On Higuchi Ghosts and Gradient Instabilities in Bimetric Gravity

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Bimetric gravity theories allow for many different types of cosmological solutions, but not all of them are theoretically allowed. In this work we discuss the conditions to satisfy the Higuchi bound and to avoid gradient instabilities in the scalar sector at linear level. We find that in expanding universes the ratio of the scale factors of the reference and observable metric has to increase at all times. This automatically implies a ghost-free helicity-2 and helicity-0 sector and enforces a phantom Dark Energy. Furthermore, the condition for the absence of gradient instabilities in the scalar sector will be analyzed. Finally, we discuss whether cosmological solutions, including exotic evolutions like bouncing cosmologies, can exist, in which both the Higuchi ghost and scalar instabilities are absent at all times.

I. INTRODUCTION

The question whether the graviton can have a mass has been asked long time ago and its answer has always been accompanied by uncertainties. The linear theory of a massive gravity was first analyzed by Fierz and Pauli [1]. Since then the vDVZ discontinuity [2] and the appearance of the Boulware-Deser (BD) ghost [3] had been challenging the theory. Recently, a theory of a massive spin-2 field was presented in which the coupling between an additional fixed tensor field and the metric has a specific structure and is free of the BD ghost [5-12] (see Refs. [13, 14] for recent reviews on massive gravity). To promote this theory of a massive gravity to a bimetric theory, Hassan and Rosen considered a dynamical tensor field $f_{\mu\nu}$, where its kinetic term has the same Einstein-Hilbert structure as $g_{\mu\nu}$ and does not introduce the BD ghost [11, 15]. This bimetric theory is described by the action

$$S = -\frac{1}{2} \int d^4x \sqrt{-g} R(g) - \frac{1}{2} \int d^4x \sqrt{-f} R(f) + \int d^4x \sqrt{-g} \sum_{n=0}^{4} \beta_n e_n(X) + \int d^4x \sqrt{-g} \mathcal{L}_m,$$

where we already set the Planck mass for $M_g$ (see Refs. [16] [17] for further explanation), absorbed $m$, the mass scale of the graviton, into $\beta_n$ and expressed masses in units of $M^2$. The interaction between both tensor fields is determined by the elementary symmetric polynomials $e_n$ of the eigenvalues of the matrices $X^\gamma_\gamma \equiv \sqrt{g} \alpha \beta f_{\mu\nu}$, multiplied by arbitrary real coupling constants $\beta_n$. It is convenient to express these free parameters in units of the present Hubble expansion rate, $H_0$.

A remarkable property of bimetric gravity theories is the possibility of non-standard, self-accelerating cosmological solutions and the ability of making predictions that are different from $\Lambda$CDM. Some of these might be useful for future measurements in order to distinguish standard $\Lambda$CDM from bigravity. To benefit from that, one has to pay the price and needs to disentangle all the non-viable models from the viable ones.

Even though this theory has five free parameters, it is not clear whether viable models exist (except for $\beta_1 = \ldots = \beta_4 = 0$ which is simply $\Lambda$CDM) and, in case, how they look like. In Ref. [18] simple criteria of viability were considered and viable background solutions were presented (see also Ref. [19]). One choice of the coupling parameters, in the following simply model, will usually lead to several different cosmological solutions [10] [18] [22]. In the following, every possible solution will be called branch. We distinguish between different types of branches, depending on how the ratio of the scale factors $r$ of the metrics $f_{\mu\nu}$ and $g_{\mu\nu}$ evolves. In solutions on finite branches the ratio evolves from zero towards a finite asymptotic value, whereas on infinite branches $r$ becomes infinitely large at early times and decreases with time. We call all other branches exotic branches, these usually describe bouncing cosmologies or a static universes in the asymptotic past or future.

So far, only finite and infinite branches were studies in the literature. While many of these are in good agreement with observational data at background level [18] [20] [23], most of them suffer from scalar instabilities [24] [27]. It seems that only one specific class of models, the infinite-branch bigravity (IBB), is free of scalar instabilities [24]. These models are specific infinite branch solutions in which $\beta_2$ and $\beta_3$ vanish. Moreover, IBB agrees very well with true standard...
observations at background and linear level [24, 28]. Unfortunately, the authors in Ref. [29] noted that the Higuchi bound is generally violated in the early time limit. This bound, first derived in Ref. [30], ensures a healthy helicity-0 mode of the graviton. A violation leads to the appearance of the Higuchi ghost, named after Higuchi who found that a spin-2 particle with mass $m$ and $0 < m^2 < 2H^2$ in a de Sitter space leads to a negative norm [31, 32] (see also Ref. [33] in which the Higuchi bound was derived for arbitrary spatially flat FLRW metrics in massive gravity). Note that even though IBB seems to be well behaved at linear level, the appearance of the Higuchi ghost may only be visible at higher orders or maybe even only in the full solution [34]. Furthermore, it was found that cosmological solutions on this infinite branch suffer from a ghost in the helicity-2 sector at early times [35]. Even though it would be probably difficult to get exotic solutions in agreement with observations, they are a priori not excluded. Moreover, poles in $r'$ or a vanishing $\beta$ in the infinite branch, are, however, based on assumptions like the existence of a matter dominated past or the absence of poles in $r'$, where the prime indicates the derivative with respect to e-folding time $t$. Even though it would probably be difficult to get exotic solutions in agreement with observations, they are a priori not excluded. Moreover, poles in $r' = \frac{d}{dt} r$ could have a very physical meaning: If $r'$ reaches a pole, then $dt$ becomes zero and the universe undergoes a bounce. Such an example model is shown in Fig. 1.

In the following, we will first briefly discuss the background evolution in Sect. II, before we then analyze conditions for the absence of the Higuchi ghost and scalar instabilities (Sect. III-IV) to draw conclusions about the viability of all theoretically possible solutions (Sect. V).

**II. EQUATIONS OF MOTION AT BACKGROUND LEVEL**

To find the cosmological background evolution, we vary the action [1] with respect to both metrics and find the equations of motion

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \frac{1}{2} \sum_{n=0}^{3} (-1)^n \beta_n \left[ g_{\mu\lambda} Y_{(n)}^{\lambda} \left( \sqrt{g^{\alpha\beta} f_{\beta\gamma}} \right) + g_{\nu\lambda} Y_{(n)}^{\lambda} \left( \sqrt{g^{\alpha\beta} f_{\beta\gamma}} \right) \right] - T_{\mu\nu} = 0, \quad (2)$$

$$\bar{R}_{\mu\nu} - \frac{1}{2}f_{\mu\nu} \bar{R} + \frac{1}{2} \sum_{n=0}^{3} (-1)^n \beta_{4-n} \left[ f_{\mu\lambda} Y_{(n)}^{\lambda} \left( \sqrt{f^{\alpha\beta} g_{\beta\gamma}} \right) + f_{\nu\lambda} Y_{(n)}^{\lambda} \left( \sqrt{f^{\alpha\beta} g_{\beta\gamma}} \right) \right] = 0, \quad (3)$$

where the overbar denotes curvature of $f_{\mu\nu}$ and $Y_{(n)}^{\lambda}$ are suitable polynomials (see Ref. [20] for their definitions). At background level, we will use a Friedmann-Lemaître-Robertson-Walker (FLRW) ansatz for both metrics with two different scale factors, $a$ and $b$, together with two different time parametrizations $t$ and $\tilde{t} \equiv Xt$. Throughout this work, $t$ represents the e-folding time and a prime denotes the derivative to it. With this ansatz for the metrics,

$$g_{\mu\nu} dx^\mu dx^\nu = a^2 \left( -H^{-2} dt^2 + dx^2 \right), \quad (4)$$

$$f_{\mu\nu} dx^\mu dx^\nu = b^2 \left( -X^2 H^{-2} dt^2 + dx^2 \right), \quad (5)$$

Figure 1: Example of a model ($\beta_i = (0, 0.3, -0.8, 1, -1)$) that describes a bouncing universe. Here, the asymptotic past of this universe is described by a root at $r \approx 0.8$. It then contracts, i.e. $dt < 0$, until $r$ reaches the pole and, finally, expands towards a root at $r \approx 1.9$, which describes a de Sitter point.

\footnote{Note that this specific model is not viable due to a negative $H^2$ and is only shown for motivation purposes.}
where $\mathcal{H}$ is the dimensionless conformal Hubble function, we obtain the $g_{00}$- and $f_{00}$-equations
\begin{align}
3\mathcal{H}^2 &= a^2 \left( \rho + \beta_0 + 3\beta_1 r + 3\beta_2 r^2 + \beta_3 r^3 \right), \quad (6) \\
3\mathcal{H}^2 &= \frac{a^2 r X^2}{(r' + r)^2} \left( \beta_1 + 3\beta_2 r + 3\beta_3 r^2 + \beta_4 r^3 \right). \quad (7)
\end{align}

Here we introduced the ratio of the scale factors $r \equiv b/a$. As usual, both the Friedmann- and acceleration equation for $g_{\mu\nu}$ are degenerated with the conservation of the energy,
\begin{equation}
\rho' = -3\rho \left( 1 + w_{\text{tot}} \right), \quad (8)
\end{equation}
where $w_{\text{tot}}$ denotes the equation of state (EOS) parameter of the total energy density, while there is no extra constraint from the acceleration equation for $f_{\mu\nu}$ due to the missing coupling to the Energy-Momentum tensor. The combination of this set of equations leads to
\begin{equation}
X = 1 + \frac{r'}{r}. \quad (9)
\end{equation}

Replacing this constraint in the equations of motion yields
\begin{align}
3\mathcal{H}^2 &= a^2 \left( \rho + \beta_0 + 3\beta_1 r + 3\beta_2 r^2 + \beta_3 r^3 \right), \quad (10) \\
3\mathcal{H}^2 &= \frac{a^2}{r} \left( \beta_1 + 3\beta_2 r + 3\beta_3 r^2 + \beta_4 r^3 \right). \quad (11)
\end{align}

The second alternative Friedmann equation is particularly interesting since it directly determines the evolution of the scale factor if the evolution of $r$ is known.

The sign of $b$ is a priori unknown and, therefore, $r$ could be negative. However, odd powers of $r$ are always proportional to either $\beta_1$ or $\beta_3$. All cosmological solutions with negative $r$ due to a negative scale factor for $f_{\mu\nu}$ are therefore equivalent to those with positive $r$ after the redefinition $\beta_{2n+1} \rightarrow -\beta_{2n+1}$. From now on, we will assume $r \geq 0$. The combination of both Friedmann equations leads to an equation for the density as a function of $r$ only,
\begin{equation}
\rho = \beta_1 r^{-1} - \beta_0 + 3\beta_2 + 3(\beta_3 - \beta_1) r + (\beta_4 - 3\beta_2) r^2 - \beta_3 r^3. \quad (12)
\end{equation}

It will be useful to study $r'$, which can be written as
\begin{equation}
r' = \frac{\rho'}{\rho_r} = -3 (1 + w_{\text{tot}}) \frac{\rho}{\rho_r}, \quad (13)
\end{equation}
where we used Eq. (8) in the last step.

### III. HIGUCHI GHOSTS

Bimetric theories are called ghost-free since the specific structure of the potential term in the Lagrangian avoids an additional degree of freedom (dof), which usually would be the BD ghost. This, however, does not imply that all dofs of the massless and massive graviton are no ghosts.

#### A. Higuchi Bound

Bimetric gravity theories describe a mixture of a massless and massive spin-2 field. The latter carries five dofs, including one helicity-0 mode. In pure massive gravity around a de Sitter spacetime, Higuchi derived a bound for the graviton mass to ensure positive norm states [31, 32]. A negative norm would imply a ghost helicity-0 mode and is

\[ 2 \text{ This assumption might only be unjustified if both positive and negative values of } r \text{ are reached at some time. In the later discussion we will find that this requires finite, non-zero values of } r' \text{ at } r = 0 \text{ in order to produce viable branches. It turns out that these specific models will not be able to produce viable cosmologies.} \]
usually dubbed Higuchi ghost. The condition for its absence in bimetric gravity theories around a FLRW background was derived in Ref. [30]. In our notations, the bound is
\[
\frac{3}{2} \left( \beta_1 + 2\beta_2 r + \beta_3 r^2 \right) \left( 1 + r^2 \right) \geq \beta_1 + 3\beta_2 r + 3\beta_3 r^2 + \beta_4 r^3 = 3r \left( \frac{H}{a} \right)^2 ,
\]
which is equivalent to
\[
\beta_1 + 3r^2 (\beta_1 - \beta_3) + 2r^3 (3\beta_2 - \beta_4) + 3r^4 \beta_3 \geq 0.
\]
Interestingly, using Eqs. (12-13) leads to the simple bound
\[
\rho_r \leq 0.
\]
This condition for the absence of the Higuchi ghost was already derived in Ref. [36] (see also Ref. [27]). Since
\[
\rho_r = -3 (1 + w_{tot}) \frac{\rho}{r}
\]
and \(\rho > 0\) together with \(1 + w_{tot} > 0\) (we are usually considering a combination of pressureless and relativistic matter), the bound is equivalent to
\[
r' \geq 0.
\]
Note that this holds even for negative values of \(r\). Therefore, in an expanding universe the ratio of the scale factors \(b\) and \(a\) has to increase at all times in order to satisfy the Higuchi bound. Since \(r'\) is negative on all infinite branches [18], this directly shows that these branches suffer from the Higuchi ghost at all times and confirms Ref. [29] who found that the bound is violated at least at early times on infinite branches, i.e. large \(r\). On the other hand, all finite branches that produce viable backgrounds are free from the Higuchi ghost since viability in these branches enforces \(r' \geq 0\) [18]. This especially includes the finite branch in the \(\beta_1\)-model, i.e. only \(\beta_1 \neq 0\), which was already shown to be free of the ghost in Ref. [30].

The RHS of the bound (14) has to be non-negative at all times. Since we already concluded that \(r \geq 0\) is a valid assumption without loss of generality, the Higuchi bound enforces
\[
B_2 \equiv \beta_1 + 2\beta_2 r + \beta_3 r^2 \geq 0,
\]
where \(B_2\) is simply the derivative of \(\rho_{mg}\), the modified part in the Friedmann equation (10), with respect to \(r\). Therefore, the Higuchi bound is related to the change of the amount of Dark Energy in our Universe with time.

### B. Phantom Dark Energy

It is often useful to study the equation of state parameter (EOS), \(w_{mg}\), i.e. the ratio between the pressure and the density, from contributions of the modification of gravity. If we know how the matter density in our Universe evolves, then the knowledge of \(w_{mg}\) enables us to draw conclusions about the acceleration and even the future of our Universe.

In Ref. [18] we showed that Eq. (19) is directly related to the EOS via
\[
w_{mg} = -1 - \frac{B_2}{\rho_{mg}} r'.
\]
If \(\rho_{mg} > 0\) (which, as observations indicate, should hold at least around present time), then the Higuchi bound enforces a phantom Dark Energy. Every cosmological solution in bimetric gravity should therefore have either a Higuchi ghost or a phantom Dark Energy.

The property of being a phantom is usually thought to come along with a future instability, the “Big Rip” [37]. Note, however, that the EOS is highly time dependent and tends to \(-1\) in the asymptotic future if it described by a root in \(r'\), e.g. in most of the finite branch models. A sufficiently fast increase of \(w_{mg}\) could then avoid this instability and guarantee a better behaved future. A phantom in bimetric gravity is therefore not as frightening as in \(\Lambda\)CDM. Thus, a model implying a phantom Dark Energy should not automatically be related with a problematic future, and much less be rejected.

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3 Note that the authors in Ref. [30] used an overall factor of \(\frac{1}{2}\) in front of the potential term in the Lagrangian, which can be compensated by a redefinition of the \(\beta\)-couplings.
C. Tensor Ghosts

Interestingly, the only factor in the lapse of $f_{\mu\nu}$ that is not strictly positive is $r + r'$, thus, the only way to get a negative lapse is a negative $r'$. Therefore, fulfilling the Higuchi bound implies a non-vanishing and especially positive lapse at all times.

It was mentioned in Ref. [35] that the relative factor between the kinetic tensor modes for $g_{\mu\nu}$ and $f_{\mu\nu}$ is the lapse function of $f_{\mu\nu}$ and, therefore, a negative lapse is responsible for a ghost in the helicity-2 sector. We conclude that the absence of the Higuchi ghost automatically implies the absence of a ghost in the helicity-2 sector.

As shown in Ref. [38], the lapse of $f_{\mu\nu}$ directly enters in the friction and mass term of the $f_{\mu\nu}$-tensor perturbation equation leading to negative values at early times, which is responsible for a fast grow of the tensor modes [35, 38]. This is already a signal of the existence of a ghost. To get such a fast grow in the tensor evolution in accordance with observations is a challenging but not undoable task [38]. The main problem, however, is the existence of the ghost itself.

D. Consequences of the Existence of Ghosts

A ghost helicity-0 or helicity-2 will have a dramatic impact on the viability of a theory. It will lead to an unbounded Hamiltonian from below and allows the existence of particles with positive and negative energies. As expounded in Ref. [34], the vacuum state will immediately decay into positive and negative energy particles. This behavior is enough to rule out the underlying theory. It is therefore not a question of how problematic the evolution of a field described by the equation of motion is. A ghost might influence its evolution in a (more or less) unacceptable way, e.g. through a negative friction. However, it is not the possibly ill behaved solution of the perturbation equations that renders the theory unphysical, but rather the absence of a stable vacuum state and interactions with negative energy particles. It is even possible that such a system seems to be completely well behaved at all orders in perturbation theory, but the perturbative solution will not converge to the exact solution. An example where perturbation theory is even able to hide the negative energy solutions, which are present in the full theory, is discussed in Ref. [34].

Since bimetric gravity is only an effective field theory, one might wonder whether a ghost could be harmless in this setup or whether a ghost is necessarily excited. This is, unfortunately, not the case. As explained in Ref. [39], only modes with positive energy are able to decouple, but not a ghost state since there is no positive energy necessary to excite a ghost (see also Ref. [40]). Even in effective field theories (and even if the mass of the ghost lies above the cutoff) one has to avoid ghosts at all costs.

IV. EIGENFREQUENCIES OF SCALAR PERTURBATIONS

After reducing the number of possible cosmological solutions with the demand of the absence of ghosts, we will analyze the behavior of scalar perturbations at linear level. Even though there are already quite a number of works in which similar properties were studied, all investigations so far based on strong assumptions, mostly a restriction in the parameter space, fixing the EOS of the matter fluid or a focus on a specific type of branches. In the majority of cases this is a consequence of the complexity of the perturbation equations. Since the aim of this work is to draw conclusions about the viability of the most general cosmological solutions in bimetric theories with a FLRW background, we will work out conditions for the absence of gradient instabilities without resigning from generality regarding the parameter space, type of branches and nature, i.e. EOS, of the fluid.

The set of scalar perturbation equations at linear level can be reduced to a system of two second order differential equations for two potentials $\Xi_i$ describing the two propagating scalar degrees of freedom [26] (see also Refs. [16, 22, 25, 29, 35, 41]),

$$\Xi_i'' + A_{ij}\Xi_j' + B_{ij}\Xi_j = 0,$$

(21)

where $A_{ij}$ and $B_{ij}$ are matrices which depend on the background quantities $r, H$ and the parameters of the models. The complexity of this system depends crucially on the choice of the gauge. A very convenient one was used in Ref. [29], leaned on Ref. [42]. In this work, we take advantage of the relatively simple perturbation equations that

\footnote{Note that there are “good ghosts”, e.g. the Faddeev-Popov ghost, which are not related to physical degrees of freedom and are therefore harmless.}

\footnote{Where “simple” means that printing these equations would fill only a couple of pages.}
the authors found in this gauge (see Ref. [29] for the derivation and printed equations) and analyze them by using the ansatz $\Xi_i \propto e^{i\omega t}$. For simplicity, we assume that the eigenfrequencies $\omega$ do not depend on time. This is a valid assumption as long as $|\omega^2/\omega^2| \ll 1$ holds and was confirmed for all models studied in Ref. [24]. In the sub-horizon limit, we obtain a surprisingly simple expression for the eigenfrequencies,

$$\omega^2 = \left( \frac{k}{\mathcal{H}} \right)^2 \left[ r'' \left( \frac{r}{\rho(r+1)} \right) - \frac{r^2 (\beta_1 - \beta_3 r^2)}{\beta_1 + 2 \beta_2 r + 3 \beta_3 r^2} \right] - 1,$$

(22)

$$= \left( \frac{k}{\mathcal{H}} \right)^2 \left[ r'' \left( -\rho_r^{-1} (r^2 + 1) \left( \beta_1 - \beta_3 r^2 \right) \frac{r^2 (\beta_1 + 2 \beta_2 r + 3 \beta_3 r^2)}{\beta_1 + 2 \beta_2 r + 3 \beta_3 r^2} \right) - 1 \right],$$

(23)

which agrees with all previous, but much more complicated, results for one- and two-parameter models that were studied in [24]. As already mentioned in Ref [24], if we assume dark matter only, then for models in which $\beta_2 = \beta_3 = 0$ this reduces to

$$\omega^2_{\beta_0 \beta_1 \beta_4} = \left( \frac{k}{\mathcal{H}} \right)^2 \frac{r''}{3 r}.$$  

(24)

In order to discuss stability, we only need to analyze the sign of $\omega^2$: A negative value would imply oscillating and therefore stable potentials $\Xi_i$. If, however, $\omega^2$ is positive, then $\Xi_i$ grows quickly with time and even faster the smaller the scales become. Such an instability is not be compatible with the structure in our Universe and needs to be avoided in a viable model.

Let us now introduce $B_2 = \beta_1 + 2 \beta_2 r + 3 \beta_3 r^2$ to obtain

$$\omega^2 = \frac{k^2}{3 \rho_r \mathcal{H}^2} \left[ r'' \left( 3 \left( r^2 + 1 \right) \frac{B_2}{r^2} - \rho_r \left( \frac{B_2}{B_2} + 1 \right) \right) - 3 \rho_r r \rho_r \right].$$

(25)

Interestingly, the condition for stability depends on how Dark Energy (and the density of the cosmic fluid) changes but not explicitly on how large it is. We observed a similar property during the analysis of the Higuchi bound. Note that $B_2$ is related to the change of the energy density, $\rho_r$, and the Hubble expansion via

$$B_2 = -\frac{r}{1 + r^2} \left( \frac{1}{3} r \rho_r - 2 \left( \frac{\mathcal{H}}{a} \right)^2 \right).$$

(26)

Together with

$$\left( \frac{B_2}{r} \right)_{,r} = r^{-2} B_2 \left( \frac{B_2}{B_2} - 1 \right),$$

(27)

we finally arrive at

$$\omega^2 = \left( \frac{k}{\mathcal{H}} \right)^2 \left[ \frac{2 r'' \left( r^2 + 1 \right) B_2 \rho_r - \left( \frac{\mathcal{H}}{a} \right)^2 \left( 3 \left( r^2 + 1 \right) B_2 + r \rho_r \right) + 6 \left( \frac{\mathcal{H}}{a} \right)^4}{r^2 \rho_r \left( \rho_r - 6 \left( \frac{\mathcal{H}}{a} \right)^2 \right) - 1} \right].$$

(28)

As we will see later, this expression for the eigenfrequencies will become very convenient when analyzing the stability around poles in $r''$, which e.g. always appear in exotic branches.

It might be useful to study an expression for $\omega^2$ which does not explicitly depend on the $\beta$-parameters but on $r$ and its derivatives, like Eq. [24]. Finding such an expression is always possible when using a set of five independent equations to eliminate all coupling parameters. One possibility is the set of equations for $r''$, $r''', H^2$ and $\rho$ (note that the result will not depend on $r'''$) which yields

$$\omega^2 = \left( \frac{k}{\mathcal{H}} \right)^2 \frac{a^2 \rho_r (w + 1) \left[ 2 (w + 1) r'' + r' \left( 6 w^2 - 2 w' + 9 w + 3 \right) \right] - 2 \mathcal{H}^2 r' \left( r' (w + 1) (r' - 3 r w) + r w' \right) - r (w + 1) r''}{3 r (w + 1) r' \left( a^2 \rho_r (w + 1) + 2 \mathcal{H}^2 r' \right)}.$$

(29)
Here, and in all the following equations, we dropped the subscript in $w_{\text{tot}}$ for simplicity. If we are interested in analyzing the eigenfrequencies at specific epochs, e.g. RDE and MDE, we can assume $w \simeq \text{const}$ and obtain

$$\omega^2 = \left( \frac{k}{\mathcal{H}} \right)^2 \frac{a^2 \rho r^2(w+1) \left[ 2r'' + 3r'(2w+1) \right] + 2\mathcal{H}^2 r' \left[ r(r'' + 3wr') - r'^2 \right]}{3rr'(a^2 \rho r(w+1) + 2\mathcal{H}^2 r')}.$$ (30)

This leads to the condition

$$r' \left[ a^2 \rho r(w+1) + 2\mathcal{H}^2 r' \right] \left[ a^2 \rho r^2(w+1) (2r'' + (6w + 3)r') + 2\mathcal{H}^2 r' (r(r'' + 3wr') - r'^2) \right] < 0$$ (31)

in order to get stable scalar perturbations, i.e. $\omega^2 < 0$. When using the Higuchi bound, $r' > 0$, the first bracket term is always positive and, thus, the second one has to be negative. This is equivalent to

$$r'' < \frac{r' \left[ 2\mathcal{H}^2 r' (r' - 3r w) - 3a^2 \rho r^2(w+1)(2w+1) \right]}{a^2 \rho r(w+1) + \mathcal{H}^2 r'},$$ (32)

where we also used $r' > 0$. Note that the denominator is always positive. If the numerator would be negative, then the bound would especially imply $r'' < 0$. However, this is not generally the case and, thus, the condition for stable scalar modes is not automatically equivalent to $r'' < 0$ in contrast to the case for $\beta_0 \beta_1 \beta_4$-models during matter domination (see Eq. (24)).

### A. Radiation Dominated Era

Even though we will not aim to exclude models which are theoretically allowed but do very likely not reproduce observational data (an example would be a nearly static universe that did not have a radiation dominated epoch), it is worthwhile to analyze the conditions when the universe is filled with either relativistic particles or pressureless matter only.

When radiation dominates, i.e. $w \simeq 1/3$, the eigenfrequencies simplify to

$$\omega^2 = \frac{k^2}{3\mathcal{H}^2 r'} \frac{a^2 \rho r^2(2r'' + 5r') + 2\mathcal{H}^2 r' (rr'' - r'^2 + rr')}{\frac{2}{3} a^2 \rho r + 2\mathcal{H}^2 r'}.$$ (33)

In the early Universe, the Hubble expansion is usually driven by radiation, i.e. $3\mathcal{H}^2 \simeq a^2 \rho$. With this approximation, the condition for stability in the scalar sector becomes

$$r'' > -\frac{r' (r' + 10r)}{r' + 4r}.$$ (34)

For large absolute values of $r'$, which is the case e.g. near a pole, we simply obtain $r'' > -r'$.

In a previous work [24], we studied the eigenfrequencies for IBB and confined ourselves to a universe filled with dark matter only. According to Eq. [24], we concluded stable scalar modes because $r'$ increases with time but stays negative until reaching the final de Sitter point. Since $r'$ is always negative in IBB, the condition [24] is not necessarily valid anymore. However, we can still use condition [31]. Here, the product of the first two terms is always positive since

$$r' \left[ a^2 \rho r(w+1) + 2\mathcal{H}^2 r' \right] \bigg|_{\text{IBB}} = 9a^2 \beta_1 r \left( r'^2 + 1 \right) \left( \frac{(w + 1) \left( \beta_1 + \beta_4 r^3 - 3\beta_1 r^2 \right)}{\beta_1 - 2\beta_4 r^3 + 3\beta_1 r^2} \right)^2 > 0.$$ (35)

Therefore, we can analyze the third factor and, assuming $w \in (-1,1)$ for simplicity, find that stable modes are guaranteed if

$$3\beta_1 r^2 < \beta_1 + \beta_4 r^3,$$ (36)

which is not only satisfied in the RDE, i.e. large $r$ (note that both $\beta_1$ and $\beta_4$ have to be positive in order to get a viable cosmological background), but, in fact, is equivalent to the condition $\rho > 0$ on that branch and, therefore, trivially satisfied at all times.
B. Matter Dominated Era

Let’s study the regime when matter dominates the universe. Now the EOS vanishes and the scalar modes are described through

$$\omega^2 = \frac{k^2}{3H^2 r^2} \frac{a^2 \rho r^2 (2r'' + 3r') + 2H^2 r' (rr'' - r'^2)}{a^2 \rho r + 2H^2 r'^2}. \quad (37)$$

For stability, we need to satisfy the condition

$$r'' < \frac{2H^2 r'' - 3a^2 \rho r^2 r'}{2a^2 \rho r^2 + 2rH^2 r'^2}. \quad (38)$$

If we assume that $\frac{a^2 \rho}{H^2} \to 0$ for late times, which should be true when Dark Energy starts to dominate, then the condition of stability reduces to

$$r'' \lesssim \frac{r'^2}{r}. \quad (39)$$

V. FINDING VIABLE BRANCHES

We will now raise the question whether branches exist that satisfy both the Higuchi bound and the condition for scalar stability. Here we will only focus on cosmological solutions that are not equivalent to ΛCDM, which of course satisfy both conditions. We therefore assume that at least one of the couplings $\beta_1, ..., \beta_4$ is non-zero. Together with conditions of physicality, $a, \rho, H^2 > 0$, we define these as criteria of viability. Note that we allow for solutions that have a very non-standard past, e.g. no matter or radiation dominated epoch, or even contracting backgrounds, even though these might be hard to compare with observational data. This extends the more restrictive background analysis of [18]. Therefore, not only the finite branch with small $r$ or the infinite one could be viable but also many solutions on exotic branches. Many different types of branches exist, some of them start from a root $r' = 0$, other may evolve from a pole or even pass a pole at some finite time. In many cases it is not directly clear whether such branches solve the equations of motion. Especially, every branch always needs to contain a solution of Eq. (11) at present time, i.e. when $H = a = 1$.

We start with focusing on finite branches with a root at $r = 0$. Let us first concentrate on models with $\beta_1 \neq 0$, which always have a root at $r = 0$ (see Eq. [38]). In Ref. [24] we generally found scalar instabilities in these type of branches. Even though this is based on the assumption of a universe filled with dark matter only, this conclusion does not change when considering arbitrary but reasonable EOS parameters. We take the same line of argument and study the simple $\beta_1$-models, i.e. models with only non-vanishing $\beta_1$, since all other models will reduce to these in the limit when $r$ gets close to $r = 0$. The eigenfrequencies in $\beta_1$-models are given by

$$\omega_{\beta_1}^2 = \frac{1 + 2w - 6r^2(w + 2) - 9r^4}{(3r^2 + 1)^2} \left( \frac{k}{H} \right)^2 \approx \frac{1 + 2w}{(3r^2 + 1)^2} \left( \frac{k}{H} \right)^2. \quad (40)$$

and therefore indicate unstable modes for small values of $r$ as long $w > -1/2$. Let’s consider the previously excluded models with $\beta_1 = 0$ and find

$$r' \bigg|_{r=0} = \frac{\beta_0 - 3\beta_2}{\beta_3} (w + 1). \quad (41)$$

Even though the combination $\beta_0 = 3\beta_2$ is able to produce a root at $r = 0$, it will not lead to viable solutions since in this case $r' = -3(1 + w) + \mathcal{O}(r^2)$ indicates a violation of the Higuchi bound. From this we conclude that

1. Finite branches with a root at $r = 0$ always lead to either unstable modes (if $\beta_1 \neq 0$) or violate the Higuchi bound (if $\beta_1 = 0$) for small $r$.

On the other hand, $r'$ could be non-zero but still finite at $r = 0$. In this case, one of the asymptotic point is either a pole or the whole branch evolves between two roots at negative and positive $r$. In the first case, we can assume that at least one of the poles is reached at $r > 0$, otherwise we are able to analyze viability in the “mirrored” model corresponding to $\beta_{2n+1} \to -\beta_{2n+1}$. If the branch does not contain any pole, then $\rho, r$ has to vanish at $r = 0$ (roots at $r \neq 0$ always indicate a vanishing density whereas a maximum of the density at $r \neq 0$ leads to a pole). The position of the maximum of $\rho$ at $r = 0$ requires $\beta_3 = 0$ and leads to $\rho, r \propto r$ which cannot be negative for both regions, $r > 0$ and $r < 0$. We can summarize that
2. All finite branches with a non-zero and finite \( r' \) at \( r = 0 \) have to have a pole either in the asymptotic past or future.

Roots, except for those at \( r = 0 \), always correspond to a vanishing density. Due to Eq. (13), we will always find a pole between two roots \( r_1 \) and \( r_2 \), if both \( r_1 \) and \( r_2 \) are non-zero. Therefore, poles could be interesting starting or final points of a branch. Whenever such a pole describes the asymptotic future, then \( r' \) has to go to zero, otherwise the pole would not be a stable asymptotic point. Since \( \rho \) vanishes, which, as we already noted, requires \( \beta_1 = \beta_3 = 0 \) and leads to \( H^2 |_{r=0} = 3\beta_2 - \beta_0 \). If the density is finite at early times, the scale factor \( a \) has to have a finite but non-zero value. In this case, \( H \) needs to be zero at early times, too, otherwise one could go backwards in time and we would not have an asymptotic past. Thus, we conclude

3. \( H \) has to become zero on a pole, if it describes an asymptotic point.

Let’s assume a pole at \( r = 0 \), which, as we already noted, requires \( \beta_1 = \beta_3 = 0 \) and leads to \( H^2 |_{r=0} = 3\beta_2 - \beta_0 \). From the previous conclusion, we need a vanishing \( H^2 \) at \( r = 0 \). Note that \( a > 0 \), otherwise this would contradict a finite density. Therefore, we need \( \beta_2 = 0 \) and, thus, obtain \( B_2 = 0 \) for all \( r \), which means that

4. A pole at \( r = 0 \) violates the Higuchi bound.

For simplicity, we will from now on assume that if there is a pole at \( r_p \), then \( r_p > 0 \). Additionally, we can exclude \( r = 0 \) from being an asymptotic point due to the previous conclusions. Furthermore, Eqs. (13) and (12) provide the limit \( r' \propto -r \) when taking \( r \to \infty \) as long as the density does not vanish (see Ref [18] more detailed explanations).

This excludes infinite branches, i.e. branches in which \( r \) evolves from or to \( r \to \infty \), from being viable due to the violation of the Higuchi bound and we find that

5. The limits \( r \to 0 \) and \( r \to \infty \) are no viable asymptotic points.

We will now consider a root at \( r \neq 0 \) as the asymptotic past. Due to Eq. (13), the density vanishes on a root. To fulfill the conservation of energy, those models require a contracting universe at early times. If this universe evolves to another root (on which again \( \rho = 0 \)), then it has to undergo a bounce at \( \rho, r = 0 \) leading to a pole at which \( H = 0 \). Employing the previous conclusions, we find the general statement

6. Every viable branch needs to contain at least one pole on which \( H \) vanishes.

This result is particularly interesting as it will allow us to draw conclusions when connecting this with the requirement of stable scalar perturbations and the absence of the Higuchi ghost. The necessary condition for a pole is \( \rho, r \to 0 \).

Then, the eigenfrequencies of scalar perturbations around the pole (28) reduce to

\[
\omega^2 \to \left( \frac{k}{H} \right)^2 \left( 2B_{2,r} \frac{r' \left( 1 + r^2 \right) r \rho_x - 3 \left( \frac{\rho}{\rho_x} \right)^2}{r^2 \rho_x} - 6 \left( \frac{\rho}{\rho_x} \right)^2 - 1 \right)
\]

\[
\approx \left( \frac{k}{H} \right)^2 \left( 2B_{2,r} \frac{r' \left( 1 + r^2 \right)}{r^2 \rho_x} - 1 \right)
\]

where we used \( \left( \frac{\rho}{\rho_x} \right)^4 < \left( \frac{\rho}{\rho_x} \right)^2 \), \( \left( \frac{\rho}{\rho_x} \right)^2 \rho_r \ll \left( \frac{\rho}{\rho_x} \right)^2 \) (and, additionally, \( B_{2,r} \neq 0 \) which, as we will see later, is justified), together with

\[
\frac{r \rho_x - 3 \left( \frac{\rho}{\rho_x} \right)^2}{r \rho_x - 6 \left( \frac{\rho}{\rho_x} \right)^2} = \frac{B_2 + \left( 1 + r^2 \right) - \left( \frac{\rho}{\rho_x} \right)^2 r}{B_2 + \left( 1 + r^2 \right)} \approx 1,
\]

which follows from Eq. (26) and \( B_2 + \left( 1 + r^2 \right) - \left( \frac{\rho}{\rho_x} \right)^2 r \approx B_2 + \left( 1 + r^2 \right) \) (note that the Higuchi bound (14) implies \( B_2 + \left( 1 + r^2 \right) > 2 \left( \frac{\rho}{\rho_x} \right)^2 > 0 \). Since \( r' \to \infty \) and \( \rho, r \to 0 \) (but still \( \rho, r < 0 \) and \( r' > 0 \), the first term in the bracket of Eq. (43) dominates unless \( B_{2,r} = 0 \).

Let’s first assume that \( B_{2,r} = 0 \) at the pole \( r_p \). We then find

\[
B_{2,r} \bigg|_{r=r_p} = 0 \quad \Rightarrow \quad \beta_2 = -\beta_3 r_p,
\]

\[
\left( \frac{H}{a} \right)^2 \bigg|_{r=r_p} = 0 \quad \Rightarrow \quad \beta_1 = -\beta_4 r_p^3,
\]

\[
\rho, r \bigg|_{r=r_p} = 0 \quad \Rightarrow \quad \beta_3 = -\beta_4 r_p,
\]
which leads to

$$3H^2 = \frac{a^2}{r} \beta_4 (r - r_p)^3,$$

as well as

$$B_2 = -\beta_4 r_p (r_p - r)^2.$$  \hspace{1cm} (49)

If $r$ increases with time (which implies that $dt > 0$ since $r' > 0$), then $\beta_4$ has to be positive in order to get a positive $H^2$. On the other hand, this would imply a negative $B_2$ and, thus, would violate the Higuchi bound. We therefore have to have a decrease of $r$ with time (which implies contraction, $dt < 0$). Now, this is only compatible with negative values for $\beta_4$. Of course, such a model would be hard to believe in since it would contract at all times. But there is more solid argument for ruling out these models: The contraction would lead to an increasing density. Since a root corresponds to a vanishing density, there must be a point of maximum density which always indicates a pole (see Eq. (13)). Note, that we already excluded both $r = 0$ and $r \to \infty$ as asymptotic states, which are the only ones that would be able to describe an infinitely large density. However, on this one, $H$ cannot vanish. Even though this second pole does not necessarily need to be an asymptotic point, $\mathcal{H} = 0$ is required due to the bounce. Both, positive and negative values for $\beta_4$ do not lead to viable solutions and we conclude that

7. Viability enforces a non-zero value for $B_{2,r}$ around a pole.

We are now allowed to assume $B_{2,r} \neq 0$. Then the term $-1$ in Eq. (43) is negligible and, thus, $B_{2,r}$ has to be positive in order to get stability, i.e. $\omega^2 < 0$.

We will now study the expansion rate around the pole $r_p$ and check whether $H^2$ is positive. Note, that $(\mathcal{H}^2)_{,r} |_{r_p}$ does not automatically vanish since $\mathcal{H}^2$ could become negative, too (which, however, would not correspond to physical solutions). However, the conditions for scalar stability ($B_{2,r} |_{r_p} > 0$), the existence of a pole ($\rho_{,r} |_{r_p} = 0$ with $\mathcal{H}^2 |_{r_p} = 0$) and physicality ($\rho |_{r_p} \geq 0$) together with the assumption that $(\mathcal{H}^2)_{,r} |_{r_p} \neq 0$ lead to an contradiction. Therefore, let’s assume $(\mathcal{H}^2)_{,r} |_{r_p} = 0$ which, together with $\mathcal{H}^2 |_{r_p} = 0$, implies

$$\beta_2 = -\frac{1}{3} r_p^{-1} (2\beta_1 - \beta_4 r_p^3),$$

$$\beta_3 = \frac{1}{3} r_p^{-2} (\beta_1 - 2\beta_4 r_p^3).$$  \hspace{1cm} (50)

If we now assume that $\mathcal{H}^2$ is positive and non-zero at second order, then we need to have

$$3 \left( \frac{\mathcal{H}}{a} \right)^2 = \frac{(r_p - r)^2}{r_p^2} (\beta_1 + \beta_4 r_p^3) = r_p^{-3} (\beta_1 + \beta_4 r_p^3) (r - r_p)^2 + O \left( (r - r_p)^3 \right) > 0.$$  \hspace{1cm} (52)

However, this would imply that

$$\rho_{,r} = 2 \frac{1 + r_p^2}{r_p^3} (\beta_1 + \beta_4 r_p^3) (r - r_p) + O \left( (r - r_p)^2 \right)$$

only becomes negative when leaving the pole, if $r$ decreases with time, i.e. in an contracting universe. We can now use the same argument that we used before and conclude that we need to reach a second pole which will either describe an asymptotic point or a bounce. Eq. (52) provides the possibility of another point $r_{p_2} = -\beta_1 / (\beta_4 r_p^3)$ at which the expansion stops but this cannot be a pole since then we would find

$$\rho_{,r} |_{r_{p_2}} = -\frac{(\beta_1 + \beta_4 r_p^3)^2 (\beta_1^2 + \beta_4^2 r_p^4)}{\beta_1 \beta_4 r_p^4} \neq 0.$$  \hspace{1cm} (54)

Our last chance are models in which $\mathcal{H}^2$ vanishes up to second order implying that

$$\beta_1 = -\beta_4 r_p^3,$$ \hspace{1cm} (55)

$$\beta_2 = \beta_4 r_p^2,$$ \hspace{1cm} (56)

$$\beta_3 = -\beta_4 r_p.$$ \hspace{1cm} (57)

These solutions lead to $B_{2,r} = 0$, which we already excluded earlier. Therefore,

8. A negative $B_{2,r}$ around a pole leads to gradient instabilities whereas a positive value violates either the Higuchi bound or leads to unstable scalar perturbations.

In combination with the requirement of a positive $B_{2,r}$ in order to get stable scalar perturbations this shows that every branch is plagued by either the Higuchi ghost or scalar gradient instabilities.
VI. CONCLUSIONS AND OUTLOOK

We analyzed general models in singly-coupled bimetric gravity around a FLRW background and found that all physical cosmological solutions that are not equivalent to $\Lambda$CDM have a period in time in which either linear scalar perturbations undergo a gradient instability or the Higuchi ghost appears. The condition for the absence of ghosts is surprisingly equivalent to $r' > 0$, which means that the ratio of the scale factors $b$ and $a$ has to increase as long as the universe expands. Moreover, satisfying this bound ensures a positive lapse of $f_{\mu\nu}$ which is related to the absence of a helicity-2 ghost.

In fact, all infinite branches suffer from the Higuchi ghost at all times and a ghost in the helicity-2 sector at early times, whereas in all finite branches, and even exotic branches that do not contain the limit $r \to 0$, there exists at least one epoch in which there is either a gradient instability in the scalar sector or a ghost appears. A schematic illustration of a typical phase space diagram with the forbidden regions is presented in Figure 2.

While already the existence of a ghost renders the model unphysical and forces us to discard this type of model, unstable scalar modes will not necessarily rule out the theory. A Vainshtein screening may be able to prevent the scalar sector from getting unstable. Furthermore, this gradient instability is not present at all times. Every finite branch has a point in time at which the instability stops and the scalar perturbations begin to oscillate. As shown in [15], a small, but natural, Planck mass for $f_{\mu\nu}$ would shift this gradient instability to very early times or even to energy scales above the cutoff of the effective field theory. In the latter case, the cosmological evolution would be very close to $\Lambda$CDM. On the other hand, if the instability would end between inflation and Big Bang nucleosynthesis, only very small scales would be affected [43]. These could in principle lead to a creation of many seeds for black holes.

All models which we don’t have to exclude due to the presence of a ghost will describe a phantom Dark Energy. Such a property would cause an anxious future in a $\Lambda$CDM model but not necessarily in bimetric theories due to the time dependence of EOS corresponding to Dark Energy. In fact, it could cause welcome signatures that might allow observations to distinguish bimetric gravity from General Relativity.

Throughout this work we assumed a very simple, but well motivated, type of bigravity. We considered a fluid that is only singly coupled to an observable metric and where both metrics are of FLRW type. Several extensions exist in the literature. One example would be the coupling of matter to both metrics $g_{\mu\nu}$ and $f_{\mu\nu}$ simultaneously [27, 45–49], which, however, would introduce the BD ghost if the same matter sector is coupled to both metrics [45, 50, 51]. Ghost-free (but not always with well behaved cosmological solutions) scenarios exist if one assumes a coupling through a composite metric [40, 50, 52–58]. But even the bimetric gravity with a standard matter coupling could allow for cosmological solutions without any gradient or ghost instabilities at the cost of giving up a FLRW background [59, 60].

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Note that in this and many previous works, the Planck mass $M_f$ was set to $M_g$, which is allowed due to a redundancy in the parameters but is, however, not the most natural choice.

Since the cosmological evolution at the time where the instability would end is not close to $\Lambda$CDM yet, the fast change in $\rho_{mg}$ might even have a stronger influence on the evolution of primordial black holes compared to standard $\Lambda$CDM.
Figure 2: Illustration of a phase space diagram of a typical model together with colored regions corresponding to different types of instabilities. While finite branches are plagued from gradient instabilities in the scalar sector (diagonal blue stripes from top-left to bottom-right) at early times, the infinite branches suffer from the Higuchi ghost (diagonal red stripes from bottom-left to top-right) at all times and a ghost in the helicity-2 sector (vertical orange stripes) at early times. Finally, all exotic branches, including e.g. bouncings, have always a pathological behavior at least around the pole in $r'$.

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