2D String Theory Coupled to Quantum Gravity

Nobuyuki Ishibashi

Theory Group, KEK, Tsukuba, Ibaraki 305, Japan

ABSTRACT

We consider self-avoiding Nambu-Goto open strings on a random surface. We have shown that the partition function of such a string theory can be calculated exactly. The string susceptibility for the disk is evaluated to be $-\frac{1}{2}$. We also consider modifications of the Nambu-Goto action which are exactly soluble on a random surface.
1 Introduction

String theory appears in many aspects of physics. It is an old idea that QCD might be represented as a string theory[1]. Also, three dimensional Ising model was argued to be described as a three dimensional superstring theory[2]. Therefore, it is important to search for consistent string theories in each dimension and classify the universality classes of them. Such string theories will be useful in describing various physical systems as point particle field theories was.

However, constructing consistent string theories appropriate for such applications is not an easy task. It is always possible to construct low dimensional string theories by compactifying critical string theories, but such theories have features like the existence of a massless spin two particle, which are not desirable in most cases. Hence we should look for noncritical string theories. However, constructing a consistent string theory preserving Lorentz or rotation invariance is not so easy. The noncritical Polyakov string was quantized by the authors of [3], but their quantization is not consistent in \( d \) dimensional space-time with \( 1 < d < 25 \). The string susceptibility becomes complex, if one naively applies the KPZ formula to these cases. Therefore one should overcome this “\( c = 1 \) barrier” in order to construct Polyakov string theories of physical interest.

There can be another approach to noncritical string theory, namely the direct quantization of the Nambu-Goto action. Usually the Nambu-Goto action is quantized by transforming it into the Polyakov action. In 26 dimensional space-time, we can quantize the Nambu-Goto string directly and it gives the same results as that of the quantization à la Polyakov. However, there is no reason to believe in their equivalence in the noncritical cases. Therefore, it is possible that the Nambu-Goto string has no “\( c = 1 \) barrier” when directly quantized. The Nambu-Goto string is manifestly Lorentz or rotation invariant and it can be useful in describing various systems. Unfortunately the Nambu-Goto string has a highly nonlinear action and intractable in general. Here let us discuss the two dimensional self-avoiding Nambu-Goto string as the simplest case in this direction.

In two dimensional string theories, the embedding of the worldsheet into the space-time is generically singular, involving folds. If one requires self-avoidingness, such folds are forbidden and the Nambu-Goto action becomes trivial for closed strings[4]. However for open strings the theory is not so
trivial. For example, the partition function corresponding to the disk graph of such an open string can be written as

\[ Z(\mu) = \sum_{\Gamma} \exp(-\mu A(\Gamma)). \]  

(1)

Here \( \Gamma \) denotes the self-avoiding loop in the space-time corresponding to the boundary of the open string graph and the sum is over such loops. \( A(\Gamma) \) is the area of the space-time \( \Gamma \) encloses. For the self-avoiding open strings, the Nambu-Goto action depends only on the boundary of the graph, which coincides with \( A(\Gamma) \). This is a nontrivial sum and it is not sure if this open string theory is consistent contrary to the usual noncritical string case.

This self-avoiding open string theory may be applied to two dimensional systems of interest, e.g. two dimensional QCD\[5\], which have been actively studied recently\[6\]. The expectation value of the Wilson loop for two dimensional pure Yang-Mills theory is known to be

\[ \langle \text{Tr}_R \mathcal{P} e^{\oint_C A_{\mu} dx^{\mu}} \rangle = e^{-g^2 C_2(R) A}, \]  

(2)

where \( g \) is the gauge coupling, \( C_2(R) \) the quadratic Casimir operator for representation \( R \) of the gauge group, and \( A \) is the area enclosed by the loop. If one tries to incorporate dynamical quarks into the theory, one should sum the above expectation value over the fluctuations of the loop \( C \). Thus we encounter a sum over loops similar to eq.(1). Eq.(1) may correspond to two dimensional QCD with self-avoiding quarks.

Anyway, it seems difficult to calculate the sum in eq.(1). In this paper, we will discuss this self-avoiding Nambu-Goto open string theory on a random surface. In other words, we will consider the string theory coupled to quantum gravity in the target space. On a random surface, it is possible to calculate the partition functions of the open string theory exactly. We will show that the string susceptibility is not complex contrary to the usual noncritical string case. This fact implies the consistency of this string theory, at least in the presence of quantum gravity in the target space. In a sense, what we are dealing with is related to two dimensional QCD with self-avoiding quarks coupled to quantum gravity.

The organization of this paper is as follows. In section 2, we consider self-avoiding random walks on a random surface as a warm-up. We will first review the techniques developed by Duplantier and Kostov\[7\] to calculate the
partition function of self-avoiding random walks on a random surface. Using their results, we express the configuration sum of a loop on a random surface in terms of the wave function of quantum gravity. In section 3, we go on to the case of self-avoiding strings and evaluate the partition function. We will show that the string susceptibility for the disk is $-\frac{1}{2}$. We also discuss modifications of the Nambu-Goto action. Section 4 contains concluding remarks.

2 Self-avoiding random walks on a random surface

The techniques needed for performing the sum in eq.(1) over random walks on a random surface was developed by Duplantier and Kostov. They considered self-avoiding walks on a random surface:

$$Z(\lambda, m) = \sum_{\Gamma, \text{metric}} e^{-ml - \lambda V}, \quad \text{(3)}$$

where the sum is over self-avoiding walks $\Gamma$ and the space-time metrics with the Boltzmann weight involving the total length $l$ of $\Gamma$ and the volume $V$ of the space-time. Let us first review their results[7].

The ensemble of random surfaces was defined by the continuum limit of the dynamical triangulation of the surfaces. If we restrict the topology of the surface to that of the sphere, the partition function $Z(\lambda, m)$ in eq.(3) is discretized as

$$Z(\beta, K) = \sum_G e^{-\beta |G|} \frac{1}{S(G)} \sum_{\Gamma} K^{\left|\Gamma\right|}. \quad \text{(4)}$$

Here $G$ denotes a planar $\phi^3$ graph, $|G|$ is the number of vertices of the graph $G$ and $S(G)$ is the symmetry factor. The dual of $G$ describes a triangulated surface. $\Gamma$ is a self-avoiding random walk on the graph $G$. $\beta$ and $K$ correspond to the parameters $\lambda$ and $m$ respectively, in the continuum limit.

Duplantier and Kostov rewrote eq.(3) in terms of the partition function $G_n$ of random graphs with $n$ external legs

$$G_n(\beta) = \sum_{\text{n leg planar } G} e^{-\beta |G|}. \quad \text{(5)}$$
Here, let us concentrate on the case where Γ in eq.(4) is a loop. Then Γ divides the surface into two parts. \( Z(\beta, K) \) can be expressed by two \( G_n \)'s representing these two parts as

\[
Z(\beta, K) = \sum_{m,n} \frac{1}{m+n} \frac{(m+n)!}{m!n!} (e^{-\beta K})^{m+n} G_m(\beta)G_n(\beta). \tag{6}
\]

\( G_n(\beta) \) was calculated using the large-\( N \) limit of the matrix model and it has an integral expression [8]:

\[
G_n(\beta) = \int_{2a}^{2b} d\lambda \rho(\lambda) \lambda^n. \tag{7}
\]

Here \( \rho(\lambda) \) is the density of eigenvalues of the \( N \times N \) matrix given in [8]. Substituting this expression, we obtain

\[
Z(\beta, K) = -\int_{2a}^{2b} d\lambda_1 \int_{2a}^{2b} d\lambda_2 \rho(\lambda_1)\rho(\lambda_2) \ln(1 - e^{-\beta K(\lambda_1 + \lambda_2)}). \tag{8}
\]

The singular behaviour of \( Z(\beta, K) \) can be seen from this integral representation. There is a critical point \( \beta_c \) for \( \beta \) which corresponds to the pure gravity critical point. The critical point for \( K \) is at

\[
K_c = e^{\beta_c}/4b_c, \tag{9}
\]

where \( b_c \) is the value of \( b \) at \( \beta = \beta_c \). When \( K \) approaches \( K_c \) simultaneously as \( \beta \) approaches \( \beta_c \), the integral in eq.(8) diverges because the point \( \lambda_1 = \lambda_2 = 2b \) in the integration region approaches the singularity of the logarithm in the integrand. Loops with infinite length dominate at this critical point.

Eq.(8) is the main result of Duplantier and Kostov, which we will use in this paper. In the rest of this section, we will calculate \( Z(\lambda, m) \) in eq.(3) using eq.(8). In order to extract the singular part and take the continuum limit, it is convenient to rewrite eq.(8) as

\[
Z(\beta, K) = \int_{2a}^{2b} d\lambda_1 \int_{2a}^{2b} d\lambda_2 \rho(\lambda_1)\rho(\lambda_2) \ln \left( \frac{1}{\Gamma(e)} \int_0^\infty dt t^{e-1} e^{-t(1-e^{-\beta K(\lambda_1+\lambda_2)})} \right), \tag{10}
\]

\footnote{Here we consider only the dilute phase in [8].}
and change the variables as
\[ e^{-\beta} = e^{-\beta_c(1-\lambda\delta^2)}, \quad K = K_c(1-m\delta), \quad t = l/\delta. \]  
\tag{11}

In the continuum limit, \( \delta \), which is supposed to be the lattice spacing, approaches zero. We use \( Z(\beta, K) \) in this limit to define the sum in eq.(3) with \( m \) and \( \lambda \) as above. As \( \delta \to 0 \),
\[ Z(\beta, K) \to \lim_{\epsilon \to 0} \epsilon \frac{\delta^5}{\Gamma(\epsilon)} \int_0^{\infty} dl l^{5-\epsilon} \Psi(l) \Psi(l) e^{-ml}, \]  
\tag{12}

where
\[ \Psi(l) = \lim_{\delta \to 0} l^{-5/2} \int_{2a}^{2b} d\mu \rho(\mu) e^{-\frac{l}{2} \frac{2b-a}{4\epsilon}}. \]  
\tag{13}

The right hand side of eq.(12) has a natural interpretation as follows. \( \Psi(l) \) is the partition function of two dimensional gravity for the disk with the boundary length \( l \), which can be regarded as the wave function of quantum gravity \[9\]. The right hand side of eq.(12) can be considered as a regularized form of
\[ \int_0^{\infty} \frac{dl}{l} \Psi(l) \Psi(l) e^{-ml}. \]  
\tag{14}

When \( m = 0 \), this formally coincides with the inner product of the wave functions. The above result shows that such an inner product gives the number of configurations of self-avoiding loops on a random surface. Since the wave function in quantum gravity does not depend on time, but rather is something integrated over the time variable, it is easy to understand intuitively that the inner product would give the number of ways of taking a time slice on a random surface. The time slice we are considering here is a self-avoiding random walk in the space-time.

Using \( \rho(\mu) \) in \[8\], we can evaluate \( \Psi(l) \) up to an overall constant factor as,
\[ \Psi(l) = l^{-5/2}(1+\sqrt{\lambda}l)e^{-\sqrt{\lambda}l}, \]  
\tag{15}

after a rescaling of the variable \( \lambda \). Substituting this expression, it is easy to calculate the right hand side of eq.(12). \( \Psi(l) \Psi(l) e^{-ml} \) in the integrand can be expressed as a finite sum of terms of the form \( l^n e^{-(m+2\sqrt{\lambda})l} \) with \( n \) an integer. The following formula is useful in evaluating the right hand side of eq.(12):
\[ \lim_{\epsilon \to 0} \frac{\delta^{-\epsilon}}{\Gamma(\epsilon)} \int_0^{\infty} dl l^{5-\epsilon} l^n e^{-(m+2\sqrt{\lambda})l} = -(l^n e^{-(m+2\sqrt{\lambda})0} \ln(m+2\sqrt{\lambda}) \]  
\tag{16a}

+ (terms analytic in \( m \)).
Here \((f(l))_0\) denotes the coefficient \(a_0\) of \(l^0\) in the Laurent expansion \(f(l) = \sum a_n l^n\). Thus we obtain as \(Z(\lambda, m)\),

\[-(\Psi(l)\Psi(l)e^{-ml})_0 \ln(m + 2\sqrt{\lambda}) + \text{terms analytic in } m. \tag{17}\]

The result includes the nonuniversal part which is analytic in \(m\) and vanishes when differentiated sufficiently many times. It is the contribution of very short loops. Therefore, the first term in eq.(17) is the universal part which should be taken as the continuum limit. We can show that it is really universal and does not change if one changes the way of discretization by making use of quadrangles etc. instead of triangles. Discarding the nonuniversal part, we obtain as the continuum limit:

\[Z(\lambda, m) = \left[ \frac{1}{5!}(m + 2\sqrt{\lambda})^5 - \frac{2}{4!}\sqrt{\lambda}(m + 2\sqrt{\lambda})^4 + \frac{\lambda}{3!}(m + 2\sqrt{\lambda})^3 \right] \ln(m + 2\sqrt{\lambda}). \tag{18}\]

### 3. Self-avoiding open strings on a random surface

It is straightforward to apply the techniques in the previous section to self-avoiding open strings on a random surface. The configuration sum we have to deal with is as follows:

\[Z(\lambda, \mu) = \sum_{\Gamma, \text{metric}} e^{-\mu A - \lambda V}. \tag{19}\]

In this case, the Boltzmann weight involves the area \(A\) enclosed by the self-avoiding walk \(\Gamma\). Let us restrict ourselves to the case where the topology of the space-time is that of the sphere. If one discretizes this sum as in the previous section, it can again be rewritten by two \(G_n\)’s each of which corresponds to the outside and the inside of the loop \(\Gamma\). However, this time the cosmological constant in the \(G_n\) corresponding to the inside is shifted because of the term \(\mu A\) in the Boltzmann weight.

\(^2(f(l))_0\) appeared in the definition of an inner product of the wave functions in \cite{9}.

\(^3\)We consider that \(\Gamma\) is oriented and we can define the inside and the outside.
Therefore, after the same procedure as in section 2, we can derive

\[ Z(\lambda, \mu) = \lim_{\epsilon \to 0} \partial_\epsilon \left( \frac{\delta - \epsilon}{\Gamma(\epsilon)} \int_0^\infty dl l^{\epsilon-1} \Psi_{\mu + \lambda}(l) \Psi_{\lambda}(l) \right). \]  

(20)

Here \( \Psi_{\lambda}(l) \) denotes the disk partition function of two dimensional quantum gravity with the cosmological constant \( \lambda \). The right hand side of eq.(20) can be considered as the regularized inner product of two wave functions with different cosmological constants. The calculation of eq.(20) goes exactly as in section 2. Substituting eq.(15), one can again use the formula eq.(16) with a slight modification. In order to select the universal part, we should insert \( e^{-ml} \) into the integrand of eq.(20), discard the terms analytic in \( m \) and take \( m \) to be 0 in the end. Eventually we obtain

\[ Z(\lambda, \mu) = \left[ -\frac{1}{30} (\sqrt{\lambda} + \sqrt{\mu + \lambda})^5 + \frac{1}{6} \sqrt{\lambda} (\sqrt{\lambda} + \mu) (\sqrt{\lambda} + \sqrt{\lambda + \mu})^3 \right] \ln(\sqrt{\lambda} + \sqrt{\lambda + \mu}). \]  

(21)

When \( \mu \gg \lambda \), the typical area of the worldsheet of the open string is much smaller than the typical area of the space-time. Then the compactness of the space-time becomes irrelevant to the fluctuations of the open string and we expect that the partition function scales as

\[ Z(\lambda, \mu) \sim \mu^{-\Gamma_{\text{disk}} + 2}. \]  

(22)

This \( \Gamma_{\text{str.}} \) should be taken as the definition of the string susceptibility for the disk. When \( \mu \gg \lambda \), eq.(21) gives

\[ Z(\lambda, \mu) \sim \mu^{\frac{3}{2}} \ln(\mu), \]  

(23)

and \( \Gamma_{\text{disk}} = -\frac{1}{2} \). Therefore the string susceptibility of this two dimensional string theory is not complex and we do not encounter the inconsistency contrary to the case of the usual noncritical string.

It is possible to generalize our self-avoiding Nambu-Goto string on a random surface as follows. The action of the Nambu-Goto string is the area of the worldsheet. However any reparametrization invariant quantity is conceivable as a string action. Let us perturb the Nambu-Goto action by reparametrization invariant operators \( O_i \). Then the sum in eq.(1) will be modified to be

\[ Z(\lambda, \mu, g_i) = \sum_{\Gamma} e^{-\mu A_{\Gamma} - \sum g_i O_i}. \]  

(24)
and we obtain instead of eq.(20),

\[ Z(\lambda, \mu, g_i) = \lim_{\epsilon \to 0} \partial_{\epsilon} \frac{\delta^{\epsilon}}{\Gamma(\epsilon)} \int_0^\infty dl l^{\epsilon-1} \Psi_{\mu+\lambda,g_i}(l) \Psi_\lambda(l). \]  

(25)

Here \( \Psi_{\mu+\lambda,g_i}(l) \) is the wave function of quantum gravity with the action perturbed by \( \sum g_i O_i \). We will show that such perturbations to the Nambu-Goto action can change the critical behaviours of the theory.

By choosing \( O_i \) to be the reparametrization invariant observables of 2d quantum gravity in [10], and fine tuning \( g_i \), \( \Psi_{\mu+\lambda,g_i}(l) \) becomes the wave function of quantum gravity in the multicritical phase[11]. The explicit form of such a wave function in the \( m \)-th multicritical point is

\[ \Psi_{\mu+\lambda,g_i}(l) = l^{-1} (\sqrt{\mu + \lambda})^{m-\frac{1}{2}} K_{m-\frac{1}{2}}(\sqrt{\mu + \lambda} l), \]  

(26)

where \( K_{m-\frac{1}{2}} \) is the Bessel function[9]. \( m = 2 \) corresponds to the pure gravity and the critical points with \( m > 2 \) can be reached starting from the pure gravity. If one perturbs the Nambu-Goto action so that the wave function \( \Psi_{\mu+\lambda,g_i}(l) \) which appears in eq.(23) is the multicritical wave function eq.(24), one can obtain a new class of string theories. We may call such a string the “multicritical string”. The string susceptibility for the disk of the multicritical string theory can be calculated using eq.(25) and eq.(26) as

\[ \Gamma_{\text{disk}} = \begin{cases} 
\frac{-m+1}{2} & \text{(if } m = 2, 4 \text{ or odd)} \\
\frac{-m+4}{2} & \text{(otherwise)} 
\end{cases}. \]  

(27)

4 Conclusions

We have shown that the two dimensional self-avoiding Nambu-Goto string is exactly solvable when it is coupled to quantum gravity. The string susceptibility can be calculated and we obtain a real value. Usually we quantize string theory à la Polyakov. If one tries to quantize the two dimensional Polyakov string, one either spoils Lorentz or rotation invariance, or encounters a complex string susceptibility. The string theory considered here is different from the usual noncritical string theory in the following three points. Firstly here

\footnote{Here we take the conformal background in [4], because we want \( \mu \) to couple to the area of the worldsheet.}
we deal with the direct quantization of the Nambu-Goto action. Second, the string here is self-avoiding. And the last point is that the string is coupled to quantum gravity. Any of these points can be the reason why the string theory has a real string susceptibility contrary to the usual noncritical string.

We have also constructed new string models on a random surface by perturbing the Nambu-Goto action. We have shown that multicritical phases can be reached by such perturbations. It is possible to calculate multiloop amplitudes and obtain the mass spectrum of such string theories. We will report on this problem elsewhere.

Acknowledgements

We would like to thank K.Higashijima, Y.Okada, N.Tsuda, Y.Yamada, T.Yukawa and other members of KEK theory group for useful discussions and encouragements.

References

[1] G. 'tHooft, Nucl. Phys. B72(1974)461; K.Wilson, Phys. Rev. D8(1974)2445.

[2] E.Fradkin, M.Srednicki and L.Susskind, Phys. Rev. D21(1980)2885; C.Itzykson, Nucl. Phys. B210(1982)477; A.Casher, D.Förster and P.Windey, Nucl. Phys. B251(1985)29; Vl.Dotsenko and A.Polyakov, in Advanced Studies in Pure Math. 15(1987).

[3] V.G.Knizhnik, A.M.Polyakov and A.B.Zamolodchikov, Mod. Phys. Lett. A3(1988)819; F.David, Mod. Phys. Lett. A3(1988)1651; J.Distler and H.Kawai, Nucl. Phys. B321(1989)509.

[4] K.Fujikawa, J.Kubo and H.Terao, Phys. Lett. B263(1991)371.

[5] A.Migdal, Sov. Phys. JETP. 42(1975)413.

[6] D.Gross, preprint PUPT 1356; D.Gross and W.Taylor, preprints, PUPT 1376; PUPT 1382; J.Minahan, preprint UVA-HET-92-10; M.R.Douglas, preprint RU-93-13; M.R.Douglas and V.A.Kazakov, preprint LPTENS-93/20.
[7] B.Duplantier and I.K.Kostov, Phys. Rev. Lett. 61(1988)1433; Nucl. Phys. B340(1990)491.

[8] E.Brézin, C.Itzykson, G.Parisi and J.B.Zuber, Commun. Math. Phys. 59(1978)35.

[9] G.Moore, N.Seiberg and M.Staudacher, Nucl. Phys. B362(1991) 665.

[10] E.Brézin and V.Kazakov, Phys. Lett. B236(1990)144; M.Douglas and S.Shenker, Nucl. Phys. B335(1990)635; D.Gross and A.Migdal, Phys. Rev. Lett. 64(1990)127; Nucl. Phys. B340(1990)333.

[11] V.Kazakov, Mod. Phys. Lett. A4(1989)2125.