Simplex, associahedron, and cyclohedron

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Introduction.

I.1. The paper deals with three types of convex polyhedra. The most classical is the \(n\)-dimensional simplex \(\Delta^n\) \[20, \S 10.1\], the basic ingredient of simplicial topology and perhaps one of the most important mathematical objects at all \[15\].

Another polyhedron is the Stasheff polytope \(K_n\), also called the associahedron, the basic tool for the study of homotopy associative Hopf spaces \[18, \text{page 277}\].

The last type is the polyhedron \(W_n\), defined as the Axelrod-Singer compactification of the configuration space of \(n\) points on the circle, and introduced by R. Bott and C. Taubes \[3, \text{page 5249}\] in connection with the study of nonperturbative link invariants, recently dubbed by J. Stasheff the cyclohedron \[19\].

I.2. The crucial property of the collection \(K = \{K_n\}_{n \geq 1}\) is that it forms a cellular operad \[18, \text{page 278}\]. J. Stasheff observed in \[19\] that the collection \(W = \{W_n\}_{n \geq 1}\) is a right module, in the sense of \[13, \text{page 1476}\], over the operad \(K\). In Theorem 3.1 we prove that also the collection \(\Delta = \{\Delta^n\}_{n \geq 0}\) is a natural right module over the operad \(\text{Ass}\) for associative algebras.

I.3. Operads were introduced to encode varieties of algebras. We show that, in the same spirit, also modules over operads describe varieties of some objects. We call these objects traces (Definition 2.6), since they naturally generalize traces on associative algebras.

For a so-called cyclic operad \(\mathcal{P}\) \[7, \text{Definition 2.1}\] we construct a natural \(\mathcal{P}\)-module \(M_\mathcal{P}\), the module associated to the operad \(\mathcal{P}\) (Definition 4.3). We show that \(M_\mathcal{P}\)-traces are exactly invariant bilinear forms in the sense of E. Getzler and M.M. Kapranov \[7, \text{Definition 4.1}\].

I.4. Algebras over the cellular chain operad \(CC_\ast(K)\) of the associahedron are \(A(\infty)\)-algebras introduced by J. Stasheff in \[18, \text{page 294}\]. They can be understood as algebras with the usual associativity condition

\[(ab)c = a(bc)\]
satisfied only up to a system of coherent homotopies. In Proposition 2.14 we show that the traces over the cellular chain complex \( CC_*(W) \) of the cyclohedron are homotopy traces on \( \Lambda(\infty) \)-algebras, for which the usual condition

\[
T(ab) = T(ba)
\]

is satisfied only up to a system of coherent homotopies. The traces over the cellular chain complex \( CC_*(\Delta) \) of the simplex are described in Theorem 3.1.

I.5. The cellular chain complex \( CC_*(K) \) of the associahedron has a very effective description – it is the operadic bar construction on the operad \( Ass \) for associative algebras \([13, Example 4.1]\). We introduce the bar construction on a module over an operad (this definition was independently made by V. Ginzburg and A.A. Voronov in \([9]\)) and show that the cellular chain complex \( CC_*(W) \) of the cyclohedron is the bar construction on the \( Ass \)-module \( Cycl \), which describes traces (ordinary, not homotopy) on associative algebras (Theorem 5.5). A fully algebraic description of \( CC_*(\Delta) \) is given in Theorem 3.3.

I.6. V. Ginzburg and M.M. Kapranov \([8, Definition 4.1.3]\) introduced so-called Koszul operads, with all expected nice properties, and the related notion of the Koszul dual of a quadratic operad \([8, \S 2.1.9]\).

We introduce analogous notions for modules over operads, i.e. we introduce quadratic modules, their Koszul (quadratic) duals and the property of Koszulness for these modules. These definitions were again independently made by V. Ginzburg and A.A. Voronov in \([9]\).

I.7. The operad \( Ass \) for associative algebras is Koszul \([8, Corollary 4.2.7]\). Since the operadic bar construction on \( Ass \) is the cellular chain complex of the associahedron, the Koszulness of \( Ass \) follows from the acyclicity of \( K \), which in turn follows from the fact that it is a convex polyhedron.

In a similar manner, we show in Theorem 5.4 that the module \( Cycl \) describing traces on associative algebras is Koszul, as a consequence of the convexity of the cyclohedron \( W \). A more general argument is to observe that \( Cycl \) is the module associated to the cyclic operad \( Ass \) (Example 4.9) and then apply Theorem 5.6 saying that a module associated to a Koszul operad is Koszul.

I.8. We show in Lemma 5.4 that, for each module over an operad, there exists a spectral sequence, converging to the homology of the bar construction. We also prove in Proposition 5.3 that for modules over Koszul operads this spectral sequence collapses.
Our spectral sequence, applied to an Ass-module $Cycl$, carries a strong geometrical message – the initial term is the cellular chain complex of the cyclohedron, while the next term is the cellular chain complex of the simplex. If we interpret the cyclohedron as the compactification of the simplex constructed by a sequence of blow-ups [3, page 5249], then the spectral sequence describes the inverse process – ‘deblowing-up’ the cyclohedron back to the simplex, see Section 6.

I.9. Some further suggestions. Consider the following ‘standard situation’ closely related to the topological quantum field theory. Let $C(S^m)$ be the Axelrod-Singer compactification [1, Section 5] of the configuration space of distinct points in the sphere $S^m$, and $F_m$ the compactification of the moduli space of configurations of distinct points in $R^m$ [3, §3.2].

It is known that $F_n$ is a topological operad [3, §3.2] and that this operad acts on the right module $C(S^m)$ [11, Theorem 5.2] (to be precise, if $m \neq 1, 3, 7$, the sphere $S^m$ is not parallelizable and we need a suitable framed versions of the objects above). The homology operad $H_*(F_m)$ describes a form of graded Poisson algebras [3, Theorem 3.1] (or Batalin-Vilkovisky algebras, in the framed case) [3, Section 4], and it is not difficult to see that $H_*(C(S^n))$ is the module associated to the cyclic operad $H_*(F_m)$ in the sense of our Definition 4.5. Our paper deals with the above situation for $n = 1$, while all the machinery cries for an application to a general situation.

Another suggestion for further research is the following. E. Getzler and M.M. Kapranov introduced in [7, Definition 5.2] the cyclic homology of an algebra over a cyclic operad as the (left, nonabelian) derived functor of the universal invariant bilinear form functor $\lambda(P, -)$. We propose to study, for a (noncyclic) operad $P$ and a $P$-module $M$, the derived functor of the universal $M$-trace as a natural generalization of the cyclic homology. The cyclic homology will be then a special case for $P$ cyclic and $M = M_P$.

There are two ways to read the paper – either as an exposition of the properties of the associahedron, cyclohedron and simplex, with some generalizations, or as a paper on general theory of modules over operads, with a special attention paid to the three examples above.

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Plan of the paper:

Section 1: Associahedron and the cyclohedron as a truncation of the simplex. We recall the convex realization of the associahedron as a truncation of the simplex, due to S. Shnider and S. Sternberg, and construct a similar realization of the cyclohedron.

Section 2: Cyclohedron as a module over the associahedron. We recall (right) modules over operads and introduce traces as algebraic objects described by these modules. We introduce the module Cycl for traces on associative algebras. We prove that the cyclohedron is a module over the associahedron and describe the corresponding traces.

Section 3: Simplex as a module over the operad for associative algebras. We show that the simplex is a module over the operad Ass for associative algebras. We prove that the associated cellular chain complex is free and describe the corresponding traces.

Section 4: Quadratic operads and modules; modules associated to cyclic operads. We present a class of operads and modules having a particularly easy description. We recall cyclic operads and introduce the module associated to a cyclic operad.

Section 5: Cyclohedron as the cobar construction. We introduce the cobar construction on a module over an operad. We define quadratic Koszul modules. We show that the cellular chain complex of the cyclohedron is the cobar construction on the module Cycl and deduce from this fact that Cycl is Koszul.

Section 6: Cyclohedron as a compactification of the simplex. We view the cyclohedron as a compactification of the simplex, constructed as a sequence of blow-ups. We show that the spectral sequence related to the cobar construction ‘deflates’ the cyclohedron back to the simplex.

Appendix: Traces versus invariant bilinear forms. We show that traces over the module associated to a cyclic operad are exactly invariant bilinear forms of E. Getzler and M.M. Kapranov.

1. Associahedron and the cyclohedron as a truncation of the simplex.

Let $B(n)$ denote the set of all meaningful bracketings of $n$ independent variables $1, \ldots, n$. The associahedron $K_n$ is a convex $(n - 2)$-dimensional polyhedron whose faces are indexed by elements of $B(n)$. To be more precise, $B(n)$ is a poset (= partially-ordered set) ordered
by saying that \( b' \prec b'' \) if \( b'' \) is obtained from \( b' \) by removing one or more pair of brackets. Then \( K_n \) is a convex polyhedron whose poset of faces is (isomorphic to) \( \mathcal{B}(n) \). See Figure 1 for \( K_3 \) and \( K_4 \). A nice picture of \( K_5 \) can be found in [14, page 151].

![Figure 1: \( K_3 \) (left) and \( K_4 \) (right).](image)

We recall a very cute ‘linear convex realization’ of \( K_n \) as a truncation of the \((n-2)\)-dimensional simplex, due to S. Shnider and S. Sternberg [17]. Our exposition follows the corrected version given in [19, Appendix B].

We need an alternative description of the poset \( \mathcal{B}(n) \). Let \( P(n) \) denote the set of all proper subintervals of the interval \([1, n-1] = \{1, \ldots, n-1\}\). Two intervals \( I, J \in P(n) \) are called \emph{compatible}, if \( I \cup J \) is not an interval properly containing both \( I \) and \( J \), i.e. if either \( J \subset I \), or \( I \subset J \), or \( I \cup J \) is not an interval. Let \( \mathcal{I}(n) \) be the set of all subsets \( \iota \) of \( P(n) \) such that \( I \) and \( J \) are compatible for any \( I, J \in \iota \). The poset structure on \( \mathcal{I}(n) \) is given by the set inclusion: \( \iota \preceq \kappa \) if \( \kappa \subset \iota \).

**Lemma 1.1.** (Shnider-Sternberg) The posets \( \mathcal{B}(n) \) and \( \mathcal{I}(n) \) are isomorphic.

**Proof.** For \( I = [i, j] \in P(n) \), let \( b(I) \) be the bracketing \( 1 \cdots (i \cdots j + 1) \cdots n \). This correspondence is easily seen to induce a poset isomorphism \( \mathcal{I}(n) \cong \mathcal{B}(n) \).

Define the function \( c : P(n) \to \mathbb{R}_{>0} \) by \( c(I) := 3^{#I} \), for \( I \in P(n) \). Let \( K_n \subset \mathbb{R}^{n-1} \) be the convex polytope defined by

\[
K_n = \left\{(t_1, \ldots, t_{n-1}) \in \mathbb{R}^{n-1}; \sum_{k=1}^{n-1} t_k = c([1, n-1]), \sum_{k \in I} t_k \geq c(I), \ I \in P(n) \right\}.
\]
Denote also, for $I \in \mathcal{P}(n)$, by $P_I$ the hyperplane

$$P_I := \left\{ (t_1, \ldots, t_{n-1}) \in \mathbb{R}^{n-1}; \sum_{k \in I} t_k = c(I) \right\}.$$

The proof of the following proposition is given in [19, Appendix B].

**Proposition 1.2.** (Shnider-Sternberg) The polytope $K_n$ has nonempty interior in the $(n - 2)$-dimensional hyperplane $\{(t_1, \ldots, t_{n-1}) \in \mathbb{R}^{n-1}; \sum_{k=1}^{n-1} t_k = 3^{n-1}\}$. The intersection

$$K_n \cap \bigcap\{P_I, I \in \iota\}$$

defines a nonempty $(n - \#I - 2)$-dimensional face of $K_n$ for any $\iota \in \mathcal{I}(n)$. All faces of $K_n$ are obtained in this way.

If we denote, for $\iota \in \mathcal{I}(n)$, by $P_\iota$ the intersection $\bigcap\{P_I, I \in \iota\}$, then the above proposition immediately implies that the correspondence $\iota \mapsto K_n \cap P_\iota$ defines an isomorphism of the poset $\mathcal{I}(n)$ and the poset of faces of $K_n$. This is the promised convex realization of $K_n$. The case $n = 4$ is illustrated on Figure 2.

![Figure 2: $(t_1, t_3)$-projection of the convex realization of $K_4$.](image-url)
As observed in [19, Appendix B], the above construction works also for other choices of the function $c : P(n) \to \mathbb{R}_{>0}$ provided it is admissible in the sense that

$$c(I) + c(J) < c(I \cup J), \text{ if } I \cup J \text{ properly contains both } I \text{ and } J.$$ 

Let us proceed to the definition of the cyclohedron $W_n$. As the associahedron, it will be a convex polyhedron characterized by the poset $BC(n)$ indexing its faces. Consider again $n$ independent formal variables, labeled by natural numbers $1, \ldots, n$. The elements of $BC(n)$ will be equivalence classes represented by bracketing of the chain $\sigma(1), \ldots, \sigma(n)$, where $\sigma \in \Sigma_n$ is a cyclic permutation. In contrast to the case of $B(n)$, we allow also the bracketing which embraces all elements. Thus, for instance, $(3(12))$ represents an element of $BC(n)$.

The equivalence relation is given as follows. Let $\sigma \in \Sigma_n$ be a cyclic permutation, let $b'$ be a bracketing of $\sigma(1) \cdots \sigma(s)$ and $b''$ a bracketing of $\sigma(s + 1) \cdots \sigma(n)$, for some $1 \leq s \leq n$. Then we identify $b'b''$ to $b''b'$. Thus, for example, $3(12) = (12)3$ in $BC(3)$ (but $(3(12)) \neq ((12)3)$). The partial order on $BC(n)$ is defined, as for $B(n)$, by deleting pairs of brackets.

Each element $b$ of $BC(n)$ can be uniquely represented by a symbol, obtained from a representative of $b$ by forcing the indeterminates into the natural order. We call such symbols cyclic bracketings. The formal definition will be obvious from the following example.

**Example 1.3.** The poset $BC(2)$ contains three elements, $(12), (21)$ and $12$, where $(21)$ is represented by the cyclic bracketing $1)(2$. The poset structure is depicted by the interval, see Figure 3.

![Figure 3: W₂.](image-url)
Below are listed elements of the poset $BC(3)$:

| elements of $BC(3)$ | cyclic bracketings | cont. | cont. |
|----------------------|--------------------|-------|-------|
| (1(23))             | (1(23))            | (231) | 1)(23 |
| ((12)3)             | ((12)3)            | 2(31) = (31)2 | 1)2(3 |
| ((23)1)             | (1)((23)        | (312) | 12)(3 |
| (2(31))             | 1))((2(3       | (12)3 = 3(12) | (12)3 |
| ((31)2)             | 1)2)((3        | (123)  | (123) |
| (3(12))             | (12))((3       | 123 = 231 = 312 | 123 |
| 1(23) = (23)1       | (1(23))           |       |       |

The poset structure of $BC(3)$ is depicted on Figure 4.

The structure of $BC(4)$ is indicated on Figure 4. The picture is already rather complicated, so we labeled only the vertices (= the minimal elements of $BC(4)$). The label of an arbitrary face can be easily found – it is the least upper bound of all vertices of the face. For example, the pentagon on the top of $W_4$ is labeled by 123)(4, the front hexagon is labeled by (12)34, etc.

We construct, mimicking the approach of Shnider and Sternberg, a convex realization of the poset $BC(n)$. First some terminology. By a cyclic subinterval of $[1, n]$ we mean either a ‘normal’ subinterval $[i, j]$, $1 \leq i \leq j \leq n$, representing the subset $\{i, \ldots, j\}$ of $\{1, \ldots, n\}$, or the symbol $i][j$, $1 \leq i < j \leq n$, representing $\{1, \ldots, i\} \cup \{j, \ldots, n\}$. We will always suppose that the corresponding sets are proper subsets of $\{1, \ldots, n\}$, i.e. we exclude the intervals

Figure 4: $W_3$. 

Figure 5: $W_4$. 

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Let us denote by $PC(n)$ the set of all cyclic subintervals in the above sense. We denote by $IC(n)$ the set of all subsets of $PC(n)$ consisting of nested subintervals, meaning that, for $I, J \in \iota \in IC(n)$, either $I \subset J$ or $J \subset I$. Again, $IC(n)$ is a poset, the order being induced by the inclusion. We have the following analog of Lemma 1.1.

**Lemma 1.4.** The posets $BC(n)$ and $IC(n)$ are isomorphic.

**Proof.** Define, for $I \in PC(n)$, the cyclic bracketing $b(I) \in BC(n)$ by

$$b(I) = \begin{cases} 1 \cdots (i \cdots j + 1) \cdots n, & \text{for } I = [i, j], \; j < n, \\ 1 \cdots (i \cdots n, & \text{for } I = [i, n], \text{ and} \\ 1 \cdots i + 1) \cdots (j \cdots n, & \text{for } I = i][j]. 
\end{cases}$$

This correspondence induces the desired poset isomorphism. 

Our convex realization of $BC(n)$, whose possibility was predicted in [19, Appendix B]), is defined as follows. Let $W_n \subset \mathbb{R}^n$ be the convex polyhedron

$$W_n = \left\{ (t_1, \ldots, t_n) \in \mathbb{R}^n, \sum_{k=1}^{n} t_k = c([1, n]), \sum_{k \in I} t_k \geq c(I), \; I \in PC(n) \right\}.$$
For $I \in P(n)$, let $P_I$ be the hyperplane

$$P_I := \left\{ (t_1, \ldots, t_n) \in \mathbb{R}^n; \sum_{k \in I} t_k = c(I) \right\}.$$ 

The proof of the following proposition is a straightforward modification of the proof of Proposition 1.2 as given in [19, Appendix B]).

**Proposition 1.5.** The polytope $W_n$ has nonempty interior in the $(n-1)$-dimensional hyperplane \(\{(t_1, \ldots, t_n) \in \mathbb{R}^n; \sum_{k=1}^n t_k = c([1,n])\}\). The intersection

$$W_n \cap \bigcap\{P_I, I \in \iota\}$$

defines a nonempty $(n-\#I-1)$-dimensional face of $W_n$ for any $\iota \in \mathcal{IC}(n)$ and all faces of $W_n$ are obtained in this way.

For $\iota \in \mathcal{IC}(n)$, let $P_\iota$ be the intersection $\bigcap\{P_I, I \in \iota\}$. Then the correspondence $\iota \mapsto W_n \cap P_\iota$ defines an isomorphism between the poset $\mathcal{IC}(n)$ and the poset of faces of the polytope $W_n$. This is our convex realization of $W_n$. The convex realization of $W_3$ is shown on Figure 3.

**Observation 1.6.** The cyclohedron $W_n$ has $n(n-1)$ codimension-one faces, represented by the bracketings

$$b_{k,n} := (\sigma(1), \ldots, \sigma(k))\sigma(k+1), \ldots, \sigma(n), \ 1 < k \leq n,$$

where $\sigma \in \Sigma_n$ is a cyclic permutation. The face represented by the bracketing $b_{k,n}$ is isomorphic to the product $W_{n-k+1} \times K_k$. For example, $W_4$ depicted on Figure 5, has

- 4 hexagonal faces, corresponding to (12)34, (23)41, (34)12 and (41)23, isomorphic to $W_3 \times K_2 = W_3 \times \text{point}$,
- 4 square faces, corresponding to (123)4, (234)1, (341)2 and (412)3, isomorphic to $W_2 \times K_3$, and
- 4 pentagonal faces, corresponding to (1234), (2341), (3412) and (4123), isomorphic to $W_1 \times K_4 = \text{point} \times K_4$.

This was observed by J. Stasheff who realized that this is a strong motivation for the existence of a module structure which we will discuss in the following Section 2.
Observation 1.7. It is clear from our constructions that the cyclohedron $W_n$ is a truncation of the associahedron $K_{n+1}$, for $n \geq 1$. This is, of course, a trivial statement – any convex polyhedron is a truncation of an arbitrary other convex polyhedron of the same dimension, so we must be more precise.

For any $n \geq 1$ there is an obvious map $P(n+1) \hookrightarrow PC(n)$ which decomposes $PC(n)$ as

$$PC(n) = P(n+1) \sqcup E(n),$$

where $E(n)$ is the subset of ‘exotic’ cyclic intervals of the form $i][j$, $1 \leq i < j \leq n$. The polyhedron $W_n$ is then the truncation of $K_{n+1}$ by hyperplanes $P_I$ indexed by the ‘exotic’ intervals $I \in E(n)$. Compare Figures 2 and 3 for $n = 3$. We do not know whether this observation has any deeper meaning.

Observation 1.8. Choose, for each $t = 1, \ldots, n$, a point $P_t$ in the interior of the codimension one face of $W_n$ corresponding to $(t, \ldots, n, 1, \ldots, t-1)$. The convex hull of the set $\{P_1, \ldots, P_n\}$ is a simplex, closely related to the ‘deblowing up’ of $W_n$ described in Section 6. We will use this simplex to introduce an orientation of $W_n$.
2. Cyclohedron as a module over the associahedron.

We believe that there is no need to give a detailed definition of an operad. Recall only that an operad (in a symmetric monoidal category $\mathcal{C} = (\mathcal{C}, \times)$) is a sequence $\mathcal{P} = \{\mathcal{P}(n); n \geq 1\}$ of objects of $\mathcal{C}$ together with morphisms

$$\gamma = \gamma_{m_1, \ldots, m_l} : \mathcal{P}(l) \times \mathcal{P}(m_1) \times \cdots \times \mathcal{P}(m_l) \to \mathcal{P}(m_1 + \cdots + m_l),$$

given for any $l, m_1, \ldots, m_l \geq 1$, satisfying the usual axioms [16, Definition 3.12]. If not stated otherwise, we assume our operads to be symmetric, i.e. we assume that each $P(n)$ has a right action of the symmetric group $\Sigma_n$, $n \geq 2$, which has again to satisfy some axioms [16, Definition 1.1]. We frequently write $p(p_1, \ldots, p_l)$ instead of $\gamma(p, p_1, \ldots, p_l)$.

One comment concerning the action of the symmetric group is in order here. Our convention is determined by the conventional choice of the multiplication in the symmetric group. We accepted the standard one with $\sigma \cdot \tau$ meaning $\sigma(\tau)$, i.e. the permutation (= a map) $\tau$ followed by $\sigma$. Then $\mathcal{P}(n)$ must be a right $\Sigma_n$-module, which is the convention used in the original definition of $P$. May quoted above.

Recall that, for any object $V \in \mathcal{C}$, there exist the so-called endomorphism operad $\mathcal{E}_V = \{\mathcal{E}_V(n)\}_{n \geq 1}$ with $\mathcal{E}_V(n) := \text{Hom}(V^{\times n}, V)$. If $\mathcal{P}$ is an operad in $\mathcal{C}$, then a $\mathcal{P}$-algebra structure on $V$ is an operad map $a : \mathcal{P} \to \mathcal{E}_V$.

Example 2.1. The collection $\mathcal{B} = \{\mathcal{B}(n)\}_{n \geq 1}$ introduced in Section 1 has a structure of a (nonsymmetric) operad in the category of posets. The composition $\gamma(b; b_1, \ldots, b_l)$ is, for $b \in \mathcal{B}(l)$ and $b_i \in \mathcal{B}(m_i)$, defined as the bracketing $b(b_1, \ldots, b_l)$ obtained by inserting $b_i$ at the $i$-th position in $b$, $1 \leq i \leq l$. We believe that it is clear what we mean by this. For example

$$\gamma_{\mathcal{B}}(12; 1(23), 12) = 1(23)45, \quad \gamma_{\mathcal{B}}((12)3; (12)3, 12, (12)(24)) = ((12)345)(67)(89),$$

everything.

A classical result of J. Stasheff [18, page 278] says that the collection of the associahedra $K = \{K_n\}_{n \geq 1}$ has a cellular (nonsymmetric) operad structure which induces on the collection $\mathcal{B} = \{\mathcal{B}(n)\}_{n \geq 1}$ of its faces the operad structure of Example 2.1. As a consequence, cellular chains on $K$ form an operad $\mathcal{A} = \{\mathcal{A}(n)\}_{n \geq 1}$, $\mathcal{A}(n) = CC_*(K_n)$, in the category of differential graded vector spaces. Algebras over the operad $\mathcal{A}$ are $A(\infty)$-algebras [18, page 294].
Probably the most effective way to describe the operad $\mathcal{A}$ is to say that $\mathcal{A} = \Omega((s\text{Ass})^*)$, the operadic cobar construction on the dual cooperad $(s\text{Ass})^*$, where $s\text{Ass}$ is the suspension of the operad for associative algebras. This is the same, since the operad $\text{Ass}$ is Koszul, as to say that $\mathcal{A}$ is the minimal model of the associative operad $\text{Ass}$. All this is explained in [13]. We are going to make a similar analysis for the cyclohedron.

**Definition 2.2.** A (right) module over an operad $\mathcal{P}$ is a collection $M = \{M(n)\}_{n \geq 1}$ such that each $M(n)$ is, for $n \geq 1$, a $\Sigma_n$-module, together with morphisms

$$\nu = \nu_{m_1, \ldots, m_l} : M(l) \times \mathcal{P}(m_1) \times \cdots \times \mathcal{P}(m_l) \to M(m_1 + \cdots + m_l),$$

given for any $l, m_1, \ldots, m_l \geq 1$. The structure maps $\nu_{m_1, \ldots, m_l}$ must satisfy the axioms which are obtained by replacing, in the May’s definition of an operad [16, Definition 3.12], the first occurrence of $\mathcal{P}$ by $M$, see [13, Definition 1.3] for details.

**Remark 2.3.** Observe the resemblance of the above definition to the definition of an operad. This is due to the fact that right modules over an operad are special cases of general modules, which are abelian groups object in a certain comma category of operads, see the discussion in [13, page 1476].

**Remark 2.4.** The structure map $\nu$ is, as in the case of operads, determined by the system of ‘comp’ maps

$$\circ_i : M(m) \otimes \mathcal{P}(n) \to M(m + n - 1), \quad m, n \geq 1, \quad 1 \leq i \leq m,$$

defined by $\circ_i(x, p) := \nu(x; 1, \ldots, 1, p, 1, \ldots, 1)$ ($p$ at the $i$-th position) which have to satisfy certain axioms [13, Formula (1)].

**Example 2.5.** An operad $\mathcal{P}$ is a module over itself. Very important nontrivial examples are provided by the Axelrod-Singer compactification of configuration spaces of points in a manifold. The result is a module over the operad of ‘local’ configurations, see [11].

There is the following analog of the endomorphism operad. Let $A, W$ be objects of the category $\mathcal{C}$. Then the collection $\mathcal{E}_{A,W} = \{\mathcal{E}_{A,W}(n)\}_{n \geq 1}$ with $\mathcal{E}_{A,W}(n) := \text{Hom}(A^{\times n}, W)$ is a module over the endomorphism operad $\mathcal{E}_A$, the module structure being given by the obvious composition of maps, as in the case of the endomorphism operad. A $\mathcal{P}$-algebra structure $a : \mathcal{P} \to \mathcal{E}_A$ on $A$ induces a $\mathcal{P}$-module structure on $\mathcal{E}_{A,W}$. 
We are going now to introduce objects described, in the similar sense as algebras are described by operads, by modules over operads. We will call them, from the reasons which will be explained in Example 2.10, traces over algebras.

**Definition 2.6.** Let $M$ be a $P$-module and let $A$ be a $P$-algebra. An $M$-trace over $A$ is a map $t : M \rightarrow E_{A,W}$ of $P$-modules, where $E_{A,W}$ has the $P$-module structure induced from the $P$-algebra structure on $A$.

**Example 2.7.** Rather dull examples of traces are given by taking $M = P$. For example, an $Ass$-trace over an associative algebra is (given by) a bilinear map $B : A \times A \rightarrow W$ such that $B(ab, c) = B(a, bc)$, $a, b, c \in A$, i.e. by an (not necessary symmetric) invariant bilinear form.

We will need the following notation. Let, for permutations $\sigma \in \Sigma_l$ and $\sigma_i \in \Sigma_{m_i}$, $1 \leq i \leq l$,

$$
(4) \quad \sigma(\sigma_1, \ldots, \sigma_l) \in \Sigma_{m_1 + \cdots + m_l}
$$

denote the permutation $\sigma(m_1, \ldots, m_l) \cdot (\sigma_1 \oplus \cdots \oplus \sigma_l)$, where the meaning of $\sigma_1 \oplus \cdots \oplus \sigma_l$ is clear and $\sigma(m_1, \ldots, m_l)$ permutes the blocks of $m_1, \ldots, m_l$-elements via $\sigma$. This defines a map $\Sigma_l \times \Sigma_{m_1} \times \cdots \times \Sigma_{m_l} \rightarrow \Sigma_{m_1 + \cdots + m_l}$.

**Example 2.8.** More interesting example of a trace can be constructed as follows. Take again the (symmetric) operad $Ass$ for associative algebras. Recall that $Ass(n) = k[\Sigma_n]$, the group ring of the symmetric group over the ground field $k$. The operad structure map $\gamma = \gamma_{Ass}$ is defined by $\gamma(\sigma; \sigma_1, \ldots, \sigma_l) = \sigma(\sigma_1, \ldots, \sigma_l)$, where $\sigma(\sigma_1, \ldots, \sigma_l)$ has the same meaning as in (4).

The group of cyclic permutations $Z_n = Z/nZ$ acts from the left on $\Sigma_n$. The group $\Sigma_{n-1}$ is imbedded in $\Sigma_n$ as permutations leaving 1 fixed. This embedding is a cross-section to the $Z_n$-action, thus we can identify $\Sigma_{n-1}$ as a set to the coset space $Z_n \backslash \Sigma_n$. The projection $\pi_n : \Sigma_n \rightarrow Z_n \backslash \Sigma_n \cong \Sigma_{n-1}$ then induce on $k[\Sigma_{n-1}]$ a structure of a right $\Sigma_n$-module. Define the collection $Cycl = \{Cycl(n)\}_{n \geq 1}$ by $Cycl(n) := k[\Sigma_{n-1}]$, $n \geq 1$. The system of maps $\{\pi_n : \Sigma_n \rightarrow \Sigma_{n-1}\}_{n \geq 1}$ is the projection $\pi : Ass \rightarrow Cycl$ of collections.

**Lemma 2.9.** The projection $\pi : Ass \rightarrow Cycl$ induces on Cycl the structure of a module over the operad $Ass$. 
Proof. The structure maps $\nu = \nu_{Cycl}$ are determined, for $\sigma \in \Sigma_l$ and $\sigma_i \in \Sigma_{m_i}$, $1 \leq i \leq l$, by

$$\nu(\pi(\sigma); \sigma_1, \ldots, \sigma_l) := \gamma_{Ass}(\sigma; \sigma_1, \ldots, \sigma_l),$$

where $\gamma_{Ass}(\sigma; \sigma_1, \ldots, \sigma_l) = \sigma(\sigma_1, \ldots, \sigma_l)$. The proof is then finished by an easy verification that, if $\sigma' \equiv \sigma'' \mod \mathbb{Z}_l$, then

$$\sigma'(\sigma_1, \ldots, \sigma_l) \equiv \sigma''(\sigma_1, \ldots, \sigma_l) \mod \mathbb{Z}_{m_1 + \cdots + m_l},$$

which we leave to the reader. \qed

Example 2.10. A $Cycl$-trace over an associative algebra $A$ is (characterized by) a map $T : A \to W$ such that

$$T(ab) = (-1)^{|a||b|}T(ba), \quad a, b \in A,$$

i.e. $T$ is a trace in the usual sense. We postpone the verification of this statement to Example 4.3.

In the rest of this section we show that the collection $W := \{W_n\}_{n \geq 1}$ of the cyclohedra is a natural cellular (right) module over the cellular operad $K = \{K_n\}_{n \geq 1}$ and describe $W$-traces on an $A(\infty)$- ($= K$)-algebra.

It is convenient to consider harmless symmetrizations. Recall that $\mathcal{B}(n)$ was the poset of all bracketings of $1, \ldots, n$. Take instead be the poset $\overline{\mathcal{B}}(n)$ of all bracketings of $\sigma(1), \ldots, \sigma(n)$, $\sigma \in \Sigma_n$. Obviously $\overline{\mathcal{B}}(n) = \Sigma_n \times \mathcal{B}(n)$ and $\overline{\mathcal{B}}(n)$ is the poset of faces of the $n!$-connected polyhedron $\overline{K}_n := \Sigma_n \times K_n$. The collection $\overline{K} := \{\overline{K}_n\}_{n \geq 1}$ is a (symmetric) cellular operad and the corresponding operad of cellular chains $\overline{A} := CC_\ast(\overline{K})$ is the (symmetric) operad for $A(\infty)$-algebras.

There is a similar symmetrization of the cyclohedron. We introduced $\mathcal{B}C(n)$ as the poset of equivalence classes of bracketings of $\sigma(1), \ldots, \sigma(n)$ with a cyclic permutation $\sigma \in \Sigma_n$. If we admit all permutations, we obtain the poset $\overline{\mathcal{B}C}(n) = \Sigma_{n-1} \times \mathcal{B}C(n)$ whose realization is the $(n - 1)!$-connected polyhedron $\overline{W}_n := \Sigma_{n-1} \times W_n$.

Lemma 2.11. The collection $\overline{\mathcal{B}C} := \{\overline{\mathcal{B}C}(n)\}_{n \geq 1}$ is a natural module over the operad $\overline{\mathcal{B}} := \{\overline{\mathcal{B}}(n)\}_{n \geq 1}$ in the symmetric monoidal category of posets.
The easiest way to define the module structure is the following. Let \( b \) be a bracketing of \( 1, \ldots, l \) representing an element \( [b] \in BC(l) \subset B(l) \). Let \( b_i \in B(m_i) \subset B(m) \) be, for \( 1 \leq i \leq l \), a bracketing of \( 1, \ldots, m_i \). Then we define \( \nu_{BC}(b; b_1, \ldots, b_l) \in BC(m) \), \( m = m_1 + \cdots m_l \), to be the element represented by the composite (in the same sense as in Example 2.1) bracketing \( b(b_1, \ldots, b_l) \) of \( m \).

The set \( BC(l) \) is \( \Sigma_l \)-generated by elements of the same form as \( b \), i.e. by elements represented by a bracketing of the ‘unpermuted’ string \( 1, \ldots, l \), and the same is true also for \( B(m_i) \), \( 1 \leq i \leq l \). Thus the formula for the composition of arbitrary elements is dictated by the equivariance of the module composition map. We leave the verification that this definition is correct to the reader.

\[ \text{Proof.} \] The proof is a modification of the proof of the existence of an operad structure on the associahedron, given by J. Stasheff in [18, page 278]. By Remark 2.4, the \( K \)-module structure on \( W \) is given by the ‘comp’ maps

\[ \circ_i : W_m \times K_n \rightarrow W_{m+n-1}, \quad m, n \geq 1, \quad 1 \leq i \leq m. \]

As a matter of fact, in our case it is enough to specify

\[ \circ_1 : W_m \times K_n \rightarrow W_{m+n-1}, \]

the remaining ‘comp’ maps are determined by the equivariance. We define \( \circ_1 \) of (6) to be the identification of the product \( W_m \times K_n \) to the face of \( W_{m+n-1} \) indexed by the bracketing \( b_{n,n+m-1} \) of (1). The second part is immediate.

\[ \text{Observation 2.13.} \] The \( K \)-module structure on the ‘symmetrized’ cyclohedron \( W \) restricts to the right action of the nonsymmetric operad \( K \) on the ‘nonsymmetric’ cyclohedron \( W \). This is the structure observed by J. Stasheff in [19, Section 4]. Another argument for the existence of the module structure of Theorem 2.12 is the interpretation of the cyclohedron to the compactification of a configuration space, see Section 6.
Let us consider the $\mathfrak{A}$-module $M := CC_*(W)$ of cellular chains on the cyclohedron. To describe traces over $M$, it is convenient to accept the following notation. For graded indeterminates $a_1, \ldots, a_n$ and a permutation $\sigma \in \Sigma_n$, the Koszul sign $\epsilon(\sigma) = \epsilon(\sigma; a_1, \ldots, a_n)$ is defined by

$$a_1 \wedge \ldots \wedge a_n = \epsilon(\sigma; a_1, \ldots, a_n) \cdot a_{\sigma(1)} \wedge \ldots \wedge a_{\sigma(n)},$$

which has to be satisfied in the free graded commutative algebra $\wedge (a_1, \ldots, a_n)$. Denote also

$$\chi(\sigma) = \chi(\sigma; a_1, \ldots, a_n) := \text{sgn}(\sigma) \cdot \epsilon(\sigma; a_1, \ldots, a_n).$$

For an expression $X(a_1, \ldots, a_n)$ in indeterminates $a_1, \ldots, a_n$, let the cyclic sum

$$(7) \quad \sum X(a_1, \ldots, a_n) := \sum_{\sigma} \chi(\sigma)X(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$$

be the summation over all cyclic permutations. A convincing example of the use of this convention is the (graded) Jacobi identity written as

$$\sum[a_1, [a_2, a_3]] = 0.$$

The following proposition, whose proof we postpone after Theorem 5.5, describes $M$-traces over $A(\infty)$-algebras.

**Proposition 2.14.** Let $A = (A; m_1 = \partial, m_2, m_3, \ldots)$ be an $A(\infty)$-algebra ([12], page 294], but we use the sign convention of [10, §1.4]). Then an $M$-trace is given by a differential graded vector space $W = (W, \delta)$, $\deg \delta = -1$, and a system $T_n : A^{\otimes n} \rightarrow W$ of degree-$(n - 1)$ linear maps, $n \geq 1$, such that, for all $a_1, \ldots, a_n \in A$,

(i) $T_n(a_1, \ldots, a_n) = \chi(\sigma) \cdot T_n(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$ for all cyclic permutations $\sigma \in \Sigma_n$, and

(ii) for all $n \geq 1$,

$$(8) \quad \delta T_n(a_1, \ldots, a_n) = \sum_{1 \leq k \leq n} (-1)^{k+n} \cdot T_{n-k+1}(m_k(a_1, \ldots, a_k), a_{k+1}, \ldots, a_n).$$

We call such objects homotopy traces over an $A(\infty)$-algebra $A$.

Let us write down the axiom (8) explicitly for small $n$. For $n = 1$ it gives

$$\partial T_1(a) = T_1(\delta(a)), \ a \in A,$$
which means that $T_1$ is a homomorphism of differential graded spaces $(A, \partial)$ and $(W, \delta)$. For $n = 2$ it becomes

$$\delta T_2(a, b) + T_2(\partial a, b) - (-1)^{|a||b|}T_2(\partial b, a) = T_1(m_2(a, b)) - (-1)^{|a||b|}T_1(m_2(b, a)), \quad a, b \in A,$$

i.e. $T_1$ is a trace for the ‘multiplication’ $m_2$ up to a homotopy $T_2$. For higher $n$’s, the axiom (8) represents ‘coherence conditions’ for the homotopy $T_2$. An important special case is when $A$ is an ordinary associative algebra, that is the only nontrivial structure map is $m_2$, which is an associative multiplication ·. The axioms for the corresponding trace are obtained by putting, in Proposition 2.14, $m_k = 0$ for $k \geq 3$. We also substitute $(-1)^{n(n-1)/2}T_n$ for $T_n$, to get rid of the overall sign $(-1)^n$. A homotopy trace is then a system

$$\{ T_n : A^\otimes n \to W \}_{n \geq 1}$$

of degree-$(n-1)$ linear maps, satisfying 2.14(i) and

$$\begin{align*}
\delta T_1(a) &= 0 \\
\delta T_2(a, b) &= T_1(a \cdot b) - (-1)^{|a||b|}T_1(b \cdot a) \\
\delta T_3(a, b, c) &= T_2(a \cdot b, c) + (-1)^{|a|-(|b|+|c|)}T_2(b \cdot c, a) + (-1)^{|c|-(|a|+|b|)}T_2(c \cdot a, b) \\
&\vdots \\
\delta T_n(a_1, \ldots, a_n) &= \sum T_{n-1}(a_1 \cdot a_2, a_3, \ldots, a_n), \quad n \geq 4.
\end{align*}$$

Equation (9) describes a trace over a certain $Ass$-module $\overline{D}$, closely related to the simplex. The following Section 3 is devoted to the study of this module.

3. Simplex as a module over the operad for associative algebras.

Let $\Delta_n$ be the standard $(n-1)$-dimensional simplex (observe that the conventional notation for our $\Delta_n$ is $\Delta^{n-1}$). An explicit description of $\Delta_n$ is the following. Denote, for $1 \leq i \leq n$, by $e_i$ the point $(0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^n$ (1 at the $i$-th position). Then $\Delta_n \subset \mathbb{R}^n$ is the convex hull of the set \{e_1, \ldots, e_n\}. Figure 7 shows $\Delta_n$ for $n = 3$. There is a classical correspondence between the poset of subsets of \{1, \ldots, n\} and the poset of faces of $\Delta_n$ given by

subset $S$ of \{1, \ldots, n\} $\longleftrightarrow$ convex hull of the set \{e_i\}_{i \in S} $\subset$ \mathbb{R}^n.

See [20, §10.1] for details. Let $\Delta := \{ \Delta_n \}_{n \geq 1}$. In fact, it is more convenient to consider the symmetrized version $\Xi := \{ \overline{\Delta}_n \}_{n \geq 1}$, where $\overline{\Delta}_n$ is the disjoint union of $(n-1)!$ copies of $\Delta_n$,
indexed by cyclic orders of its vertices. This means that the poset of faces of \( \Delta_n \) consists of elements of the form
\[
\{i_1, \ldots, i_l\} \times [\sigma],
\]
where \( \{i_1, \ldots, i_l\} \) is a subset of \( \{1, \ldots, n\} \), \( 1 \leq l \leq n \), and \([\sigma]\) is an equivalence class from the left coset \( \mathbb{Z}_n \backslash \Sigma_n \). We define the right action of \( \Sigma_n \) by
\[
(\{i_1, \ldots, i_l\} \times [\sigma]) \cdot \rho := \{\rho^{-1}(i_1), \ldots, \rho^{-1}(i_l)\} \times [\sigma \rho].
\]

Let \( e = \{i_1, \ldots, i_l\} \times [\sigma] \) be a face (= cell) of \( \Delta(n) \) as in (10). An orientation of \( e \) is given by choosing an order of elements of \( \{i_1, \ldots, i_l\} \). Two such orders induce the same orientation if and only if they differ by a permutation of signature +1. Thus the cellular cell complex \( \mathcal{D}(n) := CC_*(\Delta(n)) \) is a vector space with the basis
\[
\langle i_1, \ldots, i_l \rangle \times [\sigma]
\]
where \( \langle i_1, \ldots, i_l \rangle \) denotes the cell \( \{i_1, \ldots, i_l\} \times [\sigma] \) with the orientation induced by the order \( i_1 < \cdots < i_l \). The right action of \( \Sigma_n \) is given by
\[
(\langle i_1, \ldots, i_l \rangle \times [\sigma]) \cdot \rho := (\langle \rho^{-1}(i_1), \ldots, \rho^{-1}(i_l) \rangle) \times [\sigma \rho].
\]

**Theorem 3.1.** The collection \( \overline{\Delta} := \{\Delta_n\}_{n \geq 1} \) of cell complexxes has a natural structure of a (right) module over the operad Ass for associative algebras. The traces over the cellular chain complex \( \overline{D} := CC_*(\overline{\Delta}) \) are the objects described by (3).
**Proof.** We observed in Remark 2.4 that the action is determined by a system of ‘comp’ maps

\[ \circ_i : \bigtriangleup_n \otimes \text{Ass}(m) \to \bigtriangleup_{m+n-1}, \quad n, m \geq 1. \]

Since \( \bigtriangleup_n \) is \( \Sigma_n \)-generated by \( \Delta_n = \Delta_n \times [\mathbb{I}_n] \subset \bigtriangleup_n \) (the copy corresponding to the ‘normal’ cyclic order \( (1, \ldots, n) \)) and \( \text{Ass}(m) = k[\Sigma_m] \) is \( \Sigma_m \)-generated by the identity permutation \( \mathbb{I}_m \in \Sigma_m \), it is enough to specify \( \circ_i(t, x) \) for \( t \in \Delta_n \) and \( x = \mathbb{I}_m \). We define \( \circ_i(-, \mathbb{I}_m) : \Delta_n \to \Delta_{m+n-1} \) to be the unique simplicial map such that

\[ \circ_i(\{j\} \times [\mathbb{I}_n], [\mathbb{I}_m]) := \begin{cases} \{j\}, & \text{for } 1 \leq j \leq i, \text{ and} \\ \{j \pm 1\}, & \text{for } i < j \leq n. \end{cases} \]

In other words, \( \circ_i(-, \mathbb{I}_m) \) is the canonical inclusion \( \Delta_n \hookrightarrow \Delta_{m+n-1} \), identifying \( \Delta_n \) to the \((n-1)\)-dimensional face of \( \Delta_m \) corresponding to the subset \( \{1, \ldots, i, i + m, \ldots, n + m - 1\} \).

The induced map of the cellular chain complex satisfies

\[ \circ_i(\langle 1, \ldots, n \rangle \times [\mathbb{I}_n], [\mathbb{I}_m]) = \langle 1, \ldots, i, i + m, \ldots, n + m - 1 \rangle \times [\mathbb{I}_{m+n-1}]. \] (12)

It is a straightforward verification to prove that this really defines an Ass-action. The second part will be proved after we formulate Theorem 3.5.

We are going to give an algebraic characterization of the Ass-module \( \mathcal{D} \). To do this, we need some more or less standard notions, which we will also find useful later. From now on, if not stated otherwise, the underlying symmetric monoidal category will be the category of (differential) graded vector spaces.

For any collection \( E = \{E(n)\}_{n \geq 1} \) there exists the free operad \( \mathcal{F}(E) \) on \( E \) [8, page 226]. The operad \( \mathcal{F}(E) \) has the following very explicit description in terms of trees. Denote by \( \mathcal{T} \) the set of (labeled rooted) trees and by \( \mathcal{T}_n \) the subset of \( \mathcal{T} \) consisting of trees having \( n \) input edges. Let \( E(\mathcal{T}) \) denote, for \( \mathcal{T} \in \mathcal{T} \), the set of ‘multilinear’ colorings of the vertices of \( \mathcal{T} \) by the elements of \( E \) such that a vertex with \( k \) input edges is colored by an element of \( E(k) \). The free operad \( \mathcal{F}(E) \) on \( E \) may be then described as

\[ \mathcal{F}(E)(n) := \bigoplus_{\mathcal{T} \in \mathcal{T}_n} E(\mathcal{T}) \] (13)

with the operad structure on \( \mathcal{F}(E) \) given by the operation of ‘grafting’ trees. We will in fact always assume that \( E(1) = 0 \), thus we consider in (13) only trees whose all vertices are at least binary, i.e. they have at least two incoming edges. The details may be found in [8, 3].
Example 3.2. The set $T_2$ has only one element (Figure 8) and $F(E)(2) = E(2)$. The set $T_3$ has four elements (see again Figure 8) and $F(E)(3)$ consists of three copies of $E(2) \otimes E(2)$ which corresponds to the three binary trees in $T_3$ and one copy of $E(3)$ corresponding to the corolla (= the tree with one vertex). Compare also [8, Figure 7].

In the same manner, for each operad $P$ and for each collection $X = \{X(n)\}_{n \geq 1}$ there exists the free (right) $P$-module generated by the collection $X$, which we denote $X \circ P$. An explicit description is [12, page 312]

\begin{equation}
(X \circ P)(m) = \bigoplus \left( \text{Ind}^{\Sigma_m}_{\Sigma_{m_1} \times \cdots \times \Sigma_{m_l}} (X(l) \otimes P(m_1) \otimes \cdots \otimes P(m_l)) \right)_{\Sigma_l},
\end{equation}

where the summation is taken over all $m_1 + \cdots + m_l = n$, $l \geq 1$. On the right-hand side, $\text{Ind}^{\Sigma_m}_{\Sigma_{m_1} \times \cdots \times \Sigma_{m_l}}$ denotes the induced representation and $(-)_{\Sigma_l}$ the quotient under the obvious action of $\Sigma_l$. The term $X(l) \otimes P(m_1) \otimes \cdots \otimes P(m_l)$ on the right-hand side of (14) can be interpreted as colorings of the tree $T_{m_1, \ldots, m_l}$ from Figure 8 such that the output vertex is colored by an element of $X(l)$ and the remaining vertices by elements of $P$.

Example 3.3. We have $(X \circ P)(1) = X(1)$, corresponding to the tree $T_1$ on Figure 10. The vector space $(X \circ P)(2)$ consists of a copy of $X(2)$ corresponding to $T_{1,1}$ and a copy of $X(1) \otimes P(2)$ corresponding to $T_2$, see again Figure 10. Note that we still assume that $P(1) = k$. 
For a graded vector space $V = \bigoplus_p V_p$ let $\uparrow V$ (resp. $\downarrow V$) be the suspension (resp. the desuspension) of $V$, i.e. the graded vector space defined by $(\uparrow V)_p := V_{p-1}$ (resp. $(\downarrow V)_p := V_{p+1}$). We have the obvious natural maps $\uparrow: V \to \uparrow V$ and $\downarrow: V \to \downarrow V$. For a collection $E$, the suspension $sE$ is the collection with

$$ (sE)(n) := \text{sgn} \otimes \uparrow^{n-1} E(n), $$

$n \geq 1$, where $\uparrow^{n-1}$ is the $(n - 1)$-fold suspension introduced above and sgn is the signum representation of $\Sigma_n$ on $k$. The reason why we need the signum factor is that we intend to apply the suspension to operads and modules over operads. Without this factor, the composition induced on the suspension will not be equivariant, compare also \[6, page 8\]. There is an obvious similar notion of the desuspension $s^{-1}E$ of the collection $E$.

Let us come back to the promised algebraic characterization of the $\text{Ass}$-module $\overline{D}$. Consider the free $\text{Ass}$-module $s\text{Cycl} \circ \text{Ass}$ generated by the suspension of the collection $\text{Cycl}$ introduced in Example 2.8.

It will be useful to have an explicit description of the elements of $s\text{Cycl} \circ \text{Ass}$. Consider the free graded right $\Sigma_n$-module $H(n)$ generated by the trees $T_{m_1,\ldots,m_l}$ introduced in Figure 9, with $m_1 + \cdots + m_l = n$. The grading is given by $\deg(T_{m_1,\ldots,m_l}) := l - 1$ Thus $H(n)$ is, by definition, the graded vector space with the basis

$$ \{ T_{m_1,\ldots,m_l} \times \sigma, \sigma \in \Sigma_n, 1 \leq l \leq n, m_1 + \cdots + m_l = n \}, \deg(T_{m_1,\ldots,m_l} \times \sigma) = l - 1. $$

A neat graphical presentation of the symbol $T_{m_1,\ldots,m_l} \times \sigma$ is the tree $T_{m_1,\ldots,m_l}$ with the inputs labeled by $\sigma^{-1}(1),\ldots,\sigma^{-1}(n)$. For example, $T_{2,1} \times (123)$ can be depicted as
Define now the left action of the group $\mathbb{Z}_l$, considered as the group of cyclic permutations of order $l$, on $H(n)$ as follows. Recall that, for $\zeta \in \mathbb{Z}_l \subset \Sigma_l$, we denoted by $\zeta(m_1, \ldots, m_l) \in \Sigma_n$ the permutation which permutes the blocks of $m_1, \ldots, m_l$-elements via $\zeta$. Then we put
\[
\zeta(T_{m_1, \ldots, m_l} \times \sigma) := \text{sgn}(\zeta) \cdot T_{m_{\zeta^{-1}(1)}, \ldots, m_{\zeta^{-1}(l)}} \times \zeta(m_1, \ldots, m_l) \cdot \sigma.
\]
(16)

The following lemma is an easy consequence of definitions and formula (14).

**Lemma 3.4.** The graded vector space $(s\text{Cycl } \circ \text{Ass})(n)$ can be identified with the graded vector space with the basis given by equivalence classes of symbols
\[
T_{m_1, \ldots, m_l} \times \sigma \in H(n), \quad \text{deg}(T_{m_1, \ldots, m_l}) = l - 1,
\]
modulo the left action of the group $\mathbb{Z}_l$ defined in (16). Under this identification, the right action of $\Sigma_n$ on the equivalence class $[T_{m_1, \ldots, m_l} \times \sigma]$ is described as
\[
[T_{m_1, \ldots, m_l} \times \sigma] \cdot \rho := \text{sgn}(\sigma) \cdot [T_{m_1, \ldots, m_l} \times \sigma \rho].
\]

Let $\xi_n \in \text{Cycl}(n)$ be the generator represented by the identical permutation $\mathbb{1}_n \in \Sigma_n$ and let $\alpha_2 = \mathbb{1}_2 \in \text{Ass}(2) = k[\Sigma_2]$. Define the differential $\partial$ on $s\text{Cycl } \circ \text{Ass}$ by
\[
\partial(\uparrow^{n-1}\xi_n) := \sum_{\sigma} \text{sgn}(\sigma) \cdot \nu(\uparrow^{n-2}\xi_{n-1}; \alpha_2, 1, \ldots, 1) \cdot \sigma
\]
\[
= -\mathbf{\sum} \nu(\uparrow^{n-2}\xi_{n-1}; \alpha_2, 1, \ldots, 1) \quad \text{/the cyclic sum notation of (7)/}
\]
(18)

Using the identification of Lemma 3.4, this could be also written as
\[
\partial(T_{1, \ldots, 1} \times [\mathbb{1}_n]) = -\mathbf{\sum} (T_{2, 1, \ldots, 1} \times [\mathbb{1}_n]),
\]
or, in a diagrammatic shorthand,
\[
\partial(\begin{array}{cccc} 1 & 2 & \cdots & n \\ \end{array}) = -\mathbf{\sum} \begin{array}{cccc} 1 & 2 & \cdots & n \\ \end{array}
\]

Since the elements $\{\uparrow^{n-1}\xi_n\}_{n \geq 1}$ generate $s\text{Cycl } \circ \text{Ass}$, formula (18) is enough to determine the differential $\partial$. We leave to the reader to verify that the definition is correct and that $\partial^2 = 0$. 

\[\]
Theorem 3.5. The Ass-module $\mathcal{D} = CC_*(\Delta)$ is isomorphic to the free differential Ass-module $(sCycl \circ \text{Ass}, \partial)$ constructed above.

Proof. Any differential Ass-module map $\omega : (sCycl \circ \text{Ass}, \partial) \to (\mathcal{D}, \partial)$ is determined by the values $\omega(\uparrow^{n-1}\xi_n)$, $n \geq 1$. We define $\omega(\uparrow^{n-1}\xi_n) := e_n$, where $e_n \in \mathcal{D}(n)$ is the top $(n-1)$-dimensional oriented cell $\langle 1, \ldots, n \rangle$. We shall verify that the above defined map $\omega$ commutes with the differentials,

$$\omega(\partial \uparrow^{n-1}\xi_n) = -\omega(\sum \nu(\uparrow^{n-2}\xi_{n-1}, \alpha_2, 1, \ldots, 1)) = \partial e_n.$$  

Because $\omega$ is a module homomorphism, the above equation can be rewritten as

$$\sum \nu(e_{n-1}; \alpha_2, 1, \ldots, 1) = -\partial e_n. \quad (19)$$

The standard formula for the boundary of $\langle 1, \ldots, n \rangle$ [20, §10.1] says that

$$\partial e_n = \partial \langle 1, \ldots, n \rangle = \sum_{1 \leq i \leq n} (-1)^{i+1} \cdot \langle 1, \ldots, i-1, i+1, \ldots, n \rangle \quad (20)$$

while the defining formula (12) for the Ass-module action on $\Delta$ gives

$$\nu(e_{n-1}; \alpha_2, 1, \ldots, 1) = \langle 1, 3, \ldots, n \rangle.$$  

Now it is enough to observe that

$$\sum \langle 1, 3, \ldots, n \rangle = -\sum_{1 \leq i \leq n} (-1)^{i+1} \cdot \langle 1, \ldots, i-1, i+1, \ldots, n \rangle$$

which, together with (20), gives (19).

It remains to prove that $\omega$ is an isomorphism. To this end, we give an explicit formula for the map $\omega$. Let $T_{m_1, \ldots, m_l} \times \sigma \in H(n)$ be as in Lemma 3.4. The numbers $m_1, \ldots, m_l$ determine a sequence $i_1, \ldots, i_l$ by $i_s := m_1 + \cdots + m_{s-1} + 1$, $1 \leq s \leq l$. Consider a map $\varphi : H(n) \to \mathcal{D}(n)$ defined by

$$\varphi(T_{m_1, \ldots, m_l} \times \sigma) := \langle \sigma^{-1}(i_1), \ldots, \sigma^{-1}(i_l) \rangle \times [\sigma],$$

where we denoted elements of $\mathcal{D}(n)$ (= cells of $\Delta(n)$) as in (11). It is immediate to see that $\varphi$ is an $\Sigma_n$-equivariant epimorphism. For the left action of $\zeta \in \mathbb{Z}_l$ we have

$$\varphi(\zeta(T_{m_1, \ldots, m_l} \times \sigma) = \text{sgn}(\zeta) \cdot \varphi(T_{m_{\zeta^{-1}(1)}, \ldots, m_{\zeta^{-1}(l)}} \times \zeta(m_1, \ldots, m_l) \sigma)$$

$$= \text{sgn}(\zeta) \cdot \langle \sigma^{-1}(i_{\zeta(1)}), \ldots, \sigma^{-1}(i_{\zeta(l)}) \rangle \times [\sigma] = \langle \sigma^{-1}(i_1), \ldots, \sigma^{-1}(i_l) \rangle \times [\sigma],$$
Figure 11: A representation of oriented faces of $\Delta_1$, $\Delta_2$ and $\Delta_3$ by equivalence classes of labeled planar trees, representing elements of $H(1)$, $H(2)$ and $H(3)$. 
which shows that $\varphi(x) = \varphi(\zeta y)$. On the other hand, a moment’s reflection show that $\varphi(x) = \varphi(y)$, for $x, y \in H(n)$, implies the existence of some $\zeta \in \mathbb{Z}_l$ such that $x = \zeta y$. Thus the map $\varphi$ induces an equivariant isomorphism $(sCycl \circ \text{Ass})(n) = \mathbb{Z}_l \backslash H(n) \cong \mathcal{D}(n)$, which is exactly our map $\omega$. The nature of the map $\omega$ is illustrated on Figure 11.

A $\mathcal{D}$-trace $t : \mathcal{D} \rightarrow \mathcal{E}_{A,V}$ is, under the identification of Theorem 3.5, given by the values $T_n := t(\uparrow^{n-1} \xi_n), n \geq 1$. The axiom (9) then reflects (19). The relation between the axiom (9) and the geometry of the simplex is visualized on Figure 12.

\[
\delta T_1(a) = 0: \quad \delta( a \bullet ) = 0
\]

\[
\delta T_2(a, b) = T_1(a \cdot b) - (-1)^{|a||b|}T_1(b \cdot a): \quad \delta( \bullet \bullet ) = \bullet - \bullet
\]

\[
\delta T_3(a, b, c) = T_2(a \cdot b, c) + (-1)^{|a|(|b|+|c|)}T_2(b \cdot c, a) + (-1)^{|c|(|a|+|b|)}T_2(c \cdot a, b):
\]

\[
\delta \left( \frac{(ca)b}{(ab)c} \right) = \frac{(ab)c}{(bc)a} \rightarrow \frac{(bc)a}{(ca)b}
\]

Figure 12: Relation of the axiom (9) to the geometry of the simplex.

4. Quadratic operads and modules; modules associated to cyclic operads.

Each operad $\mathcal{P}$ can be presented as a quotient $\mathcal{F}(E)/I$, for a collection $E$ and an ‘operadic’ ideal $I$. The operad $\mathcal{P}$ is quadratic [8, page 228] if it has a presentation such that the collection $E$ is concentrated in degree 2, $E = E(2)$, and the ideal $I$ is generated by a subspace $R \subset \mathcal{F}(E)(3)$. In this case we write $\mathcal{P} = \langle E; R \rangle$.

Example 4.1. Quadratic operads are omnipresent. Just recall that $\text{Ass} = \langle E; R \rangle$ for $E = E(2) = k[\Sigma_2]$, the regular representation of $\Sigma_2$. Choosing a generator $\mu \in E$, we can write $E = \text{Span}(\mu, \mu S_{21})$, where $S_{21} \in \Sigma_2$ is the transposition. Then $R \subset \mathcal{F}(E)(3)$ is the $\Sigma_3$-subspace generated by $\mu(1, \mu) - \mu(\mu, 1)$. If we think of $\mu$ as corresponding to a multiplication, then the generator of $R$ expresses the associativity. Sometimes we simplify the notation and write $\text{Ass} = \langle \mu; \mu(1, \mu) - \mu(\mu, 1) \rangle$. 

Similarly, for each \( P \)-module \( M \) there exists a collection \( X \) such that \( M = (X \circ P)/J \) for some right submodule \( J \subset X \circ P \). The following definition was introduced independently also in \([3]\).

**Definition 4.2.** The module \( M \) is called quadratic, if, in the above presentation, the collection \( X \) is concentrated in degree 1, \( X = X(1) \), and the right submodule \( J \) is generated by a subspace \( G \subset (X \circ P)(2) \). In this case we write \( M = \langle X; P; G \rangle \).

**Example 4.3.** Let \( X = X(1) \) be generated by one element \( g \); \( X(1) = \text{Span}(g) \). Let \( G \subset (X \circ \text{Ass})(2) \) be defined as \( G = \text{Span}(g(\mu)(1 - S_{21})) \), where \( \mu \) and \( S_{21} \in \Sigma_2 \) has the same meaning as in Example \([4.1]\). Then it is not difficult to see that \( \text{Cycl} = \langle X; \text{Ass}; G \rangle \) or, in a more explicit notation,

\[
\text{Cycl} = \langle g; \text{Ass}; g(\mu) - g(\mu)S_{21} \rangle.
\]

Now we can give the characterization of \( \text{Cycl} \) traces promised in Example \([2.10]\). Since the \( \text{Ass} \)-module \( \text{Cycl} \) is generated by \( g \in \text{Cycl}(1) \), any \( \text{Cycl} \)-trace \( t : \text{Cycl} \to \mathcal{E}_{A,W} \) is determined by the image \( T := t(g) \in \mathcal{E}_{A,V}(1) = \text{Hom}(A,W) \). The symmetry \([3]\) then follows from the condition \( t(g(\mu)(1 - S_{21})) = 0 \).

In fact, we show that \( \text{Cycl} \) is a very special case of a module over the \textit{cyclic} operad \( \text{Ass} \). Cyclic operads were introduced by E. Getzler and M.M. Kapranov \([7, \text{Section 2}]\). The definition we are going to recall is based on the following convention.

We interpret the symmetric group \( \Sigma_{n+1} \) as the group of permutations of the set \( \{0, \ldots, n\} \) and \( \Sigma_n \) as the subgroup of \( \Sigma_{n+1} \) consisting of permutations \( \sigma \in \Sigma_{n+1} \) with \( \sigma(0) = 0 \). If \( \tau_n \in \Sigma_{n+1} \) is the cycle \( (0, \ldots, n) \), then \( \tau_n \) and \( \Sigma_n \) generate \( \Sigma_{n+1} \).

**Definition 4.4.** Cyclic operad is an ordinary operad \( \mathcal{P} \) such that the right \( \Sigma_n \)-action on \( \mathcal{P}(n) \) extends, for \( n \geq 1 \), to an action of \( \Sigma_{n+1} \) such that

(i) \( \tau_1(1) = 1 \), where 1 \( \in \mathcal{P}(1) \) is the unit and

(ii) for \( p \in \mathcal{P}(m) \) and \( q \in \mathcal{P}(n) \),

\[
\gamma(p; q, 1, \ldots, 1) \cdot \tau_{m+n-1} = (-1)^{|p| \cdot |q|} \gamma(q \cdot \tau_n; 1, \ldots, 1, p \cdot \tau_m).
\]
Figure 13: A ‘visualization’ of the action of $\tau_n$. The element $\tau_n$ turns $p \in \mathcal{P}(n)$, represented as a ‘thing’ with $n$ inputs and one output, a bit so that the first input becomes the output and the output becomes the last input of $p \cdot \tau_n$.

An intuitive feeling for the action of $\tau_n$ is suggested by Figure 13.

Let $\mathcal{P} = \langle E; R \rangle$ be a quadratic operad. Thus $E(2)$ is a $\Sigma_2 = \mathbb{Z}_2$ space and the homomorphism $\text{sgn} : \Sigma_3 \to \mathbb{Z}_2$ equips $E(2)$ with a $\Sigma_3$-action which induces on $\mathcal{F}(E)$ a cyclic operad structure. We say, according to [7, §3.2], that $\mathcal{P}$ is cyclic quadratic, if the subspace $R \subset \mathcal{F}(E)(3)$ is $\Sigma_4$-invariant. In this case the operad $\mathcal{P}$ carries a natural cyclic structure induced from the cyclic structure of $\mathcal{F}(E)$.

An example of a cyclic quadratic operad is the operad $\text{Ass}$, see [7, Proposition 2.4] for a very explicit description of the cyclic structure, and also for other examples of cyclic operads.

Let $\mathcal{P} = (\mathcal{P}, \gamma)$ be a cyclic operad in the sense of Definition 4.4. Define the $\mathcal{P}$-module $M_\mathcal{P}$ as follows. As a collection, $M_\mathcal{P}(n+1) = \mathcal{P}(n)$, for $n \geq 0$. The structure maps are given by

\[
\nu(x; p_0, 1, \ldots, 1) := (-1)^{|p_0| - |x|} \gamma(p_0 \cdot \tau_{m_0}; 1, \ldots, 1, x), \text{ and }
\nu(x; 1, p_1, \ldots, p_n) := \gamma(x; p_1, \ldots, p_n),
\]

for $x \in M_\mathcal{P}(n+1) = \mathcal{P}(n)$, $p_i \in \mathcal{P}(m_i)$, $0 \leq i \leq n$. These conditions, along with the module axioms (Definition 2.2), imply that

\[
\nu(x; p_0, \ldots, p_n) = (-1)^{|p_0| - |x|} \gamma(p_0 \cdot \tau_{m_0}; 1, \ldots, 1, \nu(x; 1, p_1, \ldots, p_n)).
\]

The verification of module axioms of $M_\mathcal{P}$ is routine.

**Definition 4.5.** We call the module $M_\mathcal{P}$ the module associated to the cyclic operad $\mathcal{P}$.

We will also consider unital operads which describe algebras with unit. Unital operad is an operad such that $\mathcal{P}(0)$ is nonempty, generated by an element $\vartheta$, encoding the unit $k \to A$ in the corresponding algebra $A$. The element $\vartheta$ determines, for $n \geq 1$ and $1 \leq i \leq n$, the ‘degeneracy’ maps $s_i : \mathcal{P}(n) \to \mathcal{P}(n-1)$ by $s_i(p) := \gamma(p; 1, \ldots, 1, \vartheta, 1, \ldots, 1) \ (\vartheta$ at the $i$-th
position). These maps satisfy certain commutation relations \([18, \text{page 278, Proposition 3}]\) which follow from the axioms of an operad. For us, the most important is the relation 
\[ s_1(p) = s_2(pS_2) \]
for \( p \in \mathcal{P}(2) \), which follows from the equivariance of structure maps. This, together with a natural requirement that \( s_1 = s_2 \) on \( \mathcal{P}(2) \), gives that the maps \( s_1 = s_2 \) are \( \Sigma_2 \)-equivariant on \( \mathcal{P}(2) \). We denote both maps \( s_1 \) and \( s_2 \) (which are the same) by \( s \).

**Example 4.6.** There is the operad \( UAss \) for associative algebras with unit, which is the same as the operad \( Ass \) except that \( UAss(0) = \text{Span}(\vartheta) \). The map \( s : \text{Ass}(2) = k[\Sigma_2] \to k \) is the standard augmentation of the group ring \( k[\Sigma_2] \).

Another example is the operad \( UComm \) for unital commutative algebras. It has, for \( n \geq 1 \), \( UComm(n) = \text{Comm}(n) = \mathbb{1} \) (the trivial one-dimensional representation), and \( UComm(0) = \text{Span}(\vartheta) \). The map \( s : UComm(2) = k \to k \) is the identity.

A more complicated example is the operad \( UPoiss \) for Poisson algebras with unit. Here by a Poisson algebra with unit we mean an ordinary Poisson algebra \([12, \text{Example 3.3}]\) \( P = (P, \cdot, [-,-]) \) with a distinguished element \( 1 \in P \) which is a two-sided unit for the commutative multiplication \( \cdot \), while \( [x,1] = [1,x] = 0 \), for all \( x \in P \). The operad \( UPoiss \) coincides with the operad \( Poiss \) for Poisson algebras (which is very explicitly described in \([4]\)), except that \( UPoiss(0) = \text{Span}(\vartheta) \). The component \( UPoiss(2) \) is the direct sum \( \mathbb{1} \oplus \text{sgn} \), with the trivial one-dimensional representation \( \mathbb{1} \) concentrated in degree zero, and the signum representation \( \text{sgn} \) in degree 1. The map \( s : UPoiss(2) \to k = UPoiss(1) \) is the projection on the zero-dimensional component.

We saw that a natural way to construct cyclic operads was to take a quadratic operad \( \mathcal{P} = \langle E; R \rangle \) for which the relations \( R \) were invariant under the natural \( \Sigma_4 \)-action; the operad \( \mathcal{P} \) had then a natural cyclic structure. There is a similar approach to unital operads.

So, let \( \mathcal{P} = \langle E; R \rangle \) be a quadratic operad and suppose we are given an epimorphism \( s : E \to k \). This will be a model for the two degeneracy maps \( s_1 = s_2 : \mathcal{P}(2) = E \to \mathcal{P}(1) = k \). These two maps induce degeneracy maps on the free operad \( \mathcal{F}(E) \) satisfying the correct commutation relations. The following definition expresses the conditions assuring that this structure preserves the relations \( R \).

**Definition 4.7.** Let \( \mathcal{P} = \langle E; R \rangle \) be a quadratic (resp. cyclic quadratic) operad. Suppose that we are given an epimorphism \( s : E \to k \) such that

(i) \( s \) is invariant under the \( \Sigma_2 \) (resp. \( \Sigma_3 \), in the cyclic case) action, and
(ii) the induced maps \( s_1, s_2, s_3 : \mathcal{F}(E)(3) \to \mathcal{F}(E)(2) \) send the subspace \( R \subset \mathcal{F}(E)(3) \) to zero.

Then the collection \( UP \), defined by \( UP(n) := \mathcal{P}(n) \) for \( n \geq 1 \) and \( UP(0) = k \), has a natural structure of a unital operad. We call operads of this form quadratic unital (resp. cyclic quadratic unital) operads.

All the three operads \( UAss \), \( UComm \) and \( UPoiss \) from Example 4.6 are cyclic quadratic unital operads in the sense of Definition 4.7.

**Proposition 4.8.** Let \( P = \langle E; R \rangle \) be a cyclic unital quadratic operad in the sense of Definition 4.7. Then the associated module \( M_{UP} \) is quadratic,

\[
M_{UP} = \langle \text{Span}(\vartheta); \mathcal{P}; \text{Ker}(s) : E \to k \rangle.
\]

**Proof.** Let \( X = X(1) := \text{Span}(g) \) and consider the map \( \psi : X \circ \mathcal{P} \to M_{UP} \) defined by \( p(g) := \vartheta \in M_{UP}(1) = UP(0) \). Because clearly \( X \circ \mathcal{P} = \mathcal{P} \), \( \psi \) is, by (22), given as

\[
(X \circ \mathcal{P})(n) = \mathcal{P}(n) \ni p \mapsto \nu(\vartheta; p) = \gamma(p \cdot \tau_n; 1, \ldots, 1, \vartheta) \in M_{UP}(n) = UP(n-1).
\]

Thus \( \psi \) is a sequence \( \psi(n) : \mathcal{P}(n) \mapsto UP(n-1) \) given by \( \psi(n)(p) := \gamma(p \cdot \tau_n; 1, \ldots, 1, \vartheta) \). These maps are epimorphisms, because \( E \) generates \( \mathcal{P} \) and \( s : E \to k \) (= the composition with \( \vartheta \)) is epi, by assumption.

On the other hand, \( \psi(2) : \mathcal{P}(2) = E \to UP(1) = k \) is exactly the map \( s \), thus the submodule of \( X \circ \mathcal{P} \) generated by \( \text{Ker}(s) \) is certainly contained in the kernel of \( \psi \). A moment’s reflections shows that the whole kernel of \( \psi \) is generated by \( \text{Ker}(s) \).

**Example 4.9.** If \( \mathcal{P} = UAss \) is the operad for unital associative algebras from Example 4.6, then \( \text{Ker}(s : E = k[\Sigma_2] \to k) = \text{sgn.} \) So, by Proposition 4.8, \( M_{UAss} = \langle \vartheta; \text{Ass}; \text{sgn} \rangle \), thus \( M_{UAss} = \text{Cycl} \), by (21). For the operad \( UComm \) the kernel of \( s \) is trivial, hence

\[
M_{UComm} = \text{Span}(\vartheta) \circ \text{Comm} \cong \text{Comm}.
\]

In other words, \( M_{UComm} \) is the operad \( \text{Comm} \) considered as a right module over itself. We recommend to the reader to make the similar discussion for the operad \( UPoiss \).
5. Cyclohedron as the cobar construction.

For a graded vector space $V$, let $V^*$ denote the graded dual of $V$, i.e. the graded vector space $\bigoplus_p (V^*)_p$ with $(V^*)_p = \text{Hom}^{-p}(V,k) = \text{Hom}(V^*_p,k)$, the space of linear maps from $V_p$ to $k$. If $V$ is a right $\Sigma_n$-module, then we equip $V^*$ with the transposed $\Sigma_n$-action. We believe that there is no real risk of confusion of the $^*$ indicating the dual with the star indicating the degree.

Recall also [8, §3.5.1] that a cooperad is a collection $Q = \{Q(n)\}_{n \geq 1}$ together with a system of maps

$$\omega = \omega_{m_1,\ldots,m_l} : Q(m_1 + \cdots + m_l) \to Q(l) \otimes Q(m_1) \otimes \cdots \otimes Q(m_l),$$

which satisfy the axioms which are exactly the duals of the axioms for an operad. A typical example of a cooperad is the dual $P^*$ of an operad $P$, i.e. the collection $P^* = \{(P(n))^*\}_{n \geq 1}$ with the cooperad structure defined by the dualization of the structure maps of $P$; here an obvious finite type assumption is necessary, but it will be always satisfied in the paper and we will make no explicit comments about it.

Observe that if $P$ is an operad, then both the suspension $sP$ and the desuspension $s^{-1}P$ introduced in Section 3, with the sign convention of (15), have a natural operad structure induced from the operad structure on $P$.

The structure maps of a cooperad $Q$ determine (and are determined by) a map $\varpi : Q \to \mathcal{F}(Q)$ of collections. Composing this map with the (de)suspensions gives a degree -1 map $\downarrow Q \to \mathcal{F}(\downarrow Q)$ which uniquely (because of the freeness of $\mathcal{F}(\downarrow Q)$) extends to a degree -1 derivation $\partial_\Omega$ of the operad $\mathcal{F}(\downarrow Q)$ which satisfies, as a consequence of the axioms of a cooperad, $\partial_\Omega \circ \partial_\Omega = 0$. The differential graded operad

$$\Omega(Q) := (\mathcal{F}(\downarrow Q), \partial_\Omega)$$

is called the cobar construction on the cooperad $Q$ [8, §3.2.12].

Let $P = \langle E; R \rangle$ be a quadratic operad. Take the dual $E^*$ of $E$ and let $R^\perp$ be the annihilator of the space $R \subset \mathcal{F}(E)(3)$ in $\mathcal{F}(E^* \otimes \text{sgn})(3)$. The quadratic operad $P^! := \langle E^* \otimes \text{sgn}; R^\perp \rangle$ is, according to [8, §2.1.9], called the Koszul (or quadratic) dual of the operad $P$. We always have a map $\downarrow (sP)^* \to P^!$ of collections defined as the composition

$$\downarrow (sP)^* \xrightarrow{\text{proj}} (\downarrow (sP)(2))^* = E^* \otimes \text{sgn} = P^!(2) \hookrightarrow P^!$$
which extends, by the freeness of $\mathcal{F}(\downarrow (sP)^*)$, to a differential graded operad map

$$\pi : \Omega((sP)^*) = (\mathcal{F}(\downarrow (sP)^*), \partial_\Omega) \to (P\!, \partial = 0).$$

The quadratic operad $P$ is called Koszul [8, Definition 4.1.3] if the map in (23) is a homology isomorphism. In the rest of the paper, the space of generators $E = E(2)$ of a quadratic operad will always be ungraded, concentrated in degree zero. Then the components of both the operad $P$ and its dual $P\!$ are concentrated in degree zero as well, and the Koszulness implies that the complex $(\Omega((sP)^*)(n), \partial_\Omega)$ is acyclic in positive dimensions, for all $n$. On the other hand, as shown in [8, Theorem 4.1.13], this acyclicity condition implies the Koszulness of $P$.

**Example 5.1.** The operad Ass is well-known to be Koszul self-dual, $Ass = Ass\!$ [8, Theorem 2.1.11], and Koszul [8, Corollary 4.2.7]. We have already remarked that the operad $\mathcal{A}$ of cellular chains on the (symmetrized) associahedron $K$ coincides, as a differential graded operad, with the cobar construction $\Omega((sAss)^*)$. Let us give an explicit illustration of this statement.

The $n$-th piece $(\downarrow (sAss)^*)(n)$ of the collection $(\downarrow (sAss)^*)$ is isomorphic to one copy of the regular representation $k[\Sigma_n]$ concentrated in degree $(n - 2)$. The isomorphism is not unique, but we may choose, for example, $\lambda : k[\Sigma_n] \to (sAss)^*)(n)$ given by $\lambda(\sigma)(\uparrow^{n-2}\rho) := \text{sgn}(\sigma) \cdot \delta_{\sigma,\rho}$, where $\sigma, \rho \in \Sigma_n$ and the meaning of the ‘Kronecker delta’ $\delta_{\sigma,\rho}$ is clear.

Formula (13) then describes $\mathcal{F}(\downarrow (sAss)^*)(n)$ as the vector space spanned by the set of all planar (rooted, labeled) $n$-trees, having at least binary vertices. The identification of these trees with the bracketings of $\sigma(1), \ldots, \sigma(n)$, $\sigma \in \Sigma_n$, i.e. with the elements of the set $\mathcal{B}(n)$, is classical – see [2, §1.4]; two examples are shown on Figure 14. The fact that the cellular differential coincides with $\partial_\Omega$ is a routine combinatorics. See [13, Example 4.1] for details.

**Figure 14:** Two examples of an identification of planar trees with elements of $\mathcal{B}$. 

((41)(35))2 : 1(62)(34)5 :
The identification above shows that the operad $\text{Ass}$ is Koszul. More precisely, we know that $H_*(\mathcal{K}) = H_0(\mathcal{K}) = \text{Ass}$, because $\mathcal{K}(n)$ is the union of convex polyhedra indexed by the elements of the symmetric group. On the other hand, due to the identification above, $H_*(\Omega((s\mathcal{P})^*)) = H_*(\mathcal{K})$, thus the map $\pi$ of (23) is a homology isomorphism.

A comodule over a cooperad $\mathcal{Q}$ is a collection $N = \{N(n)\}_{n \geq 1}$ together with structure maps

$$\kappa = \kappa_{m_1, \ldots, m_l} : N(m_1 + \cdots + m_l) \to N(l) \otimes Q(m_1) \otimes \cdots \otimes Q(m_l),$$

satisfying axioms dual to the axioms of a module over an operad. An example is the dual $\mathcal{M}^*$ of a $\mathcal{P}$-module $\mathcal{M}$, which is a natural comodule over the cooperad $\mathcal{P}^*$.

Let $N$ be a $\mathcal{Q}$-comodule. As in the case of cooperads, the structure maps induce a degree -1 differential $\partial_1$ on the free module $N \circ \mathcal{F}(\downarrow \mathcal{Q})$. The right differential graded $\Omega(\mathcal{Q})$-module

$$\Omega(N; \mathcal{Q}) := (N \circ \mathcal{F}(\downarrow \mathcal{Q}), \partial_1)$$

is called the (right) cobar construction on the $\mathcal{Q}$-comodule $N$.

Suppose that $M = \langle X; \mathcal{P}; G \rangle$ is a quadratic $\mathcal{P}$-module, in the sense of Definition 4.2, over a quadratic operad $\mathcal{P} = \langle E; R \rangle$. Take the dual $X^*$ and let $G^\perp$ be the annihilator of $G$ in $(X^* \circ (E^* \otimes \text{sgn}))(2)$. Then the quadratic $\mathcal{P}^d$-module $M^! := \langle X^*; \mathcal{P}^d; G^\perp \rangle$ is called the Koszul (or quadratic) dual of $M$. The above definitions were independently made in [9].

**Proposition 5.2.** The $\text{Ass}$-module Cycl is Koszul self-dual, $\text{Cycl}^! = \text{Cycl}$.

**Proof.** Under the notation of Example 4.3, the $\Sigma_2$-space $(X \circ \text{Ass})(2)$ is the direct sum $1 \oplus \text{sgn}$ of the trivial and signum representations, while clearly $R = \text{sgn}$, generated by $g(\mu) - g(\mu)S_{21}$. Then $R^\perp$ is easily seen to be sgn and the proposition follows.

Observe that, for a $\mathcal{P}$-module $M = \{M(n)\}_{n \geq 1}$, the collection $sM$ has an induced $s\mathcal{P}$-module structure. For a quadratic $\mathcal{P}$-module $\langle X; \mathcal{P}; G \rangle$ over a quadratic operad $\mathcal{P} = \langle E; R \rangle$ we have, as in (23), the map

$$\pi : \Omega((sM)^*, (s\mathcal{P})^*) \to (M^1, \partial = 0),$$

induced by the composition

$$(sM)^* \xrightarrow{\text{proj}} (sM)^*(1) = X^* = M^!(1) \hookrightarrow M^!.$$
Definition 5.3. A quadratic module $M$ over a quadratic operad $P$ is called Koszul if the map $\pi$ in (24) is a homology isomorphism.

Theorem 5.4. The module Cycl is Koszul.

The theorem will follow from Theorem 5.5 which explicitly identifies the cobar construction $\Omega((s\text{Cycl})^*, (s\text{Ass})^*)$ to the cellular chain complex $M$ of the symmetrized cyclohedron $W$. Thus the Koszulness of Cycl is, as in the case of the operad Ass, a consequence of the fact that the cyclohedron is a convex polyhedron. At the end of the section we formulate a more general statement which also implies Theorem 5.4.

Theorem 5.5. The cobar construction $\Omega((s\text{Cycl})^*, (s\text{Ass})^*)$ is isomorphic to the cellular chain complex $M = CC_*(W)$ of the cyclohedron $W$.

Proof. As we observed in Example 5.1, the collection $\downarrow (s\text{Ass})^*(n)$ consists of one copy of the regular representation $k[\Sigma_n]$ concentrated in degree $(n - 2)$. Similarly, the collection $(s\text{Cycl})^*$ is isomorphic to the collection $s\text{Cycl}$. We are going to give an explicit description of the space $((s\text{Cycl})^* \circ \downarrow (s\text{Ass})^*)(n)$, similar to that of Lemma 3.4. Suppose that $T_i$ is, for each $1 \leq i \leq l$, a planar $m_i$-tree, whose all vertices are at least binary. Let $R(T_1, \ldots, T_l)$ be the tree obtained by grafting the trees $T_1, \ldots, T_l$ at the inputs of the '$l$-rake,' or, pictorially:

\[ R(T_1, \ldots, T_l) = \]

\[
\begin{array}{c}
\cdot \\
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\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\end{array}
\]

Let $J(n)$ be the free graded right $\Sigma_n$-module generated by the symbols $R(T_1, \ldots, T_i), m_1 + \cdots + m_l = n$, with the degree defined by

\[
\deg(R(T_1, \ldots, T_i)) = n - 1 - \sum_{i=1}^{l} \#\text{vert}(T_i),
\]

where $\#\text{vert}(T_i)$ is the number of vertices of the tree $T_i$. For $\zeta \in \mathbb{Z}_l$ we put (see (16) for the notation)

\[
\zeta(R(T_1, \ldots, T_i) \times \sigma) := \text{sgn}(\zeta) \cdot R(T_{\eta^{-1}(1)}, \ldots, T_{\eta^{-1}(l)}) \times \zeta(m_1, \ldots, m_l)\sigma.
\]
Then the graded vector space \(((sCycl)^* \circ \downarrow (sAss)^*)\)(n) is spanned by equivalence classes of elements

\[ R(T_1, \ldots, T_l) \times \sigma \in J(n), \]

modulo the left action of the group \(\mathbb{Z}_l\) introduced in (23). The right action of the group \(\Sigma_n\) is given by

\[ [R(T_1, \ldots, T_l) \times \sigma] \cdot \rho := [R(T_1, \ldots, T_l) \times \sigma \rho]. \]

We may symbolize the element \(R(T_1, \ldots, T_l) \times \sigma \in J(n)\) as the tree \(R(T_1, \ldots, T_l)\) with the inputs labeled by \(\sigma^{-1}(1), \ldots, \sigma^{-1}(l)\). There is an almost obvious one-to-one correspondence between these labeled planar \(n\)-trees and cyclic bracketings of \(n\) indeterminates from \(\mathcal{B\Gamma}(n)\). This becomes absolutely clear after looking at Figure 15.

To be more formal, the isomorphism \(\varphi : ((sCycl)^* \circ \downarrow (sAss)^*) \to \mathcal{M}\) is defined by \(\varphi(\uparrow^{n-1}\xi_n) := f_n\), where \(f_n = (1, \ldots, n)\) is the top \(n\)-dimensional cell of the cyclohedron \(W_n\) and \(\xi_n\) the generator of \(Cycl(n)\) represented by \(\mathbb{1}_n \in \Sigma_n\). We must specify also the orientation of \(f_n\). In Observation 1.8 we constructed points \(P_1, \ldots, P_n\) spanning a simplex \(\Delta_{f_n} \in f_n\). We orient \(f_n\) coherently with the orientation of \(\Delta_{f_n} \in f_n\) induced by the order \(P_1 < \cdots < P_n\) of its vertices.

The cobar differential \(\partial_\Omega\) is given by

\[ (26) \quad \partial_\Omega(\uparrow^{n-1}\xi_n) = \sum_{1 \leq k \leq n} \sum_{1 \leq k \leq n} (-1)^{n+k} \cdot \nu(\uparrow^{n-k}\xi_{n-k+1} ; \alpha_k, 1, \ldots, 1), \]

where \(\alpha_k = \mathbb{1}_k \in k[\Sigma_k] = Ass\) is the generator, \(k \geq 1\). We shall compare now (26) to the geometric boundary of the top-dimensional cell \(f_n\) of the cyclohedron \(W_n\). This can be done exactly as in the proof of Theorem 3.3 and we leave it to the reader.

The proof of Theorem 2.14 is now immediate. An \(\mathcal{M}\)-trace \(T : \mathcal{M} \to \mathcal{E}_{A,W}\) is determined by a system \(\{T_n := t(\uparrow^{n-1}\xi_n) : A \otimes^n \to W\}_{n \geq 1}\). The axiom (8) then reflects (26).

We finish this section by the following theorem whose proof, based on a straightforward but involved spectral sequence argument, we omit.

**Theorem 5.6.** Let \(M_{UP}\) be a module associated to a cyclic unital quadratic operad \(\mathcal{P}\). Then \(M_{UP}\) is Koszul if and only if \(\mathcal{P}\) is.

Because \(Cycl = M_{UAss}\) (Example 1.9) and the operad \(Ass\) is Koszul [8, Corollary 4.2.7], Theorem 5.6 gives an alternative proof of Theorem 5.4.
\[ n = 1: \quad \begin{array}{c}
\ \bigcdot \\
1
\end{array} = (1) \]

\[ n = 2: \quad \begin{array}{c}
1 & 2 \\
\ \bigcdot \\
2 & 1
\end{array} = 12, \quad \begin{array}{c}
1 & 2 \\
\ \bigcdot \\
2 & 1
\end{array} = (12), \quad \begin{array}{c}
2 & 1 \\
\ \bigcdot \\
1 & 2
\end{array} = (21) \]

\[ n = 3: \quad \begin{array}{c}
1 & 2 & 3 \\
\ \bigcdot \\
2 & 3 & 1 \\
\ \bigcdot \\
3 & 1 & 2
\end{array} = 123 \]

\[ \begin{array}{c}
1 & 2 & 3 \\
\ \bigcdot \\
2 & 3 & 1 \\
\ \bigcdot \\
3 & 1 & 2
\end{array} = - \begin{array}{c}
1 & 2 & 3 \\
\ \bigcdot \\
2 & 3 & 1 \\
\ \bigcdot \\
3 & 1 & 2
\end{array} = 1(23), \quad \begin{array}{c}
2 & 3 & 1 \\
\ \bigcdot \\
3 & 1 & 2 \\
\ \bigcdot \\
1 & 2 & 3
\end{array} = - \begin{array}{c}
2 & 3 & 1 \\
\ \bigcdot \\
3 & 1 & 2 \\
\ \bigcdot \\
1 & 2 & 3
\end{array} = 2(31) \]

\[ \begin{array}{c}
3 & 1 & 2 \\
\ \bigcdot \\
1 & 2 & 3
\end{array} = - \begin{array}{c}
3 & 1 & 2 \\
\ \bigcdot \\
1 & 2 & 3
\end{array} = 3(12) \]

\[ \begin{array}{c}
1 & 2 & 3 \\
\ \bigcdot \\
3 & 1 & 2
\end{array} = (123), \quad \begin{array}{c}
3 & 1 & 2 \\
\ \bigcdot \\
1 & 2 & 3
\end{array} = (312), \quad \begin{array}{c}
1 & 2 & 3 \\
\ \bigcdot \\
3 & 1 & 2
\end{array} = (123) \]

\[ \begin{array}{c}
1 & 2 & 3 \\
\ \bigcdot \\
2 & 3 & 1
\end{array} = (1(23)), \quad \begin{array}{c}
2 & 3 & 1 \\
\ \bigcdot \\
3 & 1 & 2
\end{array} = (2(31)), \quad \begin{array}{c}
3 & 1 & 2 \\
\ \bigcdot \\
1 & 2 & 3
\end{array} = (3(12)) \]

\[ \begin{array}{c}
1 & 2 & 3 \\
\ \bigcdot \\
2 & 3 & 1
\end{array} = (1(23)), \quad \begin{array}{c}
2 & 3 & 1 \\
\ \bigcdot \\
3 & 1 & 2
\end{array} = (2(31)), \quad \begin{array}{c}
3 & 1 & 2 \\
\ \bigcdot \\
1 & 2 & 3
\end{array} = ((31)2) \]

**Figure 15.** A representation of elements of \( BC(1), BC(2) \) and \( BC(3) \) by equivalence classes of planar trees.
6. Cyclohedron as a compactification of the simplex.

For a compact Riemannian manifold $V$, let $C^0_n(V) = \{(v_1, \ldots, v_n); v_i \neq v_j\}$ be the configuration space of $n$ distinct points in $V$. Axelrod and Singer constructed in [1] a compactification $C_n(V)$ of this space, by adding to $C^0_n(V)$ the blow-ups along the diagonals. The space $C_n(V)$ is a manifold with corners, whose open part (= top-dimensional stratum) is $C^0_n(V)$.

There exists a similar compactification of the moduli space $\circ F_m(n)$ of configurations of $n$ distinct points in the $m$-dimensional Euclidean plane $\mathbb{R}^m$ modulo the action of the affine group, described by Getzler and Jones in [3, §3.2] and denoted by $F_m(n)$. The authors of [3] also observed that the collection $F_m := \{F_m(n)\}_{n \geq 1}$ has a natural structure of a topological operad. In [4] we proved that

**Theorem 6.1.** If $V$ is an $m$-dimensional parallelizable Riemannian manifold, then the collection $C(V) := \{C_n(V)\}_{n \geq 1}$ forms a right module over the operad $F_m$ in the category of manifolds with corners.

There is also a ‘framed’ version of the above theorem for manifolds which are not parallelizable, but we will not need it.

Take $V = S^1$. Then it is immediately seen that the space $C^0_n(V)$ has $(n - 1)!$ components indexed by cyclic orders of $n$ points on the circle. Each of these components is isomorphic to $\Delta^n \times S^1$, where $\Delta^n$ is the open $n$-dimensional simplex. It is ‘well-known’ (see Remark 6.2 below) that the compactification $C_n(S^1)$ is isomorphic to $W_n \times S^1$, the product of the symmetrized cyclohedron with the circle [3, page 5249].

Similarly, $\circ F_1(n)$ is easily seen to have $n!$ components indexed by orders of the set of $n$ points on the line, each component being isomorphic to $\Delta^{n-2}$. Again, it is ‘well-known’ that the compactification $F_1(n)$ is the (symmetrized) associahedron $K_n$ [3, 3.2(1)]. This assumed, our statement (Theorem 2.12) about the existence of a $K$-module structure on the cyclohedron follows from Theorem 6.1 applied on $C(S^1) = W_n \times S^1$ (the extra factor $S^1$ plays no rôle).

**Remark 6.2.** The Axelrod-Singer compactification $C_n(S^1)$ is a manifold with corners, constructed by a very explicit sequence of blow-ups. We do not know any ‘universal’ characterization of this space. Thus to prove that $C_n(S^1) \cong W_n \times S^1$ would require an explicit construction of an isomorphism of two manifolds with corners, which is certainly not a tempting challenge. But a reflection on the structure of these two object ‘proves’ the isomorphism ‘beyond any doubts’, which is the opinion shared by many authors. The same remark applies also to the isomorphism $F_1(n) \cong K_n$. 
Remark 6.3. It follows from general properties of manifold-with-corners that both $K_n$ and $W_n$ are truncations of a simplex \[11, Proposition 6.1\], but this existence statement says nothing about an explicit linear convex realization of Section \[11\].

As we observed above, the cyclohedron $W_n$ can be viewed as the simplex $\Delta^n$, some faces of whose were blown-up. In the rest of this section we show that a very natural spectral sequence related to the cobar construction can be interpreted as an inverse process – ‘deblowing-up’ of the cyclohedron back to the closed simplex.

For a collection $X$ and $p \geq 1$, let $X^{(\leq p)} \subset X$ be the subcollection defined by $X^{(\leq p)}(n) = X(n)$ for $n \leq p$ and $X^{(\leq p)}(n) = 0$ otherwise.

Let us consider, for a comodule $N$ over a cooperad $Q$ and for a natural $n$, the subspace 

$$F_p(n) := (N^{(\leq p+1)} \circ \mathcal{F}(\downarrow Q))(n) \subset (N \circ \mathcal{F}(\downarrow Q))(n).$$

It is easily seen that $F_p(n)$ is $\partial_1$-invariant, thus $\{F_p(n)\}_{p \geq 0}$ is an increasing filtration of the $n$-th piece $\Omega(N, Q)(n)$ of the cobar construction $\Omega(N, Q) = (N \circ \mathcal{F}(\downarrow Q), \partial_1)$. Let $E(n) = (E^r_{pq}(n), d^r)$ be the corresponding spectral sequence. The following lemma is an easy exercise.

Lemma 6.4. The spectral sequence $E(n) = (E^r_{pq}(n), d^r)$ constructed above converges to $H_\ast(\Omega(N, Q)(n))$. The first term $E^1$ is described as 

$$E^1_{pq}(n) = (N(p + 1) \circ H_\ast(\Omega(Q)))(n)_{p+q},$$

the space of elements of degree $p + q$ in the $n$-th piece of the free $H_\ast(\Omega(Q))$-module on the $(p + 1)$-th piece of the collection $N$.

If the cooperad $Q$ and the comodule $N$ are Koszul, the spectral sequence above collapses at the 1st term, which has a very explicit description. Since we did not formulate the Koszulness for cooperads and comodules (though the definition is an exact dual), we suppose from now on that $N = (sM)^\ast$ and $Q = (sP)^\ast$, for a module $M$ over an operad $P$. We also suppose that $M$ and $P$ are not graded, i.e. that both $M(n)$ and $P(n)$ are concentrated in degree 0, $n \geq 1$. 
Proposition 6.5. If the operad $\mathcal{P}$ is Koszul, then $E_{pq}^{1}(n) = 0$ for $q \geq 1$, while
\[ E_{p0}^{1}(n) = ((sM)^*(p+1) \circ \mathcal{P}^i)(n), \]
and the spectral sequence collapses at $E_1$. The module $M$ is Koszul if and only if the complex
\[ 0 \leftarrow E_{00}^{1}(n) \leftarrow E_{10}^{1}(n) \leftarrow E_{20}^{1}(n) \leftarrow \cdots \]
is acyclic in positive dimensions, for all $n \geq 1$.

**Proof.** If $\mathcal{P}$ is Koszul, then $H^{*}(\Omega(\mathcal{P})) = H_0(\Omega(\mathcal{P})) = \mathcal{P}!$, by definition. Thus, by Lemma 6.4, $E_{pq}^{1}(n) = ((sM)^*(p+1) \circ \mathcal{P}^i)(n)_{p+q}$, which may be nonzero only for $q = 0$, because $(sM)^*(p+1)$ is concentrated in degree $p$. The collapsing is obvious from degree reasons. The second part of the statement follows immediately from the definition of the Koszulness of a module (Definition 5.3).

Our spectral sequence has, for $\mathcal{P} = \text{Ass}$ and $M = \text{Cycl}$, a beautiful geometric meaning. The initial term $E^0 = (E_{pq}^0, d^0)$ is the cobar construction $\Omega((s\text{Cycl})^*, (s\text{Ass})^*)$ which is isomorphic, by Theorem 5.3, to the cellular chain complex of the cyclohedron $W_n$, while $E^1 = (E_{pq}^1, d^1) = (s\text{Cycl} \circ \text{Ass}, \partial)$ is isomorphic, by Theorem 3.5, to the cellular chain complex of the simplex $\Delta_n$. The passage from $E^0$ to $E^1$ can be interpreted as the ‘deblowing-up’ of the cyclohedron back to the simplex. This process is visualized on Figure 16.

**Appendix: Traces versus invariant bilinear forms.**

Let us recall the following notion of [7, Definition 4.1]. If $\mathcal{P}$ is a cyclic operad and $A$ a $\mathcal{P}$-algebra, then a bilinear form $B : A \otimes A \to W$ with values in a vector space $W$ is called invariant if, for all $n \geq 0$, the map $B_n : \mathcal{P}(n) \otimes A^{\otimes(n+1)} \to W$ defined by the formula
\[ B_n(p \otimes x_0 \otimes x_1 \otimes \cdots \otimes x_n) := (-1)^{|x_0| \cdot |p|} B(x_0, p(x_1, \ldots, x_n)), \]
is invariant under the action of the symmetric group $\Sigma_{n+1}$ on $\mathcal{P}(n) \otimes A^{\otimes(n+1)}$.

**Proposition A.1.** Let $\mathcal{P}$ be a cyclic operad and let $M_\mathcal{P}$ be the associated module introduced in Definition 4.3. Let $A$ be a $\mathcal{P}$-algebra. Then there exists a 1-1 correspondence between $M_\mathcal{P}$-traces on the $\mathcal{P}$-algebra $A$ in the sense of Definition 2.6, and invariant bilinear forms on $A$. 
Figure 16: $\Delta_3$ as deblowing-up of $W_3$. The faces of $W_3$ which are contracted by $d^0$ to a vertex are indicated by double lines, $(123)$ is contracted to $\{1\}$, $1)(23$ to $\{2\}$ and $12)(3$ to $\{3\}$.

**Proof.** Let $t : M_P \to \mathcal{E}_{A,W}$ be an $M_P$-trace. Because $M_P(n + 1) = \mathcal{P}(n)$, the trace is represented by a system $\{t_n : \mathcal{P}(n - 1) \to \text{Hom}(A^\otimes n, W)\}_{n \geq 2}$ of linear maps. We claim that $B := t_2(1) : A \otimes A \to W$, where $1 \in \mathcal{P}(1)$ is the unit, is an invariant bilinear form. To see it, observe that (27) can be rewritten as

$$B_n(p \otimes x_0 \otimes \cdots \otimes x_n) = \nu_{\mathcal{E}_{A,W}}(t_2(1); 1, p)(x_0, \ldots, x_n),$$

while

$$\nu_{\mathcal{E}_{A,W}}(t_2(1); 1, p) = \nu_{\mathcal{E}_{A,W}}(\nu_{M_P}(1; 1, p)) = t_{n+1}(p \cdot \tau_n),$$

thus

(28)  
$$B_n(p \otimes x_0 \otimes \cdots \otimes x_n) = t_{n+1}(p \cdot \tau_n)(x_0, \ldots, x_n)$$

and the equivariance of $B_n$ follows from the equivariance of $t_{n+1}$. On the other hand, if $B$ is an invariant bilinear form, then (28) defines a trace.

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