The Maximal Abelian Gauge in SU(3) Lattice Gauge Theory

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We gauge fix 600 $SU(3)$ $\beta = 6.0$ configurations on a $16^4$ lattice to a simple form of the maximal abelian gauge. We project the $SU(3)$ valued links to the $U(1) \times U(1)$ subgroup, and extract $U(1) \times U(1)$ and monopole string tensions. After gauge fixing to the indirect center gauge, the $U(1) \times U(1)$ links are projected to $Z(3)$ and a vortex string tension is measured. The vortex and magnetic current densities are measured.

The idea that the long-range physics of an $SU(N)$ gauge field can be compressed into the abelian sector of that field was suggested by 't Hooft. The simplest way to do this is to gauge fix the field so that it is as much in the abelian "direction" as possible and then to project to the abelian sector.

The process of gauge fixing to the maximum abelian gauge (MAG) has been studied extensively in $SU(2)$ lattice gauge theory. Gauge fixed configurations are projected to the $U(1)$ subgroup. From this $U(1)$ gauge field, monopole currents are located and a potential from the monopoles can be found and compared to the full $SU(2)$ case. Also studied is the indirect center gauge (ICG). Here, after going to the MAG, the $U(1)$ field is also gauge fixed and then projected to the $Z(2)$ center of the group. There are still unresolved issues such as stability under cooling and sensitivity to gauge ambiguities for the case of $SU(2)$. It is natural, however, to go on to the case of an $SU(3)$ gauge group in order to look at confinement as it occurs in QCD.

1. Maximal Abelian Gauge

In $SU(2)$ gauge theory in the continuum, the MAG is the gauge that minimizes the quantity

$$ G^\text{SU(2)}_{\text{mag}} = \int \left[ (A^1_\mu)^2 + (A^2_\mu)^2 \right] d^4 x. $$

2. Abelian Projection

Having reached a maximum of Eq. (4), it is necessary to project each $SU(3)$ link matrix to the

$$ G^\text{SU(3)}_{\text{mag}} = \int \left[ (A^1_\mu)^2 + (A^2_\mu)^2 + (A^3_\mu)^2 \right] d^4 x. $$

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The equivalent lattice functional is

$$ G^\text{SU(2)}_{\text{mag}} = \frac{1}{2N_{\text{link}}} \sum_{x,\mu} \text{Tr} \left[ U^\dagger(x) \sigma_3 U(x) \right]. $$

A maximum of Eq. (3) is reached by sweeping over the lattice and performing gauge transformations that maximize the lattice functional at a given site.

In $SU(3)$, the corresponding continuum functional is

$$ G^\text{SU(3)}_{\text{mag}} = \int \left[ (A^1_\mu)^2 + (A^2_\mu)^2 + (A^3_\mu)^2 \right] d^4 x. $$

On the lattice Eq. (3) becomes

$$ G^\text{SU(3)}_{\text{mag}} = \frac{1}{4N_{\text{link}}} \left\{ \sum_{x,\mu} \text{Tr} \left[ U^\dagger(x) \sigma_3 U(x) \lambda_3 \right] + \sum_{x,\mu} \text{Tr} \left[ U^\dagger(x) \lambda_8 U(x) \lambda_8 \right] \right\}. $$

For $SU(3)$, gauge fixing updates are done based on $SU(2)$ subgroups. We denote the three subgroups as follows: $SU(2)$ : $\lambda_1, \lambda_2, \lambda_3$, $SU(2)'$ : $\lambda_4, \lambda_5, \lambda_7$, $SU(2)''$ : $\lambda_6, \lambda_7, \lambda_8$, where $\lambda_i$ are the usual $\lambda$-matrices, and $\lambda'_i = \text{diag}(1, 0, -1)$ and $\lambda''_i = \text{diag}(0, 1, -1)$.
$U(1) \times U(1)$ subgroup. In the $SU(2)$ case, the projection to $U(1)$ is determined by the (opposite) phases of the diagonal elements of each link matrix. For $SU(3)$, the phases of the diagonal elements of the link matrices do not determine a diagonal unitary $3 \times 3$ matrix. One simple way to project to $U(1) \times U(1)$ is to choose two of the diagonal phases, and determine the third by unitarity. Alternatively, each diagonal phase can be used, with unitarity enforced by an additional overall phase. The method used here is to find a diagonal matrix $U$ such that the quantity Re $\langle \text{Tr} [UA^\dagger] \rangle$ is maximized, where $A$ is the $SU(3)$ link. The purpose of this method is to find the “best” $U(1) \times U(1)$ approximation to a given $SU(3)$ matrix. Either of the two methods just described can be used for a trial solution. The trial solution is then successively updated using matrices of the form $\text{exp}(i\phi \lambda_3)$, or matrices with $\lambda_3', \lambda_3''$ replacing $\lambda_3$. The procedure was found to give numerical accuracy in 10 iterations. As a side note, it would be the natural procedure to use for projecting spatially smeared $U(1) \times U(1)$ links back to the $U(1) \times U(1)$ manifold.

3. Indirect Center Gauge

With configurations projected to the $U(1) \times U(1)$ subgroup, it becomes a simple (and considerably less computationally intensive) matter to gauge fix to the indirect center gauge (ICG). This gauge is determined by maximizing the functional

$$G_{\text{icg}}^{SU(3)} = \frac{1}{9N_{\text{link}}} \sum_{x,\mu} |\text{Tr}[U_\mu(x)]|^2,$$

where the $U_\mu$ are the $U(1) \times U(1)$ projected links. The easiest way to reach the ICG is with gauge transformations by factors of $\lambda_8$, $\lambda_8'$, and $\lambda_8''$, where $\lambda_8 = \sqrt{3}\lambda_8 = \text{diag}(1,1,-2)$, $\lambda_8' = \text{diag}(1,-2,1)$, and $\lambda_8'' = \text{diag}(-2,1,1)$. This choice of transformation has the advantage that solving for the optimal transformation gives an easy analytical solution.

Projection to $Z(3)$ is accomplished by choosing the element $\xi_j^1 \mathbf{1}$, $j = 0,1,2$, $\xi = \exp(2\pi i/3)$ that maximizes the quantity $\text{Tr}[\xi_j^1 U_\mu^\dagger] = \xi_j^1 \text{Tr}[U_\mu^\dagger]$, for each $U(1) \times U(1)$ link. This results in a set of $Z(3)$ links. The Wilson loop is then calculated by multiplying these $Z(3)$ valued links around the loop.

4. Magnetic Currents

Magnetic currents are extracted for each $SU(3)$ color by applying the Toussaint DeGrand procedure to the $U(1) \times U(1)$ links. This produces three magnetic currents, of which only two are independent. In our method, the sum over colors of the magnetic current is non-vanishing on a link by link basis. Nevertheless, when used to calculate a Wilson loop, this total current produces a null potential, as it should. After extracting the magnetic currents, monopole Wilson loops are calculated for each color. This part of the calculation proceeds exactly as in $U(1)$ or $SU(2)$ lattice gauge theory. Finally the color average of these monopole Wilson loops is used to produce a monopole potential from which a monopole string tension is extracted. The color averaged fraction of links carrying magnetic current was also measured and found to be $7.44(4) \times 10^{-3}$.

5. Results

We gauge fixed each of 600 configurations to the MAG using Eq.(1), generating one gauge copy per configuration. The MAG gauge fixing was done using overrelaxation with overrelaxation parameter $\omega = 1.8$. Gauge fixing was continued until $\langle |X_{12}|^2 + |X_{13}'|^2 + |X_{23}'|^2 \rangle < 3 \times 10^{-11}$ was satisfied where the matrix $X(x)$ is

$$\sum_\mu U_\mu(x)\lambda_3 U_\mu^\dagger(x) + U_\mu(x-\mu)\lambda_3 U_\mu(x-\mu),$$

and the matrices $X', X''$ are similar, but have $\lambda_3', \lambda_3''$, respectively in place of $\lambda_3$.

Further gauge-fixing using Eq.(3) was performed to get to the ICG. The stopping criterion for this procedure was the same as in [3]. One gauge copy/configuration was generated, and the overrelaxation parameter for ICG was $\omega = 1.7$. Links were then projected to $Z(3)$ and Wilson loops calculated as in a $Z(3)$ gauge theory. The P-vortex density, that is the fraction of $Z(3)$ plaquettes with non-zero flux, was measured to be $2.25(1) \times 10^{-2}$. 
Wilson loops up to $R = 8, T = 8$ were generated for the various cases. Potentials were determined from Wilson loop values by fitting $-\ln(W(R,T))$ to a straight line in $T$. The $\ln(W)$ vs $T$ plots were remarkably linear in all cases, even for $T < R$. From the potentials, string tensions were determined by fitting to the form $V(R) = \sigma R + \alpha/R + V_0$. Values for these parameters are given in Table 1. As can be seen there, the MAG $U(1) \times U(1)$, ICG vortex, and MAG monopole string tensions all lie below the full $SU(3)$ value by an amount well outside of error bars. In Fig. 1, we plot the potentials determined here along with the best $SU(3)$ potential from the literature. The values of the constant $V_0$ have been adjusted for display.

Our findings for $SU(3)$ are to be contrasted with the behavior in $SU(2)$, where for one gauge copy/configuration, at couplings with comparable string tension, the MAG $U(1)$ and ICG vortex string tensions are larger than the full $SU(2)$ value, while the monopole string tension lies very close to the full $SU(2)$ value. Accounting for gauge copies with higher values of the functionals causes all the string tensions to decrease, with the $U(1)$ MAG string tension ending up $\sim 8\%$ below the full $SU(2)$ result. If the same trend holds here, the $U(1) \times U(1)$ MAG string tension would end up well below the full $SU(3)$ value, with the monopole and ICG vortex values lower still. To remedy this situation for $SU(3)$, we are studying more general forms of the MAG condition.

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