Structure of Gauge-Invariant Lagrangians

Marco Castrillón López, Jaime Muñoz Masqué and Eugenia Rosado María

Abstract. The theory of gauge fields in Theoretical Physics poses several mathematical problems of interest in Differential Geometry and in Field Theory. Below, we tackle one of these problems: the existence of a finite system of generators of gauge-invariant Lagrangians and how to compute them. More precisely, if \( p : C \rightarrow M \) is the bundle of connections on a principal \( G \)-bundle \( \pi : P \rightarrow M \), and \( G \) is a semisimple connected Lie group, then a finite number \( L_1, \ldots, L_N \) of gauge-invariant Lagrangians defined on \( J^1C \) is proved to exist such that for any other gauge-invariant Lagrangian \( L \in C^\infty(J^1C) \), there exists a function \( F \in C^\infty(\mathbb{R}^N) \), such that \( L = F(L_1, \ldots, L_N) \). Several examples are dealt with explicitly.

Mathematics Subject Classification. Primary 35F20; Secondary 53C05, 58A20, 58D19, 58E15, 58E30, 81T13.

Keywords. Bundle of connections, gauge invariance, jet bundles, curvature mapping, functionally independent gauge-invariant Lagrangians, structure of Lie algebras.

1. Introduction

The notion of gauge invariance—defined as invariance under the group of base-preserving automorphisms of a principal fibre \( G \)-bundle \( \pi : P \rightarrow M \) over an oriented space-time—is fundamental in the theory of gauge fields and their associated fields, such as Yang–Mills–Higgs fields; for example, see the classical expositions [14] or [15].

Below, we are concerned with gauge invariance of the variational problems determined by a free (i.e., without any interaction term with a particle field) gauge-invariant Lagrangian function defined on the fibre bundle \( p : C \rightarrow M \) of connections on \( \pi : P \rightarrow M \).

The fundamental step in determining such Lagrangians is the so-called geometric formulation of Utiyama’s theorem (see [4, 10.2.15 Theorem]), according to which a Lagrangian \( L \) of first order on \( C \) is gauge invariant if and only if \( L \) factors through the curvature map by means of a zero-order Lagrangian on the vector bundle of differential 2-forms on \( M \) with values in.
the adjoint bundle of $P$ (also called “the curvature bundle”), which, in turn, must be invariant under the natural representation of the gauge group on that bundle; see [3] for the generalization of Utiyama’s theorem to Lagrangians for gauge-particle field interaction.

This reduces the problem of determining gauge-invariant Lagrangians to the problem of determining the zero-order gauge-invariant Lagrangians defined on the curvature bundle. If $G$ is connected, then the second problem can infinitesimally be solved by proving that zero-order gauge-invariant Lagrangians on the curvature bundle are the first integrals of an involutive distribution $\mathcal{D}$.

In this paper, we prove that $\mathcal{D}$ is of constant rank on a dense open subset and we compute this rank. If $\mathfrak{g}$ is the Lie algebra of $G$, and $n = \dim M$, $m = \dim \mathfrak{g}$, $l = \text{rank} \mathfrak{g}$, then we obtain the following results (see Sect. 4.1 below):

(1st) If $n = 2$, then the generic rank of $\mathcal{D}$ equals $m - l$,

(2nd) If $n \geq 3$, then the generic rank is $m$.

The result for $n = 2$ explains why the theory of Yang–Mills fields on a surface presents special features. In addition, if $G$ is semisimple, then according to a classical theorem by Chevalley (e.g., see [16, Theorem 4.9.3]), there exist $l$ homogeneous algebraically independent polynomials $p_1, \ldots, p_l$, such that the algebra of polynomial functions on $\mathfrak{g}$ that are invariant with respect to the adjoint representation of $G$ on $\mathfrak{g}$, is isomorphic to the algebra of polynomials in $p_1, \ldots, p_l$, thus providing a basis with geometric meaning for the algebra of first integrals of $\mathcal{D}$, as stated in Remark 5.1.

Next, for $n \geq 3$, we also obtain a basis of first integrals spanning differentiably the ring of zero-order Lagrangians. Assuming $G$ connected and semisimple, in the local case, such a basis is deduced from Hilbert–Nagata theorem (see Theorem 4.2).

Finally, we include several worked examples in low dimensions illustrating the previous general results.

2. Preliminaries

Let $p: E \to M$ be a fibred manifold over an orientable connected smooth manifold oriented by a volume form $\mathbf{v}$. A pair of diffeomorphisms $\phi \in \text{Diff} M$, $\Phi \in \text{Diff} E$, such that $p \circ \Phi = \phi \circ p$, is said to be an automorphism of $p$; the group of all automorphisms is denoted by $\text{Aut} E$.

If $Y \in \mathfrak{X}(E)$, then there exists a unique vector field $Y^{(1)}$ in $\mathfrak{X}(J^1 E)$—the 1-jet prolongation of $Y$ to the first-order jet bundle $J^1 E$—projectable onto $Y$, such that $L_{Y^{(1)}}$ keeps invariant the module of contact 1-forms spanned by the forms $\theta^\alpha = dy^\alpha - y^\alpha_i dx^i$, $1 \leq \alpha \leq m$ on $\Omega^1(J^1 E)$, where $n = \dim M$, $m + n = \dim E$, and $(x^i, y^\alpha), 1 \leq \alpha \leq m, 1 \leq i \leq n$, is a fibred coordinate system for $p: E \to M$ and $(x^i, y^\alpha, y^\alpha_i)$ is the induced coordinate system on $J^1 E$.

The Lie algebra of $\text{Aut} E$ is the Lie subalgebra of $p$-projectable vector fields $\mathfrak{X}_p(E) \subset \mathfrak{X}(E)$, namely, if $\Phi_t$ is the local flow of $Y \in \mathfrak{X}(E)$, then $\Phi_t \in \text{Aut} E$, $\forall t$, if and only if $Y \in \mathfrak{X}_p(E)$; in this case, $\Phi_t^{(1)}$ is the flow of $Y^{(1)}$. 
If $Y$ is a $p$-vertical vector field, then the formulas of 1-jet prolongation are as follows:

$$Y = v^\alpha \frac{\partial}{\partial y^\alpha}, v^\alpha \in C^\infty(E),$$

$$Y^{(1)} = v^\alpha \frac{\partial}{\partial y^\alpha} + v^\alpha_i \frac{\partial}{\partial y^\alpha_i}, v^\alpha_i = \frac{\partial v^\alpha}{\partial x^i} + \frac{\partial v^\alpha}{\partial y^\beta} y^\beta_i.$$  \hspace{1cm} (1)

Let $G$ be a Lie group. An automorphism of a principal $G$-bundle $\pi: P \to M$ is a $G$-equivariant diffeomorphism $\Phi: P \to P$. The group of all automorphisms of $P$ is denoted by $\text{Aut}P$. Every $\Phi \in \text{Aut}P$ determines a unique diffeomorphism $\phi: M \to M$, such that $\pi \circ \Phi = \phi \circ \pi$. If $\phi$ is the identity map on $M$, then $\Phi$ is said to be a gauge transformation (cf. [4, 3.2.1]); the subgroup of all gauge transformations is denoted by $\text{Gau}P$. A vector field $X \in \mathfrak{X}(P)$ is said to be $G$-invariant if $R_g \cdot X = X$, $\forall g \in G$; if $\Phi_t$ is the flow of $X$, then $X$ is $G$-invariant if and only if $\Phi_t \in \text{Aut}P$, $\forall t \in \mathbb{R}$. The Lie subalgebra of $G$-invariant vector fields on $P$ is denoted by $\text{aut}P \subset \mathfrak{X}(P)$. Each $G$-invariant vector field on $P$ is $\pi$-projectable.

Similarly, a $\pi$-vertical vector field $X \in \mathfrak{X}(P)$ is $G$-invariant if and only if $\Phi_t \in \text{Gau}P$, $\forall t \in \mathbb{R}$. Let $\text{gau}P \subset \text{aut}P$ be the ideal of all $\pi$-vertical $G$-invariant vector fields on $P$, which is usually called the gauge algebra of $P$. The quotient $T(P)/G$ exists as a differentiable manifold and it is endowed with a vector-bundle structure over $M$ (see [1]), whose global sections can naturally be identified to $\text{aut}P$; i.e., $\text{aut}P \cong \Gamma(M, T(P)/G)$.

If $G$ acts on the left on a manifold $F$, $(g, y) \mapsto g \ast y$, then $G$ acts on the product $P \times F$ by setting $(u, y) \cdot g = (u \cdot g, g^{-1}(y))$, $\forall y \in G$, $\forall u \in P$, $\forall y \in F$. The quotient manifold $(P \times F)/G$ exists and it defines a fibre bundle $\pi_F: P \times_G F \to M$, $\pi_F((u, y) \mod G) = \pi(u)$, called the bundle associated with $P$ by the action on $F$; e.g., see [4, Sect. 3.1], [10, Sect. 35], [11, p. 54].

Every $\Phi \in \text{Aut}P$ induces a diffeomorphism $\Phi_F: P \times_G F \to P \times_G F$ by setting $\Phi_F((u, y) \mod G) = (\Phi(u), y) \mod G$, $\forall u \in P$, $\forall y \in F$ and then $\pi_F \circ \Phi_F = \phi \circ \pi_F$.

Smooth sections $s: M \to P \times G$ of $\pi_F: P \times_G F \to M$ are in one-to-one correspondence with $G$-equivariant smooth maps $\tilde{s}: P \to F$, which means in this case that $\tilde{s}(u \cdot g) = g^{-1} \ast \tilde{s}(u)$, $\forall u \in P$, $\forall g \in G$, see [10, Proposition 35.1].

If $G$ acts on itself by conjugation, $G \times G \to G$, $(g, x) \mapsto gxg^{-1}$, $\forall g, x \in G$, then the associated bundle $\pi_G: \text{Ad}P = P \times_{G} G \to M$ is a Lie-group fibre bundle. Consequently, smooth sections $\Gamma(M, \text{Ad}P)$ of $\pi_G$ admit a group structure given by $(s \cdot s')(x) = s(x) \ast s'(x)$, for all $s, s' \in \Gamma(M, \text{Ad}P)$, $x \in M$, and the group $\text{Gau}P$ is isomorphic to $\Gamma(M, \text{Ad}P)$, see [10, Proposition 35.2]. Similarly $\text{gau}P \cong \Gamma(M, \text{Ad}P)$, where $\pi_G: \text{Ad}P \to M$ denotes the adjoint bundle, i.e., the bundle associated with $P$ by the adjoint representation of $G$ on its Lie algebra $\mathfrak{g}$; namely $\text{Ad}P = (P \times \mathfrak{g})/G$, the action of $G$ on $P \times \mathfrak{g}$ being defined by $(u, A) \cdot g = (u \cdot g, \text{Ad}_{g^{-1}}(A))$, $\forall u \in P, \forall A \in \mathfrak{g}, \forall g \in G$. The $G$-orbit of $(u, A) \in P \times \mathfrak{g}$ in $\text{Ad}P$ is denoted by $(u, A)_{\text{ad}}$. We thus obtain an exact sequence of vector bundles over $M$ (the so-called Atiyah sequence, see [1, Th. 1]):

$$0 \to \text{ad}P \to T(P)/G \xrightarrow{\pi_\pi} TM \to 0,$$  \hspace{1cm} (2)
where \( T(P)/G \) is the Atiyah algebroid of the principal bundle \( P \); e.g., see [13]. The fibres \((\text{ad}P)_x\) are endowed with a Lie-algebra structure determined by \([([u, A])_{\text{ad}}, (u, B)_{\text{ad}}] = (u, -[A, B])_{\text{ad}}, \forall u \in \pi^{-1}(x), \forall A, B \in \mathfrak{g}\), where \([\cdot, \cdot]\) denotes the bracket in \( \mathfrak{g} \). The sign of the bracket in the previous equation is needed to ensure that the natural identification \( \text{gau}P \simeq \Gamma(M, \text{ad}P) \) is a Lie-algebra isomorphism, when \( \text{gau}P \) is considered as a Lie subalgebra of \( \mathfrak{X}(P) \).

3. Bundle of Connections

Let \( X^{hr} \in \mathfrak{X}(P) \) be the horizontal lift of \( X \in \mathfrak{X}(M) \) with respect to a connection \( \Gamma \) on \( \pi: P \rightarrow M \). The vector field \( X^{hr} \) is \( G \)-invariant and projects onto \( X \) (cf. [11, II. Proposition 1.2]). Hence, we have a splitting of (2), \( s_{\Gamma}: TM \rightarrow T(P)/G \), \( s_{\Gamma}(X) = X^{hr} \). Conversely, any splitting \( s: TM \rightarrow T(P)/G \) of that sequence comes from a unique connection on \( P \). Therefore, there is a natural bijection between connections on \( P \) and splittings of the sequence above. Connections on \( P \) can be identified with the global sections of a bundle \( p: C \rightarrow M \); the section of \( p \) induced by \( \Gamma \) is denoted by \( s_{\Gamma}: M \rightarrow C \), and \( C \) is an affine bundle modelled over \( \text{Hom}(TM, \text{ad}P) \cong T^*M \otimes \text{ad}P \). For the details, we refer the reader to [6].

Let \( (U; x^i) \) be a coordinate system on an open domain \( U \subset M \) over which \( \pi \) admits a section \( s: U \rightarrow P \), so that \( \pi^{-1}(U) \cong U \times G \). For every \( B \in \mathfrak{g} \) let \( \tilde{B} \) be the infinitesimal generator of the flow of gauge transformations over \( U \) defined by \( \varphi^B_t(x, g) = (x, \exp(tB \cdot g), x \in U \). As \( \pi \circ \varphi^B_t = \pi \), the vector field \( \tilde{B} \in \mathfrak{X}(\pi^{-1}(U)) \) is \( \pi \)-vertical. If \( (B_\alpha)_{\alpha=1}^m \) is a basis of \( \mathfrak{g} \), then \( (\tilde{B}_\alpha)_{\alpha=1}^m \) is a basis of \( \Gamma(U, \text{ad}P) \). The horizontal lift with respect to \( \Gamma \) of \( \partial/\partial x^i \) is given as follows: \( \partial_{\Gamma}(\partial/\partial x^i) = \partial/\partial x^i - (A^\alpha_i \circ s_{\Gamma}) \tilde{B}_\alpha, 1 \leq i \leq n \). The functions \( (x^i, A^\alpha_i) \), \( i, j = 1, \ldots, n \) \( = \dim M \), \( 1 \leq \alpha \leq m = \dim G \), induce a coordinate system on \( p^{-1}(U) = C(\pi^{-1}U) \) (cf. [6]); hence, \( \dim C = n(m+1) \).

Each automorphism \( \phi \in \text{Aut}P \) acts on connections by pulling back connection forms; i.e., \( \Gamma' = \Phi(\Gamma) \) where \( \omega_{\Gamma'} = (\Phi^{-1})^*\omega_{\Gamma} \) (cf. [11, II. Proposition 6.2-(b)]). For each \( \phi \in \text{Aut}P \), there exists a unique diffeomorphism \( \Phi_C: C \rightarrow C \), such that \( p \circ \Phi_C = \phi \circ p \), where \( \phi: M \rightarrow M \) is the diffeomorphism induced by \( \Phi \) on the ground manifold. We thus obtain a group homomorphism \( \text{Aut}P \rightarrow \text{Diff}C \), and for every connection \( \Gamma \) on \( P \), we have \( \Phi_C \circ s_{\Gamma} = s_{\Phi(\Gamma)} \). If \( \Phi_t \) is the flow of a \( G \)-invariant vector field \( X \in \text{aut}P \), then \( (\Phi_t)_C \) is a one-parameter group in \( \text{Diff}C \) with infinitesimal generator denoted by \( X_C \), and the map \( \text{aut}P \rightarrow \mathfrak{X}(C), X \mapsto X_C \) is a Lie-algebra homomorphism.

In a coordinate domain \( (U; x^i)_{i=1}^n \) on \( M \) with the basis \( (\tilde{B}_\alpha)_{\alpha=1}^m \) of \( \text{ad}\pi^{-1}(U) \), it follows that each \( X \in \text{gau}\pi^{-1}(U) \) is written as \( X = g^\alpha \tilde{B}_\alpha, g^\alpha \in C^\infty(U) \), and we have (e.g., see [6]):

$$X_C = - \left( \frac{\partial g^\alpha}{\partial x^i} - c^\alpha_{\beta\gamma} A^\beta_i \right) \frac{\partial}{\partial A^\gamma_i},$$  \hspace{1cm} (3)

where \( c^\alpha_{\beta\gamma} \) are the structure constants: \([B_{\beta}, B_\gamma] = c^\alpha_{\beta\gamma} B_\alpha \).
4. Gauge Invariance

A Lagrangian density $\Lambda = L_\nu$, $L \in C^\infty(J^1C)$, on the bundle of connections is said to be gauge invariant if $X_C^{(1)}(L) = 0$, $\forall X \in \text{gauP}$, where $X_C^{(1)}$ denotes the 1-jet prolongation of the natural representation $\text{aut}P \to \mathfrak{X}(C)$. Similarly, a Lagrangian density is said to be aut$P$-invariant if:

$$L_{X_C^{(1)}}(L) = X_C^{(1)}(L)v + L(L_{X_C^{(1)})}v = 0, \quad \forall X \in \text{aut}P.$$  

The vector field $X_C^{(1)}$ is $p_1$-projectable onto $X' = \pi_*X$, where $p_1 : J^1C \to M$ is the canonical projection. We thus have $L_{X_C^{(1)}}(L) = (X_C^{(1)}(L) + \text{Ldiv}X')v$ and the condition of aut$P$-invariance yields $X_C^{(1)}(L) + \text{Ldiv}X' = 0$, $\forall X \in \text{aut}P$. It turns out, every aut$P$-invariant Lagrangian density is variationally trivial, as it is proved in [7, Corollary 1]. Thus, the notion of aut$P$-invariance is too restrictive to be useful in Field Theory.

If $X \in \text{gauP}$, then $X' = 0$ and the definition of gauge invariance is recovered. As every $\Phi \in \text{gauP}$ induces the identity map on $M$, the function $L$ is gauge invariant if and only if the gauge group is a group of symmetries of the Lagrangian density $\Lambda = L_\nu$, where $\nu$ is the volume form on the ground manifold. For more details, we refer the reader to [6–8].

4.1. The Number of Gauge-Invariant Lagrangians

Let $\Omega : J^1C \to \bigwedge^2 T^*M \otimes \text{ad}P$, $\Omega(\sigma_{\Gamma}^1) = (\Omega_{\Gamma})_x$, be the curvature mapping. The curvature form $\Omega_{\Gamma}$ of the connection $\Gamma$ corresponding to a section $s_\Gamma$ of $p$ is seen to be a two-form on $M$ with values in the adjoint bundle ad$P$. On the vector bundle $\bigwedge^2 T^*M \otimes \text{ad}P$, we consider the coordinate systems $(x^i; R_{jk}^\alpha)$, $j < k$, induced by a coordinate system $(U; x^i)_i$ on $M$, and a basis $(B_{\alpha})_{\alpha=1}^m$ of $\mathfrak{g}$, as follows:

$$\eta_2 = \sum_{j<k} R_{jk}^\alpha(\eta_2) \left(dx^j \wedge dx^k \otimes \tilde{B}_{\alpha}\right), \quad \forall \eta_2 \in \bigwedge^2 T^*_xM \otimes (\text{ad}P)_x, \forall x \in U.$$  

(4)

With respect to the coordinate systems $(x^i, A_{j}^\alpha, A_{j,k}^\alpha)$ and $(x^i; R_{jk}^\alpha)$, $j < k$, on $J^1C$ and $\bigwedge^2 T^*M \otimes \text{ad}P$, respectively, the equations of the curvature mapping are as follows:

$$R_{jk}^\alpha \circ \Omega = A_{j,k}^\alpha - A_{k,j}^\alpha - \sum_{\beta < \gamma} c_{\beta \gamma}^\alpha \left(A_{j}^\beta A_{k}^\gamma - A_{j}^\gamma A_{k}^\beta\right),$$  

$$1 \leq j < k \leq n, \quad 1 \leq \alpha \leq m.$$  

(5)

The geometric formulation of Utiyama’s Theorem (e.g., see [4]) states that a Lagrangian $L: J^1C \to \mathbb{R}$ is gauge invariant if and only if $L$ factors through $\Omega$ as $L = \tilde{L} \circ \Omega$, where

$$\tilde{L} : \bigwedge^2 T^*M \otimes \text{ad}P \to \mathbb{R}$$  

is a $C^\infty$ function that is invariant under the adjoint representation of $G$ on the curvature bundle. As the curvature map is surjective, the function $\tilde{L}$ is unique.
Theorem 4.1. Assume that the group \(G\) is connected, then according to the formulas (3) and (1), and taking the equations of the curvature mapping (5) into account, the function \(\tilde{L}\) in the formula (6) is invariant under the adjoint representation of \(G\) on the curvature bundle if and only if:

\[
\chi_\alpha(\tilde{L}) = 0, \quad 1 \leq \alpha \leq m.
\]

\[
\chi_\alpha = \sum_{i<j} c_{\gamma \alpha}^{\beta} R_{ij}^{\gamma} \frac{\partial}{\partial R_{ij}^{\gamma}}.
\] (7)

Alternatively, this equivalence can also be deduced from the formula for \(X_C^{(1)}\) in [7, (2.10)].

**Theorem 4.1.** Assume that the group \(G\) is connected.

The distribution \(\mathcal{D}\) on \(\bigwedge^2 T^* M \otimes \text{ad} P\) generated by the vector fields \(\chi_\alpha\), \(1 \leq \alpha \leq m\), given in (7), is involutive.

(i) The Lie algebra \(g\) is abelian if and only if \(\mathcal{D} = \{0\}\).

(ii) If \(g\) is not abelian, then the rank of \(\mathcal{D}\) on a dense open subset is constant.

(iii) If \(\dim M = n = 2\) and \(\dim g = m\), \(\text{rank } g = l\), then the generic rank of \(\mathcal{D}\) is equal to \(m - l\).

(iv) If \(\dim M = n \geq 3\), \(\dim g = m\), and \(g\) is semisimple, then the generic rank of \(\mathcal{D}\) is equal to \(m\).

**Proof.** As a computation shows, we have \([\chi_\rho, \chi_\sigma] = c_{\rho \sigma}^{\gamma} \chi_\gamma\) for \(1 \leq \rho < \sigma \leq m\); hence \(\mathcal{D}\) is involutive. The previous commutation relations come from a computation, although they are consistent with the fact that \(\chi_\alpha\) are fundamental vector fields of the adjoint action of \(G\) on a local trivialization of \(\text{ad} P\).

(i) The vector fields \(\chi_\alpha\), \(1 \leq \alpha \leq m\), vanish if and only if \(c_{\gamma \alpha}^{\beta} R_{ij}^{\gamma} = 0\) for all \(\alpha, \beta = 1, \ldots, m\), \(1 \leq i < j \leq n\), and these equations are obviously equivalent to saying that \(c_{\gamma \alpha}^{\beta} = 0\) for all \(\alpha, \beta, \gamma = 1, \ldots, m\).

(ii) The rank of \(\mathcal{D}\) at a point \(\eta_2 \in \bigwedge^2 T^*_{x_0} M \otimes (\text{ad} P)_{x_0}\) equals the rank of the \(m \times m\) matrix:

\[
\Lambda(\eta_2) = \begin{pmatrix}
\lambda_{\alpha,ij}^{\beta}(\eta_2) \\
\lambda_{\beta,ij}^{\alpha}(\eta_2)
\end{pmatrix}_{1 \leq \alpha \leq m, 1 \leq \beta \leq m, 1 \leq i < j \leq n}.
\]

As the entries of \(\Lambda\) are polynomial functions in the coordinates \(R_{ij}^\alpha\), \(1 \leq \alpha \leq m\), \(1 \leq i < j \leq n\), it follows that the rank of \(\mathcal{D}\) takes its maximum value on a dense open subset, and we have \(\max_{\eta_2} \text{rank } \Lambda(\eta_2) \leq m\), \(\forall \eta_2 \in \bigwedge^2 T^* M \otimes \text{ad} P\).

(iii) If \(n = 2\), then \(\Lambda(\eta_2)\) is a square matrix of size \(m\). If \(B = R_{12}^\alpha(\eta_2)B_\alpha\), then the matrix of the linear map \(\text{ad}_B : g \to g\) in the basis \((B_\gamma)^m_{\gamma=1}\) coincides with \(\Lambda(\eta_2)\); in fact, we have \(\text{ad}_B(B_\alpha) = [B, B_\alpha] = c_{\gamma \alpha}^{\beta} R_{12}^\gamma(\eta_2)B_\beta\), \(1 \leq \alpha \leq m\). Hence, if \(B \in g\) is a regular element, then the rank of \(\Lambda(\eta_2)\) is \(m - l\) exactly.

(iv) For every pair of indices \(1 \leq j < k \leq n\), let \(\Lambda_{jk}(\eta_2)\) be the \(m \times m\) matrix \(\Lambda_{jk}(\eta_2) = \begin{pmatrix}
\lambda_{\beta,jk}^{\alpha}(\eta_2) \\
\lambda_{\beta,jk}^{\alpha}(\eta_2)
\end{pmatrix}_{\alpha,\beta = 1}^m\). Then, the \(m \times m\) matrix \(\Lambda(\eta_2)\) can be...
written in blocks as follows: \( \Lambda(\eta_2) = (\Lambda_{12}(\eta_2), \ldots, \Lambda_{1n}(\eta_2), \ldots, \Lambda_{n-1,n}(\eta_2)) \).

To prove this case, we can use a Chevalley basis (e.g., see [9, Chapter 3, Theorem 1.19]); more precisely: let \( \{\alpha_1, \ldots, \alpha_l\} \) be a system of simple roots in the set \( \Delta_g = \{\alpha_1, \ldots, \alpha_l, \alpha_{l+1}, \ldots, \alpha_{m-1}\} \), and let \( h_i = h_{\alpha_i} \) for \( 1 \leq i \leq l \). The basis \( h_i, 1 \leq i \leq l, \epsilon_\alpha, \alpha \in \Delta_g \), of \( g \) satisfies the following properties:

\[
\begin{align*}
c_{ij}^\alpha &= 0, & c_{\alpha\beta}^\alpha &= 0, & i, j = 1, \ldots, l, \alpha \in \Delta_g, \\
c_{i\alpha}^\alpha &= 0, & c_{\alpha\beta}^{\alpha\beta} &= \delta^\alpha_\beta \langle \alpha | \alpha_i \rangle, & 1 \leq i \leq l, \alpha, \beta \in \Delta_g, \\
c_{\alpha,-\alpha}^\alpha &= e_i^\alpha, & c_{\alpha,-\alpha}^\beta &= 0, & 1 \leq i \leq l, \alpha, \beta \in \Delta_g, \\
c_{\alpha\beta}^\alpha &= 0, & c_{\alpha\beta}^{\gamma\beta} &= 0, & \alpha, \beta, \gamma \in \Delta_g, \alpha + \beta \neq 0, \alpha + \beta \notin \Delta_g, \\
c_{i\beta}^\alpha &= 0, & c_{\alpha\beta}^{\gamma\beta} &= \pm \delta^\gamma_{\alpha+\beta}(p+1), & 1 \leq i \leq l, \alpha + \beta \in \Delta_g, \gamma \in \Delta_g.
\end{align*}
\]

According to the general notations (4) in this case, we can write:

\[
\eta_2 = \sum_{j<k} \sum_{i=1}^{l} \left( R_{jk}^i(\eta_2) dx^j \wedge dx^k \otimes \tilde{h}_i \right) + \sum_{j<k} \sum_{u=1}^{m-l} \left( R_{jk}^{\alpha u}(\eta_2) dx^j \wedge dx^k \otimes \tilde{e}_{\alpha u} \right)
\]

With these notations, for \( 1 \leq i < j \leq n \), we have \( \lambda_{u,j,k}(\eta_2) = 0 \) for all \( t, u, v = 1, \ldots, l \), and

\[
\begin{align*}
\lambda_{t,j,k}^l(\eta_2) &= -\langle \beta_b \beta_t \rangle R_{jk}^{\beta_b t}(\eta_2), 1 \leq t \leq l, 1 \leq b \leq m - l, \\
\lambda_{t,j,k}^{\alpha u}(\eta_2) &= -e_t R_{jk}^{\alpha u}(\eta_2), 1 \leq t \leq l, 1 \leq a \leq m - l,
\end{align*}
\]

However, it is readily seen to be equivalent to the following two formulas:

If \( \alpha_u - \alpha_v = \alpha_{r_0} \in \Delta_g \), for some \( 1 \leq r_0 \leq m - l \), then:

\[
\lambda_{u,j,k}^\alpha(\eta_2) = \sum_{t=1}^{l} \delta^\alpha_{\alpha_v} \langle \alpha_v | \alpha_t \rangle R_{jk}^t(\eta_2)
\]

\[
\pm (p_{r_0} + 1) R_{jk}^{\alpha u}(\eta_2),
\]

\( u, v = 1, \ldots, m - l \).

If \( \alpha_u - \alpha_v \notin \Delta_g \), then:

\[
\lambda_{u,j,k}^\alpha(\eta_2) = \sum_{t=1}^{l} \delta^\alpha_{\alpha_v} \langle \alpha_v | \alpha_t \rangle R_{jk}^t(\eta_2),
\]

\( u, v = 1, \ldots, m - l \).

Hence, the \( m \times m \) matrix \( \Lambda_{jk}(\eta_2) \) is given by:

\[
\Lambda_{jk}(\eta_2) = \begin{pmatrix}
O & A_{jk}(\eta_2) \\
B_{jk}(\eta_2) & C_{jk}(\eta_2)
\end{pmatrix},
\]
where $O$ denotes the $l \times l$ zero matrix, and $A_{jk}(\eta_2)$, $B_{jk}(\eta_2)$, $C_{jk}(\eta_2)$ are the matrices with sizes $l \times (m - l)$, $(m - l) \times l$, $(m - l) \times (m - l)$, respectively, given by:

\[
A_{jk}(\eta_2) = (\lambda_{a,jk}^l(\eta_2))_{1 \leq l \leq l}^{1 \leq a \leq m - l}, \quad B_{jk}(\eta_2) = (\lambda_{i,jk}^l(\eta_2))_{1 \leq a \leq m - l}^{1 \leq l \leq l}, \quad C_{jk}(\eta_2) = (\lambda_{a,b,jk}^l(\eta_2))_{1 \leq a \leq m - l}^{1 \leq b \leq m - l}.
\]

According to (8), we have:

\[
A_{jk}(\eta_2) = -\begin{pmatrix}
\langle \beta_1 | \beta_1 \rangle R_{jk}^{\beta_1} \cdots \langle \beta_{m-l} | \beta_1 \rangle R_{jk}^{\beta_{m-l}}(\eta_2) \\
\vdots & \ddots & \vdots \\
\langle \beta_1 | \beta_1 \rangle R_{jk}^{\beta_1} \cdots \langle \beta_{m-l} | \beta_1 \rangle R_{jk}^{\beta_{m-l}}(\eta_2)
\end{pmatrix},
\]

and according to (9), we have:

\[
B_{jk}(\eta_2) = -\begin{pmatrix}
e_1 R_{jk}^{-\alpha_1}(\eta_2) & \cdots & e_1 R_{jk}^{-\alpha_1}(\eta_2) \\
\vdots & \ddots & \vdots \\
e_1 R_{jk}^{-\alpha_{m-1}}(\eta_2) & \cdots & e_1 R_{jk}^{-\alpha_{m-1}}(\eta_2)
\end{pmatrix},
\]

and $C_{jk}(\eta_2) = (\lambda_{a,b,jk}^l(\eta_2))_{1 \leq a \leq m - l}^{1 \leq b \leq m - l}$ is given by the formulas (11) and (12).

Let $A \subset \Delta_\mathfrak{g}$ be the set of elements $\alpha_{r_0} \in \Delta_\mathfrak{g}$ for which there exist $\alpha_u, \alpha_v \in \Delta_\mathfrak{g}$, such that $\alpha_u - \alpha_v = \alpha_{r_0}$, and let $E_{x_0} \subset \bigwedge^2 T_{x_0} M \otimes (adP)_{x_0}$ be the closed subset of $(adP)_{x_0}$-valued 2-covectors $\eta_2^0$, such that $R_{ij}^{\alpha_{r_0}}(\eta_2^0) = 0$ for all indices $1 \leq i < j \leq n$, as long as $\alpha_{r_0} \in A$. If $\eta_2^0 \in E_{x_0}$, then $C_{ij}(\eta_2^0)$ is a diagonal square matrix of order $m - l$, whose non-vanishing entries are given by:

\[
\mu_{u,ij} = \lambda_{\alpha_{u},i,j}^l(\eta_2^0) = \sum_{t=1}^{l} \langle \alpha_u | \alpha_t \rangle R_{i,j}^t(\eta_2^0), \quad 1 \leq u \leq m - l, \quad 1 \leq i < j \leq n.
\]

Hence, by taking the values $R_{i,j}^t(\eta_2^0)$ in a suitable dense open subset in $E_{x_0}$, it follows that $det C_{ij}(\eta_2^0) \neq 0$.

We have $m > 2l$ (see Remark 5.2), as $\mathfrak{g}$ is semisimple. We can thus decompose the matrix $A_{jk}(\eta_2)$ into two blocks as follows: $A_{jk}(\eta_2) = (A'_{jk}(\eta_2), A''_{jk}(\eta_2))$, where $A'_{jk}(\eta_2)$ denotes the $l \times l$ submatrix of $A_{jk}(\eta_2)$ determined by its $l$ rows and its first $l$ columns of $A_{jk}(\eta_2)$, whereas $A''_{jk}(\eta_2)$ denotes the $l \times (m - 2l)$ submatrix of $A_{jk}(\eta_2)$ determined by its $l$ rows and its $m - 2l$ columns. As a computation shows, we have:

\[
det A'_{jk}(\eta_2^0) = R_{jk}^{\alpha_1}(\eta_2^0) \cdots R_{jk}^{\alpha_l}(\eta_2^0) \det((\langle \alpha_u | \alpha_v \rangle)^l_{u,v=1}).
\]

Hence, $A'_{jk}(\eta_2^0)$ is non-singular on a dense open subset in $E_{x_0}$.

Moreover, since $n \geq 3$, we can consider the $m \times 2m$ submatrix of $\Lambda(\eta_2^0)$ given by $\Lambda'(\eta_2^0) = (\Lambda_{12}(\eta_2^0), \Lambda_{13}(\eta_2^0))$, and also the $m \times m$ submatrix $\Lambda''(\eta_2^0)$ of $\Lambda'(\eta_2^0)$ defined by:

\[
\Lambda''(\eta_2^0) = \begin{pmatrix}
A_{12}(\eta_2^0) & A_{13}(\eta_2^0) \\
C_{12}(\eta_2^0) & C_{13}(\eta_2^0)
\end{pmatrix},
\]
where $C'_{13}(\eta_2^0)$ is the $l \times l$ matrix:

$$C'_{13}(\eta_2^0) = \begin{pmatrix} \mu_{1,13} & 0 & \ldots & 0 \\ 0 & \mu_{2,13} & 0 & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \ldots & \mu_{l,13} \end{pmatrix}.$$  

Next, the determinant of $\Lambda''(\eta_2^0)$ is evaluated by the Laplacian expansion along the $l \times l$ minors of its $l$ last columns; e.g., see [5, III, Sect. 8, formula (21)].

Let $S_l$ be the set of $l$-element subsets of $\{1, 2, \ldots, m\}$ and for every $I = \{i_1 < \cdots < i_l\} \in S_l$ let us denote by $\Delta_I$ the $l \times l$ submatrix of:

$$\begin{pmatrix} A'_{13}(\eta_2^0) \\ C'_{13}(\eta_2^0) \end{pmatrix}$$  

determined by the rows $i_1, \ldots, i_l$; for example:

$$\Delta_{\{1, \ldots, l\}} = A'_{13}(\eta_2^0), \quad \Delta_{\{m-l+1, \ldots, m\}} = C'_{13}(\eta_2^0).$$  

If $\Delta'_I$ denotes the complement of $\Delta_I$ in $\Lambda''(\eta_2^0)$, then by setting $|I| = i_1 + \cdots + i_l$, we have

$$\det \Lambda''(\eta_2^0) = \sum_{I \in S_l} (-1)^{\frac{(m-i_1+1)+m+|I|}{2}} (\det \Delta_I) (\det \Delta'_I).$$

In this formula, the functions $\det \Delta_I$ depend on the values $R_{13}^l(\eta_2^0)$ only, while the functions $\det \Delta'_I$ depend on the values $R_{12}^l(\eta_2^0)$ only. Hence, $\det \Lambda''(\eta_2^0)$ is written as a sum of double products of functions depending on disjoint values. Furthermore, by separating the first summand from the right-hand side of the previous formula, it follows:

$$\det \Lambda''(\eta_2^0) = (-1)^{|I|}(m+1) \cdot R_{13}^l(\eta_2^0) \cdots R_{13}^l(\eta_2^0) \det(\langle \alpha_u | \alpha_v \rangle)_{u,v=1}^l \mu_{1,13} \cdots \mu_{l,13}$$

$$+ \sum_{I \in S_l, I \neq \{1, \ldots, l\}} (-1)^{\frac{(m-i_1+1)+m+|I|}{2}} (\det \Delta_I) (\det \Delta'_I),$$

with which one concludes the proof.  

\[ \square \]

### 4.2. Generators for Gauge-Invariant Lagrangians

Given a vector $\alpha \in \oplus^N (\text{ad}P)x$ and an element $u \in \pi^{-1}(x)$, there exist unique elements $A_1, \ldots, A_N \in \mathfrak{g}$, such that $\alpha = ((u, A_1)_\text{ad}, \ldots, (u, A_N)_\text{ad})$. To see this, $\alpha$ can be written as $\alpha = ((u', A'_1)_\text{ad}, \ldots, (u', A'_N)_\text{ad})$, with $u'_h \in \pi^{-1}(x)$, $A'_h \in \mathfrak{g}$, $1 \leq h \leq N$. As $G$ operates freely and transitively on the fibre $\pi^{-1}(x)$, there exist unique elements $g_1, \ldots, g_N \in G$, such that $u'_h = u \cdot g_h$, $1 \leq h \leq N$. Hence:

$$\alpha = ((u \cdot g_1, A'_1)_\text{ad}, \ldots, (u \cdot g_N, A'_N)_\text{ad})$$

$$= ((u, \text{Ad}_{g_1} A'_1)_\text{ad}, \ldots, (u, \text{Ad}_{g_N} A'_N)_\text{ad}).$$

If $I: \oplus^N \mathfrak{g} \to \mathbb{R}$ is a polynomial function invariant under the diagonal action induced by the adjoint representation of $G$ on its Lie algebra $\mathfrak{g}$, then a function $\tilde{I} \in C^\infty(\oplus^N \text{ad}P)$ can be associated by setting:

$$\tilde{I}((u, A_1)_\text{ad}, \ldots, (u, A_N)_\text{ad}) = I(A_1, \ldots, A_N),$$

$$\forall A_1, \ldots, A_N \in \mathfrak{g}.$$
as the formula above makes sense, because it does not depend on the representative chosen, provided that all the factors are expressed with respect to a common element in $P$. In fact, any other representative of the element:

$$\alpha = ((u, A_1)_{\text{ad}}, \ldots, (u, A_N)_{\text{ad}})$$

is of the form $\alpha = ((u \cdot g_i^{-1}, \text{Ad}_g A_1)_{\text{ad}}, \ldots, (u \cdot g_N^{-1}, \text{Ad}_g A_N)_{\text{ad}}), \forall g_i \in G,$ $1 \leq i \leq N$. As $I$ is invariant under the diagonal action, we have:

$$\bar{I} ((u \cdot g^{-1}, \text{Ad}_g A_1)_{\text{ad}}, \ldots, (u \cdot g^{-1}, \text{Ad}_g A_N)_{\text{ad}}) = I((\text{Ad}_g A_1, \ldots, \text{Ad}_g A_N)$$

$$= I(A_1, \ldots, A_N)$$

$$= \bar{I} ((u, A_1)_{\text{ad}}, \ldots, (u, A_N)_{\text{ad}}).$$

**Theorem 4.2.** Assume that the group $G$ is connected and semisimple. If $(U; x^i)_{i=1}^n$ is a coordinate system, such that $P$ is trivial over $U$, then a $\text{GauP}|_U$-equivariant vector-bundle isomorphism $\Psi: \bigwedge^2 T^* U \otimes \text{ad}P|_U \to \oplus^N \text{ad}P|_U$, $N = \frac{1}{2}n(n - 1)$, is defined by:

$$\Psi(\eta_2) = \eta_2\left(\left(\frac{\partial}{\partial x^i}\right)_x, \left(\frac{\partial}{\partial x^j}\right)_x\right), \ldots, \eta_2\left(\left(\frac{\partial}{\partial x^n}\right)_x, \left(\frac{\partial}{\partial x^n}\right)_x\right),$$

$$\eta_2 \in \bigwedge^2 T^*_x U \otimes (\text{ad}P)_x, \ x \in U.$$ (13)

There exists a finite system of generators $I^i$, $1 \leq i \leq \nu$, of the algebra of polynomial functions $\mathcal{P}(\oplus^N g)^G$ invariant under the diagonal action induced by the adjoint representation of $G$ on $g$ and the functions $\bar{I}^i \circ \Psi$, $1 \leq i \leq \nu$, generate the algebra $C^\infty(\bigwedge^2 T^* U \otimes \text{ad}P)^{\text{GauP}|_U}$ over $C^\infty(U)$ differentiably.

Finally, if $\{(U_\kappa; (x^i_\kappa)_{i=1}^n)_{\kappa \in K}\}$, is an atlas of $M$, $P$ is trivial over every $U_\kappa$, and $(\eta_\kappa)$, $\kappa \in K$, is a partition of unity subordinate to $(U_\kappa)$, $\kappa \in K$, then the functions $J^i = \sum_{\kappa \in K} \eta_\kappa(\bar{I}_\kappa^i \circ \Psi)$, $1 \leq i \leq \nu$, generate the algebra $C^\infty(\bigwedge^2 T^* U \otimes \text{ad}P)^{\text{GauP}|_U}$ over $C^\infty(M)$ differentiably.

**Proof.** From the very definition of $\Psi$, it follows that the map $\Psi_x$ induced on every fibre $\bigwedge^2 T^*_x U \otimes (\text{ad}P)_x$, $x \in U$, is $\mathbb{R}$-linear.

If $\eta_2 = \sum_{1 \leq i < j \leq n} (dx^i)_x \wedge (dx^j)_x \otimes A_{ij}, A_{ij} \in (\text{ad}P)_x$, belongs to $\ker \Psi_x$, then $A_{ij} = 0$ for $1 \leq i < j \leq n$. Hence, $\Psi_x$ is injective and since the vector bundles $\bigwedge^2 T^* M \otimes \text{ad}P$ and $\oplus^N \text{ad}P$ have the same rank, we conclude that $\Psi$ is an isomorphism. Moreover, every $\Phi \in \text{GauP}$ acts on $\text{ad}P$ as explained in Sect. 2, namely $\Phi_g((u, v) \mod G) = (\Phi(u), v) \mod G,$ $\forall u \in P,$ $\forall v \in g$, and acts trivially on $\bigwedge^2 T^* M$, thus proving that $\Psi$ is $\text{GauP}|_U$-equivariant.

As $G$ is assumed to be connected and semisimple, according to Weyl’s Theorem every (finite dimensional) linear representation of $G$ is completely reducible. Hence, by virtue of Hilbert–Nagata Theorem, there exists a finite system of generators $I^i$, $1 \leq i \leq \nu$, of the algebra of polynomial functions $\mathcal{P}(\oplus^N g)^G$ invariant under the diagonal action induced by the adjoint representation of $G$ on $g$, as said in the statement. If $I: \oplus^N g \to \mathbb{R}^\nu$ is the map whose components are $I^1, \ldots, I^\nu$, then by virtue of the main result in [12], it follows that for every $f \in C^\infty(\oplus^N g)^G$, there exists $g \in C^\infty(\mathbb{R}^\nu)$, such that $f = g \circ I$. Moreover, as $P|_U$ is trivial by virtue of the hypothesis, we can choose a trivialization $\tau: P|_U \to U \times G$, i.e., $\tau$ is an isomorphism.
of principal $G$-bundles, such that $\mathrm{pr}_1 \circ \tau = \pi$, which induces an isomorphism of vector bundles over $U$, $\tau_{\mathrm{ad}}: \oplus^N \mathrm{ad} P|_U \to U \times \oplus^N \mathfrak{g}$. Hence, every $F \in C^\infty(\wedge^2 T^* U \otimes \mathrm{ad} P|_U)^{\mathrm{Gau} P}$ can be written as $F = (g \circ I) \circ \mathrm{pr}_2 \circ \tau_{\mathrm{ad}} \circ \Psi$, where $\mathrm{pr}_2: U \times \oplus^N \mathfrak{g} \to \oplus^N \mathfrak{g}$ denotes the canonical projection onto the second factor.

Finally, as support $\eta_\alpha \subset U_\alpha$ and $\eta_\alpha \in C^\infty(M)$, it follows that $\eta_\alpha(\tilde{J}_\alpha \circ \Psi)$ is globally defined for $1 \leq i \leq \nu$, and taking into account the fact that the vector fields $\chi_\alpha$, $1 \leq \alpha \leq m$, in (7) spanning the distribution $\mathcal{D}$, are vertical with respect to the natural projection $\wedge^2 T^* M \otimes \mathrm{ad} P \to M$, we can conclude that $X(J^i) = 0$ for every $1 \leq i \leq \nu$ and every $X \in \Gamma(M, \mathcal{D})$, thus finishing the proof.

\section{Remarks and Examples}

\begin{remark}
Let $G$ be a connected Lie group. As $M$ is also assumed to be connected and oriented, if $n = 2$, then $\wedge^2 M$ is a trivial bundle of rank 1; hence, $\wedge^2 M \otimes \mathrm{ad} P$ is isomorphic to the adjoint bundle by means of the map $\wedge^2 T^* M \otimes \mathrm{ad} P \to \mathrm{ad} P$, $\nu_x \otimes X \mapsto X$, $\forall X \in (\mathrm{ad} P)_x$, $\nu$ being the volume form on $M$. As $G$ is connected, the first integrals of $\mathcal{D}$ coincide with the $C^\infty$ functions on $\mathrm{ad} P$ invariant under the adjoint representation of $G$ on $\mathfrak{g}$, namely $C^\infty(\mathrm{ad} P)^G = \{ f \in C^\infty(\mathrm{ad} P) : X(f) = 0, \forall X \in \mathcal{D} \}$. If $p \in S^*(\mathfrak{g}^*)$ is a polynomial invariant under the adjoint representation of $G$ on its Lie algebra and if $G$ is a semisimple complex Lie group, then we can consider the function $\tilde{p} \in C^\infty(\mathrm{ad} P)$ defined as above and Chevalley’s theorem (e.g., see [16, Theorem 4.9.3]) ensures the existence of $l$ homogeneous algebraically independent polynomials $p_1, \ldots, p_l$, such that the algebra of polynomial functions on $\mathfrak{g}$ that are invariant with respect to the adjoint representation of $G$ on $\mathfrak{g}$, is isomorphic to $\mathbb{C}[p_1, \ldots, p_l]$, i.e., $S^*(\mathfrak{g}^*)^G = \mathbb{C}[p_1, \ldots, p_l]$. If $\varphi: \mathrm{ad} P \to M \times \mathbb{C}^l$ is the map given by:

$$\varphi((u, A)_{\mathrm{ad}}) = (\pi(u), \tilde{p}_1((u, A)_{\mathrm{ad}}), \ldots, \tilde{p}_l((u, A)_{\mathrm{ad}})),$$

then according to [12, THEOREM 3], we have $C^\infty(\mathrm{ad} P)^G = \varphi^* C^\infty(M \times \mathbb{C}^l)$, and by applying Frobenius theorem, we conclude that $C^\infty(\wedge^2 T^* M \otimes \mathrm{ad} P)^G$ is generated by the $l$ functions $\tilde{p}_1, \ldots, \tilde{p}_l$.

This fact explains why equations of Yang–Mills type on a surface admit a geometric treatment; see [2].

\begin{remark}
The map $\pi^{(N)}: P^{(N)} = P \times_M \cdots \times_M P \to M$ is a principal fibre bundle with structure group $G^N$, the adjoint bundle of which can be identified to $\mathrm{ad} P^{(N)} = \oplus^N \mathrm{ad} P$. If $G$ is complex and semisimple, then $G^N$ also is, and we have $\dim G^N = Nm$, $\rank \mathfrak{g}^N = Nl$, $l = \rank \mathfrak{g}$. Hence, Chevalley’s theorem can be applied to the adjoint representation of $G^N$ on its Lie algebra, thus deducing that its ring of invariants admits a basis of $Nl$ algebraically independent homogeneous polynomials that can be constructed from a basis $p_1, \ldots, p_l$ of Chevalley for $\mathfrak{g}$ by defining the polynomial $p_i^h: \mathfrak{g}^N \to \mathbb{C}$ by $p_i^h(A_1, \ldots, A_N) = p_i(A_h)$, $1 \leq i \leq l$, $1 \leq h \leq N$. In particular, such polynomials are also invariant under the diagonal action induced by the
adjoint representation of $G$ on $\mathfrak{g}^N$; but the polynomials $(p^h_{i})_{1 \leq h \leq N}$ do not generate in general the ring of invariants for the diagonal action. In fact, if $n \geq 3$, then we know that the generic rank of the distribution $\mathcal{D}$ is $m$; hence, the maximum number of functionally independent gauge-invariant functions is $mN - m = m(N - 1)$. As the number of polynomials $p^h_i$ is $Nl$, if such polynomials span the ring of gauge invariants, it should be $Nl \geq m(N - 1)$, i.e., $\frac{l}{m} \geq 1 - \frac{1}{N}$, and this inequality never occurs in the semisimple case, as in this case we have $m > 2l$. Actually, as dimension and rank are additive, it suffices to prove the last inequality for simple algebras:

$$\dim \mathfrak{sl}(l + 1, \mathbb{C}) = l(l + 2) > 2l, \quad \dim \mathfrak{so}(2l + 1, \mathbb{C}) = l(2l + 1) > 2l,$$
$$\dim \mathfrak{sp}(2l, \mathbb{C}) = l(2l + 1) > 2l, \quad \dim \mathfrak{so}(2l, \mathbb{C}) = l(2l - 1) > 2l,$$

as in the fourth case, we must have $l \geq 2$; and for the exceptional algebras:

$$\dim \mathfrak{e}_6 = 78 > 12, \quad \dim \mathfrak{e}_7 = 133 > 14, \quad \dim \mathfrak{e}_8 = 248 > 16, \quad \dim \mathfrak{g}_2 = 14 > 4.$$

**Example 5.3.** If $\dim M = n = 3$ and $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$, then $m = 3$, $l = 1$. The basic invariant is $p(X) = x_{12}x_{21} + (x_{11})^2 = - \det(X)$, where

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & -x_{11} \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{R}).$$

The maximum number of functionally independent $G$-invariant functions over the curvature bundle $\bigwedge^2 T^* (M) \otimes \ad P \cong \text{locally } \oplus^3 \ad P$ is $3(3) - 3 = 6$ in this case.

To the quadratic polynomial $p$, it corresponds a unique symmetric bilinear form $s_p: \mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{sl}(2, \mathbb{R}) \to \mathbb{R}$ obtained by polarization:

$$p(X, Y) = \frac{1}{2} [p(X + Y) - p(X) - p(Y)] = x_{11}y_{11} + \frac{1}{2}x_{12}y_{21} + \frac{1}{2}x_{21}y_{12},$$

from which the following invariants on $\oplus^3 \mathfrak{sl}(2, \mathbb{R})$ follow:

$$I_{11}(X, Y, Z) = s_p(X, X), \quad I_{22}(X, Y, Z) = s_p(Y, Y),$$
$$I_{33}(X, Y, Z) = s_p(Z, Z), \quad I_{12}(X, Y, Z) = s_p(X, Y),$$
$$I_{13}(X, Y, Z) = s_p(X, Z), \quad I_{23}(X, Y, Z) = s_p(Y, Z).$$

As a calculation shows, these 6 functions are functionally independent in the dense open subset defined by:

$$0 \neq z_{11}(x_{11}y_{21} - x_{21}y_{11})$$
$$\cdot (x_{11}y_{12}z_{21} - x_{11}y_{21}z_{12} - x_{12}y_{11}z_{21} + x_{12}y_{21}z_{11} + x_{21}y_{11}z_{12} - x_{21}y_{12}z_{11}).$$
Using the local method in Theorem 4.2 by means of the map (13), it follows:

\[ I_{11} \circ \Psi = R_{12}^1 R_{12}^3 + (R_{12}^2)^2, \]
\[ I_{12} \circ \Psi = R_{12}^2 R_{12}^3 + \frac{1}{2} (R_{12}^1 R_{12}^3 + R_{12}^1 R_{12}^3), \]
\[ I_{13} \circ \Psi = R_{12}^2 R_{23}^3 + \frac{1}{2} (R_{23}^1 R_{12}^3 + R_{12}^1 R_{23}^3), \]
\[ I_{22} \circ \Psi = R_{13}^1 R_{13}^3 + (R_{12}^3)^2, \]
\[ I_{23} \circ \Psi = R_{13}^2 R_{23}^3 + \frac{1}{2} (R_{23}^1 R_{13}^3 + R_{13}^1 R_{23}^3), \]
\[ I_{33} \circ \Psi = R_{13}^2 R_{23}^3 + (R_{23}^2)^2. \]

**Example 5.4.** If \((e_{\alpha \beta})^m_{\alpha \beta=1}\) denotes the standard basis for \(\mathfrak{gl}(m, \mathbb{R})\), then we have \([e_{\alpha \beta}, e_{\rho \sigma}] = \delta_{\beta \rho}e_{\alpha \sigma} - \delta_{\alpha \sigma}e_{\rho \beta}\). If \(\mathfrak{g} = \langle e_{12}, e_{13}, e_{23} \rangle\) denotes the Heisenberg Lie algebra and \(B_1 = e_{12}, B_2 = e_{13}, B_3 = e_{23}\), then \([B_1, B_2] = 0, [B_1, B_3] = -B_2, [B_2, B_3] = 0\), or equivalently \(c_{12}^\alpha = c_{23}^\alpha = 0, c_{13}^\alpha = -\delta_{\alpha}^\beta, 1 \leq \alpha \leq 3\); it thus follows: \(\chi_1 = \sum_{i<j} R_{ij}^3 \frac{\partial}{\partial R_{ij}^3}, \chi_2 = 0, \chi_3 = -\sum_{i<j} R_{ij}^1 \frac{\partial}{\partial R_{ij}^1}\). We distinguish two cases: first) If \(n = 2\), then \(\chi_1 = R_{12}^3 \frac{\partial}{\partial R_{12}^3}, \chi_3 = -R_{12}^3 \frac{\partial}{\partial R_{12}^3}\), and these two vectors fields span a (singular) distribution which has rank one on a dense open subset. Hence, the generic rank of \(\mathcal{D}\) is 1 in this case; 2nd) If \(n \geq 3\), then the matrix of \(\chi_1\) and \(\chi_2\) in the basis \(\frac{\partial}{\partial R_{ij}^3}, 1 \leq i < j \leq n\), is as follows:

\[
\begin{pmatrix}
R_{12}^3 & \ldots & R_{1n}^3 & R_{23}^3 & \ldots & R_{2n}^3 & \ldots & R_{n-1,n}^3 \\
-R_{11}^3 & \ldots & -R_{1n}^3 & -R_{23}^3 & \ldots & -R_{2n}^3 & \ldots & -R_{n-1,n}^3
\end{pmatrix},
\]

and on the dense open subset on which at least one of the determinants:

\[
\begin{vmatrix}
R_{hi}^3 & R_{jk}^3 \\
-R_{hi}^3 & -R_{jk}^3
\end{vmatrix}, \quad 1 \leq h < i < j < k \leq n,
\]

does not vanish, the rank of \(\mathcal{D}\) is 2 in this case. Note that the algebra \(\mathfrak{g}\) under consideration are not semisimple.

**Example 5.5.** Let us consider \(\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})\) with its standard basis \(B_1 = e_{21}, B_2 = e_{12}, B_3 = e_{12}; [B_1, B_2] = 2B_1, [B_1, B_3] = -B_2, [B_2, B_3] = 2B_3\), i.e., \(c_{12}^\alpha = 2\delta_1^\alpha, c_{13}^\alpha = -\delta_2^\alpha, c_{23}^\alpha = 2\delta_3^\alpha, 1 \leq \alpha \leq 3\). If \(n = 2\), then we have:

\[
\begin{align*}
\chi_1 &= 2R_{12}^1 \frac{\partial}{\partial R_{12}^1} + R_{12}^3 \frac{\partial}{\partial R_{12}^3}, \\
\chi_2 &= 2R_{12}^1 \frac{\partial}{\partial R_{12}^1} - R_{12}^3 \frac{\partial}{\partial R_{12}^3}, \\
\chi_3 &= -R_{12}^1 \frac{\partial}{\partial R_{12}^1} + 2R_{12}^3 \frac{\partial}{\partial R_{12}^3}, \quad 0 = R_{12}^1 \chi_1 + R_{12}^2 \chi_2 + R_{12}^3 \chi_3.
\end{align*}
\]

Consequently, the generic rank of \(\mathcal{D}\) is 2 in this case. If \(n \geq 3\), then the components \(\chi_1', \chi_2', \chi_3'\) of the vector fields \(\chi_1, \chi_2, \chi_3\), respectively, in the subspace generated by \(\frac{\partial}{\partial R_{ij}^3}, 1 \leq \alpha \leq 3, 1 \leq i < j < 3\), are:

\[
\begin{align*}
\chi_1' &= -2R_{12}^1 \frac{\partial}{\partial R_{12}^1} - 2R_{13}^1 \frac{\partial}{\partial R_{13}^1} - 2R_{23}^1 \frac{\partial}{\partial R_{23}^1} + R_{12}^3 \frac{\partial}{\partial R_{12}^3} + R_{13}^3 \frac{\partial}{\partial R_{13}^3} + R_{23}^3 \frac{\partial}{\partial R_{23}^3}, \\
\chi_2' &= 2R_{12}^1 \frac{\partial}{\partial R_{12}^1} + 2R_{13}^1 \frac{\partial}{\partial R_{13}^1} + 2R_{23}^1 \frac{\partial}{\partial R_{23}^1} + 2R_{12}^3 \frac{\partial}{\partial R_{12}^3} - 2R_{13}^3 \frac{\partial}{\partial R_{13}^3} - 2R_{23}^3 \frac{\partial}{\partial R_{23}^3}, \\
\chi_3' &= -R_{12}^1 \frac{\partial}{\partial R_{12}^1} - R_{13}^1 \frac{\partial}{\partial R_{13}^1} - R_{23}^1 \frac{\partial}{\partial R_{23}^1} + 2R_{12}^3 \frac{\partial}{\partial R_{12}^3} + 2R_{13}^3 \frac{\partial}{\partial R_{13}^3} + 2R_{23}^3 \frac{\partial}{\partial R_{23}^3},
\end{align*}
\]
or in matrix notation:

\[
\begin{pmatrix}
\frac{\partial}{\partial R_{12}} & \frac{\partial}{\partial R_{13}} & \frac{\partial}{\partial R_{23}} & \frac{\partial}{\partial R_{12}} & \frac{\partial}{\partial R_{13}} & \frac{\partial}{\partial R_{23}} & \frac{\partial}{\partial R_{12}} & \frac{\partial}{\partial R_{13}} & \frac{\partial}{\partial R_{23}} \\
2R_{12} & -2R_{13} & -2R_{23} & R_{12}^3 & R_{13}^3 & R_{23}^3 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
\chi_1' & \chi_2' & \chi_3' \\
-2R_{12} & 2R_{13} & 0 & -R_{12} & -R_{13} & -R_{23} & 2R_{12} & 2R_{13} & 2R_{23}
\end{pmatrix}
\]

and the determinant

\[
\begin{vmatrix}
-2R_{12} & -2R_{13} & R_{12}^3 \\
2R_{13} & 0 & 0 & -R_{12}
\end{vmatrix}
= 4R_{12}(R_{13}R_{12}^2 - R_{12}^2R_{13})
\]

does not vanish identically. Accordingly, for \(n \geq 3\), the generic rank of \(D\) is 3.

**Example 5.6.** If \(n = 3\), \(g = \mathfrak{so}(3, \mathbb{R})\), then \(m = 3\), \(l = 1\), and by considering the standard basis \(B_1 = e_{12} - e_{21}\), \(B_2 = e_{13} - e_{31}\), \(B_3 = e_{23} - e_{32}\), we have \(X = xB_1 + yB_2 + zB_23\), \(\det(tI_3 - X) = t^3 + (x^2 + y^2 + z^2) t\). Thus, the basic invariant is \(p(X) = x^2 + y^2 + z^2\). Hence:

\[
I_{ii}(X_1, X_2, X_3) = (x_i)^2 + (y_i)^2 + (z_i)^2,
\]

\(i = 1, 2, 3\),

\[
I_{12}(X_1, X_2, X_3) = x_1x_2 + y_1y_2 + z_1z_2,
\]

\[
I_{13}(X_1, X_2, X_3) = x_1x_3 + y_1y_3 + z_1z_3,
\]

\[
I_{23}(X_1, X_2, X_3) = x_2x_3 + y_2y_3 + z_2z_3,
\]

and one can check that these invariants are functionally independent and, therefore, build a system of generators for the ring of (local) invariants.

**Example 5.7.** If \(n = 3\), \(g = \mathfrak{so}(4, \mathbb{R})\), then \(m = 6\), \(l = 2\) and the characteristic polynomial of \(X = \sum_{1 \leq i < j \leq 4} x_{ij}(e_{ij} - e_{ji})\) is:

\[
\det(tI_4 - X) = t^4 + p_1(X)t^2 + p_2(X)^2,
\]

\[
p_1(X) = \sum_{1 \leq i < j \leq 4} (x_{ij})^2,
\]

\[
p_2(X) = x_{12}x_{34} + x_{14}x_{23} - x_{13}x_{24}.
\]

Proceeding as above, by polarizing \(p_1\) and \(p_2\), we obtain 12 generically independent invariant functions, which coincides with the corank of the distribution \(D\) in this case.

**Acknowledgements**

MCL and ERM were partially supported by MCIU (Spain) under project no. PGC2018–098321-B-I00.

**Publisher’s Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.
References

[1] Atiyah, M.F.: Complex analytic connections in fibre bundles. Trans. Am. Math. Soc. 85, 181–207 (1957)
[2] Atiyah, M.F., Bott, R.: The Yang–Mills equations over Riemann surfaces. Phil. Trans. R. Soc. Lond. A 308, 523–615 (1982)
[3] Betounes, D.: The geometry of gauge-particle field interaction: a generalization of Utiyama’s theorem. J. Geom. Phys. 6(1), 107–125 (1989)
[4] Bleecker, D.: Gauge Theory and Variational Principles, Global Analysis Pure and Applied Series A, 1. Addison-Wesley Publishing Co., Reading (1981)
[5] Bourbaki, N.: Éléments de Mathématique. Algèbre. Chapitres 1 à 3. Hermann, Paris (1970)
[6] Castrillón López, M., Muñoz Masqué, J.: The geometry of the bundle of connections. Math. Z. 236, 797–811 (2001)
[7] Castrillón López, M., Muñoz Masqué, J., Tudor, T.: Gauge invariance and variational trivial problems on the bundle of connections. Differ. Geom. Appl. 19, 127–145 (2003)
[8] Castrillón López, M., Muñoz Masqué, J.: Hamiltonian structure of gauge-invariant variational problems. Adv. Theor. Math. Phys. 16, 39–63 (2012)
[9] Gorbatsevich, V.V., Onishchik, A.L., Vinberg, E.B.: Lie Groups and Lie Algebras III. Structure of Lie Groups and Lie Algebras, Encyclopaedia of Mathematical Sciences, vol. 41. Springer, Berlin (1994). (English transl. of: A. L. Onishchik, V. V. Gorbatsevich, E. B. Vinberg, “Lie groups and Lie algebras III”, Current Problems in Mathematics. Fundamental Directions, Itogi Nauki i Tekhniki, 41, Akad. Nauk SSSR, Moscow, (1990))
[10] Guillemin, V., Sternberg, S.: Symplectic Techniques in Physics. Cambridge University Press, Cambridge (1983)
[11] Kobayashi, S., Nomizu, K.: Foundations of Differential Geometry, Vol. II & II. Wiley, New York (1963, 1969)
[12] Luna, D.: Fonctions différentiables invariantes sous l’opération d’un groupe réductif. Ann. Inst. Fourier (Grenoble) 26(1), 33–49 (1976)
[13] Mackenzie, K.: Lie Groupoids and Lie Algebroids in Differential Geometry. London Mathematical Society Lecture Note Series Book, p. 124 (1987)
[14] Marathe, K.B., Martucci, G.: The Mathematical Foundations of Gauge Theories, Studies in Mathematical Physics, vol. 5. North-Holland Publishing Co., Amsterdam (1992)
[15] Trautman, A.: Differential Geometry for Physicists, Stony Brook Lectures. Monographs and Textbooks in Physical Science, vol. 2. Bibliopolis, Naples (1984)
[16] Varadarajan, V.S.: Lie Groups, Lie Algebras, and Their Representations. Prentice-Hall Series in Modern Analysis. Prentice-Hall Inc, Englewood Cliffs (1974)
Marco Castrillón López  
Departamento de Álgebra, Geometría y Topología, Facultad de Matemáticas  
UCM  
Plaza de ciencias 3  
28040 Madrid  
Spain  
e-mail: mcastri@mat.ucm.es

Jaime Muñoz Masqué  
Instituto de Tecnologías Físicas y de la Información  
CSIC  
C/Serrano 144  
28006 Madrid  
Spain  
e-mail: jaime@iec.csic.es

Eugenia Rosado María  
Departamento de Matemática Aplicada  
Escuela Técnica Superior de Arquitectura, UPM  
Avda. Juan de Herrera 4  
28040 Madrid  
Spain  
e-mail: eugenia.rosado@upm.es

Received: April 24, 2019.  
Revised: June 24, 2019.  
Accepted: December 2, 2019.