Modelling the Particle Stress at the Dilute-intermediate-dense Flow Regimes: A Review

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Abstract

Gas-solid flow systems are found in many industrial applications such as catalytic reactions, pneumatic conveying, granulation, crystallisation, mineral classification, etc. The operational hydrodynamics can vary depending on the operation method from fast dilute flow, which is dominated by collisional particle-particle contacts, to dense slow flow, which is dominated by sustained frictional contacts. For many years, the former has been successfully modelled using the classic kinetic theory for granular flow, while the latter has been modelled based on soil mechanics principles. At the intermediate-dense regime, three different modelling approaches are identified: (1) the kinetic-frictional model using an ad hoc patching together of the stress from the two limiting regimes at a specific solid fraction (Johnson and Jackson, 1978; Ocone et al., 1993; Syamlal et al., 1993); (2) the switching from one regime to another using different solid stress formulations (Laux, 1998; Makkawi and Ocone, 2005); and (3) the new emerging fluid mechanic approach which allows for a smooth transition from one regime to another using a unified model (Tardos et al., 2003, Savage, 1998). In this study, a one-dimensional fully developed gas-solid flow model for horizontal flow will be used to review the various treatments of solid stresses, and the sensitivity of the flow predictions to the frictional stress will be assessed.

Key words: Gas-solid flow; Kinetic theory; Frictional stress; Intermediate regime; Hydrodynamic modelling

1. Introduction

The last two decades have seen a growing interest in the modelling of dense gas-solid flow where kinetic, collisional and frictional particle stresses coexist. Various forms of constitutive relations for these stresses have been reported in the literature. In 1984, Lun et al. reported the most widely used comprehensive model for slightly inelastic granular flow supported by the corresponding constitutive relations for the energy balance. The model was developed in analogy with the kinetic theory of gases, and it incorporates kinetics as well as collisional contributions to the particle stress and energy generation, conduction and dissipation. In 1987, Johnson and Jackson introduced a new constitutive frictional-collisional model for plane shearing to account for the slow deformation of particles in dense flow. The model employs a frictional stress formula adopted from the critical state theory of soil mechanics. Accordingly, the resulting particle stress was assumed as the sum of the kinetic and frictional contribution at a hypothetical critical particle concentration. In 1993, Ocone et al. and Syamlal et al. reported further development of this approach. In 1998, Laux developed an alternative frictional stress relation and employed it in the simulation of dense gas-solid flow using a complete switching approach at a critical concentration. Most recently, Tardos et al. (2003) proposed new constitutive relations for the particle shearing and energy dissipation developed for the geometry of a Couette device. These relations were used to simulate the rheological behaviour of powder flow in the so-called “intermediate regime”. The model was basically developed to replace the additive approach and presents a smooth merging between rapid-intermediate-dense flows.
In this study, we present a critical review of the various reported closure equations for particle stress. We begin by writing the mathematical form for each contributing stress; then we show the different approaches for merging these contributions in a unified particle stress constitutive equation. The different approaches for the particle stress along with the corresponding energy balance constitutive equations are then tested in the simulation of gas-particle flow in a horizontal duct using the two-fluid models. Finally, we assess and compare the various predicted hydrodynamics by each model.

2. Modelling the particle stress

2.1 Kinetic stress

At very dilute concentrations, the particles stress is dominated by the rate-dependent “viscous” contribution. According to Lun et al. (1984), this was described as kinetic stress, and its contribution to the total shear stress is given by:

\[ \tau_{\text{S-kinetic}} = 2\mu_{\text{kinetic}} S \]  

(1)

where \( S \) is the strain rate and defined as follows:

\[ S = \frac{1}{2} \left( \nabla v + (\nabla v)^T \right) \]  

(2)

and the kinetic viscosity is:

\[ \mu_{\text{kinetic}} = \frac{5\sqrt{\pi}}{96} \rho \frac{d_v}{\eta} \left( \frac{1}{(2-\eta)} + \frac{1}{9} + \frac{8}{5}\eta \alpha (3\eta - 2) \right) \]  

(3)

The kinetic contribution to the total solid pressure is given by:

\[ p_{\text{S-kinetic}} = \rho_s \alpha \theta \]  

(4)

2.2 Collisional stress

For moderate concentrations, and because of the slightly inelastic collisions between particles, a second contribution term, designated here as collisional stress, was described by Lun et al. (1984) as follows:

\[ \tau_{\text{s-collision}} = 2\mu_{\text{collision}} S \]  

(5)

where

\[ \mu_{\text{collision}} = \frac{5\sqrt{\pi}}{96} \rho \frac{d_v}{\eta} \left( \frac{1}{(2-\eta)} \left( \frac{8}{5}\eta \alpha + \frac{64}{25}\eta^2 \alpha^2 (3\eta - 2) \right) + \frac{768\alpha^2 \eta \eta_{\text{c}}}{25\pi} \right) \]  

(6)

The quantity in the first bracket of Eqs. 3 and 6 represents the shear viscosity for perfectly elastic particles. The collisional solid pressure is given by:

\[ P_{\text{collision}} = 4\rho_s \eta g_s \alpha^2 \theta \]  

(7)

2.3 Frictional stress

At very high solid concentrations, the grain inertia is negligible and sustained contacts control the flow. The majority of the available friction models are based on the critical state theory of soil mechanics (Atkinson and Bransby, 1978; Syamlal, 1993; Ocone et al., 1993). Here, the shear stress is described in terms of a “frictional viscosity” such that:

\[ \tau_{\text{S-friction}} = 2\mu_{\text{friction}} S \]  

(8)

On the basis of soil mechanics principles, Schaeffer (1978) proposed an equation for the frictional viscosity that satisfies the Coulomb yield condition in dense plane shear such that:

\[ \mu_{\text{friction}} = \frac{\sqrt{2} p_{\text{friction}} \sin \phi}{|S|} \]  

(9)

where \( \phi \) is the angle of internal friction. \( |S| \) is the magnitude of the strain rate and is given by:

\[ |S| = \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 + \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 \right]^{1/2} \]  

(10)

Johnson and Jackson (1978) proposed a critical state solid frictional pressure that allows for a slight compressibility with very limited particle concentration change, such that:

\[ p_{\text{S-friction}} = \begin{cases} 0 & \text{if } \alpha < \alpha_{\text{critical}} \\ A \left( \alpha - \alpha_{\text{critical}} \right)^n & \text{if } \alpha \geq \alpha_{\text{critical}} \end{cases} \]  

(11)

where A, n and p are empirical constants depending on the particle properties and specified by Ocone et al. (1993) as 0.05, 2 and 3, respectively. Johnson et al. (1990) and Ocone et al. (1993) specified the critical solid concentration to be 0.5, they assumed this to be the point where the frictional stress becomes significant.

Similarly, Syamlal et al. (1993) developed and implemented the following equation which they used in the Mfix hydrodynamic code:

\[ \mu_{\text{friction}} = \frac{\sqrt{2} p_{\text{friction}} \sin \phi}{\alpha S} \]  

(12)

\[ p_{\text{S-friction}} = \begin{cases} 0 & \text{if } \alpha < \alpha_{\text{critical}} \\ A \alpha (\alpha - \alpha_{\text{critical}})^n & \text{if } \alpha \geq \alpha_{\text{critical}} \end{cases} \]  

(13)
where $A$ and $n$ are equal to $10^{25}$ and 10, respectively. The magnitude of the strain rate is given here by:

$$\dot{\varepsilon} = \left[ \frac{1}{6} \left( \frac{\partial v_x}{\partial x} - \frac{\partial v_y}{\partial y} \right)^2 + \left( \frac{\partial v_x}{\partial x} \right)^2 + \left( \frac{\partial v_y}{\partial y} \right)^2 \right]^{1/2}$$

(14)

Instead of a critical solid concentration of 0.5, here Syamlal et al. (1993) assume that the frictional stress is only important above the critical value of 0.59.

Laux (1998) proposed the following frictional stress viscosity, which is only assumed significant above the critical particle concentration of 0.5:

$$\mu_{\text{friction}} = \frac{6 \sin \phi}{9 - \sin^2 \phi} \frac{|I_{11}|}{2 \sqrt{3} |I_D|}$$

(15)

the invariant $I_{11}$ is defined in terms of the particle pressure and the bulk viscosity such that:

$$I_{11} = 3 \left( \frac{\lambda s v v - \rho_s}{\alpha} \right)$$

(16)

the solid pressure, $\rho_s$, is defined as the sum of kinetic and collisional contributions (i.e. Eq. 4 + Eq. 7). The second invariant of the strain rate, $I_D$, is given by:

$$I_D = \frac{1}{2} S.S$$

(17)

3. Merging the dilute with the intermediate-dense flow

In many gas-solid transport and processes, multi-flow regimes coexist and the particle stress fluctuations may be a result of dominant kinetic, collisional, or frictional stresses or the result of all three mechanisms working together. Different modelling approaches for particle stresses have been developed in the last two decades to provide a unified model covering the whole flow regime. These approaches, which are summarised in Table 1, are discussed below in detail.

3.1 Classic kinetic theory approach

The classic kinetic theory was developed to treat the rapid flows of granular material at low and moderate concentrations. In 1984, Lun et al. reported one of the most widely used kinetic-theory-based model for the solid stress, energy flux and collisional dissipation. The theory, which was developed for slightly inelastic particles, incorporates kinetics as well as collisional contributions to the solid stress. Lun et al. (1984) described their theory as “appropriate for dilute as well as dense concentrations of solids”.

The classic kinetic theory of Lun et al. (1984) assumes the total particle shear stress as the ad hoc patching of the kinetic and collisional contribution described in the previous section:

$$\tau_{s-k} = \tau_{s-kinetic} + \tau_{s-collision}$$

(18)

Similarly, the solid pressure is assumed to arise from kinetic and collisional contributions such that:

$$p_{s-k} = p_{s-kinetic} + p_{s-collision}$$

(19)

Fig. 1 shows the kinetic and collisional contributions to the total particle stress as a function of concentration. While the kinetic stress decreases and almost vanishes close to the maximum packing condition, the collisional stress shows exactly the opposite behaviour. The simple addition of both limiting
Table 1: Summary of particle stress modelling approaches and the corresponding energy balance constitutive relations

| Modelling approach              | Sources                       | Equations                                                                 |
|---------------------------------|-------------------------------|---------------------------------------------------------------------------|
| Classic kinetic theory          | Lun et al. (1984)             | \( \tau_s = \tau_s - \text{kinetic} + \tau_s - \text{collision} \)       |
|                                 |                               | \( p_s = p_s - \text{kinetic} + p_s - \text{collision} \)                   |
|                                 |                               | \( q_s = q_s - \text{kinetic} + q_s - \text{collision} \)                   |
|                                 |                               | \( \gamma_s = \frac{48}{\sqrt{\pi}} \eta (1 - \eta) \frac{p_s}{d} g_b \theta^{3/2} \) |
| Frictional additive             | Johnson and Jackson (1978)\(^a\) | \( \tau_s = \tau_s - \text{friction} \)                                  |
|                                 | Ocone et al. (1993)\(^a\)    | \( p_s = p_s - \text{friction} \)                                         |
|                                 |                               | \( q_s = q_s - \text{friction} \)                                         |
|                                 |                               | \( \gamma_s = \frac{48}{\sqrt{\pi}} \eta (1 - \eta) \frac{p_s}{d} g_b \theta^{3/2} \) |
|                                 |                               | \( \alpha < \alpha_{\text{critical}} \Rightarrow \tau_s - \text{friction} = 0 \) |
| Syamlal et al. (1993)\(^b\)    |                               | \( \alpha < \alpha_{\text{critical}} \Rightarrow \tau_s - \text{friction} = 0 \) |
| Switching                       | Laux (1998)\(^c\)            | \( \tau_s = \max \{ \tau_s - \text{kinetic}, \tau_s - \text{friction} \} \) |
|                                 |                               | \( p_s = p_s - \text{kinetic} + p_s - \text{collision} \)                   |
|                                 |                               | \( q_s = q_s - \text{kinetic} + q_s - \text{collision} \)                   |
|                                 |                               | \( \gamma_s = \frac{48}{\sqrt{\pi}} \eta (1 - \eta) \frac{p_s}{d} g_b \theta^{3/2} \) |
| Makkawi and Ocone (2005)        |                               | \( \tau_s = \max \{ \tau_s, \tau_s - \text{Tardos} \} \)                  |
|                                 |                               | \( q_s = q_s - \text{kinetic} + q_s - \text{collision} \)                   |
|                                 |                               | \( p_s = \text{Hydrostatic pressure} \)                                   |
|                                 |                               | \( \gamma_s = \gamma_s - \text{Tardos} \)                                |
| Smooth Transition               | Tardos et al. (2003)\(^d\)   | \( \tau_s = p_s \sin \phi \tanh \left( \frac{\sqrt{K \pi} \phi}{2} \right) \) |
|                                 |                               | \( q_s = q_s - \text{kinetic} + q_s - \text{collision} \)                   |
|                                 |                               | \( \gamma_s = \sqrt{ \frac{\pi}{2} K \sigma \sin \phi \left( \frac{1}{\sin \phi} + \sec h \left( \frac{\sqrt{K \pi} \phi}{2} \right) \right) } \) |

(a) \( \alpha_{\text{critical}} = 0.5 \)
(b) \( \alpha_{\text{critical}} = 0.59 \)
(c) \( \alpha_{\text{critical}} = 0.5 \)
(d) This is tested here using the particle pressure estimated by the classic kinetic model.

stresses produces a smooth stress curve covering the whole range of particle concentration. However, this approach has a serious drawback in the simulation of very dilute flow close to empty duct condition and very dense flow close to maximum packing condition. In the former case, the total shear stress does not
reduce towards the characteristic shearing behaviour of very dilute flow (i.e. zero particle shearing); in the latter case, because of neglecting frictional shearing stress, the model fails to capture the characteristic feature of slow flow of enduring contacts (i.e. abruptly increasing stress at close to maximum packing condition). In the next section, we will review further research efforts directed towards resolving this deficiency.

3.2 Kinetic-frictional approach

3.2.1 Critical additive approach

This approach is based on the assumptions of Savage et al. (1983, 1998), which assume that the solid stress arises from kinetic, collisional and frictional contributions with each source acting separately. However, the frictional stress is only considered important at high particle concentration such that:

$$\tau_s = \begin{cases} \tau_s - k & \text{if } \alpha < \alpha_{\text{critical}} \\ \tau_s - k + \tau_s - \text{friction} & \text{if } \alpha \geq \alpha_{\text{critical}} \end{cases}$$ (20)

and similarly,

$$p_s = \begin{cases} p_s - k & \text{if } \alpha < \alpha_{\text{critical}} \\ p_s - k + p_s - \text{friction} & \text{if } \alpha \geq \alpha_{\text{critical}} \end{cases}$$ (21)

where $\tau_s - k$ and $p_s - k$ are the sum of the kinetic and collisional contributions as described by the classic kinetic approach.

Ocone et al. (1993) employed this approach for the simulation of gas-solid flow in inclined ducts. Syamlal et al. (1993) employed the same approach for the simulation of bubbling bed behaviour; but due to the dense nature of the flow, they neglected the kinetic contribution to the particle stress as well as its contribution to the energy balance. Fig. 2 shows the variation of particle stress as a function of particle concentration using the frictional additive approach of Ocone et al. (1993). The models show abruptly increasing frictional stress at the critical concentration. At the same point, it is also interesting to note the same behaviour in the classic kinetic stresses contribution, but this time in a decreasing manner. This is mainly related to the considerable increase in particle stresses, which tends to shift the material towards the packing condition. This in turn results in stiff contact between particles and considerable decrease in collisional stresses. Here, the frictional stress compensates for the loss in particle stress and increase of the shearing rate. This approach ensures a smooth transition from the dominant collisional regime to the slow frictional regime.

As described by many researchers, this simple ad hoc approach has no strong physical justification; however, it has shown limited success in the simulation of spouted and bubbling beds (Huulin et al., 2004, Patil et al., 2005). Another critical weakness of this approach lies in the hypothetical assumption of a critical concentration at which transition occurs. The exact value of the critical particle concentration, $\alpha_{\text{critical}}$, remains without any experimental proof. On the other hand, as stated by Campbell (2002), “the solid concentration at which this transition occurs as well as the magnitude of the stresses in the elastic regimes are strong functions of the particle surface friction”, thus casting extra doubts on the generalisation of a critical concentration value.
3.2.2 Switching approach

This method employs switching from the classic kinetic approach to a complete frictional model when the latter is proved dominant. This is done by selecting the maximum of the two acting stresses. Laux (1998) incorporated a frictional stress model in the simulation of dense gas-solid flow, but only considered the maximisation analysis after reaching the critical solid fraction. This is described mathematically as follows:

\[ \tau_s = \begin{cases} \tau_{s-k} & \text{if } \alpha < \alpha_{\text{critical}} \\ \max(\tau_{s-friction}) & \text{if } \alpha \geq \alpha_{\text{critical}} \end{cases} \]  

(22)

This approach was tested by Laux (1998) and Patil et al. (2005) assuming \( \alpha_{\text{critical}} = 0.5 \) with the frictional viscosity \( \mu_{s-friction} \) given by Eq. 15. It is justified by the fact that after reaching the critical state, the particles experience long-term contacts with a very limited chance of collisional and kinetic stress taking place due to the limited free space and considerable drop in the kinetic temperature. Although this approach is backed up by strong physical justification, the precise switching point must be selected with some caution.

We have recently employed a similar switching approach for the gas-solid flow in a horizontal duct. The model allows for a smooth switching between the classic kinetic approach and the intermediate flow shear model of Tardos et al. (2003) (see next section) without the need to specify a critical switching point. The predicted variation of shear stress as a function of the particle concentration is shown in Fig. 3.

\[ \langle \tau_s \rangle = \frac{p_0 \sqrt{\pi} \sin \phi \exp(-K)}{2\sqrt{2\sigma}} \left[ I_0(K) + I_1(K) \right] \]  

(23)

where \( \sigma \) is the standard deviation of the strain rate and related to the granular temperature, \( \theta \), such that \( \sigma = \alpha \theta / d_p \). \( I_0 \) and \( I_1 \) are the modified Bessel functions of the first kind. \( K \) is a parameter defined in terms of the average strain rate and its fluctuation such that:

\[ K = \left[ \frac{\langle S \rangle}{2\sigma} \right]^2 \]  

(24)

The coefficient \( \omega \) was calculated by Savage to be approximately equal to \( 1/4 \). Tardos et al. (2003) also provided further theoretical development for the energy dissipation that smoothly merges the long-term frictional and rapid flow dissipation such that:

\[ \gamma_s = \frac{p_0 \alpha \sqrt{\pi} \sin \phi \exp(-K)}{\sqrt{2}} \left[ \frac{1}{\sin \phi} - I_0(K) - I_1(K) \right] \]  

(25)

Makkawi and Ocone (2005) tested a simplified version of Tardos’ formulation in the simulation of horizontal duct flow using the particle hydrostatic pressure for the pressure term \( p_h \); this is given by:

\[ p_h = \rho_y \int_y^H \alpha dy \]  

(26)

where \( y \) is the distance measured from the lower wall towards the top wall and \( H \) is the duct’s width (see Fig. 6). Surprisingly, further analysis using the classical kinetic pressure of Eq. 19 instead of the hydrostatic pressure showed the same results with negligible differences at the dilute flow region (see Fig. 4). Therefore, in the rest of this review, we used the classical kinetic pressure with the above formulations of Tardos et al. (2003).

Fig. 5 shows the variation of shear stress predicted by Eq. 23 as a function of the particle concentration. A specially developed subroutine was used in the simulation code to compute the modified Bessel functions.
Similar to the classic kinetic theory predictions, here we note a linear increase in the shear stress up to the point of solid fraction $\frac{1}{L_{0.5}}$, whereas at the other extreme of close packing, the shear increases sharply independently of the solid concentration. In between these two limits a smooth transition is evident. This seems to prove the principal hypotheses of the original work of Savage (1984) and the recent development of Tardos et al. (2003). Qualitatively, this approach reduces to the same behaviour as that of the frictional additive approach for dense flow as discussed in Section 3.2; however, it has the advantage of smooth merging between the limiting stress without the need for specifying a critical transition point.

4. Simulation tests

To test the various particle stress modelling approaches discussed in Section 3, we incorporated those in the numerical model reported in Makkawi and Ocone (2005) and summarised here in Table 2.Briefly, this is a two-fluid model for the simulation of one-dimensional particle-gas flow in a horizontal duct. Because Eqs. 23 and 25 are mathematically difficult to use in the numerical solution of the flow momentum equation, Tardos et al. (2003) proposed a simplified version that reduces to the same solution of Eqs. 23 and 25 (see Table 1). These simpler forms will be used in the rest of this review. The schematic diagram of the flow geometry is shown in Fig. 6 and the parameters used in the simulation test are summarised in Table 3.

![Fig. 4](image1.png) Comparison between the hydrostatic pressure and classic kinetic model predictions.

![Fig. 5](image2.png) Predicted particle shear stress using Eq. 24 of Tardos et al. (2003).

![Fig. 6](image3.png) Schematic diagram of the simulation test geometry and orientation.

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![Fig. 6](image3.png) Schematic diagram of the simulation test geometry and orientation.

| Table 2 | Model equations for one-dimensional steady flow in a horizontal duct |
|-----------------|------------------------------------------------------------------|
| **Gas phase**   | X-momentum: $\frac{\partial u}{\partial y} - \frac{\partial P_s}{\partial x} - \beta(u - v) = 0$ |
| **Particle phase** | X-momentum: $\frac{\partial \sigma_s}{\partial y} + \beta(u - v) - \frac{\partial P_s}{\partial x} = 0$ |
|                | Y-momentum: $\frac{\partial \sigma_s}{\partial y} + \alpha \rho_s g = 0$ |
| **Energy balance** | $\tau_s \frac{\partial v}{\partial y} - \frac{\partial}{\partial y} \left( \kappa_s \frac{\partial P_s}{\partial y} \right) - \gamma_s = 0$ |
4.1 Flow operational map:

Fig. 7 shows the solid-gas flow map predicted for a pressure gradient of \( \frac{dP_g}{dx} = -200 \text{ N/m}^3 \). Each curve predicts an increasing solid flow rate to a maximum value, which then decreases rapidly as the solid loading increases. This behaviour is a classic feature of horizontal gas-solid flow at increasing solid loading.

The model by Ocone et al. (1993) predicts exactly the same curve as that of the classic kinetic approach except at high particle concentration, where a slight deviation is noticed and the model terminates earlier due to the incorporation of the frictional stress contribution to the total solid stress close to packing condition \((\alpha>0.5)\). The switching approach of Makkawi and Ocone (2005) shifts away from the classic kinetic predictions due to switching to the model of Tardos et al. (2003) at the intermediate regime. This approach also experiences numerical difficulties as the flow becomes much denser, and hence terminates at an earlier stage. The models of both Syamlal et al. (1993) and Tardos et al. (2003) show critical differences when compared with the classic kinetic approach; they both fail to provide a solution for dilute flow at low particle loading (i.e. close to empty duct condition). This is probably related to the kinetic stress contribution which is the dominant contribution in dilute flow. Syamlal (1993) neglects this component in order to ensure zero particle shearing at zero particle concentration (i.e. empty duct flow). On the other hand, the particle stress of Eq. 23 by Tardos et al. (2003) ideally should reduce to the “fluid-like behaviour” when the parameter \( K \) tends towards zero (i.e. particle fluctuation tends towards infinity or granular temperature tends towards zero). Since in reality such conditions cannot be fulfilled, Eq. 22 may not correctly predict the rapid flow behaviour at \( \alpha < 0.1 \).

At the other end of dense flow, the models of both Syamlal et al. (1993) and Ocone et al. (1993) terminate at a lower particle loading when compared to the classic kinetic approach; this is related to the consid-
erable high increase in particle stress due to the inclusion of frictional stress. Although the proposed frictional stress model of Syamlal et al. (1993) kicks off at a higher particle concentration (at $\alpha > 0.59$), the simulation fails at a much lower particle loading compared with the model by Ocone et al. (1993). This is because the additive frictional solid pressure of Syamlal et al. (1993) (Eq. 13) increases abruptly and reaches unrealistically high values at or just above the critical particle concentration. Detailed comparison between the hydrodynamic features of the flow predicted by each model presented in Section 3 will be discussed below.

4.2 Particle shear stress

Fig. 8 compares the predicted dimensionless shear stress, $\tau_s^* = \tau_s/\rho_p d_p^2 (dv/dy)^2$, as a function of the particle concentration for the different treatments of particle shear stress. Generally, all models show a linear increase in $\tau_s^*$ at the intermediate range between $0.1 < \alpha < 0.5$, but the Tardos model predicts a higher shearing stress. The model by Tardos et al. (2003) is mainly based on the assumptions that both collisional and frictional stresses coexist at the intermediate regime. Hence it is not surprising to see a higher stress in this range when compared with other models which mainly retain the dominant collisional stress only (kinetic stress is almost negligible at the intermediate dense flow). At the dilute region of $\alpha < 0.1$, unlike the classic kinetic model predictions, both models of Tardos et al. (2003) and Syamlal et al. (1993) show a decrease in $\tau_s^*$ as $\alpha$ decreases. This is physically justifiable as the flow approaches the empty duct flow condition; however, such an approach causes numerical difficulties and the model fails to converge, thus limiting the applicability of these two models to intermediate-dense flow. The classic kinetic model shows an increase in $\tau_s^*$ when approaching the zero particle concentration. Lun et al. (1984) discussed this in detail, and they attributed this to the monotonic increase in granular temperature as $\alpha$ approaches zero, and hence results in an undesirable increase in the dominant kinetic shear component at this limit. To clarify this behaviour, we must discuss it in the context of energy balance formulation. In the classic kinetic theory model of Lun et al. (1984), the energy dissipation is only associated with particle-particle collision and hence, in the absence of collisional dissipation and increasing energy generation due to particle fluctuation, the increase in granular temperature is not surprising. This problem was resolved in the model by Tardos et al. (2003) by the inclusion of a continuous energy dissipation model (Eq. 25) which tends to predict high and constant energy dissipation at the dilute regime. Similarly, the model by Syamlal et al. (1993) shows the same trend as a result of neglecting the kinetic contribution to the energy conduction and generation. These features are shown in Fig. 9. The classic kinetic model shows a loop in the predicted energy dissipation at the region of high shear rate, the turning point in the loop at the maximum shearing is a result of an increasing contribution.
of collisional stress as the flow gets denser, then as the shear rate decreases further, the dissipation starts to reduce due to the limited space available for any further collisions. It must be noted that the comparison with the predictions in the model by Ocone et al. (1993) is omitted here because inclusion of the additive frictional stress shows a negligible effect in energy dissipation with respect to the classic kinetic prediction.

Fig. 10 shows the shear stress to particle pressure ratio, \( \delta = \tau_s / \rho_s \), as a function of the particle concentration \( \alpha \). Within the range of \( \alpha < 0.3 \), all models predict a decrease in \( \delta \) with increasing \( \alpha \). Qualitatively, this agrees with the results reported by Lun et al. (1984) for small values of \( \alpha \). Within the range of \( 0.3 < \alpha < 0.59 \) and for all models, \( \delta \) decreases slightly (i.e. almost independent of the variations in \( \alpha \)). Beyond this range, both models of Syamlal et al. (1993) and Ocone et al. (1993) behave differently when compared with the classic kinetic predictions. As discussed earlier, this is attributed to the different treatment of kinetic stress contribution at the dilute flow and of frictional stress at the other extreme of dense flow. The approach by Makkawi and Ocone (2005) employing the smooth transition shear deviates from all other models with a lower value of \( \delta \) which decreases sharply when approaching the extreme end of close packing condition. According to observations made by Campbell and Brennen (1982) and Lun et al. (1984), the latter approach seems to predict the correct shearing to solid pressure ratio at large \( \alpha \) and this is detailed as follows. Campbell and Brennen (1982) observed that at \( \alpha > 0.3 \), the particles tend to rearrange in the form of distinct shear layers. Collisions between particles then become limited due to the limitation of free space, and particle interaction becomes restricted to (a) front and back collisions between particles in the same layer, and (b) top and bottom collisions between sheared layers. Such forms of particle-particle interaction make no effective contribution to the particle shearing but can considerably increase the solid pressure perpendicular to the shearing layer. In fact, Lun et al. (1984) clearly admit that the increase in \( \delta \) at large particle concentration is “no doubt a failure” in the kinetic theory model formulation for the shear stress at this limit.

4.3 Particle velocity, granular temperature and concentration profiles

In this section, we compare the various predicted hydrodynamic features of the flow using the different modelling approaches described in Section 3. These results will be discussed according to five different flow categories: very dilute; dilute; intermediate; dense; very dense. Since the momentum transport properties of solid-gas are strong functions of the granular temperature, the different treatment of energy balance will be discussed first.

In the classic kinetic theory of Lun et al. (1984), the energy balance incorporates both the kinetic and collisional contributions to the energy generation and conduction. The frictional additive approach of
Syamlal et al. (1993) neglects the kinetic contribution, while Ocone et al. (1993) follow the original classic kinetic theory and retain both of the kinetic and collisional contribution to the energy balance. However, both approaches neglect the effect of the additive frictional shearing on the energy balance at the limit of large particle concentration, and no further theoretical development for the frictional dissipation was considered. It must be noted that in employing the smooth merging models, we also considered the effect of possible frictional energy generation and dissipation by incorporating the simplified shear stress and the energy dissipation formulas of Tardos et al. (2003) in the energy balance equation. The particle conductivity coefficient was estimated from the existing classic kinetic theory model.

Fig. 11 shows a comparison of the predicted granular temperature, $\theta$, as a function of the particle concentration using various test models. Generally, all models predict a decreasing $\theta$ as $\alpha$ increases. Within the intermediate range of $0.3<\alpha<0.59$, the additive approach of Ocone et al. (1993) and Syamlal et al. (1993) shows no difference with respect to the kinetic theory predictions. The smooth transition approach of Tardos et al. (2003) consistently predicts a lower $\theta$, mostly due to the increase in energy dissipation. Syamlal et al. (1993) deviate from the classic kinetic predictions at dilute flow due to neglecting the kinetic contribution. At the other dense end, while all models reasonably agree when approaching the maximum packing condition, this model also clearly deviates with a very sharp decrease in granular temperature due to the inclusion of the frictional stress at $\alpha>0.59$.

A comparison between the various modelling approach predictions for the particle velocity, granular temperature and concentration profiles is shown in Fig. 12 and is discussed in the following.

Very Dilute flow (Fig. 12a):
This corresponds to $Q_s/Q_g=33$ (point 1 in the classic kinetic curve of Fig. 7). In this range of flow, it is widely believed that the classic kinetic theory captures the hydrodynamic feature of the flow with a reasonable level of accuracy. Therefore, the result produced from the classic kinetic model is used here as a benchmark for comparison with other model predictions. At this dilute flow, the frictional stress is absent and therefore the model by Ocone et al. (1993) yields identical results to that produced from the classic kinetic theory. The model by Syamlal et al. (1993) clearly overestimates the granular temperature and particle velocity for the reasons discussed in Section 4.2 above. Tardos et al. (2003) reasonably predict the various hydrodynamic features with a better level of accuracy. All models seem to predict almost the same concentration profile.

Dilute flow (Fig. 12b):
This corresponds to $Q_s/Q_g=120$ (point 2 in the classic kinetic curve of Fig. 7). The predictions in the model of Tardos et al. (2003) and the classic kinetic theory model are in good agreement to a great extent. Again, the model of Syamlal et al. (1993) that assumes negligible kinetic contributions to the parti-
Fig. 12  Hydrodynamic predictions of a wide range of gas-solid flow in a horizontal duct predicted by different shear modelling approaches.
cle stress and energy balance clearly overestimates the granular temperature and particle velocity. Also due to the small particle concentration, we do not expect to see any effect of frictional stress in this flow range. The predicted particle concentrations seem to be insensitive to the various applied shearing models in this range.

Intermediate flow (Fig. 12c):
This corresponds to \( Q_s/Q_g = 114 \) (point 3 in the classic kinetic curve of Fig. 7). Here it appears that with a further increase in particle loading, each model starts to behave differently. In this range of flow, collisional effects dominate the particle stresses, and hence the predictions by the model of Syamlal et al. (1993) come closer to the classic kinetic theory model results. On the other hand, the model by Tardos et al. (2003) starts shifting away from the other models with lower predicted granular temperature and particle velocity, and most important, a clear tendency towards asymmetric behaviour. This is mainly due to the fact that this new mechanistic approach was originally developed to take into consideration both the collisional and frictional contribution to the particle stress at the intermediate regime, while the frictional additive approach of Ocone et al. (1993) and Syamlal et al. (1993) neglects the frictional contribution at this flow limit. The asymmetric behaviour is due to the migration of particles towards the lower wall as a result of increasing gravity besides the possible increase in shear stress at the denser lower part of the duct.

Dense flow (Fig. 12d):
This corresponds to \( Q_s/Q_g = 1060 \) (point 4 in the classic kinetic curve of Fig. 7). Here, the frictional additive approach of Syamlal et al. (1993) and Ocone et al. (1993) appears to have no significant effect on the general hydrodynamic behaviour of the flow when compared with the classic kinetic model. The smooth transition model of Tardos et al. (2003) stands alone with significantly reduced granular temperature and particle velocity. Moreover, the particle concentration profile shows well-pronounced segregation behaviour as a result of particle migration towards the lower wall; this in turn is probably a result of the frictional contribution to the particle stress, which tend to slow down the particles’ velocity in the lower half of the duct.

Very dense flow (Fig. 12e):
This corresponds to \( Q_s/Q_g = 2660 \) (point 5 in the classic kinetic curve of Fig. 7). In this limit, the additive frictional stress of the models by Ocone et al. and Syamlal et al. appears to play a role in the rheology of the dense flow. It is here where these two models deviate slightly from the classic kinetic predictions due to the incorporation of dense frictional effect. It is also interesting to note that with the frictional stress taken into consideration, these models predict a higher granular temperature. This is because both models do not allow for the possible energy dissipation resulting from the high frictional stress. The predicted profiles in the model by Tardos et al. (2003) confirm a slow moving layer close to the upper wall, while most of the lower part of the duct is almost stationary with the particle concentration reaching the maximum packing condition.

5. Conclusions
In this study, we reviewed different treatments for the particle stress in gas-solid flow. These models were tested in the simulation of gas-solid flow in a horizontal duct covering the whole range of dilute-intermediate-dense flows. The limitations and applicability of each modelling approach are discussed and supported with the predicted profiles of the hydrodynamic features of the flow. The superiority of any of these modelling approaches over the other within the range of its applicability is a matter of open debate and further experimental validation.

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Notation
\[
\begin{align*}
A & \quad \text{Parameter defined in Eqs. 11 and 13} \quad (-) \\
d_p & \quad \text{Particle diameter} \quad (m) \\
e_p, e_w & \quad \text{Particle-particle and particle-wall restitution coefficient, respectively} \quad (-) \\
g & \quad \text{Gravity acceleration constant} \quad (m \cdot s^{-2}) \\
g_o & \quad \text{Radial distribution function} \quad (-) \\
H & \quad \text{Duct width} \quad (m) \\
I_0, I_1 & \quad \text{Modified Bessel functions of the first kind} \\
I_{11} & \quad \text{Invariant of particle pressure, defined in Eq. 16} \quad (N \cdot m^{-2}) \\
I_0 & \quad \text{Second invariant of the deviator of particle strain rate, defined in Eq. 17} \quad (s^{-2}) \\
k_s & \quad \text{Effective thermal conductivity of particles} \quad (kg \cdot m^{-1} \cdot s^{-1})
\end{align*}
\]
| $K$ | Parameter defined in Eq. 24 | (–) |
| $\rho$ | Parameter defined in Eqs. 11 and 13 | (–) |
| $p_s$ | Parameter defined in Eq. 11 | (N/m$^2$) |
| $P_g$ | Gas pressure | (N/m$^2$) |
| $q_{gs}$ | Pseudothermal energy flux gas flow rate | (kg·s·m$^{-3}$) |
| $Q_g$ | Gas flow rate | (kg·m$^{-2}$·s$^{-1}$) |
| $Q_s$ | Solid flow rate | (kg·m$^{-2}$·s$^{-1}$) |
| $S$ | Strain rate tensor | (s$^{-1}$) |
| $u$ | Gas velocity | (m·s$^{-1}$) |
| $v$ | Solid velocity | (m·s$^{-1}$) |
| $x$ | Axial coordinate | (m) |
| $y$ | Radial coordinate | (m) |

**Greek symbols**

| $\alpha$ | Solid volume fraction | (–) |
| $\mu$ | Viscosity | (N·m$^{-2}$·s) |
| $\rho$ | Density | (kg·m$^{-3}$) |
| $\beta$ | Gas-particle interphase drag coefficient | (–) |
| $\tau$ | Shear stress | (N·m$^{-2}$) |
| $\eta$ | Parameter function of particle-particle restitution coefficient | (–) |
| $\gamma_g$ | Dissipation of granular energy | (kg·m$^{-1}$·s$^{-3}$) |
| $\phi$ | Angle of internal friction for the particle | (degrees) |
| $\theta$ | Granular temperature | (m$^2$·s$^{-2}$) |
| $\sigma$ | Standard deviation of strain rate | (s$^{-1}$) |
| $\omega$ | Coefficient, $\omega = 1/4$ | (–) |
| $\delta$ | Particle shearing to pressure ratio, $\delta = \tau_p/p_s$ | (–) |

**Subscripts**

| $g$ | Solid phase |
| $s$ | Solid phase |
| max | Maximum |
| k | Classic kinetic theory |

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