A new type of nuclear collective motion – the spin scissors mode

E.B. Balbutsev, I.V. Molodtsova

Joint Institute for Nuclear Research, 141980 Dubna, Moscow Region, Russia

P. Schuck

Institut de Physique Nucléaire, IN2P3-CNRS,
Université Paris-Sud, F-91406 Orsay Cédex, France;
Laboratoire de Physique et Modélisation des Milieux Condensés, CNRS and Université Joseph Fourier,
25 avenue des Martyrs BP166, F-38042 Grenoble Cédex 9, France

The coupled dynamics of low lying modes and various giant resonances are studied with the help of the Wigner Function Moments method on the basis of Time Dependent Hartree-Fock equations in the harmonic oscillator model including spin-orbit potential plus quadrupole-quadrupole and spin-spin residual interactions. New low lying spin dependent modes are analyzed. Special attention is paid to the spin scissors mode.

PACS numbers: 21.10.Hw, 21.60.Ev, 21.60.Jz, 24.30.Cz
Keywords: spin; collective motion; scissors mode; giant resonances

I. INTRODUCTION

The idea of the possible existence of the collective motion in deformed nuclei similar to the scissors motion continues to attract the attention of physicists who extend it to various kinds of objects, not necessary nuclei, (for example, magnetic traps, see the review by Heyde at al [1]) and invent new sorts of scissors, for example, the rotational oscillations of neutron skin against a proton-neutron core [2].

The nuclear scissors mode was predicted [3]–[6] as a counter-rotation of protons against neutrons in deformed nuclei. However, its collectivity turned out to be small. From RPA results which were in qualitative agreement with experiment, it was even questioned whether this mode is collective at all [7, 8].

Purely phenomenological models (such as, e.g., the two rotors model [3]) and the sum rule approach [10] did not clear up the situation in this respect. Finally in a very recent review [1] it is concluded that the scissors mode is "weakly collective, but strong on the single-particle scale" and further: "The weakly collective scissors mode excitation has become an ideal test of models – especially microscopic models – of nuclear vibrations. Most models are usually calibrated to reproduce properties of strongly collective excitations (e.g. of \(J^\pi = 2^+\) or \(3^-\) states, giant resonances, ...). Weakly-collective phenomena, however, force the models to make genuine predictions and the fact that the transitions in question are strong on the single-particle scale makes it impossible to dismiss failures as a mere detail, especially in the light of the overwhelming experimental evidence for them in many nuclei [11, 12]."

The Wigner Function Moments (WFM) or phase space moments method turns out to be very useful in this situation. On the one hand it is a purely microscopic method, because it is based on the Time Dependent Hartree-Fock (TDHF) equation. On the other hand the method works with average values (moments) of operators which have a direct relation to the considered phenomenon and, thus, make a natural bridge with the macroscopic description. This makes it an ideal instrument to describe the basic characteristics (energies and excitation probabilities) of collective excitations such as, in particular, the scissors mode. Our investigations have shown that already the minimal set of collective variables, i.e. phase space moments up to quadratic order, is sufficient to reproduce the most important property of the scissors mode: its inevitable coexistence with the IsoVector Giant Quadrupole Resonance (IVGQR) implying a deformation of the Fermi surface.

Further developments of the Wigner Function Moments method, namely, the switch from TDHF to Time Dependent Hartree-Fock Bogoliubov (TDHFB) equations, i.e. taking into account pair correlations, allowed us to improve considerably the quantitative description of the scissors mode [13, 14]: for rare earth nuclei the energies are reproduced with \(\sim 10\%\) accuracy and \(B(M1)\) values were reduced by about a factor of two with respect to their non superfluid values. However, they remain about two times too high with respect to experiment. We have suspected, that the reason of this last discrepancy is hidden in the spin degrees of freedom, which were so far ignored by the WFM method. One cannot exclude, that due to spin dependent interactions some part of the force of \(M1\) transitions is shifted to the energy region of 5–10 MeV, where a \(1^+\) resonance of spin nature is observed [2].

In a recent paper [15] the WFM method was applied for the first time to solve the TDHF equations including spin dynamics. As a first step, only the spin-orbit interaction was included in the consider-
ation, as the most important one among all possible spin dependent interactions because it enters into the mean field. This allows one to understand the structure of necessary modifications of the method avoiding cumbersome calculations. The most remarkable result was the discovery of a new type of nuclear collective motion: rotational oscillations of "spin-up" nucleons with respect of "spin-down" nucleons (the spin scissors mode). It turns out that the experimentally observed group of peaks in the energy interval 2-4 MeV corresponds very likely to two different types of motion: the conventional (orbital) scissors mode and this new kind of mode, i.e. the spin scissors mode.

Three low lying excitations of a new nature were found: isovector and isoscalar spin scissors and the excitation generated by the relative motion of the orbital angular momentum and the spin of the nucleus (they can change their absolute values and directions keeping the total spin unchanged). In the frame of the same approach ten high lying excitations were also obtained: well known isoscalar and isovector Giant Quadrupole Resonances (GQR), two resonances of a new nature describing isoscalar and isovector quadrupole vibrations of "spin-up" nucleons with respect of "spin-down" nucleons, and six resonances which can be interpreted as spin flip modes of various kinds and multipolarity.

The obtained results are very interesting, however, they are only intermediate in our investigation of M1 modes. Our finite goal is to get reasonable agreement with experimental data for the conventional scissors mode, especially for its B(M1) factors which remain about two times too strong. We should keep in mind that only the standard spin-orbit potential was taken into account in the paper [15], spin dependent residual interactions being completely neglected.

The aim of this work is to get a qualitative understanding of the influence of the spin-spin force on the new states analyzed in [15], as, for instance, the spin scissors mode. As a matter of fact we will find that the spin-spin interaction does not change the general picture of the positions of excitations described in [15]. It pushes all levels up proportionally to its strength without changing their order. The most interesting result concerns the B(M1) values of both scissors modes – the spin-spin interaction strongly redistributes M1 strength in the favour of the spin scissors mode. This is a very promising fact, because it shows that after taking into account in addition pairing [16] one may achieve agreement with experiment.

One of the main points of the present work will, indeed, be that we will be able to give a tentative explanation of a recent experimental finding [17] where the B(M1) values in $^{235}$Th of the two low lying magnetic states are inverted in strength in favor of the lowest, i.e., the spin scissors mode, when cranking up the spin-spin interaction. Indeed, the explanation with respect to a triaxial deformation given in [17] yields a stronger B(M1) value for the higher lying state, contrary to observation, as remarked by the authors themselves.

The paper is organized as follows. In Sec. 2 the TDHF equations for the 2x2 density matrix are formulated and their Wigner transform is found. In Sec. 3 the model Hamiltonian is analyzed and the mean field generated by the spin-spin interaction is found. In Sec. 4 the collective variables are defined and the respective dynamical equations are derived. In Sec. 5 the results of our calculations of energies, B(M1) and B(E2) values are discussed. Lastly, remarks and the outlook are given in the conclusion section. The mathematical details are concentrated in appendices A, B.

### II. WIGNER TRANSFORMATION OF TDHF EQUATION WITH SPIN

The TDHF equation in operator form reads [10]

$$i\hbar \dot{\rho} = [\hat{h}, \rho].$$

(1)

Let us consider its matrix form in coordinate space keeping all spin indices:

$$i\hbar \langle \mathbf{r}, s|\hat{\rho}|\mathbf{r}''', s''''\rangle = \sum_{s'} \int d^3 r' \left( \langle \mathbf{r}, s|\hat{h}|\mathbf{r}', s'>\langle \mathbf{r}', s'|\hat{\rho}|\mathbf{r}'', s''\rangle + \langle \mathbf{r}, s'|\hat{\rho}|\mathbf{r}', s'>\langle \mathbf{r}', s'|\hat{h}|\mathbf{r}'', s''\rangle \right).$$

(2)

We do not specify the isospin indices in order to make the formulae more transparent. They will be re-introduced at the end.

These equations will be solved by the method of phase space (or Wigner function) moments. To this end we will rewrite the expression [2] with the help of the Wigner transformation [16]. To make the formulae more readable we will not write out the coordinate dependence $(\mathbf{r}, \mathbf{p})$ of the functions. With the
conventional notation

\[ \uparrow \text{ for } s = \frac{1}{2} \text{ and } \downarrow \text{ for } s = -\frac{1}{2} \]

the Wigner transform of (2) can be written as

\[ i\hbar \dot{f}^{\uparrow\uparrow} = i\hbar (h^{\uparrow\uparrow}, f^{\uparrow\uparrow}) + \hbar^{\uparrow\uparrow} f^{\uparrow\uparrow} \fastop h^{\uparrow\uparrow} f^{\uparrow\uparrow} + \frac{i\hbar}{2} (h^{\uparrow\downarrow}, f^{\uparrow\downarrow}) - \frac{i\hbar}{2} (f^{\uparrow\downarrow}, h^{\uparrow\downarrow}) - \frac{\hbar^2}{8} \{(h^{\uparrow\downarrow}, f^{\uparrow\downarrow})\} + \frac{\hbar^2}{8} \{(f^{\uparrow\downarrow}, h^{\uparrow\downarrow})\} + ... , \]

\[ i\hbar \dot{f}^{\uparrow\downarrow} = f^{\uparrow\downarrow} (h^{\uparrow\downarrow} - h^{\downarrow\uparrow}) + \frac{i\hbar}{2} ((h^{\uparrow\downarrow} + h^{\downarrow\uparrow}), f^{\uparrow\downarrow}) - \frac{\hbar^2}{8} \{(h^{\uparrow\downarrow} - h^{\downarrow\uparrow}), f^{\uparrow\downarrow}\} \]

\[ -h^{\downarrow\uparrow} (f^{\uparrow\downarrow} - f^{\downarrow\uparrow}) + \frac{i\hbar}{2} (h^{\downarrow\uparrow}, (f^{\downarrow\uparrow} + f^{\uparrow\downarrow})) + \frac{\hbar^2}{8} \{(h^{\downarrow\uparrow}, (f^{\downarrow\uparrow} - f^{\uparrow\downarrow})\} + .... \tag{3} \]

where the functions \( h, f \) are the Wigner transforms of \( \hat{h}, \hat{f} \) respectively, \( \{f, g\} \) is the Poisson bracket of the functions \( f \) and \( g \) and \( \{\{f, g\}\} \) is their double Poisson bracket; the dots stand for functions proportional to higher powers of \( h \). The remaining two equations are obtained by the obvious change of arrows \( \uparrow \leftrightarrow \downarrow \).

It is useful to rewrite the above equations in terms of functions \( f^+ = f^{\uparrow\uparrow} + f^{\downarrow\uparrow} \), \( f^- = f^{\uparrow\downarrow} - f^{\down\uparrow} \). By analogy with isoscalar \( f^0 + f^p \) and isovector \( f^0 - f^p \) functions one can name the functions \( f^+ \) and \( f^- \) as spin-scalar and spin-vector ones, respectively. We have:

\[ i\hbar \dot{f}^+ = \frac{i\hbar}{2} (h^+, f^+) + \frac{i\hbar}{2} (h^-, f^-) + i\hbar (h^{\uparrow\downarrow}, f^{\uparrow\downarrow}) + i\hbar (h^{\downarrow\uparrow}, f^{\downarrow\uparrow}) + ... , \]

\[ i\hbar \dot{f}^- = \frac{i\hbar}{2} (h^+, f^-) + \frac{i\hbar}{2} (h^-, f^+) - 2h^+ f^{\uparrow\downarrow} + 2h^- f^{\downarrow\uparrow} + \frac{\hbar^2}{4} \{(h^{\uparrow\downarrow}, f^{\uparrow\downarrow})\} - \frac{\hbar^2}{4} \{(h^{\uparrow\downarrow}, f^{\uparrow\downarrow})\} + ... , \]

\[ i\hbar \dot{f}^{\uparrow\downarrow} = -h^{\downarrow\uparrow} f^- - h^+ f^{\uparrow\downarrow} + \frac{i\hbar}{2} (h^{\uparrow\downarrow}, f^{\uparrow\downarrow}) + \frac{i\hbar}{2} (h^{\uparrow\downarrow}, f^{\uparrow\downarrow}) + \frac{\hbar^2}{8} \{(h^{\uparrow\downarrow}, f^{\downarrow\uparrow})\} + ... , \]

\[ i\hbar \dot{f}^{\downarrow\uparrow} = h^+ f^- - h^+ f^{\downarrow\uparrow} + \frac{i\hbar}{2} (h^{\uparrow\downarrow}, f^{\uparrow\downarrow}) + \frac{i\hbar}{2} (h^{\downarrow\uparrow}, f^{\downarrow\uparrow}) - \frac{\hbar^2}{8} \{(h^{\uparrow\downarrow}, f^{\uparrow\downarrow})\} + \frac{\hbar^2}{8} \{(h^{\downarrow\uparrow}, f^{\downarrow\uparrow})\} + ... , \tag{4} \]

where \( h^\pm = h^{\uparrow\uparrow} \pm h^{\downarrow\uparrow} \).

### III. MODEL HAMILTONIAN

The microscopic Hamiltonian of the model, harmonic oscillator with spin-orbit potential plus separable quadrupole-quadrupole and spin-spin residual interactions is given by

\[ H = \sum_{i=1}^{A} \left[ \frac{\hat{p}_i^2}{2m} + \frac{1}{2} m \omega_i^2 r_i^2 - \eta_i \hat{S}_i \right] + H_{qq} + H_{ss} \tag{5} \]

with

\[ H_{qq} = \sum_{\mu=-2}^{2} (-1)^\mu \left\{ \kappa \sum_{i=1}^{Z} \sum_{j=1}^{N} q_{2-\mu}(r_i)q_{2\mu}(r_j) + \frac{1}{2} \kappa \sum_{i \neq j}^{Z} q_{2-\mu}(r_i)q_{2\mu}(r_j) + \frac{1}{2} \kappa \sum_{i \neq j}^{Z} q_{2-\mu}(r_i)q_{2\mu}(r_j) \right\} , \tag{6} \]

\[ H_{ss} = \sum_{\mu=-1}^{1} (-1)^\mu \left\{ \chi \sum_{i=1}^{Z} \sum_{j=1}^{N} \hat{S}_{-\mu}(i)\hat{S}_{\mu}(j) + \frac{1}{2} \chi \sum_{i \neq j}^{Z} \hat{S}_{-\mu}(i)\hat{S}_{\mu}(j) + \frac{1}{2} \chi \sum_{i \neq j}^{Z} \hat{S}_{-\mu}(i)\hat{S}_{\mu}(j) \right\} \delta(r_i - r_j) , \tag{7} \]

where \( N \) and \( Z \) are the numbers of neutrons and protons and \( \hat{S}_\mu \) are spin matrices \[18]\:

\[ \hat{S}_1 = -\frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \hat{S}_0 = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \hat{S}_{-1} = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} . \tag{8} \]
The quadrupole operator \( q_{2\mu} = \sqrt{16\pi/5}r^2Y_{2\mu}(\theta, \phi) \) can be written as the tensor product: \( q_{2\mu}(r) = \sqrt{6}\{r \otimes r\}_{2\mu} \), where

\[
\{r \otimes r\}_{\lambda \mu} = \sum_{\sigma, \nu} C_{1,1,1,1}^{\lambda \mu} r_\sigma r_\nu,
\]

\( r_{-1}, r_0, r_1 \) are cyclic coordinates \(^{13}\) and \( C_{1,1,1,1}^{\lambda \mu} \) is a Clebsch-Gordan coefficient.

### A. Mean Field

Let us analyze the mean field generated by this Hamiltonian.

#### 1. Spin-orbit Potential

Written in cyclic coordinates, the spin-orbit part of the Hamiltonian reads

\[
\hat{h}_{ls} = -\eta \sum_{\mu=-1}^{1} (-)^\mu \hat{l}_{\mu} \hat{S}_{-\mu} = -\eta \left( \hat{l}_{\frac{1}{2}} - \frac{1}{\sqrt{2}} \hat{l}_{-1} - \frac{1}{\sqrt{2}} \hat{l}_{1} \right),
\]

where \(^{18}\)

\[
\hat{l}_{\mu} = -\hbar \sqrt{2} \sum_{v,\alpha} C_{1v,1\alpha}^{l\mu} r_\nu \nabla_\alpha
\]

and

\[
\hat{l}_{1} = \hbar (r_{0} \nabla_{1} - r_{1} \nabla_{0}) = -\frac{1}{\sqrt{2}} (\hat{l}_{x} + i\hat{l}_{y}), \quad \hat{l}_{0} = \hbar (r_{-1} \nabla_{1} - r_{1} \nabla_{-1}) = \hat{l}_{z},
\]

\[
\hat{l}_{-1} = \hbar (r_{-1} \nabla_{0} - r_{0} \nabla_{-1}) = \frac{1}{\sqrt{2}} (\hat{l}_{x} - i\hat{l}_{y}),
\]

\[
\hat{l}_{x} = -i\hbar (y \nabla_{z} - z \nabla_{y}), \quad \hat{l}_{y} = -i\hbar (z \nabla_{x} - x \nabla_{z}), \quad \hat{l}_{z} = -i\hbar (x \nabla_{y} - y \nabla_{x}).
\]

(10)

Matrix elements of \( \hat{h}_{ls} \) in coordinate space can be obviously written as

\[
< r_{1}, s_{1} | \hat{h}_{ls} | r_{2}, s_{2} >= -\frac{\hbar}{2} \eta(r_{1}) \{ \hat{l}_{0}(r_{1}) | \delta_{s_{1} \uparrow} \delta_{s_{2} \uparrow} - \delta_{s_{1} \downarrow} \delta_{s_{2} \downarrow} | \} + \sqrt{2} \hat{l}_{-1}(r_{1}) \delta_{s_{1} \uparrow} \delta_{s_{2} \downarrow} - \sqrt{2} \hat{l}_{1}(r_{1}) \delta_{s_{1} \downarrow} \delta_{s_{2} \uparrow} | \delta_{r_{1} - r_{2}}.
\]

(11)

The Wigner transform of \(^{11}\) reads \(^{15}\):

\[
\hat{h}_{ls}^{s_{1}s_{2}}(r, p) = -\frac{\hbar}{2} \eta(r_{1}) \{ \hat{l}_{0}(r, p) | \delta_{s_{1} \uparrow} \delta_{s_{2} \uparrow} - \delta_{s_{1} \downarrow} \delta_{s_{2} \downarrow} | \} + \sqrt{2} \hat{l}_{-1}(r, p) \delta_{s_{1} \uparrow} \delta_{s_{2} \downarrow} - \sqrt{2} \hat{l}_{1}(r, p) \delta_{s_{1} \downarrow} \delta_{s_{2} \uparrow},
\]

(12)

where \( l_{\mu} = -i\sqrt{2} \sum_{v,\alpha} C_{1v,1\alpha}^{l\mu} r_\nu p_\alpha. \)

#### 2. q-q interaction

The contribution of \( H_{qq} \) to the mean field potential is easily found by replacing one of the \( q_{2\mu} \) operators by the average value. We have

\[
V_{qq} = 6 \sum_{\mu} (-1)^\mu Z_{2-\mu}^{+} \{ r \otimes r \}_{2\mu}.
\]

(13)
Here

\[ Z_{2\mu}^{n+} = \kappa R_{2\mu}^{n+} + \bar{\kappa} R_{2\mu}^{n+}, \quad Z_{2\mu}^{p+} = \kappa R_{2\mu}^{p+} + \bar{\kappa} R_{2\mu}^{p+}, \quad R_{\lambda\mu}^{n+}(t) = \int d(p, r)\{r \otimes r\}_{\lambda\mu} f^{n+}(r, p, t) \] (14)

with \( f(d(p, r)) \equiv (2\pi\hbar)^{-3} \int d^3p \int d^3r \) and \( \tau \) being the isospin index.

3. Spin-spin interaction

The analogous expression for \( H_{ss} \) is found in the standard way, with the Hartree-Fock contribution given \( \text{16} \) by:

\[ \Gamma_{kk'}(t) = \sum_{ll'} \bar{\nu}_{kk'l'\lambda} \rho_{ll'}(t), \] (15)

where \( \bar{\nu}_{kk'l'\lambda} \) is the antisymmetrized matrix element of the two body interaction \( v(1,2) \). Identifying the indices \( k, k', l, l' \) with the set of coordinates \( (r, s, \tau) \), i.e. (position, spin, isospin), one rewrites \( \text{15} \) as

\[ V^{HF}(r_1, s_1, \tau_1; r'_1, s'_1, \tau'_1; t) = \int d^3r_2 \int d^3r'_2 \sum_{s_2, s'_2} \sum_{\tau_2, \tau'_2} \langle r_1, s_1, \tau_1; r_2, s_2, \tau_2|\hat{v}|r'_1, s'_1, \tau'_1; r'_2, s'_2, \tau'_2 \rangle > a.s. \rho(r'_2, s'_2, \tau'_2; r_2, s_2, \tau_2; t). \]

Let us consider the neutron-proton part of the spin-spin interaction. In this case

\[ \hat{v} = v(\hat{r}_1 - \hat{r}_2) \sum_{\mu=-1}^1 (-1)^\mu \hat{S}_\mu(1)\hat{S}_\mu(2)\delta_{\tau_1 p}\delta_{\tau_2 n}, \]

where \( \hat{r}_1 \) is the position operator: \( \hat{r}_1|r_1 > = r_1|r_1 >, \quad < r_1|\hat{r}_1|r'_1 > = < r_1|r'_1 > = \delta(r_1 - r'_1)r'_1. \)

For the Hartree term one finds:

\[ \sum_{\mu=-1}^1 (-1)^\mu < s_1, \tau_1; s_2, \tau_2|\hat{S}_\mu(1)\hat{S}_\mu(2)\delta_{\tau_1 p}\delta_{\tau_2 n}|s'_1, \tau'_1; s'_2, \tau'_2 >, \]

\[ V^H(r_1, s_1, \tau_1; r'_1, s'_1, \tau'_1; t) = \int d^3r_2 \int d^3r'_2 \sum_{s_2, s'_2} \sum_{\tau_2, \tau'_2} \langle r_1, s_1, \tau_1; r_2, s_2, \tau_2|\hat{v}|r'_1, s'_1, \tau'_1; r'_2, s'_2, \tau'_2 \rangle > \rho(r'_2, s'_2, \tau'_2; r_2, s_2, \tau_2; t) \]

\[ = \delta_{\tau_1 p}\delta_{\tau'_1 n} \sum_{s_2, s'_2} \sum_{\mu=-1}^1 (-1)^\mu < s_1|\hat{S}_\mu(1)|s'_1 > < s_2|\hat{S}_\mu(2)|s'_2 > \]

\[ \delta(r_1 - r'_1) \int d^3r_2 v(r_1 - r_2)\rho(r_2, s'_2, n; r_2, s_2, n; t). \]

The Fock term reads:

\[ \sum_{\mu=-1}^1 (-1)^\mu < s_1, \tau_1; s_2, \tau_2|\hat{v}|r'_2, s'_2, \tau'_2; r'_1, s'_1, \tau'_1 > = \delta(r_1 - r'_2)\delta(r_2 - r'_1)v(r'_2 - r'_1) \]

\[ \sum_{\mu=-1}^1 (-1)^\mu < s_1, \tau_1; s_2, \tau_2|\hat{S}_\mu(1)\hat{S}_\mu(2)\delta_{\tau_1 p}\delta_{\tau_2 n}|s'_2, \tau'_2; s'_1, \tau'_1 >, \]
\[ V_F(r_1, s_1, \tau_1; r'_1, s'_1, \tau'_1; t) = - \int d^3r_2 \int d^3r'_2 \sum_{s_2, s'_2, \tau_2, \tau'_2} \sum_{s_1, s'_1, \tau_1, \tau'_1} \delta_{\tau_1 \rho \tau'_1 (n)} \sum_{s_2, s'_2, \mu = -1} (-1)^\mu < s_1 | \tilde{S}_\mu (1) | s'_2 > < s_2 | \tilde{S}_\mu (2) | s'_1 > v(r_1 - r'_1) \rho(r_1, s_1, \tau_1; r'_1, s'_1, \tau'_1, t). \]

Taking into account the relations
\[ < s_1 | \tilde{S}_1 | s' > = \frac{\hbar}{\sqrt{2}} \delta_{s_1 s'} \delta_{\tau_1}, \quad < s_1 | \tilde{S}_0 | s' > = \frac{\hbar}{2} \delta_{s_1 s'} (\delta_{\tau_1} - \delta_{\tau_1}), \quad < s_1 | \tilde{S}_s | s' > = - \frac{\hbar}{\sqrt{2}} \delta_{s_1 \tau} \delta_{s_1 \tau} \]
and \( v(r - r') = \tilde{\chi} \delta(r - r') \) one finds for the mean field generated by the proton-neutron part of \( H_{ss} \):

\[
\Gamma_{pn}(r, s, \tau, s', \tau'; t) = \frac{\hbar^2}{4} \left\{ \delta_{\tau \rho \tau} \delta_{s s'} \left[ \delta_{s_1 s_1} \delta_{\tau_1 \tau_1} \rho(r, \downarrow, n; r', \uparrow, t) + \delta_{s_1 s_1} \delta_{\tau_1 \tau_1} \rho(r, \uparrow, n; r', \downarrow, t) \right] \\
+ \frac{1}{2} \delta_{\tau \rho \tau} \delta_{s s'} \left[ \delta_{s_1 s_1} \delta_{\tau_1 \tau_1} \rho(r, {\downarrow\uparrow}, n; r', \uparrow, t) - \rho(r, \uparrow, n; r', \downarrow, t) \right] \\
- \delta_{s_1 \tau} \delta_{\tau_1 \tau} \rho(r, \uparrow, p; r', \uparrow, n; t) \right\} \delta(r - r') + \frac{\hbar^2}{4} \left\{ p \leftrightarrow n \right\} \delta(r - r'). \tag{16} \]

The expression for the mean field \( \Gamma_{pp}(r, s, \tau; r', s', \tau'; t) \) generated by the proton-proton part of \( H_{ss} \) can be obtained from (16) by replacing index \( n \) by \( p \) and the strength constant \( \tilde{\chi} \) by \( \chi \). The proton mean field is defined as the sum of these two terms with \( \tau = \tau' = \rho \). Its Wigner transform can be written as

\[
V^{ss'}_p(r, s, \tau; r', s', \tau'; t) = \text{const} \left\{ \delta_{s_1 s_1} \delta_{\tau_1 \tau_1} n_1^{s_2 \tau_2} + \delta_{s_1 s_1} \delta_{\tau_1 \tau_1} n_1^{s_2 \tau_2} - \delta_{s_1 s_1} \delta_{\tau_1 \tau_1} n_1^{\tau_2 \tau_1} - \delta_{s_1 s_1} \delta_{\tau_1 \tau_1} n_1^{\tau_1 \tau_2} \right\}, \tag{17} \]

where \( n^{ss'}_1(r, t) = \int \frac{d^3 p}{(2\pi)^3} f^{ss'}_p(r, p, t) \). The Wigner transform of the neutron mean field \( V^{ss'}_n \) is obtained from (17) by the obvious change of indices \( p \leftrightarrow n \). The Wigner function \( f \) and density matrix \( \rho \) are connected by the relation

\[
f^{ss'}_p(r, p, t) = e^{-ipq/\hbar} \rho(r_1, s, \tau; r_2, s', \tau'; t), \]

with \( q = r_1 - r_2 \) and \( r = \frac{1}{2}(r_1 + r_2) \). Integrating this relation over \( p \) with \( \tau' = \tau \) one finds:

\[
n^{ss'}_n(r, t) = \rho(r, s, \tau; r, s', \tau; t). \tag{17} \]

By definition the diagonal elements of the density matrix describe the proper densities. Therefore \( n^{ss}_n(r, t) \) is the density of spin-up nucleons (if \( s = \uparrow \)) or spin-down nucleons (if \( s = \downarrow \)). Off diagonal in spin elements of the density matrix \( n^{ss'}_n(r, t) \) are spin-flip characteristics and can be called spin-flip densities.

### IV. EQUATIONS OF MOTION

Integrating the set of equations (11) over phase space with the weights

\[
W = \{r \otimes p\}_\lambda \mu, \{r \otimes r\}_\lambda \mu, \{p \otimes p\}_\lambda \mu, \text{ and } 1 \tag{18} \]

at
one gets dynamic equations for the following collective variables:

\[
L_{\lambda \mu}^\tau(t) = \int d(p, r) \{r \otimes p\}_{\lambda \mu} f^{\tau\tau}(r, p, t), \quad R_{\lambda \mu}^{\tau\tau}(t) = \int d(p, r) \{r \otimes r\}_{\lambda \mu} f^{\tau\tau}(r, p, t),
\]

\[
P_{\lambda \mu}^{\tau\tau}(t) = \int d(p, r) \{p \otimes p\}_{\lambda \mu} f^{\tau\tau}(r, p, t), \quad F^{\tau\tau}(t) = \int d(p, r) f^{\tau\tau}(r, p, t),
\]

for \(\tau = \pm, -\), \(\uparrow\downarrow, \downarrow\uparrow\). We already called the functions \(f^+ = f^{\uparrow\downarrow} + f^{\downarrow\uparrow}\) and \(f^- = f^{\uparrow\downarrow} - f^{\downarrow\uparrow}\) spin-scalar and spin-vector ones, respectively. It is, therefore, natural to call the corresponding collective variables \(X_{\lambda \mu}^+(t)\) and \(X_{\lambda \mu}^-(t)\) spin-scalar and spin-vector variables. The required expressions for \(h^{\pm, \uparrow\downarrow}\) and \(h^{\downarrow\uparrow}\) are

\[
h^+_\tau = \frac{h^2}{m} + m^2 \omega^2 r^2 + 12 \sum_{\mu} (-1)^{\mu} Z_{2\mu}^\tau(t) \{r \otimes r\}_{2-\mu} + V^+_\tau(r, t),
\]

\[
h^-_\tau = -\hbar \eta_0 + V^-_\tau(r, t), \quad h^{\uparrow\downarrow}_\tau = -\frac{\hbar}{\sqrt{2}} \eta_{-1} + V^{\uparrow\downarrow}_\tau(r, t), \quad h^{\downarrow\uparrow}_\tau = \frac{\hbar}{\sqrt{2}} \eta_{1} + V^{\downarrow\uparrow}_\tau(r, t),
\]

where according to (17)

\[
V^+_\tau(r, t) = -\frac{3 h^2}{8} \chi n^+_p(r, t), \quad V^-_\tau(r, t) = \frac{3 h^2}{8} \chi n^-_p(r, t) + \frac{h^2}{4} \bar{\chi} n^-_n(r, t),
\]

\[
V^{\uparrow\downarrow}_\tau(r, t) = \frac{3 h^2}{8} \chi n^+_{p\uparrow\downarrow}(r, t) + \frac{h^2}{4} \bar{\chi} n^+_{p\downarrow\uparrow}(r, t), \quad V^{\downarrow\uparrow}_\tau(r, t) = \frac{3 h^2}{8} \chi n^+_{p\downarrow\uparrow}(r, t) + \frac{h^2}{4} \bar{\chi} n^+_{p\uparrow\downarrow}(r, t)
\]

and the neutron potentials \(V^\tau_n\) are obtained by the obvious change of indices \(p \leftrightarrow n\).

The integration yields:

\[
\dot{L}^+_{\lambda \mu} = \frac{1}{m} P^+_{\lambda \mu} - m \omega^2 R^+_{\lambda \mu} + 12 \sqrt{5} \sum_{j=0}^2 \sqrt{2j+1} \binom{11j}{2j+1} \{Z^+_{2j} \otimes R_j^+\}_{\lambda \mu}
\]

\[
- ih \eta \left[ \frac{\hbar^2}{2} \left( L^+_{\lambda \mu} + \sqrt{(\lambda - \mu)(\lambda + \mu + 1)} L^+_{\lambda \mu+1} + \sqrt{(\lambda + \mu)(\lambda - \mu + 1)} L^+_{\lambda \mu-1} \right) \right] \nonumber
\]

\[
- \frac{h^2}{2} \eta \delta_{\lambda,1} \left[ \delta_{\lambda,0} (F^{\uparrow\downarrow} + \delta_{\mu,1} F^{\downarrow\uparrow}) - \frac{1}{2} \int d^3 r \left( 2 n^\tau \{r \otimes \nabla\}_{\lambda \mu} V^+ + n^\tau \{r \otimes \nabla\}_{\lambda \mu} V^- \right) \right]
\]

\[
- 2 \frac{i}{\hbar} \int d(p, r) \{r \otimes p\}_{\lambda \mu} \left( h^\tau \{F^{\uparrow\downarrow} - h^{\downarrow\uparrow} f^{\downarrow\uparrow}\} \right)
\]

\[
\dot{L}^+_{\lambda \mu+1} = \frac{1}{m} P^+_{\lambda \mu+1} - m \omega^2 R^+_{\lambda \mu+1} + 12 \sqrt{5} \sum_{j=0}^2 \sqrt{2j+1} \binom{11j}{2j+1} \{Z^+_{2j} \otimes R_j^+\}_{\lambda \mu+1}
\]

\[
- ih \eta \left[ \frac{\hbar^2}{4} \sqrt{(\lambda - \mu)(\lambda + \mu + 1)} L^+_{\lambda \mu} + \frac{h^2}{2} \eta \delta_{\lambda,1} \left[ \delta_{\mu,0} F^- + \frac{1}{2} \sqrt{2} \delta_{\mu,1} F^{\downarrow\uparrow} \right] \right]
\]

\[
- \frac{1}{2} \int d^3 r \left( n^\tau \{r \otimes \nabla\}_{\lambda \mu+1} V^+ + n^\tau \{r \otimes \nabla\}_{\lambda \mu+1} V^- \right) - \frac{i}{\hbar} \int d(p, r) \{r \otimes p\}_{\lambda \mu+1} \left[ h^\tau F^{\downarrow\uparrow} - h^{\downarrow\uparrow} f^{\downarrow\uparrow} \right]
\]

\[
\dot{L}^\downarrow_{\lambda \mu-1} = \frac{1}{m} P^\downarrow_{\lambda \mu-1} - m \omega^2 R^\downarrow_{\lambda \mu-1} + 12 \sqrt{5} \sum_{j=0}^2 \sqrt{2j+1} \binom{11j}{2j+1} \{Z^\downarrow_{2j} \otimes R_j^\downarrow\}_{\lambda \mu-1}
\]

\[
- ih \eta \left[ \frac{\hbar^2}{4} \sqrt{(\lambda + \mu)(\lambda - \mu + 1)} L^\downarrow_{\lambda \mu} + \frac{h^2}{4} \eta \delta_{\lambda,1} \left[ \delta_{\mu,0} F^- - \sqrt{2} \delta_{\mu,1} F^{\downarrow\uparrow} \right] \right]
\]

\[
- \frac{1}{2} \int d^3 r \left( n^\tau \{r \otimes \nabla\}_{\lambda \mu-1} V^+ + n^\tau \{r \otimes \nabla\}_{\lambda \mu-1} V^- \right) - \frac{i}{\hbar} \int d(p, r) \{r \otimes p\}_{\lambda \mu-1} \left[ h^{\downarrow\uparrow} F^- - h^{\downarrow\uparrow} f^- \right]
\]
\[ \dot{F}^- = 2\eta \left[ L_{11}^{\perp 1} + L_{11}^{\perp 1} \right], \]
\[ \dot{F}^{+1} = -\eta [L_{11}^1 - \sqrt{2}L_{10}^{1}], \]
\[ \dot{F}^{11} = -\eta \left[ L_{11}^1 + \sqrt{2}L_{10}^{1} \right], \]
\[ \dot{R}_{\lambda \mu}^+ = \frac{2m}{\hbar} L_{\lambda \mu}^+ - \frac{\hbar \eta}{2} \left[ \mu R_{\lambda \mu}^+ + \sqrt{(\lambda - \mu)(\lambda + \mu + 1)} R_{\lambda \mu}^{\perp 1} \right] + \sqrt{(\lambda + \mu)(\lambda - \mu + 1)} R_{\lambda \mu}^{\perp 1} \]
\[ \dot{R}_{\lambda \mu}^- = \frac{2m}{\hbar} L_{\lambda \mu}^- - \frac{\hbar \eta}{2} \left[ \mu R_{\lambda \mu}^- - 2 \sqrt{\frac{\lambda + \mu + 1}{\lambda - \mu + 1}} \right] \]
\[ \dot{R}_{\lambda \mu}^{\perp 1} = \frac{2m}{\hbar} L_{\lambda \mu}^{\perp 1} - \frac{\hbar \eta}{4} \left[ \lambda R_{\lambda \mu}^{\perp 1} - \frac{1}{\hbar} \int\! d(p, r) \{ r \otimes r \} \lambda \mu \right] \]
\[ \dot{R}_{\lambda \mu}^{\perp 1} = \frac{2m}{\hbar} L_{\lambda \mu}^{\perp 1} - \frac{\hbar \eta}{4} \left[ \lambda R_{\lambda \mu}^{\perp 1} - \frac{1}{\hbar} \int\! d(p, r) \{ r \otimes r \} \lambda \mu \right] \]
\[ \dot{R}_{\lambda \mu} = -2m \omega^2 L_{\lambda \mu}^+ + 24 \sqrt{5} \sum_{j=0}^{2} \sqrt{2j + 1} \left\{ \{ 1 undergoes a nonlinearity due to quadrupole-quadrupole and spin-spin interactions. We will solve them in the small amplitude approximation, by linearizing the equations. This procedure helps also to solve another problem: to represent the integral terms in \[(11)\] as the linear combination of collective variables \[(19)\], that allows to close the whole set of equations \[(21)\]. The detailed analysis of the integral terms is given in the appendix \[\text{A}\].

We are interested in the scissors mode with quantum number \(K^z = 1^+\). Therefore, we only need the part of dynamic equations with \(\mu = 1\).

### A. Linearized equations (\(\mu = 1\)), isovector, isoscalar

Writing all variables as a sum of their equilibrium value plus a small deviation
\[ R_{\lambda \mu}(t) = R_{\lambda \mu}^{eq} + R_{\lambda \mu}(t), \quad \dot{P}_{\lambda \mu}(t) = P_{\lambda \mu}^{eq} + \dot{P}_{\lambda \mu}(t), \quad L_{\lambda \mu}(t) = L_{\lambda \mu}^{eq} + L_{\lambda \mu}(t) \]
and neglecting quadratic deviations, one obtains the linearized equations. Naturally one needs to know the equilibrium values of all variables. Evident equilibrium conditions for an axially symmetric nucleus are:

\[ R_{2\pm 1}^n(eq) = R_{2\pm 2}^p(eq) = 0, \quad R_{20}^\pm(eq) \neq 0. \]  

(22)

It is obvious that all ground state properties of the system of spin up nucleons are identical to the ones of the system of nucleons with spin down. Therefore

\[ R_{\lambda \mu}^- (eq) = P_{\lambda \mu}^- (eq) = L_{\lambda \mu}^- (eq) = 0. \]  

(23)

We also will suppose

\[ L_{\lambda \mu}^+ (eq) = L_{\lambda \mu}^{1\uparrow} (eq) = L_{\lambda \mu}^{\uparrow\uparrow} (eq) = 0 \quad \text{and} \quad R_{\lambda \mu}^{1\uparrow} (eq) = R_{\lambda \mu}^{\uparrow\uparrow} (eq) = 0. \]  

(24)

Let us recall that all variables and equilibrium quantities \( R_{\lambda \mu}^\pm (eq) \) and \( Z_{20}^\pm (eq) \) in (21) have isospin indices \( \tau = n, p \). All the difference between neutron and proton systems is contained in the mean field quantities \( Z_{20}^\tau (eq) \) and \( V_{\tau}^\tau \), which are different for neutrons and protons (see eq. (13) and (20)).

It is convenient to rewrite the dynamical equations in terms of isovector and isoscalar variables

\[ R_{\lambda \mu} = R_{\lambda \mu}^n + R_{\lambda \mu}^p, \quad P_{\lambda \mu} = P_{\lambda \mu}^n + P_{\lambda \mu}^p, \quad L_{\lambda \mu} = L_{\lambda \mu}^n + L_{\lambda \mu}^p, \]

\[ \bar{R}_{\lambda \mu} = R_{\lambda \mu}^\mp - R_{\lambda \mu}^p, \quad \bar{P}_{\lambda \mu} = P_{\lambda \mu}^\mp - P_{\lambda \mu}^p, \quad \bar{L}_{\lambda \mu} = L_{\lambda \mu}^\mp - L_{\lambda \mu}^p. \]

It also is natural to define isovector and isoscalar strength constants \( \kappa_1 = \frac{1}{2}(\kappa - \bar{\kappa}) \) and \( \kappa_0 = \frac{1}{2}(\kappa + \bar{\kappa}) \) connected by the relation \( \kappa_1 = \alpha \kappa_0 \) [10]. Then the equations for the neutron and proton systems are transformed into isovector and isoscalar ones. Supposing that all equilibrium characteristics of the proton system are equal to that of the neutron system one decouples isovector and isoscalar equations. This approximations looks rather crude, nevertheless the possible corrections to it are very small, being of the order \( (\mathcal{N} - Z)^2 \). With the help of the above equilibrium relations one arrives at the following final set of equations for the isovector system:

\[ \dot{\mathcal{L}}_{21}^+ = \frac{1}{m} \mathcal{P}_{21}^+ - \left[ m \omega^2 - 4\sqrt{3} \alpha \kappa_0 R_{20}^{eq} + \sqrt{6}(1 + \alpha) \kappa_0 R_{20}^{eq} \right] \bar{R}_{21}^- + i \eta \frac{\hbar}{2} \left[ \bar{L}_{21}^- + 2 \bar{L}_{22}^- + \sqrt{6} \mathcal{L}_{20}^{\uparrow\uparrow} \right], \]

\[ \dot{\mathcal{L}}_{21}^- = \frac{1}{m} \mathcal{P}_{21}^- - \left[ m \omega^2 + \sqrt{6} \alpha R_{20}^{eq} - \sqrt{3} \frac{\hbar^2}{5} \left( \chi - \frac{3}{\bar{\chi}} \right) \left( \frac{I_1}{a_0} + \frac{I_1}{a_1} \right) \left( a_1^2 A_2 - a_0^2 A_1 \right) \right] \bar{L}_{21}^+ - i \eta \frac{\hbar}{2} \bar{L}_{21}^+ \]

\[ \dot{\mathcal{L}}_{22}^+ = \frac{1}{m} \mathcal{P}_{22}^+ - \left[ m \omega^2 - 2 \sqrt{6} \alpha R_{20}^{eq} - \sqrt{3} \frac{\hbar^2}{5} \left( \chi - \frac{3}{\bar{\chi}} \right) \frac{I_1}{A_2} \right] \bar{L}_{22}^+ - i \eta \frac{\hbar}{2} \bar{L}_{22}^+, \]

\[ \dot{\mathcal{L}}_{20}^+ = \frac{1}{m} \mathcal{P}_{20}^+ - \left[ m \omega^2 + 2 \sqrt{6} \alpha R_{20}^{eq} \right] \bar{R}_{20}^{\uparrow\uparrow} + \frac{2}{\sqrt{3}} \alpha \kappa_0 R_{20}^{eq} \bar{R}_{20}^{-} - i \eta \frac{\hbar}{2} \sqrt{3} \frac{\hbar^2}{2} \bar{L}_{21}^+, \]

\[ \dot{\mathcal{L}}_{11} = -3 \sqrt{6}(1 - \alpha) \kappa_0 R_{20}^{eq} \bar{R}_{21}^{-} - i \eta \frac{\hbar}{2} \left[ \bar{L}_{11}^- + \sqrt{2} \mathcal{L}_{10}^{\uparrow\uparrow} \right], \]

\[ \dot{\mathcal{L}}_{11}^- = -\left[ 3 \sqrt{6} \alpha R_{20}^{eq} - \sqrt{3} \frac{\hbar^2}{20} \left( \chi - \frac{3}{\bar{\chi}} \right) \left( \frac{I_1}{a_0} - \frac{I_2}{a_1} \right) \left( a_1^2 A_2 - a_0^2 A_1 \right) \right] \bar{R}_{21}^- - h \eta \frac{\hbar}{2} \left[ i \mathcal{L}_{11}^+ + \hbar \mathcal{F}^{\uparrow\uparrow} \right], \]

\[ \dot{\mathcal{L}}_{10}^+ = -\eta \frac{\hbar}{2} \left[ 3 \mathcal{L}_{11}^+ + \hbar \mathcal{F}^{\uparrow\uparrow} \right], \]

\[ \dot{\mathcal{F}}^{\uparrow\uparrow} = -\eta \left[ \mathcal{L}_{11}^- + \sqrt{2} \mathcal{L}_{10}^{\uparrow\uparrow} \right]. \]
\[\hat{\mathcal{R}}_{21}^+ = \frac{2}{m} \mathcal{C}_{21} - i\hbar \frac{n}{2} \left[ \mathcal{R}_{21}^- + 2 \mathcal{R}_{22}^+ + \sqrt{6} \mathcal{P}_{20}^+ \right], \]
\[\hat{\mathcal{R}}_{21}^- = \frac{2}{m} \mathcal{C}_{21} - i\hbar \frac{n}{2} \mathcal{P}_{21}^+, \]
\[\hat{\mathcal{R}}_{22}^+ = \frac{2}{m} \mathcal{C}_{22} - i\hbar \frac{n}{2} \mathcal{P}_{21}^+, \]
\[\hat{\mathcal{R}}_{22}^- = \frac{2}{m} \mathcal{C}_{22} - i\hbar \frac{n}{2} \mathcal{R}_{21}^+, \]
\[\hat{\mathcal{R}}_{20}^+ = \frac{2}{m} \mathcal{C}_{22} - i\hbar \frac{n}{2} \mathcal{P}_{20}^-, \]
\[\hat{\mathcal{P}}_{21}^- = -2 \left[ m \omega^2 + \sqrt{6} \kappa_0 R_{20}^{eq} \right] \mathcal{C}_{21}^+ + 6 \sqrt{6} \kappa_0 R_{20}^{eq} \mathcal{C}_{11}^+ - i\hbar \frac{n}{2} \left[ \mathcal{P}_{21}^- + 2 \mathcal{P}_{22}^- + \sqrt{6} \mathcal{P}_{20}^+ \right] + \frac{3\sqrt{3}}{4} \hbar^2 \chi \frac{I_2}{A_1 A_2} \left[ (A_1 - A_2) \mathcal{C}_{21}^+ + (A_1 + A_2) \mathcal{C}_{11}^+ \right], \]
\[\hat{\mathcal{P}}_{21}^+ = -2 \left[ m \omega^2 + \sqrt{6} \kappa_0 R_{20}^{eq} \right] \mathcal{C}_{21}^- + 6 \sqrt{6} \kappa_0 R_{20}^{eq} \mathcal{C}_{11}^- - i\hbar \frac{n}{2} \mathcal{P}_{21}^+, \]
\[\hat{\mathcal{P}}_{22}^+ = -2 \left[ m \omega^2 - 4 \sqrt{6} \kappa_0 R_{20}^{eq} - \frac{3\sqrt{3}}{2} \hbar^2 \chi \frac{I_2}{A_2} \right] \mathcal{C}_{22}^+ - i\hbar \frac{n}{2} \mathcal{P}_{21}^+, \]
\[\hat{\mathcal{P}}_{20}^- = \left[ m \omega^2 + 4 \sqrt{6} \kappa_0 R_{20}^{eq} \right] \mathcal{C}_{20}^+ + 8 \sqrt{6} \kappa_0 R_{20}^{eq} \mathcal{C}_{10}^+ - i\hbar \frac{n}{2} \sqrt{3} \mathcal{P}_{21}^+ + \frac{\sqrt{3}}{2} \hbar^2 \chi \frac{I_2}{A_1 A_2} \left[ (A_1 - 2A_2) \mathcal{C}_{20}^+ + \sqrt{2} (A_1 + A_2) \mathcal{C}_{00}^+ \right], \]
\[\hat{\mathcal{L}}_{00}^+ = \frac{1}{m} \mathcal{P}_{00}^+ - m \omega^2 \mathcal{R}_{00}^+ + 4 \sqrt{3} \kappa_0 R_{20}^{eq} \mathcal{R}_{20}^- \]
\[+ \frac{1}{2\sqrt{3}} \hbar^2 \left[ \left( \chi - \frac{\bar{\chi}}{3} \right) I_1 - \frac{9}{4} \chi I_2 \right] \frac{(2A_1 - A_2)}{A_1 A_2} \mathcal{R}_{00}^+ + \sqrt{2} (A_1 + A_2) \mathcal{R}_{20}^+, \]
\[\hat{\mathcal{R}}_{00}^+ = \frac{2}{m} \mathcal{C}_{00}^+, \]
\[\hat{\mathcal{P}}_{00}^- = -2m \omega^2 \mathcal{C}_{00}^+ + \sqrt{3} \kappa_0 R_{20}^{eq} \mathcal{C}_{20}^+ + \frac{\sqrt{3}}{2} \hbar^2 \chi I_2 \left[ \left( \frac{2}{A_2} - \frac{1}{A_1} \right) \mathcal{C}_{00}^+ + \sqrt{2} \left( \frac{1}{A_2} + \frac{1}{A_1} \right) \mathcal{C}_{20}^+ \right], \]

where \( A_1, A_2 \) are defined in appendix [8] \( \kappa_0 = -m\bar{\omega}^2/(4Q_{00}) \) [21] with \( \bar{\omega}^2 = \omega^2/(1 + \frac{2}{3} \delta) \), \( a_1 = R_0 \left( \frac{1 - (2/3)\delta}{1 + (4/3)\delta} \right)^{1/6} \) and \( a_0 = R_0 \left( \frac{1 - (2/3)\delta}{1 + (4/3)\delta} \right)^{-1/3} \) are semiaxes of ellipsoid by which the shape of nucleus is approximated, \( \delta \) – deformation parameter, \( R_0 = 1.2 A^{1/3} \) fm.

\[ I_1 = \frac{\pi}{4} \int_{-\infty}^{+\infty} dr r^4 \left( \frac{\partial n^+(r)}{\partial r} \right)^2, \quad I_2 = \frac{\pi}{4} \int_{-\infty}^{+\infty} dr r^2 n^+(r)^2, \quad n^+(r) = n^+_p + n^+_n = \frac{n_0}{1 + e^{-r/a}}. \]

The isovector set of equations is easily obtained from [24] by taking \( \alpha = 1 \) and replacing \( \bar{\chi} \to -\bar{\chi} \).

V. DISCUSSION AND INTERPRETATION OF THE RESULTS

The energies and excitation probabilities obtained by the solution of the isovector set of equations [25] are given in Table [4]. The used spin-spin interaction is repulsive, the values of its strength constants being taken from the paper [21], where the notation \( \chi = K_s/A, \bar{\chi} = q\chi \) was introduced. The results without spin-spin interaction (variant I) are compared with those performed with two sets of constants \( K_s, q \) (variants II, III). The first set of constants (variant II) was extracted by the authors of [21] from Skyrme forces following the standard procedure, the residual interaction being defined in terms of second derivatives of the Hamiltonian density \( H(\rho) \) with respect to the one-body densities \( \rho \). Different variants of Skyrme forces produce different strength constants of spin-spin interaction. The most consistent results are obtained with SG1, SG2 [22]
TABLE I: Isovector energies and excitation probabilities of $^{164}\text{Er}$. Deformation parameter $\delta = 0.25$, spin-orbit constant $\eta = 0.36$ MeV. Spin-spin interaction constants are: I $- K_s = 0$ MeV; II $- K_s = 92$ MeV, $q = -0.8$; III $- K_s = 200$ MeV, $q = -0.5$. Quantum numbers (including indices $\zeta = +, -, \uparrow, \downarrow$) of variables responsible for the generation of the present level are shown in the first column. For example: $(1,1)$ $^-$ spin scissors, $(1,1)$ $^+$ conventional scissors, etc.

| $(\lambda, \mu)^\alpha$ | $E_{W}$, MeV | $B(M1)$, $\mu^2_N$ | $B(E2)$, $B_W$ |
|--------------------------|--------------|---------------------|---------------|
|                          | I            | II                  | III           |
| $(1,1)^-$                | 1.61         | 2.02                | 2.34          |
| $(1,1)^+$                | 2.18         | 2.45                | 2.76          |
| $(0,0)^{\uparrow \downarrow}$ | 12.80       | 16.81               | 20.02         |
| $(2,1)^-$                | 14.50        | 18.52               | 21.90         |
| $(2,2)^{\uparrow \downarrow}$ | 16.18       | 20.61               | 24.56         |
| $(2,0)^{\uparrow \downarrow}$ | 16.20       | 22.65               | 27.67         |
| $(2,1)^+$                | 20.59        | 21.49               | 22.42         |
| $(1,0)^{\uparrow \downarrow}$ | 0.26i        | 0.26i               | 0.26i         |

and Sk3 forces. We use here the spin-spin constants extracted from Sk3 force. Another set of constants (variant III) was also found by the authors of [21] phenomenologically in the calculations with a Woods-Saxon potential, when there is not any self-consistency between the mean field and the residual interaction. We tentatively will use it, because in our case also there is no self-consistency. The strength of the spin-orbit interaction is taken from [24].

One can see from Table I that the spin-spin interaction does not change the qualitative picture of the positions of the excitations described in [12]. It pushes all levels up proportionally to its strength (20-30% in the case II and 40-60% in the case III) without changing their order. The most interesting result concerns the relative $B(M1)$ values of the two low lying scissors modes, namely the spin scissors $(1,1)^-$ and the conventional (orbital) scissors $(1,1)^+$ mode. As can be noticed, the spin-spin interaction strongly redistributes M1 strength in the favour of the spin scissors mode. We tentatively want to link this fact to the recent experimental finding in isotopes of Th and Pa [17]. The authors have studied deuteron and $^3$He-induced reactions on $^{232}\text{Th}$ and found in the residual nuclei $^{231,232,233}\text{Th}$ and $^{232,233}\text{Pa}$ "an unexpectedly strong integrated strength of $B(M1) = 11 - 15 \mu^2_N$ in the $E_\gamma = 1.0 - 3.5$ MeV region". The $B(M1)$ force in most nuclei shows evident splitting into two Lorentzians. "Typically, the experimental splitting is $\Delta \omega_{M1} \sim 0.7$ MeV, and the ratio of the strengths between the lower and upper resonance components is $B_L/B_U \sim 2^\circ$. (Note a misprint in that paper: it is written erroneously $B_2/B_1 \sim 2$ whereas it should be $B_1/B_2 \sim 2$. To avoid misunderstanding, we write here $B_L$ instead of $B_1$ and $B_U$ instead of $B_2$.) The authors have tried to explain the splitting by a $\gamma$-deformation. To describe the observed value of $\Delta \omega_{M1}$ the deformation $\gamma \sim 15^\circ$ is required, that leads to the ratio $B_L/B_U \sim 0.7$ in an obvious contradiction with experiment. The authors conclude that "the splitting may be due to other mechanisms". In this sense, we tentatively may argue as follows. On one side, theory [25] and experiment [26] give zero value of $\gamma$-deformation for $^{233}\text{Th}$. On the other side, it is easy to see that our theory suggests the required mechanism. The calculations performed for $^{233}\text{Th}$ give $\Delta \omega_{M1} \sim 0.32$ MeV and $B_L/B_U \sim 1.6$ for the first variant of the spin-spin interaction and $\Delta \omega_{M1} \sim 0.28$ MeV and $B_L/B_U \sim 4.1$ for second one in reasonable agreement with experimental values. The inclusion of pair correlations will affect our results, but one may speculate that the agreement between the theory and experiment will be conserved at least qualitatively.

The energies and excitation probabilities obtained by the solution of the isoscalar set of equations [25] are displayed in the Table II. The general picture of the influence of the spin-spin interaction here is quite close to that observed in the isovector case. The only difference is the low lying mode marked by $(1,1)^+$ which is practically insensitive to the spin-spin interaction. The negligibly small negative $B(M1)$ value of spin scissors appears undoubtedly due to the lack of the self consistency in our calculations. In ref [17] the assignment of the resonances to be of isovector type is only tentative based on the assumption that at such low energies there is no collective mode other than the isovector scissors mode. However, from [17] one cannot exclude that also an isoscalar spin scissors mode is mixed in. From our analysis we see that the isoscalar spin scissors where all nucleons with spin up counter-rotate with respect the ones of spin up comes more or less at the same energy as the isovector scissors. So it would be very important for the future to pin down precisely the quantum numbers of the resonances.
Let us discuss in more detail the nature of the predicted excitations. As one sees, the generalization of the WFM method by including spin dynamics allowed one to reveal a variety of new types of nuclear collective motion involving spin degrees of freedom. Two isovector and two isoscalar low lying eigenfrequencies and five isovector and five isoscalar high lying eigenfrequencies have been found.

Three low lying levels correspond to the excitation of new types of modes. For example the isovector level marked by (1,1)\(^{-}\) describes rotational oscillations of nucleons with the spin projection "up" with respect of nucleons with the spin projection "down", i.e. one can talk of a nuclear spin scissors mode. Having in mind that this excitation is an isovector scissors mode discussed in our work here. It would be interesting to study whether our suggested description is correct or not. This could for example be done in analyzing the current patterns.

One more new low lying mode (isoscalar, marked by (1, 1)\(^{+}\)) is generated by the relative motion of the orbital angular momentum and spin of the nucleus (they can change their absolute values and directions keeping the total spin unchanged).

In order to complete the picture of the low-lying states, it is important to discuss the state which is of this state has nothing to do with neither spin scissors nor with conventional scissors. It can namely be seen from the structure of our equations that this state corresponds to a spin flip induced by the spin-orbit potential. Such a state is of purely quantal character and it cannot be hoped that we can accurately describe it with our WFM approach restricting the consideration by second order moments only. For its correct treatment, we certainly should consider higher moments like fourth order moments, for instance. The spin-orbit potential is the only term in our theory which couples the second order moments to the fourth order ones. As mentioned, we decoupled the system in neglecting the fourth order moments. Therefore, it is no surprise that this particular spin flip mode is not well described. Nevertheless, one may try to better understand the origin of this mode almost at zero energy. For this, we make the following approximation of our diagonalisation procedure to get the eight eigenvalues listed in Table I. We neglect in all couplings between the set of variables \(X_{\lambda\mu}^{++}, X_{\lambda\mu}^{--}\) and the set of variables \(X_{\lambda\mu}^{↑↓}, X_{\lambda\mu}^{↓↑}\). To this end in the dynamical equations for \(X_{\lambda\mu}^{++}, X_{\lambda\mu}^{--}\) we omit all terms containing \(X_{\lambda\mu}^{↑↓}, X_{\lambda\mu}^{↓↑}\) and in the dynamical equations for \(X_{\lambda\mu}^{↑↓}, X_{\lambda\mu}^{↓↑}\) we omit all terms containing \(X_{\lambda\mu}^{++}, X_{\lambda\mu}^{--}\). In such a way we get two independent sets of dynamical equations. The first one (for \(X_{\lambda\mu}^{++}, X_{\lambda\mu}^{--}\)) was already studied in [13], where we have found that such approximation gives satisfactory (in comparison with the exact solution) results but must be used

| \((\lambda, \mu)\) | \(E_{ir}, \text{MeV}\) | \(B(M1), \mu_{N}^{2}\) | \(B(E2), B_{W}\) |
|----------------|-----------------|-----------------|-----------------|
| (1,1)\(^{-}\) | 1.73 | 2.04 | 2.40 | -0.07 | -0.05 | 0 | 1.12 | 0.65 | 0.39 |
| (1,1)\(^{+}\) | 0.39 | 0.37 | 0.24 | 0.24 | 0.24 | 117.2 | 0 | 117.9 | 118.3 |
| (0,0)\(^{↑↓}\) | 12.83 | 15.59 | 18.72 | 0 | 0 | 0 | 0.66 | 0.31 | 0.15 |
| (2,1)\(^{↑↓}\) | 14.51 | 17.40 | 20.65 | 0 | 0 | 0 | 0.12 | 0.06 | 0.03 |
| (2,2)\(^{↑↓}\) | 16.20 | 19.43 | 23.09 | 0 | 0 | 0 | 0.20 | 0.07 | 0.04 |
| (2,0)\(^{↑↓}\) | 16.22 | 20.09 | 24.80 | 0 | 0 | 0 | 0.20 | 0.07 | 0.04 |
| (2,1)\(^{+}\) | 10.28 | 11.92 | 13.60 | 0 | 0 | 0 | 66.50 | 57.78 | 50.87 |
| (1,0)\(^{↑↓}\) | 0.20i | 0.20i | 0.20i | -0.1i | -0.1i | 30.0i | 29.8i | 30.3i |
The solution of these equations is i.e. the coupling between the respective variables are very close to that of exact calculations, it was shown that the results of approximate calculations (for momentum conservation. The second set of equations of all spin up nucleons (protons together with isoscalar resonance describes out of phase oscillations to those in isospin space are also possible. That is why we did not try and the quantitative description of the conventional scissors mode. Nevertheless our results turned out to be very useful, because they demonstrate that the common effect of spin-spin interaction and pair correlations are able to push a substantial part of the M1 force out of the area of the conventional scissors mode what is required for the reasonable agreement with experimental data.

In this respect, we should mention that we did not include pairing in this work. Inclusion of pairing would have complicated the formalism quite a bit. It shall be worked out in the future. We here wanted to study the features of spin dynamics in a most transparent way staying, however, somewhat on the qualitative side. That is why we did not try to discuss in detail possible relations with experiment or to compare with the results of other theories. Nevertheless we mentioned the quite recent experimental work, where for the two low lying magnetic states a stronger B(M1) transition for experimental work failed. However, our theory can naturally predict such a scenario with a non vanishing spin-spin force. It would indeed be very exciting, if the results of had already discovered the isovector spin scissors mode. However, much deeper experimental and theoretical results must be obtained before a firm conclusion on this point is possible.

In the light of the above results, the study of spin excitations with pairing included, will be the natural continuation of this work. Pairing is important for a quantitative description of the conventional scissors mode. The same is expected for the novel spin scissors mode discussed here. The effect of pairing generally is to push up the spectrum in energy. Therefore, as just mentioned, it can be expected that the results come into better agreement with experimental data.

VI. CONCLUDING REMARKS

The inclusion of spin-spin interaction does not change qualitatively the picture concerning the spectrum of the spin modes found in. It pushes all levels up without changing their order. However, it strongly redistributes M1 strength between the conventional and spin scissors mode in the favour of the last one. Our calculations did not fully confirm the expectations mentioned in the introduction, namely that essentially only the low lying part of the spectrum will be strongly influenced by the spin-spin force. Nevertheless our results turned out to be very useful, because they demonstrate that the common effect of spin-spin interaction and pair correlations are able to push a substantial part of the M1 force out of the area of the conventional scissors mode what is required for the reasonable agreement with experimental data.

Two high lying excitations of a new nature are found. They are marked by (2,1) and following the paper can be called spin-vector giant quadrupole resonances. The isovector one corresponds to the following quadrupole motion: the proton system oscillates out of phase with the neutron system, whereas inside of each system spin up nucleons oscillate out of phase with spin down nucleons. The respective isoscalar resonance describes out of phase oscillations of all spin up nucleons (protons together with neutrons) with respect of all spin down nucleons.

Six high lying modes can be interpreted as spin-flip giant monopole (marked by (0,0)↑↓) and quadrupole (marked by (2,0)↑↑ and (2,2)↑↓) resonances.

It is a pertinent place to make following citation from the review by F. Osterfeld: "Similar oscillations to those in isospin space are also possible in spin space. Nucleons with spin up and spin down may move either in phase (spin-scalar S=0 modes) or out of phase (spin-vector S=1 modes). The latter class of states is also referred to as spin excitations or spin-flip transitions." On account of our results in this work, the latter statement that all spin excitations are of spin-flip nature should be modified. We predict in this paper the existence of spin excitations of non-spin-flip nature – the isovector and isoscalar spin scissors and the isovector and isoscalar spin-vector GQR!
I_{ss} = \int d(p,r)\{r \otimes p\} \lambda_{\mu} [V^{\uparrow \downarrow} r f^{\downarrow \downarrow} - V^{\downarrow \uparrow} r f^{\uparrow \downarrow}],

V_{\tau}^{s'} being defined in [20]. It is easy to see that the integral \( I_{so} \) generate moments of fourth order. According to the rules of the WFM method [28] this integral is neglected.

Let us analyze the integral \( I_{ss} \) (to be definite, for protons). In this case

\[
V^{\uparrow \downarrow}_p(r) = \frac{\hbar^2}{8} \chi^{\uparrow \downarrow}_p(r) + \frac{h^2}{4} \tilde{\chi}^{\uparrow \downarrow}_p(r),
\]

\[
V^{\downarrow \uparrow}_p(r) = \frac{\hbar^2}{8} \chi^{\downarrow \uparrow}_p(r) + \frac{h^2}{4} \tilde{\chi}^{\downarrow \uparrow}_p(r).
\]

It is seen that \( I_{ss} \) is split into four terms of identical structure, so it will be sufficient to analyze in detail only one part. For example

\[ I_{ss4} = \int d(p,r)\{r \otimes p\} \lambda_{\mu} n^{\uparrow \downarrow} f^{\downarrow \downarrow} = \int d^3r \{r \otimes J^{\downarrow \downarrow}\} \lambda_{\mu} n^{\uparrow \downarrow} = \sum_{\nu,\alpha} C_{\nu,\alpha}^{\lambda_{\mu}} \int d^3 r r_{\nu} J^{\downarrow \downarrow}_{\alpha} n^{\uparrow \downarrow}, \tag{A1} \]

where \( J^{\downarrow \downarrow}_{\alpha}(r,t) = \int \frac{d^3p}{(2\pi)^3} p_{\alpha} f^{\downarrow \downarrow}(r,p,t) \). The variation of this integral reads

\[
\delta I_{ss4} = \sum_{\nu,\alpha} C_{\nu,\alpha}^{\lambda_{\mu}} \int d^3 r r_{\nu} [n^{\uparrow \downarrow} (eq) \delta J^{\downarrow \downarrow}_{\alpha} + J^{\downarrow \downarrow}_{\alpha} (eq) \delta n^{\uparrow \downarrow}]. \tag{A2} \]

It is necessary to represent this integral in terms of the collective variables [19]. This problem can not be solved exactly, so we will use the approximation suggested in [28] and expand the density and current variations as a series (see appendix [3]).

Let us consider the second part of integral (A2). With the help of formula (B4) we find

\[
I_2 \equiv \sum_{\nu,\alpha} C_{\nu,\alpha}^{\lambda_{\mu}} \int d^3 r r_{\nu} J^{\downarrow \downarrow}_{\alpha} (eq) \delta n^{\uparrow \downarrow} = - \sum_{\nu,\alpha} C_{\nu,\alpha}^{\lambda_{\mu}} \int d^3 r r_{\nu} J^{\downarrow \downarrow}_{\alpha} (eq) \sum_{\beta} (-1)^{\beta} \left\{ N^{\uparrow \downarrow}_{\beta,eq} r^{\downarrow \downarrow}_{\alpha} + \sum_{\gamma} (-1)^{\gamma} N^{\downarrow \downarrow}_{\beta,eq} r^{\uparrow \downarrow}_{\alpha} \frac{1}{\partial r} \frac{\partial n^{\uparrow \downarrow}_{\beta,eq}}{\partial r} \right\}. \tag{A3} \]

Let us analyze at first the more simple part of this expression:

\[
I_{2,1} \equiv - \sum_{\beta} (-1)^{\beta} N^{\uparrow \downarrow}_{\beta,eq} r^{\downarrow \downarrow}_{\alpha} \int d^3 r \sum_{\nu,\alpha} C_{\nu,\alpha}^{\lambda_{\mu}} r_{\nu} n^{\uparrow \downarrow} = - \sum_{\beta} (-1)^{\beta} N^{\uparrow \downarrow}_{\beta,eq} X_{\lambda_{\mu}}. \tag{A4} \]

We are interested in the value of \( \mu = 1 \), therefore it is necessary to analyze two possibilities: \( \lambda = 1 \) and \( \lambda = 2 \).
In the case $\lambda = 1, \mu = 1$ we have

$$X_{11} = \int d^3 r \ n^+ \sum_{v, \alpha} C_{1v, \alpha}^{11} r_{vJ_0^+}(eq) = \int d^3 r \ n^+ \frac{1}{\sqrt{2}} \left[ r_1 J_0^+(eq) - r_0 J_1^+(eq) \right]. \quad (A5)$$

By definition

$$J_{ss'}^v = \int \frac{d^3 p}{(2\pi \hbar)^3} p_v f_{ss'}(r, p) = -\frac{i\hbar}{2} \left[ (\nabla_v - \nabla'_{v'}) p(r, s; r', s') \right]_{r'=r}
= \frac{i\hbar}{2} \sum_k v_k^2 [\phi_k(r, s)\nabla_v \phi_k^*(r, s') - \phi_k^*(r, s')\nabla_v \phi_k(r, s)], \quad (A6)$$

where $k = n, l, j, m$ is a set of oscillator quantum numbers, $v_k^2$ are occupation numbers, and

$$\phi_{nljm}(r, s) = R_{nl}(r) \sum_{\Lambda, \sigma} C_{lm, \Lambda, \frac{1}{2} \sigma}^{jm} Y_{\Lambda}(\theta, \phi) \chi_{\frac{1}{2} \sigma}^\Lambda(s) = R_{nlj}(r) C_{lm, s, \frac{1}{2} \sigma}^{jm} Y_{lm-\sigma}(\theta, \phi) \quad (A7)$$
are single particle wave functions, $\chi_\frac{1}{2}^\Lambda(s) = \delta_\sigma, s$ being spin functions. Inserting (A6) into (A5) one finds

$$X_{11} = \frac{i\hbar}{2} \frac{1}{\sqrt{2}} \sum_{nljm} v_{nljm}^2 \int d^3 r \ n^+ (r) R_{nlj}(r) C_{lm, s, \frac{1}{2} \sigma}^{jm} \sum_{\Lambda, \sigma} C_{lm, \Lambda, \frac{1}{2} \sigma}^{jm} [Y_{\Lambda}(r_1 \nabla_1 - r_0 \nabla 1) Y_{lm}^{\Lambda'}
- Y_{lm}^{\Lambda'}(r_1 \nabla_0 - r_0 \nabla 1) Y_{\Lambda}] \quad (A8)$$

with $\Lambda = m - \frac{l}{2}$ and $\Lambda' = m + \frac{l}{2}$. Remembering the definition (10) of the angular momentum $\hat{l}_1 = \hbar (r_0 \nabla_1 - r_1 \nabla 0)$ and using the relation $|\hat{l}_{1+1} Y_{\Lambda} = \pm \frac{i\hbar}{\sqrt{2}} \sqrt{(l+\Lambda)(l+\Lambda+1)} Y_{\Lambda}]$ one transforms (A8) into

$$X_{11} = \frac{i\hbar}{2} \frac{1}{\sqrt{2}} \sum_{nljm} v_{nljm}^2 \int dr n^+(r) r^2 R_{nlj}(r) C_{lm, s, \frac{1}{2} \sigma}^{jm} \sum_{\Lambda, \sigma} C_{lm, \Lambda, \frac{1}{2} \sigma}^{jm} \frac{2}{\sqrt{2}} \sqrt{(l-\Lambda)(l+\Lambda+1)}
= -i\hbar \sum_{nl} \sum_{m=\frac{l}{2}}^{\frac{l}{2}} \frac{[(l + \frac{1}{2})^2 - m^2]}{2l + 1} \int dr n^+(r) r^2 \left[ v_{nl+\frac{1}{2}m}^2 R_{nl+\frac{1}{2}m}^2(r) - v_{nl-\frac{1}{2}m}^2 R_{nl-\frac{1}{2}m}^2(r) \right]. \quad (A9)$$

As it is seen, the value of this integral is determined by the difference of the wave functions of spin-orbital partners $(vR^2)_{nl+\frac{1}{2}m} - (vR^2)_{nl\left(\frac{1}{2}m\right)}$, which is usually very small, so we will neglect it. The only remarkable contribution can appear in the vicinity of the Fermi surface, where some spin-orbital partners with $j = l + \frac{1}{2}$ and $j = |l - \frac{1}{2}|$ can be disposed on different sides of the Fermi surface. In reality such situation happens very frequently, nevertheless we will not take into account this effect, because the values of the corresponding integrals are considerably smaller than $R_{200}(eq)$, the typical input parameter of our model.

Let us consider now the integral $I_{2,1}$ (formula (A4)) for the case $\lambda = 2, \mu = 1$. We have

$$X_{21} = \int d^3 r n^+ \sum_{v, \alpha} C_{1v, \alpha}^{21} r_{vJ_0^+}(eq) = \int d^3 r n^+ C_{11, 10}^{21} \left[ r_1 J_0^+(eq) + r_0 J_1^+(eq) \right]. \quad (A10)$$

With the help of formulae (A6) and (A7) one can show by simple algebraic transformations that

$$\int d\Omega r_1 J_0^+(eq) = -\int d\Omega r_0 J_1^+(eq), \quad (A11)$$

where $\int d\Omega$ means the integration over angles. As a result $X_{21} = 0$. 

Let us consider the second, more complicated, part of integral $I_2$:

$$I_{2,2} = - \sum_{\beta, \gamma} \left[ -1 \right]^{\beta + \gamma} N_{\beta, -\gamma}^{\uparrow}(t) \sum_{\nu, \alpha} C_{1\nu, 1\alpha}^{\lambda \mu} \int d^3r r_v J_{\alpha}^{\uparrow}(eq) \frac{1}{r} \frac{\partial n^{+}}{\partial r} r_{\beta} r_{\gamma}$$

$$= - \sum_{\beta, \gamma} \left[ -1 \right]^{\beta + \gamma} N_{\beta, -\gamma}^{\uparrow}(t) X_{\lambda \mu}^{\beta, \gamma}(\beta, \gamma). \quad (A12)$$

The case $\lambda = 1, \mu = 1$:

$$X_{11}^{\beta, \gamma}(\beta, \gamma) = \frac{1}{\sqrt{2}} \int d^3r \frac{1}{r} \frac{\partial n^{+}}{\partial r} \left[ r_1 J_{0}^{\uparrow}(eq) - r_0 J_{1}^{\uparrow}(eq) \right] r_{\beta} r_{\gamma}$$

$$= - \frac{i}{4} \sum_{n lj m} v_{nljm}^2 \int d^3r \frac{1}{r} \frac{\partial n^{+}}{\partial r} R_{nlj}(r) C_{l\lambda}^{jm} \frac{C_{l\lambda}^{jm} r}{r_{\beta}} \sqrt{(l - \Lambda)(l + \Lambda + 1)} [Y_{\lambda\Lambda} Y_{\lambda\Lambda}^{*} + Y_{\lambda\Lambda}^{*} Y_{\lambda\Lambda}] r_{\beta} r_{\gamma}. \quad (A13)$$

The angular part of this integral is

$$\int d\Omega [Y_{\lambda\Lambda} Y_{\lambda\Lambda}^{*} + Y_{\lambda\Lambda}^{*} Y_{\lambda\Lambda}] r_{\beta} r_{\gamma} = \sum_{L, M} C_{L\beta, 1\gamma}^{LM} \int d\Omega [Y_{\lambda\Lambda} Y_{\lambda\Lambda}^{*} + Y_{\lambda\Lambda}^{*} Y_{\lambda\Lambda}] \{ r \otimes r \}_{LM}$$

$$= - \frac{2}{\sqrt{3}} r^{2} \gamma_{\beta, \gamma}^{2} \sum_{M} C_{1\beta, 1\gamma}^{2 M} \int d\Omega [Y_{\lambda\Lambda} Y_{\lambda\Lambda}^{*} + Y_{\lambda\Lambda}^{*} Y_{\lambda\Lambda}] Y_{2M}$$

$$= \frac{2}{3} r^{2} \delta_{\gamma, \beta} \left\{ 1 - \sqrt{\frac{5}{2}} C_{l0,20}^{1 M} \left[ C_{1\gamma,2M}^{1 \beta} + C_{1\gamma,2M}^{1 \beta} \right] \right\}. \quad (A14)$$

Therefore

$$X_{11}^{\beta, \gamma}(\beta, \gamma) = - \frac{i}{6} \delta_{\gamma, \beta} \int dr \frac{\partial n^{+}(r)}{\partial r} r^{3} \sum_{n lj m} \left\{ 1 - \sqrt{\frac{5}{2}} C_{l0,20}^{1 M} \left[ C_{1\gamma,2M}^{1 \beta} + C_{1\gamma,2M}^{1 \beta} \right] \right\} \times$$

$$\sum_{n lj m} v_{nljm}^2 R_{nlj}(r) C_{l0}^{jm} \frac{C_{l0}^{jm} r}{r_{\beta}} \sqrt{(l - \Lambda)(l + \Lambda + 1)}$$

$$= - \frac{i}{3} \delta_{\gamma, \beta} \sum_{n lj m} \left\{ 1 - \sqrt{\frac{5}{2}} C_{l0,20}^{1 M} \left[ C_{1\gamma,2M}^{1 \beta} + C_{1\gamma,2M}^{1 \beta} \right] \right\} \times$$

$$\sum_{m=\frac{1}{2}}^{\frac{l+1}{2}} \left[ ( \frac{l+1}{2} )^2 - m^2 \right] \int dr \frac{\partial n^{+}(r)}{\partial r} r^{3} \left[ v_{nl+\frac{1}{2}m}^2 R_{nl+\frac{1}{2}m}^2 (r) - v_{nl-\frac{1}{2}m}^2 R_{nl-\frac{1}{2}m}^2 (r) \right]. \quad (A15)$$

One sees that, exactly as in formula (A9), the value of this integral is determined by the difference of the wave functions of spin-orbital partners $(v R)^{2}_{nll+\frac{1}{2}m} - (v R)^{2}_{nll-\frac{1}{2}m}$ near the Fermi surface, so it can be omitted together with $X_{11}$ following the same arguments.

The case $\lambda = 2, \mu = 1$ can be analyzed in full analogy with formulae [A10 A11] that allows us to take $X_{21}^{\beta, \gamma} = 0$.

So, we have shown that the integral $I_2$ can be approximated by zero. Let us consider now the first part of the integral (A2):

$$I_1 = \sum_{\nu, \alpha} C_{1\nu, 1\alpha}^{\lambda \mu} \int d^3r r_v n^{+}(eq) \delta J_{\alpha}^{\uparrow} = \sum_{\nu, \alpha} C_{1\nu, 1\alpha}^{\lambda \mu} \int d^3r r_v n^{+}(eq) n^{+}(r) \sum_{\gamma} (-1)^{\gamma} K_{\alpha, -\gamma}^{\uparrow}(t) r_{\gamma}$$

$$= \sum_{\nu, \alpha} C_{1\nu, 1\alpha}^{\lambda \mu} \sum_{\gamma} (-1)^{\gamma} K_{\alpha, -\gamma}^{\uparrow}(t) \int d^3r n^{+}(eq) n^{+}(r) \sum_{L, M} C_{L\gamma, 1\nu, 1\alpha}^{LM} \{ r \otimes r \}_{LM}. \quad (A16)$$
This integral can be estimated in the approximation of constant density \( n^+(r) = n_0 \). Then
\[
I_1 = n_0 \sum_{\nu, \alpha} C^{\nu}_{1\nu, \alpha} \sum_{\gamma} (-1)^\gamma K^{\uparrow \downarrow}_{\alpha, -\gamma}(t) \sum_{L, M} C^{LM}_{1\nu, 1\gamma} R_{LM}^\dagger(eq) = 0. \tag{A17}
\]

It is easy to show, that \( R_{LM}^\dagger(eq) = 0 \). Let us consider, for example, the case with \( L = 2 \):
\[
R_{2M}^\dagger = \int d(p, r) \{ r \otimes r \}_{2M} f^\dagger(r, p) = \int d^3r \{ r \otimes r \}_{2M} n^\dagger(r) = \sqrt{8\pi} \int d^3r r^2 Y_{2M} n^\dagger(r). \tag{A18}
\]

By definition
\[
n^{ss'}(r) = \int \frac{d^3p}{(2\pi\hbar)^3} f^{ss'}(r, p) = \sum_k v_k^2 \phi_k(r, s) \phi_k^*(r, s') \tag{A19}
\]
with \( \phi_k \) defined in \( \text{(A7)} \). Therefore
\[
R_{2M}^\dagger = \sqrt{\frac{8\pi}{15}} \int d^3r r^2 Y_{2M} \sum_{nljm} v_{nljm}^2 R_{nlj}^2(r) C^{jm}_{l\alpha, \frac{1}{2}-\frac{3}{2}} C^{jm}_{l\beta, \frac{1}{2}+\frac{3}{2}} Y_{l\alpha}^* Y_{l\beta}^* \]
\[
= \sqrt{\frac{2}{3}} \sum_{nljm} v_{nljm}^2 \int drr^2 R_{nlj}^2(r) C^{jm}_{l\alpha, \frac{1}{2}-\frac{3}{2}} C^{jm}_{l\beta, \frac{1}{2}+\frac{3}{2}} C^{0}_{20, 0} C^{l}_{2M, l\alpha} = 0, \tag{A20}
\]
where \( \Lambda = m - \frac{1}{2} \) and \( \Lambda' = m + \frac{1}{2} \). The zero is obtained due to summation over \( m \). Really, the product
\[
C^{jm}_{l\alpha, \frac{1}{2}-\frac{3}{2}} C^{jm}_{l\beta, \frac{1}{2}+\frac{3}{2}} = \pm \frac{\sqrt{l+l'-2m^2}}{2l+1} \quad (\text{for } j = l \pm \frac{1}{2})
\]
does not depend on the sign of \( m \), whereas the Clebsh-Gordan coefficient \( C^{l}_{2M, l\alpha} \) changes its sign together with \( m \).

Summarizing, we have demonstrated that \( I_1 + I_2 \approx 0 \), hence one can neglect the contribution of the integrals \( I_h \) in the equations of motion.

- It is necessary to analyze also the integrals with the weight \( \{ p \otimes p \}_{\lambda\mu} \):
\[
I_h' = \int d(p, r) \{ p \otimes p \}_{\lambda\mu} \left[h^{\dagger \dagger} f^{\dagger \dagger} - h^{\uparrow \downarrow} f^{\dagger \downarrow}\right] = I_{ss'} + I_{ss'}'.
\]

Again we neglect the contribution of the spin-orbital part \( I_{so} \), which generates fourth order moments. For the spin-spin contribution, we have
\[
I_{ss'} = \int d(p, r) \{ p \otimes p \}_{\lambda\mu} n^{\dagger \dagger}(r, t) f^{\dagger \dagger}(r, p, t) = \int d^3r \Pi_{\lambda\mu}^{\dagger}(r, t) n^{\dagger \dagger}(r, t), \tag{A21}
\]
where \( \Pi_{\lambda\mu}^{\dagger}(r, t) = \int \frac{d^3p}{(2\pi\hbar)^3} \{ p \otimes p \}_{\lambda\mu} f^{\dagger \dagger}(r, p, t) \) is the pressure tensor. The variation of this integral reads:
\[
\delta I_{ss'} = \int d^3r \left[ n^{\dagger \dagger}(eq) \delta \Pi_{\lambda\mu}^{\dagger}(r, t) + \Pi_{\lambda\mu}^{\dagger}(eq) \delta n^{\dagger \dagger}(r, t) \right]. \tag{A22}
\]
The pressure tensor variation is defined in appendix \( \text{[13]} \). With formula \( \text{(B6)} \) one finds for the first part of \( \text{(A22)} \):
\[
I_1' = \int d^3r n^{\dagger \dagger}(eq) \delta \Pi_{\lambda\mu}^{\dagger}(r, t) \simeq T_{\lambda\mu}^{\dagger}(t) \int d^3r n^{\dagger \dagger}(eq) n^{\dagger}(r) \simeq T_{\lambda\mu}^{\dagger}(t) n_0 \int d^3r n^{\dagger}(eq) = 0. \tag{A23}
\]
The last equality follows obviously from the definition of \( n^{\dagger} \) \( \text{(A19)} \).
The second part of \([A22]\) reads:

\[
L'_2 = \int d^3r \Pi^\uparrow_{
\lambda \mu}(eq) \delta n^\uparrow(r, t) \\
= -\sum_\beta (-1)^\beta \int d^3r \Pi^\uparrow_{\lambda \mu}(eq) \left\{ N^\uparrow_{\beta, -\beta}(t) n^+ + \sum_\gamma (-1)^\gamma N^\uparrow_{\beta, \gamma}(t) \frac{1}{r} \frac{\partial n^+}{\partial r} r_{-\beta \gamma} \right\}.
\]

(A24)

Let us consider at first the simpler part of this integral

\[
-\sum_\beta (-1)^\beta N^\uparrow_{\beta, -\beta}(t) \int d^3r \Pi^\uparrow_{\lambda \mu}(eq)n^+ (r).
\]

(A25)

The value of the last integral is determined by the angular structure of the function \(\Pi^\uparrow_{\lambda \mu}(r)\). We are interested in \(\lambda = 2, \mu = 1\). By definition

\[
\Pi^\uparrow_{21}(r) = \int \frac{d^3p}{(2\pi\hbar)} (p \otimes p)_{21} f^\uparrow(r, p) = \sum_{\nu, \sigma} C^2_{\nu, 1} \int \frac{d^3p}{(2\pi\hbar)^3} p_{\nu} p_{\sigma} f^\uparrow(r, p)
\]

\[
= 2C^2_{11, 10} \int \frac{d^3p}{(2\pi\hbar)^3} p_{\nu} p_{\sigma} f^\uparrow(r, p) = -\frac{\hbar^2}{2\sqrt{2}} [(\nabla'_1 - \nabla_1)(\nabla'_{0} - \nabla_{0}) \rho(\textbf{r}' \uparrow, \textbf{r} \downarrow) |_{r' = r} \\
- \nabla_0 \phi_k(r, \uparrow) | \nabla_1 \phi_k(r, \downarrow)] + \phi_k(r, \uparrow) | \nabla_1 \phi_k(r, \downarrow)]
\]

(A26)

with \(\phi_k\) being defined by \([A7]\). Taking into account formulae \([18]\)

\[
\nabla_{\pm 1} Y_{\lambda} = -\sqrt{\frac{(l + \lambda + 1)(l + \lambda + 2)}{2(2l + 1)(2l + 3)}} \frac{l}{r} Y_{l+1, \pm 1} - \sqrt{\frac{(l + \lambda - 1)(l + \lambda)}{2(2l - 1)(2l + 1)}} \frac{l + 1}{r} Y_{l-1, \pm 1},
\]

\[
\nabla_{0} Y_{\lambda} = -\sqrt{\frac{(l + 1)^2 - \Lambda^2}{2l(l + 1)(2l + 3)} \frac{l}{r} Y_{l+1, \Lambda} + \sqrt{\frac{l^2 - \Lambda^2}{2l(l - 1)(2l + 1)}} \frac{l + 1}{r} Y_{l-1, \Lambda}}
\]

one finds that

\[
\int d^3r \Pi^\uparrow_{\lambda \mu}(eq)n^+ (r) = \hbar^2 \sum_{n l j m} v_{n l j m} \int dr n^+ (r) R^2_{n l j} (r)(\delta_{j, l + \frac{1}{2}} - \delta_{j, l - \frac{1}{2}}) \frac{l(l + 1)[(l + \frac{1}{2})^2 - m^2]}{(2l + 3)(2l + 1)} m = 0
\]

(A27)

due to summation over \(m\). The more complicated part of the integral \([A24]\) is calculated in a similar way with the same result, hence \(I'_2 = 0\).

So, we have shown that \(I'_1 + I'_2 \simeq 0\), therefore one can neglect by the contribution of integrals \(I'_h\) (together with \(I_h\)) into equations of motion.

- And finally, just a few words about the integrals with the weight \(\{r \otimes r\}_{\lambda \mu}:

\[
I''_{s o} = \int d(p, r) \{r \otimes r\}_{\lambda \mu} \left[ h^\uparrow f^\downarrow - h^\downarrow f^\uparrow \right] = I''_{s o} + I''_{ss}.
\]

The spin-orbital part \(I''_{so}\) is neglected and for the spin-spin part we have

\[
I''_{ss} = \int d(p, r) \{r \otimes r\}_{\lambda \mu} n^\uparrow(t) f^\uparrow (r, p, t) = \int d^3r \{r \otimes r\}_{\lambda \mu} n^\uparrow (r, t) n^\uparrow (r, t).
\]

(A28)

The variation of this integral reads:

\[
\delta I''_{ss} = \int d^3r \{r \otimes r\}_{\lambda \mu} [n^\uparrow (eq) \delta n^\uparrow (r, t) + n^\uparrow (eq) \delta n^\uparrow (r, t)].
\]

(A29)
With the help of formulae (A19) and (B4) the subsequent analysis becomes quite similar to that of the integral (A16) with the same result, i.e. \( I_\mu^\nu \approx 0 \).

- The integrals \( \int d(\rho, r) W_{\lambda\mu} [h - f^{1^\dagger} - h^{\dagger} f^-] \) and \( \int d(\rho, r) W_{\lambda\mu} [h - f^{1^\dagger} - h^{\dagger} f^-] \), where \( W_{\lambda\mu} \) is any of the above mentioned weights, can be analyzed in an analogous way with the same result.

**Appendix B:**

According to the approximation suggested in [28], the variations of density, current, and pressure tensor are expanded as the series

\[
\delta n^\xi(\rho, t) = - \sum_{\beta} (-1)^\beta \nabla_\beta \left\{ n^+ (\rho) \left[ N_{\lambda\gamma}^\xi (t) + \sum_{\gamma} (-1)^\gamma N_{\beta\gamma}^\xi (t) r_{\gamma} \right] + \sum_{\lambda, \mu'} (-1)^{\lambda\mu'} N_{\lambda\gamma, \lambda'\mu'}^\xi (t) (r \otimes r) \lambda'\mu' + \ldots \right\}, \tag{B1}
\]

\[
\delta J_\beta^\xi (\rho, t) = n^+ (\rho) \left[ K_{\lambda\gamma}^\xi (t) + \sum_{\gamma} (-1)^\gamma K_{\beta\gamma}^\xi (t) r_{\gamma} + \sum_{\lambda, \mu'} (-1)^{\lambda\mu'} K_{\lambda\gamma, \lambda'\mu'}^\xi (t) (r \otimes r) \lambda'\mu' + \ldots \right], \tag{B2}
\]

\[
\delta \Pi_{\lambda\mu}^\xi (\rho, t) = n^+ (\rho) \left[ T_{\lambda\mu}^\xi (t) + \sum_{\gamma} (-1)^\gamma T_{\lambda\mu, \gamma}^\xi (t) r_{\gamma} + \sum_{\lambda, \mu'} (-1)^{\lambda\mu'} T_{\lambda\mu, \lambda'\mu'}^\xi (t) (r \otimes r) \lambda'\mu' + \ldots \right]. \tag{B3}
\]

Putting these series into the integrals (A2, A22), one discovers immediately that all terms containing expansion coefficients \( N, K, T \) with odd numbers of indices disappear due to axial symmetry. Furthermore, we truncate these series omitting all terms generating higher (than second) order moments. So, finally the following expressions are used:

\[
\delta n^\xi (\rho, t) \approx - \sum_{\beta} (-1)^\beta \nabla_\beta \left\{ n^+ (\rho) \sum_{\gamma} (-1)^\gamma N_{\beta\gamma}^\xi (t) r_{\gamma} \right\}
= - \sum_{\beta} (-1)^\beta \left\{ N_{\beta, \gamma}^\xi (t) n^+ + \sum_{\gamma} (-1)^\gamma N_{\beta, \gamma}^\xi (t) \frac{1}{r} \frac{\partial n^+}{\partial r} r_{\beta} r_{\gamma} \right\}, \tag{B4}
\]

\[
\delta J_\beta^\xi (\rho, t) \approx n^+ (\rho) \sum_{\gamma} (-1)^\gamma K_{\beta, \gamma}^\xi (t) r_{\gamma} \tag{B5}
\]

and

\[
\delta \Pi_{\lambda\mu}^\xi (\rho, t) \approx n^+ (\rho) T_{\lambda\mu}^\xi (t). \tag{B6}
\]

The coefficients \( N_{\beta, \gamma}^\xi (t) \) and \( K_{\beta, \gamma}^\xi (t) \) are connected by the linear relations with collective variables \( \mathcal{R}_{\lambda\mu}^\xi (t) \) and \( \mathcal{L}_{\lambda\mu}^\xi (t) \) respectively.

\[
\mathcal{R}_{\lambda\mu}^\xi = \int d^3 r \{ r \otimes r \lambda_{\mu} \delta n^\xi (r) \} = \frac{2}{\sqrt{3}} \left[ A_1 C_{\lambda_{1010}}^\mu N_{\mu0}^{\xi} - A_2 \left( C_{\lambda_{1111}}^\mu N_{\mu11}^{\xi} + C_{\lambda_{1111}}^\mu + C_{\lambda_{1111}}^\mu N_{\mu11}^{\xi} \right) \right], \tag{B7}
\]

where

\[
A_1 = \sqrt{2} R_{200}^{eq} - R_{000}^{eq} = \frac{Q_{00}}{\sqrt{3}} \left( 1 + \frac{4}{3} \delta \right), \quad A_2 = R_{200}^{eq} / \sqrt{2} + R_{000}^{eq} = -\frac{Q_{00}}{\sqrt{3}} \left( 1 - \frac{2}{3} \delta \right). \tag{B8}
\]
\[ R_{20} = Q_{20}/\sqrt{6}, \quad R_{00} = -Q_{00}/\sqrt{3}, \quad Q_{20} = \frac{\delta Q_{00}}{Q_{00}}, \quad Q_{00} = A < r^2 > = \frac{4}{9} A R_0^2. \]

\[ N_{\delta_{-1},-1} = -\frac{\sqrt{3} R_{-2}}{2 A_2}, \quad N_{\delta_{0},0} = \frac{\sqrt{6} R_{-2}}{4 A_1}, \quad N_{\delta_{1},1} = -\frac{R_{00} + R_{20}/\sqrt{2}}{2 A_2}, \]

\[ N_{\delta_{-1},-1} = -\frac{\sqrt{6} R_{-2}}{4 A_2}, \quad N_{\delta_{0},0} = \frac{\sqrt{2} R_{00} - R_{-2}}{2 A_1}, \quad N_{\delta_{1},1} = -\frac{\sqrt{6} R_{31}}{4 A_2}, \]

\[ N_{\delta_{-1},-1} = N_{\delta_{1},1}, \quad N_{\delta_{0},0} = \frac{\sqrt{6} R_{21}}{4 A_1}, \quad N_{\delta_{1},1} = -\frac{3 R_{22}}{2 A_2}. \] (B9)

\[ \mathcal{L}_{\lambda \mu} = \int d^3 r \{ r \otimes \delta J^r \}_\lambda \mu = \frac{1}{\sqrt{3}} (1)^\lambda \left[ A_1 C^{\lambda \mu}_{1,10} K^r_{\lambda,0} - A_2 \left( C^{\lambda \mu}_{1,1,1-1} K^r_{\lambda,1} + C^{\lambda \mu}_{1,11,1} K^r_{\lambda,1} \right) \right]. \] (B10)

\[ K_{\lambda_{-1},-1} = \frac{\sqrt{3} L_{-2}}{A_2}, \quad K_{\lambda_{0},0} = \frac{\sqrt{3} (L_{-1} - L_{-2})}{\sqrt{2} A_1}, \quad K_{\lambda_{1},1} = -\frac{\sqrt{3} L_{10} + L_{20} + \sqrt{2} L_{30}}{\sqrt{2} A_2}, \]

\[ K_{\lambda_{0},-1} = \frac{\sqrt{3} (L_{-1} - L_{-2})}{\sqrt{2} A_1}, \quad K_{\lambda_{0},0} = \frac{\sqrt{2} L_{20} - L_{00}}{A_1}, \quad K_{\lambda_{0},1} = -\frac{\sqrt{3} (L_{11} + L_{31})}{\sqrt{2} A_2}, \]

\[ K_{\lambda_{1},1} = \frac{\sqrt{3} L_{10} - L_{20} - \sqrt{2} L_{00}}{A_1}, \quad K_{\lambda_{1},0} = \frac{\sqrt{3} (L_{21} - L_{11})}{\sqrt{2} A_1}, \quad K_{\lambda_{1},1} = -\frac{\sqrt{3} L_{22}}{A_2}. \] (B11)

The coefficient \( T_{\lambda \mu}(t) \) is connected with \( \mathcal{L}_{\lambda \mu}(t) \) by the relation \( \mathcal{L}_{\lambda \mu}(t) = AT_{\lambda \mu}(t) \), \( A \) being the number of nucleons.
[28] E. B. Balbutsev, Sov. J. Part. Nucl. 22 (1991) 159.