Boundedness of the solution of a higher-dimensional parabolic–ODE–parabolic chemotaxis–haptotaxis model with generalized logistic source

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Abstract
This paper deals with a quasilinear chemotaxis–haptotaxis system with generalized logistic source

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \nabla \cdot (\phi(u)\nabla u) - \nabla \cdot (u \nabla v) - \nabla \cdot (u \nabla w) + u(1 - u^{r-1} - w), \\
\frac{\partial v}{\partial t} &= \Delta v - v + u, \\
\frac{\partial w}{\partial t} &= -v w,
\end{align*}
\]

under homogeneous Neumann boundary conditions in a smooth bounded domain \(\mathbb{R}^N(N \geq 3)\), with parameter \(r > 1\), where the given function \(\phi(u)\) is the nonlinear diffusion. Besides appropriate smoothness assumptions, in this paper it is only required that \(\phi(u) \geq C_0(u + 1)^{m-1}\) for all \(u \geq 0\) with some \(C_0 > 0\) and some

\[
\begin{align*}
2 - \frac{2}{N} &\quad \text{if } 1 < r < \frac{N+2}{N}, \\
\frac{(N+2-r)^+}{N+2} &\quad \text{if } \frac{N+2}{2} \geq r > \frac{N+2}{N}, \\
1 &\quad \text{if } r \geq \frac{N+2}{2},
\end{align*}
\]
It is shown that then for all reasonably regular initial data, a corresponding initial-boundary value problem for (0.1) possesses a unique global classical solution that is uniformly bounded in \( \Omega \times (0, \infty) \).

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1. Introduction

Chemotaxis, a biological process in which cells migrate towards higher concentrations of a chemical signal, has received great interest in biological and mathematical communities (see Winkler et al [1], Hillen and Painter [9], Maini [16]). In 1970, Keller and Segel (see Keller and Segel [14, 15]) proposed a pioneering mathematical model for the sake of describing chemotaxis. In order to describe processes of cancer cell invasion of surrounding healthy tissue, a more complex cell migration mechanism was proposed by Chaplain and Lolas (Chaplain and Lolas [3, 4]) which can be an important extension of the classical Keller–Segel model. To be more precise, following [1, 3, 4, 22], we consider the initial-boundary value problem of the following quasilinear chemotaxis–haptotaxis system

\[
\left\{
\begin{array}{l}
u_t = \nabla \cdot (\phi(u) \nabla u) - \chi \nabla \cdot (u \nabla v) - \xi \nabla \cdot (u \nabla w) + u(1 - u^{r-1} - w), \quad x \in \Omega, t > 0, \\
\nu = \Delta v + u - v, \quad x \in \Omega, t > 0, \\
w_t = -vw, \quad x \in \Omega, t > 0, \\
\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, \quad x \in \partial \Omega, t > 0, \\
u(x, 0) = u_0(x), v(x, 0) = v_0(x), w(x, 0) = w_0(x), \quad x \in \Omega,
\end{array}
\right.
\tag{1.1}
\]

where \( r > 1, \Omega \) is a bounded domain in \( \mathbb{R}^N (N \geq 3) \) with smooth boundary \( \partial \Omega, \Delta = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} \) denotes the outward normal derivative on \( \partial \Omega \). The model (1.1) was proposed in [22] (see Chaplain and Lolas [4] and Winkler et al [1, 22]) to describe the cancer cell invasion into surrounding healthy tissue. The positive parameters \( \chi \) and \( \xi \) are the chemotaxis and haptotaxis coefficients, respectively. \( \phi(u) \) depicts the density-dependent motility of cancer cells through the ECM. As mentioned by [1, 22], in this modeling context, the cancer cells are also usually assumed to follow a generalized logistic growth \( u(1 - u^{r-1} - w) \) (\( r > 1 \)), denotes the proliferation rate of the cells and competing for space with healthy tissue.

If \( w \equiv 0 \), the PDE system (1.1) is reduced to the chemotaxis-only system with quasilinear chemotaxis system with (generalized) logistic source

\[
\left\{
\begin{array}{l}
u_t = \nabla \cdot (\phi(u) \nabla u) - \chi \nabla \cdot (u \nabla v) + u(1 - u^{r-1}), \quad x \in \Omega, t > 0, \\
v = \Delta v + u - v, \quad x \in \Omega, t > 0.
\end{array}
\right.
\tag{1.2}
\]

In the last four decades, considerable progress has been made in the analysis of various particular cases of (1.2). Indeed, if problem (1.2) is free of the (generalized) logistic source, there have been many papers concerned with the global boundedness or blow-up of the solutions, one can refer to Cieślak et al [5, 6], Winkler et al [10, 23, 32] and the references therein for some related works on it.

On the other hand, (generalized) logistic-type growth restrictions have been detected to prevent any chemotactic collapse in some systems closely related to (1.2). Indeed, if \( D \equiv 1 \) and
the coefficient of the standard logistic term is large enough, Winkler [31] proved that problem (1.2) possesses a unique global-in-time and bounded classical solution for all sufficiently smooth initial data. While, for generalized logistic term \( r > 1 \), one can see Zheng [33, 34] and the references therein.

Before stating our main results about the model (1.1), let us mention the following quasilinear chemotaxis–haptotaxis system (standard logistic growth, \( r = 2 \)), which is a closely related variant of (1.1)

\[
\begin{aligned}
&u_t = \nabla \cdot (\phi(u)\nabla u) - \chi \nabla \cdot (u \nabla v) - \xi \nabla \cdot (u \nabla w) + \mu u(1 - u - w), \quad x \in \Omega, t > 0, \\
v_t = \Delta v + u - v, \quad x \in \Omega, t > 0, \\
w_t = -\nu w, \quad x \in \Omega, t > 0, \\
\end{aligned}
\]

\[
\begin{aligned}
\partial u \bigg|_{\partial \Omega} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, \quad x \in \partial \Omega, t > 0, \\
u(x, 0) = u_0(x), v(x, 0) = v_0(x), w(x, 0) = w_0(x), \quad x \in \Omega,
\end{aligned}
\]

(1.3)

where \( \mu \geq 0, \tau = 0 \) or 1, the function \( \phi(u) \) fulfills

\[
\phi \in C^1([0, \infty))
\]

and there exist constants \( m \geq 1 \) and \( C_m \) such that

\[
\phi(u) \geq C_m(u + 1)^{m-1} \text{ for all } u \geq 0.
\]

There are only few results on the mathematical analysis of this quasilinear chemotaxis–haptotaxis system (1.3). In fact, when \( \tau = 0 \), MDEs diffuses much faster than cells (see [12, 25]), that is, the second equation in (1.3) can be approximated by an elliptic equation, this together with the other two equations in (1.3) and the initial and boundary conditions, makes up a parabolic–ODE–elliptic chemotaxis–haptotaxis system (see Tao and Winkler [24, 26]). For the special case \( \phi(u) = 1 \) in (1.3), Tao and Winkler [21] proved that model (1.3) possesses a unique global bounded classical solution for any \( \mu > 0 \) in two space dimensions, and for large enough \( \mu > 0 \) in three space dimensions. In [24], Tao and Winkler also gave an explicit condition on \( \mu (\mu > \chi) \) to ensure global existence and boundedness and analyzed the stability and attractivity properties of the non-trivial homogeneous equilibrium \((1,1,0)\). While in [25], Tao and Winkler studied the global boundedness for model (1.3) under the condition \( \mu > \frac{(N-2)\chi}{N} \), moreover, in additional explicit smallness on \( w_0 \), they achieved the exponential decay of \( w \) in the large time limit. Moreover, other types of model that are similar to (1.3) have been studied by some authors (see our recent paper [35]).

However, for the linear parabolic–ODE–parabolic chemotaxis–haptotaxis system (i.e. \( \tau = 1 \) and \( \phi = 1 \)), the results with regards to global existence, boundedness and large time behavior of solutions can be obtained mainly in lower dimensions (less than three dimensions) space (Cao [2], Winkler et al [17–20, 28]). In particularly, if \( N = 3 \), Stinner et al [17] showed that (1.3) exist at least one global weak solutions. Tao and Wang [20] proved that model (1.1) possesses a unique global-in-time classical solution for any \( \chi > 0 \) in one space dimension, and for small \( \mu > 0 \) in two and three space dimensions. Tao [18] improved the result of [20] for any \( \mu > 0 \) in two space dimensions.

If \( \phi \) is a nonlinear function of \( u \), then (1.3) becomes a quasilinear parabolic–ODE–parabolic chemotaxis–haptotaxis system. In two space dimension, Zheng et al [36] studied the global boundedness for model (1.3) with \( \phi \) satisfies (1.4) and (1.5) and \( m > 1 \), moreover, in additional explicit smallness on \( w_0 \), they gave the exponential decay of \( w \) in the large time limit. Moreover, if \( \phi \) satisfying (1.4) and (1.5) with \( m > \max\{1, \bar{m}\} \) and
Tao and Winkler [22] proved that model (1.1) possesses at least one nonnegative global classical solution, however, their boundedness is left as an open problem. Other authors discussed the logistic types of model which are similar to (1.3) (see our recent paper [13]).

Motivated by the above works, the aim of the present paper is to study the quasilinear chemotaxis system (1.1) under the conditions (1.4) and (1.5). We find that the existence of the solution to the system depends on the nonlinear diffusion and the generalized logistic source term. It is worth to remark the main idea underlying the proof of our results. The essential novelty in our approach, to be presented in lemma 3.5, consists of a subtle combination of entropy like estimates for

\[ \int_{\Omega} (u + 1)^p + \int_{\Omega} |\nabla v|^{2\beta} \] for any \( p > 1 \) and \( \beta > 1 \).

Finally, in section 3, we complete the proofs of theorems 1.1. For this purpose, in lemma 3.2 we shall first involve the variation-of-constants formula for variable \( v \) to gain

\[ v(t + \tau) = e^{-\tau(A + 1)}v(t) + \int_{t}^{t + \tau} e^{-(\tau - \tau')A + 1}u(s)ds, \quad t \in (0, T_{\max} - \tau) \]

for some \( \tau > 0 \) and the operator \( A \) defined below. Then with the help of boundedness of \( \int_{0}^{T} \int_{\Omega} u' dxdt \), we can obtain

\[ \int_{\Omega} |\nabla v|^p \leq C \text{ for all } t \in (0, T_{\max}) \]

for \( \sigma \in \{1, \max\{\frac{N_r}{N + 2 - r}, \frac{N}{N - 1}\}\} \), and thereby establish the a priori estimates of the functional (1.7).

**Theorem 1.1.** Assume that \( \phi \) satisfy \((1.4)\) and \((1.5)\) and the initial data \((u_0, v_0, w_0)\) fulfills

\[
\begin{align*}
&u_0 \in C(\bar{\Omega}) \text{ with } u_0 \geq 0 \text{ in } \Omega \text{ and } u_0 \neq 0, \\
&v_0 \in W^{1,\infty}(\Omega) \text{ with } v_0 \geq 0 \text{ in } \Omega, \\
&w_0 \in C^{2+\gamma}(\bar{\Omega}) \text{ with } w_0 \geq 0 \text{ in } \bar{\Omega} \text{ and } \frac{\partial w_0}{\partial \nu} = 0 \text{ on } \partial \Omega
\end{align*}
\]

with some \( \gamma \in (0, 1) \). If

\[
\begin{align*}
m &> 2 - \frac{2}{N} \text{ if } 1 < r < \frac{N + 2}{N}, \\
m &> 1 + \left(\frac{N + 2 - 2r}{N + 2}\right)^+ \text{ if } \frac{N + 2}{2} \leq r \geq \frac{N + 2}{N}, \\
m &\geq 1 \text{ if } r > \frac{N + 2}{2},
\end{align*}
\]

with some \( \gamma \in (0, 1) \). If

\[
\begin{align*}
m &> 2 - \frac{2}{N} \text{ if } 1 < r < \frac{N + 2}{N}, \\
m &> 1 + \left(\frac{N + 2 - 2r}{N + 2}\right)^+ \text{ if } \frac{N + 2}{2} \leq r \geq \frac{N + 2}{N}, \\
m &\geq 1 \text{ if } r > \frac{N + 2}{2},
\end{align*}
\]

with some \( \gamma \in (0, 1) \).
then there exists a triple \((u, v, w) \in (C^0(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty)))^3\) which solves (1.1) in the classical sense. Moreover, both \(u\), \(v\) and \(w\) are bounded on \(\Omega \times (0, \infty)\).

**Remark 1.1.**

(i) Obviously, if \(r = 2\), then \(\frac{2N}{N+2} - \frac{2}{N}(N \geq 3)\), hence theorem 1.1 extends (or partly extends) the results of theorem 1.1 of Wang [30] and theorem 1.1 of our recent paper [13].

(ii) Obviously, if \(r = 2\), then \(\frac{2N}{N+2} < \hat{m}(N \geq 3)\), therefore, theorem 1.1 extends the results of corollary 1.2 of Tao and Winkler [22], who showed the global existence of solutions the cases \(m > \hat{m}\), where \(\hat{m}\) is given by (1.6).

(iii) If \(w \equiv 0\), theorem 1.1 extends the results of theorem 1.1 of Wang et al [29].

(iv) With the help of precise estimation, the idea of our paper can also be used to deal with the three-dimensional chemotaxis-fluid system with generalized logistic source.

### 2. Preliminaries and main results

Firstly, we recall some preliminary lemmas, which play essential roles in our subsequent analysis. To begin with, let us collect some basic solution properties which essentially have already been used in [10].

**Lemma 2.1.** [10] For \(p \in (1, \infty)\), let \(A = A_p\) denote the sectorial operator defined by

\[
A_p u := -\Delta u \quad \text{for all } u \in D(A_p) := \{ \varphi \in W^{2,p}(\Omega) | \frac{\partial \varphi}{\partial \nu} |_{\partial \Omega} = 0 \}. \tag{2.1}
\]

The fact that the spectrum of \(A\) is a \(p\)-independent countable set of positive real numbers \(0 = \mu_0 < \mu_1 < \mu_2 < \cdots\) entails the following consequences: (i) The operator \(A + 1\) possesses fractional powers \((A + 1)^{-\alpha}\), the domains of which have the embedding properties

\[
D((A + 1)^{-\alpha}) \hookrightarrow W^{1,p} \text{ if } \alpha > \frac{1}{2}. \tag{2.2}
\]

(ii) Moreover, for all \(1 \leq p < q < \infty\) and \(u \in L^p(\Omega)\) the general \(L^p-L^q\) estimate

\[
\| (A + 1)^{-\alpha} e^{-tA} u \|_{L^q(\Omega)} \leq C e^{-\alpha \frac{N}{2} \frac{1}{p} - \frac{1}{q}} e^{(1-\mu)\frac{t}{p}} \| u \|_{L^p(\Omega)} \tag{2.3}
\]

for any \(t > 0\) and \(\alpha \geq 0\) with some \(\mu > 0\).

**Lemma 2.2.** [8, 11] Let \(s \geq 1\) and \(q \geq 1\). Assume that \(p > 0\) and \(a \in (0, 1)\) satisfy

\[
\frac{1}{2} - \frac{p}{N} = (1 - a) \frac{q}{s} + a \left( \frac{1}{2} - \frac{1}{N} \right) \text{ and } p \leq a.
\]

Then there exist \(c_0, c'_0 > 0\) such that for all \(u \in W^{1,2}(\Omega) \cap L^\infty(\Omega),\)

\[
\| u \|_{W^{1,2}(\Omega)} \leq c_0 \| \nabla u \|_{L^2(\Omega)} + c'_0 \| u \|_{L^\infty(\Omega)}.
\]

The following local existence result is rather standard. Since a similar reasoning in [13, 22], see for example and we omit it.
Lemma 2.3. Assume that the nonnegative functions $u_0, v_0,$ and $w_0$ satisfies (1.9) for some $\vartheta \in (0, 1), \phi$ satisfies (1.4) and (1.5). Then there exists a maximal existence time $T_{\text{max}} \in (0, \infty)$ and a triple of nonnegative functions

\[
\begin{align*}
  u &\in C^0(\Omega \times [0, T_{\text{max}})) \cap C^{1,1}(\Omega \times (0, T_{\text{max}})), \\
v &\in C^0(\Omega \times [0, T_{\text{max}})) \cap C^{1,1}(\Omega \times (0, T_{\text{max}})), \\
w &\in C^{1,1}(\Omega \times [0, T_{\text{max}})).
\end{align*}
\]

which solves (1.1) classically and satisfies $w \leq \|w_0\|_{L^\infty(\Omega)}$ in $\Omega \times (0, T_{\text{max}})$. Moreover, if $T_{\text{max}} < +\infty$, then

\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} \to \infty \text{ as } t \to T_{\text{max}}^-.
\]  (2.4)

3. The proof of theorem 1.1

In this section, we are going to establish an iteration step to develop the main ingredient of our result. The iteration depends on a series of *a priori* estimates. Firstly, based on the ideas of lemma 2.2 in [27], we recall now a well-known property of systems of type (1.1) with a generalized logistic source exhibiting a decay with respect to $u$ in the first equation.

Lemma 3.1. There exists $C > 0$ such that the solution of (1.1) satisfies

\[
\int_\Omega u \leq C \text{ for all } t \in (0, T_{\text{max}}) \tag{3.1}
\]

and

\[
\int_t^{t+\tau} \int_\Omega u' \leq C \text{ for all } t \in (0, T_{\text{max}} - \tau), \tag{3.2}
\]

where

\[
\tau := \min\{1, \frac{1}{6}T_{\text{max}}\}. \tag{3.3}
\]

Lemma 3.2. Let $q_0 \in [1, \max\left\{\frac{N}{N-1}, \frac{N}{(N+2)\vartheta}\right\}].$ Then there exists a positive constant $C$ depending on $q_0$ such that the solution of (1.1) satisfies

\[
\int_\Omega |\nabla v|^{q_0} \leq C \text{ for all } t \in (0, T_{\text{max}}). \tag{3.4}
\]

Proof. Firstly, integrating the second equation in (1.1) with respect to space, we have

\[
\frac{d}{dt} \int_\Omega v + \int_\Omega v = \int_\Omega u.
\]  (3.5)

This together with (3.1) yields that

\[
\int_\Omega v(x, t) \leq C + (\int_\Omega v_0(x) - C)e^{-t} \text{ for all } t \in (0, T_{\text{max}}). \tag{3.6}
\]
Hence, with the help of a standard regularity argument involving the variation-of-constants formula for $v$ and $L^p$-$L^q$ estimates for the heat semigroup (see e.g. lemma 4.1 of [10]), we conclude that

$$
\int_\Omega |\nabla v|^q \leq C \text{ for all } t \in (0, T_{\text{max}}),
$$

(3.7)

where $q \in [1, \frac{N}{N-1})$. Now, let us pick the above $\tau \in (0, T_{\text{max}})$. Then by the regularity principle asserted by lemma 2.3, we have $(u(\cdot, \tau), v(\cdot, \tau)) \in C^2(\Omega)$ with $\frac{d^2 u(\cdot, \tau)}{dt^2} = 0$ on $\partial \Omega$, so that in particular, one can pick $K_0 > 0$ which depends on $\tau$ such that

$$
\|u(s)\|_{L^\infty(\Omega)} \leq K_0 \text{ and } \|v(s)\|_{L^\infty(\Omega)} \leq K_0 \text{ for all } s \in [0, \tau].
$$

(3.8)

Therefore, in light of the first equation of (1.1), due to the standard parabolic regularity theory, we derive that there exists a positive constant $K := K(K_0, \tau)$ such that

$$
\|\nabla v(s)\|_{L^\infty(\Omega)} \leq K \text{ for all } s \in [0, \tau].
$$

(3.9)

Next, we fix $q < \frac{Nr}{(N+2-r)}$ and choose some $\alpha > \frac{1}{2}$ such that

$$
q < \frac{N}{2} - \frac{1}{2} + \frac{1}{2} + \frac{N}{2} - \frac{r-1}{r} < \frac{N}{2} + \frac{N}{2r} - \frac{r-1}{r} \leq \frac{Nr}{(N+2-r)^r},
$$

(3.10)

which implies that

$$
[-\alpha - \frac{N}{2r} - \frac{1}{q}] \frac{r}{r-1} > -1.
$$

(3.11)

Now, involving the variation-of-constants formula for $v$, we have

$$
v(t + \tau) = e^{-\tau(A+1)}v(t) + \int_t^{t+\tau} e^{-\tau(t+s)}(A+1)u(s)ds, \ t \in (0, T_{\text{max}} - \tau).
$$

(3.12)

Hence, it follows from (2.3), (3.6), (3.12) and the Young inequality that

$$
\|(A + 1)^\alpha v(t + \tau)\|_{L^p(\Omega)} \\
\leq C(q) \int_t^{t+\tau} (t + \tau - s)^{-\alpha - \frac{N}{2r} \frac{1}{2} + \frac{1}{2} e^{-\mu(t+s)}\|u(s)\|_{L^p(\Omega)} ds + c\tau^{-\alpha - \frac{N}{2r} \frac{1}{2} - \frac{1}{2}} v(t) + \int_t^{t+\tau} \|u(s)\|_{L^p(\Omega)} ds
$$

\leq C(q) \left( \int_t^{t+\tau} (t + \tau - s)^{-\alpha - \frac{N}{2r} \frac{1}{2} + \frac{1}{2} e^{-\mu(t+s)}\|u(s)\|_{L^p(\Omega)} ds \right)^{\frac{r-1}{r-1}} \left( \int_t^{t+\tau} \|u(s)\|_{L^p(\Omega)} ds \right)^{\frac{r}{r-1}}
$$

+ c\tau^{-\alpha - \frac{N}{2r} \frac{1}{2} - \frac{1}{2}} (C + (\int_\Omega v_0(x) - C)e^{-\tau})

\leq C(q) \left( \int_0^{t+\tau} (t + \tau - s)^{-\alpha - \frac{N}{2r} \frac{1}{2} + \frac{1}{2} e^{-\mu(t+s)}\|u(s)\|_{L^p(\Omega)} ds \right)^{\frac{r-1}{r-1}} C^{\frac{1}{r}} + c\tau^{-\alpha - \frac{N}{2r} \frac{1}{2} - \frac{1}{2}} (C + (\int_\Omega v_0(x) - C)e^{-\tau}).
$$

(3.13)

Hence, due to (3.11), (3.13) and (2.2), we have

$$
\int_\Omega |\nabla v(t + \tau)|^q \leq C \text{ for all } t \in (0, T_{\text{max}} - \tau)
$$

(3.14)
for all $q \in [1, \frac{N}{(N+2-r)/2}]$. Finally, in view of (3.7), (3.9) and (3.14), we can get (3.4).

Base on lemma 3.2, we can obtain an a priori estimate for $u(\cdot, t)$ in $L^p(\Omega)$ for any $p > 1$. To do this, employing almost exactly the same arguments as in the proof of lemma 3.2 of [30] (the minor necessary changes are left as an easy exercise to the reader), we conclude the following lemma:

**Lemma 3.3.** Let $(u, v, w)$ be a solution to (1.1) on $(0, T_{\max})$. Then for any $k > 1$, there exists a positive constant $C_1$ which is independent of $k$ such that

$$
-\xi \int_{\Omega} (u + 1)^{k-1} \nabla \cdot (u \nabla w) \leq C_1 (\int_{\Omega} (u + 1)^{k(v + 1)} + k \int_{\Omega} (u + 1)^{k-1} |\nabla u|).
$$

(3.15)

**Proof.** Here and throughout the proof of lemma 3.3, we shall denote by $M(i \in N)$ several positive constants independent of $k$. Firstly, observing that the third equation of (1.1) is an ODE, we derive that

$$
w(x, t) = w_0(x) e^{-\int_0^t \nabla v(x, s) ds}, \quad (x, t) \in \Omega \times (0, T_{\max}).
$$

(3.16)

Hence, by a basic calculation, we conclude that

$$
\nabla w(x, t) = \nabla w_0(x) e^{-\int_0^t \nabla v(x, s) ds} - w_0(x) e^{-\int_0^t \nabla v(x, s) ds} \int_0^t \nabla v(x, s) ds, \quad (x, t) \in \Omega \times (0, T_{\max}).
$$

(3.17)

and

$$
\Delta w(x, t) \geq \Delta w_0(x) e^{-\int_0^t \nabla v(x, s) ds} - 2 \nabla w_0(x) \cdot \int_0^t \nabla v(x, s) ds e^{-\int_0^t \nabla v(x, s) ds} \int_0^t \nabla v(x, s) ds - w_0(x) e^{-\int_0^t \Delta v(x, s) ds}.
$$

(3.18)

On the other hand, for any $k \geq 1$, integrating by parts yields

$$
-\xi \int_{\Omega} (u + 1)^{k-1} \nabla \cdot (u \nabla w) = -\xi \frac{k-1}{k} \int_{\Omega} (u + 1)^k \Delta w - \xi \int_{\Omega} (u + 1)^{k-1} \Delta w
$$

$$
:= J_1 + J_2,
$$

(3.19)

where

$$
J_1 = -\xi \frac{k-1}{k} \int_{\Omega} (u + 1)^k \Delta w - \xi \int_{\Omega} (u + 1)^{k-1} \Delta w
$$

(3.20)

and

$$
J_2 = -\xi \int_{\Omega} (u + 1)^{k-1} \Delta w.
$$

(3.21)

Now, inserting (3.18) into (3.20) and using $v \geq 0$ and the Young inequality, we have
Next, due to the second equality of (1.1) and $u \geq 0$, we conclude that there exists a positive constant $M_3$ such that

$$J_1 \leq -\frac{k-1}{k} \int_{\Omega} (u + 1)^{k} \Delta w_0(x)e^{-\int_{0}^{t} v(x,s)ds} + \frac{k-1}{k} \int_{\Omega} (u + 1)^{k} w_0(x)e^{-\int_{0}^{t} v(x,s)ds} \int_{0}^{t} \Delta v(x,s)ds$$

$$+ \frac{2(k-1)}{k} \int_{\Omega} (u + 1)^{k} \nabla w_0(x) \cdot \nabla e^{-\int_{0}^{t} v(x,s)ds} \int_{0}^{t} \Delta v(x,s)ds$$

$$= -\frac{k-1}{k} \int_{\Omega} (u + 1)^{k} \Delta w_0(x)e^{-\int_{0}^{t} v(x,s)ds} + \frac{k-1}{k} \int_{\Omega} (u + 1)^{k} w_0(x)e^{-\int_{0}^{t} v(x,s)ds} \int_{0}^{t} \Delta v(x,s)ds$$

$$- \frac{2(k-1)}{k} \int_{\Omega} (u + 1)^{k} \nabla w_0(x) \cdot \nabla e^{-\int_{0}^{t} v(x,s)ds} \int_{0}^{t} \Delta v(x,s)ds$$

$$\leq \|\Delta w_0\|_{L^{\infty}(\Omega)} \int_{\Omega} (u + 1)^{k} + \frac{k-1}{k} \int_{\Omega} (u + 1)^{k} w_0(x)e^{-\int_{0}^{t} v(x,s)ds} \int_{0}^{t} \Delta v(x,s)ds$$

$$+ 2 \frac{(k-1)}{k} \int_{\Omega} (u + 1)^{k-1} \nabla u \cdot \nabla w_0(x)e^{-\int_{0}^{t} v(x,s)ds} + \frac{2(k-1)}{k} \int_{\Omega} (u + 1)^{k} \Delta w_0(x)e^{-\int_{0}^{t} v(x,s)ds}$$

$$\leq M_1 \int_{\Omega} (u + 1)^{k} + \frac{k-1}{k} \int_{\Omega} (u + 1)^{k} w_0(x)e^{-\int_{0}^{t} v(x,s)ds} \int_{0}^{t} \Delta v(x,s)ds$$

$$+ M_2 \int_{\Omega} (u + 1)^{k} \nabla u + \int_{\Omega} (u + 1)^{k}. \quad (3.22)$$

Here we have used the fact that $\frac{k}{d} \leq 1$ (for all $t \geq 0$). Collecting (3.22) with (3.23), we can get

$$J_1 \leq M_3 \int_{\Omega} (u + 1)^{k}(v + 1) + k \int_{\Omega} (u + 1)^{k-1} |\nabla u|). \quad (3.24)$$

The same argument as in the derivation of (3.25) then shows that there exists a positive constant $M_5$ such that

$$J_2 \leq M_5 \int_{\Omega} (u + 1)^{k}(v + 1) + k \int_{\Omega} (u + 1)^{k-1} |\nabla u|), \quad (3.25)$$

which together with (3.19) and (3.25) yields the result.

**Lemma 3.4.** Let $\Omega \subset \mathbb{R}^{N}$ be a bounded domain with smooth boundary. Furthermore, assume that $\phi$ and $m$ satisfies (1.4) and (1.5) and (1.10), respectively. Then for all $\beta > 1$, the solution of (1.1) from lemma 2.3 satisfies
where $C$ is a positive constant.

**Proof.** Multiplying both sides of the first equation in (1.1) by $(u + 1)^{k-1}$ and integrating over $\Omega$, we get

$$
\frac{d}{dt} \left( \|u + 1\|^2_{L^2(\Omega)} + \frac{1}{2\beta} \|\nabla v\|^2_{L^2(\Omega)} \right) + \frac{3(\beta - 1)}{4\beta^2} \int_\Omega |\nabla| |\nabla v|^2 + \frac{1}{2^{1+\gamma}} \int_\Omega (u + 1)^{k+r-1} + \frac{1}{2} \int_\Omega |\nabla v|^{2\beta - 2} |D^2 v|^2 + \int_\Omega |\nabla v|^{2\beta - 2} + \int_\Omega (u + 1)^{m+k-3} |\nabla u|^2
$$

\leq C \int_\Omega (u + 1)^{k+1-m} |\nabla v|^2 + \int_\Omega u^2 |\nabla v|^{2\beta - 2} + \int_\Omega v^{k+1} + C,

(3.26)

which combined with $-u' \leq 1 - \frac{1}{\gamma} (u + 1)'$ implies that

$$
\frac{d}{dt} \left( \|u + 1\|^2_{L^2(\Omega)} + (k - 1) \int_\Omega (u + 1)^{k-2} \phi(u) |\nabla u|^2 \right)
$$

\begin{align*}
&= -\chi \int_\Omega \nabla \cdot (u \nabla v)(u + 1)^{k-1} - \xi \int_\Omega \nabla \cdot (u \nabla w)(u + 1)^{k-1} \\
& \quad + \int_\Omega (u + 1)^{k-1} |u(1 - u^{\gamma-1}- w),
\end{align*}

(3.27)

which, by (1.5), we obtain

$$
(k - 1)C_\phi \int_\Omega (u + 1)^{m+k-3} |\nabla u|^2 \leq (k - 1) \int_\Omega (u + 1)^{k-2} \phi(u) |\nabla u|^2.
$$

(3.29)

Integrating by parts to the first term on the right hand side of (3.28) and from the second equation in (1.1) we obtain

$$
-\chi \int_\Omega \nabla \cdot (u \nabla v)(u + 1)^{k-1}
$$

\begin{align*}
&= (k - 1) \chi \int_\Omega u(a + 1)^{k-2} \nabla a \cdot \nabla v \\
& \leq \frac{(k - 1)C_\phi}{2} \int_\Omega (u + 1)^{m+k-3} |\nabla u|^2 + \int_\Omega (u + 1)^{k+1-m} |\nabla v|^2.
\end{align*}

(3.30)

On the other hand, due to (3.3), we have

$$
-\xi \int_\Omega \nabla \cdot (u \nabla w)(u + 1)^{k-1}
$$

\begin{align*}
&\leq C_1 \int_\Omega (u + 1)^{k}(v + 1) + C_1 k \int_\Omega (u + 1)^{k-1} |\nabla u| \\
& \leq C_1 \int_\Omega (u + 1)^{k}(v + 1) + \frac{(k - 1)C_\phi}{4} \int_\Omega (u + 1)^{m+k-3} |\nabla u|^2 + \frac{2k^2C_1^2}{C_\phi} \int_\Omega (u + 1)^{k+1-m}.
\end{align*}

(3.31)
Furthermore, inserting (3.29)–(3.31) into (3.28), we have
\[
\frac{1}{k} \frac{d}{dt} \|u(t)\|_{L^p(\Omega)}^2 + \frac{(k-1)C_\phi}{4} \int_\Omega (u + 1)^{m+k-3} |\nabla u|^2 + \frac{1}{2r} \int_\Omega (u + 1)^{k+r-1} \\
\leq \int_\Omega (u + 1)^{k+1-m} |\nabla v|^2 + C_1 \int_\Omega (u + 1)^{k} (\nu + 1) \\
+ \frac{2k^2 C_2^2}{C_\phi} \int_\Omega (u + 1)^{k+1-m} + \int_\Omega (u + 1)^{k},
\]
which together with the Young inequality implies that
\[
\frac{1}{k} \frac{d}{dt} \|u(t)\|_{L^p(\Omega)}^2 + \frac{(k-1)C_\phi}{4} \int_\Omega (u + 1)^{m+k-3} |\nabla u|^2 + \frac{1}{2r} \int_\Omega (u + 1)^{k+r-1} \\
\leq \int_\Omega (u + 1)^{k+1-m} |\nabla v|^2 + C_2 \int_\Omega v^{k+1} + C_2.
\]
where \(C_2 > 0\) depends on \(k, m, C_\nu, \mu, |\Omega|\) and \(C_\phi\).

Using that \(\nabla \cdot \nabla \Delta v = \frac{1}{2} \Delta |\nabla v|^2 - |D^2 v|^2\), by a straightforward computation using the second equation in (1.1) and several integrations by parts, we derive
\[
\frac{1}{2\beta} \frac{d}{dt} \|v(t)\|_{L^\beta(\Omega)}^{2\beta} = \int_\Omega |\nabla v|^{2\beta-2} \nabla v \cdot \nabla (\Delta v - v + u) \\
= \frac{1}{2} \int_\Omega |\nabla v|^{2\beta-2} \Delta |\nabla v|^2 - \int_\Omega |\nabla v|^{2\beta-2} |D^2 v|^2 \\
- \int_\Omega |\nabla v|^{2\beta} - \int_\Omega u \nabla \cdot (|\nabla v|^{2\beta-2} \nabla v) \\
= -\beta - \frac{1}{2} \int_\Omega |\nabla v|^{2\beta-4} |\nabla v| |\nabla v|^2 + \frac{1}{2} \int_\Omega |\nabla v|^{2\beta-2} \frac{\partial |\nabla v|^2}{\partial \nu} \\
- \int_\Omega |\nabla v|^{2\beta-2} |D^2 v|^2 - \int_\Omega u |\nabla v|^{2\beta-2} \Delta v - \int_\Omega u \nabla v \cdot \nabla (|\nabla v|^{2\beta-2}) \\
= -\frac{2(\beta - 1)}{\beta} \int_\Omega |\nabla v|^{2\beta-2} |\nabla v| |\nabla v|^2 + \frac{1}{2} \int_\Omega |\nabla v|^{2\beta-2} \frac{\partial |\nabla v|^2}{\partial \nu} \\
- \int_\Omega u |\nabla v|^{2\beta-2} \Delta v - \int_\Omega u \nabla v \cdot \nabla (|\nabla v|^{2\beta-2}) - \int_\Omega |\nabla v|^{2\beta} 
\]
for all \(t \in (0, T_{\text{max}})\). Here, in view of \(|\Delta v| \leq \sqrt{N} |D^2 v|\) and the Young inequality, we conclude
\[
\int_\Omega u |\nabla v|^{2\beta-2} \Delta v \leq \sqrt{N} \int_\Omega u |\nabla v|^{2\beta-2} |D^2 v| \\
\leq \frac{1}{4} \int_\Omega |\nabla v|^{2\beta-2} |D^2 v|^2 + N \int_\Omega u^2 |\nabla v|^{2\beta-2} 
\]
for all \(t \in (0, T_{\text{max}})\). Moreover, by the Cauchy–Schwarz inequality, we have
\[-\int_{\Omega} u \nabla v \cdot \nabla (|\nabla v|^{2\beta-2}) = -(\beta - 1) \int_{\Omega} u |\nabla v|^{2(\beta - 2)} \nabla v \cdot \nabla |\nabla v|^2 \leq \frac{\beta - 1}{8} \int_{\Omega} |\nabla v|^{2\beta - 4} |\nabla |\nabla v|^2|^2 + 2(\beta - 1) \int_{\Omega} |u|^2 |\nabla v|^{2\beta - 2} \leq \frac{(\beta - 1)}{2\beta^2} \int_{\Omega} |\nabla |\nabla v|^2|^2 + 2(\beta - 1) \int_{\Omega} |u|^2 |\nabla v|^{2\beta - 2}. \tag{3.36}\]

Next we deal with the integration on \(\partial \Omega\). We see from lemma 2.2 that

\[\int_{\partial \Omega} \frac{\partial |\nabla v|^2}{\partial \nu} |\nabla v|^{2\beta - 2} \leq C_0 \int_{\partial \Omega} |\nabla v|^{2\beta} = C_0 \|\nabla v\|^2 \|_{L^2(\partial \Omega)}. \tag{3.37}\]

Take \(\gamma \in (0, \frac{1}{2})\). Since the embedding \(W^{\gamma + \frac{1}{2}, 2}(\Omega) \hookrightarrow L^2(\partial \Omega)\) is compact (see e.g. Haroske and Triebel [7]), we have

\[\|\nabla |\nabla v|^\beta\|_{L^2(\partial \Omega)} \leq C_\|\nabla v\|^\beta\|_{W^{\gamma + \frac{1}{2}, 2}(\Omega)}. \tag{3.38}\]

In order to apply lemma 2.2 to the right-hand side of (3.38), let us pick \(a \in (0, 1)\) satisfying

\[a = \frac{1}{N} + \beta + \frac{\beta}{7} = \frac{1}{2}. \tag{3.39}\]

Noting that \(\gamma \in (0, \frac{1}{2})\) and \(\beta > 1\) imply that \(\gamma + \frac{1}{2} \leq a < 1\), we see from the fractional Gagliardo–Nirenberg inequality (lemma 2.2) and boundedness of \(|\nabla v|^\beta\|\) (see lemma 3.2) that

\[\|\nabla |\nabla v|^\beta\|_{W^{\gamma + \frac{1}{2}, 2}(\Omega)} \leq C_0 \|\nabla v\|^\beta\|_{L^2(\Omega)} + C_0 \|\nabla v\|^\beta\|_{L^2(\Omega)} \tag{3.39}\]

Combining (3.37) and (3.38) with (3.39), we obtain

\[\int_{\partial \Omega} \frac{\partial |\nabla v|^2}{\partial \nu} |\nabla v|^{2\beta - 2} \leq C_0 \|\nabla v\|^2 \|_{L^2(\partial \Omega)} + C_5. \tag{3.40}\]

Now, inserting (3.40)–(3.36) into (3.34) and using the Young inequality we can get

\[\frac{1}{2\beta} \|\nabla v\|^2_{L^2(\Omega)} + \frac{3(\beta - 1)}{2\beta^2} \int_{\Omega} |\nabla |\nabla v|^2|^2 + \frac{1}{2} \int_{\Omega} \int_{\Omega} |\nabla v|^{2\beta - 2} |D^2v|^2 + \int_{\Omega} |\nabla v|^{2\beta} \leq C_0 \int_{\Omega} u^2 |\nabla v|^{2\beta - 2} + C_6 \text{ for all } t \in (0, T_{\text{max}}). \tag{3.41}\]

which together with (3.33) and the Young inequality implies the result.
Lemma 3.5. Let $\Omega \subset \mathbb{R}^N (N \geq 3)$ be a bounded domain with smooth boundary. Furthermore, assume that $\phi$ satisfies (1.4) and (1.5) with $m > 1 + \frac{(N+2-r^*)}{N^*}$, respectively. If $\frac{N+2}{N} \leq r \leq \frac{N+2}{2}$, then for all $\beta > 1$ and $k > 1$ there exists $C > 0$ such that

$$
\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|\nabla v(\cdot, t)\|_{L^2(\Omega)} \leq C \text{ for all } t \in (0, T_{\text{max}}).
$$

(3.42)

Proof. Since $\frac{N+2}{N} \leq r \leq \frac{N+2}{2}$, then

$$
(\frac{N}{N^*-1}) \leq (\frac{N}{N^*-1}) \leq (\frac{N}{N^*-1}) \leq (\frac{N}{N^*-1}) < N.
$$

Hence, we may choose $q_0 \in [\max\{1, \frac{N(2-m)}{m}\}, \max\{\frac{N}{N^*-1}, (\frac{N}{N^*-1})\}]$ such that $q_0 < N$. Now, let

$$
\beta = \max\{\frac{7N+4}{2}, \frac{4(Nm_0 + q_0 - N - Nq_0)}{2(m-1)(N + q_0) - (N - q_0)}\}
$$

and $k_0(\beta) = \frac{m-1}{N-q_0} (2N\beta - Nq_0 + 2q_0) + 2 - \frac{2}{N} - m$.

then due to (1.10) and $N \geq 3$, we have

$$
\min\{k_0(\beta), 2\beta - 1\} > \frac{\beta - 4}{\beta q_0 + N\beta - 4Nq_0} (2N\beta - Nq_0 + 2q_0) + 2 - \frac{2}{N} - m
$$

$$
\geq \max\{1 - m + \frac{N - 2}{4N} \beta, m - \frac{2}{N} - \frac{2}{N} - m\}
$$

(3.43)

for all $\beta \geq \beta$. Therefore, we can choose

$$
k \in \left(\frac{\beta - 4}{\beta q_0 + N\beta - 4Nq_0} (2N\beta - Nq_0 + 2q_0) + 2 - \frac{2}{N} - m, \min\{k_0(\beta), 2\beta - 1\}\right).
$$

(3.44)

Now for the above $k$, by the Hölder inequality, we have

$$
J_1 = C \int_{\Omega} (u + 1)^{k+1-m} |\nabla v|^2
$$

$$
\leq C \left( \int_{\Omega} (u + 1)^{\frac{N}{N^*-2}(k+1-m)} \right)^{\frac{N^*-2}{N}} \left( \int_{\Omega} |\nabla v|^N \right)^{\frac{N}{N^*-2}}
$$

$$
= C \|(u + 1)^{\frac{k+1-m}{N-1}} \frac{2(k+1-m)}{N-2} \|_{L^{N^*-2}(\Omega)}^{\frac{N^*-2}{N}} \|\nabla v\|_{L^2(\Omega)}^N.
$$

(3.45)

Next, based on the fact $m > 1$ and $N > 2$, we have

$$
\frac{1}{k+m-1} \leq \frac{k+1-m}{k+m-1} \frac{N}{N^*-1} < \frac{N}{N^*-2} < \frac{N}{N-2}
$$

and

$$
\mu_1 = \frac{N[k+m-1]}{2} - \frac{N}{2} \geq \frac{N}{2} (k+1-m) = \frac{k+1-m}{2} \geq \frac{N}{2} + \frac{N}{2} (k+1-m) \in (0, 1),
$$

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from the Gagliardo-Nirenberg inequality, it is evident that
\[ C \| (u + 1) \|^2 \frac{2(k+1-m)}{L^2(\Omega)} \]
\[ \leq C_1 \| \nabla (u + 1) \|^2 \frac{2(k+1-m)}{L^2(\Omega)} + (u + 1) \frac{k+m-1}{2} \| \nabla (u + 1) \|^2 \frac{2(k+1-m)}{L^2(\Omega)} \]
\[ \leq C_2 \| \nabla (u + 1) \|^2 \frac{2(k+1-m)}{L^2(\Omega)} + 1 \]
\[ = C_2 \| \nabla (u + 1) \|^2 \frac{2(k+1-m)}{L^2(\Omega)} + 1 \]
(3.46)

with some positive constants \( C_1 \) and \( C_2 \).

On the other hand, in view of (3.4), due to the Gagliardo-Nirenberg inequality and the fact that \( \bar{\beta} > \frac{q_0}{N} + 1 > \frac{2q_0}{N} \), we have
\[ \| \nabla v \|^2 \frac{2}{L^2(\Omega)} \leq C_3 \| \nabla \| \nabla v \|^2 \frac{2}{L^2(\Omega)} + \| \nabla v \|^2 \frac{2}{L^2(\Omega)} \]
\[ \leq C_4 \| \nabla \| \nabla v \|^2 \frac{2}{L^2(\Omega)} + 1 \]
\[ = C_4 \| \nabla \| \nabla v \|^2 \frac{2}{L^2(\Omega)} + 1 \]
(3.47)

where some positive constants \( C_3, C_4 \) and
\[ \mu_2 = \frac{N \beta - N \beta}{q_0} = \beta - \frac{N \beta}{q_0} + \frac{N \beta}{q_0} \in (0, 1). \]

Here we have used the fact that \( N \geq 3 \) and \( m > 1 \). In light of \( m > 1 + \frac{(N + 2 - 2)\beta}{N + 2} \), inserting (3.46) and (3.47) into (3.45) and using the Young inequality and (3.44), we conclude that for any \( \delta > 0 \), there exist positive constants \( C_5 \) and \( C_6 \) such that
\[ J_t \leq C_5 \| \nabla (u + 1) \|^2 \frac{2(k+1-m)+2}{L^2(\Omega)} + \delta \| \nabla v \|^2 \| \nabla v \|^2 \| v_\Omega(t) \| + C_6 \text{ for all } t \in (0, T_{\text{max}}). \]
(3.48)

Here we have used the fact that
\[ 0 < \frac{2N(k - m) + 4}{N(k + m - 2) + 2} + \frac{4(N - q_0)}{2q_0} = Nq_0 + 2N\beta < 2 \]
and
\[
\frac{2N(k - m) + 4}{N(k + m - 2) + 2} > 0, \quad \frac{4(N - q_0)}{2q_0 - Nq_0 + 2N\beta > 0}.
\]

Next, in light of the Hölder inequality and \(\beta \geq \tilde{\beta} > 8\), we have
\[
J_2 = C \int_{\Omega} u^2 |\nabla v|^{\beta - 2} \leq C \left( \int_{\Omega} u^2 \right)^{\frac{4}{\beta}} \left( \int_{\Omega} |\nabla v|^{\frac{\beta(2\beta - 2)}{\beta - 8}} \right)^{\frac{\beta - 8}{\beta}}.
\]

\[
= C \left\| (u + 1)^{\frac{k + m - 1}{2}} \right\|_{L^{2/(\beta - 2)}(\Omega)} \left\| \nabla v^{\frac{2\beta - 2}{\beta - 8}} \right\|_{L^{\beta - 8/\beta}(\Omega)}. \tag{3.49}
\]

On the other hand, with the help of \(\beta > \frac{N - 2}{2N\beta \leq \tilde{\beta} > k > 1 - m + \frac{N - 2}{2N}\beta \geq \tilde{\beta} > \max\{4, q_0\}\) and the Gagliardo-Nirenberg inequality we conclude that there exist positive constants \(C_7\) and \(C_8\) such that
\[
C_7\left( \int_{\Omega} (u + 1)^{\frac{k + m - 1}{2}} \right)^{\frac{4}{k + m - 1}} \leq C_7\left( \int_{\Omega} |\nabla (u + 1)^{1 - \mu_3} |_{L^{2/(\beta - 2)}(\Omega)}^{\frac{4}{\beta - 8}} \right)^{\frac{\beta - 8}{\beta}},
\]

\[
= C_8\left( \int_{\Omega} |\nabla (u + 1)^{1 - \mu_3} |_{L^{2/(\beta - 2)}(\Omega)}^{\frac{4}{\beta - 8}} \right)^{\frac{\beta - 8}{\beta}} \tag{3.50}
\]

with
\[
\mu_3 = \frac{N[(k + m - 1)] - N(k + m - 1)}{2} = \frac{N - \frac{N[k + m - 1]}{2}}{2} \in (0, 1).
\]

On the other hand, by \(\beta \geq \tilde{\beta} > \frac{7N + 2}{2}\), we derive that
\[
\mu_4 = \frac{N[k + m - 1]}{2} = \beta \frac{N}{2} + \frac{N[2k + m - 1]}{2} \in (0, 1),
\]

hence, in light of (3.4), the Gagliardo-Nirenberg inequality asserts that
\[ \|\nabla v\|^{\frac{2(\beta-2)}{\beta-2+\gamma}}_{L^{\beta+\gamma}(\Omega)} = \|\nabla v\|^2_{L^\beta(\Omega)} \leq C_5(\|\nabla |\nabla v|^\beta\|^\frac{\beta}{\beta-2+\gamma}_{L^{\beta+\gamma}(\Omega)} + \|\nabla v\|^2_{L^\beta(\Omega)} + \|\nabla v\|^\beta_{L^\beta(\Omega)}) \]
\[ \leq C_{10}(\|\nabla |\nabla v|^\beta\|^\frac{\beta}{\beta-2+\gamma}_{L^{\beta+\gamma}(\Omega)} + 1) \]
\[ = C_{10}(\|\nabla |\nabla v|^\beta\|^\frac{2-N(4\beta^2-4\beta-2q_0\beta+16q_0)}{2-Nq_0\beta+2N\beta^2} + 1), \quad (3.51) \]

where \(C_9\) and \(C_{10}\) are positive constants.

Now, since \(m > 1 + \frac{(N+2-2\gamma^+)}{N+2}, \quad N \geq 3\) and \(q_0 \in \left[\max\left\{1, \frac{N(2-m)}{m}\right\}, \max\left\{\frac{N}{N-1}, \frac{N}{N+2-\gamma^+}\right\}\right),\) by virtue of (3.44), we derive that
\[ 0 < \frac{4N(\beta-4)}{\beta N(k+m-2)+2\beta} + \frac{N(4\beta^2-4\beta-2q_0\beta+16q_0)}{(2-N)q_0\beta+2N\beta^2} < 2 \]
and
\[ \frac{4N(\beta-4)}{\beta N(k+m-2)+2\beta} > 0, \quad N(4\beta^2-4\beta-2q_0\beta+16q_0) > 0, \]
inserting (3.50) and (3.51) into (3.49) and using \(k > \frac{\beta-4}{\beta q_0 + N\beta - 4Nq_0}(2N\beta - Nq_0 + 2q_0) + 2 - \frac{2}{N} - m\) and the Gagliardo-Nirenberg inequality, we derive that for any \(\delta > 0\), there exist positive constants \(C_{11}\) and \(C_{12}\) such that
\[ J_2 \leq C_{11}(\|\nabla (u + 1)\|^{\frac{k+m-1}{2}}_{L^\beta(\Omega)} + 1)(\|\nabla |\nabla v|^\beta\|^\frac{N(4\beta^2-4\beta-2q_0\beta+16q_0)}{2-Nq_0\beta+2N\beta^2} + 1) \]
\[ \leq \delta \int_{\Omega} |\nabla (u + 1)\|^{\frac{k+m-1}{2}}_{L^\beta(\Omega)} + \frac{\beta}{2} \|\nabla |\nabla v|^\beta\|^2_{L^\beta(\Omega)} + C_{12} \text{ for all } t \in (0, T_{\text{max}}). \quad (3.52) \]

Finally, an application of with the help of the Sobolev inequality, the Young inequality and (3.44), we have
\[ J_3 = C \int_{\Omega} v^{k+1} \]
\[ \leq C_{13}(\|\nabla v\|^{\frac{k+1}{L^\beta(\Omega)} + 1)} \]
\[ \leq C_{14}(\|\nabla v\|^{\frac{k+1}{L^\beta(\Omega)}} + 1) \]
\[ \leq C_{15}(\|\nabla v\|^{\frac{k+1}{L^\beta(\Omega)}} + 1) \]
\[ \leq \frac{1}{2} \|\nabla v\|^2_{L^\beta(\Omega)} + C_{16} \quad (3.53) \]
with $\beta \geq \bar{\beta} > \frac{N+1}{2}$ and some positive constants $C_i(i = 13, \ldots, 16)$. Now, inserting (3.48), (3.52) and (3.53) into (3.26) and using the Young inequality and choosing $\delta$ small enough yields

$$\frac{d}{dt} \frac{1}{k} \|u + 1\|_{L^2(\Omega)}^2 + \frac{1}{2\beta} \|\nabla v\|_{L^2(\Omega)}^{2\beta} + \frac{3(\beta - 1)}{8\beta^2} \int_{\Omega} |\nabla |\nabla v|^{2\beta}|^2 + \frac{1}{2^{\gamma+1}} \int_{\Omega} (u + 1)^{2^{\gamma+1}} + \frac{1}{2} \int_{\Omega} \nabla v^{2\beta - 2} |D^2v|^2 + \frac{(k-1)C_2}{8} \int_{\Omega} (u + 1)^{m+k-3} |\nabla u|^2$$

$$\leq C_{17} \text{ for all } t \in (0, T_{\max})$$

with some positive constant $C_{17}$. Therefore, letting $y := \int_{\Omega} (u + 1)^{2^{\gamma+1}} + \int_{\Omega} |\nabla v|^{2\beta}$, an elementary calculus entails that there exist positive constants $C_{18}$ and $C_{19}$ such that

$$\frac{d}{dt} y(t) + C_{18} y(t) \leq C_{19} \text{ for all } t \in (0, T_{\max}).$$

Thus a standard ODE comparison argument implies boundedness of $y(t)$ for all $t \in (0, T_{\max})$. Clearly, $\|u(\cdot, t)\|_{L^2(\Omega)}$ and $\|\nabla v(\cdot, t)\|_{L^2(\Omega)}$ are bounded for all $t \in (0, T_{\max})$. Obviously,

$$\lim_{\beta \to +\infty} \frac{-m}{\beta} = \lim_{\beta \to +\infty} \frac{m}{\beta} = \lim_{\beta \to +\infty} \min\{k_0(\beta), 2\beta - 1\} = +\infty,$$

hence, the boundedness of $\|u(\cdot, t)\|_{L^2(\Omega)}$ and the Hölder inequality implies the result. \qed

\textbf{Lemma 3.6.} \textit{Let $\Omega \subset \mathbb{R}^N(N \geq 3)$ be a bounded domain with smooth boundary. Furthermore, assume that $\phi$ satisfies (1.4) and (1.5) with $m \geq 1$. If $r > \frac{N+2}{2}$, then for all $\beta > 1$ and $k > 1$ there exists $C > 0$ such that}

$$\|u(\cdot, t)\|_{L^2(\Omega)} + \|\nabla v(\cdot, t)\|_{L^2(\Omega)} \leq C \text{ for all } t \in (0, T_{\max}).$$

\textbf{Proof.} Firstly, in light of $r > \frac{N+2}{2}$, we derive that max\{\frac{N}{N-1}, \frac{N}{N + 2 - r}\} > N$. Hence, we may choose $\theta > N$ such that

$$N < \theta < \min\{2N, \max\{\frac{N}{N-1}, \frac{N}{N + 2 - r}\}\}.$$

Now, let $\bar{\beta} = \frac{7\theta+4}{2}$, then due to (1.10) and $N \geq 3$, we have

$$2\beta - 1 > 2(\beta - 4) + 2 - \frac{2}{N} - m$$

$$> \frac{\beta - 4}{\beta \theta + N\beta - 4N\theta}(2N\beta - N\theta + 2\theta) + 2 - \frac{2}{N} - m$$

$$> \frac{\beta - 4}{3} + 2 - \frac{2}{N} - m$$

$$> \max\{1 - m, \frac{N - 2}{4N}, \frac{2}{N}, \frac{2}{N} - \frac{2}{N} - m\}$$

for all $\beta \geq \bar{\beta}$. Therefore, we can choose

$$k \in \left(\frac{\beta - 4}{\beta \theta + N\beta - 4N\theta}(2N\beta - N\theta + 2\theta) + 2 - \frac{2}{N} - m, 2\beta - 1\right).$$
Now for the above $k$, thanks to the Hölder inequality, (3.4) and (3.57), we conclude that there exists a positive constant $C_1$ such that

$$J_1 = C \int_\Omega (u + 1)^{k+1-m} |\nabla v|^2$$

$$\leq C \left( \int_\Omega (u + 1)^{\frac{\theta}{\theta-2} (k+1-m)} \right)^{\theta-2} \left( \int_\Omega |\nabla v|^\theta \right)^{\frac{2}{\theta}}$$

$$= C \|(u + 1)^{\frac{k+m-1}{2}} \|_{L^\frac{2(k+1-m)}{k+m-1}(\Omega)}^2 \|\nabla v\|_{L^\theta(\Omega)}^2$$

$$\leq C_1 \|(u + 1)^{\frac{k+m-1}{2}} \|_{L^\frac{2(k+1-m)}{k+m-1}(\Omega)}^2 .$$

(3.60)

Since, $m \geq 1$ and $\theta > N > 2$, we have

$$\frac{1}{k+m-1} < \frac{k+1-m}{k+m-1} \frac{\theta}{\theta-2} < \frac{N}{N-2},$$

which together with the Gagliardo-Nirenberg inequality implies that

$$C_1 \|(u + 1)^{\frac{k+m-1}{2}} \|_{L^\frac{2(k+1-m)}{k+m-1}(\Omega)}^2 \leq C_2 \|(\nabla (u + 1)^{\frac{k+m-1}{2}} \|_{L^\frac{6(k+1-m)}{4k+4m-21}(\Omega)}^2 + \|(u + 1)^{\frac{k+m-1}{2}} \|_{L^\frac{2(k+1-m)}{k+m-1}(\Omega)}^2 \|_{L^\frac{6(k+1-m)}{4k+4m-21}(\Omega)}^2$$

$$\leq C_3 \|(\nabla (u + 1)^{\frac{k+m-1}{2}} \|_{L^\frac{6(k+1-m)}{4k+4m-21}(\Omega)}^2 + 1)$$

$$= C_3 \|(\nabla (u + 1)^{\frac{k+m-1}{2}} \|_{L^\frac{6(k+1-m)}{4k+4m-21}(\Omega)}^2 + 1)$$

(3.61)

with some positive constants $C_2, C_3$ and

$$\mu_1 = \frac{N(k+m-1)}{2} - \frac{N(k+m-1)}{2} \frac{k+m-1}{k+m-1} \frac{N}{N-2} \frac{N}{N-2} \left[ k+m-1 \right] \frac{N}{N-2} \frac{N}{N-2} \left[ k+m-1 \right] \in (0, 1).$$

Due to $\theta > N$ and $m \geq 1$, we have

$$\frac{2N(k-m) + \frac{4N}{\theta}}{N(k+m-2) + 2} < 2.$$ (3.62)

Hence (3.61) and the Young inequality yields to for any $\delta > 0$, there exists $C_4 > 0$ such that

$$J_1 \leq \delta \int_\Omega |\nabla (u + 1)^{\frac{k+m-1}{2}} |^2 + C_4 \text{ for all } t \in (0, T_{\text{max}}).$$ (3.63)
Next, due to the Hölder inequality and $\beta \geq \bar{\beta} > 8$, we have

\[ J_2 = C \int_{\Omega} u^2 |\nabla v|^{2\beta-2} \]

\[ \leq C \left( \int_{\Omega} u^\beta \left( \int_{\Omega} |\nabla v|^{\beta(2\beta-2)} \right)^{\frac{\beta-8}{\beta}} \right)^{\frac{\beta}{2}} \]

\[ \leq C \left( \int_{\Omega} (u + 1)^{\beta} \left( \int_{\Omega} |\nabla v|^{\beta(2\beta-2)} \right)^{\frac{\beta-8}{\beta}} \right)^{\frac{\beta}{2}} \]

\[ = C \| (u + 1)^{k+m-1} \|^2 \| \nabla v \|^2_{L^{\beta(2\beta-2)}(\Omega)} . \] (3.64)

Now, with the help of $k > 1 - m + \frac{N-2}{4N}$, $\beta, m \geq 1$, $\beta \geq \bar{\beta} > \max\{4, \theta\}$ and the Gagliardo-Nirenberg inequality we conclude that

\[ C \| (u + 1)^{k+m-1} \|^2 \| \nabla v \|^2_{L^{\beta(2\beta-2)}(\Omega)} \]

\[ \leq C_{\delta}(\| \nabla (u + 1)^{k+m-1} \|^2 + \| (u + 1)^{k+m-1} \|^2_{L^{\frac{\beta-8}{N(\beta-4)}}(\Omega)})^{\frac{4}{\delta}} \]

\[ = C_{\delta}(\| \nabla (u + 1)^{k+m-1} \|^2 + 1) \] (3.65)

with some positive constants $C_5, C_6$ and

\[ \mu_3 = \frac{N(k+m-1)}{2} - \frac{N(k+m-1)}{2} = [k + m - 1] - \frac{N}{2} - \frac{N}{2} \in (0, 1). \]

On the other hand, in view of (3.4) and (3.57), it then follows from $\beta \geq \bar{\beta} > \frac{7N+2}{2}$ and the Gagliardo-Nirenberg inequality that

\[ \| \nabla v \|^2_{L^{\beta(2\beta-2)}(\Omega)} = \| \nabla v \|^2 \left( \frac{2\beta-2}{2\beta-2} \right)_{L^{2\beta-2}(\Omega)} \]

\[ \leq C_{\delta}(\| \nabla v \|^2_{L^{\beta(2\beta-2)}(\Omega)} + \| \nabla v \|^2_{L^{\beta(2\beta-2)}(\Omega)} + \| \nabla v \|^2_{L^{\beta(2\beta-2)}(\Omega)}) \]

\[ \leq C_{\delta}(\| \nabla v \|^2_{L^{\beta(2\beta-2)}(\Omega)} + 1) \]

\[ = C_{\delta}(\| \nabla v \|^2_{L^{\beta(2\beta-2)}(\Omega)} + 1) \] (3.66)
where $C_7$ and $C_8$ are positive constants and

$$
\mu_4 = \frac{N/3}{1 - N/2 + N/3} = \beta \frac{N}{1 - N/2 + N/3} \in (0, 1).
$$

Observe that

$$0 < \frac{4N(\beta - 4)}{\beta N(k + m - 2) + 2\beta} + \frac{N(\frac{2\beta - 2}{\theta} - \frac{\beta - 8}{\beta})}{1 - N/2 + N/3} < 2$$

and

$$\frac{4N(\beta - 4)}{\beta N(k + m - 2) + 2\beta} > 0, \quad \frac{N(\frac{2\beta - 2}{\theta} - \frac{\beta - 8}{\beta})}{1 - N/2 + N/3} > 0.$$

Inserting (3.65) and (3.66) into (3.64) and using the Gagliardo-Nirenberg inequality, we have

$$J_2 \lesssim C_{1a}(\|\nabla(u + 1)^{N/3}\|^{(k + m - 1)/2} + \|\nabla|\nabla v|^{\frac{3}{2}}\|_{L^p(\Omega)}^{1 - \frac{N}{2} + \frac{N}{\theta}} + 1)$$

$$\lesssim \delta \int_{\Omega} |\nabla(u + 1)^{m + k - 1/2} + \delta|\nabla|\nabla v|^{\frac{3}{2}}\|_{L^p(\Omega)}^{2} + C_{1b} \text{ for all } t \in (0, T_{\max}).$$

Finally, employing the same arguments as in the proof of (3.53)–(3.55), and taking advantage of (3.57)–(3.67), we conclude the result.

\[ \Box \]

**Lemma 3.7.** Let $1 < r < \frac{N+2}{N}$ and $\Omega \subset \mathbb{R}^N (N \geq 3)$ be a bounded domain with smooth boundary. Moreover, assume that $\phi$ satisfies (1.4) and (1.5) with $m > 2 - \frac{2}{N}$. Then for all large number $\beta > 2$, there exists a positive constant $C$ such that for all $\beta > \beta$ and $k > 1$, \[ \|u(\cdot, t)\|_{L^p(\Omega)} + \|\nabla v(\cdot, t)\|_{L^{2s}(\Omega)} \leq C \text{ for all } t \in (0, T_{\max}). \]

**Proof.** Due to $m > 2 - \frac{2}{N}$ and $N \geq 3$, applying the same argument of lemma 2.1 of [23], we can conclude that there exist numbers $k \geq 1, \tilde{\beta} \geq 2, s \in [1, \frac{N}{N-1}), \theta > 1$ and $\mu > 1$ such that for the sufficiently large $k \geq k, \beta \geq \tilde{\beta}$, the following inequalities are valid

$$k > m, 2\beta > k + 1,$$

\[ \frac{(N - 2)(k + 1 - m)}{N(k + m - 1)} \leq \frac{1}{\theta} < 1 - \frac{N - 2}{\beta N}, \]

\[ \frac{2(k + 1 - m)}{N(k + m - 1)} < \frac{1}{\mu} < \frac{2}{N} + \frac{N - 2}{\beta N}. \]
as well as
\[
N(k + 1 - m - \frac{1}{\rho}) \left( 1 - \frac{N}{2} + \frac{k + m - 1}{2} \right) + \frac{N(\frac{2}{s} + \frac{1}{\rho} - 1)}{1 - \frac{N}{2} + \frac{3N}{s}} < 2
\]  
(3.72)

and
\[
N(2 - \frac{1}{\rho}) \left( 1 - \frac{N}{2} + \frac{k + m - 1}{2} \right) + \frac{N(\frac{2\beta - 1}{s} + \frac{1}{\rho} - 1)}{1 - \frac{N}{2} + \frac{3N}{s}} < 2.
\]  
(3.73)

Now for the above \( k > 1, \beta > 2, \) with the help of \( L^1 \)-boundedness of \( u \) and \( |\nabla v|^\rho \) (see lemma 3.2), one can follow the procedure as in (3.14) and (3.26) of [23] to conclude that there exist positive constants \( C_1 \) and \( C_2 \) such that
\[
\frac{d}{dt} \left( \frac{1}{k+1} \|u + 1\|^k_{L^1(\Omega)} + \frac{1}{2\beta} \|\nabla v\|_{L^2(\Omega)}^{2\beta} \right) + C_1 \int_{\Omega} (u + 1)^{k+\rho-1} + C_1 \int_{\Omega} |\nabla v|^{2\beta}
\]
\[\leq C_2 \left( \int_{\Omega} v^{k+1} + 1 \right). \]  
(3.74)

Next, in light of lemma 3.2 and (3.74), the same argument as in the derivation of (3.53)–(3.55) then shows the result. \( \square \)

Underlying the estimates established above, we can prove theorem 1.1 by invoking a Moser-type iteration (see lemma A.1 in [23]) and the standard parabolic regularity arguments.

**Proof.** The proof of theorem 1.1 Firstly, using the outcome of lemma 3.5 with suitably large \( k \) and \( \beta \) as a starting point, we may invoke lemma A.1 in [23] which by means of a Moser-type iteration applied to the first equation in (1.1) establishes
\[\|u(.,t)\|_{L^\infty(\Omega)} \leq C \text{ for all } t \in (0, \infty).\]  
(3.75)

Next, with the regularity properties from (3.75) at hand, one can readily derive
\[\|v(.,t)\|_{W^{1,\infty}(\Omega)} \leq C \text{ for all } t \in (0, \infty)\]  
(3.76)

by means of standard parabolic regularity arguments applied to the second equation in (1.1). Finally, according to lemma 2.3, this entails that \((u, v, w)\) is global in time, and that \( u \) is bounded in \( \Omega \times (0, \infty) \). \( \square \)

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