Edge states for the Kalmeyer-Laughlin wave function

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We study lattice wave functions obtained from the SU(2)1 Wess-Zumino-Witten conformal field theory. Following Moore and Read’s construction, the Kalmeyer-Laughlin fractional quantum Hall state is defined as a correlation function of primary fields. By an additional insertion of Kac-Moody currents, we associate a wave function to each state of the conformal field theory. These wave functions span the complete Hilbert space of the lattice system. On the cylinder, we study global properties of the lattice states analytically and correlation functions numerically using a Metropolis Monte Carlo method. By comparing short-range bulk correlations, numerical evidence is provided that the states with one current operator represent edge states in the thermodynamic limit. We show that the edge states with one Kac-Moody current of lowest order have a good overlap with low-energy excited states of a local Hamiltonian, for which the Kalmeyer-Laughlin state approximates the ground state. For some states, exact parent Hamiltonians are derived on the cylinder. These Hamiltonians are SU(2) invariant and nonlocal with up to four-body interactions.

PACS numbers: 71.27.+a, 73.43.-f, 11.25.Hf

I. INTRODUCTION

The discovery of the fractional quantum Hall (FQH) effect1 has revealed the physical existence of a new, strongly correlated state of matter. One surprising property of FQH phases is the fact that they cannot be classified in terms of symmetries.2 This is a fundamental difference to most other known states of matter and led to the concept of classifying states in terms of topological order.3,4 Topological phases are not characterized by a local order parameter but their properties depend on the topology of the system. For example, the ground state degeneracy of a FQH system was shown to depend on the genus of the surface on which the state is defined.2

A central characteristic of topological order is the appearance of gapless edge states. Such edge modes are known to be present in a FQH system5 even though the bulk is gapped. It was shown that these edge states can be used to characterize the topological order of a FQH state.6 Furthermore, they are of particular physical importance because they determine the transport properties of the system. The strongly correlated state formed by the edge excitations of a FQH system is that of a chiral Luttinger liquid and its theoretical properties are related to Kac-Moody algebras.7 For continuous systems, edge state wave functions were constructed as correlation functions of a conformal field theory (CFT)8,9 and in terms of Jack polynomials.10

The use of 1 + 1 dimensional CFT in describing FQH states was pioneered by Moore and Read.11 In this approach, a trial wave function for the FQH state is given as the correlation function of conformal primary fields. Laughlin’s wave function for the continuum12 as well as its SU(2) symmetric, bosonic lattice version, the Kalmeyer-Laughlin (KL) state,13,14 were shown to be of this form15,16,17. It was already conjectured by Moore and Read13 that the FQH edge modes should be described by the same CFT that defines the bulk wave function. The idea of this bulk-edge correspondence was later developed into an extension of Moore and Read’s method10,11. Within this approach, trial FQH edge states are obtained from CFT descendant states.

This paper is based on a similar ansatz for edge states in lattice systems. Our starting point is the SU(2)1 Wess-Zumino-Witten (WZW) theory, which was used previously18,19 to obtain the KL state as a correlation function of primary fields. By insertion of Kac-Moody currents, we define a tower of states, which corresponds to the CFT descendant states. Since we work in a spin formulation, the Hilbert space of the lattice system is that of N spin-1/2 degrees of freedom, where occupied sites are represented as spin-up and empty sites as spin-down.

We show that the mapping from CFT states to spin states is surjective, i.e. any state of the spin system can be written as a linear combination of states constructed from the CFT. As a consequence, not all states obtained in this way are edge states. For some of the wave functions, we carry out numerical calculations to test if they describe edge states: On the cylinder, we compare spin correlation functions in the states with one current operator to the KL state using a Metropolis Monte Carlo algorithm.20,21 We show that the nearest-neighbor bulk correlations approach each other exponentially as the thermodynamic limit is taken. This indicates that the states with one current operator indeed describe edge states.

In the past years, parent Hamiltonians of the KL state14,21 and its non-Abelian generalizations21,22 were constructed. It was also shown that the KL state10 and non-Abelian FQH lattice states20,22 have a good overlap with the ground states of certain local Hamiltonians. For one of these Hamiltonians, an implemen-
II. CFT MODEL AND SPIN-$\frac{1}{2}$ STATES

In this section, we review some properties of the SU(2)$_1$ Wess-Zumino-Witten (WZW) theory and define the correspondence between states of the CFT and states of a spin-$\frac{1}{2}$ system on the lattice. It is shown that this map of CFT states to lattice spin states is surjective.

A. SU(2)$_1$ Wess-Zumino-Witten theory

We consider the chiral sector of the SU(2)$_1$ WZW theory. In addition to conformal invariance, this theory exhibits an SU(2) symmetry generated by the currents $J^a(z)$ with $a \in \{x, y, z\}$. The modes $J^a_n$ are defined in terms of the Laurent expansion

\[ J^a(z) = \sum_{n=-\infty}^{\infty} J^a_n z^{-n-1}, \]
\[ J^a_n = \oint \frac{dz}{2\pi i} z^n J^a(z), \]

and satisfy the Kac-Moody algebra

\[ [J^m_n, J^h_n] = i\varepsilon_{abc} J^f_{m+n} + \frac{m}{2} \delta_{ab} \delta_{m+n,0}. \]

Here, $\varepsilon_{abc}$ is the Levi-Civita symbol and $\delta_{ab}$ the Kronecker delta. For indices $c \in \{x, y, z\}$, we adopt the convention that indices occurring twice are summed over, unless explicitly stated otherwise.

The SU(2)$_1$ WZW theory has two primary fields, the identity with conformal weight $h = 0$, and a two-component field $\phi_s(z)$ ($s = \pm 1$) with conformal weight $h = \frac{1}{4}$. The field $\phi_s(z)$ provides a spin-$\frac{1}{2}$ irreducible representation through its operator product expansion (OPE) with the SU(2) current

\[ J^a(z) \phi_s(w) \sim -\frac{1}{z-w} \sum_{s'=\pm 1} t^a_{ss'} \phi_s(w), \]

where $t^a = \frac{s}{2}$ are the SU(2) spin operators.

The field content of the SU(2)$_1$ WZW theory can be represented in terms of the chiral part $\varphi(z)$ of a free boson field as

\[ \phi_s(z) = e^{i\pi(q-1)(s+1)/2} e^{i\varphi(z)/\sqrt{2}}, \]
\[ J^z(z) = -\frac{i}{\sqrt{2}} \partial_z \varphi(z), \]
\[ J^\pm(z) = J^x(z) \pm i J^y(z) = e^{i\pi(q-1)} : e^{i\sqrt{2}\varphi(z)} :. \]

Here, $q = 0$ if the operators act on a state with an even number of modes of the $h = \frac{1}{4}$ primary field and $q = 1$ otherwise, and the colons denote normal ordering. The value of $s \in \{-1, 1\}$ equals two times the spin-$z$ eigenvalue of $\phi_s(z)$.

A general state in the identity sector of the CFT Hilbert space is a linear combination of states

\[ (J^{a_1}_{-n_1} \ldots J^{a_{s1}}_{-n_{s1}})(0)|0\rangle, \]

where $|0\rangle$ is the CFT vacuum and $n_i$ are positive integers. The sum $k = \sum_{i=1}^{s_1} n_i$ defines the level of the state. By means of the Kac-Moody algebra of Eq. (4), a basis can be chosen for which the mode numbers are ordered: $n_1 \geq n_{l-1} \geq \cdots \geq n_1 > 0$.

B. Spin states on the lattice

To each CFT state $(J^{a_1}_{-n_1} \ldots J^{a_{s1}}_{-n_{s1}})(0)|0\rangle$, we associate a state

\[ |\psi^{a_1 \ldots a_{s1}}_{n_1 \ldots n_{s1}}(s_1, \ldots, s_N)\rangle = \sum_{s_1, \ldots, s_N} \psi^{a_1 \ldots a_{s1}}_{n_1 \ldots n_{s1}}(s_1, \ldots, s_N)|s_1, \ldots, s_N\rangle \]

in the Hilbert space of a system of $N$ spin-$\frac{1}{2}$ degrees of freedom. Its spin wave function is defined as the CFT correlator

\[ \psi^{a_1 \ldots a_{s1}}_{n_1 \ldots n_{s1}}(s_1, \ldots, s_N) = \langle \phi_{s_1}(z_1) \ldots \phi_{s_N}(z_N)(J^{a_1}_{-n_1} \ldots J^{a_{s1}}_{-n_{s1}})(0)|0\rangle, \]

where $\langle \ldots \rangle$ denotes the expectation value of radially ordered operators in the CFT vacuum.

In the sum of Eq. (6), $s_i = \pm 1$ and $|s_1, \ldots, s_N\rangle$ is the tensor product of eigenstates $|s_i\rangle$ of the spin operator $t^i$ at position $i$ ($t^i|s_i\rangle = \frac{s_i}{2}|s_i\rangle$). The complex coordinates $z_i$ are parameters of the wave function $\psi^{a_1 \ldots a_{s1}}_{n_1 \ldots n_{s1}}(s_1, \ldots, s_N)$ and define a lattice of positions in the complex plane. Since we want to keep them fixed, we do not explicitly indicate the parametric dependence...
of \(\psi_{n_1,\ldots,n_l}(s_1,\ldots,s_N)\) on the positions \(z_i\) for simplicity of notation.

The wave function corresponding to the CFT vacuum is given by

\[
\psi_0(s_1,\ldots,s_N) \equiv \langle \phi s_1(z_1) \ldots \phi s_N(z_n) \rangle
\]

\[
= \delta_s \chi_s \prod_{1 < j} (z_i - z_j)^{s_is_j/2},
\]

where \(\delta_s\) is 1 if \(\sum_{1}^{N} s_i = 0\) and 0 otherwise. \(\chi_s\) denotes the Marshall sign factor,

\[
\chi_s = \prod_{q=1}^{N} e^{i\pi(q-1)(s_q+1)/2}.
\]

It follows from the alternating sign in the definition of \(\phi_s(z)\) [cf. Eq. (4)]. Due to the presence of the Marshall sign factor, the state \(\psi_0\) is a singlet of the total spin \(s\).

The states \(\psi_{n_1,\ldots,n_l}\) can be related to \(\psi_0\) by an application of spin operators \(^{33}\). In Eq. (7), one can write the modes \(J_{a}^{n_1}l_{a}^{n_1}\) as an integral according to Eq. (11), deform the integral contour, and apply the OPE \(^{3}\) between current operators and primary fields. Defining

\[
u_{k}^{a} \equiv \sum_{i=1}^{N} (z_i)^{k} l_{i}^{a}\]

it then follows that

\[
|\psi_{n_1,\ldots,n_l}^{a_1,\ldots,a_l}\rangle = u_{n_1}^{a_1} \ldots u_{n_l}^{a_l} |\psi_0\rangle.
\]

We note that the operators \(u_{k}^{a}\) defined in Eq. (11) satisfy

\[
[u_{m}^{a}, u_{n}^{b}] = i\epsilon_{abc} u_{m+n}^{c},
\]

which is a Kac-Moody algebra with vanishing central extension.

In addition to the states \(\psi_{n_1,\ldots,n_l}^{a_1,\ldots,a_l}\), we also consider the wave functions obtained by the insertion of two additional primary fields, one at \(z_0 = 0\) and one at \(z_\infty = \infty:\

\psi_{0,\infty}^{a_1,\ldots,a_l}(s_1,\ldots,s_N) \equiv \langle \phi s_\infty(\infty) \phi s_1(z_1) \ldots \phi s_N(z_N) \rangle\]

\[
= \delta_s \chi_s (-1)^{(1-s_\infty)/2} \prod_{n=1}^{N} z_{n}^{s_{ns_n}/2} \prod_{n<m} (z_n - z_m)^{s ns_m/2}.
\]

where \(\delta_s\) is 1 if \(s_0 + s_\infty + \sum_{j=1}^{N} s_j = 0\) and 0 otherwise. The state \(\psi_{0,\infty}^{a_1,\ldots,a_l}\) contains a singlet and a triplet obtained by the tensor product decomposition of the two additional spin-\(1/2\) fields. As we show in Sec. III B, the singlet component on the cylinder can be derived from the wave function of \(N\) primary fields on a torus in the limit where the torus becomes a cylinder.

We note that the states \(\psi_0, \psi_{n_1,\ldots,n_l}^{a_1,\ldots,a_l}\), and \(\psi_{0,\infty}^{a_1,\ldots,a_l}\) of Eqs. (7, 8), and (14) are not normalized. Whenever the norm of a state is needed, it will be explicitly included in the corresponding expression.

C. Completeness of spin states

One may ask if the linear transformation that maps CFT states \((J_{a}^{n_1}l_{a}^{n_1})(0)|0\rangle\) to spin states \(\psi_{n_1,\ldots,n_l}^{a_1,\ldots,a_l}\) is surjective, i.e. whether any state in the Hilbert space \(\mathcal{H}_{N}\) of \(N\) spin-\(1/2\) particles can be written as a linear combination of the states \(\psi_{n_1,\ldots,n_l}^{a_1,\ldots,a_l}\). We now show that this is indeed the case. Introducing the \(N \times N\) matrix

\[
Z = \begin{pmatrix}
1 & \cdots & 1 \\
(z_1)^{-1} & \cdots & (z_N)^{-1} \\
(z_1)^{-2} & \cdots & (z_N)^{-2} \\
\vdots & \ddots & \vdots \\
(z_1)^{-(N-1)} & \cdots & (z_N)^{-(N-1)}
\end{pmatrix},
\]

the definition of Eq. (11) becomes

\[
\begin{pmatrix}
u_0^{a} \\
u_1^{a} \\
\vdots \\
u_{N-1}^{a}
\end{pmatrix} = \begin{pmatrix}t_1^{a} \\
t_2^{a} \\
\vdots \\
t_{N}^{a}
\end{pmatrix}.
\]

The determinant of \(Z\) is the well known Vandermonde determinant:

\[
\det(Z) = \prod_{i<j} (z_j^{-1} - z_i^{-1}).
\]

\(Z\) is therefore nonsingular if all positions \(z_i\) are distinct, which we assume to be the case. As a consequence, the relation of Eq. (16) can be inverted, i.e. all spin operators \(t_j^{a}\) can be written as linear combinations of \(u_n^{a}\) with \(n \in \{0,\ldots,N-1\}\).

The Hilbert space \(\mathcal{H}_{N}\) is spanned by the states obtained from any nonzero state by successive application of spin operators \(t_j^{a}\). Given that \(t_j^{a}\) can be expressed in terms of \(u_n^{a}\) it follows in particular that the states \(|\psi_{n_1,\ldots,n_l}^{a_1,\ldots,a_l}\rangle = u_{n_1}^{a_1} \ldots u_{n_l}^{a_l} |\psi_0\rangle\) with \(n_i \in \{0,\ldots,N-1\}\) span \(\mathcal{H}_{N}\). Since \(\psi_0\) is a singlet and \(u_0 = \sum_{j=1}^{N} t_j^{a}\) is the total spin, a state

\[
u_{0}^{a} u_{n_1}^{a_1} \ldots u_{n_l}^{a_l} |\psi_0\rangle
\]

can be written in terms of states for which all mode numbers are greater than zero by commuting \(u_0^{a}\) to the right.
until it annihilates $\psi_0$ [cf. the commutator of Eq. (13)]. Therefore, $\mathcal{H}_N$ is spanned by the states $\psi_{n_1\ldots n_l}^{a_1}$ with $n_i > 0$.

This argument shows that not all states constructed by the insertion of current operators can be edge states compared to $\psi_0$. For the states with one current operator $J_{-n}^z$, numerical evidence will be given in Sec. III C that these represent edge states on the cylinder.

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The fact that the states $\psi_{n_1\ldots n_l}$ span $\mathcal{H}_N$ raises the question about the minimal level $k = \sum_{i=1}^{l} n_i$ needed to obtain the complete Hilbert space. We note that an upper bound is given by $k = N(N-1)$, since any product of spin operators $\mathcal{T}_i^a$ can be reduced to a product of at most $N$ spin operators. Each of these spin operators can then be expanded in terms of the operators $u^a_n$ with $n \in \{0, \ldots, N-1\}$. We carried out numerical calculations for the states $\psi_{n_1\ldots n_l}$ which indicate that the states up to level $k = (N/2)^2$ are enough to obtain the complete Hilbert space from $\psi_0$.

III. PROPERTIES OF STATES ON THE CYLINDER

In this section, we study properties of the states defined in Sec. III B on the cylinder.

We consider a square lattice with $N_x$ lattice sites in the open direction and $N_y$ lattice sites in the periodical direction of the cylinder. After mapping the cylinder to the complex plane, the coordinates assume the form

$$z_j = e^{2\pi i (j_x+i-j_y_N)} e^{-2\pi i N_y \frac{n_z+1}{2}}.$$

(19)

Here, $j_x \in \{1, \ldots, N_x\}$ is the $x$-component of the index and $j_y \in \{1, \ldots, N_y\}$ is the $y$-component, so that $j = (j_x-1)N_y + j_y$ ranges from 1 to $N = N_x N_y$. For the remainder of this paper, we adopt a two-index notation, where this is convenient, i.e. we may write $z_{j_x,j_y}$ instead of $z_j$, denoting the $x$- and $y$-components by subscripts.

Note that one has the freedom to rescale the coordinates since this changes the wave functions only by a total factor. The constant factor that we included in Eq. (19) is chosen such that the center of the cylinder is at the unit circle.

We assume that the number of sites $N$ is even. It is also possible to study the case of $N$ being odd which will show the existence of two topological sectors. However, we can already identify the two anyonic sectors for $N$ even. As will be shown below in Sec. III B the state $\psi_0$ and the singlet component $\psi_0^{sgl}$ of the state with two additional spins (one at $z_0 = 0$ and one at $z_\infty = \infty$) can be obtained from the wave function of $N$ primary fields on the torus in the limit where the torus becomes a cylinder. This argumentation shows that the two states $\psi_0$ and $\psi_0^{sgl}$ represent the two anyonic sectors in the case of an even number of spins. It would be possible to consider an odd number of sites on the cylinder by putting an additional spin either at $z_0 = 0$ or at $z_\infty = \infty$ so that the charge neutrality condition is satisfied. On the torus, however, such a construction is not possible and the argumentation that we used to identify the two sectors for even $N$ does not directly apply.

A. Global transformation properties of CFT states

In this subsection, we study global transformation properties of the states $\psi_0$, $\psi_{n_1\ldots n_l}$, and $\psi_0^{q_0,s_\infty}$. This serves two purposes: First, it allows us to conclude that states with a different momentum are orthogonal, i.e. they have different global properties. Later, we will study their local behavior numerically and compare spin correlation functions in the bulk. The symmetries derived in this subsection will be exploited in our numerical calculations to obtain efficient Monte Carlo estimates.

We consider the translation operator in the periodical direction $T_y$ and the inversion operator $I$. Their precise definition and the derivation of their action on the states $\psi_0$, $\psi_{n_1\ldots n_l}$, and $\psi_0^{q_0,s_\infty}$ are given in Appendix A. Geometrically, the translation operator rotates the system in the periodical direction and the inversion operator corresponds to a reflection of the cylinder along its two central cross sections. We call it an inversion because it acts on the coordinates defined in Eq. (19) as $z_i \rightarrow z_i^{-1}$.

Eigenstates of $T_y$ and $I$ are given in Table 1. As we show in Appendix A, applying the inversion $I$ to $\psi_{n_1\ldots n_l}$ corresponds to inserting the current operators $a_{j_x,j_y}$ instead of $z_0 = 0$. We use the notation $\psi_{n_1\ldots n_l}^{a_1}$ for these states:

$$\psi_{n_1\ldots n_l}^{a_1} (s_1, \ldots, s_N) \equiv (T_n^a_1 \ldots T_n^a_l \phi_{s_1}(z_1) \ldots \phi_{s_N}(z_1)) = (u_{s_1}^a_1 \ldots u_{s_N}^a_l \psi_0)(s_1, \ldots, s_N).$$

(20)

Since the momentum in the periodical direction $P_y$ is related to $T_y$ through the relation $T_y = e^{iP_y}$, we conclude from Table 1 that an additional insertion of a current operator $J_{-n}^z$ into the correlation function of primary fields adds a momentum of $-2\pi n/N_y$ to the state. In particular, the states $\psi_0$ and $\psi_{n_1\ldots n_l}$ have a different momentum if $k = \sum_{j=1}^{l} n_j$ is different from 0 modulo $N_y$.

B. Relation to the KL states on the torus

In this subsection, we place the system on the torus and take a limit in which the torus becomes a cylinder. We show that the wave function of $N$ primary fields on the torus gives rise to $\psi_0$ and the singlet component $\psi_0^{sgl}$ of $\psi_0^{q_0,s_\infty}$ on the cylinder.

We define the torus for $\omega_1 > 0$ and $\omega_2 = iL$ with $L > 0$ by identifying a complex number $z$ with $z + m\omega_1 + n\omega_2$ for $m, n \in \mathbb{Z}$. The two circumferences of the torus are therefore given by $\omega_1$ and $|\omega_2|$. Let us denote the positions on the torus by $v_i$, i.e. we assume that $v_i$ lie in the rectangle spanned by $\omega_1$ and $\omega_2$. Keeping the positions $v_i$ fixed


| Eigenstate | \( \mathcal{T}_\varphi \) | \( \mathcal{I} \) |
|------------|----------------|----------------|
| \( \psi_0 \) | \((-1)^{N_x} \frac{\omega}{2\pi} \) | \((-1)^{N_y} \frac{\omega}{2\pi} \) |
| \( \psi_{a_1\ldots a_N} \) | \((-1)^{N_x} \frac{\omega}{2\pi} e^{\frac{2\pi i a_1}{N_y}} \) | \((-1)^{N_y} \frac{\omega}{2\pi} \) |
| \( \psi_{n_1\ldots n_N} \) | \(-1 \) | \((-1)^{N_x} \frac{\omega}{2\pi} \) |
| \( \psi_{a_1\ldots a_N} \pm \psi_{-a_1\ldots -a_N} \) | \(-1 \) | \((-1)^{N_y} \frac{\omega}{2\pi} \) |
| \( \psi_{\downarrow} \) | \(-1 \) | \((-1)^{N_y} \frac{\omega}{2\pi} \) |
| \( \psi_{\downarrow} \) | \(-1 \) | \((-1)^{N_y} \frac{\omega}{2\pi} \) |
| \( \psi_{\uparrow} \) | \(-1 \) | \((-1)^{N_y} \frac{\omega}{2\pi} \) |
| \( \psi_{\uparrow} \) | \(-1 \) | \((-1)^{N_y} \frac{\omega}{2\pi} \) |

and taking the circumference \( L = |\omega_2| \to \infty \) transforms the torus into a cylinder, as illustrated in Fig. 1.

![Image of a torus](image.png)

**Fig. 1.** (Color online) Limit in which the torus becomes a cylinder: The circumference \( |\omega_2| \) is taken to infinity while the positions lie in the finite region of size \( N_x \times N_y \) (red patch).

On the torus, there are two states \( \psi_k^{\text{torus}} \) with \( k \in \{0, \frac{1}{2}\} \). These are given by\(^{19}\)

\[
\psi_k^{\text{torus}}(s_1, \ldots, s_N) = \langle \phi_{s_1} (v_1) \ldots \phi_{s_N} (v_N) \rangle_k
\]

\[
\propto \delta_s \chi_s \left[ \prod_{i<j} \left( \sum_{j=1}^{N} \zeta_i s_i, 2\tau \right) \prod_{i=1}^{N} \left( \prod_{j=1}^{N} \left( \zeta_i - \zeta_j, \tau \right) \right) \right]^{s_i s_j / 2}.
\]

(21)

Here, \( \zeta_i = v_i / \omega_1 \) are the rescaled positions, \( \tau = \omega_2 / \omega_1 \) is the modular parameter of the torus, and \( \theta \) the Riemann theta function defined as

\[
\theta \left[ \begin{array}{c} a \\ b \end{array} \right] (\zeta, \tau) = \sum_{n \in \mathbb{Z}} e^{i \pi (n+a)^2 + 2\pi i (n+a)(\zeta+b)}.
\]

(22)

The limit \( \omega_2 \to i\infty \), which transforms the torus into a cylinder, implies \( \tau \to i\infty \). In this case, only the terms with the smallest value of \( (n+a)^2 \) contribute to the sum of Eq. (22). These terms have \( n = 0 \) for \( a = 0 \) and \( n \in \{-1, 0\} \) for \( a = \frac{1}{2} \). Therefore,

\[
\theta \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] (\zeta, \tau) \to 1,
\]

\[
\theta \left[ \begin{array}{c} 1/2 \\ 0 \end{array} \right] (\zeta, \tau) \to e^{\frac{\pi i}{\omega} (e^{i\pi \zeta} + e^{-i\pi \zeta})},
\]

and

\[
\theta \left[ \begin{array}{c} 1/2 \\ 1/2 \end{array} \right] (\zeta_j - \zeta_j, \tau) \to i e^{-i\pi (\zeta_i + \zeta_j)} e^{\frac{\pi i}{\omega} (e^{2\pi i \zeta_i} - e^{2\pi i \zeta_j})},
\]

(23)

for \( \tau \to i\infty \). With \( \sum_{j=1}^{N} s_j = 0 \), it follows that

\[
\prod_{m=0}^{N} e^{-i\pi/2 (\zeta_m + \zeta_n)} s_m s_n = e^{i\pi/2 \sum_{n=1}^{N} \zeta_n},
\]

(24)

In the limit \( \omega_2 \to i\infty \), we therefore obtain

\[
\psi_0^{\text{torus}}(s_1, \ldots, s_N) \propto \delta_s \chi_s \prod_{m<n} \left( e^{2\pi i v_m / \omega_1} - e^{2\pi i v_n / \omega_1} \right)^{s_m s_n / 2}
\]

(25)

and

\[
\psi_{1\over 2}^{\text{torus}}(s_1, \ldots, s_N) \propto \delta_s \chi_s \left( \prod_{n=1}^{N} e^{2\pi i v_m / \omega_1} - e^{2\pi i v_n / \omega_1} \right)^{s_m s_n / 2}
\]

(26)

The exponentials \( e^{2\pi i v_m / \omega_1} \) lie on a cylinder of circumference \( \omega_1 \). We therefore identify \( z_n = e^{2\pi i v_n / \omega_1} \) and \( N_{\omega_1} = \omega_1 \). Comparing the expressions for \( \psi_k^{\text{torus}} \) in the limit \( \omega_2 \to i\infty \) to Eqs. (8) and (14), we conclude that \( \psi_0^{\text{torus}} \propto \psi_0 \) and \( \psi_{1/2}^{\text{torus}} \propto \psi_0^{\downarrow} \psi_0^{\uparrow} \equiv \psi_0^{\text{gl}} \).

It would be very interesting to investigate the relation between excited states in one and two dimensions on the circle and the plane, respectively, to those on the torus. The result for the relation between \( \psi_{1/2}^{\text{torus}} \) and \( \psi_0^{\text{gl}} \) is a first step in that direction.

### C. Spin correlation functions and edge states

We calculated two-point spin correlation functions in the states \( \psi_0, \psi_0^{\downarrow} \) and in the singlet state

\[
\psi_0^{\text{gl}} = \psi_0^{\downarrow} \psi_0^{\uparrow}
\]

(27)

using a Metropolis Monte Carlo algorithm. This allowed us to compare properties of the states numerically for large system sizes by sampling the relevant probability distributions. We furthermore exploited the translation and inversion symmetries of Table I to average over...
equivalent correlation functions, thus obtaining a faster converging Monte Carlo estimate.

In this subsection, we shall use the notation

\[ S_{\psi}^{ab}(i_x, j_x, \Delta y) = 4 \frac{\langle \psi | \sigma_i^a \sigma_j^b \sigma_{-i}^a \sigma_{-j}^b | \psi \rangle}{\langle \psi | \psi \rangle} \]  

(28)

for the two-point correlation function in a state \( \psi \). Since all wave functions that we consider have a translational symmetry in the periodical direction, their value only depends on the difference \( \Delta y \) of the positions in the \( y \)-direction.

Before comparing the wave functions with each other, we discuss the spin ordering pattern in \( \psi_0 \), which is encoded in the correlation function \( S_{\psi_0}^{zz}(i_x, j_x, \Delta y) \). (Since \( \psi_0 \) is a singlet, \( xx \)-, \( yy \)-, and \( zz \)-correlations are the same and only correlation functions with \( a = b \) are nonzero.) Our numerical results are shown in Fig. 2. In the bulk of the system, we observe a ring-like structure with an alternating magnetization. At the edge, the correlations are anti-ferromagnetic at short distances. At larger distances along the \( y \)-direction, however, the sign becomes stationary and a negative correlation remains. In the two-dimensional picture, the ordering is still characterized by an alternating magnetization with the sign of the correlation function changing along the \( x \)-direction.

We now discuss the question if the states \( \psi_i^a \) can be considered as edge states. If so, then the local properties of \( \psi_i^a \) and \( \psi_0 \) in the bulk should be the same. Since these are encoded in the spin correlation functions, we compared the nearest-neighbor two-point correlators in the bulk for different system sizes. The relative differences

\[ \left| \frac{S_{\psi_i^a}^{bc}(i_x, j_x, \Delta y) - S_{\psi_0}^{bc}(i_x, j_x, \Delta y)}{S_{\psi_0}^{bc}(i_x, j_x, \Delta y)} \right| \]  

(29)

are shown in Fig. 3 for \( a = b = c = z \) (left panels) and \( a = z, b = c = x \) (right panels). Correlation functions for other choices of \( a, b, \) and \( c \) either vanish or can be reduced to these due to the SU(2) invariance of \( \psi_0 \). In the upper panels, the correlations along the \( y \)-direction are shown and in the panels of the lower two rows along the \( x \)-direction. We find that the relative differences approach zero exponentially as a function of \( N_x \). Even though the differences tend to be larger for smaller \( N_y \), they are still exponentially suppressed as \( N_x \) is increased. This is an indication that the wave functions \( \psi_i^a \) indeed describe edge states compared to \( \psi_0 \) as the thermodynamic limit in the open direction is taken.

Our results for the comparison between \( \psi_0^{\text{sgl}} \) and \( \psi_0 \) are shown in Fig. 4. Since both \( \psi_0 \) and \( \psi_0^{\text{sgl}} \) are singlets, it is enough to compare the \( zz \)-correlations. Furthermore, the correlations in the positive and the negative \( x \)-direction are the same in the middle of the cylinder since \( \psi_0 \) and \( \psi_0^{\text{sgl}} \) are symmetric under the inversion, cf. Sec. II A. In contrast to \( \psi_i^a \), we find that the thermodynamic limit in the \( x \)-direction is not enough for the differences to
vanish. Rather, we observe that the differences become stationary if \( N_y \) is held fixed and \( N_x \) increased. As shown in the right panels of Fig. 4, the differences do, however, tend to zero exponentially as a function of \( N_y \) if \( N_x \) is chosen large enough.

**D. States at a higher level**

In the previous subsection, the states at level one in current operators were considered. We also compared spin correlations in \( \psi_n^a \) to those in \( \psi_0 \) for higher values of \( n \). For very large mode numbers \( n \), only the terms at the edge contribute to the sum in \( u_n^a \). To see this, let us consider

\[
\psi_n^a = \sum_{j=1}^N e^{\frac{2\pi i}{N_y} n_j a_j} \chi_n^a,\]

and

\[
\psi_0 = \lim_{m \to \infty} \psi_{n+mN_y}^a = \sum_{j=1}^N e^{\frac{2\pi i}{N_y} n_j a_j} \chi_0^a.
\]

For large values of \( m \), the terms with \( j_x > 1 \) are exponentially suppressed with respect to those that have \( j_x = 1 \). We denote the corresponding states with one current operator by \( \chi_n^a \):

\[
|\chi_n^a\rangle = \lim_{m \to \infty} |\psi_{n+mN_y}^a\rangle \propto \sum_{j_y=1}^{N_y} e^{-2\pi i \frac{n_j a_j}{N_y} |\chi_0^a\rangle}.
\]

Fig. 5 shows the difference in nearest-neighbor correlations along the \( y \)-direction relative to \( \psi_0 \) for \( N_x = 13 \) and \( N_y = 8 \). The three curves correspond to the states \( \psi_1^1, \psi_2^2 \) and \( \chi_1^1 \). As the position in the open direction is increased, the differences vanish exponentially for all three states. We note that the differences are large at the left edge \( (i_x = 1) \) and small at the right edge \( (i_x = 13) \). This agrees with the expectation that the operators \( u_{-n}^a \) are localized at the left edge. In contrast to the state \( \psi_0^{\text{rig}} \), the states \( \psi_n^a \) perturb \( \psi_0 \) only at one edge and their behavior is therefore expected to approach that of \( \psi_0 \) at the other edge. The results of Fig. 5 provide an indication that \( \psi_n^a \) describe edge states also for \( n > 1 \).
E. Inner products of states from current operators

In this subsection, we discuss the relation of inner products between the states $\psi_{n_1 \ldots n_1}$ on the level of the spin system and CFT inner products between states $(J_{n_1} \ldots J_{a_1}) (0 | 0)$. For edge states in the continuum that are constructed from descendant states of a CFT, the authors of Ref. [11] come to the remarkable conclusion that, in the thermodynamic limit and under the assumption of exponentially decaying correlations in the bulk, the inner products between edge states are the same as the inner products between CFT states. We now consider inner products between states constructed from current operators to test if a similar correspondence holds for the lattice states $\psi_{n_1 \ldots a_1}$ and the CFT states they are constructed from. The spin system inner products that we consider are given by

$$\langle \psi_{0} | \psi_{0} \rangle \equiv R^{k+k'} \langle \psi_{n_1 \ldots n_1} | \psi_{b_1' \ldots b_1} \rangle \langle \psi_{b_1' \ldots b_1} | \psi_{n_1 \ldots n_1} \rangle,$$

where $N$ is the number of spins,

$$R = \min_{j \in \{1, \ldots, N\}} |z_j| = e^{-\frac{1}{N} (N_x - 1)}$$

is the minimal absolute value of the positions, $k = \sum_{j=1}^{l} n_j$, and $k' = \sum_{j=1}^{l'} m_j$. The factor $R^{k+k'}$ accounts for the scaling of the operators $u_{n}^a$ with respect to a rescaling of the positions. The minimal value is chosen because the operators

$$u_{-n}^a = \sum_{j=1}^{N} \frac{t_j^a}{(z_j)^n}$$

have the highest contribution at the edge with $|z_j| = R$. We compare the inner products of the lattice system (35) to the CFT inner products

$$\langle 0 | \psi_{j_1} \ldots \psi_{j_1} J_{-m_1} \ldots J_{-m_1} \rangle | 0 \rangle.$$

If a correspondence similar to that of Ref. [11] also holds for lattice states, then the expressions of Eq. (35) should approach those of Eq. (38) in the thermodynamic limit.

Note that the inner products of the spin system are hard to evaluate for large system sizes, whereas the CFT inner product can be easily computed using the Kac-Moody algebra [2]. On the level of the spin system, the insertion of current operators corresponds to an application of spin operators to $\psi_0$ [cf. Eq. (12)]. Therefore, the inner products can be determined numerically using a Monte Carlo method if the number of current operators is small.

We calculated the inner products for the states $\psi_{1,1}^a$, $\psi_{1}^a$ and $\psi_{1,1}^{b,b}$, which are all nonzero states at levels one and
two. For these states, inner products between different states vanish because they have either a different spin or a different momentum. It is thus sufficient to compare the norm squared of a state to the norm squared of the corresponding CFT state, as summarized in the following table:

| Spin state | CFT state | Norm squared of CFT state |
|------------|-----------|---------------------------|
| \( \psi^a \) | \( J^a_1 | 0 \rangle \) | \( \langle J^a_1 | J^a_1 \rangle = \frac{1}{2} \) (no sum over \( a \)) |
| \( \psi^b \) | \( J^a_1 | 0 \rangle \) | \( \langle J^a_1 | J^a_2 \rangle = 1 \) (no sum over \( a \)) |
| \( \psi^b_{1,1} \) | \( J^a_{1,1} | 0 \rangle \) | \( \langle J^a_{1,1} | J^a_{1,1} \rangle = \frac{9}{2} \) |

For a given system size, we observe a smaller difference for the states at level \( k = 1 \) than for those at level \( k = 2 \). The computed inner products approach the CFT expectation if \( N_y \) is increased. The dependence on the number of spins in the \( x \)-direction is, however, very weak for \( N_x \geq 3 \). In particular, the CFT result is not approached if \( N_x \) is increased and \( N_y \) kept fixed. For large enough \( N_y \), our data suggest that the spin system inner products approach the CFT result with a power law in \( N_y \).

IV. LOCAL MODEL HAMILTONIAN

In the previous section, we provided numerical evidence that the states with one current operator insertion represent edge states with respect to \( \psi_0 \). In this section, we study a set of local Hamiltonians on the cylinder. For a suitable choice of parameters, the ground state of the corresponding Hamiltonian has a good overlap with \( \psi_0 \) and some of its low-energy excited states are well approximated by \( \psi^a \), the states with one current operator of order one.

We study the local Hamiltonians\(^{28}\)

\[
H = J_2 \sum_{\langle i,j \rangle} t^a_i t^a_j + J_2' \sum_{\langle\langle i,j \rangle\rangle} t^a_i t^a_j + J_3 \sum_{\langle\langle i,j,k \rangle\rangle} \varepsilon_{abc} t^a_i t^b_j t^c_k .
\]  

(40)

In these sums, the sites lie on a square lattice, \( \langle i,j \rangle \) denotes all nearest neighbors, \( \langle\langle i,j \rangle\rangle \) all next-to-nearest neighbors, and \( \langle\langle i,j,k \rangle\rangle \) all triangles of nearest neighbors for which \( i,j,k \) are oriented counter-clockwise. It was shown in a previous study\(^{28}\) that the ground state of \( H \) on the plane (open boundary conditions in both directions) and on the torus has a good overlap with the KL state for a range of parameters \( J_2, J_2' \), and \( J_3 \). Here, we study \( H \) on a cylinder of size \( N_x \times N_y \), where \( N_y \) denotes the number of sites in the open direction and \( N_y \) the number of sites in the periodical direction. In the following, we parametrize \( H \) in terms of two angles \( \theta_1 \) and \( \theta_2 \):

\[
J_2 = \cos (\theta_1) \cos (\theta_2) , \quad J_2' = \sin (\theta_1) \cos (\theta_2) , \quad J_3 = \sin (\theta_2) .
\]  

(41)

For \( N_x = 5 \) and \( N_y = 4 \), we studied the overlap between \( \psi_0 \) and the ground state \( \psi_E \) of \( H \) as a function of \( \theta_1 \) and \( \theta_2 \) using an exact numerical diagonalization method. We also computed the overlap of the states with one current operator insertion at level one \( \psi^a \) and the first excited states \( \psi^m_E \) of \( H \) that have spin one and the same momentum in the \( y \)-direction as \( \psi^a \). Here, \( m \in \{-1,0,1\} \) denotes the \( T^m \) eigenvalue of \( \psi^m_E \).

We denote the overlap between two states \( \phi_1 \) and \( \phi_2 \) as

\[
\Omega (\phi_1, \phi_2) = \frac{\langle \phi_1 | \phi_2 \rangle}{\| \phi_1 \| \| \phi_2 \|}.
\]  

(42)
The angles $\theta_1$ and $\theta_2$ parametrize the coupling constants of $H$ according to Eq. (11). In the upper panel, the overlap $\Omega(\psi_G, \psi_0) \equiv |\langle \psi_G | \psi_0 \rangle | / ||\psi_G|| ||\psi_0||$ between $\psi_0$ and the ground state $\psi_G$ of $H$ is plotted. The lower panel shows the overlap between $\psi^E_1$ and the first excited state $\psi^E_0$ of $H$ with the same spin and $y$-momentum as $\psi^E_1$ [spin one, $T^z = 0$, momentum $3/(8\pi)$]. The point marked with an open circle has $\theta_1 = 0.0275 \times 2\pi$ and $\theta_2 = 0.06 \times 2\pi$ and the highest combined overlap of $\sqrt{\Omega(\psi_G, \psi_0)^2 + \Omega(\psi^E_1, \psi^E_0)^2} \approx 1.2858$.

where $||\phi|| = \sqrt{\langle \phi | \phi \rangle}$ is the norm of a state. In Fig. 7, the overlaps $\Omega(\psi_G, \psi_0)$ and $\Omega(\psi_G, \psi^E_1)$ are shown as a function of the parameters of the Hamiltonian. Due to SU(2) invariance, it is sufficient to consider the overlap between the states $\psi^E_0$ and $\psi^E_1$, which both have $T^z = 0$:

$$|\langle \psi^E_0 | \psi^E_1 \rangle | = |\langle \psi^E_0 | \psi^E_1 \rangle | = |\langle \psi^E_0 | \psi^E_1 \rangle |,$$

(43)

where $\psi^E_1 \equiv \psi^E_1 \mp i \psi^E_1$. The best value for the combined overlap $\sqrt{\Omega(\psi_G, \psi_0)^2 + \Omega(\psi^E_1, \psi^E_0)^2}$ was obtained for the angles $\theta_1 = 0.0275 \times 2\pi$ and $\theta_2 = 0.06 \times 2\pi$:

$$\Omega(\psi_G, \psi_0) \Omega(\psi^E_0, \psi^E_1) \Omega(\psi_G, \psi^E_0) \approx 0.9829$$

The relation $T^{-1} H T = H$. The relation $T^{-1} T_y T = T^{-1}$ between $T$ and translation operator in the $y$-direction $T_y$ follows directly from their definition (cf. Appendix A.2 and A.3). Therefore, if $|\psi\rangle$ is an eigenstate of $H$ with momentum $p/(2\pi N_y)$, then $T|\psi\rangle$ is also an eigenstate with momentum $(N_y - p)/(2\pi N_y)$. This means that for $|\psi^E_0\rangle$ with $p = 3$, there is a corresponding eigenstate $T|\psi^E_0\rangle$ with $p = 1$, which satisfies

$$|\langle \psi^E_0 | \psi^E_1 \rangle | = |\langle T|\psi^E_0\rangle | \langle \psi^E_1 \rangle |.$$

Here, $|\psi^E_1\rangle = T|\psi^E_1\rangle$ is the state obtained by inserting the current operator at time $t = \infty$ instead of $t_0 = 0$ [cf. Eq. (20)].
Our results show that the state $\psi_0$ and the states with one current operator of order one are good approximations of low-energy eigenstates of $H$ for $N_x = 5$, $N_y = 4$, $\theta_1 = 0.0275 \times 2\pi$, and $\theta_2 = 0.06 \times 2\pi$. This raises the question if further eigenstates of $H$ are effectively described by states constructed as CFT correlators. We also computed the overlaps of eigenstates of $H$ with some additional states constructed from current operators for higher orders in current operators. At level two in current operators, the overlaps with the first excited states that have the same spin and momentum as our ansatz states are given by $0.5486$ for $\psi_2^a$ (0.9704 per site) and $0.3301$ for $\psi_1^{b,b}$ (0.9461 per site).

In this Section, we considered a local model and computed overlaps between ansatz states constructed from CFT and eigenstates of the model Hamiltonian. In Appendix B, we also derive exact, SU(2) invariant parent CFT and eigenstates of the model Hamiltonian. In Appendix C, we also derive exact, SU(2) invariant parent Hamiltonians for some of the states constructed from current operators. These Hamiltonians are nonlocal with up to four-body interactions.

V. CONCLUSION

This work studies trial wave functions for lattice FQH states constructed as chiral correlators of the SU(2) WZW CFT. To each CFT state, characterized by a sequence of current operators, we associated a corresponding state $\psi_{n_1\ldots n_1}^{a_1\ldots a_1}$ of the lattice system. For continuous systems, analogous states constructed from CFT were proposed as FQH edge states previously. The fact that we work on the lattice allowed us to apply Monte Carlo techniques to test a central expectation for edge states: That the local, bulk properties of different edge states should be the same.

For a system on the cylinder, we compared spin correlation functions in the states with one current operator ($\psi_1^a$) to the state with no current operators ($\psi_0$). Our numerical results show that the nearest-neighbor bulk correlations approach each other exponentially as the number of spins in the open direction ($N_y$) is increased. On the other hand, the states $\psi_1^a$ and $\psi_0$ are different globally since their spin and momentum are different. This supports the assumption that they describe edge states.

We compared inner products of lattice states at levels one and two in current operators to CFT inner products of the corresponding descendant states. For large enough $N_y$ (periodical direction), the computed inner products approach the CFT expectation with a power law in $N_y$. This suggests that there is a correspondence between inner products of states $\psi_{n_1\ldots n_1}^{a_1\ldots a_1}$ and CFT inner products in the thermodynamic limit. Such a correspondence was found for continuous wave functions in Ref. [11].

Furthermore, we compared nearest-neighbor bulk correlations in $\psi_{0}^{s}$ to those in $\psi_0$, where $\psi_{0}^{s}$ is the singlet component of the state obtained by insertion of two extra primary fields. In contrast to $\psi_1^a$, we find that the correlations do not approach each other if the thermodynamic limit is taken only in the open direction. However, if $N_x$ is chosen large enough, the difference in correlation functions vanishes exponentially as a function of $N_y$.

We showed by an exact diagonalization that $\psi_0$ has a good overlap with the ground state of a local Hamiltonian and $\psi_1^a$ with the first excited states that have the same spin and momentum as $\psi_1^a$. This could be an indication that further low-energy excitations of that local Hamiltonian are edge states described by the SU(2) WZW CFT. It would be interesting to investigate this relation in more detail for larger system sizes and different topologies.

We showed that the complete Hilbert space is covered by the linear span of the states $\psi_{n_1\ldots n_1}^{a_1\ldots a_1}$ and, therefore, only a subset of these states are edge states. For the states with one current operator, we argued that not all linear combinations of states $\psi_m^{a}$ describe edge modes because states with the same $y$-momentum can be non-orthogonal. It is possible to restrict the space of states to an orthogonal subset given by $\psi_m^{a}$ with $m \in \{1, \ldots, N_y\}$.

Taking the limit of large mode numbers could be another possibility of removing bulk states for the linear span of $\psi_{n_1\ldots n_1}$. More precisely, one can replace $n_i$ by $n_i + m_i N_y$ and then take $m_i \to \infty$. In this limit, the sum in the operators $u^a_{n_1\ldots m_i N_y}$ only extends over the edge sites because all other positions are exponentially suppressed. The fact that this class of states (and also linear combinations of such states) is obtained from $\psi_0$ by application of edge spin operators only, suggests that their complete span represents edge states. For one of these states, $\chi_1^a$, we did numerical tests that indeed indicated that $\chi_1^a$ is an edge state.

ACKNOWLEDGMENTS

We acknowledge funding from the EU Integrated Project SIQS, FIS2012-33642, the Comunidad de Madrid grant QUITEMAD+ S2013/ICE-2801 (CAM), the Severo Ochoa Program, and the Villum Foundation.

Appendix A: Translation and inversion of states on the cylinder

1. Transformation under a permutation of the spins

Both the translation operator $T_y$ and the inversion operator $I$ act on a product state as a permutation of the spins. Such a permutation operator $O_\tau$ is defined for the permutation $\tau$ of $N$ elements as

$$O_\tau |s_1, \ldots, s_N \rangle = |s_{\tau(1)}, \ldots, s_{\tau(N)} \rangle. \quad (A1)$$

The action of $O_\tau$ on $\psi_0$ and $\psi_0^{s_0,s_w}$ can be rewritten in terms of a permutation of the positions $z_i$, which will facilitate our calculations for $T_y$ and $I$. Our derivation of this transformation rule follows Ref. [19].
We consider the wave function
\[ \tilde{\psi}_{0}^{s_{1},...,s_{N}}(s_{1},...,s_{N}) = \delta_{s}(\chi_{s}) \prod_{i<j}^{N}(z_{i} - z_{j})^{(s_{i}s_{j}+1)/2}, \quad (A2) \]
which is equivalent to \( \psi_{0} \) because it only differs by a spin-independent constant [cf. Eq. (5)]. We have also explicitly written out the parametric dependence on the positions \( z_{i} \). Similarly, the wave function
\[ \tilde{\psi}_{0}^{s_{1},...,s_{N}}(s_{1},...,s_{N}) = \delta_{s}(\chi_{s}) \prod_{n=1}^{N}(z_{n} - z_{m})^{(s_{n}s_{m}+1)/2} \]
is equivalent to \( \tilde{\psi}_{0}^{s_{1},...,s_{N}} \). Let us first calculate the transformation of \( \tilde{\psi}_{0}^{s_{1},...,s_{N}} \) under a simultaneous permutation of both the spins and the coordinates. Since every permutation can be decomposed into a series of transpositions, we consider the case that \( \tau \) is a transposition:
\[ \tau(i) = \begin{cases} i, & \text{if } i \in \{m, n\}, \\ n, & \text{if } i = m, \\ m, & \text{if } i = n, \end{cases} \]
for \( m, n \in \{1, ..., N\} \) and \( m < n \). It follows that
\[ \begin{align*}
\tilde{\psi}_{0}^{s_{1},...,s_{N}}(s_{1},...,s_{N}) &= \frac{\tilde{\psi}_{0}^{s_{1},...,s_{N}}(s_{1},...,s_{N})}{\tilde{\psi}_{0}^{s_{1},...,s_{N}}(s_{1},...,s_{N})} \\
&= \frac{(-1)^{(n-m)(s_{m}-s_{n})/2}}{\text{transformation of } \chi_{s}} \prod_{i<j}^{N}(z_{i} - z_{j})^{(s_{i}s_{j}+1)/2} \\
&= \frac{(-1)^{(n-m)(s_{m}-s_{n})/2}}{\prod_{j=m+1}^{n-1}(s_{j}(s_{m}+s_{n})(s_{m}+1)/2)} \\
&= -1. \quad (A5)
\end{align*} \]
Therefore, if \( \tau \) is a general permutation corresponding to \( \mathcal{N}_{\tau} \), the subsequent transposition,
\[ \tilde{\psi}_{0}^{s_{1},...,s_{N}}(s_{\tau(1)},...,s_{\tau(N)}) = \text{sign}(\tau) \tilde{\psi}_{0}^{s_{1},...,s_{N}}(s_{1},...,s_{N}) \]
(6)
where \( \text{sign}(\tau) = (-1)^{N_{\tau}} \) is the signature of the permutation. Substituting \( s_{\tau(i)} \) in Eq. (A6), we arrive at the final transformation rule:
\[ \begin{align*}
\tilde{\psi}_{0}^{s_{1},...,s_{N}}(s_{1},...,s_{N}) &= \text{sign}(\tau) \tilde{\psi}_{0}^{s_{1},...,s_{N}}(s_{1},...,s_{N}) \\
&= \text{sign}(\tau) \tilde{\psi}_{0}^{s_{1},...,s_{N}}(s_{1},...,s_{N}) \quad (A7)
\end{align*} \]
The transformation under a permutation of the spins can therefore be calculated by considering the transformation of the coordinates and taking into account the signature of the permutation.

We note that Eq. (A7) is not valid for the original wave function \( \psi_{0} \) but only for \( \tilde{\psi}_{0}^{s_{1},...,s_{N}} \), which differs from \( \psi_{0} \) by a factor depending on \( z_{i} \). However, this factor does not depend on the spins. Therefore, if \( \tilde{\psi}_{0}^{s_{1},...,s_{N}} \) is an eigenstate of \( \mathcal{O}_{\tau} \), then this is also the case for \( \psi_{0} \).

Compared to \( \psi_{0} \), there are some additional factors present the wave function \( \tilde{\psi}_{0}^{s_{0},s_{\infty},...,s_{1},...,s_{N}} \). Since these are invariant under a permutation of both the spins and the coordinates, a formula analogous to Eq. (A7) holds for \( \tilde{\psi}_{0}^{s_{0},s_{\infty},...,s_{1},...,s_{N}} \).

2. Translation in the periodical direction

The translation operator \( \mathcal{T}_{y} \) is defined through the permutation \( \mathcal{T}_{y} \):
\[ \mathcal{T}_{y}(i, j) = \begin{cases} (i_{x}, i_{y} + 1), & \text{if } i_{y} \neq N_{y}, \\ (i_{x}, 1), & \text{if } i_{y} = N_{y}, \end{cases} \quad (A8) \]
where \( i_{x} \) is the \( x \)-component and \( i_{y} \) the \( y \)-component of an index \( i \).

The signature of this permutation is given by
\[ \text{sign}(\mathcal{T}_{y}) = (-1)^{N_{z}(N_{y}-1)} = (-1)^{N_{x}}, \quad (A9) \]
where we used that \( N = N_{y}N_{x} \) is even. In terms of the positions, the transformation corresponds to a multiplication by a phase, \( z_{\mathcal{T}_{y}(i)} = e^{2\pi i/N_{y}z_{i}} \). Therefore,
\[ \tilde{\psi}_{0}^{s_{1},...,s_{N}}(s_{\mathcal{T}_{y}(1)},...,s_{\mathcal{T}_{y}(N)}) = \text{sign}(\mathcal{T}_{y}) \tilde{\psi}_{0}^{s_{1},...,s_{N}}(s_{1},...,s_{N}) \]
\[ = (-1)^{N_{z}} \delta_{s}(\chi_{s}) \prod_{i<j}^{N} e^{2\pi i/N_{x}(z_{i} - z_{j})} (s_{i}s_{j}+1)/2 \]
\[ = (-1)^{N_{z}} N_{2} \tilde{\psi}_{0}^{s_{1},...,s_{N}}(s_{1},...,s_{N}). \quad (A10) \]
Here, we have used that
\[ \prod_{i<j}^{N} \frac{e^{2\pi i/N_{x}}(s_{i}s_{j}+1)}{s_{i}s_{j}+1} = (-1)^{N_{z}} \frac{N_{z}}{N_{x}} + N_{z}, \quad (A11) \]
which follows from \( \sum_{j=1}^{N} s_{j} = 0 \). The eigenvalue of \( \psi_{0} \) with respect to \( \mathcal{T}_{y} \) is therefore \( (-1)^{N_{z}} N_{2}/2 \).

It follows that the eigenvalue of \( \psi_{0}^{s_{0},...,s_{1}} \) is \( e^{-2\pi i/kN_{x}}(-1)^{N_{z}N_{2}/2} \), where \( k = \sum_{j=1}^{l} n_{j} \).

For \( \tilde{\psi}_{0}^{s_{0},s_{\infty},...,s_{1},...,s_{N}} \), we obtain
\[ \begin{align*}
\tilde{\psi}_{0}^{s_{0},s_{\infty},...,s_{1},...,s_{N}}(s_{\mathcal{T}_{y}(1)},...,s_{\mathcal{T}_{y}(N)}) &= (-1)^{N_{z}} \prod_{n=1}^{N} \frac{e^{2\pi i/N_{x}}(s_{n}s_{n}+1)}{s_{n}s_{n}+1} \prod_{n<m}^{N} e^{2\pi i/N_{x}}(s_{n}s_{m}+1) \\
&\times \tilde{\psi}_{0}^{s_{0},s_{\infty},...,s_{1},...,s_{N}}(s_{1},...,s_{N}) \\
&= (-1)^{N_{z}+N_{x}N} \tilde{\psi}_{0}^{s_{0},s_{\infty},...,s_{1},...,s_{N}}(s_{1},...,s_{N}). \quad (A12) \end{align*} \]
In the last equation, we have used that $\sum_{j=1}^{N} s_j + s_0 + s_\infty = 0$.

### 3. Inversion

We require that the inversion $\mathcal{I}$ acts on the positions defined in Eq. (19) as

$$z_{\mathcal{I}}(i_x,i_y) = \frac{1}{z_{i_x,i_y}}.$$  

(A14)

This leads to the definition

$$\mathcal{I}(i_x,i_y) = \begin{cases} (N_x + 1 - i_x, N_y - i_y), & \text{if } i_y \neq N_y, \\ (N_x + 1 - i_x, N_y), & \text{if } i_y = N_y. \end{cases}$$  

(A15)

We note that in our choice of $z_i$, the center of the cylinder is at the unit circle. If this is not the case, then the definition of Eq. (A15) leads to an additional factor when $\mathcal{I}$ is applied to $z_i$.

In order to determine the sign of the permutation, we arrange the state $|s_1, \ldots, s_N\rangle$ in a matrix:

$$|s_1, \ldots, s_{N_x}, N_y\rangle \equiv \left( \begin{array}{c} s_{1,N_y} \\ s_{2,N_y} \\ \vdots \\ s_{N_x,1} \end{array} \right).$$  

(A16)

The transformed state is then given by

$$\mathcal{I}|s_1, \ldots, s_{N_x}, N_y\rangle \equiv \left( \begin{array}{cccc} s_{N_x,N_y-1} & s_{N_x,N_y-2} & \cdots & s_{N_x,1} \\ s_{N_x-1,N_y-1} & s_{N_x-1,N_y-2} & \cdots & s_{N_x-1,1} \\ \vdots & \vdots & \ddots & \vdots \\ s_{1,N_y-1} & s_{1,N_y-2} & \cdots & s_{1,1} \end{array} \right).$$  

(A17)

To bring the transformed matrix back to the original form, we first reverse all $N_y$ columns and then reverse all $N_x$ rows excluding the last element of each row. A single sequence of $L$ elements can be reversed in $L(L-1)$ steps. Therefore, the sign of the permutation is given by

$$\text{sign}(\mathcal{I}) = (-1)^{N_y \frac{1}{2} N_x(N_x-1)+N_x \frac{1}{2} (N_x-1)(N_y-2)}.$$  

(A18)

We next determine the contribution from the coordinate part of the wave function $\psi_0$. Using Eq. (A17), we have

$$\begin{align*} \psi_0^{(s_0,s_\infty,z_1,\ldots,z_N)}(s_1, \ldots, s_N) &= \psi_0^{(s_0,s_\infty,z_1,\ldots,z_N)}(s_1, \ldots, s_N) \prod_{m<n} (-z_m z_n)^{-\frac{1}{2}} (s_m s_n+1) \\
&= \psi_0^{(s_0,s_\infty,z_1,\ldots,z_N)}(s_1, \ldots, s_N) e^{-\frac{1}{2} \sum_{m<n} (s_m s_n+1)(\log(z_m z_n)+\pi i)} \\
&= \psi_0^{(s_0,s_\infty,z_1,\ldots,z_N)}(s_1, \ldots, s_N) e^{-\frac{1}{2} \sum_{m<n} (s_m s_n+1)(\log(z_m z_n)+\pi i)} \\
&= \psi_0^{(s_0,s_\infty,z_1,\ldots,z_N)}(s_1, \ldots, s_N)(-1)^{\frac{N}{2} N_x+N_y}. \quad (A19) \end{align*}$$

In the last step, we have used that $s_1 + \cdots + s_N = 0$ and

$$\sum_{n=1}^{N} \log(z_n) = \pi i (N + N_x).$$  

(A20)

Therefore, the eigenvalue of $\psi_0$ with respect to $\mathcal{I}$ is

$$\text{sign}(\mathcal{I}) = (-1)^{\frac{N}{2} N_x+N_y}. \quad (A21)$$

The states $\psi_{a_1,\ldots,a_n}^{(a_1,\ldots,a_n)}$ are not eigenstates of $\mathcal{I}$, but transform as

$$\begin{align*} \mathcal{I}|\psi_{a_1,\ldots,a_n}^{(a_1,\ldots,a_n)}\rangle &= \mathcal{I} u_{a_n}^{a_1} \mathcal{I}^{-1} \ldots \mathcal{I} u_{a_1}^{a_1} \mathcal{I}^{-1} |\psi_0\rangle \\
&= (-1)^{\frac{N}{2} N_x+N_y} u_{a_1}^{a_1} \ldots u_{a_n}^{a_n} |\psi_0\rangle. \quad (A22) \end{align*}$$

Here, we have used that

$$\begin{align*} \mathcal{I} u_{a_n}^{a_1} \mathcal{I}^{-1} &= \sum_{i=1}^{N} \frac{1}{(z_i)^{N_y}} t_i^{a_n} \mathcal{I}^{-1} = \sum_{i=1}^{N} \frac{1}{(z_i)^{N_y}} t_i^{a_n} \mathcal{I}^{-1(i)} \\
&= \sum_{i=1}^{N} (z_i)^{N_y} t_i^{a_n} = u_{a_1}^{a_1}, \quad (A23) \end{align*}$$

if the center of the cylinder is at the unit circle. In terms of the states $\psi_{a_1,\ldots,a_n}^{(a_1,\ldots,a_n)}$ defined in Eq. (20), we therefore have

$$\mathcal{I}|\psi_{a_1,\ldots,a_n}^{(a_1,\ldots,a_n)}\rangle = (-1)^{\frac{N}{2} N_x} |\psi_{a_1,\ldots,a_n}^{(a_1,\ldots,a_n)}\rangle.$$  

(A24)

For the transformed states $\mathcal{I}|\psi_{a_1,\ldots,a_n}^{(a_1,\ldots,a_n)}\rangle$, the current operators are therefore inserted at $z_{\infty} = \infty$ instead of at $z_0 = 0$. Eigenstates of $\mathcal{I}$ with eigenvalues $(\pm 1)(-1)^{\frac{N}{2} N_y}$ are then given by

$$\psi_{a_1,\ldots,a_n}^{(a_1,\ldots,a_n)} \pm \psi_{a_1,\ldots,a_n}^{(a_1,\ldots,a_n)}.$$  

(A25)

Finally, we determine the transformation of $\psi_0^{(s_0,s_\infty,z_1,\ldots,z_N)}$ with respect to $\mathcal{I}$. As for $\psi_0$, there is a contribution from the sign of the permutation and from the transformation of the coordinates. The calculation is similar to that for $\psi_0$, only that now $s_0 + s_\infty + \sum_{i=1}^{N} s_i = 0$. We find

$$\mathcal{I}|\psi_0^{(s_0,s_\infty,z_1,\ldots,z_N)}\rangle = \text{sign}(\mathcal{I}) |\psi_0^{(s_0,s_\infty,z_1,\ldots,z_N)}\rangle \equiv (-1)^{N_x \frac{1}{2} N_x+N_y+1} |\psi_0^{(s_0,s_\infty,z_1,\ldots,z_N)}\rangle.$$  

(A26)
operators that are annihilated by the three operators

\[ \{ n \} \] only differ by a spin-independent factor, we also have

\[ \mathcal{I}|\psi_{0,\infty}^{a}\rangle = (-1)^{N_{y} + N_{z} + 1} |\psi_{0,\infty}^{a}\rangle. \]  
(A27)

Note that \( \mathcal{I} \) exchanges the spins \( s_{0} \) and \( s_{\infty} \) in \( \psi_{0,\infty}^{a} \).

**Appendix B: Exact parent Hamiltonians**

As shown in Sec. [IV](#) the edge states \( \psi_{l}^{a} \) have a good overlap with low-lying excited states of a local model, for which \( \psi_{0} \) approximates the ground state. In this section, we analytically construct SU(2)-invariant, nonlocal parent Hamiltonians for some linear combinations of the states \( \psi_{a_{1}...a_{i}} \), i.e. Hamiltonians for which they are exact eigenstates with the lowest energy.

1. **Construction of parent Hamiltonians**

The starting point of our construction is the operator

\[ C^{a} = \sum_{i \neq j}^{N} \frac{z_{i} + z_{j}}{z_{i} - z_{j}} (t_{j}^{a} + i \varepsilon a c t^{b} t_{j}^{b}). \]  
(B1)

In Appendix C we explicitly compute the action of \( C^{a} \) on states constructed from \( \psi_{0} \) by insertion of current operators, and show that \( C^{a} \) does not mix the states \( \psi_{a_{1}...a_{i}} \) with different levels \( k = \sum_{j=1}^{i} n_{j} \) if \( k < N_{y} \). This property is key to our construction of parent Hamiltonians: It allows us to treat the levels separately starting with the lower levels, which have fewer states. The action of \( C^{a} \) on states at level \( k \) is described by a matrix. For low \( k \), the dimension of this matrix is considerably smaller compared to that of an operator acting on the complete Hilbert space. Moreover, the size of the matrix depends only on the level \( k \) rather than the number of spins \( N \).

We next add a multiple of the total spin \( T^{a} \) to \( C^{a} \) and define the operators

\[ D_{n}^{a} = C^{a} + (n + 1 - N)T^{a}, \]  
(B2)

where \( n \) is an integer. The operator \( D_{n}^{a} \) is also closed in the subspace of states of level \( k \) if \( k < N_{y} \) since \( T^{a} \) does not mix states of different levels. For certain values of \( n \), we managed to find states constructed from current operators that are annihilated by the three operators \( D_{n}^{a} \) for \( a \in \{ x, y, z \} \). These states are then ground states of the Hamiltonian

\[ H_{n} = (D_{n}^{a})^\dagger D_{n}^{a}, \]  
(B3)

where the index \( a \) is summed over. Note that the Hamiltonian \( H_{n} \) is positive semi-definite and SU(2) invariant. \( H_{n} \) is nonlocal and contains terms with up to four-body interactions since \( D_{n}^{a} \) has terms linear and quadratic in spin operators.

Before describing our results, we note that the condition \( D_{n}^{a} |\psi\rangle = 0 \) for all \( a \) implies that the state \( \psi \) is part of the subspace on which \( T^{b}T^{b} \) and \( D_{n}^{a} \) commute. To show this, we first note that

\[ [D_{n}^{a}, T^{b}] = i \varepsilon a c D_{n}^{a}, \]  
(B4)

which is a direct consequence of the definitions of Eqs. (B1) and (B2). It then follows that

\[ [T^{b}T^{b}, D_{n}^{a}] |\psi\rangle = (-i \varepsilon a c T^{b} D_{n}^{a} - i \varepsilon a c D_{n}^{a} T^{b}) |\psi\rangle = -i \varepsilon a c T^{b} D_{n}^{a} |\psi\rangle = \varepsilon a c c b d D_{n}^{b} |\psi\rangle = 0, \]  
(B5)

where we assumed that \( D_{n}^{a} |\psi\rangle = 0 \) for all \( a \). The states satisfying \( D_{n}^{a} |\psi\rangle = 0 \) can therefore be decomposed into sectors of different total spin.

We note that the condition \([T^{b}T^{b}, D_{n}^{a}] |\psi\rangle = 0 \) is equivalent to

\[ [T^{b}T^{b}, C^{a} + (1 - N)T^{a}] |\psi\rangle = 0, \]  
(B6)

since \( T^{b}T^{b} \) and \( T^{a} \) commute. The operator \( C^{a} + (1 - N)T^{a} \) has the advantage that its matrix entries in terms of the states at level \( k \) do not depend \( N \) and \( n \) [cf. Eq. (C8) in Appendix C]. In our calculations, we found it technically easier to first determine the subspace of states on which \( T^{b}T^{b} \) and \( C^{a} + (1 - N)T^{a} \) commute and then look for states that are annihilated by \( D_{n}^{a} \) for a suitable \( n \) within that subspace.

**TABLE II. States constructed from current operators that are annihilated by \( D_{n}^{a} \)**

| \( k \) | State | Spin \( n \) |
|------|-------|--------|
| 0 | \( \psi_{0} \) | any |
| 1 | \( \psi_{1}^{a} \) | 1 1 |
| 2 | — | — |
| 3 | \( \psi_{3}^{a} + i \varepsilon a c \psi_{3}^{d} \) | 1 5 |
| 4 | Symmetric-traceless part of \( \psi_{3}^{a} \) | 2 3 |
| 5 | \( \psi_{3,1,1} + \frac{3}{2} \varepsilon a c \psi_{3,2,1} + \frac{3}{2} \varepsilon a c \psi_{3,4,1} + \frac{9}{\sqrt{2}} \psi_{5}^{b} \) | 1 9 |
| 6 | — | — |
| 7 | \( \psi_{3,1,1} + \frac{3}{2} \varepsilon a c \psi_{3,2,1} + \frac{3}{2} \varepsilon a c \psi_{3,4,1} + \frac{9}{\sqrt{2}} \psi_{5}^{b} \) | 1 13 |
| 8 | Symmetric-traceless part of \( \psi_{5,1}^{b} \) | 2 7 |
| 9 | Symmetric-traceless part of \( \psi_{5,1}^{b} \) | 3 5 |

We summarize our analytical results in Table II. The states with spin 2 and 3 appear as the symmetric-
traceless parts of states with 2 and 3 open indices, respectively. For a two-index state \( \phi^{ab} \), the symmetric-traceless part is defined as

\[
3(\phi^{ab} + \phi^{ba}) - 2\delta_{ab}\phi^{dd},
\]

and for a three-index state \( \phi^{abc} \) as

\[
5(\phi^{abc} + \phi^{bca} + \phi^{cab} + \phi^{bac} + \phi^{acb}) - 2(\delta_{ab}\phi^{cdd} + \delta_{ac}\phi^{bdc} + \delta_{bc}\phi^{adb} + \phi^{dbd} + \phi^{dcb}) + \delta_{bc}(\phi^{add} + \phi^{dab} + \phi^{dab}).
\]

Except for the levels 2 and 6, we find states and corresponding parent Hamiltonians for all levels that were considered. Note that the singlet \( \psi_0 \) is a ground state of \( H_n \) for any value of \( n \). For the additional ground states, we observe that the value of \( n \) tends to be larger at higher levels \( k \). This means that the ground state space of the Hamiltonians \( H_n \) with lower \( n \) contains states of a lower level in current operators. For example, we only find the ground states \( \psi_0 \) and \( \psi_1^a \) for \( H_1 \). Similarly, the only appearing ground states of \( H_3 \) at levels \( k \leq 9 \) are \( \psi_0 \) and the symmetric-traceless part of \( \psi_{3,1}^{a,b} \).

2. Ground-state degeneracies

In the previous subsection, we explicitly constructed analytical ground states of the Hamiltonians \( H_n \) with \( n \in \{1, 3, 5, 7, 9, 13, 17\} \) in terms of linear combinations of states \( \psi_1^{a_1...a_k} \) with levels \( k \leq 9 \). We now study the ground state spaces of the Hamiltonians \( H_n \) numerically and provide evidence for \( n \in \{1, 3, 5\} \) that the complete ground state space is spanned by the states given in Table III.

| \( n \) | \( \bar{N}_y^{\text{min}} \) | Ground state multiplet |
|---|---|---|
| 1 | 2 | \( 0 \oplus 1 \) |
| 3 | 4 | \( 0 \oplus 2 \) |
| 5 | 6 | \( 0 \oplus 1 \oplus 3 \) |
| 7 | 8 | \( 0 \oplus 2 \oplus 4 \) |
| 9 | 10 | \( 0 \oplus 1 \oplus 3 \oplus 5 \) |
| 11 | 12 | \( 0 \oplus 2 \oplus 4 \oplus 6 \) |
| 13 | 14 | \( 0 \oplus 1 \oplus 3 \oplus 5 \oplus 7 \) |

By an exact diagonalization, we numerically determined the ground state multiplets of the Hamiltonians \( H_n \) for \( n \leq 13 \) and systems with \( N = N_x N_y \leq 14 \) and \( N \) even. Our results are summarized in Table III. We observe that states with spin \( s \) occur in the ground state spaces only in systems with \( N_y \geq 2s \). Furthermore, we find that the ground state degeneracy does not grow anymore if \( N_y \) reaches a certain value \( N_{y}^{\text{min}} \). This statement is most conclusive for the lower values \( n \), where \( N_{y}^{\text{min}} \) is smaller and we are thus able to probe more systems with \( N_y \geq N_{y}^{\text{min}} \). For \( n \in \{1, 3, 5\} \), this implies that all ground states are given by the corresponding states of Table III.

Finally, let us formulate a conjecture about the structure of the states annihilated by \( D_n^a \), which are ground states of \( H_n \). Our analytical results are consistent with the following rule: For each spin sector \( s \in \{1, 2, \ldots\} \), there is a series of states at levels \( k = s^2 + 2sj \) with \( j \in \{0, 1, 2, \ldots\} \). These states are annihilated by \( D_n^a \) with \( n = 2s - 1 + 4j \). As one can show by induction, the second rule implies that the ground state space of \( H_n \) with \( n = 2s - 1 \) contains the multiplet

\[
0 \oplus \begin{cases} 1 \oplus 3 \oplus \cdots \oplus s, & \text{if } s \text{ is odd,} \\ 2 \oplus 4 \oplus \cdots \oplus s, & \text{if } s \text{ is even.} \end{cases}
\]

The numerical results of Table III are consistent with this multiplet structure and thus support the conjecture that the values of \( n \) are given by \( n = 2s - 1 + 4j \).

Appendix C: Action of \( C^a \) on states built from current operators

Our starting point is the decoupling equation for the states \( \psi_{1 \ldots 1}^{a_1 \ldots a_k} \) derived in Ref. [33]. This equation describes the action of the operator

\[
C_i^a = \sum_{j \in \{1, \ldots, N\} \setminus \{i\}} \frac{z_i + z_j}{z_i - z_j} (t_j^a + i \varepsilon_{abc} t_i^b c_j^c) \quad \text{(C1)}
\]

on the states \( \psi_{1 \ldots 1}^{a_1 \ldots a_k} \) and follows from the CFT null field

\[
(K_k^a)_{ij} (J_{-1}^b \phi_a) (z_i) \quad \text{with} \quad (K_k^a)_{ij} = \delta_{ab} - i \varepsilon_{abc} f_i^c. \quad \text{(C2)}
\]

[The definition of \( C_i^a \) used here differs from that of Ref. [33] by a factor of 2/3.]
The operator
\[ C^a = \sum_{i=1}^{N} C_i^a \]  
was used in Sec. B to construct parent Hamiltonians for states built from current operators.

The decoupling equation reads
\[ C_i^a |\psi_1^{a_k \ldots a_1} \rangle = \sum_{q=1}^{k} \frac{(K_{a q})_i z_i}{z_i} |\psi_1^{a_k \ldots a_{q+1}a_{q-1} \ldots a_1} \rangle + (K_{b q})_i T^b_i |\psi_1^{a_k \ldots a_1} \rangle \\
+ 2(K_{q b})_i \sum_{s_1, \ldots, s_N} \sum_{q=2}^{k} \frac{z_i}{z_i} \varepsilon_{b a c} \Phi_s(z)|J_{n-1}^a J_{n-1}^b \ldots J_{n-1}^a|0\rangle \langle s_1, \ldots, s_N|, \]

where
\[ \Phi_s(z) = \phi_{s_1}(z_1) \ldots \phi_{s_N}(z_N). \]

The decoupling equation for the states \( \psi_1^{a_k \ldots a_1} \), which all mode numbers are one, is enough to describe the action of \( C_i^a \) on states with general mode numbers \( \psi_1^{a_k \ldots a_1} \). Using the Kac-Moody algebra of Eq. (2), the latter can be expressed in terms of the states \( \psi_1^{a_k \ldots a_1} \) by repeated application of
\[ J_{n-1}^a = \frac{i}{2} \varepsilon_{a b c} [J_{n-1}^c, J_{n-1}^b] \quad (n \neq 0). \]

On the cylinder, we have
\[ \sum_{i=1}^{N} (z_i)^{-n} = 0, \quad \text{if } n \mod N_y \neq 0. \]

Summing over \( i \) in Eq. (C4), we therefore obtain for \( k < N_y \)
\[ C_i^a |\psi_1^{a_k \ldots a_1} \rangle = (N-1) T^a_i |\psi_1^{a_k \ldots a_1} \rangle + \sum_{q=1}^{k} i \varepsilon_{a q c} |\psi_1^{a_k \ldots a_{q+1}a_{q-1} \ldots a_1} \rangle \\
+ \sum_{s_1, \ldots, s_N} \sum_{q=2}^{k} G_{a k \ldots a_1}^{q n} (s_1, \ldots, s_N) |s_1, \ldots, s_N\rangle, \]

with
\[ G_{a k \ldots a_1}^{q n} (s_1, \ldots, s_N) = 2\Phi_s(z)|J_{n-1}^a J_{n-1}^b \ldots J_{n-1}^a J_{n-1}^a \ldots J_{n-1}^a|0\rangle \langle s_1, \ldots, s_N| \\
- 2\delta_{a q} \Phi_s(z)|J_{n-1}^c J_{n-1}^b \ldots J_{n-1}^a J_{n-1}^a \ldots J_{n-1}^a|0\rangle \langle s_1, \ldots, s_N|. \]

The first two terms on the right hand side of Eq. (C8) are of order \( k \) in current operators since
\[ T^a_i |\psi_1^{a_k \ldots a_1} \rangle = \sum_{q=1}^{k} i \varepsilon_{a q c} |\psi_1^{a_k \ldots a_{q+1}a_{q-1} \ldots a_1} \rangle. \]

In the remaining terms, the modes \( J_n^a \) and \( J_n^c \) can be commuted to the right since \( J_n^a|0\rangle = 0 \) for \( n \geq 0 \):
\[ (J_n^a J_{n-1}^{a-1} \ldots J_{n-1}^{a_1}|0\rangle|0\rangle = \sum_{r=1}^{q-1} i \varepsilon_{a a d} (J_{n-1}^{d-1} \ldots J_{n-1}^{d_{r+1}} J_{n-1}^d J_{n-1}^{d_{r-1}} \ldots J_{n-1}^d)_{a_{r+1} \ldots a_1}|0\rangle|0\rangle, \]

and similarly for \( (J_n^c J_{n-1}^{a-1} \ldots J_{n-1}^{a_1}|0\rangle|0\rangle \). Iterating this step, the current operator modes with a positive mode number can be eliminated. The resulting terms only have negative mode numbers and are all of order \( k \) in current operators.
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