Capacity of the range of branching random walks in low dimensions

by

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dedicated to the 75th anniversary of Professor Andrei M. Zubkov
and the 70th anniversary of Professor Vladimir A. Vatutin

Summary. Consider a branching random walk \((V_u)_{u \in T_{IGW}}\) in \(\mathbb{Z}^d\) with
the genealogy tree \(T_{IGW}\) formed by a sequence of i.i.d. critical Galton-Watson trees. Let \(R_n\) be the set of points in \(\mathbb{Z}^d\) visited by \((V_u)\) when the index \(u\) explores the first \(n\) subtrees in \(T_{IGW}\). Our main result
states that for \(d \in \{3, 4, 5\}\), the capacity of \(R_n\) is almost surely equal to
\(n^{\frac{d-2}{2}} + o(1)\) as \(n \to \infty\).

Keywords. Branching random walk, tree-indexed random walk, capacity.

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1 Introduction

In this paper, we continue the study in [2] on the capacity of the range of a branching
random walk in \(\mathbb{Z}^d\).

Let \(d \geq 3\) and \(\eta\) be a probability distribution on \(\mathbb{Z}^d\). The \(\eta\)-capacity of a finite set
\(A \subset \mathbb{Z}^d\) (with respect to \(\eta\)) is defined as

\[
\text{cap}_\eta A := \sum_{x \in A} P^\eta_x(\tau^+_A = \infty),
\]

where \(P^\eta_x\) denotes the law of a (discrete) random walk \((S_n)\) with jump distribution \(\eta\)
started at \(x\), and \(\tau^+_A := \inf\{n \geq 1 : S_n \in A\}\) is \((S_n)\)'s first returning time to \(A\).

Let \(\mu\) be a probability distribution on \(\mathbb{N}\). A \(\mu\)-Galton-Watson tree starts with one
initial ancestor which produces a random number of children according to \(\mu\), and these
children form the first generation. Then particles in the first generation produce their
children independently in the same way, forming the second generation. The system goes
on until infinity, or until when there is no particle in a generation. In this paper, we are interested in the critical case, i.e. the case when \( \sum_{k=0}^{\infty} k \mu(k) = 1 \). In this case, it is well-known that the Galton-Watson tree extincts (stops with no particle in finitely many generations) almost surely. To avoid extinction, we consider the Galton-Watson forest defined as follows. Let \((T_n)_{n \geq 0}\) be a sequence of independent \( \mu \)-Galton-Watson trees. As showed in Figure 1, we start with a fixed infinite ray \((w_n)_{n \geq 0}\) called spine, and attach \( T_n \) to each \( w_n \). For every \( n \geq 1 \), \( w_{n-1} \) is considered as the parent of \( w_n \) and the whole forest is rooted at \( w_0 \) which we denote by \( \varnothing \). As all \( T_n \) are finite, this Galton-Watson forest is in fact an infinite rooted tree, denoted by \( T_{IGW} \). Let \( P_{\mu} \) be the law of \( T_{IGW} \).

Let \( \theta \) be a probability distribution on \( \mathbb{Z}^d \). Given a (finite or infinite) tree \( T \), we can define a tree-indexed random walk \((V_u)_{u \in T}\) in \( \mathbb{Z}^d \) as follows: To all edges of \( T \) we attach i.i.d. random variables which are distributed as \( \theta \), independent of \( T \). Define \( V_{\varnothing} := 0 \). For each \( u \in T \setminus \{\varnothing\} \), let \( V_u \) be the sum of those random variables which are attached to the edges in the (unique) simple path relating \( u \) to the root \( \varnothing \). Clearly \( T \) describes the genealogy of \((V_u)_{u \in T}\). We may also call \((V_u)_{u \in T}\) a branching random walk when its genealogy tree is a Galton-Watson tree (or forest).

Denote by \( P_{\mu,\theta} \) the law of the branching random walk \((V_u)_{u \in T_{IGW}}\) when \( T_{IGW} \) is the Galton-Watson forest distributed as \( P_{\mu} \).

Figure 1: The Galton Watson forest \( T_{IGW} \).

Under the measure \( P_{\mu,\theta} \), let \( R_n := \{V_u, u \in \bigcup_{j=0}^{n-1} T_j\} \) be the set of points in \( \mathbb{Z}^d \) visited by the branching random walk \((V_u)\) when the index \( u \) explores the first \( n \) subtrees of \( T_{IGW} \). Our main result is:

**Theorem 1.1.** In dimensions \( d = 3, 4, 5 \), let \( \mu \) be a probability measure in \( \mathbb{N} \), let \( \theta, \eta \) be probability measures in \( \mathbb{Z}^d \), with the conditions

\[
\begin{align*}
\mu & \text{ has mean 1 and finite variance, and } \mu \not\equiv \delta_1, \\
\eta & \text{ is aperiodic, irreducible, with mean 0 and finite } (d+1)\text{-th moment,} \\
\theta & \text{ is symmetric, irreducible, with some finite exponential moments.}
\end{align*}
\]

Then almost surely under \( P_{\mu,\theta} \), as \( n \to \infty \),

\[
\text{cap}_{\eta} R_n = n^{\frac{d+2}{2}} + o_{n^2}(1),
\]

where here and in the sequel, \( o_{n^2}(1) \) denotes a quantity which converges to 0 almost surely as \( n \to \infty \).
Remark 1.2. We need the finite second moment of $\mu$ in Lemma 2.4 and Lemma 2.6, and use the symmetry and finite exponential moments of $\theta$ in Corollary 2.5, Lemma 3.3 and Lemma 4.4, whereas the finite $(d + 1)$-th moment of $\eta$ is needed in Lemma 3.1. □

A few comments are in order. First, it will be clear from our proof that Theorem 1.1 holds when $\mathcal{T}^{IGW}$ is replaced by a more general tree with one unique infinite ray, for example if we attach to each spine $w_i$, $i \geq 0$, an i.i.d. random number of independent $\mu$-Galton-Watson trees, as long as this random number has finite second moment. In particular Theorem 1.1 holds for the Kesten tree which is the $\mu$-Galton-Watson tree conditioned to survive forever if $\mu$ has finite third moment (because by the spine decomposition, the number of children of $w_i$ in the Kesten tree has the size-biased law of $\mu$).

Second, to avoid the extinction of a critical $\mu$-Galton-Watson tree $\mathcal{T}$, we may condition $\mathcal{T}$ to have $n$ vertices, thus we obtain a random tree, say $\mathcal{T}_n^{\text{cond}}$. Let $R_n^{\text{cond}} := \{V_u, u \in \mathcal{T}_n^{\text{cond}}\}$ be the range of $(V_u)_{T_n^{\text{cond}}}$ when the underlying genealogy tree is $\mathcal{T}_n^{\text{cond}}$. Le Gall and Lin [7, 8] studied in detail $\#R_n^{\text{cond}}$, the cardinality of the range $R_n^{\text{cond}}$, and obtained various scaling limits for all dimensions. In particular, their results show that the critical dimension for the range of the tree-indexed walk is $d = 4$: for $d \geq 5$, $\#R_n^{\text{cond}}$ grows linearly whereas for $d = 4$, $\#R_n^{\text{cond}}$ is sub-linear and for $d \leq 3$, $\#R_n^{\text{cond}}$ is of order $n^{d/4}$.

The study of the capacity of the range $R_n^{\text{cond}}$ was initiated in [2] where the authors proved that $\text{cap}_{\eta} R_n^{\text{cond}}$ grows linearly for $d \geq 7$ and is sub-linear for $d = 6$. This suggests, also as conjectured in [2], that $d = 6$ should be the critical dimension for the capacity of the range. The main motivation of the present work is to confirm this prediction, by giving the growth order of $\text{cap}_{\eta} R_n^{\text{cond}}$ for $d \in \{3, 4, 5\}$, this will be stated in the forthcoming Remark 2.2, see (2.1).

At last, let us mention the systematical studies on the capacity of the range for a simple random walk on $\mathbb{Z}^d$, see Asselah, Schapira and Sousi [1] and the references therein.

The rest of the paper is organized as follows: In Section 2, we order the vertices in the Galton-Watson forest $\mathcal{T}^{IGW}$ and state the corresponding result for the range of the walk indexed by the first $n$ vertices (Proposition 2.1). Then Theorem 1.1 follows as a consequence of Proposition 2.1 and Lemma 2.3. Sections 3 and 4 are devoted to the proofs of the upper and lower bound of Proposition 2.1 respectively.

Notation: Under $P_\theta^x$ (resp: $P_\eta^x$), $(S_n)_{n \geq 0}$ denotes a random walk on $\mathbb{Z}^d$ starting from $x$ with jump distribution $\theta$ (resp: $\eta$). For brevity, we call $(S_n)$ a $\theta$ (resp: $\eta$)-random walk. Finally, $C_i, 1 \leq i \leq 12$ denote some positive constants.

2 On the Galton-Watson forest

It will be more convenient to study the capacity for $n$ vertices than $n$ subtrees, then we order the vertices in the Galton-Watson forest. On $\mathcal{T}^{IGW}$, we visit the vertices in the order illustrated in Figure 2 starting with the first subtree $\mathcal{T}_0$ rooted at $w_0$, one visits every vertex in the order of Depth-First Search (lexicographical order). Then we continue
with the subtree $T_1$ rooted at $w_1$ and iterate the process. We denote the sequence of vertices in this order by $(u_i)_{i \geq 0}$.

![Figure 2: A sample of the $\mu$-Galton Watson forest. The path in bold is the spine $(w_n)$. Labels correspond to the sequence $(u_i)$. For example, $u_0 = w_0 = \emptyset$ and $u_6 = w_1$.](image)

Under the measure $P_{\mu, \theta}$, the sequence $(u_i)$ then induces a sequence of points in $\mathbb{Z}^d$, $(V_{u_i})$ the positions of $(u_i)$, and we define

$$R[0, n] = \{V_{u_0}, V_{u_1}, \cdots, V_{u_n}\}.$$ 

The main part of this paper will be devoted to prove that

**Proposition 2.1.** In dimensions $d = 3, 4, 5$, let $\mu, \theta, \eta$ be probability distributions with the conditions (1.1). Then almost surely under $P_{\mu, \theta}$,

$$\text{cap}_\eta R[0, n] = n^{d-2+o_p(1)}.$$ 

**Remark 2.2.** As for Theorem 1.1, Proposition 2.1 also holds for more general trees with one unique infinite ray: if we attach to each $w_i$ an i.i.d. random number $\nu_i$ of $\mu$-Galton-Watson tree, then the same conclusion holds as long as $E_{\mu, \theta}[\nu_i^2] < \infty$. The proof follows in the same way as that of Proposition 2.1 and we skip the details.

Now let $R_n^{\text{cond}} := \{V_u, u \in T_n^{\text{cond}}\}$ be as before the range of $(V_{u_i})_{T_n^{\text{cond}}}$, where $T_n^{\text{cond}}$ is the $\mu$-Galton-Watson tree conditioned to have $n$ vertices. Assume (1.1) and furthermore that $\mu$ has finite third moment, then in probability

$$\text{cap}_\eta R_n^{\text{cond}} = n^{d-2+o_p(1)},$$

where $o_p(1)$ denotes a quantity which converges to 0 in probability as $n \to \infty$. The conclusion (2.1) follows from the aforementioned generalized version of Proposition 2.1.
with \( \mathbb{P}_{\mu, \delta}(\nu_i = k) = \sum_{j=k+1}^{\infty} \mu(j), k \geq 0 \), and the arguments in Zhu [10], Section 5 for the coupling between the infinite tree model and \( T_n^{\text{cond}} \). Indeed, we first observe that \( \nu_i \) has finite second moment thanks to the assumption on \( \mu \). Fix \( 0 < a < 1 \). By the generalized version of Proposition 2.1 (with \( \nu_i \)) and [2, Lemma 3.6], we have
\[
\text{cap}_{\eta} R_{n}^{\text{cond}}[0, |an|] = n^{d-\frac{2}{4} + o_p(1)},
\]
where \( R_{n}^{\text{cond}}[0, |an|] \) is the range of \( (V_u T_n^{\text{cond}}) \) when \( u \) runs over the first \( 1 + |an| \) vertices of \( T_n^{\text{cond}} \) in the lexicographical order. This gives a lower bound of (2.1),
\[
\text{cap}_{\eta} R_{n}^{\text{cond}}[0, |an|] = n^{d-\frac{2}{4} + o_p(1)}.
\]
Moreover, by exploring the tree \( T_n^{\text{cond}} \) in the reversed order, we get that
\[
\text{cap}_{\eta} R_{n}^{\text{cond}}[|an|, n] = n^{d-\frac{2}{4} + o_p(1)},
\]
yielding the upper bound because
\[
\text{cap}_{\eta} R_{n}^{\text{cond}}[0, |an|] + \text{cap}_{\eta} R_{n}^{\text{cond}}[|an|, n] = n^{d-\frac{2}{4} + o_p(1)}.
\]

Admitting Proposition 2.1, we deduce Theorem 1.1 from the following lemma.

**Lemma 2.3.** Let \( \mu \neq \delta_1 \) be a probability measure on \( \mathbb{N} \) with mean 1 and finite variance, then \( \mathbb{P}_\mu \) almost surely, there are \( n^{2+o_p(1)} \) vertices in the first \( n \) subtrees rooted at \( w_0, \ldots, w_{n-1} \).

**Proof.** Denote by \( \#T \) the total vertices of a finite tree \( T \). It is well-known (see [4], Section 0.2) that there exists a random walk \( Y \) on \( \mathbb{Z} \) with \( Y_0 = 0 \) and jump distribution \( \mathbb{P}_\mu(Y_1 = k) = \mu(k+1) \) for \( k = -1, 0, 1, 2, \ldots \), such that
\[
\#T_0 + \ldots + \#T_{n-1} = \inf\{k \geq 1 : Y_k = -n\}, \quad n \geq 1.
\]
Here \( Y_k \) is the random walk on \( \mathbb{Z} \) with jump distribution \( \mathbb{P}_\mu \cdot \mathbb{P}_\mu \). By the classical Khintchine and Hirsch laws of iterated logarithm for the random walk \( (Y_k) \) (see Csáki [3] for Hirsch’s law of iterated logarithm under the second moment assumption),
\[
- \min_{0 \leq k \leq n} Y_k = n^{\frac{1}{2} + o_p(1)}, \quad \text{a.s.}
\]
It follows that \( \#T_0 + \ldots + \#T_{n-1} = n^{2+o_p(1)} \) a.s. \( \square \)

The rest of the paper is devoted to the proof of Proposition 2.1. At first, we need the following estimates on the population of the Galton-Watson forest \( T^{\IGW} \). For any \( u, v \in T^{\IGW} \), let \( d(u, v) \) be the graph distance between \( u \) and \( v \).

**Lemma 2.4.** Let \( \mu \neq \delta_1 \) be a probability measure on \( \mathbb{N} \) with mean 1 and finite variance, then \( \mathbb{P}_\mu \) almost surely,
\[
\max_{0 \leq i \leq n} d(\emptyset, u_i) = n^{\frac{1}{2} + o_p(1)}.
\]
Figure 3: The decomposition $d(\emptyset, u_n) = \zeta_n + H_n$.

**Proof.** Let

$$
\zeta_n = \max\{k \geq 0 : w_k \in R[0,n]\}, \quad H_n = d(u_n, w_{\zeta_n}),
$$

(2.3)

then as showed in Figure 3 we have

$$
d(\emptyset, u_n) = \zeta_n + H_n, \quad \forall n \geq 0.
$$

By Lemma 2.3, we have

$$
\zeta_n = n^{\frac{1}{2} + o(n^\varepsilon)},
$$

(1)

It thus suffices to show that $\mathbb{P}_\mu$-almost surely,

$$
\max_{0 \leq i \leq n} H_i \leq n^{\frac{1}{2} + o(n^\varepsilon)}.
$$

(2.4)

Note that the process $(H_n)$ is distributed as the height process in the sense of [4, Section 0.2]: Using the random walk $(Y_k)$ introduced in the proof of Lemma 2.3, we have

$$
H_n = \sum_{k=0}^{n-1} 1_{\{Y_k = \min_{0 \leq j \leq n} Y_j\}}, \quad n \geq 1.
$$

For any fixed $n$, by considering $Y_n - Y_{n-k}$, $0 \leq k \leq n$, we see that $H_n$ is distributed as $\sum_{k=1}^{n} 1_{\{Y_k = \max_{0 \leq j \leq k} Y_k\}}$. In other words, let $t_0 := 0$ and for $j \geq 1$, $t_j := \inf\{k > t_{j-1} : Y_k \geq Y_{t_j-1}\}$ be the sequence of (weak) ascending ladder epochs of $Y$. Then for all $n, \ell \geq 1$,

$$
\mathbb{P}_\mu(H_n \geq \ell) = \mathbb{P}_\mu(t_\ell \leq n) \leq \inf_{\lambda > 0} e^{\lambda n} (\mathbb{E}_\mu(e^{-\lambda t_1}))^\ell,
$$

where in the above inequality we have used the fact that $t_k - t_{k-1}, k \geq 1$ are i.i.d and distributed as $t_1$. The Laplace transform of $\mathbb{E}_\mu(e^{-\lambda t_1})$ can be computed by the Sparre-Anderson identity, whose asymptotic is given by Kersting and Vatutin ([5], proof of Theorem 4.6, Page 75):

$$
1 - \mathbb{E}_\mu(e^{-\lambda t_1}) \sim C_1 \sqrt{\lambda}, \quad \lambda \to 0.
$$

Take $\lambda = \frac{1}{n}$ we see that for all $n \geq 1$, $\mathbb{P}_\mu(H_n \geq n^{\frac{1}{2}}(\log n)^2) \leq e^{1-C_2(\log n)^2}$. It follows that $\mathbb{P}_\mu(\max_{1 \leq k \leq n} H_k \geq n^{\frac{1}{2}}(\log n)^2) \leq n e^{1-C_2(\log n)^2}$ whose sum over $n$ converges. We get (2.4) by the Borel-Cantelli lemma. \qed
Corollary 2.5. Let $\mu, \theta$ be probability measures satisfying (1.1), then $\mathbb{P}_{\mu, \theta}$-almost surely,

$$\max_{0 \leq i \leq n} |V_{u_i}| = n^{1+o_{\text{as}}(1)}.$$  

Proof. Conditionally on $u \in \mathcal{T}^{IGW}$ with $d(\emptyset, u) = k$, $V_u$ is distributed as $S_k$, where $(S_n)_{n \geq 0}$ is a $\theta$-random walk started at 0, i.e. a random walk in $\mathbb{Z}^d$ whose law is $\mathbb{P}_\emptyset^\theta$. By assumption (1.1), $\mathbb{P}_0^\emptyset(S_1) = 0$ and $S_1$ has some finite exponential moments.

Notice that $(V_{w_j}, 0 \leq j \leq \zeta_n)$ is a $\theta$-random walk on $\mathbb{Z}^d$, and $\zeta_n = n^{1+o_{\text{as}}(1)}$, we have the lower bound $\max_{0 \leq j \leq \zeta_n} |V_{w_j}| \geq \max_{0 \leq j \leq \zeta_n} |V_{w_j}| = n^{1+o_{\text{as}}(1)}$ by the same argument as in the proof of Lemma 2.3.

Below we show the upper bound $\max_{0 \leq i \leq n} |V_{u_i}| \leq n^{1+o_{\text{as}}(1)}$. Indeed, applying Petrov ([9], Theorem 2.7 and Lemma 2.2) gives that for all $n \geq 1$ and $\lambda > 0$,

$$\mathbb{P}_0^\emptyset(|S_n| \geq \lambda) \leq \max(e^{-C_3\frac{\lambda^2}{n}}, e^{-C_3\lambda}). \quad (2.5)$$

It follows that for any $\varepsilon > 0$,

$$\mathbb{P}_{\mu, \theta}\left(\max_{0 \leq i \leq n} |V_{u_i}| \geq n^{1+\varepsilon}, \max_{0 \leq i \leq n} d(\emptyset, u_i) \leq n^{1+\varepsilon}\right) \leq n \max_{0 \leq k \leq n^{1+\varepsilon}} \mathbb{P}_0^\emptyset(|S_k| \geq n^{1+\varepsilon}) \leq n e^{-C_3 n^\varepsilon},$$

whose sum over $n$ converges. By using the Borel-Cantelli lemma and Lemma 2.4, we get the corollary.

For $\varepsilon \in (0, \frac{1}{4})$, let

$$F_\varepsilon(n) := \left\{ \max_{0 \leq i \leq n} d(\emptyset, u_i) < n^{\frac{1}{2}+\varepsilon} \right\}. \quad (2.6)$$

By Lemma 2.4 almost surely $F_\varepsilon(n)$ holds for all large $n$.

Lemma 2.6. Let $\mu \neq \delta_1$ be a probability measure on $\mathbb{N}$ with mean 1 and finite second moment, then for any $k \geq 0$,

$$\mathbb{E}_\mu [\# \{(i, j) : 0 \leq i \leq j \leq n, d(u_i, u_j) = k\} 1_{F_\varepsilon(n)}] \leq (k+1)^2 n^{1+\varepsilon} + C_4(k+1)n^{1+2\varepsilon},$$

where $C_4 := \sum_{j=0}^\infty j^2 \mu(j)$.

Proof. For any $u, v \in \mathcal{T}^{IGW}$, we $u \preceq v$, if $u$ is an ancestor of $v$ and denote by $u \wedge v$ their most youngest common ancestor. We consider the two cases: $u_i \wedge u_j \in \{w_\ell : \ell \geq 0\}$, $u_i \wedge u_j \neq \{w_\ell : \ell \geq 0\}$ separately.

First case: $u_i \wedge u_j = w_\ell$ for some $\ell \geq 0$.

Note that the subtree rooted at $w_\ell$, $T_\ell = \{u : w_\ell \preceq u, w_{\ell+1} \not\preceq u\}$, is a critical Galton-Watson tree,

$$\mathbb{E}_\mu [\# \{u \in T_\ell : d(u, w_\ell) = k\}] = 1, \quad \forall k \geq 1. \quad (2.7)$$
As is shown in Figure 4(a),
\[ E_{\mu}\left[ \#\{ (i,j) : 0 \leq i \leq j \leq n, d(u_i,u_j) = k, u_i \wedge u_j \in \{ w_\ell : \ell \geq 0 \} \} 1_{F_n(\varepsilon)} \right] \]
\[ \leq \sum_{r=0}^{k} \sum_{0 \leq \ell < m \leq \ell + k - r} 1_{\{ m \leq n^{\frac{1}{2}+\varepsilon} \}} E_{\mu}\left[ \sum_{u \in T_{\ell}, u \in T_m} 1_{\{ d(u,w_\ell) = r, d(u',w_m) = k-r-(m-\ell) \}} \right] \]
\[ = \sum_{r=0}^{k} \sum_{0 \leq \ell < m \leq \ell + k - r} 1_{\{ m < n^{\frac{1}{2}+\varepsilon} \}} \]
\[ \leq (k+1)^2 n^{\frac{1}{2}+\varepsilon}, \quad (2.8) \]
where the above equality follows from (2.7) and the independence of \( T_\ell \) and \( T_m \).

**Second (and last) case:** \( u_i \wedge u_j \not\in \{ w_\ell : \ell \geq 0 \} \).

For this case, similarly as shown in Figure 4(b), let \( v = u_i \wedge u_j \). On \( F_n(\varepsilon) \), \( d(\emptyset,v) \leq n^{\frac{1}{2}+\varepsilon} \). Then
\[ E_{\mu}\left[ \#\{ (i,j) : 0 \leq i \leq j \leq n, d(u_i,u_j) = k, u_i \wedge u_j \not\in \{ w_\ell : \ell \geq 0 \} \} 1_{F_n(\varepsilon)} \right] \]
\[ \leq \sum_{r=0}^{k} \sum_{0 \leq \ell < n^{\frac{1}{2}+\varepsilon}} E_{\mu}\left[ \sum_{v \in T_\ell,d(v,w_\ell) = \ell} \sum_{u \wedge u' = v} 1_{\{ d(u,v) = r, d(u',v) = k-r \}} \right]. \]

Conditionally on the number of children of \( v \), say \( j \), by using (2.7), the expectation of
\[ \sum_{u \wedge u' = v} 1_{\{ d(u,v) = r, d(u',v) = k-r \}} \]
is dominated by \( j^2 \). Again using (2.7), we deduce from the branching property that
\[ E_{\mu}\left[ \sum_{v \in T_\ell,d(v,w_\ell) = \ell} \sum_{u \wedge u' = v} 1_{\{ d(u,v) = r, d(u',v) = k-r \}} \right] \leq \sum_{j=0}^{\infty} J^2 \mu(j), \]
which together with (2.8) yield the Lemma.

\[ \square \]

### 3 Proof of the upper bound in Proposition 2.1

Before studying the capacity, we need the basic notation of Green’s function:
\[ G_\eta(x,y) = G_\eta(y-x) := E_0^\eta \left[ \sum_{i=0}^{\infty} 1_{\{ S_i = y-x \}} \right] = \sum_{i=0}^{\infty} P_0^\eta(S_i = y-x), \quad x,y \in \mathbb{Z}^d. \]
The Green function \(G_\eta(x)\) has the following asymptotic estimate:

**Lemma 3.1** (Lawler and Limic [6, Theorem 4.3.5]). *Given an aperiodic and irreducible distribution \(\eta\) on \(\mathbb{Z}^d(d \geq 3)\) with mean 0 and covariance matrix \(\Gamma_\eta\), if it has finite \((d+1)\)-th moment \(\mathbb{E}_\eta[|S_1|^{d+1}] < \infty\), then*

\[
G_\eta(x) = \frac{C_{d,\eta}}{J_\eta(x)} + O(|x|^{1-d}),
\]

where \(C_{d,\eta} = \frac{\Gamma(\frac{d}{2})}{(d-2)\pi^{d/2} \sqrt{\det \Gamma_\eta}}\), \(\Gamma(\cdot)\) refers to the Gamma function and \(J_\eta(x) = \sqrt{x \cdot \Gamma_{\eta}^{-1} x}\).

Below is a lemma that connects the capacity with Green’s function, which is inspired from [2, Lemma 2.12].

**Lemma 3.2.** *Let \(\eta\) be a probability distribution in \(\mathbb{Z}^d, d \geq 3\). For any sequence \((x_n)_{n \geq 0} \in \mathbb{Z}^d\),*

\[
\frac{1}{n+1} \sum_{i=0}^{n} 1_{x_i \notin \{x_{i+1}, \ldots, x_n\}} \mathbb{P}_x^\eta(\tau_+^{\{x_0, \ldots, x_n\}} = \infty) \sum_{j=0}^{n} G_\eta(x_j, x_i) = 1,
\]

*where under \(\mathbb{P}_x^\eta\), \((S_n)\) is a random walk on \(\mathbb{Z}^d\) started at \(x\) and with jump distribution \(\eta\), and \(\tau_A^+: = \inf\{i \geq 1 : S_i \in A\}\) denotes as before the first returning time of \(A\), for any finite \(A \subset \mathbb{Z}^d\).*

**Proof.** Since the random walk \((S_n)\) in dimension \(d \geq 3\) is transient, for any finite set \(A \subset \mathbb{Z}^d\) and \(z \in A\), let \(\sigma_A := \sup\{i \geq 0 : S_i \in A\}\) be the last-passage time, then

\[
1 = \mathbb{P}_x^\eta(\sigma_A < \infty)
\]

\[
= \sum_{x \in A} \sum_{i=0}^{\infty} \mathbb{P}_z^\eta(S_i = x) \mathbb{P}_x^\eta(\tau_A^+ = \infty)
\]

\[
= \sum_{x \in A} G_\eta(z, x) \mathbb{P}_x^\eta(\tau_A^+ = \infty).
\]

Take \(A = \{x_0, \ldots, x_n\}\) in this equation, then

\[
\sum_{i=0}^{n} 1_{x_i \notin \{x_{i+1}, \ldots, x_n\}} \mathbb{P}_x^\eta(\tau_+^{\{x_0, \ldots, x_n\}} = \infty) G_\eta(z, x_i) = 1,
\]

and the conclusion follows by summing over \(z = x_0, \ldots, x_n\). \(\square\)

Then we estimate the sum of Green’s functions.

**Lemma 3.3.** *In dimensions \(d = 3, 4, 5\), let \(\mu, \theta, \eta\) be probability distributions with the conditions in [1.1]. Then \(\mathbb{P}_{\mu, \theta}\)-almost surely,*

\[
\min_{0 \leq i \leq n} \sum_{j=0}^{n} G_\eta(V_{u_i}, V_{u_j}) \geq n^{\frac{d-1}{d} + o_{\mathbb{R}}(1)}.
\]
Proof. By Lemma 3.1, it suffices to show that \( \mathbb{P}_{\mu, \theta} \)-almost surely,
\[
\min_{0 \leq i \leq n} \sum_{j=0}^{n} \frac{1}{(1 + |V_{u_i} - V_{u_j}|)^{d-2}} \geq n^{\frac{6-d}{4} + o_{\mathbb{P}}(1)}. \tag{3.1}
\]

Denote as before by \( d(u_i, u_j) \) the graph distance between the two vertices \( u_i, u_j \) on the tree, then
\[
V_{u_i} - V_{u_j} \overset{d}{=} S_{d(u_i, u_j)},
\]
where \( (S_n)_{n \geq 0} \) is the \( \theta \)-random walk started at 0, independent of \( d(u_i, u_j) \).

For any \( \varepsilon \in (0, \frac{1}{4}) \), using the union bound and (2.5) we get that
\[
\mathbb{P}_{\mu, \theta} \left( \bigcup_{0 \leq i, j \leq n} \{|V_{u_i} - V_{u_j}| \geq n^{\varepsilon} \sqrt{1 + d(u_i, u_j)}\} \right) \leq (n + 1)^2 e^{-C_3 n^{\varepsilon}}.
\]

By the Borel-Cantelli lemma, almost surely the above event cannot happen infinitely often. Thus to prove (3.1), it suffices to show that \( \mathbb{P}_{\mu, \theta} \)-almost surely,
\[
\min_{0 \leq i \leq n} \sum_{j=0}^{n} \frac{1}{(1 + d(u_i, u_j))^{\frac{d}{2}}} \geq n^{\frac{6-d}{4} + o_{\mathbb{P}}(1)},
\]
or more generally, for any \( \alpha > 0 \), \( \mathbb{P}_{\mu, \theta} \)-almost surely
\[
\min_{0 \leq i \leq n} \sum_{j=0}^{n} \frac{1}{(1 + d(u_i, u_j))^{\alpha}} \geq n^{1 - \frac{\alpha}{2} + o_{\mathbb{P}}(1)}. \tag{3.2}
\]

Observe that \( \min_{0 \leq i \leq n} \sum_{j=0}^{n} \frac{1}{(1 + d(u_i, u_j))} \geq (n + 1)(1 + \max_{0 \leq i, j \leq n} d(u_i, u_j))^{-\alpha} \geq (n + 1)(1 + 2 \max_{0 \leq i \leq n} d(\emptyset, u_i))^{-\alpha} \), then (3.2) follows from Lemma 2.4. \(\square\)

**Proof of the upper bound in Proposition 2.1** Applying Lemma 3.2 to \( \{V_{u_0}, \cdots, V_{u_n}\} \), we deduce from the definition of the \( \eta \)-capacity that
\[
\capa_{\eta} R[0, n] = \sum_{i=0}^{n} 1_{V_{u_i} \notin \{V_{u_{i+1}, \cdots, V_{u_n}}\}} \mathbb{P}_{V_{u_i}}^{\eta} \left( \tau^{+}_{\{V_{u_0}, \cdots, V_{u_n}\}} = \infty \mid \{V_{u_0}, \cdots, V_{u_n}\} \right)
\leq \min_{0 \leq i \leq n} \sum_{j=0}^{n} G_{\eta}(V_{u_i}, V_{u_j}),
\]
and the conclusion follows from Lemma 3.3. \(\square\)

### 4 Proof of the lower bound in Proposition 2.1

For the lower bound, our main tool is the following lemma.
Lemma 4.1 ([2] Lemma 2.11). Let \( d \geq 3 \) and \( \eta \) be any probability distribution on \( \mathbb{Z}^d \). For any finite set \( A \subset \mathbb{Z}^d \) and \( k \in \mathbb{N}_+ \),

\[
\text{cap}_\eta A \geq \frac{|A|}{k+1} - \frac{\sum_{x,y \in A} G_\eta(x,y)}{k(k+1)}.
\]

According to this lemma, the capacity \( \text{cap}_\eta R[0,n] \) can be bounded below by estimates of \( \#R[0,n] \) and the sum of Green’s functions. We start with \( \#R[0,n] \). Let

\[
L^x_n := \sum_{i=0}^{n} 1\{V_i = x\}, \quad \forall x \in \mathbb{Z}^d, \quad n \geq 0,
\]

denote the local times, then we can write the range as

\[
R[0,n] = \{x \in \mathbb{Z}^d : L_n^x \geq 1\}.
\]

The following second moment estimate for local times is inspired by the proof of Le Gall and Lin [7, Lemma 3].

Lemma 4.2. Let \( d \geq 3 \). With the conditions in (1.1), \( \mathbb{P}_{\mu,\theta} \)-almost surely, as \( n \to \infty \),

\[
\sum_{x \in \mathbb{Z}^d} (L_n^x)^2 \leq n^{\max(\frac{8-d}{4},1)+o(1)}.
\]

Proof. Let \( \varepsilon \in (0, \frac{1}{4}) \) and recall the event \( F_\varepsilon(n) \) defined in (2.6). We are going to prove that for all \( n \geq 1 \),

\[
\sum_{x \in \mathbb{Z}^d, |x| \leq n} \mathbb{E}_{\mu,\theta}[(L_n^x)^2 1_{F_\varepsilon(n)}] \leq C_5 n^{\max(\frac{8-d}{4},1)+4\varepsilon}.
\] (4.1)

Admitting for the moment (4.1) we can give the proof of Lemma 4.2. Let \( \xi_n := \sum_{x \in \mathbb{Z}^d, |x| \leq n} (L_n^x)^2, \gamma := \max(\frac{8-d}{4},1)+5\varepsilon \) and \( n_j := 2^j \) for \( j \geq 1 \). By Markov’s inequality, (4.1) implies that for all \( j \geq 1 \),

\[
\mathbb{P}_{\mu,\theta}(\xi_n \geq n_{j-1}^\gamma, F_\varepsilon(n_j)) \leq C_5 \frac{n_j^{-\varepsilon}}{n_j} \leq C_6 2^{-\varepsilon j}.
\]

The Borel-Cantelli lemma says that almost surely for all large \( j \), either \( \xi_{n_j} < n_{j-1}^\gamma \) or \( F_\varepsilon(n_j) \) does not hold. However by Lemma 2.4 almost surely \( F_\varepsilon(n_j) \) holds for all large \( j \), hence we have proved that almost surely for all large \( j \), \( \xi_{n_j} < n_{j-1}^\gamma \). On the other hand, by Corollary 2.5 almost surely for all large \( j \), \( L_n^x = 0 \) for all \( |x| > n_j \), hence \( \sum_{x \in \mathbb{Z}^d} (L_n^x)^2 = \xi_{n_j} < n_{j-1}^\gamma \). Then by monotonicity for all large \( n \), \( \sum_{x \in \mathbb{Z}^d} (L_n^x)^2 < n^\gamma \) a.s. Since \( \varepsilon \) can be arbitrarily small, we have proved Lemma 4.2.

It remains to show (4.1). To this end, we denote the transition probabilities for a \( \theta \)-walk \((S_n)_{n \geq 0}\) by

\[
\pi_m(x) := \mathbb{P}_\theta^S(S_m = x), \quad m \geq 0, \quad x \in \mathbb{Z}^d.
\]
for simplicity. Then there exists a constant \( C_7 > 0 \) depending on \( d \) and \( \theta \) such that for all \( x \in \mathbb{Z}^d \) and \( m \geq 0 \),

\[
\pi_m(x) \leq C_7 (1 + |x|)^{-d}, \quad (4.2)
\]

\[
\pi_m(x) \leq C_7 (1 + m)^{-\frac{d}{2}}, \quad (4.3)
\]

\[
\sum_{x \in \mathbb{Z}^d} \pi_m(x) = 1, \quad (4.4)
\]

where (4.2) follows from [6, Proposition 2.4.6], and (4.3) follows from [6, p.24]. (In [6], \( \theta \) is also required to be aperiodic, but since we only need an upper bound, these results can be easily extended to periodic cases.) Then we decompose the second moment in (4.1) as

\[
\sum_{x \in \mathbb{Z}^d, |x| \leq n} \mathbb{E}_{\mu,\theta} \left[ (L_n^x)^2 1_{F_i(n)} \right] = \sum_{x \in \mathbb{Z}^d, |x| \leq n} \sum_{i,j = 0}^n \mathbb{P}_{\mu,\theta} \left( V_{u_i} = V_{u_j} = x, F_x(n) \right).
\]

For notational brevity, we write \( u = u_i \wedge u_j \) for the youngest common ancestor of \( u_i, u_j \), and \( y = V_u \) for the spatial location of \( u \). We also write \( a = d(\emptyset, u) \), \( b = d(u, u_i) \) and \( c = d(u, u_j) \) for the graph distances between these particles, as shown in Figure 5.

![Figure 5: An illustration for the relative positions of \( u, u_i, u_j \).](image)

We assume without loss of generality that \( b \geq c \), then

\[
b \geq \frac{1}{2} d(u_i, u_j). \quad (4.5)
\]
Therefore (keeping in mind that $a, b, c$ depend on $u_i, u_j$),

$$
\mathbb{E}_{\mu, \theta}\left[\left.\left(\sum_{i,j=0}^{n} a(y)\pi_b(x-y)\pi_c(x-y)\right)\right| F_{e}(n)\right] \\
= \sum_{i,j=0}^{n} \mathbb{P}_{\mu, \theta}(V_{u_i} = V_{u_j} = x, F_{e}(n)) \\
= \sum_{i,j=0}^{n} \sum_{y \in \mathbb{Z}^d} \mathbb{P}_{\mu, \theta}(V_u = y, V_{u_i} = V_{u_j} = x, F_{e}(n)) \\
= \mathbb{E}_{\mu, \theta}\left[\sum_{i,j=0}^{n} \sum_{y \in \mathbb{Z}^d} \pi_a(y)\pi_b(x-y)\pi_c(x-y)\right] 1_{F_{e}(n)} \\
= A + B,
$$

where

$$
A := \mathbb{E}_{\mu, \theta}\left[\sum_{i,j=0}^{n} \sum_{|y| \geq \varepsilon/2} \pi_a(y)\pi_b(x-y)\pi_c(x-y)\right] 1_{F_{e}(n)} , \\
B := \mathbb{E}_{\mu, \theta}\left[\sum_{i,j=0}^{n} \sum_{|y| < \varepsilon/2} \pi_a(y)\pi_b(x-y)\pi_c(x-y)\right] 1_{F_{e}(n)} .
$$

For $A$, we use (4.2) for $\pi_a$, (4.3) and (4.5) for $\pi_b$ and (4.4) for $\pi_c$, then

$$
A \leq C_7^2 \mathbb{E}_{\mu, \theta}\left[\sum_{i,j=0}^{n} \left(1 + \frac{|x|}{2}\right)^{-d} \left(1 + \frac{1}{2}d(u_i, u_j)\right)^{-\frac{d}{2}} \sum_{y \in \mathbb{Z}^d} \pi_c(x-y)\right] 1_{F_{e}(n)} \\
= C_7^2 \mathbb{E}_{\mu, \theta}\left[\sum_{i,j=0}^{n} \left(1 + \frac{|x|}{2}\right)^{-d} \left(1 + \frac{1}{2}d(u_i, u_j)\right)^{-\frac{d}{2}} 1_{F_{e}(n)}\right] \\
= C_7^2 \left(1 + \frac{|x|}{2}\right)^{-d} \sum_{k=0}^{\infty} \left(1 + \frac{k}{2}\right)^{-\frac{d}{2}} \mathbb{E}_{\mu, \theta}\left[\#\{0 \leq i, j \leq n : d(u_i, u_j) = k\}\right] 1_{F_{e}(n)} .
$$

Note that

$$
\sum_{k=0}^{\infty} \left(1 + \frac{k}{2}\right)^{-\frac{d}{2}} \mathbb{E}_{\mu, \theta}\left[\#\{0 \leq i, j \leq n : d(u_i, u_j) = k\}\right] 1_{F_{e}(n)} \\
\leq \sum_{0 \leq k \leq n^{\frac{1}{2}}} \left(1 + \frac{k}{2}\right)^{-\frac{d}{2}} \mathbb{E}_{\mu, \theta}\left[\#\{0 \leq i, j \leq n : d(u_i, u_j) = k\}\right] 1_{F_{e}(n)} + \left(1 + \frac{1}{2}n^{\frac{1}{2}}\right)^{-\frac{d}{2}} n^2 ,
$$

which by Lemma 2.6 is further bounded by

$$
\sum_{0 \leq k \leq n^{\frac{1}{2}}} \left(1 + \frac{k}{2}\right)^{-\frac{d}{2}} \left[(k+1)^2 n^{\frac{1}{2}+\varepsilon} + C_4(k+1)n^{1+2\varepsilon}\right] + \left(1 + \frac{1}{2}n^{\frac{1}{2}}\right)^{-\frac{d}{2}} n^2 \leq C_8 n^{\max\left(\frac{8-d}{4}, 1\right)+3\varepsilon} .
$$
Therefore we have proved that for any \( x \in \mathbb{Z}^d \) and \( n \geq 1 \),
\[
A \leq C_7^2 C_8 \left( 1 + \frac{|x|}{2} \right)^{-d} n^{\max(\frac{8-d}{4},1)+3\varepsilon}.
\]

We may deal with the term \( B \) in a similar way. If \( |y| < \frac{|x|}{2} \), then \( |x - y| \geq \frac{|x|}{2} \), so we use (4.4) for \( \pi_a \), (4.3) and (4.5) for \( \pi_b \) and (4.2) for \( \pi_c \),
\[
B \leq C_7^2 \sum_{i,j=0}^{n} \sum_{y \in \mathbb{Z}^d} \pi_a(y) \left( 1 + \frac{1}{2} d(u_i, u_j) \right)^{-\frac{d}{2}} \left( 1 + \frac{|x|}{2} \right)^{-d} 1_{F_i(n)}
\]
\[
= C_7^2 \sum_{i,j=0}^{n} \left( 1 + \frac{|x|}{2} \right)^{-d} \left( 1 + \frac{1}{2} d(u_i, u_j) \right)^{-\frac{d}{2}} 1_{F_i(n)}
\]
\[
\leq C_7^2 C_8 \left( 1 + \frac{|x|}{2} \right)^{-d} n^{\max(\frac{8-d}{4},1)+3\varepsilon}.
\]

Then for any \( x \in \mathbb{Z}^d \), we have
\[
\mathbb{E}_{\mu,\theta}[(L_{x,n}^2)^{1_{F_i(n)}]} = A + B \leq 2C_7^2 C_8 \left( 1 + \frac{|x|}{2} \right)^{-d} n^{\max(\frac{8-d}{4},1)+3\varepsilon}.
\]

Taking the sum over \( |x| \leq n \) gives (4.1). This completes the proof of Lemma 4.2.

From this lemma we deduce an almost-sure lower bound for \( \#R[0,n] \):

**Proposition 4.3.** For \( d \geq 3 \), let \( \mu, \theta \) be probability distributions with the conditions in (1.1), then \( \mathbb{P}_{\mu,\theta} \)-almost surely for all large \( n \),
\[
\#R[0,n] \geq n^{\min(\frac{d}{2},1)+o(1)}.
\]

**Proof.** By definition,
\[
\sum_{x \in R[0,n]} L_{x,n}^2 = n + 1,
\]
then by Cauchy-Schwarz’ inequality,
\[
\#R[0,n] \geq \frac{(n + 1)^2}{\sum_{x \in \mathbb{Z}^d} (L_{x,n}^2)^2}.
\]

We conclude by Lemma 4.2.

**Lemma 4.4.** For \( d = 3, 4, 5 \), let \( \mu, \theta, \eta \) be probability distributions with the conditions in (1.1), then \( \mathbb{P}_{\mu,\theta} \)-almost surely for all large \( n \),
\[
\sum_{x,y \in R[0,n]} G_{\eta}(x,y) \leq \begin{cases} 
  n^{\frac{5}{4}+o(1)}, & d = 3 \\
  n^{\frac{11}{4}+o(1)}, & d = 4, 5 
\end{cases}.
\]
Remark 4.5. One would expect that the sum of Green’s functions is monotone decreasing in $d$ with a unified asymptotic formula. However, in dimension $d = 3$, $R[0,n]$ contains considerably less points than that in $d \in \{4,5\}$. Therefore, we have different results and proofs for the case $d = 3$ and the case $d \in \{4,5\}$.

Proof. Let $\varepsilon \in (0, \frac{1}{12})$ be small.

For $d = 3$, by Corollary 2.5, $\mathbb{P}_{\mu,\theta}$-almost surely for all large $n$,

$$\sum_{x,y \in \mathbb{R}[0,n]} G_\eta(x,y) \leq \sum_{|x|,|y| \leq n^{\frac{1}{2}+\varepsilon}} G_\eta(x,y) \leq C_9 n^{\frac{3}{2}+3\varepsilon},$$

where the last inequality follows from the asymptotic behaviors of $G_\eta$ given in Lemma 3.1. This proved the case $d = 3$.

For $d \in \{4,5\}$, recall the event $F_\varepsilon(n)$ defined in (2.6). Since $\sum_{x,y \in \mathbb{R}[0,n]} G_\eta(x,y) \leq \sum_{i,j=0}^n G_\eta(V_{u_i}, V_{u_j})$, we have

$$\mathbb{E}_{\mu,\theta} \left[ \sum_{x,y \in \mathbb{R}[0,n]} G_\eta(x,y) \mathbb{1}_{F_\varepsilon(n)} \right] \leq \sum_{i,j=0}^n \mathbb{E}_{\mu,\theta} \left[ G_\eta(V_{u_i}, V_{u_j}) \mathbb{1}_{F_\varepsilon(n)} \right].$$

Using Lemma 3.1 and the fact that $V_{u_i} - V_{u_j}$ is distributed as $S_{d(u_i, u_j)}$ with $S$ a $\theta$-random walk independent of $d(u_i, u_j)$, we deduce from the local limit theorem for $S$ that

$$\mathbb{E}_{\mu,\theta} \left[ \sum_{x,y \in \mathbb{R}[0,n]} G_\eta(x,y) \mathbb{1}_{F_\varepsilon(n)} \right] \leq C_{10} \sum_{i,j=0}^n \mathbb{E}_{\mu,\theta} \left[ \frac{1}{(1 + |V_{u_i} - V_{u_j}|)^{d-2}} \mathbb{1}_{F_\varepsilon(n)} \right]$$

$$\leq C_{11} \sum_{i,j=0}^n \mathbb{E}_{\mu,\theta} \left[ \frac{1}{(1 + d(u_i, u_j))^{\frac{d-2}{2}}} \mathbb{1}_{F_\varepsilon(n)} \right]$$

$$= C_{11} \sum_{k=0}^\infty \mathbb{E}_{\mu,\theta} \left[ \# \{0 \leq i, j \leq n : d(u_i, u_j) = k \} \mathbb{1}_{F_\varepsilon(n)} \right] \frac{1}{(1 + k)^{\frac{d-2}{2}}}.$$

The above sum over $k$ is less than

$$\sum_{0 \leq k \leq \sqrt{n}} \mathbb{E}_{\mu,\theta} \left[ \# \{0 \leq i, j \leq n : d(u_i, u_j) = k \} \mathbb{1}_{F_\varepsilon(n)} \right] \frac{n^2}{(1 + k)^{\frac{d+2}{2}}} + \frac{n^2}{(1 + \sqrt{n})^{\frac{d+2}{2}}}$$

which by Lemma 2.6 is further bounded by

$$\sum_{0 \leq k \leq \sqrt{n}} \left[ (k+1)^{3-d/2} n^{1+\varepsilon} + C_4 (k+1)^{2-d/2} n^{1+2\varepsilon} \right] + n^{\frac{10-d}{4}} \leq C_{12} n^{\frac{10-d}{4}+2\varepsilon}.$$
Similarly to the proof of Lemma 4.2, we use the Borel-Cantelli lemma and the fact that $F_n(n)$ holds eventually for all large $n$ (Lemma 2.4), to get that a.s. for all large $n$,\[
\sum_{x,y \in R[0,n]} G_\eta(x,y) \leq n^{10^{-d}+3\varepsilon}.
\]Since $\varepsilon$ can be arbitrarily small, we get the Lemma for the case $d \in \{4,5\}$. 

\textit{Proof of the lower bound in Proposition 2.1:} Let $d \in \{3,4,5\}$. Let $\varepsilon \in (0,\frac{1}{12})$ be small. By Proposition 4.3 and Lemma 4.4, we see that $\mathbb{P}_{\mu,\theta}$-almost surely for all large $n$, $\# R[0,n] \geq n^{\min(\frac{d}{4},1) - \varepsilon}$, and \[
\sum_{x,y \in R[0,n]} G_\eta(x,y) \leq \begin{cases}
 n^{\frac{5}{4}+\varepsilon}, & d = 3 \\
 n^{\frac{10-d}{4}+\varepsilon}, & d = 4,5.
\end{cases}
\]

Applying Lemma 4.1 to $A = R[0,n]$ with $k = \lfloor 2n^{\frac{1}{4}+2\varepsilon} \rfloor$ if $d = 3$ and $k = \lfloor 2n^{\frac{6-d}{4}+2\varepsilon} \rfloor$ if $d \in \{4,5\}$, we get that $\mathbb{P}_{\mu,\theta}$-almost surely for all large $n$, $\text{cap}_\eta R[0,n] \geq \frac{1}{5} n^{\frac{d-2}{4}-3\varepsilon}$. Since $\varepsilon$ can be arbitrarily small, this gives the lower bound in Proposition 2.1. □

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