Generalized Indices for $\mathcal{N} = 1$ Theories in Four-Dimensions

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Data for a QFT

A Quantum Field Theory can be constructed using a set of “fields” \( \Phi \) and a real even functional \( S[\Phi] \)

- \( \Phi \) could be sections of - or connections for - some bundles over a smooth “spacetime” \( M \). They determine a (super) Hilbert space \( \mathcal{H} \) associated to \( \partial M \).
- The charges, equivalently representations, are restricted
  - Spin-statistics: odd fields sit in spinor representations of the tangent bundle.
  - Anomaly cancelation.
- A family of \( S \)'s is parametrized by “coupling constants”. In modern language: non-dynamical background fields - \( S[\Phi, \Phi_B] \).
- \( S \) determines a linear map between \( \mathcal{H} \)'s associated to different components of \( \partial M \) in one of two ways
  - By determining an operator (the Hamiltonian) \( H \) and the propagator \( \exp itH \).
  - By providing a “measure” for the path integral.
Symmetries

A transformation $\delta$ on the fields $\Phi$ is said to be a symmetry if

$$\delta S [\Phi, \Phi_B] = 0.$$ 

Every $\delta$ determines a $U_\delta$ such that

$$[U_\delta, H] = 0.$$ 

Some standard QFT symmetries when $M = \mathbb{R}^d$ (a group $G_{\text{even}}$ with algebra $\mathfrak{g}_{\text{even}}$)

1. The Lorentz or Euclidean rotation groups ($SO (1, d - 1), SO(d)$). The central element of the double cover (e.g. Spin $(d)$) is denoted $(-1)^F$. The Poincare group also includes translations.

2. Global symmetries - do not act on $M$. Sometimes called “flavor” if they come from including duplicate fields in $\Phi$.

3. Conformal symmetry - an extension of 1.
Supersymmetry and BPS states

An \( \mathcal{N} \) extended supersymmetry algebra adds odd generators (must be Lorentz spinors)

\[
\{ Q_i, Q_j \} \subset \mathfrak{g}_{\text{even}}, \quad (-1)^F Q_i = -Q_i (-1)^F, \quad i \in \{1 \ldots \mathcal{N}\}
\]

States are paired when \( Q^2 \neq 0 \)

\[
Q^2 |\psi > = (H + \ldots) |\psi > = \lambda |\psi > \quad \Rightarrow \quad |\psi > = Q \left( \frac{Q}{\lambda} |\psi > \right).
\]

Note that

\[
|\psi > = \begin{pmatrix} B \\ F \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & \bullet \\ \bullet & 0 \end{pmatrix}, (-1)^F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Define a state \( |\psi > \) is said to be **BPS** if

\[
Q |\psi > = 0 \quad \Leftrightarrow \quad (H + \ldots) |\psi > = 0.
\]
The Witten Index

An “index” is a quantity you can calculate in a supersymmetric QFT defined on $\mathbb{R}_t \times M_{\text{space}}$.

- Example: Choose a “space” manifold $T^{d-1}$. $Q$ is odd and Hermitian

\[ Q^2 = H, \quad Q = \begin{pmatrix} 0 & M^* \\ M & 0 \end{pmatrix}. \]

The Witten index is\(^1\)

\[ \mathcal{I}_W \equiv \text{tr}_\mathcal{H} (-1)^F = \dim \ker M - \dim \ker M^* \]

If $[Q, X_i] = 0$, form a “refined” index

\[ \mathcal{I} (\{a\}) = \text{tr}_\mathcal{H} \left[ (-1)^F e^{a_i X_i} \right] \]

\(^1\text{Witten (1988)}\)
Calculating an index by deformation (localization)

Indices are deformation invariant and get contributions only from BPS ("unpaired") states

\[ A = \{ Q, V \}, \quad [Q, A] = 0 \quad \Rightarrow \]
\[ \text{tr}_\mathcal{H} \left[ (-1)^F e^{a^i X_i} \right] = \text{tr}_\mathcal{H} \left[ (-1)^F e^{a^i X_i} e^{-tA} \right]. \]

Specifically, can be calculated at weak coupling (\( \beta \to \infty \))
\[ \text{tr}_\mathcal{H} \left[ (-1)^F e^{a^i X_i} \right] = \text{tr}_\mathcal{H} \left[ (-1)^F e^{a^i X_i} e^{-\beta(H+\ldots)} \right] \]

- Note: interesting deformations (\( a^i \)) parametrize the Q-cohomology.
Path integral formula for an index of states on $M_3$

$$\text{tr} \left[ (-1)^F e^{a^i X_i} e^{-\beta (H+\ldots)} \right] = \int \mathcal{D} [\Phi] \exp (-S_{\{a\},\beta} [\Phi])$$

- The fields $\Phi$ live on $S^1 \times M_3$.
- Supersymmetry means $\delta_Q S_{\{a\},\beta} [\Phi] = 0$. Example: $(-1)^F$ picks out the spin structure on the $S^1$ such that fermions are periodic.
- The $a_i$ are coordinates on some space of supersymmetric deformations of $S$: metrics, background fluxes etc.
Atiyah-Bott-Berline-Vergne formula

**Theorem (Atiyah and Bott - 1984, Berline and Vergne - 1982)**

Let $Q$ be an equivariant differential and $\alpha$ a $Q$-closed equivariant form on a compact manifold $M$, then the following holds

$$\int_M \alpha = \int_{\mathcal{K}_Q} i_{\mathcal{K}_Q}^* \alpha e(N_{\mathcal{K}_Q})$$

where $\mathcal{K}_Q$ is the zero set of $Q$, $i_{\mathcal{K}_Q}^*$ is the pullback and $e(N_{\mathcal{K}_Q})$ is the equivariant Euler class of the normal bundle of $\mathcal{K}_Q$ in $M$.

- Example: Duistermaat-Heckman Formula (1982)

$$\alpha = \exp \left[ i \left( H + \Omega \right) \right]$$

$$\int_M \Omega^n e^{iH} = i^n \sum_{p \in \mathbb{R}} e^{i\pi \text{sgn}(\text{Hess}(H(p)))} \frac{e^{iH(p)}}{\sqrt{\text{det} \left( \text{Hess}(H(p)) \right)}}$$
Localization in supergeometry

**Theorem (Schwarz and Zaboronski - 1995)**

Let \( M \) be a compact supermanifold with volume form \( dV \). Let \( Q \) be an odd non-degenerate vector field on \( M \) such that

1. \( \text{div}_{dV} Q = 0 \) (the volume form is \( Q \) invariant)

2. \( Q^2 \) is an even compact vector field on \( M \).

Let \( \mathcal{K}_Q \) be the zero set of \( Q \) and let \( S \) be an even \( Q \)-invariant function, \( \rho(p) \) is the volume density at \( p \), and “sdet” denotes the superdeterminant (Berezinian)

\[
\int_M dV e^{is} = \sum_{p \in \mathcal{K}_Q} \frac{\rho(p)e^{iS(p)}}{\sqrt{\text{sdet}(\text{Hess}(S(p)))}}
\]

In the DH formula

\[
\int_M \Omega^n e^{iH} \rightarrow i^{-n} \int_{\prod TM} \prod_{i=1}^{2n} dx^i d\xi^i e^{i(H(x) + \Omega_{ab}(x)\xi^a\xi^b)}
\]
Localization for path integrals

**Deformation**

- Identify an appropriate conserved fermionic charge: $Q$.
- Choose $V$ such that $\{Q, V\}$ is a positive semi-definite functional ($Q$ should square to 0 on $V$).
- Deform the action by a total $Q$ variation $S \rightarrow S + t\{Q, V\}$. The resulting path integral is independent of $t$!
- Add some $Q$ closed operators (Wilson loops, defect operators).

**Localization**

- Take the limit $t \rightarrow \infty$.
- The measure $\exp(-S)$ is very small for $\{Q, V\} \neq 0$.
- The semi-classical approximation becomes exact, but there may be many saddle points to sum over ("the zero locus").
- Integrate over the zero locus of $\{Q, V\}$ (+ small fluctuations)
Setting up QFT localization

Set up an integral with the odd symmetry $Q$

1. Write down a general $S[\Phi, \Phi_B]$ such that $\delta_Q S = 0$.
2. Pick background fields $\Phi_B$ such that $\delta_Q \Phi_B = 0$.

Some susy jargon

- **Twisting**: picking $Q$ and $\Phi_B (g)$ such that $T_{EM} \equiv dS/dg = \{Q, X\}$. Under mild assumptions, the result is a (“cohomological” or “Witten type”) TQFT - changing the metric $g$ results in

$$\frac{d}{dg} \int \mathcal{D}[\Phi] \exp (-S) = \int \mathcal{D}[\Phi] \ T \exp (-S) = 0.$$

- **Moduli space** - the set $\{\Phi | \delta_Q \Phi = 0\}$.
- **One loop determinant** - the function on moduli space given by $\text{sdet}^{-1/2} [\text{Hess} (\{Q, V\})]$. 
About the model

The (dynamical) field content

1. $U(N)$ vector multiplet (SYM) - $A, \lambda, D$
2. Some chiral multiplets - $\phi_i, \psi_i, F_i$

The action functional ($S [A, \lambda, D, \phi, \psi, F]$)

- Yang Mills action - $\frac{1}{g_{YM}^2} \int \text{tr} (F \wedge \star F)$
- Kinetic terms and minimal coupling - $\int \bar{\lambda} D \lambda, \int \bar{\psi} D \psi, \int D \phi \wedge \star D \phi$
- A “superpotential” which won’t play a prominent role.
- Non-derivative terms in $D, F$. 
Parameters and symmetries of the model

Some parameters are not background fields

1. The gauge group \( G \) (I took \( U(N) \)).
2. The representations of the matter fields (chirals).

Spacetime symmetries

1. Poincare - translations + rotations + boosts.
2. \( \mathcal{N} = 1 \) supersymmetry - a fermionic symmetry with one Weyl generator.

Global symmetries

1. \( U(1)_R \) which does not commute with supersymmetry.
2. Some flavor symmetry group \( F \) acting on chirals.
General motivation for $\mathcal{N} = 1$ SYM and SQCD

- A lot in common with QCD and electroweak theory
  - Asymptotic freedom/strongly coupled IR theory, higgs mechanism.
  - Confinement of color, chiral symmetry breaking.
  - Instantons and monopoles.

- Many other interesting features
  - Some exact results: non-renormalization theorem, NSVZ $\beta$-function etc.
  - Interacting conformal phase.
  - Seiberg duality.
  - No “solution” a la Seiberg-Witten for $\mathcal{N} = 2$ (but some partial results).
Introduction to Localization in QFT

Indices of $\mathcal{N} = 1$ super-Yang-Mills

Setup

Computation

Results/Conclusions

Additional motivation

- Exact results for strongly coupled theories are hard to come by.
- Few computations for 4d $\mathcal{N} = 1$ theories using localization.
- Supersymmetric backgrounds have been worked out recently and a large class of manifolds preserving two supercharges was identified.\(^2\)
- Existing examples like the superconformal index\(^3\) $(S^1 \times S^3)$ and $T^2 \times S^2^4$ show that the two supercharge case is particularly nice.

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\(^2\) Dumitrescu, Festuccia, and Seiberg (2012)

\(^3\) Assel et al (2014)

\(^4\) Closset and Shamir (2013)
Indices and partition functions

Indices are Euclidean partition functions that can be interpreted as a supertrace over the spectrum of a theory quantized on a $d - 1$ dimensional manifold (usually compact)

- The Witten index is a partition function on $T^d$. It counts supersymmetric ground states with signs.\(^5\)
- The superconformal index counts local BPS operators in a CFT.\(^6\) In 4d

\[
\mathcal{I}(p, q, u) = \text{Tr}_{S^3} \left( (-1)^F \ p^{J_3 + J'_3 - \frac{R}{2}} q^{J_3 - J'_3 - \frac{R}{2}} u^{Q_f} \right)
\]

is equivalently the partition function on a Hopf surface (topologically $S^1 \times S^3$) and $p, q$ are complex structure moduli.\(^7\)

- The lens space index replaces $S^3$ by $L(r, 1)$.\(^8\)

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\(^5\)Witten (1982)
\(^6\)Kinney et al (2005), Romelsberger (2005)
\(^7\)Closset, Dumitrescu, Festuccia, and Komargodski (2013)
\(^8\)Benini, Nishioka and Yamazaki (2012) Razamat and Willett (2013)
Overview

The goal: compute partition functions that represent indices for 4d $\mathcal{N} = 1$ theories

- Applicability
  - The theory must have a conserved $U(1)_R$ current.
  - The manifold should admit an appropriate metric with a holomorphic torus isometry.
  - The result is an unambiguous universal quantity which characterizes the IR CFT.\(^9\)

- Method
  - Choose a topology and complex structure only. The metric doesn’t matter!\(^{10}\)
  - Calculate fluctuations using the equivariant index theorem.

\(^9\) Assel, Cassani, and Martelli (2014)
\(^{10}\) Closset, Dumitrescu, Festuccia, and Komargodski (2013)
Rigid supersymmetry in curved space

New minimal supergravity couples to the $\mathcal{R}$ multiplet$^{11}$ of a 4d $\mathcal{N} = 1$ theory with a conserved $U(1)_R$

- The SUGRA multiplet: $g_{\mu\nu}$, $A_{\mu}^{(R)}$, $B_{\mu\nu}$, $\psi_{\mu}$, $\tilde{\psi}_{\mu}$
- The $\mathcal{R}$ multiplet: $T_{\mu\nu}$, $J_{\mu}^{(R)}$, ...

Rigid supersymmetric backgrounds solve a generalized Killing spinor equation$^{12}$ ($V \propto \star dB$)

\[
\delta \psi_{\mu} = (\nabla_{\mu} - i (A_{\mu} - V_{\mu}) - iV^{\nu} \sigma_{\mu\nu}) \epsilon = 0 , \\
\delta \tilde{\psi}_{\mu} = (\nabla_{\mu} + i (A_{\mu} - V_{\mu}) + iV^{\nu} \bar{\sigma}_{\mu\nu}) \bar{\epsilon} = 0 ,
\]

The backgrounds are complex manifolds

\[
J_{\mu\nu} \equiv - \frac{2i}{|\epsilon|^2} \epsilon^{\dagger} \sigma_{\mu\nu} \epsilon , \quad J_{\mu}^{\rho} J_{\nu}^{\rho} = - \delta_{\mu}^{\nu}
\]

$^{11}$Komargodski and Seiberg (2010)
$^{12}$Dumitrescu, Festuccia, and Seiberg (2012)
Backgrounds with both \( \epsilon \) and \( \tilde{\epsilon} \)

When we restrict to backgrounds preserving an \( \epsilon \) and an \( \tilde{\epsilon} \) we get, in addition

- two commuting complex structures
  \[
  J_{\mu\nu} = -\frac{2i}{|\epsilon|^2}\epsilon^\dagger \sigma_{\mu\nu} \epsilon, \quad \tilde{J}_{\mu\nu} = -\frac{2i}{|\tilde{\epsilon}|^2}\tilde{\epsilon}^\dagger \tilde{\sigma}_{\mu\nu} \tilde{\epsilon}, \]
  \[
  [J, \tilde{J}] = 0
  \]
- a complex holomorphic Killing vector
  \[
  K^\mu = \epsilon \sigma^\mu \tilde{\epsilon}.
  \]
  \[
  \nabla_\mu K_\nu + \nabla_\nu K_\mu = 0, \quad J_\nu^\mu K^\nu = \tilde{J}_\nu^\mu K^\nu = iK^\mu,
  \]
- the backgrounds are torus fibrations over a Riemann surface.

We’ll restrict to

\[
[K, K^\dagger] = 0
\]
A simple class: $M_4 \simeq S^1 \times M_3$

Take $M_4$ to be the total space of a principal elliptic fiber bundle over a compact orientable Riemann surface $\Sigma_g$

$$T^2 \to M_4 \xrightarrow{\pi} \Sigma_g .$$

- $M$ is actually diffeomorphic to $S^1 \times M_3$ where $M_3$ is a principal $U(1)$ bundle over $\Sigma_g$. The topology is determined by two numbers: the genus ($g$) and the degree ($d$).
- $M$ is Kähler if and only if $d = 0$, in which case it is diffeomorphic to $T^2 \times \Sigma_g$.
- $M$ has interesting cohomology classes, specifically $^{13}$

$$\text{Tor} \left( H^2 (M_4, \mathbb{Z}) \right) = \pi^* \left( H^2 (\Sigma_g, \mathbb{Z}) \right) \simeq \mathbb{Z}_d .$$

$^{13}$ Teleman (1998)
Complex structure and R symmetry

The localization depends on the topological and holomorphic properties of the R symmetry line bundle $L$.

- The supersymmetry equations imply that $L$ is “locked” to the canonical bundle: $L^{-2} \times \mathcal{K}_{M_4}$ is a trivial line bundle.\(^{14}\)
- For most values of $g, d$ the manifold $M_4$ has a canonical bundle with properties\(^{15}\)
  \[
  \mathcal{K}_{M_4} = \pi^* \mathcal{K}_{\Sigma g},
  \]
  and hence
  \[
  c_1(\mathcal{K}_{M_4}) = \pi^* c_1(\mathcal{K}_{\Sigma g}) = 2g - 2 \mod d \in \mathbb{Z}_d \subset H^2(M, \mathbb{Z}) .
  \]
- For $g = 0$ and $d \geq 3$ there is a more general possibility\(^{16}\)
  \[
  \mathcal{K}_{M_4} = \begin{cases} \text{topologically trivial} & \text{I}, \\ \pi^* \mathcal{K}_{\Sigma g} & \text{II}. \end{cases}
  \]

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\(^{14}\) Dumitrescu, Festuccia, and Seiberg (2012)

\(^{15}\) Hofer (1993)

\(^{16}\) Nakagawa (1995)
At this point we assume that $M$ admits the right type of metric to support two supercharges.

- The complex Killing vector $K$ has non-vanishing components in the fiber directions and acts freely on them.
- The supersymmetry algebra is

$$\left\{ \delta_\epsilon, \delta_{\bar{\epsilon}} \right\} = \frac{1}{2} \delta K ,$$

$$\left\{ \delta_\epsilon, \delta_\epsilon \right\} = \left\{ \delta_{\bar{\epsilon}}, \delta_{\bar{\epsilon}} \right\} = 0 ,$$

$$= [\delta_K, \delta_\epsilon] = 0 ,$$

$$= [\delta_K, \delta_{\bar{\epsilon}}] = 0 ,$$

$$\delta_K \equiv \mathcal{L}_K - irK^\mu A^{(R)}_\mu - iq_{\text{flavor/gauge}} K^\mu a_\mu .$$

- Supersymmetric actions for vector/chiral multiplets are easy to write down. R charge quantization may be required if $L$ is non-trivial.
Localization on $M_4$

We choose a supercharge $Q$ which is a linear combination of transformation using $\epsilon$ and $\tilde{\epsilon}$

$$\{ Q, Q \} = \frac{1}{2} \delta_K ,$$

$$\delta_K = \mathcal{L}_K - i r K^\mu A_\mu^{(R)} - i q_{\text{flavor/gauge}} K^\mu a_\mu .$$

The localizing functionals are the curved space $D$ terms

$$\mathcal{L}_{\text{gauge}}^{(\text{loc})} = \frac{1}{2} F_{\mu\nu} F^{\mu\nu} + \lambda \sigma^\mu D_\mu \tilde{\lambda} + \tilde{\lambda} \tilde{\sigma}^\mu D_\mu \lambda + D^2 ,$$

$$\mathcal{L}_{\text{matter}}^{(\text{loc})} = D_\mu \tilde{\phi} D^\mu \phi + \frac{1}{2} \tilde{\psi} \tilde{\sigma}^\mu D_\mu \psi + \ldots$$

The path integral localizes to flat connections

$$F_{\mu\nu} = 0 , \quad D = 0 , \quad \phi = 0 , \quad F = 0 ,$$

and we’ll call a linearized operator acting on fluctuations around this $D_{oe}$. 
The partition function

\[ Z_{G,r,M_g,d}(\tau_{cs}, \xi_{\text{FI}}, a_f) = \int_{\mathcal{M}_0^G(g,d)} e^{-S_{\text{classical}}(\tau_{cs}, \xi_{\text{FI}})} \times \]

\[ Z_{g,d}^{\text{gauge}}(\tau_{cs}) Z_{g,d,r}^{\text{matter}}(\tau_{cs}, a_f) \]

- Actually an integral and sum over the moduli space of flat connections \( \mathcal{M}_0^G(g,d) \). Background flat connections are included: \( a_f \).
- Dependence on the metric is through the space of complex structures \( \tau_{CS} \).
- The determinants will be computed using the equivariant index theorem

\[ \text{ind}(D_{oe}) = \text{tr}_{\text{Ker}D_{oe}} e^{\delta K} - \text{tr}_{\text{Coker}D_{oe}} e^{\delta K} \rightarrow Z_{\text{one-loop}} = \frac{\det_{\text{Coker}D_{oe}} \delta K}{\det_{\text{Ker}D_{oe}} \delta K} \]
The fundamental group of $M_4 (g, d)$ is described by generators 

$$a_i, b_i, h, x, \quad i \in 1, \ldots, g,$$

and relations

$$[a_i, h] = [b_i, h] = [a_i, x] = [b_i, x] = [x, h] = 1, \quad \prod_{i=1}^{g} [a_i, b_i] = h^d.$$

- It's a central extension of $\pi_1 (\Sigma_g)$ plus the decoupled generator $x$. For $g \neq 1$ only the $h$ and $x$ holonomies deform $\delta_K$.
- For non-trivial values of $h^d$ this implies flux on $\Sigma_g$.\textsuperscript{17} The flux changes the bundles used in the index theorem for $D_{oe}$.

\textsuperscript{17} Atiyah and Bott (1983)
This is the simplest case: in the holonomy representation \( \mathcal{M}_{g,d}^0 \) is the set of \( N \) dim unitary representations of \( \pi_1(M_4) \)

- Commuting generators can be simultaneously put in the Cartan.
- \( \det h^d = 1 \) so the spectrum of \( h \) is discrete - the quantum number \( m \) is the flux. The effect of the degree is \( m \to m \mod d \).
- In an irrep of \( \prod_{i=1}^g [a_i, b_i] = h^d \) the additional holonomy \( x \) must be scalar. A general representation breaks

\[
U(N) \to U(N_1) \times U(N_2) \times \cdots \times U(N_p)
\]

and has \( p \) fluxes.
Gaugino zero modes

The Killing spinor equations and the eom for the gaugino are similar

\[
\bar{\sigma}^\mu \left( \nabla_\mu - i \left( A_\mu^{(R)} + \frac{1}{2} V_\mu \right) \right) \epsilon = 0, \\
\bar{\sigma}^\mu \left( \nabla_\mu - i \left( A_\mu^{(R)} + a_\mu^{\text{gauge}} - \frac{3}{2} V_\mu \right) \right) \lambda = 0.
\]

- The background has $\chi(M_4) = \sigma(M_4) = 0$ and all the gauge fields satisfy $c_1^2 = c_2 = 0$ so the index theorem for the Dirac operator gives 0.

- If $V_\mu = 0$, i.e. Kähler manifolds with $d = 0$ and $M_4 \simeq T^2 \times \Sigma_g$, then gauginos in the same Cartan as the holonomies have an obvious zero mode: $\epsilon$.

- Under some assumptions $d > 0$ guarantees no gaugino zero modes.
Equivariant index for $d > 0$

The index is a function (density) on the abelian group of “symmetries” $S$ or chemical potentials

$$\text{ind}(D_{oe}) = \text{tr} \ker D_{oe} e^{\delta_K} - \text{tr} \text{coker} D_{oe} e^{\delta_K},$$

which can be used to compute the one loop determinants by the rule

$$\text{ind}(D_{oe}) = \sum_{\alpha} c_\alpha e^{tw_\alpha} \rightarrow Z_{\text{one-loop}} = \prod_{\alpha} w_\alpha^{-c_\alpha}.$$  

- $w_\alpha$ are weights in the representation in which the field sits. $c_\alpha$ is the multiplicity.
- $S$ includes the geometric action of $\mathcal{L}_K$, dynamical/background gauge transformations, and $R$ symmetry transformations.
- The structure of $M_4$ allows us to reduce to $\Sigma_g$. For a chiral, $D_{oe}$ is the pullback of a Dirac operator on $\Sigma_g$ and its index will be calculated using the Atiyah Singer index theorem (transversally elliptic version). The gauge sector is similar.
Equivariant index - $g > 1$

The computation simplifies because there are no isometries on $\Sigma_g$.

- The holonomies on the base do not deform the equivariant complex.
- We can use the usual Atiyah Singer index theorem for the Dirac operator

$$\text{ind}(D_{\text{Dirac}}, E) = \int_X \hat{A}(TX) \text{ch}(E) = \int_{\Sigma} 1 \cdot c_1(E) = \text{deg}(E).$$

The bundle on the base is geometric+gauge+R symmetry. The index and determinant are

$$\text{ind}(D_{\text{oe}}) = \sum_{\rho \in \mathfrak{R}, n, l \in \mathbb{Z}} \left( -(r - 1) \frac{\chi(\Sigma)}{2} + dl + \rho(m) \right) x^n y^{dl-(r-1)\frac{\chi(\Sigma)}{2}} u,$$

$$Z_{\text{matter}}^{(r, \rho)} = \prod_{n, l \in \mathbb{Z}} \left( n + \tau d \left( l - (r - 1) \frac{\chi(\Sigma)}{2d} \right) + \rho(a_w) \right)^{-(r-1)\frac{\chi(\Sigma)}{2}+dl+\rho(m)}$$
Equivariant index - $g = 0$

This is the lens space index\(^{18}\) for which we use the Atiyah Bott fixed point formula on $\Sigma_0 = S^2$

$$\text{ind}_T(D) = \sum_{\rho \in F} \frac{\text{tr} E(e)(p) t - \text{tr} E_0(p) t}{\det_T x_\rho(1 - t)}. $$

The index and determinant are

$$\text{ind}(D_{oe}) = \sum_{\rho \in \mathcal{R}, n, l \in \mathbb{Z}} t^{-r/2} \frac{t^{(dl + \rho(m))/2} - t^{-(dl + \rho(m))/2}}{1 - t^{-1}} x^n y^{dl + \rho(m)} u,$$

$$Z_{\text{matter}}^{(r, \rho)}(m, u) = e^{i \pi \mathcal{E}^{(r)}(\rho(m), u)} \Gamma(u(pq)^{r/2} q^{d - \rho(m)}; q^d, pq) \begin{pmatrix} p \leftrightarrow q \\ \rho \rightarrow d - \rho \end{pmatrix}$$

- $e^{i \pi \mathcal{E}^{(r)}(\rho(m), u)}$ is an interesting zero point energy.

\(^{18}\)Benini, Nishioka and Yamazaki (2012) Razamat and Willett (2013)
Equivariant index - $g = 1$

An interesting case

- $\chi(\Sigma) = 0$ implies that there is no R charge quantization for any $d$.
- There are isometries on the base torus, but no fixed points.
- General arguments imply that the base complex structure does not affect the partition function, but it seems like the holonomies do.
Classical contributions

Fayet-Iliopoulos terms for $U(1)$ factors exist in curved space

$$\xi \int (D - iV^\mu a_\mu),$$

- After localizing to flat connections only $K^\mu a_\mu$ contributes due to
  $$V_\mu = -\frac{1}{2} \nabla^\nu J_{\nu\mu} + \kappa K_\mu, \quad K^\mu \partial_\mu \kappa = 0.$$

- $\xi$ must be quantized to keep this invariant under large gauge transformations. This may not make sense for arbitrary $g, d$ and an arbitrary complex structure.

- The result is trivial if $V = \star dB$ for a well defined $B$, hence we must have a non trivial three form flux in $H^{1,2}(M_4)$.

- The expression is equivalent to a sort of smeared supersymmetric abelian Wilson loop. Is there a non-abelian analogue?
Aspects of the partition function - I

\[ Z_{G,r,M_g,d} (\tau_{\text{cs}}, \xi_{\text{FI}}, a_f) = \frac{1}{|\mathcal{W}|} \int_{\mathcal{M}_G^0(g,d)} e^{-S_{\text{classical}}(\tau_{\text{cs}}, \xi_{\text{FI}})} \times \]
\[ Z_{gauge}^{g,d} (\tau_{\text{cs}}) Z_{\text{matter}}^{g,d,r} (\tau_{\text{cs}}, a_f) \]

- The restriction on R charges is
  \[ r (g - 1 \mod d) \in \mathbb{Z} . \]
This does not apply to the (usual) lens space index.

- \( \tau_{\text{CS}} \) consists of the complex structure parameter for the torus fiber (\( \tau \)), an additional complex number for the fibration (\( \sigma \)) when \( g = 0 \), and possibly the complex structure on the base for \( g = 1 \).

- \( Z_{\text{matter}}^{g,d,r} (\tau_{\text{cs}}, a_f) \) and \( Z_{\text{gauge}}^{g,d} (\tau_{\text{cs}}) \) are elliptic gamma (type) functions.

- An overall factor is included to account for the residual Weyl
Aspects of the partition function - II

The parameters entering the partition function are split between\textsuperscript{19}

1. Parameters and deformations of the theory
   1. The gauge/flavor groups and the matter representations. This is where the superpotential comes in.
   2. A set of admissible Fayet-Iliopoulos terms \( \xi \), one for each independent \( U(1) \) factor in \( G \).
   3. An element of the moduli space of flat connections on \( M \) of the flavor symmetry group \( F \).

2. Parameters of \( M \)
   1. The genus, \( g \), of the underlying Riemann surface and the first Chern class, \( d \), of the circle bundle whose total space is \( M_3 \).
   2. A point in the complex structure moduli space on \( M \) admitting a holomorphic Killing vector \( K \). This may include a discrete choice in the case \( g = 0 \).
   3. A choice of \( W \in H^{1,2}(M) \).

\textsuperscript{19} In agreement with Closset et al. (2014)
The interpretation of the index is complicated by

1. Accidental symmetries may prevent us from correctly identifying the IR R charge.

2. A metric supporting the necessary holomorphic Killing vector may not exist for all $g, d, \tau_{CS}$.

The computation itself has a few shortcomings

1. The integral over the moduli space of flat connections is complicated and involves an unresolved quantity

$$\int_{\mathcal{M}_G^0(g,d)} = \sum_{\text{partitions } N} \prod_{j=1}^p \left( \sum_{m_j \in 0, \ldots, dN_j - 1} V \left[ \mathcal{M}^{g}_{N_j m_j} \right] \int_0^1 \frac{dx_j}{2\pi} \right).$$

2. Exclusion of fermionic zero modes required some assumptions.
Applications

A few standard applications for exact calculations

1. Checking dualities: this involved a complicated calculation in the case of Seiberg duality and the superconformal index $(S^1 \times S^3)$. The more intricate topology of $M_4$ can help check some global issues like discrete theta angles. Mapping of operators would be more ambitious.

2. Holography and large $N$: this potentially sidesteps some of the intricacies of the moduli space of flat connections.

Some more recent applications

1. Extracting trace anomalies from supersymmetric partition functions at “high temperature”.

2. Constructing integrable lattice models.

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20 Spiridonov and Vartanov (2009)
21 Razamat and Willett (2013)
22 Di Pietro and Komargodski (2014)
23 Yamazaki (2013)
Future directions

Extending the results to include

- Manifolds where $K$ acts with finite isotropy groups. The same basic techniques can be used.
- Looking for supersymmetric operators/defects.

More challenging options

- Manifolds with gaugino zero modes.
- Backgrounds preserving one supercharge: localization to the instanton moduli space.
Thank you!