A-INFINITY STRUCTURES RELATED TO BI-KOSZUL ALGEBRAS

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ABSTRACT. Let $A$ be a bi-Koszul algebra, we describe all possible $A_\infty$-algebra structures on the Ext-algebra $E(A)$, and prove that $E(A)$ must be $[m_2, m_3]$-finitely generated. An equivalent description for a connected graded algebra to be a bi-Koszul algebra is given in terms of $A_\infty$-language. The case that $E(A)$ is endowed with minimal number of multiplications is discussed for decomposition.

INTRODUCTION

To understand certain homological properties of graded algebras whose trivial modules admit non-pure resolutions, the authors introduced what they have called bi-Koszul algebras in [10]. Any non-Koszul Artin-Schelter regular algebras generated in degree 1 of global dimension four are the examples. Different from algebras with certain pure resolutions of the trivial modules (such as Koszul algebras [12], d-Koszul algebras [2], piecewise-Koszul algebras [11, etc] and $K_2$-algebras [3], bi-Koszul algebras lose a nice homological property that their Ext-algebras are finitely generated.

This may be remodeled if one endows “generating” with an appropriate meaning. For example, Keller claimed that Ext-algebras are $A_\infty$-generated by their homogeneous components of degree 1 for a large number of graded algebras [6, Proposition 1(b)]. Though it is nice to have finite generating components on one hand, it maybe require, as a redeem, infinite multiplications to guarantee finitely generating on the other hand. One of goals of this paper is to find out the multiplications on a given Ext-algebra as less as possible to carry out the finitely generating. We prove that the Ext-algebra of any bi-Koszul algebra is $[m_2, m_3]$-finitely generated.

We examine all possible $A_\infty$-algebra structures, corresponding to a bi-Koszul algebra $A$ determined by $\Delta_d$, to get at most five multiplications $m_2, m_3, m_4, m_d, m_{d+1}$. An $A_\infty$-version duality theory for a bi-Koszul algebra is given. The case that $E(A)$ is endowed with minimal number of multiplications $m_2, m_d, m_{d+1}$ is especially interesting for decomposition. Two single $A_\infty$-algebras are obtained here and can be returned to the $A_\infty$-structure of $E(A)$ by a bridge.

2000 Mathematics Subject Classification. 16E05, 16E40, 16S37, 16W50.
Key words and phrases. Ext-algebra, $A_\infty$-algebra, bi-Koszul algebra.

The work was supported by the NSFC (Grant No. 10571152) and partially by the NSF of Zhejiang Province of China (Grant No. J20080154).
We introduce a modified concept of “generating” which reflects some balance between multiplications and elements in the $A_\infty$-algebra system and prove that there exists an $A_\infty$-algebra structure on $E(A)$ of a bi-Koszul algebra $A$ such that $E(A)$ is $[m_2, m_3]$-finitely generated by $E^1(A)$, $E^2(A)$ and $E^3(A)$. A new criterion for a bi-Koszul algebra to be strongly is given in the $A_\infty$-version. Based on the fact that the $A_\infty$-Ext-algebra $E(A)$ is unique up to quasi-isomorphism, we discuss whether an $A_\infty$-algebra $E(A)$ is generated by $E^1(A)$.

1. $A_\infty$-ALGEBRAS AND BI-KOSZUL ALGEBRAS

In this section, we review basic material necessary for the paper: Ext-algebras, $A_\infty$-algebras, and bi-Koszul algebras.

1.1. Ext-algebras. Throughout we fix a field $F$. We always assume that a graded algebra $A = F \oplus A_1 \oplus A_2 \oplus \cdots$ is locally finite, connected, and generated in degree 1. The graded Jacobson radical of $A$, denoted by $J$, is $J = A_{\geq 1}$. Let $\text{Gr}(A)$ denote the category of graded left $A$-modules. The morphisms in this category, denoted by $\text{Hom}_{\text{Gr}(A)}(M, N)$ for $M, N \in \text{Gr}(A)$, are graded $A$-module maps of degree zero.

For $M \in \text{Gr}(A)$, we denote the $n$th shift of $M$ by $M[n]$, where $M[n]_j = M_{j+n}$.

We write $\text{Ext}_A^\star$, the derived functor of the graded $\text{Hom}_A^\star$ functor $\text{Hom}_A^\star(M, N) := \bigoplus_n \text{Hom}_{\text{Gr}(A)}(M, N[n])$, and denote

$E(A) := \text{Ext}_A^\star(F, F), \quad E(M) := \text{Ext}_A^\star(M, F),$

the Koszul dual of the algebra $A$ and the Koszul dual of the module $M \in \text{Gr}(A)$, respectively. $E(A)$ is equipped with a bigraded algebra structure by the Yoneda product with the $(i, j)^{th}$ component $\text{Ext}_A^i(F, F)_{-j}$, we also call it (classical) Ext-algebra of $A$. Here, $i$ is the cohomology degree and $-j$ is the internal degree. Note that the internal degree in $E(A)$ is non-positive. For simplicity, we promise $E^j(A) := \text{Ext}_A^j(F, F)_{-j}$. Similarly, $E(M)$ is a bigraded left $E(A)$-module with the $(i, j)^{th}$ component $E^i_j(M) := \text{Ext}_A^i(M, F)_{-j}$.

The (classical) Ext-algebra $E(A)$ carries rich information about the algebra $A$ and its module category, but it does not contain enough information to recover the original algebra in general, the “hidden” information is revealed in the $A_\infty$-world.

1.2. $A_\infty$-algebras. There are different methods to give the definition of an $A_\infty$-algebra (algebraical, geometrical, operadic, etc.), but here we prefer the algebraical definition of an $A_\infty$-algebra. We refer to [7] or [9] for the details.

Definition 1.1. An $A_\infty$-algebra over a field $F$ is a $\mathbb{Z}$-graded vector space

$E = \bigoplus_{p \in \mathbb{Z}} E^p$

endowed with a family of graded $F$-linear maps

$m_n : E^\otimes n \rightarrow E, \quad (n \geq 1)$
of degree \(2 - n\) satisfying the Stasheff’s identities: for all \(n \geq 1\),

\[
\text{Sl}(n) \sum (-1)^{i+t+j} m_{i+1+j} (1^{\otimes i} \otimes m_t \otimes 1^{\otimes j}) = 0,
\]

where the sum runs over all decompositions \(n = i + t + j\) \((i, j \geq 0\) and \(t \geq 1\)).

The graded maps \(m_n\) for \(n \geq 3\) are called higher multiplications of \(E\). An \(A_\infty\)-algebra \(E\) is strictly unital if \(E\) contains an element \(1\) which acts as a two-sided identity with respect to \(m_2\), and for \(n \neq 2\), \(m_n(x_1 \otimes \cdots \otimes x_n) = 0\) if \(x_i = 1\) for some \(i\). The \(A_\infty\)-algebras in this paper are always assumed to be strictly unital. An \(A_\infty\)-algebra with zero \(m_1\) is called minimal. An \(A_\infty\)-subalgebra of \(E\) is a graded subspace \(F\) such that \(m_n\) maps \(F^{\otimes n}\) to \(F\) for all \(n \geq 1\). By an \(A_\infty\)-algebra \(E\) being generated by \(E^1\) we mean that for any \(p \geq 2\),

\[
E^p = \sum m_t(E^{i_1} \otimes \cdots \otimes E^{i_l}),
\]

where the sum runs over all decompositions \(i_1 + \cdots + i_l + 2 - l = p\) \((i_1, \cdots, i_l \geq 1)\) and \(l \geq 1\).

Let \(E\) and \(F\) be two \(A_\infty\)-algebras. A morphism of \(A_\infty\)-algebras \(f : E \to F\) is a family of graded \(F\)-linear maps

\[
f_n : E^{\otimes n} \to F, \quad n \geq 1
\]

of degree \(1 - n\) satisfying the Stasheff’s morphism identities: for all \(n \geq 1\),

\[
\text{Mi}(n) \sum (-1)^{i+t+j} f_{i+1+j} (1^{\otimes i} \otimes m_t \otimes 1^{\otimes j}) = \sum (-1)^w m_r(f_{i_1} \otimes \cdots \otimes f_{i_r}),
\]

where the first sum runs over all decompositions \(n = i + t + j\) \((i, j \geq 0\) and \(t \geq 1\)), and the second sum runs over all \(1 \leq r \leq n\) and all decompositions \(n = i_1 + \cdots + i_r\) \((all i_j \geq 1)\); the sign on the right-hand side is given by \(w = (r - 1)(i_1 - 1) + (r - 2)(i_2 - 1) + \cdots + (i_{r-1} - 1)\).

An \(A_\infty\)-morphism in this paper is also required to be strictly unital (see \[3\]). A morphism \(f\) is called a quasi-isomorphism if \(f_1\) is a quasi-isomorphism. A morphism \(f\) is called a strict isomorphism if \(f_1 = 0\) for any \(i \geq 2\) and \(f_1\) is an isomorphism.

\(A_\infty\)-algebras have been in use in topology since their introduction by Stasheff. Their applicability in an algebraic context was made clear by the minimality theorem, proven by Kadeishvili \[5 \] \& \[7\].

**Theorem 1.2.** (The minimality theorem) Let \(E\) be an \(A_\infty\)-algebra. Then the cohomology \(H^*E\) has an \(A_\infty\)-algebra structure such that \(m_1 = 0, m_2\) is induced by \(m_2^E\), and \(H^*E\) is quasi-isomorphic to \(E\) as \(A_\infty\)-algebras. \(\square\)

The techniques used to prove the minimality theorem all yield explicit methods to compute an \(A_\infty\)-algebra structure. The Ext-algebra \(\text{Ext}^*_A(F, F)\) is the cohomology of \(\text{End}_A(P)\), where \(P\) is any free resolution of \(A\). Since \(E = \text{End}_A(P)\) is a differential graded algebra, by the minimality theorem, \(\text{Ext}^*_A(F, F)\) has a natural \(A_\infty\)-structure, which is called an \(A_\infty\)-Ext-algebra of \(A\). By abuse of notation we still use \(E(A)\) to denote an \(A_\infty\)-Ext-algebra. The importance is that the information from the \(A_\infty\)-algebra \(E(A)\) is sufficient to recover \(A\).
We are mainly, in this paper, interested in the $A_{\infty}$-algebra $E(A) := \bigoplus_{p,i \in \mathbb{Z}} E_p^i(A)$ which is bigraded with the lower grading inherited from the graded algebra $A$. Each multiplication $m_n$, as well as each morphism between two bigraded $A_{\infty}$-algebras, must preserve the lower grading.

An $A_{\infty}$-algebra $E(A)$ that we consider in the paper always comes from a free resolution. Different choice of the free resolutions yields quasi-isomorphic $A_{\infty}$-algebra structures on $E(A)$. Under the assumption on $A$, any choice of such an $A_{\infty}$-algebra structure on $E(A)$ with the multiplications $\{m_n\}_{n \geq 1}$ has the following properties: $m_1 = 0$, $m_2$ is the Yoneda product of $E(A)$, and $E^2(A)$ is $A_{\infty}$-generated by $E^1(A)$; that is, $E^2_n(A) = m_n(E^1(A) \otimes \cdots \otimes E^1(A))$ for each $n \geq 2$. Moreover, there exists an $A_{\infty}$-algebra structure on $E(A)$ such that $E(A)$ is generated by $E^1(A)$ $[6]$. For more properties we refer to [6] [7] or [8] [9].

1.3. Bi-Koszul algebras. To extend Koszulity to a graded algebra with a bi-degree resolution of the ground field, the authors introduced what they have called bi-Koszul algebras in $[10]$.

**Definition 1.3.** A bi-Koszul algebra (determined by $\Delta_d$) is a connected graded algebra $A$ whose trivial module $F$ has a minimal graded free resolution $\mathcal{P}$ such that each $P_n$ is generated in degrees $\Delta_d(n)$ for all $n \geq 0$, where the degree distribution $\Delta_d : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$ is defined, for a fixed integer $d \geq 2$, by

$$
\Delta_d(n) = \begin{cases} 
\frac{n}{3}(2d,2d), & \text{if } n \equiv 0(\text{mod}3); \\
\frac{n-1}{3}(2d,2d) + (1,1), & \text{if } n \equiv 1(\text{mod}3); \\
\frac{n-2}{3}(2d,2d) + (d,d+1), & \text{if } n \equiv 2(\text{mod}3).
\end{cases}
$$

For simplicity, $\Delta_d(n)$ is used to express both of its image $(x,y)$ and of the set $\{x,y\}$. Artin-Schelter regular algebras of global dimension 4 of types $(13,31)$ and $(12,221)$ are the examples by taking $d=2$ and $d=3$, respectively. We refer to [10] for the details.

**Theorem 1.4.** [10] The following statements are equivalent:

1. $A$ is a bi-Koszul algebra determined by $\Delta_d$;
2. $E(A)$ begins with $E^1(A) = E_1^1(A)$, $E^2(A) = E_2^2(A) \oplus E_{d+1}^2(A)$, $E^3(A) = E_{2d}^3(A)$, and for each $n \geq 1$,

   a) $E^{3n}(A) = \bigoplus_1^{n} E^3(A) \cdots E^3(A)$,
   b) $E^{3n+1}(A) = E^1(A) \otimes E^{3n}(A) = E^{3n}(A) E^1(A)$,
   c) $E^{3n+2}(A) \cong E^2(A) \otimes E^{3n}(A) \oplus E^2_{2n+d+1}(J\Omega^{3n}(F))$ as $F$-spaces. [3]

In the above theorem, the obstruction $E^2_{2n+d+1}(J\Omega^{3n}(F))$ arises from the bigger degree in $\Delta_d(3n+2)$. We call a bi-Koszul algebra $A$ strongly if the obstruction is vanished. In graded algebras setting, it is clear that the Ext-algebra $E(A)$ of a strongly bi-Koszul algebra $A$ is generated by $E^1(A), E^2(A)$ and $E^3(A)$, but it is not sure that the Ext-algebra $E(A)$ of a bi-Koszul algebra $A$ is finitely generated.
There is a remedy of finitely generating on \(E(A)\) by using higher multiplications in Section 3.

2. \(A_\infty\)-Ext-algebras of bi-Koszul algebras

In this section, we examine the possible multiplications on \(E(A)\) as an \(A_\infty\)-algebra for a bi-Koszul algebra \(A\) by using information about the grading of \(E(A)\). An \(A_\infty\)-version duality theory of bi-Koszul algebras is given. In particular, we discuss a kind of bi-Koszul algebras whose Ext-algebras are endowed with the minimal number of nonzero multiplications.

2.1. \(A_\infty\)-structures on \(E(A)\). For the sake of convenience, we write

\[
m_l(E^{t_1} \cdots E^{t_l}) := m_l(E^{t_1} \otimes \cdots \otimes E^{t_l}).
\]

The following lemma gives an equivalent definition of the bi-Koszul algebra which is characterized by its Ext-algebra.

**Lemma 2.1.** \(A\) is a bi-Koszul algebra if and only if for any \(n \geq 0\), \(E^n_j(A) = 0\) for \(j \notin \Delta_d(n)\).

**Proof.** Similar to the proof in [1, Proposition 2.1.3]. \(\square\)

Before determining all possible multiplications on \(E(A)\), we claim that \(m_2, m_d\) and \(m_{d+1}\) must be non-trivial.

**Proposition 2.2.** Let \(A\) be a bi-Koszul algebra determined by \(\Delta_d\). An \(A_\infty\)-algebra \(E := E(A)\) must have nonzero multiplications \(m_2, m_d\) and \(m_{d+1}\).

**Proof.** As mentioned in the last section, \(m_2\) is the Yoneda product, so we need only to show that both \(m_d\) and \(m_{d+1}\) are non-trivial. Noting that \(E^2 = E^2_d \oplus E^2_{d+1}\) and \(E^2\) is generated by \(E^1\), we have

\[
m_d(E^1 \cdots E^1) = E^2_d, \quad m_{d+1}(E^1 \cdots E^1) = E^2_{d+1}.
\]

So we get \(m_2, m_d\) and \(m_{d+1}\) are nonzero. \(\square\)

One of main results of this section is

**Theorem 2.3.** Let \(A\) be a bi-Koszul algebra determined by \(\Delta_d\). Then all possible non-trivial multiplications on the \(A_\infty\)-Ext-algebra \(E(A)\) are \(m_2, m_3, m_4, m_d\) and \(m_{d+1}\).

**Proof.** Denote \(E := E(A)\). Let \(m_l\) be a multiplication on \(E(A)\). Since only information about the grading of \(E(A)\) is considered in the following, we can neglect the order of \(E^{t_1}, \cdots, E^{t_l}\) acted by \(m_l\). Write

\[
M := m_l(E^{3k_1+t_1} \cdots E^{3k_\alpha+t_\alpha} E^{3k_{\alpha+1}+2} \cdots E^{3k_l+2})
\]

where \(\alpha \leq l\) and \(t_j = 0\) or 1 \((1 \leq j \leq \alpha)\). Denote \(\beta = l - \alpha\). So

\[
M \subseteq E^{3(k_1+\cdots+k_l)+(t_1+\cdots+t_\alpha)+2\beta+2-l}
\]
and the lower grading of \( M \) falls into the set
\[
\{2d(k_1 + \cdots + k_l) + (t_1 + \cdots + t_\alpha) + d\beta + j \mid j = 0, 1, \cdots, \beta\},
\]
where \( 0 \leq t_1 + \cdots + t_\alpha \leq l - \beta \).

(1) If \((t_1 + \cdots + t_\alpha) + 2\beta + 2 - l = 3k \) \((k \geq 0)\), then
\[
E^{3(k_1 + \cdots + k_l) + (t_1 + \cdots + t_\alpha) + 2\beta + 2 - l} = E^{3(k_1 + \cdots + k_l) + 3k}_{2d(k_1 + \cdots + k_l) + 2dk}.
\]

We have the following inequalities:
\[
(t_1 + \cdots + t_\alpha) + d\beta \leq 2kd \leq (t_1 + \cdots + t_\alpha) + d\beta + \beta,
\]
which produce the solutions of \((k, \beta, l)\) as the following list
\[
(0, 0, 2), \ (1, 1, d), \ (1, 1, d+1), \ (1, 2, 3), \ (2, 4, 4).
\]

(2) If \((t_1 + \cdots + t_\alpha) + 2\beta + 2 - l = 3k + 1 \) \((k \geq 0)\), then
\[
E^{3(k_1 + \cdots + k_l) + (t_1 + \cdots + t_\alpha) + 2\beta + 2 - l} = E^{3(k_1 + \cdots + k_l) + 3k + 1}_{2d(k_1 + \cdots + k_l) + 2dk + 1}.
\]

The inequalities:
\[
(t_1 + \cdots + t_\alpha) + d\beta \leq 2kd + 1 \leq (t_1 + \cdots + t_\alpha) + d\beta + \beta
\]
imply the solutions of \((k, \beta, l)\) in the following list
\[
(0, 0, 2), \ (1, 2, 2), \ (1, 2, 3).
\]

(3) If \((t_1 + \cdots + t_\alpha) + 2\beta + 2 - l = 3k + 2 \) \((k \geq 0)\), then
\[
E^{3(k_1 + \cdots + k_l) + (t_1 + \cdots + t_\alpha) + 2\beta + 2 - l} = E^{3(k_1 + \cdots + k_l) + 3k + 2}_{2d(k_1 + \cdots + k_l) + 2dk + d} \oplus E^{3(k_1 + \cdots + k_l) + 3k + 2}_{2d(k_1 + \cdots + k_l) + 2dk + d + 1}.
\]

We have the following inequalities:
\[
(t_1 + \cdots + t_\alpha) + d\beta \leq 2kd + d \leq (t_1 + \cdots + t_\alpha) + d\beta + \beta,
\]
or
\[
(t_1 + \cdots + t_\alpha) + d\beta \leq 2kd + d + 1 \leq (t_1 + \cdots + t_\alpha) + d\beta + \beta.
\]
The solutions of \((k, \beta, l)\) are listed in the following
\[
(0, 1, 2), \ (1, 3, 3), \ (0, 0, d), \ (0, 1, 2), \ (0, 1, 3), \ (0, 0, d+1), \ (1, 3, 3), \ (1, 3, 4) \quad \text{ (S3)}
\]

In conclusion of (S1)-(S3), all possible solutions of \( l \) are 2, 3, 4, \( d \) and \( d + 1 \). This completes the proof. \( \square \)

**Corollary 2.4.** Let \( A \) be a bi-Koszul algebra determined by \( \Delta_d \).

(1) If \( d = 2 \) or 3, the possible non-trivial multiplications on \( E(A) \) are \( m_2, m_3 \) and \( m_4 \).

(2) If \( d = 4 \), the possible non-trivial multiplications on \( E(A) \) are \( m_2, m_3, m_4 \) and \( m_5 \).

(3) If \( d \geq 5 \), the possible non-trivial multiplications on \( E(A) \) are \( m_2, m_3, m_4, m_d \) and \( m_{d+1} \). \( \square \)
To describe what components the multiplications act on non-trivial, we denote

\[ E^{[0]} := \bigoplus_{k \geq 0} E^{3k}, \quad E^{[1]} := \bigoplus_{k \geq 0} E^{3k+1}, \quad E^{[2]} := \bigoplus_{k \geq 0} E^{3k+2} = \bigoplus_{k \geq 0} E^{2}_{(d)} \oplus E^{2}_{(d+1)}. \]

The following proposition is clear from the proof of Theorem 2.3.

**Proposition 2.5.** Let \( A \) be a bi-Koszul algebra determined by \( \Delta_d, E \) the Ext-algebra of \( A \). Then the possible nonzero components of \( m_i (i = 2, 3, 4, d, d+1) \) are:

| \( i \) | fall into \( E^{[0]} \) | fall into \( E^{[1]} \) | fall into \( E^{[2]} \) |
|-------|-----------------|-----------------|-----------------|
| \( m_2 \) | \( E^{[0]} E^{[0]} \) | \( E^{[0]} E^{[1]}, E^{[2]}_{(d)} E^{[2]}_{(d+1)} \) | \( E^{[0]} E^{[2]} \) |
| \( m_4 \) | \( E^{[2]}_{(d)} E^{[2]}_{(d)} \) | \( E^{[2]}_{(d)} E^{[2]}_{(d)} \) | \( E^{[2]}_{(d)} E^{[2]}_{(d)} \) |
| \( m_d \) | \( E^{[1]} \cdots E^{[1]} E^{[2]}_{(d)} \) | \( E^{[1]} \cdots E^{[1]} E^{[2]}_{(d+1)} \) | \( E^{[2]}_{d+1} \) |
| \( m_{d+1} \) | \( E^{[1]} \cdots E^{[1]} E^{[2]}_{(d)} \) | \( E^{[1]} \cdots E^{[1]} E^{[2]}_{(d+1)} \) | \( E^{[2]}_{d+1} \) |

*: including all permutations of the components listed above.

**Definition 2.6.** We call an \( A_\infty \)-algebra \( E = (E; m_2, m_3, m_4, m_d, m_{d+1}) \) reduced, if all possible nonzero components of multiplications are in the above table.

**Corollary 2.7.** A is a bi-Koszul algebra if and only if any \( A_\infty \)-algebra structure on \( E(A) \) is reduced.

**Proof.** The necessity is from Theorem 2.3 and Proposition 2.5. Now suppose that any \( A_\infty \)-algebra on \( E(A) \) is reduced. Take the \( A_\infty \)-algebra \( E(A) \) that is generated by \( E(A) \), then \( A \) is a bi-Koszul algebra by checking the lower grading.

**2.2. Truncated bi-Koszul algebras.** We discuss a kind of bi-Koszul algebras whose Ext-algebras are endowed with the minimal number of non-trivial multiplications \( m_2, m_d \) and \( m_{d+1} \).

**Definition 2.8.** Let \( A \) be a bi-Koszul algebra determined by \( \Delta_d, E := E(A) \) its Ext-algebra. We say that \( A \) is truncated if the \( A_\infty \)-Ext-algebra \( E \) only has the non-trivial multiplications \( m_2, m_d, m_{d+1} \) and the possible nonzero actions of \( i = 2, d, d+1 \) are on

| \( i \) | fall into \( E^{[0]} \) | fall into \( E^{[1]} \) | fall into \( E^{[2]} \) |
|-------|-----------------|-----------------|-----------------|
| \( m_2 \) | \( E^{[0]} E^{[0]} \) | \( E^{[0]} E^{[1]}, E^{[2]}_{(d)} E^{[2]}_{(d+1)} \) | \( E^{[0]} E^{[2]} \) |
| \( m_d \) | \( E^{[1]} \cdots E^{[1]} E^{[2]}_{(d)} \) | \( E^{[1]} \cdots E^{[1]} E^{[2]}_{(d+1)} \) | \( E^{[2]}_{d+1} \) |
| \( m_{d+1} \) | \( E^{[1]} \cdots E^{[1]} E^{[2]}_{(d)} \) | \( E^{[1]} \cdots E^{[1]} E^{[2]}_{(d+1)} \) | \( E^{[2]}_{d+1} \) |

*: including all permutations of the components listed above.

By abuse of notation we also say that the \( A_\infty \)-algebra \( (E; m_2, m_d, m_{d+1}) \) is truncated in this case.

**Example 2.9.** All Artin-Schelter regular algebras listed in [9, Theorem A] are truncated bi-Koszul algebras.
The following result comes from Proposition 2.7 of Section 3.

**Proposition 2.10.** A truncated bi-Koszul algebra must be strongly.

A minimal \(A_\infty\)-algebra is called single if it has only one non-trivial higher multiplication (i.e. a \((2, p)\)-algebra discussed in [4]). Single \(A_\infty\)-algebras are related to \(p\)-Koszul algebras ([4]).

A bigraded algebra is called pure in the sense that every component is supported in a single lower grading. We also say that a minimal \(A_\infty\)-algebra which is bigraded is pure if the underlying bigraded algebra itself is pure.

We need the following lemma. Consider an \(A_\infty\)-algebra \((E; m_d, m_t, m_t)\) \((2 < d < t \text{ and } 2 + t \neq 2d)\). All non-trivial Stasheff’s identities, in this case, are listed as follows:

\[
\begin{align*}
\text{SI}(3): & \quad m_2(m_2 \otimes 1) = m_2(1 \otimes m_2); \\
\text{SI}(d+1): & \quad \sum_{i+j=d-1}(-1)^i m_d(1^\otimes \otimes m_2 \otimes 1^\otimes j) = m_2(1 \otimes m_d) - (-1)^d m_2(m_2 \otimes 1^\otimes j); \\
\text{SI}(t+1): & \quad \sum_{i+j=t-1}(-1)^i m_t(1^\otimes \otimes m_2 \otimes 1^\otimes j) = m_2(1 \otimes m_t) - (-1)^1 m_2(m_t \otimes 1^\otimes j); \\
\text{SI}(2d-1): & \quad \sum_{i+j=d-1}(-1)^{i+j} m_d(1^\otimes \otimes m_d \otimes 1^\otimes j) = 0; \\
\text{SI}(d+t-1): & \quad \sum_{i+j=d-1}(-1)^{i+j} m_d(1^\otimes \otimes m_d \otimes 1^\otimes j) = \sum_{i+j=t-1}(-1)^{i+j} m_t(1^\otimes \otimes m_d \otimes 1^\otimes j); \\
\text{SI}(2t-1): & \quad \sum_{i+j=t-1}(-1)^{i+j} m_t(1^\otimes \otimes m_t \otimes 1^\otimes j) = 0.
\end{align*}
\]

**Lemma 2.11.** Let \(E\) be a connected graded algebra with three graded \(F\)-linear maps \(m_n: E^\otimes n \to E\) \((n = 2, d, t)\). Suppose \(2 < d < t\) and \(2 + t \neq 2d\). Then the following statements are equivalent.

1. \((E; m_2, m_d, m_t)\) is an \(A_\infty\)-algebra;
2. \(E\) together with \(\{m_2, m_d, m_t\}\) satisfies
   
   a. \((E; m_2, m_d)\) is single,
   b. \((E; m_2, m_t)\) is single,
   c. \(m_d\) and \(m_t\) obey \(\text{SI}(d+t-1)\).

**Proof.** It is clear by noting that all non-trivial Stasheff’s identities of a single \(A_\infty\)-algebra with the higher multiplication \(m_p\) are \(\text{SI}(3), \text{SI}(p+1), \text{SI}(2p-1)\). □

From the lemma above, one may decompose a truncated \(A_\infty\)-algebra into two single \(A_\infty\)-algebras, while the Stasheff’s identity \(\text{SI}(2d)\) serves as a bridge between two single \(A_\infty\)-algebras.

**Proposition 2.12.** Let \(A\) be a truncated bi-Koszul algebra determined by \(\Delta_d\) \((d \geq 4)\). Then both \((E; m_2, m_d)\) and \((E; m_2, m_{d+1})\) are single \(A_\infty\)-algebras, generated by \(E^1, E^2\) and \(E^3\).

**Proof.** This is a direct result of Proposition 2.10 and Lemma 2.11 □

The single \(A_\infty\)-algebra \((E; m_2, m_d)\) in the proposition above is not the \(A_\infty\)-Ext-algebra of any graded algebra, since there are no components acted by \(m_2\) and \(m_d\) that fall into \(E^2_{d+1}\); neither is \((E; m_2, m_{d+1})\).

Drawing upon the \(A_\infty\)-algebras \((E; m_2, m_d)\) and \((E; m_2, m_{d+1})\), the \(A_\infty\)-Ext-algebra \(E\) can be decomposed further into two single \(A_\infty\)-algebras which are both pure.
Theorem 2.13. Let $A$ be a truncated bi-Koszul algebra determined by $\Delta_d$ ($d \geq 4$), $E := E(A)$ its $\text{Ext}$-algebra. Set
\[
F := E^0 \oplus E^1 \oplus E^{[2]}_{(d)}, \quad G := E^0 \oplus E^1 \oplus E^{[2]}_{(d+1)}.
\]
Then
\begin{enumerate}
  \item $(F; m_2, m_d)$ is a pure and single $A_{\infty}$-subalgebra of $(E; m_2, m_d)$, where $m_d$ is determined by $m_2$ and $m_d \, |_{(E^1)^{\otimes 4}}$;
  \item $(G; m_2, m_{d+1})$ is a pure and single $A_{\infty}$-subalgebra of $(E; m_2, m_{d+1})$, where $m_{d+1}$ is determined by $m_2$ and $m_{d+1} \, |_{(E^1)^{\otimes 4+1}}$.
\end{enumerate}

Proof. It is easy to justify that both $(F; m_2, m_d)$ and $(G; m_2, m_{d+1})$ are pure and single $A_{\infty}$-subalgebras of $(E; m_2, m_d)$ and $(E; m_2, m_{d+1})$, respectively. And the nonzero actions of $m_d$ are only on $(E^1)^{\otimes d}$ and $m_{d+1}$ only on $(E^1)^{\otimes d+1}$.

Write $m_2(x, y)$ by $xy$ or $x \cdot y$. Set $t = d$ or $d + 1$. For any homogeneous elements $x_1, \ldots, x_t \in E^1$ and $u \in E^0$, we note that
\[
m_t(x_1, \ldots, x_{i-1}, ux_i, \ldots, x_t) = m_t(x_1, \ldots, x_{i-1}u, x_i, \ldots, x_t), \quad (2 \leq i \leq t),
\]
\[
m_t(ux_1, \ldots, x_t) = u \cdot m_t(x_1, \ldots, x_t),
\]
\[
m_t(x_1, \ldots, x_t \cdot u) = m_t(x_1, \ldots, x_t) \cdot u.
\]

Since $E^{3n+j} = E^j E^{3n} = E^{3n} E^j$ ($j = 1, 2$) by Proposition 2.10 for any $x \in E^{3n+j}$ ($n \geq 0$), there exist $f, g \in E^j$ and $u, v \in E^{3n}$ such that $x = fu = vg$. If $n = 0$, set $u = v = 1$.

For any $x_1, \ldots, x_d \in E^1$, choose $z_1, \ldots, z_d \in E^1$ satisfying
\[
x_d = y_d z_d, \quad x_{d-1} y_d = y_{d-1} z_{d-1}, \ldots, \quad x_2 y_3 = y_2 z_2, \quad x_1 y_2 = y_1 z_1
\]
where $y_1, \ldots, y_d \in E^0$. We have
\[
m_d(x_1, \ldots, x_d) = m_d(x_1, \ldots, x_{d-1}, y_d z_d) = m_d(x_1, \ldots, x_{d-1} y_d, z_d) = m_d(x_1, \ldots, y_d z_{d-1}, z_d) = \ldots = m_d(x_1 y_2, z_2, z_3 \ldots, z_d) = y_1 \cdot m_d(z_1, \ldots, z_d),
\]
since $m_d$ is determined by $m_2$ and $m_d \, |_{(E^1)^{\otimes 4}}$.

By the same method, $m_{d+1}$ is determined by $m_2$ and $m_{d+1} \, |_{(E^1)^{\otimes 4+1}}$. We complete the proof. \hfill \Box

In the $A_{\infty}$-Ext-algebra $(E; \{m_i\})$ of a graded algebra $A$, $m_2$ is the Yoneda product and $m_i \, |_{(E^1)^{\otimes 4}}$ can be computed out explicitly for every $i \geq 3$. This was demonstrated in [4]. More concretely, the single higher multiplication in either $(F; m_2, m_d)$ or $(G; m_2, m_{d+1})$ becomes definite in form.
Now, we turn to find a way in which a truncated $A_\infty$-algebra can be formed by jointing two single $A_\infty$-algebras together as follows.

Suppose that $(E; m_2)$ is a bigraded algebra starting with $E^1 = E_1^1$, $E^2 = E_2^2 \oplus E^2_{d+1}$, $E^3 = E_2^3$, and satisfying $E^{3n+i} = E^i E^{3n} = E^{3n} E^i$ for all $n \geq 1$, $i = 1, 2, 3$.

Define two single $A_\infty$-algebras $(E; m_2, m_d)$ and $(E; m_2, m_{d+1})$ such that

1. the nonzero actions of $m_d$ are only on $(E^{[1]} \otimes d$ and $E^{[1]} \cdots E^{[2]} (d+1 \cdots E^{[1]}$ (including all permutations);
2. the nonzero actions of $m_{d+1}$ are only on $(E^{[1]} \otimes d+1$ and $E^{[1]} \cdots E^{[2]} (d+1 \cdots E^{[1]}$ (including all permutations).

Then we have

**Proposition 2.14.** Let $(E; m_2)$ and $m_d, m_{d+1}$ be as above with $d \geq 4$. If $m_d$ is compatible with $m_{d+1}$ by $\text{Sl}(2d)$ on $(E^{[1]} \otimes 2d)$. Then $(E; m_2, m_d, m_{d+1})$ is a truncated $A_\infty$-algebra.

**Proof.** We only need to show that $m_d$ is compatible with $m_{d+1}$ by $\text{Sl}(2d)$ on $(E^{[1]} \otimes 2d)$ by Lemma 2.11 and the nonzero actions of $m_d$ and $m_{d+1}$.

Write $m_2(x, y)$ by $xy$ or $x \cdot y$. Note the nonzero actions of $m_d$ and $m_{d+1}$ and the proof of Theorem 2.13. For any $x_1, \cdots, x_{2d} \in E^{[1]}$, choose $z_1, \cdots, z_{2d} \in E^1$ satisfying

$$x_{2d} = y_{2d} z_{2d}, \ x_{2d-1} y_{2d} = y_{2d-1} z_{2d-1}, \cdots, \ x_2 y_{3} = y_2 z_{2}, \ x_1 y_{2} = y_1 z_1$$

where $y_1, \cdots, y_{2d} \in E^{[0]}$. We have

$$m_d(x_1, \cdots, x_i, m_{d+1}(x_{i+1}, \cdots, x_{i+d+1}), x_{i+d+2}, \cdots, x_{2d})$$

$$= m_d(x_1, \cdots, x_i, m_{d+1}(x_{i+1}, \cdots, x_{i+d+1}), x_{i+d+2}, \cdots, x_{2d-1} y_{2d}, z_{2d})$$

$$= m_d(x_1, \cdots, x_i, m_{d+1}(x_{i+1}, \cdots, x_{i+d+1}), x_{i+d+2} y_{i+d+3}, \cdots, z_{2d})$$

$$= m_d(x_1, \cdots, x_i, m_{d+1}(x_{i+1}, \cdots, x_{i+d+1}), y_{i+d+2}, z_{i+d+2}, \cdots, z_{2d})$$

$$= m_d(x_1, \cdots, x_i, m_{d+1}(x_{i+1}, \cdots, x_{i+d+1} y_{i+d+2}), z_{i+d+2}, \cdots, z_{2d})$$

$$= m_d(x_1, \cdots, x_i, m_{d+1}(x_{i+1} y_{i+2}, \cdots, z_{i+d+1}), z_{i+d+2}, \cdots, z_{2d})$$

$$= m_d(x_1, \cdots, x_i, y_{i+1} \cdot m_{d+1}(z_{i+1}, \cdots, z_{i+d+1}), z_{i+d+2}, \cdots, z_{2d})$$

$$= m_d(x_1 y_2, \cdots, z_i, m_{d+1}(z_{i+1}, \cdots, z_{i+d+1}), z_{i+d+2}, \cdots, z_{2d})$$

$$= y_1 \cdot m_d(z_1, \cdots, z_i, m_{d+1}(z_{i+1}, \cdots, z_{i+d+1}), z_{i+d+2}, \cdots, z_{2d})$$.

Similarly,

$$m_{d+1}(x_1, \cdots, x_i, m_d(x_{i+1}, \cdots, x_{i+d}), x_{i+d+1}, \cdots, x_{2d})$$

$$= y_1 \cdot m_{d+1}(z_1, \cdots, z_i, m_d(z_{i+1}, \cdots, z_{i+d}), z_{i+d+1}, \cdots, z_{2d})$$.

Set

$$\varphi := \sum_{i+j=d-1} (-1)^{i+j} m_d(1^{\otimes i} \otimes m_{d+1} \otimes 1^{\otimes j}) + \sum_{i+j=d} (-1)^{i+j} m_{d+1}(1^{\otimes i} \otimes m_d \otimes 1^{\otimes j})$$,
then \( \varphi(x_1 \otimes \cdots \otimes x_{2d}) = y_1 \cdot \varphi(z_1 \otimes \cdots \otimes z_{2d}) = 0 \) by the assumption. Therefore, \( \text{SI}(2d) \) holds on all \( (E^{[1]})^{\otimes 2d} \).

We complete the proof. \( \square \)

**Remark 2.1.** In the proofs of Theorem 2.13 and Proposition 2.14, we ignore the \( \sum \) when run up the multiplication \( m_2 \). This does not affect the results.

We finally give a condition under which the \( A_{\infty} \)-Ext-algebra \( E(A) \) is generated by \( E^1(A) \).

**Proposition 2.15.** Let \( A \) be a truncated bi-Koszul algebra determined by \( \Delta_d \) (\( d \geq 4 \)), \( E := E(A) \) its Ext-algebra. If either \( (E; m_2, m_d) \) or \( (E; m_2, m_{d+1}) \) is generated by \( E^1 \) and \( E^2 \), then \( (E; m_2, m_d, m_{d+1}) \) is generated by \( E^3 \).

**Proof.** By Proposition 2.10, \( E \) is generated by \( E^1, E^2, E^3 \) as an associative algebra. Clearly, \( E^2 = m_d(E^1 \otimes \cdots \otimes E^1) + m_{d+1}(E^1 \otimes \cdots \otimes E^1) \) in \( (E; m_2, m_d, m_{d+1}) \). Therefore, to prove the result, we need only to show that \( E^3 \) can be generated by \( E^1 \) and \( E^2 \) in \( (E; m_2, m_d, m_{d+1}) \), which follows either from the assumption of \( (E; m_2, m_d) \) generated by \( E^1 \) and \( E^2 \) then

\[
E^3 = \sum_{i=1}^{d} m_d(E^1 \cdot \cdots \cdot E^1)_i,
\]

or from the assumption of \( (E; m_2, m_{d+1}) \) generated by \( E^1 \) and \( E^2 \) then

\[
E^3 = \sum_{i=1}^{d+1} m_{d+1}(E^1 \cdot \cdots \cdot E^1)_i.
\]

We get the result. \( \square \)

### 3. Balanced Generating

For a bi-Koszul algebra \( A \), it is a question whether \( E(A) \) is finitely generated as a graded algebra. In this section, we generalize the concept of “generating”, and show that \( E(A) \) is \( [m_2, m_3] \)-finitely generated by \( E^1(A), E^2(A) \) and \( E^3(A) \) for any bi-Koszul algebra \( A \). An equivalent statement of a bi-Koszul algebra is given in terms of such concept.

**3.1. \([m_2, m_3]\)-Generating.** The original concept of “generating” in the associative algebra setting is defined with respect to the multiplication. When we work in the field of \( A_{\infty} \)-algebras, we need a generalized concept of “generating” to reflect certain balance between multiplications and elements.

**Definition 3.1.** Let \( E \) be an \( A_{\infty} \)-algebra. Suppose there exists a fixed integer \( l \) and multiplications \( m_{n_1}, \ldots, m_{n_t} \) such that, for any \( p > l \),

\[
E^p = \sum_{k_1+\cdots+k_{n_1}+2\cdots+n_t=p; \ k_1,\ldots,k_{n_t}\geq 1; \ 1\leq i\leq t} m_{n_i}(E^{k_1} \otimes \cdots \otimes E^{k_{n_i}}).
\]
Proposition 3.2. Let $E = [m_{n_1}, \ldots, m_{n_l}]$-finitely generated by $E^1, \ldots, E^l$.

Remark 3.1. In the case of $m_1 = 0$ and $[m_{n_1}, \ldots, m_{n_l}] = [m_2]$, the concept is the original one of finitely generating as an associative graded algebra.

Here are the examples.

If $A$ is a $p$-Koszul algebra ($p \geq 3$), then any $A_{\infty}$-algebra $(E(A); m_2, m_p)$ is $[m_2, m_p]$-finitely generated by $E^1(A)$. This is obtained by noting the facts that $E(A)$ is generated by $E^1(A)$ and $E^2(A)$, while $E^2(A) = m_p(E^1(A) \otimes \cdots \otimes E^1(A))$ ([H Theorem 2.5]).

If $A$ is a bi-Koszul algebra, then there exists an $A_{\infty}$-algebra $(E(A); m_2, m_3, m_4, m_d, m_{d+1})$ such that $E(A)$ is $[m_2, m_3, m_4, m_d, m_{d+1}]$-finitely generated by $E^1(A)$.

If we admit the set of multiplications to be infinite, Keller’s result tells that there exists an $A_{\infty}$-algebra structure on $E(A)$ which is generated by $E^1(A)$ [6 Proposition 1(b)].

It is natural to expect that the multiplications in the set $\{m_{n_1}, \ldots, m_{n_l}\}$ as less as possible. When $A$ is a bi-Koszul algebra, though we can not claim $E(A)$ is $[m_2]$-finitely generated, it does be finitely generated as long as to add one higher multiplication.

Now let $A$ be a bi-Koszul algebra determined by $\Delta_d$, and $E := E(A)$ the Ext-algebra of $A$ in the following. Denote

- $U^{3k+2}$: the sum of the actions of $m_3$ on all permutations of $E^{3k+2}$ and $E^{3k+2}_{2d+3}$ for any $k_1 \geq 1, k_2, k_3 \geq 0$ with $k_1 + k_2 + k_3 = k$;
- $V^{3k+2}$: the sum of the actions of $m_3$ on all permutations of $E^{3k+2}_{2d+3}$ and $E^{3k+2}_{2d+3}$ for any $k_1, k_2, k_3 \geq 0$ with $k_1 + k_2 + k_3 = k - 1$;
- $W^{3k+2}$: the sum of the actions of $m_4$ on all permutations of $E^{3k+2}$, $E^{3k+2}_{2d+3}$ and $E^{3k+2}_{2d+3}$ for any $k_1, k_2, k_3, k_4 \geq 0$ with $k_1 + k_2 + k_3 + k_4 = k - 1$.

Proposition 3.2. Let $A$ be a bi-Koszul algebra. Then for any $k \geq 1$,

\[
E^{3k+3} = m_2(E^3 E^{3k}) = m_2(E^{3k} E^3);
E^{3k+1} = m_2(E^1 E^{3k}) = m_2(E^{3k} E^1);
E^{3k+2}_{2d+3} = m_2(E^2 E^{3k}) = m_2(E^{3k} E^2).
\]

Proof. By Theorem [1,3] we need only to show the last equality:

\[
m_2(E^2 E^{3k}) = m_2(E^{3k} E^2).
\]

Since $m_2(E^2 E^{3k}) = m_2(m_d(E^1 \cdots E^1) E^{3k})$, to get $m_2(E^2 E^{3k}) \subseteq m_2(E^{3k} E^2)$ we need only to verify $m_2(m_d(x_1, \cdots, x_d), y) \in m_2(E^{3k} E^2)$ for any $x_1, \cdots, x_d \in E^1$ and $y \in E^{3k}$. This is performed by using the Stasheff’s identity $SI(d+1)$

\[
m_2(m_d(x_1, \cdots, x_d), y) = m_d(x_1, \cdots, x_{d-1}, m_2(x_d, y))
\]

with $m_2(x_d, y) \in E^{3k+1}$. Since $E^{3k+1} = m_2(E^{3k} E^1)$,

\[
m_d(x_1, \cdots, x_{d-1}, m_2(x_d, y)) = m_d(x_1, \cdots, x_{d-1}, m_2(y', x_d'))
= m_d(x_1, \cdots, m_2(x_{d-1}, y'), x_d')
\]
with $y' \in E^{3k}$ and $x'_d \in E^1$. We can continue the foregoing procedure to obtain

$$m_d(x_1, \ldots, x_{d-1}, m_2(x_d, y)) = m_d(m_2(z, x'_1), \ldots, x'_d) = m_2(z, m_d(x'_1, \ldots, x'_d))$$

with $z \in E^{3k}$ and $x'_1, \ldots, x'_d \in E^1$. Thus, $m_2(E^3_1 E^{3k}_d) \subseteq m_2(E^2_1 E^{3k}_d)$.

The converse $m_2(E^{3k}_1 E^{3k}_d) \subseteq E^{3k+2}_{2d+k+d} = m_2(E^{2k}_d E^{3k})$ is clear from Theorem 1.4 again.

\[\square\]

**Lemma 3.3.** Let $A$ be a bi-Koszul algebra. Assume $k_i \geq 0$ ($i = 1, \ldots, d+1$) and $k_1 + \cdots + k_{d+1} = k \geq 1$. Then

$$m_{d+1}(E^{3k_1+1} \cdots E^{3k_{d+1}+1}) \subseteq \sum_{i_1 + i_2 = k, i_1 \geq 1, i_2 \geq 0} m_2(E^{3i_1} E^{3i_2+2}) + U^{3k+2}.$$

**Proof.** There exists an integer $i$ ($1 \leq i \leq d+1$) such that $k_i \geq 1$. If $2 \leq i \leq d+1$, using $\text{Sl}(d+2)$, we get

$$m_{d+1}(E^{3k_1+1} \cdots E^{3k_{d+1}+1}) = m_{d+1}(E^{3k_1} \cdots E^{3k_{i-1}+1} E^{3k_i} E^{3k_{i+1}} \cdots E^{3k_{d+1}+1}) + U^{3k+2} \subseteq m_{d+1}(E^{3(k_1 + \cdots + k_i) + 1} \cdots E^{3k_{d+1}+1}) + U^{3k+2}$$

with $k_1 + \cdots + k_i \geq 1$.

So we can assume $k_i \geq 1$. Using $\text{Sl}(d+2)$ again,

$$m_{d+1}(E^{3k_1+1} \cdots E^{3k_{d+1}+1}) = m_{d+1}(m_2(E^{3k_1} E^1) E^{3k_2+1} \cdots E^{3k_{d+1}+1}) \subseteq m_3(E^{3k_1+1} E^{3k_2+1} \cdots E^{3k_{d+1}+1}) + m_3(E^{3k_1} E^{3k_2+1} E^{3k_{d+1}+1}) + m_2(E^{3k_1} E^{3k_2+1} E^{3k_{d+1}+1}).$$

We complete the proof. \[\square\]

**Lemma 3.4.** Let $A$ be a bi-Koszul algebra. Then

$$W^{3k+2} \subseteq \sum_{i_1 + i_2 = k, i_1 \geq 1, i_2 \geq 0} m_2(E^{3i_1} E^{3i_2+2}) + U^{3k+2} + V^{3k+2}.$$

**Proof.** By ignoring the lower grading, we may write

$$W^{3k+2} = \sum m_4(E^{3k_1+1} E^{3k_2+2} E^{3k_3+i_3} E^{3k_4+i_4})$$

where the sum runs over all $i_1 + i_2 + i_3 + i_4 = 7$ ($i_j = 1$ or 2) and $k_j \geq 0$.

First, assume $k_1 + k_2 + k_3 + k_4 = 0$. In this case, the first or last component, say the last component, must be $E^2_3$. Using $\text{Sl}(d+3)$, we have

$$m_4(E^{i_1} E^{i_2} E^{i_3} E^{i_4}) = m_4(E^{i_1} E^{i_2} E^{i_3} m_3(E^{1} \cdots E^{1})) \subseteq m_{d+1}(E^{1} E^{1} \cdots E^{1}) + m_{d+1}(E^{1} E^{2} \cdots E^{1}) + U^{3k+2}.$$
Next, consider $k_1 + k_2 + k_3 + k_4 \geq 1$.

If $k_2 \geq 1$,

$$m_4(E^{3k_1+i_1}E^{3k_2+i_2}E^{3k_3+i_3}E^{3k_4+i_4}) \subseteq m_4(E^{3k_1+3k_2+i_1}E^{i_2}E^{3k_3+i_3}E^{3k_4+i_4}) + U^{3k+2} + V^{3k+2}.$$ 

If $k_3 \geq 1$, by the similar method we get

$$m_4(E^{3k_1+i_1}E^{3k_2+i_2}E^{3k_3+i_3}E^{3k_4+i_4}) \subseteq m_4(E^{3k_1+3k_2+3k_3+i_1}E^{i_2}E^{3k_4+i_4}) + U^{3k+2} + V^{3k+2}.$$ 

Now we assume that $k_1 \geq 1$ (the case of $k_4 \geq 1$ is symmetrical). Whether $E^{3k_1+i_1} = E^{3k_1+1}$ or $E^{3k_1+2}$, by $\text{S}(5)$ we get

$$m_4(E^{3k_1+i_1}E^{3k_2+i_2}E^{3k_3+i_3}E^{3k_4+i_4}) \subseteq U^{3k+2} + V^{3k+2}.$$ 

We complete the proof. \hfill \Box

Examining the table in Proposition 2.7 again, the following result is clear from Proposition 2.2 the lemmas 3.3 and 3.4

**Corollary 3.5.** Let $A$ be a bi-Koszul algebra. Then the actions of $m_4, m_d, m_{d+1}$ which fall into $E^{\geq 4}$ are determined by the actions of $m_2$ and $m_3$. \hfill \Box

Now we can state an equivalent statement of the bi-Koszul algebra.

**Theorem 3.6.** Let $A$ be a locally finite, connected graded algebra generated in degree 1. Then $A$ is a bi-Koszul algebra if and only if there exists a reduced $A_\infty$-algebra $(E(A);\{m_i\})$ which is $[m_2, m_3]$-finitely generated by $E^1(A), E^2(A)$ and $E^3(A)$ with $E^1(A) = E^1_d(A), E^2(A) = E^2_d(A) \oplus E^2_{d+1}(A), E^3(A) = E^3_{2d}(A)$.

**Proof.** Assume that $A$ is a bi-Koszul algebra. Take an $A_\infty$-algebra $(E(A);\{m_i\})$ that is generated by $E^1(A)$, then $(E(A);\{m_i\})$ is a reduced $A_\infty$-algebra by Corollary 2.7. By Proposition 3.2 the lemmas 3.3 and 3.4 we get $E(A)$ begins with $E^1(A) = E^1_d(A), E^2(A) = E^2_d(A) \oplus E^2_{d+1}(A), E^3(A) = E^3_{2d}(A)$, and for each $k \geq 1$,

$$E^{3k+3} = \sum_{k_1+k_2=k} m_2(E^{3k_1}E^{3k_2}),$$

$$E^{3k+1} = \sum_{k_1+k_2=k} m_2(E^{3k_1}E^{3k_2+1}) = \sum_{k_1+k_2=k} m_2(E^{3k_1+1}E^{3k_2}),$$

$$E^{3k+2} = \sum_{k_1+k_2=k} m_2(E^{3k_1}E^{3k_2+2}) + \sum_{k_1+k_2=k} m_2(E^{3k_1+2}E^{3k_2}) + U^{3k+2} + V^{3k+2},$$

which implies that $(E(A);\{m_i\})$ is $[m_2, m_3]$-finitely generated by $E^1(A), E^2(A)$ and $E^3(A)$.

The converse is straightforward by comparing the lower grading. \hfill \Box

The result above tells that the obstruction in Theorem 1.4 can be described by the multiplications $m_2$ and $m_3$. Using the multiplications $m_2$ and $m_3$, we may also give a criteria for a bi-Koszul algebra to be strongly.
Proposition 3.7. Let $A$ be a bi-Koszul algebra. Then $A$ is strongly if and only if for any $A_{\infty}$-algebra $(E(A); \{m_i\})$,

$U^{3k+2} \subseteq m_2(E_{d+1}^2 E_{d+1}^{3k})$, and

$V^{3k+2} \subseteq \sum_{k_1+k_2=k} m_2(E_{d+1}^{3k_1} E_{d+1}^{3k_2+2}) + \sum_{k_1+k_2=k} m_2(E_{d+1}^{3k_1+2} E_{d+1}^{3k_2}).$

Proof. The necessity is obvious. To show the condition being sufficient, we need only to check

$E_{2dk+d+1}^{3k+2} = m_2(E_{d+1}^{2} E_{d+1}^{3k})$, for any $k \geq 1$. (*)

The reason is that the obstruction arises only from the bigger degree $2dk + d + 1$ in $\Delta_d(3k+2)$ as we pointed before.

We first show that

$m_2(E_{d+1}^{2} E_{d+1}^{3k}) + U^{3k+2} = m_2(E_{d+1}^{2} E_{d+1}^{3k}) + U^{3k+2}. (**)$

In fact, by the Stasheff’s identity $SI(d+2)$

$m_2(E_{d+1}^{3k} E_{d+1}^{2}) = m_2(E_{d+1}^{3k} m_{d+1}(E_1 \cdots E_1))$

$\subseteq m_{d+1}(m_2(E_{d+1}^{3k} E_1) \cdots E_1) + U^{3k+2}$

$= m_{d+1}(m_2(E_1 E_{d+1}^{3k}) \cdots E_1) + U^{3k+2}$

$\subseteq m_{d+1}(E_1 m_2(E_{d+1}^{3k} E_1) \cdots E_1) + U^{3k+2}$

$\subseteq \ldots \ldots$

$\subseteq m_2(E_{d+1}^{2} E_{d+1}^{3k}) + U^{3k+2},$

which implies one inclusion relation, the opposite inclusion is similar to prove.

Using (**) and the condition on $U^{3k+2}$, we have $m_2(E_{d+1}^{3k} E_{d+1}^{2}) \subseteq m_2(E_{d+1}^{2} E_{d+1}^{3k}).$

Again from the conditions on $U^{3k+2}$, $V^{3k+2}$ and the proof of Theorem 3.6 we get

$E_{2dk+d+1}^{3k+2} = \sum m_2(E_{d+1}^{3k_1} E_{d+1}^{3k_2+2}) + \sum m_2(E_{d+1}^{3k_1+2} E_{d+1}^{3k_2}).$

We prove (*) by induction on $k \geq 1$.

$E_{3d+1}^{5} = m_2(E_{d+1}^{3} E_{d+1}^{2}) + m_2(E_{d+1}^{2} E_{d+1}^{2}) = m_2(E_{d+1}^{2} E_{d+1}^{3}).$

Suppose that $E_{2dk+d+1}^{3k+2} = m_2(E_{d+1}^{2} E_{d+1}^{3k})$ for all $1 \leq i < k$. Now, for any $k_1 + k_2 = k \geq 2 (k_1 \geq 1$ and $k_2 \geq 1)$,

$m_2(E_{2dk_1+d+1}^{3k_1} E_{2dk_2+d+1}^{3k_2+2}) = m_2(E_{d+1}^{2} E_{d+1}^{3k_1+2} E_{d+1}^{3k_2}) \subseteq m_2(E_{d+1}^{2} E_{d+1}^{3k_1+2} E_{d+1}^{3k_2})$

and

$m_2(E_{2dk_1+d+1}^{3k_1+2} E_{2dk_2+d+1}^{3k_2}) = m_2(E_{d+1}^{2} E_{d+1}^{3k_1} E_{d+1}^{3k_2}) \subseteq m_2(E_{d+1}^{2} E_{d+1}^{3k_1} E_{d+1}^{3k_2}).$

This proves, for any $k \geq 1$, $E_{2dk+d+1}^{3k+2} \subseteq m_2(E_{d+1}^{2} E_{d+1}^{3k})$, the opposite inclusion is trivial, so we have (*).

Hence the bi-Koszul algebra $A$ is strongly. \qed

Corollary 3.8. If $A$ is a bi-Koszul algebra with $gl.dim(A) \leq 4$, or its Ext-algebra $E(A)$ with $m_3 = 0$, then $A$ is a strongly bi-Koszul algebra.
**Proof.** It is clear since each assumption implies $U^{3k+2} = V^{3k+2} = 0$ for any $k > 0$ by Proposition 3.7.

**Example 3.9.** Any truncated bi-Koszul algebra is strongly.

3.2. **Generated by $E^1(A)$**. Let $A$ be a bi-Koszul algebra, Theorem 3.6 tells that any $A_\infty$-algebra $E(A)$ is $[m_2, m_3]$-finitely generated by $E^1(A)$, $E^2(A)$ and $E^3(A)$. On the other hand, Keller has claimed that there exists an $A_\infty$-algebra structure on $E(A)$ which is generated by $E^1(A)$. In this subsection, we discuss the universality of the property that $E(A)$ is generated by $E^1(A)$ as an $A_\infty$-algebra.

For example, let $A$ be an Artin-Schelter regular algebra listed in [9, Theorem A], then any $A_\infty$-algebra $E(A)$ is generated by $E^1(A)$.

Before discussing, we give a general result which points out that a strict isomorphism of $A_\infty$-algebras can be obtained from a quasi-isomorphism of $A_\infty$-algebras. This was found in [4] for single $A_\infty$-algebras.

**Lemma 3.10.** Let $(E; \{m_i\})$ and $(E'; \{m'_i\})$ be two minimal $A_\infty$-algebras, and $(f_i): (E; \{m_i\}) \to (E'; \{m'_i\})$ a quasi-isomorphism between them. Then

1. $(E'; \{m''_i\})$ is a minimal $A_\infty$-algebra where $m''_i := f_1 m_i f_1^{-1} \otimes \cdots \otimes f_1^{-1}$ with $m''_0 = m'_1$.
2. $(g_i) : (E; \{m_i\}) \to (E'; \{m'_i\})$ is a strict isomorphism of $A_\infty$-algebras where $g_1 = f_1$ and $g_i = 0$ for all $i \geq 2$.

**Proof.** Since $m_1 = m'_1 = 0$, $f_1 : (E; m_2) \to (E'; m'_2)$ is an isomorphism. To prove the first statement, we need the Stasheff’s morphism identities $SI(n)$ ($n = 1, 2, \cdots$) for $\{m''_i\}$. Note that the degrees of both $f_1$ and $f_1^{-1}$ are zero, the Koszul sign convention can be neglected in the following. For any $i + t + j = n$ and $l = i + 1 + j$,

\[
m''_l (1^{\otimes i} \otimes m''_t \otimes 1^{\otimes j}) = f_1 m_i (f_1^{-1} \otimes \cdots \otimes f_1^{-1}) (1^{\otimes i} \otimes m_i (f_1^{-1} \otimes \cdots \otimes f_1^{-1}) \otimes 1^{\otimes j}) = f_1 m_l (1^{\otimes i} \otimes m_t \otimes 1^{\otimes j}) (f_1^{-1} \otimes \cdots \otimes f_1^{-1}),
\]

hence

\[
\sum_{n=i+t+j} (-1)^{i+t} m''_i (1^{\otimes i} \otimes m''_t \otimes 1^{\otimes j}) = f_1 \left( \sum_{n=i+t+j} (-1)^{i+t} m_i (1^{\otimes i} \otimes m_t \otimes 1^{\otimes j}) (f_1^{-1} \otimes \cdots \otimes f_1^{-1}) \right) (f_1^{-1} \otimes \cdots \otimes f_1^{-1}) = 0.
\]

Moreover, $m''_l = f_1 m_2 (f_1^{-1} \otimes f_1^{-1}) = m'_2 (f_1 \otimes f_1) (f_1^{-1} \otimes f_1^{-1}) = m'_2$.

Clearly, $f_1 m_i = m''_l (f_1 \otimes \cdots \otimes f_1)$. So $\{g_i\}$ is a strict isomorphism between $(E; \{m_i\})$ and $(E'; \{m'_i\})$. \qed
Let \( A \) be a bi-Koszul algebra determined by \( \Delta_d, E := E(A) \) the Ext-algebra of \( A \). There is a quasi-isomorphism between two \( A_\infty \)-algebra structures on \( E(A) \):

\[
\{ f_i \} : (E(A); \{ m_i \}) \rightarrow (E(A); \{ m_i' \}).
\]

Now assume that \( (E(A); \{ m_i \}) \) is generated by \( E^1(A) \), it is a natural question that whether the same claim is true for \( (E(A); \{ m_i' \}) \).

The following facts are immediately:

(i) if \( n \geq 2, f_n(E^1 \cdots E^1) = 0; \)

(ii) if \( d \geq 3, m_2(E^1 E^1) = m_2(E^1 E^2) = m_2(E^2 E^1) = 0; \)

(iii) there are only \( m_d' \) and \( m_d' + 1 \) whose actions can fall into \( E^3 \) in \( (E(A); \{ m_i' \}) \);

(iv) \( f_1 : E(A) \rightarrow E(A) \) is an isomorphism.

Combining with Lemma \ref{lem:2.1} and Proposition \ref{prop:2.5} we can write the Stasheff’s morphism identities, for small \( n \) or in some special cases, more clearly.

(1) MI(2): \( f_1m_2 = m_2'(f_1 \otimes f_1); \)

(2) MI(3): \( f_1m_3 + f_2(m_2 \otimes 1) - f_2(1 \otimes m_2) = m_3'(f_1 \otimes f_1 \otimes f_1) + m_2'(f_1 \otimes f_2) - m_2'(f_2 \otimes f_1); \)

(3) MI(d) acting on \( E^1 \cdots E^1 \) or \( E^1 \cdots E_{d+1} \cdots E^1 \) can be reduced as

\[
f_1 m_d = m_d'(f_1 \otimes \cdots \otimes f_1);
\]

(4) MI(d+1) acting on \( E^1 \cdots E^1 \) can be reduced as

\[
f_1 m_{d+1} + (-1)^d f_2(m_d \otimes 1) + (-1)^{d+1} f_2(1 \otimes m_d) = m_{d+1}'(f_1 \otimes \cdots \otimes f_1);
\]

(5) MI(d+1) acting on \( E^1 \cdots E^2 \cdots E^1 \) can be reduced as

\[
f_1 m_{d+1} + (-1)^d f_2(m_d \otimes 1) + (-1)^{d+1} f_2(1 \otimes m_d)
= m_{d+1}'(f_1 \otimes \cdots \otimes f_1) + \sum_{1 \leq j \leq d} (-1)^{d-j} m_d'(f_1 \otimes \cdots \otimes f_2 \otimes \cdots \otimes f_1).
\]

**Proposition 3.11.** Assume \( E^3(A) \) is generated by \( E^1(A) \) and \( E^2(A) \) with \( \{ m_i \} \).

If \( f_2(m_d \otimes 1) = f_2(1 \otimes m_d) \), then \( E^3(A) \) is also generated by \( E^1(A) \) and \( E^2(A) \) with \( \{ m_i' \} \).

**Proof.** The hypothesis on \( E^3(A) \) tells us that

\[
E^3 = \sum m_d(E^1 \cdots E_{d+1} \cdots E^1) + \sum m_{d+1}(E^1 \cdots E^2 \cdots E^1).
\]

So \( E^3 = f_1(E^3) = \sum f_1m_d(E^1 \cdots E_{d+1} \cdots E^1) + \sum f_1m_{d+1}(E^1 \cdots E^2 \cdots E^1). \)

By the Stasheff’s morphism identities listed above, we have

\[
\begin{align*}
f_1m_d(E^1 \cdots E_{d+1} \cdots E^1) \\
\subseteq m_d'(f_1 \otimes \cdots \otimes f_1)(E^1 \otimes \cdots \otimes E^2_{d+1} \otimes \cdots \otimes E^1) \\
= m_d'(E^1 \cdots E^2_{d+1} \cdots E^1),
\end{align*}
\]
and
\[ f_1 m_{d+1}(E^1 \cdots E^2_d \cdots E^1) \]
\[ \subseteq m'_{d+1} (f_1 \otimes \cdots \otimes f_1)(E^1 \otimes \cdots \otimes E^2_d \otimes \cdots \otimes E^1) \]
\[ + \sum_{1 \leq j \leq d} (-1)^{d-j} m'_d(f_1 \otimes \cdots \otimes f_j \otimes \cdots \otimes f_1)(E^1 \otimes \cdots \otimes E^2_d \otimes \cdots \otimes E^1). \]
\[ \subseteq m'_{d+1} (E^1 \cdots E^2_d \cdots E^1) + \sum m'_d(E^1 \cdots E^2_d \cdots E^1). \]

Thus, \( E^3 \subseteq \sum m'_d(E^1 \cdots E^2_{d+1} \cdots E^1) + \sum m'_{d+1}(E^1 \cdots E^2_d \cdots E^1). \) We complete the proof. \( \square \)

Now, we can prove the main theorem of this subsection.

**Theorem 3.12.** Let \( \{f_i\} : (E(A); \{m_i\}) \to (E(A); \{m'_i\}) \) be a quasi-isomorphism with \( f_2(m_i \otimes 1) = f_2(1 \otimes m_i) \) for \( i = 2, d \), suppose that \( (E(A); \{m_i\}) \) is generated by \( E^1(A) \). Then \( (E(A); \{m'_i\}) \) is also generated by \( E^1(A) \).

**Proof.** Since \( m_1 = m'_1 = 0, f_1 : (E(A), m_2) \to (E(A), m'_2) \) is an isomorphism with degree zero. By the proof of Theorem 3.11 and MI(2), we have

\[ E^{3k+3} = f_1(E^{3k+3}) = f_1(\sum m_2(E^{3k_1}E^{3k_2})) \]
\[ \subseteq \sum m'_2(f_1 \otimes f_1)(E^{3k_1} \otimes E^{3k_2}) = \sum m'_2(E^{3k_1}E^{3k_2}); \]
\[ E^{3k+1} = f_1(E^{3k+1}) = f_1(\sum m_2(E^{3k_1+1}E^{3k_2})) \]
\[ \subseteq \sum m'_2(f_1 \otimes f_1)(E^{3k_1+1} \otimes E^{3k_2}) = \sum m'_2(E^{3k_1+1}E^{3k_2}); \]
\[ E^{3k+2} = f_1(E^{3k+2}) \]
\[ = f_1(\sum m_2(E^{3k_1}E^{3k_2} + \sum m_2(E^{3k_1+2}E^{3k_2}) + U^{3k+2} + V^{3k+2}) \]
\[ \subseteq \sum m'_2(E^{3k_1}E^{3k_2} + \sum m'_2(E^{3k_1+2}E^{3k_2}) + f_1(U^{3k+2}) + f_1(V^{3k+2}). \]

For any \( E^{3i_1}E^{3i_2+1}E_{2d i_3+d} \) \((i_1 + i_2 + i_3 = k, i_1 \geq 1, i_2, i_3 \geq 0)\), the assumption \( f_2(m_2 \otimes 1) = f_2(1 \otimes m_2) \) and MI(3) imply that

\[ f_1 m_3(E^{3i_1}E^{3i_2+1}E_{2d i_3+d}) \]
\[ \subset m'_3(E^{3i_1}E^{3i_2+1}E_{2d i_3+d}) + m'_2(f_2(E^{3i_1}E^{3i_2+1})f_1(E^{3i_3+2} + \sum m'_2(E^{3i_1+1}E_{2d i_3+d})) \]
\[ + m'_2(f_1(E^{3i_1})f_2(E^{3i_2+1}E_{2d i_3+d}))) \]
\[ \subseteq m'_2(E^{3i_1}E^{3i_2+1}E_{2d i_3+d}) + m'_2(E^{3i_1+1}E^{3i_2+2}E_{2d i_3+d}) \]
\[ + m'_2(E^{3i_1}E^{3i_2+1}E_{2d i_3+d}). \]

By the same method, we obtain that the action of \( f_1 \) on every component of \( U^{3k+2} \) or \( V^{3k+2} \) falls into \( \sum m'_2(E^{3k_1}E^{3k_2} + \sum m'_2(E^{3k_1+2}E^{3k_2}) + U^{3k+2} + V^{3k+2} \) where \( U^{3k+2} \) and \( V^{3k+2} \) correspond to \( U^{3k+2} \) and \( V^{3k+2} \), respectively, changing the multiplication \( m_3 \) to \( m'_3 \). Thus,

\[ E^{3k+2} = \sum m'_2(E^{3k_1}E^{3k_2} + \sum m'_2(E^{3k_1+2}E^{3k_2}) + U^{3k+2} + V^{3k+2}. \]

Since \( E^2 \) is generated by \( E^1 \), and \( E^3 \) is generated by \( E^1 \) and \( E^2 \) by Proposition 3.11, we get \( (E(A); \{m'_i\}) \) is also generated by \( E^1 \). \( \square \)
Corollary 3.13. Assume the bi-Koszul algebra $A$ is strongly. If $(E(A); \{m_i\})$ is generated by $E^1$, and $f_2(m_d \otimes 1) = f_2(1 \otimes m_d)$, then $(E(A); \{m'_i\})$ is also generated by $E^1$.

Continuing to consider the quasi-isomorphism between two $A_\infty$-structures on $E(A)$, $\{f_i\} : (E(A); \{m_i\}) \to (E(A); \{m'_i\})$, some extra hypothesis will make $\{f_i\}$ to be a strict isomorphism which guarantees the properties of such two $A_\infty$-algebras identify with each other.

Theorem 3.14. If $f_2(m_i \otimes 1) = f_2(1 \otimes m_i)$ for $i = 2, d$, and

$$\sum_{1 \leq j \leq d} (-1)^{i-j} m'_i(f_1 \otimes \cdots \otimes f_j \otimes \cdots f_1) = 0,$$

then $g = f_1 : (E(A); \{m_i\}) \to (E(A); \{m'_i\})$ is a strict isomorphism.

Proof. By Lemma 3.10, $g = f_1 : (E(A); \{m_i\}) \to (E(A); \{m'_i\})$ is a strict isomorphism and $m''_2 = m'_d = m_2$. The assumption directly implies $m''_i = f_1m_3(f_{i-1} \otimes f_{i-1}) = m'_3$. For any $x_1, \cdots, x_d \in E^1$, or one of $x_j$’s in $E^1_{d+1}$ and the others in $E^1$, we have

$$m''_d(x_1, \cdots, x_d) = f_1m_d(f_{i-1}^{-1}(x_1), \cdots, f_{i-1}^{-1}(x_d)) = m'_d(x_1, \cdots, x_d)$$

by the Stasheff’s morphism identities listed above. So $m''_d|_{E^1 \otimes d} = m'_d|_{E^1 \otimes d}$ and $m''_d|_{E^1 \cdots E^1_{d+1} \cdots E^1} = m'_d|_{E^1 \cdots E^1_{d+1} \cdots E^1}$. By the same method, we check that $m''_{d+1}|_{E^1 \otimes (d+1)} = m'_{d+1}|_{E^1 \otimes (d+1)}$ and $m''_{d+1}|_{E^1 \cdots E_{d+1} \cdots E^1} = m'_{d+1}|_{E^1 \cdots E_{d+1} \cdots E^1}$. Thus, $(E(A); \{m_i\})$ and $(E(A); \{m'_i\})$ are the same. The result follows immediately. □

Corollary 3.15. Assume the bi-Koszul algebra $A$ is strongly. If $f_2(m_d \otimes 1) = f_2(1 \otimes m_d)$ and $\sum_{1 \leq j \leq d} (-1)^{d-j} m'_d(f_1 \otimes \cdots \otimes f_j \otimes f_1) = 0$. Then

$$g = f_1 : (E(A); \{m_i\}) \to (E(A); \{m'_i\})$$

is a strict isomorphism.

Proof. By the proof of Theorem 3.14. □

References

1. A. Beilinson, V. Ginzburg and W. Soergel, Koszul duality patterns in representation theory, J. Amer. Math. Soc. 9 (1996), no. 2, 473-52
2. R. Berger, Koszulity of nonquadratic algebras, J. Algebra, 239 (2001), 705-734.
3. T. Cassidy and B. Shelton, Generalizing the notion of a Koszul algebra, Math. Z., 260 (2008), 93-114.
4. J.-W. He and D.-M. Lu, Higher Koszul algebras and $A$-infinity algebras, J. Algebra, 293 (2005), 335-362.
5. T. V. Kadeishvili, On the theory of homology of fiber spaces, (Russian) International Topology Conference (Moscow State Univ., Moscow, 1979). Uspekhi Mat. Nauk 35 (1980), no. 3 (213), 183–188. The English translation was published in Russian Math. Surveys, 35, no.3 (1980), 231-238.
6. B. Keller, *A-infinity algebras in representation theory*, Contribution to the Proceedings of ICRA IX. Beijing: Peking University Press, 2000.
7. B. Keller, *Introduction to A-infinity algebras and modules*, Homology Homotopy Appl., 3 (2001), 1-35 (electronic).
8. D.-M. Lu, J. H. Palmieri, Q.-S. Wu and J. J. Zhang, *A-infinity algebras for ring theorists*, Algebra Colloq., 11 (2004), 91-128.
9. D.-M. Lu, J. H. Palmieri, Q.-S. Wu and J. J. Zhang, *Regular algebras of dimension 4 and their A∞-Ext-algebras*, Duke Math. J., 137 (2007), 537-584.
10. D.-M. Lu and J.-R. Si, *Koszulity of algebras with non-pure resolutions*, Comm. Alg., accepted for publication.
11. J.-F. Lü, J.-W. He and D.-M. Lu, *Piecewise-Koszul algebras*, Sci. China Ser. A, 50 (2007), 1795-1804.
12. S. Priddy, *Koszul resolutions*, Trans. Amer. Math. Soc., 152 (1970), 39-60.

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