STRONG FILLABILITY AND THE WEINSTEIN CONJECTURE

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Abstract. Extending work of Chen, we prove the Weinstein conjecture in dimension three for strongly fillable contact structures with either non-vanishing first Chern class or with strong and exact filling having non-trivial canonical bundle. This implies the Weinstein conjecture for certain Stein fillable contact structures obtained by the Eliashberg-Gompf construction. For example we prove the Weinstein conjecture for the Brieskorn homology spheres $\Sigma(2, 3, 6n - 1)$, $n \geq 2$, oriented as the boundary of the corresponding Milnor fibre. Furthermore, for tight contact structures on odd lens spaces, non-contractible closed Reeb orbits are found.

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1. Basic definitions and main results

In order to formulate our results, we need to make precise certain well-known notions.

Co-orientable contact 3-manifolds. Let $(M, \xi)$ be a co-orientable contact 3-manifold, this is a 3-manifold $M$ equipped with a co-orientable contact structure $\xi$; the latter is by definition the kernel of some 1-form $\lambda$ on $M$ such that the 3-form $\lambda \wedge d\lambda$ is a volume form. We call each such $\lambda$ a $\xi$-defining contact form. We equip $M$ with the orientation induced by $\xi$, i.e., with the orientation defined by the non-vanishing top-rank form $\lambda \wedge d\lambda$, where $\lambda$ is any $\xi$-defining contact form. Moreover, we equip $\xi$ with a (bundle) orientation and denote the resulting oriented plane field by $\xi_+$. (Given the orientation of $M$, defining an orientation of $\xi$ is equivalent to defining a co-orientation of $\xi$, i.e., an orientation of the line bundle $TM/\xi_+$. ) There exists a $\xi$-defining contact form $\lambda$ such that the top-rank form $d\lambda|_\xi$ induces the given orientation on $\xi$; we call
each such \( \lambda \) a \( \xi_+ \)-defining contact form. We can identify \( \xi_+ \) with the positive conformal class \( \text{PC}(\xi_+) \) consisting of all \( \xi_+ \)-defining contact forms. The group \( C^\infty(M, \mathbb{R}_{>0}) \) of all positive-valued functions on \( M \) acts freely and transitively on \( \text{PC}(\xi_+) \) by multiplication.

**\( \lambda \)-Reeb orbits and links.** For each element \( \lambda \) of \( \text{PC}(\xi_+) \), the \( \lambda \)-Reeb vector field \( X_\lambda \) is defined to be the unique vector field \( X \) on \( M \) with \( \iota_X d\lambda = 0 \) and \( \lambda(X) = 1 \). The integral curves of \( X_\lambda \) are the \( \lambda \)-Reeb orbits. A finite, non-empty union of (necessarily embedded and disjoint) closed \( \lambda \)-Reeb orbits (i.e., \( \lambda \)-Reeb orbits diffeomorphic to \( S^1 \)) is called a \( \lambda \)-Reeb link.

**Hypersurfaces of contact type.** Consider a compact symplectic 4-manifold \((W, \omega)\) (possibly with boundary). We orient it by the volume form \( \omega \wedge \omega \). A closed hypersurface \( M \) in \( W \) is said to be of contact type if there exists a (necessarily (co-)oriented) contact structure \( \xi_+ \) on \( M \) and a contact form \( \lambda \in \text{PC}(\xi_+) \) satisfying \( d\lambda = i^* \omega \), where \( i : M \rightarrow W \) denotes the inclusion map. We write \((M, \lambda)\) for the hypersurface of contact type to indicate a particular choice of a contact form \( \lambda \) and we write \((M, \xi_+)\) if only the co-oriented contact structure is used.

**Strong fillings.** If the boundary \( \partial W \) of \( W \) is non-empty, we orient \( \partial W \) by \( \iota_\nu(\omega \wedge \omega) \), where \( \nu \) is an outward pointing vector field. If \( \partial W = M \) as oriented manifolds we call \( M \) strongly convexly fillable and \((W, \omega)\) a strong convex filling of \((M, \lambda)\). If \( \partial W = M \), where \( M \) is the manifold \( M \) with reverse orientation, we call \( M \) strongly concavely fillable and \((W, \omega)\) a strong concave filling of \((M, \lambda)\). We wish to point out that both concepts, being of ‘contact type’ and being a ‘strong filling’, correspond to the same convexity condition, the difference between the two concepts is only topological (hypersurface vs. oriented boundary).

Denoting by \( c_1(\xi_+) \in H^2(M, \mathbb{Z}) \) the first Chern class of the oriented 2-plane bundle \( \xi_+ \) on \( M \) (i.e., the first Chern class with respect to any complex structure on \( \xi_+ \) which is compatible with the orientation), we state our first result as:

**Theorem 1.1.** Let \((M, \xi_+)\) be a closed co-oriented strongly convexly fillable contact manifold and \( \lambda \) a \( \xi_+ \)-defining contact form in \( \text{PC}(\xi_+) \). If \( c_1(\xi_+) \neq 0 \) then there is a \( \lambda \)-Reeb link whose homology class is Poincaré dual to \(-c_1(\xi_+)\).

Before we come to the second result we shall make the following remark: the first Chern class \( c_1(W, \omega) \) of the symplectic manifold \((W, \omega)\) is by definition the first Chern class of the tangent bundle of \( W \) equipped with any \( \omega \)-compatible almost complex structure. Fixing an \( \omega \)-compatible almost complex structure \( J \), the canonical bundle \( K \) on \( W \) is defined to be \( \Lambda^2 T^{1,0} \), where \( T^* W \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1} \) is the eigenspace decomposition with respect to to eigenvalues \( i \) and \(-i \) of the induced action of \( J \). Notice that \( c_1(K) = -c_1(W, \omega) \) (see Example 21.7).

**Theorem 1.2.** Let \((M, \xi_+)\) be a closed co-oriented contact manifold and \((W, \omega)\) a strong convex filling. Suppose that \( c_1(W, \omega) \neq 0 \) and \( \omega \) is exact. Then for any \( \xi_+ \)-defining contact form \( \lambda \in \text{PC}(\xi_+) \) there exists a closed \( \lambda \)-Reeb orbit.
Remark 1.3. In fact, in the situation of Theorem 1.2 there is a \( \lambda \)-Reeb link whose homology class is Poincaré dual to \(-c_1(\xi_+)\) regardless whether the first Chern class of the contact structure vanishes or not. If \((W, \omega)\) is a strong convex filling of \((M, \xi_+))\), as in the situation of Theorem 1.1, then the identity \(i^*c_1(W, \omega) = c_1(\xi_+)\) holds, where \(i : M \to W\) denotes the inclusion map, because of the isomorphism \(TW|_M \cong \xi \oplus \mathbb{C}\) as complex vector bundles over \(M\). In particular, if \(c_1(\xi_+) \neq 0\) then also \(c_1(W, \omega) \neq 0\), meaning that if additionally \(\omega\) is exact then Theorem 1.2 follows from Theorem 1.1. But it may be possible that \(c_1(\xi_+) = 0\) without \(c_1(W, \omega)\) being zero.

Theorems 1.1 and 1.2 confirm certain cases of the Weinstein conjecture in the 3-dimensional case (cf. [47]); it asks for a closed \(\lambda\)-Reeb orbit for every contact form \(\lambda\) on a closed 3-manifold. We will say that the Weinstein conjecture holds true for \(M\) (for the oriented manifold \(M\), for the co-orientable contact structure \(\xi\)) if there is a closed \(\lambda\)-Reeb orbit for all contact forms \(\lambda\) on \(M\) (for all contact forms \(\lambda\) on \(M\) such that \(\lambda \wedge d\lambda\) induces the given orientation on \(M\), for all \(\xi_+\)-defining contact forms \(\lambda \in PC(\xi_+)\)) respectively. In this language Theorem 1.1 and Theorem 1.2 show that under the posed conditions the Weinstein conjecture holds true for the contact structure \(\xi\).

If \((M, \lambda)\) is a hypersurface of contact type in \((W, \omega)\) then one can consider the characteristic line bundle \(L_M = \ker(i^*\omega)\), where \(i : M \to W\) denotes the inclusion map. In this case the Weinstein conjecture asks for a closed characteristic of \(L_M\). For more about the extrinsic form of the conjecture and the state of the art of this problem we refer to [32, 19, 25].

In [30] Hofer proved the Weinstein conjecture for \(S^3\) (using Rabinowitz’ periodic orbit theorem, see [11, 47], and his results for overtwisted contact structures in [30]) and for all closed 3-manifolds \(M\) with non-trivial second homotopy group. Notice that any 3-manifold \(B\) which is covered by such an \(M\) satisfies the Weinstein conjecture. For example the Weinstein conjecture holds true for all lens spaces \(L_{p, q}\) for \(p > q \geq 1\) coprime (notice that \(L_{0, 1} = S^1 \times S^2\) and \(L_{1, 1} = S^3\)) and the Poincaré homology sphere \(\Sigma(2, 3, 5)\), which are universally covered by \(S^3\). Further, Hofer [30] verified the Weinstein conjecture for all overtwisted contact structures \(\xi\) on closed 3-manifolds. In particular the Weinstein conjecture holds true for all virtually overtwisted contact structures, i.e., for all tight contact structures for which the lift to a finite cover becomes overtwisted.

A co-orientable contact structure is called planar if there exists a supporting open book decomposition of the underlying closed 3-manifold (in the sense of Giroux, cf. [14]) which has genus zero pages (cf. [11, 15]). In [11] Abbas, Cieliebak and Hofer verified the Weinstein conjecture for planar contact structures. In Section 3 we will prove:

**Theorem 1.4.** The Weinstein conjecture holds true for the positively oriented Brieskorn homology spheres \(+\Sigma(2, 3, 6n - 1), n \geq 2\).

All closed \(\lambda\)-Reeb orbits found in [30] are contractible (a covering induces an injection on \(\pi_1\)) and the \(\lambda\)-Reeb links obtained in [11] are zero in integral homology of the underlying closed 3-manifold. In Section 4 below we will establish non-contractible (and in fact not null-homologous)
\(\lambda\)-Reeb orbits for so-called odd lens spaces (proving Theorem 1.5 below). We call the lens space \(L_{p,q}\) odd, and will write \(L_{p,q}^{\text{odd}}\), if there is at least one odd integer \(n_i\) in the associated continued fraction expansion \([n_1, \ldots, n_k]\) of the fraction \(-\frac{p}{q}\).

**Theorem 1.5.** Let \(p > q \geq 1\) be coprime integers. For all odd lens spaces \(L_{p,q}^{\text{odd}}\) and all tight contact structures \(\xi\) there exists a non-contractible \(\lambda\)-Reeb orbit for all \(\lambda \in \text{PC}(\xi)\).

The proof of Theorem 1.1 and Theorem 1.2 has two main ingredients, which we shall state next. The first ingredient is due to Chen [5]. He proved the Weinstein conjecture for particular classes of contact type hypersurfaces in 4-manifolds using work of Taubes [45] on Seiberg-Witten equations and pseudo-holomorphic curves as well as stretching the neck, which is due to Hofer, Wysocki and Zehnder [31]. Some ideas in [5] are borrowed from [30, 13].

For a compact 4-dimensional manifold \(W\) (possibly with boundary) denote by \(b^+_2(W)\) the number of positive eigenvalues of the intersection form \(Q_W\) of \(W\) (see [28, Definition 1.2.1]).

**Theorem 1.6 (Chen).** Let \((W, \omega)\) be a closed connected symplectic 4-manifold with \(b^+_2(W) > 1\) and let \((M, \lambda)\) be a hypersurface of contact type in \((W, \omega)\). Set \(\xi := \ker \lambda\).

(1) If \(c_1(\xi) \neq 0\) then there exists a \(\lambda\)-Reeb link whose homology class is Poincaré dual to \(-c_1(\xi)\).

(2) If \(M\) bounds a submanifold \(\widehat{W}\) of \(W\) with \(c_1(\widehat{W}) \neq 0\) and \(\omega|_{\widehat{W}}\) is exact, then there exists a closed \(\lambda\)-Reeb orbit. In fact there is a \(\lambda\)-Reeb link whose homology class is Poincaré dual to \(-c_1(\xi)\).

Notice that Theorems 1.1 and 1.2 remove the assumption \(b^+_2(W) > 1\) in this result and establish its conclusion for all \(\xi\)-defining contact forms \(\lambda\); not only for those with \(d\lambda = i^*\omega\), where \(i : M \rightarrow W\) denotes the inclusion map (in the sequel we will also write \(\omega|_M\) for \(i^*\omega\)).

The second ingredient is that under certain circumstances a closed co-orientable contact 3-manifold \((M, \xi)\) can be realized as a hypersurface of contact type in a closed symplectic 4-manifold or in a compact symplectic 4-manifold with boundary equal to \((M, \xi)\). The following result is due to Etnyre and Honda [18, Theorem 1.3], see also [35, 20, 2, 12, 17] for previous work in this direction.

**Theorem 1.7 (Etnyre, Honda).** Any closed connected contact 3-manifold admits a connected strong concave filling. In fact there are infinitely many strong concave fillings which are mutually not homotopy equivalent and not related by a sequence of blow-downs and blow-ups.

This article is organized as follows: in Section 2 we translate our criterion to decide whether the Weinstein conjecture holds true for a Stein fillable contact structure given by Theorem 1.1 or Theorem 1.2 into the language of Legendrian \((-1)\)-surgery (see Corollary 2.3). This leads to a proof of the Weinstein conjecture for the positively oriented Brieskorn homology spheres \(\pm \Sigma(2, 3, 6n - 1), n \geq 2\), (see Section 3). In Section 4 we treat the odd lens spaces and prove Theorem 1.5. The main Theorems are established in two steps. The first one is made in Section...
Acknowledgements. The research presented in this article was carried out while I was supported by the DFG through the Graduiertenkolleg - Analysis, Geometrie und ihre Verbindung zu den Naturwissenschaften at the Universität Leipzig. I am grateful to my supervisor Prof. M. Schwarz for his steady help and encouragement. I would like to thank Paolo Ghiggini and Stephan Schönenberger for the stimulating e-mail conversation as well as John Etnyre for explaining me the proof of Theorem 1.7. I am indebted to Casim Abbas, Peter Albers, Shahram Biglari, Dragomir Dragnev, Frank Klinker, Otto van Koert, Matthias Kurzke, Rainer Munck, Marc Nardmann, Klaus Niederkrüger, Burak Özbağcı, Ferit Öztürk and Felix Schlenk for many helpful suggestions and comments.

2. Application to Stein fillable contact structures

In this section we describe the Eliashberg-Gompf construction of Stein fillable contact structures. It provides us with examples having computable topological invariants useful to decide whether the constructed contact structure satisfies the Weinstein conjecture.

A Stein 4-manifold with boundary (or simply a Stein surface with boundary) is a triple \((W, J, \varphi)\) consisting of a smooth 4-manifold \(W\) with non-empty boundary, a complex structure \(J\) on \(W\) such that there exists \(N \in \mathbb{N}\) with the property that \((\text{Int}(W), J)\) is biholomorphically equivalent to a complex submanifold of \(\mathbb{C}^N\), and a Morse function \(\varphi : W \to \mathbb{R}\) such that \(\varphi\) \mid_{\partial W}\) is constant and the 2-form 
\[
\omega_\varphi = -dJ^*d\varphi
\]
defines a symplectic structure on \(W\), where \(J^*\alpha = \alpha \circ J\) for all 1-forms \(\alpha\) on \(W\). For any non-empty regular level set \(\varphi^{-1}(c)\) the 1-form \(\lambda = -J^*d\varphi\mid_{\varphi^{-1}(c)}\) defines a co-oriented contact structure \(\xi_J\) on \(\varphi^{-1}(c)\). If the strong convex filling \((W, \omega)\) of \((M, \lambda)\) carries a Stein structure, that is, there exists a Stein 4-manifold with boundary \((W, J, \varphi)\) such that \(\omega = \omega_\varphi\) and \(\lambda = -J^*d\varphi\mid_{\partial W}\), we call \((W, \omega)\) a Stein filling of \((M, \lambda)\). It is unknown whether strong convex fillability implies Stein fillability of connected contact 3-manifolds. But there are examples of disconnected strongly convexly fillable contact manifolds, such that their corresponding fillings cannot carry any Stein structure (see \cite{3S} \cite{21}).

Suppose that the Stein surface with boundary \((W, J, \varphi)\) is a handlebody with only one 0-handle and \(m\) 1-handles where \(m \geq 0\). The induced contact structure \((\partial W, \xi_J)\) is contactomorphic to \((\#mS^1 \times S^2, \xi_0)\) where \(\xi_0\) denotes the standard contact structure on \(\#mS^1 \times S^2\), and \((W, J, \varphi)\) is the unique Stein filling of \((\#mS^1 \times S^2, \xi_0)\) \cite{27} \cite{9}.

In the remainder of this section we assume that the reader is familiar with \cite{27}. We call a link \(L = (K_1, \ldots, K_n)\), \(n \in \mathbb{N}\), in a contact 3-manifold \((M, \xi)\) Legendrian if the knots \(K_i\) are tangent to \(\xi\). Any Legendrian link in \((\#mS^1 \times S^2, \xi_0)\) is contact isotopic to a Legendrian link in so called standard form (see \cite{27} Definition 2.1 and Theorem 2.2). If a Legendrian knot \(K\) is in standard form one can define its Thurston-Bennequin invariant \(\text{tb}(K)\). In the special case that \(K\) is null-homologous, \(\text{tb}(K)\) is the linking-number of the knot \(K\) and the parallel push-off knot determined by the canonical framing, this is any vector field along \(K\) transverse to \(\xi_0\) respecting the co-orientation. In exactly the same way any normal vector field to \(K\) defines a framing.
With a Legendrian link \( L \) in standard form one can associate a second invariant – the rotation number \( \text{rot}(L) \) – defined in [27, Section 2] or [27, Formula 1.2]. In the case of a Legendrian link in \( (\mathbb{S}^3, \xi_0) \) the rotation number \( \text{rot}(L) \) equals the relative Chern number \( (c_1(\xi_0, \tau), [F]) \) of \( \xi_0 \) relative to a tangent vector field \( \tau \) along \( L \), evaluated on a Seifert surface \( F \) (i.e., \( \partial F = L \) and \( c_1(\xi_0, \tau) \in H^2(\mathbb{S}^3, L; \mathbb{Z}) \)). In other words, \( \text{rot}(L) \) is the degree of \( \tau \) with respect to any trivialisation of \( \xi_0|_F \). (This is independent of the choice of a Seifert surface \( F \) because \( c_1(\xi_0) \) vanishes.) The following Theorem is due to Eliashberg [8, 10] and Gompf [27].

**Theorem 2.1** (Eliashberg, Gompf). Let \( W \) be an oriented compact connected 4-manifold with non-empty boundary. Then \( W \) admits the structure of a Stein surface with boundary if and only if it carries a handlebody decomposition with the following properties:

1. There are no 3- and 4-handles.
2. \( W \) is built from the unique Stein filling of \((\#m\mathbb{S}^1 \times \mathbb{S}^2, \xi_0)\) by attaching 2-handles \( h_i \), \( i = 1, \ldots, n \), to \( K_i \) with framing \( \text{tb}(K_i) - 1 \), where \( L = (K_1, \ldots, K_n) \), \( n \in \mathbb{N} \), is a Legendrian link in \((\#m\mathbb{S}^1 \times \mathbb{S}^2, \xi_0)\) in standard form.

The handle decomposition of the Stein structure \((W, J, \varphi)\) is induced by \( \varphi \) and the first Chern class \( c_1(W, \omega_\varphi) \in H^2(W; \mathbb{Z}) \) is represented by a cocycle whose value on \([D]_1\) (the class of the core of \( h_i \) in \( H_2(W; \mathbb{Z}) \) oriented as at the end of [27, Section 1]) is equal to \( \text{rot}(K_i) \).

In the situation of Theorem 2.1 we will say that \((W, J, \varphi)\) is obtained from \((\#m\mathbb{S}^1 \times \mathbb{S}^2, \xi_0)\) by Legendrian \((-1)\)-surgery along \( L \). The surgered contact manifold \((\partial W, \xi_J)\) is Stein cobordant to \((\#m\mathbb{S}^1 \times \mathbb{S}^2, \xi_0)\) (see Section 5) and Stein filled by \((W, \omega_\varphi)\). Notice that [35, Theorem 1.2] implies that if \( J_1 \) and \( J_2 \) are two Stein structures on \( W \) with \( c_1(W, J_1) \neq c_1(W, J_2) \) then the induced contact structures \( \xi_{J_1} \) and \( \xi_{J_2} \) on \( \partial W \) are not isotopic.

**Remark 2.2.** Using [31, p. 49] and [27, p. 658] we have in more explicit terms that

\[
\text{PD}(c_1(W, \omega_\varphi)) = \sum_{i=1}^{n} \text{rot}(K_i)[C_i],
\]

where \([C_i]\) is the class of the cocore of the 2-handle \( h_i \) in \( H_2(W, \partial W; \mathbb{Z}) \) provided with the orientation mentioned in Theorem 2.1. By Remark 1.3 we also get

\[
\text{PD}(c_1(\xi_J)) = \sum_{i=1}^{n} \text{rot}(K_i)[\partial C_i],
\]

which is an element of \( H_1(\partial W; \mathbb{Z}) \). If the surgery is performed on \( \mathbb{S}^3 \) then the \([C_i]\) freely generate \( H_2(W, \partial W; \mathbb{Z}) \) and the \([\partial C_i]\) generate \( H_1(\partial W; \mathbb{Z}) \) with relations

\[
\sum_{j=1}^{n} \text{lkd}(K_i, K_j)[\partial C_j] = 0,
\]

where \((\text{lkd}(K_i, K_j))\) is the linking matrix of the link \( L \). In this special case (surgery along \( \mathbb{S}^3 \)), the orientation mentioned in Theorem 2.1 can be described as follows: orienting the knot \( K_j \) (for example as the boundary of an oriented Seifert surface \( F_j \)) together with the canonical
orientation of the ambient manifold $S^3$ gives an orientation of a small normal disc $N_j$ to $K_j$. We choose the orientation of the cocore $C_j$ such that the normal disc $N_j$ will represent the class $[C_j]$ with the same sign. Analogously, the boundary orientation of $\partial N_j$ determines the sign of $[\partial C_j]$. It turns out to be useful to remark that the change of the orientation of $K_j$ induces the multiplication of $[C_j]$, $[\partial C_j]$ and $\text{rot}(K_j)$, respectively, by the factor $-1$. Therefore, the signs of the summands in both equations (1) and (2) remain unchanged. But the sign of $\text{tb}(K_j)$ and the diagonal elements in $(\text{lk}(K_i, K_j))$ are not effected by orientation-reversing $K_j$. This means that the signs of the summands in equation (3) do not change either (of course $\text{lk}(K_i, K_j)$ changes sign for $i \neq j$).

Putting Theorems 1.2 and 2.1 together we obtain

**Corollary 2.3.** Let $(W, J, \varphi)$ be a Stein surface with boundary obtained from the unique Stein filling of $(\# n S^1 \times S^2, \xi_0)$ via Legendrian $(-1)$-surgery along $L = (K_1, \ldots, K_n)$, $n \in \mathbb{N}$. Suppose that the rotation number $\text{rot}(K_i)$ does not vanish for at least one $i \in \{1, \ldots, n\}$. Then the Weinstein conjecture holds true for the induced contact structure $\xi_J$ on $\partial W$. In fact there exists a $\lambda$-Reeb link whose homology class equals

$$-\sum_{i=1}^{n} \text{rot}(K_i)[\partial C_i]$$

for all $\lambda \in \text{PC}(\xi_J)$.

### 3. The Weinstein conjecture for the Brieskorn spheres

$\Sigma(2, 3, 6n - 1), \ n \geq 2$

Let $n$ be a natural number and let $\varepsilon \in \{0 < |z| < 1\} \subset \mathbb{C}$ be a fixed parameter. Following [28, p. 74] we define the Brieskorn manifold $\Sigma(2, 3, 6n - 1)$ as the oriented boundary of the compactified Milnor fibre

$$M_{\varepsilon}(2, 3, 6n - 1) = \{(x, y, z) \in \mathbb{C}^3 \mid x^2 + y^3 + z^{6n-1} = \varepsilon\} \cap \mathbb{D}^6.$$ 

Alternatively, $-\Sigma(2, 3, 6n - 1) = \Sigma(2, 3, 6n - 1)$ is the oriented boundary of the nucleus $N(n)$ (cf. [26]). The intersection form $Q_{N(n)}$ is unimodular (see [28, Figure 8.14]). Therefore, by [28, Corollary 5.3.12 and Remark 1.2.11] the Brieskorn manifold $\pm \Sigma(2, 3, 6n - 1)$ is in fact a homology sphere (see Remark 6.2 and is therefore called a Brieskorn homology sphere.

If $n \geq 2$ (the case $n = 1$ corresponds to the Poincaré homology sphere) then $\pi_1(\Sigma(2, 3, 6n - 1))$ is infinite (see [12, Section 1.1.3]) and not Abelian. Because $-\Sigma(2, 3, 6n - 1)$ admits a description as Seifert fibred homology sphere $M(-\frac{1}{2}, \frac{1}{3}, \frac{n}{6n - 1})$ (see [12, Section 1.1.4] and [29]) it follows that $\Sigma(2, 3, 6n - 1)$ is irreducible (see [29, Proposition 1.12]) and $\pi_i(\Sigma(2, 3, 6n - 1)) = 0$ for all $i > 1$ (see [29, Corollary 3.9]). In particular, $\Sigma(2, 3, 6n - 1)$ cannot be covered by a homotopy sphere. In fact, it follows from [39, Section 1] that the universal cover of $\Sigma(2, 3, 6n - 1)$ is the universal cover of $\text{PSL}(2, \mathbb{R})$ and hence equal to $\mathbb{R}^3$ (use that $\text{SL}(2, \mathbb{R})$ is diffeomorphic to $S^1 \times \mathbb{R}^2$).

Let $n = 2$ and consider $\Sigma(2, 3, 11)$ (which is equal to $M(-\frac{1}{2}, -\frac{1}{3}, -\frac{2}{11})$ in the notation above). By
Theorem 4.4] there exist, up to isotopy, exactly two tight contact structures $\xi_{\pm}$ on $\Sigma(2, 3, 11)$, both of which are Stein fillable. The tight contact structures $\xi_{\pm}$ are obtained by Legendrian $(-1)$-surgery along a Legendrian link $L$ in $S^3$ with Legendrian knots having $\text{rot} = 0$, except for exactly one Legendrian knot which has $\text{rot} = \pm 1$ (see [24 Section 4.1.4]). As Stephan Schönenberger pointed out to author, for $n \geq 3$ a similar statement is true, as can be seen by using the methods developed in [24]. On $\Sigma(2, 3, 6n - 1)$ there exist, up to isotopy, exactly two tight contact structures, both constructed by Legendrian $(-1)$-surgery along a Legendrian link $L$ in $S^3$ having $\text{rot} = \pm 1$. Combining Corollary 2.3 with the verified Weinstein conjecture for overtwisted contact structures confirmed in [30] and the above mentioned classification theorem (notice that if the Weinstein conjecture holds true for $\xi$ then the Weinstein conjecture follows trivially for all contact structures contactomorphic to $\xi$) we have

**Corollary 3.1.** The Weinstein conjecture holds true for $+\Sigma(2, 3, 6n - 1)$, $n \geq 2$.

By the above discussion this result is not covered by [30]. Indeed, as Paolo Ghiggini pointed out to author, the tight contact structures on $+\Sigma(2, 3, 6n - 1)$, $n \geq 2$, are universally tight: by [36] Theorem 1.3(a) and Corollary 2.2 there exists a universally tight contact structure on $+\Sigma(2, 3, 6n - 1)$, which must be isotopic to either $\xi_-$ or $\xi_+$. The contact structure $\xi_+$, which is the contact structure $\xi_+$ with orientation reversed, is isotopic to $\xi_+$. Because this operation, called conjugation, preserves (universal) tightness, the claim follows. (In fact, the non-isotopic tight contact structures $\xi_-$ and $\xi_+$ on $+\Sigma(2, 3, 11)$ are contactomorphic.)

Furthermore, at least one of the contact structures $\xi_{\pm}$ on $+\Sigma(2, 3, 6n - 1)$, $n \geq 2$, is not planar, and hence Corollary 3.1 does not follow from the result in [1]. Indeed, $M_c(2, 3, 6n - 1)$ carries a Stein structure (see [4] Section 1), which induces a contact structure on $\Sigma(2, 3, 6n - 1)$ and $b^+_2(M_c(2, 3, 6n - 1)) = 2(n - 1)$ (see [28] p. 74)). (An alternative strategy for $n = 2$ can be found in [44] Proof of Theorem 1.9.) With [15] Theorem 4.1, which tells us that $b^+_2(M_c(2, 3, 6n - 1))$ must vanish in the planar case, the claim follows.

On $-\Sigma(2, 3, 11)$ there exists exactly one tight contact structure $\xi_0$, and the contact structure $\xi_0$ is Stein fillable (see [24] Theorem 4.9]). It is not known whether there exists a Stein filling of $(-\Sigma(2, 3, 11), \xi_0)$ with non trivial Chern class. In [13] Remark 3.3.3 it is conjectured that any Stein filling of $(-\Sigma(2, 3, 11), \xi_0)$ is diffeomorphic to the nucleus $N(2)$. Notice that the only possible Stein structure on $N(2)$ has trivial first Chern class (use that $N(2)$ is simply connected, [28] Figure 12.81] and Theorem 2.1). Therefore we do not know whether our criterion applies in this situation or not. Further, by [36] Theorem 1.3(c) and Corollary 2.2] $\xi_0$ is universally tight (use that $1/2 < 3/5$, $1/3 < 2/5$ and $2/11 < 1/5$) and not planar (use again [15] Theorem 4.1] and that $b^+_2(N(2)) = 1$; see argumentation after Remark 6.2).

For a discussion of $-\Sigma(2, 3, 17)$ we refer the reader to [22]. Further we remark that our technique may apply to small Seifert fibred manifolds as studied for example in [23] [49].
4. Non-contractible Reeb orbits on odd lens spaces

For coprime natural numbers \( p \) and \( q \) satisfying \( p > q \geq 1 \) the lens space \( L_{p,q} \) is defined as the quotient \( \mathbb{S}^3/G_{p,q} \) of the unit sphere in \( \mathbb{C}^2 \) by the discrete subgroup \( G_{p,q} \) of all diagonal matrices \( \text{diag}(\zeta, \zeta^q) \in \text{U}(2) \) with \( \zeta^p = 1 \). Notice that \( \pi_2(L_{p,q}) = 0 \) and \( \pi_1(L_{p,q}) = \mathbb{Z}_p = H_1(L_{p,q}; \mathbb{Z}) \). There exists a unique continued fraction expansion

\[
\frac{-p}{q} = n_1 - \frac{1}{n_2 - \frac{1}{\ldots - \frac{1}{n_k}}},
\]

of \( \frac{-p}{q} \) which will be shortly denoted by \([n_1, \ldots, n_k]\). The classification theorem due to Honda [33] for tight contact structures on lens spaces \( L_{p,q} \) states that there exist exactly

\[ |(n_1 + 1)(n_2 + 1) \cdots (n_k + 1)| \]

tight contact structures up to isotopy. All the tight contact structures on \( L_{p,q} \) are obtained by Legendrian \((-1)\)-surgery on Legendrian links \( L \) in \( \mathbb{S}^3 \) (see [33]). These links \( L \) are linked chains \(((K_1, n_1), \ldots, (K_k, n_k))\) of framed unknots (i.e., they admit a Seifert surface diffeomorphic to \( \mathbb{D}^2 \)) with \( n_i = \text{tb}(K_i) - 1 \) for all \( i \in \{1, \ldots, k\} \) as shown in [33, Figure 16]. Therefore, all tight contact structures on \( L_{p,q} \) are Stein fillable by Theorem 2.1. The rotation number \( \text{rot}(K_i) \), for each \( i = 1, \ldots, k \), can have any of the following values

\[ n_i + 2, n_i + 4, \ldots, n_i + 2|n_i + 1|, \]

which are exactly the values allowed by the Bennequin-inequality \( \text{tb}(K_i) + |\text{rot}(K_i)| \leq -1 \) and the condition \( \text{tb}(K_i) + \text{rot}(K_i) \equiv 1 \) (mod 2), cf. [10].

Recall, that the lens space \( L_{p,q} = L^{\text{odd}}_{p,q} \) is called odd if there exists at least one odd integer \( n_i \) in the associated continued fraction expansion \([n_1, \ldots, n_k]\). As we already remarked in the introduction, the Weinstein conjecture holds true for all lens spaces. Alternatively, for odd lens spaces the Weinstein conjecture follows from Corollary 2.3 with the above mentioned isotopy classification theorem (notice that if the Weinstein conjecture holds true for \( \xi \) then the Weinstein conjecture follows trivially for all contact structures contactomorphic to \( \xi \)) and the verified Weinstein conjecture for overtwisted contact structures (see [30]). The following result is not covered by Hofer’s approach in [30].

**Proposition 4.1.** For all odd lens spaces \( L_{p,q}^{\text{odd}} \) and all tight contact structures \( \xi \) there exists a \( \lambda \)-Reeb link not homologous to zero in \( H_1(L_{p,q}^{\text{odd}}; \mathbb{Z}) \) for all \( \lambda \in \text{PC}(\xi) \).

**Proof of Theorem 1.5.** By Proposition 4.1 for all \( \lambda \in \text{PC}(\xi) \) there exists a \( \lambda \)-Reeb link not homologous to zero in \( H_1(L_{p,q}^{\text{odd}}; \mathbb{Z}) \). In particular, there exists a component of the \( \lambda \)-Reeb link which is not contractible. \( \square \)
Proof of Proposition 4.1. Let $W$ be the Stein filling manifold of $(L_{p,q}^{\text{odd}}, \xi_1, \ldots, \xi_k)$ obtained from $\mathbb{D}^4$ via Legendrian $(-1)$-surgery along the framed link $((K_1, n_1), \ldots, (K_k, n_k))$ in $\mathbb{S}^3 = \partial \mathbb{D}^4$ as described above, with

$$r_j := \text{rot}(K_j) \quad \text{for all } j \in \{1, \ldots, k\}.$$ \(\text{Recall that } n_j \leq -2 \text{ and notice that } n_j + r_j \equiv 0 \pmod{2} \text{ and } |r_j| \leq -n_j - 2.\)

We orient the knots $K_j$ in such a way that the linking matrix takes the form

$$(\text{lk}(K_i, K_j)) = \begin{pmatrix} n_1 & 1 \\ 1 & n_2 & 1 \\ & \ddots & \ddots \\ 1 & \cdots & 1 & n_k \end{pmatrix}$$

with respect to the (free) basis $[C_1], \ldots, [C_k]$ of $H_2(W, \partial W; \mathbb{Z})$, cf. Remark 2.2. The equations (4) as well as the equations (1), (2) and (3) in Remark 2.2 are valid regardless of which orientations we choose.

The image of $[C_j]$ in $H_1(L_{p,q}^{\text{odd}}; \mathbb{Z})$ under the connecting homomorphism is denoted by $c_j$. Because at least one of the $n_j$’s is odd the corresponding $r_j$ does not vanish. Therefore, by Corollary 2.3, there exists a $\lambda$-Reeb link in $L_{p,q}^{\text{odd}}$ representing the integral 1-homology class

$$x := -\sum_{j=1}^k r_j c_j,$$

for all $\lambda \in \text{PC}(\xi_{r_1}, \ldots, r_k)$. Using the relations (3) in Remark 2.2 and $c_0 = 0 = c_{k+1}$ we get

$$e_{j+1} = -e_{j-1} - n_{j+1}e_j, \quad \text{for } j = 1, \ldots, k.$$ \(\text{Hence, there are integers } e_j, j = 1, \ldots, k, \text{ unique modulo } p \text{ such that } c_{k+1-j} = e_jc_k. \text{ With } e_0 = 0 = e_{k+1} \text{ we get}\)

$$e_1 = 1 \quad \text{and} \quad e_{j+1} = -e_{j-1} - n_{k+1-j}e_j, \quad \text{for } j = 1, \ldots, k,$$

as well as

$$x = \left(\sum_{j=1}^k r_j e_{k+1-j}\right)(-c_k) \in H_1(L_{p,q}^{\text{odd}}; \mathbb{Z}).$$

Further, there are uniquely determined coprime integers $p_j > q_j \geq 1$ defined by

$$\frac{p_j}{q_j} = [n_{k+1-j}, \ldots, n_k] \quad \text{for } j = 1, \ldots, k.$$ \(\text{Notice that with } q_0 = 0 \text{ we get}\)

$$q_1 = 1 \quad \text{and} \quad q_{j+1} = -q_{j-1} - n_{k+1-j}q_j \quad \text{for } j = 1, \ldots, k,$$
as well as $q_k = q$, $q_{k+1} = p$ and $q_{j+1} > q_j \geq 1$ for all $j = 1, \ldots, k$. By (5) and (6) we find
\[ e_j \equiv q_j \pmod{p}, \quad \text{for } j = 0, \ldots, k + 1. \]
The claim is equivalent to $x \neq 0$ in $H_1(L_{p,q}^{\text{odd}}, \mathbb{Z})$ or $\sum_{j=1}^{k} r_j e_{k+1-j} \not\equiv 0 \pmod{p}$. Arguing by contradiction we suppose that $x = 0$. Then, representing the residual classes $e_j$ by the integers $q_j$ for all $j = 0, \ldots, k + 1$, either
\[ \sum_{j=1}^{k} r_j q_{k+1-j} = 0 \quad \text{or} \quad p \leq \left| \sum_{j=1}^{k} r_j q_{k+1-j} \right|. \]
Supposing that the latter is true we get, using (4) and (6),
\[ p \leq \sum_{j=1}^{k} r_j |q_{k+1-j}| \leq \sum_{j=1}^{k} (-n_j - 2)q_{k+1-j} \]
\[ = -2 \sum_{j=1}^{k} q_{k+1-j} + \sum_{j=1}^{k} q_{k-j} + \sum_{j=1}^{k} q_{k+2-j} \]
\[ = -2 \sum_{j=1}^{k} q_{k+1-j} + \sum_{j=2}^{k+1} q_{k+1-j} + \sum_{j=0}^{k-1} q_{k+1-j} \]
\[ = -2q_k - 2q_1 + q_1 + q_0 + q_{k+1} + q_k \]
\[ = -q - 1 + p. \]
This leads to $q \leq -1$ which is a contradiction. Therefore, with the first equation in (7) we get
\[ -r_1 q_k = \sum_{j=2}^{k} r_j q_{k+1-j}, \]
and so $q_k$ divides the right hand side of (8). Then either
\[ \sum_{j=2}^{k} r_j q_{k+1-j} = 0 \quad \text{or} \quad q_k \leq \left| \sum_{j=2}^{k} r_j q_{k+1-j} \right|. \]
Supposing that the latter is true we get with a similar reasoning as above
\[ q_k \leq -2 \sum_{j=2}^{k} q_{k+1-j} + \sum_{j=2}^{k} q_{k-j} + \sum_{j=2}^{k} q_{k+2-j} \]
\[ = -2q_k - 2q_1 + q_1 + q_0 + q_k + q_{k-1} \]
\[ = -q_k - 1 + q_k. \]
This leads to $q_{k-1} \leq -1$ which is a contradiction. So the first case in (9) is left. Therefore, (8) gives $r_1 q_k = 0$, i.e., $r_1 = 0$ and hence, by (4), $n_1 \equiv 0 \pmod{2}$. If we repeat this argument we end up with $r_k = 0$ and hence $n_k \equiv 0 \pmod{2}$. This shows that all $n_j$’s are even, contradicting our assumption on $L_{p,q}^{\text{odd}}$ to be odd. \qed
5. A directed cobordism

A hypersurface $M$ of a connected manifold $W$ is called \textit{separating} if $W \setminus M$ is not connected. In the case of a separating hypersurface of contact type Theorem 1.6 can be slightly extended to the following

\textbf{Proposition 5.1.} Let $(W, \omega)$ be a closed connected symplectic 4-manifold satisfying $b_2^+(W) > 1$ and let $(M, \xi_+)$ be a separating hypersurface of contact type in $(W, \omega)$. Then the implications (1) and (2) of Theorem 1.6 hold for any $\xi_+$-defining contact form $\lambda \in \text{PC}(\xi_+)$. 

We shall prove Proposition 5.1 by a gluing argument used before in \cite{5, 11, 16} and by directed cobordisms. Let $(W, \omega)$ be a compact symplectic 4-manifold with boundary $\partial W = M_1 \cup M_2$ and $M_1, M_2 \neq \emptyset$. If $(M_1, \lambda_1)$ and $(M_2, \lambda_2)$ are hypersurfaces of contact type, then $(W, \omega)$ is called a \textit{directed symplectic cobordism} from $(M_1, \lambda_1)$ to $(M_2, \lambda_2)$ and we will write $(M_1, \lambda_1) \prec_{\omega} (M_2, \lambda_2)$ instead of $(W, \omega)$ in this case. Notice that our terminology is borrowed from \cite{18} and differs from the one used in \cite{13}. Again, if the $\xi_+$-defining contact form $\lambda$ is not needed, we will use $\xi_+$ in our notation. Similar to the case of Stein fillings we call a directed symplectic cobordism $(M_1, \lambda_1) \prec_{\omega} (M_2, \lambda_2)$ a \textit{Stein cobordism} if it carries a Stein structure.

\textbf{Lemma 5.2.} Let $(M, \xi_+)$ be a closed contact manifold and $\lambda_j \in \text{PC}(\xi_+)$, $j = 1, 2$, $\xi_+$-defining contact forms. Then there exists a positive constant $c_{12} > 0$ and a directed symplectic cobordism $(M, \lambda_1) \prec_{\omega_{12}} (M, c_{12} \lambda_2)$ diffeomorphic to $[-1, 1] \times M$.

\begin{proof}
We use a construction from Ustilovsky’s thesis \cite[Section 3.6]{46}. We consider the \textit{symplectisation} $(\mathbb{R} \times M, d(e^\theta \lambda_1))$ of $(M, \lambda_1)$. There exists a function $f_{12} \in C^\infty(M, \mathbb{R}_{>0})$ on the compact manifold $M$ such that $\lambda_1 = f_{12} \lambda_2$. Let $R > 0$ be a constant which will be chosen later and $\beta \in C^\infty(\mathbb{R}, [0, 1])$ such that $\beta_{(-\infty, -1]} = 0$, $\beta_{[1, \infty)} = 1$ and $\beta' \geq 0$. Define a function $f \in C^\infty(\mathbb{R} \times M, \mathbb{R}_{>0})$ by

$$f(\theta, p) := e^{\theta + R} \left( (1 - \beta(\frac{\theta}{R})) f_{12}(p) + \beta(\frac{\theta}{R}) \right), \quad (\theta, p) \in \mathbb{R} \times M.$$ 

Note that

$$\partial_\theta f(\theta, p) = e^{\theta + R} \left( \frac{1}{R} \beta'(\frac{\theta}{R}) (1 - f_{12}(p)) + (1 - \beta(\frac{\theta}{R})) f_{12}(p) + \beta(\frac{\theta}{R}) \right)$$

for all $(\theta, p) \in \mathbb{R} \times M$. There exists $R_{12} > 0$ such that for all $R \geq R_{12}$ we have $\partial_\theta f > 0$ on $\mathbb{R} \times M$. Consider the closed 2-form $\omega_{12} := d(f \lambda_2)$ on $\mathbb{R} \times M$ with $f$ defined using $R = R_{12}$. Then

$$\omega_{12} \wedge \omega_{12} = 2 f df \wedge \lambda_2 \wedge d\lambda_2$$

and hence $\iota_{\partial_\theta f}(\omega_{12} \wedge \omega_{12}) = 2 f(\partial_\theta f) \lambda_2 \wedge d\lambda_2$.

Therefore, $\omega_{12}$ is a symplectic form on $\mathbb{R} \times M$ which equals $d(e^{\theta + R_{12} \lambda_1})$ on $(-\infty, -R_{12}] \times M$ and $d(e^{\theta + R_{12} \lambda_2})$ on $[R_{12}, \infty) \times M$. The symplectic manifold $([-R_{12}, R_{12}] \times M, \omega_{12})$ defines the claimed cobordism $(M, \lambda_1) \prec_{\omega_{12}} (M, c_{12} \lambda_2)$ with $c_{12} = e^{2R_{12}}$. \qed


Proposition 6.1. Let \((M, \lambda_1)\) be a separating hypersurface of contact type in \((W, \omega)\) and \(\lambda = \lambda_2 \in \text{PC}(\xi_+).\) Denote by \(W_\pm\) the closures of the components of \(W \setminus M\) and \(\omega_\pm := \omega |_{W_\pm},\) where the sign is chosen such that \((W_-, \omega_-)\) is the strong convex filling of \((M, \lambda_1)\) and \((W_+, \omega_+)\) is the strong concave filling of \((M, \lambda_1).\) There exist collar neighbourhoods \(U_\pm\) of \(M\) in \(W_\pm\) such that in the notation of the proof of Lemma 5.2 we have that \((U_-, c^{-1}_1 \omega_-)\) is symplectomorphic to \((-\varepsilon, 0] \times M, c^{-1}_2 d(e^\theta \lambda_1))\) and \((U_+, c_2 \omega_+)\) is symplectomorphic to \(([0, \varepsilon) \times M, c_2 \omega_1)\) for some \(\varepsilon > 0\) (cf. [16, Section 2]). Denote the corresponding symplectomorphisms by \(\varphi_\pm.\) Gluing \((W_-, c^{-1}_1 \omega_-), (M, c^{-1}_2 \lambda_1) \prec_{c_2 \omega_1} (M, \lambda_2), (M, \lambda_2) \prec_{\omega_1} (M, c_2 \lambda_1)\) and \((W_+, c_2 \omega_+)\) along the boundaries using the symplectomorphisms \(\varphi_\pm\) yields a symplectic manifold \((W', \omega')\) such that \((M, \lambda_2)\) is a hypersurface of contact type in \((W', \omega').\) Proposition 5.1 follows now from Theorem 1.6 because \(W\) and \(W'\) are homotopy equivalent. □

The construction in the proof of Proposition 5.1 can be used to glue directed symplectic cobordisms or both kinds of strong fillings along orientation-reversing contactomorphic contact manifolds appearing as boundary components. For that one must allow rescaling by positive constants of the corresponding symplectic or contact forms. Therefore, only the involved co-oriented contact structures and positive conformal classes of symplectic forms are respected. We recall the gluing construction from [11] (which also follows from the proof of Proposition 5.1):

Definition 5.3 (Gluing along boundaries of contact type). Let \((W_j, \omega_j)\) be a symplectic manifold with nonempty boundary and let \((M_j, \xi_+^j)\) be a hypersurface of contact type in \((W_j, \omega_j), j = 1, 2,\) such that \(M_j\) is a boundary component of \(W_j.\) Suppose that there exists an orientation-reversing contactomorphism \(\varphi_{12} : (M_1, \xi_+^1) \to (M_2, \xi_+^2).\) The manifold \(W_1 \cup_{\varphi_{12}} W_2\) obtained by gluing along \(M_1\) via \(\varphi_{12}\) carries a symplectic form whose restriction to \(W_1\) coincides with \(\omega_1.\) The resulting symplectic manifold is denoted by \((W_1 \cup_{\varphi_{12}} W_2, \omega_{\xi_+^1}).\)

6. Realisation and proof of the main theorems

An embedding \(f : (W_1, \omega_1) \to (W_2, \omega_2)\) of a symplectic manifold \((W_1, \omega_1)\) into a symplectic manifold \((W_2, \omega_2)\) is called iso-symplectic if \(f^* \omega_2 = \omega_1.\)

Proposition 6.1. Let \((M, \xi_+)\) be a closed contact 3-manifold and \((W, \omega)\) be a strong convex filling of it. Then for every \(n \in \mathbb{N}\) there exists a closed connected symplectic 4-manifold \((W(n), \omega(n))\) with \(b_2^+ (W(n)) \geq n\) such that \((W, \omega)\) admits an iso-symplectic embedding into \((W(n), \omega(n)).\) In particular \((M, \xi_+)\) is contactomorphic to a separating hypersurface of contact type in \((W(n), \omega(n)).\)

Proof. Set \((M_0, \xi_+^0) = (M, \xi_+)\) and \((W_0, \omega_0) = (W, \omega).\) Let \((M_j, \xi_+^j), j \in \{1, \ldots, n\},\) be a finite collection of closed connected contact manifolds, which we specify below. The iterated connected sum \(M_{\#^n} = \#_{j=0}^n M_j\) carries a co-oriented contact structure denoted by \(\xi_{\#^n}.\) There exists a directed symplectic cobordism

\[(W_{\#^n}, \omega_{\#^n}) := (M_{\#^n}, \xi_{\#^n}^0) \prec (M_{\#^n}, \xi_{\#^n}^n),\]
where \((M_{\mathcal{n}}, \xi_{\mathcal{n}}) = \bigsqcup_{j=0}^{\mathcal{n}} (M_j, \xi_j)\) (see [48]). (If \(M_0\) is not connected, perform additional connected sum surgeries along the ordered components. Hence \((M_0, \xi_0)\) is directed cobordant to a connected contact manifold. For simplicity we assume that \(M_0\) is already connected.) Let \((W_{\text{cav}}, \omega_{\text{cav}})\) be a strong concave filling of \((M\#^{\mathcal{n}} n, \xi\#^{\mathcal{n}} n)\) ensured by Theorem 1.7 and set
\[
(\widehat{W}, \widehat{\omega}) := W_0 \cup_{\xi_0} W_{\#^{\mathcal{n}} n} \cup_{\xi\#^{\mathcal{n}} n} W_{\text{cav}}.
\]
Then \(\widehat{W}\) is connected by construction. Suppose that for each \(j \in \{1, \ldots, n\}\) there exists a connected strong convex filling \((W_j, \omega_j)\) of \((M_j, \xi_j)\). Set
\[
(W(n), \omega(n)) := \bigsqcup_{j=1}^{\mathcal{n}} W_j \cup_{\xi_j} \widehat{W}.
\]

**Remark 6.2.** A closed orientable connected 3-manifold \(Y\) is called an (integral) homology sphere if the first integral homology vanishes, i.e., \(H_1(Y; \mathbb{Z}) = H_1(S^3; \mathbb{Z})\). If \(M_j\) is a homology sphere for each \(j \in \{1, \ldots, n\}\) then
\[
b_2^+(W(n)) = b_2^+(\widehat{W}) + \sum_{j=1}^{\mathcal{n}} b_2^+(W_j),
\]
because the intersection form splits as \(Q_{W(n)} = Q_{\widehat{W}} \oplus Q_{W_1} \oplus \ldots \oplus Q_{W_n}\) (see [28] Exercise 1.3.5.(b)*)).

For example we take the homology sphere \(M_j := \partial N(2)\); the boundary of the Gompf nucleus \(N(2)\) (see [26]). \(N(2)\) carries the structure of a Stein manifold with boundary inducing a contact structure on \(\partial N(2)\) and satisfies \(b_2^+(N(2)) = 1\) (see [43] Theorem 1.7 and Remark 3.3.1) or [35, p. 515]). Proposition 6.1 follows from Remark 6.2. \(\square\)

**Proof of Theorem 1.1 and Theorem 1.2** The claims follow from Proposition 6.1 (with \(n = 2\)) and Proposition 5.1. \(\square\)

We remark that it is known (in principle) how to construct symplectic 4-manifolds with \(b_2^+ > 1\) via surgery. For example the proof of the second part of Theorem 1.7 includes a method via fibre sum with elliptic surfaces along symplectic tori. In [34, 40] similar statements can be found.

We sketch a further approach. Proposition 6.1 can be obtained by the following argument whose germ was already used in [35] Theorem 3.2] (see [17] Lemma 3.1 and [44] Lemma 3.1]). If \(M\) is not a homology sphere then glue a directed cobordism \((M, \xi_+) \prec (M_1, \xi_+^1)\) to \((W, \omega)\) along \((M, \xi_+)\) where \(M_1\) is a homology sphere depending on \(M\) (we used the notation from Proposition 6.1 and assumed for simplicity that \(M\) is connected). Using Legendrian \((-1)\)-surgery one can find inductively directed cobordisms \((W_n, \omega_n)\) from \((M_n, \xi_n^0)\) to \((M_{n+1}, \xi_{n+1}^0)\) with \(b_2^+(W_n) \geq 1\), \(n \in \mathbb{N}\), where \(M_{n+1}\) is again a homology sphere. Using a concave filling of \((M_{n+1}, \xi_{n+1}^{a+1})\), gluing along the contact type boundaries yields a closed connected symplectic 4-manifold \((W(n), \omega(n))\) with \(b_2^+(W(n)) \geq n\) (see Remark 6.2).
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