An inhomogeneous perturbation for a class of nonlinear scalar field equations

Xin Sun¹, Yu Duan¹ and Jiu Liu²,3*  
*Correspondence: jiuliu2011@163.com
²School of Mathematics and Statistics, Qiannan Normal University for Nationalities, Duyun, Guizhou 558000, People’s Republic of China
³Key Laboratory of Complex Systems and Intelligent Computing of Qiannan, Duyun, Guizhou 558000, People’s Republic of China
Full list of author information is available at the end of the article

Abstract  
This paper deals with a class of nonlinear scalar field equations with an inhomogeneous perturbation. Two positive solutions were obtained using the variational methods.  
MSC: 35J20; 35B20; 35B09
Keywords: Nonlinear scalar field equations; Inhomogeneous perturbation; Variational methods; Positive solution

1 Introduction and main result  
In this paper, we consider the following nonlinear scalar field equations with an inhomogeneous perturbation

\[-\Delta u = g(u) + h(x), \quad u \in H^1(\mathbb{R}^N), N \geq 3. \tag{1.1}\]

Actually, \( g \) satisfies the Berestycki–Lions conditions:

\( (g_1) \) \( g \in C(\mathbb{R}, \mathbb{R}); \)
\( (g_2) \) \( -\infty < \lim \inf_{s \to 0^+} \frac{g(s)}{s} \leq \lim \sup_{s \to 0^+} \frac{g(s)}{s} = -m < 0; \)
\( (g_3) \) \( \lim_{s \to +\infty} \frac{g(s)}{s^{2^* - 1}} = 0, \) where \( 2^* = \frac{2N}{N-2}; \)
\( (g_4) \) there exists \( \xi > 0 \) such that \( G(\xi) := \int_0^\xi g(\tau) \, d\tau > 0; \)

and \( h \) satisfies
\( (h_1) \) there exists \( p \in [\frac{2N}{N-2}, 2] \) such that \( h \in L^p(\mathbb{R}^N); \)
\( (h_2) \) \( h \) is nonnegative and \( h \not\equiv 0; \)
\( (h_3) \) \( h \) is radially symmetric;
\( (h_4) \) \( (\nabla h, x) \in L^{\frac{2N}{N+2}}(\mathbb{R}^N), \) where \( (\cdot, \cdot) \) denotes scalar product in \( \mathbb{R}^N. \)

When \( h \equiv 0, \) Eq. (1.1) reduces to the following nonlinear scalar field equations

\[-\Delta u = g(u), \quad u \in H^1(\mathbb{R}^N), N \geq 3. \tag{1.2}\]

Equation (1.2) possesses strong physical background as introduced in [1, 6] and has been extensively studied, for example, in [3, 5, 10, 12]. Especially in [3], Berestycki and Lions gave nearly optimal conditions known as the Berestycki–Lions conditions.

© The Author(s) 2022. This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.
In [2, 8, 9], the authors studied Eq. (1.2) with a homogeneous perturbation and obtained a positive solution using the variational method. In this paper, we consider the effect of an inhomogeneous perturbation. In other words, we investigate the existence of positive solutions of Eq. (1.1). With regard to Eq. (1.1), there are some results, for example, [4, 14, 15]. Compared with those results, in the present paper, the nonlinearity $g$ is almost optimal.

Set $|\cdot|_s = (\int_{\mathbb{R}^N} |\cdot|^s \, dx)^{\frac{1}{s}}$. Using the variational method, we get the following

**Theorem 1.1** Suppose that $(g_1)$–$(g_3)$ and $(h_1)$–$(h_3)$ hold. Then there exists $\Lambda > 0$ such that when $|h|_p < \Lambda$, Eq. (1.1) has a positive solution. If we add $(g_4)$ and $(h_4)$, then when $|h|_p < \Lambda$, Eq. (1.1) has another positive solution.

We have something to say about the perturbation $h$. The assumptions $(h_1)$ and $(h_2)$ are necessary, and $(h_3)$ is to overcome the lack of compactness. Moreover, to prove the second positive solution, we need to use the Pohožaev identity, and then $(h_4)$ seems appropriate. Set $f(s) = g(s) + ms$, then Eq. (1.1) equals to the following equation

$$-\Delta u + mu = f(u) + h(x), \quad u \in H^1(\mathbb{R}^N), N \geq 3. \quad (1.3)$$

where $f$ satisfies

- $(f_1)$ $f \in C(\mathbb{R}, \mathbb{R})$;
- $(f_2)$ $-\infty < \liminf_{s \to -1} \frac{f(s)}{s} \leq \limsup_{s \to -1} \frac{f(s)}{s} = 0$;
- $(f_3)$ $\lim_{s \to -\infty} \frac{f(s)}{s} = 0$;
- $(f_4)$ there exists $\zeta > 0$ such that $F(\zeta) := \int_0^\zeta f(\tau) \, d\tau > \frac{1}{2} \zeta^2$.

We only need to prove the following

**Theorem 1.2** Suppose that $(f_1)$–$(f_3)$ and $(h_1)$–$(h_3)$ hold. Then there exists $\Lambda > 0$ such that when $|h|_p < \Lambda$, Eq. (1.3) has a positive solution. If we add $(f_4)$ and $(h_4)$, then when $|h|_p < \Lambda$, Eq. (1.3) has another positive solution.

**Remark 1.3** (i) $f$ can be sign-changing. (ii) There exist some functions that satisfy $(h_1)$–$(h_4)$. For example,

$$h_1(x) = \frac{\Lambda}{2\sqrt{\omega_N}(1 + |x|^N)}, \quad h_2(x) = \frac{\Lambda e^{-|x|}}{\sqrt{N\omega_N[1 + (N + 1)!]}},$$

where $\omega_N$ denotes the volume of the unit ball in $\mathbb{R}^N$. By computing, we have $h_i \in L^2(\mathbb{R}^N)$, $(\nabla h_i, x) \in L^{\frac{2N}{N+2}}(\mathbb{R}^N)$ and $|h_i|_2 < \Lambda$, $i = 1, 2$.

The rest of the paper is organized as follows: In Sect. 2, we introduce some preliminaries. In Sect. 3, we give the proof of the first positive solution. Section 4 is devoted to obtaining the second positive solution.

**2 Preliminaries**

From now on, $C, C_1, C_2, \ldots$, denotes various positive constant, $u^\pm = \max\{\pm u, 0\}$ and $(H, \| \cdot \|)$ is a Hilbert space, where

$$H = \{ u \in H^1(\mathbb{R}^N) : u(x) = u(|x|) \}, \quad \| \cdot \| = \left( \int_{\mathbb{R}^N} (|\nabla \cdot |^2 + m |\cdot|^2) \, dx \right)^{\frac{1}{2}}.$$
To ensure the positivity of solutions and for simplicity, we always take \( f(s) = 0 \) for all \( s \leq 0 \). As is well known, the solutions of Eq. (1.3) correspond to the critical points of the following energy functional

\[
I(u) = \frac{1}{2} \| u \|^2 - \int_{\mathbb{R}^N} F(u) \, dx - \int_{\mathbb{R}^N} h(x)u \, dx.
\]

By Principle of symmetric criticality [13], we know that if \( u \) is a critical point of \( I \) restricted to \( H \), then \( u \) is a critical point of \( I \) in \( H^1(\mathbb{R}^N) \). Set

\[
F_1(s) = \int_0^s f'(t) \, dt \quad \text{and} \quad F_2(s) = \int_0^1 f'(t) \, dt,
\]

then \( F_1(s) \geq 0, F_2(s) \geq 0, F(s) = F_1(s) - F_2(s) \) for all \( s \in \mathbb{R} \),

\[
I(u) = \frac{1}{2} \| u \|^2 + \int_{\mathbb{R}^N} F_2(u) \, dx - \int_{\mathbb{R}^N} F_1(u) \, dx - \int_{\mathbb{R}^N} h(x)u \, dx
\]

and by \((f_1)-(f_3)\), we have

\[
\lim_{s \rightarrow 0^+} \frac{f'(s)}{s} = \lim_{s \rightarrow +\infty} \frac{f'(s)}{s^{2^* - 1}} = 0. \tag{2.1}
\]

### 3 The first positive solution of Eq. (1.3)

In this section, we prove that Eq. (1.3) has a local minimal solution.

**Lemma 3.1** Suppose that \((f_1)-(f_3)\) and \((h_1)\) hold. Then there exist \( \rho > 0, \Lambda > 0, \alpha > 0 \) such that when \( |h|_p < \Lambda, I(u) \geq \alpha \) for all \( \| u \| = \rho \).

**Proof** From (2.1), it follows that

\[
F_1(s) \leq \frac{m}{4} |s|^2 + C_1 |s|^{2^*} \quad \text{for all} \quad s \in \mathbb{R}.
\]

Combining with Hölder’s inequality and Sobolev’s inequality, we get

\[
I(u) \geq \frac{1}{2} \| u \|^2 - \frac{m}{4} \int_{\mathbb{R}^N} u^2 \, dx - C_1 \int_{\mathbb{R}^N} |u|^{2^*} \, dx - |h|_p |u|_{\frac{2^*}{2}} - (C_2 \| u \| - C_3 \| u \|^{2^* - 1} - C_4 |h|_p) \| u \|.
\]

Define \( k(t) = C_2 t - C_3 t^{2^* - 1} \) for \( t > 0 \), then there exists \( \rho > 0 \) such that \( k(t) \) is increasing in \([0, \rho]\), \( k(t) \) is decreasing in \([\rho, +\infty)\), and \( k(\rho) = \max_{t > 0} k(t) \). Hence when \( |h|_p < \Lambda := \frac{k(\rho)}{C_4} \), we have \( I(u) \geq \alpha := [k(\rho) - C_4 |h|_p] \rho \) for all \( \| u \| = \rho \). \( \square \)

Define \( \overline{B}_\rho = \{ u \in H : \| u \| \leq \rho \} \) and \( m = \inf_{u \in \overline{B}_\rho} I(u) \), then we have

**Lemma 3.2** Suppose that \((f_1)-(f_3)\) and \((h_1)-(h_2)\) hold. Then \( m \in (-\infty, 0) \).

**Proof** It follows from \((f_1)-(f_3)\) that there exist \( M > 0 \) and \( \theta > 0 \) such that

\[
F(s) \geq -Ms^2 \quad \text{for all} \quad s \in [0, \theta).
\]
By \((h_3)\), there exist \(L \in (0, \theta)\) and \(\varphi \in H\) such that \(\int_{\mathbb{R}^N} h \varphi \, dx > 0\) and \(0 \leq \varphi(x) \leq L\) for all \(x \in \mathbb{R}^N\). Then we have
\[
\lim_{t \to 0^+} \frac{I(t \varphi)}{t} \leq \lim_{t \to 0^+} \left[ t \frac{\| \varphi \|^2}{2} + Mt \int_{\mathbb{R}^N} \varphi^2 \, dx - \int_{\mathbb{R}^N} h \varphi \, dx \right] = -\int_{\mathbb{R}^N} h \varphi \, dx < 0,
\]
which implies that there exists \(t_0 > 0\) such that \(t_0 \varphi \leq \rho\) and \(I(t_0 \varphi) < 0\). Hence \(m < 0\). It is obvious that \(m > -\infty\). \qed

**Lemma 3.3** Suppose that \((f_1)-(f_3)\) and \((h_1)-(h_3)\) hold. Then \(m\) is achieved.

**Proof** By the definition of \(m\), there exists a sequence \(\{u_n\} \subset H\) such that \(\|u_n\| \leq \rho\) and \(I(u_n) = m + o(1)\). Then there exists \(u \in H\) such that up to a subsequence,
\[
\begin{align*}
&u_n \rightharpoonup u, \quad \text{in } H, \\
&u_n \to u, \quad \text{in } L^{2^*_s}_{\text{loc}}(\mathbb{R}^N), 2 < s < 2^*, \\
&u_n(x) \to u(x), \quad \text{a.e. in } \mathbb{R}^N.
\end{align*}
\tag{3.1}
\]

The weakly lower semicontinuity of the norm infers
\[
\|u\| \leq \liminf_{n \to \infty} \|u_n\|. \tag{3.2}
\]

Thus \(\|u\| \leq \rho\). Fatou’s lemma [11] and Strauss’s compactness lemma [3] yield
\[
\int_{\mathbb{R}^N} F_2(u) \, dx \leq \liminf_{n \to \infty} \int_{\mathbb{R}^N} F_2(u_n) \, dx \tag{3.3}
\]
and
\[
\int_{\mathbb{R}^N} F_1(u_n) \, dx = \int_{\mathbb{R}^N} F_1(u) \, dx + o(1). \tag{3.4}
\]

Since \((h_1)\) holds,
\[
\int_{\mathbb{R}^N} hu_n \, dx = \int_{\mathbb{R}^N} hu \, dx + o(1). \tag{3.5}
\]

By (3.2)–(3.5), we get
\[
\begin{align*}
m &= \liminf_{n \to \infty} I(u_n) \\
&\geq \frac{1}{2} \|u\|^2 + \int_{\mathbb{R}^N} F_2(u) \, dx - \int_{\mathbb{R}^N} F_1(u) \, dx - \int_{\mathbb{R}^N} h(x)u \, dx \\
&= I(u) \\
&\geq m.
\end{align*}
\]

Hence \(I(u) = m\). \qed
The proof of the first positive solution \( \) From Lemma 3.3, there exists \( u \in H \) such that \( \| u \| \leq \rho \) and \( I(u) = m \). Lemma 3.1 infers \( \| u \| < \rho \). Thus for any \( v \in H \),
\[
\langle I'(u), v \rangle = \lim_{t \to 0^+} \frac{I(u + tv) - I(u)}{t} \geq 0
\]
and
\[
\langle I'(u), v \rangle = \lim_{t \to 0^-} \frac{I(u + tv) - I(u)}{t} \leq 0,
\]
which imply \( I'(u) = 0 \). By \( \langle I'(u), u^- \rangle = 0 \), we know \( u^- = 0 \). The strong maximum principle deduces \( u > 0 \) in \( \mathbb{R}^N \).
\( \square \)

4 The second positive solution of Eq. (1.3) \( \) In this section, we prove that Eq. (1.3) has another positive solution. In order to obtain a bounded Palais–Smale sequence, we use the following Jeanjean's theorem [7].

**Theorem 4.1** Let \( X \) be a Banach space equipped with a norm \( \| \cdot \|_X \) and let \( J \subset \mathbb{R}^+ \) be an interval. We consider a family \( \{ \Phi_\mu \}_{\mu \in J} \) of \( C^1 \)-functionals on \( X \) of the form
\[
\Phi_\mu(u) = A(u) - \mu B(u), \quad \forall \mu \in J,
\]
where \( B(u) \geq 0 \) for all \( u \in X \) and such that either \( A(u) \to +\infty \) or \( B(u) \to +\infty \) as \( \| u \|_X \to +\infty \). We assume that there are two points \( v_1, v_2 \) in \( X \) such that
\[
c_\mu = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \Phi_\mu(\gamma(t)) > \max \{ \Phi_\mu(v_1), \Phi_\mu(v_2) \},
\]
where
\[
\Gamma = \{ \gamma \in C([0,1],X) : \gamma(0) = v_1, \gamma(1) = v_2 \}.
\]
Then for almost every \( \mu \in J \), there is a sequence \( \{ u_\mu \} \subset X \) such that
(i) \( \{ u_\mu \} \) is bounded in \( X \),
(ii) \( \Phi_\mu(u_\mu) \to c_\mu \) and
(iii) \( \Phi'_\mu(u_\mu) \to 0 \) in the dual \( X^* \) of \( X \).
Moreover, the map \( \mu \to c_\mu \) is non-increasing and continuous from the left.

From [3], we know that Eq. (1.2) has a positive ground state solution \( \omega \in H \) and
\[
\int_{\mathbb{R}^N} F_1(\omega) \, dx - \int_{\mathbb{R}^N} F_2(\omega) \, dx - \frac{m}{2} \int_{\mathbb{R}^N} \omega^2 \, dx > 0.
\]
Then there exists \( \delta \in (0,1) \) such that
\[
\delta \int_{\mathbb{R}^N} F_1(\omega) \, dx - \int_{\mathbb{R}^N} F_2(\omega) \, dx - \frac{m}{2} \int_{\mathbb{R}^N} \omega^2 \, dx > 0.
\]
(4.1)

In Theorem 4.1, we set
\[
X = H, \quad \| \cdot \|_X = \| \cdot \|, \quad \Phi_\mu = I_\mu, \quad J = [\delta, 1], \quad B(u) = \int_{\mathbb{R}^N} F_1(u) \, dx.
and
\[ A(u) = \frac{1}{2} \|u\|^2 + \int_{\mathbb{R}^N} F_2(u) \, dx - \int_{\mathbb{R}^N} hu \, dx. \]

By Hölder’s inequality and Sobolev’s inequality, we have
\[ \frac{A(u)}{\|u\|} \geq \frac{1}{2} \|u\| - \int_{\mathbb{R}^N} hu \, dx \geq \frac{1}{2} \|u\| - C|h|_p \to +\infty \quad \text{as} \quad \|u\| \to +\infty, \]
which implies \( A(u) \to +\infty \) as \( \|u\| \to +\infty \). Note that
\[ I_{\mu}(u) = A(u) - \mu B(u) = \frac{1}{2} \|u\|^2 + \int_{\mathbb{R}^N} F_2(u) \, dx - \mu \int_{\mathbb{R}^N} F_1(u) \, dx - \int_{\mathbb{R}^N} hu \, dx. \]

In the following, \( I_1 \) will always replace \( I \). The next lemma is to verify the assumptions of Theorem 4.1.

**Lemma 4.2** Suppose that \((f_1)-(f_4)\) and \((h_1)-(h_2)\) hold. Then when \( |h|_p < \Lambda \), there exist \( v_1, v_2 \in E \) such that for any \( \mu \in J \), \( c_\mu \geq \alpha > \max\{I_{\mu}(v_1), I_{\mu}(v_2)\} \), where \( \Lambda, \alpha \) are from Lemma 3.1.

**Proof** From Lemma 3.1, it follows that for any \( \mu \in J \), \( I_{\mu}(u) \geq I_1(u) \geq \alpha \) for all \( \|u\| = \rho \). Define
\[ \omega_t(x) = \begin{cases} 0, & t = 0, \\ \omega(t^{-1}x), & t > 0, \end{cases} \]
where \( \omega \) satisfies (4.1). For any \( \mu \in J \), one has
\[ I_{\mu}(\omega_t) \leq \frac{t^{N-2}}{2} \int_{\mathbb{R}^N} |\nabla \omega|^2 \, dx - t^N \int_{\mathbb{R}^N} \left[ \delta F_1(\omega) - F_2(\omega) - \frac{m}{2} \omega^2 \right] \, dx. \]
Note that
\[ \|\omega_t\|^2 = t^{N-2} \int_{\mathbb{R}^N} |\nabla \omega|^2 \, dx + mt^N \int_{\mathbb{R}^N} |\omega|^2 \, dx. \]
Thus, there exists \( t_0 > 0 \) such that \( \|\omega_t\| > \rho \) and \( I_{\mu}(\omega_{t_0}) < 0 \). Set \( v_1 = 0, v_2 = \omega_{t_0} \). Hence for any \( \gamma \in \Gamma \), \( \max_{t \in [0,1]} I_{\mu}(\gamma(t)) \geq \alpha > 0 \). Consequently, \( c_\mu \geq \alpha > 0 = \max\{I_{\mu}(v_1), I_{\mu}(v_2)\} \). \( \square \)

From Theorem 4.1, we know that for almost every \( \mu \in J \), there is a sequence \( \{u_n\} \subset H \) such that
\begin{enumerate}
  \item \( \{u_n\} \) is bounded in \( H \),
  \item \( I_{\mu}(u_n) \to c_\mu \), \hspace{1cm} (4.2)
  \item \( I_{\mu}'(u_n) \to 0 \) in the dual \( H^* \) of \( H \).
\end{enumerate}

Moreover, the map \( \mu \to c_\mu \) is non-increasing and continuous from the left.
Lemma 4.3 Fix $\mu \in J$. Suppose that $(f_1) - (f_4)$ and $(h_1) - (h_3)$ hold. Assume that $\{u_n\} \subset H$ satisfies (4.2). Then there exists a positive function $u \in H$ such that $I_\mu(u) = c_\mu$ and $I'_\mu(u) = 0$.

**Proof** Since $\{u_n\} \subset H$ satisfies (4.2), there exists $u \in H$ such that up to a subsequence, (3.1)–(3.5) hold, and Fatou’s lemma [11] and Strauss’s compactness lemma [3] yield

$$\int_{\mathbb{R}^N} f^-(u) u \, dx \leq \liminf_{n \to \infty} \int_{\mathbb{R}^N} f^-(u_n) u_n \, dx \quad (4.3)$$

and

$$\int_{\mathbb{R}^N} f^+(u_n) u_n \, dx = \int_{\mathbb{R}^N} f^+(u) u \, dx + o(1). \quad (4.4)$$

Obviously, $I'_\mu(u) = 0$. That is

$$\|u\|^2 - \mu \int_{\mathbb{R}^N} f^+(u) u \, dx + \int_{\mathbb{R}^N} f^-(u) u \, dx - \int_{\mathbb{R}^N} h u \, dx = 0. \quad (4.5)$$

Note that

$$0 = \liminf_{n \to \infty} \langle I'_\mu(u_n), u_n \rangle \geq \liminf_{n \to \infty} \|u_n\|^2 - \limsup_{n \to \infty} \left[ \mu \int_{\mathbb{R}^N} f^+(u_n) u_n \, dx - \int_{\mathbb{R}^N} f^-(u_n) u_n \, dx + \int_{\mathbb{R}^N} h u_n \, dx \right].$$

Using (3.1)–(3.5) and (4.3)–(4.5), we obtain

$$\|u\|^2 \leq \liminf_{n \to \infty} \|u_n\|^2 \leq \mu \int_{\mathbb{R}^N} f^+(u) u \, dx - \int_{\mathbb{R}^N} f^-(u) u \, dx + \int_{\mathbb{R}^N} h u \, dx = \|u\|^2.$$  

It is easy to know that $\|u_n\| \to \|u\|$. Combining with (3.1), we get $u_n \to u$ in $H$. Therefore, $I_\mu(u) = c_\mu$ and $I'_\mu(u) = 0$. By $\langle I'_\mu(u), u^- \rangle = 0$, we know $u^- = 0$. The strong maximum principle deduces $u > 0$ in $\mathbb{R}^N$. \hfill \Box

The proof of the second positive solution We choose $\mu_n \in J$ and $\mu_n \nearrow 1$. Lemma 4.3 implies that there exists a positive sequence $\{u_{\mu_n}\} \subset H$ such that $I_{\mu_n}(u_{\mu_n}) = c_{\mu_n}$ and $I'_{\mu_n}(u_{\mu_n}) = 0$. Note that $(h_4)$ holds. Then we have

$$0 = \langle I'_{\mu_n}(u_{\mu_n}), u_{\mu_n} \rangle = \|u_{\mu_n}\|^2 + \int_{\mathbb{R}^N} f^-(u_{\mu_n}) u_{\mu_n} \, dx - \mu_n \int_{\mathbb{R}^N} f^+(u_{\mu_n}) u_{\mu_n} \, dx - \int_{\mathbb{R}^N} h(x) u_{\mu_n} \, dx \quad (4.6)$$

and the following Pohožaev identity

$$0 = P_{\mu_n}(u_{\mu_n})$$
\[ \begin{align*}
N - 2 \int_{\mathbb{R}^N} |\nabla u_{\mu_n}|^2 \, dx &+ N \frac{m}{2} \int_{\mathbb{R}^N} u_{\mu_n}^2 \, dx + N \int_{\mathbb{R}^N} F_2(u_{\mu_n}) \, dx \\
-N \mu_n \int_{\mathbb{R}^N} F_1(u_{\mu_n}) \, dx &- N \int_{\mathbb{R}^N} hu_{\mu_n} \, dx - \int_{\mathbb{R}^N} (\nabla h, x)u_{\mu_n} \, dx.
\end{align*} \]

Combining with Hölder’s inequality and Sobolev’s inequality, we get

\[ c_1 \geq c_{\mu_n} \]

\[ = I_{\mu_n}(u_{\mu_n}) - \frac{1}{N} \int_{\mathbb{R}^N} P_{\mu_n}(u_{\mu_n}) \]

\[ = \frac{1}{N} \int_{\mathbb{R}^N} |\nabla u_{\mu_n}|^2 \, dx + \frac{1}{N} \int_{\mathbb{R}^N} (\nabla h, x)u_{\mu_n} \, dx \]

\[ \geq \frac{1}{N} \int_{\mathbb{R}^N} |\nabla u_{\mu_n}|^2 \, dx - C |(\nabla h, x)|_2 \left( \int_{\mathbb{R}^N} |\nabla u_{\mu_n}|^2 \, dx \right) \frac{1}{2}. \]

So

\[ \int_{\mathbb{R}^N} |\nabla u_{\mu_n}|^2 \, dx \leq C \quad \text{for all } n \in \mathbb{N}. \quad (4.7) \]

By (2.1), (4.6), (4.7), Hölder’s inequality, and Sobolev’s inequality, we yield

\[ m \int_{\mathbb{R}^N} u_{\mu_n}^2 \, dx \leq \|u_{\mu_n}\|^2 + \int_{\mathbb{R}^N} f^*(u_{\mu_n})u_{\mu_n} \, dx \]

\[ = \mu_n \int_{\mathbb{R}^N} f^*(u_{\mu_n})u_{\mu_n} \, dx + \int_{\mathbb{R}^N} h(x)u_{\mu_n} \, dx \]

\[ \leq m \int_{\mathbb{R}^N} u_{\mu_n}^2 \, dx + C_1 \int_{\mathbb{R}^N} u_{\mu_n}^p \, dx + C_2 |h|_p \|u_{\mu_n}\| \]

\[ \leq m \int_{\mathbb{R}^N} u_{\mu_n}^2 \, dx + C_3 + C_2 |h|_p \left( C + m \int_{\mathbb{R}^N} u_{\mu_n}^2 \, dx \right) \frac{1}{2}. \]

So

\[ \int_{\mathbb{R}^N} u_{\mu_n}^2 \, dx \leq C \quad \text{for all } n \in \mathbb{N}. \]

Hence \( \{u_{\mu_n}\} \) is bounded in \( H \). Note that \( \mu_n \nrightarrow 1 \),

\[ I_1(u_{\mu_n}) = I_{\mu_n}(u_{\mu_n}) + (\mu_n - 1) \int_{\mathbb{R}^N} F_1(u_{\mu_n}) \, dx = c_{\mu_n} + (\mu_n - 1) \int_{\mathbb{R}^N} F_1(u_{\mu_n}) \, dx \]

and

\[ \|I_1'(u_{\mu_n})\|_* = \|I_1'(u_{\mu_n}) - I_1'(u_{\mu_n})\|_* = \sup_{|v|_* = 1} \left( \mu_n - 1 \right) \int_{\mathbb{R}^N} f^*(u_{\mu_n}) v \, dx, \]

where \( \| \cdot \|_* \) denotes the norm in \( H^* \). Therefore, \( I_1(u_{\mu_n}) \rightarrow c_1 \) and \( \|I_1(u_{\mu_n})\|_* \rightarrow 0 \). According to Lemma 4.2 and Lemma 4.3, we get that there exists a positive function \( u \in H \) such that \( I_1(u) = c_1 > 0 \) and \( I_1'(u) = 0 \). \( \square \)

By Sect. 3 and Sect. 4, we complete the proof of Theorem 1.2.
Acknowledgements
The authors wish to thank the referees and the editor for their valuable comments and suggestions.

Funding
This research was supported by NNSFC (11861052), Natural Science Foundation of Education of Guizhou ([2019]065, KY[2020]144), Science and Technology Foundation of Guizhou ([2019]5653) and Funds of QNUN (QNYSTJD2018012).

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare that they have no competing interests.

Authors’ contributions
All authors contributed equally and significantly to writing this paper. All authors wrote, read, and approved the final manuscript.

Author details
1 College of Science, Guizhou University of Engineering Science, Bijie, Guizhou 551700, People’s Republic of China.
2 School of Mathematics and Statistics, Qiannan Normal University for Nationalities, Duyun, Guizhou 558000, People’s Republic of China. 3 Key Laboratory of Complex Systems and Intelligent Computing of Qiannan, Duyun, Guizhou 558000, People’s Republic of China.

Publisher’s Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 18 January 2022 Accepted: 1 April 2022 Published online: 18 April 2022

References
1. Anderson, D., Derrick, G.: Stability of time dependent particle like solutions in nonlinear field theories. J. Math. Phys. 11, 1336–1346 (1970)
2. Azzollini, A., Pomponio, A.: On the Schrödinger equation in $\mathbb{R}^n$ under the effect of a general nonlinear term. Indiana Univ. Math. J. 58(3), 1361–1378 (2009)
3. Berestycki, H., Lions, P.-L.: Nonlinear scalar field equations. I. Existence of a ground state. Arch. Ration. Mech. Anal. 82(4), 313–345 (1983)
4. Cao, D.-M., Zhou, H.-S.: Multiple positive solutions of nonhomogeneous semilinear elliptic equations in $\mathbb{R}^N$. Proc. R. Soc. Edinb., Sect. A 126(2), 443–463 (1996)
5. Coleman, S., Glaser, V., Martin, A.: Action minima among solutions to a class of Euclidean scalar field equations. Commun. Math. Phys. 58(2), 211–221 (1978)
6. Frampton, P.H.: Consequences of vacuum instability in quantum field theory. Phys. Rev. D 15(10), 2922–2928 (1977)
7. Jeanjean, L.: On the existence of bounded Palais–Smale sequences and application to a Landesman–Lazer-type problem set on $\mathbb{R}^N$. Proc. R. Soc. Edinb., Sect. A 129(4), 787–809 (1999)
8. Li, G.-D., Li, Y.-Y., Tang, C.-L.: Existence and asymptotic behavior of ground state solutions for Schrödinger equations with Hardy potential and Berestycki–Lions type conditions. J. Differ. Equ. 275, 77–115 (2021)
9. Liu, J., Liu, T., Liao, J.-F.: A perturbation of nonlinear scalar field equations. Nonlinear Anal., Real World Appl. 45, 531–541 (2019)
10. Pohožaev, S.I.: On the eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$. Dokl. Akad. Nauk SSSR 165, 36–39 (1965) (Russian)
11. Rudin, W.: Real and Complex Analysis, 3rd edn. McGraw-Hill, New York (1987)
12. Strauss, W.A.: Existence of solitary waves in higher dimensions. Commun. Math. Phys. 55(2), 149–162 (1977)
13. Willem, M.: Minimax Theorems. Progress in Nonlinear Differential Equations and Their Applications, vol. 24. Birkhäuser, Boston (1996)
14. Zhu, X.-P.: A perturbation result on positive entire solutions of a semilinear elliptic equation. J. Differ. Equ. 92(2), 163–178 (1991)
15. Zhu, X.-P., Zhou, H.-S.: Existence of multiple positive solutions of inhomogeneous semilinear elliptic problems in unbounded domains. Proc. R. Soc. Edinb., Sect. A 115(3–4), 301–318 (1990)