CONSTRUCTION OF AN INVARIANT FOR INTEGRAL HOMOLOGY 3-SPHERES VIA COMPLETED KAUFFMAN BRACKET SKEIN ALGEBRAS

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Abstract. We construct an invariant \( z(M) = 1 + a_1(A^4 - 1) + a_2(A^4 - 1)^2 + a_3(A^4 - 1)^3 + \cdots \in \mathbb{Q}[[A^4 - 1]] = \mathbb{Q}[[A + 1]] \) for an integral homology 3-sphere \( M \) using a completed skein algebra and a Heegaard splitting. The invariant \( z(M) \bmod ((A + 1)^{n+1}) \) is a finite type invariant of order \( n \). In particular, \( -a_1/6 \) equals the Casson invariant. If \( M \) is the Poincaré homology 3-sphere, \( (z(M))|_{A^4=q} \bmod (q+1)^{14} \) is the Ohtsuki series \([10]\) for \( M \).

1. Introduction

Heegaard splitting theory clarifies a relationship between mapping class groups on surfaces and closed oriented 3-manifolds. In particular, there exists some equivalence relation \( \sim \) of Torelli groups of a surface \( \Sigma_{g,1} \) with genus \( g \) and non-empty connected boundary, and the well-defined bijective map

\[
\lim_{g \to \infty} \mathcal{I}(\Sigma_{g,1})/\sim \to \mathcal{H}(3)
\]

plays an important role, where we denote by \( \mathcal{I}(\Sigma_{g,1}) \) the Torelli group of \( \Sigma_{g,1} \) and by \( \mathcal{H}(3) \) the set of integral homology 3-spheres, i.e. closed oriented 3-manifolds whose homology groups are isomorphic to the homology group of \( S^3 \). For details, see Fact 2.2 in this paper. This bijective map makes it possible to study integral homology 3-spheres using the structure of Torelli groups. See, for example, Morita [9] and Pitsch [12] [13].

On the other hand, in our previous papers [14] [15] [16], we study some new relationship between the Kauffman bracket skein algebra and the mapping class group of a surface. It gives us a new way of studying the mapping class group. For example [16], we reconstruct the first Johnson homomorphism in terms of the skein algebra. Since the Kauffman bracket skein algebra comes from link theory, we expect that this relationship brings us a new information of 3-manifolds.

The aim of this paper is to construct an invariant \( z(M) \) for an integral homology 3-sphere \( M \) using completed skein algebras and the above bijective map. In other words, the aim of this paper is to prove the following main theorem.

Theorem 1.1 (Theorem 3.3). The map \( Z : \mathcal{I}(\Sigma_{g,1}) \to \mathbb{Q}[[A + 1]] \) defined by

\[
Z(\xi) \overset{\text{def}}{=} \sum_{i=0}^{\infty} \frac{1}{(-A + A^{-1})^i i!} e_*(((\zeta(\xi)))^i)
\]

induces an invariant

\[
z : \mathcal{H}(3) \to \mathbb{Q}[[A + 1]], M(\xi) \to Z(\xi),
\]
where $e_\ast$ is the $\mathbb{Q}[[A + 1]]$-module homomorphism induced by standard embedding. Here $\zeta : \mathcal{I}(\Sigma_{g,1}) \to \mathcal{S}(\Sigma_{g,1})$ is an embedding defined in Theorem 3.2.

We remark we do not rely on number theory for constructing the invariant.

Let $V$ be a $\mathbb{Q}$-vector space. In our paper, a map $z' : \mathcal{H}(3) \to V$ is called a finite type invariant of order $n$ if and only if the $\mathbb{Q}$-linear map $z' : \mathbb{Q}\mathcal{H}(3) \to V$ induced by $z' : \mathcal{H}(3) \to V$ satisfies the condition that

$$
\sum_{\epsilon_i \in \{0,1\}} (-1)^{i}z'(M(\prod_{i=1}^{2n+2} \xi_i^{\epsilon_i})) = 0.
$$

for any $\xi_1, \xi_2, \cdots, \xi_{2n+2} \in \mathcal{I}(\Sigma_{g,1})$. The above condition and the condition that

$$
\sum_{\epsilon_i \in \{0,1\}} (-1)^{i}z'(M(\prod_{i=1}^{n+1} \xi_i^{\epsilon_i})) = 0.
$$

for any $\xi_1, \xi_2, \cdots, \xi_{n+1} \in \mathcal{K}(\Sigma_{g,1})$ are equivalent to each other. This follows from [2] Theorem 1 and [3] subsection 1.8. Furthermore, in our paper, a finite type invariant $z' : \mathcal{H}(3) \to V$ of order $n$ is called nontrivial if and only if the $\mathbb{Q}$-linear map $z' : \mathbb{Q}\mathcal{H}(3) \to V$ induced by $z' : \mathcal{H}(3) \to V$ satisfies the condition that there exists $\xi_1, \xi_2, \cdots, \xi_{2n} \in \mathcal{I}(\Sigma_{g,1})$ such that

$$
\sum_{\epsilon_i \in \{0,1\}} (-1)^{i}z'(M(\prod_{i=1}^{2n} \xi_i^{\epsilon_i})) \neq 0.
$$

By [2] Theorem 1 and [3] subsection 1.8, the above condition and the condition that there exists $\xi_1, \xi_2, \cdots, \xi_{n} \in \mathcal{K}(\Sigma_{g,1})$ such that

$$
\sum_{\epsilon_i \in \{0,1\}} (-1)^{i}z'(M(\prod_{i=1}^{n} \xi_i^{\epsilon_i})) \neq 0.
$$

are equivalent to each other. The invariant $z : \mathcal{H}(3) \to \mathbb{Q}[[A + 1]]$ defined in this paper induces a finite type invariant $z(M) \in \mathbb{Q}[[A + 1]]/((A + 1)^n)$ of order $n + 1$ for $M \in \mathcal{H}(3)$ (Corollary 3.12). In Proposition 3.13 we prove the finite type invariant $z(M) \in \mathbb{Q}[[A + 1]]/((A + 1)^n)$ of order $n + 1$ is nontrivial, where we use a connected sum of the Poincaré spheres.

Furthermore, we give some computations of this invariant $z$ for some integral homology 3-spheres. As a corollary of this computation, the coefficient of $(A^4 - 1)$ in $z$ is $(-6)$ times the Casson invariant. On the other hand, Ohtsuki [10] defined the Ohtsuki series $\tau : \mathcal{H}(3) \to \mathbb{Z}[q]$. If $M$ is the Poincaré homology 3-sphere, $\tau(M)$ mod $((A + 1)^{14})$ is equal to $\tau(M) |_{\eta = A^4}$ mod $((A + 1)^{14})$. This lead us the following.

**Expectation 1.2.** Using the change of variables $A^4 = q$, the invariant $z$ induces the Ohtsuki series $\tau$, in other words, we have $z(M) = \tau(M) |_{\eta = A^4}$ for any $M \in \mathcal{H}(3)$.

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2. Mapping class groups and closed 3-manifolds

Let $\Sigma_g$ denote an closed oriented surface of genus $g$ standardly embedded in the oriented 3-sphere $S^3$. The embedded surface $\Sigma_g$ separates $S^3$ into two handle bodies of genus $g$, $S^3 = H_+^g \cup \varphi H_-^g$ where $\varphi : \Sigma_g = \partial H_+^g \to \partial H_-^g$ is a diffeomorphism. We fix an closed disk $D$ in $\Sigma_g$ and denote by $\Sigma_g, 1$ the closure of $\Sigma_g \setminus D$. The embedding $\Sigma_g \hookrightarrow S^3$ determines two natural subgroups of $\mathcal{M}(\Sigma_g, 1) \overset{\text{def}}{=} \text{Diff}^+(\Sigma_g, 1, \partial \Sigma_g, 1)/\text{Diff}_0(\Sigma_g, 1, \partial \Sigma_g, 1)$, namely $\mathcal{M}(H_\epsilon^g, 1) \overset{\text{def}}{=} \text{Diff}^+(H_\epsilon^g, D)/\text{Diff}_0(H_\epsilon^g, D)$ for $\epsilon \in \{+,-\}$. We denote $M(\xi) \overset{\text{def}}{=} H_\epsilon^g \cup \varphi_\epsilon \xi H_-^g$. Let $I(\Sigma_g, 1)$ be the Torelli group of the surface $\Sigma_g, 1$, which is the set consisting of all elements of $\mathcal{M}(\Sigma_g, 1)$ acting trivially on $H_1(\Sigma_g, \partial \Sigma_g)$. We remark that there is a natural injective stabilization map $\mathcal{M}(\Sigma_g, 1) \mapsto \mathcal{M}(\Sigma_g+1, 1)$, which is compatible with the definitions of the above two subgroups.

Definition 2.1. For $\xi_1$ and $\xi_2 \in I(\Sigma_g, 1)$, we define $\xi_1 \sim \xi_2$ if there exist $\eta^+ \in \mathcal{M}(H_{\epsilon}^g, 1)$ and $\eta^- \in \mathcal{M}(H_{\epsilon}^g, 1)$ satisfying $\xi_1 = \eta^- \xi_2 \eta^+$.

Fact 2.2 (For example, see [9] [12] [13]). The map
\[
\text{lim}_{g \to \infty}(I(\Sigma_g, 1)/\sim) \to \mathcal{H}(3), \xi \mapsto M(\xi)
\]
is bijective, where $\mathcal{H}(3)$ is the set of integral homology 3-spheres, i.e., closed oriented 3-manifolds whose homology group is isomorphic to the homology group of $S^3$.

We denote $I(\mathcal{M}(H_{\epsilon}^g, 1), \mathcal{M}(H_{\epsilon}^g, 1)) \overset{\text{def}}{=} \mathcal{M}(H_{\epsilon}^g, 1) \cap \mathcal{M}(H_{\epsilon}^g, 1)$ for $\epsilon \in \{+,-\}$.

Lemma 2.3 (Pitsch [13], Theorem 9, P.295, Omori [11]). For $\epsilon \in \{+,-\}$, the subgroup $I(\mathcal{M}(H_{\epsilon}^g, 1))$ is generated by
\[
\{t_{\xi(c_a)c'_a}(c'_b) | \xi \in \mathcal{M}(H_{\epsilon}^g, 1)\} \cup \{t_{\xi(c_a)c'_a}(c'_b) | \xi \in \mathcal{M}(H_{\epsilon}^g, 1)\},
\]
where the simple closed curves $c_a, c'_a, c_b$ and $c'_b$ are as in Figure 1.
Fig 1. $c_{a}, c'_{a}, c_{b}$ and $c'_{b}$

Proof. We prove the lemma in the case $\epsilon$ is +. Let $\IAut\pi_{1}(H_{g}^{+}, \star)$ be the kernel of $\Aut\pi_{1}(H_{g}^{+}, \star) \to \Aut(H_{1}(H_{g}))$. By [7], Theorem N4, p.168, $\IAut\pi_{1}(H_{g}^{+}, \star)$ is generated by

$$\{x_{\star} \in \IAut\pi_{1}(H_{g}^{+}, \star) | x \in \{t_{\xi(c_{a}), \xi(c'_{a})}^{\star} | \xi \in \mathcal{M}(H_{g,1}^{+})\}\},$$

where we denote by $x_{\star}$ the element of $\IAut\pi_{1}(H_{g}^{+}, \star)$ induced by $x \in \{t_{\xi(c_{a}), \xi(c'_{a})}^{\star} | \xi \in \mathcal{M}(H_{g,1}^{+})\}$. We denote by $\mathcal{LTM}(H_{g,1}^{+})$ the Luft-Torelli group which is the kernel of $\mathcal{I}\mathcal{M}(H_{g,1}^{+}) \to \IAut\pi_{1}(H_{g}^{+}, \star)$. Pitsch [13], Theorem 9, P.295 proves that $\mathcal{LTM}(H_{g,1}^{+})$ is generated by

$$\{t_{\xi(c_{a}), \xi(c'_{a})}^{\star} | \xi \in \mathcal{M}(H_{g,1}^{+})\}.$$

This proves the case that $\epsilon$ is +. If $\epsilon$ is −, replacing $a$ by $b$, the same proof works. This finishes the proof.

Lemma 2.4 ([13], Lemma 4, p.285). Let $G$ be a subgroup of $\mathcal{M}(H_{g,1}^{+}) \cap \mathcal{M}(H_{g,1}^{-})$ such that the natural map $G \to \Aut(H_{1}(H_{g,1}^{+}))$ is onto. For two elements $\xi_{1}$ and $\xi_{2} \in \mathcal{I}(\Sigma_{g,1})$, $\xi_{1} \sim \xi_{2}$ if and only if there exist $\eta_{G} \in G$, $\eta^{+} \in \mathcal{I}\mathcal{M}(H_{g,1}^{+})$ and $\eta^{-} \in \mathcal{I}\mathcal{M}(H_{g,1}^{-})$ satisfying $\eta^{-}\eta_{G}\eta\eta_{G}^{-1}\eta^{+} = \xi_{2}$.

Proof. Pitsch proved the above claim in the case $G = \mathcal{M}(H_{g,1}^{+}) \cap \mathcal{M}(H_{g,1}^{-})$. The proof is based on the fact that the natural map $\mathcal{M}(H_{g,1}^{+}) \cap \mathcal{M}(H_{g,1}^{-}) \to \Aut(H_{1}(H_{g,1}^{+}))$ is onto. Therefore, the proof of [13] Lemma 4 works for this lemma.

We construct a subgroup of $\mathcal{M}(H_{g,1}^{+}) \cap \mathcal{M}(H_{g,1}^{-})$ satisfying the above condition. Let $G \subset \mathcal{M}(H_{g,1}^{+}) \cap \mathcal{M}(H_{g,1}^{-})$ be the subgroup generated by

$$\{h_{i} | i \in \{1, 2, \cdots, g\}\} \cup \{s_{ij} | i \neq j\}$$

where we denote by $h_{i}$ and $s_{ij}$ the half twist along $c_{h,i}$ as in Figure 2 and the element $t_{c_{a}, c_{a}^{-1}} t_{c_{a}, c_{a}^{-1}}^{-1}$ as in Figure 3 and Figure 4. Since this subgroup $G$ satisfies the condition in the above lemma, we have the following.

Lemma 2.5. The equivalence relation $\sim$ in $\mathcal{I}(\Sigma_{g,1})$ is generated by $\xi \sim \eta_{G}\xi\eta_{G}^{-1}$ for $\eta_{G} \in \{h_{i}, s_{ij}\}$, $\xi \sim \xi\eta^{+}$ for $\eta^{+} \in \{t_{\xi(c_{a}), \xi(c'_{a})}^{\star} | \xi \in \mathcal{M}(H_{g,1}^{+})\} \cup \{t_{\xi(c_{a}), \xi(c'_{a})}^{\star} | \xi \in \mathcal{M}(H_{g,1}^{-})\}$.
\[ \mathcal{M}(H_{g,1}^-) \text{ and } \xi \sim \eta^- \xi \text{ for } \eta^- \in \{ t_{\xi(c_a)c_b} | \xi \in \mathcal{M}(H_{g,1}^-) \} \cup \{ t_{\xi(c_a)c_b} | \xi \in \mathcal{M}(H_{g,1}^-) \} \]
3. Proof of main theorem

3.1. Completed Kauffman bracket skein algebras and Torelli groups. Let \( \Sigma \) be a compact connected oriented surface. We denote by \( \mathcal{T}(\Sigma) \) the set of un-oriented framed tangles in \( \Sigma \times I \). Let \( \mathcal{S}(\Sigma) \) be the quotient of \( \mathbb{Q}[A,A^{-1}]/\mathcal{T}(\Sigma) \) by the skein relation and the trivial knot relation as in Figure 3. We consider the product of \( \mathcal{S}(\Sigma) \) as in Figure 6 and the Lie bracket \([x,y] \) satifies \([x,y] = \frac{1}{A+A^{-1}} (xy -yx) \) for \( x,y \in \mathcal{S}(\Sigma) \). The completed Kauffman bracket skein algebra is defined by

\[
\hat{\mathcal{S}}(\Sigma) \overset{\text{def}}{=} \lim_{i \to \infty} \mathcal{S}(\Sigma)/(\ker \varepsilon)^i
\]

where the augmentation map \( \varepsilon : \mathcal{S}(\Sigma) \to \mathbb{Q} \) is defined by \( A+1 \mapsto 0 \) and \( |L| - (-2)^{|L|} \mapsto 0 \) for \( L \in \mathcal{T}(\Sigma) \). In [15], we define the filtration \( \{F^n\hat{\mathcal{S}}(\Sigma)\}_{n \geq 0} \) satisfying

\[
F^n\hat{\mathcal{S}}(\Sigma), F^n\hat{\mathcal{S}}(\Sigma) \subset F^{n+m}\hat{\mathcal{S}}(\Sigma),
\]

\[
[F^n\hat{\mathcal{S}}(\Sigma), F^n\hat{\mathcal{S}}(\Sigma)] \subset F^{n+m-2}\hat{\mathcal{S}}(\Sigma),
\]

\[
F^{2n}\hat{\mathcal{S}}(\Sigma) = (\ker \varepsilon)^n.
\]

By the second equation, we can consider the Baker Campbell Hausdorff series

\[
\text{bch}(x,y) \overset{\text{def}}{=} (-A + A^{-1}) \log(\exp\left(\frac{x}{-A + A^{-1}}\right) \exp\left(\frac{y}{-A + A^{-1}}\right))
\]

on \( F^3\hat{\mathcal{S}}(\Sigma) \). As elements of the associated Lie algebra \((\hat{\mathcal{S}}(\Sigma), [ , ]\), bch has a usual expression. For example,

\[
\text{bch}(x,y) = x + y + \frac{1}{2} [x,y] + \frac{1}{12} ([x,[x,y]] + [y,[y,x]]) + \cdots.
\]

Furthermore, we have the following.

**Proposition 3.1** ([15] Corollary 5.7.). For any embedding \( i : \Sigma \times I \to S^3 \) inducing \( i_* : \hat{\mathcal{S}}(\Sigma) \to \mathbb{Q}[|A + 1|] \), we have \( i_* (F^n\hat{\mathcal{S}}(\Sigma)) \subset ((A + 1)^{|a| - 1}) \), where \(|x| \) is the greatest integer not greater than \( x \) for \( x \in \mathbb{Q} \).

In our previous papers [14] [15] [16], we study a relationship between the Kauffman bracket skein algebra and the mapping class group on a surface \( \Sigma \). Let \( \hat{\mathcal{S}}(\Sigma) \) be the completed Kauffman bracket skein algebra on \( \Sigma \) and \( \hat{\mathcal{S}}(\Sigma, J) \) the completed Kauffman bracket skein module with base point set \( J \times \{\frac{1}{2}\} \) for a finite subset \( J \subset \partial \Sigma \). In [14], we prove the formula of the Dehn twist \( t_c \) of a simple closed curve \( c \)

\[
t_c(\cdot) = \exp(\sigma(L(c)))(\cdot) = \sum_{i=0}^{\infty} \frac{1}{i!} (\sigma(L(c)))^i(\cdot) \in \text{Aut}(\hat{\mathcal{S}}(\Sigma, J))
\]

where

\[
L(c) \overset{\text{def}}{=} -A + A^{-1} \left( \arccosh\left(\frac{c}{2}\right)^2 - (-A + A^{-1}) \log(-A) \right).
\]

We obtain the formula by analogy of the formula of the completed Goldman Lie algebra [3] [6] [8]. We define the filtration \( \{F^n\hat{\mathcal{S}}(\Sigma)\}_{n \geq 0} \) in [15]. We consider \( F^3\hat{\mathcal{S}}(\Sigma) \) as a group using the Baker Campbell Hausdorff series bch where \( g > 1 \). We remark that \( \mathcal{I}(\Sigma, g, 1) \) is generated by \( \{t_{c_1c_2} \overset{\text{def}}{=} t_{c_1}t_{c_2}^{-1} | (c_1, c_2) : \text{BP}\} \) where a BP (bounding pair) is a pair of two simple closed curves bounding a submanifold of \( \Sigma_{g,1} \). By analogy of [6] 6.3, we have the following.
The skein relation

\[ \begin{array}{c}
\begin{array}{ccc}
\text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ }
\end{array}
\begin{array}{c}
\text{ } \\
\text{ } \\
\text{ }
\end{array}
\end{array} = A \begin{array}{c}
\begin{array}{c}
\text{ } \\
\text{ }
\end{array}
\end{array} + A^{-1} \begin{array}{c}
\begin{array}{c}
\text{ } \\
\text{ }
\end{array}
\end{array} \]

The trivial knot relation

\[ \begin{array}{c}
\begin{array}{c}
\text{ }
\end{array}
\end{array} = (-A^2 - A^{-2}) \begin{array}{c}
\begin{array}{c}
\text{ } \\
\text{ }
\end{array}
\end{array} \]

Fig 5. Definition of Kauffman bracket skein module

\[ xy \overset{\text{def.}}{=} \begin{array}{c|c}
1 & x \\
\hline
0 & y \\
\hline
\Sigma & \text{for } x, y \in S(\Sigma)
\end{array} \]

Fig 6. Definition of the product

**Theorem 3.2** ([16] Theorem 3.13. Corollary 3.14.). The group homomorphism \( \zeta : I(\Sigma, g, 1) \to (F^3 \widehat{S}(\Sigma, g, 1), \bch) \) defined by \( \zeta(t_{c_1c_2}) = L(c_1) - L(c_2) \) for a BP \((c_1, c_2)\) is injective where \( g > 1 \). Furthermore, we have

\[ \xi(\cdot) = \exp(\sigma(\zeta(\xi))(\cdot)) \in \text{Aut}(\widehat{S}(\Sigma, g, 1, J)) \]

for any \( \xi \in I(\Sigma, g, 1) \) and any finite subset \( J \subset \partial\Sigma_{g, 1} \).

We remark that \( \zeta(t_c) = L(c) \) for a separating simple closed curve \( c \).

Let \( e \) be an embedding \( \Sigma_{g, 1} \times [0, 1] \) satisfying the following conditions

\[ e|_{\Sigma_{g, 1} \times \{\frac{1}{2}\} : \Sigma \times \{\frac{1}{2}\} \to \Sigma, (x, \frac{1}{2}) \mapsto x,} \]

\[ e(\Sigma \times [0, \frac{1}{2}]) \subset H^+_g, \]

\[ e(\Sigma \times [\frac{1}{2}, 1]) \subset H^-_g. \]

We call this embedding a standard embedding. We denote by \( e_* \) the \( \mathbb{Q}[A + 1] \)-module homomorphism \( S(\Sigma_{g, 1}) \to \mathbb{Q}[A + 1] \) induced by \( e \). The following is our main theorem.
Theorem 3.3. The map $Z : \mathcal{I}(\Sigma_{g,1}) \to \mathbb{Q}[A + 1]$ defined by

$$Z(\xi) \overset{\text{def}}{=} \sum_{i=0}^{\infty} \frac{1}{(-A + A^{-1})i!} e_*(\xi(\xi)^i)$$

induces

$$z : \mathcal{H}(3) \to \mathbb{Q}[A + 1], M(\xi) \to Z(\xi).$$

3.2. Main theorem and its proof. The aim of this subsection is to prove Theorem 3.3.

By Proposition 3.1, the map $Z : \mathcal{I}(\Sigma_{g,1}) \to \mathbb{Q}[A + 1]$ is well-defined.

For $\epsilon \in \{+,-\}$, let $\mathcal{S}(H_g^\epsilon)$ be the quotient of $\mathbb{Q}[A,A^{-1}]T(H_g^\epsilon)$ by the skein relation and the trivial knot relation, where $T(H_g^\epsilon)$ is the set of unoriented framed link in $H_g^\epsilon$. We can consider its completion $\hat{\mathcal{S}}(H_g^\epsilon)$, for details see [14] Theorem 5.1. We denote the embedding $\iota^+ : \Sigma_{g,1} \times [0,1/2] \to H_g^+$ and the embedding $\iota^- : \Sigma_{g,1} \times [1/2,1] \to H_g^-$. The embeddings

\begin{align*}
\iota^+ : \Sigma_{g,1} \times I &\to H_g^+, (x,t) \mapsto \iota^+(x,t/2), \\
\iota^- : \Sigma_{g,1} \times I &\to H_g^-, (x,t) \mapsto \iota^+(x,(t+1)/2)
\end{align*}

induces

$$\iota^+ : \hat{\mathcal{S}}(\Sigma_{g,1}) \to \hat{\mathcal{S}}(H_g^+), \quad \iota^- : \hat{\mathcal{S}}(\Sigma_{g,1}) \to \hat{\mathcal{S}}(H_g^-).$$

By definition, we have the followings.

Proposition 3.4. (1) The kernel of $\iota^+$ is a right ideal of $\hat{\mathcal{S}}(\Sigma_{g,1})$.

(2) The kernel of $\iota^-$ is a left ideal of $\hat{\mathcal{S}}(\Sigma_{g,1})$.

(3) We have $e_*(\ker \iota^+) = \{0\}$ for $\epsilon \in \{+,-\}$.

Proposition 3.5. We have $Z(\xi_1\xi_2) = \sum_{i,j \geq 0} \frac{1}{(-A + A^{-1})i^j} e_*(\xi_1(\xi_1)^i(\xi_2)^j)$.

Lemma 3.6. (1) Let $c_a$, $c_a'$, $c_b$, and $c'_b$ be simple closed curves as in Figure 1. For $\epsilon \in \{+,-\}$, we have

$$\iota^+(\xi(L(c_a) - L(c_a'))) = 0, \quad \iota^-(\xi(L(c_b) - L(c'_b))) = 0$$

for $\xi \in \mathcal{M}(H_3^I)$.

(2) Let $c_{a,i}$, $c_{b,j}$, and $c_{i,j}$ be simple closed curves as in Figure 3 or Figure 4. For $\epsilon \in \{+,-\}$, we have

$$\iota^+(L(c_{a,i}) - L(c_{a,i})) = 0$$

for $i \neq j$.

By Lemma 2.3 in order to prove Theorem 3.3 it is enough to check the following lemmas.

Lemma 3.7. For any $i$, we have $e_* \circ h_i = e_*$. Furthermore, we have $Z(h_i\xi h_i^{-1}) = Z(\xi)$.

Proof. The embeddings $e \circ h_i$ and $e$ are isotopic. This proves the first claim. Using this, we have $e_*(\xi(h_i\xi h_i^{-1})) = e_*(\xi h_i(\xi))$. This proves the second claim. This proves the lemma.
Lemma 3.8. For any $i \neq j$, we have $e_* \circ s_{ij} = e_*$. Furthermore, we have $Z(s_{ij} \xi s_{ij}^{-1}) = Z(\xi)$.

Proof. We fix an element $x$ of $\mathcal{S}(\Sigma_{g,1})$. We have $s_{ij}(x) = \exp(\sigma(L(c_{i,j}) - L(c_{a,i}) - L(c_{b,j})))x$. Using Lemma 3.6(2) and Proposition 3.4(1)(2)(3), we have $e_*(\exp(\sigma(L(c_{i,j}) - L(c_{a,i}) - L(c_{b,j})))x) = e_*(x)$. This proves the first claim. Using this, we have $e_*(\xi(s_{ij} \xi s_{ij}^{-1})) = e_* \circ s_{ij}(\xi(\xi))$. This proves the second claim. This proves the lemma.

Lemma 3.9. (1) We have $Z(\xi \eta^+) = Z(\xi)$ for any $\xi \in \mathcal{I}(\Sigma_{g,1})$ and any $\eta^+ \in \{t_{\xi(c_{a})\xi(c_{b})}\xi \in \mathcal{M}(H^+_{g,1})\} \cup \{t_{\xi(c_{a})\xi(c_{b})}\xi \in \mathcal{M}(H^+_{g,1})\}$.

(2) We have $Z(\eta^- \xi) = Z(\xi)$ for any $\xi \in \mathcal{I}(\Sigma_{g,1})$ and any $\eta^- \in \{t_{\xi(c_{a})\xi(c_{b})}\xi \in \mathcal{M}(H^-_{g,1})\} \cup \{t_{\xi(c_{a})\xi(c_{b})}\xi \in \mathcal{M}(H^-_{g,1})\}$.

Proof. We prove only (1) (because the proof of (2) is almost the same.) By Proposition 3.5, we have $Z(\xi \eta^+) = \sum_{i,j \geq 0} \frac{1}{(-A + A^{-1})^{2ij}ij!} e_*(\xi(\xi)^{ij}(\xi(\eta^+)^i))$. Using Lemma 3.6 and Proposition 3.4(1)(3), we obtain $Z(\xi \eta^+) = \sum_{i,j \geq 0} \frac{1}{(-A + A^{-1})^{2ij}ij!} e_*(\xi(\xi)^{ij}(\eta^+)) = \sum_{i \geq 0} \frac{1}{(-A + A^{-1})^{2ij}ij!} e_*(\xi(\xi)^{ij}) = Z(\xi)$. This proves the lemma.

Proof of Theorem 3.3. By Fact 2.2 and Lemma 2.5, it is enough to check the following:

\begin{align*}
Z(h_i \xi h_i^{-1}) &= Z(\xi), \\
Z(s_{ij} \xi s_{ij}^{-1}) &= Z(\xi), \\
Z(\xi \eta^+) &= Z(\xi), \\
Z(\eta^- \xi) &= Z(\xi),
\end{align*}

for any $\xi \in \mathcal{I}(\Sigma_{g,1})$, any $\eta^+ \in \{t_{\xi(c_{a})\xi(c_{b})}\xi \in \mathcal{M}(H^+_{g,1})\} \cup \{t_{\xi(c_{a})\xi(c_{b})}\xi \in \mathcal{M}(H^+_{g,1})\}$, any $\eta^- \in \{t_{\xi(c_{a})\xi(c_{b})}\xi \in \mathcal{M}(H^-_{g,1})\} \cup \{t_{\xi(c_{a})\xi(c_{b})}\xi \in \mathcal{M}(H^-_{g,1})\}$ and any $i \neq j$. The above lemmas prove these equations. This proves the Theorem.

This invariant satisfies the following conditions.

Proposition 3.10. For $M_1, M_2 \in \mathcal{H}(3)$, we have $z(M_1 \sharp M_2) = z(M_1)z(M_2)$ where $M_1 \sharp M_2$ is the connected sum of $M_1$ and $M_2$.

Proof. Let $\iota_1 : \Sigma^1 \rightarrow \Sigma_{g,1}$ and $\iota_2 : \Sigma^2 \rightarrow \Sigma_{g,1}$ be the embedding maps as in Figure 7. The embedding maps induces $\iota_1 : M(\Sigma^1) \rightarrow M(\Sigma_{g,1})$, $\iota_2 : M(\Sigma^2) \rightarrow M(\Sigma_{g,1})$, $\iota_1 : S(\Sigma^1) \rightarrow S(\Sigma_{g,1})$, $\iota_2 : S(\Sigma^2) \rightarrow S(\Sigma_{g,2})$.\]
By Proposition 3.1, we have

\[ e_\ast(t_1(x_1)t_2(x_2)) = e_\ast(t_1(x_1))e_\ast(t_2(x_2)) \]

for \( x_1 \in S(\Sigma^1) \) and \( x_2 \in S(\Sigma^2) \).

For \( \xi_1 \in \iota_1(M(\Sigma^1)) \) and \( \xi_2 \in \iota_2(M(\Sigma^2)) \) we have

\[ z(M(\xi_1)z(M(\xi_2)) = z(M(\xi_1 \circ \xi_2)) = Z(\xi_1 \circ \xi_2) = \sum_{i=1}^{\infty} \frac{1}{(-A + A^{-1})i!} e_\ast((\zeta(\xi_1 \circ \xi_2))^i) \]

\[ = \sum_{i,j \geq 0} \frac{1}{(-A + A^{-1})^{i+j}i!j!} e_\ast((\zeta(\xi_1))^i(\zeta(\xi_2))^j) = \sum_{i,j \geq 0} \frac{1}{(-A + A^{-1})^{i+j}i!j!} e_\ast((\zeta(\xi_1))^i) e_\ast((\zeta(\xi_2))^j) \]

\[ = (\sum_{i=0}^{\infty} \frac{1}{(-A + A^{-1})^i} e_\ast((\zeta(\xi_1))^i))(\sum_{j=0}^{\infty} \frac{1}{(-A + A^{-1})^j} e_\ast((\zeta(\xi_2))^j)) = z(M(\xi_1))z(M(\xi_2)). \]

This proves the proposition.

\[ \square \]

**Proposition 3.11.** For \( \xi_1 \in \zeta^{-1}(F^{n_1+2}\mathcal{S}(\Sigma_{g,1})) \), \( \cdots \), \( \xi_k \in \zeta^{-1}(F^{n_k+2}\mathcal{S}(\Sigma_{g,1})) \), we have

\[ \sum_{\epsilon_i \in \{1,0\}} (-1)^{\sum \epsilon_i} z(M(\xi_1^{\epsilon_1} \cdots \xi_k^{\epsilon_k})) \in (A + 1)^{(n_1 + \cdots + n_k + 1)/2} Q[[A + 1]]. \]

We remark that \( \zeta^{-1}(F^d\mathcal{S}(\Sigma_{g,1})) \) equals \( \mathcal{I}(\Sigma_{g,1}) \) and that \( \zeta^{-1}(F^d\mathcal{S}(\Sigma_{g,1})) \) equals the Johnson kernel.

**Proof.** We have

\[ \sum_{\epsilon_i \in \{1,0\}} (-1)^{\sum \epsilon_i} z(M(\xi_1^{\epsilon_1} \cdots \xi_k^{\epsilon_k})) \]

\[ = e_\ast((1 - \exp(\frac{\zeta(\xi_1)}{-A + A^{-1}})) \cdots (1 - \exp(\frac{\zeta(\xi_k)}{-A + A^{-1}}))). \]

By Proposition 3.1, we have

\[ \sum_{\epsilon_i \in \{1,0\}} (-1)^{\sum \epsilon_i} z(M(\xi_1^{\epsilon_1} \cdots \xi_k^{\epsilon_k})) \in (A + 1)^{(n_1 + \cdots + n_k + 1)/2} Q[[A + 1]]. \]

This proves the proposition.

\[ \square \]
Corollary 3.12. The invariant $z(M) \in \mathbb{Q}[[A+1]]/(A+1)^{n+1}$ is a finite type invariant for $M \in \mathcal{H}(3)$ order $n$.

Proposition 3.13. The invariant $z(M) \in \mathbb{Q}[[A+1]]/(A+1)^{n+1}$ is a nontrivial finite type invariant for $M \in \mathcal{H}(3)$ order $n$.

Proof. It is enough to show that there exists $\xi_1, \cdots, \xi_n \in K(\Sigma_g, 1)$ such that

$$\sum_{\epsilon_i \in \{0,1\}} (-1)^{\sum \epsilon_i} M(\prod_{i=1}^{n} \xi_i^{\epsilon_i}) \neq 0 \mod ((A+1)^{n+1}).$$

Let $c_1, c_2, \cdots, c_L$ be simple closed curves as in Figure 8. We denote $c_i \overset{\text{def}}{=} t_{c_i} \circ t_{c_i}(c_L)$ and $t_i \overset{\text{def}}{=} t_{c_i}$ for $i = 1, \cdots, n$. We remark that $M(t_1 \circ \cdots \circ t_k) = \#^k M(t_1)$ for $1 \leq t_1 < \cdots < t_k \leq n$ and that $M(t_1)$ is the Poincaré homology 3-sphere. By the computation in section 4 and Proposition 3.10

$$\sum_{\epsilon_i \in \{0,1\}} (-1)^{\sum \epsilon_i} M(\prod_{i=1}^{n} t_i^{\epsilon_i}) = (1 - z(M(t_1)))^n = 6^n (A^4 - 1)^n \mod ((A+1)^{n+1}).$$

This proves the proposition. \hfill \Box

By the computation in section 4 the coefficient of $A^4 - 1$ in the invariant $z$ for the Poincaré homology 3-sphere is $-6$. Since the casson invariant is the unique nontrivial finite type invariant of order 1 up to a scalar, we have the following.

Corollary 3.14. For any $M \in \mathcal{H}(3)$, we have the coefficient of $A^4 - 1$ in $z(M)$ is $(-6)$ times the Casson invariant.

4. Example

Let $c_1$, $c_2$ and $c_L$ be simple closed curves in $\Sigma_{g,1}$ as in Figure 9. We consider integral homology 3-spheres $M(c_1, c_2, c_3) \overset{\text{def}}{=} M((t_1 t_2 t_3)^{\epsilon_3})$ for $\epsilon_1, \epsilon_2, \epsilon_3 \in \{\pm 1\}$, where $t_1 \overset{\text{def}}{=} t_{c_1}$ and $t_2 \overset{\text{def}}{=} t_{c_2}$. For $\epsilon \in \{\pm 1\}$, the manifold $M(1, -1, \epsilon)$ is the integral homology 3-sphere obtained from $S^3$ performing the $\epsilon$-surgery on the figure eight knot 41, which is $e(t_1^{-1} \circ t_2(c_L))$. For $\epsilon \in \{\pm 1\}$, the
manifold $M(1, 1, \epsilon)$ is the integral homology 3-sphere obtained from $S^3$ performing the $\epsilon$-surgery on the trefoil knot $3_1$, which is $e(t_1 \circ t_2(c_L))$. For $\epsilon \in \{ \pm 1 \}$, the manifold $M(-1, -1, \epsilon)$ is the integral homology 3-sphere obtained from $S^3$ performing the $\epsilon$-surgery on the mirror $-3_1$ of the trefoil knot, which is $e(t_1^{-1} \circ t_2^{-1}(c_L))$. In particular, $M(1, 1, 1)$ is the Poincaré homology sphere. We remark that we compute the computations using Habiro’s formula [4] for colored Jones polynomials of the trefoil knot and the figure eight knot, we have the following. We remark that we compute $z(M(1, -1, 1)) = z(M(1, 1, -1))$ (rep. $z(M(1, -1, -1)) = z(M(-1, -1, 1))$) by two ways $Z(t_{t_1 ot_2^{-1}}(c_L)) = Z((t_{t_1 ot_2^{-1}}(c_L))^{-1})$ (resp. $Z((t_{t_1 ot_2^{-1}}(c_L))^{-1}) = Z(t_{t_1^{-1} ot_2^{-1}}(c_L))$.

**Proposition 4.1.** We have

$$z(M(1, 1, 1)) = [1, -6, 45, -464, 6224, -102816, 2015237, -45679349, 1175123730, -33819053477, 1076447743008, -37544249290614, 142385123295885, -58335380481272941],$$

$$z(M(1, -1, 1)) = z(M(1, 1, -1)) = [1, 6, 63, 932, 17779, 145086, 11461591, 365340318, 13201925372, 533298919166, 23814078531737, 1164804017792623, 61932740213389942, 3556638330023177088],$$

$$z(M(-1, -1, -1)) = z(M(1, -1, -1)) = [1, -6, 39, 380, 4961, 80530, 1558976, 35012383, 894298109, 25591093351, 81078512236, 28169720107881, 1064856557864671, 43506118043043092],$$

$$z(M(1, -1, 1)) = z(M(-1, -1, -1)) = [1, -6, 69, -1064, 20770, -492052, 13724452, -440706098, 16015171303, -649815778392, 29121224693198, -1428607184648931, 76147883907835312, -438222160786508572].$$

Here we denote

$$1 + a_1(A^4 - 1) + a_2(A^4 - 1)^2 + \cdots + a_{13}(A^4 - 1)^{13} + o(14) = [1, a_1, a_2, \cdots, a_{13}]$$

where $o(14) \in (A + 1)^{14}\mathbb{Q}[[A + 1]].$

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Fig 9. simple closed curves $c_1$, $c_2$ and $c_L$

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