EXTENSION THEOREM FOR NONLOCAL OPERATORS

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ABSTRACT. We solve the extension problem in Sobolev spaces for nonlocal operators under minimal regularity of the exterior values. The extension with the smallest value of the quadratic form is given by a suitable Poisson integral and is the weak solution of the corresponding Dirichlet problem. We express the Sobolev form of the extension as a weighted Sobolev form of the exterior data.

1. INTRODUCTION

Let \( d = 1, 2, \ldots \). Let \( \nu : [0, \infty) \to (0, \infty] \) be nonincreasing and denote \( \nu(z) = \nu(|z|) \) for \( z \in \mathbb{R}^d \). In particular, \( \nu(z) = \nu(-z) \). We assume that \( \int_{\mathbb{R}^d} \nu(z)dz = \infty \) and

\[
\int_{\mathbb{R}^d} (|z|^2 \wedge 1) \nu(z)dz < \infty.
\]

Summarizing, \( \nu \) is a strictly positive density function of an isotropic infinite unimodal Lévy measure on \( \mathbb{R}^d \). In short we will say \( \nu \) is unimodal. For \( x \in \mathbb{R}^d \) and \( u : \mathbb{R}^d \to \mathbb{R} \) we let

\[
Lu(x) = \lim_{\epsilon \to 0^+} \int_{|x-y| > \epsilon} (u(y) - u(x))\nu(y-x)dy
\]

\[
= \lim_{\epsilon \to 0^+} \frac{1}{2} \int_{|x-y| > \epsilon} (u(x+z) + u(x-z) - 2u(x))\nu(z)dz.
\]

Here and in what follows all the considered sets, functions and measures are assumed to be Borel.

The limit in (1.1) exists, e.g., for \( u \in C_\infty^\infty(\mathbb{R}^d) \), the smooth functions with compact support. We note that \( L \) is a non-local symmetric translation-invariant linear operator on \( C_\infty^\infty(\mathbb{R}^d) \) satisfying the positive maximum principle, cf. [13, Section 2]. For example, if \( 0 < \alpha < 2 \) and

\[
\nu(z) = \frac{2^\alpha \Gamma((d + \alpha)/2)}{\pi^{d/2} \Gamma(-\alpha/2)} |z|^{-d-\alpha}, \quad z \in \mathbb{R}^d,
\]

then \( L \) is the fractional Laplacian, denoted by \( \Delta^{\alpha/2} \). In what follows we write \( \nu(x, y) = \nu(y-x), \quad x, y \in \mathbb{R}^d \).

Let \( D \) be a nonempty open set in \( \mathbb{R}^d \). Motivated by Dipierro, Ros-Oton and Valdinoci [19], Felsinger, Kassmann and Voigt [22] and Millot, Sire and Wang [32] for \( u : \mathbb{R}^d \to \mathbb{R} \) we consider the quadratic form,

\[
\mathcal{E}_D(u, u) = \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus D^c \times D^c} (u(x) - u(y))^2\nu(x, y)dxdy.
\]

Similar expressions are used by Caffarelli, Roquejoffre and Savin [14, Section 7] in the study of minimal surfaces. For more general Lévy measures we refer the reader to Rutkowski [35], see also...
Ros-Oton [34]. The quadratic form measures the smoothness of $u$ in a Sobolev fashion by integrating the squared increments of $u$. The corresponding Sobolev space is defined as

\begin{equation}
V^D = \{ u : \mathbb{R}^d \to \mathbb{R} \text{ such that } \mathcal{E}_D(u, u) < \infty \}.
\end{equation}

We also consider

\begin{equation}
V^D_0 = \{ u \in V^D : u = 0 \text{ a.e. on } D^c \}.
\end{equation}

Recall [24] that the classical Dirichlet form of $L$ is

\begin{equation}
\mathcal{E}_{\mathbb{R}^d}(u, u) = \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} (u(x) - u(y))^2 \nu(x, y) \, dx \, dy.
\end{equation}

Therefore $\mathcal{E}_{\mathbb{R}^d}(u, u) = \mathcal{E}_D(u, u) + \frac{1}{2} \iint_{D^c \times D^c} (u(x) - u(y))^2 \nu(x, y) \, dx \, dy \geq \mathcal{E}_D(u, u)$, and $\mathcal{E}_D(u, u) = \mathcal{E}_{\mathbb{R}^d}(u, u)$ if $u = 0$ on $D^c$, in particular if $D = \mathbb{R}^d$. For $u, v \in V^D$, we let

\begin{equation}
\mathcal{E}_D(u, v) = \frac{1}{2} \iint_{D^c \times D^c} (u(x) - u(y))(v(x) - v(y)) \nu(x, y) \, dx \, dy,
\end{equation}

and we have $\mathcal{E}_D(u, \phi) = \mathcal{E}_{\mathbb{R}^d}(u, \phi) = \int_{\mathbb{R}^d} \phi(x) Lu(x) \, dx$ if, e.g., $u \in C_c^\infty(\mathbb{R}^d)$ and $\phi \in C_c^\infty(D)$ [24, Example 1.4.1]. We emphasize that the increments of $u$ between points $x, y \in D^c$ do not appear in (1.4), which is why the Sobolev space $V^D$ is suitable for solving the Dirichlet problem

\begin{equation}
\begin{cases}
Lu = 0 & \text{on } D, \\
u = g & \text{on } D^c,
\end{cases}
\end{equation}

under minimal smoothness assumptions on the exterior condition $g : D^c \to \mathbb{R}$. Here by a solution of (1.6) we mean the weak solution, that is any function $u \in V^D$ equal to $g$ a.e. on $D^c$ (an extension of $g$) such that for all $\phi \in V^D_0$,

\begin{equation}
\mathcal{E}_D(u, \phi) = 0
\end{equation}

or, equivalently, $\mathcal{E}_{\mathbb{R}^d}(u, \phi) = 0$, cf. [22] [35]. For geometrically regular sets $D$ considered below, $V^D_0$ can be approximated by the bona fide test functions, $C_c^\infty(D)$. This is proved in Theorem 3.4 in the Appendix, and we refer the reader to Fukushima, Oshima and Takeda [24, Section 2.3] for the larger context. In passing we would like to advocate for using $\mathcal{E}_D$ as adequate Sobolev setting of the Dirichlet and Neumann boundary problems for nonlocal operators. We also refer to Servadei and Valdinoci [38], to Klimsiak and Rozkosz [29] and to Dlotko, Kania and Sun [20] for discussions of various notions of solutions to nonlocal equations, and to Barles, Chasseigne, Georgelin and Jakobsen [3] for several approaches to the nonlocal Neumann problem. We also note that some care should be exercised when interpreting (1.6) pointwise or in terms of the generator. For instance, in general even the test functions need not be in the domain of the generator of the Dirichlet heat kernel corresponding to $L$, see Baeumer, Luks and Meerschaert [2, Section 2]. We refer to Bogdan and Byczkowski [6, Lemma 5.3] to indicate a safe formulation of (1.6) that in fact uses the operator.

We say that the extension problem for $g$, $D$ and $\nu$ (or $L$) has a solution if the exterior condition $g$ has an extension $u \in V^D$. If this is so, then the existence and uniqueness of the solution to (1.6) comfortably follow from the general Lax-Milgram theory (see [22] [35] and Section 5 below). We thus focus on the extension problem. Here the main difficulty is to define $u$ on $D$ and control the Sobolev smoothness of $u$ by that of $g$. In this connection we mention the important recent result of Dyda and Kassmann [28, Theorem 2 and 4], who use the Whitney decomposition to solve the extension problem for $\Delta^{\alpha/2}$ under the assumptions $g \in L^2_{loc}(D^c)$ and

\[ \int_{D^c} \int_{D^c} \frac{(g(z) - g(w))^2}{r(z, w)^{2\alpha}} \, dz \, dw < \infty. \]
Here and below \( r(z,w) = \delta_D(z) + |z - w| + \delta_D(w) \) and \( \delta_D(z) = \text{dist}(z, \partial D) \), for \( z,w \in \mathbb{R}^d \). For information on classical (local) extension theorems we refer the reader to Kalajdzievski and Dhara [15], Koskela, Soto and Wand [30] and to the book of Adams and Fournier [1].

In this paper we characterize the existence of the solution to the extension problem by the finiteness of a quadratic Sobolev form \( \mathcal{H}_D(g,g) \) with a specific weight \( \gamma_D \) on \( D^c \times D^c \), called the interaction kernel and defined below. This is an analogue of the result of Kassmann and Dyda, but for operators \( L \) much more general than \( \Delta^{\alpha/2} \).

The structure of the paper is as follows. In Section 2 we present the notation, definitions and main results. In Section 3 we state auxiliary results. In Section 4 we discuss harmonic functions of \( L \) and we give for \( L \) the quadratic Hardy-Stein identity, generalizing the formula given by Bogdan, Dyda and Luks [7] (6) for the Hardy spaces of \( \Delta^{\alpha/2} \). Section 5 provides the proof of the extension theorem for \( \nabla^D \) and related results, notably the so-called Sobolev-Hardy-Stein identity. In Section 6 we estimate the interaction kernel \( \gamma_D \) for bounded \( C^{1,1} \) sets and half-spaces under mild conditions on \( \nu \). In Section 7 we give specific examples of \( \nu \) to which our results apply. In the Appendix we prove auxiliary facts needed to treat \( \nu \) and \( L \) in the present generality. The reader only interested in the Sobolev-Hardy-Stein identity and the estimates of \( \gamma_D \) in the simplest possible setting, may focus on \( \Delta^{\alpha/2} \). Even in this case we obtain a new remarkable conservation law, or a sweeping-out formula, for squared increments of harmonic functions.

In the sequel we will often use the probabilistic language and results from the potential theory of Lévy stochastic processes. This is avoidable but dramatically reduces the effort needed to define and handle such objects as harmonic functions, Green function and Poisson kernel for general operators \( L \). Furthermore, the probabilistic setting facilitates integration in spaces with many coordinates and proving the convergence of approximations by subdomains. Therefore we ask the analytic-oriented reader to bear with us. In particular, probability in not essential to formulate the main results.

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2. Main results

Here are additional assumptions on \( \nu : [0,\infty) \to (0,\infty] \) which are sometimes made in what follows.

**A1:** \( \nu \) is twice continuously differentiable and there is a constant \( C_1 \) such that
\[
|\nu'(r)|, |\nu''(r)| \leq C_1 \nu(r) \quad \text{for } r > 1.
\]

**A2:** There exist constants \( \beta \in (0,2) \) and \( C > 0 \) such that
\[
\begin{align*}
\nu(\lambda r) & \leq C \lambda^{-d-\beta} \nu(r), & 0 < \lambda, r \leq 1, \\
\nu(r) & \leq C \nu(r+1), & r \geq 1.
\end{align*}
\]

**A3:** There exist constants \( \alpha \in (0,2) \) and \( c > 0 \) such that
\[
\begin{align*}
\nu(\lambda r) & \geq c \lambda^{-d-\alpha} \nu(r), & 0 < \lambda, r \leq 1.
\end{align*}
\]

Here and below by a constant we mean a strictly positive number. Recall that \( \nu(z, w) = \nu(z - w) = \nu(|z - w|) \) for \( z, w \in \mathbb{R}^d \). We also denote \( \nu(A) = \int_A \nu(|z|) \, dz \) and \( \nu(z, A) = \nu(A - z) \) for \( z \in \mathbb{R}^d, A \subset \mathbb{R}^d \). Let \( L \) and unimodal \( \nu \) be related by (1.1). Clearly, A1, A2 and A3 hold true if \( L = \Delta^{\alpha/2} \).

Further examples of Lévy measure densities \( \nu \) satisfying these assumptions are given in Section 7.

The condition A1 is used for the proof of the fact that harmonic functions of \( L \) (see Definition 1.1) are twice continuously differentiable. Should we assume similar condition for the derivatives of \( \nu \) of order up to \( N \), we would obtain that \( L \)-harmonic functions are \( N \) times continuously differentiable
EgV Spaces similar to integral. We define the intensity of interaction of Let accordingly we define $P$ Then In particular, $\gamma$ Let set $A_1$ (see the proof of Theorem 4.6). We note that $A_1$ implies that for every $s > 0$ there is a (positive finite) constant $C_s$ such that

$$|\nu'(r)|, |\nu''(r)| \leq C_s \nu(r), \quad r \geq s.$$  

(2.4)

The condition $2.1$ in $A_2$ is equivalent to the assumption that $r^{d+\beta} \nu(r)$ is almost increasing on $(0,1]$ in the sense of $8$ Section 3, and $2.3$ means that $r^{d+\alpha} \nu(r)$ is almost decreasing on $(0,1]$. Let $G_D(x,y)$ be the Green function of $D$ for $L$ and let $\omega_D^L(\cdot)$ be the harmonic measure of $D$ for $L$ (for details see Section 3). Our first result, a main tool in the sequel, is a Hardy-Stein type identity for $L$–harmonic functions. It generalizes $7$ (6) from $\Delta^{\alpha/2}$ to $L$ and identifies the Hardy-type squared norm (on the left) with a Sobolev-type squared norm weighted by $G_D$ (on the right).

Theorem 2.1. If $u$ is $L$-harmonic in $D$ and $x \in D$, then

$$\sup_{x \in U \subset D} \int_{U{x}} u^2(z) \omega^L_D(dz) = u(x)^2 + \int_D G_D(x,y) \int_{\mathbb{R}^d} (u(z) - u(y))^2 \nu(z,y)dzdy.$$  

(2.5)

Theorem 2.1 is proved in Section 4 by using recent regularity results of Grzywny and Kwaśnicki [25] for $L$-harmonic functions. We next define

$$P_D(x,z) = \int_D G_D(x,y) \nu(y,z)dy, \quad x \in D, \quad z \in D^c,$$

(2.6)

the Poisson kernel of $D$ for $L$. For $g : D^c \to \mathbb{R}$ we let $P_D[g](x) = g(x)$ for $x \in D^c$ and

$$P_D[g](x) = \int_{D^c} g(y) P_D(x,y)dy \quad \text{for} \quad x \in D,$$

(2.7)

if the integrals exists. This is the Poisson extension of $g$ and $\int_{D^c} g(y) P_D(x,y)dy$ is the Poisson integral. We define the intensity of interaction of $w, z \in D^c$ via $D$, in short, the interaction kernel,

$$\gamma_D(w,z) = \int_D \int_D \nu(w,x) G_D(x,y) \nu(y,z) dxdy = \int_D \nu(w,x) P_D(x,z) dx = \int_D \nu(z,x) P_D(x,w) dx.$$

In particular, $\gamma_D(z,w) = \gamma_D(w,z)$. The reader may directly verify the following result.

Example 2.2. Let $d = 1$, $D = (0, \infty) \subset \mathbb{R}$ and $\nu(w,x) = \pi^{-1} |x - w|^{-2}$, $x, w \in \mathbb{R}$, i.e. $L = \Delta^{1/2}$. Then $P_{(0,\infty)}(x,z) = \pi^{-1} x^{-1/2} |z|^{-1/2} (x - z)^{-1}$ for $x > 0, z < 0$ [7, (3.40)], and

$$\gamma_{(0,\infty)}(z,w) = \int_0^\infty \frac{1}{\pi^2} \frac{\sqrt{x}}{|z| (x-z)^2} dx = \frac{1}{2\pi \sqrt{z}w (|z| + \sqrt{|w|^2})}, \quad z, w < 0.$$  

(2.8)

Part of our development calls for geometric assumptions on $D$, which we detail in Section 8. In particular, the $C^{1,1}$ condition for the “smoothness” of $D$ and the volume density condition (VDC) for the “fatness” of $D^c$ are defined there. For $g : D^c \to \mathbb{R}$ we let

$$H_D(g,g) = \frac{1}{2} \int_{D^c \times D^c} (g(w) - g(z))^2 \gamma_D(w,z) dw dz.$$  

(2.9)

Accordingly we define

$$\mathcal{X}^D = \{ g : D^c \to \mathbb{R} : H_D(g,g) < \infty \}.$$  

Spaces similar to $\mathcal{V}^D$ and $\mathcal{V}_0^D$ were considered in [22, 33] and $\mathcal{X}^D$ is new ($\mathcal{X}$ stands for eXterior). Here is our second main result, which we call the Sobolev-Hardy-Stein identity. It resembles $2.1$ but identifies two Sobolev-type norms, on $\mathbb{R}^d \times \mathbb{R}^d \setminus D^c \times D^c$ (left) and on $D^c \times D^c$ (right).

Theorem 2.3. Let set $D \subset \mathbb{R}^d$ be open, $D^c$ satisfy VDC and let unimodal $\nu$ satisfy $A_1, A_2$. If $g : D^c \to \mathbb{R}$ is such that $P_D[g](x) < \infty$ for some $x \in D$, in particular if $H_D(g,g) < \infty$, then

$$E_D(P_D[g], P_D[g]) = H_D(g,g).$$  

(2.9)
Thus, \( u = P_D[g] \), if finite, satisfies
\[
\iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus D^c \times D^c} (u(x) - u(y))^2 \nu(x, y) \, dx \, dy = \iint_{D^c \times D^c} (g(w) - g(z))^2 \gamma_D(w, z) \, dw \, dz.
\]

The proof of Theorem 2.3 is given in Section 5 by using Theorem 2.1. The result is a close analogue of the Douglas integral for the trace of a Markov process on a subset of the state space, as discussed by Fukushima and Chen [15, Sections 5.5-5.8 and 7.2], see also [24 (1.2.18)].

**Example 2.4.** In the setting of Example 2.2 we have \( u(x) = g(x) \) for \( x \leq 0 \), and
\[
u_u(x) = \int_0^x \frac{\sqrt{g(z)}}{\pi(x - z) \sqrt{|z|}} \, dz \quad \text{for} \quad x > 0.
\]

If the above integral is absolutely convergent, then
\[
\iint_{x > 0 \text{ or } y > 0} \frac{(u(x) - u(y))^2}{\pi(x - y)^2} \, dx \, dy = \iint_{z < 0 \text{ and } w < 0} \frac{(g(z) - g(w))^2}{2\pi \sqrt{|w|^2 + |w|^2} \, dw \, dz}.
\]

We note that \( \mathcal{H}_D(g, g) \) in Theorem 2.3 may be finite even for rather rough functions. Indeed,
\[
\mathcal{H}_D(g, g) = \int_{D^c} \int_{D^c} (g(z) - g(w))^2 \gamma_D(z, w) \, dz \, dw \leq 2 \int_{D^c} \int_{D^c} g^2(z) \gamma_D(z, w) \, dz \, dw = 2 \int_{D^c} \int_{D^c} g^2(z) \rho(z) \, dz,
\]
where \( \rho(z) = \int_D \nu(z, x) \, dx \). If \( g \) is \( L^2 \)-integrable and \( \text{dist}(D, \text{supp}(g)) > 0 \), then \( \mathcal{H}_D(g, g) < \infty \) and so \( g \) has an extension in \( \nu^{D} \). Also, if \( L = \Delta^{\alpha/2} \) and \( D \) is a bounded \( C^{1,1} \) set, then \( \rho(z) \approx \delta_D(z)^{-\alpha} (1 + |z|)^{-d} \), and so \( \mathcal{H}_D(g, g) < \infty \) if, e.g., \( g \) is bounded and \( \alpha < 1 \).

For full analysis precise estimates of \( \gamma_D \) are necessary. Hereby we propose sharp explicit estimates of \( \gamma_D(z, w) \) for bounded open sets \( D \) of class \( C^{1,1} \). To this end for \( r > 0 \) we let
\[
(2.10) \quad K(r) = \int_{|z| \leq r} \frac{|z|^2}{r^2} \nu(z) \, dz, \quad h(r) = K(r) + \nu(B_r) = \int_{\mathbb{R}^d} \left( \frac{|z|^2}{r^2} + 1 \right) \nu(z) \, dz,
\]
\[
(2.11) \quad V(r) = \frac{1}{\sqrt{h(r)}}.
\]

**Example 2.5.** For \( \Delta^{\alpha/2} \) we have \( \nu(r) = cr^{-d-\alpha} \) and \( V(r) = cr^{\alpha/2} \).

Here is our third main result.

**Theorem 2.6.** Let \( \nu \) be unimodal and assume A2, A3. Let \( D \) be a bounded \( C^{1,1} \) set. Then,
\[
\gamma_D(z, w) \approx \begin{cases} 
\nu(\delta_D(w)) \nu(\delta_D(z)), & \text{if } \text{diam}(D) \leq \delta_D(z), \delta_D(w), \\
\nu(\delta_D(w)) / V(\delta_D(z)), & \text{if } \delta_D(z) < \text{diam}(D) \leq \delta_D(w), \\
\nu(r(z, w)) [V(r(z, w)) / \nu(\delta_D(z)) V(\delta_D(w))], & \text{if } \delta_D(z), \delta_D(w) < \text{diam}(D).
\end{cases}
\]

As usual in the boundary potential theory, it is a challenge to handle unbounded or less regular sets \( D \). In Theorem 6.1 below we give estimates for \( \gamma_H(z, w) \), where \( H \) is the half-space in dimensions \( d \geq 3 \). Other extensions are left to the interested reader.

3. Preliminaries

3.1. **Functions and constants.** Above and below we write \( f(x) \approx g(x) \), or write that functions \( f \) and \( g \) are comparable, if \( f, g \geq 0 \) and there is a number \( C \in (0, \infty) \), called the comparability constant, such that \( C^{-1} f(x) \leq g(x) \leq C f(x) \) for all the considered arguments \( x \). Such estimates are also called sharp. Similarly, \( f(x) \lessapprox g(x) \) means that \( f(x) \leq C g(x) \), the same as \( g(x) \geq f(x) \).
We write $C = C(a, \ldots, z)$ if the constant $C$ may be so chosen to depend only on $a, \ldots, z$ and we write $C_a$ to emphasize that $C$ may depend on $a$. In comparisons and inequalities constants may change values from line to line.

We let $C_c(D)$ be the class of continuous functions: $\mathbb{R}^d \to \mathbb{R}$ with compact support contained in $D$ and we let $C_0(D)$ be the closure of $C_c(D)$ in the supremum norm. By $C^\infty_c(D)$ we denote the class of compactly supported and infinitely differentiable functions on $D$. We write $f \in C^2(\overline{U})$ if $f : \overline{U} \to \mathbb{R}$ extends to a twice continuously differentiable function in a neighborhood of $\overline{U}$.

### 3.2. Geometry.

Recall that $D$ is an open nonempty subset of $\mathbb{R}^d$, the Euclidean space of dimension $d \in \mathbb{N}$. We write $U \subset D$ if $U$ is an open set, its closure $\overline{U}$ is bounded, hence compact, and $\overline{U} \subset D$. Let $B(x, r) = \{ y \in \mathbb{R}^d : |x - y| < r \}$, the open ball with radius $r > 0$ and center at $x \in \mathbb{R}^d$. We write $B_r = B(0, r)$ and consider the Poisson kernel of the ball $P_{B_r}(z) := P_{B_r}(0, z)$, $z \in B_r^c$. Let $B_r^c = (B(0, r))^c$ and $\overline{B}_r = (\overline{B}(0, r))^c = \{ y \in \mathbb{R}^d : |x - y| > r \}$. Let $\omega_d = 2\pi^{d/2}/\Gamma(d/2)$, the surface measure of the unit sphere in $\mathbb{R}^d$.

**Definition 3.1.** We say that $D^c$ satisfies the volume density condition (VDC) if there is $c > 0$ such that for every $r > 0$ and $x \in \partial D$,

$$|D^c \cap B(x, r)| \geq cr^d. \tag{3.1}$$

This is a fatness-type condition for $D^c$, uniform in $r$ and $x$. For instance, VDC holds if $D$ satisfies the exterior cone condition. We say that VDC holds locally for $D^c$ if VDC holds for $(D \cap B)^c$ for every ball $B$. For instance if $D = \{ x \in \mathbb{R}^d : |x| > 1 \}$, then VDC holds locally for $D^c$. Naturally, if VDC holds for $D^c$, then VDC holds locally for $D^c$ because $\partial(D \cap B) \subset \partial D \cup \partial B$ and $(D \cap B)^c = D^c \cup B^c$ for every ball $B$. If VDC holds locally for $D$, then the Lebesgue measure of $\partial D$ is zero [40].

**Definition 3.2.** We say that an open set $D \subset \mathbb{R}^d$ has continuous boundary if $\partial D$ is compact and there exist open sets $U_1, \ldots, U_m, \Omega_1, \ldots, \Omega_m \subset \mathbb{R}^d$, continuous functions $f_1, \ldots, f_m : \mathbb{R}^{d-1} \to \mathbb{R}$, and rigid motions $T_1, \ldots, T_m : \mathbb{R}^d \to \mathbb{R}^d$, such that $\partial D \subset \bigcup_{i=1}^m U_i$, and for $i = 1, \ldots, m$, we have $T_i(\Omega_i) = \{(x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}^d : f_i(x') < x_d \}$ and $D \cap U_i = \Omega_i \cap U_i$.

A bounded open set is Lipschitz if the functions $f_i$ are Lipschitz: $|f_i(x') - f_i(y')| \leq \lambda|x' - y'|$ for $x', y' \in \mathbb{R}^{d-1}$, $i = 1, \ldots, m$. If $D$ is a bounded Lipschitz open set, then VDC holds for $D^c$.

**Definition 3.3.** Let $D \subset \mathbb{R}^d$ be open. If number $R > 0$ exists such that for every $Q \subset \partial D$ there are balls $B(x', R) \subset D$ and $B(x'', R) \subset D^c$ mutually tangent at $Q$, then $D$ is $C^{1,1}$ (at scale $R$).

If $D$ is of class $C^{1,1}$, then it is Lipschitz and the defining functions $f_i$ can be so chosen that their gradient is Lipschitz, see [11] for more on the geometry of $C^{1,1}$ open sets.

### 3.3. Completeness.

**Lemma 3.4.** If $D$ is bounded, then $\mathcal{V}^D \subset L^2(D)$.

**Proof.** Let $u \in \mathcal{V}^D$. Since $\nu$ is unimodal, we have $\nu \geq c > 0$ on $D \times D$. Consequently, there is $y \in D$ for which $\int_D |u(x) - u(y)|^2 \, dx < \infty$. For $a, b \in \mathbb{R}$ we have $a^2 \leq 2(a - b)^2 + 2b^2$, so $\int_D u(x)^2 \, dx \leq 2 \int_D |u(x) - u(y)|^2 \, dx + 2|D|u(y)^2 < \infty$. \hfill \Box

In view of Lemma 3.4 for $|D| < \infty$ it is plausible to let

$$||u||_{\mathcal{V}^D} = \sqrt{||u||_{L^2(D)}^2 + \mathcal{E}_D(u, u)}. \tag{3.2}$$

This is a seminorm, actually a norm on $\mathcal{V}^D$, because if a nonzero function $u$ vanishes a.e. in $D$, then by the positivity of $\nu$, the increments between $D^c$ and $D$ yield a positive value of $\mathcal{E}_D(u, u)$. Furthermore, $\mathcal{V}^D_D$ is a Hilbert space with this norm [22Lemma 2.3], [35 Lemma 3.4]. The completeness of $\mathcal{V}^D_D$ is also a consequence of the completeness of $\mathcal{V}^D$. The latter is not given in [22, 35], but it was verified in [19] for the fractional Laplacian. We present a short proof which only uses the local strict positivity of $\nu$. 

Lemma 3.5. If $|D| < \infty$, then $\mathcal{V}^D$ is complete with the norm $\| \cdot \|_{\mathcal{V}^D}$.

Proof. If $\emptyset \neq U \subset\subset D$, then
\[
\int_{D^c} u(y)^2 \nu(y, U) dy = \int_U \int_{D^c} u(y)^2 \nu(x, y) dy dx
\leq 2 \int_U \int_{D^c} (u(x) - u(y))^2 \nu(x, y) dy dx + 2 \int_U \int_{D^c} u(x)^2 \nu(x, y) dy dx
\leq 4\mathcal{E}_D(u, u) + 2 \int_U u(x)^2 \nu(x, D^c) dz \lesssim \|u\|_{\mathcal{V}^D}.
\]

We note that $y \mapsto \nu(y, U)$ is locally bounded from below on $\mathbb{R}^d$. Therefore each Cauchy sequence $u_n$ in $\mathcal{V}^D$ has a subsequence such that $u_{n_k} \to u$ a.e. as $k \to \infty$. By Fatou’s lemma $\|u\|_{\mathcal{V}^D} < \infty$, in fact $\|u_n - u\|_{\mathcal{V}^D} \to 0$, as $n \to \infty$, cf. [22] Lemma 2.3. □

Here is another simple result on the $L^2$-integrability implied by the $L^2$-integrability of increments.

Lemma 3.6. For all $g \in \mathcal{X}^D$ and $x \in D$ we have $\int_{D^c} g(z)^2 P_D(x, z) dz < \infty$.

Proof. By the definition of $\gamma_D$,
\[
\mathcal{H}_D(g, g) = \frac{1}{2} \int_{D^c} \int_D \int_D (g(z) - g(w))^2 \nu(w, x) P_D(x, z) dz dw dz < \infty.
\]

Since $\nu > 0$, for almost all $(x, w) \in D \times D^c$ we obtain
\[
\int_{D^c} g(z)^2 P_D(x, z) dz \leq 2 \int_{D^c} (g(w) - g(z))^2 P_D(x, z) dz + 2g(w)^2 < \infty.
\]

Thus $\int_{D^c} g(z)^2 P_D(x, z) dz < \infty$ for some $x \in D$, hence for all $x \in D$, cf. the proof of Lemma 4.2 □

We fix an arbitrary (reference) point $\xi_0 \in D$. For $g \in \mathcal{X}^D$, we let
\[
\|g\|_{D^c}^2 = \int_{D^c} g(z)^2 P_D(\xi_0, z) dz,
\]
which is finite by Lemma 3.6 (we omit $\xi_0$ from the notation). We define a norm on $\mathcal{X}^D$:
\[
\|g\|_{\mathcal{X}^D} = \sqrt{\|g\|_{D^c}^2 + \mathcal{H}_D(g, g)}.
\]

Arguing as in the last part of the proof of Lemma 3.5 we see that $\mathcal{X}^D$ is complete with this norm. In what follows we denote by $\mathcal{V}_0^{D*}$ the dual space of $\mathcal{V}_0^D$.

3.4. Stochastic process. We define
\[
\psi(\xi) = \int_{\mathbb{R}^d} (1 - \cos \xi \cdot x) \nu(|x|) dx, \quad \xi \in \mathbb{R}^d,
\]
the Lévy-Khinchine exponent. Since $\nu(\mathbb{R}^d) = \infty$, by [36, Theorem 27.7] and [31, Lemma 2.5], for every $t > 0$ there is a continuous function $p_t(x) \geq 0$ on $\mathbb{R}^d \setminus \{0\}$ such that
\[
\int_{\mathbb{R}^d} e^{\xi \cdot x} p_t(x) dx = e^{-t \psi(\xi)}, \quad \xi \in \mathbb{R}^d.
\]

Measures $\mu_t(dx) = p_t(x) dx$ form a weakly continuous convolution semigroup on $\mathbb{R}^d$. Accordingly,
\[
P_t f(x) = \int_{\mathbb{R}^d} f(y) p_t(y - x) dy, \quad t > 0, \quad x \in \mathbb{R}^d,
\]
is a strongly continuous semigroup of operators on $C_0(\mathbb{R}^d)$ and its generator has Fourier symbol $\psi(\xi)$ [36]. On Borel sets in the space $\Omega$ of càdlàg functions $\omega : [0, \infty) \mapsto \mathbb{R}^d$ we consider the
probability measures $\mathbb{P}^x$, $x \in \mathbb{R}^d$, constructed from the Kolmogorov’s extension theorem and the finite-dimensional distributions

$$\mathbb{P}^x_{t_1,t_2,\ldots,t_n}(A_1,\ldots,A_n) = \int_{A_1} \cdots \int_{A_n} \prod_{i=1}^n p_{t_i-t_{i-1}}(x_i - x_{i-1}) \, dx_n \cdots dx_1,$$

where $0 = t_0 \leq t_1 \leq \ldots \leq t_n$, $x_0 = x$, $A_1,\ldots,A_n \subset \mathbb{R}^d$ and $n = 1,2,\ldots$. The process $X(t) := \omega(t)$ on $\Omega$ is a convenient tool to handle $\mathbb{P}^x$. In particular, $\mathbb{P}^x(X_t \in A) = \int_A p_t(y - x) \, dy$ and $\mathbb{P}^x(X_{t_1} \in A_1,\ldots,X_{t_n} \in A_n) = \mathbb{P}^x_{t_1,t_2,\ldots,t_n}(A_1,\ldots,A_n)$. We call $\mathbb{P}^x$ the distribution of the process starting from $x \in \mathbb{R}^d$ and we let $\mathbb{E}^x$ be the corresponding integration. By the construction, $X = \{X_t\}_{t\geq 0}$ is a symmetric Lévy process in $\mathbb{R}^d$ with $(0,\nu,0)$ as the Lévy triplet [36 Section 11]. We let, as usual, $X_{t-} = \lim_{s \to t^-} X_s$ for $t > 0$ and $X_{0-} = X_0$. We introduce the time of the first exit of $X$ from $D$,

$$\tau_D = \inf\{t > 0 : X_t \notin D\}.$$ 

The Dirichlet heat kernel $p_t^D(x,y)$, is determined by the identity

$$\int_{\mathbb{R}^d} f(y)p_t^D(x,y) \, dy = \mathbb{E}^x[f(X_t) ; \tau_D > t], \quad t > 0, \quad x \in \mathbb{R}^d,$$

where $f : \mathbb{R}^d \to [0,\infty]$, cf. [17]. The Green function of $D$ is

$$G_D(x,y) = \int_0^\infty p_t^D(x,y) \, dt, \quad x,y \in \mathbb{R}^d,$$

and for functions $f \geq 0$ we have

$$\int_{\mathbb{R}^d} G_D(x,y)f(y) \, dy = \int_0^\infty \int_{\mathbb{R}^d} f(y)p_t^D(x,y) \, dy = \mathbb{E}^x \int_0^{\tau_D} f(X_t) \, dt, \quad x \in \mathbb{R}^d.$$ 

Accordingly, $G_D(x,y)$ is interpreted as the occupation time density of $X_t$ prior to the first exit from $D$. The following Ikeda-Watanabe formula defines the joint distribution of $(\tau_D, X_{\tau_D-}, X_{\tau_D})$ restricted to the event $\{\tau_D < \infty, X_{\tau_D-} \neq X_{\tau_D} \} :$ if $x \in D$, then

$$\mathbb{P}^x[\tau_D \in I, A \ni X_{\tau_D-} \neq X_{\tau_D} \in B] = \int_I \int_B \int p_u^D(x,y) \nu(y,z) \, dz \, dy \, du,$$

see, e.g., [12 Section 4.2] for a direct proof. Thus, if $\text{dist}(B,D) > 0$, then

$$\mathbb{P}^x[X_{\tau_D} \in B] = \int_B P_D(x,z) \, dz \leq 1, \quad x \in D,$$

where $P_D$ is the Poisson kernel [26]. The $L$-harmonic measure of $D$ for $x \in \mathbb{R}^d$, denoted $\omega^D_x$, is the distribution of the random variable $X_{\tau_D}$ with respect to $\mathbb{P}^x$. Thus,

$$\omega^D_x(dx) = \mathbb{P}^x[X_{\tau_D} \in dx].$$

From [3.5] we see that $P_D(x,z) \, dz$ is the part of $\omega^D_x(dx)$ which results from the discontinuous exit (by a jump) from $D$. The reader may easily obtain other marginal distributions of $(\tau_D, X_{\tau_D-}, X_{\tau_D})$. For instance,

$$\mathbb{P}^x[X_{\tau_D-} \in D] = \int_D G_D(x,y) \kappa_D(y) \, dy, \quad x \in D,$$

where

$$\kappa_D(x) = \int_{D^c} \nu(x,z) \, dz, \quad x \in D.$$

Formula [3.5] allows to interpret $p_t^D(x,y)$ as the density function of the distribution of $X_u$ for the process killed at time $\tau_D$. We interpret $\kappa_D(x)$ as the intensity of escape (or killing) outside $D$. For $U \subset D$ we have inequalities $p^U \leq p^D$, $G^U \leq G_D$. Also, $P_U(x,z) \leq P_D(x,z)$ for $x \in U$, $z \in D^c$ and $\gamma_U(z,w) \leq \gamma_D(z,w)$ for $z,w \in D^c$. These inequalities are referred to as domain monotonicity.
4. Harmonic functions

Suppose that \( \nu \) satisfies A1 and A2. Let \( L \) be the operator given by (1.1) and let \( (X_t, \mathbb{P}_x)_{t \geq 0, x \in \mathbb{R}^d} \) be the symmetric pure-jump Lévy process in \( \mathbb{R}^d \) constructed above. As before, \( D \) denotes nonempty open subset of \( \mathbb{R}^d \).

**Definition 4.1.** We say that \( u : \mathbb{R}^d \to \mathbb{R} \) is \( L \)-harmonic (or harmonic, if \( L \) is understood) in \( D \) if it has the mean value property, that is for all (open) \( U \subset D \) and \( x \in U \),

\[
u \text{ follows by letting } u = 0.\]

\[
\text{We say that } u \text{ is regular } L \text{-harmonic in } D \text{ if } u(x) = \mathbb{E}^x u(X_{\tau_D}) \text{ for } x \in D. \text{ Here we assume that the integrals are well-defined, in fact absolutely convergent.}
\]

**Lemma 4.2.** If \( u \) is regular \( L \)-harmonic in \( D \), then it is \( L \)-harmonic in \( D \).

**Proof.** We first assume that \( u \) is nonnegative. Let \( g : D^c \to [0, \infty] \). Let \( u(x) = \mathbb{E}^x g(X_{\tau_D}) \) for \( x \in D \) and \( u(x) = g(x) \) for \( x \in D^c \). Let \( U \) be an arbitrary open set such that \( U \subset D \). Note that \( \tau_U \leq \tau_D \). Let \( x \in U \). By the strong Markov property of \( X \) and the regular harmonicity of \( u \) in \( D \),

\[
u \text{ is regular } L \text{-harmonic in } D \text{ if } u(x) = \mathbb{E}^x u(X_{\tau_D}) \text{ for } x \in D. \text{ Here we assume that the integrals are well-defined, in fact absolutely convergent.}
\]

\[
\text{Therefore } u \text{ is } L \text{-harmonic. We note for clarity that by the (boundary) Harnack inequality of Theorem 1.9, if } u(x) < \infty \text{ for some } x \in D, \text{ then } u(x) < \infty \text{ for all } x \in D. \text{ The case of the general } u \text{ follows by letting } g(x) = u(x) \text{ for } x \in D^c \text{ and considering the positive and negative parts of } g.
\]

In fact, the above proof shows that \( \{u(X_{\tau_U}), U \subset D\} \) is a martingale ordered by inclusion of open subsets of \( D \) and closed by \( u(X_{\tau_D}) \).

\[
\square
\]

**Lemma 4.3.** If \( u \) is \( L \)-harmonic in \( D \), then \( u \in L^1_{\text{loc}}(\mathbb{R}^d) \).

**Proof.** If \( x \in D \) and \( 0 < \epsilon < \text{dist}(x, D^c) \), then \( u(x) = \int_{B^c(x)} u(z) B_z(x, z) \text{d}z \). By unimodality, Ikeda-Watanabe formula and strict positivity of the Lévy measure, we know that the Poisson kernel of the ball is strictly positive and radially decreasing, see e.g. [28] for details. Hence, for every \( R > \epsilon \) there is \( C > 0 \), such if \( |z - x| < R \), then \( B_z(x, z) \geq C \). Since \( x, \epsilon, R \) were arbitrary, \( u \in L^1_{\text{loc}}(\mathbb{R}^d) \).

The next result is due to Grzywny and Kwaśnicki [25].

**Lemma 4.4.** Let \( X_t \) be an isotropic, unimodal Lévy process in \( \mathbb{R}^d \) and let \( 0 \leq q < r < \infty \). There is a radial kernel \( P_{q,r}(z) \), a constant \( C = C(X, q, r) > 0 \) and a probability measure \( \mu_{q,r} \) on the segment \([q, r] \), such that

\[
u \text{ is regular } L \text{-harmonic in } D \text{ if } u(x) = \mathbb{E}^x u(X_{\tau_D}) \text{ for } x \in D. \text{ Here we assume that the integrals are well-defined, in fact absolutely convergent.}
\]

\[
0 = \int_{B_q} P_{q,r}(z) \text{d}z \leq C \text{ in } \mathbb{R}^d, \quad P_{q,r} = C \text{ in } B_r \setminus B_q \text{ and } P_{q,r} \text{ decreases radially on } B_r^c.
\]

Furthermore, \( P_{q,r}(z) \leq P_{r, r}(z) \), for \( |z| > r \), and if \( f \) is \( L \)-harmonic in \( B_r \), then

\[
0 = \int_{\mathbb{R}^d \setminus B_q} f(z) P_{q,r}(z) \text{d}z.
\]

**Corollary 4.5.** If \( f \) is \( L \)-harmonic in \( B_{2r} \), then \( f = f * \mathbb{P}_{0,r} \) in \( B_r \).

We will use the representation from Lemma 4.4 to prove that Poisson extensions are at least twice continuously differentiable. In the proof we closely follow the arguments from Theorem 1.7 and Remark 1.8 b) in [25] except that we do not assume the boundedness of \( u \).
Theorem 4.6. Suppose that \( L \) satisfies A1 and A2 and let \( D \subset \mathbb{R}^d \) be an open set. If \( u : \mathbb{R}^d \to \mathbb{R} \) is \( L \)-harmonic in \( D \), e.g., if \( u(x) = \int_{D^c} u(z) P_D(x,z) \, dz \) for \( x \in D \), then \( u \in C^2(D) \).

Proof. We are in a position to apply Lemma 4.4. Let \( x \in D \), and let \( r > 0 \) be such that \( B_{2r}(x) \subset D \). Since \( \nu(z) \) is continuous, we get from (4.1) that kernels \( \overline{P}_{q,r} \) are continuous as well and by Corollary 4.5 \( u \) is continuous in \( B_r(x) \). Next we fix a nonnegative smooth radial function \( \kappa \) such that \( 0 \leq \kappa \leq 1 \), \( \kappa \equiv 1 \) in \( B_{3r} \) and \( \kappa \equiv 0 \) outside \( B_{2r} \). As in [25], denote \( \pi_r(z) = \overline{P}_{0,r}(z) \kappa(z) \) and \( \Pi_r(z) = \overline{P}_{0,r}(z)(1 - \kappa(z)) \). Obviously, \( u = u \ast \pi_r + u \ast \Pi_r \) in \( B_r(x) \). In particular, both terms are well-defined. Iterating, we get

\[
\begin{align*}
u &= (\pi_r + \pi_r \ast \pi_r + \pi_r \ast 2 \ast \pi_r + \ldots \pi_r \ast (k-1) \ast \pi_r + \pi_r \ast k) \ast u \\
&= (\delta_0 + \pi_r \ast 2 \ast \ldots + \pi_r \ast (k-1) \ast \pi_r + \pi_r \ast k \ast u.
\end{align*}
\]

Using an argument based on the Fourier transform as in [25, Proof of Theorem 1.7], we get that for every \( N \) and sufficiently large \( k \), \( \pi_r \ast k \) is \( N \) times continuously differentiable. It is also compactly supported. Since \( u \in L^1_{\text{loc}}(\mathbb{R}^d) \), it follows that \( \pi_r \ast k \ast u \) has \( N \) continuous derivatives in \( D \), but for our purposes it suffices to take \( N = 2 \).

We will now handle the first summand in the above expansion of \( u \). First, observe that for every \( \theta > r \), \( |z| > \theta > r \), and \( |\alpha| = 1 \) or 2 we have

\[
|\partial^\alpha \overline{P}_{0,r}(z)| \leq C_{\theta,r} \overline{P}_{0,r}(z).
\]

Indeed, by the definition of \( \overline{P}_{0,r} \) and the Ikeda-Watanabe formula we can write

\[
\overline{P}_{0,r}(z) = \int_{[0,r]} P_{B_s}(z) \mu_{0,r}(ds) = \int_{[0,r]} \int_{B_s} \nu(z,y) G_{B_s}(0,dy) \mu_{0,r}(ds)
\]

and further

\[
\partial^\alpha \overline{P}_{0,r}(z) = \int_{[0,r]} \int_{B_s} \partial^\alpha \nu(z,y) G_{B_s}(0,dy) \mu_{0,r}(ds).
\]

For \( z \) as above and \( y \in B_s \subset B_r \) we have \( |z - y| \geq \theta - r \). By A1, \( |\partial^\alpha \overline{P}_{0,r}(z)| \leq C_{\theta,r} \overline{P}_{0,r}(z) \), so that

\[
|\partial^\alpha \overline{P}_{0,r}(z)| \leq C_{\theta,r} \int_{[0,r]} \int_{B_s} \nu(z,y) G_{B_s}(0,dy) \mu_{0,r}(ds) = C_{\theta,r} \overline{P}_{0,r}(z).
\]

Since \( \operatorname{supp} \Pi_r \subset B_{\frac{3r}{2}} \), and \( \kappa \) is smooth, from the Leibniz rule and (4.2) we see that for all \( z \in \mathbb{R}^d \),

\[
|\partial^\alpha \Pi_r(z)| \leq C_r |\Pi_r(z)|. 
\]

Therefore if \( |\alpha| \leq 2 \), then

\[
\int_{\mathbb{R}^d} |\partial^\alpha \Pi_r(x - z) u(z)| \, dz < \infty,
\]

which allows to differentiate under the integral sign and so \( \partial^\alpha \Pi_r \ast u(x) \) is well-defined. Continuity of the derivative follows from the continuity of \( \partial^\alpha \nu \) and the dominated convergence.

\[\square\]

Lemma 4.7. If \( u \) is \( L \)-harmonic in \( D \), then \( Lu = 0 \) on \( D \).

Proof. By Theorem 4.6 \( u \in C^2(D) \). Let \( x \in U \subset \subset D \). Let \( \phi \in C^2_c(\mathbb{R}^d) \) be such that \( u = \phi \) on \( U \). Let \( w = u - \phi \). We recall that on \( C^2_c(\mathbb{R}^d) \), \( L \) coincides with the generator of the semigroup \( \{P_t\} \) [36 Theorem 31.5] and also with the Dynkin characteristic operator [21 Chapter V.3]

\[
\mathcal{U} \phi(x) = \lim_{r \to 0} \frac{\mathbb{E}^x \phi(X_{\tau_{B(x,r)}}) - \phi(x)}{\mathbb{E}^x \tau_{B(x,r)}}.
\]
Lemma 4.8. Let \( u \) be a bounded continuous function near \( x \). The first integral in (4.4) is not greater than \( \nu \). By [25, Theorem 1.9], \( \int_{B(x,r)} \nu(x,y) u(y) |dy| < \infty \) for \( r > 0 \). It follows that \( z \mapsto \int_{U_c} \nu(z,y) u(y) |dy| \) is bounded for \( r > 0 \). Therefore, \( \nu \int_{U_c} \nu(z,y) u(y) |dy| \) converges weakly to the Dirac mass at \( x \) as \( r \to 0 \). Then \( \int U_c \nu(x,y) u(y) |dy| \) is bounded by a constant depending only on \( r \). Then \( \nu \int_{U_c} \nu(x,y) u(y) |dy| \) is bounded by a constant depending only on \( r \).

Proof. First, we prove that \( \int_{U_c} \nu(x,y) u(y) |dy| \) is bounded by a constant depending only on \( U \). To this end, choose small \( \epsilon > 0 \) so that \( u \in C^2(dU) \) on \( U + B_2 \). In particular \( u \) all of its second-order partial derivatives are bounded on \( U + B_\epsilon \). By Taylor’s formula, \( \phi(x) \) is well-defined and

\[
\begin{align*}
L \phi(x) = L \phi(x) + Lu(x) - \phi(x) = \int U_c \nu(x,y) u(y) |dy|.
\end{align*}
\]

We should warn the reader that for more general operators, \( L \)-harmonic functions may lack sufficient regularity to calculate \( Lu \) pointwise, see remarks after [13, Corollary 20].

We note that if \( u \in L_{loc}^1(\mathbb{R}^d) \) and \( \int_{B_\rho} |u(y)| |\nu(y)| dy < \infty \) for some \( \rho > 0 \), then it is so for all \( \rho > 0 \).

**Lemma 4.8.** Let \( U \subset \subset D \) and let \( u \in C^2(\overline{U}) \cap L_{loc}^1(\mathbb{R}^d) \). Assume that \( \int_{B_\rho} |u(y)| |\nu(y)| dy < \infty \) for some \( \rho > 0 \). Then \( Lu \) is bounded on \( \overline{U} \) and for every \( x \in \mathbb{R}^d \),

\[
\begin{align*}
\mathbb{E}^x u(X_{\tau_U}) - u(x) = \int_U G_U(x,y) Lu(y) |dy|.
\end{align*}
\]

Proof. Both sides of (4.3) are equal to zero for \( x \notin \overline{U} \), let \( x \in \overline{U} \). First, we prove that \( Lu(x) \) is bounded by a constant depending only on \( U \). To this end, choose small \( \epsilon > 0 \) so that \( u \in C^2(\overline{U + B_\epsilon}) \). In particular \( u \) all of its second-order partial derivatives are bounded on \( U + B_\epsilon \). By Taylor’s formula, \( Lu(x) \) is well-defined and

\[
\begin{align*}
|Lu(x)| &= \left| \frac{1}{2} \int_{\mathbb{R}^d} \left( 2u(x) - u(x + y) - u(x - y) \right) \nu(y) |dy| \right| \\
&\leq \frac{1}{2} \sup_{\xi \in U + B_\epsilon} |D^2 u(\xi)| \int_{B_{\epsilon}} |y|^2 \nu(y) |dy| + \frac{1}{2} \int_{B_{\epsilon}^c} \left| (2u(x) - u(x + y) - u(x - y)) \nu(y) |dy| \right| \\
&\leq C_{\epsilon} \int_{B_{\epsilon}} |y|^2 \nu(y) |dy| + |u(x)| \nu(B_{\epsilon}^c) + \int_{B_{\epsilon}^c} |u(x + y)| \nu(y) |dy|.
\end{align*}
\]

To estimate the last integral, let \( R = \sup_{x \in U} |x| + \epsilon \). Then,

\[
\begin{align*}
\int_{B_{\epsilon}^c} |u(x + y)| \nu(y) |dy| &= \int_{B_{\epsilon}^c} |u(z)| \nu(z, x) |dz| \\
&= \int_{B_{\epsilon}^c \cap B_{2R}} |u(z)| \nu(z, x) |dz| + \int_{B_{\epsilon}^c \cap B_{2R}^c} |u(z)| \nu(z, x) |dz|.
\end{align*}
\]

The first integral in (4.4) is not greater than \( \nu(\epsilon) \int_{B_{2R}} |u(z)| |dz| < \infty \). For the second integral we note that \( x \in \overline{U}, z \notin B_{2R} \) imply \( |z - x| \geq |z| - |x| \geq |z| - R \) thus from (A2) there is \( C_R > 0 \) such that \( \nu(z, x) \leq C_R \nu(z) \) and so the integral is bounded by \( C_R \int_{B_{2R}^c} |u(z)| \nu(z) |dz| < \infty \).
Collecting all the bounds together we see that
\[
|Lu(x)| \leq \sup_{\xi \in \overline{U} + B_1} |D^2u(\xi)| \int_{\overline{U} + B_1} (|y^2| \wedge 1) \nu(y) dy + 2|u(x)|\nu(B_2^c)
\]
\[
+ 2C_\epsilon \int_{B_{2R}} |u(z)|dz + 2M \int_{B_{2R}^c} |u(z)|\nu(z)dz.
\]

For the second part of the statement, recall that by Dynkin’s formula \([21 \text{ (5.8)}]\) for \(u \in C_c^\infty(\mathbb{R}^d)\),
\[
(4.5) \quad \mathbb{E}^x u(x_{\tau_U}) - u(x) = \mathbb{E}^x \int_0^{\tau_U} Lu(X_t)dt = \int_0^\infty \mathbb{E}^x|Lu(X_t); \tau_U > t| dt.
\]
Here the change the order of integration was justified because \(Lu\) is bounded on \(\overline{U}\) and \(\mathbb{E}^x \tau_U < \infty\), cf. \([10, 33]\). As usual, we let \(p^U\) denote the transition density of the process killed upon leaving \(U\). Since \(Lu\) is measurable and bounded on \(\overline{U}\),
\[
\mathbb{E}^x |Lu(X_t); \tau_U > t| = \int_U p^U_t(x, y)Lu(y)dy.
\]
We conclude the case of \(u \in C_c^\infty(\mathbb{R}^d)\) by writing
\[
\mathbb{E}^x u(x_{\tau_U}) - u(x) = \int_0^\infty \int_U p^U_t(x, y)Lu(y)dydt = \int_U G_U(x, y)Lu(y)dy.
\]
For the general \(u\) satisfying the assumptions of Lemma 4.8 we use approximation. We consider \(\phi_n \in C_c^\infty(\mathbb{R}^d)\) such that \(\phi_n \to u\) at the same time in \(C^2(\overline{U})\), in \(L^1\) on a sufficiently large ball and in \(L^1(\nu(0, \cdot)1_{B_\rho})\) with \(\rho > 0\) so small that \(u \in C^2(U + B_2\rho)\). Arguing as in \(\text{[7 \text{ (7)}]}\) and performing calculations similar to \(\text{[4 \text{.4]}}\), we get that \(L\phi_n \to Lu\) uniformly in \(U\). From \([28]\) we have \(P_U(x, y) \preceq \nu(x, y)\) if \(x \in U\) and \(\text{dist}(y, U) > \rho\). It follows that \(\mathbb{E}^x \phi_n(x_{\tau_U}) \to \mathbb{E}^x u(x_{\tau_U})\).

We next generalize the Hardy-Stein formula of Bogdan, Dyda and Luks \([7,\text{ Lemma 3}].\)

**Lemma 4.9.** If \(u : \mathbb{R}^d \to \mathbb{R}\) is \(L\)-harmonic in \(D\) and \(U \subset \subset D\), then
\[
(4.6) \quad \mathbb{E}^x u^2(x_{\tau_U}) = u(x)^2 + \int_U G_U(x, y) \int_{\mathbb{R}^d} (u(z) - u(y))^2 \nu(z, y)dzdy.
\]

**Proof.** For \(\epsilon > 0\) we denote \(U_\epsilon = \{z \in \mathbb{R}^d : d(z, U) < \epsilon\}\). Let \(x \in U\).

CASE 1. Assume that \(\int_{U_c} u(z)^2 \nu(z, x)dz = \infty\). Then \(\mathbb{E}^x u(x_{\tau_U})^2 = \infty\) as well. Indeed, take \(\delta > 0\) such that \(\overline{B}(x, \delta) \subset U\). By domain monotonicity we have \(\nu_U(x, z) \geq \nu_B(x, z)\), \(z \in U_c\).

For \(z \in U^c\) we have \(|z - x| \geq \delta\) and so \(\nu_B(x, z) \geq \nu(x, z)\), with a constant depending only on \(x\) and \(\delta\) (cf. Lemma 2.2 in \([25]\)). It follows that
\[
\mathbb{E}^x u^2(x_{\tau_U}) = \int_{U^c} u(z)^2 \nu_U(x, z)dz \geq \int_{U^c} u(z)^2 \nu_B(x, z)dz \geq \int_{U^c} u(z)^2 \nu(x, z)dz = \infty.
\]

Further, we claim that in this case the right-hand side of \(\text{(4.6)}\) is also infinite. Namely, we will check that \(\int_{\mathbb{R}^d} u(z) - u(y)\nu(z, y)dy = \infty\) for \(y \in B(x, \frac{\delta}{2})\), and then use the positivity of the Green function in \(U\) \([25]\). If \(x, y, z\) are as above, then \(|z - y| \leq C_{\delta, \epsilon}|z - x|\) with some \(C_{\delta, \epsilon} > 0\), therefore by \(A2\) we have that \(\int_{U^c} u(z)^2 \nu(x, z)dz = \infty\) yields \(\int_{U^c} u(z)^2 \nu(y - z)dz = \infty\). Let \(A = A(y) = \{z \in U^c : (u(y) - u(z))^2 \geq \frac{1}{2}u(z)^2\}\). Note that \(|u(z)| \leq C|u(y)|\) for \(z \in U^c \setminus A\) and thus the integral over
$U \setminus A$ is finite. It follows that $\int_A u(z)^2 \nu(y, z) dz = \infty$, in particular $A(y)$ is nonempty. Therefore,

$$\infty = \frac{1}{2} \int_A u(z)^2 \nu(y, z) dz \leq \int_A (u(y) - u(z))^2 \nu(y, z) dz,$$

and the claim follows.

Case 2. Now assume that $\int_{U^c} u(y)^2 \nu(x, y) dy < \infty$ for every $\epsilon > 0$. Since $U \subset D$, Theorem 4.6 implies that $u^2 \in C^2(\overline{U})$, hence $u^2 \in L^1_{loc}(\mathbb{R}^d)$. By Lemma 4.8, $Lu^2$ is bounded in $\overline{U}$ and

$$\mathbb{E}^u u^2(X_{\tau_U}) = u(x)^2 + \int_U G_U(x, y) Lu^2(y) dy. \tag{4.7}$$

We can now compute $Lu^2(y)$ for $y \in U$. The $L$-harmonicity of $u$ yields (cf. [7, proof of Lemma 3])

$$Lu^2(y) = \int_{\mathbb{R}^d} (u(z) - u(y))^2 \nu(z, y) dz, \quad y \in U. \tag{4.8}$$

The latter integral is convergent because $u$ is Lipschitz near $y$, and far from $y$ we can use the assumed integrability condition (we may take smaller $\epsilon$ to ensure that $U_\epsilon \subset D$). Inserting (4.8) into (4.7) yields the desired result. \hfill \Box

**Proof of Theorem 2.1.** If $u$ is $L$-harmonic in $D$ and $x \in D$, then

$$\sup_{x \in U \subset D} \mathbb{E}^u u(X_{\tau_U})^2 = u(x)^2 + \int_D G_D(x, y) \int_{\mathbb{R}^d} (u(z) - u(y))^2 \nu(z, y) dz dy. \tag{4.9}$$

Indeed, since $\sup_{U \subset D} G_U(x, y) = G_D(x, y)$, (4.9) follows from Lemma 4.9 and the monotone convergence theorem. Clearly, (4.9) is a reformulation of (2.5). \hfill \Box

5. **Solving the Dirichlet problem**

5.1. **Sobolev regularity of Poisson integrals.**

**Proof of Theorem 2.3.** Assume that $g \in L^1(D(x, \cdot))$ for some $x \in D$, cf. the proof of Lemma 4.2. By Lemma 3.6 this is true if $\mathcal{H}_D(g, g) < \infty$. We are going to prove that $\mathcal{H}_D(g, g) = \mathcal{E}_D(u, u)$, where $u = P_D[g]$ is the Poisson extension of $g$. By (4.7), $P_D(x, z) dz$ is a probability measure on $D^c$ for every $x \in D$. The integral of $g$ against this measure is equal to $u(x)$. Recall that if $Y$ is a (real-valued) random variable and $\mathbb{E}|Y| < \infty$, then for every $a \in \mathbb{R}$,

$$\mathbb{E}(Y - a)^2 = \mathbb{E}(Y - \mathbb{E}Y)^2 + (\mathbb{E}Y - a)^2,$$

also when $\mathbb{E}Y^2 = \infty$. In particular, for $Y = u(X_{\tau_D})$ we have

$$\mathbb{E}_x Y = u(x) \quad \text{and} \quad \mathbb{E}_x Y^2 = \int_{D^c} u^2(z) P_D(x, z) dz, \quad x \in D.$$

Therefore,

$$\int_{D^c} (u(w) - u(z))^2 P_D(x, z) dz = \int_{D^c} (u(x) - u(z))^2 P_D(x, z) dz + (u(w) - u(x))^2, \quad x \in D, w \in D^c,$$

and further, by Lemma 3.6,

$$2\mathcal{H}_D(g, g) = \int_D \int_{D^c} \int_{D^c} ((u(x) - u(z))^2 P_D(x, z) \nu(x, w) dz dwdx$$

$$+ \int_D \int_{D^c} (u(w) - u(x))^2 \nu(x, w) dwdx$$

$$= \int_D \int_{D^c} (u(x) - u(z))^2 P_D(x, z) \kappa_D(x) dz dx + \int_D \int_{D^c} (u(w) - u(x))^2 \nu(x, w) dwdx. \tag{5.1}$$
We next derive a simple consequence of Proposition 5.1. By formula (4.9) applied to the function \( z \mapsto u(z) = u(z) - u(x) \), for each \( x \in D \) we get

\[
(5.2) \quad \sup_{x \in U \subseteq D} \int_{U^c} (u(x) - u(z))^2 P_U(x, z) dz = \int_D G_D(x, y) \int_{\mathbb{R}^d} (u(y) - u(z))^2 \nu(y, z) dz dy.
\]

By (A.7), for \( U^c \) satisfying VDC we have

\[
\int_{U^c} (u(x) - u(z))^2 P_U(x, z) dz = \mathbb{E}^x (u(X_{T_D}) - u(x))^2, \quad x \in U.
\]

By the proof of Lemma 4.2, \( \{u(X_{T_D})\}_{U \subseteq D} \) is a closed martingale. Therefore the Hardy-Stein formula (4.9) remains valid if we replace \( \sup_{x \in U \subseteq D} \) by \( \lim_{x \in U \uparrow D} \) with \( U \subseteq D \) increasing to \( D \). By (A.8), for almost every trajectory of \( X \), there exists \( U \subseteq D \) such that \( X_{T_D} = X_{T_D} \), so \( u(X_{T_D}) \to u(X_{T_D}) = g(X_{T_D}) \) as \( U \uparrow D \). By the martingale convergence theorem and (5.2),

\[
\int_{D^c} (u(x) - u(z))^2 P_D(x, z) dz = \int_D G_D(x, y) \int_{\mathbb{R}^d} (u(y) - u(z))^2 \nu(y, z) dz dy, \quad x \in D.
\]

We now turn our attention to the first integral in (5.1). By Fubini-Tonelli,

\[
\int_D \int_{D^c} ((u(x) - u(z))^2 P_D(x, z) \kappa_D(x) dz dx
\]

\[
= \int_D \int_{\mathbb{R}^d} G_D(x, y) ((u(y) - u(z))^2 \nu(y, z) \kappa_D(x) dz dy dx
\]

\[
= \int_{\mathbb{R}^d} \int_D \kappa_D(x) G_D(x, y) dx ((u(y) - u(z))^2 \nu(y, z) dz dy.
\]

Since \( D^c \) satisfies VDC, by (3.8) and Lemma A.1,

\[
(5.3) \quad \int_D \kappa_D(x) G_D(x, y) dx = \mathbb{P}_y (X_{T_D} \in D) = 1, \quad y \in D.
\]

Therefore,

\[
\int_D \int_{D^c} (u(x) - u(z))^2 P_D(x, z) \kappa_D(x) dz dx = \int_{\mathbb{R}^d} \int_D (u(y) - u(z))^2 \nu(y, z) dz dy.
\]

By this and (5.1),

\[
2H_D(g, g) = \int_{\mathbb{R}^d} \int_D (u(x) - u(y))^2 \nu(x, y) dx dy + \int_D \int_{D^c} (u(x) - u(y))^2 \nu(x, y) dy dx
\]

\[
= \int_{\mathbb{R}^d \times \mathbb{R}^d \setminus D \times D^c} (u(x) - u(y))^2 \nu(x, y) dy dx = 2\mathcal{E}_D(u, u).
\]

In the setting of Theorem 2.3 we immediately obtain the following consequences.

**Corollary 5.1.** \( \text{Ext} (g) = P_D[g] \) defines an extension operator from \( \mathcal{X}^D \) into \( \mathcal{V}^D \).

**Corollary 5.2.** The Poisson extension \( u = P_D[g] \) satisfies

\[
\mathcal{E}_{\mathbb{R}^d}(u, u) = \frac{1}{2} \int_{D \times D^c} (g(w) - g(z))^2 (\gamma_D(w, z) + \nu(w, z)) dw dz.
\]

Corollary 5.2 and the Sobolev-Hardy-Stein identity in Theorem 2.3 may be considered as analogues of the Douglas integral [24] (1.2.18).

In the next section we get the minimality of the Poisson extension, for \( \mathcal{E}_D \) and \( \mathcal{E}_{\mathbb{R}^d} \).
5.2. **Weak and variational solutions.** The next proposition shows that weak solutions coincide with the variational solutions of \((1.6)\). The proof is classical but we include it here to make our argument self-contained, cf. [34, 35].

**Proposition 5.3.** Let \(g \in \mathcal{X}^D\) and let \(u\) be a weak solution of \((1.6)\). If \(\overline{g} : \mathbb{R}^d \to \mathbb{R}\) is another measurable function equal to \(g\) a.e. on \(D^c\), then

\[
(5.4) \quad \mathcal{E}_D(u, u) \leq \mathcal{E}_D(\overline{g}, \overline{g}).
\]

The converse is also true.

**Proof.** Note that \((5.4)\) holds trivially when either \(\mathcal{E}_D(\overline{g}, \overline{g}) = +\infty\) or \(\mathcal{E}_D(u, u) = 0\). Therefore we may assume otherwise. We have \(\overline{g} - u \in \mathcal{V}^D_0\). Since \(u\) is a weak solution, \(\mathcal{E}_D(u, \overline{g} - u) = 0\), hence \(\mathcal{E}_D(\overline{g}, u) = \mathcal{E}_D(u, u)\) and

\[
\mathcal{E}_D(u, u) = \mathcal{E}_D(\overline{g}, u) \leq (\mathcal{E}_D(\overline{g}, \overline{g}))^{1/2} (\mathcal{E}_D(u, u))^{1/2}.
\]

Canceling out \(\mathcal{E}_D(u, u) > 0\), we obtain \((5.4)\).

For the second part, let \(\phi \in \mathcal{V}^D_0\). Since \(u\) is a minimizer, we have

\[
0 \leq \mathcal{E}_D(u + \lambda \phi, u + \lambda \phi) - \mathcal{E}_D(u, u) = 2\lambda \mathcal{E}_D(u, \phi) + \lambda^2 \mathcal{E}_D(\phi, \phi), \quad \lambda \in \mathbb{R}.
\]

This necessitates that \(\mathcal{E}_D(u, \phi) = 0\), hence \(u\) is a weak solution. \(\square\)

By Theorem [2.3], if \(g \in \mathcal{X}^D\), then its Poisson extension belongs to \(\mathcal{V}^D\). In fact, the Poisson extension \(P_D[g]\) is the weak solution of \((1.6)\), as we will shortly see.

**Theorem 5.4.** Suppose \(L\) satisfies \(A_1, A_2\). Let nonempty open \(D \subset \mathbb{R}^d\) have continuous boundary and let \(D^c\) fulfill VDC. If \(g \in \mathcal{X}^D\), then \(u = P_D[g]\) is a solution to the Dirichlet problem \((1.6)\). Furthermore, for bounded \(D\) the solution is unique.

**Proof.** The uniqueness for bounded \(D\) follows from [35, Theorem 4.2], therefore it is enough to show that \(P_D[g]\) is a weak solution.

If \(g \in \mathcal{X}^D\), then \(u = P_D[g]\) is well-defined, \(u = g\) on \(D^c\) and by Theorem [2.3, 3] \(u \in \mathcal{V}^D\). By Theorem [A.4] we only need to verify \((5.7)\) for \(\phi \in C_0^\infty(D)\). Let \(\epsilon > 0\), \(\nu_\epsilon(x, y) = \nu(x, y)1_{|x-y|>\epsilon}\) and

\[
2\mathcal{E}_D^\epsilon(u, \phi) = \int_{\mathbb{R}^d \times \mathbb{R}^d \setminus (D^c \times D^c) \setminus \{|x-y|>\epsilon\}} (u(x) - u(y))(\phi(x) - \phi(y))\nu_\epsilon(x, y)dxdy
\]

\[
= \int_{\mathbb{R}^d \times \mathbb{R}^d \setminus (D^c \times D^c)} (u(x) - u(y))(\phi(x) - \phi(y))\nu_\epsilon(x, y)dxdy.
\]

Since \(u, \phi \in \mathcal{V}^D\), the integral \(\mathcal{E}_D(u, \phi)\) is absolutely convergent and \(\mathcal{E}_D^\epsilon(u, \phi) \to \mathcal{E}_D(u, \phi)\) as \(\epsilon \to 0\). We claim that \(\mathcal{E}_D^\epsilon(u, \phi) \to 0\) when \(\epsilon \to 0\). Indeed,

\[
\int_A (u(x) - u(y))(\phi(x) - \phi(y))\nu(x, y)dxdy
\]

is an absolutely convergent integral for every set \(A \subset \mathbb{R}^d \times \mathbb{R}^d \setminus D^c \times D^c\). We have

\[
2\mathcal{E}_D^\epsilon(u, \phi) = \int_D \int_D (u(x) - u(y))(\phi(x) - \phi(y))\nu_\epsilon(x, y)dxdy
\]

\[
+ \int_D \int_{D^c} (u(x) - u(y))(\phi(x) - \phi(y))\nu_\epsilon(x, y)dxdy
\]

\[
+ \int_{D^c} \int_D (u(x) - u(y))(\phi(x) - \phi(y))\nu_\epsilon(x, y)dxdy =: I + II + III.
\]
By the symmetry of \( \nu \) and the fact that \( \phi \equiv 0 \) on \( D^c \), we readily see that \( \mathbb{I} = \mathbb{II} = \int \int (u(x) - u(y)) \phi(x) \nu_\epsilon(x, y) dy dx \). Also, \( I = 2 \int \int (u(x) - u(y)) \phi(x) \nu_\epsilon(x, y) dy dx \), which converges absolutely by the Cauchy-Schwarz inequality and the fact that \( \phi(x) \nu_\epsilon(x, y) dy dx \) is a finite measure. Thus,

\[
2\mathcal{E}_D^\epsilon(u, \phi) = 2 \int_{\mathbb{R}^d} \int_D (u(x) - u(y)) \phi(x) \nu_\epsilon(x, y) dy dx \\
= 2 \int_D \phi(x) \left( \int_{\mathbb{R}^d} (u(x) - u(y)) \nu_\epsilon(x, y) dy \right) dx \\
= -2 \int_D \phi(x) L_\epsilon u(x) dx = -2 \int_{\text{supp} \phi} \phi(x) L_\epsilon u(x) dx,
\]

where

\[
L_\epsilon u(x) := \int_{\mathbb{R}^d} (u(y) - u(x)) \nu_\epsilon(x, y) dy.
\]

The function \( u \) is regular \( L \)-harmonic (see Definition 4.1). By Theorem 4.6, \( u \in C^2(D) \) and by Lemma 4.7 \( Lu(x) = 0 \) for \( x \in D \). In particular, \( L_\epsilon u(x) \to Lu(x) = 0 \) for \( x \in D \). We will prove that the convergence is uniform on the support of \( \phi \). For \( x \in D \), \( 0 < \eta < \epsilon \),

\[
(5.5) \quad L_\eta u(x) - L_\epsilon u(x) = \int_{|y-x| < \eta} (u(y) - u(x)) \nu(x, y) dy.
\]

Let \( \delta = \text{dist}(\text{supp} \phi, D^c) > 0 \) and let \( \epsilon < \delta/2 \). If \( x \in \text{supp} \phi \), then points \( y \) appearing in (5.5) belong to the compact set \( K := \text{supp} \phi + \overline{B}_\delta/d \subset D \). By Theorem 4.3, \( u \in C^2(D) \), in particular

\[
C_K := \sup_{x \in K, i,j \leq d} (|u(x)|, |\partial_i^j u(x)|) < \infty.
\]

By the symmetry \( \nu(x, y) = \nu(y, x) \),

\[
L_\eta u(x) - L_\epsilon u(x) = \int_{|\eta-x| < \eta} (u(y) - u(x)) \nu(x, y) dy \\
= \int_{|\eta-x| < \epsilon} (u(y) - u(x) - \nabla u(x) \cdot (y-x)) \nu(x, y) dy.
\]

From Taylor’s formula we obtain

\[
|Lu(x) - L_\epsilon u(x)| \leq \frac{C_K}{2} \lim_{\eta \to 0} \int_{|\eta-x| < \epsilon} |y-x|^2 \nu(x, y) dy \\
\leq \frac{C_K}{2} \int_{|x-y| < \epsilon} |y-x|^2 \nu(x, y) dy = C_K \epsilon, \quad x \in \text{supp} \phi,
\]

and \( C_\epsilon \to 0 \) as \( \epsilon \to 0 \), so \( \int \phi(x) L_\epsilon u(x) dx \to \int_{\text{supp} \phi} \phi(x) Lu(x) dx = 0 \). Our claim follows: \( \mathcal{E}_D^\epsilon(u, \phi) \to -2 \int_{\text{supp} \phi} \phi(x) Lu(x) dx = 0 \). Therefore \( \mathcal{E}_D(u, \phi) = 0 \), as needed.

We can resolve the non-homogeneous Dirichlet problem, too. For \( g \in X^D, f \in L^2(D) \) let

\[
(5.6) \quad \left\{ \begin{array}{cc} Lu = -f & \text{on } D, \\ u = g & \text{on } D^c. \end{array} \right.
\]

This is interpreted as the requirement that \( u \in V^D, u = g \text{ a.e. on } D^c \), and

\[
(5.7) \quad \mathcal{E}_D(u, \phi) = \int_{\mathbb{R}^d} f(x) \phi(x) dx, \quad \phi \in V^D_0.
\]

The next result follows from Corollary 5.1 and [35, Theorem 4.2].
Corollary 5.5. Suppose $L$ satisfies A1, A2. Let nonempty open bounded set $D \subset \mathbb{R}^d$ have continuous boundary and let $D^c$ fulfill VDC. Then the non-homogeneous Dirichlet problem (5.6) has a unique solution for arbitrary $g \in \mathcal{X}^D$ and $f \in L^2(D)$.

6. Estimates of the interaction kernel

In this section we prove sharp estimates of $\gamma_D$ for the half-space and bounded $C^{1,1}$ open sets. In the proof of the result for the half-space we often use the following global scalings.

**A4:** There exist constants $\alpha, \beta \in (0, 2)$ and $c, C > 0$ such that

\begin{equation}
\nu(\lambda r) \leq C\lambda^{-d-\beta} \nu(r), \quad 0 < \lambda \leq 1, r > 0. \tag{6.1}
\end{equation}

\begin{equation}
\nu(\lambda r) \geq c\lambda^{-d-\alpha} \nu(r), \quad 0 < \lambda \leq 1, r > 0. \tag{6.2}
\end{equation}

Note that (6.1) is but a global version of (2.1), equivalent to $r^{d+\beta} \nu(r)$ being almost increasing on $(0, \infty)$, cf. [3] Section 3. Clearly, A4 holds true if $L = \Delta^{\alpha/2}$.

We start with some basic observations. If (6.2) holds, then $K(s) \approx s^d \nu(s)$, $s > 0$, hence

\begin{equation}
\nu(s) \approx \frac{K(s)}{s^d}, \quad s > 0. \tag{6.3}
\end{equation}

For $a \in (0, 2]$ we denote

$$U_a(s) = \frac{K(s)}{h^a(s)s^d}, \quad s > 0.$$ 

Due to [25] Theorem 1.2, unimodality of $\nu$ and (6.3), $U_a$ is almost decreasing, i.e. there is a constant $c_0 > 0$ such that for all $0 < s_1 < s_2$ we have $U_a(s_1) \geq c_0 U_a(s_2)$, in short $U_a(s_1) \geq U_a(s_2)$. It is known [10] (3.5)) that $h'(r) = -2K(r)/r$. In particular, $h$ is decreasing and $V$ is increasing. A direct calculation gives

\begin{equation}
-\left(\frac{1}{V(s)}\right)' = \frac{V(s)K(s)}{s} \approx V(s)\nu(s)s^{d-1}, \quad [V^2]'(s) = 2s^{d-1}U_2(s). \tag{6.4}
\end{equation}

The factor $s^{d-1}$ will be useful for integrations in spherical coordinates. It is also easy to verify that $s^2h(s)$ is nondecreasing, hence $V(s)/s$ is nonincreasing and for every $a < 1$ we have

\begin{equation}
V(s) \geq V(as) \geq aV(s), \quad s > 0. \tag{6.5}
\end{equation}

Here is our main result for the half-space

$$H = \{x \in \mathbb{R}^d : x_d > 0\}.$$ 

**Theorem 6.1.** Let $d \geq 3$ and assume that (6.1) holds true. Then,

$$\gamma_H(z, w) \leq C \frac{V^2(r(z, w))\nu(r(z, w))}{V(\delta_H(z))V(\delta_H(w))}.$$ 

If we additionally assume (6.2), then

$$\gamma_H(z, w) \approx \frac{V^2(r(z, w))\nu(r(z, w))}{V(\delta_H(z))V(\delta_H(w))}.$$ 

The proof of Theorem 6.1 requires the following lemma.

**Lemma 6.2.** Let $d \geq 3$. Assume that (6.1) holds true. Then,

$$P_H(x, z) \approx \frac{V\delta_H(x)}{V(\delta_H(z))}V^2(|x - z|)\nu(|x - z|), \quad x \in H, \ z \in \overline{H}.$$
Proof. By [25, Theorem 1.13],

\[
G_H(x, y) \approx \frac{V(\delta_H(x))}{V(\delta_H(x) + |x - y|)} \frac{V(\delta_H(y))}{V(\delta_H(y) + |x - y|)} U_2(|x - y|), \quad x, y \in H.
\]

From the Ikeda-Watanabe formula and the monotonicity properties of \(V, U_a, \nu,\)

\[
P_H(x, z) \leq \int_{H \cap \{|x-z| \leq 2|x-y|\}} \frac{V(\delta_H(y))}{V^2(\frac{|x-z|}{2})} U_2(\frac{|x-z|}{2}) \nu(z, y) dy
\]

\[
+ \int_{|x-z| > 2|x-y|} \frac{1}{V(|x-y|)} U_2(|x-y|) \nu((x-z)/2) dy
\]

\[
\leq U_1(\frac{|x-z|}{2}) \int_{B(z, \delta_H(z))} V(|y-z|) \nu(z, y) dy
\]

\[
+ \nu(x, z) \int_{B(x-z)\backslash B(z, \delta_H(z))} U_{3/2}(|y|) dy
\]

\[
\leq U_1(\frac{|x-z|}{2}) \int_{B(x-z)\backslash B(z, \delta_H(z))} V(|y|) \nu(|y|) dy + \nu(x, z) \int_{B(x-z)\backslash B(z, \delta_H(z))} U_{3/2}(|y|) dy
\]

\[
\leq U_1(\frac{|x-z|}{2}) \frac{V(\delta_H(z))}{V(\delta_H(z))} + U_1(\frac{|x-z|}{2}) = 2 \frac{U_1(|x-z|)}{V(\delta_H(z))}.
\]

In the last inequality we use (6.3) and the formula \(h'(r) = -2K(r)/r,\) which result in

\[
\int_r^\infty \frac{K(s)}{h^{1/2}(s)} ds = h^{1/2}(r) \quad \text{and} \quad \int_0^r \frac{K(s)}{sh^{3/2}(s)} ds = \frac{1}{h^{1/2}(r)}.
\]

We next prove a matching lower estimate. Using repeatedly the monotonicity properties of \(U_a, V,\) formula (6.5) and the scaling of \(\nu\) we see that up to a multiplicative constant, \(P_H(x, z)/V(\delta_H(x))\) is not less than

\[
\int_{H \cap \{|y-z| \leq 2|x-z|\}} \frac{V(\delta_H(y))}{V(\delta_H(y) + |x-y|)V(3|x-z|)} U_2(|x-y|) \nu(3|x-z|) dy
\]

\[
+ \int_{H \cap \{|y-z| \leq 2|x-z|\}} \frac{V(\delta_H(y))}{V(\delta_H(y) + |x-y|)V(3|x-z|)} U_2(|x-y|) \nu(3|x-z|) dy
\]

\[
\geq U_1(5|x-z|) I + \frac{\nu(|x-z|)}{V(3|x-z|)} \geq U_1(|x-z|) \left( I + \frac{1}{V^3(2|x-z|)} \right),
\]

where

\[
I = \int_{H \cap \{|y-z| \leq 2|x-z|\}} V(\delta_H(y)) \nu(z, y) dy,
\]

\[
\Pi = \int_{H \cap \{|y-z| \leq 2|x-z|\}} \frac{V(\delta_H(y))}{V(\delta_H(y) + |x-y|)} U_2(|x-y|) dy.
\]

First we estimate the integral \(I.\) Without loss of generality we may and do assume that \(z = (0, \ldots, 0, z_d)\) with \(z_d < 0.\) Let \(\Gamma = \{(\tilde{y}, y_d) : |\tilde{y}| < y_d\}.\) Then, for \(y \in \Gamma,\) we have \(2\delta_H(y) \geq \)
\[ |y - z| - \delta_H(z). \] Hence, by the rotational invariance of \( \nu \) and (6.5) we obtain
\[
I \geq \int_{H \cap \{|y-z| \leq 2|x-z|\}} V(|y-z| - \delta_H(z))/2 \nu(|y-z|)dy
\]
\[
\geq c(d) \int_{3\delta_H(z)/2 \leq |y-z| \leq |x-z|} V(|y-z| - \delta_H(z)) \nu(y,z)dy
\]
\[
\geq \int_{3\delta_H(z)/2}^{2|x-z|} V(s)\nu(s)s^{d-1}ds \approx \frac{1}{V(3\delta_H(z)/2)} - \frac{1}{V(2|x-z|)}.
\]
Similarly,
\[
II \geq \int_{|y-x| \leq 2|y-x| \wedge y_d} \frac{V(\delta_H(y))}{V(3\delta_H(y))} U_2(|x-y|)dy
\]
\[
\geq \int_{|y-x| \leq 2|x-z|} U_2(|x-y|)dy \approx V^2(2|x-z|),
\]
where in the second inequality we use the isotropy of \( U_2 \) and the inclusion
\[
\{ y : |y-x| \leq 2y_d \} \supset \{ y : |y-x| \leq 2(y_d - x_d) \} \supset x + \Gamma.
\]
Hence, up to a multiplicative constant, \( P_H(x,z)/V(\delta_H(x)) \) is not less than
\[
U_1(|x-z|) \left( \frac{1}{V(3\delta_H(z)/2)} - \frac{1}{V(2|x-z|)} + \frac{1}{V(2|x-z|)} \right) \geq \frac{U_1(|x-z|)}{V(\delta_H(z))}.
\]
Since \( U_1(s) \approx \nu(s)V^2(s) \), the proof is complete. \( \square \)

**Proof of Theorem 6.1** We have
\[
\gamma_H(z,w) \approx \frac{1}{V(\delta_H(z))} \int_H V(\delta_H(x))V^2(|x-z|)\nu(|z-x|)\nu(|w-x|)dx.
\]
Let \( \tilde{z} \in H \) be the reflection of \( z \) in the hyperplane \( \{x_d = 0\} \). Then \( |w - \tilde{z}| \approx r(z,w) \) and for \( x \in H \) we have \( |x - \tilde{z}| < |x - z| \), and \( \delta_H(\tilde{z}), \delta_H(x) \leq |x - z| \). Consequently, the estimates of the Green function (6.6) and Lemma 6.2 imply
\[
\gamma_H(z,w)V^2(\delta_H(\tilde{z})) \approx \int_H V(\delta_H(x))V(\delta_H(\tilde{z}))V^4(|x-z|)\nu(|z-x|)\nu(|w-x|)dx
\]
\[
\leq P_H(\tilde{z}, w) \approx \frac{V(\delta_H(\tilde{z}))}{V(\delta_H(w))} V^2(|\tilde{z} - w|)\nu(|\tilde{z} - w|).
\]
We next assume (6.2) and prove the matching lower bound. It suffices to replace \( z \) with \( \tilde{z} \) in the right-hand side of (6.7) because then we have approximation \( \approx \) instead of inequality \( \leq \) in (6.8). To this end we again use (6.5) and obtain
\[
\int_{B(\tilde{z},\delta_H(\tilde{z}))/2} V(\delta_H(x))V(\delta_H(\tilde{z}))V^4(|x-z|)\nu(|z-x|)\nu(|w-x|)dx
\]
\[
\approx V^4(\delta_H(z))\nu(\delta_H(z))\nu(|w - \tilde{z}|)\delta_H(z)^d.
\]
For the integrand with \( \tilde{z} \) we have
\[
\int_{B(\tilde{z},\delta_H(\tilde{z}))/2} V(\delta_H(x))V(\delta_H(\tilde{z}))V^2(\delta_H(\tilde{z}) + |x - \tilde{z}|)V^4(|x-z|)\nu(\tilde{z} - x)\nu(w,x)dx
\]
\[
\approx \nu(|w - \tilde{z}|) \int_{B_{\delta_H(z)/2}} V^4(|x|)\nu(|x|)dx \approx V^2(\delta_H(z))\nu(r(z,w)).
\]
The last comparison follows from \( V^4(s)\nu(s)s^{d-1} \approx [V^2]'(s) \). Since (6.2) gives \( \nu(r)r^dV^2(r) \approx 1 \), the right-hand sides of (6.9) and (6.10) are comparable. We have \(|x - \tilde{z}| \approx |x - z|\), for \( x \in H \) such that \(|x - \tilde{z}| \geq \delta_H(z)/2\). Therefore we can replace \( z \) by \( \tilde{z} \) in the integrand in (6.7), and so

\[
\gamma_H(z, w) \approx \frac{P_H(\tilde{z}, w)}{V^2(\delta_H(z))} \approx \frac{V^2(r(z, w))}{V(\delta_H(z))V(\delta_H(w))}\nu(r(z, w)).
\]

\( \square \)

The result for the bounded \( C^{1,1} \) open sets has a similar proof, so we will be brief.

**Proof of Theorem 2.6.** Let \( D \) be \( C^{1,1} \) at scale \( R > 0 \).

(i) First we let \( \delta_D(z), \delta_D(w) \geq R \). Since

\[
\int_D G_D(x, y)dy = E^x\tau_D,
\]

by the radial monotonicity of \( \nu \) we get

\[
\nu(\delta_D(w) + \text{diam}(D))E^x\tau_D \leq P_D(x, w) \leq \nu(\delta_D(w))E^x\tau_D.
\]

By (2.2),

\[
\nu(\delta_D(w) + \text{diam}(D)) \approx \nu(\delta_D(w)).
\]

These imply

\[
\gamma_D(z, w) \approx \nu(\delta_D(z))\nu(\delta_D(w)) \int_D E^x\tau_D dx,
\]

which ends the proof of the first case.

(ii) We next assume that \( \delta_D(z) \leq R \leq \delta_D(w) \). We get

\[
\gamma_D(z, w) \approx \nu(\delta_D(w)) \int_D E^x\tau_D \nu(z, x) dx.
\]

Let \( A = B(z, 2\text{diam}(D)) \setminus B(z, \delta_D(z)) \). By [10, Theorem 4.6 and Proposition 5.2],

\[
\int_D E^x\tau_D \nu(z, x) dx \leq \int_A E^x\tau_A \nu(z, x) dx \leq c_1 \int_A V(\delta_A(x)) \nu(z, x) dx
\]

\[
\leq c_1 \int_{B(0, \delta_D(z))} V(|y|) \nu(y) dy.
\]

Using [10, Lemma 3.5] we obtain

\[
\gamma_D(z, w) \leq c\nu(\delta_D(w)) \frac{1}{V(\delta_D(z))}.
\]

Since \( D \) is \( C^{1,1} \), there is \( x_0 \in D \) such that \( B = B(x_0, R) \subset D \). By [27, Theorem 2.6],

\[
\int_D E^x\tau_D \nu(z, x) dx = \int_D P_D(x, z) dx
\]

\[
\geq c_2 \int_{B(x_0, R/2)} \frac{V(\delta_D(x))}{V(\delta_D(z))} V^2(|x - z|) \nu(x, z) dx \geq c_2 (R/\text{diam}(D))^d \frac{1}{V(\delta_D(z))}.
\]

Therefore,

\[
\gamma_D(z, w) \approx \nu(\delta_D(w)) \frac{1}{V(\delta_D(z))}.
\]

(iii) Finally, let \( \delta_D(z), \delta_D(w) < R \). By [9, Proposition 4.4 and Theorem 4.5] the Dirichlet heat kernel of \( D \) satisfies

\[
p_D(t, x, y) \approx e^{-\lambda(D)t} \left( \frac{V(\delta_D(x))}{\sqrt{t} \wedge V(r)} \wedge 1 \right) \left( \frac{V(\delta_D(x))}{\sqrt{t} \wedge V(r)} \wedge 1 \right) p(t \wedge V^2(r), x, y), \quad t > 0, \: x, y \in D,
\]
where \( \lambda(D) \approx 1/V^2(R) \). Integrating against time we get rather standard estimates of the Green function, cf. [16, proof of Theorem 7.3]. For instance if \( d \geq 2 \), then
\[
G_D(x, y) \approx U_2(|x - y|) \left( \frac{V(\delta_D(x))V(\delta_D(y))}{V^2(|x - y|)} \wedge 1 \right), \quad x, y \in D.
\]
The Ikeda-Watanabe formula yields estimates for the Poisson kernel, cf. [27, Theorem 2.6],
\[
P_D(x, z) \approx \frac{V(\delta_D(x))}{V(\delta_D(z))} \frac{1}{|x - z|^d}, \quad x \in D, \delta_D(z) < R.
\]
By similar calculation as in the proof of Theorem 6.1, we obtain the estimate in Theorem 2.6. \( \square \)

7. Examples

In this section we provide examples of Lévy measures other than (1.2) which satisfy A1 and A2.

**Example 7.1.** By inspection, A1 and A2 are satisfied when the Lévy density is
\[
\nu(z) = \frac{1}{|z|^d \ln(2 + |z|)}, \quad z \in \mathbb{R}^d.
\]
Due to mild singularity of \( \nu \) at the origin the resulting operator \( L \) may be considered of "0-order".

We next consider \( \nu \) at the origin the resulting operator \( L \) may be considered of "0-order".

The function is nonnegative and its derivative is completely monotone, i.e. it is a Bernstein function.

The Lévy-Khinchine exponent corresponding to \( \nu \) is
\[
\psi(\lambda) = \int_0^\infty (1 - e^{-\lambda t})\eta(dt), \quad \lambda \geq 0.
\]

The function is nonnegative and its derivative is completely monotone, i.e. it is a Bernstein function.

The Lévy-Khinchine exponent corresponding to \( \nu \) is
\[
\psi(\lambda) = \int_0^\infty (1 - e^{-\lambda t})f(t)dt, \quad \lambda \geq 0,
\]
with a completely monotone \( f \). See Schilling, Vondraček and Song [37] for details.

**Proposition 7.2.** The Lévy density \( \nu \) is smooth on \((0, \infty)\). If \( \nu(r + 1) \approx \nu(r), \) for \( r \geq 1 \), then
\[
(7.2) \quad \left| \left( \frac{d}{dr} \right)^n \nu(r) \right| \leq C_n \nu(r), \quad r \geq 1, \ n \in \mathbb{N},
\]
in particular A1 holds true.

**Proof.** Using (7.1) we get, for \( h > 0 \),
\[
(7.3) \quad \frac{1}{h} \left( \nu(r + h) - \nu(r) \right) = \int_0^\infty g_t(r) \frac{1}{h} \left( e^{\frac{2rh - h^2}{4t}} - 1 \right) \eta(dt).
\]
Let \( 0 < h < r/4 \). Since \( e^u - 1 = \int_0^u e^sds \leq u(1 + e^u) \) for \( u \geq 0 \), we get
\[
0 < \frac{1}{h} \left( e^{\frac{2rh - h^2}{4t}} - 1 \right) \leq \frac{r}{2t} \left( 1 + e^{\frac{r^2}{4t}} \right)
\]
and this quantity, multiplied by \(g_t(r)\) is integrable with respect to \(\eta\). Letting \(h \to 0\) in (7.3) by the dominated convergence, we see that the derivative of \(\nu\) exists and

\[
(7.4) \quad \nu'(r) = - \int_0^\infty r g_t(r) \eta(dt), \quad r > 0,
\]

so the integration and differentiation commute. Continuity of the derivative is evident from (7.4). Higher order differentiability of \(\nu\) can be established in the same way.

To see the estimate (7.2) we first observe that for all \(t > 0, r \geq 1\) we have \(|g_t^{(n)}(r)| \leq W_n(r/2t)g_t(r)\), where \(W_n\) is a polynomial of degree \(n\) with nonnegative coefficients. When \(r \geq 2\) and \(t \geq 0\), then \(g_t(r) (r/2t)^n \leq C_n g_t(r - 1)\), so that \(|\nu^{(n)}(r)| \leq C_n \nu(r - 1) \approx \nu(r)\) for \(r \geq 2\). The estimate can be extended to \(r \in [1, 2]\) by continuity.

**Proposition 7.3.** If \(\int_0^\infty t^{-d/2} \eta(dt) \geq c_1 e^{-c_2 r} \) for \(r \geq 1\), then \(\nu(r + 1) \approx \nu(r), r \geq 1\).

**Proof.** Assume that \(\int_0^\infty t^{-d/2} \eta(dt) \geq c_1 e^{-c_2 r} \) for \(r \geq 1\). By monotonicity of \(\nu\), for \(r \geq 1\) and \(\lambda > 0\),

\[
\int_0^\lambda g_t(r) \eta(dt) \leq e^{-r^2/(8\lambda)} \int_0^\lambda g_t(r/\sqrt{2}) \eta(dt) \leq e^{-r/(8\lambda)} \nu(1/\sqrt{2})
\]

and

\[
\int_0^\infty g_t(r) \eta(dt) \geq (4\pi)^{-d/2} e^{-r/4} c_1 e^{-c_2 r} = c e^{-r(c_2 + 1/4)}.
\]

Hence, for \(\lambda_0 = (8c_2 + 2)^{-1}\) we have

\[
\int_0^{r\lambda_0} g_t(r) \eta(dt) \leq c \int_r^\infty g_t(r) \eta(dt), \quad r \geq 1.
\]

Since \(\lambda_0 < 1\) we obtain

\[
\nu(r) \approx \int_{r\lambda_0}^\infty g_t(r) \eta(dt), \quad r \geq 1.
\]

This yields

\[
\nu(r + 1) \geq \int_0^\infty g_t(r) e^{-3r/(4t)} \eta(dt) \geq e^{-3/(4\lambda_0)} \int_{r\lambda_0}^\infty g_t(r) \eta(dt) \approx \nu(r).
\]

**Lemma 7.4.** If \(\varphi(\lambda) = \int_0^\infty (1 - e^{-t\lambda}) \eta(dt)\) is complete Bernstein, then \(\nu(r + 1) \approx \nu(r), r \geq 1\).

**Proof.** By our assumptions \(\eta(dt) = f(t)dt\) and there is a measure \(\mu\) such that

\[
f(t) = \int_0^\infty e^{-ts} \mu(ds), \quad s > 0.
\]

Let \(s_0 > 0\) be such that \(\mu((0, s_0]) > 0\). Then,

\[
f(t) \geq e^{-t s_0} \mu((0, s_0]), \quad t > 0,
\]

hence for some constants \(c_1, c_2,\)

\[
\int_r^\infty t^{-d/2} \eta(dt) \geq c \int_r^\infty t^{-d/2} e^{-s_0 t} dt \geq c_1 e^{c_2 r}, \quad r \geq 1.
\]

The lemma follows from Proposition 7.3.

By [26, Theorem 5.18], the inequality (2.1) of A2 is satisfied if the derivative of \(\varphi\) satisfies

\[
(7.5) \quad c^{-1} \varphi(r) \lambda^{-d/2 - 1 + \beta} \leq \varphi'(\lambda r) \leq c \lambda^{-\alpha} \varphi(r), \quad \lambda, r \geq 1,
\]

for some \(c, \alpha, \beta > 0\). Next we discuss (2.2) in A2. The simplest situation arises when inequalities (7.5) hold for every \(r > 0\). Then (2.1) holds for every \(r > 0\) and therefore (2.2) holds as well. Hence the assumption A2 is satisfied in that case.
Example 7.5. Assumptions A1 and A2 hold for the following operators:

\[ L = \Delta^\alpha \log^\beta (1 + \Delta^\gamma) \] if \( \gamma, \alpha, \alpha + 2\beta \in [0, 1) \), \( \gamma \beta + \alpha > 0 \), 

\[ L = \Delta^{\alpha_1} + \Delta^{\alpha_2} \] if \( \alpha_1, \alpha_2 \in (0, 1) \).

The corresponding Bernstein functions and more examples are discussed in detail in [37].

Appendix A.

A.1. Not hitting the boundary. The boundary effects are easier to handle if the Lévy process \( X \) does not hit \( \partial D \) at \( \tau_D \). This motivates the following development. Assume that the Lévy measure \( \nu \) satisfies A2. Then for every \( R \in (0, \infty) \),

\[ \nu(r) = r \leq c\lambda^{d-\beta} \nu(r), \quad 0 < \lambda \leq 1, \ 0 < r \leq R. \] (A.1)

Indeed, for \( r \in (0, 1] \) we can take \( c = C \) and if \( 1 < r \leq R \), then

\[ \nu(r) \leq \nu(1) \leq c \lambda^{d-\alpha} \nu(1) \leq C \frac{\nu(1)}{\nu(R)} \lambda^{d-\alpha} \nu(r). \]

Let \( K, h \) be the functions defined by (2.10). Clearly, \( K > 0, h > 0 \) and \( h \) is strictly decreasing, but \( r^2h(r) \) is increasing. Thus for \( a \geq 1 \) and \( r > 0 \),

\[ h(r) \geq h(ar) = (ar)^2h(ar)/(ar)^2 \geq r^2h(r)/(ar)^2 = h(r)/a^2. \] (A.2)

Recall that \( \omega_d = 2\pi^{d/2}/\Gamma(d/2) \), the surface area of the unit sphere in \( \mathbb{R}^d \). We obtain

\[ K(r) = r^{-2} \int_0^r \omega_d s^{d-1+2} \nu(s) ds \geq \nu(r) r^d \omega_d/(d + 2), \quad r > 0. \]

By (A.1), for every \( R < \infty \) we get

\[ K(r) = r^{-2} \int_0^r \omega_d s^{d+1} \nu(s) ds \leq cr^{-2} \int_0^r \omega_d s^{d+1} \nu(s) (s/r)^{-d-\beta} ds = \nu(r) r^d c\omega_d/(2 - \beta), \quad r \leq R. \]

Therefore, for every \( R \in (0, \infty) \),

\[ \nu(r) \approx \frac{K(r)}{r^d}, \quad 0 < r \leq R. \] (A.3)

If \( 0 < r \leq R/2 \), then by (A.1) we have

\[ \nu(B^c_r) = \int_r^\infty \omega_d s^{d-1} \nu(s) ds \geq c \int_r^R s^{d-1} K(s) ds/s^d \]

\[ = c \int_r^R h'(s) ds \geq c h(r - h(R)) \geq c \left( 1 - \frac{h(R)}{h(R/2)} \right) h(r) \geq c \nu(B^c_r) + K(r) \geq c\nu(B^c_r). \]

Thus for every \( R \in (0, \infty) \),

\[ \nu(B^c_r) \approx h(r), \quad 0 < r \leq R/2. \] (A.4)

Lemma A.1. If VDC holds locally for \( D^c \), then \( \mathbb{P}^x(X_{\tau_D} \in \partial D) = 0 \), \( x \in D \).

For the narrower class of Lipschitz open sets and all isotropic pure-jump Lévy processes with infinite Lévy measure the result is stated after Theorem 1 in [39]. Our proof follows the argument given for the fractional Laplacian by Wu [40, Theorem 1].
Proof of Lemma A.1. The trajectories of $X$ are càdlàg, so locally bounded, therefore
\[ \mathbb{P}^x(X_{\tau_D} \in \partial D) = \mathbb{P}^x(\tau_D < \infty, X_{\tau_D} \in \partial D) = \lim_{R \to \infty} \mathbb{P}^x(\tau_D < \tau_{B_R}, X_{\tau_D} \in \partial D). \]
We have $\mathbb{P}^x(\tau_D < \tau_{B_R}, X_{\tau_D} \in \partial D) \leq \mathbb{P}^x(X_{\tau_D \wedge \tau_{B_R}} \in (\partial D \cap B_R))$ for every $R > 0$. Indeed, if $\tau_D < \tau_{B_R}$, then $X_{\tau_D} \in B_R$ and it suffices to note that $\partial D \cap B_R \subset \partial (D \cap B_R)$. Therefore in what follows we may assume that $D$ is bounded and (3.1) holds. Let $a = \max\{(2|B_1|/c)^{1/d}, 2\}$, where $c$ is the constant from (3.1). By (A.4),
\[ \nu(B_c^r) \approx h(r), \quad r \leq a^2 \text{diam}(D). \]
Here, as usual,
\[ \text{diam}(D) = \sup\{|x - y| : x, y \in D\}. \]
For $x \in D$ we let $r_x = \delta_D(x)/2$ and $B_x = B(x, r_x)$. If $Q \in \partial D$ is such that $|x - Q| = \delta_D(x)$, then by (3.1),
\[ (A.5) \quad |D_c \cap (B(Q, ar) \setminus B(Q, r))| \geq |B(Q, r)|, \quad r > 0. \]
By unimodality of $\nu$, (A.5) and then (A.2) we get
\[ \nu(x, D_c) \geq \sum_{k \geq 1} \nu \left( D_c \cap \left( B(Q, a^k r_x) \setminus B(Q, a^{k-1} r_x) \right) - x \right) \]
\[ \geq \sum_{k \geq 1} \nu(a^k r_x + 2r_x)|B(Q, a^{k-1} r_x)| \]
\[ \geq \sum_{k \geq 1} \nu(a^{k+1} r_x)|B(Q, a^{k+2} r_x) \setminus B(Q, a^{k+1} r_x)| \]
\[ = \sum_{k \geq 1} \nu(a^{k+1} r_x)|B(0, a^{k+2} r_x) \setminus B(0, a^{k+1} r_x)| \]
\[ \geq a^{-3d} \nu(B_a^r \setminus P^r) \approx h(a^2 r_x) \approx h(r_x). \]
The estimates for Poisson kernel for the ball [25, Lemma 2.2] give
\[ P_{B_x}(0, z) \geq \frac{\nu(z)}{h(r)} \quad |z| > r > 0. \]
By (A.6) we have $\omega_{B_x}(x, A) := \mathbb{P}^x(X_{\tau_{B_x}} \in A) = \int_A P_{B_x}(0, z - x) dz$, if dist$(D, A) > 0$, hence
\[ (A.6) \quad \mathbb{P}^x(X_{\tau_{B_x}} \in D_c) \geq \frac{\nu(x, D_c)}{h(r_x)} \geq c, \]
where $c > 0$ does not depend on $x$. Following [40], we write
\[ \mathbb{P}^x(X_{\tau_D} \in \partial D) = \mathbb{P}^x(X_{\tau_{B_x}} \in \partial D) + \mathbb{P}^x(X_{\tau_{B_x}} \in D, X_{\tau_D} \in \partial D) \]
The first term vanishes because $|\partial D| = 0$. By the strong Markov property and (A.6), the second term is equal to
\[ \int_{D \setminus B_x} \mathbb{P}^y(X_{\tau_D} \in \partial D) \omega_{B_x}(x, dy) \leq \sup_{y \in D} \mathbb{P}^y(X_{\tau_D} \in \partial D) \mathbb{P}^x(X_{\tau_{B_x}} \in D \setminus B_x) \]
\[ \leq (1 - c) \sup_{y \in D} \mathbb{P}^y(X_{\tau_D} \in \partial D). \]
Thus, for every $x \in D$ we have
\[ \mathbb{P}^x(X_{\tau_D} \in \partial D) \leq (1 - c) \sup_{y \in D} \mathbb{P}^y(X_{\tau_D} \in \partial D). \]
This implies that $\sup_{x \in D} \mathbb{P}^x(X_{\tau_D} \in \partial D) = 0$. \hfill \qed
Corollary A.2. If $D$ is a bounded open set and VDC holds for $D^c$ and $x \in D$, then
\begin{equation}
\mathbb{P}^x(X_{\tau_D} \in \text{int}(D^c)) = \int_{D^c} P_D(x, y) dy = 1,
\end{equation}
\begin{equation}
\mathbb{P}^x(X_{\tau_D-} \in D) = 1.
\end{equation}

Proof. Clearly, $\mathbb{P}^x(\tau_D < \infty) = 1$, so (A.7) follows from Lemma A.1. If $X_{\tau_D-} \in \partial D$ a.s., then $X_{\tau_D} \in \partial D$ a.s. [1] proof of Lemma 17, proving (A.8). \qed

Corollary A.3. If VDC holds locally for $D^c$, then for nonnegative or integrable $u$,
\[ \mathbb{E}^x u(X_{\tau_D}) = \int_{D^c} u(y) P_D(x, y) dy, \quad x \in D. \]

A.2. Approximation by smooth functions. The following theorem is an extension of the result by Valdinoci et al. [23], where it was proven for the fractional Laplacian. In fact, in this section we let $\nu$ be an arbitrary Lévy measure on $\mathbb{R}^d$, i.e. we only assume that $\int_{\mathbb{R}^d} (1 + |y|^2) \nu(dy) < \infty$ and $\nu(\{0\}) = 0$. In this general case the quadratic form is best defined as
\[ \mathcal{E}_D(u, u) = \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} (u(x) - u(x + y))^2 (1_D(x) \land 1_D(x + y)) \nu(dy) dx, \]
and $\mathcal{V}_0^D$ is defined as before, cf. [35].

Theorem A.4. Let $\nu$ be an arbitrary Lévy measure. If $D$ has continuous boundary and $u \in \mathcal{V}_0^D$, then there are functions $\phi_n \in C_0^\infty(D)$ such that $\mathcal{E}_D(u - \phi_n, u - \phi_n) \to 0$ as $n \to \infty$.

We may construct the approximating functions $\phi_n$ in the same way as in [23] provided we check that the mollification, translation and cut-off are continuous in the seminorm $\sqrt{\mathcal{E}_D(\cdot, \cdot)}$. We do this below. Let $\eta \in C_0^\infty(B_1)$ be a nonnegative radial function on $\mathbb{R}^d$ satisfying $\int_{\mathbb{R}^d} \eta(x) dx = 1$ and let $\eta_\epsilon(x) = \epsilon^{-d} \eta(\frac{x}{\epsilon})$ for $\epsilon > 0$, $x \in \mathbb{R}^d$. Here $B_r = B(0, r)$.

In the sequel we write $f_n \to f$ to denote $\lim_{n \to \infty} \mathcal{E}_D(f - f_n, f - f_n) = 0$.

Lemma A.5 (Mollification). For every $u \in \mathcal{V}_0^D$, $\eta_\epsilon \ast u \to u$ as $\epsilon \to 0$.

Proof. Note that $\mathcal{E}_{\mathbb{R}^d}(u, u) = \mathcal{E}_D(u, u) < \infty$. It suffices to verify that
\[ I := \int_{\mathbb{R}^d \times \mathbb{R}^d} (u \ast \eta_\epsilon(x) - u(x) - u \ast \eta_\epsilon(x + y) + u(x + y))^2 \nu(dy) dx \to 0 \text{ as } \epsilon \to 0. \]
By Fubini-Tonelli theorem and Jensen’s inequality,
\begin{align}
I & = \int_{\mathbb{R}^d \times \mathbb{R}^d} \left( \int_{B_1} (u(x - \epsilon z) - u(x + y - \epsilon z) - u(x) + u(x + y)) \eta(z) dz \right)^2 \nu(dy) dx \\
& \leq \int_{B_1} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left( u(x - \epsilon z) - u(x + y - \epsilon z) - u(x) + u(x + y) \right)^2 dx \nu(dy) \eta(z) dz.
\end{align}
We will use the dominated convergence theorem to the integral over $B_1 \times \mathbb{R}^d$. By the translation invariance of the Lebesgue measure
\[ \eta(z) \int_{\mathbb{R}^d} (u(x - \epsilon z) - u(x + y - \epsilon z) - u(x) + u(x + y))^2 dx \leq 4\eta(z) \int_{\mathbb{R}^d} (u(x) - u(x + y))^2 dx, \]
which is integrable against $\nu(dy)dz$. Furthermore, by the continuity of translations in $L^2(\mathbb{R}^d)$ the integrand converges to 0 for every $z \in B_1, y \in \mathbb{R}^d$, which ends the proof. \qed

If we fix $z \in B_1$ in the integral over $\mathbb{R}^d \times \mathbb{R}^d$ in (A.9) we can use the same reasoning to get the following fact.

Corollary A.6 (Translation). For every $u \in \mathcal{V}_0^D$, $z \in B_1$, $u(\cdot + \epsilon z) \to u(\cdot)$ as $\epsilon \to 0$. 

Consider a collection of smooth functions \( q_j, j \in \mathbb{N} \) satisfying \( 0 \leq q_j \leq 1 \), \( q_j = 1 \) in \( B_j \) and \( q_j = 0 \) in \( B_{j+1}^c \) for which there is \( M > 0 \) such that \( |\nabla q_j(x)| < M, x \in \mathbb{R}^d, j = 1, 2, \ldots \).

**Lemma A.7** (Cut-off). For every \( u \in \mathcal{V}_0^D \), \( q_j u \to u \) as \( j \to \infty \).

**Proof.** Since \( |(q_j u)(x) - (q_j u)(x + y) - u(x) + u(x + y)| \leq |(q_j(x) - 1)(u(x + y) - u(x))| + |(q_j(x) - q_j(x + y))u(x + y)| \), we get

\[
E_{\mathbb{R}^d} (q_j u - u, q_j u - u) \leq \int_{\mathbb{R}^d} (q_j(x) - 1)^2 (u(x) - u(x + y))^2 \nu(\mathrm{d}y) \mathrm{d}x
\]

\[
+ \int_{\mathbb{R}^d} (q_j(x) - q_j(x + y))^2 u(x + y)^2 \nu(\mathrm{d}y) \mathrm{d}x.
\]

The integrands in (A.10) and (A.11) converge to 0 a.e. as \( j \to \infty \). For (A.10) we have \( (q_j(x) - 1)^2 (u(x) - u(x + y))^2 \leq (u(x) - u(x + y))^2 \), which is integrable against \( \nu(\mathrm{d}y) \mathrm{d}x \) since \( u \in \mathcal{V}_0^D \). For (A.11) we use the smoothness of \( q_j \):

\[
(q_j(x) - q_j(x + y))^2 u(x + y)^2 \leq C(1 \wedge |y|^2) u(x + y)^2.
\]

Then,

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (1 \wedge |y|^2) u(x + y)^2 \nu(\mathrm{d}y) = \int_{\mathbb{R}^d} (1 \wedge |y|^2) \nu(\mathrm{d}y) \int_{\mathbb{R}^d} u(x)^2 \mathrm{d}x < \infty.
\]

By the dominated convergence theorem we obtain the desired result. \( \square \)

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