Boundary Value Problems for Parabolic Operators in a Time-Varying Domain

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We prove the existence of unique solutions to Dirichlet boundary value problems for linear second-order uniformly parabolic operators in either divergence or non-divergence form with boundary blowup low-order coefficients. The domain is possibly time varying, non-smooth, and satisfies an exterior measure condition.

Keywords Blowup low-order coefficients; Exterior measure condition; Parabolic Dirichlet boundary value problems; Time-varying domain; Vanishing mean oscillation.

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1. Introduction and Main Results

In this paper we consider parabolic operators in divergence form

\[ Lu = D_t(u) - \sum_{i,j} a_{ij}D_{ij}u + \sum_i b_i D_i u - \sum_i c_i u + c_0 u \]  \hspace{1cm} (D)

and in non-divergence form

\[ Lu = D_t u - \sum_{i,j} a_{ij}D_{ij}u + \sum_i b_i D_i u + c_0 u \]  \hspace{1cm} (ND)

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in a time-varying domain $Q$ in $\mathbb{R}^{n+1}$, $n \geq 1$, with boundary blowup low-order coefficients. Here and in the sequel,

$$D_i := \frac{\partial}{\partial x_i}, \quad D_{ij} := D_i D_j, \quad i, j = 1, \ldots, n, \quad D_t := \frac{\partial}{\partial t},$$

some derivatives in parentheses in divergence form are understood in the weak sense, and summation over repeated indices is assumed. For convenience of notation, in the sequel we set $c_i = 0$, $i = 1, \ldots, n$, in the non-divergence case.

With the operator $L$ in (D) or (ND), we study the following boundary value problems of Dirichlet type:

$$\begin{cases}
Lu = f & \text{in } Q, \\
u = 0 & \text{on } \partial_p Q,
\end{cases} \quad (DP)$$

where $f = f_0 - D_t f$, for the divergence case and $\partial_p Q$ is the parabolic boundary of $Q$ (see Definition 1.1 below). We prove that there exist unique solutions to the Dirichlet problems (DP) when the domain $Q$ satisfies an exterior measure condition (see Definition 1.3). In the non-divergence case, solutions satisfy the equation in the strong sense, and are locally in $W^{1,2}_p$, $p > (n + 2)/2$. While in the divergence case, they are understood in the weak sense. In both cases, solutions are continuous up to the boundary; thus, the boundary condition in (DP) is understood in the pointwise sense. The coefficients which we consider have two features. First, concerning the leading coefficients, while in the divergence case we do not impose any regularity assumptions, in the non-divergence case we assume that they have vanishing mean oscillations (VMO) with respect to the spacial variables and merely measurable with respect to the time variable. Second, the lower-order coefficients may blow up near the boundary with a certain optimal growth condition.

Indeed, there is an extensive literature on the existence and uniqueness of solutions to the boundary value problem (DP) in a straight cylindrical domain with lower-order coefficients which are bounded or in certain Sobolev spaces. See, for instance, [1, 21, 25, 26, 29, 31] and the references therein.

Regarding non-divergence form parabolic equations in time-varying domains (or more general degenerate elliptic-parabolic equations) one may find related results in Kohn and Nirenberg [19], Krylov [23], and references therein, where the existence, uniqueness, and regularity of solutions were discussed for equations with smooth coefficients. The solutions to parabolic equations in non-divergence form considered here are called $L_p$-strong solutions in Crandall et al. [8], where the authors treated various types of solutions to nonlinear equations. We note that in [8] they considered equations in a cylindrical domain satisfying a uniform exterior cone condition and the $L_p$-strong solutions are locally in $W^{1,2}_p$, $p > p_0$, where $p_0 > (n + 2)/2$ is a number close to $n + 1$. We also mention that in [32] Lieberman treated a similar problem for a non-divergence elliptic operator in a cylindrical domain with blowup lower-order coefficients in weighed Hölder spaces.

As noted above, in the non-divergence case we assume that the leading coefficients are in a class of VMO functions. The study of elliptic and parabolic equations with VMO coefficients was initiated by Chiarenza et al. [4], and continued in [5] and [1]. The class of leading coefficients in this paper was introduced by Krylov [24] in the context of parabolic equations in the whole space. See also [9, 10, 17, 18] for further development, the results of which we shall use in the proof.
For divergence form equations in time-varying domains, Yong [39] proved the unique existence of weak solutions by using a penalization method when the domain satisfies the exterior measure condition (Definition 1.3) and its cross section at time $t$ is simply connected. He considered equations with a non-zero initial condition, and coefficients and the data $f_0, f_1$ are in some suitable Lebesgue spaces, so that the weak solutions are actually Hölder continuous up to the boundary. Later in [2] Brown et al. obtained a similar solvability result in a parabolic Lipschitz time-varying domain with bounded measurable coefficients and more general square integrable data. One may refer to Lions [33] for existence results in an abstract framework. Recently, Byun and Wang [3] obtained certain $L^\infty$-estimates for equations in time-varying $\delta$-Reifenberg domains. We also refer the reader to [15, 28, 32, 34, 35] and the references therein for other results about boundary value problems in time-varying domains. Of course, the boundary value problem in curvilinear cylinders can be deduced from the estimates in “straight cylinders” by using a change of variables as long as the domains are sufficiently regular.

For the Laplace operator, we recall that a necessary and sufficient condition for the solvability of the corresponding boundary value problem to (DP) is the celebrated Wiener’s criterion. See, for example, [16, 27, 38]. As to the heat equation, an analogous result was established by Evans and Gariepy [12]. We are going to use this result in our proofs below.

To formulate our main results, we introduce some notation, function spaces, and assumptions. A typical point in $\mathbb{R}^{n+1}$ is denoted by $X = (x, t)$, where $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. The parabolic distance between points $X = (x, t)$ and $Y = (y, s)$ in $\mathbb{R}^{n+1}$ is

$$|X - Y| := \max \left\{ |x - y|, |t - s|^{1/2} \right\}.$$  

For any $Y = (y, s) \in \mathbb{R}^{n+1}$ and $r > 0$, we set  

$$B_r(y) := \{ x \in \mathbb{R}^n : |x - y| < r \}$$

and

$$C_r(Y) := B_r(y) \times (s - r^2, s) = \{ X = (x, t) \in \mathbb{R}^{n+1} : |X - Y| < r, \ t < s \}$$

to be a ball in $\mathbb{R}^n$ and a standard parabolic cylinder in $\mathbb{R}^{n+1}$, respectively. We also set  

$$\bar{C}_r(Y) = B_r(y) \times (s - r^2, s + r^2).$$

Let $|\Gamma| := |\Gamma|_{n+1}$ be the $n + 1$-dimensional Lebesgue measure of a set $\Gamma$ in $\mathbb{R}^{n+1}$. For any real number $c$, denote $c_+ := \max(c, 0)$ and $c_- := \min(-c, 0)$. We use the following standard definition for $\partial_\rho Q$.

**Definition 1.1.** Let $Q$ be an open set in $\mathbb{R}^{n+1}$. The parabolic boundary $\partial_\rho Q$ of $Q$ is the set of all points $X_0 = (x_0, t_0) \in \partial Q$ such that there exists a continuous function $x = x(t)$ on an interval $[t_0, t_0 + \delta)$ with values in $\mathbb{R}^n$ satisfying $x(t_0) = x_0$ and $(x(t), t) \in Q$ for all $t \in (t_0, t_0 + \delta)$. Here $x = x(t)$ and $\delta > 0$ depend on $X_0$.

The function $d(X)$ stands for the distance to the parabolic boundary of $Q$, i.e.,

$$d(X) = \text{dist}(X, \partial_\rho Q).$$
We write \( N = N(n, v, \cdots) \) meaning that \( N \) is a constant depending only on the prescribed quantities, such as \( n, v, \) etc., in the parentheses, and the value of \( N \) may vary from line to line.

We assume that the coefficients \( a_{ij} \) are defined on \( \mathbb{R}^{n+1} \) and satisfy the following uniform ellipticity condition: there exists a constant \( \nu \in (0, 1) \) such that

\[
v|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(X) \xi_i \xi_j, \quad \sup_{i,j} |a_{ij}(X)| \leq \nu^{-1}\tag{UE}
\]

for all \( X \in \mathbb{R}^{n+1} \) and \( \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \). In the non-divergence case, we impose the following vanishing mean oscillation (VMO) condition on \( a_{ij} \) with respect to \( x \).

We denote

\[
\omega_a(R) := \sup_{r \in (0, R]} \sup_{(x_0, t_0) \in \mathbb{R}^{d+1}} \int_{C_{r}(t_0, x_0)} \int_{B_{r}(x_0)} |a_{ij}(x, t) - \bar{a}_{ij}(t_0)| \, dy \, dX,
\]

where \( \bar{f} \) is the average of \( f \) over \( C \).

**Assumption 1.2.** We have

\[
\omega_a(R) \to 0 \quad \text{as} \quad R \to 0^+.
\]

Note that, under this assumption, no regularity is required for \( a_{ij} \) as functions of \( t \). For instance, \( \omega_a(R) = 0 \) if \( a_{ij} = a_{ij}(t) \). In the non-divergence case, without loss of generality, we may assume \( a_{ij} = a_{ji} \). However, we do not impose this condition in the divergence case.

For lower-order coefficients, we assume the following:

\[
\int_{Q} (c_0(X) \phi(X) + c_i(X) D_i \phi(X)) \, dX \geq 0, \quad \forall \phi \geq 0, \quad \phi \in C_0^\infty(Q) \tag{1}
\]

in the divergence case, and

\[
c_0(X) \geq 0 \quad \text{in} \quad Q
\]

in the non-divergence case. Note that the above two conditions can be collectively referred to the following unified condition:

\[
L1 \geq 0, \tag{2}
\]

which implies the maximum principle for \( L \). The lower-order coefficients \( b_i \) and \( c_i \) are allowed to blow up near the boundary under a certain growth condition, stated in the theorem below. In light of the example before the proof of Theorem 1.4, this growth condition is optimal.

We impose the following exterior measure condition (or condition (A)) on the domain.
Definition 1.3. An open set \( Q \subset \mathbb{R}^{n+1} \) is said to satisfy the condition (A) if there exists a constant \( \theta_0 \in (0, 1) \) such that for any \( X = (x, r) \in \partial_{\rho}Q \) and \( r > 0 \), we have \( |C_r(X) \setminus Q| > \theta_0 |C_r(X)| \).

We deal simultaneously with both divergence (D) and non-divergence (ND) form. In fact, the first author and Safonov took a unified approach and obtained global a priori Hölder estimates in [7, Corollary 3.6, Theorem 3.10] for elliptic equations without lower-order terms and with locally bounded right-hand side, and in [6, Theorem 3.4, Theorem 4.2] for the parabolic case. For this approach, it is convenient to introduce the solution space \( W(Q) \), which varies according to (D) and (ND). Let \( p_0 \in (\frac{2n+2}{n}, \infty) \) be a fixed constant. We use the notation:

\[
W(Q) := W^{ND}(Q) := W^{2,1}_{p_0, \text{loc}}(Q) \cap C(\hat{Q}) \text{ in the case (ND)},
\]

\[
W(Q) := W^D(Q) := H(Q) \cap C(\hat{Q}) \text{ in the case (D)},
\]

where \( \hat{Q} := Q \cup \partial Q \) and

\[
H(Q) = \{ u \in L_2(Q) \mid D_j u \in L_{2, \text{loc}}(Q), \quad D_j(u) = g_0 + D_j g_i \text{ for some } g_0 \in L_{1, \text{loc}}(Q), g_i \in L_{2, \text{loc}}(Q), i = 1, \ldots, n \}.
\]

Here \( f \in L_{p_0, \text{loc}}(Q), \ p > 0 \) if and only if \( f \in L_p(Q) \) for any open set \( Q' \Subset Q \). Throughout the paper we use \( Q' \Subset Q \) to indicate that \( Q' \subset Q \) and \( \text{dist}(Q', \partial Q) > 0 \).

In both cases, functions \( u \in W \) are continuous on \( \hat{Q} \). In addition, in the non-divergence case (ND), the functions \( u \) have derivatives \( D_j u, \ D_j u, \ D_j u \) in the Lebesgue space \( L_{p_0, \text{loc}}(Q) \). In this case, the relations \( Lu = f \) or \( Lu \leq f \) are understood in the almost everywhere sense in \( Q \). In the divergence case (D), the functions \( u \in W \) have weak (generalized) derivatives \( D_j u \) and \( D_i u \), and \( Lu = (\leq, \geq) f_0 - D_i f_i \) for \( f_0, f_i \in L_{1, \text{loc}}(Q) \) is understood in the following weak sense:

\[
\int_Q (-u D_j \phi + a_{ij} D_i u D_j \phi + b_j D_j u \phi + c_{ij} u D_i \phi + c_0 u \phi - f_0 \phi - f_j D_j \phi) \, dX
\]

\[
= (\leq, \geq) 0
\]

for any nonnegative function \( \phi \in C_c^\infty(Q) \).

Regarding the data, we consider more general function spaces than those in [7] and [6]. We first define, for \( \beta \in (0, 2), \ p > 0 \),

\[
\| f \|_{F_{\beta,p}(Q)} := \sup_{Y_0 \in \partial_{\rho}Q, r > 0} \left( \int_{C_r(Y_0) \cap Q} |d^{2-\beta}(X)f(X)|^p \, dX \right)^{1/p},
\]

and we say \( f \in F_{\beta,p}(Q) \) when \( \| f \|_{F_{\beta,p}(Q)} < \infty \). Note that if \( f \in F_{\beta,p}(Q) \), then \( f \in L_{p, \text{loc}}(Q) \). For some \( \beta \in (0, 1) \), we set

\[
F(Q) = F_{\beta}(Q)
\]

\[
:= \{ f = (f_0, f_1, \ldots, f_n) : f_0 \in F_{\beta,p_0}(Q), f_i \in F_{1+\beta,2p_0}(Q), i = 1, \ldots, n \},
\]

\[
\| f \|_{F(Q)} := \| f_0 \|_{F_{\beta,p_0}(Q)} + \sum_{i=1}^n \| f_i \|_{F_{1+\beta,2p_0}(Q)}
\]

for the divergence case, and \( F(Q) = F_{\beta}(Q) := F_{\beta,p_0}(Q) \) for the non-divergence case.
Now we are ready to state the main results of this paper. Note that, although not specified, the operators throughout the paper in the form (D) or (ND) always satisfy the ellipticity condition (UE).

**Theorem 1.4.** Let \( p_0 \in \left( \frac{n+2}{2}, \infty \right) \), \( L \) be an operator in either divergence (D) or non-divergence form (ND) satisfying \( (UE), (2) \), and \( Q \) be a bounded domain in \( \mathbb{R}^{n+1} \) satisfying the measure condition \( (A) \) with a constant \( \theta_0 \in (0, 1) \). Suppose that \( b_i, c_i, c_0 \in L_{\infty,0}(Q) \) and

\[
|b_i|, |c_i| = o(d^{-1}), \quad d = d(X),
\]

i.e., there exists a nondecreasing function \( \gamma \) on \( [0, \infty) \) such that \( \gamma(d) \to 0 \) as \( d \to 0 \) and

\[
|b_i|, |c_i| \leq d^{-1} \gamma(d).
\]

For the non-divergence case, we further assume that the coefficients \( a_{ij} \) satisfy Assumption 1.2. Let \( \beta_1 := \beta_1(n, \nu, \theta_0) \) be the constant from Proposition 4.1 below.

Then, for any \( f \in F_{\beta}(Q), \beta \in (0, \beta_1) \), there exists a unique solution \( u \in W(Q) \) to the Dirichlet problem \( (DP) \) (the boundary datum 0 is assumed in the pointwise sense).

It is worth noting that from the proofs below the solution \( u \) is globally Hölder continuous in \( \bar{Q} \) and, in the divergence case, \( D_i(u) = g_0 + D_i g_i \) for some \( g_0, g_i \in L_{2,\text{loc}}(Q) \). The corresponding results for elliptic operators are also obtained by following the proofs in this paper.

Here we illustrate the idea in the proof of Theorem 1.4. Our proof relies on the growth lemma, from which we deduce an a priori uniform boundary estimate in Proposition 4.1. To prove the existence result, in the non-divergence case, first we approximate the operator \( L \) by a sequence of operators \( L^k \), each of which becomes the heat operator near the boundary and coincide with the original operator \( L \) in the interior of the domain. We then find a sequence of solutions \( u_k \) corresponding to the operators \( L^k \) by Perron’s method, which requires barrier functions and the solvability in cylindrical domains. We construct certain barrier functions by using the result of Evans and Gariepy [12] mentioned above and an idea in Krylov [22]. Under Assumption 1.2, the \( W^{1,2}_p \) solvability of non-divergence form parabolic equations in cylindrical domains is also available in the literature. By using the a priori boundary estimate and the interior \( W^{1,2}_p \) estimate, we are able to show that along a subsequence \( u_k \) converge locally uniformly to a solution \( u \in W(Q) \) of the original equation. The divergence case is a bit more involved. We additionally take mollifications of the coefficients and data, and rewrite the approximating equations into non-divergence form equations, for which the solvability has already been proved. We then show the convergence of a subsequence of \( u_k \) to a solution \( u \in W(Q) \) of the original equation by again using the a priori boundary estimate and the interior De Giorgi–Nash–Moser estimate.

The remaining part of the paper is organized as follows. We present several auxiliary results in the next section including a version of the maximum principle for solutions in \( W(Q) \). Section 3 is devoted to a growth lemma (Lemma 3.1) and a pointwise estimate (Lemma 3.3). In Section 4, we obtain an a priori boundary estimate which is crucial in our argument, and in Section 5 we complete the proof of the main results. In the Appendix, we show that any domains satisfying the exterior
measure condition also satisfy Wiener’s criterion, which is used in the construction of the barrier functions.

2. Auxiliary Results

This section is devoted to some auxiliary results. Throughout the paper we denote \( \overline{\partial}_p Q \) to be the closure of \( \partial_p Q \) in \( \partial Q \). By the continuity, it is easily seen that the condition (A) is satisfied for any \( X \in \overline{\partial}_p Q \). The next lemma follows from Lemma 2.3 of [6], which reads that \( \partial Q \setminus \overline{\partial}_p Q \) is locally flat.

**Lemma 2.1.** Let \( Q \) be an open set in \( \mathbb{R}^{n+1} \) and \( X_0 = (x_0, t_0) \in \partial Q \setminus \overline{\partial}_p Q \). Then there exists \( r > 0 \) such that \( B_r(x_0) \times \{t = t_0\} \subset \partial Q \setminus \overline{\partial}_p Q \) and

\[
(1) \quad B_r(x_0) \times (t_0, t_0 + r^2) \subset \mathbb{R}^{n+1} \setminus \overline{Q}, \quad B_r(x_0) \times (t_0 - r^2, t_0) \subset Q.
\]

For any interior point \( X \in Q \), we have the following measure condition for sufficiently large \( r \).

**Lemma 2.2.** Assume that \( Q \) satisfies the measure condition (A) with a constant \( \theta_0 \in (0, 1) \). Let \( X = (x, t) \in Q \) and denote \( \rho = d(X) \). Then for any \( r \geq 4\rho/\theta_0 \), we have

\[
|C_r(X) \setminus Q| > \theta_0 2^{n-2}|C_r(X)|. \tag{3}
\]

**Proof.** By a scaling argument, without loss of generality we may assume that \( \rho = 1 \). Let \( Y_0 = (y, s) \) be a point on \( \partial Q \) such that \( d(X, Y_0) = 1 \). In the case when \( s \leq t \), we have \( C_{r/2}(Y_0) \subset C_r(X) \). By using the condition (A) at \( Y_0 \),

\[
|C_r(X) \setminus Q| \geq |C_{r/2}(Y_0) \setminus Q| > \theta_0|C_{r/2}(Y_0)| = \theta_0 2^{n-2}|C_r(X)|, \tag{4}
\]

which gives (3).

Next we consider the case when \( s > t \). We claim that \( C_{r/2}(X) \setminus Q \) is not empty. Otherwise, we would have \( C_{r/2}(X) \subset Q \). Since \( \rho = 1 \) and \( r \geq 4/\theta_0 \),

\[
|C_{r/4}(Y_0) \setminus Q| \leq |C_{r/4}(Y_0) \setminus C_{r/2}(X)| \leq \theta_0|C_{r/4}(Y_0)|,
\]

which contradicts with the condition (A) at \( Y_0 \). Now we fix a point \( Y_1 \in C_{r/2}(X) \setminus Q \). Denote \( Y_\tau = (1 - \tau)X + \tau Y_1 \), \( \tau \in [0, 1] \) to be the line segment connecting \( X \) and \( Y_1 \). Let \( \tau^* \) be the smallest number in \((0, 1)\) such that \( Y_\tau \in \mathbb{R}^{d+1} \setminus Q \). Clearly, we have \( Y_\tau \in \partial Q \) and \( C_{r/2}(Y_\tau) \subset C_r(X) \). By using the condition (A) at \( Y_\tau \), we obtain (4) with \( Y_\tau \) in place of \( Y_0 \). The lemma is proved. \( \Box \)

In the remaining part of this section, we do not impose the condition (A) on \( Q \). The following lemma is useful in approximating \( u_+ \) by smooth functions.

**Lemma 2.3.** Let \( G \in C^\infty(\mathbb{R}) \) and \( u \in W(Q) \). Then \( v := G(u) \in W(Q) \). In addition, assume \( G', G'' \geq 0 \) on \( \mathbb{R} \) and \( G(0) = 0 \). Then, for a function \( f \) defined in \( Q \) such that
$$f \in L_{p_0,\text{loc}}(Q)$$ in the case (ND), or \( f = f_0 - D_i f_i, \) \( f_0 \in L_{1,\text{loc}}(Q), \) \( f_i \in L_{2,\text{loc}}(Q) \) in the case (D), satisfying \( Lu \leq f, \) we have \( Lv \leq F \) in \( Q \) where

$$F := F_0 - D_i F_i,$$

$$F_0 = G'(u)f_0 - G''(u) a_{ij} D_i u D_j u + G''(u) f_i D_i u \in L_{1,\text{loc}}(Q),$$

$$F_i = G'(u) f_i \in L_{2,\text{loc}}(Q),$$

and

$$F := G'(u) f - G''(u) a_{ij} D_i u D_j u \quad \text{for (ND).}$$

In particular, we have \( Lv \leq 0 \) in \( Q \) provided that \( Lu \leq 0 \) in \( Q \).

**Proof.** Clearly, in both cases \( v \in C(\bar{Q}). \) First we consider the non-divergence case. We have

$$D_i v = G'(u) D_i u, \quad v_i = G'(u) u_i, \quad D_{ij} v = G'(u) D_{ij} u + G''(u) D_i D_j u.$$

Since \( G'(u), G''(u) \in C(\bar{Q}) \), we get \( D_i v, v_i \in L_{p_0,\text{loc}}(Q) \). By the parabolic Sobolev embedding theorem (see, for instance, [26, Chap. II]), \( D_i u \in L_{p_0,\text{loc}}(Q) \) with

$$p = p_0(n + 2)/(n + 2 - p_0) > 2p_0,$$

which implies that \( D_{ij} v \in L_{p_0,\text{loc}}(Q) \). Therefore, \( v \in W(Q) \). Since \( G' \geq 0 \) and \( c_0 \geq 0 \), we get

$$Lv = LG(u) = G'(u)Lu - G''(u) a_{ij} D_i u D_j u + c_0 (G(u) - G'(u) u) \leq F \quad \text{in} \quad Q,$$

where we have used the simple inequality

$$G(u) \leq G'(u) u \quad \text{(5)}$$

because \( G(0) = 0 \) and \( G \) is convex.

In the divergence case, by the definition of the space \( H(Q) \), we have \( D_i v = G'(u) D_i u \in L_{2,\text{loc}}(Q) \) and

$$D_i (v) = G'(u) u_i = G'(u) (g_0 + D_i g_i)$$

provided that \( D_i (u) = g_0 + D_i g_i \), for some \( g_0 \in L_{1,\text{loc}}(Q) \) and \( g_i \in L_{2,\text{loc}}(Q) \). Then

$$D_i (v) = \tilde{g}_0 + D_i \tilde{g}_i,$$

where

$$\tilde{g}_0 = G'(u) g_0 - G''(u) g_i D_i u, \quad \tilde{g}_i = G'(u) g_i.$$

It is easily seen that \( \tilde{g}_0 \in L_{1,\text{loc}}(Q) \) and \( \tilde{g}_i \in L_{2,\text{loc}}(Q) \). Therefore, \( v \in W(Q) \). To show the desired inequality, it suffices to prove that for any nonnegative \( \phi \in C_0^\infty(Q), \)

$$\int_Q -G(u) \phi_x + G'(u) (a_{ij} D_i u D_j \phi + b_i D_i u \phi) + c_i G(u) D_i \phi$$

$$+ c_0 G(u) \phi - F_0 \phi - F_i D_i \phi \, dX \leq 0.$$
From (1) and (5), we have
\[ \int_Q c_i G(u)D_i \phi + c_0 G(u) \phi \, dX = \int_Q c_i D_i(G(u)\phi) - c_i G'(u) \phi D_i u + c_0 G(u) \phi \, dX \]
\[ \leq \int_Q c_i D_i(G'(u)u\phi) - c_i G'(u) \phi D_i u + c_0 G'(u)u\phi \, dX \]
\[ = \int_Q c_i u D_i(G'(u)\phi) + c_0 G'(u)u\phi \, dX. \]

Therefore, we only need to show that
\[ \int_Q -G(u)\phi + a_{ij} D_i D_j \psi + b_i \psi D_i u + c_i u D_i \psi + c_0 u \psi - f_0 \psi - f_i D_i \psi \, dX \leq 0, \quad (6) \]
where \( \psi := G'(u)\phi \). We use a standard mollification argument. Define \( \psi^\varepsilon = G'(u^\varepsilon)\phi \), where \( u^\varepsilon \) is the standard mollification of \( u \). By the definition of a weak solution,
\[ \int_Q -u D_i \psi^\varepsilon + a_{ij} D_i D_j \psi^\varepsilon + b_i \psi^\varepsilon D_i u + c_i u D_i \psi^\varepsilon + c_0 u \psi^\varepsilon - f_0 \psi^\varepsilon - f_i D_i \psi^\varepsilon \, dX \leq 0. \quad (7) \]

Let \( Q' = \text{supp} \psi^\varepsilon \subset Q \). Since \( u^\varepsilon \to u \) in \( C(\tilde{Q}) \) and \( D u^\varepsilon \to D u \) in \( L^2(Q') \), we have
\[ \psi^\varepsilon \to \psi \text{ in } C(\tilde{Q}), \quad D \psi^\varepsilon \to D \psi \text{ in } L^2(Q'). \]

From this together with \( u \in W(Q) \), we see that the left-hand side of (7) converges to that of (6) as \( \varepsilon \to 0 \). In particular,
\[ -\int_Q u D_i \psi^\varepsilon \, dX = \int_Q g_{ij} \phi \psi^\varepsilon \, dX - \int_Q \tilde{g}_i D_j \psi^\varepsilon \, dX \]
\[ = \int_Q (G'(u^\varepsilon)g_{ij} - G''(u^\varepsilon)\tilde{g}_i D_j u^\varepsilon) \phi \, dX - \int_Q G'(u^\varepsilon)\tilde{g}_i D_j \phi \, dX \]
\[ \to \int_Q (\tilde{g}_0 \phi - \tilde{\tilde{g}}_i D_i \phi) \, dX = -\int_Q G(u) \phi \, dX. \]

This completes the proof of (6). The second assertion follows from the first one by taking \( f = 0 \) and using \( G'' \geq 0 \) and the ellipticity condition. The lemma is proved. \( \Box \)

The following lemma allows us to reduce our consideration to functions defined on a standard cylinder \( C_r(X_0) \) rather than on a general open set \( Q \subset \mathbb{R}^{n+1} \). For operators without lower-order terms, a similar result is claimed in Theorem 2.6 of [6], the proof of which, however, contains a flaw.

**Lemma 2.4.** Let \( Q \) be an open set in \( \mathbb{R}^{n+1} \) and \( u \in W(Q) \) satisfy \( Lu \leq 0 \) in \( Q \) with an operator \( L \) in the form (ND) or (D). Suppose \( u \leq 0 \) on \( C_r \cap \partial Q \), where \( C_r := C_r(X_0) \), \( X_0 \in Q \). Then for any \( \varepsilon > 0 \), there exists a function \( u_\varepsilon \in W(C_r) \) which vanishes in a neighborhood of \( C_r \cap \partial Q \) and satisfies
\[ u_\varepsilon \geq 0, \quad Lu_\varepsilon \leq 0 \text{ in } C_r, \quad u_\varepsilon \equiv 0 \text{ in } C_r \setminus Q. \]
and

\[(u - \varepsilon)_+(X_0) \leq u_+(X_0), \quad u_\varepsilon \leq u_+ \text{ in } Q.\]

**Proof.** The idea of the proof is to modify \(u\) so that it vanishes near \(\bar{C}_r \cap \partial Q\). For \(\varepsilon > 0\), we choose a convex non-decreasing nonnegative function \(G_\varepsilon \in C^\infty(\mathbb{R})\) such that \((s - \varepsilon)_+ \leq G_\varepsilon(s) \leq (s - \varepsilon/2)_+\) on \(\mathbb{R}\). We first modify \(u\) near \(\bar{C}_r \cap \partial Q\). By Lemma 2.3, we have \(v_\varepsilon := G_\varepsilon(u) \in W(Q)\) and it satisfies

\[v_\varepsilon \geq 0, \quad L v_\varepsilon \leq 0, \quad (u - \varepsilon)_+ \leq v_\varepsilon \leq (u - \varepsilon/2)_+ \text{ in } Q.\]

Since \(u \leq 0\) on \(\bar{C}_r \cap \partial_p Q\), \(v_\varepsilon\) vanishes in a neighborhood of \(\bar{C}_r \cap \partial Q\) in \(\tilde{Q}\). Now we modify \(v_\varepsilon\) near \(\bar{C}_r \cap (\tilde{Q} \setminus \partial_p Q)\). Thanks to Lemma 2.1,

\[\partial Q \setminus \partial_p Q = \bigcup_{s \in \Lambda} S_s \times \{t = t_s\},\]

where \(\Lambda\) is an index set. Here, for each \(s \in \Lambda\), \(S_s\) is an open set in \(\mathbb{R}^n\), \(t_s \in \mathbb{R}\), and

\[\partial S_s \times \{t = t_s\} \subset \partial_p Q.\]

In fact, \(\Lambda\) is at most countable (see Remark 2.5). However, we will not use this in the proof below. Since \(u\) is uniformly continuous in \(\tilde{Q}\) and \(u \leq 0\) on \(\bar{C}_r \cap \partial Q\), we can find \(\delta > 0\) such that \(\varepsilon/2 \in [X \in \bar{C}_r \cap \partial Q | \text{dist}(X, \bar{C}_r \cap \partial Q) < \delta]\). Clearly, the set

\[\{X \in \bar{C}_r \cap \partial Q | \text{dist}(X, \bar{C}_r \cap \partial Q) \geq \delta/2\} \subset \partial Q \setminus \partial_p Q\]

is compact, which has a finite covering by \(S_{s_1} \times \{t = t_{s_1}\}\), where \(s_1, \ldots, s_M \in \Lambda\) and \(M \in \mathbb{N}\). Denote

\[S^d_s = \{x \in \mathbb{R}^n | (x, t_s) \in S_s \times \{t = t_s\} \cap \bar{C}_r, \text{dist}((x, t_s), \bar{C}_r \cap \partial_p Q) \geq \delta/2\}.\]

By Lemma 2.1, there is a small constant \(\delta_1 \in (0, \delta/2)\) such that

\[S^d_s \times \{t_{s_k} - \delta_1^2, t_{s_k}\} \subset Q, \quad k = 1, \ldots, M,\]

and these sets do not intersect each other. We choose a smooth function \(\eta = \eta(x, t)\) in \(Q\) satisfying the following three properties:

(i) \(0 \leq \eta \leq 1\) in \(Q\);
(ii) For each \(k = 1, \ldots, M\), in \(S^d_s \times (t_{s_k} - \delta_1^2, t_{s_k})\) the function \(\eta\) is independent of \(x\), non-increasing in \(t\), and \(\eta = 0\) in \(S^d_s \times (t_{s_k} - \delta_1^2/2, t_{s_k})\);
(iii) \(\eta = 1\) in \(Q \setminus \bigcup_{k=1}^M (S^d_s \times (t_{s_k} - \delta_1^2, t_{s_k}))\), which contains \(X_0\).

Observe that \(D_\eta \eta \leq 0\) and \(D_\eta \eta = 0\) in

\[\{X \in \bar{C}_r \cap Q | \text{dist}(X, \bar{C}_r \cap \partial_p Q) \geq \delta\} \supset \{X \in \bar{C}_r \cap Q | v_\varepsilon(X) > 0\}.\]

Consequently, \(u_\varepsilon := v_\varepsilon \eta\) satisfies \(Lu_\varepsilon = \eta L v_\varepsilon + v_\varepsilon D_\eta \eta \leq 0\) in \(\bar{C}_r \cap Q\). Noting that \(u_\varepsilon\) vanishes in a neighborhood of \(\bar{C}_r \cap \partial Q\) in \(\tilde{Q}\), we can extend \(u_\varepsilon\) to be zero in
It is now straightforward to check that \( u_\varepsilon \) satisfies all the properties in the lemma.

**Remark 2.5.** One example of space-time domains with infinitely many flat portions of the non-parabolic boundary can be obtained by connecting a sequence of shrinking cubes by triangular prisms as in Figure 1 infinitely many times. Note that \( \partial Q \setminus \partial_p Q \) is a countable union of the top surfaces of these cubes and the domain satisfies the exterior measure condition.

Finally, we prove a version of the maximum principle for solutions in \( W(\Omega) \). For later use in the proof of Theorem 1.4, the domain \( Q \) is assumed to be bounded in the lemma and corollary below.

**Lemma 2.6 (Maximum principle).** Let \( u \in W(Q) \) and \( Lu \leq 0 \) in \( Q \), where \( L \) is an operator in either divergence (D) or non-divergence form (ND) satisfying (2) with locally bounded lower order coefficients (i.e., bounded on sets \( Q' \supset P Q \)). For the non-divergence case, we further assume that the coefficients \( a_{ij} \) satisfy Assumption 1.2. Then

\[
\sup_{\Omega} u \leq \sup_{\partial_p Q} u \vee 0.
\]  

(8)

Similarly, if \( Lu \geq 0 \), then

\[
\inf_{\Omega} u \geq \inf_{\partial_p Q} u \wedge 0.
\]

**Proof.** In the divergence case, this is classical. See, for instance, [31, Sec. 6.7].

Next we treat the non-divergence case. It suffices to prove (8). Due to (2), we may assume \( \sup_{\partial_p Q} u \leq 0 \). We first consider the special case when \( Q = \Omega \times (0, T) \) is a cylindrical domain, \( \Omega \) is a bounded \( C^{1,1} \) domain in \( \mathbb{R}^n \), and the coefficients are all bounded. In this case, similar estimates can be found in [25, Theorem 11.8.1] for elliptic operators with continuous coefficients. The proof there is based on the \( W^2 \) estimate for elliptic operators. Since the \( W^{2,1}_p \) estimate for parabolic operators in cylindrical domains with VMO coefficients is available (cf. [18] and [10]), by the same argument one can derive the maximum principle for \( L \). It is standard to extend to the general case by a contradiction argument. Suppose that \( \sup_{Q} u > 0 \). Since \( Q \) is bounded, we may assume that \( Q \subset \{ t > t_0 \} \) for some \( t_0 \in \mathbb{R} \). Then for \( \delta > 0 \) sufficiently small, \( v := u - \delta(t - t_0) \) attains its maximum \( M > 0 \) at a point \( X_0 \in \bar{Q} \setminus \partial Q \).
Take a small $r > 0$ such that $C_r(X_0) \Subset_p Q$ and a smooth function $\eta$ such that $\eta(X_0) = 1$ and $\eta = 0$ on $\partial_p C_r(X_0)$. It is easily seen that for sufficiently small $\varepsilon > 0$, we have

$$L(v + \varepsilon \eta) \leq 0 \text{ in } C_r(X_0), \quad v + \varepsilon \eta \leq M \text{ on } \partial_p C_r(X_0).$$

By the maximum principle in cylindrical domains proved above, we have

$$M + \varepsilon = v(X_0) + \varepsilon \eta(X_0) \leq M;$$

a contradiction. The lemma is proved. \qed

**Corollary 2.7 (Comparison principle).** Let $u, v \in W(Q)$ and, under the same assumptions on $L$ as in Lemma 2.6, $Lu \geq Lv$ in $Q$, $u \geq v$ on $\partial_p Q$. Then $u \geq v$ in $Q$.

**Proof.** We apply Lemma 2.6 to $u - v$. \qed

By the comparison principle, it is immediate to see that a solution to (DP) is unique if it exists.

### 3. Growth Lemma and Pointwise Estimates

The following growth lemma is in the spirit of the book by Landis [29, Section 1.4]. In one form or another, the growth lemma was used in the proofs of Harnack inequalities for solutions to elliptic and parabolic equations. See, for instance, [36] and the references therein.

**Lemma 3.1 (Growth Lemma).** Let $r_0 \in (0, \infty)$ be a constant and $Q \subset \mathbb{R}^{n+1}$ be a bounded domain. Suppose $X_0 = (x_0, t_0) \in \bar{Q} \setminus \partial_p Q$ and

$$|C_r \setminus Q| > \theta |C_r|, \quad 0 < \theta < 1, \quad C_r \ := C_r(X_0) \text{ for some } r \in (0, r_0].$$

Also, let $L$ be the operator in (D) or (ND) satisfying (2) with $|b_j|, |c_i| \leq K_0$.

Then for any function $u \in W(Q_r)$, $Q_r := C_r \cap Q_r$, satisfying

$$Lu \leq 0 \text{ in } Q_r \text{ and } u \leq 0 \text{ on } C_r \cap \partial_p Q_r,$$

we have

$$u(X_0) \leq \beta_0 \sup_{C_r} u +$$

with a constant

$$\beta_0 = \beta_0(n, \nu, \theta, K_0, r_0) \in (0, 1).$$

Assume that $u$ is extended as $u \equiv 0$ on $C_r \setminus Q$, so that the right-hand side of (9) is always well defined.
Proof. First we make some reductions. By considering slightly smaller cylinders $C_{r-\varepsilon}(x_0, t_0 - \varepsilon^2)$ and letting $\varepsilon \to 0$, without loss of generality we may assume that $u \leq 0$ on $\tilde{C} \cap \tilde{C}_p Q$ and $X_0 \in Q$. In particular, if $X_0 \in \tilde{C} \setminus \tilde{C}_p Q$, then by Lemma 2.1 $(x_0, t_0 - \varepsilon^2) \in \tilde{Q}$ with a sufficiently small $\varepsilon > 0$. Next by a scaling and a translation, we further assume $X_0 = (0, 0)$ and $r = 1$. Note that the dependence of $\beta_0$ on $r_0$ comes from the scaling argument. Thanks to Lemma 2.4 with $Q = Q_1$, there exists $u_\varepsilon$ such that

$$(u - \varepsilon)_+(X_0) \leq u_\varepsilon(X_0), \quad u_\varepsilon \leq u_+ \quad \text{in } Q_1$$

$$u_\varepsilon \in W(C_1), \quad u_\varepsilon \geq 0, \quad Lu_\varepsilon \leq 0 \quad \text{in } C_1, \quad u_\varepsilon \equiv 0 \quad \text{in } C_1 \setminus Q.$$

We claim that $u_\varepsilon(X_0) \leq \beta_0 \sup_{C_1}(u_\varepsilon)_+$. Once this is proved, using $u_\varepsilon \leq u_+$ in $Q_1$ and $u_\varepsilon \equiv 0$ in $C_1 \setminus Q$, we obtain $\sup_{C_1}(u_\varepsilon)_+ \leq \sup_{C_1} u_\varepsilon$. Taking $\varepsilon \to 0^+$, we will eventually get (9). Now $u_\varepsilon$ will be denoted by $u$ for the rest of proof. Moreover, since $u \geq 0$ in $C_1$ and $L1 \geq 0$, by considering

$$L - c_0 + D_i c_i = D_i - D_j (a_{ij} D_i) + (b_i - c_i) D_i$$

instead of $L$, we only need to treat the case $c_0 = c_i = 0$.

We first treat the case when $u$ and the coefficients are smooth. In this case, for operators without the lower-order terms, the lemma was proved in [13] under a slightly weaker condition. The general case follows by using an idea in Remark 6.2 there. Indeed, we introduce a new variable $x_0 \in \mathbb{R}$ and define

$$v(x_0, x, t) = e^{x_0 + A t} \left( \sup_{C_1} u - u(x, t) \right), \quad \tilde{C}_r = (-r, r) \times C_r, \quad \tilde{Q} = \mathbb{R} \times Q.$$

Set

$$a_{0i} = 0, \quad a_{0i} = -b_i, \quad i = 1, \ldots, n, \quad a_{00} = A$$

in the divergence case, and

$$a_{0i} = a_{00} = -b_i/2, \quad i = 1, \ldots, n, \quad a_{00} = A$$

in the non-divergence case. Here we choose $A = A(n, v, K_0) > 0$ sufficiently large such that the $(n + 1) \times (n + 1)$ matrix $(a_{ij})_{i,j=0}^n$ satisfies (UE) with a possibly smaller constant $v$. Then it is easy to check that $v$ satisfies

$$L^0 v = v_i - \sum_{i,j=0}^n D_j(a_{ij} D_i v) \geq 0$$

in the divergence case, and

$$L^0 v = v_i - \sum_{i,j=0}^n a_{ij} D_i v \geq 0$$

in the non-divergence case.
in the non-divergence case. Now [13, Corollary 5.4] is applicable to \( v \). Indeed, we can find a small constant \( \delta_0 \in (0, 1/4) \) depending only on \( n \) and \( \theta \) such that

\[
2\delta_0^2 + (1 - \delta_0)^2 \leq 1, \quad |\tilde{C}_{1-\delta_0}(0, -s) \setminus \tilde{Q}| > \theta |\tilde{C}_1|/2, 
\]

where \( s = 1 - (1 - \delta_0)^2 \). Moreover, since \( u = 0 \) in \( C_1 \setminus Q \), we have \( v \geq e^{-1-A} \sup_{C_1} u \) in \( \tilde{C}_{1-\delta_0}(0, -s) \setminus \tilde{Q} \). Thus we get

\[
v \geq \beta e^{-1-A} \sup_{C_1} u \quad \text{in} \quad \tilde{C}_{\delta_0}
\]

for some \( \beta = \beta(n, v, \theta, K_0) \in (0, 1) \), which implies (9) with \( \beta_0 = 1 - \beta e^{-1-A} \). We note that the proof above only requires

\[
Lu \leq 0 \quad \text{in} \quad C_1, \quad u \leq 0 \quad \text{in} \quad C_1 \setminus Q. 
\]  
(10)

Next we remove the smoothness assumption on \( u \) and the coefficients by using an approximation argument. We take a small \( \delta \) depending on \( n \) and \( \theta \) such that \( |C_{1-\delta} \setminus Q| \geq \theta |C_1|/2 \). For \( \varepsilon \in (0, \delta) \), let \( u^{(\varepsilon)} \), \( a_{ij}^{(\varepsilon)} \), and \( b_i^{(\varepsilon)} \) be the standard mollifications of \( u \), \( a_{ij} \), and \( b_i \).

By the Lebesgue lemma,

\[
(a_{ij}^{(\varepsilon)}, b_i^{(\varepsilon)}) \to (a_{ij}, b_i) \quad \text{a.e. in} \quad C_{1-\delta}.
\]

**Non-divergence case:** Clearly, \( Lu \in L_{p_0, \text{loc}}(C_1) \). Since \( a_{ij}^{(\varepsilon)} \), \( b_i^{(\varepsilon)} \), and \( u^{(\varepsilon)} \) are smooth, the Dirichlet problem

\[
L^\varepsilon v^\varepsilon := v_i^{\varepsilon} - a_{ij}^{(\varepsilon)} D_{ij} v^\varepsilon + b_i^{(\varepsilon)} D_i v^\varepsilon = (Lu)^{(\varepsilon)} \quad \text{in} \quad C_{1-\delta}
\]

with the boundary condition \( v^\varepsilon = u^{(\varepsilon)} \) on \( \partial NC_{1-\delta} \) has a unique smooth solution \( v^\varepsilon \) in \( C_{1-\delta} \). Moreover,

\[
L^\varepsilon (v^\varepsilon - u^{(\varepsilon)}) = (Lu)^{(\varepsilon)} - Lu + Lu - L^\varepsilon u^{(\varepsilon)} \to 0 \quad \text{in} \quad L_{p_0}(C_{1-\delta}).
\]

By Lemma 3.3 below and the uniform continuity of \( u \), for any \( h > 0 \) we have \( |v^\varepsilon - u| \leq h \) in \( C_{1-\delta} \) provided that \( \varepsilon \) is sufficiently small. Because

\[
L^\varepsilon (v^\varepsilon - h) = (Lu)^{(\varepsilon)} \leq 0 \quad \text{in} \quad C_{1-\delta}
\]

and

\[
v^\varepsilon - h \leq u = 0 \quad \text{in} \quad C_{1-\delta} \setminus Q,
\]

(10) is satisfied with \( v^\varepsilon - h \) in place of \( u \) and \( C_{1-\delta} \) in place of \( C_1 \). Since \( v^\varepsilon - h \), \( a_{ij}^{(\varepsilon)} \), and \( b_i^{(\varepsilon)} \) are smooth, by the proof above, we get

\[
u - 2h \leq v^\varepsilon - h \leq \beta_0 \sup_{C_{1-\delta}} (v^\varepsilon - h) \leq \beta_0 \sup_{C_1} u \quad \text{in} \quad C_{\delta_0(1-\delta)}.
\]

Letting \( h \to 0 \) and \( \delta \to 0 \) gives (9).
Divergence case: Let
\[ D_i(u) - D_i(a_{ij}D_ju) + b_iD_iu =: f_0 - D_if_i. \]
By the property of convolution, we have
\[ D_iu^{(e)} - D_i((a_{ij}D_ju)^{(e)}) + (b_iD_iu)^{(e)} = f_0^{(e)} - D_if_i^{(e)}. \]
Similar to the non-divergence case, let \( v^e \) be the weak solution of
\[ L^e v^e := D_i v^e - D_i \left( a_{ij}^{(e)} D_j v^e \right) + b_i^{(e)} D_i v^e = f_0^{(e)} - D_if_i^{(e)} \quad \text{in } C_{1-\delta} \]
with the boundary condition \( v^e = u^{(e)} \) on \( \partial_p C_{1-\delta} \). Note that
\[ L^e u^{(e)} = D_i \left( (a_{ij}D_ju)^{(e)} - (b_iD_iu)^{(e)} + f_0^{(e)} - D_if_i^{(e)} \right) = D_j \left( a_{ij}^{(e)} D_j u^{(e)} \right) + b_i^{(e)} D_i u^{(e)}. \]
Thus,
\[ L^e (v^e - u^{(e)}) = D_j \left( a_{ij}^{(e)} D_j u^{(e)} - (a_{ij}D_ju)^{(e)} \right) + (b_iD_iu)^{(e)} - b_i^{(e)} D_i u^{(e)}. \]
It is easy to see that
\[ a_{ij}^{(e)} D_j u^{(e)} - (a_{ij}D_ju)^{(e)} \rightarrow 0, \quad (b_iD_iu)^{(e)} - b_i^{(e)} D_i u^{(e)} \rightarrow 0 \]
in \( L_2(C_{1-\delta}) \). Due to the energy inequality, we have \( v^e \rightarrow u \) in \( L_2(C_{1-\delta}) \), and, after extracting a subsequence, \( v^{e_k} \rightarrow u \) (a.e.) in \( C_{1-\delta} \) for a decreasing sequence \( \epsilon_k \searrow 0 \).
Fix a small \( h > 0 \), denote
\[ S_{\epsilon,h,1-\delta} := \{ v^\epsilon > h, u \leq 0 \} \cap C_{1-\delta}. \]
By Chebyshev’s inequality, as \( \epsilon \rightarrow 0 \),
\[ |S_{\epsilon,h,1-\delta}| \leq \frac{1}{h^2} \int_{C_{1-\delta}} (v^\epsilon - u)^2 \, dX \rightarrow 0. \]
Thus, for \( \epsilon \) sufficiently small, we have
\[ |\{ v^\epsilon \leq h \} \cap C_{1-\delta}| \geq |\{ u \leq 0 \} \cap C_{1-\delta}| - |S_{\epsilon,h,1-\delta}| > \frac{\theta}{4}|C_{1-\delta}|. \]
Since \( v^\epsilon, (a_{ij})^{(e)}, \) and \( (b_i)^{(e)} \) are smooth, and \( (Lu)^{(e)} \leq 0 \) in \( C_{1-\delta} \), from the proof above and the maximum principle, we reach
\[ \sup_{C_{\epsilon,h,1-\delta}} (v^\epsilon - h) \leq \beta_0 \sup_{C_{1-\delta}} (v^\epsilon - h) \leq \beta_0 \sup_{C_{1-\delta}} u^{(e)} - h \leq \beta_0 \sup_{C_1} u. \]
Bearing in mind that \( v^{e_k} \rightarrow u \) (a.e.) in \( C_{1-\delta} \) as \( k \rightarrow \infty \) and \( u \) is continuous, we get
\[ \sup_{C_{\epsilon,h,1-\delta}} (u - h) \leq \beta_0 \sup_{C_1} u. \]
Since \( h \) is an arbitrary positive constant, we obtain the desired estimate. \( \square \)
Remark 3.2. If we additionally assume \( u = 0 \) on \( \partial \varrho \Omega \), then under the conditions of Lemma 3.1, from \( u(X_0) \leq \beta_0 \sup_{C_r} u_+ \) one can derive
\[
\sup_{Q \in \hat{C}_{\varrho_0}(X_0)} u \leq \beta_0 \sup_{Q \in C_r(X_0)} u_+
\]
with constants \( \delta_0 = \delta_0(n, \varrho) \in (0, 1/4) \), \( \beta_0 = \beta_0(n, \varrho, \varrho/2, K_0, r_0) \in (0, 1) \).

Moreover, in this case \( X_0 \) can be any point on \( \hat{Q} \). For this, note that the measure condition
\[
|C_r(X) \setminus Q| > \frac{\theta}{2} |C_r(X)|
\]
holds for any \( X \in \hat{C}_{\varrho_0}(X_0) \) with small \( \delta_0 \) depending on \( n \) and \( \varrho \).

Lemma 3.3 (Pointwise estimate). Let \( p_0 \in (\frac{n+2}{2}, \infty) \), \( R_1 > 0 \), and \( L \) be an operator (in the form (D) or (ND)) defined on a cylinder \( C_R := C_R(X_0) \subset \mathbb{R}^{n+1} \), \( R \leq R_1 \), which satisfies (2) and \( |b|, |c|, |\omega_n| \leq K_0 \). In the non-divergence case we further assume that \( a_{ij} \) satisfy Assumption 1.2. Then for any \( f = f_0 - D_i f_i \), where \( f_0 \in L_{p_0}(C_R) \) and \( f_i \in L_{2p_0}(C_R) \) for the divergence case, and \( f \in L_{p_0}(C_R) \) for the non-divergence case, there exists a unique solution \( w \in W(C_R) \) to the equation
\[
Lw = f \text{ in } C_R, \quad w = 0 \text{ on } \partial \varrho C_R.
\]

Moreover, \( w \) is Hölder continuous in \( C_R \) and
\[
|w| \leq N_1 \left( \|f_0\|_{L_{p_0}(C_R)} + \|f_i\|_{L_{2p_0}(C_R)} \right) \quad \text{for the divergence case,}
\]
where
\[
N_1 = N_1(n, \nu, K_0, R_1, p_0),
\]
and
\[
|w| \leq N_2 \|f\|_{L_{p_0}(C_R)} \quad \text{for the non-divergence case,}
\]
where \( N_2 = N_2(n, \nu, K_0, R_1, p_0, \omega_n) \).

Proof. In the divergence case, the unique existence of \( w \) in \( H(C_R) \) is classical by noting that \( f_i \in L_2(C_R) \) and \( f_0 \in L_{2(n+2)/(n+4)}(C_R) \). The Hölder continuity and (11) are due to the parabolic De Giorgi–Nash–Moser estimate. See, for instance, [31, Chap. 6]. In the non-divergence case, for operators without the lower-order terms, the unique solvability was first established in [1] under the assumption that the leading coefficients are VMO with respect to both \( x \) and \( t \). In the general case, the unique solvability in \( W^{1,2}_{p_0}(C_R) \) can be found in [10]; see Theorem 6 and Remark 1 there. Recall that \( p_0 > (n+2)/2 \). The Hölder continuity and (12) then follow from the parabolic Sobolev embedding theorem. The lemma is proved.

We remark that if \( p_0 \geq n+1 \), by the parabolic Alexandrov–Bakelman–Pucci estimate (cf. [8, 20, 37]), the constant \( N_2 \) can be taken to be independent of the regularity of \( a_{ij} \) and the \( L_{p_0} \) norm over \( C_R \) can be replaced by the \( L_{p_0} \) norm over the
so called upper contact set. Moreover, there exists $\tilde{p} \in ((n + 2)/2, n + 1)$ such that the estimate (12) holds true for any $p_0 > \tilde{p}$ with $N_2$ independent of the regularity of $a_{ij}$.

### 4. Weighted Uniform Estimate

The *a priori* estimate in Proposition 4.1 below is a key ingredient in the proof of the existence of solutions. The idea of the proof is based on [7, Theorem 3.5]. Our case is much more involved because the coefficients of lower-order terms may blow up near the boundary. We use a re-scaling argument, for which we introduce some additional notation. For any $\rho > 0$, we denote

$$u^\rho(X) = u(\rho x, \rho^2 t) \quad \text{in} \quad \rho^{-1}Q := \{(\rho^{-1}x, \rho^{-2}t) \mid (x, t) \in Q\}.$$  

For the divergence case (D), by a simple scaling,

$$L^\rho u^\rho := D_i(a_{ij}^\rho D_j u^\rho) + \rho b_i^\rho D_i u^\rho - \rho D_i(c_i^\rho u^\rho) + \rho^2 c_0^\rho u^\rho = \rho^2 f_0^\rho - \rho D_i f_i^\rho,$$

where $a_{ij}^\rho(X) = a_{ij}(\rho x, \rho^2 t)$, etc. For the non-divergence case (ND), we have

$$L^\rho u^\rho := u^\rho - a_{ij}^\rho D_i D_j u^\rho + \rho b_i^\rho D_i u^\rho + \rho^2 c_i^\rho u^\rho = \rho^2 f^\rho.$$  

Denote

$$d_\rho(X) = \text{dist}(X, \hat{c}_\rho(\rho^{-1}Q)) = \rho^{-1}d(\rho x, \rho^2 t).$$  

Next we modify the operators near the parabolic boundary. For any $\rho > 0$, take a smooth non-negative function $\eta^\rho$ such that $\eta^\rho = 0$ in $\{d < \rho/2\}$, $\eta^\rho = 1$ in $\{d > \rho\}$, and its modulus of continuity $\omega_\eta(r) \leq Nr/\rho$. For $\varepsilon \in (0, 1)$, denote

$$a_{ij}^{\rho, \varepsilon} = \eta^{\rho, \varepsilon}(\rho x, \rho^2 t)a_{ij}^\rho + (1 - \eta^{\rho, \varepsilon}(\rho x, \rho^2 t)) \delta_{ij},$$

which satisfy (UE), and the modulus of continuity $\omega_{a_{ij}^{\rho, \varepsilon}}$ has an upper bound

$$\omega_{a_{ij}^{\rho, \varepsilon}}(r) \leq N(\omega_{a^\rho}(\rho r) + r/\varepsilon).$$

Note that this is independent of $\rho$ for $\rho \in (0, 1)$. Let

$$b_{ij}^{\rho, \varepsilon} = \eta^{\rho, \varepsilon}(\rho x, \rho^2 t)b_{ij}^\rho, \quad c_i^{\rho, \varepsilon} = \eta^{\rho, \varepsilon}(\rho x, \rho^2 t)c_i^\rho,$$

$$c_0^{\rho, \varepsilon} = \eta^{\rho, \varepsilon}(\rho x, \rho^2 t)c_0^\rho + D_i\eta^{\rho, \varepsilon}(\rho x, \rho^2 t)c_i^\rho, \quad f^{\rho, \varepsilon} = \eta^{\rho, \varepsilon}(\rho x, \rho^2 t)f^\rho.$$  

Then $b_{ij}^{\rho, \varepsilon}, c_i^{\rho, \varepsilon}, c_0^{\rho, \varepsilon} \in L_\infty(\mathbb{R}^{n+1})$. We define an operator $L^{\rho, \varepsilon}$ on $\mathbb{R}^{n+1}$:

$$L^{\rho, \varepsilon} w := D_j(a_{ij}^{\rho, \varepsilon} D_j w) + \rho b_{ij}^{\rho, \varepsilon} D_i w - \rho D_j(c_i^{\rho, \varepsilon} w) + \rho^2 c_0^{\rho, \varepsilon} w$$

in the divergence case, and

$$L^{\rho, \varepsilon} w := w_i - a_{ij}^{\rho, \varepsilon} D_j w + \rho b_{ij}^{\rho, \varepsilon} D_i w + \rho^2 c_0^{\rho, \varepsilon} w$$

in the non-divergence case. It is easily seen that $L^{\rho, \varepsilon} 1 \geq 0$. 

Proposition 4.1. Let \( p_0 \in (\frac{\alpha+2}{2}, \infty) \) be a constant, \( Q \) be a bounded domain in \( \mathbb{R}^{n+1} \) satisfying the measure condition (A) with a constant \( \theta_0 > 0 \), and \( L \) be an operator in the form of (D) or (ND) satisfying the same assumptions for the coefficients \( a_{ij}, b_i, c_i \), and \( c_0 \) as in Theorem 1.4. Then there exists a constant \( \beta_1 = \beta_1(n, v, \theta_0) \in (0, 1] \) such that the following hold true:

(i) For \( \beta \in (0, \beta_1) \) and an open subset \( Q' \Subset Q \), if \( u \in W(Q') \), \( f \in F(Q) \), and

\[
Lu = f \text{ in } Q', \quad u = 0 \text{ on } \partial_p Q',
\]

then

\[
\sup_{Q'} d^{-\beta} u \leq N \|f\|_{F(Q)},
\]

where \( d = d(X) \), \( N = N(n, v, \theta_0, \gamma, \text{diam}(Q), \beta, p_0) \), which also depends on \( \omega^*_\alpha \) in the non-divergence case. Here

\[
\omega^*_\alpha = \max \left\{ \sup_{\rho \in (0, p_0)} \omega^{4\law}_{\text{diam}(Q), \beta, p_0} \right\},
\]

where \( \varepsilon \in (0, 1) \) depending only on \( n, v, \theta_0, \) and \( \beta \), and \( \rho_0 \in (0, 1) \) depending only on the same parameters and \( \gamma \).

(ii) The same estimate holds true for \( Q' = Q \) if, in addition, we assume that \( f \) vanishes near \( \partial_p Q \), and \( b_i \) and \( c_i \) are bounded (for instance, by 1) near \( \partial_p Q \).

Proof. We first prove the assertion (i). We assume that the set \( Q' \cap \{ u > 0 \} \) is nonempty; otherwise there is nothing to prove. The assumption \( \text{dist} (Q', \partial_p Q) > 0 \) allows us to claim that \( d^{-\beta} u \in C(Q') \), and hence there is a point \( X_0 \in Q' \) at which

\[
d^{-\beta}(X_0)u(X_0) = M := \sup_{Q'} d^{-\beta} u > 0.
\]

Set \( \rho := d(X_0) := \text{dist}(X_0, \partial_p Q) > 0 \), and choose a point \( Y_0 \in \partial_p Q \) for which \( d(X_0) = |X_0 - Y_0| \). Without loss of generality, we assume \( X_0 = (0, 0) \) and \( Y_0 = (y_0, s_0) \). The cylinder, for instance, \( C_{r} \) in the proof below is to be understood as \( C_{r}(0, 0) \).

Set \( R := 4/\theta_0 \) and \( \beta_1 := -\log_{2R} \beta_0 > 0 \), where

\[
\beta_0 = \beta_0(n, v, \theta_0/2^{n+2}, 1, 4/\theta_0)
\]

is the constant from Lemma 3.1. Since \( 0 < \beta < \beta_1 \) and \( R > 1 \), we have

\[
1 - (R + 1)^{\beta} \beta_0 > 1 - (R + 1)^{\beta - \beta_1} > 0.
\]

Now we take \( \varepsilon \in (0, 1) \), depending only on \( n, v, \theta_0, \) and \( \beta \), such that

\[
\varepsilon^\beta + (R + 1)^\beta \beta_0 < 1.
\]
Denote $\rho^{-1}Q := \{(\rho^{-1}x, \rho^{-2}t) | (x, t) \in Q\}$ and $Q_\rho = C_\rho \cap (\rho^{-1}Q)$. Observe that
\[
|\rho b_\nu^\rho(X)|, \ |\rho c_\nu^\rho(X)| \leq \rho o \left(d^{-1}(\rho x, \rho^2 t)\right) = \rho o(\rho^{-1}d^{-1}_\rho(x, t)) \leq \varepsilon^{-1}/\gamma ((R + 1)\rho)
\]
in $\{X \in Q_\rho | d_\rho(X) \geq \varepsilon\}$, where we used
\[
Q_\rho \subset \{X \in \rho^{-1}Q | d_\rho(X) < R + 1\}.
\]
We choose $\rho_0 \in (0, 1)$ depending only on $n, \nu, \theta_0, \beta$, and $\gamma$ such that
\[
\varepsilon^{-1}/\gamma ((R + 1)\rho_0) \leq 1.
\]
Next, we consider two cases: (i) $\rho < \rho_0$ and (ii) $\rho \geq \rho_0$.

Case 1: $\rho < \rho_0$. We note that $f^{\rho, \varepsilon}_0 \in L^{p, \varepsilon}_\rho(C_\rho)$ in the non-divergence case, and $f^{\rho, \varepsilon}_1 \in L^{p, \varepsilon}_\rho(C_\rho)$, $f^{\rho, \varepsilon}_2 \in L^{2, \rho}_\rho(C_\rho)$ in the divergence case. By Lemma 3.3, there exists a unique solution $w \in W(C_\rho)$ to $L^{p, \varepsilon}w = \rho^2 f^{\rho, \varepsilon}$ in the non-divergence case (or $L^{p, \varepsilon}w = \rho^2 f^{\rho, \varepsilon} - \rho D_i(f^{\rho, \varepsilon}_i)$ in the divergence case) with the zero boundary condition $w = 0$ on $\partial_\rho C_\rho$. Moreover, $w$ satisfies the property (11) (or (12)).

Consider the function
\[
v^\rho(X) := u^\rho(X) - (\varepsilon\rho)^\beta M - w(X) - \sup_{C_\rho} |w| \text{ on } \rho^{-1}Q \cap C_\rho,
\]
which may be extended continuously outside $\rho^{-1}Q \cap C_\rho$, and define
\[
V := \rho^{-1}Q \cap \{d_\rho > \varepsilon\} \cap \{v^\rho > 0\}.
\]
We see that
\[
M = \sup_{X \in \rho^{-1}Q} d^{-\beta}_\rho(X)u(X) = \rho^{-\beta} \sup_{X \in \rho^{-1}Q} d^{-\beta}_\rho(X)u^\rho(X),
\]
which implies that
\[
u^\rho = d^\beta_\rho d^{-\beta}_\rho u^\rho \leq d^\beta_\rho \rho^\beta M \leq (\varepsilon\rho)^\beta M, \text{ when } d_\rho \leq \varepsilon.
\]
We first assume $v^\rho(0) > 0$. By Lemma 2.2 applied to $\rho^{-1}Q$, we have
\[
|C_\rho \setminus V| \geq |C_\rho \setminus \rho^{-1}Q| \geq 2^{-n-2}\theta_\rho|C_\rho|.
\]
From $u^\rho = 0$ on $\partial_\rho (\rho^{-1}Q)$ and $u^\rho \leq (\varepsilon\rho)^\beta M$ on $\{d_\rho \leq \varepsilon\}$, it follows $v^\rho = 0$ on $C_\rho \cap \partial_\rho V$. Moreover, $0 \in \tilde{V} \setminus \partial_\rho V$ because $0 \in \rho^{-1}Q \cap \{d_\rho > \varepsilon\} \cap \{u^\rho > 0\}$ and $0 \notin \partial_\rho (\rho^{-1}Q)$. We also observe that $L^{p, \varepsilon}u^\rho \leq 0$ in $V$, where we use the condition (2), the definition of $w$, and fact that $L^{p, \varepsilon}$ coincides with $L^p$ in $\{d_\rho \geq \varepsilon\} \cap \rho^{-1}Q$. Then by Lemma 3.1 with $Q = V$, we obtain
\[
v^\rho(0) \leq \beta_0 \sup_{V \cap C_\rho} v^\rho \leq \beta_0 \sup_{V \cap C_\rho} u^\rho,
\]
where $\beta_0 = \beta_0(n, \nu, \theta_0/2n+2, 1, R) \in (0, 1)$. Since $d_\rho(X) \leq R + 1$ on $C_\rho$, we see that $u^\rho \leq ((R + 1)\rho)^\beta M$ on $V \cap C_\rho$, which implies $v^\rho(0) \leq \beta_0 ((R + 1)\rho)^\beta M$. Of course,
the last estimate also holds in the case \( v^0 (0) \leq 0 \). From this estimate, together with (13) and (15), it follows that

\[
\rho^\beta M = u(0) = u^0 (0) \leq \rho^\beta \left[ e^\beta + (R + 1)^\beta \beta_0 \right] M + 2 \sup_{C\varepsilon} |w|.
\]

By the property (11) or (12) of the function \( w \) on \( C\varepsilon \) we have

\[
|w| \leq N_1 \| \rho^2 \tilde{f}_0 \|_{L^p (C\varepsilon)} + N_1 \sum_{i=1}^n \| \rho \tilde{f}_i^\rho \|_{L^2 (C\varepsilon)}
\]

in the divergence case, where \( N_1 = N_1 (n, \nu, \theta_0, \rho_0) > 0 \), and

\[
|w| \leq N_1 \| \rho^2 \tilde{f}_0 \|_{L^p (C\varepsilon)}
\]

in the non-divergence case, where \( N_1 = N_1 (n, \nu, \theta_0, \rho_0, \omega_{\mu, \nu}) > 0 \).

A direct computation shows that

\[
\| \rho^2 \tilde{f}_0 \|_{L^p (C\varepsilon)} = \rho^2 \left( \int_{Q_{2r} \cap \{ d_{e, \rho} \geq \varepsilon \}} |f^\rho (Y)|^p \, dY \right)^{1/p_0}
\]

\[
= \rho^2 \left( \int_{Q_{2r} \cap \{ d_{e, \rho} \geq \varepsilon \}} |f (X)|^p \, dX \right)^{1/p_0}
\]

\[
\leq \rho^2 \left( \int_{Q_{2r} \cap \{ d_{e, \rho} \geq \varepsilon \}} |f (X)|^p \, dX \right)^{1/p_0}
\]

\[
\leq \rho^2 \left( \int_{Q_{2r} \cap \{ d_{e, \rho} \geq \varepsilon \}} |f (X)|^p \, dX \right)^{1/p_0}
\]

Similarly, in the divergence case,

\[
\| \rho^2 f_0 \|_{L^p (Q_0)} \leq \rho^\beta \| \tilde{f}_0 \|_{L^2 (Q_0)} (2|C_{(R+1)p}|)^{1/p_0},
\]

\[
\| \rho f_i^\rho \|_{L^2 (Q_0)} \leq \rho^\beta \| \tilde{f}_i \|_{L^2 (Q_0)} (2|C_{(R+1)p}|)^{1/2p_0}.
\]

Therefore,

\[
M \leq \left[ e^\beta + (R + 1)^\beta \beta_0 \right] M + N_1 \| \rho^2 \tilde{f}_0 \|_{L^1 (Q)}.
\]

From this inequality and (14), it follows that \( M \leq N \| \tilde{f} \|_{L^p (Q)} \), where \( N = N(n, \nu, \theta_0, \beta, \rho_0) \), which also depends on \( \omega_{\mu, \nu} \) in the non-divergence case.

Case 2: \( \rho \geq \rho_0 \). In this case, it suffices to bound \( |u(0)| \) in terms of \( f \). Let \( R_0 = \text{diam}(Q) \) and

\[
Q' = Q' \cap \{ X \in Q : \text{dist}(X, \partial \rho Q) > \rho/4 \}.
\]

In the non-divergence case, let \( w \) be the unique solution in \( W(C_{R_0}) \) of

\[
L^1_{\rho_0/4} w = f^1_{\rho_0/4} \text{ in } C_{R_0}, \quad w = 0 \text{ on } \partial \rho C_{R_0}.
\]
By Lemma 3.3, we have

\[ |w| \leq N \|f\|_{L^p_0(Q^o)}, \tag{16} \]

where \( N = N(n, \nu, \theta_0, \gamma, \beta, \rho_0, \text{diam}(Q), \omega_{n, n/4}) \).

Because \( L^{1, n/4} \) coincides with \( L \) in \( Q^o \),

\[ L \left( \sup_{\partial p Q^o} u^+ + w + N \|f\|_{L^p_0(Q^o)} \right) \geq f \text{ in } Q^o. \]

Moreover,

\[ \sup_{\partial p Q^o} u^+ + w + N \|f\|_{L^p_0(Q^o)} \geq u \text{ on } \partial_p Q^o. \]

By the comparison principle Corollary 2.7 and (16), we have

\[ u(0) \leq \sup_{\partial p Q^o} u^+ + N \|f\|_{L^p_0(Q^o)}. \]

Since \( u = 0 \) on \( \partial_p Q^o \), we observe that

\[ \sup_{\partial p Q^o} u^+ \leq (\rho/4)^\beta M. \]

Thus,

\[ M = \rho^{-\beta} u(0) \leq 4^{-\beta} M + N \rho^{-\beta} \|f\|_{L^p_0(Q^o)} \]
\[ \leq 4^{-\beta} M + N \rho^{-\beta} \left( \frac{n}{4} \right)^{\beta/2} \|f\|_{L^p_0} |Q|^{1/n} \]
\[ \leq 4^{-\beta} M + N \rho^{-2} \|f\|_{L^p_0}, \]

which gives the desired estimate.

The divergence case is similar. This finishes the proof of the first assertion.

For the assertion (ii), we find \( r \in (0, 1) \) such that \( f = 0 \) and \( |b|, |c| \leq 1 \) on \( \{X \in Q : d(X) \leq r\} \). Then we take \( \beta_1 := \log_{\rho_{01}} \rho_{0} \), where \( \beta_0 = \beta_0(n, \nu, \theta_0, 1, 1) \) and \( \delta_0 = \delta_0(n, \theta_0) \) are the constants from Remark 3.2. We observe that by Remark 3.2 and an iteration argument, we have

\[ |u(X)| \leq N(d(X))^\beta_1 \quad \forall X \in \bar{Q}. \]

Thus, for any \( \beta \in (0, \beta_1) \) and \( X_0 \in \partial_p Q \),

\[ \lim_{Q \ni X \to X_0} d^{-\beta} u(X) = 0, \]

and \( d^{-\beta} u \in C(\bar{Q}) \). Then we argue as in the proof of the first assertion with \( Q \) in place of \( Q^o \) with the smaller \( \beta_1 \) between the above \( \beta_1 \) and the one from the proof of the assertion (i). The lemma is proved.
The following 1-D example suggests that the growth condition on $b_i$ and $c_i$ near the boundary in the proposition above is in some sense optimal even in the elliptic case.\(^1\)

**Example 4.2.** We take a nonnegative smooth function $\eta$ on $\mathbb{R}$ such that $\eta(x) = 1$ for $x \leq 1/3$ and $\eta = 0$ for $x \geq 1/2$. Consider $u(x) := -\eta(x) (\ln x)^{-1}$. Clearly, $u \in C([0, 1/2]) \cap C^\infty((0, 1/2))$ and $u(0) = u(1/2) = 0$. By a direct computation, it is easily seen that $u$ satisfies

$$u'' + bu' = f \quad \text{in } (0, 1/2),$$

where $b(x) := x^{-1} \left(1 + 2(\ln x)^{-1}\right)$ for $x \in (0, 1/3)$ and $f \in L_\infty([0, 1])$. Note that $|b| \sim x^{-1}$ near $0$ and $\lim_{x \downarrow 0} d^{-\beta} u = \infty$ for any $\beta > 0$.

5. **Proofs of the Main Theorems**

Now we are ready to prove the main results of the paper, Theorem 1.4.

*Proof of Theorem 1.4.* We first prove the non-divergence case. Recall the notation introduced at the beginning of the previous section. For $k = 1, 2, \ldots$, denote

$$Q_k = \{ X \in Q \mid \text{dist}(X, \partial_p Q) > 1/k \},$$

$L_k = L^{1,1/k}$, and $f_k = f^{1,1/k}$. We shall find a unique solution $u_k$ to the equation

$$L_k u_k = f_k \quad \text{in } Q, \quad u_k|_{\partial_p Q} = 0 \quad (17)$$

using Perron’s method. Since the $W^{2,1}_p$-solvability in cylindrical domains for parabolic operators with VMO coefficients is available (cf. [10]) and $f^k$ are bounded in $L^p_0(Q)$, in order to find a unique solution $u_k \in W(Q)$ to the equation (17) using Perron’s method, it suffices to construct a barrier function $\psi_k \in W(Q)$ satisfying

$$L_k \psi_k \geq 1 \quad \text{in } Q, \quad \psi_k = 0 \quad \text{on } \partial_p Q. \quad (18)$$

For this purpose, we use an idea in [22]. Let $w \in W(Q)$ be the solution to the heat equation

$$w_t - \Delta w = 1 \quad \text{in } Q, \quad w = 0 \quad \text{on } \partial_p Q. \quad (19)$$

The existence of such $w$ is due to [12] and the fact that any domain satisfying the exterior measure condition also satisfies Wiener’s criterion. See Appendix A. By the maximum principle, $w$ is strictly positive in $Q$. Denote $R = \text{diam}(Q)$ and without loss of generality, we assume $Q \subset C_R$. Let $v = \cosh(\mu R) - \cosh(\mu |x|)$. Then $v \geq 0$ in $Q$, and a straightforward calculation shows that

$$L^1 v \geq 1 \quad \text{in } Q \quad (20)$$

\(^1\)The authors thank an anonymous referee who pointed out to us this example.
for $\mu$ sufficiently large. Denote $F(x, y) := \min\{x, y\}$ and let $F^{(\varepsilon)}$ be the standard mollification of $F$. Clearly, for any $\varepsilon > 0$, $F^{(\varepsilon)}$ is a smooth function and

$$D_x F^{(\varepsilon)} \geq 0, \quad D_y F^{(\varepsilon)} \geq 0, \quad D_x F^{(\varepsilon)} + D_y F^{(\varepsilon)} = 1, \quad D^2 F^{(\varepsilon)} \leq 0. \quad (21)$$

Now we choose $\lambda$ sufficiently large such that $\lambda w \geq v$ in $Q_{2\lambda}$, and let

$$\psi_k = F^{(\lambda)}((1 + \lambda)w, v).$$

Then using the definition of $L^k$, (19), (20), and (21), it is easily seen that $\psi_k$ satisfies (18) if $\varepsilon$ is sufficiently small.

By Proposition 4.1 (ii), there exists $N = N(n, v, \theta_0, \gamma, \text{diam}(Q), \beta, \rho_0, \omega_{u_k})$ such that, for any $\beta \in (0, \beta_1)$,

$$\sup_Q d^{-\beta} u_k \leq N\|f\|_{L^p(Q)},$$

where $\beta_1 = \beta_1(n, v, \theta_0) \in (0, 1]$ is the constant from Proposition 4.1. Note that by the definition of $L^k$ the choice of $\beta_1$ can be made independent of $k \in \mathbb{N}$. By using the simple equality

$$\chi_2(\chi_1A + (1 - \chi_1)B) + (1 - \chi_2)B = \chi_1\chi_2A + (1 - \chi_1\chi_2)B$$

and the definition of $\omega_{u_k}^\ast$, we see that $\omega_{u_k}^\ast$ has an upper bound

$$\omega_{u_k}^\ast(r) \leq N(\omega_{u_k}(r) + r/\varepsilon + r/\rho_0),$$

which is independent of $k$. Here $\varepsilon$ and $\rho_0$ are constants from Proposition 4.1. With $-u_k$ in place of $u_k$, we obtain

$$\inf_Q d^{-\beta} u_k \geq -N\|f\|_{L^p(Q)}.$$ 

Therefore, for any $X \in Q$, we have

$$|u_k| \leq Nd^{\beta}\|f\|_{L^p(Q)} \quad (22)$$

where $N$ is independent of $k$.

For an arbitrary parabolic cylinder $C \subset Q$, thanks to the local $W^{1,2}_p$ estimate and the Sobolev embedding theorem, each $u_k$ is Hölder continuous in $C$ with a uniform Hölder norm. Then by applying the Arzela–Ascoli theorem to the sequence $\{u_k\}$ on each $Q_m, m \in \mathbb{N}$, and using Cantor’s diagonal argument, we find a subsequence, still denoted by $\{u_k\}$, converging locally uniformly to a function $u$ in $Q$. Moreover, by the inequality (22), $u \in C(Q)$ with $u = 0$ on $\partial_p Q$. Now we show that $Lu = f$ in $C$. One can find another parabolic cylinder $C'$ such that $C \subset C' \subset Q$. Then for sufficiently large $k$, we have

$$L(u_{k+1} - u_k) = 0$$

in $C'$. By the interior $W^{1,2}_p$ estimate, we have

$$\|u_{k+1} - u_k\|_{W^{1,2}_p(C')} \leq N\|u_{k+1} - u_k\|_{L^p(C')}.$$
From this inequality and the fact that $u_k$ converges uniformly to $u$ in $C$, it follows that $u \in W^{1,2}_{p_0}(C)$ and $Lu = f$ in $C$. Since $C \subseteq \bar{Q}$ is arbitrary, we have proved that $u \in W(Q)$ and $Lu = f$ in $Q$. The uniqueness assertion follows immediately from the comparison principle, Corollary 2.7.

Now we deal with the divergence case. Take the standard convolutions of coefficients and data functions $f$ with a non-negative mollifier, so that the mollifications $a^{(k)}_{ij}$, $b_i^{(k)}$, $c_i^{(k)}$, $e^{(k)}_{ij}$, $f_0^{(k)}$, and $f_i^{(k)}$ are in $C^\infty(\bar{Q}_2)$. In particular, the mollified coefficients converge to their original ones locally in $L^2(\bar{Q})$. Define

$$
a^{(k)}_{ij} := \left(a^{(k)}_{ij}\right)^{1,1/k}, \quad b_i^{(k)} := \left(b_i^{(k)}\right)^{1,1/k},
$$

$$
c_i^{(k)} := \left(c_i^{(k)}\right)^{1,1/k}, \quad e^{(k)}_{ij} := \left(e^{(k)}_{ij}\right)^{1,1/k},
$$

$$
f_0^{(k)} := \left(f_0^{(k)}\right)^{1,1/k}, \quad f_i^{(k)} := \left(f_i^{(k)}\right)^{1,1/k},
$$

where each term is in $C^\infty(\bar{Q})$. Let $L^k$ be the divergence type operator with the coefficients $a^{(k)}_{ij}$, $b_i^{(k)}$, $c_i^{(k)}$, and $e^{(k)}_{ij}$. Then we turn $L^k$ into a non-divergence type operator $\tilde{L}^k$ by defining

$$
\tilde{L}^k = \frac{\partial}{\partial t} - \tilde{a}^k_{ij}D_{ij} + \tilde{b}^k_iD_i + \tilde{c}^k_0,
$$

where

$$
\tilde{a}^k_{ij} = a^{(k)}_{ij}, \quad \tilde{b}^k_i = -D_ja^{(k)}_{ij} + b_i^{(k)} - c_i^{(k)}, \quad \tilde{c}^k_0 = c^{(k)}_0 - D_i e_i^{(k)}.
$$

Note that, due to the condition (1), we have $\tilde{c}^{(k)}_0 \geq 0$. Indeed, for any nonnegative function $\phi \in C^\infty_0(\bar{Q}),$

$$
\int_Q \tilde{c}^{(k)}_0 \phi \, dX = \int_Q \left(c^{(k)}_0 \phi + c^{(k)}_i D_i \phi \right) \, dX = \int_Q \left(c^{(k)}_0 \eta_0^{1/k} \phi + c^{(k)}_i D_i \left(\eta_0^{1/k} \phi \right) \right) \, dX \geq 0,
$$

which implies $\tilde{c}^{(k)}_0 \geq 0$ on $Q$. Then using the argument described above for the non-divergence case, we find a unique $u_k$ satisfying $\tilde{L}^k u_k = f_0^{(k)} - D_i f_i^{(k)}$ in $Q$ and $u_k |_{\partial_\nu Q} = 0$. Since the coefficients and $f_0^{(k)} - D_i f_i^{(k)}$ are infinitely differentiable, $u_k$ is infinitely differentiable in $Q$ and continuous on $\bar{Q}$. Furthermore, $u_k$ also satisfies the divergence type equation $L^k u_k = f_0^{(k)} - D_i f_i^{(k)}$ in $Q$. That is,

$$
\int_Q \left(-u_k D_t \phi + a^{(k)}_{ij}D_{ij} u_k \phi + b_i^{(k)} D_i u_k \phi + e^{(k)}_{ij} u_k D_{ij} \phi + e^{(k)}_0 u_k \phi \right) \, dX = \int_Q \left(f_0^{(k)} \phi + f_i^{(k)} D_i \phi \right) \, dX
$$

(23)

for any $\phi \in C^\infty_0(\bar{Q})$. Similarly as in the non-divergence case above, with the De Giorgi–Nash–Moser estimate we find $u \in C(\bar{Q})$ with $u = 0$ on $\partial_\nu Q$, which is the local
uniform limit of \( \{u_k\} \). Let us now prove that \( u \) belongs to \( H(Q) \) and satisfies the desired equation. For cylinders \( C \subseteq_p C' \subseteq_p Q \), by the standard local \( L_2 \)-estimate,

\[
\|Du_k\|_{L_2(C)} \leq N\|u_k\|_{L_2(C)} + N\|f_0^k\|_{L_2(\mathbb{R}^{n+2(m+4)}(C))} + N\|f_i^k\|_{L_2(C)},
\]

where \( N \) is independent of \( k \in \mathbb{N} \). Moreover, for sufficiently large \( k \), the right-hand side is dominated by a quantity independent of \( k \), so after taking a subsequence, \( Du_k \) converges to \( Du \) weakly. Then

\[
\int_Q a_{ij}^k D_i u_k D_j \phi \, dX \to \int_Q a_{ij} D_i u D_j \phi \, dX,
\]

where \( \phi \in C_0^\infty(C) \). More precisely,

\[
\left| \int_Q a_{ij}^k D_i u_k D_j \phi \, dX - \int_Q a_{ij} D_i u D_j \phi \, dX \right| \leq \|a_{ij}^k - a_{ij}\|_{L_2(C)} \|Du_k D_j \phi\|_{L_2(C)} \to 0
\]

because \( a_{ij}^k \) converges to \( a_{ij} \) locally in \( L_2 \) and the \( L_2(C) \)-norm of \( Du_k D_j \phi \) is uniformly bounded, and

\[
\int_Q a_{ij} D_i u_k D_j \phi \, dX \to \int_Q a_{ij} D_i u D_j \phi \, dX
\]

because \( Du_k \) converges to \( Du \) weakly and \( a_{ij} D_j \phi \in L_2(C) \). The same reasoning is applied to the other terms in (23). Therefore, by letting \( k \to \infty \) in (23), we see that \( u \) satisfies the divergence type equation \( Lu = f_0 - D_i f_i \). The fact that \( u \in H(Q) \) follows from \( Du \in L_2,\text{loc}(Q) \) and the equation. The uniqueness follows from Corollary 2.7 as in the non-divergence case. \( \square \)

**Remark 5.1.** For the non-divergence case, Assumption 1.2 is needed for the existence of the solution \( u_k \) during the proof. If one can generalize the solvability in smooth domains with more general coefficients (see, for instance, \([9, 17, 18]\)), our main results can also be improved.

**Remark 5.2.** Following the proof of Theorem 1.4, one can obtain a corresponding result for elliptic equations in a domain \( \Omega \) in \( \mathbb{R}^n \) satisfying the exterior measure condition. The solution space is \( W(\Omega) := W^{2,p}_0(\Omega) \cap C(\overline{\Omega}) \) in the non-divergence case for \( p_0 \in (n/2, \infty) \), and \( W(\Omega) := W^{1,p}_0(\Omega) \cap C(\overline{\Omega}) \) in the divergence case. We leave the details to the interested reader.

**Appendix A. Regularity of (A) Domains**

In this appendix, we will show that any (A) domain is regular, i.e., it satisfies Wiener’s criterion. We recall the definition of (A) domains:

**Definition A.1.** An open set \( Q \subset \mathbb{R}^{n+1} \) satisfies the condition (A) if there exists a constant \( \theta_0 \in (0, 1) \) such that for any \( Y_0 \in \partial_p Q \) and \( r > 0 \), we have \( |C(Y_0) \setminus Q| > \theta_0 |C(Y_0)| \).

Now we introduce some standard notation and definitions. One may consult, for instance, \([11]\).
Let $F$ be the fundamental solution of the heat equation, namely,

$$
F(x, t) = \begin{cases} 
(4\pi t)^{-n/2} \exp \left( -\frac{|x|^2}{4t} \right), & t > 0, \\
0, & t \leq 0,
\end{cases}
$$

and $E(x, t, r)$ be a heat ball of level set $r$ centered at $(x, t)$,

$$
E(x, t, r) := \{ (y, s) \in \mathbb{R}^{n+1} \mid F(x - y, t - s) \geq r \}.
$$

Now we list some useful properties of heat balls:

$$
E(r) := E(0, 0, r) = \{ (x, -t) \in \mathbb{R}^{n+1} \mid F(x) \geq r \},
$$

$$
F(r^{-1/n} x, r^{-2/n} t) = rF(x, t), \quad |E(r)| = r^{-1-2/n}|E(1)| < \infty,
$$

$$(y, s) \in E(1) \iff (x, t) \in E(r), \quad (x, t) = (r^{-1/n} y, r^{-2/n} s), \quad (24)$$

$$
E(1) = \left\{ (x, t) \in \mathbb{R}^{n+1} \mid t < 0, (-4\pi t)^{-n/2} e^{\frac{|x|^2}{4}} \geq 1 \right\}.
$$

**Definition A.2.** An open set $Q \subset \mathbb{R}^{n+1}$ satisfies the condition (B) if there exists a constant $\theta_1 \in (0, 1)$ such that, for any $Y_0 \in \partial \rho Q$ and $\lambda > 1$, there exists $k_0 \in \mathbb{N}$ satisfying

$$
\| (\dot{x}^k)^{k+1} \| \geq F(Y_0 - Y) \geq (\dot{x}^k)^k \setminus Q \| > \theta_1 \| (\dot{x}^k)^{k+1} \| \geq F(Y_0 - Y) \geq (\dot{x}^k)^k \|
$$

for all $k \in \mathbb{N}$.

We will show in Theorem A.5 that the condition (A) implies the condition (B).

For each $-\frac{1}{4\pi} < t < 0$, $(t, x) \in E(1)$ if and only if

$$
|x|^2 \leq 2nt \ln(-4\pi t).
$$

This implies that in a neighborhood of the origin, the boundary of $E(1)$ can be represented as $t = \Phi(x)$, where $\Phi$ is a $C^2$ function and satisfies

$$
\Phi(0) = |D\Phi(0)| = |D^2\Phi(0)| = 0. \quad (25)
$$

In the sequel, $C_r$ means $C((0, 0))$.

**Lemma A.3.** For any $\theta_0 > 0$, there exists $\delta = \theta(n, \theta_0) \in (0, 1)$ such that, for any $r > 0$,

$$
|C_{\delta r^{-1/\alpha}} \setminus E(r)| \leq \frac{\theta_0}{4} |C_{\delta r^{-1/\alpha}}|. \quad (26)
$$
Thus, it suffices to prove (26) when $r = 1$. In this case, (26) follows immediately from (25). \hfill \square

Lemma A.4. Let $\lambda > 1$ and $\theta \in (0, 1)$ be fixed numbers. Then one can choose a large $k_0 = k_0(n, \lambda, \theta, \theta_0) \in \mathbb{N}$ such that, for any $k \in \mathbb{N}$,

$$|E (f_k^{(n)})^{k+1}| \leq \frac{\theta_0}{4} |C_{\theta_0^{-1}/4}|.$$ 

Proof. Set $r := \lambda^{k_0}$. Then

$$|E (f_k^{(n)})^{k+1}| = |E f^{k+1}| = r^{(k+1)(-1-2/n)}|E(1)|$$

and

$$|C_{\theta_0^{-1}/4}| = |C_{\theta_0^{-1}/4}| = (\theta r^{-k/n})^{n+2} |C_1|.$$ 

Thus,

$$\frac{|E f^{k+1}|}{|C_{\theta_0^{-1}/4}|} = r^{1+2/n} |E(1)| = \frac{1}{(\lambda^{k_0})^{1+2/n}} |E(1)|$$

which can be made small if $\lambda > 1$ and $k_0$ is sufficiently large. The lemma is proved. \hfill \square

Theorem A.5 (Condition (A) implies Condition (B)). Let $Q$ be a $(A)$-domain in $\mathbb{R}^{n+1}$. For any $Y_0 \in \partial \rho_Q$ and $\lambda > 1$, there exist $k_0 = k_0(n, \lambda, \theta_0) \in \mathbb{N}$ and $\theta_1 = \theta_1(n, \theta_0) \in (0, 1)$ such that

$$|\{(f_k^{(n)})^{k+1} \geq F(0) - Y \geq (\lambda^{k_0})^k\} \setminus Q| \geq \theta_1 |\{(f_k^{(n)})^{k+1} \geq F(0) - Y \geq (\lambda^{k_0})^k\}|$$

for all $k \in \mathbb{N}$.

Proof. Without loss of generality, we assume that $Y_0 = (0, 0) \in \partial \rho_Q$. For a given $\lambda > 1$, take $k_0$ from Lemma A.4, where $\theta = \theta(n, \theta_0)$ is a number from Lemma A.3. Set $r = \lambda^{k_0}$. Then

$$\{f^{(n)} \geq F(-Y) \geq (\lambda^{k_0})^k\} = \{r^{k+1} \geq F(Y_0 - Y) \geq r^k\} = E(r^k) \setminus E(r^{k+1}).$$

Thus

$$\{(f_k^{(n)})^{k+1} \geq F(-Y) \geq (\lambda^{k_0})^k\} \setminus Q = [E(r^k) \setminus Q] \setminus [E(r^{k+1}) \setminus Q]$$

and

$$|\{(f_k^{(n)})^{k+1} \geq F(-Y) \geq (\lambda^{k_0})^k\} \setminus Q| \geq |E(r^k) \setminus Q| - |E(r^{k+1}) \setminus Q|$$

$$\geq |E(r^k) \setminus Q| - |E(r^{k+1})| \geq |E(r^k) \setminus Q| - \frac{\theta_0}{4} |C_{\theta_0^{-1}/4}|,$$

(27)
where the last inequality is due to Lemma A.4. On the other hand,
\[
|C_{\theta r^{-k/n}} \setminus Q| = \left| \left( C_{\theta r^{-k/n}} \cap E(r^k) \right) \setminus Q \right| + \left| \left( C_{\theta r^{-k/n}} \setminus E(r^k) \right) \setminus Q \right|
\]
\[
\leq \left| E(r^k) \setminus Q \right| + \left| C_{\theta r^{-k/n}} \setminus E(r^k) \right| \leq \left| E(r^k) \setminus Q \right| + \frac{\theta_0}{4} \left| C_{\theta r^{-k/n}} \right|
\]
where the last inequality is due to Lemma A.3. Along with (27) and Definition 1.3,
\[
\left| \left\{ (\dot{\lambda}_k)^{t+1} \geq F(Y) \geq (\dot{\lambda}_k)^t \right\} \setminus Q \right| \geq \left| C_{\theta r^{-k/n}} \setminus Q \right| - \frac{\theta_0}{2} \left| C_{\theta r^{-k/n}} \right|
\]
\[
\geq \frac{\theta_0}{2} \left| C_{\theta r^{-k/n}} \right| = \frac{\theta_0}{2} \left| E(r^k) \right| \left( |E(r^k)| \right) = \frac{\theta_0 \rho^{n+2}}{2} \frac{|C(1)|}{|E(1)|} \left| E(r^k) \right|
\]
\[
\geq \theta_1 \left| \left\{ (\dot{\lambda}_k)^{t+1} \geq F(Y_0 - Y) \geq (\dot{\lambda}_k)^t \right\} \right|
\]
where
\[
\theta_1 = \frac{\theta_0 \rho^{n+2}}{2} \frac{|C(1)|}{|E(1)|}.
\]
That is, \( \theta_1 \) depends only on \( n \) and \( \theta_0 \). \( \square \)

We denote
\[
V_2(\mathbb{R}^{n+1}) = \{ u \mid \nabla u \in L_2(\mathbb{R}^{n+1}), \| u(\cdot, t) \|_{L^2(\mathbb{R}^n)} \in L_\infty(\mathbb{R}) \}.
\]
For a compact set \( K \) in \( \mathbb{R}^{n+1} \), recall the thermal capacity:
\[
cap(K) = \sup\{ \mu(\mathbb{R}^{n+1}) \mid \mu \in M(K), F \ast \mu \leq 1 \},
\]
where \( M(K) \) is the set of all nonnegative Radon measure supported in \( K \), and the parabolic capacity:
\[
\Gamma(K) = \inf \left\{ \sup_i \int_\mathbb{R} u^2(x, t) \, dx + \int_\mathbb{R} \int_\mathbb{R} |\nabla u|^2 \, dX \right\},
\]
where the function \( u \) is taken over all functions in \( V_2(\mathbb{R}^{n+1}) \) with compact support such that \( K \subset \text{int}\{ X : u(X) \geq 1 \} \).

The following result can be found in [14].

**Lemma A.6.** For any compact set \( K \) in \( \mathbb{R}^{n+1} \),
\[
cap(K) \geq \frac{1}{2} \Gamma(K).
\]

As a consequence, we have

**Lemma A.7.** For any compact set \( K \) in \( \mathbb{R}^{n+1} \), we have, for some \( N > 0 \),
\[
cap(K) \geq N|K|^{\frac{n}{n+2}}.
\]
Proof. From Lemma A.6,
\[ \text{cap}(K) \geq \frac{1}{2} \Gamma(K). \]
By the parabolic type Gagliardo–Nirenberg–Sobolev inequality (see, for instance, [31, Theorem IV.6.9]), for any \( u \in V_2(\mathbb{R}^{n+1}) \) with compact support such that \( K \subset \text{int}\{X : u(X) \geq 1\} \),
\[ \sup_t \int_{\mathbb{R}^n} u^2(x, t) \, dx + \int_{\mathbb{R}^n} |\nabla u|^2 \, dX \geq N \left( \int |u|^{\frac{2(n+1)}{n}} \, dX \right)^{\frac{n}{n+2}} \geq N|K|^{\frac{n}{n+2}}. \]
By the definition of \( \Gamma(K) \), the lemma follows. \( \square \)

Finally, we show that the condition (A) implies Wiener’s criterion.

**Theorem A.8.** Let \( Q \) satisfy the condition (A). Then any point on \( \partial \rho Q \) is regular. Namely, \( X_0 \in \partial \rho Q \) satisfies the following Wiener’s criterion from [12, Theorem 1]: for any \( \lambda > 1 \),
\[ \sum_{k=1}^{\infty} \lambda^k \text{cap}(Q^c \cap \{\lambda^{k+1} \geq F(X_0 - X) \geq \lambda^k\}) = \infty. \]

Proof. We take the constant \( k_0 \) from Theorem A.5. It then follows from Lemma A.7, Theorem A.5, and (24) that
\[ \sum_{k=1}^{\infty} \lambda^k \text{cap}(Q^c \cap \{\lambda^{k+1} \geq F(X_0 - X) \geq \lambda^k\}) \]
\[ \geq \frac{1}{\lambda^{k_0}} \sum_{k=1}^{\infty} \tilde{\lambda}^{k_0} \text{cap}(Q^c \cap \{\tilde{\lambda}^{k_0(k+1)} \geq F(X_0 - X) \geq \tilde{\lambda}^{k_0k}\}) \]
\[ \geq \frac{N}{\lambda^{k_0}} \sum_{k=1}^{\infty} \tilde{\lambda}^{k_0} |Q^c \cap \{\tilde{\lambda}^{k_0(k+1)} \geq F(X_0 - X) \geq \tilde{\lambda}^{k_0k}\}|^{\frac{n}{n+2}} \]
\[ \geq \frac{N}{\lambda^{k_0}} \sum_{k=1}^{\infty} \tilde{\lambda}^{k_0} \theta_1^{\frac{n}{n+2}} |\{\tilde{\lambda}^{k_0(k+1)} \geq F(X_0 - X) \geq \tilde{\lambda}^{k_0k}\}|^{\frac{n}{n+2}} \]
\[ = \frac{N}{\lambda^{k_0}} \sum_{k=1}^{\infty} \theta_1^{\frac{n}{n+2}} |\{\tilde{\lambda}^{k_0} \geq F(X_0 - X) \geq 1\}|^{\frac{n}{n+2}} = \infty. \]

The theorem is proved. \( \square \)

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