Optimal Threshold Design for Quanta Image Sensor

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Abstract—Quanta Image Sensor (QIS) is a binary imaging device envisioned as a candidate for the next generation imaging device after CCD and CMOS. Equipped with a massive number of single photon detectors, the sensor has a threshold \( q \) above which the number of arriving photons will trigger a binary response “1”. Existing methods in the device literature typically assume that \( q = 1 \) for circuit simplicity. We argue that a spatially varying threshold can significantly improve the signal to noise ratio of the reconstructed image. In this paper, we present an optimal threshold design method. We make two contributions. First, we derive a set of oracle threshold results to inform the maximally achievable performance. We show that the oracle threshold should match exactly with the underlying pixel intensity. Second, we show that around the oracle threshold there exists a set of thresholds that give asymptotically unbiased reconstructions. The asymptotic unbiasedness has a phase transition behavior which allows us to develop a practical threshold update scheme using a bisection method. Experimentally, the new threshold design method achieves better rate of convergence than existing methods.

Index Terms—Quanta image sensor, threshold adaptation, high dynamic range, quantization map, maximum likelihood.

I. INTRODUCTION

A. Threshold Design for Quanta Image Sensor

Quanta Image Sensor (QIS) is a class of solid-state image sensors envisioned as the next generation imaging device after CCD and CMOS. Originally proposed by Eric Fossum in 2005 [1], the sensor has gained significant momentum in the past decade, both in terms of hardware design [2]–[4] and image processing [5]–[9]. The advantage of QIS over the mainstream CCD and CMOS is attributed to its high spatial resolution (e.g., 10⁹ pixels per sensor with 200nm pitch per pixel [10]) and high speed (e.g., 100k fps as reported in [11]). However, in order to simplify its circuit, minimize power consumption and reduce the amount of data transfer, QIS is operated in a binary mode: When the number of photons arriving at the sensor exceeds a threshold \( q \), the sensor generates a binary bit “1”. When the number of photons is less than \( q \), the sensor generates a “0”. The goal of this paper is address the question of how to optimally choose \( q \).

Optimal threshold design for QIS is important as it directly affects the dynamic range of an image. Figure 1 illustrates an example. In this figure, we simulate the raw binary data acquired by a QIS using certain threshold \( q \). When \( q \) is low, most of the bits in the raw input are “1”. The reconstructed image is therefore an over-exposed image. On the other hand, when \( q \) is high, most of the bits in the raw input are “0”. The reconstructed image is then under-exposed. In either case, the simulation result illustrates the fact that a constant threshold has a limited performance. We argue that a better way is to allow \( q \) to vary spatially so that a pixel (or a group of pixels) has its own threshold value. By optimally determining the thresholds we can maximize the signal-to-noise ratio and the dynamic range of the reconstructed image.

B. Scope and Contributions

The goal of this paper is to present an optimal threshold design methodology. We make two contributions. First, we provide a rigorous theoretical analysis of the performance limit of the image reconstruction as a function of the threshold. These results form the basis of our subsequent discussions of the threshold update scheme. Some results are known previously, e.g., the signal to noise ratio is a function of the Fisher Information [16], [17]. Several new results are established: (i) We show that the maximum likelihood estimate has a closed-form expression in terms of the incomplete Gamma function (Section III.B); (ii) We show that the oracle threshold can be derived in closed-form by maximizing the signal to noise ratio (Section III.C); (iii) We show that the image reconstruction has a phase transition behavior (Section IV.A - Section IV.B).
Second, we propose an efficient threshold update scheme based on our theoretical results. The new scheme adopts a bisection method which iteratively updates the threshold without the need of reconstructing the image. By checking whether the proportion of one’s and zero’s approaches 0.5 in a spatial-temporal block, the threshold is guaranteed to be near optimal. Compared to other existing threshold update schemes such as [18] and [19]–[21], the new scheme offers significantly faster rate of convergence (Section IV.C). We also demonstrate how the dynamic range can be extended for high dynamic range (HDR) imaging (Section IV.D).

The preliminary work of this paper was presented in ICIP 2016 [8]. This journal version contains significantly more details. In particular, we provide complete proofs of major results. We make comprehensive comparisons with existing methods and we include discussions of HDR imaging.

II. BACKGROUND

A. Current State of QIS

Quanta Image Sensor (QIS) belongs to the family of photon-counting devices. Photon-counting devices have been known for a long time. Examples include electron-multiplying charge-coupled device (EMCCD) [22], [23], single-photon avalanche diode (SPAD) [11], [14], [24], Geiger-mode avalanche photodiode (GMAPD) [13], etc. The common feature of these devices is that their single photon sensitivity. Many of these devices are used in medical imaging [25]–[27], astronomy [28], defense [29], nuclear engineering [30], depth and reflectivity reconstruction [31], and recently in quantum random number generation used in cryptography [32], [33].

The concept of QIS was first proposed by Fossum in 2005 as a solution for sub-diffraction limit pixels. The sensor was called the digital film sensor, and later the quanta image sensor [15], [34], [35]. After the introduction of QIS, researchers in EPFL developed a similar concept called the Gigavision camera [6], [36], [37]. Recently, teams at the University of Edinburgh [14], [24], [38] and EPFL [32], [39] have made new progresses in QIS using binary single photon detectors.

In the industry, Rambus Inc. (Sunnyvale, CA) has developed binary image sensors for high dynamic range imaging [19]–[21]. Table I lists several recent QIS prototypes that are available or are currently being developed. As a comparison we also show a Canon 5D Mark III CMOS camera. Among many different features, the most noticeable is the frame rate. For example, SPS SPAD can be operated at 20k fps. SwissSPAD can even achieve 156k fps. Both are significantly faster than a standard CMOS camera.

B. Related Work on Threshold Design

Existing work on QIS threshold design study can be summarized into two classes of methods.

- Markov Chain [18]. The Markov Chain method developed by Hu and Lu [18] is a time-sequential update scheme. A Markov Chain probability is used to control how easy the threshold should be increased or decreased. While the method has provable convergence, the threshold of each single photon detector of the QIS has to be updated sequentially in time. In contrast, our proposed method allows a group of single photon detectors to share the same threshold. As a result, our proposed method has significantly faster rate of convergence.

- Conditional Reset [19]–[21]. The conditional reset method is a hardware solution proposed by Vogelsang and colleagues. The idea is to take a sequence of images with ascending (or descending) thresholds, and digitally integrate the sequence to form an image. The drawback of the method, besides the additional hardware cost of the per-pixel reset transistors, is the limited quality of the reconstructed image. For the same number of frames, our proposed method produces better images.

C. QIS Imaging Model

In this subsection we provide an overview of the QIS imaging model. The model has been previously discussed in several papers, e.g., [6]–[9]. Readers interested in details can refer to these papers for further explanations.

1) Spatial Oversampling: We denote the discrete version of the light intensity as a vector $c = [c_0, \ldots, c_{N-1}]^T$, where $n = 0, \ldots, N - 1$ specify the spatial coordinates. We assume that $c_n$ is normalized to the range $[0, 1]$ for all $n$ so that there is no scaling ambiguity. To model the actual light intensity, we multiply $c_n$ by a constant $\alpha$ to yield $\alpha c_n$, where $\alpha > 0$ is a fixed scalar constant.

Given the $N$-dimensional vector $c$, QIS uses $M \gg N$ tiny pixels called jots to sample $c$. The ratio $K \overset{\text{def}}{=} M/N$ is known as the spatial oversampling factor. The oversampling process is illustrated in Figure 2, where it first upsamples the vector $c$ by a factor of $K$, and then filters the output by a lowpass filter $\{g_k\}$. Mathematically, the process can be expressed as

$$\theta = \alpha G c,$$

where $\theta = [\theta_0, \ldots, \theta_{M-1}]^T$ denotes the light intensity sam-
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\[ G = \frac{1}{K} I_{N \times N} \otimes 1_{K \times 1}, \]

where \( 1_{K \times 1} \) is a vector of all ones and \( \otimes \) denotes the Kronecker product. Note that the choice of \( G \) in (2) is the result of a simplified model by assuming that the lowpass filter is \( g_k = 1/K \) for all \( k \). It is possible to use a general lowpass filter as discussed in [6], [7]. However, in order to simplify the theoretical derivation, we will focus on this special \( G \).

\section*{2) Truncated Poisson Process:} We assume that the operating speed of QIS is significantly faster than the scene motion. Therefore, for a given scene \( c \) (and also \( \theta \)), we are able to acquire a set of \( T \) independent measurements. We illustrate this using the \( T \) channels in Figure 2.

The oversampled signal \( \theta \) generates a sequence of Poisson random variables according to the distribution

\[ P(Y_{m,t} = y_{m,t}) = \frac{\theta_{m,t}^{y_{m,t}} e^{-\theta_{m,t}}}{y_{m,t}!}, \]

where \( m = 0, 1, \ldots, M-1 \) denotes the \( m \)-th jot of the QIS and \( t = 0, 1, \ldots, T-1 \) denotes the \( t \)-th independent measurement in time. Denoting \( q \in \mathbb{N} \) as the quantization threshold, the final observed binary measurement \( B_{m,t} \) is a truncation of \( Y_{m,t} \):

\[ B_{m,t} = \begin{cases} 0, & \text{if } Y_{m,t} < q, \\ 1, & \text{if } Y_{m,t} \geq q. \end{cases} \]

The probability mass function of \( B_{m,t} \) is given by

\[ P(B_{m,t} = b_{m,t}) = \begin{cases} \sum_{k=0}^{q-1} \frac{\theta_{m,t}^{k} e^{-\theta_{m,t}}}{k!}, & \text{if } b_{m,t} = 0, \\ \sum_{k=q}^{\infty} \frac{\theta_{m,t}^{k} e^{-\theta_{m,t}}}{k!}, & \text{if } b_{m,t} = 1. \end{cases} \]

\section*{3) Properties of Truncated Poisson Processes:} The probability mass function of \( B_{m,t} \) in (4) shows that \( B_{m,t} \) is a Bernoulli random variable. However, the right hand side of (4) involves an infinite sum which is difficult to interpret. To alleviate the infinite sum, we consider the upper incomplete Gamma function \( \Psi_q : \mathbb{R}_+ \to [0, 1] \) defined in [40] as:

\[ \Psi_q(\theta) = \frac{1}{\Gamma(q)} \int_0^\infty t^{q-1} e^{-t} dt, \quad \text{for } \theta > 0, q \in \mathbb{N}. \]

where \( \Gamma(q) = (q-1)! \) is the standard Gamma function. The incomplete Gamma function allows us to rewrite the infinite sum in (4) as follows:

\[ \Psi_q(\theta) = \sum_{k=0}^{q-1} \frac{\theta_{m,t}^{k} e^{-\theta_{m,t}}}{k!}. \]

Consequently, the probabilities in (4) become

\[ P(B_{m,t} = 0) = \Psi_q(\theta_{m,t}), \]
\[ P(B_{m,t} = 1) = 1 - \Psi_q(\theta_{m,t}). \]

\section*{Example 1.} In the special case of \( q = 1 \), we obtain:

\[ P(B_{m,t} = 0) = \frac{1}{\Gamma(1)} \int_0^\infty t^{0} e^{-t} dt = e^{-\theta_{m,t}}, \]

which coincides with the results shown in [6] and [7].

The incomplete Gamma function \( \Psi_q(\theta) \) is a decreasing function of \( \theta \) because the first order derivative of \( \Psi_q(\theta) \) with respect to \( \theta \) is negative:

\[ \frac{d}{d\theta} \Psi_q(\theta) = -\theta^{q-1} e^{-\theta} < 0, \quad \forall q \in \mathbb{N}, \text{ and } \theta > 0. \]

The limiting behavior of \( \Psi_q(\theta) \) is important. For a fixed \( q \), the function \( \Psi_q(\theta) \to 1 \) as \( \theta \to 0 \) and \( \Psi_q(\theta) \to 0 \) as \( \theta \to \infty \). While \( \Psi_q^{-1} \) still exists in these situations because \( \Psi_q \) is monotonically decreasing, for a given \( z \) the value \( \Psi_q^{-1}(z) \) could be numerically very difficult to evaluate. To characterize the sets of \( \theta \) and \( q \) that \( \Psi_q \) is (numerically) invertible, we define the \( \theta \)-admissible set and the \( q \)-admissible set.

\section*{Definition 1.} The \( \theta \)-admissible set and \( q \)-admissible set of the incomplete Gamma function are

\[ \Theta_q \overset{\text{def}}{=} \{ \theta \mid \varepsilon \leq \Psi_q(\theta) \leq 1 - \varepsilon \}, \]
\[ Q_\theta \overset{\text{def}}{=} \{ q \mid \varepsilon \leq \Psi_q^{-1}(\theta) \leq 1 - \varepsilon \}, \]

respectively, where \( 0 < \varepsilon < 1 \) is a constant.

More discussions of the incomplete Gamma function can be found in the Supplementary material.

\section*{III. OPTIMAL THRESHOLD: THEORY}

In this section we present a set of oracle results.

\subsection*{A. Image Reconstruction by MLE}

We begin by discussing image reconstruction because the optimality of the threshold is measured with respect to the reconstructed image. The goal of image reconstruction is to recover the underlying image \( c \) from the binary measurements \( B = \{ B_{m,t} \mid m = 0, \ldots, M-1, \text{ and } t = 0, \ldots, T-1 \} \). Image reconstruction for QIS is a known problem and there are several existing methods, such as gradient descent [6], dynamic programming [41], ADMM [7], and Transform-Denoise method [9]. In order to ensure analytic tractability, we shall focus on the maximum likelihood estimation (MLE) approach, which has a closed-form expression.

Given \( B \), MLE solves the following optimization problem:

\[ \hat{c} \overset{(a)}{=} \arg \max_c \prod_{t=0}^{T-1} \prod_{m=0}^{M-1} P[B_{m,t} = 1 ; \theta_m]^{b_{m,t}} \]
\[ \times P[B_{m,t} = 0 ; \theta_m]^{1-b_{m,t}} \]
\[ \overset{(b)}{=} \arg \max_c \sum_{t=0}^{T-1} \sum_{m=0}^{M-1} \left[ b_{m,t} \log(1 - \Psi_q(\theta_m)) \right. \]
\[ \left. + (1 - b_{m,t}) \log \Psi_q(\theta_m) \right], \]

subject to the constraint that \( \theta = \alpha Gc \). Here, the right hand side of (a) is the likelihood function of a Bernoulli
random variable, and (b) follows from taking the logarithm. With the $G$ defined in (2), we can partition $B$ into $N$ blocks 
$\{B_1, \ldots, B_N\}$ where each block is

$$B_n \triangleq \{B_{Kn+k,t} \mid k = 0, \ldots, K-1, t = 0, \ldots, T-1\}.$$ 

Then, the pixel $\hat{c}_n$ can be estimated as follows.

**Proposition 1** (Closed-form ML Estimate). The solution of the MLE in (9) is

$$\hat{c}_n = \frac{K}{\alpha c} \Psi_q^{-1} \left(1 - \frac{S_n}{KT}\right),$$

where $S_n \triangleq \sum_{t=0}^{T-1} \sum_{k=0}^{K-1} B_{Kn+k,t}$ is the sum of bits in the $n$-th block $B_n$.

**Proof:** See [9].

**B. Signal-to-Noise Ratio of ML Estimate**

In order to determine the optimal threshold, we need to quantify the performance of the ML estimate. The performance metric we use is the signal-to-noise ratio of the ML estimate at every pixel $\hat{c}_n$. Considering each $\hat{c}_n$ individually is allowed here because they are independently determined according to (10). For notation simplicity we drop the subscript $n$ in the subsequent discussions.

**Definition 2.** The signal-to-noise ratio (SNR) of the ML estimate $\hat{c}$ is defined as

$$\text{SNR}_q(c) \triangleq 10 \log_{10} \frac{c^2}{E[(\hat{c} - c)^2]},$$

where the expectation is taken over the probability mass function of the binary measurements in (6).

The difficulty of working with $\text{SNR}_q(c)$ is that it does not have a simple closed-form expression. In view of this, Lu [17] showed that the SNR is asymptotically linear to the log of the Fisher Information.

**Proposition 2.** As $KT \to \infty$,

$$\text{SNR}_q(c) \approx 10 \log_{10} (c^2 I_q(c)) + 10 \log_{10} KT,$$

where $I_q(c)$ is the Fisher Information of the probability mass function defined in (6).

**Proof:** See [17].

While the asymptotic result shown in Proposition 2 has significantly simplified the SNR, we still need to determine the Fisher Information. The following proposition gives a new result of the Fisher Information with arbitrary $q$.

**Proposition 3.** The Fisher Information $I_q(c)$ of the probability mass function in (6) under a threshold $q$ is:

$$I_q(c) = \left(\frac{\alpha c}{K}\right)^2 \frac{e^{-2(\frac{c}{K})} (\frac{\alpha c}{K})^{2q-2}}{\Gamma^2(q) \Psi_q \left(\frac{\alpha c}{K}\right) (1 - \Psi_q \left(\frac{\alpha c}{K}\right))}.$$  

**Proof:** See Appendix A-A.

Substituting (13) into (12), we observe that the SNR can be approximated as

$$\text{SNR}_q(c) \approx 10 \log_{10} \frac{KT e^{-2(\frac{c}{K})} (\frac{\alpha c}{K})^{2q}}{\Gamma(q)^2 \Psi_q \left(\frac{\alpha c}{K}\right) (1 - \Psi_q \left(\frac{\alpha c}{K}\right))},$$

which is characterized by the unknown pixel value $c$, the threshold $q$, the spatial oversampling ratio $K$ and the number of temporal measurements $T$. To understand the behavior of (14), we show in Figure 3 SNR$_q(c)$ as a function of $c$ for different thresholds $q \in \{1, \ldots, 16\}$. For a fixed $q$, SNR$_q(c)$ is a convex function with a unique maximum. The goal of optimal threshold design is to determine a $q$ which maximizes SNR$_q(c)$ for a fixed $c$.

**Remark 1.** The SNR$_q(c)$ in (14) can also be derived from a concept in the device literature called the exposure-referred SNR [42]. See Supplementary material for discussions.

**C. Oracle Threshold**

We now discuss the optimal threshold design in an oracle setting. We call the result oracle because the optimal threshold depends on the unknown pixel intensity $c$. The practical threshold design scheme will be discussed in Section IV.

Using the definition of the signal-to-noise ratio, the optimal threshold is determined by maximizing SNR$_q(c)$ with respect to $q$:

$$q^* = \arg\max_{q \in \mathbb{N}} \text{SNR}_q(c) = \arg\max_{q \in \mathbb{N}} \log(c^2 I_q(c)).$$

The second equality follows from Proposition 2. Substituting (13) yields an expression of the right hand side of (15). To further simplify the expression we derive the following lower bound.

**Proposition 4.** The function $\log(c^2 I_q(c))$ is lower bounded as follows.

$$\log(c^2 I_q(c)) \geq 2 \left(\log 2 - \frac{\alpha c}{K} + q \log \frac{\alpha c}{K} - \log \Gamma(q)\right),$$

**Proof:** See Appendix A-B.

Using this lower bound, we can derive the optimal threshold $q$ as follows.

![Fig. 3. SNR$q(c)$ for different thresholds $q \in \{1, \ldots, 16\}$. In this experiment, we set $\alpha = 400$, $K = 4$, and $T = 30$. For fixed $q$, SNR$q(c)$ is always a convex function.](image-url)
Proposition 5. The optimal threshold \( q^*(c) \) is
\[
q^*(c) = \arg \max_{q \in \mathbb{N}} L_q(c) = \left[ \frac{c}{K} \right] + 1,
\]
where \( \lfloor \cdot \rfloor \) denotes the flooring operator that returns the largest integer smaller than or equal to the argument.

Proof: See Appendix A-C.

The result of Proposition 5 is important. It states that the oracle threshold is exactly the same as the light intensity observed at a jot, i.e., \( \alpha c / K \). The flooring operation and the addition of a constant 1 ensure that the threshold is an integer. In fact, a special where \( \alpha = 1 \) has been experimentally demonstrated in [18]. The new result in Proposition 5 provides a theoretical justification.

IV. OPTIMAL THRESHOLD: PRACTICE

The oracle threshold derived in the previous section provides a theoretical foundation but is practically not feasible. In this section, we present a practical threshold design that achieves a relaxed optimality criteria. The insight of the proposed method is drawn from the following asymptotic unbiasedness and phase transition phenomenon.

A. Asymptotic Unbiasedness

Recall in Proposition 1 we showed that for a spatial-temporal block \( B_n \), the maximum likelihood (ML) estimate \( \hat{c} \) has to satisfy the condition
\[
\Psi_q \left( \frac{\alpha \hat{c}}{K} \right) = 1 - \frac{S}{KT},
\]
where \( S = \sum_{l,k} B_{K_n+k,l} \) is the sum of bits in \( B_n \). We define the right hand side of the equation as
\[
\gamma_q(c) \overset{\text{def}}{=} 1 - \frac{S}{KT}.
\]
In the device literature (e.g., [42]), the term \( 1 - \gamma_q(c) \) is known as the bit-density as it is the proportion of ones in \( B_n \).

There are a few important properties about \( \gamma_q(c) \). First of all, \( \gamma_q(c) \) is a random variable. The mean and variance of \( \gamma_q(c) \) are given by the following proposition.

Proposition 6. The mean and variance of \( \gamma_q(c) \) are
\[
\mathbb{E}[\gamma_q(c)] = \Psi_q \left( \frac{\alpha c}{K} \right), \quad \text{and}
\]
\[
\text{Var}[\gamma_q(c)] = \frac{1}{KT} \Psi_q \left( \frac{\alpha c}{K} \right) \left[ 1 - \Psi_q \left( \frac{\alpha c}{K} \right) \right],
\]
respectively.

Proof: See Appendix A-D.

Applying strong law of large number to \( \gamma_q(c) \), we show that
\[
\gamma_q(c) \xrightarrow{a.s.} \mathbb{E}[\gamma_q(c)] = \Psi_q(\alpha c / K),
\]
as \( KT \to \infty \). As a result, the ML estimate \( \hat{c} \), which is also a random variable, should have its mean \( \mathbb{E}[\hat{c}] \) converging to \( c \):
\[
\mathbb{E}[\hat{c}] = \frac{K}{\alpha} \mathbb{E} \left[ \Psi_q^{-1}(\gamma_q(c)) \right] \overset{(a)}{=} \frac{K}{\alpha} \mathbb{E} \left[ \Psi_q^{-1} \Psi_q \left( \frac{\alpha c}{K} \right) \right] \overset{(b)}{=} c.
\]

In the above derivation, (a) follows from the definition of \( \hat{c} \), (b) follows from (20), and (c) holds because \( \Psi_q \) and \( \Psi_q^{-1} \) cancel each other.

The result we have just shown is the asymptotic unbiasedness of the ML estimate \( \hat{c} \). Asymptotic unbiasedness means that as the number of independent measurements grows, the ML estimate \( \hat{c} \) approaches to the ground truth \( c \). In other words, as long as \( KT \) is large enough, the random variable \( \hat{c} \) would be a very good estimate of the ground truth.

Does an asymptotically unbiased estimate maximize the SNR? The answer is no, because Proposition 5 states that if \( q^* \) is the optimal threshold, then \( \text{SNR}_{q^*}(c) \geq \text{SNR}_{q}(c) \) for any \( q \neq q^* \). Therefore, moving from the exact optimal \( q^* \) to a threshold that leads to an asymptotically unbiased estimate is a relaxation of the optimality criteria.

In order to understand the difference between the two optimality criteria, we make a quick detour to first understand the limitation of the incomplete Gamma function. For finite \( KT \), it is possible that \( \gamma_q(c) = 0 \) or \( \gamma_q(c) = 1 \), i.e., all bits in \( B_n \) are 1’s or 0’s. When either case happens, the inversion of the incomplete Gamma function becomes problematic because every incomplete Gamma function has a \( q \)-admissible set \( Q_\theta \) (See Definition 1). Putting into the QIS problem, the tolerance level \( \varepsilon \) defining \( Q_\theta \) should be set as follows.

Proposition 7. Let \( 0 < \delta < 1 \) be a constant. Then, for any \( q \in Q_\theta \defeq \left\{ q \mid 1 - \left( \frac{\delta}{2} \right)^{\frac{1}{KT}} \leq \Psi_q(\theta) \leq \left( \frac{\delta}{2} \right)^{\frac{1}{KT}} \right\} \), (21)
the random variable \( \gamma_q(c) \) will not attain a value of 0 or 1 with probability at least \( 1 - \delta \), i.e.,
\[
\mathbb{P}[0 < \gamma_q(c) < 1] > 1 - \delta.
\]

Proof: See Appendix A-E.

Example 2. We consider an example to illustrate the magnitude of the quantities in Proposition 7. In this example, we let the ground truth pixel value be \( c = 0.5 \). The sensor parameters are set as \( T = 50, K = 4, \alpha = 300 \). For a constant \( \delta = 2 \times 10^{-4} \), the tolerance level is \( \varepsilon = 1 - (\delta/2)^{1/KT} = 0.045 \). Therefore, as long as \( q \in \{ q \mid 0.045 \leq \Psi_q(\theta) \leq 1 - 0.045 \} \), which is the set \( \{ q \mid 0.045 \leq \Psi_q(\theta) \leq 1 - 0.045 \} \), the probability that \( \gamma_q(c) \) equals to 0 or 1 is upper bounded by \( \delta = 2 \times 10^{-4} \).

B. Phase Transition Phenomenon

The combination of the above asymptotic unbiasedness and the \( q \)-admissibility of the incomplete Gamma function leads to an important phenomenon. Figure 4 shows the plot of a typical experiment with experimental setup discussed in Example 2. We generate 10,000 random realizations, where each realization is a spatial-temporal block \( B_n \) containing \( KT = 200 \) binary bits. In the plot, we show the average bit density \( 1 - \gamma_q(c) \). This average is taken over the 10,000 realizations. We also overlay the average with the true expectation \( 1 - \Psi_q(\theta) \) (where \( \theta = \alpha c / K \)) to verify the validity of our equations. In the same plot, we mark the \( q \)-admissible set \( Q_\theta \) with the tolerance level \( \varepsilon \) calculated in Example 2.
The asymptotic unbiasedness of \( \hat{c} \) can be understood from the ratio \( \mathbb{E}[\hat{c}]/c \). Let \( Q_\theta = \{ q \mid q_L \leq q \leq q_H \} \), where \( q_L \) and \( q_H \) are the smallest and the largest integers in \( Q_\theta \) respectively. There are three distinct phases:

- When \( q < q_L \), the threshold is low and so most bits become 1. Therefore, \( \gamma_c(q) \rightarrow 0 \) and hence \( \hat{c} \rightarrow \infty \). Thus, \( \mathbb{E}[\hat{c}]/c \rightarrow \infty \) as \( q \) decreases.
- When \( q > q_H \), the threshold high and so most bits become 0. Therefore, \( \gamma_c(q) \rightarrow 1 \) and hence \( \hat{c} \rightarrow 0 \). Thus, \( \mathbb{E}[\hat{c}]/c \rightarrow 0 \) as \( q \) increases.
- When \( q_L \leq q \leq q_H \), the ML estimate \( \hat{c} \) is asymptotically unbiased. Therefore, \( \mathbb{E}[\hat{c}]/c = 1 \).

These three phases are indicated by the coloring in Figure 4.

We can now come back to the question of how an asymptotically unbiased estimate is compared to the oracle estimate. Or more precisely, how much SNR drop will there be if we choose a \( q \in Q_\theta \) but not necessarily \( q = q^* \)? As shown in Figure 4, \( SNR_q(c) \) stays close to \( SNR_{q^*}(c) \) if \( q \) is close to \( q^* \). However, as \( q \) moves away from \( q^* \), the average bit density \( 1 - \mathbb{E}[\gamma_c(q)] \) also changes. This is a very useful phenomenon, because now we can control \( q \) to aim for a specific level of the average bit density. For example, in the case of Figure 4, we observe that for a set of \( q \) such that \( 0.25 \leq 1 - \mathbb{E}[\gamma_c(q)] \leq 0.6 \) (which is reasonably wide), the SNR stays in the range \( 36.15dB \leq SNR_q(c) \leq 36.65dB \) (which is reasonably narrow). In practice, we observe that as long as \( q \) is chosen to ensure \( \gamma_c(q) \approx 0.5 \), the final SNR is close to the optimal SNR.

Choosing the level 0.5 can be justified from an information theoretic perspective. The goal of threshold update is to find a \( q \) such that the information available in \( B_n \) is maximized. When \( q \) is too high or too low, information is lost because most bits are either 1 or 0. The entropy is maximized when the probability of obtaining a zero and the probability of obtaining a one is equal, i.e., 0.5. Additional discussions of the choice of 0.5 and the variation of \( SNR_q(c) \) can be found in the Supplementary material.

Algorithm 1 Bisection Threshold Update Scheme

Initial thresholds \( q_A \) and \( q_B \) such that \( 1 - \gamma_{q_A} > 0.5 \) and \( 1 - \gamma_{q_B} < 0.5 \).

Compute \( q_M = \lceil(q_A + q_B)/2 \rceil \), where \( \lceil \cdot \rceil \) denotes the ceiling operator.

while \( |\gamma_{q_M} - 0.5| < tol \) do
  If \( \gamma_{q_M} < 0.5 \), then set \( q_A = q_M \). Else, set \( q_B = q_A \).
  Compute \( q_M = \lceil(q_A + q_B)/2 \rceil \).
end while

return \( q_M \)

C. Bisection Threshold Update Scheme

The above analysis of the ML estimate leads to our proposed threshold update scheme. The idea is to make sure that the measured bit density \( \gamma_q(c) \) stays close to 0.5. To achieve this goal, we propose a bisection method illustrated in Figure 5 and Algorithm 1. Starting with initial thresholds \( q_A \) and \( q_B \), we check whether the bit density satisfies \( 1 - \gamma_{q_A} > 0.5 \) and \( 1 - \gamma_{q_B} < 0.5 \). If this is the case, then we find a mid point \( q_M = (q_A + q_B)/2 \) and check whether \( 1 - \gamma_{q_M} \) is greater or less than 0.5. If \( 1 - \gamma_{q_M} > 0.5 \), we replace \( q_A \) by \( q_M \), otherwise we replace \( q_B \) by \( q_M \). The process repeats until \( 1 - \gamma_{q_M} \) is sufficiently close to 0.5.

In our proposed threshold update scheme, we assume that the image has been partitioned into \( N \) blocks \( \{B_n \mid n = 1, \ldots, N\} \). Each \( B_n \) contains \( KT \) binary bits and is used to estimate one pixel value \( c_n \). This setting results in \( N \) different thresholds, one for every pixel. To generalize the setting, it is also possible to allow multiple pixels to share a common threshold. Figure 6 shows an example. The advantage of sharing threshold for multiple pixels is that circuits associated with the sensor can be simplified. In terms of performance, since neighboring pixels are typically correlated, sharing the threshold causes little drop in the resulting SNR.

The price that the proposed bisection algorithm has to pay is the number of frames it requires to determine a good \( q \). For every evaluation of \( \gamma_{q_M} \), the sensor has to physically acquire one frame and compute the bit density in each of the
Fig. 6. Threshold \( q \) at different iterations of the bisection algorithm. Note that \( q \) is spatially varying. In this example, one threshold value is shared by \( 16 \times 16 \) shots. The maximum threshold level is \( q_{\text{max}} = 16 \).

\( N \) blocks. Therefore, the more bisection steps we need, the more frames that the sensor has to physically acquire. The rate of convergence of the proposed method and existing methods will be compared in Section V.

D. Extension to High Dynamic Range

While QIS is a photon counting device, it is not designed to count an excessive number of photons. As a result, the maximum number of threshold levels is also limited. When using QIS for high dynamic range (HDR) imaging, appropriate dynamic range extension methods are needed.

We present two ways to enable QIS for HDR imaging:

- **Reduce Duty Cycle.** In the signal processing block diagram shown in Figure 2, we can replace the constant \( \alpha \) by a fraction as \( \alpha \tau \), where \( 0 \leq \tau \leq 1 \) determines the duty cycle of each QIS exposure. For very bright scenes, a low duty cycle will prevent QIS from saturating early.

- **Multiple Measurements.** For dark scenes, multiple measurements can be taken to ensure enough photons over the measurement period. This, however, is different from conventional HDR imaging. In conventional HDR imaging, the multiple shots are taken at different shutter speeds, e.g., 1/8192, 1/2048, 1/512, 1/128, 1/32, 1/8, 1/2 seconds [43], which is redundant. QIS’s multiple shot functions are more similar to burst photography [44]. The amount of acquisition time is significantly less than the conventional HDR imaging.

Both methods can be used for any threshold scheme. The benefit of using our proposed threshold scheme is the amount of dynamic range extension it can support. In Figure 7, we illustrate the total dynamic range that can be covered using 4 multiple measurements at duty cycles \( \tau = 1, \tau = 0.2, \tau = 0.04 \), and \( \tau = 0.008 \). The maximum threshold level is \( q_{\text{max}} = 25 \), and the minimum threshold level is \( q_{\text{min}} = 1 \). It can be seen from the figure that with the optimal threshold \( q^* \), the dynamic range is significantly more than the non-optimal ones. In particular, we observe a 16dB and a 54dB improvement compared to \( q_{\text{min}} = 1 \) and \( q_{\text{max}} = 25 \), respectively.

V. Experimental Results

In this section we evaluate the proposed threshold update scheme by comparing it with existing methods. We consider two evaluation metrics: (1) convergence rate of the threshold update methods; (2) quality of the reconstructed images. The dataset we use is the Berkeley segmentation and benchmark dataset [45], which contains 100 test images of resolution 481 × 321. We also test the threshold update scheme for high dynamic range images using the HDR-Eye dataset [46], [47]. In all experiments, we fix the spatial over-sampling factor as \( K = 4 \times 4 = 16 \), and number of temporal frames as \( T = 13 \). The maximum threshold level is set as \( q_{\text{max}} = 16 \) to ensure that it is realistic for today’s QIS.

A. Convergence

We compare the proposed threshold update scheme with the Markov Chain (MC) adaptation proposed by Hu and Lu [18]. The Markov Chain adaptation models the threshold as a variable with \( 2^L \) states. These \( 2^L \) states can be regarded as \( 2^L \) steps before reaching to the next threshold level. The probability of changing from one state to another is controlled by a parameter \( 1 - \beta \) with \( 0 < \beta < 1 \). When a bit arrives, the state will be updated (increased or decreased) or will stay. Once the state is increased by \( 2^L \) times, the threshold will be increased by one.

When comparing Markov Chain adaptation with the proposed bisection algorithm, one should be aware of the difference between the two methods. Markov Chain adaptation is a per-jot update scheme whereas the proposed bisection algorithm is a per-pixel update scheme. For a pixel with \( K \times K \) jots, Markov Chain adaptation needs \( K^2 \) iterations to update each jot sequentially. In contrast, the proposed bisection algorithm updates a common threshold for all \( K^2 \) jots simultaneously. Thus in practice our bisection algorithm is significantly less complex to implement in hardware than the Markov Chain. In order to take into account of the different forms of updates, we treat the \( K^2 \) iterations of Markov Chain adaptation as one “major iteration” and compare it with the one bisection step of the proposed algorithm.

The first comparison we make is to check the threshold at different jots. Figure 8 shows the results of three typical runs with ground truth threshold \( q^* = 1, 8, 16 \). In this experiment, we generate 100 random binary blocks of size \( K \times K \) and
estimate the threshold at each major iteration. We report the average of these 100 estimates to minimize the randomness of the data. The results show that one iteration of the proposed bisection algorithm works as good as the $K^2$ iterations of the Markov Chain adaptation. In some cases, Markov Chain tends to oscillate whereas the bisection result is stable.

The second comparison we make is to check how close the estimated threshold is compared to the optimal threshold. The optimal threshold $q^*$ is obtained using the oracle scheme. In Figure 9, we plot the mean squared error between the estimated threshold and the oracle threshold. For fairness we show the results of the MSE average over all threshold values in the 100 images from the Berkeley segmentation dataset. One threshold is shared by $K \times K$ jots, and each $K \times K$ jots correspond to one pixel. The result is consistent with the ones shown in Figure 8.

B. Image Reconstruction Quality

The convergence comparison in the previous subsection is only useful to compare threshold update methods that actually return a threshold. In the QIS literature, there are methods that implicitly update the threshold, e.g., the conditional reset method [21]. For comparison with these methods, we have to compare the quality of the image reconstructed from the binary raw data. The image reconstruction is done using the closed-form ML estimate in Section III-A.

We consider three classes of methods:

- Fixed Threshold. Fixed threshold is commonly used in the device literature [5]–[7]. A fixed threshold is a single threshold applied to all pixels in the image. In this experiment, we consider the following choices of fixed thresholds: $q = 1$, $q = 5$, $q = 10$ and $q = 16$.
- Conditional Reset [21]. Conditional reset counts the number of photons and is reset when it is above the threshold. The threshold in conditional reset is sequentially increasing or decreasing. The reconstructed image is obtained by digitally integrating the raw binary frames.
- Proposed Method. As we discussed in Section IV-C, the proposed method can be implemented to let multiple pixels share a common threshold. Thus, in this experiment we consider three sharing strategies: (1) Share a threshold between a neighborhood of $K \times K$ jots (i.e., one threshold for one pixel); (2) Share a threshold between a neighborhood of $K^2 \times K^2$ jots (i.e., one threshold for $K \times K$ pixel); (3) Share a threshold between a neighborhood of $2K^2 \times 2K^2$ jots (i.e., one threshold for $2K \times 2K$ pixels).

The result of the experiment is shown in Table II. The PSNR values reported are averaged over 100 testing images in the Berkeley segmentation dataset. Each image generates 50 random realizations, and the PSNR of an image is averaged over these 50 random realizations to minimize the randomness. As shown in the table, while conditional reset generally performs better than a fixed threshold, it performs significantly worse than the proposed threshold update scheme.

C. Influence of QIS Threshold on HDR Imaging

Since QIS does not have sufficient full well capacity to accumulate photons for HDR imaging, we apply the dynamic

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**Table II**

| Configuration | Average PSNR | Std |
|---------------|--------------|-----|
| Fixed Threshold | q = 1 | 10.30 | 0.01 |
| | q = 5 | 28.80 | 0.04 |
| | q = 10 | 23.22 | 0.02 |
| | q = 16 | 12.95 | 0.01 |
| Conditional Reset [21] | Ascending q sequence | 23.77 | 0.52 |
| | Descending q sequence | 24.95 | 0.53 |
| Proposed Method | $2K^2 \times 2K^2$ | 30.14 | 0.06 |
| | $K^2 \times K^2$ | 31.18 | 0.06 |
| | $K \times K$ | 32.78 | 0.02 |

---
Fig. 10. Bracketed images with different exposure settings. From Left to Right: −2.7, −2, −1.3, −0.7, 0, 0.7, 1.3, 2, and 2.7 EV.

$q = 1$, PSNR = 17.94 dB

$q = 16$, PSNR = 20.77 dB

Proposed, PSNR = 31.46 dB

Fig. 11. The reconstructed HDR images using different thresholds. See supplementary material for additional results.

The result of this experiment is shown in Figure 11. With the proposed threshold update scheme, the reconstructed images achieve the highest PSNR value and visual quality. When $q = 1$, which is too low, the image appears under-exposed. When $q = 16$, which is too high, the image appears over-exposed. The spatially varying property of the proposed method mitigates the issue by allowing multiple thresholds.

As for comparison with the existing HDR imaging methods using CMOS, we argue that the major difference would be the total amount of data acquisition time. However, a fair comparison can only be made when an actual QIS becomes available for the experiment. We are optimistic about the QIS hardware development, and we believe such experimental results can be demonstrated in a near future.

VI. CONCLUSION

Quanta Image Sensor is a new image sensor for high speed, high resolution and high dynamic range imaging. The sensor has a threshold which needs to be carefully adjusted so that the dynamic range can be maximized. We studied the threshold design problem by establishing several theoretical results. First, we showed that an oracle threshold can be obtained assuming that we know the underlying pixel value. Our result showed that the oracle threshold must match with the pixel value in order to maximize the signal to noise ratio. Second, we showed that around the oracle threshold, there exists a set of thresholds that can produce asymptotically unbiased estimates of the pixel value. Within this set of threshold, the signal to noise ratio stays very close to the oracle case. Third, we developed a bisection method to update the threshold scheme. Experimental results showed the effectiveness of our proposed approach compared to the standard approach that uses fixed threshold for all pixels.

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APPENDIX A

A. Proof of Proposition 3

The Fisher Information metric is defined as:

$$I_q(c) \equiv \mathbb{E}_B \left[ -\frac{\partial^2}{\partial c^2} \log \mathbb{P}(B = b; \theta, q) \right],$$

(22)

where $\theta = \alpha c / K$. Using the chain rule, we can derive the Fisher Information as follows

$$I_q(c) = \left( \frac{\alpha}{K} \right)^2 \mathbb{E}_B \left[ -\frac{\partial^2}{\partial \theta^2} \log \mathbb{P}(B = b; \theta, q) \right].$$

(23)

The expectation can be calculated as follows

$$I_q(c) = \left( \frac{\alpha}{K} \right)^2 \left[ -\frac{\partial^2}{\partial \theta^2} \log \mathbb{P}(B = 1; \theta, q) \right] \mathbb{P}(B = 1; \theta, q)$$

$$+ \left( \frac{\alpha}{K} \right)^2 \left[ -\frac{\partial^2}{\partial \theta^2} \log \mathbb{P}(B = 0; \theta, q) \right] \mathbb{P}(B = 0; \theta, q)$$

(24)
Using (7) to differentiate the 1st term, we get:
\[
\frac{\partial^2}{\partial \theta^2} \log P(B = 1; \theta, q) = \frac{\partial^2}{\partial \theta^2} \log (1 - \Psi_q(\theta))
= \frac{R'(1 - \Psi_q(\theta)) - R^2 / \Gamma(q)}{\Gamma(q)(1 - \Psi_q(\theta))^2},
\]
(25)
where \( R = e^{-\theta^{2\gamma-1}} \) and \( R' = \partial R / \partial \theta \). Similarly, the second term is
\[
\frac{\partial^2}{\partial \theta^2} \log P(B = 0; \theta, q) = \frac{\partial^2}{\partial \theta^2} \log \Psi_q(\theta)
= \frac{-R' \Psi_q(\theta) + R^2 / \Gamma(q)}{\Gamma(q)(\Psi_q(\theta))^2}.
\]
(26)
Substitute (25) and (26) in (24) yields
\[
I_q(\theta) = \delta^2 \left[ - \frac{R' \Gamma(q)(1 - \Psi_q(\theta)) - R^2}{\Gamma(q)(1 - \Psi_q(\theta))},
+ \frac{R' \Gamma(q) \Psi_q(\theta) + R^2}{\Gamma^2(q) \Psi_q(\theta)} \right]
= \delta^2 \left( \frac{\alpha}{K} \right)^2 \frac{1}{\Gamma^2(q) \Psi_q(\theta)(1 - \Psi_q(\theta))}.
\]

**B. Proof of Proposition 4**

The lower bound is obtained by observing that the product \( \Psi_q(\theta)(1 - \Psi_q(\theta)) \) attains its maximum value when \( \Psi_q(\theta) = 1/2 \). Substituting with the upper bound \( \Psi_q(\theta)(1 - \Psi_q(\theta)) \leq 1/4 \), we get:
\[
\log(c^2 I_q(c)) = \log \left( \left( \frac{\alpha}{K} \right)^2 \frac{e^{-2\theta^{2\gamma-2}}}{\Gamma^2(q) \Psi_q(\theta)(1 - \Psi_q(\theta))} \right)
= \log \delta^2 \frac{1}{\Gamma^2(q) \Psi_q(\theta)(1 - \Psi_q(\theta))}
\geq \log \delta^2 \frac{4e^{-2\theta^{2\gamma}}}{\Gamma^2(q)}
= 2 \log 2 - 2\theta + 2q \log \theta - 2 \log \Gamma(q)
= 2 \left( \log 2 - \frac{\alpha}{K} + q \log \frac{\alpha}{K} - \log \Gamma(q) \right).
\]

**C. Proof of Proposition 5**

Using the definition of Gamma function \( \Gamma(q) = (q - 1)! \) and \( \theta = \frac{\alpha}{K} \), we can rewrite the lower bound in Proposition 4 as follows.
\[
L_q(c) = 2 \left( \log 2 - \theta + q \log \theta - \log(q - 1)! \right)
= 2 \left( \log 2 - \theta + (q - 1) \log \theta + \log \theta - \log \prod_{k=1}^{q-1} \right)
= 2 \left( \log 2 - \theta - \sum_{k=1}^{q-1} \log(\theta/k) + \log \theta \right)
\]
The only dependence on \( q \) is in the second term, so we take a closer look at it. When \( q - 1 < [\theta] \), all summands \( \log(\theta/k) \) are positive because \( k < [\theta] \). Hence, the total sum increases by increasing \( q \). On the other hand, when \( q - 1 > [\theta] \), we start to add negative summands \( \log(\theta/k) \) because \( k > [\theta] \). Therefore, the total sum decreases on increasing \( q - 1 \) over \([\theta]\). Thus, maximum is obtained at \( q = \lfloor \theta \rfloor + 1 = \lfloor \frac{\alpha}{K} \rfloor + 1 \).

**D. Proof of Proposition 6**

By definition, \( S = \sum_{i=0}^{K-1} \sum_{k=0}^{K-1} B_{k,i} \) is the summation of \( KT \) independent i.i.d. Bernoulli random variables. Therefore, \( S \) is a binomial random variable with parameters \( n = KT \) and \( p \) def \( = 1 - \Psi(\alpha/\delta) \). The mean and variance of a binomial random variable is \( E[S] = np \), and \( Var[S] = np(1 - p) \). Therefore, we have
\[
E[\gamma_q(c)] = 1 - \frac{E[S]}{KT} = \Psi_q \left( \frac{\alpha}{K} \right), \quad \text{and}
\]
\[
Var[\gamma_q(c)] = \frac{Var[S]}{KT^2} = \frac{1}{KT} \Psi_q \left( \frac{\alpha}{K} \right) \left( 1 - \Psi_q \left( \frac{\alpha}{K} \right) \right).
\]

**E. Proof of Proposition 7**

The probability \( P[0 < \gamma_q(c) < 1] \) can be evaluated by checking the complement where \( \gamma_q(c) = 0 \) or \( \gamma_q(c) = 1 \):
\[
P[0 < \gamma_q(c) < 1] = 1 - P[\gamma_q(c) = 0] - P[\gamma_q(c) = 1]
= 1 - P[S = 0] - P[S = KT]
= 1 - \Psi_q(0)^KT - [1 - \Psi_q(\theta)]^KT,
\]
where (a) follows from the fact that \( S \), which is a sum of i.i.d. Bernoulli random variables, is a binomial random variable.

Let \( 0 < \delta < 1 \). If
\[
1 - \left( \frac{\delta}{2} \right)^{1/\delta} \leq \Psi_q(\theta) \leq \left( \frac{\delta}{2} \right)^{1/\delta},
\]
then we have
\[
\Psi_q(\theta)^{KT} < \frac{\delta}{2} \quad \text{and} \quad \left[ 1 - \Psi_q(\theta) \right]^{KT} < \frac{\delta}{2}
\]
Thus, it holds that
\[
1 - \Psi_q(\theta)^{KT} - [1 - \Psi_q(\theta)]^{KT} > 1 - \delta.
\]

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Optimal Threshold Design for Quanta Image Sensor
(Supplementary Material)

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Abstract

This supplementary report provides the following additional information of the main article.

- Derivation of $\text{SNR}_q(c)$ from exposure-referred SNR.
- Properties of the incomplete Gamma function.
- Phase transition under different configurations.
- Additional results for HDR image reconstruction.

I. DERIVATION OF $\text{SNR}_q(c)$ FROM EXPOSURE-REFERRED SNR

In the literature of QIS devices, one metric to quantify the image quality is the exposure-referred signal-to-noise [1]. In image processing, however, exposure-referred SNR is not commonly used. The goal of this section is to show that the SNR we showed in the main article is equivalent to the exposure-referred SNR.

![Block diagram illustrating a QIS with input-output relation output = $F$(input)](image)

To understand the exposure-referred SNR, we have to first understand two common ways of defining a signal to noise ratio. Consider the truncated Poisson part of the QIS model shown in Figure 1. The input to this model is the over-sampled measurement $\theta$. The truncated Poisson process can be considered as a black box function $F$ which takes an input $\theta$ and generates an output $S$, defined as

$$S = \sum_{t=0}^{T-1} \sum_{k=0}^{K-1} B_{k,t},$$

where $B_n = \{B_{k,t} \mid k = 0, 1, \ldots, K - 1, \ t = 0, 1, \ldots, T - 1\}$ is the spatial-temporal block containing all binary bits corresponding to $\theta$. As shown in the main article, the mean and variance of $S$ are

$$\mathbb{E}[S] = KT(1 - \Psi_q(\theta)), \quad \text{Var}[S] = K T \Psi_q(\theta)(1 - \Psi_q(\theta)),$$

respectively.

The first notion of signal-to-noise, which is the one used in CCD and CMOS, is called the output-referred SNR. $\text{SNR}_{\text{OR}}$ is defined as the ratio between the output signal and the photon shot noise. Referring to Figure 1, this is

$$\text{SNR}_{\text{OR}} = \frac{\text{output signal}}{\text{noise}} = \frac{\mathbb{E}[S]}{\sqrt{\text{Var}[S]}} = \sqrt{KT\frac{1 - \Psi_q(\theta)}{\Psi_q(\theta)}},$$

However, $\text{SNR}_{\text{OR}}$ fails to work for QIS because the shot noise is arbitrarily small if all bits are 1 or 0. In [1], Fossum called it squeezing of the noise. If we plot $\text{SNR}_{\text{OR}}$ as a function of $\theta$, then we observe that $\text{SNR}_{\text{OR}}$ approaches to infinity as $\theta$ grows.

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The second notion of signal-to-noise, which is a modification of SNR\textsubscript{OR}, is the exposure-referred SNR. SNR\textsubscript{ER} is the ratio between the exposure signal $\theta$ and the exposure-referred noise. This noise is defined as [1]:

$$\text{Exposure-referred noise} = \frac{d\theta}{d\mathbb{E}[S]} \sqrt{\text{Var}[S]}$$

The factor $\frac{d\theta}{d\mathbb{E}[S]}$ can be considered as the “inverse” transfer function from the output to the input. $\frac{d\theta}{d\mathbb{E}[S]}$ can be determined by taking derivative of the expectation in (2) with respect to $\mathbb{E}[S]$

$$\frac{d\mathbb{E}[S]}{d\mathbb{E}[S]} = \frac{dKT(1 - \Psi_q(\theta))}{d\mathbb{E}[S]}$$

Using chain rule, we observe that

$$1 = -KT \frac{d\Psi_q(\theta)}{d\mathbb{E}[S]} \frac{d\theta}{d\mathbb{E}[S]}$$

Since $\frac{d\Psi_q(\theta)}{d\theta} = \frac{e^{-\theta q - 1}}{\Gamma(q)}$, it holds that

$$1 = -KT \left(\frac{e^{-\theta q - 1}}{\Gamma(q)}\right) \frac{d\theta}{d\mathbb{E}[S]}$$

Hence,

$$\frac{d\theta}{d\mathbb{E}[S]} = \frac{\Gamma(q)}{KT e^{-\theta q - 1}}$$

The exposure-referred SNR is defined as

$$\text{SNR}_{\text{ER}} = \frac{\text{exposure signal}}{\text{exposure-referred noise}} = \frac{\theta}{\sqrt{\text{Var}[S]} \frac{d\theta}{d\mathbb{E}[S]}} = \frac{e^{-\theta q}}{\Gamma(q)} \sqrt{\frac{KT}{\Psi_q(\theta)(1 - \Psi_q(\theta))}}.$$ 

Taking logarithm shows that SNR\textsubscript{ER} is identical to the SNR derived from the Fisher Information shown in the main article.
II. PROPERTIES OF THE INCOMPLETE GAMMA FUNCTION

In the main article, we used the incomplete Gamma function for QIS analysis. In this section, we provide more details about the properties of the incomplete Gamma function.

First, we recall that the normalized upper incomplete Gamma function $\Psi_q : \mathbb{R}^+ \rightarrow [0, 1]$ is defined as

$$\Psi_q(\theta) \overset{\text{def}}{=} \frac{1}{\Gamma(q)} \int_{\theta}^{\infty} t^{q-1} e^{-t} dt, \quad \text{for } \theta > 0, \ q \in \mathbb{N}. \quad (4)$$

where $\Gamma(q) = (q-1)!$ is the standard Gamma function.

In this equation, we note that $\Psi_q(\theta)$ depends on two variables: $q$ and $\theta$.

- As a function of $\theta$, as we showed in the main article, $\Psi_q(\theta)$ is a monotonically decreasing function of $\theta$ because the derivative is negative:

$$\frac{d}{d\theta} \Psi_q(\theta) = \frac{-\theta^{q-1} e^{-\theta}}{\Gamma(q)} < 0.$$

However, $\Psi_q(\theta)$ is very close to 1 when $\theta$ is small, and is very close to 0 when $\theta$ is large. Therefore, there exists a range of $\theta$ in which $\Psi_q(\theta)$ can attain a reasonably good inverse. We define this set as the $\theta$-admissible set

$$\Theta_q \overset{\text{def}}{=} \{ \theta \mid \epsilon \leq \Psi_q(\theta) \leq 1 - \epsilon \}, \quad (5)$$

for any fixed $q$ and a tolerance level $\epsilon$. An illustration of $\Theta_q$ is shown in Figure 3.

- As a function of $q$. The incomplete Gamma function $\Psi_q(\theta)$ can also be considered as a function of $q$. In this case, $\Psi_q(\theta)$ is only defined for integer values of $q$. We illustrate the behavior of $\Psi_q(\theta)$ as a function of $q$ in Figure 3. The set of $q$ in which $\Psi_q(\theta)$ is sufficiently away from 0 and 1 is defined as the $q$-admissible set

$$Q_q \overset{\text{def}}{=} \{ q \mid \epsilon \leq \Psi_q(\theta) \leq 1 - \epsilon \}. \quad (6)$$

III. PHASE TRANSITION UNDER DIFFERENT CONFIGURATIONS

In the main article, we showed the phase transition behavior of the ML estimate using $K = 4$, $T = 50$, and $\delta = 2 \times 10^{-4}$. In this section, we study the effect of changing $K$, $T$, and $\delta$ on the phase transition region width.

As a function of $T$. Figure 4-Figure 5 illustrate the phase transition behavior when $T = 10, 25, 50, 100$. As $T$ increases, the width of the green region increases. However, if we fix the range of the bit density $1 - \mathbb{E}[\gamma_q(c)]$, we observe that the SNR does not vary significantly even as $T$ changes.

As a function of $K$. The spatial oversampling $K$ affects both the threshold $q^*(c) = \lfloor \alpha c / K \rfloor + 1$ and the phase transition width. Figure 6(a) illustrates the behavior of the threshold $q^*$ as a function of $K$. As $K$ increases, $q^*$ decreases. However, the optimal $q^*$ still stays within the set $Q_q$. 

Fig. 3. $\Psi_q(\theta)$ as a function of $\theta$ and $q$. In defining, $Q_\theta$ and $\Theta_q$, we set $\epsilon = 0.01$. 

(a) $\Psi_q(\theta)$ vs. $\theta$  
(b) $\Psi_q(\theta)$ vs. $q$
As a function of $\delta$. The constant $\delta$ is used to define the set $Q_{\theta}$:

$$Q_{\theta} \overset{\text{def}}{=} \left\{ q \mid 1 - \left( \frac{\delta}{2} \right)^{\frac{1}{KT}} \leq \Psi_q(\theta) \right\}.$$  \hspace{1cm} (7)

The constant $\delta$ is the tolerance level. When $\delta$ increases, the size of the set $Q_{\theta}$ should also increase. This result is shown in Figure 6(b).

Using the closed form expression of the average bit density $1 - \Psi_q(\theta)$, we can calculate the average bit density at the optimal threshold $q^* = \lceil \theta \rceil + 1$, which is shown in Figure 7. We notice that as long as $\theta \geq 1$, the average bit density is between 0.264 and 0.630. Within this range, we observe from Figure 4–Figure 5 that the SNR does not vary significantly if the estimated threshold is deviated from the optimal threshold. This observation relaxes the requirement of the bisection method from obtaining the exact optimal threshold to obtaining a threshold that make the bit density equal to 0.5. Since $0.5 \in [0.264, 0.630]$, we guarantee to achieve an SNR which is sufficiently close to the optimal SNR.

Controlling $\theta \geq 1$ can be achieved by tuning the constant $\alpha$. Tuning $\alpha$ can be hardware-implemented by increasing the exposure period. Intuitively what $\theta \geq 1$ requires is that the average number of impinging photons per jot must be at least one. If $\theta$ is less than one, then most bits will become zeros. Increasing exposure period (i.e., increasing $\alpha$) will ensure sufficient number of photons.

In this section, we show more results for HDR image reconstruction using our method compared to the fixed threshold approach. In addition, we compare with the checkerboard Q-map proposed in [2]. This Q-map is simply an alternation of $q = 1$ and $q = q_{\text{max}}$ in a checkerboard pattern. Figures 8, 9, and 10 show reconstructed HDR images using adapted Q-map by the bisection algorithm, and fixed Q-maps with low threshold ($q = 1$) and high threshold ($q_{\text{max}} = 16$). The spatial and temporal oversampling factors are $K = 4$, and $T = 13$, respectively. Sensor gain is $\alpha = K^2/(q_{\text{max}} - 1)$.

IV. SUPPLEMENTARY HDR RESULTS

In this section, we show more results for HDR image reconstruction using our method compared to the fixed threshold approach. In addition, we compare with the checkerboard Q-map proposed in [2]. This Q-map is simply an alternation of $q = 1$ and $q = q_{\text{max}}$ in a checkerboard pattern. Figures 8, 9, and 10 show reconstructed HDR images using adapted Q-map by the bisection algorithm, and fixed Q-maps with low threshold ($q = 1$) and high threshold ($q_{\text{max}} = 16$). The spatial and temporal oversampling factors are $K = 4$, and $T = 13$, respectively. Sensor gain is $\alpha = K^2/(q_{\text{max}} - 1)$.

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Fig. 5. Phase transition for $T = 50$ and $T = 100$. SNR range is shown for average bit density $1 - E[\gamma_q(c)]$ in the range $[0.264, 0.630]$. For all cases, we set $\delta = 2 \times 10^{-4}$, and $K = 4$.

Fig. 6. (a) The threshold $q$ and $Q_\theta$ as $K$ increases. (b) The width of $Q_\theta$ as $KT$ and $\delta$ changes.

Fig. 7. Average bit density $1 - E[\gamma_q(c)]$ calculated at optimal threshold $q^* = \lfloor \theta \rfloor + 1$. 
Fig. 8. Reconstructed HDR images using different threshold maps

- Ground Truth
  - $q = 1$, PSNR = 15.94 dB
  - $q = 16$, PSNR = 20.77 dB

- Checkerboard
  - PSNR = 24.91 dB

- Proposed
  - PSNR = 29.97 dB
Fig. 9. Reconstructed HDR images using different threshold maps

Ground Truth

$q = 1$, PSNR = 17.94 dB

$q = 16$, PSNR = 20.77 dB

Checkerboard, PSNR = 28.27 dB

Proposed, PSNR = 31.46 dB

Fig. 9. Reconstructed HDR images using different threshold maps
Fig. 10. Reconstructed HDR images using different threshold maps

- Ground Truth
  - $q = 1$, PSNR = 15.74 dB
  - $q = 16$, PSNR = 20.01 dB

- Checkerboard
  - PSNR = 27.17 dB

- Proposed
  - PSNR = 31.65 dB

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Fig. 10. Reconstructed HDR images using different threshold maps

- Ground Truth
  - $q = 1$, PSNR = 15.74 dB
  - $q = 16$, PSNR = 20.01 dB

- Checkerboard
  - PSNR = 27.17 dB

- Proposed
  - PSNR = 31.65 dB