Non-uniqueness of the solution of one mathematical model of an autocatalytic reaction with diffusion and the Showalter – Sidorov condition

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Abstract. The article is devoted to the numerical study of the phase space of one mathematical model of an autocatalytic reaction with diffusion, based on a degenerate system of equations for a distributed brusselator. We will obtain conditions for the existence, uniqueness or multiplicity of solutions to the Showalter – Sidorov problem and reveal the dependence of these conditions on the parameters of the system. The approach used in the numerical research of this problem is based on the reduction of the semilinear Sobolev-type equation to a system of algebraic-differential equations with the subsequent solution of this system using the Runge – Kutta method of order 4-5. The article also provides the result of a computational experiment which illustrates the operation of a program complex based on an algorithm for the numerical solution of the problem. The results of numerical simulation in the case of existence of two solutions of the investigated model are presented.

1. Introduction
Currently, studies of self-organization phenomena in various nonequilibrium systems, consisting in the emergence and evolution of ordered space-time structures. An example of the latter is autowaves, which are formed in excitable media in response to external disturbance. There are many examples of excitable media: nerves and muscles tissues, colonies of microorganisms, a number of chemical solutions and gels, magnetic superconductors with current, some solid-state systems [1–4] and others.

Researching the mechanism of reactions resulting in the formation of ordered temporal and (or) spatial structures, mathematical models were obtained, one of which is the autocatalytic reaction with diffusion

\[
\begin{align*}
\varepsilon_1 \psi_t &= \alpha_1 \psi_{ss} + \gamma - (\delta + 1) \psi + \psi^2 \omega, \\
\varepsilon_2 \omega_t &= \alpha_2 \omega_{ss} + \delta \psi - \psi^2 \omega.
\end{align*}
\]

(1)

Here \(\psi = \psi(s,t)\) and \(\omega = \omega(s,t)\) are functions, characterizing the concentration of the reactants, elements \(\alpha_1 \psi_{ss}, \alpha_2 \omega_{ss}\) characterize the diffusion of reagents, according to Fickey’s law (\(\alpha_1, \alpha_2 \in \mathbb{R}_+\) are diffusion coefficients), parameters \(\gamma, \delta \in \mathbb{R}_+\) characterize concentrations starting reagents that are assumed to be constant.

System of equations (1), which was named by researchers as a distributed Brusselator, was studied in various aspects, and in many works, along with the case \(\varepsilon_1 > 0\) or \(\varepsilon_2 > 0\) \([5,6]\), the case \(\varepsilon_1 = 0\) or \(\varepsilon_2 = 0\). The reason to study these cases is that the rate of change of one of the
components is much higher than the other, which is typical for all reaction-diffusion models. The case $\varepsilon_2 = 0$ was considered in the papers [7] and it was established that in this case the phase space of system of equations (1) is simple, which means that the solution will be unique. In this paper, only the case $\varepsilon_1 = 0$ will be considered, and it will be shown that the phase space of system of equations (1) may contain singularities of the Whitney assembly type, which leads to non-uniqueness of the solution. In this case, the system of equations (1) takes the form

$$\begin{cases}
0 = \alpha_1 v_{ss} + \gamma - (\delta + 1)v + v^2 w, \\
w_t = \alpha_2 w_{ss} + \delta v - v^2 w.
\end{cases}$$  \hspace{1cm} (2)$$

Consider a degenerate system of equations (2) in a cylinder $Q = \Omega \times \mathbb{R}_+$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary of class $C^\infty$ with boundary conditions

$$v(s, t) = 0, \quad w(s, t) = 0, \quad (s, t) \in \partial \Omega \times \mathbb{R}_+$$  \hspace{1cm} (3)

and the Showalter – Sidorov initial condition

$$w(0) = w_0.$$  \hspace{1cm} (4)

The problem (2) – (4) in Banach spaces is reduced to the initial Showalter – Sidorov problem

$$L(x(0) - x_0) = 0$$  \hspace{1cm} (5)

for a semilinear Sobolev-type equation

$$L\dot{x} = Mx + N(x).$$  \hspace{1cm} (6)

Here $L \in \mathcal{L}(\Omega, \mathcal{F})$, $M \in \mathcal{C}(\Omega, \mathcal{F})$, $N$ is a nonlinear operator. The initial condition (5) was first formulated explicitly by R.E. Showalter in 1975 [8]. To research it, he had to construct “semi-Hilbert spaces with non-Hausdorff metric”. Independently and in a different way, N.A. Sidorov [9] arrived at the initial condition (5) in 1984. As it turned out later, many authors [10–12] considered that the Showalter – Sidorov problem is more natural for Sobolev-type equations than the Cauchy problem.

In this paper, based on the theory of relatively sectorial operators [13], conditions were found for the unique solvability of the problem (2) – (4). Namely, if the operator $M$ is $L$-sectorial, then any solution (5), (6) will be a quasi-stationary semi-trajectory [14]. If the phase space of equation (6) lies on a smooth Banach manifold with singularities such as Whitney assemblies, Showalter – Sidorov problem (5) for equation (6) may have several solutions. It was shown that the phase space of the system of equations (2) contains singularities of Whitney 1-assemblies type, therefore, the Showalter – Sidorov problem for such an equation can have one or more solutions or the solution may not exist in the works [15–17]. In the course of this research, we will find conditions for the existence and uniqueness or multiplicity of solutions to problem (2) – (4) depending on the parameters of the system.

In addition to theoretical research, we carried out a numerical study of problem (2) – (4) and developed an algorithm based on the modified projection Galerkin method. Based on this method, we will represent the sought functions in the form of the Galerkin sum

$$v_m(s, t) = \sum_{k=1}^{m} a_k(t) \varphi_k(s), \quad w_m(s, t) = \sum_{k=1}^{m} b_k(t) \varphi_k(s),$$

{$\{\varphi_k(s)\}$} are the eigenfunctions of the homogeneous Dirichlet problem of the Laplace operator ($-\Delta$) in the domain $\Omega$, $a_k(t)$, $b_k(t)$ satisfy the corresponding system algebraic differential equations and the corresponding initial conditions. For the first time the Galerkin method for Sobole-type equations was considered in the work [18]. This method is especially effective for degenerate equations or systems of equations [19–21]. The developed algorithm served as the basis for the development of a complex of programs for finding one or more solutions to the problem (2) – (4).
2. Mathematical model of autocatalytic reaction with diffusion

Reduce problem (2), (3) to semilinear Sobolev-type equation (6). At the first step of the reduction, take the spaces \( \mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2 = W^{1,2}_{\text{loc}}(\Omega) \times W^{1,2}_{\text{loc}}(\Omega) \), \( \mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 = L^2(\Omega) \times L^2(\Omega) \). Note that \( \mathcal{U} \) is a Hilbert space with scalar product \( [x, \zeta] = \langle v, \xi \rangle + \langle w, \eta \rangle \), where \( x = (v, w), \zeta = (\xi, \eta) \), and by \( \langle \cdot, \cdot \rangle \) we denote the scalar product in \( L^2(\Omega) \). Let \( \mathcal{F} \) is the dual space of \( \mathcal{H} \) with respect to the duality \( \langle \cdot, \cdot \rangle \). By the Sobolev embedding theorems, there are dense and continuous embeddings 

\[
\mathcal{H} \hookrightarrow \mathcal{U} \hookrightarrow \mathcal{F}.
\]

At the second step of the reduction, we define the linear operators \( L, M : \mathcal{U} \to \mathcal{F} \) by the formulas

\[
[Lx, \zeta] = \langle w, \xi \rangle, \quad x, \zeta \in \mathcal{U},
\]

\[
[Mx, \zeta] = -\alpha_1 (v_s, \xi_s) - \alpha_2 (w_s, \eta_s), \quad u, \zeta \in \mathcal{U}, \text{dom } M = \mathcal{H}.
\]

(Einstein’s convention on summation over repeated indices holds everywhere.) By construction, the operator \( L \in \mathcal{L}(\mathcal{U}; \mathcal{F}), M \in \mathcal{L}(\mathcal{U}; \mathcal{F}) \).

We denote by \( \{\nu_k\} \) the sequence of eigenvalues of the following spectral problem:

\[
-\Delta \varphi = \nu \varphi, \quad s \in \Omega,
\]

\[
\varphi(s) = 0, \quad s \in \partial \Omega,
\]

where the eigenvalues are numbered in non-decreasing order taking into account their multiplicity. We denote by \( \{\varphi_k\} \) the corresponding eigenfunctions, orthonormal in the sense of the scalar product \( \langle \cdot, \cdot \rangle \) in \( L^2(\Omega) \).

Note that \( \mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2, \mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2 \), then

\[
\mathcal{U}^0 = \ker L = \mathcal{U}_1 \times \{0\},
\]

\[
\mathcal{U}^1 = \{0\} \times \mathcal{U}_2,
\]

\[
\mathcal{F}^1 = \text{im} L = \{0\} \times \mathcal{F}_2,
\]

\[
\mathcal{F}^0 = M[\mathcal{U}^0 \cap \text{dom } M] = \mathcal{F}_1 \times \{0\}.
\]

**Lemma 2.1** [22] **For any** \( \alpha_1, \alpha_2 \in \mathbb{R} \setminus \{0\} \) **the operator** \( M \) **is** \( L \)-sectorial, **and the conditions**

\[
\mathcal{U}^0 \oplus \mathcal{U}^1 = \mathcal{U} \quad (\mathcal{F}^0 \oplus \mathcal{F}^1 = \mathcal{F})
\]

**and**

\[
\text{operator } L_1^{-1} \in \mathcal{L}(\mathcal{F}^1; \mathcal{U}^1).
\]

At the third step, we define a nonlinear operator by the formula

\[
[N(x), \zeta] = (\gamma - (\delta + 1)v + v^2w, \xi) + \langle \delta v - v^2w, \eta \rangle
\]

and put \( \text{dom } N = \mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 = L^4(\Omega) \times L^4(\Omega) \). Denote by \( \mathcal{B}^* \) is the dual space of \( \mathcal{B} \) with respect to duality \( [, , \cdot ] \). This space is topolinearly isomorphic to the space \( L^4(\Omega) \times L^4(\Omega) \). By the Sobolev embedding theorem, there are dense and continuous embeddings for \( n \leq 4 \)

\[
\mathcal{H} \hookrightarrow \mathcal{B} \hookrightarrow \mathcal{U} \hookrightarrow \mathcal{B}^* \hookrightarrow \mathcal{F}.
\]

**Lemma 2.2** **For any** \( \gamma, \delta \in \mathbb{R}, n \leq 4 \) **operator** \( N \in C^\infty(\mathcal{B}; \mathcal{F}) \).
For a semilinear Sobolev-type equation (6), where the operators $L, M, N$ problem (2) – (4) for any $x$ which in this particular case will have the form (4). Thus, we are interested in the solvability of a solution of the equation. The solution $x$ is called the solution of problem (2) – (4) based on theory of relatively sectorial operators $G. A. Sviridyuk [14,17].$

Proof. Let us show first that $N \in C^\infty(\mathcal{B}; \mathcal{B}^*)$. Indeed, by virtue of Hilder’s inequality and the continuity of the embedding $\mathcal{B} \hookrightarrow \mathcal{U} \hookrightarrow \mathcal{B}^*$ we have

$$
\begin{align*}
|N(x, \zeta)| & \leq (a_1 + a_2 \|x\|_B + a_3 \|x\|_B^2) \cdot \|\zeta\|_B, \\
|N'(x_1, x_2)| & \leq (b_1 + b_2 \|x\|_B) \cdot \|\zeta_1\|_B \cdot \|\zeta_2\|_B, \\
|N''(x_1, x_2, x_3)| & \leq c \|x\|_B \cdot \|\zeta_1\|_B \cdot \|\zeta_2\|_B \cdot \|\zeta_3\|_B, \\
|N'''(x_1, x_2, x_3, x_4)| & \leq d \cdot \|\zeta_1\|_B \cdot \|\zeta_2\|_B \cdot \|\zeta_3\|_B \cdot \|\zeta_4\|_B,
\end{align*}
$$

where the constants $a_1, a_2, a_3, b_1, b_2, c, d \in \mathbb{R}_+$ does not depend on any of the $x$, or by $\zeta, \zeta_1, \zeta_2, \zeta_3, \zeta_4$, $N_4$ is derivative Frechet operator $N$ a point $x$. The remaining derivatives equal to zero.

So, the inclusion $N \in C^\infty(\mathcal{B}; \mathcal{B}^*)$ is proved. The assertion of the theorem takes place due to the continuous embedding $\mathcal{B}^* \rightarrow \mathfrak{H}$.

Following [23], we introduce the interpolation space $\mathfrak{U}_\alpha$ for this, consider the sectorial operator $A \in Cl(\mathfrak{U})$ and $\text{Re}e(A) < 0$, $\text{dom} A = W_2^1(\Omega)$. For any $\alpha > 0$, put $A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1}e^{-At}dt$

Define $A^\alpha$ as the inverse operator to $A^{-\alpha}$, $\text{dom} A^\alpha = \text{im} A^{-\alpha}$, $A^0 = I$. Further, we put that for each $\alpha \geq 0$, $\Omega^\alpha = \text{dom} A^\alpha$, and we endow the space $\Omega^\alpha$ graph norm $\|x\|_\alpha = \|A^\alpha x\|$, $x \in \Omega$. According to the Sobolev embedding theorem for $\alpha \in (\frac{1}{2}, 1]$ space $\mathfrak{U} \hookrightarrow L_\infty(\Omega)$. Take $\mathfrak{U}_\alpha = \mathfrak{U}_0^\alpha \oplus \mathfrak{U}_1^\alpha$, where $\mathfrak{U}_0^\alpha = \{0\} \times \mathfrak{U}^\alpha$, $\mathfrak{U}_1^\alpha = W_2^1(\Omega) \times \{0\}$. There are dense and continuous embeddings $\mathfrak{U} \hookrightarrow \mathfrak{U}_\alpha \hookrightarrow \mathfrak{B} \hookrightarrow \mathfrak{U}$. Operator $N \in C^\infty(\mathfrak{U}_\alpha; \mathfrak{F})$.

So, we have reduced problem (2), (3) to semilinear Sobolev-type equation (6). Next, using the phase space method, we study the question of the unique solvability of problem (2) – (4).

3. Non-uniqueness and non-existence of solutions to the Showalter – Sidorov problem

For a semilinear Sobolev-type equation (6), where the operators $L, M, N$ and spaces $\mathfrak{U}, \mathfrak{U}_\alpha, \mathfrak{F}, \mathfrak{B}, \mathfrak{F}$ are given as in the previous section, consider Showalter – Sidorov problem (5), which in this particular case will have the form (4). Thus, we are interested in the solvability of the problem (2) – (4) for any $x_0 = (v_0, w_0) \in \mathfrak{U}_\alpha$.

Definition 3.1 Vector-function $x \in C^4((0, \tau); \mathfrak{U}) \cap C((0, \tau); \mathfrak{U}_\alpha)$ satisfying equation (6) is called a solution of the equation. The solution $x = x(t)$ of equation (6) is called the solution of problem (5), (6), if $\lim_{t \rightarrow 0^+} \|Lx(t) - x_0\|_\mathfrak{F} = 0$.

In our case, all solutions system of equations (2) lie pointwise in the set

$$
\mathfrak{M} = \{x \in \mathfrak{U}_\alpha : \langle \alpha_1 v_s, \xi_{s_i} \rangle = \langle \gamma - (\delta + 1)v - v^2w, \xi \rangle\}. 
$$

In [15], the following statement was obtained and proved

Lemma 3.1 [15] Let $\alpha_1, \alpha_2 \in \mathbb{R}\{0\}$, $\gamma \in \mathbb{R}$, $\beta \in \mathbb{R}\{\alpha_1 v_k - 1\}$ and $x_0 \in \mathfrak{U}_\alpha$ then the set $\mathfrak{M}$ at the point $x_0$ is a simple $C^\infty$-manifold.

We research the existence of a solution to problem (2) – (4) based on theory of relatively sectorial operators G.A.Sviridyuk [14,17].

Lemma 3.2 For any $\alpha_1, \alpha_2 \in \mathbb{R}\{0\}$, $\gamma \in \mathbb{R}$, $\beta \in \mathbb{R}\{\alpha_1 v_k - 1\}$ and $x_0 \in \mathfrak{U}_\alpha$ there is a solution to problem (2) – (4).

Proof. This statement is true by virtue of Lemmas 2.1, 2.2, 3.1 and Theorem 2 in [17].
Lemma 3.3 Let $\alpha_1, \alpha_2 \in \mathbb{R} \setminus \{0\}$, $\gamma \in \mathbb{R}$, $\beta \in \mathbb{R} \setminus \{\alpha_1 \nu_k - 1\}$ and $x_0 \in \mathcal{U}_\alpha$ then the phase space of system of equation (2) is the set $\mathfrak{M}$.

Proof. The validity of this statement follows from the definition of the set $\mathfrak{M}$, the definition of the phase space for Sobolev type equations [13], and the Lemma 3.2.

Lemma 3.4 For any $\alpha_1, \alpha_2 \in \mathbb{R} \setminus \{0\}$, $\gamma \in \mathbb{R}$, $\beta \in \mathbb{R} \setminus \{\alpha_1 \nu_k - 1\}$ and $w_0 \in \mathcal{U}_\alpha$ there is a unique solution to problem (2) – (4).

Proof. The validity of this statement follows from Lemmas 2.1, 3.2, 3.3 and Lemma 2 in [17].

Now consider the case where $\delta = \alpha_1 \nu_k - 1$, where $\nu_k$ is the eigenvalue of spectral problem (10). Take an arbitrary point $x = (v, w) \in \mathcal{U}_\alpha$, represent as $x = (v^\perp + r \varphi_k, w^\perp)$, where vectors $x^\perp = (v^\perp, w^\perp) \in \mathcal{H}_\alpha^\perp$, $v^\perp \in \mathcal{H}_1^\perp$, $w^\perp \in (\mathcal{U}_\alpha^\perp)^\perp$, $\mathcal{H}_1^\perp = \{v \in \mathcal{H}_1 : (v^\perp, \varphi_k) = 0\}$, $(\mathcal{U}_\alpha^\perp)^\perp = \{w^\perp \in \mathcal{U}_\alpha^\perp : (w^\perp, \varphi_k) = 0\}$, $\varphi_k$ is an eigenfunction of problem (10) corresponding to the eigenvalue $\nu_k$ and normalized in the sense of $L_2(\Omega)$. In this case, set $\mathfrak{M}$ (15) is determined by a system of two equations

$$\mathfrak{M} = \left\{ x = (v, w) \in \mathcal{U}_\alpha : \begin{array}{l}
\langle \alpha_1 v^\perp, \xi^\perp \rangle = \langle \gamma - (\delta + 1) v^\perp, \xi^\perp \rangle - (v^1 + r \varphi_1)^2 w^\perp, \xi^\perp \\
\langle \gamma, \varphi_k \rangle = (v^1 + r \varphi_1)^2 w^\perp, \varphi_k \end{array} \right\}. \tag{16}$$

Note that system of equations defining set (16) is obtained from the equation defining set (15) if we put $\delta = \alpha_1 \nu_k - 1$, in it and then instead of $\xi$ first substitute $\xi^\perp$, and then $\varphi_k$.

Let us turn to the second equation of the system defining set (16). Transforming the resulting equation, we get:

$$r^2 \int_\Omega w^\perp \varphi_k^3 ds + 2r \int_\Omega (v^\perp)^2 w^\perp \varphi_k^2 ds + \int_\Omega (v^\perp)^2 w^\perp \varphi_k ds + \int_\Omega \gamma \varphi_k ds = 0. \tag{17}$$

Note that equation (17) is a quadratic equation of the type $ar^2 + br + c = 0$ with respect to $r$, where

$$a = \int_\Omega w^\perp \varphi_k^2 ds, \quad b = 2 \int_\Omega (v^\perp)^2 w^\perp \varphi_k^2 ds,$$
$$c = \int_\Omega (v^\perp)^2 w^\perp \varphi_k ds + \int_\Omega \gamma \varphi_k ds.$$

Let us introduce the functional

$$\Delta(v^\perp) = 4 \left( \int_\Omega (v^\perp)^2 w^\perp \varphi_k^2 ds \right)^2 - 4 \int_\Omega w^\perp \varphi_k^2 ds \cdot \int_\Omega (v^\perp)^2 w^\perp \varphi_k ds + \int_\Omega \gamma \varphi_k ds, \tag{18}$$

$\Delta(v^\perp) : \mathcal{H}_1^\perp \rightarrow \mathbb{R}$, and construct the set

$$\mathcal{H}_1^\perp = \{ v^\perp \in \mathcal{H}_1^\perp : \Delta(v^\perp) > 0 \}.$$
Take the point \( v^\perp \in (\mathcal{H}_1^\perp)_+ \), then equation (17) has two solutions

\[
\begin{align*}
    r_- &= -2 \int_{\Omega}(v^\perp)^2 w^\perp \varphi_k^2 ds - \sqrt{\Delta(v^\perp)} \bigg|_{\Omega} \\
    r_+ &= -2 \int_{\Omega}(v^\perp)^2 w^\perp \varphi_k^2 ds + \sqrt{\Delta(v^\perp)} \bigg|_{\Omega}
\end{align*}
\] (19)

We construct the sets

\[
\mathcal{M}_- = \left\{ x = (v, w) \in \mathfrak{U}_\alpha : \langle (\alpha_1 v^\perp_k, \xi^\perp_k), (\gamma - (\delta + 1)v^\perp, (v^\perp + r\varphi_1)^2 w^\perp, \xi^\perp) \rangle, v = r_-(v^\perp) \varphi_k + v^\perp, \alpha \in \mathfrak{U}_\alpha \cap \mathfrak{U}_+ \right\},
\]

\[
\mathcal{M}_+ = \left\{ x = (v, w) \in \mathfrak{U}_\alpha : \langle (\alpha_1 v^\perp_k, \xi^\perp_k), (\gamma - (\delta + 1)v^\perp, (v^\perp + r\varphi_1)^2 w^\perp, \xi^\perp) \rangle, v = r_+(v^\perp) \varphi_k + v^\perp, \alpha \in \mathfrak{U}_\alpha \cap \mathfrak{U}_+ \right\}.
\]

**Lemma 3.5** Let \( \alpha_1, \alpha_2 \in \mathbb{R} \setminus \{0\}, \gamma \in \mathbb{R}, \beta = \alpha_1 \nu_k - 1 \) and \( x_0 \in \mathfrak{U}_\alpha \) then the phase space of system of equation (2) is the set \( \mathcal{M}_- \cup \mathcal{M}_+ \).

**Proof.** With taking into account the fact, that \( \mathcal{M}_- \cup \mathcal{M}_+ \subset \mathfrak{M} \) in the proof requires only the approval of the bijective projection. Indeed, for any vector \( x_0 \in (\mathcal{H}_1^\perp)_+ \times (\mathfrak{U}^\perp) \) there are exactly two vectors \( x_{0_+} = (v_{0+}, w_{0+}, 0) \in \mathcal{M}_+ \) and \( x_{0_-} = (v_{0-}, -v_{0-}, w_0) \in \mathcal{M}_- \). On the other hand, for any vector \( x_{0_+}(x_{0_-}) \in \mathcal{M}_+(\mathcal{M}_-) \) there is only one vector \( x_0 \in (\mathcal{H}_1^\perp)_+ \times (\mathfrak{U}^\perp) \).

In the case when \( \beta = \alpha_1 \nu_k - 1 \) and point \( x_0 \notin \mathcal{M}_- \) then it is obvious that a solution to problem (2) – (4). In the case when the initial values lie in the set \( \mathcal{M}_- \cap \mathcal{M}_+ \), they are usually called the point of the Whitney fold of the phase space problem (2), (3). The work [15] showed

**Theorem 3.1** [15] For \( \alpha_1, \alpha_2 \in \mathbb{R} \setminus \{0\}, \gamma \in \mathbb{R}, \beta = \alpha_1 \nu_k - 1 \) and \( x_0 \in \mathfrak{U}_\alpha \) the phase space \( \mathcal{M}_- \cup \mathcal{M}_+ \) contains the Whitney 1-assembly.

**Lemma 3.6** For \( \alpha_1, \alpha_2 \in \mathbb{R} \setminus \{0\}, \gamma \in \mathbb{R}, \beta = \alpha_1 \nu_k \) and \( x_0 \in (\mathcal{H}_1^\perp)_+ \times (\mathfrak{U}^\perp) \) there are two different solutions to problem (2) – (4).

**Proof.** Choose arbitrarily the point \( x_0 \in \mathcal{M}_- \cup \mathcal{M}_+ \). Then its projection \( v^\perp_0 \in (\mathcal{H}_1^\perp)_+ \). In this case, the point \( v^\perp_0 \) is the image of two points \( x_{0_+} = (v_{0+}, a_+(v^\perp_0) + v^\perp, w_0) \in \mathcal{M}_+ \) and \( x_{0_-} = a_-(v^\perp_0) + v^\perp, w_0) \in \mathcal{M}_- \). This means that there are two different solutions to the problem (2) – (4).

We construct the sets

\( (\mathcal{H}_1^\perp)_- = \{ v^\perp \in (\mathcal{H}_1^\perp) : \Delta(v^\perp) < 0 \} \)

and

\( (\mathcal{H}_1^\perp)_0 = \{ v^\perp \in (\mathcal{H}_1^\perp) : \Delta(v^\perp) = 0 \} \).

**Lemma 3.7** For any \( \alpha_1, \alpha_2 \in \mathbb{R} \setminus \{0\}, \gamma \in \mathbb{R}, \beta = \alpha_1 \nu_k \) and \( x_0 \in (\mathcal{H}_1^\perp)_0 \times (\mathfrak{U}^\perp) \) there is one solution to problem (2) – (4).

**Proof.** Take a vector-function \( v^\perp_0 \in (\mathfrak{U}^\perp)_0 \), then the point \( x_0 = (v^\perp_0 + r\varphi_k, w^\perp_0) \in \mathcal{M}_- \cap \mathcal{M}_+ \), and thus, can be said, that there is one single solution (2) – (4).

**Lemma 3.8** For any \( \alpha_1, \alpha_2 \in \mathbb{R} \setminus \{0\}, \gamma \in \mathbb{R}, \beta = \alpha_1 \nu_k \) and \( x_0 \in (\mathcal{H}_1^\perp)_- \times (\mathfrak{U}^\perp) \) does not exist or a solution of problem (2) – (4).
**Proof.** Take \( v_0^1 \in (U_{0}^1)\). Then the problem (2) – (4) will not have a single solution, since under no \( r \in \mathbb{R} \) point \( x_0 = (v_0^1 + r\varphi_k, w_0^1) \) will not lie in set \( \mathcal{M} \), that is, it will not satisfy the system of equations (2).

In view of the above, the following theorem is true.

**Theorem 3.2** For any \( \alpha_1, \alpha_2 \in \mathbb{R}\{0\}, \gamma \in \mathbb{R}, \) and

(i) \( \beta \in \mathbb{R}\{\alpha_1 \nu_k - 1\} \) and \( x_0 \in \mathcal{U}_\alpha \) there is a unique solution to problem (2) – (4);

(ii) \( \beta = \alpha_1 \nu_k \) and \( x_0 \in (\mathcal{D}_1^1) \times (\mathcal{U}_{\alpha})^\perp \) there are two solutions to problem (2) – (4);

(iii) \( \beta = \alpha_1 \nu_k \) and \( x_0 \in (\mathcal{D}_1^1) \times (\mathcal{U}_{\alpha})^\perp \) there is one solution to problem (2) – (4);

(iv) \( \beta = \alpha_1 \nu_k \) and \( x_0 \in (\mathcal{D}_1^1) \times (\mathcal{U}_{\alpha})^\perp \) there is no solution to problem (2) – (4).

### 4. Algorithm and computational experiment

Here is an algorithm for finding an approximate solution to problems (2) – (4):

**Stage 1.** Find the eigenvalues \( \{\nu_k\} \) and the eigenfunctions \( \{\varphi_k(s)\} \) of the homogeneous Dirichlet problem of the Laplace operator \((-\Delta)\) in the domain \( \Omega \).

**Stage 2.** Let’s represent the required functions in the form of the Galerkin sum

\[
v_m(s,t) = \sum_{k=1}^{m} a_k(t)\varphi_k(s), \quad w_m(s,t) = \sum_{k=1}^{m} b_k(t)\varphi_k(s),
\]

where \( a_k(t), b_k(t) \) satisfy the system of algebraic differential equations

\[
\begin{align*}
&-\alpha_1 \sum_{k=1}^{m} a_k(t)\nu_k\langle \varphi_k, \varphi_i \rangle + \gamma - (\delta + 1) \sum_{k=1}^{m} a_k(t)\langle \varphi_k, \varphi_i \rangle + \\
&\left( \sum_{k=1}^{m} a_k(t)\langle \varphi_k, \varphi_i \rangle \right)^2 \sum_{k=1}^{m} b_k(t)\langle \varphi_k, \varphi_i \rangle = 0, \\
&\sum_{k=1}^{m} \frac{d}{dt} b_k(t)\langle \varphi_k, \varphi_i \rangle - \alpha_2 \sum_{k=1}^{m} b_k(t)\nu_k\langle \varphi_k, \varphi_i \rangle - \\
&\delta \sum_{k=1}^{m} a_k(t)\langle \varphi_k, \varphi_i \rangle - \left( \sum_{k=1}^{m} a_k(t)\langle \varphi_k, \varphi_i \rangle \right)^2 \sum_{k=1}^{m} b_k(t)\langle \varphi_k, \varphi_i \rangle = 0, \\
&i = 1, \ldots, m,
\end{align*}
\]

and initial conditions

\[
\langle w(0) - w_0, \varphi_i \rangle = 0.
\]

**Stage 3.** Taking for \( r = a_1(0) \), \( q = b_1(0) \), \( v^\perp = \sum_{k=2}^{m} a_k(t)\varphi_k \), \( w^\perp = \sum_{k=2}^{m} b_k(t)\varphi_k \), and substituting the obtained values into the formulas (18), check the uniqueness or multiplicity of the solution to the Showalter – Sidorov problem under the given initial conditions and the obtained \( a_k(0), b_k(0) \). In the case when the required problem has two solutions \( v_{01}(s,t) \) and \( v_{02}(s,t) \), therefore, the system algebraic differential equations will have two solutions and two sets \( a_k(t) \) and \( b_k(t) \) for each of the solutions, respectively. In this case, all subsequent steps must be done twice for each of the sets \( a_k(t) \) and \( b_k(t) \).

**Stage 4.** Find \( a_k(0) \) by scalar multiplying in \( L_2(\Omega) \) the initial condition (23) by the eigenfunctions \( \varphi_i(s), i = 1, \ldots, m \).

**Stage 5.** Solving the system of algebraic equations (21) for \( a_k(0) \), we obtain the values of \( b_k(0) \).

**Stage 6.** Using the Runge – Kutta method of 4-5 orders, we find a solution to the system of differential equations (22) with initial conditions (23).

Let us carry out a computational experiment to illustrate the operation of the program package “Numerical research of the non-uniqueness of the Showalter – Sidorov problem for the model of an autocatalytic reaction with diffusion”, based on the above algorithm.
Example 4.1 For a distributed Brusselator system of equations simulating an autocatalytic reaction with diffusion
\[
\begin{align*}
0 &= v_{ss} + 1 - v + v^2 w, \\
w_t &= w_{ss} - v^2 w, \ s \in \Omega, \ t \in (0,1),
\end{align*}
\]

it is required to find a solution to the Showalter – Sidorov problem (4) with initial function
\[
w(s,0) = \sqrt{\frac{2}{\pi}} \sin(s) + \sqrt{\frac{2}{\pi}} \sin(2s), \ s \in \Omega,
\]

with the Dirichlet boundary condition
\[
v(s,t) = w(s,t) = 0, \ s \in \partial \Omega, \ t \in [0,1],
\]

where \( \Omega = (0,\pi) \).

The eigenfunctions of the homogeneous Dirichlet problem for the Laplace operator \((-\Delta)\) on the segment \((0,\pi)\), have the form:
\[
\varphi_k(s) = \sqrt{\frac{2}{\pi}} \sin(ks), \ k = 1, 2.
\]
We represent the sought functions as a Galerkin sum:

\[ v(s,t) = \sqrt{\frac{2}{\pi}} (a_1(t) \sin(s) + a_2(t) \sin(2s)), \]
\[ w(s,t) = \sqrt{\frac{2}{\pi}} (b_1(t) \sin(s) + b_2(t) \sin(2s)). \]  

Taking for \( r = a_1(0), v^\perp = a_2(t)\sqrt{\frac{2}{\pi}} \sin(2s), \ w^\perp = \sqrt{\frac{2}{\pi}} \sin(s) + \sqrt{\frac{2}{\pi}} \sin(2s), \) and substituting the obtained values into the formulas (18) we obtain \( \Delta(v^\perp) = 10.195082164. \) As follows from the Theorem 3.2 Showalter – Sidorov – Dirichlet problem (4), (26) for system of equations (24) for the given parameters of the system and initial data will have two solutions \((v_01(s,t), w_0(s,t))\) and \((v_02(s,t), w_0(s,t))\). These two solutions are presented in the figure 1 and figure 2.

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