THREEFOLD TRIPLE SYSTEMS WITH NONSINGULAR $N_2$

Peter J. Dukes and Kseniya Garaschuk

August 29, 2014

Abstract. There are various results connecting ranks of incidence matrices of graphs and hypergraphs with their combinatorial structure. Here, we consider the generalized incidence matrix $N_2$ (defined by inclusion of pairs in edges) for one natural class of hypergraphs: the triple systems with index three. Such systems with nonsingular $N_2$ (over the rationals) appear to be quite rare, yet they can be constructed with PBD closure. In fact, a range of ranks near $\binom{v}{2}$ is obtained for large orders $v$.

1. Introduction

We consider hypergraphs with the possibility of repeated edges. Let $v$ and $\lambda$ be positive integers, and suppose $K \subset \mathbb{Z}_{\geq 2} := \{2, 3, 4, \ldots\}$. A pairwise balanced design $\text{PBD}_\lambda(v, K)$ is a hypergraph $(V, B)$ with $v$ vertices, edge sizes belonging to $K$, and such that

- any two distinct vertices in $V$ appear together in exactly $\lambda$ edges.

In this context, vertices are also called points and edges are normally called blocks. The parameter $\lambda$ is the index; often it is taken to be 1 and suppressed from the notation. We remark that $K$ could contain unused block sizes.

There are numerical constraints on $v$ given $\lambda$ and $K$. An easy double-counting argument on pairs of points leads to the global condition

$$\lambda v(v - 1) \equiv 0 \pmod{\beta(K)},$$

where $\beta(K) := \gcd\{k(k - 1) : k \in K\}$. Similarly, counting incidences with any specific point leads to the local condition

$$\lambda(v - 1) \equiv 0 \pmod{\alpha(K)},$$

where $\alpha(K) := \gcd\{k - 1 : k \in K\}$. Wilson’s theory, [9], asserts that (1.1) and (1.2) are sufficient for large $v$.

In the case $K = \{3\}$, we obtain a ($\lambda$-fold) triple system or $\text{TS}_\lambda(v)$. When $\lambda = 1$ we have a Steiner triple system and it is well known that these exist for all $v \equiv 1, 3 \pmod{6}$. In this article we are especially interested in the case $\lambda = 3$. The divisibility conditions (1.1) and (1.2) simply reduce to $v$ being odd. There are $3v(v - 1)/6 = \binom{v}{2}$ blocks. For a comprehensive reference on triple systems, the reader is referred to Colbourn and Rosa’s book [4].

Given any hypergraph $H = (V, E)$, we may define its incidence matrix $N = N(H)$ as the zero-one inclusion matrix of points versus edges. That is, $N$ has rows indexed by $V$, columns indexed by $E$,
and where, for $x \in V$, $e \in E$,

$$N(x, e) = \begin{cases} 
1 & \text{if } x \in e; \\
0 & \text{otherwise.}
\end{cases}$$

Linear algebraic properties of incidence matrices have received a lot of attention. Especially interesting are connections with the underlying combinatorial structure. We give two classical examples. First, in the case of ordinary graphs, in which $E \subseteq \binom{V}{2}$, it is known [8] that $N$ has full rank (over $\mathbb{R}$) if and only if every connected component is non-bipartite. As a different example, the rank of a Steiner triple system over the binary field $\mathbb{F}_2$ is connected in [5] with its ‘projective dimension’. This measures the length of the lattice of largest possible proper subsystems.

Let $s$ be a positive integer. The higher incidence matrix $N_s$ has a similar definition, but where rows are indexed by $\binom{V}{s}$ (the $s$-subsets of vertices), columns are again indexed by blocks, and entries are defined by inclusion. That is, for $S \subseteq V$, $|S| = s$, and $e \in E$, we have

$$N_s(S, e) = \begin{cases} 
1 & \text{if } S \subseteq e; \\
0 & \text{otherwise.}
\end{cases}$$

Higher incidence matrices were used by Ray-Chaudhuri and Wilson in [7] to extend Fisher’s inequality to designs of ‘higher strength’. In a little more detail, suppose we have a system $(V, B)$ of $v$ points, blocks of a fixed size $k$, and every $t$-subset of points belongs to exactly $\lambda$ blocks. These are sometimes denoted $S_\lambda(t, k, v)$. Suppose further that $t$ is even, say $t = 2s$, and $v \geq k + s$. Then the conclusion is that $|B| \geq \binom{k}{s} \cdot \binom{v}{t}$, and it comes with a strong structural condition for equality. The matrix $N_k$ plays a key role in the proof. Incidentally, a new result of Keevash in [6] proves that, for large $v$, the divisibility conditions $\binom{k-s}{t-i} | \binom{v}{t-i}$ for $i = 0, \ldots, t$ (which are the analogs of (1.1-1.2)) suffice for the existence of $S_\lambda(t, k, v)$.

Returning to pairwise balanced designs, higher incidence matrices are of limited use when $\lambda = 1$. In this case, the matrix $N_2$ is only slightly interesting; each of its rows has exactly one nonzero entry. The matrix $N_k$ is just, under a reordering of rows, the identity matrix on top of the zero matrix. In between, $N_s$ for $2 < s < k$ has many zero rows and not much structure.

We would like to consider $N_2$ for what is perhaps the first natural case: threefold triple systems $TS_3(v)$. For such designs, $N_2$ is square of order $\binom{v}{3}$. In general, we observe that the property of a design having full rank $N_2$ is ‘PBD-closed’. From this and some small designs, we have the following main result.

**Theorem 1.1.** There exists a $TS_3(v)$ with $N_2$ nonsingular over $\mathbb{R}$ for all odd $v \geq 5$ except possibly for $v \in E_{579} := \{v : v \equiv 1 \pmod{2}, v \geq 5, \text{ and } \not\exists \text{ PBD}(v, \{5, 7, 9\})\}$.

It is known (see [1] and the summary table entry at [2], page 252) that

$$E_{579} \subseteq \{11, 19, 23, 27, 33, 39, 43, 51, 59, 71, 75, 83, 87, 95, 99, 107, 111, 113, 115, 119, 139, 179\},$$

and therefore Theorem 1.1 settles the existence question for all but a finite set of values $v$.

The next section sets up and completes the proof. Then, we conclude with a short discussion of some related topics, including a brief look at such ranks in characteristic $p$. 

2
To clarify, we are working in characteristic zero (rank computed over $\mathbb{Q}$).

Lemma 2.1. Suppose there exists a PBD($v, L$) and, for each $u \in L$, there exists a PBD$_\lambda(v, K)$ having $N_2$ square and full rank over $\mathbb{F}$. Then there exists a PBD$_\lambda(v, K)$ having $N_2$ square and full rank over $\mathbb{F}$.

Proof. Suppose our PBD($v, L$) is $(V, A)$. Construct a PBD$_\lambda(v, K)$ with points $V$ and block collection

$$B = \bigcup_{U \in A} B[U],$$

where $B[U]$ denotes the blocks of a PBD$_\lambda(|U|, K)$ on $U$ having full rank $N_2$. (Note (2.1) should be interpreted as a formal sum or ‘multiset union’.) It is clear that $(V, B)$ is a PBD$_\lambda(v, K)$.

Consider its incidence matrix $N_2(B)$. If columns are ordered respecting some ordering $U_1, U_2, \ldots$ of $A$ and the union in (2.1), and rows are ordered respecting $(U_1^T), (U_2^T), \ldots$, then we obtain a block-diagonal structure

$$N_2(B) = N_2(B[U_1]) \oplus N_2(B[U_1]) \oplus \ldots.$$

Since each block is nonsingular, so is $N_2(B)$. □

To clarify, we are working in characteristic zero (rank computed over $\mathbb{Q}$) throughout the remainder of the section.

Lemma 2.2. For $v = 5, 7, 9$, there exists a TS$_\lambda(v)$ having nonsingular $N_2$.

Proof. The unique TS$_3(5)$ is just the complete design $\binom{5}{3}$. Accordingly, for this design, we have $N_2N_2^T = 3I + A$, where $A$ is the adjacency matrix of the line graph of $K_5$ (or complement of the Petersen graph). Since $A$ is known to have eigenvalues $(-2)^5, 1^4, 6^1$, it follows that $N_2$ has full rank.

Examples for $v = 7, 9$ are given below as a list of blocks on $\{0, \ldots, v - 1\}$.

$$v = 7: \quad \{0, 1, 2\}, \{0, 1, 3\}, \{0, 1, 4\}, \{0, 2, 3\}, \{0, 2, 5\}, \{0, 3, 6\}, \{0, 4, 5\}, \{0, 4, 6\}, \{0, 5, 6\}, \{1, 2, 4\}, \{1, 2, 5\}, \{1, 3, 5\}, \{1, 3, 6\}, \{1, 4, 5\}, \{1, 5, 6\}, \{2, 3, 4\}, \{2, 3, 5\}, \{2, 4, 6\}, \{2, 5, 6\}, \{3, 4, 5\}, \{3, 4, 6\}.$$

$$v = 9: \quad \{0, 1, 2\}, \{0, 1, 3\}, \{0, 1, 4\}, \{0, 2, 3\}, \{0, 2, 5\}, \{0, 3, 6\}, \{0, 4, 6\}, \{0, 4, 7\}, \{0, 5, 7\}, \{0, 5, 8\}, \{0, 6, 8\}, \{0, 7, 8\}, \{1, 2, 4\}, \{1, 2, 5\}, \{1, 3, 6\}, \{1, 3, 8\}, \{1, 4, 7\}, \{1, 5, 6\}, \{1, 5, 8\}, \{1, 6, 7\}, \{1, 7, 8\}, \{2, 3, 4\}, \{2, 3, 7\}, \{2, 4, 8\}, \{2, 5, 6\}, \{2, 6, 7\}, \{2, 6, 8\}, \{2, 7, 8\}, \{3, 4, 5\}, \{3, 4, 8\}, \{3, 5, 7\}, \{3, 5, 8\}, \{3, 6, 7\}, \{4, 5, 6\}, \{4, 5, 7\}, \{4, 6, 8\}.$$

It is straightforward to confirm these have full rank $N_2$; for example, the following sage code can be used given $v$ and a list of sets $B$ as above. □

```python
T = Set(range(v)).subsets(2)
N2 = matrix(QQ, binomial(v,2))
for i in range(binomial(v,2)):
    for j in range(binomial(v,2)):
        if Set(T[i]).issubset(B[j]):
            N2[i,j] += 1
N2.rank()
```

2. PBD Closure and Proof of the Main Result
Remark. Of the ten non-isomorphic $TS_3(7)$, exactly one has nonsingular $N_2$. Of the 22521 $TS_3(9)$, exactly 27 have nonsingular $N_2$.

The proof of our main result is now an easy combination of the preceding lemmas.

Proof of Theorem 1.1. Take a $PBD(v,\{5,7,9\})$ and replace its blocks as in Lemma 2.1 by $TS_3(u)$ for $u = 5, 7, 9$ having nonsingular $N_2$, the latter existing by Lemma 2.2. The result is a $TS_3(v)$ having nonsingular $N_2$, as desired.

3. Discussion

A (pairwise) trade is a 2-edge-colored hypergraph $(T, A_1, A_2)$ such that each color class $A_i$ covers, counting multiplicity, the same pairs in $\binom{T}{2}$. A nontrivial example is the ‘quadrilateral’ $\{u, v, a\}, \{x, y, a\}, \{u, x, b\}, \{v, y, b\}$ together with its image under permuting $a, b$. Suppose a (multi-)hypergraph $H = (V, B)$ contains a trade $(T, A_1, A_2)$ with $T \subseteq V$ and $A_1, A_2$ as different (multiset) subsets of $B$. Then the trade induces a $\{±1, 0\}$-vector in the kernel of $N_2(H)$. It follows that some design has $N_2$ of full (column) rank only if it is ‘trade-free’, and in particular, has no repeated blocks. Accordingly, we have the following direct consequence of Theorem 1.1.

Corollary 3.1. There exist trade-free $TS_3(v)$ for all odd integers $v \geq 5$, $v \notin E_{579}$.

It may be of interest to compute the set of all possible ranks of $N_2$ over $TS_3(v)$ for a fixed $v$. When $v \equiv 1, 3 \pmod{6}$, one such $TS_3(v)$ comes from three copies of a Steiner triple system, which has rank $\binom{v}{3}/3$. It is clear that this is the minimum possible rank. In the case $v = 7$, the complete list of ranks (with repetition) is

$$7, 10, 12, 13, 15, 16, 18, 21.$$ 

For $v = 9$ we compute the list of distinct ranks as $12, 17, 19, \ldots, 36$. Now, a result of Colbourn and Rödl in [3] guarantees the existence of a $PBD(v,\{5,7,9\})$ for large odd $v$ with many blocks of size 9. It follows with a similar argument as in the proof of Theorem 1.1 that all ranks in the interval $[c\binom{v}{2}, \binom{v}{2}]$ are realizable for $c \approx \frac{49}{56}$ and large $v$.

Finally, we briefly consider $p$-ranks (that is, over $F_p$, the field of order $p$). Here are two easy facts.

Proposition 3.2. The 2-rank of $N_2$ for a $TS_3(v)$ is at most $\binom{v-1}{2}$.

Proof. Consider the $v-1$ pairs incident with some point, say $x$. Every block intersects either zero or two such pairs, and hence the corresponding vector in $\mathbb{R}^{\binom{v}{2}}$ lies in the left kernel of $N_2$ over $F_2$. There are $v-1$ such independent relations over $F_2$, and therefore the kernel has dimension at least $v-1$.

Proposition 3.3. The 3-rank of $N_2$ for a $TS_3(v)$ is at most $\binom{v}{2} - 1$.

Proof. Observe that $N_2 N_2^\top$ has constant rowsum equal to 9. So the all-ones vector is in the kernel of $N_2 N_2^\top$ over $F_3$. □
In our searches for $v = 7, 9$, we found that both of the above bounds can be met with equality. Also, it appears likely that, for our problem, $Q$-rank always agrees with $p$-rank for primes $p > 3$. We presently see no easy argument to confirm this.

Acknowledgements

The authors would like to thank Patric R.J. Östergård for providing a data file of all $TS_3(9)$ up to isomorphism. We would also like to acknowledge Felix Goldberg’s question on MathOverflow (and Yuichiro Fujiwara’s thoughtful answer) at http://mathoverflow.net/questions/151702/ which confirmed our belief that ranks of higher incidence matrices could be of interest to the broader community.

References

[1] F.E. Bennett, C.J. Colbourn, and R.C. Mullin, Quintessential pairwise balanced designs. J. Stat. Plann. Infer. 72 (1998), 15–66.
[2] C.J. Colbourn and J.H. Dinitz, eds., The CRC Handbook of Combinatorial Designs, 2nd edition, CRC Press, Inc., 2006.
[3] C.J. Colbourn and V. Rödl, Percentages in pairwise balanced designs. Discrete Math. 77 (1989), 57–63.
[4] C.J. Colbourn and A. Rosa, Triple Systems, Oxford Univ. Press, 1999.
[5] J. Doyen, X. Hubaut and M. Vandensavel, Ranks of incidence matrices of Steiner triple systems. Math. Z. 163 (1978), 251–259.
[6] P. Keevash, The existence of designs, arXiv preprint http://arxiv.org/pdf/1401.3665v1.pdf, 2014.
[7] D.K. Ray-Chaudhuri and R.M. Wilson, On $t$-designs. Osaka J. Math. 12 (1975), 737–744.
[8] C. Van Nuffelen, On the incidence matrix of a graph. IEEE Trans. Circuits and Systems 9 (1976), 572.
[9] R.M. Wilson, An existence theory for pairwise balanced designs III: Proof of the existence conjectures. J. Combin. Theory Ser. A 18 (1975), 71–79.

Peter J. Dukes: Mathematics and Statistics, University of Victoria, Victoria, Canada

E-mail address: dukes@uvic.ca

Kseniya Garaschuk: Mathematics and Statistics, University of Victoria, Victoria, Canada

E-mail address: kgarasch@uvic.ca