A FUNCTIONAL EXPRESSION FOR THE CURVATURE OF
HYPER-DIMENSIONAL RIEMANNIAN SPACES

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Abstract. Analogously to the concept of a curvature of curve and surface, in the differential geometry, in the main part of this paper the concept of the curvature of the hyper-dimensional vector spaces of Riemannian metric is generally defined. The defined concept of the curvature of Riemannian spaces of higher dimensions \( M : M \geq 2 \), in the further text of the paper, is functional related to the fundamental parameters of an internal geometry of space, more exactly, to components of Riemann-Christoffel’s tensor of curvature. At the end, analogously to the concept of lines of curvature in the differential geometry, the concept of sub-spaces of curvature of Riemannian hyper-dimensional vector spaces is also generally defined.

1. Introduction

The well-known Riemannian mathematical model of defining the curvature of hyper-dimensional curvilinear metric spaces, via Gaussian concept of the two-dimensional surface curvature, [1]; [2]; and [4], is an imperfect in spite of that it functional related to internal geometry of a space:

1. Firstly, for a reason that the concept of the curvature of hyper-dimensional metric spaces reduced to the concept of Gaussian curvature of two-dimensional geodesic surface, it cannot be said that Riemannian concept of the curvature of hyper-dimensional spaces is general one because it is an inapplicable to one-dimensional space.

2. As secondly, since there are more than one geodesic surface in vector spaces of higher dimensions, it is obvious that Riemannian concept of the curvature of hyper-dimensional spaces, in the general case, is not uniquely defined one.

3. Finally, at any an individual, concrete case, from the practical point of view, it is not simple to come to the quantitatively usable functional expression for Riemannian curvature of an analyzed hyper-dimensional metric space.

Hence, the one other mathematical model of defining the curvature of the hyper-dimensional metric spaces, which essentially differs from Riemannian model, is presented in this paper. Concretely, the mathematical model is being discussed, which can be said to be generalization of well-known model of defining the curvature of curve and surface in the differential geometry.

Received by the editors April 26, 2000.

1991 Mathematics Subject Classification. Primary 53A35; Secondary 53A45.

Key words and phrases. space, curvature of space, sub-space of curvature.
1.1. Basic characteristics of space continuum. The concept of geometrical point is one of fundamental concepts. Closely related to the concept of geometrical points is the system of values \((a_1, a_2, ..., a_N)\) of some an arbitrary \(N\) variables \((x_1, x_2, ..., x_N)\) such that a set of all geometrical points, and for all real values of the variables, is a real \(N\)-dimensional space of a space continuum, \([1]\). The geometrical point \(O\), defined by system of zero values \((0, 0, ..., 0)\), is an origin of system of reference (of co-ordinate system) of the space. The vector \(\vec{r}(x^i)\), defined with respect to the origin \(O\), is a position vector. Note that the concept of a vector, in the vector hyper-dimensional spaces \((N > 3)\), should be conditionally comprehend in the sense of its geometrical presentation in a form of segments, hence it bears a name linear tensor, \([4]\).

Covariant vectors \(\vec{e}_i: \vec{e}_i = \partial_{x^i} \vec{r}(x^j)\), where \(\partial_{x^i} = \frac{\partial}{\partial x^i}\), form the covariant vector basis \(\{\vec{e}_i\}_{i=1}^N\) of the tangent space of a space continuum. The vectors \(\vec{e}_i\), such that at each point of a space: \(\vec{e}_i \cdot \vec{e}_k = \delta^k_i\), where the second order system of the unit values \(\delta^k_i\) - is an unit \(N \times N\) matrix (Kronecker’s delta-symbol), \([1]\); \([2]\) and \([3]\), form a basis \(\{e^a\}_{a=1}^N\), which is called the dual basis of the covariant vector basis \(\{\vec{e}_i\}_{i=1}^N\). This is so-called natural isomorphism from \(\{\vec{e}_i\}_{i=1}^N\) onto \(\{e^a\}_{a=1}^N\). The infinitesimal value \(d\vec{r} = dx^i\vec{e}_i = dx_i\vec{e}_i\), where the well-known Einstein’s convention is applied to summation with respect to the repetitive indexes (uppers and lowers), \([4]\) and \([5]\), herein as well as in the further text of the paper.

2. The main results

2.1. A curvature of hyper-dimensional Riemannian spaces. By the following transformation low: \(x^i = x^i(q^\alpha)\); \(i = 1, 2, ..., N, \alpha = 1, 2, ..., M \leq N\), in the general case, an arbitrary \(M\)-dimensional metric space is defined, \([6]\)

\[ds^2 = \partial_{x^i} \vec{r} \partial_{q^\alpha} x^i \cdot \partial_{x^j} \vec{r} \partial_{q^\beta} x^j dq^\alpha dq^\beta = e_{ij} \partial_{q^\alpha} x^i \partial_{q^\beta} x^j dq^\alpha dq^\beta = g_{\alpha \beta} dq^\alpha dq^\beta,\]

where the positive definite symmetric square matrices \(e_{ij}; e_{ij} = \vec{e}_i \cdot \vec{e}_j\) and \(g_{\alpha \beta}:

\[g_{\alpha \beta} = \partial_{q^\alpha} \vec{r} \cdot \partial_{q^\beta} \vec{r} = \vec{g}_\alpha \cdot \vec{g}_\beta = \partial_{x^i} \vec{r} \partial_{q^\alpha} x^i \cdot \partial_{x^j} \vec{r} \partial_{q^\beta} x^j = \vec{e}_i \cdot \vec{e}_j \partial_{q^\alpha} x^i \partial_{q^\beta} x^j,\]

of degree: \(N\) and \(M\), respectively, are fundamental (metric) tensors of an ambient \(N\)-dimensional Euclidean (more exactly Cartesian) space \(x^i\), as well as, in the general case of Riemannian covering map \((M < N)\), of an internal \(M\)-dimensional Riemannian curvilinear space \(q^\alpha\) immersed in it. As it is well-known if \(M = N\) a smooth map \(x^i \rightarrow q^\alpha\) is called an isometric immersion.

The smallest possible dimensional difference \(C\): \(C = N - M\), is said to define a class of the Riemannian vector space \(q^\alpha\), \([6]\). Hence, Riemannian space \(q^\alpha; q^\alpha = q\), of class \(C\): \(C = N - 1\), is an arbitrary curve of the ambient \(N\)-dimensional Euclidean (Cartesian) space \(x^i\).

Since the vector \(dq \vec{g}\), where \(dq = \frac{d}{dq}\), as derivative of the fundamental vector \(\vec{g}\):

\[\vec{g} = dq \vec{r},\]

lying in the tangent space of Riemannian space \(q\) of class \(C\): \(C = N - 1\) (on the tangent of curve), along the curve, from the point of view of the interior of an ambient space \(x^i\), is a vector of the ambient space \(x^i\), then at each points of the curve there exist \(C\): \(C = N - 1\), the linearly independent and mutually orthogonal unit vectors: \(i^\lambda; \Lambda = 1, 2, ..., N - 1\), being orthogonal on the unit vector of tangent.
\[ \vec{t} \cdot \vec{e} = d_q \vec{g} \cdot \vec{n} \] is a vector of curvature of curve, more exactly, from the point of view of the internal geometry of an ambient Euclidean space \( x^i \). it is a vector of geodesic curvature of curve, more exactly, from the point of view of the internal geometry of a Riemannian space \( q \), the vector \( \vec{\kappa} \):

\[ \vec{\kappa} = (d_q \vec{g} \cdot \vec{n}^\Lambda) \vec{n}_\Lambda, \]

is a vector of curvature of curve, more exactly, from the point of view of the internal geometry of an ambient Euclidean space \( x^i \). it is a vector of geodesic curvature of curve, more exactly, from the point of view of the internal geometry of a Riemannian space \( q \), the vector \( \vec{\kappa} \):

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is a vector of curvature of curve, more exactly, from the point of view of the internal geometry of an ambient Euclidean space \( x^i \). it is a vector of geodesic curvature of curve, more exactly, from the point of view of the internal geometry of a Riemannian space \( q \), the vector \( \vec{\kappa} \):

\[ \vec{\kappa} = (d_q \vec{g} \cdot \vec{n}^\Lambda) \vec{n}_\Lambda, \]
proportionality which is equal to the determinant of matrix of fundamental tensor \(g_{\alpha\beta}\)

\[
\begin{vmatrix}
\hat{K}_{\alpha\beta} \cdot \hat{K}_\gamma^\beta
\end{vmatrix} = \left| \tau^\Lambda_{\alpha\beta} \tau^\Sigma_{\gamma\delta} g^{\delta\beta} n_{\Lambda\Sigma} \right| = |g_{\alpha\beta}| \kappa^2,
\]

more exactly,

\[
\kappa^2 = \left| \frac{\tau^\Lambda_{\alpha\beta} \tau^\Sigma_{\gamma\delta} g^{\delta\beta} n_{\Lambda\Sigma}}{|g_{\alpha\beta}|} \right|.
\]

Comment: If \(N\) - dimensional ambient space \(x^i\) is either Riemannian curvilinear space or Euclidean curvilinear space of class \(C\): \(C \geq 1\), both with normal vector space \(\vec{w}_P\), \(P = 1, 2, ..., C\), then the matrix scheme of vectors

\[
\hat{K}_{\alpha\beta} = \left( \partial_{q^\phi} \tilde{g}_\alpha \cdot \tilde{n}^\Lambda \right) \tilde{n}_\Lambda + \left( \partial_{q^\phi} \tilde{g}_\alpha \cdot \tilde{w}^P \right) \tilde{w}_P,
\]

is a matrix scheme of vectors of curvature of Riemannian space \(q^\alpha\) of class \(\hat{C}\): \(C = C + N - M\).

From the point of view of the internal geometry of the ambient space \(x^i\), the partial matrix schemes of vectors of curvature: \(\hat{N}_{\alpha\beta} = \left( \partial_{q^\phi} \tilde{g}_\alpha \cdot \tilde{w}^P \right) \tilde{w}_P = x^P_{\alpha\beta} \tilde{w}_P\) and \(\hat{G}_{\alpha\beta} = \left( \partial_{q^\phi} \tilde{g}_\alpha \cdot \tilde{n}^\Lambda \right) \tilde{n}_\Lambda = \tau_{\alpha\beta} \tilde{n}_\Lambda\), are matrix schemes of vectors of normal curvature, as well as of geodesic curvature, of Riemannian space \(q^\alpha\) of class \(\hat{C}\), respectively.

The square of intensity of curvature \(\kappa\) of Riemannian space of class \(\hat{C}\), in this case is equal to the sum of square of the normal curvature \(\kappa_n\):

\[
\kappa_n = \left| \frac{x^P_{\alpha\beta} x^L_{\gamma\delta} g^{\beta\delta} w_{PL}}{|g_{\alpha\beta}|} \right|
\]

and of the geodesic curvature \(\kappa_g\):

\[
\kappa_g = \left| \frac{\tau^\Lambda_{\alpha\beta} \tau^\Sigma_{\gamma\delta} g^{\delta\beta} n_{\Lambda\Sigma}}{|g_{\alpha\beta}|} \right|,
\]

of Riemannian space \(q^\alpha\) of class \(\hat{C}\):

\[
\kappa^2 = \kappa_n^2 + \kappa_g^2.
\]

The space \(q^\alpha\) of class \(\hat{C}\), as well as of the geodesic curvature zero \(\kappa_g\): \(\kappa_g = 0\), with respect to the ambient space \(x^i\) of class \(C\), is a geodesic sub-space of that ambient space.▼

2.1.1. Functional expression for the curvature of hyper-dimensional Riemannian spaces, from the point of view of the internal geometry of space. If the vector functional equality (2.1)

\[
\partial_{q^\phi} \tilde{g}_\alpha = \left( \partial_{q^\phi} \tilde{g}_\alpha \cdot \tilde{g}_\gamma \right) \tilde{g}_\gamma + \left( \partial_{q^\phi} \tilde{g}_\alpha \cdot \tilde{n}^\Lambda \right) \tilde{n}_\Lambda,
\]

in the general case of the ambient Riemannian space \(q^\alpha\) of class \(C\): \(C = N - M\), is partially differentiated with respect to dual co-ordinates \(q^\delta\), it is obtained that

\[
(2.4) \quad \partial^2_{q^\phi q^\delta} \tilde{g}_\alpha = \partial_{q^\phi} \Gamma^\gamma_{\alpha\beta} \tilde{g}_\gamma + \Gamma^\gamma_{\alpha\beta} \partial_{q^\delta} \tilde{g}_\gamma + \partial_{q^\phi} \tau^\Lambda_{\alpha\beta} \tilde{n}_\Lambda + \tau^\Lambda_{\alpha\beta} \partial_{q^\phi} \tilde{n}_\Lambda,
\]

that is,

\[
(2.5) \quad \partial^2_{q^\phi q^\delta} \tilde{g}_\alpha = \partial_{q^\phi} \Gamma^\gamma_{\alpha\beta} \tilde{g}_\gamma + \Gamma^\gamma_{\alpha\beta} \partial_{q^\delta} \tilde{g}_\gamma + \partial_{q^\phi} \tau^\Lambda_{\alpha\beta} \tilde{n}_\Lambda + \tau^\Lambda_{\alpha\beta} \partial_{q^\phi} \tilde{n}_\Lambda.
\]
Since differentials \( d\vec{g}_\alpha = \partial_{q^\beta} \vec{g}_\alpha dq^\beta \) are an absolute ones, more exactly, the covariant vector basis \( \{ \vec{g}_\alpha \} \) of a Riemannian space \( q^\alpha \) of class \( C: C = N - M \), is uniquely defined at each point of a space, and in accordance with that the condition of integrability: 
\[
\partial_{q^\beta} \vec{g}_\alpha - \partial_{q^\gamma} \vec{g}_\alpha = 0,
\]
is satisfied, then by projection of difference of the respectable vector functional expressions on both sides of the previous equations: \( (2.4) \) and \( (2.5) \), onto covariant vector basis \( \{ \vec{n}_\alpha \} \), the functional relation is obtained
\[
\partial_{q^\beta} \Gamma^\gamma_{\alpha\beta} g_{\gamma\rho} - \partial_{q^\gamma} \Gamma^\gamma_{\alpha\beta} g_{\gamma\rho} + \Gamma^\lambda_{\alpha\beta} \Gamma^\gamma_{\lambda\rho} g_{\gamma\rho} - \Gamma^\lambda_{\alpha\rho} \Gamma^\gamma_{\lambda\beta} g_{\gamma\rho} = \tau^\Lambda_{\alpha\beta} \tau^\delta_{\beta\rho,\Lambda} - \tau^\Lambda_{\alpha\delta} \tau^\beta_{\beta\rho,\Lambda},
\]
having in mind both the equation \( (2.2) \) and orthogonality of vectors: \( \vec{g}_\alpha \) and \( \vec{n}_\Lambda \). On account of the fact that the functional expression on the left hand side of preceding relation represents \textit{Riemann-Christoffel’s} tensor of \textit{Riemannian} space \( q^\alpha \) of class \( C: C = N - M \). \( \[1\] \); \( \[2\] \) and \( \[3\] \), it finally follows that
\[
(2.6)
\]
By application of \textit{Gauss-Chiò’s} procedure for condensation of determinants, \( \[1\] \), and on the basis of tensorial relation \( (2.4) \), as well as of functional relation \( (2.3) \), it is possible the curvature of \textit{Riemannian} space \( q^\alpha \) of class \( C: C = N - M \), to come into functional relation to internal geometry of a space, more exactly to the components of \textit{Riemann-Christoffel’s} tensor of the curvature \( R_{\alpha\delta\beta\gamma} \).

Namely, if in addition to the matrix scheme of vectors: \( K_{\alpha\beta} = \tau^\Lambda_{\alpha\beta} \vec{n}_\Lambda \), the matrix schemes of vectors:
\[
\vec{\Phi}_{\alpha\beta} = \begin{bmatrix}
\phi^\Lambda \vec{n}_\Lambda & 0 & \cdots & 0 & 0 \\
\tau^\Lambda_{12} \vec{n}_\Lambda & \tau^\Lambda_{11} \vec{n}_\Lambda & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\tau^\Lambda_{1(M-1)} \vec{n}_\Lambda & 0 & \cdots & \tau^\Lambda_{11} \vec{n}_\Lambda & 0 \\
\tau^\Lambda_{1M} \vec{n}_\Lambda & 0 & \cdots & 0 & \tau^\Lambda_{11} \vec{n}_\Lambda \\
\end{bmatrix};
\]
\[
\vec{Z}_{\alpha\beta} = \begin{bmatrix}
\tau^\Lambda_{12} \vec{n}_\Lambda & 0 & \cdots & 0 & 0 \\
\tau^\Lambda_{11} \vec{n}_\Lambda & \zeta^\Lambda \vec{n}_\Lambda & \cdots & \tau^\Lambda_{1(M-1)} \vec{n}_\Lambda & \tau^\Lambda_{1M} \vec{n}_\Lambda \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \tau^\Lambda_{12} \vec{n}_\Lambda & 0 \\
0 & 0 & \cdots & 0 & \tau^\Lambda_{12} \vec{n}_\Lambda \\
\end{bmatrix};
\]
\[
\vec{\Xi}_{\alpha\beta} = \begin{bmatrix}
\xi^\Lambda \vec{n}_\Lambda & 0 & \cdots & 0 & 0 \\
\tau^\Lambda_{22} \vec{n}_\Lambda & \tau^\Lambda_{21} \vec{n}_\Lambda & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\tau^\Lambda_{2(M-1)} \vec{n}_\Lambda & 0 & \cdots & \tau^\Lambda_{21} \vec{n}_\Lambda & 0 \\
\tau^\Lambda_{2M} \vec{n}_\Lambda & 0 & \cdots & 0 & \tau^\Lambda_{21} \vec{n}_\Lambda \\
\end{bmatrix};
\]
\[
\vec{\Psi}_{\alpha\beta} = \begin{bmatrix}
\tau^\Lambda_{22} \vec{n}_\Lambda & 0 & \cdots & 0 & 0 \\
\tau^\Lambda_{21} \vec{n}_\Lambda & \psi^\Lambda \vec{n}_\Lambda & \cdots & \tau^\Lambda_{2(M-1)} \vec{n}_\Lambda & \tau^\Lambda_{2M} \vec{n}_\Lambda \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \tau^\Lambda_{22} \vec{n}_\Lambda & 0 \\
0 & 0 & \cdots & 0 & \tau^\Lambda_{22} \vec{n}_\Lambda \\
\end{bmatrix};
\]
are also introduced into analysis, where: \( \phi^\Lambda \vec{n}_\Lambda, \zeta^\Lambda \vec{n}_\Lambda, \xi^\Lambda \vec{n}_\Lambda \) and \( \psi^\Lambda \vec{n}_\Lambda \) are arbitrary vector functions of the normal vector space \( \vec{n}_\Lambda \) of \textit{Riemannian} space \( q^\alpha \) of class \( C: C = N - M \), then having in view the fact that determinant of the product of
matrices, as it is well-known, is equal to the product of determinants of any matrix separately, \( R_{1212} \) and \( R_{3212} \), it follows that

\[
(2.7) \quad \left( \Phi_{1\gamma} \cdot \tilde{K}_{\delta}^\gamma \right) \begin{vmatrix} R_{1212} & R_{1213} & \cdots & R_{121M} \\ R_{1312} & R_{1313} & \cdots & R_{131M} \\ \cdots & \cdots & \cdots & \cdots \\ R_{1M12} & R_{1M13} & \cdots & R_{1M1M} \end{vmatrix} = \begin{vmatrix} R_{1212} & R_{1213} & \cdots & R_{121M} \\ R_{1312} & R_{1313} & \cdots & R_{131M} \\ \cdots & \cdots & \cdots & \cdots \\ R_{1M12} & R_{1M13} & \cdots & R_{1M1M} \end{vmatrix}
\]

more exactly,

\[
\begin{aligned}
\left| K_{\alpha\delta} \cdot R_{\beta}\right| & \begin{vmatrix} R_{1212} & R_{1213} & \cdots & R_{121M} \\ R_{1312} & R_{1313} & \cdots & R_{131M} \\ \cdots & \cdots & \cdots & \cdots \\ R_{1M12} & R_{1M13} & \cdots & R_{1M1M} \end{vmatrix} = \frac{1}{|g_{\alpha\beta}|} \begin{vmatrix} R_{1212} & R_{1213} & \cdots & R_{121M} \\ R_{1312} & R_{1313} & \cdots & R_{131M} \\ \cdots & \cdots & \cdots & \cdots \\ R_{1M12} & R_{1M13} & \cdots & R_{1M1M} \end{vmatrix}
\end{aligned}
\]

considering the fact that the functional expression in the relation (2.7), which is an independent of the choice of arbitrary vector functions: \( \phi^\Lambda R_{\Lambda}^\lambda \psi^\Lambda R_{\Lambda}^\lambda \) (the specially interesting case is one in which \( \phi^\Lambda = \psi^\Lambda = \tau^\Lambda_{12} \)), defines the determinant \( \left| K_{\alpha\delta} \cdot R_{\beta}\right| \).

Similarly

\[
\begin{aligned}
\left| K_{\alpha\delta} \cdot R_{\beta}\right| & \begin{vmatrix} R_{1212} & R_{1213} & \cdots & R_{121M} \\ R_{1312} & R_{1313} & \cdots & R_{131M} \\ \cdots & \cdots & \cdots & \cdots \\ R_{1M12} & R_{1M13} & \cdots & R_{1M1M} \end{vmatrix} = \frac{1}{|g_{\alpha\beta}|} \begin{vmatrix} R_{1212} & R_{1213} & \cdots & R_{121M} \\ R_{1312} & R_{1313} & \cdots & R_{131M} \\ \cdots & \cdots & \cdots & \cdots \\ R_{1M12} & R_{1M13} & \cdots & R_{1M1M} \end{vmatrix}
\end{aligned}
\]

1

\[
\left| \Phi_{\alpha\delta} \cdot \tilde{\Psi}_{\beta}\right| = \frac{1}{|g_{\alpha\beta}|} \begin{vmatrix} \phi^\Lambda \tau_{22,1} & \psi^\Lambda \tau_{12,1} & \cdots & \psi^\Lambda \tau_{22,1} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ -\phi^\Lambda \tau_{22,1} & \psi^\Lambda \tau_{12,1} & \cdots & \psi^\Lambda \tau_{22,1} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ -\phi^\Lambda \tau_{22,1} & \psi^\Lambda \tau_{12,1} & \cdots & \psi^\Lambda \tau_{22,1} \\ \cdot & \cdot & \cdots & \cdot \end{vmatrix}
\]

\[
= \frac{1}{|g_{\alpha\beta}|} \begin{vmatrix} \phi^\Lambda \tau_{22,1} & \psi^\Lambda \tau_{12,1} \\ \cdot & \cdot \\ \cdot & \cdot \\ -\phi^\Lambda \tau_{22,1} & \psi^\Lambda \tau_{12,1} \\ \cdot & \cdot \\ \cdot & \cdot \\ -\phi^\Lambda \tau_{22,1} & \psi^\Lambda \tau_{12,1} \end{vmatrix} \left( \phi^\Lambda \tau_{22,1} \right) \left( \psi^\Lambda \tau_{12,1} \right) \left( \tau^\Lambda_{11} \tau_{22,1} \right)^{M-2}.
\]
On the basis of the functional formulation of the curvature of Riemannian space $q^\alpha$ of class $C = N - M$, the relation (2.3): $\kappa^2 = \left| \frac{K_{\alpha\beta}}{g_{\alpha\beta}} \right|^2$, as well as of the two previously derived relations, and for $(M > 2)$, it finally follows that

$$\kappa^2 = \frac{1}{|g_{\alpha\beta}|^2 (R_{1212})^{M-2}} \left[ \left( \Delta_{t_1}^R \Delta_{t_2}^R \right) \frac{1}{M} - \left( \Delta_{i_1}^R \Delta_{i_2}^R \right) \frac{1}{M-2} \right]^{M-2}.$$

Clearly, in the case of Riemannian space $q^\alpha$ of class $C = N - M$ $(M = 2)$, functional expression (2.3) for the surface is reduced to the well-known Gaussian curvature of surface in the theory of surfaces, $C_{\alpha\beta}$ and $C_{\alpha\beta}$.

Comment: In the case when the component of Riemann-Christoffel’s tensor is equal to zero, it is possible, in the functional expression (2.8) for the curvature of Riemannian spaces, to take any another combination of the components of the matrix scheme of the curvature vectors of space: $K_{\alpha\beta} = \tau_{\alpha\beta}^q \delta_{\alpha\beta}$, as the support elements of Gauss-Chiò’s procedure for condensation of determinants, and for which some of the components of Riemann-Christoffel’s tensor of curvature are not equal to zero. Clearly, if all components of Riemann-Christoffel’s tensor of the curvature of space are identically equal to zero, then the space is Riemannian space of curvature zero (Euclidean space).

2.2. Sub-spaces of curvature of hyper-dimensional spaces. Let $N$ - dimensional ambient space of space continuum $x^i$, just as in the first Comment of preceding Section 2.1 of this paper, be either Riemannian space of class $C$ or Euclidean curvilinear space of class $C$: $C \geq 1$, both with the normal vector space $\vec{w}_P$; $P = 1, 2, ..., C$.

If and only if the vector component:

$$\left( \partial_{q^i} \vec{N}_{\alpha\beta} g^{\alpha\beta} \cdot \vec{e}_i \right) \hat{e}^2 = \kappa^2 \left( \kappa^2 \vec{g}_\beta \cdot \vec{e}_i \right) \hat{e}^2,$$

of vectors obtained by a partial differentiation of the matrix scheme of vector of normal curvature: $\vec{N}_{\alpha\beta} = \left( \partial_{q^i} \vec{g}_\alpha \cdot \vec{w}_P \right) \vec{w}_P = \kappa^2 \vec{w}_P$, of Riemannian space $q^\alpha$ of class $\hat{C}$: $\hat{C} = C + N - M$, immersed in ambient space $x^i$ of class $C$, are vectors of the tangent vector space $\vec{g}_\alpha$ of Riemannian space $q^\alpha$ of class $\hat{C}$, in other words if and only if the mutually equivalent conditions:

$$- \left( \partial_{q^i} \vec{N}_{\alpha\beta} g^{\alpha\beta} \cdot \vec{e}_i \right) \hat{e}^2 = \kappa^2 e_{ij} \partial_{q^\beta} x^j \hat{e}^2 = \kappa^2 \left( \vec{g}_\beta \cdot \vec{e}_i \right) \hat{e}^2 = \kappa^2 \vec{g}_\beta$$

and

$$- \kappa^2 \left( \partial_{q^i} \vec{w}_P \cdot \vec{e}_i \right) = \kappa^2 e_{ij} \partial_{q^\beta} x^j = \kappa^2 \vec{g}_\beta \partial_{q^\beta} x^j,$$

are satisfied, then Riemannian sub-space $q^\alpha$ of class $\hat{C}$, is a sub-space of curvature of the ambient space $x^i$ of class $C$. In the case in which a Riemannian

2The second of the two relations of equality is reduced to: $t^i_{k,P} t^j_{l,P} \partial_{q^\beta} x^k = \kappa^2 e_{ij} \partial_{q^\beta} x^j$, clearly on the condition that all vectors of vector components $\left( \partial_{q^i} \vec{w}_P \cdot \vec{e}_i \right)$ are also vectors of the tangent vector space $\vec{g}_\beta$. Namely, since: $t^i_{k,P} \partial_{q^\beta} x^k = \partial_{q^\beta} \vec{w}_P \cdot \vec{e}_i$ and $\kappa^2 \partial_{q^\beta} x^j = \left( \partial_{q^i} \vec{w}_P \cdot \vec{g}_\alpha \right) \vec{g}^\alpha \cdot \vec{g}_\beta \left( \vec{g}_\alpha \cdot \vec{g}_\beta \right)$, then $\kappa^2 \partial_{q^\beta} x^j = t^i_{k,P} \partial_{q^\beta} x^k$. 

\]
space \( q^a \) of class \( \hat{C} \): \( \hat{C} = C + N - M \), is an arbitrary curve \((M = 1)\), the preceding conditions are reduced to the conditions: 
\[- \left( \partial_q \mathbf{N} \mathbf{g} \right) \partial_q x^i \mathbf{e}^i = \hat{\kappa}^2 e_{ij} \partial_q x^j \mathbf{e}^i \]
and
\[
d \mathbf{g} \mathbf{w}^p \cdot x^1 \mathbf{e}^i = \hat{\kappa}^2 e_{ij} \partial_q x^j \mathbf{e}^i, \]
as well as
\[
d \mathbf{t} \mathbf{w}^p \cdot x^1 \mathbf{e}^i = k_{ij} \partial_q x^j \mathbf{e}^i = \hat{\kappa}^2 e_{ij} \partial_q x^j \mathbf{e}^i.
\]

The last of them is the well-known condition for the curve to be a line of the curvature of the ambient space \( x^1 \) of class \( C \). It is obvious from this that conditions: \((2.9)\) and \((2.10)\), are a generalization of the preceding conditions.

By projection of the condition \((2.9)\), onto covariant vector basis \( \{ \mathbf{g}, \} \), it is obtained that
\[\mathbf{x}^{\alpha \beta} g^{\delta i} \partial_{q^i} \mathbf{x}^j \partial_q x^i = \mathbf{x}^{\alpha \beta} g^{\delta i} \partial_{q^i} x^j \partial_q x^i = \hat{\kappa}^2 g_{\beta \gamma},\]
more exactly,
\[\left( \kappa_n \right)^2 = \frac{\left| \mathbf{x}^{\alpha \beta} g^{\delta i} \partial_{q^i} x^j \partial_q x^i \right|}{\left| g_{\beta \gamma} \right|} = \hat{\kappa}^2 M,\]
with regard to the relation \((2.3)\).

**Comment:** Riemannian space of curvature: \( q^a \), of class \( \hat{C} \), as a sub-space of the ambient space \( x^1 \) of class \( C \), is said to be the principal, if and only if
\[(2.11)\]
\[t^k_{j, p^{i}k} \partial_{q^i} x^j = \hat{\kappa}^2 e_{ij} \partial_q x^j,\]
in other words, if all vectors of the vector components: \( \left( \partial_q \mathbf{w}^p \cdot \mathbf{e}^i \right) \mathbf{e}_i \), are vectors of the tangent vector space (see the Footnote 2 of the paper).

In order to exist nontrivial solutions of the previous homogeneous linear system \((2.11)\) of ordinary differential equations with respect to the unknowns \( \partial_{q^i} x^j \), the following condition
\[(2.12)\]
\[\left| t^k_{j, p^{i}k} - \hat{\kappa}^2 e_{ij} \right| = 0,\]
must be satisfied, \((2)\) and \((3)\).

On the basis of the developed form of the preceding condition \((2.12)\), expressed by polynomial of \( N \)-th degree with respect to the unknown \( \hat{\kappa} \), as well as of Viète’s formulas \((2)\), and taking the relation \((2.3)\) into consideration, it finally follows that
\[\hat{\kappa}^2 = \frac{\left| t^k_{j, p^{i}k} \right|}{\left| e_{ij} \right|} = \prod_{i=1}^{N} \hat{\kappa}_i^2.\]

If the varied of all eigenvalues \( \hat{\kappa}_i \) of the matrix: \( t^k_{j, p^{i}k} \), is an unit, then the principal Riemannian sub-spaces of curvature: \( q^a \), of class \( \hat{C} \), are one-dimensional ones \((M = 1)\), in other words there exist \( N \) principal directions of curvature with the normal curvatures: \( \left( \kappa_n \right)_i = \hat{\kappa}_i \), of the ambient space \( x_i \) of class \( C \). On the condition that there exists at least one of all eigenvalues \( \hat{\kappa} \) with the varied \( M \), then there exists at least one principal \( M \)-dimensional Riemannian sub-space of curvature: \( q^a \), of class \( \hat{C} \), with the normal curvature: \( \kappa_n = \hat{\kappa}^M \), as well as \( N - M \) mutually orthogonal principal directions of curvature with the normal curvatures: \( \left( \kappa_n \right)_i = \hat{\kappa}_i \), such that are orthogonal.
onto the principal $M$ - dimensional Riemannian sub-spaces of curvature: $q^\alpha$, of class $C$. In this emphasized case

$$k = \hat{k}^M \prod_{i=1}^{N-M} \hat{\kappa}_i.$$  

Furthermore, if the vector components:

$$\left( \partial q^i \tilde{N}_{\alpha \beta} \cdot \hat{e}_i \right) \hat{e}^\alpha = \mathcal{X}_{\alpha \beta}^P \left( \partial q^i \hat{w}_P \cdot \hat{e}_i \right) \hat{e}^\alpha,$$

are also vectors of the tangent vector space $\hat{g}_\delta$ of Riemannian sub-space $q^\alpha$ of class $\hat{C}$

$$- \left( \partial q^i \tilde{N}_{\alpha \beta} \cdot \hat{e}_i \right) \hat{e}^\alpha = \hat{k}^2 g_{\alpha \beta} e_{ki} \partial_{q^k} x^i \hat{e}^\alpha = \hat{k}^2 g_{\alpha \beta} (\hat{g}_\delta \cdot \hat{e}_i) \hat{e}^\alpha = \hat{k}^2 g_{\alpha \beta} \hat{g}_\delta,$$

more exactly,

$$\mathcal{X}_{\alpha \beta}^P \left( t_{ij}, p \partial_{q^\gamma} x^i \partial_{q^\delta} x^j \right) = \mathcal{X}_{\alpha \beta}^P \mathcal{X}_{\gamma \delta}^P = \hat{k}^2 g_{\alpha \beta} g_{\gamma \delta},$$

then, the normal curvature $\kappa_n$ of sub-space of curvature: $q^\alpha$, of class $\hat{C}$, of the ambient space $x^i$ of class $C$, is determined by functional form

$$(\kappa_n)^2 = \frac{\mathcal{X}_{\alpha \beta}^P \mathcal{X}_{\gamma \delta}^P}{|g_{\alpha \beta}|} = \hat{k}^{2M},$$

more exactly,

$$M \sqrt{(\kappa_n)^2} = \hat{k}^2 = \frac{(t_{ij} t_{kl}, P - t_{ij} t_{kj}, P) \partial_{q^\alpha} x^i \partial_{q^\beta} x^j \partial_{q^\gamma} x^k}{(e_{ij} e_{kl} - e_{ij} e_{kj}) \partial_{q^\alpha} x^i \partial_{q^\beta} x^j \partial_{q^\gamma} x^k};$$

$$M \sqrt{(\kappa_n)^2} = \hat{k}^2 = \frac{R_{ijkl} \partial_{q^\alpha} x^i \partial_{q^\beta} x^j \partial_{q^\gamma} x^k}{(e_{ij} e_{kl} - e_{ij} e_{kj}) \partial_{q^\alpha} x^i \partial_{q^\beta} x^j \partial_{q^\gamma} x^k} = \frac{R}{g_{\alpha \beta} g_{\gamma \delta} - g_{\alpha \gamma} g_{\beta \delta}} = \frac{R}{M (M - 1)},$$

where the scalar invariant $R$: $R = g^{\alpha \gamma} g^{\beta \delta} R_{\alpha \beta \gamma \delta}$, is so-called invariant of curvature (scalar curvature). [1]

It can be proved by an application of the result of Schur Theorem, [1], that Riemannian sub-spaces of curvature: $q^\alpha$, of class $\hat{C}$, of the ambient space $x^i$ of class $C$, whether they are principals or not, in this emphasized case are isotropic spaces of the constant normal curvature $\kappa_n$. [1]

2.2.1. Example. The curvature of Riemannian spaces:

$$ds^2 = e^{\nu(\rho)} d\rho^2 + \rho^2 d\theta^2 + \sin^2 \theta d\varphi^2 + e^{\nu(\rho)} d\tau^2;$$

$$\mu(\rho) = - \nu(\rho) = - \ln \left( 1 - \frac{2m}{\rho} \right); m = \text{const.}$$

and

$$d\tilde{s}^2 = e^{\mu(\rho)} \left( d\rho^2 + \rho^2 d\theta^2 + \sin^2 \theta d\varphi^2 + d\tau^2 \right);$$

$$\mu(\rho) = - \frac{2m}{\rho}; m = \text{const.},$$

with spherical symmetry.
Components of Christoffel’s symbols:
\[ \Gamma_{\alpha\beta\gamma} = \frac{1}{2} \left( \partial_{\rho^\alpha} g_{\beta\gamma} + \partial_{\rho^\beta} g_{\alpha\gamma} - \partial_{\rho^\gamma} g_{\alpha\beta} \right), \]
as well as of Reimann-Christoffel’s tensor of curvature:
\[ R_{\alpha\beta\gamma\delta} = \partial_{\rho^\delta} \Gamma_{\beta\gamma\alpha} - \partial_{\rho^\alpha} \Gamma_{\beta\gamma\delta} + g^{\lambda\sigma} \left( \Gamma_{\beta\delta\lambda} \Gamma_{\lambda\alpha\gamma} - \Gamma_{\beta\gamma\lambda} \Gamma_{\alpha\delta\lambda} \right), \]
which are not identically zeros, for these sub-classes of general class of Riemannian spaces with spherical symmetry, are the following forms:
\[ \Gamma_{11,1} = \bar{\Gamma}_{11,1} = \frac{1}{2} \rho \partial_{\rho^\mu} \mu; \Gamma_{22,1} = -\Gamma_{12,2} = -\rho; \bar{\Gamma}_{22,1} = -\bar{\Gamma}_{12,2} = -\frac{\rho e^\mu}{2} (2 + \rho \partial_{\rho^\mu}); \]
\[ \Gamma_{33,1} = -\Gamma_{13,3} = -\rho \sin^2 \theta; \bar{\Gamma}_{33,1} = -\bar{\Gamma}_{13,3} = -\frac{\rho \sin^2 \theta e^\mu}{2} (2 + \rho \partial_{\rho^\mu}); \]
\[ \Gamma_{33,2} = -\Gamma_{23,3} = -\rho \sin \theta \cos \theta; \bar{\Gamma}_{33,2} = -\bar{\Gamma}_{23,3} = -\rho^2 e^\mu \sin \theta \cos \theta; \]
\[ \Gamma_{44,1} = -\Gamma_{14,4} = -\frac{1}{2} \rho e^\nu \partial_{\rho^\nu}; \bar{\Gamma}_{44,1} = -\bar{\Gamma}_{14,4} = -\frac{1}{2} e^\mu \partial_{\rho^\mu}, \]
as well as
\[ R_{1212} = \frac{1}{2} \rho \partial_{\rho^\mu} \mu; \bar{R}_{1212} = \frac{1}{2} \rho e^\mu \left( \partial_{\rho^\mu} + \partial_{\rho^\rho} \partial_{\rho^\mu} \right); \]
\[ R_{1313} = \frac{1}{2} \rho \sin^2 \theta \partial_{\rho^\rho} \mu; \bar{R}_{1313} = \frac{1}{2} \rho e^\mu \sin^2 \theta \left( \partial_{\rho^\rho} + \rho \rho^2 \partial_{\rho^\rho} \right); \]
\[ R_{2323} = \rho^2 \sin^2 \theta (1 - e^{-\mu}); \bar{R}_{2323} = -\rho^3 e^\mu \partial_{\rho^\rho} \mu \sin^2 \theta \left( 1 + \frac{\rho}{4} \partial_{\rho^\rho} \mu \right); \]
\[ R_{1414} = \frac{1}{2} e^\nu \left\{ \frac{1}{2} \left[ \partial_{\rho^\mu} \partial_{\rho^\nu} - (\partial_{\rho^\nu})^2 \right] - \partial_{\rho^\rho} \partial_{\rho^\nu} \right\} ; \bar{R}_{1414} = -\frac{1}{2} e^\mu \partial_{\rho^\rho} \mu; \]
\[ R_{2424} = -\frac{1}{2} \rho e^{-\mu} \partial_{\rho^\rho} \mu; \bar{R}_{2424} = -\frac{1}{2} \rho e^\mu \partial_{\rho^\rho} \mu \left( 1 + \frac{\rho}{2} \partial_{\rho^\rho} \mu \right). \]
By the functional relation (3.8), it follows that:
\[ \kappa = \frac{1}{R_{1212} g} \sqrt{(R_{1212})^2 R_{1313} R_{1414} R_{2323} R_{2424}} = \frac{1}{g} \sqrt{R_{1313} R_{1414} R_{2323} R_{2424}}, \]
\[ \kappa = \left\{ \frac{1}{4} \rho^4 (1 - e^{-\mu}) e^{2\mu - \mu} \sin^4 \theta \partial_{\rho^\mu} \partial_{\rho^\rho} \mu \left[ \frac{\partial^2_{\rho^\rho} \mu}{\rho^4 e^{\mu + \nu}} \sin^2 \theta \right] \right\}^{1/4} = \frac{2 m^2}{\rho^5} \]
and
\[ \tilde{\kappa} = \frac{1}{R_{1212} g} \sqrt{(R_{1212})^2 R_{1313} R_{1414} R_{2323} R_{2424}} = \frac{1}{g} \sqrt{R_{1313} R_{1414} R_{2323} R_{2424}}, \]
\[ \tilde{\kappa} = \left\{ \frac{1}{4} \rho^5 e^{\mu} \sin^4 \theta \left( \partial_{\rho^\rho} \mu \right)^2 \left( \partial_{\rho^\mu} + \rho \partial_{\rho^\rho} \mu \right) \left[ \frac{\partial_{\rho^\rho} \mu}{\rho^4 e^{\mu + \nu}} \sin^2 \theta \right] \right\}^{1/2} = \]
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\[ = 2 \frac{m^2}{\rho^6} \sqrt{\left(1 + \frac{m}{2\rho}\right) \left(1 + \frac{m}{\rho}\right)} \nabla \]

3. CONCLUSION

By functional form (2.8), derived from general functional form (2.3) generalizing the concept of the curvature of Riemannian both one and two dimensional spaces to the general concept of the curvature of Riemannian vector spaces \( q^\alpha \) of class \( C: C = N - M \), the concept itself of the curvature of Riemannian vector spaces of higher dimensions \( (M \geq 2) \), directly related to internal geometry of space, more exactly, to components of Reimann-Christoffel’s tensor of curvature \( R_{\alpha\delta\beta\gamma} \).

The process of generalization of fundamental concepts of the differential geometry, presented in this paper, gives the solid base to further generalization other, whether they are fundamentals or not, concepts and theorems of the differential geometry of a surface, and what may be the subject of separated analysis.

The one of such concepts is that of Codazzi’s equations (formula) of a surface, 4 and 5, which can be, in the general case of Riemannian space \( q^\alpha \) of class \( C: C = N - M \), obtained by projection of vector condition of integrability:

\[ \partial^2_{q^\alpha q^\beta} \tilde{g}_\alpha - \partial^2_{q^\alpha q^\beta} \tilde{g}_\alpha = 0, \]

of absolute differentials of the fundamental vectors \( \tilde{g}_\alpha \); \( d\tilde{g}_\alpha = \partial_{q^\beta} \tilde{g}_\alpha dq^\beta \), onto the normal vector space of Riemannian space \( q^\alpha \) of class \( C \).

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