LINEAR-QUADRATIC OPTIMAL CONTROL FOR DISCRETE-TIME STOCHASTIC DESCRIPTOR SYSTEMS

YADONG SHU* AND BO LI

School of Mathematics and Statistics
Nanjing University of Information Science and Technology
Nanjing 210044, China
School of Applied Mathematics
Nanjing University of Finance and Economics
Nanjing 210023, China

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Abstract. In this paper, an optimal control model ruled by a class of linear discrete-time stochastic descriptor systems is considered under quadratic index performance. Employing dynamic programming method, a recurrence equation to simplify the optimal control problem is presented provided that the descriptor systems are both regular and impulse-free. When the objective function is quadratic, according to the recurrence equation, a discrete-time linear-quadratic optimal control problem is completely settled, that is, optimal controls and optimal values of the problem are both obtained through analytical expressions. At last, a numerical example about linear-quadratic optimal control for a discrete-time stochastic descriptor system is provided to illustrate the validness of the results derived.

1. Introduction. Descriptor systems, also known as singular systems and generalized state-space systems, are usually formulated by difference-algebraic equations (DAEs) or differential-algebraic equations. Descriptor systems[5, 11, 18] have been widely investigated among last several decades because a large number of natural phenomena and practical systems can be modeled through descriptor systems in many fields, for example, power industry[21], management science[26] and robot technology[25]. Cobb[8] considered three internal properties including controllability, observability and duality for a class of descriptor systems. Ishihara and Terra[11] analyzed Lyapunov stability for descriptor systems according to two generalized Lyapunov equations. Shu and Zhu[27] presented several theorems to judge stability in measure, stability in mean and almost sure stability for uncertain descriptor systems. These theoretical results about descriptor systems are able to provide the precondition for discussing the optimal control problems ruled by descriptor systems.

In the middle of last century, optimal control theory which is an important branch of modern control field began to attract the attention of so many mathematicians. Bellman's principle of optimality, the minimum principle proposed by Pontryagin

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* Corresponding author.
and Kalman’s filtering method laid a solid foundation for optimal control theory. Then with the rapid progresses of mathematics and computer science, optimal control theory has made lots of remarkable developments in theory\cite{2, 14}, and also in practice\cite{24, 1} such as investment, production engineering and space technology. Seeking for the best input to optimize a given objective function defines an optimal control problem. An optimal control problem subject to linear dynamic systems with quadratic performance index which is generally called a linear-quadratic (LQ) control problem.

The linear-quadratic control plays a very significant role in optimal control field on account of that quadratic objective function is frequently used and state feedback control is convenient to realize in practical situations. The research on deterministic LQ optimal control problem was initiated by Kalman\cite{12}, and then plenty of researchers started to investigate LQ control problems. By using the maximum principle, a necessary optimality condition is derived in \cite{3} to simplify an optimal linear-quadratic model ruled by a dynamic system with time-delay controls. In 2009, Zhang et al.\cite{31} discussed a LQ optimal control model for discrete-time switched systems with a specific value function in depth. Due to the complexity of real world, random factors were introduced into optimal control models and their applications, such as in \cite{4} for economics. Wonham\cite{29} first considered stochastic LQ control problem and obtained a matrix Riccati equation. Chen and Wu\cite{7} discussed an optimal control model for continuous-time stochastic systems with delay, and settled a production and consumption choice problem. An aircraft trajectory optimization problem was transformed into a stochastic LQ control model and was thoroughly solved in \cite{20}. The above results of optimal control are obtained subject to normal systems, while LQ control model for descriptor systems will be studied in this paper.

In past 40 years, several meaningful works about LQ optimal control problems ruled by (stochastic) descriptor systems have been published. Mantas and Krikels\cite{23} considered the LQ optimal control problem subject to discrete descriptor systems under some proper boundary conditions. In \cite{16}, the generalized Euler-Lagrange equations were employed to analyze LQ control problems for descriptor systems under variable coefficients. In \cite{15}, optimal controls were approximately presented by using a numerical method called neural network algorithm for a LQ optimal control model subject to stochastic descriptor systems. Under two premises about the rank, Feng et al.\cite{10} proposed a singular LQ control problem for discrete-time stochastic descriptor systems in finite-time horizon. A linear-quadratic Pareto optimal control problem ruled by stochastic descriptor systems was discussed in \cite{32}, and the existence of optimal solutions was demonstrated by generalized differential Riccati equations. Optimal controls and optimal values derived in these papers are either imprecise or just satisfy a matrix Riccati equation, while in this work they are both given through analytical expressions. The results obtained in Section 3 and Section 4 indicate that the realization of optimal controls is very simple in practice.

The first contribution of this paper is that a recurrence equation is derived to simplify optimal control models ruled by a class of discrete-time stochastic descriptor systems. Compared with the results in \cite{13, 30}, the recurrence equation can be employed to investigate more types of discrete-time optimal control problems and has more applications in practical areas. The second one is that a stochastic LQ optimal control problem is considered, and optimal solutions including optimal controls and optimal values are both presented in analytical form that is easy to put
into operation, while in lots of related papers [15, 32] only approximate solutions are obtained. The last contribution is that the optimal feedback controls decided by initial sub-state \( x_1(k) \), terminal sub-state \( x_2(N) \) and expected values of some random variables reveal an essential difference between LQ optimal control models subject to stochastic descriptor systems and stochastic normal systems. In short, this work makes a contribution to broadening the horizon of stochastic optimal control and its applications.

In this work, optimal control models for linear discrete-time stochastic descriptor systems will be investigated with quadratic performance index. This paper is organized as follows: in the next section, under the condition that the stochastic descriptor system is both regular and impulse-free, a type of equivalent form including two sub-systems will be introduced for the descriptor system. In Section 2, an optimal control model ruled by linear stochastic descriptor systems will be discussed, and according to the principle of dynamic programming a recurrence equation is proposed to simplify the model. In Section 3, an optimal control model subject to discrete-time stochastic descriptor systems will be settled by employing the recurrence equation, and optimal controls and optimal values are presented through analytical expressions. In Section 4, on the premise of several discussions, and according to the principle of dynamic programming a recurrence equation is proposed to simplify the model. In Section 5, a numerical example of a discrete-time LQ optimal control problem will be solved to display the effectiveness of the results derived.

2. Discrete-time stochastic descriptor systems. In this section, a linear discrete-time stochastic descriptor system is introduced as the following matrix difference equation:

\[
\begin{cases}
Fx(j + 1) = Ax(j) + Bu(j) + D\xi_j, \\
\quad j = 0, 1, 2, \cdots, N - 1,
\end{cases}
\tag{1}
\]

where \( x(j) \in \mathbb{R}^n \) is the state vector of the system at stage \( j \), \( u(j) \in U_j \subset \mathbb{R}^m \) is the input variable at stage \( j \) with the constraint domain \( U_j \), and \( A, F \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m} \) are known coefficient matrices associated with \( x(j) \) and \( u(j) \), respectively. \( F \) is a given (singular) matrix with \( \text{rank}(F) = q \leq n \), and \( \text{deg}(\text{det}(zF - A)) = r \) where \( z \) is a complex variable. \( D \in \mathbb{R}^n \) is a vector, and \( \xi_0, \xi_1, \cdots, \xi_{N-1} \) are random variables representing the disturbance of the system.

**Definition 2.1.** \( \text{(9)} \)

(a) System (1) is said to be regular if \( \text{det}(zF - A) \) is not identically zero;
(b) System (1) is called impulse-free when \( \text{deg}(\text{det}(zF - A)) = \text{rank}(F) \).

**Lemma 2.1.** \( \text{(9)} \) If system (1) is both regular and impulse-free, it is equivalent to system (2):

\[
\begin{cases}
x_1(j + 1) = A_1x_1(j) + B_1u(j) + D_1\xi_j, \\
0 = x_2(j) + B_2u(j) + D_2\xi_j,
\end{cases}
\tag{2}
\]

where \( x(j) = [x_1(j) \ x_2(j)] \), \( x_1(j) \in \mathbb{R}^r, x_2(j) \in \mathbb{R}^{n-r}, r = q, \) and \( A_1 \in \mathbb{R}^{r \times r}, B_1 \in \mathbb{R}^{r \times m}, B_2 \in \mathbb{R}^{(n-r) \times m}, D_1 \in \mathbb{R}^r, D_2 \in \mathbb{R}^{n-r} \).

**Remark 2.1.** Matrices \( Q, A_1, B_1, B_2 \) and vectors \( D_1, D_2 \) listed in Lemma 2.1 are obtained by a method called Weierstrass decomposition: since system (1) is
both regular and impulse-free, there exist two invertible matrices $P \in \mathbb{R}^{n \times n}$ and $Q \in \mathbb{R}^{n \times n}$ for the pair $(F, A)$ such that the following equalities are satisfied:

\[ P F Q = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \quad P A Q = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{bmatrix}, \]

where $r = \text{rank}(F) = q$, and $PB$, $PD$ can be written as follows:

\[ PB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad PD = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}. \]

**Remark 2.2.** According to the equalities $0 = x_2(j) + B_2 u(j) + D_2 \xi_j$, $j = 0, 1, 2, \cdots, N - 1$ in Lemma 2.1, we obviously obtain that

\[ x_2(j) = -B_2 u(j) - D_2 \xi_j, \quad j = 0, 1, 2, \cdots, N - 1. \]

These equations clearly show that sub-state $x_2(j)$ is with respect to the present input $u(j)$ and random variable $\xi_j$, while it has nothing to do with its former sub-state $x_2(j-1)$, and there does not exist any restricted conditions on the terminal sub-state $x_2(N)$. To sum up, this peculiarity of stochastic descriptor systems essentially differs from it of stochastic normal systems which may bring more difficulties in analysis and more extensive applications in some practical fields.

**Remark 2.3.** In Lemma 2.1, system (2) is proposed as a equivalent system for stochastic descriptor system (1), and it is composed of two sub-systems which are expressed by stochastic difference equation and stochastic algebraic equation. In brief, Lemma 2.1 and Lemma 3.1 (similar with Lemma 2.1) can provide a solid foundation to settle the main problems in this work, so they will play significant roles in Section 3 and Section 4.

### 3. Optimal control model for linear discrete-time stochastic descriptor systems

Throughout this article, system (1) is always supposed to be both regular and impulse-free. Under such assumption, an optimal control model subject to a linear discrete-time stochastic descriptor system will be considered in the following:

\[
\begin{aligned}
J(0, x(0)) &= \inf_{u(\cdot) \in U} \mathbb{E} \left[ \sum_{j=0}^{N} f(x(j), u(j), j) \right] \\
\text{subject to} \quad Fx(j + 1) &= Ax(j) + Bu(j) + D\xi_j, \\
&= 0, 1, 2, \cdots, N - 1,
\end{aligned}
\]

where $\mathbb{E}[\cdot]$ is the expectation operator of a random variable, and it is commonly employed to rank different random variables.

According to Lemma 2.1, the following conclusion is plainly derived.

**Lemma 3.1.** Problem (3) can be equivalently transformed into problem (4):

\[
\begin{aligned}
J(0, x_1(0), x_2(N)) &= \inf_{u(\cdot) \in U} \mathbb{E} \left[ \sum_{j=0}^{N-1} g_j(x_1(j), u(j), \xi_j, j) + g_N(x_1(N), x_2(N), u(N), N) \right] \\
\text{subject to} \quad x_1(j + 1) &= A_1 x_1(j) + B_1 u(j) + D_1 \xi_j, \\
0 &= x_2(j) + B_2 u(j) + D_2 \xi_j, \\
&= 0, 1, 2, \cdots, N - 1,
\end{aligned}
\]
where
\[ x(j) = Q \begin{bmatrix} x_1(j) \\ x_2(j) \end{bmatrix}, \]
for \( j = 0, 1, \cdots, N \), and
\[
\begin{align*}
f(x(N), u(N), N) & = f(Q_1 x_1(N) + Q_2 x_2(N), u(N), N) \\
& \triangleq g_N(x_1(N), x_2(N), u(N), N), \\
f(x(j), u(j), j) & = f(Q_1 x_1(j) + Q_2 x_2(j), u(j), j) \\
& = f(Q_1 x_1(j) - Q_2 B_2 u(j) - Q_2 D_2 \xi_j, u(j), j) \\
& \triangleq g_j(x_1(j), u(j), \xi_j, j),
\end{align*}
\]
for \( j = 0, 1, \cdots, N - 1 \), and \( Q = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}, \ Q_1 \in \mathbb{R}^{n \times r}, \ Q_2 \in \mathbb{R}^{n \times (n-r)}. \)

Now let \( J(k, x_1(k), x_2(N)) \) be the optimal reward obtainable in \([k, N]\) for any \( 0 \leq k \leq N \) with the condition at stage \( k \) that we are in the sub-states \( x_1(k) \) and \( x_2(N). \)

**Remark 3.1.** Lemma 3.1 implies that optimal solutions of problem (3) can be acquired by settling problem (4). Because the initial state
\[
x(0) = Q \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = [Q_1 \ Q_2] \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = Q_1 x_1(0) + Q_2 x_2(0),
\]
and the equality
\[ x_2(0) = -B_2 u(0) - D_2 \xi_0 \]
according to Remark 2.2, we know that \( J(0, x(0)) \) is not related to the sub-state \( x_2(0). \) By Remark 2.2, we easily get that terminal sub-state \( x_2(N) \) is arbitrary. To sum up, \( J(0, x(0)) \) is decided by initial sub-state \( x_1(0) \) and terminal sub-state \( x_2(N) \), so it can be equivalently rewritten as a new symbol \( J(0, x_1(0), x_2(N)) \). Such property reveals a very significant distinction between discrete-time optimal control problems for stochastic descriptor systems and stochastic normal systems.

Optimal controls of problem (4) are denoted by \( u^*(k), k = 0, 1, \cdots, N. \) Based on the principle of dynamic programming, the following recurrence equation to simplify problem (4) can be deduced.

**Theorem 3.1.** For problem (4), we have the following recurrence equation
\[
\begin{align*}
J(N, x_1(N), x_2(N)) &= \inf_{u(N) \in U_N} E\left[ g_N(x_1(N), x_2(N), u(N), N) \right], \\
J(k, x_1(k), x_2(N)) &= \inf_{u(k) \in U_k} E\left[ g_k(x_1(k), u(k), \xi_k, k) + J(k + 1, x_1(k + 1), x_2(N)) \right], \quad (5)
\end{align*}
\]
for \( k = N - 1, N - 2, \cdots, 1, 0. \)

**Proof.** The first equality of Equation (5) is obviously true. Denote the right side of the second equality in (5) by \( \tilde{J}(k, x_1(k), x_2(N)) \). It follows from the definition of
Taking the infimum with respect to \( u(j) \), \( j = k + 1, k + 2, \cdots, N \) in Equation (6), and then \( u(k) \) in (6), we obtain that 
\[
J(k, x_1(k), x_2(N)) \leq \tilde{J}(k, x_1(k), x_2(N)).
\]

On the other hand, for every \( u(j), j = k, k + 1, \cdots, N \), we have
\[
E\left[ \sum_{j=k}^{N-1} g_j(x_1(j), u(j), \xi_j, j) + g_N(x_1(N), x_2(N), u(N), N) \right] = E\left[ g_k(x_1(k), u(k), \xi_k, k) + \sum_{j=k+1}^{N-1} g_j(x_1(j), u(j), \xi_j, j) + g_N(x_1(N), x_2(N), u(N), N) \right] \geq E\left[ g_k(x_1(k), u(k), \xi_k, k) + J(k + 1, x_1(k + 1), x_2(N)) \right] \geq \inf_{u(k) \in U_k} E\left[ g_k(x_1(k), u(k), \xi_k, k) + J(k + 1, x_1(k + 1), x_2(N)) \right],
\]
that is, 
\[
J(k, x_1(k), x_2(N)) \geq \tilde{J}(k, x_1(k), x_2(N)).
\]

In conclusion, \( J(k, x_1(k), x_2(N)) = \tilde{J}(k, x_1(k), x_2(N)) \) is established. This completes the proof. \( \square \)

Theorem 3.1 indicates us that to simplify problem (4) can be transformed to settle the easier problems (5) step by step from the last stage to the initial stage in reverse order. This result lays theoretical foundation for the research on a linear-quadratic optimal control problem in next section.

4. Linear-quadratic optimal control for discrete-time stochastic descriptor systems. On the basis of the discussions in previous sections, the LQ optimal control problem ruled by a discrete-time stochastic descriptor system will be investigated as the following:

\[
\begin{aligned}
J(0, x(0)) &= \inf_{u(0) \in U_{ad}} E \left\{ x^T(N)S_Nx(N) + \sum_{j=0}^{N-1} \left[ x^T(j)S_jx(j) + u^T(j)R_ju(j) \right] \right\} \\
\text{subject to} & \\
F x(j + 1) &= Ax(j) + Bu(j) + D\xi_j, \\
j &= 0, 1, 2, \cdots, N - 1,
\end{aligned}
\]

where \( R_j > 0, S_j \geq 0, j = 0, 1, \cdots, N - 1 \) are known symmetric matrices, and \( S_N > 0 \) is a given terminal penalty matrix, and \( U_{ad} = R^m \) represents the admissible set of input vectors. Note that \( R_j > 0 \) means that matrix \( R_j \) is positive definite, and \( S_j \geq 0 \) tells us that \( S_j \) is a positive semi-definite matrix. For each \( j \in \{0, 1, \cdots, N - 1\} \), \( \xi_j \sim N(e_j, \sigma_j^2) \) is a normal random variable with expected value \( e_j \) and standard deviation \( \sigma_j \).
In order to make problem (7) simpler, the following proposition is presented by employing Lemma 2.1.

**Lemma 4.1.** Problem (7) is equivalent to problem (8):

\[
J(0, x_1(0), x_2(N)) = \inf_{u(t) \in U_{ad}} \mathbb{E} \left\{ \left( Q_1 x_1(N) + Q_2 x_2(N) \right)^T S_N \left( Q_1 x_1(N) + Q_2 x_2(N) \right) \right\} \\
+ \sum_{j=0}^{N-1} \left[ \left( Q_1 x_1(j) + Q_2 x_2(j) \right)^T S_j \left( Q_1 x_1(j) + Q_2 x_2(j) \right) + u^T(j) R_j u(j) \right]
\]

subject to

\[
x_1(j+1) = A_1 x_1(j) + B_1 u(j) + D_1 \xi_j, \\
x_2(j) = B_2 u(j) + D_2 \xi_j, \\
\text{for } j = 0, 1, 2, \ldots, N - 1,
\]

where

\[
x(j) = Q \begin{bmatrix} x_1(j) \\ x_2(j) \end{bmatrix},
\]

for \( j = 0, 1, \ldots, N \), and \( Q = [Q_1 \ Q_2], \ Q_1 \in \mathbb{R}^{n \times r}, \ Q_2 \in \mathbb{R}^{n \times (n-r)} \).

Applying Equation (5), we are able to obtain the exact solutions for problem (8), which means that problem (7) is also settled according to Lemma 4.1.

**Theorem 4.1.** If the inequality

\[
R_k - B_1^T H_{k+1}^{-1} \tilde{R}_{k+1}^{-1} H_{k+1} B_1 > 0
\]

is satisfied for each \( k \in \{N - 2, N - 3, \ldots, 0\} \), where

\[
\tilde{R}_{N-1} = R_{N-1} + B_1^T Q_1^T S_N Q_1 B_1 + B_2^T Q_2^T S_{N-1} Q_2 B_2,
\]

\[
\tilde{R}_k = R_k + B_1^T \left[ \sum_{i=k}^{N-1} (A_1^i)^{i-k} Q_1^T S_i Q_1 (A_1)^{i-k} \right] B_1 + B_2^T Q_2^T S_k Q_2 B_2
\]

\[
- B_1^T H_{k+1}^T \tilde{R}_{k+1}^{-1} H_{k+1} B_1,
\]

for \( k = N - 2, N - 3, \ldots, 1, 0 \), and

\[
H_{N-1} = B_1^T Q_1^T S_N Q_1 A_1 - B_2^T Q_2^T S_{N-1} Q_1,
\]

\[
H_k = B_1^T \left[ \sum_{i=k}^{N-1} (A_1^i)^{i-k} Q_1^T S_i Q_1 (A_1)^{i-k} \right] A_1 - B_2^T Q_2^T S_k Q_1 - B_1^T H_{k+1}^T \tilde{R}_{k+1}^{-1} H_{k+1} A_1,
\]

for \( k = N - 2, N - 3, \ldots, 1, 0 \), then optimal controls of problem (8) exist. And the optimal controls are provided by

\[
u^*(N) \in \mathbb{R}^n,
\]

\[
u^*(k) = -\tilde{R}_k^{-1} \left[ G_k x_2(N) + H_k x_1(k) + \sum_{i=k}^{N-1} \beta_i^{i-k} e_i \right],
\]

for \( k = N - 1, N - 2, \ldots, 1, 0 \), where

\[
G_{N-1} = B_1^T Q_1^T S_N Q_2,
\]

\[
G_k = B_1^T (A_1^N)^{N-1-k} Q_1^T S_N Q_2 - B_1^T H_{k+1}^T \tilde{R}_{k+1}^{-1} H_{k+1} G_{k+1}, \quad \text{for } k = N - 2, N - 3, \ldots, 1, 0,
\]
and
\[ \beta_{N-1}^l = B_1^T Q_1^T S_N Q_1 D_1 + B_2^T Q_2^T S_{N-1} Q_2 D_2, \]
\[ \beta_{N-1}^l = B_1^T \left( A_1^T Q_1^T S_N Q_1 D_1 - Q_1^T S_{N-2} Q_2 D_2 - H_{N-1}^T R_{N-1}^{-1} \beta_{N-1}^{l-1} \right), \]
for \( l = 1, 2, \ldots, N - 1 \), and
\[ \beta_0^k = B_1^T \left[ \sum_{i=k}^{N-1} (A_1^T)^i Q_1^T S_{i+1} Q_1 (A_1)^{i-k} \right] D_1 + B_2^T Q_2^T S_k Q_2 D_2 \]
\[ - B_1^T H_{k+1}^T \bar{R}_{k+1}^{-1} H_{k+1} D_1, \]
\[ \beta_k^k = B_1^T \left( A_1^T Q_1^T S_k Q_1 D_1 - Q_1^T S_{k-1} Q_2 D_2 - H_k^T \bar{R}_k^{-1} \beta_{k-1}^k \right), \]
for \( k = N - 2, N - 3, \ldots, 1, 0 \), and \( l = 1, 2, \ldots, k \).
The expressions of optimal values are
\[ J(N, x_1(N), x_2(N)) = x_1^T (N) Q_1^T S_N Q_1 x_1 (N) + 2 x_1^T (N) Q_1^T S_N Q_2 x_2 (N) + x_2^T (N) Q_2^T S_N Q_2 x_2 (N), \]
\[ J(N - 1, x_1(N - 1), x_2(N)) = x_1^T (N - 1) \left( Q_1^T S_{N-1} Q_1 + A_1^T Q_1^T S_N Q_1 A_1 \right) x_1 (N - 1) \]
\[ + 2 x_1^T (N - 1) A_1^T Q_1^T S_N Q_2 x_2 (N) \]
\[ + x_2^T (N) Q_2^T S_N Q_2 x_2 (N) - \left[ H_{N-1} x_2 (N) + H_{N-1} x_1 (N - 1) + \beta_0^0 \right] (x_1(N) - x_2(N)) \]
\[ - B_1^T H_{N-1}^T \bar{R}_{N-1}^{-1} \beta_{N-1} \right] + 2 x_1^T (N - 1) \left( A_1^T Q_1^T S_{N-1} Q_1 D_1 - Q_1^T S_{N-2} Q_2 D_2 \right) + x_2^T (N) Q_2^T S_1 Q_1 D_1 \]
\[ + \left( D_1^T Q_1^T S_N Q_1 D_1 + D_2^T Q_2^T S_{N-1} Q_2 D_2 \right) \]
Proof. Denote the optimal controls for problem (8) by \( u^*(0), u^*(1), \ldots, u^*(N) \). According to the recurrence equation (5), we have

\[
J(N, x_1(N), x_2(N)) = \inf_{u^{(N)} \in \mathbb{R}^m} E \left\{ \left[ Q_1 x_1(N) + Q_2 x_2(N) \right]^T S_N \left[ Q_1 x_1(N) + Q_2 x_2(N) \right] \right\} \\
= x_1^T(N) Q_1^T S_N x_1(N) + 2x_1^T(N) Q_1^T S_N x_2(N) + x_2^T(N) Q_2^T S_N x_2(N),
\]

where \( u^*(N) \in \mathbb{R}^m \).

For \( k = N - 1 \), by using the recurrence equation (5), we have

\[
J(N - 1, x_1(N - 1), x_2(N)) = \inf_{u^{(N-1)} \in \mathbb{R}^m} E \left\{ \left[ Q_1 x_1(N - 1) + Q_2 x_2(N - 1) \right]^T S_N \left[ Q_1 x_1(N - 1) + Q_2 x_2(N - 1) \right] \right\} \\
= x_1^T(N - 1) Q_1^T S_N x_1(N - 1) + 2x_1^T(N - 1) Q_1^T S_N x_2(N - 1)
\]

+ \( x_2^T(N - 1) Q_2^T S_N x_2(N - 1) + u^T(N - 1) R_{N-1} u(N - 1) + x_1^T(N) Q_1^T S_N x_1(N) \)

+ \( x_2^T(N) Q_2^T S_N x_2(N) + x_2^T(N) Q_2^T S_N x_2(N) \)

\[
= \inf_{u^{(N-1)} \in \mathbb{R}^m} E \left\{ \left[ Q_1 x_1(N - 1) + Q_2 x_2(N - 1) \right]^T S_N \left[ Q_1 x_1(N - 1) + Q_2 x_2(N - 1) \right] \right\} \\
- 2x_1^T(N - 1) Q_1^T S_N x_2(N) [B_2 u(N - 1) + D_2 \xi_{N-1}]
\]

+ \( [B_2 u(N - 1) + D_2 \xi_{N-1}]^T Q_2^T S_{N-1} Q_2 [B_2 u(N - 1) + D_2 \xi_{N-1}] + u^T(N - 1) R_{N-1} u(N - 1) \)

+ \( A_1 x_1(N - 1) + B_1 u(N - 1) + D_1 \xi_{N-1} \]^T \( Q_1^T S_N x_1 \left[ A_1 x_1(N - 1) + B_1 u(N - 1) + D_1 \xi_{N-1} \right] \)

+ \( 2 \left[ A_1 x_1(N - 1) + B_1 u(N - 1) + D_1 \xi_{N-1} \right]^T Q_1^T S_N Q_2 x_2(N) + x_2^T(N) Q_2^T S_N x_2(N) \)

\[
= x_1^T(N - 1) \left( Q_1^T S_{N-1} Q_1 + A_1^T Q_1^T S_N x_1 A_1 \right) x_1(N - 1) + 2x_2^T(N - 1) A_1^T Q_2^T S_N x_2(N)
\]

+ \( x_2^T(N) Q_2^T S_N x_2(N) \)

+ \( \inf_{u^{(N-1)} \in \mathbb{R}^m} \left\{ u^T(N - 1) \left[ R_{N-1} + B_1^T Q_1^T S_N Q_1 B_1 + B_2^T Q_2^T S_{N-1} Q_2 B_2 \right] \right\} \\
\cdot u(N - 1) + 2u^T(N - 1) \left[ B_1^T Q_1^T S_N x_2(N) + B_2^T Q_2^T S_N x_2(N) \right] x_1(N - 1)
\]

+ \( B_1^T Q_1^T S_N Q_1 D_1 + B_2^T Q_2^T S_{N-1} Q_2 D_2 \) \( E(\xi_{N-1}) \).
Obviously, optimal control $u$ because

Now denote

Hence, the optimal value from this stage to final stage is

because

Now denote

and because

Obviously, optimal control $u^*(N-1)$ satisfies the following equation:

and because

therefore we obtain that

Hence, the optimal value from this stage to final stage is
\[ R_{N-1}^{-1} \left[ G_{N-1} x_2(N) + H_{N-1} x_1(N-1) + \beta_{N-1}^0 e_{N-1} \right] \\
+ 2 \left[ x_1^T(N-1) \left( A_1^T Q_1^2 S_N Q_1 D_1 - Q_1^2 S_{N-1} Q_2 D_2 \right) + x_2^T(N) Q_2^2 S_N Q_1 D_1 \right] e_{N-1} \\
+ \left( D_1^T Q_1^2 S_N Q_1 D_1 + D_2^T Q_2^2 S_{N-1} Q_2 D_2 \right) (\sigma_{N-1}^2 + e_{N-1}^2) \]

For \( k = N - 2 \), with the help of recurrence equation (5), we get that

\[ J(N-2, x_1(N-2), x_2(N)) = \inf_{u(N-2) \in \mathbb{R}^m} \mathbb{E} \left\{ \left[ Q_1 x_1(N-2) + Q_2 x_2(N-2) \right]^T S_{N-2} \left[ Q_1 x_1(N-2) + Q_2 x_2(N-2) \right] \\
+ u^T(N-2) R_{N-2} u(N-2) + J(N-1, x_1(N-1), x_2(N)) \right\} \]

\[ = \inf_{u(N-2) \in \mathbb{R}^m} \mathbb{E} \left\{ x_1^T(N-2) Q_1^2 S_{N-2} Q_1 x_1(N-2) + 2 x_1^T(N-2) Q_1^2 S_{N-2} Q_2 x_2(N-2) \\
+ x_1^T(N-1) (Q_1^T S_{N-1} Q_1 + A_1^T Q_1^T S_N Q_1 A_1) x_1(N-1) + 2 x_1^T(N-1) A_1^T Q_1^T S_N Q_2 x_2(N) \\
- \left[ G_{N-1} x_2(N) + H_{N-1} x_1(N-1) + \beta_{N-1}^0 e_{N-1} \right]^T R_{N-1}^{-1} \left[ G_{N-1} x_2(N) \\
+ H_{N-1} x_1(N-1) + \beta_{N-1}^0 e_{N-1} \right] + x_2^T(N) Q_2^2 S_N Q_2 x_2(N) \\
+ 2 \left[ x_1^T(N-1) (A_1^T Q_1^T S_N Q_1 D_1 - Q_1^T S_{N-1} Q_2 D_2) + x_2^T(N) Q_2^2 S_N Q_1 D_1 \right] e_{N-1} \\
+ \left( D_1^T Q_1^2 S_N Q_1 D_1 + D_2^T Q_2^2 S_{N-1} Q_2 D_2 \right) (\sigma_{N-1}^2 + e_{N-1}^2) \right\} \]

\[ = \inf_{u(N-2) \in \mathbb{R}^m} \mathbb{E} \left\{ x_1^T(N-2) Q_1^2 S_{N-2} Q_1 x_1(N-2) \\
- 2 x_1^T(N-2) Q_1^2 S_{N-2} Q_2 \left[ B_2 u(N-2) + D_2 \xi_{N-2} \right] \\
+ \left[ B_2 u(N-2) + D_2 \xi_{N-2} \right]^T Q_2^2 S_{N-2} Q_2 \left[ B_2 u(N-2) + D_2 \xi_{N-2} \right] + u^T(N-2) R_{N-2} u(N-2) \\
+ \left[ A_1 x_1(N-2) + B_1 u(N-2) + D_1 \xi_{N-2} \right]^T \left[ Q_1^T S_{N-1} Q_1 + A_1^T Q_1^T S_N Q_1 A_1 \right] \left[ A_1 x_1(N-2) + B_1 u(N-2) + D_1 \xi_{N-2} \right] \\
+ \left[ B_1 u(N-2) + D_1 \xi_{N-2} \right] + 2 \left[ A_1 x_1(N-2) + B_1 u(N-2) + D_1 \xi_{N-2} \right]^T A_1^T Q_1^T S_N Q_2 \xi_{N-2} \\
+ x_1^T(N) Q_1^2 S_N \xi_{N-2} + \left[ A_1 x_1(N-2) + B_1 u(N-2) + D_1 \xi_{N-2} \right]^T H_{N-1}^T R_{N-1}^{-1} H_{N-1}^{-1} \\
\cdot \left[ A_1 x_1(N-2) + B_1 u(N-2) + D_1 \xi_{N-2} \right] \\
- 2 \left[ A_1 x_1(N-2) + B_1 u(N-2) + D_1 \xi_{N-2} \right]^T H_{N-1}^T R_{N-1}^{-1} G_{N-1} x_2(N) + \beta_{N-1}^0 e_{N-1} \\
- \left[ G_{N-1} x_2(N) + \beta_{N-1}^0 e_{N-1} \right]^T R_{N-1}^{-1} \left[ G_{N-1} x_2(N) + \beta_{N-1}^0 e_{N-1} \right] \\
+ \left( D_1^T Q_1^2 S_N Q_1 D_1 + D_2^T Q_2^2 S_{N-1} Q_2 D_2 \right) (\sigma_{N-1}^2 + e_{N-1}^2) + 2 x_2^T(N) Q_2^2 S_N Q_1 D_1 e_{N-1} \\
+ \left[ A_1 x_1(N-2) + B_1 u(N-2) + D_1 \xi_{N-2} \right]^T \left[ A_1^T Q_1^T S_N Q_1 D_1 - Q_1^T S_{N-1} Q_2 D_2 \right] e_{N-1} \right\} \]

\[ = x_1^T(N-2) \left[ Q_1^T S_{N-2} Q_1 + A_1^T Q_1^T S_N Q_1 + A_1^T Q_1^T S_N Q_1 A_1 \right] A_1 \\
- A_1^T H_{N-1}^T R_{N-1}^{-1} H_{N-1} \cdot x_1(N-2) + 2 x_1^T(N-2) \left( A_1^T Q_1^T S_N Q_2 - A_1^T H_{N-1}^T R_{N-1}^{-1} G_{N-1} \right) x_2(N) \\
+ x_2^T(N) \left( Q_2^2 S_N Q_2 - G_{N-1}^2 R_{N-1}^{-1} G_{N-1} \right) x_2(N) + \inf_{u(N-2) \in \mathbb{R}^m} \left\{ u^T(N-2) R_{N-2} \right\} \]
is true based on the inequality (9). Then denote

\[
\begin{align*}
&+ B_1^T \left( Q_1^T S_{N-1} Q_1 + A_1^T Q_1^T S N Q_1 A_1 \right) B_1 + B_2^T Q_2^T S_{N-2} Q_2 B_2 \\
&- B_1^T H_{N-1}^T \tilde{R}_{N-1}^{-1} H_{N-1} B_1 \right) u(N-2) \\
&+ 2u^T(N-2) \left[ B_1^T A_1^T Q_1^T S N Q_2 - B_1^T H_{N-1}^T \tilde{R}_{N-1}^{-1} G_{N-1} \right] x_2(N) \\
&+ 2u^T(N-2) \left[ B_1^T \left( Q_1^T S_{N-1} Q_1 \right) \\
&+ A_1^T Q_1^T S_{N-1} A_1 \right) A_1 - B_2^T Q_2^T S_{N-2} Q_2 - B_2^T H_{N-1}^T \tilde{R}_{N-1}^{-1} H_{N-1} A_1 \right] x_1(N-2) \\
&+ 2u^T(N-2) \left[ B_1^T \left( Q_1^T S_{N-1} Q_1 + A_1^T Q_1^T S N Q_1 A_1 \right) D_1 + B_2^T Q_2^T S_{N-2} Q_2 D_2 \\
&- B_1^T H_{N-1}^T \tilde{R}_{N-1}^{-1} H_{N-1} A_1 \right) E(\xi_{N-2}) + 2u^T(N-2) B_1^T \left( A_1^T Q_1^T S_{N-1} Q_2 - Q_1^T S_{N-2} Q_2 D_2 \\
&- H_{N-1}^T \tilde{R}_{N-1}^{-1} \beta_{N-1} \right) e_{N-1} \right) + 2u^T(N-2) \left[ A_1^T \left( Q_1^T S_{N-1} Q_1 + A_1^T Q_1^T S N Q_1 A_1 \right) D_1 \\
&- Q_1^T S_{N-2} Q_2 D_2 - A_1^T H_{N-1}^T \tilde{R}_{N-1}^{-1} H_{N-1} D_1 \right) E(\xi_{N-2}) \\
&+ 2u^T(N-2) \left( Q_2^T S_{N-1} Q_1 D_1 - G_{N-1}^T \tilde{R}_{N-1}^{-1} H_{N-1} D_1 \right) E(\xi_{N-2}) \\
&+ 2u^T(N-2) \left( Q_2^T S_{N-1} Q_1 D_1 - G_{N-1}^T \tilde{R}_{N-1}^{-1} \beta_{N-1} \right) e_{N-1} \right) + 2u^T(N-2) \left( A_1^T Q_1^T S_{N-1} Q_2 - Q_1^T S_{N-2} Q_2 D_2 \\
&- H_{N-1}^T \tilde{R}_{N-1}^{-1} \beta_{N-1} \right) e_{N-1} \right) E(\xi_{N-2}) \\
&+ 2u^T(N-2) \left( A_1^T Q_1^T S_{N-1} Q_2 - Q_1^T S_{N-2} Q_2 D_2 - H_{N-1}^T \tilde{R}_{N-1}^{-1} \beta_{N-1} \right) e_{N-1} \right) E(\xi_{N-2}) \\
&+ 2u^T(N-2) \left( A_1^T Q_1^T S_{N-1} Q_2 - Q_1^T S_{N-2} Q_2 D_2 - H_{N-1}^T \tilde{R}_{N-1}^{-1} \beta_{N-1} \right) e_{N-1} \right) E(\xi_{N-2}) \\
&+ \left[ D_1^T \left( Q_1^T S_{N-1} Q_1 + A_1^T Q_1^T S N Q_1 A_1 - H_{N-1}^T \tilde{R}_{N-1}^{-1} H_{N-1} \right) D_1 \\
&+ D_2^T Q_2^T S_{N-2} Q_2 D_2 \right) E(\xi_{N-2}) \\
&+ \left( D_1^T Q_1^T S_{N-1} Q_1 D_1 + D_2^T Q_2^T S_{N-1} Q_1 D_2 \right) \right) \left( \sigma_{N-1}^2 + \varepsilon_{N-1}^2 \right) - \left( \beta_{N-1} \right) e_{N-1} \right) \right)
\end{align*}
\]

Some symbols are introduced for simplification of writing as follows:

\[
\begin{align*}
\tilde{R}_{N-2} &= R_{N-2} + B_1^T \left( Q_1^T S_{N-1} Q_1 + A_1^T Q_1^T S N Q_1 A_1 \right) B_1 + B_2^T Q_2^T S_{N-2} Q_2 B_2 \\
&- B_1^T H_{N-1}^T \tilde{R}_{N-1}^{-1} H_{N-1} B_1, \\
G_{N-2} &= B_1^T A_1^T Q_1^T S N Q_2 - B_1^T H_{N-1}^T \tilde{R}_{N-1}^{-1} G_{N-1}, \\
H_{N-2} &= B_1^T \left( Q_1^T S_{N-1} Q_1 + A_1^T Q_1^T S N Q_1 A_1 \right) A_1 - B_2^T Q_2^T S_{N-2} Q_1 \\
&- B_1^T H_{N-1}^T \tilde{R}_{N-1}^{-1} H_{N-1} A_1, \\
\beta_{N-2}^0 &= B_1^T \left( Q_1^T S_{N-1} Q_1 + A_1^T Q_1^T S N Q_1 A_1 \right) D_1 + B_2^T Q_2^T S_{N-2} Q_2 D_2 \\
&- B_1^T H_{N-1}^T \tilde{R}_{N-1}^{-1} H_{N-1} D_1, \\
\beta_{N-1}^1 &= B_1^T \left( A_1^T Q_1^T S_{N-1} Q_1 D_1 - Q_1^T S_{N-2} Q_2 D_2 - H_{N-1}^T \tilde{R}_{N-1}^{-1} \beta_{N-1} \right),
\end{align*}
\]

where

\[
\tilde{R}_{N-2} \succeq R_{N-2} - B_1^T H_{N-1}^T \tilde{R}_{N-1}^{-1} H_{N-1} B_1 \succ 0
\]

is true based on the inequality (9). Then denote

\[
\begin{align*}
h \left( u(N-2) \right) &= u^T(N-2) \tilde{R}_{N-2} u(N-2) \\
&+ 2u^T(N-2) \left[ G_{N-2} x_2(N) + H_{N-2} x_1(N-2) + \beta_{N-2}^0 e_{N-2} + \beta_{N-1}^1 e_{N-1} \right].
\end{align*}
\]
We know that optimal control $u^*(N - 2)$ satisfies the following equation:
\[
\frac{\partial h}{\partial u(N - 2)} = 2\tilde{R}_{N-2}u^*(N - 2)
\]
\[
+ 2\left[G_{N-2}x_2(N) + H_{N-2}x_1(N - 2) + \beta_{N-2}^\alpha e_{N-2} + \beta_{N-1}^\alpha e_{N-1}\right] = 0,
\]
and because
\[
\frac{\partial^2 h}{\partial u^2(N - 2)} = 2\tilde{R}_{N-2} > 0,
\]
therefore the optimal control can be expressed by
\[
u^*(N - 2) = -\tilde{R}_{N-2}^{-1}\left[G_{N-2}x_2(N) + H_{N-2}x_1(N - 2) + \beta_{N-2}^\alpha e_{N-2} + \beta_{N-1}^\alpha e_{N-1}\right].
\]

Finally, we obtain the optimal value from this stage, that is,
\[
J(N-2,x_1(N - 2),x_2(N))
\]
\[
=x_1^T(N - 2)\left[Q_2^T S_{N-2}Q_1 + A_1^T \left(Q_2^T S_{N-1}Q_1 + A_1^T Q_2^T S_{N}Q_1A_1\right)A_1
\]
\[
- A_1^T H_{N-1}^{-1}R_{N-1}^{-1} H_{N-1}A_1\right]
\]
\[
\cdot x_1(N - 2) + 2x_1^T(N - 2)\left[A_1^T Q_2^T S_{N}Q_2 - A_1^T H_{N-1}^{-1} G_{N-1}^{-1} G_{N-1}\right]x_2(N) + x_2^T(N)
\]
\[
\cdot \left(Q_2^T S_{N}Q_2 - G_{N-1}^{-1} G_{N-1}\right)\nu_{N-2}^T \tilde{R}_{N-2}^{-1}\left[G_{N-2}x_2(N) + H_{N-2}x_1(N - 2) + \beta_{N-2}^\alpha e_{N-2} + \beta_{N-1}^\alpha e_{N-1}\right]
\]
\[
+ 2x_1^T(N - 2)\left[A_1^T \left(Q_2^T S_{N-1}Q_1 + A_1^T Q_2^T S_{N}Q_1A_1\right)D_1 - Q_2^T S_{N-2}Q_2D_2
\]
\[
- A_1^T H_{N-1}^{-1}R_{N-1}^{-1} H_{N-1}D_1\right]\nu_{N-2} + 2x_1^T(N)\left(Q_2^T S_{N}Q_1A_1D_1 - G_{N-1}^{-1} G_{N-1}\right)\nu_{N-1}
\]
\[
+ 2D_1^T \left(A_1^T Q_2^T S_{N}Q_1D_1 - Q_2^T S_{N-1}Q_1D_1 - H_{N-1}^{-1} R_{N-1}^{-1} \beta_{N-1}^\alpha\right)\nu_{N-1}e_{N-2}
\]
\[
+ 2x_2^T(N - 2)A_1^T \left(Q_2^T S_{N-1}Q_1D_1 - Q_2^T S_{N-1}Q_2D_2 - H_{N-1}^{-1} R_{N-1}^{-1} \beta_{N-1}^\alpha\right)\nu_{N-1}
\]
\[
+ \left(D_1^T Q_2^T S_{N}Q_1D_1 - H_{N-1}^{-1} R_{N-1}^{-1} H_{N-1}\right)\nu_{N-1} + D_2^T Q_2^T S_{N-2}Q_2D_2
\]
\[
- \beta_{N-2}^\alpha R_{N-1}^{-1} \beta_{N-1}^\alpha e_{N-1}.\]

By mathematical induction, the conclusions of this theorem are easily deduced. This completes the proof.

\[\square\]

**Remark 4.1.** Note that when the matrix $F$ is invertible, system (1) will degrade into a stochastic normal system. Optimal controls of problem (8) are obviously substate feedback controls that are effortless to operate in practical situations. Furthermore, this type of feedback control is the linear combinations of initial sub-state $x_1(k)$, terminal sub-state $x_2(N)$ and several expected values of random variables. Such fact is totally different from the case of LQ optimal control for stochastic normal systems. Because of the results in Theorem 4.1, LQ optimal control is expanded from stochastic normal system to stochastic descriptor systems which makes the investigation on optimal control theory deeper than before.

**Remark 4.2.** In many application fields, physical systems can be formulated by (stochastic) descriptor systems such as the dynamic input-output Leontief system[22]
and the self-excited oscillation phenomenon[28]. Therefore, the LQ optimal control model considered in this section can be employed to investigate the dynamical optimization problems as such cases.

Theorem 4.1 reveals a fact that the optimal controls are sub-state feedback controls for LQ optimal control problems similar with problem (8). The stochastic LQ optimal control model considered in this section has plenty of applications in many important horizons.

5. Numerical example. In Section 4, optimal controls and optimal values of the LQ optimal control problem ruled by stochastic descriptor systems are presented in accurate forms. For this problem, the optimal solutions can be obtained provided that the initial sub-state $x_1(0)$ and the terminal sub-state $x_2(N)$ are both given. As a numerical example, we investigate the following stochastic LQ optimal control problem:

$$J(0, x(0)) = \inf_{u(i) \in U_{ad}} \mathbb{E} \left\{ x^T(7)S_7 x(7) + \sum_{j=0}^{6} \left[ x^T(j)S_j x(j) + u^T(j)R_j u(j) \right] \right\}$$

subject to

$$Fx(j + 1) = Ax(j) + Bu(j) + D\xi_j,$$

$$j = 0, 1, 2, \cdots, 6,$$

(11)

where $U_{ad} = R^3$, uncertain factors $\xi_i \sim N(\varepsilon_i, \sigma_i^2)$ ($i = 0, 1, \cdots, 6$) are normal random variables as follows:

$$\xi_0 \sim N(1.2, 1^2), \ \xi_1 \sim N(1.5, 0.7^2), \ \xi_2 \sim N(2, 1.5^2),$$

$$\xi_3 \sim N(0.5, 2^2), \ \xi_4 \sim N(1.5, 1^2), \ \xi_5 \sim N(0.8, 1.2^2), \ \xi_6 \sim N(1.7, 0.6^2);$$

and

$$F = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}, \ D = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}. $$

The symmetric matrices in the performance index are listed in the following:

$$R_0 = \begin{bmatrix} 7 & 3 & -1 \\ 3 & 8 & 0 \\ -1 & 0 & 2 \end{bmatrix}, \ R_1 = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{bmatrix}, \ R_2 = \begin{bmatrix} 6 & 2 & -1 \\ 2 & 10 & -2 \\ -1 & -2 & 7 \end{bmatrix},$$

$$R_3 = \begin{bmatrix} 8 & 2 & 0 \\ 2 & 5 & 1 \\ 0 & 1 & 6 \end{bmatrix}, \ R_4 = \begin{bmatrix} 6 & -2 & 0 \\ -2 & 9 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \ R_5 = \begin{bmatrix} 4 & 0 & -2 \\ 0 & 9 & 1 \\ -2 & 1 & 5 \end{bmatrix},$$

$$R_6 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 5 & 0 \\ 1 & 0 & 3 \end{bmatrix};$$

and

$$S_0 = \begin{bmatrix} 3 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \ S_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ S_2 = \begin{bmatrix} 4 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. $$
Through calculating, we get that
\[
\det(zF - A) = \det \begin{bmatrix} z & -1 & 0 & 0 \\ -1 & 0 & z & 0 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & -1 & -1 \end{bmatrix} = z^2 + z + 1.
\]

Obviously, \(\det(zF - A)\) is not identically zero and \(\text{deg}(\det(zF - A)) = \text{rank}(F) = 2\), namely, the stochastic descriptor system in problem (11) is both regular and impulse-free. According to Lemma 2.1, by deduction it is able to obtain a pair of invertible matrices

\[
P = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix},
\]

such that

\[
PFQ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad PAQ = \begin{bmatrix} -1 & -1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
\]

\[
PQ = \begin{bmatrix} 0 & 2 & -1 \\ -1 & 1 & 0 \\ 1 & -2 & 2 \\ 0 & 1 & -1 \end{bmatrix}, \quad PD = \begin{bmatrix} -1 \\ 2 \\ 2 \\ 1 \end{bmatrix},
\]

which indicates that

\[
A_1 = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & -1 \end{bmatrix}, \quad D_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix};
\]

and

\[
Q_1 = \begin{bmatrix} 1 & 0 \\ -1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.
\]
Similar to Section 4, problem (11) can be equivalently transformed into problem (12) by applying Lemma 4.1:

\[
J(0, x_1(0), x_2(7)) = \inf_{u(i) \in \mathbb{R}^3} \mathbb{E}\left\{ \left( Q_1x_1(7) + Q_2x_2(7) \right)^T S_7 \left( Q_1x_1(7) + Q_2x_2(7) \right) \right\}
+ \sum_{j=0}^{6} \left[ \left( Q_1x_1(j) + Q_2x_2(j) \right)^T S_j \left( Q_1x_1(j) + Q_2x_2(j) \right) + u^T(j)R_j u(j) \right]
\]

subject to

\[
x_1(j + 1) = A_1x_1(j) + B_1u(j) + D_1\xi_j,
0 = x_2(j) + B_2u(j) + D_2\xi_j,
\]
\[j = 0, 1, 2, \ldots, 6.\]

(12)

According to Theorem 4.1 and using Matlab 2018a (each number is rounded up to the third decimal place), the matrices we need are presented as follows:

\[
\tilde{R}_6 = \begin{bmatrix}
8 & -13 & -6 \\
-13 & 45 & -18 \\
6 & -18 & 14
\end{bmatrix} > 0,
\]

\[
\tilde{R}_5 = \begin{bmatrix}
6.145 & -3.526 & 0.191 \\
-3.526 & 32.084 & -12.279 \\
0.191 & -12.279 & 14.044
\end{bmatrix},
\]

\[\tilde{R}_4 = \begin{bmatrix}
15.055 & -18.168 & 3.556 \\
-18.168 & 73.501 & -24.169 \\
3.556 & -24.167 & 15.305
\end{bmatrix} > 0,
\]

\[
\tilde{R}_3 = \begin{bmatrix}
16.022 & -11.062 & 4.020 \\
-11.062 & 52.048 & -18.993 \\
4.020 & -18.993 & 16.986
\end{bmatrix} > 0,
\]

\[\tilde{R}_2 = \begin{bmatrix}
17.150 & -18.582 & 5.216 \\
-18.582 & 97.745 & -39.081 \\
5.216 & -39.081 & 25.933
\end{bmatrix} > 0,
\]

\[
\tilde{R}_1 = \begin{bmatrix}
22.556 & -26.522 & 11.483 \\
-26.522 & 111.919 & -50.698 \\
11.483 & -50.698 & 36.108
\end{bmatrix} > 0,
\]

\[\tilde{R}_0 = \begin{bmatrix}
20.337 & -21.679 & 7.671 \\
-21.679 & 111.075 & -46.198 \\
7.671 & -46.198 & 27.764
\end{bmatrix} > 0;
\]

and

\[
H_6 = \begin{bmatrix}
-1 & 4 \\
-5 & -13 \\
6 & 6
\end{bmatrix},
\]

\[
H_5 = \begin{bmatrix}
-0.954 & 1.191 \\
-7.753 & -9.279 \\
4.854 & 5.044
\end{bmatrix},
\]

\[H_4 = \begin{bmatrix}
-5.499 & 3.556 \\
-7.999 & -24.167 \\
6.749 & 10.305
\end{bmatrix},
\]

\[H_3 = \begin{bmatrix}
-4.002 & 3.020 \\
-6.931 & -17.993 \\
6.966 & 8.986
\end{bmatrix},
\]

\[
H_2 = \begin{bmatrix}
-4.934 & 5.216 \\
-16.499 & -34.081 \\
12.717 & 15.933
\end{bmatrix},
\]

\[H_1 = \begin{bmatrix}
-1.073 & 8.483 \\
-24.176 & -43.698 \\
17.624 & 22.108
\end{bmatrix},
\]

\[H_0 = \begin{bmatrix}
-3.665 & 6.671 \\
-23.519 & -40.198 \\
19.092 & 19.764
\end{bmatrix};
\]

and

\[
G_6 = \begin{bmatrix}
3 & 0 \\
-9 & 2 \\
3 & -1
\end{bmatrix},
\]

\[
G_5 = \begin{bmatrix}
-0.454 & 0.421 \\
-0.180 & -0.896 \\
0.317 & 0.238
\end{bmatrix},
\]

\[G_4 = \begin{bmatrix}
0.064 & -0.738 \\
5.502 & 0.521 \\
-2.783 & 0.109
\end{bmatrix},
\]

\[G_3 = \begin{bmatrix}
0.693 & 0.006 \\
-3.023 & 1.146 \\
1.165 & -0.576
\end{bmatrix};
\]
\[ G_2 = \begin{bmatrix} -1.943 & 0.551 \\ 1.125 & -1.908 \\ 0.409 & 0.678 \end{bmatrix}, \quad G_1 = \begin{bmatrix} -0.110 & -0.364 \\ 3.881 & -0.005 \\ -1.866 & 0.184 \end{bmatrix}, \quad G_0 = \begin{bmatrix} 1.374 & 0.073 \\ -4.921 & 1.453 \\ 1.773 & -0.763 \end{bmatrix}. \]

The vectors appearing in the optimal controls are also obtained:

\[
\beta_6^0 = \begin{bmatrix} -5 \\ 10 \\ -1 \end{bmatrix}, \quad \beta_5^0 = \begin{bmatrix} -3.625 \\ 6.857 \\ -2.851 \end{bmatrix}, \quad \beta_6^2 = \begin{bmatrix} -2.913 \\ 8.614 \\ -2.851 \end{bmatrix}, \quad \beta_5^2 = \begin{bmatrix} -1.095 \\ 8.227 \\ -4.260 \end{bmatrix},
\]

\[
\beta_6^4 = \begin{bmatrix} -5.603 \\ -4.991 \\ -9.125 \end{bmatrix}, \quad \beta_5^4 = \begin{bmatrix} -9.305 \\ 14.912 \\ -4.161 \end{bmatrix}, \quad \beta_6^6 = \begin{bmatrix} -2.917 \\ 14.048 \\ -4.416 \end{bmatrix}, \quad \beta_5^6 = \begin{bmatrix} -1.095 \\ -8.227 \\ 6.663 \end{bmatrix},
\]

\[
\beta_5^2 = \begin{bmatrix} -14.555 \\ -11.74 \\ -8.204 \end{bmatrix}, \quad \beta_5^3 = \begin{bmatrix} -6.019 \\ 25.713 \\ -9.125 \end{bmatrix}, \quad \beta_5^4 = \begin{bmatrix} -11.573 \\ 29.823 \\ -9.501 \end{bmatrix}, \quad \beta_5^5 = \begin{bmatrix} -12.768 \\ 31.769 \\ -2.144 \end{bmatrix},
\]

\[
\beta_4^0 = \begin{bmatrix} 8.169 \\ 3.193 \end{bmatrix}, \quad \beta_4^0 = \begin{bmatrix} -14.282 \\ 55.373 \end{bmatrix}, \quad \beta_4^1 = \begin{bmatrix} -11.810 \\ 40.466 \end{bmatrix}, \quad \beta_4^2 = \begin{bmatrix} -20.272 \\ 55.434 \end{bmatrix}, \quad \beta_4^3 = \begin{bmatrix} -23.994 \\ 17.632 \end{bmatrix}, \quad \beta_4^4 = \begin{bmatrix} -15.720 \end{bmatrix},
\]

\[
\beta_3^0 = \begin{bmatrix} -10.023 \\ 2.131 \end{bmatrix}, \quad \beta_3^0 = \begin{bmatrix} -1.318 \\ 3.425 \end{bmatrix}, \quad \beta_3^2 = \begin{bmatrix} -4.102 \\ 20.369 \end{bmatrix}, \quad \beta_3^3 = \begin{bmatrix} -10.223 \\ 32.447 \end{bmatrix}, \quad \beta_3^4 = \begin{bmatrix} -11.112 \end{bmatrix},
\]

\[
\beta_2^0 = \begin{bmatrix} -14.805 \\ 8.501 \end{bmatrix}, \quad \beta_2^0 = \begin{bmatrix} -9.561 \\ 4.561 \end{bmatrix}, \quad \beta_2^2 = \begin{bmatrix} -7.923 \\ 25.037 \end{bmatrix}, \quad \beta_2^3 = \begin{bmatrix} -8.691 \end{bmatrix},
\]

\[
\beta_1^0 = \begin{bmatrix} -7.629 \\ 16.141 \end{bmatrix}, \quad \beta_1^0 = \begin{bmatrix} 0.196 \\ 1.679 \end{bmatrix}, \quad \beta_1^0 = \begin{bmatrix} -3.554 \\ 12.421 \end{bmatrix}, \quad \beta_1^0 = \begin{bmatrix} -0.840 \end{bmatrix}.
\]

Because \( \tilde{R}_i \) \((i = 0, 1, \cdots, 6)\) are positive definite matrices, each of them is certainly invertible. Their inverse matrices are listed in the following:

\[
\tilde{R}_6^{-1} = \begin{bmatrix} 0.241 & 0.058 & -0.028 \\ 0.058 & 0.060 & 0.052 \\ -0.028 & 0.052 & 0.150 \end{bmatrix}, \quad \tilde{R}_5^{-1} = \begin{bmatrix} 0.178 & 0.028 & 0.022 \\ 0.028 & 0.051 & 0.044 \\ 0.022 & 0.044 & 0.110 \end{bmatrix}, \quad \tilde{R}_4^{-1} = \begin{bmatrix} 0.102 & 0.036 & 0.034 \\ 0.036 & 0.041 & 0.057 \\ 0.034 & 0.057 & 0.147 \end{bmatrix},
\]

\[
\tilde{R}_3^{-1} = \begin{bmatrix} 0.073 & 0.016 & 0.000 \\ 0.016 & 0.036 & 0.036 \\ 0.000 & 0.036 & 0.099 \end{bmatrix}, \quad \tilde{R}_2^{-1} = \begin{bmatrix} 0.076 & 0.021 & 0.016 \\ 0.021 & 0.032 & 0.043 \\ 0.016 & 0.043 & 0.101 \end{bmatrix}, \quad \tilde{R}_1^{-1} = \begin{bmatrix} 0.062 & 0.016 & 0.003 \\ 0.016 & 0.029 & 0.035 \\ 0.003 & 0.035 & 0.076 \end{bmatrix}, \quad \tilde{R}_0^{-1} = \begin{bmatrix} 0.063 & 0.016 & 0.010 \\ 0.016 & 0.034 & 0.051 \\ 0.010 & 0.051 & 0.119 \end{bmatrix}.
\]

The optimal controls and optimal values are provided by Theorem 4.1 and written in Table 1. The data in the table are obtained with initial sub-state \( x_1(0) = (1, 2)^T \) and terminal sub-state \( x_2(7) = (1, 1)^T \).
Table 1. The optimal results of problem (11)

| Stage | \((x^*(t))^T\) | \(x^*_1(k)\) | \(x^*_2(k)\) | \(x^*(k)\) | \(J(k, x_1(k), x_2(k))\) |
|-------|----------------|--------------|--------------|-------------|-----------------------------|
| 0     | (3.567, -2.324, -3.406) | 2.517 | (1.2) | (1.000, -5.434, 2.600, -4.600) | -112.722  |
| 1     | (0.438, -2.541, -0.797) | 1.434 | (-4.756, 0.145) | (-4.756, -2.099, 0.145, -4.756) | -65.708  |
| 2     | (1.796, -2.599, -0.303) | 2.998 | (-1.416, -4.148) | (-1.416, 5.019, -4.148, -1.218) | 20.114   |
| 3     | (0.167, -1.831, 0.460) | 0.202 | (-1.429, -1.014) | (-1.429, -1.769, -1.014, 0.490) | 41.336   |
| 4     | (0.035, -1.184, 0.582) | 1.165 | (-0.934, -2.843) | (-0.934, -2.743, -2.843, -0.320) | 35.270   |
| 5     | (0.362, -0.338, -0.291) | 0.572 | (-0.362, -0.436) | (-0.362, 0.838, -0.436, -1.147) | 22.059   |
| 6     | (0.030, -0.274, -0.403) | 1.494 | (-0.505, 0.184) | (-0.505, -2.505, 0.182, -2.027) | 45.318   |
| 7     | \(u^*(t) \in R^3\) | [-1.217, 1.619] | [-1.217, 0.098, 1.619, -0.217] | 7.363     |

Remark 5.1. In Column 4 of Table 1, sub-states \(x_1(k)\) \((k = 1, 2, \cdots, 7)\) are derived according to the matrix equalities

\[
x_1(k + 1) = A_1 x_1(k) + B_1 u(k) + D_1 \mu_k, \quad k = 0, 1, \cdots, 6,
\]

under initial sub-state \(x_1(0) = (1, 2)^T\), where \(\mu_k\) is the realization of random variable \(\xi_k\) in the intervals \([\varepsilon_k - 3\sigma_k, \varepsilon_k + 3\sigma_k]\) for \(k = 0, 1, \cdots, 6\). And state vectors \(x(k)\) \((k = 0, 1, \cdots, 7)\) in the fifth column are generated from the following equations:

\[
x(k) = Q_1 x_1(k) + Q_2 x_2(k), \quad k = 0, 1, \cdots, 7.
\]

In order to illustrate above optimal results more intuitively, Figure 1 and Figure 2 are drawn to describe the trajectories of optimal controls and state vectors for problem (11), respectively.

![Figure 1. Trajectories of optimal controls \(u^*(k)\) for problem (11).](image_url)

6. Conclusions. In this work, an optimal control model was proposed subject to linear discrete-time stochastic descriptor systems that are regular and impulse-free. To make the model easier to settle, a recurrence equation was obtained according to the principle of dynamic programming. When the objective function is quadratic, by employing the recurrence equation, optimal controls and optimal values were both expressed analytically by deduction. These expressions indicates that each optimal control is a sub-state feedback control made up of the linear combinations of initial sub-state \(x_1(k)\), terminal sub-state \(x_2(N)\) and expected values of some random variables. This property is a essential difference between this problem and
a LQ optimal control problem for stochastic normal systems. Finally, a numerical example was provided as an application of the results derived.

Throughout this paper, the descriptor systems are disturbed by random factors, while sometimes subjective uncertainties cannot be neglected in practice. Therefore, stability and stabilization issues of uncertain descriptor systems based on the results in [6] and [19], and the LQ optimal control problem ruled by uncertain descriptor systems inspired by the paper[17] may be considered in the future.

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E-mail address: 972718523@qq.com (Y. Shu)
E-mail address: 003012@nuist.edu.cn (Y. Shu)
E-mail address: libnust@163.com (B. Li)