BOUNDS OF AUTOMORPHISM GROUPS OF GENUS 2 FIBRATIONS

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Abstract. For a relatively minimal fibration of genus 2, the best bounds of the orders of its automorphism group, abelian automorphism group and cyclic automorphism group are obtained as a linear function of the self-intersection number of the canonical divisor.

It is well known that the automorphism group of a surface of general type is finite and bounded by a function of $K^2$ (cf. [1]). Since then, several authors worked on this subject and found better upper bounds of the group. Recently Xiao [11], [12] obtained a linear bound for this group. Hence it is natural to investigate the upper bounds for particular classes of surfaces. Here we are interested in the upper bounds of various automorphism groups of surfaces with genus 2 pencils. As a first step, in the present paper, we will study the upper bounds of automorphism groups of genus 2 fibrations.

We always assume that $S$ is a smooth projective surface over the complex number field. A genus 2 fibration is a morphism $f: S \to C$ where $C$ is a projective curve such that a general fiber of $f$ is a smooth curve of genus 2.

DEFINITION 0.1. An automorphism of the fibration $f: S \to C$ is a pair of automorphisms $(\delta, \sigma)$ with $\delta \in \text{Aut}(S)$, $\sigma \in \text{Aut}(C)$ such that the diagram

$$
\begin{array}{ccc}
S & \xrightarrow{\delta} & S \\
f & & f \\
C & \xrightarrow{\sigma} & C 
\end{array}
$$

commutes.

The automorphism group of fibration $f$ will be denoted by $\text{Aut}(f)$. The main results of this paper are the following:

THEOREM 0.1. Suppose $S$ is a surface of general type over the complex number field with a relatively minimal genus 2 fibration $f: S \to C$. Then

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If $f$ is not locally trivial, then

$$|\text{Aut}(f)| \leq 288K_S^2.$$  

More precisely,

$$|\text{Aut}(f)| \leq \begin{cases} 
126K_S^2, & \text{if } g(C) \geq 2; \\
144K_S^2, & \text{if } g(C) = 1; \\
120K_S^2 + 960, & \text{if } g(C) = 0. 
\end{cases}$$

These bounds are the best possible.

**Theorem 0.2.** Suppose $S$ is a surface of general type over the complex number field with a relatively minimal genus 2 fibration $f : S \to \mathbb{C}$. Then an abelian automorphism group $G$ of $f$ satisfies

$$|G| \leq 12.5K_S^2 + 100.$$  

This bound is the best possible.

**Theorem 0.3.** Suppose $S$ is a surface of general type over the complex number field with a relatively minimal genus 2 fibration $f : S \to \mathbb{C}$. Then a cyclic automorphism group $G$ of $f$ satisfies

$$|G| \leq \begin{cases} 
5K_S^2, & \text{if } g(C) = 1, \ K_S^2 \geq 12; \\
12.5K_S^2 + 90, & \text{if } g(C) = 0. 
\end{cases}$$

These bounds are the best possible.

**Theorem 0.4.** Suppose $S$ is a minimal surface of general type over the complex number field with a genus 2 fibration $f : S \to \mathbb{C}$ with $g(C) \geq 2$. Then a cyclic automorphism group $G$ of $f$ satisfies

$$|G| \leq 5K_S^2 + 30.$$  

Theorem 0.1 will be obtained as a consequence of several propositions in Section 3. In Section 4, we discuss abelian and cyclic automorphism groups of the fibration $f$. The propositions proved there imply Theorems 0.2, 0.3 and 0.4. We remark that Xiao [7] has obtained a bound for abelian automorphism groups of $f$. Our theorem is an improvement of his. Examples are given in Section 5 to show that most of these bounds are the best possible.

1. **Preliminaries.** The surfaces with genus 2 pencils have been studied by many authors. The facts we need in this paper appeared mostly in [3], [6], [9], [10]. In particular, Xiao’s book [10] gave a systematic description of the properties of genus 2
fibrations which are just what we need here. Unfortunately, this book has not been translated into English yet, hence it is not available for most readers. For this reason, we will recall some materials in this section.

Let $f : S \to C$ be a relatively minimal fibration of genus 2 and $\omega_{S/C} = \omega_S \otimes f^*\omega_C$ the relative canonical sheaf of $f$. For a sufficiently ample invertible sheaf $\mathcal{L}$ on $C$, the natural homomorphism $f^*(f_*\omega_{S/C} \otimes \mathcal{L}) \to \omega_{S/C} \otimes f^*\mathcal{L}$ defines a natural map $\Phi$:

\[
\begin{array}{ccc}
S & \overset{\Phi}{\longrightarrow} & P = P(f_*\omega_{S/C} \otimes \mathcal{L}) \\
\downarrow f & & \downarrow \pi \\
C, & &
\end{array}
\]

$\Phi$ is called a relative canonical map. By a succession of blow-ups, we can obtain the following commutative diagram:

\[
\begin{array}{ccc}
\tilde{S} & \overset{\tilde{\rho}}{\longrightarrow} & \tilde{P} \\
\downarrow \rho & & \downarrow \psi \\
S & \overset{\Phi}{\longrightarrow} & P \\
\downarrow f & & \downarrow \pi \\
C, & &
\end{array}
\]

where $\rho$ and $\psi$ are composites of finitely many blow-ups, $\tilde{\rho}$ is a double cover. Then we get the branch loci $\tilde{R}$ on $\tilde{P}$ and $R$ on $P$ such that $\tilde{R}$ is the minimal even resolution of $R$ (i.e., the canonical resolution of the double cover). If $\mathcal{L}$ is sufficiently ample, then all the singularities of $R$ must be located in one of the six types 0), I), II), III), IV) and V) of singular fibers defined by Horikawa [3].

$P$ is a relatively minimal ruled surface. We denote a section which has the least self-intersection number by $C_0$ with $C_0^2 = -e$. We use $F$ to denote both the fiber of $f$ and $\pi$.

A singular point of the branch locus is said to be negligible if this point itself and all its infinitely near points are double points or triple points with at least two different tangents. By the minimal even resolution, the inverse image of a negligible singular point is composed of $(-2)$-curves. All other singular points are said to be non-negligible. The singular fiber of type 0) in the classification of Horikawa is nothing else but the fiber which does not contain any non-negligible singular points.

The minimal even resolution $\psi : \tilde{P} \to P$ can be decomposed into $\tilde{\psi} : \tilde{P} \to \hat{P}$ followed by $\tilde{\psi} : \hat{P} \to P$, where $\tilde{\psi}$ and $\psi$ are composed respectively of negligible and non-negligible blow-ups. The image of $\tilde{R}$ in $\hat{P}$ is denoted by $\hat{R}$.
If we take away all the isolated vertical \((-2)\)-curves from the reduced divisor \(\tilde{R}\), we get a new reduced divisor \(\tilde{R}_p\), which is called the principal part of the branch locus \(\tilde{R}\). Then for any fiber \(F\) of \(\pi: P\to C\), the second and third singularity index \(s_2(F), s_3(F)\) of \(F\) is defined as follows:

If \(R\) has no quadruple singularities on \(F\), then \(s_3(F)\) equals the number of \((3\to3)\) type singularities of \(R\) on \(F\). Otherwise \(s_3(F)\) equals the number of \((3\to3)\) type singularities of \(R\) on \(F\) plus one. Hence \(s_3(F)=0\) if and only if \(R\) has no non-negligible singularities on \(F\).

Let \(\phi: \tilde{R}_p\to \mathbb{C}\) be the natural projection induced by \(\pi \circ \tilde{\psi}: \hat{P}\to \mathbb{C}\). Then the second singularity index \(s_2(F)\) of \(F\) is the ramification index of the divisor \(\tilde{R}_p\) on \(f(F)\) with respect to the projection \(\phi\). If \(\tilde{R}_p\) has singularities (which must be negligible) on \(F\), the singularity index \(s_2(F)\) can be calculated as follows:

For a smooth point \(p\in \tilde{R}_p\cap F\), the ramification index of \(\phi\) at \(p\) can be defined as that for an ordinary smooth curve. If \(p\in \tilde{R}_p\cap F\) is a singular point of \(\tilde{R}_p\), then the ramification index of \(\phi\) at \(p\) is defined as the sum of ramification indices of the normalization of \(\tilde{R}_p\) at the pre-image of \(p\) with respect to its projection to \(\mathbb{C}\) plus the double of the contribution to the arithmetic genus of \(\tilde{R}_p\) during its normalization at the singular point \(p\). If the normalization of \(\tilde{R}_p\) contains an isolated vertical component \(E\), then the contribution of \(E\) to the ramification index of \(\phi\) is equal to \(2g(E)-2\).

Since there are a finite number of fibers \(F\) with \(s_i(F)\neq 0\), we define the \(i\)-th singularity index \(s_i(F)\) of \(f\) to be the sum of \(s_i(F)\) for all fibers, when \(i=2, 3\). If we take away from the branch locus \(R\) all the fibers \(F\) with odd \(s_3(F)\), we obtain a divisor \(R_p\) which is called the principal part of \(R\). Suppose that

\[ R_p \sim -3K_{P/C} + nF, \]

where \(K_{P/C}\) is the relative canonical divisor of \(\pi\) and \(\sim\) represents the numerical equivalence. With these definitions, the formula for the relative invariants of a genus 2 fibration can be stated as follows:

**Theorem 1.1** (Xiao [10]). Let \(f: S\to C\) be a relatively minimal fibration of genus 2. Then

\[ K_{S/C}^2 = K_S^2 - 8(g(C) - 1) = \frac{1}{5} s_2(f) + \frac{7}{5} s_3(f) = 2n - s_3(f), \]

\[ \chi_f = \chi(\mathcal{O}_S) - (g(C) - 1) = \frac{1}{10} s_2(f) + \frac{1}{5} s_3(f) = n - s_3(f). \]

2. **Local cases.** We begin with a local fibration \(f: S_d\to \Delta\) where \(f\) is an analytic mapping onto the unit disk \(\Delta\), \(S_d\) is a 2-dimensional analytic smooth manifold and the fibers of \(f\) are projective curves. We assume that the fiber over the zero is singular and all the fibers over \(\Delta^* = \Delta - \{0\}\) are smooth curves of genus 2.
Similarly, we have a commutative diagram:

\[
\begin{array}{ccc}
S_\Delta & \xrightarrow{\overline{\vartheta}} & \overline{P}_\Delta \\
\downarrow{\rho} & & \downarrow{\psi} \\
S_\Delta & \xrightarrow{f} & P_\Delta \\
\downarrow{\pi} & & \\
\Delta & & \\
\end{array}
\]

Denote the branch locus in \( P_\Delta \) by \( R_\Delta \). We also denote the horizontal part of \( R_\Delta \) by \( R'_\Delta \), that is, \( R'_\Delta = \{ R_\Delta - F \_0 \} \) if \( R_\Delta \) contains \( F_\_0 \), otherwise

\[
R'_\Delta = \begin{cases} 
R_\Delta - F_\_0 , & \text{if } R_\Delta \text{ contains } F_\_0 , \\
R_\Delta & \text{otherwise} .
\end{cases}
\]

Let \( F_0 = \pi^{-1}(0) \), \( F_t = \pi^{-1}(t) \), \( t \in \Delta^* \), and \( K_\Delta = \{ \sigma \in \text{Aut}(S_\Delta) \mid f \circ \sigma = f \} \). Any automorphism \( \sigma \in K_\Delta \) induces an automorphism \( \sigma \) of \( P_\Delta \) satisfying \( \pi \circ \sigma = \pi \) and \( \sigma(R_\Delta) = R_\Delta \). If we denote the image of \( K_\Delta \) by \( K_\Delta^\text{AutP_\Delta} \), then

\[
| K_\Delta | = 2| \overline{K}_\Delta | .
\]

Note that any finite automorphism group of \( P^1 \) must be those in Table 1.

| Table 1. |
|-----------------------------|
| \( G \subseteq \text{Aut}(P^1) \) | \( | G | \) | Number of points in an orbit |
|-----------------------------|
| Cyclic group | \( Z_\_n \) | \( n \) | \( 1, n \) |
| Dihedral group | \( D_{2n} \) | \( 2n \) | \( 2, n, 2n \) |
| Tetrahedral group | \( T_{12} \) | 12 | \( 4, 6, 12 \) |
| Octahedral group | \( O_{24} \) | 24 | \( 6, 8, 12, 24 \) |
| Icosahedral group | \( I_{60} \) | 60 | \( 12, 20, 30, 60 \) |

For any \( \sigma \in \overline{K}_\Delta \), its restriction \( \sigma|_{F_t} \) to \( F_t \cong P^1 \) must preserve the set of six points contained in \( F_t \cap R_\Delta \). Hence \( \overline{K}_\Delta \) can be isomorphic to one of the following groups \( O_{24}, T_{12}, D_{12}, D_6, Z_6, Z_5, D_4, Z_4, Z_3, Z_2 \) and \{1\}.

**Lemma 2.1.** If \( \overline{K}_\Delta \cong O_{24}, T_{12} \) or \( D_{12} \), then \( F_0 \) is contained in \( R_\Delta \), and \( R_\Delta \) has six ordinary double points on \( F_0 \). In this case, we have \( s_2(F_0) = 10 \) and \( s_3(F_0) = 0 \).

**Proof.** Since \( \overline{K}_\Delta \cong O_{24}, T_{12} \) or \( D_{12} \), \( R_\Delta \cap F_t (t \in \Delta^*) \) consists respectively of six vertices of a regular octahedron, of six points corresponding to the centers of edges of a regular tetrahedron, or of sixth roots of unity. These six horizontal branches of \( R_\Delta \)
cannot intersect when \( t \to 0 \). Since \( R_d \) must have some singularities by assumption, \( F_0 \) is contained in \( R_d \).

Since \( R_d \) does not contain non-negligible singularities, one has \( s_3(F_0)=0 \) and \( R_d = \hat{R}_d.(\hat{R}_d) \). On \( F_0 \), \( R_d \) has six ordinary double points, the contribution of each double point to the arithmetic genus of \( R_d \) during its normalization being equal to one. The pre-image of \( F_0 \) in the normalization of \( R_d \) is a smooth vertical rational curve which does not meet any other branches, so its contribution to the index \( s_2(F_0) \) is equal to \(-2\). Therefore \( s_2(F_0)=2 \times 6 + (-2) = 10 \). \( \square \)

We list the following useful lemmas, whose proofs are evident. Since local equations are used for calculation of singularity indices, they are given in simplified form, omitting some higher order terms. All the non-negligible singularities here are canonical, i.e., those defined by Horikawa.

**Lemma 2.2.** If \( k_d \cong D_6 \) and \( R'_d \) is not étale over \( \Delta \), then up to coordinate transformation we have:

1. The equation of \( R'_d \) is \( (x^3-t^k)(t^k x^3-1) \), \( k \geq 0 \). In this case, \( s_3(F_0)=0 \) implies \( s_2(F_0) \geq 4 \).
2. The equation of \( R'_d \) is \( (x^3-1)^2-t^k(x^3+1)^2 \), \( k \geq 0 \). In this case, we have \( s_3(F_0)=0 \) and \( s_2(F_0) \geq 3 \).

**Lemma 2.3.** If \( k_d \cong Z_6 \) and \( R'_d \) is not étale over \( \Delta \), then up to coordinate transformation, the equation of \( R'_d \) is \( x^6-t^k \), \( 1 \leq k \leq 3 \). If \( k=3 \), it has a non-negligible singularity with \( s_3(F_0)=1 \) and \( s_2(F_0)=3 \). Otherwise \( s_2(F_0) \geq 5 \).

**Lemma 2.4.** If \( k_d \cong Z_5 \) and \( R'_d \) is not étale over \( \Delta \), then up to coordinate transformation, we have:

1. The equation of \( R'_d \) is \( x(x^5-t^k) \), \( k \geq 1, 2 \). In this case, \( s_3(F_0)=0 \) and \( s_2(F_0) \geq 6 \).
2. The equation of \( R'_d \) is \( x(t^k x^5-1) \), \( k \geq 1, 2 \). In this case, \( s_3(F_0)=0 \) and \( s_2(F_0) \geq 4 \).

**Lemma 2.5.** If \( k_d \cong D_4 \) and \( R'_d \) is not étale over \( \Delta \), then up to coordinate transformation, we have:

1. The equation of \( R'_d \) is \( (x^2-1)((x-1)^2-t^k(x+1)^2)(t^k x^2-1)-x(x+1)^2) \), \( k \geq 0 \). In this case, \( s_3(F_0)=0 \) implies \( s_2(F_0) \geq 6 \).
2. The equation of \( R'_d \) is \( (x-1)(x^2-t^k)(t^k x^2-1) \), \( k \geq 0 \). In this case, we have \( s_3(F_0)=0 \) and \( s_2(F_0) \geq 2 \).

**Lemma 2.6.** If \( k_d \cong Z_4 \) and \( R'_d \) is not étale over \( \Delta \), then up to coordinate transformation, the equation of \( R'_d \) is \( x(x^4-t^k) \), \( k \geq 1, 2 \). In this case, we have \( s_3(F_0)=0 \) and \( s_2(F_0) \geq 5 \).

**Lemma 2.7.** If \( k_d \cong Z_3 \) and \( R'_d \) is not étale over \( \Delta \), then up to coordinate transformation, we have:

1. The equation of \( R'_d \) is \( (x^3-t^k)(t^k x^3-a(t)) \), \( k_1, k_2 > 0, a(0) \neq 0 \). In this case, \( s_3(F_0)=0 \) implies \( s_2(F_0) \geq 4 \).
The equation of $R'_{\Delta}$ is $x^6 + a(t)x^3 + t^k$, $1 \leq k \leq 3$. In this case, $s_3(F_0) = 0$ implies $s_2(F_0) \geq 5$.

(3) The equation of $R'_{\Delta}$ is $(x^3 - b - t^k)(x^3 - b - t^ka(t))$, $k_1, k_2 > 0$, $a(0) \neq 0$ and $b \neq 0$. In this case, we have $s_3(F_0) = 0$ and $s_2(F_0) \geq 6$.

(4) The equation of $R'_{\Delta}$ is $(x^3 - t^k)(x^3 - a(\eta))$, $1 \leq k \leq 3$, $a(0) \neq 0$. In this case, we have $s_3(F_0) = 0$ and $s_2(F_0) > 2$.

(5) The equation of $R'_{\Delta}$ is $((x - b)^2 - t^k a(t))(x - b\omega^2 - \omega^2 t^k a(t))(x - b\omega^2 - \omega t^k a(t))$, $k > 0$, $a(0) \neq 0$, $b \neq 0$, $\omega = \exp(2\pi i/3)$. In this case, we have $s_3(F_0) = 0$ and $s_2(F_0) \geq 3$.

We summarize the results of Lemmas 2.2 through 2.7 in Table 2 where we assume that $R'_{\Delta}$ has only negligible singularities or ramifications on $F_0$.

**Table 2.**

| $\bar{K}_A$ | $|K_A|$ | $s_2(F_0)$ | $|K_A|/s_2(F_0)$ |
|------------|-------|------------|----------------|
| $D_6$      | 12    | $\geq 3$   | $\leq 4$       |
| $Z_6$      | 12    | $\geq 5$   | $\leq 2.4$     |
| $Z_5$      | 10    | $\geq 4$   | $\leq 2.5$     |
| $D_4$      | 8     | $\geq 2$   | $\leq 4$       |
| $Z_4$      | 8     | $\geq 5$   | $\leq 1.6$     |
| $Z_3$      | 6     | $\geq 2$   | $\leq 3$       |
| $Z_2$      | 4     | $\geq 1$   | $\leq 4$       |
| $1$        | 2     | $\geq 1$   | $\leq 2$       |

**Lemma 2.8.** If $R'_{\Delta}$ has only negligible singularities or ramifications on $F_0$, then $|K_A|/s_2(F_0) \leq 2.5$. Moreover, if $\bar{K}_A \cong Z_6, Z_5, Z_4$ or $\{1\}$, then $|K_A|/s_2(F_0) \leq 2.5$.

3. Bounds of automorphism groups. Let $G = \text{Aut}(f)$ be the automorphism group of the fibration $f: S \to C$ of genus two. Then we have an exact sequence

$$1 \to K \to G \to H \to 1,$$

$$(\tilde{\sigma}, \sigma) \mapsto \sigma,$$

where $H \leq \text{Aut}(C)$, $K = \{ (\tilde{\sigma}, \text{id}) \in G \} = \{ \tilde{\sigma} \in \text{Aut}(S) \mid f \circ \tilde{\sigma} = f \}$. Thus $|G| = |K||H|$. The elements of $H$ are often regarded as transformations of the fibers of $f$ or $\pi$.

**Proposition 3.1.** If $f: S \to C$ is a relatively minimal fibration of genus 2 with $g(C) \geq 2$, then

$$|\text{Aut}(f)| \leq 504K_5^2 .$$

**Proof.** Since $|K| \leq 48$, $|H| \leq |\text{Aut}(C)| \leq 84(g(C) - 1)$, we have

$$|G| = |K||H| \leq 4032(g(C) - 1) .$$
On the other hand, $K_{S/C}^2 \geq 0$ and the equality holds if and only if $f$ is locally trivial. Hence

$$K_S^2 \geq 8(g - 1)(g(C) - 1) = 8(g(C) - 1),$$

and $|G| \leq 504 K_S^2$.

**Proposition 3.2.** If $f : S \to C$ is a relatively minimal fibration of genus 2 with $g(C) \geq 2$ which is not locally trivial, then

$$|\text{Aut}(f)| \leq 126 K_S^2.$$

**Proof.** Let $R'$ denote the horizontal part of the branch locus $R$. If $R'$ is not étale over $C$, then by the lemmas in Section 2, we have $|K| \leq 12$. Since $|H| \leq 84(g(C) - 1) \leq 10.5 K_S^2$, $|G| \leq 12 |H| \leq 126 K_S^2$.

Now assume that $R'$ is étale. Since $f$ is not locally trivial, we must have $K_{S/C}^2 > 0$, i.e., either $s_3(f) > 0$ or $s_2(f) > 0$. So $R$ must contain some fiber $F_0$. By Lemma 2.1, $s_3(F_0) = 0$ and $s_2(F_0) = 10$. Let $p = f(F_0)$, $n = |H|$. Since $H$ is a subgroup of $\text{Aut}(C)$, $H$ determines a finite morphism $\tau : C \to X = C/H$. Denote the ramification index of $p \in C$ with respect to $\tau$ by $r$ and the other ramification indices by $r_i$. Then Hurwitz's theorem implies that

$$2g(C) - 2 = n(2g(X) - 2) + n \sum \left(1 - \frac{1}{r_i}\right).$$

Since the $H$-orbit of the point $p$ has $n/r$ points, this implies that $s_2(f) \geq 10n/r$. Hence

$$K_S^2 \geq \frac{1}{5} s_2(f) + 8(g(C) - 1) = \frac{2n}{r} + 4n \left[2g(X) - 2 + \sum \left(1 - \frac{1}{r_i}\right)\right]$$

$$= 4n \left[2g(X) - 2 + \frac{1}{2r} + \sum \left(1 - \frac{1}{r_i}\right)\right].$$

It is not difficult to see that the expression $2g(X) - 2 + 1/2r + \sum(1 - 1/r_i)$ reaches its minimal value $2/21$ (under the condition $2g(X) - 2 + \sum(1 - 1/r_i) > 0$) when $g(X) = 0$, $r_1 = 2$, $r_2 = 3$, and $r = r_3 = 7$, that is,

$$K_S^2 \geq \frac{8}{21} n = \frac{8}{21} |H|.$$

Thus

$$|G| \leq 48 |H| \leq 126 K_S^2.$$

**Remark.** It is not difficult to see that if $g(C) \geq 2$, $f$ is not locally trivial and $|\text{Aut}(f)| = 126 K_S^2$, then $|\text{Aut}(C)| = 84(g(C) - 1)$, $|\text{Aut}(F)| = 48$ for any smooth fiber $F$ and $\text{Aut}(f) \cong \text{Aut}(C) \times \text{Aut}(F)$. We will give an example later. In this case, the fibration
LEMMA 3.1. Let $S$ be a surface of general type which has a relatively minimal genus 2 fibration $f : S \to \mathbb{C}$. If the third singularity index $s_3(f) \neq 0$, then
\[ |\text{Aut}(f)| \leq \frac{60}{7} rK_{S/C}^2, \]
where
\[ r = \min_{s_3(F) \neq 0} |\text{Stab}_{H} f(F)|, \]
$\text{Stab}_{H} f(F)$ being the stabilizer of $f(F)$ in $H$.

PROOF. Let $F_0$ be a singular fiber such that $s_3(F_0) \neq 0$ and $r = |\text{Stab}_{H} f(F_0)|$. Then
\[ K_{S/C}^2 \geq \frac{7}{5} s_3(f) \geq \frac{7s_3(F_0)}{5r} |H|, \]
and we get
\[ |G| = |K| |H| \leq \frac{r}{s_3(F_0)} \cdot \frac{60}{7} K_{S/C}^2 \leq \frac{60}{7} rK_{S/C}^2. \]

LEMMA 3.2. Let $S$ be a surface of general type which has a relatively minimal genus 2 fibration $f : S \to \mathbb{C}$. If the horizontal part $R'$ of the branch locus $R$ is not étale and has only negligible singularities or ramifications, then
\[ |\text{Aut}(f)| \leq 20rK_{S/C}^2, \]
where
\[ r = \min \{|\text{Stab}_{H} f(F)| \mid F \text{ singular fiber}\}. \]

PROOF. Let $F_0$ be a singular fiber with $r = |\text{Stab}_{H} f(F_0)|$. Since here
\[ K_{S/C}^2 \geq \frac{1}{5} s_2(f) \geq \frac{s_2(F_0)}{5r} |H|, \]
we have
\[ |G| = |K| |H| \leq \frac{r|K|}{s_2(F_0)} \cdot 5K_{S/C}^2 \leq 20rK_{S/C}^2, \]
by Lemma 2.8. \qed

LEMMA 3.3. Let $S$ be a surface of general type which has a relatively minimal genus 2 fibration $f : S \to \mathbb{C}$. If the horizontal part $R'$ of the branch locus $R$ is étale, then
where

\[ r = \min \{|\text{Stab}_H f(F)| : F \text{ singular fiber} \} . \]

**Proof.** Let \( F_0 \) be a singular fiber with \( r = |\text{Stab}_H f(F_0)| \). By assumption, we have \( s_2(F_0) = 10 \). Hence

\[ |G| = |K||H| \leq \frac{r|K|}{s_2(F_0)} \cdot 5K_{S/C}^2 \leq 24rK_{S/C}^2 . \]

Let \( \bar{K} \) denote the subgroup in \( \text{Aut}(P) \) which is induced by \( K \). If \( \sigma \in \bar{K} \), then \( \pi \circ \sigma = \pi \) and \( \sigma(R) = R \). Let \( K_1 \) be a cyclic subgroup of order \( m \) of \( \bar{K} \), and let \( Q = P/K_1 \) be the quotient surface. Then \( Q \) is a ruled surface. We have a commutative diagram:

\[
\begin{array}{ccc}
P & \longrightarrow & Q \\
\pi & \downarrow & \pi' \\
C & \Downarrow & \end{array}
\]

Let \( C_0 \) and \( C_\infty \sim C_0 + \epsilon F \) be the reduced ramification divisors of \( K_1 \). Let \( C'_0 \) be a section of \( \pi' \) with the least self-intersection number \( C'_0^2 = -e' \), and let \( F' \) be a general fiber of \( \pi' \). Then \( \alpha^* C'_0 = mC_0 \), \( \alpha^* C'\infty = mC_\infty \), \( \alpha^* F' = F \) and \( e' = me \). Let \( D = \alpha(R') \), and let \( C' = C'_0 + C_\infty \) be the branch locus. Then \( C' \sim 2C_0 + e'F' \sim -K_{Q/C} \).

**Lemma 3.4.** Assume \( \bar{K} \cong D_6 \). If \( R' \) is not étale and has only negligible singularities or ramifications, then \( f \) has more than one \( H \)-orbits of singular fibers.

**Proof.** Let \( K_1 \) be the unique cyclic subgroup of order 3 of \( \bar{K} \). There are two types of singular fibers as listed in Lemma 2.2. Let \( F_0 \) be a singular fiber. Then the local equations of \( D \) near \( F_0 \) are (1) \( (x - t^4)(t^kx - 1) \), \( k \leq 3 \), (2) \( (x - 1)^2 - t^k(x + 1)^2 \), \( k > 0 \). In Case (1), \( D \) meets \( C' \) at two points in \( F_0 \). In Case (2), \( D \) does not meet \( C' \) in \( F_0 \).

If all the singular fibers of \( f \) are of type (1), then \( D \) is an étale cover of \( C \). This means that \( a = e' \) and \( C' \sim D \). Hence \( DC' = 0 \), which is impossible because \( D \) and \( C' \) meet in \( F_0 \).

If all the singular fibers of \( f \) are of type (2), then \( DC' = 0 \). Hence \( D \sim C' \) and \( D(D + K_{Q/C}) = 0 \). This means that \( D \) is étale over \( C \), a contradiction.

**Lemma 3.5.** Assume \( \bar{K} \cong D_4 \). If \( R' \) is not étale, then \( f \) has more than one \( H \)-orbits of singular fibers. If \( H \) is cyclic and \( g(C) = 0 \), then

\[ |\text{Aut}(f)| \leq 12.5K_{S/C}^2 . \]
Proof. In this case, there are four sections in \( P \) which do not meet one another. Hence \( e = 0 \). \( R' \) contains two of these sections denoted by \( C_0 \) and \( C_\infty \). Let \( K_1 \) be a cyclic subgroup of \( K \) with \( C_0 \) and \( C_\infty \) are ramifications. Assume that there is only one \( H \)-orbit of singular fibers. If these singular fibers are all of type (1) in Lemma 2.5, then the local equation of \( D = \alpha(R' - C_0 - C_\infty) \) is \( (x - t^k(x - 1)) \), namely, \( D \) is étale. Therefore \( D \sim 2C_0', DC_0 = DC_\infty = 0 \), a contradiction. If the singular fibers are of type (2) in Lemma 2.5, then \( D \) does not meet \( C_0' \) and \( C_\infty' \). Hence \( D \sim 2C_0' \), \( D^2 = 0 \), a contradiction. Hence there are at least two \( H \)-orbits.

Now suppose \( H \) is cyclic. Let \( h = \lvert H \rvert \). An \( H \)-orbit is said to be big if it contains \( h \) fibers. If there is a big \( H \)-orbit whose singular fibers are of type (1), then \( s_2(F_0) \geq 6 \), so \( |G| \leq (20/3)K_{5/C}^2 \). If \( |G| > (20/3)K_{5/C}^2 \), then the singular fibers in a big \( H \)-orbit must be of type (2) with \( k \leq 2 \). Let \( F_2 \) and \( F_3 \) denote two fibers fixed by \( H \). Then at least one of them is of type (1). The structure of types (1) and (2) implies that the normalization of \( D = \alpha(R' - C_0 - C_\infty) \) is étale with respect to \( \pi'' \). Hence \( D \) must be decomposed into two isomorphic sections \( D_1 \) and \( D_2 \) with \( D_1 \sim D_2 \sim C_0' + aF' \). Since both \( D_1 \) and \( D_2 \) meet \( C_0' \) and \( C_\infty' \), \( F_2 \) and \( F_3 \) are all singular of type (1). Since \( D_1D_2 = 2a = kh \), we get \( D_1C_0 = a = kh/2 \). Hence the local equation of \( R' \) near \( F_2 \) or \( F_3 \) is \( (x - t^k(x - 1)) \). When \( h \geq 6 \), these are non-negligible singularities. If \( F_i \) (\( i = 2, 3 \)) is a singular fiber of type I), then \( s_3(F_i) = 2[\lfloor (kh - 2)/8 \rfloor + 1] \geq (kh - 1)/4 \). If \( F_i \) is of type II), then \( s_3(F_i) = 2\lceil (kh/8) \rceil \geq (kh - 6)/4 \). So

\[
K_{5/C}^2 \geq \frac{1}{5} \times 2 \times h + \frac{7}{5} \times \frac{h - 6}{4} \times 2 = \frac{11}{10} h - \frac{21}{5}.
\]

\[
|G| = 8h \leq \frac{80}{11} \left( K_{5/C}^2 + \frac{21}{5} \right) < 12.5K_{5/C}^2.
\]

If there are more than one big \( H \)-orbits, it can be similarly shown that \( |G| \leq 12.5K_{5/C}^2 \).

Lemma 3.6. Assume \( K \cong Z_3 \). If \( R' \) is not étale and has only negligible singularities or ramifications and \( f \) has only one \( H \)-orbit of singular fibers, then

\[
|\text{Aut}(f)| \leq 6rK_{5/C}^2,
\]

where

\[
r = \min \{ |\text{Stab}_H f(F)| \mid F \text{ singular fiber} \}.
\]

Proof. Let \( K_1 = K \). If the singular fibers are of types (1) or (4) in Lemma 2.7, then \( D \sim 2C_0' + aF' \) is étale. \( D(K_{Q/C} + D) = 0 \) implies \( a = e' \). Hence \( D(C_0' + C_\infty') = 0 \), a contradiction. If the singular fiber \( F_0 \) is of type (5) with \( k = 1 \), then \( D \) is irreducible and smooth near \( F_0 \). This implies \( DC_\infty \neq 0 \), a contradiction. Therefore \( s_2(F_0) \geq 5 \) for any singular fiber \( F_0 \). So \( |G| \leq 6rK_{5/C}^2 \).
LEMMA 3.7. Assume $K \cong \mathbb{Z}_2$. If $R'$ is not étale and $f$ has only one $H$-orbit of singular fibers, then

$$|\text{Aut}(f)| \leq 5rK^{3}_{S/C},$$

where

$$r = \min\{|\text{Stab}_{H}f(F)| \mid F \text{ singular fiber} \}.$$

PROOF. Let $F_0$ be a singular fiber. $|G| > 5rK^{3}_{S/C}$ implies $s_2(F_0) \leq 3$. We distinguish between two cases.

Case I. $R'$ contains $C_0$ and $C_{\infty}$. Then the local equation of $R'$ near $F_0$ must be

1. $x(x^2 - t)(x^2 - a(t)), \ a(0) \neq 0, \ s_2(F_0) = 3, \text{ or } 2. \ x((x^2 - a_2)^2 - t), \ a \neq 0, \ s_2(F_0) = 2.$

Let $D = \alpha(R' - C_0 - C_{\infty}) \sim 2C_0 + aF'$. If all the singular fibers are of type (1), then $D$ is étale. This is impossible. If the singular fibers are of type (2), then $D$ is irreducible and does not meet $C'$. This is impossible.

Case II. $R'$ does not contain $C_0$ and $C_{\infty}$. Then the local equation of $R'$ may be

1. $(x^2 - t)(x^2 - a(t))(x^2 - b(t)), \ a(0)b(0) \neq 0, \ s_2(F_0) = 1; \text{ or } 2. \ (x^2 - t)(x^2 - 1)(x^2 - b(t)), \ a(0)b(0) \neq 0, \ s_2(F_0) = 2; \text{ or } 3. \ ((x^2 - a^2)^2 - t)(x^2 - b(t)), \ a(0) \neq 0, \ s_2(F_0) = 2; \text{ or } 4. \ ((x^2 - a^2)^2 - t)(x^2 - b(t)), \ b(0) \neq 0, \ s_2(F_0) = 3.$

Let $D = \alpha(R') \sim 3C_0 + aF'$. If $F_0$ is of type (1) or (2), then $D$ is étale and smooth. $D$ must be decomposed into three disjoint components. This means $e' = 0$, a contradiction. If $F_0$ is of type (3) or (4), then $D$ is smooth. The ramification index is $D(D + K_{S/C}) = 4a - 6e' = |H|/r$. Hence $DC' = 2a - 3e' = |H|/2r$. This is a contradiction because we have $DC' = 0$ for type (3) and $DC' = |H|/r$ for type (4).

PROPOSITION 3.3. If $S$ is a minimal surface of general type which has a genus 2 fibration $f: S \to C$ with $g(C) = 1$, then

$$|\text{Aut}(f)| \leq 144K^2_{S}.$$

PROOF. In this case, we have

$$K^2_{S} = K^2_{S/C} = \frac{1}{5} s_2(f) + \frac{7}{5} s_3(f) > 0.$$

Thus either $s_3(f) > 0$ or $s_2(f) > 0$.

Let $j(C)$ be the $j$-invariant of the elliptic curve $C$. Let $m$ denote the number of points contained in a smallest $H$-orbit of $C$. Since $H$ is a finite subgroup of $\text{Aut}(C)$, we have

$$m = \begin{cases} |H|/2 & \text{if } j(C) \neq 0, 1728, \\ |H|/4 & \text{if } j(C) = 1728, \\ |H|/6 & \text{if } j(C) = 0. \end{cases}$$

Since $r \leq 6$, by Lemmas 3.1, 3.2 and 3.3, the conclusion is immediate. □
PROPOSITION 3.4. If \( S \) is a surface of general type which has a relatively minimal fibration \( f : S \to C \) of genus 2 with \( g(C) = 0 \), then
\[ |\text{Aut}(f)| \leq 120(K_S^2 + 8). \]
Moreover, we have
\[ |\text{Aut}(f)| \leq 48(K_S^2 + 8) \]
for \( K_S^2 \geq 33 \), and when \( K_S^2 \leq 32 \), there are only four exceptions.

PROOF. In this case, we have
\[ K_S^2 + 8 = K_{S/C}^2 = \frac{1}{5} s_2(f) + \frac{7}{5} s_3(f) > 0. \]
Hence either \( s_3(f) > 0 \) or \( s_2(f) > 0 \).

Case I. Assume that \( R \) is étale over \( C \). If \( r \leq 5 \), then by Lemma 3.3
\[ |G| \leq 24rK_{S/C}^2 \leq 120(K_S^2 + 8). \]
If \( r \geq 6 \), then \( H \) must be a cyclic or a dihedral group. In this case, there are at most two singular fibers. Hence \( K_{S/C}^2 \leq 4 \) by Theorem 1.1. This means that \( S \) is not of general type [10, Theorem 4.2.5, p. 90].

Case II. Assume that \( R \) is not étale. Then \( f \) is a fibration of variable moduli. Hence \( f \) must contain more than two singular fibers (cf. [2]). This implies \( r \leq 5 \). The conclusion follows from Lemmas 3.1 and 3.2.

In the preceding argument, we can see that \( |G| \leq 48(K_S^2 + 8) \) holds if \( r \leq 2 \). If \( |G| > 48(K_S^2 + 8) \), we must have \( r > 3 \). Then \( H \) is one of \( T_{12}, O_{24} \) and \( I_{60} \).

If \( f \) has more than one \( H \)-orbit of singular fibers, then
\[
\frac{K_{S/C}^2}{|G|} \geq \frac{1}{5r} \left( \frac{s_2(F_0)}{|K|} + \frac{7s_3(F_0)}{|K|} \right) + \frac{1}{5r_1} \left( \frac{s_2(F_1)}{|K|} + \frac{7s_3(F_1)}{|K|} \right)
\geq \frac{1}{25} \times 1 \times 1 + \frac{1}{20} \times \frac{1}{4} = \frac{9}{400} > \frac{1}{48}.
\]
Therefore \( f \) has only one \( H \)-orbit.

If the singular fibers has non-negligible singularities, then by Lemma 3.1, \( |G| \leq (60/7)rK_{S/C}^2 \leq (300/7)K_{S/C}^2 < 48K_{S/C}^2 \). Suppose that the horizontal part \( R' \) of the branch locus has only negligible singularities or ramifications. Then by Lemmas 3.4, 3.5, 3.6 and 3.7, we have
\[ |G| \leq 12.5rK_{S/C}^2. \]
Thus \( |G| > 48K_{S/C}^2 \) implies that \( r \geq 4 \) and \( \bar{K} \) is \( Z_6 \) or \( Z_5 \). If \( \bar{K} \cong Z_6 \), then \( r = 5 \) and \( H \cong I_{60} \).
To ensure \( |G| > 48K_{S/C}^2 \), we have \( s_2(F_0) = 5 \), i.e., \( R = R' \sim -3K_{p/C} + nF \) is a smooth irreducible divisor. As a multiple cover on \( C \), the ramification index of \( R \) is equal to
$R(R + K_{P_C}) = 12n$. On the other hand, this ramification index is equal to $5 \times (60/5) = 60$, i.e., $n = 5$. However, $2n = 10 = K_{S/C}^2 \neq s_2(f)/5 = 12$, a contradiction.

If $\bar{K} \cong \mathbb{Z}_3$, then $|G| > 48K_{S/C}^2$ implies $s_2(F_0) = 4$. In this case $R = R' = C_0 + R_1$, where $R_1 \sim 5C_0 + (n + 3e)F$ is a smooth irreducible divisor and $R_1C_0 = 0$, i.e., $n = 2e$. Computing the ramification index of $R_1$ we get $R_1(R_1 + K_{P/C}) = 10n = 4|H|/r$. Thus $5r$ divides $|H|$, a contradiction. Hence $|G| > 48(K_S^2 + 8)$ implies that $R'$ is étale over $C$. There are only a finite number of possibilities. We list the possible fibrations with $|G| > 48(K_S^2 + 8)$ in Table 3.

| $H$ | $r$ | $|G|$ | $K_S$ | $|K||K_S + 8|$ | $|K||/K_S^2$ |
|-----|-----|-------|-------|-----------------|----------------|
| $\mathbb{I}_{60}$ | 5   | 2880  | 16    | 120             | 180            |
| $\mathbb{I}_{60}$ | 3   | 2880  | 32    | 72              | 90             |
| $\mathbb{O}_{24}$ | 4   | 1152  | 4     | 96              | 288            |
| $\mathbb{O}_{24}$ | 3   | 1152  | 8     | 72              | 144            |

In Section 5 we will show the existence.

**Corollary 3.5.** If $S$ is a minimal surface of general type which has a genus 2 fibration $f : S \to C$ with $g(C) = 0$, then

$$|\text{Aut}(f)| \leq 288K_S^2.$$  

**Proof.** If $K_S^2 \geq 2$, then $48(K_S^2 + 8) < 288K_S^2$. By Proposition 3.4 we need only check the four exceptional examples.

**4. Abelian automorphism groups.** Let $G \subseteq \text{Aut}(f)$ be an abelian group. Then it is well known that $|K| \leq 12$.

**Proposition 4.1 (Xiao [7, Lemma 8]).** Let $f : S \to C$ be a relatively minimal fibration of genus 2 with $g(C) \geq 2$. Then an abelian automorphism group $G$ of $S$ satisfies

$$|G| \leq 6K_S^2 + 96.$$  

Let $\bar{G} \subseteq \text{Aut}(P)$ be the induced automorphism group of a commutative group $G$. Then

$$1 \to \bar{K} \to \bar{G} \to H \to 1.$$

**Lemma 4.1.** Assume that $\bar{K} \cong \mathbb{Z}_3$ and $g(C) = 0$. Let $p \in C$ be a fixed point of the cyclic group $H$, and let $F = \pi^{-1}(p)$. If there is a $\bar{K}$-orbit containing three points in $F$, then $s_2(F) \geq 3|H|$.  

**Proof.** Since $p$ is a fixed point of $H$, the induced action of $\bar{G}$ on $F$ forms
a commutative subgroup $G|_F \subseteq \text{Aut}(F) \cong \text{Aut}(P^1)$. Since $G|_F$ stabilizes this $K|_F$-orbit, we have $G|_F = K|_F \cong Z_3$, i.e., $H|_F = 1$. Hence the local equation of $R'$ near $F$ has the form $f(x^3, t^h)$ where $h = |H|$. More explicitly, the local equation of $R'$ is (3) $(x^3 - b - t^{kh}a_1(t^h))((x^3 - b - t^{kh}a_2(t^h))$ or (5) $((x-b)^2 - t^{kh}(t^h))((x-b\omega)^2 - \omega t^{kh}(t^h))$, $b \neq 0$. Thus $s_3(F) \geq 3h = 3|H|$.

**Proposition 4.2.** If $S$ is a surface of general type which has a relatively minimal fibration $f : S \to C$ of genus 2 with $g(C) \leq 1$, then an abelian automorphism group $G$ of $f$ satisfies

$$|G| \leq 12.5(K^2 + 8).$$

**Proof.** It is well known that $H$ must be a cyclic group or a dihedral group $D_4 \cong Z_2 \oplus Z_2$.

If $g(C) = 1$ and $H$ does not act freely on $C$, then $|H| \leq 72 < 12.5(K^2 + 8)$. If $g(C) = 0$ and $H \cong D_4$, then $|G| \leq 48$ and the claim holds too. So we can assume that $H$ is a cyclic group and that there exists a singular fiber $F_0$ with $|\text{Stab}_H f(F_0)| = 1$.

Case I. Suppose that the horizontal part $R'$ of the branch locus $R$ is étale over $C$. Then $|G| \leq 6K^2_C$.

Case II. Suppose that $R'$ is not étale. If there is a big $H$-orbit with $s_3(F_0) \neq 0$, then

$$K^2_{Z'/C} \geq \frac{7}{5} s_3(f) \geq \frac{7}{5} |H|,$$

so

$$|G| \leq \frac{60}{7} K^2_{Z'/C} < 12.5(K^2 + 8).$$

Now suppose that on the big $H$-orbits $R'$ has only negligible singularities or ramifications. If $\bar{R} \cong Z_6, Z_5, Z_4$ or $\{1\}$, then by Lemma 2.8, we have

$$|G| \leq \frac{|K|}{s_2(F_0)} \cdot 5K^2_{Z'/C} \leq 12.5K^2_{Z'/C} \leq 12.5(K^2 + 8).$$

Suppose that $\bar{R} \cong D_4, Z_3$ or $Z_2$ and that $|G| > 12.5(K^2 + 8)$. Then Lemmas 3.5, 3.6 and 3.7 imply that $f$ must have more than one $H$-orbits of singular fibers. To ensure $|G| > 12.5(K^2 + 8)$, $f$ cannot have more than one big $H$-orbits. Thus we have $g(C) = 0$. Lemma 3.5 excludes the case of $\bar{R} \cong D_4$.

If $\bar{R} \cong Z_3$, then $s_2(F_0) \leq 2$. Hence $F_0$ must be of type (4) of Lemma 2.7 with $k = 1$. Taking $K_1 = \bar{R}$ we construct the quotient surface $Q = P/K_1$ as in §3. Then $D = \alpha(R')$ is étale near $F_0$. But $D$ cannot be étale. Hence at least one of the $H$-stabilized fibers $F_2$ and $F_3$ is of type (2) $k = 1$ or type (5) $k = 1$. Lemma 4.1 excludes the case of type (5). Suppose one of the $F_i$ is of type (2). Then $D \sim 2C'_0 + aF''$ is irreducible and smooth. As
a smooth double cover of $C \cong P^1$, the ramification index of $D$ is at least 2. So $F_2$ and $F_3$ are all of type (2). Then $DC' = D(D + K_{Q/C}) = 2(a - c') = 2$, a contradiction.

If $R \cong Z$, then $s_2(F_0) = 1$. Hence the local equation of $R'$ near $F_0$ is $(x^2 - t)(x^2 - a(t))(x^2 - b(t))$, $a(0)b(0) \neq 0$, $a(0) \neq b(0)$. So $D = \mathcal{A}(R')$ is étale near $F_0$.

If $F_2$ and $F_3$ have no ramifications, then $D$ can be decomposed into three components $D_i \sim C_0 + a_iF'$, $i = 1, 2, 3$. These three components must meet one another on $F_2$ and $F_3$. So there exists at least one point on $F_i$ where three components intersect. The local equation of $R'$ is $(x^2 + a(t)x + t^2)(x^2 + t^2b(t))$. Since $D_i(C_\infty - C_0) = e'$, we have $|H| \leq 1$.

If $F_2$ or $F_3$ has ramifications, the equation of $R'$ near $F_i$ must be one of (1) $x^6 - t$; (2) $(x^4 - t)(t^4a(t)x^2 - b(t))$, $a(0) \neq 0$; (3) $(x^2 - a^2)^2 - t(x^2 - t^2)$, $a \neq 0$; (4) $(x^2 - a^2)^2 - t(x^2 - b(t))$, $b(t) \neq 0$. If $F_2$ is of type (1), then $D$ is irreducible and smooth. As a smooth triple cover of $C \cong P^1$, the ramification index of $D$ is at least 4. Hence $F_3$ is of type (1) as well. Let $D \sim 3C_0 + aF'$. Then $2DC' = D(D + K_{Q/C}) = 4$, impossible. If $F_2$ is of type (2), then $D$ is smooth and cannot be irreducible. $D$ has two components $D_1 \sim 2C_0 + aF'$ and $D_2 \sim 2C_0 + bF'$. By the same argument, we have $D_1C_0 = D_1(D_1 + K_{Q/C}) + 2$. Hence $D_1C_0 = 0$ and $D_1D_2 = 0$, which is impossible.

Suppose that $G$ is a cyclic automorphism group of $f$. Similarly, there is an exact sequence

$$
1 \longrightarrow K \xrightarrow{\alpha} G \xrightarrow{\beta} H \longrightarrow 1,
$$

where $H \subseteq \text{Aut}(C)$, $K = \{(\sigma, \text{id}) \in G\}$. It is known that $|K| \leq 10$.

**Lemma 4.2.** Suppose that $f : S \rightarrow C$ is a fibration and that $G$ is a cyclic automorphism group of $f$. Suppose there exists a point $p \in C$ such that

1. $\sigma|_{f^{-1}(p)} \in K|_{f^{-1}(p)}$, for $\sigma \in G$ and $\sigma$ stabilize $f^{-1}(p)$;
2. $K \rightarrow \text{Aut}(f^{-1}(p))$ is injective.

Then $|K|$ and $|\text{Stab}_H(p)|$ are coprime.

**Proof.** Let $H_1 = \text{Stab}_H(p)$, $F = f^{-1}(p)$. Let $h = |H_1|$, $k = |K|$, $d = (h, k)$. Assume that $\sigma$ is a generator of $\beta^{-1}(H_1)$. Then $\beta((\sigma^{(n/d)^k})) = 1$ implies $\sigma^{(k/d)} \in K$. On the other hand, since $\sigma|_F \in K|_F$ by (1), we obtain $(\sigma^{h/d})|_F = \text{id}_F$. Thus $\sigma^{kh} = 1$ by (2). This is impossible.

**Proposition 4.3.** If $S$ is a surface of general type which has a relatively minimal fibration $f : S \rightarrow C$ of genus 2 with $g(C) = 1$. Then a cyclic automorphism group $G$ of $f$ satisfies

$$
|G| \leq 5K_5^2
$$

for $K_5^2 \geq 12$.

**Proof.** If $H$ does not act freely on $C$, then $|H| \leq 6$. Hence $|G| \leq 60$ and the
conclusion holds. Therefore we assume that $H$ acts freely. So $G \cong K \times H$ and $G$ is cyclic if and only if $(|K|, |H|) = 1$. We distinguish two cases.

Case I. Suppose that the horizontal part $R'$ of the branch locus $R$ is étale over $C$. There exists a singular fiber $F_0$ with $|\text{Stab}_H(f(F_0))| = 1$. It is not difficult to show that in this case $|G| \leq 5K_5^2$.

Case II. Suppose that $R'$ is not étale.

(a) $\bar{K} \cong Z_5$. Let $F_0$ be a singular fiber. The local equation of $R'$ near $F_0$ is (1) $x(x^5 - t^k)$ or (2) $x(t^k x^5 - 1)$, $k = 1, 2$. We use the same construction as in Case (a). Then $R'$ must contain one of the sections $C_0$ and $C_\infty$. We take this section away from $R'$, and get a reduced divisor $R_1$ with $R_1 F = 5$. Let $D = \alpha(R_1)$. Then $D \sim C_0 + aF'$. Since $DC_0 = 0$, we have $a = e' = 5e$. Thus $R_1 \sim SC_0 + 5eF$ and $R_1 C_\infty = 5e$. Since the intersection number of $R_1$ and $F$ on the fiber $F_0$ is equal to $k \leq 2$, the number of singular fibers must be a multiple of 5. But $|H|$ cannot be divisible by 5, hence the singular fibers are located in different $H$-orbits. This means $|G| \leq 5K_5^2$.

(b) $\bar{K} \cong Z_4$. The local equation of $R'$ near a singular fiber $F_0$ is $x(x^4 - t^k)$, $k = 1, 2$. We use the same construction as in Case (a). Then $R'$ must contain $C_0$ and $C_\infty$. Let $R_1 = R' - C_0 - C_\infty$ and $D = \alpha(R_1)$. Then $D \sim C_0 + aF'$. Similarly we deduce $R_1 C_\infty = 4e$. Since $|H|$ cannot be even, there are more than one singular $H$-orbits. So $|G| \leq 5K_5^2$.

(c) $\bar{K} \cong Z_3$. If $f$ has only one $H$-orbit of singular fibers and if $|G| > 5K_5^2$, then $s_2(F_0) = 5$, namely, the local equations of $R'$ is $x^6 + a(t)x^3 + t$. Constructing the quotient surface $Q = P/K$, we see that $D = \alpha(R') - 2C_0 + aF'$ is a smooth irreducible curve and $r \neq |H|$. Since $DC_0 = 0$ and $DC_\infty = |H|$, we get $a = e' = 3e = |H|$, i.e., $(|H|, |K|) = 3$, a contradiction.

(d) $\bar{K} \cong Z_2$. Lemma 3.7 ensures $|K| \leq 5K_5^2$.

(e) $\bar{K} = 1$. If $s_2(F_0) \geq 2$, then $|G| \leq 5K_5^2$. If $s_2(F_0) = 1$, there is only one situation, i.e., the local equation of $R'$ near $F_0$ is $(x^2 - t)(x - a_1(t))(x - a_2(t))(x - a_3(t))(x - a_4(t))(x - a_5(t))$, $a_i(0) \neq 0$. Suppose that there is only one singular $H$-orbit. Then $R'$ is a smooth sextuple cover of $C$. The contribution of each singular fiber to the ramification index equals 1. By Hurwitz's formula,

$$2g(R') - 2 = 6(2g(C) - 2) + |H|.$$ 

So $|H|$ is even, a contradiction.

\begin{proof}
If $R'$ is étale, we have $|G| \leq 5K_5^2$. If there is a singular fiber in a big $H$-orbit with $s_5(F) > 0$, then $|G| \leq (50/7)K_5^2$. Now assume that $R'$ has only negligible singularities or ramifications in big $H$-orbits. If $\bar{K} \cong Z_4$ or $\{1\}$, we have $|G| \leq 10K_5^2$ by

\text{PROPOSITION 4.4.} If $S$ is a surface of general type which has a relatively minimal fibration $f : S \to C$ of genus 2 with $g(C) = 0$. Then a cyclic automorphism group $G$ of $f$ satisfies

$$|G| \leq 12.5K_5^2 + 90.$$ 

\text{PROOF.} If $R'$ is étale, we have $|G| \leq 5K_5^2$. If there is a singular fiber in a big $H$-orbit with $s_5(F) > 0$, then $|G| \leq (50/7)K_5^2$. Now assume that $R'$ has only negligible singularities or ramifications in big $H$-orbits. If $\bar{K} \cong Z_4$ or $\{1\}$, we have $|G| \leq 10K_5^2$ by
Lemma 2.8. When $\bar{K} \cong \mathbb{Z}_3$ or $\mathbb{Z}_2$, if $f$ has only one $H$-orbit of singular fibers, then Lemmas 3.6 and 3.7 ensure $|G| \leq 6K^2_{\bar{S}_C}$. Otherwise, by the proof of Proposition 4.2, $f$ has at least two big $H$-orbits of singular fibers, hence $|G| \leq 10K^2_{\bar{S}_C}$.

There remains the case of $\bar{K} \cong \mathbb{Z}_5$. The proof of Proposition 4.3 tells us that if $f$ has only one big $H$-orbit of singular fibers, then $f$ has another singular fiber which is stabilized by $H$. By Lemma 2.4, we have

$$K^2_{\bar{S}_C} \geq \frac{4}{5}(|H| + 1),$$

so

$$|G| + 10|H| \leq 12.5K^2_{\bar{S}_C} - 10 = 12.5K^2_{\bar{S}} + 90.$$  

When $g(C) \geq 2$, we need the following lemma on the order of some automorphisms of a curve. The proof of the lemma is just a slight modification of that of the theorem of Wiman [5]. For the convenience of the reader, we include its proof here which is a modified copy of the version given in [8, Lemma B].

**Lemma 4.3.** Let $H$ be a cyclic group of automorphisms of a curve $C$ of genus $g \geq 2$ such that the order of $|\text{Stab}_H(p)|$ is odd for any $p \in C$. Then

$$|H| \leq 3g + 3.$$  

**Proof.** Let $x$ be a non-zero element in $H$ with the maximal number of fixed points, $H'$ the subgroup of $H$ generated by elements fixing all fixed points of $x$, $n$ the number of fixed elements of $x$, and $k$ the order of $H'$. Then $k$ must be odd. Let $C' = C/H'$, $g' = g(C')$, and let $\Sigma$ be the image of the set of fixed points of $H'$ on $C'$. We have

$$2g - 2 = 2kg' - 2k + n(k - 1)$$

and the quotient group $H'' = H/H'$ is a cyclic group of automorphisms of $C'$ which satisfies the same condition imposed on $H$, i.e., $|\text{Stab}_{H''}(p)|$ is odd for any $p \in C'$.

If $n = 0$, then $g' \geq 2$ and $|H| \leq g - 1$. If $n = 2$, then because every non-zero element of $H''$ induces a non-trivial translation on $\Sigma$, we must have $|H''| \leq 2$, so $|H| \leq 2k$. Then $|H| \leq 2g$ by (1) (note that $g' \neq 0$ in this case). So we may assume $n \geq 3$.

Suppose $g' = 1$ and $H''$ acts freely on $C'$. Considering the induced action $H''$ on $\Sigma$, we see that $|H''| \leq n$. So (1) gives $|H| \leq 2g + n - 2$. On the other hand, since $k \geq 3$, (1) also gives $n \leq g - 1$, therefore $|H| \leq 3g - 3$ in this case.

Suppose $g' = 1$ and $H''$ does not act freely on $C'$. Then $H''$ has a fixed point. By assumption, $|H''|$ must be odd. This implies $|H''| \leq 3$. So (1) gives $|H| \leq 2g + 1$.

Now suppose that $C'$ is a rational curve. Then the action of $H''$ has exactly two fixed points. So $|H''|$ must be odd. If one of these two points is in $\Sigma$, then $|H''| \leq n - 1$ in view of the action of $H''$ on $\Sigma$. Since $|H''|$ is odd, we have $n \geq 4$. So $|H| \leq 3g + 3$.  

Suppose that Σ and the two fixed points ξ, η of \( H'' \) are disjoint. Let \( H_1 \subset H \) be the stabilizer of a point in the inverse image of ξ. Then \([H : H_1] = k\). Since the stabilizer of a point in the inverse image of η is also of index \( k \) in \( H \), we see that any non-zero element in \( H_1 \) fixes exactly \( 2k \) points, i.e., the inverse image of ξ and η. Now we can replace \( H' \) by \( H_1 \) and repeat the arguments above (note that the only conditions we used are that non-trivial elements in \( H' \) have the same fixed point set and that \( H/H' \) acts faithfully on \( Σ \)). But then \( Σ \) is composed of two orbits of \( H'' \), so \(|H''| \leq n/2\), whereby

\[ |H| \leq \frac{3}{2} g + 3 \]

by (1).

Finally, we use induction on \( g \). Suppose that \( g' \geq 2 \) and \(|H'| \leq 3g' + 3\). (1) gives

\[ 3g + 3 - (n - 4) \frac{3(g - g')}{2g' - 2 + n} \geq |H|. \]

If \( n \geq 4 \), we are done. If \( n = 3 \), by assumption, we must have \(|H''| \leq 3\). Therefore

\[ |H| \leq \frac{3(2g + 1)}{2g' + 1} \leq \frac{3}{5} (2g + 1) \leq 3g + 3. \]

PROPOSITION 4.5. If \( f: S \to C \) is a relatively minimal fibration of genus 2 with \( g(C) \geq 2 \), then a cyclic automorphism group \( G \) of \( f \) satisfies

\[ |G| \leq 5K_5^2 + 30 \]

for \( K_5^2 \geq 48 \).

PROOF. (1) Assume that \(|H| = 4g(C) + 2\) and \(|K| = 10\). Let \( g = g(C) \). By the theorem of Wiman (see the version given in \[8, \text{Lemma B}\]), \( C \) is a cyclic cover of \( \mathbb{P}^1 \) with ramification indices \( r_1 = 2, r_2 = 2g + 1, r_3 = 4g + 2 \) or \( r_1 = 3, r_2 = 6, r_3 = (4g + 2)/3 \). In fact, these \( r_i \) are the orders of \( \text{Stab}_{H}(p) \) for \( p \in C \). Since \( Z_{10} \) is a maximal cyclic automorphism subgroup of a smooth curve of genus 2, by Lemma 4.2 we have \(|\text{Stab}_{H}(p)\), \(|K|)| = 1\) if \( f^{-1}(p) \) is a smooth fiber. But in Case 1, \( r_1 \) and \( r_3 \) are even, while in Case 2, \( r_2 \) and \( r_3 \) are even. So \( f \) has at least \((2g + 10)/3\) singular fibers. By Lemma 2.4, we have \( s_2(F) \geq 4 \) for a singular fiber \( F \). Hence

\[ K_5^2 - 8(g - 1) = K_{S/C}^2 \geq \frac{4}{5}, \quad \frac{2g + 10}{3} = \frac{8(g + 5)}{15}, \]

\[ |G| = 10 |H| = 40g + 20 \leq \frac{75}{16} K_5^2 + 45 \leq 5K_5^2 + 30 \]

when \( K_5^2 \geq 48 \).
If \(|K| \leq 8\) and \(|K|\) is even, then by Lemma 4.3 there exist points \(p \in C\) with \(|\text{Stab}_H(p)| = 2\) \(\neq 1\). Hence \(K^2 - 8(g - 1) = K^2_{SC} \geq 1\) and
\[
|G| \leq 8|H| = 32g + 16 \leq 4K^2 + 44 \leq 5K^2 + 30
\]
when \(K^2 \geq 14\).

If \(|K|\) is odd, then \(|K| \leq 5\). The inequality is immediate.

(2) Assume that \(|H|\) is odd. By Lemma 4.3, we have \(|H| \leq 3g + 3\). So
\[
|G| \leq 10|H| \leq 30g + 30 \leq \frac{15}{4} K^2 + 60 \leq 5K^2 + 30
\]
when \(K^2 \geq 24\).

(3) Assume that \(|H|\) is even and \(|H| < 4g + 2\). If \(|K| = 10\), \(f\) must have more than one singular fibers by Lemma 2.4. So \(K^2_2 - 8(g - 1) = K^2_{SC} \geq 2\). We get
\[
|G| = 10|H| \leq 40g \leq 5K^2 + 30.
\]
If \(|K| \leq 8\), it is not difficult to obtain this inequality. □

It seems that this bound is not the best possible. In Section 5 we will give an example to show that there are infinitely many fibrations which has an automorphism with order \(3.75K^2 + 60\).

5. Examples.

Example 5.1. Fibration with \(|G| = 50K^2_5\).

Let \(C\) be a Hurwitz curve, i.e., \(|\text{Aut}(C)| = 84(g(C) - 1)\), and let \(F\) be a curve of genus 2 with \(|\text{Aut}(F)| = 48\). Let \(S = C \times F\) with \(f = \text{pr}_1 : S \to C\). Then \(K^2 = 8(g(C) - 1)\),
\[
|\text{Aut}(f)| = |\text{Aut}(C)| \cdot |\text{Aut}(F)| = 504K^2_5.
\]

Example 5.2. Fibrations with \(|G| = 126K^2_5\) which is not locally trivial.

Let \(F = P^1\). Let \(p_1 = 0\), \(p_2 = \infty\), \(p_3 = 1\), \(p_4 = \sqrt{-1}\), \(p_5 = -\sqrt{-1}\), \(p_6 = -\sqrt{-1}\) be six points on \(F\). Let \(C\) be a Hurwitz curve. Then \(C\) has an \(H\)-orbit \(\{q_1, \ldots, q_m\}\) which contains \(m = 12(g(C) - 1)\) points. Let \(P = C \times F\). Taking \(R = \text{pr}_1^*(q_1 + \cdots + q_m) + \text{pr}_2^*(p_1 + \cdots + p_6)\) as the branch locus, we construct a double cover of \(P\). After desingularization, we get a smooth surface \(S\) with a genus 2 fibration \(f : S \to C\). By computation, we obtain \(K^2 = 32(g(C) - 1)\), and \(|G| = 48 \times 84(g(C) - 1) = 126K^2_5\).

Example 5.3. Fibrations with \(|G| = 144K^2_5\) and \(g(C) = 1\).

Let \(F\) and \(p_1, \ldots, p_6\) be as in Example 5.2. Let \(C\) be an elliptic curve with the \(j\)-invariant \(j(C) = 0\). Fix a \(q_1 \in C\). Then the order of the group of automorphisms \(\text{Aut}(C, q_1)\) of \(C\) leaving \(q_1\) fixed is equal to 6. Let \(H_1 \cong \mathbb{Z}_m \oplus \mathbb{Z}_m\) be a subgroup of
translations of $\text{Aut}(C)$. Take an extension subgroup $H_1 \subset H \subset \text{Aut}(C)$ such that $H/H_1 \cong \text{Aut}(C, q_1)$. Then $|H| = 6m^2$. Let $q_1, \ldots, q_m$ be the orbit of $q_1$ under $H$. Let $P = C \times F$. Using $R = \text{pr}_1^*(q_1 + \cdots + q_m) + \text{pr}_2^*(p_1 + \cdots + p_6)$ as the branch locus, we construct a double cover of $P$. After desingularization, we get a smooth surface $S$ with a genus 2 fibration $f : S \to C$. By computation, we get $K \cong 2m$. On the other hand, $|K| = 48$ gives $|G| = 288m^2 = 144K$.  

**Example 5.4.** Rational fibration with $|G| = 120(K^2 + 8)$.  

Let $F$ and $p_1, \ldots, p_6$ be as in Example 5.2. Let $C = P^1$, $q_1, \ldots, q_{12}$ be the twelve vertices of an icosahedron. Let $P = C \times F$. Using $R = \text{pr}_1^*(q_1 + \cdots + q_{12}) + \text{pr}_2^*(p_1 + \cdots + p_6)$ as the branch locus, we construct a double cover of $P$. After desingularization, we get a genus 2 fibration $f : S \to C$ with $K \cong 16$, $|H| = 60$, $|K| = 48$, $|G| = 2880 = 120(K^2 + 8)$.  

**Example 5.5.** Rational fibrations with $|G| = 48(K^2 + 8)$.  

Let $F$ and $p_1, \ldots, p_6$ be as in Example 5.2. Let $C = P^1$ and let $q_1, \ldots, q_m$ be the $m$-th roots of unity. Then using the same construction as in Example 5.2, we get a genus 2 fibration with $K \cong 2(m - 4)$, $|K| = 48$, $|H| = 2m$, $|G| = 96m = 48(K^2 + 8)$.  

**Example 5.6.** Exceptional rational fibrations listed in the proof of Proposition 3.4.  

Using the same construction as in Example 5.2, take $q_1, \ldots, q_{20}$ as the twenty vertices of a dodecahedron. We get a fibration with $K \cong 32$ and $|G| = 2880 = 90K$. If we take $q_1, \ldots, q_6$ as the six vertices of an octahedron, we get a fibration with $K \cong 4$ and $|G| = 1152 = 288K$. If we take $q_1, \ldots, q_8$ as the eight vertices of a cube, we get a fibration with $K \cong 8$ and $|G| = 1152 = 144K$.  

**Example 5.7.** Fibrations the order of whose abelian automorphism group is 12.5($K^2 + 8$).  

Let $x_0, \ldots, x_{2m}, x_{2m+1}$ be the homogeneous coordinates in $P^{2m+1}$, and let $P^{2m}$ be the hyperplane defined by $x_{2m+1} = 0$. Let $\varphi : t \mapsto (1, t, \ldots, t^{2m}, 0)$ be a $2m$-ple embedding of $P^1$ in $P^{2m}$ and denote its image by $Y$. Then $Y$ is a rational normal curve of degree $2m$. Let $X$ be the cone over $Y$ in $P^{2m+1}$ with vertex $P_0 = (0, 0, \ldots, 0, 1)$. Denote $\eta = \exp(2\pi i/10m)$. Then the automorphism $\sigma : (x_0, \ldots, x_{2m+1}) \mapsto (x_0, x_1 \eta, \ldots, x_{2m} \eta^{2m}, x_{2m+1})$ of $P^{2m+1}$ is of order $10m$. The automorphism $\tau : (x_0, \ldots, x_{2m+1}) \mapsto (x_0, \ldots, x_{2m}, x_{2m+1} \eta^{2m})$ of $P^{2m+1}$ is of order 5. The cone $X$ is stabilized by these automorphisms $\sigma$ and $\tau$. Take a hypersurface $H$ defined by $x_0^5 + x_{2m}^5 + x_{2m+1}^5$ which is also stabilized by $\sigma$ and $\tau$. Moreover, $P_0 \notin H$. How blowing up the cone $X$ at the vertex $P_0$, we get the Hirzebruch surface $P = F_{2m}$ which has an automorphism $\tilde{\sigma}$ of order 10$^m$ induced by $\sigma$ and an automorphism $\tilde{\tau}$ of order 5 induced by $\tau$. The pull-back of the intersection $H \cap X$ is a smooth divisor $R_1$ on $P$ which is linearly equivalent to $5C_0 + 10mF$. Taking $R = R_1 + C_0 + 6C_0 + 10mF$, which is a smooth even divisor and stabilized under $\tilde{\sigma}$ and
As the branch locus, we can construct a double cover $S$ of $P$ which has a natural genus 2 fibration $f : S \to P^1$. Since $K_p = -2C_0 - (2m + 2)F$, we have $K_2^2 = 2(K_p + R/2)^2 = 8(m - 1)$. The pull-back of $\sigma$ to $S$ can generate a cyclic automorphism subgroup $H$ of order $10m$. The pull-back of $\tau$ to $S$ together with the hyperelliptic involution of the fibration $f$ generates a cyclic automorphism subgroup $K \simeq \mathbb{Z}_{10}$. Since $H$ and $K$ commute, $G = KH \simeq \mathbb{Z}_{10} \oplus \mathbb{Z}_{10m}$ is an abelian automorphism group of $f$ with order $|G| = 100m = 12.5K_2^2 + 8$.

**Example 5.8.** Rational fibrations which has an automorphism of order $12.5K_2^2 + 90$.

Let $x_0, \ldots, x_{2m}, x_{2m+1}$ be the homogeneous coordinates in $P^{2m+1}$, and let $p^{2m}$ be the hyperplane defined by $x_{2m+1} = 0$. Let $\varphi : t \mapsto (t^m, 0)$ be a 2m-ple embedding of $P^1$ in $P^{2m}$ and denote its image by $Y$. Then $Y$ is a rational normal curve of degree $2m$. Let $X$ be the cone over $Y$ in $P^{2m+1}$ with vertex $P_0 = (0, 0, \ldots, 0, 1)$. Denote $\eta = \exp(2\pi i/(50m - 5))$. Then the automorphism $\sigma : (x_0, \ldots, x_{2m+1}) \mapsto (x_0, x_1\eta, \ldots, x_{2m+1}\eta^{10m})$ of $P^{2m+1}$ is of order $50m - 5$. The cone $X$ is stabilized by this automorphism $\sigma$. Take a hypersurface $H$ defined by $x_0^2 + x_1^2 + x_{2m+1}^2 + x_{2m+1}^2$ which is also stabilized by $\sigma$ and $P_0 \notin H$. Now blowing up the cone $X$ at the vertex $P_0$, we get the Hirzebruch surface $P = F_{2m}$ which has an automorphism $\sigma$ of order $50m - 5$ induced by $\sigma$. The pull-back of the intersection $H \cap X$ is a smooth divisor $R_1$ on $P$ which is linearly equivalent to $5C_0 + 10mF$. Taking $R = R_1 + C_0 = 6C_0 + 10mF$, which is a smooth even divisor and stabilized under $\sigma$, as the branch locus, we can construct a double cover $S$ of $P$ which has a natural genus 2 fibration $f : S \to P^1$. Since $K_p \equiv -2C_0 - (2m + 2)F$, we have $K_2^2 = 2(K_p + R/2)^2 = 8(m - 1)$. The pull-back of $\sigma$ to $S$ can generate a cyclic automorphism group $G_1$ of order $50m - 5$. Since $|G_1|$ is odd, $G_1$ and the hyperelliptic involution of the fibration $f$ generate a cyclic automorphism group $G$ of $S$. Therefore $|G| = 100m - 10 = 12.5K_2^2 + 90$.

**Example 5.9.** Fibrations which has an automorphism of order $5K_2^2$.

Let $p_0 = 0, p_1 = \exp(2k\pi i/5), k = 1, \ldots, 5$, be six points in $F$. Let $C$ be an elliptic curve, $\{q_1, \ldots, q_m\}$ an orbit of a cyclic translation group $H \subseteq \text{Aut}(C)$ of order $m$, where $m$ is an odd prime different from 5. Then using the same construction as in Example 5.2, we obtain a genus 2 fibration with $K_2^2 = 2m, K \simeq \mathbb{Z}_{10}$. Let $G = K \times H \simeq \mathbb{Z}_{10m}$. Then $|G| = 10m = 5K_2^2$.

**Example 5.10.** Fibrations which has an automorphism of order $3.75K_2^2 + 60$.

Let $p_0 = 0, p_k = \exp(2k\pi i/3), k = 1, 2, 3$ be four points in $C' = P^1$. For any odd prime $m \neq 3, 5$, taking $D = p_0 + p_1 + (m-1)p_2 + (m-1)p_3$ as a branch locus, we can construct a cyclic cover $\sigma : C \to C'$ of degree $m$. Then $g(C) = m - 1$. $H'' = \{x \mapsto x \exp(2k\pi i/3)\}$ $k = 1, 2, 3 \} \subset \mathbb{Z}_3$ is a cyclic automorphism group of $C'$ which stabilizes the set $\{p_0, p_1, p_2, p_3\}$. On the other hand, the Galois group $H'$ of the cyclic cover $\sigma$ is isomorphic
to $Z_m$. We obtain an extension

$$1 \to H' \to H \to H'' \to 1$$

such that $Z_{3m} \cong H \subseteq \text{Aut}(C)$.

Let $q_0 = 0$, $q_k = \exp(2k\pi i/5)$, $k = 1, \ldots, 5$, be six points in $F \cong \mathbb{P}^1$. Let $P = C \times F$. Taking $R = \text{pr}^1_2(q_0 + q_1 + \cdots + q_5)$ as branch locus, we can construct a double cover $\theta : S \to P$ which is also a genus 2 fibration $f = p_1 \circ \theta : S \to C$. $F$ has a cyclic automorphism group $K_1 = \{ y \mapsto y \exp(2k\pi i/5) \mid k = 1, \ldots, 5 \} \cong Z_5$ which stabilizes the set $\{ q_0, \ldots, q_5 \}$ and can be lifted to $P$. It is not difficult to see that we can get $K \cong Z_{10}$ by adding the involution of the double cover. Then $G = K \times H \cong Z_{30m}$ is a cyclic automorphism group of $f$ which satisfies

$$|G| = 30m = 30(g(C) + 1) = \frac{15}{4} K^2 + 60,$$

because $K_2 = 8(g(C) - 1)$.

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