Reheating in $R^2$ Palatini inflationary models

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We consider $R^2$ inflation in the Palatini gravity assuming the existence of scalar fields, coupled to gravity in the most general manner. These theories, in the Einstein frame, and for one scalar field $h$, share common features with $K$ - inflation models. We apply this formalism for the study of popular inflationary models, whose potentials are monomials, $V \sim h^n$, with $n$ a positive even integer. We also study the Higgs model non-minimally coupled to gravity. Although these have been recently studied, in the framework of the Palatini approach, we show that the scalar power spectrum severely constrains these models. Although we do not propose a particular reheating mechanism, we show that the quadratic $\sim h^2$ and the Higgs model can survive these constraints with a maximum reheating temperature as large as $\sim 10^{15}$ GeV, when reheating is instantaneous. However, this can be only attained at the cost of a delicate fine-tuning of couplings. Deviations from this fine-tuned values can still yield predictions compatible with the cosmological data, for couplings that lie in very tight range, giving lower reheating temperatures.

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I. INTRODUCTION

It has been long known that the Palatini formulation of General Relativity (GR), or first-order formalism, is an alternative to the well-known metric formulation, or second-order formalism. In the latter the space time connection is determined by the metric while in the Palatini approach the connection $\Gamma^\mu_{\lambda\sigma}$ is treated as an independent variable [1–8], (see also [9], and references therein). It is through the use of the equations of motion that $\Gamma^\mu_{\lambda\sigma}$ receive the well known form of the Christoffel symbols, describing thus a metric connection. Within the context of GR the two formulations are equivalent. However in the presence of fields that are coupled in a non-minimal manner to gravity this no longer holds, [1–3]. In that case the two formulations describe different physical theories.

Encompassing the popular inflation models into Palatini Gravity, in an effort to describe the cosmological evolution of the Universe, leads to different cosmological predictions, from the metric formulation, due to the fact that the dynamics of the two approaches differ. A notable example is the Starobinsky model, for instance, where except the graviton there exists an additional propagating scalar degree of freedom, the scalaron, whose mass is related to the coefficient of the $R^2$ term. In the Einstein frame this shows up as a dynamical scalar field, the inflaton, moving in a potential, the celebrated Starobinsky potential, [10–12]. Within the framework of the Palatini Gravity, in any $f(R)$ theory [3], there are no

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extra propagating degrees of freedom, that can play the role of the inflaton, and hence the inflaton has to be put in by hand as an additional field coupled to $f(R)$ gravity.

The differences between metric and Palatini formulation in the cosmological predictions, as far as inflation is concerned, arise from the non-minimal couplings of the scalars, that take-up the role of the inflaton. These couplings are different in the two approaches. This has been first pointed out in [13] and has attracted the interest of many authors since, [14–42], with still continuing activity, [43–49].

The measurements of the cosmological parameters, by various collaborations, has tighten the allowed limits of these observables which in turn constrain severely, or even exclude, particular inflationary models, [50–52]. In particular, the spectral index $n_s$ and the bounds on the tensor-to-scalar ratio $r$ impose severe restrictions and not all models can be compatible with the observational data. Within the class of $f(R)$ theories the Starobinsky model, which is an $R^2$- theory, is singled out, although other popular models can also successfully pass the tests provided by the recent cosmological observations. The measurements of the primordial scalar perturbations, and of the associated power spectrum amplitude $A_s$, constrains the scale of inflation in models encompassed in the framework of the metric formulation. We shall show that in the Palatini formalism this imposes restrictions that are more stringent, at least in some cases, than the ones arising from the observables $n_s, r$ and should be duly taken into account. In this work we shall consider $R^2$ theories, in the framework of the Palatini Gravity, and study the cosmological predictions of some of the popular models existing in literature. We will show that these do not comfortably stand, unless the parameters describing the models are fine-tuned, the main source of this fine-tuning being the power spectrum amplitude.

This paper is organized as follows:

In section II, we present the salient features and give a general setup of $f(R)$ - theory, in the presence of an arbitrary number of scalar fields, coupled to gravity in a non-minimal manner, in general. Although this is not new, as this effort has been undertaken by other authors as well, we think that the general, and model-independent, expressions we arrive at, are worth being discussed. We shall then focus on the case of $R^2$ theories for which the passage to the Einstein frame is easily implemented. These theories have a gravity sector, specified by two arbitrary functions, sourcing in general non-minimal couplings of the scalars involved in Palatini Gravity and a third function which is the scalar potential. In the Einstein frame, and when a single field is present, these models have much in common with the $K$ - inflation models [54, 55].

In section III, we discuss the arising equations of motions and the slow-roll mechanism, and give the pertinent slow-roll parameters, adapted to the particular setup. This is necessary since it is our aim to employ a scheme in which the passage to canonically normalized fields is not mandatory. This we find it more convenient especially because the use of canonically normalized fields results, in most of the cases, to expressions that cannot be cast in closed forms.

The discussion of the cosmological observables is the subject of section IV. We focus, in particular, to the power spectrum amplitude which, as already advertised, puts severe constraints on the inflation models that we are going to discuss. We find it necessary to calculate the first order corrections, in the slow-roll parameters, of the power spectrum, since these account for contributions comparable, in magnitude, to the errors associated with the power spectrum. Although we shall not adopt a particular reheating mechanism, the dependence of the number of e-folds on the reheating temperature is of paramount importance, for the study of the cosmological predictions. This is, also, reviewed in section IV.

In section V, we consider particular inflation models, namely the class of models in which the scalar field $h$, is characterized by monomial potentials $\sim h^n$, with $n$ a positive even integer, and the Higgs model. Although these have been much studied, we shall show that the cosmological data put severe restrictions on the associated couplings leading to fine-tuned adjustments of the parameters involved, when the power spectrum data are taken into account. The reheating mechanism can be instantaneous, at the cost of an unnatural fine-tuning of the couplings pertinent to the potential, describing the aforementioned

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1 In this work, standard assumptions are made for neutrino masses and their effective number. Relaxing these it induces substantial shifts in $n_s$ [53].

2 Throughout this paper we use different symbols for the Ricci scalar which in the metric formulation we denote by $R$, and in the Palatini approach denoted by $R$. 
models. For the models discussed, the instantaneous reheating temperature $T_{\text{ins}}$, which sets the maximum temperature, can be as large as $\sim 10^{15}\text{ GeV}$. Departing from these fine-tune values, we can still be in agreement with all data, with temperatures that are significantly lower than the instantaneous reheating temperature. This requires that the coupling of the potential lies within a very tight range. Outside this range these models cannot be made compatible with the power spectrum data for any value of the equation of state parameter $w$ in the range $-1/3 < w < 1$.

In sections VI, we end up with our conclusions.

II. THE MODEL

We shall consider an action where scalar fields $h^J$ are coupled to gravity in the following manner

$$S = \int d^4x \sqrt{-g} \left( f(\mathcal{R},h) + \frac{1}{2} G_{IJ}(h) \partial h^I \partial h^J - V(h) \right).$$

(1)

In it $\mathcal{R}$ is the scalar curvature in the Palatini formalism and $f(\mathcal{R},h)$ and arbitrary function of the scalars $h^J$ and $\mathcal{R}$. This action is reminiscent of an $f(\mathcal{R})$ theory in which scalar fields are involved with kinetic terms written in the most general way resembling $\sigma$ - models. Following standard procedure we write this action in the following manner, introducing the auxiliary field $\Phi$.

$$S = \int d^4x \sqrt{-g} \left( f(\Phi,h) + f'(\Phi,h) (\mathcal{R} - \Phi) + \frac{1}{2} G_{IJ}(h) \partial h^I \partial h^J - V(h) \right).$$

(2)

In this $f'(\Phi,h)$ denotes the derivative with respect $\Phi$. One can define $\psi$ in the following way

$$\psi = \frac{\partial f(\Phi,h)}{\partial \Phi}, \quad \text{with inverse} \quad \Phi = \Phi(\psi,h),$$

(3)

so the action is written as follows,

$$S = \int d^4x \sqrt{-g} \left( \psi \mathcal{R} + \frac{1}{2} G_{IJ}(h) \partial h^I \partial h^J - \psi \Phi + f(\Phi,h) - V(h) \right).$$

(4)

One can go to the Einstein frame by performing a Weyl transformation of the metric $g_{\mu\nu} = \lambda \bar{g}_{\mu\nu}$, with $\lambda = \frac{1}{2}$.

(5)

That done the theory in the Einstein frame receives the following form,

$$S = \int d^4x \sqrt{-\bar{g}} \left( \frac{\mathcal{R}}{2} + \frac{1}{4\psi} G_{IJ}(h) \partial h^I \partial h^J - \frac{1}{4\psi^2} (\psi \Phi - f(\Phi,h) + V(h)) \right).$$

(6)

The last step is to eliminate the field $\psi$ whose equation of motion is trivially found to be

$$\psi (\partial h)^2 = \psi \Phi - 2 f(\Phi,h) + 2V(h),$$

(7)

where, in order to speed up notation, we have denoted $G_{IJ}(h) \partial h^I \partial h^J = (\partial h)^2$.

Note that (7) is not solvable, in general, but we shall exemplify it in $\mathcal{R}^2$-theories where this can be analytically solved. In the following we shall focus on such theories which can be considered as generalizations of the Starobinsky action. However there are two major differences, first the coefficients of the linear and quadratic, in the curvature $\mathcal{R}$, terms are not in general constants, and second the framework is the Palatini formalism in which the connection is not the well-known Christoffel connection but it is treated as an independent field.
We shall apply the previous formalism when only a single scalar, \( h \), is present and \( f(h, R) \) is quadratic in the curvature having the form

\[
f(R, h) = \frac{g(h)}{2} R + \frac{R^2}{12M^2(h)}.
\]

Since a single scalar field is assumed its kinetic term can be always brought to the form \((\partial h)^2/2\), that is in the action (1) the field can be taken canonically normalized. Therefore in this theory there are three arbitrary functions, namely \( g(h), M^2(h), V(h) \), and any choice of them specifies a particular model. We have set the reduced Planck mass \( m_p = (8\pi G_N)^{-1/2} \) dimensionless and equal to unity and thus all quantities in (8) are dimensionless. When we reinstate dimensions the functions \( g, V \) have dimensions mass\(^2\), mass\(^4\), respectively, while \( M^2 \) is dimensionless. Note that a non-trivial field dependence of the functions \( g(h) \) and / or \( M^2(h) \) is a manifestation of non-minimal coupling of the scalar \( h \) to Palatini Gravity. Note that since we employ Palatini formalism, there is no a scalaron field, associated with an additional propagating degree of freedom, which in the Einstein frame of the metric formulation play the role of the inflaton.

With the function \( f(R, h) \) as given by (8) we get from Eq. (3),

\[
\psi = \frac{g(h)}{2} + \frac{\Phi}{6M^2(h)},
\]

whose inverse is,

\[
\Phi = 6M^2(h) \left( \psi - \frac{g(h)}{2} \right).
\]

Using these we can solve (7) in terms of \( \psi \) in a trivial manner,

\[
\psi = \frac{4V + 3M^2 g^2}{2(\partial h)^2 + 6M^2 g},
\]

that is \( \psi \), an hence \( \Phi \) from (10), are expressed in terms of \( h, (\partial h)^2 \). Plugging \( \psi, \Phi \) into (6) we get, in a straightforward manner,

\[
S = \int d^4 x \sqrt{g} \left( \frac{R}{2} + \frac{K(h)}{2}(\partial h)^2 + \frac{L(h)}{4}(\partial h)^4 - V_{eff}(h) \right).
\]

In this action we have suppressed the bar in the the scalar curvature and, also, \( \sqrt{-g} \), and in order to simplify notation we have denoted \( \partial_\mu h \partial^\mu h \) by \((\partial h)^2\) and \((\partial_\mu h \partial^\mu h)^2\) by \((\partial h)^4\). Note the appearance of quartic terms \((\partial h)^4\) in the action. As for the functions \( K, L, V_{eff} \), appearing in (12), they are analytically given by

\[
L(h) = (3M^2 g^2 + 4V)^{-1}, \quad K(h) = 3M^2 gL, \quad V_{eff} = 3M^2 VL.
\]

Observe that since terms up to \( R^2 \) have been considered, in the \( f(R) \) - gravity, higher than \((\partial h)^4\) terms do not appear in the action.

The above Lagrangean may feature, under conditions, \( K \) - inflation models [54, 55], which involve a single field, described by an action whose general form is

\[
S = \int \sqrt{-g} \left( \frac{R}{2} + p(h, X) \right) d^4 x.
\]

In this, \( p(h, X) = F(X) - V(h) \), with \( F(X) \) an arbitrary function of \( X \), the latter being defined by \( X \equiv (1/2)(\partial_\mu h \partial^\mu h) \). The cosmological predictions of these models has been studied in detail [56]. In (12) the Lagrangean density involving the scalar field is identified with \( p(h, X) \), but the function \( F(X) \) is now replaced by \( K(h) X + L(h) X^2 \), which depends, in addition to \( X \), on the field \( h \), as well, through \( K(h), L(h) \).
In a flat Robertson-Walker metric, where the background field $h$ is only time dependent, the energy density and pressure are given by

$$\rho(h, X) = K(h)X + 3L(h)X^2 + V_{\text{eff}}(h) , \quad p(h, X) = K(h)X + L(h)X^2 - V_{\text{eff}}(h), \quad (15)$$

with $X$ being, in this case, half of the velocity squared, $X = \dot{h}^2/2$.

We shall assume that the function $L(h)$ is always positive to avoid phantoms, which may lead to an equation of state with $w < -1$. This may occur when $L < 0$ and $X$ becomes sufficiently large. However, there is no restriction for the sign of $K(h)$ which may be negative in some regions of the field space, signaling that the kinetic term has the wrong sign in those regions. Obviously the sign of $K(h)$ should be positive at the minimum of the potential. Options where $K$ is negative in some regions, although interesting, will not be pursued in this work. Besides, we shall assume that the potential is positive $V_{\text{eff}}(h) \geq 0$ and it has a Minkowski vacuum. This ensures that the energy density is positive definite. When inflation models are considered, the inflaton will roll down towards this minimum signaling the end of inflation and beginning of Universe thermalization. These are rather mild conditions.

Concerning the potential $V_{\text{eff}}$, appearing in the Lagrangian (12) in the Einstein frame, from the last of (13) we see that due to the fact that we have assumed $L, M^2 > 0$, positivity of $V_{\text{eff}} \geq 0$ entails $V \geq 0$. Moreover one can trivially show, from (13), that $V_{\text{eff}}$ be cast in the following form,

$$V_{\text{eff}} = \frac{3M^2}{4} \left(1 - \frac{K^2}{3M^2L}\right). \quad (16)$$

From this it is seen that besides being positive the potential is bounded from above by,

$$V_{\text{eff}} \leq \frac{3M^2}{4}. \quad (17)$$

This upper bound can be easily saturated, for large $h$, by choosing appropriately the functions involved, namely $g, M^2$ and $V$. Actually the asymptotic values of these functions, for large $h$, control the behavior of the potential in this regime \(^3\). If we opt that the function $M^2$ approaches a plateau, or is constant, so does the potential which may therefore drive successful inflation. The requirement to have a Minkowski vacuum can be, also, easily satisfied, and therefore many options are available for potentials bearing the characteristics demanded for the inflationary slow-roll mechanism to be implemented. This will be exemplified in specific models, to be discussed later.

Concluding this section, we presented a general, and model independent, framework of $R^2$ - theories, in the Palatini formulation of Gravity, which may be useful for the study of inflation models and may support slow-roll inflation. In the Einstein frame these theories have much in common with the $K$-inflation models. This formalism will be implemented, for the study of various models of inflation in the following sections.

III. THE EQUATIONS OF MOTION AND THE SLOW-ROLL

When non canonical kinetic terms are present the equations of motions for the would be inflaton scalar field $h$ differ from their standard form. As a result, the cosmological parameters describing the slow-roll evolution should be modified appropriately. Certainly one can normalize the kinetic term of the scalar field appropriately but this is not always very convenient. Actually the integrations needed, in order to pass from the non canonical to the canonical field, are not easy, in most of the cases, to be carried and the results cannot be presented in a closed form. Therefore it proves easier to work directly with the non

\(^3\) It is fairly easy to see that saturation of the bound (17), for large field values, is easily obtained if $g^2 \to 0$, and $g^2 M^2 \ll 1$, as $h \to \text{large}$. 
canonical fields and express the pertinent cosmological observables in a manner that is appropriate for this treatment.

It is not hard to see that the field $h$ satisfies the equation of motion given by

$$\left(K + 3L\dot h^2\right)\ddot h + 3H(K + LH^2)\dot h + V_{\text{eff}}'(h) + \frac{1}{4}(2K' + 3L'\dot h^2)\dot h^2 = 0,$$

in it all primes denote derivatives with respect $h$. If the field were canonical, $K = 1$, and there were no quartic in the velocity terms, that is $L = 0$, then the equation above receives its well-known form. In this, the effect of using a non canonical, in general, field $h$ is encoded in the function $K$. The effect of the presence of terms $(\partial h)^4$ in the action is encoded within the function $L$. The terms that depend on $L$ are multiplied by an extra power of the velocity squared, as compared to the $K$-terms, and hence are expected to be small in a slow-roll evolution unless in some region(s) the function $L$ is much larger than the function $K$. In this case it cannot be neglected. This has to be watched in each particular model. For the models that are considered in the following chapters the effect of the $L$ is indeed small.

We can gain more insight is we momentarily use a canonically normalized field, say $\phi$, defined by

$$\phi = \int \sqrt{K(h)} \, dh.$$  \hfill (19)

To avoid ghosts we shall assume that $K > 0$, so that the integration above makes sense. Actually if $K$ is negative the kinetic term of the field $\phi$ will have the wrong sign, i.e. $-(\partial \phi)^2/2$. It could happen however that this function is negative in some region but at the Minkowski vacuum is strictly positive. In this way ghosts are also avoided. This case, interesting as might be, is not discussed and we prefer to take a rather conservative view point of having $K > 0$ in the whole region. Then in terms of the field $\phi$ the equation of motion (18) takes up the form

$$\left(1 + \frac{3L}{K^2}\dot \phi^2\right)\ddot \phi + 3H\left(1 + \frac{L}{K^2}\dot \phi^2\right)\dot \phi + \frac{dV_{\text{eff}}}{d\phi} + \frac{3L}{4K^2} \frac{d\ln (L/K^2)}{d\phi} \phi^4 = 0.$$  \hfill (20)

From this form it appears that the smallness of the $(\partial h)^4$ terms in the action is quantified by the smallness of the ratio $\frac{L}{K^2}\dot \phi^2 \ll 1$, which is equivalent to $\frac{L}{K}\dot h^2 \ll 1$. Neglecting this in the equation above we recover the well-known form of the equation of motion for the canonical field $\phi$. The effect of $L$ terms is negligible, during slow-roll, as we found in our numerical approach, and thus we shall neglect these in the discussion that follows. In fact we have found, at least in the models that we consider, that they play little role. This has been also pointed out in [31–33, 38].

Neglecting $L$, and with $K > 0$, the first slow-roll parameters, as defined in terms of the potential are given by, in terms of the non canonical field $h$,

$$\epsilon_V = \frac{1}{2K(h)} \left(\frac{V'}{V_{\text{eff}}}ight)^2, \quad \eta_V = \frac{\left(K^{-1/2}V'_{\text{eff}}\right)'}{K^{1/2}V_{\text{eff}}}. \hfill (21)$$

In these equations primes denote derivatives with respect the field $h$. It is trivial to show that these definitions indeed coincide with the well-known definitions if the canonically normalized field $\phi$ of Eq. (19), is used. As for the the number of e-folds, left to the end of inflation, this is given by

$$N_* = \int_{h_{\text{end}}}^{h_*} K(h) \frac{V_{\text{eff}}(h)}{V_{\text{eff}}'(h)} dh. \hfill (22)$$

In this $h_*$ is the pivot value and $h_{\text{end}}$ the value of the field at the end of inflation.

### IV. COSMOLOGICAL OBSERVABLES

Concerning the cosmological observables, we start by discussing first the scalar power spectrum which will play an important role in our analysis when considering specific models. In fact, we shall later show
that the CMB observational data restrict considerably the inflationary models, in the framework of the Palatini formulation.

The scalar power spectrum, following standard steps [55], is found to have the following expression

\[ P_\zeta(k) = \frac{1}{8\pi m_P^2} \left( \frac{2^{\nu-3/2}}{\Gamma(3/2)} \right)^2 \left( \frac{1 - \epsilon_1}{1 + \epsilon_1 \epsilon_2} \right)^{2\nu-1} \left( \frac{H^2}{\epsilon_1 c_s} \right) \left( \frac{c_s k}{a_H} \right)^{3-2\nu}. \]  

(23)

In order to compare with existing formulas in literature we have reinstated the dimensions so that the reduced Planck mass appears in the result above. In this expression we have kept terms that are of higher order in the slow-roll parameters. Their contribution is small but, as we have seen, in some cases is comparable to the observational error accompanying the power spectrum measurements.

The slow-roll parameters appearing in (23), and in the equations that follow, are defined in the usual manner, in terms of the Hubble rate,

\[ \epsilon_1 \equiv -\frac{d\ln H}{dN} = -\frac{\dot{H}}{H^2}, \quad \epsilon_2 \equiv \frac{d\ln \epsilon_1}{dN} = \frac{\dot{\epsilon}_1}{\epsilon_1 H}, \quad s_1 \equiv \frac{d\ln c_s}{dN} = \frac{\dot{c}_s}{c_s H}, \]  

(24)

where \( dN = H dt \). In terms of these the parameter \( \nu \) in (23) is given by

\[ \nu = \frac{3}{2} + \frac{\epsilon_1}{2} + \frac{\epsilon_2}{2} - s_1, \]  

(25)

and the power \( 3 - 2\nu \) of the last term is \( n_s - 1 \), where \( n_s \) is the spectral index, given by,

\[ n_s = 1 - 2 \epsilon_1 - \epsilon_2 + 2 s_1. \]  

(26)

The expression (23) coincides with those given by [55] and [56], when the third term in Eq. (23) is put equal to unity. Its appearance is due to the fact that conformal time is related to Hubble co-moving length as

\[ -\tau = \frac{1}{aH} \frac{1 + \epsilon_1 \epsilon_2}{1 - \epsilon_1}, \]  

(27)

which is valid when second order terms in the slow roll parameters are kept.

The appearance of the sound of speed parameter \( c_s \) is due to the fact that in the Palatini formulation of \( R^2 \) gravity higher in the velocity \( h \) terms unavoidably appear, and its value deviates from unity. In fact \( c_s \) is defined by

\[ c_s^2 = \frac{\partial p}{\partial X} \frac{\partial \rho}{\partial X}, \]  

(28)

where \( X \), defined after Eq. (14), is half the velocity squared. In terms of the field \( h \) and its velocity \( \dot{h} \) this receives the form

\[ c_s^2 = \frac{1 + L \dot{h}^2/K}{1 + 3L \dot{h}^2/K}. \]  

(29)

\( c_s \) is controlled by \( L \dot{h}^2/K \), the same ratio that appears in the equation of motion for the field \( h \), and approaches unity when \( L \dot{h}^2/K \ll 1 \).

If we use a certain pivot scale \( k^* \), defined by \( k^* = a(t^*)H(t^*) \), then the spectrum (23) can be cast in the form

\[ P_\zeta(k) = A_s \left( \frac{k}{k^*} \right)^{n_s-1}. \]  

(30)

Note that \( t^* \) is not exactly the time the mode \( k^* \) exits the cosmological horizon, due to the fact \( c_s \neq 1 \). We shall comment upon it in the sequel. The amplitude \( A_s \) in (30) is then given by,

\[ A_s = \frac{1}{8\pi^2 m_P^2} 2^{1-n_s} \left( \frac{\Gamma(2-n_s/2)}{\Gamma(3/2)} \right)^2 c_s^{n_s} \left( \frac{H_0^2}{\epsilon_1 c_s} \right) \left( 1 - \frac{\epsilon_1}{1 + \epsilon_1 \epsilon_2} \right)^{3-n_s}. \]  

(31)
The last factor on the right hand side (rhs) of this equation induces corrections that are of the order 3%, or so, which are sizable, given the errors of $A_s$.

Although in our numerical analysis, when considering specific models, we shall use the precise expression given by (31), it is useful to comment on the magnitude of $A_s$ by simplifying the above expression retaining the leading terms. In particular ignoring first order slow-roll correction terms and putting $n_s \approx 1$ everywhere, except the velocity dependent terms, this takes the form

$$A_s \approx \frac{1}{8\pi^2 m_P^2} \left( \frac{H^2}{\epsilon_1^* c_s^*} \right) c_s^{s(n_s-1)}.$$  \hspace{1cm} (32)

This is a much simpler expression that yields the correct order of magnitude and may be used in order to locate the range of the parameters, in each particular model, for which the amplitude $A_s$ is in the allowed range according to the latest cosmological observations. Note that (32) differs slightly from the corresponding expression given in [56], by the appearance of the last factor, $c_s^{s(n_s-1)}$. The difference is due to the fact that a different pivot scale is used here. In [56] $c_s k^* = a^* H^*$ is employed, where now the starred quantities refer to the time of the horizon crossing. Replacing $k^*$ in (30) by $k^* = a^* H^*/c_s$ results to the same expression for the power spectrum with an amplitude having the form of (32) in which, however, the term $c_s^{s(n_s-1)}$ is absent. We point out that during inflation the effect of the function $L(h)$ is negligible, as our numerical study reveals, and therefore the sound of speed turns out to be very close to unity. Therefore using either value for the pivot scale $k^*$ yields same results.

The Planck 2018 data [50, 51], yield a value for $A_s$

$$\log(10^{10} A_s) \approx 3.04,$$  \hspace{1cm} (33)

at a pivot scale $k^* = 0.05 Mpc^{-1}$. In slow-roll inflation the pivot values correspond to times $t^*$ that are well within the slow-roll regime. With this in mind we can further simplify (32) and write

$$A_s \approx \frac{1}{24\pi^2 m_P^2} \frac{V_{eff}^*}{\epsilon_1^* c_s^*},$$  \hspace{1cm} (34)

where the spectral index $n_s$ has been set to unity. Then in this approximation, and using (33), we get

$$\frac{V_{eff}^*}{m_P^2} = 4.97 \times 10^{-7} \epsilon_1^* c_s^*.$$  \hspace{1cm} (35)

The corrections that have been omitted little affect this result, they are first order corrections to the slow-roll parameters and the sound of speed which is close to unity, at least in the models under consideration, in the slow-roll region.

The amplitude for the tensor perturbations, in the same approximation, is found to be

$$A_t \approx \frac{2 V_{eff}^*}{3\pi^2 m_P^2},$$  \hspace{1cm} (36)

resulting to a tensor to scalar ratio

$$r = \frac{A_t}{A_s} = 16 \epsilon_1^* c_s^*,$$  \hspace{1cm} (37)

which yields, on account of (34)

$$\frac{V_{eff}^*}{m_P^2} = \frac{3\pi^2}{2} A_s r,$$  \hspace{1cm} (38)

a well-known result. The Planck 2018 data, when combined with the BICEP2/Keck Array BK15 data, see [51, 52], yield an upper bound $r < 0.063$, which when the pivot scale $k^* = 0.002 Mpc^{-1}$ is used,
decreases to \( r_{0.002} < 0.058 \). With \( r < 0.063 \) we have an upper bound on the value of the potential given by
\[
\frac{V_{eff}^*}{m_p^2} \lesssim 2.0 \times 10^{-9},
\]  
which constrains the scale of inflation. In terms of the Hubble rate this is actually the bound \( H_*/m_p < 2.5 \times 10^{-5} \) quoted in [50, 51].

The bound \( r < 0.063 \) translates to a bound on the combination \( \epsilon_1^* c_s^* \), which is actually follows from (37), given by
\[
\epsilon_1^* c_s^* < 0.004.
\]  
Note that separate lower bounds on \( c_s \) are obtained from absence of non gaussianities, which are however satisfied in all models considered in this work, due to the fact that \( c_s^* \) is very close to unity, as we have already remarked.

Equation (35) and the bound (39) restrict severely the inflation models that can be embedded within the Palatini formalism, something that has been overlooked in other studies. We shall return to this point, when discussing particular inflation models in this formulation.

The value of \( V_{eff}^*/m_p^2 \), as given in (35), is known to play an important role for the number of e-folds, \( N_\star \), which are left until horizon crossing. For a given comoving wave number \( k^* \), \( N_\star \) is given by [57, 58], see also [59],
\[
N_\star = 66.89 - \ln \left( \frac{k^*}{a_0 H_0} \right) + \frac{1}{4} \left( \ln \frac{3H_0^2}{m_p^2} + \ln \frac{3H_0^2 m_p^2}{\rho_{end}} \right) - \frac{1 - 3w}{4} \Delta N_{reh} - \frac{1}{12} \ln g_s^{(reh)}. 
\]  
The term before the last in Eq. (41), \( \Delta N_{reh} \), is the number of e-folds covered in the reheating period,
\[
\Delta N_{reh} = \ln \frac{a_{reh}}{a_{end}} = - \frac{1}{3(1 + w)} \ln \frac{\rho_{reh}}{\rho_{end}}. 
\]  
The subscripts (reh), (end) in the cosmic scale factor and the energy densities denote these quantities at the end of the reheating period and inflation respectively. In (42) we have assumed a constant value for the effective equation of state parameter \( w \) in the reheating period, whose value we consider as a free parameter. At the end of inflation \( w = -1/3 \) and in the canonical reheating scenario \( w = 0 \), although values in the range \( 0.0 - 0.25 \), or larger, right after inflation, are also possible in some models [70, 71]. The value \( w = 1/3 \) corresponds to the onset of radiation dominance.

In the last term of (41) the quantity \( g_s^{(reh)} \) is the number of effective entropy degrees of freedom at reheating. Assuming the SM content this is given by \( g_s^{(reh)} = 106.75 \). This term contributes little, less than \( 0.1% \). The fourth term is not large. The largest contributions stem from the third and the reheating term. Their uncertainties yield values of \( N(k^*) \) in the range 50 – 60, usually quoted in the literature. For any given value of \( N_\star \), we have a prediction of \( \Delta N_{reh} \) and in this sense (41) serves as a probe of the reheating process. Inversely, given a reheating mechanism, within the context of any particular inflationary model, the value of \( \Delta N_{reh} \) is fixed, and hence \( N_\star \) is predicted. In terms of \( \Delta N_{reh} \), if a constant value for \( w \) is used, one has, see for instance [66]
\[
T_{reh} = \left( \frac{30}{\pi^2} \frac{\rho_{end}}{g_s^{(reh)}} \right)^{1/4} \exp \left( -\frac{3(1 + w)\Delta N_{reh}}{4} \right). 
\]
Since in general $g^*(reh)$ and $g_\ast^*(reh)$ are close to each other, in our numerical studies we shall adopt the common value $g^*(reh) = g_\ast^*(reh) = 106.75$, corresponding to the SM content as discussed above \(^4\). Note that since $a_{reh} > a_{end}$ we have that $\Delta N_{reh} \geq 0$, and therefore due to $w > -1$ the reheating temperature $T_{reh}$ is bounded from above

$$T_{reh} \leq \left( \frac{30}{\pi^2} \frac{\rho_{end}}{g^*(reh)} \right)^{1/4}. \quad (44)$$

The bound on the right hand side of (44) defines the instantaneous reheating temperature, $T_{ins}$. The temperature $T_{reh}$ reaches this upper bound when the reheating process is instantaneous, in which case $\Delta N_{reh} = 0$. Note that for rapid thermalization we have $\rho_{end} = \rho_{reh}$, from Eq. (42). The reheating temperature should be larger than $\simeq \text{MeV}$ so that Big Bang Nucleosynthesis (BBN) is not upset. Lower values on $T_{reh}$ have been established in [72] and recently in [73].

Using (43) we get, from (41),

$$N_* = 66.89 - \ln \left( \frac{k^*}{a_0 H_0} \right) + \frac{1}{4} \left( \ln \frac{3H_0^2}{m_p^2} + \ln \frac{3H_0^2 m_p^2}{\rho_{end}} \right) - \frac{1}{12} \ln g_\ast^*(reh) + \frac{1 - 3w}{3(1 + w)} \left( \ln \frac{T_{reh}}{m_p} - \frac{1}{4} \ln \frac{\rho_{end}}{m_p^4} - \frac{1}{4} \ln \frac{30}{\pi^2} + \frac{\ln g_\ast^*(reh)}{4} \right). \quad (45)$$

which we shall use in the following.

V. MODELS

A. Minimally coupled models with potentials $\sim h^n$

In this section we consider specific models using the formalism presented in previous sections, and discuss their predictions. An interesting class of models is the one in which the potential $V$ is a monomial in the field $h$, $V \sim h^n$, with $n$ even integer, and $g, M^2$ are constants, that is the scalar $h$ couples to gravity in a minimal manner. We set $g = 1$ \(^5\) and hence these models are described by

$$g(h) = 1, \quad M^2(h) = \frac{1}{3a}, \quad V(h) = \frac{\lambda}{n} h^n \quad \text{with} \quad n = \text{positive even integer}. \quad (46)$$

Therefore two parameters, $a$ and $\lambda$ are involved which are in principle unknown. Cosmological data will constrain their allowed values as we shall see shortly. In order to facilitate the analysis we define the parameter $c$ defined by the combination,

$$c = \frac{4\lambda a}{n}. \quad (47)$$

Then the functions $K, L$ are given by

$$K(h) = (1 + ch^n)^{-1}, \quad L(h) = a (1 + ch^n)^{-1}, \quad (48)$$

while the potential $V_{eff}$ receives the form

$$V_{eff}(h) = \frac{1}{4a} \frac{ch^n}{1 + ch^n}. \quad (49)$$

---

\(^4\) With $g^*(reh) = 100$ Eq. (43) coincides with that given in [66].

\(^5\) When Planck mass is reinstated in the action this corresponds to $g = m_p^2$. 

For large values of $h$ this is $\simeq 1/4a$ therefore $1/a$, which is proportional to $M^2$, sets actually the inflation scale.

In order to find the region of the parameters $a, \lambda$, or equivalently $a, c$, which are consistent with cosmological data, we shall first consider the amplitude of the power spectrum $A_s$. It suffices, for this purpose, to consider the simplified form given by (34), take $c_1^* \simeq 1$ and replace $\epsilon_1^*$ by $\epsilon_V$ as given by (21). Then from the analytic form of the potential, given before, and from (21) the amplitude $A_s$ of Eq. (34) takes the form, putting $m_P = 1$,

$$A_s \simeq \frac{1}{24\pi^2} \frac{1}{2n^2} \left( \frac{c}{a} \right) h_*^{n+2} = \frac{1}{12\pi^2} \frac{\lambda}{n^3} h_*^{n+2},$$

(50)

where $h_*$ is the value of the field at $t^*$. One sees immediately that it is the ratio $c/a$, or equivalently the parameter $\lambda$, that controls the magnitude of the amplitude $A_s$. For the central value of $A_s$, which is $A_s \simeq 2.1 \times 10^{-9}$, on account of (50), we have

$$\lambda h_*^{n+2} \simeq (2.49 \times 10^{-7}) n^3 \quad \text{or} \quad \left( \frac{c}{a} \right) h_*^{n+2} \simeq (9.95 \times 10^{-7}) n^2.$$  

(51)

To further quantify the allowed range of the parameters we also need have an estimate for $h_*$. To this goal we use (22) from which it follows that

$$N_* = \frac{1}{2n} (h_*^2 - h_{end}^2),$$

(52)

which yields

$$h_*^2 = 2nN_* + h_{end}^2.$$  

(53)

$h_{end}$ is defined as the value for which $\epsilon_V = 1$. For the specific models

$$\epsilon_V = \frac{n^2}{2} \frac{1}{h^2(1 + ch^n)},$$

(54)

therefore $h_{end}^2$ is solution of the equation

$$ch_{end}^{n+2} + h_{end}^2 - \frac{n^2}{2} = 0.$$  

(55)

For $c = 0$ the solution is exactly $h_{end}^2 = n^2/2$ while for any $c > 0$ the only real and positive solution for $h_{end}$ is easily found to be bounded by $n^2/2$. From this bound on $h_{end}$ and using the fact that $N_*$ is $\sim 50$, or so, it follows from (53) that $h_*$ is well approximated by

$$h_* = \sqrt{2nN_*},$$

(56)

provided that $n << 4N_*$. This covers a large class of models ranging from $n = 2$ up to $n = 10$ or even larger. Using $h_*$, given above, $A_s$ of Eq. (50) is written, in terms of $N_*$, as

$$A_s \simeq \frac{1}{12\pi^2} \frac{\lambda}{n^3} (2nN_*)^{(n/2+1)}.$$  

(57)

For $A_s \simeq 2.1 \times 10^{-9}$ we have that the coupling $\lambda$ is constrained to be

$$\lambda \simeq (4.97 \times 10^{-7}) \frac{k^2}{(4k)^k} \frac{1}{N_*^{k+1}} \quad \text{where} \quad n = 2k.$$  

(58)

Note that this is inverse proportional to $N_*^{k+1}$. For $N_* = 55$ and for $n = 2$, that is $V \sim h^2$, this yields $\lambda \simeq 4.11 \times 10^{-11}$ while for $k = 2$, that is $V \sim h^4$ we get $\lambda \simeq 1.87 \times 10^{-13}$. Note that for the $n = 4$ case Eq.
(57) coincides with that given in [38]. In that work a small value of the quartic coupling, \( \lambda \simeq 2.0 \times 10^{-13} \), is also quoted, quite close to ours given before.

As for the parameter \( a \) a lower bound can be established from the bound (39), that is from the observational bound on the tensor to scalar ratio \( r \). Using the analytic form of the potential one finds

\[
\frac{1}{4a} \frac{ch_n^2}{1 + ch_n^2} < 2.0 \times 10^{-9}.
\]

Replacing \( c \) in terms of \( a \) from (47), and using the value of \( h_* \) given before in (56), we have from (59), after some trivial manipulations,

\[
a \gtrsim 10^8 \left( 1.25 - \frac{N_*}{50n} \right) .
\]

For instance, for the quartic potential \( V \sim h^4 \) and for \( N_* = 55 \) this yields \( a \geq 0.97 \times 10^8 \), resulting to an inflationary scale, lower than \( \sim 10^{-5} \), or so. Note that (60) is the lowest allowed value of \( a \) consistent with the power spectrum and the bound on the potential imposed by the tensor to scalar ratio \( r < 0.063 \).

The constraints on the parameters given before arise from the amplitude of the power spectrum, in combination with the bound on \( r \), and set the range where acceptable values for \( A_s \) can be obtained. However the primordial tilt \( n_s \) puts additional constraints and in order to have an estimate of it we use the approximate formula given by

\[
n_s = 1 - 6\epsilon_V + 2\eta_V .
\]

The parameter \( \epsilon_V \) is given by (54) and for \( \eta_V \) we employ (21) from which it follows that

\[
\eta_V = \frac{n (n - 1 - (n/2 + 1)ch^n)}{h^2(1 + ch^n)} .
\]

From this, and \( \epsilon_V \) of Eq. (54), we get, on account of (61),

\[
n_s = 1 - \frac{n^2 + 2n}{h^2} .
\]

Replacing \( h \) by \( h_* = \sqrt{2nN_*} \) a rather simple expression for \( n_s \) is obtained given by

\[
n_s = 1 - \frac{n + 2}{2N_*} .
\]

Note that for \( n = 2 \) and \( N_* = 55 \) the above formula yields \( n_s = 0.9636 \) which is well within observational limits but for \( n = 4 \) a rather large value of \( N_* \) is needed to have an acceptable value for \( n_s \). In fact \( N_* > 76 \) is required to have \( n_s = 0.9607 \), the lowest allowed if the data \( n_s = 0.9649 \pm 0.0042 \) is used. This is a rather large value for the number of e-folds \( N_* \). The situation becomes even worse for models with \( n > 4 \).

It is important, in the framework of this qualitative discussion, to have estimates of the variations of the quantities of interest with varying the parameters of the models at hand. Starting from the power spectrum amplitude, given by (57), it is a trivial task to see that such a variation yields

\[
\delta A_s = \left( \frac{\delta \lambda}{\lambda} + \frac{n + 2}{2} \frac{\delta N_*}{N_*} \right) A_s .
\]

The first term stems from the explicit dependence of \( A_s \) on \( \lambda \). For fixed \( \lambda \), and varying only \( a \), it is only the second term that contributes. In this case it can be seen that, if the variation of e-folds is of order unity or so, it may produce a substantial change in \( A_s \), of the same order of the errors accompanying the measurements of \( A_s \). Due to the prefactor \((n + 2)/2\), on the right hand side of (65), this is larger for models with larger \( n \).
On the other hand, the corresponding variation of the spectral index $n_s$ is found, from (64),

$$
\delta n_s = \frac{n + 2}{2N_*^2} \delta N_* .
$$

(66)

This is proportional to the relative change $\delta N_*/N_*$ but is accompanied by an extra $N_*$ in the denominator. Due to that one expects that $n_s$ little varies with changing the number of e-folds.

In order to estimate the variations $\delta N_*$ and hence $\delta \lambda$, $\delta n_s$, with varying the couplings involved, namely $a$ and $\lambda$ for the models under investigation, one should start from Eq (45), and for a fixed value of the reheating temperature, vary $N_*$ with respect to $a$, $\lambda$. The only dependence on these is through the logarithm of $3H_*^2$, which in the slow-roll regime equals to $V_{eff}(h_*)$, and the logarithm involving $\rho_{end}$. We skip the details of such an analysis. We merely state that the final result is of the form

$$
\delta N_* = \frac{\delta a}{a} f_a + \frac{\delta \lambda}{\lambda} f_{\lambda} ,
$$

where the factors $f_{a,\lambda}$ depend on the model under consideration.

A last comment regards the instantaneous reheating temperature $T_{ins}$. This is determined once we know $\rho_{end}$, see Eq. (44) and discussion following it. With $g_s^{s(rech)} = 106.75$, which we have been using, we have

$$
T_{ins} = 0.411 \rho_{end}^{1/4} ,
$$

(68)

which holds in general. However $\rho_{end}$ depends on the details of the model under consideration. Independently of the model, $\rho_{end} = \sigma V_{eff}(h_{end})$, where $\sigma = 1.5$, and therefore $\rho_{end}$ is known once $h_{end}$ is given. For the class of models we are considering in this section, the latter follows from the solution of (55) which depends only on the combination $c$. Using the analytic form of the potential it is found, in a straightforward manner, that

$$
\rho_{end} = \frac{\sigma}{4a} \left( 1 - \frac{2}{n^2} h_{end}^2 \right) .
$$

(69)

$\rho_{end}$, and hence $T_{ins}$, cannot be quantified further, at this stage, since for this purpose the value of $h_{end}$ is needed. However a first conclusion can be reached when the parameter $c$ happens to be large. In that case $h_{end}$ is small, as we have discussed, and $\rho_{end}$ turns out to be inverse proportional to $a$. Then the instantaneous reheating temperature is proportional to $a^{-1/4}$. This argument does not hold, however, for small $c$. We shall return to this point when discussing specific models.

In the following we shall analyze in detail the predictions for this class of models. In doing this, we shall not follow the aforementioned approximate formulas, which neglect the temperature dependence of the number of e-folds. These are only used simply to delineate regions of the parameters involved which can be consistent with limits imposed by $A_s$, $r$ and $n_s$. Instead we will solve Friedmann’s equations numerically, for values of the parameters that fall within the right ball park, and use the accurate expressions, for all observables involved, presented in previous sections. In our procedure the time corresponding to the end of inflation, $t_{end}$, is determined, as usual, by $\epsilon_1(t_{end}) = 1$, or same $\ddot{a}(t_{end}) = 0$. The time $t^*$, corresponding to a scale $k^*$, for any given reheating temperature $T_{reh}$ is found by solving Eq. (45), which is a fairly easy task to implement numerically. That done all quantities at $t^*$, and therefore corresponding to the scale $k^*$, are easily extracted.

**Model I**:

We first consider the model (Model I) in which the functions $g, M^2$ and $V$ are as given by (46) with $n = 2$, that is the potential $V$ is quadratic in the field $h$,

$$
V(h) = \frac{m^2}{2} h^2 .
$$

(70)

For this case we prefer to use $m^2$, instead of $\lambda$, since it carries dimension of mass$^2$ when $m_P$ is reinstated. This models has been discussed in [32] and belongs to the class of the cosmological attractors [74], which
is clearly seen if one uses the canonically normalized field $\phi$, see (19). However no need to do that as we prefer to work directly with the non canonical field $h$ instead. Following the previous findings we define, see Eq. (47), the constant $c$ as the combination

$$c = 2m^2a.$$  \hspace{1cm} (71)

The value of $h_*$ in this case is given by, using (53),

$$h_* \simeq 2\sqrt{N_*}.$$  \hspace{1cm} (72)

Then from (58), which arose from the power spectrum amplitude, we get, for values $N_* = 50 - 60$,

$$m \simeq (6.5 \pm 0.5) \times 10^{-6} \text{ or } \frac{c}{a} \simeq (8.5 \pm 1.5) \times 10^{-11}.\hspace{1cm} (73)$$

The lowest (largest) limits correspond to $N_* = 60 (N_* = 50)$. Therefore by using reasonable approximations we derived rather tight limits for the parameter $m$. Recall that $m^2 \equiv \lambda$ and therefore $\lambda$ is of the order of $10^{-11}$. From the bound (60), which actually arises from the tensor to scalar ratio bound $r < 0.063$, we get, for $N_* = 50 - 60$, a lower bound which is estimated to be in the range,

$$a \geq (0.65 - 0.75) \times 10^8.\hspace{1cm} (74)$$

In this the lowest value corresponds to $N_* = 60$ and the largest to $N_* = 50$. Therefore the parameter $a$ cannot be chosen at will. It should be $\sim 10^8$ or larger. In the following, due to (74), we shall take the largest value as the bound set on $a$, i.e. $a > 0.75 \times 10^{8}$, which is valid for any $N_*$ in the range of interest.

Concerning the instantaneous reheating temperature, in this case, by solving analytically (55), and replacing $h_{\text{end}}$ into (69), we get

$$\rho_{\text{end}} = \frac{\sigma}{4a} \left(1 - \frac{1 + 8c}{4c} \right).\hspace{1cm} (75)$$

We can consider two separate regimes, the small $c$ and the large $c$, for which $\rho_{\text{end}}$, and consequently $T_{\text{ins}}$, have different dependencies on the parameters involved, as we shall see. Since from (73) the ratio $c/a$ should be of the order of $\sim 10^{-10}$, small $c$ values are obtained when $a < 10^{10}$. On the other hand large $c$ values are obtained when $a > 10^{10}$.

For small $c$ - values one can expand (75), and using the fact that $\sigma = 1.5$, the instantaneous temperature, as given by (68), it receives the form,

$$\rho_{\text{end}} \simeq \frac{\sigma}{2a} \frac{c}{2a} = \sigma m^2 \rightarrow T_{\text{ins}} = 0.455 \times \sqrt{m}.\hspace{1cm} (76)$$

This, on account of (73), results to a temperature which is $T_{\text{ins}} \simeq 2.82 \times 10^{15} \text{ GeV}$, for $m = 6.5 \times 10^{-6}$. As we shall see this estimate is not far from the one we get in our numerical treatment. What is more important, perhaps, is the fact that in the regime of small $c$ the power spectrum amplitude, which forces $m$ to be within the limits suggested by (73), also determines the maximum reheating temperature.

In the case of large $c$, $\rho_{\text{end}}$, and hence $T_{\text{ins}}$, have a completely different behavior. In fact in this case, from (75) and (68), we get

$$\rho_{\text{end}} \simeq \frac{\sigma}{4a} \rightarrow T_{\text{ins}} = 0.321 \times a^{-1/4},\hspace{1cm} (77)$$

that is, $T_{\text{ins}}$ is controlled by the value of $a$, being proportional to $a^{-1/4}$, and therefore it decreases with increasing $a$. Due to the fact $a > 10^{10}$, for being within the large $c$ regime, $T_{\text{ins}}$ turns out to have values lower than in the small $c$ case. For instance for $a = 10^{12}$ we get from (77) a temperature $T_{\text{ins}} \simeq 0.783 \times 10^{15} \text{ GeV}$ and certainly even lower temperatures for larger values of $a$. Therefore for having the largest possible value for the instantaneous temperature, of the order $\simeq 10^{15} \text{ GeV}$, we had better used values $a < 10^{10}$ so that we are within the small $c$ regime.
FIG. 1: The spectral index $n_s$ (top) and the number of e-folds $N_*$ (bottom), versus the reheating temperature $T_{reh}$, in GeV, for a scale $k^* = 0.05 \text{ Mpc}^{-1}$, and for different values of the equation of state parameter, for the cases A (left) and C (right) of Model I, discussed in the text. The shaded region marks the allowed values for the spectral index $n_s = 0.9649 \pm 0.0042$ while the vertical dotted line the instantaneous reheating temperature.

For the model at hand, predictions for three different inputs are presented named A, B and C, in the following. These correspond to values of the parameters $a$ and $c$ given by $(a, c) = (0.75 \times 10^8, 0.006), (2 \times 10^8, 0.016)$ and $(2 \times 10^9, 0.16)$. These have not been randomly chosen. In fact, for the case A the parameter $a$ touches its lower bound, discussed before, and $c$ has been taken so that $m$ falls well within the range suggested by (73). In fact we choose $m \simeq 6.32 \times 10^{-6}$. The reasoning behind this particular choice for $m$ will be discussed later.

For the other cases larger values of $a$’s were chosen but the values of $c$ are tuned so that in all cases we have the same value of $m$, i.e. $m \simeq 6.32 \times 10^{-6}$. In this way we can check how predictions vary with changing the parameter $a$ since we have kept a fixed $m$. Note that from all cases presented, the case A has the lowest allowed value of $a$ and therefore the Planck upper bound on the tensor to scalar ratio parameter $r$ is almost saturated. The other cases $B, C$ are expected to yield smaller values for $r$.

In Figure 1, at the top, we display, for the cases A (left) and C (right), the spectral index $n_s$ versus the reheating temperature $T_{reh}$, for various values of the equation of state parameter ranging from $w = -1/3$ to $w = 1.0$. The shaded region marks the range $n_s = 0.9649 \pm 0.0042$ allowed by observations. All lines intersect at a common temperature, the instantaneous reheating temperature $T_{ins}$, marked by thin vertical dashed lines, which for case A equals to $T_{ins} = 2.337 \times 10^{15}$ GeV, and for case C to $T_{ins} = 2.099 \times 10^{15}$ GeV. Values of reheating temperatures beyond that point, although displayed, are not allowed. The data shown correspond to a pivot scale $k^* = 0.05 \text{ Mpc}^{-1}$. Note that $n_s$ data by themselves do not impose any restriction on the reheating temperature, as long as the equation of state parameter is in the range from 0.25 to values slightly lower than $\simeq 1.0$. For these values of $w$ any
temperature is allowed. For $w < 0.25$ a lower reheating temperature is imposed which is larger for smaller values of $w$. For instance for the canonical reheating scenario, $w = 0$, this is $\approx 10^7 - 10^9 \text{ GeV}$ while for $w = -1/3$ this is $\approx 10^{13} \text{ GeV}$. At the bottom of the same figure, and for the same set of inputs, the corresponding numbers of e-folds, $N_*$, are shown, for the A (left) and the C (right) cases respectively.

Note that both $n_s$ and $N_*$, shown in the figures, are very similar for the two cases, A and C. In particular both observables move slightly downwards in going from A (left) to C (right), that is by increasing the value of $a$ from $0.75 \times 10^8$ to $2.0 \times 10^9$, keeping the other parameter fixed. In fact, varying only the parameter $a$, keeping $\lambda = m^2$ fixed, which is the case for the inputs we are using, we get from (67),

$$\delta N_* = \frac{\delta a}{a} f_a .$$

(78)

For our input values, we find that the factor $f_a$ is of order unity and negative. The result is that by increasing the value of the parameter $a$, the relative change $\delta N_*/N_*$ is negative and therefore, due to (66), $n_s$ decreases. This decrease is small, as we have already discussed, what is indeed imprinted on this figure.

The power spectrum amplitude imposes more stringent bounds on $T_{\text{reh}}$ than $n_s$, as shown in Figure 2. In this figure we plot the amplitude $10^9 \times A_s$ versus the reheating temperature $T_{\text{reh}}$, in GeV, for $k^* = 0.05 \text{ Mpc}^{-1}$, and for different values of the equation of state parameter, as in the previous figure. The shaded region marks the allowed range $10^9 \times A_s = 2.10 \pm 0.03$. On the left the case A is shown and on the right the case C. The lines are as in Figure 1. One notices that for the A-case values $w \geq 1/3$ are totally excluded by $A_s$ data while for $w \lesssim 0.25$ limits on the minimum and maximum allowed temperature are imposed. In this case the maximum temperature, for any allowed value of $w$, can never reach the instantaneous temperature. For the C-case, right panel, one sees, by comparing this figure with the $n_s$ plot, top and right pane of Figure 1, that the bounds set on the reheating temperature are more constrained. In particular, for values of $w$, which deviate from $w \simeq 1/3$, a lower reheating temperature is established, which is much higher than this imposed by $n_s$ data.

Comparing the two cases, A and C, we observe that $A_s$ also decreases in accord with (65), and the fact that $\lambda$, or equivalently $m$, is fixed and $\delta N_*/N_*$ is negative. However the change in $A_s$ is relatively large, unlike $n_s$, in the sense that its variation reaches the order of magnitude of the observational error of $A_s$, as has been previously discussed.

It is worth mentioning that given a fixed value for the parameter $a$ there is a fine-tuned value of $m$, in the range suggested by (73), for which the case $w = 1/3$ falls within the allowed region by $A_s$.

FIG. 2: The amplitude $10^9 A_s$ versus the reheating temperature $T_{\text{reh}}$, in GeV, for $k^* = 0.05 \text{ Mpc}^{-1}$, for different values of the equation of state parameter. The shaded region marks the allowed range $10^9 \times A_s = 2.10 \pm 0.03$. On the left the case A is shown and on the right the case C of Model I. The instantaneous temperatures, in each case, are marked by thin dotted vertical lines.
observations\textsuperscript{6}. In this case the instantaneous reheating temperature is attained for any value of \(w\) in the range \(-1/3 \leq w \leq 1\). However in this case a lowest temperature is determined, which is close to the instantaneous temperature, for any \(w\) that deviates from the value 1/3. This includes the values 0.0 \(\lesssim w \lesssim 0.25\) which are favored in some reheating scenarios. This can be clearly seen, for instance, in the case C where for \(a = 2.0 \times 10^9\) the value \(m = 6.32 \times 10^{-6}\) forces the line \(w = 1/3\) to be within \(A_s\) limits, as shown on the right panel of Figure 2. Keeping \(a\) fixed, any slight change in the value of the parameter \(m\), which essentially controls \(A_s\), will move, downwards or upwards, the line \(w = 1/3\), off the allowed range, and in this case the instantaneous reheating scenario is no longer supported. At the same time, depending on the value of \(m\), lower and higher limits of reheating temperatures are imposed, different for each \(w\). However some values of \(w\) are totally excluded. For instance, by increasing \(m\), the line \(w = 1/3\) will be uplifted and move above the upper observational limit set on \(A_s\). In this case all values in the range \(1/3 \leq w \leq 1\) are excluded. If, on the other hand \(m\) is decreased, the line \(w = 1/3\) will move below the lower limit of \(A_s\) and values \(-1/3 \leq w \leq 1/3\) are excluded. Increasing, or decreasing, further the value of \(m\) will exclude all possible cases \(-1/3 \leq w \leq 1\). Therefore, there is a range of \(m\) outside of which agreement with \(A_s\) data can not be obtained, for any value of the equation of state parameter, in the interval \(-1/3 \leq w \leq 1\). This range is actually very tight and falls within the suggested range given by (73). Within this range there are fine-tuned values for which reheating can be instantaneous. Note that the sensitivity of the spectral index \(n_s\) on the value of \(m\) is not that dramatic and \(n_s\) data leave more amber space for the observational requirements to be satisfied. Therefore, the conclusion is that given \(a\), the value of \(m\) should lie in a very narrow range, in order to comply with power spectrum data. Moreover if reheating is instantaneous it should be fine-tuned accordingly. This, as we shall see, holds for other popular models as well, notably the Higgs model that will be discussed later.

Following the already outlined numerical procedure, in Table I we display sample outputs of the model under consideration for the choice of the parameters corresponding to the inputs \(A_s\) and \(C\) for a pivot scale \(k^* = 0.05 \text{ Mpc}^{-1}\). The predicted cosmological observables \(n_s, r, A_s\) are displayed, for various values of the equation of state parameter \(w\), corresponding to the minimum (upper rows) and maximum (lower rows) allowed reheating temperatures \(T_{\text{reh}}\), when the limits \(A_s \simeq (2.10 \pm 0.03) \times 10^{-9}\), and \(n_s = 0.9649 \pm 0.0042\) are observed. The corresponding predictions for the number of e-folds \(N_s\), are also shown. Blanc entries indicate that there are no values compatible with observational bounds put on \(n_s, A_s\), for the specific value of \(w\). Note that for the \(C\) case, the maximum reheating temperature reaches the instantaneous reheating temperature, \(T_{\text{ins}} = 2.099 \times 10^{15} \text{ GeV}\). At this temperature predictions are independent of \(w\), due to the fact that \(T_{\text{ins}}\) marks the intersection of all \(w\)-lines. For the same case, the lower limits on \(T_{\text{reh}}\) are also shown. For the cases \(w = 0.0, 0.25\) and \(w = 1.0\), these are not very far from the \(T_{\text{ins}}\), as already discussed, in agreement with Figure 2, right panel. For the case \(A\), on the other hand, the minimum and maximum reheating temperatures are both smaller than the corresponding ones of the \(C\) case. Note in particular the predictions for \(w = 0.25\) for which the range of temperatures, allowed by all observations, is \(T_{\text{reh}} \simeq (1.7 \times 10^9 - 2.0 \times 10^8) \text{ GeV}\).

In Figure 3 the tensor-to-scaler ratio \(r_{0.002}\) versus the spectral index \(n_s\) is plotted, for the Model I, for the data set A (red-line), B (green-line) and C (blue-line). In drawing this figure a pivot scale \(k^* = 0.002 \text{ Mpc}^{-1}\), was used so that it can be directly compared to the corresponding Planck 2018 bounds [50, 51], which are also drawn. The tiny circle (in magenta), the small (in orange) and the large (in green) correspond to reheating temperatures close to BBN, Electroweak and Leptogenesis scenarios, given by \(T_b = 1 \text{ MeV}, T_{\text{ew}} = 10^2 \text{ GeV}\) and \(T_{\text{lep}} = 10^9 \text{ GeV}\), respectively. The largest circle (in yellow) marks the instantaneous reheating temperature, see Eq. (44), for each case displayed. The number close to each circle indicates the corresponding number of e-folds left, at the pivot scale \(k^* = 0.002 \text{ Mpc}^{-1}\). The value of the equation of state parameter for the figure on top is \(w = 0.0\), while for the one at the bottom \(w = 0.25\). In the latter only the e-folds corresponding to \(T_b\) and the instantaneous reheating are shown, to be clearly visible. In both cases shown, \(w = 0\) and \(w = 0.25\), the smallest values for the tensor-to-scaler ratio \(r\) are obtained in the C-case, that is for the largest values of the parameters \(a, c\).

\textsuperscript{6} This requires that the case \(w = 1/3\) is compatible with \(N_s\) in the range \(50 - 60\), which is always the case provided the parameter \(a\) does not take extremely high values.
Model I  ( pivot scale $k^* = 0.05 \, Mpc^{-1}$ )

| $w$ - value | $w = 0.0$ | $w = 0.25$ | $w = 1.0$ | $w = 0.0$ | $w = 0.25$ | $w = 1.0$ |
|-------------|-----------|-----------|-----------|-----------|-----------|-----------|
| $10^9 A_s$  | 2.07      | 2.07      |           | 2.07      | 2.07      | 2.13      |
| $n_s$       | 0.9637    | 0.9637    |           | 0.9636    | 0.9636    | 0.9641    |
| $r$         | 0.0629    | 0.0629    |           | 0.0041    | 0.0041    | 0.0040    |
| $N_*$       | 55.24     | 55.24     |           | 55.63     | 55.63     | 56.41     |
| $T_{reh}$   | $8.738 \times 10^{12}$ | $1.737 \times 10^{7}$ |           | $1.113 \times 10^{15}$ | $8.756 \times 10^{13}$ | $3.744 \times 10^{14}$ |

| $10^9 A_s$  | 2.13      | 2.13      | 2.08      | 2.08      | 2.08      |
| $n_s$       | 0.9642    | 0.9642    | 0.9637    | 0.9637    | 0.9637    |
| $r$         | 0.0615    | 0.0615    | 0.0040    | 0.0040    | 0.0040    |
| $N_*$       | 56.02     | 56.02     | 55.84     | 55.84     | 55.84     |
| $T_{reh}$   | $9.063 \times 10^{15}$ | $2.047 \times 10^8$ | 2.099 $\times 10^{15}$ | 2.099 $\times 10^{15}$ | 2.099 $\times 10^{15}$ |

TABLE I: Sample outputs for the Model I, for inputs corresponding to cases A, C (see main text), for the cosmological observables $n_s, r, A_s$ and $N_*$, for various values of the equation of state parameter $w$. The values shown for the reheating temperature $T_{reh}$, in GeV, correspond to the minimum (upper rows) and maximum (lower rows) allowed, when the observational limits for $A_s \simeq (2.10 \pm 0.03) \times 10^{-9}$ and $n_s = 0.9649 \pm 0.0042$ are imposed. Blank entries indicate that there are no values compatible with the observational bounds put on $n_s$ and $A_s$, for the specific value of $w$.

Recall that the ratio $c/a$ has been kept fixed. For smaller values of the parameters, $r$ gets larger and saturates the Planck upper bound in the A - case, corresponding to the lowest allowed values of $a, c$, as we have already remarked.

We point out that in drawing Figure 3 the $A_s$ constraints have not been taken into account. Including these will shrink considerably the allowed line segments, displayed on the figure, since $T_{reh}$ is further constrained by $A_s$ data. For instance, for the C - case, which is well within the region allowed by all observations, yielding also the smallest value for $r$, a large portion of the segment, with ends corresponding to temperatures $T_b$ and the minimum allowed temperature, as this is read from Table I for each $w$-case, will be excised. Only a tiny part of it, close to the maximum reheating temperature $T_{ins}$, will be left.

Model II:

As a second model (Model II) worth studying, is the one in which the functions $g, M^2$ are as in (46), as in the Model I, but the potential is quartic in the scalar field involved, i.e.

$$V(h) = \frac{\lambda}{4} h^4,$$

that is $n = 4$. We have already remarked, based on the qualitative arguments presented earlier, that this model, as well as all with $n > 4$, fails to satisfy the observations on the spectral index unless one has a large number of e-folds, probably larger than $N_* > 76$, or so. However a more detailed study is required to get a firm conclusion which also takes into account the reheating temperature.

Using the general results, given at the beginning of this section, when applied to this model, we get,

$$h_* \simeq \sqrt{8 N_*}.$$  

Also on account of (58) the coupling $\lambda$ is

$$\lambda \simeq 10^{-8} \frac{3.11}{N_*^3},$$

which for e-folds in the range $N_* = 50 - 60$, yields

$$\lambda \simeq (1.45 - 2.50) \times 10^{-13},$$
FIG. 3: The tensor-to-scalar ratio $r_{0.002}$ versus the spectral index $n_s$ for the Model I, for the data set A (red-line), B (green-line) and C (blue-line) corresponding to different inputs of the parameters (see main text). A pivot scale $k^* = 0.002\, Mpc^{-1}$ is used so that a direct comparison with the corresponding Planck 2018 data is possible. The value of the equation of state parameter for the figure on top is $w = 0.0$, while for the figure at the bottom is $w = 0.25$. The tiny circle (in magenta), the small (in orange) and the large (in green) correspond to reheating temperatures close to BBN, Electroweak and Leptogenesis scenarios, while the largest (in yellow) marks the instantaneous reheating temperature (see main text). The numbers indicate the e-folds, in each case, when $k^* = 0.002\, Mpc^{-1}$.

the lowest value corresponding to $N_* = 60$. Therefore the coupling $\lambda$ must be quite small in order to satisfy the constraints put by observations. As for the parameter $a$, which sets the inflation scale, employing (60), we have a lower bound given by

$$a \gtrsim (0.95 - 1.00) \times 10^8,$$

not much different from the bounds given in (74).
FIG. 4: The spectral index $n_s$, versus the reheating temperature $T_{\text{reh}}$, in GeV, for a scale $k^*=0.05\, \text{Mpc}^{-1}$, and for various values of the equation of state parameter, for the cases A (left) and B (right) of Model II discussed in the text. The shaded regions marks the allowed values for the spectral index $n_s=0.9649\pm 0.0042$ and the vertical dotted lines the instantaneous reheating temperatures.

FIG. 5: As in Figure 4 for the number of e-folds $N_*$. 

For $T_{\text{ins}}$ we have to calculate $h_{\text{end}}$, as in the $n=2$ case, and use (69) adapted to the case $n=4$. Although analytic solution for $h_{\text{end}}$ is feasible, through Eq. (55), we will not present it. Instead we shall discuss its behavior for small and large $c$-values. For small $c$, omitting $O(c^2)$ terms, we find $h_{\text{end}}^2 \approx 8(1-64c)$. Then from (69) the leading contribution is,

$$\rho_{\text{end}} \approx \sigma \frac{16c}{a} = 16\sigma \lambda \quad \rightarrow \quad T_{\text{ins}} = 0.909 \times \lambda^{1/4}. \quad (84)$$

With $\lambda = 2 \times 10^{-13}$, the central value in the range (82), this yields an instantaneous reheating temperature around $T_{\text{ins}} \approx 1.48 \times 10^{15}\, \text{GeV}$. As in the previously studied model, $n=2$ case, the power spectrum determines the maximum reheating temperature, in the regime of small $c$. In the case of large $c$, $h_{\text{end}}^2$ behaves as $c^{-1/3}$, and hence little contributes to Eq. (69). Then keeping only the leading term in $\rho_{\text{end}}$, we get the same result (77), as in the previous model, and $T_{\text{ins}}$ is again proportional to $a^{-1/4}$.

For this model we shall also present sample outputs of our numerical treatment, considering a fixed value $\lambda = 2.0 \times 10^{-13}$, in the middle of the range suggested by (82), and values of $a$ in the range $a=10^8-10^{10}$, respecting therefore the bound (83). The value $a=10^8$ corresponds to the lowest allowed
value, and for future reference we name it A - case, while $10^{10}$ is arbitrarily taken to be two orders of magnitude larger, which we name B - case. Although, in principle, one can consider larger $a$ values there is no need to do this for reasons that will be shortly explained.

On the left panel of Figure 4 the predictions for the spectral index $n_s$, for the cases A (left) and B (right), are shown versus the reheating temperature for various values of the equation of state parameter $w$. Note that there is no much difference between the two cases, although the parameter $a$ differs by two orders of magnitude. The explanation is the same as that discussed for Model I. Notice that on the right the lines have been moved imperceptibly lower. That is, the tendency is to get lower values as the parameter $a$ increases. Concerning the instantaneous reheating temperature, for the values taken for $a, \lambda$, for the A - case it is $T_{\text{ins}} = 1.223 \times 10^{15}$ GeV, while for for B - case this is $T_{\text{ins}} = 1.129 \times 10^{15}$ GeV. These are marked by vertical thin dotted lines, as in previous figures. As already remarked for this model agreement with $n_s$ observational data is hard to achieve. In both cases it is clearly seen from this figure, that only for very small reheating temperatures, and only for $w = 1$, values of $n_s$ that are marginally acceptable can be obtained. In this case the number of e-folds is large $N_s > 70$, as is shown in Figure 5 where the number of e-folds is displayed. This agrees with the general arguments given before, see discussion following Eq. (64). We have not considered larger values of $a$, since as explained, they would predict lower $n_s$, resulting to larger deviations from the data.

Although agreement with $n_s$ data cannot be obtained, in this model, for reasons of completeness we shall give a brief account for the predictions for the power spectrum amplitude. Agreement with $A_s$ data requires values of $w$ that are smaller than 0.25, for the A - case, while for the B - case the value $w = 0.25$ is marginally accepted. Values lower than $w \simeq 0.25$ are allowed. Whatever the case, such values for the equation of state parameter lead, according to Figure 4, to even smaller values of $n_s$, less than $\simeq 0.945$ or so, and hence unacceptable.

These results are in agreement with [38] where small $n_s$ are also obtained, and indicate that the quartic potential ($n = 4$) is in tension with cosmological data. Models with $n > 4$ yield predictions that according to our general arguments are, also, hard to reconcile with the data.

Therefore the conclusion is that, from the class of the models whose initial potential is of the monomial form $V \sim h^n$, and with constant values for the coefficients of $R$ and $R^2$ terms in the Palatini gravity, only the case $n = 2$, which belongs to the class of the cosmological attractors [74], can lead to successful inflation, if all observational constraints are taken into account.

B. Non-minimally coupled models

A non-minimal coupling arises if in the previously studied models the constants $g$ and/or $M^2$ are promoted to be field dependent. A particularly interesting case is the model in which

$$g(h) = 1 + \xi h^2, \quad M^2(h) = \frac{1}{3a}, \quad V(h) = \frac{\lambda}{4} h^4. \quad (85)$$

This belongs to the class of models (46), with quartic potential, however the scalar $h$ is non-minimally coupled to the scalar curvature $R$, in the Palatini framework, since $g$ is field dependent, in the particular way shown above. This model arises actually from the Higgs coupling to Palatini gravity

$$\frac{m_P^2 + 2 \xi H^1 H}{2} R + \frac{a}{4} R^2 + |DH|^2 - \lambda \left( H^2 - \frac{u^2}{2} \right)^2, \quad (86)$$

where $u \simeq 246$ GeV is the Electroweak scale. In Planck units, $m_P = 1$, this is very small $u \sim 10^{-16}$ and plays no significant role in inflation. Setting therefore $u = 0$ and working in the unitary gauge, $H^1 = (0, h/\sqrt{2})$, (86) is actually the model described by $g, M^2$ and quartic potential as given in (85).

The Higgs coupling to gravity and its role as the inflaton, in the metric formulation, has been proposed in [75, 76] and it has been widely studied since then [29, 32, 33, 36, 38, 44, 77–98] both in the context of the metric formulation and in Palatini formulation. The importance of the $R^2$ term in (86) has been discussed in [32, 33, 36, 38]. In this work we shall show that the quartic coupling $\lambda$, as in the minimally coupled quartic model studied previously, corresponding to $\xi = 0$, is constrained considerably by cosmological
data, especially the power spectrum amplitude $A_s$. This limits the available options especially when the reheating of Universe after inflation is taken into account.

The functions $K, L$ and $V_{\text{eff}}$ in this model are given below, in the limit $u = 0$,

$$K(h) = \frac{1 + \xi h^2}{(1 + \xi h^2)^2 + ch^4}, \quad L(h) = \frac{a}{(1 + \xi h^2)^2 + ch^4},$$  \hspace{1cm} (87)

while the potential $V_{\text{eff}}$ receives the form

$$V_{\text{eff}}(h) = \frac{1}{4a} \frac{ch^4}{(1 + \xi h^2)^2 + ch^4}.$$  \hspace{1cm} (88)

As in the simple quartic potential the parameter $c$ is the combination $c = a\lambda$. Notice however that a non-trivial $\xi$-dependence exists and therefore the Higgs model differs from the simple quartic model studied previously. Evidently when $\xi = 0$ the functions (87), (88) smoothly go into (48), (49).

For large values of $h$ the potential (88) approaches a plateau $\simeq 1/4(a + \xi^2/\lambda)$, and therefore an inflation scale $\mu$ can be set. In particular, reinstating units, this is defined by

$$\mu \equiv m_P/\sqrt{3(\xi^2/\lambda + a)}.$$

For comparison, in the Starobinsky model the inflaton potential reaches the value $3\mu_S^2 m_P^2/4$, where $\mu_S$ is the scalaron mass, and in that case cosmological data determine its magnitude, given by $\mu_S \simeq 10^{-5} m_P$. In the model under consideration, the magnitude of $\mu$ will be discussed later, when imposing limits on the parameters $\xi, \lambda$ and $a$.

Proceeding in the same manner, as in the previously studied models, the slow - roll parameters $\epsilon_V, \eta_V$ are given by, as functions of $h$,

$$\epsilon_V = \frac{8(1 + \xi h^2)}{h^2(1 + 2\xi h^2 + (\xi^2 + c)h^4)}, \quad \eta_V = \frac{4}{h^2} \left( -\frac{3 + 2\xi h^2}{1 + \xi h^2} + \frac{6(1 + \xi h^2)}{(1 + \xi h^2)^2 + ch^4} \right).$$  \hspace{1cm} (89)

Although the parameter $\eta_V$ has a rather complicated form both the spectral index and the power spectrum amplitude have rather simple expressions. In fact they are given by

$$n_s = 1 - \frac{16}{h_s^2} - \frac{8}{h_s^2(1 + \xi h_s^2)}$$  \hspace{1cm} (90)

and

$$A_s = \frac{\lambda}{24\pi^2} \frac{h_s^6}{32(1 + \xi h_s^2)},$$  \hspace{1cm} (91)

where we have replaced the field $h$ by its pivot value $h_s$. These coincide with (63) and (50), respectively, for $n = 4$, when $\xi = 0$, as they should. However, the presence of the $\xi$ alters the predictions for the cosmological observables, as we shall see.

In order to proceed further we need the pivot value $h_s$. In this case the number of e-folds $N_s$ is given by

$$N_s = \frac{1}{8}(h_s^2 - h_{\text{end}}^2).$$  \hspace{1cm} (92)

This does not depend explicitly on the parameter $\xi$ and is identical with (52) when $n = 4$. Therefore

$$h_s^2 = 8N_s + h_{\text{end}}^2,$$  \hspace{1cm} (93)

which is functionally the same as (53) but the value of $h_{\text{end}}$ differs. The latter depends on both $\xi$ and the combination $c = a\lambda$, being determined as solution of the equation

$$c h_{\text{end}}^6 + (1 + \xi h_{\text{end}}^2)(h_{\text{end}}^2(1 + \xi h_{\text{end}}^2) - 8) = 0.$$  \hspace{1cm} (94)

This is a cubic equation in $h_{\text{end}}^2$, which we prefer to cast it in the form (94) for reasons that will become clear in the following. Notice that in the limit $\xi = 0$ this equation becomes (55), when in the latter we put
n = 4. In the form presented by (94) we see that when c = 0 the solution for $h_{\text{end}}^2$ is easily obtained since it becomes a quadratic equation for $h_{\text{end}}^2$. This observation is useful if we want to study the predictions of the model for small c, and in doing that we expand in powers of c about the zeroth order solution.

Being a cubic equation for $h_{\text{end}}^2$, analytic solution can be obtained, and in our case there is only one real and positive solution. The value of this solution, for $h_{\text{end}}^2$, can never exceed 8. In fact this value is reached when $\xi, c$ are smaller than $\sim 10^{-3}$, or so. For larger values the root of this equation is smaller. The conclusion is that $h_{\text{end}}^2$ can be neglected in (93) and $h_*$ can be approximated by

$$h_* \simeq \sqrt{8N_*},$$

as in the simple quartic model. Replacing this value in (90) and (91) we get

$$n_s = 1 - \frac{2}{N_*} - \frac{1}{N_*(1 + 8\xi N_*)},$$

and

$$A_* = \frac{2\lambda}{3\pi^2} \frac{N_*^3}{(1 + 8\xi N_*)}.$$  

As expected in the limit $\xi = 0$ these smoothly go to (64) and (57) when in the latter we put $n = 4$.

However the role of the parameter $\xi$ is very important and can improve the case, as far as the spectral index $n_s$ is concerned. In the simple quartic model the predictions for $n_s$ are hard to comply with the cosmological observations, unless large values of the e-folds are considered, $N_* > 70$ or so, as already discussed. This has been also pointed out in [32, 33]. Such large values of e-folds may not be acceptable, since they demand very low values for the reheating temperature, at least in the standard reheating scenarios. Accepting large number of e-folds, $N_* > 70$, it may be consistent with alternative reheating scenarios, which may be interesting per se, however, in this work we would prefer to keep a more conservative attitude.

Concerning $\xi$, we shall assume that it is positive. Then one sees from (96) that $n_s$ is larger than the one obtained in the quartic potential studied before, which corresponds to $\xi = 0$. Moreover, for any $N_*$ the observable $n_s$ increases as $\xi$ grows and therefore values within limits may be obtained for sufficiently large values of $\xi$. From (96) it can be seen that for values $\xi \simeq 0.06$ the spectral index can be within observational limits, for e-folds in the range $N_* \simeq 52.0 - 60.0$. That is for this value of $\xi$ a large portion of e-folds, in the range 50.0 − 60.0, is covered, which is broadened for larger $\xi$ allowing, also, for values of $N_*$ lower than 52.0. Values of $\xi < 0.06$ are also acceptable, at the cost of shrinking considerably the range of the allowed e-folds, that are compatible with the observational limits imposed by $n_s$. For instance for $\xi \simeq 0.004$ one obtains $n_s = 0.9607$, at the edge of the lower observational limit, pushing $N_*$ to $N_* \simeq 60$. From these arguments it is obvious that a reasonable range to deal with in our numerical procedure is to focus on values of $\xi$ of the order of $\mathcal{O}(10^{-2})$, or larger. In the following we shall take $\xi \gtrsim 0.06$ on the grounds that is likely to cover a wider range of e-folds, as we explained above.

From (97), and accepting that $A_* \simeq 2.1 \times 10^{-9}$, the quartic coupling is found to be constrained by

$$\lambda \simeq 3.11 \times 10^{-8} \frac{1 + 8\xi N_*}{N_*^3}. $$  

In the limit $\xi = 0$ this coincides with (81), as it should. From this it is seen that the allowed values for $\lambda$ depend on the parameter $\xi$, and also that larger values of the coupling $\lambda$, as compared to the simple quartic model, are available in this case. However even in this case the quartic coupling is small. For $\xi = 0.06$ it is of order $\sim 10^{-12}$. In order for $\lambda$ to reach values of order $\gtrsim 10^{-6}$ one needs large values $\xi \gtrsim 10^4$, when $N_* \simeq 50.0 - 60.0$.

Concerning the parameter $a$, by the same token, as discussed in previous models, a lower limit on it can be established by (39), given by

$$a \gtrsim 10^8 \left( 1.25 - \frac{N_*(1 + 8\xi N_*)}{200} \right). $$  

as we have remarked. Therefore in this case the results are independent of the parameter $a$, as defined before (see discussion following Eq. (88)), becomes
\[ \mu \xi \]
where $R$ was taken, as usual, in the range $N = 50 - 60$. In creating this table the values of $\xi$ and $N_*$ were taken, as usual, in the range $N = 50 - 60$. In the fourth (fifth) column the lower (upper) bound, set on the parameter $a$, is displayed for having $a > \xi^2/\lambda$ ($a < \xi^2/\lambda$). The power $\nu$ is positive, for $\xi > 1$, and negative for $\xi < 1$.

This bound on $a$ depends on $\xi$, it is quadratic in $N_*$, and there is a critical value of $\xi$ beyond which it becomes negative, signaling that in this case any positive value of $a$ is actually allowed. As we prefer to work with values $\xi > 0.06$ the rhs of (99) is negative, for $N_* \simeq 50 - 60$, and practically for our purposes there is no lower limit imposed on the parameter $a$. The absence of a lower bound may be important since in this case $a$ can be chosen either larger, or smaller, than the ratio $\xi^2/\lambda$. In the regime
\[ \xi^2 > a \lambda, \]
an upper bound on $a$ is imposed, for given $\xi, \lambda$. Of particular interest, within this regime, is the case where $\xi^2 \gg a \lambda$ In this limit, it is seen from (87) and (88) that the functions $K(h)$ and the potential $V_{eff}(h)$ do not depend on the parameter $a$. In fact $K(h)$ depends only on $\xi$ and $V_{eff}(h)$ on $\xi, \lambda$. The function $L(h)$ does depend on $a$, however, its effect in the equations of motion is imperceptibly small, as we have remarked. Therefore in this case the results are independent of the parameter $a$, as long as $\xi^2 \gg a \lambda$ holds. This we have verified in our numerical procedure. In this case the inflation scale $\mu$, as defined before (see discussion following Eq. (88)), becomes $\mu \simeq \sqrt{3} \xi^2 m_P$ and lies in the range $\sim (2 \times 10^{-5} - 5 \times 10^{-7}) m_P$, for values of $\xi$ in the range $0.06 - 100.0$ and for $N_*$ between $50 - 60$, the smaller (larger) scales being attained for higher (lower) $\xi$ and $N_*$ values.

Evidently the arguments given before are no longer valid if the parameters are chosen in the regime
\[ \xi^2 < a \lambda. \]
Then we have a lower bound on $a$, for given $\xi, \lambda$. In this case the predictions depend on $a$ and $\xi, \lambda$ as well. In particular, when $\xi^2 \ll a \lambda$ the inflation scale is $\mu \simeq m_P/\sqrt{3} \xi$, that is it is determined solely by $a$.

For facilitating the discussion, in Table II we present order of magnitude estimates of the quartic coupling $\lambda$, as these arise from (98), and the corresponding $\xi^2/\lambda$ for given value of $\xi = 10^\nu$, where $\nu < 0$ or $\nu > 0$, corresponding to $\xi < 1$ or $\xi > 1$, respectively. We see that the coupling $\lambda$ increases with increasing $\xi$, or same, increasing $\nu$. Although not displayed in the table, we remark that for $\nu \geq 0$ the coupling $\lambda$ lies within $0.7 - 1.0 \times 10^{-10+\nu}$. The lower and upper bounds on the parameter $a$, for having $a > \xi^2/\lambda$ and $a < \xi^2/\lambda$, are shown in the fourth and fifth column, respectively. In creating this table the values of $N_*$ were taken, as usual, in the range $N_* \simeq 50 - 60$.

In order to have an estimate of the instantaneous reheating temperature, $T_{ins}$, which is given by (68), we need know the energy density at the end of inflation. Following similar arguments, as for the models studied previously, we find that in this case it is given by
\[ \rho_{end} = \frac{\sigma}{4a} \left( 1 - \frac{h_{end}^2 (1 + \xi h_{end}^2)}{8} \right) = \frac{\sigma}{4a} F(\xi, c). \]

Recall that $\sigma = 1.5$. The function $F(\xi, c)$ is too complicated to be presented, although analytic expression for the unique positive solution $h_{end}^2$ of Eq. (94) does exist. This we shall actually use for the calculation of $\rho_{end}$ through (102). Replacing $a$ by $c/\lambda$, with $\lambda$ as given by (98), we get from (68),
\[ T_{ins} = (0.968 \times 10^{-3}) \left( \frac{55}{N_*} \right)^{1/2} \left( \xi + 2.27 \times 10^{-3} \frac{55}{N_*} \right)^{1/4} R^{1/4}(\xi, c), \]
where $R(\xi, c) = F(\xi, c)/c$. This it gets a very simple form in particular regions, and interestingly enough this includes the region where $T_{ins}$ gets its largest value.
The first region of interest is when \( c/\xi^2 < 1 \). As we have already remarked, Eq. (94) is easily solved when \( c \) vanishes, since in that case it is reduced to a quadratic equation for \( h_{\text{end}}^2 \). For non-vanishing \( c \), within the regime \( c/\xi^2 < 1 \), we can treat this ratio as a small parameter, in order to find the desired solution as deviation from the zeroth-order solution, corresponding to \( c = 0 \). This is easily implemented, resulting to a function \( R(\xi, c) \), which to the lowest order in \( c/\xi^2 \), is independent of \( c \). In particular it is found that,

\[
R(\xi, c) = \left( \frac{1 + 16\xi - \sqrt{1 + 32\xi}}{16\xi^2} \right)^2 = P(\xi). \tag{104}
\]

The function \( P(\xi) \) is regular at \( \xi = 0 \), with limit \( P(0) = 64 \). Using this, we find from (103)

\[
T_{\text{ins}} = (0.968 \times 10^{-3}) \left( \xi P(\xi) \right)^{1/4}. \tag{105}
\]

In this we have set \( 55/N_\ast \simeq 1 \), and, besides, we assume that \( \xi > 0.01 \), which is actually the region we are interested in. Note that (105) is valid in the regime \( c/\xi^2 < 1 \) and it is a very handy relation. Within the \( c < \xi^2 \) regime the maximum temperature is attained when \( \xi P(\xi) \) reaches its maximum. This occurs at \( \xi = 3/32 \), that is very close to \( \simeq 0.094 \), and for this value \( T_{\text{ins}} \simeq 2.47 \times 10^{15} \text{ GeV} \), in natural units. This is independent of \( c \) as long as \( c \) is much smaller than \( \xi^2 \). Away from this maximum \( T_{\text{ins}} \) drops, as \( \xi \) increases, behaving as \( T_{\text{ins}} \simeq (0.968 \times 10^{-3}) \xi^{-1/4} \).

Another region of interest is when \( c \) is large and \( c \gg \xi^2 \). In this region the function \( F(\xi, c) \), that controls \( \rho_{\text{end}} \) in (102), is very close to unity. Note that the largeness of \( c \) by itself is not adequate to have \( F(\xi, c) \simeq 1 \), despite the fact that \( h_{\text{end}}^2 \) is small. We must require, in addition, that \( c \gg \xi^2 \). Then \( \rho_{\text{end}} \) turns out to be inverse proportional to \( a \), and hence the instantaneous reheating temperature is proportional to \( a^{-1/4} \), or same proportional to \( (\lambda/c)^{1/4} \). The latter is proportional to \( (\xi/c)^{1/4} \), when (98) is used. Then the analytic result for \( T_{\text{ins}} \), in this case, is trivially found from (103),

\[
T_{\text{ins}} \simeq (0.968 \times 10^{-3}) \left( \frac{\xi}{c} \right)^{1/4}. \tag{106}
\]

This holds for large \( c \) values, satisfying \( c \gg \xi^2 \), and therefore it cannot be arbitrarily large. The largest value within this regime is about \( \simeq 10^{15} \text{ GeV} \), which is slightly smaller than the corresponding temperature of the \( c << \xi^2 \) region. This is obtained for \( c \simeq 10^2 \), which is relatively large, and values of \( \xi^2 \) about an order of magnitude smaller than \( c \). Any other pair of values, for these parameters, within this particular regime, results to lower values of \( T_{\text{ins}} \).

Unfortunately, outside the aforementioned regions there are not simple mathematical expressions to deal with, and we shall rely on a numerical treatment of (103). In fact scanning the two dimensional parameter space \( c, \xi^2 \) we found, that the approximate formulae given before in the appropriate regions, agree to a very good accuracy with the values obtained from (103). In Figure 6 we display the instantaneous reheating temperature, as given by Eq. (103), for \( N_\ast = 55 \). Light colors correspond to higher temperatures. From this figure it is clearly seen that the larger temperatures are obtained for values of the parameters within the small yellow region, located at the bottom and left. The region with the largest temperature \( T_{\text{ins}} \) is centered about \( \xi \simeq 0.1 \), and values \( c \lesssim 10^{-4} \), having as boundary the blue dashed line corresponding to \( T_{\text{ins}} = 2.47 \times 10^{15} \text{ GeV} \). The maximum temperature attained is very close to it, confirming, therefore, our previous arguments. Within this region \( \xi \simeq 0.1 \), and since Eq. (98) is used, \( \lambda \simeq 10^{-12} \). Therefore \( a = c/\lambda \lesssim 10^8 \) is needed for having the largest possible \( T_{\text{ins}} \). This is also seen by drawing the locus of points for which the parameter \( a \) has a constant value, \( a = 10^8 \). This lies just above the aforementioned region. Lower values, \( a < 10^8 \), will move this line downwards, crossing the largest \( T_{\text{ins}} \) region, and thus the maximum \( T_{\text{ins}} \) is obtainable.

Note that the discussion for \( T_{\text{ins}} \), given so far, serve as an estimate of the magnitude of the instantaneous temperature. The precise values for \( T_{\text{ins}} \) will be extracted by solving Friedmann’s equations, in order to know the value of the field \( h \) at any instant, and through it \( h_{\text{end}} \) and \( \rho_{\text{end}} \), and hence \( T_{\text{ins}} \), are determined. However the numerical analysis reveals that these estimates are quite accurate.

Our numerical study can be summarized by selecting the following representative inputs:
For the value $\xi = 0.06$, which according to preceding discussion sets the threshold for having sufficient number of e-folds, we choose the quartic coupling $\lambda = 4.875 \times 10^{-12}$. From (98) one can see that for $N_* = 50 - 60$ the quartic coupling is between $4.29 \times 10^{-12}$ (for $N_* = 60$) and $6.22 \times 10^{-12}$ (for $N_* = 50$), so that the value chosen is indeed within the appropriate range. However, this fine-tuned value has been chosen so that the predicted amplitude $A_*$ is within observational limits, in such a way that instantaneous reheating is feasible. It should be remarked that the approximate formula used for $A_*$ may differ from the one that the numerical procedure returns. The latter yields more accurate results, since the exact numerical solution for the field $h$ is used, and also, because it incorporates corrections, see Eq. (31), that although small in some cases are of the same order of magnitude with the observational errors. It is for this reason that fine-tune adjustments are necessary to make the instantaneous reheating mechanism a viable possibility.

For these inputs $\xi^2/\lambda \simeq 7.38 \times 10^8$, and therefore for values $a \ll 10^8$ we are in the regime $a \ll \xi^2/\lambda$ and, as we have discussed, predictions are insensitive to the choice of $a$. Therefore any value of $a$ yields the same results, provided $a \ll 10^8$. This we have verified by our numerical code. For definiteness we take $a = 10^6$ which is three orders of magnitude smaller than $\xi^2/\lambda$ given above.

In Figure 7, at the top, we display the spectral index and the power spectrum amplitude. We see that agreement with $n_s$ data is obtained for any temperature when the parameter $w$ is $\simeq 0.25$ or larger, but smaller than 1.0. For canonical reheating, $w = 0.0$, however a lower bound is imposed $T_{\text{reh}} \gtrsim 10^{10} \text{GeV}$, while for $w = 1.0$ the lower bound is about $T_{\text{reh}} \gtrsim 100 \text{GeV}$. Looking at $A_*$ plot we observe, as advertised, that instantaneous reheating can occur, for the given $\xi, \lambda$ inputs. We also observe that the constraints are more stringent than those imposed by $n_s$. In fact values of $w > 1/3$, allow for temperatures which are very close to $T_{\text{ins}}$. At the same time a lower reheating temperature is imposed for the $w = 0.25$ case, $T_{\text{reh}} \gtrsim 10^{11} \text{GeV}$, while for the canonical scenario the lower limit imposed by $A_*$ is pushed to a much higher value, close to $T_{\text{ins}}$. At the bottom of the same figure the number of e-folds is shown. Although
values of e-folds $N_*$ as large as $\simeq 70$ for low $T_{reh}$ are allowed, by $n_s$ data, when $1 > w \geq 0.25$, the $A_s$ measurements restrict the allowed temperature range in such a way that $N_*$ is forced to be in the range $\simeq 55.70 - 56.30$, as shown in Table III. In this table the predictions for $A_s, n_s, r, N_*$, corresponding to the minimum ( upper rows ) and maximum ( bottom rows ) reheating temperature, are also shown. The maximum reheating temperature is the instantaneous temperature, $T_{ins} = 2.027 \times 10^{15}$ GeV, and for this reason the predictions for the various $w$, in that case, coincide.

As a second sample we consider values of $\xi$ in the range $\xi = 0.06 - 10.0$ when the parameter $a$ is increased to $a = 10^{12}$. These cases fall in the regime $a > \xi^2/\lambda$ when $\lambda$ is within the range suggested by (98). Following the same reasoning, we may consider values for the quartic coupling so that agreement with $A_s$ data is obtained, requiring, at the same time, the maximum reheating temperature can reach the instantaneous temperature $T_{ins}$. For the lowest value of $\xi$ in this range, $\xi = 0.06$, the quartic coupling can be taken $\lambda = 5.60 \times 10^{-12}$ while for the largest, $\xi = 10$, the value $\lambda = 8.85 \times 10^{-10}$ suits our needs. For reference, these cases we shall name A and B, respectively.

Note that by changing $a$, from $a = 10^6$ to $a = 10^{12}$, the predicted values for the cosmological parameters change as well, and thus readjustments of $\lambda$ are necessary, in order to obtain agreement with $A_s$ data, and have, at the same time, $T_{ins}$ as the maximum temperature. This is the reason the values of $\lambda$, for the case $\xi = 0.06$, are slightly different for $a = 10^6$ and $a = 10^{12}$.

In Figures 8 and 9 we display the predictions for the spectral index and the power spectrum amplitude for the cases A and B, respectively, discussed before. Comparing Figure 8 with Figure 7 (on top) we
Higgs Model (pivot scale $k^* = 0.05 \, Mpc^{-1}$)

| Input values | $\xi = 0.06$ | $\lambda = 4.875 \times 10^{-12}$ | $a = 10^6$ |
|---------------|--------------|----------------------------------|--------------|
| $w$-value     | $w = 0.0$   | $w = 0.25$ | $w = 1.0$ |
| $10^9 A_s$    | 2.07        | 2.07    | 2.13 |
| $n_s$         | 0.9633      | 0.9633  | 0.9639 |
| $r$           | 0.0105      | 0.0105  | 0.0102 |
| $N_*$         | 55.66       | 55.66   | 56.44 |
| $T_{reh}$     | $2.521 \times 10^{14}$ | $6.188 \times 10^{14}$ | $1.594 \times 10^{15}$ |

TABLE III: Predictions of the Higgs Model, for the input values shown on the top, for the cosmological observables $n_s, r, A_s, N_*$ and for various values of the equation of state parameter $w$. The values shown for the reheating temperature $T_{reh}$, in GeV, correspond to the minimum (upper rows) and maximum (lower rows) allowed, when the observational limits for $A_s$ and $n_s$ are imposed.

first observe that $T_{ins}$ is lowered, in comparison to the A-case. In fact, from $T_{ins} = 2.027 \times 10^{15}$ GeV it slides down to $6.522 \times 10^{14}$ GeV. Also the lowest reheating temperatures change a little. For instance for $w = 0.25$ this is $1.525 \times 10^{11}$, i.e. it has been slightly increased from the corresponding $a = 10^6$ case, which was $6.188 \times 10^{10}$ (see Table III). In Figure 9 the corresponding predictions for the B-case are shown. In this case $T_{ins} = 6.647 \times 10^{14}$ GeV. That is, it is slightly larger than the case A. Keeping $a$ fixed the tendency for $T_{ins}$ is to decrease, with increasing the parameter $\xi$, as long as $a > \xi^2/\lambda$, while tuning the quartic coupling to have agreement with $A_s$ data.

In Figure 10, we show the tensor-to-scalar ratio $r_{0.002}$ versus the spectral index $n_s$ for the Higgs model. The numbers of the $e$-folds are shown, and the circles designate different reheating temperatures, exactly as in Figure 3. The upper line (in red) corresponds to parameters $a = 10^6$, $\xi = 0.06$ and $\lambda = 4.875 \times 10^{-12}$ while for the one at the bottom (in blue) the parameters are $a = 10^{12}$, $\xi = 0.06$ and $\lambda = 5.60 \times 10^{-12}$. Only the cases for the canonical reheating are shown, i.e. $w = 0$. Note that in
FIG. 9: The same as in Figure 8, for inputs $\xi = 10.0$, $\lambda = 8.85 \times 10^{-10}$ and $a = 10^{12}$ ( case B ).

FIG. 10: The tensor-to-scalar ratio $r_{0.002}$ versus the spectral index $n_s$ for the Higgs. As in Figure 3 the numbers shown correspond to the e-folds and the circles designate different reheating temperatures. For the line on top ( in red ) the parameters are $a = 10^6$, $\xi = 0.06$ and $\lambda = 4.875 \times 10^{-12}$ while for the one at the bottom ( in blue ) $a = 10^{12}$, $\xi = 0.06$ and $\lambda = 5.60 \times 10^{-12}$. Only the cases for the canonical scenario are shown, $w = 0$.

drawing this figure the constraints arising from $A_s$ have not been taken into account. When they are a small segment including the $T_{\text{ins}}$ temperature is left. In any case, we observe from these figures that by increasing the parameter $a$ the tensor-to-scalar ratio gets smaller and the predictions move lower and the instantaneous reheating temperature mechanism is in full agreement with Planck 2018 cosmological constraints.
VI. CONCLUSIONS

In this work we have considered $\mathcal{R}^2$ theories in the framework of the Palatini formulation. Although this is not new, we have presented a general setup within which inflation models can be studied. The actions, in the Einstein frame, resemble $K$-inflation models, however additional terms, that are quartic in the derivatives of the fields involved, emerge. These have little impact on the inflationary evolution as we have verified numerically, in accord with the findings of other authors. This formulation is model independent and can be applied to any inflationary model. These theories are described by three arbitrary functions. Two of them are associated with the coupling of the scalars to the linear and the quadratic terms, with respect the Palatini curvature $\mathcal{R}$, and the third is a scalar potential. Inflation can be studied in this framework without the need of using canonically normalized fields.

We have applied this for the study of popular inflationary models that are minimally coupled to gravity, with monomial potentials of the form $V \sim h^n$, with the power $n$ a positive and even integer. We also considered the Higgs potential non-minimally coupled to gravity. These models have been put under scrutiny over the years, in the metric formalism, and recently have been extensively studied in the nonmetric, or Palatini, formalism. However the stringent constraints arising from the scalar power spectrum measurements have not been duly taken into account, in most of the studies, in conjunction with the reheating temperature of the Universe. In [38] such a study has been undertaken, in the context of the quartic Higgs model that is minimally coupled to gravity.

In this work, without invoking any particular reheating temperature mechanism, we have undertaken this study, and show that the measurements of the primordial power spectrum amplitude imposes very stringent constraints. These, in combination with the restrictions arising from the measurements of other cosmological observables, in particular the primordial tilt $n_s$ and the tensor to scalar ratio $r$, restrict considerably these models.

For the quadratic model $V = m^2 h^2$ we have seen that the scalar power spectrum amplitude $A_s$ puts constraints on the parameter $m$, and agreement with data is obtained for values of it that lie in a tight range. The maximum reheating, or instantaneous, temperature $T_{\text{ins}}$, is of order $\sim 10^{15} \text{GeV}$, and this is attained for fine-tuned values of $m$, within this range. For these fine-tuned values, the range of the allowed temperatures is rather narrow, and depends on the effective equation of state parameter $w$, with a lowest temperature not far from the instantaneous temperature. For the canonical scenario, although smaller, this is of the same order of magnitude with $T_{\text{ins}}$. If we allow for small deviations, from this fine-tuned values, agreement with data is still feasible. However these deviations, although they do not disturb substantially the observable $n_s$, should lie in a narrow range, outside which agreement with $A_s$ data is hard to achieve. In these cases the allowed temperatures are well below $T_{\text{ins}}$ and rapid thermalization is not possible. Besides, depending on the value of $m$, not any value of $w$ in the range $-1/3 < w < 1$ is allowed. The conclusion, concerning this model, is that, agreement with all cosmological data is possible for values of the potential coupling $m$ that lie in a narrow range. Instantaneous reheating is possible at the cost of a very fine-tuned values of $m$.

The model with the quartic potential $V \sim h^4$ is in conflict with the spectral index $n_s$ data. Only marginal agreement with the primordial tilt can be obtained, with $n_s \simeq 0.960$, but this occurs for very low reheating temperatures close to Nucleosynthesis $T_{\text{reh}} \sim \text{MeV}$, and for values $w$ close to $w = 1.0$. On the other hand, the amplitude $A_s$ prefers smaller values of the equation of state parameter $w \lesssim 0.25$. The conclusion is that, this model is hard to reconcile with $n_s$, the scalar power spectrum measurements and reheating temperatures that are reasonably larger than $T_{\text{reh}} \sim \text{MeV}$ so that we do not run into problems with Big Bang Nucleosynthesis. As our qualitative arguments have shown, for the descendant models, $V \sim h^n$ with $n > 4$, the situation is even worse.

The situation with the quartic potential is rescued in the Higgs model when the scalar field couples in a non-minimal manner to gravity, specified by a parameter $\xi$. This helps in that, as we have explicitly shown, the value of $n_s$ depends on $\xi$ allowing for larger values of $n_s$. Agreement with $n_s$ observations demands that $\xi$ is not smaller than about $\sim 0.06$. Given $\xi$, the primordial spectrum measurements in the Higgs model restricts severely the quartic coupling $\lambda$. The larger the value of $\xi$ is the largest the values of the allowed $\lambda$ are. The quartic coupling is small, smaller than $\sim 10^{-6}$, for values $\xi$ that do not exceed $\sim 10^4$. Higher $\lambda$ values are in principle allowed but these require very large values of $\xi$ leading to instantaneous reheating temperatures, lower than $\sim 10^{15} \text{GeV}$. Note that in the Higgs case there is no
bound on the parameter $a$ specifying the coupling of the scalar field to the gravity term $R^2$, which, unlike the previous models is unrestricted. Thus both large and low values of $a$ are allowed. Due to that, and for given $\xi$ and $\lambda$ in the appropriate range, two regimes can be distinguished. The small-$a$, when $a < \xi^2/\lambda$, and the large $a > \xi^2/\lambda$ regime. In the small $a$-regime, and particularly when $a \ll \xi^2/\lambda$, the predictions are independent of the parameter $a$, provided that $a$ stays much smaller than $\xi^2/\lambda$. The inflationary scale in this case is $\mu \sim \sqrt{(\lambda/\xi^2)}$ and lies in the range $10^{-5} - 10^{-7}$ Planck masses, for $\xi$ between 0.06 and 0.1. The instantaneous reheating temperature $T_{\text{ins}}$, in this case, is larger for smaller values of the parameter $\xi$ and receives its largest possible value, $\simeq 2.5 \times 10^{15}$ GeV, when $\xi$ is in the vicinity of $\xi \approx 0.1$.

In the large-$a$ regime, on the other hand, the inflationary scale is $\mu \sim a^{-1/2}$. At the same time $T_{\text{ins}}$ behaves as $\sim a^{-1/4}$. Unless $a$ is not exceedingly large, $T_{\text{ins}}$ can be as large as $\sim 10^{15}$ GeV, and this requires values of $\xi$ of order unity or so. In both regimes there are values of the parameters for which all cosmological data can be satisfied. However for given $\xi$, as in the models discussed previously, the quartic coupling $\lambda$ should lie in a tight range, as the power spectrum observations dictate. Moreover for instantaneous reheating $\lambda$ should be fine-tuned. In that case the allowed temperatures are close to $T_{\text{ins}}$ for the canonical scenario, $w = 0$, while a broader range of $T_{\text{reh}}$ is allowed, bracketing values $T_{\text{reh}} \sim 10^9$ GeV or so, for values of the equation of state parameter $w$ in the vicinity of $\simeq 0.25$.

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