ABSTRACT. A coordinate-free definition for Wick-type symbols is given for symplectic manifolds by means of the Fedosov procedure. The main ingredient of this approach is a bilinear symmetric form defined on the complexified tangent bundle of the symplectic manifold and subject to some set of algebraic and differential conditions. It is precisely the structure which describes a deviation of the Wick-type star-product from the Weyl one in the first order in the deformation parameter. The geometry of the symplectic manifolds equipped by such a bilinear form is explored and a certain analogue of the Newlander-Nirenberg theorem is presented. The 2-form is explicitly identified which cohomological class coincides with the Fedosov class of the Wick-type star-product. For the particular case of Kähler manifold this class is shown to be proportional to the Chern class of a complex manifold. We also show that the symbol construction admits canonical superextension, which can be thought of as the Wick-type deformation of the exterior algebra of differential forms on the base (even) manifold. Possible applications of the deformed superalgebra to the noncommutative field theory and strings are discussed.

1. Introduction

The deformation quantization as it was originally defined in [1], [2] has now been well established for every symplectic manifold through the combined efforts of many authors (for review see [3]). The question of existence of the formal associative deformation for the commutative algebra of smooth functions, so-called star product, has been solved by De Wilde and Lecomte [4]. The classification of the equivalence classes of deformation quantization by formal power series in the second De Rham cohomology has been carried out in several works [5], [6], [7], [8]. Finally, in the seminal paper [9] Fedosov has given an explicit geometric construction for the star product on an arbitrary symplectic manifold making use of the symplectic connection. As it was shown by Xu [10] every deformation quantization on a symplectic manifold is equivalent to that obtained by Fedosov’s method. Recently, the relationship has been established between the Fedosov quantization and BRST method for the constrained dynamical systems [11].

In parallel with the general theory of deformation quantization some special types of star-products, possessing additional algebraic/geometric properties, have been studied as well. Motivated by the constructions of geometric quantization and symbol calculus on the Kähler manifolds, a particular emphasis has been placed on the deformation quantization of symplectic manifolds admitting two transverse polarizations. The respective quantization constructions may be thought of as a generalization of the Wick or qp-symbol calculus, known for the linear symplectic spaces, much as the Fedosov deformation quantization may be regarded as a generalization of the Weyl-Moyal star-product construction. At present, there is a large amount of literature concerning the deformation quantization on polarized symplectic manifolds [12], [13], [14], [15], [16], [17], [18], [19], [20], beginning with the pioneering paper by Berezin [21] on the quantization in complex symmetric spaces.

It should be emphasized that in all the papers cited above the construction of the Wick-type star-products is based on the explicit use of a special local coordinate system adapted to the polarization (separation of variables in terminology of work [15]). This does not seem entirely adequate for the physical applications as the most of interesting physical theories are formulated in general covariant way, i.e. without resorting to a particular choice of coordinates. So, it is desirable to relate the polarization with an additional geometric structure (tensor field) on the symplectic manifold in such a way that the resulting star-product construction would not imply to use any particular

Key words and phrases. Deformation quantization, Wick symbol, supersymplectic manifolds.
choice of the coordinates. One may further treat this structure, if necessary, as dynamical field encoding all the polarizations, just as the Einstein’s equations for metric field govern the geometry of Riemannian manifolds, and try to assign this structure with a physical interpretation. In this form the Wick symbols may enjoy some interesting physical applications, two of which (noncommutative field theories on curved symplectic manifolds and nonlinear sigma-models) are discussed in the concluding section.

Although it is a commonly held belief that, leaving aside global geometrical aspects, all the quantizations are (unitary or formally) equivalent to each other, this equivalence can be spoiled because of quantum divergences which might appear as soon as one deals with infinite-dimensional symplectic manifolds (field theories). This is best illustrated by a concrete physical example of the free bosonic string. Formally, the theory may be quantized by means of both Weyl and Wick symbols, but the critical dimension of space-time and nontrivial physical spectrum are known to arise only for the Wick symbol. This is due to the infinities that appear when one actually tries to apply the equivalence transformation to the operators of physical observables. Thus the different quantization schemes may lead to essentially different quantum mechanics and the Wick symbols are recognized to be more appropriate for the field-theoretical problems. On the other hand, the global aspects of quantization cannot be surely ignored in a mathematically rigor treatment of the question. It is the account of a phase-space topology that gives a deep geometrical insight into the quantization of such paramount physical characteristics as the spin, magnetic and electric charge of the particle, energy levels of hydrogen atom and harmonic oscillator and so on, both in the frameworks of geometric or deformation quantizations [22, 23]. Finally, even though two quantization are equivalent in a mathematical sense this does not yet imply their physical equivalence since, for instance, the same classical observables being quantized in either case will have different spectra in general [24].

In this paper we give a constructive coordinate-free definition for the star-product of Wick type in the framework of the Fedosov deformation quantization. After the paper [25], a symplectic manifold equipped by a torsion-free symplectic connection is usually called the Fedosov manifold because precisely these data, symplectic structure and connection, do enter the Fedosov star-product [26]. The Wick deformation quantization involves one more geometric structure - a pair of transverse polarizations, and, by analogy to the previous case, the underlying manifold may be called as the Fedosov manifolds of Wick type or FW-manifold for short.

Let us briefly outline the key idea underlying our approach to the construction of Wick-type symbols on general symplectic manifolds. Hereafter the term “Wick-type” will be used in a reference to a broad class of symbols incorporating, along with the ordinary (genuine) Wick symbols, the so-called qp-symbols as well as various mixed possibilities commonly regarded as the pseudo-Wick symbols. To give a more precise definition of what is meant here, consider first the linear symplectic manifold $\mathbb{R}^{2n}$ equipped with the canonical Poisson brackets $\{y^i, y^j\} = \omega^{ij}$. Then the usual Weyl-Moyal product of two observables, defined as

$$a \ast b(y) = \exp \left( \frac{i\hbar}{2} \omega^{ij} \frac{\partial}{\partial y^i} \frac{\partial}{\partial z^j} \right) a(y) b(z) |_{z = y},$$

turns the space of smooth functions in $y$ to the noncommutative associative algebra with a unit, which is called the algebra of Weyl symbols. Note that all the coordinates $y$’s enter uniformly in the above formula. Contrary to this, the construction of Wick-type symbols always implies some (real, complex or mixed) polarization [22] splitting the $y$’s into two sets of (canonically) conjugated variables. For example, the qp-symbol construction is based on separation of phase-space variables on the “coordinates” $q$ and “momenta” $p$ (that corresponds to the choice of some real polarization) and the standard ordering prescription, “first $q$, then $p$”, for any polynomial in $y$’s observable. For the complex polarization the same role is played by pairs of oscillatory variables $q \pm ip$. Formally, the transition from the Weyl to Wick-type symbols is achieved by adding a certain complex-valued symmetric tensor $g$ to the Poisson tensor $\omega$ in the formula for the Weyl-Moyal $\ast$-product,

$$a \ast_g b(y) = \exp \left( \frac{i\hbar}{2} \Lambda^{ij} \frac{\partial}{\partial y^i} \frac{\partial}{\partial z^j} \right) a(y) b(z) |_{z = y},$$

$$\Lambda^{ij} = \omega^{ij} + g^{ij}, \quad \Lambda^{\dagger} = -\Lambda, \quad i, j = 1, 2, \ldots, 2n.$$
Although the associativity of modified product holds for any constant \( g \), the Wick-type symbols are extracted by the additional condition
\[
\text{rank}\Lambda = \text{corank}\Lambda = n. \tag{1.2}
\]
In particular, the genuine Wick symbol corresponds to a pure imaginary \( g \) while the real \( \Lambda \) gives a qp-symbol. In either case (including the mixed polarization) the square of the matrix
\[
I^i_j = \omega^{ik}g_{kj} \tag{1.3}
\]
is equal to 1 and hence the operator \( I \) generates a polarization splitting the complexified phase space \( \mathbb{C}^{2n} \) in a direct sum of two transverse subspaces related to the eigen values \( \pm 1 \).

The formula (1.1) may serve as the starting point for the covariant generalization of the notion of Wick symbol to general symplectic manifolds. Turning to the curved manifold \( M \), \( \dim M = 2n \), we just replace the constant matrix \( \Lambda \) by a general complex-valued bilinear form \( \Lambda(x) = \omega(x) + g(x) \) with the closed and non-degenerate antisymmetric part \( \omega(x) \) and satisfying to the above half-rank condition (1.2). Then each tangent space \( T_xM \), \( x \in M \), turns to the symplectic vector space w.r.t. \( \omega(x) \) and may be quantized by means of Wick-type product (1.1). Taking the union of all tangent spaces equipped with the star-product we get the bundle of Wick symbols, which is a sort of “quantum tangent bundle”. Then, following Fedosov’s idea, we introduce a flat connection on it by adding some quantum correction to the usual affine connection preserving \( \Lambda(x) \). The flat sections of this connection can be naturally identified with the space of quantum observables \( C^\infty(M)[[\hbar]] \).

Finally, the pull back of the bundle star-product via the Fedosov connection induces a star-product on \( C^\infty(M)[[\hbar]] \). The only crucial point of this program is the existence of a torsion-free linear connection \( \nabla \) preserving \( \Lambda \). As we show bellow the necessary and sufficient condition for such a connection to exist is the integrability of the right and left kernel distribution of \( \Lambda(x) \). When the latter condition is fulfilled \( \nabla \) is just the Levi-Civita connection associated to the symmetric and non-degenerate form \( g(x) \) and the right and left kernel distributions define the transverse polarization of the symplectic manifold \( (M,\omega) \). Under specified conditions we refer to the pair \( (M,\Lambda) \) as the FW-manifold.

The paper is organized as follows. In Section 2 we define the Fedosov manifolds of Wick type, discuss their geometry and give some examples. The main tool we use here is the integrable involution structure (1.3) associated to any FW-manifold structure \( \Lambda \). The deformation quantization on the FW-manifolds by Fedosov’s method is presented in Section 3. In Section 4 we pose the question about equivalence between original Fedosov star-product (generalized Weyl symbols) and star-product of Wick-type. The answer is follows: The only obstruction for the equivalence is associated with a non-zero De Rham class of a certain (in general complex) 2-form, which explicit expression is given by contraction of the curvature tensor of the FW-manifold and the corresponding involution tensor. In the Kähler case, this 2-form represents the first Chern class \( c_1 \) of a complex manifold. Section 5 is devoted to the superextension of the previous constructions to supersymplectic manifold, which, as we show, can be canonically associated to the tangent bundle of any FW-manifold. This superextension is a particular example of more general construction of super-Poisson brackets and their quantization proposed some time ago by Bordemann [20]. Here we also study the relationship between the algebra of quantum observables on initial FW-manifold and that on its superextension. This relationship is not so evident because the canonical projection of superextended FW-manifold to the base \( (M,\Lambda) \) or the natural embedding of the FW-manifold to its superextension do not induce homomorphism of the corresponding algebras.

2. Fedosov manifolds of Wick type.

Consider a \( 2n \)-dimensional real manifold \( M \) equipped with a complex-valued bilinear form \( \Lambda \) (not necessarily symmetric or antisymmetric). In local coordinates \( \{x^i\} \) on \( M \) the form \( \Lambda \) is completely determined by its components \( \Lambda_{ij} = \Lambda(\partial_i, \partial_j) \), where \( \partial_i \equiv \partial/\partial x^i \). Having the form \( \Lambda \) one can define two maps from the complexified tangent bundle to the complexified cotangent one: for any \( X \in TM \) the corresponding linear forms are
\[
\Lambda(\cdot, X), \quad \Lambda'(\cdot, X) = \Lambda(X, \cdot) \tag{2.1}
\]
\(\Lambda^t\) being transpose to \(\Lambda\). Denote by \(\ker \Lambda\) and \(\ker \Lambda^t\) the right and left kernel distributions of \(\Lambda\). Obviously, \(\dim \ker \Lambda = \dim \ker \Lambda^t\). It is convenient to decompose \(\Lambda\) into the sum of symmetric and antisymmetric parts

\[
\Lambda = \omega + g, \quad \omega = \frac{1}{2}(\Lambda - \Lambda^t), \quad g = \frac{1}{2}(\Lambda + \Lambda^t)
\] (2.2)

**Definition 1.1** The pair \((M, \Lambda)\) will be called by an almost Fedosov manifold of Wick type (or almost FW-manifold for short) if at each point \(p \in M\)

i) \(\omega = \frac{1}{2}(\Lambda - \Lambda^t)\) is a real non-degenerate two-form,

ii) \(\dim_C \ker \Lambda = \frac{1}{2} \dim M\).

The first condition merely implies that the antisymmetric part of \(\Lambda\) defines on \(\omega\) a structure tensor field of skew-symmetric bilinear maps \(N\). Definition 1.2

Additionally, we have proved that the form \(g\) is non-degenerate and the automorphism \(I\) is involutive, i.e.

\[
I^2 = \text{id} \ (\text{identical transformation})
\] (2.4)

That is why we will call \(I\) by an almost involution structure. Note that for an anti-Hermitian \(\Lambda\), that is \(\Lambda^t = -\Lambda\), the components of tensor \(I\) are purely imaginary

\[
I = \sqrt{-1}J, \quad J^2 = -\text{id}
\] (2.5)

and hence they define (and defined by) an almost complex structure \(J\). As is well known the almost complex structure becomes the complex one if it defines a structure of the complex manifold. Due to the Newlander-Nirenberg identification theorem for the complex manifolds \[27\] it is equivalent to vanishing of the Nijenhuis tensor associated to \(J\) providing the latter is sufficiently smooth. The vanishing of the Nijenhuis tensor is known to be equivalent, in turn, to the integrability of vector distributions belonging to the eigenvalues \(\pm \sqrt{-1}\) of \(J\). As we will see below the analog of the latter condition can be advocated for the almost involution structure \(I\) as well, with precisely the same definition of the Nijenhuis tensor.

**Definition 1.2** The Nijenhuis tensor associated to the almost involution structure \(I\) is a smooth tensor field of skew-symmetric bilinear maps \(N : T^2_M \to T^2_M\), which can be defined in two equivalent ways:

i) for every pair of smooth vector fields \(X\) and \(Y\)

\[
N(X, Y) = [X, Y] - I[I X, Y] - I[X, I Y] + [I X, I Y],
\] (2.6)

where the brackets \([., .]\) stand for the commutator of vector fields;

ii) let \(\nabla\) be an arbitrary torsion-free connection, then in local coordinates \(\{x^i\}\) the components of \(N\) are given by

\[
N^k_{ij} = \nabla_i I^k_j - \nabla_j I^k_i - I^k_j (\nabla_i I^j - \nabla_j I^i)
\] (2.7)
One may easily check that the relations (2.6,2.7) do really define (the same) tensor as they do not actually depend on derivatives of \( X,Y \) and on the choice of connection \( \nabla \). Before examining the geometric consequences of the condition \( N = 0 \), let us introduce the key ingredient of our construction.

**Definition 1.3** The almost FW-manifold \((M,\Lambda)\) will be called the FW-manifold if there exist a torsion-free connection \( \nabla \) preserving \( \Lambda \), i.e. \( \nabla \Lambda = 0 \).

Obviously, the form \( \Lambda \) is covariantly constant iff both its symmetric and antisymmetric part is the constant, i.e.

\[
\nabla \Lambda = 0 \iff \nabla \omega = \nabla g = 0.
\]

As there is the only torsion-free connection compatible with a given non-degenerate symmetric form \( g \), the existence of such a connection \( \nabla \) implies its uniqueness. On the other hand, the totally antisymmetric part of the equation \( \nabla \omega = 0 \) written in some local coordinates suggests that the form \( \omega \) is closed, i.e. \( d\omega = 0 \), and therefore any FW-manifold is a symplectic manifold as well. Emphasize that in general we deal with the complex-valued form \( g \), so the connection \( \nabla \) exists, when it is supposed to act in the complexified tangent bundle and determined, in each coordinate chart, by complex-valued Christoffel symbols. The following theorem gives the explicit criteria for an almost FW-manifold \((M,\Lambda)\) to admit a torsion-free connection preserving \( \Lambda \).

**Theorem 1.1** Given an almost FW-manifold \((M,\Lambda)\) with the closed antisymmetric part of \( \Lambda \), then the following statements are equivalent:

i) \( \Lambda \) defines the structure of FW-manifold,

ii) the involution \( I \) associated to \( \Lambda \) has the vanishing Nijenhuis tensor,

iii) the kernel distributions \( \ker \Lambda \) and \( \ker \Lambda^t \) are integrable.

Although these assertions are quite elementary we prove them below as they have a direct bearing on the deformation quantization of FW-manifolds, which will be considered in the next sections.

**Proof.** We will proceed following the scheme: \( i) \Leftrightarrow ii) \), \( ii) \Leftrightarrow iii) \).

Implication \( i) \Rightarrow ii) \) straightforwardly follows from the second definition of Nijenhuis tensor (2.7) in which \( \nabla \) is taken to be compatible with \( \Lambda \) (and hence with \( I \) ). Conversely, let \( \nabla \) be the torsion-free connection preserving \( g \). Accounting that \( d\omega = 0 \), the expression (2.7) for the Nijenhuis tensor can be reduced to

\[
N_{ijk}^i = 2\omega^l \nabla_l \omega_{jk},
\]

from which the implication \( ii) \Rightarrow i) \) is immediately follows. Now let \( X \) and \( Y \) be two eigenvector fields of the involution tensor with the same eigenvalue \( \alpha \); of course, \( \alpha^2 = 1 \). Evaluating \( N \) on these vector fields with the help of first definition (2.6) we get

\[
N(X,Y) = 2([X,Y] - \alpha I[X,Y])
\]  

(2.8)

So, if \( N = 0 \) then \( I[X,Y] = \alpha [X,Y] \) and the eigen distributions of \( I \) are involutive. This proves the implication \( iii) \Rightarrow ii) \). The relation (2.8) also implies that \( N \) comes to zero on each pair of tangent vectors belonging to the same kernel distribution, \( \ker \Lambda \) or \( \ker \Lambda^t \), whenever they are integrable. In the case of \( X \) and \( Y \) belonging to the different distributions associated to the eigenvalues \( \alpha \) and \( -\alpha \), respectively, the value of the Nijenhuis tensor on them is equal to zero identically:

\[
N(X,Y) = [X,Y] - \alpha I[X,Y] - I[X,Y] + [IX,IY] =
\]

\[
[X,Y] - \alpha I[X,Y] + \alpha I[X,Y] + [X,Y] = 0
\]

Since the vectors from left and right kernel distributions form the basis of \( T_xM, \forall \ x \in M \), the last conclusion proves the implication \( ii) \Rightarrow iii) \) and the theorem.
Examples. The extended list of examples is provided by the Kähler manifolds. In this case the form $\Lambda$ is anti-Hermitian and the integrable involution structure $I$ is identified with the complex one multiplied by $\sqrt{-1}$. In the frame $\{\partial/\partial z^a, \partial/\partial \overline{z}^\beta\}$ associated to the local holomorphic coordinates $\{z^a\}, a = 1, \ldots, n$, on $M$ the matrix of the form $\Lambda$ looks like

$$
\Lambda = \begin{pmatrix}
0 & h_{\overline{a} \overline{\beta}} \\
0 & 0
\end{pmatrix}
$$

where $h = h_{\overline{a} \overline{\beta}} dz^a \wedge d\overline{z}^\beta$ is the Kähler $(1,1)$-form, $dh = 0$. The form $\Lambda$ is preserved by the Kähler connection with non-zero components of the Christoffel symbols being $\Gamma^a_{bc} = h_{\overline{a} \overline{\gamma}} \partial h_{\overline{\gamma} \overline{\beta}} / \partial z^c = (\Gamma^a_{\overline{\beta} \overline{\gamma}})$, the matrix $h^{-1}$ is inverse to $h_{\overline{a} \overline{\beta}}$. In particular, all two-dimensional orientable manifolds admit the structure of FW-manifold as they known to admit the Kähler structure. More generally, let $g$ be a (pseudo-)Riemannian metric on a two-dimensional real manifold and let $\omega = \sqrt{|\det g|} dx^1 \wedge dx^2$ be the corresponding volume form. Then the form

$$
\Lambda = g + \alpha \omega
$$

is obviously preserved by the metric connection for any $\alpha \in \mathbb{C}$. In order for $\Lambda$ to define the structure of FW-manifold we have to require

$$
\det \Lambda = \det g + \alpha^2 |\det g| = 0,
$$

that fixes $\alpha = \pm \sqrt{-1}$ for the Riemannian metric ($\det g > 0$) and $\alpha = \pm 1$ for the pseudo-Riemannian one ($\det g < 0$). The former case corresponds to the Kähler manifolds, while the latter is concerned with the manifolds endowed by a real structure. The last situation is strikingly illustrated by the example of one-sheet hyperboloid embedded into three dimensional Minkowsky space as the surface

$$
(x, y, z) \text{ being linear coordinates in } R^{2,1}.
$$

The induced metric is of the pseudo-Euclidean type and it has a constant negative curvature. For the corresponding real structure, the integral leaves of the eigen distributions coincide with the two transverse sets of linear elements of the hyperboloid. Obviously, these linear elements are nothing but the isotropic geodesics of metric $g$.

Let us recall that the integrable complex distribution $P$ on a symplectic manifold $(M, \omega)$ is called polarization if $\dim_{\mathbb{C}} P = \dim M/2$ and $\omega|_P = 0$. In other words, at each point $x \in M$, $P$ extracts a (complex) Lagrangian subspace $P_x \subset T^*_x M$ in the complexified tangent bundle. It is easy to see that the right and left kernel distributions on the FW-manifold $(M, \Lambda)$ are Lagrangian and hence define a pair of transversal polarizations $P_R, P_L$. These polarizations are highly important for physical applications both in the framework of the deformation quantization and geometric one as it allows one to introduce the notion of a state of quantum-mechanical system. The quantization on symplectic manifolds endowed with a pair of transversal polarizations was intensively studied in two limiting cases: $P_R = \overline{P}_L$, $P_R \cap \overline{P}_L = 0$. The first possibility is realized for the Kähler manifolds with holomorphic-anti-holomorphic polarizations, while the second implies the existence of a pair of transversal Lagrangian foliations on $M$, as a consequence of the Frobenius theorem. It is interesting to note that for the real $\Lambda = g + \omega$ the leaves of foliations $P_R, P_L$, being Lagrangian submanifolds with respect to the symplectic structure $\omega$, turn out to be totally geodesic submanifolds with respect to pseudo-Riemannian structure $g$. The proof of the last fact is straightforward and we omit it.

### 3. Deformation quantization on FW-manifolds.

Let, as before, $(M, \Lambda)$ be an FW-manifold of dimension $2n$. We are reminded that $\Lambda$ can be splitted into the sum of symmetric and antisymmetric parts (2.2) which are both non-degenerate. Introduce the second-rank contravariant tensor field $\Lambda^{ij}(x) \partial_5 \otimes \partial_7$ defined as follows:

$$
\Lambda^{ij} = \omega^{im} \Lambda_{mn} \omega^{nj} = g^{ij} + \omega^{ij},
$$

where $\omega^{ij}$ and $g^{ij}$ are matrices inverse to $\omega_{ij}$ and $g_{ij}$, respectively. By construction,

$$
\text{rank}(\Lambda^{ij}) = \text{rank}(\Lambda_{ij}) = n
$$
and \(\omega^{ij}\) is a Poisson tensor. Under the deformation quantization of the FW-manifold \((M, \Lambda)\) we will mean the construction of an associative multiplication operation \(*\) of two functions, which is an one-parametric deformation of the ordinary pointwise multiplication in the algebra \(C^\infty(M)\) and which meets the “boundary condition”:

\[
a * b = ab - \frac{ih}{2} \Lambda^{ij} \partial_i a \partial_j b + \ldots
\]  

(3.1)

where \(h\) is the formal deformation parameter (“Plank constant”), and dots mean the terms of higher orders in \(h\). The condition (3.1) is compatible with so-called correspondence principle of quantum mechanics:

\[
\lim_{h \to 0} \frac{i}{h}(a * b - b * a) = \{a, b\},
\]

(3.2)

where \(\{\cdot, \cdot\}\) means the Poisson brackets associated to \(\omega^{ij}\). The boundary condition should also be added by the requirement of locality

\[
supp(f * g) = supp(f) \cap supp(g)
\]

(3.3)

To meet the latter condition the coefficient of each power of \(h\) in (3.1) are usually restricted to be a finite-order differential expression bilinear in \(a\) and \(b\). Note that when \(h\) is treated as a formal (not numerical) parameter the \(*\)-product of two functions is not a function but is an element of a more wide space \(C^\infty(M)[[h]]\) consisting of the formal series: \(a(x, h) = a_0(x) + h a_1(x) + h^2 a_2(x) + \ldots\), \(a_i(x) \in C^\infty(M)\). The space \(C^\infty(M)[[h]]\) is closed already with respect to \(*\)-multiplication and it may be regarded as the algebra of quantum observables, much as the Poisson algebra of smooth functions \(C^\infty(M)\) on symplectic manifold \((M, \omega)\) is identified with the space of classical observables. The problem now is to construct an associative \(*\)-product starting from the ansatz (3.1). In this section we show how this the problem can be resolved by a minimal modification of Fedosov’s geometric approach to the deformation quantization [4].

**Definition 2.1** The formal symbol algebra \(\mathcal{A} = \bigoplus_{m,n=0}^\infty \mathcal{A}_{m,n}\) is the bi-graded associative algebra over \(\mathbb{C}\) with a unit whose elements are formal series

\[
a(x, y, dx, h) = \sum_{2k+p,q \geq 0} h^k a_{k,i_1 \ldots i_p,j_1 \ldots j_q}(x) y^{i_1} \ldots y^{i_p} dx^{j_1} \wedge \ldots \wedge dx^{j_q},
\]

(3.4)

Here the expansion coefficients \(a_{k,i_1 \ldots i_p,j_1 \ldots j_q}(x)\) are the components of covariant tensor fields on \(M\) symmetric with respect to \(i_1, \ldots, i_p\) and antisymmetric in \(j_1, \ldots, j_q\); \(h\) is a formal (deformation) parameter; and \(\{y^i\}\) are variables transforming as components of tangent vector, hence the whole expression (3.4) does not depend on the choice of coordinates. The general term of the expansion (3.4) is assigned by a bi-degree \((2k + p, q)\) and thus belongs to the subspace \(\mathcal{A}_{2k+p,q}\). The product of two elements \(a, b \in \mathcal{A}\) is defined by the rule

\[
a \circ b = \exp \left( i h \frac{\Lambda^{ij}(x)}{2} \frac{\partial}{\partial y^i} \frac{\partial}{\partial y^j} \right) a(x, y, dx, h) \wedge b(x, z, dx, h)|_{z=y},
\]

(3.5)

where \(\wedge\) stands for the ordinary exterior product of differential forms.

It is easily seen that the multiplication (3.5) is associative and bi-graded, i.e. \(\mathcal{A}_{m,n} \circ \mathcal{A}_{k,l} \subset \mathcal{A}_{m+k,n+l}\). In what follows we will refer to subscripts \(m\) and \(n\) labelling the graded subspace \(\mathcal{A}_{m,n}\) as the first and second degree respectively. The natural filtration in \(\mathcal{A}\) with respect to the first degree,

\[
\mathcal{A} \supset \mathcal{A}_1 \supset \mathcal{A}_2 \supset \ldots, \quad \mathcal{A}_m = \bigoplus_{k \geq m} \mathcal{A}_{k,n},
\]

(3.6)

defines the topology and convergence in the space of infinite formal series (3.4). Being associative algebra, \(\mathcal{A}\) can be turned to a differential graded Lie algebra with respect to the second degree: the commutator of two homogeneous elements \(a \in \mathcal{A}_{m,n}, b \in \mathcal{A}_{k,l}\) is given by \([a, b] = a \circ b - (-1)^{kl} b \circ a\), and extends to whole \(\mathcal{A}\) by linearity; the nilpotent differential \(\delta: \mathcal{A}_{m,n} \to \mathcal{A}_{m-1,n+1}\) acts as

\[
\delta a = dx^k \wedge \frac{\partial a}{\partial y^k}, \quad \delta(a) = 0, \quad \forall a \in \mathcal{A}
\]

(3.7)
Alternatively, one can write
\[ \delta a = -\frac{1}{i\hbar} [\omega_{ij} y^i dx^j, a], \]  
and hence \( \delta \) is an inner derivation of the superalgebra and of the \( \circ \)-product as well:
\[ \delta (a \circ b) = (\delta a) \circ b + (-1)^n a \circ (\delta b), \quad \forall a \in \mathcal{A}_{*, n}, \quad \forall b \in \mathcal{A} \]  
Note that \( \delta \) acts “algebraically” in a sense that it does not involve derivatives with respect to \( x \). The nontrivial cohomologies of \( \delta \) corresponds to the subspace of quantum observables \( C^\infty(M)[[\hbar]] \subset \mathcal{A} \) whose elements are independent of \( y \) and \( dx \); for the complementary subspace one can construct a homotopy operator \( \delta^{-1} : \mathcal{A}_{m,n} \rightarrow \mathcal{A}_{m+1,n-1} \) of the form
\[ \delta^{-1} a = y^k i \left( \frac{\partial}{\partial x^k} \right) \int_0^1 a(x, ty, tdx, \hbar) \frac{dt}{t}, \]  
where \( i(\partial/\partial x^k) \) means the contraction of the vector field \( \partial/\partial x^k \) and a form. Extending the action of \( \delta^{-1} \) to \( C^\infty(M)[[\hbar]] \) by zero, we get the “Hodge-De Rham decomposition” holding for any \( a \in \mathcal{A} \),
\[ a = \sigma(a) + \delta \delta^{-1} a + \delta^{-1} \delta a, \]  
where \( \sigma(a) = a(x, 0, 0, \hbar) \) denotes the projection of \( a \) onto \( C^\infty(M)[[\hbar]] \).

Given the torsion-free connection \( \nabla \) preserving FW-manifold structure \( \Lambda \), it induces the covariant derivative on elements of \( \mathcal{A} \) which will be denoted by the same symbol,
\[ \nabla : \mathcal{A}_{n,m} \rightarrow \mathcal{A}_{n,m+1}, \quad \nabla = dx^i \wedge \left( \frac{\partial}{\partial x^i} - y^j \Gamma^i_{jk}(x) \frac{\partial}{\partial y^k} \right), \]  
\( \Gamma^i_{jk} \) are Christoffel symbols of the connection \( \nabla \). Definition 1.3 implies the following property of the covariant derivative:
\[ \nabla (a \circ b) = \nabla a \circ b + (-1)^n a \circ \nabla b, \quad \forall a \in \mathcal{A}_{*, n}, \forall b \in \mathcal{A} \]  
The next Lemma is a counterpart of Lemma 2.4 in [9] for FW-manifolds.

**Lemma 2.1.** Let \( \nabla \) be the covariant derivative of \( \mathcal{A} \). Then
\[ \nabla \delta a + \delta \nabla a = 0, \]  
\[ \nabla^2 a = \nabla(\nabla a) = \frac{1}{i\hbar} [R, a], \quad \nabla = \frac{1}{4} R_{ijkl} y^i dx^k \wedge dx^l \]  
where \( R_{ijkl} = \omega_{im} R^m_{jkl} \) is the curvature tensor of the connection \( \nabla \).

**Proof.** The first identity holds because \( \nabla \) is a symmetric connection. It follows from the definition (3.12) that
\[ \nabla^2 a = \frac{1}{2} dx^k \wedge dx^l R_{ijkl} y^i \frac{\partial a}{\partial y^j}. \]  
So, \( \nabla^2 \) is an algebraic operator. On the other hand,
\[ \frac{1}{i\hbar} [R, a] = -\frac{1}{4} (R_{ijkl} y^i \Lambda^jn \frac{\partial a}{\partial y^n} - R_{ijkl} y^i \Lambda^mj \frac{\partial a}{\partial y^n}) dx^k \wedge dx^l + \]
\[ + \frac{i\hbar}{8} (R_{ijkl} \Lambda_{im} \Lambda_{jn} \frac{\partial^2 a}{\partial y^m \partial y^n} - R_{ijkl} \Lambda_{mj} \Lambda_{ni} \frac{\partial^2 a}{\partial y^m \partial y^n}) dx^k \wedge dx^l \]
\[ = -\frac{1}{2} dx^k \wedge dx^l R_{ijkl} y^i \omega^m_{nj} \frac{\partial a}{\partial y^n} = \nabla^2 a \]  
Here we have used the Ricci identity, \( g_{in} R^m_{ijkl} = -g_{jn} R^m_{ikl}, \) and its symplectic analog \( \omega_{im} R^m_{jkl} = \omega_{jm} R^m_{ikl} \) (for the proof see [24]). The Lemma proved suggests that the square of external derivative.
\(\nabla\) is a derivative again and, moreover, it is an inner derivative of the algebra \(\mathcal{A}\). This fact is of primary importance for the \(\ast\)-product construction along the line of the Fedosov approach.

Following Fedosov, we define a more general derivative in \(\mathcal{A}\) of the form

\[
D = \nabla - \delta + \frac{1}{i\hbar}[r, \cdot] = \nabla + \frac{1}{i\hbar}[\omega_{ij}y^i dx^j + r, \cdot],
\]

(3.17)

where \(r = r_i(x, y, dx, \hbar)dx^i\) belongs to \(\mathcal{A}_3^1\) and satisfies the Weyl normalization condition \(r_i(x, 0, dx, \hbar)dx^i = 0\). A simple calculation yields that

\[
D^2a = \frac{1}{i\hbar}[\Omega, a], \quad \forall a \in \mathcal{A},
\]

(3.18)

where

\[
\Omega = -\frac{1}{2}\omega_{ij}dx^i \wedge dx^j + R - \delta r + \nabla r + \frac{1}{i\hbar}r \circ r
\]

(3.19)

is the curvature of \(D\). A connection of the form (3.17) is called Abelian if two-form \(\Omega\) does not contain \(y^i\)‘s, i.e. belongs to the center of \(\mathcal{A}\). It is clear, that the kernel subspace of an Abelian connection \(D\) is automatically a subalgebra in \(\mathcal{A}\).

Theorem 2.1. There exists a unique \(r \in \mathcal{A}_3^3\) obeying condition \(\delta^{-1}r = 0\) such that \(D\), given by (3.17), is an Abelian connection with the curvature \(\Omega = -(1/2)\omega_{ij}dx^i \wedge dx^j\).

Theorem 2.2. For any observable \(a \in C^\infty(M)[[\hbar]]\) there is a unique element \(\tilde{a} \in \mathcal{A}_D\) such that \(\sigma(\tilde{a}) = a\). Therefore, \(\sigma\) establishes an isomorphism between \(\mathcal{A}_D\) and \(C^\infty(M)[[\hbar]]\).

Corollary 2.1. The pull-back of \(\circ\)-product via \(\sigma^{-1}\) induce an associative \(\ast\)-product on the space of physical observables \(C^\infty(M)[[\hbar]]\), namely

\[
a \ast b = \sigma(\sigma^{-1}(a) \circ \sigma^{-1}(b))
\]

(3.20)

Besides the fact of existence these theorems provide an effective procedure for the construction of the lifting map \(\sigma^{-1}\) by iterating a pair of coupled equations

\[
r = \delta^{-1}(R + \nabla r + \frac{1}{i\hbar}r \circ r),
\]

\[
\sigma^{-1}(a) = a + \delta^{-1}(\nabla \sigma^{-1}(a) + \frac{1}{i\hbar}[r, \sigma^{-1}(a)])
\]

(3.21)

Since the operator \(\nabla\) preserves the filtration and \(\delta^{-1}\) raises it by 1, the iteration of the system (3.21) converges in the topology (3.6) and define the unique solution for \(\sigma^{-1}(a)\). Thus the \(\ast\)-product of two functions can be computed with any prescribed accuracy in \(\hbar\).

It should be noted that for the anti-Hermitian \(A\)‘s (the case of Kähler manifolds) the introduced multiplication possesses the following property of reality:

\[
\overline{a \ast b} = \overline{b} \ast \overline{a}, \quad \forall a, b \in C^\infty(M)[[\hbar]]
\]

(3.22)

In particular, the formal functions whose coefficients at each power of \(\hbar\) are real-valued smooth functions form a closed Lie subalgebra with respect to the \(\ast\)-commutator multiplied by \(i\). In the standard quantum-mechanical interpretation, this algebra is usually referred to as an algebra of physical observables corresponding to the algebra of self-adjoint operators. The formula (3.22) trivially follows from the analogous relation for \(\circ\)-product,

\[
\overline{a \circ b} = (-1)^{mn}\overline{b} \circ \overline{a}, \quad \forall a \in \mathcal{A}_{*m}, \ b \in \mathcal{A}_{*,n}
\]

(3.23)

and from the structure of the equations (3.20), (3.22).
Remark. As is seen, the rank condition imposed on $\Lambda$ by Definition 1.1 is certainly inessential for the construction of associative $\ast$-product obeying (3.1). The only fact we have used here is that of existence of a torsion-free connection preserving $\Lambda$. So, the construction described above works in a more general situation including the case of degenerate $g$, when $g = 0$ we get the Fedosov quantization. It would be interesting to formulate the necessary and sufficient conditions for a torsion-free connection preserving a given form $\Lambda$ to exist and to describe all such connections.

In conclusion let us introduce one more important ingredient of deformation quantization - a trace functional on the algebra of quantum observables.

Let $C^\infty_0(M)[\hbar] \subset C^\infty(M)[\hbar]$ be an ideal consisting of compactly supported quantum observables. Recall, that linear functional on $C^\infty_0(M)[\hbar]$ with values in formal constants $\mathbb{C}[\hbar]$ is called a trace if it vanishes on commutators, that is

$$tr(a) = \sum_{k=0}^{\infty} \hbar^k c_k, \quad c_k \in \mathbb{C}$$

and

$$tr(a \ast b) = tr(b \ast a)$$

Let $d\mu = d\mu_0 + \hbar d\mu_1 + \ldots$ be a formal smooth density on $M$, then integration by $d\mu$ delivers a continuous functional of the form $C^\infty_0(M)[\hbar] \ni a \rightarrow \int_M a \cdot d\mu$. A formal density is called a trace density for $\ast$-product if the functional $\int_M a \cdot d\mu$ is a trace. The work of Nest and Tsygan suggests that any continuous trace functional is defined by a trace density. For the deformation quantization on FW-manifolds a more profound statement is true.

**Theorem 2.3.** Up to an overall constant factor there exist a unique trace density associated to the algebra of quantum observables $C^\infty(M)[\hbar]$. It has the form

$$d\mu = (1 + \hbar t_1(x) + \hbar^2 t_2(x) + \ldots) \cdot \Omega$$

where $\Omega = \omega^n/n! = \sqrt{|\det g|} dx^1 \wedge \ldots \wedge dx^{2n}$ is symplectic (Riemannian) volume form on $M$ and coefficients $t_i(x)$ are polynomials in curvature tensor $R_{ijkl}$ and its covariant derivatives.

**Proof.** This theorem is the analogue of Theorem 5.6.6 in [23] and it can be proved following the same idea based on a localization principle. Besides, in the next section we will present the explicit form for an operator establishing a local isomorphism between the deformation quantization on FW-manifolds and that on the Fedosov quantization on the corresponding symplectic manifold $(M, \omega)$. Using this local isomorphism one may derive the trace density (3.26) from Fedosov’s trace density for symplectic manifold $(M, \omega)$.

It is pertinent to note that for a homogeneous FW-manifold the corresponding symmetry group, acting on $M$ transitively by symplectomorphisms, defines the symmetry group of deformation quantization. In this case one may see that $d\mu = \Omega$.

4. THE QUESTION OF EQUIVALENCE.

The rich geometry of the FW-manifolds offers at least two different schemes for their quantization: the Fedosov quantization, which exploits only the skew-symmetric part of the form $\Lambda$, and the deformation quantization involving entire form $\Lambda$ via ansatz (3.1). For the reasons mentioned in the Introduction we refer to these quantizations as those of Weyl and Wick type, respectively. The natural question aroused is that of whether we have essentially different quantizations or an equivalence transform may be found to establish a global isomorphism between both algebras of quantum observables. Below we formulate the necessary and sufficient conditions for such an isomorphism to

1Our definition defers from the standard one by the lack of inessential normalization multiplier $1/(2\pi \hbar)^n$. 
exist. As in the general case, the obstruction for equivalence of two star-products lies in the second De Rahm cohomology of symplectic manifold and we identify a certain 2-form as its representative.

In order to distinguish Wick-type star product from the Weyl one, all the constructions related to the former product will be attributed by the additional symbol $g$ (pointing on non-zero symmetric part $g$ in $\Lambda$). In particular, through this section the fibrewise multiplication \[ (3.5) \] will be denoted by $\circ_g$, while $\circ$ will be reserved for the Fedosov $\circ-$product \[ (3.5) \] resulting from \[ (3.5) \] if put $g = 0$.

We start by noting that fibrewise $\circ$ and $\circ_g$ products are equivalent in the following sense:

**Lemma 3.1.** For $a, b \in A$ we have
\[ \circ_g b = G^{-1}(G a \circ G b) \] (4.1)
where the formal operator $G$ and its formal inverse are given by
\[ G = \exp\left(\frac{\i}{\hbar} g^{ij} \frac{\partial}{\partial y^i} \frac{\partial}{\partial y^j}\right), \quad G^{-1} = \exp\left(\frac{\i}{\hbar} g^{ij} \frac{\partial}{\partial y^j} \frac{\partial}{\partial y^i}\right) \] (4.2)
In other words, the map $G : A \to A$, being considered as an automorphism of a linear space, establishes the isomorphism of algebras $(A, \circ)$ and $(A, \circ_g)$.

The proof is obvious from the direct substitution \[ (4.2) \] to \[ (4.1) \].

The operator $G$ satisfies following simple properties:
\[ \nabla G = G \nabla, \quad \delta G = G \delta \] (4.3)

The automorphism $G$ defines a new Abelian connection $\tilde{D} = GD_gG^{-1}$ which in virtue of relations \[ (4.3) \] can be written as
\[ \tilde{D} = \nabla - \delta + \frac{1}{\i \hbar} [\tilde{r}, \cdot], \quad \tilde{r} = Gr_g, \] (4.4)
and the bracket stands for $\circ$-commutator. The elements $\tilde{r}$ and $r$ fulfill the equations
\[ GR + \nabla \tilde{r} - \delta \tilde{r} + \frac{1}{\i \hbar} \tilde{r} \circ \tilde{r} \] (4.5)
\[ R + \nabla r - \delta r + \frac{1}{\i \hbar} r \circ r \] (4.6)

Thus we have two star-products $*$ and $\bar{*}$ corresponding to the pair of Abelian connections $D$ and $\tilde{D}$. Since $D \neq \tilde{D}$, in general, the action of the fibrewise isomorphism $G$ establishing the equivalence between $\circ$ and $\circ_g$-products (and hence between star products $*$ and $\bar{*}$) did not automatically followed by the equality $* = \bar{*}$. Indeed, evaluating lowest orders in $\hbar$ we get:
\[ a \star b = ab + \frac{\i}{\hbar} \omega^{ij} \nabla_i a \nabla_j b - \frac{\hbar^2}{4} \omega^{ij} \omega^{kl} \nabla_i \nabla_j a \nabla_k \nabla_l b + \mathcal{O}(\hbar^3) \] (4.7)
\[ a \bar{\star} b = ab + \frac{\i}{\hbar} \omega^{ij} \nabla_i a \nabla_j b - \frac{\hbar^2}{4} \omega^{ij} \omega^{kl} \nabla_i \nabla_j a \nabla_k \nabla_l b + \mathcal{O}(\hbar^3) \]
where
\[ \bar{\omega}^{ij} = \omega^{ij} + \hbar \omega_1^{ij}, \] (4.8)
and
\[ \omega_1^{ij} = \omega^{ij} \Omega_{kl} \omega^{kl}, \quad \Omega = \frac{i}{\hbar} (GR - R) = \frac{1}{8} R_{ijkl} g^{ij} dx^k \wedge dx^l \] (4.9)

The 2-form $\Omega$ is closed in virtue of the Bianchi identity for the curvature tensor. In fact, refs. \[ [13] \], \[ [18] \] say that the second star product $\bar{*}$ is a so-called 1-differentiable deformation of the first one $*$. This deformation is known to be trivial iff the 2-form $\Omega$ is exact \[ [4] \], \[ [28] \]. Now supposing that $\Omega = d\omega$ let us try to establish an equivalence between $*$ and $\bar{*}$ by means of a fibrewise conjugation automorphism.
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\[ a \rightarrow U \circ a \circ U^{-1}, \]  
(4.10)

where \( U \) is an invertible element of \( A_{*,0} \). The element \( U \) is so chosen that (4.10) turns \( D \) to \( \tilde{D} \). In other words, we subject \( U \) to the condition

\[ D(U \circ a \circ U^{-1}) = U \circ (\tilde{D}a) \circ U^{-1}, \quad \forall a \in A \]

or, equivalently,

\[ [U^{-1} \circ DU, a] = \frac{1}{i\hbar} [\Delta r, a], \]  
(4.11)

where \( \Delta r = \tilde{r} - r \). The last condition means that

\[ U^{-1} \circ DU = \frac{1}{i\hbar} \Delta r + \frac{1}{i\hbar} \psi, \]  
(4.12)

where \( \psi \) is a globally defined 1-form on \( M \). The compatibility condition for equation (4.12) resulting from the identity \( D^2 = 0 \) requires that

\[ DU^{-1} \circ DU = DU^{-1} \circ U \circ U^{-1} \circ DU = -\frac{1}{(i\hbar)^2} \Delta r \circ \Delta r = \frac{1}{i\hbar} D\Delta r + \frac{1}{i\hbar} d\psi. \]  
(4.13)

That is

\[ D\Delta r + \frac{1}{i\hbar} \Delta r \circ \Delta r + d\psi = 0 \]  
(4.14)

The analogous relation is obtained if we subtract (4.7) from (4.6)

\[ D\Delta r + \frac{1}{i\hbar} \Delta r \circ \Delta r + \Omega = 0 \]  
(4.15)

Comparing (4.14) with (4.15) we conclude that the compatibility condition holds provided \( \Omega \) is exact. Now rewrite (4.12) in the form

\[ \delta U = \nabla U + \frac{1}{i\hbar} [r, U] - \frac{1}{i\hbar} U \circ (\Delta r + \psi) \]  
(4.16)

and apply the operator \( \delta^{-1} \) to both sides of the equation. Using the Hodge-De Rham decomposition (3.11) and taking \( \sigma(U) = 1 \), we get

\[ U = 1 + \delta^{-1}(\nabla U + \frac{1}{i\hbar} [r, U] - \frac{1}{i\hbar} U \circ (\Delta r + \psi)) \]  
(4.17)

In [9, Theorem 4.3] it was proved that iterations of the last equation yield a unique solution for (4.16) provided the compatibility condition (4.14) is fulfilled. Starting from 1, this solution defines an invertible element of \( A_{*,0} \). Then the equivalence transform \( T : (C^\infty(M), *) \rightarrow (C^\infty(M), *_g) \) we are looking for is defined as the sequence of maps

\[ Ta(a) = (U \circ G(\sigma^{-1}_g(a)) \circ U^{-1})|_{y=0}, \]  
(4.18)

so that

\[ a *_g b = T^{-1}(Ta * Tb) \]

The main results of this section can be summarized as follows:

**Theorem 3.1.** The obstruction to equivalence between Weyl and Wick type deformation quantizations lies in the second De Rham cohomology \( H^2(M) \). The quantizations are equivalent iff the 2-form \( R_{ijkl}g^{ij}dx^k \wedge dx^l \) is exact.

**Remark.** For the anti-Hermitian \( \Lambda \) the 2-form \( \Omega \) is nothing but the Ricci form of the Kähler manifold \( (M, \Lambda) \). In this case the cohomology class of \( \Omega \), being proportional to the first Chern class \( c_1(M) \), is known to depend only on the complex structure of the manifold [29]. Since, for example, \( c_1(CP^n) \neq 0 \) and for any Kähler manifold \( M \) a topological equivalence \( M \sim CP^n \) implies a bi-holomorphic one [30], the Weyl and Wick quantizations on \( CP^n \) are not equivalent for any \( \Lambda \).
5. Superextension.

In this section we show that the Wick-type deformation quantization of FW-manifolds described above admits a surprisingly simple generalization to a certain class of supersymplectic supermanifolds. According to the Rothstein theory \[31\] any supersymplectic supermanifold can be completely specified (up to isomorphism) by the set \((M, \omega, \mathcal{E}, g, \nabla^g)\), where \((M, \omega)\) is the symplectic manifold, and \(\mathcal{E}\) is a vector bundle over \(M\) with metric \(g\) and \(g\)-compatible connection \(\nabla^g\). When \(\mathcal{E} = TM\) and \(M\) is an FW-manifold all Rothstein's data are already contained in the FW-structure \(\Lambda\), and thus, one may speak about canonical superextension for the FW-manifolds.

As is well known, the geometry of a manifold can be recovered from the commutative algebra of functions on it. A supermanifold is defined by extending of such an algebra to a supercommutative superalgebra denoted below by \(C\). Geometrically, the elements of \(C\) can be viewed as the sections of a Grassmann bundle associated to some vector bundle \(\mathcal{E} \rightarrow M\) over a given manifold \(M\), that is \(\mathcal{C} = \Gamma(\Lambda \mathcal{E})\), and the supercommutative multiplication is given by the pointwise wedge product,

\[
a \wedge b = (-1)^{d_1 d_2} b \wedge a \tag{5.1}
\]

Here \(a, b \in \Gamma(\Lambda \mathcal{E}), a\) of degree \(d_1\) and \(b\) of degree \(d_2\). The elements of superalgebra \(\mathcal{C}\) will be called superfunctions depending on the commuting coordinates \(\{x^i\}\) (i.e. local coordinates on the base \((M, \Lambda)\)) and “anticommuting coordinates” \(\{\theta^j\}\), so that the general element of superalgebra looks like

\[
\mathcal{C} \ni a = a(x, \theta) = \sum_{k=1}^{2n} a(x)_{i_1 \ldots i_k} \theta^{i_1} \ldots \theta^{i_k}, \tag{5.2}
\]

where \(a_k = a(x)_{i_1 \ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k}\) transforms as a \(k\)-form on \(M\). As usual the symbols \(\frac{\partial a}{\partial \theta^j}\) and \(\frac{\partial a}{\partial \theta^j}\) will denote the left and right partial derivatives in \(\theta^j\)’s, which definition is as follows:

\[
\frac{\partial a}{\partial \theta^j} = \sum_{k=1}^{2n} k a(x)_{i_2 \ldots i_k} \theta^{i_2} \ldots \theta^{i_k}, \quad \frac{\partial a}{\partial \theta^j} = \sum_{k=1}^{2n} k a(x)_{i_1 \ldots i_{k-1} j} \theta^{i_1} \ldots \theta^{i_{k-1}} \tag{5.3}
\]

In terms of supercoordinates \((x^i, \theta^j)\) the Rothstein supersymplectic form on the superextended FW-manifold \((M, \Lambda)\) is given by

\[
\Omega = \omega_{ij} dx^i dx^j + \frac{1}{2} g_{ij} R^k_{mkl} \theta^m \theta^l dx^k dx^l + g_{ij} D\theta^i D\theta^j
\]

\[
D\theta^j \equiv d\theta^j - \Gamma^j_{ik} \theta^k dx^i
\]

Hereafter we use the following conventions:

\[
dx^i dx^j = -dx^j dx^i, \quad \theta^j \theta^i = -\theta^i \theta^j, \quad dx^i \theta^j = -\theta^j dx^i
\]

\[
d\theta^j d\theta^i = d\theta^j d\theta^i, \quad \theta^j d\theta^i = \theta^i d\theta^j
\]

The deformation quantization of general supersymplectic supermanifolds defined by Rothstein’s data was first performed by Bordemann \[26\]. The remarkable feature of this construction is that the deformation quantization for the algebra of superfunctions is performed first and the super Poisson bracket arises here \(a \text{ posteroiri}\) as \(\hbar\)-linear term of the supercommutator. Below we show how this quantization can be extended to generate a super Wick symbols associated to the super FW-manifold. In view of natural bijection between superfunctions and inhomogeneous differential forms,

\[
a(x)_{i_1 \ldots i_k} \theta^{i_1} \ldots \theta^{i_k} \leftrightarrow a(x)_{i_1 \ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k},
\]

this construction may also be thought of as a deformation of the exterior algebra of differential forms on \(M\).

To begin with, we define a superextension of the formal symbol algebra \(\mathcal{A}\) introduced in the Definition 2.1.

\[\footnote{It is pertinent to note that in the flat case the deformation of exterior form calculus was previously considered in \[22, 23\].} \]
Definition 4.1. The bi-graded superalgebra $\mathcal{S}A = \oplus_{m,n=0}^{\infty}(\mathcal{S}A)_{m,n}$ with unit over $\mathbb{C}$ is a space of formal series,

$$a(x, \theta, y, dx, h) = \sum_{2k+p \geq 0} h^k a_{k, i_1 \ldots i_p j_1 \ldots j_q} (x)y^{i_1} \ldots y^{i_p} \theta^{j_1} \ldots \theta^{j_q} dx^{i_1} \ldots dx^{i_p} dx^{j_1} \ldots dx^{j_q}, \quad (5.5)$$

multiplied with the help of associative $\circ$-product of the form

$$a \circ b =\exp \frac{ih}{2} \Lambda(x) \left( \frac{\partial}{\partial y^j} + \frac{\partial}{\partial \theta^{j'}} \right) a(x, y, \theta, dx, h)b(x, \chi, dx, h)|_{y = z, \theta = \chi}. \quad (5.6)$$

A general term of the series (5.5) is assigned by the bi-degree $(2k + p + q', q'')$.

The associativity of this $\circ$-product is well known (see e.g. [26, Proposition 2.1.]) and may be checked by straightforward computation. As before the expansion coefficients $a_{k, i_1 \ldots i_p j_1 \ldots j_q} (x)$ are considered to be components of covariant tensors on $M$ symmetric in $i_1, \ldots, i_p$ and antisymmetric with respect to $j_1, \ldots, j_q$ and $l_1, \ldots, l_q$'. In accordance with (5.4) one may regard $dx^i$ and $\theta^j$ as the set of $2n$ anti-commuting variables, that turns $\mathcal{S}A$ to $\mathbb{Z}_2$-graded algebra with respect to the Grassmann parity $q = q' + q''$. The supercommutator of two homogeneous elements $a, b \in \mathcal{S}A$ with the parities $q_1$ and $q_2$ is defined as

$$[a, b] = a \circ b - (-1)^{q_1 q_2} b \circ a. \quad (5.7)$$

The nilpotent operator $\delta$ defined by the equation (5.7) is still the inner derivation of the superalgebra $\mathcal{S}A$ and $\delta^{-1} (\mathcal{S}A)_{n,m}$ is the partial homothopy operator for $\delta$ in the sense of “Hodge-De Rham” decomposition (5.11), where now $\sigma(a) = a(x, \theta, 0, h)$. Thus the nontrivial cohomology of $\delta$ coincides with the space of superobservables $C[[h]]$. Obviously, $\mathcal{S}A$ contains $A$ as a subalgebra. The superextension of covariant derivative (5.12) from $A$ to $\mathcal{S}A$ is defined as follows:

$$\nabla : (\mathcal{S}A)_{n,m} \to (\mathcal{S}A)_{n,m+1}, \quad (5.8)$$

$$\nabla = dx^i \left( \frac{\partial}{\partial x^i} - y^j \Gamma^k_{ij}(x) \frac{\partial}{\partial y^k} - \theta^j \Gamma^k_{ij}(x) \frac{\partial}{\partial \theta^k} \right),$$

$\Gamma^k_{ij}$ are Christoffel symbols of the connection associated to $\Lambda$. Then

$$\nabla (a \circ b) = \nabla a \circ b + (-1)^q a \circ \nabla b, \quad (5.9)$$

with $q$ being the Grassmann parity of $a$. It is easy to check that $\nabla$ anti-commutes with $\delta$ and that its curvature is given by

$$\nabla^2 a = \nabla (\nabla a) = \frac{1}{ih} [R, a], \quad (5.10)$$

$$R = \frac{1}{4} R_{ijkl} y^i dx^k dy^j + \frac{1}{4} R_{ijkl} \theta^i dx^k dy^j, \quad (5.11)$$

where we have used the notations $R_{ijkl} = \omega_{im} R^m_{nk}$ and $R_{ijkl} = g_{im} R^m_{nk}$. By analogy with nonsuper case, one can combine $\nabla$ with an inner derivative to get the Abelian connection of the form

$$D = \nabla - \delta + \frac{1}{ih} [r, \cdot] = \nabla + \frac{1}{ih} [\omega_{ij} y^i dx^j + r, \cdot], \quad r = r_i (x, \theta, y, h) dx^i, \quad (5.12)$$

$$D^2 a = \frac{1}{ih} [\Omega, a] = 0, \quad \forall a \in \mathcal{S}A, \quad \Omega = -\frac{1}{2} \omega_{ij} dx^i dx^j,$$

Denote $\mathcal{S}A_D = \ker D \cap \mathcal{S}A_{i,0}$. The next assertion is the super counterpart of that stated in Theorems 2.1, 2.2. and Corollary 2.1.
Theorem 4.1. With the above definitions and notations we have:

i) there is a unique Abelian connection $D$ (5.12) for which

$$\delta^{-1} r = 0, \quad r_i(x, \theta, 0, \hbar) = 0$$

and $r$ consists of monomials whose first degree is no less then 3;

ii) $SA_D$ is a subalgebra of $SA$ and the map $\sigma$ being restricted to $SA_D$ defines a linear bijection onto $C[[\hbar]]$;

iii) the formula

$$a * b = \sigma^{-1}(a) \circ \sigma^{-1}(b)$$

defines the associative multiplication on $C[[\hbar]]$.

Proof can be directly read off from [26, Theorems 2.1, 2.2].

As in nonsuper case the explicit construction for the lifting map $\sigma^{-1}$ results from iterations of two coupled equations (3.21) with respect to the first degree, with the only difference that $R$ is now determined by (5.11) and $a \in C[[\hbar]]$.

Using two transverse polarizations on $(M, \Lambda)$, associated to the left and right kernel distributions of $\Lambda$, one may introduce yet another graded structure on $SA$. In order to present this gradation in explicit form, let us introduce a frame of (complex) 1-forms

$$\{ e^\alpha_i = e^{\alpha_i} dx^i, f^{\beta_j} = f^{\beta_j} dx^j \}, \quad \alpha, \beta = 1, \ldots, \dim M/2$$

spanned the complexified cotangent bundle over a contractible coordinate chart $U$.

The 1-forms can be chosen to satisfy

$$I e^\alpha = e^\alpha, \quad I f^{\beta} = -f^{\beta}, \quad (5.13)$$

$I$ being the integrable involution structure defined by (2.3). Then one can introduce the new (polarized) basis in the space of Grassmann generating elements $\theta^i$:

$$\vartheta^\alpha_i = e^\alpha_i \theta^i, \quad \vartheta^\beta_j = f^{\beta_j} \theta^j \quad (5.14)$$

With respect to this basis the superalgebra is decomposed onto the direct sum of its subspaces

$$SA = \bigoplus_{k=-\infty}^{\infty} SA^{(k)} \quad (5.15)$$

$$SA^{(k)} \ni a = \sum_{m-n=k} a_{\alpha_1 \ldots \alpha_m \beta_1 \ldots \beta_n}(x, y, \hbar, dx) \vartheta^{\alpha_1} \ldots \vartheta^{\alpha_m} \vartheta^{\beta_1} \ldots \vartheta^{\beta_n}$$

The easiest way to see that the decomposition (5.15) does really define a $\mathbb{Z}$-gradation with respect to $\circ$-product is to introduce an inner derivative of the form

$$\widehat{N} a = \frac{1}{i\hbar} [N, a], \quad N = -\frac{1}{2} \omega_{ij} \theta^i \theta^j \quad (5.16)$$

The main properties of the derivative $\widehat{N}$ are collected in the following

Proposition 4.1. Let all notations be as above, then

i) $\widehat{N} a = \theta^i \partial_i \vartheta^a / \partial \theta^i$;

ii) $\widehat{N} a = n a$, \quad $\forall a \in SA^{(n)}$, in particular $\widehat{N} \vartheta^a = \vartheta^a, \widehat{N} \vartheta^\beta = -\vartheta^\beta$;

iii) $\widehat{N} D = D \widehat{N} \Rightarrow r \in SA^{(0)}$;

iv) $\sigma^{-1} \widehat{N} = \widehat{N} \sigma^{-1}$.

The second assertion presents the intrinsic definition of the $\mathbb{Z}$–gradation (5.15) without resorting to the special local frame (5.13).

Proof. i) Straightforward computation leads to
\[ \hat{N}a = \frac{1}{\hbar} [N, a] = \theta^i \frac{\partial}{\partial \theta^k} + \frac{i}{\hbar} \omega_{ij}(\Lambda^k \Lambda^l \Lambda^i) \frac{\partial}{\partial \theta^k} \partial \theta^l a. \]  

(5.17)

The second term in (5.17) vanishes due to the identities \( \omega_{ij} \Lambda^k \Lambda^l \Lambda^i = \omega_{ij} \Lambda^k \Lambda^l = 0. \)

ii) This immediately follows from i) and definition (5.14).

iii) We have to show that \( DN = 0 \) or, more explicitly,

\[ \nabla N - \delta N + \frac{1}{\hbar} [r, N] = 0 \]  

(5.18)

The first two terms vanish since \( \nabla \) preserves the symplectic structure \( \omega \) and \( N \) does not depend on \( y \)'s. So, it remains to prove that \( N \) commutes with \( r \) or, in other words, that \( r \in \mathcal{S}A^{(0)} \). The element \( r \) is determined from the equation (3.21), where \( R \) is given by (5.11). Expanding \( r \) with respect to the first degree, \( r = \sum_{i=0}^{\infty} r_i, r_i \in \mathcal{S}A_{i,1} \), and substituting to the first equation (3.21) we get a recursive definition for \( r \):

\[ r_3 = \delta^{-1} R = \frac{1}{8} R_{ijkl} y^j y^k dx^l + \frac{1}{4} R_{ijkl} \theta^i y^j dx^l, \]  

(5.19)

\[ r_{3+k} = \delta^{-1} \left( \nabla r_{2+k} + \frac{1}{8} \sum_{j=1}^{k-1} r_{j+2} \circ r_{k-j+2} \right), \quad k = 1, 2, \ldots. \]  

(5.20)

Since \( \hat{N} \) commutes with \( \nabla \) and \( \delta^{-1} \) and differentiates \( \circ \)-product the assertion may be proved by induction on the degree. The starting point of induction is

\[ \hat{N} r_3 = \frac{1}{\hbar} [N, \delta^{-1} R] = -\frac{1}{2} R_{ijkl} \theta^i \theta^j dx^l = 0. \]  

(5.21)

The latter equality holds because \( R_{ijkl} = R_{jikl} \).

iv) Let \( \bar{a} = \sigma^{-1}(a) \). We expand \( a \) and \( \bar{a} \) with respect to the first degree:

\[ a = \sum_{k=0}^{\infty} a_k, \quad \bar{a} = \sum_{k=0}^{\infty} \bar{a}_k, \quad a_k = \hbar^k \alpha_k, \quad \alpha_k \in \mathcal{C}, \quad \bar{a}_k \in \mathcal{S}A_{k,0}. \]  

(5.22)

Given \( r \), then the sequence \( \{\bar{a}_k\} \) is recursively determined via \( \{a_k\} \) from the equation (3.21) as follows:

\[ \bar{a}_k = a_k + \delta^{-1} \left( \nabla (\bar{a}_{k-1}) + \frac{i}{\hbar} \sum_{n=1}^{k-2} (r_{n+2}, \bar{a}_{k-n-1}) \right), \quad k = 1, 2, \ldots, \]  

(5.23)

where of course an empty sum is assumed to be zero and \( \bar{a}_0 = a_0 \). Applying \( \hat{N} \) to both sides of (5.23) and using the fact that \( \hat{N} r = 0 \) we get a chain of equations determining the lift of \( \hat{N}a \). But this means iv).

Now let us clarify the relationship between the algebra of quantum observables on FW-manifold \( (C^\infty(M), [\cdot, \cdot], *) \), introduced in Section 3, and its superextension \( (\mathcal{C}[\hbar], *) \). In order to avoid a confusion all the constructions related to the former algebra will be labelled by prime. First of all, we note that the subspace \( C^\infty(M[[\hbar]], \) \( \subset \mathcal{C}[[\hbar]] \) is not closed with respect to \( * \)-product. The reason is that even though elements \( a \) and \( b \) do not depend on \( \theta \)'s there lifts \( \sigma^{-1}(a) \) and \( \sigma^{-1}(b) \), defined with respect to (5.12), are in general do and do not induce a homomorphism of algebras; instead, the following relation holds true:

\[ a * b = \pi(a * b), \quad \forall a, b \in C^\infty(M[[\hbar]], \]  

(5.24)

where \( \pi : \mathcal{C}[[\hbar]] \to C^\infty(M[[\hbar]] \) is the canonical projection defined by the rule: \( \pi a(x, y, h, \theta) = a(x, y, \hbar, 0) \). This relation is the particular case of a more general
Proposition 4.2. For $\forall a \in C^\infty(M)[[\hbar]]$ and for $\forall b \in C[[\hbar]]$ we have:

$$a \ast' (\pi b) = \pi(a \ast b), \quad (\pi b) \ast' a = \pi(b \ast a)$$

(5.25)

Before proving this proposition consider first the fibrewise analog of relations (5.25).

Proposition 4.3. Let $\pi : \mathcal{SA} \to \mathcal{A}$ be a canonical projection defined as $\pi a(x, y, \theta, dx, \hbar) = a(x, y, 0, dx, \hbar)$. Consider the algebras $\mathcal{SA}$ and $\mathcal{A}$ as left (right) moduli over $\mathcal{SA}^{(0)}$ and $\mathcal{A}$, respectively. Then $\pi$ defines a homomorphism of the moduli introduced, that is for $\forall a \in \mathcal{SA}^{(0)}$ and $\forall b \in \mathcal{SA}$

$$\pi(a \circ b) = (\pi a) \circ (\pi b), \quad \pi(b \circ a) = (\pi b) \circ (\pi a)$$

(5.26)

Proof. In view of (5.15) we have the expansion $a = \sum_{n=0}^\infty a_n$, where

$$a_n = a_{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n}(x, y, \hbar, dx) \partial^{\alpha_1} \cdots \partial^{\alpha_n} \hbar \cdots \hbar^{\beta_n}.$$  

Now it suffices to prove that for $n > 0$, $\pi(a_n \circ b) = \pi(b \circ a_n) = 0$. But this directly follows from the identities

$$b \circ \partial^{\alpha} = b \partial^{\alpha}, \quad \hbar \circ b = \hbar b, \quad \forall b \in \mathcal{SA}. \quad (5.27)$$

To prove Proposition 4.2, we need the following

Lemma 4.1. $\pi \sigma^{-1}(a) = (\sigma^{-1})' \pi a$, $\forall a \in C[[\hbar]]$.

Proof. Denote as before $\tilde{a} = \sigma^{-1}(a)$. The element $\tilde{a}$, being expanding as in (5.23), is recursively determined from the pair of coupled equations (5.20,5.23). Note that $\pi r_3 = r_3'$ and $\pi$ commutes with $\nabla$ and $\delta^{-1}$. Now apply the canonical projection $\pi$ to both sides of equations (5.20,5.23). In doing so, we can use the identities (5.26) since $r \in \mathcal{SA}^{(0)}$ in view of Proposition 4.1. By induction we have

$$r'_{3+k} = \delta^{-1} \left[ \nabla r'_{2+k} + \frac{i}{\hbar} \sum_{j=1}^{k-1} r'_{j+2} \circ r'_{k-j+2} \right],$$

(5.28)

$$\pi \tilde{a}_n = \pi a_n + \delta^{-1} \left[ \nabla (\pi \tilde{a}_{n-1}) + \frac{i}{\hbar} \sum_{k=1}^{n-2} [r'_{k+2}, \pi \tilde{a}_{n-1-k}] \right].$$

(5.29)

But this is precisely the recursive definition of $(\sigma^{-1})' \pi a$. Thus the theorem is proved.

Now the Proposition 4.2. follows from the sequence of equalities

$$\pi(a \ast b) = \pi \sigma(\sigma^{-1} a \circ \sigma^{-1} b) = \sigma \pi(\sigma^{-1} a \circ \sigma^{-1} b) = \sigma(\pi \sigma^{-1} a \circ \pi \sigma^{-1} b) =$$

$$= \sigma((\sigma^{-1})' \pi a \circ (\sigma^{-1})' \pi b) = \sigma((\sigma^{-1})' \pi a \circ (\sigma^{-1})' \pi b) = a \ast b$$

and analogous relations for the second equality of (5.25).
6. Discussion.

The deformation quantization, as it was originally formulated, was aimed to provide the rigor mathematical groundwork for what physicists called quantization. Nowadays the methods of deformation quantization go far beyond the quantization problem by itself and constitute an integral and the most elaborated part of more general concept – noncommutative geometry \[34\]. The fresh interest to the subject was provoked by the recent developments in the M-theory \[35, 36\], where a certain limit of nonperturbative string dynamics was recognized as the noncommutative Yang-Mills theory (NYM) \[37\]. By now, however, only the restricted class of such models has been intensively studied, namely, the models formulated on linear symplectic space \( \mathbb{R}^n \) or tori \( T^n \) endowed with canonical Poisson brackets and Euclidean metric. Turning to the more general manifolds (such as ALE and K3 spaces, Calabi-Yau orbifolds, higher genus Riemann surfaces etc., which all of physical interest) one has to deal with the non-constant symplectic and metric structures. It is clear, that the naive covariantization of the flat NYM Lagrangian destroys the associativity of star-multiplication and thus it may break the gauge invariance of the theory. So, the problem is to unify consistently three different ingredients of the NYM theory: the noncommutativity of the coordinates, governed by some Poisson structure; the metric properties of the manifold and the tensor nature of the YM fields. This can be also considered as a part of more general mathematical problem: given a Poisson manifold, how to define the associative deformation of the corresponding tensor algebra and fundamental tensor operations? In the special case of functions, forming the subalgebra in the full tensor algebra, it is just the problem which is studied by the deformation quantization.

In the seminal paper \[35\] the Fedosov deformation quantization was suggested as a possible tool for constructing the NYM on general symplectic manifolds. So far as we know, only two attempts have been undertaken to proceed in this direction \[38, 39\], and both are not too successful. Without going into details we note that the action functional for NYM proposed in \[38\] is not in fact gauge invariant (except the trivial flat case) because the Fedosov covariant derivative has not been properly extended to the tensor observables. In the work \[39\] a local frame of closed 1-forms is introduced to convert the tensor observables into the scalar functions, which may be then quantized by the Fedosov method. However, the technical restrictions imposed by the authors actually imply the existence of global Darboux coordinates on the manifold that makes all the construction trivial.

We hope that the super FW-manifolds introduced in this work may provide a geometrical background for constructing NYM theories in nontrivial symplectic manifolds. Indeed, the natural bijection between differential forms and superfunctions,

\[
a(x)_{i_1 \ldots i_k} \theta^{i_1} \ldots \theta^{i_k} \leftrightarrow a(x)_{i_1 \ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k},
\]

give rise to the associative deformation of the exterior algebra on the base FW-manifold. The NYM fields are embedded in this algebra as 1-forms. The gauge transformations are then associated with the internal automorphisms of the superalgebra,

\[
\delta_\varepsilon a = [a, \varepsilon].
\]

Note, that subspace of 1-forms is not in general invariant under these transformations even when the gauge parameter \( \varepsilon \) is restricted to be a function (0-form). Thus a hypothetical NYM theory being consolidated in this way would necessarily incorporate the entire multiplet of antisymmetric tensor fields. In the flat limit, however, the dynamics of NYM field is expected to decouple from the dynamics of higher-rank forms. The main advantage of this approach is that the symplectic and metric structures enter to the formalism on equal footing from the very beginning, that ensures the general covariance of the theory in question. Finally, the construction of action functional for NYM theory requires to define the trace functional on the algebra of superobservables. This is still an open question and we postpone it to a future work.

The covariant Wick-type symbol construction, given in this paper, may probably find an application in the problems related to quantization of the nonlinear field theory models like strings in the curved spaces. The naive canonical quantization, being based on the Weyl type symbol, is usually inadequate in this case (as is seen even from the simple case of the bosonic string in the flat space), while the nontrivial operator formulation is known only for special examples of nonlinear
theories, like WZWN sigma models. The perturbative quantization of the nonlinear theory might be sometimes possible by means of the expansion over the linear background making use of the Wick symbol defined with respect to the linear approximation (examples of the perturbative construction of the Wick symbols see in ref. [40, 11, 13]). However, the expansion over the linear vacuum cannot always be an efficient method, e.g., it would be a hopeless attempt to construct the string theory in the AdS space taking it as a weak perturbation to the Minkowskian string vacuum. As one can expect, the presented Wick symbol formalism may have a potential for constructing an adequate deformation quantization procedures for the nonlinear field theoretical models. However, the practice of the general method application should be developed in separate works for the actual problems of this type. In particular, the physically relevant models are usually subject to the phase space constraints, so the symbol construction should be first adapted to the case of a constrained system. Second, it should be understood in the nonlinear field theory how one can identify the symmetric tensor which defines the Wick symbol in the phase space. We expect that in many of physically relevant cases, the second problem can be reduced to the first one, because it frequently occurs that a nonlinear theory can be equivalently represented as a linear one subject to constraints (e.g.: the AdS string can be thought of as a string on the constrained hypersurface the in flat space with an additional time dimension).

7. ACKNOWLEDGMENTS

We are grateful to I. Gorbunov, M. Grigoriev, M. Henneaux, A. Karabegov, R. Marinelius, A. Nersessian and M. Vasiliev for valuable discussions. S.L.L appreciates partial financial support from the RFBR grant No 99-01-00980, the work of VAD and AAS is partially supported by the RFBR grant No 00-02-17-956.

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