Spin-transfer torque magnetization reversal in uniaxial nanomagnets with thermal noise

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We consider the general Landau-Lifshitz-Gilbert (LLG) dynamical theory underlying the magnetization switching rates of a thin film uniaxial magnet subject to spin-torque effects and thermal fluctuations. After discussing the various dynamical regimes governing the switching phenomena, we present analytical results for the mean switching time behavior. Our approach, based on explicitly solving the first passage time problem, allows for a straightforward analysis of the thermally assisted, low spin-torque, switching asymptotics of thin film magnets. To verify our theory, we have developed an efficient Graphics Processing Unit (GPU)-based micromagnetic code to simulate the stochastic LLG dynamics out to millisecond timescales. We explore the effects of geometrical tilts between the spin-current and uniaxial anisotropy axes on the thermally assisted dynamics. We find that even in the absence of axial symmetry, the switching times can be functionally described in a form virtually identical to the collinear case.

I. INTRODUCTION

More than a decade has passed since spin-torque effects were demonstrated experimentally by the switching of the magnetization of a thin ferromagnetic film when current is passed between it and a pinned ferromagnetic layer.1–4 A spin-polarized current passing through a small magnetic conductor will deposit spin-angular momentum into the magnetic system. This in turn causes the magnetic moment to precess and in some cases even switch direction. This has led to sweeping advances in the field of spintronics through the development and study of spin-valves and magnetic tunnel junctions (see, for example, Ref. 6). The theoretical approach to such a problem has conventionally been to treat the thin ferromagnetic film as a single macrospin in the spirit of Brown.5 Spin-torque effects are taken into account phenomenologically by modifying the macrospin’s Landau-Lifshitz-Gilbert (LLG) dynamical equation.1 A thorough understanding of the phenomena, however, cannot proceed without taking into account the effect of thermal fluctuations. This is of particular experimental relevance since spin-transfer effects on nanomagnets are often conducted at low currents, where noise is expected to dominate. Recent debate in the literature over the proper exponential scaling behavior between mean switching time and current shows how the thermally assisted properties of even the simplest magnetic setups leave much to be understood.7–12 The interplay between spin-torque and thermal effects determine the dynamical properties of recent experimental studies on nanopillar devices.13 Except at very high currents where the dynamics are predominantly deterministic, the switching appears to be thermal in nature. Fitting to experimental data requires accurate knowledge of the energetics, which, in the realm of spin-torque, are hard to come by due to the inherently non-conservative nature of the spin-torque term.

Theoretical progress has been hindered by the computational power needed to run numerical simulations to the desired degree of accuracy. The LLG equation modified into its set of coupled stochastic equations can be studied in one of two ways: either by concentrating on the associated Focker-Planck equation or by constructing a stochastic Langevin integrator to be used enough times to gather sufficient statistics on the phenomena.14 The latter approach, however, has been unable to extrapolate to long enough times to capture the dynamical extent of the thermal regime. Recent papers by Taniguchi and Imamura15 and Butler,16 following arguments first made by Suzuki,17 suggest that previous analytics of the thermally assisted dynamics should be revisited. Nonetheless, no numerical simulation has yet been able to fully evaluate the accuracy of the Taniguchi and Imamura results without resorting to comparison with the field switching model.18 In fact, as will be apparent in what follows, the applied current in the spin-torque model cannot always be interpreted mathematically as an applied magnetic field. In our paper, we will show that simulations run harnessing the vast computational parallelization capabilities intrinsic in Graphics Processing Unit (GPU) technology for numerical modeling can allow a deeper probing of such a thermally activated regime.

II. GENERAL FORMALISM

A simple model of a ferromagnet uses a Stoner-Wohlfarth monodomain magnetic body with magnetization \( \mathbf{M} \). The body is assumed to have a size \( l_m \) along the \( \mathbf{e}_z \) direction, and size \( a \) in both the \( \mathbf{e}_x \) and \( \mathbf{e}_y \) directions. The total volume of the object is then \( V = a^2 l_m \). The energy landscape experienced by \( \mathbf{M} \) is generally described by three terms: an applied field \( \mathbf{H} \), a uniaxial anisotropy energy \( U_K \) with easy
axis along \( \mathbf{n}_K \) in the \( \mathbf{e}_x - \mathbf{e}_z \) making an angle \( \omega \) with the \( \mathbf{e}_z \) axis, and an easy-plane anisotropy \( U_p \) in the \( \mathbf{e}_z - \mathbf{e}_e \) plane, with normal direction \( \mathbf{n}_p \) perpendicular to \( \mathbf{n}_K \). The magnetization \( \mathbf{M} \) is assumed to be constant in magnitude and for simplicity normalized into a unit direction vector \( \mathbf{m} = \mathbf{M}/|\mathbf{M}| \). A spin-polarized current \( J \) enters the magnetic body in the \( -\mathbf{e}_z \) direction, with spin polarization factor \( \eta \) and spin direction along \( \mathbf{e}_s \). The current exits in the same direction, but with its average spin direction aligned to that of \( \mathbf{M} \). The self-induced magnetic field of the current is ignored here as the dimension \( a \) is considered to be smaller than 100nm, where the spin-current effects are expected to become dominant over the current induced magnetic field. The standard LLG equation used to describe the dynamics is then written as

\[
\mathbf{\dot{m}} = -\gamma' \mathbf{m} \times \mathbf{H}_{\text{eff}} - 2\gamma \mathbf{m} \times (\mathbf{m} \times \mathbf{H}_{\text{eff}}) - \gamma' \mathbf{j} \mathbf{m} \times (\mathbf{m} \times \mathbf{n}_p) + \gamma z \mathbf{j} \mathbf{m} \times \mathbf{n}_p,
\]

where \( \gamma' = \gamma/(1 + z^2) \) is the Gilbert ratio, \( \gamma \) is the usual gyromagnetic ratio \( 1.76 \times 10^{11} \text{rad T}^{-1} \text{s}^{-1} \), and \( j = (h/2e)\eta J \) is the spin-angular momentum deposited per unit time with \( \eta = (J_1 - J_2)/(J_1 + J_2) \) the spin-polarization factor of incident current \( J \). The last two terms describe a vector torque generated by current polarized in the direction \( \mathbf{n}_p \). These are obtained by assuming that the macrospin absorbs angular momentum from the spin-polarized current only in the direction perpendicular to \( \mathbf{m} \).

To write \( \mathbf{H}_{\text{eff}} \) explicitly, we must construct a proper energy landscape for the magnetic body. There are three main components that need to be considered: a uniaxial anisotropy energy \( U_K \), an easy-plane anisotropy \( U_P \), and an external field interaction \( U_H \). These are written as follows:

\[
\begin{align*}
U_K &= -K V (\mathbf{n}_K \cdot \mathbf{m})^2, \\
U_P &= K_P V (\mathbf{n}_P \cdot \mathbf{m})^2, \\
U_H &= -M_S V \mathbf{m} \cdot \mathbf{H}_{\text{ext}}.
\end{align*}
\]

In these equations, \( M_S \) is the saturation magnetization, \( K_P \) is the easy-plane anisotropy, \( K = (1/2)M_S H_K \), and \( H_K \) is the Stoner-Wohlfarth switching field (in units of Teslas). In what follows, we will consider a simplified model in which the effects of easy-plane anisotropy are ignored and all external magnetic fields are absent. However, we retain contributions due to magnetic fields in our derivations for reasons which will be apparent in Sec. III. The full energy landscape then becomes \( U(\mathbf{m}) = U_K + U_H \) and reads

\[
U(\mathbf{m}) = -K V (\mathbf{n}_K \cdot \mathbf{m})^2 + 2h \cdot \mathbf{m},
\]

where \( h = \mathbf{H}_{\text{ext}}/H_K \), \( \mathbf{n}_K \) is the unit vector pointing in the orientation of the uniaxial anisotropy axis. Such an energy landscape generally selects stable magnetic configurations parallel and anti-parallel to \( \mathbf{n}_K \). The effective interaction field \( \mathbf{H}_{\text{eff}} \) is then given by

\[
\mathbf{H}_{\text{eff}} = - \frac{1}{M_S V} \nabla U(\mathbf{m}) = H_K (\mathbf{n}_K \cdot \mathbf{m}) \mathbf{n}_K + \mathbf{h}.
\]

The symmetries of the problem lead to slightly simplified equations and the deterministic LLG dynamics can then be expressed as

\[
\mathbf{\dot{m}} = - \mathbf{m} \times [(\mathbf{n}_K \cdot \mathbf{m}) \mathbf{n}_K + \mathbf{h}] - 2\gamma \mathbf{m} \times (\mathbf{m} \times \mathbf{H}_{\text{eff}}) - \gamma' \mathbf{j} \mathbf{m} \times (\mathbf{m} \times \mathbf{n}_p) + \gamma z \mathbf{j} \mathbf{m} \times \mathbf{n}_p - 2z \mathbf{j} \mathbf{m} \times (\mathbf{m} \times \mathbf{k}) + 2x^2 \mathbf{j} \mathbf{m} \times \mathbf{k},
\]

where we have defined \( I = j/(\alpha H_K) \), conveniently chose \( \mathbf{n}_p \) as the orientation of our \( z \)-axis (\( \mathbf{k} \)) and introduced the natural timescale \( \tau = \gamma' H_K t \).

### III. THERMAL EFFECTS

Thermal effects are included by considering uncorrelated fluctuations in the effective interaction field: \( \mathbf{H}_{\text{eff}} \to \mathbf{H}_{\text{eff}} + \mathbf{h}_{\text{th}} \). These transform the LLG equation into its Langevin form upon performing the substitution \( \mathbf{h} \to \mathbf{H}_{\text{th}} \) in (4). We model the stochastic contribution \( \mathbf{H}_{\text{th}} \) by specifying its correlation properties, namely

\[
\begin{align*}
\langle \mathbf{H}_{\text{th}} \rangle &= 0, \\
\langle \mathbf{H}_{\text{th}}(\mathbf{m}) \mathbf{H}_{\text{th}}(\mathbf{m})' \rangle &= 2D \delta(\mathbf{m} \cdot \mathbf{m}') \delta(t - t').
\end{align*}
\]

The effect of the random torque \( \mathbf{H}_{\text{th}} \) is to produce a diffusive random walk on the surface of the \( \mathbf{M} \)-sphere. An associated Fokker-Planck equation describing such dynamics was constructed by Brown.\(^5\) At long times, the system attains thermal equilibrium; and by the fluctuation-dissipation theorem, we have

\[
D = \frac{z k_B T}{2KV(1 + x^2)} = \frac{\alpha}{2(1 + x^2)^2}.
\]

It is convenient to introduce the notation \( KV/k_B T = \zeta \), representing the natural barrier height of the uniaxial anisotropy energy.

Considering only thermal fluctuations, the stochastic LLG equation reads

\[
\mathbf{\dot{m}}_t = A(m) + B_h(m) \cdot \mathbf{H}_{\text{th},k},
\]

where

\[
\begin{align*}
A(m) &= x(2x^2 - 3) + \alpha (\mathbf{n}_K \cdot \mathbf{m}) \mathbf{n}_K - 2z \mathbf{m} \times (\mathbf{m} \times \mathbf{k}), \\
B_h(m) &= -\epsilon_{ijk} m_j - x (m_i m_k - \delta_{ik}),
\end{align*}
\]

and ""\( H_{\text{th},k} \)"" means to interpret our stochastic dynamics in the sense of Stratonovich\(^3\)\(^\text{\textsuperscript{33}}\)\(\text{\textsuperscript{33}}\) calculus in treating the multiplicative noise terms.\(^19\)

We numerically solve the above Langevin equations by using a standard second order Heun scheme to ensure proper convergence to the Stratonovich calculus. At each time step, the strength of the random kicks is given by the fluctuation-dissipation theorem. Statistics were gathered from an ensemble of 5000 events with a natural integration stepsize of 0.01. For concreteness, we set the Landau damping constant \( z = 0.04 \). A magnetic ensemble was considered “switched” when half the members of the ensemble have reversed their
direction of magnetization. Different barrier heights were explored although the main results in this paper are shown for a barrier height of $\xi = 80$. To explore the simulations for long time regimes, the necessary events were simulated in parallel on an NVidia Tesla C2050 graphics card. To generate the large number of necessary random numbers, we chose a proven combination$^{20}$ of the three-component combined Tausworthe “taus88”$^{21}$ and the 32-bit “Quick and Dirty” Linear Congruential Generator (LCG)$^{22}$. The hybrid generator provides an overall period of around $2^{121}$.

**IV. SWITCHING DYNAMICS**

In experiments, one is generally interested in understanding how the interplay between thermal noise and spin torque effects switch an initial magnetic orientation from parallel to anti-parallel and vice versa. The role of spin-torque effects switch an initial magnetic orientation from parallel on an NVidia Tesla C2050 graphics card. To generate the large number of necessary random numbers, we chose a proven combination of the three-component combined Tausworthe “taus88” and the 32-bit “Quick and Dirty” Linear Congruential Generator (LCG). The hybrid generator provides an overall period of around $2^{121}$.

**V. COLLINEAR SPIN-TORQUE MODEL**

Having derived the necessary expressions for our macro-spin model dynamics, it is useful to consider the following simplification. Let us take the uniaxial anisotropy and spin-current axes to be collinear, namely, $\hat{n}_K \equiv \hat{n}_z \equiv \hat{k}$. In such a scenario, the stochastic LLG equation simplifies significantly. In particular, (10) reduces to the simplified form

$$\hat{q} = \sigma [(n_z I + q)(1 - q^2) + n_z J_{qp}] + \zeta^2 I m_x m_y + \frac{\zeta}{\xi} (1 - q^2) \cdot \hat{W}. \quad (10)$$

In the above equation, we have defined $q \equiv \hat{m} \cdot \hat{n}_K$, $p$ the analogous projection onto an axis perpendicular to $\hat{n}_K$. Furthermore, $\hat{n}_K$ is taken to lie in the x-z plane (such that we are projecting $\hat{m}$ on axes rotated about the y-axis); $n_z$ and $n_x$ are the projections of the uniaxial anisotropy axis, respectively, on the $z$ and $x$ axes. Furthermore, the multiple stochastic contributions are assumed to be Gaussian random variables with identical average and space dependent variance, $\hat{W}$ is a standard mean zero, variance 1, Wiener process, and its prefactor explicitly expresses the strength of the stochastic contribution.$^{19}$ As will be shown in the following Subsections, (10) is a convenient analytical tool in specific scenarios. In general, it is not useful as it explicitly depends on the dynamics of both the $m_z$ and $m_p$ components of the magnetization.

Numerically, we can solve (7) directly. In all our simulations, the initial ensemble of magnetizations was taken to be Boltzmann-distributed along the anti-parallel orientation. We assume that the energy barrier height is so large that, before current effects are activated, thermalization has only been achieved within the antiparallel energy well and no states have had time to thermally switch to the parallel orientation on their own. A typical histogram of magnetic orientations at a given time is shown in Figure 1.

We now turn on a current and allow the system to evolve for a fixed amount of time. Once this time has passed, we use the projection rule expressed above to evaluate what fraction of the ensemble has effectively switched from the anti-parallel to the parallel state.

![Graph](image)
\[ \dot{q} = a(I + q)(1 - q^2) + \sqrt{\frac{2}{\xi}}(1 - q^2) \cdot W. \] (11)

In this symmetric scenario, \( q \) coincides with \( m_z \) and magnetization reversal has been reduced to a straightforward 1-D problem. For a general value of \( I < 1 \), the evolution of \( q \) has two local minima and a saddle. The two stable configurations are at \( q = -1 \) and \( q = 1 \), while the saddle is located at \( q = -I \). For currents \( I > I_c = 1 \), there is only one stable minimum. Above critical current, spin torque pushes all magnetic configurations toward the parallel \( q = 1 \) state. This regime is particularly important not just for its simplicity but also for its similarity to the pure field switching model. The collinear spin-torque model is, in fact, mathematically identical to a field switching model with applied field of intensity \( I \) applied parallel to the uniaxial anisotropy axis of the magnetic system.\(^{17} \)

**A. Collinear high current regime**

In the high current regime \( I \gg I_c \), we expect the deterministic dynamics to dominate over thermal effects. We refer to this also as ballistic evolution interchangeably. The deterministic contribution of (11) can then be solved analytically given an initial configuration \( q = m_z = -m_0 \). The switching time \( \tau_s \) will simply be the time taken to get from some \( m_z = -m_0 < 0 \) to \( m_z = 0 \) and reads

\[
\tau_s(m_0) = \frac{1}{2a} \int_{-m_0}^{m_0} \frac{dm}{(I + m)(1 - m^2)}
= \frac{1}{2a(I^2 - 1)} \left\{ \frac{\log[1 + m_0]}{[1 - m_0]} - \log[1 - m_0]^2 \right\} - 2 \log \left( \frac{I}{I - m_0} \right). \quad (12)
\]

Since the magnetic states are considered to be in thermal equilibrium before the current is turned on, one should average the above result over the equilibrium Boltzmann distribution in the starting well to obtain the average switching time \( \langle \tau_s \rangle_B \). For \( \xi \) large enough, such an initial distribution will be

\[
\rho_B(m) = \frac{\sqrt{\xi} \exp[-\xi m^2]}{F[\sqrt{\xi}]} \exp[\xi m^2], \quad (13)
\]

where \( F[x] = \exp(-x^2) \int_0^\infty \exp(y^2)dy \) is Dawson’s integral. This expression can be used to compute the average switching time numerically.

As the intensity of spin-currents becomes closer to \( I_c \), thermal effects increasingly contribute. Moreover, diffusion gradients add to the deterministic drift, which can be shown explicitly by writing (11) in its equivalent Ito form. Doing so leads us to a first correction of the ballistic dynamics due to thermal influences. The \( z \)-component behavior then reads

\[
m_z = a(I + m_z)(1 - m_z^2) - \frac{a}{\xi} m_z + \sqrt{\frac{2}{\xi}}(1 - m_z^2) \cdot W. \quad (14)
\]

The first term on the right hand side is still the ballistic flow that we have just discussed. The second term is the desired diffusion-gradient drift term. The contribution of such a term generates a net motion away from the stable minima of the ballistic equations as one expects to see under the influence of thermal effects. Again, we can solve the drift dominated flow analytically to compute the switching time. Considering diffusion-gradient drift, this reads

\[
\tau_s(m_0) = \frac{1}{2a} \int_{-m_0}^{m_0} \frac{dm}{(I + m)(1 - m^2) - (m/2\xi)^2}
= \frac{1}{2a} \sum_j \log[1 + (m_0/w_j)] - \log[1 - (m_0/w_j)], \quad (15)
\]

where the \( w_j \) are the three zeros of the cubic equation \( w^3 + Iw^2 - (1 - 1/\xi^2)w - I = 0 \). As before, the average switching \( \langle \tau_s \rangle_B \) time will simply be given by averaging numerically over the Boltzmann distribution \( \rho_B \). In Figure 2, the reader can see how well these two limiting results fit the simulation data. As expected, both expressions coincide in the limit of high currents.

**VI. UNIAXIAL TILT**

In the high current regime \( (I \gg I_c) \), where \( \hat{n}_K = \hat{k} \) (i.e., the uniaxial axis is aligned with the \( z \)-axis), the ballistic equation for \( m_z \) was shown to decouple from the other components, and the dynamics became one dimensional and deterministic. For the more general case where the uniaxial anisotropy axis may have any tilt with respect to the \( z \)-axis, such a critical current is not as intuitively defined. Unlike the collinear limit, a critical current, above which all magnetic states perceive a net flow towards an increasing projection, does not exist. One can plot \( \langle \dot{q} \rangle = \mathbf{m} \cdot \mathbf{n}_K \) over the unit sphere to see what regions allow for an increasing and decreasing projection as the current is changed. An example of this is shown in Figure 3.

Unfortunately, regions characterizing negative projection flow can be shown to persist at all currents. The
approach is refined by requiring that on average, over constant-energy precessional trajectories, the flow is toward the positive uniaxial anisotropy axis: \( \langle \mathbf{m} \cdot \mathbf{n}_K \rangle > 0 \). Such trajectories are found by solving the flow equations with \( I = \alpha = 0 \). Solutions correspond to circular libations about the uniaxial anisotropy axis. The critical current is then redefined to be the minimum current at which the average projectional flow is positive at all possible precessional energies. This is easily done and results in

\[
I \geq \max_{\alpha} \left[ \frac{-\epsilon}{\cos \alpha} \right] = \frac{1}{\cos(\alpha)} = I_{\text{crit}}, \tag{16}
\]

thus allowing for a direct comparison of dynamical switching results between different angular configurations of uniaxial tilt. In our discussion of (10), we mentioned how, in the general case, presenting uniaxial tilt, there is no way to reduce the dimensionality of the full dynamical equations. In fact, in the presence of tilt, precessional trajectories might allow for a magnetization state to temporarily transit through a \( q > 0 \), “switched” configuration, even though it might spend the majority of its orbit in a \( q < 0 \), “unswitched” configuration. This allows for a much richer mean switching time behavior, especially for currents greater than the critical current, as shown in Figure 4 and discussed more in depth later.

VII. THERMALLY ACTIVATED REGIME

For currents \( I < I_c \), the switching relies on thermal effects to stochastically push the magnetization from one energy minimum to the other. It is of interest to understand how switching probabilities and switching times depend on temperature and applied current. This is easily done in stochastic systems with gradient flow. In such cases, an energy landscape exists and a steady state probability distribution can be constructed to be used via Kramer’s theory in deriving approximate low-noise switching probabilities.

Unfortunately, spin-torque effects introduce a non-gradient term, and the resulting LLG equation does not admit an energy landscape in the presence of applied current. The collinear simplification, however, is an exception. As already described, in the absence of uniaxial tilt, the dynamics become effectively one dimensional since the \( m_z \) component decouples from the other magnetization components. Consider then (11): because it is decoupled from the other degrees of freedom, we can construct a corresponding one-component Focker-Planck equation. The evolution in time of the distribution of \( q \) is then

\[
\frac{\partial}{\partial t} \rho(q,t) = \hat{L}[\rho](q,t), \tag{17}
\]

where

\[
\hat{L}[f] = -2\partial_q \left[ (q + I)(1 - q^2) - \frac{1}{2\xi}(1 - q^2)\partial_q \right] f. \tag{18}
\]

For high energy barriers and low currents, the switching events from one basin to the other are expected to be rare. The probability of a double reversal should be even smaller. We therefore model the magnetization reversal as a mean first passage time (MFPT) problem with absorbing boundaries at the saddle point. The MFPT will then be given by the solution of the adjoint equation (\( L^\dagger \langle \tau \rangle(q) = -1 \))

\[
\frac{\partial}{\partial q} \left[ \frac{2}{2\xi} \exp(-\xi(q + I)^2)\partial_q [(1 - q^2) \exp(\xi(q + I)^2)] \right] \partial_q \langle \tau \rangle(q) = -1 \tag{19}
\]

subject to the boundary condition \( \langle \tau \rangle(0) = 0 \). This can be solved to give

\[
\langle \tau \rangle(q) = \frac{2}{\alpha} \int_0^q du \exp\left[ -\frac{1-\xi(u+I)^2}{1-u^2} \right] ds \exp(\xi(s+I)^2). \tag{20}
\]
The rightmost integral can be computed explicitly. Retaining only dominant terms, the final integral can be computed by saddlepoint approximation to give

$$\langle \tau \rangle \simeq 2 \frac{\sqrt{\pi} \exp(\zeta(1-I)^2)F(\sqrt{\zeta}(1-I))}{1-I^2}. \quad (21)$$

Such a square exponential dependence has recently been derived by Taniguchi and Imamura\textsuperscript{15,24} as well as Butler et al.,\textsuperscript{16} although a $\tau \propto \exp(\zeta(1-I))$ dependence, proposed elsewhere in the literature,\textsuperscript{7,9,10} has also been successfully used to fit experimental data.\textsuperscript{13,34}

To decide between these experimental dependences, we fit the scaling behaviors in Figure 5, along with the theoretical prediction from (24). The square exponential dependence fits the data better, confirming analytical results. Furthermore, comparison of the asymptotic expression (21) to the full theoretical prediction obtained by solving (20) numerically demonstrates that even for mean switching times of the order $10^{-1}$ T · ms, asymptotically still is not fully achieved.

All that remains is to consider the effects of angular tilt on the switching properties in the thermally activated regime. Insight into this problem can be obtained by invoking (12) again. For small values of $\alpha$, the term in square brackets is of leading order over the second ballistic term depending on $m_a$. This allows us, in the small $\alpha$ regime, to neglect the second ballistic term altogether.

We now concentrate on the behavior of the term in square brackets. For low sub-critical currents, switching will depend on thermal activation for the most part. We expect an initially anti-parallel configuration to not diffuse very far away from its local energy minima. It will remain that way until a strong enough thermal kick manages to drive it across the energy barrier. Because of this, the second term appearing in the square brackets will generally be close to zero as the particle awaits thermal switching. To make the statement more precise, one can imagine the magnetic state precessing many times before actually making it over the saddle. The second term can then be averaged over a constant energy trajectory where $p$ will vanish identically. Hence, in the subcritical regime, (12) can be rewritten in the following approximate form:

$$\dot{q} \simeq \alpha(n_z I + q)(1-q^2) + \sqrt{\frac{\beta}{\zeta}(1-q^2)} \cdot \dot{W}. \quad (22)$$

This is reminiscent of the 1D projectional dynamics discussed in relation to the collinear limit and shown explicitly in (11). The only difference between the two is the substitution $I \rightarrow n_z I$; recall because $n_z = \cos(\omega)$, $n_z I = 1/I_c$. In other words, the thermally activated dynamics are the same for all angular separations up to a rescaling by the critical current. We then expect that the mean switching time dependences remain functionally identical to the collinear case for all uniaxial tilts. We have confirmed this by comparison with data from our simulations, and the results are shown in Figure 6. As predicted, all mean switching time data from different uniaxial tilts collapses on top of each other after a rescaling by each tilt’s proper critical current.

As already hinted in the introduction to the collinear model, a strong mathematical analogy exists with the field switching model studied in the literature.\textsuperscript{25–31} By considering a macrospin model with external magnetic field applied in the direction of the uniaxial anisotropy axis, one obtains a dynamical equation for the magnetization vector analogous to (11) with the field strength in place of the applied current. In fact, one can think of writing an effective energy landscape $U(m_z) = -K(m_z^2 - 2m_c)$, in terms of which the equilibrium Boltzmann distribution has precisely the same form as that given by Brown.\textsuperscript{5} The thermally activated behavior discussed in the literature also reproduces an exponent 2
scaling dependence such as what we have shown here. One must understand, that spin torque effects are generally non conservative and it is only in this collinear scenario that they may be interpreted in this way. Upon introducing an angular tilt between uniaxial and spin-polarization axes, the analogy with the field switching model will generally break (see (10)). It is interesting, to quantify the crossover between the spin torque and field switched macrospin model. Coffey\textsuperscript{25,26} has already discussed the effect of angular tilt between anisotropy and applied switching field axes. We introduced noise in the macrospin model by considering a random applied magnetic field. In (7), we showed the full form of the dynamical equations. To write the dynamical equations for the field switched model, we need to suppress current effects and simply introduce a term identical to the stochastic contribution with the exception that now the applied field will not be random but fixed at a specific angular separation from the uniaxial anisotropy axis. Writing the dynamical equations for the field switched model is straightforward

\[
\dot{q} = x[(n_x h + q)(1 - q^2) + n_y h(m_y - xq)] + \sqrt{\frac{\gamma}{\kappa}} (1 - q^2) \cdot \dot{W},
\]

(23)

where instead of an applied current \(I\), the dynamics depend on an applied field with strength \(h\). Comparison with (10) shows how, in general, the two evolution equations are very different from each other. In the thermally activated regime, however, where one is able to average over constant energy trajectories due to their timescales being much smaller than those required for actual diffusion or magnetic torque (\(h < h_{\text{coll}}\)), the second term in square brackets will be averaged away and one is left with a stochastic evolution equation identical to (22). All thermally activated switching will then again be functionally identical for all angular tilts up to a trivial rescaling of the applied field.

In comparing our scaling relationships between current and mean switching time with the previous literature, a subtle issue must be addressed. Results obtained by Apalkov and Visscher\textsuperscript{9} rely upon an initial averaging of the dynamics in energy space over constant energy trajectories (limit for small damping) and only subsequently applying weak noise methods to extrapolate switching time dependences. The small damping and weak noise limits are singular and the order in which they are taken is important. Both limits radically alter the form of the equations: whereas both limits suppress thermal effects, the first also severely restrains the deterministic evolution of the magnetic system. Our approach considers the weak noise limit and, only in discussing the effects of an angular tilt between polarization and easy axes do we employ the small damping averaging technique to obtain functional forms for the mean switching time. The switching time data shown seem to justify, in this particular case, an interchangeability between these two limits. More generally, however, one should not expect the two limits to commute.

VIII. SWITCHING TIME PROBABILITY CURVES

Up until now, we have analyzed the main properties of spin-torque induced switching dynamics by concentrating solely on the mean switching times. In experiments, one generally constructs full switching time probability curves. The probability that a given magnetic particle has a switching time \(\tau_s \leq \tau\) can be explicitly written as

\[
P[\tau_s \leq \tau] = \exp[-\xi(1 - m(\tau)^2)] \frac{F[\sqrt{\xi}m(\tau)]}{F[\sqrt{\xi}]},
\]

(24)

where \(m(\tau)\) is the initial magnetization that is switched deterministically in time \(\tau\). Once one has evaluated \(m(\tau)\), the probability curve follows. Ideally, in the ballistic regime, one would like to invert the ballistic equations. Unfortunately, the solutions of such ballistic equations are generally transcendental and cannot be inverted analytically. Even in the simpler collinear case, as can be seen from Eqs. (12) and (15), no analytical inversion is possible. One must instead compute the inversion numerically.\textsuperscript{35} Nonetheless, one can construct appropriate analytical approximations by inverting the dominant terms in the expressions. In the case of (12), for example, one has that for currents much larger than the critical current, \(\tau(m) \simeq \frac{I}{2\xi(I^2 - 1)} \log \frac{1 + m}{1 - m}\),

(25)

which can be inverted to give

\[
m(\tau) = \tanh \left[\frac{\tau I^2 - 1}{I}\right].
\]

(26)

Plugging into expression (24) for the \(\tau\) probability curve, one has

\[
P[\tau_s \leq \tau] = \exp[-\xi \frac{\tanh \left[\frac{\tau I^2 - 1}{I}\right]}{\cosh^2 \left[\frac{\tau I^2 - 1}{I}\right]}] \frac{F[\sqrt{\xi} \tanh \left[\frac{\tau I^2 - 1}{I}\right]]}{F[\sqrt{\xi}]},
\]

(27)

This expression can be truncated to a simpler form by noting that the leading exponential term dominates over the ratio of Dawson functions. Furthermore, if one considers the limit of large values for \(\tau\) (or, analogously, \(I \gg 1\)), the “cosh” can be also approximated by its leading exponential term. We are finally left with

\[
P[\tau_s \leq \tau] \simeq \exp[-\xi \frac{\tanh \left[\frac{\tau I^2 - 1}{I}\right]}{\cosh^2 \left[\frac{\tau I^2 - 1}{I}\right]}] \sim \exp[-4\xi \exp[-2\tau I^2 - 1]],
\]

(28)

which is very similar in form to what has already been reported and used for fitting in the literature.\textsuperscript{8,13}

In the low current regime, one constructs probability curves by considering the mean switching time and modeling a purely thermal reversal as a decay process with rate given by Eq. (21). The fraction of switched states then vary in time as
Upon introducing uniaxial tilt, precessional effects can be witnessed directly on the switching probability curves in the super-critical regime. One expects that in the initial phases of switching, the fraction of switched states is sensitive to the time at which the current is turned off. One may accidentally turn off the current during a moment of transient passage through the switched region along the precessional orbit. This was checked and verified from our numerical simulations (see Figure 7). More generally, effects similar to the “waviness” seen in the mean switching time curves can be seen in the probability curves as well as the angle of uniaxial tilt is allowed to vary (see Figure 8). Only a numerical solution of the LLG equation can bring such subtleties to light.

\[ P(m > 0) = 1 - \exp\left(-t/\langle \tau \rangle\right). \]  

(29)

\( P(m > 0) \) is the switching probability for magnitude greater than zero.

**FIG. 7.** Influence of precessional orbits on transient switching as seen from the switching time probability curve in the supercritical current regime. The case shown is that of an angular tilt of \( \pi/3 \) subject to a current intensity of 2.0 times the critical current. Data were gathered by numerically solving (7). The non-monotonicity in the probability curve shows the existence of transient switching. Times are shown in units of \( (T \cdot s) \) where \( T \) stands for Tesla: real time is obtained upon division by \( H_s \).

**FIG. 8.** Spin-torque induced switching time probability curves for various angular configurations of uniaxial tilt (a sample normalized current of 10 was used) obtained by numerically solving (7). A log-log y-axis is used following (28) to make the tails of the probability distributions visible.

**IX. CONCLUSION**

We have constructed the theory underlying the dynamics of a uniaxial macrospin subject to both thermal fluctuation and spin-torque effects. We then studied the subtle interplay between these two effects in aiding magnetization reversal between energy minima in a magnetically bistable system. Two regimes stand out in such a theory: a ballistic regime dominated by the deterministic flow and a thermally activated regime where reversal is dominated by noise. In the ballistic regime, we discussed how to approximate the mean switching time behavior and found that corrections due to the diffusion-gradient term, arising from the stochastic equations, allow one to model the dynamics more accurately.

In the thermally activated regime, we solved the relevant mean first passage time problem and obtained an expression for the dependence of mean switching time on applied current. In doing so, the correct scaling was shown to be \( \langle \tau \rangle \propto \exp(\xi(1-I)^2), \) in contrast to the prevailing view that \( \langle \tau \rangle \propto \exp(\xi(1-I)). \) Analytical results were then compared with detailed numerical simulations of the stochastic LLG equation. The massively parallel capabilities of our GPU devices have allowed us to explore the behavior of macrospin dynamics over six orders of temporal magnitude. Comparing to our analytical results, we suggest that the thermal asymptotic behavior is achieved fairly slowly in comparison to the switching timescales that have been probed experimentally.

Different geometrical configurations of the uniaxial anisotropy axis with respect to the spin-current axis were shown to influence the thermally activated regime very minimally inasmuch as the currents were rescaled by the proper critical current of the angular setup. Only in the super-critical regime were distinctions shown to exist due to complex precessional and transient switching behavior.

These results have important implications for the analysis of experimental data in which measurements of the switching time versus current pulse amplitude are used to determine the energy barrier to magnetization reversal. Clearly, use of the correct asymptotic scaling form is essential to properly determine the energy barrier to reversal. The energy barrier, in turn, is very important in assessing the thermal stability of magnetic states of thin film elements that are being developed for long term data storage in STT-MRAM. Further work should address how these results extend to systems with easy plane anisotropy and situations in which the nanomagnet has internal degrees of freedom, leading to a break down of the macrospin approximation.

We also note that current flow is a source of shot noise, which at low frequencies acts like a white-noise source in much the same way as thermal noise. It is therefore interesting to understand when this additional source of noise plays a role. For a magnetic layer coupled to unpolarized leads, the current induced noise on the magnetization dynamics was found to be \( \langle \Gamma_L/\Gamma_R \rangle V^2, \) where \( V \) is the voltage drop across the magnetic layer, while \( \Gamma_L/\Gamma_R \) is a dimensionless ratio characterizing the coupling strength of the magnetic layer to the left (L) and right (R) leads. Thus, the noise is maximal.
(V/4) for perfectly symmetrical couplings, and is smaller in the limit of highly asymmetric contacts. This basic behavior, and the order of magnitude of the effect, is not likely to be modified by polarized leads. We argue that the temperatures at which the experiments have been performed current noise effects are not important. The experiments are performed at room temperature where \( T = 300 \text{ K} \). For an all-metallic device, such as a spin-valve nanopillar, the couplings are nearly symmetrical; and at the critical current, a typical voltage drop across the magnetic layer is less than 10 \( \mu \text{V} \) or, equivalently, 1 K. For a magnetic tunnel junction device, \( V \) can be \( \sim 1 \text{ V} \). However, in this case, the coupling is asymmetric. One lead (L) forms a magnetic tunnel junction with the nanomagnet, while the other (R) a metallic contact. This gives \( \Gamma_R/\Gamma_L > 10^4 \) and a relevant energy \( \sim 1 \text{ K} \), again far lower than room temperature. It appears that current induced noise can only be important at room temperature for a nanomagnet coupled symmetrically between two tunnel barriers.

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33. In modeling thermal effects through stochastic contributions in a system like ours, we are interested in considering the white noise limit of a potentially colored noise process. Stratonovich calculus is then to be preferred over Ito calculus as ascertained by the Wong-Zakai theorem.
34. It is important to note that experiments at fixed temperature are by necessity performed over a limited range of \( I/I_c \), and thus cannot truly distinguish an exponent of 1 or 2.
35. Easily achieved thanks to the monotonicity of their form.