UNIQUENESS FOR THE NONLOCAL LIOUVILLE EQUATION IN $\mathbb{R}$

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Abstract. We prove uniqueness of solutions for the nonlocal Liouville equation

$$(-\Delta)^{1/2} w = Ke^w \quad \text{in } \mathbb{R},$$

with finite total $Q$-curvature $\int_{\mathbb{R}} Ke^w \, dx < +\infty$. Here the prescribed $Q$-curvature function $K = K(|x|) > 0$ is assumed to be a positive, symmetric-decreasing function satisfying suitable regularity and decay bounds. In particular, we obtain uniqueness of solutions in the Gaussian case with $K(x) = \exp(-x^2)$.

Our uniqueness proof exploits a connection of the nonlocal Liouville equation to ground state solitons for Calogero–Moser derivative NLS, which is a completely integrable PDE recently studied by P. Gérard and the second author.

1. Introduction and Main Results

In this paper, we consider the following nonlocal Liouville equation

$$(1.1) \quad (-\Delta)^{1/2} w = Ke^w \quad \text{in } \mathbb{R},$$

subject to the finiteness condition that

$$(1.2) \quad \Lambda := \int_{\mathbb{R}} K(x)e^{w(x)} \, dx < +\infty.$$  

Here $K : \mathbb{R} \to \mathbb{R}$ denotes a given function, which geometrically can be seen as a prescribed $Q$-curvature problem in one space dimension. More precisely, if $w$ is a solution of (1.1) then $g = e^{2w} |dx|^2$ is a metric on $\mathbb{R}$ that is conformal to the standard metric $g_0 = |dx|^2$ on $\mathbb{R}$ having constant $Q$-curvature equal to 1. The quantity $\Lambda$ then corresponds to the total $Q$-curvature of the metric $g$ on $\mathbb{R}$. We note that, by means of the stereographic projection, the nonlocal Liouville equation (1.1) can also be related to a prescribed $Q$-curvature problem on the unit circle. We refer to [10, 11] for more details on the geometric background on (1.1) and its relation to the generalized Riemann mapping theorem in the complex plane $\mathbb{C}$.

In fact, existence and non-existence results of prescribed $Q$-curvatures problems in $\mathbb{R}^n$ for general dimensions $n \geq 1$ have recently attracted a great deal of attention, leading to the class of Liouville type equations given by

$$(1.3) \quad (-\Delta)^{n/2} w = Ke^n w \quad \text{in } \mathbb{R}^n.$$  

In the case of $n = 2$ space dimensions, equation (1.3) then becomes the well-known Liouville equation which is a central object in nonlinear elliptic PDEs and geometric analysis; see [4, 7, 8, 19, 18, 20].

From an analytic point of view, a particularly challenging situation for equation (1.3) arises in odd space dimensions $n \in \{1, 3, 5, \ldots\}$ due to the nonlocal nature of the pseudo-differential operator $(-\Delta)^{n/2}$. Apart from the important special case of constant positive constant $K > 0$, where solutions $w$ are known in closed form (see (1.9) below), the question of uniqueness of solutions $w$ have been entirely out of scope so far in odd dimensions. In the present paper, we address the case of $n = 1$ space dimension. In fact, our analysis is strongly inspired by a surprisingly close connection to solitons of continuum limits of completely integrable Calogero-Moser systems; see below for more details on this.

Before we state our main results on (1.1), we introduce some basic notions as follows. Throughout this paper, we assume that the solutions $w : \mathbb{R} \to \mathbb{R}$ belong to the space

$$L_{1/2}(\mathbb{R}) := \left\{ f \in L^1_{\text{loc}}(\mathbb{R}) : \mathbb{R} \frac{|f(x)|}{1 + x^2} \, dx < +\infty \right\}.$$  

which is the natural space to define distributional solutions of \(1.1\), see, e.g., [17] for details. We remark that we always deal with real-valued functions throughout this paper.

In order to state the main results, we will impose the following conditions on the \(Q\)-curvature function \(K\), where we employ the commonly used short-hand notation \(\langle x \rangle = \sqrt{1 + x^2}\).

**Assumption (A).** We assume that \(K : \mathbb{R} \to \mathbb{R}\) has the following properties.

(i) \(K\) is strictly positive, even and monotone decreasing in \(|x|\).

(ii) \(K\) is continuously differentiable and satisfies the pointwise bound

\[
\sqrt{K(x)} + |x\partial_x \sqrt{K(x)}| \leq C \langle x \rangle^{-1/2 - \delta}
\]

for all \(x \in \mathbb{R}\) with some constants \(C > 0\) and \(\delta > 0\).

Important examples for admissible \(Q\)-curvatures are the Gaussian function \(K(x) = e^{-x^2}\) and \(K(x) = (x)^{-1 - 2\delta}\) for some \(\delta > 0\). Further below, we will see that imposing regularity and decay conditions on the square root \(\sqrt{K}\) of the \(Q\)-curvature function becomes natural due to our approach that is based on a connection to solitons for the Calogero-Moser derivative NLS discussed below.

The first main result of this paper is now as follows.

**Theorem 1.** Suppose \(K\) satisfies Assumption (A) and let \(w \in L_{1/2}(\mathbb{R})\) be a solution of \(1.1\) satisfying \(\smaller{1.2}\). Then the following properties hold.

(i) **Regularity and Total \(Q\)-Curvature Bound:** We have \(w \in C^{1,1/2}_\text{loc}(\mathbb{R})\) and the total \(Q\)-curvature \(\Lambda = \int_{\mathbb{R}} Ke^w dx\) satisfies \(0 < \Lambda < 2\pi\).

(ii) **Symmetry and Monotonicity:** \(w\) is even and decreasing in \(|x|\), i.e., it holds \(w(-x) = w(x)\) for all \(x \in \mathbb{R}\) and \(w(x) \geq w(y)\) whenever \(|x| \leq |y|\).

(iii) **Existence:** For every \(w_0 \in \mathbb{R}\), there exists a solution \(w \in L_{1/2}(\mathbb{R})\) of \(1.1\) with \(w(0) = w_0\) such that \(\smaller{1.2}\) holds.

The next main result establishes uniqueness of solutions for the fractional Liouville equation \(\smaller{1.1}\). In fact, despite the nonlocal nature of the problem, we obtain the following Cauchy-Lipschitz ODE type uniqueness result stating that the ‘initial value’ \(w(0) = w_0\) completely determines the solution \(w\) in all of \(\mathbb{R}\).

**Theorem 2 (Uniqueness).** Suppose \(K\) satisfies Assumption (A). If \(w, \tilde{w} \in L_{2}(\mathbb{R})\) are solutions of \(\smaller{1.1}\) satisfying \(\smaller{1.2}\), then it holds

\[
\tilde{w}(0) = w(0) \implies \tilde{w} \equiv w.
\]

**Remarks.**

1) In view of existing techniques, we consider Theorem \(\smaller{2}\) to be the most original contribution of the present paper. Further below, we will comment in more detail on the strategy behind its proof.

2) It remains an interesting open question whether – instead of prescribing the ‘initial value’ \(w(0)\) – we also have uniqueness of solutions \(w\) determined by the value of the total \(Q\)-curvature \(\Lambda\). That is, if for two solutions \(\tilde{w}, w \in L_{1/2}(\mathbb{R})\) of \(\smaller{1.1}\) such that

\[
\int_{\mathbb{R}} Ke^{\tilde{w}} dx = \int_{\mathbb{R}} Ke^w dx
\]

we necessarily have that the identity \(\tilde{w} \equiv w\) holds. We hope to address this open problem in the future.

3) We remark that the uniqueness result in Theorem \(\smaller{2}\) is non-perturbative, since no smallness condition on either \(K\) nor the ‘initial value’ \(w(0)\) is imposed.

**Comments on the Uniqueness Proof.** Let us briefly describe the strategy behind proving the uniqueness result stated in Theorem \(\smaller{2}\) above. The starting point rests on recasting the problem by introducing the positive function \(v : \mathbb{R} \to \mathbb{R}_{>0}\) given by

\[
(1.4) \quad v = \sqrt{Ke^w}.
\]

In terms of \(v\), the nonlocal Liouville equation \(\smaller{1.1}\) acquires the form

\[
(1.5) \quad \partial_s v + Wv + \frac{1}{2}H(v^2)v = 0 \quad \text{in} \quad \mathbb{R}.
\]
Here $H$ denotes the Hilbert transform on the real line and the function
\[ W = -\partial_x (\log \sqrt{K}) \]
plays the role of a given external potential. In fact, equation (1.5) and its solutions $v$ naturally arise in the study of solitons for the Calogero-Moser derivative NLS; see below.

Despite the nonlocality of the Hilbert transform $H$, it turns out that (1.5) becomes more amenable to the study of uniqueness for solutions $v$ parametrized by its "initial value" $v_0 = v(0)$. To this end, we recast (1.5) once more into a corresponding integral equation stated in (1.6) below, where $v_0 > 0$ enters as a parameter. As a next essential step towards proving Theorem 2, we establish a local uniqueness result around any given solution $v$ of (1.5) by constructing a locally unique branch parametrized by $v_0$ using the implicit function theorem. To achieve this, we show that the invertibility of the relevant linearized (and nonlocal) operator is tantamount to ruling out non-trivial solutions $\psi \in \dot{H}^1_{\text{even}}(\mathbb{R})$ with $\psi(0) = 0$ that satisfy
\[ (-\Delta)^{1/2} \psi - v^2 \psi = 0 \quad \text{in } \mathbb{R}. \]

Here the use of a monotonicity formula for the fractional Laplacian $(-\Delta)^{1/2}$ found in [6, 9] (and applied for the spectral analysis related to nonlinear ground states in [14]) becomes the key ingredient. However, in contrast to these works, we develop a different approach which completely avoids the use of the harmonic extension to the upper half-plane $\mathbb{R}^2_+$. Instead, we directly work with the singular integral expression for $(-\Delta)^{1/2}$ and we thus obtain expressions which relate to the classical theory of Carleman-Hankel operators on the half-line; see Section 5 and Appendix B for more details. We believe that this novel approach for monotonicity formulas can lead to further general insights into spectral and uniqueness problems involving the fractional Laplacian $(-\Delta)^s$ with $s \in (0, 1)$ and other suitable pseudo-differential operators (but which may not be seen as a Dirichlet-to-Neumann map).

Once the local uniqueness for solutions $v$ of (1.5) is established, we complete the proof of Theorem 2 by a-priori bounds allowing us to make a global continuation argument linking to the limit $v_0 \to 0^+$, which finally shows that there exists only one global branch of solutions $v$ parametrized by $v(0) = v_0$.

**Solitons for the Calogero-Moser Derivative NLS.** We now sketch the connection between the nonlocal Liouville equation (1.1) and solitons for the Calogero-Moser derivative NLS, which is a Hamiltonian PDE that can be written as
\[ i\partial_t \psi = -\partial_{xx} \psi + V \psi - ((-\Delta)^{1/2} |\psi|^2) \psi + \frac{1}{4} |\psi|^4 \psi \]
for the complex-valued field $\psi : [0, T) \times \mathbb{R} \to \mathbb{C}$. For the external potential $V$, the choices $V(x) = x^2$ (external harmonic potential) and $V(x) \equiv 0$ (no external potential) both arise naturally in the physical context of continuum limits of completely integrable many-body systems of Calogero-Moser type. For a formal derivation of (CM) in the physics literature, we refer to [11, 12]. A striking feature, indicating that a completely integrable nature of the problem, is that (CM) stems from a Hamiltonian energy functional $\mathcal{E}(\psi)$ which admits a factorization into a complete square of first-order terms; see [15, 3]. More precisely, the Hamiltonian energy (up to an inessential additive term) is found to be
\[ \mathcal{E}(\psi) = \frac{1}{2} \int_\mathbb{R} |\partial_x \psi + \sqrt{\mathcal{V}} \psi + \frac{1}{2} H(|\psi|^2) \psi|^2 dx. \]
Minimizers of $\mathcal{E}(\psi)$ provide soliton solutions for (1.5); see [15, 4] again for more details. Evidently, solutions $v$ of (1.5) above with $W = \sqrt{\mathcal{V}}$ are minimizers for $\mathcal{E}(\psi)$. For an external harmonic potential when $V = x^2$, this corresponds to the choice of a prescribed $Q$-curvature in (1.1) given by the Gaussian function $K(x) = e^{-x^2}$. which clearly falls under the scope of Assumption (A).

In the case of no external potential $V \equiv 0$ and hence $K \equiv 1$ is a positive constant, it was recently shown in [15] by Hardy-space techniques that the real-valued minimizers
We have that
\[ v \in H^1(\mathbb{R}) \]
must be of the explicit form
\[ v(x) = \pm \sqrt{\frac{2\lambda}{1 + \lambda^2(x - x_0)^2}} \]
with arbitrary \( \lambda > 0 \) and \( x_0 \in \mathbb{R} \). Translating this back via \((1.1)\) and using regularity theory, this shows that all solutions \( w \in L_{1/2}(\mathbb{R}) \) of the nonlocal Liouville equation \((1.1)\) with \( K \equiv 1 \) and \( e^w \in L^1(\mathbb{R}) \) are given by
\[ w(x) = \log \left( \frac{2\lambda}{1 + \lambda^2(x - x_0)^2} \right). \]
This uniqueness result for \((1.1)\) with constant \( K \equiv 1 \) has also been obtained in \([7, 20, 10]\)
by different techniques. However, the approach taken in \([15]\) provides yet another self-contained and independent proof of this fact by exploiting the relation to solitons for \((1.1)\).

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2. Preliminaries

We first collect some results that can be deduced by adapting known arguments. In particular, the results in this section will imply that item (i) in Theorem 1 holds true.

Throughout this section, we always assume that \( w \in L_{1/2}(\mathbb{R}) \) solves \((1.1)\) subject to \((1.2)\), where the \( Q \)-curvature function \( K \) satisfies Assumption \((A)\).

2.1. Regularity, Asymptotics, and Universal Bound on \( \Lambda \).

We start by collecting some immediate facts about the \( Q \)-curvature function \( K \) satisfying Assumption \((A)\).

Lemma 2.1. It holds that \( \sqrt{K}, K \in H^1(\mathbb{R}) \) and \( K \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}) \).

Proof. Since \( \sqrt{K} \lesssim (x)^{-1/2-\delta} \) for some \( \delta > 0 \), we readily see that \( \sqrt{K} \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}) \), whence it follows that \( K \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \). Furthermore, by the bound \( |x\partial_x \sqrt{K}(x)| \lesssim (x)^{-1/2-\delta} \) for some \( \delta > 0 \) and the fact that \( \partial_x \sqrt{K} \) is continuous and hence locally bounded, we deduce that \( \partial_x \sqrt{K} \in L^2(\mathbb{R}) \). Thus \( \partial_x K = 2\sqrt{K} \partial_x \sqrt{K} \in L^2(\mathbb{R}) \) since \( \sqrt{K} \in L^\infty(\mathbb{R}) \). This shows that \( \sqrt{K} \) and \( K \) both belong to \( H^1(\mathbb{R}) \).

Next, we derive the following regularity result for solutions \( w \in L_2^1(\mathbb{R}) \) of \((1.1)\) subject to the integrability condition \( Ke^w \in L^1(\mathbb{R}) \).

Lemma 2.2. It holds that \( w \in C^{0,1/2}_{loc}(\mathbb{R}) \).

Proof. We can adapt the arguments presented in \([17]\), where regularity for the equation \((-\Delta)^{1/2} u = |x|\alpha e^{\alpha u} \) in \( \mathbb{R}^n \) with \( \alpha > -1 \) subject to the integrability condition \( \int_{\mathbb{R}^n} |x|^{\alpha} e^{\alpha u} < +\infty \) is discussed.

For the reader’s convenience, we sketch the necessary modifications for our case. First, we show that \( e^w \in L^p_{loc}(\mathbb{R}) \) for any \( p \in (1, \infty) \) by an ‘\( e \)-regularity trick’ as follows. Indeed, for any such \( p \geq 1 \), we can take \( 0 < \epsilon < \frac{1}{p} \) and we split \( Ke^w = f_1 + f_2 \) with \( f_1, f_2 \geq 0 \) such that \( f_1 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) and \( \|f_2\|_{L^1} \leq \epsilon \). Next, we write
\[ w = w_1 + w_2 + w_3 \]
with the functions
\[ w_i(x) := \frac{1}{\pi} \int_{\mathbb{R}} \log \left( \frac{1 + |y|}{|x - y|} \right) f_i(y) \, dy \quad \text{for } i = 1, 2, \quad w_3 := w - w_1 - w_2. \]
We have that \( w_1 \in C^0(\mathbb{R}) \) and \( w_3 \in C^\infty(\mathbb{R}) \) since \( w_3 \) is \( \frac{1}{x} \)-harmonic. (In fact, we can deduce that \( w_3 \) must be a constant function; see also the proof of Lemma 2.3 below.) For
any \( R > 0 \) be given, we apply Jensen's inequality to find
\[
\int_{B_R} e^{u(x)} \, dx = \int_{B_R} \exp \left( \int_{\mathbb{R}} \frac{p|f_2|_{L^1}}{\pi} \log \left( \frac{1 + |y|}{|x - y|} \right) \frac{f_2(y)}{|f_2|_{L^1}} \, dy \right) \, dx \\
\leq \int_{B_R} \int_{\mathbb{R}} \exp \left( \frac{p|f_2|_{L^1}}{\pi} \log \left( \frac{1 + |y|}{|x - y|} \right) \frac{f_2(y)}{|f_2|_{L^1}} \, dy \right) \, dx \\
= \frac{1}{|f_2|_{L^1}} \int_{B_R} f_2(y) \int_{B_R} \left( \frac{1 + |y|}{|x - y|} \right)^{\frac{p|f_2|_{L^1}}{\pi}} \, dx \, dy < +\infty,
\]
without \( p|f_2|_{L^1} \leq p e < \pi \) holds. This shows that \( e^{u(x)} \in L^p_{\text{loc}}(\mathbb{R}) \) and hence \( u(x) \in L^p_{\text{loc}}(\mathbb{R}) \) by the regularity of \( u_1 \) and \( u_2 \).

Since \( K \in L^\infty(\mathbb{R}) \) and thus \( K e^w \in L^p_{\text{loc}}(\mathbb{R}) \), we see that \( w \in W^{1,p}_{\text{loc}}(\mathbb{R}) \) for all \( p \in (1, \infty) \). By Sobolev embeddings, this implies the Hölder continuity \( w \in C^\alpha_{\text{loc}}(\mathbb{R}) \) for any \( \alpha \in (0,1) \).

Recall that \( K \in H^1(\mathbb{R}) \subset C^{0,1/2}(\mathbb{R}) \), whence it follows that \( K e^w \in C^{0,1/2}_{\text{loc}}(\mathbb{R}) \). By local Schauder-type estimates for fractional Laplacians, we deduce that \( w \in C^{1,1/2}_{\text{loc}}(\mathbb{R}) \).

**Lemma 2.3.** It holds that
\[
w(x) = \frac{1}{\pi} \int_{\mathbb{R}} \log \left( \frac{1 + |y|}{|x - y|} \right) K(y) e^{w(y)} \, dy + C
\]
with some constant \( C \in \mathbb{R} \). Moreover, we have \( K e^w \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) and the asymptotics
\[
w(x) = -\frac{\Lambda}{\pi} \log |x| + O(1) \quad \text{as} \quad |x| \to +\infty
\]
holds, where \( \Lambda = \int_{\mathbb{R}} K e^w \, dx > 0 \). Finally, we have that \( w_+ = \max\{w, 0\} \in L^\infty(\mathbb{R}) \) and
\[
\int_{\mathbb{R}} \log(1 + |x|) K(x) e^{w_+(x)} \, dx < +\infty.
\]

**Proof.** Define the function \( \tilde{w}(x) := w(x) - \frac{1}{\pi} \int_{\mathbb{R}} \log \left( \frac{1 + |y|}{|x - y|} \right) K(y) e^{w(y)} \, dy \). Then \( \tilde{w} \in L^1_{\text{loc}}(\mathbb{R}) \) satisfies \( (-\Delta)^{1/2} \tilde{w} = 0 \) in \( \mathbb{R} \). From [12] we deduce that \( \tilde{w} : \mathbb{R} \to \mathbb{R} \) is an affine function and thus \( \tilde{w} = \text{const.} \), since we have \( w \in L^1_{\text{loc}}(\mathbb{R}) \). This proves the integral formula for \( w \) stated above.

From the discussion in [17][Remark 3.2] we deduce that
\[
\lim_{|x| \to +\infty} \frac{w(x)}{\log |x|} = -\frac{\Lambda}{\pi}
\]
with \( \Lambda = \int_{\mathbb{R}} K e^w \, dx > 0 \). Clearly, the above limit implies that \( w(x) < 0 \) for \( |x| \geq R \) with \( R > 0 \) sufficiently large. Since \( w \in L^\infty_{\text{loc}}(\mathbb{R}) \) by Lemma 2.2, we thus find \( w^+ = \max\{w, 0\} \in L^\infty(\mathbb{R}) \) and hence \( e^{w^+} \in L^\infty(\mathbb{R}) \). By our assumptions on \( K \), this implies
\[
0 < K(x) e^{w(x)} \leq C(x)^{-1-\delta} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})
\]
with some \( \delta > 0 \). Thus the function \( f := K e^w \) satisfies \( \log(1 + |\cdot|) f \in L^1(\mathbb{R}) \). Hence we can rewrite the integral formula for \( w \) as
\[
w(x) = -\frac{1}{\pi} \int_{\mathbb{R}} \log |x - y| f(y) \, dy + C = -\frac{\Lambda}{\pi} \log |x| - \frac{1}{\pi} \int_{\mathbb{R}} \log \left| \frac{x - y}{x} \right| f(y) \, dy + C
\]
with some constant \( C \in \mathbb{R} \). The asymptotic formula for \( w \) now follows from
\[
\lim_{|x| \to +\infty} \int_{\mathbb{R}} \log \left| \frac{x - y}{x} \right| f(y) \, dy = 0.
\]
Indeed, this can be seen by splitting the integration into the sets \(|x - y| \geq |x|/2\) and \(|x - y| \leq |x|/2\) and by using that \( f \in L^1(\mathbb{R}) \cap L^1(1 + \log(1 + |x|)) \) and dominated convergence. We omit the details. \( \square \)

**Lemma 2.4.** The total \( Q \)-curvature satisfies \( 0 < \Lambda < 2\pi \).

**Proof.** By the positivity of \( K(x) > 0 \), it is clear that \( \Lambda > 0 \) holds. The upper bound \( \Lambda < 2\pi \) follows by Pohozaev-type argument adapted to Liouville type equations; see, e.g., [20] [11]. For the reader’s convenience, we state the proof adapted to our case.
If we differentiate the integral equation for $w$ and multiply with $xK(x)$, we obtain
\begin{equation}
(2.2) \quad xK(x)\frac{\partial w}{\partial x} = -\frac{1}{\pi} \text{PV} \int_{\mathbb{R}} xK(x)K(y)e^{w(y)} \, dy.
\end{equation}

Multiplication with $e^{w(x)}$ and integration of the left-hand side over $[-R, R]$ yields
\begin{align*}
I := \int_{-R}^{R} xK(x)e^{w(x)} \frac{\partial w}{\partial x} \, dx &= xK(x)e^{w(x)} \bigg|_{x=-R}^{x=R} - \int_{-R}^{R} \frac{\partial}{\partial x} (xK(x))e^{w(x)} \, dx \\
&\rightarrow -\Lambda - \int_{\mathbb{R}} x(\partial_x K)(x)e^{w(x)} \, dx \quad \text{as} \quad R \to +\infty.
\end{align*}

Note that $xK(x)e^{w(x)} \bigg|_{x=-R}^{x=R} \to 0$ as $R \to \infty$ since $|xK(x)e^{w(x)}| \leq C|x|^{-2\delta}$ for some $\delta > 0$ in view of $e^{w} \in L_{\infty}^{\text{loc}}(\mathbb{R})$ and our assumption on $K$. Furthermore, we notice that $x(\partial_x K)e^{w(x)} \in L^{1}(\mathbb{R})$ by the assumption on $K$.

On the other hand, if we use the right-hand side in (2.2) we deduce
\begin{align*}
II := -\frac{1}{\pi} \int_{-R}^{R} xK(x) e^{w(x)}K(y)e^{w(y)} \, dy \, dx &= -\frac{1}{2\pi} \int_{-R}^{R} \int_{\mathbb{R}} K(x)e^{w(x)}K(y)e^{w(y)} \, dy \, dx \\
&\quad - \frac{1}{2\pi} \int_{-R}^{R} \int_{x-R}^{x+R} K(x)e^{w(x)}K(y)e^{w(y)} \, dy \, dx \\
&\rightarrow -\frac{\Lambda^2}{2\pi} + 0 \quad \text{as} \quad R \to \infty.
\end{align*}

Since $I = II$, we deduce that
\begin{equation*}
\frac{\Lambda}{2\pi}(\Lambda - 2\pi) = \int_{\mathbb{R}} (x\partial_x K)e^{w} \, dx.
\end{equation*}

Because $K$ is monotone decreasing in $|x|$ and non-constant, we see that $\int_{\mathbb{R}} (x\partial_x K)e^{w} \, dx < 0$. This implies that $\Lambda < 2\pi$ must hold. \hfill \Box

\textbf{Proof of Theorem 1 (i).} The proof directly follows from Lemmas 2.2 and 2.4. \hfill \Box

3. Even Symmetry

This section is devoted to the proof of item (ii) in Theorem 1. We implement the method of moving planes; actually, it is a ‘moving point’ argument since we are in one space dimension. Because of the nonlocal nature of the problem, it is expedient to work with the equation for $w(x)$ written in integral form. We then adapt the moving plane method generalized to integral equations, which was initiated in the work of [9].

For $\lambda > 0$ and $x \in \mathbb{R}$, we set
\begin{equation*}
\Sigma_{\lambda} := [\lambda, \infty), \quad x_{\lambda} := 2\lambda - x, \quad w_{\lambda}(x) := w(x_{\lambda}), \quad \Sigma_{\lambda}^w := \{x \in \Sigma_{\lambda} : w(x) > w_{\lambda}(x)\}.
\end{equation*}

From the proof of Lemma 2.3 we recall the integral representation
\begin{equation}
w(x) = \int_{\mathbb{R}} G(x - y)K(y)e^{w(y)} \, dy + C
\end{equation}
with some constant $C \in \mathbb{R}$ and we denote
\begin{equation*}
G(x) = -\frac{1}{\pi} \log |x|.
\end{equation*}

We first initiate the moving plane method by showing that $\Sigma_{\lambda}^w$ is empty in the regime of sufficiently large $\lambda$.

\textbf{Proposition 3.1.} \textit{There exists $\lambda_0 > 0$ such that $\Sigma_{\lambda}^w = \emptyset$ for all $\lambda > \lambda_0$.}

\textbf{Proof.} Let $\lambda > 0$. Using (3.1) and the fact $G$ is an even function, a calculation yields
\begin{equation*}
w(x) - w_{\lambda}(x) = \int_{\Sigma_{\lambda}} (G(x - y) - G(x_{\lambda} - y))(K(y)e^{w(y)} - K(y_{\lambda})e^{w_{\lambda}(y)}) \, dy.
\end{equation*}

Since $G(x)$ and $K(x)$ are monotone decreasing functions in $|x|$, we deduce
\begin{equation*}
G(x - y) \geq G(x_{\lambda} - y) \quad \text{and} \quad 0 < K(y) \leq K(y_{\lambda}) \quad \text{for} \ x, y \in \Sigma_{\lambda}.
\end{equation*}
For any \( x \in \Sigma^w_\lambda \subset \Sigma_\lambda \), we thus estimate
\[
w(x) - w_\lambda(x) \leq \int_{\Sigma_\lambda} (G(x - y) - G(x_\lambda - y)) K(y) \left( e^{w(y)} - e^{w_\lambda(y)} \right) dy
\]
\[
\leq \int_{\Sigma^w_\lambda} (G(x - y) - G(x_\lambda - y)) K(y) \left( e^{w(y)} - e^{w_\lambda(y)} \right) dy
\]
\[
\leq \int_{\Sigma^w_\lambda} (G(x - y) - G(x_\lambda - y)) F_\lambda(y) dy,
\]
where we denote
\[
F_\lambda(y) := K(y) e^{w(y)} (w(y) - w_\lambda(y)).
\]
Note that \( F_\lambda(y) > 0 \) for \( y \in \Sigma^w_\lambda \). Next, we observe the upper bounds
\[
-G(x_\lambda - y) = \frac{1}{\pi} \log |2\lambda - x - y| \leq \frac{1}{\pi} (\log 2 + |\log x| + |\log y|) \quad \text{for } x, y \in \Sigma_\lambda,
\]
and
\[
G(x - y) \leq 0 \quad \text{for } |x - y| \geq 1.
\]
Thus we find, for \( x \in \Sigma^w_\lambda \),
\[
0 < w(x) - w_\lambda(x) \leq \int_{\Sigma^w_\lambda \cap \{|x - y| \leq 1\}} G(x - y) F_\lambda(y) dy
\]
\[
+ \frac{1}{\pi} \int_{\Sigma^w_\lambda} (\log 2 + |\log x| + |\log y|) F_\lambda(y) dy.
\]
Next, we let \( \alpha := 1 + \delta > 1 \) with \( \delta > 0 \) taken from Assumption (A). Thus we deduce
\[
\| (x)^{-\alpha} (w - w_\lambda) \|_{L^1(\Sigma^w)} \leq \int_{\Sigma^w_\lambda} \left( \int_{\Sigma^w_\lambda \cap \{|x - y| \leq 1\}} (x)^{-\alpha} G(x - y) F_\lambda(y) dy \right) dx
\]
\[
+ \frac{1}{\pi} \int_{\Sigma^w_\lambda} \left( \int_{\Sigma^w_\lambda} (x)^{-\alpha} (\log 2 + |\log x| + |\log y|) F_\lambda(y) dy \right) dx
\]
\[
= \int_{\Sigma^w_\lambda} \left\{ C_1, \Sigma^w_\lambda(y) + C_2, \Sigma^w_\lambda(y) \right\} F_\lambda(y) dy,
\]
where we have
\[
(3.2) \quad C_1, \Sigma^w_\lambda := \int_{\Sigma^w_\lambda \cap \{|x - y| \leq 1\}} (x)^{-\alpha} G(x - y) dx \leq \int_{y-1}^{y+1} G(x - y) dx = \frac{2}{\pi},
\]
\[
(3.3) \quad C_2, \Sigma^w_\lambda := \int_{\Sigma^w_\lambda} (x)^{-\alpha} (\log 2 + |\log x| + |\log y|) dx \leq C (|\log y| + 1)
\]
with some constant \( C = C(\alpha) > 0 \) independent of \( \lambda > 0 \). Therefore, we have found that
\[
\| (x)^{-\alpha} (w - w_\lambda) \|_{L^1(\Sigma^w_\lambda)} \leq C \int_{\Sigma^w_\lambda} (|\log y| + 1) K(y) e^{w(y)} (w(y) - w_\lambda(y)) dy
\]
\[
\leq C \sup_{y \geq \lambda} (|\log y| + 1) K(y) e^{w(y)} \| (x)^{-\alpha} (w - w_\lambda) \|_{L^1(\Sigma^w_\lambda)}
\]
where the constant \( C > 0 \) is independent of \( \lambda \). Since \( e^w \in L^\infty(\mathbb{R}) \) and by Assumption (A) we have \( K \leq C (x)^{-1-\delta} \), there exists a constant \( \lambda_0 > 0 \) sufficiently large such that
\[
\| (x)^{-\alpha} (w - w_\lambda) \|_{L^1(\Sigma^w_\lambda)} \leq \frac{1}{2} \| (x)^{-\alpha} (w - w_\lambda) \|_{L^1(\Sigma^w_\lambda)} \quad \text{for } \lambda > \lambda_0.
\]
Thus the set \( \Sigma^w_\lambda \) has measure zero for \( \lambda > \lambda_0 \), which implies that \( \Sigma^w_\lambda = \emptyset \) for \( \lambda > \lambda_0 \) by continuity of \( w - w_\lambda \).

As a next step, we establish the following continuation property.

**Proposition 3.2.** Suppose that \( \lambda_0 > 0 \) satisfies \( \Sigma^w_\lambda = \emptyset \) for all \( \lambda > \lambda_0 \). Then there exists \( \epsilon > 0 \) such that \( \Sigma^w_\lambda = \emptyset \) for all \( \lambda > \lambda_0 - \epsilon \).
Proof. We divide the proof into the following steps.

Step 1. By assumption, we have \( w(x) \leq w_\lambda(x) \) for all \( x \geq \lambda \) and \( \lambda > \lambda_0 \). By continuity, we conclude that \( w(x) \leq w_{\lambda_0}(x) \) for all \( x \geq \lambda_0 \). This shows that \( \Sigma_{\lambda_0}^w = \emptyset \) holds.

To show that we indeed have \( \Sigma^w_\lambda = \emptyset \) for any \( \lambda > \lambda_0 - \epsilon \) with some \( \epsilon > 0 \), we argue as follows. First, we claim that the strict inequality holds:

\[
(3.4) \quad w(x) < w_{\lambda_0}(x) \quad \text{for all } x > \lambda_0.
\]

We argue by contradiction. Suppose that \( w(x) = w_{\lambda_0}(x) \) for some \( x > \lambda_0 \). Since \( K(x) \) is monotone decreasing in \( |x| \), we find

\[
(-\Delta)^{1/2}(w_{\lambda_0} - w)(x) = (K(x_{\lambda_0}) - K(x)) e^{w(x)} \geq 0.
\]

On the other hand, by using the singular integral expression for \((-\Delta)^{1/2}\), we conclude

\[
(-\Delta)^{1/2}(w_{\lambda_0} - w)(x) = \frac{1}{\pi} \text{PV} \int_{\mathbb{R}} \frac{(w_{\lambda_0} - w)(x) - (w_{\lambda_0} - w)(y)}{(x-y)^2} \, dy
\]

\[
= \frac{1}{\pi} \text{PV} \int_{\mathbb{R}} \frac{(w_{\lambda_0} - w)(y)}{(x-y)^2} \, dy
\]

\[
= \frac{1}{\pi} \text{PV} \int_{\lambda_0}^{\infty} \left( \frac{1}{(x-y)^2} - \frac{1}{(x-y_{\lambda_0})^2} \right) (w_{\lambda_0} - w)(y) \, dy \leq 0.
\]

Since we must have equality and \( x > \lambda_0 \), we deduce that \( w_{\lambda_0} - w \equiv 0 \) on \( \Sigma_{\lambda_0} \). Therefore,

\[
0 = (-\Delta)^{1/2}(w_{\lambda_0} - w)(y) = (K(y_{\lambda_0}) - K(y)) e^{w(y)} \quad \text{for } y \in \Sigma_{\lambda_0}.
\]

Thus \( K(y) = K(y_{\lambda_0}) \) for every \( y \in \Sigma_{\lambda_0} \), which means that \( K : \mathbb{R} \to \mathbb{R} \) is symmetric with respect to reflection at \( \{y = \lambda_0\} \) with some \( \lambda_0 > 0 \). Since \( K \) is also symmetric with respect to the origin by Assumption (A), we conclude that \( K \) is constant. But this contradicts Assumption (A). This completes our proof of claim (3.4).

Step 2. From the proof of Proposition 3.3, we recall the estimate

\[
\|\langle x \rangle^{-\alpha}(w - w_\lambda)\|_{L^1(\Sigma^w_\lambda)} \leq C_{\Sigma^w_\lambda} \sup_{\alpha} \left( \|\langle \log y \rangle + 1\|_{L^1(\Sigma^w_\lambda)} \right) \|\langle x \rangle^{-\alpha}(w - w_\lambda)\|_{L^1(\Sigma^w_\lambda)}.
\]

Now, in view of (3.4), we conclude that the set

\[
\Sigma^w_{\lambda_0} := \{ x \in \Sigma_\lambda : w(x) \geq w_{\lambda_0}(x) \} = \{ x = \lambda_0 \}
\]

has Lebesgue measure zero. Since \( \lim_{\lambda \searrow \lambda_0} \Sigma^w_\lambda \subset \Sigma^w_{\lambda_0} \) and, by inspecting the expression for \( C_{\Sigma^w_\lambda} \) (see (3.4) and (3.3)), we deduce that \( C_{\Sigma^w_\lambda} \to 0 \) as \( \lambda \searrow \lambda_0 \). Thus for some \( \epsilon > 0 \) sufficiently small, we find that

\[
\|\langle x \rangle^{-\alpha}(w - w_\lambda)\|_{L^1(\Sigma^w_\lambda)} \leq \frac{1}{2} \|\langle x \rangle^{-\alpha}(w - w_\lambda)\|_{L^1(\Sigma^w_\lambda)} \quad \text{for } \lambda > \lambda_0 - \epsilon,
\]

which implies that \( \Sigma^w_\lambda \) as Lebesgue measure zero and hence is empty for \( \lambda > \lambda_0 - \epsilon \) by the continuity of \( w - w_\lambda \).

This completes the proof of Proposition 3.3. \( \square \)

Finally, we show the following closedness property.

Proposition 3.3. If \( \Sigma^w_\lambda = \emptyset \) for all \( \lambda > \lambda_0 \) and \( \lambda_n \to \lambda_* \), then \( \Sigma^w_\lambda = \emptyset \) for all \( \lambda > \lambda_* \).

Proof. This property holds trivially by construction. Indeed, let \( \lambda > \lambda_* \), be arbitrary. Since \( \lambda_n \to \lambda_* \), there exists \( n \in \mathbb{N} \) such that \( \lambda > \lambda_n \) and hence \( \Sigma^w_\lambda = \emptyset \). \( \square \)

Proof of Theorem 1 (ii): Symmetry. We are now ready to give the proof of Theorem 1 (ii). By combining Propositions 3.3, 3.4 and using a standard open-closed argument, we deduce that \( \Sigma^w_\lambda = \emptyset \) for all \( \lambda > 0 \). By continuity of \( w \) and passing to the limit \( \lambda \to 0^+ \), we thus obtain

\[
w(x) \leq w(-x) \quad \text{for all } x \geq 0.
\]

On the other hand, replacing \( w(x) \) by \( \tilde{w}(x) = w(-x) \) yields another solution of (1.1). By re-running the moving plane argument above, we find the opposite inequality \( w(-x) \leq w(x) \) for \( x \geq 0 \). This shows that \( w(x) = w(-x) \) for all \( x \in \mathbb{R} \).

Finally, we show that \( w \) is a decreasing function of \( |x| \). Since \( w \) is even, it suffices to prove that \( w(y) \leq w(x) \) for \( y > x \geq 0 \). In fact, let \( y > x \geq 0 \) and define \( \lambda = \frac{y-x}{2} > 0 \),
which directly yields \( x = 2\lambda - y = y_1 \) and \( y \in \Sigma_\lambda \). Since \( \Sigma_{\lambda_2} = \emptyset \), we see that \( w(y) \leq w_1(y) = w(y_1) = w(x) \).

The proof of Theorem 1 (ii) is now complete. \( \square \)

4. Compactness and A-Priori Estimates

In this section, we derive results which will be used to prove Theorem 2 (iii) about existence of solutions. Some estimates will also be later needed in the rest of the proof of Theorem 2 below.

4.1. Preliminaries. We start by recasting the nonlocal Liouville equation (1.1) into a more convenient form as follows. Recall that we will always assume that the \( Q \)-curvature function \( K(x) > 0 \) satisfies Assumption (A).

Suppose that \( w \in L^1_1(\mathbb{R}) \) is a solution of (1.1) with \( Ke^w \in L^1(\mathbb{R}) \). We define the positive function \( \varphi : \mathbb{R} \rightarrow \mathbb{R}_{>0} \) by setting

\[
\varphi(x) = K(x)e^{w(x)}.
\]

From Lemmas 2 and 3, we see that \( \varphi \) is of class \( C^1 \) and \( \varphi \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \). Furthermore, by applying the Hilbert transform \( H \) on both sides of (1.1) and using that \( H(-\Delta)^{\frac{1}{2}} = -\partial_x \), we obtain

\[
- \partial_x w = H(\varphi).
\]

Since \( w \) is \( C^1 \), this is equivalent to the integral equation

\[
w(x) = w(0) - \int_0^x H(\varphi(y)) \, dy.
\]

Using that \( \varphi(x) = K(x)e^{w(x)} \) and \( K(x) > 0 \), we can rewrite this equation as

\[
\varphi(x) = \varphi(0) \frac{K(x)}{K(0)} e^{-\int_0^x H(\varphi(y)) \, dy}.
\]

Next, by using that \( \varphi(x) > 0 \) is a positive \( C^1 \)-function, we can introduce the positive \( C^1 \)-function \( \nu : \mathbb{R} \rightarrow \mathbb{R}_{>0} \) with \( \nu(x) = \sqrt{\varphi(x)} \). By taking the square root in the previous equation, we obtain

\[
\nu(x) = \nu(0) \left( \frac{K(x)}{K(0)} \right)^{\frac{1}{2}} e^{-\int_0^x \frac{1}{2} H(\nu^2(y)) \, dy}.
\]

By differentiating, we readily check that \( \nu \) solves the nonlinear equation

\[
\partial_x \nu - (\partial_x \log \sqrt{K}) \nu + \frac{1}{2} H(\nu^2) \nu = 0.
\]

We remark that, for the special case \( K(x) = e^{-x^2} \), we retrieve the ground state soliton equation for the harmonic Calogero–Moser DNLS. Of course, the following analysis will allow for more general \( K(x) \) that satisfy Assumption (A). We also note that \( \nu = \sqrt{\nu} \in L^2(\mathbb{R}) \). Since \( \nu \) is even and decreasing in \( |x| \), the same holds for \( \nu(x) = \sqrt{\varphi(x)} = \sqrt{K(x)e^{w(x)}} \) due to the symmetric-decreasing property of \( K(x) \). Since \( \nu = \nu^* \) is also symmetric-decreasing, we deduce \( e^{-\int_0^x \frac{1}{2} H(\nu^2) \, dy} \leq 1 \) for all \( x \in \mathbb{R} \) by Lemma 2.3. Now, a glimpse at (1.6) yields the bound

\[
\left\| \partial_x \nu \right\|_{L^2} \leq \left\| \nu(0) \right\|_{L^2} \left\| \partial_x \sqrt{K} \right\|_{L^2} \nu + \frac{1}{2} \left\| H(\nu^2) \right\|_{L^2} \leq \left\| \nu(0) \right\|_{L^2} + \left\| \nu \right\|_{L^2} \nu \left\| \nu^2 \right\|_{L^2} \leq \left\| \nu(0) \right\| + \left\| \nu \right\|_{L^2} < +\infty.
\]

by also using \( \nu = \sqrt{\nu} \in L^4(\mathbb{R}) \) in view of the fact that \( \nu \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \). This show that \( \nu \in H^1(\mathbb{R}) \) holds. Moreover, by Lemma 2.3, we notice that

\[
\int_{\mathbb{R}} \log(1 + |x|) \nu(x)^2 \, dx < +\infty.
\]

On the other hand, it is easy to see that any solution \( \nu \in H^1(\mathbb{R}) \) with \( \nu(0) > 0 \) of (1.3) is automatically \( C^1 \) and gives rise to a solution \( w \in L_{1/2}(\mathbb{R}) \) of (1.1) by setting \( w = \log(K^{-1/2} \nu^2) \) satisfying the finiteness condition \( \int_{\mathbb{R}} Ke^w \, dx = \int_{\mathbb{R}} \nu^2 \, dx < +\infty \). In
summarize and view of the symmetry result shown in Theorem 1(ii), we have established the following result.

**Proposition 4.1.** A function \( w \in L^2_+(\mathbb{R}) \) solves \((1.1)\) with \( Ke^w \in L^1(\mathbb{R}) \) if and only if 
\[
\int_{\mathbb{R}} \log(1 + |x|)K(x)e^{w(x)} \, dx = \int_{\mathbb{R}} \log(1 + |x|)|v(x)|^2 \, dx < +\infty.
\]

Finally, every solution \( v \in H^1(\mathbb{R}) \) of \((1.5)\) must be symmetric-decreasing, i.e., it holds that \( v = v^* \).

Based on the discussion above, we make the following definitions. We let \( X \) denote the real Hilbert space of real-valued and even functions on \( \mathbb{R} \) given by
\[
X := \{ u : \mathbb{R} \to \mathbb{R} \mid u(-x) = u(x) \text{ and } \| u \|_X < +\infty \},
\]
where we define the norm via
\[
\| u \|_X^2 := \| u \|_{H^1}^2 + \int_{\mathbb{R}} \log(1 + |x|)|u(x)|^2 \, dx.
\]
Moreover, we define the set
\[
X^* := \{ u \in X \mid u = u^* \}
\]
which is the set of symmetric-decreasing functions that belong to the space \( X \). We note that \( X^* \) is a closed and convex subset of \( X \).

4.2. **Compactness, A-Priori Bounds and Existence.** For \( \lambda > 0 \) and \( u \in X^* \) given, we set
\[
T_\lambda(u)(x) := \lambda \sqrt{K(x)}e^{-\frac{1}{2} \int_{x}^{\infty} H(u^2)(y) \, dy}.
\]
In view of the Proposition 1.1 and the symmetry result in Theorem 2(ii), we note that
\[(4.7) \quad T_\lambda(v) = u \iff w = \log(K^{-1}v^2) \text{ solves } (1.1) \text{ with } \int_{\mathbb{R}} Ke^w = \int_{\mathbb{R}} v^2,
\]
where, of course, we always assume that \( K \) satisfies Assumption \((A)\). We record the following fact.

**Lemma 4.1.** The map \( T_\lambda : X^* \to X^* \) is well-defined and continuous.

**Proof.** We first show that \( T_\lambda \) maps \( X^* \) into itself. Indeed, let \( \lambda > 0 \) and \( u \in X^* \) be given. By Lemma 3.1 we deduce that
\[
\psi_u(x) := e^{-\frac{1}{2} \int_{x}^{\infty} H(u^2)(y) \, dy}
\]
is a symmetric-decreasing function with \( 0 < \psi_u(x) \leq 1 \) for all \( x \in \mathbb{R} \). Using Assumption \((A)\) we find that \( T_\lambda(u) = \lambda \sqrt{K} \psi_u \) is symmetric-decreasing and belongs to \( L^2(\mathbb{R}) \). Next, we verify that \( \partial_x T_\lambda(u) \in L^2(\mathbb{R}) \) by using Assumption \((A)\) as well as \( H(u^2) \in H^1(\mathbb{R}) \subset L^\infty(\mathbb{R}) \). Finally, by using that \( 0 < \psi_u(x) \leq 1 \) again together with the decay properties of \( K \), we directly see that \( \log(1 + |\cdot|)|T_\lambda(u)|^2 \in L^1(\mathbb{R}) \), whence it follows that \( T_\lambda(u) \in X^* \) holds.

Next, we show that \( T_\lambda : X^* \to X^* \) is a continuous map. Suppose that \( u_k \in X^* \) satisfies \( u_k \to u \) in \( X \). We readily check that \( \psi_{u_k}(x) \to \psi_u(x) \) for almost every \( x \in \mathbb{R} \). By dominated convergence and \( 0 < \psi_{u_k}(x) \leq 1 \), we deduce that \( T_\lambda(u_k) \to T_\lambda(u) \) in \( X \). \( \square \)

Next, we establish the following local Lipschitz estimate.

**Lemma 4.2.** The map \( T_\lambda : X^* \to X^* \) satisfies the estimate
\[
\| T_\lambda(v) - T_\lambda(u) \|_X \lesssim \lambda (\| u \|_X + \| v \|_X)(1 + \lambda \| u \|_X^2 + \lambda \| v \|_X^2)\| u - v \|_X^2.
\]

**Proof.** We first establish some auxiliary estimate as follows. For \( f \in X^* \), we use again the short-hand notation
\[
\psi_f(x) := e^{-\frac{1}{2} \int_{x}^{\infty} H(f^2)(y) \, dy}.
\]
By Lemma 4.1 it holds that $0 < \psi_f(x) \leq 1$ for any $f \in X^*$. Furthermore, by using the boundedness of the Hilbert transform $H$ on $L^p(R)$ for any $p \in (1, \infty)$, we deduce, for any $x \in \mathbb{R}$, the pointwise bound
\[
|\psi_u(x) - \psi_v(x)| \leq \frac{1}{2} \left( \int_0^{|x|} H(u^2) \, dy - \int_0^{|x|} H(v^2) \, dy \right) \leq \frac{1}{2} \left( \int_0^0 H(u^2 - v^2) \, dy \right)
\]
\[
\leq \frac{1}{2} |x|^{1/8} \|H(u^2 - v^2)\|_{L^p} \leq C|x|^{1/8} \|u^2 - v^2\|_{L^p}
\]
\[
\leq C|x|^{1/8} \|u + v\|_{L^\infty} \|u - v\|_{L^2} \leq C|x|^{1/8} \left( \|u\|_{X} + \|v\|_{X} \right) \|u - v\|_{L^2}
\]
with some constant $C = C(p) > 0$ and $1/q + 1/p = 1$, $1 < p \leq 2$, $1/2 + 1/r = 1/p$, $r > 2$. Since $|x|^{1/8} \sqrt{K} \in L^2(\mathbb{R})$ by Assumption (A), provided we take $q \gg 1$ sufficiently large (and thus $p > 1$ is sufficiently close to 1), we obtain
\[
\|\mathcal{T}_\lambda(u) - \mathcal{T}_\lambda(v)\|_{L^2} \leq C\lambda(\|u\|_{X} + \|v\|_{X}) \|x|^{1/8} \sqrt{K}\|_{L^2} \|u - v\|_{L^2}
\]
\[
\leq C\lambda(\|u\|_{X} + \|v\|_{X}) \|u - v\|_{L^2}.
\]
Likewise, we use $\sqrt{\log(1 + |x|)} |x|^{1/8} \sqrt{K} \in L^2(\mathbb{R})$ for $q \gg 1$ to conclude
\[
\|\sqrt{\log(1 + |x|)}(\mathcal{T}_\lambda(u) - \mathcal{T}_\lambda(v))\|_{L^2} \leq C\lambda(\|u\|_{X} + \|v\|_{X}) \|u - v\|_{L^2}.
\]
Next, we notice that
\[
\partial_x \mathcal{T}_\lambda(u) = \lambda \left( \partial_x \sqrt{K} \psi_u - \frac{1}{2} \sqrt{K} \psi_u H(u^2) \right).
\]
Using that $|x|^{1/8} \sqrt{K} \in L^2(\mathbb{R})$, $\sqrt{K} \in L^\infty(\mathbb{R})$, and $|\psi_u|, |\psi_v| \leq 1$, we find
\[
\|\partial_x (\mathcal{T}_\lambda(u) - \mathcal{T}_\lambda(v))\|_{L^2} \leq C\lambda \left( \|\partial_x \sqrt{K}(\psi_u - \psi_v)\|_{L^2} + \|\sqrt{K}(\psi_u H(u^2) - \psi_v H(v^2))\|_{L^2} \right)
\]
\[
\leq C\lambda(\|u\|_{X} + \|v\|_{X}) \left( 1 + \|H(u^2)\|_{L^\infty} + \|H(v^2)\|_{L^\infty} \right) \|u - v\|_{L^2}
\]
\[
+ C\lambda(\|u\|_{X} + \|v\|_{X}) \|H(u^2) - H(v^2)\|_{L^2}
\]
\[
\leq C\lambda(\|u\|_{X} + \|v\|_{X}) \left( 1 + \|u\|_{X}^2 + \|v\|_{X}^2 \right) \|u - v\|_{L^2}.
\]
Here we also used that, by Sobolev embeddings and the boundedness of $H$ on $H^1(\mathbb{R})$ that we have
\[
\|H(f^2)\|_{L^\infty} \leq C\|H(f^2)\|_{L^1} \leq C\|f^2\|_{H^1} \leq C\|f\|_{X}^2
\]
as well as
\[
\|H(u^2) - H(v^2)\|_{L^2} = \|u^2 - v^2\|_{L^2} \leq (\|u\|_{X} + \|v\|_{X}) \|u - v\|_{L^2}.
\]
In view of the estimates above, we complete the proof. \hfill \Box

Next, we establish the following result.

**Lemma 4.3.** The map $T_\lambda : X^* \to X^*$ is compact.

**Proof.** Let $(u_k)_{k=1}^\infty$ be a bounded sequence in $X^*$. By Proposition 4.1 and passing to a subsequence if necessary, it holds that $(u_k)$ converges strongly in $L^2(\mathbb{R})$ and thus forms a Cauchy sequence in $L^2(\mathbb{R})$. From the estimate in Lemma 12 we deduce that $(T_\lambda(u_k))_{k=1}^\infty$ is a Cauchy sequence in $X$, which converges to some element in $X^*$ due to the closedness of this subset. \hfill \Box

As a next step, we show the following a-priori bound for fixed points of the compact map $T_\lambda : X^* \to X^*$.

**Lemma 4.4.** For any $v \in X^*$ with $v = T_\lambda(v)$, it holds that
\[
\|T_\lambda(v)\|_{X} \leq \lambda^2 + \lambda.
\]

**Proof.** Suppose that $v \in X^*$ satisfies $T_\lambda(v) = v$. By the Pohozaev-type result in Lemma 2.3 we deduce the a-priori bound
\[
\|v\|_{L^2}^2 < 2\pi \leq 1.
\]
Next, since $v \in X^*$ is symmetric-decreasing, an application of Lemma 1.1 yields that $e^{-\frac{\lambda}{2} \int_0^x H(v^2)} \leq 1$ for all $x \in \mathbb{R}$. Hence we directly obtain the pointwise bounds
\[
0 < v(x) \leq \lambda \sqrt{K(x)} \leq \lambda(x)^{-1/2 - \delta},
\]
for some $\delta > 0$. From this we readily deduce the bounds
\[
\|v\|_{L^2} \lesssim \lambda, \quad \|\sqrt{\log(1 + |\cdot|)} v\|_{L^2} \lesssim \lambda.
\]
To bound $\partial_x v = \partial_x T_\lambda(v)$, we use the bound $e^{-\frac{1}{2} \int_0^1 H(v^2)} \leq 1$ once again together with the Gagliardo-Nirenberg interpolation estimate and the $L^2$-boundedness of $H$. This yields
\[
\|\partial_x v\|_{L^2} \lesssim \|\partial_x \sqrt{K} e^{-\frac{1}{2} \int_0^1 H(v^2)}\|_{L^2} + \|vH(v^2)\|_{L^2} \\
\lesssim \lambda(\|\partial_x \sqrt{K}\|_{L^2} + \|v\|_{L^\infty} \|v^2\|_{L^2}) \lesssim \lambda(1 + \|v\|_{L^2}^2) \\
\lesssim \lambda \left(1 + \|v\|_{L^2}^{3/2}\|\partial_x v\|_{L^2}^{1/2}\right) \lesssim \lambda \left(1 + \|\partial_x v\|_{L^2}^{1/2}\right),
\]
where in the last step we also used that $\|v\|_{L^2} \lesssim 1$. An elementary argument now yields $\|\partial_x v\|_{L^2} \lesssim \lambda^2 + \lambda$. By recalling the definition of $\|v\|_X$ and the bounds found above, we deduce
\[
\|v\|_X \lesssim \lambda^2 + \lambda,
\]
which completes the proof. \hfill $\Box$

We are now in the position to show the following existence result.

**Proposition 4.2.** For any $\lambda > 0$, the map $T_\lambda : X^* \to X^*$ has a fixed point. Consequently, for any $v_0 > 0$ there exists a solution $v \in X^*$ of equation (4.5) with $v(0) = v_0 = \lambda\sqrt{K}(0)$.

Likewise, for any $w_0 \in \mathbb{R}$, there exists a solution $w \in L_{1/2}(\mathbb{R})$ of $\Box$ with $Ke^w \in L^1(\mathbb{R})$ and $w(0) = w_0$.

**Proof.** Let $\lambda > 0$ be given. By Lemma 4.4 there exists a constant $M = M(\lambda) > 0$ such that the following implication holds true:
\[
(4.9) \quad \exists v \in X^* \text{ and } \exists \sigma \in (0, 1] \text{ with } v = \sigma T_\lambda(v) \implies \|v\|_X \leq M.
\]
We consider the closed convex set
\[
S := \{u \in X^* : \|u\|_X \leq M\} \subset X
\]
along with the mapping $T^* : S \to S$ defined as
\[
T^*(v) := \begin{cases}
T_\lambda(v) & \text{if } \|T_\lambda(v)\| \leq M, \\
M \frac{T_\lambda(v)}{\|T_\lambda(v)\|_X} & \text{if } \|T_\lambda(v)\| > M.
\end{cases}
\]
Clearly, the map $T^* : S \to S$ is well-defined and continuous. Furthermore, since $T_\lambda(S)$ is precompact by the compactness of $T_\lambda : X^* \to X^*$, we see that the image $T^*(S)$ is precompact in the Banach space $X$. By a suitable version of Schauder’s fixed point theorem (see [18] Corollary 11.2), we conclude the map $T^* : S \to S$ has a fixed point $v \in S$. We claim that $v \in S$ is a fixed point of $T_\lambda$ as well. To show this, let us suppose that $\|T_\lambda(v)\| \geq M$. Then $v = T^*(v) = \sigma T_\lambda(v)$ with $\sigma = M/\|T_\lambda(v)\| \leq 1$. But this contradicts (4.9).

Thus we have proven that there exists $v \in S \subset X^*$ such that $T_\lambda(v) = v$, whence $v$ solves (4.5) with $v(0) = \lambda\sqrt{K}(0)$. Finally, by the discussion in Subsection 4.1, the existence of $v$ with $v(0) = v_0 > 0$ given is equivalent to the fact that $w = \log(K^{-1}v^2) \in L_{1/2}(\mathbb{R})$ is a solution of (4.1) with $w(0) = w_0 = \log(K^{-1}(0)v(0)^2) \in \mathbb{R}$ and $\int_{\mathbb{R}} Ke^w dx = \int_{\mathbb{R}} v^2 dx < +\infty$. \hfill $\Box$

**Proof of Theorem (iii): Existence.** This directly follows from Proposition 4.2. \hfill $\Box$

5. **Uniqueness**

In this section, we prove the uniqueness result stated in Theorem 2. This will be the main result of this paper.
5.1. Nondegeneracy and Local Uniqueness. Recalling the definition of the Banach space $X$ above, we define the map $F : X \times (0, \infty) \to X$ by setting

$$F(u, \lambda)(x) := \lambda \sqrt{K(x)} e^{-\frac{1}{2} \int_0^y h(u(z)) \, dz} - u(x).$$

Thus, in terms of the map $T_{\lambda}$ introduced in the previous section, we can write

$$F(u, \lambda) = T_{\lambda}(u) - u.$$

However, the reader should be aware of the fact that we extend the map $T_{\lambda}$ from $X^* \subset X$ to all of $X$ here. By standard estimates, it is straightforward to check that the mapping $F : X \times (0, \infty) \to X$ is indeed well-defined and of class $C^1$; see Lemma A.2 below. By construction, we have the equivalence

$$F(v, \lambda) = 0 \iff v \in X \text{ solves } \tag{5.2}$$

$$\lambda = \frac{v(0)}{\sqrt{K(0)}}.$$

Our next goal is to apply the implicit function theorem in order to construct a locally unique $C^1$-branch $\lambda \mapsto \lambda_0$ around a given solution $(v, \lambda)$ satisfying $F(v, \lambda) = 0$. As a key result, we shall need to prove that the Fréchet derivative $\partial_\lambda F$ has a bounded inverse on $X$. Indeed, we notice that

$$\partial_\lambda F(v, \lambda) = K - I,$$

where $K : X \to X$ denotes the bounded linear operator given by

$$Kf(x) = -v(x) \int_0^x H(vf)(y) \, dy.$$  

We record the following basic fact.

**Lemma 5.1.** Suppose that $F(v, \lambda) = 0$. Then the linear operator $K$ extends to a bounded map from $L^2_{\text{even}}(\mathbb{R})$ into $X$. As a consequence, the linear operator $K : L^2_{\text{even}}(\mathbb{R}) \to L^2_{\text{even}}(\mathbb{R})$ is compact.

**Proof.** We show that the linear operator $K$ extends to a bounded map from $L^2_{\text{even}}(\mathbb{R})$ into $X$ as follows. Let $f \in L^2_{\text{even}}(\mathbb{R})$ be given. Using that $v \in H^1(\mathbb{R})$ and by Sobolev embeddings, we deduce from Hölder’s inequality together with the boundedness of $H$ on $L^p(\mathbb{R})$ when $p \in (1, \infty)$ that we have the pointwise bound

$$\left| \int_0^x H(vf)(y) \, dy \right| \leq |x|^{1/4} \|H(vf)\|_{L^p} \leq C|x|^{1/4} \|vf\|_{L^p} \leq C|x|^{1/4} \|vf\|_{H^1} \|f\|_{L^2}$$

where $\frac{1}{2} + \frac{1}{q} = 1$, $1 < p \leq 2$, and with some constant $C = C(p) > 0$. Next, we use that $F(v, \lambda) = 0$ holds and thus $v = v^* \in X^*$ is symmetric-decreasing (see Proposition 4.1). By Lemma A.1, this implies that $e^{-\frac{1}{2} \int_0^y h(v^z) \, dz} \leq 1$ for all $x \in \mathbb{R}$. In particular, this shows that $0 < v(x) \leq \lambda \sqrt{K(x)}$, which implies the pointwise bound

$$|Kf(x)| \leq C\|f\|_{L^2} \sqrt{K(x)} |x|^{1/4} \leq C\|f\|_{L^2} \sqrt{K(x)} x^{-\frac{1}{4} - \epsilon},$$

for some $\epsilon > 0$, where the last inequality follows from the assumed bound for $K$ and by taking $q \gg 1$ sufficiently large (and thus $p > 1$ sufficiently close to 1). Clearly, the bound leads to

$$\int_{\mathbb{R}} |Kf(x)|^2 \, dx + \int_{\mathbb{R}} \log(1 + |x|)|Kf(x)|^2 \, dx \leq C\|f\|_{L^2}^2$$

with some constant $C > 0$ independent of $f$.

Next, by differentiating and using the equation satisfied by $v$, we observe that

$$\|\partial_\lambda Kf\|_{L^2} \leq \|\partial_\lambda v\|_{L^2} \|H(vf)\|_{L^2} + \|vfH(vf)\|_{L^2}$$

$$\leq C \left( \|\partial_\lambda \sqrt{K} \|_{L^2} \|x|^{1/4}\|_{L^2} + \|\sqrt{K} \|_{L^2} \|x|^{1/4}\|_{L^2} + \|vf\|_{L^2} \right) \|f\|_{L^2} \leq C\|f\|_{L^2}$$

with some constant $C > 0$ independent of $f$. Again, we have chosen $q \gg 1$ sufficiently large and we have used the pointwise bounds for $\sqrt{K}$ and $\partial_\lambda \sqrt{K}$. In summary, we have shown that

$$\|Kf\|_{X} \leq C\|f\|_{L^2}$$
with some constant $C > 0$ independent of $f$. Since $v$ and $f$ are even functions, we readily check that $Kf$ is even as well. Hence we have proven that the linear map $K : L_{	ext{even}}^2(\mathbb{R}) \to X$ is bounded.

Finally, we note that the map $K : L_{	ext{even}}^2(\mathbb{R}) \to L_{	ext{even}}^2(\mathbb{R})$ is compact due to fact that the embedding $X \subset L_{	ext{even}}^2(\mathbb{R})$ is compact; see Proposition A.1 below. □

Next, we establish the following key result.

**Lemma 5.2.** Let $F(v, \lambda) = 0$ hold. Then the Fréchet derivative $\partial_v F = K - 1$ is invertible on $X$ with bounded inverse.

**Proof.** Since $K$ maps $L_{	ext{even}}^2(\mathbb{R})$ into $X$, it suffices to show $K - 1$ is invertible on $L_{	ext{even}}^2(\mathbb{R})$. By the compactness of $K$ on $L_{	ext{even}}^2(\mathbb{R})$ and the Fredholm alternative, this amounts to showing the implication

$$Kf = f \quad \text{and} \quad f \in L_{	ext{even}}^2(\mathbb{R}) \Rightarrow f = 0.$$  

Indeed, let us assume that $f \in L_{	ext{even}}^2(\mathbb{R})$ solves $Kf = f$. Since $f \in \text{ran} \ K$, we obtain that $f \in X$ and in particular the function $f$ is continuous. Next, we note that the equation $Kf = f$ can be written as

$$v\psi = f,$$

where we define the even and continuous function $\psi : \mathbb{R} \to \mathbb{R}$ by setting

$$\psi(x) := -\int_0^x H(vf)(y) \, dy.$$  

Since $H(vf) \in L^2(\mathbb{R})$, we see that $\partial_x \psi \in L^2(\mathbb{R})$. On the other hand, we see that $(-\Delta)^{1/2} \psi = H\partial_x \psi = -H^2(vf) = v^2 f \in L^2(\mathbb{R})$. Therefore, we find that $\psi \in \dot{H}_{	ext{even}}^1(\mathbb{R})$ solves the equation

$$(-\Delta)^{1/2} \psi - v^2 \psi = 0 \quad \text{in} \mathbb{R}.$$  

Now, by using that $W = -v^2$ is $C^1$ and monotone increasing on $[0, \infty)$ and $\psi \in \dot{H}_{	ext{even}}^1(\mathbb{R})$ with $\psi(0) = 0$, we obtain that $\psi \equiv 0$ by Lemma 5.3 below. Thus $f = v\psi = 0$ is the zero function. This shows \(^{13}.\)

In summary, we have shown that the bounded linear operator $\partial_v F = K - 1$ is invertible on $X$. By bounded inverse theorem, its inverse $(\partial_v F)^{-1} : X \to X$ is bounded as well. □

**Lemma 5.3** (Key Lemma). Let $W : \mathbb{R} \to \mathbb{R}$ be a $C^1$-function with $W'(x) \geq 0$ for $x \geq 0$. Assume that $\psi \in \dot{H}_{	ext{even}}^1(\mathbb{R})$ solves

$$(-\Delta)^{1/2} \psi + W \psi = 0 \quad \text{in} \mathbb{R}$$

with $W(x)\psi(x)^2 \to 0$ as $x \to +\infty$. Then $\psi(0) = 0$ implies that $\psi \equiv 0$.

**Remarks.** 1) The assumption above that $\psi$ is even function is essential. For example, the odd function $\psi(x) = \frac{2}{\pi x} \in H^1(\mathbb{R})$ (and hence $\psi(0) = 0$) solves the equation

$$(-\Delta)^{1/2} \psi - \frac{2}{1 + x^2} \psi = 0 \quad \text{in} \mathbb{R}.$$  

2) We refer to Appendix B for a discussion which relate the arguments below to monotonicity formulas for $(-\Delta)^s$ in \cite{6, 5, 14}, which involve the $s$-harmonic extension.

**Proof.** By integrating the equation on $[0, \infty)$ against $\partial_x \psi \in L^2(\mathbb{R})$, we find

$$I + II := \int_0^{+\infty} ((-\Delta)^{1/2} \psi(x)\partial_x \psi(x) \, dx + \int_0^{+\infty} W(x)\psi(x)\partial_x \psi(x) \, dx = 0.$$  

We remark that $(-\Delta)^{1/2} \psi, W\psi \in L^2(\mathbb{R})$ holds and hence the integrals above are absolutely convergent. To analyze the first term, we set $g := \partial_x \psi$ and we notice that

$$I = \int_0^{+\infty} (Hg)(x)g(x) \, dx = \frac{1}{\pi} \int_0^{+\infty} \left( PV \int_{-\infty}^{+\infty} \frac{g(y)}{x - y} \, dy \right) g(x) \, dx$$

$$= -\frac{1}{\pi} \int_0^{+\infty} \int_0^{+\infty} \frac{g(x)g(y)}{x + y} \, dx \, dy,$$
Thus there exists $v$ such that

$$u_{decreasing} \text{ by Proposition 4.1, we see that}$$

the interval is unique for

$$0 < \lambda \leq 0.5.$$ 

There exists $N$ such that

$$\exists \text{ a-priori bounds.}$$

Let

$$\psi(0) = 0 \text{ and } W(x)\psi(x)^2 \to 0$$

as $x \to +\infty$, we obtain from $W'(x) \geq 0$ for $x \geq 0$ that

$$II = \int_0^{+\infty} W(x)\psi(x)dx = -\frac{1}{2} \int_0^{+\infty} W'(x)\psi(x)^2dx \leq 0.$$ 

Because of $I + II = 0$, we conclude that $I = 0$ must hold and thus $g = \partial_x \psi \equiv 0$. Hence

$\psi$ is a constant function. Since $\psi(0) = 0$ by assumption, this implies that $\psi \equiv 0$. 

By applying the implicit function theorem together with Lemma 5.2, we can construct a unique local branch around any given solution of the equation $F(v, \lambda) = 0$ as follows.

**Proposition 5.1.** Let $(v, \lambda) \in X \times (0, \infty)$ solve $F(v, \lambda) = 0$. Then there exists an open interval $I = (\lambda - \epsilon, \lambda + \epsilon) \cap (0, \infty)$ with some $\epsilon = \epsilon(\|v\|X) > 0$ and a $C^1$-map

$$I \to X, \quad t \mapsto v_t$$

such that $v_{t=\lambda} = v$ and $F(v_t, t) = 0$ for all $t \in I$. Moreover, there exists a neighborhood $N \subset X$ around $v$ such that $F(u, t) = 0$ with $(u, t) \in N \times I$ implies that $u = v_t$.

Below, we will see that we can extend every such local branch $t \mapsto v_t$ to all of $t \in (0, \lambda]$ thanks to a-priori bounds.

5.2. **Global Uniqueness and Proof of Theorem 2.** First, we notice that we must have global uniqueness of solutions of $F(v, \lambda) = 0$ for sufficiently small $\lambda > 0$.

**Proposition 5.2.** There exists $0 < \lambda_* \ll 1$ such that the solution $v \in X$ of $F(v, \lambda) = 0$ is unique for $0 < \lambda \leq \lambda_*$. 

**Proof.** Let $u, v \in X$ both solve $F(u, \lambda) = F(v, \lambda) = 0$. Since $u, v \in X$ are symmetric-decreasing by Proposition 4.1, we see that $u$ and $v$ are fixed points of the map $\Gamma : X^* \to X^*$, if we combine the a-priori bounds in Lemma 4.2 with Lipschitz estimates in Lemma 1.3, we deduce that

$$\|u - v\|_{L^2} \lesssim \lambda(\lambda^6 + \lambda^\lambda)\|u - v\|_{L^2}.$$ 

Thus there exists $0 < \lambda_* \ll 1$ sufficiently small such that $\lambda \in (0, \lambda_*)$ implies $\|u - v\|_{L^2} \leq \frac{\lambda}{2}\|u - v\|_{L^2}$ and hence $u \equiv v$. 

We are now ready to prove Theorem 2 as follows. Let $w, \tilde{v} \in L_{1/2}(\mathbb{R})$ be two solutions of

$$\int_{\mathbb{R}} K^w e^{\tilde{v}} \in L^1(\mathbb{R}).$$

Correspondingly, we define the functions $v = \sqrt{K^w e^v} \in X$ and $\tilde{v} = \sqrt{K^w e^{\tilde{v}}} \in X$. Suppose now that $w(0) = \tilde{v}(0)$ holds, which implies $v(0) = \tilde{v}(0)$. By the previous discussion, we have $F(v, \lambda) = F(\tilde{v}, \lambda) = 0$ with $\lambda = v(0) = \tilde{v}(0) > 0$. By Proposition 5.1 and thanks to the a-priori bounds in Lemma 4.2, we can construct two branches

$$t \mapsto v_t \quad \text{and} \quad t \mapsto \tilde{v}_t$$

for all $t \in (0, \lambda]$ such that $F(v_t, t) = F(\tilde{v}_t, t) = 0$ for $t \in (0, \lambda]$ and $v_{t=\lambda} = v$ and $\tilde{v}_{t=\lambda} = \tilde{v}$. Suppose now $w \neq \tilde{v}$ and hence $v \neq \tilde{v}$. By the local uniqueness property in Proposition 5.1, the branches can never intersect, i.e., we have $v_t \neq \tilde{v}_t$ for all $t \in (0, \lambda]$. But this contradicts the uniqueness result in Proposition 5.2 whenever $0 < t \ll 1$ is sufficiently small. Therefore, we conclude that $v = \tilde{v}$ and hence $w = \log(K^{-t}v^2) = \log(K^{-t}\tilde{v}^2) = \tilde{w}$ holds true.

The proof of Theorem 2 is now complete. 

\[\square\]
Appendix A. Some Technical Facts

Lemma A.1. Let $f \in X$ be symmetric-decreasing, i.e., we have $f = f^*$. Then its Hilbert transform satisfies

$$H(f)(x) \geq 0 \quad \text{for} \quad x \geq 0 \quad \text{and} \quad H(f)(x) \leq 0 \quad \text{for} \quad x \leq 0.$$ 

Proof. The proof is elementary. Since $f = f^*$ is an even function, we note that its Hilbert transform

$$H(f)(x) = \frac{1}{\pi} \text{PV} \int_{-\infty}^{\infty} \frac{f(y)}{x-y} \, dy = \frac{1}{\pi} \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} \frac{f(x-y) - f(x+y)}{y} \, dy,$$

is an odd function in $x \in \mathbb{R}$. Thus it suffices to show that $H(f)(x) \geq 0$ for $x \geq 0$. Since $|x-y| \leq |x+y|$ for $x,y \geq 0$ and $f = f^*$ is symmetric-decreasing, we see that $f(x-y) \geq f(x+y)$ for all $x,y \geq 0$. Thus the claim directly follows from the integral expression for $H(f)(x)$.

Proposition A.1. The embedding $X \subset L^2_{\text{even}}(\mathbb{R})$ is compact.

Proof. Suppose that $(f_n)$ is a bounded sequence in $X$. In particular, the sequence $(f_n)$ is bounded in $H^1(\mathbb{R})$. Hence, by local Rellich compactness, we can assume that $(f_n)$ converges in $L^2_{\text{even}}(\mathbb{R})$ after passing to a subsequence if necessary. Next, by the uniform bound $\int_{\mathbb{R}} \log(1 + |x|) |f_n(x)|^2 \, dx \leq C$ with some constant $C > 0$ independent of $n$, the property $\log(1 + |x|) \to +\infty$ as $|x| \to +\infty$ readily implies that $(f_n)$ strongly converges in $L^2(\mathbb{R})$. Since $L^2_{\text{even}}(\mathbb{R})$ is a closed subspace of $L^2(\mathbb{R})$, the proof is complete.

Lemma A.2. The map $F : X \times (0, \infty) \to X$ given in (5.1) is well-defined and of class $C^1$.

Proof. Let $(u, \lambda) \in X \times (0, \infty)$ be given. First we show that $F(u, \lambda) \in X$ as follows. We set

$$h_u(x) := -\int_0^x \frac{H(u^2)(y)}{\pi} \, dy.$$ 

We claim that

$$h_u(x) \leq A \|u\|^2_X$$ 

for all $x \in \mathbb{R}$, with some universal constant $A > 0$. Indeed, using that $H_\delta = (-\Delta)^{1/2}$, we note that $(-\Delta)^{1/2} h_u = u^2$ in $\mathbb{R}$. By adapting the arguments in the proof of Lemma 2, and using that $\log(1 + |x|) u^2 \in L^1(\mathbb{R})$, we deduce that

$$h_u(x) = \frac{1}{\pi} \int_{\mathbb{R}} \log |x-y| u(y)^2 \, dy + C_0$$ 

with some constant $C_0 \in \mathbb{R}$. Since $h_u(0) = 0$, we find the upper bound

$$C_0 = \frac{1}{\pi} \int_{\mathbb{R}} (\log |y|) u(y)^2 \, dy \leq \frac{1}{\pi} \int_{\mathbb{R}} (1 + |y|) u(y)^2 \, dy \leq \frac{1}{\pi} \|u\|^2_X.$$ 

Furthermore, we estimate

$$\frac{1}{\pi} \int_{\mathbb{R}} \log |x-y| u(y)^2 \, dy \leq \frac{1}{\pi} \int_{B_1(x)} \log |x-y| u(y)^2 \, dy \leq \frac{1}{\pi} \log |x| \|u\|^2_{L^2(B_1(x))} \leq B \|u\|^2_X$$ 

with some constant $B > 0$. This completes the proof of (A.1).

Using (A.1), we see that $e^{hi} \in L^\infty(\mathbb{R})$ and thus, by our assumptions on $K$, we deduce that $F(u, \lambda) = \lambda \sqrt{K} e^{hi u} - u$ satisfies

$$\int_{\mathbb{R}} |F(u, \lambda)(x)|^2 + \int_{\mathbb{R}} \log(1 + |x|) |F(u, \lambda)(x)|^2 \, dx < +\infty.$$ 

Similarly, we show that $\partial_x F(u, \lambda) \in L^2(\mathbb{R})$. Finally, it is easy to see that $F(u, \lambda)(-x) = F(u, \lambda)(x)$ using that $u(x) = u(-x)$ holds for $u \in X$. This proves that $F(u, \lambda) \in X$.

The fact that the map $F : X \times (0, \infty) \to X$ is continuous follows by using dominated convergence together with previous bounds, standard estimates, and the fact that $u_n \to u$ in $X$ implies that $h_{u_n} \to h_u$ pointwise almost everywhere. Furthermore, it is straightforward to verify that $F$ is of class $C^1$. We omit the details.
Appendix B. Relation to the Monotonicity Formula

In this section, we explain how the arguments in the proof of the key Lemma 5.3 can be seen from the perspective of monotonicity formulas for the fractional Laplacian \((-\Delta)^s\) in [6, 5, 14].

To simplify the presentation and to focus on the main ideas, we will work on a purely calculational level and we thus omit any technicalities. Furthermore, we will exclusively consider \(n = 1\) space dimension and \((-\Delta)^s\) with \(s = 1/2\) (which is the relevant case in this paper). Suppose that \(u : \mathbb{R} \rightarrow \mathbb{R}\) solves

\[
(-\Delta)^{1/2}u + Vu = 0 \quad \text{in} \ \mathbb{R},
\]

where \(V : \mathbb{R} \rightarrow \mathbb{R}\) is a given potential of class \(C^1\), say. Following [14] we introduce the function \(H\) on \(\mathbb{R}\) defined as

\[
H(x) = \frac{1}{2} \int_0^\infty \left\{ u_x(x,t)^2 - u_t(x,t)^2 \right\} dt - \frac{1}{2} V(x)u(x)^2.
\]

Here \((x,t)\) denotes the harmonic extension of \(u(x)\) to the upper half-plane \(\mathbb{R}^+_t = \mathbb{R} \times \{t > 0\}\). Using the classical fact \(-\partial_t u(0,x) = (-\Delta)^{1/2}u(x)\) and [5, 14], a calculation yields that

\[
H'(x) = -\frac{1}{2} V'(x)u(x)^2.
\]

Thus if \(V\) is monotone increasing, we see that \(H'(x) \leq 0\) for \(x \geq 0\), showing that \(H\) is a monotone decreasing quantity on the half-line \([0, \infty)\). See [14] for applications to show uniqueness results.

To make a link with \(H(x)\) to the arguments in the proof of Lemma 5.3, let us consider the expression

\[
\tilde{H}(x) = \int_x^\infty ((-\Delta)^{1/2}u(y))u_x(y) \ dy + \int_x^\infty V(y)u(y)u_x(y) \ dy.
\]

Note that we have \(\tilde{H}(x) \equiv 0\), since \((-\Delta)^{1/2}u + Vu = 0\) holds. But by separately analyzing the two integrals above, we find the following relation to \(H(x)\). First, we claim

\[
\int_x^\infty ((-\Delta)^{1/2}u(y))u_x(y) \ dy = \frac{1}{2} \int_0^\infty \left\{ u_x(x,t)^2 - u_t(x,t)^2 \right\} dt
\]

for any \(x \in \mathbb{R}\). To see this, we introduce the vector field \(F : \mathbb{R}^+_t \rightarrow \mathbb{R}^2\) with

\[
F(x,t) = \left( \begin{array}{c} -\frac{u_t}{2} u_x \\ \frac{u_t}{2} \end{array} \right),
\]

where \(u = u(x,t)\) denotes the harmonic extension \(u\) to \(\mathbb{R}^+_t\). In view of \(u_{xx} + u_{tt} = 0\) in \(\mathbb{R}^+_t\), we readily check that \(F : \mathbb{R}^+_t \rightarrow \mathbb{R}^2\) is curl-free, i.e.,

\[
\text{curl} \ F = \partial_t F_2 - \partial_x F_1 = 0.
\]

By Stokes’ theorem and assuming sufficient decay of \(F\) at infinity, we can deduce

\[
0 = \int_D \text{curl} \ F \ dx \ dt = \oint\partial D \ F \cdot \ ds = \int_x^\infty F_1(y,0) \ dy - \int_0^\infty F_2(x,t) \ dt
\]

with the region \(D = [x, \infty) \times [0, \infty) \subset \mathbb{R}^+_t\). Because of \(F_1|_{t=0} = ((-\Delta)^{1/2}u)u_x\), we see that [15.5] holds. On the other hand, by integration by parts and assuming that \(V u^2\) vanishes at infinity, we immediately find

\[
\int_x^\infty V(y)u(y)u_x(y) \ dy = -\frac{1}{2} V(x)u(x)^2 - \frac{1}{2} \int_x^\infty V'(y)u(y)^2 \ dy
\]

In summary, we have derived that following identity

\[
\tilde{H}(x) = H(x) - \frac{1}{2} \int_x^\infty V'(y)u(y)^2 \ dy
\]

relating \(\tilde{H}\) and \(H\). In view of \(\tilde{H}(x) \equiv 0\), we obtain (B.3) as a direct consequence.
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