Some algebraic surfaces with canonical map of degree 10, 12, 14

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**ABSTRACT**

Surfaces of general type with canonical map of degree $d$ bigger than 8 have bounded geometric genus and irregularity. In particular the irregularity is at most 2 if $d \geq 10$. In the present paper, the existence of surfaces with $d = 10$ and all possible irregularities, surfaces with $d = 12$ and irregularity 1 and 2, and surfaces with $d = 14$ and irregularity 0 and 1 is proven, by constructing these surfaces as $\mathbb{Z}_3^2$-covers of certain rational surfaces. These results together with the construction by C. Rito of a surface with $d = 12$ and irregularity 0 show that all the possibilities for the irregularity in the cases $d = 10, d = 12$ can occur, whilst the existence of a surface with $d = 14$ and irregularity 2 is still an open problem.

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1. Introduction

The problem of constructing examples of surfaces of general type with canonical map of high degree has been studied by many authors in the last decades. Let $X$ be a minimal smooth complex surface of general type and denote by $\varphi_{|K_X|} : X \rightarrow \mathbb{P}^{[p_g(X)-1]}$ the canonical map of $X$, where $K_X$ is the canonical divisor of $X$ and $p_g(X) = \dim H^0(X,K_X)$ is the geometric genus. In 1979, A. Beauville proved that the degree $d$ of the canonical map is at most 9 if the surface has holomorphic Euler-Poincaré characteristic bigger than 30 [3, Proposition 4.1]. Later in 1986, G. Xiao showed that the degree of the canonical map is at most 8 if the surface has geometric genus bigger than 132 [24, Theorem 3]. Only few surfaces with $d$ greater than 8 have been known so far, such as: S. L. Tan’s example [23] with $d = 9$, U. Persson’s example [18] with $d = 16$. In the last decade, some surfaces with $d = 12, 16, 20, 24, 27, 32, 36$ were constructed by C. Rito [19–22], C. Gleissner, R. Pignatelli and C. Rito [11], Ching-Jui Lai and Sai-Kee Yeung [12], [25] and the author [4, 6]. There are recent preprints [9, 10] of F. Fallucca and C. Gleissner constructing surfaces with $d = 10, 11, 12, 13, 14, 15, 18$.

When the canonical map has degree $d = 10, 12, 14$, by the Bogomolov-Miyaoka-Yau inequality the irregularity $q$ is at most 2. Indeed, we have

$$d(p_g - 2) \leq d \deg (\text{im}(\varphi_{|K_X|})) \leq K_X^2 \leq 9 \chi(O_X) = 9p_g - 9q + 9.$$  

This implies that $q \leq 2$ since $p_g \geq 3$. In this paper, by using $\mathbb{Z}_3^2$-covers of del Pezzo surfaces of degree 5, we construct three surfaces with $d = 10$ and with all possible irregularities $q = 0, 1, 2$. By using $\mathbb{Z}_3^2$-covers of del Pezzo surfaces of degree 6, we construct two surfaces with $d = 12$ and with $q = 1, 2$. These two examples together with C. Rito’s with $d = 12$ and with $q = 0$ show the existence of surfaces with $d = 12$ for every possibility of the irregularity. By using $\mathbb{Z}_3^2$-covers of del Pezzo surfaces of degree 7, we construct two surfaces with $d = 14$ and with $q = 0, 1$. We do not have any example with $d = 14$ and with $q = 2$. The following theorem is the result of this paper:

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Theorem 1. There exist minimal surfaces of general type $X$ satisfying the following

| $d$ | $q$ ($X$) | $p_g$ ($X$) | $K_X^2$ | $|K_X|$ |
|-----|-----------|-----------|--------|-------|
| 10  | 0         | 3         | 10     | base point free |
| 10  | 1         | 3         | 10     | base point free |
| 10  | 2         | 3         | 10     | has a non-trivial fixed part |
| 12  | 0         | 3         | 12     | base point free |
| 12  | 1         | 3         | 12     | base point free |
| 12  | 2         | 3         | 12     | base point free |
| 14  | 0         | 3         | 14     | base point free |
| 14  | 1         | 3         | 14     | base point free |

Furthermore, the canonical map corresponding to each surface in the above table is a morphism.

The idea of our constructions is the following: we first construct the surfaces with $d = 14$ and with $q = 0, 1$ by taking specific $\mathbb{Z}_2^3$-covers $X$ of del Pezzo surface $Y_2$ of degree 7 (see Notation 1). In particular, we take the covers to be in such a way that the following diagram commutes

\[
\begin{array}{ccc}
X & \xrightarrow{g} & Y_2 \\
\downarrow & & \downarrow \\
Z & \xrightarrow{\varphi} & \mathbb{P}^2 \\
\end{array}
\]

where the intermediate surface $Z$, which is a $\mathbb{Z}_2$-quotient of $X$, is a surface of general type whose only singularities are of type $A_1$. Moreover, we require that:

1. the canonical map of $Z$ is of degree 7,
2. the canonical map of $X$ factors through the quotient map $X \rightarrow Z$.

More precisely, we take the $\mathbb{Z}_2^3$-covers such that the $\mathbb{Z}_2$-quotient $Z := X/(0, 0, 1)$ is the bidouble cover of $Y_2$ with the building data \{D_1, D_2, D_3, L_1, L_2, L_3\} determined as follows:

\[
\begin{align*}
D_1 & := D_{010} + D_{011} \equiv -K_{Y_2} \\
D_2 & := D_{100} + D_{101} \equiv -K_{Y_2} \\
D_3 & := D_{110} + D_{111} \equiv -K_{Y_2} \\
L_1 & := L_{100} \equiv -K_{Y_2} \\
L_2 & := L_{010} \equiv -K_{Y_2} \\
L_3 & := L_{110} \equiv -K_{Y_2}.
\end{align*}
\]

The branch divisors $D_\sigma$ are chosen so that the intermediate surface $Z$ has only singularities of type $A_1$ and that the canonical linear system $|K_Z|$ is base point free. So the canonical map of $Z$ is of degree 7. Furthermore, we take the $\mathbb{Z}_2^3$-covers with $h^0(L_X + K_Y) = 0$ for all $\chi \in \{\chi_{001}, \chi_{101}, \chi_{011}, \chi_{111}\}$ so that the canonical map of $X$ factors through the quotient map $X \rightarrow Z$. In fact, by Proposition 5, we have the following decomposition:

\[
H^0(X, K_X) = H^0(Y_2, K_{Y_2}) \oplus \bigoplus_{\chi \neq \chi_{000}} H^0(Y_2, K_{Y_2} + L_{\chi}).
\]

We consider the subgroup $\Gamma := \langle (0, 0, 1) \rangle$ of $\mathbb{Z}_2^3$. Let $\Gamma^\perp$ denote the kernel of the restriction map $(\mathbb{Z}_2^3)^* \rightarrow \Gamma^*$, where $\Gamma^*$ is the character group of $\Gamma$. We have $\Gamma^\perp = \langle \chi_{100}, \chi_{010}, \chi_{110} \rangle$. The subgroup
An abelian cover of a smooth surface $Y$ with finite abelian group $G$ is a finite map $f : X \longrightarrow Y$ together with a faithful action of $G$ on $X$ such that $f$ exhibits $Y$ as the quotient of $X$ via $G$.

The construction of abelian covers was studied by R. Pardini in [16]. For detail about the building data of abelian covers and their notations, we refer the reader to Sections 1 and 2 of R. Pardini’s work.

2. Notation and $\mathbb{Z}_2^3$-coverings

2.1. Notation and conventions

Throughout this paper all surfaces are projective algebraic over the complex numbers. Linear equivalence of divisors is denoted by $\equiv$. A character $\chi$ of the group $\mathbb{Z}_2^3$ is a homomorphism from $\mathbb{Z}_2^3$ to $\mathbb{C}^*$, the multiplicative group of the non-zero complex numbers. We also use the following notation of del Pezzo surfaces:

**Notation 1.** We denote by $Y_2$ the blow-up of $\mathbb{P}^2$ at two distinct points $P_1, P_2$, by $l$ the pull-back of a general line in $\mathbb{P}^2$, by $e_1, e_2$ the exceptional divisors corresponding to $P_1, P_2$, respectively, by $f_1, f_2$ the strict transforms of a general line through $P_1, P_2$, respectively and by $h_{12}$ the strict transform of the line $P_1P_2$. The anti-canonical class $-K_{Y_2} \equiv f_1 + f_2 + l$ is very ample and the linear system $|-K_{Y_2}|$ embeds $Y_2$ as a smooth del Pezzo surface of degree 7 in $\mathbb{P}^7$.

**Notation 2.** We denote by $Y_3$ the blow-up of $\mathbb{P}^2$ at three distinct non-collinear points $P_1, P_2, P_3$. Let us denote by $l$ the pull-back of a general line in $\mathbb{P}^2$, by $e_1, e_2, e_3$ the exceptional divisors corresponding to $P_1, P_2, P_3$, respectively, by $f_1, f_2, f_3$ the strict transforms of a general line through $P_1, P_2, P_3$, respectively and by $h_{12}, h_{23}, h_{31}$ the strict transforms of the lines $P_1P_2, P_2P_3, P_3P_1$, respectively. The anti-canonical class $-K_{Y_3} \equiv f_1 + f_2 + f_3$ is very ample and the linear system $|-K_{Y_3}|$ embeds $Y_3$ as a smooth del Pezzo surface of degree 6 in $\mathbb{P}^6$.

**Notation 3.** We denote by $Y_4$ the blow-up of $\mathbb{P}^2$ at four points in general position $P_1, P_2, P_3, P_4$. Let us denote by $l$ the pull-back of a general line in $\mathbb{P}^2$, by $e_1, e_2, e_3, e_4$ the exceptional divisors corresponding to $P_1, P_2, P_3, P_4$, respectively, by $f_1, f_2, f_3, f_4$ the strict transforms of a general line through $P_1, P_2, P_3, P_4$, respectively and by $h_{ij}$ the strict transforms of the line $P_iP_j$, for all $i, j \in \{1, 2, 3, 4\}$, respectively. The anti-canonical class $-K_{Y_4} \equiv f_1 + f_2 + f_3 + f_4 - e_4$

$$\equiv f_1 + f_2 + f_4 - e_3$$

$$\equiv f_1 + f_3 + f_4 - e_2$$

$$\equiv f_2 + f_3 + f_4 - e_1$$

is very ample and the linear system $|-K_{Y_4}|$ embeds $Y_4$ as a smooth del Pezzo surface of degree 5 in $\mathbb{P}^5$.

2.2. $\mathbb{Z}_2^3$-coverings

An abelian cover of a smooth surface $Y$ with finite abelian group $G$ is a finite map $f : X \longrightarrow Y$ together with a faithful action of $G$ on $X$ such that $f$ exhibits $Y$ as the quotient of $X$ via $G$.

The construction of abelian covers was studied by R. Pardini in [16].
For the sake of completeness, we recall some facts on \( \mathbb{Z}_2^3 \)-covers, in a form which is convenient for our later constructions.

We will denote by \( \chi_{j_1j_2j_3} \) the character of \( \mathbb{Z}_2^3 \) defined by
\[
\chi_{j_1j_2j_3}(a_1, a_2, a_3) := e^{a_1j_1}e^{a_2j_2}e^{a_3j_3}
\]
for all \( j_1, j_2, j_3, a_1, a_2, a_3 \in \mathbb{Z}_2 \). A \( \mathbb{Z}_2^3 \)-cover \( X \to Y \) can be determined a collection of non-trivial divisors \( L_\chi \) labeled by characters of \( \mathbb{Z}_2^3 \) and effective divisors \( D_\sigma \) labeled elements of \( \mathbb{Z}_2^3 \) of the surface \( Y \).

More precisely, from [16, Theorem 2.1] we can define \( \mathbb{Z}_2^3 \)-covers as follows:

**Proposition 4.** Given \( Y \) a smooth projective surface with no non-trivial 2-torsion, let \( L_\chi \) be divisors of \( Y \) such that \( L_\chi \not\equiv \mathcal{O}_Y \) for all non-trivial characters \( \chi \) of \( \mathbb{Z}_2^3 \) and let \( D_\sigma \) be effective divisors of \( Y \) for all \( \sigma \in \mathbb{Z}_2^3 \setminus \{(0,0,0)\} \) such that the total branch divisor \( B := \sum_{\sigma \neq 0} D_\sigma \) is reduced. Then \( \{L_\chi, D_\sigma\}_{\chi,\sigma} \) is the building data of a \( \mathbb{Z}_2^3 \)-cover \( f : X \to Y \) if and only if
\[
\begin{align*}
2L_{100} & \equiv D_{100} + D_{101} + D_{110} + D_{111} \\
2L_{010} & \equiv D_{010} + D_{011} + D_{101} + D_{111} \\
2L_{001} & \equiv D_{001} + D_{011} + D_{101} + D_{111} \\
2L_{110} & \equiv D_{101} + D_{100} + D_{110} + D_{111} \\
2L_{101} & \equiv D_{001} + D_{010} + D_{101} + D_{111} \\
2L_{011} & \equiv D_{001} + D_{010} + D_{101} + D_{111} \\
2L_{111} & \equiv D_{001} + D_{010} + D_{101} + D_{111}.
\end{align*}
\]

The condition \( L_\chi \not\equiv \mathcal{O}_Y \) for all non-trivial characters \( \chi \) assures that the surface \( X \) is irreducible.

By [16, Theorem 3.1] if each branch component \( D_\sigma \) is smooth and the total branch locus \( B \) is a simple normal crossings divisor, the surface \( X \) is smooth.

Also from [16, Lemma 4.2, Proposition 4.2] we have:

**Proposition 5.** If \( Y \) is a smooth surface and \( f : X \to Y \) is a smooth \( \mathbb{Z}_2^3 \)-cover with the building data \( \{L_\chi, D_\sigma\}_{\chi,\sigma} \), the surface \( X \) satisfies the following:
\[
2K_X \equiv f^* \left( 2K_Y + \sum_{\sigma \neq 0} D_\sigma \right); \quad f_* \mathcal{O}_X = \mathcal{O}_Y \oplus \bigoplus_{\chi \neq \chi_{000}} L_\chi^{-1}.
\]

This implies that
\[
\begin{align*}
H^0(X, K_X) & = H^0(Y, K_Y) \oplus \bigoplus_{\chi \neq \chi_{000}} H^0(Y, K_Y + L_\chi); \\
K_X^2 & = 2 \left( 2K_Y + \sum_{\sigma \neq 0} D_\sigma \right) \\
p_g(X) & = p_g(Y) + \sum_{\chi \neq \chi_{000}} h^0(L_\chi + K_Y); \\
\chi(\mathcal{O}_X) & = 8\chi(\mathcal{O}_Y) + \sum_{\chi \neq \chi_{000}} \frac{1}{2} L_\chi (L_\chi + K_Y).
\end{align*}
\]

We notice that most of Proposition 5 does not require the smoothness assumption on \( X \). For a more general statement, we direct the reader to the work of Bauer and Pignatelli [2, Section 2].
From [16, Proposition 4.1.c 2.1], the generators of the canonical linear system $|K_X|$ are obtained as follows:

**Proposition 6.** If $Y$ is a smooth surface and $f : X \to Y$ is a smooth $\mathbb{Z}_2^3$-cover with building data $\{L_\chi, D_\sigma\}_{\chi, \sigma}$, the canonical linear system $|K_X|$ is generated by

$$f^*\left(|K_Y + L_\chi|\right) + \sum_{\chi(\sigma) = 1} R_\sigma, \forall \chi \in J$$

where $J := \{ \chi : |K_Y + L_\chi| \neq 0 \}$ and $R_\sigma$ is the reduced divisor supported on $f^* (D_\sigma)$.

For the explanation of this proposition, we refer the reader to [11, p. 3], [15, Remark 3.16] or [13, Section 3.4].

### 2.3. Resolution of singularities

Let $Y$ be a smooth surface, and let $f : X \to Y$ be a smooth $\mathbb{Z}_2^3$-cover with building data $\{L_\chi, D_\sigma\}_{\chi, \sigma}$. We consider $L'_\chi \equiv L_\chi, D'_\sigma \equiv D_\sigma$ so that $\{L'_\chi, D'_\sigma\}_{\chi, \sigma}$ is a building data of a $\mathbb{Z}_2^3$-cover $f : X' \to Y$ such that $X'$ is a normal surface. Assume the moment that there is only one point $P$ of $B' := \sum_{\sigma \neq 0} D'_\sigma$ giving rise to singularities on $X'$ and that the branch components $D'_\sigma$ meet transversally at $P$. This section is devoted to presenting a resolution of these singularities, which is similar to the resolution of singular bidouble cover [8]. The results in this section are taken from the work of C. Liedtke [13, Section 3].

Let $\pi : \tilde{Y} \to Y$ be the blow-up at $P$, and let $E$ be the exceptional divisor. We write $P = (\mu_{001}, \mu_{010}, \mu_{011}, \mu_{100}, \mu_{101}, \mu_{110}, \mu_{111})$, where $\mu_\sigma$ is the multiplicity of $D'_\sigma$ at $P$. Let $\tilde{\mu}_\sigma := \mu_\sigma$ for all $\sigma$, except for at most one case where $\tilde{\mu}_\sigma := \mu_\sigma - 1$ so that the following equations has a solution $(a_{001}, a_{010}, a_{011}, a_{100}, a_{101}, a_{110}, a_{111})$:

$$
\begin{align*}
\tilde{\mu}_{001} + \tilde{\mu}_{011} + 2\tilde{\mu}_{101} + 2\tilde{\mu}_{110} + \tilde{\mu}_{111} &= 2a_{100} \\
\tilde{\mu}_{001} + \tilde{\mu}_{011} + \tilde{\mu}_{101} + \tilde{\mu}_{110} + \tilde{\mu}_{111} &= 2a_{010} \\
\tilde{\mu}_{001} + \tilde{\mu}_{011} + \tilde{\mu}_{101} + \tilde{\mu}_{110} + \tilde{\mu}_{111} &= 2a_{001} \\
\tilde{\mu}_{001} + \tilde{\mu}_{011} + \tilde{\mu}_{101} + \tilde{\mu}_{110} + \tilde{\mu}_{111} &= 2a_{111}.
\end{align*}
$$

We consider

$$
\tilde{D}_\sigma := \pi^* (D'_\sigma) - \tilde{\mu}_\sigma E, \quad \tilde{L}_\chi := \pi^* (L'_\chi) - a_\chi E,
$$

for all non-trivial $\sigma, \chi$. We notice that in the case where $\tilde{\mu}_\sigma := \mu_\sigma - 1$, we set $\tilde{D}_\sigma := (\pi^* (D'_\sigma) - \mu_\sigma E) + E$. This means that the exceptional divisor $E$ is added into the strict transform of $D'_\sigma$. Then $\{\tilde{L}_\chi, \tilde{D}_\sigma\}_{\chi, \sigma}$ determines a canonical $\mathbb{Z}_2^3$-cover $\tilde{f} : \tilde{X} \to \tilde{Y}$. In [13, Section 3.2], the effect of point $P$ on $\chi(\mathcal{O}_{\tilde{X}})$ and $K_{\tilde{X}}^2$ is thoroughly analyzed. To ensure a comprehensive understanding of our construction, we provide restatements of some of the cases we utilize. Specifically, our construction involves the utilization of two types of triple points, which we describe below.

**Proposition 7.** Let $\mu_{\sigma_1} = \mu_{\sigma_2} = \mu_{\sigma_3} = 1$ and $\mu_\sigma = 0$ for all remaining $\sigma$.

1. If $(\sigma_1, \sigma_2, \sigma_3) \equiv \mathbb{Z}_2^3$, then $K_{\tilde{X}}^2 = K_X^2 - 2$ and $\chi(\mathcal{O}_{\tilde{X}}) = \chi(\mathcal{O}_X)$. 


2. If \((\sigma_1, \sigma_2, \sigma_3) \cong \mathbb{Z}_2^3\), then \(K_X^2 = K_Y^2\) and \(\chi(\mathcal{O}_Y) = \chi(\mathcal{O}_X)\).

**Proof.** 1. By reordering indices, without loss of generality, we may assume that \(\sigma_1 = (0, 0, 1), \sigma_2 = (0, 1, 0), \sigma_3 = (0, 1, 1)\). We have that

\[
\begin{align*}
(\hat{\mu}_{001}, \hat{\mu}_{010}, \hat{\mu}_{011}, \hat{\mu}_{100}, \hat{\mu}_{101}, \hat{\mu}_{110}, \hat{\mu}_{111}) &= (1, 1, 1, 0, 0, 0, 0) \\
(a_{001}, a_{010}, a_{011}, a_{100}, a_{101}, a_{110}, a_{111}) &= (1, 1, 1, 0, 1, 1, 1).
\end{align*}
\]

By Proposition 5 or [13, Section 3.2], we get \(K_Y^2 = K_X^2 - 2\) and \(\chi(\mathcal{O}_Y) = \chi(\mathcal{O}_X)\).

2. Analogously, we may assume that \(\sigma_1 = (0, 0, 1), \sigma_2 = (0, 1, 0), \sigma_3 = (1, 0, 0)\). We have that

\[
\begin{align*}
(\hat{\mu}_{001}, \hat{\mu}_{010}, \hat{\mu}_{011}, \hat{\mu}_{100}, \hat{\mu}_{101}, \hat{\mu}_{110}, \hat{\mu}_{111}) &= (1, 1, 0, 1, 0, 0, -1) \\
(a_{001}, a_{010}, a_{011}, a_{100}, a_{101}, a_{110}, a_{111}) &= (0, 0, 1, 0, 1, 1, 1).
\end{align*}
\]

By Proposition 5 or [13, Section 3.2], we get \(K_Y^2 = K_X^2\) and \(\chi(\mathcal{O}_Y) = \chi(\mathcal{O}_X)\). \(\square\)

In our construction, we apply a type of quadruple points in order to decrease \(\chi(\mathcal{O}_X)\). These quadruple points belong to the branch locus of a double cover ramified on \(2L_X\) for some \(\chi\).

**Proposition 8.** Fix a character \(\chi\) and let \(\sigma_1, \sigma_2, \sigma_3, \sigma_4\) be non-trivial elements in \(\mathbb{Z}_2^3\) such that \(\chi(\sigma_1) = \chi(\sigma_2) = \chi(\sigma_3) = \chi(\sigma_4) = -1\). Let \(\mu_{\sigma_1} = 2, \mu_{\sigma_2} = \mu_{\sigma_3} = 1, \mu_{\sigma_4} = 0\) and \(\mu_{\sigma} = 0\) for all remaining \(\sigma\). Then

\[K_X^2 = K_Y^2 - 2\text{ and }\chi(\mathcal{O}_Y) = \chi(\mathcal{O}_X) - 1.\]

**Proof.** By reordering indices, without loss of generality, we may assume that \(\chi = \chi_{100}\) and \(\sigma_1 = (1, 0, 0), \sigma_2 = (1, 0, 1), \sigma_3 = (1, 1, 0), \sigma_4 = (1, 1, 1)\). We have that

\[
\begin{align*}
(\hat{\mu}_{001}, \hat{\mu}_{010}, \hat{\mu}_{011}, \hat{\mu}_{100}, \hat{\mu}_{101}, \hat{\mu}_{110}, \hat{\mu}_{111}) &= (0, 0, -1, 2, 1, 1, 0) \\
(a_{001}, a_{010}, a_{011}, a_{100}, a_{101}, a_{110}, a_{111}) &= (0, 0, 1, 2, 1, 1, 1).
\end{align*}
\]

By Proposition 5 or [13, Section 3.2], we obtain \(K_Y^2 = K_X^2 - 2\) and \(\chi(\mathcal{O}_Y) = \chi(\mathcal{O}_X) - 1\). \(\square\)

### 3. Constructions of the regular surfaces

In this section, we present the constructions of the regular surfaces with \(d = 10, 12, 14\) listed in Theorem 1. We first construct a surface with \(d = 14\) and \(q = 0\) by using a \(\mathbb{Z}_3^2\)-cover of a del Pezzo surface of degree 7. To obtain regular surfaces with \(d = 10, 12\), we impose triple points in the total branch locus and resolve the singularities.

#### 3.1. A surface with \(d = 14\) and \(q = 0\).

Let \(Y_2\) be a del Pezzo surface of degree 7 (see Notation 1). We consider the following smooth divisors:

\[
\begin{align*}
D_{010} &:= f_{11} + f_{12} & D_{011} &:= f_{21} + e_1 \\
D_{100} &:= f_{22} & D_{101} &:= C_1 \in [l + f_1] \\
D_{110} &:= C_2 \in [l + f_1] & D_{111} &:= f_{23},
\end{align*}
\]

and \(D_{001} = 0\) where \(f_{11}, f_{12} \in [f_1], f_{21}, f_{22}, f_{23} \in [f_2]\) and \(C_1, C_2\) are distinct divisors of \(Y_2\) such that no more than two of these divisors \(D_{\sigma}\) go through the same point. We consider the following non-trivial
divisors of $Y_2$:

\[
\begin{align*}
L_{100} & := f_1 + f_2 + l \\
L_{010} & := f_1 + f_2 + l \\
L_{001} & := f_2 + l \\
L_{110} & := f_1 + f_2 + l \\
L_{101} & := f_2 + l \\
L_{011} & := 2f_1 + l \\
L_{111} & := f_1 + f_2.
\end{align*}
\]

These divisors $D_\sigma, L_X$ satisfy the following relations:

\[
\begin{align*}
2L_{100} & \equiv D_{100} + D_{101} + D_{110} + D_{111} \equiv 2f_1 + 2f_2 + 2l \\
2L_{010} & \equiv D_{010} + D_{011} + D_{110} + D_{111} \equiv 2f_1 + 2f_2 + 2l \\
2L_{001} & \equiv D_{001} + D_{011} + D_{101} + D_{111} \equiv 2f_2 + 2l \\
2L_{110} & \equiv D_{101} + D_{011} + D_{100} + D_{111} \equiv 2f_1 + 2f_2 + 2l \\
2L_{101} & \equiv D_{001} + D_{010} + D_{101} + D_{111} \equiv 2f_1 + 2f_2 + 2l \\
2L_{111} & \equiv D_{001} + D_{010} + D_{100} + D_{111} \equiv 4f_1 + 2l \\
\end{align*}
\]

Thus by Proposition 4, the divisors $D_\sigma, L_X$ define a $\mathbb{Z}_2^3$-cover $g : X \rightarrow Y_2$. Because each branch component $D_\sigma$ is smooth and the total branch locus $B$ is a normal crossings divisor, the surface $X$ is smooth. Moreover, by Proposition 5, the surface $X$ satisfies the following:

\[2K_X \equiv g^*(f_1 + f_2 + l)\]

We notice that a surface is of general type and minimal if the canonical divisor is big and nef (see e.g. [14, Section 2]). Since the divisor $2K_X$ is the pull-back of a nef and big divisor, the canonical divisor $K_X$ is nef and big. Thus, the surface $X$ is of general type and minimal. Furthermore, from Proposition 5, the surface $X$ possesses the following invariants:

\[K_X^2 = 2(f_1 + f_2 + l)^2 = 14,\]

\[\chi_0 (X) = p_g (Y_2) + \sum_{X \neq X_{000}} h^0 (L_X + K_{Y_2}) = h^0 (O_{Y_2}) + h^0 (O_{Y_2}) + h^0 (O_{Y_2}) = 3,\]

\[\chi (O_{X}) = 8\chi (O_{Y_2}) + \sum_{X \neq X_{000}} \frac{1}{2} L_X (L_X + K_{Y_2}) = 4,\]

\[q (X) = 1 + p_g (X) - \chi (O_{X}) = 0.\]

We show that the canonical map $\varphi_{|K_X|}$ has degree 14. By Proposition 6, the linear system $|K_X|$ is generated by the three following divisors:

\[
\bar{f}_{11} + \bar{f}_{12} + \bar{f}_{21} + \bar{f}_{22} + \bar{C}_1, \bar{C}_2 + \bar{f}_{23},
\]

where $\bar{C}_1 := g^* (C_1)_{\text{red}}, \bar{C}_2 := g^* (C_2)_{\text{red}}, \bar{e}_1 := g^* (e_1)_{\text{red}}$ and $\bar{f}_{ij} := g^* (f_{ij})_{\text{red}}$. Because the three divisors $\bar{f}_{11} + \bar{f}_{12} + \bar{f}_{21} + \bar{e}_1, \bar{f}_{22} + \bar{C}_1, \bar{C}_2 + \bar{f}_{23}$ have no common intersection, the linear system $|K_X|$ is base point free. This together with $K_X^2 = 14 > 0$ implies that the linear system $|K_X|$ is not composed with a pencil. Thus, the canonical image is $\mathbb{P}^3$ and the canonical map is of degree 14. Therefore, we obtain the surface described in the seventh row of Theorem 1.

Remark 1. The surface $X$ has two pencils of genus 5 corresponding to the fibers $f_1, f_2$.

Infact, the $\mathbb{Z}_2^3$-cover $g : X \rightarrow Y_2$ can be written as the composition of the three double covers:

\[
X \xrightarrow{g_3} X_2 \xrightarrow{g_2} X_1 \xrightarrow{g_1} Y_2.
\]
where the double covers \( g_1, g_2, \) and \( g_3 \) correspond to \( 2L_{001}, 2L_{010}, \) and \( 2L_{100} \), respectively. The pull-backs \( g^* (f_1) = g_3^* (g_2^* (f_1)) \) of a general fiber \( f_1 \) is an irreducible curve since the pull-backs of a general fiber \( f_1 \) intersect the branch locus in every double cover. By applying the Hurwitz formula, we find that the genus of \( g^* (f_1) \) is 5.

### 3.2. Variations

We now consider variations of the construction presented in Section 3.1 in order to obtain the remaining regular surfaces listed in Theorem 1. To achieve this, we impose ordinary triple points on the branch locus and we resolve the singularities. We apply triple points that arise from the intersection of three divisors \( D_{\sigma_1}, D_{\sigma_2}, \) and \( D_{\sigma_1 + \sigma_2} \). By introducing these triple points, we can achieve surfaces with the same Euler characteristic. According to Proposition 7, imposing one such triple point will result in a surface with \( K^2 = 12 \) and \( \chi = 4 \). On the other hand, if we simultaneously impose two such triple points, we will obtain a surface with \( K^2 = 10 \) and \( \chi = 4 \). More precisely, the specific types of triple points we impose and the corresponding surfaces obtained are described in the following table:

| \( P = (\mu_{001}, \mu_{010}, \mu_{011}, \mu_{100}, \mu_{101}, \mu_{110}, \mu_{111}) \) | \( K^2 \) | \( \chi (\mathcal{O}_X) \) | \( p_g (X) \) | \( q (X) \) | \( d \) |
|---|---|---|---|---|---|
| \( P_1 = (0, 0, 1, 0, 1, 1, 0) \) | 12 | 4 | 3 | 0 | 12 |
| \( P_3 = (0, 0, 1, 0, 1, 1, 0) \) and \( P_4 = (0, 1, 0, 1, 0, 1, 1) \) | 10 | 4 | 3 | 0 | 10 |

In our construction, we consistently have \( L_{100} \equiv L_{010} \equiv L_{110} \equiv -K_{Y_1} \), where \( Y_1 \) is the del Pezzo we take the cover. This condition ensures that \( p_g (X) = 3 \). The degree of the canonical map is proven similarly in Section 3.1. For further information regarding these constructions, we provide the building data of the \( \mathbb{Z}^3_2 \)-covers.

#### 3.2.1. The building data of the surface with \( d = 12 \) and \( q = 0 \)

Let us denote by \( Y_3 \) the blow-up of \( Y_2 \) at a point \( P_3 \). The surface \( Y_3 \) is a del Pezzo surface of degree 6 (see Notation 2). We consider the following smooth divisors:

\[
\begin{align*}
D_{100} :&= f_{11} + f_{12} & D_{011} :&= h_{23} + e_1 \\
D_{100} :&= f_{21} & D_{101} :&= C_1 \in |f_1 + f_3|
\end{align*}
\]

and \( D_{001} = 0 \), where \( f_{11}, f_{12} \in |f_1|, f_{21}, f_{22} \in |f_2| \) and \( C_1, C_2 \) are distinct divisors in \( Y_3 \) such that no more than two of these divisors \( D_{\sigma} \) go through the same point. In addition, we consider the following non-trivial divisors in \( Y_3 \):

\[
\begin{align*}
L_{100} :&= f_1 + f_2 + f_3 \\
L_{010} :&= f_1 + f_2 + f_3 \\
L_{001} :&= f_2 + f_3 \\
L_{110} :&= f_1 + f_2 + f_3 \\
L_{011} :&= 2f_1 + f_3 \\
L_{111} :&= f_1 + f_2.
\end{align*}
\]

These divisors \( D_{\sigma}, L_\chi \) define a \( \mathbb{Z}^3_2 \)-cover \( g : X \longrightarrow Y_3 \). Therefore we obtain the surface in the fourth row of Theorem 1.

#### 3.2.2. The building data of the surface with \( d = 10 \) and \( q = 0 \)

We denote by \( Y_4 \) the blow-up of \( Y_2 \) at two distinct points \( P_3 \) and \( P_4 \). The surface \( Y_4 \) is a del Pezzo surface of degree 5 (see Notation 3). We consider the following smooth divisors:

\[
\begin{align*}
D_{010} :&= f_{11} + h_{14} & D_{011} :&= h_{23} + e_1 \\
\end{align*}
\]
and $D_{001} = 0$, where $f_{11} \in \left| f_{1} \right|$, $f_{21} \in \left| f_{2} \right|$ and $C_1, C_2$ are divisors of $Y_4$ such that no more than two of these divisors $D_\sigma$ go through the same point. We consider the following divisors:

$$
\begin{align*}
L_{100} &:= f_1 + f_2 + f_3 - e_4 \\
L_{010} &:= f_1 + f_2 + f_3 - e_4 \\
L_{001} &:= f_2 + f_3 - e_4 \\
L_{110} &:= f_1 + f_2 + f_3 - e_4 \\
L_{101} &:= f_2 + f_3 \\
L_{011} &:= 2f_1 + f_3 - e_4 \\
L_{111} &:= f_1 + f_2 - e_4.
\end{align*}
$$

These divisors $D_\sigma, L_\chi$ define a $\mathbb{Z}_2^3$-cover $g : X \to Y_4$. Therefore we obtain the surface in the first row of Theorem 1.

**Remark 2.** The surface $X$ has four pencils of genus 5 corresponding to the fibers $f_1, f_2, f_3, f_4$.

**Remark 3.** Taking the $\mathbb{Z}_2^3$-cover of $Y_2$ ramified on the above branch locus with two triple points $P_3, P_4$, we would obtain a surface with four singular points of type $\frac{1}{2} (1, 1)$ (cf. [1, Table 1, Section 3.3]). The preimage of each $P_i$ is two $(-4)$-curves. The surface $X$ is the minimal resolution of this singular surface.

In fact, we consider the following commutative diagram

\[
\begin{array}{ccc}
\phantom{g} & X & \overset{\mathbb{Z}_2^3}{\longrightarrow} & Y_2 \\
\Downarrow & & & \Downarrow \\
\phantom{g} & X & \overset{\mathbb{Z}_2^3}{\longrightarrow} & Y_4 \\
\end{array}
\]

where the map $\overline{g}$ is the $\mathbb{Z}_2^3$-cover of $Y_2$ ramified on the above branch locus with two triple points $P_3, P_4$ and the map $\pi$ is the blow-up at $P_3, P_4$. Each divisor $g^*(\pi(P_i))$ are two $(-4)$-curves. The map $\epsilon$ is the contraction of these four $(-4)$-curves.

### 4. Constructions of the irregular surfaces

In this section, we present the constructions of the surfaces with $d = 10, 12, 14$ and $q = 1, 2$ listed in Theorem 1. We first construct a surface with $d = 14$ and $q = 1$ by using a $\mathbb{Z}_2^3$-cover of a del Pezzo surface of degree 7. To obtain the other irregular surfaces described in Theorem 1, we impose triple or quadruple points in the total branch locus and resolve the singularities.

#### 4.1. A surface with $d = 14$ and $q = 1$

The construction of the surface with $d = 14$ and $q = 1$ is essentially similar to the construction of the regular surface with $d = 14$ described in Section 3.1. We construct this surface as a $\mathbb{Z}_2^3$-cover of a del Pezzo surface $Y_2$ of degree 7. However, in order to achieve $q = 1$, we take the branch locus in such a way there is a quotient $X/\mathbb{Z}_2^3$, which is a double cover of $Y_2$ ramified in four fibers $f_2$. More precisely, we consider the following smooth divisors:

$$
\begin{align*}
D_{010} &:= f_{11} + f_{12} \\
D_{100} &:= l_1 + e_2 \\
D_{011} &:= f_{21} + e_1 \\
D_{101} &:= f_{22} + h_{12}
\end{align*}
$$
Consequently, applying the universal property, the image of the Albanese morphism of \(X\) goes through the same point. We consider the following non-trivial divisors in \(Y\):

\[
\begin{align*}
L_{100} & := f_1 + f_2 + l \\
L_{010} & := f_1 + f_2 + l \\
L_{001} & := 2f_2 \\
L_{110} & := f_1 + f_2 + l \\
L_{101} & := 2l \\
L_{111} & := 2f_2 + f_2 + l.
\end{align*}
\]

Since the divisors \(D_\sigma, L_X\) satisfy the relations in Proposition 4, these divisors \(D_\sigma, L_X\) define a \(\mathbb{Z}_2^3\)-cover \(g : X \rightarrow Y_2\). Because each branch component \(D_\sigma\) is smooth and the total branch locus is a normal crossings divisor, the surface \(X\) is smooth. Moreover, we have \(q(X) = 1\) based on the given building data of \(2L_{001} = 4f_2\), and \(p_g(X) = 3\) since \(L_{100} \equiv L_{010} \equiv L_{110} \equiv -K_Y\). Infact, according to Proposition 5, the surface \(X\) possesses the following invariants:

\[
K_X^2 = 14, \quad p_g(X) = 3, \quad \chi(\mathcal{O}_X) = 3, \quad q(X) = 1.
\]

Moreover, since

\[
2K_X \equiv g^*(f_1 + f_2 + l),
\]

the canonical divisor \(K_X\) is nef and big. Thus, the surface \(X\) is of general type and minimal.

We prove that the canonical map \(\varphi|_{K_X}\) is of degree 14. By Proposition 6, the linear system \(|K_X|\) is generated by the three following divisors:

\[
\tilde{f}_{11} + \tilde{f}_{12} + \tilde{f}_{21} + \tilde{e}_1, \tilde{l}_1 + \tilde{e}_2 + \tilde{f}_{22} + \tilde{f}_{12}, \tilde{C}_2 + \tilde{f}_{23},
\]

where \(l_1 := g^*(l_1)_{\text{red}}, \tilde{C}_2 := g^*(C_2)_{\text{red}}, \tilde{e}_1 := g^*(e_1)_{\text{red}}, \tilde{f}_{12} := g^*(f_{12})_{\text{red}}\) and \(\tilde{f}_{ij} := g^*(f_{ij})_{\text{red}}\). Because the three divisors \(\tilde{f}_{11} + \tilde{f}_{12} + \tilde{f}_{21} + \tilde{e}_1, \tilde{l}_1 + \tilde{e}_2 + \tilde{f}_{22} + \tilde{f}_{12}, \tilde{C}_2 + \tilde{f}_{23}\) have no common intersection, the linear system \(|K_X|\) is base point free. This together with \(K_X^2 = 14 > 0\) implies that the linear system \(|K_X|\) is not composed with a pencil. Thus, the canonical image is \(\mathbb{P}^2\) and the canonical map is of degree 14. Therefore we obtain the surface in the last row of Theorem 1.

Remark 4. The surface \(X\) has one pencil of genus 5 corresponding to the fibers \(f_1\). The Albanese pencil of \(X\), which comes from the fibration \(|f_2|\), has genus 3.

In fact, we consider the following commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\mathbb{Z}_2^3} & Y_2 \\
\downarrow g_1 & & \downarrow g_2 \\
\mathbb{Z}_2^2 & \xrightarrow{g_2} & \mathbb{Z}_2
\end{array}
\]

where \(g_2 : Z \rightarrow Y_2\) is the double cover ramified on \(2L_{001} \equiv 4f_2\). The intermediate surface \(Z\) has \(p_g(Z) = 0\) and \(q(Z) = 1\), indicating that the Albanese morphism of \(Z\) maps its image to a curve. Consequently, applying the universal property, the image of the Albanese morphism of \(X\) is also a curve. By the Hurwitz formula, we find that the genus of \(g^*(f_1)\) is 5, while the genus of \(g^*(f_2)\) is 3.
4.2. Variations

We now consider variations of the construction presented in Section 4.1 in order to obtain the remaining irregular surfaces listed in Theorem 1. Initially, we utilize triple points that arise from the intersection of three divisors $D_{\sigma_1}, D_{\sigma_2},$ and $D_{\sigma_1+\sigma_2}$. By introducing these triple points, we can get surfaces with the same Euler characteristic. According to Proposition 7, a such single triple point results in a surface with $K^2 = 12$ and $\chi = 3$. Conversely, the imposition of two such triple points simultaneously yields a surface with $K^2 = 10$ and $\chi = 3$. To decrease the Euler characteristic by one, we impose one quadruple point as described in Proposition 8. By doing so, we can obtain a surface with $K^2 = 12$ and $\chi = 2$. Finally, we consider the application of triple points arising from the intersection of three divisors $D_{\sigma_1}, D_{\sigma_2},$ and $D_{\sigma_3}$ with $(\sigma_1, \sigma_2, \sigma_3) \cong \mathbb{Z}_2^3$. By Proposition 7, the imposition of a such single triple point results in a surface with the same Euler characteristic and self-intersection of canonical class. Moreover, by simultaneously imposing one such triple point and one quadruple point, we obtain a surface with $K^2 = 12$ and $\chi = 2$.

The specific types of singular points we impose and the corresponding surfaces obtained are detailed in the following table:

| $P = (\mu_{001}, \mu_{010}, \mu_{011}, \mu_{100}, \mu_{101}, \mu_{110}, \mu_{111})$ | $K^2_X$ | $\chi (\mathcal{O}_X)$ | $p_g (X)$ | $q (X)$ | $d$ |
|---|---|---|---|---|---|
| $P_3 = (0, 1, 0, 1, 0, 1, 0)$ | 12 | 3 | 3 | 1 | 12 |
| $P_3 = (0, 1, 0, 1, 0, 1, 0)$ and $P_4 = (0, 1, 0, 1, 0, 1, 0)$ | 10 | 3 | 3 | 1 | 10 |
| $P_3 = (0, 0, 1, 1, 0, 1, 0)$ | 12 | 2 | 3 | 2 | 12 |
| $P_3 = (0, 0, 1, 1, 0, 1, 0)$ and $P_4 = (0, 1, 0, 1, 0, 1, 0)$ | 12 | 2 | 3 | 2 | 10 |

In our construction, we consistently maintain $L_{100} \equiv L_{010} \equiv L_{110} \equiv -K_Y$, where $Y_i$ is the del Pezzo we take the cover. This condition guarantees that $p_g (X) = 3$. With the exception of the surface with $d = 10, q = 2$, the degree of the canonical map can be proven in a similar manner as described in Section 4.1. The exceptional case will prove in the last section. For further details regarding these constructions, we provide the building data of the $\mathbb{Z}_2^3$-covers.

4.2.1. The building data of the surface with $d = 12$ and $q = 1$

Let us denote by $Y_3$ the blow-up of $Y_2$ at a point $P_3$. The surface $Y_3$ is a del Pezzo surface of degree 6 (see Notation 2). We consider the following smooth divisors:

$$D_{010} := f_{11} + h_{13}$$
$$D_{100} := f_{31} + e_2$$
$$D_{110} := C_2 \in [f_1 + f_3]$$

and $D_{001} = 0$, where $f_{11} \in [f_1], f_{21}, f_{22}, f_{23} \in [f_2], f_{31} \in [f_3]$ and $C_2$ are distinct divisors in $Y_3$ such that no more than two of these divisors $D_\sigma$ go through the same point. In addition, we consider the following non-trivial divisors in $Y_3$:

$$_{L_{100}} := f_1 + f_2 + f_3$$
$$_{L_{010}} := f_1 + f_2 + f_3$$
$$_{L_{001}} := 2f_2$$
$$_{L_{110}} := f_1 + f_2 + f_3$$
$$_{L_{101}} := l + f_3$$
$$_{L_{011}} := 2f_1 + f_2 - e_3$$
$$_{L_{111}} := f_1 + f_3.$$

These divisors $D_\sigma, L_\chi$ define a $\mathbb{Z}_2^3$-cover $g : X \longrightarrow Y_3$. Moreover, the surface $X$ possesses the following invariants:

$$K_X^2 = 12, p_g (X) = 3, \chi (\mathcal{O}_X) = 3, q (X) = 1, d = 12$$

Remark 5. The surface $X$ has two pencils of genus 5 corresponding to the fibers $f_1, f_3$. The Albanese pencil of $X$, which comes from the fibration $[f_2]$, has genus 3.
Remark 6. Taking the $\mathbb{Z}_2^3$ cover of $Y_2$ ramified on the above branch locus with a triple point $P_3$, we would obtain a surface with two singular points of type $\frac{1}{3} (1,1)$ coming from the point $P_3$ (cf. [1, Table 1, Section 3.3]). The surface $X$ is the minimal resolution of this singular surface.

4.2.2. The building data of the surface with $d = 10$ and $q = 1$

Let us denote by $Y_4$ the blow-up of $Y_2$ at two distinct points $P_3$ and $P_4$. The surface $Y_4$ is a del Pezzo surface of degree 5 (see Notation 3). We consider the following smooth divisors of $Y_4$:

$$D_{010} := h_{13} + h_{14}$$
$$D_{100} := h_{34} + e_2$$
$$D_{110} := C_2 \in \{f_1 + h_{34}\}$$

and $D_{001} = 0$, where $f_{21}, f_{22}, f_{23} \in \{f_2\}$ and $C_2$ are distinct divisors such that no more than two of these divisors $D_\sigma$ go through the same point. We consider the following non-trivial divisors of $Y_4$:

$$L_{100} = f_1 + f_2 + f_3 - e_4$$
$$L_{010} = f_1 + f_2 + f_3 - e_4$$
$$L_{001} = 2f_2$$
$$L_{110} = f_1 + f_2 + f_3 - e_4$$
$$L_{101} = f_3 + f_4$$
$$L_{011} = 2f_1 + f_2 - e_3 - e_4$$
$$L_{111} = f_1 + f_3 - e_4.$$

These divisors $D_\sigma, L_\chi$ define a $\mathbb{Z}_2^3$-cover $g : X \rightarrow Y_4$. Moreover, the surface $X$ possesses the following invariants:

$$K_X^2 = 10, p_g (X) = 3, \chi (O_X) = 3, q (X) = 1, d = 10.$$

Remark 7. The surface $X$ has three pencils of genus 5 corresponding to the fiber $f_1, f_3, f_4$. The Albanese pencil of $X$, which comes from the fibration $|f_2|$, has genus 3.

Remark 8. Taking the $\mathbb{Z}_2^3$-cover of $Y_2$ ramified on the above branch locus with two ordinary triple points $P_3, P_4$, we would obtain a surface with four points of type $\frac{1}{3} (1,1)$ coming from $P_3$ and $P_4$ (cf. [1, Table 1, Section 3.3]). The surface $X$ is the minimal resolution of this singular surface.

4.2.3. The building data of the surface with $d = 12$ and $q = 2$

We denote by $Y_3$ the blow-up of $Y_2$ at a point $P_3$. The surface $Y_3$ is a del Pezzo surface of degree 6 (see Notation 2). We consider the following divisors of $Y_3$:

$$D_{010} := f_{11} + f_{12}$$
$$D_{100} := f_{31} + e_2$$
$$D_{110} := f_{32} + h_{13}$$

and $D_{001} = 0$, where $f_{11}, f_{12} \in \{f_1\}, f_{21}, f_{22} \in \{f_2\}$ and $f_{31}, f_{32} \in \{f_3\}$ are distinct divisors such that no more than two of these divisors $D_\sigma$ go through the same point. We consider the following non-trivial divisors of $Y_3$:

$$L_{100} := f_1 + f_2 + f_3$$
$$L_{010} := f_1 + f_2 + f_3$$
$$L_{001} := 2f_2$$
$$L_{110} := f_1 + f_2 + f_3$$
$$L_{101} := 2f_3$$
$$L_{011} := 2f_1 + f_2 - e_3$$
$$L_{111} := f_1 + l.$$
These divisors \( D_\sigma, L_\chi \) define a \( \mathbb{Z}_2^3 \)-cover \( g : X \longrightarrow Y_3 \). Moreover, the surface \( X \) possesses the following invariants:

\[
K_X^2 = 12, p_g (X) = 3, \chi (\mathcal{O}_X) = 2, q (X) = 2, d = 12.
\]

**Remark 9.** Taking the \( \mathbb{Z}_2^3 \)-cover of \( Y_2 \) ramified on the above branch locus with a quadruple point \( P_3 \), we would obtain a surface with a singular point coming from the point \( P_3 \). This point is Gorenstein elliptic singularity whose minimal resolution are elliptic curves with self-intersection \(-2\) (cf. [1, Table 1, Section 3.3]). The surface \( X \) is the minimal resolution of this singular surface.

**Remark 10.** The Albanese morphism of this surface is of degree 2.

In fact, we consider the following commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\mathbb{Z}_2^3} & Y_3 \\
\downarrow g_1 & & \downarrow g_2 \\
\mathbb{Z}_2 & \xrightarrow{g_2} & \mathbb{Z}_2 \\
\end{array}
\]

where \( g_2 : Z \longrightarrow Y_3 \) is the bidouble cover with the following building data:

\[
\begin{aligned}
D_1 &:= D_{011} = h_{23} + e_1 \\
D_2 &:= D_{101} + D_{111} = f_{21} + h_{12} + f_{22} + e_3 \\
D_3 &:= D_{100} + D_{110} = f_{31} + e_2 + f_{32} + h_{13} \\
L_1 &:= L_{100} \\
L_2 &:= L_{101} \\
L_3 &:= L_{001}
\end{aligned}
\]

\[
\begin{aligned}
D_001 &:= e_4 \\
D_{010} &:= h_{14} + f_{11} \\
D_{100} &:= h_{34} + e_2 \\
D_{110} &:= h_{13} + f_{31} \\
D_{011} &:= h_{23} + e_1 \\
D_{101} &:= h_{12} + f_{21} \\
D_{111} &:= h_{24} + e_3.
\end{aligned}
\]

We have that \( p_g (Z) = 1, q (Z) = 2 \) and

\[
2K_Z \equiv g_2^* (h_{23} + e_1).
\]

We notice that the surface \( Z \) contains \((-1)\)-curves. Let \( c : Z \longrightarrow \overline{Z} \) be the contraction map. The minimal surface \( \overline{Z} \) satisfies \( p_g (\overline{Z}) = 1, q (\overline{Z}) = 2 \) and \( 2K_{\overline{Z}} \equiv 0 \). So the surface \( \overline{Z} \) is an abelian surface.

By the universal property of the Albanese morphism of the surface \( X \), the following diagram commutes:

\[
\begin{array}{ccc}
X & \xrightarrow{\mathbb{Z}_2} & Z \\
\downarrow \alpha & & \downarrow c \\
& \xrightarrow{\text{Alb}(X)} & \\
& & \overline{Z}
\end{array}
\]

Because the surface \( X \) is of general type, the Albanese morphism \( \alpha \) is of degree 2 and \( \text{Alb}(X) \) is isomorphic to \( \overline{Z} \).

**4.2.4. The building data of the surface with \( d = 10 \) and \( q = 2 \).**

Let us denote by \( Y_4 \) the blow-up of \( Y_2 \) at two distinct points \( P_3 \) and \( P_4 \). \( Y_4 \) is a del Pezzo surface of degree 5 (see Notation 3). We consider the following smooth divisors of \( Y_4 \):

\[
\begin{aligned}
D_{001} &:= e_4 \\
D_{010} &:= h_{14} + f_{11} \\
D_{100} &:= h_{34} + e_2 \\
D_{110} &:= h_{13} + f_{31} \\
D_{011} &:= h_{23} + e_1 \\
D_{101} &:= h_{12} + f_{21} \\
D_{111} &:= h_{24} + e_3.
\end{aligned}
\]
where \( f_{11} \in |f_1|, f_{21} \in |f_2|, f_{31} \in |f_3| \) are divisors such that no more than two of these divisors \( D_\sigma \) go through the same point. We consider the following non-trivial divisors of \( Y_4 \):

\[
\begin{align*}
L_{100} & := f_1 + f_2 + f_3 - e_4 \\
L_{010} & := f_1 + f_2 + f_3 - e_4 \\
L_{001} & := 2f_2 \\
L_{110} & := f_1 + f_2 + f_3 - e_4 \\
L_{101} & := 2f_3 \\
L_{011} & := 2f_1 + f_2 - e_3 \\
L_{111} & := f_1 + f_4.
\end{align*}
\]

These divisors \( D_\sigma, L_X \) define a \( \mathbb{Z}_2^3 \)-cover \( g : X \longrightarrow Y_4 \). Moreover, the surface \( X \) possesses the following invariants:

\[
K_X^2 = 12, p_g(X) = 3, \chi(O_X) = 2, q(X) = 2.
\]

We show that the canonical map \( \varphi_{|K_X|} \) has degree 10 and the linear system \( |K_X| \) has a non-trivial fixed component. By Proposition 6, the linear system \( |K_X| \) is generated by the three following divisors:

\[
\bar{e}_4 + \bar{h}_{14} + \bar{f}_{11} + \bar{h}_{23} + \bar{e}_1, \bar{e}_4 + \bar{h}_{34} + \bar{e}_2 + \bar{h}_{12}, \bar{e}_4 + \bar{h}_{13} + \bar{f}_{31} + \bar{h}_{24} + \bar{e}_3,
\]

where \( \bar{C}_i := g^\ast(C_i)_{\text{red}}, \bar{e}_i := g^\ast(e_i)_{\text{red}}, \bar{h}_{ij} := g^\ast(h_{ij})_{\text{red}} \) and \( \bar{f}_{ij} := g^\ast(f_{ij})_{\text{red}} \). Because the \((-2)\)-curve \( \bar{e}_4 \) is the common part of these three above divisors, the \((-2)\)-curve \( \bar{e}_4 \) is a fixed component of \( |K_X| \).

On the other hand, since the three divisors \( \bar{h}_{14} + \bar{f}_{11} + \bar{h}_{23} + \bar{e}_1, \bar{h}_{34} + \bar{e}_2 + \bar{h}_{12} + \bar{f}_{21}, \bar{h}_{13} + \bar{f}_{31} + \bar{h}_{24} + \bar{e}_3 \) have no common intersection, the linear system \( |M| \) is base point free, where \( |M| := |\bar{h}_{14} + \bar{f}_{11} + \bar{h}_{23} + \bar{e}_1| \). This together with \( M^2 = 10 > 0 \) implies that the linear system \( |K_X| \) is not composed with a pencil. Thus, the canonical image is \( \mathbb{P}^2 \) and the canonical map is of degree 10. Therefore we obtain the surface described in the third row of Theorem 1.

Remark 11. Taking the \( \mathbb{Z}_2^3 \)-cover of \( Y_1 \) ramified on the above branch locus with singular points \( P_3, P_4 \), we would obtain a surface with one singular point of type \( A_1 \) and one Gorenstein elliptic singular point whose minimal resolution is an elliptic curve with self-intersection \(-2\). The \( A_1 \) point comes from \( P_4 \) and the Gorenstein elliptic singular point comes from \( P_3 \). The surface \( X \) is the minimal resolution of this singular surface.

Remark 12. Similarly to the construction of the surface with \( d = 2, p_g = 3, q = 2 \), the Albanese morphism of this surface is of degree 2.

Remark 13. Let \( H \) be the subgroup generated by \((0, 0, 1)\) of the group \( \mathbb{Z}_2^3 \). In the all above constructions, the canonical map \( \varphi_{|K_X|} \) is the composition of the quotient map \( X \longrightarrow X/H \) with the canonical map of the quotient surface \( X/H \) because \( h^0(L_X + K_Y) = 0 \) for all \( \chi \in \{ \chi_{001}, \chi_{101}, \chi_{011}, \chi_{111} \} \).

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