DYNAMICS OF OBSERVABLES IN RANK-BASED MODELS AND
PERFORMANCE OF FUNCTIONALLY GENERATED PORTFOLIOS

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Abstract. In the seminal work [9], several macroscopic market observables have
been introduced, in an attempt to find characteristics capturing the diversity of a
financial market. Despite the crucial importance of such observables for investment
decisions, a concise mathematical description of their dynamics has been missing.
We fill this gap in the setting of rank-based models and expect our ideas to extend
to other models of large financial markets as well. The results are then used to study
the performance of multiplicatively and additively functionally generated portfolios,
in particular, over short-term and medium-term horizons.

1. Introduction

A key characteristic of an equity market is its diversity which, on an intuitive level,
describes how evenly the investors distribute their capital among the publicly traded
companies. In the seminal work [9] (see also [10], [11], [12]), FERNHOLZ has initiated
the program of capturing the concept of diversity mathematically and, thus, quanti-
fying its implications on the performance of investment portfolios. Given the market
weights $\mu_1(t), \mu_2(t), \ldots, \mu_n(t)$ at a time $t \geq 0$ (i.e. the fractions of market capital
invested in the $n$ publicly traded companies at that time), he suggested to measure
the market diversity by

$$(1.1) \quad D_p(t) := \left(\sum_{i=1}^{n} \mu_i(t)^p\right)^{1/p} \quad \text{for some} \quad p \in (0, 1)$$

(see [9, Example 3.4.4]). The choice $p \in (0, 1)$ ensures that the right-hand side of
(1.1) is a concave function of $\mu_1(t), \mu_2(t), \ldots, \mu_n(t)$ and attains its maximum for the
uniform capital distribution $\mu_1(t) = \mu_2(t) = \cdots = \mu_n(t) = \frac{1}{n}$. The limiting case

$$(1.2) \quad H(t) := \lim_{p \uparrow 1} \frac{p}{1-p} \log D_p(t) = -\sum_{i=1}^{n} \mu_i(t) \log \mu_i(t),$$

known as the market entropy, retains the latter two properties and can therefore be
regarded as an alternative measure of the market diversity (cf. [9, Section 2.3]). We
refer to [9, Figures 6.7, 7.3, and 6.2] for plots of the process $D_{1/2}(\cdot)$ for the largest 1000
companies in the U.S., the process $D_{0.76}(\cdot)$ for the companies forming the S&P 500
index, and the process $H(\cdot)$ for the companies in the Center for Research in Securities
Prices (CRSP) database of major U.S. stock exchanges (including the NYSE, the
AMEX and the NASDAQ), respectively.

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Despite the considerable interest in the quantities \( D_p(\cdot), p \in (0, 1) \) and \( H(\cdot) \) (and the associated functionally generated portfolios, see below), a concise mathematical description of their dynamics has been missing so far. The challenge lies thereby in the fact that, while the vector of market weights \((\mu_1(\cdot), \mu_2(\cdot), \ldots, \mu_n(\cdot))\) is typically modeled by a Markov process, the Markov property is generally not inherited by \( D_p(\cdot), p \in (0, 1) \) or \( H(\cdot) \). Our first main goal in this paper is to capture the dynamics of non-linear macroscopic observables of the point process of logarithmic market capitalizations, such as \( D_p(\cdot) \) and \( H(\cdot) \), in the context of rank-based (a.k.a. first-order) models and when the number of companies \( n \) is large. We choose to work with rank-based models because they are known to form the simplest class of market models that is able to reproduce the true long-term average capital distribution of a financial market (see [9, Chapter 5], [2] and [11, Chapter 13]). We point to [9, Figure 5.1] for a plot of the latter for the stocks in the CRSP database. It is worth stressing that, even though the details of our proofs rely on the specifics of rank-based models, the high-level ideas of our work can be applied to any model of a large financial market.

The term rank-based model refers to the unique weak solution of the system of stochastic differential equations (SDEs)

\[
\text{d}X_i^{(n)}(t) = b(F_{\rho^{(n)}}(X_i^{(n)}(t))) \, \text{d}t + \sigma(F_{\rho^{(n)}}(X_i^{(n)}(t))) \, \text{d}B_i^{(n)}(t), \quad i = 1, 2, \ldots, n,
\]

with coefficient functions \( b : [0, 1] \to \mathbb{R} \) and \( \sigma : [0, 1] \to (0, \infty) \), the empirical cumulative distribution functions \( F_{\rho^{(n)}}(x) := \frac{1}{n} \sum_{i=1}^{n} 1_{(X_i^{(n)}(t)) \leq x} \), and independent standard Brownian motions \( B_1^{(n)}, B_2^{(n)}, \ldots, B_n^{(n)} \). The system (1.3) is a special case of the systems of SDEs studied by Bass and Pardoux in [3], who were motivated by the piecewise linear filtering problem. In particular, the main result of [3] shows the weak uniqueness for (1.3) (the weak existence for (1.3) falls under the classical result of [28, Exercise 12.4.3]). More recently, the interacting particle system described by (1.3) and its variants have attracted much attention due to their appearance in stochastic portfolio theory and an open problem of Aldous [11] (see [9, Section 5.5], [2], [11, Section 13], [16], [27], [17], [15], [18], [25] for the former and [23], [24], [30], [31], [26], [6], [7] for the latter).

The \( n \to \infty \) asymptotics of non-linear macroscopic observables that we derive herein rely on the law of large numbers for rank-based models in [17, Corollary 2.13] (see also [3 Corollary 1.6], [27, Theorem 1.2]) and the associated central limit theorem in [21, Theorem 1.2]. Both of these results hold under the following (stronger than the original) assumption.

**Assumption 1.1.** (a) There exists a probability measure \( \lambda \) on \( \mathbb{R} \) possessing a bounded density function and satisfying

\[
\forall \theta > 0: \int_{\mathbb{R}} e^{\theta |x|} \lambda(\text{d}x) < \infty
\]

such that the initial locations of the particles \( X_1^{(n)}(0), X_2^{(n)}(0), \ldots, X_n^{(n)}(0) \) are i.i.d. according to \( \lambda \) for all \( n \in \mathbb{N} \).

(b) The coefficient functions \( b \) and \( \sigma \) are differentiable with locally Hölder continuous derivatives.

We now state the versions of [17, Corollary 2.13] and [21, Theorem 1.2] used in this paper for future reference. Hereby, we write \( M_1(\mathbb{R}) \) for the space of probability
measures on \( \mathbb{R} \) equipped with the topology of weak convergence, \( C([0, \infty), M_1(\mathbb{R})) \) for the space of continuous functions from \([0, \infty) \) to \( M_1(\mathbb{R}) \) endowed with the topology of locally uniform convergence, as well as \( M_{\text{fin}}(\mathbb{R}) \) and \( M_{\text{fin}}([0, t] \times \mathbb{R}), t > 0 \) for the spaces of finite signed measures on \( \mathbb{R} \) and \([0, t] \times \mathbb{R}, t > 0 \) viewed as the duals of \( C_0(\mathbb{R}) \) and \( C_0([0, t] \times \mathbb{R}), t > 0 \) with the associated weak-* topologies, respectively.

**Proposition 1.2** (cf. [17], Corollary 2.13). Under Assumption [17] the processes of empirical measures \( g^{(n)}(\cdot) = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i^{(n)}(\cdot)}, n \in \mathbb{N} \) converge in probability in \( C([0, \infty), M_1(\mathbb{R})) \) to a deterministic limit \( g(\cdot) \). Moreover, the corresponding process of cumulative distribution functions \( R(t, \cdot) := F_{\phi(t)}(\cdot), t \geq 0 \) forms the unique generalized solution of the Cauchy problem for the porous medium equation

\[
R_t = -B(R)_x + \Sigma(R)_{xx}, \quad R(0, \cdot) = F_\lambda(\cdot)
\]

in the sense of [17, Definition 3], where \( B(r) := \int_0^r b(a) da, \Sigma(r) := \int_0^r \frac{1}{2} \sigma(a)^2 da, \) and \( F_\lambda(\cdot) \) is the cumulative distribution function of \( \lambda \).

**Proposition 1.3** (cf. [21], Theorem 1.2). Let Assumption [17] be satisfied and \( G \) be the mild solution of the stochastic partial differential equation (SPDE)

\[
G_t = -g(R)G_x + \left( \frac{\sigma(R)}{2} \right)_x G + \sigma(R) R_x^{1/2} \dot{W}, \quad G(0, \cdot) = \beta(F_\lambda(\cdot)),
\]

with the function \( R \) from Proposition [17] the space-time white noise \( \dot{W} \), and a standard Brownian bridge \( \beta \) independent of \( \dot{W} \). In other words,

\[
G(t, x) = \int_\mathbb{R} \beta(F_\lambda(y)) p(0, y; t, x) \, dy + \int_0^t \int_\mathbb{R} \sigma(R(s, y)) R_x(s, y)^{1/2} p(s, y; t, x) \, dW(s, y),
\]

\((t, x) \in [0, \infty) \times \mathbb{R},\)

where \( p \) is the transition kernel associated with the solution of the martingale problem for the operators \( b(R(t, \cdot)) \frac{d}{dx} + \frac{\sigma(R(t, \cdot))^2}{2} \frac{d^2}{dx^2}, t \geq 0 \) and the double integral is taken in the Itô sense. Then, the sequences of processes

\[
t \mapsto \sqrt{n}(F_{\phi(n)}(t_x) - R(t, x)) \, dx, n \in \mathbb{N} \quad \text{and} \quad t \mapsto \sqrt{n}(F_{\phi(n)}(s_x) - R(s, x)) \, dx \, ds, n \in \mathbb{N},
\]

with values in \( M_{\text{fin}}(\mathbb{R}) \) and \( M_{\text{fin}}([0, t] \times \mathbb{R}), t > 0 \), respectively, converge jointly in the finite-dimensional distribution sense to

\[
t \mapsto G(t, x) \, dx \quad \text{and} \quad t \mapsto G(s, x) 1_{[0, t] \times \mathbb{R}}(s, x) \, ds \, dx.
\]

We are now ready to give the first two main results of the present work, which yield a comprehensive description of the large \( n \) asymptotic dynamics for non-linear macroscopic observables of the form

\[
\mathcal{J}_{f_1, \ldots, f_k}(\alpha(\cdot)) := J\left( \int_\mathbb{R} f_1 \, d\alpha(\cdot), \ldots, \int_\mathbb{R} f_k \, d\alpha(\cdot) \right),
\]

where \( \alpha(\cdot) \in C([0, \infty), M_1(\mathbb{R})) \), \( J \) is a continuously differentiable function, and

\[
f_1, \ldots, f_k \in \mathcal{E}_f := \left\{ f \in C^3(\mathbb{R}) : \left| \frac{d^3 f}{dx^3}(x) \right| \leq Ce^{C|x|}, x \in \mathbb{R} \text{ for some } C \geq 0 \right\},
\]
with \( \ell = 1 \) in Theorem \[1.4\] and \( \ell = 3 \) in Theorem \[1.5\] below. For simplicity, we use henceforth the bilinear form notation \( \langle f, \nu \rangle \) for \( \int _{\mathbb{R}} f \, d\nu \) and write \((f_1, \ldots, f_k) \in \mathcal{E}^U_\ell \) for a \( U \subseteq \mathbb{R}^k \) whenever \( f_1, \ldots, f_k \in \mathcal{E}_\ell \) and

\[
\langle f_1, \nu \rangle, \ldots, \langle f_k, \nu \rangle \in U \text{ for all } \nu \in M_1(\mathbb{R}) \text{ fulfilling } \int _{\mathbb{R}} e^{\theta |x|} \nu(dx) < \infty, \ \theta > 0.
\]

**Theorem 1.4.** Let Assumption \[1.4\] be satisfied and \((f_1, \ldots, f_k) \in \mathcal{E}^U_1 \) for a convex open \( U \subseteq \mathbb{R}^k \). Then, for all \( J \in C^1(U) \), one has

\[
\sqrt{n}\left( \mathcal{J}_{J;f_1,\ldots,f_k}(\phi^{(n)}(\cdot)) - \mathcal{J}_{J;f_1,\ldots,f_k}(\phi(\cdot)) \right) \xrightarrow{n \to \infty} - \sum_{j=1}^{k} \mathcal{J}_{J_x;f_1,\ldots,f_k}(\phi(\cdot)) \int _{\mathbb{R}} f'_j(x) G(\cdot, x) \, dx
\]

in the finite-dimensional distribution sense.

**Theorem 1.5.** Let Assumption \[1.4\] be satisfied, \((f_1, \ldots, f_k) \in \mathcal{E}^U_3 \) for a convex open \( U \subseteq \mathbb{R}^k \), and \( J \in C^1(U) \). Suppose \( a \in \mathbb{R} \) is such that

\[
\tau := \inf \{ t \geq 0 : \mathcal{J}_{J;f_1,\ldots,f_k}(\phi(t)) = a \} < \infty \quad \text{and} \quad \frac{d\mathcal{J}_{J;f_1,\ldots,f_k}(\phi(\cdot))}{dt}(\tau) \neq 0.
\]

Then, the sequence of hitting times

\[
\tau^{(n)} := \inf \{ t \geq 0 : \mathcal{J}_{J;f_1,\ldots,f_k}(\phi^{(n)}(t)) = a \}, \quad n \in \mathbb{N}
\]

converges in distribution when properly rescaled:

\[
\sqrt{n}(\tau^{(n)} - \tau) \xrightarrow{n \to \infty} \sum_{j=1}^{k} \frac{\mathcal{J}_{J_x;f_1,\ldots,f_k}(\phi(\tau)) \int _{\mathbb{R}} f'_j(x) G(\tau, x) \, dx}{\frac{d\mathcal{J}_{J;f_1,\ldots,f_k}(\phi(\cdot))}{dt}(\tau)}.
\]

**Remark 1.6.** We emphasize that Theorems \[1.4\] and \[1.5\] apply, in particular, to the (appropriately normalized) processes \( D_p(\cdot), p \in (0, 1) \) and \( H(\cdot) \) of \[1.1\] and \[1.2\], respectively, allowing to approximate them by Gaussian processes (Theorem \[1.4\]) and their hitting times by Gaussian random variables (Theorem \[1.5\]). For more details, please see Section \[5\] below.

**Remark 1.7.** The condition of finiteness of all exponential moments on \( \lambda \) in Assumption \[1.1(a)\] enters naturally in the context of market observables from stochastic portfolio theory, which often (e.g. in the case of \( D_p(\cdot), p \in (0, 1) \)) involves market capitalizations that, in turn, are images of \( X^{(n)}_1(\cdot), X^{(n)}_2(\cdot), \ldots, X^{(n)}_n(\cdot) \) under the exponential function.

Theorem \[1.5\] can be used further to get estimates on the performance of multiplicatively and additively functionally generated portfolios in the sense of \[9, \text{ Chapter 3}\] and \[19\], respectively. Consider a function

\[
\Psi: \mathbb{R}^n \to \mathbb{R}, \quad x \mapsto J\left( \frac{1}{n} \sum_{i=1}^{n} f_1(x_i), \ldots, \frac{1}{n} \sum_{i=1}^{n} f_k(x_i) \right)
\]

with the homogeneity property

\[
\forall x \in \mathbb{R}^n, \ r \in \mathbb{R} : \ \Psi(x) = \Psi(x_1 + r, x_2 + r, \ldots, x_n + r).
\]

The latter is equivalent to the existence of the representation

\[
\Psi(x) = \Psi\left( \frac{e^{x_1}}{\sum_{i=1}^{n} e^{x_i}}, \frac{e^{x_2}}{\sum_{i=1}^{n} e^{x_i}}, \ldots, \frac{e^{x_n}}{\sum_{i=1}^{n} e^{x_i}} \right), \quad x \in \mathbb{R}^n.
\]
If \( \overline{\Psi} \) can be extended to a twice continuously differentiable function on an open neighborhood of the open unit simplex \( \{ x \in (0,1)^n : \sum_{i=1}^n x_i = 1 \} \subset \mathbb{R}^n \), then one can formally define the weights (i.e. the fractions invested in the different companies) of the portfolios \( \pi^{\Psi\times} \) and \( \pi^{\Psi+:} \) multiplicatively and additively generated by \( \overline{\Psi} \) via
\[
\pi^{\Psi\times}_i(t) = \left( (\log \overline{\Psi})_{x_i}(\mu(\cdot)) + 1 - \sum_{j=1}^n \mu_j(\cdot) (\log \overline{\Psi})_{x_j}(\mu(\cdot)) \right) \mu_i(\cdot), \quad i = 1, 2, \ldots, n,
\]
\[
\pi^{\Psi+:}_i(t) = \left( \frac{\overline{\Psi}_{x_i}(\mu(\cdot)) - \sum_{j=1}^n \mu_j(\cdot) \overline{\Psi}_{x_j}(\mu(\cdot))}{\overline{\Psi}(\mu(\cdot)) - \frac{1}{2} \sum_{i,j=1}^n \int_0^1 \overline{\Psi}_{x_i,x_j}(\mu(s)) d[\mu_i, \mu_j](s)} + 1 \right) \mu_i(\cdot), \quad i = 1, 2, \ldots, n,
\]
respectively. Here, \( \mu(\cdot) = (\mu_1(\cdot), \mu_2(\cdot), \ldots, \mu_n(\cdot)) \) is the process of the market weights, which in the context of a rank-based model amounts to
\[
\left( \frac{e^{X_{1}(t)}}{\sum_{i=1}^n e^{X_{i}(t)}}, \ldots, \frac{e^{X_{n}(t)}}{\sum_{i=1}^n e^{X_{i}(t)}} \right),
\]
and \([\cdot, \cdot]\) denotes the quadratic covariation process.

Functionally generated portfolios \( \pi^{\Psi\times}(\cdot) \), \( \pi^{\Psi+:}(\cdot) \) have the remarkable property that their values \( V^{\Psi\times}(\cdot) \), \( V^{\Psi+:}(\cdot) \) relative to that of the market portfolio \( \mu(\cdot) \) admit pathwise representations, which under the usual convention \( V^{\Psi\times}(0) = V^{\Psi+:}(0) = 1 \) read
\[
(1.19) \quad V^{\Psi\times}(t) = \frac{\overline{\Psi}(\mu(t))}{\overline{\Psi}(\mu(0))} \exp \left( -\frac{1}{2} \sum_{i,j=1}^n \int_0^t \overline{\Psi}_{x_i,x_j}(\mu(\cdot)) d[\mu_i, \mu_j](\cdot) \right), \quad t \geq 0,
\]
\[
(1.20) \quad V^{\Psi+:}(t) = 1 + \overline{\Psi}(\mu(t)) - \overline{\Psi}(\mu(0)) - \frac{1}{2} \sum_{i,j=1}^n \int_0^t \overline{\Psi}_{x_i,x_j}(\mu(\cdot)) d[\mu_i, \mu_j](\cdot), \quad t \geq 0
\]
(cf. [11] equation (11.2), [19] equation (4.3)). We assume henceforth that the function \( \overline{\Psi} \) is positive and concave in the setting of (1.19) or concave in the setting of (1.20), since then the respective excess growth process 
\[-\frac{1}{2} \sum_{i,j=1}^n \int_0^t \overline{\Psi}_{x_i,x_j}(\mu(\cdot)) d[\mu_i, \mu_j](\cdot), \quad t \geq 0
\] or 
\[-\frac{1}{2} \sum_{i,j=1}^n \int_0^t \overline{\Psi}_{x_i,x_j}(\mu(\cdot)) d[\mu_i, \mu_j](\cdot), \quad t \geq 0\]
are non-decreasing (see [19] Example 3.5). In particular, the associated value process \( V^{\Psi\times}(\cdot) \) or \( V^{\Psi+:}(\cdot) \) reaches levels \( v > 1 \) before the hitting times \( \tau^{(n)} \) of levels \( a = v \overline{\Psi}(\mu(0)) \) or \( a = v - 1 + \overline{\Psi}(\mu(0)) \), respectively, whose asymptotics are described by Theorem 1.5.

More precise estimates on the processes \( V^{\Psi\times}(\cdot) \) and \( V^{\Psi+:}(\cdot) \) can be obtained under the additional assumptions
\[
(1.21) \quad \forall n \in \mathbb{N}, \quad i = 1, 2, \ldots, n - 1 : \quad \frac{1}{i} \sum_{j=1}^i b \left( \frac{j}{n} \right) > \frac{1}{n-i} \sum_{j=i+1}^n b \left( \frac{j}{n} \right) \quad \text{and} \quad \sigma(\cdot) = 1
\]
(see [24] Remark on p. 2187) for a detailed discussion of the first assumption; in addition, note that the constant 1 in the second assumption can be turned into any other positive constant by a deterministic time change. Indeed, under the assumptions in (1.21), [15] Corollary 8 applies and can be naturally combined with Theorem 1.5.
Corollary 1.8. Let Assumption 1.1, the assumptions in (1.21), and for all \( n \in \mathbb{N} \), (1.22)

\[
\text{ess sup}_{\omega,t} \sum_{i,j=1}^{n} \frac{\Psi_{x,x_j}(\mu(\cdot))}{\Psi(\mu(\cdot))} d[\mu_i, \mu_j](\cdot) = \text{ess inf}_{\omega,t} \sum_{i,j=1}^{n} \frac{\Psi_{x,x_j}(\mu(\cdot))}{\Psi(\mu(\cdot))} d[\mu_i, \mu_j](\cdot) \in (0, \infty)
\]

or

(1.23)

\[
\text{ess sup}_{\omega,t} \sum_{i,j=1}^{n} \frac{\Psi_{x,x_j}(\mu(\cdot))}{\Psi(\mu(\cdot))} d[\mu_i, \mu_j](\cdot) = \text{ess inf}_{\omega,t} \sum_{i,j=1}^{n} \frac{\Psi_{x,x_j}(\mu(\cdot))}{\Psi(\mu(\cdot))} d[\mu_i, \mu_j](\cdot) \in (0, \infty)
\]

be satisfied. Then, for the functions \( J \) and \( f_1, \ldots, f_k \) of (1.16), (1.18), \( a, \tau \) fulfilling the conditions in (1.13), \( r^\times := -\lim_{t \to \infty} \frac{1}{\sqrt{2}} \int_{0}^{t} \frac{\Psi_{x,x_j}(\mu(\cdot))}{\Psi(\mu(\cdot))} d[\mu_i, \mu_j](\cdot) \) or \( r^+ := -\lim_{t \to \infty} \frac{1}{\sqrt{2}} \sum_{i,j=1}^{n} \int_{0}^{t} \frac{\Psi_{x,x_j}(\mu(\cdot))}{\Psi(\mu(\cdot))} d[\mu_i, \mu_j](\cdot) \) and \( r, s > 0 \), the stopping times

\[
\eta_{r^\times} := \inf \left\{ t \geq 0 : V\Psi_{r^\times}(t) = \frac{a}{\Psi(\mu(0))} e^{(r^\times-r)(\tau-s/\sqrt{n})} \right\},
\]

\[
\eta_{r^+} := \inf \left\{ t \geq 0 : V\Psi_{r^+}(t) = 1 + a - \Psi(\mu(0)) + (r^+ - r)(\tau - s/\sqrt{n}) \right\}
\]

satisfy for all \( \varepsilon > 0 \) the respective estimates

(1.24) \[ \mathbb{P}(\eta_{r^\times} \geq \tau + s/\sqrt{n}) \leq 2\Phi(s/\chi)(1 + o_n(1)) + \left\| \frac{d\kappa(n)}{d\zeta(n)} \right\|_{L^2(\zeta(n))} e^{-c(r,\varepsilon)(\tau-s/\sqrt{n})}, \]

(1.25) \[ \mathbb{P}(\eta_{r^+} \geq \tau + s/\sqrt{n}) \leq 2\Phi(s/\chi)(1 + o_n(1)) + \left\| \frac{d\kappa(n)}{d\zeta(n)} \right\|_{L^2(\zeta(n))} e^{-c(r,\varepsilon)(\tau-s/\sqrt{n})}. \]

Hereby, \( \Phi \) is the standard normal tail cumulative distribution function; \( \chi \) is the standard deviation of the random variable on the right-hand side of (1.13); \( o_n(1) \) is a quantity tending to 0 as \( n \to \infty \); \( \kappa(n) \) and \( \zeta(n) \) are the laws of the vector of differences between the consecutive order statistics of \( (X_1^{(n)}, X_2^{(n)}, \ldots, X_n^{(n)}) \) at time 0 and in stationarity, respectively; and

(1.26) \[ c^\times(r, \varepsilon) = \min_{1 \leq j \leq n-1} \left( \frac{\sum_{i=1}^{j} b(\frac{j}{n}) - \frac{j}{n} \sum_{i=1}^{n} b(\frac{i}{n})}{2 - 2 \cos \frac{\pi}{n}} \right)^2 \cdot \max \left( \frac{r^2}{(C^\times)^2}, 4\varepsilon(\varepsilon + v^\times) \left( 1 + \frac{r^2}{2\varepsilon(\varepsilon + v^\times)^2} \max(\left| C^\times \right|, \left| C^\times \right|^2) - 1 \right) \right), \]

(1.27) \[ c^+(r, \varepsilon) = \min_{1 \leq j \leq n-1} \left( \frac{\sum_{i=1}^{j} b(\frac{j}{n}) - \frac{j}{n} \sum_{i=1}^{n} b(\frac{i}{n})}{2 - 2 \cos \frac{\pi}{n}} \right)^2 \cdot \max \left( \frac{r^2}{(C^+)^2}, 4\varepsilon(\varepsilon + v^+) \left( 1 + \frac{r^2}{2\varepsilon(\varepsilon + v^+)^2} \max(\left| C^+ \right|, \left| C^+ \right|^2) - 1 \right) \right) \]

with \( C^\times, C^{\times \uparrow}, C^{\times \downarrow}, \) and \( v^\times (C^+, C^{\uparrow}, C^{\downarrow}, \) and \( v^+ \) resp.) being the overall expression, the essential supremum, the essential infimum, and the variance under \( \zeta(n) \) of the expression inside the essential supremum in (1.22) (1.23) resp.)
Remark 1.9. The inequalities (1.24), (1.25) can be interpreted as follows. If one invests in the portfolio generated multiplicatively (or additively resp.) by a function \( \tilde{\Psi} \) satisfying the condition (1.22) (or (1.23) resp.) and aims for the associated process \( \tilde{\Psi}(\mu(\cdot)) \) to reach an admissible value of \( a \) (i.e. one for which (1.13) holds), then one will achieve a logarithmic (or arithmetic resp.) return relative to the market portfolio \( \mu(\cdot) \) of 
\[
\log a - \log \tilde{\Psi}(\mu(0)) + (r^\times - r)(\tau + s/\sqrt{n})
\]
(resp.) before time \( \tau + s/\sqrt{n} \) with a confidence probability of at least one minus the right-hand side of (1.24) (or (1.25) resp.).

The rest of the paper is structured as follows. In Section 2, we collect some results from [14], [21] and [4] that are used repeatedly in the proofs of Theorems 1.4 and 1.5. Section 3 is then devoted to the proof of Theorem 1.4. The latter is based on Proposition 1.3 but requires significant additional work due to the exponential growth at infinity of the derivatives of functions in \( E_1 \) and the non-linearity of \( J \). In particular, the proof invokes the mean stochastic comparison of [14] and the quantitative propagation of chaos result of [21, Theorem 1.6]. Subsequently, we give the proof of Theorem 1.5 in Section 4, which relies on the previously mentioned tools and a creative reduction to the estimate on the expected Wasserstein distance \( W_1 \) between the empirical measure of an i.i.d. sample and the underlying distribution in [4, Theorem 3.2]. Next, in Section 5, we apply Theorems 1.4 and 1.5 to the main examples of diversity measures from stochastic portfolio theory. Lastly, in Section 5.3, we provide the proof of Corollary 1.8.

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2. Preliminaries

The general propagation of chaos paradigm (see [29]) suggests that under Assumption 1.1 for large values of \( n \), the weak solution of (1.3) should be well-approximated by the strong solution of

\[
\begin{align*}
    dX^{(n)}_i(t) &= b(R(t, X^{(n)}_1(t))) \, dt + \sigma(R(t, X^{(n)}_1(t))) \, dB^{(n)}_i(t), \\
    X^{(n)}_i(0) &= X^{(n)}_i(0),
\end{align*}
\]

with the initial condition \( X^{(n)}_i(0) = X^{(n)}_i(0), i = 1, 2, \ldots, n \), the function \( R \) from Proposition 1.2 and the same Brownian motions \( B^{(n)}_1, B^{(n)}_2, \ldots, B^{(n)}_n \) as in (1.3). Indeed, under Assumption 1.1 the coefficient functions \( (t, x) \mapsto b(R(t, x)) \), \( (t, x) \mapsto \sigma(R(t, x)) \) are uniformly Lipschitz in \( x \) on any strip of the form \([0, T] \times \mathbb{R}\) by [21, Proposition 2.5], so that the strong existence and uniqueness for (2.1) readily follow. To quantify the term “well-approximated” we introduce the process of empirical measures

\[
\bar{\nu}^{(n)}(\cdot) = \frac{1}{n} \sum_{i=1}^{n} \delta_{X^{(n)}_i(\cdot)}
\]
and recall that the Wasserstein distance of order $p \geq 1$ is defined for any $\nu_1, \nu_2 \in M_1(\mathbb{R})$ with finite $p$-th moments by

$$W_p(\nu_1, \nu_2) = \inf_{Z_1 \sim \nu_1, Z_2 \sim \nu_2} \mathbb{E}[|Z_1 - Z_2|^p]^{1/p}.$$  

(2.3)

Then, under Assumption 1.1 the following quantitative propagation of chaos estimates from [21, Theorem 1.6] apply.

**Proposition 2.1** (cf. [21], Theorem 1.6). Let Assumption 1.1 be satisfied. Then, for all $p, T > 0$, one can find a constant $C = C(p, T) < \infty$ such that

$$\mathbb{E} \left[ \sup_{t \in [0, T]} |X_i^{(n)}(t) - \overline{X}_i^{(n)}(t)|^p \right] \leq C n^{-p/2}, \quad i = 1, 2, \ldots, n, \ n \in \mathbb{N}. \tag{2.4}$$

In particular, for $p \geq 1$, it holds

$$\mathbb{E} \left[ \sup_{t \in [0, T]} W_p(\varrho^{(n)}(t), \overline{\varrho}^{(n)}(t)) \right] \leq C n^{-p/2}, \quad n \in \mathbb{N}. \tag{2.5}$$

Moreover, each $\overline{\varrho}^{(n)}(t)$ constitutes the empirical measure of an i.i.d. sample from the probability measure $\varrho(t)$ introduced in Proposition 1.2. Hence, we may aim to bound the associated expected $W_1$-distance $\mathbb{E}[W_1(\overline{\varrho}^{(n)}(t), \varrho(t))]$ by means of [4, Theorem 3.2], which requires a moment estimate for $\varrho(t)$. The latter, in turn, can be obtained under Assumption 1.1 from the mean stochastic comparison results of [14] as follows. With $C_b := \max_{a \in [0,1]} b(a)$, $C_b^k := \min_{a \in [0,1]} b(a)$ and $C_\sigma := \max_{a \in [0,1]} |\sigma(a)|$, consider the Brownian motions

$$dY^+(t) = C_b^+ dt + C_\sigma dB_1^{(1)}(t), \quad Y^+(0) = \lambda, \tag{2.6}$$

$$dY^-(t) = C_b^- dt + C_\sigma dB_1^{(1)}(t), \quad Y^-(0) = \lambda. \tag{2.7}$$

The next proposition is then a direct consequence of [14] inequality (1.5) and p. 318, Remark (4).

**Proposition 2.2.** Let Assumption 1.1 be satisfied. Then, for all $i = 1, 2, \ldots, n, \ n \in \mathbb{N}$, $T > 0, \ M \in \mathbb{R}$ and $\theta > 0$, one has the comparison results

$$\mathbb{P} \left( \sup_{t \in [0, T]} |X_i^{(n)}(t)| \geq M \right) \vee \mathbb{P} \left( \sup_{t \in [0, T]} |\overline{X}_i^{(n)}(t)| \geq M \right) \leq 2 \mathbb{P} \left( \sup_{t \in [0, T]} Y^+(t) \geq M \right)$$

$$+ 2 \mathbb{P} \left( \sup_{t \in [0, T]} (-Y^-(t)) \geq M \right), \tag{2.8}$$

$$\mathbb{E} \left[ \sup_{t \in [0, T]} e^{\theta |X_i^{(n)}(t)|} \right] \vee \mathbb{E} \left[ \sup_{t \in [0, T]} e^{\theta |\overline{X}_i^{(n)}(t)|} \right] \leq \mathbb{E} \left[ \sup_{t \in [0, T]} e^{\theta Y^+(t)} \right] + \mathbb{E} \left[ \sup_{t \in [0, T]} e^{-\theta Y^-(t)} \right] < \infty. \tag{2.9}$$

In particular, the bound of (2.9) allows us to use [4, Theorem 3.2] to estimate each of the quantities $\mathbb{E}[W_1(\overline{\varrho}^{(n)}(t), \varrho(t))]$. Hereby, we keep in mind the alternative representation of the $W_1$-distance as the $L^1$-distance between the cumulative distribution functions (see e.g. [4, Theorem 2.9]):

$$W_1(\nu_1, \nu_2) = \int_{\mathbb{R}} |F_{\nu_1}(x) - F_{\nu_2}(x)| \, dx. \tag{2.10}$$
Proposition 2.3. Let Assumption [1.4] be satisfied. Then, for all $T > 0$, one can find a constant $C = C(T) < \infty$ such that
\begin{equation}
\sup_{t \in [0,T]} \mathbb{E} \left[ W_1(\hat{\nu}^{(n)}(t), \varrho(t)) \right] \leq Cn^{-1/2}, \quad n \in \mathbb{N}.
\end{equation}

3. Proof of Theorem [1.4]

Our starting point for the proof of Theorem 1.4 is the identity
\begin{equation}
\sqrt{n} \left( J_{f_1, \ldots, f_k} (\varrho^{(n)}(\cdot)) - J_{f_1, \ldots, f_k} (\varrho(\cdot)) \right) = \sqrt{n} \left( \langle f_1, \varrho^{(n)}(\cdot) - \varrho(\cdot) \rangle, \ldots, \langle f_k, \varrho^{(n)}(\cdot) - \varrho(\cdot) \rangle \right) \nabla J \left( \langle f_1, \varrho^{(n)}(\cdot) \rangle, \ldots, \langle f_k, \varrho^{(n)}(\cdot) \rangle \right)
\end{equation}
due to the mean value theorem, where $\varrho^{(n)}(\cdot) = \xi^{(n)}(\cdot)\varrho^{(n)}(\cdot) + (1 - \xi^{(n)}(\cdot))\varrho(\cdot)$ and $\xi^{(n)}(\cdot)$ can be chosen as stochastic processes with values in $M_1([R])$ and $[0,1]$, respectively, by the Borel selection result of [5, Theorem 6.9.6]. The proof of Theorem 1.4 is carried out by studying the convergence of the vector-valued stochastic processes
\begin{align}
I_1^{(n)}(\cdot) &:= \sqrt{n} \left( \langle f_1, \varrho^{(n)}(\cdot) - \varrho(\cdot) \rangle, \ldots, \langle f_k, \varrho^{(n)}(\cdot) - \varrho(\cdot) \rangle \right), \\
I_2^{(n)}(\cdot) &:= \left( \langle f_1, \varrho^{(n)}(\cdot) \rangle, \ldots, \langle f_k, \varrho^{(n)}(\cdot) \rangle \right)
\end{align}
as $n \to \infty$. In both cases, it is helpful to introduce, for each $M > 0$, an auxiliary function $h_M \in C^\infty([R])$ with values in $M_1([R])$ such that $h_M(x) = 1$ if $|x| \leq M$, $h_M(x) = 0$ if $|x| > M + 1$, and
\begin{equation}
\sup_{M > 0} \sup_{x \in [R]} |h'_M(x)| \vee \sup_{M > 0} \sup_{x \in [R]} |h''_M(x)| < \infty.
\end{equation}

In addition, we denote $(1 - h_M)$ by $\hat{h}_M$ for each $M > 0$.

Convergence of $I_1^{(n)}(\cdot)$. With the mild solution $G$ of the SPDE (1.6), we claim that
\begin{equation}
I_1^{(n)}(\cdot) \xrightarrow{n \to \infty} \left( \int_{[R]} f_1'(x) G(\cdot, x) \, dx, \ldots, \int_{[R]} f_k'(x) G(\cdot, x) \, dx \right)
\end{equation}
in the finite-dimensional distribution sense. To this end, we write each component $\sqrt{n} \langle f_j, \varrho^{(n)}(\cdot) - \varrho(\cdot) \rangle$ of $I_1^{(n)}(\cdot)$ as
\begin{equation}
\sqrt{n} \int_{[R]} f_j'(x) h'_M(x) (F_{\rho^{(n)}}(\cdot)) (x - R(\cdot, x)) \, dx + \sqrt{n} \int_{[R]} f_j'(x) h'_M(x) (F_{\rho^{(n)}}(\cdot)) (x - R(\cdot, x)) \, dx \\
+ \sqrt{n} \int_{[R]} f_j'(x) \hat{h}_M(x) (F_{\rho^{(n)}}(\cdot)) (x - R(\cdot, x)) \, dx + \sqrt{n} \int_{[R]} f_j'(x) \hat{h}_M(x) (F_{\rho^{(n)}}(\cdot)) (x - R(\cdot, x)) \, dx
\end{equation}
using $f_j = f_j h_M + f_j \hat{h}_M$ and integration by parts (observe that the boundary terms thereby vanish thanks to $f_j \hat{h}_M \in \mathcal{E}_0$, the estimate (2.9) and Markov’s inequality).

Taking first the $n \to \infty$ limit and then the $M \to \infty$ limit of the first summand in (3.6) for $j = 1, 2, \ldots, k$ gives
\begin{equation}
\left( \int_{[R]} f_1'(x) G(\cdot, x) \, dx, \ldots, \int_{[R]} f_k'(x) G(\cdot, x) \, dx \right)
\end{equation}
in the finite-dimensional distribution sense. Indeed, by Proposition 1.3 the \( n \to \infty \) limit results in the mean zero Gaussian process

\[
(3.8) \quad \left( \int_{\mathbb{R}} f_1'(x) h_M(x) G(\cdot, x) \, dx, \ldots, \int_{\mathbb{R}} f_k'(x) h_M(x) G(\cdot, x) \, dx \right).
\]

For its convergence in finite-dimensional distribution as \( M \to \infty \) to the mean zero Gaussian process in (3.7), it suffices to verify the convergence of the corresponding covariance functions. Upon a decomposition of \( f_1', f_2', \ldots, f_k' \) into the positive and negative parts, the positivity of the covariance function of \( G \) (see [21, Remark 1.4]) and the monotone convergence theorem allow to reduce the convergence of the covariance functions to a statement about the uniform boundedness of the variances involved.

**Lemma 3.1.** Let Assumption 1.1 be satisfied. Then, for all \( t \geq 0 \) and \( j \in \{1, 2, \ldots, k\} \),

\[
(3.9) \quad \sup_{M > 0} \mathbb{E} \left[ \left( \int_{\mathbb{R}} f_j'(x_+) h_M(x) G(t, x) \, dx \right)^2 \right] \leq \liminf_{n \to \infty} \mathbb{E} \left[ \left( \int_{\mathbb{R}} f_j'(x_+) h_M(x) \sqrt{n}(F_{\phi^{(n)}}(t)(x) - R(t, x)) \, dx \right)^2 \right] < \infty.
\]

Assuming Lemma 3.1 the proof of (3.8) hinges on the next lemma, which shows that the contributions of the second, third and fourth summands in (3.6) to the \( n \to \infty \) limit of \( I_1^{(n)}(\cdot) \) become negligible as \( M \) tends to infinity.

**Lemma 3.2.** Let Assumption 1.1 be satisfied. Then, for any \( \varepsilon > 0 \), \( t \geq 0 \), \( f_0 \in \mathcal{E}_0 \) and uniformly bounded family of functions \( g_M: \mathbb{R} \to \mathbb{R} \), \( M > 0 \) such that \( g_M(x) = 0 \), \( x \in [-M, M] \) for each \( M > 0 \),

\[
(3.10) \quad \limsup_{M \to \infty} \limsup_{n \to \infty} \mathbb{P} \left( \left| \sqrt{n} \int_{\mathbb{R}} f_0(x) g_M(x) (F_{\phi^{(n)}}(t)(x) - R(t, x)) \, dx \right| > \varepsilon \right) = 0.
\]

We proceed to the proofs of the two lemmas.

**Proof of Lemma 3.1** For all \( M > 0 \), we have

\[
(3.11) \quad \mathbb{E} \left[ \left( \int_{\mathbb{R}} f_j'(x_+) h_M(x) G(t, x) \, dx \right)^2 \right] \leq \liminf_{n \to \infty} \mathbb{E} \left[ \left( \int_{\mathbb{R}} f_j'(x_+) h_M(x) \sqrt{n}(F_{\phi^{(n)}}(t)(x) - R(t, x)) \, dx \right)^2 \right]
\]

by Proposition 1.3 Skorokhod’s representation theorem and Fatou’s lemma. With \( f_{j,M;+}(x) := \int_0^x f_j'(y) h_M(y) \, dy \), integration by parts yields for the term inside the latter limit inferior

\[
(3.12) \quad n \mathbb{E} \left[ \left( f_{j,M;+}, \phi^{(n)}(t) - \phi(t) \right)^2 \right] \leq 2n \mathbb{E} \left[ \left( f_{j,M;+}, \phi^{(n)}(t) - \bar{\phi}^{(n)}(t) \right)^2 \right] + 2n \mathbb{E} \left[ \left( f_{j,M;+}, \bar{\phi}^{(n)}(t) - \phi(t) \right)^2 \right].
\]

Next, we insert the definitions of \( \phi^{(n)}(t), \bar{\phi}^{(n)}(t) \), apply the Cauchy-Schwarz inequality, and exploit the independence of \( X_i^{(n)}(t) \), \( \frac{d}{d t} X_i^{(n)}(t) \) \( \frac{d}{d t} \ldots \frac{d}{d t} X_i^{(n)}(t) = \frac{d}{d t} \phi(t) \) to get

\[
(3.13) \quad 2 \mathbb{E} \left[ \sum_{i=1}^n \left( f_{j,M;+}(X_i^{(n)}(t)) - f_{j,M;+}(X_i^{(n)}(t)) \right)^2 \right] + 2 \mathbb{E} \left[ \left( f_{j,M;+}(X_1^{(n)}(t)) - \left( f_{j,M;+}(X_1^{(n)}(t)) \right)^2 \right)^2 \right].
\]

Since \( f_1, \ldots, f_k \in \mathcal{E}_1 \), we can pick a constant \( C < \infty \) independent of \( j \) and \( M \) such that \( |f_{j,M;+}(x)| \leq Ce^{C|x|}, x \in \mathbb{R} \) and \( |f_{j,M;+}(x)| \leq Ce^{C|x|}, x \in \mathbb{R} \). This, the convexity of the
absolute value function, and the observation \( (X_1^{(n)}(t), \overline{X}_1^{(n)}(t)) \overset{d}{=} (X_2^{(n)}(t), \overline{X}_2^{(n)}(t)) \overset{d}{=} \cdots \overset{d}{=} (X_n^{(n)}(t), \overline{X}_n^{(n)}(t)) \) allow to bound the expression in (3.13) from above by

\[
2nC^2 \mathbb{E} \left[ \mathbf{1}_{\{|X_1^{(n)}(t)| \geq \overline{X}_1^{(n)}(t)\}} \exp \left[ 2C|X_1^{(n)}(t)| (X_1^{(n)}(t) - \overline{X}_1^{(n)}(t))^2 \right] \right] + 2nC^2 \mathbb{E} \left[ \mathbf{1}_{\{|X_1^{(n)}(t)| > |X_1^{(n)}(t)|\}} \exp \left[ 2C|X_1^{(n)}(t)| (X_1^{(n)}(t) - \overline{X}_1^{(n)}(t))^2 \right] \right] + 2C^2 \mathbb{E} \left[ \exp \left[ 2C|X_1^{(n)}(t)| \right] \right].
\]

By dropping the indicator random variables, using the Cauchy-Schwarz inequality twice, and invoking the estimate (2.9) and the \( p = 4 \) version of the inequality (2.4) we conclude that the quantity in (3.11) is uniformly bounded in \( n \) and \( M \). An analogous argument for the second expectation in (3.9) completes the proof of the lemma. \( \square \)

**Proof of Lemma 3.2.** With \( F_{0,M}(x) := \int_0^x f_0(y) g_M(y) \, dy \), we integrate by parts to rewrite the probability in (3.10) as

\[
\mathbb{P} \left( \left| \sqrt{n} \left\langle F_{0,M}, \varphi^{(n)}(t) - \varphi(t) \right\rangle \right| > \varepsilon \right) \leq \mathbb{P} \left( \left| \sqrt{n} \left\langle F_{0,M}, \varphi^{(n)}(t) - \overline{\varphi^{(n)}(t)} \right\rangle \right| > \varepsilon/2 \right) + \mathbb{P} \left( \left| \sqrt{n} \left\langle F_{0,M}, \overline{\varphi^{(n)}(t)} - \varphi(t) \right\rangle \right| > \varepsilon/2 \right)
\]

(note that the boundary terms in the integration by parts vanish thanks to \( F_{0,M} \in \mathcal{E}_0 \) and the estimate (2.3) in conjunction with Markov’s inequality).

Now, we employ Markov’s inequality, plug in the definitions of \( \varphi^{(n)}(t), \overline{\varphi^{(n)}(t)} \), and recall \( (X_1^{(n)}(t), \overline{X}_1^{(n)}(t)) \overset{d}{=} (X_2^{(n)}(t), \overline{X}_2^{(n)}(t)) \overset{d}{=} \cdots \overset{d}{=} (X_n^{(n)}(t), \overline{X}_n^{(n)}(t)) \) to control the first probability on the right-hand side of (3.15) by

\[
\frac{2\sqrt{n}}{\varepsilon} \mathbb{E} \left[ \left| F_{0,M}(X_1^{(n)}(t)) - F_{0,M}(\overline{X}_1^{(n)}(t)) \right| \right].
\]

In view of the assumptions on \( f_0 \) and \( g_M \), \( M > 0 \), we can find a constant \( C < \infty \) independent of \( M \) such that \( |f_0(x)| \leq C e^{C|x|}, \ x \in \mathbb{R} \) and \( |g_M(x)| \leq C 1_{\{|x| > M\}}, \ x \in \mathbb{R}, \ M > 0 \). This and the convexity of the absolute value function show that the expression in (3.10) is not greater than

\[
\frac{2\sqrt{n}C^2}{\varepsilon} \mathbb{E} \left[ \exp \left[ 2C|X_1^{(n)}(t)| \right] \mathbf{1}_{\{|X_1^{(n)}(t)| > M\}} \mathbf{1}_{\{|X_1^{(n)}(t)| \geq |\overline{X}_1^{(n)}(t)|\}} |X_1^{(n)}(t) - \overline{X}_1^{(n)}(t)| \right] + \frac{2\sqrt{n}C^2}{\varepsilon} \mathbb{E} \left[ \exp \left[ 2C|\overline{X}_1^{(n)}(t)| \right] \mathbf{1}_{\{|X_1^{(n)}(t)| > |\overline{X}_1^{(n)}(t)|\}} \left| X_1^{(n)}(t) - \overline{X}_1^{(n)}(t) \right| \right].
\]

Leaving out the second indicator random variables from both expectations and applying Hölder’s inequality twice we end up with

\[
\frac{2\sqrt{n}C^2}{\varepsilon} \mathbb{E} \left[ \exp \left[ 4C|X_1^{(n)}(t)| \right] \right]^{1/3} \mathbb{P} \left( |X_1^{(n)}(t)| > M \right)^{1/3} \mathbb{E} \left[ |X_1^{(n)}(t) - \overline{X}_1^{(n)}(t)|^3 \right]^{1/3} + \frac{2\sqrt{n}C^2}{\varepsilon} \mathbb{E} \left[ \exp \left[ 4C|\overline{X}_1^{(n)}(t)| \right] \right]^{1/3} \mathbb{P} \left( |\overline{X}_1^{(n)}(t)| > M \right)^{1/3} \mathbb{E} \left[ |X_1^{(n)}(t) - \overline{X}_1^{(n)}(t)|^3 \right]^{1/3},
\]

which tends to 0 when one takes the limits superior \( n \to \infty, M \to \infty \) due to the estimates (2.9), (2.8) and the \( p = 3 \) version of the inequality (2.4).

An appeal to Markov’s inequality and the independence of \( \overline{X}_1^{(n)}(t) \overset{d}{=} \overline{X}_2^{(n)}(t) \overset{d}{=} \cdots \overset{d}{=} \overline{X}_n^{(n)}(t) \overset{d}{=} \varphi(t) \) reveal that the second probability on the right-hand side of (3.15)
is at most
\[
(3.19) \quad \frac{4}{\varepsilon^2} \mathbb{E}\left[\left| F_{0,M}(X^{1(n)}_t) - \langle F_{0,M}, \varrho(t) \rangle \right|^2 \right].
\]
Moreover, by the definition of $F_{0,M}$, $M > 0$ and the assumptions on $f_0$ and $g_M$, $M > 0$ we have $F_{0,M}(x) = 0$, $x \in [-M, M]$, $M > 0$ and $|F_{0,M}(x)| \leq C e^{C|x|}$, $|x| > M$, $M > 0$, which allows to upper bound the latter expectation by relying on the estimates (2.8), (2.9) one more time.

To finish the proof of the lemma we pass to the limits superior $n \to \infty$, $M \to \infty$ relying on the estimates (2.8), (2.9) one more time.

**Convergence of $I_2^{(n)}(\cdot)$.** We claim that, for all $t \geq 0$, it holds
\[
(3.21) \quad I_2^{(n)}(t) \xrightarrow{n \to \infty} \left( \langle f_1, \varrho(t) \rangle, \ldots, \langle f_k, \varrho(t) \rangle \right)
\]
in probability, which together with (3.1) and (3.5) yields Theorem 1.4. To obtain (3.21), we need to establish $\lim_{n \to \infty} \langle f_j, \tilde{\varrho}^{(n)}(t) \rangle = \langle f_j, \varrho(t) \rangle$ in probability for every fixed $t \geq 0$ and $j \in \{1, 2, \ldots, k\}$. Consider the decomposition
\[
(3.22) \quad \langle f_j, \tilde{\varrho}^{(n)}(t) \rangle = \langle f_j, \varrho(t) \rangle + \xi^{(n)}(t) \langle f_j h_M, \varrho^{(n)}(t) - \varrho(t) \rangle + \xi^{(n)}(t) \langle f_j \hat{h}_M, \varrho^{(n)}(t) - \varrho(t) \rangle,
\]
valid for any $M > 0$. We have $\lim_{n \to \infty} \xi^{(n)}(t) \langle f_j h_M, \varrho^{(n)}(t) - \varrho(t) \rangle = 0$ in probability due to $|\xi^{(n)}(t)| \leq 1$ and Proposition 1.2 (note that $f_j h_M$ is continuous and bounded). Finally, $|\xi^{(n)}(t)| \leq 1$, integration by parts (in which the boundary terms vanish thanks to the estimate (2.9) and Markov’s inequality), the union bound, and Lemma 3.2 give
\[
(3.23) \quad \limsup_{M \to \infty} \limsup_{n \to \infty} \mathbb{P}\left( |\xi^{(n)}(t) \langle f_j h_M, \varrho^{(n)}(t) - \varrho(t) \rangle| > \varepsilon \right)
\]
\[
\leq \limsup_{M \to \infty} \limsup_{n \to \infty} \mathbb{P}\left( \left| \int_{\mathbb{R}} f_j'(x) \hat{h}_M(x) \left( F_{\varrho^{(n)}(t)}(x) - R(t, x) \right) \, dx \right| > \varepsilon/2 \right)
\]
\[
+ \limsup_{M \to \infty} \limsup_{n \to \infty} \mathbb{P}\left( \left| \int_{\mathbb{R}} f_j'(x) \hat{h}_M(x) \left( F_{\varrho^{(n)}(t)}(x) - R(t, x) \right) \, dx \right| > \varepsilon/2 \right) = 0
\]
for all $\varepsilon > 0$, so that $\lim_{n \to \infty} \langle f_j, \tilde{\varrho}^{(n)}(t) \rangle = \langle f_j, \varrho(t) \rangle$ in probability as desired. \hfill \Box

4. PROOF OF THEOREM 1.5

It is convenient to introduce the truncated versions $\tau^{(n)} := \tau^{(n)} \wedge (\tau + 1)$, $n \in \mathbb{N}$ of the hitting times $\tau^{(n)}$, $n \in \mathbb{N}$. The convergence in distribution of $\sqrt{n} \left( \tau^{(n)} - \tau \right)$ to a limit is then equivalent to the convergence in distribution of $\sqrt{n} \left( \tau^{(n)} - \tau \right)$ to the same limit thanks to the following proposition, which is proved further below in this section.

**Proposition 4.1.** In the setting of Theorem 1.5, $\tau^{(n)} \xrightarrow{n \to \infty} \tau$ in probability.

With the simplified notations
\[
(4.1) \quad Z^{(n)}(\cdot) := J_{J_{f_1}, \ldots, f_k}(\varrho^{(n)}(\cdot)) \quad \text{and} \quad Z(\cdot) := J_{J_{f_1}, \ldots, f_k}(\varrho(\cdot)),
\]
our starting point for the proof of the convergence of $\sqrt{n} \left( \tau^{(n)} - \tau \right)$ is the identity
\[
(4.2) \quad 1_{\{\tau^{(n)} \leq \tau + 1\}} Z^{(n)}(\tau^{(n)}) = 1_{\{\tau^{(n)} \leq \tau + 1\}} Z(\tau).
\]
The latter stems from the continuity of \( Z^{(n)}(\cdot) \) and \( Z(\cdot) \): for \( Z^{(n)}(\cdot) \), it is a direct consequence of the definitions and, for \( Z(\cdot) \), one can write \( \langle f_1, \varrho(\cdot) \rangle, \ldots, \langle f_k, \varrho(\cdot) \rangle \) as \( \mathbb{E}[f_1(X_1^{(1)}(\cdot))], \ldots, \mathbb{E}[f_k(X_1^{(1)}(\cdot))] \) and conclude by taking the expectation in Itô’s formula and using Fubini’s theorem (recall \( f_1, \ldots, f_k \in \mathcal{E}_3 \subset \mathcal{E}_2 \) and the estimate (2.9)). We observe in passing that, for the same reasons in conjunction with the dominated convergence theorem, \( Z(\cdot) \) is actually continuously differentiable.

Next, we expand (4.2) into

\[
1_{\{\tau^{(n)} \leq \tau + 1\}} \sqrt{n} \left( Z(\hat{\tau}^{(n)}) - Z(\tau) \right)
\]

(4.3) \[= -1_{\{\tau^{(n)} \leq \tau + 1\}} \sqrt{n} \left( Z^{(n)}(\tau) - Z(\tau) \right)
\]

\[+ 1_{\{\tau^{(n)} \leq \tau + 1\}} \sqrt{n} \left( Z^{(n)}(\tau) - Z(\tau) \right) - 1_{\{\tau^{(n)} \leq \tau + 1\}} \sqrt{n} \left( Z^{(n)}(\hat{\tau}^{(n)}) - Z(\hat{\tau}^{(n)}) \right).\]

In view of the continuous differentiability of \( Z(\cdot) \), the mean value theorem and Proposition 4.1, the left-hand side of (4.3) converges in distribution as \( n \to \infty \) if and only if \( \sqrt{n} (\tau^{(n)} - \tau) \) converges in distribution as \( n \to \infty \), and the two limits differ by a factor of \( Z'(\tau) \neq 0 \) (cf. (1.13)). Concurrently, the first line on the right-hand side of (4.3) tends to \( \sum_{j=1}^{k} J_{x_j} f_j(\varrho(\tau)) \int_{\mathbb{R}} f_j'(x) G(\tau, x) \, dx \) in distribution as \( n \to \infty \) by Proposition 4.1 and Theorem 1.4.

To obtain Theorem 1.5 it now suffices to verify that the second line on the right-hand side of (4.3) converges to 0 in probability as \( n \to \infty \). As a result of (3.1)-(3.3), the desired convergence follows from

\[
\sqrt{n} \left( Z^{(n)}(\tau) - Z(\tau) \right) - \sqrt{n} \left( Z^{(n)}(\hat{\tau}^{(n)}) - Z(\hat{\tau}^{(n)}) \right)
\]

(4.4) \[= I_1^{(n)}(\tau) J \left( I_2^{(n)}(\tau) - I_1^{(n)}(\hat{\tau}^{(n)}) J \left( I_2^{(n)}(\hat{\tau}^{(n)}) \right) \right)
\]

\[= I_1^{(n)}(\tau) J \left( I_2^{(n)}(\tau) - I_1^{(n)}(\hat{\tau}^{(n)}) J \right) + \left( I_1^{(n)}(\tau) - I_1^{(n)}(\hat{\tau}^{(n)}) \right) J \left( I_2^{(n)}(\hat{\tau}^{(n)}) \right),\]

(3.5), (3.21), and the next two lemmas.

**Lemma 4.2.** In the setting of Theorem 1.5

\[
\nabla J \left( I_2^{(n)}(\hat{\tau}^{(n)}) \right) \xrightarrow{n \to \infty} \nabla J \left( \langle f_1, \varrho(\tau) \rangle, \ldots, \langle f_k, \varrho(\tau) \rangle \right)
\]

in probability.

**Lemma 4.3.** In the setting of Theorem 1.5 \( I_1^{(n)}(\tau) - I_1^{(n)}(\hat{\tau}^{(n)}) \xrightarrow{n \to \infty} 0 \) in probability.

We complete the proof of Theorem 1.5 by establishing Proposition 4.1, Lemma 4.2, and Lemma 4.3.

**Proof of Proposition 4.1** Our proof of the proposition relies on the following lemma that extends the convergence result of Proposition 1.2 to test functions in \( \mathcal{E}_0 \).

**Lemma 4.4.** Let Assumption 1.1 be satisfied. Then, for all \( f_0 \in \mathcal{E}_0 \), \( T \geq 0 \) and \( \varepsilon > 0 \),

\[
\lim_{n \to \infty} \mathbb{P} \left( \sup_{t \in [0, T]} |\langle f_0, \varrho^{(n)}(t) \rangle - \langle f_0, \varrho(t) \rangle| > \varepsilon \right) = 0.
\]

Given Lemma 4.4, the uniform continuity of the function \( J \) on compact neighborhoods of the set \( \{(f_1, \varrho(t)), \ldots, (f_k, \varrho(t)) : t \in [0, \tau + 1]\} \subset \mathbb{R}^k \) implies

\[
\lim_{n \to \infty} \mathbb{P} \left( \sup_{t \in [0, \tau + 1]} |Z^{(n)}(t) - Z(t)| > \varepsilon \right) = 0, \quad \varepsilon > 0.
\]
Since $Z'(\tau) \neq 0$ (cf. (4.13)) and $Z'(\cdot)$ is continuous, there exist $[0,1] \ni \nu_m \downarrow 0$ such that $Z(\tau + \nu_m) > Z(\tau)$ for all $m \in \mathbb{N}$ if $Z'(\tau) > 0$, or $Z(\tau + \nu_m) < Z(\tau)$ for all $m \in \mathbb{N}$ if $Z'(\tau) < 0$. Applying (4.7) with $\varepsilon := |Z(\tau + \nu_m) - Z(\tau)|/2 = |Z(\tau + \nu_m) - a|/2$ consecutively, we find that

$$\lim_{n \to \infty} \mathbb{P}(\tau(n) > \tau + \nu_m) = 0$$

for all $m$. At the same time, for all $\nu > 0$, we have by (4.7):

$$\lim_{n \to \infty} \mathbb{P}(\tau(n) \leq \tau - \nu) \leq \lim_{n \to \infty} \mathbb{P}\left( \sup_{t \in [0,\tau - \nu]} |Z(n)(t) - Z(t)| \geq \min_{t \in [0,\tau - \nu]} |a - Z(t)| \right) = 0.$$

We conclude the proof of the proposition by showing Lemma 4.4. Recalling the auxiliary functions $\mathcal{h}_M$, $M > 0$ and $\mathcal{h}_M$, $M > 0$ from the beginning of Section 5, we know from Proposition 1.2 that, for any $M > 0$ and $\varepsilon > 0$,

$$\lim_{n \to \infty} \mathbb{P}\left( \sup_{t \in [0,T]} |\langle f_0 \mathcal{h}_M, \varphi(n)(t) \rangle - \langle f_0 \mathcal{h}_M, \varphi(t) \rangle | > \varepsilon \right) = 0.$$

Therefore, it is enough to check that

$$\limsup_{M \to \infty} \limsup_{n \to \infty} \mathbb{P}\left( \sup_{t \in [0,T]} |\langle f_0 \mathcal{h}_M, \varphi(n)(t) \rangle | > \varepsilon \right) = 0, \quad \limsup_{M \to \infty} \sup_{t \in [0,T]} |\langle f_0 \mathcal{h}_M, \varphi(t) \rangle | = 0.$$

For the first assertion in (4.11), we use the definition of $\varphi(n)(t)$, the observation $X_1^{(n)}(\cdot) \overset{d}{=} X_2^{(n)}(\cdot) \overset{d}{=} \cdots \overset{d}{=} X_n^{(n)}(\cdot)$, the estimate $|f_0(x) \mathcal{h}_M(x)| \leq C e^{C|x|} 1_{\{|x| > M\}}$, $x \in \mathbb{R}$, and the Cauchy-Schwarz inequality to deduce that

$$\mathbb{E}\left[ \sup_{t \in [0,T]} |\langle f_0 \mathcal{h}_M, \varphi(n)(t) \rangle | \right] = \mathbb{E}\left[ \sup_{t \in [0,T]} \left| \frac{1}{n} \sum_{i=1}^{n} f_0(X_i^{(n)}(t)) \mathcal{h}_M(X_i^{(n)}(t)) \right| \right]$$

$$\leq \mathbb{E}\left[ \sup_{t \in [0,T]} \left| f_0(X_1^{(n)}(t)) \mathcal{h}_M(X_1^{(n)}(t)) \right| \right]$$

$$\leq C \mathbb{E}\left[ \sup_{t \in [0,T]} \left( e^{C|X_1^{(n)}(t)|} 1_{\{|X_1^{(n)}(t)| > M\}} \right) \right]$$

$$\leq C \mathbb{E}\left[ \sup_{t \in [0,T]} e^{2C|X_1^{(n)}(t)|} \right]^{1/2} \mathbb{E}\left[ \sup_{t \in [0,T]} 1_{\{|X_1^{(n)}(t)| > M\}} \right]^{1/2}.$$

Since $\sup_{t \in [0,T]} 1_{\{|X_1^{(n)}(t)| > M\}} = 1_{\{\sup_{t \in [0,T]} |X_1^{(n)}(t)| > M\}}$, the first assertion in (4.11) now follows from Markov’s inequality and the estimates (2.9), (2.8).

For the second assertion in (4.11), we recall that $\overline{X}_1^{(n)}(t) \overset{d}{=} \varphi(t)$, $t \geq 0$, allowing us to bound $\sup_{t \in [0,T]} |\langle f_0 \mathcal{h}_M, \varphi(t) \rangle |$ by

$$\mathbb{E}\left[ \sup_{t \in [0,T]} \left| f_0(\overline{X}_1^{(n)}(t)) \mathcal{h}_M(\overline{X}_1^{(n)}(t)) \right| \right]$$

$$\leq C \mathbb{E}\left[ \sup_{t \in [0,T]} e^{2C|\overline{X}_1^{(n)}(t)|} \right]^{1/2} \mathbb{P}\left( \sup_{t \in [0,T]} |\overline{X}_1^{(n)}(t)| > M \right)^{1/2}.$$

via the procedure in the last paragraph. The estimates (2.9), (2.8) yield the result. □

**Proof of Lemma 4.2** In view of the continuity of $\nabla J$, it suffices to show that

$$\langle f_j, \tilde{\varphi}^{(n)}(\tilde{\tau}^{(n)}) \rangle \overset{n \to \infty}{\to} \langle f_j, \varphi(\tau) \rangle, \quad j = 1, 2, \ldots, k.$$
in probability. Since $\tilde{g}^{(n)}(\tilde{\tau}^{(n)})$ is a convex combination of $g^{(n)}(\tilde{\tau}^{(n)})$ and $g(\tilde{\tau}^{(n)})$, we may swap $\tilde{g}^{(n)}(\tilde{\tau}^{(n)})$ for $g(\tilde{\tau}^{(n)})$ on the left-hand side of \eqref{4.14} by Lemma 4.4 with $T := \tau + 1$. Then, Proposition 4.1 and the continuity of $(f_j, g(\cdot))$ give the lemma. □

**Proof of Lemma 4.3.** We need to verify that

\begin{equation}
\sqrt{n} \left< f_j, g^{(n)}(\tau) - g^{(n)}(\tilde{\tau}^{(n)}) \right> - \sqrt{n} \left< f_j, g(\tau) - g(\tilde{\tau}^{(n)}) \right> \xrightarrow{n \to \infty} 0, \quad j = 1, 2, \ldots, k
\end{equation}

in probability. For a fixed $j \in \{1, 2, \ldots, k\}$, we start by establishing the corresponding convergence under the assumption that $f_j \in C^3_0(\mathbb{R}) \subset \mathcal{E}_3$.

**Step 1: convergence \eqref{4.15} for $f_j \in C^3_0(\mathbb{R})$.** Inserting the definition of $g^{(n)}(\cdot)$ and applying Itô’s formula we find for the first term in \eqref{4.15}:

\begin{equation}
\sqrt{n} \left< f_j, g^{(n)}(\tau) - g^{(n)}(\tilde{\tau}^{(n)}) \right> = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( f_j(X_i^{(n)}(\tau)) - f_j(X_i^{(n)}(\tilde{\tau}^{(n)})) \right)
\end{equation}

\begin{align*}
&= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \int_{\tilde{\tau}^{(n)}}^{\tau} b(F_{g^{(n)}}(t)(X_i^{(n)}(t))) f_j'(X_i^{(n)}(t)) + \frac{\sigma(F_{g^{(n)}}(t)(X_i^{(n)}(t)))^2}{2} f_j''(X_i^{(n)}(t)) \, dt \\
&\quad + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \int_{\tilde{\tau}^{(n)}}^{\tau} \sigma(F_{g^{(n)}}(t)(X_i^{(n)}(t))) f_j'(X_i^{(n)}(t)) \, dB_i^{(n)}(t).
\end{align*}

To simplify the second line in \eqref{4.16} we introduce the discrete antiderivatives

\begin{equation}
(\mathcal{I}_n b)(r) := \frac{1}{n} \sum_{i=1}^{n} b(i/n) \mathbf{1}_{\{r \geq i/n\}}, \quad \left( \mathcal{I}_n \frac{\sigma^2}{2} \right)(r) := \frac{1}{n} \sum_{i=1}^{n} \frac{\sigma(i/n)^2}{2} \mathbf{1}_{\{r \geq i/n\}}, \quad r \in [0, 1].
\end{equation}

Since the order statistics $X_{(1)}^{(n)}(t) \leq X_{(2)}^{(n)}(t) \leq \cdots \leq X_{(n)}^{(n)}(t)$ are almost surely distinct for Lebesgue almost every $t \geq 0$ by [22, theorem on p. 439] for the function $x \mapsto \sum_{1 \leq i < i+1 \leq n} \mathbf{1}_{\{x_{i+1} = x_i\}}$, we can now use summation by parts, the piecewise constant nature of $(\mathcal{I}_n b)(F_{g^{(n)}}(t)(\cdot))$, and the convention $X_{(n+1)}^{(n)}(t) = \infty$ to compute

\begin{equation}
\frac{1}{n} \sum_{i=1}^{n} b(F_{g^{(n)}}(t)(X_i^{(n)}(t))) f_j'(X_i^{(n)}(t)) = \frac{1}{n} \sum_{i=1}^{n} b(i/n) f_j'(X_i^{(n)}(t))
\end{equation}

\begin{align*}
&= \sum_{i=1}^{n} \left( (\mathcal{I}_n b)(i/n) - (\mathcal{I}_n b)((i-1)/n) \right) f_j'(X_i^{(n)}(t)) \\
&= \sum_{i=1}^{n} (\mathcal{I}_n b)(i/n) \left( f_j'(X_i^{(n)}(t)) - f_j'(X_{(i+1)}^{(n)}(t)) \right) \\
&\quad = - \int_{\mathbb{R}} (\mathcal{I}_n b)(F_{g^{(n)}}(x)(\cdot)) f_j''(x) \, dx.
\end{align*}

Similarly, we see that

\begin{equation}
\frac{1}{n} \sum_{i=1}^{n} \frac{\sigma(F_{g^{(n)}}(t)(X_i^{(n)}(t)))^2}{2} f_j''(X_i^{(n)}(t)) = - \int_{\mathbb{R}} \left( \mathcal{I}_n \frac{\sigma^2}{2} \right) (F_{g^{(n)}}(x)(\cdot)) f_j'''(x) \, dx.
\end{equation}
Consequently, we arrive at
\[
\sqrt{n} \left< f_j, \varrho^{(n)}(\tau) - \varrho^{(\tau(n))} \right> = -\sqrt{n} \int_{\tau(n)}^T \int_{\mathbb{R}} \left( \mathcal{I}_n b \left(F_{\varrho^{(n)}}(x)\right) f'''_j(x) + \left( \mathcal{I}_n \sigma^2 \right) \left( F_{\varrho^{(n)}}(x) \right) f'''_j(x) \, dx \, dt \right. \\
\left. + \mathcal{N}^{(n)}(\tau) - \mathcal{N}^{(\tau(n))} \right),
\]
where
\[
\mathcal{N}^{(n)}(t) := \frac{1}{\sqrt{n}} \sum_{i=1}^n \int_0^t \sigma \left( F_{\varrho^{(n)}}(X^{(n)}_i(s)) \right) f''_j \left( X^{(n)}_i(s) \right) dB^{(n)}_i(s), \quad t \geq 0.
\]

On the other hand, integration by parts and the notion of a generalized solution for the PDE (1.5) (see [13], Definition 3) imply that
\[
\sqrt{n} \left< f_j, \varrho(\tau) - \varrho^{(\tau(n))} \right> = -\sqrt{n} \int_{\tau(n)}^T \int_{\mathbb{R}} B \left(R(t,x)\right) f''_j(x) + \Sigma \left(R(t,x)\right) f'''_j(x) \, dx \, dt,
\]
which can be combined with (4.20) to
\[
\sqrt{n} \left< f_j, \varrho^{(n)}(\tau) - \varrho^{(\tau(n))} \right> - \sqrt{n} \left< f_j, \varrho(\tau) - \varrho^{(\tau(n))} \right> = -\sqrt{n} \int_{\tau(n)}^T \int_{\mathbb{R}} \left( \mathcal{I}_n b \left(F_{\varrho^{(n)}}(x)\right) - B \left(R(t,x)\right) \right) f''_j(x) \\
+ \left( \mathcal{I}_n \sigma^2 \right) \left( F_{\varrho^{(n)}}(x) \right) - \Sigma \left(R(t,x)\right) \right) f'''_j(x) \, dx \, dt \\
+ \mathcal{N}^{(n)}(\tau) - \mathcal{N}^{(\tau(n))}.
\]

To prove that the right-hand side of (4.23) converges to 0 in probability we note that the Lipschitz property of \( b, \frac{\sigma^2}{2} \) (cf. Assumption 1.1(b)) yields
\[
\lim_{n \to \infty} \sqrt{n} \sup_{r \in [0,1]} \left| (\mathcal{I}_n b)(r) - B(r) \right| = \lim_{n \to \infty} \sqrt{n} \sup_{r \in [0,1]} \left| \left( \mathcal{I}_n \sigma^2 \right) (r) - \Sigma(r) \right| = 0.
\]
Since, in addition, \( B, \Sigma \) are Lipschitz (cf. Assumption 1.1(b)) and \( f''_j, f'''_j \) are bounded, it suffices to obtain the limits in probability
\[
\lim_{n \to \infty} \sqrt{n} \int_{\tau(n)}^T \int_{\mathbb{R}} \left| F_{\varrho^{(n)}}(x) - R(t,x) \right| \, dx \, dt = 0 \quad \text{and} \quad \lim_{n \to \infty} \left( \mathcal{N}^{(n)}(\tau) - \mathcal{N}^{(\tau(n))} \right) = 0.
\]

For the first convergence in (4.23), we recall the representation of the \( W_1 \)-distance in (2.10) and apply the triangle inequality for the latter together with Markov’s inequality and Fubini’s theorem to find, for all \( \varepsilon, \varepsilon' > 0 \),
\[
\mathbb{P} \left( \sqrt{n} \int_{\tau(n)}^T W_1(\varrho(t), \varrho(t)) \, dt \right. > \varepsilon \left. \right) \\
\leq \mathbb{P} \left( \left| \tau(n) - \tau \right| > \varepsilon' \right) + \mathbb{P} \left( \sqrt{n} \int_{\tau - \varepsilon'}^{\tau + \varepsilon'} W_1(\varrho(t), \varrho(t)) \, dt > \varepsilon \right) \\
\leq \mathbb{P} \left( \left| \tau(n) - \tau \right| > \varepsilon' \right) + \frac{\sqrt{n}}{\varepsilon} \int_{\tau - \varepsilon'}^{\tau + \varepsilon'} \mathbb{E} \left[ W_1(\varrho(t), \varrho(t)) \right] + \mathbb{E} \left[ W_1(\varrho(t), \varrho(t)) \right] \, dt.
\]
In view of Propositions 3.1, 2.1 and 2.3 this estimate tends to 0 for all $\varepsilon > 0$ when we take $n \to \infty$ and then $\varepsilon' \downarrow 0$.

For the second convergence in (4.25), we compute the quadratic variation process

$$\int \frac{1}{n} \sum_{i=1}^{n} \sigma(F_{\theta}(X_{i}(s)))^{2} f_{j}(X_{i}(s))^{2} ds, \quad t \geq 0,$$

bound the resulting integrand by a constant $C < \infty$, and use the martingale representation theorem (see e.g. [20, Chapter 3, Theorem 4.6 and Problem 4.7]) to conclude

$$\limsup_{n \to \infty} \left| \frac{\sqrt{n}}{\varepsilon} \left( f_{j} h_{M}, \theta(n) \right) \right| \leq \limsup_{n \to \infty} \left| \frac{\sqrt{n}}{\varepsilon} \left( f_{j} \hat{h}_{M}, \theta(n) \right) \right|.$$
With \( A_t := b(R(t, \cdot)) \frac{dR}{d\tau} + \frac{\sigma(R(t, \cdot))^2}{2} \frac{d^2}{d\tau^2} \), \( t \geq 0 \), we may now replace \( \varrho^{(n)}(\cdot) \) by \( \varrho(\cdot) \) in the second line of (4.29) and deduce by means of Itô’s formula that

\[
(4.33) \quad \sqrt{n} \langle f_j \hat{h}_M, \varrho^{(n)}(\tau) - \varrho(\tau) \rangle = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( \langle f_j \hat{h}_M \rangle (X_i^{(n)}(\tau)) - \langle f_j \hat{h}_M \rangle (X_i^{(n)}(\tau)) \right)
\]

\[
= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \int_{\tau}^{\tau} \left( A_t(f_j \hat{h}_M) \right) (X_i^{(n)}(t)) \ dt + \mathcal{N}(\tau) - \hat{\mathcal{N}}^{(n)}(\tau),
\]

where

\[
(4.34) \quad \mathcal{N}^{(n)}(t) := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \int_{0}^{t} \sigma(R(s, X_i^{(n)}(s))) \left( f_j \hat{h}_M \right)' (X_i^{(n)}(s)) \ dB_i^{(n)}(s), \quad t \geq 0.
\]

Next, we take the expectation in Itô’s formula for \( \langle f_j \hat{h}_M, \varrho(\tau) - \varrho(\hat{\tau}^{(n)}) \rangle = \int_{\tau}^{\tau} \mathbb{E} \left[ \left( A_t(f_j \hat{h}_M) \right) (X_1^{(1)}(t)) \right] \ dt.

We proceed using the union bound, Markov’s inequality, and \( \hat{\tau}^{(n)} \in [0, \tau + 1] \):

\[
(4.35) \quad \mathbb{P} \left( \sqrt{n} \langle f_j \hat{h}_M, \varrho^{(n)}(\tau) - \varrho(\tau) \rangle - \sqrt{n} \langle f_j \hat{h}_M, \varrho(\tau) - \varrho(\hat{\tau}^{(n)}) \rangle \right) > \varepsilon
\]

\[
\leq \frac{2}{\varepsilon} \mathbb{E} \left[ \int_{0}^{\tau+1} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( \langle A_t(f_j \hat{h}_M) \rangle (X_i^{(1)}(t)) - \mathbb{E} \left[ \langle A_t(f_j \hat{h}_M) \rangle (X_i^{(1)}(t)) \right] \right) \right| dt \right]
\]

\[
+ \mathbb{P} \left( \left| \mathcal{N}(\tau) - \hat{\mathcal{N}}^{(n)}(\tau) \right| > \varepsilon/2 \right)
\]

\[
\leq \frac{2}{\varepsilon} \int_{0}^{\tau+1} \mathbb{SD} \left( \left( A_t(f_j \hat{h}_M) \right) (X_1^{(1)}(t)) \right) \ dt + \mathbb{P} \left( \left| \mathcal{N}(\tau) - \hat{\mathcal{N}}^{(n)}(\tau) \right| > \varepsilon/2 \right),
\]

where we have applied Fubini’s theorem, Jensen’s inequality, and the independence of \( X_1^{(n)}(t) = \frac{d}{d\tau} X_2^{(n)}(t) = \cdots = \frac{d}{d\tau} X_n^{(n)}(t) = \varrho(t) \) and have written \( \mathbb{SD} \) for the standard deviation operator. Since the standard deviation of a random variable does not exceed its \( L^2 \)-norm, the boundedness of \( b, \sigma \) (cf. Assumption \( \square \)), \( f_j \in \mathcal{E}_2 \), and the properties of \( \hat{h}_M \) imply an estimate of the form

\[
(4.36) \quad \mathbb{SD} \left( \left( A_t(f_j \hat{h}_M) \right) (X_1^{(1)}(t)) \right) \leq \mathbb{E} \left[ C e^{C \left| X_1^{(1)}(t) \right|} \mathbf{1}_{\left( \left| X_1^{(1)}(t) \right| > M \right)} \right]^{1/2}.
\]

Moreover, its right-hand side tends to 0 as \( M \to \infty \) uniformly in \( t \in [0, \tau] \) by the Cauchy-Schwarz inequality and the estimates (2.9), (2.8).

To finish the proof we need to analyze the last probability in (4.36). For this purpose, we compute

\[
(4.37) \quad \mathcal{N}^{(n)}(t) = \frac{1}{n} \sum_{i=1}^{n} \int_{0}^{t} \sigma(R(s, X_i^{(n)}(s))) \left( f_j \hat{h}_M \right)' (X_i^{(n)}(s))^2 ds, \quad t \geq 0.
\]
Thus, the martingale representation theorem (see e.g. [20], Chapter 3, Theorem 4.6 and Problem 4.7) and the union bound give, for all $\varepsilon' > 0$,

$$
P\left(\left| \overline{N}(\tau) - \hat{N}(\tau) \right| > \varepsilon / 2 \right)
$$

(4.39)

$$
\leq P\left(\left| \overline{N}'(\tau) \right| - \left| \hat{N}'(\tau) \right| > \varepsilon' \right) + P\left( \sup_{t \in [0, \varepsilon]} |B^{(1)}_t| > \varepsilon / 2 \right).
$$

Due to $\hat{N}(n) \in [0, \tau + 1]$, Markov’s inequality, $\overline{X}^{(n)}_1(\cdot) \equiv \overline{X}^{(n)}_2(\cdot) \equiv \cdots \equiv \overline{X}^{(n)}_n(\cdot) \equiv \overline{X}^{(1)}_1(\cdot)$, Fubini’s theorem, the boundedness of $\sigma$ (cf. Assumption L1(b)), $f_j \in \mathcal{E}_1$, and the properties of $\hat{N}_M$ we have an estimate of the type

$$
P\left(\left| \overline{N}'(\tau) \right| - \left| \hat{N}'(\tau) \right| > \varepsilon' \right) \leq \frac{1}{\varepsilon'} \int_0^{\tau+1} \mathbb{E}\left[ C_{\varepsilon'}(X^{(1)}_1(t)) 1_{\{|X^{(1)}_1(t)| > M\}} \right] dt.
$$

(4.40)

The latter converges to 0 as $M \to \infty$ thanks to the Cauchy-Schwarz inequality and the estimates (2.9), (2.8). It remains to observe that the second probability on the right-hand side of (4.39) vanishes as $\varepsilon' \downarrow 0$.  

5. Applications in stochastic portfolio theory

5.1. Dynamics of the market diversity. Consider a stock market with $n$ companies, as described by the market weight processes $\mu_1(\cdot), \mu_2(\cdot), \ldots, \mu_n(\cdot)$, i.e. the fractions of the total market capital invested in the different companies at any given time. In this context, a concept that has attracted much interest, both for scientific reasons and its importance in investment decisions, is the market diversity. Informally speaking, a market is thought of as diverse when one can be certain that no single company will end up with the vast majority of the market capital. In [9], FERNHOLZ has proposed to formalize the notion of diversity as follows.

**Definition 5.1** ([9], Definition 2.2.1). A market is called diverse if for some $\varepsilon > 0$ it holds $\max_{1 \leq i \leq n} \mu_i(t) \leq 1 - \varepsilon$ for all $t \geq 0$ almost surely. A market is referred to as weakly diverse on a finite time interval $[0, T]$ if for some $\varepsilon > 0$ one has

$$
\frac{1}{T} \int_0^T \max_{1 \leq i \leq n} \mu_i(t) \, dt \leq 1 - \varepsilon
$$

(5.1)

almost surely.

Subsequently, it is noticed in [9] that the vector of the market weight processes $\mu(\cdot) := (\mu_1(\cdot), \mu_2(\cdot), \ldots, \mu_n(\cdot))$ takes values in the closed unit simplex

$$
\overline{\Delta}^n := \left\{ x \in [0, 1]^n : \sum_{i=1}^n x_i = 1 \right\},
$$

(5.2)

whereas the diversity condition $\max_{1 \leq i \leq n} \mu_i(t) \leq 1 - \varepsilon$ is violated when $\mu(\cdot)$ enters the corresponding open neighborhoods of the vertices of $\overline{\Delta}^n$. Hence, it is natural to use a symmetric concave function on $\overline{\Delta}^n$, which necessarily attains its minimum at the vertices, to quantify the diversity of a market (or the lack thereof). The main examples of such functions discussed in [9] are:

(i) the entropy function $H(t) = -\sum_{i=1}^n \mu_i(t) \log \mu_i(t)$, $t \geq 0$, 
(ii) the $L^p$-norms $D_p(t) = \left( \sum_{i=1}^n \mu_i(t)^p \right)^{1/p}$, $t \geq 0$ for $p \in (0, 1)$, 
(iii) and the geometric mean $S(t) = \left( \prod_{i=1}^n \mu_i(t) \right)^{1/n}$, $t \geq 0$. 

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In particular, the entropy function and the $\ell^p$-norms for $p \in (0, 1)$ can be employed to test if a market is diverse in the sense of Definition 5.1 (cf. [9, Proposition 2.3.2]).

**Proposition 5.2.** A market is diverse if and only if for some $\varepsilon' > 0$ it holds $H(t) \geq \varepsilon'$ for all $t \geq 0$ almost surely or, equivalently, for some $p \in (0, 1)$ and $\varepsilon'' > 0$ one has $D_p(t) \geq 1 + \varepsilon''$ for all $t \geq 0$ almost surely.

Our Theorem [14] can be utilized to capture the dynamics of the entropy $H(\cdot)$, the $\ell^p$-norms $D_p(\cdot)$, $p \in (0, 1)$, and the geometric mean $S(\cdot)$ in rank-based models with a large number $n$ of companies. In that setting, the market weight processes are defined in terms of the solution to [13] by

$$
\mu_i(\cdot) = \frac{e^{X_i(n)(\cdot)}}{e^{X_1(n)(\cdot)} + e^{X_2(n)(\cdot)} + \ldots + e^{X_n(n)(\cdot)}}, \quad i = 1, 2, \ldots, n
$$

and give rise to the associated entropy, $\ell^p$-norm and geometric mean processes via the items (i), (ii), (iii) above.

**Corollary 5.3.** Under Assumption [17] the following convergences hold in the finite-dimensional distribution sense:

(a) for the entropy process $H(\cdot)$,

$$
\sqrt{n} \left( H(\cdot) - \log n - \log \left( e^x, \varrho(\cdot) \right) + \left( x e^x, \varrho(\cdot) \right) \right)
\xrightarrow{n \to \infty} \frac{1}{\langle e^x, \varrho(\cdot) \rangle} + \frac{\langle x e^x, \varrho(\cdot) \rangle}{\langle e^x, \varrho(\cdot) \rangle^2} \int_\mathbb{R} e^x G(\cdot, x) \, dx - \frac{1}{\langle e^x, \varrho(\cdot) \rangle} \int_\mathbb{R} (e^x + x e^x) G(\cdot, x) \, dx,
$$

(b) for an $\ell^p$-norm process $D_p(\cdot)$ with $p \in (0, 1)$,

$$
\sqrt{n} \left( n^{\frac{1}{p}} D_p(\cdot) - \frac{\langle e^{p x}, \varrho(\cdot) \rangle^{1/p}}{\langle e^x, \varrho(\cdot) \rangle} \right)
\xrightarrow{n \to \infty} \frac{\langle e^{p x}, \varrho(\cdot) \rangle^{1/p-1}}{\langle e^x, \varrho(\cdot) \rangle} \int_\mathbb{R} e^{p x} G(\cdot, x) \, dx + \frac{\langle e^{p x}, \varrho(\cdot) \rangle^{1/p}}{\langle e^x, \varrho(\cdot) \rangle^2} \int_\mathbb{R} e^x G(\cdot, x) \, dx,
$$

(c) for the geometric mean process $S(\cdot)$,

$$
\sqrt{n} \left( n S(\cdot) - \frac{e^{(x, \varrho(\cdot))}}{e^x, \varrho(\cdot)} \right)
\xrightarrow{n \to \infty} \frac{e^{(x, \varrho(\cdot))}}{e^x, \varrho(\cdot)} \int_\mathbb{R} G(\cdot, x) \, dx + \frac{e^{(x, \varrho(\cdot))}}{e^x, \varrho(\cdot)^2} \int_\mathbb{R} e^x G(\cdot, x) \, dx.
$$

**Proof.** The corollary is a direct consequence of Theorem [14]. For the sake of completeness, we write out the functions $J$, $f_1$, $\ldots$, $f_k$ in each of the three cases.

(a) For the normalized entropy process $H(\cdot) - \log n$, take

$$
J : (0, \infty) \times \mathbb{R} \to \mathbb{R}, \quad (x_1, x_2) \mapsto \log x_1 - \frac{x_2}{x_1}, \quad f_1(x) = e^x, \quad f_2(x) = xe^x.
$$

(b) For every normalized $\ell^p$-norm process $n^{\frac{1}{p-1}} D_p(\cdot)$, define

$$
J : (0, \infty) \times (0, \infty) \to \mathbb{R}, \quad (x_1, x_2) \mapsto \frac{x_1^{1/p}}{x_2}, \quad f_1(x) = e^{p x}, \quad f_2(x) = e^x.
$$

(c) For the normalized geometric mean process $\frac{S(\cdot)}{n}$, pick

$$
J : \mathbb{R} \times (0, \infty) \to \mathbb{R}, \quad (x_1, x_2) \mapsto \frac{e^{x_1}}{x_2}, \quad f_1(x) = x, \quad f_2(x) = e^x.
It is elementary to check the assumptions of Theorem 1.5 for all of these functions. □

5.2. Hitting times of the market diversity. In this subsection, we discuss the implications of Theorem 1.5 for the measures of diversity from Subsection 5.1. To this end, we denote by $H^*(\cdot)$, $D^*_p(\cdot)$, $p \in (0,1)$, and $S^*(\cdot)$ the limiting entropy, $\ell^p$-norms, and geometric mean processes, respectively:

\[
H^*(\cdot) = \log \langle e^x, g(\cdot) \rangle - \frac{\langle xe^x, g(\cdot) \rangle}{\langle e^x, g(\cdot) \rangle}, \quad D^*_p(\cdot) = \frac{(\ell^p x, g(\cdot))^{1/p}}{\ell^p x, g(\cdot)}, \quad p \in (0,1), \quad S^*(t) = \frac{\langle e^{x,t}, g(\cdot) \rangle}{\langle e^x, g(\cdot) \rangle}.
\]

The functions $J$, $f_1$, $\ldots$, $f_k$ in the three cases, given explicitly in (5.7), (5.8), and (5.9), respectively, satisfy the assumptions in Theorem 1.5, so that one only needs to verify the statements in (1.13) for the coefficients $b$, $\sigma$ and levels $a$ of interest. The next proposition provides the dynamics of $H^*(\cdot)$, $D^*_p(\cdot)$, $p \in (0,1)$, and $S^*(\cdot)$, thus, yielding a sufficient condition on $b$, $\sigma$, and $a$ for Theorem 1.5 to apply.

**Proposition 5.4.** Under Assumption [4] consider

\[
\rho_p(\cdot) := \frac{\langle e^{x,t}, g(\cdot) \rangle}{\langle e^x, g(\cdot) \rangle} \in C([0,\infty), M_1(\mathbb{R})), \quad p \in [0,1).
\]

Then, one has for the processes $H^*(\cdot)$, $D^*_p(\cdot)$, $p \in (0,1)$, and $S^*(\cdot)$ of (5.10):

\[
\begin{align*}
\frac{dH^*(t)}{dt} &= -\frac{1}{2}\langle \sigma(R(t,\cdot))^2, \varphi_1(t) \rangle - \text{cov}_{\varphi_1(t)}\left(x, b(R(t,\cdot)) + \frac{\sigma(R(t,\cdot))^2}{2}\right), \\
\frac{dD^*_p(t)}{dt} &= D^*_p(t)\left(b(R(t,\cdot)) + \frac{p\sigma(R(t,\cdot))^2}{2}, \rho_p(t)\right) - \left(b(R(t,\cdot)) + \frac{\sigma(R(t,\cdot))^2}{2}, \varphi_1(t)\right), \\
\frac{dS^*(t)}{dt} &= S^*(t)\left(b(R(t,\cdot)), g(t)\right) - \left(b(R(t,\cdot)) + \frac{\sigma(R(t,\cdot))^2}{2}, \varphi_1(t)\right).
\end{align*}
\]

In particular, whenever $b + \frac{\sigma^2}{2}$ is an increasing function and $a \in (-\infty, H^*(0)]$, $a \in (0, D^*_p(0)]$, $p \in (0,1)$, or $a \in (0, S^*(0)]$, the assertions in (1.13) hold and, hence, also the conclusion of Theorem 1.5 for the resulting hitting times of the normalized processes $H(\cdot) - \log n$, $n^{1/p} D^*_p(\cdot)$, $p \in (0,1)$, or $n S(\cdot)$.

We prepare the following continuous version of Chebyshev’s sum inequality for the proof of Proposition 5.4.

**Lemma 5.5.** For all $\nu \in M_1(\mathbb{R})$ and increasing functions $f, g$ on $\mathbb{R}$ integrable with respect to $\nu$,

\[
\langle fg, \nu \rangle \geq \langle f, \nu \rangle \langle g, \nu \rangle.
\]

**Proof.** Since $g$ is increasing, there exists an $x_0 \in \mathbb{R}$ such that $g(x) \leq \langle g, \nu \rangle$ if $x < x_0$ and $g(x) \geq \langle g, \nu \rangle$ if $x > x_0$. By distinguishing between $x < x_0$ and $x > x_0$ and using that $f$ is increasing we deduce

\[
f(x)\langle g(x) - \langle g, \nu \rangle \rangle \geq f(x_0)\langle g(x) - \langle g, \nu \rangle \rangle, \quad x \in \mathbb{R}.
\]

Integrating both sides with respect to $\nu$ and rearranging we arrive at (5.15). □

We are now ready to present the proof of Proposition 5.4.

**Proof of Proposition 5.4.** We recall the notation $A_t = b(R(t, \cdot)) \frac{d}{dx} + \frac{\sigma(R(t, \cdot))^2}{2} \frac{d^2}{dx^2}$, $t \geq 0$ and that, for any $f \in \mathcal{E}_2$,

\[
\langle f, \varphi(t) \rangle - \langle f, \varphi(0) \rangle = \int_0^t \langle A_s f, \varphi(s) \rangle ds, \quad t \geq 0
\]
(cf. \eqref{assumption1}). Due to the boundedness of $b$, $\sigma$ (cf. Assumption \ref{assumption1}(b)) and $f \in \mathcal{E}_2$ the dominated convergence theorem implies that the function $s \mapsto \langle A_t f, g(s) \rangle$ is continuous on $[0, \infty)$ and, thus,

\begin{equation}
\frac{d(f, g(t))}{dt} = \langle A_t f, g(t) \rangle, \quad t \geq 0.
\end{equation}

Therefore, in the setting of Theorem \ref{mainresult}

\begin{equation}
\frac{d\mathcal{J}_{f_1, \ldots, f_k}(\varphi(t))}{dt} = \sum_{j=1}^{k} \mathcal{J}_{f_1, \ldots, f_k}(\varphi(t)) \langle A_t f_j, \varphi(t) \rangle, \quad t \geq 0.
\end{equation}

To obtain the differential equations \eqref{eq1}, \eqref{eq2}, and \eqref{eq3} it suffices to insert into \eqref{5.19} the formulas from \eqref{5.7}, \eqref{5.8}, and \eqref{5.9}, respectively, and to simplify the result.

Supposing, in addition, that $b + \frac{a^2}{2}$ is increasing we can first employ Lemma \ref{lem1} with $\nu = \varphi_1(t)$, $f(x) = x$ and $g(x) = b(R(t, x)) + \frac{\sigma(R(t, x))^2}{2}$ to find

\begin{equation}
\text{cov}_{\varphi_1(t)} \left( x, b(R(t, \cdot)) + \frac{\sigma(R(t, \cdot))^2}{2} \right) \geq 0, \quad t \geq 0.
\end{equation}

Consequently, we read off from \eqref{5.12} that

\begin{equation}
\frac{dH^*(t)}{dt} \leq -\frac{1}{2} \min_{r \in [0, 1]} \sigma(r)^2, \quad t \geq 0,
\end{equation}

so \eqref{1.4} must hold for all $a \in (-\infty, H^*(0)]$, and the conclusion of Theorem \ref{mainresult} applies to the hitting times of such $a$ by $H(t) - \log u$.

Now, we take $\nu = \varphi_p(t)$, $f(x) = b(R(t, x)) + \frac{\sigma(R(t, x))^2}{2}$ and $g(x) = e^{(1-p)x}$, with $p \in [0, 1)$, in Lemma \ref{lem1} to get

\begin{equation}
\left( b(R(t, \cdot)) + \frac{\sigma(R(t, \cdot))^2}{2}, \varphi_1(t) \right) \geq \left( b(R(t, \cdot)) + \frac{\sigma(R(t, \cdot))^2}{2}, \varphi_p(t) \right), \quad t \geq 0.
\end{equation}

The values of $p \in (0, 1)$ and $p = 0$ reveal

\begin{equation}
\frac{d\log D^*_p(\cdot)}{dt} \leq -\frac{1 - p}{2} \min_{r \in [0, 1]} \sigma(r)^2, \quad p \in (0, 1) \quad \text{and} \quad \frac{d\log S^*_p(\cdot)}{dt} \leq -\frac{1}{2} \min_{r \in [0, 1]} \sigma(r)^2,
\end{equation}

respectively, yielding the remaining assertions. \hfill \square

**Remark 5.6.** A verification of the conditions in \eqref{1.4} beyond the setup in Proposition \ref{prop1} seems to require information on $\varphi(\cdot)$ or, equivalently, $R$ that needs to be deduced on a case-by-case basis. This is possible, for example, when \eqref{assumption1} is of the special form

\begin{equation}
dX^{(n)}_i(t) = \left( 2C_1 F_{\varphi^{(n)}(t)}(X^{(n)}_i(t)) + C_2 \right) dt + \sigma dB^{(n)}_i(t), \quad i = 1, 2, \ldots, n
\end{equation}

for some $C_1 \neq 0$, $C_2 \in \mathbb{R}$ and $\sigma > 0$. Indeed, then the Cauchy problem for the porous medium equation \eqref{assumption1} reduces to the one for the generalized Burgers equation

\begin{equation}
R_t = -(C_1 R^2 + C_2 R)_x + \frac{\sigma^2}{2} R_{xx}, \quad R(0, \cdot) = F_\lambda(\cdot).
\end{equation}

The solution of the latter is provided by the Cole-Hopf transformation $R = \frac{-\varphi}{\sigma^2}(\log \varphi)_x$, where $\varphi$ is the solution of the Cauchy problem for the heat equation

\begin{equation}
\varphi_t = -C_2 \varphi_x + \frac{\sigma^2}{2} \varphi_{xx}, \quad \varphi(0, x) = e^{-\frac{\sigma^2}{2C_1} \int_0^x F_\lambda(y) \, dy}.
\end{equation}
For any fixed \( \lambda \in M_1(\mathbb{R}) \) (perhaps retrieved from the observed market capitalizations), \( \varphi \) is given explicitly by a convolution with the heat kernel, and one can check if the conditions in (1.13) are valid for the resulting \( \theta(\cdot) = R_x(\cdot, x) \, dx \).

5.3. Performance of functionally generated portfolios. This last subsection is devoted to a discussion of the performance of multiplicatively and additively generated portfolios \( \pi \Psi; x \) and \( \pi \Psi; + \), as defined in the introduction. We focus initially on their associated non-decreasing excess growth processes

\[
\frac{1}{2} \sum_{i,j=1}^{n} \int_{0}^{t} \frac{\Psi_{x,ix,j}(\mu(\cdot))}{\Psi(\mu(\cdot))} \, d[\mu_i, \mu_j](\cdot), \; t \geq 0 \quad \text{and} \quad \frac{1}{2} \sum_{i,j=1}^{n} \int_{0}^{t} \bar{\Psi}_{x,ix,j}(\mu(\cdot)) \, d[\mu_i, \mu_j](\cdot), \; t \geq 0
\]

which enter the value processes \( V^{\Psi; x}(\cdot) \) and \( V^{\Psi; +}(\cdot) \) relative to that of the market portfolio \( \mu(\cdot) \) according to (1.19) and (1.20), respectively. Under the assumptions in (1.21), as well as (1.22) or (1.23), respectively, the former obey the concentration of measure estimate from [15, Corollary 8] (note the symmetry of \( \bar{\Psi} \) due to (1.18), (1.16); the strong law of large numbers for the process \( \mu(\cdot) \) in [2] equation (4.5)); and that the derivation of [15] Corollary 8 for multiplicatively generated portfolios carries over mutatis mutandis to the case of additive generation).

**Proposition 5.7.** Suppose the assumptions in (1.21), as well as (1.22) or (1.23) are satisfied for some \( n \in \mathbb{N} \). Then, for all \( r, t, \varepsilon > 0 \) and in the notation of Corollary 1.8

\[
\mathbb{P} \left( -\frac{1}{2t} \sum_{i,j=1}^{n} \int_{0}^{t} \frac{\Psi_{x,ix,j}(\mu(\cdot))}{\Psi(\mu(\cdot))} \, d[\mu_i, \mu_j](\cdot) \leq r^x - r \right) \leq \left\| \frac{d\kappa(n)}{d\zeta(n)} \right\|_{L^2(\zeta(n))} e^{-c^x(r, \varepsilon)t}
\]

or

\[
\mathbb{P} \left( -\frac{1}{2t} \sum_{i,j=1}^{n} \int_{0}^{t} \bar{\Psi}_{x,ix,j}(\mu(\cdot)) \, d[\mu_i, \mu_j](\cdot) \leq r^+ - r \right) \leq \left\| \frac{d\kappa(n)}{d\zeta(n)} \right\|_{L^2(\zeta(n))} e^{-c^+(r, \varepsilon)t},
\]

respectively.

If, in addition, Assumption 1.4 holds, one can combine Proposition 5.7 with Theorem 1.4 by using the union bound and obtain, for all \( r, s, t, \varepsilon > 0 \), the performance estimates

\[
\mathbb{P} \left( V^{\Psi; x}(t) \leq \frac{\mathcal{J}_{j_1, \ldots, j_k}(g(t)) - s/\sqrt{n}}{\Psi(\mu(0))} \right) \leq \Phi(s/\chi_t)(1 + o_n(1)) + \left\| \frac{d\kappa(n)}{d\zeta(n)} \right\|_{L^2(\zeta(n))} e^{-c^x(r, \varepsilon)t}
\]

or

\[
\mathbb{P} \left( V^{\Psi; +}(t) \leq 1 + \mathcal{J}_{j_1, \ldots, j_k}(g(t)) - s/\sqrt{n} - \bar{\Psi}(\mu(0)) + (r^+ - r)t \right) \leq \Phi(s/\chi_t)(1 + o_n(1)) + \left\| \frac{d\kappa(n)}{d\zeta(n)} \right\|_{L^2(\zeta(n))} e^{-c^+(r, \varepsilon)t},
\]

respectively, where \( \Phi \) is the standard normal tail cumulative distribution function, \( \chi_t \) is the standard deviation of the time \( t \) value of the Gaussian process on the right-hand side of (1.12), and \( o_n(1) \) is a quantity tending to 0 as \( n \to \infty \). Complementary to the performance estimates (5.30), (5.31) for fixed times, Corollary 1.8 which is proved next, provides a bound on the random time it takes for a multiplicatively or additively generated portfolio to reach the desired performance.
Proof of Corollary 1.8. We only give the proof of (1.24), as (1.25) can be shown in the same way. Our starting point is the observation that

\[
\mathbb{P}(\eta \bar{V}^n(x) \geq \tau + s/\sqrt{n}) \leq \mathbb{P}(\eta \bar{V}^n(x) \geq \tau(n)) + \mathbb{P}(\tau(n) \geq \tau + s/\sqrt{n}).
\]

By the definition of \(\eta \bar{V}^x\), the first of the latter two summands is less or equal to

\[
\mathbb{P}\left(V \bar{V}^x(\tau(n)) \leq \frac{a}{\Psi(\mu(0))} e^{(r^x-r)(\tau-s/\sqrt{n})}\right)
\]

\[
= \mathbb{P}\left(-\frac{1}{2} \sum_{i,j=1}^{n} \int_{0}^{\tau(n)} \frac{\bar{V}^x_{i,j}(\mu(\cdot))}{\Psi(\mu(\cdot))} \, d[\mu_i, \mu_j](\cdot) \leq (r^x-r)(\tau-s/\sqrt{n})\right)
\]

(recall (1.19) and \(\tilde{\Psi}(\mu(\tau(n))) = a\)). Since the excess growth process is non-decreasing, the probability on the right-hand side of (5.33) is at most

\[
\mathbb{P}(\tau(n) \leq \tau - s/\sqrt{n}) + \mathbb{P}\left(-\frac{1}{2} \sum_{i,j=1}^{n} \int_{0}^{\tau - s/\sqrt{n}} \frac{\bar{V}^x_{i,j}(\mu(\cdot))}{\Psi(\mu(\cdot))} \, d[\mu_i, \mu_j](\cdot) \leq (r^x-r)(\tau-s/\sqrt{n})\right).
\]

Using (1.15) for the second summand on the right-hand side of (5.32) and the first summand in (5.34), then (5.28) for the second summand in (5.34) we get (1.24). \(\square\)

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