Spin-Hall effect in a disordered 2D electron-system

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We calculate the spin-Hall conductivity for a two-dimensional electron gas within the self-consistent Born approximation, varying the strength and type of disorder. In the weak disorder limit we find both analytically and numerically a vanishing spin-Hall conductivity even when we allow a momentum dependent scattering. Separating the reactive from the dissipative current response, we find the universal value \( \sigma_{sH} = e/8\pi \) for the reactive response, which cancels however with the dissipative part \( \sigma_{sH}^D = -e/8\pi \).

Spin-orbit coupling in two dimensional electron systems allows a number of unconventional transport phenomena, since charge current and the spin degrees of freedom are coupled. In particular the spin-Hall effect in two-dimensional electron systems, i.e. a spin-current which flows in the plane but perpendicular to the electrical current, and which is polarized perpendicular to the plane, has been discussed intensively over the last year. The spin-Hall conductivity connects the spin-current with an electric field, \( j_y^z = \sigma_{sH} E_x \), where \( j_y^z \) denotes a current in \( y \)-direction with spin-polarization in \( z \)-direction. In a clean two dimensional electron gas, the spin-Hall conductivity was predicted to have a universal value \( \sigma_{sH} = e/8\pi \), independent of the strength of the spin-orbit scattering. Several publications have addressed the issue of whether this result is modified in the presence of impurity scattering. Murakami analyzed the Luttinger-Hamiltonian, which applies to two-dimensional hole gases, and concluded that the spin-Hall conductivity in the limit of weak impurity scattering reproduces the intrinsic value (at least when restricting to \( s \)-wave impurity scattering). For the Rashba model, which applies to two-dimensional electron gases, conflicting results exist in the literature: By applying the standard Green’s function techniques Inoue et al. and Mishchenko et al. concluded that \( s \)-wave impurities suppress the spin-Hall effect in bulk samples even when the disorder broadening of the energy levels is small compared to the spin-orbit splitting. On the other hand, Dimitrova and Chalaev et al., starting from the same model Hamiltonian and applying similar methods, found a non-zero spin-Hall conductivity. Even direct numerical evaluations of the effect do not fully agree with each other: Xiong and Xie found within a scattering matrix approach the universal value of the spin-Hall conductance \( G_{sH} = e/8\pi \) over a large parameter range. Nomura et al. on the other hand found a spin-Hall conductivity of the order of but not identical to the universal value.

In this paper we calculate the spin-Hall conductivity for a bulk sample within the self-consistent Born approximiation. We confirm Refs.\(^8,11,15\), i.e., we find that even a weak disorder suppresses the spin-Hall conductivity. For \( s \)-wave scatterers we calculate the impurity self-energy and the dressed current vertex numerically. This allows us to obtain results beyond the limit \( \epsilon_F \tau \to \infty \), which is accessible analytically. We find that a non-zero spin-Hall conductivity is, in principle, possible although it remains much smaller than \( e/8\pi \).

Our calculations are based on our previous work\(^20,21\), where a number of technical details can be found. In the following we sketch the derivation of the spin-Hall conductivity. The starting point is the Hamiltonian

\[
H = \frac{\hbar^2}{2m} + \alpha \sigma \cdot p \times \mathbf{e}_z,
\]

where the parameter \( \alpha \) describes the strength of the spin-orbit coupling, \( \sigma \) is a vector of Pauli matrices, and \( \mathbf{e}_z \) is a unit vector perpendicular to the two dimensional system. The spin-Hall conductivity is obtained by the standard linear response theory as

\[
\sigma_{sH} = \lim_{\omega \to 0} \frac{e}{2\pi} \int d\epsilon \left[ \text{Tr} \left( \tilde{j}_{sH}^R G^>(\epsilon) j_{sH}^z G^A(\epsilon - \omega) \right) + j_{sH}^R G^>(\epsilon) \tilde{j}_{sH}^z G^<(\epsilon - \omega) \right],
\]

with \( G^>(\epsilon) = f(\epsilon) (G^R - G^A) \), \( f(\epsilon) \) being the Fermi function. In Eq. \#2, the spin- and charge-current operators are given by \( \tilde{j}_s = (1/4) \{ \sigma_j \mathbf{v} + \mathbf{v} \sigma_j \} \) and \( j_s = \mathbf{v} \), respectively. The velocity operator, \( \mathbf{v} \), is obtained from the Hamiltonian \((\#1)\) and reads \( v^y = p^y / m \mp \alpha \sigma_y \). We choose the electron charge as \( -e \) \((\epsilon > 0)\). The trace in Eq. \#2 is over the eigenstates of the Hamiltonian, and the bar indicates that the expression must be averaged over the disorder configurations.

When performing the disorder average, we rely on the self-consistent Born approximation. To begin with, we consider point-like, i.e., pure \( s \)-wave scatterers. The retarded/advanced impurity self-energy is then given by

\[
\Sigma^{R,A} = \frac{1}{2\pi N_0 \tau} \sum_{\mathbf{p}} G^{R,A}(\mathbf{p}).
\]
Due to the spin-orbit coupling, the Green’s functions have a non-trivial structure in the spin-space, although the self-energy remains diagonal. Explicitly, one finds that $\Sigma_{sx'} = \Sigma_0 \delta_{sx'}$, $G_{ss'} = G_0 \delta_{ss'} + G_1 \sigma^x_{ss'} + G_2 \sigma^y_{ss'}$, with

$$G_0(p) = \frac{1}{\pi}(G_+ + G_-) \quad (4)$$
$$G_1(p) = \frac{p_x}{2p}(G_+ - G_-) \quad (5)$$
$$G_2(p) = -\frac{1}{2p}(G_+ - G_-) \quad (6)$$
$$G_\pm = \left(\epsilon + \mu - \frac{p^2}{2m} \pm \alpha p - \Sigma_0\right)^{-1}. \quad (7)$$

By taking the zero frequency limit of Eq. (2), the spin-Hall conductivity reads

$$\sigma_{sH} = \frac{\epsilon}{4\pi} \sum_p \text{Tr}_\sigma \left[ 2j_y^y G^R(p) J^x_\text{c} G^A(p) \right], \quad (8)$$

since terms of the type $G^R G^R$ and $G^A G^A$ contribute only in the order $(1/\epsilon_F \tau)^2 (\alpha/\tau)^2$ and can be safely neglected in the limit $\alpha p_F \ll \epsilon_F$ and/or for weak disorder $\epsilon_F \tau \gg 1$. The charge current $J^x_\text{c}$ has to be calculated including the vertex corrections, $J^x_\text{c} = p^x/m + \Gamma^x$, compare Eq. (33) of Ref. 9. In the case of s-wave impurity scattering, the momentum dependent part of the current vertex is not renormalized, while the momentum independent, but spin-dependent part, $\Gamma^x$, is obtained by solving the set of equations

$$\Gamma^x_{ss'} = \gamma^x_{ss'} + \frac{1}{2\pi N_0 \tau} \sum_{p, a} G^R_{sa} \Gamma_{ab} G^A_{bs'}, \quad (9)$$

with the effective bare vertex given by

$$\gamma^x_{ss'} = -\alpha \sigma^y_{ss'} + \frac{1}{2\pi N_0 \tau} \sum_{p, a} G^R_{sa}(p) \frac{p_x}{m} G^A_{as'}(p). \quad (10)$$

By expanding $\Gamma^x_{ss'} = \sum_{\mu} \Gamma^x_{\mu} \sigma^\mu_{ss'}$ in Pauli matrices we obtain the spin-Hall conductivity as

$$\sigma_{sH} = -\frac{e}{\pi} \Gamma^x \text{Im} \sum_p \frac{p_x}{m} G^R_0(p) G^A_1(p). \quad (11)$$

Performing the momentum integration Eq. (11) under the restriction that $\alpha p_F \ll \epsilon_F$ and $\epsilon_F \tau \gg 1$ leads to

$$\sigma_{sH} = -\frac{e}{\pi} \Gamma^x \pi N_0 \tau \frac{\alpha p_F \epsilon_F \tau}{1 + 4\alpha^2 p_F^2 \tau^2}. \quad (12)$$

Apparently $\sigma_{sH}$ goes to zero when the spin-splitting, $\alpha p_F$, is small compared to the disorder broadening of the levels, $1/\tau$. If we neglect vertex corrections, i.e., if we insert in Eq. (11) the bare vertex $\Gamma^x = -\alpha$, we find

$$\sigma_{sH}|_{\text{bare vertex}} = \frac{e}{8\pi} \frac{4\alpha^2 p_F^2 \tau^2}{1 + 4\alpha^2 p_F^2 \tau^2}. \quad (13)$$

i.e., the universal value $\sigma_{sH} = \epsilon/8\pi$ is recovered in the weak disorder limit. On the other hand, by inserting the dressed vertex $\Gamma^x = 0$ as calculated in Refs. 20, 21 one finds that $\sigma_{sH} \approx 0$. Our result then agrees with that found in Refs. 11, 15.

As explained in Ref. 21, the vanishing of the dressed vertex $\Gamma^x = 0$ is due to the fact that the integral on the right-hand side of Eq. (11) gives $\approx \alpha \sigma^y$, making the effective bare vertex $\gamma$ itself to vanish. A more careful numerical evaluation of the integral for arbitrary disorder strength actually shows that the compensation of the two terms in Eq. (11) is exact only in the weak disorder limit, i.e., $\epsilon_F \tau \gg 1$. In Fig. (1) we show the dressed vertex as a function of disorder. $\Gamma^x$ goes to zero as $1/\epsilon_F \tau \rightarrow 0$, nothing special is observed as $\alpha p_F \sim 1/\tau$, and even in the strong disorder limit $\Gamma^x$ remains much smaller than its bare value ($-\alpha$). We conclude that although, in principle, a non-zero spin-Hall conductivity may be obtained, one expects a much smaller value than the universal one.

Next we address the question whether the spin-Hall effect is sensitive to the type of disorder potential. Inoue et al. 11 argued that $\sigma_{sH}$ may be non-zero for long-range defect potentials, although an explicit result has not been given. In our calculation we follow again closely 21 where all the details can be found. Here we assume weak disorder, so that the inequalities $\epsilon_F \gg \alpha p_F \gg 1/\tau$ hold. In the following, we work in the eigenstate basis of the Hamiltonian \[1\]

$$|p\pm\rangle = \frac{1}{\sqrt{2}} \{\pm \exp(-i\varphi)|p\uparrow\rangle + |p\downarrow\rangle\} \quad (14)$$

where $\tan(\varphi) = p_y/p_x$ and the corresponding eigenvalues are $E_{\pm} = \epsilon^2/2m \pm \alpha p$. In this basis, the matrix elements
of the current operators read
\[ \langle p \pm | j_e^p | p \pm \rangle = 0 \quad (15) \]
\[ \langle p \pm | j_e^p | p \mp \rangle = -\frac{1}{2m} p \sin(\varphi) \quad (16) \]
\[ \langle p \pm | j_e^p | p \mp \rangle = \left( \frac{p}{m} \pm \alpha \right) \cos(\varphi) \quad (17) \]
\[ \langle p \mp | j_e^p | p \mp \rangle = \mp i \alpha \sin(\varphi). \quad (18) \]

To use Eq. (8) we need the dressed charge operator, \( J_e^x \). Since, as seen from the above equations, the spin-current operator is off-diagonal in the eigenstate basis we get the spin-Hall conductivity in the form
\[ \sigma_{sH} = -\frac{e}{\pi} \sum_p \text{Re} \left[ \langle p + | j_e^p | p - \rangle \langle p - | J_e^x | p + \rangle G^R(p) G^A(p) \right]. \quad (19) \]

To calculate the dressed current operator we make use of the assumption that the spin-orbit splitting is large compared to the impurity broadening of the levels, \( \alpha_{PF} \gg 1/\tau \). The off-diagonal matrix elements of the current operator are then obtained in terms of the diagonal ones,
\[ \langle p \mp | j_e^x | p \pm \rangle = \langle p \mp | j_e^x | p \pm \rangle + \sum_{p', m} \left[ \langle p \mp | V | p' m \rangle \langle p' m | V | p \pm \rangle \times G^R_m(p') G^A_m(p') \langle p' m | J_e^x | p' m \rangle \right]. \quad (20) \]

The diagonal matrix elements on the other hand were already considered in Ref.\(^\text{23}\), and are obtained from the equation
\[ \langle p \pm | j_e^x | p \pm \rangle = \langle p \pm | j_e^x | p \pm \rangle + \sum_{p', m} |\langle p \pm | V | p' m \rangle|^2 \times G^R_m(p') G^A_m(p') \langle p' m | J_e^x | p' m \rangle. \quad (21) \]

We consider impurity scattering which conserves spin, but allow the scattering amplitude to be momentum-transfer dependent. Such a dependence appears as a product of two contributions. The first is due to the type of disorder potential one considers, \( V_{p,p'} \), while the second is induced by the transformation to the eigenstate basis. The latter gives rise to the following matrix elements
\[ \langle p - | V | p' \pm \rangle \langle p' \pm | V | p + \rangle = \pm \frac{1}{2} \sin(\varphi - \varphi') |V_{p,p'}|^2 \quad (22) \]
and
\[ |\langle p \pm | V | p' \pm \rangle|^2 = \frac{1}{2} |V_{p,p'}|^2 (1 + \cos(\varphi - \varphi')) \quad (23) \]
\[ |\langle p \pm | V | p' \mp \rangle|^2 = \frac{1}{2} |V_{p,p'}|^2 (1 - \cos(\varphi - \varphi')) \quad (24) \]

We assume that the scattering amplitude \( V_{p,p'} \) depends weakly on the momentum transfer, so that the scattering probability depends only on the angle between the incoming and scattered particle. Under this condition we can expand the scattering probability as
\[ |V_{p,p'}|^2 = V_0 + 2V_1 \cos(\varphi - \varphi') + 2V_2 \cos(2\varphi - 2\varphi') + \ldots. \quad (25) \]

To obtain the spin-Hall conductivity the two momentum integrations over \( p \) and \( p' \) have to be performed. We split the momentum integration in an integral over the energy \( \xi = p^2/2m - \mu \) and the angular variable \( \varphi \),
\[ \sum_p \rightarrow N_0 \int \frac{d\xi}{\tau} \int \frac{d\varphi}{2\pi}. \quad (26) \]
and find
\[ N_0 \int d\xi G^R_{\pm} G^A_{\pm} = \frac{2\pi N_0}{\alpha_{PF} + 1/\tau} \approx \frac{i\pi N_0}{\alpha_{PF}} \quad (27) \]
\[ N_0 \int d\xi G^R_{\pm} G^A_{\pm} = 2\pi N_0 \tau_{\pm}, \quad (28) \]
where \( N_\pm \) and \( \tau_{\pm} \) are the density of states and the lifetime in the two subbands. Notice that the first of the two integrals is correct only to the lowest order in the (small) parameter \( \alpha_{PF}/\epsilon_F \), whereas the second integration is valid beyond that limit. Finally the spin-Hall conductivity is determined as
\[ \sigma_{sH} = \frac{e}{8\pi} \left[ 1 - \frac{1}{4\alpha} \frac{N_+ \tau_+ J_e^x - N_- \tau_- J_e^x}{V_0} \right]. \quad (29) \]
where the second term is due to vertex corrections. When calculating the product \( N_\pm \tau_{\pm} J_e^x \) to zero order in \( \alpha_{PF}/\epsilon_F \) the vertex corrections disappear. Expanding the density of states, the scattering time and to dressed current operator to first order yields
\[ N_\pm \approx N_0 \left( 1 + \frac{\alpha_{PF}}{2\epsilon_F} \right) \quad (30) \]
\[ \tau_{\pm} \approx \tau \left( 1 \pm \frac{V_1}{V_0} \frac{\alpha_{PF}}{2\epsilon_F} \right) \quad (31) \]
\[ J_e^x \approx \frac{V_0}{V_0 - V_1} \frac{\alpha_{PF}}{m} \pm \alpha \frac{V_0 + V_2}{V_0 - V_2}. \quad (32) \]

Notice that the dressed current operator, to the leading order in \( \alpha \), is of the familiar form \( J = j_{\tau x} / \tau \), where \( \tau_{\tau x} \) is the transport scattering time. By combining all the terms, one then finds that the spin-Hall conductivity vanishes as in the case of pure s-wave scattering, \( \sigma_{sH} = 0 \).

As a last useful observation, we separate the reactive and dissipative contributions to the current response, \( \sigma_{sH} = \sigma^R_{sH} + \sigma^D_{sH} \) where
\[ \sigma^R_{sH} = \lim_{\omega \rightarrow 0} \frac{e}{\omega} \int \frac{d\epsilon}{2\pi} \text{Tr} \left[ j^R_\text{R}(\epsilon) j^x_\text{R}(\epsilon) G^A(\epsilon - \omega) \right] \]
\[ + j^R_\text{R} \text{Re} G^R(\epsilon) j^x_\text{R} G^<(\epsilon - \omega) \quad (33) \]
\[ \sigma^D_{sH} = -\frac{e}{\pi} \text{Tr} \left[ j^R_\text{s} \text{Im} G^R j^x_\text{s} \text{Im} G^R \right]. \quad (34) \]
Since the zero frequency spin-Hall conductivity is real, the terms with imaginary (real) current matrix elements contribute to $\sigma_{R}^{sH}$ ($\sigma_{D}^{sH}$), respectively. It then follows that the first term on the right hand side of Eq. (29) corresponds to $\sigma_{R}^{sH} = e/8\pi$, whereas the second term (the vertex corrections) is the dissipative response with

$$\sigma_{D}^{sH} = -\frac{e}{8\pi} \frac{1}{4\alpha} \frac{N_{+}\tau_{+}J_{x}^{z} - N_{-}\tau_{-}J_{x}^{z} V_{2} - V_{0}}{N_{0}\tau V_{0}} = -\frac{e}{8\pi}$$

and only the sum of the reactive and dissipative response is zero.

In summary, we calculated the spin-Hall conductivity in a two dimensional electron gas within the self-consistent Born approximation, including the vertex corrections in the ladder approximation. We remark that, although a number of similar studies exist in the recent literature, the final conclusions are often contradictory. This may be due to the fact that the relevant integrals depend in a very subtle way on the type of the physical limit considered. For this reason in this work we evaluated all the relevant integrals both analytically and numerically. This allowed us to confirm the conclusions of Refs. 8,11,15. In particular, we find that the spin-Hall conductivity is strongly suppressed below the universal value of $e/8\pi$. Furthermore we have demonstrated that the result is not only valid for pure $s$-wave scattering, but is robust upon the inclusion of a weak momentum dependence of the scattering probability.

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