Torus fibrations and localization of index

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ICM Satellite Conference, Daejeon, Korea, August 2014
Joint work with Hajime Fujita and Mikio Furuta:

1. H. Fujita, M. Furuta, and T. Y, *Torus fibrations and localization of index I*, J. Math. Sci. Univ. Tokyo 17 (2010), no. 1, 1-26.
2. _____, *Torus fibrations and localization of index II*, Comm. Math. Phys. 326 (2014), no. 3, 585-633.
3. _____, *Torus fibrations and localization of index III*, Comm. Math. Phys. 327 (2014), no. 3, 665-689.

These joint works are concerned with an index theory for a Dirac-type operator on a possibly noncompact Riemannian manifold.

Purpose of this talk

1. *an overview of the index theory for a Dirac-type operator on a possibly noncompact Riemannian manifold*

2. *applications to symplectic geometry*
In the context of the geometric quantization of Lagrangian fibrations and Hamiltonian T-actions, it is often observed

- $RR(M, \omega) = \# \text{ of Bohr-Sommerfeld fibers}$
  - $RR(M, \omega) \cdot \cdots \dim \text{ of the quantum Hilbert space of } Spin^c \text{ quantization}$
  - $\# \text{ of BS} \cdot \cdots \dim \text{ of the quantum Hilbert space of the geometric quantization using real polarization}$
- $RR_T(M, \omega)$ can be written as the sum of the contributions from the lattice points in the moment map image. (Ex. Danilov’s formula concerning the equivariant Riemann-Roch index for a projective toric variety)

One of the motivation is to understand the mechanism underlying these phenomena from the viewpoint of index theory.
Local index theory

Input data

- \((M, g)\): possibly non-compact Riemannian manifold
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- \(p: W \to M\): \(\mathbb{Z}_2\)-graded \(Cl(TM)\)-module bundle

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\begin{align*}
def & \iff W = W^0 \oplus W^1 \\
& \exists c: TM \to \text{End}_{\mathbb{C}}(W) \text{ such that } c(u)c(v) + c(v)c(u) = -2g(u, v)id_W
\end{align*}
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- \(V\): open subset of \(M\) whose complement \(M \setminus V\) is compact
- \(\pi: V \to U\): fiber bundle
  - a fiber is a torus
  - \(\pi\) is Riemannian submersion w.r.t. \(g\) and some Riem. metric on \(U\)
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- \(\pi: V \to U\): fiber bundle
  - a fiber is a torus
  - \(\pi\) is Riemannian submersion w.r.t. \(g\) and some Riem. metric on \(U\)
- \(D_{\text{fiber}} \circ \Gamma (W|_V)\): 1st order formally self-adjoint differential operator of degree-one satisfying
  - \(D_{\text{fiber}}\) contains only derivatives along fibers of \(\pi\)
  - \(\forall b \in U\ D_{\text{fiber}}|_{\pi^{-1}(b)} \circ \Gamma (W|_{\pi^{-1}(b)})\) is a Dirac-type operator of \(\pi^{-1}(b)\)
  - \(\ker (D_{\text{fiber}}|_{\pi^{-1}(b)}) = 0 \ \forall b \in U\)
  - The Clifford multiplication of \(TB\) anti-commutes with \(D_{\text{fiber}}\).
Main Theorem (Fujita-Furuta-Y.)

For these input data, \( \exists \, \text{ind}(M, V : W) \in \mathbb{Z} \) satisfying the following properties:

1. \( \text{ind}(M, V : W) \) is invariant under continuous deformation of the data.
2. If \( M \) is closed, then \( \text{ind}(M, V : L) \) is equal to the index of a Dirac-type operator.
3. For an open subset \( V' \subset V \) with \( M \setminus V' \) compact
   \[
   \text{ind}(M, V : W) = \text{ind}(M, V' : W).
   \]
4. Excision property. For an open neighborhood \( M' \) of \( M \setminus V \)
   \[
   \text{ind}(M, V : W) = \text{ind}(M', V \cap M' : W|_{M'}).\]
5. Gluing formula
6. Product formula
Outline of Proof

For \( t \geq 0 \) consider the following perturbation of the Dirac-type operator \( D \)

\[
D_t := D + t \rho D_{\text{fiber}},
\]

where \( \rho \) is a cut off function on \( V \). Then one can show

1. If \( M \) is compact and \( V = M \), then \( \ker D_t = 0 \) for a sufficiently large \( t \).
2. If \( V = N \times (0, \infty) \) and all the data are translation invariant on \( V \). From 1 one can deduce the following
   1. On \( V \) \( D_t = \alpha(\partial_r + D_{N,t}) \). Then, \( \ker D_{N,t} = 0 \) for a sufficiently large \( t \).
   2. \( \dim \ker D_t|_{W_0} \cap L^2 - \dim \ker D_t|_{W_1} \cap L^2 \) is independent of sufficiently large \( t \). (\( \therefore \) Atiyah-Patodi-Singer boundary condition)

Definition

\[
\text{ind}(M, V : L) := \dim \ker D_t|_{W_0} \cap L^2 - \dim \ker D_t|_{W_1} \cap L^2 \text{ for } \forall t \gg 0.
\]

3. For a general case,
   1. Deform \( V \) cylindrically so that all the data are translation invariant, and come down to the cylindrical case.
   2. Check the definition is independent of a choice of a cut locus.
Remarks on Main Theorem

Remark

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Remarks on Main Theorem

Remark

1. The perturbation used in the proof can be understood as an infinite dimensional analogue of Witten’s deformation.

2. Theorem can be modified to the following case where

   - $V = \bigcup_i V_i$,
   - each $V_i$ is equipped with a torus bundle $\pi_i : V_i \to U_i$,
   - the ranks of $\pi_i$’s can vary according to $i$,
   - $\pi_i$’s satisfy a certain compatibility condition on $V_i \cap V_j$.

   It is necessary to formulate a product formula. Such a case will be mentioned later.
Question

*Which situation provides such input data?*
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We will explain the following two cases provide such input data.

1. Lagrangian fibrations
2. Hamiltonian $S^1$-actions
The first Case - Lagrangian fibration

- \((M, \omega)\): symplectic manifold

Definition (Lagrangian fibration) \(f: (M, \omega) \to B\) is a Lagrangian fibration if

\[ f: \text{fiber bundle } f^{-1}(b): \text{Lagrangian} \quad \forall b \in B \]

Example \(f_0:\mathbb{R}^n \times \mathbb{R}/\mathbb{Z}^n, p_i = \frac{1}{2}dx_i \wedge dy_i \to \mathbb{R}^n\)

Theorem (Arnold-Liouville) A Lagrangian fibration \(f: (M, \omega) \to B\) with closed connected fibers is locally modeled on \(f_0: \mathbb{R}^n \times \mathbb{R}/\mathbb{Z}^n, p_i dx_i \wedge dy_i \to \mathbb{R}^n\).
The first Case - Lagrangian fibration

- \((M, \omega)\): symplectic manifold

**Definition (Lagrangian fibration)**

\[ f : (M, \omega) \to B \] is a Lagrangian fibration \(\iff\)

\[
\begin{cases} 
  f : \text{fiber bundle} \\
  f^{-1}(b) : \text{Lagrangian} \quad \forall \, b \in B
\end{cases}
\]
The first Case - Lagrangian fibration

- \((M, \omega)\): symplectic manifold

**Definition (Lagrangian fibration)**

\[ f: (M, \omega) \rightarrow B \text{ is a Lagrangian fibration} \mapsto \begin{cases} f: \text{fiber bundle} \\ f^{-1}(b): \text{Lagrangian} \quad \forall \ b \in B \end{cases} \]

**Example**

\[ f_0: (\mathbb{R}^n \times (\mathbb{R}/\mathbb{Z})^n, \sum_{i=1}^n dx_i \wedge dy_i) \cup (x_1, \ldots, x_n, y_1, \ldots, y_n) \rightarrow \mathbb{R}^n \cup (x_1, \ldots, x_n) \]
The first Case - Lagrangian fibration

Let $(M, \omega)$ be a symplectic manifold.

**Definition (Lagrangian fibration)**

A function $f: (M, \omega) \to B$ is a Lagrangian fibration if

$$f: \text{fiber bundle} \iff f^{-1}(b): \text{Lagrangian for all } b \in B$$

**Example**

$$f_0: (\mathbb{R}^n \times (\mathbb{R}/\mathbb{Z})^n, \sum_{i=1}^n dx_i \wedge dy_i) \to \mathbb{R}^n \cup (x_1, \ldots, x_n, y_1, \ldots, y_n) \mapsto (x_1, \ldots, x_n)$$

**Theorem (Arnold-Liouville)**

A Lagrangian fibration $f: (M, \omega) \to B$ with closed connected fibers is locally modeled on $f_0: (\mathbb{R}^n \times (\mathbb{R}/\mathbb{Z})^n, \sum_i dx_i \wedge dy_i) \to \mathbb{R}^n$. 
Bohr-Sommerfeld fiber

- $f: (M, \omega) \to B$: Lagrangian fibration with fiber a torus
Bohr-Sommerfeld fiber

- \( f: (M, \omega) \to B \): Lagrangian fibration with fiber a torus
- \((L, \nabla^L) \to (M, \omega)\): prequantum line bundle

\[
\begin{align*}
\text{def} \quad & \begin{cases} 
L \to M \text{ Hermitian line bundle} \\
\nabla^L \text{ connection of } L \text{ with } \frac{\sqrt{-1}}{2\pi} F_{\nabla^L} = \omega
\end{cases} \\
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L \to M & \text{Hermitian line bundle} \\
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\end{cases}
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- \( \forall b \in B, (L, \nabla^L)|_{f^{-1}(b)} \to f^{-1}(b) \) is a flat line bundle.
Bohr-Sommerfeld fiber

- $f: (M, \omega) \to B$: Lagrangian fibration with fiber a torus
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- $\forall b \in B$, $(L, \nabla^L)|_{f^{-1}(b)} \to f^{-1}(b)$ is a flat line bundle.

Definition (Bohr-Sommerfeld (BS) fiber)

$f^{-1}(b)$ is Bohr-Sommerfeld $\defeq H^0 \left( f^{-1}(b); (L, \nabla^L)|_{f^{-1}(b)} \right) \neq 0$
Bohr-Sommerfeld fiber

- $f: (M, \omega) \to B$: Lagrangian fibration with fiber a torus
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$$\text{def} \begin{cases} L \to M & \text{Hermitian line bundle} \\ \nabla^L & \text{connection of } L \text{ with } \frac{\sqrt{-1}}{2\pi} F_{\nabla^L} = \omega \end{cases}$$

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**Definition (Bohr-Sommerfeld (BS) fiber)**

$f^{-1}(b)$ is Bohr-Sommerfeld $\text{def} \iff H^0 (f^{-1}(b); (L, \nabla^L)|_{f^{-1}(b)}) \neq 0$

**Example**

$f_0: (\mathbb{R}^n \times (\mathbb{R}/\mathbb{Z})^n, \sum_{i=1}^n dx_i \wedge dy_i) \to \mathbb{R}^n$

$(\mathbb{R}^n \times (\mathbb{R}/\mathbb{Z})^n \times \mathbb{C}, d - 2\pi \sqrt{-1} \sum_{i=1}^n x_i dy_i) \to (\mathbb{R}^n \times (\mathbb{R}/\mathbb{Z})^n, \sum_{i=1}^n dx_i \wedge dy_i)$

Then

$$f_0^{-1}(x) \text{ is BS } \iff x \in \mathbb{Z}^n.$$
The local index theory for Lagrangian fibrations

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- $W := \bigoplus_q \Lambda^{0,q} T^* M \otimes L$
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  - $f^{-1}(b)$: non BS $\iff H^\bullet(f^{-1}(b); (L, \nabla^L)|_{f^{-1}(b)}) = 0$ ($\because f^{-1}(b)$ is a torus)
  - $\iff$ The de Rham operator of $f^{-1}(b)$ with coefficients in $(L, \nabla^L)|_{f^{-1}(b)}$ has zero kernel.
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**Theorem (Andersen, Fujita-Furuta-Y, Kubota)**

$$RR(M, \omega) = \#BS\text{ fibers}.$$
In this case we assume $M$ is closed, and the input data are given by

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**Theorem (Andersen, Fujita-Furuta-Yamaguchi, Kubota)**

\[ RR(M, \omega) = \#\text{BS fibers}. \]

\[ \therefore RR(M, \omega) = \text{ind}(M, V : W) (: : M \text{ : closed}) \]
\[ = \text{sum of contributions from BS fibers} (: : \text{excision}) \]
\[ = \# \text{of BS} (: : \text{computation of 2-dim. cylinder & product formula}) \]
The second Case - Hamiltonian $S^1$-action

- $(M, \omega)$: effective Hamiltonian $S^1$-space
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  - Each orbit $O$ is isotropic, namely, $\omega|_O \equiv 0$. 
    $\Rightarrow (L, \nabla^L)|_O$ is a flat line bundle. ($\therefore \frac{\sqrt{-1}}{2\pi} F_{\nabla^L} = \omega$)
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Definition ($L$-acyclic orbit)

$O : L$-acyclic $\iff H^0(O; (L, \nabla^L)|_O) = 0$
(\(M, \omega\)): effective Hamiltonian \(S^1\)-space

\((L, \nabla^L) \rightarrow (M, \omega)\) \(S^1\)-equivariant prequantum line bundle

- Each orbit \(\mathcal{O}\) is isotropic, namely, \(\omega|_{\mathcal{O}} \equiv 0\).

\[\Rightarrow (L, \nabla^L)|_{\mathcal{O}}\] is a flat line bundle. (\(\because \frac{\sqrt{-1}}{2\pi} F_{\nabla^L} = \omega\))

**Definition (L-acyclic orbit)**

\[\mathcal{O} : L\text{-acyclic} \overset{\text{def}}{\iff} H^0(\mathcal{O}; (L, \nabla^L)|_{\mathcal{O}}) = 0\]

**Example**

\[(\mathbb{R} \times \mathbb{R}/\mathbb{Z} \times \mathbb{C}, d - 2\pi \sqrt{-1} x dy) \rightarrow (\mathbb{R} \times \mathbb{R}/\mathbb{Z}, dx \wedge dy)\]

\(S^1 = \mathbb{R}/\mathbb{Z} \hookrightarrow \mathbb{R} \times \mathbb{R}/\mathbb{Z} \times \mathbb{C},\ t(x, y, w) = (x, y + t, e^{2\pi \sqrt{-1} mt} w)\ (m \in t^*_\mathbb{Z} \cong \mathbb{Z})\)

Then

\[\mathcal{O}_x := \{x\} \times \mathbb{R}/\mathbb{Z}\] is \(L\)-acyclic \(\Leftrightarrow x \notin \mathbb{Z}\).
For the $S^1$-action on the prequantum line bundle $(L, \nabla^L) \to (M, \omega)$, the moment map $\mu: M \to t^*$ ($t := \text{Lie}(S^1)$) is defined by the Kostant formula

$$\mathcal{L}_{X_{\xi}} s = \nabla_{X_{\xi}} s + 2\pi \sqrt{-1} \langle \mu, \xi \rangle s$$

for $\forall \xi \in t$ and $\forall s \in \Gamma(L)$, where $X_{\xi}$ is the infinitesimal action of $\xi$. 

**Lemma**

Let $O$ be an orbit. If $O$ is non $L$-acyclic, i.e., $H^0(O; (L, \nabla^L)) \neq 0$, then $O \subset \mu^{-1}(t^* \mathbb{Z})$. 


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**Lemma**

*Let $O$ be an orbit. If $O$ is NON $L$-acyclic, i.e. $H^0 (O; (L, \nabla^L) |_O) \neq 0$, then $O \subset \mu^{-1}(t^*_\mathbb{Z})$.***
The local index theory for Hamiltonian $S^1$-actions

We assume $\mu$ is proper and the cardinality of $\mu(M) \cap t_\mathbb{Z}^*$ is finite. In this case, the input data are given as follows.
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The local index theory for Hamiltonian $S^1$-actions

We assume $\mu$ is proper and the cardinality of $\mu(M) \cap t_Z^*$ is finite. In this case, the input date are given as follows

- $W := \bigoplus q \wedge^0 q \ T^*M \otimes L$
- $V := M \setminus \mu^{-1}(t_Z^*)$
The local index theory for Hamiltonian $S^1$-actions

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- $W := \bigoplus_q \wedge^0_q T^*M \otimes L$
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- $\pi : V \to U := V/S^1$
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- $\mathcal{O}$: $L$-acyclic $\iff$ The de Rham operator of $\mathcal{O}$ with coefficients in $(L, \nabla^L)|_{\mathcal{O}}$ has zero kernel.
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- $D_{\text{fiber}} := \{\text{de Rham operators of } \pi^{-1}(b) \text{ with coeff. in } L\}_{b \in U}$
- $O : L$-acyclic $\iff$ The de Rham operator of $O$ with coefficients in $(L, \nabla L)|_O$ has zero kernel.

**Theorem (Fujita-Furuta-Y)**

For above data $\exists \text{ind}_{S^1}(M, V : W) \in R(S^1)$ satisfying the properties in Main Theorem. In particular, for each $\gamma \in \mu(M) \cap t^*_Z$ let $V_\gamma$ be a sufficiently small $S^1$ invariant neighborhood of $\mu^{-1}(\gamma)$ so that $\{V_\gamma\}_{\gamma \in \mu(M) \cap t^*_Z}$ are mutually disjoint. Then

$$\text{ind}_{S^1}(M, V : W) = \bigoplus_{\gamma \in \mu(M) \cap t^*_Z} \text{ind}_{S^1}(V_\gamma, V_\gamma \cap V : L|_{V_\gamma}) \in R(S^1).$$
The local index theory for Hamiltonian $S^1$-actions

Example (cylinder)

$$(M, \omega) := ((i - \varepsilon, i + \varepsilon) \times \mathbb{R}/\mathbb{Z}, dx \wedge dy) \quad (0 < \varepsilon < 1, i \in \mathbb{Z})$$

$$(L, \nabla^L) := (M \times \mathbb{C}, d - 2\pi \sqrt{-1} x dy) \to (M, \omega)$$

$S^1 = \mathbb{R}/\mathbb{Z} \ltimes (L, \nabla^L), \quad t(x, y, w) = (x, y + t, e^{2\pi \sqrt{-1}mt}w) \quad (m \in t^*_\mathbb{Z} \cong \mathbb{Z})$$

$V := M \setminus \{i\} \times \mathbb{R}/\mathbb{Z}$

Then

$$\text{ind}_{S^1} (M, V : W) = \mathbb{C}_{i-m}$$
For $\sigma \in t_\mathbb{Z}^*$ and $U \in R(S^1)$ we denote the multiplicity of the irreducible representation of $S^1$ with weight $\sigma$ in $U$ by

$$U^\sigma := \dim \Hom_{S^1}(\mathbb{C}_\sigma, U).$$
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\[
U^\sigma := \dim \text{Hom}_{S^1}(\mathbb{C}_\sigma, U).
\]

**Theorem (Fujita-Furuta-Y)**

For each \( \gamma \in \mu(M) \cap t^*_\mathbb{Z} \) and \( \sigma \in t^*_\mathbb{Z} \) with \( \gamma \neq \sigma \)
\[
\text{ind}_{S^1}(V_\gamma, V_\gamma \cap V : L|_{V_\gamma})^\sigma = 0.
\]
Suppose \( \gamma \in \mu(M) \cap t_\mathbb{Z}^* \) is a regular value of \( \mu \). Then, a new symplectic manifold \((M_\gamma, \omega_\gamma)\) with prequantum line bundle \((L_\gamma, \nabla^{L_\gamma})\) is obtained by

\[
(L_\gamma, \nabla^{L_\gamma}) := \left( (L, \nabla^L) \otimes \mathbb{C}_\gamma |_{\mu^{-1}(\gamma)} \right) / S^1
\]

\[
(M_\gamma, \omega_\gamma) := \left( \mu^{-1}(\gamma), \omega |_{\mu^{-1}(\gamma)} \right) / S^1.
\]

**Theorem (Fujita-Furuta-Y)**

*Let \( \gamma \in \mu(M) \cap t_\mathbb{Z}^* \) be a regular value of \( \mu \). Then*

\[
\text{ind}_{S^1}(V_\gamma, V_\gamma \cap V : L|_{V_\gamma})^\gamma = RR(M_\gamma, \omega_\gamma).
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\]

\[
\downarrow
\]

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**Theorem (Fujita-Furuta-Y)**

Let $\gamma \in \mu(M) \cap t^*_Z$ be a regular value of $\mu$. Then

$$\text{ind}_{S^1}(V_\gamma, V_\gamma \cap V : L|_{V_\gamma})^\gamma = RR(M_\gamma, \omega_\gamma).$$

**Corollary (Guillemin-Sternberg, Meinrenken, Vergne, Tian-Zhang, . . .)**

Let $(L, \nabla^L) \to (M, \omega)$ be as above. Assume $M$ is closed and $\gamma \in \mu(M) \cap t^*_Z$ is a regular value of $\mu$. Then

$$RR(M, \omega)^\gamma = RR(M_\gamma, \omega_\gamma).$$
Let us consider $L := \mathcal{O}(1) \to M := \mathbb{C}P^2$ with moment map $\mu : M \to \mathbb{R}^2$. We put
Generalization

Let us consider $L := O(1) \to M := CP^2$ with moment map $\mu : M \to \mathbb{R}^2$. We put

- $V_1 := \mu^{-1}(\mu(M)^\circ)$,
- $V_2 := \mu^{-1}(U'_2)$,
- $V_3 := \mu^{-1}(U'_3)$,
- $V_4 := \mu^{-1}(U'_4)$.
Let us consider $L := \mathcal{O}(1) \to M := \mathbb{C}P^2$ with moment map $\mu : M \to \mathbb{R}^2$. We put

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- $S^1 \circ V_2, t[z_0 : z_1 : z_2] = [z_0 : z_1 : tz_2] \Rightarrow \pi_2 : V_2 \to V_2/S^1$
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- $\pi_1 := \mu|_{V_1}: V_1 \to \mu(M)^\circ$: $(S^1)^2$-bundle
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Figure: $\mu(M)$
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Let us consider $L := \mathcal{O}(1) \to M := \mathbb{CP}^2$ with moment map $\mu : M \to \mathbb{R}^2$. We put

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- On $V_i \cap V_j \neq \emptyset$, $\pi_i^{-1}(\pi_i(V_i \cap V_j)) = \pi_j^{-1}(\pi_j(V_i \cap V_j))$

![Diagram](image-url)
Let us consider $L := O(1) \to M := \mathbb{C}P^2$ with moment map $\mu : M \to \mathbb{R}^2$. We put

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induce the notion of “compatible fibration and acyclic compatible system” and we can generalize the local index theory for compatible acyclic systems. (These will be explained in the next talk.)
As an application of this generalization, we obtain the following result.
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**Theorem (Fujita-Furuta-Y)**

*For a prequantized four-dimensional closed locally toric Lagrangian fibration*

\[ RR(M, \omega) = \# \text{(both singular and nonsingular)BS-fibers}. \]
As an application of this generalization, we obtain the following result.

**Theorem (Fujita-Furuta-Y)**

For a prequantized four-dimensional closed locally toric Lagrangian fibration

\[ RR(M, \omega) = \# \text{(both singular and nonsingular)BS-fibers}. \]

By the local index theory we often obtain a localization formula for the index. But, in general, it is difficult to compute the local contributions. The condition “four-dimension” is a technical condition to compute the contribution from singular fibers. The next talk by Fujita may become a breakthrough to get a systematic computation of the contribution from singular fibers.
Thank you for your attention!