Rate Splitting, Superposition Coding and Binning for Groupcasting over the Broadcast Channel: A General Framework

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Abstract—A general inner bound is given for the discrete memoryless broadcast channel with an arbitrary number of users and general message sets, a setting that accounts for the most general form of concurrent groupcasting, with messages intended for any set of subsets of receivers. Achievability is based on superposition coding and rate-splitting without and with binning, where each receiver jointly decodes both its desired messages as well as the partial interference assigned to it via rate-splitting. The proof of achievability builds on the techniques for the description and analysis of superposition coding recently developed by the authors for the multiple access channel with general messages as well as a new recursive mutual covering lemma for the analysis of the more general achievable scheme with binning.

I. INTRODUCTION

A fundamental feature of wireless transmission is its broadcast nature. On the one hand, this feature can be seen as a detriment insofar as it inhibits a receiver from decoding its desired messages when undesired messages interfere at that receiver. On the other hand, it facilitates the distribution of one message to many receivers, when all receivers desire that message.

A model suited for the analysis of the benefits and detractions of this broadcast nature of wireless transmission is the broadcast channel (BC) with general message sets. With general message sets, each distinct message is groupcasted to a distinct group of receivers. That group may contain only a single receiver, as is the case for a unicast message, or as many as all receivers, as is the case for a multicast message, or any range of intermediate options. In general, our model of the broadcast channel with general messages permits multiple such messages to be concurrently groupcasted.

Much of the work on broadcast channels has focused on multiple unicast (i.e., private messages), dating to the seminal paper on two-user binary symmetric and scalar Gaussian BCs by Cover [4]. There, a coding strategy known as superposition coding was proposed. Its extension to general BCs, with an arbitrary number of users, was developed in Bergmans [1]. This inner bound is tight for the BC with degraded receivers1, as was established by Gallager [10] in the discrete memoryless case, and by Bergmans [2] in the scalar Gaussian case. Rate-splitting was first proposed by Carleial in [3] in the context of the two-user interference channel, a technique later used in two- and some three-user broadcast channels (as well as in other problems cf. [8]). Rate-splitting and superposition coding are combined with binning in [14] (see [8, Theorem 8.4]) for the two-user broadcast channel with private and common messages. This idea is extended to the three-user broadcast channel with three degraded messages in [15] (but with one receiver employing indirect joint decoding) and, more generally, to the diamond message set groupcasting in the $K$-user broadcast channel in [20]. The mutual covering lemma (cf. [8]) suffices for the analyses of all these schemes. Moreover, in [15] and [20], the most economical choice $F = E$ is made. Here, we considerably generalize these prior inner bounds to the discrete memoryless (DM) BC with an arbitrary number of users and with arbitrary message sets. Our first inner bound employs generalized notions of superposition coding, rate-splitting, and the joint decoding of desired messages along with partial decoding of undesired messages. Our second, more general inner bound combines these notions with binning (referred to as multicoding and joint typicality codebook generation in [8]).

To characterize the rates achievable by superposition coding, we use the order-theoretic framework developed by the authors for the multiple-access channel (MAC) with general message sets [18]. In so doing, we succinctly characterize the rates achievable by superposition coding, and provide a connection to polyhedral combinatorics.

In some cases, it may be beneficial to decode interference. To allow for both treating interference as noise and fully decoding interference, and a range of intermediate options, we consider partial interference decoding in a very general way. This is described by rate-splitting prior to superposition coding, which splits messages into sub-messages (in one of many ways), and relabels each sub-message as being intended by its original intended receivers and by additional receivers. Geometrically, this inner bound is equal to the non-negative rates which lie within the Minkowski sum of a polytope, representing the rates achievable through superposition coding and a rate-splitting cone of vectors, representing the enlargements achievable through partial interference decoding. The polytopes have combinatorial structure as they are the intersection of $K$ unbounded polyhedra, whose bounded component represents a polymatroid.

Next, our achievable scheme based on rate-splitting and superposition coding is combined with binning. This more
general scheme requires for its analysis a generalization of the mutual covering lemma of [9] (see also [8, Lemma 8.1]) which we prove here and call the recursive mutual covering lemma. This lemma helps succinctly characterize the conditions under which the probability of encoding errors due to unavailability of jointly typical codewords at the encoder can be made vanishingly small. Here too, we provide a connection to polyhedral combinatorics.

II. PRELIMINARIES

A. The Discrete Memoryless Broadcast Channel

The DM BC consists of one transmitter $X \in X'$, $K$ receivers $Y_i \in Y_i$, for $1 \leq i \leq K$, and a transition function $W(y_1, \ldots, y_K|x)$. If $X_1, Y_1, \ldots, Y_K$ are the channel input and output at the $t$th channel use, then the conditional probability of a sequential block of $n$ channel outputs, conditioned on the corresponding $n$ channel inputs, factors as $p(y_1^n, x^n) = \prod_{t=1}^{n} W(y_1, \ldots, y_K|x)$. The transmitter may send multiple independent messages, each of which is intended for a group of receivers. Each independent message $M_S$, and its rate $R_S$, are indexed by the subset $S \subseteq [1:K]^2$ of the set of receivers that it is intended for.

We collect the indices of all the messages into the set $E$, the message index set. As $E$ contains non-empty subsets of $[1:K]$, it is a subset of $2^{[1:K]}$, the power set of $[1:K]$. Each receiver only desires to receive all the messages sent; we denote the set of indices of messages desired by the $j$th receiver as

$$W_j^E = \{ S : j \in S \in E \}$$

B. Notation

Let $P$ be an ordered set of sets. If the order on $P$ satisfies $S \leq S'$ only when $S \subseteq S'$, so that $S$ and $S'$ are incomparable if neither $S \subseteq S'$ nor $S' \subseteq S$, then we call that order a superposition order. A subset $B$ of $P$ is a down-set if, for every $S \in B$ and $S' \leq S$, $S' \in B$. Similarly, $B$ is an up-set if, for every $S \in B$ and $S \leq S'$, $S' \in B$. Let $\uparrow Q$ (resp., $\downarrow Q$) be the smallest down-set (resp., up-set) of $P$ containing $Q$ [6]. Let $\mathcal{F}(P)$ be the set of all down-sets of $P$, which is closed under intersections and unions, and is referred to as the down-set lattice. $^3$ If $\mathcal{I}$ contains the indices for a collection of random variables $(A_i : i \in \mathcal{I})$, we succinctly denote any subcollection $(A_i : i \in \mathcal{J})$, with $\mathcal{J} \subseteq \mathcal{I}$, of those random variables as $A_\mathcal{J}$.

In the context of examples, albeit with an abuse of notation, we denote the index of a message or its rate by a string of the elements of that set. For example, the message $M_{\{1,2,4\}}$ and its rate $R_{\{1,2,4\}}$ are denoted as $M_{124}$ and $R_{124}$, respectively. Similarly, the message index set $E = \{ \{1\}, \{2\}, \{1,2\}, \{1,2,3\} \}$ will be succinctly denoted by $E = \{1, 2, 12, 123\}$.

\footnote{For any positive integer $M > 0$, we denote $\{1, \ldots, M\}$ by $[1:M]$.}

\footnote{The set of down-sets is a lattice whose join and meet are given by union and intersection, respectively, as the set of down-set is closed under union and intersection.}

III. SUPERPOSITION-BASED INNER BOUND

Consider an arbitrary $K$-receiver DM BC, whose channel input takes values in $X'$ and whose message index set is $E$. For any subset $F$ of $E$ within $P = 2^{[1:K]}$, let $X'_F = (X, U_F, Q)$ refer to a collection of random variables which includes the channel input $X$, $|F|$ auxiliary random variables $U_F = (U_S : S \in F)$, and a coded time-sharing random variable $Q$. Let $F$ be ordered by a superposition order (denoted as $\leq$).

Definition 1: $X'_F = (X, U_F, Q)$ is superposition-admitting if $X$ is a deterministic function of $(U_F, Q)$, whose joint probability mass function factors as

$$p(U_F, Q) = p(Q) \prod_{S \in F} p(U_S | U_{\{1\} \setminus \{S\}}, Q)$$

Let $A_{E, \leq}$ contain all superposition-admitting random variables with respect to the ordered set $F$. For any set of random variables within $X'_F \subseteq A_{E, \leq}$, define

$$\mathcal{P}^j_{\downarrow}(X'_F) = \{ R \in \mathbb{R}^{+} : \sum_{S \in B} R_S \leq I(U_B; Y_j | U_{W^F_B}, Q) \text{ for all } B \in \mathcal{F}_j(W^F) \}$$

a subset of $\mathbb{R}^{+}$. Note that this polyhedron imposes no constraints on the nonnegative rates $(R_S : S \in F \setminus W^F)$. Our principal result is the following inner bound.

Theorem 1: For the $K$-receiver DM BC with general message sets, the non-negative rates $(R_S : S \in E)$ are achievable if, for a message index superset $F$ with $E \subseteq F$ equipped with a superposition order, there exist non-negative split-rates $(r_{S \rightarrow S'} : S' \in E, S' \subseteq F, S \subseteq S')$, where the desired rates satisfy

$$R_S = \sum_{S \subseteq S', S' \in F} r_{S \rightarrow S'}$$

while the reconstructed rates

$$\hat{R}_{S'} = \sum_{S \subseteq S', S' \in F} r_{S \rightarrow S'}$$

are constrained to be within the rate region

$$\bigcup_{X' \in A_{E, \leq}} \left( \mathcal{P}^j_{\downarrow}(X'_F) \cap \cdots \cap \mathcal{P}^K_{\downarrow}(X'_F) \right)$$

Furthermore, the projection of each polyhedron $\mathcal{P}^j_{\downarrow}(X'_F)$ onto the cone of non-negative rates with indices in $W^F_j$ is a polymatroid.

Proof: We sketch the achievability proof in two parts. First, we show that a set of rates $(\hat{R}_{S'} : S' \in F)$ are achievable for the enlarged message index set $F$ if they are within the rate region (6). This follows from superposition encoding at the single transmitter, where we use the order-theoretic framework we developed in [18] to analyze the probability of error at each receiver. Details for this argument are provided in the Appendix.

\footnote{For a set $E$, we use the notation $\mathbb{R}^{E}$ to refer to a vector of real numbers, where the components are labeled by the elements of $E$. Thus, if $E$ has $M$ elements, this space can be identified with $\mathbb{R}^{M}$. Replacing $\mathbb{R}$ with $\mathbb{R}^{+}$ denotes the positive orthant of the previously described spaces.}
Next, we allow for partially decoding interference through rate-splitting. Divide each message $M_S$ into a collection of split-messages $(m_{S \to S'} : S \in \mathcal{E}, S' \in \mathcal{F}, S \subseteq S')$. In this collection, the partial message $m_{S \to S'}$ is to be treated as though it were intended for all messages within the set $S'$ rather than only for the messages within the set $S$. Whenever the receiver decodes a partial message $m_{S \to S'}$, with $S$ as a strict subset of $S'$, that receiver is partially decoding the interfering message $m_S$. With $r_{S \to S'}$ as the rate of the split message $m_{S \to S'}$, this rate-split decomposes the target rate $R_S$ according to (4). Each receiver within the group $S' \in \mathcal{F}$ must decode the reconstructed message $M_{S'} = (m_{S \to S'} : S \in \mathcal{E} : S \subseteq S')$, whose rate $\hat{R}_S$ is given by (5). These reconstructed rates in turn can be reliably transmitted to their desired receivers if they satisfy (6), as previously mentioned. Finally, the fact that the projection of $P_\perp(j)(X'; F)$ onto the cone of non-negative rates with indices in $\mathcal{W}_j^F$ is a polymatroid follows from Theorem 3 of [18].

### A. Inner bound of Theorem 1 as a Minkowski sum

To delineate the two strategies that comprise Theorem 1, and to better understand the geometric structure of the associated inner bound, we express that inner bound as the Minkowski\(^5\) sum of a polytope, representing the rates achievable through superposition coding, and a cone, representing the rate gains possible through partial interference decoding. Observe that if $R_E$ is in the inner bound of Theorem 1, then for each $S \in \mathcal{E}$, 

$$R_S = r_{S \to S} + \sum_{S' \in \mathcal{F}, S' \supset S} r_{S \to S'} = (\hat{R}_S - \sum_{S' \in \mathcal{F}, S' \subset S} T_{S' \to S} - \sum_{S' \in \mathcal{F}, S' \subset S} r_{S' \to S}) + \sum_{S' \in \mathcal{F}, S' \supset S} r_{S \to S'} \quad (7)$$

where 

$$\Delta_S = \sum_{S' \in \mathcal{F}, S' \subset S} r_{S \to S'} - \sum_{S' \in \mathcal{F}, S' \subset S} r_{S' \to S} \quad (8)$$

These equations reveal that the rate-splitting effectively exchanges groupcasting rates between different groupcast labels. As some of these exchanges occur not just among the rates indexed by the message index set $\mathcal{E}$, but among the rates within the enlarged message index set $\mathcal{F}$, (7) is incomplete. A complete description embeds the original set of rate demands, which live in $\mathbb{R}^+_\times$, into $\mathbb{R}^+_\times$, by setting $R_S = 0$ for each $S \in \mathcal{F} \setminus \mathcal{E}$. For each $S \in \mathcal{F}$, the analogous statement to (7) is that $R_S = \hat{R}_S + \Delta_S$, where 

$$\Delta_S = -\sum_{S' \in \mathcal{F}, S' \subset S} r_{S' \to S} \quad (9)$$

To translate these observations into a geometrical characterization of the inner bound in Theorem 1, we introduce a few additional definitions. For each pair of distinct message indices $S \in \mathcal{E}, S' \in \mathcal{F}$ with $S \subset S'$, let the vector $e_{S \to S'}$ be the vector in $\mathbb{R}^F_+$ for which 

$$(e_{S \to S'})_A = \begin{cases} 1 & \text{if } A = S \\ -1 & \text{if } A = S' \\ 0 & \text{else} \end{cases} \quad (10)$$

Let $C^F_+$ denote the cone of vectors generated by the collection of vectors $\{e_{S \to S'} : S \in \mathcal{E}, S' \in \mathcal{F} \text{ and } S \subset S'\}$. Then the vector $\Delta = (\Delta_S : S \in \mathcal{F})$, whose elements are defined by (8), if $S \in \mathcal{E}$, or by (9), if $S \in \mathcal{F} \setminus \mathcal{E}$, is within the cone $C^F_+$. We will refer to this vector as the exchange-rate vector, as its elements reveal the effective rate exchange that rate-splitting entails. Recalling that the reconstructed rate point $R = (R_S : S \in \mathcal{F})$ is within the union of polytopes described by (6), these observations lead to the following geometrical characterization of Theorem 1.

**Theorem 2:** For the $K$-receiver broadcast channel with message index set $\mathcal{E}$, if $\mathcal{F}$ is a superset of $\mathcal{E}$ and is equipped with a superposition order, then the non-negative rates $(\hat{R}_S : S \in \mathcal{E})$ within 

$$\bigcup_{X' \in \mathcal{A}^F_{(\leq)}} \left( \bigcap_{j=1}^K P_\perp(j)(X'; F) + C^F_+ \right) \cap \mathbb{R}^F_+ \quad (11)$$

are achievable, where 

$$\mathbb{R}^F_+ = \{ R \in \mathbb{R}^F_+ : R_S = 0 \text{ if } S \in \mathcal{F} \setminus \mathcal{E} \} \quad (12)$$

### B. Specializations

We highlight two specializations of Theorem 1, corresponding to two alternative choices of superposition order on the superset $\mathcal{F}$ of $\mathcal{E}$. In general, the choice of superposition order corresponds to a choice in dependency in the generation of auxiliary codewords. One possibility is to generate auxiliary codewords that are conditionally dependent whenever possible. This corresponds to equipping $\mathcal{F}$ with the superposition order of set inclusion. In this case, the set of superposition-admitting random variables, $A^F_{(\subseteq)}$, contains those random variables $X' = (X, U_F, Q)$ where the density of the auxiliary random variables and time-sharing random variable factors as $p(Q) \prod_{S \in \mathcal{F}} p(U_S)(U_S : S \subseteq S')$, $Q$. The constituent polyhedra are denoted by $P(\subseteq)(X'; F)$, as defined in (3), where a subset $B$ of $\mathcal{W}_j^F$ is within $F_{(\subseteq)}(\mathcal{W}_j^F)$ if, for every $S \in \mathcal{B}$, and $S' \subseteq S$ with $S' \in \mathcal{F}$, $S' \in \mathcal{B}$. With this superposition order, Theorem 1 yields

**Corollary 1:** For the $K$-receiver broadcast channel with message index set $\mathcal{E}$, if $\mathcal{F}$ is a superset of $\mathcal{E}$ ordered by set inclusion, then the non-negative rates $(\hat{R}_S : S \in \mathcal{E})$ within the rate region 

$$\bigcup_{X' \in \mathcal{A}^F_{(\subseteq)}} \left( \bigcap_{j=1}^K P(j)(X'; F) + C^F_+ \right) \cap \mathbb{R}^F_+ \quad (13)$$

where $X' \in \mathcal{A}^F_{(\subseteq)}$, are achievable.

Another possibility is to generate auxiliary codewords that are all conditionally independent given a coded time-sharing
sequence. This corresponds to equipping $F$ with the discrete order, so that every pair of sets within $F$ are incomparable. The set of superposition admitting random variables, $\mathcal{A}_c^F$, contains those random variables $X' = (X, U, Q)$ for which the auxiliary random variables $(U_S : S \in F)$ are mutually independent, conditioned on the time-sharing random variable $Q$. The constituent polyhedra are denoted by $\mathcal{P}_c^{(j)}(X'; F)$, where the down-set lattice $F_{c \in \mathcal{F}}(W_j^F)$ contains every subset $B \subseteq W_j^F$. With this superposition order, Theorem 1 yields Corollary 2: For the $K$-receiver broadcast channel with message index set $E$, if $F$ is a superset of $E$ ordered by the discrete order, then the non-negative rates $(R_S : S \in E)$ within the rate region

$$\bigcup_{X' \in \mathcal{A}_c^F} \left(\bigcap_{j=1}^K \mathcal{P}_c^{(j)}(X'; F) + C_1^F\right) \cap R_+^{c \rightarrow E}$$ (14)

where $X' \in \mathcal{A}_c^F$, are achievable.

IV. EXAMPLES OF RATE REGIONS FOR TWO- AND THREE-RECEIVER BCs

In this section, we show that the inner bound in Theorem 1 is sufficiently general to include previous characterizations of superposition coding in the BC.

A. Two-receiver BC

1) Degraded Messages: Consider the two-user BC in the special case where $R_2 = 0$. This is the degraded message case, with $E = \{1, 12\}$. Korner and Marton [13] determine that the capacity region contains the non-negative rates for which

$$R_{12} \leq I(U_1; U_{12}; Y_2)$$

(15a)

$$R_1 \leq I(U_1; Y_1 | U_{12})$$

(15b)

$$R_1 + R_{12} \leq I(U_1; Y_1)$$

(15c)

for some pair $(U_1, U_{12})$, where $X = U_1$ and $U_{12} \rightarrow U_1 \rightarrow Y$ form a Markov chain. Achievability of this rate region follows from Corollary 1 when $F = E = \{1, 12\}$ and both rate-splitting and time-sharing are omitted.

2) Cover’s Inner Bound: For the two-receiver DM BC with message index set $E = \{1, 2, 12\}$, Cover [5] determines that any non-negative rate point $(R_1, R_2, R_{12})$ is achievable if

$$R_1 \leq I(U_1; U_{12}, Y_1 | Q)$$

(16a)

$$R_{12} \leq I(U_{12}; U_1, Y_1 | Q)$$

(16b)

$$R_1 + R_{12} \leq I(U_1, U_{12}; Y_1 | Q)$$

(16c)

$$R_2 \leq I(U_2; U_{12}, Y_2 | Q)$$

(16d)

$$R_{12} \leq I(U_{12}; U_2, Y_2 | Q)$$

(16e)

$$R_2 + R_{12} \leq I(U_2, U_{12}; Y_2 | Q)$$

(16f)

for a channel input $X$ that is a deterministic function of a time-sharing random variable $Q$ and a set of auxiliary random variables $U_1, U_2, U_{12}$ that are independent when conditioned on $Q$.

This inner bound follows from Corollary 2 when $F = E$ and rate-splitting is omitted. By the conditional independence of the auxiliary random variables, $I(U_B; U_{\{1,12\}}; Y_1 | Q) = I(U_B; Y_1 | U_{\{1,12\}}; B, Q)$ for each non-empty subset $B \subseteq \{1, 2\}$, and analogously for the terms involving the second receiver. Hence, the conditions that $(R_1, R_2, R_{12})$ be within the polyhedron $\mathcal{P}_{c \in \mathcal{F}}^{(1)}(X'; \{1, 2\})$ and $\mathcal{P}_{c \in \mathcal{F}}^{(2)}(X'; \{1, 2\})$, when $X' = (X, U_1, U_2, U_{12}, Q)$, are the conditions (16a)-(16c) and (16d)-(16f), respectively.

3) Rate-Splitting and Partial Interference Decoding for Two-Receiver BC: The inner bound of Theorem 1 is implicitly described in terms of split-rates as well as the desired rates. In principle, the split-rates can be projected away with Fourier Motzkin, but in practice, this is only possible for very small settings. For example, in the two-user BC with message index set $E = \{1, 2, 12\}$, with the order relation taken to be that of subset inclusion, for each set of input, auxiliary, and coded time-sharing random variables $X' = (X, U_1, U_2, U_{12}, Q) \in \mathcal{A}_c^F$, the polyhedron $\mathcal{P}_{c \in \mathcal{F}}^{(1)}(X'; E) \cap \mathcal{P}_{c \in \mathcal{F}}^{(2)}(X'; E)$ consists of those non-negative rates which satisfy

$$R_1 + R_{12} \leq I(U_1, U_{12}; Y_1 | Q)$$

(17a)

$$R_2 + R_{12} \leq I(U_2, U_{12}; Y_2 | Q)$$

(17b)

$$R_1 + R_2 + R_{12} \leq I(U_2, U_{12}; Y_2 | Q) + I(U_1; Y_1 | U_{12}, Q)$$

(17c)

$$R_1 + R_2 + R_{12} \leq I(U_2; Y_2, U_{12}, Q) + I(U_1, U_{12}; Y_1 | Q)$$

(17d)

Note that in this two-user setting, the message index set $E$ is as large as it can be, so necessarily $F = E$.

B. Three-receiver BCs

1) A class with with degraded message sets: For the three-receiver BC with message index set $E = \{1, 123\}$ and the degraded receiver pair $X \rightarrow Y_1 \rightarrow Y_2$, Nair and El Gamal [15] determine that the capacity region is given by the non-negative rates for which

$$R_{123} \leq \min\{I(U; Y_2), I(V; Y_3)\}$$

(18a)

$$R_1 \leq I(X; Y_1 | U)$$

(18b)

$$R_1 + R_{123} \leq I(V; Y_3) + I(X; Y_1 | V)$$

(18c)

for some pair $U, V$ for which $U \rightarrow V \rightarrow X$. Further, by optimizing over the choice of auxiliary random variables, it can be shown that every rate in the capacity region satisfies (18) and the additional inequality

$$R_1 \leq I(X; Y_1 | V) + I(V; Y_3 | U)$$

(19)

for some pair $U, V$ for which $U \rightarrow V \rightarrow X$ [15].

The inner bound of Corollary 1 is equal to this second capacity characterization. To achieve this rate region, it suffices to omit coded time-sharing and consider rate-splitting in the context of the enlarged message index set $F = \{1, 13, 123\}$. With these restrictions, Theorem 1 states that the rates $(R_1, R_{123})$ are achievable if there exist non-negative split-rates $r_{1 \rightarrow 1} = R_1, r_{1 \rightarrow 13} = R_{13}$, and $r_{123 \rightarrow 123} = R_{123}$ so
that $R_1 = \hat{R}_1 + \hat{R}_{13}$, $R_{123} = \hat{R}_{123}$, and
\[
\begin{align*}
\hat{R}_1 &\leq I(U_1; Y_1 | U_{13}, U_{123}) & (20a) \\
\hat{R}_{13} &\leq I(U_1, U_{13}; Y_1 | U_{123}) & (20b) \\
\hat{R}_1 + \hat{R}_{13} &\leq I(U_1, U_{13}; U_{123}; Y_1) & (20c) \\
\hat{R}_{123} &\leq I(U_{123}; Y_2) & (20d) \\
\hat{R}_{13} &\leq I(U_{13}; Y_3 | U_{123}) & (20e) \\
\hat{R}_{13} + \hat{R}_{123} &\leq I(U_{13}; U_{123}; Y_3) & (20f)
\end{align*}
\]
for some triple $(U_1, U_{13}, U_{123})$ where $X = U_1$ and $U_{123} \rightarrow - U_{13} \rightarrow - U_1$ form a Markov chain. With Fourier-Motzkin, and the degraded assumption on the first two receivers, this demonstrates the achievability of any rate pair that satisfies (18) and (19), with $U = U_{123}$ and $V = U_{13}$.

C. Combination Networks

Consider a noiseless $K$-user broadcast network where the input is comprised of the $2^K - 1$ components $(V_S : \emptyset \subseteq S \subseteq [1:K])$, indexed by the non-empty subsets of $[1:K]$, and the $j$th output has noiseless access to those components that are indexed by $W_j$ so that $Y_j = (V_S : S \in W_j)$. For each $S \subseteq [1:K]$, the component $V_S$ is assumed to be within a finite alphabet $\mathcal{V}_S$; let $C_S = \log_2 |\mathcal{V}_S|$. When $K = 3$, and the message set demand contains all possible messages, so that $E = 2^{[1:K]}$, the capacity of this channel was determined in [12]. Corollary 1 provides an alternative proof of achievability of this result for a particular choice of auxiliary random variables, as discovered in [16].

That choice assigns
\[
U_S \sim \text{Uniform}(\mathcal{V}_S) \quad V_S = U_S \quad (21)
\]
so that $I(U_B; Y_j | U_{W_j \setminus B}) = \sum_{S \in B} C_S$. With computer-aided Fourier-Motzkin [11], the split-rates can be projected away, yielding the 15 defining inequalities of the capacity region. When the number of users is arbitrary, but a symmetry assumption of $C_S = C_{\{S\}}$ and $R_S = R_{\{S\}}$ is imposed across all $S \subseteq [1:K]$, then the capacity region was determined in [21] and [19]. In this setting, the specific auxiliary variable choice (21) within Corollary 1 yields an alternative proof of the achievability of the capacity region, as detailed in [16].

V. BINNING

A last element is to consider binning, which allows consideration of arbitrary input distributions. The central idea is to create an excessively large codebook, with rates $\hat{R}_S \geq R_S$ where each message has a list of codewords of exponential size $(2^{n(\hat{R}_S - R_S)})$, rather than a single codeword. If the rate excesses $\hat{R}_S - R_S$ are sufficiently large, then every message can jointly select a set of codewords that appear as though they were jointly generated with respect to an arbitrary joint distribution, rather than according to its recursive marginal distributions.

In particular, the excess rates will be $\hat{R}_S$, with the excess over the desired rate being $r_S = \hat{R}_S - R_S$ for each $S \in \mathcal{E}$. The key result is a recursive generalization of the mutual covering lemma of El Gamal and van der Meulen in [9] (see also [7, Lemma 8.1]).

**Lemma 1 (Recursive Mutual Covering Lemma):** Let $(U_S : S \in \mathcal{E})$ have arbitrary joint distribution $p(u_S : S \in \mathcal{E})$. Pick an order on $\mathcal{E}$. With respect to this order, recursively generate length-$n$ vectors
\[
u^n_S(m_S) \sim \prod_{i=1}^n p(u_{Si} | u_{Ri} : R \in \uparrow \{S\} \setminus \{S\})
\]
for each $m_S \in [1:2^{n r_S}]$ and each $S \in \mathcal{E}$. Then the probability that $(u_S(m_S) : S \in \mathcal{E})$ is jointly $\epsilon$-typical for some $(m_S : S \in \mathcal{E})$ tends to one as $n \to \infty$ if the non-negative rates $(r_S : S \in \mathcal{E})$ satisfy
\[
\gamma(G) \geq \sum_{S \in \mathcal{G}} H(U_S | U_{\uparrow(S)} \setminus \{S\}) - H(U_G) \triangleq \gamma(G), \quad (22)
\]
for all up-sets $G \subseteq \mathcal{E}$.

**Proof:** See Appendix II.

This unbounded polyhedron is a contra-polytoematid, defined only over the up-set lattice $\mathcal{F}_\uparrow$. This follows as
- $\gamma(G)$ is supermodular: $\gamma(F \cap G) + \gamma(F \cup G) \geq \gamma(F) + \gamma(G)$, a consequence of the submodularity of entropy.
- $\gamma(G)$ is non-increasing, a consequence of the fact that conditioning reduces entropy: for $F \subseteq G$,
\[
\gamma(G) - \gamma(F) = \left( \sum_{S \in G \setminus F} H(U_S | U_{\uparrow(S)} \setminus \{S\}) \right) - H(U_G | U_F) \\
\geq \sum_{S \in G \setminus F} \left( H(U_S | U_{\uparrow(S)} \setminus \{S\}) - H(U_S | U_F) \right) \\
\geq \sum_{S \in G \setminus F} \left( H(U_S | U_F) - H(U_S | U_F) \right) \\
= 0
\]
- $\gamma(G)$ is normalized: $\gamma(\emptyset) = 0$ as the sum is vacuous.

This new recursive mutual covering lemma provides a basis for a significant generalization of the Marton achievable scheme for the broadcast channel with private messages (given in [8, Section 8.6]) to a scheme that incorporates Marton-type coding with general forms of rate-splitting and superposition coding and is applicable to arbitrary group-casting message sets $\mathcal{E}$. Indeed, such a scheme was given in special cases (in number of users and/or message sets), all of which require only the mutual covering lemma, namely, in [14, Theorem 5] for the two-user BC with two private and one common message, in [15] (see [8, Proposition 8.2]) for the three-user broadcast channel with degraded messages, and in [20] for the $K$-user broadcast channel with the so-called diamond groupcasting message set. While these three rate regions were given as unions of explicit polytopes in message rate space (i.e., after Fourier Motzkin elimination) we provide here an implicit description of the inner bound in terms of the split-rates. Explicit descriptions of the inner bound of the following theorem may be possible in other special cases of the general groupcasting message set $\mathcal{E}$ (and choices of $F$).
Note that even the private message case where $E = \{1, 2, \ldots, K\}$ and any choice of $F \supseteq E$ would as such constitute a generalization of Marton’s inner bound (which corresponds to the no rate-splitting choice $F = E$), with that no rate-splitting bound thought to be the best known inner bound for the DM broadcast channel with private messages (see [8, Section 8.6]).

**Theorem 3 (Generalization of Marton Coding):** For the $K$-receiver DM BC with general message sets, the non-negative rates $(R_S : S \in E)$ are achievable if, for a message index superset $F$ with $E \subseteq F$ and with $F$ equipped with a superposition order $\leq$, there exist non-negative split-rates $(r_{S \rightarrow S'} : S \in E, S' \in F, S \leq S')$, where the desired rates satisfy

$$R_S = \sum_{S \subseteq S': S' \in F} r_{S \rightarrow S'}$$

for all $S \in E$ (23)

while the reconstructed rates

$$\hat{R}_S = \sum_{S \subseteq S': S' \in E} r_{S \rightarrow S'}$$

for all $S' \in F$ (24)

satisfy the binning constraints

$$\sum_{S \subseteq E} (\hat{R}_S - \hat{R}_S) \geq \gamma(G)$$

for every up-set $G \in \mathcal{F}_r(F; \leq)$ with the rates $(\hat{R}_S : S \in F)$ constrained to be within the rate region (with $X \triangleq (X, U, W)$ with respect to the join distribution on $\mathcal{X} \times U_F \times W$).

The general notion of superposition coding in [18] allows for a unified treatment of the achievability of the reconstructed rate regions in Theorem 1. That is, with the order-theoretic tools described therein, we will demonstrate that the reconstructed rates $(\hat{R}_S : S \in F)$ are achievable if they are within

$$\bigcup_{X' \in \mathcal{A}^E_{\leq}} \Big( \mathcal{P}_{\downarrow}^{(1)}(X'; F) \cap \cdots \cap \mathcal{P}_{\downarrow}^{(K)}(X'; F) \Big)$$

for some arbitrary distribution $p(u_F)$ and $X$ a deterministic function of $U_F$.

**Proof:** An outline of the proof is given. The proof involves rate-splitting and message reconstruction as in the achievable scheme used to prove Theorem 1 with a similar codebook generation scheme using superposition coding according to the general superposition order $\leq$ but each codebook is generated to have excess codewords (with the $U_S$ codebook having $2^{nR_S}$ codewords) which are assigned to $2^{nR_S}$ equi-sized bins (of size $2^{n(R_S - R_S)}$ codewords) indexing the reconstructed messages $m_S$. The selection of the set of codewords from the bins that are jointly typical incurs an error whenever such a set of jointly typical codewords does not exist and this encoding error can be made vanishingly small provided that the binning constraints (25) hold as dictated by the recursive mutual covering Lemma 1.

Prior works that combine superposition coding, rate-splitting with binning include the two-user broadcast channel with two private and one common messages [14] (see [8, Theorem 8.4]), the three-user case with three degraded messages in [15] (but with one receiver employing indirect joint decoding) and, more generally, to the diamond message set groupcasting in the $K$-user broadcast channel in [20]. However, in all of these schemes, the classical mutual covering lemma proved by El Gamal and van der Meulen [9] (cf. [8]) suffices for their analyses. Moreover, because in these problems Fourier Motzkin elimination was possible, direct descriptions of the rate regions are given therein as unions of polytopes in the space of the rates of the messages.

**VI. Conclusion**

With the tools of order theory, we provide a general inner bound for the DM BC with an arbitrary number of users and an arbitrary set of message demands, a setting which encompasses any type of concurrent groupcasting.

Prior results on the capacity region of the two-user broadcast channel with degraded messages and a three-user multi-level broadcast channel with two degraded messages can be seen through the lens of the general inner bound proposed in this work, as can other superposition coding based achievable schemes with joint desired message and partial interference decoding proposed in multiple unicast or private message settings.

**Appendix I**

**Achievability of reconstructed rates in Theorem 1**

The general notion of superposition coding in [18] allows for a unified treatment of the achievability of the reconstructed rate regions in Theorem 1. That is, with the order-theoretic tools described therein, we will demonstrate that the reconstructed rates $(\hat{R}_S : S \in F)$ are achievable if they are within

$$\bigcup_{X' \in \mathcal{A}^E_{\leq}} \Big( \mathcal{P}_{\downarrow}^{(1)}(X'; F) \cap \cdots \cap \mathcal{P}_{\downarrow}^{(K)}(X'; F) \Big)$$

where $\mathcal{A}^E_{\leq}$ is defined in Definition 1, and the polyhedra $\mathcal{P}_{\downarrow}^{(j)}(X'; F)$, for $j \in [1:K]$, were defined in (3). Superposition coding is a random coding strategy in which the dependencies between the codewords between distinct message sources follow an order on the message index set.

To prove the achievability of (27), equip the set $F$ with a superposition order. Fix a set of input, auxiliary, and coded time-sharing random variables $(X, U, W)$ such that $X$ is a deterministic function of the auxiliary and coded time-sharing random variables $(U, W)$, whose joint distribution factors as $p(Q) \prod_{S \in F} p(U_S)(U_S' : S' \in F, S' > S, Q)$. Let $\{S_1, \ldots, S_M\}$ be an non-increasing enumeration of $F$. First, generate the coded time-sharing sequence $q^n \sim \prod_{i=1}^n p(q_i)$. Then, for each $i \in [1:M]$, and each collection of messages $m_{T_i}(S_i) = (m_{S'} : S' \in F, S' > S_i)$, generate the codewords through $u_{S_i}^n(m_{T_i}(S_i)) \sim \prod_{i=1}^n p(u_{S_i}(u_{S',i} : S' \in F, S' > S), q_i)$. This process can be carried out iteratively from $i = 1$ to $i = M$ as $\{S_i\} \subseteq \{S_1, \ldots, S_M\}$ for each $i \in [1:M]$.

The $j$th receiver jointly decodes the messages $m_{W_j}$, while treating all other messages as noise. To describe and analyze this joint decoding rule and its error probabilities, let $\delta > 0$ be an infinitesimal typicality parameter and $\epsilon > 0$ be a target error probability guarantee. With $m \equiv m_{W_j}$, let $T(m)$ be the event that $(q^n, u_{S_i}^n(m_{T_i}(S_i)) : S \in W_j, y^n)$ is $\delta$-jointly typical with respect to the join distribution on $(Q, U_{W_j}, Y)$. The $j$th receiver declares the message estimates $\hat{m}$ to be the sent messages if and only if it is the unique set of messages for which $T(\hat{m})$ occurs. Without loss of generality, assume that the set message was $m = 1$, which is to say that $(m_{S} = 1 : S \in F)$. An error occurs if either a) the event $T(1)$ does not occur, or b) the event $T(\hat{m})$ occurs from some $\hat{m} \neq 1$. The law large numbers assures that the probability
of the first event vanishes in the block length $n$. For the second error event, we show that the probability of certain categories of error events must vanish if a certain partial sum-rate condition holds. With the union bound, we extend this argument to show that if all such possible partial sum-rate bounds hold, then the probability if any error must vanish. Let $B$ be any subset of $W_f^j$. By the joint typicality lemma [7] one can show that the probability that $T(m)$ occurs for a wrong message of the form $\hat{m}_S \neq 1$ for $S \in B$ and $\hat{m}_S = 1$ for $S \not\in B$ is bounded by $2^{-n(\sum_{U \in B} Y[U_{W_f^j} \setminus U]) - \epsilon_4}$ for a parameter $\epsilon_4$ that tends to zero as $\delta$ vanishes. Here, $C$ is the smallest down-set within $W_f^j$ containing $B$, succinctly denoted by $C = \downarrow B$ for the remainder of this proof [18]. Together with the union bound, a finite summation yields that the probability of error vanishes if $\sum_{S \in B} R_S \leq I(U_{W_f^j \setminus U} : Y[U_{W_f^j} \setminus U])$ for every $B \subseteq W_f^j$. By removing redundant inequalities, these conditions are equivalent to the requirement that $\sum_{S \in B} R_S \leq I(U_{W_f^j} : Y[U_{W_f^j} \setminus U])$ for every down-set $B$ in $W_f^j$, with respect to the particular order chosen on $F$. These are the defining inequalities within (3). Replacing this argument over all superposition-admitting random variables yields that the rate region (27) is achievable.

APPENDIX II

RECURSIVE MUTUAL COVERING LEMMA PROOF

Let $T_{\epsilon}(U_E)$ be the set of jointly $\epsilon$-typical length-$n$ sequences $\{u^n_S : S \in E\}$ with respect to the joint distribution of $U_E$. Let

$$A = \{(m_S : S \in E) : (u^n_S(m_S) : S \in E) \in T_{\epsilon}(U_E)\}$$

be the set of all independently randomly generated vectors which appear as though they were jointly generated (by being jointly typical). Chebyshev’s inequality supplies $P(|A| = 0) \leq \text{Var}(|A|)/E(|A|)^2$. The probability mass function governing the distribution of a codeword tuple $(u^n_S(m_S) : S \in E)$ is independent of the message tuple $m \equiv (m_S : S \in E)$. Thus each codeword tuple has the same probability of being jointly typical, which we define to be

$$P(U^n_E(m) \in T_{\epsilon}(U_E)) = P(U^n_E(1) \in T_{\epsilon}(U_E)) = p.$$  

By linearity of expectation, $E(|A|) = 2^n r(E) p$, where $r(E) \triangleq \sum_{S \in E} r_S$. With $[.]$ denoting the indicator function, introduce

$$B(m(E), m'(E)) = \left\{(u^n_{\delta}(m_E) : m_E \in T_{\epsilon}(U_E)) \right\}.$$ 

Then we may write

$$E(|A|^2) = \sum_{m(E), m'(E)} E[B(m(E), m'(E))]$$

$$= \sum_{m \in E} \sum_{m' \in E} E[B(m(E), m'(E))]$$

$$\leq \sum_{m \in E} 2^n (2(n(D) + r(E,D))) p_D,$$

where $p_D = E[B(m(E), m'(E))]$ with $(m_S = 1 : S \in E)$ and $(m'_S = 1 : S \not\in D)$ but $(m'_S = 2 : S \in D)$, more succinctly stated as $m = 1$ and $m' = 1 + 1(D)$ where $1(D) = (\{S \in D : S \in E\})$ is the indicator vector for the subset $D$ of $E$. Note that the terms corresponding to $D = E$ or $E = \emptyset$ are easily computed: $p_E = p^2$ and $p_E = p$. In fact, the term corresponding to $D = E$ is simply $|E| |A||^2$, and thus

$$\text{Var}(|A|) \leq \sum_{D \subset E} 2^n (2(n(D) + r(E,D))) p_D,$$

where the inequality in the summation indices is strict. By the recursive generating procedure, if $m_S \neq m'_S$ for some $S \in E$, then all the vectors corresponding to $R \subseteq \{S\}$ appear as though they were independently generated, even if the indices $m_R$ and $m'_R$ match. In other words,

$$p\left(u^n(1) = u^n, U^n(1 + 1(D)) = v^n\right) = \prod_{i=1}^n \prod_{S \in E} p(u_{S_{\uparrow}}|u_{S_{\downarrow}}) p(v_{S_{\downarrow}}|v_{S_{\downarrow}}) \times \prod_{S \in E \setminus D} p(u_{S_{\uparrow}}|u_{S_{\downarrow}})$$

(28)

for all potential sequences $u^n, v^n$ with $u_S = v_S$ when $S \not\in D$. If these potential sequences are in addition also jointly $\epsilon$-typical (denote the set of such sequences with $\mathcal{T}(\epsilon, D)$), then

$$p\left(u^n(1) = u^n, U^n(1 + 1(D)) = v^n\right) \leq 2^{-n(d_0 - \delta_e/6)},$$

where

$$d_0 = 2 \sum_{S \in E} H(U_S|U_{S_{\downarrow}}) + \sum_{S \in E \setminus D} H(U_S|U_{S_{\downarrow}}).$$

Consider the set of sequences $v^n$ that are jointly $\epsilon$-typical with respect to the joint distribution on $U_E$ and have components in $E \setminus D$ fixed to $v_S = u_S$ for $S \in E \setminus D$, which we denote by $\mathcal{S}(\epsilon, u_{E \setminus D})$. Then by standard arguments, $\log |\mathcal{S}(\epsilon, u_{E \setminus D})| \leq nH(U_E|U_{E \setminus D})$. Combining the above observations yields a bound on the probability that the pair of sequences $U^n(1) = u^n, U^n(1 + 1(D)) = v^n$ are jointly typical:

$$p_D = \sum_{u^n_E, v^n_E \in \mathcal{T}(\epsilon, D)} p\left(U^n(1) = u^n, U^n(1 + 1(D)) = v^n\right)$$

$$= \sum_{u^n_E, v^n_E \in \mathcal{T}(\epsilon, D)} 2^{-n d_0}$$

$$= \sum_{u^n_E} \sum_{v^n_E \in \mathcal{S}(\epsilon, u_{E \setminus D})} 2^{-n d_0}$$

$$\leq 2^n H(U_E) + n H(U_D|U_{E \setminus D}) - n d_0 - \delta_e^2/2.$$ 

This bound, combined with the bound

$$\frac{1}{n} \log p \geq H(U_E) - \sum_{S \in E} H(U_S|U_{S_{\downarrow}}) + \delta_e/4$$

yields

$$\frac{\text{Var}(|A|)}{E(|A|)^2} \leq \sum_{D \subset E} 2^{-n \triangle(D)}$$

for some $\triangle(D)$.
where

\[ \triangle(D) = r(E \setminus D) - \left( \sum_{S \in E \setminus D} H(U_S | U_{\uparrow 1 \setminus \{S\}}) - H(U_{E \setminus D}) \right) \]

Thus the error tends to zero if the conditions

\[ r(G) \geq \sum_{S \in \mathcal{X}'(G)} H(U_S | U_{\uparrow 1 \setminus \{S\}}) - H(U_G), \]

where \( \mathcal{X}'(G) \) is the largest up-set contained within \( G \), hold for all subsets \( G \subseteq E \). But not all of these inequalities are necessary. Observe that as the rates are non-negative, and \( \mathcal{X}(G) \) is a subset of \( G \),

\[ r(G) \geq r(\mathcal{X}(G)) \geq \sum_{S \in \mathcal{X}'(G)} H(U_S | U_{\uparrow 1 \setminus \{S\}}) - H(U_{\mathcal{X}(G)}) \]

\[ \geq \sum_{S \in \mathcal{X}'(G)} H(U_S | U_{\uparrow 1 \setminus \{S\}}) - H(U_G). \]  

(29)

Thus, the inequality corresponding to \( \mathcal{X}'(G) \) implies the inequality corresponding to \( G \). As such, it suffices to only enforce those inequalities corresponding to the up-sets of \( E \).

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