ČECH COHOMOLOGY OF SEMIRING SCHEMES

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Abstract. A semiring scheme generalizes a scheme in such a way that the underlying algebra is that of semirings. We generalize Čech cohomology theory and invertible sheaves to semiring schemes. In particular, when $X = \mathbb{P}^1_{\mathbb{Q}_{\text{max}},}$ the (suitably defined) projective line over the semifield $\mathbb{Q}_{\text{max}} = (\mathbb{Q} \cup \{-\infty\}, \max, +)$, this generalized framework provides the result which is coherent with the classical Čech cohomology computation of a projective line.

1. Introduction

In this paper, our main interest is Čech cohomology theory of a semiring scheme which is a generalization of a scheme based on commutative semirings. A notion of semiring schemes has been known (cf. [2], [8]), however there are very few results on semiring schemes. In particular, sheaves and homological methods on semiring schemes have been never considered.

In [1], J. Giansiracusa and N. Giansiracusa proved that one can associate a semiring scheme $X$ to a tropical variety $Y$ in such a way that $Y$ can be identified with $X(\mathbb{R}_{\max},)$, the set of ‘$\mathbb{R}_{\max}$-rational points’ of $X$, where $\mathbb{R}_{\max} = (\mathbb{R} \cup \{-\infty\}, \max, +)$ is a tropical semifield with a maximum convention. This opens the door to approach tropical geometry by means of semiring schemes and, to this end, one needs to better comprehend semiring schemes in perspective of both $\mathbb{F}_1$-geometry and tropical geometry. This paper is organized as follows: In §2, we quickly review basic properties of semiring schemes and then in §3 we use a tensor product of semimodules defined in [9] to confirm that a construction of Picard groups can be generalized to semiring schemes. Finally, in §4, we generalize Čech cohomology theory to semiring schemes by appealing to the framework of A. Patchkoria in [10]. The basic idea is to replace a coboundary map with a pair of coboundary maps. The following is the main result of the paper.

Theorem. (cf. Proposition 4.4, Theorem 4.16, Example 4.17) Let $X$ be a semiring scheme.

1. $\Gamma(X, \mathcal{O}_X) \simeq \check{H}^0(X, \mathcal{O}_X)$.
2. $\text{Pic}(X) \simeq \check{H}^1(X, \mathcal{O}_X^*)$.
3. Let $X$ be the (suitably defined) projective line $\mathbb{P}^1_{\mathbb{Q}_{\max}}$ over the semifield $\mathbb{Q}_{\max}$. Then:

   \[
   \check{H}^0(X, \mathcal{O}_X) \simeq \mathbb{Q}_{\max}, \quad \check{H}^n(X, \mathcal{O}_X) = 0 \text{ for } n \geq 2, \quad \text{and } \check{H}^1(X, \mathcal{U}, \mathcal{O}_X^*) \simeq \mathbb{Z},
   \]

   where $\mathcal{U}$ is an open covering of $X$ by two affine charts.

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2. Review: Construction of semiring schemes

Throughout this section, all semirings are assumed to be commutative. A (multiplicatively) cancellative semiring $M$ is a semiring such that: $\forall x, y, z \in M$, $xy = xz$ implies $y = z$ if $x \neq 0_M$. Note that this is different from $M$ having no (multiplicative) zero-divisor due to the absence of additive inverses. For an introduction to semiring theory, we refer the reader to [2] (also, see Appendix A for the basic definitions).

Recall that for a semiring $M$, by a prime ideal $p$ of $M$ we mean an ideal $p$ of a semiring $M$ such that if $xy \in p$, then $x \in p$ or $y \in p$. The set $X = \text{Spec } M$ is a topological space equipped with Zariski topology. Then, as in the classical case, we can implement the structure sheaf $\mathcal{O}_X$ of $X$ to obtain a semiring scheme. The following is well known in the theory of semiring schemes (cf. [8], [11]).

**Proposition 2.1.** Let $M$ be a semiring and $X = \text{Spec } M$ be an affine semiring scheme.

1. For a non-zero element $f \in M$, we have $M_f \simeq \mathcal{O}_X(D(f))$. In particular, $M \simeq \mathcal{O}_X(X)$.
2. For $p \in X = \text{Spec } M$, the stalk $\mathcal{O}_{X,p}$ of the sheaf $\mathcal{O}_X$ is isomorphic to the local semiring $M_p$.
3. The opposite category of affine semiring schemes is equivalent to the category of semirings.

**Remark 2.2.** In the papers, [4], [5], [6], P. Lescot considered a topological space of prime congruences instead of prime ideals. Let $M$ be a semiring. A congruence on $M$ is an equivalence relation preserving operations of $M$. More precisely, if $x \sim y$ and $a \sim b$, then $xa \sim yb$ and $x + a \sim y + b \ \forall x, y, a, b \in M$. A prime congruence is a congruence $\sim$ which satisfies the following condition: if $xy \sim 0$, then $x \sim 0$ or $y \sim 0$. In the theory of commutative rings, there is a one to one correspondence between congruences on a commutative ring $A$ and ideals of $A$. However, such correspondence no longer holds for semirings. In general, one only obtains an ideal from a congruence as follows:

$$I_\sim := \{a \in M \mid a \sim 0\}.$$  

(1)

The main advantage of a congruence over an ideal is that in the theory of semirings a quotient by an ideal does not behave well, however, a quotient by a congruence behaves well.

Similar to the construction of a prime spectrum $\text{Spec } M$, one can define the set $X$ of prime congruences and impose Zariski topology on $X$. Each ideal $I_\sim$ arises from a congruence $\sim$ as in [11] is called a saturated ideal. In his papers, Lescot had not considered a structure sheaf on the topological space $X$. However, one can mimic the construction of a structure sheaf on semiring schemes by using saturated prime ideals to construct a structure sheaf for a topological space of congruence spectra. This might give a notion of a congruence semiring scheme $(X, \mathcal{O}_X)$. It appears, however, that a semiring $\mathcal{O}_X(X)$ of global sections of an ‘affine congruence semiring scheme $(X, \mathcal{O}_X)$’ might not be isomorphic to a semiring $M$ since a naive generalization of Hilbert’s Nullstellensatz (which is the main ingredient in the proof of the classical case) does not hold in the case of congruences. If every ideal of a semiring $M$ is saturated, then an affine semiring scheme induced from $M$ and an affine congruence semiring scheme induced from $M$ are isomorphic as locally semiringed spaces. For example, this is the case when $M$ is a commutative ring.

3. Picard group of a semiring scheme

For a given semiring scheme $X$, one defines a sheaf of $\mathcal{O}_X$-semimodules to be a sheaf $\mathcal{F}$ of sets on $X$ such that $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-semimodule, and restriction maps $\mathcal{F}(U) \rightarrow \mathcal{F}(V)$ and $\mathcal{O}_X(U) \rightarrow \mathcal{O}_X(V)$ are compatible for open sets $V \subseteq U$ of $X$. A morphism of sheaves
of $\mathcal{O}_X$-semimodules is also defined as in the classical case. In particular, we call a sheaf $\mathcal{L}$ of $\mathcal{O}_X$-semimodules invertible if $\mathcal{L}$ is locally isomorphic to $\mathcal{O}_X$.

**Example 3.1.** Clearly, a structure sheaf $\mathcal{O}_X$ is a sheaf of $\mathcal{O}_X$-semimodules. Furthermore, let $\mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}_X$-semimodules. Then, as in the classical case, the sheaf $\text{Hom}(\mathcal{F}, \mathcal{G})$ becomes a sheaf of $\mathcal{O}_X$-semimodules.

Next, we construct the tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ of sheaves of $\mathcal{O}_X$-semimodules. Note that when we define a tensor product of semimodules, we need to be careful. There are several ways one can generalize the classical construction of a tensor product to semimodules, and some generalizations might not work well. For example, the generalization as in the Golan’s book [2] is not a proper generalization. In fact, if we follow the generalization of a tensor product in [2], for a semiring $A$ and an $A$-semimodule $M$, we have

$$A \otimes A M \simeq (M/\sim),$$

(2)

where $\sim$ is a congruence relation on $M$ such that $a \sim b$ if and only if $\exists c \in M$ such that $a + c = b + c$. When $A$ is an idempotent semiring (in which our main interest lies), the tensor product of [2] does not behave well. For example, we have

$$\mathbb{Z}_{\text{max}} \otimes \mathbb{Z}_{\text{max}} \mathbb{R}_{\text{max}} \simeq \{0\}.$$

Furthermore, we have

$$\{0\} = \text{Hom}(\mathbb{Z}_{\text{max}} \otimes \mathbb{Z}_{\text{max}}, \mathbb{Z}_{\text{max}}) \neq \text{Hom}(\mathbb{Z}_{\text{max}}, \text{Hom}(\mathbb{Z}_{\text{max}}, \mathbb{Z}_{\text{max}})) = \mathbb{Z}_{\text{max}}.$$

This implies that we cannot have the Hom-Tensor duality at the level of sheaves of $\mathcal{O}_X$-semimodules with the Golan’s notion. Therefore, one cannot generalize directly the construction of Picard groups. To this end, we use the definition of a tensor product which is proposed in [9]. Then we recover useful isomorphisms which one can expect from a tensor product. In particular, we have $R \otimes_R M \simeq M \otimes_R R \simeq M$ and $\text{Hom}(M \otimes_R N, P) \simeq \text{Hom}(M, \text{Hom}(N, P))$ for a semiring $R$ and $R$-semimodules, $M, N, P$. By appealing to such results, we define the Picard group $\text{Pic}(X)$ of a semiring scheme $X$.

**Definition 3.2.** Let $X$ be a semiring scheme and $\mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}_X$-semimodules. We define $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ to be the sheafification of the presheaf $\mathcal{H}$, where $\mathcal{H}(U) = \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$ for each open set $U$ of $X$ and the tensor product is as in [9].

**Remark 3.3.** Note that one can easily observe that $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is indeed a sheaf of $\mathcal{O}_X$-semimodules.

The following are statements which can be directly generalized from the classical statements (mainly due to the fact that the existence of additive inverses is not used in the classical proofs).

**Lemma 3.4.** Let $X$ be a semiring scheme. Let $\mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}_X$-semimodules and $\mathcal{L}$ be an invertible sheaf of $\mathcal{O}_X$-semimodules on $X$.

(1) \hspace{1cm} (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_p \simeq \mathcal{F}_p \otimes_{\mathcal{O}_X,p} \mathcal{G}_p, \hspace{0.5cm} \forall p \in X$

(2) \hspace{1cm} \text{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{L} \simeq \text{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L})$

(3) \hspace{1cm} \text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) \simeq \mathcal{O}_X$

(4) \hspace{1cm} \text{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}) \simeq \mathcal{O}_X.$
4. Čech cohomology

In [10], A.Patchkoria generalized the notion of a chain complex of modules to semimodules by realizing that an alternating sum can be written as the sum of two sums in such a way that one stands for a positive sum and the other a negative sum. In this section, we use this idea to define Čech cohomology with values in sheaves of semimodules. Then we compute the simple example of the projective line $\mathbb{P}^1$ over $\mathbb{Q}_{\text{max}}$.

**Remark 4.1.** One might be also interested in developing the sheaf cohomology for semiring schemes via derived functors. In [3], we proved that an idempotent semimodule (as well as a sheaf of idempotent semimodules on a semiring scheme) has a (properly defined) injective resolution. However, different from the classical case, the global section functor is not left exact. Moreover, it is unclear whether any two injective resolutions are homotopic (in a suitable sense) or not. There is some evidence that the derived functor approach to the sheaf cohomology might not be a good direction to pursue. More precisely, in [7], Lorscheid computed the sheaf cohomology of the projective line $\mathbb{P}^1$ over $\mathbb{F}_1$ via an injective resolution and found that the computation is not in accordance with the classical result. For example, $\text{H}^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$ is an infinite-dimensional $\mathbb{F}_1$-vector space whereas classically, we have $\text{H}^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0$. Although this is the case of a monoid scheme, this suggests that one might have to look for other possible approaches.

**Definition 4.2.** (cf. [10, Definition 1.10])

1. Let $R$ be a semiring. A cochain complex (of $R$-semimodules) $X = \{X^n, \partial^+_n, \partial^-_n\}_{n \in \mathbb{Z}}$ consists of $R$-semimodules $X^n$ and $R$-homomorphisms $\partial^+_n, \partial^-_n$ as follows:

$$X : \cdots \xrightarrow{\partial^-_{n-2}} X^{n-1} \xrightarrow{\partial^+_{n-1}} X^n \xrightarrow{\partial^-_{n-1}} X^{n+1} \xrightarrow{\partial^+_{n}} \cdots, \quad n \in \mathbb{Z},$$

which satisfies the following condition:

$$\partial^+_{n+1} \circ \partial^+_n + \partial^-_{n+1} \circ \partial^-_n = \partial^-_{n+1} \circ \partial^+_n + \partial^+_{n+1} \circ \partial^-_n, \quad n \in \mathbb{Z}. \quad (4)$$

2. For a cochain complex $X$, one defines the following $R$-semimodule:

$$Z^n(X) := \{x \in X^n \mid \partial^+_n(x) = \partial^-_n(x)\}$$

as $n$-cocycles, and the $n$-th cohomology as an $R$-semimodule

$$H^n(X) := Z^n(X)/\rho^n,$$

where $\rho^n$ is the equivalence relation generated by the boundary relations.

(5) The sheaf $\text{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$ is also an invertible sheaf of $\mathcal{O}_X$-semimodules. Furthermore, we have the following isomorphism:

$$\text{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{L} \simeq \mathcal{O}_X. \quad (3)$$

It follows from the above that the set $\text{Pic}(X)$ of isomorphism classes of invertible sheaves (of $\mathcal{O}_X$-semimodules) on a semiring scheme $X$ is indeed a group with a group operation $\otimes_{\mathcal{O}_X}$ as in the classical case. In other words, in a monoid of sheaves (with a binary operation given by a tensor product) of $\mathcal{O}_X$-semimodules, the group of invertible elements are indeed sheaves which are locally isomorphic to $\mathcal{O}_X$. This justifies our term of an invertible sheaf on a semiring scheme. In the next section, we will construct Čech cohomology theory for a semiring scheme $X$, and derive the following classical result:

$$\text{Pic}(X) \simeq \check{H}^1(X, \mathcal{O}_X^*).$$
where \( \rho^n \) is a congruence relation on \( Z^n(X) \) such that \( x\rho^ny \) if and only if

\[
x + \partial_{n-1}^+(u) + \partial_{n-1}^-(v) = y + \partial_{n-1}^+(v) + \partial_{n-1}^-(u) \text{ for some } u, v \in X^{n-1}.
\]  

(5)

Suppose that \( X = \{X^n, d^+_n, d^-_n\} \) and \( Y = \{Y^n, \partial^+_n, \partial^-_n\} \) are cochain complexes of semimodules. Then, by a \( \pm \)-morphism from \( X \) to \( Y \) one means a collection \( f = \{f^n\} \) of homomorphisms of semimodules which satisfies the following condition:

\[
f^{n+1} \circ d^+_n = \partial^+_n \circ f^n, \quad f^{n+1} \circ d^-_n = \partial^-_n \circ f^n.
\]  

(6)

In [10], it is proven that a \( \pm \)-morphism \( f = \{f^n\} \) from \( X = \{X^n, d^+_n, d^-_n\} \) to \( Y = \{Y^n, \partial^+_n, \partial^-_n\} \) induces a canonical homomorphism \( H^n(f) \) of cohomology semimodules as follows:

\[
H^n(f) : H^n(X) \to H^n(Y), \quad [x] \mapsto [f^n(x)], \quad n \in \mathbb{Z},
\]  

(7)

where \([x]\) is the equivalence class of \( x \in Z^n(X) \) in \( H^n(X) \).

**Remark 4.3.** As pointed out in [10], a sequence \( G = \{G^n, d^+_n, d^-_n\} \) of modules is a cochain complex in the sense of Definition 4.2 if and only if \( G' = \{G^n, \partial^n := d^+_n - d^-_n\} \) is a cochain complex of modules in the classical sense. Clearly, in this case, the cohomology semimodules of \( G \) as in Definition 4.2 is the cohomology modules of \( G' \) in the classical sense.

By means of Definition 4.2 we introduce Čech cohomology with values in sheaves of semimodules which generalizes the classical construction. Let \( R \) be a semiring, \( X \) be a topological space, and \( F \) be a sheaf of \( R \)-semimodules on \( X \). Suppose that \( \mathcal{U} = \{U_i\}_{i \in I} \) is an open covering of \( X \), where \( I \) is a totally ordered set. Let \( U_{i_0,i_1,...,i_p} := U_{i_0} \cap ... \cap U_{i_p} \). We define the following set:

\[
C^n = C^n(\mathcal{U}, \mathcal{F}) := \prod_{i_0 < i_1 < \cdots < i_n} \mathcal{F}(U_{i_0,i_1,...,i_n}), \quad n \in \mathbb{N}.
\]  

(8)

Let \( x_{i_0,...,i_n} \) be the coordinate of \( x \in C^n \) in \( \mathcal{F}(U_{i_0,i_1,...,i_n}) \). The differentials are given as follows:

\[
(d^+_n(x))_{i_0,i_1,...,i_{n+1}} = \sum_{k=0, k \text{ even}}^{n+1} x_{i_0,...,\hat{i}_k,...,i_{n+1}}|_{U_{i_0,i_1,...,i_{n+1}}},
\]  

(9)

\[
(d^-_n(x))_{i_0,i_1,...,i_{n+1}} = \sum_{k=0, k \text{ odd}}^{n+1} x_{i_0,...,\hat{i}_k,...,i_{n+1}}|_{U_{i_0,i_1,...,i_{n+1}}},
\]  

(10)

where the notation \( \hat{i}_k \) means that we omit that index. One can directly use the classical computation to show that \( C = \{C^n, d^+_n, d^-_n\} \) is a cochain complex in the sense of Definition 4.2. We denote the \( n \)-th cohomology semimodule (with respect to an open covering \( \mathcal{U} \)) of \( C \) by \( \check{H}^n(\mathcal{U}, \mathcal{F}) \).

**Proposition 4.4.** Let \( R \) be a semiring, \( X \) be a topological space, and \( \mathcal{F} \) be a sheaf of \( R \)-semimodules on \( X \). Let \( \mathcal{U} \) be an open covering of \( X \). Then we have

\[
\check{H}^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X).
\]

**Proof.** By the definition, we have \( \check{H}^0(\mathcal{U}, \mathcal{F}) := Z^0(\mathcal{U}, \mathcal{F})/\rho^0 \). Moreover, \( x\rho^0y \iff x + d^+_1(u) + d^-_1(v) = y + d^+_1(v) + d^-_1(u) \) for some \( u, v \in C^{-1} \). Since \( C^{-1} := 0 \), we have \( x\rho^0y \iff x = y \). It follows that \( \check{H}^0(\mathcal{U}, \mathcal{F}) = Z^0(\mathcal{U}, \mathcal{F}) \). Consider the following:

\[
C^0 = \prod_{i \in I} \mathcal{F}(U_i) \xrightarrow{d^+_0} C^1 = \prod_{i < j \in I} \mathcal{F}(U_{ij}),
\]

where \( d^+_0 \) is the product of maps \( \mathcal{F}(U_j) \to \mathcal{F}(U_{ij}) \) induced by the inclusion \( U_{ij} \to U_i \) and \( d^-_0 \) is the product of maps \( \mathcal{F}(U_i) \to \mathcal{F}(U_{ij}) \) induced by the inclusion \( U_{ij} \to U_i \). Clearly, we
have \( Z^0(\mathcal{U}, \mathcal{F}) \subseteq C^0 \). It follows from the inclusion \( U_i \hookrightarrow X \) that we have a homomorphism 
\[ r = (r_i) : \mathcal{F}(X) \rightarrow \mathcal{F}(U_i), \]
hence the following homomorphism:
\[ s = (s_i) : \mathcal{F}(X) \rightarrow \mathcal{F}(U_i), \]
Since \( \mathcal{F} \) is a sheaf, we have \( \text{Img}(r) \subseteq Z^0(\mathcal{U}, \mathcal{F}) \). Conversely, suppose that 
\[ y = (y_i) \in Z^0(\mathcal{U}, \mathcal{F}) = \{ y \in C^0 = \prod_{i \in I} \mathcal{F}(U_i) \mid d_0^1(y) = d_0^0(y) \}. \]
Then we have \( y_i|_{U_{ij}} = y_j|_{U_{ij}} \). It follows that there exists a unique global section \( y_X \in \mathcal{F}(X) \) such that \( (y_X)|_{U_i} = y_i \). Consider the following map:
\[ s : Z^0(\mathcal{U}, \mathcal{F}) \rightarrow \mathcal{F}(X), \quad y \mapsto y_X. \]
Then \( s \) is clearly an \( R \)-homomorphism. Furthermore, \( r \circ s \) and \( s \circ r \) are identity maps. This shows that \( \hat{H}^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X) \) for an open covering \( \mathcal{U} \) of \( X \).

**Proposition 4.5.** Let \( R \) be a semiring, \( X \) be a topological space, and \( \mathcal{F} \) be a sheaf of \( R \)-semimodules on \( X \). Let \( \mathcal{U} \) be an open covering of \( X \) which consists of \( n \) proper open subsets of \( X \). Then \( \hat{H}^m(\mathcal{U}, \mathcal{F}) = 0 \) for \( m \geq n \).

**Proof.** The proof is identical to that of the classical case since \( C^m = 0 \) for \( m \geq n \).

Recall that a covering \( \mathcal{V} = \{V_j\}_{j \in J} \) of a topological space \( X \) is a refinement of a covering \( \mathcal{U} = \{U_i\}_{i \in I} \) if there exists a map \( \sigma : J \rightarrow I \) such that \( V_j \subseteq U_{\sigma(j)} \) for each \( j \in J \). Suppose that \( X^n := C^n(\mathcal{U}, \mathcal{F}) \) and \( Y^n := C^n(\mathcal{V}, \mathcal{F}) \). Then the map \( \sigma \) induces the following \( \pm \)-morphism:
\[ \sigma^n : X^n \rightarrow Y^n, \quad \sigma^n(x)_{j_0, \ldots, j_n} = x_{\sigma(j_0), \ldots, \sigma(j_n)}|_{V_{j_0} \ldots, j_n}. \tag{11} \]
In fact, let \( X = \{X^n, d_n^+, d_n^-\} \) and \( Y = \{Y^n, \partial_n^+, \partial_n^-\} \). We have 
\[ (\sigma^n \circ d_n^+(x))_{j_0, \ldots, j_n+1} = (d_n^+(x))_{\sigma(j_0), \ldots, \sigma(j_n+1)}|_{V_{j_0} \ldots, j_n+1} \]
\[ = \left( \sum_{k=0, k \text{ is even}}^{n+1} x_{\sigma(j_0), \ldots, \sigma(j_k), \sigma(j_{n+1})}|_{U_{\sigma(j_0), \ldots, \sigma(j_{n+1})}} \right)_{V_{j_0} \ldots, j_{n+1}} \]
\[ = \left( \sum_{k=0, k \text{ is even}}^{n+1} x_{\sigma(j_0), \ldots, \sigma(j_k), \sigma(j_{n+1})} \right)_{V_{j_0} \ldots, j_{n+1}} \]
\[ = (\partial_n^+ \circ \sigma^n(x))_{j_0, \ldots, j_{n+1}}. \]
Hence, we obtain \( \sigma^{n+1} \circ d_n^+ = \partial_n^+ \circ \sigma^n \). Similarly one can prove that \( \partial_n^{n+1} \circ d_n^- = \partial_n^- \circ \sigma^n \).

The \( \pm \)-morphism \( \sigma = \{\sigma^n\} \) induces a homomorphism, \( \hat{H}^n(\mathcal{U}, \mathcal{F}) \rightarrow \hat{H}^n(\mathcal{V}, \mathcal{F}) \).

The collection of open coverings of a topological space \( X \) becomes a directed system (with a refinement as a partial order). Since (co)limits exist in the category of semimodules, the following definition is well defined.

**Definition 4.6.** Let \( R \) be a semiring. Let \( X \) be a topological space and \( \mathcal{F} \) be a sheaf of \( R \)-semimodules on \( X \). We define the \( n \)-th Čech cohomology of \( X \) with values in \( \mathcal{F} \) as follows:
\[ \hat{H}^n(X, \mathcal{F}) := \lim_{\mathcal{U}} \hat{H}^n(\mathcal{U}, \mathcal{F}). \]

Note that from Proposition 4.4 we have \( \hat{H}^0(X, \mathcal{F}) = \mathcal{F}(X) \).
Example 4.7. Consider the projective line $X = \mathbb{P}^1_{\mathbb{Q}_{\text{max}}}$ over $\mathbb{Q}_{\text{max}}$. More precisely, we consider $X$ as the semiring scheme with two open affine charts $U_0 := \text{Spec } \mathbb{Q}_{\text{max}}[T]$ and $U_1 := \text{Spec } \mathbb{Q}_{\text{max}}[\frac{1}{T}]$ glued along $T \mapsto \frac{1}{T}$. As in the classical case, one observes that $\mathcal{O}_X(X) = \mathbb{Q}_{\text{max}}$.

From Proposition 4.4, we have $\check{H}^0(X, \mathcal{O}_X) = \mathbb{Q}_{\text{max}}$. Furthermore, since $X$ has the open covering $\mathcal{U} = \{U_0, U_1\}$ which consists of two proper open subsets of $X$, we have $\check{H}^1(X, \mathcal{O}_X) = 0$ for $n \geq 2$ from Proposition 4.3. Finally, with respect to the covering $\mathcal{U} = \{U_0, U_1\}$, we have

$$C : \mathcal{O}_X(U_0) \oplus \mathcal{O}_X(U_1) \xrightarrow{d_0} \mathcal{O}_X(U_0) \xrightarrow{d_1} 0.$$ 

In other words, we have

$$C : \mathbb{Q}_{\text{max}}[T] \oplus \mathbb{Q}_{\text{max}}[\frac{1}{T}] \xrightarrow{d_0} \mathbb{Q}_{\text{max}}[\frac{1}{T}] \xrightarrow{d_1} 0,$$

where $d_0^+(a, b) = b$ and $d_0^-(a, b) = a$. It follows that $Z^1(\mathcal{U}, \mathcal{O}_X) = \mathbb{Q}_{\text{max}}[T, \frac{1}{T}]$. Let $x, y \in Z^1(\mathcal{U}, \mathcal{O}_X)$. Then, we can write $x = x_0 + x_1, y = y_0 + y_1$, where $x_0, y_0 \in \mathbb{Q}_{\text{max}}[T]$ and $x_1, y_1 \in \mathbb{Q}_{\text{max}}[\frac{1}{T}]$. Let $u = (x_0, y_1), v = (y_0, x_1)$. Then, we have

$$x + d_0^+(u) + d_0^-(v) = y + d_0^+(v) + d_0^-(u).$$

It follows that $xp^0y$ and hence $\check{H}^1(\mathcal{U}, \mathcal{O}_X) = 0$. However, since this computation depends on the specific covering $\mathcal{U}$, we do not know yet whether $\check{H}^1(X, \mathcal{O}_X) = \check{H}^1(\mathcal{U}, \mathcal{O}_X)$ or not. We remark that the above computation is also valid when we replace $\mathbb{Q}_{\text{max}}$ with other totally ordered semifields.

Next, we prove that the Picard group $\text{Pic}(X)$ of a semiring scheme $X$ is isomorphic to the first Čech cohomology group of the sheaf $\mathcal{O}_X^\times$. The proof is not much different from the classical case, but we include the proof for completeness. Note that $\mathcal{O}_X^\times$ is the sheaf such that $\mathcal{O}_X^\times(U) = \{a \in \mathcal{O}_X(U) \mid ab = 1 \text{ for some } b \in \mathcal{O}_X(U)\}$ for an open subset $U$ of $X$. Even though $\mathcal{O}_X$ is a sheaf of semirings, $\mathcal{O}_X^\times$ is a sheaf of (multiplicative) abelian groups. Hence, $\check{H}^1(\mathcal{U}, \mathcal{O}_X^\times)$ is an abelian group. We use the multiplicative notation for $\mathcal{O}_X^\times$.

In what follows, let $X$ be a semiring scheme, $\mathcal{L}$ be an invertible sheaf of $\mathcal{O}_X$-semimodules on $X$, and $\mathcal{U} = \{U_i\}_{i \in I}$ be a covering of $X$ such that $\varphi_i : \mathcal{O}_X|_{U_i} \simeq \mathcal{L}|_{U_i} \forall i \in I$. Let $e_i \in \mathcal{L}(U_i)$ be the image of $1 \in \mathcal{O}_X(U_i)$ under $\varphi_i(U_i)$. Through the following lemmas, we define a corresponding cocycle in $\check{H}^1(X, \mathcal{O}_X^\times)$ for an invertible sheaf $\mathcal{L}$ on $X$.

Lemma 4.8. For $i < j \in I$ and $U_{ij} = U_i \cap U_j$, there exists $f_{ij} \in \mathcal{O}_X^\times(U_{ij})$ such that $e_i|_{U_{ij}} = (e_j|_{U_{ij}})f_{ij}$.

Proof. This is clear since $e_i|_{U_{ij}}$ and $e_j|_{U_{ij}}$ are invertible elements in $\mathcal{O}_X^\times(U_{ij})$. \hfill $\Box$

We fix $f_{ij}$ in Lemma 4.8. We have the following:

Lemma 4.9. Let $f := (f_{ij}) \in C^1(\mathcal{U}, \mathcal{O}_X^\times)$. Then we have $d_0^+(f) = d_1^-(f)$ and hence $f \in Z^1(\mathcal{U}, \mathcal{O}_X^\times)$. In particular, $f$ has a canonical image in $\check{H}^1(\mathcal{U}, \mathcal{O}_X^\times)$.

Proof. For $i < j < k$, we have $e_i|_{U_{ijk}} = (e_j|_{U_{ijk}})f_{ij}, e_j|_{U_{ijk}} = (e_k|_{U_{ijk}})f_{jk}$. Thus we have

$$e_i|_{U_{ijk}} = (e_j|_{U_{ijk}})(f_{ij})|_{U_{ijk}} = (e_k|_{U_{ijk}})(f_{jk})(f_{ij})|_{U_{ijk}} = (e_k|_{U_{ijk}})(f_{ik})|_{U_{ijk}}.$$ 

This implies that $(f_{jk})|_{U_{ijk}} = (f_{ik})|_{U_{ijk}}$. It follows that $(d_1^-(f))|_{U_{ijk}} = (d_1^+(f))|_{U_{ijk}}$ and hence $f = (f_{ij}) \in Z^1(\mathcal{U}, \mathcal{O}_X^\times)$. Therefore, $f$ has a canonical image in $\check{H}^1(\mathcal{U}, \mathcal{O}_X^\times)$. \hfill $\Box$
**Lemma 4.10.** The canonical image of \( f \in C^1(\mathcal{U}, \mathcal{O}_X^*) \) in \( \tilde{H}^1(\mathcal{U}, \mathcal{O}_X^*) \) as in Lemma 4.9 does not depend on the choice of \( e_i \).

**Proof.** Let \( \{e'_i\}_{i \in I} \) be another choice with \( \{f'_j\}_{j \in J} \). We can take \( \{g_i\}_{i \in I} \), where \( g_i \in \mathcal{O}_X^*(U_i) \) such that \( e'_i = g_i e_i \). Then, we have \( e_i |_{U_{ij}} = f_{ij} e_j |_{U_{ij}} \). It follows that \( g_i |_{U_{ij}} e_i |_{U_{ij}} = f'_{ij} e'_j |_{U_{ij}} \). Therefore, \( f_{ij} g_i |_{U_{ij}} = f'_{ij} g_i |_{U_{ij}} \). This implies that \( f \cdot d^+_0(g) = f' \cdot d^+_0(g) \). In other words, \( f \) and \( f' \) give the same canonical image in \( \tilde{H}^1(\mathcal{U}, \mathcal{O}_X^*) \).

We denote the canonical image of \( f \in C^1(\mathcal{U}, \mathcal{O}_X^*) \) in \( \tilde{H}^1(\mathcal{U}, \mathcal{O}_X^*) \) by \( \phi_\mathcal{U}(\mathcal{L}) \). Let \( \mathcal{U} = \{U_i\}_{i \in I} \) and \( \mathcal{U}' = \{V_j\}_{j \in J} \) be two open coverings of \( X \) such that \( \mathcal{L}|_{U_i} \simeq \mathcal{O}_X|_{U_i} \) and \( \mathcal{L}|_{V_j} \simeq \mathcal{O}_X|_{V_j} \) for all \( i \in I \) and \( j \in J \). We define a new covering \( \mathcal{U} \cap \mathcal{U}' := \{U_i \cap V_j\}_{(i,j) \in I \times J} \) of \( X \). Then, clearly \( \mathcal{U} \cap \mathcal{U}' \) is a refinement of \( \mathcal{U} \). It follows that \( \phi_\mathcal{U}(\mathcal{L}) \) has a canonical image in \( \tilde{H}^1(\mathcal{U} \cap \mathcal{U}', \mathcal{O}_X^*) \).

**Lemma 4.11.** Let \( \mathcal{U} = \{U_i\}_{i \in I} \) and \( \mathcal{U}' = \{V_j\}_{j \in J} \) be two open coverings of \( X \) such that \( \mathcal{L}|_{U_i} \simeq \mathcal{O}_X|_{U_i} \) and \( \mathcal{L}|_{V_j} \simeq \mathcal{O}_X|_{V_j} \) for all \( i \in I \) and \( j \in J \). Let \( f \in C^1(\mathcal{U}, \mathcal{O}_X^*) \) and \( f' \in C^1(\mathcal{U}', \mathcal{O}_X^*) \) (as in Lemma 4.9). Then, the canonical images of \( f \) and \( f' \) are same in \( \tilde{H}^1(\mathcal{U} \cap \mathcal{U}', \mathcal{O}_X^*) \). In particular, each invertible sheaf \( \mathcal{L} \) determines a unique element \( \phi(\mathcal{L}) \) in \( \tilde{H}^1(X, \mathcal{O}_X^*) \).

**Proof.** Let \( \{e_i\}_{i \in I} \), \( \{f_j\}_{j \in J} \) be \( \mathcal{L} \) and \( \mathcal{L}' \) for \( \mathcal{U} \) and \( \mathcal{U}' \) as in Lemma 4.8. We claim that the images of \( \phi_\mathcal{U}(\mathcal{L}) \) and \( \phi_{\mathcal{U}'}(\mathcal{L}') \) in \( \tilde{H}^1(\mathcal{U} \cap \mathcal{U}', \mathcal{O}_X^*) \) are equal. Indeed, we can find \( g_k \in \mathcal{O}_X^*(U_i \cap V_k) \) such that \( \phi_k |_{U_i \cap V_k} = (g_k) e_i |_{U_i \cap V_k} \). Hence, from the relation \( \phi_k |_{U_i \cap V_k} = (g_k) e_i |_{U_i \cap V_k} \), we have that \( (g_k)|_{U_i \cap V_k} \cdot e_i |_{U_i \cap V_k} = (f_{i_k}) |_{U_i \cap V_k} \cdot e_i |_{U_i \cap V_k} \). It follows that

\[
(f_{i_k}) |_{U_i \cap V_k} \cdot (g_k) |_{U_i \cap V_k} = (f'_{i_k}) |_{U_i \cap V_k} \cdot (g_k) |_{U_i \cap V_k}.
\]

(12)

Let \( g = (g_k) \) for \( i \in I, k \in J \). Then, we have \( g \in C^0(\mathcal{U} \cap \mathcal{U}', \mathcal{O}_X^*) \). Give the set \( I \times J \) a dictionary order. Then we have

\[
(d^+_0(g))_{(i,k) \times (j,l)} = g_{j,l} |_{U_i \cap V_k} \quad \text{and} \quad (d^+_0(g))_{(i,k) \times (j,l)} = g_{i,k} |_{U_i \cap V_k}.
\]

(13)

Let \( \alpha : Z^1(\mathcal{U}, \mathcal{O}_X^*) \to Z^1(\mathcal{U} \cap \mathcal{U}', \mathcal{O}_X^*) \) be the \( \pm \)-morphism as in (11). Then \( \alpha \) induces the map \( \tilde{\alpha} : \tilde{H}^1(\mathcal{U}, \mathcal{O}_X^*) \to \tilde{H}^1(\mathcal{U} \cap \mathcal{U}', \mathcal{O}_X^*) \). Similarly, for \( \mathcal{U}' \), we obtain

\[
\beta : Z^1(\mathcal{U}', \mathcal{O}_X^*) \to Z^1(\mathcal{U} \cap \mathcal{U}', \mathcal{O}_X^*), \quad \beta : \tilde{H}^1(\mathcal{U}', \mathcal{O}_X^*) \to \tilde{H}^1(\mathcal{U} \cap \mathcal{U}', \mathcal{O}_X^*).
\]

In particular, if \( \phi_\mathcal{U}(\mathcal{L}) = [f] \), then \( \tilde{\alpha}(\\phi_\mathcal{U}(\mathcal{L})) = [\alpha(f)] \), where \([f] \) is the equivalence class of \( f \in Z^1(\mathcal{U}, \mathcal{O}_X^*) \) in \( \tilde{H}^1(\mathcal{U}, \mathcal{O}_X^*) \). We complete the proof, we have to show that \( [\alpha(f)] = [\beta(f')] \).

We know that \( \alpha(f)_{(i,k) \times (j,l)} = f_{i,j} |_{U_i \cap V_k} \) and \( \beta(f')_{(i,k) \times (j,l)} = f'_{i,j} |_{U_i \cap V_k} \). It follows from (12) and (13) that

\[
(\alpha(f) \cdot d^+_0(g))_{U_i \cap V_k} = (\beta(f') \cdot d^+_0(g))_{U_i \cap V_k}.
\]

This proves that \([\alpha(f)] = [\beta(f')] \). Thus, \( f \) and \( f' \) have the same image in \( \tilde{H}^1(X, \mathcal{O}_X^*) \). We denote this image by \( \phi(\mathcal{L}) \).

Consider the following map:

\[
\phi : \text{Pic}(X) \to \tilde{H}^1(X, \mathcal{O}_X^*), \quad \mathcal{L} \mapsto \phi(\mathcal{L}),
\]

where \([\mathcal{L}]\) is the isomorphism class of \( \mathcal{L} \) in \( \text{Pic}(X) \).

**Lemma 4.12.** \( \phi \) is well defined.
Next, let $A$ and therefore one can glue $L$. We can find an open covering $U = \{U_i\}_{i \in I}$ of $X$ such that on $U_i$ both $L$ and $L'$ are isomorphic to $O_X$. Let $\{e_i\}$ and $\{f_{ij}\}$ be as in Lemma 4.8 for $L$. Then we have $\varphi_{U_i} (e_i) |_{U_{ij}} = f_{ij} \cdot \varphi_{U_i} (e_j) |_{U_{ij}}$. Since $\phi (L')$ does not depend on the choice of $\{e_i\}$, we let $e_i' = \varphi_{U_i} (e_i)$ as in Lemma 4.8 for $L'$. Then the desired property follows. 

\textbf{Lemma 4.13.} $\phi$ is a group homomorphism.

Proof. Suppose that $L$ and $L'$ are invertible sheaves of $O_X$-semimodules. Then so is $L \otimes_{O_X} L'$ (this directly follows from Lemma 3.6). Therefore, we can find an affine open covering $U = \{U_i = \text{Spec } R_i\}_{i \in I}$ of $X$ such that $(L \otimes_{O_X} L') (U_i) \simeq O_X (U_i) \simeq L (U_i) \simeq L' (U_i) \simeq R_i$. In particular, we have $(L \otimes_{O_X} L')(U_i) \simeq (L(U_i) \otimes_{O_X} L'(U_i))$. Let $\{e_i\}_{i \in I}$, $\{f_{ij}\}_{i,j \in I}$ for $L$ and $\{e_i'\}_{i \in I}$, $\{f_{ij}'\}_{i,j \in I}$ for $L'$ as in Lemma 4.8 on the open covering $U$. Then we can take $\{e_i \otimes e_i'\}$ as a basis for $(L \otimes_{O_X} L')(U_i)$ and the corresponding transition map is $F = (f_{ij} : f_{ij}')$. It follows that $\phi (L \otimes_{O_X} L') = \phi (L) \phi (L')$.

\textbf{Lemma 4.14.} $\phi ([L]) = 1$ if and only if $[L]$ is the isomorphism class of $O_X$. In particular, $\phi$ is injective.

Proof. Suppose that $\phi (L) = 1$. Let $U = \{U_i\}_{i \in I}$ be an open covering of $X$ such that $L|_{U_i} \simeq O_X |_{U_i}$, $\forall i \in I$ and let $f$ and $e_i$ be as in Lemma 4.8. Since the canonical image of $f$ does not depend on the choice of an open covering $U$, we may assume that $[f] = [1] \in \tilde{H}^1 (U, O_X)$. This implies that there exists $g \in C^0 (U, O_X^*)$ such that $d_0^+ (g) = f \cdot d_0^-(g)$. Hence, $(d_0^+ (g))_{ij} = (f \cdot d_0^- (g))_{ij}$ and $f_{ij} : g_i |_{U_{ij}} = g_j |_{U_{ij}}$. It follows that $(g_i e_i |_{U_{ij}} = g_i |_{U_{ij}} e_i |_{U_{ij}} = g_i |_{U_{ij}} f_{ij} e_j |_{U_{ij}} = g_j |_{U_{ij}} e_j |_{U_{ij}} = (g_j e_j) |_{U_{ij}}$. Thus, $e_i g_i$ and $e_j g_j$ agree on $U_{ij}$ and hence we can glue them to obtain the global isomorphism $\varphi : L \mapsto O_X$. Conversely, if $L \simeq O_X$, then clearly $\phi (L) = 1$. In fact, one can take $e_i = e |_{U_i}$, where $e$ is the identity in $O_X (X)$.

\textbf{Lemma 4.15.} $\phi$ is surjective.

Proof. Notice that $\alpha \in \tilde{H}^1 (X, O_X^*)$ for each $[f] \in \tilde{H}^1 (U, O_X^*)$ for an open covering $U = \{U_i\}_{i \in I}$ of $X$. Let $L_i := O_X |_{U_i}$ for each $i \in I$. Let $f = (f_{ij}) \in Z^1 (U, O_X^*)$. Then, for $i < j$, each $f_{ij}$ defines the following isomorphism:

$$\phi_{ij} : L_i |_{U_{ij}} \mapsto L_j |_{U_{ij}}, \quad s \mapsto f_{ij} \cdot s.$$ 

We define $\phi_i := id$. Since $f \in Z^1 (U, O_X^*)$, we have $d_0^+ (f) = d_0^- (f)$. It follows that $(d_0^+ (f))_{ij} = f_{jk} \cdot f_{ij} = (d_0^- (f))_{ij} = f_{jk}$, and $f_{ij} \cdot f_{jk} = f_{ik}$. This implies that $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ and therefore one can glue $\phi_i$ to obtain the invertible sheaf $L$. Let $e_i$ be the image of 1 under the isomorphism $O_X (U_i) \simeq L(U_i)$. Then we obtain the corresponding $f = (f_{ij})$. This implies that $\phi ([L]) = \alpha$, hence $\phi$ is surjective. 

Finally, we conclude the following theorem via the isomorphism $\phi$.

\textbf{Theorem 4.16.} Pic ($X$) $\simeq \tilde{H}^1 (X, O_X^*)$ for a semiring scheme $(X, O_X)$.

\textbf{Example 4.17.} We compute the Picard group of $\mathbb{P}^1_{\mathbb{Q}_{\text{max}}}$, the projective line over $\mathbb{Q}_{\text{max}}$. We first compute the invertible elements of the semiring $B := \mathbb{Q}_{\text{max}} [T]$. If $f (T) = \sum_{i=0}^n a_i T^i$ is an invertible element of $B$, then there exists $g (T) = \sum_{i=0}^m b_i T^i$ such that $f (T) \odot g (T) = \sum_{i=0}^{n+m} (\max\{a_r + b_l\}) T^i = 1_B = 0$.

This implies that $\max_{r+l=i} \{a_r + b_l\} = c_i = -\infty$ for $i \geq 1$. Hence, $a_j = b_j = -\infty$ for $j \geq 1$ and $f (T) \in \mathbb{Q}$. Next, let $A := \mathbb{Q}_{\text{max}} [T, \frac{1}{T}]$ and $A^*$ be the set of elements in $A$ which is multiplicatively
invertible (in particular, $A^\ast$ is an abelian group). If $f(T) \in A^\ast$, then there exists $k \in \mathbb{N}$ such that $T^k f(T) \in B$. This implies that $T^k f(T) \in \mathbb{Q}$ from the first case. Since $T^k$ for $k \in \mathbb{Z}$ is invertible in $A$, we conclude that $A^\ast = \{qT^n \mid q \in \mathbb{Q}, n \in \mathbb{Z}\}$.

Let $X := \mathbb{P}_\text{Qmax}^1$ and $\mathcal{U} = \{U_1, U_2\}$ be an open covering of $X$ such that $U_1 \simeq \text{Spec} \mathbb{Q}\text{max}[T]$ and $U_2 \simeq \text{Spec} \mathbb{Q}\text{max}[\frac{1}{T}]$. From the above computation, we have $\mathcal{O}_X^\ast(U_1) = \mathbb{Q}$ and $\mathcal{O}_X^\ast(U_1 \cap U_2) = A^\ast = \{qT^n \mid q \in \mathbb{Q}, n \in \mathbb{Z}\}$. Then, we have the following Čech complex:

$$C : C^0 = \mathbb{Q} \times \mathbb{Q} \xrightarrow{d_0} C^1 = A^\ast \xrightarrow{d_1} \mathbb{Q},$$

where $d_0^{-1}(a, b) = b$, $d_0^{-1}(a, b) = a$. Clearly, we have $C^1 = \mathbb{Z}(\mathcal{U}, \mathcal{O}_X^\ast)$. Two elements $qT^n$ and $q'T^{n'}$ in $A^\ast$ are equivalent if and only if there exist $c = (a, b)$, $c' = (a', b') \in C^0$ such that

$$qT^n \circ d_0^{-1}(c) \circ d_0^{-1}(c') = q'T^{n'} \circ d_0^{-1}(c') \circ d_0^{-1}(c).$$

However, (14) holds if and only if $n = n'$. Therefore, we have

$$\tilde{H}^1(\mathcal{U}, \mathcal{O}_X^\ast) = C^1 / \rho^1 = \{T^n \mid n \in \mathbb{Z}\} \simeq \mathbb{Z}.$$

This is coherent with the classical result.

**Example 4.18.** Note that, different from the classical case, $\mathbb{Q}\text{max}[T]$ is not multiplicatively cancellative. Therefore the canonical map, $S^{-1} : \mathbb{Q}\text{max}[T] \rightarrow S^{-1}\mathbb{Q}\text{max}[T]$ does not have to be injective. In tropical geometry, rather than working directly with $\mathbb{Q}\text{max}[T]$, one works with the semiring $\mathbb{Q}\text{max}[T] := \mathbb{Q}\text{max}[T]/\sim$, where $\sim$ is a congruence relation such that $f(T) \sim g(T) \Leftrightarrow f(x) = g(x) \forall x \in \mathbb{Q}\text{max}$. Let $B := \mathbb{Q}\text{max}[T]$. If $f(T) \in B$ is multiplicatively invertible, then there exists $g(T)$ such that $f(T) \circ g(T) = 1_B = 0$. However, for $l \in \mathbb{Q}\text{max}$, the set $\mathbb{Q}(l)$ consists of a single element $l$. It follows that $f(T) \circ g(T) = 0$. From Example 4.17, this implies that $f(T) \in \mathbb{Q}$ and hence $B^\ast = \mathbb{Q}$. Let $S = \{T, T^2, \ldots\}$ be a multiplicative subset of $B$, and $A := S^{-1}B$. Since $B$ is multiplicatively cancellative (cf. [3]), $B$ is canonically embedded into $A$. Moreover, similar to Example 4.17, one can observe that $A^\ast = \{T^n \mid q \in \mathbb{Q}, n \in \mathbb{Z}\}$.

Suppose that the projective line $X := \mathbb{P}^1$ over $\mathbb{Q}\text{max}$ is the semiring scheme such that two affine semiring schemes $\text{Spec} \mathbb{Q}\text{max}[T]$ and $\text{Spec} \mathbb{Q}\text{max}[\frac{1}{T}]$ are glued along $\text{Spec} A$. The exact same argument as in Example 4.17 shows the following:

$$\tilde{H}^1(\mathcal{U}, \mathcal{O}_X^\ast) = \mathbb{Z}.$$

**Remark 4.19.** One may mimic the classical construction to define $\mathcal{O}_{\mathbb{P}^1}(n)$. If one can show that the generalized Čech cohomology of a semiring scheme is independent from a choice of an open covering, then Example 4.17 implies that any invertible sheaf $\mathcal{L}$ on $\mathbb{P}_\text{Qmax}^1$ should be isomorphic to $\mathcal{O}_{\mathbb{P}^1}(n)$ for some $n \in \mathbb{Z}$. This classifies all invertible sheaves on $\mathbb{P}_\text{Qmax}^1$ as in the classical case.

**Remark 4.20.** Since differential maps of many (co)homology theories are defined by alternating sums, it seems that many of those theories can be directly generalized by using the above framework. For example, if $k$ is a semifield, then Hochschild homology can be computed via the above framework and the result is same as classical case, i.e. $HH_0(k) = k$ and $HH_n(k) = 0$ for all $n > 0$.

**Appendix A. Basic definitions of semirings**

In this section, we provide the basic definitions of semirings which are frequently used in the paper.
Definition A.1. A set $T$ equipped with a binary operation $\cdot$ is called a monoid if for $a, b, c \in T$, we have $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ and there exists $1 \in T$ such that $1 \cdot a = a \cdot 1 = a$. When $a \cdot b = b \cdot a \forall a, b \in T$, we say that $T$ is a commutative monoid. When $T$ does not have $1$, $T$ is called a semigroup.

Definition A.2. A semiring $(M, +, \cdot)$ is a non-empty set $M$ endowed with an addition $+$ and a multiplication $\cdot$ such that

1. $(M, +)$ is a commutative monoid with the neutral element $0$.
2. $(M, \cdot)$ is a monoid with the identity $1$.
3. $r(s + t) = rs + rt$ and $(s + t)r = sr + tr \quad \forall r, s, t \in M$.
4. $r \cdot 0 = 0 \cdot r = 0 \quad \forall r \in M$.
5. $0 \neq 1$.

If $(M, \cdot)$ is a commutative monoid, then we call $M$ a commutative semiring. If $(M \setminus \{0\}, \cdot)$ is a group, then a semiring $M$ is called a semifield.

Definition A.3. (cf. [2]) Let $M_1, M_2$ be semirings. A map $f : M_1 \to M_2$ is a homomorphism of semirings if $f$ satisfies the following conditions:

$f(a + b) = f(a) + f(b), \quad f(ab) = f(a)f(b), \quad f(0) = 0, \quad f(1) = 1 \quad \forall a, b \in M_1$.

Definition A.4. Let $M$ be a commutative semiring and $T$ be a commutative monoid. We say that $T$ is a $M$-semimodule if there exists a map $\varphi : M \times T \to T$ which satisfies the following properties: $\forall m, m_1, m_2 \in M, \forall t, t_1, t_2 \in T$,

1. $\varphi(1, t) = t$.
2. If $t = 0$ or $m = 0$, then $\varphi(m, t) = 0$.
3. $\varphi(m_1 + m_2, t) = \varphi(m_1, t) + \varphi(m_2, t), \quad \varphi(m, t_1 + t_2) = \varphi(m, t_1) + \varphi(m, t_2)$.
4. $\varphi(m_1m_2, t) = \varphi(m_1, \varphi(m_2, t)), \quad \varphi(m_1t_1t_2) = \varphi(m_1, \varphi(t_2, \varphi(t_1, m_2)))$.

By an idempotent semiring, we mean a semiring $M$ such that $x + x = x \quad \forall x \in M$.

Example A.5. Let $\mathbb{B} := \{0, 1\}$. We define an addition as: $1 + 1 = 1, \ 1 + 0 = 0 + 1 = 1$, and $0 + 0 = 0$. A multiplication is defined by $1 \cdot 1 = 1, \ 1 \cdot 0 = 0, \ 0 \cdot 0 = 0$. Then, $\mathbb{B}$ becomes the initial object in the category of idempotent semirings.

Example A.6. The tropical semifield $\mathbb{R}_{\max}$ is $\mathbb{R} \cup \{-\infty\}$ as a set. An addition $\oplus$ is given by: $a \oplus b := \max\{a, b\} \quad \forall a, b \in \mathbb{R}_{\max}$, where $-\infty \leq a \forall a \in \mathbb{R}_{\max}$. A multiplication $\odot$ is defined as the usual addition of $\mathbb{R}$ as follows: $a \odot b := a + b \forall a, b \in \mathbb{R}$ and $(-\infty) \odot a = a \odot (-\infty) = (-\infty) \forall a \in \mathbb{R}_{\max}$. We denote by $\mathbb{Q}_{\max}, \mathbb{Z}_{\max}$ the sub-semifields of $\mathbb{R}_{\max}$ with the underlying sets $\mathbb{Q} \cup \{-\infty\}, \mathbb{Z} \cup \{-\infty\}$ respectively.

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