Ladder Operators in Repulsive Harmonic Oscillator with Application to the Schwinger Effect

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ABSTRACT: The ladder operators in harmonic oscillator are a well-known strong tool for various problems in physics. In the same sense, it is sometimes expected to handle the problems of repulsive harmonic oscillator in a similar way to the ladder operators in harmonic oscillators, though their analytic solutions are well known. In this paper, we discuss a simple algebraic way to introduce the ladder operators of the repulsive harmonic oscillators, which can reproduce well-known analytic solutions. Applying this formalism, we discuss the charged particles in a constant electric field in relation to the Schwinger effect; the discussion is also made on a supersymmetric extension of this formalism.

KEYWORDS: Repulsive (inverted) harmonic oscillator, SUSY QM, Schwinger effect

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1 Introduction

The algebraic approaches to the potential problems in quantum mechanics are commonly used ways from the early state of those fields[1]. In particular, the harmonic oscillators (h.o.) give a good operative example of algebraic approach to the eigenvalue problems in terms of the ladder operators, the annihilation and creation operators ($\hat{a}, \hat{a}^\dagger$) characterized by $[\hat{a}, \hat{a}^\dagger] = 1$. In such a dynamical system, the eigenvalue problem of Hamiltonian can be solved exactly by use of those ladder operators without depending on the representation of the eigenstates[1, 2]; and, if we take the coordinate representation of those states, then the eigenstates will be reduced to the well-known analytic solutions expressed in terms of Hermite polynomials. The use of ladder operators also provides necessary tools in the field theories, since the dynamical degrees of freedom of bosonic-free fields are decomposed into those of infinite harmonic oscillators.

In comparison with h.o., the physical applications of the repulsive harmonic oscillators (r.h.o.) are limited, since the Hamiltonian of r.h.o. is parabolic; and, its eigenstates are scattering states. The algebraic approaches to r.h.o., however, have been tried from a few different viewpoints: the dynamical groups including r.h.o. [3, 4], the analytic continuation of angular velocity $\omega \rightarrow \pm i \omega$ in h.o.[5, 6], the Bose systems in SUSY quantum mechanics[7–9], and so on.

On the other hand, it is known that the eigenvalue problems of r.h.o.-Hamiltonian are reduced to solve Weber’s equation, which has analytic solutions so-called parabolic cylinder functions or the Weber functions[10, 11]. The relation between the algebraic approaches to r.h.o and the analytic solutions, however, is not always clear. It is also important to study the completeness of the states

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1 The inverted oscillator or reversed oscillator, in other words.
constructed out of the algebraic approaches, since the trace calculations in physical applications require such a property of those states.

The purpose of this paper is, thus, to give a simple algebraic approach to the eigenvalue problems of r.h.o. by introducing Hermitian ladder operators \([A, \bar{A}] = i\). We can show that the dynamical variables of r.h.o. can be represented in the functional spaces constructed out of \((A, \bar{A})\) with two ground states \((\phi_0, \bar{\phi}_0)\) satisfying \(A\phi_0 = \bar{A}\bar{\phi}_0 = 0\). The states generated out of \(\phi_0\) and \(\bar{\phi}_0\) form dual spaces each other so that the complete basis can be constructed based on the inner products between those states belonging to respective dual spaces.

In the next section, we study some types of complete bases, a discrete basis and continuous bases, representing r.h.o. in terms of the ladder operators with their ground states; there, the completeness of those bases is discussed carefully. The discussions are also made on the eigenvalue problems of r.h.o.-Hamiltonian by considering the relation between the ladder operator formalism and the well-known analytic solutions.

In section 3, we discuss the applications of the present ladder operator formalism to two topics: one is a problem of charged particles under a constant electric field, the problem of the Schwinger effect\(^{[13]}\). This dynamical system is equivalent to r.h.o.; and, the discrete basis in the ladder operator formalism is shown to be a useful to evaluate that effect. As another topic, we study an extension of r.h.o. to a model of SUSY quantum mechanics by taking the advantage of the ladder operator formalism, though such an extension has been discussed from the early stages of r.h.o.. We focus our attention on that the Schwinger effect for fermions is closely related to such an extended model.

Section 4 is devoted to the summary of our results. In appendices, some mathematical problems used in the text are discussed: the analytic solutions of Hamiltonian eigenstates, a proof of completeness, and the evaluation of the Schwinger effect for fermions.

2 Ladder operators in repulsive harmonic oscillators

2.1 Summary of standard harmonic oscillators

To begin with, we summarize the ladder operator approach to the problems of the usual harmonic oscillator, to which the Hamiltonian operator of a mass \(m\) particle with the characteristic frequency \(\omega\) of the oscillation in one-dimensional space is given by

\[
\hat{H} = \frac{1}{2m} \hat{p}^2 + \frac{m\omega^2}{2} \hat{x}^2 = \frac{\hbar \omega}{2} (\hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger) = \hbar \omega \left( \hat{N} + \frac{1}{2} \right),
\]

where \(\hat{N} = \hat{a}^\dagger \hat{a}\) and

\[
\hat{a} = \sqrt{\frac{m\omega}{2\hbar}} \hat{x} + \frac{i}{\sqrt{2m\hbar\omega}} \hat{p},
\]

\[
\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \hat{x} - \frac{i}{\sqrt{2m\hbar\omega}} \hat{p}.
\]

Then, because of \([\hat{a}, \hat{a}^\dagger] = 1\), one can verify that \([\hat{H}, \hat{a}^\dagger] = \hbar \omega \hat{a}^\dagger\), \([\hat{H}, \hat{a}] = -\hbar \omega \hat{a}\), and \(\|\hat{H}\Phi\| \geq \frac{\hbar \omega}{2}\) on a state \(\Phi\) normalized so that \(\|\Phi\|^2 = 1\). This means that starting from the ground state \(\Phi_0\) defined by \(\hat{a}\Phi_0 = 0\) with \(\|\Phi_0\|^2 = 1\), the states

\[
\Phi_n = \frac{1}{\sqrt{n!}} \hat{a}^{2n}\Phi_0 \quad (n = 0, 1, 2, 3, \cdots)
\]

\(\cdots\)
satisfy the eigenvalue equations
\[ \hat{H} \Phi_n = \hbar \omega \left( n + \frac{1}{2} \right) \Phi_n \quad (n = 0, 1, 2, 3, \ldots), \quad (2.4) \]
and the normalization \( \langle \Phi_n | \Phi_m \rangle = \delta_{n,m} \). The importance is that the states \( \{ \Phi_n \} \) really form a complete basis of the functional space \( V \), in which the canonical operators \( (\hat{x}, \hat{p}) \) are represented. Namely, in terms of the bra and the ket states, the operator
\[ \hat{I} = \sum_{n=0}^{\infty} |\Phi_n \rangle \langle \Phi_n| \quad (2.5) \]
is the unit operator in the functional space \( V \); and, one can verify
\[ \langle x | \hat{I} | x' \rangle = \delta(x - x'), \quad (2.6) \]
where \( \{ |x \rangle \} \) are the eigenstates of \( \hat{x} \) characterized by \( \hat{x} |x \rangle = x |x \rangle \) and \( \langle x | x' \rangle = \delta(x - x'), \ (x, x' \in \mathbb{R}) \). Furthermore, if it is necessary, the \( x \)-representation of \( \Phi_n \) can be written explicitly in terms of the Hermitian polynomial \( H_n(x) \) so that \( \Phi_n(x) = \langle x | \Phi_n \rangle = \sqrt{\frac{1}{2\pi \hbar \omega}} e^{-m \omega x^2 / 2\hbar} H_n(x \sqrt{m \omega / \hbar}) \).

2.2 The case of repulsive harmonic oscillators

Now, for a repulsive harmonic oscillator, the Hamiltonian operator \( \hat{H}_r \) is given from \( \hat{H} \) in eq.(2.1) by changing the sign of \( -\frac{\hbar \omega}{2} \frac{\hat{x}^2}{\hat{p}^2} \); and, a complete basis in the same functional space \( V_r \) by means of new ladder operators can be constructed in roughly parallel with equations (2.1) \( \sim \) (2.6). Namely, one can start with the expression
\[ \hat{H}_r = \frac{1}{2m} \hat{p}^2 - \frac{m \omega^2}{2} \hat{x}^2 = -\frac{\hbar \omega}{2} (\hat{A} A + A \hat{A}), \quad (2.7) \]
where
\[ A = \sqrt{\frac{m \omega}{2\hbar}} \hat{x} - \frac{1}{\sqrt{2m \hbar \omega}} \hat{p}, \]
\[ \hat{A} = \sqrt{\frac{m \omega}{2\hbar}} \hat{x} + \frac{1}{\sqrt{2m \hbar \omega}} \hat{p}. \quad (2.8) \]

By definition, \( A \) and \( \hat{A} (\neq \hat{A}^\dagger) \) are Hermitian operators themselves; however, they satisfy a similar algebra as that of \( (\hat{a}, \hat{a}^\dagger) \) such as \( [A, \hat{A}] = -[\hat{A}, A] = i \). Further, in terms of \( (A, \hat{A}) \), the Hamiltonian operator \( \hat{H}_r \) can be written as \(^2\)
\[ \hat{H}_r = -i \hbar \omega \left( \Lambda + \frac{1}{2} \right) = -i \hbar \omega \left( \hat{\Lambda} - \frac{1}{2} \right), \quad (2.9) \]
where
\[ \Lambda = -i \hat{A} A \quad \text{and} \quad \hat{\Lambda} = -i A \hat{A} \ (= \Lambda + 1). \quad (2.10) \]

\(^2\)In terms of the ladder operator \( (\hat{a}, \hat{a}^\dagger) \) defined in eq.(2.1), the Hamiltonian operator (2.7) can be represented as \( \hat{H}_r = \frac{-\hbar \omega}{2} (\hat{a}^\dagger \hat{a} + \hat{a}^2) \). From this expression, carrying out the successive canonical (\# unitary) transformations by \( U_1 = e^{i \frac{\hbar \omega}{2} \hat{x}^2} \) and \( U_2 = e^{-\frac{\hbar \omega}{4} \hat{x}^2} \), one can find the relation between \( \hat{H}_r \) and \( \hat{H} \) such that \( U_2 U_1 \hat{H} U_1^{-1} U_2^{-1} = i \hat{H} \). The eigenvalue problem of \( \hat{H}_r \), thus, can also be solved in terms of \( (\hat{a}, \hat{a}^\dagger) \) and these canonical transformations.
Since $\Lambda^\dagger = -\Lambda - 1$, the Hermiticity of the operator $\hat{H}_r$ given in eq. (2.9) is formally guaranteed. The eigenvalue problem of $\hat{H}_r$ is, thus, reduced to those of the operators $\Lambda$ and $\bar{\Lambda}$ ($\neq \Lambda^\dagger$), which are commutable each other.

In order to solve the eigenvalue problem of $\Lambda$ and $\bar{\Lambda}$, let us introduce eigenstates $(\phi_\sigma, \bar{\phi}_\sigma)$ defined by

$$A\phi_\sigma = \left( \frac{m \omega}{2\hbar} \hat{x} - \frac{1}{\sqrt{2m\hbar \omega}} \hat{\rho} \right) \phi_\sigma = \sigma \phi_\sigma,$$

$$\bar{A}\bar{\phi}_\sigma = \left( \frac{m \omega}{2\hbar} \hat{x} + \frac{1}{\sqrt{2m\hbar \omega}} \hat{\rho} \right) \bar{\phi}_\sigma = \sigma \bar{\phi}_\sigma,$$  \hspace{1cm} (2.11) \hspace{1cm} (2.12)

where the $\sigma$ is a real parameter. Then, the particular states $(\phi_0, \bar{\phi}_0)$ defined by $A\phi_0 = \bar{A}\bar{\phi}_0 = 0$ should be regarded as the counterparts of $\Phi_0$ in the h.o. It should be noticed that in spite of the similarity of eq. (2.11) to the coherent state equation in h.o., the index $\sigma$ of $\phi_\sigma$ runs over real continuous spectrum due to the Hermiticity of $A$; and, the same is true for $\bar{\phi}_\sigma$.

In the $x$-representation, eqs. (2.11) and (2.12) can be solved explicitly, and we obtain

$$\phi_\sigma(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{i m \omega}{2\hbar} x^2 - i \sqrt{\frac{2m\hbar}{\omega}} \sigma x},$$

$$\bar{\phi}_\sigma(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{i m \omega}{2\hbar} x^2 + i \sqrt{\frac{2m\hbar}{\omega}} \sigma x},$$  \hspace{1cm} (2.13) \hspace{1cm} (2.14)

where the normalizations of those states are $\langle \phi_\sigma | \phi_\sigma' \rangle = \langle \bar{\phi}_\sigma | \bar{\phi}_\sigma' \rangle = \delta(\sigma - \sigma')$. In this representation, because of $\phi_\sigma(x) = \phi_\sigma(x)^*$, the “bar” becomes simply complex conjugation, and one can find the completeness of $(\phi_\sigma, \bar{\phi}_\sigma)$ in the form

$$\int d\sigma \langle \phi_\sigma | \phi_\sigma \rangle = \int d\sigma \langle \bar{\phi}_\sigma | \bar{\phi}_\sigma \rangle = \delta(x - x').$$  \hspace{1cm} (2.15)

The meaning of the “bar”, the definition of dual states, however, depends on the representation. In the ladder operator formalism for r.h.o., the meaning of dual spaces can be given without depending on the representation as in the case of ladder operator formalism for h.o.. To make clear this point, let us write the $(\phi_0, \bar{\phi}_0)$, the ground states in the sense of $A\phi_0 = \bar{A}\bar{\phi}_0 = 0$, as $(\phi_{(0)}, \bar{\phi}_{(0)})$. Then, the discrete states defined by

$$\phi_{(n)} = A^n \phi_{(0)} \quad (\bar{\phi}_{(n)} = A^n \bar{\phi}_{(0)}), \quad (n = 0, 1, 2, \cdots)$$  \hspace{1cm} (2.16)

are able to satisfy the eigenvalue equations

$$\Lambda \phi_{(n)} = n \phi_{(n)} \quad (\bar{\Lambda} \bar{\phi}_{(n)} = -n \bar{\phi}_{(n)}), \quad (n = 0, 1, 2, \cdots).$$  \hspace{1cm} (2.17)

Thus, on the states $(\phi_{(n)}, \bar{\phi}_{(n)})$, the Hamiltonian operator $\hat{H}_r$ takes discrete eigenvalues (figure 1) such that

$$\hat{H}_r \phi_{(n)} = -i\hbar \omega \left( n + \frac{1}{2} \right) \phi_{(n)}, \quad (n = 0, 1, 2, \cdots).$$

Now, the inner product of those states can be determined from the algebra of $(A, \bar{A})$ and the normalization $\langle \bar{\phi}_{(0)} | \phi_{(0)} \rangle \equiv N_0 = \frac{\sqrt{\pi}}{2\pi}$ only. Indeed for $m = n + l$ ($l > 0$), one can verify

$$\langle \bar{\phi}_{(m)} | \phi_{(n)} \rangle = i^n \langle \bar{\phi}_{(0)} | A_l^* A^n A_l | \phi_{(0)} \rangle = i^n \langle \bar{\phi}_{(0)} | A_l^* A^{n-1} A_l | \phi_{(0)} \rangle$$

$$= \cdots = i^n n! \langle \bar{\phi}_{(0)} | A_l^* | \phi_{(0)} \rangle,$$  \hspace{1cm} (2.18) \hspace{1cm} (2.19)
which leads to \( \langle \bar{\phi}_{(m)} | \phi_{(n)} \rangle = 0 \) \((m > n)\); the same is true for the case \( m < n \). Thus, the inner products between any \( m, n \) states can be represented as

\[
\langle \bar{\phi}_{(m)} | \phi_{(n)} \rangle = \delta_{m,n} N_n \quad (N_n \equiv i^n n! N_0).
\] (2.20)

This equation says that the \( \{ \langle \bar{\phi}_{(n)} | \} \) is a basis of dual space of \( \{ | \phi_{(n)} \rangle \} \) normalized by eq. (2.20), though \( | \phi_{(n)} \rangle^\dagger \neq \text{const} \times \langle \bar{\phi}_{(n)} | \) in general.

Further eq. (2.20) suggests that the operator

\[
\hat{I}_r = \sum_{n=0}^{\infty} \frac{1}{N_n} | \phi_{(n)} \rangle \langle \bar{\phi}_{(n)} |
\] (2.21)

plays the role of a unit operator in \( \{ | \phi_{(n)} \rangle \} \) space. The expectation \( \hat{I}_r = 1 \), can be confirmed through the equation

\[
A \hat{I}_r = \sum_{n=1}^{\infty} \frac{in}{N_n} | \phi_{(n-1)} \rangle \langle \bar{\phi}_{(n)} | = \sum_{n=0}^{\infty} \frac{i(n+1)}{N_{n+1}} | \phi_{(n)} \rangle \langle \bar{\phi}_{(n+1)} |
\]
\[
= \sum_{n=0}^{\infty} \frac{1}{N_n} | \phi_{(n)} \rangle \langle \bar{\phi}_{(n)} | A = \hat{I}_r A,
\] (2.22)

which can be verified using \( \frac{i(n+1)}{N_{n+1}} = \frac{1}{N_n} \). In a similar way, one can derive \( \bar{A} \hat{I}_r = \hat{I}_r \bar{A} \). Since \( A \) and \( \bar{A} \) are composing elements of dynamical variables in r.h.o., one can say \( \hat{I}_r = c1 \), \((c = \text{const.})\) in the sense of Schur’s lemma. Here, the constant in the right-hand side is necessary to be \( c = 1 \) because of \( \hat{I}_r | \phi_{(0)} \rangle = | \phi_{(0)} \rangle \) by eq. (2.20). Another proof of \( \hat{I}_r = 1 \) is to show directly

\[
\langle x | \hat{I}_r | x' \rangle = \delta(x - x'),
\] (2.23)

which will be given in appendix B.

It should also be emphasize that the discrete basis \( \{ | \phi_{(n)} \rangle, \bar{\phi}_{(n)} \rangle \} \) are closely related to Weber’s functions as the energy eigenstates of \( \hat{H}_r \) by means of the analytic continuation with respect to \( n \). In

\[\begin{array}{c|c|c|c}
\hline
\hline
\frac{iE}{\hbar \omega} & \phi(2) & \phi(1) & \phi(0) \\
\hline
\frac{4}{2} & & & \\
\frac{3}{2} & & & \\
\frac{1}{2} & & & \\
\hline
\bar{\phi}(0) & -\frac{1}{2} & & \\
\bar{\phi}(1) & -\frac{3}{2} & & \\
\hline
\end{array}\]

Figure 1. There are many types of complete bases in the representation space of \( \hat{H}_r \). On the discrete bases \( \{ | \phi_{(n)} \rangle, \bar{\phi}_{(n)} \rangle \} \), the \( \hat{H}_r \) takes the eigenvalues shown in the figure on the left of vertical axis.
order to verify this, we take notice the formula for a complex $\lambda$:

$$\tilde{A}^\lambda = \frac{1}{\Gamma(-\lambda)} \int_0^\infty dt e^{-\bar{A}t} t^{-(\lambda+1)}$$

$$= \frac{1}{\Gamma(-\lambda)} \int_0^\infty dt t^{-(\lambda+1)} e^{-\frac{1}{4}t^2} e^{-\frac{1}{2}\sqrt{\frac{m\omega}{\hbar}} t} e^{-\frac{1}{2}\sqrt{\frac{m\omega}{\hbar}} \bar{t}}. \quad (2.25)$$

Here, eq.(2.24) seems to hold on the states such as $\{\tilde{\phi}_\sigma; \sigma > 0\}$, on which $\tilde{A}$ becomes an operator with positive eigenvalues. Applying eq.(2.25) to $\tilde{\phi}_{(0)}(x)$, such a constraint will fade away in the sense of analytic continuation; and, we obtain the expression

$$\tilde{A}^\lambda \tilde{\phi}_{(0)}(x) = e^{\frac{im\omega}{2\hbar \pi^2} \lambda} \frac{1}{\Gamma(-\lambda)} \int_0^\infty dt t^{-(\lambda+1)} e^{-\frac{1}{4}t^2} e^{-\frac{1}{2}\sqrt{\frac{m\omega}{\hbar}} x} e^{i\frac{1}{2}\sqrt{\frac{m\omega}{\hbar}} (x+it\sqrt{\frac{\hbar}{m\omega}})^2}$$

$$= e^{\frac{im\omega}{2\hbar \pi^2} \lambda} \frac{1}{\Gamma(-\lambda)} \int_0^\infty dt \tilde{t}^{-(\lambda+1)} e^{-\frac{1}{4}\tilde{t}^2} e^{-\frac{1}{2}\sqrt{\frac{m\omega}{\hbar}} \tilde{t}} e^{i\frac{1}{2}\sqrt{\frac{m\omega}{\hbar}} \tilde{t} \frac{\sqrt{\hbar}}{\sqrt{m\omega}}}, \quad (2.26)$$

where $\tilde{t} = e^{i\frac{\pi}{2}} t$ and $z = e^{i\frac{\pi}{2}} \sqrt{\frac{2m\omega}{\hbar}} x$. The last equality in eq.(2.26) shows the relationship[14] between $\tilde{A}^\lambda \tilde{\phi}_{(0)}(x)$ and Weber’s function $D_{\lambda}(z)$ (appendix A). In a similar manner, one can verify that

$$A^\rho \tilde{\phi}_{(0)}(x) = e^{-\frac{i\pi}{2} \rho} \sqrt{\frac{m\omega}{2\hbar \pi^2}} D_{\rho}(iz), \quad (2.27)$$

which can be regarded as the analytic continuation of the relation $\tilde{\phi}_{(n)}(x) = \phi_{(n)}^*(x)$ with respect to $n$. We note that if the $\lambda$ in eq.(2.26) and the $\rho$ in eq.(2.27) give the same eigenvalue of $\frac{i\hbar r}{\hbar}$, then $\lambda + \frac{1}{2} = -(\rho + \frac{1}{2})$ or $\rho = -\frac{1}{2}(\lambda + 1)$. Therefore, $D_{\lambda}(z)$ and $D_{-(\lambda+1)}(iz)$ are independent eigenstates of $\frac{i\hbar r}{\hbar}$ belonging to the same eigenvalue $\lambda + \frac{1}{2}$. This is a well-known result of discrete eigenstates in the eigenvalue problem of r.h.o. [10, 11].

Furthermore, in terms of Weber’s D-function, the completeness condition (2.23) can also be represented as

$$\langle x|\tilde{r}_r|x'\rangle = \sum_{n=0}^\infty \frac{1}{N_n} \phi_{(n)}(x)\tilde{\phi}_{(n)}(x')^*$$

$$= \sum_{n=0}^\infty i^n \frac{(m\omega)^{\frac{1}{2}}}{2\hbar \pi^2} D_{n}(z)D_{n}(iz')^*. \quad (2.28)$$

In summary, the complete basis $\{\phi_{\sigma}(x)\}$ are eigenstates of $A$ belonging to continuous eigenvalues $\{\sigma \in \mathbb{R}\}$, but those are not eigenstates of $\tilde{H}_r$. On the other hand, the complete basis $\{\phi_{(n)}(x)\}$ are eigenstates of $\tilde{H}_r$ belonging to discrete eigenvalues labeled by the non-negative integers $n (= 0, 1, 2, \cdots)$. The $D_{\lambda}(z)$ is an analytic continuation of $\phi_{(n)}(x)$ with respect to $n$, on which $\tilde{H}_r$ takes the continuous eigenvalue $-i\hbar \omega (\lambda + \frac{1}{2})$.

3 Topics related to the present r.h.o. formalism

The complete bases $\{\phi_{\sigma}\}$ or $\{\phi_{(n)}, \tilde{\phi}_{(n)}\}$ based on ladder operator $(A, \tilde{A})$ gives us useful ways to handle the problems related to r.h.o.; in what follows, we exhibit two simple examples:
3.1 Schwinger effect

We note that the r.h.o. is effectively realized by a particle interacting with a specific gauge field. Let us consider the scalar field $\Phi$ in 4-dimensional spacetime for a mass $m$ particles under gauge fields $A^\mu$ satisfying \(^3\)

$$\left[\hat{\Pi}_\mu(A)\hat{\Pi}^\mu(A) + (mc)^2\right] \Phi(x) = 0,$$  \tag{3.1}

where $\hat{\Pi}^\mu(A) = \hat{\pi}^\mu - \frac{2}{c^2} A^\mu$ and $g = \pm |e|$. We, here, setup the gauge potentials in such a way that $(A^\mu_0(x), A^c(x)) = (-E x^1, 0)$ ($E = \text{const.} > 0$), which produces the uniform electric field $E$ along $x^1$ direction. Then,

$$\hat{\Pi}(A_c)^2 = 2m\hat{H}_01 + \hat{p}_\perp^2,$$  \tag{3.2}

where $\hat{p}_\perp = (\hat{p}_2, \hat{p}_3)$ and

$$\hat{H}_01 = \frac{1}{2m} \hat{p}_1^2 - \frac{1}{2m} \left(\frac{|e|E}{c}\right)^2 \left(x^1 + \frac{c}{gE} \hat{p}_0^0\right)^2.$$  \tag{3.3}

Further, in terms of the canonical variables defined by the unitary transformation $U_E = e^{\hat{\pi}((\frac{e}{c})p^0 p^1)}$ so that

$$(X^\mu) = (U_E x^\mu U_E^{-1}) = \left(x^0 - \frac{c}{gE} \hat{p}_1^1, x^1 + \frac{c}{gE} \hat{p}_0^0, x^2, x^3\right),$$  \tag{3.4}

$$(\hat{P}^\mu) = (U_E \hat{p}^\mu U_E^{-1}) = (\hat{p}^\mu),$$  \tag{3.5}

the Hamiltonian operator (3.3) can be written as

$$\hat{H}_01 = \frac{1}{2m} \hat{p}_1^2 - \frac{m\omega^2}{2} X_1^2,$$  \tag{3.6}

where the angular frequency is defined by $\omega = \frac{|e|E}{mc}$. This means that the $H_{01}$ is just the Hamiltonian of r.h.o. defined in the phase space $(X^\mu, P^\mu)$.

Now, the classical action of gauge field under consideration is $S_G[A_c] = \frac{1}{2}\int d^4 x E^2$; and, the one loop correction due to the scalar field $\Phi$ adds the quantum effect $S_Q[A_c] = -i\hbar \log \{\det(\hat{\Pi}(A_c)^2 + (mc)^2)\}^{-1}$ to $S_G[A_c]$ \(^4\); the resultant effective action of gauge fields $S_{\text{eff}}[A_c] = S_G[A_c] + S_Q[A_c]$ becomes

$$S_{\text{eff}}[A_c] = S_G[A_c] - i\hbar \int_0^\infty \frac{dt}{\tau} e^{-it((mc)^2-i\epsilon)} \text{tr} \left( e^{-it\hat{\Pi}(A_c)} \right) \text{ (const.)}$$  \tag{3.7}

disregarding unimportant additional constant. Namely, under the classical background gauge field $A^\mu_0$, the scalar QED gives rise to the transition amplitude $\langle 0_{\text{in}} | 0_{\text{out}} \rangle \sim N e^{\frac{i}{\hbar} S_{\text{eff}}[A_c] (|N|^2 = 1)}$, which defines an unitary S-matrix element for a real $S_{\text{eff}}[A_c]$. If the S-matrix contains pair productions under that the state of electric field is constant in time, then Im$S_{\text{eff}}[A_c] \neq 0$, and we have $|\langle 0_{\text{in}} | 0_{\text{out}} \rangle|^2 \sim e^{-\frac{i}{\hbar} \text{Im} S_{\text{eff}}[A_c]} \neq 1$. This ratio, the Schwinger effect, can be evaluated by calculating the “trace” in eq.(3.7).

\(^3\) diag($\eta_{\mu\nu}$) = ($- + + +$).

\(^4\) In the expression of $S_Q[A_c]$, the use has been made of the well-known formulas $\{\det(M)\}^{-1} = e^{-\text{tr} \log M}$ and $\text{tr} \log M = -\text{tr} \int_0^\infty \frac{dt}{\tau} e^{-it(M-ic)}$ (+const.).
For this purpose, it is convenient to use \( \{|X^0\rangle \otimes |\phi_{(n)}(X^1)\rangle \otimes |X_\perp\rangle\) as the base states in the trace calculation. Then by taking \(1 = \sum_{n=0}^{\infty} \frac{1}{\sqrt{N_n}} |\phi_{(n)}\rangle \langle \phi_{(n)}|\) and \(\int dX^1 \phi_{(n)}(X^1)^* \phi_{(n)}(X^1) = N_n\) into account, we obtain

\[
\frac{1}{\hbar} \text{Im} \text{S}_{\text{eff}}[A_c] = -\text{Re} \int_0^\infty \frac{d\tau}{\tau} e^{-ir((mc)^2 - i\epsilon)} \text{tr} \left( e^{-i\tau(2m\hbar_0 + \hat{P}^2)} \right) = -\text{Re} \int_0^\infty \frac{d\tau}{\tau} e^{-ir((mc)^2 - i\epsilon)} \sum_{n=0}^{\infty} e^{-\tau 2m\hbar(n + \frac{1}{2})} \times \int dX^0 d^2X_\perp \delta(0) \left( \sqrt{\frac{1}{4\pi\hbar^2\tau}} \right)^2.
\] (3.8)

Putting, here, \(V_0 \sim \int dX^0, V_\perp \sim \int d^2X_\perp\) as cutoff volumes in \(X^0, X_\perp\) spaces respectively, the right-hand side of this equation becomes

\[
\text{r.h.s.} = \delta(0)V_0 V_\perp \text{Re} \left( \frac{i}{4\pi\hbar^2} \int_\tau^\infty \frac{d\tau}{\tau^2} \frac{e^{-ir((mc)^2 - i\epsilon)}}{2 \sinh(\tau m\hbar)} \right), \quad (\epsilon = +0),
\]

\[
= \delta(0)V_0 V_\perp \frac{i(mc)^2}{16\pi\hbar^2} P \int_{-\infty}^\infty \frac{dz}{z^2} \frac{e^{-iz}}{\sinh \left( \frac{z}{2m\hbar} \right)}, \quad (z = \tau (mc)^2),
\]

\[
= \delta(0)V_0 V_\perp \frac{m^2}{8\pi^2c^2} \left( |\epsilon|E \right) \left( \frac{mc}{m^2c^2} \right)^2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} e^{-\frac{m^2\hbar c^2}{mc} \left( \frac{n\pi}{m\hbar} \right)^2}}{n^2},
\] (3.9)

where the \(P\) denotes the principal value in \(z\) integral. Further, since \(mc\) and \(\Lambda mc\) play respectively the roles of typical momentum and the length in this system, we may put

\[
\delta(0) = \int \frac{dP^0}{2\pi\hbar} e^{iP^0} \sim \frac{mc}{2\pi\hbar}, \quad V_\perp \frac{mc}{\hbar} \sim 1, \quad \left( V_\perp \sim \int dX^1 \right)
\] (3.10)

; and so, \(\delta(0)V_0 V_\perp \sim V(4) \frac{(mc)^2}{2\pi\hbar^2}\) with \(V(4) = V_0 V_\perp\). Therefore, we finally arrive at the expression

\[
\frac{1}{\hbar} \text{Im} \text{S}_{\text{eff}}[A_c] \sim V(4) \frac{m^4}{16\pi^4\hbar^2} \left( |\epsilon|E \right) \left( \frac{mc}{m^2c^2} \right)^2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} e^{-\frac{m^2\hbar c^2}{mc} \left( \frac{n\pi}{m\hbar} \right)^2}}{n^2}.
\] (3.11)

The result just coincides with the formula of pair creation given by Schwinger for scalar QED[13, 16].

### 3.2 Extension to SUSY quantum mechanics

The present ladder operator formalism of r.h.o. is easily extended to one of SUSY quantum mechanics[17–20]. To this end, let us introduce the Fermi oscillators characterized by \(\{b, b^\dagger\} = 1, \ b^2 = b^{12} = 0,\) which can be represented in 2-dimensional vector space so that

\[
b = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad b^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\] (3.12)

In terms of \((b, b^\dagger)\), the supersymmetric extension of \(\hat{H}_r\) should be

\[
\hat{H}_r = -i\hbar \omega \left( -i\hat{A} + b^\dagger b \right) = -i\hbar \omega \begin{pmatrix} \hat{A} & 0 \\ 0 & \hat{A} \end{pmatrix}.
\] (3.13)
Then the generators of SUSY transformation defined by
\[ Q = -i\sqrt{\hbar\omega}Ab^1 = \sqrt{\hbar\omega}\begin{pmatrix} 0 & -iA \\ 0 & 0 \end{pmatrix}, \quad \bar{Q} = -i\sqrt{\hbar\omega}b = \sqrt{\hbar\omega}\begin{pmatrix} 0 & 0 \\ -iA & 0 \end{pmatrix}, \] (3.14)
are characterized by the algebras
\[ [Q, A] = 0, \quad [Q, \bar{A}] = \sqrt{\hbar\omega}b^1, \quad [Q, b] = -i\sqrt{\hbar\omega}A, \quad [Q, b^1] = 0, \] (3.15)
\[ [\bar{Q}, A] = -\sqrt{\hbar\omega}, \quad [\bar{Q}, \bar{A}] = 0, \quad [\bar{Q}, b] = 0, \quad [\bar{Q}, b^1] = -i\sqrt{\hbar\omega}A, \] (3.16)
and
\[ [Q, \hat{H}_r] = [Q, \hat{H}_r] = 0, \quad \{Q, \bar{Q}\} = \hat{H}_r. \] (3.17)
If we introduce \( Q_1 = \frac{1}{\sqrt{2}}(\bar{Q} + Q) \) and \( Q_2 = \frac{1}{\sqrt{2}}(\bar{Q} - Q) \), the last equations can also be written as
\[ \{Q_i, Q_j\} = \delta_{ij}\hat{H}_r, \quad (i, j = 1, 2). \] (3.18)
Those algebras should be compared with that of \( N = 2 \) SUSY quantum mechanics, though \( Q_i (i = 1, 2) \) are not Hermitian operators. The zero-point oscillation of \( \hat{H}_r \) is removed by this supersymmetry.

In spite of formal resemblance of the present dynamical system to SUSY quantum mechanics of h.o., the true nature of both dynamical systems are fairly different as can be seen from \( \hat{H} \neq \hat{H}^1 \), non-positive structure of \( \hat{\gamma}_r \), and so on. On the discrete complete basis \{\( \phi_{(n)} \), \( \bar{\phi}_{(n)} \}\}, the eigenvalue equation \( \hat{H}_r|\phi_E\rangle = E|\phi_E\rangle \) can be solved easily: for \( n = 1, 2, \ldots \),
\[
E_n^+ = -i\hbar\omega n \quad \text{doublet} \quad E_n^- = i\hbar\omega n \quad \text{doublet} \] (3.19)
\[
|\phi_n^-\rangle = \begin{pmatrix} 0 \\ \phi_{(n)} \end{pmatrix} = |-\rangle \otimes |\phi_{(n)}\rangle \quad |\bar{\phi}_n^-\rangle = \begin{pmatrix} 0 \\ \bar{\phi}_{(n)} \end{pmatrix} = |+\rangle \otimes |\bar{\phi}_{(n)}\rangle
\]
\[
|\phi_n^+\rangle = \frac{Q}{\sqrt{\hbar\omega}}|\phi_n^-\rangle = \left( -\frac{i}{\sqrt{n}}A\phi_{(n)} \right) \quad |\bar{\phi}_n^+\rangle = \frac{Q}{\sqrt{\hbar\omega}}|\bar{\phi}_n^-\rangle = \left( -\frac{i}{\sqrt{n}}A\bar{\phi}_{(n)} \right)
\]
, where \(|-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\) and \(|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\). Here, the mapping \( |\phi_n^+\rangle = \frac{Q}{\sqrt{\hbar\omega n}}|\phi_n^-\rangle, (n \geq 1) \) can be inverted by \( |\phi_n^-\rangle = i\frac{Q}{\sqrt{\hbar\omega n}}|\phi_n^+\rangle \); and so, the states \( |\phi_n^\pm\rangle, (n = 1, 2, \ldots) \) form a tower of super pairs. In the same sense, the states \( |\bar{\phi}_n^\pm\rangle, (n = 1, 2, \ldots) \) form another tower of super pairs.

In contrast, the states \( |\phi_0^\pm\rangle \) and \( |\bar{\phi}_0^\pm\rangle \) belonging to the same eigenvalue \( E_0^+ = 0 \) are two super singlets, which satisfy \( Q_1|\phi_0^\pm\rangle = Q_2|\bar{\phi}_0^\pm\rangle = 0, (i = 1, 2) \). Therefore in the space of states \( \{|\phi_0^-\rangle, |\phi_0^+\rangle\} \), the supersymmetry is realized as a good symmetry; that is, SUSY is not broken. The same is true for the space of states \( \{|\bar{\phi}_0^+\rangle, |\bar{\phi}_0^+\rangle\} \). In each space, the operators \( Q_i \) work as the generators of supersymmetry; however, there arises no mapping between those two spaces by \( Q_i \) (figure 2).

In the context of this SUSY quantum mechanics, we emphasize the following: in the Schwinger effect for fermions, the SUSY quantum mechanics of the r.h.o. plays an effective role in its background; that is, the topics 3.1 and 3.2 are not independent in this effect.

The Dirac field \( \Psi \) interacting with an external gauge fields \( A^\mu \) obeys the \( U(1) \) symmetry field equation \( ^5 \)
\[
(\gamma \cdot \hat{\Pi}(A) + mc) \Psi = 0 \quad (\gamma \cdot \Pi(A) = \gamma_\mu \Pi(A)^\mu). \] (3.20)
\(^5\) The gamma matrices are normalized so that \( \{\gamma^\mu, \gamma^\nu\} = -2\eta^\mu\nu. \)
When we multiply this equation by \(- (\gamma \cdot \hat{\Pi}(A) - mc)\) from the left, the field equation becomes the second order form such that
\[
\left[ - (\gamma \cdot \hat{\Pi}(A))^2 + (mc)^2 \right] \Psi = 0.
\] (3.21)

Here, if we use the configuration of gauge potentials \((A_0(x), A_c(x)) = (-Ex^1, 0)\) as in eq.(3.2), then with \(\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]\) and \(F^{\mu\nu} = \partial_\mu (A_c)_\nu\), we obtain
\[
- (\gamma \cdot \hat{\Pi}(A_c))^2 = \hat{\Pi}(A_c)^2 - \frac{\hbar g}{2c} \sigma^{\mu\nu} F^{\mu\nu} = \hat{\Pi}(A_c)^2 - \frac{\hbar g}{c} \sigma^{01} E
= 2m \hat{H}_{01} + \hat{p}_1^2 - \frac{\hbar g}{c} (i\sigma_1 \otimes \sigma_1) E.
\] (3.22)

Carrying out the unitary transformation in 4-spinor space by \(U = e^{i\pi/2} \sigma_2 \otimes e^{i\pi/2} \sigma_2\), the eq.(3.22) becomes
\[
-U (\gamma \cdot \hat{\Pi}(A_c)) U^\dagger = 2m \left[ -i\hbar \omega \left( -i\AA + \frac{1}{2} \right) \right] - m\hbar \omega \sigma_3 \otimes \sigma_3 + \hat{p}_1^2
= 2m \left[ -i\hbar \omega \left( -i\AA + b^\dagger b \right) \right] + \hat{p}_1^2,
\] (3.23)

where we have used \(\sigma_3 \otimes \sigma_3 = \sigma_3 \otimes [b^\dagger, b]\) and \(\omega = \frac{|E|}{mc}\) as before. The result implies that the spectra of upper components of \(\tilde{\Psi} = U \Psi\) are those of the supersymmetric Hamiltonian \(\hat{\mathcal{H}}_r\); on the other side, the spectra of lower components of \(\tilde{\Psi}\) are governed by \(\hat{\mathcal{H}}'_r\), which is obtained from \(\hat{H}_r\) changing the role of \((b, b^\dagger)\). Thus, one can evaluate the Schwinger effect for fermions again according to the procedure of eq.(3.8)-eq.(3.9) (appendix C).

4 Summary

In this paper, we have discussed the eigenvalue problems of r.h.o. in terms of ladder operators \((A, \bar{A})\) introduced by an analogous way to the ladder operator \((\hat{a}, \hat{a}^\dagger)\) in h.o.. The non-positive property of the Hamiltonian operator \(\hat{H}_r\) in r.h.o. is a result of the property of ladder operators such as \(A^\dagger = A\),
$\bar{A}^\dagger = \bar{A}$, and $\bar{A} \neq A^\dagger$. Then the states defined by $A\phi_0 = 0$ and $\bar{A}\phi_0 = 0$ play the role of different ground states, which can be normalized by $\langle \phi_0 | \phi_0 \rangle = \sqrt{2\pi}$ in spite of $\langle \phi_0 | \phi_0 \rangle = \langle \phi_0 | \phi_0 \rangle = \infty$.

The complete bases representing r.h.o. can be constructed by the combination of $(A, \phi_0)$ or that of $(\bar{A}, \bar{\phi}_0)$; the aspects of the complete basis are various depending on the ways of construction. For instance, the $\{\phi_\sigma(x)\}$ are eigenstates of $A$ with continuous eigenvalues $\{\sigma \in \mathbb{R}\}$, though those are not eigenstates of $\hat{H}_r$. On the other hand, the $\{\phi_{(n)}(x)\}$ are eigenstates of $\hat{H}_r$ with discrete eigenvalues $\{-i\hbar\omega (n + \frac{1}{2}); (n = 0, 1, 2, \cdots)\}$; the Hamiltonian operator $\hat{H}_r$ is able to have continuous eigenvalues $\{-i\hbar\omega (\lambda + \frac{1}{2}); \lambda \in \mathbb{R}\}$ on the states of Weber’s D-function $D_\lambda(z)$, where the D-function can be shown to be an analytic continuation of $\phi_{(n)}(x)$ with respect to $n$.

As good applications of this ladder operator formalism, we have shown two topics: the Schwinger effect in scalar QED and an extension of r.h.o. to SUSY quantum mechanics. In the first, the Hamiltonian of particles interacting with a constant electric field is shown to be canonically equivalent to one of r.h.o.; and so, the knowledge of r.h.o. is useful to handle the problem of pair production by the electric field. Indeed, it has been shown that the discrete complete bases $\{\phi_{(n)}, \bar{\phi}_{(n)}\}$ characterized by eq. (2.21) gives a simple way to evaluate such a production rate within the framework of quantum mechanics.

Secondly, we have tried to extend the present r.h.o. system to a supersymmetric dynamical system; the extended Hamiltonian $\hat{H}_r$ is again a non-positive Hermitian operator constructed out of fermionic oscillators $(b, b^\dagger)$ and ladder operators $(A, \bar{A})$. The ladder operator formalism gives rise to two towers of super-pair states $|\phi^+_n\rangle$ and $|\bar{\phi}^+_n\rangle (n = 1, 2, \cdots)$, which belong to the eigenvalues $E^+_n = -i\hbar\omega_n$ and $E^-_n = i\hbar\omega_n$ respectively. In addition to this, the $n = 0$ ground states exist as two singlets $|\phi^-_0\rangle$ and $|\bar{\phi}^-_0\rangle$, which satisfy $Q_i|\phi^-_0\rangle = Q_i|\bar{\phi}^-_0\rangle = 0, (i = 1, 2)$. Namely, in each space of super-pair tower states, SUSY is realized as a good symmetry, though the SUSY in this model is an extended concept from the standard one as can be seen from $Q_i^\dagger \neq Q_i$.

Furthermore, we have brought up the following: if we consider the Dirac fields interacting with an external electric field, then the supersymmetric structure of r.h.o. will be implicitly included in a loop effect of those Dirac fields. According to this line of approach, we have shown the way to evaluate the Schwinger effect for fermions in appendix C.

The knowledge on the complete bases in r.h.o. under the ladder operator formalism is expected to give useful tools in various problems other than the topics discussed in this paper. For example, the Hamiltonian $\hat{H}_r$ is able to take continuous eigenvalues on the states $(\phi_\sigma, \bar{\phi}_\sigma)$; in the space of those eigenstates, the SUSY may show a different feature from the standard analysis. Those are interesting future problems.

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A Weber’s functions as the energy eigenvalue functions for the r.h.o.

We here summarize the standard way to make the eigenvalue functions of the r.h.o. reduce to Weber’s functions.
In the $x$-representation with $\hat{p} = -i\hbar \frac{d}{dx}$, the eigenvalue equation of $\hat{H}_r$ can be written as

$$
\left( -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} - \frac{m\omega^2}{2} x^2 - E \right) \psi_E(x) = 0. \tag{A.1}
$$

Introducing, here, the variable $z$ defined by

$$
x = e^{it} \sqrt{\frac{\hbar}{2m\omega}} z, \quad \left( \frac{d^2}{dz^2} = \frac{2m\omega}{i\hbar} \frac{d^2}{dz^2} \right), \tag{A.2}
$$
eq)

eq.(A.1) with $\psi_E(x(z)) = w_E(z)$ gives rise to

$$
-\frac{i}{\hbar \omega} \times \text{eq.}(A.1) = \left( \frac{d^2}{dz^2} + \frac{iE}{\hbar \omega} - \frac{1}{4} z^2 \right) w_E(z) = 0. \tag{A.3}
$$

Writing $\frac{iE}{\hbar \omega} = \lambda + \frac{1}{2}$ and $w_E(z) = w_\lambda(z)$, the eq.(A.3) becomes standard form of Weber’s equation

$$
\frac{d^2 w_\lambda(z)}{dz^2} + \left( \lambda + \frac{1}{2} - \frac{z^2}{4} \right) w_\lambda(z) = 0. \tag{A.4}
$$

For $\tilde{w}_\lambda(z) = e^{-\frac{1}{4}z^2} w_\lambda(z)$, eq.(A.4) can also be written as

$$
\left( \frac{d^2}{dz^2} - z \frac{d}{dz} + \lambda \right) \tilde{w}_\lambda(z) = 0. \tag{A.5}
$$

To solve eq.(A.5), let us use the Fourier-Laplace representation

$$
\tilde{w}_\lambda(z) = \int_{\Gamma} dt e^{-zt} f_\lambda(t), \tag{A.6}
$$

where $\Gamma$ is a path from $a$ to $b$ in complex $t$ plane. Then under the integration by parts with respect to $t$, the eq.(A.5) with eq.(A.4) gives

$$
\frac{d}{dt} \left\{ tf_\lambda(t) \right\} + \left( t + \lambda \right) \left\{ tf_\lambda(t) \right\} = 0 \tag{A.7}
$$
on condition that $[e^{-zt} \{ tf_\lambda(t) \}]_a^b = 0$. The eq.(A.7) can be solved easily so that $f_\lambda(t) = \text{const.} e^{-\frac{1}{4}t^2} t^{-(\lambda+1)}$; since the boundary conditions are satisfied by $(a, b) = (0, \infty)$ for $\text{Re}\lambda < 0$ on the real $t$ axis, we finally obtain the integral representation for $w_\lambda(z) = e^{-\frac{1}{4}z^2} \tilde{w}_\lambda(z) (= D_\lambda(z))$ in such a form as [14, 15]

$$
D_\lambda(z) = \frac{e^{-\frac{1}{4}z^2}}{\Gamma(-\lambda)} \int_0^\infty dt e^{-zt} t^{\frac{1}{2}t^2} t^{-(\lambda+1)} \quad (\text{Re}\lambda < 0) \tag{A.8}
$$

$$
= -\frac{\Gamma(\lambda+1)}{2\pi i} e^{-\frac{1}{4}z^2} \int_C dt e^{-zt} t^{\frac{1}{2}t^2} t^{-(\lambda+1)}, \tag{A.9}
$$

where $C$ is the contour given in (figure 3). It is not difficult to rewrite the contour integral in eq.(A.9) to the path integral in eq.(A.8) by taking into account $\Gamma(\lambda+1) \sin(-\pi\lambda) = \frac{\pi e}{\Gamma(-\lambda)}$.

The function $D_\lambda(z)$ is Weber’s D-function [6] (Parabolic cylinder function)[15], by which the independent solutions of eq.(A.1) for $\frac{iE}{\hbar \omega} = \lambda + \frac{1}{2}$ are given as $D_\lambda(z)$ and $D_{-\lambda-1}(iz)$.

---

6The D-function is normalized so that $D_n(z), (n = 0, 1, \cdots)$ reduces to $e^{-\frac{1}{4}z^2} H_n(z)$, where $\{H_n(z)\}$ are the Chebyshev-Hermite polynomials.
Therefore, \( \hat{I}_c = 1 \).

By definition of \( \phi(n) \) and \( \bar{\phi}(n) \), we obtain the expression

\[
\langle x|\hat{I}_c|x'\rangle = \sum_{n=0}^{\infty} \frac{1}{N_n} \phi(n)(x) \bar{\phi}(n)(x')^*
\]

\[
= \sqrt{\frac{2\pi}{i}} \int \frac{m\omega}{2\pi k^2} \sum_{n=0}^{\infty} \frac{1}{n!} \left( e^{-\frac{i}{\hbar} \tilde{A}^{\alpha} x^\alpha} \right)^n e^{i \frac{m\omega}{2\pi} x^2} e^{i \frac{m\omega}{2\pi} x'^2}
\]

\[
= \sqrt{\frac{2\pi}{i}} \int \frac{m\omega}{2\pi k^2} e^{-\frac{i}{\hbar} \tilde{A}^{\alpha} x^\alpha} e^{i \frac{m\omega}{2\pi} x^2} e^{i \frac{m\omega}{2\pi} x'^2}
\]

\[
= \sqrt{\frac{2\pi}{i}} \int \frac{m\omega}{2\pi k^2} \frac{i}{\hbar} \tilde{A}^{\alpha} x^\alpha \frac{m\omega}{2\pi} x
\]

\[
\left. \sqrt{\left( A^{\alpha} \right)^2} \right| = 0, \text{ with } \hat{a} = e^{-\frac{i}{\hbar} \tilde{A}^{\alpha} x^\alpha} \sqrt{\frac{\hbar}{2m\omega}} \hat{a}. \text{ Remembering, further,}
\]

\[
e^{\hat{a} + \hat{b}} = e^{\hat{a}} e^{\hat{b}} e^{-\frac{i}{2}[\hat{a}, \hat{b}]} \text{ for } [[\hat{a}, \hat{b}], \hat{a}] = [[\hat{a}, \hat{b}], \hat{b}] = 0, \text{ with } \hat{a} = e^{-\frac{i}{\hbar} \tilde{A}^{\alpha} x^\alpha} \sqrt{\frac{\hbar}{2m\omega}} x
\]

\[
\text{and using again } e^{\hat{a} + \hat{b}} = e^{\hat{a}} e^{\hat{b}} e^{-\frac{i}{2}[\hat{a}, \hat{b}], \hat{b}} \text{, we arrive at}
\]

\[
\langle x|\hat{I}_c|x'\rangle = \sqrt{\frac{m\omega}{2\pi \hbar^2}} \int_{-\infty}^{\infty} dk e^{-\frac{i}{\hbar} k^2} e^{-i \left( \sqrt{\frac{m\omega}{2\pi \hbar^2}} x - k \right)^2} e^{i \frac{m\omega}{2\pi \hbar^2} x'^2}
\]

\[
= \sqrt{\frac{m\omega}{2\pi \hbar^2}} \int_{-\infty}^{\infty} dk e^{-\frac{i}{\hbar} k^2} e^{-i \left( \sqrt{\frac{m\omega}{2\pi \hbar^2}} x - k \right)^2} x' \left( \sqrt{\frac{m\omega}{2\pi \hbar^2}} x - k \right)^2
\]

\[
\times e^{i \frac{m\omega}{2\pi \hbar^2} \left( x' - \sqrt{\frac{m\omega}{2\pi \hbar^2}} x - k \right) \sqrt{\frac{m\omega}{2\pi \hbar^2}}}
\]

\[
= \sqrt{\frac{m\omega}{2\pi \hbar^2}} e^{i \frac{m\omega}{2\pi \hbar^2} x^2} \times e^{-i \frac{m\omega}{2\pi \hbar^2} x x' + i \frac{m\omega}{2\pi \hbar^2} x^2 + i \frac{m\omega}{2\pi \hbar^2} x'^2} \int_{-\infty}^{\infty} dk e^{i k \sqrt{\frac{m\omega}{2\pi \hbar^2}} (x - x')}
\]

\[
= \delta(x - x').
\]

Therefore, \( \hat{I}_c \) is nothing but the unit operator for the present r.h.o. system.

### C The Schwinger effect for fermions

The action of the Dirac field \( \Psi \) obeying the eq.(3.20) with the gauge fields \( A^\alpha_\mu \) is \( S_D[\Psi, A_\mu] = \int d^4x \bar{\Psi} \left( \gamma \cdot \Pi(A_\mu) + mc \right) \Psi, \left( \bar{\Psi} = \Psi^\dagger \gamma^0 \right). \) Then the path integral result \( S_Q[A_\mu] = -i\hbar \log \int D\Psi \bar{\Psi} e^{\frac{i}{\hbar} S_D} = \)
\[-i\hbar \text{Tr} \log(\gamma \cdot \hat{\Pi} + mc) + \text{const.} \text{ is the quantum correction to } S_G[A_c] = \frac{1}{2} \int d^4x E^2 \text{ so that } S_{\text{eff}}[A_c] \equiv S_G[A_c] + S_Q[A_c] \text{ becomes the effective action of } A_c. \text{ Here, the } "\text{Tr}" \text{ involves the trace over 4-component spinor space. To evaluate the } "\text{Tr}" \text{ in } S_Q[A_c], \text{ we take notice that}
\]
\[
\frac{d}{da} \text{Tr} \log \left[ \gamma \cdot \hat{\Pi} + (mc) + a \right] = \text{Tr} \frac{-i}{\hbar} \left[ (mc) + a \right] + \text{Tr} \frac{-i}{\hbar} \left[ (mc) + a \right] = \text{Tr} \frac{-i}{\hbar} \left[ (mc) + a \right] = \text{Tr} \frac{-i}{\hbar} \left[ (mc) + a \right] - (\gamma \cdot \Pi)^2 + \left( (mc) + a \right)^2 = \frac{d}{da} \left( \frac{1}{2} \text{Tr} \int_0^\infty \frac{dr}{\tau} e^{-ir[(\gamma \cdot \Pi)^2+(mc) + a] - (mc) + a} \right), \tag{C.1} \]
\[
\text{in consideration of that the trace of odd powers of } \gamma \text{ matrices vanishes. Integrating this equation with respect to } a \text{ from } a_1 \text{ to } a_2, \text{ we obtain}
\]
\[
\text{Tr} \log \left[ \gamma \cdot \hat{\Pi} + (mc) + a_2 \right] - \text{Tr} \log \left[ \gamma \cdot \hat{\Pi} + (mc) + a_1 \right] = \frac{1}{2} \text{Tr} \int_0^\infty \frac{dr}{\tau} e^{ir[(\gamma \cdot \Pi)^2 + (mc) + a] - (mc) + a} \left[ e^{ir((mc) + a_2)^2} - e^{ir((mc) + a_1)^2} \right]. \tag{C.2} \]
\[
\text{Setting } a_2 = 0 \text{ and } a_1 = -(\gamma \cdot \hat{\Pi} + mc) + 1, \text{ we get the expression}
\]
\[
\text{Tr} \log(\gamma \cdot \hat{\Pi} + mc) = \frac{1}{2} \times 4\text{tr} \int_0^\infty \frac{dr}{\tau} e^{-ir(\gamma \cdot \Pi)^2} e^{-ir((mc)^2 - i\epsilon)} \cos \left( \frac{h \xi E}{\epsilon} \right) \tag{C.4} \]
\[
\text{by virtue of the trace of } \sigma^{01} \text{ vanishes. Here, the } "\text{tr}" \text{ denotes the trace in the functional space, which yields } \text{tr} e^{-ir\hat{\Pi}} = \delta(0) V_0 V_L \left( \frac{1}{4\pi i\hbar^2} \frac{1}{2 \sinh(\tau \hbar \omega)} \right) \text{ with } \omega = \frac{|E| E}{mc} \text{ as in the case of scalar QED. Therefore, we arrive at the expression}
\]
\[
\text{Tr} \log(\gamma \cdot \hat{\Pi} + mc) = -\delta(0) V_0 V_L \left( \frac{mc}{4\pi i\hbar^2} \right)^2 \int_0^\infty \frac{dz}{z^2} \int_0^\infty e^{-iz} \sinh(\tau \hbar \omega) \right) (\epsilon = +0), \tag{C.5} \]
\[
\text{where } z = \tau (mc)^2. \text{ The integration with respect to } z \text{ in eq.}(C.5) \text{ can be carried out in the same manner as eq.(3.9) except replacing the residue } (-1)^n \text{ of } 1/\sinh(z\hbar \omega/mc^2) \text{ by } 1/\tanh(z\hbar \omega/mc^2). \text{ The resultant formula corresponding to eq.(3.11) in the case of Dirac fields becomes}
\]
\[
\frac{1}{\hbar} \text{Im } S_{\text{eff}}[A_c] = -\text{Re } \text{Tr} \log \left[ \gamma \cdot \hat{\Pi}(A_c) + mc \right] \approx V_4 \left( \frac{m}{8\pi^3 \hbar^2} \right)^2 \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{\pi^2}{2} \left( \frac{mc}{m} \right)^2 \left( \frac{m}{mc} \right)^2. \tag{C.6} \]
\[
\text{This formula is nothing but the one given originally by Schwinger[13].}
\]

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