KP Solitons from Tropical Limits
joint with Agostini D., Fevola C., and Sturmfels B.

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The Kadomtsev-Petviashvili equation

The KP equation is a PDE that describes the motion of water waves

$$\frac{\partial}{\partial x} (4p_t - 6pp_x - p_{xxx}) = 3p_{yy}$$

where $p = p(x, y, t)$

Taken in Nuevo Vallarta, Mexico by Mark J. Ablowitz
Connection to Algebraic Curves

We seek solutions of the form

\[ p(x, y, t) = 2 \frac{\partial^2}{\partial x^2} \log \tau(x, y, t) \]

where \( \tau(x, y, t) \) satisfies the Hirota’s differential equation

\[ \tau \tau_{xxxx} - 4 \tau_{xxx} \tau_x + 3 \tau_{xx}^2 + 4 \tau_x \tau_t - 4 \tau \tau_{xt} + 3 \tau \tau_{yy} - 3 \tau_y^2 = 0 \]

- One can construct \( \tau \)-functions from an algebraic curve \( C \) of genus \( g \)
Definition

The **Riemann theta function** is the complex analytic function

$$\theta = \theta(z | B) = \sum_{c \in \mathbb{Z}^g} \exp \left[ \frac{1}{2} c^T B c + c^T z \right]$$

where $z \in \mathbb{C}^g$ and $B$ is a Riemann matrix, a $g \times g$ symmetric matrix normalized to have negative definite real part.
Connection to Algebraic Curves

In 1997, Krichever proved that the KP equation has solutions of the form

\[ p(x, y, t) = 2 \frac{\partial^2}{\partial x^2} \log \theta(u x + v y + w t, B) \]

for certain vectors \( u = (u_1, \ldots, u_g), v = (v_1, \ldots, v_g), w = (w_1, \ldots, w_g) \in \mathbb{C}^g \).
Connection to Algebraic Curves

In 1997, Krichever proved that the KP equation has solutions of the form

\[ p(x, y, t) = 2 \frac{\partial^2}{\partial x^2} \log \theta(u x + v y + w t, B) \]

for certain vectors \( u = (u_1, \ldots, u_g), v = (v_1, \ldots, v_g), w = (w_1, \ldots, w_g) \in \mathbb{C}^g. \)

Now, for a specific curve \( C \) of genus \( g \) with Riemann matrix \( B \), we can look for \( \tau \) of the form

\[ \tau(x, y, t) = \theta(u x + v y + w t, B). \]
Consider \((u_1, \ldots, u_g, v_1, \ldots, v_g, w_1, \ldots, w_g)\) as a point in \(\mathbb{P}^{3g-1}\) such that
\[
\text{deg}(u_i) = 1, \quad \text{deg}(v_i) = 2, \quad \text{deg}(w_i) = 3 \quad \text{for } i = 1, 2, \ldots, g
\]
Connection to Algebraic Curves

Consider \((u_1, \ldots, u_g, v_1, \ldots, v_g, w_1, \ldots, w_g)\) as a point in \(\mathbb{WP}^{3g-1}\) such that

\[
\deg(u_i) = 1, \quad \deg(v_i) = 2, \quad \deg(w_i) = 3 \quad \text{for } i = 1, 2, \ldots, g
\]

**Definition (Agostini-Çelik-Sturmfels, 2020)**

The **Dubrovin threefold** \(\mathcal{D}_C\) comprises all points \((u, v, w)\) in \(\mathbb{WP}^{3g-1}\) such that \(\tau(x, y, t)\) satisfies the Hirota’s differential equation.
Soliton Solutions

Fix $k < n$ and a vector of parameters $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_n) \in \mathbb{R}^n$ and consider

$$
\tau(x, y, t) = \sum_{I \in \binom{[n]}{k}} p_I \cdot \prod_{i < j \in I} (\kappa_j - \kappa_i) \cdot \exp \left[ x \cdot \sum_{i \in I} \kappa_i + y \cdot \sum_{i \in I} \kappa_i^2 + t \cdot \sum_{i \in I} \kappa_i^3 \right]
$$
Soliton Solutions

Fix \( k < n \) and a vector of parameters \( \kappa = (\kappa_1, \kappa_2, \ldots, \kappa_n) \in \mathbb{R}^n \) and consider

\[
\tau(x, y, t) = \sum_{I \in \binom{[n]}{k}} p_I \cdot \prod_{i,j \in I, i < j} (\kappa_j - \kappa_i) \cdot \exp \left[ x \cdot \sum_{i \in I} \kappa_i + y \cdot \sum_{i \in I} \kappa_i^2 + t \cdot \sum_{i \in I} \kappa_i^3 \right]
\]

Proposition (Sato)

The function \( \tau \) is a solution to Hirota’s differential equation if and only if the point \( p = (p_I)_{I \in \binom{[n]}{k}} \) lies in the Grassmannian \( \text{Gr}(k, n) \).

Definition

We define a \( (k, n) \)-soliton to be any function \( \tau(x, y, t) \) where \( \kappa \in \mathbb{R}^n \) and \( p \in \text{Gr}(k, n) \).
Main Idea

We study solutions to the KP equations arising from algebraic curves defined over a non-archimedean field $\mathbb{K}$, like $\mathbb{Q}(\epsilon)$ or $\mathbb{C}\{\epsilon}\}$. 

A curve over $\mathbb{K}$ can be thought of as a family of curves depending on a parameter $\epsilon$

\[
\theta(z) = \sum_{c \in \mathbb{Z}^g} \exp \left[ \frac{1}{2} c^T B c + c^T z \right] \quad \sim \quad \theta_{\mathcal{C}}(x) = \sum_{c \in \mathcal{C}} a_c \exp \left[ c^T z \right]
\]
Main Idea

We study solutions to the KP equations arising from algebraic curves defined over a non-archimedean field $\mathbb{K}$, like $\mathbb{Q}(\epsilon)$ or $\mathbb{C}\{\epsilon}\}$. 

For $\epsilon \to 0$

- The theta function becomes a finite sum of exponentials
- The function

$$p(x, y, t) = 2 \frac{\delta^2}{\delta x^2} \log \tau (x, y, t)$$

becomes a soliton solution of the KP equation
Degenerations of Theta Functions

Let $X$ be a smooth curve of genus $g$ over $\mathbb{K}$. The metric graph is $\text{Trop}(X)$.

The metric graph $\Gamma = (V, E)$ of a genus 2 hyperelliptic curve

- $e := |E|$
- $\Lambda := g \times e$ matrix whose $i$-th row records the coordinate of $\gamma_i$ with respect to the standard basis of $\mathbb{Z}^e$
- $\Delta := \text{diagonal } e \times e$ matrix that records edge lengths of the metric graph.

Definition

The **Riemann matrix** of $\Gamma = (V, E)$ is

$$Q = \Lambda \Delta \Lambda^T$$
Example (g=2)

Consider $X := \{ y^2 = f(x) \}$ where

$$f(x) = (x - 1)(x - 1 - \epsilon)(x - 2)(x - 2 - \epsilon)(x - 3)(x - 3 - \epsilon)$$

The six roots determine a subtree with six leaves which has a unique hyperelliptic covering by a metric graph of genus 2
Example \((g=2)\)

Consider \(X := \{ y^2 = f(x) \}\) where

\[
f(x) = (x - 1)(x - 1 - \epsilon)(x - 2)(x - 2 - \epsilon)(x - 3)(x - 3 - \epsilon)
\]

The six roots determine a subtree with six leaves which has a unique hyperelliptic covering by a metric graph of genus 2

From the graph we can read off the tropical Riemann matrix \(Q\)

\[
Q = \Lambda \Delta \Lambda^T = \begin{bmatrix}
1 & -1 & 0 \\
0 & 1 & -1 \\
0 & 0 & 2
\end{bmatrix} \begin{bmatrix}
2 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
-1 & 1 \\
0 & -1
\end{bmatrix} = \begin{bmatrix}
4 & -2 \\
-2 & 4
\end{bmatrix}
\]
Degenerations of Theta Functions

Consider

\[ B_{\epsilon} = -\frac{1}{\epsilon} Q + R(\epsilon) \]

Fix \( a \in \mathbb{R}^{g} \)

\[ \theta(z + \frac{1}{\epsilon} Qa | B_{\epsilon}) = \sum_{c \in \mathbb{Z}^{g}} \exp \left[ -\frac{1}{2\epsilon} c^T Q c + \frac{1}{\epsilon} c^T Q a \right] \cdot \exp \left[ \frac{1}{2} c^T R(\epsilon) c + c^T z \right] \]
Degenerations of Theta Functions

Consider

\[ B_\epsilon = -\frac{1}{\epsilon} Q + R(\epsilon) \]

Fix \( \mathbf{a} \in \mathbb{R}^g \)

\[ \theta(z + \frac{1}{\epsilon} Q \mathbf{a} | B_\epsilon) = \sum_{\mathbf{c} \in \mathbb{Z}^g} \exp \left( -\frac{1}{2\epsilon} \mathbf{c}^T Q \mathbf{c} + \frac{1}{\epsilon} \mathbf{c}^T \mathbf{a} \right) \cdot \exp \left[ \frac{1}{2} \mathbf{c}^T R(\epsilon) \mathbf{c} + \mathbf{c}^T z \right] \]

Let \( \epsilon \to 0 \). This converges provided

\[ \mathbf{c}^T Q \mathbf{c} - 2 \mathbf{c}^T Q \mathbf{a} \geq 0 \text{ for all } \mathbf{c} \in \mathbb{Z}^g \]

or equivalently

\[ \mathbf{a}^T Q \mathbf{a} \leq (\mathbf{a} - \mathbf{c})^T Q (\mathbf{a} - \mathbf{c}) \text{ for all } \mathbf{c} \in \mathbb{Z}^g \]
Voronoi and Delaunay

The condition

\[ a^T Q a \leq (a - c)^T Q (a - c) \quad \text{for all} \quad c \in \mathbb{Z}^g \]

holds if and only if \( a \) belongs to the **Voronoi cell** for \( Q \).

For \( a \) in the Voronoi cell for \( Q \), consider the associated **Delaunay set**:

\[ \mathcal{D}_{a,Q} = \{ c \in \mathbb{Z}^g : a^T Q a = (a - c)^T Q (a - c) \} \]
\[ \theta(z + \frac{1}{\epsilon} Qa \mid B) = \sum_{c \in \mathbb{Z}^g} \exp \left[ -\frac{1}{2\epsilon} c^T Qc + \frac{1}{\epsilon} c^T Qa \right] \cdot \exp \left[ \frac{1}{2} c^T R(\epsilon)c + c^T z \right] \]

**Theorem (Agostini-Fevola-M.-Sturmfels, 2021)**

*Fix a in the Voronoi cell of the tropical Riemann matrix Q. For \( \epsilon \to 0 \), the series

\[ \theta(z + \frac{1}{\epsilon} Qa \mid B) \]

converges to the theta function supported on the Delaunay set \( C = \mathcal{D}_{a,Q} \), namely

\[ \theta_C(x) = \sum_{c \in C} a_c \exp \left[ c^T z \right], \quad \text{where} \quad a_c = \exp \left[ \frac{1}{2} c^T R(0)c \right] \]
Example (g=2)

Theorem (Agostini-Fevola-M.-Sturmfels, 2021)

Fix $a$ in the Voronoi cell of $Q$ and let $C = \mathcal{D}_{a,Q}$ be the Delaunay set. As $\epsilon \to 0$,

$$\theta(z + \frac{1}{\epsilon} Qa | B_\epsilon) \to \theta_C(x) = \sum_{c \in C} a_c \exp[c^Tz],$$

where $a_c = \exp[\frac{1}{2} c^T R(0) c]$
Theorem (Agostini-Fevola-M.-Sturmfels, 2021)

Fix $a$ in the Voronoi cell of $Q$ and let $\mathcal{C} = \mathcal{D}_{a,Q}$ be the Delaunay set. As $\epsilon \to 0$,

$$\theta(z + \frac{1}{\epsilon} Qa \mid B_\epsilon) \to \theta_{\mathcal{C}}(x) = \sum_{c \in \mathcal{C}} a_c \exp[c^T z],$$

where $a_c = \exp[\frac{1}{2} c^T R(0)c]$.

Example

For $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\mathcal{C} = \mathcal{D}_{a,Q} = \{(0,0), (1,0), (0,1), (1,1)\}$$

The associated theta function is

$$\theta_{\mathcal{C}} = a_{00} + a_{10} \exp[z_1] + a_{01} \exp[z_2] + a_{11} \exp[z_1 + z_2]$$
The Hirota Variety

Let $\mathcal{C} = \{c_1, c_2, \ldots, c_m\} \subset \mathbb{Z}^g$

$$\theta_\mathcal{C}(z) = a_1 \exp[c_1^T z] + a_2 \exp[c_2^T z] + \cdots + a_m \exp[c_m^T z]$$

Consider

$$\tau(x, y, t) = \theta_\mathcal{C}(ux + vy + wt) = \sum_{i=1}^{m} a_i \exp\left[\left(\sum_{j=1}^{g} c_{ij} u_j\right) x + \left(\sum_{j=1}^{g} c_{ij} v_j\right) y + \left(\sum_{j=1}^{g} c_{ij} w_j\right) t\right]$$
The Hirota Variety

Let \( \mathcal{C} = \{ c_1, c_2, \ldots, c_m \} \subset \mathbb{Z}^g \)

\[
\theta_\mathcal{C}(z) = a_1 \exp[c_1^T z] + a_2 \exp[c_2^T z] + \cdots + a_m \exp[c_m^T z]
\]

Consider

\[
\tau(x, y, t) = \theta_\mathcal{C}(ux + vy + wt) = \sum_{i=1}^{m} a_i \exp\left[ \left( \sum_{j=1}^{g} c_{ij}u_j \right) x + \left( \sum_{j=1}^{g} c_{ij}v_j \right) y + \left( \sum_{j=1}^{g} c_{ij}w_j \right) t \right]
\]

Definition

The Hirota variety \( \mathcal{H}_\mathcal{C} \) consists of all points \((a, (u, v, w))\) in \((\mathbb{K}^*)^m \times \mathbb{WP}^{3g-1}\) such that \(\tau(x, y, t)\) satisfies Hirota’s differential equation
Polynomials defining the Hirota Variety

Remark

Hirota’s differential equation can be written via the Hirota differential operators as

\[ P(D_x, D_y, D_t) \tau \cdot \tau = 0 \]

where \( P(x, y, t) = x^4 - 4xt + 3y^2 \) gives the *soliton dispersion relation*
Remark

Hirota’s differential equation can be written via the Hirota differential operators as

\[ P(D_x, D_y, D_t) \tau \cdot \tau = 0 \]

where \( P(x, y, t) = x^4 - 4xt + 3y^2 \) gives the soliton dispersion relation.

For any two indices \( k, \ell \) in \( \{1, \ldots, m\} \)

\[ P_{k\ell}(u, v, w) := P((c_k - c_\ell) \cdot u, (c_k - c_\ell) \cdot v, (c_k - c_\ell) \cdot w) \]

is a hypersurface in \( \mathbb{WP}^{3g-1} \)
The polynomials defining $\mathcal{H}_\ell$ correspond to points in

$$\mathcal{C}^2 = \{ c_k + c_\ell : 1 \leq k < \ell \leq m \} \subset \mathbb{Z}^g$$

**Definition**

A point $d$ in $\mathcal{C}^2$ is **uniquely attained** if there exists precisely one index pair $(k, \ell)$ such that $c_k + c_\ell = d$. In that case, $(k, \ell)$ is a **unique pair**.
The Hirota variety $\mathcal{H}_C$ is defined by the quartics

$$\mathcal{P}_{k\ell}(u,v,w) := \mathcal{P}\left((c_k - c_\ell) \cdot u, (c_k - c_\ell) \cdot v, (c_k - c_\ell) \cdot w\right)$$

for all unique pairs $(k, \ell)$ and by the polynomials

$$\sum_{1 \leq k < \ell \leq m} \mathcal{P}_{k\ell}(u,v,w) \ a_k a_\ell$$

for all non-uniquely attained points $d \in C^{[2]}$.

If all points in $C^{[2]}$ are uniquely attained then $\mathcal{H}_C$ is defined by the $\binom{m}{2}$ quartics $\mathcal{P}_{k\ell}(u,v,w)$. 
Example (The Square)

Let \( g = 2 \) and \( \mathcal{C} = \{0, 1\}^2 \)

\[
\mathcal{C}^{[2]} = \{(0, 1), (1, 0), (1, 1), (1, 2), (2, 1)\}
\]

There are four unique pairs \((k, \ell)\)

\[
P_{13} = P_{24} = u_1^4 - 4u_1w_1 + 3v_1^2
\]

\[
P_{12} = P_{34} = u_2^4 - 4u_2w_2 + 3v_2^2
\]

The point \( d = (1, 1) \) is not uniquely attained in \( \mathcal{C}^{[2]} \)

\[
P(u_1 + u_2, v_1 + v_2, w_1 + w_2) a_{00}a_{11} + P(u_1 - u_2, v_1 - v_2, w_1 - w_2) a_{01}a_{10}
\]

For any point in \( \mathcal{H}_{\mathcal{C}} \subset (\mathbb{K}^*)^4 \times \mathbb{WP}^5 \), we can write \( \tau(x, y, t) \) as a (2, 4)-soliton

\[
A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}
\]
Example (The Cube)

Let $g = 3$ and consider the tropical degeneration of a smooth plane quartic $C$ to a rational quartic

$$
\theta_C = a_{000} + a_{100} \exp[z_1] + a_{010} \exp[z_2] + a_{001} \exp[z_3] + a_{110} \exp[z_1 + z_2] + a_{101} \exp[z_1 + z_3] + a_{011} \exp[z_2 + z_3] + a_{111} \exp[z_1 + z_2 + z_3].
$$
Example (The Cube)

Let $g = 3$ and consider the tropical degeneration of a smooth plane quartic $C$ to a rational quartic

$$\theta_C = a_{000} + a_{100} \exp[z_1] + a_{010} \exp[z_2] + a_{001} \exp[z_3] + a_{110} \exp[z_1 + z_2]$$
$$+ a_{101} \exp[z_1 + z_3] + a_{011} \exp[z_2 + z_3] + a_{111} \exp[z_1 + z_2 + z_3].$$

We compute the Hirota variety in $(\mathbb{K}^*)^8 \times WP^8$. The set $\mathcal{C}^{[2]}$ consists of 19 points.

- 12 pts uniquely attained, one for each edge of the cube $\rightarrow$ quartics $u_j^4 - 4u_j w_j + 3v_j^2$, one for each edge direction $\mathbf{c}_k - \mathbf{c}_\ell$
- 6 pts attained twice $\rightarrow$ contribute 6 equations, one for each facet
- $(1, 1, 1)$ four times $\rightarrow P(u_1 + u_2 + u_3, v_1 + v_2 + v_3, w_1 + w_2 + w_3) a_{000} a_{111}$
  $$+ P(u_1 + u_2 - u_3, v_1 + v_2 - v_3, w_1 + w_2 - w_3) a_{001} a_{110}$$
  $$+ P(u_1 - u_2 + u_3, v_1 - v_2 + v_3, w_1 - w_2 + w_3) a_{010} a_{101}$$
  $$+ P(-u_1 + u_2 + u_3, -v_1 + v_2 + v_3, -w_1 + w_2 + w_3) a_{100} a_{011}$$
The Sato Grassmannian: Some Intuition

Classical Grassmannian $\text{Gr}(k, n)$
- parametrizes $k$-dimensional subspaces of $\mathbb{K}^n$
- projective variety in $\mathbb{P}^{\binom{n}{k}-1}$, cut out by quadratic relations known as Plücker relations
- Plücker coordinates $p_I = c_\lambda$ are indexed by $k$-element subsets of $[n]$, identified with partitions $\lambda$ that fit into a $k \times (n-k)$ rectangle. These are the maximal minors of a $k \times n$ matrix $M$ of unknowns

The Sato Grassmannian (SGM)
- the zero set of the Plücker relations in the unknowns $c_\lambda$, where we now drop the constraint that $\lambda$ fits into a $k \times (n-k)$-rectangle.
- we now allow arbitrary partitions $\lambda$

The SGM is a device for encoding all solutions to the KP equation!
The Sato Grassmannian

Let $V = \mathbb{K}((z))$ and consider the natural projection map

$$\pi: V \rightarrow \mathbb{K}[z^{-1}]$$

Definition

Points in the Sato Grassmannian $SGM$ correspond to $\mathbb{K}$-subspaces $U \subset V$ such that

$$\dim \ker \pi|_U = \dim \operatorname{coker} \pi|_U < \infty$$
The Sato Grassmannian

Let \( V = \mathbb{K}((z)) \) and consider the natural projection map

\[
\pi: V \rightarrow \mathbb{K}[z^{-1}]
\]

**Definition**

Points in the **Sato Grassmannian** SGM correspond to \( \mathbb{K} \)-subspaces \( U \subset V \) such that

\[
\dim \ker \pi|_U = \dim \text{coker } \pi|_U < \infty
\]

**How do we represent a point in SGM?**

For any basis \((f_1, f_2, f_3, \ldots)\) of \( U \in \text{SGM} \)

\[
f_j(z) = \sum_{i=-\infty}^{+\infty} \xi_{i,j} z^{i+1}.
\]

Then \( U \) is the column span of the infinite matrix \( \xi = (\xi_{i,j}) \).
The Sato Grassmannian

A subspace $U$ of $V$ gives a point in SGM if and only if there is a basis whose corresponding matrix has the shape

$$\xi = \begin{pmatrix}
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\
\cdots & 1 & 0 & 0 & 0 & \cdots & 0 & \\
\cdots & * & 1 & 0 & 0 & \cdots & 0 & \\
\cdots & * & * & \xi_{-\ell, \ell} & \xi_{-\ell, \ell-1} & \cdots & \xi_{-\ell, 1} & \\
\cdots & * & * & \xi_{-\ell+1, \ell} & \xi_{-\ell+1, \ell-1} & \cdots & \xi_{-\ell+1, 1} & \\
\cdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\
\cdots & * & * & \xi_{-1, \ell} & \xi_{-1, \ell-1} & \cdots & \xi_{-1, 1} & \\
\cdots & * & * & \xi_{0, \ell} & \xi_{0, \ell-1} & \cdots & \xi_{0, 1} & \\
\cdots & * & * & \xi_{1, \ell} & \xi_{1, \ell-1} & \cdots & \xi_{1, 1} & \\
\cdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\
\end{pmatrix}.$$

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Connections of the SGM to solutions to the KP equation

Definition

A Maya diagram is an infinite sequence \( M = (m_1, m_2, m_3, \ldots) \) of integers \( m_1 > m_2 > m_3 > \ldots \) such that \( m_i = -i \) for all \( i \) large enough.

- Maya diagrams correspond to partitions \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \)
- It makes sense to define

\[
\xi_\lambda := \xi_M = \det(\xi_{m_{ij}})
\]

The \( \tau \)-function associated to \( \xi \) is

\[
\tau(t_1, t_2, t_3, \ldots) = \sum_\lambda \xi_\lambda \sigma_\lambda(t_1, t_2, t_3, \ldots)
\]
Connections of the SGM to solutions to the KP equation

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A Maya diagram is an infinite sequence $M = (m_1, m_2, m_3, \ldots)$ of integers $m_1 > m_2 > m_3 > \ldots$ such that $m_i = -i$ for all $i$ large enough.

- Maya diagrams correspond to partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$
- It makes sense to define

$$\xi_{\lambda} := \xi_M = \det(\xi_{m_i,j})$$

The $\tau$-function associated to $\xi$ is

$$\tau(t_1, t_2, t_3, \ldots) = \sum_{\lambda} \xi_{\lambda} \sigma_{\lambda}(t_1, t_2, t_3, \ldots)$$

Theorem (Sato)

The function $\tau(t; \xi)$ is a solution to the KP hierarchy. Conversely, any solution of the KP hierarchy is of this form.
Soliton Solutions & Sato’s Solutions

Recall

\[ \tau(x, y, t) = \sum_{I \in \binom{[n]}{k}} p_I \cdot \prod_{i < j} (\kappa_j - \kappa_i) \cdot \exp \left[ x \cdot \sum_{i \in I} \kappa_i + y \cdot \sum_{i \in I} \kappa_i^2 + t \cdot \sum_{i \in I} \kappa_i^3 \right] \]

and

\[ \tau(t_1, t_2, t_3, \ldots) = \sum_{\lambda} \xi_{\lambda} \sigma_{\lambda}(t_1, t_2, t_3, \ldots) \]
Recall

\[ \tau(x, y, t) = \sum_{I \in \binom{[n]}{k}} p_I \cdot \prod_{i,j \in I, i < j} (\kappa_j - \kappa_i) \cdot \exp \left[ x \cdot \sum_{i \in I} \kappa_i + y \cdot \sum_{i \in I} \kappa_i^2 + t \cdot \sum_{i \in I} \kappa_i^3 \right] \]

and

\[ \tau(t_1, t_2, t_3, \ldots) = \sum_{\lambda} \xi_{\lambda} \sigma_{\lambda}(t_1, t_2, t_3, \ldots) \]

**Proposition (Agostini-Fevola-M.-Sturmfels)**

The \((k, n)\)-soliton has the following expansion into Schur polynomials

\[ \tau(x, y, t) = \sum_{\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0} c_{\lambda} \cdot \sigma_{\lambda}(x, y, t), \quad \text{where} \quad c_{\lambda} = \sum_{I \in \binom{[n]}{k}} p_I \cdot \Delta_{\lambda}(\kappa_i, i \in I) \]
Tau Functions from Algebraic Curves

Let $X$ be a smooth projective curve of genus $g$ defined over a field $\mathbb{K}$. Fix a divisor $D$ of degree $g - 1$ on $X$ and a point $p \in X$

For $m < n$

$$H^0(X, D + mp) \subseteq H^0(X, D + np) \subseteq \cdots \subseteq H^0(X, D + \infty p)$$

Let $z$ denote a local coordinate on $X$ at $p$ and $m = \text{ord}_p(D)$

$$\iota : H^0(X, D + \infty p) \rightarrow V, \quad s = \sum_{n \in \mathbb{Z}} s_n z^n \mapsto \sum_{n \in \mathbb{Z}} s_n z^{n+m+1}$$

Proposition (Segal-Wilson)

The space $U = \iota(H^0(X, D + \infty p)) \subset V$ lies in SGM.
The case of genus 2 hyperelliptic curves

Let

\[ X = \{ y^2 = (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_6) \} \]

Let \( p \) be one of the two preimages of \( p_\infty \) under the double cover \( X \rightarrow \mathbb{P}^1 \).

Let \( z = \frac{1}{x} \) around \( p \)

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y = \pm \sqrt{(x - \lambda_1) \cdots (x - \lambda_6)} = \pm \frac{1}{z^3} \sum_{n=0}^{+\infty} \alpha_n z^n
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We consider three kinds of divisors:

\[ D_0 = p, \quad D_1 = p_1, \quad D_2 = p_1 + p_2 - p \]

For \( m \geq 3 \), consider the functions

\[ g_m(x) = \sum_{j=0}^{m} \alpha_j x^{m-j} \]

\[ f_m(x, y) = \frac{1}{2} \left( x^{m-3} y + g_m(x) \right) \]

\[ h_j(x, y) = \frac{f_3(x, y) - f_3(c_j, -y_j)}{x-c_j} = \frac{y + g_3(x) - (-y_j + g_3(c_j))}{2(x-c_j)} \quad \text{for } j = 1, 2 \]
We write $U_i = \iota(H^0(X, D_i + \infty p))$

**Proposition (Nakayashiki)**

The set \{1, f_3, f_4, f_5, \ldots\} is a basis of $U_0$, the set \{1, f_3, f_4, f_5, \ldots\} \cup \{h_1\} is a basis of $U_1$, and \{1, f_3, f_4, f_5, \ldots\} \cup \{h_1, h_2\} is a basis of $U_2$.

**Computation**

We implemented the method in Maple for $D_0 = p$ on hyperelliptic curves over $\mathbb{K} = \mathbb{Q}(\epsilon)$

\[
\tau[n] := \sum_{i=1}^{n} \sum_{\lambda \vdash i} \xi_{\lambda} \sigma_{\lambda}(x, y, t),
\]

You can find it here:

https://mathrepo.mis.mpg.de
Thank you!