A $T_1$ THEOREM FOR WEAKLY SINGULAR INTEGRAL OPERATORS

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Abstract. We establish conditions in the spirit of the $T_1$ theorem of David and Journé which guarantee the boundedness of $\nabla T$ on $L^p(\mathbb{R}^n)$, where $T$ is an integral transformation and $1 < p < \infty$. These are natural size and regularity conditions for the kernel of the integral transformation, along with the sharp condition $T_1^1 T_1^1 \in T^1(\text{BMO})$. A simple example satisfying these conditions is the Riesz potential denoted by $T_1^1$.

1. Introduction

The purpose of this paper is to explore boundedness properties of certain weakly singular integral operators, and to study their connections to so called almost diagonal potential operators. Given a linear integral operator

\begin{equation}
Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy,
\end{equation}

where $|K(x,y)| \leq C_K|x - y|^{1-n}$, one is often interested in conditions under which $\nabla T$ has a bounded extension to $L^p(\mathbb{R}^n)$ or to some other relevant function space. To state this more precisely, we assume that $T : S_0 \to S'/\mathcal{P}$ is continuous and that $Tf$ is the regular distribution given by (1.1) for all $f \in S_0$; this is the space of Schwartz functions with all vanishing moments and $S'/\mathcal{P}$ is its topological dual space. Then we want to know when $\partial_j T$ has the extension for all $j \in \{1, 2, \ldots, n\}$.

Weakly singular integral operators emerge in the context of pseudodifferential operators with symbols in $\dot{S}^{-1,1}_{1,1}$ [GT99]. This class of homogeneous symbols consists of smooth functions $a : \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{C}$ satisfying

$$|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \leq C_{\alpha,\beta} |\xi|^{-1-|\alpha|+|\beta|}$$

for all multi-indices $\alpha, \beta \in \mathbb{N}_0^n$. The Pseudodifferential operator $T_a$ associated with symbol $a \in \dot{S}^{-1,1}_{1,1}$ is the linear integral operator defined for $f \in S_0$ pointwise by

$$T_a f(x) = \int_{\mathbb{R}^n} a(x,\xi) \hat{f}(\xi) e^{ix\cdot\xi} d\xi.$$

It is known that such an operator has an alternative weakly singular representation with highly smooth kernel $K_a$. On the other hand, boundedness properties of $\nabla T_a$ are not automatically guaranteed. Singular integral methods are available when treating with pseudodifferential operators as $\nabla T_a$ is such an operator when the differentiation is taken under the integral sign. As a simple example, well known

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Riesz transforms, that are bounded operators on $L^p$-spaces for $p \in (1, \infty)$, can be realized in such a form:

$$\mathcal{R} = (\mathcal{R}_1, \ldots, \mathcal{R}_n) = \nabla \mathcal{I}^1.$$  

Here $\mathcal{I}^1$ stands for the first order Riesz potential, which is a weakly singular integral operator associated to the kernel $K(x, y) = c_n|x - y|^{1-n}$.

Our approach is to treat operators without first differentiating them under the integral sign. This approach is also adopted in [Tor91], where certain results about boundedness of operators with singular kernels of different sizes are available. Recent work [Väh09] of present author deals directly with boundedness properties of weakly singular integral operators on Euclidean domains, whereas the present work is oriented towards sharp connections between weakly singular integral operators and so called almost diagonal operators of Frazier and Jawerth [FJ90].

Let us next quantify certain kernel classes and weakly singular integral operators. A continuous function $K : \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, x)\} \to \mathbb{C}$ is a standard kernel of order $-1$, if $|K(x, y)| \leq C_K|x - y|^{1-n}$ for distinct $x, y \in \mathbb{R}^n$, and there exists $\delta \in (0, 1]$ such that

$$|K(x + h, y) + K(x - h, y) - 2K(x, y)| + |K(x + y + h) + K(x, y - h) - 2K(x, y)| \leq C_K|h|^{1+\delta}|x - y|^{-n-\delta}$$  

(1.2)

when $0 < |h| \leq |x - y|/2$. If $T$ as in (1.1) is associated with a standard kernel $K$ of order $-1$, we denote $T \in \text{SK}^{-1}(\delta)$, and we say that $T$ is a weakly singular integral operator (of order $-1$). As an example, $T_a \in \text{SK}^{-1}(1)$ if $a \in S_{1,1}^-$. Below is our main result. Formulation below is a rather concrete one, and we will later give formulations based on homogeneous Triebel–Lizorkin spaces.

1.3. Theorem. Assume that $T \in \text{SK}^{-1}(\delta)$. Then the following three conditions are equivalent:

- $T^1, T^1 \in \mathcal{I}^1(\text{BMO})$, where $\mathcal{I}^1$ is the first order Riesz potential;
- $\nabla T$ and $\nabla T^\dagger$ have bounded extension to $L^2$;
- $\nabla T$ and $\nabla T^\dagger$ have bounded extension to $L^p$ for all $1 < p < \infty$.

As another result we characterize operators $T \in \text{SK}^{-1}(\delta)$ satisfying the condition $T^1 = 0 = T^1$ by a slightly modified almost diagonality condition of Frazier and Jawerth [FJ90].

It is only natural that almost diagonality or almost orthogonality has a crucial role here. Corresponding decompositions of operators appeared already in the proof of the classical $T^1$ theorem of David and Journé [DJS84], and have been the standard technique to deal with non-convolution type singular integral operators on very general settings ever since [FJHW89, HS94, MC97, Tor91].

To remind the reader we state the classical $T^1$ theorem here. Let $T : \mathcal{S} \to \mathcal{S}'$ be a continuous linear operator that is associated with a kernel $K : \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, x)\} \to \mathbb{C}$ so that (1.1) holds for every $x \notin \text{supp} f$. Assume further that $K$ satisfies $|K(x, y)| \leq C_K|x - y|^{-n}$ for distinct $x, y \in \mathbb{R}^n$, and there is $\delta \in (0, 1]$ such that

$$|K(x + h, y) - K(x, y)| + |K(x, y + h) - K(x, y)| \leq C_K|h|^{\delta}|x - y|^{-n-\delta}$$

whenever $0 < |h| \leq |x - y|/2$. Then we say that $T$ is associated with a standard kernel and denote this by $T \in \text{SK}(\delta)$. Here is the classical $T^1$ theorem [DJS84]:

1.4. Theorem. Let $T \in \text{SK}(\delta)$. Then the following two conditions are equivalent:

- $T^1, T^1 \in \text{BMO}$ and $T$ has a certain weak boundedness property;
- $T$ has a bounded extension to $L^2$ i.e., $T$ is a Calderón–Zygmund operator.
The main issue here is the $L^2$-boundedness. The $L^p$-boundedness for $1 < p < \infty$ follows from the $L^2$-boundedness and this was known before the celebrated result of David and Journé. The additional weak boundedness property is required because the kernel is in some sense too singular near the diagonal $\{(x, x)\} \subset \mathbb{R}^n \times \mathbb{R}^n$ and therefore does not specify the operator $T$ uniquely. This undesired phenomenon is ruled out in the context of standard kernels of order $-1$.

Here is the organization of this paper. In Section 2 we review homogeneous Triebel–Lizorkin spaces along with the $\varphi$-transform. The almost diagonality is also presented. Section 3 contains definitions of potential operators and associated kernels. Section 4 and Section 5 are devoted to the special $T1$ theorem ($T1 = 0 = T^t1$) and to its converse: such special operators correspond to almost diagonal potential operators on sequence spaces. Section 6 deals with the full $T1$ theorem ($T1, T^t1 \in \mathcal{I}(\text{BMO})$). We prove that this condition is not only sufficient but also necessary for the boundedness properties of $T$.

2. Preliminaries

2.1. Notation. We denote $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and $\mathbb{N} = \{1, 2, \ldots\}$. If $a, b \in \mathbb{R}$, then we denote $a \lor b = \max\{a, b\}$ and $a \land b = \min\{a, b\}$. We work on the Euclidean space $\mathbb{R}^n$ with $n \geq 2$. We denote $X = X(\mathbb{R}^n)$ and usually omit the “$\mathbb{R}^n$” whenever $X$ is a function space, whose members are defined on $\mathbb{R}^n$.

Let $\nu \in \mathbb{Z}$, $y \in \mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{C}$. Then we denote $f_\nu(x) = 2^{\nu n} f(2^n x)$, $\tau_y f(x) = f(x-y)$, and $\hat{f}(x) = f(-x)$ for all $x \in \mathbb{R}^n$. The second order difference is denoted by $\Delta_y f = \tau_{-y} f + \tau_y f - 2f$. The Fourier transform is defined by

$$\hat{f}(\xi) = \mathcal{F}(f)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-ix\xi} dx$$

whenever $f \in L^1$.

The Schwartz class of rapidly decaying smooth functions, equipped with its usual topology, is denoted by $\mathcal{S}$. Its topological dual space is the space of tempered distributions $\mathcal{S}'$. Let $\mathcal{S}_0$ be the space of all Schwartz functions $\varphi \in \mathcal{S}$ such that $\partial^\alpha \varphi(0) = 0$ for all multi-indices $\alpha \in \mathbb{N}_0^n$. Then $\mathcal{S}_0$ is a closed subspace of $\mathcal{S}$. If $\mathcal{S}_0'$ denotes the topological dual space of $\mathcal{S}_0$, then $\mathcal{S}_0'$ and $\mathcal{S}'/\mathcal{P}$ are isomorphic [Tri83 5.1.2]. The Riesz $s$-potential, $s \in \mathbb{R}$, is defined by

$$\mathcal{I}^s(\varphi) = \mathcal{F}^{-1}(|\xi|^{-s}\hat{\varphi}).$$

These linear operators map $\mathcal{S}_0$ to itself continuously and extend to $\mathcal{S}'/\mathcal{P}$ by duality: if $\Lambda \in \mathcal{S}'/\mathcal{P}$ then $\mathcal{I}^s(\Lambda) \in \mathcal{S}'/\mathcal{P}$, where $\langle \mathcal{I}^s(\Lambda), \varphi \rangle = \langle \Lambda, \mathcal{I}^s(\varphi) \rangle$. For $0 < s < n$ holds the integral formulation

$$\mathcal{I}^s(\varphi)(x) = C_{n,s} \int_{\mathbb{R}^n} \frac{\varphi(y)}{|x-y|^{n-s}} dy$$

if $\varphi \in \mathcal{S}_0$.

The parameters $\nu \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$ define a dyadic cube

$$Q_{\nu k} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : k_i \leq 2^\nu x_i < k_i + 1 \text{ for } i = 1, \ldots, n\}.$$ Denote the “lower left-corner” $2^{-\nu} k$ of $Q = Q_{\nu k}$ by $x_Q$ and the sidelength $2^{-\nu}$ by $l(Q)$. Denote by $\mathcal{D}$ the set of all dyadic cubes in $\mathbb{R}^n$ and by $\mathcal{D}_0 \subset \mathcal{D}$ the set of all dyadic cubes with sidelength $2^{-\nu}$. Given a function $f \in L^2$, we denote

$$f_Q(x) = |Q|^{-1/2} f(2^{\nu} x - k) = |Q|^{1/2} f_\nu(x - x_Q).$$
2.2. Homogeneous Triebel–Lizorkin spaces. For the convenience of the reader we provide some details about homogeneous Triebel–Lizorkin spaces. These details are mostly based on [FJ90]. For further properties about these spaces, see the fundamental reference [Tri83].

First of all, we need the Littlewood–Paley functions:

2.1. Definition. A function \( \varphi \in \mathcal{S}_0 \) is a Littlewood–Paley function if it satisfies the following properties:
- \( \hat{\varphi} \) is a real valued radial function;
- \( \text{supp} \hat{\varphi} \subseteq \{ \xi \in \mathbb{R}^n : 1/2 < |\xi| < 2 \} \);
- \( |\hat{\varphi}(\xi)| \geq c > 0 \) if \( 3/5 \leq |\xi| \leq 5/3 \).

The function \( \psi \) defined by \( \hat{\psi} = \hat{\varphi}/\rho, \rho(\xi) = \sum_{\nu \in \mathbb{Z}} (\hat{\varphi}(2^\nu \xi))^2 \), is a Littlewood–Paley dual function related to \( \varphi \). It is a Littlewood–Paley function itself and we have the identity
\[
\sum_{\nu \in \mathbb{Z}} \hat{\varphi}(2^{-\nu} \xi) \hat{\psi}(2^{-\nu} \xi) = 1, \quad \text{if } \xi \neq 0.
\]

2.2. Definition. Fix a Littlewood–Paley function \( \varphi \). Let \( 1 \leq p, q \leq \infty \) and \( \alpha \in \mathbb{R} \). The homogeneous Triebel–Lizorkin space \( \dot{F}^{\alpha q}_p \) is the normed vector space of all \( f \in \mathcal{S}'/\mathcal{P} \) such that
\[
||f||_{\dot{F}^{\alpha q}_p} = \left( \left( \sum_{\nu \in \mathbb{Z}} \left( 2^{\nu \alpha} |\varphi_\nu * f| \right)^q \right)^{1/q} \right)_{L_p} < \infty,
\]
if \( p < \infty \), and
\[
||f||_{\dot{F}^{\alpha q}_p} = \sup_{P \in \mathcal{D}} \left( \frac{1}{|P|} \int_P \sum_{\nu = -\log_2 l(P)}^\infty \left( 2^{\nu \alpha} |\varphi_\nu * f(x)| \right)^q dx \right)^{1/q} < \infty.
\]
If \( 1 \leq p < q = \infty \), we use sup-norms instead of \( l^q \)-norms. Define also \( \dot{F}^{\alpha \infty}_\infty = \sup_{\nu \in \mathbb{Z}} 2^{\nu \alpha} ||\varphi_\nu * f||_{L^\infty} \).

The definition of homogeneous Triebel–Lizorkin spaces does not depend on the choice of Littlewood–Paley function \( \varphi \); for a proof, see [FJ90, Remark 2.6.]. An easy consequence of this is the following:

2.3. Theorem. Let \( \alpha, s \in \mathbb{R} \) and \( 1 \leq p, q \leq \infty \). Then \( \mathcal{I}^s \) is an isomorphism of \( \dot{F}^{\alpha q}_p \) onto \( \dot{F}^{s+\alpha q}_p \).

The homogeneous Triebel–Lizorkin spaces cover many interesting function spaces; here \( \approx \) indicates that the corresponding norms are equivalent:
- \( \dot{F}^{02}_p \approx L^p \), where \( 1 < p < \infty \);
- \( \dot{F}^{12}_p \approx H^1 \);
- \( \dot{F}^{\alpha \infty}_\infty \approx \text{BMO} \);
- \( \dot{F}^{\alpha q}_p \approx W^{\alpha,p} \), where \( 1 < p < \infty \).

Here \( W^{\alpha,p} \) is the homogeneous Sobolev space defined as the space of all \( f \in \mathcal{S}'/\mathcal{P} \) with \( \mathcal{I}^{-\alpha} f \in L^p \). This space is normed with \( ||f||_{W^{\alpha,p}} = ||\mathcal{I}^{-\alpha} f||_p \). Using the Riesz transforms, we have \( ||f||_{W^{1,p}} \approx ||\nabla f||_p \) for all \( 1 < p < \infty \). Detailed proofs or further references of these identifications can be found in [Gra04].
2.3. **Sequence spaces and the \( \varphi \)-transform.** Here we define the sequence spaces \( \hat{f}_{pq}^a \) and \( \varphi \)-transform following still the treatment in [FJ90].

2.4. **Definition.** Let \( \alpha \in \mathbb{R} \), \( 1 \leq p, q \leq \infty \), and \( \hat{\chi}_Q = |Q|^{-1/2}\chi_Q \). The sequence space \( \hat{f}_{pq}^a \) is the collection of all complex valued sequences \( s = \{s_Q\}_{Q \in \mathcal{D}} \) such that

\[
||s||_{\hat{f}_{pq}^a} = \left( \left\| \left( \sum_{Q \in \mathcal{D}} (|Q|^{-\alpha/n}|s_Q|\hat{\chi}_Q)^q \right)^{1/q} \right\|_{L_p} \right)^{1/q} < \infty,
\]

if \( 1 \leq p, q < \infty \). If \( 1 \leq p < q = \infty \), we use sup-norm instead of \( l^q \)-norm. Also,

\[
||s||_{\hat{f}_{pq}^a} = \sup_{P \in \mathcal{D}} \left( \frac{1}{|P|} \sum_{Q \subset P} (|Q|^{-\alpha/n-1/2+1/q}|s_Q|)^q \right)^{1/q},
\]

if \( 1 \leq q < \infty \). Finally we define \( ||s||_{\hat{f}_{pq}^{\infty}} = \sup_{Q \in \mathcal{D}} \{ |Q|^{-\alpha/n-1/2}|s_Q| \} \).

2.5. **Remark.** If \( 1 \leq p < \infty \) and \( s \in \hat{f}_{pq}^a \), then the series \( \sum_Q s_Q \psi_Q \) converges unconditionally in the weak* topology of \( \mathcal{S}'/\mathcal{P} \); for a proof, see [Kyr03, pp. 152–154]. If also \( 1 \leq q < \infty \), then sequences with finite support are dense in \( \hat{f}_{pq}^a \).

Then we define the \( \varphi \)-transform and its left inverse.

2.6. **Definition.** For a Littlewood–Paley function \( \varphi \), the \( \varphi \)-transform \( S_\varphi \) is the operator taking each \( f \in \mathcal{S}'/\mathcal{P} \) to the sequence \( S_\varphi f = \{(S_\varphi f)_Q = \langle f, \varphi_Q \rangle : Q \in \mathcal{D}\} \). Let \( \psi = \psi_\varphi \) be a dual Littlewood–Paley function with respect to \( \varphi \). Then the left inverse \( \varphi \)-transform \( T_\psi \) is the operator taking the sequence \( s = \{s_Q\}_{Q \in \mathcal{D}} \) to \( T_\psi s = \sum_Q s_Q \psi_Q \).

The technical definition of the summation associated with \( T_\psi \) is given by the following theorem. Its detailed proof can be found in [FJS5].

2.7. **Theorem.** Suppose that \( \varphi \) and \( \psi = \psi_\varphi \) is a dual pair of Littlewood–Paley functions. Assume that \( f \in \mathcal{S}'/\mathcal{P} \). Then for all \( \gamma \in \mathcal{S}_0 \) we have

\[
\langle f, \gamma \rangle = \sum_{\nu \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} \langle f, \varphi_{\nu k} \rangle \langle \psi_{\nu k}, \gamma \rangle.
\]

Here the inner sum converges absolutely for any fixed \( \nu \). In particular, we recover the \( \varphi \)-transform identity: \( I = T_\psi \circ S_\varphi \) on \( \mathcal{S}'/\mathcal{P} \); that is,

\[
f = \sum_{\nu \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} \langle f, \varphi_{\nu k} \rangle \psi_{\nu k},
\]

with convergence in the weak* topology of \( \mathcal{S}'/\mathcal{P} \).

Relying on the \( \varphi \)-transform identity and on maximal operators, Frazier and Jawerth prove the following technical result in [FJ90] Theorem 2.2., Theorem 5.2.:

2.8. **Theorem.** Let \( \alpha \in \mathbb{R} \) and \( 1 \leq p, q \leq \infty \). Then operators \( S_\varphi : \hat{f}_{pq}^a \to \hat{f}_{pq}^a \) and \( T_\psi : \hat{f}_{pq}^a \to \hat{f}_{pq}^a \) are bounded and \( T_\psi \circ S_\varphi \) is the identity on \( \hat{f}_{pq}^a \).

2.4. **Almost diagonal potential operators.** We continue our recapitulation of [FJ90] and move to almost diagonal operators. But there is a slight modification: we are interested in potential operators and this is reflected in the following definitions and results. Let \( \varepsilon > 0 \), \( P, Q \in \mathcal{D} \), and denote

\[
\omega_{P,Q}(\varepsilon) = \frac{(l(P) \land l(Q))^{1+(n+\varepsilon)/2}}{(l(P) \lor l(Q))^{(n+\varepsilon)/2}} \left( 1 + \frac{|x_P - x_Q|}{l(P) \lor l(Q)} \right)^{-(n+\varepsilon)}.
\]
2.9. **Definition.** Let $T : S_0 \to S'/\mathcal{P}$ be a continuous linear operator whose transpose $T^t : S_0 \to S'/\mathcal{P}$ is also continuous; the transpose is defined by the dual operation $\langle Tf, g \rangle = \langle f, T^t g \rangle$. Assume also that there exists $\varepsilon > 0$ such that

$$
\sup \left\{ \frac{|\langle T(\psi_P), \varphi_Q \rangle|}{\omega_PQ(\varepsilon)} : P, Q \in \mathcal{D} \right\} < \infty.
$$

Then $T$ is an *almost diagonal potential operator* and we denote $T \in \text{ADP}(\varepsilon)$.

2.10. **Remark.** Note that $\text{ADP}(\varepsilon)$ is a vector space and that $\text{ADP}(\varepsilon) \subset \text{ADP}(\varepsilon')$ if $\varepsilon \geq \varepsilon'$. Note that, a priori, any given dual pair of Littlewood–Paley functions $\varphi$ and $\psi_\varphi$ may appear in the definition of $\text{ADP}(\varepsilon)$. However, later it turns out that the union $\bigcup_{\varepsilon > 0} \text{ADP}(\varepsilon)$ does not depend on the chosen dual pair of Littlewood–Paley functions.

Almost diagonal potential operators are bounded operators $\hat{F}_p^{\alpha q} \to \hat{F}_p^{1+\alpha,q}$ for $1 \leq p, q < \infty$ and $-1 \leq \alpha \leq 0$. This follows from the following lemma, whose proof is a straightforward modification of [FJ90, pp. 54–55].

2.11. **Lemma.** Assume $-1 \leq \alpha \leq 0$, $1 \leq p, q < \infty$, $T \in \text{ADP}(\varepsilon)$, and $f \in S_0$. Define for every dyadic cube $Q \in \mathcal{D}$

$$
(A(S_\varphi f))_Q = \sum_{P \in \mathcal{D}} \langle T(\psi_P), \varphi_Q \rangle (S_\varphi f)_P.
$$

This sum converges absolutely. Moreover, the operator $A$ is the discrete representative of $T$ in the sense that $T = T_\psi \circ A \circ S_\varphi$ and we have the estimate $\|A(S_\varphi f)\|_{\hat{F}_p^{1+\alpha,q}} \leq C\|S_\varphi f\|_{F_p^{\alpha q}}$.

As a consequence, we have the following:

2.12. **Corollary.** Let $-1 \leq \alpha \leq 0$, $1 \leq p, q < \infty$, and $T \in \text{ADP}(\varepsilon)$. Then the operators $T$ and $T^t$ have unique bounded extensions $\hat{F}_p^{\alpha q} \to \hat{F}_p^{1+\alpha,q}$.

## 3. Potential Operators and Associated Standard Kernels

All the required machinery is now at our disposal and we begin with the main theme of this paper, namely with first order potential operators associated with standard kernels of order $-1$. First of all, we use the following terminology corresponding to Calderón–Zygmund operators in the Introduction:

3.1. **Definition.** We say that a linear operator $T : S_0 \to S'/\mathcal{P}$ is a potential operator if it has a bounded extension $\hat{W}^{\alpha,2} \to \hat{W}^{1+\alpha,2}$ for all $-1 \leq \alpha \leq 0$.

Next we define the classes $\text{SK}^{-1}(\delta)$, already mentioned in the Introduction. Their role resembles that of the standard kernel classes $\text{SK}(\delta)$ from the Introduction:

3.2. **Definition.** Let $T : S_0 \to S'/\mathcal{P}$ be a linear operator such that

$$
Tf(x) = \int_{\mathbb{R}^n} K(x, y) f(y) dy.
$$

Assume that the kernel $K : \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, x)\} \to \mathbb{C}$ is continuous and satisfies the following conditions for some $0 < \delta \leq 1$ and whenever $|h| \leq |x - y|/2$

\begin{align}
|K(x, y)| &\leq C_K |x - y|^{1-n}, \\
|K(x + h, y) + K(x - h, y) - 2K(x, y)| &\leq C_K \frac{|h|^{1+\delta}}{|x - y|^{n+\delta}}, \\
|K(x, y + h) + K(x, y - h) - 2K(x, y)| &\leq C_K \frac{|h|^{1+\delta}}{|x - y|^{n+\delta}}.
\end{align}
Then T is associated with a standard kernel K of order \(-1\). We denote this by \(T \in \text{SK}^{-1}(\delta)\).

3.6. **Remark.** Note that \(Tf\) is well defined for \(f \in S_0\) because of property (3.3). Also, \(\text{SK}^{-1}(\delta)\) is a vector space and \(\text{SK}^{-1}(\delta) \subset \text{SK}^{-1}(\delta')\) if \(\delta \geq \delta'\). The transpose \(T^t\) of an operator \(T \in \text{SK}^{-1}(\delta)\) is defined by \(\langle T^t f, g \rangle = \langle f, Tg \rangle\). It is associated with the kernel \(K^t : (x, y) \mapsto K(y, x)\), which also possesses the properties (3.3)–(3.5). The transpose satisfies \(T^t \in \text{SK}^{-1}(\delta')\).

Let us also clarify the terminology concerning “T1”:

3.7. **Definition.** Fix a Littlewood–Paley function \(\varphi\). Let \(T \in \text{SK}^{-1}(\delta)\), \(f \in S'/P\), and denote \(n^j(x) = \varphi(x/2^j)/\varphi(0)\). Assume that for all dyadic cubes \(Q \in D\) we have \(\lim_{j \to \infty} \langle T(n^j), \varphi_Q \rangle = \langle f, \varphi_Q \rangle\) and the same with \(\varphi\) replaced in the brackets by the dual function \(\psi = \psi_{\varphi}\). Then we denote \(T1 = f\).

3.8. **Remark.** If \(T1\) exists, then Theorem 2.7 implies that it is uniquely defined in \(S'/P\) with respect to a fixed dual pair of Littlewood–Paley functions. On the other hand, the definition of \(T1\) seems at first sight to depend on the Littlewood–Paley function \(\varphi\) and its dual function \(\psi\). However, we will not address this dependence unless explicitly needed. Instead, we prefer to work with a fixed dual pair of Littlewood–Paley functions.

The central question we shall address in this paper is: Under what conditions \(T \in \text{SK}^{-1}(\delta)\) is a potential operator? It turns out that this is equivalent to the requirement \(T1, T^41 \in \mathfrak{F}_\infty^{12}\).

Let us discuss the implications of condition (3.3) occurring in the definition of standard kernels of order \(-1\). It also allows the operator \(T \in \text{SK}^{-1}(\delta)\) to inherit all the properties of Riesz potential that involve no cancellation but size only. To be more precise, let \(1 \leq p < q < \infty\) satisfy \(1/p - 1/q = 1/n\). Then it is well known [Gra04, pp. 415–416] that for all \(f \in S_0\) and \(x \in \mathbb{R}^n\), we have

\[
|Tf(x)| \leq C_K T^1(|f|)(x) \leq C_{K,n,p} M(f)(x)^{p/q} ||f||_p^{1-p/q}.
\]

This maximal inequality allows us to conclude that \(T\) and \(T^4\) are continuous \(S_0 \to S'/P\).

Another consequence of (3.3) is the following uniqueness result:

3.10. **Proposition.** Let \(T \in \text{SK}^{-1}(\delta)\) be associated with standard kernel K of order \(-1\). If \(Tf = 0\) in \(S'/P\) for all \(f \in S_0\), then \(K(x, y) = 0\) for every \((x, y) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, x)\}\).

**Proof.** Let \(Q \in D\). Then \(\varphi_Q = 0\) in \(S'/P\). Therefore there exists a polynomial \(P_Q\) such that \(T\varphi_Q(x) = P_Q(x)\) for almost every \(x \in \mathbb{R}^n\). Applying (3.3) and estimate (3.9), we have for almost every \(x \in \mathbb{R}^n\)

\[
|P_Q(x)| \leq C_K \int_{\mathbb{R}^n} \frac{\varphi_Q(y)}{|x - y|^{n-1}} dy \leq C_{K,n,p} M\varphi_Q(x)^{(n-1)/n} ||\varphi_Q||_1^{1/n} \in L^{2/(n dx)}.
\]

Thus \(P_Q \equiv 0\). Denote \(A = \bigcup_{Q \in D} \{x : T\varphi_Q(x) \neq 0\}\). Then \(m_n(A) = 0\). Fix \(x_A \in \mathbb{R}^n \setminus A\). Then \(T\varphi_Q(x_A) = \int_{\mathbb{R}^n} K(x_A, y)\varphi_Q(y) dy = 0\) for every \(Q \in D\). Using Theorem 2.7, we see that \(y \mapsto K(x_A, y)\) induces a tempered distribution; thus \(K(x_a, y) = P_{x_A}(y)\) for every \(y \in \mathbb{R}^n \setminus \{x_A\}\), where \(P_{x_A}\) is a polynomial depending on \(x_A\). Using (3.3) again, we have \(|P_{x_A}(y)| \leq C_K |x_A - y|^{1-n} \forall y \in \mathbb{R}^n \setminus \{x_A\}\). Therefore \(P_{x_A} \equiv 0\) and \(K(x_A, y) = 0\) for every \(y \in \mathbb{R}^n \setminus \{x_A\}\). Because \(m_n(A) = 0\) and \(K\) is continuous, we see that \(K \equiv 0\).
3.1. Motivation for further conditions. Recall the homogeneous pseudodifferential operators in the Introduction, or see [GT99] for more details. These operators are of interest to us mainly because $T_a \in \mathcal{SK}^{-1}(1)$ if $a \in \dot{S}^{-1}_{1,1}$; see [Ste93] p. 271 for the proof of a similar relation. These operators provide us motivation for further conditions: we present a construction of a symbol $a \in \dot{S}^{-1}_{1,1}$ such that $T_a$ has no bounded extension $\hat{F}_2^{-12} \rightarrow \hat{F}_2^{02}$. This construction is a small modification of [Ste93] pp. 272–273. 

Fix a Littlewood–Paley function $\varphi$ with the additional properties 

$$\mathcal{F}(\varphi_{-1})|B(e_1, \varepsilon) \equiv 0 \equiv \mathcal{F}(\varphi_1)|B(e_1, \varepsilon)$$

and $\hat{\varphi}|B(e_1, \varepsilon) \equiv 1$ for some $\varepsilon > 0$ (here $e_1$ is the first base vector on $\mathbb{R}^n$). Define then a symbol as follows 

$$a(x, \xi) = \sum_{\nu \in \mathbb{Z}} 2^{-\nu} e^{-i2^\nu e_1 \cdot x} \hat{\varphi}(2^{-\nu} \xi).$$

It is easy to verify that $a \in \dot{S}^{-1}_{1,1}$ and that for $f \in \mathcal{S}_0$ we have the representation 

$$T_0 f(x) = (2\pi)^{-n/2} \sum_{\nu \in \mathbb{Z}} 2^{-\nu} e^{-i2^\nu e_1 \cdot x} (\varphi_{\nu} \ast f)(x).$$

For the proof of the following unboundedness result see the references above. Note, however, that $T_a$ is bounded operator $\hat{F}_2^{0,2} \rightarrow \hat{F}_2^{12}$, see Theorem 1.1 [GT99].

3.11. Proposition. Let $a \in \dot{S}^{-1}_{1,1}$ be as above. Then the operator $T_a \in \mathcal{SK}^{-1}(1)$ has no bounded extension $\hat{F}_2^{-1,2} \rightarrow \hat{F}_2^{02}$ and it is not a potential operator.

4. A special $T1$ theorem

As indicated above, we need further conditions on the operator $T \in \mathcal{SK}^{-1}(\delta)$ to ensure that it is a potential operator. In this section we pose the cancellation conditions $T1 = 0 = T^t1$; these are automatically satisfied by convolution type operators whose standard kernel, of order $-1$, is of the form $K(x, y) = k(x - y)$ for some $k : \mathbb{R}^n \rightarrow \mathbb{C}$. Later these conditions are relaxed with the aid of this special case and paraproducts, which constitute a prime example of potential operators.

We prove the special $T1$ theorem (where $T1 = 0 = T^t1$) using the theory of almost diagonal potential operators. This involves an almost diagonality estimate and this is the very essence of this section in the form of a tedious but elementary computation performed in Lemma 4.1 and in Theorem 4.10. This is a typical almost diagonality estimate; for instance it resembles the wavelet proof for the classical $T1$ theorem, see [MC97] pp. 51–57

4.1. Lemma. Let $T \in \mathcal{SK}^{-1}(\delta)$ and $Q = Q_{bk}$. Then for all $x \in \mathbb{R}^n$, we have 

$$|T(\psi_Q)(x)| + |T^t(\varphi_Q)(x)| \leq C_{K, \psi, \psi}\|Q\|^{-1/2} \left( 1 + \frac{|x - x_Q|}{l(Q)} \right)^{-{(n+\delta)}}.$$

Proof. Let us prove the following inequalities 

$$|T(\psi_Q)(x)| \leq C_{K, \psi}\|Q\|^{-1/2},$$

$$|T^t(\varphi_Q)(x)| \leq C_{K, \psi}\|Q\|^{-1/2} (l(Q)^{-1}|x - x_Q|)^{-(n+\delta)}.$$

The claim concerning $T(\psi_Q)$ follows from the inequalities above. Similar proof shows that the estimate holds also for $T^t(\varphi_Q)$. Let us first prove the inequality (4.2); using
the maximal inequality (3.9) with $p = 1$ it is obtained as follows

$$|T(\psi_Q)(x)| \leq C_K M(\psi_Q)(x)^{(n-1)/n}||\psi_Q||_{1/n}^{1/n}$$
$$\leq C_K |Q|^{-(n-1)/2n+1/2n}||\psi||_{\infty}^{(n-1)/n}||\psi||_{1/n}^{1/n} = C_{K,\psi}|Q||Q|^{-1/2}.$$  

Then we prove the inequality (4.3). We can assume that $x \neq x_Q$. Since $\psi$ is radial and its integral vanishes, we obtain

$$|T(\psi_Q)(x)| = |Q|^{-1/2} \left| \int K(x, y) \psi(2^\nu y - 2^\nu x_Q)dy \right|$$
$$= |Q|^{-1/2} \left| \int K(x, x_Q + \omega) \psi(2^\nu \omega)d\omega \right|$$
$$\leq |Q|^{-1/2} \left| \int (K(x, x_Q + \omega) + K(x, x_Q - \omega) - 2K(x, x_Q)) \psi(2^\nu \omega)d\omega \right|$$
$$\leq |Q|^{-1/2} \int |K(x, x_Q + \omega) + K(x, x_Q - \omega) - 2K(x, x_Q)| |\psi(2^\nu \omega)|d\omega.$$  

Denote $\sigma = x - x_Q$ and $A = \{\omega : |\omega| \leq |\sigma|/2\}$, $B = \{\omega : |\sigma - \omega| < |\sigma|/2\}$, $C = \{\omega : |\sigma + \omega| < |\sigma|/2\}$, $D = \mathbb{R}^n \setminus (A \cup B \cup C)$. It suffices to show the desired upper bound while integrating with respect to each of the sets above.

Set $A$. Using the property (3.3), we obtain

$$\int_A |K(x, x_Q + \omega) + K(x, x_Q - \omega) - 2K(x, x_Q)| |\psi(2^\nu \omega)|d\omega$$
$$\leq C_K |\sigma|^{-(n+\delta)} \int_{\mathbb{R}^n} |\omega|^{1+\delta} |\psi(2^\nu \omega)|d\omega$$
$$\leq C_K |\sigma|^{-(n+\delta)} 2^{-\nu(n+1+\delta)} \int_{\mathbb{R}^n} |\rho|^{1+\delta} |\psi(\rho)|d\rho$$
$$\leq C_{K,\psi}|Q|(I(Q)^{-1}|x - x_Q|)^{-(n+\delta)}.$$  

Set $B$.

$$\int_B |K(x, x_Q + \omega) + K(x, x_Q - \omega) - 2K(x, x_Q)| |\psi(2^\nu \omega)|d\omega$$
$$\leq \int_B (|K(x, x_Q + \omega)| + |K(x, x_Q - \omega)| + |2K(x, x_Q)|) |\psi(2^\nu \omega)|d\omega.$$  

Two latter summands are dealt with like the set $D$. Therefore it suffices to estimate the first summand using the property (3.3) as follows

$$\int_B |K(x, x_Q + \omega)||\psi(2^\nu \omega)|d\omega$$
$$\leq C_{K,\psi} \int_B |\sigma - \omega|^{1-n} |2^\nu \omega|^{-(n+1+\delta)}d\omega$$
$$\leq C_{K,\psi} 2^{-\nu(n+1+\delta)} |\sigma|^{-(n+1+\delta)} \int_B |\sigma - \omega|^{1-n}d\omega$$
$$= C_{K,\psi} 2^{-\nu(n+1+\delta)} |\sigma|^{-(n+\delta)} = C_{K,\psi}|Q|(I(Q)^{-1}|x - x_Q|)^{-(n+\delta)}.$$
Set $C$. This is estimated as the integral over the set $B$.

Set $D$. Using the property (3.3), we obtain

\[
\int_{D} |K(x, x_Q + \omega) + K(x, x_Q - \omega) - 2K(x, x_Q)| \psi(2^\nu \omega) d\omega
\]

\[
\leq C_K \int_{D} |\sigma|^{1-n} |\psi(2^\nu \omega)| d\omega
\]

\[
= C_K |\sigma|^{1-n} \int_{D} |\omega|^{-(1+\delta)} |\omega|^{1+\delta} |\psi(2^\nu \omega)| d\omega
\]

\[
\leq C_K |\sigma|^{-(n+\delta)} \int_{\mathbb{R}^n} |\omega|^{1+\delta} |\psi(2^\nu \omega)| d\omega
\]

\[
\leq C_K, \psi l(Q) (l(Q)^{-1} |x - x_Q|)^{-(n+\delta)}.
\]

The last inequality is dealt with like the set $A$. The estimate (4.3) follows. \qed

4.4. **Remark.** Let us now make the following remark regarding the Definition 3.7 of $T_1$. Assume that $T \in \text{SK}^{-1}(\delta)$. Then transposing and using Lemma 4.1, we get the identities $\lim_{j \to \infty} \langle T(\eta^j), \varphi_Q \rangle = \int T^t(\varphi_Q)$ and similarly for $\psi_Q$’s. However, in many cases it is easier to work with this regularization given by the family $\{\eta^j : j \in \mathbb{N}\}$ and therefore we have embedded it already in the definition of $T_1$.

The following lemma is a special case of [FJ90, Lemma B.1]. Note the small misprint in the original formulation where the inequality “$j \geq k$” should be replaced by the inequality “$j \leq k$”.

4.5. **Lemma.** Let $\nu, \mu \in \mathbb{Z}$, $\nu \leq \mu$, $x_1 \in \mathbb{R}^n$, $\delta > 0$, and $g, h \in L^1$ be such that

\[
|g(x)| \leq C_g 2^{\nu n/2} (1 + 2^\nu |x|)^{-(n+\delta)},
\]

\[
|g(x) - g(y)| \leq C_g 2^{(n/2+\delta/2)} |x - y|^\delta \sup_{|z| \leq |x-y|} (1 + 2^\nu |z - x|)^{-(n+\delta)},
\]

\[
|h(x)| \leq C_h 2^{\mu n/2} (1 + 2^\mu |x - x_1|)^{-(n+\delta)},
\]

\[
\int_{\mathbb{R}^n} h(x) dx = 0.
\]

Then $|g \ast h(x)| \leq C_{g,h,\delta} 2^{-(\mu-\nu)(n/2+\delta/2)} (1 + 2^\nu |x - x_1|)^{-(n+\delta)}$.

The following special $T_1$ theorem is one of our main results. We make a remark concerning the vanishing integrals assumption after the proof.

4.10. **Theorem.** Let $T \in \text{SK}^{-1}(\delta)$ be such that for every dyadic cube $Q \in \mathcal{D}$ we have

\[
\int_{\mathbb{R}^n} T(\psi_Q) = \int_{\mathbb{R}^n} T^t(\varphi_Q) = 0.
\]

Then $T \in \text{ADP}(\delta)$; in particular, the integral operators $T$ and $T^t$ have bounded extensions $\hat{T}^o_p \to \hat{F}^{1+\alpha,q}_p$ for all $1 \leq p, q < \infty$ and $-1 \leq \alpha \leq 0$.

**Proof.** The boundedness results follow from Corollary 2.12 after we have verified the claim concerning the almost diagonality. For this purpose let $P = P_{\mu,\nu}$ and $Q = Q_{\nu,k}$. We do case study with respect to the relative order of $l(P)$ and $l(Q)$.

Case $l(P) = 2^{-\mu} \leq 2^{-\nu} = l(Q)$. It suffices to show that

\[
|\langle T(\psi_P), \varphi_Q \rangle| \leq C_{T,\psi,\nu} \frac{l(P)^{1+(n+\delta)/2}}{l(Q)^{(n+\delta)/2}} \left(1 + \frac{|x_P - x_Q|}{l(Q)}\right)^{-(n+\delta)}.
\]
Applying Lemma 4.11 we obtain

\[ |T(\psi_P)(x)| \leq C_{K,\psi}l(P)2^{n/2}(1 + 2^\mu|x - x_P|)^{-(n + \delta)}. \]

Denote \( h(x) = l(P)^{-1}T(\psi_P)(x) \) and \( g(x) = \varphi_Q(x_Q - x) \). Then

\[ |g(x)| = |Q|^{1/2} |\varphi_{\nu}(-x)| = 2^{-\nu/2}2^{\nu |\varphi(-2^\nu x)|} \leq C_\varphi 2^{\nu/2}(1 + 2^\nu|x|)^{-(n + \delta)}, \]

and also, since \( 0 < \delta \leq 1 \), we have

\[ |g(x) - g(y)| = 2^{\nu/2}|\varphi(-2^\nu x) - \varphi(-2^\nu y)| \leq C_\varphi 2^{\nu/2}|2^\nu(x - y)|^{\delta/2} \sup_{|z| \leq 2^\nu|x - y|} (1 + |z - 2^\nu x|)^{-(n + \delta)} \]

\[ = C_\varphi 2^{\nu(n/2 + \delta/2)}|x - y|^{\delta/2} \sup_{|\omega| \leq 2^\nu|\omega - x|} (1 + 2^\nu|\omega - x|)^{-(n + \delta)}. \]

Using these estimates and Lemma 4.3 we obtain

\[ |(T(\psi_P), \varphi_Q)| = \left| \int T(\psi_P)(x)\varphi_Q(x)\text{d}x \right| \]

\[ = l(P)\left| \int g(x_Q - x)h(x)\text{d}x \right| \]

\[ = l(P)|g * h(x_Q)| \leq C_{K,\psi,\nu,\delta}(P)2^{-(n/2)(n/2 + \delta/2)} |Q|^{2\nu/2}(1 + 2^\nu|x_Q - x_P|)^{-(n + \delta)} \]

\[ = C_{K,\psi,\nu,\delta}l(P)^{1+(n/2 + \delta/2)}/(Q)^{(n/2 + \delta/2)} \left( 1 + \frac{|x_P - x_Q|}{l(Q)} \right)^{-(n + \delta)}. \]

This is the desired estimate in this case. Case \( l(P) = 2^{-\nu} > 2^{-\nu} = l(Q) \). It suffices to show that

\[ |\langle \psi_P, T^t(\varphi_Q) \rangle| = |\langle T(\psi_P), \varphi_Q \rangle| \leq C_{T,\psi,\nu,\delta}l(P)^{1+(n/2 + \delta/2)}/(Q)^{(n/2 + \delta/2)} \left( 1 + \frac{|x_P - x_Q|}{l(P)} \right)^{-(n + \delta)}. \]

Computation here are similar to the first case and we omit the details.

4.11. Remark. If \( T \) satisfies the assumptions of Theorem 4.10, then \( T1 = 0 = T^t1 \). This is true because Theorem 4.10 implies that \( Tf, T^t f \in F^{02}_1 \approx H^1 \) for every \( f \in S_0 \subset F^{12}_1 \) and the integrals of \( H^1 \)-functions vanish. The converse is also true: if \( T \in SK^{-1}(\delta) \) is such that \( T1 = 0 = T^t1 \), then \( T \) satisfies all the assumptions of Theorem 4.10. It is also worth noting that the property \( T1 = 0 = T^t1 \) for \( T \in SK^{-1}(\delta) \) is independent of the dual pair of Littlewood–Paley functions; this also follows from the boundedness properties as above.

We conclude this section with another formulation of Theorem 4.10. Its proof uses a duality result from [FJ01] pp. 76–79 stating that \((f_{pq})' \approx f_{-pq}^\prime \) for \( 1 \leq p, q < \infty \) and \( \alpha \in \mathbb{R} \).

4.12. Corollary. Let \( T \in SK^{-1}(\delta) \) be such that \( T1 = 0 = T^t1 \). Then \( T \) has an extension as a bounded operator \( \tilde{F}^{pq}_p \rightarrow \tilde{F}^{1+n,q}_p \) for all \(-1 \leq \alpha \leq 0 \) and \( 1 \leq p, q \leq \infty \) excluding the cases \((p, q) = (1, \infty) \) and \((p, q) = (\infty, 1) \).

5. A converse to the special \( T1 \) theorem

Here we prove a converse result to the special \( T1 \) theorem by adapting computations performed in [HL03] to our context of potential operators. To be more specific, we show that if \( T \in ADP(\varepsilon) \), then \( T \in SK^{-1}(\delta) \) for suitable \( \delta \) and \( T1 = 0 = T^t1 \) (see Definition 3.7). Combining the results above we find that \( T \in ADP(\varepsilon) \) if and
only if $T \in SK^{-1}(\delta)$ and $T_1 = 0 = T'1$. Thus the role of almost diagonal potential operators is twofold: first they are used to establish the special $T1$ theorem and second they are naturally occurring examples of operators associated with standard kernels of order $-1$.

From the ADP point of view the characterization above states that $\cup_{\varepsilon > 0} ADP(\varepsilon)$ is independent of the dual pair of Littlewood–Paley functions. Here it becomes crucial that the property $T1 = 0 = T'1$ for $T \in SK^{-1}(\delta)$ is independent of the dual pair of Littlewood–Paley functions.

First we recall some notation. Let $\beta, \gamma > 0$, $P \in D_\mu$, and $Q \in D_{\nu}$. Then we denote

$$W_{P,Q}(\beta, \gamma) = 2^{-|n-\mu|(n+\gamma)/2} \left(1 + \frac{|x_P - x_Q|}{l(Q) \vee l(P)}\right)^{-(n+\beta)}.$$ 

Note that $\omega_{P,Q}(\varepsilon) = (l(P) \wedge l(Q))^2 W_{P,Q}(\varepsilon, \varepsilon)$; recall Definition 2.9.

The following lemma, yielding a control to the matrix product of two almost diagonal matrices, is a crucial tool here. Its proof can be found in [FJ90, p. 158].

5.1. Lemma. Let $P, Q \in D$. Let $\beta, \gamma_1, \gamma_2 > 0$, $\gamma_1 \neq \gamma_2$, and $\gamma_1 + \gamma_2 > 2\beta$. Then

$$\sum_{R \in D} W_{P,R}(\beta, \gamma_1) W_{R,Q}(\beta, \gamma_2) \leq C_{\beta,\gamma_1,\gamma_2} W_{P,Q}(\beta, \gamma_1 \wedge \gamma_2).$$

We also use the following homogeneity estimate for a double sum. The reader will find it easy to construct the omitted proof.

5.2. Lemma. Let $\beta > \alpha > 0$ and $\lambda, \varepsilon > 0$. Then

$$\sum_{\nu \in \mathbb{Z}} \sum_{\mu \in \mathbb{Z}} 2^{-|n-\mu|2\varepsilon}(1 + (2^\mu \wedge 2^\nu)\lambda)^{-\beta} \leq C_{\alpha,\beta,\varepsilon}\lambda^{-\alpha}.$$ 

We are ready for our converse result; recall Definitions 3.2 and 2.9.

5.3. Theorem. Suppose that the operator $T : S_0 \to S'/P$ is an almost diagonal potential operator in the sense of Definition 2.9, that is, $T \in ADP(\varepsilon)$. Then $T$ is associated with a standard kernel of order $-1$ so that for every $f \in S_0$, we have

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y)f(y)dy.$$ 

Here $K : \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, x)\} \to \mathbb{C}$ is continuous and satisfies (3.3)–(3.5) with any $\delta > 0$ such that $\delta < \varepsilon/2$ and $\delta \leq 1$; that is, $T \in SK^{-1}(\delta)$. Also, we have $T1 = 0 = T'1$ in the sense of Definition 3.7.

Proof. Let $x, y \in \mathbb{R}^n$ and denote

$$K(x, y) = \sum_{Q \in D} \sum_{P \in D} \langle T(\psi_P), \varphi_Q \rangle \varphi_P(y)\psi_Q(x).$$

We first prove the estimate (3.3) for this kernel. This proof will also show that the series above converges absolutely for any $x \neq y$. For $x, y \in \mathbb{R}^n$ with $x \neq y$, we have

$$|K(x, y)| \leq C_T \sum_{Q \in D} \sum_{P \in D} (l(P) \wedge l(Q)) W_{Q,P}(\varepsilon, \varepsilon)|\varphi_P(y)||\psi_Q(x)|.$$ 

Let $P \in D_\mu$ and $R \in D_\mu$ be such that $y \in R$. Using $|\varphi_P(y)| \leq C_{\varphi, \varepsilon}2^{\mu/2}(1 + |x_P - y|)/(l(P))^{-n-\varepsilon}$, we get

$$|\varphi_P(y)| \leq C_{\varphi, \varepsilon}2^{\mu/2}\left(1 + \frac{|x_P - x_R|}{l(P) \vee l(R)}\right)^{-n-\varepsilon} = C_{\varphi, \varepsilon}2^{\mu/2}W_{P,R}(\varepsilon, 1 + \varepsilon).$$
Using the estimates above and Lemma 5.1, we obtain for any given \( Q \in \mathcal{D}_\nu \) and fixed \( \mu \in \mathbb{Z} \) the estimates
\[
\sum_{P \in \mathcal{D}_\nu} W_{Q,P}(\varepsilon, \varepsilon)|\varphi_P(y)| \leq C_{\varphi,\varepsilon} 2^{\mu/2} \sum_{P \in \mathcal{D}} W_{Q,P}(\varepsilon, \varepsilon) W_{P,R}(\varepsilon, 1 + \varepsilon)
\]
\[
\leq C_{\varphi,\varepsilon} 2^{\mu/2} W_{Q,R}(\varepsilon, \varepsilon).
\]
Let \( Q \in \mathcal{D}_\nu \) and \( S \in \mathcal{D}_\nu \) be such that \( x \in S \); computing as above, we find that
\[
|\psi_Q(x)| \leq C_{\psi,\varepsilon} 2^{\mu/2} W_{S,Q}(\varepsilon, 1 + \varepsilon)
\]
and then
\[
2^{\mu/2} \sum_{Q \in \mathcal{D}_\nu} |\psi_Q(x)| W_{Q,R}(\varepsilon, \varepsilon) \leq C_{\psi,\varphi,\varepsilon} 2^{\mu/2 + \nu/2} W_{S,R}(\varepsilon, \varepsilon).
\]
Write \( \sum_{P \in \mathcal{D}} \sum_{Q \in \mathcal{D}} = \sum_{\nu \in \mathbb{Z}} \sum_{\mu \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_\nu} \sum_{P \in \mathcal{D}_\mu} \) and \( l(P) \land l(Q) = 2^{-(\mu \lor \nu)} \) in (5.4). Using the two estimates above and the identities \( \mu \lor \nu \leq \mu \lor \nu \) and \( \mu + \nu - |\nu - \mu| = 2(\mu \lor \nu) \), we arrive at
\[
|K(x,y)| \leq C_{T,\varphi,\psi,\varepsilon} \sum_{\nu \in \mathbb{Z}} \sum_{\mu \in \mathbb{Z}} 2^{-(\mu \lor \nu) + \mu/2 + \nu/2} W_{S,R}(\varepsilon, \varepsilon)
\]
\[
= C_{T,\varphi,\psi,\varepsilon} \sum_{\nu \in \mathbb{Z}} \sum_{\mu \in \mathbb{Z}} 2^{-(\mu \lor \nu) + \mu/2 + \nu/2} \left( 1 - \frac{|x_S - x_R|}{l(S) \lor l(R)} \right)^{-n-\varepsilon}
\]
\[
\leq C_{T,\varphi,\psi,\varepsilon} \sum_{\nu \in \mathbb{Z}} \sum_{\mu \in \mathbb{Z}} 2^{-|\nu - \mu/2|} (1 + (2^\mu \land 2^\nu) |x - y|)^{-n-\varepsilon}.
\]
Note that \( n + \varepsilon > n - 1 > 0 \) and \( \varepsilon/2 > 0 \); therefore using Lemma 5.2, we get
\[
|K(x,y)| \leq C_{T,\varphi,\psi,\varepsilon} |x - y|^{-1-n}.
\]
This is the estimate (3.3).

Then we prove (3.5) for \( K \); fix \( x, y, h \in \mathbb{R}^n \) such that \( x \neq y \) and \( |h| \leq |x - y|/2 \). Then
\[
|K(x,y + h) + K(x,y - h) - 2K(x,y)|
\]
\[
\leq C_T \sum_{Q \in \mathcal{D}} \sum_{P \in \mathcal{D}_\mu} (l(P) \land l(Q)) W_{Q,P}(\varepsilon, \varepsilon) |(\Delta_h \varphi_P)(y)| |\psi_Q(x)|.
\]
Assume first that \( P \in \mathcal{D}_\mu \) is such that \( |h| \leq 2^{-\mu} = l(P) \). If \( R \in \mathcal{D}_\mu \) is such that \( y \in R \), then a short computation shows that
\[
|(\Delta_h \varphi_P)(y)| \leq C_{\varphi,\varepsilon} 2^{\mu(n/2 + 1 + \delta)} |h|^{1+\delta} W_{P,R}(\varepsilon, 1 + \varepsilon).
\]
Let \( S \in \mathcal{D}_\nu \) be such that \( x \in S \); then computing as before and using Lemma 5.2 with assumption \( 2\delta < \varepsilon \), we find that
\[
\sum_{\nu \in \mathbb{Z}} \sum_{\mu \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_\nu} \sum_{P \in \mathcal{D}_\mu} (l(P) \land l(Q)) W_{Q,P}(\varepsilon, \varepsilon) |(\Delta_h \varphi_P)(y)| |\psi_Q(x)|
\]
\[
\leq C_{T,\varphi,\psi,\varepsilon} |h|^{1+\delta} \sum_{\nu \in \mathbb{Z}} \sum_{\mu \in \mathbb{Z}} 2^{-|\nu - \mu/2| - (\mu \land \nu)(n+\delta)} (1 + (2^\mu \land 2^\nu) |x - y|)^{-n-\varepsilon}
\]
\[
\leq C_{T,\varphi,\psi,\varepsilon} |h|^{1+\delta} |x - y|^{-n-\delta}.
\]
Assume then that $P \in \mathcal{D}_\mu$ is such that $|h| > 2^{-\mu} = l(P)$; then $1 < (2^\mu|h|)^{1+\delta}$ and therefore
\[
|\langle \Delta_h \varphi_P \rangle(y) \rangle | \leq C_{\varphi, \epsilon} 2^{\mu(2n/2+2+1+\delta)} |h|^{1+\delta} (1 + 2^\mu |y + h - x_P|)^{-n-\epsilon}
+ C_{\varphi, \epsilon} 2^{\mu(n/2+2+1+\delta)} |h|^{1+\delta} (1 + 2^\mu |y - h - x_P|)^{-n-\epsilon}
+ C_{\varphi, \epsilon} 2^{\mu(n/2+2+1+\delta)} |h|^{1+\delta} (1 + 2^\mu |y - x_P|)^{-n-\epsilon}.
\]

Computing further as in the case $|h| \leq l(P)$ we get
\[
\sum_{\nu \in \mathbb{Z}_+} \sum_{|h| > 2^{-\mu}} \sum_{Q \in \mathcal{D}_\nu} \sum_{P \in \mathcal{D}_\mu} \omega_{QP}(\epsilon; \nu) |\langle \Delta_h \varphi_P \rangle(y) ||\psi_Q(x)|
\leq C_{T, \varphi, \psi, \epsilon, \delta} |h|^{1+\delta} |x - y - h|^{-n-\epsilon} + |x - y + h|^{-n-\epsilon} + |x - y|^{-n-\epsilon}
\leq C_{T, \varphi, \psi, \epsilon, \delta} |h|^{1+\delta} |x - y|^{-n-\epsilon},
\]
because $|h| \leq |x - y|/2$. Inserting the above estimates in (5.5), we have
\[
|K(x, y + h) + K(x, y - h) - 2K(x, y)| \leq C_{T, \varphi, \psi, \epsilon, \delta} |h|^{1+\delta} |x - y|^{-n-\epsilon}.
\]
This concludes the proof of (5.5). The proof of (3.1) and for the continuity of the kernel is similar to this; in the latter use first order differences.

It remains to show that operator $T$ is associated with the standard kernel $K$ of order $-1$, and $T1 = 0 = T^*1$. Fix $f, g \in S_\eta$. Then using Theorem 2.7 and Lemma 2.11 we obtain
\[
\langle Tf, g \rangle = \sum_{Q \in \mathcal{D}} \sum_{P \in \mathcal{D}} \langle T(\psi_P), \varphi_Q \rangle \langle \varphi_P, f \rangle \langle \psi_Q, g \rangle.
\]
Using dominated convergence theorem with (3.9) and above kernel size estimate beginning from (5.4), we have
\[
\langle Tf, g \rangle = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x, y) f(y) g(x) dy dx.
\]
Thus $T$ is associated with the standard kernel $K$ of order $-1$. The conclusion $T1 = 0 = T^*1$ follows using Corollary 2.12 and the properties of $\hat{F}_0^\omega \approx H^1$. \qed

5.6. Remark. Assume that $T \in \mathcal{S}^k_k^{-1}(\delta)$ is associated with standard kernel $K$ of order $-1$, and that $T^*1 = 0 = T^*1$. Theorem 4.10 states that $T \in \text{ADP}(\delta)$ and Theorem 5.3 combined with the kernel uniqueness, Proposition 3.10 implies that we have the identity
\[
K(x, y) = \sum_{Q \in \mathcal{D}} \sum_{P \in \mathcal{D}} \langle T(\psi_P), \varphi_Q \rangle \varphi_P(y) \psi_Q(x)
\]
for every $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, x)\}$. Using this representation it is possible to prove different regularity results for the kernel $K$; as an example one can obtain a homogeneity estimate for the first order differences of the kernel $K$.

6. A FULL $T1$ THEOREM VIA PARAPRODUCTS

Here the special $T1$ theorem is generalized so that the assumption $T1 = 0 = T^*1$ is replaced by a weaker assumption $T1, T^*1 \in \hat{F}_0^{12}$. Under this weaker assumption the operator $T \in \mathcal{S}^k_k^{-1}(\delta)$ has a bounded extension $\hat{W}^{\alpha,p} \rightarrow \hat{W}^{1+\alpha,p}$ for all $1 < p < \infty$ and $-1 \leq \alpha \leq 0$.

We also obtain a sharpness result stating that if $T \in \mathcal{S}^k_k^{-1}(\delta)$ is a potential operator in the sense of Definition 3.1 then $T1, T^*1 \in \hat{F}_0^{12}$. Combining the results
above, we find that if $T \in SK^{-1}(\delta)$ is a potential operator, then it has a bounded extension $\hat{W}^{\alpha, p} \to \hat{W}^{1+\alpha, p}$ for all $-1 \leq \alpha \leq 0$ and $1 < p < \infty$.

6.1. **A full T1 theorem with a converse.** The full T1 theorem is obtained via reduction to Theorem 4.10 with the aid of potential operators $\Pi_b$ and $\Pi^b$, $b \in \hat{F}^{12}_{\infty}$, satisfying

- $\Pi_b1 = b$ and $\Pi^b1 = 0$;
- $\Pi_b$ and $\Pi^b$ are bounded $\hat{F}^{\alpha, 2}_p \to \hat{F}^{1+\alpha, 2}_p$ for all $1 < p < \infty$ and $-1 \leq \alpha \leq 0$;
- $\Pi_b, \Pi^b \in SK^{-1}(1)$.

The very end of this section is devoted to the construction of such potential operators, known as paraproduct operators. But assuming these properties for now, we can recover the full T1 theorem:

6.1. **Theorem.** Let $T \in SK^{-1}(\delta)$ and $T1, T'1 \in \hat{F}^{12}_{\infty}$. Then $T$ has an extension as a bounded operator $\hat{F}^{\alpha, 2}_p \to \hat{F}^{1+\alpha, 2}_p$ for all $1 < p < \infty$ and $-1 \leq \alpha \leq 0$.

**Proof.** Let $a = T1$, $b = T'1$, and $S = T - \Pi_a - \Pi^b$. Because $\Pi_a, \Pi^b \in SK^{-1}(1)$, we see that $S \in SK^{-1}(\delta)$. Moreover using the assumptions and the properties of paraproducts above, we have for any $Q \in D$

$$\int_{\mathbb{R}^n} S(\psi_Q)(x)dx = \lim_{j \to \infty} \langle \eta^j, S(\psi_Q) \rangle = \lim_{j \to \infty} \langle S^j(\eta^j), \psi_Q \rangle = 0.$$ 

In a similar way, we get

$$\int_{\mathbb{R}^n} S^j(\varphi_Q)(x)dx = \lim_{j \to \infty} \langle S(\eta^j), \varphi_Q \rangle = 0.$$ 

Theorem 4.10 implies that $S : \hat{F}^{\alpha, 2}_p \to \hat{F}^{1+\alpha, 2}_p$ is a bounded operator. Using the boundedness properties of $\Pi_b$, we see that $T = S + \Pi_a + \Pi^b : \hat{F}^{\alpha, 2}_p \to \hat{F}^{1+\alpha, 2}_p$ is a bounded operator.

6.2. **Remark.** Choose $b = \overline{\varphi_Q} \in \hat{F}^{12}_{\infty}$. Then $\langle b, \varphi_Q \rangle = ||\varphi||^2 \neq 0$ and therefore $\Pi_b1 \neq 0$. This shows the existence of a potential operator $\Pi_b$ not satisfying all the assumptions of Theorem 4.10.

Recall that the assumption $T1, T'1 \in BMO$ is sharp in classical T1 theorem [DJ84]. The following result describes a similar sharpness result in our context:

6.3. **Theorem.** Assume that $T \in SK^{-1}(\delta)$ is a potential operator. In other words, assume that it has a bounded extension $\hat{F}^{\alpha, 2}_2 \to \hat{F}^{1+\alpha, 2}_2$ for all $-1 \leq \alpha \leq 0$. Then $T1, T'1 \in \hat{F}^{12}_{\infty}$.

**Proof.** We show that $T1 \in \hat{F}^{12}_{\infty}$; the proof for $T'$ is completely analogous. We first prove that the family $\{T(\eta^j)\}_{j \in \mathbb{N}}$ is bounded in $\hat{F}^{12}_{\infty}$. Let $R = R_{j, 0} \in D$; that is, $l(R) = 2^j$ and $x_R = 0$. Then $\varphi(0)|R|^{-1/2}\eta^j = \varphi_R$ and using Theorem 2.8, we have

$$||T(\varphi_R)||_{\hat{F}^{12}_{\infty}} \leq C||\{T(\varphi_R), \varphi_Q\}||_{\hat{F}^{12}_{\infty}}$$

$$= C \sup_{P \in \mathcal{D}} \left( \frac{1}{|P|} \sum_{Q \in P} (|Q|^{-1/2n}|T(\varphi_R), \varphi_Q|)^2 \right)^{1/2}.$$
Fix $P \in D_{\mu}$; assume first that $|R| \leq |P|$. Then using the boundedness assumption and Theorem 2.8 we obtain the estimate

\[
\left( \frac{1}{|P|} \sum_{Q \subset P} \left( |Q|^{-1/n} |\langle T(\varphi_R), \varphi_Q \rangle| \right)^2 \right)^{1/2} \leq \| \{ \langle T(|P|^{-1/2} \varphi_R), \varphi_Q \rangle \}_{Q \in D} \|_{f^2_2} \\
\leq C \| T(|P|^{-1/2} \varphi_R) \|_{f^2_2} \leq C |P|^{-1/2} \| \varphi_R \|_{\dot{f}^2_2} \leq C |R|^{-1/2}.
\]

Assume then that $|P| \leq |R|$ and denote $B = B(x_P, 2\sqrt{n}l(P))$. Using Theorem 2.8 we have

\[
\left( \frac{1}{|P|} \sum_{Q \subset P} \left( |Q|^{-1/n} |\langle T(\varphi_R), \varphi_Q \rangle| \right)^2 \right)^{1/2} = \left( \frac{1}{|P|} \sum_{Q \subset P} \left( |Q|^{-1/n} |\langle T(\chi_B \varphi_R + (1 - \chi_B) \varphi_R), \varphi_Q \rangle| \right)^2 \right)^{1/2} \\
\leq C_{\varphi, \psi} |P|^{-1/2} \| T(\chi_B \varphi_R) \|_{F^2_2} + \left( \frac{1}{|P|} \sum_{Q \subset P} \left( |Q|^{-1/n} |\langle (1 - \chi_B) \varphi_R, T'(\varphi_Q) \rangle| \right)^2 \right)^{1/2}.
\]

First term on the right hand side is easier to estimate; using the boundedness assumption for $T$ and the estimate $\| \varphi_R \|_{\infty} \leq C_{\varphi} |R|^{-1/2}$, we get

\[
|P|^{-1/2} \| T(\chi_B \varphi_R) \|_{F^2_2} \leq C_T |P|^{-1/2} \| \chi_B \varphi_R \|_{F^2_2} \leq C_{T, \varphi} |P|^{-1/2} |R|^{-1/2} |P|^{1/2} = C_{T, \varphi} |R|^{-1/2}.
\]

Then we look at the second term. Fix a dyadic cube $Q$ with $Q \subset P$. Then using Lemma 4.1 and the estimate $\| \varphi_R \|_{\infty} \leq C_{\varphi} |R|^{-1/2}$, we have

\[
|\langle (1 - \chi_B) \varphi_R, T'(\varphi_Q) \rangle| \leq \int_{\mathbb{R}^n \setminus B} |\varphi_R(x)||T'(\varphi_Q)(x)|dx \\
\leq C_{T, \varphi, \psi} |R|^{-1/2} l(Q) |Q|^{-1/2} \int_{\mathbb{R}^n \setminus B} \left( 1 + \frac{|x - x_Q|}{l(Q)} \right)^{-(n+\delta)} dx.
\]

Note that if $x \in \mathbb{R}^n \setminus B$, then $|x - x_Q| \geq l(P)$ and therefore $1 + \frac{|x - x_Q|}{l(Q)} \geq \frac{l(P)}{l(Q)}$. Using this we can estimate the right hand side of (6.4) from above by

\[
C_{T, \varphi, \psi} |R|^{-1/2} l(Q)^{1/2} l(P)^{-\delta/2} \int_{\mathbb{R}^n} \left( 1 + \frac{|x - x_Q|}{l(Q)} \right)^{-(n+\delta/2)} dx \\
\leq C_{T, \varphi, \psi, \delta} |R|^{-1/2} |Q|^{1/2} l(Q)^{1+\delta/2} l(P)^{-\delta/2}.
\]
Denote \( l(P) = 2^{-\nu} \). Then we get the following estimate for our second term

\[
\left( \frac{1}{|P|} \sum_{Q \subset P} \left( |Q|^{-1/\alpha}|(1 - \chi_B)\varphi_R, T^d(\varphi_Q)| \right)^2 \right)^{1/2} \leq C_{T,\varphi,\psi,\delta} |R|^{-1/2} l(P)^{-\delta/2} \left( \frac{1}{|P|} \sum_{Q \subset P} |Q| l(Q)^\delta \right)^{1/2} \leq C_{T,\varphi,\psi,\delta} |R|^{-1/2} l(P)^{-\delta/2} \left( \frac{1}{|P|} \sum_{\nu=\mu}^\infty \frac{|P|}{2^{-\nu} 2^{-\nu\delta}} \right)^{1/2} \leq C_{T,\varphi,\psi,\delta} |R|^{-1/2}.
\]

Combining the uniform estimates above, we see that

\[ ||T(\varphi_R)||_{\dot{F}^1_\infty} \leq C |R|^{-1/2} \]

with \( C \) independent of \( R \). Multiplying both sides by \( |R|^{1/2}/\varphi(0) \), we see that the family \( \{T(\eta^j)\}_{j \in \mathbb{N}} \) is bounded in \( \dot{F}^1_\infty \).

Recall that \( (\dot{F}^{-1/2}_1)^* \approx \dot{F}^{1/2}_\infty \); for a proof, see [F.J90, pp. 79–80]. Denote \( X = \dot{F}^{-1/2}_1 \). Then \( X \) is separable and Banach–Alaoglu theorem [Rud91, p. 70] implies that there exists a subsequence \( \{\eta^{jk}\}_{k \in \mathbb{N}} \) such that \( \lim_{k \to \infty} (T(\eta^{jk}), \varphi_Q) = b \in \dot{F}^{1/2}_\infty \) in the weak* topology of \( \dot{F}^{1/2}_\infty \). In particular, we have \( \lim_{k \to \infty} (T(\eta^{jk}), \varphi_Q) = \langle b, \varphi_Q \rangle \) for all \( Q \in D \); on the other hand, we have for all \( Q \in D \)

\[
\lim_{j \to \infty} \langle T(\eta^j), \varphi_Q \rangle = \int T^d(\varphi_Q) = \lim_{k \to \infty} \langle T(\eta^{jk}), \varphi_Q \rangle = \langle b, \varphi_Q \rangle.
\]

The same conclusions hold true for \( \psi_Q \)'s and therefore \( T1 = b \in \dot{F}^{1/2}_\infty \). \( \square \)

Recall that the \( L^2 \)-boundedness of a Calderón–Zygmund operator implies its \( L^p \)-boundedness for \( 1 < p < \infty \); the proof is based on a weak type \((1,1)\) estimate and interpolation [Gra01, pp. 584–585]. Combining theorems 6.1 and 6.3 we immediately get the following analogous result:

**6.5. Corollary.** Assume that \( T \in \text{SK}^{-1}(\delta) \) is a potential operator. In other words, assume that it has a bounded extension \( \dot{F}_2^{1/2} \to \dot{F}_2^{1+\alpha,2} \) for all \( -1 \leq \alpha \leq 0 \). Then \( T \) has a bounded extension \( \dot{F}_2^{1+\alpha,2} \to \dot{F}_2^{1+\alpha,2} \) for all \( 1 < p < \infty \) and \( -1 \leq \alpha \leq 0 \).

**6.2. About the construction of paraproduct operators.** As seen already the paraproduct operators establish a reduction from the full \( T1 \) theorem to the special \( T1 \) theorem. Here we construct these paraproduct operators to complete the proof of the full \( T1 \) theorem, Theorem 6.1. The treatment here follows closely [Wan99] and for the convenience of the reader we provide some of the shorter details here.

**6.6. Definition.** Let \( \Phi \in \mathcal{S} \) be such that \( \text{supp} \Phi \subset B(0,1) \) and \( \hat{\Phi}(0) = 1 \). For \( b \in \dot{F}^{1/2}_\infty \) we define the paraproduct operator \( \Pi_b \) by

\[
(6.7) \quad \Pi_b(f) = \sum_{Q \in D} \langle b, \varphi_Q \rangle |Q|^{-1/2} \langle f, \Phi_Q \psi_Q \rangle,
\]

where \( f \in \mathcal{S}_0 \).
6.8. **Theorem.** The series (6.7) defining $\Pi_b(f)$ converges unconditionally in the weak* topology of $S'/\mathcal{P}$ and absolutely pointwise (and these limits coincide). Also, we have $\Pi_b \in SK^{-1}(1)$.

**Proof.** For the proof recall the notation from Section 5. Denote $\pi_{Q,Q} = \langle b, \varphi_Q \rangle |Q|^{-1/2}$ and $\pi_{P,Q} = 0$ for every $P, Q \in \mathcal{D}$ such that $P \neq Q$. It is easy to verify that $|\pi_{Q,Q}| \leq C_{b,\varphi}(l(P) \wedge l(Q))W_{P,Q}(3,3)$.

Denote for $x \neq y$

$$K(x, y) := \sum_{Q \in \mathcal{D}} \langle b, \varphi_Q \rangle |Q|^{-1/2} \Phi_Q(y) \psi_Q(x) = \sum_{Q \in \mathcal{D}} \sum_{P \in \mathcal{D}} \pi_{Q,P} \Phi_P(y) \psi_Q(x).$$

Computing as in the proof of Theorem 5.3 we see that $K : \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, x)\} \to \mathbb{C}$ is continuous and it satisfies properties (3.3)–(3.5) with $\delta = 1$. Actually the same proof shows that, for any given $x \in \mathbb{R}^n$, we have the estimate

$$\sum_{Q \in \mathcal{D}} |\langle b, \varphi_Q \rangle| |Q|^{-1/2} |\langle f, \Phi_Q \rangle| |\psi_Q(x)|$$

$$\leq \int_{\mathbb{R}^n} \sum_{Q \in \mathcal{D}} |\langle b, \varphi_Q \rangle| |Q|^{-1/2} |\Phi_Q(y)| |\psi_Q(x)| |f(y)| \, dy$$

$$\leq C \int_{\mathbb{R}^n} \frac{|f(y)|}{|x - y|^{n-1}} \, dy \leq C |f|_{\infty}^{(n-1)/n} |f|_1^{1/n} < \infty;$$

the second last inequality follows from the maximal inequality (3.9). Using dominated convergence we see that $\Pi_b$ is associated with kernel $K$. Also we see that the series (6.7) converges absolutely pointwise and unconditionally in $S'/\mathcal{P}$. \( \square \)

6.9. **Theorem.** Let $b \in \dot{F}_{12}^{\infty}$. Then $\Pi_b 1 = b$ and $\Pi_b^\dagger 1 = 0$.

**Proof.** Recall that $\eta^j(x) = \varphi(x/2^j)/\varphi(0)$ and fix $P \in \mathcal{D}_p$. Then we use the inequality

$$\sup_{Q \in \mathcal{D}} |Q|^{-1/2} |\langle \eta^j, \Phi_Q \rangle| < \infty,$$

both conclusions of Theorem 2.7 and Theorem 6.8 in order to obtain

$$\lim_{j \to \infty} \langle \Pi_b(\eta^j), \varphi_P \rangle = \lim_{j \to \infty} \sum_{\nu = -1}^{\mu + 1} \sum_{k \in \mathbb{Z}^n} \langle b, \varphi_{Q^j_k} \rangle |Q^j_k|^{-1/2} |\eta^j, \Phi_{Q^j_k} \rangle |\psi_{Q^j_k}, \varphi_P \rangle$$

$$= \sum_{\nu = -1}^{\mu + 1} \sum_{k \in \mathbb{Z}^n} \langle b, \varphi_{Q^j_k} \rangle |\psi_{Q^j_k}, \varphi_P \rangle = \langle b, \varphi_P \rangle.$$

On the other hand, we have

$$\lim_{j \to \infty} \langle \Pi_b^\dagger(\eta^j), \varphi_P \rangle = \lim_{j \to \infty} \sum_{\nu \geq \mu - 3} \sum_{k \in \mathbb{Z}^n} \langle b, \varphi_{Q^j_k} \rangle |Q^j_k|^{-1/2} |\varphi_P, \Phi_{Q^j_k} \rangle |\psi_{Q^j_k}, \eta^j \rangle = 0$$

since $\langle \psi_{Q^j_k}, \eta^j \rangle = 0$ for $j \geq 5 - \mu$ and $\nu \geq \mu - 3$. These combined with similar computations for $\psi$ imply the desired conclusion. \( \square \)

The $\dot{F}_p^{\alpha_2} \to \dot{F}_p^{1+\alpha,2}$ boundedness of the paraproduct can be reduced to the boundedness of certain matrix on the corresponding sequence spaces. This matrix factors as a product of two matrices $T_c$ and $G$, whose boundedness properties can then be established separately. This program is carried out in detail by [Wan99], see Corollary 3.3., Lemma 3.4., and the proof of Theorem 4.1.(a) therein.
6.10. **Theorem.** Let $b \in \dot{F}^{12}_{\infty}$. If $1 < p < \infty$ and $\alpha \leq 0$, then the paraproduct operator $\Pi_b$ has a bounded extension $\dot{F}^{\alpha 2}_p \to \dot{F}^{1+\alpha, 2}_p$.

Due to [FJ90, pp. 76–79] we have $(\dot{f}^{\beta q}_p)' \approx \dot{f}^{-\beta q}_p$ for $\beta \in \mathbb{R}$, $1 \leq p < \infty$, and $1 \leq q < \infty$. This duality combined with the previous theorem gives us:

6.11. **Corollary.** Let $b \in \dot{F}^{12}_{\infty}$. If $1 < p < \infty$ and $\beta \geq -1$, then the transposed paraproduct operator $\Pi^t_b$ has a bounded extension $\dot{F}^{\beta 2}_p \to \dot{F}^{1+\beta, 2}_p$.

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