A LAX PAIR STRUCTURE FOR THE HALF-WAVE MAPS EQUATION

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Abstract. We consider the half-wave maps equation
\[ \partial_t \vec{S} = \vec{S} \wedge |\nabla| \vec{S}, \]
where \( \vec{S} = \vec{S}(t, x) \) takes values on the two-dimensional unit sphere \( S^2 \) and \( x \in \mathbb{R} \) (real line case) or \( x \in \mathbb{T} \) (periodic case). This is an energy-critical Hamiltonian evolution equation recently introduced in [8, 16], which formally arises as an effective evolution equation in the classical and continuum limit of Haldane–Shastry quantum spin chains. We prove that the half-wave maps equation admits a Lax pair and we discuss some analytic consequences of this finding. As a variant of our arguments, we also obtain a Lax pair for the half-wave maps equation with target \( H^2 \) (hyperbolic plane).

Keywords: Integrable Systems, Half-Wave Maps, Haldane–Shastry model, Calogero–Moser–Sutherland model.

1. Introduction and Main Results

Spin chains – both in quantum and classical versions – arise as fundamental models in the study of exactly solvable and completely integrable systems. For instance, the classical Heisenberg model (HM) for ferromagnets in one space dimension provides a prototype of a completely integrable classical spin system; see [3, 7, 15].

In this note, we are concerned with a new evolution equation for classical spins recently introduced in [8, 16], which we will refer to as the half-wave maps equation following [8]. This equation has some similarities to (HM) and yet it shows a completely different mathematical features in many aspects (e.g., traveling solitary waves given by rational functions and energy-criticality of the evolution problem). In fact, the half-wave maps equation can be – formally, at least – obtained from taking a combined classical and continuum limit from a quantum spin chain of Haldane–Shastry (HS) type introduced in [6, 13]. Our main result shown below will yield a Lax pair for the half-wave maps equation, which will involve certain suitable nonlocal operators.

Let us now introduce the mathematical framework to formulate the problem at hand. We consider a time-dependent field of classical spins \( \vec{S} = \vec{S}(t, x) \in \mathbb{R}^3 \) defined for either \( x \in \mathbb{R} \) (real line case) or \( x \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z} \) (periodic case). Without loss of generality, we assume that the classical spins are normalized to unit length, i.e., we have that \( \vec{S}(t, x) \in S^2 \) holds. For the spin field \( \vec{S} = \vec{S}(t, x) \), we consider the half-wave maps equation given by
\[ (HWM) \quad \partial_t \vec{S} = \vec{S} \wedge |\nabla| \vec{S}. \]

Here \( \wedge \) denotes the usual vector product in \( \mathbb{R}^3 \). The (pseudo-differential) operator \( |\nabla| \) is defined via its corresponding multiplication symbol in Fourier space, i.e.,
\[ (|\nabla|f)(\xi) = \begin{cases} \langle \xi \rangle \hat{f}(\xi), & (x \in \mathbb{R}, \text{real line case}) \\ \text{sn} \hat{f}(n), & (x \in \mathbb{T}, \text{periodic case}) \end{cases} \]

Here \( \hat{\cdot} \) denotes the Fourier transform for functions either defined on \( \mathbb{R} \) or \( \mathbb{T} \), respectively. The formal derivation of (HWM) from a quantum spin chain model of Haldane–Shastry type will be sketched in Section 2 below, followed by a brief summary about the traveling solitary waves for (HWM) recently studied in [8].
In analogy to the classical Heisenberg model, the half-wave maps equation comes with a Hamiltonian structure where energy functional in our case reads
\[ E[S] = \frac{1}{2} \int \bar{S} : |\nabla|S \, dx. \tag{1.1} \]

The corresponding Poisson bracket for the $S^2$-valued function $\bar{S} = (S_1, S_2, S_3)$ is given by
\[ \{S_i(x), S_j(y)\} = \bar{\varepsilon}_{ijk} S_k(x) \delta(x-y), \tag{1.2} \]
where $\varepsilon_{ijk}$ denotes the standard anti-symmetric Levi-Civitè symbol. As a consequence, the \textit{HWM} can be (formally) written as $\partial_t \bar{S} = \{S, E\}$ in analogy to the Heisenberg model. Furthermore, it is straightforward to check that \textit{HWM} exhibits formal conservation of total spin and linear momentum, due to the rotational invariance (on the target $S^2$) and translational invariance on the domain; see [3] for details.

From the point of view of PDE analysis, the evolution problem \textit{HWM} is energy-critical since the scaling transform
\[ \bar{S}(t,x) \mapsto \lambda \bar{S}(\lambda t, \lambda x) \]
maps solutions into solutions, whereas the energy $E[S] = E[\bar{S}]$ stays invariant under this transformation. Such a critical scaling behavior is in striking contrast to the one-dimensional Heisenberg model (HM), which is energy-subcritical. As a consequence, the existence of unique global-in-time solutions for the half-wave maps equation is much more delicate. In particular, a possible singularity formation (blowup) of smooth solutions cannot be simply ruled out by using energy conservation.

Let us now turn to the issue of complete integrability for the half-wave maps equation, where we will show below that \textit{HWM} admits a Lax pair. As in the study of the classical Heisenberg model, it turns out to be expedient to first formulate the problem by using the standard Pauli matrices $\sigma_1, \sigma_2, \sigma_3 \in \mathfrak{su}(2)$. For a given vector $\bar{X} \in \mathbb{R}^3$, we define the $2 \times 2$-matrix given by
\[ X = \bar{X} \cdot \sigma = \sum_{j=1}^3 X_j \sigma_j = \begin{pmatrix} X_3 & X_1 - iX_2 \\ X_1 + iX_2 & -X_3 \end{pmatrix}. \]
with the usual notation $\sigma = (\sigma_1, \sigma_2, \sigma_3)$. Clearly, we have that $X^* = X$ is a Hermitian $2 \times 2$-matrix with vanishing trace $\text{Tr} X = 0$. For the reader’s convenience, we recall some standard facts when dealing with Pauli matrices. From the fundamental relation $\sigma_j \sigma_k = \delta_{jk} 1 + i \varepsilon_{jkl} \sigma_l$ (where $1$ denotes $2 \times 2$-unit matrix) we obtain
\[ (\bar{X} \cdot \sigma)(\bar{Y} \cdot \sigma) = (\bar{X} \cdot \bar{Y}) 1 + i(\bar{X} \wedge \bar{Y}) \cdot \sigma. \tag{1.3} \]
for arbitrary vectors $\bar{X}, \bar{Y} \in \mathbb{R}^3$. As a direct consequence of this, we readily deduce that
\[ S^2 = 1 \quad \text{if and only if} \quad |S|^2 = 1. \tag{1.4} \]
Furthermore, we see that \textit{HWM} can be equivalently written as
\[ \partial_t S = -\frac{i}{2} [S, |\nabla|S], \tag{1.5} \]
where $[A, B] = AB - BA$ denotes the commutator of $A$ and $B$. Clearly, this matrix-valued formulation as a commutator equation bears a strong resemblance to the equation $\partial_t \bar{S} = -\frac{i}{2} [\bar{S}, |\nabla|\bar{S}]$ used to prove the complete integrability of the one-dimensional Heisenberg model (HM) by Takthajan in [15]. However, the presence of the nonlocal pseudo-differential operator $|\nabla|$ in equation (1.5) makes the search for a Lax pair a rather different task.

To formulate the operators for the Lax pair of the half-wave maps equation, we introduce the following notation in order to avoid any potential ambiguities in the expressions below. Suppose that $\bar{A} = \bar{A}(t,x) \in \mathbb{R}^3$ is a given function and let $A = \bar{A} \cdot \sigma$ denote the corresponding function with values in the Lie algebra $\mathfrak{su}(2)$. We use $\mu_A$ to denote the multiplication operator for $A$ acting on functions $\varphi = \varphi(x)$ with values in $\mathbb{C}^2$, i.e., we set
\[ (\mu_A \varphi)(t,x) = A(t,x) \varphi(x). \]
Given a spin field $\vec{S} = \vec{S}(t,x)$, we are now ready to define the following pair of operators $L_S$ and $B_S$ as follows

$$L_S = [H, \mu_S] \quad \text{and} \quad B_S = -\frac{i}{2} (\mu_S |\nabla| + |\nabla| \mu_S) + \frac{i}{2} \mu_S \nabla |\mu_S|.$$  

The operators $L_S$ and $B_S$ are formally defined on the complex Hilbert space $L^2(X; \mathbb{C}^2)$, where either $X = \mathbb{R}$ (real line case) or $X = \mathbb{T}$ (periodic case). Furthermore, the operator $H$ denotes the Hilbert transform defined through the principal value expression given by

$$(H f)(x) = \left\{ \begin{array}{ll}
P.V. - \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(y)}{x-y} dy & (x \in \mathbb{R}, \text{ real line case}) \\
P.V. \int_{\mathbb{T}} f(y) \cot \left( \frac{x-y}{2} \right) dy & (x \in \mathbb{T}, \text{ periodic case}). \end{array} \right.$$  

In (1.6) above and throughout the following, the Hilbert transform $H$ has to be understood to act as $H \varphi = (H \varphi_1, H \varphi_2)$ on each of the components of $\varphi \in L^2(X; \mathbb{C}^2)$. Since $H^* = -H$ is skew-symmetric and $(\mu A)^* = \mu A$ is symmetric, we readily deduce the (formal) properties

$$(L_S)^* = L_S \quad \text{and} \quad (B_S)^* = -B_S.$$  

In fact, we will see below that $L_S$ is a Hilbert–Schmidt operator if and only if $\vec{S}$ has finite energy. On the other hand, the operator $B_S$ is clearly an unbounded operator to be defined on some suitable dense subset of $L^2(X; \mathbb{C}^2)$.

The main result of this paper shows that $L_S$ and $B_S$ provide a Lax pair for the half-wave maps equation.

**Theorem 1** (Lax Pair of the Half-Wave Maps Equation). Let $\vec{S} = \vec{S}(t,x)$ be a sufficiently regular solution of $\text{(HWM)}$ with either $x \in \mathbb{R}$ (real line case) or $x \in \mathbb{T}$ (periodic case). Then the following Lax equation holds true:

$$\frac{d}{dt} L_S = [B_S, L_S],$$  

where the operators $L_S$ and $B_S$ are defined in (1.6) above.

**Proof.** We first recall that $d\mu_S/dt = -\frac{i}{2} [\mu_S, \mu_S |\nabla|]$ from (1.6). By using Jacobi’s identity $[A, [B, C]] + [C, [A, B]] + [B, [C, A]] = 0$, we get

$$\frac{d}{dt} L_S = -\frac{i}{2} [H, [\mu_S, \mu_S |\nabla|]] = \frac{i}{2} \left( [\mu_S, [\mu_S |\nabla|, H]] + [\mu_S |\nabla|, [H, \mu_S]] \right).$$

Next, from Cotlar’s product identity $H(fg) = Hfg + fhg + H(fHg)$ for the Hilbert transform we deduce

$$[\mu_S |\nabla|, H] = -[\mu_S |\nabla| + H \mu_S |\nabla|, H] = \mu_\partial_S + H \mu_\partial_S H,$$

where we also used that $H |\nabla| = -\partial_x$ holds. Hence can write the time derivative of $L_S$ as

$$\frac{d}{dt} L_S = \frac{i}{2} [\mu_S, \mu_\partial_S + H \mu_\partial_S H] + \frac{i}{2}[\mu_S |\nabla|, L_S].$$

It remains to show that the first commutator on right-hand side can be written as a commutator with $L_S$. This can be seen as follows. Using $|\nabla| = H \partial_x$ again, we get

$$[\mu_S |\nabla|, H] = [\mu_S H \partial_x, H] = \mu_S H [\partial_x, H] = \mu_S H \mu_\partial_S H,$$

where in the last step we used $[\partial_x, \mu_S] = \mu_\partial_S$ by Leibniz’ rule. On the other hand, in view of $\mu \mu_S = \#$ by (1.3) and (1.4) together with $H \partial_x H = H^2 \partial_x = -\partial_x$, we see that

$$[\mu_S |\nabla|, H] = [\mu_S H \partial_x, H \mu_S] = \mu_S H \partial_x H \mu_S - H \mu_\partial_S H \partial_x = -\mu_\partial_S \mu_S + \partial_x = -\mu_\partial_S [\partial_x, \mu_S] = -\mu_\partial_S \mu_\partial_S.$$

Recall now the definition $L_S = H \mu_S - \mu_S H$. Thus if we now combine (1.10) with the identity found above, we deduce the identity

$$[\mu_S |\nabla|, L_S] = -\mu_\partial_S \mu_\partial_S - \mu_S H \mu_\partial_S H.$$  

Since $(L_S)^* = L_S$, we can take adjoints to calculate $[|\nabla| \mu_S, L_S]$. However, it is also interesting to make the computation directly, namely

$$[|\nabla| \mu_S, H \mu_S] = [\partial_x H \mu_S, H \mu_S] = [\partial_x, H \mu_S] H \mu_S = H \mu_\partial_S H \mu_S.$$
and
\[ [\nabla |\mu_\ast, \mu_\ast H] = [H \partial_x \mu_\ast, \mu_\ast H] = H \partial_x \mu_\ast \mu_\ast H - \mu_\ast H \partial_x \mu_\ast \mu_\ast \]
where last identity follows from differentiating \(\mu_\ast \mu_\ast = \mathbb{I}\). Thus we conclude that
\[ (1.12) \quad [\nabla |\mu_\ast, L_\ast] = \mu_\ast \mu_\ast + H \mu_\ast \partial_x H \mu_\ast. \]
By adding (1.11) and (1.12), we arrive at
\[ [\mu_\ast \nabla] + [\nabla |\mu_\ast, L_\ast] = -[\mu_\ast, \mu_\ast, L_\ast] + H \mu_\ast \partial_x H \mu_\ast \]

Corollary 1. We have the following (formal) conservation laws:
\[ \text{Tr}([L_\ast]^p) = \text{const.} \]
for any \(1 \leq p < \infty\).

Remarks. 1) Consider the real line case when \(x \in \mathbb{R}\). Then the operator \(L_\ast\) has the symmetric kernel
\[ K_\ast(x, y) = \frac{1}{\pi} \frac{S(x) - S(y)}{x - y} \in \mathbb{C}^{2 \times 2}. \]

Using that \(\text{Tr}_c(\mathbf{A}^\ast \mathbf{A}^\ast) = 2|\mathbf{A}|^2\) thanks to (1.3), we find that the squared Hilbert–Schmidt norm of the Lax operator \(L_\ast\) is given by
\[ (1.13) \quad \text{Tr}([L_\ast]^2) = \frac{2}{\pi^2} \int_{\mathbb{R} \times \mathbb{R}} \frac{|\hat{S}(x) - \hat{S}(y)|^2}{|x - y|^2} \, dx \, dy = \frac{8}{\pi} E[\hat{S}], \]
where we used the well-known identity
\[ \int_{\mathbb{R}} f|\nabla| f \, dx = \frac{1}{2\pi} \int_{\mathbb{R} \times \mathbb{R}} \frac{|f(x) - f(y)|^2}{|x - y|^2} \, dx \, dy. \]

Hence \(\text{Tr}([L_\ast]^2)\) is equal to the conserved energy \(E[\hat{S}]\) for (HWM) (up to a multiplicative constant). Moreover, we deduce that \(L_\ast\) is Hilbert–Schmidt if and only if \(E[\hat{S}] < +\infty\).

2) In the periodic case when \(x \in \mathbb{T}\), an analogous calculation yields that
\[ \text{Tr}([L_\ast]^2) = 2 \int_{\mathbb{T} \times \mathbb{T}} |\hat{S}(x) - \hat{S}(y)|^2 \cot^2 \left(\frac{x - y}{2}\right) \, dx \, dy \]
\[ = 2 \int_{\mathbb{T} \times \mathbb{T}} \frac{|\hat{S}(x) - \hat{S}(y)|^2}{\sin^2 \left(\frac{1}{2} (x - y)\right)} \, dx \, dy - 2 \int_{\mathbb{T} \times \mathbb{T}} \frac{1}{\pi} S(x) \, dx \, dy \]
\[ = 8\pi E[\hat{S}] + 4 \int_{\mathbb{T}} S(x) \, dx^2 - 16\pi^2, \]
which amounts to a combination of the conserved energy and the square of the conserved total spin \(\int_{\mathbb{T}} \hat{S}\), plus some numerical constant. Again, we see that \(L_\ast\) is Hilbert-Schmidt if and only if \(\hat{S}\) has finite energy.

3) More generally, by decomposing \(L^2(X, \mathbb{C}) = L^2_\mathrm{H}(X, \mathbb{C}) \oplus L^2_\mathrm{H}(X, \mathbb{C})\) according to the sign of the Fourier spectrum, it is easy to make the link between operators \(L_\ast\) and Hankel operators with matrix symbols given by symbols \(\hat{S}\); see Peller [11]. As a consequence from Peller’s theorem [11], we conclude (both in real line and the periodic case) that the following norm equivalence to homogeneous Besov norms of \(\hat{S}\) holds:
\[ (1.14) \quad \text{Tr}([L_\ast]^p) \sim \|\hat{S}\|^p_{\mathcal{B}^p_{2,p}}. \]
In particular, applying this result for \(p = 1\), if the initial value of \(\hat{S}\) is smooth enough, then \(\nabla |\hat{S}, \partial_x \hat{S}\) and \(\partial_x \hat{S}\) are uniformly bounded in \(L^1(X, \mathbb{R}^3)\) for \(t\) in the interval of existence of the solution.
4) As an instructive example, let us explicitly compute the Lax operator for the profile \( \hat{Q}_v : \mathbb{R} \rightarrow S^2 \) of a traveling solitary wave of degree \( m = 1 \); see Section 2 below for more details on solitary waves for the half-wave maps equation. As a traveling solitary wave profiles of degree \( m = 1 \) with velocity \( v \in \mathbb{R} \) and \( |v| < 1 \), we can take (without loss of generality) the following function

\[
\hat{Q}_v(x) = (\alpha_v f(x), \alpha_v g(x), v) = \left( \frac{x^2 - 1}{1 + x^2}, \frac{-2x}{1 + x^2}, v \right) \quad \text{with} \quad \alpha_v = \sqrt{1 - v^2}.
\]

With the help of the singular integral expression for \( H \), we verify the commutator formulas

\[
[H, f]u = \langle \varphi, u \rangle \varphi + \langle \varphi, u \rangle \psi, \quad [H, g]u = -\langle \varphi, u \rangle \varphi + \langle \psi, u \rangle \psi,
\]

with the functions \( \varphi(x) = \sqrt{\frac{2}{\pi(1 + x^2)}}, \psi(x) = \sqrt{\frac{2}{\pi(1 + x^2)}}, \) and \( \langle \cdot, \cdot \rangle \) being the scalar product on \( L^2(\mathbb{R}; \mathbb{C}) \). Hence it follows that the corresponding Lax operator for \( \hat{Q}_v \) is found to be

\[
L_{Q_v} = [H, \alpha_v f] \sigma_1 + [H, \alpha_v g] \sigma_2 + [H, v] \sigma_3 = \alpha_v \left[ \begin{array}{ccc} 0 & 0 & \frac{[H, f] - i [H, g]}{\alpha_v} \\ \frac{[H, f] + i [H, g]}{\alpha_v} & 0 & 0 \\ 0 & 0 & 0 \end{array} \right]
\]

acting on \( L^2(\mathbb{R}; \mathbb{C}^2) = L^2(\mathbb{R}; \mathbb{C}) \oplus L^2(\mathbb{R}; \mathbb{C}) \). It is evident that the range of \( L_{Q_v} \) belongs to the four-dimensional space spanned by orthonormal basis \( \{ (\zeta_1, \zeta_2, \zeta_3), (\zeta_4, \zeta_5, \zeta_6) \} \). With respect to this basis, it is easy to check that \( L_{Q_v} \) has the corresponding matrix

\[
M = \alpha_v \left[ \begin{array}{ccc} 0 & 0 & +i \sqrt{2} \\ 0 & 1 & -i \\ -i & 1 & 0 \end{array} \right] + \alpha_v \left[ \begin{array}{ccc} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{array} \right].
\]

Its eigenvalues (counting multiplicities) are found to be \( \lambda_1 = -2 \alpha_v, \lambda_2 = \lambda_3 = 0, \lambda_4 = +2 \alpha_v \). Thus the spectrum of the corresponding Lax operator of \( \hat{Q}_v \) is found to be

\[
\text{spec} (L_{Q_v}) = \{ 0, -2 \alpha_v, +2 \alpha_v \} \quad \text{with} \quad \alpha_v = \sqrt{1 - v^2}.
\]

Also note that \( \text{Tr}(L_{Q_v})^2 = 8 \alpha_v^2 = (1 - v^2) E(\hat{Q}_v)/\pi \), which is of course in accordance with relation \([13, 20]\) above and identity \([20]\) below.

Using Kronecker’s theorem characterizing finite rank Hankel operators \([11]\), another fundamental consequence of Theorem \([13]\) is as follows.

**Corollary 2.** Let \( \tilde{S} = \tilde{S}(t, x) \) solve \([\text{HWM}]\) for every \( t \) in an interval \( I \) containing 0.

- If \( x \in \mathbb{R} \) and \( \tilde{S}(0, x) \) is a rational function of \( x \), then so is \( \tilde{S}(t, x) \) for every \( t \in I \).
- If \( x \in T \) and \( \tilde{S}(0, x) \) is a rational function of \( e^{ix} \), then so is \( \tilde{S}(t, x) \) for every \( t \in I \).

**Remark.** In fact, the rank of \( L_S \) (which is constant in time by the Lax equation) can be used to bound the number of poles of the rational functions in the components of \( \tilde{S}(t, x) \). In particular, we expect multi-soliton solutions for the half-wave maps equation in the subset of rational solutions.

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2. Link to Haldane–Shastry Models and Solitary Waves

2.1. Relation to Haldane–Shastry Models. Following \([3\) and \(10]\), we explain how the evolution equation \([\text{HWM}]\) can be formally obtained from a discrete quantum spin system of Haldane–Shastry (HS) type by taking the classical (large-spin) limit followed by taking the continuum limit. Let us also mention that HS-type model have a close connection to Calogero–Moser–Sutherland models which have been intensively studied in the past decades; see \([14]\) for a review on these models.

To illustrate this procedure, we consider the periodic setting on \( \mathbb{Z} \). (The formal arguments here carry over to the real line case \( \mathbb{R} \) with some modifications.) Let \( N \geq 2 \) be
an integer and divide \( T \) by introducing equally spaced lattice points \( x_j = 2\pi i k/N \) with \( k = 1, \ldots, N \) and \( x_0 \equiv x_N \) mod \( 2\pi \). At each site \( x_j \in \mathbb{T} \), we attach a quantum spin of size \( s \in \frac{1}{2}\mathbb{N} \) (integer or half-integer) and we consider the quantum Hamiltonian \( H_{HS} \) defined as

\[
H_{HS} = \sum_{j<k}^{N} \frac{1 - \vec{S}(x_j) \cdot \vec{S}(x_k)}{\sin^2 [\frac{1}{2}(x_j - x_k)]},
\]

acting on the Hilbert space \( \mathcal{H} = (\mathbb{C}^{2s+1})^\otimes N \) (the \( N \)-fold tensor product of \( \mathbb{C}^{2s+1} \)). Note that \( \sin^2 [\frac{1}{2}(e^{i\epsilon x_j} - e^{i\epsilon x_k})] = \frac{1}{4}|e^{i\epsilon x_j} - e^{i\epsilon x_k}|^2 \) is proportional to the squared chord distance between the points \( e^{i\epsilon x_j} \) and \( e^{i\epsilon x_k} \) on the unit circle \( S^1 \). Here \( \vec{S}(x_j) = (\hat{S}_1(x_j), \hat{S}_2(x_j), \hat{S}_3(x_j)) \) is the quantum spin operator associated to site \( x_j \), where its entries are given by the generators of the spin-\( s \)-representation of \( su(2) \) acting on \( \mathbb{C}^{2s+1} \), rescaled by \( s^{-1} \) for later convenience. (For \( s = 1/2 \), the operators \( \hat{S}_i \) are given by the Pauli matrices \( \sigma_i \) acting on \( \mathbb{C}^2 \).) We have the general commutation relations

\[
[S_a(x_j), S_b(x_k)] = \frac{i}{s} \varepsilon_{a,b,c} S_c(x_l) \delta_{jk}.
\]

In summary, the above Hamiltonian \( H_{HS} \) defines a Haldane–Shastry type quantum spin chain with long-range \( 1/|x|^d \) interactions of ferromagnetic type (because aligning the spins in the same direction is energetically favorable).

Now we study the classical (large-spin) limit by passing to \( s \to +\infty \) (which can also be viewed as a semi-classical limit with parameter \( \hbar = s^{-1} \to 0 \)). In heuristic terms, this passage amounts to replacing the quantum spins by classical spin variables \( \hat{S}(x_j) \in \mathbb{S}^2 \), i.e., unit vectors in \( \mathbb{R}^3 \). We refer to e.g. to [10] where semi-classical spin limits were rigorously studied in the context spin dynamics (with smooth short-ranged interactions) and partition functions for spin systems, respectively. In summary, we (formally at least) obtain as a classical limit of \( H_{HS} \) the following Hamiltonian

\[
H_{HS}^{(\text{classical})} = \sum_{j<k}^{N} \frac{1 - \hat{S}(x_j) \cdot \hat{S}(x_k)}{\sin^2 [\frac{1}{2}(x_j - x_k)]},
\]

which is defined on the classical phase space \( \Gamma = \prod_{j \in \mathbb{T}} S^2 \) (i.e. the \( N \)-fold cartesian product of \( S^2 \) with itself). On the space \( \Gamma \), we have the canonical Poisson bracket which reads

\[
\{S_a(x_j), S_b(x_k)\} = \varepsilon_{a,b,c} S_c(x_l) \delta_{jk}.
\]

Using that \( |\hat{S}(x_j)|^2 = 1 \), we readily check that the equation of motions \( \partial_t \hat{S}(t, x_k) = \{\hat{S}(t, x_k), H_{HS}^{(\text{classical})} \} \) are found to be

\[
\partial_t \hat{S}(t, x_k) = \hat{S}(t, x_k) \wedge \left( \sum_{j \neq k}^{N} \frac{\hat{S}(t, x_j) - \hat{S}(t, x_j)}{\sin^2 [\frac{1}{2}(x_j - x_k)]} \right)
\]

for every \( k = 1, \ldots, N \). Now if we pass to the continuum limit \( N \to +\infty \) so that the lattices site \( x_j \in T \) range over all of \( \mathbb{S}^2 \), we formally arrive (after a suitable time rescaling \( t \to \text{const} N^{-1/t} \)) at the half-wave maps equation

\[
\partial_t \hat{S} = \hat{S} \wedge |\nabla| \hat{S}
\]

posed on \( T \), where we recall that the singular integral expression for \( |\nabla| \) in the periodic setting is \( (|\nabla| f)(x) = \frac{1}{2\pi} \text{p.v.} \int \frac{f(y) - f(x)}{\sin^2(\pi(x-y)/2)} dy \) for \( x \in T \). A rigorous investigation of this continuum limit procedure passing from \( 2.4 \) to \( 2.0 \) will be addressed in [1].

With regard to complete integrability, let us mention that the quantum Hamiltonian \( H_{HS} \) is known to admit a (quantum) Lax pair; see [14] for a review on Haldane–Shastry models. For instance, if we take \( s = 1/2 \), the (quantum) Lax operator \( \hat{L} \) has operator-valued entries that read

\[
\hat{L}_{jk} = i(1 - \delta_{jk}) \frac{1 + \hat{S}(x_j) \cdot \hat{S}(x_k)}{x_j - x_k} \in \mathbb{C}^{2 \times 2} \quad \text{for } k, l = 1, \ldots, N.
\]

However, it seems not to be a straightforward procedure (not even formally) to deduce the right expression for a Lax operator for the classical models given by \( H_{HS}^{(\text{classical})} \) (discrete
LAX PAIR FOR THE HALF-WAVE MAPS EQUATION

2.2. Traveling Solitary Waves. A remarkable fact recently found in [8, 16] is that the half-wave maps equation admits non-trivial traveling solitary wave solutions
\[ S(t, x) = \vec{Q}_v(x - vt), \]
where the parameter \( v \in \mathbb{R} \) denotes the velocity. It is easy to check that the profile \( \vec{Q}_v = \vec{Q}_v(x) \) has to satisfy the nonlinear equation
\[ \vec{Q}_v \wedge |\nabla|\vec{Q}_v - v\partial_x \vec{Q}_v = 0. \]

For the special case of vanishing velocity \( v = 0 \), we obtain the so-called half-harmonic maps equation \( \vec{Q} \wedge |\nabla|\vec{Q} = 0 \). In fact, this equation was recently introduced in [2] by a completely different motivation coming from conformally invariant problems in the study of PDE.

From [8] we recall the following explicit classification result for profiles \( \vec{Q}_v : \mathbb{R} \to S^2 \) with finite energy.

- If \( |v| < 1 \), then any profile \( \vec{Q}_v : \mathbb{R} \to S^2 \) with finite energy of the form
  \[ \vec{Q}_v(x) = \left( \sqrt{1 - v^2} \text{Re} B(x), \sqrt{1 - v^2} \text{Im} B(x), v \right) \]
  up to rotations on \( S^2 \) and a complex conjugation symmetry. Here \( B = B(x + iy) \) is a finite Blaschke product defined on the (closed) upper complex plane \( \mathbb{C}_+ \), i.e., we have
  \[ B(z) = \prod_{k=1}^{m} \frac{z - z_k}{z - \overline{z}_k}, \]
  with some \( m \in \mathbb{N} \) and \( z_1, \ldots, z_m \in \mathbb{C}_+ \). Note that \( m = 0 \) corresponds to the case of trivial constant profile \( \vec{Q}_v \).

- If \( |v| \geq 1 \), then any profile \( \vec{Q}_v : \mathbb{R} \to S^2 \) with finite energy is trivial, i.e.,
  \[ \vec{Q}_v(x) \equiv \vec{P} \]
for some \( \vec{P} \in S^2 \).

The arguments in [8] exploit a close connection to minimal surfaces inside the unit ball, where \( \vec{Q}_v(x) \) arise as a boundary curve on \( S^2 \). We remark that the dependence on \( v \) in the expression for \( \vec{Q}_v \) can be indeed be regarded as a Lorentz boost implemented by the conformal group (i.e. the Möbius group) acting on the target sphere \( S^2 \); see [8]. Furthermore, the energy of \( \vec{Q}_v \) is found to be quantized by multiples of \( \pi \) such that
\[ E[\vec{Q}_v] = (1 - v^2) \cdot \pi m. \]
Thus there exist traveling solitary waves for (HWM) with arbitrarily small energy. This is in stark contrast to other energy-critical dispersive geometric PDEs (e.g. energy-critical Schrödinger maps, wave maps, and Yang–Mills equations) where a certain energy threshold exists, below which finite-energy solutions scatter to “free” solutions.

Finally, we refer to [8] for a complete spectral analysis of the linearized operator \( \vec{Q}_v \) in the static case \( v = 0 \).

3. Extension of Results to Target \( \mathbb{H}^2 \)

A variant of geometric interest of the half-wave maps equation occurs when the target two-sphere \( S^2 \) is replaced by the hyperbolic plane \( \mathbb{H}^2 \), which is a non-compact Kähler manifold. To formulate the corresponding evolution equation, we regard \( \mathbb{H}^2 \) as embedded into Minkowski three-space \( \mathbb{R}^{1,2} \) as a unit pseudosphere with positive component \( X_1 > 0 \), i.e., we set
\[ \mathbb{H}^2 = \left\{ X \in \mathbb{R}^{1,2} : -X_1^2 + X_2^2 + X_3^2 = -1, \ X_1 > 0 \right\}. \]
In particular, we find that

\[ \tilde{\mathbf{X}} \cdot \eta \mathbf{Y} = \eta \tilde{\mathbf{X}} \cdot \mathbf{Y} = -X_1 Y_1 + X_2 Y_2 + X_3 Y_3. \]

Likewise, we introduce the cross-type product for vectors \( \tilde{\mathbf{X}}, \tilde{\mathbf{Y}} \in \mathbb{R}^{1,2} \) by setting

\[ \tilde{\mathbf{X}} \wedge \eta \tilde{\mathbf{Y}} = \eta \tilde{\mathbf{X}} \wedge \tilde{\mathbf{Y}} = (-X_2 Y_3 - X_3 Y_2, X_3 Y_1 - X_1 Y_3, X_1 Y_2 - X_2 Y_1). \]

Now the half-wave maps equation on \( \mathbb{R} \) or \( \mathbb{T} \) with hyperbolic plane target \( \mathbb{H}^2 \) is given by

\[ \partial_t \hat{\mathbf{S}} = \tilde{\mathbf{S}} \wedge \eta |\nabla| \hat{\mathbf{S}}. \]

This is a Hamiltonian equation with the corresponding conserved energy

\[ E_\eta[\hat{\mathbf{S}}] = \frac{1}{2} \int \tilde{\mathbf{S}} \cdot \eta |\nabla| \hat{\mathbf{S}} - \frac{1}{2} \int (-\eta \mathbf{S}_1 |\nabla| \mathbf{S}_1 + \eta \mathbf{S}_2 |\nabla| \mathbf{S}_2 + \eta \mathbf{S}_3 |\nabla| \mathbf{S}_3). \]

Note that \( E_\eta \) is not positive definite due to indefinite scalar product \( \cdot \eta \).

To show that this equation admits a Lax pair, we proceed as follows. Consider the matrices

\[ \rho_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \]

which span the Lie algebra \( \mathfrak{su}(1,1) \). Now given a vector \( \tilde{\mathbf{X}} \in \mathbb{R}^{1,2} \), we define the complex \( 2 \times 2 \)-matrix given by

\[ \hat{\mathbf{X}} = \tilde{\mathbf{X}} \cdot \rho = \sum_{j=1}^{3} X_j \rho_j = \begin{pmatrix} iX_1 & X_2 + iX_3 \\ X_2 - iX_3 & -iX_1 \end{pmatrix}. \]

It is straightforward to verify that

\[ (\hat{\mathbf{X}} \cdot \rho)(\tilde{\mathbf{Y}} \cdot \rho) = (\tilde{\mathbf{X}} \cdot \eta \tilde{\mathbf{Y}}) \mathbb{1} + (\tilde{\mathbf{X}} \wedge \eta \tilde{\mathbf{Y}}) \cdot \rho. \]

In particular, we find that \( \hat{\mathbf{S}}^2 = -\mathbb{1} \) if and only if \( \tilde{\mathbf{S}} \cdot \eta \tilde{\mathbf{S}} = -1 \). Now, we can recast \( \text{HWM}_{\mathbb{H}^2} \) into the following form

\[ \partial_t \hat{\mathbf{S}} = \frac{1}{2} \hat{\mathbf{S}}, |\nabla| \hat{\mathbf{S}}. \]

**Theorem 2** (Lax Pair for Half-Wave Maps with Target \( \mathbb{H}^2 \)). Let \( \hat{\mathbf{S}} = \hat{\mathbf{S}}(t,x) \) be a sufficiently regular solution of the half-wave maps equation \( \text{HWM}_{\mathbb{H}^2} \), where either \( x \in \mathbb{R} \) or \( x \in \mathbb{T} \). Then the following Lax equation holds true:

\[ \frac{d}{dt} L \hat{\mathbf{S}} = i[B \hat{\mathbf{S}}, L \hat{\mathbf{S}}], \]

where the operators \( L \hat{\mathbf{S}} \) and \( B \hat{\mathbf{S}} \) are given by \( \text{HWM}_{\mathbb{H}^2} \) with \( \mathbf{S} \) replaced by \( \hat{\mathbf{S}} \).

Note the factor of \( i \) on the right-hand side. In general, the Lax operator \( L \hat{\mathbf{S}} \) is neither symmetric nor skew-symmetric anymore due to the fact that \( \hat{\mathbf{S}} \) is neither Hermitian nor anti-Hermitian in general.

**Proof.** The proof of Theorem 2 carries over mutatis mutandis. \( \Box \)

### 4. Summary and Conclusion

We have proved that the half-wave maps equation

\[ \partial_t \hat{\mathbf{S}} = \tilde{\mathbf{S}} \wedge |\nabla| \hat{\mathbf{S}}, \]

where \( \tilde{\mathbf{S}} = \tilde{\mathbf{S}}(t,x) \) is valued into the two-dimensional sphere \( \mathbb{S}^2 \) or the two-dimensional hyperbolic space \( \mathbb{H}^2 \), and \( x \in \mathbb{R} \) or \( x \in \mathbb{T} \), enjoys a Lax pair. The Lax operator is connected to Hankel operators with special matrix symbols. This allowed us to find new conservation laws equivalent to the homogeneous Besov norms \( \| \cdot \|_{p^1/p} \). As another consequence of the Lax equation, we establish that the subclass rational functions of \( x \in \mathbb{R} \) on the line or of \( e^{it} \) on the circle is conserved by the dynamics. More generally, it is expected that the inverse spectral theory of Hankel operators developed in \( [5] \) for studying dynamics of the cubic Szegő equation will be of valuable help in the study of dynamics of this
energy-critical half-wave maps equation. We hope to come back to these questions in a forthcoming paper.

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