Nilpotent Graph in Ring $Z_n$

F Tahya, E R Persulessy*
Mathematics Department, Faculty of Mathematics and Natural Sciences University of Pattimura, Jalan Ir. M. Putuhena, Ambon

* E-mail: er.persulessy@gmail.com

Abstract. The nilpotent graph is a graph's structure motivated by the characteristics of some elements of a ring. If $G$ is a graph, $G(N)$ is a nilpotent graph in which the set of all non-nilpotent elements of a ring taken as a vertex set. For each $x, y$ in $R\backslash nil(R)$, $x$ and $y$ are said to be connected or adjacent to an edge, if the sum of the vertices $x$ and $y$ is equal to the nilpotent element in the ring. This research aims to explain how to construct a nilpotent graph $G(N)$ from a commutative ring. The results of this research are the characteristic of the nilpotent graph of a ring $Z_n$ for $n$ prime numbers, a nilpotent graph for the ring $Z_{2^k}$, and ring $Z_n$ with $n = 6m$ where $1 \leq m < 5$ can create 3 nilpotent subgraphs, they are a complete subgraph $(k_m)$ and 2 regular subgraphs in which the degree of vertices is equal to $m$.

1. Introduction
Mathematics is essential to be studied. Algebra is a branch of mathematics that studies the structure, relationships and quantitative. Algebra usually uses symbols (e.g. letters) to represent numbers in general as a means of simplification.

The ring is a concept known in Algebra, which to form a ring a set must not an empty set and equipped with two operations, addition and multiplication. The elements in the set with that both operations must satisfy the nine axioms which divided into three major parts, abelian group axioms for an addition operation, semigroup axioms for a multiplication operation and multiplication over addition operation must satisfy the left and right distributive properties [1]. There exist a special of the element in the ring. The element is called the nilpotent element which an element is said to be a nilpotent element. If the element multiplied in $n$ times, then the result is a neutral element in the ring [2].

The graph is a concept in discrete mathematics that is remarkably developed at this time. The graph is a set consist of vertices and edges in which the vertex cannot be empty [3]. Each edge has either one or two vertices associated with it, called its endpoints. An edge is said to connect its endpoints. A graph with an infinite vertex set is called an infinite graph, whereas a graph with a finite vertex set is called a finite graph. Based on the presence or absence of double sides, graphs can be divided into simple graphs and non-graphs simple, while based on the presence or absence of direction orientation on the side, graphs can be divided into directed and undirected graphs. There are several individual graphs namely complete graphs, circle graphs, regular graphs, and bipartite graphs[3-4].

Algebra and graphs are often understood as two different concepts because algebraic is an abstract structure but the graph is an application structure. Florentin can
successfully construct a graph in which its vertices are taken from elements in the group [9]. Another research from Basnet et al. can examine the elements which are not nilpotent can be formed nilpotent graphs [10].

This research focus to examine the nilpotent graph which the vertices on the graph are taken from elements that are not nilpotent in a ring $\mathbb{Z}_n$.

2. Research Method

2.1 Research Type
The type of this research is a literature review that reviews articles, books, etc. Therefore, the materials needed are books (including e-books), journals, etc.

2.2 Research Procedures
In general, the stages in this study can be described as in the following figure.

![Diagram of research procedures]

- Identify the structure of ring and graph
- Determine the ring
- Determine the nilpotent elements and non-nilpotent elements of the ring
- Construct the nilpotent graph from the ring
- Conclude the result of the research
3. Results and Discussion

Definition:
Nilpotent graph of ring $R$ denoted by $G(R)$ is defined by vertex set on $R \setminus \text{Nil}(R)$, and two distinct vertices $x$ and $y$ are adjacent if sum $x$ and $y$ are nilpotent element on $R$ [10].

Example 1:
Let $(\bar{Z}, +, \times)$ is a ring which $\bar{Z} = \{0, 1, 2, 3, 4, 5, 6, 7\}$. The following table sum operation on a ring $(\bar{Z}, +, \times)$.

|   | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 0 |
| 2 | 2 | 3 | 4 | 5 | 6 | 7 | 0 | 1 |
| 3 | 3 | 4 | 5 | 6 | 7 | 0 | 1 | 2 |
| 4 | 4 | 5 | 6 | 7 | 0 | 1 | 2 | 3 |
| 5 | 5 | 6 | 7 | 0 | 1 | 2 | 3 | 4 |
| 6 | 6 | 7 | 0 | 1 | 2 | 3 | 4 | 5 |
| 7 | 7 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |

Because $\bar{2^3} = \bar{0}$, $\bar{4^3} = \bar{0}$, $\bar{6^3} = \bar{0}$, so $\text{nil}(\bar{Z}) = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$.

Based on the definition, we obtain a nilpotent graph for $\bar{Z}$ as follows:

![Nilpotent graph in ring $\bar{Z}$](image)

Example 2:
Let $(\bar{Z}, +, \times)$ is the ring which $\bar{Z} = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$. The following table sum operation on the ring $(\bar{Z}, +, \times)$.
Because $3^2 = 0$, so $nil(Z_9) = \{0, 3, 6\}$.

Based on the definition, we obtain a nilpotent graph for $Z_9$ as follows:

![Nilpotent Graph](image)

The nilpotent graph in ring $Z_9$

**Lemma:**
Let $N(G(R))$ be the nilpotent graph of a ring $R$. If $x$ is any vertices on this graph, we have the following:
If $2x \in nil(R)$, then $\text{deg}(x) = |\text{nil}(R)| - 1$
If $2x \notin \text{nil}(R)$, then $\text{deg}(x) = |\text{nil}(R)|$

**Corollary:**
For any graph $N(G(Z_n))$ with $n = 2^k$ where $k \geq 1$ is a complete graph.
Theorem 1:
For any graph \( Z_n \) with \( n \) prime can be formed \( \left\lfloor \frac{n}{2} \right\rfloor \) complete subgraph \( k_2 \).

Proof:
We know that \( Z_n = \{0, 1, \ldots, (n-1)\} \) with \( \text{nil}(Z_n) = \{0\} \). Let \( x \in Z_n \). Based on the previous lemma, we have \( 2x \notin \text{nil}(Z_n) \), then \( \text{deg}(x) = |\text{nil}(Z_n)| = 1 \). So, we can have \( \left\lfloor \frac{n}{2} \right\rfloor \) complete subgraph \( k_2 \).

Theorem 2:
For any graph \( (Z_n, +, \times) \) with \( n = 6m \) where \( 1 \leq m < 5 \) can be formed nilpotent graph which consists of a complete subgraph \( (k_m) \) and two regular subgraphs with the degree of the vertices equals \( m \).

4. Conclusions
Based on the results and discussion that had been stated in the previous chapter, the following research conclusions are: for any finite commutative ring can be formed nilpotent graph. The nilpotent graph on the ring \( Z_n \) have some special characteristics such as for \( n \) prime can be formed \( \left\lfloor \frac{n}{2} \right\rfloor \) complete subgraph, for \( n = 2^k \) can be formed complete subgraph, and if \( Z_{6m} \) is a ring with \( 1 \leq m < 5 \), it can form a complete subgraph \( (k_m) \) and two regular subgraphs with the degree of the vertices equals \( m \).

References
[1] Robinson D 2003 \textit{An introduction to abstract algebra} Walter de Gruyter GmbH&Co. Berlin.
[2] Rotman J J 2005 \textit{First course in abstract algebra (Third Edition)} Prentice hall, Upper Saddle River. New Jersey.
[3] Rossen K H 2012 \textit{Discrete mathematics and its applications (Seventh Edition) Book} The McGraw-Hill Companies, Inc. New York.
[4] Diestel R 2000 \textit{Graph theory (Electronic Edition)} Springer-Verlag. New York
[5] Ashrafi N, Maimani H R, Pournaki M R and Yassemi S 2010 \textit{Unit graphs associated with rings} Taylor & Francis Group, LLC.
[6] Asir T and Chelvam T T 2013 \textit{On the total graph and its complement of a commutative ring} Taylor & Francis Group, LLC.
[7] Sharma A, Bhattacharyya J and Basnet D K 2017 Weakly nil clean graphs of rings \textit{Algebra Colloquium} \textbf{24} doi:10.1142/S1005386717000311.
[8] Basnet D K and Bhattacharyya J 2017 Nil clean graph of rings \textit{Algebra Colloquium} \textbf{24} 481-492
[9] Kandasamy W B V and Smarandache F 2009 \textit{Group as graphs} Era CdituuArt
[10] Basnet D K, Sharma A and Dutta R 2018 \textit{Nilpotent graph} https://arxiv.org/abs/1804.08937