Robust graphical lasso based on multivariate Winsorization

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Abstract

We propose the use of a robust covariance estimator based on multivariate Winsorization in the context of the Tarr–Müller–Weber framework for sparse estimation of the precision matrix of a Gaussian graphical model. Likewise Croux–Öllerer’s precision matrix estimator, our proposed estimator attains the maximum finite sample breakdown point of 0.5 under cellwise contamination. We conduct an extensive Monte Carlo simulation study to assess the performance of ours and the currently existing proposals. We find that ours has a competitive behavior, regarding the the estimation of the precision matrix and the recovery of the graph. We demonstrate the usefulness of the proposed methodology in a real application to breast cancer data.

Key words: Gaussian graphical model; Precision matrix; Sparse robust estimation; Cellwise contamination; Winsorization.
1 Introduction

Let \( X = (X_1, \ldots, X_p)' \) be a \( p \)-variate random vector with Gaussian distribution with mean vector \( \mu \) and covariance matrix \( \Sigma \). We assume that \( \Sigma \) is positive definite (\( \Sigma \succ 0 \)) and its inverse, the precision matrix, will be denoted by \( \Omega = (\omega_{ij})_{i,j=1,\ldots,p} \); furthermore, we assume that \( \mu = 0 \). Abbreviated, the model is

\[
X \sim N(0, \Sigma). \tag{1.1}
\]

Given \( V = \{1, \ldots, p\} \), let \( V^2 = V \times V \) and \( V^2_{-d} = \{(i,l) \in V^2 : i \neq l\} \). For a given pair \( (i,l) \in V^2_{-d} \), let \( V \setminus \{i,l\} = \{j \in V : i \neq j \neq l\} \) and \( X_{V\setminus\{i,l\}} = \{X_j : j \in V \setminus \{i,l\}\} \).

For a random vector \( X \) satisfying (1.1), a Gaussian graphical model (GGM) is the undirected graph \((V,E)\) where \( V \) is the set of nodes and \( E \) is the set of edges, which is defined by

\[
(i,l) \notin E \text{ if and only if } \text{corr} \left( X_i, X_l | X_{V\setminus\{i,l\}} \right) = 0, \tag{1.2}
\]

where \( \text{corr} \left( X_i, X_l | X_{V\setminus\{i,l\}} \right) \) is the conditional correlation coefficient of \( X_i \) and \( X_l \) given \( X_{V\setminus\{i,l\}} \).

For brevity, (1.2) can be rewritten as

\[
(i,l) \notin E \text{ if and only if } X_i \independent X_l | X_{V\setminus\{i,l\}}. \tag{1.3}
\]

So, the set of edges \( E \) can be expressed as

\[
E = \{ (i,l) \in V^2_{-d} : \text{corr} \left( X_i, X_l | X_{V\setminus\{i,l\}} \right) \neq 0 \}. \tag{1.4}
\]

It is well known that there exists a characterization of the conditional correlation in terms of the elements of the precision matrix. More specifically

\[
\forall (i,l) \in V^2_{-d} : \text{corr} \left( X_i, X_l | X_{V\setminus\{i,l\}} \right) = -\frac{\omega_{il}}{\sqrt{\omega_{ii}\omega_{ll}}}. \tag{1.5}
\]

Hence, we have the following parametrization for \( E \):

\[
E = \{ (i,l) \in V^2_{-d} : \omega_{i,l} \neq 0 \}. \tag{1.6}
\]

Given a sample of \( X \), the goal of covariance selection is to estimate the conditional dependence structure by determining the set of nonzero entries of the precision matrix \( \Omega \) (see Dempster 1972, Lauritzen 1996, Edwards 2000). Generally, in high-dimensional statistics, it is assumed that there are just a few entries of \( \Omega \) which are different from zero, that is, that \( \Omega \) is sparse.

Until a few decades ago, statistical procedures assumed that datasets included many observations of a few and carefully chosen variables. Nowadays, datasets contain a large number of variables relative to the sample size, producing blessings but also curses of dimensionality (Donoho, 2000, 2017). Therefore, in a high-dimensional setting, the estimation of precision matrices faces significant challenges.

Let \( \mathbf{X} = (\mathbf{x}_1', \ldots, \mathbf{x}_n')' \) be a \( n \times p \) data matrix where \( \mathbf{x}_1, \ldots, \mathbf{x}_n \) is a sample. If \( n > p \) then the sample covariance matrix \( S = \frac{1}{n} \sum_{i=1}^{n} (\mathbf{x}_i - \bar{\mathbf{x}})(\mathbf{x}_i - \bar{\mathbf{x}})' \) is well conditioned and a well known optimal
estimate of $\Sigma$. In this case $[(n - p - 1)/(n - 1)]S^{-1}$ is an unbiased estimate of $\Omega$. On the other hand, when $p > n$ the sample covariance matrix is not invertible.

To deal with this problem several covariance selection procedures based on regularization have been developed under the assumption that $\Omega$ is sparse. For instance, if $\hat{\Sigma}$ is an estimator of $\Sigma$, the Graphical lasso (Glasso) proposed by Friedman et al. (2008) is defined by

$$\hat{\Omega} = \arg\min_{\{U: U' = U, U > 0\}} \left\{ \text{tr}(U\hat{\Sigma}) - \log\det(U) + \lambda \|U\|_1 \right\}$$  \hspace{1cm} (1.7)

where the optimization is over the set of symmetric positive definite matrices,

$$\|U\|_1 = \sum_{i,j} |u_{ij}| \text{ for } i, j = 1, \ldots, p,$$  \hspace{1cm} (1.8)

is the $\ell_1$ norm of the matrix $U = (u_{ij})_{i,j=1,p}$ and $\lambda \geq 0$ is a regularization or penalty parameter usually determined by crossvalidation. Note that the larger the value of $\lambda$ is, the more sparse the precision matrix estimate becomes.

For $\lambda = 0$, if $\hat{\Sigma} = S > 0$ then the solution of (1.7) is the classical maximum likelihood estimate of $\Omega$. On the other hand Banerjee et al. (2008) proved that, for any symmetric and positive semidefinite matrix $\hat{\Sigma}$ and $\lambda > 0$, the equation (1.7) has a strictly positive definite solution $\hat{\Omega}$ even if $p > n$.

In contrast to univariate data sets, in multivariate settings, outliers can appear in complex ways. In this regard, two types of contamination mechanisms have been introduced in the robustness literature: the Tukey-Huber contamination model (THCM) and the independent contamination model (ICM). In the THCM it is assumed that a relative large proportion $\epsilon$ ($\epsilon > 0.5$) of the rows in the data table are contaminated. In the ICM, introduced by Alqallaf et al. (2009), each cell of the data matrix has a probability to be independently contaminated. This second mechanism is a better fit for the high-dimensional setting where the variables are likely to be obtained from different sources and measured separately (Agostinelli et al., 2015).

The vast majority of the work in the area of robust statistics has concentrated on the estimation of the covariance matrix under these two types of contamination models. Robust conditional correlation coefficient estimation has been studied when $p$ is small. Rao and Sievers (1995) introduced a measure that uses residuals based on rank estimates of regression parameters when $p = 3$. Only recently a few papers have focused on estimation of the precision matrix in the context of ICM.

Tarr et al. (2016) and Öllerer and Croux (2015) showed that Glasso is not robust in the presence of cellwise outliers. Therefore, in order to obtain a robust estimate of the precision matrix, they proposed a plug-in approach, using a robust covariance matrix estimator $\hat{\Sigma}$ in equation (1.7). There are several robust estimators of $\Sigma$ but, unfortunately, their computation is very time-consuming and may not be possibly well defined when the dimension $p$ is high (Khan et al., 2007).

To overcome this problem, resistant pairwise procedures can be used to avoid sensitivity to two-dimensional outliers, like in Tarr et al. (2016) and Öllerer and Croux (2015) proposals. Tarr et al. (2016) proposed to use pairwise robust covariances estimates, whereas Öllerer and Croux (2015) use pairwise robust correlation estimates.

Huber (2011) proposed a robust estimator of the correlation coefficient by using one-dimensional Winsorization. Alqallaf et al. (2002) proposed the use of Huberized pairwise correlation coefficients.
based on one dimensional Winsorization. A limitation of this approach is that the pairwise Huberized estimates and covariance estimates do not take into account the orientation of the (pairwise) bivariate data. To overcome this limitation, Khan et al. (2007) developed an adjusted bivariate Winsorization estimation, obtaining a robust estimator of the correlation matrix under cellwise contamination. Here, we use this estimator to introduce a new robust Graphical lasso procedure, RGlassoWinsor. We compare the performance of our method with other existing approaches under cellwise and casewise contamination.

Section 2 discusses the main differences between the THCM and ICM. Section 3 introduces our proposal. Section 4 presents the results of an extensive simulation experiment comparing the currently existing estimators of the robust precision matrix with our new robust Graphical lasso procedure. Section 5 contains an application to breast cancer data. Section 6 concludes with some remarks. The Appendix gives some additional simulation results.

2 Outliers in high-dimensional data

In this section, we briefly outline the main differences between THCM and ICM.

Consider a set of $n$ independent observations of the multivariate Gaussian vector $X = (X_1, \ldots, X_p)'$ satisfying (1.1), let $\epsilon \in (0, 1)$ be the fraction of contamination and define the random vector $B = (B_1, \ldots, B_p)'$ with $B_j \sim \text{Bernoulli}(\epsilon), j = 1, \ldots, p$. (2.1)

Suppose that instead of $X$ we observe $Y = (I - D)X + DZ$ (2.2)

where $I$ is the $p \times p$ identity matrix, $Z$ is a $p$-variate random vector with an arbitrary and unspecified outlier generating distribution and $D$ is a diagonal matrix with diagonal elements $B_1, \ldots, B_p$. Moreover, we assume that $X, B$ and $Z$ are independent.

The classical THCM assumes that the random vector $B = (B_1, \ldots, B_p)'$ satisfy $P(B_1 = B_2 = \ldots = B_p) = 1$. So, we either see a perfect realization of the random vector $X$, with probability $1 - \epsilon$, or a realization of the random vector $Z$, with probability $\epsilon$.

Motivated by the THCM, robust procedures identify and downweight possibly contaminated cases. However, in a high-dimensional setting, this strategy is inconvenient for two reasons. The most obvious is that in high-dimension, when $n$ is relatively small compared with $p$, discarding a single observation may result in a substantial loss of information. A perhaps less obvious reason was highlighted by Alqallaf et al. (2009), where they argued that there are situations where the contaminating mechanism may be independent for different variables. Consequently, they proposed the ICM that assumes that $B_1, \ldots, B_p$ are independent random variables and satisfy

$$P(B_1 = 1) = \ldots = P(B_p = 1) = \epsilon.$$ (2.3)

Hence, a case is uncontaminated, $Y = X$, with probability $P(B = 0) = (1 - \epsilon)^p$, which quickly decreases below 1/2 as $p$ increases. Equivalently, the probability that at least one component of $Y$ is contaminated is $1 - (1 - \epsilon)^p$. For example if $p = 60$ and $\epsilon = 0.05$ this probability equals to 0.95. If $p \geq 200$ (not an uncommon case these days) this probability becomes nearly 1.
The indicator matrix $D$, whose diagonal is a sequence of Bernoulli random variables, determines the structure of the contamination model. Figure 1 shows a representation of a sample of size $n = 100$ of $B$ with dimension $p = 60$, contamination fraction $\epsilon = 0.10$, under both contamination models: THCM in panel (a) and ICM in panel (b). On each panel, uncontaminated cells are in color white and contaminated cells are in color black. For THCM the actual proportion of contaminated cells is 0.08, coinciding with the percentage of contaminated observations (rows). But, for ICM the proportion of contaminated cells is 0.10 but all the observations have at least one contaminated cell ($\approx 1 - (0.9)^{60}$), hence the totality of the cases or rows are contaminated. This phenomenon is called “propagation of outliers” in Alqallaf et al. (2009).

THCM is also called casewise contamination model, where a minority of observations or cases (rows) of the data matrix contains outliers and the size of this minority does not depend on the number $p$ of variables. ICM is also denominated cellwise contamination model because the contamination is produced randomly affecting the cells of the data table.

The classical robustness theory based on the affine equivariant Tukey-Huber contamination model relays and enforces the concept of equivariance. Alqallaf et al. (2009) showed that under the cellwise contamination model, a standard high-breakdown affine equivariant estimators propagate outliers, and this causes their very poor performance when $p$ is large. The reason is that affine equivariant robust estimators depend on linear combinations of the observations which have a very high probability of being contaminated under ICM for moderate and large $p$. Notice that under ICM, the majority of cases will have at least some contaminated component.
et al. (2015) addressed the problem of robust estimation of location and scatter under the two contamination models.

3 Robust lasso for covariance selection

3.1 Plug-in strategy

Hereafter $y_i = (y_{i1}, \ldots, y_{ip})'$, $i = 1, \ldots, n$ denotes a sample of observations of a $p$-multivariate random vector $Y = (Y_1, \ldots, Y_p)'$ satisfying (2.2) and let $Y = (y_1, \ldots, y_n) \in \mathbb{R}^{n \times p}$ be the corresponding data table. Let $R$ denote the correlation matrix; i.e. if $\Sigma = (\Sigma_{ij})$ then $R = (R_{ij})$ with $R_{ij} = \Sigma_{ij} / \sqrt{\Sigma_{ii} \Sigma_{jj}}$.

Following Tarr et al. (2016) and using (1.7) we will construct a robust estimation procedure of the precision matrix as follows:

$$\hat{\Omega} = \arg\min_{\{U: U' = U, U \succ 0\}} \text{tr}(U\hat{\Sigma}) - \log\det U + \lambda \| U \|_1$$

(3.1)

where $\hat{\Sigma}$ is a robust estimator of the covariance matrix.

3.2 Adjusted multivariate Winsorization

To control the effect of bivariate outliers on the pairwise estimation of $\Sigma$, we apply the procedure proposed by Khan (2006). In this procedure, the robust estimator $\hat{R}^W$ of the correlation matrix $R$ is defined in two steps by first computing the pairwise correlation matrix, $\hat{R}^0$, using an adjusted Winsorization scheme, which takes into consideration the orientation of bidimensional data. Later, based on $\hat{R}^0$, a robust estimator of the covariance matrix $\Sigma$ is defined.

The two steps to compute $\hat{R}^W$ are given below:

1) Initial estimate $\hat{R}^0$.

Given $j, k \in \{1, \ldots, p\}$, with $j \neq k$, let consider the bivariate sample $\{(y_{ij}, y_{ik})', i = 1, \ldots, n\}$ and compute for every $l = j, k$

$$m_l = \text{median}(y_{1l}, \ldots, y_{nl}), s_l = \text{mad}(y_{1l}, \ldots, y_{nl}),$$

(3.2)

where “mad” denotes the median absolute deviation. Define now the the standardized samples

$$\tilde{y}_{il} = \frac{y_{il} - m_l}{s_l}, i = 1, \ldots, n$$

(3.3)

for every $l = j, k$.

As Khan et al. (2007) noted, one dimensional Winsorization does not account for the orientation of the bidimensional data and does not address the effect of bivariate outliers. Therefore, they propose a bivariate adjusted Winsorization that uses two tuning constants denoted $c_1$ and $c_2$. The constant $c_1$ is used on the two quadrants that contain the majority
of the standardized data and the constant $c_2$, smaller than $c_1$, is used on the other two
quadrants. Typically $c_1 = 2$ or $2.5$ and $c_2 = \sqrt{h c_1}$ with $h = n_2/n_1$, where $n_1$ is the number
of observations in the two major quadrants and $n_2 = n - n_1$.

The bivariate Winsorized data $(v_{ij}, v_{ik})'$, $i = 1, \ldots, n$ are computed as follows. If $(\tilde{y}_{ij}, \tilde{y}_{ik})$
lies in one of the major (more populated) quadrants, let
\begin{equation}
    v_{il} = \psi_{c_1}(\tilde{y}_{il}), \quad i = 1, \ldots, n; \ l = j, k,
\end{equation}
where $\psi_{c_1}$ is the Huber function $\psi_c(x) = \min\{\max\{-c, x\}, c\}$ with tuning constant $c = c_1$. On the other hand, if $(\tilde{y}_{ij}, \tilde{y}_{ik})$
lies in one of the minor (less populated) quadrants then
\begin{equation}
    v_{il} = \psi_{c_2}(\tilde{y}_{il}), \quad i = 1, \ldots, n; \ l = j, k.
\end{equation}

The elements $\tilde{R}_{jk}^0$ of the matrix $\tilde{R}^0$ are now defined as follows. For $j = k$ we set $\tilde{R}_{jj}^0 = 1,$
and for $j \neq k$ we set
\begin{equation}
    \tilde{R}_{jk}^0 = \text{corr}(v_j, v_k)
\end{equation}
where $v_j = (v_{1j}, \ldots, v_{nj})'$ and $v_k = (v_{1k}, \ldots, v_{nj})'$.

2) Final estimate $\tilde{R}^W$.

As before, consider $\{(y_{ij}, y_{ik})', \ i = 1, \ldots, n\}$ a bivariate sample of the two variables $Y_j$ and
$Y_k$, with $j \neq k$ (columns $j$ and $k$ of the data table). Let
\begin{equation}
    A_{jk} = \begin{pmatrix} 1 & \tilde{R}_{jk}^0 \\ \tilde{R}_{kj}^0 & 1 \end{pmatrix}
\end{equation}
be the $2 \times 2$ submatrix of $\tilde{R}^0$. Perform now, for every $l = j, k$, the following bivariate transformation
\begin{equation}
    u_{il} = y_{il} \min\left(\sqrt{c/D_{jk}(y_{ij}, y_{ik})}, 1\right), \ i = 1, \ldots, n; \ l = j, k,
\end{equation}
where $D_{jk}$ is the Mahalanobis distance based on the correlation matrix $A_{jk}$ and evaluated
in $(y_{ij}, y_{ik})$. The tuning constant $c = 5.99$ corresponds to the 95% quantile of a $\chi^2_2$
distribution. By this transformation the outliers are shrunken to the border of an ellipse,
including the majority of the data.

We now define the Winsorized correlation estimate $\tilde{R}^W = (\tilde{R}_{jk}^W)$ as follows. For $j \neq k$, we set
\begin{equation}
    \tilde{R}_{jk}^W = \text{corr}(u_j, u_k),
\end{equation}
where $u_j = (u_{1j}, \ldots, u_{nj})'$ and $u_k = (u_{1k}, \ldots, u_{nj})'$ and, for $j = k$, we set $\tilde{R}_{jj}^W = 1.$
Finally, based on $\hat{R}^W$, a robust estimator of $\Sigma$ is defined as

$$\hat{\Sigma}^W = \text{diag}(s_1, \ldots, s_p) \hat{R}^W \text{diag}(s_1, \ldots, s_p)$$

(3.7)

where $s_j$ is the robust estimator of the dispersion introduced in (3.2). In order to guarantee positive definiteness of $\hat{\Sigma}^W$ we compute the nearest positive definite matrix [Higham, 2002]. Finally, the robust Glasso estimator of the precision matrix based on bivariate adjusted Winsorization, called RGlassoWinsor and denoted by $\hat{\Omega}^W$, is defined by (3.1) with $\hat{\Sigma} = \hat{\Sigma}^W$.

**Remark 3.1** By Theorem 19.1 and Proposition 19.1 in Öllerer and Croux (2015) the finite sample breakdown point under ICM of $\hat{\Omega}^W$ satisfies

$$\epsilon_n(\hat{\Omega}^W) \geq \epsilon_n^+(\hat{\Sigma}^W) \geq \max_{j=1,\ldots,p} \epsilon_n^+(s_j) = 1/2$$

where $\epsilon_n^+(\hat{\Sigma}^W)$ is the explosion finite-sample breakdown point (EBP) under ICM contamination of $\hat{\Sigma}^W$ and $\epsilon_n^+(s_j)$ is the EBP of the univariate scale estimator scale $s_j$, $j = 1, \ldots, p$.

### 4 Simulation experiment and numerical results

We conducted a Monte Carlo simulation experiment to investigate the performance of RGlassoWinsor compared with other procedures.

#### 4.1 Simulation settings

In the following, we describe the precision matrix models, the contamination scenarios and the precision matrix estimation procedures considered in our simulation study.

**Precision matrix models**

We consider two dimension values ($p = 60, 200$) and five $\Omega$ models.

**Model 1.** Autoregressive model of order 1, denoted AR(1). In this case we set $\Sigma_{ij} = 0.4^{|i-j|}$ for $i, j = 1, \ldots, p$ and $\Omega = \Sigma^{-1}$.

**Model 2.** Block diagonal matrix model, denoted BG. In this case the precision matrix $\Omega$ has $q$ blocks of size $p/q$. Each block has diagonal elements equal to 1 and off-diagonal elements equal to 0.5. For $p = 60, 200$ we use $q = 10$ and 40 blocks, respectively.

**Model 3.** Random model, denoted Rand. [Jiang et al., 2020], in the R package huge, compute the $\Omega$ matrix of this model as follows. First they consider $\Theta = (\theta_{ij})$ an adjacency matrix of dimension $p$ such that every diagonal entry $\theta_{ii} = 0$, each pair of off-diagonal elements is randomly set $\theta_{ij} = \theta_{ji} = 1$ with probability $\text{prob} = 3/p$ (the default value) and defined as 0 otherwise. Then they define the set of edges of the graph, establishing that two different nodes, $i$ and $j$, are connected if and only if $\theta_{ij} = 1$. Finally, given $\Theta$, is possibly to choose real constants $v$ and $s$ such that $\Omega = v\Theta + sI_p$ is positive definite, with $I_p$ the identity matrix.
**Model 4.** Nearest neighbors model of order 2, denoted NN(2). For each node we randomly select two neighbors and choose a pair of symmetric entries of $\Omega$ using the “NeighborOmega” function of the R package *Tlasso* ([Sun et al., 2016]).

**Model 5.** Hub model, denoted Hub. As in Model 3, consider $\Theta = (\theta_{ij})$ an adjacency matrix defined as follows. The row/columns are evenly partitioned into 3 (10) disjoint groups if $p = 60$ (if $p = 200$). Each group is associated with a “center” row $i$ in that group. Each pair of off-diagonal elements, $i \neq j$, are set $\theta_{ij} = \theta_{ij} = 1$ if $j$ also belongs to the same group as $i$ and 0 otherwise. It results in 57 (190) edges in $E$ if $p = 60$ (if $p = 200$). The precision matrix $\Omega$ is defined as in Rand Model and computed using the same R package *huge* ([Jiang et al., 2020]).

Figure 2 displays graphs from Models 1-5 with $p = 60$.

![Graphs of Models 1-5](image)

**Contamination scenarios**

As in (2.2), let $Y = (I - B)X + BZ$ and consider the following scenarios.
i) **Clean data.** $Y = X \sim N(0, \Sigma)$ correspondind to $\epsilon = 0$.

ii) **Cellwise or ICM.** Here $Z \sim N(\mu_1, \sigma^2 \Sigma)$ where $\mu_1 = (10, \ldots, 10)'$, $\sigma = 0.2$ and contamination fractions $\epsilon : 0.01, 0.05, 0.10$.

iii) **Casewise or THCM.** Let $Z = z$ with $z = k\upsilon$, $\upsilon$ is the eigenvector corresponding to the smallest eigenvalue of $\Sigma$ satisfying $\upsilon'\Sigma^{-1}\upsilon = 1$ and $k = 100$. We consider $\epsilon : 0.05$ and 0.10.

For every $p$, $\epsilon$ and $\Omega$ model we generate $N = 100$ random samples $Y_1, \ldots, Y_N$, of size $n = 100$ of $Y$.

**Precision matrix estimators**

We will compare the performance of the following estimators of $\Omega$.

1. The classical Glasso estimator defined by (1.7).

2. RGlassoQn and RGlassoTau. [Tarr et al. (2016)] estimates a robust initial covariance matrix based on the approach proposed by [Gnanadesikan and Kettenring (1972)]. Noting that the covariance of two random variables $X$ and $Y$ can be written as

$$\text{Cov}(X, Y) = \frac{1}{4\alpha\beta} \left[ \text{Var}(\alpha X + \beta Y) - \text{Var}(\alpha X - Y) \right],$$

(4.1)

where $\alpha = 1/\sqrt{\text{Var}(X)}$ and $\beta = 1/\sqrt{\text{Var}(Y)}$, a robust estimate of the bivariate covariance $\hat{\Sigma}_{ij}$ can be obtained by replacing $\text{Var}$ in (4.1) with a robust variance estimator like $Q_n$ or $\tau$-scale estimators defined by [Maronna and Zamar (2002) and Rousseeuw and Croux (1993)]. Based on these robust estimators of the covariance matrix, using (3.1), [Tarr et al. (2016)] derived a robust estimator of $\Omega$, denoted by RGlassoQn and RGlassoTau. We use the R package robustbase to compute the robust variance estimators $Q_n$ and $\tau$-scale ([Todorov and Filzmoser, 2009]).

3. RGlassoGauss, RGlassoSpearman and RGlassoQuadrant. [Öllerer and Croux (2015)] propose a robust estimator $\hat{\Sigma}_{ij}^R$ of the bivariate correlations

$$\hat{\Sigma}_{ij}^R = \text{scale}(y_l)\text{scale}(y_j)r(y_l, y_k)$$

(4.2)

where $r(\cdot)$ and scale(\cdot) are robust correlation and scale estimators, respectively. For instance scale(\cdot) is $Q_n$ (or the mad) and for $r(\cdot)$ there are different possibilities, like Gaussian rank correlation, Spearman correlation and Quadrant correlation. This proposal leads, using (3.1), to three robust estimators called RGlassoGauss, RGlassoSpearman and RGlassoQuadrant.

4. Our proposal, RGlassoWinsor estimator. To compute the robust bivariate adjusted correlation estimator defined in steps 1 and 2 of Section 3, we use the function “corhuber” of the R package robustHD.
In proposals 1) and 3), to make the pairwise correlation matrices positive-definite, we compute the nearest positive definite matrix using the function “nearPD” of the R package Matrix (Bates and Maechler, 2019). To solve the regularized equation (3.1) we use the R-package huge. There are different alternatives to select the optimal regularization parameter and we use 5-fold cross-validation as it is indicated by Zhao et al. (2014) and Öllerer and Croux (2015).

**Estimation performance evaluation**

We wish to evaluate two different features of the procedures: (i) their performance as estimates of Ω; and (ii) how well they recover the true graphical model graph.

The numerical performance of $\hat{\Omega}$ is measured by the mean squared error (MSE) defined by the Frobenius norm of the difference between Ω and the predicted precision matrix $\hat{\Omega}$

$$m_F = \| \hat{\Omega} - \Omega \|_F = \sqrt{\sum_{ij} |\omega_{ij} - \hat{\omega}_{ij}|^2}$$

and also quantified by the Kullback-Leibler divergence

$$D_{KL} = \frac{1}{2} \left( \text{tr} \{ \hat{\Omega} \Omega^{-1} \} - \log \{ \text{det} \left[ \hat{\Omega} \Omega^{-1} \right] \} - p \right).$$

To evaluate the graph recovery or classification performance we compute the true positive and true negative rates- also called sensitivity and specificity, respectively- defined by

$$\text{TPR} = \frac{\# \{ (i,j) \in V^2_d : \hat{\omega}_{ij} \neq 0 \land \omega_{ij} \neq 0 \} }{\# E} \quad \text{and} \quad \text{TNR} = \frac{\# \{ (i,j) \in V^2_d : \hat{\omega}_{ij} = 0 \land \omega_{ij} = 0 \} }{\# NE}$$

where $E = \{ (i,j) \in V^2_d : \omega_{ij} \neq 0 \}$ is the set of edges, $NE = \{ (i,j) \in V^2_d : \omega_{ij} = 0 \}$ is the set of non connected nodes and

$$\text{TP} = \# \{ (i,j) \in V^2_d : \hat{\omega}_{ij} \neq 0 \land \omega_{ij} \neq 0 \} \quad \text{and} \quad \text{TN} = \# \{ (i,j) \in V^2_d : \hat{\omega}_{ij} = 0 \land \omega_{ij} = 0 \}$$

denotes the size of the sets of true positives and true negatives, respectively.

A related measure is the Matthews correlation coefficient (MCC) given by

$$\text{MCC} = \frac{\text{TP} \times \text{TN} - \text{FP} \times \text{FN}}{\sqrt{(\text{TP} + \text{FP})(\text{TP} + \text{FN})(\text{TN} + \text{FP})(\text{TN} + \text{FN})}},$$

where

$$\text{FP} = \# \{ (i,j) \in V^2_d : \hat{\omega}_{ij} \neq 0 \land \omega_{ij} = 0 \} \quad \text{and} \quad \text{FN} = \# \{ (i,j) \in V^2_d : \hat{\omega}_{ij} = 0 \land \omega_{ij} \neq 0 \}$$

denotes the number of false positives and false negatives sets, respectively.

Note that larger values of TPR, TNR and MCC indicate better performances (Fan et al., 2009; Baldi et al., 2000).

Heatmaps are useful to visualize the graph recovery performance of a given procedure. As an example, for $p = 60$, the axes in the panels of Figure display the graph nodes in a given order.
Panel (a) shows \( N = 100 \) estimated Rand models by Glasso from simulation replicates where each cell displays a gray level proportional to how frequently the corresponding pair of nodes appear in the estimated graph in the \( N \) simulation replicates. So, a white color in a given cell \((i, j)\) means that nodes \( i \) and \( j \) are never adjacent in the simulated graphs and, a pair of nodes that are always adjacent in the simulated graphs are represented by a black coloured cell. The heatmap of Panel (a) is compared with the figure of Panel (b) that represents the graph of true model Rand where a black or white cell correspond to a pair of connected or non connected nodes, respectively.

Figure 3: Heatmap for the frequency of adjacency for each pair of nodes with \( p = 60 \) and true graph of model Rand. The axes display the graph \( p \)-nodes in a given order.

Finally, Figure 4 represent the five true models graphs.

Figure 4: True models graphs with \( p = 60 \). The axes display the graph \( p \)-nodes in a given order.

4.2 Estimation and graph recovery performances

In this section we analyze the numerical and graph recovery performances of the estimation of the different GGM, represented by its precision matrix \( \Omega \), for clean data and under both contamination scenarios. To abbreviate we will group RGlassoQn and RGlassotau under the name of Group 2, and RGlassoGauss, RGlassoSpearman and RGlassoQuadrant will be named Group 3.

To fix some ideas we first focus on the estimation results for the AR(1) model. Tables 1 to 6 show the estimation performance under ICM and Tables 40 to 45 in the Appendix B, under THCM.
In terms of numerical performance, Glasso is slightly better than other methods for clean data, but it is clearly non robust under both contamination models for all positive contamination fractions. In both contamination models, our proposal, RGlassoWinsor, has the best numerical performance. Note that the mean squared error, $m_F$, and the Kullback-Leibler divergence, $D_{KL}$, grow when the dimension $p$ increases, for both, clean and contaminated data. $D_{KL}$ and $m_F$ are higher for cellwise contamination model than the caseewise contamination model.

| $r = 0$ | $r = 0.01$ | $r = 0.05$ | $r = 0.10$ |
|--------|--------|--------|--------|
| TPR    | TNR    | TPR    | TNR    | TPR    | TNR    | TPR    | TNR    |
| RGlassoWinsor | 0.591 (0.001) | 0.885 (0.003) | 0.962 (0.001) | 0.887 (0.003) | 0.799 (0.002) | 0.934 (0.002) |
| Glasso | 0.997 (0.001) | 0.998 (0.002) | 0.998 (0.002) | 0.998 (0.002) | 0.999 (0.002) | 0.999 (0.002) |
| RGlassoQc | 0.605 (0.001) | 0.788 (0.004) | 0.903 (0.001) | 0.903 (0.001) | 0.903 (0.001) | 0.903 (0.001) |
| RGlassoTau | 0.786 (0.002) | 0.914 (0.003) | 0.914 (0.003) | 0.914 (0.003) | 0.914 (0.003) | 0.914 (0.003) |
| RGlassoSpearman | 0.996 (0.001) | 0.997 (0.002) | 0.997 (0.002) | 0.997 (0.002) | 0.998 (0.002) | 0.998 (0.002) |
| RGlassoQuad | 0.729 (0.002) | 0.868 (0.003) | 0.853 (0.002) | 0.853 (0.002) | 0.853 (0.002) | 0.853 (0.002) |

| $r = 0$ | $r = 0.01$ | $r = 0.05$ | $r = 0.10$ |
|--------|--------|--------|--------|
| TPR    | TNR    | TPR    | TNR    |
| RGlassoWinsor | 0.594 (0.002) | 0.963 (0.002) | 0.994 (0.001) | 0.988 (0.001) |
| Glasso | 0.997 (0.001) | 0.997 (0.002) | 0.998 (0.002) | 0.998 (0.002) |
| RGlassoQc | 0.607 (0.001) | 0.806 (0.003) | 0.901 (0.001) | 0.901 (0.001) |
| RGlassoTau | 0.787 (0.002) | 0.915 (0.003) | 0.915 (0.003) | 0.915 (0.003) |
| RGlassoSpearman | 0.997 (0.001) | 0.997 (0.002) | 0.997 (0.002) | 0.997 (0.002) |
| RGlassoQuad | 0.729 (0.002) | 0.868 (0.003) | 0.853 (0.002) | 0.853 (0.002) |

Table 3: Model AR(1) under ICM. Comparison of means and standard deviations (in brackets) of TPR and TNR over $N = 100$ replicates. $p = 60$, $n = 100$.
by RGlassoWinsor and those produced by Group 3 estimators remain almost constant and even slightly increase when the contamination fractions increase as it shown in Tables 4 and 6. Conversely, MCC means of the Group 2 estimators dramatically decrease when the contamination fraction $\epsilon$ increases. A better explanation can be found by looking at the Tables 3 and 5, while the mean of TPR remains relatively high for RGlassoWinsor and for the estimators of Group 3, the mean of TPR goes to zero for the estimators of Group 2. Note that, although not so extreme, a similar phenomena occurs under THCM, as shown in Tables 14 and 15.

![Figure 5](image5.png)

**Figure 5:** Graph of true model AR(1) and heatmaps for the frequency of adjacency for each pair of nodes over $N = 100$ replicates. $p = 60$ and $n = 100$. ICM with $\epsilon = 0.01$. The axes display the graph $p$-nodes in a given order.

![Figure 6](image6.png)

**Figure 6:** Graph of true model AR(1) and heatmaps for the frequency of adjacency for each pair of nodes over $N = 100$ replicates. $p = 60$ and $n = 100$. ICM with $\epsilon = 0.05$. The axes display the graph $p$-nodes in a given order.

![Figure 7](image7.png)

**Figure 7:** Graph of true model AR(1) and heatmaps for the frequency of adjacency for each pair of nodes over $N = 100$ replicates. $p = 60$ and $n = 100$. ICM with $\epsilon = 0.10$. The axes display the graph $p$-nodes in a given order.
Figures 5, 6 and 7 show the performance of Glasso and RGlassoWinsor for contaminated data under ICM. Notice that for contaminated data, Glasso cannot recover the true set of edges, introducing a large number of false negatives. Although RGlassoWinsor introduces false positives, it better recovers the true set of edges.

In the following paragraphs we set general conclusions about the behavior of the estimators for all analyzed Ω models, based on Tables 1 to 6 of this section, Tables 16 to 39 in Appendix A (ICM), Tables 40 to 69 in Appendix B (THCM).

Tables 7 to 14 below report the average ranks for all the compared estimation methods, evaluated across all the considered precision matrix models. Rank 1 and rank 7 corresponds to the best and worst performing method, respectively. The average ranks of the best two performing methods are shown in bold face.

For $\epsilon = 0$, Glasso performs slightly better than the other estimators and shows a non-robust performance, being the worst ranked for contaminated data. For almost all contamination scenarios RGlassoWinsor is the best ranked and the estimators of Group 2, specially RGlassoSpearman and RGlassoGauss, have the closest rankings. Note that for dimension $p = 200$ RGlassoSpearman has a slightly better average rank than RGlassoWinsor, under THCM.

### Table 7: Average rank of the estimation methods based on $m_F$ and $D_{KL}$ under ICM. $p = 60, n = 100$.

| $\epsilon$ | $m_F$ | $D_{KL}$ | $m_F$ | $D_{KL}$ | $m_F$ | $D_{KL}$ | $m_F$ | $D_{KL}$ |
|------------|-------|----------|-------|----------|-------|----------|-------|----------|
| RGlassoWinsor | 3.4   | 1.6      | 2     | 1        | 1     | 1        | 1     | 1        |
| Glasso     | 1     | 6.4      | 6.4   | 7        | 7     | 6.6      | 6.6   | 6.6      |
| RGlassoQuad | 6.4   | 5.8      | 6     | 6        | 6     | 5.2      | 5.2   | 5.2      |
| RGlassoGauss | 5.4   | 4.2      | 4.8   | 5        | 5     | 6.2      | 6.2   | 6.2      |
| RGlassoSpearman | 2.2   | 2.2      | 1.2   | 2.4      | 2.6   | 2.6      | 2.4   | 2.6      |
| RGlassoQuad | 3.4   | 3       | 2.6   | 2.8      | 2.6   | 2.4      | 2.4   | 2.6      |
| RGlassoQuad | 6.2   | 6.2      | 5.2   | 5        | 4     | 4        | 4     | 4        |

### Table 8: Average rank of the estimation methods based on MCC under ICM. $p = 60, n = 100$.

| $\epsilon$ | $m_F$ | $D_{KL}$ | $m_F$ | $D_{KL}$ | $m_F$ | $D_{KL}$ | $m_F$ | $D_{KL}$ |
|------------|-------|----------|-------|----------|-------|----------|-------|----------|
| RGlassoWinsor | 3.2   | 1.4      | 1     | 1        | 1     | 1        | 1     | 1        |
| Glasso     | 3.6   | 6.2      | 6.2   | 6.6      | 6.6   | 6.6      | 6.6   | 6.6      |
| RGlassoQuad | 6.8   | 6.8      | 6     | 6.2      | 5.6   | 5.6      | 5.4   | 5        |
| RGlassoTau | 5.6   | 5.2      | 5     | 5.4      | 5.2   | 6.2      | 6.2   | 6.2      |
| RGlassoGauss | 2     | 2        | 1.6   | 1.2      | 2     | 2        | 2     | 2        |
| RGlassoSpearman | 3.4   | 3.4      | 2.8   | 3        | 3     | 3        | 2.8   | 2.8      |
| RGlassoQuad | 5.6   | 5.8      | 4.8   | 4.6      | 4.4   | 4.4      | 4     | 4        |

### Table 9: Average rank of the estimation methods based on $m_F$ and $D_{KL}$ under ICM. $p = 200, n = 100$.

| $\epsilon = 0.05$ | $m_F$ | $D_{KL}$ | $\epsilon = 0.1$ | $m_F$ | $D_{KL}$ |
|-------------------|-------|----------|-------------------|-------|----------|
| RGlassoWinsor     | 1.2   | 1.2      | Glasso            | 7     | 7        |
| RGlassoSpearman   | 5.2   | 5.8      | RGlassoGauss      | 6.8   | 6.8      |
| RGlassoQuad       | 4.6   | 4.4      | RGlassoQuad       | 4.8   | 4.8      |

### Table 10: Average rank of the estimation methods based on MCC under ICM. $p = 200, n = 100$.

| $\epsilon = 0.05$ | $m_F$ | $D_{KL}$ | $\epsilon = 0.1$ | $m_F$ | $D_{KL}$ |
|-------------------|-------|----------|-------------------|-------|----------|
| RGlassoWinsor     | 2.6   | 2        | Glasso            | 6.8   | 7        |
| RGlassoSpearman   | 3.6   | 6.8      | RGlassoGauss      | 4.6   | 4.8      |
| RGlassoQuad       | 3.8   | 3.8      | RGlassoQuad       | 2.6   | 2.6      |

### Table 11: Average rank of the estimation methods based on $m_F$ and $D_{KL}$ under THCM. $p = 60, n = 100$.

| $\epsilon = 0.05$ | $m_F$ | $D_{KL}$ | $\epsilon = 0.1$ | $m_F$ | $D_{KL}$ |
|-------------------|-------|----------|-------------------|-------|----------|
| RGlassoWinsor     | 2.6   | 2        | Glasso            | 6.8   | 7        |
| RGlassoSpearman   | 3.6   | 6.8      | RGlassoGauss      | 4.6   | 4.8      |
| RGlassoQuad       | 3.8   | 3.8      | RGlassoQuad       | 2.6   | 2.6      |

### Table 12: Average rank of the estimation methods based on MCC under THCM. $p = 60, n = 100$.

| $\epsilon = 0.05$ | $m_F$ | $D_{KL}$ | $\epsilon = 0.1$ | $m_F$ | $D_{KL}$ |
|-------------------|-------|----------|-------------------|-------|----------|
| RGlassoWinsor     | 2.6   | 2        | Glasso            | 6.8   | 7        |
| RGlassoSpearman   | 3.6   | 6.8      | RGlassoGauss      | 4.6   | 4.8      |
| RGlassoQuad       | 3.8   | 3.8      | RGlassoQuad       | 2.6   | 2.6      |
5 Real data example

In preoperative chemotherapy when all invasive cancer cells are eradicated, the patient is said to have reached the state of *pathological complete response*, abbreviated as pCR. This pCR is associated with the long-term cancer-free survival of a person. On the contrary, residual disease (RD) indicates that the disease has not been eradicated. Measurements of the expression level (activity) of genes may be able to predict if a patient can reach a pCR.

Hess et al. (2006) use normalized gene expression data of patients in stages I-III of breast cancer, to identify patients that may achieve pCR under preoperative chemotherapy. Their data base has 22283 gene expression levels for 133 patients, with 34 pCR and 99 RD. Hess et al. (2006) and Natowicz et al. (2008) identify 26 important genes for predicting survival and response to adjuvant chemotherapy. Following Ambroise et al. (2009) and Tang et al. (2021), we estimate the precision matrix for the 26 key genes on the two classes pCR and RD.

Raymaekers and Rousseeuw (2020) proposed a method that detect cellwise outliers, implemented in the R package *cellWise* (Raymaekers and Rousseeuw, 2021). The function “cellHandler” of *cellWise* flags cellwise outliers in the data matrix, based on robust estimates of the mean $\mu$ and covariance matrix $\Sigma$ with 0.95% as cutoff used in the detection of cellwise outliers. We compare the performance of RGlassoWinsor and RGlassoGauss because both have shown similar rankings. Using the sample median and a robust estimates of $\Sigma$ provided by Winsorization, $\hat{\Sigma}^W$, and Gaussian rank correlations, $\hat{\Sigma}^G$, (see (3.7) and (4.2)) we first detect outliers in the data set.

Figure 8 illustrates cellwise outliers flagged by “cellHandler” based on both robust covariance estimates for the RD class. The rows represent the patients or cases and the columns represent the variables or genes expressions. A black colored cell indicates that its value is an outlier.
Figure 8: Cellwise outliers detected by “cellHandler” for the RD class based on a) $\hat{\Sigma}^W$ and b) $\hat{\Sigma}^G$. A black colored cell indicates an outlier.

Of the total of 2574 $(99 \times 26)$ cells of the data matrix of the RD class, 384 (15%) are contaminated according to “cellHandler” based on $\hat{\Sigma}^W$. The first five most contaminated variables correspond to genes RRM2, SCUBE2, MELK, CA12 and BTG3. Using $\hat{\Sigma}^G$, 409 (almost 16 %) are flagged as contaminated and the first five most contaminated variables corresponds to genes BTG3, RRM2, MELK, SCUBE2, MAPT.

A similar procedure shows that for the pCR group, using $\hat{\Sigma}^W$, of the total of 884 $(34x26)$ cells of the data matrix, “cellHandler” flags 166 (19%) cells as contaminated and the first five most contaminated variables correspond to genes PDGFRA, CA12, SCUBE2, BBS4 and IGFBP4. Using $\hat{\Sigma}^G$, 108 (almost 12 %) cells are flagged as contaminated and the first five most contaminated variables corresponds to genes CA12, SCUBE2, IGFBP4, KIAA1467 and MTRN.

Figures 9 and 10 display the resulting network obtained using Glasso, RGlassoWinsor, RGlassoTau and RGlassoGauss, the latter two representing Groups 2 and 3 of procedures. Table 15 exhibits the estimated network density for the 26 genes for each class, for all procedures, using a regularization parameter chosen by 5-fold cross-validation.

Excluding the estimated networks by RGlasso$Q_n$ and RGlassoTau, the undirected graphs differ according to the class membership which may suggest that genes regulation differs according the participants response to the treatment (Ambroise et al., 2009).

In the pCR class, RGlassoWinsor produce a less sparse network than Glasso and RGlassoSpearman, but a similar structure. But, in the RD class while Glasso and RGlassoTau does not detect any conditional relationship between nodes (genes), RGlassoWinsor and the procedures of Group 3 detect several edges between genes.
Figure 9: Estimated graph of the GGM for the 26 genes corresponding to RD class.

|                  | pCR class | RD class |
|------------------|-----------|----------|
| RGlassoWinsor    | 0.280     | 0.169    |
| Glasso           | 0.243     | 0.003    |
| RGlasso$_Q_n$    | 0.000     | 0.000    |
| RGlassoTau       | 0.000     | 0.000    |
| RGlassoGauss     | 0.203     | 0.249    |
| RGlassoSpearman  | 0.197     | 0.237    |
| RGlassoQuad      | 0.117     | 0.234    |

Table 15: Estimated network density for the 26 genes from breast cancer gene expressions data.
Figure 10: Estimated graph of the GGM for the 26 genes corresponding to PCR class.
6 Concluding remarks

This paper introduces a new robust graphical lasso procedure called RGlassoWinsor based on adjusted bivariate Winsorization estimation of the covariance matrix for high-dimension covariance selection or precision matrix estimation.

RGlassoWinsor is compared with the currently existing robust estimators of the precision matrix, introduced by Tarr et al. (2016) and Öllerer and Croux (2015), by using different performance measures regarding graph recovery and sparse estimation of the precision matrix.

Our proposal shows a good performance for all the precision models, dimensions and contamination scenarios considered in this research. For clean data Glasso is slightly better than other methods, but it is clearly non robust. Under contamination and for almost all performance measures, our proposal, RGlassoWinsor, has the best overall performance.

Moreover our procedure attains the maximum finite sample breakdown point of 0.5 under cellwise contamination.

Finally, we demonstrate the usefulness of RGlassoWinsor in an application to the analysis of breast cancer data.

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Appendix A

This section contains numerical and performance classification performance results for the seven precision matrix estimators applied to BG, NN(2), Rand and Hub models. We compare two scenarios.

Figures 12, 14 compare the heatmaps for the true graph and the estimated graphs using Glasso and RGlassoWinsor for clean and 5% contaminated data, for p = 60.

### Table 16: Model BG under ICM. Comparison of means and standard deviations (in brackets) of mF and DKL over N = 100 replicates. p = 60, n = 100.

| Model         | mF (p) | DKL (p) |
|---------------|--------|---------|
| RGlassoWinsor | (0.008) | (0.004) |
| Glasso        | 0.67    | 0.010   |
| GlassoQ       | 0.006   | 0.000   |
| GlassoQuad    | 0.001   | 0.000   |
| GlassoGauss   | 0.544   | 0.000   |
| GlassoSpearman| 0.486   | 0.000   |
| GlassoQuad    | 0.095   | 0.000   |

### Table 17: Model BG under ICM. Comparison of means and standard deviations (in brackets) of mF and DKL over N = 100 replicates. p = 200, n = 100.

| Model         | mF (p) | DKL (p) |
|---------------|--------|---------|
| RGlassoWinsor | 0.323   | 0.002   |
| Glasso        | 0.336   | 0.002   |
| GlassoQ       | 0.044   | 0.005   |
| GlassoQuad    | 0.000   | 0.000   |
| GlassoGauss   | 0.000   | 0.000   |
| GlassoSpearman| 0.332   | 0.000   |
| GlassoQuad    | 0.124   | 0.000   |

### Table 18: Model BG under ICM. Comparison of means and standard deviations (in brackets) of TPR and TNR over N = 100 replicates. p = 60, n = 100.

| Model         | TPR (p) | TNR (p) |
|---------------|---------|---------|
| RGlassoWinsor | 0.192   | 0.196   |
| Glasso        | 0.547   | 0.201   |
| GlassoQ       | 0.000   | 0.000   |
| GlassoQuad    | 0.124   | 0.012   |
| GlassoGauss   | 0.482   | 0.113   |
| GlassoSpearman| 0.000   | 0.395   |
| GlassoQuad    | 0.113   | 0.102   |

### Table 19: Model BG under ICM. Comparison of means and standard deviations (in brackets) of MCC over N = 100 replicates. p = 60, n = 100.

| Model         | MCC (p) |
|---------------|---------|
| RGlassoWinsor | 0.425   |
| Glasso        | 0.472   |
| GlassoQ       | 0.000   |
| GlassoQuad    | 0.124   |
| GlassoGauss   | 0.000   |
| GlassoSpearman| 0.332   |
| GlassoQuad    | 0.124   |

### Table 20: Model BG under ICM. Comparison of means and standard deviations (in brackets) of TPR and TNR over N = 100 replicates. p = 200, n = 100.

| Model         | TPR (p) | TNR (p) |
|---------------|---------|---------|
| RGlassoWinsor | 0.200   | 0.196   |
| Glasso        | 0.547   | 0.201   |
| GlassoQ       | 0.000   | 0.000   |
| GlassoQuad    | 0.124   | 0.012   |
| GlassoGauss   | 0.482   | 0.113   |
| GlassoSpearman| 0.000   | 0.395   |
| GlassoQuad    | 0.113   | 0.102   |

### Table 21: Model BG under ICM. Comparison of means and standard deviations (in brackets) of MCC over N = 100 replicates. p = 200, n = 100.

| Model         | MCC (p) |
|---------------|---------|
| RGlassoWinsor | 0.425   |
| Glasso        | 0.472   |
| GlassoQ       | 0.000   |
| GlassoQuad    | 0.124   |
| GlassoGauss   | 0.000   |
| GlassoSpearman| 0.332   |
| GlassoQuad    | 0.124   |
Table 22: Model NN(2). Comparison of means and standard deviations (in brackets) of $m_p$ and $D_{KL}$ over $N = 100$ replicates. $p = 60, n = 100$.

| Model | $m_p$ (Mean, SD) | $D_{KL}$ (Mean, SD) |
|-------|-----------------|--------------------|
| RGlassoWinsor | 0.284 (0.999) | 0.265 (0.999) |
| Glasso | 0.258 (0.999) | 0.170 (0.999) |
| RGlasso | 0.079 (0.000) | 0.004 (0.000) |
| RGlassoTau | 0.005 (0.000) | 0.000 (0.000) |
| RGlassoGauss | 0.023 (0.000) | 0.000 (0.000) |
| RGlassoSpearman | 0.042 (0.000) | 0.000 (0.000) |
| RGlassoQuad | 0.091 (0.000) | 0.000 (0.000) |

Table 23: Model NN(2). Comparison of means and standard deviations (in brackets) of $m_p$ and $D_{KL}$ over $N = 100$ replicates. $p = 200, n = 100$.

| Model | $m_p$ (Mean, SD) | $D_{KL}$ (Mean, SD) |
|-------|-----------------|--------------------|
| RGlassoWinsor | 0.284 (0.999) | 0.265 (0.999) |
| Glasso | 0.258 (0.999) | 0.170 (0.999) |
| RGlasso | 0.079 (0.000) | 0.004 (0.000) |
| RGlassoTau | 0.005 (0.000) | 0.000 (0.000) |
| RGlassoGauss | 0.023 (0.000) | 0.000 (0.000) |
| RGlassoSpearman | 0.042 (0.000) | 0.000 (0.000) |
| RGlassoQuad | 0.091 (0.000) | 0.000 (0.000) |

Table 24: Model NN(2) under ICM. Comparison of means and standard deviations (in brackets) of TPR and TNR over $N = 100$ replicates. $p = 60, n = 100$.

| Model | TPR (Mean, SD) | TNR (Mean, SD) |
|-------|----------------|----------------|
| RGlassoWinsor | 0.369 (0.997) | 0.566 (0.997) |
| Glasso | 0.641 (0.997) | 0.420 (0.996) |
| RGlasso | 0.265 (0.999) | 0.254 (0.999) |
| RGlassoTau | 0.007 (0.000) | 0.000 (0.000) |
| RGlassoGauss | 0.630 (0.997) | 0.604 (0.996) |
| RGlassoSpearman | 0.623 (0.996) | 0.608 (0.996) |
| RGlassoQuad | 0.530 (0.987) | 0.513 (0.987) |

Table 25: Model NN(2). Comparison of means and standard deviations (in brackets) of MCC over $N = 100$ replicates. $p = 60, n = 100$.

| Model | MCC (Mean, SD) |
|-------|----------------|
| RGlassoWinsor | 0.284 (0.999) |
| Glasso | 0.258 (0.999) |
| RGlasso | 0.079 (0.000) |
| RGlassoTau | 0.005 (0.000) |
| RGlassoGauss | 0.023 (0.000) |
| RGlassoSpearman | 0.042 (0.000) |
| RGlassoQuad | 0.091 (0.000) |

Table 26: Model NN(2) under ICM. Comparison of means and standard deviations (in brackets) of TPR and TNR over $N = 100$ replicates. $p = 200, n = 100$.

| Model | TPR (Mean, SD) | TNR (Mean, SD) |
|-------|----------------|----------------|
| RGlassoWinsor | 0.369 (0.997) | 0.566 (0.997) |
| Glasso | 0.641 (0.997) | 0.420 (0.996) |
| RGlasso | 0.265 (0.999) | 0.254 (0.999) |
| RGlassoTau | 0.007 (0.000) | 0.000 (0.000) |
| RGlassoGauss | 0.630 (0.997) | 0.604 (0.996) |
| RGlassoSpearman | 0.623 (0.996) | 0.608 (0.996) |
| RGlassoQuad | 0.530 (0.987) | 0.513 (0.987) |

Table 27: Model NN(2). Comparison of means and standard deviations (in brackets) of MCC over $N = 100$ replicates. $p = 200, n = 100$.

| Model | MCC (Mean, SD) |
|-------|----------------|
| RGlassoWinsor | 0.284 (0.999) |
| Glasso | 0.258 (0.999) |
| RGlasso | 0.079 (0.000) |
| RGlassoTau | 0.005 (0.000) |
| RGlassoGauss | 0.023 (0.000) |
| RGlassoSpearman | 0.042 (0.000) |
| RGlassoQuad | 0.091 (0.000) |
Table 28: Model Rand under ICM. Comparison of means and standard deviations (in brackets) of $m_p$ and $D_{KL}$ over $N = 100$ replicates. $p = 60$, $n = 100$.

| $p$ | $n$ | $m_p$ | $D_{KL}$ | $m_p$ | $D_{KL}$ | $m_p$ | $D_{KL}$ |
|-----|-----|-------|---------|-------|---------|-------|---------|
| table 28 | table 28 | table 28 | table 28 | table 28 | table 28 | table 28 | table 28 |

Table 29: Model Rand under ICM. Comparison of means and standard deviations (in brackets) of $m_F$ and $D_{KL}$ over $N = 100$ replicates. $p = 200$, $n = 100$.

| $p$ | $n$ | $m_F$ | $D_{KL}$ | $m_F$ | $D_{KL}$ | $m_F$ | $D_{KL}$ |
|-----|-----|-------|---------|-------|---------|-------|---------|
| table 29 | table 29 | table 29 | table 29 | table 29 | table 29 | table 29 | table 29 |

Table 30: Model Rand under ICM. Comparison of means and standard deviations (in brackets) of TPR and TNR over $N = 100$ replicates. $p = 60$, $n = 100$.

| $p$ | $n$ | TPR | TNR | TPR | TNR | TPR | TNR |
|-----|-----|-----|-----|-----|-----|-----|-----|
| table 30 | table 30 | table 30 | table 30 | table 30 | table 30 | table 30 | table 30 |

Table 31: Model Rand under ICM. Comparison of means and standard deviations (in brackets) of MCC over $N = 100$ replicates. $p = 60$, $n = 100$.

| $p$ | $n$ | MCC | MCC | MCC | MCC | MCC | MCC |
|-----|-----|-----|-----|-----|-----|-----|-----|
| table 31 | table 31 | table 31 | table 31 | table 31 | table 31 | table 31 | table 31 |

Table 32: Model Rand under ICM. Comparison of means and standard deviations (in brackets) of TPR and TNR over $N = 100$ replicates. $p = 200$, $n = 100$.

| $p$ | $n$ | TPR | TNR | TPR | TNR | TPR | TNR |
|-----|-----|-----|-----|-----|-----|-----|-----|
| table 32 | table 32 | table 32 | table 32 | table 32 | table 32 | table 32 | table 32 |

Table 33: Model Rand under ICM. Comparison of means and standard deviations (in brackets) of MCC over $N = 100$ replicates. $p = 200$, $n = 100$.

| $p$ | $n$ | MCC | MCC | MCC | MCC | MCC | MCC |
|-----|-----|-----|-----|-----|-----|-----|-----|
| table 33 | table 33 | table 33 | table 33 | table 33 | table 33 | table 33 | table 33 |
Table 34: Model Hub. $p = 60, n = 100$. Comparison of means and standard deviations (in brackets) of $m$ and $D_{KL}$ over $N = 100$ replicates. $p = 60, n = 100$.

| $c = 0$ | $c = 0.01$ | $c = 0.05$ | $c = 0.10$ |
|---------|------------|------------|------------|
| RGlassoWinsor | D$_{KL}$ | m | m | m |
| Glasso | | | | |
| RGlassoQ$_{4}$ | | | | |
| RGlassoTau | | | | |
| RGlassoGauss | | | | |
| RGlassoSpearman | | | | |
| RGlassoQuad | | | | |

Table 35: Model Hub under ICM. Comparison of means and standard deviations (in brackets) of $m$ and $D_{KL}$ over $N = 100$ replicates. $p = 200, n = 100$.

| $c = 0$ | $c = 0.01$ | $c = 0.05$ | $c = 0.10$ |
|---------|------------|------------|------------|
| RGlassoWinsor | D$_{KL}$ | m | m | m |
| Glasso | | | | |
| RGlassoQ$_{4}$ | | | | |
| RGlassoTau | | | | |
| RGlassoGauss | | | | |
| RGlassoSpearman | | | | |
| RGlassoQuad | | | | |

Table 36: Model Hub under ICM. Comparison of means and standard deviations (in brackets) of TPR and TNR over $N = 100$ replicates. $p = 60, n = 100$.

| $c = 0$ | $c = 0.01$ | $c = 0.05$ | $c = 0.10$ |
|---------|------------|------------|------------|
| RGlassoWinsor | TPR | TNR | TPR | TNR |
| Glasso | | | | |
| RGlassoQ$_{4}$ | | | | |
| RGlassoTau | | | | |
| RGlassoGauss | | | | |
| RGlassoSpearman | | | | |
| RGlassoQuad | | | | |

Table 37: Model Hub under ICM. Comparison of means and standard deviations (in brackets) of MCC over $N = 100$ replicates. $p = 60, n = 100$.

| $c = 0$ | $c = 0.01$ | $c = 0.05$ | $c = 0.10$ |
|---------|------------|------------|------------|
| RGlassoWinsor | MCC | MCC | MCC | MCC |
| Glasso | | | | |
| RGlassoQ$_{4}$ | | | | |
| RGlassoTau | | | | |
| RGlassoGauss | | | | |
| RGlassoSpearman | | | | |
| RGlassoQuad | | | | |

Table 38: Model Hub under ICM. Comparison of means and standard deviations (in brackets) of TPR and TNR over $N = 100$ replicates. $p = 200, n = 100$.

| $c = 0$ | $c = 0.01$ | $c = 0.05$ | $c = 0.10$ |
|---------|------------|------------|------------|
| RGlassoWinsor | TPR | TNR | TPR | TNR |
| Glasso | | | | |
| RGlassoQ$_{4}$ | | | | |
| RGlassoTau | | | | |
| RGlassoGauss | | | | |
| RGlassoSpearman | | | | |
| RGlassoQuad | | | | |

Table 39: Model Hub. Comparison of means and standard deviations (in brackets) of MCC over $N = 100$ replicates. $p = 200, n = 100$.

| $c = 0$ | $c = 0.01$ | $c = 0.05$ | $c = 0.10$ |
|---------|------------|------------|------------|
| RGlassoWinsor | MCC | MCC | MCC | MCC |
| Glasso | | | | |
| RGlassoQ$_{4}$ | | | | |
| RGlassoTau | | | | |
| RGlassoGauss | | | | |
| RGlassoSpearman | | | | |
| RGlassoQuad | | | | |
Figure 11: Heatmaps for the frequency of adjacency for each pair of nodes over $N = 100$ replicates. ICM.

Figure 12: Heatmaps for the frequency of adjacency for each pair of nodes over $N = 100$ replicates under ICM. $p = 60$ and $n = 100$. The axes display the graph $p$-nodes in a given order.
Figure 13: Heatmaps for the frequency of adjacency for each pair of nodes over $N = 100$ replicates under ICM. $p = 60$ and $n = 100$. The axes display the graph $p$-nodes in a given order.

Figure 14: Heatmaps for the frequency of adjacency for each pair of nodes over $N = 100$ replicates under ICM. $p = 60$ and $n = 100$. The axes display the graph $p$-nodes in a given order.

Appendix B

This section contains numerical and performance classification performance results for the seven precision matrix estimators applied to AR(1), BG, NN(2), Rand and Hub models. We compare two scenarios $p = 60$ and $p = 200$ with $n = 100$ under THCM.
Table 40: Model AR(1) under THCM. Comparison of means and standard deviations (in brackets) of $m_F$ and $D_{KL}$ over $N = 100$ replicates. $p = 60, n = 100$.

Table 41: Model AR(1) under THCM. Comparison of means and standard deviations (in brackets) of $m_F$ and $D_{KL}$ over $N = 100$ replicates. $p = 200, n = 100$.

Table 42: Model AR(1) under THCM. Comparison of means and standard deviations (in brackets) of TPR and TNR over $N = 100$ replicates. $p = 60, n = 100$.

Table 43: Model AR(1) under THCM. Comparison of means and standard deviations (in brackets) of MCC over $N = 100$ replicates. $p = 60, n = 100$.

Table 44: Model AR(1) under THCM. Comparison of means and standard deviations (in brackets) of TPR and TNR over $N = 100$ replicates. $p = 200, n = 100$.

Table 45: Model AR(1) under THCM. Comparison of means and standard deviations (in brackets) of MCC over $N = 100$ replicates. $p = 200, n = 100$. 

| $\epsilon = 0.05$ | $\epsilon = 0.10$ |
|------------------|------------------|
| $|\epsilon|_2$ | $m_F$ | $D_{KL}$ | $m_F$ |
| RGlassoWinsor | 8.947 | 5.265 | 6.964 | 5.286 |
| Glasso | 63.652 | 10.374 | 63.578 | 10.372 |
| RGlassoQn | 1.355 | 0.940 | 1.388 | 0.948 |
| RGlassoQuad | 21.979 | 8.151 | 16.634 | 8.215 |
| RGlassoSpearman | 0.959 | 0.107 | 1.674 | 0.118 |
| RGlassoTau | 13.671 | 6.854 | 31.968 | 8.836 |
| RGlassoGauss | 0.740 | 0.110 | 1.512 | 0.312 |
| RGlassoGauss | 29.055 | 7.776 | 14.582 | 6.696 |
| RGlassoSpearman | 0.288 | 0.055 | 0.234 | 0.049 |
| RGlassoQuad | 16.301 | 7.666 | 22.257 | 8.356 |
| RGlassoSpearman | 0.315 | 0.047 | 0.469 | 0.042 |

| $\epsilon = 0.05$ | $\epsilon = 0.10$ |
|------------------|------------------|
| $|\epsilon|_2$ | $m_F$ | $D_{KL}$ | $m_F$ |
| RGlassoWinsor | 8.947 | 5.265 | 6.964 | 5.286 |
| Glasso | 63.652 | 10.374 | 63.578 | 10.372 |
| RGlassoQn | 1.355 | 0.940 | 1.388 | 0.948 |
| RGlassoQuad | 21.979 | 8.151 | 16.634 | 8.215 |
| RGlassoSpearman | 0.959 | 0.107 | 1.674 | 0.118 |
| RGlassoTau | 13.671 | 6.854 | 31.968 | 8.836 |
| RGlassoGauss | 0.740 | 0.110 | 1.512 | 0.312 |
| RGlassoGauss | 29.055 | 7.776 | 14.582 | 6.696 |
| RGlassoSpearman | 0.288 | 0.055 | 0.234 | 0.049 |
| RGlassoQuad | 16.301 | 7.666 | 22.257 | 8.356 |
| RGlassoSpearman | 0.315 | 0.047 | 0.469 | 0.042 |

| $\epsilon = 0.05$ | $\epsilon = 0.10$ |
|------------------|------------------|
| $|\epsilon|_2$ | $m_F$ | $D_{KL}$ | $m_F$ |
| RGlassoWinsor | 8.947 | 5.265 | 6.964 | 5.286 |
| Glasso | 63.652 | 10.374 | 63.578 | 10.372 |
| RGlassoQn | 1.355 | 0.940 | 1.388 | 0.948 |
| RGlassoQuad | 21.979 | 8.151 | 16.634 | 8.215 |
| RGlassoSpearman | 0.959 | 0.107 | 1.674 | 0.118 |
| RGlassoTau | 13.671 | 6.854 | 31.968 | 8.836 |
| RGlassoGauss | 0.740 | 0.110 | 1.512 | 0.312 |
| RGlassoGauss | 29.055 | 7.776 | 14.582 | 6.696 |
| RGlassoSpearman | 0.288 | 0.055 | 0.234 | 0.049 |
| RGlassoQuad | 16.301 | 7.666 | 22.257 | 8.356 |
| RGlassoSpearman | 0.315 | 0.047 | 0.469 | 0.042 |

| $\epsilon = 0.05$ | $\epsilon = 0.10$ |
|------------------|------------------|
| $|\epsilon|_2$ | $m_F$ | $D_{KL}$ | $m_F$ |
| RGlassoWinsor | 8.947 | 5.265 | 6.964 | 5.286 |
| Glasso | 63.652 | 10.374 | 63.578 | 10.372 |
| RGlassoQn | 1.355 | 0.940 | 1.388 | 0.948 |
| RGlassoQuad | 21.979 | 8.151 | 16.634 | 8.215 |
| RGlassoSpearman | 0.959 | 0.107 | 1.674 | 0.118 |
| RGlassoTau | 13.671 | 6.854 | 31.968 | 8.836 |
| RGlassoGauss | 0.740 | 0.110 | 1.512 | 0.312 |
| RGlassoGauss | 29.055 | 7.776 | 14.582 | 6.696 |
| RGlassoSpearman | 0.288 | 0.055 | 0.234 | 0.049 |
| RGlassoQuad | 16.301 | 7.666 | 22.257 | 8.356 |
| RGlassoSpearman | 0.315 | 0.047 | 0.469 | 0.042 |
Table 46: Model BG under THCM. Comparison of means and standard deviations (in brackets) of $m_F$ and $D_{KL}$ over $N = 100$ replicates. $p = 60$, $n = 100$.

| $\epsilon$ | $m_F$ mean (SD) | $D_{KL}$ mean (SD) |
|------------|-----------------|-------------------|
| $0.05$     | 12.665 (0.004)  | 35.383 (0.011)    |
| $0.10$     | 46.680 (0.005)  | 2.142 (0.004)     |
| Glasso     | RGlassoWinsor   | RGlassoQuad       |
| $0.05$     | 0.995 (0.005)   | 0.765 (0.005)     |
| $0.10$     | 4.011 (0.006)   | 0.011 (0.006)     |
| RGlasso    | RGlassoQua      | RGlassoSpearman   |
| $0.05$     | 23.439 (0.021)  | 0.123 (0.004)     |
| $0.10$     | 13.361 (0.011)  | 0.123 (0.004)     |
| RGlassoQuad| RGlassoSpearman | RGlassoQuad       |

Table 47: Model BG under THCM. Comparison of means and standard deviations (in brackets) of $m_F$ and $D_{KL}$ over $N = 100$ replicates. $p = 200$, $n = 100$.

| $\epsilon$ | $m_F$ mean (SD) | $D_{KL}$ mean (SD) |
|------------|-----------------|-------------------|
| $0.05$     | 12.665 (0.004)  | 35.383 (0.011)    |
| $0.10$     | 46.680 (0.005)  | 2.142 (0.004)     |
| Glasso     | RGlassoWinsor   | RGlassoQuad       |
| $0.05$     | 0.995 (0.005)   | 0.765 (0.005)     |
| $0.10$     | 4.011 (0.006)   | 0.011 (0.006)     |
| RGlasso    | RGlassoQua      | RGlassoSpearman   |
| $0.05$     | 23.439 (0.021)  | 0.123 (0.004)     |
| $0.10$     | 13.361 (0.011)  | 0.123 (0.004)     |
| RGlassoQuad| RGlassoSpearman | RGlassoQuad       |

Table 48: Model BG under THCM. Comparison of means and standard deviations (in brackets) of TPR and TNR over $N = 100$ replicates. $p = 60$, $n = 100$.

| $\epsilon$ | TPR mean (SD) | TNR mean (SD) |
|------------|--------------|---------------|
| $0.05$     | 0.214 (0.004)| 0.011 (0.006)|
| $0.10$     | 0.000 (0.004)| 0.001 (0.005)|
| Glasso     | RGlassoQua   | RGlassoSpearman|
| $0.05$     | 0.214 (0.004)| 0.011 (0.006)|
| $0.10$     | 0.000 (0.004)| 0.001 (0.005)|
| RGlasso    | RGlassoQua   | RGlassoSpearman|
| $0.05$     | 0.214 (0.004)| 0.011 (0.006)|
| $0.10$     | 0.000 (0.004)| 0.001 (0.005)|
| RGlassoQuad| RGlassoSpearman | RGlassoQuad   |

Table 49: Model BG under THCM. Comparison of means and standard deviations (in brackets) of MCC over $N = 100$ replicates. $p = 60$, $n = 100$.

| $\epsilon$ | MCC mean (SD) |
|------------|---------------|
| $0.05$     | 0.214 (0.004)|
| $0.10$     | 0.000 (0.004)|
| Glasso     | RGlassoQua   |
| $0.05$     | 0.214 (0.004)|
| $0.10$     | 0.000 (0.004)|
| RGlasso    | RGlassoQua   |
| $0.05$     | 0.214 (0.004)|
| $0.10$     | 0.000 (0.004)|
| RGlassoQuad| RGlassoSpearman | RGlassoQuad |

Table 50: Model BG under THCM. Comparison of means and standard deviations (in brackets) of TPR and TNR over $N = 100$ replicates. $p = 200$, $n = 100$.

| $\epsilon$ | TPR mean (SD) | TNR mean (SD) |
|------------|--------------|---------------|
| $0.05$     | 0.214 (0.004)| 0.011 (0.006)|
| $0.10$     | 0.000 (0.004)| 0.001 (0.005)|
| Glasso     | RGlassoQua   | RGlassoSpearman|
| $0.05$     | 0.214 (0.004)| 0.011 (0.006)|
| $0.10$     | 0.000 (0.004)| 0.001 (0.005)|
| RGlasso    | RGlassoQua   | RGlassoSpearman|
| $0.05$     | 0.214 (0.004)| 0.011 (0.006)|
| $0.10$     | 0.000 (0.004)| 0.001 (0.005)|
| RGlassoQuad| RGlassoSpearman | RGlassoQuad   |

Table 51: Model BG under THCM. Comparison of means and standard deviations (in brackets) of MCC over $N = 100$ replicates. $p = 200$, $n = 100$.

| $\epsilon$ | MCC mean (SD) |
|------------|---------------|
| $0.05$     | 0.214 (0.004)|
| $0.10$     | 0.000 (0.004)|
| Glasso     | RGlassoQua   |
| $0.05$     | 0.214 (0.004)|
| $0.10$     | 0.000 (0.004)|
| RGlasso    | RGlassoQua   |
| $0.05$     | 0.214 (0.004)|
| $0.10$     | 0.000 (0.004)|
| RGlassoQuad| RGlassoSpearman | RGlassoQuad |

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### Table 52: Model NN(2) under THCM. Comparison of means and standard deviations (in brackets) of $m_F$ and $D_{KL}$ over $N = 100$ replicates. $p = 60, n = 100$.

| $\epsilon = 0.05$ | $\epsilon = 0.10$ |
|------------------|------------------|
| $\mu_{m_F}$ | $\mu_{D_{KL}}$ | $\sigma_{m_F}$ | $\sigma_{D_{KL}}$ |
| RGlassoWinsor | 0.390 0.998 | 0.257 0.999 | 0.015 0.003 | 0.016 0.000 |
| Glasso | 0.202 0.892 | 0.167 0.888 | 0.013 0.006 | 0.010 0.006 |
| RGlassoQuad | 0.087 1.000 | 0.092 0.999 | 0.005 0.000 | 0.005 0.000 |
| RGlassoGauss | 0.283 0.999 | 0.275 0.997 | 0.012 0.000 | 0.013 0.000 |
| RGlassoSpearman | 0.295 0.999 | 0.292 0.998 | 0.013 0.000 | 0.013 0.000 |
| RGlassoQuad | 0.276 0.990 | 0.279 0.985 | 0.012 0.003 | 0.012 0.003 |

### Table 53: Model NN(2) under THCM. Comparison of means and standard deviations (in brackets) of $m_F$ and $D_{KL}$ over $N = 100$ replicates. $p = 200, n = 100$.

| $\epsilon = 0.05$ | $\epsilon = 0.10$ |
|------------------|------------------|
| $\mu_{m_F}$ | $\mu_{D_{KL}}$ | $\sigma_{m_F}$ | $\sigma_{D_{KL}}$ |
| RGlassoWinsor | 0.390 0.998 | 0.257 0.999 | 0.015 0.003 | 0.016 0.000 |
| Glasso | 0.202 0.892 | 0.167 0.888 | 0.013 0.006 | 0.010 0.006 |
| RGlassoQuad | 0.087 1.000 | 0.092 0.999 | 0.005 0.000 | 0.005 0.000 |
| RGlassoGauss | 0.283 0.999 | 0.275 0.997 | 0.012 0.000 | 0.013 0.000 |
| RGlassoSpearman | 0.295 0.999 | 0.292 0.998 | 0.013 0.000 | 0.013 0.000 |
| RGlassoQuad | 0.276 0.990 | 0.279 0.985 | 0.012 0.003 | 0.012 0.003 |

### Table 54: Model NN(2) under THCM. Comparison of means and standard deviations (in brackets) of TPR and TNR over $N = 100$ replicates. $p = 60, n = 100$.

| $\epsilon = 0.05$ | $\epsilon = 0.10$ |
|------------------|------------------|
| $\mu_{TPR}$ | $\mu_{TNR}$ | $\sigma_{TPR}$ | $\sigma_{TNR}$ |
| RGlassoWinsor | 0.564 0.996 | 0.564 0.996 | 0.011 0.000 | 0.011 0.000 |
| Glasso | 0.595 0.985 | 0.600 0.982 | 0.013 0.005 | 0.014 0.006 |
| RGlassoQuad | 0.237 0.999 | 0.219 0.999 | 0.008 0.000 | 0.006 0.000 |
| RGlassoGauss | 0.247 0.999 | 0.260 0.998 | 0.008 0.000 | 0.009 0.000 |
| RGlassoSpearman | 0.303 0.997 | 0.620 0.996 | 0.009 0.000 | 0.008 0.000 |
| RGlassoQuad | 0.499 0.985 | 0.519 0.984 | 0.008 0.001 | 0.008 0.001 |

### Table 55: Model NN(2) under THCM. Comparison of means and standard deviations (in brackets) of MCC over $N = 100$ replicates. $p = 60, n = 100$.

| $\epsilon = 0.05$ | $\epsilon = 0.10$ |
|------------------|------------------|
| $\mu_{MCC}$ | $\mu_{MCC}$ | $\sigma_{MCC}$ | $\sigma_{MCC}$ |
| RGlassoWinsor | 0.390 0.998 | 0.257 0.999 | 0.015 0.003 | 0.016 0.000 |
| Glasso | 0.202 0.892 | 0.167 0.888 | 0.013 0.006 | 0.010 0.006 |
| RGlassoQuad | 0.087 1.000 | 0.092 0.999 | 0.005 0.000 | 0.005 0.000 |
| RGlassoGauss | 0.283 0.999 | 0.275 0.997 | 0.012 0.000 | 0.013 0.000 |
| RGlassoSpearman | 0.295 0.999 | 0.292 0.998 | 0.013 0.000 | 0.013 0.000 |
| RGlassoQuad | 0.276 0.990 | 0.279 0.985 | 0.012 0.003 | 0.012 0.003 |

### Table 56: Model NN(2) under THCM. Comparison of means and standard deviations (in brackets) of TPR and TNR over $N = 100$ replicates. $p = 200, n = 100$.

| $\epsilon = 0.05$ | $\epsilon = 0.10$ |
|------------------|------------------|
| $\mu_{TPR}$ | $\mu_{TNR}$ | $\sigma_{TPR}$ | $\sigma_{TNR}$ |
| RGlassoWinsor | 0.564 0.996 | 0.564 0.996 | 0.011 0.000 | 0.011 0.000 |
| Glasso | 0.595 0.985 | 0.600 0.982 | 0.013 0.005 | 0.014 0.006 |
| RGlassoQuad | 0.237 0.999 | 0.219 0.999 | 0.008 0.000 | 0.006 0.000 |
| RGlassoGauss | 0.247 0.999 | 0.260 0.998 | 0.008 0.000 | 0.009 0.000 |
| RGlassoSpearman | 0.303 0.997 | 0.620 0.996 | 0.009 0.000 | 0.008 0.000 |
| RGlassoQuad | 0.499 0.985 | 0.519 0.984 | 0.008 0.001 | 0.008 0.001 |

### Table 57: Model NN(2) under THCM. Comparison of means and standard deviations (in brackets) of MCC over $N = 100$ replicates. $p = 200, n = 100$.

| $\epsilon = 0.05$ | $\epsilon = 0.10$ |
|------------------|------------------|
| $\mu_{MCC}$ | $\mu_{MCC}$ | $\sigma_{MCC}$ | $\sigma_{MCC}$ |
| RGlassoWinsor | 0.390 0.998 | 0.257 0.999 | 0.015 0.003 | 0.016 0.000 |
| Glasso | 0.202 0.892 | 0.167 0.888 | 0.013 0.006 | 0.010 0.006 |
| RGlassoQuad | 0.087 1.000 | 0.092 0.999 | 0.005 0.000 | 0.005 0.000 |
| RGlassoGauss | 0.283 0.999 | 0.275 0.997 | 0.012 0.000 | 0.013 0.000 |
| RGlassoSpearman | 0.295 0.999 | 0.292 0.998 | 0.013 0.000 | 0.013 0.000 |
| RGlassoQuad | 0.276 0.990 | 0.279 0.985 | 0.012 0.003 | 0.012 0.003 |
Table 58: Model Rand under THCM. Comparison of means and standard deviations (in brackets) of $m_F$ and $D_{KL}$ over $N = 100$ replicates. $p = 60$, $n = 100$.

|               | $c = 0.05$ | $c = 0.10$ |
|---------------|------------|------------|
|               | $\mu_F$  | $SD_F$  | $\mu_{KL}$  | $SD_{KL}$  |
| RGlassoWinsor| 9.394     | 4.037     | 7.187       | 4.497      |
| Glasso        | 61.668    | 7.162     | 79.630      | 8.502      |
| RGlassoQn     | 6.077     | 3.344     | 8.033       | 0.314      |
| RGlassoTau    | 0.118     | 0.036     | 0.113       | 0.038      |
| RGlassoGauss  | 6.358     | 4.995     | 7.071       | 4.922      |
| RGlassoSpearman| 0.063   | 0.028     | 0.082       | 0.024      |
| RGlassoQuad   | 5.146     | 4.406     | 5.755       | 4.320      |
| RGlassoWinsor| 0.050     | 0.029     | 0.062       | 0.025      |
| Glasso        | 5.455     | 4.679     | 5.974       | 4.466      |
| RGlassoQuad   | 0.048     | 0.027     | 0.076       | 0.022      |

Table 59: Model Rand under THCM. Comparison of means and standard deviations (in brackets) of $m_F$ and $D_{KL}$ over $N = 100$ replicates. $p = 200$, $n = 100$.

|               | $c = 0.05$ | $c = 0.10$ |
|---------------|------------|------------|
|               | $\mu_F$  | $SD_F$  | $\mu_{KL}$  | $SD_{KL}$  |
| RGlassoWinsor| 44.109    | 12.349    | 44.848      | 12.369     |
| Glasso        | 98.156    | 15.942    | 103.344     | 15.908     |
| RGlassoQn     | 6.016     | 0.222     | 9.142       | 0.244      |
| RGlassoTau    | 116.645   | 17.964    | 116.103     | 17.934     |
| RGlassoGauss  | 19.781    | 5.881     | 25.47       | 0.894      |
| RGlassoSpearman| 0.284   | 0.046     | 0.320       | 0.067      |
| RGlassoQuad   | 4.523     | 12.948    | 45.451      | 12.967     |

Table 60: Model Rand under THCM. Comparison of means and standard deviations (in brackets) of TPR and TNR over $N = 100$ replicates. $p = 60$, $n = 100$.

|               | $c = 0.05$ | $c = 0.10$ |
|---------------|------------|------------|
|               | $\mu_{TPR}$  | $SD_{TPR}$  | $\mu_{TNR}$  | $SD_{TNR}$  |
| RGlassoWinsor| 0.909     | 0.552     | 0.870       | 0.861      |
| Glasso        | 61.668    | 7.162     | 79.630      | 8.502      |
| RGlassoQn     | 5.795     | 0.358     | 5.958       | 0.356      |
| RGlassoTau    | 0.141     | 0.019     | 0.141       | 0.019      |
| RGlassoGauss  | 0.917     | 0.805     | 0.742       | 0.881      |
| RGlassoSpearman| 0.906   | 0.003     | 0.000       | 0.003      |
| RGlassoQuad   | 0.927     | 0.825     | 0.927       | 0.825      |
| RGlassoWinsor| 0.003     | 0.005     | 0.000       | 0.000      |
| Glasso        | 5.505     | 0.845     | 5.677       | 0.856      |
| RGlassoQuad   | 0.481     | 0.339     | 0.355       | 0.449      |
| RGlassoSpearman| 0.010   | 0.002     | 0.013       | 0.003      |

Table 61: Model Rand under THCM. Comparison of means and standard deviations (in brackets) of MCC over $N = 100$ replicates. $p = 60$, $n = 100$.

|               | $c = 0.05$ | $c = 0.10$ |
|---------------|------------|------------|
|               | $\mu_{MCC}$  | $SD_{MCC}$  | $\mu_{MCC}$  | $SD_{MCC}$  |
| RGlassoWinsor| 0.411     | 0.856     | 0.306       | 0.856      |
| Glasso        | 0.079     | 0.007     | 0.007       | 0.007      |
| RGlassoQn     | 0.000     | 0.010     | 0.010       | 0.009      |
| RGlassoTau    | 0.000     | 0.010     | 0.010       | 0.009      |
| RGlassoGauss  | 0.000     | 0.010     | 0.010       | 0.009      |
| RGlassoSpearman| 0.000   | 0.010     | 0.010       | 0.009      |
| RGlassoQuad   | 0.000     | 0.010     | 0.010       | 0.009      |

Table 62: Model Rand under THCM. Comparison of means and standard deviations (in brackets) of TPR and TNR over $N = 100$ replicates. $p = 200$, $n = 100$.

|               | $c = 0.05$ | $c = 0.10$ |
|---------------|------------|------------|
|               | $\mu_{TPR}$  | $SD_{TPR}$  | $\mu_{TNR}$  | $SD_{TNR}$  |
| RGlassoWinsor| 116.103    | 17.934    | 116.645     | 17.964     |
| Glasso        | 5.455     | 4.679     | 5.974       | 4.466      |
| RGlassoQn     | 0.090     | 0.028     | 0.082       | 0.024      |
| RGlassoTau    | 0.048     | 0.027     | 0.076       | 0.022      |
| RGlassoGauss  | 0.050     | 0.029     | 0.062       | 0.025      |
| RGlassoSpearman| 0.048   | 0.027     | 0.076       | 0.022      |
| RGlassoQuad   | 0.048     | 0.027     | 0.076       | 0.022      |

Table 63: Model Rand under THCM. Comparison of means and standard deviations (in brackets) of MCC over $N = 100$ replicates. $p = 200$, $n = 100$.
Table 64: Model Hub under THCM. Comparison of means and standard deviations (in brackets) of $m_F$ and $D_{KL}$ over $N = 100$ replicates. $p = 60$, $n = 100$.

| Method      | $\epsilon = 0.05$ | $\epsilon = 0.10$ |
|-------------|-------------------|-------------------|
|             | $m_F$ | $\pm m_F$ | $D_{KL}$ | $\pm D_{KL}$ |
| RGlassoWmmer| 5.679 | 6.277 | 6.579 | 6.580 |
| Glasso      | 12.000 | 12.000 | 12.000 | 12.000 |
| RGlassoQn   | 6.294 | 6.188 | 7.679 | 7.149 |
| RGlassoQnu  | 0.222 | 0.943 | 0.384 | 0.055 |
| RGlassoTau  | 6.631 | 6.818 | 8.080 | 7.238 |
| RGlassoGauss| 0.195 | 0.034 | 0.144 | 0.037 |
| RGlassoSpearman | 5.509 | 6.484 | 6.449 | 6.776 |
| RGlassoQuad | 0.959 | 0.025 | 0.072 | 0.021 |

Table 65: Model Hub under THCM. Comparison of means and standard deviations (in brackets) of $m_F$ and $D_{KL}$ over $N = 100$ replicates. $p = 200$, $n = 100$.

Table 66: Model Hub under THCM. Comparison of means and standard deviations (in brackets) of TPR and TNR over $N = 100$ replicates. $p = 60$, $n = 100$.

Table 67: Model Hub under THCM. Comparison of means and standard deviations (in brackets) of MCC over $N = 100$ replicates. $p = 60$, $n = 100$.

Table 68: Model Hub under THCM. Comparison of means and standard deviations (in brackets) of TPR and TNR over $N = 100$ replicates. $p = 200$, $n = 100$.

Table 69: Model Hub under THCM. Comparison of means and standard deviations (in brackets) of MCC over $N = 100$ replicates. $p = 200$, $n = 100$.

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