Symmetry enhancement of Killing horizons in $D = 6$ supergravity

U. Kayani

Abstract

We investigate the near-horizon geometry of supersymmetric extremal black holes in 6-dimensional gauged supergravity. We solve the Killing spinor equations along the lightcone and establish the independent differential and algebraic conditions which are given as the naive restriction of the KSEs on $S$. By analyzing the global properties of the Killing spinors, we prove that the near-horizon geometries undergo a (super)symmetry enhancement. This follows from generalized Lichnerowicz-type theorems for the zero modes of the Dirac operator and an index theory argument. We also prove that horizons with non-trivial fluxes admit an $\mathfrak{sl}(2, \mathbb{R})$ symmetry group.
1 Introduction

The enhancement of (super)symmetry near to brane and black hole horizons is well known. In the context of branes, many solutions are known which exhibit supersymmetry enhancement near to the brane [1, 2]. For example, the geometry of D3-branes doubles its supersymmetry to become the maximally supersymmetric $AdS_5 \times S^5$ solution. The bosonic symmetry of the near-horizon region is given by $SO(2, 4) \times SO(6)$. Similarly for the $M2$- and $M5$ branes, we have an $AdS_4 \times S^7$ near-horizon geometry for the $M2$ and a $AdS_7 \times S^4$ for the $M5$. Both these near-horizon geometries have enhanced supersymmetry and allow for 32 real supercharges. The $M2$-brane has a bosonic symmetry group $SO(2, 3) \times SO(8)$ while the $M5$-branes near-horizon geometry has $SO(2, 6) \times SO(5)$.

Black hole solutions are also known to exhibit supersymmetry enhancement; for example the $\mathbb{R} \times SO(3)$ isometry group of the Reissner-Nordström black hole in the extremal near-horizon limit enhances to $SL(2, \mathbb{R}) \times SO(3)$ with near-horizon geometry $AdS_2 \times S^2$. In addition, viewing the extreme Reissner-Nordström black hole as a solution of the $\mathcal{N} = 2, D = 4$ minimal supergravity, the $\mathcal{N} = 4$ supersymmetry of the solution also enhances to $\mathcal{N} = 8$ near the horizon. Other black hole solutions which exhibit (super)symmetry enhancement include the five-dimensional BMPV black hole [3, 4, 5]. The essential features of this (super)symmetry enhancement mechanism have been described in the form of the horizon conjecture.

This phenomenon played a crucial role in the early development of the AdS/CFT correspondence [6]. Further recent interest in the geometry of black hole horizons has arisen in the context of the BMS-type symmetries associated with black holes, following [7, 8, 9, 10]. In particular, the analysis of the asymptotic symmetry group of Killing horizons was undertaken in [11]. In that case, an infinite dimensional symmetry group is obtained, analogous to the BMS symmetry group of asymptotically flat solutions.

The black hole horizon topology is important in establishing black hole uniqueness theorems. In $D = 4$ these imply that the Einstein equations admit a unique class of asymptotically flat black hole solutions, parametrized by $(M, Q, J)$. A key step is to establish the horizon topology theorem, which proves that the event horizon of a stationary black hole must have $S^2$ topology [12]. This relies on the Gauss-Bonnet theorem applied to the 2-manifold spatial horizon section, and therefore does not generalize to higher dimensions. Indeed, the first example of how the classical uniqueness theorems break down in higher dimensions is given by the five-dimensional black ring solution [13, 14]. There exist black ring solutions with the same asymptotic conserved charges as BMPV black holes, but with a different horizon topology. Even more exotic solutions in five dimensions are now known to exist, such as the solutions obtained in [15], describing asymptotically flat black holes which possess a non-trivial topological structure outside the event horizon, but whose near-horizon geometry is the same as that of the BMPV solution.

The understanding of supersymmetric black holes is facilitated by the recent progress that has been made towards understanding the geometry of all supersymmetric backgrounds of supergravity theories. In particular, we shall exploit the fact that an extremal black hole has a well-defined near-horizon limit which solves the same field equations as the full black hole solution. We shall consider the validity of the following conjecture...
concerning the properties of regular near-horizon geometries, in $N = (1, 0), D = 6$ supergravity, for which all fields are smooth and the spatial cross section of the event horizon, $S$, is smooth and compact without boundary:

- the number of Killing spinors $N$, $N \neq 0$, of Killing horizons in supergravity is given by
  \[ N = 2N_- + \text{Index}(D_E), \]
  where $N_- \in \mathbb{N}_{>0}$ and $D_E$ is a Dirac operator twisted by a vector bundle $E$, defined on the spatial horizon section $S$, which depends on the gauge symmetries of the supergravity theory in question,

- that horizons with non-trivial fluxes and $N_- \neq 0$ admit an $\mathfrak{sl}(2, \mathbb{R})$ symmetry sub-algebra.

Establishing the horizon conjecture relies on establishing Lichnerowicz-type theorems and an index theory argument. A similar analysis has been conducted for IIA, Roman’s Massive IIA and IIB, $D = 5$ gauged with vector multiplets and $D = 4$ gauged $[16, 17, 18, 19, 20, 21]$. We shall also establish the $\mathfrak{sl}(2, \mathbb{R})$ symmetry algebra for near-horizon geometries. In general we find that the orbits of the generators of $\mathfrak{sl}(2, \mathbb{R})$ are 3-dimensional, though in some special cases they are 2-dimensional. In these special cases the geometry is a warped product $\text{AdS}_2 \times_w S$. The properties of $\text{AdS}_2$ and their relationship to black hole entropy have been examined in $[22, 23]$. Our result, together with those of our previous calculations, implies that the $\mathfrak{sl}(2, \mathbb{R})$ symmetry is a universal property of supersymmetric black holes.

Another important observation in the study of black holes is the attractor mechanism $[24]$. This states that the entropy is obtained by extremizing an entropy function which depends only on the near-horizon parameters and conserved charges, and if this admits a unique extremum then the entropy is independent of the asymptotic values of the moduli. In the case of 4-dimensional solutions the analysis of $[25]$ implies that if the solution admits $SO(2, 1) \times U(1)$ symmetry, and the horizon has spherical topology, then such a mechanism holds. In $D = 4, 5$ it is known that all known asymptotically flat black hole solutions exhibit attractor mechanism behaviour which follow from near-horizon symmetry theorems $[26]$ for any Einstein-Maxwell-scalar-CS theory. In particular, a generalization of the analysis of $[25]$ to five dimensions requires the existence of a $SO(2, 1) \times U(1)^2$ symmetry, where all the possibilities have been classified for $D = 5$ minimal ungauged supergravity $[27]$. Near-horizon geometries of asymptotically $\text{AdS}_5$ supersymmetric black holes admitting a $SO(2, 1) \times U(1)^2$ symmetry have been classified in $[28, 29]$. It remains to be determined if all supersymmetric near-horizon geometries fall into this class. There is no general proof of an attractor mechanism for higher dimensional black holes ($D > 5$) as it depends largely on the properties of the geometry of the horizon section e.g for $D = 10$ heterotic, it remains undetermined if there are near-horizon geometries with non-constant dilaton $\Phi$.

We can obtain $N = (1, 0), D = 6$ supergravity coupled to matter multiplets by reducing type IIB supergravity on $T^4$ or $K_3$; a large number of six dimensional gauged supergravities has been constructed $[30, 31, 32, 33, 34]$. Such chiral supergravities with 8
supersymmetries give rise to a small cosmological constant when compactified to $D = 4$ via the vacuum solution $\mathbb{R}^{1,3} \times S^2$; which has motivated the study of the theory \cite{39, 40, 41, 42}. Starting from a minimal model from Einstein-Maxwell supergravity and an exponential potential for the dilaton from the gauging of the $U(1)-R$-symmetry; the bosonic fields in addition to the metric are the 2-form $B_{(2)}$, dilaton $\Phi$ and Maxwell gauge field $A_{(1)}$. For this theory the only possibility for a maximally-symmetric solution turns out to be the same as the Minkowski vacuum without the need for any apparent fine tuning.

In six dimensions, $(1,0)$ near horizon geometries have been classified in minimal (ungauged) supergravity \cite{43} with either $\mathbb{R}^{1,1} \times T^4$, $\mathbb{R}^{1,1} \times K^3$ or $AdS_3 \times S^3$. When coupled to an arbitrary number of tensor and hypermultiplets \cite{44}, the near horizon geometry is locally $AdS_3 \times \Sigma^3$ where $\Sigma^3$ is a homology 3-sphere or $\mathbb{R}^{1,1} \times S^4$ where $S^4$ is a 4-manifold whose geometry depends on hypermultiplet scalars. The near horizon geometries preserving 8 supersymmetries are locally isometric to either $AdS_3 \times S^3$ or $\mathbb{R}^{1,1} \times T^4$. When the hypermultiplets are zero \cite{45}, it yields near horizon geometries locally given by those classified in the minimal theory. These near horizon geometries also follow from uplifting a 5d solution. The most studied case is the 6d BPS black string with horizon $S^1 \times S^3$ and near horizon geometry $AdS_3 \times S^3$, arising from the uplift of both a 5d spherical black hole and the 5d black ring.

In this paper, we shall be focusing on the simplest example in $D = 6$, for which the field content comprises a graviton multiplet with bosonic fields $(g^{MN}, B_{+}^{MN})$ and chiral (complex) gravitino superpartner $\psi_M$, a tensor multiplet with bosonic field $B_{-}^{MN}$ and chiral spin-1/2 superpartner $\chi$ and a vector multiplet with bosonic field $A_M$ and chiral superpartner $\lambda$ \cite{35, 46}. We shall investigate the mechanism by which supersymmetry is enhanced for supersymmetric extremal black hole near-horizon geometries in both gauged and ungauged $N = (1,0)$, $D = 6$ supergravity. Analysis and solutions of the KSEs of 6-dimensional supergravities have been investigated before in various cases \cite{46, 43, 44, 47, 48, 49, 50}.

Unlike most previous investigations of near horizon geometries, e.g \cite{51, 52, 53, 43}, we do not assume the vector bilinear matching condition, which is the identification of the stationary Killing vector field of a black hole with the vector Killing spinor bilinear; in fact we prove this is the case for the theories under consideration. In particular, we find that the emergence of an isometry generated by the spinor, from the solution of the KSEs, is proportional to Killing vector which generates the Killing horizon. Thus previous results which assumed the bilinear condition automatically follow for the theory that we consider. By analysing the conditions on the geometry, we are also able to eliminate certain solutions.

We will assume that the black hole event horizon is a Killing horizon. Rigidity theorems have been constructed which imply that the black hole horizon is Killing for both non-extremal and extremal black holes, under certain assumptions, have been constructed, e.g. \cite{54, 55, 56, 57}. The assumption that the event horizon is Killing enables the introduction of Gaussian Null co-ordinates \cite{58, 55} in a neighbourhood of the horizon. The analysis of the near-horizon geometry is significantly simpler than that of the full black hole solution, as the near-horizon limit reduces the system to a set of equations on a co-dimension 2 surface, $\mathcal{S}$, which is the spatial section of the event horizon.

The new Lichnerowicz type theorems established in this paper are of interest because
they have certain free parameters appearing in the definition of various connections and Dirac operators on $\mathcal{S}$. Such freedom to construct more general types of Dirac operators in this way is related to the fact that the minimal set of Killing spinor equations consists not only of parallel conditions on the spinors but also certain algebraic conditions. These algebraic conditions do not arise in the case of $D = 11$ supergravity. Remarkably, the Lichnerowicz type theorems imply not only the parallel transport conditions but also the algebraic ones as well. The solution of the KSEs is essential to the investigation of geometries of supersymmetric horizons. We show that the enhancement of the supersymmetry produces a corresponding symmetry enhancement, and describe the resulting conditions on the geometry.

The content in this paper is organised in the following way. In section 2, we state the key properties for $N = (1,0), D = 6$ gauged supergravity. We give the bosonic part of the action, the field equations and the fermionic supersymmetry variations (the vanishing of which are the KSEs). In section 3, we state the near-horizon data and solve the KSEs by appropriately decomposing the gauge fields and integrating along two lightcone directions, and we identify the independent KSEs. In section 4 we present some details of the calculations used to find the minimal set of independent KSEs on the spatial horizon section. In section 5, we establish a generalized Lichnerowicz-type theorem in order to show that, on spatial cross-sections of the event horizon, the zero modes certain Dirac operators $\mathcal{D}^{(\pm)}$ are in a 1-1 correspondence with the Killing spinors. In section 6, we prove the supersymmetry enhancement, and we analyse the relationship between positive and negative lightcone chirality spinors which gives rise to the doubling of the supersymmetry. We also prove that horizons with non-trivial fluxes admit an $\mathfrak{sl}(2,\mathbb{R})$ symmetry subalgebra.

In appendix A, we state the supersymmetry conventions. In appendix B, we state the spin connection and the Ricci curvature tensor. In appendix C, we state the independent horizon Bianchi identities and field equations. In section D, we state the independent horizon Bianchi identities and field equations for the gauge decomposition given in section 3.

## 2 $N = (1,0), D = 6$ gauged supergravity

In this section we will review the basics of the $N = (1,0), D = 6$ gauged supergravity from \[35, 46\]. It is a chiral theory with 8 real supersymmetries with the $U(1)$-$R$ symmetry gauged. The fermions carry the doublet index of the $R$-symmetry group $Sp(1)_{R}$. All the fermions are chiral, which means $\Gamma_{s} \lambda = \pm \lambda$ where $\Gamma_{s} = \Gamma_{0} \cdots \Gamma_{5}$. We can choose the plus sign and hence consider left handed spinors. We have the following multiplets,

$$(e_{M}^{a}, \psi_{M}, B_{MN}^{\pm}) \quad \text{graviton}$$

$$(\Phi, \chi, B_{MN}^{-}) \quad \text{tensor/dilaton}$$

$$(A_{M}, \lambda) \quad U(1)\text{-vector} \quad (2.1)$$

Where $B^{\pm}$ gives rise to self dual/anti-self dual 3-form field strengths. $\lambda, \chi$ are spin-$\frac{1}{2}$ particles, $\psi_{M}$ is the spin-$\frac{3}{2}$ gravitino, $A_{M}$ is the vector gauge field from the $U(1)$ symmetry
and $\Phi$ is a dilaton. The Lagrangian is given by,

$$\mathcal{L} = R \ast 1 - \frac{1}{4} d\Phi \wedge d\Phi - \frac{1}{2} e^\Phi H_{(3)} \wedge H_{(3)} - \frac{1}{2} e^\frac{\Phi}{2} \ast F_{(2)} \wedge F_{(2)} - 8 g^2 e^{-\frac{\Phi}{2}} \ast 1 \quad (2.2)$$

The field strengths $F_{(2)}$ and $H_{(3)}$ are defined by,

$$F_{(2)} = dA_{(1)}$$
$$H_{(3)} = dB_{(2)} + \frac{1}{2} F_{(2)} \wedge A_{(1)} \quad (2.3)$$

These give rise to the Bianchi identities $dF_{(2)} = 0$ and $dH_{(3)} = \frac{1}{2} F_{(2)} \wedge F_{(2)}$ which in coordinates can be expressed as,

$$BF_{MNP} \equiv \nabla_{[M} F_{NP]} = 0$$
$$BH_{MNPQ} \equiv \nabla_{[M} H_{NPQ]} - \frac{3}{4} F_{[MN} F_{PQ]} = 0 \quad (2.4)$$

Now we will give the field equations for the bosonic fields. The Einstein equation is given by,

$$E_{MN} \equiv R_{MN} - \frac{1}{4} \nabla_M \Phi \nabla_N \Phi - \frac{1}{2} e^{\frac{\Phi}{2}} (F_{MP} F_{NP} - \frac{1}{8} F^2 g_{MN})$$
$$- \frac{1}{4} e^\Phi \left( H_{MPQ} H_N^{PQ} - \frac{1}{6} H^2 g_{MN} \right) - 2 g^2 e^{-\frac{\Phi}{2}} g_{MN} = 0 \quad (2.5)$$

The dilaton field equation,

$$F \Phi \equiv \nabla^M \nabla_M \Phi - \frac{1}{4} e^{\frac{\Phi}{2}} F^2 - \frac{1}{6} e^\Phi H^2 + 8 g^2 e^{-\frac{\Phi}{2}} = 0 \quad (2.6)$$

and the field equations for the fluxes,

$$d(e^{\frac{\Phi}{2}} \ast F_{(2)}) = e^\Phi \ast H_{(3)} \wedge F_{(2)} \quad (2.7)$$
$$d(e^\Phi \ast H_{(3)}) = 0 \quad (2.8)$$

In coordinates these can be expressed as,

$$FH_{MN} \equiv \nabla^P H_{MNP} + H_{MNP} \nabla^P \Phi = 0$$
$$FF_M \equiv \nabla^N F_{MN} + \frac{1}{2} F_{MN} \nabla^N \Phi + \frac{1}{2} F^{NP} H_{MNP} = 0 \quad (2.9)$$

The KSEs are given as the vanishing of the supersymmetry transformations of the fermionic fields,

$$\delta \psi_M \equiv D_M \epsilon = \left( \nabla_M - ig A_M + \frac{1}{48} e^{\frac{\Phi}{2}} H_{NPQ}^{+} \Gamma_{NPQ} \Gamma_M \right) \epsilon = 0 \quad (2.10)$$
\[ \delta \chi \equiv A \epsilon = \left( \Gamma^N \nabla_N \Phi - \frac{1}{6} e^{\frac{\Phi}{2}} H_{NPQ}^N \Gamma^{NPQ} \right) \epsilon = 0 \]  
(2.11)

\[ \delta \lambda \equiv F \epsilon = \left( e^{\frac{\Phi}{2}} F_{NM} \Gamma^{NM} - 8 i e^{-\frac{\Phi}{4}} \right) \epsilon = 0 \]  
(2.12)

where \( \epsilon \) is the supersymmetry parameter which from now on is taken to be a commuting symplectic Majorana-Weyl spinor of \( \text{Spin}(5,1) \). Note that the \( \pm \) superscripts appearing on the 3-form \( H_{NPQ} \) in these expressions are redundant, since the chirality of \( \epsilon \) already implies projections onto the self-dual or anti-self-dual parts. The integrability conditions of the KSEs are given by,

\[ \Gamma^N [\mathcal{D}_M, \mathcal{D}_N] \epsilon + \mu_M \mathcal{A} \epsilon + \lambda_M \mathcal{F} \epsilon = \left( \frac{1}{2} E_{MN} \Gamma^N + \frac{1}{12} e^{\frac{\Phi}{2}} B H_{NPQ} \Gamma^{NPQ} \right) \epsilon \]

\[ - \frac{1}{48} e^{\frac{\Phi}{2}} B H_{NPQR} \Gamma^M \Gamma^{NPQR} + \frac{1}{8} e^{\frac{\Phi}{2}} F H_{MN} \Gamma^N \]

\[ - \frac{1}{16} e^{\frac{\Phi}{2}} F H_{NP} \Gamma^N \]

\[ \epsilon \]  
(2.13)

where,

\[ \mu_M = \frac{1}{8} \nabla_M \Phi + \frac{1}{96} e^{\frac{\Phi}{2}} H_{NPQ} \Gamma^{NPQ} \Gamma_M \]

\[ \lambda_M = \frac{1}{64} e^{\frac{\Phi}{2}} F_{NP} \Gamma_M \Gamma^{NP} - \frac{1}{8} e^{\frac{\Phi}{2}} F_{MN} \Gamma^N + i e^{-\frac{\Phi}{4}} g \Gamma_M \]  
(2.14)

we see that if the \( H \) field equation, Bianchi identity and the Killing spinor conditions are satisfied, and given that the Ricci tensor is diagonal, the Einstein equation is then satisfied as well. Additional integrability conditions may be derived from the from the algebraic conditions as follows,

\[ \Gamma^M [\mathcal{D}_M, \mathcal{A}] \epsilon + \lambda \mathcal{A} \epsilon + \mu \mathcal{F} \epsilon = \left( F \Phi - \frac{1}{6} e^{\frac{\Phi}{2}} B H_{NPQ} \Gamma^{NPQ} - \frac{1}{2} e^{\frac{\Phi}{2}} F H_{NP} \Gamma^N \right) \epsilon \]

\[ \Gamma^M [\mathcal{D}_M, \mathcal{F}] \epsilon - \lambda \mathcal{F} \epsilon - 2 \mu \mathcal{A} \epsilon = \left( e^{\frac{\Phi}{2}} B F_{MNP} \Gamma^{MNP} - 2 e^{\frac{\Phi}{2}} F F_{M} \Gamma^M \right) \epsilon \]  
(2.15)

where

\[ \lambda = - \frac{1}{24} e^{\frac{\Phi}{2}} H_{NPQ} \Gamma^{NP} \]

\[ \mu = \frac{1}{8} e^{\frac{\Phi}{2}} F_{NP} \Gamma^{NP} + i e^{-\frac{\Phi}{4}} g \]  
(2.16)

The first shows once the \( H \) field equation and Bianchi identity and the Killing spinor conditions are satisfied, then the dilaton field equation is satisfied as well. The second is automatically satisfied as a result of the \( F \) field equation and the Killing spinor equations.

\(^1\)\( \epsilon \) also has an \( Sp(1) \) index which we will suppress
3 Near-horizon Data and Solution to the KSEs

In order to study near-horizon geometries we need to introduce a coordinate system which is regular and adapted to the horizon. We will consider a five-dimensional stationary black hole metric, for which the horizon is a Killing horizon, and the metric is regular at the horizon. A set of Gaussian Null coordinates \( \{ u, r, y^I \} \) will be used to describe the metric, where \( r \) denotes the radial distance away from the event horizon which is located at \( r = 0 \) and \( y^I, I = 1, \ldots, 3 \) are local co-ordinates on \( S \). The metric components have no dependence on \( u \), and the timelike isometry \( \frac{\partial}{\partial u} \) is null on the horizon at \( r = 0 \). The black hole metric in a patch containing the horizon is given by

\[
\begin{align*}
\frac{ds^2}{\Delta(y)} &= 2dudr + 2rh_I(r, y)dudy^I - rf(r, y)du^2 + \gamma_{IJ}(y)dy^I dy^J. \\
&= 2dudr + 2rh_I(y)dudy^I - rf(r, y)du^2 + \frac{1}{2}r^2\Delta(y)du^2 + \frac{3}{2}\gamma_{IJ}(y)dy^I dy^J.
\end{align*}
\]

The spatial horizon section \( S \) is given by \( u = \text{const}, r = 0 \) with the metric

\[
\begin{align*}
\frac{ds^2}{\Delta(y)} &= \gamma_{IJ}(y)dy^I dy^J.
\end{align*}
\]

We assume that \( S \) is compact, connected and without boundary. The 1-form \( h \), scalar \( \Delta \) and metric \( \gamma \) are functions of \( r \) and \( y^I \); they are analytic in \( r \) and regular at the horizon. The surface gravity associated with the Killing horizon is given by \( \kappa = \frac{1}{2}f(y, 0) \). The near-horizon limit is a particular decoupling limit defined by

\[
\begin{align*}
r &\to \epsilon r, \quad u \to \epsilon^{-1}u, \quad y^I \to y^I, \quad \text{and} \quad \epsilon \to 0.
\end{align*}
\]

This limit is only defined when \( f(y, 0) = 0 \), which implies that the surface gravity vanishes, \( \kappa = 0 \). Hence the near horizon geometry is only well defined for extremal black holes, and we shall consider only extremal black holes here. After taking the limit (3.19) we obtain,

\[
\frac{ds^2}{\Delta(y)} = 2dudr + 2rh_I(y)dudy^I - r^2\Delta(y)du^2 + \gamma_{IJ}(y)dy^I dy^J.
\]

In particular, the form of the metric remains unchanged from (3.17), however the 1-form \( h \), scalar \( \Delta \) and metric \( \gamma \) on \( S \) no longer have any radial dependence. For \( N = (1, 0), \) \( D = 6 \) supergravity, in addition to the metric, there are also gauge field strengths and scalars. We will assume that these are also analytic in \( r \) and regular at the horizon, and that there is also a consistent near-horizon limit for these matter fields:

\[
\begin{align*}
A &= -rae^+ + \tilde{A} \\
F &= e^+ \wedge e^- + re^+ \wedge T + T^F \\
H &= e^+ \wedge e^- \wedge L + re^+ \wedge M + M^H
\end{align*}
\]

where \( F^I = dA^I \) and we have introduced the frame

\[
\begin{align*}
e^+ &= du, \quad e^- = dr + rh - \frac{1}{2}r^2\Delta du, \quad e^i = e^I dy^I,
\end{align*}
\]

in which the metric is

\[
\frac{ds^2}{\Delta(y)} = 2e^+ e^- + \delta_{ij}e^i e^j.
\]

\[\text{The near-horizon metric } (3.20) \text{ also has a new scale symmetry, } r \to \lambda r, \ u \to \lambda^{-1}u \text{ generated by the Killing vector } L = u\partial_u - r\partial_r. \text{ This, together with the Killing vector } V = \partial_u \text{ satisfy the algebra } [V, L] = V \text{ and they form a 2-dimensional non-abelian symmetry group } G_2. \text{ We shall show that this further enhances into a larger symmetry algebra, which will include a } \mathfrak{sl}(2, \mathbb{R}) \text{ subalgebra.} \]

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3.1 Solving the KSEs along the Lightcone

For supersymmetric near-horizon horizons we assume there exists an \( \epsilon \neq 0 \) which is a solution to the KSEs. In this section, we will determine the necessary conditions on the Killing spinor. To do this we first integrate along the two lightcone directions i.e. we integrate the KSEs along the \( u \) and \( r \) coordinates. To do this, we decompose \( \epsilon \) as

\[
\epsilon = \epsilon_+ + \epsilon_- , \tag{3.24}
\]

where \( \Gamma_\pm \epsilon_\pm = 0 \), and find that

\[
\epsilon_+ = \phi_+(u, y) , \quad \epsilon_- = \phi_- + r \Gamma_- \Theta_+ \phi_+ , \tag{3.25}
\]

and

\[
\phi_- = \eta_- , \quad \phi_+ = \eta_+ + u \Gamma_+ \Theta_- \eta_- , \tag{3.26}
\]

where

\[
\Theta_\pm = \frac{1}{4} h_i \Gamma^i \pm \frac{1}{8} e^\frac{\Phi}{2} L_i \Gamma^i + \frac{1}{48} e^\frac{\Phi}{2} \tilde{H}_{ijk} \Gamma^{ijk} \tag{3.27}
\]

and \( \eta_\pm \) depend only on the coordinates of the spatial horizon section \( S \). Substituting the solution \([3.25]\) of the KSEs along the light cone directions back into the gravitino KSE \((2.10)\), and appropriately expanding in the \( r \) and \( u \) coordinates, we find that for the \( \mu = \pm \) components, one obtains the additional conditions

\[
\left( \frac{1}{2} \Delta - \frac{1}{8} (dh)_{ij} \Gamma^{ij} + ig \alpha \right) \phi_+ + 2 \left( \frac{1}{4} h_i \Gamma^i - \frac{1}{8} e^\frac{\Phi}{2} L_i \Gamma^i + \frac{1}{48} e^\frac{\Phi}{2} \tilde{H}_{ijk} \Gamma^{ijk} \right) \tau_+ = 0 , \tag{3.28}
\]

\[
\left( \frac{1}{4} \Delta h_i \Gamma^i - \frac{1}{4} \partial_i \Delta \Gamma^i \right) \phi_+ + \left( - \frac{1}{8} (dh)_{ij} \Gamma^{ij} + \frac{1}{8} e^\frac{\Phi}{2} M_{ij} \Gamma^{ij} \right) \tau_+ = 0 , \tag{3.29}
\]

\[
\left( - \frac{1}{2} \Delta - \frac{1}{8} (dh)_{ij} \Gamma^{ij} + ig \alpha + \frac{1}{8} e^\frac{\Phi}{2} M_{ij} \Gamma^{ij} - 2 \Theta_+ \Theta_- \right) \phi_- = 0 . \tag{3.30}
\]

Similarly the \( \mu = i \) component of the gravitino KSEs gives

\[
\tilde{\nabla}_i \phi_+ + \left( - \frac{1}{4} h_i - ig \tilde{A}_i \mp \frac{1}{8} e^\frac{\Phi}{2} L_j \Gamma^j \Gamma_i + \frac{1}{48} e^\frac{\Phi}{2} \tilde{H}_{jkl} \Gamma^{jkl} \Gamma_i \right) \phi_\pm = 0 , \tag{3.31}
\]

and

\[
\tilde{\nabla}_i \tau_+ + \left( - \frac{3}{4} h_i - ig \tilde{A}_i + \frac{1}{8} e^\frac{\Phi}{2} L_j \Gamma^j \Gamma_i + \frac{1}{48} e^\frac{\Phi}{2} \tilde{H}_{jkl} \Gamma^{jkl} \Gamma_i \right) \tau_+ + \left( - \frac{1}{4} (dh)_{ij} \Gamma^j + \frac{1}{16} e^\frac{\Phi}{2} M_{jk} \Gamma^j \Gamma_i \right) \phi_+ = 0 , \tag{3.32}
\]
where we have set
\[ \tau_+ = \Theta_+ \phi_+ . \tag{3.33} \]

Similarly, substituting the solution of the KSEs (3.25) into the algebraic KSE (2.11) and expanding appropriately in the \( u \) and \( r \) coordinates, we find
\[
\left( \Gamma^i \nabla_i \Phi \pm e^{\Phi/2} L_i \Gamma^i - \frac{1}{6} e^{\Phi/2} \tilde{H}_{ijk} \Gamma^{ijk} \right) \phi_\pm = 0 , \tag{3.34}
\]
\[-\left( \Gamma^i \nabla_i \Phi - e^{\Phi/2} L_i \Gamma^i - \frac{1}{6} e^{\Phi/2} \tilde{H}_{ijk} \Gamma^{ijk} \right) \tau_+ - \frac{1}{2} e^{\Phi/2} M_{ij} \Gamma^{ij} \phi_+ = 0 . \tag{3.35}
\]
and (2.12),
\[
\left( e^{\Phi/4}(\mp 2\alpha + \tilde{F}_{jk} \Gamma^{jk}) - 8ige^{\Phi/4} \right) \phi_\pm = 0 \tag{3.36}
\]
\[
\left( e^{\Phi/4}(2\alpha + \tilde{F}_{jk} \Gamma^{jk}) - 8ige^{-\Phi/4} \right) \tau_+ + 2e^{\Phi/4} T_i \Gamma^i \phi_+ = 0 \tag{3.37}
\]

In the next section, we will demonstrate that many of the above conditions are redundant as they are implied by the independent KSE\(\text{s}\) (4.73), upon using the field equations and Bianchi identities.

## 4 Simplification of KSEs on \( S \)

The integrability conditions of the KSEs in any supergravity theory are known to imply some of the Bianchi identities and field equations. Also, the KSEs are first order differential equations which are usually easier to solve than the field equations which are second order. As a result, the standard approach to find solutions is to first solve all the KSEs and then impose the remaining independent components of the field equations and Bianchi identities as required. We will take a different approach here because of the difficulty of solving the KSEs and the algebraic conditions which include the \( \tau_+ \) spinor given in (3.33). Furthermore, we are particularly interested in the minimal set of conditions required for supersymmetry, in order to systematically analyse the necessary and sufficient conditions for supersymmetry enhancement.

In particular, the conditions (3.28), (3.29), (3.32), and (3.35) which contain \( \tau_+ \) are implied from those containing \( \phi_+ \), along with some of the field equations and Bianchi identities. Furthermore, (3.30) and the terms linear in \( u \) in (3.31), (3.34) and (3.36) from the + component are implied by the field equations, Bianchi identities and the − component of (3.31), (3.34) and (3.36).

\(^3\)These are given by the naive restriction of the KSEs on \( S \).
A particular useful identity is obtained by considering the integrability condition of (3.31), which implies that

\[
(\hat{\nabla}_j \hat{\nabla}_i - \hat{\nabla}_i \hat{\nabla}_j) \phi_{\pm} = \left( \pm \frac{1}{4} \hat{\nabla}_j (h_i) + ig \hat{\nabla}_j (A_i) \pm \frac{1}{8} \hat{\nabla}_j (e^{\frac{\phi}{2}} L^i) \Gamma^j \Gamma_i \right) \phi_{\pm} + \left( \pm \frac{1}{4} h_j + ig \tilde{A}_j \pm \frac{1}{8} e^{\frac{\phi}{2}} L_{\ell} \Gamma^j \Gamma_j \right) \phi_{\pm} - \frac{1}{48} e^{\frac{\phi}{2}} H_{\ell \iota \iota} \Gamma_\ell \Gamma_\iota \Gamma_i \phi_{\pm}
\]

\[
\pm \frac{1}{8} h_{ij} \phi_{\pm} + \frac{1}{8} \tilde{A}_i \phi_{\pm} \pm \frac{1}{16} e^{\frac{\phi}{2}} L_k \Gamma^k \phi_{\pm} + \frac{1}{16} e^{\frac{\phi}{2}} H_{\ell \kappa \kappa} \Gamma_\ell \Gamma_\kappa \Gamma_i \phi_{\pm} - (i \leftrightarrow j)
\]

This will be used in the analysis of (3.28), (3.30), (3.32) and the positive chirality part of (3.31) which is linear in \( u \). In order to show that the conditions are redundant, we will be considering different combinations of terms which vanish as a consequence of the independent KSEs. However, non-trivial identities are found by explicitly expanding out the terms in each case. Let us also define,

\[
A_1 = \left( \Gamma^i \nabla_i \Phi + e^{\frac{\phi}{2}} L_i \Gamma^i - \frac{1}{6} e^{\frac{\phi}{2}} H_{\ell \kappa \kappa} \Gamma_\ell \Gamma_\kappa \right) \phi_+ .
\]

\[
B_1 = \left( \Gamma^i \nabla_i \Phi - e^{\frac{\phi}{2}} L_i \Gamma^i - \frac{1}{6} e^{\frac{\phi}{2}} H_{\ell \kappa \kappa} \Gamma_\ell \Gamma_\kappa \right) \eta_.
\]

\[
F_1 = \left( e^{\frac{\phi}{2}} (-2\alpha + \tilde{F}_{jk} \Gamma^{jk}) - 8 i g e^{-\frac{\phi}{2}} \right) \phi_+.
\]

\[
G_1 = \left( e^{\frac{\phi}{2}} (2\alpha + \tilde{F}_{jk} \Gamma^{jk}) - 8 i g e^{-\frac{\phi}{2}} \right) \eta_-
\]

4.1 The condition (3.28)

It can be shown that the algebraic condition on \( \tau_+ \) (3.28) is implied by the independent KSEs. Let us define,

\[
\xi_1 = \left( \frac{1}{2} \Delta - \frac{1}{8} (dh)_{ij} \Gamma^{ij} + ig \alpha \right) \phi_+ + 2 \left( \frac{1}{4} h_i \Gamma^i - \frac{1}{8} e^{\frac{\phi}{2}} L_i \Gamma^i + \frac{1}{48} e^{\frac{\phi}{2}} H_{ij \kappa} \Gamma^{ij \kappa} \right) \tau_+ ,
\]

where \( \xi_1 = 0 \) is equal to the condition (3.28). It is then possible to show that this expression for \( \xi_1 \) can be re-expressed as

\[
\xi_1 = \left( - \frac{1}{4} \tilde{R} - \Gamma^{ij} \hat{\nabla}_i \hat{\nabla}_j \right) \phi_+ + \mu A_1 + \lambda F_1 = 0
\]
where the first two terms cancel as a consequence of the definition of curvature, and

\[
\mu = \frac{1}{16} \tilde{\nabla}_i \Phi \Gamma^i + \frac{1}{8} e^\Phi L_i \Gamma^i + \frac{1}{48} e^\Phi \tilde{H}_{ijk} \Gamma^{ijk}
\]

\[
\lambda = -\frac{3}{64} e^\Phi \tilde{F}_{ij} \Gamma^{ij} - \frac{5}{32} e^\Phi \alpha + \frac{1}{8} e^{-\Phi} g_i
\]

the scalar curvature is can be written as

\[
\tilde{R} = -2 \Delta - \frac{1}{2} h^2 + \frac{1}{4} \nabla^i \Phi \nabla_i \Phi + \frac{5}{4} e^\Phi \alpha^2 + \frac{3}{8} e^\Phi \tilde{F}^2 + e^\Phi L^2 + \frac{1}{6} e^\Phi \tilde{H}^2 + 4 e^{-\Phi} g^2,
\]

The expression appearing in (4.39) vanishes because \( A_1 = \mathcal{F}_1 = 0 \) is equivalent to the positive chirality part of (3.34) and (3.36). Furthermore, the expression for \( \xi_1 \) given in (4.62) also vanishes. We also use (4.38) to evaluate the terms in the first bracket in (4.62) and explicitly expand out the terms with \( A_1 \). In order to obtain (3.28) from these expressions we make use of the Bianchi identities (C.2), the field equations (C.4) and (C.5). We have also made use of the \( + \) component of the Einstein equation (C.6) in order to rewrite the scalar curvature \( \tilde{R} \) in terms of \( \Delta \). Therefore (3.28) follows from (3.31), (3.34) and (3.36) together with the field equations and Bianchi identities mentioned above.

### 4.2 The condition (3.29)

Here we will show that the algebraic condition on \( \tau_+ \) (3.29) follows from (3.28). It is convenient to define

\[
\xi_2 = \left( \frac{1}{4} \Delta h_i \Gamma^i - \frac{1}{4} \partial_i \Delta \Gamma^i \right) \phi_+ + \left( \frac{1}{8} (dh)_{ij} \Gamma^{ij} + \frac{1}{4} e^\Phi M_{ij} \Gamma^{ij} \right) \tau_+ ,
\]

where \( \xi_2 = 0 \) equals the condition (3.29). One can show after a computation that this expression for \( \xi_2 \) can be re-expressed as

\[
\xi_2 = -\frac{1}{4} \Gamma^i \tilde{\nabla}_i \xi_1 + \frac{7}{16} h_j \Gamma^j \xi_1 = 0 ,
\]

which vanishes because \( \xi_1 = 0 \) is equivalent to the condition (3.28). In order to obtain this, we use the Dirac operator \( \Gamma^i \tilde{\nabla}_i \) to act on (3.28) and apply the Bianchi identities (C.2) with the field equations (C.4) and (C.5) to eliminate the terms which contain derivatives of the fluxes, and we can also use (3.28) to rewrite the \( dh \)-terms in terms of \( \Delta \). We then impose the algebraic conditions (3.31) and (3.35) to eliminate the \( \tilde{\nabla}_i \Phi \)-terms, of which some of the remaining terms will vanish as a consequence of (3.28). We then obtain the condition (3.29) as required, therefore it follows from section 4.1 above that (3.29) is implied by (3.31) and (3.34) together with the field equations and Bianchi identities mentioned above.

### 4.3 The condition (3.32)

Here we will show the differential condition on \( \tau_+ \) (3.32) is not independent. Let us define

\[
\lambda_i = \tilde{\nabla}_i \tau_+ + \left( -\frac{3}{4} h_i - ig \tilde{A}_i + \frac{1}{8} e^\Phi L_j \Gamma^j \Gamma_i + \frac{1}{48} e^\Phi \tilde{H}_{jkl} \Gamma^{jkl} \Gamma_i \right) \tau_+ + \left( -\frac{1}{4} (dh)_{ij} \Gamma^j + \frac{1}{16} e^\Phi M_{jk} \Gamma^{jkl} \Gamma_i \right) \phi_+ ,
\]
where $\lambda_i = 0$ is equivalent to the condition (3.32). We can re-express this expression for $\lambda_i$ as

$$
\lambda_i = \left( -\frac{1}{4} \tilde{R}_{ij} \Gamma^j + \frac{1}{2} \Gamma^j (\tilde{\nabla}_j \tilde{\nabla}_i - \tilde{\nabla}_i \tilde{\nabla}_j) \right) \phi_+ + \mu_i A_1 + \lambda_i F_1 = 0 ,
$$

(4.50)

where the first terms again cancel from the definition of curvature, and

$$
\mu_i = \frac{1}{16} \tilde{\nabla}_i \Phi + \frac{1}{192} e^\Phi \bar{H}_{\ell_1 \ell_2 \ell_3} \Gamma^{\ell_1 \ell_2 \ell_3} \Gamma_i - \frac{1}{32} e^\Phi L_\ell \Gamma^\ell \Gamma_i
$$

(4.51)

and

$$
\lambda_i = \frac{1}{128} e^\Phi \bar{F}_{\ell_1 \ell_2} \Gamma^{\ell_1 \ell_2} \Gamma_i - \frac{1}{16} e^\Phi \bar{F}_\ell \Gamma^\ell - \frac{1}{64} e^\Phi \alpha \Gamma_i + \frac{1}{16} e^\Phi g_i \Gamma_i
$$

(4.52)

This vanishes as $A_1 = F_1 = 0$ is equivalent to the positive chirality component of (3.34) and (3.36). The identity (4.50) is derived by making use of (4.38), and explicitly expanding out the $A_1$ and $F_1$ terms. We can also evaluate (3.32) by substituting in (3.33) to eliminate $\tau_+$, and use (3.31) to evaluate the supercovariant derivative of $\phi_+$. Then, on adding this to (4.50), one obtains a condition which vanishes identically on making use of the Einstein equation (C.6). Therefore it follows that (3.32) is implied by the positive chirality component of (3.31), (3.33) (3.34), the Bianchi identities (C.2) and the gauge field equations (C.4) and (C.5).

### 4.4 The condition (3.35)

Here we will show that the algebraic condition containing $\tau_+$ (3.35) follows from the independent KSEs. We define

$$
\mathcal{A}_2 = -\left( \Gamma^i \tilde{\nabla}_i \Phi - e^\Phi L_i \Gamma^i - \frac{1}{6} e^\Phi \bar{H}_{ijk} \Gamma^{ijk} \right) \tau_+ - \frac{1}{2} e^\Phi M_{ij} \Gamma^{ij} \phi_+
$$

(4.53)

where $\mathcal{A}_2 = 0$ equals the expression in (3.35). The expression for $\mathcal{A}_{1,2}$ can be rewritten as

$$
\mathcal{A}_2 = -\frac{1}{2} e^\Phi \tilde{\nabla}_i (A_1) + \Phi_1 A_1 + \Phi_2 F_1
$$

(4.54)

where,

$$
\Phi_1 = \frac{3}{8} h_\ell \Gamma^\ell + \frac{ig}{2} A_\ell \Gamma^\ell - \frac{1}{8} e^\Phi L_\ell \Gamma^\ell + \frac{1}{48} e^\Phi \bar{H}_{\ell_1 \ell_2 \ell_3} \Gamma^{\ell_1 \ell_2 \ell_3}
$$

(4.55)

and

$$
\Phi_2 = -\frac{1}{16} e^\Phi \bar{F}_{\ell_1 \ell_2} \Gamma^{\ell_1 \ell_2} + \frac{1}{8} \alpha e^\Phi - \frac{ig}{2} e^{-\frac{\Phi}{2}}
$$

(4.56)

In evaluating the above conditions, we have made use of the + component of (3.31) in order to evaluate the covariant derivative in the above expression. In addition we have made use of the Bianchi identities (C.2) and the field equations (C.4), (C.5) and (C.8).

It follows from (4.56) that $\mathcal{A}_2 = 0$ as a consequence of the condition $A_1 = F_1 = 0$, which as we have already noted is equivalent to the positive chirality part of (3.34).
4.5 The condition (3.37)

Here we will show that the algebraic condition containing $\tau_+$ (3.37) follows from the independent KSEs. We define

$$\mathcal{F}_2 = \left( e^\frac{\Phi}{2}(2\alpha + \bar{F}_{jk} \Gamma^{jk}) - 8i e^{-\frac{\Phi}{2}} \right) \tau_+ + 2e^\frac{\Phi}{2} T_i \Gamma^i \phi_+$$

(4.57)

where $\mathcal{F}_2 = 0$ equals the expression in (3.35). The expression for $\mathcal{F}_2$ can be rewritten as

$$\mathcal{F}_2 = -\frac{1}{2} \Gamma^i \nabla_i (\mathcal{F}_1) + \Phi_1 \mathcal{F}_1 + \Phi_2 \mathcal{A}_1$$

(4.58)

where,

$$\Phi_1 = \frac{3}{8} h_\ell \Gamma^\ell + \frac{ig}{2} A_\ell \Gamma^\ell + \frac{1}{8} e^\frac{\Phi}{2} L_\ell \Gamma^\ell - \frac{1}{48} e^\frac{\Phi}{2} \tilde{H}_{\ell_1 \ell_2 \ell_3} \Gamma^{\ell_1 \ell_2 \ell_3}$$

(4.59)

and

$$\Phi_2 = \frac{1}{8} e^\frac{\Phi}{2} \bar{F}_{\ell_1 \ell_2} \Gamma^{\ell_1 \ell_2} - \frac{1}{4} \alpha e^\frac{\Phi}{2} + ig e^{-\frac{\Phi}{2}}$$

(4.60)

In evaluating the above conditions, we have made use of the + component of (3.31) in order to evaluate the covariant derivative in the above expression. In addition we have made use of the Bianchi identities (C.1) and the field equation (C.3).

It follows from (4.58) that $\mathcal{F}_2 = 0$ as a consequence of the conditions $\mathcal{A}_1 = \mathcal{F}_1 = 0$, which as we have already noted is equivalent to the positive chirality part of (3.34) and (3.36).

4.6 The condition (3.30)

In order to show that (3.30) is implied by the independent KSEs, we define

$$\kappa = \left( -\frac{1}{2} \Delta - \frac{1}{8} (dh)_{ij} \Gamma^{ij} + ig \alpha + \frac{1}{8} e^\frac{\Phi}{2} M_{ij} \Gamma^{ij} - 2\Theta_+ \Theta_- \right) \phi_- = 0$$

(4.61)

where $\kappa$ equals the condition (3.30). Again, this expression can be rewritten as

$$\xi_1 = \left( \frac{1}{4} \tilde{R} + \Gamma^{ij} \nabla_i \nabla_j \right) \phi_+ - \mu \mathcal{B}_1 - \lambda \mathcal{G}_1 = 0$$

(4.62)

where we use the (4.38) to evaluate the terms in the first bracket, and

$$\mu = \frac{1}{16} \nabla_i \Phi \Gamma^i - \frac{1}{8} e^\frac{\Phi}{2} L_i \Gamma^i + \frac{1}{48} e^\frac{\Phi}{2} \tilde{H}_{ijk} \Gamma^{ijk}$$

$$\lambda = -\frac{3}{64} e^\frac{\Phi}{2} \bar{F}_{ij} \Gamma^{ij} + \frac{5}{32} e^\frac{\Phi}{2} \alpha + \frac{1}{8} e^{-\frac{\Phi}{2}} g_i$$

(4.63)

The expression above vanishes identically since the negative chirality component of (3.34) and (3.36) is equivalent to $\mathcal{B}_1 = \mathcal{G}_1 = 0$. In order to obtain (3.30) from these expressions we make use of the Bianchi identities (C.2) and the field equations (C.5), (C.6) and (C.7). Therefore (3.30) follows from (3.31), (3.34) and (3.36) together with the field equations and Bianchi identities mentioned above.
4.7 The positive chirality part of (3.31) linear in $u$

Since $\phi_+ = \eta_+ + u\Gamma_+ \Theta \eta_-$, we must consider the part of the positive chirality component of (3.31) which is linear in $u$. We then determine that $B_1$ satisfies the following expression

$$
\left(\frac{1}{2} \Gamma^j (\tilde{\nabla}_j \tilde{\nabla}_i - \tilde{\nabla}_i \tilde{\nabla}_j) - \frac{1}{4} \tilde{R}_{ij} \Gamma^j\right) \eta_- + \mu_i B_1 + \lambda_i G_1 = 0 ,
$$

(4.64)

where

$$
\mu_i = \frac{1}{16} \tilde{\nabla}_i \Phi + \frac{1}{192} e^\Phi \tilde{H}_{\ell_1 \ell_2 \ell_3} \Gamma^{\ell_1 \ell_2 \ell_3} \Gamma_i + \frac{1}{32} e^\Phi L_i \Gamma^\ell \Gamma_i
$$

(4.65)

and

$$
\lambda_i = \frac{1}{128} e^\Phi F_{\ell_1 \ell_2} \Gamma^{\ell_1 \ell_2} \Gamma_i - \frac{1}{16} e^\Phi \tilde{F}_{\ell i} \Gamma^\ell + \frac{1}{64} e^\Phi \alpha \Gamma_i + \frac{1}{16} e^{-\Phi} g \Gamma_i
$$

(4.66)

We note that $B_1 = G_1 = 0$ is equivalent to the negative chirality component of (3.34) and (3.36). Next, we use (4.38) to evaluate the terms in the first bracket in (4.64) and explicitly expand out the terms with $B_1$ and $G_1$. The resulting expression corresponds to the expression obtained by expanding out the $u$-dependent part of the positive chirality component of (3.31) by using the negative chirality component of (3.31) to evaluate the covariant derivative. We have made use of the Bianchi identities (C.2) and the gauge field equations (C.4) and (C.5).

4.8 The positive chirality part of condition (3.34) linear in $u$

Again, as $\phi_+ = \eta_+ + u\Gamma_+ \Theta \eta_-$, we must consider the part of the positive chirality component of (3.34) which is linear in $u$. One finds that the $u$-dependent part of (3.34) is proportional to

$$
-\frac{1}{2} \Gamma^i \tilde{\nabla}_i (B_1) + \Phi_1 B_1 + \Phi_2 G_1 ,
$$

(4.67)

where,

$$
\Phi_1 = \frac{1}{8} h_{\ell} \Gamma^\ell + \frac{i g}{2} A_\ell \Gamma^\ell + \frac{1}{8} e^\Phi L_\ell \Gamma^\ell + \frac{1}{48} e^\Phi \tilde{H}_{\ell_1 \ell_2 \ell_3} \Gamma^{\ell_1 \ell_2 \ell_3}
$$

(4.68)

and

$$
\Phi_2 = -\frac{1}{16} e^\Phi \tilde{F}_{\ell_1 \ell_2} \Gamma^{\ell_1 \ell_2} - \frac{1}{8} \alpha e^\Phi - \frac{i g}{2} e^{-\Phi}
$$

(4.69)

and where we use the (4.38) to evaluate the terms in the first bracket. In addition we have made use of the Bianchi identities (C.2) and the field equations (C.4), (C.5) and (C.8).
4.9 The positive chirality part of condition (3.36) linear in $u$

Finally, we must consider the part of the positive chirality component of (3.36) which is linear in $u$. One finds that the $u$-dependent part of (3.36) is proportional to

$$-\frac{1}{2} \Gamma^i \tilde{\nabla}_i (\Phi_i) + \Phi_1 B_1 + \Phi_2 G_1$$  \hspace{1cm} (4.70)

where,

$$\Phi_1 = \frac{1}{8} \hbar \ell \Gamma^\ell + \frac{ig}{2} A_\ell \Gamma^\ell - \frac{1}{8} e^\Phi L_\ell \Gamma^\ell - \frac{1}{48} e^\Phi H_{\ell_1 \ell_2 \ell_3} \Gamma^{\ell_1 \ell_2 \ell_3}$$  \hspace{1cm} (4.71)

and

$$\Phi_2 = \frac{1}{8} e^\Phi \tilde{F}_{\ell_1 \ell_2} \Gamma^{\ell_1 \ell_2} + \frac{1}{4} \alpha e^\Phi + i ge^{-\Phi}$$  \hspace{1cm} (4.72)

In evaluating the above conditions, we have made use of the + component of (3.31) in order to evaluate the covariant derivative in the above expression. In addition we have made use of the Bianchi identities (C.1) and the field equation (C.3).

4.10 The Independent KSEs on $S$

On taking the previous sections into account, it follows that, on making use of the field equations and Bianchi identities, the independent KSEs are

$$\nabla_i^{(\pm)} \eta_\pm = 0, \quad A^{(\pm)} \eta_\pm = 0, \quad F^{(\pm)} \eta_\pm = 0$$  \hspace{1cm} (4.73)

where

$$\nabla_i^{(\pm)} = \tilde{\nabla}_i + \Psi_i^{(\pm)}$$  \hspace{1cm} (4.74)

with

$$\Psi_i^{(\pm)} = \mp \frac{1}{4} \hbar_i - ig \tilde{A}_i \mp \frac{1}{8} e^\Phi L_j \Gamma^j i + \frac{1}{48} e^\Phi \tilde{H}_{jk i} \Gamma^{jk} i,$$  \hspace{1cm} (4.75)

and

$$A^{(\pm)} = \Gamma^i \nabla_i \Phi \pm e^\Phi L_i \Gamma^i - \frac{1}{6} e^\Phi \tilde{H}_{ijk i} \Gamma^{ijk},$$  \hspace{1cm} (4.76)

$$F^{(\pm)} = e^\Phi (\mp 2\alpha + \tilde{F}_{jk i} \Gamma^{jk i}) - 8ige^{-\Phi}$$  \hspace{1cm} (4.77)

These are derived from the naive restriction of the supercovariant derivative and the algebraic KSE on $S$. Furthermore, if $\eta_-$ solves (4.73) then

$$\eta_+ = \Gamma_+ \Theta_- \eta_-,$$  \hspace{1cm} (4.78)

also solves (4.73). However, further analysis using global techniques, is required in order to determine if $\Theta_-$ has a non-trivial kernel.
5 Global Analysis: Lichnerowicz Theorems

In this section, we shall establish a correspondence between parallel spinors \( \eta_{\pm} \) satisfying (4.73), and spinors in the kernel of appropriately defined horizon Dirac operators. We define the horizon Dirac operators associated with the supercovariant derivatives following from the gravitino KSE as

\[
\mathcal{D}^{(\pm)} \equiv \Gamma^i \tilde{\nabla}_i^{(\pm)} = \Gamma^i \tilde{\nabla}_i + \Psi^{(\pm)},
\]

where

\[
\Psi^{(\pm)} \equiv \Gamma^i \Psi_i^{(\pm)} = \mp \frac{1}{4} h_{ij} \Gamma^i - ig A_i \Gamma^i \pm \frac{1}{4} e^2 L_i \Gamma^i + \frac{1}{24} e^2 \tilde{H}_{ijk} \Gamma^{ijk}.
\]

To establish the Lichnerowicz type theorems, we begin by calculating the Laplacian of \( \| \eta_{\pm} \|^2 \). Here we will assume throughout that \( \mathcal{D}^{(\pm)} \eta_{\pm} = 0 \), so

\[
\tilde{\nabla}^i \tilde{\nabla}_i \| \eta_{\pm} \|^2 = 2 \text{Re} \langle \eta_{\pm}, \tilde{\nabla}^i \tilde{\nabla}_i \eta_{\pm} \rangle + 2 \text{Re} \langle \tilde{\nabla}^i \eta_{\pm}, \tilde{\nabla}_i \eta_{\pm} \rangle.
\]

To evaluate this expression note that

\[
\tilde{\nabla}^i \tilde{\nabla}_i \| \eta_{\pm} \|^2 = \Gamma^i \tilde{\nabla}_i (\Gamma^j \tilde{\nabla}_j \eta_{\pm}) - \Gamma^i \tilde{\nabla}_i \tilde{\nabla}_j \eta_{\pm}
\]

\[
= \Gamma^i \tilde{\nabla}_i (\Gamma^j \tilde{\nabla}_j \eta_{\pm}) + \frac{1}{4} \tilde{R} \eta_{\pm}
\]

\[
= \Gamma^i \tilde{\nabla}_i (-\Psi^{(\pm)} \eta_{\pm}) + \frac{1}{4} \tilde{R} \eta_{\pm}.
\]

Therefore the first term in (5.3) can be written as,

\[
\text{Re} \langle \eta_{\pm}, \tilde{\nabla}^i \tilde{\nabla}_i \eta_{\pm} \rangle = \frac{1}{4} \tilde{R} \| \eta_{\pm} \|^2 + \text{Re} \langle \eta_{\pm}, \Gamma^i \tilde{\nabla}_i (-\Psi^{(\pm)}) \eta_{\pm} \rangle + \text{Re} \langle \eta_{\pm}, \Gamma^i (-\Psi^{(\pm)}) \tilde{\nabla}_i \eta_{\pm} \rangle.
\]

(5.5)

For the second term in (5.3) we write,

\[
\text{Re} \langle \tilde{\nabla}^i \eta_{\pm}, \tilde{\nabla}_i \eta_{\pm} \rangle = \| \nabla^{(\pm)} \eta_{\pm} \|^2 - 2 \text{Re} \langle \eta_{\pm}, \Psi^{(\pm)\dagger} \tilde{\nabla}^i \eta_{\pm} \rangle - \text{Re} \langle \eta_{\pm}, \Psi^{(\pm)\dagger} \Psi_i^{(\pm)} \eta_{\pm} \rangle.
\]

(5.6)

We remark that \( \dagger \) is the adjoint with respect to the Spin\(_c\)(4)-invariant inner product \( \text{Re} \langle , \rangle \)\(^4\). Therefore using (5.5) and (5.6) with (5.3) we have,

\[
\frac{1}{2} \tilde{\nabla}^i \tilde{\nabla}_i \| \eta_{\pm} \|^2 = \| \nabla^{(\pm)} \eta_{\pm} \|^2 + \text{Re} \langle \eta_{\pm}, \left( \frac{1}{4} \tilde{R} + \Gamma^i (-\Psi^{(\pm)}) - \Psi^{(\pm)\dagger} \Psi_i^{(\pm)} \right) \eta_{\pm} \rangle
\]

\[
+ \text{Re} \langle \eta_{\pm}, \left( \Gamma^i (-\Psi^{(\pm)}) - 2 \Psi^{(\pm)\dagger} \right) \tilde{\nabla}_i \eta_{\pm} \rangle.
\]

(5.7)

In order to simplify the expression for the Laplacian, we observe that the second line in (5.7) can be rewritten as

\[
\text{Re} \langle \eta_{\pm}, \left( \Gamma^i (-\Psi^{(\pm)}) - 2 \Psi^{(\pm)\dagger} \right) \tilde{\nabla}_i \eta_{\pm} \rangle = \text{Re} \langle \eta_{\pm}, \mathcal{K}^{(\pm)} \Gamma^i \tilde{\nabla}_i \eta_{\pm} \rangle \pm \frac{1}{2} h_i \tilde{\nabla}_i \| \eta_{\pm} \|^2,
\]

(5.8)

\(^4\)This inner product is positive definite and symmetric.
where
\[ K^{(\pm)} = \mp \frac{1}{4} h_j \Gamma^j - ig \tilde{A}_i \Gamma^i \]  
(5.9)

We also have the following identities
\[ \text{Re}\langle \eta_+, \Gamma^{\ell_1 \ell_2} \eta_+ \rangle = \text{Re}\langle \eta_+, \Gamma^{\ell_1 \ell_2} \eta_+ \rangle = 0 \]  
(5.10)

and
\[ \text{Re}\langle \eta_+, i \Gamma^{\ell} \eta_+ \rangle = 0 . \]  
(5.11)

It follows that
\[ \frac{1}{2} \tilde{\nabla}^i \tilde{\nabla}_i \| \eta_\pm \|^2 = \| \nabla^{(\pm)} \eta_\pm \|^2 \mp \frac{1}{2} h_i \tilde{\nabla}_i \| \eta_\pm \|^2 + \text{Re}\langle \eta_\pm, \left( \frac{1}{4} \tilde{R} + \Gamma^i \tilde{\nabla}_i (-\Psi^{(\pm)}) - \Psi^{(\pm)} i \Psi^{(\pm)} + K^{(\pm)} (-\Psi^{(\pm)}) \right) \eta_\pm \rangle , \]  
(5.12)

It is also useful to evaluate \( \tilde{R} \) using (C.6); we obtain
\[ \tilde{R} = -\tilde{\nabla}^i(h_i) + \frac{1}{2} h^2 + \frac{1}{4} \tilde{\nabla}^i \Phi \tilde{\nabla}_i \Phi + \frac{1}{4} \epsilon^i \tilde{F}^2 + \frac{1}{2} \epsilon^{ij} \alpha^2 + \frac{1}{12} \epsilon^{ijk} + \frac{1}{12} \epsilon^{ij} \tilde{H}^2 + \frac{1}{2} \epsilon^{ij} L^2 + 8 \epsilon^{ij} g^2 , \]  
(5.13)

One obtains, upon using the field equations and Bianchi identities,
\[ \left( \frac{1}{4} \tilde{R} + \Gamma^i \tilde{\nabla}_i (-\Psi^{(\pm)}) - \Psi^{(\pm)} i \Psi^{(\pm)} + K^{(\pm)} (-\Psi^{(\pm)}) \right) \eta_\pm = \left[ i \tilde{\nabla}^i \tilde{A}_i \pm \frac{ig}{4} e^{\frac{3}{2} \tilde{F}} L_i \tilde{\nabla}_i (h_{\ell_1} h_{\ell_2} - h_{\ell_1} h_{\ell_2} - \frac{1}{16} \epsilon^{ij} \tilde{H}_{\ell_1 \ell_2} \tilde{\nabla}_i \tilde{\nabla}_j) \right] \eta_\pm + \left( \frac{1}{16} \tilde{\nabla}^i \Phi \tilde{\nabla}_i \Phi \pm \frac{1}{8} e^{\frac{3}{2} \tilde{F}} L_i \tilde{\nabla}_i \Phi + \frac{1}{48} e^{\frac{3}{2} \tilde{F}} \tilde{H}_{\ell_1 \ell_2 \ell_3} \tilde{\nabla}_4 \Phi \Gamma^{\ell_1 \ell_2 \ell_3 \ell_4} + \frac{1}{16} e^{\frac{3}{2} \tilde{F}} L^2 \right) \pm \frac{1}{48} e^{\frac{3}{2} \tilde{F}} \tilde{H}_{\ell_1 \ell_2 \ell_3} L_{\ell_4} \Gamma^{\ell_1 \ell_2 \ell_3 \ell_4} - \frac{1}{64} e^{\frac{3}{2} \tilde{F}} \tilde{H}_{\ell_1 \ell_2} \tilde{H}_{\ell_3 \ell_4} \Gamma^{\ell_1 \ell_2 \ell_3 \ell_4} + \frac{1}{96} e^{\frac{3}{2} \tilde{F}} \tilde{H}^2 \right) \eta_\pm + \left( \frac{1}{8} e^{\frac{3}{2} \tilde{F}} \alpha^2 - \frac{1}{32} e^{\frac{3}{2} \tilde{F}} \tilde{F}_{\ell_1 \ell_2} \tilde{F}_{\ell_3 \ell_4} \Gamma^{\ell_1 \ell_2 \ell_3 \ell_4} + \frac{1}{16} e^{\frac{3}{2} \tilde{F}} \tilde{F}^2 + \frac{ig}{2} \tilde{F}_{\ell_1 \ell_2} \Gamma^{\ell_1 \ell_2} + 2 e^{-\frac{3}{2} \tilde{F}} g^2 \right) \eta_\pm - \frac{1}{4} (1 \mp 1) \tilde{\nabla}^i (h_i) \eta_\pm . \]  
(5.14)

One can show that the fourth and fifth line in (5.14) can be written in terms of the algebraic KSE (4.76), in particular we find,
\[ \frac{1}{16} A^{(\pm)\dagger} A^{(\pm)} \eta_\pm = \frac{1}{16} \tilde{\nabla}^i \Phi \tilde{\nabla}_i \Phi \mp \frac{1}{8} e^{\frac{3}{2} \tilde{F}} L^i \tilde{\nabla}_i \Phi + \frac{1}{48} e^{\frac{3}{2} \tilde{F}} \tilde{H}_{\ell_1 \ell_2 \ell_3} \tilde{\nabla}_4 \Phi \Gamma^{\ell_1 \ell_2 \ell_3 \ell_4} + \frac{1}{16} e^{\frac{3}{2} \tilde{F}} L^2 \pm \frac{1}{48} e^{\frac{3}{2} \tilde{F}} \tilde{H}_{\ell_1 \ell_2 \ell_3} L_{\ell_4} \Gamma^{\ell_1 \ell_2 \ell_3 \ell_4} - \frac{1}{64} e^{\frac{3}{2} \tilde{F}} \tilde{H}_{\ell_1 \ell_2} \tilde{H}_{\ell_3 \ell_4} \Gamma^{\ell_1 \ell_2 \ell_3 \ell_4} + \frac{1}{96} e^{\frac{3}{2} \tilde{F}} \tilde{H}^2 \right) \eta_\pm \]  
(5.15)
and the sixth line,
\[
\frac{1}{32} \mathcal{F}^{(+)} \mathcal{F}^{(-)} \eta_{\pm} = \frac{1}{8} e^{\frac{\Phi}{2}} \alpha^2 - \frac{1}{32} e^{\frac{\Phi}{2}} \tilde{F}_{\ell_1 \ell_2} \tilde{F}_{\ell_3 \ell_4} \Gamma^{\ell_1 \ell_2 \ell_3 \ell_4} + \frac{1}{16} e^{\frac{\Phi}{2}} \tilde{F}^2 + \frac{i g}{2} \tilde{F}_{\ell_1 \ell_2} \Gamma^{\ell_1 \ell_2} + 2 e^{-\frac{\Phi}{2}} g^2
\]
(5.16)

Note that on using (5.10) and (5.11) all the terms on the RHS of the above expression, with the exception of the final four lines, vanish in the second line of (5.12) since all these terms in (5.14) are anti-Hermitian. Also, for \( \eta^+ \) the final line in (5.14) also vanishes and thus there is no contribution to the Laplacian of \( \| \eta^+ \|^2 \) in (5.12). For \( \eta^- \) the final line in (5.14) does give an extra term in the Laplacian of \( \| \eta^- \|^2 \) in (5.12). For this reason, the analysis of the conditions imposed by the global properties of \( S \) is different in these two cases and thus we will consider the Laplacians of \( \| \eta^\pm \|^2 \) separately.

For the Laplacian of \( \| \eta^+ \|^2 \), we obtain from (5.12):
\[
\hat{\nabla}^i \hat{\nabla}_i \| \eta^+ \|^2 - h^{\ast} \hat{\nabla}_i \| \eta^+ \|^2 = 2 \| \nabla^{(+)} \eta^+ \|^2 + \frac{1}{8} \| A^{(+)} \eta^+ \|^2 + \frac{1}{16} \| \mathcal{F}^{(+)} \eta^+ \|^2
\]
(5.17)

The maximum principle thus implies that \( \eta^+ \) are Killing spinors on \( S \) assuming that it is compact, connected and without boundary, i.e.
\[
\nabla^{(+)} \eta^+ = 0, \quad A^{(+)} \eta^+ = 0, \quad \mathcal{F}^{(+)} \eta^+ = 0
\]
(5.18)

and moreover \( \| \eta^+ \| = \text{const} \).

The Laplacian of \( \| \eta^- \|^2 \) is calculated from (5.12), on taking account of the contribution to the second line of (5.12) from the final line of (5.14). One obtains
\[
\hat{\nabla}^i (W_i) = 2 \| \nabla^{(-)} \eta^- \|^2 + \frac{1}{8} \| A^{(-)} \eta^- \|^2 + \frac{1}{16} \| \mathcal{F}^{(-)} \eta^- \|^2
\]
(5.19)

where \( W = d \| \eta^- \|^2 + \| \eta^- \|^2 h. \) On integrating this over \( S \) and assuming that \( S \) is compact and without boundary, the LHS vanishes since it is a total derivative and one finds that \( \eta^- \) are Killing spinors on \( S \), i.e.
\[
\nabla^{(-)} \eta^- = 0, \quad A^{(-)} \eta^- = 0, \quad \mathcal{F}^{(-)} \eta^- = 0
\]
(5.20)

This establishes the Lichnerowicz type theorems for both positive and negative chirality spinors \( \eta^\pm \) which are in the kernels of the horizon Dirac operators \( \mathcal{D}^{(\pm)} \); i.e.
\[
\{ \nabla^{(\pm)} \eta = 0, \quad A^{(\pm)} \eta = 0, \quad \text{and} \quad \mathcal{F}^{(\pm)} \eta = 0 \} \iff \mathcal{D}^{(\pm)} \eta = 0.
\]
(5.21)

### 6 (Super)symmetry Enhancement

In this section we will consider the counting of the number of supersymmetries, which will differ slightly in the ungauged and gauged case. We will denote by \( N^\pm \) the number of linearly independent \( \eta^\pm \) Killing spinors i.e.
\[
N^\pm = \dim \text{Ker}\{ \nabla^{(\pm)}, A^{(\pm)}, \mathcal{F}^{(\pm)} \}.
\]
(6.1)
In terms of the spinors $\eta_{\pm}$ restricted to $\mathcal{S}$, for the ungauged theory the spin bundle $\mathbb{S}$ decomposes as $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ where the signs refer to the projections with respect to $\Gamma_{\pm}$, and $\mathbb{S}^\pm$ are $Spin(4)$ bundles. For the gauged theory, the spin bundle $\mathbb{S} \otimes \mathcal{L}$, where $\mathcal{L}$ is a $U(1)$ bundle on $\mathcal{S}$, decomposes as $\mathbb{S} \otimes \mathcal{L} = \mathbb{S}^+ \otimes \mathcal{L} \oplus \mathbb{S}^- \otimes \mathcal{L}$ where $\mathbb{S}^\pm \otimes \mathcal{L}$ are $Spin_c(4) = Spin(4).U(1)$. The number of supersymmetries of near horizon geometries is $N = N_+ + N_-$ where $N_\pm$ is the number of linearly independent $\eta_\pm$ Killing spinors. In particular, utilizing the Lichnerowicz type theorems which we have established previously, we have

$$N_\pm = \dim \ker \mathcal{D}^{(\pm)}.$$  \hfill (6.2)

Next let us focus on the index of the $\mathcal{D}^{(+)}$ operator. Since $\mathcal{D}^{(+)}$ is defined on the even dimensional manifold $\mathcal{S}$, the index is given as \cite{59},

$$\text{Index}(\mathcal{D}^{(+)}) = \dim \ker \mathcal{D}^{(+) -} - \dim \ker (\mathcal{D}^{(+)})^{\dagger}$$  \hfill (6.3)

where $(\mathcal{D}^{(+)})^{\dagger}$ is the adjoint of $\mathcal{D}^{(+)}$. Furthermore observe that

$$\Gamma_-(\mathcal{D}^{(+)}) = \mathcal{D}^{(-)} \Gamma_- ,$$  \hfill (6.4)

and so

$$N_- = \dim \ker (\mathcal{D}^{(-)}) = \dim \ker (\mathcal{D}^{(+)})^{\dagger} .$$  \hfill (6.5)

Therefore, we conclude that

$$\text{Index}(\mathcal{D}^{(+)}) = N_+ - N_-$$  \hfill (6.6)

and so the number of supersymmetries of such horizons is

$$N = N_+ + N_- = 2N_- + \text{Index}(\mathcal{D}^{(+)})$$  \hfill (6.7)

It is not a priori apparent that $\text{Index}(\mathcal{D}^{(+)})$ will be an even number but in all examples investigated so far the index it is either an even number e.g IIB supergravity \cite{19} or it vanishes for odd-dimensional manifolds \cite{59} e.g in $D = 5$ \cite{20} and it is expected to vanish for non-chiral even-dimensional supergravities e.g IIA supergravity \cite{18, 17}. Also if $N_- = 0$, then $N = \text{Index}(\mathcal{D}^{(+)})$ and the number of Killing spinors is completely determined by the topology of $\mathcal{S}$; but it turns out that such near horizon geometries are rather restricted and likely all the fluxes vanish and the scalars become constant. The near-horizon geometries, up to discrete identifications, are products of the form $\mathbb{R}^{1,1} \times \mathcal{S}$, where $\mathcal{S}$ is a product of Berger manifolds and the formula $N = \text{Index}(\mathcal{D}^{(+)})$ becomes a well-known relation between the index of the Dirac operator and the number of parallel spinors.

### 6.1 Algebraic Relationship between $\eta_+$ and $\eta_-$ Spinors

We shall exhibit the existence of the $\mathfrak{sl}(2, \mathbb{R})$ symmetry of gauged $D = 6$ horizons by directly constructing the vector fields on the spacetime which generate the action of
\(\mathfrak{sl}(2, \mathbb{R})\). The existence of these vector fields is a direct consequence of the doubling of the supersymmetries. We have seen that if \(\eta_-\) is a Killing spinor, then \(\eta_+ = \Gamma_+ \Theta_- \eta_-\) is also a Killing spinor provided that \(\eta_+ \neq 0\). It turns out that under certain conditions this is always possible. To consider this we must investigate the kernel of \(\Theta_-\). We assume that \(\mathcal{S}\) and the fields satisfy the requirements for the maximum principle to apply. In previous theories we have been able to establish that the near horizon data is trivial when there is a non-trivial kernel.

Suppose \(\text{Ker} \Theta_- \neq \{0\}\), i.e. there is \(\eta_- \neq 0\) such that \(\Theta_- \eta_- = 0\). In such a case, (3.30) gives \(\Delta \text{Re}\langle \eta_-, \eta_- \rangle = 0\). Thus \(\Delta = 0\), as \(\eta_-\) is no-where vanishing. Next, the gravitino KSE \(\nabla^{(-)} \eta_- = 0\), together with \(\text{Re}\langle \eta_-, \Gamma_- \eta_- \rangle = 0\), imply that

\[
\tilde{\nabla}_i \parallel \eta_- \parallel^2 = -h_i \parallel \eta_- \parallel^2 .
\]

This implies that \(d h = 0\), and then (C.9) implies that \(T = M = 0\). On taking the divergence of (6.8), eliminating \(\nabla_i h_i\) upon using (C.5), and after setting \(\Delta = 0\), one finds

\[
\tilde{\nabla}_i \tilde{\nabla}_i \parallel \eta_- \parallel^2 = \left( \frac{3}{8} e^{\frac{\Phi}{2}} \alpha^2 + \frac{1}{16} e^{\frac{\Phi}{2}} \tilde{F}^2 + \frac{1}{4} e^{\Phi} L^2 + \frac{1}{12} e^{\Phi} \tilde{H}^2 - 2 e^{\frac{\Phi}{2}} g^2 \right) \parallel \eta_- \parallel^2 .
\]

Clearly the maximum principle can’t be applied for the gauged theory due to the \(-2 e^{\frac{\Phi}{2}} g^2\) term. For the ungauged theory with \(g = 0\); the maximum principle implies that \(\parallel \eta_- \parallel^2\) is constant. We conclude that \(\alpha = \tilde{F} = L = \tilde{H} = 0\) and from (3.31) that \(\Phi\) is constant. Finally, integrating (C.3) over the horizon section implies that \(h = 0\). Thus, all the fluxes vanish, and the scalars are constant.

We remark that in the ungauged theory, if \(\text{Ker} \Theta_- \neq \{0\}\), triviality of the near-horizon data implies that the spacetime geometry is \(\mathbb{R}^{1,1} \times T^4\). Hence, to exclude both the trivial \(\mathbb{R}^{1,1} \times T^4\) solution in the ungauged theory, and as an assumption in the gauged theory, we shall henceforth take \(\text{Ker} \Theta_- = \{0\}\).

### 6.2 The \(\mathfrak{sl}(2, \mathbb{R})\) Symmetry

Having established how to obtain \(\eta_+\) type spinors from \(\eta_-\) spinors, we next proceed to determine the \(\mathfrak{sl}(2, \mathbb{R})\) spacetime symmetry. First note that the spacetime Killing spinor \(\epsilon\) can be expressed in terms of \(\eta_{\pm}\) as

\[
\epsilon = \eta_+ + u \Gamma_+ \Theta_- \eta_- + \eta_- + r \Gamma_- \Theta_+ \eta_+ + ru \Gamma_- \Theta_+ \Gamma_+ \Theta_- \eta_- .
\]

Since the \(\eta_-\) and \(\eta_+\) Killing spinors appear in pairs for supersymmetric horizons, let us choose a \(\eta_-\) Killing spinor. Then from the previous results, horizons with non-trivial fluxes also admit \(\eta_+ = \Gamma_+ \Theta_- \eta_-\) as a Killing spinor. Taking \(\eta_-\) and \(\eta_+ = \Gamma_+ \Theta_- \eta_-\), one can construct two linearly independent Killing spinors on the spacetime as

\[
\epsilon_1 = \eta_- + u \eta_+ + ru \Gamma_- \Theta_+ \eta_+ , \quad \epsilon_2 = \eta_+ + r \Gamma_- \Theta_+ \eta_+ .
\]

It is known from the general theory of supersymmetric \(D = 6\) backgrounds that for any Killing spinors \(\zeta_1\) and \(\zeta_2\) the dual vector field \(K(\zeta_1, \zeta_2)\) of the 1-form bilinear

\[
\omega(\zeta_1, \zeta_2) = \text{Re}\langle (\Gamma_+ - \Gamma_-) \zeta_1, \Gamma_\alpha \zeta_2 \rangle e^\alpha
\]

(6.12)
is a Killing vector which leaves invariant all the other bosonic fields of the theory. Evaluating the 1-form bilinears of the Killing spinor $\epsilon_1$ and $\epsilon_2$, we find that

$$
\omega_1(\epsilon_1, \epsilon_2) = (2r \text{Re} (\Gamma_+ \eta_-, \Theta_+ \eta_+) + 4w^2 \parallel \Theta_+ \eta_+ \parallel^2) \mathbf{e}^+ - 2u \parallel \eta_+ \parallel^2 \mathbf{e}^- + \text{Re} (\Gamma_+ \eta_-, \Gamma_+ \eta_+) \mathbf{e}^i,
$$

$$
\omega_2(\epsilon_2, \epsilon_2) = 4r^2 \parallel \Theta_+ \eta_+ \parallel^2 \mathbf{e}^+ - 2 \parallel \eta_+ \parallel^2 \mathbf{e}^- + 4r \text{Re} (\eta_+, \Gamma_i \Theta_+ \eta_+) \mathbf{e}^i,
$$

$$
\omega_3(\epsilon_1, \epsilon_1) = (2 \parallel \eta_- \parallel^2 + 4ru \text{Re} (\Gamma_+ \eta_-, \Theta_+ \eta_+) + 4r^2u^2 \parallel \Theta_+ \eta_+ \parallel^2) \mathbf{e}^+ - 2u^2 \parallel \eta_+ \parallel^2 \mathbf{e}^- + (2u \text{Re} (\Gamma_+ \eta_-, \Gamma_i \Theta_+ \eta_+) + 4ru \text{Re} (\eta_+, \Gamma_i \Theta_+ \eta_+)) \mathbf{e}^i.
$$

(6.13)

We can establish the following identities

$$
-\Delta \parallel \eta_+ \parallel^2 + 4 \parallel \Theta_+ \eta_+ \parallel^2 = 0 , \quad \text{Re} (\eta_+, \Gamma_i \Theta_+ \eta_+) = 0 ,
$$

(6.14)

which follow from the first integrability condition in (3.28), $\parallel \eta_+ \parallel = \text{const}$ and the KSEs of $\eta_+$. Further simplification to the bilinears can be obtained by making use of (6.14). We then obtain

$$
\omega_1(\epsilon_1, \epsilon_2) = (2r \text{Re} (\Gamma_+ \eta_-, \Theta_+ \eta_+) + 4w^2 \parallel \Theta_+ \eta_+ \parallel^2) \mathbf{e}^+ - 2u \parallel \eta_+ \parallel^2 \mathbf{e}^- + \tilde{V}_i \mathbf{e}^i,
$$

$$
\omega_2(\epsilon_2, \epsilon_2) = r^2 \Delta \parallel \eta_+ \parallel^2 \mathbf{e}^+ - 2 \parallel \eta_+ \parallel^2 \mathbf{e}^- ,
$$

$$
\omega_3(\epsilon_1, \epsilon_1) = (2 \parallel \eta_- \parallel^2 + 4ru \text{Re} (\Gamma_+ \eta_-, \Theta_+ \eta_+) + r^2u^2 \Delta \parallel \eta_+ \parallel^2) \mathbf{e}^+ + 2u^2 \parallel \eta_+ \parallel^2 \mathbf{e}^- + 2u \tilde{V}_i \mathbf{e}^i ,
$$

(6.15)

where we have set

$$
\tilde{V}_i = \text{Re} (\Gamma_+ \eta_-, \Gamma_i \Theta_+ \eta_+) .
$$

(6.16)

To uncover explicitly the $\mathfrak{sl}(2, \mathbb{R})$ symmetry of such horizons it remains to compute the Lie bracket algebra of the vector fields $K_1$, $K_2$ and $K_3$ which are dual to the 1-form spinor bilinears $\omega_1$, $\omega_2$ and $\omega_3$. In simplifying the resulting expressions, we shall make use of the following identities

$$
-2 \parallel \eta_+ \parallel^2 - h_i \tilde{V}^i + 2\text{Re} (\Gamma_+ \eta_-, \Theta_+ \eta_+) = 0 , \quad i\varphi (dh) + 2d\text{Re} (\Gamma_+ \eta_-, \Theta_+ \eta_+) = 0 , \quad 2\text{Re} (\Gamma_+ \eta_-, \Theta_+ \eta_+) - \Delta \parallel \eta_- \parallel^2 = 0 , \quad \tilde{V}_i + \parallel \eta_- \parallel^2 h + d \parallel \eta_- \parallel^2 = 0 .
$$

(6.17)

We then obtain the following dual Killing vector fields:

$$
K_1 = -2u \parallel \eta_+ \parallel^2 \partial_u + 2r \parallel \eta_+ \parallel^2 \partial_r + \tilde{V} ,
$$

$$
K_2 = -2 \parallel \eta_+ \parallel^2 \partial_u ,
$$

$$
K_3 = -2u^2 \parallel \eta_+ \parallel^2 \partial_u + (2 \parallel \eta_- \parallel^2 + 4ru \parallel \eta_+ \parallel^2) \partial_r + 2u \tilde{V} .
$$

(6.18)

As we have previously mentioned, each of these Killing vectors also leaves invariant all the other bosonic fields in the theory. It is then straightforward to determine the algebra satisfied by these isometries:

**Theorem:** The Lie bracket algebra of $K_1$, $K_2$ and $K_3$ is $\mathfrak{sl}(2, \mathbb{R})$. 

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Proof: Using the identities summarised above, one can demonstrate after a direct computation that

\[ [K_1, K_2] = 2 \| \eta_+ \|^2 K_2, \quad [K_2, K_3] = -4 \| \eta_+ \|^2 K_1, \quad [K_3, K_1] = 2 \| \eta_+ \|^2 K_3. \]

(6.19)

A special case arises for \( \tilde{V} = 0 \), where the group action generated by \( K_1, K_2 \) and \( K_3 \) has only 2-dimensional orbits. A direct substitution of this condition in (6.17) reveals that

\[ \Delta \| \eta_- \|^2 = 2 \| \eta_+ \|^2, \quad h = \Delta^{-1} d \Delta. \]

(6.20)

Since \( h \) is exact, such horizons are static. A coordinate transformation \( r \to \Delta r \) reveals that the geometry is a warped product of \( AdS_2 \times S \).

6.3 Isometries of \( S \)

It is known that the vector fields associated with the 1-form Killing spinor bilinears given in (6.12) leave invariant all the fields of gauged \( D = 5 \) supergravity with vector multiplets. In particular suppose that \( \tilde{V} \neq 0 \). The isometries \( K_a \) \( (a = 1, 2, 3) \) leave all the bosonic fields invariant:

\[ \mathcal{L}_{K_a} g = 0, \quad \mathcal{L}_{K_a} F = 0, \quad \mathcal{L}_{K_a} H = 0, \quad \mathcal{L}_{K_a} \Phi = 0. \]

(6.21)

Imposing these conditions and expanding in \( u, r \), and also making use of the identities (6.17), one finds that

\[ \tilde{\nabla}(i \tilde{V}_j) = 0, \quad \mathcal{L}_{\tilde{V}} h = \mathcal{L}_{\tilde{V}} \Delta = 0, \quad \mathcal{L}_{\tilde{V}} \Phi = 0, \quad \mathcal{L}_{\tilde{V}} \tilde{F} = \mathcal{L}_{\tilde{V}} \alpha = \mathcal{L}_{\tilde{V}} L = \mathcal{L}_{\tilde{V}} \tilde{H} = 0. \]

(6.22)

Therefore \( \tilde{V} \) is an isometry of \( S \) and leaves all the fluxes on \( S \) invariant. In fact, \( \tilde{V} \) is a spacetime isometry as well. Furthermore, the conditions (6.17) imply that \( \mathcal{L}_{\tilde{V}} \| \eta_- \|^2 = 0. \)

6.4 Conditions on the geometry

Here we will consider the further restrictions on the geometry of \( S \). We begin by explicitly expanding out the identities established in (6.14), which follow from the first integrability condition in (3.22), \( \| \eta_+ \| = \text{const} \) and the KSEs of \( \eta_+ \), in terms of bosonic fields and using (6.17) along with the field equations (C.3)-(C.8) and Bianchi identities (C.1) and (C.2). On expanding (6.14) we obtain,

\[ \Delta \| \eta_+ \|^2 = \text{Re}(\eta_+, \left( \frac{1}{4} h^2 + \frac{1}{4} e^\Phi h_i L^i + \frac{1}{16} e^\Phi L^2 + \frac{1}{96} e^\Phi H^2 + (-\frac{1}{24} e^\Phi \tilde{H}_{\ell_1 \ell_2 \ell_3 \ell_4} h_{\ell_4} \\
- \frac{1}{48} e^\Phi \tilde{H}_{\ell_1 \ell_2 \ell_3 \ell_4} L_{\ell_4} - \frac{1}{64} e^\Phi \tilde{H}^k_{\ell_1 \ell_2} \tilde{H}_{\ell_3 \ell_4} \Gamma^{\ell_1 \ell_2 \ell_3 \ell_4} \right) \eta_+ \). \]

(6.23)

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and
\[ \text{Re} \langle \eta_+, \Gamma_i \Theta_+ \eta_+ \rangle = \text{Re} \langle \eta_+, \left( \frac{1}{4} h_i + \frac{1}{8} e^\Phi L_i + \frac{1}{48} e^\Phi \tilde{H}_{i_l i_{l_2} i_{l_3} \Gamma_i} \right) \eta_+ \rangle = 0. \] (6.24)

On contracting and substituting this in (6.23) we can write,
\[ \Delta \parallel \eta_+ \parallel^2 = \text{Re} \langle \eta_+, \left( -\frac{1}{4} h_i^2 - \frac{1}{4} e^\Phi L_i h_i - \frac{1}{16} e^\Phi L^2 + \frac{1}{96} e^\Phi \tilde{H}^2 \right. \]
\[ \left. - \frac{1}{64} e^\Phi \tilde{H}^{k_{i_l} i_{l_2} i_{l_3} \Gamma_i} \right) \eta_+ \rangle \] (6.25)

From the algebraic KSE (4.76) we have,
\[ \text{Re} \langle \eta_\pm, A(\pm) \eta_\pm \rangle = (\tilde{\nabla}_i \Phi \pm e^\Phi L_i) \text{Re} \langle \eta_\pm, \Gamma_i \eta_\pm \rangle = 0 \]
\[ \text{Re} \langle \eta_\pm, \Gamma_i A(\pm) \eta_\pm \rangle = \text{Re} \langle \eta_\pm, \left( \tilde{\nabla}_i \Phi \pm e^\Phi L_i - \frac{1}{6} e^\Phi \tilde{H}_{i_l i_{l_2} i_{l_3} \Gamma_i} \right) \eta_\pm \rangle = 0 \] (6.26)

From this and (6.24) we obtain,
\[ \text{Re} \langle \eta_+, \left( \Gamma_i \Theta_+ + \frac{1}{8} \Gamma_i A(\mp) \right) \eta_+ \rangle = \left( \frac{1}{4} h_i + \frac{1}{4} e^\Phi L_i + \frac{1}{8} \tilde{\nabla}_i \Phi \right) \parallel \eta_+ \parallel^2 = 0 \] (6.27)

since \( \eta_+ \neq 0 \) the norm is non-vanishing and we can write,
\[ h_i = -\left( e^\Phi L_i + \frac{1}{2} \tilde{\nabla}_i \Phi \right) \] (6.28)

On taking the divergence of this expression and using the field equations (C.4), (C.6) and (C.8) and substituting back (6.28), we obtain the condition,
\[ \Delta = \frac{1}{2} e^\Phi \alpha^2 \] (6.29)

On considering the algebraic KSE (5.2) we have,
\[ \text{Re} \langle \eta_\pm, F^{(\pm)} \eta_\pm \rangle = \pm 2 \alpha \parallel \eta_\pm \parallel^2 = 0 \]
\[ \text{Re} \langle \eta_\pm, \Gamma_i F^{(\pm)} \eta_\pm \rangle = 2 e^\Phi \tilde{F}_{i_l} \text{Re} \langle \eta_\pm, \Gamma^k \eta_\pm \rangle = 0 \] (6.30)

Thus we obtain \( \alpha = 0 \) and from (6.29) this implies \( \Delta = 0 \) which from (6.17) implies \( \text{Re} \langle \Gamma_+ \eta_-, \Theta_+ \eta_+ \rangle = 0 \). The other identities in (6.17) become,
\[ -2 \parallel \eta_+ \parallel^2 - h_i \tilde{V}^i = 0, \quad i_{\tilde{V}}(dh) = 0, \quad \tilde{V}_+ \parallel \eta_- \parallel^2 = h + d \parallel \eta_- \parallel^2 = 0. \] (6.31)

Using these identities it is straightforward to show that there are no near-horizon geometries for which \( h = 0 \) or \( \tilde{V} = 0 \) since this would lead to a contradiction to our assumption that \( \eta_+ \neq 0 \).
7 Conclusion

We have investigated the supersymmetry preserved by horizons in $N = (1, 0), D = 6$ gauged, and ungauged supergravity. First we solved the KSEs along the lightcone, obtaining a large number of additional differential and algebraic conditions on the spinors $\eta_{\pm}$; we were able to prove the redundancy of most of these conditions thus stating the independent KSEs on $S$. Making use of global techniques by establishing Lichnerowicz-type theorems for the Dirac operator, we have demonstrated that such horizons always admit $N = 2N_+ + \text{Index}(\mathcal{D}^{(+)})$ supersymmetries. We have also shown that the near-horizon geometries possess a $\mathfrak{sl}(2, \mathbb{R})$ symmetry group. The analysis that we have conducted is further evidence that this type of symmetry enhancement is a generic property of supersymmetric black holes. By analysing the conditions on the geometry, we demonstrated that there are no static solution in the gauged theory for $h = 0$ and there exists at least one Killing vector on the horizon section as we have also shown there are no solutions with $\tilde{V} = 0$.

In order to establish the $\mathfrak{sl}(2, \mathbb{R})$ symmetry for the gauged theory we assumed that $\text{Ker} \Theta_- = \{0\}$, while in the ungauged theory we were able to prove this for non-trivial fluxes. In all theories prior where the horizon conjecture has been established, assuming a non-trivial kernel either leads to a trivial solution where all the fluxes vanish and the scalars become constant or a contradiction; which is the case for the ungauged theory with $g = 0$. However for the gauged theory, assuming a non-trivial kernel leads to $dh = 0$ and $M = T = 0$ but the application of the maximum principle requires the positive semi-definiteness of certain terms which depend on the fluxes and the existence of a negative cosmological constant in the theory has invalidated these arguments as it has contributed with the opposite sign in the expressions required for the application of the maximum principle i.e in (6.9) due to the $-2e^{-\phi}g^2$ term, although the Lichnerowicz-type theorems and the conditions on the geometry still hold for the gauged theory. There may be non-trivial solutions in the gauged theory with $\text{Ker} \Theta_- \neq \{0\}$; details of this will be given elsewhere. Apart from exhibiting an $\mathfrak{sl}(2, \mathbb{R})$ symmetry, $D = 6$ horizons are further geometrically restricted which we touched upon in section 6.4. This is because we have not explored all the restrictions imposed by the KSEs and the field equations of the theory, in this paper we only explored enough to establish the symmetry enhancement and it will be explored elsewhere.

Another avenue for future research would be to extend the analysis for an arbitrary number of tensor and vector multiplets; similar to the calculation in $D = 4, 5$ with vector multiplets [21, 20]. In six dimensions, near horizon geometries have been classified in $N = (1, 0)$ for many cases and for those with an arbitrary number of tensor multiplets [45] have near horizon geometries locally given by $\mathbb{R}^{1,1} \times T^4$, $\mathbb{R}^{1,1} \times K_3$ or $AdS_3 \times S^3$. It would be interesting to obtain this classification from the conditions on the geometry we obtained from our analysis and to extend it to the more general theory.
Appendix A  Supersymmetry Conventions

We follow the spinor conventions of [30, 60] with mostly positive signature. The $8 \times 8$ Dirac matrices in six dimensions obey the clifford algebra,

$$\{\Gamma_M, \Gamma_N\} = 2g_{MN}$$  \hspace{1cm} (A.1)

The chirality projector is defined as,

$$\Gamma_* = \Gamma_0 \cdots \Gamma_5, \quad \Gamma_*^2 = 1, \quad \Gamma_*^\dagger = -\Gamma_*$$  \hspace{1cm} (A.2)

The gamma matrices also satisfy the duality relation,

$$\Gamma^{A_1 \cdots A_n} = (-1)^{n/2} \epsilon^{A_1 \cdots A_nB_1 \cdots B_{6-n}} \Gamma_{B_1 \cdots B_{6-n}} \Gamma_*$$  \hspace{1cm} (A.3)

with $\epsilon^{012345} = 1$. For a product of two anti-symmetrized gamma matrices we have,

$$\Gamma^{A_1 \cdots A_n} \Gamma^{B_1 \cdots B_m} = \sum_{k=0}^{\min(n,m)} \frac{m!n!}{(m-k)!(n-k)!k!} \Gamma^{[B_{k+1} \cdots B_m} \delta^{A_{n-k} \cdots A_n]} \Gamma_{A_1 \cdots A_{n-k}} \delta_{A_{n-k+1}]}.$$  \hspace{1cm} (A.4)

All the spinors are symplectic Majorona,

$$\chi^\alpha = \epsilon^{\alpha \beta} (\bar{\chi}_\beta)^T, \quad \bar{\chi}_\alpha = (\chi^\alpha)^\dagger \Gamma_0$$  \hspace{1cm} (A.5)

where $\bar{\chi}^\alpha = (\chi^\alpha)^T$ and $\alpha, \beta$ are $Sp(1)$ indices. It will be convenient to decompose the spinors into positive and negative chiralities with respect to the lightcone directions as

$$\epsilon = \epsilon_+ + \epsilon_-,$$  \hspace{1cm} (A.6)

where

$$\Gamma_{\pm} \epsilon_\pm = \pm \epsilon_\pm, \quad \text{or equivalently} \quad \Gamma_{\pm} \epsilon_\mp = 0.$$  \hspace{1cm} (A.7)

The representation of $Spin(5,1)$ decomposes under $Spin(4) = SU(2) \times SU(2)$ specified by the lightcone projections $\Gamma_{\pm}$. We have also made use of the $Spin(4)$-invariant inner product $Re\langle , \rangle$ which is identified with the standard Hermitian inner product. In particular, note that $(\Gamma_{ij})^\dagger = -\Gamma_{ij}$.

Appendix B  Spin Connection and Curvature

The non-vanishing components of the spin connection in the frame basis [322] are

$$\Omega_{-,+i} = -\frac{1}{2} h_i, \quad \Omega_{+,+-} = -r \Delta, \quad \Omega_{+,+i} = \frac{1}{2} r^2 (\Delta h_i - \partial_i \Delta),$$

$$\Omega_{+,i-} = -\frac{1}{2} h_i, \quad \Omega_{+,ij} = -\frac{1}{2} r d h_{ij}, \quad \Omega_{i,+-} = \frac{1}{2} h_i, \quad \Omega_{i,+j} = -\frac{1}{2} r d h_{ij}.$$  

\[5\] Explicit representations are not needed for the calculations.
where $\tilde{\Omega}$ denotes the spin-connection of the 3-manifold $S$ with basis $e^i$. If $f$ is any function of spacetime, then frame derivatives are expressed in terms of co-ordinate derivatives as

$$\partial_+ f = \partial_u f + \frac{1}{2} r^2 \Delta \partial_r f, \quad \partial_- f = \partial_r f, \quad \partial_i f = \tilde{\partial}_i f - r \partial_r f h_i. \quad (B.2)$$

The non-vanishing components of the Ricci tensor in the basis (3.22) are

$$R_{+\pm} = \frac{1}{2} \tilde{\nabla}^i h_i - \Delta - \frac{1}{2} h^2, \quad R_{ij} = \tilde{R}_{ij} + \tilde{\nabla}_{(ih)} - \frac{1}{2} h_i h_j$$

$$R_{++} = r^2 \left( \frac{1}{2} \tilde{\nabla}^i \Delta - \frac{3}{2} h^i \tilde{\nabla}_i \Delta - \frac{1}{2} \Delta \tilde{\nabla}^i h_i + \Delta h^2 + \frac{1}{4} (dh)_{ij} (dh)^{ij} \right)$$

$$R_{+i} = r \left( \frac{1}{2} \tilde{\nabla}^j (dh)_{ij} - (dh)_{ij} h^j - \tilde{\nabla}_i \Delta + \Delta h_i \right), \quad (B.3)$$

where $\tilde{\nabla}$ denotes the Levi-Civita connection of $S$, and $\tilde{R}$ is the Ricci tensor of the horizon section $S$, and $i,j$ denote $e^i$ frame indices.

## Appendix C  Horizon Bianchi Identities and Field Equations

Substituting the fields (3.23) into the the Bianchi identity $dF = 0$ and $dH = \frac{1}{2} F \wedge F$ implies

$$T = (d_h \alpha), \quad dF = 0 \quad (C.1)$$

and

$$M = (d_h L) - \alpha \tilde{F}, \quad d\tilde{H} = \frac{1}{2} \tilde{F} \wedge \tilde{F} \quad (C.2)$$

Similarly, the independent field equations of the near horizon fields are as follows. The 2-form field equation (2.7) gives,

$$\tilde{\nabla}^\ell (e^\phi \tilde{F}_{\ell \ell}) - e^\phi \tilde{F}_{\ell \ell} h^\ell - e^{\phi} T_\ell - e^\phi L_\ell \alpha + \frac{1}{2} e^\phi \tilde{F}_{\ell_1 \ell_2} \tilde{H}_{\ell_1 \ell_2} = 0 \quad (C.3)$$

the 3-form field equation (2.8) gives,

$$\tilde{\nabla}^\ell (e^\phi L_\ell) = 0 \quad (C.4)$$

and

$$\tilde{\nabla}^\ell (e^\phi \tilde{H}_{ij \ell}) - e^\phi h^\ell \tilde{H}_{ij \ell} + e^\phi M_{ij} = 0 \quad (C.5)$$

The $+\pm$ and $ij$-component of the Einstein equation (2.5) gives

$$-\Delta - \frac{1}{2} h^2 + \frac{1}{2} \tilde{\nabla}^i (h_i) = \frac{1}{2} e^\phi \left( -\frac{3}{4} \alpha^2 - \frac{1}{8} \tilde{F}^2 \right) + \frac{1}{4} e^\phi \left( -L^2 - \frac{1}{6} \tilde{H}^2 \right) + 2 g^2 e^{-\frac{\phi}{2}} \quad (C.6)$$
\[ \tilde{R}_{ij} = -\tilde{\nabla}_i h_j + \frac{1}{2} h_i h_j + \frac{1}{4} e^\Phi \left( \tilde{F}_{ij} \tilde{F}^\ell - \frac{1}{8} \tilde{F}^2 \delta_{ij} \right) + \frac{1}{8} e^\Phi \alpha^2 \delta_{ij} \]
\[ + \frac{1}{4} e^\Phi \left( \tilde{H}_{i\ell, \ell} - \tilde{H}^2 \delta_{ij} \right) + \frac{1}{4} e^\Phi \left( -2 \tilde{L}_i \tilde{L}_j + \tilde{L}^2 \delta_{ij} \right) + 2 g^2 e^{-\frac{\Phi}{2}} \delta_{ij} \] (C.7)

The scalar field equation (2.6) gives
\[ \tilde{\nabla}_i \tilde{\nabla}_i \Phi - h_i \tilde{\nabla}_i \Phi = -\frac{1}{2} e^\frac{\Phi}{2} \alpha^2 + \frac{1}{4} e^\frac{\Phi}{2} \tilde{F}^2 - e^\Phi L^2 + \frac{1}{6} e^\Phi \tilde{H}^2 - 8 g^2 e^{-\frac{\Phi}{2}} \] (C.8)

We remark that the ++ and +i components of the Einstein equations are given by
\[ \frac{1}{2} \tilde{\nabla}^2 \Delta - \frac{3}{2} h^i \tilde{\nabla}_i \Delta - \frac{1}{2} \Delta \tilde{\nabla}^i h_i + \Delta h^2 + \frac{1}{4} (dh)_{ij} (dh)^{ij} = \frac{1}{2} e^\frac{\Phi}{2} T^i T_i + \frac{1}{4} e^\Phi M^{ij} M_{ij} \] (C.9)

and
\[ \frac{1}{2} \tilde{\nabla}^j (dh)_{ij} - (dh)_{ij} h^j - \tilde{\nabla}_i \Delta + \Delta h_i = \frac{1}{2} e^\frac{\Phi}{2} (-\alpha T_i + T^j \tilde{F}_{ij}) + \frac{1}{4} e^\Phi (-2 L_j M_{ij} + M^{jk} \tilde{H}_{ijk}) \] (C.10)

These are implied by (C.3), (C.4), (C.5), (C.6), (C.7) and (C.8) together with (C.3), and the Bianchi identities (C.1) and (C.2).

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