A Derivation of the Classical Einstein-Dirac-Maxwell Equations
From a Model of an Elastic Medium

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Abstract
Starting from a model of an elastic medium, partial differential equations with the form of the coupled Einstein-Dirac-Maxwell equations are derived. The form of these equations describes particles with mass and spin coupled to electromagnetic and gravitational type of interactions. A two dimensional version of these equations is obtained by starting with a model in three dimensions and deriving equations for the dynamics of the lowest fourier modes assuming one dimension to be periodic. Generalizations to higher dimensions are discussed.
I. INTRODUCTION

Dirac’s equation describes the behavior of particles with mass and spin and how they couple to the electromagnetic field. The usual form of Dirac’s equation (with $\hbar$, and $c$ set to unity) is

$$(\gamma^\mu \partial_\mu + m)\Psi(x) = 0$$

where $\Psi$ is a spinor field, $m$ is the particle mass and the gamma matrices $\gamma^\mu$ satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2I\delta_{\mu\nu}.$$  

The electromagnetic field is introduced by the minimal coupling prescription $\partial_\mu \rightarrow D_\mu$, with

$$D_\mu = \partial_\mu + ieA_\mu(x)$$

where $A_\mu$ is the electromagnetic vector potential. Dirac’s equation can be further coupled to gravity (at the classical level) and the equation then takes the form

$$\tilde{\gamma}^\mu [\partial_\mu - \Gamma_\mu + ieA_\mu]\Psi(x) + m\Psi(x) = 0$$

where $\Gamma_\mu$ is known as the spin connection, $A_\mu$ is the electromagnetic vector potential, and $m$ is the mass. The gravitational coupling enters through the modified dirac matrices $\tilde{\gamma}^\mu$ which satisfy the anticommutation relation

$$\{\tilde{\gamma}^\mu, \tilde{\gamma}^\nu\} = 2Ig^{\mu\nu}. \quad (2)$$

and the spin connection satisfies the additional constraint

$$\frac{\partial \gamma^\mu_\beta}{\partial x^\nu} + \gamma^{\beta\gamma} \Gamma^\mu_\beta\gamma^\nu - \Gamma^\nu_\beta \gamma^\mu + \gamma^\mu \Gamma^\nu = 0 \quad (3)$$

where $\Gamma^\mu_\beta\gamma^\nu$ are the usual Christoffel symbols.

The above form of Dirac’s equation describes the dynamics of the spinor field $\Psi$ when coupled to the scalar fields $A_\mu$ and gravity. There are two additional equations which describe the dynamics of $A_\mu$ and $g_{\mu\nu}$, these are the Einstein field equation

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu}$$

(4)
\[ \nabla_\mu F^{\mu\nu} = 4\pi e \bar{\Psi} \gamma^\nu \Psi \]  

The Equations (1), (4) and (5) are collectively known as the Einstein-Dirac-Maxwell equations. This paper will show that there exists a "Dirac-like" equation, with the form of Equation (1), that determines the dynamics of the lowest Fourier modes of an elastic medium and furthermore the dimensional reduction involved in the Fourier transform implies Equations (4) and (5).

II. ELASTICITY THEORY

The theory of elasticity is usually concerned with the infinitesimal deformations of an elastic body. We assume that the material points of a body are continuous and can be assigned a unique label \( \vec{a} \). For definiteness the elastic body can be taken to be a three dimensional object so each point of the body may be labeled with three coordinate numbers \( a^i \) with \( i = 1, 2, 3 \).

If this three dimensional elastic body is placed in a large ambient three dimensional space then the material coordinates \( a^i \) can be described by their positions in the 3-D fixed space coordinates \( x^i \) with \( i = 1, 2, 3 \). In this description the material points \( a^i(x^1, x^2, x^3) \) are functions of \( \vec{x} \). A deformation of the elastic body results in infinitesimal displacements of these material points. If before deformation, a material point \( a^0 \) is located at fixed space coordinates \( x^{01}, x^{02}, x^{03} \) then after deformation it will be located at some other coordinate \( x^1, x^2, x^3 \). The deformation of the medium is characterized at each point by the displacement vector

\[ u^i = x^i - x^{0i} \]

which measures the displacement of each point in the body after deformation.

It is our aim to take this model of an elastic medium and derive from it equations of motion that have the same form as Dirac’s equation. In doing so we have to distinguish between the intrinsic coordinates of the medium which we will call "internal" coordinates and the fixed space coordinates which facilitates our derivation of the equations of motion. In the undeformed state we may take the external coordinates to coincide with the material coordinates \( a^i = x^{0i} \). The approach that we will use in this paper is to derive equations of
motion using the fixed space coordinates and then translate this to the internal coordinates of our space.

We first consider the effect of a deformation on the measurement of distance. After the elastic body is deformed, the distances between its points changes as measured with the fixed space coordinates. If two points which are very close together are separated by a radius vector \( dx^0 \) before deformation, these same two points are separated by a vector \( dx^i = dx^0 + du^i \) afterwards. The square distance between the points before deformation is then \( ds^2 = (dx^0)^2 + (dx^2)^2 + (dx^3)^2 \). Since these coincide with the material points in the undeformed state, this can be written \( ds^2 = (da^1)^2 + (da^2)^2 + (da^3)^2 \). The squared distance after deformation can be written \( ds'^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 = \sum_i (dx^i)^2 = \sum_i (da^i + du^i)^2 \). The differential element \( du^i \) can be written as \( du^i = \sum_i \frac{\partial u^i}{\partial a^k} da^k \), which gives for the distance between the points

\[
\begin{align*}
    ds'^2 &= \sum_i \left( da^i + \sum_k \frac{\partial u^i}{\partial a^k} da^k \right) \left( da^i + \sum_l \frac{\partial u^i}{\partial a^l} da^l \right) \\
    &= \sum_i \sum_k \left( \delta_{ik} + \left( \frac{\partial u^i}{\partial a^k} + \frac{\partial u^k}{\partial a^i} + \sum_l \frac{\partial u^l}{\partial a^i} \frac{\partial u^i}{\partial a^k} \right) da^i da^k \right) \\
    &= \sum_{ik} (\delta_{ik} + 2\epsilon'_{ik}) da^i da^k
\end{align*}
\]

where \( \epsilon'_{ik} \) is

\[
\epsilon'_{ik} = \frac{1}{2} \left( \frac{\partial u^i}{\partial a^k} + \frac{\partial u^k}{\partial a^i} + \sum_l \frac{\partial u^l}{\partial a^i} \frac{\partial u^l}{\partial a^k} \right) \tag{6}
\]

The quantity \( \epsilon'_{ik} \) is known as the strain tensor. It is fundamental in the theory of elasticity. In most treatments of elasticity it is assumed that the displacements \( u^i \) as well as their derivatives are infinitesimal so the last term in Equation (6) is dropped. This is an approximation that we will not make in this work.

The quantity

\[
\begin{align*}
    g_{ik} &= \delta_{ik} + \frac{\partial u^i}{\partial a^k} + \frac{\partial u^k}{\partial a^i} + \sum_l \frac{\partial u^l}{\partial a^i} \frac{\partial u^l}{\partial a^k} \\
    &= \delta_{ik} + 2\epsilon'_{ik} \tag{7}
\end{align*}
\]

is the metric for our system and determines the distance between any two points.
That this metric is simply the result of a coordinate transformation from the flat space metric can be seen by writing the metric in the form:

\[
\begin{pmatrix}
\frac{\partial x^1}{\partial a^1} & \frac{\partial x^2}{\partial a^1} & \frac{\partial x^3}{\partial a^1}
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\frac{\partial x^1}{\partial a^1} & \frac{\partial x^2}{\partial a^1} & \frac{\partial x^3}{\partial a^1}
\end{pmatrix}
= J^T I J
\]

where

\[
\frac{\partial x^\mu}{\partial a^\nu} = \delta_{\mu\nu} + \frac{\partial u^\mu}{\partial a^\nu},
\]
and \( J \) is the Jacobian of the transformation. Later in section VI A we will show that the metric for the Fourier modes of our system is not a simple coordinate transformation.

The inverse matrix \((g^{ik}) = (g_{ik})^{-1}\) can be obtained by direct inversion of Equation (7) which would yield components of \(g^{ik}\) in terms of derivatives of \(u^i\) with respect to the internal coordinates \(a^i\). Later it will be useful however to have the inverse metric stated in terms of derivatives of \(u^i\) with respect to the external coordinates \(x^i\).

To accomplish this we write the inverse metric as \((g^{ik}) = (J^{-1})(J^{-1})^T\) where

\[
J^{-1} = \begin{pmatrix}
\frac{\partial a^1}{\partial x^1} & \frac{\partial a^1}{\partial x^2} & \frac{\partial a^1}{\partial x^3} \\
\frac{\partial a^2}{\partial x^1} & \frac{\partial a^2}{\partial x^2} & \frac{\partial a^2}{\partial x^3} \\
\frac{\partial a^3}{\partial x^1} & \frac{\partial a^3}{\partial x^2} & \frac{\partial a^3}{\partial x^3}
\end{pmatrix}
\]

This yields for the inverse metric

\[
g^{ik} = \delta_{ik} - \frac{\partial u^i}{\partial x^k} - \frac{\partial u^k}{\partial x^i} + \sum_l \frac{\partial u^l}{\partial x^i} \frac{\partial u^l}{\partial x^k}
\]

\[
= \delta_{ik} + 2 \epsilon_{ik}
\]

and \(\epsilon_{ik}\) is the strain tensor in fixed space coordinates. We see that the metric components can be written in terms of either sets of coordinates, internal or fixed space.
III. EQUATIONS OF MOTION

In the following we will use the notation

\[ u_{\mu\nu} = \frac{\partial u^\mu}{\partial x^\nu} \]

and therefore the strain tensor is

\[ \epsilon_{\mu\nu} = \frac{1}{2} \left( -u_{\mu\nu} - u_{\nu\mu} + \sum_\beta u_{\beta\mu} u_{\beta\nu} \right). \]

We work in the fixed space coordinates and take the strain energy as the lagrangian density of our system. This approach leads to the usual equations of equilibrium in elasticity theory\textsuperscript{9,13}. The strain energy is quadratic in the strain tensor \( \epsilon^{\mu\nu} \) and can be written as

\[ E = \sum_{\mu\nu\alpha\rho} C_{\mu\nu\alpha\rho} \epsilon_{\mu\nu} \epsilon_{\alpha\rho} \]

The quantities \( C_{\mu\nu\alpha\rho} \) are known as the elastic stiffness constants of the material\textsuperscript{10}. For an isotropic space most of the coefficients are zero and in 3 dimensions, the lagrangian density reduces to

\[ L = (\lambda + 2\mu) \left[ \epsilon_{11}^2 + \epsilon_{22}^2 + \epsilon_{33}^2 \right] + 2\lambda \left[ \epsilon_{11}\epsilon_{22} + \epsilon_{11}\epsilon_{33} + \epsilon_{22}\epsilon_{33} \right] + 4\mu \left[ \epsilon_{12}^2 + \epsilon_{13}^2 + \epsilon_{23}^2 \right] \]  

(10)

where \( \lambda \) and \( \mu \) are known as Lamé constants\textsuperscript{10}.

The usual Lagrange equations,

\[ \sum_\nu \frac{d}{dx^\nu} \left( \frac{\partial L}{\partial u_{\rho\nu}} \right) - \frac{\partial L}{\partial u^\rho} = 0, \]

apply with each component of the displacement vector treated as an independent field variable. Since our Lagrangian contains no terms in the field \( u^\rho \), Lagrange’s equations reduce to

\[ \sum_\nu \frac{d}{dx^\nu} \left( \frac{\partial L}{\partial u_{\rho\nu}} \right) = 0. \]

which we shall denote

\[ V^\rho \equiv \sum_\nu \frac{d}{dx^\nu} \left( \frac{\partial L}{\partial u_{\rho\nu}} \right) = 0. \]

Using the above form of the Lagrangian one can write

\[ \frac{\partial L}{\partial u_{\rho\nu}} = 2\lambda \left( \sum_\alpha \epsilon_{\alpha\alpha} \right) (\delta_{\rho\nu} - u_{\rho\nu}) + 4\mu \sum_\alpha \epsilon_{\alpha\nu} (\delta_{\rho\alpha} - u_{\rho\alpha}) \]

\[ = -2\lambda \sigma - 2\mu (u_{\rho\nu} + u_{\nu\rho}) + \lambda \left[ \sum_\alpha (u_{\alpha1}^2 + u_{\alpha2}^2) \delta_{\rho\nu} - 2 (\epsilon_{11} + \epsilon_{22}) u_{\rho\nu} \right] \]
\[ + 2\mu \left[ \sum_\alpha \sum_\beta u_{\beta\alpha} u_{\beta\nu} \delta_{\rho\alpha} - 2 \sum_\alpha \epsilon_{\alpha\nu} u_{\rho\alpha} \right] \]

(11)
where the divergence of the displacement field is \( \sigma \equiv (u_{11} + u_{22} + u_{33}) \).

The first two terms in Equation (11) are first order in the components \( u_{\mu\nu} \) while the last terms (in square brackets) are second order and higher in \( u_{\mu\nu} \).

Let us denote by \( E_{\rho\nu} \) the last two bracketed terms in Equation (11). This allows us to write

\[
V^\rho = -2\lambda \frac{\partial \sigma}{\partial x^\rho} - 2\mu \sum_\nu \frac{\partial}{\partial x^\nu}(u_{\rho\nu} + u_{\nu\rho}) + \sum_\nu \frac{\partial E_{\rho\nu}}{\partial x^\nu} = 0. \tag{12}
\]

The quantity \( V^\rho \) is a vector and every term in Equation (12) transforms as a vector. As such the quantity

\[
E_{\rho} \equiv \sum_\nu \frac{\partial E_{\rho\nu}}{\partial x^\nu}
\]

can always be written as the gradient of a scalar plus the curl of a vector

\[
\vec{E} = \nabla \alpha + \nabla \times \chi.
\]

This allows us to write

\[
V^\rho = -2\lambda \frac{\partial \sigma}{\partial x^\rho} - 2\mu \left( \nabla^2 u^\rho + \frac{\partial \sigma}{\partial x^\rho} \right) + \frac{\partial \alpha}{\partial x^\rho} + (\nabla \times \chi)^\rho \tag{13}
\]

or

\[
\nabla^2 \phi = 0, \tag{14}
\]

where

\[
\phi = \left[ (2\lambda + 4\mu)\sigma - \alpha \right]
\]

and notation \((\nabla \times \chi)^\rho\) represents the \( \rho \) component of the vector \((\nabla \times \chi)\). We see therefore that there exists a scalar quantity, in the medium that obeys Laplace’s equation. Given that \( \alpha \) is second order in the strain quantities this implies that \( \phi \) reduces to the divergence of the displacement field, \( \sigma \), in the infinitesimal strain approximation.

A. Small Strain Approximation

In the usual treatment of elasticity theory the strain components \( u_\mu \) and their derivatives \( u_{\mu\nu} \) are taken to be infinitesimal. In this approximation, only the first order terms in any equation are kept. We will not make this assumption in this paper. Where necessary we will adopt a small strain approximation where the strain components are small but not infinitesimal. In using this small strain approximation we will always keep our equations of
motion to at least order one order of magnitude higher in the strain components than the
infinitesimal approximation.

B. Internal Coordinates

In sections V and VI we will need to translate the equations of motion from the fixed
space coordinates to the internal coordinates. For clarity and to adopt a more consistent
convention, in the remainder of this text we change notation slightly and write the internal
coordinates not as \( a^i \) but as \( x'^i \) and the fixed space coordinates will continue to be unprimed
and denoted \( x^i \). Now using \( u^i = x^i - x'^i \) we can write

\[
\frac{\partial}{\partial x^i} = \sum_j \frac{\partial x'^j}{\partial x^i} \frac{\partial}{\partial x'^j}
\]

\[
= \sum_j \left( \frac{\partial x^j}{\partial x^i} - \frac{\partial u^j}{\partial x^i} \right) \frac{\partial}{\partial x'^j}
\]

\[
= \sum_j \left( \delta_{ij} - \frac{\partial u^j}{\partial x^i} \right) \frac{\partial}{\partial x'^j}
\]

\[
= \frac{\partial}{\partial x'^i} - \sum_j \frac{\partial u^j}{\partial x^i} \frac{\partial}{\partial x'^j}
\]  

Equation (15) relates derivatives in the fixed space coordinates \( x^i \) to derivatives in the
material coordinates \( x'^i \). As mentioned earlier, in the standard treatment of elastic solids
the displacements \( u^i \) as well as their derivatives are assumed to be infinitesimal and so the
second term in Equation (15) is dropped and there is no distinction made between the \( x^i \) and
the \( x'^i \) coordinates. In this paper we will keep the nonlinear terms in Equation (15) when
changing coordinates. Hence we will make a distinction between the two sets of coordinates
and this will be pivotal in the derivations to follow.

C. Vector Transformation

The partial derivatives in Equation (15) transform as covectors. Therefore from Equation (15) we see that any covector transforms as

\[
V_\mu = V'_\mu - \sum_j \frac{\partial u^j}{\partial x^i} V'_j
\]  

(16)
when going between the unprimed Euclidean space and the primed coordinate system. In a similar manner the components of a vector change as

\[ dx^i = \sum_j \frac{\partial x^i}{\partial x^j} dx^j \]

\[ = \sum_j \left( \frac{\partial x^i}{\partial x^j} - \frac{\partial u^i}{\partial x^j} \right) dx^j \]

\[ = \sum_j \left( \delta_{ij} + \frac{\partial u^i}{\partial x^j} \right) dx^j \]

or for any vector

\[ V^i = V^i - \sum_j \frac{\partial u^i}{\partial x^j} V^j \]  \hspace{1cm} (17)

where the "up" and "down" notation is used to distinguish components of a vector from those of a covector. These transformations will be useful in later sections.

**D. Decomposition of Strain**

The transformation in Equation (17) has the form \( \vec{V} \rightarrow \vec{V} - \delta \vec{V} \) where

\[ \delta \vec{V}^\nu = \sum_\mu \frac{\partial u^\nu}{\partial x^\mu} V^\mu. \]

This can be decomposed as\(^\text{10}\):

\[ \delta V^\mu = \sum_\mu \left( \frac{u^\mu_\nu + u_\nu^\mu}{2} + \frac{u^\mu_\nu - u_\nu^\mu}{2} \right) V^\mu \]

\[ = \sum_\mu (e_{\mu\nu} + \omega_{\mu\nu}) V^\mu. \]  \hspace{1cm} (18)

The quantity \( \omega_{\mu\nu} = 1/2(u_{\mu\nu} - u_{\nu\mu}) \) represents a local rigid body motion of the medium\(^\text{10}\). The quantity \( e_{\mu\nu} = 1/2(u_{\mu\nu} + u_{\nu\mu}) \) represents what is usually termed "pure deformation" and for sufficiently small \( u_{\mu\nu} \) we have \( e_{\mu\nu} \approx \epsilon_{\mu\nu} \). This decomposition will be useful later in identifying the electromagnetic field vector.

We will now demonstrate a new method for reducing Laplace’s Equation (14) to Dirac’s equation and compare this method to the traditional Dirac reduction.
IV. CARTAN’S SPINORS

The concept of Spinors was introduced by Eli Cartan in 1913. In Cartan’s original formulation spinors were motivated by studying isotropic vectors which are vectors of zero length. In three dimensions the equation of an isotropic vector is

\[(x^1)^2 + (x^2)^2 + (x^3)^2 = 0\]  

(19)

for complex quantities \(x^i\). A closed form solution to this equation is realized as

\[x^1 = -2\xi_0\xi_1, \quad x^2 = i(\xi_0^2 + \xi_1^2), \quad \text{and} \quad x^3 = \xi_0^2 - \xi_1^2\]  

(20)

where the two quantities \(\xi_i\) are

\[\xi_0 = \pm \sqrt{\frac{x^3 - ix^2}{2}} \quad \text{and} \quad \xi_1 = \pm \sqrt{-\frac{x^3 - ix^2}{2}}.\]

The two component object \(\xi = (\xi_0, \xi_1)\) is a spinor and any equation of the form [19] has a spinor solution.

In the following we use the notation \(\partial_\mu \equiv \partial/\partial x^\mu\) and Laplace’s equation is written

\[\left(\partial_1^2 + \partial_2^2 + \partial_3^2\right)\phi = 0.\]

This equation can be viewed as an isotropic vector in the following way. The components of the vector are the partial derivative operators \(\partial/\partial x^\mu\) acting on the quantity \(\phi\). As long as the partial derivatives are restricted to acting on the scalar field \(\phi\) it has a spinor solution given by

\[\hat{\xi}_0^2 = \frac{1}{2} \left(\frac{\partial}{\partial x^3} - i\frac{\partial}{\partial x^2}\right) = \frac{\partial}{\partial z^0}\]  

(21)

and

\[\hat{\xi}_1^2 = -\frac{1}{2} \left(\frac{\partial}{\partial x^3} + i\frac{\partial}{\partial x^2}\right) = \frac{\partial}{\partial z^1}\]  

(22)

where

\[z^0 = x^3 + ix^2 \quad \text{and} \quad z^1 = -x^3 + ix^2\]

and the ”hat” notation indicates that the quantities \(\hat{\xi}\) are operators. The equations

\[\hat{\xi}_0^2 = \frac{\partial}{\partial z^0}\]  

and

\[\hat{\xi}_1^2 = \frac{\partial}{\partial z^1}\]
are equations of fractional derivatives of order 1/2 denoted \( \hat{\xi}_0 = D_{z^0}^{1/2} \) and \( \hat{\xi}_1 = D_{z^1}^{1/2} \). Fractional derivatives have the property that\(^{16}\)

\[
D_{z^0}^{1/2} D_{z^1}^{1/2} = \frac{\partial}{\partial z}
\]

and various methods exist for writing closed form solutions for these operators\(^{16}\). The exact form for these fractional derivatives however, is not important here. The important thing to note is that a solution to Laplace’s equation can be written in terms of spinors which are fractional derivatives.

**A. Spinor Properties**

If we assume that the fractional derivatives \( \hat{\xi}_0 \) and \( \hat{\xi}_1 \) commute then we also have

\[
(\hat{\xi}_0 \hat{\xi}_1)^2 = \hat{\xi}_0 \hat{\xi}_0 \hat{\xi}_1 \hat{\xi}_1 = \frac{\partial}{\partial z^0} \frac{\partial}{\partial z^1} = -\frac{1}{4} \left( \partial_3 - i \partial_2 \right) \left( \partial_3 + i \partial_2 \right) = -\frac{1}{4} \left( \partial_2^2 + \partial_3^2 \right) = \frac{1}{4} \partial_1^2
\]

Using this result combined with Equations (21) and (22) we may write for the components of our vector

\[
\frac{\partial}{\partial x_1} = -2\hat{\xi}_0 \hat{\xi}_1
\]

(23)

\[
\frac{\partial}{\partial x_2} = i(\hat{\xi}_0^2 + \hat{\xi}_1^2)
\]

(24)

\[
\frac{\partial}{\partial x_3} = \hat{\xi}_0^2 - \hat{\xi}_1^2.
\]

(25)

Upon complex conjugation we observe that

\[
(\hat{\xi}_0^2)^* = -\hat{\xi}_1^2 \quad \text{and} \quad (\hat{\xi}_1^2)^* = -\hat{\xi}_0^2.
\]

which implies

\[
\hat{\xi}_0^* = i\hat{\xi}_1 \quad \text{and} \quad \hat{\xi}_1^* = i\hat{\xi}_0.
\]

(26)
Under complex conjugation therefore our vector becomes
\[
\left( \frac{\partial}{\partial x^1} \right)^* = -\frac{\partial}{\partial x^1}, \quad \left( \frac{\partial}{\partial x^2} \right)^* = \frac{\partial}{\partial x^2} \quad \text{and} \quad \left( \frac{\partial}{\partial x^3} \right)^* = \frac{\partial}{\partial x^3}.
\]

We see therefore that our solution is consistent only for an elastic medium embedded in a pseudo-Euclidean space in which the vector \( \partial/\partial x^1 \) is pure imaginary. If we write
\[
\frac{\partial}{\partial x^1} = \imath \frac{\partial}{\partial t}
\]
then Laplace’s equation becomes the wave equation
\[
\left( -\frac{\partial^2}{\partial t^2} + \vec{\nabla}^2 \right) \phi = 0
\]
We will continue to work with \( dx \) rather than \( dt \) to avoid carrying the minus sign in the computations.

B. Matrix Form

It can be readily verified that our spinors satisfy the following equations
\[
\begin{align*}
\left[ \hat{\xi}_0 \frac{\partial}{\partial x^1} + \hat{\xi}_1 \left( \frac{\partial}{\partial x^3} - \imath \frac{\partial}{\partial x^2} \right) \right] \phi &= 0 \\
\left[ \hat{\xi}_0 \left( \frac{\partial}{\partial x^3} + \imath \frac{\partial}{\partial x^2} \right) - \hat{\xi}_1 \frac{\partial}{\partial x^1} \right] \phi &= 0
\end{align*}
\]
and in matrix form
\[
\begin{pmatrix}
\frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^3} - \imath \frac{\partial}{\partial x^2} \\
\frac{\partial}{\partial x^3} + \imath \frac{\partial}{\partial x^2} & -\frac{\partial}{\partial x^1}
\end{pmatrix}
\begin{pmatrix}
\hat{\xi}_0 \\
\hat{\xi}_1
\end{pmatrix}
\phi = 0 \tag{27}
\]

The matrix
\[
X = \begin{pmatrix}
\frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^3} - \imath \frac{\partial}{\partial x^2} \\
\frac{\partial}{\partial x^3} + \imath \frac{\partial}{\partial x^2} & -\frac{\partial}{\partial x^1}
\end{pmatrix}
\]
is equal to the dot product of the vector \( \partial_\mu \equiv \partial/\partial x^\mu \) with the pauli spin matrices
\[
X = \frac{\partial}{\partial x^1} \gamma^1 + \frac{\partial}{\partial x^2} \gamma^2 + \frac{\partial}{\partial x^3} \gamma^3
\]
where
\[
\begin{align*}
\gamma_1 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
\gamma_2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \\
\gamma_3 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\end{align*}
\]
are the Pauli matrices which satisfy the anticommutation relations
\[
\{\tilde{\gamma}^\mu, \tilde{\gamma}^\nu\} = 2I\delta_{\mu\nu}.
\] (28)
where \(I\) is the identity matrix.

Equation (27) can be written
\[
\sum_{\mu=1}^3 \partial_\mu \gamma_\mu \xi = 0.
\] (29)
where we have used the notation \(\xi \equiv \xi \phi\). This equation has the form of Dirac’s equation in 3 dimensions for a noninteracting massless field \(\xi\).

**C. Relation to the Dirac Decomposition**

The fact that Laplace’ equation and Dirac’s equation are related is not new. However the decomposition used here is not the same as that used by Dirac. The usual method of connecting the second order Laplace Equation to the first order Dirac equation is to operate on Equation (29) from the left with \(\sum_{\nu=1}^3 \gamma_\nu \partial_\nu\) giving

\[
0 = \sum_{\mu,\nu=1}^3 \gamma_\nu \gamma_\mu \partial_\nu \partial_\mu \Psi(x)
\]
\[
= \sum_{\mu,\nu=1}^3 \frac{1}{2} (\gamma_\nu \gamma_\mu + \gamma_\mu \gamma_\nu) \partial_\nu \partial_\mu \Psi(x)
\]
\[
= (\partial_1^2 + \partial_2^2 + \partial_3^2) \Psi(x)
\] (30)
where \(\Psi = (\alpha_1, \alpha_2)\) is a two component spinor and Equation (28) has been used in the last step.

This shows that Dirac’s equation does in fact imply Laplace’s equation. The important thing to note about Equation (30) however, is that the three dimensional Dirac’s equation implies not one Laplace equation but two in the sense that each component of the spinor \(\Psi\) satisfies this equation. Explicitly stated, Equation (30) reads
\[
\begin{pmatrix}
\partial_1^2 + \partial_2^2 + \partial_3^2 & 0 \\
0 & \partial_1^2 + \partial_2^2 + \partial_3^2
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2
\end{pmatrix} = 0
\]
for the independent scalars $\alpha_1, \alpha_2$.

Conversely, if one starts with Laplace’s equation and tries to recover Dirac’s equation, it is necessary to start with two independent scalars each independently satisfying Laplace’s equation. In other words, using the usual methods, it is not possible to take a single scalar field that satisfies Laplace’s equation and recover Dirac’s equation for a two component spinor.

What has been demonstrated in the preceding sections is that starting with only one scalar quantity satisfying Laplace’s equation Dirac’s equation for a two component spinor may be derived. Furthermore any medium (such as an elastic solid) that has a single scalar that satisfies Laplace’s equation must have a spinor that satisfies Dirac’s equation and such a derivation necessitates the use of fractional derivatives.

V. TRANSFORMATION TO INTERNAL COORDINATES

In section VI A we will take the $x^\beta$ coordinate to be periodic and we will derive equations for the Fourier components of our fields. Since the elastic solid is assumed to be periodic in the internal coordinates we need to translate our equations of motion from fixed space coordinates to internal coordinates. Using Equation (15) we can rewrite Equation (29), as

$$\sum_{\mu=1}^{3} \gamma^\mu \left( \partial'_\mu - \sum_{\nu} \frac{\partial u^\nu}{\partial x^\mu} \partial'_\nu \right) \xi = 0 \quad (31)$$

or

$$\sum_{\mu=1}^{3} \gamma'^\mu \partial'_\mu \xi = 0$$

where $\partial'_\mu = \partial/\partial x'_\mu$ and $\gamma'^\mu$ is given by

$$\gamma'^\mu = \gamma^\mu - \sum_{\alpha=1}^{3} u_{\mu\alpha} \gamma^\alpha. \quad (32)$$

By Equation (17) these are the gamma matrices expressed in the primed coordinate system.

The anticommutator of these matrices is

$$\{ \gamma'^\mu, \gamma'^\nu \} = \{ \gamma^\mu - \sum_{\alpha} u_{\mu\alpha} \gamma^\alpha, \gamma^\nu - \sum_{\beta} u_{\nu\beta} \gamma^\beta \}$$

$$= \{ \gamma^\mu, \gamma^\nu \} - \sum_{\beta} u_{\nu\beta} \{ \gamma^\mu, \gamma^\beta \} - \sum_{\alpha} u_{\mu\alpha} \{ \gamma^\alpha, \gamma^\nu \} + \sum_{\alpha\beta} u_{\mu\alpha} u_{\nu\beta} \{ \gamma^\alpha, \gamma^\beta \}$$
\begin{align*}
= 2I \left( \delta_{\mu\nu} - \sum_{\beta} u_{\nu\beta} \delta_{\mu\beta} - \sum_{\alpha} u_{\mu\alpha} \delta_{\alpha\nu} + \sum_{\alpha} \sum_{\beta} u_{\mu\alpha} u_{\nu\beta} \delta_{\alpha\beta} \right) \\
= 2I \left( \delta_{\mu\nu} - u_{\nu\mu} - u_{\mu\nu} + \sum_{\alpha} u_{\mu\alpha} u_{\nu\alpha} \right) \\
\equiv 2I g^{\mu\nu}
\end{align*}

This shows that the gamma matrices have the form of the usual Dirac’s matrices in a curved space\(^4\). To further develop the form of Equation (31) we have to transform the spinor properties of \(\xi\). As currently written \(\xi\) is a spinor with respect to the \(x_i\) coordinates not the \(x'_i\) coordinates. To transform its spinor properties we use a real similarity transformation\(^4\) and write \(\xi = S \tilde{\xi}\) where \(S\) is a similarity transformation that takes our spinor in \(x_\mu\) to a spinor in \(x'_\mu\).

We then have

\[ \partial'_\mu \xi = (\partial'_\mu S) \tilde{\xi} + S \partial'_\mu \tilde{\xi}. \]

Equation (31) then becomes

\[ 0 = \gamma'^\mu [S \partial'_\mu \tilde{\xi} + (\partial'_\mu S) \tilde{\xi}] \]
\[ = \gamma'^\mu S[\partial'_\mu \tilde{\xi} + S^{-1}(\partial'_\mu S) \tilde{\xi}] \]
\[ = S^{-1}\gamma'^\mu S[\partial'_\mu \tilde{\xi} + S^{-1}(\partial'_\mu S) \tilde{\xi}] \]

Using \((\partial'_\mu S^{-1})S = -S^{-1}(\partial'_\mu S)\). This can finally be written

\[ \tilde{\gamma}'^\mu [\partial'_\mu - \Gamma_\mu] \tilde{\xi} = 0 \quad (33) \]

where \(\Gamma_\mu = (\partial'_\mu S^{-1})S\) and \(\tilde{\gamma}'^\mu = S^{-1}\gamma'^\mu S\).

Equation (33) has the form of the Einstein-Dirac equation in 3 dimensions for a free particle of zero mass. The quantity \(\partial'_\mu - \Gamma_\mu\) is the covariant derivative for an object with spin in a curved space\(^4\). In order to make this identification, the field \(\Gamma_\mu\) must satisfy the additional equation\(^4,5\)

\[ \frac{\partial \tilde{\gamma}'^\mu}{\partial x^\nu} + \tilde{\gamma}'^\beta \Gamma'^\mu_{\beta\nu} - \Gamma_\nu \tilde{\gamma}'^\mu + \tilde{\gamma}'^\mu \Gamma_\nu = 0 \]

where \(\Gamma'^\mu_{\beta\nu}\) is the usual Christoffel symbol. That this equation holds can be seen by considering the equation

\[ \frac{\partial \gamma'^\mu}{\partial x^\nu} = 0 \]
true in the unprimed coordinate system. But since the unprimed coordinate system is Euclidean space, the Christoffel symbols are identically zero. This allows us to write

\[ \frac{\partial \gamma^\mu}{\partial x^\nu} + \gamma^\beta \Gamma^\mu_{\beta \nu} = 0 \]

Since this is a tensor equation true in all frames, in the primed coordinate system we can immediately write

\[ \partial'_\nu \gamma'^\mu + \gamma'^\beta \Gamma'^\mu_{\beta \nu} = 0 \]

Using \( \gamma'^\mu = S \tilde{\gamma}^\mu S^{-1} \), we have

\[ \partial'_\nu (S \tilde{\gamma}^\mu S^{-1}) + (S \tilde{\gamma}^\beta S^{-1}) \Gamma'^\mu_{\beta \nu} = 0 \]

or

\[ (\partial'_\nu S) \tilde{\gamma}^\mu S^{-1} + S(\partial'_\nu \tilde{\gamma}^\mu) S^{-1} + S \tilde{\gamma}^\mu (\partial'_\nu S^{-1}) S + (S \tilde{\gamma}^\beta S^{-1}) \Gamma'^\mu_{\beta \nu} = 0. \]

Multiplying by \( S^{-1} \) on the left and \( S \) on the right yields

\[ S^{-1}(\partial'_\nu S) \tilde{\gamma}^\mu + (\partial'_\nu \tilde{\gamma}^\mu) + \tilde{\gamma}^\mu (\partial'_\nu S^{-1}) S + \tilde{\gamma}^\beta \Gamma'^\mu_{\beta \nu} = 0 \]

Finally, using \( \Gamma^\nu = (\partial'_\nu S^{-1}) S \) and again noting that \( \partial'_\nu S^{-1} S = -S^{-1} \partial'_\nu S \) we have,

\[ \tilde{\gamma}^\mu \Gamma^\nu - \Gamma^\nu \tilde{\gamma}^\mu + \left( \partial'_\nu \tilde{\gamma}^\mu + \tilde{\gamma}^\beta \Gamma'^\mu_{\beta \nu} \right) = 0 \quad (34) \]

We have just demonstrated that in the internal coordinates, the equations of motion of an elastic medium have the same form as the free-field Einstein Dirac equation for a massless particle in three dimensions. Of course this form is trivial in the sense that it is due solely to a coordinate change and can be removed by simply changing back to the unprimed coordinates. In the next section we will show that if one of the dimensions in our problem is periodic, then the equations of motion for the fourier modes is not trivial. Furthermore the introduction of the fourier components will generate extra terms in Equation (31) that implies a series of equations relevant for particles with mass coupled to fields that can be associated with electromagnetism and classical gravity.

VI. INTERACTING PARTICLES WITH MASS

In this section we again consider a three dimensional elastic solid but we take the third dimension to be compact with the topology of a circle. All field variables then become
periodic functions of $x^3$ and can be Fourier transformed. The act of Fourier transforming
the equations of motion will effectively reduce the dimensionality of our problem from three
dimensions ($x^1, x^2, x^3$) to two dimensions ($x^1, x^2$). This dimensional reduction will in some
cases create two of the same type of objects (for instance a two dimensional metric and a three
dimensional metric). We will need to take care to distinguish between the two dimensional
and three dimensional quantities and use explicit limits in most summations.

Throughout the derivation we will assume the small strain approximation of section (III A). In doing so, we will keep the equations of motion of the system to one order of
magnitude higher in the strain components than the infinitesimal results. For the Dirac
equation the infinitesimal result would yield $\gamma^\mu \partial_\mu \Psi = 0$ which is a linear operator that is
zero'th order in $u_{\mu\nu}$, acting on $\Psi$. In our derivation extra terms will appear in this equation
of motion and we will keep the linear operator to first order in $u_{\mu\nu}$.

A. Fourier Transform

In preparation for Fourier Transforming we isolate the terms involving $x^3$ and rewrite
Equation (31) as,

$$
\sum_{\mu=1}^{2} \gamma^\mu \left( \partial_\mu - \sum_{\nu=1}^{2} u_{\nu\mu} \partial_\nu - u_{3\mu} \partial_3 \right) \xi + \gamma^3 \left( \partial_3 - \sum_{\nu=1}^{2} u_{\nu3} \partial_\nu - u_{33} \partial_3 \right) \xi = 0
$$

Similar to Equation (32) we define

$$
\gamma'^\mu = \gamma^\mu - \sum_{\beta=1}^{2} u_{\mu\beta} \gamma^\beta
$$

to obtain

$$
\sum_{\mu=1}^{2} \gamma'^\mu \left( \partial_\mu - u_{3\mu} \partial_3 \right) \xi + \left( \sum_{\beta=1}^{2} \gamma^\beta u_{\mu3} \partial_3 \right) \xi + \gamma^3 \left( \partial_3 - \sum_{\nu=1}^{2} u_{\nu3} \partial_\nu - u_{33} \partial_3 \right) \xi = 0
$$

In keeping with the small strain approximation the second term in parenthesis may be
neglected as being second order in the small strain quantities. We are left with

$$
\sum_{\mu=1}^{2} \gamma'^\mu \left( \partial_\mu - u_{3\mu} \partial_3 \right) \xi + \gamma^3 \left( \partial_3 - \sum_{\nu=1}^{2} u_{\nu3} \partial_\nu - u_{33} \partial_3 \right) \xi = 0
$$

Next we transform the spinor

$$
\xi = S \tilde{\xi}
$$
Which yields
\[ \sum_{\mu=1}^{2} \gamma^\mu \left( S \left[ \partial'_\mu - u_{3\mu} \partial'_3 \right] + \left[ (\partial'_\mu S) - u_{3\mu}(\partial'_3 S) \right] \right) \xi \\
+ \gamma^3 \left( S \left[ \partial'_3 - \sum_{\nu=1}^{2} u_{\nu3} \partial'_\nu - u_{33} \partial'_3 \right] + (\partial'_3 S) - \sum_{\nu=1}^{2} u_{\nu3}(\partial'_\nu S) - u_{33}(\partial'_3 S) \right) \tilde{\xi} = 0 \]

In the absence of deformation there is no distinction between primed and unprimed coordinates and the similarity transformation therefore reduces to the identity matrix as the displacement field \( u_\mu \) goes to zero. This implies that \( S \) has the form
\[ S = I + f(u_\mu) \]
where \( f(u_\mu) \) is a matrix that is a function of the strain components and \( f \to 0 \) as the strain components go to zero. This gives
\[ u_{\alpha\beta} \partial'_\mu S = u_{\alpha\beta} \frac{\partial f(u_\nu)}{\partial x'_\mu} \]
\[ = u_{\alpha\beta} \sum_{\nu} \frac{\partial f}{\partial u_\nu} \frac{\partial u_\nu}{\partial x'_\mu} \]
\[ \sim u_{\alpha\beta} u_{\nu\nu} \frac{\partial f}{\partial u_\nu} \]
which implies that these terms are second order in the strain components and can be neglected. We are left with
\[ \sum_{\mu=1}^{2} \gamma^\mu S \left( \partial'_\mu - u_{3\mu} \partial'_3 + S^{-1}(\partial'_3 S) \right) \tilde{\xi} \\
+ \gamma^3 S \left( \partial'_3 - \sum_{\nu=1}^{2} u_{\nu3} \partial'_\nu - u_{33} \partial'_3 + S^{-1}(\partial'_3 S) \right) \tilde{\xi} = 0 \]

Now we multiply by \(-iS^{-1}\gamma^3\) on the left to obtain
\[ \sum_{\mu=1}^{2} \tilde{\gamma}^\mu \left( \partial'_\mu - u_{3\mu} \partial'_3 + S^{-1}(\partial'_3 S) \right) \tilde{\xi} \\
- i \left( \partial'_3 - \sum_{\nu=1}^{2} u_{\nu3} \partial'_\nu - u_{33} \partial'_3 + S^{-1}(\partial'_3 S) \right) \tilde{\xi} = 0 \quad (35) \]
where
\[ \tilde{\gamma}^\mu = -iS^{-1}\gamma^3\gamma^\mu S. \]

This translation to the internal coordinates has introduced several fields into the equations of motion. In addition to the spinor field \( \tilde{\xi} \) there are the fields \( u_{\mu\nu} \) and \( S^{-1}(\partial'_3 S) \). We are
now in a position to Fourier transform these field quantities. We first transform the fields $u_{\nu\mu}$ and $S^{-1}(\partial_\mu S)$ in equation (35) to obtain

$$u_{\nu\mu} = \sum_k u_{\nu\mu,k} e^{ikx_3'}$$

and

$$S^{-1}(\partial_\mu S) = \left[ S^{-1}(\partial_\mu S) \right]_k e^{ikx_3'}$$

where $u_{\nu\mu,k}$ and $[S^{-1}(\partial_\mu S)]_k$ are the $k^{th}$ Fourier modes of the relevant quantities and $k = 2\pi n/a$ with $a$ the length of the circle formed by the elastic solid in the $x'_3$ direction and $n$ is an integer. Equation (35) now becomes,

$$\sum_k e^{ikx_3'} \left[ \sum_{\mu=1}^2 \tilde{\gamma}^\mu \left( \partial_\mu' \delta_{k,0} - u_{3\mu,k} \partial_3' + \left[ S^{-1}(\partial_\mu' S) \right]_k \right) \tilde{\xi} 
- i \left( \partial_3' \delta_{k,0} - \sum_{\nu=1}^2 u_{3\nu,k} \partial_\nu' - u_{33,k} \partial_3' \right) \left[ S^{-1}(\partial_3' S) \right]_k \right] = 0$$

Next we transform the spinor (remembering that it is periodic in 4$\pi$)

$$\tilde{\xi} = \sum_q \tilde{\xi}_q e^{i(q/2)x'_3}$$

and the gamma matrices

$$\tilde{\gamma}^\mu = \sum_l \tilde{\gamma}_l^\mu e^{lx'_3}$$

with $q = 2\pi j/a$, $l = 2\pi j'/a$ and $j, j'$ integers. This yields,

$$\sum_{q,k,l} e^{i(l+k+q/2)x'_3} \left[ \sum_{\mu=1}^2 \tilde{\gamma}_l^\mu \left( \partial_\mu' \delta_{k,0} - \frac{i}{2} u_{3\mu,k} \right) \left[ S^{-1}(\partial_\mu' S) \right]_k \right] \tilde{\xi}_{q/2}$$

$$+ \delta_{l,0} \left( \frac{q}{2} \delta_{k,0} + \sum_{\nu=1}^2 u_{3\nu,k} \partial_\nu' - u_{33,k} \partial_3' + \left[ S^{-1}(\partial_3' S) \right]_k \right) \tilde{\xi}_{q/2} = 0$$

This equation is true for each distinct value of $l + k + q/2 = m/2$ with $m = 2\pi n/a$ and $n$ an integer. Writing $q = (m - 2k - 2l)$ yields finally,

$$\sum_{l,k} \left[ \sum_{\mu=1}^2 \tilde{\gamma}_l^\mu \left( \partial_\mu' \delta_{k,0} - \frac{i}{2} \frac{(m - 2k - 2l)}{2} u_{3\mu,k} \right) \left[ S^{-1}(\partial_\mu' S) \right]_k \right] \tilde{\xi}_{(m-2k-2l)/2}$$

$$+ \delta_{l,0} \left( \frac{(m - 2k - 2l)}{2} \delta_{k,0} - \sum_{\nu=1}^2 u_{3\nu,k} \partial_\nu' - \frac{m - 2k - 2l}{2} u_{33,k} \right) \tilde{\xi}_{(m-2k-2l)/2} = 0$$

(36)

This is an infinite series of equations describing the dynamics of the fields $\xi_{m/2}$.
B. Spectrum of Lowest modes

In quantum mechanical systems a low energy approximation can often be made in which only the lowest Fourier modes of a system are present at low energies (or low temperatures). For instance, in condensed matter systems the mean number of phonons of wavevector \( k \) present in a system at temperature \( T \) is

\[
n_k = \frac{1}{\kappa \omega_k} \left( e^{\kappa \omega_k / T} - 1 \right)
\]

where \( \hbar \) is Planck’s constant, \( k_B \) is Boltzmann’s constant, \( T \) is temperature and \( \omega_k \) for acoustic phonons can be approximated by \( \omega_k = ck \) with \( c \) a constant. The important aspect of Equation (37) is that as \( T \) approaches 0 only the \( k = 0 \) mode is occupied. So for spin zero fields (such as \( u_{\mu,0} \)) in the low energy limit in a quantum system only the \( k = 0 \) modes are present. In this paper we are not considering a quantum mechanical system. Nevertheless we shall examine this ”low energy” approximation but will not attempt to give a rigorous justification, focusing instead on examining the form of the equations of motion that result.

We therefore consider a theory in which only the lowest \( k \) and \( l \) modes are present and Equation (36) reduces to

\[
\sum_{\mu=1}^{2} \tilde{\gamma}_{\mu} \left( \partial' - \sum_{\nu=1}^{2} u_{\nu,0,0} \partial'_{\nu} - \frac{i}{2} u_{3,0,0} + \left[ S^{-1}(\partial' S) \right]_{0} \right) \tilde{\varphi}_{m/2} + \left( \frac{m}{2} + i \sum_{\nu=1}^{2} u_{\nu,0} \partial'_{\nu} - \frac{m}{2} u_{33,0} - i \left[ S^{-1}(\partial'_{3} S) \right]_{0} \right) \tilde{\varphi}_{m/2} = 0
\]

true for each value of \( m \).

We are now in a position to examine in detail the form of the equations for the spinor fields \( \xi_{m/2} \). 

1. \( m_1 \) mode

We use the notation \( m_q = 2\pi q/a \) corresponding to the equation of motion of the fields \( \xi_{q/2} \). For clarity we focus on the equation of motion of the \( q = 1 \) field which yields

\[
\sum_{\mu=1}^{2} \tilde{\gamma}_{\mu} \left( \partial' - \frac{m_1}{2} u_{3,0} + \left[ S^{-1}(\partial' S) \right]_{0} \right) \tilde{\varphi}_{1/2} + \left( \frac{m_1}{2} + i \sum_{\nu=1}^{2} u_{\nu,0} \partial'_{\nu} - \frac{m_1}{2} u_{33,0} - i \left[ S^{-1}(\partial'_{3} S) \right]_{0} \right) \tilde{\varphi}_{1/2} = 0
\]
Let us focus on the first three terms in Equation (38)

\[
\sum_{\mu=1}^{2} \tilde{\gamma}_0^\mu \left( \partial_{\mu} - i \frac{m_1}{2} u_{3\mu,0} + \left[ S^{-1}(\partial_{\mu} S) \right]_0 \right) \tilde{\xi}_{1/2}. \tag{39}
\]

This portion of the equation of motion is very suggestive when compared with Equation (1) and suggests we associate \(\tilde{\gamma}^\mu\) with the 2 dimensional gamma matrices and that we associate \(u_{3\mu,0}\) and \(\left[ S^{-1}(\partial_{\mu} S) \right]_0\) with the electromagnetic vector potential and the spin connection. In order to make these identifications we must show that they are consistent with Maxwell’s equations, and the auxiliary Equation (3). We will now demonstrate that these definitions are consistent.

C. Spin Connection

We propose to make the following definition for the 2D spin connection

\[
\Gamma_\mu = \left[ (\partial_\mu S^{-1}) S \right]_0 + i \left( \frac{m_1}{2} \right) \omega_{3\mu} \tag{40}
\]

where \(\omega_{3\mu} = (1/2)(u_{3\mu} - u_{\mu 3})\). It must now be shown that this is consistent with Equation (3). A solution to Equation (3) is unique only up to an additive multiple of the unit matrix\(^\text{[4]20}\). It is sufficient therefore to consider only the first term on the right hand side of Equation (40).

First let us note that the gamma matrices satisfy the anticommutation relations

\[
\{ \tilde{\gamma}^\mu, \tilde{\gamma}^\nu \} = 2I \left( \delta_{\mu\nu} - (u_{\mu\nu} + u_{\nu\mu}) + 2 \sum_{\beta=1}^{n} u_{\mu\beta} u_{\nu\beta} \right).
\]

Upon Fourier transforming the gamma matrices this becomes

\[
\sum_{k,k'} e^{i\varphi_3(k+k')} \{ \tilde{\gamma}_k^\mu, \tilde{\gamma}_{k'}^\nu \} = \sum_{k,k'} e^{i\varphi_3(k+k')} 2I \left( \delta_{\mu\nu} \delta_{k,k'} - (u_{\mu\nu,k} + u_{\nu\mu,k}) \delta_{k,0} + 2 \sum_{\beta=1}^{n} u_{\mu\beta,k} u_{\nu\beta,k'} \right).
\]

and using the ansatz that only the \(k, k' = 0\) modes are present yields

\[
\{ \tilde{\gamma}_0^\mu, \tilde{\gamma}_0^\nu \} = 2I \left( \delta_{\mu\nu} - (u_{\mu\nu,0} - u_{\nu\mu,0}) + 2 \sum_{\beta=1}^{n} u_{\mu\beta,0} u_{\nu\beta,0} \right). \tag{41}
\]

If we insist that the metric of our system is equal to twice the anticommutator of the gamma matrices then we are led to define

\[
2I g^{\mu\nu} \equiv \{ \tilde{\gamma}^\mu, \tilde{\gamma}^\nu \}. \tag{42}
\]
which becomes

\[ 2Ig^{\mu\nu} \rightarrow \{\tilde{\gamma}_0^\mu, \tilde{\gamma}_0^\nu\} \]

\[ = \delta_{\mu\nu} - (u_{\mu\nu,0} + u_{\nu\mu,0}) + \sum_{\beta=1}^2 u_{\mu\beta,0}u_{\nu\beta,0}. \]  

(43)

This is the metric for our two dimensional subspace and it does not have the form of a simple coordinate transformation on a flat space metric like that of section II. In other words there is no transformation involving the coordinates \(x'^1\) and \(x'^2\) that will remove the Fourier transform in this equation.

The auxiliary Equation (3) can now be shown to hold in a similar manner to the construction in three dimensions. First we note that the definition of the two dimensional metric Equation (43) can be obtained from a transformation on 2D Euclidean space

\[ g^{\mu\nu} = \delta_{\mu\nu} - (u_{\mu\nu} + u_{\nu\mu}) + \sum_{\beta=1}^2 u_{\mu\beta}u_{\nu\beta}, \]  

(44)

\[ \begin{pmatrix}
\frac{\partial x^1}{\partial x'^1} & \frac{\partial x^2}{\partial x'^1} \\
\frac{\partial x^1}{\partial x'^2} & \frac{\partial x^2}{\partial x'^2}
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\frac{\partial x^1}{\partial x'^1} & \frac{\partial x^1}{\partial x'^2} \\
\frac{\partial x^2}{\partial x'^1} & \frac{\partial x^2}{\partial x'^2}
\end{pmatrix}
\]

\[ = J^T I J \]  

(45)

where

\[ \frac{\partial x^\mu}{\partial x'^\nu} = \delta_{\mu\nu} - \sum_{\nu=1}^2 \frac{\partial u_{\mu\nu}}{\partial x'^\nu}. \]  

(46)

In other words the two dimensional metric in Equation (43) is the Fourier Transform of a matrix obtained by a coordinate transformation from two dimensional Euclidean space. Given this transformation we note, similar to Equation (17), the components of any vector \(V'^\mu\) transform in the same manner as the differential element \(dx'^\mu\) or

\[ V'^\mu = V^\mu - \sum_{\beta=1}^2 u_{\mu\beta}V^\beta. \]

We now start with the following tensor equation valid in the Euclidean frame

\[ \frac{\partial \gamma^\mu}{\partial x'^\nu} = \partial_\nu \gamma^\mu + \sum_{\beta=1}^2 \gamma^\beta \Gamma^\mu_{\beta\nu} = 0 \]
which implies
\[ \partial' \gamma'^\mu + \sum_{\beta=1}^{2} \gamma'^\beta \Gamma'^\mu_{\beta\nu} = 0 \]
or
\[ \partial'(-\gamma'^3 \gamma'^\mu) + \sum_{\beta=1}^{2} (-\gamma'^3 \gamma'^\beta) \Gamma'^\mu_{\beta\nu} = 0 \]
valid in the primed coordinate system. We now use \(-\gamma'^3 \gamma'^\mu = S \tilde{\gamma}'^\mu S^{-1}\) to obtain,
\[ -(\partial' S^{-1}) S \tilde{\gamma}'^\mu + \tilde{\gamma}'^\mu (\partial' S^{-1}) S + \left( \partial' \tilde{\gamma}'^\mu + \sum_{\beta=1}^{2} \tilde{\gamma}'^\beta \Gamma'^\mu_{\beta\nu} \right) = 0 \]

We now Fourier transform the fields \(u_{\mu\nu}\) and \((\partial_{\nu} S^{-1}) S\) and use the ansatz that only the lowest fourier mode is retained. This gives
\[ (\partial_{\nu} S^{-1}) S \to \left[ (\partial_{\nu} S^{-1}) S \right]_0 \]
\[ \tilde{\gamma}'^\mu \to \tilde{\gamma}'^\mu_0 \]
\[ g'^{\mu\nu} \to \delta'_{\mu\nu} - (u_{\mu\nu,0} + u_{\nu\mu,0}) + \sum_{\beta=1}^{2} u_{\beta\mu,0} u_{\beta\nu,0} \]

This demonstrates that Equation (48) is satisfied.

We next examine Maxwell’s equations and the Einstein Field equations by comparing our dimensional reduction to that used in Kaluza Klein theory.

VII. KALUZA KLEIN THEORY

In Kaluza Klein Theories one starts with an \(N\) dimensional system (usually 5 dimensional) and uses dimensional reduction to reduce the dimensionality of the system to \(N - 1\) dimensions. The starting point is the assumption that the Ricci tensor is identically zero in \(N\) dimensions where the Ricci tensor is defined as
\[ R_{\alpha\beta} = \frac{\partial \Gamma_{\alpha\beta}^\rho}{\partial x^\rho} - \frac{\partial \Gamma_{\alpha\rho}^\beta}{\partial x^\beta} + \Gamma_{\alpha\beta}^\sigma \Gamma_{\rho\sigma}^\rho - \Gamma_{\alpha\rho}^\sigma \Gamma_{\beta\sigma}^\rho \] (47)

and \(\Gamma_{\alpha\beta}^\rho\) are the Christoffel symbols.

The ansatz is then made that all field variables are independent of the \(N\)th coordinate effectively reducing the dimensionality of the problem to \(N - 1\) dimensions. The key outcome of this dimensional reduction is that the equation \(R_{\alpha\beta} = 0\) in \(N\) dimensions now becomes three sets of equations in \(N - 1\) dimensions. Two of these three equations has the form of Maxwell’s equations and the Einstein Field equations.
In the next section we will compare our dimensional reduction with the Kaluza Klein methods. In doing so we will continue to use the small strain approximation of Section III A. Our approach will be to keep the field equations to at least second order in the strain quantities. From Equation (17) this implies that we need to keep the Christoffel symbols to second order in \( u_{\mu \nu} \). Given the form of the Christoffel symbols

\[
\Gamma_{\alpha \beta}^{\rho} = g^{\lambda \rho} \left( \frac{\partial g_{\lambda \alpha}}{\partial x^\beta} + \frac{\partial g_{\lambda \beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha \beta}}{\partial x^\lambda} \right)
\]

we see that it is sufficient to keep the metric tensor \( g_{\alpha \beta} \) to second order in \( u_{\alpha \beta} \) and the inverse metric \( g^{\alpha \beta} \) to first order in \( u_{\alpha \beta} \).

**VIII. THE EINSTEIN FIELD AND MAXWELL’S EQUATIONS**

In preparation for a comparison with Kaluza Klein theory, we wish to invert the matrices defined in Equation (9) and Equation (13) keeping only terms that are to second order and lower in \( u_{\mu \nu} \).

In the following we will use the notation that 3 dimensional quantities are denoted with a carat (e.g. \( \hat{g}_{\mu \nu} \)) and two dimensional quantities do not have a carat. Our two dimensional inverse metric from Equation (13) is

\[
g^{\mu \nu} = \delta_{\mu \nu} + 2\epsilon_{\mu \nu} = \left( \begin{array}{cc} 1 + 2\epsilon_{11} & -2\epsilon_{12} \\ -2\epsilon_{12} & 1 + 2\epsilon_{22} \end{array} \right)
\]

where

\[
\epsilon_{\mu \nu} = \frac{1}{2} \left( -u_{\mu \nu} - u_{\nu \mu} + \sum_{\beta=1}^{2} u_{\beta \nu} u_{\beta \mu} \right)
\]

and \( \mu, \nu \) take values 1 or 2.

The inverse of Equation (18) is

\[
g_{\mu \nu} = \frac{1}{D} \left( \begin{array}{cc} 1 + 2\epsilon_{22} & -2\epsilon_{12} \\ -2\epsilon_{12} & 1 + 2\epsilon_{11} \end{array} \right)
\]

The determinant of the inverse metric \( g^{\mu \nu} \) is

\[
D = 1 + 2(\epsilon_{11} + \epsilon_{22}) + 4(\epsilon_{11}\epsilon_{22} - \epsilon_{12}^2)
\]
Using the expansion
\[
\frac{1}{1 + x} = 1 - x + x^2 + O(x^3)
\]
we have
\[
\frac{1}{D} = 1 - 2(\epsilon_{11} + \epsilon_{22}) + 4(\epsilon_{11}\epsilon_{22} + \epsilon_{12}^2 + \epsilon_{11}^2 + \epsilon_{22}^2) + O(\epsilon^3)
\]
This gives the following form for the metric tensor correct to second order in \(u_{\mu\nu}\)
\[
g_{\mu\nu} = \delta_{\mu\nu} - 2\epsilon_{\mu\nu} + 4 \sum_{\alpha=1}^{2} \epsilon_{\alpha\mu}\epsilon_{\alpha\nu} \tag*{(49)}
\]
In a similar manner the three dimensional metric in Equation (9) can be inverted to yield
\[
\hat{g}_{\mu\nu} = \delta_{\mu\nu} - 2\hat{\epsilon}_{\mu\nu} + 4 \sum_{\alpha=1}^{3} \hat{\epsilon}_{\alpha\mu}\hat{\epsilon}_{\alpha\nu} \tag*{(50)}
\]
with
\[
\hat{\epsilon}_{\mu\nu} = \frac{1}{2} \left( -u_{\mu\nu} - u_{\nu\mu} + \sum_{\beta=1}^{3} u_{\beta\nu} u_{\beta\mu} \right).
\]
and \(\mu, \nu\) taking on the values 1 to 3. The combination of Equations (9), (43), (49) and (50) allows us to write the three dimensional metric in terms of the two dimensional metric
\[
\hat{g}_{\alpha\beta} = \begin{pmatrix} g_{\alpha\beta} + 4\hat{\epsilon}_{3\alpha}\hat{\epsilon}_{3\beta} & -2\hat{\epsilon}_{\alpha3} + 4 \sum_{\nu=1}^{3} \hat{\epsilon}_{\nu\alpha}\hat{\epsilon}_{\nu3} \\ -2\hat{\epsilon}_{3\beta} + 4 \sum_{\nu=1}^{3} \hat{\epsilon}_{\nu3}\hat{\epsilon}_{\nu\beta} & 1 - 2\hat{\epsilon}_{33} + 4 \sum_{\nu=1}^{3} \hat{\epsilon}_{\nu3}^2 \end{pmatrix} + O(u^3)
\]
and
\[
\hat{g}^{\alpha\beta} = \begin{pmatrix} g^{\alpha\beta} + u_{3\alpha}u_{3\beta} & -u_{\alpha3} - u_{3\alpha} + \sum_{\nu=1}^{3} u_{\nu\alpha}u_{\nu3} \\ -u_{\beta3} - u_{3\beta} + \sum_{\nu=1}^{3} u_{\nu\beta}u_{\nu3} & 1 - 2u_{33} + \sum_{\nu=1}^{3} u_{\nu3}^2 \end{pmatrix} + O(u^3) \tag*{(51)}
\]
we can now compare this dimensional reduction to the usual Kaluza Klein result. The Kaluza Klein dimensional reduction from \(N\) dimensions to \(N - 1\) dimensions takes the form
\[
\hat{g}(kk)_{\alpha\beta} = \begin{pmatrix} g_{\alpha\beta} - \Phi^2 A_\alpha & -\Phi^2 A_\alpha \\ -\Phi^2 A_\beta & -\Phi^2 \end{pmatrix} \quad \hat{g}(kk)^{\alpha\beta} = \begin{pmatrix} g^{\alpha\beta} & -A^\alpha \\ -A^\beta & (-\Phi^2 + A^\mu A_\mu) \end{pmatrix} \tag*{(52)}
\]
where $\vec{A}$ is the electromagnetic vector potential and $\Phi$ is a scalar. Comparison of Equations (51) and (52) suggests that we make the following definitions

$$\begin{align*}
A^\alpha &= -\left(-u_{3\alpha} - u_{\alpha 3} + \sum_{\beta=1}^{3} u_{\beta \alpha} u_{\beta 3}\right) \\
\Phi^2 &= -\left(1 - 2\hat{e}_{33} + 4 \sum_{\beta=1}^{3} \hat{e}_{33}^{2}\right)
\end{align*}$$

which implies

$$A_\nu = \sum_{\mu=1}^{2} g_{\mu \nu} A^\mu$$

$$= \sum_{\mu=1}^{2} \left(\delta_{\mu \nu} - 2\epsilon_{\mu \nu} + 4 \sum_{\beta=1}^{2} \epsilon_{\beta \mu} \epsilon_{\beta \nu}\right) \left(u_{3\mu} + u_{\mu 3} - \sum_{\alpha=1}^{3} u_{\alpha 3} u_{\alpha \mu}\right)$$

$$= \sum_{\mu=1}^{2} \left(\delta_{\mu \nu} - 2\epsilon_{\mu \nu} + 4 \sum_{\beta=1}^{2} \epsilon_{\beta \mu} \epsilon_{\beta \nu}\right) (-2\epsilon_{3\mu} - u_{33} u_{3\mu})$$

$$= -2\epsilon_{3\nu} + 4 \sum_{\mu=1}^{2} \epsilon_{\mu \nu} \epsilon_{3\mu} - u_{33} u_{3\nu} + O(u^3)$$

and

$$-\left(\Phi^2\right)^{-1} = \frac{1}{1 + \left(-2\hat{e}_{33} + 4 \sum_{\beta=1}^{3} \hat{e}_{33}^{2}\right)}$$

$$= 1 + 2\hat{e}_{33} - 4 \sum_{\beta=1}^{2} \hat{e}_{33}^{2}$$

This now allows us to put Equation (51) into the same form as the Kaluza Klein decomposition,

$$\tilde{g}_{\alpha \beta} = \begin{pmatrix} g_{\alpha \beta} - \Phi^2 A_\alpha A_\beta & -\Phi^2 A_\alpha \\
-\Phi^2 A_\beta A_\alpha & -\Phi^2 \end{pmatrix} + O(u^3)$$

$$\tilde{g}^{\alpha \beta} = \begin{pmatrix} g^{\alpha \beta} & -A_\alpha \\
-A_\beta & -\Phi^{-2} + A^\mu A_\mu \end{pmatrix} + O(u^2).$$

The final form is obtained by taking the Fourier transform and keeping only the zero’th term of $u_{\mu \nu}$

$$A^\alpha \rightarrow \left(u_{3\alpha,0} + u_{\alpha 3,0} - \sum_{\beta=1}^{3} u_{\beta \alpha,0} u_{\beta 3,0}\right)$$
\[
g^{\mu\nu} \rightarrow \delta_{\mu\nu} - (u_{\mu\nu,0} + u_{\nu\mu,0}) + \sum_{\beta=1}^{2} u_{\beta\mu,0}u_{\beta\nu,0}
- \Phi^2 \rightarrow \left(1 - 2\hat{\epsilon}_{33,0} + 4\sum_{\beta=1}^{3} \hat{\epsilon}_{\beta33,0}^2\right)
\]

with
\[
\hat{\epsilon}_{\mu\nu,0} \equiv \frac{1}{2} \left( -u_{\mu\nu,0} - u_{\nu\mu,0} + \sum_{\beta=1}^{3} u_{\beta\nu,0}u_{\beta\mu,0} \right)
\]

The Christoffel symbols and the Ricci scalar can now be written in terms of the two-dimensional metric, the electromagnetic vector potential and the scalar field \(\phi\). The details of this reduction can be found in reference 19 and here we simply quote the results. Using Equation (57), the Christoffel symbols are
\[
\hat{\Gamma}_{\beta\gamma}^{\alpha} = \Gamma_{\beta\gamma}^{\alpha} + \frac{1}{2} \Phi^2 \left( A_{\beta\gamma} F_{\alpha}^{\alpha} + A_{\gamma} F_{\beta}^{\alpha} \right) + \Phi F_{\beta} A_{\gamma} A_{\alpha}
\]
\[
\hat{\Gamma}_{33}^{\alpha} = \Phi F_{\alpha}
\]
\[
\hat{\Gamma}_{33}^{\beta} = \frac{1}{2} \Phi^2 F_{\beta}^{\alpha} + \Phi F_{\alpha} A_{\beta}
\]
\[
\hat{\Gamma}_{33}^{3} = -\Phi F_{\gamma} A_{\gamma}
\]
\[
\hat{\Gamma}_{3\beta}^{3} = \frac{1}{2} \Phi^2 A_{\gamma} F_{\lambda\beta} - \Phi F_{\lambda} A_{\beta} + \Phi - 1 \Phi F_{\beta}
\]
\[
\hat{\Gamma}_{\alpha\beta}^{3} = \frac{1}{2} (A_{\alpha;\beta} + A_{\beta;\alpha}) - \frac{1}{2} \Phi^2 A_{\lambda} \left( A_{\alpha} F_{\lambda\beta} + A_{\beta} F_{\lambda\alpha} \right) - A_{\alpha} A_{\beta} \Phi F_{\gamma} A_{\lambda} + \Phi^{-1} (A_{\alpha} F_{\beta} + A_{\beta} F_{\alpha})
\]
(59)

and the components of the Ricci tensor are,
\[
\hat{R}_{44} = \frac{1}{4} \Phi^4 F_{\alpha\beta} F_{\alpha\beta} + \Phi F_{\alpha;\alpha}
\]
\[
\hat{R}_{4\alpha} = \frac{1}{2} \Phi^2 F_{\alpha;\lambda}^{\lambda} + \frac{3}{2} \Phi F_{\lambda\alpha} + A_{\alpha} \left( \frac{1}{4} \Phi^4 F_{\mu\nu} F_{\mu\nu} + \Phi F_{\alpha;\mu} \right)
\]
\[
\hat{R}_{\alpha\beta} = R_{\alpha\beta} + \frac{1}{2} \Phi^2 F_{\alpha\lambda} F_{\beta}^{\lambda} - \Phi^{-1} F_{\alpha;\beta} + \frac{1}{2} A_{\alpha} \left( \Phi^2 F_{\gamma;\gamma} + 3 \Phi F_{\gamma} F_{\gamma\beta} \right)
\]
\[
\quad + \frac{1}{2} A_{\beta} \left( \Phi^2 F_{\gamma;\gamma} + 3 \Phi F_{\gamma} F_{\gamma\alpha} \right) + A_{\alpha} A_{\beta} \left( \frac{1}{4} \Phi^4 F_{\mu\nu} F_{\mu\nu} + \Phi F_{\alpha;\mu} \right).
\]
(60)

where we have used the notation where a comma indicates an ordinary derivative and a semicolon indicates covariant differentiation.

Using the fact that \(\hat{R}_{\alpha\beta} = 0\) gives now three sets equations for the scalar \(\phi\), the electromagnetic field \(A_{\mu}\) and the 2D Ricci tensor \(R_{\alpha\beta}\). We again quote the results of this transformation 19.

\[
\Phi_{\alpha} = \frac{1}{4} \Phi^3 F_{\alpha\beta} F_{\alpha\beta}
\]
\[ F_{\alpha\lambda} = -3\Phi^{-1}\Phi^{\lambda}F_{\lambda\alpha} \]
\[ R_{\alpha\beta} = -\frac{1}{2}\Phi^2F_{\lambda\beta}^{\lambda} + \Phi^{-1}\Phi_{\alpha\beta} \]

These equations can be interpreted as a wave equation for the quantity $\Phi$, Maxwell’s equations for the fields $A_\mu$ and the Einstein field equations, the latter of which may be written

\[ G_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = 8\pi \left( T_{em}^{\alpha\beta} + T_s^{\alpha\beta} \right) \quad (61) \]

where the quantities $T_{em}^{\alpha\beta}$ and $T_s^{\alpha\beta}$ are effective energy momentum tensors given by

\[ T_{em}^{\alpha\beta} = -\frac{1}{2}\Phi^2 \left( F_{\lambda}^{\alpha} F_{\beta}^{\lambda} - \frac{1}{4} g_{\alpha\beta} F_{\mu\nu}^{\mu
u} F_{\mu\nu} \right) \]
\[ T_s^{\alpha\beta} = \Phi^{-1} \left( \pi_{\alpha\beta}^{\mu} - g^{\alpha\beta} \Phi_{\mu} \right) \]

These results indicate that the definition of the electromagnetic vector potential given in Equations (53) is consistent.

**IX. DIRAC’S EQUATION**

Given the definitions of the spin connection (40) and the electromagnetic field vector (53) we now pick up the derivation of Dirac’s equation from Equation (38) which reads

\[ \sum_{\mu=1}^{2} \gamma_{0}\left( \partial_{\mu} - i \frac{m_1}{2} u_{3\mu,0} + \left[ S^{-1}(\partial_{\mu} S) \right]_0 \right) \xi_{1/2} \]
\[ + \left( \frac{m_1}{2} + i \sum_{\nu=1}^{2} u_{\nu3,0} \partial_{\nu} - \frac{m_1}{2} u_{33,0} - i \left[ S^{-1}(\partial_{3} S) \right]_0 \right) \xi_{1/2} = 0 \quad (62) \]

Using Equation (18) the quantity $u_{3\mu}$ can be written

\[ u_{3\mu} = \epsilon_{3\mu} + \omega_{3\mu} + O(u^2) \]
\[ = \frac{1}{2} A_\mu + \omega_{3\mu} + O(u^2) \]

and the quantity $u_{\mu3}$ is

\[ u_{\mu3} = \epsilon_{3\mu} - \omega_{3\mu} + O(u^2) \]
\[ = \frac{1}{2} A_\mu - \omega_{3\mu} + O(u^2) \]
We also write the spinor field as
\[ \Psi = \tilde{\xi}_{1/2}. \]

Equation (62) now becomes
\[
\sum_{\mu=1}^{2} \tilde{\gamma}^\mu \left( \partial_\mu + i \frac{m_1}{4} A_\mu - \Gamma_\mu \right) \Psi \\
+ \left( \frac{m_1}{2} (1 - u_{33,0}) + \frac{i}{2} \sum_{\nu=1}^{2} \omega_{3\nu} \partial'_\nu - \frac{i}{2} \sum_{\nu=1}^{2} (A^\nu \partial'_\nu) + \left[ S^{-1}(\partial'_3 S) \right]_0 \right) \Psi = 0
\]

A. Mass Term

The term \( \frac{m_1}{2} (1 - u_{33,0}) \) can be written in terms of the scalar \( \Phi^2 \) as
\[
\frac{m_1}{2} (1 - u_{33,0}) = \frac{m_1}{4} (1 - \Phi^2) + O(u^3)
\]
This scalar term can be interpreted as a mass term in Equation (63) for a particle with mass \( m \), where \( m \) is
\[
m = \frac{m_1}{4} (1 - \Phi^2)
\]
Equation (63) now becomes
\[
\sum_{\mu=1}^{2} \tilde{\gamma}^\mu \left( \partial_\mu + i \frac{m_1}{4} A_\mu - \Gamma_\mu \right) \Psi + m \Psi \\
+ \left( \frac{1}{2} \sum_{\nu=1}^{2} \omega_{3\nu} \partial'_\nu - \frac{1}{2} \sum_{\nu=1}^{2} (A^\nu \partial'_\nu) + \left[ S^{-1}(\partial'_3 S) \right]_0 \right) \Psi = 0
\]
With the exception of the last three terms, this equation has the same form as Dirac’s Equation (1). This is the main result of this work which shows that a Dirac-like equation is present in this system.

B. Additional Interaction Terms

The last three terms in Equation (64) can be interpreted as interaction terms between the spinor field \( \Psi \) and the vector fields \( A^\nu \), and \( \omega_{3\nu} \), and the matrix field \( [S^{-1}(\partial'_3 S)]_0 \). These terms do not normally appear in Dirac’s equation but the existence of these types of terms is not forbidden in modern quantum field theories.
In addition to the equation given in (64), each of these fields satisfies additional constraint equations that determines its dynamics. The electromagnetic field satisfies Maxwell’s Equation and has been discussed in section VIII. The interaction with the matrix $S^{-1}(\partial'_3 S)\partial_0$ results from the spin connection in 3 dimensions and its solution is determined from Equation (34). An equation of motion for the field $\omega_{3\mu}$ can be derived from Equation (13). If we take the divergence of the vector $V^\rho$ from this equation we have

$$\nabla^2 u^\rho = -(2\lambda + 2\mu) \frac{\partial \sigma}{\partial x^\rho} + \frac{\partial \alpha}{\partial x^\rho} + (\nabla \times \chi)^\rho$$

which implies

$$\nabla^2 \frac{\partial u^\rho}{\partial x^\nu} = -(2\lambda + 2\mu) \frac{\partial^2 \sigma}{\partial x^\nu \partial x^\rho} + \frac{\partial^2 \alpha}{\partial x^\nu \partial x^\rho} + \frac{\partial}{\partial x^\nu} (\nabla \times \chi)^\rho$$

We can now write the equation of motion of the field $\omega_{\rho\nu}$ as

$$\nabla^2 \omega_{\rho\nu} \equiv \left( \frac{\partial u^\rho}{\partial x^\nu} - \frac{\partial u^\nu}{\partial x^\rho} \right) = \frac{\partial}{\partial x^\nu} (\nabla \times \chi)^\rho - \frac{\partial}{\partial x^\rho} (\nabla \times \chi)^\nu.$$  

We see that the scalar $\phi$ (and hence the spinor field $\xi$) is completely determined by the scalars $\alpha$ and $\sigma$ in Equation (13) and the field $\omega_{\rho\nu}$ is completely determined by the vector field $\chi$.

We will not attempt to identify these three additional terms with known interactions, simply noting again that interactions of this type are not forbidden to exist and that the central result of this work is Equation (64) which shows that there is a Dirac-like equation that exists in a classical elastic solid.

X. HIGHER DIMENSIONS

Throughout our derivation we assumed that we were working in three dimensional space. This was convenient since explicit solutions to Laplace’s equation could be obtained in terms of fractional derivatives. The formalism however extends to any number of dimensions. The main difference is that there is no explicit solution of the spinors in terms of fractional derivatives. Nevertheless it should be noted that the basic equations hold in higher dimensions and therefore the usual four dimensional Dirac equations could in principle be derived by starting with an elastic solid in 5 dimensions.
XI. CONCLUSIONS

We have taken a model of an elastic medium and derived an equation of motion that has the same form as Dirac’s equation in the presence of electromagnetism and gravity. We derived our equation by using the formalism of Cartan to reduce the quadratic form of Laplace’s equation to the linear form of Dirac’s equation. We further assumed that one coordinate was compact and upon Fourier transforming this coordinate we obtained a mass term and an electromagnetic interaction term in the equations of motion. The formalism demonstrates that the classical Einstein-Dirac-Maxwell equations can be derived as the Equations of motion of the Fourier modes of an elastic solid in the small strain approximation.

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