Designing the sound of a cut-off drum

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The spectral action in noncommutative geometry naturally implements an ultraviolet cut-off, by counting the eigenvalues of a (generalized) Dirac operator lower than an energy of unification. Inverting the well known question “how to hear the shape of a drum”, we ask what drum can be designed by hearing the truncated music of the spectral action? This makes sense because the same Dirac operator also determines the metric, via Connes distance. The latter thus offers an original way to implement the high-momentum cut-off of the spectral action as a short distance cut-off on space. This is a non-technical presentation of the results of [8].
1. Introduction

Cut-off are generally used to avoid undesirable divergencies occurring at small or very large scales. The scale is usually an energy scale $E$ and the cut-off is implemented either on momentum $p$ or on the wavelength $\lambda$. In various senses, these two implementations are dual to each other: a high momentum cut-off is equivalent to a short wavelength cut-off and vice-versa, as can be read e.g. in de Broglie relation $p = \frac{h}{\lambda}$. In this note we explore another possibility to give sense to this duality, based on the double nature of the (generalized) Dirac operator $D$ which is at the heart of Connes approach to noncommutative geometry [6]. This is indeed the same operator $D$ which

- provides an action which naturally incorporates a high energy cut-off. This is the spectral action [1]

$$\text{Tr} f \left( \frac{D}{\Lambda} \right)$$

(1.1)

where $f$ is the characteristic function of the interval $[0,1]$ and $\Lambda$ a energy scale of unification. We refer to [2] for details on the choice of the operator $D$ and how the asymptotic expansion $\Lambda \to \infty$ yields the standard model of elementary particles minimally coupled with (Euclidean) general relativity (see also [13] for an highly readable introduction to the subject, [8, 3] and [14, 11] for recent developments).

- defines a metric on the space $\mathcal{S}(\mathcal{A})$ of states\(^1\) of an algebra $\mathcal{A}$, provided the later acts on the same Hilbert space $\mathcal{H} = D$ in such a way that

$$L_D(a) := ||[D, \pi(a)]||$$

(1.2)

is finite for any $a \in \mathcal{A}$ ($\pi$ denotes the representation of $\mathcal{A}$ as bounded operators on $\mathcal{H}$). This is the spectral distance [5]

$$d_{\mathcal{A},D}(\varphi, \psi) := \sup_{a \in \mathcal{A}} \{|\varphi(a) - \psi(a)|, L_D(a) \leq 1\}$$

(1.3)

for any $\varphi, \psi$ in $\mathcal{S}(\mathcal{A})$. For $\mathcal{A} = C^\infty(\mathcal{M})$ the algebra of smooth functions on a manifold $\mathcal{M}$ and

$$D = \partial$$

the usual Dirac operator of quantum field theory, the spectral distance computed between pure states (which are nothing but the points of $\mathcal{M}$, viewed as the application $\delta_x : f \to f(x)$) gives back the geodesic distance [7],

$$d_{C^\infty(\mathcal{M}),\partial}(\delta_x, \delta_y) = d_{\text{geo}}(x, y).$$

(1.4)

The action (1.1) counts the eigenvalues of the Dirac operator smaller than the energy scale $\Lambda$, which amounts to cut-off the Fourier modes with energy greater than $\Lambda$. In other terms the spectral action naturally implements an ultraviolet cut-off as a high momentum cut-off. The question we adress in this note is: can this be read as a short wave-length cutoff in the distance formula (1.3)? More precisely, by cutting-off the spectrum of the Dirac operator in the distance formula, does one transform the high-momentum cut-off $\Lambda$ in a short distance cut-off $\lambda$? Reversing the well known question on how to retrieve the shape of a drum from its vibration modes, our point here is to understand what drum can one design from hearing the spectral action? We have shown in [8] that the answer is not obvious, and asks for a careful discussion on the nature of the points of the “cut-off drum”. We report here some of these results, in a non technical manner.

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\(^1\)A state $\varphi$ of $\mathcal{A}$ is a linear application $\mathcal{A} \to \mathbb{C}$ which is positive ($\varphi(a^*a) \in \mathbb{R}^+$) and of norm 1 (in case $\mathcal{A}$ is unital, this means $\varphi(I) = 1$).
2. Cutting-off the geometry

We implement a cut-off by the conjugate action of a projection $P_\Lambda$ acting on $\mathcal{H}$:

$$D \rightarrow D_\Lambda := P_\Lambda DP_\Lambda. \tag{2.1}$$

Typically $P_\Lambda$ is a projection on the eigenspaces of $D$, but for the moment it is simply a projection acting on $\mathcal{H}$. We are interested in computing the spectral distance on a manifold $\mathcal{M}$ induced by this cut-off, that is formula (1.3) for $\mathcal{A} = C^\infty(\mathcal{M})$ but with $D$ substituted with $D_\Lambda$.

Suppose that $D_\Lambda$ is a bounded operator with norm $\Lambda \in \mathbb{R}^+$. Then one has $\text{[8, Prop. 5.1]}$

$$d_{C^\infty(\mathcal{M}),D_\Lambda}(\delta_x, \delta_y) \geq \Lambda^{-1}. \tag{2.2}$$

This seems to be precisely the answer one was expecting: by cutting-off the spectrum of $D$ below $\Lambda$, one is not able to probe space with a resolution better than $\Lambda^{-1}$. Unfortunately (2.2) is an inequality, not an equality. There is as expected a lower bound to the resolution on the position space, but nothing guarantees that this bound is optimal. In particular if $D_\Lambda$ has finite rank, then the distance is actually infinite [8, Prop. 5.4]. For $\mathcal{M}$ compact, this happens for instance when $P_\Lambda = P_N$ is the projection on the first $N$ Fourier modes for some $N \in \mathbb{N}$. Then $D_\Lambda = D_N := P_N DP_N$ has finite rank and the distance between any two points is infinite,

$$d_{C^\infty(\mathcal{M}),D_N}(\delta_x, \delta_y) = \infty. \tag{2.3}$$

In other terms, cutting-off all but a finite number of Fourier modes destroys the metric structure of the manifold. It is an open question whether the distance remains finite for a bounded $D_\Lambda$ with infinite rank.

Eq. (2.3) illustrates the tension between truncating the Dirac operator while keeping the usual notion of points. A solution to maintain a metric structure is to truncate point as well. This actually makes sense in full generality, that is for $\mathcal{A}$ non-necessarily commutative, acting on some Hilbert space $\mathcal{H}$ together with an operator $D$ such that $L_D(a)$ is finite for any $a \in \mathcal{A}$. Given a finite rank projection $P_N$ in $\mathcal{B}(\mathcal{H})$, we then define

$$\mathcal{O}_N := P_N \pi(\mathcal{A}_{sa}) P_N \tag{2.4}$$

where $\mathcal{A}_{sa}$ is the set of selfadjoint elements of $\mathcal{A}$. The set $\mathcal{O}_N$ has no reason to be an algebra but it has the structure of ordered unit space, which is sufficient to define its state space $\mathcal{S}(\mathcal{O}_N)$ and to give sense to formula (1.3) (substituting the seminorm $L_D$ with the seminorm

$$L_N := ||[D_N, \cdot]|| \tag{2.5}$$

and $\mathcal{A}$ with $\mathcal{O}_N$). In addition to the original distance $d_{\mathcal{A},D}$, one thus inherits from the cut-off two “truncated” distances: $d_{\mathcal{A},D_N}$ on $\mathcal{S}(\mathcal{A})$ and $d_{\mathcal{O}_N,D_N}$ on $\mathcal{S}(\mathcal{O}_N)$. To make the comparison between these distances possible, we use the injective map $\varphi^\sharp := \varphi \circ \text{Ad}P_N$ that sends a state $\varphi$ of $\mathcal{O}_N$ to a state $\varphi^\sharp$ of $\mathcal{A}$. By pull back, one obtains three distances on $\mathcal{S}(\mathcal{O}_N)$:

$$d_{\mathcal{O}_N,D_N}(\varphi, \psi), \quad d_{\mathcal{A},D}(\varphi, \psi) := d_{\mathcal{O}_N,D_N}(\varphi^\sharp, \psi^\sharp), \quad d_{\mathcal{A},D}(\varphi, \psi) := d_{\mathcal{O}_N,D_N}(\varphi^\sharp, \psi^\sharp). \tag{2.6}$$
A sufficient condition that makes the truncated distance \( d_{\mathcal{O},D_N} \) equivalent to the “bi-truncated” distance \( d_{\mathcal{O},D} \) is that [8, Prop. 3.5] the seminorm \( L_N := ||[D_N, \cdot]|| \) is Lipschitz [14], meaning that \( L_N(a) = 0 \) if and only if \( a = \mathbb{C}I \). If in addition the non-truncated seminorm \( L_D \) is Lipschitz, or \( P_N \) is in the commutant \( \mathcal{A}' \) of the algebra \( \mathcal{A} \), or \( P_N \) commutes with \( D \), then one also has that \( d_{\mathcal{A},D} \) is equivalent to \( d_{\mathcal{O},D} \).

In the commutative case, the set \( \mathcal{F}(\mathcal{O}_N) \) permits to give a precise meaning to the notion of truncated points. By this we mean a sequence of states of \( \mathcal{O}_N \) that tends to some \( \delta_x \) as \( N \to \infty \). For instance on the circle, that is \( \mathcal{A} = C^\infty(S^1) \), the Fejer transform

\[
\Psi_{x,N}(f) = \sum_{n=-N}^{N} (1 - \frac{|n|}{N+1}) f_n e^{inx}, \quad N \in \mathbb{N}
\] (2.7)

is a state of \( \mathcal{O}_N \) for \( P_N \) the projection on the first \( N \) negative and \( N \) positive Fourier modes [8, Lem. 5.10]. It is an approximation of the point \( x \in S^1 \) in that

\[
\lim_{N \to \infty} \Psi_{x,N}(f) = f(x) \quad \forall f \in C^\infty(S^1).
\] (2.8)

Moreover this approximation deforms the metric structure of the circle but does not destroy it, since - with \( D \) the usual Dirac operator of \( S^1 \) - the bi-truncated distance between any two Fejer transforms is finite for any \( N \) [8, Prop. 5.11],

\[
d_{\mathcal{O},D_N}(\Psi_{x,N},\Psi_{y,N}) \leq d_{\text{geo}}(x,y),
\] (2.9)

and tends towards the geodesic distance for large \( N \),

\[
\lim_{N \to \infty} d_{\mathcal{O},D_N}(\Psi_{x,N},\Psi_{y,N}) = d_{\text{geo}}(x,y) \quad \forall x,y \in S^1.
\] (2.10)

A similar example has been worked out on the real line [8, Prop. 5.7].

3. Convergence of truncations

Let us study in a more systematic way the idea introduced in the previous section of approximating a state by a sequence of truncated states. To this aim, take \( \mathcal{A} \), \( \mathcal{H} \) and \( D \) satisfying the conditions of the precedent section, and let us consider a sequence \( \{P_N\}_{N \in \mathbb{N}} \) of increasing finite-rank projections, weakly converging to 1. Under which conditions can states of \( \mathcal{A} \) be approximated by states of \( \mathcal{O}_N \) in such a way that the metric structure is preserved?

For normal states \(^2\varphi \in \mathcal{F}(\mathcal{A}) \) is normal if and only if there exists a positive traceclass operator \( R \) on \( \mathcal{H} \) (the density matrix) such that \( \varphi(a) = \text{Tr}(Ra) \), \( \forall a \in \mathcal{A} \).

\[
\lim_{N \to \infty} d_{\mathcal{A},D}(\varphi, \varphi_N^\sharp) = 0
\] (3.1)

where \( \varphi_N \) is the state with density matrix \( Z_N^{-1}R \) where \( Z_N := \text{Tr}(P_N R) \). One also has the convergence in the sense of metric spaces [8, Prop. 4.3]: \( \langle \mathcal{F}(\mathcal{O}_N), d_{\mathcal{A},D} \rangle \) converges to \( \langle \mathcal{N}(\mathcal{A}), d_{\mathcal{A},D} \rangle \) for the Gromov–Hausdorff distance.

\(^2\varphi \in \mathcal{F}(\mathcal{A}) \) is normal if and only if there exists a positive traceclass operator \( R \) on \( \mathcal{H} \) (the density matrix) such that \( \varphi(a) = \text{Tr}(Ra), \forall a \in \mathcal{A} \).
In case the topology of the spectral distance is not the weak* topology, the answer is more challenging. A preliminary step is to work out a class of states at finite distance from one another. In the commutative case, for $\mathcal{M}$ connected and complete, such a class is given by states with finite moment of order 1. Recall that there is a 1-to-1 correspondence between states $\varphi$ of $C_0^*(\mathcal{M})$ and probability measures $\mu$ on $\mathcal{M}$, 

$$\varphi(f) = \int_{\mathcal{M}} f(x) \, d\mu(x) \quad \forall f \in C_0(\mathcal{M}).$$

For $\mathcal{M}$ connected, the finiteness of the moment of order 1 of $\varphi$,

$$\mathcal{M}_1(\varphi, x') := \int_{\mathcal{M}} d_{\text{geo}}(x, x') \, d\mu(x)$$

(3.2)

does not depend on the choice of $x' \in \mathcal{M}$ and so is intrinsic to the state. If furthermore $\mathcal{M}$ is complete, one has that the spectral distance $d_\varphi$ between states with finite moment of order 1 is finite (see e.g. [9]).

In the noncommutative case, the correspondence between states and probability measure on the pure state space is no longer 1-1, as can be seen on easy examples such as $M_2(C)$. However in [8, §4.2] we proposed to give meaning to the notion of “finite moment of order 1” for normal states in the following way. Let $\varphi$ be a normal state with density matrix $R$. Fix an orthonormal basis $\mathcal{B} = \{\psi_n\}_{n \in \mathbb{N}}$ of $\mathcal{H}$ made of eigenvectors of $R$, with eigenvalues $p_n \in \mathbb{R}^+$. Denote $\Psi_n(a) := \langle \psi_n, a \psi_n \rangle$ the corresponding vector states in $S(A)$ so that $\varphi(a) = \sum_{n \geq 0} p_n \Psi_n(a) \quad \forall a \in A$.

We call moment of order 1 of $R$ with respect to the eigenbasis $\mathcal{B}$ and to a state $\Psi_k$ (induced by a vector $\psi_k \in \mathcal{B}$) the quantity

$$\mathcal{M}_1(R, \mathcal{B}, \Psi_k) := \sum_{n \geq 0} p_n d_{\mathcal{A}}(\Psi_k, \Psi_n).$$

(3.3)

Unlike the commutative case the finiteness of (3.3) is not intrinsic to the density matrix (hence even less to the state), because for the same density matrix $R$ one may have that $\mathcal{M}_1(R, \mathcal{B}, \Psi_k)$ is infinite for a given basis while $\mathcal{M}_1(R, \mathcal{B}', \Psi_k')$ is finite for another one [8, Ex. 4.6]. However, once fixed $\mathcal{B}$, the finiteness of $\mathcal{M}_1(R, \mathcal{B}, \Psi_k)$ does not depend on $\Psi_k$. We write $\mathcal{N}_0(\mathcal{A})$ the set of normal states for which there exists at least one density matrix $R$ with an eigenbasis $\mathcal{B} = \{\psi_n\}$ such that

$$\mathcal{M}_1(R, \mathcal{B}, \Psi_n) < \infty.$$ 

(3.4)

Consider then an increasing sequence $\{P_N\}_{N \in \mathbb{N}}$ of projections weakly convergent to 1. For any $\varphi \in \mathcal{N}_0(\mathcal{A})$ such that (3.4) holds for an eigenbasis $\mathcal{B}$ in which the $P_N$’s are all diagonal, there exists a sequence $\varphi_N \in \mathcal{S}(O_N)$ such that

$$\lim_{N \to \infty} d_{\mathcal{A}}(\varphi, \varphi_N^\sharp) = 0.$$

(3.5)

In other terms, any $\varphi \in \mathcal{N}_0(\mathcal{A})$ can be approximated in the metric topology by a truncation $\varphi_N$. However unlike (3.1) where the truncating-projections $P_N$ where fixed once for all, in case the metric topology is not the weak* the truncating procedure may depend on the state.
4. An unbounded example: Berezin quantization of the plane

We conclude by an example where the truncated Dirac operator is not bounded: the Berezin quantization of the plane. We omit the details that can be found in [8, §6.2]. For other applications of noncommutative geometry to Berezin quantization, see [12].

One starts with $H = L^2(\mathbb{C}, d\pi)$ and, for $\theta > 0$, define $P_\theta$ as the projection on the subspace

$$H_\theta := \text{Span}\left\{ h_n(z) := \frac{z^n}{\sqrt{\theta^{n+1}n!}} e^{-|z|^2/2\theta} \right\}_{n \in \mathbb{N}}. \quad (4.1)$$

For $D$ the Dirac operator of the Euclidean plane, the truncated Dirac operator

$$D_\theta := (P_\theta \otimes \mathbb{I}_2) D (P_\theta \otimes \mathbb{I}_2) = \frac{2}{\sqrt{\theta}} \left( \begin{array}{cc} 0 & a^+ \\ a & 0 \end{array} \right) \quad (4.2)$$

is unbounded ($a^+, a$ are the creation, annihilation operators on the $h_n$'s).

Let $\mathcal{A} = \mathcal{S}(\mathbb{R}^2)$ denote the algebra of Schwartz functions on the plane, and denote $\mathcal{O}_\theta$ the order unit space generated by $P_\theta f P_\theta$ (for $f = f^* \in \mathcal{A}$). Both act on $H \otimes \mathbb{C}^2$. For any states $\varphi, \psi$ of $\mathcal{A}$, define

$$d^{(\theta)}_{\mathcal{A}, D}(\varphi, \psi) := \sup_{f = f^* \in \mathcal{A}} \left\{ \varphi(f) - \psi(f), \|[D, B_\theta(f)]\| \leq 1 \right\} \quad (4.3)$$

where

$$B_\theta(f) : z \rightarrow \langle \psi_z, P_\theta f P_\theta \psi_z \rangle \quad \text{where} \quad \psi_z = e^{-|z|^2/2\theta} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\theta^{n+1}n!}} h_n \quad (4.4)$$

is the Berezin transform of $f$. One gets [8]

$$d_{\mathcal{A}, D}(\varphi^z, \psi^z) \leq d_{\mathcal{O}_\theta, D_\theta}(\varphi, \psi) \leq d^{(\theta)}_{\mathcal{A}, D}(\varphi^z, \psi^z). \quad (4.5)$$

In particular, the distance between coherent states $\Psi_z, \Psi_{z'}, z, z' \in \mathbb{C}$, is

$$d_{\mathcal{O}_\theta, D_\theta}(\Psi_z, \Psi_{z'}) = |z - z'|. \quad (4.6)$$

A similar result was found in [13] from a completely different perspective, based on the construction of the element that attains the supremum in the distance formula. This illustrates that the cut-off procedure could be an efficient tool to make explicit calculations of the distance.

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