Stable First-order Particle-frame Relativistic Hydrodynamics for Dissipative Systems

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We propose a stable first-order relativistic dissipative hydrodynamic equation in the particle frame (Eckart frame) for the first time. The equation to be proposed was in fact previously derived by the authors and a collaborator from the relativistic Boltzmann equation. We demonstrate that the equilibrium state is stable with respect to the time evolution described by our hydrodynamic equation in the particle frame. Our equation may be a proper starting point for constructing second-order causal relativistic hydrodynamics, to replace Eckart’s particle-flow theory.

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Relativistic hydrodynamics (RHD) is a useful tool for analyzing slow and long wavelength behavior of relativistic many-particle systems in terms of static and dynamical thermodynamic properties. In fact, RHD is widely used in astrophysics \cite{1} and the phenomenology of relativistic heavy ion collisions \cite{2, 3, 4}. Since works demonstrating the success of perfect hydrodynamics in describing the phenomenology of the Relativistic Heavy Ion Collider (RHIC) at BNL \cite{2, 3, 4}, we are witnessing a growing interest in RHD for relativistic many-particle systems in terms of static and dynamic thermodynamic properties. In fact, RHD is widely used in astrophysics \cite{1} and the phenomenology of relativistic heavy ion collisions \cite{2, 3, 4}. Since works demonstrating the success of perfect hydrodynamics in describing the phenomenology of the Relativistic Heavy Ion Collider (RHIC) at BNL \cite{2, 3, 4}, we are witnessing a growing interest in RHD for relativistic many-particle systems in terms of static and dynamic thermodynamic properties. In fact, RHD is widely used in astrophysics \cite{1} and the phenomenology of relativistic heavy ion collisions \cite{2, 3, 4}. Since works demonstrating the success of perfect hydrodynamics in describing the phenomenology of the Relativistic Heavy Ion Collider (RHIC) at BNL \cite{2, 3, 4}, we are witnessing a growing interest in RHD for relativistic many-particle systems in terms of static and dynamic thermodynamic properties.

In relativistic many-particle systems, the resulting energy density and energy flow are independent of the specific form of the hydrodynamic equation. The so-called thermodynamic properties of a relativistic system can be determined by requiring the energy density and energy flow to be independent of the specific form of the hydrodynamic equation. The so-called thermodynamic properties of a relativistic system can be determined by requiring the energy density and energy flow to be independent of the specific form of the hydrodynamic equation. The so-called thermodynamic properties of a relativistic system can be determined by requiring the energy density and energy flow to be independent of the specific form of the hydrodynamic equation. The so-called thermodynamic properties of a relativistic system can be determined by requiring the energy density and energy flow to be independent of the specific form of the hydrodynamic equation. The so-called thermodynamic properties of a relativistic system can be determined by requiring the energy density and energy flow to be independent of the specific form of the hydrodynamic equation.

We identify the following three fundamental problems regarding relativistic hydrodynamic equations (RHDEs) for dissipative fluids \cite{11}: (a) ambiguities in the forms of the equations \cite{3, 7, 10, 12, 13, 14}; (b) the unphysical instability of the equilibrium state in the theory of the so-called first-order equations, in particular in the Eckart frame \cite{12}, defined below; (c) the lack of causality in the first-order equations \cite{14, 16, 17, 18}. The present paper is concerned with the first two problems. Although the unphysical instability of the equilibrium state may be attributable to the lack of causality, and the Israel-Stewart equations with second-order time-derivative are presently being examined in connection to this problem \cite{3, 7, 10, 12}, we emphasize that the first two problems and the third one have different origins, and the first two must be resolved before the third is addressed. Note that the causality problem also exists in non-relativistic cases and is in essence a problem of how to incorporate the space-time scales shorter than those corresponding to the mean-free-path, beyond those in the usual hydrodynamic regime. We also remark that the proper form of Israel-Stewart-type equations has not yet been definitely determined \cite{3, 7}. Let us represent the flow velocity by $u^\mu$, with $u_\mu u^\mu = g^{\mu\nu} u_\mu u_\nu = 1$ ($g^{\mu\nu} = \text{diag}(+1, -1, -1, -1)$). In the relativistic theory, the rest frame of the fluid and the flow velocity $u^\mu$ cannot be uniquely defined when there exist viscosity and heat conduction. In the phenomenological theories \cite{10, 12}, the ambiguity of the flow velocity $u^\mu$ is resolved by placing constraints on the dissipative part of the energy-momentum tensor, $\delta T^{\mu\nu}$, and the particle current, $\delta N^\mu$. Landau and Lifshitz defined $u^\mu$ such that there is no dissipative energy density, energy flow nor particle density; i.e., we have the constraints $\delta T^{\mu\nu} u_\nu = 0$ (referred to as ET) and $u_\mu \delta N^\mu = 0$ (EN). This frame is called the energy frame. Contrary to Eckart, since Eckart chose the particle frame, in which there is no dissipative contribution to the particle current; i.e., we have $\delta N^\mu = 0$ (PN), together with $u_\mu \delta T^{\mu\nu} = 0$ (PT): These conditions imply that there is no dissipative contribution to the energy density in this frame. However, it should be noted that the seemingly plausible constraint PT on $\delta T^{\mu\nu}$ is problematic, as shown in \cite{11} and explained below.

Recently, Tsumura, Kunihiro (the present authors) and Ohnishi (abbreviated as TKO) \cite{11} derived generic covariant hydrodynamic equations for a viscous fluid through a reduction of the dynamics described by the relativistic Boltzmann equation in a systematic manner, with no heuristic arguments, on the basis of the so-called renormalization group (RG) method \cite{15, 20, 21}. This was done by introducing the macroscopic frame vector $\alpha^\mu$ that defines the macroscopic Lorentz frame, in which the slow dynamics are described. The generic equation derived by TKO can produce a relativistic dissipative hydrodynamic equation in any frame with the appropriate choice of $\alpha^\mu$: the resulting equation in the energy frame coincides with that of Landau and Lifshitz \cite{12}, while that in the particle frame is similar to, but slightly different from, the Eckart equation. Interestingly, the TKO equation in the particle frame does not satisfy the constraints PT on $\delta T^{\mu\nu}$ but, instead, satisfies $\delta T^{\mu}_{\mu} = 0$, which we call PT', together with PN. It should be noted that the new constraints, PT', are identical to a matching condition postulated by Marle and Stewart (MS) in the derivation of the RHD from the Boltzmann equation with use of Grad’s moment theory \cite{22}. We call the constraints PT', together with PN, the Grad-Marle-Stewart...
(GMS) constraints. In [11], TKO proved that the simultaneous constraints PT and PN cannot be compatible with the underlying Boltzmann equation if the hydrodynamic equation describes the slow, long wavelength limit of the solutions of the Boltzmann equation. This is interesting in connection to problem (b), i.e., the fact that the solutions of the Eckart equation around the thermal equilibrium are unstable [15], while the Landau theory is stable.

An immediate question is whether the solutions of the new equations in the particle frame are stable around the thermal equilibrium. In fact, the hydrodynamic equations of MS and TKO in the particle frame are of different forms, although both satisfy the constraints PT and PN. In the present paper, we examine the stability problem for the new equations in the particle frame. Because second-order equations, such as the Israel-Stewart equations, are usually constructed in the particle frame, as an extension of the Eckart equation, finding a stable first-order equation in the particle frame is of fundamental significance. As the RG method has been employed to construct the slow dynamics of various systems through the explicit construction of the slow, stable manifold of the Boltzmann equation on the basis of the RG method, we believe that the MS equations are usually constructed in the particle frame, as an answer to problem (b), i.e., the fact that the solutions of MS and TKO in the particle frame are of different forms, although both satisfy the constraints PT and PN.

The existence condition of a solution reads \( \text{det}(\beta M_{1\beta}) = 0 \). Here we note that the independent variables are the five quantities \( \delta \mu(x), \delta \mu(x), \delta u(x), \delta \mu(x), \delta \mu(x) \). The solution is given by \( \delta u(x) = 0 \), due to the constraint \( u_{\mu}(x) \).

In terms of the Fourier components \( \tilde{\Phi}_\alpha(k) \equiv \{ (\delta \mu(k), \delta u^2(k), \delta u^3(k), \delta T(k), \delta \mu(k)) \}, \) defined through \( \Phi(x) = \int \frac{dk}{2 \pi} \tilde{\Phi}_\alpha(k) e^{i k \cdot x} \), the linearized hydrodynamic equation reduces to the algebraic equation \( \sum_{\beta=1}^5 M_{\alpha \beta} \Psi = 0 \), with

\[
M_{\alpha \beta} \equiv \begin{pmatrix}
L_1 & 0 & 0 & 0 & 0 \\
0 & L_2 & 0 & 0 & 0 \\
0 & -i L_5 & k^3 & i L_4 & k^3 \\
0 & -i L_5 & k^3 & L_6 & L_7 \\
0 & -i L_5 & k^3 & L_9 & L_{10}
\end{pmatrix},
\]

where we have set \( k^\mu = (k^0, 0, 0, k^3) \) without loss of generality. The first and second components of \( \tilde{\Phi}_\alpha \) describe the transverse mode, while the third component the longitudinal one. Here \( L_{1\sim 10} \) are given by \( L_1 \equiv (\epsilon + p)(-i k^0) + \eta |k|^2 \), \( L_2 \equiv -\eta \gamma/3 - \epsilon \), \( L_3 \equiv \partial \Phi/\partial T \), \( L_4 \equiv \partial \Phi/\partial \mu \), \( L_5 \equiv (-i k^0) \), \( L_6 \equiv \partial \Phi/\partial \mu \), \( L_7 \equiv \partial \Phi/\partial \mu \), \( L_8 \equiv -n \), \( L_9 \equiv \partial \Phi/\partial T (-i k^0) \), and \( L_{10} \equiv \partial \Phi/\partial \mu (-i k^0) \), with \( \epsilon \gamma \equiv \gamma (3 \gamma - 4)^{-2} \) being the effective bulk viscosity in the particle frame. In the above, the quantities \( \epsilon, p, n, \eta, \gamma, \lambda, \eta \) are all the same as in the Landau equation and opposite that in the MS equation. We can trace the two characteristic features of our theory back to the simple ansatz that only the spatial inhomogeneity, over distances of the order of the mean free path, is the origin of the dissipation. It should be noted that the same ansatz for the non-relativistic case leads naturally to the Navier-Stokes equation, as shown in [21], and hence our framework can be interpreted as the most natural covariantization of the non-relativistic case.

The thermal equilibrium state is given by \( u^\mu(x) = (1, 0, 0, 0) \equiv u_0^\mu, T(x) = T_0 \) and \( \mu(x) = \mu_0 \), with \( T_0 \) and \( \mu_0 \) being constant. This is a trivial solution to the equations. Let us investigate the linear stability of the equilibrium solution. Writing \( T(x) = T_0 + \delta T(x), \mu(x) = \mu_0 + \delta \mu(x) \) and \( u^\mu(x) = u_0^\mu + \delta u^\mu(x), \) we examine the time evolution of the deviations in the linear approximation using the evolution equation given by \( \partial_\tau T^\mu = 0 \) and \( \partial_\tau N^\mu = 0 \). Here we note that the independent variables are the five quantities \( \delta \mu(x), \delta \mu(x), \delta u(x), \delta \mu(x), \delta \mu(x) \). The solution is given by \( \delta u(x) = 0 \), due to the constraint \( u_{\mu}(x) \).

We see the dispersion relation for the transverse mode reads

\[
L_2 \left[ (L_1 - |k|^2 L_3)(L_6 L_{10} - L_7 L_9) - |k|^2 L_5 (L_3 L_{10} - L_4 L_9) - |k|^2 L_8 (L_4 L_6 - L_3 L_7) \right] \equiv 0.
\]

This equation gives the dispersion relation \( k^0 = k^0(|k|) \) for the hydrodynamic modes, and the stability condition for the equilibrium state reads \( \text{Im} k^0 \leq 0, \forall |k| \).

We see the dispersion relation for the transverse mode is given by \( L_1 = 0 \), whose solution is \( k^0 = -i \eta |k|^2/\epsilon(p) \).

Thus, we find that the transverse mode is stable.
Here we again stress that the equation we study does not contain a term proportional to $Du^\mu$ in the thermal force for the heat flow. What would happen if such a term were present in the thermal forces, as in the case of the MS and the Eckart theories? In this case, the corresponding equation becomes $\mathcal{L}_1 = (\epsilon + p)(-ik^0) - T \lambda (-ik^0)^2 + \eta |k|^2 = 0$, which possesses a root with a positive imaginary part, and hence an unstable transverse mode appears. We emphasize that this instability is inevitable if the heat flow term contains $Du^\mu$.

Next, we examine the dispersion relations of the longitudinal modes. We first consider the simple but interesting case in which the heat conductivity vanishes (i.e., $\lambda = 0$), but the bulk and the shear viscosities may be positive (i.e., $\zeta \neq 0$ and $\eta \neq 0$). This simple case is often studied in the literature. We subsequently carry out a full analysis in which all the transport coefficients, including $\lambda$, may be positive.

In the simple case with $\lambda = 0$, the equation has the root $k^0 = 0$ and those satisfying $a_0 \equiv (\epsilon + p) \{\epsilon, n\}, b_0 \equiv |k|^2 = 3 \zeta p \{\epsilon, n\} - 3 \zeta p \{p, n\}$ and $c_0 \equiv |k|^2 \{\epsilon + p\} \{p, n\} + n \{\epsilon, p\}$, where we have written the Jacobian as $\{F, G\} = \partial(F, G)/\partial(T, \mu)$. Now, a simple analysis of the algebraic equation $a_0 \omega^2 + b_0 \omega + c_0 = 0$ with $\omega = -ik^0$ shows that the necessary and sufficient condition for $3 \lambda^0$ with $\Im \lambda^0 \leq 0$ is that $b_0/a_0 \geq 0$ and $c_0/a_0 \geq 0$. Owing to the properties of the Jacobian and the thermodynamic relations, the last inequality generally holds, because the l.h.s can be rewritten as $c_0/a_0 = |k|^2 \partial p/\partial \epsilon|_s = |k|^2 c_s^2$, with $c_s$ being the sound velocity. Then, the stability condition reduces to $b_0/a_0 \geq 0$, which is equivalent to

$$4\eta/3 + \zeta p \{1 - 3 (\partial p/\partial \epsilon)_s\} \geq 0. \quad (4)$$

This is a new condition that involves not only the EOS but also the bulk and shear viscosities. It can be shown analytically that this inequality is satisfied at least for a rarefied gas in the massless limit. To see this, first notice that $\epsilon = 3p$ for a relativistic gas composed of massless particles. Then the inequality reduces to the trivial one $\eta \geq 0$, because the second term with a bracket on the l.h.s vanishes, provided that the effective bulk viscosity, $\zeta p = \zeta (3 \gamma - 4)^{-2}$, is finite in the massless limit. In fact, it can be shown that this is the case using the microscopic formula for $\zeta$ [23], although $3 \gamma - 4 \rightarrow 0$ in the massless limit. We also remark that numerical calculations using the viscosities $\zeta$ and $\eta$ obtained from the Boltzmann equation reveal that the inequality (4) is always satisfied, even for a rarefied gas of massive particles. Instead of presenting the numerical results for this limiting case, we present the results for the general case, i.e., that in which $\lambda \neq 0$, $\zeta \neq 0$ and $\eta \neq 0$, below.

We now demonstrate that the thermal equilibrium state is stable with respect to the dynamics described by our equation, even when the heat conductivity, $\lambda$, is finite. The dispersion equation for the longitudinal modes is obtained from the roots of the cubic equation $a_0^3 x^3 + b_0 x^2 + c_0 x + d_0 = 0$, with $\omega = -ik^0$, where the coefficients are given by $a = a_0 + |k|^2 3 \zeta p \lambda (\partial n/\partial \mu)_T$, $b = b_0 + n \lambda |k|^2 (\partial \epsilon/\partial \mu)_T$, $c = c_0 + |k|^4 (4\eta/3 + \zeta p) \lambda (\partial n/\partial \mu)_T$ and $d = |k|^4 n \lambda (\partial p/\partial \mu)_T$. The condition $\Im \lambda^0 \leq 0$ implies that the above equation for $\omega$ has roots only in the left half plane or on the imaginary axis in the complex $\omega$ plane. An elementary analysis shows that this condition is given by

$$a \geq 0, b \geq 0, d \geq 0 \text{ and } bc - ad \geq 0. \quad (5)$$

Here, the equality holds in the case that the imaginary part of $k^0$ vanishes. Note that the above equalities imply that $c \geq 0$.

Now we demonstrate that these inequalities are satisfied for rarefied systems. For a relativistic free gas, we have $n = (2\pi)^{-3} 4 \pi m^3 e^\sigma z^2 K_3(z)\], \quad \epsilon = m n [K_3(z)/K_2(z) − z^{-1}], \quad p = n T$ and $\gamma = 1 + [z^2 + 3 h - (h - 1)^2]^{-1}$ with $z = m/T$ and $h = (\epsilon + p)/n T$ being the reduced enthalpy. Here, $K_2(z)$ and $K_3(z)$ denote the second and third modified Bessel functions, respectively. It is seen that the positivity condition $d > 0$ holds from the formula $p = n T$ with $n \epsilon e^\sigma$, which implies that $(\partial p/\partial \mu)_T > 0$. It remains to demonstrate the rest of the inequalities, i.e., $a \geq 0, b \geq 0$ and $bc - ad \geq 0$, for which we need explicit forms of the transport coefficients as well. The transport coefficients $\zeta, \lambda$ and $\eta$ for a rarefied gas can be obtained from the collision term in the Boltzmann equation. The Galerkin approximation using the Ritz polynomial expansion [17] with a constant cross section $\sigma$ in the collision integral gives $\zeta = \frac{15}{32\pi} T^2 e^\mu/T [z^2 K_3^2(z)] (5 - 3 \gamma) h - 3 \gamma^2] / [2 K_2(2 z) + z K_3(2 z)], \quad \lambda = \frac{15}{32\pi} T^2 e^\mu/T [z^2 K_3^2(z) \gamma] / (\gamma - 1)^2 \{[z^2 + 2] K_2(2 z) + 5 z K_3(2 z)\}$ and $\eta = \frac{15}{32\pi} T^2 e^\mu/T [z^2 K_3^2(z) \gamma] / ([5 z^2 + 2] K_2(2 z) + (3 z^3 + 49 z) K_3(2 z)]$. Note that all the transport coefficients are proportional to the inverse of the cross section, $\sigma$. This implies that a strongly (weakly) interacting system has small (large) transport coefficients. The numerical results for $a, b$ and $bc - ad$ are displayed in Fig.11 where the $z = m/T$ dependence is shown using $\sigma T^2 = 1$ for a wide range of values of the three momentums: $|k|^2/T = 0.1 \cdot 10$. We have confirmed that the positivity of these quantities holds for a wide range of values of the cross section: $\sigma T^2 = 0.01 \cdot 10$. We point out that a rarefied gas is a system in which dissipative effects are most significant. Thus, we have demonstrated that the thermal equilibrium solution is stable within the description provided by our hydrodynamic equation in the particle frame. Obviously, a solution with flow is unstable in a viscous fluid, as it must relax to the equilibrium state.
In this Letter, we first pointed out that the constraint proposed by Eckart, $u_{\mu} u_{\nu} \delta T^{\mu\nu} = 0$ (PT), is incompatible with the underlying relativistic Boltzmann equation. This important point has been largely unnoticed. We then showed that the reduction of the Boltzmann equation employing the RG method leads to an RHDE in the particle frame, that satisfies the constraint $\delta T^{\mu\nu} = 0$, while $u_{\mu} T^{\mu\nu} = \epsilon - 3 \zeta \nabla \cdot u$, which includes a contribution from the flow as well as the internal energy. This equation might imply that the energy density of an expanding system extracted from the hydrodynamic analysis can be erroneous. We have demonstrated that the solution around the equilibrium state in the new equation is stable. This was done by carrying out a linear stability analysis using the EOS and the transport coefficients for a rarefied gas. We conclude that Eq. (1) represents the first viable possibility as a stable, first-order, particle-frame RHDE for a viscous fluid. This is significant because the Israel-Stewart causal equation is usually constructed in the particle frame with PT. A detailed presentation of this work and applications of the new equation studied here will be reported elsewhere.

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