Grassmann–Gaussian integrals and generalized star products

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Abstract
In quantum scattering on networks, there is a nonlinear composition rule for on-shell scattering matrices which serves as a replacement for the multiplicative rule of transfer matrices valid in other physical contexts. In this paper, we show how this composition rule is obtained using Berezin integration theory with Grassmann variables.

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In memory of Al B Zamolodchikov

1. Introduction

Potential scattering for one particle Schrödinger operators on the line possesses a remarkable property concerning its (on-shell) scattering matrix given as a $2 \times 2$ matrix-valued function of the energy. Let the potential $V$ be given as the sum of two potentials $V_1$ and $V_2$ with disjoint support. Then the scattering matrix for $V$ at a given energy is obtained from the two scattering matrices for $V_1$ and $V_2$ at the same energy by a certain nonlinear, noncommutative but associative composition rule. This fact in quantum scattering theory on the line has been discovered independently by several authors (see, e.g. [1, 21, 24, 27, 28] and is an easy consequence of the multiplicative property of the transfer matrix of the Schrödinger equation (see, e.g. [17]). It has also been well known in the theory of mesoscopic systems and multichannel conductors (see, e.g. [2, 6–10, 12, 23, 31, 32]). In higher space dimensions a similar rule is unlikely to exist due to the defocusing of wave packets under propagation. However, for large separation between the supports of the potentials the scattering matrix at a given energy may asymptotically be expressed in terms of the scattering matrices for $V_1$ and $V_2$ at the same energy [15–17].
To the best of our knowledge the composition rule for $2 \times 2$ scattering matrices was first observed in the context of electric network theory by Redheffer [25, 26], who called it the star product. Now there are situations, where the concept of a transfer matrix cannot always be introduced but where the (on-shell) scattering matrix nevertheless exists. An important example is given by quantum-dynamical models on graphs, that is quasi-one-dimensional quantum systems and which are described by Schrödinger operators. Such systems are nowadays a subject of intensive study (see, e.g. [4, 11, 22] and references quoted there). In the article [18], a composition rule, called the generalized star product, was introduced and further analyzed in [19]. This composition rule extends the star product of Redheffer and allows one to obtain the on-shell scattering matrix on a given graph from the on-shell scattering matrices associated with sub-graphs. The generalized star product is defined for arbitrary matrices but for unitary matrices the outcome is also unitary.

In this paper, we provide a new way of obtaining the generalized star product. The method is based on the Grassmannian (fermionic) integration theory given by Berezin [3] and it evaluates certain Gauss–Grassmann integrals. In addition, we also show how one can arrive at the generalized star product using ordinary Gaussian (bosonic) distributions. Then, however, one has to work with a restriction, the covariances have to be positive-definite matrices.

This technique of using Gaussian integration with fermionic fields permits an action formulation of some network models. Thus for example it can be applied to the Chalker–Coddington network model [5] to describe plateau transitions in the quantum Hall effect [29]. The method is also very convenient for an investigation of models with a large number of scattering centers. In this limit, as well as at the critical point, one can give an equivalent quantum field theoretic formulation of network models [30].

The paper is organized as follows. In the following section in order to establish notation and for the convenience of the reader we briefly recall the basic notions of Berezin’s theory. Though most of the material can be found in standard text books of quantum field theory, see, e.g. [14, 33], the relations we need seem not to be so easily accessible. In section 4, we show how to obtain the generalized star product using Gauss–Grassmann integrals. In the appendix we briefly discuss the corresponding bosonic version, that is how the generalized star product can be obtained from the standard theory of Gaussian distributions.

2. Preliminaries

In this section, we first briefly review the Grassmann integration and then we recall some concepts from graph theory that we will need.

2.1. Grassmann integration

In this subsection we briefly recall the basic notions concerning Grassmann variables and the associated integration theory, see [3], and which we will need. Let $\bar{a}_i, a_i$ be Grassmann variables, which means they anti-commute

$$\bar{a}_i \bar{a}_j = -\bar{a}_j \bar{a}_i, \quad \bar{a}_i a_j = -a_j \bar{a}_i, \quad a_i a_j = -a_j a_i.$$  

We denote the associated (complex) Grassmann algebra with unit $\mathbb{1}$ by $A_f$. Elements $\alpha$ in $A_f$ have a unique representation in the form

$$\alpha = \alpha(\bar{a}, a) = \sum_{J,K \subseteq I} c_{J,K} \bar{a}_J a_K, \quad c_{J,K} \in \mathbb{C}$$  

(2.1)
with the anti-ordered and ordered products
\[ \overline{a}_j = \prod_{j \in J} a_j, \quad a_K = \prod_{j \in K} a_j, \quad J \neq \emptyset, K \neq \emptyset \]
and \( \overline{a}_\emptyset = a_\emptyset = I \). By definition
\[ \alpha(\overline{a} = 0, a = 0) = c_{J=\emptyset, K=\emptyset}. \] (2.2)

Correspondingly the subalgebra generated by the elements \( a_j, a_i \) with \( i \in J \) will be denoted by \( A_J \). In addition we introduce symbols \( d\overline{a}_i \) and \( da_i \) which anti-commute among themselves and with \( a_j, a_i \) and define the anti-ordered and ordered products
\[ d\overline{a}_J = \prod_{j \in J} da_j, \quad da_K = \prod_{j \in K} da_j. \]

(Berezin-)Integration over \( L \subseteq I \) is defined as
\[
\int d\overline{a}_J d\overline{a}_K d\overline{a}_L da_L = \begin{cases} 0 & \text{if } L \nsubseteq J \text{ or } L \nsubseteq K \\ (-1)^{|L|} \text{sign}_{J,K}^L & \text{if } L \subseteq J \text{ and } L \subseteq K \end{cases}
\] (2.3)

where \( \text{sign}_{J,K}^L = \pm 1 \) is defined when \( L \subseteq J \) and \( L \subseteq K \) and is such that
\[ \overline{a}_J a_K = \text{sign}_{J,K}^L \overline{a}_J a_K' a_L a_L \]
holds. We call \( d\overline{a}_L da_L \) the volume form associated with the index set \( L \). From definition (2.3) it is easy to see that integration may be performed in steps (Fubini’s theorem), an observation which will become important in our discussion. As a special case of (2.3) and with \( \alpha \in A_I \) written in the form (2.1)
\[
\int \alpha d\overline{a}_I da_I = (-1)^{|I|} c_{I,I}. \] (2.4)

In analogy to the Lebesgue integral over \( \mathbb{R}^n \) this integral exhibits a translation invariance in the following form. Introduce additional Grassmann variables \( \overline{b}_j, b_j \), again with the index \( i \) in \( I \) and which anti-commute with all the previous Grassmann variables. With \( \alpha(\overline{a}, a) \) as in (2.1), by \( \alpha(\overline{a} - \overline{b}, a - b) \) we understand the expression
\[
\alpha(\overline{a} - \overline{b}, a - b) = \sum_{J,K \subseteq I} c_{J,K} (\overline{a} - \overline{b})_J (a - b)_K \] (2.5)

with the anti-ordered and ordered products
\[ (\overline{a} - \overline{b})_j = \prod_{j \in J} (\overline{a}_j - \overline{b}_j), \quad (a - b)_j = \prod_{j \in K} (a - b)_j. \]

Expanding each \( (\overline{a} - \overline{b})_J (a - b)_K \) into a sum of monomials \( \overline{a}_j a_K \) it is easy to establish translation invariance of the integral in the form
\[ \int \alpha(\overline{a} - \overline{b}, a - b) d\overline{a}_I da_I = \int \alpha(\overline{a}, a) d\overline{a}_I da_I \]
or even more generally
\[ \int \alpha(\overline{a} - \overline{b}, a - b; b, b, \overline{\tau}, c) d\overline{a}_I da_I = \int \alpha(\overline{a}, a; b, \overline{\tau}, c) d\overline{a}_I da_I, \] (2.6)
valid as a relation in \( B \), the Grassmann algebra generated by \( \overline{b}_j \) and \( b_j \) and possibly additional Grassmann variables \( \overline{c}_k \) and \( c_k \). This seemingly trivial relation will also become very useful
below. Let $A$ be any complex $n \times n$ matrix and choose $I_n = \{1, 2, \ldots, n\}$ to be the index set. Set

$$\overline{a} \cdot Aa = \sum_{1 \leq i, j \leq n} \overline{a}_i A_{ij} a_j.$$ 

The Gauss–Grassmann integral can be calculated as

$$\int e^{-\pi Aa} \, d\overline{a}_I \, da_I = \det A,$$  

(2.7)

the Gaussian distribution analogue of which is relation (A.2) in the appendix. In a further analogy to ordinary Gaussian distributions, see (A.3) below, we obtain

$$\int a_i e^{-\pi Aa} \, d\overline{a}_I \, da_I = 0$$  

(2.8)

$$\int a_i a_j e^{-\pi Aa} \, d\overline{a}_I \, da_I = \det A A^{-1}_{ij}.$$ 

We need an extension of (2.7) when $\det A \neq 0$. Set

$$\overline{b} \cdot a = -a \cdot \overline{b} = \sum_{1 \leq i \leq n} \overline{b}_i a_i, \quad \overline{a} \cdot b = -b \cdot \overline{a} = \sum_{1 \leq i \leq n} \overline{a}_i b_i.$$ 

Then one can prove

$$\int e^{-\pi Aa + \overline{b} \cdot a + \overline{a} \cdot b} \, d\overline{a}_I \, da_I = \det A e^\overline{b} A^{-1} \overline{b},$$ 

(2.9)

by using the translation invariance (2.6) of the integral and by completing the square in the exponent. Finally we provide a variant of (2.9), which will become important below. For any $1 \leq p \leq n$ consider $I_p$ as a subset of $I_n$ and let $I_p^c = \{p+1, \ldots, n\}$ be its complement. Let $\overline{a}^{(1)}, a^{(1)}$ denote the set of Grassmann variables $\overline{a}_i, a_i$ with $1 \leq i \leq p$ and $\overline{a}^{(2)}, a^{(2)}$ those with $p+1 \leq i \leq n$. For a matrix $A$ as before, consider the corresponding matrix block decomposition

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}. $$ 

(2.10)

where $A_{11}$ is a $p \times p$ matrix, $A_{12}$ a $p \times (n-p)$ matrix, etc and set

$$\overline{a}^{(1)} \cdot A_{11} a^{(1)} = \sum_{1 \leq i, j \leq p} \overline{a}_i A_{ij} a_j, \quad \overline{a}^{(1)} \cdot A_{12} a^{(2)} = \sum_{1 \leq i \leq p, p+1 \leq j \leq n} \overline{a}_i A_{ij} a_j$$

$$\overline{a}^{(2)} \cdot A_{21} a^{(1)} = \sum_{p+1 \leq i \leq n, 1 \leq j \leq p} \overline{a}_i A_{ij} a_j, \quad \overline{a}^{(2)} \cdot A_{22} a^{(2)} = \sum_{p+1 \leq i, j \leq n} \overline{a}_i A_{ij} a_j$$ 

(2.11)

such that

$$\overline{a} \cdot Aa = \overline{a}^{(1)} \cdot A_{11} a^{(1)} + \overline{a}^{(1)} \cdot A_{12} a^{(2)} + \overline{a}^{(2)} \cdot A_{21} a^{(1)} + \overline{a}^{(2)} \cdot A_{22} a^{(2)}$$ 

(2.12)

holds.

**Lemma 2.1.** Let the $(n-p) \times (n-p)$ matrix $A_{22}$ be invertible and set

$$\tilde{A} = A_{11} - A_{12} A^{-1}_{22} A_{21},$$

a $p \times p$ matrix. Then

$$\int e^{-\pi Aa} \, d\overline{a}_I \, da_I = \det A_{22} \cdot e^{\overline{a}^{(1)} \cdot \tilde{A} a^{(1)}}$$ 

(2.13)
holds. Therefore the relation
\[ \det A = \det A_{22} \cdot \det \hat{A} \] (2.14)

is valid. If in addition \( \hat{A} \) is invertible, the matrix elements of its inverse are the corresponding ones for \( A^{-1} \)
\[ \hat{A}_{ij}^{-1} = A_{ij}^{-1}, \quad 1 \leq i, j \leq p. \] (2.15)

**Proof.** We use the decomposition (2.12) and complete the square to obtain
\[ e^{-\pi A a} = e^{-\pi (A_{22} A_{21} \alpha + \alpha^T A_{21}^T)} A_{22} e^{A_{12}^T A_{21}^T} e^{-\pi \alpha^T \hat{A} \alpha} \]
and (2.13) follows from (2.7) and translation invariance. Equation (2.14) in turn follows from (2.7) by integrating (2.13) out over the Grassmann variables \( \alpha_i, \alpha \), with \( i \in I_p \). As for the last claim, if \( \hat{A} \) is invertible, then so is \( A \) by (2.14) (and conversely under the present assumption that \( A_{22} \) is invertible). By the last relation in (2.8) and by (2.13) for all \( 1 \leq i, j \leq p \)
\[ \det \hat{A} \cdot \hat{A}_{ij}^{-1} = \int a_i \alpha_j e^{-\pi A a} \, da_i \, da_j = \frac{\det \hat{A}}{\det \hat{A}_{22}} \cdot \hat{A}_{ij}^{-1} \]
and (2.15) follows from (2.14). \( \square \)

**Remark 2.2.** If \( A_{22} \) is not invertible, the left-hand side of (2.13) is still well defined but not the right-hand side. Relation (2.14) also follows from the factorization of a block matrix. It involves the **Schur complement** of \( A_{22} \), which is just \( \hat{A} \), see, e.g. [13, 34]. In addition the inverse of the Schur complement enters the inverse of \( A \) as one block part and this is just relation (2.15)
\[ \left( \begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array} \right)^{-1} = \left( \begin{array}{cc} \hat{A}^{-1} & -\hat{A}^{-1} A_{12} A_{22}^{-1} \\ -A_{22}^{-1} A_{21} \hat{A}^{-1} & A_{22}^{-1} + A_{21} A_{22}^{-1} \hat{A}^{-1} A_{12} A_{22}^{-1} \end{array} \right). \] (2.16)

### 2.2. Some basic concepts from graph theory

We first recall some notions, which will be useful in the following. A finite noncompact graph is a 4-tuple \( G = (V, I, E, \partial) \), where \( V \) is a finite set of **vertices**, \( I \) is a finite set of **internal edges** and \( E \) is a finite set of **external edges**. Set
\[ n(V) = |V|, \quad n(I) = |I|, \quad n(E) = |E|, \]
the number of elements in these sets. We assume each of these sets \( V, E, I \) to be ordered in some arbitrary but fixed way. This induces an ordering in \( E \cup I \), where by definition elements in \( E \) come first. On the product set \( (E \cup I) \times V \) by definition the induced ordering \( \preceq \) is such \((i, v) \preceq (i', v')\) if and only if \( v < v' \) or \( v = v' \) and \( i \leq i' \). Elements in \( I \cup E \) are called **edges**. The map \( \partial \) assigns to each internal edge \( i \in I \) an ordered pair of (possibly equal) vertices \( \partial(i) := \{ v_1, v_2 \} \) and to each external edge \( e \in E \) a single vertex \( v \). The vertices \( v_1 =: \partial^-(i) \) and \( v_2 =: \partial^+(i) \) are called the **initial** and **final** vertices of the internal edge \( i \), respectively. The vertex \( v = \partial(e) \) is the initial vertex of the external edge \( e \). If \( \partial(i) = \{ v, v' \} \), that is, \( \partial^-(i) = \partial^+(i) \) then \( i \) is called a **tadpole**. However, to facilitate our exposition, we will exclude tadpoles. Two vertices \( v \) and \( v' \) are **adjacent**, if there is \( i \in I \) such that these vertices form \( \partial(v) \). To any \( v \in V \) we associate the set of edges terminating at \( v \), \( \partial(v) = \{ i \in E \cup I \mid v \in \partial(i) \} \).
We set \( n(v) = |I(v)| \), the number of edges terminating at \( v \). Each of these sets inherits an ordering from the ordering of \( E \cup I \). Also
\[
n(v, v') = n(v', v) = |I(v) \cap I(v')|, \quad v \neq v'
\]
is the number of (internal) edges connecting \( v \) with \( v' \), so \( n(v, v') \leq \min(n(v), n(v')) \). \( n(v, v') \) is called the connectivity matrix of the graph \( G \). \( I(v) \cap I(v') \subseteq I \) holds for \( v \neq v' \) and likewise this set inherits an ordering from the ordering of \( I \). As (unordered) sets each \( I(v) \) is a disjoint union
\[
I(v) = (I(v) \cap E) \cup_{v \neq v'} (I(v) \cap I(v')).
\]
By definition, a graph is compact if \( E = \emptyset \), otherwise it is noncompact. Throughout the whole work we will assume that the graph \( G \) is connected, that is, for any \( v, v' \in V \) there is an ordered sequence \( \{v_1 = v, v_2, \ldots, v_n = v'\} \) such that any two successive vertices in this sequence are adjacent. In particular, this implies that any vertex of the graph \( G \) has nonzero degree, i.e., for any vertex \( v \) there is at least one edge with which it is incident, \( n(v) > 0 \). In addition \( n(I) \geq n(V) - 1 \) is valid. By definition a star graph is a connected graph which has no internal edges, only one vertex and at least one external edge.

A graph can be equipped as follows with a metric structure. To any internal edge \( i \in I \) we associate an interval \([0, a_i]\) with \( a_i > 0 \) such that the initial vertex of \( i \) corresponds to \( x = 0 \) and the terminal one—to \( x = a_i \). Any external edge \( e \in E \) will be associated with a semi-line \([0, \infty)\) such that \( \partial(e) \) corresponds to \( x = 0 \). We call the number \( a_i \) the length of the internal edge \( i \). The set of lengths \( \{a_i\}_{i \in E} \), which will also be treated as an element of \( \mathbb{R}_{\geq 0}^{(E)} \), will be denoted by \( a \). A compact or noncompact graph \( G \) endowed with a metric structure is called a metric graph \((G, a)\).

3. The fermionic construction

Given a graph \( G \), we introduce the following data. To each vertex \( v \) we associate a complex \( n(v) \times n(v) \) matrix \( X(v) \) indexed by the set \( I(v) \). We call \( X(v) \) a vertex matrix. In addition, complex, invertible \( n(v, v') \times n(v, v') \) matrices \( V(v, v') \) are given for any pair \( v \neq v' \). The associated index sets are \( I(v) \cap I(v') \) and they are supposed to satisfy
\[
V(v, v') = V(v', v)^{-1}.
\]
(3.1)
\( V(v, v') \) is called a connecting matrix between \( v \) and \( v' \). We write \( X_G = \{X(v)\}_{v \in V} \) and \( V_G = \{V(v, v')\}_{v \neq v' \in V} \) for these two sets of data. Figure 1 gives a pictorial description for a graph with three vertices.

![Figure 1: A graph with three vertices, seven external and six internal edges.](image-url)
Remark 3.1. Within the context of scattering on quantum graphs, see [18, 19] $X(v)$ is the unitary scattering matrix $S(v; E)$ at energy $E > 0$ on a single-vertex graph with $n(v)$ external edges and $V(v, v')$ is a unitary diagonal matrix

$$V(v, v') = \text{diag}(e^{i\sqrt{V}u})_{u \in \mathcal{I}(v) \cap \mathcal{I}(v')},$$

(3.2)

We also introduce a total of $2(|\mathcal{E}| + 2|\mathcal{I}|)$ anti-commuting Grassmann variables

$$
\bar{\eta}_{e,v}, \eta_{e,v}, d\bar{\eta}_{e,v}, d\eta_{e,v}, \quad e \in \mathcal{E}, v = \partial(e) \\
\bar{\eta}_{i,v}, \eta_{i,v}, d\bar{\eta}_{i,v}, d\eta_{i,v}, \quad i \in \mathcal{I}, v \in \partial(i)
$$

(3.3)

and correspondingly we set

$$
\mathcal{I}(v) \cap \mathcal{E} = \{v \in \mathcal{E} ; v \in \partial(e) \} \cap \mathcal{I}(v)
$$

(3.4)

Recall that $\mathcal{I}(v) \cap \mathcal{E}$ may be non-empty, so in order to have a compact notation we have added the index $v = \partial(e)$ in the definition of $\bar{\eta}_{e,v}$. On the other hand, $\mathcal{I}(v) \cap \mathcal{I}(v') \cap \mathcal{E}$ is always empty. For further reference we write

$$(\mathcal{E} \cup \mathcal{I}) \cup \mathcal{V} = \{v, v' \in \mathcal{E} \cup \mathcal{I} \cup \mathcal{V} \subset (\mathcal{E} \cup \mathcal{I}) \times \mathcal{V}$$

(3.5)

for this index set and with the ordering induced by that of $(\mathcal{E} \cup \mathcal{I}) \times \mathcal{V}$. Set

$$
\bar{\eta} \cdot \mathcal{L}(v) \eta = \sum_{i,j \in \mathcal{E}(v)} \bar{\eta}_{e,v} X(v)_{ij} \eta_{j,v}$$

$$
\bar{\eta} \cdot \mathcal{L}(v, v') \eta = - \sum_{i,j \in \mathcal{E}(v) \cup \mathcal{E}(v')} \bar{\eta}_{e,v} X(v, v')_{ij} \eta_{j,v}
$$

(3.6)

with the convention that $\bar{\eta} \cdot \mathcal{L}(v, v') \eta = 0$ if $\mathcal{I}(v) \cap \mathcal{I}(v') = \emptyset$. Define the quadratic, fermionic Lagrangian as

$$
\bar{\eta} \cdot \mathcal{L}_0 \eta = \sum_{v} \bar{\eta} \cdot \mathcal{L}(v) \eta + \sum_{v \neq v'} \bar{\eta} \cdot \mathcal{L}(v, v') \eta.
$$

(3.7)

We decompose the set of Grassmann variables into exterior and interior variables

$$
\bar{\eta}_{\mathcal{E}} = \{\bar{\eta}_{e,v} ; e \in \mathcal{E}, v = \partial(e)\}, \quad \bar{\eta}_{\mathcal{I}} = \{\bar{\eta}_{i,v} ; i \in \mathcal{I}, v = \partial(i)\}, \quad \eta_{\mathcal{E}} = \{\eta_{e,v} ; e \in \mathcal{E}, v = \partial(e)\}, \quad \eta_{\mathcal{I}} = \{\eta_{i,v} ; i \in \mathcal{I}, v = \partial(i)\}
$$

and correspondingly we set

$$
\bar{\eta} \cdot \mathcal{L}_0 \eta = \bar{\eta}_{\mathcal{E}} \cdot \mathcal{L}_{\mathcal{E}} \eta_{\mathcal{E}} + \bar{\eta}_{\mathcal{I}} \cdot \mathcal{L}_{\mathcal{I}, \mathcal{E}} \eta_{\mathcal{I}} + \bar{\eta}_{\mathcal{E}} \cdot \mathcal{L}_{\mathcal{I}, \mathcal{E}} \eta_{\mathcal{I}} + \bar{\eta}_{\mathcal{I}} \cdot \mathcal{L}_{\mathcal{I}} \eta_{\mathcal{I}}
$$

(3.8)

In analogy to (2.10) the first line just corresponds to the following block matrix representation:

$$
\mathcal{L}_0 = \begin{pmatrix}
\mathcal{L}_{\mathcal{E}} & \mathcal{L}_{\mathcal{E}, \mathcal{I}} \\
\mathcal{L}_{\mathcal{I}, \mathcal{E}} & \mathcal{L}_{\mathcal{I}}
\end{pmatrix}
$$

(3.9)

up to a different index ordering. Provided the matrix $\mathcal{L}_{\mathcal{I}}$ is invertible, we may define its Schur complement

$$
\mathcal{K}_0 = \mathcal{L}_{\mathcal{E}} - \mathcal{L}_{\mathcal{E}, \mathcal{I}} \mathcal{L}_{\mathcal{I}}^{-1} \mathcal{L}_{\mathcal{I}, \mathcal{E}}.
$$

(3.10)
It will also be convenient to introduce the notation

$$T_G = \mathcal{L}_I$$

in order to indicate the dependence on $\mathcal{G}$.

**Theorem 3.2.** If $\mathcal{L}_I$ is invertible, then

$$\int e^{-\pi e_{\mathcal{G}} d\mathcal{I}_2 d\mathcal{I}_2} = \det T_G \cdot e^{-\pi e_{\mathcal{G}} K_G H}$$

and thus also $\det L_G = \det T_G \cdot \det K_G$ hold. In particular, if in addition $L_G$ is invertible, then $K_G$ is also invertible and the matrix elements of its inverse are those of the corresponding ones for $L_G^{-1}$ itself

$$(K_G^{-1})_{t', t=\bar{\partial}(e); \bar{e}', v'=\bar{\partial}(e')} = (L_G^{-1})_{t', t=\bar{\partial}(e); \bar{e}', v'=\bar{\partial}(e')}.$$  

**Proof.** In view of (3.8), this theorem is a direct consequence of lemma 2.1. \hfill \Box

### 3.1. The generalized star product and two-vertex graphs

In this subsection we will show, how in the case of any two-vertex graph $\mathcal{G}_2$ the above construction of $K_{\mathcal{G}_2}$ out of $L_{\mathcal{G}_2}$ is also obtained from the generalized star product introduced in [18, 19]. The vertices are denoted as $v_1$ and $v_2$. Figure 2 serves as an illustration. Let the matrix $X(v_1)$, indexed by $\mathcal{I}(v_1)$, be given in a $2 \times 2$ block form

$$X(v_1) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where $A$ is an $n_1 \times n_1$ matrix, $B$ an $n_1 \times p$ matrix, $C$ an $p \times n_1$ matrix and finally $D$ a $p \times p$ matrix. Here $n_1 = |\mathcal{E} \cap \mathcal{I}(v_1)|$ is the number of external edges $e$ terminating at $v_1$, that is $\partial(e) = v_1$. $p = n(\mathcal{I})$ the number of internal lines such that $n_1 + p = n(v_1)$. Thus for example $A$ is indexed by $\mathcal{E} \cap \mathcal{I}(v_1)$ while $D$ is indexed by $\mathcal{I} \cap \mathcal{I}(v_1)$. Similarly, write $X(v_2)$, indexed by $\mathcal{I}(v_2) = (\mathcal{I}(v_1))$, in a $2 \times 2$ block matrix form

$$X(v_2) = \begin{pmatrix} E & F \\ G & H \end{pmatrix}$$

where $H$ is a $p \times p$ matrix, $G$ is a $p \times m_1$ matrix, etc. Here $m_1 = |\mathcal{E} \cap \mathcal{I}(v_2)|$ is the number of external edges $e$ terminating at $v_2$, that is $\partial(e) = v_2$. Thus for example $E$ is indexed by $8$
$E \cap I(v_2)$. Then the $(n_1 + p + m_1 + p) \times (n_1 + p + m_1 + p)$ matrix $L_{\mathring{G}}$, which is obtained from $X(v_1)$, $X(v_2)$ and $V = V(v_1, v_2)$, takes the form

$$L_{\mathring{G}} = \begin{pmatrix} A & B & 0 & 0 \\ C & D & 0 & -V^{-1} \\ 0 & 0 & E & F \\ 0 & -V & G & H \end{pmatrix}$$

which in a canonical way is indexed by the set $(E \cup I) \ni V$. This gives

$$L_{E} = \begin{pmatrix} A & 0 \\ 0 & E \end{pmatrix}, \quad L_{E,I} = \begin{pmatrix} B & 0 \\ 0 & F \end{pmatrix}$$

$$L_{I,E} = \begin{pmatrix} C & 0 \\ 0 & G \end{pmatrix}, \quad T_{G} = L_{E} = \begin{pmatrix} D & -V^{-1} \\ -V & H \end{pmatrix}$$

where $L_{E}$ is a $|E| \times |E|$ matrix, etc. Therefore by (3.10)

$$K_{\mathring{G}} = \begin{pmatrix} A & 0 \\ 0 & E \end{pmatrix} - \begin{pmatrix} B & 0 \\ 0 & F \end{pmatrix} \begin{pmatrix} D & -V^{-1} \\ -V & H \end{pmatrix}^{-1} \begin{pmatrix} C & 0 \\ 0 & G \end{pmatrix}$$

$$= \begin{pmatrix} A - B(D - V^{-1}H^{-1}V)^{-1}C & -B(D - V^{-1}H^{-1}V)^{-1}V^{-1}H^{-1}G \\ -F(H - VD^{-1}V^{-1})^{-1}VD^{-1}C & E - F(H - VD^{-1}V^{-1})^{-1} \end{pmatrix} \begin{pmatrix} D & -V^{-1} \\ -V & H \end{pmatrix}^{-1}$$

We have used (2.16) by which

$$\begin{pmatrix} D & -V^{-1} \\ -V & H \end{pmatrix}^{-1} = \begin{pmatrix} (D - V^{-1}H^{-1}V)^{-1} & (D - V^{-1}H^{-1}V)^{-1}V^{-1}H^{-1} \\ (H - VD^{-1}V^{-1})^{-1}VD^{-1} & (H - VD^{-1}V^{-1})^{-1} \end{pmatrix}$$

and we have assumed the matrix $T_{G} = L_{E}$ to be invertible. Alternatively, by theorem 3.2, if $L_{\mathring{G}}^{-1}$ is written in $4 \times 4$ block form like $L_{G}$

$$L_{\mathring{G}}^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

then

$$K_{\mathring{G}}^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

holds. This can be checked by calculating the inverse of $L_{\mathring{G}}$, a tedious but straightforward calculation using (2.16) iteratively. Moreover we have

**Lemma 3.3.** Assume $T_{G} = L_{E}$ to be invertible, such that $K_{G}$ is well defined. Then the generalized star product $X(v) \star_{V(v,v')} X(v')$ as introduced in [18] is also well defined and both these quantities are equal.

**Proof.** In the present notation, where we recall $V = V(v, v')$

$$X(v) \star_{V(v,v')} X(v') = \begin{pmatrix} A + BK_{2}HVC & BK_{2} \\ FK_{1} & E + FK_{1}DV^{-1}G \end{pmatrix}$$

holds with

$$K_{1} = (I - VDV^{-1}H)^{-1}V = V(I - DV^{-1}HV)^{-1}$$

$$K_{2} = (I - V^{-1}HVD)^{-1}V^{-1} = V^{-1}(I - HVDV^{-1})^{-1}$$
see section 4 in [18] and section 3 in [19]. We use the relations
\[
(D - V^{-1} H^{-1} V)^{-1} = -(I - V^{-1} H V D^{-1}) V^{-1} H V = -K_2 H V
\]
\[
(H - V D^{-1} V)^{-1} = -(I - V D V^{-1} H)^{-1} V D V^{-1} = -K_1 D V^{-1},
\]
insert this into (3.18). Comparison with (3.20) gives the claim. □

3.2. The generalized star product and arbitrary graphs

We are now able to extend this comparison to the case where the graph has more than two vertices. The idea for this alternative is to carry out the integrations in (3.12) in steps while using iteratively the Grassmannian–Gaussian construction of the star product as given in the previous subsection. This proof will give a more explicit representation of the star product as given in [18] and section 3 in [19]. We use the relations
\[
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\]
\[
\begin{align*}
\mathcal{G}_l &= \mathcal{G}_{l-1} \oplus \mathcal{T}^l \\
\mathcal{T}_l &= \mathcal{I}_l \cup \mathcal{I}_l \\
\mathcal{I}_l &= \bigcup_{1 \leq k \neq k' \leq l} (\mathcal{I}(v_k) \cap \mathcal{I}(v_k'))
\end{align*}
\]
To obtain \( \mathcal{G}_{l+1} \) from \( \mathcal{G}_l \), observe there is a vertex in \( \mathcal{G}_l \), denoted \( v_{l+1} \), such that
\[
\tilde{n}(v_{l+1}) \geq \sum_{k=1}^{l} n(v_k, v_{l+1}) > 0.
\]
\( \mathcal{G}_{l+1} \) is obtained as follows. Take \( \mathcal{G}_l \) and a single-vertex graph with vertex denoted by \( v_{l+1} \) and with \( \tilde{n}(v_{l+1}) \) edges emanating. Call this graph \( \tilde{\mathcal{G}}(v_{l+1}) \). Label its edges by the elements in \( \bigcup_{k \leq l} (\mathcal{I}(v_k) \cap \mathcal{I}(v_k')) \). Glue each edge \( i \in \mathcal{I}(v_k) \cap \mathcal{I}(v_k) \) in \( \tilde{\mathcal{G}}(v_{l+1}) \) to the edge with the same index in \( \mathcal{G}_l \). In case \( \mathcal{G} \) is a metric graph with set of internal edge lengths \( \mathcal{G}_l \), give the resulting internal edge \( i \) in \( \mathcal{G}_{l+1} \) the length \( q_i \). To sum up, the resulting graph \( \tilde{\mathcal{G}}_{l+1} \) has \( n(v_k, v_{l+1}) \) edges connecting \( v_k \) with \( v_{l+1} \) in \( \mathcal{G}_{l+1} \). In total \( \tilde{\mathcal{G}}_{l+1} \) thus obtained has edges which also are of the form (3.23) with \( l \) replaced by \( l + 1 \). This concludes the inductive construction of the graphs and gives \( \tilde{\mathcal{G}} \) as \( \tilde{\mathcal{G}}_{\text{out}(V)} \).

\( \mathcal{V} \) can be viewed as a subset of \( \mathcal{V} \) and that by (3.23) \( \mathcal{E}_l \cup \mathcal{I}_l \) can be viewed as a subset of both \( \mathcal{E}_{l+1} \cup \mathcal{I}_{l+1} \) and \( \mathcal{E} \cup \mathcal{I} \). Similarly, \( \mathcal{I}_l \) can be viewed as a subset of both \( \mathcal{I}_{l+1} \) and \( \mathcal{I} \). For the
In particular, \( E_0 \) is obtained from \( G \) by cutting any internal edge, which connects any vertex \( v_k (1 \leq k \leq l) \) to any vertex \( v \) different from \( v_k (1 \leq k' \leq l) \). Any such edge is then replaced by an infinite half-line. As a consequence of this discussion

\[
\hat{I}_l \supset \{ E_l, l \geq 1 \} \supset \{ E_{l+1}, l \geq 1 \} \supset \{ V(1), (\cup (E) \cup I) \}, \quad 1 \leq l \leq n(V) - 1,
\]

which induces an ordering on these sets. In order to construct the \( K_l \) we introduce the matrices

\[
\hat{V}(v_l) = V(v_1, v_l) \oplus V(v_2, v_l) \ldots \oplus V(v_l-1, v_l), \quad l \geq 2
\]

which are invertible. Thus in the example given by figure 1 \( \hat{V}(v_{l-3}) \) is a 3 \( \times \) 3 matrix.

Set \( K_1 = X(v_1) \) and inductively

\[
K_{l+1} = K_l \star \tau_{(v_{l+1})} X(v_{l+1}).
\]

In particular, \( K_2 \) is just as given by lemma 3.3. We note that the invertibility of a certain matrix is necessary, see the discussion of \( T_{G_l} \) in section 3.1. So if the invertibility of certain matrices holds—see also below—the associativity of the generalized star product [18] gives

\[
K_l = X(v_1) \star \tau_{(v_1, v_2)} X(v_2) \star \tau_{(v_2, v_3)} X(v_3) \ldots \star \tau_{(v_{l-2}, v_{l-1})} \tau_{(v_{l-1}, v_l)} X(v_l).
\]

We now repeat this construction by performing Grassmann integration over

\[
e^{-\pi \cdot \mathcal{L}^{\tau_{(v_{l+1})}}}.
\]

in steps. In order to do this we view \( G_l \) as a single-vertex graph with a vertex denoted by \( \tilde{v}_l \) and with edges labeled by \( \tilde{E}_l \). Combine it with the single-vertex graph \( \tilde{G}(v_{l+1}) \) and with the connecting matrix given as \( V(\tilde{v}_l, v_{l+1}) = \hat{V}(v_{l+1}) \). Correspondingly we take as data for \( G_{l+1} \)

\[
\hat{X}_{l+1} = \{ K_l, X(v_{l+1}) \}, \quad \hat{V}_{l+1} = \{ V(\tilde{v}_l, v_{l+1}) = \hat{V}(v_{l+1}) \}
\]

and out of this we form the fermionic Lagrangean

\[
\tilde{\eta} \cdot \mathcal{L}_{l+1} \eta = \tilde{\eta} \cdot \mathcal{L}_{G_l} \eta + \tilde{\eta} \cdot X(v_{l+1}) \eta - \tilde{\eta} \cdot \hat{V}(v_{l+1}) \eta.
\]

In order not to burden the notation, we have not stated explicitly, which Grassmann variables out of the set (3.3) are involved. Indeed, those which are involved can be read off the index set associated with the matrices \( K_l, X(v_{l+1}) \) and \( \hat{V}(v_{l+1}) \).

With this notational convention and by lemma 3.3 we obtain

\[
\det \mathcal{T}^{l+1} \cdot e^{-\pi \cdot \mathcal{L}^{\tau_{(v_{l+1})}}} = \int e^{-\pi \cdot \mathcal{L}_{l+1} \eta} d\tilde{\eta}_{l+1} d\eta_{l+1}.
\]

where \( \mathcal{T}^{l+1} = (1)_{l+1} T_{l+1} \) in an adaption of the notation used in (3.8)—is invertible. We recall that by definition \( \mathcal{T}^l = 1 \). This concludes the recursive construction of \( G_l, K_l \) and

\[
T_l = \oplus_{k=1}^l T^k.
\]
We iterate recursion (3.32) in combination with recursion (3.31), use (3.22), (3.25) and (3.29) to obtain
\[
\det T_{I_{\text{mon}(V)}} \cdot e^{-\pi K_{I_{\text{mon}(V)}}} = \int e^{-\pi \cdot \mathcal{L}_F} \eta \, d\eta_I \, dI.
\] (3.34)

Comparison with (3.12), while keeping remark 1 in mind, gives the main result of this paper.

**Theorem 3.4.** Assume that the matrices \( T_G \) and \( T^T \) are all invertible, such that all \( T_I \) are also invertible. Then the quantities \( K_G \) and \( \det T_G \) as given by (3.12) are equal to \( K_{I_{\text{mon}(V)}} \) and \( \det T_{I_{\text{mon}(V)}} \), respectively, where \( K_{I_{\text{mon}(V)}} \) and \( T_{I_{\text{mon}(V)}} \) are defined by (3.29) and (3.33).

Without proof we state that under the assumptions stated actually \( T_G \simeq T_{I_{\text{mon}(V)}} \) holds.

**Corollary 3.5.** For a given graph \( G \) let the data \( X_G \) and \( V_G \) consist of unitaries. Under the corresponding invertibility assumptions, \( K_G \) is also unitary as are in fact all \( K_{I_{\text{mon}(V)}} \).

**Proof.** This follows from the theorem and the results in [18, 19]. □

4. Conclusions

The result presented in this article is relevant in the context of quantum scattering theory on metric graphs. So the main purpose of this article is mainly pedagogical. The aim was to show how the scattering matrix can be obtained from basic building blocks by simple and well known mathematical techniques. Indeed, in this context \( K_G \) is the scattering matrix at a fixed energy \( E \) associated with the entire metric graph \( G \). The metric enters through the connecting matrices given in the form (3.2). The other building blocks, that is the vertex matrices, are the single vertex scattering matrices at the same energy \( E \), see Remark 3.1. These scattering matrices have been used to study spin transport and conductance on quantum graphs, see [35–37]. More recently, another way of obtaining the scattering matrix by an iterative procedure, again equivalent to the generalized star product procedure, has been given in [37]. It involves the use of RT-algebras. Relation (3.36) in [20] provides yet another way to obtain this scattering matrix in terms of the single vertex scattering matrices and connecting matrices. In addition in [20] a series expansion of every matrix element is given. This expansion is indexed by so called walks \( w \) with length \( |w| \) and has expansion coefficients of the form \( \exp(i \sqrt{E} |w|) \) times a monomial in the single vertex scattering matrix elements. It has been used to formulate a new approach to the traveling salesman problem.

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Appendix A. The generalized star product and Gaussian integrals

In this appendix we give a bosonic discussion using Gaussian distributions. We start by recalling some basic facts about Gaussian distributions, also in order to establish notation and for the sake of comparison with our fermionic discussion. Let \( x = (x_1, x_2, \ldots, x_n) \) and \( y = (y_1, y_2, \ldots, y_n) \) denote elements in \( \mathbb{R}^n \) and set
\[
y \cdot x = \sum_{i=1}^n y_i x_i = x \cdot y
\]
and for integration $dx = \prod_{i=1}^{n} dx_i$ denotes the infinitesimal volume element on $\mathbb{R}^n$. Let $A$ be a real symmetric matrix, which in addition is positive (definite)

$$x \cdot Ax = \sum_{i,j=1}^{n} x_i X_{ij} x_j > 0, \quad x \neq 0$$

and then we write $A > 0$. Then also $\det A > 0$ and in addition $A^{-1}$ exists, is real, symmetric and positive. $I$ denotes the unit matrix in the given context and $A > A'$ means $x \cdot Ax > x \cdot A'x$ for all $x \neq 0$. If $\kappa \|x < \mu I$, then $\mu^{-1} \| < A^{-1} < \kappa^{-1} I$.

We make the following notational convention. Here and in what follows, whenever $x \in \mathbb{R}^n$ stands to the right of an $m \times n$ matrix $A$, then it is viewed as a column vector. The outcome $Ax$ will also be interpreted as a column vector in $\mathbb{R}^m$. When $x$ stands to the left of $A$ it will be viewed as a row vector.

Define the Gauss distribution with covariance $A > 0$ via its probability measure

$$d\mu_A(x) = \frac{(\det A)^{1/2}}{(2\pi)^{n/2}} e^{-\frac{1}{2} x' Ax} \, dx \quad (A.1)$$
on $\mathbb{R}^n$. That this is a probability measure follows from

$$\int_{x \in \mathbb{R}^n} e^{-\frac{1}{2} x' Ax} \, dx = \frac{(2\pi)^{n/2}}{(\det A)^{1/2}} \quad (A.2)$$

and is the analogue of (2.7).

The first two moments of the measure $d\mu_A(x)$ are

$$\int_{\mathbb{R}^n} x_i \, d\mu_A(x) = 0 \quad \int_{\mathbb{R}^n} x_i x_j \, d\mu_A(x) = A_{ij}^{-1} \quad (A.3)$$

which are the analogues of (2.8).

Write any $x \in \mathbb{R}^n$ as $x = (x^{(1)}, x^{(2)})$ with $x^{(1)} = (x_1, \ldots, x_p) \in \mathbb{R}^p, x^{(2)} = (x_{p+1}, \ldots, x_n) \in \mathbb{R}^{n-p}$ and set

$$x^{(1)} \cdot A_{11} x^{(1)} = \sum_{i,j \leq p} x_i A_{ij} x_j, \quad x^{(1)} \cdot A_{11} x^{(2)} = \sum_{i \leq p, p+1 \leq j} x_i A_{ij} x_j$$

$$x^{(2)} \cdot A_{21} x^{(1)} = \sum_{p+1 \leq i, j \leq p} x_i A_{ij} x_j, \quad x^{(2)} \cdot A_{22} x^{(2)} = \sum_{p+1 \leq i, j \leq p} x_i A_{ij} x_j$$

such that we have the decomposition

$$x \cdot Ax = x^{(1)} \cdot A_{11} x^{(1)} + x^{(1)} \cdot A_{12} x^{(2)} + x^{(2)} \cdot A_{21} x^{(1)} + x^{(2)} \cdot A_{22} x^{(2)}.$$ 

In other words, we use the $2 \times 2$ block decomposition (2.10) of the matrix $A$. Since $A$ is assumed to be positive definite, so are $A_{11}$ and $A_{22}$ and their inverses. Also $A_{21}$ is the transpose of $A_{12}$. So $A = A_{11} - A_{12} A_{22}^{-1} A_{21}$ is a well-defined and symmetric $p \times p$ matrix. In fact it is positive definite, see, e.g. [34]. Actually we need a stronger result.

**Lemma A.1.** If $A > \kappa \|x > 0$ holds, then also $\widehat{A} > \kappa \|$ is valid for the Schur complement of $A_{22}$.

**Proof.** Under the assumption $0 < A^{-1} < \kappa^{-1} \|$, hence also $0 < \widehat{A}^{-1} < \kappa^{-1} \|$ due to (2.16). Taking the inverse gives $A > \kappa \|$. \qed

We leave the proof of the following lemma to the reader. It is the analogue of lemma 2.1.
Lemma A.2. The following relation holds
\[
\int_{x^{(2)} \in \mathbb{R}^p} e^{-\frac{1}{2} x^A x} \, dx^{(2)} = \frac{(2\pi)^{n-p/2}}{(\det A_{22})^{1/2}} e^{-\frac{1}{2} x^{(4)} \hat{A} x^{(4)}}. \tag{A.4}
\]
Since \(\hat{A}\) is positive definite, we may integrate (A.4) over \(x^{(1)}\) and \(A = \det A_{22} \cdot \det \hat{A}\) follows, which is relation (2.14), however, in the restricted context of positive \(A\).

We turn to the generalized star product and a way to obtain it through Gaussian integrals. Consider \(X(v_1)\) and \(X(v_2)\) of the form (3.14) and (3.15). We use the notation employed in this context.

Proposition A.3. Assume \(X(v_1) > \kappa I\), \(X(v_2) > \kappa I\) with \(\kappa > 1\) and let \(V(v_1, v_2)\) be an orthogonal matrix. Then \(X(v_1) \ast_{V(v_1,v_2)} X(v_2) > (\kappa - 1)I\).

Proof. Let \(L_{G_v}\) be as in (3.16) but now indexed from 1 to \(n_1 + p + m_1 + p\). Also let the orthogonal matrix \(V = V(v_1, v_2) = (v_2, v_1)^{-1} = V(v_2, v_1)^T\) be indexed from 1 to \(p\).

For \(0 \neq x \in \mathbb{R}^{n_1+m_1+2p}\) by Schwarz inequality
\[
x \cdot L_{G_v} x = \sum_{1 \leq i,j \leq n_1+p} x_i X(v_1)_{ij} x_j + \sum_{n_1+p+1 \leq i,j \leq n_1+m_1+2p} x_i X(v_1)_{ij} x_j - \sum_{1 \leq i,j \leq n_1+p+1} x_i V_{ij} x_{n_1+p+m_1+j} - \sum_{1 \leq i,j \leq n_1+p+1} x_i V_{ij} x_{n_1+p+m_1+j} > (\kappa - 1)x \cdot x.
\]

But \(X(v_1) \ast_{V(v_1,v_2)} X(v_2)\) is a Schur complement by the discussion in subsection 3.1, that is \(X(v_1) \ast_{V(v_1,v_2)} X(v_2) = K_{G_v}\), and so the claim follows from lemma A.1. \(\square\)

Write \(x = (z^{(1)}, z^{(2)}) \in \mathbb{R}^{n_1+m_1+2p}\) with \(z^{(1)} = (x_1, \ldots, x_{n_1}, x_{n_1+p+1}, \ldots, x_{n_1+2p+m_1}) \in \mathbb{R}^{n_1+m_1}\) and \(z^{(2)} = (x_{n_1+1}, \ldots, x_{n_1+2p}) \in \mathbb{R}^{2p}\). Then with the notation used in section 3.1 and in the proof of the lemma we obtain

Lemma A.4. With the assumptions as in lemma A.2 the relation
\[
\int_{z^{(2)} \in \mathbb{R}^{2p}} e^{-\frac{1}{2} x^{(2)} L_{G_v} z^{(2)}} \, dz^{(2)} = \frac{(2\pi)^{p}}{(\det T_{G_v})^{1/2}} e^{-\frac{1}{2} x^{(1)}K_{G_v} x^{(1)}} \tag{A.5}
\]
holds.

We turn to an arbitrary graph \(G\) with data \(X_G\) and \(V_G\) with the property that each \(X(v)\) is positive definite and each \(V(v, v')\) is orthogonal. Introduce the variable
\[x = \{x_{i,v} \mid i,v \in E(G)\} = \{x_{i,v} \mid i \in E(G), v \in V(G)\} \subseteq \mathbb{R}^{E(G) + V(G)},\]
let the matrices \(L_G\) and \(L(G, v, v')\) be as in (3.6) and set
\[
x \cdot L_G x = \sum_{i,j \in E(G)} x_{i,v} X_G(v)_{ij} x_{j,v}\]
\[
x \cdot L(v, v') x = - \sum_{i,j \in E(G) \cap \{v,v'\}} x_{i,v} V_G(v, v')_{ij} x_{j,v}. \tag{A.6}
\]
Define the quadratic, bosonic Lagrangian as
\[
x \cdot L_G x = \sum_v x \cdot L_G x - \sum_{v \neq v'} x \cdot L_G x = \sum_{v \neq v'} x \cdot L_G x. \tag{A.7}
\]

We decompose \(x\) into its exterior and interior components, that is \(x = (x_E, x_I)\) with
\[x_E = \{x_{e,v} \mid e \in \hat{E}\}, \quad x_I = \{x_{i,v} \mid i \in E(G), v \in V(G)\}\]
and correspondingly we get
\[
x \cdot \mathcal{L}_G x = x_T \cdot \mathcal{L}_E x_T + x_T \cdot \mathcal{L}_{E,T} x_T + x_T \cdot \mathcal{L}_T x_T
\]
\[
= (x_T + (\omega^2 I_{\mathcal{E},T})^{-1} (\omega^2 I_{\mathcal{E}})^T x_T) \cdot \mathcal{L}_T (x_T + (\omega^2 I_{\mathcal{E}})^{-1} \mathcal{L}_{T,E} x_T)
\]
\[
+ x_T \cdot (\omega^2 I_{\mathcal{E}} - \omega^2 I_{\mathcal{E},T} (\omega^2 I_{\mathcal{E}})^{-1} \mathcal{L}_{T,E} x_T).
\]  
(A.8)

The following extension of lemma A.4 to arbitrary graphs is valid.

**Proposition A.5.** Given data \( X_G = \{ X(v) \}_{v \in G} \) and \( V_G = \{ V(v, v') \}_{v, v' \in V} \) with \( X(v) > (n(V) - 1)I \) and orthogonal \( V(v, v') \), the \( |\mathcal{E}| \times |\mathcal{E}| \) matrix \( K_G \) defined by

\[
\int_{x_T \in \mathbb{R}^{|\mathcal{E}|}} e^{-\frac{i}{2} x_T \omega^2} dx_T = \frac{(2\pi)^{|\mathcal{E}|}}{(\det T_G)^{1/2}} e^{-\frac{1}{2} x_T K_G x_T}
\]  
(A.9)

is positive definite.

**Proof.** We use representation (3.29) and lemma A.1 repeatedly. Observe that \( V(v_{n+1}) \) defined by (3.27) is also orthogonal.  

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and correspondingly we get
\[
x \cdot \mathcal{L}_G x = x_T \cdot \mathcal{L}_E x_T + x_T \cdot \mathcal{L}_{E,T} x_T + x_T \cdot \mathcal{L}_T x_T
\]
\[
= (x_T + (\omega^2 I_{\mathcal{E},T})^{-1} (\omega^2 I_{\mathcal{E}})^T x_T) \cdot \mathcal{L}_T (x_T + (\omega^2 I_{\mathcal{E}})^{-1} \mathcal{L}_{T,E} x_T)
\]
\[
+ x_T \cdot (\omega^2 I_{\mathcal{E}} - \omega^2 I_{\mathcal{E},T} (\omega^2 I_{\mathcal{E}})^{-1} \mathcal{L}_{T,E} x_T).
\]  
(A.8)

The following extension of lemma A.4 to arbitrary graphs is valid.

**Proposition A.5.** Given data \( X_G = \{ X(v) \}_{v \in G} \) and \( V_G = \{ V(v, v') \}_{v, v' \in V} \) with \( X(v) > (n(V) - 1)I \) and orthogonal \( V(v, v') \), the \( |\mathcal{E}| \times |\mathcal{E}| \) matrix \( K_G \) defined by

\[
\int_{x_T \in \mathbb{R}^{|\mathcal{E}|}} e^{-\frac{i}{2} x_T \omega^2} dx_T = \frac{(2\pi)^{|\mathcal{E}|}}{(\det T_G)^{1/2}} e^{-\frac{1}{2} x_T K_G x_T}
\]  
(A.9)

is positive definite.

**Proof.** We use representation (3.29) and lemma A.1 repeatedly. Observe that \( V(v_{n+1}) \) defined by (3.27) is also orthogonal.  

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