POSITIVE ENERGY REPRESENTATIONS
OF THE CONFORMAL QUANTUM ALGEBRA

L. Dąbrowski*, V.K. Dobrev++, R. Floreanini† and V. Husain++*

* International School for Advanced Studies, SISSA, Trieste, Italy
++ International Center for Theoretical Physics, Trieste, Italy
† Istituto Nazionale di Fisica Nucleare, Sezione di Trieste, Dipartimento di Fisica Teorica, Università di Trieste, Strada Costiera 11, 34014 Trieste, Italy

Abstract

The positive-energy unitary irreducible representations of the $q$-deformed conformal algebra $\mathcal{C}_q = \mathcal{U}_q(su(2, 2))$ are obtained by appropriate deformation of the classical ones. When the deformation parameter $q$ is $N$-th root of unity, all these unitary representations become finite-dimensional. For this case we discuss in some detail the massless representations, which are also irreducible representations of the $q$-deformed Poincaré subalgebra of $\mathcal{C}_q$. Generically, their dimensions are smaller than the corresponding finite-dimensional non-unitary representation of $su(2, 2)$, except when $N = 2$, $h = 0$ and $N = 2|h| + 1$, where $h$ is the helicity of the representations. The latter cases include the fundamental representations with $h = \pm 1/2$. 

* Permanent address and after 31 July 1992: Bulgarian Academy of Sciences, Institute of Nuclear Research and Nuclear Energy, 72 Tsarigradsko Chaussee, 1784 Sofia, Bulgaria.
++ Address after 30 September 1992: Theoretical Physics Institute, University of Alberta, Edmonton, Canada T6G 2J1.
1. The positive energy unitary irreducible representations (UIRs) of the conformal algebra $su(2, 2)$ of four dimensional Minkowski space are certainly of physical relevance. They are given in Ref.[1]. In this letter we consider the $q$-deformed conformal algebra $C_q \equiv U_q(su(2, 2))$ [2] and present the $q$-analogues of the UIRs of [1].

For generic complex $q$, the positive energy UIRs $V_q$ of $C_q$ are deformations of the respective representations of $su(2, 2)$. Here, our considerations amount to an introduction of the correct scalar product in the representations $V_q$. It turns out that we can use the same scalar product as for the undeformed representations. However, we have to extend the hermitian conjugation used in [1] to a hermitian conjugation of the complexification $U_q(sl(4, C))$ of $U_q(su(2, 2))$. For $|q| = 1$, this is done in Section 3.

When the deformation parameter $q$ is a root of unity, the picture of the representations changes drastically. All irreducible representations of $U_q(su(2, 2))$, as inherited from $U_q(sl(4, C))$, are finite-dimensional. In Section 4 we give a list of the positive energy UIRs. We emphasize that, unlike the classical case, these unitary representations are finite-dimensional. We discuss in more detail the massless finite-dimensional representations of $C_q$; these are also UIRs of the $q$-deformed Poincaré subalgebra of $C_q$.

2. The physically relevant representations of the 4-dimensional conformal algebra $su(2, 2)$ may be labelled by $\chi = [j_1, j_2, d]$, where $2j_1, 2j_2$ are non-negative integers fixing finite dimensional irreducible representations of the Lorentz subalgebra $so(3, 1)$, and $d$ is the conformal dimension (or energy). We would like to label the representations of $U_q(su(2, 2))$ by the same set of indices $\chi$ and because of this we shall use the $q$-deformed conformal algebra $U_q(su(2, 2))$ (cf. [2]) which has the $q$-deformed Lorentz algebra $U_q(so(3, 1))$ as a Hopf subalgebra. Since $U_q(su(2, 2))$ is a real form of $U_q(sl(4, C))$ we need first to recall the latter deformation.

The $q$-deformation $U_q(sl(4, C))$ of the universal enveloping algebra $U(sl(4, C))$ is defined [3,4] as the associative algebra over $C$ with Chevalley generators $X_j^{\pm}$, $H_j$, $j = 1, 2, 3$ and with relations:

$$[H_j, H_k] = 0, \quad [H_j, X_k^\pm] = \pm a_{jk} X_k^\pm, \quad (1a)$$

$$[X_j^+, X_k^-] = \delta_{jk} \frac{q^{H_j/2} - q^{-H_j/2}}{q^{1/2} - q^{-1/2}} = \delta_{jk} [H_j]_q, \quad (1b)$$

$$(X_j^\pm)^2 X_k^\pm - [2]_q X_j^\pm X_k^\pm X_j^\pm + X_k^\pm (X_j^\pm)^2 = 0, \quad (jk) = (12), (21), (23), (32)$$

$$[X_1^\pm, X_3^\mp] = 0, \quad (1c)$$

where $[x]_q = (q^{x/2} - q^{-x/2})/(q^{1/2} - q^{-1/2})$, $a_{jk} = (2(\alpha_j, \alpha_k)/(\alpha_j, \alpha_j))$, $j, k = 1, 2, 3$, is the Cartan matrix of $sl(4, C)$; $\alpha_1, \alpha_2, \alpha_3$ are the simple roots; the non-zero products between the simple roots are: $\langle \alpha_j, \alpha_j \rangle = 2$, $j = 1, 2, 3$, $\langle \alpha_1, \alpha_2 \rangle = (\alpha_2, \alpha_2) = -1$. The non-simple positive roots are: $\alpha_{12} = \alpha_1 + \alpha_2$, $\alpha_{23} = \alpha_2 + \alpha_3$, $\alpha_{13} = \alpha_1 + \alpha_2 + \alpha_3$. The elements $H_j$ span the Cartan subalgebra $\mathcal{H}$ while the elements $X_j^\pm$ generate the subalgebras $\mathcal{G}^\pm$.

The algebra $U_q(sl(4, C))$ is a Hopf algebra [5] with co-multiplication $\delta$, co-unit $\epsilon$ (homomorphisms) and antipode $\gamma$ (antihomomorphism) defined on the generators as follows.
\[ \delta(H_j) = H_j \otimes 1 + 1 \otimes H_j, \quad \delta(X^\pm_j) = X^\pm_j \otimes q^{H_j/4} + q^{-H_j/4} \otimes X^\pm_j, \quad (2a) \]
\[ \epsilon(H_j) = \epsilon(X^\pm_j) = 0, \quad \gamma(H_j) = -H_j, \quad \gamma(X^\pm_j) = -q^{\pm 1/2} X^\pm_j. \quad (2b) \]

The Cartan-Weyl basis for the non-simple roots is given by (cf. [4,6]):
\[ X_{jk}^\pm = \pm (q^{1/4} X_j^\pm X_k^\pm - q^{-1/4} X_k^\pm X_j^\pm), \quad (jk) = (12), (23), \quad (3a) \]
\[ X_{13}^\pm = \pm (q^{1/4} X_1^\pm X_{23}^\pm - q^{-1/4} X_{23}^\pm X_1^\pm) = \pm (q^{1/4} X_{12}^\pm X_3^\pm - q^{-1/4} X_3^\pm X_{12}^\pm). \quad (3b) \]

All other commutation relations and Hopf algebra relations for the generators in (3) follow from these definitions.

Let us consider the conformal algebra \( su(2,2) \cong so(4,2) \). It has 15 generators \( Y_{AB} = -Y_{BA} \), \( A, B = 1, 2, 3, 5, 6, 0 \) satisfying
\[ [Y_{AB}, Y_{CD}] = \eta_{BC} Y_{AD} - \eta_{AC} Y_{BD} - \eta_{BD} Y_{AC} + \eta_{AD} Y_{BC}, \quad (4) \]
where \( \eta_{AB} = \text{diag}(- - - - + +) \). Since \( su(2,2) \) is the conformal algebra of 4-dimensional Minkowski space-time we use a deformation [2] consistent with the subalgebra structure relevant for the physical applications. These subalgebras are: the Lorentz subalgebra \( so(3,1) \) generated by \( Y_{\mu\nu} \), \( \mu, \nu = 1, 2, 3, 0 \), the subalgebra of translations generated by \( P_\mu = Y_{\mu5} + Y_{\mu6} \), the subalgebra of special conformal transformations generated by \( K_\mu = Y_{\mu5} - Y_{\mu6} \), the dilatations subalgebra generated by \( D = Y_{56} \). For a deformation of \( su(2,2) \) one uses the expressions for its generators as complex linear combinations of the generators of its complexification \( sl(4, \mathbb{C}) \) compatible with the reality structure. The deformation \( \mathcal{U}_q(su(2,2)) \) introduced in [2] uses essentially the same linear combinations as in [1] (cf. (2.21)) and we omit these explicit expressions for the lack of space.

3. In this section we consider the representations of \( \mathcal{C}_q = \mathcal{U}_q(su(2,2)) \) in the generic case when the deformation parameter is not a root of unity. In this case the representations of \( \mathcal{C}_q \) we use are irreducible lowest weight modules \( M^x \) (in particular, Verma modules \( V^x \)) of \( \mathcal{U}_q(sl(4, \mathbb{C})) \) together with the reality condition necessary for the construction of the scalar product in \( M^x \) (or \( V^x \)).

We use the standard decomposition \( \mathcal{G} = sl(4, \mathbb{C}) = \mathcal{G}^+ \oplus \mathcal{H} \oplus \mathcal{G}^- \), where \( \mathcal{H} \) and \( \mathcal{G}^\pm \) were introduced in Section 2. A lowest weight module \( M^x \) of \( \mathcal{U}_q(sl(4, \mathbb{C})) \) is given by its lowest weight \( \Lambda_x \in \mathcal{H}^* \) (\( \mathcal{H}^* \) being the dual of \( \mathcal{H} \)) and lowest weight vector \( v_0 \equiv v_0(\chi) \), such that \( v_0 \) is annihilated by the lowering generators, \( X v_0 = 0, X \in \mathcal{U}_q(\mathcal{G}^-) \), and \( H v_0 = \Lambda_x(H) v_0 \) for any Cartan generator \( H \).

In particular, we use the Verma modules \( V^x \) which are the lowest weight modules such that \( V^x = \mathcal{U}_q(\mathcal{G}^+) v_0 \). Thus the Poincaré-Birkhoff-Witt theorem tells us that the basis of \( V^x \) consists of monomial vectors
\[ \Psi_{\{k\}} = (X_1^+)^{k_1} (X_2^+)^{k_2} (X_3^+)^{k_3} (X_{12}^+)^{k_{12}} (X_{23}^+)^{k_{23}} (X_{13}^+)^{k_{13}} v_0 \quad k_j, k_{jk} \in \mathbb{Z}_+, \quad (5) \]
where $X_{jk}^+ \equiv X_{jj}^+$ and $X_{jk}^+$ are the six raising generators of $sl(4, \mathbb{C})$ (cf. Section 2). In order to consider $V^\chi$ as a representation of the real form it is not enough to express the generators of $U_q(sl(4, \mathbb{C}))$ as linear combinations of generators of $U_q(su(2, 2))$. We have to introduce, as in the $q = 1$ case, a hermiticity condition invariant with respect to $U_q(su(2, 2))$. Such a condition is well known in the undeformed case and is given by (cf. [1] (4.8)):

\[
\omega(X_{jk}^+) = \begin{cases} X_{jk}^+, & (jk) = (11), (33) \\ -X_{jk}^+, & \text{otherwise} \end{cases}
\]

(6a)

\[
\omega(H) = H, \quad \forall H \in \mathcal{H}.
\]

(6b)

The problem in the $q$-deformed case is to extend this conjugation to $U_q(sl(4, \mathbb{C}))$ as an anti-linear anti-involution thus making $U_q(sl(4, \mathbb{C}))$ a $*$-Hopf algebra. For this extension it is enough to postulate (6) for the simple root vectors and the corresponding Cartan subalgebra elements, i.e., for $X_k^\pm \equiv X_{kk}^\pm$ and $H_k$ for $k = 1, 2, 3$. Then (6) follows for the non-simple root vectors. Indeed, take for example $X_{12}^+ \equiv q^{1/4} X_1^+ X_2^+ - q^{-1/4} X_2^+ X_1^+$. Then we have (iff $|q| = 1$):

\[
\omega(X_{12}^+) = q^{1/4} \omega(X_2^+) \omega(X_1^+) - q^{-1/4} \omega(X_1^+) \omega(X_2^+) = -q^{1/4} X_2^+ X_1^- + q^{-1/4} X_1^- X_2^- = - (q^{-1/4} X_2^- X_1^- - q^{1/4} X_1^- X_2^-) = -X_{12}^+.
\]

The same considerations go for the other non-simple root vectors. For the other commutation relations $\omega$ acts as an anti-linear anti-involution. Finally one can check that $\omega$ is an anti-linear coalgebra anti-involution. Thus $U_q(sl(4, \mathbb{C}))$ is a $*$-Hopf algebra with $|q| = 1$.

As in the undeformed case, the conjugation $\omega$ will be used to introduce a $U_q(su(2, 2))$-invariant scalar product, i.e., a scalar product such that the generators $C_k$ of $\mathcal{C}_q$ are skew hermitian. As in the undeformed case (cf. [1] (2.17)) it is convenient to make a basis transformation so that the generators of $\mathcal{C}_q$ obey: $\omega(X) = -X$. A set of such generators is given by:

\[
iH_k, \quad k = 1, 2, 3, \quad X_k^+ - X_k^-, \quad i(X_k^+ + X_k^-), \quad k = 1, 3,
\]

\[
X_{jk}^+ + X_{jk}^-, \quad i(X_{jk}^+ - X_{jk}^-), \quad (jk) = (22), (12), (23), (13).
\]

(7)

In particular, the conformal Hamiltonian $H_0$ [1] is given in the two bases (4) and (7) by:

\[
H_0 = \frac{1}{2}(P_0 + K_0) = Y_{05} = \frac{1}{2}(H_1 + H_3) + H_2.
\]

(8)

For the scalar product of two vectors of the form (5) we take, as in [1],

\[
(\Psi_{\{k'\}}, \Psi_{\{k\}}) = \left(v_0^*, \omega((X_{13}^+)^{k_1} v_0) \omega((X_{23}^+)^{k_2} v_0) \omega((X_{12}^+)^{k_3} v_0) \omega((X_3^+)^{k_4} v_0) \times \omega((X_1^+)^{k_5} v_0) \omega((X_2^+)^{k_6} v_0) \omega((X_3^+)^{k_7} v_0) \omega((X_1^+)^{k_8} v_0) \times \omega((X_1^-)^{k_9} v_0) \omega((X_2^-)^{k_{10}} v_0) \omega((X_3^-)^{k_{11}} v_0) \omega((X_1^-)^{k_{12}} v_0) \times \omega((X_1^-)^{k_{13}} v_0) \right. \]
\[
\left. \times \omega((X_2^-)^{k_{14}} v_0) \omega((X_3^-)^{k_{15}} v_0) \omega((X_1^-)^{k_{16}} v_0) \omega((X_2^-)^{k_{17}} v_0) \omega((X_3^-)^{k_{18}} v_0) \right). \quad (9)
\]
with \((v_0^*, v_0) = 1\). (Note that (9) is an adaptation of the classical contravariant Schapovalov form.) Calculation of (9) is performed in the standard manner by moving the lowering (raising) generators to the right (left) where they annihilate \(v_0\) \((v_0^*)\). Finally, the result is some polynomial in \(\Lambda_\chi(H)\). Thus we have to specify how \(\Lambda_\chi(H)\) is fixed by the representation \(\chi\) which we take as in [1], i.e. we have

\[
\Lambda_\chi(H_1) = -2j_1 \, , \quad \Lambda_\chi(H_2) = d + j_1 + j_2 \, , \quad \Lambda_\chi(H_3) = -2j_2 \, .
\] (10)

Note that our generators \(H_1, H_2, H_3\) correspond to \(2H_1, H_0 - H_1 - H_2, 2H_2\), respectively, of [1]. In particular, using (8) we see that \(d\) is the eigenvalue of the conformal Hamiltonian \(H_0\); hence \(d\) is called the conformal energy (or dimension).

Given the scalar product (9), we have to determine whether it provides an UIR of \(U_q(su(2, 2))\). It is clear that the conditions on \(j_1, j_2\) and \(d\) will be the same as in [1], and below we specify precisely the UIR spaces applying new results in the representation theory of Verma modules ([6] and references therein).

Generically, the Verma modules \(V^\chi\) are irreducible. A Verma module \(V^\chi\) is reducible [6] iff there exists a positive root \(\alpha\) and a positive integer \(m_\alpha\) such that the following equality holds

\[
[(\Lambda_\chi - \rho)(H_\alpha) + m_\alpha]_q = 0 \, ,
\] (11)

where \(H_\alpha\) is a linear combination of \(H_k\), specifically, if \(\alpha = \sum_k n_k \alpha_k, n_k \in \mathbb{Z}_+, \alpha_k\) are simple roots, then \(H_\alpha = \sum_k n_k H_k\), and \(\rho\) is half the sum of positive roots; note that \(\rho(H_k) = 1\). For the six positive roots of the root system of \(sl(4, \mathbb{C})\), one has (see Ref.[7]):

\[
m_1 = -\Lambda_\chi(H_1) + 1 = 2j_1 + 1 \, ,
\] (12a)
\[
m_2 = -\Lambda_\chi(H_2) + 1 = 1 - d - j_1 - j_2 \, ,
\] (12b)
\[
m_3 = -\Lambda_\chi(H_3) + 1 = 2j_2 + 1 \, ,
\] (12c)
\[
m_{12} = -\Lambda_\chi(H_{12}) + 2 = m_1 + m_2 = 2 - d + j_1 - j_2 \, ,
\] (12d)
\[
m_{23} = -\Lambda_\chi(H_{23}) + 2 = m_2 + m_3 = 2 - d - j_1 + j_2 \, ,
\] (12e)
\[
m_{13} = -\Lambda_\chi(H_{13}) + 3 = m_1 + m_2 + m_3 = 3 - d + j_1 + j_2 \, .
\] (12f)

Whenever (11) is fulfilled there exists a singular (null) vector \(v_s\) in \(V^\chi\) such that \(v_s \neq v_0, X v_s = 0, X \in U_q(G^-)\) and \(H_\alpha v_s = (\Lambda_\chi + m_\alpha)(H_\alpha) v_s\).

To obtain an irreducible lowest weight module we have to factor out all singular vectors. First of all, we have that \(m_1\) and \(m_3\) are positive, since \(2j_1\) and \(2j_2\) are non-negative integers. The corresponding singular vectors are

\[
v_1 = (X_1^+)^{2j_1+1} v_0 \, , \quad v_3 = (X_3^+)^{2j_2+1} v_0 \, ,
\] (13)

and these are present for all representations we discuss. Next, it is clear that depending on the value of \(d\) there may be other singular vectors. Since we are interested in the positive-energy UIRs, we recall the list of these representations for \(su(2, 2)\) [1]:

1) \(d > j_1 + j_2 + 2\), \(j_1 j_2 \neq 0\)
2) \(d = j_1 + j_2 + 2\), \(j_1 j_2 \neq 0\)
3) \(d > j_1 + j_2 + 1\), \(j_1 j_2 = 0\)
4) \(d = j_1 + j_2 + 1\), \(j_1 j_2 = 0\)
5) \(d = j_1 = j_2 = 0\)
As we have already said, the same list is valid for \( \mathcal{U}_q(su(2, 2)) \). In case 1) there are no additional singular vectors. If \( d = j_1 + j_2 + 2 \), which is case 2) and is also possible in case 3), then \( m_{13} = 1 \) and there is an additional singular vector:

\[
v_{13}^{(1)} = \left( [2j_1][2j_2]X_1^+X_3^+X_2^+ - [2j_1][2j_2 + 1]X_1^+X_2^+X_3^+ - [2j_1 + 1][2j_2]X_3^+X_2^+X_1^+ + [2j_1 + 1][2j_2 + 1]X_2^+X_1^+X_3^+ \right) v_0 ,
\]

where \([x] = [x]_q = (q^{x/2} - q^{-x/2})/(q^{1/2} - q^{-1/2})\). In case 4) we have \( m_{13} = 2 \). Moreover, \( m_{23} = 1 \) if \( j_1 = 0 \) and \( m_{12} = 1 \) if \( j_2 = 0 \). Thus there are two singular vectors if \( j_1 + j_2 > 0 \), and three singular vectors if \( j_1 = j_2 = 0 \). These vectors are

\[
v_{13}^{(2)} = \left( [2j_1][2j_1 - 1][2j_2][2j_2 - 1](X_1^+)^2(X_3^+)^2(X_2^+)^2 - [2][2j_1][2j_1 - 1][2j_2 + 1][2j_2 - 1](X_1^+)^2X_3^+(X_2^+)^2X_3^+ + [2j_1][2j_1 - 1][2j_2 + 1][2j_2](X_1^+)^2(X_3^+)^2X_2^+ - [2][2j_1 + 1][2j_1 - 1][2j_2][2j_2 - 1]X_1^+(X_3^+)^2(X_2^+)^2X_1^+ + [2][2j_1 + 1][2j_1 - 1][2j_2 + 1][2j_2 - 1]X_1^+X_3^+(X_2^+)^2X_3^+X_1^+ - [2][2j_1 + 1][2j_1 - 1][2j_2 + 1][2j_2 - 1]X_1^+X_3^+(X_2^+)^2X_3^+X_1^+ - [2][2j_1 + 1][2j_1][2j_2 + 1][2j_2 - 1]X_1^+X_3^+(X_2^+)^2X_3^+X_1^+ + [2j_1 + 1][2j_1][2j_2 + 1][2j_2 - 1]X_1^+(X_3^+)^2(X_2^+)^2 - [2][2j_1 + 1][2j_1][2j_2][2j_2 - 1]X_1^+(X_3^+)^2(X_2^+)^2 + [2j_1 + 1][2j_1][2j_2 + 1][2j_2 - 1]X_1^+(X_3^+)^2X_3^+(X_1^+)^2 + [2][2j_1 + 1][2j_1][2j_2 + 1][2j_2 - 1]X_1^+(X_3^+)^2X_3^+(X_1^+)^2 \right) v_0 ,
\]

\[
d = j_1 + j_2 + 1 , \quad j_1j_2 = 0 , \quad m_{13} = 2 ,
\]

and all expressions in (16) are valid also when \( j_1 = j_2 = 0 \). Finally, in case 5), \( m_1 = m_2 = m_3 = 1 \), there are three singular vectors \( X_k^+v_0 , k = 1, 2, 3 \), and when factored out, the whole \( \mathcal{U}_q(G^+) \) gives zero contribution, yielding the one-dimensional representation. Factoring out all the singular vectors together with their descendents, one can now explicitly build the positive-energy representations.

4. In this section we consider the case where the deformation parameter is a root of unity, namely, \( q = e^{2\pi i/N} , N = 2, 3, \ldots \).

In this case all Verma modules \( V^x \) are reducible [6] and all irreducible representations are finite dimensional [8]. There are singular vectors for all positive roots \( \alpha \) [6]. Condition (11) also has more content now because if \( (\Lambda_\chi - \rho)(H_\alpha) = -m \in \mathbb{Z} \), then (11) will be fulfilled for all \( m + kN , k \in \mathbb{Z} \). In particular, there will be an infinite series of positive integers \( m \) such that (11) is true [6]. For identical reasons, there is an infinite number of lowest weights \( \Lambda_{\chi} \) such that (11) is satisfied for the same set of positive integers \( m = m_\alpha \).
Let us take a representation such that for each simple root $\alpha$ (11) is satisfied with a positive integer $m = m_{\alpha} \leq N$. Then the three numbers $m_1, m_2, m_3$ which characterize the representation $V^\chi$ will be

$$m_1 = \{2j_1 + 1\}_N$$

$$m_2 = \begin{cases} \{-d - j_1 - j_2 + 1\}_N, & \text{if } d \in \mathbb{Z}, \\ N, & \text{if } d \notin \mathbb{Z}, \end{cases}$$

$$m_3 = \{2j_2 + 1\}_N,$$

(17)

where $\{x\}_N$ is the smallest positive integer equal to $x \pmod{N}$; thus we have $m_k \in \mathbb{N}$ and $0 < m_k \leq N$, $k = 1, 2, 3$.

The weights $\chi$ such that (17) is satisfied are divided into six classes [9] depending on the values of $m_{12} = m_1 + m_2$, $m_{23} = m_2 + m_3$, and $m_{13} = m_1 + m_2 + m_3$:

a) $m_{jk} \leq N$ ,

b) $m_{12}, m_{23} \leq N$, $N < m_{13} \leq 2N$ ,

c) $m_{12} \leq N$, $N < m_{23}, m_{13} \leq 2N$ ,

d) $m_{23} \leq N$, $N < m_{12}, m_{13} \leq 2N$ ,

e) $N < m_{12}, m_{13}, m_{23} \leq 2N$ ,

f) $N < m_{12}, m_{23} \leq 2N$, $2N < m_{13} \leq 3N$.

(18)

These representations inherit all the structure from their $U_q(sl(4, \mathbb{C}))$ counterparts. Thus the classification of the unitarizable lowest weight representations of $U_q(su(2, 2))$ proceeds as follows.

Imposing the conditions of positive energy UIRs (14), we see that cases 1), 2), 3) are in one to one correspondence with the finite dimensional representations of $U_q(sl(4, \mathbb{C}))$ for $q^N = 1$ and all possibilities listed in (18) are admissible. The same list of representations is valid for $U_q(su(4))$.

The classification of the massless case 4) of (14) is more interesting since not all cases in (18) are admissible. Since $j_1 j_2 = 0$, let us choose for definiteness $j_2 = 0$. Then from (17) we have

$$m_1 = \{2j_1 + 1\}_N$$

$$m_2 = \{-2j_1\}_N$$

$$m_3 = 1$$

$$m_{12} = N + 1$$

$$m_{13} = N + 2$$

$$m_{23} = \begin{cases} \{1 - 2j_1\}_N \leq N, & \text{if } j_1 \neq 0, \\ N + 1, & \text{if } j_1 = 0. \end{cases}$$

(19)

Therefore the admissible cases are (18d) when $j_1 \neq 0$, and (18e) when $j_1 = 0$. For the dimension $d(N, J_1)$ of these representations we have, adapting a result of Ref.[9],

$$d(N, J_1) = \frac{1}{3} \left[ 2N^3 - N(12J_1^2 - 1) + 3J_1(4J_1^2 - 1) \right],$$

(20)

where $J_1$ is such that $2J_1 + 1 = (2j_1 + 1) \pmod{2N}$ and $1 \leq 2J_1 + 1 \leq N$. We recall that in the classical case the massless unitary representations are infinite-dimensional.
However, we may compare our representations with the undeformed non–unitary finite–dimensional representations which have the same quantum numbers \((m_1, m_2, m_3) = (2J_1 + 1, N - 2J_1, 1)\). We note that the dimension of the former is generically smaller than the dimension of the latter, which is given by

\[
d_c(m_1, m_2, m_3) = \frac{1}{12} (2J_1 + 1)(N - 2J_1)(N + 1)(N + 1 - 2J_1)(N + 2) .
\]  

(21)

These two dimensions can coincide only in the exceptional cases \(N = 2, J_1 = 0, d(2, 0) = d_c = 6\) and \(N = 2J_1 + 1\) where we have

\[
d_0 \equiv d(2J_1 + 1, J_1) = d_c = \frac{1}{3} (J_1 + 1)(2J_1 + 1)(2J_1 + 3) = \frac{1}{6} N(N + 1)(N + 2) .
\]

(22)

Until now we have discussed the case \(j_2 = 0, j_1 \geq 0\). The other case, \(j_1 = 0, j_2 \geq 0\), is obtained trivially from this. In fact, if we introduce the helicity \(h = j_1 - j_2\), then all the formulae above may be written in terms of \(|h|\); in particular, for the exceptional case \(N = 2|h| + 1\) we have (cf. (22)):

\[
d_0 = \frac{1}{3} (|h| + 1)(2|h| + 1)(2|h| + 3) = \frac{1}{6} N(N + 1)(N + 2) .
\]

(23)

We give now some explicit examples to illustrate the above classification. Take the massless case \(d = j_1 + 1, j_2 = 0\), and consider the three examples \(j_1 = 1/2\) and \(j_1 = 1\) for \(N = 3\), and \(j_1 = 1/2\) for \(N = 2\).

- \(N = 3, j_1 = 1/2, q = e^{2\pi i/3}\)

According to (19) and (20) we have: \(m_1 = 2, m_2 = 2, m_3 = 1,\) and \(d(3, 1/2) = 16\). Note that the classical dimension for this case is \(d_c = 20\). The singular vectors corresponding to the simple roots are: \((X^+_1)^2 v_0\), \((X^+_2)^2 v_0\) and \(X^+_3 v_0\); thus, in the irreducible representation with vacuum state \(|\rangle\), we have

\[
(X^+_1)^2 |\rangle = 0, \quad (X^+_2)^2 |\rangle = 0, \quad X^+_3 |\rangle = 0 .
\]

(24)

The sixteen basis states are the vacuum \(|\rangle\) and

\[
\begin{align*}
&X^+_1 |\rangle & &X^+_2 |\rangle & &X^+_1 X^+_2 |\rangle \\
&X^+_3 X^+_2 |\rangle & &X^+_1 X^+_2 X^+_1 |\rangle & &X^+_2 X^+_1 X^+_2 |\rangle \\
&X^+_3 X^+_2 X^+_1 |\rangle & &X^+_2 X^+_1 X^+_2 X^+_1 |\rangle & &X^+_3 X^+_1 X^+_2 X^+_1 |\rangle \\
&X^+_3 X^+_2 X^+_1 X^+_2 |\rangle & &X^+_2 X^+_1 X^+_2 X^+_1 X^+_1 |\rangle & &X^+_3 X^+_2 X^+_1 X^+_2 X^+_1 |\rangle & &X^+_2 (X^+_3)^2 X^+_2 X^+_1 X^+_2 X^+_1 |\rangle \\
&X^+_2 X^+_3 X^+_2 X^+_1 X^+_2 X^+_1 |\rangle & &X^+_2 (X^+_3)^2 X^+_2 X^+_1 X^+_2 X^+_1 |\rangle & &X^+_2 X^+_3 X^+_2 X^+_1 X^+_2 X^+_1 X^+_1 |\rangle .
\end{align*}
\]

(25)

One can explicitly check that the norms with respect to the \(U_q(su(2, 2))\)-invariant scalar product are positive (in fact, are all equal to 1). All other vectors have zero-norm and are decoupled from the representation. The same happens in the other two examples below.
• $N = 3$, $j_1 = 1$, $q = e^{2\pi i/3}$

In this case one has: $m_1 = 3$, $m_2 = 1$, $m_3 = 1$, and $d_0 = d_c = 10$. Note that now the dimension of the representation coincides with the classical one. The singular vectors corresponding to the simple roots are easily obtained: $(X_1^+)^3 v_0$, $X_2^+ v_0$ and $X_3^+ v_0$. In the irreducible representation one thus has

$$(X_1^+)^3 | \rangle = 0, \quad X_2^+ | \rangle = 0, \quad X_3^+ | \rangle = 0. \quad (26)$$

The ten states spanning this representation are given by the vacuum $| \rangle$ and

$$
\begin{align*}
X_1^+ | \rangle & \quad (X_1^+)^2 | \rangle & \quad X_2^+ X_1^+ | \rangle \\
X_1^+ X_2^+ X_1^+ | \rangle & \quad X_3^+ X_2^+ X_1^+ | \rangle & \quad X_2^+ X_1^+ X_2^+ X_1^+ | \rangle \\
X_3^+ X_1^+ X_2^+ X_1^+ | \rangle & \quad X_3^+ (X_1^+)^2 (X_1^+)^2 | \rangle & \quad (X_3^+)^2 (X_2^+)^2 (X_1^+)^2 | \rangle.
\end{align*}
$$

Also in this case one can explicitly check that the norms with respect to the $\mathcal{U}_q(sl(4, \mathbb{C}))$ invariant scalar product are all equal to unity.

• $N = 2$, $j_1 = 1/2$, $q = e^{i\pi} = -1$

This is a $q$-deformation of the fundamental representation. According to (19) and (22) we have now: $m_1 = 2$, $m_2 = 1$, $m_3 = 1$, and $d_0 = d_c = 4$, so that also in this case the dimension of the representation coincides with the classical one. The singular vectors corresponding to the simple roots are $(X_1^+)^2 v_0$, $X_2^+ v_0$ and $X_3^+ v_0$, and in the irreducible representation one has

$$(X_1^+)^2 | \rangle = 0, \quad X_2^+ | \rangle = 0, \quad X_3^+ | \rangle = 0. \quad (28)$$

The remaining four basis vectors are given by:

$$
| \rangle , \quad X_1^+ | \rangle , \quad X_2^+ X_1^+ | \rangle , \quad X_3^+ X_2^+ X_1^+ | \rangle , \quad (29)
$$

and all have unit norm. In this case one can easily work out the $4 \times 4$ matrices representing the generators of $\mathcal{U}_q(sl(4, \mathbb{C}))$. With

$$
\begin{align*}
\sigma_+ & = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, & \sigma_- & = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, & \sigma_3 & = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & e_1 & = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & e_2 & = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\end{align*}
$$

one finds:

$$
\begin{align*}
H_1 & = -\begin{pmatrix} \sigma_3 & 0 \\ 0 & 0 \end{pmatrix}, & H_2 & = \begin{pmatrix} e_2 & 0 \\ 0 & -e_1 \end{pmatrix}, & H_3 & = -\begin{pmatrix} 0 & 0 \\ 0 & \sigma_3 \end{pmatrix} \\
X_1^+ & = \begin{pmatrix} \sigma_- & 0 \\ 0 & 0 \end{pmatrix}, & X_2^+ & = \begin{pmatrix} 0 & 0 \\ \sigma_+ & 0 \end{pmatrix}, & X_3^+ & = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_- \end{pmatrix} \\
X_1^- & = \begin{pmatrix} \sigma_+ & 0 \\ 0 & 0 \end{pmatrix}, & X_2^- & = -\begin{pmatrix} 0 & \sigma_- \\ 0 & 0 \end{pmatrix}, & X_3^- & = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_+ \end{pmatrix} \\
X_{12}^+ & = -i \begin{pmatrix} 0 & 0 \\ e_1 & 0 \end{pmatrix}, & X_{23}^+ & = i \begin{pmatrix} 0 & 0 \\ e_2 & 0 \end{pmatrix}, & X_{13}^+ & = -\begin{pmatrix} 0 & 0 \\ \sigma_- & 0 \end{pmatrix} \\
X_{12}^- & = -i \begin{pmatrix} 0 & e_1 \\ 0 & 0 \end{pmatrix}, & X_{23}^- & = i \begin{pmatrix} 0 & e_2 \\ 0 & 0 \end{pmatrix}, & X_{13}^- & = \begin{pmatrix} 0 & \sigma_+ \\ 0 & 0 \end{pmatrix}
\end{align*}
$$

(31)
Note that in this basis the anti-linear anti-involution \( \omega \) (cf. (6)) is represented by matrix hermitean conjugation. The matrix form of the skew-hermitean generators of \( \mathcal{U}_q(su(2, 2)) \) can be easily obtained from the above expressions with the help of Eq. (7). One can now explicitly check that this representation of \( \mathcal{U}_q(su(2, 2)) \) is unitary.

As a final remark, we note that it may be possible to apply the finite dimensional UIRs given above to the theory and classification of elementary particles. This is suggestive because there is a simple relation between the four \( \times \) four representation given above and the fundamental representation matrices of \( su(4) \) (e.g. [10]).

Acknowledgments

V.K.D. and V.H. would like to thank Professor Abdus Salam for hospitality and financial support at the ICTP. V.K.D. was also partially supported by the Bulgarian National Foundation for Science, Grant \( \Phi - 11 \).

References

1. G. Mack, Comm. Math. Phys. 55 (1977) 1.
2. V.K. Dobrev, Canonical \( q \)-deformations of noncompact Lie (super-) algebras, Göttingen University preprint, (July 1991), to appear in J. Phys. A; cf. also: \( q \)-Deformations of noncompact Lie (super-) algebras: the examples of \( q \)-deformed Lorentz, Weyl, Poincaré and (super-) conformal algebras, ICTP preprint IC/92/13 (1992), to appear in the Proceedings of the Quantum Groups Workshop of the Wigner Symposium (Goslar, July 1991).
3. V.G. Drinfeld, Soviet. Math. Dokl. 32 (1985) 2548; in Proceedings of the International Congress of Mathematicians, Berkeley (1986), Vol. 1 (The American Mathematical Society, Providence, 1987) p.798.
4. M. Jimbo, Lett. Math. Phys. 10 (1985) 63; Lett. Math. Phys. 11 (1986) 247.
5. E. Abe, Hopf Algebras, Cambridge Tracts in Math., N 74, (Cambridge Univ. Press, 1980).
6. V.K. Dobrev, ICTP Trieste internal report IC/89/142 (June 1989) and in Proceedings of the International Group Theory Conference (St. Andrews, 1989), Eds. C.M. Campbell and E.F. Robertson, Vol. 1, London Math. Soc. Lect. Note Ser. 159 (1991) p. 87.
7. V.K. Dobrev, Rep. Math. Phys. 25 (1988) 159.
8. C. De Concini and V.G. Kac, Progress in Math. 92 (Birkhäuser, Boston, 1990) p. 471.
9. H.H. Andersen, P. Polo and K. Wen, Representations of quantum algebras, Aarhus University, Math. Inst. preprint series 1989/1990 No. 24 (1990).
10. D.B. Lichtenberg, Unitary Symmetry and Elementary Particles (2nd ed.), (Academic Press, NY, 1978).