Stochastic transport equation
in bounded domains

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Abstract

This paper is concerned with the initial-boundary value problems for stochastic transport equation in bounded domains. For a given stochastic perturbation of the drift vector field, and the initial-boundary data in $L^\infty$, we prove existence and uniqueness of weak $L^\infty$ solutions with non-regular coefficients. The existence result, which is by no means a trivial adaptation, relies on a strong stochastic trace theorem established in this paper. Moreover, the uniqueness of weak solutions is obtained under suitable conditions, which allow vacuum.

1 Introduction

In this article we establish global existence and uniqueness of solution for the stochastic transport linear equations in bounded domains. Namely, we consider the following initial-boundary value problem: Given an initial-boundary data $u_0, u_b$, find $u(t, x; \omega) \in \mathbb{R}$, satisfying

$$\begin{cases}
\partial_t u(t, x; \omega) + \left( b(t, x) + A(x) \frac{dB_t}{dt}(\omega) \right) \cdot \nabla u(t, x; \omega) = 0 \quad \text{in } U_T, \\
u = u_0 \quad \text{in } \{t = 0\} \times U, \\
u = u_b \quad \text{on } \Gamma_T,
\end{cases}
$$

(1.1)

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\[(t, x) \in U_T, \omega \in \Omega\), where \(U_T = [0, T] \times U\), for \(T > 0\) be any fixed real number, \(U\) be an open and bounded domain of \(\mathbb{R}^d\) (\(d \in \mathbb{N}\), \(b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d\) is a given vector field (called drift), \(B_t = (B^1_t, ..., B^d_t)\) is a standard Brownian motion in \(\mathbb{R}^d\) and the stochastic integration is taken (unless otherwise mentioned) in the Stratonovich sense. The real \(d \times d\) matrix value function \(A\) is symmetric, and nonsingular, so that \(\det A(x) \neq 0\).

Moreover, we denote by \(\Gamma\) the \(C^2\)—smooth boundary of \(U\), with the outside normal field to \(U\) at \(r \in \Gamma\) denoted by \(n(r)\). In fact, Lipschitz boundary is enough for all the results establish here, we take smooth for simplicity of the presentation. Finally, we denote \(\Gamma_T = (0, T) \times \Gamma\), and define the influx boundary zone

\[\Gamma^-_T := \{(t, r) \in \Gamma_T : (b \cdot n)(t, r) < 0\}. \quad (1.2)\]

The problem (1.1) has been treated for the case \(U = \mathbb{R}^d\) by many authors, see for instance [1], [8], [14], [17] and [19]. Most of the above cited works are supported by the theory of renormalized solutions of the linear transport equation, as it was used by DiPerna, Lions in the celebrated paper [12], which is to say, deterministic case. In fact, it was the first time that, the Lagrangean formalism of the continuity equation comes after the Eulerian point of view. DiPerna-Lions deduced the existence, uniqueness and stability results for ordinary differential equations with rough coefficients from corresponding results on the associated linear transport equation. Following the same strategy in [12], Ambrosio [2] generalized the results to the case where the coefficients have only bounded variation regularity by considering the continuity equation. Moreover, he establish the important concept of Regular Lagrangian Flows, see also Definition 1.1 in Bogachev, Wolf [6]. More recently, new results about stochastic linear transport equations \((U = \mathbb{R}^d)\) have been obtained without the commutators framework, in particular using the so called Ladyzhenskaya-Prodi-Serrin condition for the drift, see Neves, Olivera [23], and Beck, Flandoli, Gubinelli, Maurelli [5].

It seems that, one of the premieres studies of linear transport equations (deterministic case) in bounded domains was done by Bardos [3]. On that extended paper Bardos consider the regular case, where the vector field \(b\) has Lipschitz regularity, and it was introduced the correct notion of the Dirichlet boundary condition. Then, we mention the work of Mischler [21], who consider weak solutions for the Vlasov equation (instead of the transport
equation) posed in bounded domains. On that paper, the trace problem for linear transport type equations is discussed in details. One observes that, if $u$ is not sufficiently regular, in particular we are seeking for measurable bounded functions, the restriction to negligible Lebesgue sets is not, a priori, defined. Therefore, one has to deal with the traces theory to ensure the correct notion of Dirichlet boundary condition. In the same direction as Mischler [21], Boyer [7] establish the trace theorems with respect to the measure $\mu$ see (2.4), and show the existence and uniqueness of solutions for the transport equation using the Sobolev framework of DiPerna, Lions [12]. More recently, Crippa, Donadello, Spinolo [10] studied the initial-boundary value problems for continuity equations with total bounded variation coefficients, hence analogue framework to [2].

Let us now focus on the stochastic case. First, Funaki in [15] studied the random transport equation with very regular coefficients for bounded domains. To the knowledge of the authors, nothing has already been done for stochastic transport equations with low regularity coefficients in bounded domains. In this article, we deal with the problem (1.1) and show the existence and uniqueness of weak $L^\infty$-solutions for Dirichlet data. The initial-boundary value problem is much harder to solve than the Cauchy one, hence we exploit in this paper new improvements due the perturbation of the drift vector field by a Brownian motion. For instance, the solvability in the weak sense for the Cauchy problem is easily established under the mild assumption of local integrability for $b$ and div$b$, see [23]. On the other hand, the existence result establish here on bounded domains, see Section 2.3 considering BV regularity to the drift, relies strongly on the passage from the Stratonovich formulation (2.24) into Itô’s one (2.32), which is a new deeply result. It is also core for the existence’s proof, the strong stochastic trace result obtained in Section 2.2.

We stress that, working with BV regularity for the drift (commutators framework), the most improvement introduced by the noise is the stochastic trace, which is more refined than the determinist one studied by Boyer [7]. More precisely, we have established in Section 2.2 a strong trace result, which means that, the trace of a distributional solution $u \in L^\infty(U_T \times \Omega)$, it is a bounded and measurable function with respect to the Hausdorff measure, i.e. $\gamma u \in L^\infty([0, T] \times \Gamma \times \Omega)$, see Definition 2.6 and Proposition 2.7.

The uniqueness result obtained in this paper does not assumed $L^\infty$ control.
on the divergence of the vector field $b$, as it is used to be in the deterministic case. We have assumed just a boundedness from above, which means that vacuum is allowed to occur. Moreover, we just assume a boundedness of $b$ w.r.t. the spatial variable, see \((3.39)\). Despite we have used some special technics to show uniqueness for the stochastic case, in particular the features of the Hamilton-Jacobi-Belmann equation, the uniqueness result under the same assumptions follows as well in the deterministic case. Although, we recall that, the noise could improve the uniqueness result where the deterministic fails, we address the reader to \([1, 16, 23, 5]\).

Finally, it should be noted that, one of the main interests of the authors to study stochastic transport equations in bounded domains, is concerned with the onset learning of turbulence. The phenomenon of turbulence which occurs in the study of the Navier-Stokes equations, shall be related to the randomness of the velocity field of the fluid. Therefore, the outstanding problem of turbulence in fluid dynamics maybe treat by this approach. We address for instance our cited article \([23]\), where general stochastic balance equations in Continuum Physics are proposed, and the references cited therein related to this discussion.

**Remark 1.1.** Here in this paper, we assume the constant homogeneous isotropic case, that is, for $i, j = 1, \ldots, d$, $A_{ij}(x) = \sigma \delta_{ij}$, for some $\sigma \in \mathbb{R}$ ($\sigma \neq 0$, w.l.g. we take $\sigma = 1$), where $\delta_{ij}$ stands for the Kronecker delta. We leave for future work the more general case.

Moreover, it seems very important to be studied the degenerated case, that is to say, whenever $A(x)$ is singular on (lower dimensional) subsets of $U$.

## 2 Existence of Solutions SPDE

The main issue in this section is to establish the solvability of system \((1.1)\). We shall always assume that

$$b \in L^1((0, T); BV_{\text{loc}}(\mathbb{R}^d)) \quad \text{and} \quad \text{div} b \in L^1_{\text{loc}}((0, T) \times \mathbb{R}^d). \quad (2.3)$$

Recall from the Appendix the well known result about trace for functions of bounded variation. For convenience, let us denote by $dr$ the $(d-1)$-Hausdorff measure on $\Gamma$, and also consider the measure

$$d\mu := (b \cdot n) \, dr \, dt \quad (2.4)$$
on $\Gamma_T$. We have $\mu = \mu^+ - \mu^-$, where $d\mu^\pm := (b \cdot n)^\pm \, drdt$, from Jordan’s decomposition theorem.

As mentioned at the introduction, the initial-boundary value problem (hereupon denoted IBVP) of the stochastic linear transport equation is much more effortful than the Cauchy problem. Indeed, first considering that the drift vector field has enough regularity, we establish the classical characteristic method for stochastic differential equations. Then, by construction of a time reverse stopped process, and assuming that the initial-boundary data have also enough regularity (satisfying compatibility conditions), we obtain the representative function $u(t, x)$, which it will be a weak solution for regular coefficients. In fact, this representative function is not necessarily a classical solution to the IBVP (1.1), since the stopped backward process is usually discontinuous in the spatial variable.

Passing to rough coefficients, we consider a distributional solution to (1.1), and reeling on that, we establish a stochastic trace result, which should be more refined than deterministic one. The requirement is due the stochastic boundary terms, which are integrated with respect to $drdt$ instead of the measure $d\mu$, see for instance (2.24). Then, we give the correct notion of solution and show the solvability in this sense. Indeed, a stochastic process which satisfies the IBVP (1.1) accord Definition 2.8.

To begin, let us consider the random differential equation on $\mathbb{R}^d$, that is to say, given $s \in [0, T]$ and $x \in \mathbb{R}^d$, we consider

$$X_{s,t}(x) = x + \int_s^t b(r, X_{s,r}(x)) \, dr + B_t - B_s,$$

where $X_{s,t}(x) = X(s, t, x)$, also $X_t(x) = X(0, t, x)$. In particular, for $m \in \mathbb{N}$ and $0 < \alpha < 1$, let us consider

$$b \in L^1((0, T); C^{m,\alpha}(\mathbb{R}^d)).$$

(2.6)

It is well know that, under condition (2.6), the stochastic flow $X_{s,t}$ is a $C^m$-diffeomorphism (see for example [16] and [17]). Moreover, the inverse $Y_{s,t} := X_{t,s}^{-1}$ satisfies the following backward stochastic differential equations,

$$Y_{s,t} = y - \int_s^t b(r, Y_{r,t}) \, dr - (B_t - B_s),$$

(2.7)
for $0 \leq s \leq t$, see for instance [14] and [16] pp. 234. Usually, $Y$ is called the time reversed process of $X$.

One observes that, the Cauchy problem for the associated differential equation (2.5), that is,

$$u(t, x) = u_0(x) - \int_0^t b(s, x) \cdot \nabla_x u(s, x) \, ds - \int_0^t \nabla_x u(s, x) \cdot \circ dB_s,$$

(2.8)

is much more simple than the IBVP (1.1), where the boundary condition on $\Gamma_T$ must be considered. Given $(t, x) \in U_T$ and the time reversed process $Y_{s,t}$ given by (2.7), we consider

$$S = \{ s \in (0,t)/ Y(s, t, x) \notin U_T \}$$

and define

$$\tau(t, x) := \sup S.$$  \hspace{1cm} (2.9)

Clearly $S$ could be an empty set, and in this case we set $\tau = 0$. Now, we define $\bar{Y}_{s,t}$ on $[0,t] \times \bar{U}$ as

$$\bar{Y}_{s,t}(x) := Y_{s,t}(x) \quad \text{for} \ s \in [\max\{0, \tau\}, t],$$  \hspace{1cm} (2.10)

which is called a stopped backward process.

From the above considerations, we may apply a straightforward computation (see [15]) to prove the following

**Lemma 2.1.** For $m \geq 3$, $0 < \alpha < 1$, let $u_0 \in C^{m,\alpha}(U)$, $u_b \in C^{m,\alpha}(\Gamma_T^-)$ be respectively initial-boundary bounded data satisfying compatibility conditions, and assume (2.6). Then, the IBVP problem (1.1) has a weak-regular coefficients $L^\infty$-solution $u(t, \cdot)$ for $0 \leq t \leq T$, given by

$$u(t, x) := \begin{cases} 
  u_0(\bar{Y}_{\tau,t}(x)), & \text{if } \tau(t, x) = 0, \\
  u_b(\tau, \bar{Y}_{\tau,t}(x)), & \text{if } \tau(t, x) > 0,
\end{cases}$$  \hspace{1cm} (2.11)

where $\bar{Y}$ is the stopped backward process defined in (2.10), which satisfies:

- **Stochastic transport equation:** For each test function $\varphi \in C^\infty_c(U)$, the real value process $\int_U u(t, x) \varphi(x) \, dx$ has a continuous modification which is a
\[ \mathcal{F}_t\text{-semimartingale, and for all } t \in [0,T], \text{ we have } \mathbb{P}\text{-almost sure} \]
\[ \int_U u(t,x)\varphi(x)dx = \int_U u_0(x)\varphi(x) dx + \int_0^t \int_U u(s,x) b^j(s,x) \partial_j \varphi(x) dxds \]
\[ + \int_0^t \int_U u(s,x) \text{div } b(s,x) \varphi(x) dxds \]
\[ + \int_0^t \int_U u(s,x) \partial_i \varphi(x) dx dB^i_s. \]
\[ (2.12) \]

- **Initial condition:** We have \( \mathbb{P}\text{-almost sure} \)
\[ \lim_{t \to 0^+} \int_U |u(t,x) - u_0(x)| dx = 0. \]
\[ (2.13) \]

- **Boundary condition:** We have \( \mathbb{P}\text{-almost sure} \)
\[ \lim_{\tau \to 0^+} \int_{\Gamma} |u(\Psi_\tau(r)) - u_b(r)| dr = 0, \]
\[ (2.14) \]

where \( \Psi_\tau(\cdot) \) is the deformation boundary function, see the Appendix.

**Remark 2.2.**
1. The representative function \( u \) given by \((2.11)\) is uniformly bounded up to the boundary.
2. The conditions \((2.12)-(2.14)\) can be equivalently reformulated (see for instance \([22]\)) as: For each test function \( \varphi \in C^\infty_c(U) \), the real value process \( \int_U u(t,x)\varphi(x) dx \) has a continuous modification which is a \( \mathcal{F}_t \)-semimartingale, and satisfies for all \( t \in [0,T] \)
\[ \int_U u(t,x)\varphi(x)dx = \int_U u_0(x)\varphi(x) dx + \int_0^t \int_U u(s,x) b^j(s,x) \partial_j \varphi(x) dxds \]
\[ + \int_0^t \int_U u(s,x) \text{div } b(s,x) \varphi(x) dxds - \int_0^t \int_{\Gamma} u(s,r) \varphi(r) (b^j n_j)_+ dr ds \]
\[ + \int_0^t \int_{\Gamma} u_b(s,r) \varphi(r) (b^j n_j)_- dr ds \]
\[ + \int_0^t \int_U u(s,x) \partial_i \varphi(x) dx dB^i_s. \]
\[ (2.15) \]
2.1 Distributional solution

We begin considering a distributional solution to problem (1.1), more precisely we have the following

Definition 2.3. Let \( u_0 \in L^\infty(U) \), \( u_b \in L^\infty(\Gamma_T; \mu^-) \) be given. A stochastic process \( u \in L^\infty(U_T \times \Omega) \) is called a distributional \( L^\infty \)-solution of the IBVP (1.1), when for each test function \( \varphi \in C^\infty(U) \), the real value process \( \int_U u(t,x) \varphi(x) dx \) has a continuous modification which is a \( \mathcal{F}_t \)-semimartingale, and for all \( t \in [0,T] \), we have \( \mathbb{P} \)-almost sure

\[
\int_U u(t,x) \varphi(x) dx = \int_U u_0(x) \varphi(x) dx + \int_0^t \int_U u(s,x) b^i(s,x) \partial_x^i \varphi(x) dx ds \\
+ \int_0^t \int_U u(s,x) \text{div} b(s,x) \varphi(x) dx ds \\
+ \int_0^t \int_U u(s,x) \partial_x^i \varphi(x) dx dB^i_s.
\]

(2.16)

Hereafter the usual summation convention is used.

Remark 2.4. Following Flandoli, Gubinelli, Priola [14], see Lemma 13, we can reformulate equation (2.16) in Itô’s form as follows: A stochastic process \( u \in L^\infty(U_T \times \Omega) \) is a distributional \( L^\infty \)-solution of the SPDE (1.1) if, and only if, for every test function \( \varphi \in C^\infty_c(U) \), the process \( \int u(t,x) \varphi(x) dx \) has a continuous modification, which is a \( \mathcal{F}_t \)-semimartingale, and satisfies the following Itô’s formulation for all \( t \in [0,T] \)

\[
\int_U u(t,x) \varphi(x) dx = \int_U u_0(x) \varphi(x) dx + \int_0^t \int_U u(s,x) b^i(s,x) \partial_x^i \varphi(x) dx ds \\
+ \int_0^t \int_U u(s,x) \text{div} b(s,x) \varphi(x) dx ds \\
+ \int_0^t \int_U u(s,x) \partial_x^i \varphi(x) dx dB^i_s + \frac{1}{2} \int_0^t \int_U u(s,x) \Delta \varphi(x) dx ds.
\]

(2.17)

Lemma 2.5. Under condition (2.3), there exists a distributional \( L^\infty \)-solution \( u \) of the stochastic IBVP (1.1).
Proof. 1.(Regulatization) First, let us denote by $u_0^\epsilon$, $u_b^\epsilon$ respectively the standard mollifications of $u_0$ and $u_b$, satisfying compatibility conditions. Similarly, $b^\epsilon$ the mollification of $b$. Let $X_t^\epsilon$ be the associated flow given by (2.5), and define conform Section 2 (see Lemma 2.1)

$$u^\epsilon(t, x) := \begin{cases} 
    u_0^\epsilon(\bar{Y}_{t,t}^\epsilon(x)), & \text{if } \tau(t, x) = 0, \\
    u_b^\epsilon(\tau, \bar{Y}_{t,t}^\epsilon(x)), & \text{if } \tau(t, x) > 0.
\end{cases}$$

(2.18)

Thus $u^\epsilon(t, x)$ is uniformly bounded, with respect to $\epsilon > 0$, and satisfies for each test function $\varphi \in C_c^\infty(U)$

$$\int_U u^\epsilon(t, x) \varphi(x) dx = \int_U u_0^\epsilon(x) \varphi(x) dx + \int_0^t \int_U u^\epsilon(s, x) b^\epsilon_i(s, x) \partial_{x_i} \varphi(x) dxds$$

$$+ \int_0^t \int_U u^\epsilon(s, x) \text{div } b^\epsilon(s, x) \varphi(x) dxds$$

$$+ \int_0^t \int_U u^\epsilon(s, x) \partial_{x_i} \varphi(x) dx dB^i_s,$$

or equivalently the equation

$$\int_U u^\epsilon(t, x) \varphi(x) dx = \int_U u_0^\epsilon(x) \varphi(x) dx + \int_0^t \int_U u^\epsilon(s, x) b^\epsilon_i(s, x) \partial_{x_i} \varphi(x) dxds$$

$$+ \int_0^t \int_U u^\epsilon(s, x) \text{div } b^\epsilon(s, x) \varphi(x) dxds$$

$$+ \int_0^t \int_U u^\epsilon(s, x) \partial_{x_i} \varphi(x) dx dB^i_s + \frac{1}{2} \int_0^t \int_U u^\epsilon(s, x) \Delta \varphi(x) dxdx.$$

(2.19)

2.(Limit transition) Since $u^\epsilon$ is uniformly bounded, there exists a subsequence of $u^\epsilon$ converging weakly star in $L^\infty(U_T \times \Omega)$ and weakly in $L^2(U_T \times \Omega)$ to some $u$, which belongs to these spaces.

Now, we observe that the process $\int_U u(t, x) \varphi(x) dx$ is adapted, since it is the weak limit in $L^2((0, T) \times \Omega)$ of adapted processes. Moreover, the Itô’s integral is weakly continuous in $L^2((0, T) \times \Omega)$, see for instance Chap. III in [24] for details. Thus passing to the limit as $\epsilon$ goes to $0^+$ in (2.19), it follows that, $u(t, x)$ is a distributional $L^\infty$—solution of the SPDE (1.1). \qed
2.2 Stochastic Trace

Now, we prove the existence and uniqueness of the stochastic trace by the existence of distributional $L^\infty$-solution of the IBVP (1.1).

Definition 2.6. Let $u$ be a distributional $L^\infty$-solution of the IBVP problem (1.1). A stochastic process $\gamma u \in L^\infty([0,T] \times \Gamma \times \Omega)$ is called the stochastic trace of the distributional solution $u$, if for each test function $\varphi \in C^\infty_c(\mathbb{R}^d)$,

$$\int_{\Gamma} \varphi(r) \, dr$$

is an adapted real value process, which satisfies for any $\beta \in C^2(\mathbb{R})$ and all $t \in [0,T]$,

$$\int_{U} \beta(u(t,x)) \varphi(x) \, dx = \int_{U} \beta(u_0(x)) \varphi(x) \, dx$$

$$+ \int_{0}^{t} \int_{U} \beta(u(s,x)) b(s,x) \cdot \nabla \varphi(x) \, dx \, ds$$

$$+ \int_{0}^{t} \int_{U} \beta(u(s,x)) \text{div} b(s,x) \varphi(x) \, dx \, ds$$

$$- \int_{0}^{t} \int_{\Gamma} \beta(\gamma u) \varphi(r) b(s,r) \cdot n(r) \, dr \, ds$$

$$+ \int_{0}^{t} \int_{U} \beta(u(s,x)) \partial_{x_i} \varphi(x) \, dx \circ dB^i_s$$

$$- \int_{0}^{t} \int_{\Gamma} \beta(\gamma u) \varphi(r) n_i(r) \, dr \circ dB^i_s.$$  \hfill (2.20)

Proposition 2.7. Assume condition (2.3), and let $u$ be a distributional $L^\infty$-solution of the IBVP problem (1.1). Then, there exits the stochastic trace $\gamma u$.

Proof. 1. Let $u$ be a distributional solution of the transport equation (1.1), and set $u_\varepsilon(t,\cdot)$ the global approximation of $u$ related to the standard mollifier $\rho_\varepsilon$, see Appendix. Then, we take conveniently $\rho_\varepsilon$ as a test function in (2.16),
that is
\[ u_\varepsilon(t, y) = (u_0 * n \rho_\varepsilon)(y) \]
\[ + \int_0^t \int_U u(s, z) b(s, z) \cdot \nabla \rho_\varepsilon(y^\varepsilon - z) \, dz \, ds \]
\[ + \int_0^t \int_U u(s, z) \text{div} b(s, z) \rho_\varepsilon(y^\varepsilon - z) \, dz \, ds \]
\[ + \int_0^t \int_U u(s, z) \partial_i \rho_\varepsilon(y^\varepsilon - z) \, dz \circ dB^i_s. \]

Let \( \beta \in C^2(\mathbb{R}) \), and applying Itô-Ventzel-Kunita Formula (see Appendix), we obtain from the above equation
\[ \beta(u_\varepsilon(t, x)) = \beta(u_0 * n \rho_\varepsilon)(x) \]
\[ + \int_0^t \beta'(u_\varepsilon(s, x)) \int_U u(s, z) b(s, z) \cdot \nabla \rho_\varepsilon(x^\varepsilon - z) \, dz \, ds \]
\[ + \int_0^t \beta'(u_\varepsilon(s, x)) \int_U u(s, z) \text{div} b(s, z) \rho_\varepsilon(x^\varepsilon - z) \, dz \, ds \]
\[ + \int_0^t \beta'(u_\varepsilon(s, x)) \int_U u(s, z) \partial_i \rho_\varepsilon(x^\varepsilon - z) \, dz \circ dB^i_s. \]

Following the renormalization procedure, nowadays well known, we obtain from an algebraic manipulation
\[ \beta(u_\varepsilon(t, x)) - \beta(u_0 * n \rho_\varepsilon)(x) \]
\[ + \int_0^t b(s, x) \cdot \nabla \beta(u_\varepsilon(s, x)) \, ds + \int_0^t \partial_i \beta(u_\varepsilon(s, x)) \circ dB^i_s \]
\[ = \int_0^t \beta'(u_\varepsilon(s, x)) \mathcal{R}_\varepsilon(b, u) \, ds + \int_0^t \partial_i \beta(u_\varepsilon(s, x)) \mathcal{P}_\varepsilon(u) \circ dB^i_s, \]

where \( \mathcal{R}_\varepsilon(b, u), \mathcal{P}_\varepsilon(u) \) are commutators type, defined respectively by
\[ \mathcal{R}_\varepsilon(b, u) := (b \nabla)(\rho_\varepsilon * n u) - \rho_\varepsilon * n ((b \nabla)u), \]
\[ \mathcal{P}_\varepsilon(u) := \nabla(\rho_\varepsilon * n u) - \rho_\varepsilon * n (\nabla u). \]

2. Now, we show that \( \{\beta(u^\varepsilon)\} \) is a Cauchy sequence in \( L^2([0, T] \times \Gamma \times \Omega) \). For any \( \varepsilon_1, \varepsilon_2 > 0 \), setting \( w_{\varepsilon_1, 2} = \beta(u_{\varepsilon_1}) - \beta(u_{\varepsilon_2}) \), we get from equation
\[ w_{\varepsilon,1,2}(t, x) - w_{\varepsilon,1,2}(0, x) \]
\[ = \int_0^t b(s, x) \cdot \nabla w_{\varepsilon,1,2}(s, x) \, ds + \int_0^t \partial_i w_{\varepsilon,1,2}(s, x) \circ dB^i_s \]

where
\[ R_{\varepsilon,1,2}(b, u) = \beta'(u_{\varepsilon_1}) R_{\varepsilon_1}(b, u) - \beta'(u_{\varepsilon_2}) R_{\varepsilon_2}(b, u), \]
and
\[ P_{\varepsilon,1,2}(u) = \beta'(u_{\varepsilon_1}) P_{\varepsilon_1}(u) - \beta'(u_{\varepsilon_2}) P_{\varepsilon_2}(bu). \]

Similarly to item 1, we apply in the above equation the Itô-Ventzel-Kunita Formula, now for \( \theta(z) = z^2 \). Then, we obtain

\[
|w_{\varepsilon,1,2}(t, x)|^2 - |w_{\varepsilon,1,2}(0, x)|^2 + \int_0^t b(s, x) \cdot \nabla w_{\varepsilon,1,2}^2(s, x) \, ds + \int_0^t \partial_i w_{\varepsilon,1,2}^2(s, x) \circ dB^i_s + 2 \int_0^t w_{\varepsilon,1,2} R_{\varepsilon,1,2}(b, u) \, ds + 2 \int_0^t w_{\varepsilon,1,2} P_{\varepsilon,1,2}(u) \, dB^i_s.
\]

Then, we multiply the above equation by a test function \( \varphi \in C_c^\infty(\mathbb{R}^d) \), and integrating in \( U \), we obtain

\[
\int_U |w_{\varepsilon,1,2}(t, x)|^2 \varphi(x) \, dx - \int_U |w_{\varepsilon,1,2}(0, x)|^2 \varphi(x) \, dx
\]
\[ - \int_0^t \int_U w_{\varepsilon,1,2}^2(s, x) b(s, x) \cdot \nabla \varphi(x) \, dx \, ds - \int_0^t \int_U w_{\varepsilon,1,2}^2(s, x) \text{div}(b(s, x)) \varphi(x) \, dx \, ds
\]
\[ - \int_0^t \int_U w_{\varepsilon,1,2}^2(s, x) \partial_i \varphi(x) \, dx \circ dB^i_s
\]
\[ + \int_0^t \int_\Gamma w_{\varepsilon,1,2}^2(s, x) b(s, r) \cdot \mathbf{n}(r) \varphi(r) \, dr \, ds + \int_0^t \int_\Gamma w_{\varepsilon,1,2}(s, x) \mathbf{n}(r) \varphi(r) \, dr \circ dB^i_s
\]
\[ = 2 \int_0^t \int_U w_{\varepsilon,1,2} R_{\varepsilon,1,2}(b, u) \varphi(x) \, dx \, ds + 2 \int_0^t \int_U w_{\varepsilon,1,2} P_{\varepsilon,1,2}(u) \varphi(x) \, dx dB^i_s
\]
and taking covariation with respect to \( B_i \), we have for each \( i = 1, \ldots, d \),

\[
\begin{align*}
&\left[ \int_U |w_{\varepsilon,1,2}(t, x)|^2 \varphi(x) \, dx, B_i \right] - \int_0^t \int_U w_{\varepsilon,1,2}^2(s, x) \partial_i \varphi(x) \, dx ds \\
&+ \int_0^t \int_{\Gamma} w_{\varepsilon,1,2}(s, x) n_i(r) \varphi(r) \, dr ds = 2 \int_0^t \int_U w_{\varepsilon,1,2} \mathcal{P}_{\varepsilon,1,2}(u) \varphi(x) \, dx ds.
\end{align*}
\]

Moreover, taking the expectation

\[
\begin{align*}
&\int_0^t \int_{\Gamma} \mathbb{E}|w_{\varepsilon,1,2}(s, x)|^2 n_i(r) \varphi(r) \, dr ds = -\mathbb{E}\left[ \int_U |w_{\varepsilon,1,2}(t, x)|^2 \varphi(x) \, dx, B_i \right] \\
&+ \int_0^t \int_U \mathbb{E}|w_{\varepsilon,1,2}(s, x)|^2 \partial_i \varphi(x) \, dx ds + 2 \int_0^t \int_U \mathbb{E}[w_{\varepsilon,1,2} \mathcal{P}_{\varepsilon,1,2}(u)] \varphi(x) \, dx ds,
\end{align*}
\]

and also \( \varphi(x) = \partial_i h(x) \) (see Appendix), we obtain

\[
\begin{align*}
&\int_0^t \int_{\Gamma} \mathbb{E}|w_{\varepsilon,1,2}(s, x)|^2 \, dr ds = \mathbb{E}\left[ \int_U |w_{\varepsilon,1,2}(t, x)|^2 \partial_i h(x) \, dx, B_i \right] \\
&- \int_0^t \int_U \mathbb{E}|w_{\varepsilon,1,2}(s, x)|^2 \Delta h(x) \, dx ds \\
&- 2 \sum_{i=1}^d \int_0^t \int_U \mathbb{E}[w_{\varepsilon,1,2} \mathcal{P}_{\varepsilon,1,2}(u)] \partial_i h(x) \, dx ds.
\end{align*}
\]

Therefore, \( w_{\varepsilon,1,2} \) is a Cauchy sequence in \( L^2([0, T] \times \Gamma \times \Omega) \), and thus \( \beta(u^\varepsilon) \) converges to, say, \( \tilde{\gamma} \) in \( L^2([0, T] \times \Gamma \times \Omega) \) as \( \varepsilon \to 0 \). In particular, taking \( \beta(u) = u \), there exists a subsequence of \( u^\varepsilon \), which converges almost sure on \([0, T] \times \Gamma \times \Omega\), which limit we denote by \( \gamma u \). Therefore, by the uniqueness of the limit, \( \tilde{\gamma} = \beta(\gamma u) \).

3. Now, we show that \( \gamma u \in L^\infty([0, T] \times \Gamma \times \Omega) \), and also \( \text{(2.20)} \). We denote \( M = \|u\|_\infty \) and consider a nonnegative \( \beta \) such that \( \beta(u) = 0 \) in \([-M, M]\). Multiplying \( \text{(2.21)} \) by a test function \( \varphi \in C_\infty(\mathbb{R}^d) \), and after integration in
We obtain
\[
\int_U \beta(u_\varepsilon(t,x)) \varphi(x) \, dx - \int_U \beta(u_\varepsilon(0,x)) \varphi(x) \, dx
\]
\[- \int_0^t \int_U \beta(u_\varepsilon(s,x)) b(s,x) \cdot \nabla \varphi(x) \, dx \, ds - \int_0^t \int_U \beta(u_\varepsilon(s,x)) \text{div} b(s,x) \varphi(x) \, dx \, ds
\]
\[- \int_0^t \int_U \beta(u_\varepsilon(s,x)) \partial_i \varphi(x) \, dx \, dB_i^s
\]
\[+ \int_0^t \int_U \beta(u_\varepsilon(s,x)) b(s,r) \cdot \mathbf{n}(r) \varphi(r) \, dr \, ds + \int_0^t \int \beta(u_\varepsilon(t,x)) \mathbf{n}_i(r) \varphi(r) \, dr \, dB_i^s
\]
\[= \int_0^t \int_U \beta'(u_\varepsilon(t,x)) \mathcal{R}_\varepsilon(b) \varphi(x) \, dx \, ds + \int_0^t \int U \beta'(u_\varepsilon(t,x)) \mathcal{P}_\varepsilon(u) \varphi(x) \, dx \, dB_i^s.
\]

Then, we pass to the limit as \( \varepsilon \to 0 \), and similarly to (2.22), we take the
covariation with respect to \( B^i \), to obtain
\[
\int_0^t \int \beta(u_\varepsilon(s,r)) \mathbf{n}_i(r) \varphi(r) \, dr \, ds = 0
\]
for each \( i = 1, \ldots, d \), where we have used that \( \beta(u) = 0 \) in \([-M,M]\). Therefore, taking \( \varphi(x) = \partial_i h(x) \) and since \( \beta > 0 \) in \( \mathbb{R} \setminus [-M,M] \), it follows that
\[
\gamma u(t,r;\omega) \in [-M,M] \text{ almost sure in } [0,T] \times \Gamma \times \Omega.
\]
Similar procedure to (2.23) may be establish now for any \( \beta \in C^2 \), and then we are allowed to pass to the limit as \( \varepsilon \to 0 \) to obtain (2.20).

4. Finally, we show the uniqueness of the trace. If \( \gamma_1 u \) and \( \gamma_2 u \) are two measurable and bounded functions satisfying (2.20), then we have for each test function \( \varphi \in C^\infty(\mathbb{R}^d) \) and \( \beta \) the identity function
\[
\int_0^t \int_\Gamma \gamma_1 u \varphi(r) b(s,r) \cdot \mathbf{n}(r) \, dr \, ds + \int_0^t \int_\Gamma \gamma_1 u \varphi(r) \mathbf{n}_i(r) \, dr \, dB_i^s
\]
\[= \int_0^t \int_\Gamma \gamma_2 u \varphi(r) b(s,r) \cdot \mathbf{n}(r) \, dr \, ds + \int_0^t \int_\Gamma \gamma_2 u \varphi(r) \mathbf{n}_i(r) \, dr \, dB_i^s.
\]
Taking the covariation with respect to \( B^i \), we obtain for each \( i = 1, \ldots, d \)
\[
\int_0^t \int_\Gamma \gamma_1 u \varphi(r) \mathbf{n}_i(r) \, dr \, ds = \int_0^t \int_\Gamma \gamma_2 u \varphi(r) \mathbf{n}_i(r) \, dr \, ds,
\]
from which follows the uniqueness of the trace, and hence the thesis of the proposition.

2.3 Stochastic transport equation IBVP

In this section we give the solvability of the stochastic initial-boundary value problem (1.1), for measurable and bounded data. We follow our strategy used to show existence of distributional \( L^\infty \)-solutions, that is to say applying the stochastic characteristics. Now the proof become showier, since we have to deal with the boundary terms, hence following some ideas used in [22] the Co-area and Area Formulas are used to handle them.

**Definition 2.8.** Let \( u_0 \in L^\infty(U) \), \( u_b \in L^\infty(\Gamma_T; \mu^-) \) be given. A stochastic process \( u \in L^\infty(U_T \times \Omega) \) is called a weak \( L^\infty \)-solution of the IBVP (1.1), when for each test function \( \varphi \in C_\infty_c(\mathbb{R}^d) \), the process \( \int_U u(t, x) \varphi(x) \, dx \) has a continuous modification which is a \( \mathcal{F}_t \)-semimartingale, and satisfies for all \( t \in [0, T] \)

\[
\int_U u(t, x) \varphi(x) \, dx = \int_U u_0(x) \varphi(x) \, dx + \int_0^t \int_U u(s, x) b^j(s, x) \partial_j \varphi(x) \, dx \, ds \\
+ \int_0^t \int_U u(s, x) \text{div} b(s, x) \varphi(x) \, dx \, ds - \int_0^t \int_\Gamma \gamma u(s, r) \varphi(r) (b^j n_j)^+ \, dr \, ds \\
+ \int_0^t \int_\Gamma u_b(s, r) \varphi(r) (b^j n_j)^- \, dr \, ds - \int_0^t \int_\Gamma \gamma u(s, r) \varphi(r) n_j(r) \, dr \, dB^j_s \\
+ \int_0^t \int_U u(s, x) \partial_j \varphi(x) \, dx \, dB^j_s.
\]

Then, we pass to prove a main general existence result.

**Theorem 2.9.** Under condition (2.3), there exists a weak \( L^\infty \)-solution \( u \) of the IBVP (1.1).

**Proof.** 1. First, we may follow item 1 in the proof of Lemma 2.5 and obtain for any test function \( \varphi \in C_\infty_c(\mathbb{R}^d) \) the equation (2.24), with respectively regularized terms. We omit this standard procedure. Moreover, for simplicity
of exposition, we do not write the regularizing parameter ($\varepsilon > 0$) throughout this proof (whenever necessary). Then, we have for all $t \in [0, T]$

$$
\int_U u(t, x) \varphi(x) dx = \int_U u_0(x) \varphi(x) dx + \int_0^t \int_U u(s, x) b^j(s, x) \partial_j \varphi(x) dx ds
$$

$$
+ \int_0^t \int_U u(s, x) \text{div} b(s, x) \varphi(x) dx ds - \int_0^t \int_\Gamma u(s, r) \varphi(r) (b^j_n)_+ dr ds
$$

$$
+ \int_0^t \int_\Gamma u_b(s, r) \varphi(r) (b^j_n)_- dr ds - \int_0^t \int_\Gamma u(s, r) \varphi(r) n_j(r) dr \circ dB^j_s
$$

$$
+ \int_0^t \int_U u(s, x) \partial_j \varphi(x) dx \circ dB^j_s.
$$

(2.25)

2. Now, we conveniently reformulate equation (2.25) in Itô’s form. One remarks that, differently from Remark 2.4 we have to deal with boundary terms, which have never been done before in the literature, and it is one of the major difficulties here. To begin, let us consider the relation between Itô and Stratonovich integrals in (2.25), that is

$$
\int_0^t \int_U u(s, x) \partial_j \varphi(x) dx \circ dB_s^j = \int_0^t \int_U u(s, x) \partial_j \varphi(x) dx \circ dB_s^j + \frac{I_1}{2},
$$

(2.26)

$$
\int_0^t \int_\Gamma u(s, r) \varphi(r) n_j(r) dr \circ dB_s^j = \int_0^t \int_\Gamma u(s, r) \varphi(r) n_j(r) dr \circ dB_s^j + \frac{I_2}{2},
$$

(2.27)

where

$$
I_1 := \left[ \int_U u(., x) \partial_j \varphi(x) dx, B^j_s \right]_t, \quad I_2 := \left[ \int_\Gamma u(., r) \varphi(r) n_j(r) dr, B^j_s \right]_t,
$$

and [.,.] denotes the joint quadratic variation, which is a bounded variation term (see Appendix for more details). In fact, we compute these two joint quadratic variations above, from equation (2.25) with specific test functions, and observe that, only the martingale part have to be considered.

To compute $I_1$, we replace $\varphi$ in (2.25) by $\partial_j \varphi$. Then, for each $j = 1, \ldots, d$, the martingale part of $\int_U u(t, x) \partial_j \varphi(x) dx$ is

$$
\int_0^t \int_U u(s, x) \partial_i (\partial_j \varphi(x)) dx d B^i_s - \int_0^t \int_\Gamma u(s, r) \partial_j \varphi(r) n_i(r) dr d B^i_s.
$$
Thus, we have
\[ I_1 = \int_0^t \int_U u(s, x) \partial_2^2 \varphi(x) \, dx \, ds - \int_0^t \int_\Gamma u(s, r) \partial_j \varphi(r) \, n_j(r) \, dr \, ds. \tag{2.28} \]
Now, we compute \( I_2 \). Then, we take in equation (2.25) \( \varphi(x) \partial_j \zeta_\mu(h(x)) \) as a test function, where for \( \mu > 0 \), \( \zeta_\mu : \mathbb{R} \to [-1, 1] \) is given by
\[ \zeta_\mu(\tau) := \begin{cases} 
\text{sgn } \tau, & \text{if } |\tau| > \mu, \\
\frac{\tau}{\mu}, & \text{if } |\tau| \leq \mu,
\end{cases} \]
with \( h(x) \) the given function at the Appendix. Certainly, we have to mollify \( \zeta_\mu \) by a standard mollifier \( \rho_n \) to have the necessary regularity, and then (first) pass to the limit as \( n \to \infty \). Again, we omit this procedure (see for instance Section 2 in [22]). We consider first the left hand side of (2.25), then we pass to the martingale terms in the right hand side of it.

Claim 1: For each \( t \in [0, T] \), and \( j = 1, \ldots, d \), it follows that
\[ \text{ess lim}_{\mu \to 0} \int_U u(t, x) \varphi(x) \partial_j \zeta_\mu(h(x)) \, dx = - \int_\Gamma u(t, r) \varphi(r) \, n_j(r) \, dr. \tag{2.29} \]
Proof of Claim 1: Applying the Coarea Formula for the function \( h \), we have
\[ \int_U u(t, x) \varphi(x) \partial_j \zeta_\mu(h(x)) \, dx = \frac{1}{\mu} \int_0^\mu \int_\Gamma u(t, r) \varphi(r) \frac{\partial_j h(r)}{|\nabla h|} \, dr \, d\tau. \]
Now, we intend to pass to the limit as \( \mu \to 0^+ \). Albeit before that, we apply the Area Formula for the function \( \Psi_\tau \) (see Appendix), in the right hand side of the above equation, observing that
\[ \text{ess lim}_{\tau \to 0} J\Psi_\tau = 1, \]
where \( J\Psi_\tau \) as usual denotes the Jacobian of the map \( \Psi_\tau \). Hence we may replace 1 by \( J\Psi_\tau \) with an error which goes to zero as \( \mu \to 0^+ \). Therefore, we obtain the desired result, that is
\[ \text{ess lim}_{\mu \to 0} \int_U u(t, x) \varphi(x) \partial_j \zeta_\mu(h(x)) \, dx \]
\[ = \text{ess lim}_{\mu \to 0} \left( \frac{1}{\mu} \int_0^\mu \int_\Gamma u(t, \Psi_\tau(r)) \varphi(\Psi_\tau(r)) \frac{\partial_j h}{|\nabla h|} \, dr \, d\tau + O(\mu) \right) \]
\[ = - \int_\Gamma u(t, r) \varphi(r) \, n_j(r) \, dr, \]
where we used the Dominated Convergence Theorem. Now, let us consider for $j = 1, \ldots, d$, the following terms

\[
\int_0^t \int_U u(s, x) \partial_i \left( \varphi(x) \partial_j \zeta_\mu(h(x)) \right) \, dx \, dB_s^i
\]

\[-\int_0^t \int_\Gamma u(s, r) \varphi(r) \partial_j \zeta_\mu(h(r)) \, n_i(r) \, dr \, dB_s^i
\]
or after some computations

\[
\int_0^t \int_U u(s, x) \partial_i \varphi(x) \zeta_\mu'(h(x)) \partial_j h(x) \, dx \, dB_s^i
\]

\[+ \int_0^t \int_U u(s, x) \varphi(x) \zeta_\mu'(h(x)) \partial_i \partial_j h(x) \, dx \, dB_s^i
\]

\[+ \int_0^t \int_U u(s, x) \varphi(x) \zeta_\mu''(h(x)) \partial_i h(x) \partial_j h(x) \, dx \, dB_s^i
\]

\[-\int_0^t \int_\Gamma u(s, r) \varphi(r) \zeta_\mu'(0) \partial_j (h(r)) \, n_i(r) \, dr \, dB_s^i.
\]

Therefore, taking the variation in the above terms, we obtain

\[
\int_0^t \int_U u(s, x) \partial_i \varphi(x) \zeta_\mu'(h(x)) \partial_i h(x) \, dx \, ds
\]

\[+ \int_0^t \int_U u(s, x) \varphi(x) \zeta_\mu'(h(x)) \partial_i^2 h(x) \, dx \, ds
\]

\[+ \int_0^t \int_U u(s, x) \varphi(x) \zeta_\mu''(h(x)) |\partial_i h(x)|^2 \, dx \, ds \tag{2.30}
\]

\[-\int_0^t \int_\Gamma u(s, r) \varphi(r) \zeta_\mu'(0) \partial_i h(r) \, n_i(r) \, ds \]

\[=: J_1 + J_2 + J_3,
\]

where $J_1$ is the first integral, $J_2$ is the second, and $J_3$ is the difference of the remainder two terms.
Claim 2: For each \( t \in [0, T] \), it follows that:

\[
\begin{align*}
    i) \text{ess lim }_{\mu \to 0} J_1 &= - \int_0^t \int_{\Gamma} u(s, r) \nabla \varphi(r) \cdot \mathbf{n}(r) \ dr ds, \\
    ii) \text{ess lim }_{\mu \to 0} J_2 &= (d - 1) \int_0^t \int_{\Gamma} u(s, r) \varphi(r) H(r) \ dr ds, \\
    iii) \text{ess lim }_{\mu \to 0} J_3 &= 0,
\end{align*}
\]

(2.31)

where \( H \) is the mean curvature of \( \Gamma \), satisfying (at least) \( H \in L^1(\Gamma) \), see [4].

Proof of Claim 2: Assertion \((i)\) and \((ii)\) follow similarly to the proof of Claim 1. Thus, let us show item \((iii)\). Again, we apply the Coarea Formula for the function \( h \) and then, use the Area Formula for the map \( \Psi_\tau \) to write \( J_3 \) conveniently as

\[
J_3 = \frac{1}{\mu} \int_0^t \int_0^\mu \int_{\Gamma} u(s, \Psi_\tau(r)) \varphi(\Psi_\tau(r)) (-\delta_\mu(\tau)) |\partial_i h(\Psi_\tau(r))|^2 \ dr d\tau ds \\
+ \frac{1}{\mu} \int_0^t \int_0^\mu \int_{\Gamma} u(s, \Psi_\tau(r)) \varphi(\Psi_\tau(r)) (\delta_0(\tau)) |\partial_i h(\Psi_\tau(r))|^2 \ dr d\tau ds + O(\mu),
\]

where \( \delta_\mu(\tau) \) is the Dirac measure concentrated in \( \mu \). Then, making the difference in \( J_3 \) and passing to the limit as \( \mu \to 0 \), with the helpness of the Dominated Convergence Theorem, also recall that we have omitted the standard regularization procedure, the proof of Claim 2 follows.

Now, we are in position to write equation (2.25) in the equivalent Itô’s
form (bounded domains), that is
\[
\int_U u(t, x) \varphi(x) dx = \int_U u_0(x) \varphi(x) dx + \int_0^t \int_U u(s, x) b^j(s, x) \partial_j \varphi(x) dx ds
\]
\[
+ \int_0^t \int_U u(s, x) \text{div } b(s, x) \varphi(x) dx ds - \int_0^t \int_{\Gamma} u(s, r) \varphi(r) (b^j n_j)^+ dr ds
\]
\[
+ \int_0^t \int_{\Gamma} u_b(s, r) \varphi(r) (b^j n_j)^- dr ds - \int_0^t \int_{\Gamma} u(s, r) \varphi(r) n_j(r) dr dB^j_s
\]
\[
- \int_0^t \int_{\Gamma} \nabla \varphi(r) \cdot n(r) dr ds + \frac{(d - 1)}{2} \int_0^t \int_{\Gamma} u(s, r) \varphi(r) H(r) dr ds
\]
\[
+ \int_0^t \int_U u(s, x) \partial_j \varphi(x) dx dB^j_s + \frac{1}{2} \int_0^t \int_U u(s, x) \Delta \varphi(x) dx ds.
\]

(2.32)

3. Analogously to item 2 in the proof of Lemma 2.5, we have the limit transition. Since the family \( \{u^\varepsilon\} \) by our construction given by Lemma 2.1 is uniformly bounded up to the boundary, and satisfies the weak Ito’s form (2.32), we may pass to he limit as \( \varepsilon \) goes to zero, and obtain a stochastic process \( u \) in \( L^\infty(U_T \times \Omega) \), such that, satisfies equation (2.24), or equivalently (2.32), where instead of \( u(s, r) \) we have \( \gamma u(s, r) \), which is the weak limit of \( u^\varepsilon \) restrict to the boundary.

4. Finally, we show that \( \gamma u(s, r) = \tilde{\gamma} u(s, r) \). We observe that \( u \) is also a distributional \( L^\infty \) - solution of (1.1). Then, from equations Proposition (2.20) and equation (2.24) we have that
\[
- \int_0^t \int_{\Gamma} \gamma u(s, r) \varphi(r) (b^j n_j)^+ dr ds - \int_0^t \int_{\Gamma} u_b(s, r) \varphi(r) (b^j n_j)^- dr ds
\]
\[
- \int_0^t \int_{\Gamma} \gamma u(s, r) \varphi(r) n_j(r) dr dB^j_s =
\]
\[
- \int_0^t \int_{\Gamma} \gamma u(s, r) \varphi(r) b(r)n_j(r) dr ds - \int_0^t \int_{\Gamma} \gamma u(s, r) \varphi(r) n_j(r) dr dB^j_s.
\]

(2.33)

Taking covariation with respect to \( B^j \) we have that
\[
- \int_0^t \int_\Gamma \gamma u(s, r) \varphi(r) n_j(r) \, dr \, ds = - \int_0^t \int_\Gamma \gamma' u(s, r) \varphi(r) n_j(r) \, dr \, ds. \tag{2.34}
\]

Thus we conclude that \( \gamma u(s, r) = \gamma' u(s, r) \).

3 Uniqueness

In this section, we present the uniqueness theorem for the SPDE (1.1). We prove uniqueness following the concept of renormalized solutions introduced by DiPerna, Lions. The BV framework is the one adopted in the sequel, where we make extensive use of the ideas from \cite{2}.

Lemma 3.1. Assume condition (2.3). Let \( u \) be a distributional \( L^\infty \)-solution of the stochastic IBVP (1.1), and define \( v := E(\beta(u)) \) for any \( \beta \in C^2(\mathbb{R}) \). Then, for each \( u_0 \in L^\infty(U) \) the function \( v \) satisfies

\[
\partial_t v(t, x) + b(t, x) \cdot \nabla v(t, x) = \frac{1}{2} \Delta v(t, x) \quad \text{in} \quad \mathcal{D}'([0, T) \times U). \tag{3.35}
\]

Proof. 1. For \( \varepsilon > 0 \), we define \( U_\varepsilon := \{ x \in U : \text{dist}(x, \partial U) > \varepsilon \} \). Let \( \phi_\varepsilon \) be a standard symmetric mollifier and \( u \) a distributional \( L^\infty \)-solution of (1.1). Then, for each \( t \in [0, T] \), and \( x \in U_\varepsilon \), taking \( \phi_\varepsilon \) as a test function in (2.16), we obtain

\[
u_\varepsilon(t, x) \equiv \int_U u(t, z) \phi_\varepsilon(x - z) \, dz = u_0 * \phi_\varepsilon(x) + \int_0^t \int_U u(s, z) b^i(s, z) \partial_i \phi_\varepsilon(x - z) \, dz \, ds + \int_0^t \int_U u(s, z) \text{div} b(s, z) \phi_\varepsilon(x - z) \, dz \, ds + \int_0^t \int_U u(s, z) \partial_i \phi_\varepsilon(x - z) \, dz \circ dB^i_s.
\]
For $\beta \in C^2(\mathbb{R})$, we apply the Itô-Ventzel-Kunita formula (see Theorem 8.3 of [16] in the above equation, hence we have

$$
\begin{align*}
\beta(u_\varepsilon(t, x)) &= \beta(u_0 \ast \phi_\varepsilon(x)) \\
&+ \int_0^t \beta'(u_\varepsilon(s, x)) \int_U u(s, z) b^i(s, z) \partial_i \phi_\varepsilon(x - z) \, dz \, ds \\
&+ \int_0^t \beta'(u_\varepsilon(s, x)) \int_U u(s, z) \text{div} b(s, z) \phi_\varepsilon(x - z) \, dz \, ds \\
&+ \int_0^t \beta'(u_\varepsilon(s, x)) \int_U u(s, z) \partial_t \phi_\varepsilon(x - z) \, dz \, dB^i_s. 
\end{align*}
$$

(3.36)

2. Now it becomes clear our strategy, which follows the renormalization procedure. Then, we take a test function $\phi \in C_c^\infty(U)$, multiply equation (3.36) by it, and integrate in $U$ to obtain

$$
\int_U \beta(u_\varepsilon(t)) \varphi \, dx = \int_U \beta(u_0 \ast \phi_\varepsilon(x)) \varphi(x) \, dx \\
+ \int_0^t \int_U \beta'(u_\varepsilon(s, x)) u(s, z) b^i(s, z) \partial_i \phi_\varepsilon(x - z) \varphi(x) \, dz \, dx \, ds \\
+ \int_0^t \int_U \beta'(u_\varepsilon(s, x)) u(s, z) \text{div} b(s, z) \phi_\varepsilon(x - z) \varphi(x) \, dz \, dx \, ds \\
+ \int_0^t \int_U \beta'(u_\varepsilon(s, x)) u(s, z) \partial_t \phi_\varepsilon(x - z) \varphi(x) \, dz \, dx \circ dB^i_s,
$$

where we have used Fubini’s Stochastic Theorem, see for instance [23]. Since $\phi_\varepsilon$ is a symmetric mollifier, from an algebraic convenient manipulation and
integration by parts, we obtain

$$\int_U \beta(u_\varepsilon(t)) \varphi \, dx - \int_U \beta(u_0 \ast \phi_\varepsilon(x)) \varphi(x) \, dx$$

$$- \int_0^t \int_U \beta(u_\varepsilon(s,x)) b^i(s,x) \partial_i \varphi(x) \, dxds$$

$$- \int_0^t \int_U \beta(u_\varepsilon(s,x)) \text{div} b(s,x) \varphi(x) \, dxds$$

$$- \int_0^t \int_U \beta(u_\varepsilon(s,x)) \partial_i \varphi(x) \, dx \circ dB_s^i = \int_0^t \int_U \beta'(u_\varepsilon(s,x)) \varphi(x) \mathcal{R}_\varepsilon(b,u) \, dxds,$$

where $\mathcal{R}_\varepsilon(b,u)$ is the commutator defined as

$$\mathcal{R}_\varepsilon(b,u) = (b\nabla)(\phi_\varepsilon \ast u) - \phi_\varepsilon \ast ((b\nabla)u).$$

One remarks that, the commutator above is exactly the same one establish by DiPerna, Lions in [12]. Moreover, by the regularity assumptions on $b$ and $u$, applying the Commuting Lemma (see [2] or Theorem 9 of [1]), it follows that

$$\lim_{\varepsilon \to 0} \mathcal{R}_\varepsilon(b,u) = 0, \quad \mathbb{P} \ a.s \ in \ L^1([0,T];L^1_{\text{loc}}(\mathbb{R}^d)),$$

Therefore, since $u$ is measurable and bounded, $u_\varepsilon$ converges to $u$ in $L^1_{\text{loc}}$, we obtain from (3.37) passing to the limit as $\varepsilon \to 0$

$$\int_U \beta(u(t,x)) \varphi(x) \, dx = \int_U \beta(u_0(x)) \varphi(x) \, dx$$

$$+ \int_0^t \int_U \beta(u(s,x)) b^i(s,x) \partial_i \varphi(x) \, dxds$$

$$+ \int_0^t \int_U \beta(u(s,x)) \text{div} b(s,x) \varphi(x) \, dxds$$

$$+ \int_0^t \int_U \beta(u(s,x)) \partial_i \varphi(x) \, dx \circ dB_s^i,$$

where we have used the Dominated Convergence Theorem.
3. Recall Remark 2.4 and taking the expectation, it follows from (3.38) that, the function $v(t, x) = \mathbb{E}(\beta(u(t, x)))$ satisfies

$$\int_U v(t, x) \varphi(x) \, dx = \int_U \beta(u_0(x)) \varphi(x) \, dx$$

$$+ \int_0^t \int_U v(s, x) b^i(s, x) \partial_i \varphi(x) \, dx \, ds$$

$$+ \int_0^t \int_U v(s, x) \text{div} b(s, x) \varphi(x) \, dx \, ds$$

$$+ \frac{1}{2} \int_0^t \int_U v(s, x) \Delta \varphi(x) \, dx \, ds.$$

Finally, for $\zeta \in C^\infty_c([0, T])$ we multiply the above equation by $\zeta'(t)$, and integrating in $[0, T)$, we obtain that

$$\int_0^T \int_U v(t, x) \zeta'(t) \varphi(x) \, dx \, dt = - \int_U \beta(u_0(x)) \zeta(0) \varphi(x) \, dx$$

$$- \int_0^T \int_U v(s, x) b^i(s, x) \zeta(t) \partial_i \varphi(x) \, dx \, dt$$

$$- \int_0^T \int_U v(t, x) \text{div} b(t, x) \zeta(t) \varphi(x) \, dx \, dt$$

$$- \frac{1}{2} \int_0^T \int_U v(t, x) \zeta(t) \Delta \varphi(x) \, dx \, dt.$$

Since finite sums of function $\zeta_i(t) \varphi_i(x)$, $(\zeta_i \in C^\infty_c([0, T]), \varphi_i \in C^\infty_c(U))$ are dense in the space of test functions $D([0, T) \times U)$, the thesis of the lemma follows by a standard density argument.

Next, we pass to the uniqueness theorem. It should be assumed more conditions on the vector function $b$, which are all explicit in the following

**Theorem 3.2.** Let $b$ be a vector field satisfying condition (2.3), and for $L^1$-a.e. $t \in [0, T]$

$$|b(t, x)| \leq \alpha(t), \quad \text{div} b(t, x) \leq \gamma(t), \quad (3.39)$$
for some nonnegative functions $\alpha, \gamma \in L^1_{\text{loc}}(\mathbb{R})$. If $u, v \in L^\infty(U_T \times \Omega)$ are two weak $L^\infty$-solutions of the IBVP (1.1), with the same initial-boundary data $u_0 \in L^\infty(U)$, $u_b \in L^\infty(\Gamma_T; \mu^\cdot)$, then $u \equiv v$ almost sure in $U_T \times \Omega$.

**Proof.** 1. First, by linearity it is enough to show that, a weak $L^\infty$-solution of the IBVP (1.1), say $u(t, x)$, with initial-boundary condition $u_0 = 0$ and $u_b = 0$ vanishes identically. Since $u$ is a weak solution, for each $\varphi \in C^\infty_c(\mathbb{R}^d)$, and $t \in [0, T]$, we have

$$
\int_U u(t, x) \varphi(x) dx = \int_0^t \int_U u(s, x) b^i(s, x) \partial_i \varphi(x) \ dx ds \\
+ \int_0^t \int_U u(s, x) \text{div} b(s, x) \varphi(x) \ dx ds - \int_0^t \int_\Gamma \gamma u(s, r) \varphi(r) (b^i n_j)^+ \ dr \ ds \\
- \int_0^t \int_\Gamma \gamma u(s, r) \varphi(r) n_j(r) \ dr \circ dB_t^j + \int_0^t \int_U u(s, x) \partial_{x_j} \varphi(x) \ dx \circ dB_t^j.
$$

(3.40)

Taking $\varphi \in C^\infty_c(U)$, it follows that $u$ is a distributional $L^\infty$-solution of the stochastic IBVP (1.1). Therefore, applying Lemma 3.1 we have that, for all $\psi \in C^\infty([0, T] \times U)$ and any $\beta \in C^2(\mathbb{R})$, with $\beta(0) = 0$, $v(t, x) = \mathbb{E}(\beta(u(t, x)))$ satisfies

$$
\int_0^T \int_U v(t, x) \partial_t \psi(t, x) \ dx dt = - \int_0^T \int_U v(t, x) b^i(t, x) \partial_i \psi(t, x) \ dx dt \\
- \int_0^T \int_U v(t, x) \text{div} b(t, x) \psi(t, x) \ dx dt \\
- \frac{1}{2} \int_0^T \int_U v(t, x) \Delta \psi(t, x) \ dx dt.
$$

(3.41)

2. Fix $K \subset U$ be any compact set. By Lusin Theorem, see Evans and Gariepy [13], for each $\theta > 0$, there exists a compact set $J_\theta \subset [-2T, 2T]$, such that, $\mathcal{L}^1([-2T, 2T] - J_\theta) < \theta$ and $\alpha|_{3\theta} =: \alpha_\theta$ is a nonnegative continuous functions. Hence we define $k_\theta := \max_{t \in J_\theta} \alpha_\theta(t)$. Now, we consider a nonnegative function $\varphi$, such that, $\text{supp} \varphi = K$, and satisfies the following Hamilton-Jacobi-Belmann equation in $[0, 2T] \times U$

$$
\partial_t \varphi + k_\theta |\nabla \varphi| + \frac{1}{2} \Delta \varphi \leq 0,
$$

(3.42)
that is to say, a sub-solution of a constant coefficients equation. The existence of a such function \( \varphi \), which satisfies the above assumptions follows from the parabolic theory in [18]. Also, if \( \varphi \) satisfies equation (3.42), then we have in \( ([0, 2T] \cap J_\theta) \times U \) that

\[
\partial_t \varphi + \alpha_\theta(t) |\nabla \varphi| + \frac{1}{2} \Delta \varphi \leq 0.
\]

Let \( \zeta \in C^\infty_c([0, 2T]) \) be a nonnegative test function (with respect to time variable). Then, taking \( \psi(t, x) = \zeta(t) \varphi(t, x) \) in (3.41), we have

\[
\begin{align*}
\int_0^T \int_U v(t, x) \zeta'(t) \varphi(t, x) \, dx \, dt \\
= - \int_0^T \int_U v(t, x) \zeta(t) \left( \partial_t \varphi(t, x) + b(t, x) \cdot \nabla \varphi(t, x) + \frac{1}{2} \Delta \varphi(t, x) \right) \, dx \, dt \\
- \int_0^T \int_U v(t, x) \text{div}b(t, x) \zeta(t) \varphi(t, x) \, dx \, dt \\
\geq - \int_{[0, T] \cap \gamma} \int_U v(t, x) \zeta(t) \left( \partial_t \varphi(t, x) + \alpha(t) |\nabla \varphi(t, x)| + \frac{1}{2} \Delta \varphi(t, x) \right) \, dx \, dt \\
- \int_0^T \int_U v(t, x) \gamma(t) \zeta(t) \varphi(t, x) \, dx \, dt,
\end{align*}
\]

where we have used (3.39) and the above assumptions on \( \varphi \). If \( \varphi \) as above considered is not regular enough \( (W^{2, \infty}) \) to be as a test function, a standard regularization procedure may be used. Analogously, we take \( \zeta(t) \) the characteristic function of the interval \([\delta, t_0 - \delta]\) for any \( \delta > 0 \) and \( t_0 \in [0, T] \). Then, from the above inequality passing to the limit as \( \delta \to 0 \), and also \( \theta \to 0 \), we obtain

\[
\int_U v(t, x) \varphi(t, x) \, dx \leq \int_0^T \gamma(t) \int_U v(t, x) \varphi(t, x) \, dx \, dt.
\]

From which, applying the Gronwall Inequality implies that \( v \varphi = 0 \) almost everywhere in \( U \). Since \( \varphi > 0 \) on \( K \), which is arbitrary, taking \( \beta(z) = z^2 \), we conclude that \( u = 0 \) almost sure in \( U_T \times \Omega \).

3. Finally, since \( u = 0 \) almost sure in \( U_T \times \Omega \), it follows from (3.40) for any test function \( \varphi \in C^\infty_c(\mathbb{R}^d) \), and all \( t \in [0, T] \)

\[
\int_0^t \int_{\Gamma} \gamma u(s, r) \varphi(r)(b' \mathbf{n}_r)^+ \, dr \, ds + \int_0^t \int_{\Gamma} \gamma u(s, r) \varphi(r) \mathbf{n}_r \, dr \, dB^Z_s = 0. \tag{3.43}
\]

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Therefore, taking the covariation with respect to $B^j$, we obtain
\[ \int_0^t \int_{\Gamma} \gamma u(s, r) \varphi(r) \, n_j \, dr \, ds = 0, \]
which implies that $\gamma u = 0$ almost sure in $[0, T] \times \Gamma \times \Omega$. \qed

4 Appendix

At this point we fix some notation and material used in the paper. Moreover, recall some well known results concerning fine properties of functions.

First, through out of this paper, we fix a stochastic basis with a $d$-dimensional Brownian motion $(\Omega, \mathcal{F}, \{\mathcal{F}_t : t \in [0, T]\}, \mathbb{P}, (B_t))$. Then, we recall to help the intuition, the following definitions

\begin{align*}
\text{Itô:} & \quad \int_0^t X_s dB_s = \lim_{n \to \infty} \sum_{t_i \in \pi_n, t_i \leq t} X_{t_i}(B_{t_{i+1}} - B_{t_i}), \\
\text{Stratonovich:} & \quad \int_0^t X_s \circ dB_s = \lim_{n \to \infty} \sum_{t_i \in \pi_n, t_i \leq t} \frac{(X_{t_i} + X_{t_i + 1})(B_{t_{i+1}} - B_{t_i})}{2}, \\
\text{Covariation:} & \quad [X, Y]_t = \lim_{n \to \infty} \sum_{t_i \in \pi_n, t_i \leq t} (X_{t_i} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i}),
\end{align*}

where $\pi_n$ is a sequence of finite partitions of $[0, T]$ with size $|\pi_n| \to 0$ and elements $0 = t_0 < t_1 < \ldots$. The limits are in probability, uniformly in time on compact intervals. Details about these facts can be found in Kunita [16]. Also we address from that book, Itô’s formula, the chain rule for the stochastic integral, for any continuous d-dimensional semimartingale $X = (X_1, X_2, \ldots, X_d)$, and twice continuously differentiable and real valued function $f$ on $\mathbb{R}^d$.

One remarks that, if $U$ is bounded, $\partial U$ Lipschitz, then any bounded variation function has trace in the strong $L^1$ sense on $\partial U$. We address for instance Evans, Gariepy [13] Chapter 5.3, Theorem 1, to this well known result, and further we are not going to distinguish the notation between the BV function itself and the trace of it.
Now, let us consider some auxiliary functions. Let \( \Psi : \Gamma \times [0, 1] \to U \) be a deformation of \( \Gamma \), such that, for each \((r, \tau) \mapsto \Psi(r, \tau) \in U \). Moreover, for all \( \tau \in [0, 1] \), we denote \( \Psi_\tau(\cdot) = \Psi(\cdot, \tau) \), thus \( \Psi_0(\cdot) = Id_\Gamma \), i.e. the identity map over \( \Gamma \), and
\[
\Gamma^\tau = \Psi_\tau(\Gamma), \quad \Gamma_T^\tau = \Psi_\tau(\Gamma_T).
\]
Also, let \( h : \mathbb{R}^d \to [0, 1] \) be defined by setting \( h(x) = \tau \) if \( x \in \Gamma^\tau \), \( h(x) = 0 \) if \( x \notin U \) and \( h(x) = 1 \) otherwise. Therefore, \( h \) is a globally Lipschitz function with compact support in \( \mathbb{R}^d \), and \( \nabla h \) is parallel to \( \mathbf{n} \) almost everywhere on \( \Gamma^\tau \). For \( \tau \) sufficiently small, we may consider \( h \) the distance function from the point \( x \in U \) to the boundary \( \Gamma \).

Finally, given a function \( f \in L^1(U) \), we recall also the global approximation by smooth functions, that is, \( f_\varepsilon \in L^1(U) \cap C^\infty(\overline{U}) \), such that, \( f_\varepsilon \to f \) in \( L^1 \), see Evans, Gariepy [13] Chapter 4.2, Theorem 1 and Theorem 3. In fact, this result follows from a convenient modification of the standard mollification of \( f \) by a standard (symmetric) mollifier \( \rho_\varepsilon \). For convenience, that is to fix the notation, let us give the main idea. For any \( \varepsilon > 0 \) fixed, \( 0 \leq \tau \leq \varepsilon \), and \( y \in U \), we define
\[
y^\varepsilon := y + \lambda \varepsilon \nabla h(y),
\]
for \( \lambda > 0 \) sufficiently large. Then, we take a standard mollifier \( \rho_\varepsilon \), and for any \( u \in L^1_{loc}(U_T) \), we define the following (space) global approximation
\[
u_\varepsilon(t,y) \equiv (u \ast \rho_\varepsilon)(t,y) := \int_U u(t,z)\rho_\varepsilon(y^\varepsilon - z) \, dz.
\]
Therefore, \( u_\varepsilon \in L^1_{loc}([0,T]; C^\infty(\overline{U})) \) and converges to \( u \) in \( L^1_{loc} \).

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