Iterative Algorithms for Assessing Network Resilience Against Structured Perturbations

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Abstract—This paper studies network resilience against structured additive perturbations to its topology. We consider dynamic networks modeled as linear time-invariant systems subject to perturbations of bounded energy satisfying specific sparsity and entry-wise constraints. Given an energy level, the structured pseudospectral abscissa captures the worst-possible perturbation adversary could employ to destabilize the network, and the structured stability radius is the maximum energy in the structured perturbation that the network can withstand without becoming unstable. Building on a novel characterization of the worst-case structured perturbation, we propose iterative algorithms that efficiently compute the structured pseudospectral abscissa and structured stability radius. We provide theoretical guarantees of the local convergence of the algorithms and illustrate their efficacy and accuracy on several network examples.

I. INTRODUCTION

The resilience of dynamic networks against perturbations and attacks is key across engineering, scientific, and military domains, including the operation of cyberphysical infrastructure, the distributed control of autonomous robots, and time-critical missions. Despite important advances in designing distributed coordination, cooperation, and decision-making algorithms, dynamic networks have proven fragile to targeted attacks, as local and well-orchestrated actions have rapidly cascaded into network-wide destructive perturbations. Because of this, it is critical to develop techniques and notions that characterize network resilience and allow us to understand strengths and vulnerabilities against adversaries and unforeseen failures. However, obtaining such characterizations is difficult because resilience is a complex function of the operator’s and adversary’s capabilities, knowledge, and resources, the topology of the network, and the physical limitations on remedial and adversarial actions. Motivated by these observations, this paper studies the relationship between network resilience and structured topological perturbations, with the ultimate goal of enabling the deployment of dynamic networks with quantifiable resilience guarantees.

Literature review: Network resilience, understood as the ability of the system to carry out its goals under unexpected failures or malfunctions in its components, is a rich research area. Multiple layers of network activity are involved in ensuring resilience (e.g., detection of failures or attacks, secure communications, injection of false data or actuation signals), which naturally gets reflected in the variety of disciplines employed in its study, e.g., computer security [1], [2].

In practice, perturbations to the system matrix might be constrained by physical modeling, specific cyber vulnerabilities, or sparsity patterns. The study of the pseudospectra in linear algebra [14] and, in particular, the sign of the pseudospectral abscissa (the real part of the rightmost eigenvalue in the pseudospectrum). The work [15] shows that the pseudospectral abscissa is associated with a low-rank perturbation to the matrix. The works [16] and [17] propose criss-cross algorithms to numerically compute the value of the complex and real, respectively, pseudospectral abscissa. Both algorithms rely on the method in [18] to compute the distance to instability, which is impractical for large-scale systems. Instead, iterative algorithms [19], [20] quickly approximate complex or real pseudospectral abscissa of large matrices with sparse structures. However, due to their gradient-based nature, these algorithms possess local convergence properties and are only guaranteed to yield good approximations of the pseudospectral abscissa when the magnitude of the system perturbation is sufficiently small. Another closely-related concept is that of stability radius, which is the critical value of the magnitude of the perturbation that makes the pseudospectral abscissa become 0. While the works [19], [20], [21] propose efficient iterative algorithms for approximating stability radii of sparse matrices, the work [22] provides a formula for directly computing the stability radius when the perturbation is an arbitrary complex matrix and its magnitude is measured by its induced 2-norm. A similar stability radius formula is given in [23] when the perturbation is a real matrix.

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LTI system so that it loses controllability, observability, or stability. The work proposes an algorithm to obtain a locally optimal perturbation based on the identification of necessary conditions which, given the problem generality, are expressed implicitly in terms of abstract linear maps. Closely related to our work, [40] considers bounded-energy perturbations with sparse structure and studies the structured stability radius. The treatment relaxes the sparsity constraints by incorporating them into a cost function and relies on increasingly large penalties to satisfy them with increasing accuracy. This increasing accuracy comes at the expense of reducing the convergence speed of the proposed gradient-based algorithm, which makes it not well suited for large-scale networks. In contrast, our approach here presents a general characterization of the worst-case structured perturbation that includes the possibility of element-wise constraints. This serves as the basis for our design of efficient iterative algorithms to compute both the structured pseudospectral abscissa and stability radius that are able to deal with large-scale systems.

Statement of contributions: We model the network as a linear-time invariant system and study the effect on stability of additive perturbations to the system matrix of bounded energy and subject to sparsity and element-wise saturation constraints. Our contributions address the questions of whether an adversary can destabilize the network by employing such perturbations and characterizing the maximum amount of energy in the perturbation that the network can withstand without becoming unstable. Our first contribution is a novel necessary condition prescribing that the worst-case structured perturbation to the network must solve an implicit optimization problem. The implicitness arises because of the dependence on the right and left eigenvectors associated to the structured pseudospectral absissa and, if they were known instead, the optimization would become explicit and convex. We provide a complete description of the solution to the explicit optimization, show it is Lipschitz with respect to the problem parameters, and provide an incremental method to compute it. The observation about the implicit character of the optimization problem is the basis for our second contribution, which is an iterative algorithm alternating between finding the right and left eigenvectors given an estimate of the worst-case structured perturbation and solving the corresponding explicit optimization problem to refine said estimate. We show that the proposed algorithm is guaranteed to converge to the structured pseudospectral absissa at a linear rate for sufficiently close initial conditions. Our final contribution concerns the structured stability radius, and builds on the fact that this radius corresponds to the zero-crossing of the structured pseudospectral absissa when viewed as a function of the perturbation energy. We establish the locally Lipschitzness of this function, explicitly compute its gradient at the points of differentiability, and employ Newton’s method to design an iterative algorithm that provably approximates the structured stability radius. We illustrate in simulation the efficiency and accuracy of the proposed algorithms on several network examples, including a class of stable large-scale systems.

II. Preliminaries

Here, we introduce the notation and basic notions from linear algebra used in the paper.

Notation: For any vector \( x \in \mathbb{C}^n \) or matrix \( A \in \mathbb{C}^{n \times m} \), let \( x^*, A^* \) be their conjugate transpose. Let \( \| \cdot \| \) be the 2-norm of vectors in \( \mathbb{C}^n \), that is, \( |x| := \sqrt{x^*x} \). In addition, let \( \| \cdot \|_2, \| \cdot \|_F \) be the induced 2-norm and Frobenius norm, respectively, on \( \mathbb{C}^{n \times n} \). We let \( \|A\|_2 \) denote the closed ball of radius \( r \) in \( \mathbb{C}^{n \times n} \). We say vectors \( x, y \in \mathbb{C}^n \) are \( RP \)-compatible, cf. [19], if \( |x| = |y| = 1 \) and \( x^*y \) is real and positive. Given any \( x, y \) with \( x^*y \neq 0 \), one can obtain a pair of \( RP \)-compatible vectors \( \hat{x}, \hat{y} \) by scaling \( x \) and \( y \) as follows:

\[
\hat{x} = \frac{x}{|x|}, \quad \hat{y} = \frac{y^*}{|y^*|} y.
\]

The inner product \( \langle \cdot, \cdot \rangle : \mathbb{C}^{n \times n} \times \mathbb{C}^{n \times n} \to \mathbb{C} \) of matrices \( A = [a_{ij}] \) and \( B = [b_{ij}] \) is \( \langle A, B \rangle := \text{Tr}(A^*B) = \sum_{i,j=1}^n a_{ij}^* b_{ij} \). Note that \( \langle A, A \rangle = \|A\|_F^2 \). In addition, for any \( x, y \in \mathbb{C}^n, M \in \mathbb{R}^{n \times n} \),

\[
\text{Re}(x^*My) = \langle M, \text{Re}(xy^*) \rangle.
\]

The group inverse of \( A \), denoted \( A^\# \), is the unique matrix satisfying \( AA^\#A = A^\#A^\# = A^\# \), and \( AA^\# = A \), cf. [31]. The group inverse is different from the Moore–Penrose pseudoinverse. For a function \( f : \mathbb{R} \to \mathbb{R} \), define the right derivative of \( f \) at \( x \) to be

\[
\partial_+ f(x) := \lim_{\delta \to 0^+} \frac{f(x + \delta) - f(x)}{\delta}.
\]

For functions \( f, g : \mathbb{R}_{\geq 0} \to \mathbb{R} \), we denote \( f(t) = O(g(t)) \) if there exists \( k, \delta > 0 \) such that \( |f(t)| \leq kg(t) \) for \( t < \delta \).

Spectral absicssa and stability radius: We denote by \( \Lambda(A) \) the spectrum (i.e., set of eigenvalues) of a square matrix \( A \). The spectral absicssa of \( A \) is

\[
\alpha(A) := \max_{\lambda \in \Lambda(A)} \text{Re} \lambda.
\]

We refer to a maximizer \( \lambda_{\text{opt}} \) of this function as a rightmost eigenvalue of \( \Lambda(A) \). For \( \epsilon > 0 \) and a closed set \( \mathcal{H} \subseteq \mathbb{C}^{n \times n} \), the structured \( \epsilon \)-pseudospectrum of \( A \) (with respect to the Frobenius norm) is

\[
\Lambda_{\epsilon, \mathcal{H}}(A) := \{ \lambda \in \mathbb{C} : \lambda \in \Lambda(A + \Delta) \} \quad \text{for} \quad \Delta \in \mathcal{H} \cap \mathbb{B}_2.
\]

Note that when \( \mathcal{H} = \mathbb{C}^{n \times n} \) or \( \mathcal{H} = \mathbb{R}^{n \times n} \), \( \Lambda_{\epsilon, \mathcal{H}} \) reduces to the usual \( \epsilon \)-pseudospectrum \([19]\) or real \( \epsilon \)-pseudospectrum \([15]\), respectively. Similar to the spectral absicssa, we also define \( \alpha_{\epsilon, \mathcal{H}}(A) \) as the structured \( \epsilon \)-pseudospectral absicssa of \( A \),

\[
\alpha_{\epsilon, \mathcal{H}}(A) := \max_{\lambda \in \Lambda_{\epsilon, \mathcal{H}}(A)} \text{Re} \lambda.
\]

We refer to a maximizer \( \lambda_{\text{opt}} \) of this function as a rightmost eigenvalue of \( \Lambda_{\epsilon, \mathcal{H}}(A) \). Using (2), one can equivalently express the structured \( \epsilon \)-pseudospectral absicssa of \( A \) as

\[
\alpha_{\epsilon, \mathcal{H}}(A) = \max_{\Delta \in \mathcal{H} \cap \mathbb{B}_2} \alpha(A + \Delta).
\]

We refer to a maximizer \( \Delta_{\text{opt}} \) of this function as a worst-case structured perturbation of energy \( \epsilon \). The structured stability radius of \( A \) is

\[
r_H(\epsilon) := \min_{\epsilon \geq 0} \{ \epsilon : \alpha_{\epsilon, \mathcal{H}}(A) \geq 0 \}.
\]
Clearly, if $A$ is not Hurwitz, $r_{\mathcal{H}}(A) = 0$. When $\mathcal{H} = \mathbb{C}^{n \times n}$ or $\mathcal{H} = \mathbb{R}^{n \times n}$, $r_{\mathcal{H}}(A)$ coincides with the definition of stability radius [22] or real stability radius [23], respectively. Note also that if $\mathcal{H}_1 \subseteq \mathcal{H}_2$, then $r_{\mathcal{H}_1}(A) \ge r_{\mathcal{H}_2}(A)$.

Matrix perturbation theory: Here we gather two useful results on matrix perturbations. The first describes the derivative of a simple eigenvalue of a matrix that depends linearly on time.

**Lemma II.1** ([22] Lemma 6.3.10 and Theorem 6.3.12]). Consider a $n \times n$ matrix trajectory $\mathbb{R} \ni t \to C(t) = C_0 + tC_1$. Let $\lambda(t)$ be an eigenvalue of $C(t)$ converging to a simple eigenvalue $\lambda_0$ of $C_0$ as $t \to 0$. Let $x_0$ and $y_0$ be, respectively, right and left eigenvectors of $C_0$ corresponding to $\lambda_0$, that is, $(C_0 - \lambda_0 I)x_0 = 0$ and $y_0^*(C_0 - \lambda_0 I) = 0$. Then $y_0^*x_0 \ne 0$ and $\lambda(t)$ is analytic near $t = 0$ with

$$
\frac{d\lambda(t)}{dt} \bigg|_{t=0} = \frac{y_0^*C_1x_0}{y_0^*x_0}.
$$

The following result describes the time derivative of the product of the right and left eigenvectors corresponding to a simple eigenvalue of a complex-valued time-dependent matrix.

**Theorem II.2** ([19] Theorem 5.2]). Consider a $n \times n$ complex-analytic matrix trajectory $\mathbb{C} \ni t \to C(t) = C_0 + tC_1 + O(t^2)$. Let $\lambda(t)$ be a simple eigenvalue of $C(t)$ in a neighborhood $\mathcal{N}$ of $t = 0$, with corresponding RP-compatible right and left eigenvectors $x(t)$ and $y(t)$. Then $Q(t) = x(t)y(t)^*$ is $C^\infty$ on $\mathcal{N}$ and its derivative at $t = 0$ is

$$
\frac{dQ(t)}{dt} \bigg|_{t=0} = \text{Re}(\beta + \gamma)Q(0) - GC_1Q(0) - Q(0)C_1G,
$$

where $G = (C_0 - \lambda I)^\#$, $\beta = x^*GC_1x$, $\gamma = y^*C_1Gy$, $\lambda = \lambda(0)$, $x = x(0)$, and $y = y(0)$.

### III. Problem Statement

We provide here a formal mathematical description of the problem of interest. Let $\mathcal{G} := (\mathcal{V}, \mathcal{E})$ denote a network graph, where $\mathcal{V} = \{1, \ldots, n\}$ is the set of nodes and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges. The network dynamics is described by a linear differential equation

$$
\dot{x} = Ax,
$$

where the components of $x \in \mathbb{R}^n$ correspond to the states of the nodes, and $A = [a_{ij}] \in \mathbb{R}^{n \times n}$, with $a_{ij} = 0$ for all $(i, j) \notin \mathcal{E}$, is the weighted adjacency matrix. We assume the matrix $A$ is Hurwitz. An adversary seeks to destabilize the dynamics by attacking the network interconnections. Such attacks are structured, in the sense that the adversary is limited to perturbing only certain edges and within some budget. Formally, let $\mathcal{E}^p \subseteq \mathcal{E}$, denote the perturbation edge set and for all $(i, j) \in \mathcal{E}^p$, let $\Delta_{ij} \in \mathbb{R}_{\ge 0} \cup \{-\infty\}$, $\overline{\Delta}_{ij} \in \mathbb{R}_{\ge 0} \cup \{+\infty\}$ be parameters specifying saturation constraints. We denote the set of allowed perturbations whose sparsity pattern is compatible with $\mathcal{E}^p$ by

$$
\mathcal{H} := \{\Delta = [\Delta_{ij}] \in \mathbb{R}^{n \times n} : \Delta_{ij} = 0 \text{ if } (i, j) \notin \mathcal{E}^p, \Delta_{ij} \in [\Delta_{ij}, \overline{\Delta}_{ij}] \text{ if } (i, j) \in \mathcal{E}^p\}.
$$

Note that we always have $0 \in \mathcal{H}$. When $\Delta_{ij} = -\infty$ or $\overline{\Delta}_{ij} = +\infty$, then there is no lower or upper bound constraint on the perturbation size of the corresponding edge. We define the energy of an attack $\Delta$ to be its Frobenius norm.

After an attack $\Delta \in \mathcal{H}$, the network dynamics changes to

$$
\dot{x} = (A + \Delta)x.
$$

We are interested in answering the following questions:

1. Given the network dynamics (7), can an adversary destabilize it by employing perturbations of bounded energy $\epsilon > 0$ in $\mathcal{H}$?
2. What edges are most important to protect against perturbations of bounded energy $\epsilon > 0$ in order to preserve stability?
3. If by resilience of the network dynamics we understand the maximum amount of energy in the perturbation that it can withstand without becoming unstable, what is the network resilience against the adversary?

Each of these questions can be transcribed into a mathematically precise statement. In fact, keeping in mind that GAS if $A + \Delta$ is Hurwitz, we can equivalently say that

- question 1) refers to determining whether the structured $\varepsilon$-pseudoabsissa $\alpha_{\varepsilon, \mathcal{H}}(A)$ of $A$ is positive;
- with regards to question 2), assume the worst-case structured perturbation $\Delta_{\text{opt}}(\varepsilon)$ of energy $\varepsilon$ is unique. Notice that the larger $|\Delta_{\text{opt}}(\varepsilon)|_{ij}$, the larger proportion of weight modification is done on the edge $(i, j)$ to destabilize the system. Hence, the magnitude of the elements in the perturbation matrix provides an ordering of the relative importance of edges against the attack; and
- question 3) refers to determining the value of the structured stability radius $r_{\mathcal{H}}(A)$.

In our ensuing discussion, we address questions 1) and 2) concurrently by introducing an iterative algorithm that finds both the value of $\alpha_{\varepsilon, \mathcal{H}}(A)$ and a maximizer $\Delta_{\text{opt}}$ of (4).

### IV. Network Stability Against Perturbations: Structured Pseudospectral Abscissa

In this section, we study the answers to questions 1) and 2) of our problem statement. We begin by characterizing the worst-case structured perturbation of a given energy. We build on this characterization later to propose an algorithm that computes iteratively the structured pseudospectral abscissa.

#### A. Characterization of the worst-case structured perturbation

Our first result states that any worst-case structured perturbation which gives rise to a simple rightmost eigenvalue is a solution to an implicit optimization problem.

**Theorem IV.1** (First-order necessary condition for optimality). Let $\mathcal{H}' \subseteq \mathbb{C}^{n \times n}$ be compact and convex, and let

$$
\Delta_{\text{opt}} \in \arg\max_{\Delta \in \mathcal{H}'} \alpha(A + \Delta).
$$

Let $\lambda_{\text{opt}}(A + \Delta_{\text{opt}})$ be a rightmost eigenvalue of $A + \Delta_{\text{opt}}$ and suppose it is simple. Then $\Delta_{\text{opt}}$ must satisfy

$$
\Delta_{\text{opt}} \in \arg\max_{\Delta \in \mathcal{H}'} (\Delta, \text{Re}(yx^*))
$$

where $x, y \in \mathbb{C}^n$ are right and left eigenvectors, respectively, of $A + \Delta_{\text{opt}}$.
where \( x, y \in \mathbb{C}^n \) are the RP-compatible right and left eigenvectors associated with \( \lambda_{opt}(A + \Delta_{opt}) \).

**Proof.** To study the first-order necessary condition of optimality, we set the feasible directions at \( \Delta \in \mathcal{H}' \),

\[
\mathcal{F} := \{ C \in \mathbb{C}^{n \times n} : \exists \tau > 0 \text{ s.t. } \Delta + tC \in \mathcal{H}', \forall t \in [0, \tau] \}.
\]

The condition for optimality states that, if \( \Delta_{opt} \) is the optimizer, then \( \frac{d}{dt}(x'Ax + x'y'C \Delta + \Delta C'x) \bigg|_{t=0} \leq 0 \), for all \( C \in \mathcal{F} \). Applying Lemma [13], we deduce

\[
0 \geq \Re\left( \frac{d\lambda_{opt}(A + \Delta_{opt} + tC)}{dt} \bigg|_{t=0} \right) = \Re\left( \frac{y'Cx}{y'x} \right) = \frac{\Re(y'Cx)}{\Re(y^*x)} = \frac{(C, \Re(y^*x))}{\Re(y^*x)},
\]

where we have used the RP-compatibility for the second last equality and the identity (11) for the last equality. Now, one can see that \( \langle C, \Re(y^*x) \rangle \leq 0 \) for all \( C \in \mathcal{F} \) corresponds indeed to the first-order necessary condition for optimality of the maximization problem (11). Since this problem is convex, the condition is also sufficient to characterize an optimizer. Thus, satisfying the condition of (11) is a necessary condition for being a maximizer of (10).

It is worth pointing out that, since (11) is a necessary condition, it needs to be satisfied by any worst-case perturbation. We make use of this fact to design an algorithm based on Theorem [14] to find a worst-case perturbation and compute the structured pseudospectral abscissa.

Comparing with [4], we observe that \( [10] \) with \( \mathcal{H}' = \mathcal{H} \cap \mathbb{B}_\epsilon \) corresponds exactly to a worst-case structured perturbation of energy \( \epsilon \) (in fact, \( \mathcal{H} \) as defined in [8] is closed and convex, and hence \( \mathcal{H}' \) is compact and convex). The optimization problem in (10) is nonconvex and hence difficult to solve in general. Instead, Theorem [14] facilitates finding the structured pseudospectral abscissa by providing a characterization (11) of the worst-case structured perturbations.

For known \( x, y \in \mathbb{C}^n \), the optimization in (11) is a convex problem of the form

\[
\text{maximize } \langle \Delta, M \rangle \quad \text{subject to } \Delta \in \mathcal{H} \cap \mathbb{B}_\epsilon,
\]

when we set \( M := [m_{ij}] := \Re(yx^*) \), and can therefore be solved efficiently. However, we should note that the vectors \( x \) and \( y \) in (11) are unknown, since they are the eigenvectors of \( A + \Delta_{opt} \), making equation (11) implicit in \( \Delta_{opt} \). Before we address this issue, we finish the exposition here describing the properties of the solution of (12) for a given known \( M \).

We make the next assumption regarding the worst-case perturbation.

**Assumption 1. (Non-saturation at optimizers).** No optimizer of (12) is fully saturated, i.e., if \( \Delta_{opt} \) is an optimizer of (12), then there exists \( (i, j) \in \mathcal{E}_p \) such that \( \langle \Delta_{opt}, i,j \rangle \). Furthermore, \( m_{ij} \neq 0 \).

Assumption [1] is reasonable: in case it does not hold, i.e., a worst-case perturbation is fully saturated, then it must be at a vertex of \( \mathcal{H} \) and since this constraint set has finitely many vertices, the worst-case perturbation can be found by exhaustion. Meanwhile, if \( m_{ij} = 0 \), then the value of \( \langle \Delta_{opt}, i,j \rangle \) does not affect the optimal value and, consequently, one can construct other optimizers that are saturated at edge \( (i, j) \). This is the reason why we explicitly require \( m_{ij} \neq 0 \) in Assumption [1]. Since (12) has a non-trivial linear objective function with a convex constraint set, the optimum is achieved on its boundary and hence, if the maximizer \( \Delta_{opt} \) is not fully saturated, it must verify \( \| \Delta_{opt} \| _F = \epsilon \). In addition, by employing the KKT conditions for optimality, we are able to characterize the solution of (12).

**Proposition IV.2. (Characterization of solution of (12)).** Let \( M \in \mathbb{C}^{n \times n} \) and \( \epsilon > 0 \). Under Assumption [7] the optimization (12) has a unique optimizer \( \Delta_{opt} \) given by

\[
\langle \Delta_{opt}, i,j \rangle = \begin{cases} 
  m_{ij} \theta_{opt}, & \text{if } (i, j) \in \mathcal{E}_p \setminus (\mathcal{S} \cup \mathcal{S}_1), \\
  \Delta_{ij}, & \text{if } (i, j) \in \mathcal{S}, \\
  \Delta_{ij}, & \text{if } (i, j) \in \mathcal{S}_1, \\
  0, & \text{if } (i, j) \notin \mathcal{E}_p
\end{cases}
\]

where \( \mathcal{S} := \mathcal{S}(\epsilon, M), \mathcal{S}_1 := \mathcal{S}(\epsilon, M) \) are the unique subsets of \( \mathcal{E}_p \) such that

\[
m_{ij} \theta_{opt} \in (\Delta_{ij}, \Delta_{ij}) \quad \forall (i, j) \in \mathcal{E}_p \setminus (\mathcal{S} \cup \mathcal{S}_1),
\]

\[
m_{ij} \theta_{opt} \geq \Delta_{ij} \quad \forall (i, j) \in \mathcal{S_1},
\]

\[
m_{ij} \theta_{opt} \leq \Delta_{ij} \quad \forall (i, j) \in \mathcal{S},
\]

and \( \theta_{opt} \) is shorthand notation for \( \theta_{opt}(\epsilon, M) := \theta(\epsilon, M, \mathcal{S}(\epsilon, M), \mathcal{S}(\epsilon, M)) \), where the function \( \theta \) is

\[
\theta(\epsilon, M, \mathcal{S}, \mathcal{S}_1) := \frac{\epsilon^2 - \sum_{(i,j) \in \mathcal{S}} \Delta_{ij}^2 - \sum_{(i,j) \in \mathcal{S}_1} \Delta_{ij}^2}{\sum_{(i,j) \in \mathcal{E}_p \setminus (\mathcal{S} \cup \mathcal{S}_1)} m_{ij}^2}.
\]

In addition, there is a neighborhood \( D \) around \( (\epsilon, M) \in \mathbb{R}_0 \times \mathbb{R}^{n \times n} \) such that \( \theta_{opt} \) is Lipschitz on \( D \).

The proof of Proposition IV.2 is in the Appendix. According to this result, the element of the optimizer \( \Delta_{opt} \) corresponding to \( (i, j) \in \mathcal{E}_p \) is either saturated or proportional to \( m_{ij} \), with ratio given by \( \theta_{opt} \). We refer to \( \mathcal{S}, \mathcal{S}_1 \) as the index sets of saturation as \( \langle \Delta_{opt}, i,j \rangle \) attains either of its boundary values for all \( (i, j) \in \mathcal{S} \cup \mathcal{S}_1 \). Note that, by Assumption [1], \( \mathcal{S}(\epsilon, M) \subseteq \mathcal{E}_p \) and \( \theta_{opt} \) given by (15) is well defined since there exists \( (i, j) \in \mathcal{E}_p \setminus (\mathcal{S} \cup \mathcal{S}_1) \) such that \( m_{ij} \neq 0 \).

**Remark IV.3. (Comparison with the literature).** Proposition IV.2 is a generalization of the results available in the literature [15, 30]. When there are neither sparsity constraints nor saturation constraints, \( \mathcal{H} = \mathbb{R}^{n \times n} \) and \( \theta_{opt} = \frac{\epsilon}{\| M \| _F} \), so

\[
\Delta_{opt} = \frac{\epsilon \Re(yx^*)}{\| \Re(yx^*) \| _F},
\]

as stated in [15] Theorem 2.2. On the other hand, when \( \mathcal{H} \) contains sparsity constraints but no saturation constraints, we deduce that there exists \( c \geq 0 \) such that

\[
\langle \Delta_{opt}, i,j \rangle = \begin{cases} 
  c \Re(yx^*), & \text{if } (i, j) \in \mathcal{E}_p, \\
  0, & \text{otherwise}
\end{cases}
\]

as stated in [30] Theorem 3.2.

We note that \( \theta_{opt} \) and \( \mathcal{S}, \mathcal{S}_1 \) are inter-dependent, which means that Proposition IV.2 does not provide an explicit
expression of $\Delta_{\text{opt}}$. However, the result provides the basis for a simple method, which we summarize in Algorithm 1 to find the solution of (12) by growing the index sets of saturation $\mathcal{S}$, $\mathcal{S}$ if they do not meet the conditions (14a)–(14c) for the corresponding value of $\theta_{\text{opt}}$ determined by (15). The next result shows that Algorithm 1 finds the solution of (12).

**Lemma IV.4.** (Algorithm 1 solves (12). Under Assumption 1, Algorithm 1 finds the solution of the optimization (12) in a finite number of steps.

The proof of Lemma IV.4 is provided in the Appendix. In contrast to generic convex optimization solvers, Algorithm 1 is tailored to problem (12) and takes advantage of the characterization (13) of its optimizer. We use later the ratio $\theta_{\text{opt}}$ in Algorithm 1 to compute the structured stability radius.

We conclude this section by presenting a result which shows that the optimizer of (12) is locally Lipschitz when viewed as a function of the matrix defining the objective function. To establish this, we use the Frobenius norm and show that the Lipschitz constant is proportional to the energy of matrix perturbations. The proof is given in the Appendix.

**Lemma IV.5.** (Sensitivity of the optimizer of (12) with respect to parameters). Let $\epsilon > 0$, $M_1, M_2 \in \mathbb{R}^{n \times n}$ and suppose $\Delta_k$ are the optimizers of (12) with parameters $(\epsilon, M_k)$, $k = 1, 2$. Also assume that Assumption 1 holds for $(\epsilon, M_1)$. Then, there exist $\delta = \delta(M_1) > 0$, $\ell = \ell(M_1) > 0$ such that

$$\|\Delta_1 - \Delta_2\|_F \leq \ell \epsilon \|M_1 - M_2\|_F$$

as long as $\|M_1 - M_2\|_F \leq \delta$.

**B. Iterative computation of structured pseudospectral abscissa**

What we have unveiled about the optimization problem (12) and the structure of its solution in Section IV-A is not directly applicable to the determination of the worst-case structured perturbation and the structured pseudospectral abscissa. This is because, as we mentioned earlier, the characterization (11) is implicit in $\Delta_{\text{opt}}$, i.e., the matrix $M = \text{Re}(yx^*)$ required to set up (12) is not a priori known, and in fact depends on the optimizer itself. To address this obstacle, we propose in Algorithm 2 an iterative strategy that proceeds by repeatedly solving instances of problem (12), in each case taking the right and left eigenvectors corresponding to the previous iterate.

**Algorithm 1** Incremental construction of index sets

**Input:** $\epsilon, M, \mathcal{H}$

**Output:** $\theta_{\text{opt}}, \Delta_{\text{opt}}$

1: $\mathcal{S} \leftarrow \emptyset$, $\mathcal{S} \leftarrow \emptyset$
2: NotDone $\leftarrow$ false
3: if $\mathcal{S} \cup \mathcal{S} = \mathcal{E}_p$ then break with error “the optimizer is fully saturated, Assumption 1 is violated”
4: Compute $\theta_{\text{opt}}$ as in (15)
5: for all $(i, j) \in \mathcal{E}_p \setminus (\mathcal{S} \cup \mathcal{S})$ do
6: if $m_{ij}\theta_{\text{opt}} \geq \Delta_{ij}$ then
7: $\mathcal{S} \leftarrow \mathcal{S} \cup \{(i, j)\}$
8: $(\Delta_{\text{opt}})_{ij} \leftarrow \Delta_{ij}$
9: NotDone $\leftarrow$ true
10: else if $m_{ij}\theta_{\text{opt}} \leq \Delta_{ij}$ then
11: $\mathcal{S} \leftarrow \mathcal{S} \cup \{(i, j)\}$
12: $(\Delta_{\text{opt}})_{ij} \leftarrow \Delta_{ij}$
13: NotDone $\leftarrow$ true
14: if NotDone $=$ true then go back to Step 2
15: else $(\Delta_{\text{opt}})_{ij} \leftarrow m_{ij}\theta_{\text{opt}}$ for all $(i, j) \in \mathcal{E}_p \setminus (\mathcal{S} \cup \mathcal{S})$

The logic of Algorithm 2 can be described as follows. At each step, we consider a candidate worst-case perturbation $\Delta_k$, followed by computing the RP-compatible right and left eigenvectors $x_k, y_k$ of a rightmost eigenvalue of $A + \Delta_k$. We then solve the optimization problem (12) using $M = \text{Re}(yx_k^*)$, and set the new optimizer to be $\Delta_{k+1}$. This process is repeated until the sequence of possible worst-case perturbations converges. From its design, it is clear that a fixed point of Algorithm 2 is a solution to the maximization problem (11) (and hence, by Theorem IV.1, satisfies the first-order necessary condition for being the maximizer of (10)).

The next result establishes the local asymptotic convergence of Algorithm 2 to the structured pseudospectral abscissa.

**Theorem IV.6** (Local convergence of Algorithm 2). Let $\Delta_{\text{opt}}$ be a worst-case structured perturbation (i.e., $\Delta_{\text{opt}}$ satisfies (10) with $\mathcal{H}' = \mathcal{H} \cap \mathbb{B}_x$) and assume the rightmost eigenvalue $\lambda$ of $A + \Delta_{\text{opt}}$ is simple, with RP-compatible right and left eigenvector pair $x, y$. Let Assumption 1 hold for $\Delta_{\text{opt}}$ and $M = \text{Re}(yx^*)$. Define

$$r = \frac{4\sqrt{\ell} \epsilon}{\sigma_{n-1}(A - \lambda I)(y^*x)^2}$$

where $\ell = \ell(\text{Re}(yx^*))$ is given in Lemma IV.5 and $\sigma_{n-1}$ denotes the second smallest singular value. Let $\Delta_k, \text{Re} \lambda_k$ be the sequences generated by Algorithm 2. If $r < 1$, $r \in (r, 1)$ is arbitrary and $\|\Delta_0 - \Delta_{\text{opt}}\|_2$ is sufficiently small, then

$$\|\Delta_k - \Delta_{\text{opt}}\|_2 \leq (r^1)^k \|\Delta_0 - \Delta_{\text{opt}}\|_2$$

for all $k = 0, 1, \ldots$, and $\text{Re} \lambda_k$ converges to the structured pseudospectral abscissa $\alpha_{e, \mathcal{H}}(A)$. In addition, the output of Algorithm 2 satisfies

$$|\alpha - \alpha_{e, \mathcal{H}}(A)| = \frac{\sqrt{r}r^1 \|\text{Re}(yx^*)\|_F}{(1 - r^1) y^*x} \text{tol}_\Delta + O(\text{tol}_\Delta^2).$$
Proof. Let \( L := yx^* \) and \( L_k := y_k x_k^* \) for \( k = 0, 1, \ldots \), where \( x_k, y_k \) come from Step 4 in the \( k \)-th iteration of Algorithm 2. In addition, define
\[
E_k := \Delta_k - \Delta_{opt}, \quad F_k := L_k - L.
\]
We first find a relation between \( \|F_k\|_2 \) and \( \|E_k\|_2 \). Consider the matrix trajectory
\[
C(t) = A + \Delta_{opt} + t \frac{E_k}{\|E_k\|_2}. \tag{19}
\]
Let \( x(t), y(t) \) be right and left eigenvectors of \( C(t) \) associated with its rightmost eigenvalue \( \lambda(t) \) such that \( \lambda(t) = 0, x(0) = x \) and \( y(0) = y \). Define \( Q(t) := y(t)x(t)^* \). Invoking Theorem 4.2 and taking the conjugate of \( 6 \),
\[
\frac{dQ(t)}{dt} \bigg|_{t=0} = \text{Re}(\beta + \gamma)Q(0) - Q(0)C_1^* G^* - G^* C_1^* Q(0),
\]
where \( G = (A + \Delta_{opt} - \lambda I)^\# \), \( \beta = x^* G C_1 x \) with \( C_1 = E_k/\|E_k\|_2 \) and \( \gamma = y^* C_1 G y \). Note that, since \( C(0) = A + \Delta_{opt} \) and \( C(\|E_k\|_2) = A + \Delta_k \), we have \( Q(0) = L \) and \( \dot{Q}(\|E_k\|_2) = L_k \). Therefore using Taylor’s expansion, we have
\[
F_k = Q(\|E_k\|_2) - Q(0) = \frac{dQ(t)}{dt} \bigg|_{t=0} E_k^2 + R(\|E_k\|_2)
= \text{Re}(x^* G E_k x + y^* E_k G y) - LE_k^2 G^* - G^* E_k^2 L
+ R(\|E_k\|_2), \tag{20}
\]
where \( R(\|E_k\|_2) \) is the Taylor remainder of order 1 such that \( R(t) = O(t^2) \), which implies the existence of a class \( K \) function \( \xi \) such that \( R(t) \leq st \) for any \( s > 0 \) and \( t \in [0, \xi(s)] \). Now notice that because \( x, y \) are RP-compatible and \( L = yx^* \), \( |x| = |y| = 1 \) and \( \|L\|_F \leq \|L\|_F = 1 \). In addition, from \([19] \) Theorem 5.5, \( \|G\|_2 \leq \frac{1}{\sigma_{n-1}(A-\lambda I)(y^*x)^2} \). Hence it follows from \( 20 \) that
\[
\|F_k\|_2 \leq 4\|G\|_2 \|E_k\|_2 \|L\|_2 + R(\|E_k\|_2)
\leq \frac{4\|E_k\|_2}{\sigma_{n-1}(A-\lambda I)(y^*x)^2} + R(\|E_k\|_2)
\leq \frac{r}{\sqrt{n} \epsilon} \|E_k\|_2 + R(\|E_k\|_2), \tag{21}
\]
where we have used the definition \( \|E_0\|_2 < \delta \) in the last equality.

We are ready to establish \( 17 \) by induction. Define \( r_1 := \frac{r}{\sqrt{n} \epsilon} \) and let \( \|E_0\|_2 < \min \left\{ \frac{r_1}{\sqrt{n} \epsilon}, \delta, \xi(1) \right\} \), \( \delta \) is given in Lemma \([17] \). The base case \( k = 0 \) for \( 17 \) is trivially true. Since \( r_1 < 1 \), the induction assumption for index \( k \) implies \( \|E_k\|_2 < \min \{\frac{r_1}{\sqrt{n} \epsilon}, \delta, \xi(1)\} \) as well. Hence, from \( 21 \),
\[
\|F_k\|_2 \leq \left( \frac{r}{\sqrt{n} \epsilon} + r_1 \right) \|E_k\|_2 = \frac{r}{\sqrt{n} \epsilon} \|E_k\|_2 < \delta. \tag{22}
\]
Using Lemma \([17] \), we deduce
\[
\|E_{k+1}\|_2 \leq \|E_k\|_F + \|\Delta_{k+1} - \Delta_{opt}\|_F
\leq \ell \epsilon \|\text{Re}(yx_k^*) - \text{Re}(yx^*)\|_F + \ell \epsilon \|F_k\|_F \leq \sqrt{n} \epsilon \|F_k\|_2,
\]
where we have used \( \|M\|_2 \leq \|M\|_F \leq \sqrt{n} \|M\|_2 \). Combining this bound with the first inequality in \( 22 \), and using the induction hypothesis,
\[
\|E_{k+1}\|_2 \leq r_1 \|E_k\|_2 \leq (r_1)^{k+1} ||E_0||_2, \tag{23}
\]
establishing \( 17 \). It follows that \( \alpha(A + \Delta_k) = \text{Re} \lambda_k \) converges to \( \alpha(A + \Delta_{opt}) = \alpha_{\epsilon, \mathcal{H}}(A) \). To prove \([18] \), we apply Lemma \([11] \) to the matrix trajectory \( 19 \) and conclude
\[
\frac{d}{dt} \alpha(C(t)) \bigg|_{t=0} = \frac{\langle E_k, \text{Re}(yx^*) \rangle}{\|E_k\|_2 y^* x}.
\]
Again because \( C(0) = A + \Delta_{opt} \) and \( C(\|E_k\|_2) = A + \Delta_k \), we conclude from the Taylor expansion that
\[
|\alpha(A + \Delta_k) - \alpha_{\epsilon, \mathcal{H}}(A)| = \frac{\langle E_k, \text{Re}(yx^*) \rangle}{y^* x} + O(\|E_k\|_2^2)
\leq \frac{\sqrt{n} \|E_k\|_2 \|E_k\|_2}{y^* x} + O(\|E_k\|_2^2).
\]
On the other hand, when the loop in Algorithm 2 terminates at the \( k \)-th iteration,
\[
(1 - r^1)(\|E_{k+1}\|_2 \leq r^1(\|E_k\|_2 - \|E_{k+1}\|_2)\leq r^1(\|E_{k+1}\|_2 - \|E_{k+1}\|_2) \leq r^1 \text{tol}_\Delta,
\]
where we use the first inequality in \( 23 \), the triangle inequality and the stopping criteria of Algorithm 2. Hence \( \|E_{k+1}\|_2 \leq \frac{r^1}{1-r} \text{tol}_\Delta \), completing the proof.

According to Theorem \([16] \), the algorithm’s convergence requires \( r \) to be smaller than 1: the smaller \( r \) is, the smaller \( r^1 \) can be, leading to faster convergence of Algorithm 2 and a smaller approximation error \( |\alpha - \alpha_{\epsilon, \mathcal{H}}(A)| \). The value of \( r \) depends on various system parameters: it is proportional to the square root of the dimension of \( A \), inversely proportional to the second smallest singular value of \( A - \lambda I \), and increases as the left and right eigenvectors of \( A + \Delta_{opt} \) associated with \( \lambda \) get closer to being orthogonal. Using arguments similar to those in the proof of \([19] \) Theorem 5.7, we can conclude that \( r = O(\epsilon) \) and hence, as long as the energy of the structured perturbation is small enough, \( r < 1 \) is ensured. A smaller value of \( \text{tol}_\Delta \) results in a more accurate approximation of the structured pseudospectral absicssa, cf. \([18] \), at the cost of more iterations in Algorithm 2.

V. MEASURING NETWORK RESILIENCE: STRUCTURED STABILITY RADIUS

Here we introduce an iterative algorithm to find the structured stability radius \( r_{\mathcal{H}}(A) \), providing a metric of network resilience, corresponding to question 3 of our problem statement. Our strategy makes repeated use of Algorithm 2 to find the structured pseudospectral absicssa \( \alpha_{\epsilon, \mathcal{H}}(A) \) for a given energy \( \epsilon \), since the zero-crossing of this function corresponds to the structured stability radius.

A. Structured pseudospectral absicssa as a function of perturbation energy

From the definition \([3] \), we observe that \( \epsilon \mapsto \alpha_{\epsilon, \mathcal{H}}(A) \) is an increasing function and \( r_{\mathcal{H}}(A) \) is its zero-crossing. We claim that the map \( \epsilon \mapsto \alpha_{\epsilon, \mathcal{H}}(A) \) is locally Lipschitz and hence differentiable almost everywhere by Rademacher’s theorem \([33] \). To show Lipschitzness, we note that eigenvalues are Lipschitz functions with respect to perturbations of matrix entries, cf. \([34] \) and in fact differentiable when the eigenvalue
Lemma V.1 (Derivative of optimal value of \( (\Delta) \) with respect to energy of the perturbation). Let \( \epsilon > 0 \) and \( M \in \mathbb{R}^{n \times n} \). Suppose Assumption \( I \) holds for \( \Delta_{\text{opt}}(\epsilon) = [(\Delta_{\text{opt}}(\epsilon))_{ij}] \in \mathcal{H} \cap \mathbb{B}_e \) and let \( \eta(\epsilon, M) \) be the optimal value of the optimization problem \( (\Delta) \). Let \( S := \mathcal{S}(\epsilon, M) \), \( S := \mathcal{S}(\epsilon, M) \) be the index sets of saturation as in Proposition \( IV.2 \). Then \( \epsilon \mapsto \eta(\epsilon, M) \) is Lipschitz and wherever it is differentiable, its derivative is

\[
\frac{\partial}{\partial \epsilon} \eta(\epsilon, M) = \frac{\epsilon}{\theta_{\text{opt}}(\epsilon, M)},
\]

where \( \theta_{\text{opt}} \) is given by \( (15) \).

The proof is in the Appendix. If the perturbation is unstruc-
tured and without saturation constraints, Lemma \( V.1 \) simplifies to \( \frac{\partial}{\partial \epsilon} \eta(\epsilon, M) = \|M\|_F \), recovering \( [21 \text{ Lemma 4}] \). Using Lemma \( II.1 \) when \( \Im_{\mathcal{H}}(A) \) has a simple, unique rightmost eigenvalue associated with the worst-case perturbation \( \Delta_{\text{opt}}(\epsilon) \) with RP-compatible right and left eigenvectors \( x(\epsilon), y(\epsilon) \).

\[
\frac{d}{d \epsilon} \alpha_{\epsilon, \mathcal{H}}(A) = \frac{d}{d \epsilon} \text{Re}(x(\epsilon) \star y^*(\epsilon)) = \frac{1}{y^*(\epsilon)^* y(\epsilon)} \frac{d}{d \epsilon} \text{Re}(y(\epsilon) x(\epsilon)^*)
\]  

From Theorem \( IV.1 \) \( \Delta_{\text{opt}}(\epsilon) \) is the maximizer of \( (\Delta) \) with \( M = \text{Re}(y(\epsilon) x(\epsilon)^*) \). Thus, using Lemma \( V.1 \) we conclude

\[
\frac{d}{d \epsilon} \alpha_{\epsilon, \mathcal{H}}(A) = \frac{1}{y^*(\epsilon)^* y(\epsilon)} \text{Re}(y(\epsilon) x(\epsilon)^*) \to 0 \text{ as } \epsilon \to 0
\]  

B. Iterative computation of structured stability radius

We use Newton’s method \( [21] \)

\[
\epsilon_{i+1} = \epsilon_i - \frac{\alpha_{\epsilon, \mathcal{H}}(A)}{\frac{d}{d \epsilon} \alpha_{\epsilon, \mathcal{H}}(A)}|_{\epsilon = \epsilon_i}
\]  

(25)

to find the zero-crossing of \( \epsilon \mapsto \alpha_{\epsilon, \mathcal{H}}(A) \), which by definition \( (5) \) is the stability radius \( r_{\mathcal{H}}(A) \). Note from our discussion above that \( \frac{d}{d \epsilon} \alpha_{\epsilon, \mathcal{H}}(A) \) is not well defined when \( \Im_{\mathcal{H}}(A) \) has multiple rightmost eigenvalues. In this case, we compute the value of the right-hand of \( (24) \) for each rightmost eigenvalue and take the minimum to be the “gradient” (which is in fact the subgradient of \( \epsilon \mapsto \alpha_{\epsilon, \mathcal{H}}(A) \) with smallest norm). Consequently, substituting \( (24) \) into \( (25) \), we update \( \epsilon \) using

\[
\epsilon_{i+1} = \epsilon_i - \frac{(y^T x_i) \text{Re}(y^T x_i) \alpha_{\epsilon_i, \mathcal{H}}(A)}{\epsilon_i}
\]  

(26)

where \( x_i, y_i \) denote the right and left eigenvectors associated with the rightmost eigenvalue in \( \Im_{\mathcal{H}}(A) \) giving the smallest value of right-hand side of \( (24) \). We also observe from \( (26) \) that a fixed point of the iteration corresponds to either \( y_i^T x_i = 0 \), which is ruled out if the corresponding rightmost eigenvalue

is simple, or \( \alpha_{\epsilon_i, \mathcal{H}}(A) = 0 \), in which case the iteration has found the structured stability radius. Algorithm \( 3 \) summarizes the procedure written in pseudocode. Notice that the matrix

\[
\Delta_{\text{init}} \in \mathcal{H} \cap \mathbb{B}_{\epsilon_i} \text{ selected in Step 2 is used as the initial guess of the worst-case perturbation for Algorithm 2} \text{ (cf. Step 3).}
\]

In our simulations, cf. Section \( VI \) we use either the zero matrix, a random matrix taken from the constraint set or the result from the previous algorithm iteration. In addition, in Step 4 we let \( \epsilon_{i+1} \) be lower bounded by \( \zeta \epsilon_i \), for some \( \zeta \in (0, 1) \), in order to prevent \( \epsilon_{i+1} \) from becoming negative.

The next result characterizes the output of Algorithm \( 3 \).

Theorem V.2 (Error bound for Algorithm 3). Let \( \lambda \) be a simple rightmost eigenvalue for the structured pseudospectrum corresponding to the structured stability radius, with RP-compatible right and left eigenvector pair \( x, y \). Let \( r^* \in (0, 1) \) and suppose that for each iteration \( \ell \) it holds that \( r < r^* \), where the parameter \( r \) is defined by \( (16) \). Further assume that for every iteration \( \ell \), the rightmost eigenvalue of \( \Im_{\epsilon_i, \mathcal{H}}(A) \) is simple and the initial guess \( \Delta_{\text{init}} \) in Step 2 is close enough to \( \Delta_{\text{opt}}(\epsilon_i) \) so that it falls in the region of convergence of Algorithm 2. Then, if Algorithm 3 terminates at the \( \ell \)-th iteration, it holds that

\[
|\epsilon_{\ell} - r_{\mathcal{H}}(A)| = O \left( \frac{\sqrt{\epsilon_i r^*} \text{Re}(y^T x^*)}{(1 - r^*)^y^T x} \text{ tol}_\Delta + \text{tol}_\alpha \right). \]  

(27)

Proof. Since the rightmost eigenvalue is simple, \( \frac{d}{d \epsilon} \alpha_{\epsilon, \mathcal{H}}(A) \big|_{\epsilon = r_{\mathcal{H}}(A)} \) exists and is given by \( (24) \). Hence, using a first-order approximation on the inverse map of \( \epsilon \mapsto \alpha_{\epsilon, \mathcal{H}}(A) \),

\[
\epsilon_{\ell} = r_{\mathcal{H}}(A) + \frac{y^T x^0 |\alpha_{\epsilon_{\ell}}(A)\big|}{r_{\mathcal{H}}(A)} + O(\alpha_{\epsilon_{\ell}}, \mathcal{H}(A)^2), \]  

(28)

where \( \theta^0 := \text{opt}(r_{\mathcal{H}}(A), \text{Re}(y^T x^*)) \) is given by \( (15) \). In addition, \( |\alpha_{\epsilon_{\ell}}(A)| \leq |\alpha_{\epsilon_{\ell}}, \mathcal{H}(A) - \alpha_{\epsilon_{\ell}}(A)| + |\alpha_{\ell} - \alpha_{\ell} - \alpha_{\ell}, \mathcal{H}(A)| = \frac{\sqrt{\epsilon_i r^*} |\text{Re}(y^T x^*)|}{(1 - r^*)^y^T x} \text{ tol}_\Delta + O(\text{tol}_\Delta^2) + \text{tol}_\alpha \), where we have used the bound \( (18) \) from Theorem \( IV.6 \). The result follows from using this fact in equation \( (28) \). \qed

Notice that \( I_f \) does not appear on the right-hand side of \( (27) \), which indicates that the final error does not depend on the total number of iterations. In other words, the error introduced by using the approximate solution computed by Algorithm 2 at each iteration does not accumulate. We also make a final remark here that while in most cases Newton’s method has quadratic convergence rate, for some particular initial guesses
it may yield cyclic orbits and not converge. Nevertheless, such cyclic orbits are unstable and in all our simulations we observe Algorithm \[3\] terminates after a finite number of timesteps.

**VI. EXAMPLES**

We illustrate here the use of the proposed algorithms to find the structured pseudospectral abscissa and structured stability radius. In all examples, we use the parameters \(\epsilon_0 = 1, \text{tol}_A = \text{tol}_n = 10^{-3}\), and \(\zeta = 0.1\). Our algorithms are implemented in MATLAB on a personal computer with 4 cores at 2.71GHz.

**A. Perturbation to edges of single node of sparse network**

Consider a 5-node network system with graph given in Fig. 1a. Let system matrix in \((7)\) be given by

\[
A = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
-150 & -260 & -187 & -69 & -13 \\
\end{pmatrix},
\]

whose eigenvalues are \(-2 \pm i, -3, -3 \pm i\). We consider additive perturbations to the edges of the node 5, i.e., \(E_p = \{(5, i) : i = 1, \ldots, 5\}\). All rows of the perturbation matrix are then zero, except for the last one which has entries \(\delta = (\delta_5, \ldots, \delta_1)\). We consider three scenarios: (i) the entries \(\delta_i\) are unconstrained, (ii) \(\delta_i \leq 0\) for all \(i = 1, \ldots, 5\); and (iii) \(\delta_i \geq 0\), for all \(i = 1, \ldots, 5\). Since \(A\) is in controllable canonical form, its eigenvalues are the roots of the 5-degree polynomial \(p_a(x) = x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5\), with \(a = (a_1, \ldots, a_5)\) \(= (13, 69, 187, 260, 150)\). Hence, the problem of determining the structured stability radius of \(A\) is equivalent to finding the smallest perturbation on the non-leading coefficients of \(p_a(x)\) such that \(p_{a-\delta}(x)\) becomes non-Hurwitz. Even though the roots of a high-degree polynomial are in general sensitive to its coefficients, cf. \[13\], the proposed algorithms efficiently compute their outputs for \(A\).

1) **No constraints on perturbation:** Consider the case when \(\mathcal{H} = \{\Delta \in \mathbb{R}^{5 \times 5} : \Delta_{ij} = 0\text{ if } (i,j) \not\in E_p\}\). Algorithm \[3\] using \(\Delta_{\text{init}} = 0^{5 \times 5}\) in Step 2, finds 10.1465 as the structured stability radius, with worst-case perturbation \(\delta_{\text{opt}} = 0\text{.}1341, 0.4743, -0.1571, -0.0074, 0.0024\). Fig. 2c shows the corresponding structured \(\epsilon\)-psuedospectrum of \(A\) using random samples from \(\mathcal{H} \cap \mathbb{B}_\epsilon\). Note that structured pseudospectrum is indeed touching the imaginary axis, showing that the computed value is the true structured stability radius. We also note that the first element \((\delta_{\text{opt}})_{1}\) has the largest magnitude in the worst-case perturbation, indicating that the element \(A_{5,5}\) is the most critical for preserving network stability (equivalently, the coefficient \(a_1\) in \(p_a(x)\) is the most critical for preserving the Hurwitzness of the polynomial). If we eliminate the possibility of perturbing \(A_{5,5}\), our algorithm computes the new structured stability radius 94.3512, which is significantly larger. Fig. 2a shows the locally Lipschitzness nature of \(\epsilon \to \alpha_{\epsilon,\mathcal{H}}(A)\). The non-smooth corners in the map \(\epsilon \to \alpha_{\epsilon,\mathcal{H}}(A)\) corresponds to case when \(\Lambda_{\mathcal{H}}(A)\) has multiple rightmost eigenvalues. The evolution of \(\Lambda_{\mathcal{H}}(A)\) with respect to \(\epsilon\) is appreciated in Figs. 2b (with \(\epsilon = 5\)) through 2d (with \(\epsilon = 35\)).

2) **Non-positive constraints on perturbation:** We next consider the case when \(\mathcal{H} = \{\Delta \in \mathbb{R}^{5 \times 5} : \Delta_{ij} \leq 0\text{ if } (i,j) \not\in E_p\and \Delta_{5j} \leq 0\text{ if } j = \{1, \ldots, 5\}\}.\) This corresponds to increasing the value of the coefficients of \(p_a(x)\). Algorithm \[3\] finds 24.1733 as the structured stability radius, with a worst-case perturbation \(\delta_{\text{opt}} = (-23.9583, 0, 0, -2.6928, -1.4657)\). Fig. 3b shows the corresponding structured \(\epsilon\)-psuedospectrum of \(A\) using random samples from \(\mathcal{H} \cap \mathbb{B}_\epsilon\). In this case, we observe that using \(\Delta_{\text{init}} = 0^{5 \times 5}\) in Step 2 of Algorithm \[3\] does not always lead to convergence to a global optimizer of problem \[11\] when executing Algorithm \[3\] (cf. blue curve in Fig. 3). Instead, we make the selection

\[
\Delta_{\text{init}} = \min_{\Delta \in \mathcal{H} \cap \mathbb{B}_\epsilon} ||\Delta - \Delta_{\text{init}}||_F,
\]

which takes the previous estimated worst-case perturbation as an initial guess corresponding to the updated value of structured pseudospectral abscissa. In Fig. 3 we observe that with the selection \[29\], the first and second iterations of
Algorithm 3 correspond to local rightmost eigenvalues, but to global ones from the 3rd iteration onwards, thereby allowing the algorithm to determine the structured stability radius.

If the perturbation is constrained with $\Delta_{1,1} = 0.1, \Delta_{1,2} = -1$, and $\Delta_{1,10} = 1$, the sign of each term in the control law (50) is preserved and Algorithm 3 initialized with $\Delta_{\text{init}} = 0^{10 \times 10}$ finds the structured stability radius $1.4177$ and a worst-case perturbation $\delta_{\text{opt}} = (0.1, -1, 1)$. With this worst-case perturbation, the control $u_1$ vanishes and the network becomes marginally stable. This can also be seen in Fig. 4(b), where the structured pseudospectrum touches the imaginary axis.

Fig. 3: Structured stability radius with non-positive constraints on perturbation. (a) The blue (respectively, yellow) curve corresponds to the output of Algorithm 2 with uniform samples of $\epsilon$ in [0, 40] with step size 0.1 and the initial guess $\Delta_{\text{init}} = 0^{5 \times 5}$ (respectively, 1000 random $\Delta_{\text{init}} \in \mathcal{H} \cap \mathbb{B}_{\epsilon}$ and taking the maximum). The red dots correspond to the values $\alpha_0$ of the stability radius computed by Algorithm 3 which terminates after 6 iterations; (b) Estimated $\Lambda_{\mathcal{H}}(A)$ with $\epsilon = 24.1733$, generated with 1000 random samples of $\Delta \in \mathcal{H} \cap \mathbb{B}_{\epsilon}$.

3) Non-negative constraints on perturbation: Lastly, the case when $\mathcal{H} = \{\Delta \in \mathbb{R}^{5 \times 5} : \Delta_{ij} = 0$ if $(i, j) \notin \mathcal{E}_p$ and $\Delta_{lj} \geq 0, (i, j) \in \{1, \ldots, 5\}\}$ yields almost identical result as the scenario without constraints, with $10.1478$ as the structured stability radius and $\delta_{\text{opt}} = (10.1367, 0.474, 0, 0, 0.0024)$ as worst-case perturbation.

B. Circulant network

Consider a network system of 10 agents with circulant graph given in Fig. 1a. Each agent’s state is 1-dimensional and obeys the simple dynamics $\dot{x}_i = u_i$, where $u_i$ is the control input. Here, we consider

$$u_i = x_{i+1} - x_{i-1} - 0.1x_i,$$

(30)

where we abuse notation by identifying $(i + 1) = 1$ if $i = 10$ and $(i - 1) = 10$ if $i = 1$. The closed-loop system is then

$$\dot{x}_i = -0.1x_i + x_{i+1} - x_{i-1},$$

which is a typical formation control problem of agents running in a circle when the state $x_i$ corresponds to the angle of agent $i$ with respect to the center of rotation. The corresponding $10 \times 10$ circulant matrix $A$ is

$$A = \begin{pmatrix}
-0.1 & 1 \\
-1 & -0.1 & 1 \\
& & \ddots & \ddots \\
& & -1 & -0.1 & 1 \\
& & 1 & -1 & -0.1
\end{pmatrix}.$$

and the network graph is given in Fig. 1b.

Suppose the first agent is compromised by an adversary so that instead of (30), it implements $u_1 = (1 + \delta_1)x_2 - (1 - \delta_2)x_{10} - (0.1 - \delta_3)x_1$. This corresponds to perturbations in the non-zero elements of the first row of $A$. If $\delta = (\delta_1, \delta_2, \delta_3)$ is unconstrained, Algorithm 5 initialized at $\Delta_{\text{init}} = 0^{10 \times 10}$ finds the structured stability radius $0.4727$ and a worst-case perturbation $\delta_{\text{opt}} = (-0.0889, 0.0889, 0.4556)$. Fig. 4a shows the structured pseudospectrum.

In scenarios where several, instead of just one, consecutive agents are compromised, Fig. 5a shows the stability radius as a function of the number of compromised agents. As this number grows, the set $\mathcal{H}$ becomes larger and, for a fixed $\epsilon$, $\alpha_{\epsilon, \mathcal{H}}(A)$ increases. As a result, $r_{\mathcal{H}}(A)$ decreases, which is correctly captured in Fig. 5a. Notice also that when 5 or more consecutive agents are subject to adversaries, the structured stability radii are the same independently of whether the perturbation is unconstrained or sign preserving. This is because the worst-case perturbation is distributed over all compromised edges, with entries on each edge small enough that they do not violate the boundary constraints. It is also worth mentioning that when all agents are subject to adversaries, the estimated pseudospectrum generated by randomly sampling of perturbations in $\mathcal{H} \cap \mathbb{B}_{\epsilon}$, does not accurately reflect the true pseudospectrum, cf. Fig. 5b, because of the sensitivity of the rightmost eigenvalue with respect to the perturbation. Nevertheless, our iterative algorithms are still able to correctly find the pseudospectral abscissa and stability radius.

C. Tolosa networks

Lastly we consider large-scale systems where the network matrix in (7) is given by a Tolosa matrix [36]. These matrices
are sparse, asymmetric, and Hurwitz, and here we consider the cases of dimension $n = 90$, 340, and 1090. For each network, we consider additive perturbations to the entries of the 19th to the 36th rows ($E_p = \{(i, j) : 19 \leq i \leq 36\}$), since these are the ones with the most non-zero elements. We consider two scenarios for the constraints corresponding, respectively, to non-negative and non-positive perturbations, in both cases bounded in magnitude by 10,

$$
H_1 = \{ \Delta \in [0, 10]^{n \times n} : \Delta_{ij} = 0 \forall (i, j) \notin E_p \},
$$

$$
H_2 = \{ \Delta \in [-10, 0]^{n \times n} : \Delta_{ij} = 0 \forall (i, j) \notin E_p \}.
$$

We run Algorithm 3 where at each iteration we take the initial condition $\Delta_{\text{init}} = 0.1^{n \times n}$ and 10 random initial guesses $\Delta_{\text{init}} \in H \cup \mathbb{B}_{\epsilon_1}$, and select the one that results in the maximum structured pseudospectral abscissa. Tables I and II summarize the results for each scenario.

| Matrix   | Dimension | $r_H$      | Iterations | Time (secs) |
|----------|-----------|------------|------------|-------------|
| Tolosa90 | 90        | 0.50251    | 2          | 0.915406    |
| Tolosa340| 340       | 0.066404   | 13         | 58.732560   |
| Tolosa1090| 1090     | 0.15919    | 7          | 312.649106  |

TABLE I: Structured stability radii under perturbations in $H_1$ obtained by Algorithm 3. The columns, from left to right, correspond to the matrix name, its dimension, the structured stability radius, the number of algorithm iterations, and the total computation time.

| Matrix   | Dimension | $r_H$      | Iterations | Time (secs) |
|----------|-----------|------------|------------|-------------|
| Tolosa90 | 90        | 4.6737     | 3          | 2.896226    |
| Tolosa340| 340       | 0.35407    | 17         | 45.508739   |
| Tolosa1090| 1090     | 0.19163    | 7          | 213.961736  |

TABLE II: Structured stability radii under perturbations in $H_2$ obtained by Algorithm 3. The adescription of the columns is the same as in Table I.

In our simulations, we notice that the worst-case perturbations have a structure that is even sparser than the one specified by $E_p$. For example, for the case of Tolosa90 with perturbation in $H_1$, the worst-case perturbation $\Delta_{\text{opt}}$ only has 8 elements with magnitude larger than 0.001, and the largest element is $(\Delta_{\text{opt}})_{21, 21} = 0.5021$, which takes about 99.84% of the total energy ($r_H(A) = 0.50251$) of $\Delta_{\text{opt}}$, suggesting the importance of protecting edge $(21, 21) \in E_p$ against structured additive topological perturbations. We also observe that for these large-scale systems, around 90% of the total computation time corresponds to the computation of the left and right eigenvectors using the MATLAB command eigs. Hence, there is significant room for improving the computation time by optimizing the computation of eigenvectors of large-scale sparse matrices.

VII. CONCLUSIONS

We have studied the stability of linear dynamical systems systems against additive perturbations of the system matrix. We have formalized questions about whether an adversary can destabilize the network with perturbations of a given energy and determining what is the maximum amount of perturbation energy the network can withstand without becoming unstable using the concepts of structured pseudospectral abscissa and structured stability radius. We have proposed iterative algorithms that asymptotically compute both quantities for a given network along with the corresponding worst-case structured perturbations. Future work will study the global convergence of the algorithms and their extension to consider arbitrary values of the perturbation energy, develop distributed strategies for the computation of the structured pseudospectral abscissa and the corresponding worst-case perturbation to help individual agents assess their relative value in ensuring network stability, and examine resilience in scenarios where adversaries only have partial knowledge of the network structure.

APPENDIX

Proof of Proposition IV.2 Let $\lambda_0 \geq 0, \lambda_{ij} \geq 0, \lambda_{ij} \geq 0$ be the Lagrange multipliers for the constraints. Writing (12) element-wise, the Lagrangian is

$$L(\Delta, \lambda_0, \lambda_{ij}, \lambda_{ij}) = - \sum_{(i, j) \in \mathcal{E}_p} \Delta_{ij} m_{ij} + \lambda_0 \left( \sum_{(i, j) \in \mathcal{E}_p} \Delta_{ij}^2 - \epsilon^2 \right) + \sum_{(i, j) \in \mathcal{E}_p} \Delta_{ij} (\Delta_{ij} - \lambda_{ij}) + \sum_{(i, j) \in \mathcal{E}_p} \lambda_{ij} (\Delta_{ij} - \lambda_{ij}).$$

Define the sets $\mathcal{S} := \{(i, j) \in \mathcal{E}_p : (\Delta_{\text{opt}})_{ij} = \lambda_{ij}\}$ and $\mathcal{S} := \{(i, j) \in \mathcal{E}_p : (\Delta_{\text{opt}})_{ij} = \lambda_{ij}\}$. Note that since the sets are defined for the worst-case perturbation $\Delta_{\text{opt}}$, they are exactly the index sets of saturation as defined in Proposition IV.2. At the optimum $\lambda_{\text{opt}}, \lambda_{0, \text{opt}}, \lambda_{\text{opt}}, \lambda_{\text{opt}}$, the Lagrangian is maximized. Because $L$ is linear in $\Delta_{ij}, \lambda_{ij}, \lambda_{ij}$, and $(\Delta_{\text{opt}})_{ij}$’s do not reach boundary values for any $(i, j) \in \mathcal{E}_p \setminus \mathcal{S}$, we have $\lambda_{\text{opt}, ij} = \lambda_{\text{opt}, ij} = \lambda_{\text{opt}, ij} = 0$ for all those edges. Thus, for all $(i, j) \in \mathcal{E}_p \setminus (\mathcal{S} \cup \mathcal{S})$,

$$0 = \frac{\partial}{\partial \Delta_{ij}} L(\Delta, \lambda_{0, \text{opt}}, \lambda_{\text{opt}}, \lambda_{\text{opt}}) = -\lambda_{ij} + 2 \lambda_{0, \text{opt}} (\Delta_{\text{opt}})_{ij}.$$  

Because of Assumption 1, $\lambda_{0, \text{opt}} \neq 0$ or otherwise the above equation does not hold for that particular unsaturated element with $m_{ij} \neq 0$. This implies $\sum_{(i, j) \in \mathcal{E}_p} (\Delta_{\text{opt}})_{ij} = \epsilon^2$ and

$$\lambda_{\text{opt}, ij} = \frac{m_{ij}}{2 \lambda_{0, \text{opt}}}, \quad \text{(31)}$$

for all $(i, j) \in \mathcal{E}_p \setminus \mathcal{S}$, which yields the formula for $\Delta_{\text{opt}}$ given in Proposition IV.2 if $\theta_{\text{opt}} = \frac{\lambda_{0, \text{opt}}}{2 \lambda_{0, \text{opt}}}$. The latter is verified by squaring both sides of (31) and summing all the terms whose indices are not in the saturation index set, which gives

$$\epsilon^2 - \sum_{(i, j) \in \mathcal{S}} \Delta_{ij}^2 - \sum_{(i, j) \in \mathcal{S}} \Delta_{ij}^2 = \sum_{(i, j) \in \mathcal{E}_p \setminus (\mathcal{S} \cup \mathcal{S})} (\Delta_{\text{opt}})_{ij} = \frac{\sum_{(i, j) \in \mathcal{E}_p \setminus (\mathcal{S} \cup \mathcal{S})} m_{ij}^2}{4 \lambda_{0, \text{opt}}}.$$

Rearranging the terms and comparing with (15), we conclude

$$\theta_{\text{opt}} = \sqrt{\frac{\epsilon^2 - \sum_{(i, j) \in \mathcal{S}} \Delta_{ij}^2 - \sum_{(i, j) \in \mathcal{S}} \Delta_{ij}^2}{\sum_{(i, j) \in \mathcal{E}_p \setminus (\mathcal{S} \cup \mathcal{S})} m_{ij}^2}} = \frac{1}{2 \lambda_{0, \text{opt}}}.$$

Finally, we show the local Lipschitzness of $\theta_{\text{opt}}$ over some neighborhood $D \ni (\epsilon, M)$. Let $\tau > \epsilon, \mathcal{M} > \|M\|_F, \delta > 0$. Define $D := \{\epsilon', M' \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\times}^{n \times n} : \epsilon \in [0, \epsilon], \|M\|_F \leq \mathcal{M}, \|\epsilon' - \epsilon\|_2 + \|M' - M\|_F \leq \delta\}$. Let $(\epsilon_1, M_1) = (\epsilon, M)$ and pick arbitrary $(\epsilon_2, M_2) \in D$. Denote $\mathcal{S}_k := \mathcal{S}_{(\epsilon_k, M_k)} = \mathcal{S}_k$.
following abbreviations:

\[ a := \sum_{(i,j) \in S_1 \cap S_1} \Delta_{ij}^2 + \sum_{(i,j) \in S_1 \setminus S_1} \Delta_{ij}^2, \]

\[ b := \sum_{(i,j) \in S_1 \setminus S_2} \Delta_{ij}^2 + \sum_{(i,j) \in S_2 \setminus S_1} \Delta_{ij}^2, \]

\[ c := \sum_{(i,j) \in S_1 \setminus S_1} \Delta_{ij}^2 + \sum_{(i,j) \in S_2 \setminus S_1} \Delta_{ij}^2 \]

and for \( k = 1, 2 \), let

\[ d_k := \sum_{(i,j) \in S_p} (M_k)_{ij}^2, \]

\[ e_k := \sum_{(i,j) \in S_1 \setminus S_2} (M_k)_{ij}^2 + \sum_{(i,j) \in S_2 \setminus S_1} (M_k)_{ij}^2, \]

\[ f_k := \sum_{(i,j) \in S_2 \setminus S_1} (M_k)_{ij}^2 + \sum_{(i,j) \in S_1 \setminus S_2} (M_k)_{ij}^2, \]

\[ g_k := \sum_{(i,j) \in S_2 \setminus S_1} (M_k)_{ij}^2 + \sum_{(i,j) \in S_1 \setminus S_2} (M_k)_{ij}^2. \]

Using the formula (15), we have

\[ \theta_1^2 = \frac{e_2^2 - a - b}{d_1 - e_1 - f_1}, \quad \theta_2^2 = \frac{e_2^2 - a - c}{d_2 - e_2 - g_2}. \]

By making \( \delta \) sufficiently small, \( (\epsilon_2, M_2) \) is close enough to \( (\epsilon_1, M_1) \) and hence the denominators of the expressions for \( \theta_1^2, \theta_2^2 \) are close and have a common positive lower bound. In other words, there exists \( m > 0 \) only depending on \( \delta \) and \( M \) such that \( d_1 - e_1 - f_1 \geq m^2 \) and \( d_2 - e_2 - g_2 \geq m^2 \).

In addition, by properties of index sets of saturation [14], \( a \leq \theta_1^2 c_1, \ a \leq \theta_2^2 c_2, \) and

\[ \theta_1^2 f_2 \leq b \leq \theta_1^2 f_1, \quad \theta_2^2 g_1 \leq c \leq \theta_2^2 g_2, \]

which further implies that

\[ \epsilon_k \leq \theta_k d_k \leq \theta_k ||M_k||_F \leq \theta_k m \]

for both \( k = 1, 2 \). We show the Lipschitzness of \( \theta_{op} (\eta, M) \) by considering two cases, 1) \( f_1 = 0 \) or \( g_2 = 0 \), and 2) both \( f_1, g_2 > 0 \). In case 1), when \( f_1 = 0 \), the inequality (32) implies \( b = f_2 = 0 \). Meanwhile, it can also be deduced from (33) that

\[ (e_2^2 - a)(d_2 - e_2 - g_2) \leq (e_2^2 - a - c)(d_2 - e_2). \]

Denote \( \kappa_1 := (d_1 - e_1)(d_2 - e_2 - g_2) \geq m^4 \). Note that

\[ \theta_2^2 - \theta_1^2 \]

\[ = \kappa_1 (e_2^2 - a - c)(d_1 - e_1) - (e_2^2 - a)(d_2 - e_2 - g_2) \]

\[ \leq \kappa_1 (e_2^2 - a)(d_1 - e_1) - (e_2^2 - a)g_1 \]

\[ - (e_2^2 - a)(d_2 - e_2 - g_2) \]

\[ \leq \kappa_1 [(e_2^2 - e_1^2)(d_1 - e_1) \]

\[ + (e_2^2 - a)(d_1 - e_1 - g_1) - (d_2 - e_2 - g_2)) \]

\[ \leq \kappa_1 (|e_2^2 - e_1^2| m^2 + e_2^2 (d_1 - e_1 - g_1) - (d_2 - e_2 - g_2)). \]

Note that (34) implies that \( \epsilon_k^2 \leq \epsilon_1^2 + \epsilon_2^2 \leq m \). Hence

\[ \theta_2 - \theta_1 \leq \frac{1}{(\theta_1 + \theta_2) \kappa_1} (m^2(e_1 + e_2)e_2 - e_1) \]

\[ + e_1^2 (\sqrt{d_1 - e_1 - g_1} + \sqrt{d_2 - e_2 - g_2}) \]

\[ \cdot |\sqrt{d_1 - e_1 - g_1} - \sqrt{d_2 - e_2 - g_2}) | \]

\[ \leq \frac{m^2}{\kappa_1} \theta_1 + \theta_2 |e_2 - e_1| \]

\[ + \frac{e_1}{\kappa_1} (\sqrt{d_1 - e_1 - g_1} + \sqrt{d_2 - e_2 - g_2}) \]

\[ \cdot |\sqrt{d_1 - e_1 - g_1} - \sqrt{d_2 - e_2 - g_2}) | \]

\[ \leq \frac{m^2}{m^4} |e_2 - e_1| + \frac{m^2}{m^4} \|M_2 - M_1\|_F. \]

On the other hand,

\[ \theta_1^2 - \theta_2^2 \]

\[ = \kappa_1^{-1} ((e_2^2 - a)(d_2 - e_2 - g_2) - (e_2^2 - a - c)(d_1 - e_1)) \]

\[ = \kappa_1^{-1} ((e_2^2 - e_1^2)(d_2 - e_2 - g_2) \]

\[ + (e_2^2 - a)(d_2 - e_2 - g_2) - (e_2^2 - a - c)(d_1 - e_1)) \]

\[ \leq \kappa_1^{-1} ((e_2^2 - e_1^2)(d_2 - e_2 - g_2) \]

\[ + (e_2^2 - a - c)((d_2 - e_2 - g_2) - (d_1 - e_1)) \]

\[ \leq \kappa_1^{-1} ((e_2^2 - e_1^2) m^2 + e_2^2 (d_1 - e_1 - (d_2 - e_2)). \]

and again we can conclude the same upper bound on \( \theta_1 - \theta_2 \). Therefore \( \theta_{op} \) is Lipschitz on \( D \). Similar arguments hold when \( g_2 = 0 \).

In case (2), we have \( f_2 > 0, g_1 > 0 \) by picking \( \delta \) small enough. Hence there exist \( m_{f}, m_{g} > 0 \) only depending on \( \delta \) and \( M \) such that \( f_k \geq m_{f}^2, g_k \geq m_{g}^2 \) for both \( k = 1, 2 \). Consequently

\[ \frac{1}{\sqrt{f_1}} - \frac{1}{\sqrt{f_2}} \leq \frac{\sqrt{f_1} - \sqrt{f_2}}{\sqrt{f_1} f_2} \leq \frac{M_1 - M_2}{\sqrt{f_1} f_2}, \]

where we have used the triangle inequality. Similarly we also have

\[ \frac{1}{\sqrt{g_1}} - \frac{1}{\sqrt{g_2}} \leq \frac{M_1 - M_2}{\sqrt{g_1} g_2}. \]

Under Assumption [1] Algorithm [1] terminates if and only if NotDone \( = \) false, i.e., when [14a] holds. In addition, when Algorithm [1] terminates, \( \Delta_{op} \) is given by (13) for the computed \( \mathbb{S}, \mathbb{S} \). Thus,
to prove the statement, we are left to show that (14b) and (14c) hold. Let \( \theta_{opt,1}, \theta_{opt,2}, \ldots, \theta_{opt,k} \) be the sequence of \( \theta_{opt} \) generated by Algorithm \[1\]. From (15), it is clear that these variables are non-negative. Next we show that the sequence is non-decreasing. Take any consecutive terms \( \theta_{opt,l}, \theta_{opt,l+1} \) and let \( \delta S \) (resp. \( \delta S \)) be the difference between the set \( S \) (resp. \( S \)) computed in the \( l \)-th iteration and the one computed in the next iteration. In other words, \( m_{ij} \theta_{opt,l} \geq \delta_{ij} \) for all \( (i,j) \in \delta S \) and \( m_{ij} \theta_{opt,l} \leq \delta_{ij} \) for all \( (i,j) \in \delta S \). To simplify the presentation, for the \( l \)-th iteration, let

\[
\begin{align*}
a &:= \sum_{(i,j) \in S} \delta_{ij}^2 + \sum_{(i,j) \in S} \delta_{ij}^2, \\
b &:= \sum_{(i,j) \in S} \delta_{ij}^2 + \sum_{(i,j) \in S} \delta_{ij}^2, \\
c &:= \sum_{(i,j) \in S} m_{ij}^2, \\
d &:= \sum_{(i,j) \in S} m_{ij}^2 + \sum_{(i,j) \in S} m_{ij}^2, \\
e &:= \sum_{(i,j) \in S} m_{ij}^2 + \sum_{(i,j) \in S} m_{ij}^2.
\end{align*}
\]

Note that (15) implies

\[
\begin{align*}
\theta_{opt,l}^2 &= \frac{c^2 - a}{c - d}, \\
\theta_{opt,l+1}^2 &= \frac{c^2 - a - b}{c - d - e}.
\end{align*}
\]

Plugging (35a) into \( b \leq \theta_{opt,l}^2 c \), we have \( b(c - d) \leq (c^2 - a) c \). Subtracting (35a) from (35b),

\[
\theta_{opt,l+1}^2 - \theta_{opt,l}^2 = \frac{(c^2 - a)(c - d) - b(c - d)}{(c - d - e)(c - d)} \geq 0,
\]
as claimed. Observe that in the execution of Algorithm \[1\] an edge \((i,j) \in \delta S \) (resp. \( S \)) is added to the index set of saturation at some iteration \( l \leq k \), when \( m_{ij} \theta_{opt,l} \geq \delta_{ij} \geq 0 \) (resp. \( m_{ij} \theta_{opt,l} \leq \delta_{ij} \leq 0 \)). Since \( \{ \theta_{opt,l} \} \) is non-decreasing, we deduce \( m_{ij} \theta_{opt,l,k} \geq \delta_{ij} \) (resp. \( m_{ij} \theta_{opt,l,k} \leq \delta_{ij} \)), thereby verifying (14b) and (14c).

**Proof of Lemma \[IV.2\]** Let \( \theta_k = \theta_{opt}(\epsilon, M_k) \), \( k = 1, 2 \). Notice that for some \((i,j) \in E_p \), if \((i,j) \) belongs to neither index sets of saturation for the two optimization problems, then \( (\Delta_k)_{ij} = (M_k)_i j \theta_k \) for both \( k = 1, 2 \) so

\[
| (\Delta_1)_{ij} - (\Delta_2)_{ij} | \leq | (M_1)_ij \theta_1 - (M_2)_ij \theta_2 | 
\]

holds with equality. If \((i,j) \) only belongs to one index set of saturation, say \((i,j) \in S(\epsilon, M_1) \) but \((i,j) \not\in S(\epsilon, M_2) \), then \( (M_1)_ij \theta_1 \geq (\Delta_1)_{ij} = \delta_{ij} \geq (\Delta_2)_{ij} = (M_2)_ij \theta_2 \) and hence again the inequality (36) holds. This is also true if \((i,j) \) only belongs to one of the other index sets of saturation. If \((i,j) \) belongs to the index sets of saturation for both optimization problems, then \( (\Delta_1)_{ij} = (\Delta_2)_{ij} \) and (36) holds again.

Let \( \delta > 0 \) be the one picked in the proof of Proposition \[IV.2\] for defining the neighborhood \( D \ni (\epsilon, M_1) \). We have \( (\epsilon, M_2) \in D \) as well and using the Lipschitzness of \( \theta_{opt} \),

\[
| (\Delta_1)_{ij} - (\Delta_2)_{ij} | \leq | (M_1)_ij \theta_1 - (M_2)_ij \theta_2 | \leq \kappa \epsilon \| M_1 - M_2 \| \theta + \frac{\epsilon}{m} | (M_1)_ij - (M_2)_ij |,
\]

where \( \kappa = \max \{ \frac{m_1^2}{m_1}, \min \{ m_1, m_2, m_3, m_4, m_5, m_6 \} \} \) and \( m, m_1, m_2, m_3, m_4, m_5, m_6 \) come from the proof of Proposition \[IV.2\] and only depend on \( \delta, M_1 \). As a result,

\[
\| (\Delta_1 - \Delta_2) \|_F \leq \sqrt{\sum_{(i,j) \in E_p} | (\Delta_1)_{ij} - (\Delta_2)_{ij} |^2 } \leq \sqrt{\epsilon^2 \| p \| + \frac{\epsilon}{m} \| M_1 - M_2 \| \theta}
\]

and hence the statement holds with \( \epsilon := \sqrt{\epsilon^2 \| p \| + \frac{\epsilon}{m} \} \).

**Proof of Lemma \[IV.7\]** For fixed \( M \in \mathbb{R}^{n \times n} \), \( \eta(\epsilon, M) = (\Delta, M) \) is linear in \( \Delta \), and hence Lipschitz. In addition, \( \theta_{opt}(\epsilon, M) \mapsto \Delta, \) given by (13)-(14) is also Lipschitz. Lastly, \( \epsilon \mapsto \theta_{opt}(\epsilon, M) \) is Lipschitz by Proposition \[IV.2\]. Hence, the composition \( \epsilon \mapsto \eta(\epsilon, M) \) is locally Lipschitz.

To find the derivative of \( \epsilon \mapsto \eta(\epsilon, M) \) when it exists, we first conclude from the continuity of \( \epsilon \mapsto \theta_{opt}(\epsilon, M) \) and the criteria for index sets of saturation (14) that for \( \delta \in \mathbb{R} \) with sufficiently small \( |\delta| \), \( \bar{S}(\epsilon + \delta, M) \subseteq \bar{S}(\epsilon, M) \) and \( \bar{S}(\epsilon + \delta, M) \subseteq \bar{S}(\epsilon, M) \). Meanwhile, the linearity of the objective function (12) implies that when \( \epsilon \) grows, the saturated elements in the optimizer remain saturated. In other words, if \( \delta \geq 0 \), then \( \bar{S}(\epsilon + \delta, M) \subseteq \bar{S}(\epsilon, M) \) and \( \bar{S}(\epsilon + \delta, M) \subseteq \bar{S}(\epsilon, M) \). Therefore, the index sets of saturation are the same for sufficiently small \( \delta > 0 \); i.e., \( \bar{S}(\epsilon + \delta, M) = \bar{S}(\epsilon, M) = S \) and \( \bar{S}(\epsilon + \delta, M) = \bar{S}(\epsilon, M) = S \). Thus the difference between \( \eta(\epsilon + \delta, M) \) and \( \eta(\epsilon, M) \) can be expressed as

\[
\eta(\epsilon + \delta, M) - \eta(\epsilon, M) = (\theta_{opt}(\epsilon, M) - \theta_{opt}(\epsilon + \delta, M)) \sum_{(i,j) \in E_p \setminus (S \cup S)} m_{ij}^2.
\]

This equation is useful for computing the right one-sided derivative of \( \epsilon \mapsto \eta(\epsilon, M) \), which equals to the derivative of this map when it exists,

\[
\frac{d\eta(t, M)|_{t=\epsilon}}{dt} = \lim_{\delta \rightarrow 0^+} \frac{\eta(\epsilon + \delta, M) - \eta(\epsilon, M)}{\delta} = \lim_{\delta \rightarrow 0^+} \frac{\theta_{opt}(\epsilon + \delta, M) - \theta_{opt}(\epsilon, M)}{\delta} \sum_{(i,j) \in E_p \setminus (S \cup S)} m_{ij}^2
\]

\[
= \frac{d\theta_{opt}(t, M)|_{t=\epsilon}}{dt} \sum_{(i,j) \in E_p \setminus (S \cup S)} m_{ij}^2
\]

\[
= \epsilon \sqrt{c^2 - 2 \sum_{(i,j) \in S \cup S} \delta_{ij}^2 - \sum_{(i,j) \in S \cup S} \delta_{ij}^2} \frac{m_{ij}^2}{\theta_{opt}}.
\]

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