A NUMERICAL RENORMALIZATION METHOD FOR QUASI–CONSERVATIVE PERIODIC ATTRACTORS

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Abstract. We describe a renormalization method in maps of the plane \((x, y)\), with constant Jacobian \(b\) and a second parameter \(a\) acting as a bifurcation parameter. The method enables one to organize high period periodic attractors and thus find hordes of them in quasi-conservative maps (i.e. \(|b| = 1 - \varepsilon\)), when sharing the same rotation number. Numerical challenges are the high period, and the necessary extreme vicinity of many such different points, which accumulate on a hyperbolic periodic saddle. The periodic points are organized, in the \((x, y, a)\) space, in sequences of diverging period, that we call “branches”. We define a renormalization approach, by “hopping” among branches to maximize numerical convergence. Our numerical renormalization has met two kinds of numerical instabilities, well localized in certain ranges of the period for the parameter \(a\) (see [3]) and in other ranges of the period for the dynamical plane \((x, y)\). For the first time we explain here how specific numerical instabilities depend on geometry displacements in dynamical plane \((x, y)\). We describe how to take advantage of such displacement in the sequence, and of the high period, by moving forward from one branch to its image under dynamics. This, for high period, allows entering the hyperbolicity neighborhood of a saddle, where the dynamics is conjugate to a hyperbolic linear map.

The subtle interplay of branches and of the hyperbolic neighborhood can hopefully help visualize the renormalization approach theoretically discussed in [7] for highly dissipative systems.

1. Introduction. In this paper we deal with renormalization in deterministic dynamical systems of the plane. Specifically, we study dynamics obtained by iterating a map of constant Jacobian on the plane \((x, y)\), and involving two parameters: the Jacobian \(b\), representing the area-contraction factor, and a parameter \(a\) acting as bifurcation parameter.

The renormalization for families converging to a homoclinic onset is studied in literature, in the dissipative case, and when the onset is quadratic and non degenerate [5] and also [8].

We are interested in the quasi–conservative case, that is for \(|b| = 1 - \varepsilon\) for small positive values of \(\varepsilon\), describing a numerical algorithm to detect families of quasi-conservative periodic attractors (fixing \(\varepsilon\)) by a renormalization argument in \((x, y, a)\), with conditions of periodicity and of saddle-node bifurcation values in \(a\).

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We start introducing, in Table 1, the detailed example of our results on high period attractors of the quasi-conservative Hénon map in the form

\[ T_{a,b}(x,y) = (a - x^2 - b y, x) \]

for \( b = \bar{b} = 1 - 10^{-6} \) and rotation number \( \frac{1}{7} \). We list the values \( x_k, y_k, a_k \) of their saddle-node bifurcation, together with the differences among the interpolating values of the sequences \( \{x_k\} \) \( (g_k(x) = \frac{x_k - x_{k-1}}{x_k - x_{k-1}}) \) and \( \{a_k\} \) \( (g_k(a) = \frac{a_k - a_{k-1}}{a_k - a_{k-1}}) \).

These values are used as a “first guess” to initialize Newton’s method and find all values relative to the next periodic orbit of period \( q_{k+1} \) solving the system:

\[
\begin{align*}
(x,y) - T^{q_{k+1}}_{(a,b)}(x,y) &= 0 \\
1 + \bar{b}q_{k+1} - tr(J) &= 0
\end{align*}
\]

(1)

where \( J \) is the product of the Jacobian matrix along the \( q_{k+1} \)-periodic orbit.

Note the changes marked in the boxes for \( g_{k+1}(x) - g_k(x) \) around \( q_k = 14 \) and in \( g_{k+1}(a) - g_k(a) \) around \( q_k = 45 \). The sequences \( \{x_k\} \), \( \{y_k\} \), \( \{a_k\} \) converge and some numerical problems in the convergence are highlighted by the boxes. These sequences represent periodic orbits at their birth through a codimension-1 bifurcation: the saddle-node.

We therefore present, in Table 2, a reformulation of the same result. The list of data is on the same families of periodic orbit as in Table 1, but it is the representative inside each periodic orbit that changes: sequences again converge but the convergence is now much smoother.

This paper describes how we moved from the effort made in obtaining Table 1 to the smooth convergence of Table 2.

We think it is important because such convergence indicates an automatic method to find periodic orbits of higher and higher period with rotation number organized in a sequence. We illustrate the method in the case \( \frac{1}{7} \) for the sake of clarity, but we have already used the method in more complicated families.

In fact, convergence can be generalized on sequences of periodic orbits of diverging period, with a common geometrical pattern. This pattern characterizes the rotation number of the orbits in the sequence of type \( \frac{p_k}{q_k} \), with \( L, M, N, G \) fixed integer numbers for each sequence.

A very well-known and studied method of renormalization involving periodic points in the plane is Greene’s method for the conservative standard map [6]. In that case, we have a family of periodic orbits, of diverging period, whose rotation number \( \frac{F_k}{F_{k+1}} \) (with integer \( F_k \) the \( k \)th Fibonacci number) converge to the irrational number \( \frac{\sqrt{5} - 1}{2} \). The limiting set is thus an infinite period orbit of irrational rotation number, dense on an invariant curve. The value of the bifurcation parameter is chosen to guarantee the same condition of dynamical stability.

When the sequence of rotation numbers converges to a rational number, as in our case, the limiting set is an orbit (of infinite period), characterized by rational rotation number. So it really is an orbit accumulating on a periodic saddle, i.e. a homoclinic orbit.

Similar schemes are known rigorously in highly dissipative systems ([7]). In our case the limiting set has rational rotation number on a homoclinic orbit of a hyperbolic periodic saddle.
The obstacles are described in [3]. Here we will describe in detail the numerical difficulties of localizing periodic orbits of higher and higher periods. One numerical difficulty is the high period itself. It is necessary to reach very high periods, due to a new phenomenon while approaching the conservative case, in particular obstacles to convergence that do not seem to appear in dissipative systems. The obstacles are described in [3]. Here we concentrate on a method that overcomes such obstacles.

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A rescaling method is proposed, similar to what happens in the case of a non-degenerate homoclinic onset, except we start with the orbits instead of the onset, so there is no theorem guaranteeing its convergence.

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$M = 1$ and $O\{q_k\}$ for $q_k$-periodic points at the corresponding value $a = a_{sn(q_k)}$ of their saddle-node bifurcation, together with the interpolating values of the sequences $\{x_k\}$ and $\{a_k\}$. Note the monotone convergence of the interpolating values of $x_k$ and $y_k$. Sequence of periodic orbits is the same as in Table 1, but representative points in each orbit are picked by our “hopping method”.

The present paper is organized as follows: in Section 2 we show some pictures and illustrate the problem. In Section 3 we present a renormalization procedure, as applied to bifurcation values, to find high period attractors interpolating values of $x, y, a$, keeping $b$ fixed, for increasing period. In Section 4 we discuss the procedure and how its application lead to the results of Table 1 and Table 2.

2. The task. In this paper, for the sake of clarity, we illustrate our renormalization procedure in the case of the Hénon map $T_{a,b}(x,y) = (a - x^2 - b, y, x)$ for $b = 1 - 10^{-6}$ and we fix the sequence of rotation numbers with $\frac{\bar{a}}{q_k} = \frac{1}{k}$, i.e. we fix the values $M = G = 1$ and $L = N = 0$. The sequence $\{O_k\}$ of periodic orbits with period $q_k$ in such a case is a “period-adding machine”.

| $q_k$ | $x_k$ | $y_k$ | $a_k$ | $\frac{a_k - x_k}{x_k} - 1$ | $\frac{a_k - y_k}{y_k} - 1$ |
|-------|-------|-------|-------|--------------------------|--------------------------|
| 12    | -0.58592448190766 | -0.61922830479677 | -0.9540226060813 |                     |                     |
| 464   | CORRADO FALCONI AND LAURA TEDESCHINI-LALLI |
| 2     | $L = 1$ and $O\{q_k\}$ for $q_k$-periodic points at the corresponding value $a = a_{sn(q_k)}$ of their saddle-node bifurcation, together with the interpolating values of the sequences $\{x_k\}$ and $\{a_k\}$. Note the monotone convergence of the interpolating values of $x_k$ and $y_k$. Sequence of periodic orbits is the same as in Table 1, but representative points in each orbit are picked by our “hopping method”.

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Figure 1. Sequence of periodic orbits. Each periodic orbit of period $q_j$ is shown on plane $a = a_{sn(q_j)}$. We call “Branch” the sequence of blue dots $Q_j^{(0)}$. In Table 1 all points lie on Branch.

For the same map and other rotation number sequences the geometrical pattern is different, however we have already applied our renormalization scheme to find that it converges nicely.

We find it convenient to represent our data set in the three-dimensional space $(x,y,a)$ (see Fig. 1): each orbit $Q_k$ is on a different plane $a = a_{sn(q_k)}$. Table 1 refers to the branch $B^{(0)}$ marked in red in Fig. 1.

The two boxes on Table 1 indicate two obstacles for the algorithm convergence: one is related to the variable $x$ and the other to the variable $a$. In this paper we take care of the first box, the effort in $(x,y)$. We took care of the second difficulty, related to the $a$ variable, in a previous paper (see [3]).

The task is to build (if it exists) a sequence of periodic orbits with diverging period, sharing a geometrical pattern. Notice that as $k \to \infty$, the points accumulate on an orbit of infinite period. In our computations, the point on the orbit of infinite period accumulate on a periodic saddle. In fact, $\frac{p_k}{q_k} \to \frac{L}{M}$ if $L \neq 0$, and the limiting saddle has rotation number $\frac{L}{M}$, and period $M$. If $L = 0$ then $\frac{p_k}{q_k} \to 0$ and the limiting saddle is a fixed point. This latter case is usually regarded as “simple” homoclinic orbits.

Following is the description of our method.

3. The renormalization method: “Hopping” from branch to branch. 1) Given a 2-parameter map $T_{a,b}(x,y)$ with constant Jacobian $b$, dynamical plane $(x,y)$ and bifurcation parameter $a$, let $O_k(a,b)$ be a periodic orbit stemming from $(x^{(0)},y^{(0)})$ of period $q_k$, with $T_{a,b}^{q_k}(x^{(0)},y^{(0)}) = (x^{(0)},y^{(0)})$. $O_k(a,b) = \{P_k^{(0)}(a,b),...,P_k^{(q_k-1)}(a,b)\}$ is therefore a set in the dynamical plane with $P_k^{(j)}(a,b) = T_{a,b}^{j}(x^{(0)})$,
which there are at least three coexisting periodic orbits

2) In the conservative case, |b| = 1, let us find a value of the parameter, a = \bar{a}, at which there are at least three coexisting periodic orbits \(O_k(\bar{a}, 1), O_{k+1}(\bar{a}, 1), O_{k+2}(\bar{a}, 1)\) of elliptic type, which appear to share the same rotation pattern. This can be formalized by requiring that the three periodic orbits have period \(q_k = Mk + N\) and rotation number \(\frac{\nu_k}{\nu} = \frac{Mk + N}{Mk + N} \).

3) For topological reasons each elliptic periodic orbit is companion to a saddle periodic orbit. We call \(S_j\) the three saddles \(\{S_k(\bar{a}, \bar{b}), S_{k+1}(\bar{a}, \bar{b}), S_{k+2}(\bar{a}, \bar{b})\}\) they have period \(q_j\) for \(j = k, k+1, k+2\). We follow the three coexisting saddles varying \(b\) (i.e. \(b < 1\)) down to \(b = \bar{b}\) by analytic continuation methods. This is possible because saddles are robust and always continuable. Now our map has Jacobian \(\bar{b} < 1\) and thus we are in the area-contracting case.

4) Let us fix \(b = \bar{b}\) and for each period \(q_j\) \((j = k, k+1, k+2)\) we select the value \(a_{sn(q_j)} = a_{sn(q_j)}(\bar{b})\) at which the orbit appears by saddle-node bifurcation, where \(S_j(a_{sn(q_j)}, \bar{b}) = O_j(a_{sn(q_j)}, \bar{b})\).

5) At \(b = \bar{b}\) we thus have three periodic orbits \(S_j(a_{sn(q_j)}, \bar{b})\), for \(j = k, k+1, k+2\), each one consisting of \(q_j\) points in the \((x, y)\) plane. Let us consider, for \(j = k, k+1, k+2\), the orbit \(O_j = \{(x_j, y_j, a_{sn(q_j)})\}\), in the three-dimensional space \((x, y, a)\), by taking as third coordinate, for each point of \(S_j(a_{sn(q_j)}, \bar{b})\), the value \(a_{sn(q_j)}\). We now want to define a sequence picking one point from each orbit. On each of the three orbits \(O_j\), for \(j = k, k+1, k+2\), we choose one point \(Q_j^{(0)} = (x_j^{(0)}, y_j^{(0)}, a_{sn(q_j)})\) with \((x_j^{(0)}, y_j^{(0)})\) at the maximum distance from the hyperbolic fixed point of the map \(\mathbf{P}(a_{sn(q_j)}, \bar{b})\). These three points are vertically aligned (see Fig. 1).

6) Given the three points \(Q_k^{(0)}, Q_{k+1}^{(0)}, Q_{k+2}^{(0)}\), we then look for the next periodic point \(Q_{k+3}^{(0)}\) of period \(q_{k+3} = M(k + 3) + N\). We use an automatic Newton method (in the three-dimensional space \((x, y, a)\)) with initial guess taken by interpolation on the given three points. This procedure can be iterated (as long as the Newton method works, say up to \(k = k^*\)) yielding a sequence that we call \(\text{“branch”}\).

7) Let us define as \(\text{“branch”}\) the set \(B^{(0)} = \{Q_j^{(0)}\}_{j \in \mathbb{N}}\). Note that this set can in principle be transformed by the map \(T\) by moving each point one step along its orbit: using this procedure we can define other branches as the sets \(B^{(i)} = \{T^iQ_j^{(0)}\}_{j \in \mathbb{N}}\). This simple iteration, though, yields a number of branches equal to the period \(q_k\) of the first periodic orbit.

8) Let us define a \(\text{“segment”} T_k^{(0)} = (Q_k^{(0)}, Q_{k+1}^{(0)}, Q_{k+2}^{(0)})\), consisting of three consecutive points on the branch \(B^{(0)}\), and a transformation \(\text{“Hop”}\) acting on it by

\[
\text{Hop}(T_k^{(0)}) = T_{k+1}^{(1)}.
\]

This transformation acts first on \(T_k^{(0)}\) by moving its three points by one step along their orbit, leading to \(T_k^{(1)}\) on the new branch \(B^{(1)}\). Then it acts on \(T_k^{(1)}\), by an interpolation on its three points as an initial guess for a Newton method, to find a new periodic point. This new point \(Q_{k+1}^{(1)}\) will have period \(q_{k+1}\) given by the next value in the sequence \(\{q_k\}\). The new segment is \(T_{k+1}^{(1)} = (Q_{k+1}^{(1)}, Q_{k+2}^{(1)}, Q_{k+3}^{(1)})\).

9) If we now iterate \(\text{Hop}\) transformation \(n\) times, starting from segment \(T_k^{(0)}\), we get new segments up to \(T_{k+n}^{(n)}\). We can finally extract the point of higher period from
Figure 2. Hopping among branches. From segment $(Q_8^{(0)}, Q_9^{(0)}, Q_{10}^{(0)})$ we pass to segment $(Q_9^{(1)}, Q_{10}^{(1)}, Q_{11}^{(1)})$. Every segment, thus one point on every branch, defining a sequence of \( n \) periodic points that move closer and closer to the hyperbolic fixed point of the map (see Table 2).

4. Discussion. Our “hopping” among branches method (see Section 3) overcomes obstacles in convergence while the family of periodic orbits on which it acts remains the same.

The first part of the method, steps 1)–6), proceed along the same branch \((B^{(0)}\) in Fig. 1) searching for successive periodic orbits and selecting on each orbit a special representative.

Locally in the dynamical plane, following a renormalization procedure, one can fix a particular branch, and find a sequence spatially converging to an orbit of \( \infty \) period. Seen globally in the dynamical plane, necessarily more and more branches appear as the period diverges, each branch containing the dynamic image of another. In [8] the limiting orbit is bi-asymptotic, i.e. homoclinic. This infinite period periodic orbit accumulates on the fixed point of the map.

Usually, when dealing with dissipative systems, numerically it is convenient to choose a branch converging to a very isolated tangency value. In working in quasi-conservative regimes, we found it is instead necessary to change branch along the process, so as to enter the neighborhood of the saddle point where the dynamics is conjugate to a linear hyperbolic one, in order to possibly taking advantage of hyperbolicity. This is possible only by going very high with the period.

The second part of the method, steps 7)–9), (see Fig. 2), develops a way of “hopping” from branch to branch, while augmenting period, and thus changing \( a \)-planes.

The two parts of the method act on the same family of periodic orbits. In this second part (Table 2) the method converges smoothly to the fixed point, which again is the accumulation point of the limiting infinite period orbit.
For higher period the shape of the branch changes and gains significant curvature that prevents the use of a simple interpolation. A tangent method interpolation, often used to overcome this kind of difficulty, would be sufficient to reach a slightly higher period but the numerical algorithm remains unstable: small irregular adjustments on some of the three variables are needed at each new period to make Newton’s method converge (see Table 1).

In this case we became aware that changing branch along an orbit increases the convergence of the algorithm: the reason of such regularization is that by doing this, we enter the neighborhood of the fixed point $P$, where the dynamics is conjugate to a hyperbolic linear map, so that successive images of curves are expected to be more regular and stretched. To enter this neighborhood, on the other hand, we need high period periodic points. Going up with the period we need to move along the orbit (and we are able to do it because we deal with high period) with the dynamics so as to come closer and closer to the fixed point $P$ in such a way that the chosen sequence of points $P_k^{(n)}$, one on each consecutive branch, straightens up and the interpolation becomes numerically stable again.

In Table 2, we apply our “hopping” method, described in Fig. 1, at points of Table 1. We show values of $x_k, y_k, a_k$ for the sequence $Q_k^{(n)}$ with $k = k^*, k^* + 1, \ldots, k^* + n$ for $k^* = 12$: note the monotone convergence of the interpolating values of $x_k$ and $y_k$.

We remark that, for $b = 1 - 10^{-6}$ and $a = a_{130}$, the rate of convergence of $x_k$ and $y_k$ is shown to be $10^{-4}$ close to one of the eigenvalue of the Jacobian matrix at the hyperbolic fix point $P(a, b) = \left( \frac{b - 1 - \sqrt{4a + (-1 + b)^2}}{2}, \frac{b - 1 + \sqrt{4a + (-1 + b)^2}}{2} \right)$ which is $\lambda_1 = 1.879713977185823$.

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