Chiral Fermions on the Lattice

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Abstract. In the last century the non-perturbative regularization of chiral fermions was a long-standing problem. We review how this problem was finally overcome by the formulation of a modified but exact form of chiral symmetry on the lattice. This also provides a sound definition of the topological charge of lattice gauge configurations. We illustrate a variety of applications to QCD in the p-, the $\varepsilon$- and the $\delta$-regime, where simulation results can now be related to Random Matrix Theory and Chiral Perturbation Theory. The latter contains Low Energy Constants as free parameters, and we comment on their evaluation from first principles of QCD.

Keywords: lattice regularization, chiral symmetry, Random Matrix Theory, Chiral Perturbation Theory, topological susceptibility

PACS: 11.15.Ha, 11.30.Rd, 12.38.Gc, 12.39.Fe, 14.40.Be

CHIRAL SYMMETRY

Chiral Perturbation Theory. Fermion fields can be decomposed into a left- and a right-handed component by means of the chiral projectors, $\Psi_{L,R} = \frac{1}{2}(1 \pm \gamma_5)\Psi$, $\bar{\Psi}_{L,R} = \frac{1}{2}(1 \mp \gamma_5)\bar{\Psi}$. In massless (bilinear) theories these two spinor components decouple. In particular the QCD Lagrangian at zero quark masses takes the structure

$$\mathcal{L}_{\text{QCD}} = \bar{\Psi}_L D\Psi_L + \bar{\Psi}_R D\Psi_R + \mathcal{L}_{\text{gauge}},$$

where $D$ is the Dirac operator, and the quark fields $\bar{\Psi}$, $\Psi$ capture the $N_f$ flavors and 3 colors involved. This $\mathcal{L}_{\text{QCD}}$ is invariant under global $U(N_f)$ transformations of the quark fields, which can be performed independently in the left- and right-handed sector. The two complex phases represent baryon number conservation and an axial symmetry, which breaks under quantization (axial anomaly). One assumes the remaining chiral flavor symmetry to break spontaneously, reducing the symmetry to a unbroken group of simultaneous transformations in both sectors,

$$SU(N_f)_L \otimes SU(N_f)_R \rightarrow SU(N_f)_{L+R}.$$  \hspace{1cm} (2)

Chiral Perturbation Theory ($\chi$PT) deals with a field in the corresponding coset space, $U \in SU(N_f)$, which dominates the low energy behavior. As we add small quark masses $m_q$ to $\mathcal{L}_{\text{QCD}}$ (which is allowed, since QCD is a vector theory), $U$ represents $N_f^2 - 1$ light quasi-Nambu-Goldstone bosons, which are identified with the lightest mesons involved. For simplicity we consider only two (degenerate) flavors, $u$ and $d$, so that the field $U$
represents the pion triplet. \(^1\) \(\chi\)PT now uses an effective Lagrangian of the form \([2]\)

\[
\mathcal{L}_{\text{eff}}[U] = \frac{F_{\pi}^2}{4} \text{Tr} [\partial_\mu U^\dagger \partial_\mu U] - \frac{\Sigma m_q}{2} \text{Tr} [U + U^\dagger] + \ldots
\]  

(3)

where the dots represent terms with more derivatives and/or higher powers of the explicit symmetry breaking parameter \(m_q\). All terms, which are compatible with the symmetries, are put into an energetic hierarchy; eq. (3) displays the leading terms. Each term comes with a coefficient, which is denoted as a *Low Energy Constant* (LEC), such as the *pion decay constant* \(F_\pi\) (which can be measured experimentally) and the *chiral condensate* \(\Sigma\) (the order parameter of chiral symmetry breaking).

As we have seen, \(\chi\)PT does have a direct link to QCD, hence it describes low energy hadron physics in a way manifestly related to the fundamental theory (in contrast to many other effective approaches). However, the LECs are free parameters in \(\chi\)PT; in this sense the low energy description is incomplete. If we manage to determine LECs directly from QCD, we obtain a more complete low energy theory. This is obviously a non-perturbative task, and therefore a challenge for *lattice QCD*: its simulation in Euclidean space is the only method to tackle QCD (and other quantum fields theories) systematically beyond perturbation theory.

**Lattice fermions.** The lattice discretization of the gluon fields is conceptually unproblematic: the gauge action can be expressed in terms of small Wilson loops in a gauge invariant way. It has been a longstanding issue, however, to formulate lattice fermions such that they keep track of (approximate) chiral symmetry. For one flavor, the standard chirality condition is given by the anti-commutator \(\{D, \gamma_5\} = 0\). The “naïve” discretization of the Dirac operator yields for a free, massless fermion in momentum space the form \(D_n(p) = i\gamma_\mu \sin p_\mu\) (in lattice units, *i.e.* if we set the lattice spacing \(a = 1\)). It is chirally symmetric, but it gives rise to \(2^d - 1\) artificial poles of the propagator (inside the first Brillouin zone), in addition to the physical one at \(p = 0\) (in \(d\) dimensional Euclidean space). The Nielsen-Ninomiya Theorem states essentially that chirality and locality inevitably entail fermion doublers, which would distort the result in lattice studies \([3]\). (Here “locality” means that the coupling between \(\bar{\Psi}_x\) and \(\Psi_y\) falls off at least exponentially in \(|x - y|\); this assures a safe continuum limit).

K. Wilson subtracted a discrete Laplacian \(\Delta\) to construct the Wilson Dirac operator \([4]\), \(D_W = D_n - \frac{1}{2} \Delta\). It is still local, and it sends the doubler masses to the cutoff scale, as desired. However, the additional term breaks chiral symmetry explicitly. Under gauge interaction it leads to (highly undesired) additive mass renormalization. Thus the chiral limit can only be approximated by a tedious fine-tuning of a negative bare quark mass.

Conceptual progress was achieved at the end of the last century by deviating from the continuum form of chiral symmetry in a specifically harmless way: instead of inserting a local term for \(\{D, \gamma_5\}\) (as Wilson did), one now does so for \(\{D^{-1}, \gamma_5\}\) — this does not

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\(^1\) In this case, the symmetry breaking pattern (2) is locally isomorphic to \(O(4) \rightarrow O(3)\). This property is very specific: among all conceivable types of chiral symmetry breaking only very few can be expressed by orthogonal Lie groups \([1]\).
shift the poles in the propagator — e.g. by setting
\[ \{D^{-1}, \gamma_5\} = \gamma_5 \Rightarrow \{D, \gamma_5\} = D\gamma_5D. \] (4)

This is now known as the (simplest form of the) Ginsparg-Wilson Relation (GWR). Lattice Dirac operators with this structure are generated by block spinor Renormalization Group Transformations as the blocking factor diverges (perfect fermion action) [5, 6, 7], or by Domain Wall Fermions [8] (with the chiral modes attached to two domain walls, which are pulled apart in an extra “dimension”). By integrating out this extra “dimension” one obtains the overlap fermion [9]. The latter can easily be re-derived from the GWR (4): assume \( D \) to be \( \gamma_5 \)-Hermitian, \( D^\dagger = \gamma_5D\gamma_5 \) (which holds e.g. for \( D_W \)), so that the GWR can be written as \( D + D^\dagger = D^\daggerD \). This corresponds to the condition that \( A := D - 1 \) be unitary. For \( A_W = D_W - 1 \) this is not the case, but we can enforce it by the transformation
\[ A_{ov} = A_W / \sqrt{\lambda_{A_W}} , \quad D_{ov} = A_{ov} + 1 , \] (5)
which yields the “overlap Dirac operator” \( D_{ov} \) as a solution to the GWR [10]. Its spectrum is located on a circle in the complex plane, \( |\lambda - 1| = 1 \), which reveals the absence of additive mass renormalization. (This also holds for the generalization to \( A = D - \rho, |\lambda - \rho| = \rho, \rho \gtrsim 1 \), which has practical advantages in the interacting case.)

The overlap operator is manifestly local as long as the gauge background is sufficiently smooth [11]. This property gets lost on very coarse lattices (a \( \sim 0.17 \) fm), but if we simulate in the safe regime, a continuum extrapolation can be taken.

Zero modes of a Ginsparg-Wilson Dirac operator are exact and they have a definite chirality. Hence the topological charge \( \nu \) of a gauge configuration can be defined [12] by adapting the Atiyah-Singer Index Theorem from the continuum. Thus \( \nu \) is defined as the difference between the number of zero modes with positive and negative chirality. (In actual Monte Carlo generated configurations only zero modes with one chirality occur.) In contrast to the formulation using \( D_W \), operator mixing (on the regularized level) is under control [13]. This is very helpful for numerical measurements, for instance if one performs a fully non-perturbative Operator Product Expansion [14].

As a reason for these fantastic properties, M. Lüscher pointed out that the Lagrangian is actually invariant under a lattice modified, chiral transformation with the infinitesimal form [15]
\[ \Psi \rightarrow \Psi(1 + \epsilon\gamma_5[1 - \frac{1}{2}D])\psi , \quad \Psi \rightarrow (1 + \epsilon\gamma_5[1 - \frac{1}{2}D])\psi \] (to \( O(\epsilon) \)). (6)

In the continuum limit it turns into the standard chiral transformation, \( \Psi \rightarrow \Psi(1 + \epsilon\gamma_5)\psi, \Psi \rightarrow (1 + \epsilon\gamma_5)\psi \). However, the fermionic Lagrangian \( \bar{\Psi}D\Psi \) is invariant to \( O(\epsilon) \) even on the lattice, if the GWR holds. On the other hand, the functional measure \( D\bar{\Psi}D\Psi \) is not invariant, which is exactly what it takes to reproduce the axial anomaly correctly [15, 16].

For applications also further properties — beyond chirality — matter. This motivates the substitution of the Wilson kernel \( D_W \) in the overlap formula (5) by an extended “hypercube fermion”, which is constructed from Renormalization Group Transformations and truncations [17]; this yields improved locality and scaling, as well as approximate rotation symmetry [18, 19]. In particular locality now persists on coarser lattices, which is profitable in studies of QCD at finite temperature [20].
APPLICATIONS TO TOPOLOGY AND \( p \), \( \varepsilon \) - AND \( \delta \)-REGIME

We mentioned that the topological charge is well-defined when we deal with chiral lattice fermions (although all lattice gauge configurations can be continuously deformed into one another). This enables a sound numerical measurement of the topological susceptibility \( \chi_t = \langle \nu^2 \rangle / V \), where \( \nu \) is the (aforementioned) topological charge, and \( V \) is the space-time volume. A high statistics study was presented in Refs. [21], which is very well compatible with our results [19], see Figure 1 (left). The continuum extrapolation amounts to \( \chi_t = (191(5) \text{ MeV})^4 \). This supports the Witten-Veneziano scenario [22], which explains the heavy mass of the \( \eta' \)-meson in part as a topological effect. This conjecture involves indeed the quenched value of \( \chi_t \). Latest direct measurements of \( m_\eta \) and \( m_{\eta'} \) were performed with 2+1 flavors of dynamical Domain Wall quarks [23].

\( \chi^\text{PT} \) has been formulated in different regimes depending on the volume, which affects the counting rules for the energy hierarchy.

**\( p \)-regime.** The standard setting, where finite size affects are small, is denoted as the \( p \)-regime [24]: \( L \gg 1/m_\pi \) (\( L \) is the 4d box length, and the inverse pion mass is the correlation length). Figure 1 (right) shows a measurement of \( F_\pi \) that we performed (quenched) on a \( 12^3 \times 24 \) lattice, where the gauge coupling was chosen such that the lattice spacing corresponds to \( a \simeq 0.123 \) fm. The pion mass (and other hadron masses) can be measured from the exponential decay of correlation functions. Over a broad range, the measured value of \( F_\pi \) is clearly too high, but just at our lowest pion mass, \( m_\pi = 279(32) \) MeV, the right trend sets in, \( i.e. \) a decrease towards the value in Nature \( (F_\pi = 92.4 \text{ MeV}) \). This calls for clarification with yet lighter pion masses, still closer to its phenomenological value of 135 MeV. However, at our lowest data point we already have \( Lm_\pi \approx 2 \), hence at even smaller \( m_\pi \) we are certainly outside the \( p \)-regime. Recovering it takes a much larger volume, and therefore much more computational effort.

**\( \varepsilon \)-regime.** As an alternative, we can simulate QCD in the \( \varepsilon \)-regime [25], where \( Lm_\pi < 1 \). This setting is unphysical — experimentally we can’t squeeze pions into such a tiny box. However, the finite size effects can be computed by \( \chi^\text{PT} \), and they are parameterized by \( \text{LECs as they occur in infinite volume} \). Hence we can extract physical results nevertheless, avoiding the apparent quest for a huge volume [26]. In the \( \varepsilon \)-regime the topological sectors play an essential rôle.

Chiral Random Matrix Theory (RMT) provides a prediction for the density of the lowest non-zero Dirac eigenvalues \( \lambda_i (i = 1, 2, 3 \ldots) \) in the \( \varepsilon \)-regime [27]. More precisely it predicts the densities of the dimensionless variables \( z_i = \lambda_i \Sigma V \). If our data match the predicted shape, we can tune \( \Sigma \) for optimal agreement, and in this way evaluate \( \Sigma \). Figure 2 (left) shows the cumulative densities for \( z_1 \) as predicted by RMT in the sectors of topological charge \( |\nu| = 0, 1 \) and 2 (curves). Our data points, obtained in \( V \simeq (1.23 \text{ fm})^4 \), are in excellent agreement, if we insert \( \Sigma \simeq (253 \text{ MeV})^3 \) [28]. Also this result was obtained in the quenched approximation (which neglects sea quark contributions), but our result showed for the first time that this method to measure \( \Sigma \) is in fact applicable with chiral fermions in various topological sectors — see also Refs. [29, 19]. Recent
FIGURE 1. Left: The topological susceptibility of quenched QCD, measured with the index of chiral quarks. Our results [19] are consistent with the continuum limit of Refs. [21], which supports the Witten-Veneziano scenario for $m_{\eta'}$. Right: $F_\pi$ measured by two methods (straight pseudoscalar density correlator, and subtraction of scalar density correlator) at different pion masses in the $p$-regime (at $a \approx 0.123$ fm). For our lightest pion, $m_\pi = 279(32)$ MeV, a trend towards the value in Nature sets in [19].

FIGURE 2. Left: The cumulative density of the dimensionless variable $z_1 = \lambda_1 \Sigma V$, where $\lambda_1$ is the leading non-zero Dirac eigenvalue. We compare our data for $D_{ov}$ eigenvalues in the $\epsilon$-regime (mapped stereographically onto $\mathbb{R}^+\mathbb{R}I^+$) to RMT predictions (lines) in the sectors with topological charge $|\nu| = 0, 1$ and 2. We obtain very good agreement if we insert $\Sigma = (253$ MeV)$^3$ [28]. Right: The residual pion mass in the $\delta$-regime as a function of the spatial box size $L_s$. The chiral extrapolation of our measured pion masses (with dynamical Wilson quarks) [40] follows closely the theoretical prediction of $\chi$PT [38].

studies with dynamical quarks [30] obtain with the same method a very similar value, $\Sigma = (251(7)$ MeV)$^3$ (renormalized in the $\overline{MS}$ scheme at 2 GeV).

However, simulations of dynamical overlap fermions are not only computationally very expensive (the inverse square root in eq. (5) has to be computed by polynomials up to degree $O(100)$ in order to attain chirality close to machine precision), but they also face conceptual problems: the standard algorithm for dynamical quarks (“Hybrid Monte Carlo”) changes the topological sector only very rarely, so that direct measurements of full observables are difficult. Measurements can be performed in fixed topological sectors. A method to derive from them an approximate result for the physical value (properly summed over all sectors) has been suggested in Ref. [31] and tested successfully in the 2-flavor Schwinger model [32]. The tremendous efforts to simulate QCD with dynamical overlap quarks are reviewed in Ref. [33]. In particular $\chi_t$ is hard to measure in this case; for an indirect method we refer to Refs. [34].
We add that also $F_\pi$ can be evaluated in the $\epsilon$-regime, in particular by matching measured correlators \cite{35, 19} to $\chi$PT predictions, or by considering only their zero-mode contributions \cite{36, 19}.

**$\delta$-regime.** Let us finally mention yet a third regime where $\chi$PT has been worked out, namely the $\delta$-regime \cite{37}. Here the Euclidean time extent is long, but the 3d spatial volume, say $L_s^3$, is small ($L_s < 1/m_\pi$). This prevents spontaneous symmetry breaking, hence even at vanishing quark mass — where we can refer to the current quark mass measured through the PCAC relation — the pion mass remains finite. The formula for the residual pion mass $m_\pi^{\text{res}}(L_s)$ in the chiral limit has been computed recently to next-to-next-to-leading (NNL) order \cite{38}.

We performed p-regime measurements of $m_\pi$ in spatial volumes in the range $L_s \simeq (1.6...3.0)$ fm, and extrapolated the pion masses to the chiral limit \cite{40}. The results are in remarkably good agreement with the formula for $m_\pi^{\text{res}}(L_s)$. The latter involves $F_\pi$ again, so in principle this is yet another way to measure a physical LEC in a unphysical regime (although in our case we already used $F_\pi$ for the extrapolation).

Moreover the NNL order also involves sub-leading LECs of $\chi$PT (coefficients to terms symbolized with dots in eq. (3)). Hence from precision results for $m_\pi^{\text{res}}(L_s)$ in the $\delta$-regime one could determine even sub-leading LECs. That is useful in particular for the LEC denoted as $\bar{l}_3$ \cite{40}, the value of which is quite uncertain \cite{41}.

**CONCLUSIONS**

Chiral fermions can be regularized on the lattice such that they obey a lattice modified version of chiral symmetry. Therefore they are now well-defined and tractable non-perturbatively, at least in vector theories. Thus the existence of light quarks — with masses far below $\Lambda_{\text{QCD}}$ — is not that mysterious anymore (for reviews, see e.g. Refs. \cite{42}). This formulation also provides a sound definition of a topological charge, which enables a neat measurement of the topological susceptibility in quenched QCD. The results support the Witten-Veneziano conjecture about the $\eta'$-mass.

We sketched applications of chiral lattice fermions in the p-regime, where finite size effects are small. Here we can measure for instance the light hadron spectrum as well as the PCAC quark mass, and $F_\pi$ — one of the leading LECs in the $\chi$PT Lagrangian. The LEC determination from the underlying theory (QCD) improves the status of $\chi$PT as a description of the low energy hadronic world in a way linked to first principles.

In the $\epsilon$-regime we discussed the measurement of $\Sigma$ — the other leading LEC — by relating the microscopic Dirac spectrum to Random Matrix Theory. Also here, and in the $\delta$-regime, $F_\pi$ can be measured, i.e. it is possible to obtain physical results even from unphysically small volumes. Moreover the measurement of the residual pion mass in the $\delta$-regime even has the potential to determine sub-leading LECs.

**Acknowledgments:** I thank my collaborators in the works summarized here, and the organizers of the pleasant workshop in Mazatlán.

\footnote{This is very different from the p-regime, where finite size effects are suppressed exponentially \cite{39}.}
