Anharmonic oscillations of the single dust particle trapped in a standing striation

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Abstract. The nonlinear features of the vibrational motion of a single dust particle trapped in a standing strip are investigated. The excitation of the nonlinear oscillations with the large amplitude of the order of 1.5 mm is due to the periodic movement of the standing striation caused by the square-wave modulation of the discharge current. The frequency responses are investigated depending on value of the modulation depth. The anharmonic effects of the dust particle oscillations such as parametric instabilities and hysteresis are obtained. The theory of the anharmonic oscillator provides a good quantitative description of the experimental data. The values of the thresholds of excitation of parametric instabilities, the anharmonic coefficients and the critical values of the oscillation amplitude for the hysteresis are calculated. The potential energy curve of the single dust particle trapped in the striation is calculated using the values of anharmonic coefficients.

1. Introduction
Dusty plasma [1–5], like ordinary multicomponent plasma, has a feature that is characteristic of nonlinear systems in which energy transfer and dissipation are especially intense. So dusty plasma is an ideal model of the open dissipative system, in which nonlinear phenomena can be investigated at the kinetic level [6]. The diversity of such phenomena in complex plasma is due to the large number of types of oscillations. For example, in the case of an intense dynamic influence, when the value of the external force becomes comparable with the magnitude of the interparticle interaction, the nonlinear dust particle dynamics occur. The examples of such dynamics in dusty plasmas are solitons, shock waves, Mach cones, and others [7, 8].

The study of dust particle oscillations, induced by themselves or by external drive forces, is important for understanding the dynamic processes in dusty plasma systems. For example, the oscillatory motion of dust particles was exploited to obtain insight into the mechanism for the efficient energy transfer between degrees of freedom of the dusty plasma system. In [9] it was shown that the particle charge fluctuation is the reason for the appearance of forced resonance, which heats vertical oscillations.

Obviously, the excitation of the large-amplitude oscillations and the appearance of nonlinear effects are most easily realized near the eigenfrequency and resonant frequency of the systems. The features peculiar to anharmonic oscillator such as hysteresis of the frequency response curve, secondary resonances were investigated under the conditions of rf discharge dusty plasma in [10–12]. For instance, in paper [11], the evaluation of the potential energy of a dust particle
in the sheath field was made on the base of the experimental and theoretical investigations of the nonlinear vertical oscillations of the dust particle.

The investigation of the dust particle oscillation on a stratified glow discharge was carried out in [13, 14]. The relaxation oscillations of a single dust particle caused by low-frequency discharge current modulation were observed in [13]. The calculation of the dust particle charge with the help of the eigenfrequency and damping coefficient was made. In [14] the method of the discharge current modulation was used to obtain the amplitude-frequency characteristics (AFC) of the forced harmonic oscillations of a single dust particle trapped in the striation. The main resonance peaks at the frequency, which is close to the eigenfrequency of the dusty plasma system, were observed. Maxima at multiple of the resonant frequencies were obtained. The calculation of the Q-factor of the dusty-plasma oscillatory system was made.

In the present paper, the nonlinear vertical oscillations of a single dust particle induced by square wave modulation of current discharge are investigated in a stratified glow discharge. Amplitude-frequency characteristics of dust particle oscillations in dependence of modulation depth are measured at pressure $p = 0.16$ Torr. The detailed measurements of nonlinear features of the forced oscillations such as parametric instability and hysteresis are made. The theory of the anharmonic oscillator provides a good quantitative description of the experimental data.

2. Experimental setup

Nonlinear oscillations of the dust particle were caused by the imposition of an external force of large amplitude. The particle of dust was placed in the potential trap created by a standing striation. The low-frequency square-wave discharge current modulation led to the displacement of the striation. The nonlinear forced oscillations were caused by periodic movement of the dust particle equilibrium position.

The current discharge modulation was realized with the help of the experimental technique suggested in [13, 14]. The detailed description of the experimental setup is presented in figures 1 published in [13, 14]. A dc glow discharge was produced in neon at pressure $p = 0.16$ Torr and currents 2–3 mA. We conduct the observations in the standing striation, which was formed in the discharge tube with a radius $R = 1$ cm. We used the calibrated spherical melamine formaldehyde particles with diameters $d = 4.10 \pm 0.14 \mu m$. In the experiments the dust particles were injected into the discharge, and then fell to their equilibrium position. Levitating grain was visualized by illumination of a 30 mW diode laser. The laser beam width exceeded the dust particle size. The light scattered by dust grain was detected by a CCD video camera with the temporal resolution of 40 ms and the resolution of $960 \times 720$ pixels, located on the side of the tube. The error in measurement was about 12% as estimated in [13].

The discharge current modulator provided square wave output signals with on–off ratio of $\theta = 3/8$ with different values of modulation depth $\mu$. Under the experimental conditions $\mu$ was determined as follows $\mu = 1 - (i_{\text{min}}/i_{\text{max}})$. This definition of the modulation depth allowed uniformly changing the parameter $\mu$ to measure its influence on the amplitude-frequency characteristic. The current switching from $i_{\text{min}} = (1 - \mu)i_{\text{max}}$ to $i_{\text{max}} = 2.6$ mA leads to the rigid shift of all striations. In figure 1 (a) the photos of striation with the single dust particle are presented at two values of discharge current. The change of modulation depth in the range of $0.13 < \mu < 0.41$ results in the displacement of the striation by the order $\Delta Z = 1.5–3.0$ mm. When the current switches back from $i_{\text{max}}$ to $i_{\text{min}}$ the striations return to the initial positions. The shift of the striation by the order $\Delta Z$, which causes the dust particle movement from one equilibrium position to another and leads to the manifestation of nonlinear effects of forced oscillations.

In inert gas under low pressure and small current different types of striations (S, P, R) were observed [15]. These types of striation differ in lengths and the potential drop in one period. Under our experimental conditions P-type strata with length $L_p = 2.08$ cm was formed [13].
Figure 1. Illustration of the dust particle displacement caused by modulation of the discharge current: (a) image of striation with a single dust particle at two different values of discharge current $i_{\text{min}} = (1 - \mu)i_{\text{max}}$ (right) and $i_{\text{max}} = 2.6$ mA (left); $Z$-axis is directed from the cathode (C) to the anode (A); the double ended arrow shows the periodical displacement of the dust particle by an order $\Delta Z = 2$ mm; (b) discharge current modulated by square waves with on–ratio of $\theta = 3/8$ as a function of time.

The value of the average field was $E_0 = 4.25$ V/cm. The values of the dust particle charge $q(z_0)$ and the electric field profile at $E(z_0)$ at equilibrium position $z_0$ remained constant at two values of discharge current as discussed in [14].

The modulation of the discharge current was used to excite the forced vertical oscillations of the dust particle. The frequency of the driving force coincided with that of current modulation. Under the square wave modulation of discharge current the driving force took the form

$$f_{\text{sq}}(t) = \begin{cases} f_{\text{max}}, & \text{if } 0 < t < \theta T; \\ f_{\text{min}}, & \text{if } \theta T < t < T; \\ f[t + (j + 1)T] = f(t + jT), & \text{if } j = 0, 1, 2, \ldots \end{cases} \quad (1)$$

Here, $f_{\text{max}}$ is the maximum value of the driving force amplitude corresponding to the value of discharge current $i_{\text{max}}$, $f_{\text{min}}$ is the minimum value of the driving force amplitude corresponding to the value of discharge current $i_{\text{min}} = (1 - \mu)i_{\text{max}}$, $T$ is the period, $\theta$ is the on–off ratio.

3. Results on nonlinear oscillations of the single dust particle

Figure 2 shows the fragments of AFC of dust particle oscillations obtained under the square wave discharge current modulation with on–off ratio of $\theta = 3/8$ at pressure $p = 0.16$ Torr. Under the variation of modulation depth in the range $0.13 < \mu < 0.3$ with a granularity of 0.5 the frequency responses were measured close to the resonance frequencies $\nu_{\text{res}}$, $2\nu_{\text{res}}$. At lowest value of $\mu = 0.15$ the well-defined resonance peaks were observed at frequencies $\nu_{\text{res}} = 22$ Hz,
Figure 2. AFC fragments of forced vibrations of a single dust particle, measured near resonance maxima at frequencies (a) $\nu_{res}$ and (b) $2\nu_{res}$. AFC were obtained under the square wave discharge current modulation with on–off ratio $\theta = 3/8$ at four different values of modulation depth $\mu$ at $p = 0.16$ Torr.

the resonance at double frequency was not presented. The increase in modulation depth leads to the rise of the amplitude of oscillation, which in turn results in the manifestation of nonlinear effects. Let us discuss them in details.

3.1. Parametric instability
As one can see in figure 2(a), the frequency of the main resonance shifts toward lower frequencies with an increase in the amplitude of oscillations of dust particles, from the value of $\nu_{res} = 21$ Hz at $\mu = 0.15$ up to the value of $\nu_{res} = 19$ Hz at $\mu = 0.3$. The observed asymmetry of the shapes of the resonance peak rises with increase in the value of the modulation depth. At highest $\mu$ the amplitude jumps upward and thereafter decreases monotonically as $\nu$ is further increased.

In figure 2(b), it is shown that at lowest value of the modulation depth $\mu = 0.15$ the oscillation amplitude measured closed to $\nu_{res}$ takes the minimal values, and there is no characteristic maximum. When $\mu$ reaches the value of $\mu = 0.2$ the amplitude jumps upward at $2\nu_{res} = 43$ Hz thereby illustrating the threshold behavior of the onset of resonance at the double frequency.

The distinctive feature of dust particle oscillations such as the shift of the resonance frequency and secondary resonances, see figure 2, are the features peculiar to anharmonic oscillator as suggested above. So the quantitative description of the experimentally obtained dust particle oscillatory motion was made on the theory of the forced anharmonic oscillator. The equation describing the anharmonic oscillations with damping constant $\gamma$ under the action of the driving
The dependence of the amplitude $a$ with the help of the formula

$$
\ddot{z} + 2\gamma \dot{z} + \omega_0^2 z = \frac{f(t)}{M_d} - \alpha z^2 - \beta z^3,
$$

(2)

where $\omega_0$ is the eigenfrequency of dusty plasma oscillatory system, $\alpha$ and $\beta$ are the anharmonic coefficients. The driving force $f(t)$ given by the equation (1) corresponds to the discharge current modulation with on–off ratio of $\theta = 3/8$. We make the theoretical description of the resonance peaks at frequencies $\nu_{res}$ and $2\nu_{res}$ taking into account only the first term in the Fourier series of the driving force. The force amplitude is determined by the modulation depth as follow $(f_{\text{max}} - f_{\text{min}})/(2M_d) = F_0 = l\mu_{\text{max}}/2$, where $l$ is the coefficient of proportionality between the amplitude of the driving force and the change in discharge current caused by the imposition of modulating signals. The coefficient will be used further to calculate the threshold and critical values of the modulation depth $\mu_{\text{cr}}$ and $\mu_{\text{th}}$. We describe the dust particle oscillations separately in the vicinity of the main resonant frequency $\omega_0$ and double resonant frequency $2\omega_0$ as suggested in theory given in [16].

First, let us consider the main resonance $\omega = \omega_0$. The quantitative description of nonlinear AFC is made in a narrow region $\varepsilon = \omega - \omega_0$ where $|\varepsilon| \ll \omega_0$. This approach is normally valid in the limit $\gamma \ll \omega_0$ and is justified for our case. The value of eigenfrequency obtained under the same experimental conditions in [17] is $\omega_0/2\pi = 21$ Hz, the damping constant calculated with the help of the Epstein formula [18] takes the value of $\gamma = 15$ s$^{-1}$ at pressure $p = 0.16$ Torr. The dependence of the amplitude $a$ on $\varepsilon$ and $F_0$ is given by equation

$$
A^2 \left[ (\varepsilon - \alpha A)^2 + \gamma^2 \right] = \frac{F_0^2}{4\omega_0^2};
$$

(3)

$\kappa = 3\beta/(8\omega_0) - 5\alpha^2/(12\omega_0^3)$ characterizes nonlinear shift of the primary resonance frequency. The AFC of nonlinear dust particle oscillations measured close to main resonance, see figure 2(a), were described by equation (3) with the value of $\kappa = -1130$ s$^{-1}$ cm$^{-2}$. In figure 3(a) it is observed that resonance frequency shift by the order of 2 Hz caused by increasing modulation depth from 0.15 to 0.3 is well described by the theoretical curve. As in the case of linear forced oscillations, the maximum value of the amplitude of anharmonic oscillation can be calculated with the help of the formula

$$
A_{\text{max}} = \frac{F_0}{2\omega_0\gamma}.
$$

(4)

The coefficient $l$ was determined using the values of $A_{\text{max}}$ experimentally obtained at four values of modulation depth (see figure 2); $l = 1100$ cm s$^{-2}$/mA.

Second, let us consider the double resonance $2\omega_0$ in a narrow region $\varepsilon = \omega - 2\omega_0$. The amplitude of oscillation is given by the equation

$$
A^2 \left[ \left( \frac{\varepsilon}{2} - \kappa A^2 \right)^2 + \gamma^2 \right] = \frac{\alpha^2 A^2 F_0^2}{36\omega_0^6}.
$$

(5)

The resonance at double frequency caused by the parametric instability appears when the force amplitude takes the threshold value given by expression $F_{\text{th}} = 6\gamma\omega_0^3/\alpha$. Under the condition of $F_0 > F_{\text{th}}$ the oscillation amplitude is different from the minimum in the range of $\varepsilon_1 < \varepsilon < \varepsilon_2$, where

$$
\varepsilon_1 = -\varepsilon_2 = -\sqrt{\left[ \alpha F_0/(3\omega_0^3) \right]^2 - 4\gamma^2}.
$$

(6)

The AFC of nonlinear oscillations of the dust particle measured close to double resonance [see figure 2(b)] allows determining the boundary values $\varepsilon_1$, $\varepsilon_2$ and then to calculate the anharmonic coefficient $\alpha = -5.8 \times 10^5$ rad$^2$ s$^{-2}$/cm. The threshold value of the force amplitude is equal to $F_{\text{th}} = 400$ cm s$^{-2}$, which corresponds to threshold value of the modulation depth $\mu_{\text{th}} = 0.27$. 

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Figure 3. AFC of forced oscillations of the single dust particle measured close to the resonance maxima at frequency (a) $\nu_{\text{res}}$ and (b) $2\nu_{\text{res}}$. AFC were obtained under the square wave discharge current modulation with on–off ratio $\theta = 3/8$ at four different values of modulation depth $\mu$ at $p = 0.16$ Torr. The symbols represent experimental data, the solid line corresponds to approximation curve, described (a) by equation (3) and (b) by equation (5).

Figure 3(b) shows the amplitude-frequency characteristics measured close to the resonance maximum at double frequency $2\nu_{\text{res}}$ at four values of modulation depth described by equation (5). It is observed that the experimental data coincides with the calculated boundary values and fit quite well on the stable branches of the solution of equation (5) as follows from the theory of anharmonic oscillator [16].

3.2. Hysteresis

The asymmetry of the shapes of the resonance peak at frequency $\nu_{\text{res}}$ [see figure 2(a)] points to occurrence of amplitude hysteresis. In order to obtain the hysteresis resonance curve the frequency response of dust particle oscillations was measured in detail with increments of 0.5 Hz. The resonance curves were investigated for increasing and decreasing frequency of excitation. Figure 4 shows the AFC obtained close to the resonance maximum at frequency $\nu_{\text{res}} = 21$ Hz under the square wave discharge current modulation with modulation depth of $\mu = 0.36$. The hysteresis resonance curve shown in figure 4 makes it possible to determine the width of hysteresis zone, which takes the value of $\Delta\nu = 1.5$ Hz.
Figure 4. AFC of forced oscillations of the single dust particle obtained with increasing (black line) and with decreasing (red lines) frequency of excitation. AFC were obtained close to the resonance maxima at frequency $\nu_{res}$ under the square wave discharge current modulation with $\mu = 0.36$. The vertical arrows show the values of the resonance frequencies $\nu_+$ for increasing and $\nu_-$ for decreasing frequency of excitation, the double ended arrows indicates the width of hysteresis zone $\Delta \nu = \nu_+ - \nu_-$. Symbols represent experimental data; solid line corresponds to approximation curve, described by equation (3).

The quantitative description of the hysteresis phenomenon is made on the theory of the forced anharmonic oscillator. Figure 4 shows the resonance curve measured close to the main resonance and the theoretical description the experimental data of the help of equation (3). It is observed that the jump of the oscillation amplitude at $\nu_+$ for increasing excitation frequency and the drop of the oscillation amplitude at $\nu_-$ for decreasing excitation frequency are well described by the theoretical curve. The experimentally obtained value of the resonance amplitude at frequency $\nu_+$ coincides with the value of amplitude of resonance peak of theoretical curve in the error level. Hysteresis appears when the amplitude of the force exceeds the critical value defined as $F_{cr}^2 = 32\omega_0^2 \gamma^3/3\sqrt{3}|\kappa|$. The critical value of the force amplitude is equal to $F_{cr} = 600 \text{ cm s}^{-2}$, which corresponds to critical value of the modulation depth $\mu_{cr} = 0.41$.

3.3. Potential energy curve

The anharmonic coefficients determined by fitting the experimental data in subsection 3.1 make it possible to reconstruct the shape of the potential well, in which the dust particle oscillates. Figure 5 shows the potential energy curve calculated using the equation

$$
U(z) = M_d \left[ \frac{\omega_0^2 (z - z_0)^2}{2} + \frac{\alpha (z - z_0)^3}{3} + \frac{\beta (z - z_0)^4}{4} \right],
$$

where $\omega_0 = 139 \text{ rad/s}$, $\alpha = -5.8 \times 10^5 \text{ rad}^2 \text{s}^{-2}/\text{cm}$ and $\beta = 1.9 \times 10^7 \text{ rad}^2 \text{s}^{-2}/\text{cm}^2$. In figure 5, the dashed line represents the potential energy of simple harmonic oscillator. It is observed that the dust particle oscillations are harmonic when the dust particle displacement towards the cathode is of order $\Delta Z < 0.2$ and towards the anode is of order $\Delta Z < 0.55$. Considering the anharmonic terms $\alpha$ and $\beta$ [equation (7)] the potential well takes the form of an asymmetric parabola: the cathode branch of which becomes steeper than the anodic one. The nonlinearity
Figure 5. The potential energy of the single particle of dust at $p = 0.16$ Torr: solid line—the anharmonic oscillator; dashed line—the simple harmonic oscillator.

of the dusty-plasma oscillatory system results in an increase of the well spring constant: the same amount of energy would allow smaller dust grain oscillations.

4. Conclusion
In this paper, the nonlinear forced oscillations of a single dust particle trapped in a standing striation are investigated. The dust particle oscillations associated with the displacement of the striation were induced by the square wave discharge current modulation. The nonlinear effects of dust particle oscillations such as resonance at double frequency and hysteresis of the frequency response curve are measured. The theory of the anharmonic oscillator provides a good quantitative description of the experimental data. The values of the threshold of excitation of parametric instability at double resonance frequency, the anharmonic coefficients and the critical value of the oscillation amplitude for the hysteresis were calculated. The potential energy of the single dust particle trapped in a standing striation was calculated.

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