LOAD-SHARING DEPENDENCE MODELS AND CONSTRUCTION OF VOTING SITUATIONS FOR ANY ARBITRARY RANKING SCHEMES

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Abstract. In this paper we present a study about minima among random variables, about the context of voting theory, and about paradoxes related with such topics. In the field of reliability theory, the term load-sharing model is commonly used to designate a special type of multivariate survival models. We demonstrate the effectiveness that such dependence models can have also in some other fields, such as those of interest here. Several important, and by now classic, papers have been devoted to single out and to prove general conclusions in the field of voting theory. We reformulate and achieve such conclusions by developing a method of proof, alternative to the existing ones, and completely probabilistic in nature. As main features of this method, we focus attention on ranking schemes associated to \( m \)-tuples of non-negative random variables and suitably single out a special subclass of load-sharing models. Then we show that all possible ranking schemes can be conveniently obtained by only considering such a special family of survival models. This result leads to some new insight about the construction of voting situations which give rise to all possible types of voting paradoxes. Our method and related implications will be also illustrated by means of some examples and informative remarks.

Keywords: Minima among random variables, Voting theory, Aggregation paradoxes, Condorcet paradox, Voting situations, Preference patterns, Ranking schemes, Time-Homogeneous Load-Sharing models, Reallocation-symmetry in hazards property.

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1. Introduction

In the field of reliability theory, the term load-sharing model is mostly used to designate a special type of multivariate survival models. The latter is described by a simplifying condition of stochastic dependence among lifetimes of units which work simultaneously in a same environment and which are supposed to support one another.

In this paper we demonstrate the effectiveness that such dependence models have also in some other fields different from reliability and even far from probabilistic analysis of lifetimes. Actually, we deal with a study about minima among random variables, about the context of voting theory, and about paradoxes related with such topics. Some results of conceptual interest will be obtained, concerning both voting theory and topics of applied probability.

Let \( X_1, \ldots, X_m \) be \( m \) non-negative random variables defined on a same probability space and satisfying the no-tie condition \( \mathbb{P}(X_1 \neq \ldots \neq X_m) = 1 \). For any subset \( A \) of the set of indexes \( \{1, \ldots, m\} \), furthermore, and for any \( j \in A \), let \( \alpha_j(A) \) be the probability that \( X_j \) takes on the minimum value among all the other variables \( X_i \) with \( i \in A \).
When $A$ exactly contains two elements, $A := \{i_1, i_2\}$ say, the inequality $\alpha_{i_1}(A) \geq \frac{1}{2}$ translates the condition that $X_{i_1}$ stochastically precedes $X_{i_2}$. But one can also be interested in detecting the indexes, say $\hat{j}(A)$, which maximize $\alpha_j(A)$ for sets $A$ containing more than two elements.

Concerning the analysis of properties of the set of values $\hat{j}(A)$, two different types of paradoxes have been studied in the literature:

i. the non-transitivity property of stochastic precedence between random variables.
ii. aggregation paradoxes which can be related to the behavior of $\alpha_j(A)$ and $\hat{j}(A)$, for sets $A$ with cardinality $|A| > 2$.

Already at a first glance, such phenomena respectively reveal to be parallel to non-transitivity of collective preferences within pairs of candidates and to voting-related aggregation paradoxes. Classical references can be given where these topics have been treated, also under different types of languages and notation. See e.g. [3], [21], [28], [29], [3], for item i. and [2], [18] for item ii; see also [6]. Non-transitivity of stochastic precedence amounts to the fact that for the variables $X_1, \ldots, X_m$ and for three indexes $j, i, h$ one can simultaneously have

$$\alpha_j(\{i, j\}) > \frac{1}{2}, \alpha_i(\{i, h\}) > \frac{1}{2}, \alpha_h(\{h, j\}) > \frac{1}{2}.$$ 

We remind that, in the frame of voting theory, similar aspects of non-transitivity are shown by the Condorcet’s paradox. Consider three candidates $c_1, c_2, c_3$ who participate in elections where the winner is determined according to a majority rule. The paradox demonstrates that, at the collective level of all the voters, $c_1$ may be preferred to $c_2$ even if $c_2$ is preferred to $c_3$ and $c_3$ is preferred to $c_1$. As well-know, a very rich literature has been devoted to this specific topic (see in particular [15], [9], [1], [16]) and, more generally, to the field of paradoxes related with voting theory.

Very briefly, we just remind the following voting context created by a set of $m$ individuals, who all are admissible candidates in different elections, and a family of $n$ voters. The individual preferences of single voters are complete, transitive, and are such that indifference between any two candidates is not admitted. The ranking expressed by any single voter, thus, is simply a permutation of the set of candidates and the collection of voters’ individual ranking gives rise to a voting situation. The latter is a description of the number of voters who share a same ranking, for all possible rankings (or permutations). Furthermore, each voter is supposed to cast her/his own vote in any possible election. When $A$ is the subset of candidates who actually participate in a specific election, any voter will cast a vote for her/his preferred candidate within $A$, according to her/his pre-established ranking. A preference pattern is a description of the voting results, corresponding to a voting situations, which could respectively be observed in all the possible, matches between two different candidates. Such a voting context allows for a clear explanation of the Condorcet’s paradox. In fact, the latter amounts to the circumstance that some voting situation gives rise to non-transitive preference patterns over candidates, even though transitivity has been assumed for the individual ranking of each voter. In the description of this paradox, then, one only compares results of elections involving pairs of candidates.
One can however be interested in analyzing more detailed descriptions of the possible consequences triggered by a voting situation. In fact, one can also focus attention on the voting results that can be obtained in all the possible elections, also in the elections involving whatever subsets of the whole set of candidates. The complete description of all such results leads to the concept of ranking scheme, associated to a voting situation. The concepts of voting situation, preference pattern, and ranking scheme are very well-known in the frame of voting theory. See, in particular, exhaustive explanations and references in the monographs [11], [19]. Related definitions will also be recalled in the next section using a notation convenient for our purposes. The possibly paradoxical aspects related with the analysis of a ranking scheme are those of the type originally pointed out by Borda. A useful and informative special case in this direction is the one designed by P. Fishburn (see in particular the discussion in [10] and [18]). On the other hand, also the aggregation paradoxes mentioned in the item ii. above have similar forms. Actually, for a given \(m\)-tuple of random variables, they are related to the behavior of the family \(\{\alpha_j(A); A \subset \{1, \ldots, m\}, j \in A\}\), and the latter also determines a precise ranking scheme.

These topics are strictly related also to the classic stochastic game between two players, who respectively bet on the occurrence of different events in a sequence of one-dependent trials. The winner of the game is the one who first sees the occurrence of the event on which she/he had bet. In fact, some paradoxical phenomena can emerge in such a context as well. Relevant special cases are the possible paradoxes which are met in the analysis of times of first occurrence for different words of fixed length in random sampling of letters from an alphabet. See [13], [12], and also [7], [8]. This special field had initially motivated our own interest toward these topics, toward the concept of stochastic precedence, and toward the family of probabilities \(\{\alpha_j(A); A \subset \{1, \ldots, m\}, j \in A\}\), more in general. Special aspects of these topics have been analyzed in [6] and in [4], recently. The occurrence times studied in the latter paper can be seen as first passage times of suitable 1-dependent Markov chains, which makes the considered problems interesting from a probabilistic viewpoint.

Concerning with the family \(\{\alpha_j(A); A \subset \{1, \ldots, m\}, j \in A\}\), it will be pointed out below (see Proposition 1 and Corollary 1) that the basic bricks to be used in the computation of the \(\alpha_j(A)\)’s are the probabilities of the type

\[
P \left( X_{1:m} = X_{\pi(1)}, \ldots, X_{m:m} = X_{\pi(m)} \right),
\]

where \(\pi \equiv (\pi(1), \ldots, \pi(n))\) denotes a permutation of the indexes \(1, \ldots, m\) and \(X_{1:m}, \ldots, X_{m:m}\) denote the order statistics of \((X_1, \ldots, X_m)\).

At the best of our knowledge, only at the end of the last century the phenomena, respectively occurring in the analysis of ranking schemes for voting situations and for \(m\)-tuples of random variables, have been formally studied in unified way. See in particular the papers [16], [17] by Donald G. Saari, who developed a common approach for analyzing and comparing the paradoxes respectively arising in the different levels i. and ii., and in the two different contexts.

Saari proved that any ranking scheme can be actually observed, even when arbitrarily paradoxical, provided that a suitable voting situation (or voting profile) is considered.
Furthermore, and equivalently, the same result can be translated into the language related to comparisons among random variables, or lifetimes. The results in [16, 17] hinge on refined algebraic and geometric tools.

As a main purpose of this paper we show how general results, leading to the same consequences as the one by Saari, can be obtained by taking into account the mentioned connection between voting theory and ranking of random variables and by exploiting characteristic features of the load-sharing models. The method of proof resulting from this approach is completely constructive and based on arguments of probabilistic character. Furthermore it is only based on detailed, and rather elementary, computations.

One difference between the result by Saari and our results (Theorem 2 and Corollary 1) is in that the latter only deal with the construction of voting situations which generate strict ranking schemes (see Definition 1). More general ranking schemes are on the contrary considered in the former result where indifference is allowed, at a collective level, between any two candidates. We preferred to consider a more restricted setting for the sake of a better illustration of the basic ideas about the employment of load-sharing models. However, at the cost of adding a further step to our proof, the same method can also be used to obtain a stronger result whose implications on voting theory are completely equivalent to those in [16, 17].

By means of Theorem 3 and Corollary 3 we will show later on how our method can provide a completely different type of proof for the classic theorem by McGarvey [15]. In fact, in the same direction of Theorem 2 and Corollary 1, Theorem 3 and Corollary 3 will show how one can construct voting situations able to produce arbitrary preference patterns, in place of ranking schemes.

In the literature devoted to voting theory, many papers really aim to obtain refined results concerning the number of voters sufficient to produce a given voting situation. Besides [15], [9], see in particular [27], [1], [25]. The interest in the present paper is focused instead only on the conceptual possibility to construct voting situation and on the related role of load-sharing models.

More precisely, the plan of the paper is as follows.

In Section 2 we point out preliminary results concerning relations between voting theory and the study of the probabilities $\alpha_j(A)$ for $m$-tuples of lifetimes. In particular we introduce formalized notation, convenient to deal with the concepts of preference pattern and ranking scheme in the present setting. Furthermore, the probabilities of the type $\alpha_j(A)$ are analyzed and their connections with probabilities of permutations as in [11] and with ranking schemes are established. A couple of related examples is provided.

In Section 3, we recall the definition of load-sharing models, which are very special cases of absolutely continuous multivariate distributions for $(X_1, \ldots, X_m)$. In the absolutely continuous case, a possible tool to describe a joint distribution is provided by the set of the multivariate conditional hazard rate (m.c.h.r.) functions and load-sharing models just arise by imposing a remarkably simple form to such functions. About the latter functions, we will briefly provide definitions and some bibliographic references. We will then give some definitions concerning time-homogeneous load-sharing models and show a related property (see Proposition 1), useful in the present context.
In Section 4 we formulate and prove our main result (Theorem 2) which shows, for any strict ranking scheme \( \sigma^{(m)} \), the existence of a probabilistic model for an \( m \)-tuple \((X_1, \ldots, X_m)\) which generates \( \sigma^{(m)} \). Such a model can be found within a special subclass of time-homogeneous load-sharing models. Probabilistic aspects of this result will be illustrated by means of some related comments and a pertinent example, whereas the discussion of conceptual implications in the frame of voting theory will be deferred to Section 5. We also point out (see Theorem 1) that any set of probabilities of permutations as in (1) for an \( m \)-tuple \((X_1, \ldots, X_m)\) is shared by a corresponding time-homogeneous load-sharing model. Based on the preliminary results presented in Sections 2 and 3 and on the main results of Section 4, Section 5 will be then devoted to the construction of voting situations which give rise to pre-established preference patterns and ranking schemes. On this purpose we analyze and discuss the effects, on some aspects of voting theory, induced by the correspondence between minima among random variables and ranking schemes. In particular, as mentioned, Theorem 3 and Corollary 3 show how the classic result by McGarvey can be reobtained by means of Load-Sharing models. Finally, a short discussion containing some concluding remarks will be presented in Section 6.

2. Notation and preliminary results

This section, devoted to formalizing necessary definitions and to presenting preliminary results, is divided into three separated subsections. For \( m \)-tuples of non-negative random variables \((X_1, \ldots, X_m)\), we analyze in the first subsection the probabilities \( \alpha_j(A) \) defined in (1), the probabilities in (9) of the possible permutations triggered by \((X_1, \ldots, X_m)\), and corresponding relations among them. Next, after briefly recalling some basic notions from voting theory, in Subsection 2.2 we formalize some definitions suitably for our purposes and point out some related aspects. In particular we concentrate attention on a concept of ranking scheme. In Subsection 2.3 the arguments of the two preceding subsections will be compared and combined. This will lead us to describe the formal correspondence between the two contexts of voting theory and of minima among random variables.

For \( m \in \mathbb{N} \), the symbol \([m]\) denotes the set \( \{1, 2, \ldots, m\} \). When \( m > 1 \), we denote by \( \hat{P}(m) \) the family of subsets of \([m]\) having cardinality greater than one. Some further notation, convenient for our purposes, will be gradually introduced below.

2.1. Minima among non-negative random variables. Let \( X_1, \ldots, X_m \) be non-negative random variables defined on the same probability space and satisfying the no-tie condition, i.e.

\[
\mathbb{P}(X_1 \neq X_2 \neq \ldots \neq X_m) = 1.
\]

We denote by \( X_{1:m}, \ldots, X_{m:m} \) their order statistics. The assumption of no-tie property allow us to define the random indices \( J_1, \ldots, J_m \) through the following position

\[
J_r = i \iff X_i = X_{r:m}
\]

for any \( i, r \in [m] \). Henceforth, \( X_1, \ldots, X_m \) will be often referred to as the lifetimes.
For $A \in \hat{\mathcal{P}}(m)$ we consider the probabilities $\alpha_i(A)$, which we now define formally by setting
\[
\alpha_i(A) := \mathbb{P}(X_i = \min_{j \in A} X_j), \ i \in A.
\] (4)

When $A = [m]$ one obviously has $\alpha_i(A) = \alpha_i([m]) = \mathbb{P}(J_1 = i)$. For $A \in \hat{\mathcal{P}}(m)$ and $k \in [m - |A|]$, denote by $\mathcal{D}(m, A, k)$ the set
\[
\{(i_1, \ldots, i_k) : i_1, \ldots, i_k \in [m] \setminus A \text{ and } i_1 \neq i_2 \neq \ldots \neq i_k\}.
\] (5)
of ordered samples of size $k$ from $[m] \setminus A$. Coherently with this notation, the symbol $\mathcal{D}_m = \mathcal{D}(m, \emptyset, m)$ will denote the set of all the permutations of the elements of $[m]$.

Concerning the probabilities $\alpha_i(A)$, with $A \in \hat{\mathcal{P}}(m)$, the following claim will be useful for our purposes below.

**Proposition 1.** Let $X_1, \ldots, X_m$ be non-negative random variables satisfying the no-tie condition. Let $A \in \hat{\mathcal{P}}(m)$ and $\ell = |A|$. Then for any $i \in A$ one has
\[
\alpha_i(A) = \mathbb{P}(J_1 = i) + \sum_{k=1}^{m-\ell} \sum_{(i_1, \ldots, i_k) \in \mathcal{D}(m, A, k)} \mathbb{P}(J_1 = i_1, J_2 = i_2, \ldots, J_k = i_k, J_{k+1} = i).
\] (6)

**Proof.** The claim can be easily checked by partitioning the event $\{X_i = \min_{j \in A} X_j\}$ as follows.
\[
\{X_i = \min_{j \in A} X_j\} = \{J_1 = i\} \cup \left(\bigcup_{k=1}^{m-\ell} \{J_1 \notin A, J_2 \notin A, \ldots, J_k \notin A, J_{k+1} = i\}\right) = \{J_1 = i\} \cup \left(\bigcup_{k=1}^{m-\ell} \bigcup_{(i_1, \ldots, i_k) \in \mathcal{D}(m, A, k)} \{J_1 = i_1, J_2 = i_2, \ldots, J_k = i_k, J_{k+1} = i\}\right).
\] (7)

We set, for $k \in [m]$ and $(j_1, \ldots, j_k) \in \mathcal{D}(m, \emptyset, k)$,
\[
p_k^{(m)}(j_1, \ldots, j_k) := \mathbb{P}(J_1 = j_1, J_2 = j_2, \ldots, J_k = j_k).
\] (9)

We notice that, for $k < m$, the probability $p_k^{(m)}(j_1, \ldots, j_k)$ depends on
\[
(p_m^{(m)}(j_1, \ldots, j_k, i_{k+1}, \ldots, i_m) : (i_{k+1}, \ldots, i_m) \in \mathcal{D}(m, \{j_1, \ldots, j_k\}, m - k)
\]through the formula
\[
p_k^{(m)}(j_1, \ldots, j_k) = \sum_{(i_{k+1}, \ldots, i_m) \in \mathcal{D}(m, \{j_1, \ldots, j_m\}, m - k)} p_m^{(m)}(j_1, \ldots, j_k, i_{k+1}, \ldots, i_m).
\] (10)

As a consequence of Proposition 1, we thus obtain

**Corollary 1.** The probabilities $(\alpha_i(A) : A \in \hat{\mathcal{P}}(m), i \in A)$ are determined by the probabilities $(p_m^{(m)}(j_1, \ldots, j_m) : (j_1, \ldots, j_m) \in \mathcal{D}_m)$. 


Suppose now that, for any given \( A \in \hat{\mathcal{P}}(m) \), a ranking is fixed among the elements of \( A \). In order to describe such a ranking operation we define the ranking functions \( \sigma^{(m)}(A, \cdot) \) as follows. A function \( \sigma^{(m)}(A, \cdot) \) is a mapping from the elements of \( A \) to \( \{1, 2, \ldots, |A|\} \) with the property that its image is \( |\bar{w}| = \{1, \ldots, \bar{w}\} \) for some \( \bar{w} \leq |A| \).

For \( i, j \in A \), we say that, according to the ranking function \( \sigma^{(m)}(A, \cdot) \), \( i \) precedes \( j \) in \( A \) if and only if \( \sigma^{(m)}(A, i) < \sigma^{(m)}(A, j) \).

We admit the possibility that two elements are equivalent in \( A \) with respect to \( \sigma^{(m)} \), namely \( \sigma^{(m)}(A, i) = \sigma^{(m)}(A, j) \).

When we do not admit any possibility of equivalence between two elements we say that \( \sigma^{(m)}(A, \cdot) \) is a strict ranking function. Thus, a strict ranking function

\[
\sigma^{(m)}(A, \cdot) : A \rightarrow \{1, 2, \ldots, |A|\}
\]

is a bijection between the set \( A \) and \( \{|A|\} = \{1, 2, \ldots, |A|\} \). Hence, \( \sigma^{(m)}(A, \cdot) \) describes a permutation of the elements of \( A \).

**Definition 1.** For \( m \geq 2 \), a ranking scheme over \([m]\) is the family of the ranking functions

\[
\sigma^{(m)} = (\sigma^{(m)}(A, \cdot) : A \in \hat{\mathcal{P}}(m)).
\]

The collection of all the ranking schemes over \([m]\) will be denoted by \( \Sigma^{(m)} \). A ranking scheme only composed with strict ranking functions is called a strict ranking scheme. The collection of all the strict ranking schemes will be denoted by \( \hat{\Sigma}^{(m)} \subset \Sigma^{(m)} \).

In terms of the probabilities \( (\alpha_i(A) : A \in \hat{\mathcal{P}}(m), i \in A) \) in (4) one can associate a ranking scheme to the \( m \)-tuple \( X_1, \ldots, X_m \), as formalized by the following definition.

**Definition 2.** We say that the ranking scheme \( \sigma^{(m)} \) is \( p \)-concordant with the \( m \)-tuple \( (X_1, \ldots, X_m) \) whenever, for any \( A \in \hat{\mathcal{P}}(m) \), and \( i, j \in A \) with \( i \neq j \)

\[
\sigma^{(m)}(A, i) < \sigma^{(m)}(A, j) \iff \alpha_i(A) > \alpha_j(A), \tag{13}
\]

\[
\sigma^{(m)}(A, i) = \sigma^{(m)}(A, j) \iff \alpha_i(A) = \alpha_j(A). \tag{14}
\]

**Example 1.** For sake of brevity within this example we write \( p(\cdot) \) in place of \( p^{(3)}(\cdot) \).

With \( m = 3 \), consider non-negative random variables \( X_1, X_2, X_3 \) such that

\[
p(1, 2, 3) = \frac{2}{18}, \quad p(2, 1, 3) = \frac{2}{18}, \quad p(3, 2, 1) = \frac{5}{18},
\]

\[
p(3, 1, 2) = \frac{3}{18}, \quad p(2, 3, 1) = \frac{2}{18}, \quad p_3(1, 3, 2) = \frac{4}{18}.
\]

Thus we have

\[
\alpha_1([3]) = \frac{2}{18} + \frac{4}{18} = \frac{1}{3}, \quad \alpha_1([1, 2]) = \frac{2}{18} + \frac{3}{18} + \frac{4}{18} = \frac{1}{2}, \quad \alpha_1([1, 3]) = \frac{2}{18} + \frac{2}{18} + \frac{4}{18} = \frac{4}{9}.
\]

Similarly,

\[
\alpha_2([3]) = \frac{2}{9}, \quad \alpha_2([1, 2]) = \frac{1}{2}, \quad \alpha_2([2, 3]) = \frac{1}{3}, \quad \alpha_3([3]) = \frac{4}{9}, \quad \alpha_3([1, 3]) = \frac{5}{9}, \quad \alpha_3([2, 3]) = \frac{2}{3}.
\]

Then, the ranking scheme \( \sigma^{(3)} \), given below, is \( p \)-concordant with the triple \( (X_1, X_2, X_3) \):

\[
\sigma^{(3)}([3], 3) = 1, \quad \sigma^{(3)}([3], 1) = 2, \quad \sigma^{(3)}([3], 2) = 3.
\]
\[ \sigma^{(3)}(\{1, 2\}, 1) = \sigma^{(3)}(\{1, 2\}, 2) = 1 \] (namely, 1, 2 are equivalent in \{1, 2\}),

\[ \sigma^{(3)}(\{1, 3\}, 3) = 1, \sigma^{(3)}(\{1, 3\}, 1) = 2, \sigma^{(3)}(\{2, 3\}, 3) = 1, \sigma^{(3)}(\{2, 3\}, 3) = 2. \]

**Remark 1.** Of course, a same ranking scheme \( \sigma^{(m)} \) can be \( p \)-concordant with several different \( m \)-tuples of random variables.

A sufficient condition guaranteeing that a same ranking scheme is \( p \)-concordant with two different \( m \)-tuples \( X_1, \ldots, X_m \) and \( X'_1, \ldots, X'_m \) is, with obvious meaning of notation,

\[ p^{(m)}_m(j_1, \ldots, j_m) = p^{(m)}_m(j_1, \ldots, j_m), \text{ for any } (j_1, \ldots, j_m) \in D_m. \]

Such condition is not necessary.

### 2.2. Voting theory

We now switch to the context of voting theory and we refer to the language used e.g. in the monograph [III]. We denote by \([m] \) a set of candidates and by \( \mathcal{V}^{(n)} = \{v_1, \ldots, v_n\} \) a set of voters.

We assume that the individual preferences of the voter \( v_l \), for \( l = 1, \ldots, n \), gives rise to a linear preference ranking \( r_l \), which means that those preferences are complete, transitive and indifference is not allowed between any two candidates. Thus \( r_l \) is a strict ranking function \( \sigma^{(m)}_l([m], \cdot) \), namely any voter triggers a permutation over the set \([m] \).

The matrix \( R^{(n)} \) composed of all the \( r_l \)'s represents the voter profile.

Each voter \( v_i \) is supposed to cast her/his own vote in any possible election. When \( A \in \hat{P}(m) \) is the set of the candidates actually participating in a specific election, \( v_l \) will cast a vote for her/his preferred candidate within \( A \), according to her/his established linear preference ranking \( r_l \).

For \( h \in [m] \) and \( (j_1, \ldots, j_h) \in D(m, \emptyset, h) \), we denote by \( N^{(m)}_h(j_1, \ldots, j_h) \) the number of all the voters who respectively rank the candidates \( j_1, \ldots, j_h \) in the positions \( 1, \ldots, h \). Namely, for those voters, \( j_1, \ldots, j_h \) are the \( h \) most preferred candidates and

\[ N^{(m)}_h(j_1, \ldots, j_h) := |\{i \in [n] : r_i(j_k) = k \text{ for any } k \in [h]\}|. \]

In particular, \( N^{(m)}_m(j_1, \ldots, j_m) \) is the number of voters who share the same linear preference ranking. In the language of voting theory, the set of numbers

\[ \mathcal{N}^{(m)} = (N^{(m)}_m(j_1, \ldots, j_m) : (j_1, \ldots, j_m) \in D_m) \]

is called voting situation. Notice that the number of voters is given by

\[ n = \sum_{(j_1, \ldots, j_m) \in D_m} N^{(m)}_m(j_1, \ldots, j_m). \]  \( (15) \)

In an election where \( A \in \hat{P}(m) \) is the set of candidates, \( n_i(A) \) denotes the total number of votes obtained by the candidate \( i \in A \) according to the afore-mentioned voting rule. Thus \( \sum_{i \in A} n_i(A) = n \).
Similarly to Proposition 1 we can claim that, for \(A \in \hat{\mathcal{P}}(m)\) and \(i \in A\), the numbers \(n_i(A)\) are determined by the numbers \(N^{(m)}_h\), for \(h \in [m - 1]\). More precisely

\[
n_i(A) = N^{(m)}_1(i) + \sum_{h=1}^{m-|A|} \sum_{(j_1, \ldots, j_h) \in D(m, A, h)} N^{(m)}_{h+1}(j_1, \ldots, j_h, i). \tag{16}
\]

On the other hand the numbers \(N^{(m)}_h\) can be obtained from the knowledge of the numbers of the type \(N^{(m)}_m(j_1, \ldots, j_m)\). In fact

\[
N^{(m)}_h(j_1, \ldots, j_h) = \sum_{(i_{h+1}, \ldots, i_m) \in D(m, \{j_1, \ldots, j_h\}, h)} N^{(m)}_m(i_1, \ldots, i_{h+1}, \ldots, i_m). \tag{17}
\]

Therefore we see that the numbers \(n_i(A)\) are determined by the numbers of the type \(N^{(m)}_m\). In other words, the voting situation determines the number of votes for the candidate \(i \in A\) in an election limited to the members of \(A\).

We see that the voting scenario described so far gives rise to a ranking scheme \(\tau^{(m)} \in \Sigma^{(m)}\) in the sense of the following definition.

**Definition 3.** We say that a ranking scheme \(\tau^{(m)} \in \Sigma^{(m)}\) is \(N\)-concordant with the voting situation \(\mathcal{N}^{(m)}\) if, for any \(A \in \hat{\mathcal{P}}(m)\), and \(i, j \in A\) with \(i \neq j\)

\[
n_i(A) > n_j(A) \iff \tau^{(m)}(A, i) < \tau^{(m)}(A, j), \tag{18}
\]

\[
n_i(A) = n_j(A) \iff \tau^{(m)}(A, i) = \tau^{(m)}(A, j). \tag{19}
\]

A ranking scheme \(\tau^{(m)} \in \Sigma^{(m)}\), \(N\)-concordant with a voting situation \(\mathcal{N}^{(m)}\), in particular indicates for any subset of candidates \(A \in \hat{\mathcal{P}}(m)\) the candidate who obtains the plurality support.

### 2.3. Connections between voting theory and minima of random variables.

So far, we have been separately dealing with the two contexts of ranking non-negative random variables and voting theory, respectively.

Along the above presentation, a number of analogies and of different features of similarity have arisen at various levels. Notice in particular that formulas (16) and (17) are the analogue of formulas (10) and (11). The following two propositions highlight that the similarity existing between the two contexts is much more than an analogy. Actually, they show how questions arising from voting theory can be analyzed from the viewpoint of ranking lifetimes and pave the way for the results in the last section.

**Lemma 1.** Let \(\mathcal{N}^{(m)} = (N^{(m)}_m(j_1, \ldots, j_m) : (j_1, \ldots, j_m) \in D_m)\) be a voting situation and let \((X_1, \ldots, X_m)\) be such that

\[
\mathbb{P}(X_{j_1} < X_{j_2} < \ldots < X_{j_m}) = p^{(m)}_m(j_1, \ldots, j_m) = \frac{N^{(m)}_m(j_1, \ldots, j_m)}{K}, \tag{20}
\]

where \(K = \sum_{(j_1, \ldots, j_m) \in \mathcal{D}_m} N^{(m)}_m(j_1, \ldots, j_m)\). Then a ranking scheme \(\sigma^{(m)}\) is \(p\)-concordant with \((X_1, \ldots, X_m)\) if and only if it is \(N\)-concordant with \(\mathcal{N}^{(m)}\).
Proof. In view of Definition [2] and Definition [3] we must show that, for a set $A \in \mathcal{P}(m)$ with $i, j \in A$, 
\[ \alpha_i(A) < \alpha_j(A) \iff n_i(A) < n_j(A) \] 
(21)
and
\[ \alpha_i(A) = \alpha_j(A) \iff n_i(A) = n_j(A). \] 
(22)
By combining Proposition 1 with formula (10) and by plugging (20) into both equation (16) and equation (17), one obtains
\[ n_i(A) = K\alpha_i(A). \]
Whence (21) and (22) are immediately obtained. \qed

Let $\sigma^{(m)} \in \Sigma^{(m)}$ be a given ranking scheme. As a direct consequence of Lemma [1] we obtain an equivalence between the existence of $\mathcal{N}^{(m)}$ and the existence of $(X_1, \ldots, X_m)$, such that $\sigma^{(m)}$ is, respectively, $N$-concordant with $\mathcal{N}^{(m)}$ and $p$-concordant with $(X_1, \ldots, X_m)$.

**Proposition 2.** The following claims are equivalent.

i. There exists a voting situation $\mathcal{N}^{(m)}$ such that $\sigma^{(m)}$ is $N$-concordant with $\mathcal{N}^{(m)}$.

ii. There exists a $m$-tuple $(X_1, \ldots, X_m)$ such that $\sigma^{(m)}$ is $p$-concordant with $(X_1, \ldots, X_m)$
and $(p^{(m)}_m(j_1, \ldots, j_m) : (j_1, \ldots, j_m) \in \mathcal{D}_m)$ are all rational numbers.

Proof. ii. $\Rightarrow$ i. Let us consider a $m$-tuple $(X_1, \ldots, X_m)$ such that $\sigma^{(m)}$ is $p$-concordant with $(X_1, \ldots, X_m)$ and the associated quantities $(p^{(m)}_m(j_1, \ldots, j_m) : (j_1, \ldots, j_m) \in \mathcal{D}_m)$ are all rational numbers. Thus there exists $K \in \mathbb{N}$ such that $Kp^{(m)}_m(j_1, \ldots, j_m) \in \mathbb{N}$, for any $(j_1, \ldots, j_m) \in \mathcal{D}_m$. Let us define
\[ N^{(m)}_m(j_1, \ldots, j_m) = Kp^{(m)}_m(j_1, \ldots, j_m), \] for all $(j_1, \ldots, j_m) \in \mathcal{D}_m$.
Hence by Lemma [1] we obtain a voting situation $\mathcal{N}^{(m)} = (N^{(m)}_m(j_1, \ldots, j_m) : (j_1, \ldots, j_m) \in \mathcal{D}_m)$ such that $\sigma^{(m)}$ is $N$-concordant with $\mathcal{N}^{(m)}$.

i. $\Rightarrow$ ii. Starting from $\mathcal{N}^{(m)}$ and $\sigma^{(m)}$ that is $N$-concordant with $\mathcal{N}^{(m)}$, there are several different methods to built a $m$-tuple $X = (X_1, \ldots, X_m)$ for which $\sigma^{(m)}$ is $p$-concordant with $X$. A simple one goes as follows. We consider a discrete random variable $L$ uniformly distributed over $[n]$. From the population of all the voters we draw the single voter $r_L$, characterized by her/his linear preference ranking $r_L(\cdot) = \sigma^{(m)}([m], \cdot)$. We define, for any $i \in [m]$, $X_i = r_L(i)$. It is immediate to check that $p^{(m)}_m(j_1, \ldots, j_m) = N^{(m)}_m(j_1, \ldots, j_m)/n$. By applying Lemma [1] we conclude the proof. \qed

**Example 2.** Continuing Example [4], we consider the voting situation $\mathcal{N}^{(3)}$ obtained by setting
\[ N^{(3)}_3(j_1, j_2, j_3) = 18 \cdot p(j_1, j_2, j_3), \] for $(j_1, j_2, j_3) \in \mathcal{D}_3,$
where the probabilities $p(j_1, j_2, j_3)$ are given in the previous example. By applying Lemma [4] we obtain, without performing any computation, that the ranking scheme $\sigma^{(3)}$, defined in the previous example, is $N$-concordant with $\mathcal{N}^{(3)}$.

In the next section attention will be concentrated to the case of absolutely continuous joint probability distributions and, even more in particular, to the survival models of the type Time-Homogeneous Load-Sharing.
3. Time-homogeneous Load-Sharing model

In this section attention will be limited to the case of lifetimes admitting an absolutely continuous joint probability distribution. Such a joint distribution could then be described by means of the corresponding joint density function. In such a case, however, an alternative description can also be made in terms of the family of the Multivariate Conditional Hazard Rate functions.

The two methods are in principle equivalent, from a purely analytical viewpoint. However, they turn out to be respectively convenient to highlight different features of stochastic dependence. As it will be seen below, the description of dependence in terms of the Multivariate Conditional Hazard Rate functions is more convenient for our purposes. This choice, in particular, leads us to single out Load-Sharing dependence models and to appreciate their role in the present context.

Definition 4. Let $X_1, \ldots, X_m$ be non-negative random variables with an absolutely continuous joint probability distribution. For fixed $k \in [m - 1]$, let $(i_1, \ldots, i_k, j) \in \mathcal{D}(m, \emptyset, k + 1)$, and for an ordered sequence

$$0 < t_1 < \cdots < t_k < t,$$

the multivariate conditional hazard rate function $\lambda_j(t|i_1, \ldots, i_k; t_1, \ldots, t_k)$ is defined as follows:

$$\lambda_j(t|i_1, \ldots, i_k; t_1, \ldots, t_k) := \lim_{\Delta t \to 0^+} \frac{1}{\Delta t} \mathbb{P}(X_j \leq t + \Delta t | X_{i_1} = t_1, \ldots, X_{i_k} = t_k, X_{k+1:m} > t).$$

(23)

Furthermore, we put

$$\lambda_j(t|\emptyset) := \lim_{\Delta t \to 0^+} \frac{1}{\Delta t} \mathbb{P}(X_j \leq t + \Delta t | X_{1:m} > t).$$

(24)

For remarks, details, and for general aspects see e.g. [22], [23], [26], the review paper [24], and references cited therein.

For lifetimes $X_1, \ldots, X_m$, Load-Sharing is a very simple and natural condition of stochastic dependence which is defined in terms of the m.c.h.r. functions and which has a long history in the applied field of Reliability Theory. See e.g. the recent papers [26], [6] for references and some more detailed discussions and demonstrations.

Such a class of multivariate probability distributions is defined by means of the condition that the m.c.h.r. functions $\lambda_j(t|i_1, \ldots, i_k; t_1, \ldots, t_k)$ do not depend on the arguments $t_1, \ldots, t_k$. More precisely

Definition 5. The $m$-tuple $(X_1, \ldots, X_m)$ is distributed according to a Load-Sharing Model when, for $k \in [m - 1]$, $(i_1, \ldots, i_k, j) \in \mathcal{D}(m, \emptyset, k + 1)$, and for an ordered sequence $0 < t_1 < \cdots < t_k < t$, one has

$$\lambda_j(t|i_1, \ldots, i_k; t_1, \ldots, t_k) = \mu_j(t|i_1, \ldots, i_k),$$

for suitable functions $\mu_j(\cdot|i_1, \ldots, i_k)$. 

A Load-Sharing Model is, furthermore, time-homogeneous when there exist non-negative numbers $\mu_j(i_1, \ldots, i_k)$ and $\mu_j(\emptyset)$ such that, for any $t \geq 0,$
\begin{equation}
\mu_j(t|i_1, \ldots, i_k) = \mu_j(i_1, \ldots, i_k), \quad \mu_j(t|\emptyset) = \mu_j(\emptyset).
\end{equation}

For a fixed family $\mathcal{M}$ of the coefficients $\mu_j$'s appearing in (25), define, for $k \in [m-1]$ and $(i_1, \ldots, i_k) \in \mathcal{D}(m, \emptyset, k),$
\begin{equation}
M(i_1, \ldots, i_k) := \sum_{j \in [m]:(i_1, \ldots, i_k, j) \in \mathcal{D}(m, \emptyset, k+1)} \mu_j(i_1, \ldots, i_k) \text{ and } M(\emptyset) = \sum_{j \in [m]} \mu_j(\emptyset).
\end{equation}

In the present context, a first reason of interest for time-homogeneous load-sharing models is due to the simple form of the corresponding functions $p_k^{(m)}(j_1, \ldots, j_k).$ In fact we have

**Lemma 2.** Let $(X_1, \ldots, X_m)$ follow a THLS model with the family of coefficients $\mathcal{M}.$ Let $k \in [m]$ and let $(i_1, \ldots, i_k) \in \mathcal{D}(m, \emptyset, k).$ Then
\begin{equation}
\mathbb{P}(J_1 = i_1, J_2 = i_2, \ldots, J_k = i_k) = \frac{\mu_{i_1}(\emptyset) \mu_{i_2}(i_1) \mu_{i_3}(i_1, i_2)}{M(\emptyset) M(i_1) M(i_1, i_2)} \cdots \frac{\mu_{i_k}(i_1, i_2, \ldots, i_{k-1})}{M(i_1, i_2, \ldots, i_{k-1})}.
\end{equation}

**Proof.** See Proposition 2 in [26], see also [6].

We notice that, for fixed $j \in [m],$ the functions $\mu_j(i_1, \ldots, i_k)$ in (25) may in principle depend on the order according to which $i_1, \ldots, i_k$ are listed. When this happen, we will use the term *time-homogeneous load-sharing models in a broad sense.*

We will, on the contrary, use the term *time-homogeneous load-sharing models in a strict sense* when $\mu_j(i_1, \ldots, i_k)$ only depend on the set $\{i_1, \ldots, i_k\}$ and do not depend on the order in which the elements $i_1, \ldots, i_k$ are listed. In this situation we put, for $(i_1, \ldots, i_k, j) \in \mathcal{D}(m, \emptyset, k+1),$
\begin{equation}
\hat{\mu}_j(i_1, \ldots, i_k) = \mu_j(i_1, \ldots, i_k).
\end{equation}

This notation will be occasionally used in what follows, when convenient on the purpose of emphasizing that the argument in the function $\hat{\mu}$ is a set rather than an ordered sequence of indices. For fixed $\{i_1, \ldots, i_k\},$ consider the set formed by all the values $\hat{\mu}_j(\{i_1, \ldots, i_k\}), j \notin \{i_1, \ldots, i_k\}.$ Generally, for a THLS model in the strict sense, such a set actually depends on $\{i_1, \ldots, i_k\}.$ In this respect, we also need the following definition

**Definition 6.** A *time-homogeneous load-sharing model in a strict sense* has the property of Reallocation-Symmetry in Hazards if for any non empty set $A \subset [m]$ the collection of coefficients
\begin{equation}
\{\hat{\mu}_j([m] \setminus A) : j \in A\}
\end{equation}
depends on $A$ only through its cardinality $|A|.$ Such models will be indicated by acronym RSLS.

For RSLS models the condition
\begin{equation}
M(i_1, \ldots, i_k) = M(1, 2, \ldots, k)
\end{equation}
holds for any $(i_1, \ldots, i_k) \in \mathcal{D}(m, \emptyset, k).$
The RSLS models, even if very special, have a fundamental role in our paper. This will be attested by Theorem 2 that will be obtained in the next section.

4. MAIN RESULT

The results in this section generally aim to demonstrate the role of load-sharing models for the topics dealt with in this paper, as anticipated so far, and in Section 2 in particular.

It has also been pointed out therein that, for a vector of lifetimes \((X_1, \ldots, X_m)\), probabilities of permutations \(P(J_1 = i_1, \ldots, J_m = i_k)\) are the basic elements to determine the \(p\)-concordant ranking scheme.

Consider now an arbitrary probability distribution \(\rho\) on the set of permutations \(D_m\). The forthcoming result aim to show that it there exists a load-sharing model such that

\[
P(J_1 = i_1, \ldots, J_m = i_m) = \rho(i_1, \ldots, i_m).
\]

Next, we shall turn to facing the problem of main interest for this paper. Namely, we shall show, in Theorem 2, that for any arbitrary strict ranking scheme \(\sigma^{(m)}\), we can suitably construct a load-sharing model able to produce the condition of \(p\)-concordance.

**Theorem 1.** For \(m \geq 2\) let the function \(\rho : D_m \rightarrow [0, 1]\) satisfy the condition

\[
\sum_{(j_1, \ldots, j_m) \in D_m} \rho(j_1, \ldots, j_m) = 1.
\]

Then there exists a family of coefficients

\[
\mathcal{M} = \{\mu_j(j_1, \ldots, j_k) : k = 0, \ldots, m-1, (j_1, \ldots, j_k) \in D(m, \emptyset, k), j \notin \{j_1, \ldots, j_k\}\}
\]

such that the \(m\)-tuple \((X_1, \ldots, X_m)\), distributed according to a THLS model with parameters \(\mathcal{M}\), satisfies

\[
P(J_1 = i_1, \ldots, J_m = i_m) = \rho(i_1, \ldots, i_m).
\]

**Proof.** For the fixed function \(\rho\) and for \((j_1, \ldots, j_k) \in D(m, \emptyset, k)\) we set

\[
w(j_1, \ldots, j_k) = \sum_{(i_1, \ldots, i_{m-k}) \in D(m, \{j_1, \ldots, j_k\}, m-k)} \rho(j_1, \ldots, j_k, i_1, \ldots, i_{m-k}). \quad (30)
\]

Fix now coefficients as follows

\[
\mu_j(\emptyset) = w(j), \quad \mu_{j_2}(j_1) = \frac{w(j_1, j_2)}{w(j_1)}, \quad \mu_{j_3}(j_1, j_2) = \frac{w(j_1, j_2, i_3)}{w(j_1, j_2)}, \ldots,
\]

\[
\ldots, \mu_{j_{n-1}}(j_1, j_2, \ldots, j_{n-2}) = \frac{w(j_1, \ldots, j_{n-1})}{w(j_1, \ldots, j_{n-2})}.
\]

In the previous formula we tacitly understand \(0/0 = 0\). By considering a THLS model characterized by the coefficients above, the proof can now be concluded by applying Lemma 2. \(\square\)

The dependence model for \((X_1, \ldots, X_m)\) occurring in the proof of the above result is a broad-sense Load-Sharing model. It is, instead, sufficient to resort on the construction of a RSLS model in order to state and to prove Theorem 2 below. See also Remark 2.
Let $\sigma^{(m)} \in \hat{\Sigma}^{(m)}$ be an assigned strict ranking scheme. Starting from $\sigma^{(m)}$ we confine our attention on a RSLS model parametrized by a finite collection of positive numbers 

\[ (\varepsilon^{(m)}(u) \in (0, 1) : u = 2, \ldots, m). \]

The values $(\varepsilon^{(m)}(k) : k = 2, \ldots, m)$ will be suitably fixed along the proof of Theorem 2. The following condition is however required:

\[ \varepsilon^{(m)}(2) < \frac{1}{4} \text{ and } 2(u - 1)\varepsilon^{(m)}(u) < (u - 2)\varepsilon^{(m)}(u - 1), \text{ for } u = 2, \ldots, m. \tag{31} \]

Since here the value of $m$ is fixed once for ever, in what follows we will simply write $\varepsilon^{(i)}$ in place of $\varepsilon^{(m)}(i)$. We also set, for $u = 2, \ldots, m$,

\[ \rho(u) = \varepsilon(u)\frac{(u - 1)}{2}. \tag{32} \]

Sometimes, the parametrization based on the $\rho$’s will be used in place of the one based on the $\varepsilon$’s for the sake of obtaining more compact formulas. Written in terms of the $\rho$’s, the condition in (31) becomes

\[ \rho(2) < \frac{1}{8} \text{ and } 2\rho(u) < \rho(u - 1), \text{ for } u = 2, \ldots, m. \tag{33} \]

In what follows, we shall use several times a simple consequence of (33), namely

\[ \sum_{u=k}^{m} \rho(u) < 2\rho(k), \tag{34} \]

where $k = 2, \ldots, m$.

We now choose a special form for the coefficients of our RSLS model. For fixed $\sigma^{(m)} \in \hat{\Sigma}^{(m)}$, we denote by $\mathcal{M}(\sigma^{(m)})$ the collection of the coefficients as follows

\[ \mu_i([m] \setminus A) = 1 - (\sigma^{(m)}(A, i) - 1)\varepsilon(|A|), \text{ } A \in \hat{\mathcal{P}}^{(m)}, \text{ } i \in A. \tag{35} \]

For RSLS models $M(i_1, \ldots, i_k)$ do not depend on the indexes $i_1, \ldots, i_k$ (see Eq. (29)). With the above choice and in view of (20) above we even obtain that, for $k \in [m - 2]$, $M(i_1, \ldots, i_k)$ does not depend on $\sigma^{(m)}$. More precisely,

\[ M(i_1, \ldots, i_k) = \sum_{u=1}^{m-k} [1 - (u - 1)\varepsilon(m - k)] = m - k - \frac{(m - k)(m - k - 1)}{2}\varepsilon(m - k) \tag{36} \]

and

\[ M(\emptyset) = \sum_{u=1}^{m} [1 - (u - 1)\varepsilon(m)] = m - \frac{m(m - 1)}{2}\varepsilon(m). \tag{37} \]

We are ready to present some inequalities in the lemma below.

**Lemma 3.** Let $m \geq 2$, $\sigma^{(m)} \in \hat{\Sigma}^{(m)}$ and let the $m$-tuple $(X_1, \ldots, X_m)$ be distributed according to a RSLS model with coefficients $\mathcal{M}(\sigma^{(m)})$ defined in (35). Then, for any $k \in [m]$,

\[ \mathbb{P}(J_1 = i_1, J_2 = i_2, \ldots, J_k = i_k) \leq \frac{(m - k)!}{m!} \left(1 + 2 \sum_{u=m-k+1}^{m} \rho(u)\right) \tag{38} \]
and
\[ \mathbb{P}(J_1 = i_1, J_2 = i_2, \ldots, J_k = i_k) \geq \frac{(m-k)!}{m!} \left( 1 - 2 \sum_{u=m-k+1}^{m} \rho(u) \right). \] (39)

Proof. We start by proving the inequality (38). For \( k \in [m] \), by taking into account formula (27), (35) and (32), we obtain the following upper bound
\[ \mathbb{P}(J_1 = i_1, J_2 = i_2, \ldots, J_k = i_k) = \]
\[ = \frac{\mu_{i_1}(\emptyset)}{m - \frac{m(m-1)}{2} \epsilon(m)} \times \frac{\mu_{i_2}(i_1)}{m - 1 - \frac{(m-1)(m-2)}{2} \epsilon(m-1)} \times \ldots \times \frac{\mu_{i_k}(i_1, i_2, \ldots, i_{k-1})}{m - k + 1 - \frac{(m-k+1)(m-k)}{2} \epsilon(m - k + 1)} \leq \]
\[ \leq \frac{1}{m - \frac{m(m-1)}{2} \epsilon(m)} \times \frac{1}{m - 1 - \frac{(m-1)(m-2)}{2} \epsilon(m-1)} \times \ldots \times \frac{1}{1 - \rho(m - k + 1)} . \] (40)

By (34) one has \( \sum_{k=2}^{m} \rho(k) < \frac{1}{4} < \frac{1}{2} \). Furthermore, for \( a \in (0, \frac{1}{2}) \), the inequality \( 1/(1-a) < 1 + 2a \) holds.

Hence, we can conclude that the quantity in (40) is less or equal than
\[ \frac{1}{m(m-1) \cdots (m-k+1)} \times \frac{1}{1 - \sum_{u=m-k+1}^{m} \rho(u)} \leq \frac{1 + 2 \sum_{u=m-k+1}^{m} \rho(u)}{m(m-1) \cdots (m-k+1)} = \]
\[ = \frac{(m-k)!}{m!} \left( 1 + 2 \sum_{u=m-k+1}^{m} \rho(u) \right). \]

We now prove the inequality (39). By taking again into account formulas (27), (35) and (32), we also can give the following lower bound
\[ \mathbb{P}(J_1 = i_1, J_2 = i_2, \ldots, J_k = i_k) \geq \frac{\mu_{i_1}(\emptyset) \times \mu_{i_2}(i_1) \times \ldots \times \mu_{i_k}(i_1, i_2, \ldots, i_{m-1})}{m(m-1) \cdots (m-k+1)} \]
\[ \geq \frac{[1 - (m-1) \epsilon(m)] \times [1 - (m-2) \epsilon(m-1)] \times \ldots \times [1 - (m-k) \epsilon(m-k+1)]}{m(m-1) \cdots (m-k+1)} \]
\[ \geq \frac{1 - \sum_{u=m-k+1}^{m} \epsilon(u)}{m(m-1) \cdots (m-k+1)} = \frac{(m-k)!}{m!} \left( 1 - 2 \sum_{u=m-k+1}^{m} \rho(u) \right). \]

□

For an \( m \)-tuple \((X_1, \ldots, X_m)\) distributed according to a THLS model with coefficients \( \mathcal{M}(\sigma^m) \) defined in (35), the probabilities in (27) depend on the family \( \mathcal{M}(\sigma^m) \). Then also the probabilities \( \alpha_i(A) \) in (3) are determined by \( \sigma^m \). We now aim to give the probabilities \( \alpha_i(A) \) an expression convenient for what follows. We shall use the symbol \( \alpha_i(A, \sigma^m) \) and, in order to apply Proposition \( \Box \) we introduce the following notation. Fix \( A \in \mathcal{P}(m) \), for \( i \in A \) and \( \ell = |A| \leq m-1 \),
\[ \beta_i(A, \sigma^m) := \mathbb{P}(J_1 = i), \text{ if } \ell = m-1, \]
\[ \beta_i(A, \sigma^{(m)}) := \mathbb{P}(J_1 = i) + \sum_{k=1}^{m-\ell-1} \sum_{(i_1, \ldots, i_k) \in D(m, A, k)} \mathbb{P}(J_1 = i_1, J_2 = i_2, \ldots, J_k = i_k, J_{k+1} = i), \]

if \( 2 \leq \ell \leq m - 2 \). Also

\[ \gamma_i(A, \sigma^{(m)}) := \sum_{(i_1, \ldots, i_{m-\ell}) \in D(m, A, m-\ell)} \mathbb{P}(J_1 = i_1, J_2 = i_2, \ldots, J_{n-\ell} = i_{m-\ell}, J_{m-\ell+1} = i), \]

for any \( 2 \leq \ell \leq m - 1 \).

In terms of this notation and recalling Proposition 11, we can now write

\[ \alpha_i(A, \sigma^{(m)}) = \beta_i(A, \sigma^{(m)}) + \gamma_i(A, \sigma^{(m)}), \]

for \( A \in \hat{\mathcal{P}}(m) \) with \( |A| \leq m - 1 \).

By recalling Lemma 2 and the position \((45)\), we see that \( \beta_i(A, \sigma^{(m)}) \) only depends on \( \{\sigma^{(m)}(B, \cdot) : |B| \geq \ell\} \),

\[ \alpha_i(A, \sigma^{(m)}) \] only depends on \( \{\sigma^{(m)}(B, \cdot) : |B| \geq \ell - 1\} \).

Consider now, for fixed \( A \in \hat{\mathcal{P}}(m) \) such that \( |A| = \ell \),

\[ B(\ell) := \max\{\beta_i(A, \sigma^{(m)}) - \beta_j(A, \sigma^{(m)})\}, \]

where the maximum is computed with respect to all the strict ranking schemes \( \sigma^{(m)} \in \hat{\Sigma}(m) \). It follows that \( B(\ell) \) only depends on the quantities in \((44)\). Furthermore, such a quantity has the same value for any pair of distinct elements \( i, j \in A \) and, also recalling \((45)\), it takes the same value \( B(\ell) \) for all the subsets \( A \) with \( |A| = \ell \).

On the other hand, we also consider the quantities

\[ C(\ell) := \min\{\gamma_i(A, \sigma^{(m)}) - \gamma_j(A, \sigma^{(m)})\} > 0, \]

where the minimum is computed over the family of all the strict ranking schemes \( \sigma^{(m)} \in \hat{\Sigma}(m) \) such that \( \sigma^{(m)}(A, i) < \sigma^{(m)}(A, j) \). It follows that \( C(\ell) \) only depends on the quantities in \((45)\). Again \( C(\ell) \) only depends on the cardinality \( \ell = |A| \).

**Lemma 4.** For any \( m \geq 3 \) and any \( \sigma^{(m)} \in \hat{\Sigma}(m) \) one has, for \( \ell = 2, \ldots, m - 1 \), that

\[ B(\ell) \leq 8\ell\varepsilon(\ell + 1) \]

and

\[ C(\ell) \geq \frac{(m-\ell)!(\ell-1)!}{2 \cdot m!} \varepsilon(\ell). \]

**Proof.** For \( A \in \hat{\mathcal{P}}(m) \) such that \( |A| = \ell < m \), one has

\[ B(\ell) = \max_{\sigma^{(m)} \in \hat{\Sigma}(m)} \{\beta_i(A, \sigma^{(m)}) - \beta_j(A, \sigma^{(m)})\} \leq \]

\[ \leq \max_{\sigma^{(m)} \in \hat{\Sigma}(m)} |p_1(i) - p_1(j)| + \]

[Further details and equations follow, completing the proof.]
By (38) and (39) in Lemma 3, one has

\[ \max_{\sigma(m) \in \Sigma(m)} |p_{k+1}(i_1, \ldots, i_k, i) - p_{k+1}(i_1, \ldots, i_k, j)|. \]

(50)

By (38) and (39) in Lemma 3, one has

\[ \max_{\sigma(m) \in \Sigma(m)} |p_1(i) - p_1(j)| \leq \frac{4\rho(m)}{m} \]

and

\[ \max_{\sigma(m) \in \Sigma(m)} |p_{k+1}(i_1, \ldots, i_k, i) - p_{k+1}(i_1, \ldots, i_k, j)| \leq 4\frac{(m - k - 1)!}{m!} \sum_{u=m-k}^{m} \rho(u). \]

We can thus conclude by writing

\[ B(\ell) \leq \frac{4\rho(m)}{m} + 4 \sum_{k=1}^{m-\ell-1} \frac{(m - k - 1)!|D(m, A, k)|}{m!} \sum_{u=m-k}^{m} \rho(u). \]

(51)

For |A| = \ell, notice that \( |D(m, A, k)| = \frac{(m - \ell)!}{(m - \ell - k)!} < \frac{m!}{(m - k)!} \).

When |A| = \ell, one obtains that the r.h.s. in inequality (51) is smaller than

\[ 4\rho(m) + 4 \sum_{k=1}^{m-\ell-1} \sum_{u=m-k}^{m} \rho(u). \]

(52)

By (38), the quantity in (52) is smaller than

\[ 4\rho(m) + 8 \sum_{k=1}^{m-\ell-1} \rho(m - k) \leq 8 \sum_{k=0}^{m-\ell-1} \rho(m - k) \leq 16\rho(\ell + 1). \]

In conclusion

\[ B(\ell) \leq 16\rho(\ell + 1) = 8\ell \varepsilon(\ell + 1). \]

We now prove the inequality (49), for any \( \ell = 2, \ldots, m - 1 \).

\[ C(\ell) \geq \sum_{(i_1, \ldots, i_{m-\ell}) \in D(m, A, m-\ell)} \min_{\sigma(m) \in \Sigma(m)} \left\{ p_{m-\ell+1}(i_1, \ldots, i_{m-\ell}, i) - p_{m-\ell+1}(i_1, \ldots, i_{m-\ell}, j) \right\} \]

(53)

\[ \geq \sum_{(i_1, \ldots, i_{m-\ell}) \in D(m, A, m-\ell)} \min_{\sigma(m) \in \Sigma(m)} \left\{ p_{m-\ell}(i_1, \ldots, i_{m-\ell}) \right\} \cdot \min_{\sigma(m) \in \Sigma(m)} \left\{ \mathbb{P}(J_{m-\ell+1} = i | J_1 = i_1, \ldots, J_{m-\ell} = i_{m-\ell}) \right. \]

\[ \left. - \mathbb{P}(J_{m-\ell+1} = j | J_1 = i_1, \ldots, J_{m-\ell} = i_{m-\ell}) \right\}. \]

(54)

By (39) in Lemma 3, one has

\[ \min_{\sigma(m) \in \Sigma(m)} \left\{ p_{m-\ell}(i_1, \ldots, i_{m-\ell}) \right\} \geq \left[ \frac{\ell!}{m!} \left( 1 - 2 \sum_{u=\ell+1}^{m} \rho(u) \right) \right]. \]

(55)
Furthermore, see the proof of Lemma 2 and the identity (36), we have
\[
\min \left\{ \mathbb{P}(J_{m-\ell+1} = i | J_1 = i_1, \ldots, J_{m-\ell} = i_{m-\ell}) \right\} = \\
-\mathbb{P}(J_{m-\ell+1} = j | J_1 = i_1, \ldots, J_{m-\ell} = i_{m-\ell}) \\
: \sigma^{(m)}(A, i) < \sigma^{(m)}(A, j) \text{ with } \sigma^{(m)}(A, i) \in \hat{\Sigma}^{(m)}
\]
where the last inequality follows by the position (35).

By combining (55) and (56), we obtain
\[
\mathcal{C}(\ell) \geq \sum_{(i_1, \ldots, i_{m-\ell}) \in \mathcal{D}(m, A, m-\ell)} \left[ \frac{\ell!}{m!} (1 - 2 \sum_{u=\ell+1}^{m} \rho(u)) \right] \cdot \frac{1}{\ell - \frac{\ell(\ell-1)}{2} \varepsilon(\ell)} \cdot \varepsilon(\ell),
\]
and then
\[
\mathcal{C}(\ell) \geq (m-\ell)! \left[ \frac{(\ell-1)!}{m!} (1 - 2 \sum_{u=\ell+1}^{m} \rho(u)) \right] \varepsilon(\ell) \geq \frac{(m-\ell)!}{m!} (1 - 4 \rho(\ell + 1)) \varepsilon(\ell) \geq \frac{(m-\ell)!}{2 \cdot m!} \varepsilon(\ell),
\]
where we are exploiting inequality \(\rho(2) < \frac{1}{8}\) from (33).

Let \(\sigma^{(m)} \in \hat{\Sigma}^{(m)}\) be an arbitrary strict ranking scheme. We are now in a position to prove that it is possible to suitably construct a load-sharing model for an \(m\)-tuple \((X_1, \ldots, X_m)\), such that \(\sigma^{(m)}\) is \(p\)-concordant with \((X_1, \ldots, X_m)\).

\textbf{Theorem 2.} For any \(m \in \mathbb{N}\) and for any strict ranking scheme \(\sigma^{(m)} \in \hat{\Sigma}^{(m)}\) there exists an \(m\)-tuple \((X_1, \ldots, X_m)\) satisfying the following two conditions

i) \((X_1, \ldots, X_m)\) is distributed according to a RSLS model;

ii) \(\sigma^{(m)}\) is \(p\)-concordant with \((X_1, \ldots, X_m)\).

\textit{Proof.} We consider \((X_1, \ldots, X_m)\) distributed according to the THLS model defined in (35). We now show that, for a suitable choice of the parameters \(\varepsilon\)'s and for any strict ranking scheme \(\sigma^{(m)} \in \hat{\Sigma}^{(m)}\) one has, for any \(A \in \hat{\mathcal{P}}(m)\) and \(i, j \in A\), the equivalence
\[
\alpha_i(A, \sigma^{(m)}) > \alpha_j(A, \sigma^{(m)}) \iff \sigma^{(m)}(A, i) < \sigma^{(m)}(A, j),
\]
i.e. \(\sigma^{(m)}\) is \(p\)-concordant with \((X_1, \ldots, X_m)\). In order to prove such an equivalence it is enough, in view of the special choice in (35), to show that there exist \(\varepsilon(2), \ldots, \varepsilon(m)\), satisfying (31) and such that
\[
\alpha_i([m], \sigma^{(m)}) > \alpha_j([m], \sigma^{(m)}) \iff \sigma^{(m)}([m], i) < \sigma^{(m)}([m], j),
\]
and
\[
\mathcal{C}(\ell) > \mathcal{B}(\ell), \text{ for } \ell = 2, \ldots, m - 1.
\]
In fact, the relation (58) corresponds to (57) when $A = [m]$. Moreover, when $|A|$ belongs to $\{2, \ldots, m - 1\}$, one has, under the condition $\sigma^{(m)}(A, i) < \sigma^{(m)}(A, j)$,

$$0 < C(\ell) - B(\ell) \leq \alpha_i(A) - \alpha_j(A).$$

Therefore (58) and (59) imply (57), namely $\sigma^{(m)}$ is $p$-concordant with $X$.

It remains to prove (58) and (59). By Lemma 2 and the choice (35), one has

$$\alpha_i([m]) = \mathbb{P}(J_1 = i) = \frac{2 - 2(\sigma^{(m)}([m], i) - 1)\varepsilon(m)}{2m - m(m - 1)\varepsilon(m)},$$

then for any $\varepsilon(m) \in (0, \frac{1}{m - 1})$ the equivalence in (58) holds true. Therefore, we can defer to the end of the proof the suitable choice of the value of $\varepsilon(m)$.

In view of the inequalities (49) and (48) in Lemma 4, one can obtain the inequalities $C(\ell) > B(\ell)$ for $\ell = 2, \ldots, m - 1$ by choosing $\varepsilon(2), \ldots, \varepsilon(m)$ in such a way that

$$\frac{(m - \ell)!((\ell - 1)!}{2 \cdot m!} \varepsilon(\ell) > 8\varepsilon(\ell + 1).$$

This latter inequality can be obtained by simply letting

$$\varepsilon(\ell) = (17 \cdot m \cdot m!)^{m-\ell} \varepsilon(m).$$

In order to satisfy also condition (31), we set $\varepsilon(m) = (17 \cdot m \cdot m!)^{m-1}$. Thus, as a consequence of the last choice and of the one in (61), it has been proven that (58) and (59) hold true whence (57) holds as well.

As a result of the arguments in the above proof one can conclude as follows:

Let $\sigma^{(m)} \in \hat{\Sigma}^{(m)}$ be a given strict ranking scheme, then it is $p$-concordant with an $m$-tuple $(X_1, \ldots, X_m)$ distributed according to a RSLS model with coefficients

$$\hat{\mu}_i([m] \setminus A) = 1 - (\sigma^{(m)}(A, i) - 1)(17 \cdot m \cdot m!)^{1-|A|}, \quad i \in A,$$

where $A$ is a non-empty subset of $[m]$.

**Remark 2.** Theorem 1 and Theorem 2 respectively achieve two different goals. The former aims to show that, for any arbitrary probability distribution $\rho$ over the set of permutations, one can construct a broad-sense load-sharing model such that the corresponding vector $(J_1, \ldots, J_n)$ is distributed according to $\rho$. This result shows the general interest of load-sharing models, but it leaves unsolved the problem whether, for an arbitrary ranking scheme $\sigma^{(m)} \in \Sigma^{(m)}$, it is possible to find a distribution $\rho$ able to generate $\sigma^{(m)}$. Such a problem is solved by Theorem 2 and the solution is obtained in terms of RSLS models.

5. Ranking schemes and voting paradoxes

In this section we switch to applying the arguments of the previous section to the theme of elections and voting paradoxes. Recalling the notation introduced in Subsection 2.2, we think of a set of $m$ voters and aim to the construction of a voting situation $N^{(m)}$ able to generate a specific ranking scheme $\sigma^{(m)}$, $N$-concordant with $N^{(m)}$. The number of voters associated to a voting situation is generally denote by the symbol $n$. Later on,
we will also deal with the problem of constructing a voting situation able to generate a specific preference pattern.

First of all we show that the following result can be obtained, as a direct consequence of Theorem 2

**Corollary 2.** For any strict ranking scheme \( \sigma^{(m)} \in \hat{\Sigma}^{(m)} \), there exists a voting situation \( N^{(m)} \) with a number of voters \( n(m) \) only depending on \( m \) and such that \( \sigma^{(m)} \) is \( N \)-concordant with \( N^{(m)} \).

**Proof.** Fix \( \sigma^{(m)} \in \hat{\Sigma}^{(m)} \) and consider an \( m \)-tuple \((X_1, \ldots, X_m)\) distributed according to the THLS model with coefficients \( \hat{\mu}_i([m] \setminus A) \) as constructed by (35). Thus \( \sigma^{(m)} \) is \( p \)-concordant with \((X_1, \ldots, X_m)\).

Recalling (35) and (61), the coefficients \( \hat{\mu}_i([m] \setminus A) \) are all rational numbers. For any \((j_1, \ldots, j_m) \in D_m\), by Lemma 2, the probabilities \( p^{(m)}_m(j_1, \ldots, j_m; \sigma^{(m)}) \) are all rational numbers as well and will be denoted by \( p^{(m)}_m(j_1, \ldots, j_m; \sigma^{(m)}) \) in order to emphasize their dependence on \( \sigma^{(m)} \). Thus one can find \( n \in \mathbb{N} \) such that \( n p^{(m)}_m(j_1, \ldots, j_m; \sigma^{(m)}) \) are natural numbers, irrespectively on \( \sigma^{(m)} \). Consider now the voting situation \( N^{(m)} \) defined by

\[
N^{(m)}_m(j_1, \ldots, j_m; \sigma^{(m)}) = n p^{(m)}_m(j_1, \ldots, j_m; \sigma^{(m)}). \tag{63}
\]

By recalling (15), the total number of voters is given by

\[
n = \sum_{(j_1, \ldots, j_m) \in D_m} n p^{(m)}_m(j_1, \ldots, j_m; \sigma^{(m)}).
\]

The voting situation \( N^{(m)} \) coincides with the one given in the proof of Proposition 2 and thus \( \sigma^{(m)} \) is \( N \)-concordant with \( N^{(m)} \). \( \square \)

**Example 3.** For \( m = 3 \) consider the strict ranking scheme \( \sigma^{(3)} = \sigma \) defined as follows:

\[
\sigma([3], 1) = 1, \quad \sigma([3], 2) = 2, \quad \sigma([3], 3) = 3,
\]

\[
\sigma([1, 2], 1) = 2, \quad \sigma([1, 2], 2) = 1,
\]

\[
\sigma([1, 3], 1) = 2, \quad \sigma([1, 3], 3) = 1,
\]

\[
\sigma([2, 3], 2) = 2, \quad \sigma([2, 3], 3) = 1.
\]

Namely, when the set of candidates is \([3]\), 1 is the best candidate according to \( \sigma \) and 3 is the worst one. At the same time, still according to \( \sigma \), 3 is the best candidate and 1 is the worst when they respectively participate in a direct match against a single opponent. By applying Theorem 2 and Corollary 2 and by performing some manipulations we obtain the voting situation:

\[
N(1, 2, 3) = 2, \quad N(1, 3, 2) = 15, \quad N(2, 1, 3) = 1,
\]

\[
N(2, 3, 1) = 13, \quad N(3, 1, 2) = 0, \quad N(3, 2, 1) = 12.
\]

The following procedure shows in details how to obtain the above voting situation. In view of the positions in (35), the set of coefficients \( M \) is given by:

\[
\mu_1(\emptyset) = 1, \quad \mu_2(\emptyset) = 1 - \varepsilon(3), \quad \mu_3(\emptyset) = 1 - 2\varepsilon(3)
\]

\[
\mu_1(\{2\}) = 1 - \varepsilon(2), \quad \mu_3(\{2\}) = 1, \quad \mu_2(\{1\}) = 1 - \varepsilon(2),
\]
\[ \mu_3(\{1\}) = 1, \mu_1(\{3\}) = 1 - \varepsilon(2), \mu_2(\{3\}) = 1. \]

By the choices (61) and in the subsequent line one has \( \varepsilon(2) = \frac{1}{17 \times 2^2} = \frac{1}{306}, \varepsilon(3) = \varepsilon(2)^2 = \frac{1}{306}^2. \) For brevity sake we set \( a = \frac{1}{306}. \) The corresponding function \( \hat{M} \) is

\[ M(\emptyset) = 3(1 - a^2), \quad M(\{1\}) = M(\{2\}) = M(\{3\}) = 2 - a. \]

By Lemma 2 concerning the probabilities of permutations, we obtain

\[ p(1, 2, 3) = \frac{1}{3(1 + a)(2 - a)}, \quad p(1, 3, 2) = \frac{1}{3(1 - a)^2(2 - a)}, \]
\[ p(2, 1, 3) = \frac{1 - a}{3(2 - a)}, \quad p(2, 3, 1) = \frac{1}{3(2 - a)}, \]
\[ p(3, 1, 2) = \frac{1 - 2a^2}{3(1 + a)(2 - a)}, \quad p(3, 2, 1) = \frac{1 - 2a^2}{3(1 - a)^2(2 - a)}. \]

By multiplying all the probabilities \( p \) ’s by \( 3(1 - a)^2(2 - a) \) we obtain polynomials of degrees at most equal to three. Then we multiply by \( (306)^3 \) in order to obtain a set of all natural numbers \( \hat{N} = (\hat{N}(j_1, j_2, j_3) : (j_1, j_2, j_3) \in D_3). \) In view of Lemma 1 the ranking scheme \( \sigma \) is \( \hat{N} \)-concordant with the voting situation \( \hat{N}. \) From the voting situation \( \hat{N} \) we pass now to a different one by subtracting to all the elements \( \hat{N}(j_1, j_2, j_3) \) their minimum value. For \( A \in P(3) \) and \( i, j \in A, \) this operation does not modify the differences \( n_i(A) - n_j(A) \) and then the inequalities \( n_i(A) < n_j(A). \) A further useful operation on these numbers amounts to dividing all of them by a same natural number \( H \) in such a way that the inequalities \( [n_i(A)/H] < [n_j(A)/H] \) are maintained.

We notice that Theorem 2 and Corollary 2 with Example 3 focus attention to the case of strict ranking schemes. However the logic of construction of voting situations and of probabilities of permutations, as presented in the previous sections, can be usefully applied to different cases of interest even for general ranking schemes.

In particular we can deal with the problem solved by McGarvey in the classical paper [15], where the concept of preference pattern had been introduced. Here, we reformulate such a concept coherently with the notation used in this paper. For any \( A \subset [m] \) with \( |A| = 2, \) set

\[ \xi(A, \cdot) : A \to \{1, 2\}, \]

where

\[ \xi(\{i, j\}, i) = 1, \xi(\{i, j\}, j) = 2 \text{ when } iPj, \]
\[ \xi(\{i, j\}, j) = 1, \xi(\{i, j\}, i) = 2 \text{ when } jPi, \]
\[ \xi(\{i, j\}, i) = \xi(\{i, j\}, j) = 1 \text{ when } iIj, \]

and where, as in [15], \( iPj \) means that \( i \) is preferred to \( j \) while \( iIj \) means that \( i \) and \( j \) are equivalent.

One can now say that the collection of functions \( \xi^{(m)} = (\xi(A, \cdot) : A \subset [m], |A| = 2) \) is a preference scheme on the set \([m].\) Such a definition has then been given in a form analogous to that of ranking scheme. Similarly, it is also natural to give related definitions of \( p \)-concordance and \( N \)-concordance in the same line with Definition 2 and Definition 3.
Definition 7. We say that the preference pattern $\xi^{(m)}$ is $p$-concordant with the $m$-tuple $(X_1, \ldots, X_m)$ whenever, for any $i, j \in [m]$ with $i \neq j$ and $A = \{i, j\}$

\[
\xi^{(m)}(A, i) < \xi^{(m)}(A, j) \iff \alpha_i(A) > \alpha_j(A), \tag{65}
\]

\[
\xi^{(m)}(A, i) = \xi^{(m)}(A, j) \iff \alpha_i(A) = \alpha_j(A). \tag{66}
\]

$\xi^{(m)}$ is $N$-concordant with the voting situation $N^{(m)}$ if for any $i, j \in [m]$ with $i \neq j$ and $A = \{i, j\}$

\[
\xi^{(m)}(A, i) < \xi^{(m)}(A, j) \iff n_i(A) > n_j(A), \tag{67}
\]

\[
\xi^{(m)}(A, i) = \xi^{(m)}(A, j) \iff n_i(A) = n_j(A). \tag{68}
\]

By the use of Load-Sharing models we can easily construct probability distributions that are compatible with any preference pattern.

Theorem 3. For any preference pattern $\xi^{(m)}$ there exists an $m$-tuple $(X_1, \ldots, X_m)$ satisfying the following two conditions

i) $(X_1, \ldots, X_m)$ is distributed according to a strict sense time-homogeneous load-sharing model;

ii) $\xi^{(m)}$ is $p$-concordant with $(X_1, \ldots, X_m)$.

Proof. For brevity sake, along the proof we will write $\xi$ in place of $\xi^{(m)}$. Let us define the THLS model with coefficients given as follows. For $B \subset [m]$,

\[
\hat{\mu}_i(B) = 1, \text{ when } |B| \leq m - 3, \text{ or } |B| = m - 1,
\]

and, for $B = [m] \setminus \{i, j\}$,

\[
\hat{\mu}_i(B) = \hat{\mu}_j(B) = 1, \text{ when } \xi(\{i, j\}, i) = \xi(\{i, j\}, j) = 1,
\]

\[
\hat{\mu}_i(B) = 2, \hat{\mu}_j(B) = 1, \text{ when } \xi(\{i, j\}, i) = 1, \xi(\{i, j\}, j) = 2. \tag{69}
\]

Concerning the coefficients $M(i_1, \ldots, i_k)$ in (26) we obtain

\[
M(\emptyset) = \sum_{j \in [m]} \hat{\mu}_j(\emptyset) = m,
\]

and for $k \in [m - 3]$ and $(i_1, \ldots, i_k) \in D(m, \emptyset, k)$,

\[
M(i_1, \ldots, i_k) = \sum_{j \in [m] \setminus \{i_1, \ldots, i_k\}} \hat{\mu}_j(i_1, \ldots, i_k) = \sum_{j \in [m] \setminus \{i_1, \ldots, i_k\}} \mu_j(i_1, \ldots, i_k) = m - k.
\]

We now turn to computing the value of $M(i_1, \ldots, i_{m-2})$, which depends on $\xi$.

For $(i_1, \ldots, i_{m-2}, i_{m-1}, i_m) \in D_m$

\[
M(i_1, \ldots, i_{m-2}) = \sum_{j \in [m] \setminus \{i_1, \ldots, i_{m-2}\}} \mu_j(i_1, \ldots, i_{m-2}) = \mu_{i_{m-1}}(i_1, \ldots, i_{m-2}) + \mu_{i_m}(i_1, \ldots, i_{m-2}) = 2,
\]

if $\xi(\{i_{m-1}, i_m\}, i_{m-1}) = \xi(\{i_{m-1}, i_m\}, i_m) = 1$,

wheras

\[
M(i_1, \ldots, i_{m-2}) = \sum_{j \in [m] \setminus \{i_1, \ldots, i_{m-2}\}} \mu_j(i_1, \ldots, i_{m-2}) = \mu_{i_{m-1}}(i_1, \ldots, i_{m-2}) + \mu_{i_m}(i_1, \ldots, i_{m-2}) = 3,
\]

if $\xi(\{i_{m-1}, i_m\}, i_{m-1}) \neq \xi(\{i_{m-1}, i_m\}, i_m)$. 

Thus, for \( k \in [m - 2] \) and \((i_1, \ldots, i_k) \in \mathcal{D}(m, \emptyset, k)\),
\[
p_k^{(m)}(i_1, \ldots, i_k) = \frac{1}{m(m-1)\ldots(m-k+1)}.
\]

For any \( A = \{i, j\} \subset [m] \) (\(|A| = 2\)) we need to compare the probabilities \( \alpha_i(A) \) and \( \alpha_j(A) \). Thus, by (6),
\[
\alpha_i(A) = \frac{1}{m + \sum_{k=1}^{m-3} \sum_{(i_1, \ldots, i_k) \in \mathcal{D}(m, A, k)} \frac{1}{m(m-1)\ldots(m-k)} + \sum_{(i_1, \ldots, i_{m-2}) \in \mathcal{D}(m, A, m-2)} p_{m-1}^{(m)}(i_1, \ldots, i_{m-2}, i).}
\]

One can see that the probability \( \alpha_i(A) \) depends on \( i \) only due to the rightmost sum in formula (70). We also notice that,
\[
p_{m-1}^{(m)}(i_1, \ldots, i_{m-2}, i) = p_{m-1}^{(m)}(i_1, \ldots, i_{m-2}, j) = \frac{1}{m \cdot (m-1)\ldots\cdot 3 \cdot 2}
\]
if \( \xi(\{i, j\}, i) = \xi(\{i, j\}, j) = 1 \).

On the other hand,
\[
p_{m-1}^{(m)}(i_1, \ldots, i_{m-2}, i) = \frac{2}{3m \cdot (m-1)\ldots\cdot 3}, \quad p_{m-1}^{(m)}(i_1, \ldots, i_{m-2}, j) = \frac{1}{3m \cdot (m-1)\ldots\cdot 3}
\]
if \( \xi(\{i, j\}, i) = 1, \xi(\{i, j\}, j) = 2 \).

By using the formulas (70), (71) and (72) we can now compare the probabilities \( \alpha_i(A) \) and \( \alpha_j(A) \), for any \( A = \{i, j\} \subset [m] \).

By (70), (71) and recalling Definition 7 one has \( \alpha_i(A) - \alpha_j(A) = 0 \) when \( \xi(\{i, j\}, i) = \xi(\{i, j\}, j) = 1 \).

Whereas, by (70) and (72)
\[
\alpha_i(A) - \alpha_j(A) = \sum_{(i_1, \ldots, i_{m-2}) \in \mathcal{D}(m, A, m-2)} (p_{m-1}^{(m)}(i_1, \ldots, i_{m-2}, i) - p_{m-1}^{(m)}(i_1, \ldots, i_{m-2}, j))
\]
\[
= \frac{2}{3m(m-1)} > 0
\]
if and only if \( \xi(\{i, j\}, i) = 1, \xi(\{i, j\}, j) = 2 \). This concludes the proof.

From the proof of Theorem 2 we obtain the following conclusion concerning the construction of voting situations for \( m \) candidates: for any given preference pattern \( \xi^{(m)} \), there exists a voting situation \( N^{(m)} \) with \( n = 3m! \) voters such that \( \xi^{(m)} \) is \( N \)-concordant with \( N^{(m)} \). More precisely, we have

**Corollary 3.** Given \( m \) candidates and any given preference pattern \( \xi^{(m)} \), a voting situation \( N^{(m)} \), such that \( \xi^{(m)} \) is \( N \)-concordant with \( N^{(m)} \), is given by
\[
N_m^{(m)}(i_1, \ldots, i_{m-2}, i, j) = N_m^{(m)}(i_1, \ldots, i_{m-2}, j, i) = 3
\]
if \( \xi^{(m)}(\{i, j\}, i) = \xi^{(m)}(\{i, j\}, j) = 1 \), whereas
\[
N_m^{(m)}(i_1, \ldots, i_{m-2}, i, j) = 4, \quad N_m^{(m)}(i_1, \ldots, i_{m-2}, j, i) = 2
\]
if $\xi^{(m)}(\{i, j\}, i) = 1$, $\xi^{(m)}(\{i, j\}, j) = 2$.

Proof. All the probabilities $p^{(m)}_m(j_1, \ldots, j_{m-1}, j_m) = p^{(m)}_{m-1}(j_1, \ldots, j_{m-1})$ with $p^{(m)}_{m-1}(j_1, \ldots, j_{m-1})$ appearing in (71) and (72) give respectively rise to integer numbers when multiplied by $3m!$. By setting

$$N^{(m)}_m(j_1, \ldots, j_m) = 3m!p^{(m)}_m(j_1, \ldots, j_m),$$

we then obtain a voting situation. The latter coincides with $N^{(m)}$ and $\xi^{(m)}$ is $N$-concordant with it. \hfill \square

Besides the preference pattern $\xi^{(m)}$, we could also look at the ranking schemes produced by the THLS models in the proof of Theorem 3. We point out that a general property valid for such a ranking scheme is

$$\sigma^{(m)}(A, i) = 1, \text{ if } |A| \geq 3 \text{ and } \sigma^{(4)}(A, i) = \xi^{(4)}(A, i), \text{ if } |A| = 2.$$  \hspace{1cm} (75)

Namely, in any election all the participating candidates are equivalent except for the matches between only two candidates.

We also notice that the previous construction of voting situations can lead to a number of voters smaller than $3m!$. These two aspects are illustrated by the following example, motivated by the goal of constructing voting situations for cyclic preference patterns.

**Example 4.** We consider the following case with $m = 4$ where, in any match between $i, j \in [4]$, $i$ is the winner against $j$ if $i < j$ with the exception for the pair $1, 4$ in which 4 is the winner. Namely the preference pattern is as follows

$$\xi^{(4)}(\{1, 4\}, 1) = 2, \quad \xi^{(4)}(\{1, 4\}, 4) = 1,$$

in all the other cases

$$\xi^{(4)}(\{i, j\}, i) = 1, \quad \xi^{(4)}(\{i, j\}, j) = 2, \quad \text{for } i < j.$$  

The load sharing model considered in the proof of Theorem 3 is characterized by a set of coefficients as follows

$$\mu_i(A) = 1, \text{ for any } |A| \leq 1 \text{ and } i \not\in A,$$

$$\mu_1(\{2, 3\}) = 1, \quad \mu_4(\{2, 3\}) = 2,$$

$$\mu_i([m] \setminus \{i, j\}) = 2, \quad \mu_j([m] \setminus \{i, j\}) = 1,$$

when $i, j \in [4]$ with $i < j$ and $(i, j) \neq (1, 4)$. By formula (72) one obtains that all the probabilities of the permutations $p^{(4)}_4(j_1, j_2, j_3, j_4)$ are equal to the value $1/36$ or to the value $2/36$. As a consequence, and by formula (70), one obtains that the probabilities $\alpha_i(\{i, j\})$ are equal to $17/36$ or $19/36$. More precisely $\alpha_i(\{i, j\}) = 19/36$ if and only if $\xi^{(4)}(\{i, j\}, i) = 1$.

A voting situation giving rise to such a cyclic preference pattern is obtained by setting

$$N^{(4)}(j_1, j_2, j_3, j_4) = 36 \cdot p^{(4)}_4(j_1, j_2, j_3, j_4).$$

Hence 36 is a sufficient number of voters. The corresponding ranking scheme $\sigma^{(4)}$ has the form

$$\sigma^{(4)}(A, i) = 1, \text{ if } |A| = 3 \text{ or } |A| = 4 \text{ and } \sigma^{(4)}(A, i) = \xi^{(4)}(A, i), \text{ if } |A| = 2,$$

which in this special case is (75).
Remark 3. Both Theorem 2 and Theorem 3 are respectively based on the construction of suitable RSLS and THLS models. Those models are used (in Corollary 2 and in Corollary 3) to single out voting situations, respectively giving rise to assigned ranking schemes and to assigned preference patterns. In this respect, the first type of voting situation is more detailed in that it permits to specify assigned rankings over all the subsets $A \in \mathcal{P}(m)$. On the other hand, Corollary 3 allows us to deal with assigned preference patterns which also contain equivalent pairs of candidates.

6. Summary and concluding remarks

The main results in this paper (Theorem 2 and Theorem 3) show how Time-Homogeneous Load-Sharing models can be employed in the analysis of different types of ranking situations and, in particular, for applications in the field of voting theory (Corollary 2 and Corollary 3). By exploiting strict connections between voting theory and the topic of minima among random variables, we could hinge on the use of probabilistic tools, substantially elementary in nature, and aimed to develop a method directly related to the conceptual aspects of problems under consideration. From an analytic viewpoint, the specific role of THLS models can be explained as follows. First of all explicit formulas can be given (Lemma 2) for the computation of probabilities of all the $k$-permutations $p_k^{(m)}(j_1, \ldots, j_k) = \mathbb{P}(J_1 = j_1, \ldots, J_k = j_k)$. Such formulas become even simpler in the special case of RSLS models. Furthermore, for a given $m$-tuple $(Y_1, \ldots, Y_m)$ with any type of joint distribution, one can construct a corresponding THLS model for an $m$-tuple $(X_1, \ldots, X_m)$, in such a way that the two $m$-tuples share the same set of probabilities $p_k^{(m)}$ (Theorem 1). From an heuristic point of view, an appropriate differentiation among the coefficients $\mu_j(I)$, for given $I$ and for different $j \notin I$, dwells at the root of the construction of suitable THLS models. More precisely, in the selection of such coefficients it is convenient to reinforce (for $i, j \notin I$, $i \neq j$) the difference between $\mu_i(I)$ and $\mu_j(I)$ at the increase of the cardinality $|I|$. In the construction considered in Corollary 3 for example, the coefficients $\mu_j(I)$’s are even constant w.r.t. $j$, as far as $|I| < m - 2$.

The possibility to restrict attention on jointly absolutely continuous models permits the use of m.c.h.r. functions in the description of stochastic dependence. In the absolutely continuous case, such a type of description is in principle equivalent, but in a sense it is alternative, to the more general one based on the concept of copula. Also for what concerns the condition of stochastic precedence, and related aspects of intransitivity, some studies have been based on the concept of copula (see in particular [3], [5]). The analysis based on the m.c.h.r. functions of the same topics, as initiated in [6] and developed here, may lead to useful comparisons and interactions between the two methods.

Our results can also find an application to the field of system’s reliability, where the definition of Load-Sharing models had originally arisen. In such a field, more and more interest has been focused on the notion of signature of a system (see in particular [20]). For a system with structure function $\phi$ and with $m$ components having lifetimes $X_1, \ldots, X_m$, the signature is a probability distribution $s$ over $[m]$ which depends on $\phi$, and on the density function $f_X$ of $(X_1, \ldots, X_m)$. The dependence of $s$ on $f_X$ is encoded in the relative quality function $q(A) = \mathbb{P}(\max_{i \notin A} X_i < \min_{j \in A} X_j) (A \subset [m])$. See [14].
Here we observe that the function $q(\cdot)$ is determined, on its turn, by the probabilities $p_m^{(m)}$. Theorem 1 then also highlights the possible role of THLS models in the analysis of the signature and of related concepts.

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