic and molecular population from the left well to the right well when it is chosen to be positive in nature. In absence of coherent coupling ($\chi = 0$) the effect of dipole-dipole interaction on the dynamics is demonstrated in figures 1(b) and 4(b). Comparison of these two figures shows that the long duration of self trapping of atoms in left well in Fig.4(b) is shortened due to the effect of long range dipole-dipole interaction which prefers to transmit the atoms from left to right well. Comparison of dynamical behaviour of molecules (lower panel of figures 1(b) and 4(b)) shows transient transmission effect is present (in the lower panel of Fig.1(b)) which is the signature of presence of dipole-dipole interaction for ($\chi = 0$).

Moreover in presence of coherent coupling the transient transmission of
atomic population from left to right well is present both in the presence and absence of dipole-dipole interaction as shown in Fig.1(a) and Fig.4(a) respectively. However duration of transmission is reduced in absence of dipole-dipole interaction. Similarly for molecules the transient transmission effect is prominent in presence of dipole-dipole interaction (lower panel of figure 1(a)).

4.4 Effect of coherent coupling on the dynamics of atomic density profiles:

To investigate the dynamical behaviour of density profile of atoms in presence and absence of coherent coupling, we have plotted the atomic density as a function of $z$ in the time range of $t=76$ to 120.

Figure 7 compares the nature of atomic density profile in two cases for $a_{dd} \neq 0$ and $\chi \neq 0$ at times $t=80$, 101 and 118 (in the upper panel) and $a_{dd} \neq 0$ and $\chi = 0$ at times $t=80$, 100 and 120 (in the lower panel). It is found that the contribution of density of atoms is greater in right well than that in the left well for non-zero coherent coupling (upper panel). Whereas the density in left well is greater than that in right well for $\chi = 0$ (lower panel).

Similarly Fig.8 shows the dynamical behaviour of atomic density profile for $a_{dd} = 0$ and $\chi \neq 0$ at times $t=76$, 96 and 120 (in the upper panel) and $a_{dd} = 0$ and $\chi = 0$ at times $t=80$, 100 and 120 (in the lower panel). As in the presence of dipole-dipole interaction shown in Fig.7, the effect of coherent coupling (i.e. $\chi \neq 0$) leads to more contribution in right well than that in the left well even in absence of dipole-dipole interaction. Whereas for $\chi = 0$ the atomic density has more contribution in left well compared to that in the right well. Therefore these two figures show that in the time range of $t=76$ to 120 coherent coupling
Figure 8: Effect of coherent coupling on the dynamics of atomic density for $a_{dd} = 0$. Upper panel in presence of coherent coupling $\chi$ at times (a) $t=80$, (b) $t=101$, (c) $t=118$ and lower panel in absence of it at times (d) $t=80$, (e) $t=100$ and (f) $t=120$.

keeps its signature on the dynamics of atomic density profiles leading to more contribution in the right well than that in the left well and this effect persists both in presence and absence of dipole-dipole interaction.

5 Conclusion:

In the present model calculation we have studied numerically the dynamics of coupled dipolar atomic and molecular BECs trapped in a double well potential. The effect of coherent coupling between atoms and molecules (through magnetic Feshbach coupling) on the dynamics of atomic and molecular population between two wells has been investigated in presence and absence of long range dipole-dipole interaction. We have solved four time-dependent coupled GP like equations two for left well and two for right well (for atoms and molecules) which includes both the long range dipole-dipole interaction and short range contact interaction. These equations have been derived from the variation of energy functional as it is done for the derivation of GP equations. Then the three dimensional equations have been reduced to single dimension (in the axial direction) by considering the strong confinement in the radial direction to study the dynamics of population in the double well which is considered to be in the axial direction. To demonstrate the effect of coherent coupling and the dipole-dipole interaction the evolution of total number of atoms and molecules in the left and in the right well has been shown. Corresponding population imbalance as a function of time has been studied to support the results. In the present work dipole-dipole interaction has been chosen to be positive and since it is long range in nature, it leads to transient transmission of atomic and molecular populations from the left to the right well. Moreover the coherent coupling inherent in this
coupled system is also positive and hence intensifies the transient transmission of population from the left to the right well. As a result the absence of anyone of these two positive interactions leads to reduction of transient transmission of population. It is shown that in the absence of these two positive interactions the self trapping of population in the left well is prominent for a long duration which is disturbed in presence of anyone of these positive interactions either the long range dipole-dipole interaction or the coherent interaction between atoms and molecules. These effects are also demonstrated by the dynamics of population imbalance for atoms and molecules in each case. The effect of coherent coupling on the density profile of atoms has been demonstrated by plotting the atomic density as a function of z at three instant of time from t=76 to 120 (t in the units of $\omega_z^{-1}$). It is shown that the presence of coherent coupling leads to concentration of population in the right well more than that in the left well both in presence and absence of dipole-dipole interaction.

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