Vertex Degree of Random Intersection Graph

by

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Abstract

A random intersection graph is constructed by independently assigning a subset of a given set of objects $W$, to each vertex of the vertex set $V$ of a simple graph $G$. There is an edge between two vertices of $V$, iff their respective subsets in $W$ have at least one common element. The strong threshold for the connectivity between any two arbitrary vertices of vertex set $V$, is derived. Also we determine the almost sure probability bounds for the vertex degree of a typical vertex of graph $G$.

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1 Introduction

The development in this paper is in continuation of Dudley Stark [2], in which distribution of the degree of a typical vertex is given. Dudley studied the model given in [3] defined as:

Let us consider a set \( V \) with \( n \) vertices and another set of objects \( W \) with \( m \) objects. Define a bipartite graph \( G^*(n, m, p) \) with independent vertex sets \( V \) and \( W \). Edges between \( v \in V \) and \( w \in W \) exists independently with probability \( p \). The random intersection graph \( G(n, m, p) \) derived from \( G^*(n, m, p) \) is defined on the vertex set \( V \) with vertices \( v_1, v_2 \in V \) are adjacent if and only if there exists some \( w \in W \) such that both \( v_1 \) and \( v_2 \) are adjacent to \( w \) in \( G^*(n, m, p) \). Also define \( W_v \) be a random subset of \( W \) such that each element of \( W_v \) is adjacent to \( v \in V \). Any two vertices \( v_1, v_2 \in V \) are adjacent if and only if \( W_{v_1} \cap W_{v_2} \neq \emptyset \), and edge set \( E(G) \) is defined as

\[
E(G) = \{ \{v_i, v_j\} : v_i, v_j \in V, W_{v_i} \cap W_{v_j} \neq \emptyset \}.
\]

2 Definitions and Supporting Results

**Lemma 2.1** Let \( X \) be a random variable having binomial distribution with parameters \( n \) and \( p > 0 \), i.e., \( X \sim Bi(n, p) \). Then

\[
P[X \geq k] \leq \left( \frac{np}{k} \right)^k \exp(k - np), \quad k \geq np,
\]

and

\[
P[X \leq k] \leq \left( \frac{np}{k} \right)^k \exp(k - np), \quad 0 < k \leq np.
\]

For convenience, the bounds (2.1) and (2.2) can be expressed as

\[
\exp \left( npH \left( \frac{np}{k} \right) \right),
\]

where

\[
H(t) = \frac{1}{t} \log t + \frac{1}{t} - 1, \quad 0 < t < \infty,
\]
and \( H(\infty) = -1 \). Note that \( H(t) < 0, \forall t \in (0, \infty) \setminus \{1\} \); \( H \) is increasing on \((0, 1)\) and decreasing on \((1, \infty)\). (See Lemma 1.2 on page 25, Penrose [4].)

Let \( X = X(n, m, p) \) be the number of vertices of \( V - \{v\} \) adjacent in \( G(n, m, p) \) to a vertex \( v \in V \), i.e., \( X \) be the vertex degree of a vertex \( v \in V \) in \( G(n, m, p) \). Then \( X \) follows \( \text{Bi}(n-1, q_n) \), where \( q_n \) is the probability of \( v_1, v_2 \in V \) are adjacent.

**Proposition 2.2** Let \( v_i, v_j \in V, i \neq j \) and \( i, j = 1, 2, \ldots, n \). Then \( v_i \) and \( v_j \) are adjacent with probability \( q_n \), such that

\[
q_n \sim mp^2,
\]

for sufficiently small \( p \).

**Proof.** Consider

\[
q_n = P[v_i, v_j \text{ are adjacent.}]
\]

\[
= P[W_{v_i} \cap W_{v_j} \neq \emptyset]
\]

\[
= P[|W_{v_i} \cap W_{v_j}| \geq 1]
\]

\[
= 1 - P[w \notin (W_{v_i} \cap W_{v_j})]^m
\]

\[
= 1 - [1 - P[w \in (W_{v_i} \cap W_{v_j})]^m
\]

\[
= 1 - [1 - P[w \in W_{v_i}, w \in W_{v_j}]]^m
\]

\[
= 1 - [1 - p^2]^m
\]

Using the Taylor’s series expansion up to second term, we get

\[
q_n = mp^2 - \zeta_m, \tag{2.3}
\]

where, \( \zeta_m = \frac{m(m-1)}{2!}(1 - c)^2 p^4 \), and \( c \in (0, p^2) \).

Now if we take \( p \) is sufficiently small. Then

\[
q_n \sim mp^2. \tag{2.4}
\]
This completes the proof. 

**Definition 1** Let graphs $A$ and $B$ share the same vertices and the edge set of $A$ is a subset of the edge set of $B$, we write $A \leq B$. Also let $\Theta$ be a property of a random graphs such that if $A \leq B$ and $A \in \Theta$, then $B \in \Theta$. (Here $A \in \Theta$ is used to denote that graph $A$ has property $\Theta$.) Then $\Theta$ is called an upwards-closed property. If $B \in \Theta$ implies $A \in \Theta$, then $\Theta$ is said to be a downwards-closed property. If property $\Theta$ is upwards-closed or downwards-closed, then $\Theta$ is called monotone property. 

Fix a monotone property $\Theta$. For any two functions $\delta, \gamma : \mathbb{Z}^+ \rightarrow \mathbb{R}^+$, we write $\delta \ll \gamma$ (resp. $\delta \gg \gamma$) if $\delta(n)/\gamma(n) \rightarrow 0$, (resp. $\gamma(n)/\delta(n) \rightarrow 0$) as $n \rightarrow \infty$. We will write $\delta$ for $\delta(n)$. Let $G_n(r)$, be any random graph on $n$ vertices. 

**Definition 2** A function $\delta_{\Theta} : \mathbb{Z}^+ \rightarrow \mathbb{R}^+$ is a weak threshold function for $\Theta$ if the following is true for every function $\delta : \mathbb{Z}^+ \rightarrow \mathbb{R}^+$,

- if $\delta(n) \ll \delta_{\Theta}(n)$, then $P[G_n(\delta) \in \Theta] = 1 - o(1)$, and
- if $\delta(n) \gg \delta_{\Theta}(n)$ then $P[G_n(\delta) \in \Theta] = o(1)$.

A function $\delta_{\Theta} : \mathbb{Z}^+ \rightarrow \mathbb{R}^+$ is a strong threshold function for $\Theta$ if the following is true for every fixed $\epsilon > 0$,

- if $P[G_n(\delta_{\Theta} - \epsilon) \in \Theta] = 1 - o(1)$, and
- if $P[G_n(\delta_{\Theta} + \epsilon) \in \Theta] = o(1)$.

**3 Main Results**

We now derive the threshold probability for the connectivity of graph $G(n,m,p)$. 


Let $\Theta$ be the connectivity of graph $G(n, m, p)$ and define the probability that a vertex $v \in V$ is connected to $w \in W$, as follows

$$p := p(\alpha) = \frac{1}{(mn^\alpha)^{1/2}}.$$

**Theorem 3.1** Let $G(n, m, p)$ be the random intersection graph with $p(\alpha) = \frac{1}{(mn^\alpha)^{1/2}}$ and $v_i, v_j \in V(G)$ for $i \neq j = 1, 2, \ldots, n$. Then $p(2) = \frac{1}{m^{1/2}n}$, is a strong threshold for the random intersection graph $G(n, m, p)$, i.e.,

$$P[G(n, m, p(2 - \epsilon)) \in \Theta] = 1 - o(1),$$

$$P[G(n, m, p(2 + \epsilon)) \in \Theta] = o(1). \quad (3.5)$$

**Proof.** Let $v_i, v_j \in V(G)$ for $i \neq j, i, j = 1, 2, \ldots, n$. Then from (2.3), we have

$$P[v_i, v_j \text{ are adjacent}] = q_n$$

$$\leq mp^2. \quad (3.6)$$

Since we have $p(\alpha) = \frac{1}{(mn^\alpha)^{1/2}}$. Then

$$P[v_i, v_j \text{ are adjacent}] \leq n^{-\frac{1}{2}}. \quad (3.7)$$

Hence, for $\alpha \leq 2$ the above probability (3.7) is not summable, i.e.,

$$\sum_{n=0}^{\infty} P[v_i, v_j \text{ are adjacent}] = \infty.$$ 

Since the adjacency between any pair of vertices is independent from the adjacency between any other pair of vertices. Then by the Borel-Cantelli Lemma, we have

$$P[v_i, v_j \text{ are adjacent}, \ i.o.] = 1.$$ 

Therefore, for $\alpha \leq 2$, any pair of vertices $v_i, v_j \in V, i \neq j, i, j = 1, 2, \ldots, n$, are adjacent almost surely. Hence,

$$P[G(n, m, p(2 - \epsilon)) \in \Theta] = 1 - o(1).$$
For $\alpha > 2$ the above probability is (3.7) summable, i.e.,

$$\sum_{n=0}^{\infty} P[v_i, v_j \text{ are adjacent}] < \infty.$$ 

Then by the Borel-Cantelli’s Lemma, we have

$$P[v_i, v_j \text{ are adjacent}, \ i.o.] = o.$$

Therefore, for $\alpha > 2$, any pair of vertices $v_i, v_j \in V$ are adjacent only finitely many times. Hence,

$$P[G(n, m, p(2 + \epsilon)) \in \Theta] = o(1).$$

This completes the proof. \[ \square \]

We now state strong law results for vertex degree of a typical vertex.

Let $X = X(n, m, p)$ be the number of vertices of $V - \{v\}$ adjacent in $G(n, m, p)$ to a vertex $v \in V$, $i, j = 1, 2, \ldots, n$, $i \neq j$ i.e., $X$ be the degree of vertex $v \in V$ in $G(n, m, p)$.

**Theorem 3.2** Let $G(n, m, p)$ be the random intersection graph with $p = (mn^\alpha)^{-1/2}$, where $0 < \alpha < 1$. Also let $X$ be the degree of a typical vertex $v \in V$ in $G(n, m, p)$. Then

$$\limsup_{n \to \infty} \frac{X}{n^\delta} \geq a(c), \quad a.s.,$$

where $\delta = 1 - \alpha$ and $a(c)$ is the root in $[1, \infty)$ of

$$a \log a - a + 1 = c,$$

with $a(\infty) = 1$.

**Proof.** We know that the vertex degree of a vertex in graph $G(n, m, p)$ is distributed according to $Bi(n - 1, q_n)$. Hence by Lemma 2.1, we have

$$P[X \leq K] \leq \exp \left( (n-1)q_n H \left( \frac{(n-1)q_n}{K} \right) \right), \quad K \leq nq_n.$$

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From Proposition 2.2, we have for sufficiently small $p$, we have $q_n \sim mp^2$. Then

$$P[X \leq K] \leq \exp \left( (n-1)mp^2H \left( \frac{(n-1)mp^2}{K} \right) \right)$$

$$\sim \exp \left( n^{1-\alpha}H \left( \frac{n^{1-\alpha}}{K} \right) \right)$$

$$= \exp \left( -cn^{\delta} \right),$$

(3.11)

where $H((n/mK^2)^{1/2}) = -c$ and $\delta = 1 - \alpha$. The above expression is summable if $K = a(c)n^{\delta}$, where $a(c)$ is increasing in $[1, \infty)$ and defined as in (3.9). Then by the Borel-Cantelli Lemma we have

$$P \left[ X \leq a(c)n^{\delta}, \ i.o. \right] = 0.$$  

(3.12)

This implies that

$$\limsup_{n \to \infty} \frac{X}{n^{\delta}} \geq a(c), \quad a.s.$$  

(3.13)

This completes the proof.

**Theorem 3.3** Let $G(n, m, p)$ be the random intersection graph with $p = (mn^\alpha)^{-1/2}$, where $0 < \alpha < 1$. Also let $X$ be the degree of a typical vertex $v \in V$ in $G(n, m, p)$. Then

$$\liminf_{n \to \infty} \frac{X}{n^{\delta}} \leq a(c), \quad a.s.,$$  

(3.14)

where $\delta = 1 - \alpha$ and $a(c)$ is the root in $(0,1)$ of

$$a \log a - a + 1 = c,$$  

(3.15)

with $a(\infty) = 1$.

**Proof.** We know that the vertex degree of a vertex in graph $G(n, m, p)$ is distributed according to $\text{Bi}(n - 1, q_n)$. Hence by Lemma 2.1 we have

$$P[X \geq K] \leq \exp \left( (n-1)q_nH \left( \frac{(n-1)q_n}{K} \right) \right), \quad K \geq nq.$$  

(3.16)
From Proposition 2.2, we have for sufficiently small $p$, we have $q_n \sim mp^2$. Then

$$P[X \geq K] \leq \exp \left( (n-1)mp^2 H \left( \frac{(n-1)mp^2}{K} \right) \right)$$

$$\sim \exp \left( n^{1-\alpha} H \left( \frac{n^{1-\alpha}}{K} \right) \right)$$

$$= \exp \left( -cn^\delta \right), \quad (3.17)$$

where $H((n/mK^2)^{1/2}) = -c$ and $\delta = 1 - \alpha$. The above expression is summable if $K = a(c)n^\delta$, where $a(c)$ is decreasing in $(0, 1)$ and defined as in (3.15). Then by the Borel-Cantelli Lemma, we have

$$P \left[ X \geq a(c)n^\delta, \quad i.o. \right] = 0. \quad (3.18)$$

This implies that

$$\liminf_{n \to \infty} \frac{X}{n^\delta} \leq a(c), \quad a.s. \quad (3.19)$$

This completes the proof. \[\square\]

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