Finite and Infinite energy solutions of singular elliptic problems: Existence and Uniqueness

Francescantonio Oliva Francesco Petitta

Abstract

We establish existence and uniqueness of solution for the homogeneous Dirichlet problem associated to a fairly general class of elliptic equations modeled by

\[-\Delta u = h(u)f \text{ in } \Omega,\]

where \(f\) is an irregular datum, possibly a measure, and \(h\) is a continuous function that may blow up at zero. We also provide regularity results on both the solution and the lower order term depending on the regularity of the data, and we discuss their optimality.

Keywords. Nonlinear elliptic equations, Singular elliptic equations, Uniqueness, Measure data

Mathematics Subject Classification (2010). 35J60, 35J61, 35J75, 35A05, 35R06

1 Introduction

Let \(\Omega\) be a bounded and smooth open subset of \(\mathbb{R}^N\), and consider, as a model, the following singular elliptic boundary-value problem

\[
\begin{cases}
-\text{div}(A(x)\nabla u) = \frac{f}{u^\gamma} & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\]

(1.1)

where \(\gamma > 0\), \(A(x)\) is a bounded elliptic matrix, and \(f\) is nonnegative.

Physical motivations in the study of problems as (1.1) arise, for instance, in the study of thermoconductivity where \(u^\gamma\) represents the resistivity of the material (\cite{22}), in signal transmissions (\cite{38}), and in the theory non-Newtonian pseudoplastic fluids (\cite{37}). See also \cite{46} (and references therein) for a precise description of a model of boundary layers in which equations as in (1.1) also appear.

From the purely theoretical point of view, after the first pioneering existence and uniqueness result given in \cite{22}, a systematic treatment of problems as (1.1) was developed starting from \cite{43} \cite{15}.

Consider, for simplicity, \(A(x) = I\), i.e. the case of the laplacian as principal operator; if \(f\) is smooth enough (say Hölder continuous) and bounded away from zero on \(\Omega\) then the existence and uniqueness of a classical solution to (1.1) is proven by desingularizing the problem and then by applying a suitable sub- and super-solution method. Some remarkable refinements of the previous results were given in \cite{34};
here, the authors proved, in particular, that $u \notin C^1(\Omega)$ if $\gamma > 1$ and it has finite energy, i.e. $u \in H^1_0(\Omega)$, if and only if $\gamma < 3$ (see also [30] for further insights).

Classical theory for equations as in (1.1), also called singular Lane–Emden–Fowler equations, has been also extended to the case in which the term $s^{-\gamma}$ is replaced by a $C^1$ nonincreasing nonlinearity $h(s)$ that blows up at zero at a given rate (see [43] [15] [33] [48]).

More general situations can be considered. Let $f$ be a nonnegative function belonging to some $L^m(\Omega)$, $m \geq 1$, or even, possibly, a measure. If $f \in L^1(\Omega)$, in [6], the existence of a distributional solution $u$ to (1.1) is proved. In particular the authors prove that a locally strictly positive function $u$ exists such that equation in (1.1) is satisfied in the sense of distributions: moreover $u \in W_0^{1,1}(\Omega)$ if $\gamma < 1$, $u \in H^1_0(\Omega)$ if $\gamma = 1$, and $u \in H^1_{loc}(\Omega)$ if $\gamma > 1$, where, in the latter case, the boundary datum is only assumed in a weaker sense than the usual one of traces, i.e. $u_{\gamma+1}^{1/2} \in H^1_0(\Omega)$. Note that, if $\gamma > 1$, then solutions with infinite energy do exist, even for smooth data ([34]). Let us also mention [25] [26] where, in order to deal with homogenization issues, existence and uniqueness of finite energy solutions are considered for $f \in L^m(\Omega)$, $m > \frac{N}{2}$, also in the case of a continuous nonlinearity $h(s)$ that mimics $s^{-\gamma}$.

In the case of $f$ being a measure the situation becomes striking different. Nonexistence of solutions to problem (1.1) is proven (at least in the sense of approximating sequences) in [6] if the measure is too concentrated, while in [39] sharp existence results are obtained in the measure is diffuse enough; here concentration and diffusion is intended in the sense of capacity. For general, possibly singular, measures data existence of a distributional/renormalized solution is considered in [17] also in the case of a more general (not necessarily monotone) nonlinearity.

Without the aim to be exhaustive we also refer the reader to the following papers and to the references cited therein in which various relevant extensions and refinements (nonhomogeneous case, variational approach, nonlinear principal operators, natural growth terms, etc, ...) are considered: [45] [24] [11] [10] [2] [28] [27] [12] [14] [44] [18] [16] [13].

Consider now the boundary-value problem

\[
\begin{aligned}
Lu &= h(u)f \quad \text{in } \Omega, \\
0 &= \quad \text{on } \partial\Omega, 
\end{aligned}
\]

(1.2)

where $L$ is a linear elliptic operator in divergence form, $h : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function that may blow up at $s = 0$ and possesses a limit at infinity, and $f$ is a nonnegative function in $L^1(\Omega)$ (or, possibly, a bounded Radon measure on $\Omega$).

Besides the one arising from the presence of possibly a measure datum, new difficulties have to be taken into account in this general framework: even if $f$ is only a nonnegative function, then the solutions do not belong in general to $H^1_0(\Omega)$ even for small $\gamma$ (see Example 1 below) nor the lower order term in (1.2) need to belong to $L^1(\Omega)$ (Example 2).

In particular, as also observed in [44], the question of the summability properties of the lower order term in (1.2) plays a crucial role in order to deal with uniqueness of solutions (see [39] [40] for some partial results in this direction). In general, in fact, only finite energy solutions are known to be unique, at least in the model case (1.1) (see [4] [13]).

In the present paper, under fairly general assumptions we introduce a natural notion of distributional solution to problem (1.2) for which existence can be shown to hold. Moreover, uniqueness holds provided $h$ is nonincreasing. As we already mentioned, particular care will be addressed on how the homogeneous boundary datum for the solution $u$ is (weakly) attained.
If $f$ is a function in $L^m(\Omega)$, $m \geq 1$, we also investigate the question of whether the solution to problem (1.2) has finite energy. We provide several instances of this occurrence depending on the regularity of the datum and on the behavior of $h(s)$ both at zero and at infinity. Additionally, we obtain sharp thresholds for the lower order term to belong to $L^1(\Omega)$. The results and their optimality are discussed through appropriate examples.

Also considering the obstructions given by the above mentioned examples, we will finally be lead to establish a weighted summability estimate on the lower order term $h(u)f$; this will be a key tool in order to prove uniqueness, which will be obtained by mean of a suitable Kato type inequality.

The plan of the paper is as follows: in Section 2 we precise the structural assumptions we are going to work with, we give our definition of solution for problem (1.2), and we state our existence (Theorem 2.3) and uniqueness (Theorem 2.4) results; we then provide some useful preliminary tools we will need. In particular, Section 2.2 contains some outcomes concerning the linear case and a Kato type inequality with measures. Section 3 consists of an extensive account on the case of finite energy solutions; moreover, in Section 3.3 we also present a prototypical example and we address the question of the integrability of the lower order term $h(u)f$. Section 4 is then devoted to the proof of Theorems 2.3 and 2.4. Finally, in Section 5 we discuss some further examples and possible extensions.

1.1 Notations and auxiliary functions
We will use the following well known auxiliary functions defined for fixed $k > 0$

$$T_k(s) = \max(-k, \min(s, k)), \quad G_k(s) = (|s| - k)^+ \text{ sign}(s),$$

with $s \in \mathbb{R}$. Observe that $T_k(s) + G_k(s) = s$, for any $s \in \mathbb{R}$ and $k > 0$.

We denote the distance of a point $x \in \Omega$ from the boundary of the set in the following way

$$\delta(x) := \text{dist}(x, \partial\Omega).$$

As $\partial\Omega$ is smooth enough we will use systematically the fact that

$$\int_{\Omega} \delta(x)^r \, dx < \infty,$$

if and only if $r > -1$. Moreover, with a little abuse of notation, in order to avoid technicalities and without loosing generality, we can refer to $\delta(x)$ as a suitable positive smooth (say $C^1$) modification of the distance function which agrees with $\delta(x)$ in a neighbourhood of $\partial\Omega$.

We also need to define the following $\varepsilon$-neighborhood of $\partial\Omega$:

$$\Omega_\varepsilon = \{x \in \Omega : \delta(x) < \varepsilon\},$$

that we will always assume to be smooth (up to the choice of a suitable small $\varepsilon$).

The space of bounded Radon measures will be denoted by $\mathcal{M}(\Omega)$, while we will also made use of the following weighted spaces

$$L^1(\Omega, \delta) = \left\{ f \in L^1_{\text{loc}}(\Omega) : \int_{\Omega} |f| \delta \, dx < \infty \right\} \quad \text{and} \quad \mathcal{M}(\Omega, \delta) = \left\{ \mu \in \mathcal{M}_{\text{loc}}(\Omega) : \int_{\Omega} \delta \, d|\mu| < \infty \right\}.$$

For an integer $j$,

$$C^j_0(\overline{\Omega}) := \{ \phi \in C^j(\overline{\Omega}) : \phi = 0 \text{ on } \partial\Omega \}.$$
will denote the space of $C^j$ functions that vanishes at $\partial \Omega$. As usual, subscript $c$ will indicate a space of function with compact support in $\Omega$, e.g. $C_c(\Omega)$.

If no otherwise specified, we will denote by $C$ several constants whose value may change from line to line and, sometimes, on the same line. These values will only depend on the data but they will never depend on the indexes of the sequences we will often introduce. Moreover for the sake of simplicity we use the simplified notation

$$\int f := \int f(x) \, dx,$$

when referring to integrals when no ambiguity is possible.

## 2 Setting of the problem and some preliminary tools

### 2.1 Main assumptions and setting of the problem

Let $\Omega$ be an open and bounded domain of $\mathbb{R}^N$, $N \geq 2$, with smooth boundary, and consider the following homogeneous boundary-value problem

$$\begin{cases}
Lu = h(u)\mu & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
$$

where $L$ is a strictly elliptic linear operator in divergence form, that is

$$Lu := -\text{div}(A(x)\nabla u),$$

where $A$ is a matrix with Lipschitz continuous coefficients such that

$$A(x)\xi \cdot \xi \geq \alpha |\xi|^2 \text{ and } |A(x)| \leq \beta,$$

for every $\xi$ in $\mathbb{R}^N$, for almost every $x$ in $\Omega$ and for $\alpha, \beta > 0$. As we will see, the smoothness we assume on both $\partial \Omega$ and the coefficients of the matrix $A(x)$ are necessary in order to use standard $L^p$ elliptic regularity theory, Hopf’s boundary point lemma, and some further technical devices; we shall stress as some of the results we present require less regularity assumptions.

On the nonlinearity $h : \mathbb{R}^+ \to \mathbb{R}^+$ we assume that it is continuous, such that

$$\lim_{s \to 0^+} h(s) \in (0, \infty], \quad \text{and} \quad \lim_{s \to \infty} h(s) := h(\infty) < \infty.\quad (2.3)$$

We also assume the following growth condition near zero

$$\exists \ K_1, \omega > 0 \text{ such that } h(s) \leq \frac{K_1}{s^\gamma} \text{ if } s < \omega,\quad (2.4)$$

with $\gamma > 0$. We explicitly remark that we do not assume any control from below on the function $h$ so that the case of a bounded and continuous function satisfying (2.3) is allowed.

Finally, $\mu$ will be a nonnegative measure in $\mathcal{M}(\Omega)$. In particular, recall that $\mu$ can be (uniquely) decomposed as

$$\mu = \mu_d + \mu_c,$$

where, resp. $\mu_d \geq 0$ is absolutely continuous with respect to the harmonic capacity (also called 2-capacity) and $\mu_c \geq 0$ is concentrated on a set of zero 2-capacity (see for instance [23]). We refer the
reader to [31] for an exhaustive introduction to the theory of capacity (see also [19] which contains a nice introduction to the topic which is sufficient to our purposes). Also, we will always assume, without loss of generality, that \( \mu_d \neq 0 \). Otherwise, in fact, the singularity of the problem virtually disappears and one is brought to nonsingular problem (see Remark 2.2 below, and the discussion in [17, Section 5]).

If \( \mu_c = 0 \) we will call \( \mu \) a diffuse measure in \( M(\Omega) \). Recall (see [5]) that, if \( \mu \) is diffuse, then the following decomposition holds in the sense of distributions

\[
\mu = g - \text{div } G,
\]

where \( g \in L^1(\Omega) \) and \( G \in L^2(\Omega)^N \). It is proven in [19] that such a \( \mu \) can be approximated in the narrow topology of measures by smooth functions \( \nu_n \), that is

\[
\lim_{n \to \infty} \int_{\Omega} \nu_n \phi = \int_{\Omega} \phi \, d\mu, \quad \forall \phi \in C(\bar{\Omega}),
\]

such that

\[
\nu_n = g_n - \text{div } G_n,
\]

where \( g_n \) weakly converges in \( L^1(\Omega) \) to \( g \) and \( G_n \) strongly converges in \( L^2(\Omega)^N \) to \( G \).

Concerning the boundary datum, our solutions are not expected in general to belong to \( W^{1,1}_0(\Omega) \) (see for instance [6, 13, 39]). Due to this fact, we shall need to infer that our (positive) solution \( u \in L^1(\Omega) \) assumes the value zero at \( \partial \Omega \) in the following sense

\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\Omega_{\varepsilon}} u = 0,
\]

which is known to be weaker than the classical sense of traces for functions in \( W^{1,1}_0(\Omega) \) (see for instance [41, 1]).

To deal with both existence and uniqueness for solutions to our problem we give the following variant of the definition of distributional solution to (2.1) given in [17] (and inspired by [36]).

**Definition 2.1.** A positive function \( u \in L^1(\Omega) \cap W^{1,1}_{\text{loc}}(\Omega) \) is a *distributional solution* to problem (2.1) if \( h(u) \in L^1_{\text{loc}}(\Omega, \mu_d) \), (2.7) holds, and following is satisfied

\[
\int_{\Omega} A(x) \nabla u \cdot \nabla \varphi = \int_{\Omega} h(u) \varphi \mu_d + h(\infty) \int_{\Omega} \varphi \mu_c \quad \forall \varphi \in C^1_c(\Omega).
\]

**Remark 2.2.** Asking for \( h(u) \in L^1_{\text{loc}}(\Omega, \mu_d) \) gives sense to the right hand side of (2.8), while, as \( u \) is in \( W^{1,1}_{\text{loc}}(\Omega) \), the left hand side makes sense as well.

Recall we are assuming \( \mu_d \neq 0 \); if this is not the case, in fact, (2.8) becomes linear and it can be treated with classical tools (see Section 2.2).

If \( h(\infty) = 0 \) then the concentrated part of the measure disappears in (2.8), so we are just solving problem (2.1) with datum \( \mu_d \); this reflects a well known concentration type phenomenon of the approximating sequences of solutions when the datum is too concentrated. With this sort of nonexistence result in mind (see [6, 39, 17] for further details on these phenomena), we will always understand that \( h(\infty) \neq 0 \) if \( \mu_c \neq 0 \) in order to be consistent with the fact that we are solving (2.1).

Our main existence and uniqueness results are the content of the following theorems whose proofs will be given in Section 4.
Theorem 2.3. Let \( h \) be a continuous function satisfying (2.3) and (2.4), and let \( \mu \in \mathcal{M}(\Omega) \) be a nonnegative measure. Then a solution to problem (2.1) in the sense of Definition 2.1 does exist.

Theorem 2.4. Under the same assumptions of Theorem 2.3, if \( h \) is nonincreasing, then the distributional solution to problem (2.1) is unique.

2.2 The linear case and a Kato type lemma

Here we briefly discuss the linear case (say \( h \equiv 1 \)) of problem (2.1) with data in a suitable weighted class. We first recall the existence and uniqueness of a solution to this problem obtained by a transposition argument (see [47], see also [35, 21]). Let \( A \) be a matrix with Lipschitz continuous coefficients satisfying assumption (2.2) and consider the problem

\[
\begin{aligned}
- \text{div}(A(x)\nabla u) &= \mu \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

(2.9)

where \( \mu \) belongs to \( \mathcal{M}(\Omega, \delta) \). With the symbol \( L^* \) we will indicate the transposed operator defined by

\[
L^*\varphi := -\text{div}(A^*(x)\nabla \varphi) \quad \text{in } \mathcal{D}'(\Omega).
\]

Definition 2.5. A very weak solution of problem (2.9) is a function \( u \in L^1(\Omega) \) such that the following holds

\[
\int_{\Omega} uL^*\varphi = \int_{\Omega} \varphi d\mu,
\]

for every \( \varphi \in C^1_0(\Omega) \) such that \( L^*\varphi \in L^\infty(\Omega) \).

We have the following two results whose proofs can be found respectively in [47, Corollary 2.8] and [47, Theorem 2.9].

Lemma 2.6. Let \( f \in L^1_{\text{loc}}(\Omega) \) and let \( u \in L^1_{\text{loc}}(\Omega) \) such that

\[
\int_{\Omega} uL^*\varphi = \int_{\Omega} f \varphi,
\]

for every \( \varphi \in C^1_0(\Omega) \) such that \( L^*\varphi \in L^\infty(\Omega) \). Then for every open subsets \( G \subset \subset G' \subset \subset \Omega \), and for every \( 1 \leq q < \frac{N}{N-1} \), we have

\[
||u||_{W^{1,q}(G)} \leq C \left(||f||_{L^1(G')} + ||u||_{L^1(G')}\right).
\]

Theorem 2.7. Let \( \mu \in \mathcal{M}(\Omega, \delta) \) then there exists a unique very weak solution to problem (2.9).

In order to prove uniqueness for the singular problem (2.1) one also needs a Kato type inequality for variable coefficients operators and diffuse measures as data, which is inspired by [8] (see also [32] for a related result in the context of Dirichlet forms). First we need the following:
Lemma 2.8. Let $\mu \in \mathcal{M}(\Omega, \delta)$ and let $u$ be the very weak solution to problem (2.9), then $T_k(u) \in H^1_{loc}(\Omega)$. Moreover, for any $E \subset \subset \Omega$, one has
\[
\|T_k(u)\|_{H^1(E)} \leq C,
\]
where $C$ is a constant that only depends on $\alpha, E, \Omega$ and $k$.

Proof. We consider the scheme of approximation used in [47], that is
\[
\begin{aligned}
L_{u_n} = f_n & \quad \text{in } \Omega, \\
\phi_n = 0 & \quad \text{on } \partial \Omega,
\end{aligned}
\tag{2.10}
\]
where $f_n$ is a sequence of smooth functions bounded in $L^1(\Omega, \delta)$ converging to $\mu$ in the following sense
\[
\lim_{n \to \infty} \int_{\Omega} f_n \phi = \int_{\Omega} \phi d\mu, \quad \forall \phi : \phi \delta \in C(\overline{\Omega}).
\tag{2.11}
\]
In [47, Theorem 2.9] the author shows that the classical solutions $u_n$ satisfy
\[
\|u_n\|_{L^1(\Omega)} \leq C,
\tag{2.12}
\]
and that $u_n$ converges a.e. towards the solution to problem (2.9). Let $E \subset \subset \Omega$ and consider a function $\varphi \in C^\infty_c(\Omega)$ such that $0 \leq \varphi \leq 1$ and $\varphi = 1$ on $E$. Testing (2.10) with $T_k(u_n)\varphi$ we have
\[
\begin{aligned}
\alpha \int_{E} |\nabla T_k(u_n)|^2 & \leq \alpha \int_{\Omega} |\nabla T_k(u_n)|^2 \varphi \leq \int_{\Omega} A(x) \nabla u_n \cdot \nabla T_k(u_n)\varphi \\
& = \int_{\Omega} T_k(u_n) f_n \varphi - \int_{\Omega} T_k(u_n) A(x) \nabla u_n \cdot \nabla \varphi.
\end{aligned}
\tag{2.13}
\]
On one hand, the first term on the right hand side satisfies
\[
\left| \int_{\Omega} T_k(u_n) f_n \varphi \right| \leq k \int_{\text{supp } \varphi} f_n \varphi \leq C k.
\]
On the other hand, the second term on the right hand side of (2.13) gives
\[
\begin{aligned}
\int_{\Omega} T_k(u_n) A(x) \nabla u_n \cdot \nabla \varphi &= \int_{\Omega} T_k(u_n) u_n L^* \varphi - \int_{\Omega} T_k(u_n) \nabla T_k(u_n) \cdot A^*(x) \nabla \varphi \\
& = \int_{\Omega} T_k(u_n) u_n L^* \varphi - \frac{1}{2} \int_{\Omega} |\nabla T_k(u_n)|^2 \cdot A^*(x) \nabla \varphi \\
& = \int_{\Omega} T_k(u_n) u_n L^* \varphi - \frac{1}{2} \int_{\Omega} |T_k(u_n)|^2 L^* \varphi \\
& = \int_{\Omega} T_k(u_n) (u_n - \frac{1}{2} T_k(u_n)) L^* \varphi \leq k \int_{\Omega} |u_n| |L^* \varphi| \leq C k \int_{\Omega} |u_n|.
\end{aligned}
\]
Gathering the previous estimate with (2.13), recalling (2.12) and also using the weak lower semicontinuity, we finally obtain
\[
\int_{E} |\nabla T_k(u)|^2 \leq \liminf_{n \to \infty} \int_{E} |\nabla T_k(u_n)|^2 \leq C k.
\]
\[\square\]
Note, in particular, that, if $\mu$ is diffuse, then, by standard capacity properties, $T_k(u)$ (and so $u$) is defined $|\mu|$-a.e. The following is a version of Kato inequality in the case of a diffuse measure as datum.

**Lemma 2.9.** Let $u$ be the very weak solution to problem (2.9) where $\mu$ is a diffuse measure in $\mathcal{M}(\Omega, \delta)$. Then

$$\int_\Omega u^+ L^* \varphi \leq \int_{\{ u \geq 0 \}} \varphi d\mu,$$

for any nonnegative $\varphi \in C^1_c(\Omega)$ such that $L^* \varphi \in L^\infty(\Omega)$.

**Proof.** We consider again the approximating scheme defined by (2.10) but, to our purposes, we need to specify the structure of the approximating sequence $f_n$. As $\delta \mu$ is a diffuse measure in $\mathcal{M}(\Omega)$, by (2.5), we have that $\delta \mu = g - \text{div} G$ where $g \in L^1(\Omega)$ and $G \in L^2(\Omega)^N$, and it can be approximated in the narrow topology of measures by $g_n - \text{div} G_n$, where $g_n$ weakly converges in $L^1(\Omega)$ to $g$ and $G_n$ strongly converges in $L^2(\Omega)^N$ to $G$. Recalling (2.6), it is not difficult to establish that, if we define $f_n$ through

$$\delta f_n = g_n - \text{div} G_n$$

then $f_n$ is an approximation of $\mu$ in the sense of (2.11).

Now, let $\Phi : \mathbb{R} \to \mathbb{R}$ be a $C^2$-convex function with $0 \leq \Phi' \leq 1$ and $\Phi''$ with compact support (also $\Phi(0) = 0$); letting $\varphi \in C^1_c(\Omega)$ be a nonnegative function such that $L^* \varphi \in L^\infty(\Omega)$, we have

$$\int_\Omega \Phi(u_n) L^* \varphi = \int_\Omega \nabla u_n \cdot A^*(x) \nabla \varphi \Phi'(u_n) \leq \int_\Omega A(x) \nabla u_n \cdot \nabla (\varphi \Phi'(u_n))$$

$$- \alpha \int_\Omega \Phi''(u_n) |\nabla u_n|^2 \varphi \leq \int_\Omega \varphi \Phi'(u_n) f_n,$$

where we used both (2.2) and the convexity of $\Phi$.

Now we pass to the limit with respect to $n$ in

$$\int_\Omega \Phi(u_n) L^* \varphi \leq \int_\Omega \varphi \Phi'(u_n) f_n. \tag{2.14}$$

Thanks to Lemma 2.6 we have local strong convergence in $L^1(\Omega)$ (at least) and then, using also that $\Phi''$ has compact support, we pass to the limit by dominated convergence theorem on the left hand side. Now observe that $\Phi'(u_n)$ converges to $\Phi'(u)$ in $L^\infty(\Omega)$ -weak and a.e. so that, due to the structure of $\delta f_n$, in order to pass to the limit in the right hand side of (2.14) it suffices to check that $\Phi'(u_n)$ is locally bounded in $H^1_{\text{loc}}(\Omega)$ (and then using weak compactness). To do that only observe that, as $\Phi''$ has compact support, one has

$$\nabla \Phi'(u_n) = \Phi''(u_n) \nabla u_n = \Phi''(u_n) \nabla T_k(u_n),$$

for some level $k$, then we use Lemma 2.8 to conclude.

We can then deduce that

$$\int_\Omega \Phi(u) L^* \varphi \leq \int_\Omega \varphi \Phi'(u) d\mu \tag{2.15}.$$

Hence, we conclude by taking a sequence of regular convex functions $\Phi_\varepsilon(t)$ such that $\Phi_\varepsilon(t) = t$ if $t \geq 0$ and $|\Phi_\varepsilon(t)| \leq \varepsilon$ if $t < 0$. One can pass to the limit in (2.15) (with $\Phi_\varepsilon$ instead of $\Phi$) finally getting

$$\int_\Omega u^+ L^* \varphi \leq \int_{\{ u \geq 0 \}} \varphi d\mu.$$
2.3 Further useful tools

In Section 3, we will deal with self-adjoint operators $L$ and we shall use the properties of the first eigenfunction related to $L$ defined as the function $\varphi_1 \in H_0^1(\Omega) \cap W^{2,p}(\Omega)$ ($1 \leq p < \infty$) such that

\[
\begin{aligned}
L\varphi_1 &= \lambda_1 \varphi_1 \quad \text{in } \Omega, \\
\varphi_1 &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

Moreover, it is possible to prove the following consequence of Hopf’s boundary point lemma (see for instance [20, Lemma 2]); there exist two positive constants $c_1$ and $c_2$ such that

\begin{equation}
\frac{c_1 \delta(x)}{\varphi_1(x)} \leq \varphi_1(x) \leq \frac{c_2 \delta(x)}{\varphi_1(x)}, \quad \text{for } x \in \Omega.
\end{equation}

Given a continuous function $h$ satisfying assumptions (2.3) and (2.4), in order to use some comparison arguments, we will need to construct two nonincreasing auxiliary continuous functions $h^-, h^+: \mathbb{R}^+ \to \mathbb{R}^+$ such that

\begin{equation}
h^-(s) \leq h(s) \leq h^+(s) \quad \text{for any } s > 0.
\end{equation}

The construction of $h$ is given in [17] in such a way that it also satisfies $h^-(s) \leq T_n(h(s))$, for any positive $s$ and any $n \in \mathbb{N}$. The construction of $h^+$ is also easy; for instance one can pick $\rho \leq \omega$ such that

\begin{equation}
\frac{K_1}{\rho^\gamma} \geq \sup_{s \in [\omega, \infty)} h(s),
\end{equation}

one can let

\begin{equation}
i_0 = \frac{K_1}{\rho^\gamma}, \quad i_m = \sup_{s \in [\rho+m-1, \infty)} h(s), \quad m \geq 1,
\end{equation}

and define

\begin{equation}
\overline{h}(s) := \frac{K_1}{s^\gamma} \chi_{\{0, \rho\}}(s) + \sum_{m \geq 1} \left(2(i_m - i_{m-1}) (s - \rho - m + 1) + i_{m-1}\right) \chi_{\{\rho+m-1, \rho+m-\frac{1}{2}\}}(s) + i_m \chi_{\{\rho+m-\frac{1}{2}, \rho+m\}}(s).
\end{equation}

Observe that, by construction, also $\overline{h}$ satisfies (2.4) with constants $\gamma$, $K_1$, and $\rho$ instead of $\omega$.

3 Finite energy solutions

Here we analyze the issue of whether (2.1) admits a solution in $H_0^1(\Omega)$. The results will depend on both the regularity of the data and the behavior of the nonlinearity $h$. Although, as we will stress below, some instances of finite energy solutions can also be considered in the case of measure data and more general operators, in this section, for simplicity, we restrict our attention to the case of a nonnegative datum in some $L^m(\Omega)$, with $m \geq 1$, and a self-adjoint operator $L$ (i.e. we assume $A(x)$ is symmetric). Moreover, if $h(\infty) \neq 0$ only truncations belonging in the energy space are expected ([17]), so that we will also assume throughout this section that (2.3) is satisfied with $h(\infty) = 0$.

Therefore, we look for solutions in $H_0^1(\Omega)$ for the following problem

\[
\begin{aligned}
-\text{div}(A(x)\nabla u) &= h(u)f \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

(3.1)
where $A(x)$ is a symmetric function with Lipschitz continuous coefficients satisfying (2.2).

We start recalling some known instances of solutions in $H^1_0(\Omega)$ to problem (3.1) that are already present in the literature. Consider, for simplicity, the model case $h(s) = s^{-\gamma}$; as we mentioned, if $A(x) = I$ and $f$ is an H"older continuous function on $\overline{\Omega}$ which is bounded away from zero on $\Omega$, then a classical solution to problem (3.1) is in $H^1_0(\Omega)$ if and only if $\gamma < 3$ ([34]). Switching to weak solutions with nonnegative data in $L^m(\Omega)$, in [6] the authors proved the existence of a solution $u$ to (3.1) in $H^1_0(\Omega)$ if either $\gamma = 1$ and $f \in L^1(\Omega)$ or $\gamma < 1$ and $f \in L^{(2\gamma')/(\gamma'-1)}(\Omega)$. In the case $\gamma > 1$ solutions are always in $H^{1}_{loc}(\Omega)$; in [4] the authors prove the existence of a solution in $H^1_0(\Omega)$ if $f \geq C > 0$ is a function in $L^m(\Omega)$ ($m > 1$) and $\gamma < \frac{3m-1}{m+1}$. See also [33, 48, 14, 18, 7, 40], and references therein, for further refinements and extensions.

Following [6, 4], if $f \in L^m(\Omega)$, a function $u \in H^1_0(\Omega)$ is a distributional solution to problem (3.1) if it satisfies $u \geq c_\omega > 0$, for any $\omega \subset \subset \Omega$, and

$$\int_{\Omega} A(x) \nabla u \cdot \nabla \varphi = \int_{\Omega} h(u) f \varphi, \quad \forall \varphi \in C^1_c(\Omega). \tag{3.2}$$

Observe that in this case one has $h(u)f \in L^1_{loc}(\Omega)$ and this notion is equivalent to the one given in Definition 2.1 for general data provided $u \in H^1_0(\Omega)$. A first important remark is that finite energy distributional solutions to problem (3.1) are in fact solutions in the usual weak sense, that is they satisfy

$$\int_{\Omega} A(x) \nabla u \cdot \nabla \phi = \int_{\Omega} h(u) f \phi, \quad \forall \phi \in H^1_0(\Omega). \tag{3.3}$$

Moreover, $H^1_0$-distributional solutions to our singular problem are unique provided $h$ is nonincreasing; this was already observed in [4, 13] in the model case $h(s) = s^{-\gamma}$ (see also [33] for some related preliminary remarks). Indeed, we have the following result whose proof strictly follows the lines of the one of Theorems 2.2 and 2.4 in [4] with minor modifications. For completeness we shall sketch it in the Appendix.

**Theorem 3.1.** Let $f$ in $L^1(\Omega)$ be a nonnegative function and let $u \in H^1_0(\Omega)$ be a distributional solution to (3.1), then $u$ satisfies (3.3). Moreover, if $h$ is nonincreasing then problem (3.1) admits a unique distributional solution in $H^1_0(\Omega)$.

In the rest of this section we then answer the question of whether problem (3.1) admits a (unique, provided $h$ is not increasing) finite energy solution. As we already said the existence results for problem (3.1) are essentially based on an approximation scheme. Thus, we consider solutions $u_n \in H^1_0(\Omega) \cap L^\infty(\Omega)$ of

$$\begin{aligned}
-\text{div}(A(x) \nabla u_n) &= h_n(u_n)f_n \quad \text{in } \Omega,

u_n &= 0 \quad \text{on } \partial\Omega,
\end{aligned} \tag{3.4}$$

where $h_n := T_n(h)$, $f_n := T_n(f)$. One has (17) that a positive constant $c_\omega$ exists such that

$$u_n \geq c_\omega > 0, \quad \forall \omega \subset \subset \Omega. \tag{3.5}$$

Our aim will then consist in look for estimates on the sequence $u_n$ in $H^1_0(\Omega)$. In order to simplify the exposition observe that it is not restrictive to assume that

$$n > \max(h(\omega), \max_{[\omega, \infty]} h(s))$$
so that we are only possibly truncating $h$ near $s = 0$.

As we will see, a major role in the regularizing effect for the solutions is played by the behavior of $h$ at infinity. Therefore, in order to present the results, we will also need to assume the following:

$$\exists \ K_2, \omega > 0 \text{ such that } h(s) \leq \frac{K_2}{s^\theta} \text{ if } s > \omega,$$

(3.6)

for some $\theta > 0$.

We distinguish between the cases $\gamma \leq 1$ and $\gamma > 1$ as, in the former case the presence of a possibly singular $h$ is essentially negligible, while in the latter case a control near zero will be needed.

### 3.1 The case $\gamma \leq 1$

Assuming a strong control on $h$ at infinity then solutions to (3.1) have always finite energy for any integrable data.

**Theorem 3.2.** Let $h$ satisfy (2.4) and (3.6), with $\gamma \leq 1$ and $\theta \geq 1$. Then for any nonnegative $f \in L^1(\Omega)$ there exists a solution $u \in H^{1,0}(\Omega)$ to problem (3.1).

**Proof.** As we said we only need some a priori estimates on the sequence of approximating solutions $u_n$ to (3.4) in $H^{1,0}(\Omega)$. To this aim, we take $u_n$ as a test function in (3.4) and we use (2.4), and (3.6) obtaining

$$\alpha \int \Omega |\nabla u_n|^2 \leq \int \Omega h_n(u_n)f_n u_n \leq K_1 \int \{u_n < \omega\} f_n u_n^{1-\gamma} + \max_{\{\omega \leq u_n \leq \omega\}} h(s) \int \{\omega \leq u_n \leq \omega\} f_n u_n$$

$$+ K_2 \int \{u_n > \omega\} f_n u_n^{1-\theta} \leq K_1 \omega^{1-\gamma} \int \{u_n < \omega\} f + \omega \max_{\{\omega \leq u_n \leq \omega\}} h(s) \int \{\omega \leq u_n \leq \omega\} f + K_2 \omega^{(1-\theta)} \int \{u_n > \omega\} f \leq C,$$

and the proof is complete. \qed

Observe that the previous proof only made use of the ellipticity condition (2.2), on (2.4), and on (3.6); therefore, it easily extends to more general, possibly nonlinear, operators in not necessarily smooth domains, and to the case of measure data as considered for instance in [16, 17, 40, 13].

A milder control on $h$ at infinity (namely $\theta < 1$) is not enough to ensure, in general, finite energy solutions as the following example shows.

**Example 1.** For $N > 2$ we fix $\gamma < 1$ and we let $q = \frac{N(\gamma+1)}{N+2\gamma}$. As $q < (2^*)' = \frac{2N}{N+2}$, we can consider the positive solution $u$ to

$$\begin{cases}
-\Delta u = f & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}$$

where $0 \leq f \in L^q(\Omega)$ is such that $u$ is $W^{1,q}_0(\Omega)$ but $u \notin H^1_0(\Omega)$. We have that $u$ is a distributional solution to

$$\begin{cases}
-\Delta u = \frac{g}{u^\gamma} & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}$$

where $g = f u^\gamma$. We claim that $g$ in $L^1(\Omega)$; indeed, by Hölder’s inequality one has

$$\int \Omega f u^\gamma \leq \left( \int \Omega f^q \right)^{\frac{1}{q}} \left( \int \Omega u^{q'} \right)^{\frac{1}{q'}}$$

and, by the choice of $q$, the last integral is finite since $\gamma q' = q^{**}$. 


The previous example shows, at least in a model case, that if \( f \) is asked to merely belong to \( L^1(\Omega) \) and \( \theta < 1 \) (here \( \gamma = \theta < 1 \)) then it is possible to find a solution \( u \) not belonging to \( H^1_0(\Omega) \).

Anyway, also to recover the standard model in which \( \gamma = \theta < 1 \), the general case \( \theta > 0 \) can be treat by assuming some further requests on \( f \), namely more regularity inside \( \Omega \) and a control near the boundary of the type

\[
f(x) \leq \frac{c}{\delta(x)} \quad \text{a.e. in } \Omega_\varepsilon,
\]

where \( \varepsilon \) is small enough in order to guarantee that \( \Omega \setminus \overline{\Omega_\varepsilon} \) is a smooth subset compactly contained in \( \Omega \).

**Theorem 3.3.** Let \( 0 \leq f \in L^1(\Omega) \cap L^p(\Omega \setminus \overline{\Omega_\varepsilon}) \) with \( p > \frac{N}{\gamma} \) satisfying (3.7), and let \( h \) satisfying (2.4) and (3.6) with \( \gamma \leq 1 \) and \( \theta > 0 \). Then there exists a solution \( u \) to (3.1) belonging to \( H^1_0(\Omega) \).

**Proof.** Without loss of generality we assume \( \theta < 1 \) otherwise we can apply Theorem 3.2 to conclude.

First of all, in view of (3.5) and on the local regularity of \( f \), we can apply classical De Giorgi-Stampacchia regularity theory to get that \( u_n \in C(\Omega \setminus \Omega_\varepsilon) \) and

\[
||u_n||_{L^\infty(\Omega \setminus \Omega_\varepsilon)} \leq C,
\]

where \( C \) depends on \( \varepsilon \) but not on \( n \). In particular,

\[
||h_n(u_n)\omega||_{L^\infty(\Omega \setminus \Omega_\varepsilon)} \leq C||h(s)||_{L^\infty([c_{\Omega \setminus \Omega_\varepsilon}^{\infty})}.
\]

Therefore, testing (3.4) with \( u_n \) and using both (2.4) and (3.6), we have

\[
\alpha \int_{\Omega} |\nabla u_n|^2 \leq \int_{\Omega} h_n(u_n)f_n u_n \leq K_1 \int_{\Omega_\varepsilon \cap \{u_n < \omega\}} f_n u_n^{1-\gamma} + \int_{\Omega_\varepsilon \cap \{\omega \leq u_n \leq \overline{\omega}\}} h_n(u_n) f_n u_n
+ K_2 \int_{\Omega_\varepsilon \cap \{u_n > \overline{\omega}\}} f_n u_n^{1-\gamma} + \int_{\Omega \setminus \overline{\Omega}} h_n(u_n) f_n u_n \leq K_1 \omega^{1-\gamma} \int_{\Omega_\varepsilon \cap \{u_n < \omega\}} f
+ K_2 \int_{\Omega_\varepsilon \cap \{u_n > \overline{\omega}\}} f_n u_n^{1-\theta} + \omega \max_{[\omega, \overline{\omega}]} h(s) \int_{\Omega_\varepsilon \cap \{\omega \leq u_n \leq \overline{\omega}\}} f + K_2 \int_{\Omega_\varepsilon \cap \{u_n > \overline{\omega}\}} f_n u_n^{1-\theta} + C||h(s)||_{L^\infty([c_{\Omega \setminus \Omega_\varepsilon}^{\infty})} \int_{\Omega \setminus \overline{\Omega}} f.
\]

What is needed to conclude is then the control of the term

\[
\int_{\Omega_\varepsilon \cap \{u_n > \overline{\omega}\}} f_n u_n^{1-\theta}.
\]

Consider the smooth solutions \( w_n \) to the auxiliary problem

\[
\begin{cases}
- \text{div}(A(x) \nabla w_n) = \overline{h}_n(w_n) f_n & \text{in } \Omega, \\
w_n = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( \overline{h}_n(s) \) is the truncation at level \( n \) of the function \( \overline{h} \) defined in (2.18). By comparison \( u_n \) is a subsolution to problem (3.10) and so \( u_n \leq w_n \) for any \( n \). We now look for a super-solution to problem
\((3.10)\) in \(\Omega_\varepsilon\) of the form \(M\varphi_1^t\), for some \(M, t > 0\), in order to get, by comparison (see for instance [29, Theorem 10.7] and the discussion at the end of its proof), that
\[M\varphi_1^t \geq w_n \geq u_n, \quad \text{in } \Omega_\varepsilon. \tag{3.11}\]

We fix \(t = \frac{1}{\gamma+1}\). We need to check that the first inequality in
\[-\text{div}(A(x)\nabla M\varphi_1^t) = \overline{h}(M\varphi_1^t) \left(Mt(1-t)\frac{\varphi_1^{t-2}}{h(M\varphi_1^t)}A(x)\nabla \varphi_1 \cdot \nabla \varphi_1 + Mt\lambda_1\frac{\varphi_1^t}{h(M\varphi_1^t)}\right) \]
\[\geq f\overline{h}(M\varphi_1^t) \geq f_n\overline{h}_n(M\varphi_1^t)\]
holds in \(\Omega_\varepsilon\). Dropping a positive term and recalling both \((2.16)\) and \((3.7)\) the previous is implied by the first inequality in
\[\alpha Mt(1-t)\frac{\varphi_1^{t-2}}{h(M\varphi_1^t)}|\nabla \varphi_1|^2 \geq \frac{\varepsilon_2 c}{\varphi_1} \geq \frac{c}{\delta} \geq f \quad \text{in } \Omega_\varepsilon,\]
that, in view of the Hopf’s lemma, essentially reduces (up to normalization of not relevant constants) in proving that there exists a positive constant \(M\) such that
\[\overline{h}(M\varphi_1^t)(M\varphi_1^t)^\gamma \leq M^{1+\gamma} \quad \text{a.e. in } \Omega_\varepsilon. \tag{3.12}\]
We have
\[\overline{h}(M\varphi_1^t)(M\varphi_1^t)^\gamma \leq \max(K_1', \max_{[\rho, \infty)} \overline{h}(s)(M\varphi_1^t)^\gamma),\]
so that \((3.12)\) is satisfied up the following choice
\[M \geq \max(K_1', \max_{[\rho, \infty)} \overline{h}(s)||\varphi_1^t||_{L^\infty(\Omega_\varepsilon)})\, , \]
Observe that, by standard regularity theory, \(\varphi_1^t\) is continuous up to the boundary and smooth inside \(\Omega\). By possibly increase the value of \(M\) we also assume
\[M\varphi_1^t \geq w_n \geq u_n \quad \text{in } \partial(\Omega \setminus \partial \Omega_\varepsilon),\]
and we can apply the comparison principle in \(\Omega_\varepsilon\) obtaining \((3.11)\).

Therefore, coming back to \((3.9)\), using also \((3.7)\) and \((2.16)\), we have
\[\int_{\Omega_\varepsilon \cap \{u_n > \overline{w}\}} f_n u_n^{1-\theta} \leq M^{1-\theta} \int_{\Omega_\varepsilon} \frac{c}{\delta} \overline{h}_n \varphi_1^t \leq C \int_{\Omega_\varepsilon} \delta \frac{\theta+\gamma}{\theta+\varepsilon+\gamma} < \infty, \]
since \(\theta < 1\), and the proof is complete recalling \((3.8)\).

\[\square\]

**Remark 3.4.** Although assumption \((3.7)\) is not too restrictive, it seems to be only technical and it could be removed. On the other hand some stronger local summability inside \(\Omega\) seems to be needed (compare also with Example 3 later).
3.2 The case $\gamma > 1$

Unless for the case of compactly supported data (see the discussion following the proof of Theorem 3.5 below), it is known that boundedness away from zero of the datum $f$ (at least near the boundary) is very useful in order to obtain sharp regularity result as the one we look for ([34, 3, 48]). For merely nonnegative data we have the following:

**Theorem 3.5.** Let $f$ be a nonnegative function in $L^m(\Omega)$ with $m > 1$, and let $h$ satisfies (2.4) and (3.6), with $\theta \geq 1$. Then there exists a solution $u$ to (3.1) that belongs to $H^1_0(\Omega)$ provided

$$1 < \gamma < 2 - \frac{1}{m}.$$  

**Proof.** We take $u_n$ as a test function in (3.4); we have

$$\alpha \int_{\Omega} |\nabla u_n|^2 \leq \int_{\Omega} h_n(u_n) f_n u_n \leq K_1 \int_{\{u_n < \omega\}} f_n u_n^{1-\gamma} + K_2 \int_{\max{\omega\leq u_n \leq \omega}} h_n(u_n) f_n u_n$$

$$+ K_2 \int_{\{u_n > \omega\}} f_n u_n^{1-\theta} \leq K_1 \int_{\{u_n < \omega\}} f_n u_n^{1-\gamma} + \omega \max{\omega\leq u_n \leq \omega} h(s) \int_{\max{\omega\leq u_n \leq \omega}} f_n u_n^{1-\theta} + K_2 \int_{\{u_n > \omega\}} f_n u_n^{1-\theta}.$$  

(3.13)

In order to conclude we need to estimate the first term on the right hand side of (3.13). To do that, we consider the nonincreasing and continuous function $\underline{h} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ given by (2.17). Recall that

$$\underline{h}(s) \leq h_n(s), \quad \forall s > 0, \ n \in \mathbb{N}.$$  

(3.14)

We then consider $v_n \in H^1_0(\Omega)$ as the solutions to

$$\begin{cases}
-\text{div}(A(x)\nabla v_n) = \underline{h}(v_n) f_n & \text{in } \Omega, \\
v_n = 0 & \text{on } \partial \Omega.
\end{cases}$$

As $\underline{h}$ is nonincreasing, it is known that $v_n$ is nondecreasing with respect to $n$ (see [6, 17]) and, using (3.14), by comparison $u_n \geq v_n \geq v_1$. We can apply Hopf’s lemma to $v_1$ (see for instance [18, 42]) to deduce that

$$v_1(x) \geq C\delta(x), \quad \text{for } x \in \Omega.$$  

(3.15)

Thus, it follows from the Hölder inequality, from (3.15) and from the fact that $u_n \geq v_1$, that for the first term in the left hand side of (3.13) we have

$$\int_{\{u_n < \omega\}} f_n u_n^{1-\gamma} \leq \int_{\Omega} f v_1^{1-\gamma} \leq C^{1-\gamma} \|f\|_{L^m(\Omega)} \left( \int_{\Omega} \frac{1}{\delta^{(\gamma-1)m'}} \right)^{\frac{1}{m'}} < \infty,$$

since $\gamma < 2 - \frac{1}{m}$. This concludes the proof. \[\square\]

**Remark 3.6.** Observe that the previous proof works also for $\gamma = 1$, in particular, as $m \rightarrow 1^+$, one recovers the case in which finite energy solutions always exist for $f \in L^1(\Omega)$ in continuity with the result of Theorem 3.2.
Though, in this generality, it seems not so easy to improve it, the (upper) threshold on $\gamma$ given in Theorem 3.5 turns out to be not the optimal one. Consider, for instance, the model case $h(s) = s^{-\gamma}$ and a function $f$ in $L^m(\Omega)$ with $m > 1$. In [3] the authors prove the existence of a solution $u \in H^1_0(\Omega)$ if $1 < \gamma < \frac{3m-1}{m+1}$ provided $f$ is strictly bounded away from zero. Prototypical examples show however that finite energy solutions can be found up to $\gamma < 3 - \frac{2}{m}$ (see Example 2 below). Observe that, $3 - \frac{2}{m} > \frac{3m-1}{m+1}$ and that, as $m$ tends to infinity, one formally recovers the Lazer-McKenna threshold $\gamma = 3$ for bounded (also away from zero, and smooth) data.

Apart from explicit examples, we also refer to [48, 7] in which, as already suggested in [34, Section 4], one sees that these limit values can be reached in the case of the laplacian and a smooth and bounded away from zero datum $f$ that blows up uniformly at $\partial \Omega$ at a precise rate. Let us only mention the opposite case of $f$ having compact support on $\Omega$. If this is the case in fact, if $\gamma, \theta \geq 1$ then, in view of $(3.5)$, $h_n(u_n)u_n$ is uniformly bounded on the support of $f$ and the estimate on $u_n$ in $H^1_0(\Omega)$ is for free for any $f \in L^1(\Omega)$.

Let us consider
\begin{equation}
\begin{cases}
-\text{div}(A(x)\nabla u) = \frac{f}{u^\gamma} & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\tag{3.16}
\end{equation}
for $\gamma > 1$ and $f \in L^m(\Omega)$, $m > 1$, a positive function. We have the following

**Theorem 3.7.** Let $m, \gamma > 1$. Then the solution $u$ to problem (3.16) belongs to $H^1_0(\Omega)$ for any positive $f \in L^m(\Omega)$ if and only if $\gamma < 3 - \frac{2}{m}$.

The sufficient condition in Theorem 3.7 is the easiest part as it essentially follows by a (highly non-trivial) result proved, with variational tools, in [44]. More precisely, a line by line re-adaptation to the case a bounded matrix of the proof of [44, Theorem 1] allows us to state the following

**Theorem 3.8.** Let $\gamma > 1$ and let $f$ be a positive function in $L^1(\Omega)$. Then there exists a unique solution $u \in H^1_0(\Omega)$ to problem (3.16) if and only if there exists a function $u_0 \in H^1_0(\Omega)$ such that
\begin{equation}
\int_\Omega f u_0^{1-\gamma} < +\infty.
\tag{3.17}
\end{equation}

**Proof of Theorem 3.7.** To prove that the solution $u$ belongs to $H^1_0(\Omega)$ only plug $u_0(x) = \delta(x)^t$ into (3.17). Using Hölder inequality one has
\begin{equation}
\int_\Omega f u_0^{1-\gamma} \leq C \left( \int_\Omega \delta(1-\gamma)m' \right)^{\frac{m}{m'}}.
\end{equation}
Since $\gamma < 3 - \frac{2}{m}$ it is possible to choose $t$ greater than $\frac{1}{2}$ so that $t(1-\gamma)m' > -1$ and one can apply the result of Theorem 3.8.

In order to prove optimality we let $\gamma \geq 3 - \frac{2}{m}$, we define
\begin{equation}
f(x) := \max \left( \frac{1}{\delta(x)^{\frac{1}{m}} \log \left( \frac{1}{\delta(x)} \right)}, 1 \right),
\tag{3.18}
\end{equation}
and $f_n = T_n(f)$. One can prove that a suitable positive constant $M$ exists such that $M\varphi^t_1$
Francescantonio Oliva, Francesco Petitta

(observe that $0 < t \leq \frac{1}{2}$) is a strong super-solution of the approximating problems

\[
\begin{cases}
-\text{div}(A(x)\nabla u_n) = \frac{f_n}{(u_n + \frac{1}{n})^\gamma} & \text{in } \Omega, \\
u_n = 0 & \text{on } \partial\Omega.
\end{cases}
\]

Indeed, since

\[
\frac{f}{M^{\gamma}\varphi_1^t} \geq \frac{f_n}{(M\varphi_1^t + \frac{1}{n})^\gamma},
\]

we only need to show that

\[
-\text{div}(A(x)\nabla M\varphi_1^t) = \frac{1}{M^{\gamma}\varphi_1^t} \left( M^{1+\gamma}t(1-t)A(x)\nabla \varphi_1 \cdot \nabla \varphi_1 + M^{1+\gamma}\lambda_1t\varphi_1^{t+\gamma} \right) \geq \frac{f}{M^{\gamma}\varphi_1^t},
\]

that is implied by

\[
\alpha M^{1+\gamma}t(1-t)|\nabla \varphi_1|^2\varphi_1^{t-2+\gamma} + M^{1+\gamma}\lambda_1t\varphi_1^{t+\gamma} \geq f.
\]

(3.19)

Let $\varepsilon < e^{-1}$ be a small enough positive number such that $\Omega \setminus \Omega_\varepsilon$ is smooth and compactly contained in $\Omega$ and observe that both terms on the left hand side of (3.19) are nonnegative. Therefore, as $\varphi_1$ is locally strictly away from zero, it is possible to choose $M$ large enough such that $M^{1+\gamma}\lambda_1t\varphi_1^{t+\gamma} \geq f$, for any $x \in \Omega \setminus \Omega_\varepsilon$.

Otherwise, if $x \in \Omega_\varepsilon$, it suffices to prove the first inequality in

\[
M^{1+\gamma}t(1-t)|\nabla \varphi_1|^2\varphi_1^{t-2+\gamma} \geq \frac{1}{\delta(x)} \frac{1}{m} \geq \frac{1}{\delta(x)} \frac{1}{m} \log \left( \frac{1}{\delta(x)} \right) = f(x),
\]

where we used that $\delta(x) < \frac{1}{\varepsilon}$. Using Hopf’s lemma (and (2.16)) the previous reduces by

\[
\frac{M^{1+\gamma}}{\delta(x)^{-t+2-\gamma}} \geq \frac{K}{\delta(x)^{\frac{1}{m}}},
\]

(3.20)

where $K$ is a constant that only depends on $\Omega, \varphi_1, m,$ and $\gamma$. Thanks to the choice of $t$ (3.20) is satisfied for $M$ large enough.

Therefore, we can apply the comparison principle between $M\varphi_1^t$ and $u_n$ (as in the proof of Theorem 3.3) to obtain $M\varphi_1^t \geq u_n$ and so $M\varphi_1^t \geq u$ passing to the a.e. limit. Now suppose by contradiction that $u \in H_0^1(\Omega)$. Then, by Theorem 3.1, we can use $u$ as test function in (3.16); recalling (2.16), (3.18), and that $\gamma \geq 3 - \frac{2}{m}$, we then have

\[
\beta \int_\Omega |\nabla u|^2 \geq \int_\Omega f u^{1-\gamma} \geq M^{t(1-\gamma)} \int_\Omega f \varphi_1^{t(1-\gamma)} = \infty,
\]

a contradiction.

\[\square\]

### 3.3 On the summability of the lower order term

We discuss a prototypical example of solutions to problems as in (3.16).
Example 2. Consider problem (3.16) with $A \equiv I$ (i.e. the case of the laplacian). Let $\Omega = B_1(0)$ and $u = (1 - |x|^2)^\eta$, with $\eta > 0$. Then, if
\begin{equation}
\frac{1}{1 + \gamma} < \eta < 1,
\end{equation}
u solves (3.16) with
$$f \sim \frac{1}{(1 - |x|^2)^{2\eta - \eta \gamma}} \in L^1(\Omega).$$

First of all observe that, as $\eta < 1$, then $-\Delta u / \in L^1(\Omega)$, while, $-\Delta u \in L^1(\Omega, \delta)$ for any $\eta > 0$. This latter remark should be compared with Lemma 4.2 below. Using (3.21), we also have
(i) if $\gamma = 1$ the solution is always in $H_0^1(\Omega)$ as expected,
(ii) if $\gamma > 1$ then $f$ is in $L^m(\Omega)$ provided
$$\eta > \frac{2 - \frac{1}{m}}{\gamma + 1},$$
In particular, $u \in H_0^1(\Omega)$ if $\gamma < 3 - \frac{2}{m}$.
(iii) if $\gamma < 1$ then $u$ is always in $H_0^1(\Omega)$ and $f \in L^m(\Omega)$ for any
$$m < \frac{1}{2 - \eta - \eta \gamma}.$$

We observe that
\begin{equation}
\frac{1}{2 - \eta - \eta \gamma} \nearrow \frac{1}{1 - \gamma} \quad \text{as } \eta \to 1^-.
\end{equation}

The previous example shows that one cannot expect in general the lower order term $fu^{-\gamma}$ to belong to $L^1(\Omega)$, even for small $\gamma$. Suitable weighted summability of the lower order term will be given in the next section (see Lemma 4.2). Although observe that in (iii) $m$ needs to be small enough. For $m = \infty$ (i.e. $f$ is a nonnegative bounded function) and a mild blow up at $0$ (i.e. $\gamma < 1$), as a consequence of Theorem 1.2 in [30], one has $fu^{-\gamma} \in L^1(\Omega)$. In general, we have the following result in which the threshold $\frac{1}{1 - \gamma}$ cannot be improved in view of (3.22) and one can allow $h(\infty) \neq 0$.

Theorem 3.9. Let $h$ satisfy (2.4) with $\gamma < 1$, and let $f$ be a nonnegative function in $L^m(\Omega)$ with $m > \frac{1}{1 - \gamma}$. Then there exists a solution to problem (3.1) such that $h(u)f \in L^1(\Omega)$.

Proof. Let $v_n$ be the solution to the problem
$$\begin{cases}
-\text{div}(A(x)\nabla v_n) = h(v_n)f_n & \text{in } \Omega, \\
v_n = 0 & \text{on } \partial \Omega,
\end{cases}$$
defined as in the proof of Theorem 3.2. Notice that the fact that $u_n \geq v_n \geq v_1 \geq C\delta$ in $\Omega$ (for suitable constant $C$) is independent of the value of $\gamma$ and so it holds true still in the case $\gamma < 1$. Passing to the a.e. limit one then obtains $u \geq v_1 \geq C\delta$ and so, by Hölder’s inequality, one has
$$\int_\Omega h(u)f \leq K_1 \int_{\{u < \omega\}} fu^{-\gamma} + \sup_{\omega, \infty} h(s) \int_{\{u \geq \omega\}} f$$
$$\leq C \int_\Omega f v_1^{-\gamma} + \sup_{\omega, \infty} h(s) \int_\Omega f \leq C \left( \int_\Omega v_1^{-m'\gamma} \right)^{\frac{1}{m}} + C$$
and the last integral is finite as $m > \frac{1}{1 - \gamma}$.
\[\square\]
4 Infinite energy solutions

4.1 Existence of a solution

In this section we prove Theorem 2.3.

Proof of Theorem 2.3. Following [17], the existence of a solution to (2.1) can be proven by approximating with the solutions to the desingularized problems

\[
\begin{cases}
Lu_n = h_n(u_n)f_n & \text{in } \Omega, \\
u_n = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( h_n(s) = T_n(h(s)) \) and \( f_n \) is a sequence of smooth functions suitably converging to \( \mu \). The following a priori estimates holds:

\[
\|T_k(u_n)\|_{H^1_0(\Omega)} \leq C \text{ if } \gamma \leq 1, \quad \text{and } \|T_k(u_n)\|_{H^1_0(\Omega)}^{\gamma+1} \leq C \text{ if } \gamma > 1,
\]

for any \( k > 0 \). Moreover, (3.5) holds and, up to subsequences, \( u_n \) a.e. converges towards a positive function \( u \in W^{1,1}_{loc}(\Omega) \) such that

\[
\begin{cases}
T_k(u) \in H^1_0(\Omega) & \text{if } \gamma \leq 1 \\
T_k^{\gamma+1}(u) \in H^1_0(\Omega) & \text{if } \gamma > 1,
\end{cases}
\]

(4.2)

\( h(u) \in L^{\infty}_{loc}(\Omega, \mu_d) \), and (2.8) is satisfied (see [17 Theorem 3.3]). One also has that, for any \( \gamma > 0 \) and for any \( k > 0 \),

\[
\|G_k(u_n)\|_{W^{1,1}_{loc}(\Omega)} \leq C, \quad \text{for all } q < \frac{N}{N-1},
\]

(4.3)

where \( C \) does not depend on \( n \); in particular \( G_k(u) \in W^{1,1}_{0}(\Omega) \). The proof of (4.3) is standard and it amounts to choose \( T_r(G_k(u_n)) \) as test function in (4.1); this leads to the following estimate

\[
\int_\Omega |\nabla T_r(G_k(u_n))|^2 \leq Cr, \quad \text{for any } r > 0,
\]

which is known to imply (4.3). So that, recalling (4.2), if \( \gamma \leq 1 \) we have that \( u \in W^{1,1}_{0}(\Omega) \) and the proof is complete.

If \( \gamma > 1 \) as \( u = T_1(u) + G_1(u) \) this implies in particular that \( u \in L^1(\Omega) \) and, moreover, using Hölder’s inequality one has

\[
\frac{1}{\varepsilon} \int_{\Omega_\varepsilon} u \leq \frac{1}{\varepsilon} \int_{\Omega_\varepsilon} T_1(u)^{\frac{\gamma+1}{2}} \frac{2}{\gamma+1} |\Omega_\varepsilon|^\frac{\gamma+1}{\gamma+1} + \frac{1}{\varepsilon} \int_{\Omega_\varepsilon} G_1(u)
\]

\[
\leq C \left( \frac{1}{\varepsilon} \int_{\Omega_\varepsilon} T_1(u)^{\frac{\gamma+1}{2}} \right)^\frac{2}{\gamma+1} + \frac{1}{\varepsilon} \int_{\Omega_\varepsilon} G_1(u) \varepsilon \to 0
\]

in view of (4.2), that is (2.7) holds.

\[\square\]

Remark 4.1. We stress that, as can be deduced by the previous proof, all weak solutions in the sense considered, for instance, in [6 39 40 13] also satisfy problem (2.1) in the sense of Definition 2.1. We also remark that the previous proof easily extends to the case of a matrix \( A(x) \) with bounded measurable coefficients in Lipschitz domains.
4.2 Uniqueness of solutions

We now prove Theorem 2.4. The proof will rely on an application of the Kato’s type inequality given in Lemma 2.9.

The following is a suitable extension of a property which is well known for superharmonic functions (see for instance [9, 47, 35] and references therein). It will be a key ingredient in order to prove suitable weighted summability on the lower order term of our singular problem.

Lemma 4.2. Let \( u \in L^1(\Omega) \cap W^{1,1}_{\text{loc}}(\Omega) \) be such that \( Lu = \mu \) in the sense of distributions for some nonnegative measure \( \mu \in \mathcal{M}_{\text{loc}}(\Omega) \). Then \( \mu \in \mathcal{M}(\Omega, \delta) \).

Proof. Let \( \Phi \) be a convex smooth function with bounded \( \Phi' \) and which vanishes in a neighborhood of 0. For instance, for a fixed \( k > 0 \), a good choice is to consider \( \Phi \) as a convex smooth function that agrees with \( |G_k(s)| \) for every \( s \) but \( |s| \in [k, 2k] \). Moreover, let \( \xi \) be the solution to

\[
\begin{align*}
L^*\xi &= 1 \quad \text{in} \ \Omega, \\
\xi &= 0 \quad \text{on} \ \partial\Omega.
\end{align*}
\]

(4.4)

In order to conclude it suffices to prove that

\[
\int_\Omega \xi \mu \leq C,
\]

as, by Hopf’s lemma, one can deduce that \( \xi \geq c\delta \) on \( \Omega \).

Let \( \varphi_n := \frac{1}{n} \Phi(n\xi(x)) \). It is easy to check that \( \varphi_n \) has compact support in \( \Omega \) and that \( \varphi_n \) converges to \( \xi \) a.e. in \( \Omega \). Using (2.2) and the convexity of \( \Phi \), we have, in the sense of distributions

\[
L^*\varphi_n = \Phi'(n\xi)L^*\xi - A^*(x)\nabla \xi \cdot \nabla \xi \Phi''(n\xi) \leq \Phi'(n\xi).
\]

Then

\[
\int_\Omega \varphi_n \mu \leq \int_\Omega u L^*\varphi_n \leq \|\Phi'\|_{\infty} \int_\Omega u \leq C,
\]

and the proof is complete using the Fatou lemma.

Remark 4.3. First important remark is that Lemma 4.2 applies to distributional solutions to our problem (2.1) yielding that both \( h(u)\mu_d \) and \( \mu_c \) belongs to \( \mathcal{M}(\Omega, \delta) \). Observe that this fact is quite general and, for instance, it does not require any monotonicity on \( h \). Also remark that, if \( \mu \in L^1(\Omega) \) then the same argument shows that \( h(u)\mu \in L^1(\Omega, \delta) \) (see also [18] where analogous estimates were proven in the case \( \mu \in L^\infty(\Omega) \) and \( h(s) = s^{-\gamma} \)).

We are now in the position to prove Theorem 2.4. We stress that no control on the function \( h \) at infinity nor at zero is required for the following proof to hold.

Proof of Theorem 2.4 It follows from Lemma 4.2 that \( h(u)\mu_d + h(\infty)\mu_c \) belongs to \( \mathcal{M}(\Omega, \delta) \). In particular being \( \mu_d \) a diffuse measure and \( h(u)\delta \) measurable with respect to \( \mu_d \) then \( h(u)\mu_d \) is itself a diffuse measure in \( \mathcal{M}(\Omega, \delta) \).

We consider a distributional solution \( u \) in the sense of Definition 2.1 that is it satisfies

\[
\int_\Omega A(x)\nabla u \cdot \nabla \varphi = \int_\Omega h(u)\varphi \mu_d + h(\infty) \int_\Omega \varphi \mu_c, \quad \forall \varphi \in C^1_c(\Omega).
\]
First step is observing that taking in the previous $\varphi = \eta_k \phi$ where $\phi \in C^1_0(\overline{\Omega})$ such that $L^* \phi \in L^\infty(\Omega)$, and $\eta_k \in C^\infty_c(\Omega)$ is such that $0 \leq \eta_k \leq 1$, $\eta_k = 1$ when $\delta(x) > \frac{1}{k}$, $\|\nabla \eta_k\|_{L^\infty(\Omega)} \leq k$, and $\|L^* \eta_k\|_{L^\infty(\Omega)} \leq C k^2$, we have

$$
\int_\Omega A(x) \nabla u \cdot \nabla (\eta_k \phi) = \int_\Omega h(u) \eta_k \phi, \mu_d + h(\infty) \int_\Omega \eta_k \phi, \mu_c. \quad (4.5)
$$

We want to pass to the limit in $k$ the previous formulation. Since $h(u) \mu_d + h(\infty) \mu_c$ belongs to $\mathcal{M}(\Omega, \delta)$ and $\phi \in C^1_0(\overline{\Omega})$ we use dominated convergence theorem to pass to the limit on the right hand side of (4.5). Concerning the left hand side of (4.5) we have

$$
\int_\Omega A(x) \nabla u \cdot \nabla (\eta_k \phi) = - \int_\Omega A^*(x) \nabla \eta_k \cdot \nabla \phi u + \int_\Omega u \eta_k L^* \phi - \int_\Omega A^*(x) \nabla \phi \cdot \nabla \eta_k u + \int_\Omega \phi L^* \eta_k. \quad (4.6)
$$

Since $L^* \phi \in L^\infty(\Omega)$, again by dominated convergence we have

$$
\int_\Omega u \eta_k L^* \phi \xrightarrow{k \to \infty} \int_\Omega u L^* \phi.
$$

It follows from (2.7) that

$$
\lim_{k \to \infty} \int_\Omega |A^*(x) \nabla \eta_k \cdot \nabla \phi u| \leq \lim_{k \to \infty} \beta C k \int_{\{\delta < \frac{1}{k}\}} u = 0,
$$

so that the first and the third term on the right hand side of (4.6) vanish, while the last one can be treated as follows

$$
\lim_{k \to \infty} \int_\Omega \phi L^* \eta_k \leq \lim_{k \to \infty} \beta C k^2 \int_{\{\delta < \frac{1}{k}\}} u \phi \leq \lim_{k \to \infty} \beta C k \int_{\{\delta < \frac{1}{k}\}} u = 0.
$$

Collecting the previous we deduce that

$$
\int_\Omega u L^* \phi = \int_\Omega h(u) \phi, \mu_d + h(\infty) \int_\Omega \phi, \mu_c, \forall \phi \in C^1_0(\overline{\Omega}). \quad (4.7)
$$

Now, by applying (4.7) to the difference of two distributional solutions $v$ and $w$ of (2.1) we have

$$
\int_\Omega (v - w) L^* \xi = \int_\Omega (h(v) - h(w)) \xi, \mu_d,
$$

for every $\xi \in C^1_0(\overline{\Omega})$ such that $L^* \xi \in L^\infty(\Omega)$. As $(h(v) - h(w)) \mu_d$ is a diffuse measure in $\mathcal{M}(\Omega, \delta)$, we can apply Lemma 2.9 to obtain

$$
\int_\Omega (v - w)^+ L^* \phi \leq \int_{\{v \geq w\}} (h(v) - h(w)) \phi, \mu_d \leq 0,
$$

for every $0 \leq \phi \in C^1_0(\Omega)$ such that $L^* \phi \in L^\infty(\Omega)$. Now we reason again as in the proof of (4.7); we consider $\xi \eta_k$ where $\eta_k$ is defined as before and $\xi$ is defined by (4.4), and we deduce that

$$
\int_\Omega (v - w)^+ L^* \xi \leq 0,
$$

that, by definition of $\xi$, allows us to conclude that $v \leq w$. The proof is completed by interchanging the roles of $v$ and $w$. \qed
5 Some further remarks, extensions, and open problems

In Section 3 we analyzed some instances of finite energy solutions to problem (3.1) depending on both the regularity of the datum \( f \) and on the behavior of the nonlinearity \( h \) both at zero and at infinity. Observe that, the strongly singular case at infinity (i.e. \( \theta < 1 \)) of Theorem 3.5 can not be treated in general. The pathological phenomenon shown by Example 1, in fact, is essentially due to the behavior at infinity of \( h \); this fact is highlighted by the following suitable extension of this example.

Example 3. For \( N > 2 \) and a fixed \( \theta < 1 \), we choose two parameters \( m \) and \( q \) such that

\[
1 \leq m < q < \frac{2N}{N + 2} \quad \text{and} \quad 0 < \theta \leq \frac{N(q - m)}{m(N - 2q)}.
\]

Notice that this choice is always possible since the function \( \nu(m, q) = \frac{N(q - m)}{m(N - 2q)} \) is continuous around \((m, q) = (1\; \frac{2N}{N + 2})\) and \( \nu(1\; \frac{2N}{N + 2}) = 1 \). Now for suitable \( 0 \leq f \in L^q(\Omega) \), one can consider again the solution \( u \) to

\[
\begin{aligned}
-\Delta u &= f \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

such that \( u \) is \( W^{1, q^*}(\Omega) \) but \( u \notin H^{1}_{0}(\Omega) \). Let \( h \) satisfying both (2.4) and (3.6) with, resp., \( \gamma > 0 \) and \( \theta < 1 \). Then \( u \) satisfies

\[
\begin{aligned}
-\Delta u &= h(u)g \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

where \( g = f h(u)^{-1} \). We claim that \( g \) in \( L^m(\Omega) \); indeed, recalling that \( m < q \) and also using Hölder’s inequality, one has

\[
\int_{\Omega} g^m = \int_{\{u < \omega\}} f^m u^\gamma m + \int_{\{\omega \leq u \leq \varpi\}} f^m h(u)^{−m} + \int_{\{u > \varpi\}} f^m u^{\theta m}
\]

\[
\leq \omega^{\gamma m} \int_{\Omega} f^m + \max_{[\omega, \varpi]}(h(s)^{-m}) \int_{\Omega} f^m + C \left( \int_{\Omega} \frac{\theta m q}{q - m} \right)^{\frac{q - m}{q}} \leq C
\]

since, thanks to the choices of both \( q \) and \( m \), we have \( \frac{\theta m q}{q - m} < \frac{Nq}{N - 2q} = q^{**} \). We then have nonnegative data in \( L^m(\Omega) \) for which solutions to our problem need not to belong to \( H^{1}_{0}(\Omega) \), for \( \theta < 1 \). Observe that the behavior at zero (i.e. \( \gamma \)) plays no role and that \( m \) needs to be close to 1 (to be compared with the extra assumptions required in Theorem 3.3).

As we saw, the model case (i.e. \( h(s) = s^\gamma \), \( \gamma > 1 \)) of problem (2.1) is covered by Theorem 3.7. The proof is based on Theorem 3.8 which is not known for a general nonlinearity \( h \); though the proof in [44] can not be directly generalized due to some technical homogeneity issues, one can conjecture that it still holds, at least if \( h(s)s \) is nonincreasing. That is, in this case, one could have that (2.1) has a finite energy solution if and only if there exists \( u_0 \in H^{1}_{0}(\Omega) \) such that

\[
\int_{\Omega} f h(u_0) u_0 < \infty.
\]

We showed that, for strongly singular (at infinity) \( h \)'s then solutions need not to have finite energy in general. In the model case, one also has the following natural regularity criterion:
Proposition 5.1. Let $\gamma < 1$, $f \in L^1(\Omega)$, and $u$ solves (3.16) then $u \in H^1_0(\Omega)$ if and only if

$$\int_{\Omega} f u^{1-\gamma} < \infty.$$ 

Proof. $(\Rightarrow)$ Apply Theorem 3.1 and use $u$ as test in (3.16).

$(\Leftarrow)$ One considers the approximating solutions $u_n$ to

$$\begin{cases}
-\text{div}(A(x)\nabla u_n) = \frac{f_n}{(u_n + \frac{1}{n})^{1-\gamma}} & \text{in } \Omega, \\
u_n = 0 & \text{on } \partial \Omega,
\end{cases}$$

and use $u_n$ as test, to obtain

$$\alpha \int_{\Omega} |\nabla u_n|^2 \leq \int_{\Omega} \frac{f_n}{(u_n + \frac{1}{n})^{1-\gamma}} u_n \leq \int_{\Omega} f_n u_n^{1-\gamma} \leq \int_{\Omega} f u^{1-\gamma},$$

where in the last step we also used that $u_n$ increases with respect to $n$, and we conclude by weak lower semicontinuity.

One could conjecture that this kind of regularity principle, specializing in some sense the one given by Theorem 3.8, extends for any $\gamma > 0$ (or, more, for any $h$). That is one expects that, given a nonnegative $f \in L^1(\Omega)$ and the solution $u$ to (3.1), then

$$u \in H^1_0(\Omega) \text{ if and only if } \int_{\Omega} fh(u)u < \infty.$$ 

Notice that solutions in Example 2 satisfy (sharply) this criterion. In fact, if $\gamma > 1$ then $\int_{\Omega} f u^{1-\gamma} < \infty$ if and only if $\gamma < 3 - \frac{2}{m}$ if and only if $u \in H^1_0(\Omega)$.

Consider now the threshold $\frac{1}{\gamma}$ given in Theorem 3.9. As we said it can not be improved (at Lebesgue’s scale) in view of Example 2. Although, the borderline case remains open. A straightforward modification in the proof of Theorem 3.9 using suitable generalized Hölder’s inequalities, shows that the result still holds true if $f$ belongs to the Lorentz space $L^{\frac{1}{\gamma-1}, 1}(\Omega)$. The following example contains an explicit instance of this occurrence in the case of problem (3.1).

Example 4. Let $h$ satisfying (2.4) with $\gamma < 1$, $a > 1$, and consider

$$f(x) = \frac{1}{\delta(x)^{1-\gamma}(-\log(\delta(x)))^a}. \tag{5.1}$$

Without loss of generality we are assuming $\text{diam}(\Omega) < e^{-1}$; otherwise we define $f$ as in (5.1) near the boundary and we suitably truncate it inside of $\Omega$. Reasoning as in the proof of Theorem 3.9 one can show that $u \geq v_1 \geq C\delta$ where $u$ is the solution to (5.1). Then, as $a > 1$, one has

$$\int_{\Omega} h(u)f \leq \int_{\{u \leq \bar{u}\}} f u^{-\gamma} + \int_{\{u \geq \bar{u}\}} h(u)f \leq \int_{\{u \leq \bar{u}\}} f v_1^{-\gamma} + \sup_{[\bar{u}, \infty)} h(s) \int_{\{u \geq \bar{u}\}} f$$

$$\leq C^{-\gamma} \int_{\Omega} \frac{1}{\delta(-\log(\delta))^a} + C < \infty.$$
Finite and Infinite energy solutions of singular elliptic problems

One last interesting remark concerns the way the homogeneous boundary datum is assumed. As we observed, in general, one is not able to prove that the solution has a trace in the classical sense (at least if \( \gamma > 1 \)) and we get rid of this fact by introducing the relaxed boundary condition \( (6.1) \). Anyway, in view of Lemma \([4.2]\) we are again in presence of a borderline case. Consider for simplicity a function \( f \) as datum; in \([20]\) the authors prove that, if the lower order term \((h(u)f)\) belongs to \( L^1(\Omega, \delta^\alpha) \), for some \( 0 < \alpha < 1 \), then \( u \in W_0^1(\Omega) \), for some \( q > 1 \). Moreover, if \( h(u)f \in L^1(\Omega, |\log(\delta)|) \), then \( u \in W_0^{1,1}(\Omega) \). The solutions of Example \([2]\) are continuous up to the boundary and we are far from this pathological behavior; although, only observe that in that case one always has that \( f u^{-\gamma} \in L^1(\Omega, \delta^\alpha) \), for \( \alpha > \frac{\gamma}{1+\gamma} \).

6 Appendix

6.1 Proof of Theorem \([3.1]\)

Let \( \phi \in H_0^1(\Omega) \) and consider \( \psi_n \) a sequence in \( C_c^1(\Omega) \) that converges to \( \phi \) in \( H_0^1(\Omega) \). For \( \varepsilon > 0 \) we take \( \varphi = \sqrt{\varepsilon^2 + |\psi_n - \psi_k|^2} - \varepsilon \) as a test function in the distributional formulation of \( u \). Using Hölder’s inequality one gets

\[
\int_\Omega h(u)f(\sqrt{\varepsilon^2 + |\psi_n - \psi_k|^2} - \varepsilon) = \int_\Omega A(x)\nabla u \cdot \frac{\nabla(\psi_n - \psi_k)(\psi_n - \psi_k)}{\sqrt{\varepsilon^2 + |\psi_n - \psi_k|^2}} \\
\leq \beta ||u||_{H_0^1(\Omega)} \left( \int_\Omega \left| \frac{|\nabla(\psi_n - \psi_k)|^2 (\psi_n - \psi_k)^2}{\varepsilon^2 + |\psi_n - \psi_k|^2} \right|^{\frac{1}{2}} \right) \leq \beta ||u||_{H_0^1(\Omega)} ||\psi_n - \psi_k||_{H_0^1(\Omega)}.
\]

Therefore, by Fatou’s Lemma

\[
\int_\Omega |h(u)f\psi_n - h(u)f\psi_k| \leq \beta ||u||_{H_0^1(\Omega)} ||\psi_n - \psi_k||_{H_0^1(\Omega)}.
\] (6.1)

Estimate \([6.1]\) implies that the sequence \( h(u)f\psi_n \) is a Cauchy sequence in \( L^1(\Omega) \), and that converges to its a.e. limit \( h(u)f\phi \). We then take \( \psi_n \) as test in \([3.2]\) and we can pass to the limit obtaining that \( u \) satisfies \([3.3]\).

To prove uniqueness we consider two solutions \( u_1 \) and \( u_2 \) of \([3.1]\) in \( H_0^1(\Omega) \). We are allowed to choose \((u_1 - u_2)^-\) as a test function in \([3.3]\) for both \( u_1, u_2 \). Taking the difference we get

\[
\int_\Omega A(x)\nabla (u_1 - u_2) \cdot \nabla (u_1 - u_2)^- = \int_\Omega (h(u_1)f - h(u_2)f)(u_1 - u_2)^- \geq 0
\]

that implies, using the ellipticity of \( A \), that \( u_1 \geq u_2 \). The opposite inequality is obtained by interchanging the roles of \( u_1 \) and \( u_2 \). \( \square \)

Remark 6.1. Note that the proof of Theorem \([3.1]\) is essentially based on the ellipticity of the operator and so it can be easily extended to fairly general classes of problems as, for instance, the ones involving Leray-Lions type nonlinear operators in general domains. We also remark that, one do not need the knowledge of the behavior of \( h \) neither at zero nor at the infinity (we only use that \( h(u) \) is locally bounded).

Acknowledgements

We would like to warmly thank both Luigi Orsina and Augusto C. Ponce for fruitful discussions and, in particular, the ones concerning, resp., Example \([1]\) and Lemma \([4.2]\).
References

[1] L. Ambrosio, N. Fusco, D. Pallara, Functions of Bounded Variation and Free Discontinuity Problems, Oxford Mathematical Monographs, 2000.

[2] D. Arcoya, J. Cártona, T. Leonori, P. J. Martínez-Aparicio, L. Orsina, F. Petitta, Existence and nonexistence results for singular quasilinear equations, Journal of Differential Equations 246 (10) (2009) 4006–4042.

[3] D. Arcoya, L. Moreno-Merida, Multiplicity of solutions for a Dirichlet problem with a strongly singular nonlinearity, Nonlinear Analysis 95 (2014) 281–291.

[4] L. Boccardo, J. Casado-Díaz, Some properties of solutions of some semilinear elliptic singular problems and applications to the G-convergence, Asymptotic Analysis 86 (2014) 1–15.

[5] L. Boccardo, T. Gallouët, L. Orsina, Existence and uniqueness of entropy solutions for nonlinear elliptic equations with measure data, Ann. de l’Ins. Henri Poincare (C) Non Linear Analysis 13 (1996) 539–551.

[6] L. Boccardo, L. Orsina, Semilinear elliptic equations with singular nonlinearities, Calc. Var. and PDEs 37 (3-4) (2010) 363–380.

[7] B. Bougherara, J. Giacomoni, J. Hernández, Existence and regularity of weak solutions for singular elliptic problems, Proceedings of the 2014 Madrid Conference on Applied Mathematics in honor of Alfonso Casal, 19–30, Electron. J. Differ. Equ. Conf., 22, Southwest Texas State Univ., San Marcos, TX, 2015.

[8] H. Brezis, A. C. Ponce. Kato’s inequality when $\Delta u$ is a measure. Comptes Rendus Mathematique 338 (8) (2004) 599–604.

[9] X. Cabré, Y. Martel, Weak eigenfunctions for the linearization of extremal elliptic problems, Journal of Functional Analysis 156 (1) (1998) 30–56

[10] A. Canino, Minimax methods for singular elliptic equations with an application to a jumping problem, Journal of Differential Equations 221 (1) (2006) 210–223.

[11] A. Canino, M. Degiovanni, A variational approach to a class of singular semilinear elliptic equations, Journal of Convex Analysis 11 (1) (2004) 147–162.

[12] A. Canino, M. Grandinetti, B. Sciuinzi, Symmetry of solutions of some semilinear elliptic equations with singular nonlinearities, Journal of Differential Equations 255 (12) (2013) 4437–4447.

[13] A. Canino, B. Sciuinzi, A. Trombetta, Existence and uniqueness for $p$-Laplace equations involving singular nonlinearities, Nonlinear Differ. Equ. Appl. (2016) 23: 8.

[14] G. M. Coclite, M. M. Coclite, On a Dirichlet problem in bounded domains with singular nonlinearity, Discrete Contin. Dyn. Syst. 33 (11-12) (2013) 4923–4944.

[15] M. G. Crandall, P. H. Rabinowitz, L. Tartar, On a dirichlet problem with a singular nonlinearity, Comm. Part. Diff. Eq. 2 (2) (1977) 193–222.
[16] L. M. De Cave, Nonlinear elliptic equations with singular nonlinearities, Asymptotic Analysis 84 (2013) 181–195.

[17] L. M. De Cave, F. Oliva, Elliptic equations with general singular lower order term and measure data, Nonlinear Analysis 128 (2016) 391–411.

[18] N. El Berdan, J. I. Díaz, J. M. Rakotoson, The uniform Hopf inequality for discontinuous coefficients and optimal regularity in BMO for singular problems, J. Math. Anal. Appl. 437 (1) (2016) 350–379.

[19] G. Dal Maso, F. Murat, L. Orsina, A. Prignet, Renormalized solutions of elliptic equations with general measure data, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 28 (1999) 741–808.

[20] J. I. Díaz, J. M. Rakotoson, On the differentiability of very weak solutions with right-hand side data integrable with respect to the distance to the boundary, Journal of Functional Analysis 257 (2009) 807–831.

[21] J. I. Díaz, J. M. Rakotoson, On very weak solutions of semi-linear elliptic equations in the framework of weighted spaces with respect to the distance to the boundary, Discrete Contin. Dyn. Syst. 27 (3) (2010) 1037–1058.

[22] W. Fulks, J. S. Maybee, A Singular Non-Linear Equation, Osaka Journal Mathematics 12 (1960) 1–19.

[23] M. Fukushima, K. Sato, S. Taniguchi, On the closable part of pre-Dirichlet forms and the fine support of the underlying measures, Osaka Journal Mathematics 28 (1991), 517–535.

[24] J. A. Gatica, V. Oliker and P. Waltman, Singular nonlinear boundary-value problems for second-order ordinary differential equations, Journal of Differential Equations 79 (1989) 62–78.

[25] D. Giachetti, P. J. Martínez-Aparicio, F. Murat, A semilinear elliptic equation with a mild singularity at $u = 0$: Existence and homogenization, Journal de Mathématiques Pures et Appliquées, in press.

[26] D. Giachetti, P. J. Martínez-Aparicio, F. Murat, Definition, existence, stability and uniqueness of the solution to a semilinear elliptic problem with a strong singularity at $u = 0$, preprint

[27] D. Giachetti, F. Petitta, S. Segura de Leon, Elliptic equations having a singular quadratic gradient term and a changing sign datum, Comm. Pure Appl. Anal. 11 (5) (2012) 1875–1895.

[28] D. Giachetti, F. Petitta, S. Segura de Leon, A priori estimates for elliptic problems with a strongly singular gradient term and a general datum, Differential and Integral Equations 26 (9/10) (2013) 913–948.

[29] D. Gilbarg, N. S. Trudinger, Elliptic partial differential equations of second order, Springer-Verlag, Berlin, 1983.

[30] C. Gui, F-H. Lin, Regularity of an elliptic problem with a singular nonlinearity, Proc. Roy. Soc. Edinburgh Sect. A 123 (6) (1993) 1021–1029.

[31] J. Heinonen, T. Kilpeläinen, O. Martio, Nonlinear potential theory of degenerate elliptic equations, Oxford University Press, Oxford, 1993.
[32] T. Klimsiak, Reduced measures for semilinear elliptic equations involving Dirichlet operators, Calc. Var. Partial Differential Equations 55: 78 (2016) doi:10.1007/s00526-016-1023-6

[33] A. V. Lair, A. W. Shaker, Classical and weak solutions of a singular semilinear elliptic problem, J. Math. Anal. Appl. 211 (2) (1997) 371–385.

[34] A. C. Lazer, P. J. McKenna, On a singular nonlinear elliptic boundary-value problem, Proc. Amer. Math. Soc. 111 (3) (1991) 721–730.

[35] M. Marcus, L. Veron, Nonlinear Second Order Elliptic Equations Involving Measures, Berlin, Boston: De Gruyter, 2013

[36] F. Murat, A. Porretta, Stability properties, existence and nonexistence of renormalized solutions for elliptic equations with measure data, Comm. Partial Differential Equations 27 (2002) 2267–2310.

[37] A. Nachman, A. Callegari, A nonlinear singular boundary value problem in the theory of pseudoplastic fluids, SIAM J. Appl. Math. 28 (1986) 271–281.

[38] P. Nowosad, On the integral equation $kf = 1/f$ arising in a problem in communication, J. Math. Appl. 14 (1966) 484–492.

[39] L. Orsina, F. Petitta, A Lazer-McKenna type problem with measures, Differential Integral Equations 29 (1-2) (2016) 19–36.

[40] F. Oliva, F. Petitta, On singular elliptic equations with measure sources, ESAIM Control Optim. Calc. Var. 22 (1) (2016) 289–308.

[41] A. C. Ponce, Selected problems on elliptic equations involving measures, [arXiv:1204.0668v2]

[42] J. C. Sabina de Lis, Hopf Maximum principle revisited, Electronical Journal of Differential Equations 115 (2015) 1–9.

[43] C. A. Stuart, Existence and approximation of solutions of non-linear elliptic equations, Math. Z. 147 (1976) 53–63.

[44] Y. Sun, D. Zhang, The role of the power 3 for elliptic equations with negative exponents, Calc. Var. and PDEs 49 (3-4) (2014) 909–922.

[45] S. Taliaferro, A nonlinear singular boundary value problem, Nonlinear Analysis 3 (1979) 897–904.

[46] K. Vajravelu, E. Soewono, R. N. Mohapatra, On solutions of some singular nonlinear differential equations arising in the theory of boundary layers, J. Math. Anal. Appl. 155 (2) (1991) 499–512.

[47] L. Veron, Elliptic Equations Involving Measures, M. Chipot, P. Quittner. Stationary Partial Differential equations, Vol. 1, Elsevier, Handbook of Differential Equations (2004) 593–712.

[48] Z. Zhang, J. Cheng, Existence and optimal estimates of solutions for singular nonlinear Dirichlet problems, Nonlinear Analysis 57 (3) (2004) 473–484.