Effective potential for the conformal sector of quantum gravity with torsion

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Abstract

The effective potential which describes the conformal dynamics of quantum gravity with torsion is discussed. The phase transitions induced by the combination of torsion and curvature are investigated. The mechanism for fixing the vacuum expectation values of the metric and torsion is presented.
1. The concept of effective potential plays a very important role in modern particle physics [1]. The number and the variety of applications of effective potentials is growing continuously [1] and even the quite complicated two-loop effective potential for the standard model could be found [2], summing also the leading logarithms.

From another point of view, the scalar field effective potential has many application to early universe considerations. For instance, such a potential is required for the analysis of the inflationary universe (see [3], for a general review), where curvature effects are of the essence. It is also rather well-known [4], that the effective potential in curved spacetime shows very interesting features, such as dynamical symmetry breaking induced by curvature [4] and gravitational phase transitions (for a general review see [5]).

Recently it has been found that the effective potential concept may be important also for fixing up the vacuum expectation value in quantum gravity. One of such scenarios is based on the conformal dynamics of quantum gravity, as developed in papers [6] very recently. The conformal dynamics of quantum gravity are induced by the conformal anomaly and can be reduced to the description of infrared quantum gravity, i.e., of quantum gravity at very large scales [6].

The conformal dynamics of quantum gravity with torsion have been further discussed in refs. [7], which develop the approach of ref. [6]. The inclusion of torsion in the discussion seems quite reasonable nowadays, since we certainly know that torsion appears in string theory in a natural way (as an axion). Furthermore, there is an interest for torsion in some other contexts, stemming from the search of the so-called fifth force.

In a recent work, [8], an attempt has been made at studying the phase structure of the effective potential corresponding to the conformal factor, in the effective theory of infrared quantum gravity [6]. For instance, the curvature-induced (or gravitational) phase transitions which take place in such a situation can be considered (in some context) as the analog of the $c = 1$ phase transition of non-critical string theory.

In the present work we study the phase structure of the conformal-factor effective potential for quantum gravity with torsion. The general expression describing the conformal dynamics of quantum gravity with torsion in a curved fiducial background is obtained. The one-loop effective potential for the conformal factor (up to terms linear on the curvature and up to second order in torsion) is also calculated. Then, the phase transitions induced by the combination of background torsion and curvature are investigated. As a consequence, the relevant mechanism for fixing the vacuum expectation values of the metric and torsion, at the critical point corresponding to the phase transition, is given. Notice that a similar idea consisting in fixing the metric at the minimum of the effective potential corresponding to the conformal factor —for the case of a flat background in the approach [9]— have been discussed.
2. Let us start from the trace anomaly for a free conformal invar iant theory in curved spacetime with torsion (see refs. [5,11] for details)

\[ T_{\mu}^\mu = bC_{\mu\alpha\beta}^2 + b' \left( G - \frac{2}{3} \square R \right) + \left[ b'' + \frac{2}{3} (b + b') \right] \square R + a_1 F_{\mu\nu}^2 \\
+ a_2 (S_{\mu} S_{\nu})^2 + a_3 \Box (S_{\mu} S_{\nu}) + a_4 \nabla_{\mu} (S_{\nu} \nabla^\nu S_{\mu} - S_{\mu} \nabla^\nu S_{\nu}), \]  

(1)

where \( G \) is the Gauss-Bonnet invariant, \( F_{\mu\nu} = \nabla_{\mu} S_{\nu} - \nabla_{\nu} S_{\mu} \), \( b, b' \) and \( b'' \) are known (see, for example, [11]), and where the coefficients \( a_i \) relevant for non-zero torsion are given by [5]

\[ a_1 = -\frac{2}{3 (4\pi)^2} \Sigma \eta^2, \quad a_2 = \frac{1}{2 (4\pi)^2} \Sigma \zeta^2, \]
\[ a_3 = \frac{1}{3 (4\pi)^2} \Sigma \left( 2 \eta^2 - \frac{1}{2} \zeta^2 \right), \quad a_4 = -\frac{2}{3 (4\pi)^2} \Sigma \eta^2. \]  

(2)

Notice that the coupling constants that appear in (2) come from the free conformally invariant theory (spinors and scalars) in torsionful spacetime:

\[ S_0 = \frac{1}{2} \int d^4 x \sqrt{-g} \left( g^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi + \frac{1}{6} R \varphi^2 + \zeta_1 S_\mu S^\mu \varphi^2 \right), \]
\[ S_{1/2} = i \int d^4 x \sqrt{-g} \left[ \bar{\psi} (\gamma^\mu \nabla_\mu - \eta \gamma_5 \gamma^\mu S - \mu) \psi \right]. \]  

(3)

These theories (3) are conformally invariant, for any \( \zeta_1 \) and \( \eta \), and the minimal coupling corresponds to \( \zeta_1 = 0 \) and \( \eta = 1/8 \). Notice that vectors do not interact at all minimally with torsion (see, for example, ref. [5]).

A further remark has to do with quantum gravity effects themselves. Generally speaking, the coefficients of the trace anomaly (1) contain also a contribution from spin 2 modes. Such contribution to \( b \) and \( b'' \) (coming from the Einstein or Weyl gravity action) have been calculated explicitly in the last one of refs. [6]. It is to be expected that some new contributions from gravity to \( a_1, \ldots a_4 \) will appear, because we now consider quantum gravity with torsion. However, it turns out that quantum gravity contributions to \( a_1, \ldots a_4 \) remain absent, owing to the fact that torsion shows up without derivatives (at least in the Einstein-Cartan theory). Hence, the torsion field looks non-dynamical. Of course, a direct calculation is necessary in order to prove this statement. Moreover, the situation will certainly change in higher derivative quantum gravity with torsion, in which case torsion is actually dynamical, already at the classical level.

As the next step of our procedure, we choose the conformal parametrization

\[ g_{\mu\nu} = e^{2\sigma(x)} \bar{g}_{\mu\nu}, \quad S_{\mu} = \bar{S}_{\mu}, \]  

(4)
where $\sigma$ is the conformal factor, $\bar{g}_{\mu\nu}$ is a fixed fiducial metric, and $\bar{S}_\mu$ is an arbitrary torsion background. Then one can integrate over the trace anomaly in order to get the trace-anomaly-induced action $S_{\text{anom}}$ (see [5-7] for details).

Adding the classical gravity action

$$S_{\text{cl}} = \frac{1}{2k} \int d^4x \sqrt{-g} \left( R + hS_\mu S^\mu - 2\Lambda \right)$$

in the parametrization (4) to $S_{\text{anom}}$, we get the total effective action which describes the quantum conformal factor dynamics

$$S_{\text{eff}} = S_{\text{anom}} + S_{\text{cl}} = \int d^4x \sqrt{-g} \left\{ -\frac{\theta^2}{(4\pi)^2} \sigma \Box^2 \sigma + \sigma \left[ 2 \left( \zeta - \frac{\theta^2}{(4\pi)^2} \right) R^\mu_\nu \nabla_\mu \nabla_\nu - \left( \frac{2\theta^2}{3(4\pi)^2} \right) R \Box - \frac{1}{3(4\pi)^2} \frac{\theta^2}{(4\pi)^2} (\nabla_\mu R) \nabla_\mu \right] \sigma 
- \zeta \left[ 2\alpha(\nabla_\mu \sigma)(\nabla_\mu \sigma) \Box - \alpha^2 ((\nabla_\mu \sigma)(\nabla_\mu \sigma))^2 \right]
+ \frac{\gamma}{6\alpha^2} e^{2\alpha\sigma} R + \left[ b'' + \frac{2}{3}(b + b') \right] R(\nabla_\mu \sigma)(\nabla_\mu \sigma) + \gamma e^{2\alpha\sigma}(\nabla_\mu \sigma)(\nabla_\mu \sigma) - \frac{\lambda}{\alpha^2} e^{4\alpha\sigma}
+ \frac{h}{2k\alpha^2} e^{2\alpha\sigma} S^2 + \left( a_3 + \frac{a_4}{2} \right) S^2(\nabla_\mu \sigma)(\nabla_\mu \sigma) + a_4 S_\mu S^\nu (\nabla_\mu \sigma)(\nabla_\nu \sigma) \right\},$$

where $\theta^2/(4\pi)^2 = 2b + 3b''$, $\zeta = 2b + 2b' + 3b''$, $\gamma = 3/k$ and $\lambda = \Lambda/k$, and where the transformations $\sigma \to \alpha \sigma$ and $S_{\text{eff}} \to \alpha^{-2} S_{\text{eff}}$ have been performed. One should notice that, when calculating $S_{\text{anom}}$ we have dropped the $\sigma$-independent terms and also the terms linear on $\sigma$ (as is usually done in quantum field theory). Moreover, the bar over $g^{\mu\nu}$ and $S_\mu$ has been omitted in (6) —as will be done in what follows. The only quantum field in Eq. (6) is $\sigma$.

The following remark is in order. In two dimensions, a conformal parametrization, such as (4), completely fixes the metric. Hence, the metric can be chosen to be flat, for simplicity. On the contrary, in four dimensions this is of course not the case, and we need to consider the curved fiducial background $\bar{g}_{\mu\nu}$. As we will see, this makes things rather non-trivial and leads to a very complicated effective potential.

3. The one-loop $\beta$-functions in the purely gravitational sector of the theory defined by the action (6) have been calculated in refs. [6], and in the case of a torsionful sector, in refs. [7]. In what follows, we will investigate the theory given by (6) around the infrared stable fixed point $\zeta = 0$, which presumably describes 4$d$ quantum gravity at large distances (i.e., infrared quantum gravity [6]). As is clear, the action (6) becomes much simpler at $\zeta = 0$. Let us now study the effective potential of the composite field $\Phi = e^{\alpha\sigma}$ (one should have in mind that the classical scaling dimension of $\Phi$ is $1 - \alpha$).
As first example, we shall choose $\bar{S}_\mu = 0$ and $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$. Then, one finds that the Coleman-Weinberg like effective potential is given by

$$V^{(1)}(\Phi) = \frac{\lambda}{\alpha^2} \Phi^4 + \frac{1}{2} \left[ \frac{\gamma^2(4\pi)^2}{2\theta^4} - \frac{8\lambda}{\theta^2} \right] \Phi^4 \left( \ln \frac{\Phi^2}{\mu^2} - \frac{25}{6} \right).$$

(7)

By simple inspection of the form of this potential we already see that its classical minimum corresponds to the singularity $\Phi = 0$, in terms of the original metric.

As a result of the Coleman-Weinberg symmetry breaking ($\mu^2 = e^{2\alpha\sigma_0}$),

$$\Phi \frac{\mu^2}{\mu^2} = \exp \left[ \frac{11}{3} - \frac{2\lambda}{\alpha^2} \left( \frac{\gamma^2(4\pi)^2}{2\theta^4} - \frac{8\lambda}{\theta^2} \right) \right].$$

(8)

We thus find that the singularity has been avoided, because now a non-singular value of $\Phi$ corresponds to the ground state (minimum) of the potential. Notice that similar type of (logarithmic) corrections to the conformal-field effective potential in quantumb $R^2$-gravity have been discussed in ref. [10]. The appearance of logarithmic corrections which, generally speaking, destroy the general covariance in terms of the original metric $g_{\mu\nu}$ has been explained there as being a result of the regularization procedure. Some indications were found also pointing out at the fact that the physical metric to be used should be actually $\bar{g}_{\mu\nu}$, and not $g_{\mu\nu}$.

As the next step, we will now calculate the effective potential for non-zero background curvature and torsion. For such a calculation we shall adopt the approach used in [14] (see also [5]), in which we only consider linear dependences on invariants of the curvature and torsion, namely we suppose that $\Phi^2 >> |R|$ and $\Phi^2 >> S^2$ (here $R$ and $S$ are the fiducial curvature and torsion, respectively). Using the techniques of ref. [14] and the explicit form of the one-loop $\beta$-functions [6,7] of the theory (6), we obtain the following effective potential

$$V^{(1)}(\Phi) = \frac{\lambda}{\alpha^2} \Phi^4 + \frac{1}{2} \left[ \frac{\gamma^2(4\pi)^2}{2\theta^4} - \frac{8\lambda}{\theta^2} \right] \Phi^4 \left( \ln \frac{\Phi^2}{\mu^2} - \frac{25}{6} \right)$$

$$- \frac{h\gamma}{6\alpha^2} \Phi^2 S^2 + \left[ \frac{\gamma(4\pi)^2}{2\theta^4} \left( a_3 + \frac{3}{4} a_4 \right) - \frac{h\gamma}{6\theta^2} \right] \Phi^2 S^2 \left( \ln \frac{\Phi^2}{\mu^2} - 3 \right)$$

$$- \frac{\gamma}{6\alpha^2} \Phi^2 R + \frac{\gamma}{6\theta^2} \Phi^2 R \left( \ln \frac{\Phi^2}{\mu^2} - 3 \right),$$

(9)

where Coleman-Weinberg normalization conditions have been used (in the absence of torsion, gravitational phase transitions have been discussed in ref. [8]).

4. The rest of the work is devoted to the investigation of phase transitions induced by curvature and torsion. It is interesting to notice that the effective potential (9) should be relevant also in the discussion of the problem of the cosmological constant [13].
Starting from Eq. (9) and viewing the potential $V(1)$ as a function of $\Phi^2$, $R$ and $S^2$, it is not difficult to see that a critical point satisfying the conditions of our approximation (in particular, of course, different from the trivial one at 0) can be obtained only in the case that the following equation is satisfied:

$$a_3 + \frac{3}{4} a_4 = \frac{h \theta^2}{24 \pi^2},$$

and also that $\theta^2 > \alpha^2 \ln(7/6)$. Provided that this is the case, than a whole plane of critical points is obtained at

$$\frac{\Phi_{cr}^2}{\mu^2} = 3 + e^{\theta^2/\alpha^2},$$

$$\frac{h R_{cr} + S_{cr}^2}{\mu^2} = -e^{\theta^2/\alpha^2} \left\{ \frac{2 \lambda}{\alpha^2} + 4 \left( \frac{2 \pi^2 \gamma^2}{\theta^4} - \frac{\gamma}{\theta^2} \right) \left[ 2 \ln \left( e^{\theta^2/\alpha^2} - 7/6 \right) + \frac{e^{\theta^2/\alpha^2} + 3}{e^{\theta^2/\alpha^2} - 7/6} \right] \right\},$$

all of them with the same value for the potential:

$$V_{cr}^{(1)} = \mu^4 \left( 3 + e^{\theta^2/\alpha^2} \right)^2 \left[ \frac{\lambda}{\alpha^2} + 4 \left( \frac{2 \pi^2 \gamma^2}{\theta^4} - \frac{\gamma}{\theta^2} \right) \ln \left( e^{\theta^2/\alpha^2} - 7/6 \right) \right].$$

The conditions of our approximation are satisfied provided that

$$\theta^2 >> \frac{2 \pi^2 \gamma^2}{\lambda}.$$  

As second case, we will now investigate the situation in which the $\beta$-functions vanish [6]—hence, the trace of the energy-momentum tensor for the $\sigma$-field sector vanishes as well, as in $2d$ gravity [15]. The conditions for the vanishing of the exact beta functions for $\gamma$ and $\lambda$ are:

$$\alpha_\pm = \frac{\theta^2}{2} \left( 1 \pm \sqrt{1 - 4/\theta^2} \right), \quad \frac{\lambda}{\gamma^2} = \frac{2 \pi^2}{\theta^2} \left( 1 + \frac{4 \alpha^2}{\theta^2} + \frac{6 \alpha^4}{\theta^4} \right).$$

The first equation (14) determines, at the fixed point, the anomalous scaling dimension $\alpha$ in terms of the central charge $\theta^2$. Using the conditions (14), we see that the one-loop $\beta$-function for $h$ vanishes at [7]

$$a_3 = -\frac{3}{4} a_4.$$  

This corresponds to putting $\eta^2 = \zeta^2$. Using these conditions, (14) and (13), in the effective potential (9), we can investigate again the possibility of a phase transition. It is easy to see, from the analysis carried out before for the general case, that here no critical point is obtained, due to the fact that the necessary condition (14) is not satisfied (unless $h$ or $\theta$ is equal to zero, what would be unphysical). However, in the extreme case of very small $h$ we again obtain a critical point, which must necessarily be considered as driven by the torsion.
$S^2$, and which is obtained as a particular case of the first analysis above (the same equations are valid, just setting $h = 0$).

5. In conclusion, we have discussed in this paper the phase structure of the effective potential for the conformal sector of quantum gravity with torsion. A concrete mechanism that defines the vacuum expectation values of the metric and torsion as the ones which yield a corresponding minimum of the effective potential has been presented. For instance, in the simplest case when $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ and $\bar{S}_\mu = 0$ the expectation value of the conformal factor (8) defines the expectation value of the original metric $g_{\mu\nu} = D^{2/\alpha} \eta_{\mu\nu}$.

It is known that for $\Lambda = 0, 1/\kappa = 3...$? there exist classical time dependent solutions for the conformal factor, in theories of the De Sitter type. These theories may be understood, for example, in terms of four dimensional conformal gravity, as corresponding to inflationary universes. Hence, it would be interesting to study the effective action for $\sigma$. In fact, the time dependent solutions of this effective action may find important applications in inflationary cosmology.

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