GEOMETRY OF POINTS SATISFYING CAYLEY-BACHARACH CONDITIONS 
AND APPLICATIONS

NICOLA PICOCO

Abstract. In this paper, we study the geometry of points in complex projective space that satisfy the Cayley-Bacharach condition with respect to the complete linear system of hypersurfaces of given degree. In particular, we improve a result by Lopez and Pirola and we show that, if \( k \geq 1 \) and \( \Gamma = \{ P_1, \ldots, P_d \} \subset \mathbb{P}^n \) is a set of distinct points satisfying the Cayley-Bacharach condition with respect to \( |O_{\mathbb{P}^n}(k)| \), with \( d \leq h(k - h + 3) - 1 \) and \( 3 \leq h \leq 5 \), then \( \Gamma \) lies on a curve of degree \( h - 1 \). Then we apply this result to the study of linear series on curves on smooth surfaces in \( \mathbb{P}^3 \). Moreover, we discuss correspondences with null trace on smooth hypersurfaces of \( \mathbb{P}^m \) and on codimension 2 complete intersections.

1. Introduction

Let \( D \) be a complete linear system on a smooth projective variety \( X \). We say that a set of points \( \Gamma = \{ P_1, \ldots, P_d \} \) satisfies the Cayley-Bacharach condition with respect to \( D \) if for every \( i = 1, \ldots, d \) and for any effective divisor \( D \in D \) passing through \( P_1, \ldots, \hat{P}_i, \ldots, P_d \), we have \( P_i \in D \) as well. In particular, if a finite set of points \( \Gamma \subset \mathbb{P}^n \) satisfies the Cayley-Bacharach condition with respect to the complete linear system \( |O_{\mathbb{P}^n}(k)| \) of hypersurfaces of degree \( k \), we will say that \( \Gamma \) is \( CB(k) \).

This definition of the Cayley-Bacharach condition is the modern version of a very classical property that can be found since some milestone results of ancient geometry (see [EGH] for a detailed historical background). A turning point towards a modern point of view was the celebrated theorem of Cayley and Bacharach; it asserts that if \( \Gamma = \{ P_1, \ldots, P_{de} \} \subset \mathbb{P}^2 \) is the collection of intersection points between two plane curves of degree \( d \) and \( e \) respectively, then the set \( \Gamma \) is \( CB(d + e - 3) \).

Although the Cayley-Bacharach condition is linked to classical issues, it represents a very fruitful field with many open questions and various applications. For instance, in the last decade there has been a new and growing interest for the Cayley-Bacharach property due to its applications to the study of measures of irrationality for projective varieties (cf. [LP], [B], [BCD], [BDELU] and [GK]). Furthermore, this property has been recently studied from an algebraic viewpoint in terms of 0-dimensional affine algebras over arbitrary base fields (cf. [KLR]).

In this paper we are concerned with the geometry of points \( P_1, \ldots, P_d \in \mathbb{P}^n \) satisfying the Cayley-Bacharach condition. An important result in this direction is a theorem by Lopez and Pirola that states that if \( d \leq h(k - h + 3) - 1 \) for some \( h \geq 2 \), then \( \Gamma \) lies on a curve of degree \( h - 1 \). Furthermore, they prove the same result also in \( \mathbb{P}^3 \) but only for \( 2 \leq h \leq 4 \) (cf. [LP] Lemma 2.5).

It is therefore natural to investigate whether it remains true in any \( \mathbb{P}^n \) and for all \( h \geq 2 \). More precisely, we can formulate the following question.
**Question 1.1.** Let $\Gamma$ be a set of $CB(k)$ distinct points in $\mathbb{P}^n$ such that $|\Gamma| \leq h(k - h + 3) - 1$. Does $\Gamma$ lie on a curve of degree $h - 1$ for any $h \geq 2$?

The case $h = 2$ is completely understood. Namely, in [BCD, Lemma 2.4] the authors prove that if $\Gamma \subset \mathbb{P}^n$ is a set of distinct points $CB(k)$ and $|\Gamma| \leq 2k + 1$, then $\Gamma$ lies on a line.

We deal with the cases $h = 3, 4$ and $5$ proving the following theorem which summarizes Theorems 3.1, 3.2 and 4.1.

**Theorem A.** Let $\Gamma = \{P_1, \ldots, P_d\} \subset \mathbb{P}^n$ be a set of distinct points satisfying the Cayley-Bacharach condition with respect to $|O_{\mathbb{P}^n}(k)|$, with $k \geq 1$. For any $3 \leq h \leq 5$, if
\[
d \leq h(k - h + 3) - 1
\]
then $\Gamma$ lies on a curve of degree $h - 1$.

We recall that cases $h = 3, 4, 5$ in $\mathbb{P}^2$ and cases $h = 3, 4$ in $\mathbb{P}^3$ are covered by [LP, Lemma 2.5], so Theorem [A] is an extension of this result. As a byproduct of Theorem [A] we get some improvements of results in [SU] and [LU]. Let $\Gamma$ be a set of points $CB(k)$. In [SU, Theorem 1.9] the authors prove that $\Gamma$ lies on a curve of degree at most $2$ provided that $|\Gamma| \leq \frac{5}{2}k + 1$. We note that this bound is smaller than $3k - 1$ (i.e. the bound (1.1) with $h = 3$) as soon as $k \geq 4$, so Theorem [A] does provide an improvement. Moreover, by Theorem [A1] we get a partial improvement of [LU, Theorem 1.3 (iii)] which ensures that if $\Gamma$ is $CB(k)$ and $|\Gamma|$ is bounded by a value depending on $k$ and on an integer $d$, then $\Gamma$ lies on a union of positive-dimensional linear subspace of $\mathbb{P}^n$ whose dimensions sum to $d$ (cf. Remark [5.3]).

Concerning the proof of Theorem [A] the cases $h = 3, 4$ are achieved by induction on the dimension of the ambient space $\mathbb{P}^n$, using the fact that the assertion for $n = 2, 3$ is covered by [LP, Lemma 2.5]. On the other hand, the case $h = 4$ is more complicated, because the induction argument requires to prove separately the cases $n = 3$ and $n = 4$. To this aim we extend to this setting the argument of [LP]. In particular, we first prove that $\Gamma$ lies on a reduced curve $C$ of degree at most $9$. Then we distinguish several cases (depending on the irreducible components of $C$) and combining [G, Theorem 1.5], the main result of [GLP] and properties of points $CB(k)$ (see e.g. [LU]) we show that the sum of the degrees of the irreducible components $C_i$ of $C$ such that $C_i \cap \Gamma \neq \emptyset$ is at most $4$, as wanted.

In the last section we present various applications of Theorem [A]. One of those concerns the study of linear series on curves lying on smooth surfaces in $\mathbb{P}^3$. Following the very same argument in [LP], we prove the following slight improvement of [LP, Theorem 1.5].

**Theorem B.** Let $S \subset \mathbb{P}^3$ be a smooth surface of degree $d \geq 5$, $C$ an integral curve on $S$ such that $|O_C \otimes O_S(C)|$ is base point free and $g^r_d$ a base point free special linear series on $C$ that is not composed with an involution if $r \geq 2$. If $n \leq 5d - 31$ there exists an integer $h$, with $1 \leq h \leq 4$, such that
\[
h(d - h - 1) \leq n \leq \min\{hd, (h + 1)(d - h - 2) - 1\}
\]
and the general divisor of the $g^r_d$ lies on a curve of degree $h$.

Our contribution is the improvement of the upper bound on $n$ (that previously was $n \leq 4d - 21$) allowing the case $h = 4$. The crucial point in this theorem is that the support of a general divisor
on suitable linear series on a curve that moves on a smooth surface $S$ in $\mathbb{P}^3$ satisfies the Cayley-
Bacharach condition with respect to $|K_S|$ (cf. [LP, Lemma 3.1]).

Next we present an improvement of [LP, Theorem 1.3] concerning the so-called correspondences with null trace on surfaces in $\mathbb{P}^3$ (see Subsection 5.2) and we extend it to any hypersurfaces in $\mathbb{P}^n$. It is worth noticing that correspondences with null trace on algebraic varieties have been recently discussed in [LM], together with some interesting results on their degree in the case of very general hypersurfaces. We prove the following.

**Theorem C.** Let $n \geq 3$ and let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree $d \geq n + 2$. Let $\Gamma$ a correspondence of degree $m$ with null trace on $Y \times X$. Then if $m \leq 5(d - n) - 16$, the only possible values of $m$ are $d - n + 1 \leq m \leq d$, $2(d - n) \leq m \leq 2d$, $3(d - n - 1) \leq m \leq 3d$ and $4(d - n - 2) \leq m \leq 4d$.

Actually we prove a slightly finer statement, that is, if $m \leq h(d - n - h + 2) - 1$ for $2 \leq h \leq 5$, then the only possible values of $m$ are $(s - 1)(d - n - s + 3) \leq m \leq (s - 1)d$ for $2 \leq s \leq h$ (cf. proof of Proposition 5.2). Moreover, we point out that the bound $m \geq d - n + 1$ was already obtained in [BCD, Theorem 1.2].

Finally, we give a partial extension of [LU, Theorem 1.4] about the geometry of fibers of low degree maps to projective space from certain codimension two complete intersections (cf. Corollary 5.5) by studying this problem in terms of correspondences with null trace on these varieties (cf. Proposition 5.4).

The paper is organized as follows. In Section 2 we recall some basic properties of sets of points in $\mathbb{P}^n$ that satisfies Cayley-Bacharach condition with respect to the complete linear system $|O_{\mathbb{P}^n}(k)|$ and we prove some technical results. Section 3 is devoted to prove case $h = 3$ and case $h = 4$ of Theorem A. In Section 4 we deal with the proof of case $h = 5$ of Theorem A. Finally, in Section 5, we are concerned with applications of Theorem A to linear series on curves, correspondences with null trace and plane configurations of points in $\mathbb{P}^n$.

## 2. Properties of points satisfying Cayley-Bacharach condition

We collect some useful properties of sets that satisfies the Cayley-Bacharach condition with respect to the complete linear system $|O_{\mathbb{P}^n}(k)|$ of hypersurfaces of degree $k$.

**Lemma 2.1.** Let $\Gamma = \{p_1, \ldots, p_m\} \subset \mathbb{P}^n$ be a set of distinct points that is $CB(k)$. If $\Gamma$ is non-empty, then $m \geq k + 2$.

*Proof.* See [BCD, Lemma 2.4].

**Proposition 2.2.** Let $\Gamma = \{p_1, \ldots, p_m\} \subset \mathbb{P}^n$ be a set of distinct points $CB(k)$ and let $\mathcal{P} = L_1 \cup \cdots \cup L_r$ be a union of positive-dimensional linear spaces. Then $\Gamma \setminus \mathcal{P}$, if non-empty, is $CB(k-r)$. In particular, the complement of $\Gamma$ by a single linear space is $CB(k-1)$.

*Proof.* See [LU, Proposition 2.5].

**Remark 2.3.** In the introduction we mentioned the notion of plane configuration. We will return to this topic in Subsection 5.3. Here we anticipate that a plane configuration of length $r$ is a union
$\mathcal{P} = L_1 \cup \cdots \cup L_r$ of positive-dimensional linear spaces. Therefore, the previous proposition can be restate using this terminology.

**Proposition 2.4.** Let $\Gamma = \{p_1, \ldots, p_m\} \subset \mathbb{P}^n$ be a set of distinct points $CB(k)$. If $m \leq 2k + 1$, then $\Gamma$ lies on a line.

**Proof.** See [BCD, Lemma 2.4]. □

In [SU], Stapleton and Ullery proved that following result.

**Proposition 2.5.** Let $\Gamma = \{p_1, \ldots, p_m\} \subset \mathbb{P}^n$ be a set of distinct points $CB(k)$, with $k \geq 1$. If $m \leq \frac{5}{2}k + 1$, then $\Gamma$ lies on a reduced curve of degree 2.

**Proof.** See [SU, Theorem 1.9]. □

The following proposition, that is crucial in Section 4, gives conditions under which a set of points lying on an integral curve can not be $CB(k)$ for some $k \geq 1$.

**Proposition 2.6.** Let $\Gamma = \{P_1, \ldots, P_m\} \subset \mathbb{P}^n$ be a set of points. Let $C \subset \mathbb{P}^n$ be an integral curve passing through all the points of the set $\Gamma$ and with the dualizing sheaf $\omega_C$ invertible. Let $Z = P_1 + \cdots + P_m$ and $E = P_1 + \cdots + P_{m-1}$. If $k \geq 1$ is an integer such that $H^1(C, \mathcal{O}_C(k)(-E)) = 0$ and $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) \to H^0(C, \mathcal{O}_C(k))$ is surjective, then $\Gamma$ cannot be $CB(k)$.

**Proof.** We denote $\mathbb{P}^n$ by $\mathbb{P}$. An integral curve $C$ with the dualizing sheaf invertible is Gorenstein (cf. [HI, V, §9]). Moreover, for any 0-dimensional subscheme $Z$ of length $d$ on the integral Gorenstein curve $C$, Riemann-Roch Theorem (cf. [HI89, Theorem 1.3]) ensures that

$$h^0(L(Z)) - h^1(L(Z)) = d + 1 - p_a$$

where $p_a$ is the arithmetic genus of $C$.

Thus, for $kH - Z$ and $kH - E$, we get

$$h^0(\mathcal{O}_C(k)(-Z)) - h^1(\mathcal{O}_C(k)(-Z)) = \deg(kH - Z) + 1 - p_a,$n

$$h^0(\mathcal{O}_C(k)(-E)) - h^1(\mathcal{O}_C(k)(-E)) = \deg(kH - E) + 1 - p_a.$$

By assumption, $h^1(\mathcal{O}_C(k)(-Z)) = h^1(\mathcal{O}_C(k)(-E)) = 0$ and $\deg(kH - E) > \deg(kH - Z)$. Thus $h^0(\mathcal{O}_C(k)(-Z)) \neq h^0(\mathcal{O}_C(k)(-E))$.

We claim that $H^1(\mathbb{P}, \mathcal{O}_\mathbb{P}(k)\gamma(-C)) = 0$. In order to see this, let us consider the exact sequence

$$0 \to \mathcal{O}_\mathbb{P}(-C) \to \mathcal{O}_\mathbb{P} \to \mathcal{O}_C \to 0,$$

Twisting by $k$ and passing to cohomology, we get

$$0 \to H^0(\mathbb{P}, \mathcal{O}_\mathbb{P}(k)\gamma(--C)) \to H^0(\mathbb{P}, \mathcal{O}_\mathbb{P}(k)) \to H^0(C, \mathcal{O}_C(k)) \to H^1(\mathbb{P}, \mathcal{O}_\mathbb{P}(k)(-C)) \to 0,$$

where the last element of the sequence is 0 since $H^1(\mathbb{P}, \mathcal{O}_\mathbb{P}(k)) = 0$ for any $k \geq 1$. Thus $\gamma$ is surjective and $\text{Ker} \gamma = \text{Im} \beta$. Moreover, $\beta$ is surjective by assumption and hence $\text{Ker} \gamma = H^0(C, \mathcal{O}_C(k))$. Then the claim follows.

Let us now consider the following exact sequences

$$0 \to \mathcal{O}_\mathbb{P}(-C) \to \mathcal{O}_\mathbb{P}(-Z) \to \mathcal{O}_C(-Z) \to 0,$n

$$0 \to \mathcal{O}_\mathbb{P}(-C) \to \mathcal{O}_\mathbb{P}(-E) \to \mathcal{O}_C(-E) \to 0.$$
From these, we obtain
\[ 0 \to H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(k)(-C)) \xrightarrow{\alpha_k} H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(k)(-Z)) \xrightarrow{\beta_k} H^0(C, \mathcal{O}_C(k)(-Z)) \to 0, \]
\[ 0 \to H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(k)(-C)) \xrightarrow{\alpha_k} H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(k)(-E)) \xrightarrow{\beta_k} H^0(C, \mathcal{O}_C(k)(-E)) \to 0, \]
where the last element of both sequences is \( H^1(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(k)(-C)) = 0. \)

It follows that
\[ H^0(C, \mathcal{O}_C(k)(-Z)) \cong \frac{H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(k)(-Z))}{H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(k)(-C))} \]
and
\[ H^0(C, \mathcal{O}_C(k)(-E)) \cong \frac{H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(k)(-E))}{H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(k)(-C))}. \]
Thus \( H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(k)(-Z)) \not\cong H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(k)(-E)), \) since we saw that \( h^0(\mathcal{O}_C(k)(-Z)) \neq h^0(\mathcal{O}_C(k)(-E)). \) Therefore the set \( \Gamma = \{P_1, \ldots, P_m\} \subset \mathbb{P}^n \) does not satisfy the Cayley-Bacharach condition with respect to the complete linear system \( |\mathcal{O}_{\mathbb{P}^n}(k)|. \)

The previous proposition has an abstract formulation that does not lend itself to a practical application. The next corollary achieves exactly this goal.

**Corollary 2.7.** Let \( \Gamma = \{P_1, \ldots, P_m\} \subset \mathbb{P}^n \) be a set of points and let \( C \subset \mathbb{P}^n \) be a non degenerate integral complete intersection curve passing through all the points of the set \( \Gamma. \) If
\begin{align*}
    i) \quad k > \frac{m + 2p_a - 2}{d} \quad \text{and} \quad ii) \quad k \geq d + 1 - n
\end{align*}
where \( d \) is the degree of the curve \( C \) and \( p_a \) its arithmetic genus, then \( \Gamma \) can not be \( CB(k). \)

**Proof.** If \( C \) is a complete intersection, then \( \omega_C \) is invertible (cf. \[12\] Theorem III.7.11]). Let \( Z = P_1 + \cdots + P_m \) and \( E = P_1 + \cdots + P_{m-1}. \) Condition i) implies \( H^1(C, \mathcal{O}_C(k)(-Z)) = H^1(C, \mathcal{O}_C(k)(-E)) = 0 \) (cf. \[3\] Theorem 1.5]). Condition ii) implies \( H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) \to H^0(C, \mathcal{O}_C(k)) \) is surjective (cf. \[CLP\] Corollary p.492]). So the hypotheses of Proposition 2.6 are satisfied and therefore the thesis follows.

3. Sets of points lying on curves of degree 2 and degree 3

We recall that Lopez and Pirola proved in \[LP\] Lemma 2.5 that if a set of distinct points in \( \mathbb{P}^n \) with \( n \in \{2, 3\} \) is \( CB(k) \) and has cardinality at most \( 3k - 1, \) then it lies on a curve of degree 2. If, instead, its cardinality is at most \( 4k - 5, \) then it lies on a curve of degree 3. The two theorems we are going to prove in this section generalize this result to any \( \mathbb{P}^n. \)

Let us start with the first case.

**Theorem 3.1.** Let \( \Gamma = \{p_1, \ldots, p_m\} \subset \mathbb{P}^n \) be a set of distinct points that satisfies the Cayley-Bacharach condition with respect to the complete linear system \( |\mathcal{O}_{\mathbb{P}^n}(k)|, \) with \( k \geq 1. \) If \( m \leq 3k - 1, \) then \( \Gamma \) lies on a reduced curve of degree 2.
\begin{proof}
If $k = 1$, then $m \leq 2$ and the theorem is trivial. If $k = 2$, then $m \leq \frac{5}{2} \cdot 2 + 1$ so the theorem follows from \cite[Theorem 1.9]{SU} (cf. Remark 2.4). If $n = 2$, the assertion is included in \cite[Lemma 2.5]{LP}. We can therefore suppose $k \geq 3$ and $n \geq 3$. Let us define
\[ \alpha := \text{max number of points of } \Gamma \text{ lying on a same 2-plane.} \] (3.1)

Obviously $\alpha \geq 3$. Let us fix a plane $H$ that contains $\alpha$ points of $\Gamma$. If $\alpha = m$, this is equivalent to the case $n = 2$. Hence we may assume $\alpha < m$ and we have $m - \alpha$ points of $\Gamma$ outside from $H$. Let us denote by $\Gamma'$ the set of these points. The set $\Gamma'$ is $CB(k - 1)$ by Proposition 2.2 and then $m - \alpha \geq k + 1$ by Remark 2.4. Furthermore, $m - \alpha \geq 3k - 1 - \alpha \leq 3k - 1 - 3 = 3(k - 1) - 1$ and so, by induction on $k$, $\Gamma'$ lies on a reduced curve of degree 2, i.e. either on a reduced conic or on two skew lines.

If $\Gamma'$ lies on a reduced conic, it lies on a plane. Thus, by definition of $\alpha$, it must be $m - \alpha \leq \alpha$, i.e. $\alpha \geq \frac{m}{2}$. We have $m - \alpha \leq \frac{m}{2} \leq \frac{2k - 1}{2} \leq 2(k - 1) + 1$, so $\Gamma'$ lies on a line $\ell$ (by Remark 2.4). Hence, by Lemma 2.1 $\ell$ contains at least $k + 1$ points of $\Gamma$ and so $|\Gamma \setminus \Gamma'| \leq m - k - 1 \leq 3k - 1 - k - 1 < 2(k - 1) + 1$. Since by Proposition 2.2 also the set $\Gamma \setminus \Gamma'$ is $CB(k - 1)$, it follows that $\Gamma \setminus \Gamma'$ lies on a line $\ell'$, too. So the whole $\Gamma$ lies on $\ell \cup \ell'$.

Let us suppose now that $\Gamma'$ lies on two skew lines $\ell_1$ and $\ell_2$. If one of the two lines does not intersect $\Gamma'$, then $\Gamma'$ lies on a plane and we argue as in the previous case. Moreover, since $\Gamma'$ is $CB(k - 1)$, then $\Gamma_1 = \Gamma' \cap \ell_1$ and $\Gamma_2 = \Gamma' \cap \ell_2$ are $CB(k - 1)$ (always by Proposition 2.2). Thus Remark 2.4 yields $q_i := |\Gamma_i| \geq k$ for $i = 1, 2$. But, by definition, $\alpha > \max\{q_1, q_2\} \geq k$. Therefore we have $3k < \alpha + q_1 + q_2 = m \leq 3k - 1$, which is impossible. Hence this configuration does not occur. \hfill \Box

By a similar argument, but with more cases to analyse, we get the the second result.

\textbf{Theorem 3.2.} Let $\Gamma = \{p_1, \ldots, p_m\} \subset \mathbb{P}^n$ be a set of distinct points that satisfies the Cayley-Bacharach condition with respect to the complete linear system $|\mathcal{O}_{\mathbb{P}^n}(k)|$, with $k \geq 1$. If $m \leq 4k - 5$, then $\Gamma$ lies on a reduced curve of degree 3.

\begin{proof}
If $n = 2$ or $n = 3$ the theorem is proved in \cite[Lemma 2.5]{LP}. Moreover, if $k \leq 4$ we have $m \leq 4k - 5 \leq 3k - 1$ so the theorem follows by Theorem 3.1. Let us suppose $n \geq 4$ and $k \geq 5$ and let us define
\[ \alpha := \text{max number of points of } \Gamma \text{ lying on a same 3-plane.} \] (3.2)

Obviously $\alpha \geq 4$. Let us fix a linear space $H$ of dimension 3 that contains $\alpha$ points of $\Gamma$. If $\alpha = m$ we conclude by the assertion for $n = 3$. So we assume that $\alpha < m$ and we have $m - \alpha$ points of $\Gamma$ not lying on $H$. Let us denote by $\Gamma'$ the set of these points. The set $\Gamma'$ is $CB(k - 1)$ by Proposition 2.2 and then $m - \alpha \geq k + 1$ by Remark 2.4. Furthermore, $m - \alpha \leq 4k - 5 - \alpha \leq 4(k - 1) - 5$ and so, by induction on $k$, $\Gamma'$ lies on a reduced curve of degree 3. We have the following three possibilities.

(a) The set $\Gamma'$ lies on an irreducible cubic curve, so in particular it lies on a $\mathbb{P}^3$. Thus, by definition of $\alpha$, it must be $m - \alpha \leq \alpha$, i.e. $\alpha \geq \frac{m}{2}$. We have $m - \alpha \leq \frac{m}{2} \leq 2k - \frac{5}{2} < 2(k - 1) + 1$, so $\Gamma'$ lies on a line $\ell$ (by Remark 2.4). Thus, by Lemma 2.1 $\ell$ contains at least $k + 1$ points of $\Gamma$ and so $|\Gamma \setminus \Gamma'| \leq m - k - 1 \leq 4k - 5 - k - 1 < 3(k - 1) - 1$. Since by Proposition 2.2 also the set $\Gamma \setminus \Gamma'$ is $CB(k - 1)$, it follows that $\Gamma \setminus \Gamma'$ lies on a reduced curve $C_2$ of degree 2 by Theorem 3.1. So the whole $\Gamma$ lies on $C_2 \cup \ell$, a curve of degree 3.

(b) The set $\Gamma'$ lies on $C \cup \ell$, where $C$ is an irreducible conic and $\ell$ is a line. If either $\Gamma' \cap \ell = \emptyset$ or $\Gamma' \cap C = \emptyset$, then the set $\Gamma'$ lies on a 3-plane and we are in the case $n = 3$. The same holds if
$C \cap \ell \neq \emptyset$. Thus we assume that $q_1 := |\Gamma' \cup \ell| \neq 0$, $q_2 := |\Gamma' \cap C| \neq 0$ and $q_1 + q_2 = m - \alpha$. By definition $\alpha \geq \max\{q_1 + 2, q_2 + 1\}$. Moreover, since $\Gamma'$ is $CB(k - 1)$, $\Gamma' \cap \ell$ and $\Gamma' \cap C$ are both $CB(k - 2)$ (by Proposition 2.2) and thus $q_i \geq k$ for $i = 1, 2$, we have that $m - \alpha = q_1 + q_2 \geq 2k$. Therefore $\alpha \leq m - 2k \leq 4k - 5 - 2k < 2(k - 2) + 1$ and, by Remark 2.4, it follows that $\Gamma \setminus \Gamma'$ lies on a line. But we have also that $\alpha \geq k + 2$ and then $q_2 = m - \alpha - q_1 \leq 4k - 5 - k - 2 - k < 2(k - 2) + 1$. Thus also $\Gamma' \cap C$ lies on a line. In conclusion, the set $\Gamma$ lies on three lines.

(c) The set $\Gamma'$ lies on the union of three distinct lines $\ell_1, \ell_2, \ell_3$. If $\Gamma'$ does not intersect one of the lines $\ell_i$, then $\Gamma'$ lies on a 3-plane and we are in the case $n = 3$. The same holds if any $\ell_i$ intersects at least one of the others two lines. Thus $q_i := |\Gamma' \cap \ell_i| \neq 0$, for $i = 1, 2, 3$, and $m - \alpha \in \{q_1 + q_2 + q_3, q_1 + q_2 + q_3 - 1\}$. Moreover, any $\Gamma' \cap \ell_i$ is $CB(k - 3)$ by Proposition 2.2 so $q_i \geq k - 1$ for any $i = 1, 2, 3$. But, by definition, $\alpha > q_i$ for any $i = 1, 2, 3$, thus $\alpha > k - 1$. Then we have $4k - 5 < \alpha + q_1 + q_2 + q_3 - 1 \leq m \leq 4k - 5$, a contradiction. Hence configuration (c) does not occur. 

\[ \square \]

4. Sets of points lying on curves of degree 4

Aim of this section is to prove the following theorem, that is the main result of this paper.

**Theorem 4.1.** Let $\Gamma = \{p_1, \ldots, p_m\} \subset \mathbb{P}^n$ be a set of distinct points that satisfies the Cayley-Bacharach condition with respect to the complete linear system $|O_{\mathbb{P}^n}(k)|$, with $k \geq 1$. If $m \leq 5k - 11$, then $\Gamma$ lies on a reduced curve of degree 4.

This theorem generalizes to any $\mathbb{P}^n$ the analogue result that Lopez and Pirola proved in [LP, Lemma 2.5] just in $\mathbb{P}^2$. We point out that, in contrast with the theorems of the previous section, in this case the assertion have to be proved even in $\mathbb{P}^3$.

**Remark 4.2.** If $k < 7$, we have $5k - 11 < 4k - 5$, so, by Theorem 3.2, $\Gamma$ lies on a reduced curve of degree 3. Moreover, for $k = 7$ we have $4 \cdot 7 - 5 = 23$ and $5 \cdot 7 - 11 = 24$. Thus for $k = 7$ Theorem 4.1 has to be proved only in the case $m = 24$.

Based on the previous remark, we can just focus on the case $k \geq 7$.

In order to prove Theorem 4.1 we need to deal with some particular configurations for which the general approach does not work. Let $\Gamma = \{p_1, \ldots, p_m\} \subset \mathbb{P}^n$ be as in Theorem 4.1 and let us define

\[ \alpha := \max \text{ number of points of } \Gamma \text{ lying on a same hyperplane}. \] (4.1)

In the following subsection we prove Theorem 4.1 in $\mathbb{P}^3$ and $\mathbb{P}^4$ and for values of $\alpha$ at most 4. In the next one we complete the proof for all the remaining cases.

4.1. Lower dimensions and $\alpha \leq 4$. Goal of this subsection is proving the following proposition.

**Proposition 4.3.** Theorem 4.1 holds in $\mathbb{P}^3$ and $\mathbb{P}^4$ when $\alpha$, defined as in (4.1), is at most 4.

We start with some remarks.

**Remark 4.4.** Condition $\alpha \leq 4$ implies that in $\mathbb{P}^3$ we can have at most 3 points of $\Gamma$ on a line and at most 4 points of $\Gamma$ on a plane. Instead, in $\mathbb{P}^4$ this condition implies that we can have at most 2 points of $\Gamma$ on a line, at most 3 points of $\Gamma$ on a plane and at most 4 points of $\Gamma$ on a 3-space.
Remark 4.5. Since by Lemma 2.1 a set of points $CB(k)$ must have cardinality at least $k + 2$, it follows by Remark 4.4 that, if $\alpha \leq 4$, on a line in $\mathbb{P}^3$ we can have only a subset $CB(1)$ of points of $\Gamma$, while on a plane we can have only a subset $CB(2)$ of points of $\Gamma$. For the same reasons, in $\mathbb{P}^4$ points of $\Gamma$ on a line cannot satisfy any Cayley-Bacharach condition, they can be only $CB(1)$ on a plane and at most $CB(2)$ on a 3-space. Thus in $\mathbb{P}^3$ points of $\Gamma$ on a line cannot be $CB(k - s)$ with $s \leq 5$ (if $k \geq 7$) or $s \leq 6$ (if $k \geq 8$) and points of $\Gamma$ on a plane cannot be $CB(k - s)$ with $s \leq 4$ (if $k \geq 7$) or $s \leq 5$ (if $k \geq 8$). Similarly, if $k \geq 7$, in $\mathbb{P}^4$ points of $\Gamma$ on a line cannot be $CB(k - s)$ with $s \leq 6$, points of $\Gamma$ on a plane cannot be $CB(k - s)$ with $s \leq 5$ and points of $\Gamma$ on a 3-space cannot be $CB(k - s)$ with $s \leq 4$.

Remark 4.6. Corollary in [GLP, p.492] ensures that an integral non-degenerate curve of degree $d$ in $\mathbb{P}^n$ is cut out by hypersurfaces of degree $d + 2 - n$. As a consequence, if a set of points $A$ is $CB(k)$ and a proper subset $B \subset A$ lies on an integral non-degenerate curve of degree $d$, the set $A \setminus B$ is $CB(k + n - d - 2)$. In particular $A \setminus B$ is $CB(k - d + 1)$ in $\mathbb{P}^3$ and $CB(k - d + 2)$ in $\mathbb{P}^4$.

We prove Proposition 4.3 by a series of lemmas. For sake of clarity, we recall that $\Gamma = \{p_1, \ldots, p_m\} \subset \mathbb{P}^n$ is a set of distinct points that satisfies the Cayley-Bacharach condition with respect to the complete linear system $|O_{\mathbb{P}^n}(k)|$, $m \leq 5k - 11$ and $\alpha$, defined in (4.1), is at most 4. Moreover, we focus on the case $k \geq 7$ (see Remark 4.2).

Lemma 4.7. Under the hypotheses of Proposition 4.3, $\Gamma \subset \mathbb{P}^n$ lies on a complete intersection curve $C$ of degree $d$, where

- a) $d = 8$ if $n = 3$ and $k = 7$,
- b) $d = 9$ if $n = 3$ and $k \geq 8$,
- c) $d = 8$ if $n = 4$ and $k \geq 7$.

Proof. Case a). By Remark 4.2, we have $m = 24$. Let us fix 9 points of $\Gamma$ and let us consider a quadric $Q_2$ passing through them. We claim that $Q_2$ passes through all the point of $\Gamma$. If not, the points of $\Gamma$ outside from $Q_2$ would be $CB(5)$ and then they would be at least 7. Now, if they are at most 14, by Theorem 3.1 they must lie on a curve of degree 2 (i.e. they lie either on a plane or on two skew lines), but this is not possible by Remark 4.4. Otherwise they must be exactly 15 and then they must lie on a cubic curve $C_3$ by Theorem 3.2. Moreover, the curve $C_3$ must be irreducible and non-degenerate, again by Remark 4.3. If $C_3$ passes trough all the points of $\Gamma$, then Proposition 4.3 follows. Otherwise, the points outside from $C_3$ should be $CB(5)$ by Remark 4.6, hence they must be at least 7 by Lemma 2.1 and must lie on a curve of degree 2 by Theorem 3.1; this is not possible, once again by Remark 4.3. Thus $Q_2$ passes through all the points of $\Gamma$, as claimed. Now, by dimensional reasons, we can consider a quartic $Q_4$ passing through all the points of $\Gamma$ and not containing $Q_2$. The complete intersection $Q_2 \cap Q_4$ is a curve of degree 8 containing $\Gamma$.

Case b). We fix 18 points of $\Gamma$ and we consider two independent cubics passing through them. Their complete intersection is a curve of degree 9 that we denote by $C$. Let $\Gamma_C = \Gamma \cap C$. Clearly $\Gamma = \Gamma_C \cup \Gamma'$ with $\Gamma' \cap C = \emptyset$. We claim that $\Gamma' = \emptyset$. In fact, $|\Gamma'| \leq 5k - 23 < 5(k - 3) - 11$ and $\Gamma'$, if non-empty, is $CB(k - 3)$ and hence it has cardinality at least 7 (so it can not lie on a plane by Remark 4.4). By induction on $k$, $\Gamma'$ lies on a curve $C_4$ of degree 4. If $C_4$ passes through all the points of $\Gamma$, then Proposition 4.3 follows. Let us therefore suppose $C_4$ does not pass through all the points of $\Gamma$. If $C_4$ is irreducible, then the subset $\Gamma_C'$ of points of $\Gamma$ not on $C_4$ is $CB(k - 3)$ by Remark 4.6. But not both the sets $\Gamma'_C$ and $\Gamma'$ can have cardinality greater than $3(k - 3)$ (otherwise
6(k − 3) > 5k − 11 for k ≥ 8). So, by Theorem \[3.1\] at least one between \( \Gamma' \) and \( \Gamma'_C \) must lie either on a conic or on two skew lines. By Remark \[4.5\] this is impossible for a set that is \( CB(r) \) with \( r ≥ 5 \). Thus \( C_4 \) must be reducible. But in this case, by Proposition \[2.2\] the points of \( \Gamma' \) on any line in the decomposition would be at least \( CB(k − 6) \) and those on any conic at least \( CB(k − 5); \) impossible in any case. So we can have just a non-empty irreducible component \( C_3 \) of degree 3. But this is also impossible since the argument used for the case \( C_4 \) irreducible works a fortiori for the case \( C_3 \) irreducible. In conclusion, \( \Gamma' = \emptyset \) as wanted and therefore \( \Gamma \subset C \).

Case c). Let us fix 12 points of \( \Gamma \) and let us consider three independent quadrics passing through them. Their complete intersection is a curve of degree 8 that we denote by \( C \). As before, let \( \Gamma_C = \Gamma \cap C \). Clearly \( \Gamma = \Gamma_C \cup \Gamma' \), with \( \Gamma' \cap C = \emptyset \). Let us show that \( \Gamma' = \emptyset \). In fact, \( |\Gamma'| ≤ 5k − 13 = 5(k − 2) − 11 \) and in this case \( \Gamma' \), if non-empty, is \( CB(k − 2) \). So, by induction on \( k \), \( \Gamma' \) lies on a curve \( C_4 \) of degree 4. We can now follow the same argument used in point b), since (although \( k \) may be 7) the sets \( \Gamma'_C \) and \( \Gamma' \) are now \( CB(k − 2) \) and in \( P^4 \) we can have at most 2 or 3 points on lines and planes respectively. Then also in this case \( \Gamma' = \emptyset \), which implies \( \Gamma \subset C \). □

For the next lemmas, we need the following remark.

Remark 4.8. In \( P^3 \) the Castelnuovo bound (see e.g. [Ha, Theorem 3.7]) becomes \( p_a ≤ m(m − 1)+m \varepsilon \) with \( d − 1 = 2m + \varepsilon \) and \( \varepsilon = 0, 1 \). In any case we get

\[
p_a ≤ \frac{d^2 − 4d + 4}{4}.
\]

In \( P^4 \) the bound becomes \( p_a ≤ \frac{3}{2}m(m − 1)+m \varepsilon \) with \( d − 1 = 3m + \varepsilon \) and \( \varepsilon = 0, 1, 2 \), so in this case we get

\[
p_a ≤ \frac{d^2 − 5d + 6}{6}.
\]

Moreover, we note that, for a fixed degree \( d \), the maximum \( p_a \) in \( P^4 \) is smaller than the maximum \( p_a \) in \( P^3 \).

The following lemma shows that the curve \( C \) of Lemma \[4.7\] can not be irreducible.

Lemma 4.9. Under the hypotheses of Proposition \[4.3\] \( \Gamma \) can not lie on an irreducible curve \( C \) of degree \( d ≥ 5 \).

Proof. This follows from Corollary \[2.7\] Let us suppose, by contradiction, that \( \Gamma \) lies on an irreducible curve \( C \) of degree \( d ≥ 5 \). Since \( C \) is irreducible we can consider it is also reduced, otherwise any reduced component would have degree at most 4 and Proposition \[4.3\] would follow. So the curve \( C \) is integral. Condition ii) of Corollary \[2.7\] is clearly verified. For condition i) we have in any case (see relations \[4.2\] and \[4.3\] in Remark 4.8)

\[
\frac{m + p_a − 2}{d} ≤ \frac{5k − 11 + \frac{d^2 − 4d + 4}{4} − 2}{d} = \frac{10k − 22 + d^2 − 4d}{2d}.
\]

Let us check when the last term is less than \( k \). This happens when \( 2k(d − 5) > d^2 − 4d − 22 \), which is verified for any \( d \) between 5 and 9 (for any \( k ≥ 3 \)). Therefore Corollary \[2.7\] leads us to the absurd conclusion that the set \( \Gamma \) can not be \( CB(k) \). □
The curve $C$ containing $\Gamma$ must therefore be reducible. Some components of $C$ could not intersect $\Gamma$. We point out that, if the sum of the degrees of the components with non-empty intersection with $\Gamma$ is lower than or equal to 4, then we get the thesis of Proposition 4.3 In the following lemmas we examine the other cases. Let us start with a remark.

Remark 4.10. If some points of $\Gamma$ are contained in the union of 4 lines $\ell_1, \ell_2, \ell_3, \ell_4$, then these points lie on 3 planes. Indeed, since $\ell_4$ contains at most three points of $\Gamma$ (see Remark 4.4), then there exists 3 planes $\pi_1, \pi_2, \pi_3$ containing respectively $\ell_1, \ell_2, \ell_3$ and one of the points of $\ell_4$.

Lemma 4.11. Under the hypotheses of Proposition 4.3, let $C$ be the curve of Lemma 4.7, and suppose $C = \tilde{C} \cup \bigcup_{i \in I} C_i$ where $\tilde{C}$ and the $C_i$’s are irreducible and the component $\tilde{C}$ has degree $\tilde{d} \geq 5$. Let $C' = \tilde{C} \cup \bigcup_{i \in I'} C_i$ where $I'$ indexes the components $C_i$ of $C$ with non-empty intersection with $\Gamma$. Then the set $\tilde{\Gamma} = \Gamma \cap \tilde{C}$ is $CB(r)$ with

(i) $r \geq k - 3$ if $\tilde{d} = 5$,

(ii) $r \geq k - 3$ if $\tilde{d} = 6$ and $\bigcup_i C_i$ is the union of three lines,

(iii) $r \geq k - 2$ otherwise.

Proof. First of all it is $5 \leq \tilde{d} \leq 8$ since $C$ is reducible by Lemma 4.9. Moreover, we have $|I'| \leq 4$ and $\sum_{i \in I'} d_i \leq 4$ with $d_i := \deg C_i$.

If $\tilde{d} = 6$ and $\bigcup_i C_i$ is the union of three lines then $\tilde{\Gamma}$ is at least $CB(k - 3)$ by Proposition 2.2 so we get case (ii).

It follows by the same proposition that if, instead, $\bigcup_{i \in I'} C_i$ is one line, one irreducible conic or two lines (which are the only possibilities if $\tilde{d} \in \{7, 8\}$) then $\tilde{\Gamma}$ is at least $CB(k - 2)$. Moreover, the same is true either if $\bigcup_i C_i$ is one line and an irreducible conic or, by Remark 4.6 if it is just an irreducible cubic curve. This proves case (iii).

For $\tilde{d} = 5$ the situation is more complicated. However, in $\mathbb{P}^4$ (where $\tilde{d} + \sum_i d_i \leq 8$) the possibilities for $\sum_i C_i$ are the same as those in the case $\tilde{d} = 6$. In $\mathbb{P}^3$ it remains to consider the cases in which $\sum_i d_i = 4$. If $\bigcup_{i \in I'} C_i$ is either two irreducible conics or one irreducible conic and two lines, we have that $\tilde{\Gamma}$ is $CB(r)$ with $r \geq k - 3$ by Proposition 2.2. Moreover, this is still true if $\bigcup_{i \in I'} C_i$ is the union of four lines since in this case (see Remark 4.10) the points of $\Gamma$ on these lines actually lie on three planes. Finally, since when $\bigcup_{i \in I'} C_i$ is just an irreducible cubic curve $\tilde{\Gamma}$ is $CB(k - 2)$, if we have also a line, $\tilde{\Gamma}$ is $CB(k - 2)$ by Proposition 2.2.

Lemma 4.12. Under the hypotheses of Proposition 4.3 let $C$ be the curve of Lemma 4.7, Then $C$ can not decompose as in Lemma 4.11.

Proof. We proceed by contradiction supposing that the curve $C$ decomposes as in Lemma 4.11. With the same notation of this lemma, in $\mathbb{P}^3$ we have that the set $\Gamma' = \Gamma \setminus \Gamma$ of points of $\Gamma$ outside from $\tilde{C}$ is $CB(k - \tilde{d} + 1)$, whereas in $\mathbb{P}^4$ the set $\Gamma'$ is $CB(k - \tilde{d} + 2)$ (see Remark 4.5). In any case $|\Gamma'| \geq k - \tilde{d} + 3$. Let us analyse the following two cases.

1) If $\tilde{d} \in \{5, 6\}$, then $|\Gamma'| \geq 4$ when $k \geq 7$ and thus $\Gamma'$ can not lie on a line. Then, by Remark 2.3 $\Gamma'$ must have in fact cardinality at least $2(k - \tilde{d} + 1) + 2 = 2k - \tilde{d} + 4$. This implies $|\tilde{\Gamma}| \leq 5k - 11 - 2k + \tilde{d} - 4 = 3k + 2\tilde{d} - 15$. By Lemma 4.11 we know that $\tilde{\Gamma}$ is at least $CB(k - 3)$ with $\tilde{d} = 5$. Now we want to use Corollary 2.7 Condition ii) is clearly satisfied. Let us verify condition
i). We get that

\[ k - 3 > \frac{3k + 2\tilde{d} - 15 + \frac{d^2 - 4d + 4}{2} - 2}{d} \iff k > \frac{d^2 + 6\tilde{d} - 30}{2d - 6} \]  (4.4)

and the last one is true for \( \tilde{d} = 5 \) and for any \( k \geq 7 \), whereas if \( \tilde{d} = 6 \) the last relation in (4.4) is true only for \( k > 7 \). But if \( k = 7 \) and \( \tilde{d} = 6 \), the components \( C_i \) can not be three lines and so, always by Lemma 4.11 \( \tilde{C} \) is at least \( CB(k - 2) \). If we now substitute \( k - 2 \) in the left-hand side of (4.4), we get that it holds for \( k \geq 6 \). Thus in any case we find a contradiction by Corollary 2.7.

2) If \( \tilde{d} \in \{7, 8\} \), by Lemma 4.11 \( \tilde{C} \) is at least \( CB(k - 2) \). Moreover \( |\tilde{\Gamma}| \leq 5k - 11 - k + \tilde{d} - 3 = 4k + \tilde{d} - 14 \). So this time we get

\[ k - 2 > \frac{4k + \tilde{d} - 14 + \frac{d^2 - 4d + 4}{2} - 2}{d} \iff k > \frac{d^2 + 2\tilde{d} - 28}{2d - 8} \]

that is true for \( \tilde{d} \in \{7, 8\} \) and for any \( k \geq 7 \). So, again by Corollary 2.7 we find a contradiction. \( \square \)

Accordingly, in the decomposition of \( C \) can appear only components of degree at most 4. In the following lemmas we exclude the remaining cases.

**Lemma 4.13.** Under the hypotheses of Proposition 4.3, let \( C \) be the curve of Lemma 4.7. Then \( C \) can not decompose only in lines and irreducible conics.

**Proof.** In \( \mathbb{P}^4 \), by Remark 4.4 we can have at most 16 points on lines and on irreducible conics. As \( |\Gamma| \geq 24 \), this proves the lemma in \( \mathbb{P}^4 \).

In \( \mathbb{P}^3 \), by the same reason, if we denote by \( r \) the number of lines and by \( c \) the number of irreducible conics, the only possible cases are \( (r, c) \in \{(9, 0), (8, 0), (7, 1)\} \). Let us start with the case \( (r, c) = (8, 0) \). The only configuration allowed is that with \( |\Gamma| = 24 \) and three points on each line. By Remark 4.10 the points lie on 6 planes with 4 points on each of them. If we consider a line passing through two points on one of these planes, the remaining two points on the same plane are \( CB(k - 6) \) and this is not possible by Remark 4.5. As for the cases \( (r, c) \in \{(9, 0), (7, 1)\} \), these are possible only for \( k \geq 8 \). But, always by Remark 4.10 we have that any line would be \( CB(k - 6) \) and thus would contain \( k - 4 \geq 4 \) points, which is impossible. Hence the lemma follows in \( \mathbb{P}^3 \), too. \( \square \)

**Lemma 4.14.** Under the hypotheses of Proposition 4.3, let \( C \) be the curve of Lemma 4.7. Then \( C \) can not decompose in \( C = \tilde{C} \cup \bigcup_{i \in I} C_i \) with \( \tilde{C} \) and the components \( C_i \)'s irreducible, \( \tilde{d} := deg\tilde{C} \in \{3, 4\}, \ d_i := degC_i \leq 2 \) for any \( i \in I \) and \( \Gamma \cap \bigcup_{i \in I} C_i \neq \emptyset \).

**Proof.** Let us start analysing the situation in \( \mathbb{P}^4 \). An irreducible curve of degree 3 lies on a 3-space, which can contain at most 4 points of \( \Gamma \) in this configuration (see Remark 4.4). So, if \( \tilde{d} = 3 \) we could have at most 14 points of \( \Gamma \), in contrast with the assumption \( |\Gamma| \geq 24 \). If, instead, \( \tilde{d} = 4 \) then the set \( \Gamma \cap \bigcup_{i \in I} C_i \) is \( CB(k - 2) \) by Remark 4.6, so the points of \( \Gamma \) on any \( C_i \) have to be at least \( CB(k - 5) \) by Proposition 2.2.2 and hence contains at least \( k - 3 \geq 4 \) points of \( \Gamma \) by Lemma 2.1. But this is not possible since on a plane we can have at most three points of \( \Gamma \).

In \( \mathbb{P}^3 \) the components \( C_i \) can not contain irreducible conics. In order to see this, we shall use Remark 4.5 consistently. If \( \bigcup_{i \in I} C_i \) were only irreducible conics, arguing as usual, the points of \( \Gamma \) on any of these conics would be at least \( CB(k - 4) \). In the same way, one sees that if in \( \bigcup_{i \in I} C_i \) there appear a line and at least an irreducible conic, then the points of \( \Gamma \) on the line would be at least
configuration is possible only for $k$ if $k \geq 8$ (since this is the only case in which $C$ can have degree 9) and at least $CB(k-5)$ if $k = 7$. Thus the $C_i$’s must be all lines.

If $\deg \bar{C} = 3$, then the points of $\Gamma$ on $\bigcup_i C_i$ are $CB(k-2)$ by Remark 4.6. Moreover, remembering Remark 4.10 if $|I| \leq 5$ it follows that any $\Gamma \cap C_i$ is at least $CB(k-5)$ by Proposition 4.2.1 and this is not possible by Remark 4.5. But the same argument works for the case $|I| = 6$. In fact, this configuration is possible only for $k \geq 8$ (degree of $C$ equal to 9) and in this case any set $\Gamma \cap C_i$ would be at least $CB(k-6)$, which is still impossible by Remark 4.5.

If $\deg \bar{C} = 4$, then the points of $\Gamma$ on $\bigcup_i C_i$ are $CB(k-3)$ by Remark 4.6. Now, if $|I| = 5$ (which is possible only in the case $k \geq 8$), by Remark 4.10 and Proposition 4.2.1 we would again conclude that $\Gamma \cap C_i$ is $CB(k-6)$. Whereas, if $|I| \leq 3$, we would get directly by Proposition 4.2.1 that $\Gamma \cap C_i$ is $CB(k-5)$. So these cases are impossible and it remains to study only the case of four lines. By the same reasons of above, the points of $\Gamma$ on each of these lines are $CB(k-6)$. This implies that $k = 7$ and that on any of these lines we must have at least 3 points of $\Gamma$, that is, exactly 3, since on a line we can not have more than 3 points of $\Gamma$. Thus $|\Gamma| = 12$. Now, the set $\bar{\Gamma}$ is $CB(3)$ and we can use Corollary 2.7 to find a contradiction. Indeed, $3 > 2 = 4 - 2$ (condition ii) of Corollary 2.7 and from (4.2) we get $p_a \leq 1$, so $3 > \frac{11}{4} = \frac{12 + 1 - 2}{4}$ and condition i) of Corollary 2.7 is verified, too.

**Lemma 4.15.** Under the hypotheses of Proposition 4.3, let $C$ be the curve of Lemma 4.4. Then in the decomposition of $C$ can not appear two curves of degree 4 or two curves of degree 3 containing points of $\Gamma$.

**Proof.** Let us start with the case of two curves of degree 4, $C_1$ and $C_2$. By Remark 4.6 in $\mathbb{P}^4$ the sets $C_i \cap \Gamma$, $i = 1, 2$, are $CB(k-2)$. Moreover, not both of them can have cardinality greater than or equal to $3(k-2)$, otherwise for any $k \geq 7$ we would have $m \geq 6(k-2) > 5k-11 \geq |\Gamma|$. Then, by Theorem 3.1 at least one among $C_1 \cap \Gamma$ and $C_1 \cap \Gamma$ must lie on a curve of degree 2, but this is not possible by Lemma 4.14. In $\mathbb{P}^3$, if $k \geq 8$, in the decomposition of $C$ it can appear also a line $\ell$. But the set $\Gamma \cap \ell$ would be $CB(k-6)$ and this is impossible by Remark 4.3. So we may have only the two curves of degree 4, $C_1$ and $C_2$, and the sets $C_i \cap \Gamma$, $i = 1, 2$, are $CB(k-3)$. If $k \geq 8$ these two sets can not have both cardinality greater than or equal to $3(k-3)$; otherwise we would have $m \geq 6(k-3) > 5k-11$ and, as before, we would conclude that at least one of these sets lies on a curve of degree 2. If $k = 7$ ($m = 24$), we can find again a contradiction by Corollary 2.7. The computation is the same as the one at the end of Lemma 4.14 since now $k-3 = 4 > 3$ and at least one of the two curves must contains at most 12 points of $\Gamma$.

Let us consider the case of two irreducible curves of degree 3, $D_1$ and $D_2$. It is easy to see that in $\mathbb{P}^4$ this is not possible. Indeed, on a such curve we can have at most 4 points of $\Gamma$, so $|\Gamma|$ would be at most 8.

In order to deal with this configuration in $\mathbb{P}^3$, let us consider the set $\Gamma \setminus (D_1 \cup D_2)$ of points of $\Gamma$ not on these two curves. If $\Gamma \setminus (D_1 \cup D_2) \neq \emptyset$, then it is $CB(k-4)$ and so it must contain at least 5 points. Thus it can not lie neither on a line nor on an irreducible conic. Furthermore, it can not lie even on two lines, otherwise the set of points of $\Gamma$ on each of them would be $CB(k-5)$. If $k \geq 8$ (that is, $\deg C = 9$) we could have also the following further possibilities: (a) three lines $\ell_1, \ell_2, \ell_3$; (b) a line $\ell$ and an irreducible conic $C_2$; (c) another irreducible curve $D_3$ of degree 3.

Case (a) does not occur since, by Proposition 2.2, any set $\Gamma \cap \ell_i$, for $i = 1, 2, 3$, would be $CB(k-6)$. Also case (b) does not occur by the same reason. Namely, the set $\Gamma \cap \ell$ would be
$CB(k - 5)$. As for case (c), we note that the sets $\Gamma \cap D_3$, $\Gamma \cap D_3$ and $\Gamma \cap D_3$ are $CB(k - 4)$ by Remark 4.16 and not all of them can have cardinality greater then or equal to $2(k - 4) + 2$ (since, otherwise, $|\Gamma| \geq 6(k - 4) + 6 > 5k - 11 \geq |\Gamma|$). Thus, by Proposition 2.4 at least one of these sets would lie on a line and we would have the configuration of one of the previous cases.

Finally, let us suppose $\Gamma \setminus (D_1 \cup D_2) = \emptyset$, i.e. the non-empty components of $C$ (with respect to the points of $\Gamma$) are only the two irreducible curves of degree 3. We can prove this case does not occur by the same argument in case (c). More precisely, the sets $\Gamma \cap D_i$, for $i = 1, 2$, are $CB(k - 2)$ and, arguing as before, we see that at least one of these sets must lie on a curve of lower degree. But this is not possible by Lemma 4.14.

**Lemma 4.16.** Under the hypotheses of Proposition 4.3 let $C$ be the curve of Lemma 4.7. Then in the decomposition of $C$ can not appear simultaneously an irreducible curve $C_3$ of degree 3 and an irreducible curve $C_4$ of degree 4 containing points of $\Gamma$.

**Proof.** This configuration does not occur in $\mathbb{P}^4$ since, by Remark 4.6 and Proposition 2.2 the set $\Gamma \cap C_3$ would be at least $CB(k - 3)$, which is impossible by Remark 4.5.

In $\mathbb{P}^3$ with $k = 7$ the curve $C$ has degree 8 (see Lemma 4.7). Let us suppose that $C = C_3 \cup C_4 \cup \ell$, with $\ell$ a line. If $\Gamma \cap \ell \neq \emptyset$, then this set would be $CB(k - 5)$ by Remark 4.6 but this is not possible by Remark 4.5. If $k \geq 8$, then the curve $C$ has degree 9. If in the decomposition of $C$ it appears an irreducible conic that intersects $\Gamma$, then the same argument made for case $k = 7$ shows that this intersection must be $CB(k - 5)$; again impossible by Remark 4.5.

If, instead, in the decomposition of $C$ there appear two skew lines, $\ell_1$ and $\ell_2$, intersecting $\Gamma$, then each set $\Gamma \cap \ell_i$, for $i = 1, 2$, would be $CB(k - 6)$; once again impossible by Remark 4.5.

So we can have only points of $\Gamma$ on $C_3$ and on $C_4$. It follows that the points on $C_3$ are $CB(k - 3)$ and that the points on $C_4$ are $CB(k - 2)$. These sets of points can not have at the same time cardinality greater than or equal to $3(k - 3)$ and $4(k - 3)$ respectively, otherwise their sum would overcome the maximal cardinality of $\Gamma$. Thus, either the points on $C_3$ lie on a curve of degree lower that 3, or the points on $C_4$ lie on a curve of degree lower that 4. Both these situations are not allowed by one of the previous lemmas.

Putting all these lemmas together, we can now prove Proposition 4.3.

**Proof of Proposition 4.3.** Lemma 4.17 ensures that $\Gamma$ lies on a curve $C$ of degree 8 or 9, but Lemmas 4.9 and 4.12 imply that $\Gamma$ can not intersect $C$ in an irreducible component of degree greater than 4 (in particular, $C$ can not be irreducible). The components of $C$ containing points of $\Gamma$ can not be only lines and conics by Lemma 4.13. So we have that among these components it must appear at least one of degree 3 or 4. Actually, by Lemmas 4.14, 4.15 and 4.16 $\Gamma$ is contained in exactly one of these components.

**4.2. Proof of the main theorem.** We can finally prove Theorem 4.1.

**Proof of Theorem 4.1.** If $n = 2$ the assertion follows by [LP, Lemma 2.5]. The cases $n = 3$ and $n = 4$ with $\alpha \leq 4$, where $\alpha$ is defined as in (4.1), are proved in Proposition 4.3. Let us suppose now $n \geq 3$ and let us define $\alpha$ as in (4.1) if $n \in \{3, 4\}$ (i.e. the maximum number of points of $\Gamma$ lying on a hyperplane). If $n \geq 5$, let us define $\alpha$ as

$$\alpha := \max \text{ number of points of } \Gamma \text{ lying on a same 4-plane.}$$

(4.5)
In this last case obviously \( \alpha \geq 5 \), but we can now assume \( \alpha \geq 5 \) even in the cases \( n \in \{3, 4\} \).

Let us fix a linear subspace \( H \), of the right dimension, that contains \( \alpha \) points of \( \Gamma \) and let \( \Gamma_H = \Gamma \cap H \). If \( \alpha = m \) we are in one of the cases already dealt with. We can therefore assume \( \alpha < m \) and then we have \( m - \alpha \) points of \( \Gamma \) outside from \( H \). Let us denote by \( \Gamma' \) the set of these points. The set \( \Gamma' \) is \( CB(k - 1) \) by Proposition 2.2 and then \( m - \alpha \geq k + 1 \) by Remark 2.4. Furthermore, \( m - \alpha \leq 5k - 11 - \alpha \leq 5(k - 1) - 11 \) and so, by induction on \( k \), \( \Gamma' \) lies on a reduced curve \( C_4 \) of degree 4.

We claim that \( \Gamma_H \) is \( CB(k - s) \) with \( s \leq 4 \). Indeed, if \( C_4 \) is irreducible, then it is contained in a linear subspace of dimension 2 \( t \leq 4 \). By Remark 4.6 the curve \( C_4 \) is cut out by hypersurfaces of degree 6 \( t \leq 4 \), thus \( \Gamma_H \) is \( CB(k + t - 6) \). If the curve \( C_4 \) decomposes in an irreducible curve \( C_3 \) of degree 3 and a line \( \ell \), by the same reason, the set \( \Gamma_H \) is either \( CB(k + t - 5) \), with \( 2 \leq t \leq 3 \), if \( \Gamma' \) does not intersect \( \ell \), or \( CB(k + t - 6) \) otherwise. Finally, if \( C_4 \) decomposes in lines and conics, then the set \( \Gamma' \) lies at most on 4 distinct linear subspaces and hence, by Proposition 2.2 \( \Gamma_H \) is at least \( CB(k - 4) \). This proves the claim.

Now, if \( m - \alpha \geq 3(k - 1) \), then \( \alpha = m - (m - \alpha) \leq 2k - 8 < 2(k - s) + 1 \) with \( s \leq 4 \), thus \( \Gamma_H \) lies on a line by Remark 2.4. But, by the very definition of \( \alpha \), it must be \( m = \alpha \) and this is impossible since we have points of \( \Gamma \) outside from \( H \). If, instead, \( m - \alpha \leq 3(k - 1) - 1 \), by Theorem 3.1 the set \( \Gamma' \) lies on a reduced curve of degree 2 and thus, by Proposition 2.2 \( \Gamma_H \) is \( CB(k - 1) \) or \( C(k - 2) \), depending on whether \( \Gamma' \) lies on one or two linear subspaces. If also \( \Gamma_H \) lies on a curve of degree 2 the theorem is proved. Otherwise, by Theorem 3.1 it must be at least \( \alpha \geq 3(k - 2) \) and then \( m - \alpha \leq 2k - 5 < 2(k - 1) + 1 \), so \( \Gamma' \) lies on a line. This implies that \( \Gamma_H \) is certainly \( CB(k - 1) \) and it must lie on a curve of degree 3, because otherwise, by Theorem 3.2 we would have \( \alpha \geq 4(k - 1) - 4 \) and consequently \( m - \alpha \leq k - 3 \); this is impossible since we pointed out that \( m - \alpha \geq k + 1 \). Thus \( \Gamma \) lies on a curve of degree 4.

\( \square \)

Remark 4.17. As [SU] Theorem 1.9], compared with Theorem 3.1 shows, bound (1.1) for \( h = 3 \) is not sharp, at least not for all the values of \( k \). In fact, we already noted that for \( k \in \{1, 2\} \) the bound \( \frac{5}{2}k + 1 \) is better. Besides, the latter is been found on the trace of the bound \( 2k + 1 \) in [BCD], Lemma 2.4], that, in this case, coincides with (1.1) for \( h = 2 \). It is therefore natural suppose that it is possible to find a bound of the form \( \alpha \cdot k + 1 \), with \( \alpha \) depending on \( h \), for the cardinality of a set \( \Gamma \) that is \( CB(k) \), which forces \( \Gamma \) to lie on a curve of degree \( h - 1 \). In addition, this bound should improve (1.1) for a range of low values of \( k \) which should enlarge as \( h \) growing. It is moreover an interesting question understanding, in case of affirmative answer, if there exists a function of \( h \) that describe \( \alpha(h) \) (see also [LU] Questions 7.3 and 7.4]).

Remark 4.18. We expect that the answer to Question 1.1 would be affirmative in any \( \mathbb{P}^n \) even for values of \( h \) higher than 5. The techniques used in this paper involve, on the one hand, the study of cases in which a curve of degree \( h - 1 \) may reduce and, on the other hand, they require the study of a suitable curve of higher degree passing though all the points of \( \Gamma \). For both of these situations the number of cases to be analysed grows very quickly with the increase of \( h \). For this reason, we believe that these techniques, although they could work, are not the most appropriate to deal with this issue.
5. Applications

Aim of this section is to prove Theorems \[\text{B}, \text{C}\] and to state and prove the results about complete intersection varieties we mentioned in the introduction.

5.1. Linear series on curves. In order to prove Theorem \[\text{B}\] we need the following lemma about the Cayley-Bacharach property for a general divisor on a suitable linear series on a curve that moves on a smooth surface in \(\mathbb{P}^3\).

Lemma 5.1. Let \(S \subset \mathbb{P}^3\) be a smooth surface, \(C\) an integral curve on \(S\) and \(g_n^r\) a base point free special linear series on \(C\) that is not composed with an involution if \(r \geq 2\). Then the general divisor \(D \in g_n^r\) satisfies the Cayley-Bacharach property with respect to the dualizing sheaf of \(C\). Moreover, if \(|\mathcal{O}_C \otimes \mathcal{O}_S(C)|\) is base point free then \(D\) also satisfies the Cayley-Bacharach property with respect to \(|K_S|\).

Proof. See \[\text{LP},\ \text{Lemma 3.1}\].

Combining the previous lemma with Theorem \[\text{A}\] we can extend \[\text{LP},\ \text{Theorem 1.5}\] proving Theorem \[\text{B}\].

Proof of Theorem \[\text{B}\]. By Lemma 5.1, the general divisor \(D \in g_n^r\) is \(CB(d - 4)\). By \[\text{LP},\ \text{Lemma 2.5}\] and Theorem 4.1 there exists an integer \(h\), with \(1 \leq h \leq 4\), such that \(h(d - h - 1) \leq n \leq (h + 1)(d - h - 2) - 1\) and \(D\) lies on a curve \(E\) of degree \(h\). Since \(S\) is smooth and of general type, it does not contain infinitely many curves of degree \(h \leq 3\). Thus, by Bezout’s theorem, we also have \(n \leq hd\) when \(h \leq 3\). Indeed, if \(n \geq hd + 1\), then a component of \(E\) would be contained in \(S\) and, since the \(g_n^r\) is base point free, \(S\) would be covered by the family of such components. Moreover, the same argument holds in the case \(h = 4\) when the curves of the family covering the surface \(S\) are not degenerate. Finally, if the degree is \(h = 4\) and the curves of the family are degenerate, then the gonality of these curves is at most \(3\); i.e., using the terminology of \[\text{BDELU},\ \text{cov.gon}(X) \leq 3\]. By \[\text{BDELU},\ \text{Theorem A}\] it follows that \(\text{cov.gon}(X) \geq d - 2\), and hence \(d \leq 5\). But since we are in the case \(h = 4\), \[\text{[L2]}\] implies that \(d \geq 11\), a contradiction.

5.2. Correspondences with null trace. Theorem \[\text{C}\] concerns correspondences with null trace. Before presenting the proof of the theorem, we briefly describe this notion referring the reader to \[\text{LP}\] and \[\text{B}\] for further details.

Let \(X, Y\) be two projective varieties of dimension \(n\), with \(X\) smooth and \(Y\) integral. A correspondence of degree \(d\) on \(Y \times X\) is an integral \(n\)-dimensional variety \(\Sigma \subset Y \times X\) such that the projections \(\pi_1: \Sigma \to Y, \pi_2: \Sigma \to X\) are generically finite dominant morphisms and \(\deg \pi_1 = d\). Let \(U \subset Y_{\text{reg}}\) be an open subset such that \(\dim \pi_1^{-1}(y) = 0\) for every \(y \in U\). Associate to \(\Sigma\) there is a map \(\gamma: U \to X^{(d)}\), where \(X^{(d)}\) is the \(d\)-fold symmetric product\(^1\), defined by \(\gamma(y) = P_1 + \cdots + P_d\), where \(\pi_1^{-1}(y) = \{(y, P_i) | i = 1, \ldots, d\}\). In \[\text{M},\ \text{Section 2}\] Mumford defines a trace map \(\text{Tr}_\gamma: H^{n,0}(S) \to H^{n,0}(U)\) (see also \[\text{LP},\ \text{Section 2}\] and \[\text{B},\ \text{Section 4}\]) linked to the map \(\gamma\). We say that \(\Sigma\) is a correspondence with null trace if \(\text{Tr}_\gamma = 0\).

We are now ready to prove the following proposition that is a slightly stronger version of Theorem \[\text{C}\].

\(^1\)Let \(S_d\) be the symmetric group of \(d\) elements. The \(d\)-fold symmetric product of a variety \(X\) is \(X^{(d)} = X^d/S_d\).
Proposition 5.2. Let $n \geq 3$ and let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree $d \geq n + 2$. Let $\Gamma$ be a correspondence of degree $m$ with null trace on $X \times X$. If $m \leq h(d - n - h + 2) - 1$ for $2 \leq h \leq 5$, then the only possible values of $m$ are $(s - 1)(d - n - s + 3) \leq m \leq (s - 1)d$ for $2 \leq s \leq h$.

Proof. The crucial point is that having null trace imposes a Cayley-Bacharach condition upon a correspondence. More precisely, since $K_X = (d - n - 1)H$, where $H$ is an hyperplane section, then by [B] Proposition 4.2] the set $\Gamma = \pi_2(\pi_1^{-1}(y)) = \{P_1, \ldots, P_m\}$ is $CB(d - n - 1)$, thus we get $m \geq d - n + 1$ by Lemma 2.1. Let us suppose now, by contradiction, that

$$(h - 1)d + 1 \leq m \leq h(d - h - n + 2) - 1 \quad (5.1)$$

for $2 \leq h \leq 5$. By Theorem A, $\Gamma$ lies on a curve of degree $h - 1$. By Bezout’s theorem this curve must have a component $E$ of degree $e \leq 4$ contained in $X$. As the fiber $\pi_1^{-1}(y)$ is general, $X$ is covered by a family of curves of degree $e$. If $e \leq 3$, the curves of the family would be either rational or elliptic and this is non possible since $X$ is smooth and of general type. The same holds if $e = 4$ and the curves of the family are non-degenerate. On the other hand, if $e = 4$ and the curves of the family are degenerate, these have gonality at most 3, i.e., using as above the terminology of [BDELU], cov.gon$(X) \leq 3$. By [BDELU] Theorem A, it follows that cov.gon$(X) \geq d - n + 1$, and hence $d \leq n + 2$. But since we are in the case $h = 5$, (5.1) implies that $d \geq 5n + 17$, a contradiction. □

5.3. Plane configurations. Followig [LU], we call a union $\mathcal{P} = P_1 \cup \cdots \cup P_r \subset \mathbb{P}^n$ of positive-dimensional linear spaces a plane configuration of dimension $\dim(\mathcal{P}) = \sum \dim(P_i)$ and length $\ell(\mathcal{P}) = r$.

Remark 5.3. In [LU] Conjecture 1.2], Levinson and Ullery conjectured that given a set $\Gamma \subset \mathbb{P}^n$ of distinct points $CB(k)$, if $|\Gamma| \leq (t + 1)k + 1$, then $\Gamma$ lies on a plane configuration $\mathcal{P} = P_1 \cup \cdots \cup P_r$ of dimension $t$. In the same paper they proved the conjecture for some lower values of $t$ and $k$, given at the same time an upper bound for the length of $\mathcal{P}$. In particular, they proved the conjecture for any $k \leq 2$ and any $t \leq 3$ (cf. [LU] Theorem 1.3 (i) and (ii)) and in the case $t = 4$ and $k = 3$ (cf. [LU] Theorem 1.3 (iii)). Theorems 4.1 extends [LU] Theorem 1.3]. Indeed, for $k \geq 13$, it ensures that if $|\Gamma| \leq 5k - 11$, then $\Gamma$ lies on a plane configuration of dimension $4$ (with $k \leq 12$ we have $5k - 11 \leq 4k + 1$ and so $\Gamma$ would lie, a fortiori, on a plane configuration of dimension $3$ by [LU] Theorem 1.3(iii)]). However, we would like to point out that Theorem 4.1 does not prove [LU] Conjecture 1.2] in the cases $t = 4$ and $k \geq 13$. In fact, with $t = 4$ we have $5k - 11 < 5k + 1$, that is, the upper bound for $|\Gamma|$ in our theorem is less than the one that appears in the conjecture (even if, by contrast, we get a stronger thesis).

Applying [LU] Conjecture 1.2] in the case $t = 3$ (which has been proved in [LU] Theorem 1.3 (ii)), in [LU] Theorem 1.4] the authors study the geometry of fibers of some maps from complete intersections varieties. Namely, they prove that if $X \subset \mathbb{P}^{n+2}$ is a complete intersection of a quartic and a hypersurface of degree $a \geq 4n - 5$ and $f : X \longrightarrow \mathbb{P}^n$ is a finite rational map of degree at most $3a$, then the general fiber of $f$ lies on a plane configuration of dimension $3$. By leveraging the extension of [LU] Theorem 1.3] stated in Remark 5.3, we can give some conditions for the 4-dimensional plane configuration case. In fact, we can prove the following more general result.

Proposition 5.4. Let $X \subset \mathbb{P}^{n+2}$ be a smooth complete intersection of hypersurfaces $Y_a$ and $Y_b$, of degrees $a$ and $b$ respectively. Then for any correspondence on $X$ of degree $m$ with null trace such that $m \leq 5(a + b - n) - 26$, the general fiber lies on a plane configuration of dimension 4.
Proof. Let \( \Gamma \) be a general fiber of the correspondence. Then, by [B] Proposition 4.2], \( \Gamma \) satisfies the Cayley-Bacharach condition with respect to \( K_X \); that is, \( \Gamma \) is \( CB(a + b - n - 3) \) since \( X \) is the complete intersection of \( Y_a \) and \( Y_b \). Hence the proposition follows by Remark 5.3.

In particular, since a dominant rational map \( X \dashrightarrow \mathbb{P}^n \) of degree \( m \), from an \( n \)-dimensional variety \( X \), gives rise to a correspondence of degree \( m \) with null trace (see [B] Example 4.6), we get the following corollary that partially extends [LU, Theorem 1.4(a)].

**Corollary 5.5.** Let \( n \geq 6 \) and let \( X \subset \mathbb{P}^{n+2} \) be the complete intersection of a quartic hypersurface and a hypersurface of degree \( \left[ \frac{5}{2}n + 3 \right] \leq a \leq 4n - 6 \). If \( f : X \dashrightarrow \mathbb{P}^n \) is a finite rational map of degree at most \( 3a \), then the general fiber of \( f \) lies on a plane configuration of dimension 4.

**Proof.** Let \( \Gamma \) be a general fiber of \( f \). Then, by [B] Proposition 4.2], \( \Gamma \) satisfies the Cayley-Bacharach condition with respect to \( K_X \); that is, \( \Gamma \) is \( CB(a - n + 1) \) since \( X \) is a complete intersection. The assumption \( a \geq \left[ \frac{5}{2}n + 3 \right] \) ensures that \( 3a \leq 5(a - n + 1) - 11 \). Hence the corollary follows by Proposition 5.4.

**Remark 5.6.** The previous corollary even holds without the assumption \( a \leq 4n - 6 \). Nevertheless, if \( a \geq 4n - 5 \), then [LU] Theorem 1.4 (a)] ensures that the fiber \( \Gamma \) lies on a plane configuration of dimension 3. Moreover, condition \( n \geq 6 \) just needs to guarantee that \( \left[ \frac{5}{2}n + 3 \right] \leq 4n - 6 \).

**Remark 5.7.** The assumptions \( a \geq \left[ \frac{5}{2}n + 3 \right] \) and \( n \geq 6 \) in Corollary 5.5 ensure that \( a - n + 1 \geq 13 \). Hence [LU] Theorem 1.4(a)] does not imply that the fiber of \( f \) lies on a plane configuration of dimension 3 (cf. Remark 5.3). Corollary 5.5 is therefore a real extension of [LU, Theorem 1.4(a)].

**Remark 5.8.** Clearly, Corollary 5.5 holds for any \( X \subset \mathbb{P}^{n+2} \) complete intersection of hypersurfaces \( Y_a \) and \( Y_b \) of degrees \( a \) and \( b \) respectively and for any finite rational map \( f : X \dashrightarrow \mathbb{P}^n \) of degree \( m \leq 5(a + b - n) - 26 \). However, the formulation of Corollary 5.5 allows to deduce an analogues of [LU, Theorem 1.4(b)]. Namely, in this setting the conclusion of the corollary holds for any dominant rational map \( f : X \dashrightarrow \mathbb{P}^n \) of minimum degree. In fact, since \( n \geq 6 \) the quartic hypersurface contains a line \( \ell \) (see e.g. [CK]). Now, if the hypersurface \( Y \) of degree \( a \) does not contain the line \( \ell \), then \( X \cap \ell \) has length \( a \), so projection from \( \ell \) yields a dominant rational map \( X \dashrightarrow \mathbb{P}^n \) of degree \( 3a \). If, instead, \( \ell \subset Y \), we can still consider the projection from \( \ell \), but in this case it has degree \( 3(a - 1) \); in any case less that \( 3a \).

**Remark 5.9.** Analogously, following [B] Example 4.7], it is possible to define a correspondence of degree \( d \) with null trace from a family of \( d \)-gonal irreducible curves covering an \( n \)-dimensional variety \( X \). This leads us to apply Proposition 5.4 in the following way. Let \( X \subset \mathbb{P}^{n+2} \) be the complete intersection of hypersurfaces \( Y_a \) and \( Y_b \), of degrees \( a \) and \( b \) respectively. Let \( T \) be an \( (n - 1) \)-dimensional smooth variety and let \( \mathcal{E} = \{ E_t \}_{t \in T} \) be a family of irreducible \( d \)-gonal curves covering \( X \), i.e. for any general point \( x \in X \) there exists \( t \in T \) such that \( x \in E_t \) and for any \( t \in T \) there exists an holomorphic map \( f_t : E_t \rightarrow \mathbb{P}^1 \) of degree \( d \). If \( d \leq 5(a + b - n) - 26 \), then the fiber \( f_t^{-1}(z) = \{ P_1, \ldots, P_d \} \) lies on a plane configuration of dimension 4 for any \( t \in T \) and for any \( z \in \mathbb{P}^1 \).

\(^2\)Here the example is provided for \( C^{(k)} \), but the argument works for the wider \( n \)-dimensional variety.
Acknowledgements

I am grateful to my Ph.D. supervisor Francesco Bastianelli for getting me interested in these problems and for his patient support. I would also like to thank Andreas Leopold Knutsen for helpful suggestions.

References

[B] F. Bastianelli, On the symmetric products of curves, *Trans. Amer. Math. Soc.*, **364**(5) (2012), 2493–2519.
[BDELU] F. Bastianelli, P. De Poi, L. Eil, R. Lazarsfeld and B. Ullery, Measure of irrationality for hypersurfaces of large degree, *Compos. Math.*, **153**(11) (2017), 2358–2393.
[BCD] F. Bastianelli, R. Cortini and P. De Poi, The gonality theorem of Noether for hypersurfaces, *J. Algebraic Geom.*, **23**(2) (2014), 313–339.
[CK] C. Ciliberto and M. Zaidenberg, On Fano schemes of complete intersections, *arXiv: 1903.11294v2* (2019).
[EGH] D. Eisenbud, M. Green and J. Harris, Cayley-Bacharach theorems and conjectures, *Bull. Amer. Math. Soc. (N.S.)*, **33**(3) (1996), 295–324.
[G] S. Greco, Remarks on the postulation of zero-dimensional subschemes of projective space, *Math. Ann.*, **284** (1989), 343–351.
[GK] F. Gounelas and A. Kouvidakis, Measures of irrationality of the Fano surface of a cubic threefold, *Trans. Amer. Math. Soc.*, **371**(10) (2019), 7111–7133.
[GLP] L. Gruson, R. Lazarsfeld and C. Peskine, On a theorem of Castelnuovo and the equations defining space curves, *Invent. Math.*, **72** (1983), 491–506.
[H1] R. Hartshorne, *Residues and duality*, Lecture Notes in Math., vol. 20, Springer, 1966.
[H2] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 52, Springer, New York, NY-Heidelberg-Berlin, 1983.
[H86] R. Hartshorne, Generalized divisors on Gorenstein curves and a theorem of Noether, *J. Math. Kyoto Univ.*, **26**(3) (1986), 375–386.
[Ha] J. Harris, *Curves in projective space*, Les Presses de l’Université de Montréal, 1982.
[KLR] M. Kreuzer, L. N. Long and L. Robbiano, On the Cayley-Bacharach property, *Comm. Algebra*, **47**(1) (2019), 328–354.
[LM] R. Lazarsfeld and O. Martin, Measures of association between algebraic varieties, *arXiv:2112.00785v1* (2021).
[LP] A. F. Lopez and G. P. Pirola, On the curves through a general point of smooth surface in $\mathbb{P}^3$, *Math. Z.*, **219** (1994), 93–106.
[LU] J. Levinson and B. Ullery, A Cayley-Bacharach theorem and plane configurations, *Proc. Amer. Math. Soc.* to appear.
[M] D. Mumford, Rational equivalence of 0-cycles on surfaces, *J. Math. Kyoto Univ.*, **9**(2) (1969),195–204.
[SU] D. Stapleton and B. Ullery, The degree of irrationality of hypersurfaces in various Fano varieties, *Manuscripta Math.*, **161**(3-4) (2020), 377–408.

Dipartimento di Matematica, Università degli Studi di Bari “Aldo Moro”, Via Edoardo Orabona 4, 70125 Bari – Italy

E-mail address: nicola.picoco@uniba.it

Dipartimento di Matematica, Università degli Studi di Bari, Via Edoardo Orabona 4, 70125 Bari – Italy

Email address: nicola.picoco@uniba.it