ERLANGEN PROGRAM AT LARGE—2: INVENTING A WHEEL. THE PARABOLIC ONE

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Dedicated to 300th anniversary of Leonhard Euler's birth

ABSTRACT. We discuss parabolic versions of Euler’s identity

\[ e^{it} = \cos t + i \sin t. \]

A purely algebraic approach based on dual numbers is known to produce a very trivial relation \( e^{\varepsilon t} = 1 + \varepsilon t \). Therefore we use a geometric setup of parabolic rotations to recover the corresponding non-trivial algebraic framework. Our main tool is Möbius transformations which turn out to be closely related to induced representations of the group \( SL_2(\mathbb{R}) \).

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1. INTRODUCTION: A PARABOLIC WHEEL—AN ALGEBRAIC APPROACH

A mathematical picture of a wheel, which uniformly rotates around its axis, is given by the following “model”:
\[(1.1) \quad x = \cos t, \quad y = \sin t,\]
where \(x\) and \(y\) denote the coordinates of a point on the unit distance from the axis of rotation. The principal ingredients are \(sine\) and \(cosine\) functions, which are known for more than two thousand years.

1.1. Complex Numbers. A later invention of complex numbers \(z = x + iy, i^2 = -1\) allows to write down two identities (1.1) as a single one:
\[(1.2) \quad z = \cos t + i \sin t.\]
The next big advance is known as Euler’s formula, which expresses trigonometric functions through the exponent of an imaginary number:
\[(1.3) \quad e^{it} = \cos t + i \sin t.\]
Thus the geometrical meaning of multiplication by \(e^{it}\) is an isometric rotation of the plane \(\mathbb{R}^2\), see Fig. 1(E) with the (elliptic) metric given by:
\[(1.4) \quad x^2 + y^2 = (x + iy)(x - iy).\]
What are possible extensions of this results?

1.2. Double Numbers. Complex numbers is not the only possible extension of the reals. There are other variants of imaginary units, for example \(\epsilon^2 = 1\). Replacing \(i\) by \(\epsilon\) in (1.3) we get a key to hyperbolic trigonometry:
\[(1.5) \quad e^{\epsilon t} = \cosh t + \epsilon \sinh t.\]
Here expressions \(x + \epsilon y\) form the algebra of double numbers—the simplest case of hypernumbers. Multiplication by \(e^{\epsilon t}\) is a map of double numbers into itself which preserves the hyperbolic metric, cf. (1.4):
\[(1.6) \quad x^2 - y^2 = (x + \epsilon y)(x - \epsilon y).\]
Geometrically this may be viewed as hyperbolic rotation, see Fig. 1(H), in contrast to the elliptic case (1.3).

\(^1\)Here \(\epsilon\) is not a real number—a clarification which may be omitted in the case of \(i^2 = -1\).
1.3. Dual number. To make the picture complete we may wish to add the parabolic case through the imaginary unit of dual numbers defined by $\varepsilon^2 = 0$. Since $\varepsilon^n = 0$ for any integer $n > 1$ we get from Taylor’s series of the exponent function the following identity:

$$e^{\varepsilon t} = 1 + \varepsilon t.$$  

Then parabolic rotations associated with $e^{\varepsilon t} \cdot a \in \mathbb{C} \mapsto a + \varepsilon(ax + b)$.

This links the parabolic case with the Galilean group [20].

Should we conclude, cf. [4,20], from here that:

- the parabolic trigonometric functions are trivial:
  
  $$\cosh t = 1, \quad \sinh t = t?$$  

- the parabolic distance is independent from $y$, cf. (1.4) and (1.6):
  
  $$x^2 = (x + \varepsilon y)(x - \varepsilon y)?$$

- the polar decomposition of a dual number is defined by [20, App. C(30')]:
  
  $$u + \varepsilon v = u(1 + \varepsilon), \quad \text{thus} \quad |u + \varepsilon v| = u, \quad \arg(u + \varepsilon v) = \frac{v}{u}.$$

- the parabolic wheel looks rectangular, see Fig. 1(P0)?

The analogies (1.3)−(1.5)−(1.7) and (1.4)−(1.6)−(1.9) are quite explicit and widely accepted as an ultimate source for parabolic trigonometry [4,15,20]. However we will see shortly that there exists a less trivial form as well.

Remark 1.1. The parabolic imaginary unit $\varepsilon$ is a close relative to the infinitesimal number $\varepsilon$ from non-standard analysis [2,19]. The former has the property that its square is exactly zero, meanwhile the square of the later is almost zero at its own scale.

Remark 1.2. Introduction of double and dual numbers is not as artificial as it may looks from the traditional viewpoint, see Rem. 4.1.

Remark 1.3. In cases when we need to consider simultaneously several imaginary units we use $\iota$ to denote any of $i$, $\varepsilon$, $\epsilon$.

2. A Parabolic Wheel—a Geometrical Viewpoint

We make a second attempt to describe parabolic rotations. If multiplication (linear transformation) is not sophisticated enough for this we would advance to the next level of complexity: linear-fractional.

2.1. Matrices. Imaginary units do not need to be seen as abstract quantities. We may realise them through zero-trace $2 \times 2$ matrices as follows:

$$i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \varepsilon = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \epsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

with the parabolic $\epsilon$ nicely sitting between the elliptic $i$ and hyperbolic $\varepsilon$. Then the matrix multiplication implies $i^2 = -I, \varepsilon^2 = 0 \cdot I, \epsilon^2 = I$, where $I$ is the $2 \times 2$ identity matrix. Correspondingly we have a matrix form of the identities (1.3)−(1.5):

$$\exp \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}, \quad \exp \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}.$$
However the above pattern is only partially reproduced in the matrix form of (1.7):

\[ \exp \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}. \]

There is also some arbitrariness in our choice of a matrix representation for \( \varepsilon \), it may be equally well given by the lower-triangular form:

\[ \varepsilon' = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \]

implying
\[ \exp \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}. \]

On the first glance this is not a radical difference, however, it does have some implications for the Möbius transform from Section 2.3.

2.2. Cayley Transform. Another matrix form of the identity (1.3) is provided by the Cayley transform:

\[ \frac{1}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}, \]

where the matrix

\[ C_i = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \]

is the Cayley transform from the upper-half plane to the unit disk. It has its hyperbolic cousin

\[ C_\varepsilon = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \varepsilon \\ -\varepsilon & 1 \end{pmatrix}, \]

which produces a matrix form of (1.5):

\[ \frac{1}{2} \begin{pmatrix} 1 & \varepsilon \\ -\varepsilon & 1 \end{pmatrix} \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \begin{pmatrix} 1 & -\varepsilon \\ \varepsilon & 1 \end{pmatrix} = \begin{pmatrix} e^{\varepsilon t} & 0 \\ 0 & e^{-\varepsilon t} \end{pmatrix}. \]

In the parabolic case we use the same pattern as in (2.6) and (2.7):

\[ C_\varepsilon = \begin{pmatrix} 1 & -\varepsilon \\ -\varepsilon & 1 \end{pmatrix} \]

The Cayley transform of matrix (2.3) is:

\[ \frac{1}{\varepsilon} \begin{pmatrix} 1 & \varepsilon \\ -\varepsilon & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{pmatrix} = \begin{pmatrix} 1 + \varepsilon t & t \\ 0 & 1 - \varepsilon t \end{pmatrix} = \begin{pmatrix} e^{\varepsilon t} & t \\ 0 & e^{-\varepsilon t} \end{pmatrix}. \]

This is again not far from the previous identities (2.5) and (2.8), however, the off-diagonal \((1, 2)\)-term destroys harmony.

Remark 2.1. It is not senseless to consider three matrices (2.1), which materialise the imaginary units, together. In fact those trace-less matrices form a basis of the \( sl_2 \) Lie algebra of the group \( SL_2(\mathbb{R}) \) [14]. Moreover they are generators of the one-parameter subgroups \( K, N, A \) correspondingly, which form the Iwasawa decomposition \( SL_2(\mathbb{R}) = ANK \) of the group \( SL_2(\mathbb{R}) \), see Section 4.

2.3. Möbius maps. The matrix version of Euler’s identity from the previous section can be folded back to numbers through the linear-fractional (or Möbius) transformations. Indeed any \( 2 \times 2 \) matrix define a map:

\[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az + b}{cz + d} \text{ where } z = u + iv. \]

Notably this is a group homomorphism of invertible matrices under multiplication (or the group \( SL_2(\mathbb{R}) \)) into transformations of conformally completed plane [5, 13]. More precisely, see [9] or [11] (an easy-reading), real-valued matrices (2.2) and (2.3) act as transformations of the “upper half-plane”; matrices (2.5), (2.8) and (2.9) act...
as transformations of the respective “unit disk”, see Fig. 2. Those unit disks are images of the upper half-planes under respective Cayley transforms [9, § 8].

The actions of diagonal matrices from the right-hand side of identities (2.5) and (2.8) are straightforward: they are multiplications by $e^{-2it}$ and $e^{-2\epsilon t}$ correspondingly. Notably the images of the point $-i$ are:

(2.11)  
\[
\begin{pmatrix}
  e^{it} & 0 \\
  0 & e^{-it}
\end{pmatrix}
: -i \mapsto -i \sin 2t - i \cos 2t;  \\
\begin{pmatrix}
  e^{\epsilon t} & 0 \\
  0 & e^{-\epsilon t}
\end{pmatrix}
: -\epsilon \mapsto -\sinh 2t - \epsilon \cosh 2t.
\]

However the parabolic action of matrix (2.9) in formula (2.10) is not such a simple one.

2.3.1. The Upper-Triangular Subgroup. The parabolic version of the relations (2.11) with the upper-triangular matrices from $\mathbb{N}$ becomes:

(2.12)  
\[
\begin{pmatrix}
  e^{\epsilon t} & t \\
  0 & e^{-\epsilon t}
\end{pmatrix}
: -\epsilon \mapsto t + \epsilon(t^2 - 1).
\]

This coincides with the cyclic rotations defined in [20, § 8]. A comparison of this result with (2.11) seemingly confirms that $\sin p t = t$ but suggest a new expression for $\cos p t$:

\[\cos p t = 1 - t^2, \quad \sin p t = t.\]

Therefore the parabolic Pythagoras’ identity would be:

(2.13)  
\[\sin^2 p t + \cos^2 p t = 1,\]

which nicely fits in between of the elliptic and hyperbolic versions:

\[\sin^2 t + \cos^2 t = 1, \quad \sinh^2 t - \cosh^2 t = -1.\]

The identity (2.13) is also less trivial than the version $\cos^2 p t = 1$ from [4] (see also (1.8), (1.9)).

2.3.2. The Lower-Triangular Subgroup. There is also the second option to define parabolic rotations, it is generated by the lower-triangular variant of the above construction, cf. (2.4). The important difference now is: the reference point cannot be $-\epsilon$ since it is a fixed point—as well as any point on the vertical axis. Instead we take $\epsilon^{-1}$, which is an ideal element (a point at infinity [20, App. C]) since $\epsilon$ is a divisor of zero. The proper compactifications by ideal elements for all three cases were discussed in [9, 13].
We denote the subgroup of lower-triangular matrices by $N'$. We obtain with it:

\[
\left( \begin{array}{cc} e^{\epsilon t} & 0 \\ t & e^{-\epsilon t} \end{array} \right) : \frac{1}{\epsilon} \mapsto \frac{1}{t} + \epsilon \left( \frac{1}{t^2} - 1 \right).
\]

A comparison with (2.12) shows that this form is obtained by the change $t \mapsto t^{-1}$. The same transformation gives new expressions for parabolic trigonometric functions. The parabolic “unit circle” (or cycle [9, 20]) is defined by the equation $x^2 - y = 1$ in both cases. However other orbits are different and we will give their description in the next Section.

3. REBUILDING ALGEBRAIC STRUCTURES FROM GEOMETRY

Rotations in elliptic and hyperbolic cases are given by products of complex or double numbers correspondingly, however the multiplication of dual numbers produces only the trivial parabolic rotation from Fig. 1(P) rather than more interesting ones from Fig. 2(P) or Fig. 2(P'). Also the coordinate-wise addition of vectors on the plane is invariant under elliptic and hyperbolic rotations but is not under the parabolic one. Can we find such algebraic operations for vectors which will be compatible with parabolic rotations?

It is common in mathematics to “revert a theorem into a definition” and we will use it systematically in this section to recover a compatible algebraic structure.

3.1. Modulus and Argument. In the elliptic and hyperbolic cases orbits of rotations are points with the constant norm (modulus): either $x^2 + y^2$ or $x^2 - y^2$. In the parabolic case we employ this point of view as well:

**Definition 3.1.** Orbits of actions (2.12) and (2.14) are contour line for the following functions which we call respective moduli (norms):

(3.1) for $N$: $|u + \epsilon v| = u^2 - v$, for $N'$: $|u + \epsilon v|' = \frac{u^2}{v + 1}$.

**Remark 3.2.**

(i) The expression $||u, v|| = u^2 - v$ represents a parabolic distance from $(0, \frac{1}{2})$ to $(u, v)$, see [9, Lem. 8.4] and is in line with the “parabolic Pythagoras’ identity” (2.13).

(ii) Modulus for $N'$ expresses the parabolic focal length from $(0, -1)$ to $(u, v)$ as described in [9, Lem. 8.5].

The only straight lines preserved by the both parabolic rotations $N$ and $N'$ are vertical lines, thus we will treat them as “spokes” for parabolic wheels. Elliptic spokes in mathematical terms are “points on the complex plane with the same argument”, thus we again use it for the parabolic definition:

**Definition 3.3.** Parabolic arguments are defined as follows:

(3.2) for $N$: $\arg(u + \epsilon v) = u$, for $N'$: $\arg'(u + \epsilon v) = \frac{1}{u}$.

Both Definitions 3.1 and 3.3 possess natural properties with respect to parabolic rotations:

**Proposition 3.4.** Let $w_s$ is a parabolic rotation of $w$ by angle $s = t$ in (2.12) or $s = t^{-1}$ in (2.14). Then:

$|w_s|' = |w|$, $\arg'(w) = \arg(w') w + s$,

where primed versions are used for subgroup $N'$.

**Remark 3.5.** Note that in the commonly accepted approach [20, App. C(30')] parabolic modulus and argument are given by expressions (1.10), which are in a sense opposite to our conclusions.
3.2. **Rotation as Multiplication.** We revert again theorems into definitions to assign multiplication. In fact we require an extended version of properties stated in Proposition 3.4:

**Definition 3.6.** The product of vectors $w_1$ and $w_2$ is defined by the following two conditions:

(i) $\arg(w_1w_2) = \arg w_1 + \arg w_2$;
(ii) $|w_1w_2| = |w_1| |w_2|$.

We also need a special form of parabolic conjugation.

**Definition 3.7.** Parabolic conjugation is given by $u + \varepsilon v = -u + \varepsilon v$.

Combination of Definitions 3.1, 3.3 and 3.6 uniquely determine expressions for products.

**Proposition 3.8.** The parabolic product of vectors is defined by formulae:

(3.3) for $N$:

$$ (u, v) * (u', v') = (u + u', (u + u')^2 - (v - u^2)(v' - u'^2)); $$

(3.4) for $N'$:

$$ (u, v) * (u', v') = \left( \frac{uu'}{u + u'}, \frac{v + 1}{v' + 1} \right). $$

Although both expressions looks unusual they have many familiar properties:

**Proposition 3.9.** Both products (3.3) and (3.4) satisfy to the following conditions:

(i) They are commutative and associative;
(ii) The respective rotations (2.12) and (2.14) are given by multiplications with a dual number with the unit norm.
(iii) The product $w_1\bar{w}_2$ is invariant under respective rotations (2.12) and (2.14).
(iv) The second component of the product $w\bar{w}$ is $|w|^2$.

3.3. **Invariant Linear Algebra.** Now we wish to define a linear structure on $\mathbb{R}^2$ which would be invariant under point multiplication from the previous Subsection (and thus under the parabolic rotations, cf. Prop.3.9(ii)). Multiplication by a scalar is straightforward (at list for a positive scalar): it should preserve the argument and scale the norm of vectors. Thus we have formulae for $a > 0$:

(3.5) $a \cdot (u, v) = (u, av + u^2(1 - a))$ for $N$,

(3.6) $a \cdot (u, v) = \left( u, \frac{v + 1}{a} - 1 \right)$ for $N'$.

On the other hand addition of vectors can be done in several different ways. We present two solutions: one is tropical and another—exotic.

3.3.1. **Tropical form.** Let us introduce the lexicographic order on $\mathbb{R}^2$:

$$(u, v) < (u', v') \text{ if and only if } \begin{cases} \text{either } & u < u'; \\ \text{or } & u = u', v < v'. \end{cases}$$

One can define functions $\min$ and $\max$ of a pairs of points from on $\mathbb{R}^2$ correspondingly. An addition of two vectors can be defined either as their minimum or maximum. A similar definition is used in tropical mathematics, also known as Maslov dequantisation or $\mathbb{R}_{\min}$ and $\mathbb{R}_{\max}$ algebras, see [16] for a comprehensive survey. It is easy to check that such an addition is distributive with respect to vector multiplications (3.5)—(3.6) and consequently is invariant under parabolic rotations. Although it looks promising to investigate this framework we do not study it further for now.
3.3.2. **Exotic form.** Addition of vectors for both subgroups $N$ and $N'$ can be defined by the common rules, where subtle differences are hidden within corresponding Definitions 3.1 (norms) and 3.3 (arguments).

**Definition 3.10.** Parabolic addition of vectors is defined by the following formulae:

\[
\begin{align*}
\arg^{(r)}(w_1 + w_2) &= \frac{\arg^{(r)} w_1 \cdot |w_1|^{(r)} + \arg^{(r)} w_2 \cdot |w_2|^{(r)}}{|w_1 + w_2|^{(r)}}, \\
|w_1 + w_2|^{(r)} &= |w_1|^{(r)} + |w_2|^{(r)},
\end{align*}
\]

primed versions are used for the subgroup $N'$.

The rule for the norm of sum (3.8) may looks too trivial at a first glance. We should say in its defence that it nicely sits in between of the elliptic $|w + w'| \leq |w| + |w'|$ and hyperbolic $|w + w'| \geq |w| + |w'|$ inequalities for norms.

Both formulae (3.7)–(3.8) together uniquely define explicit expressions for additions of vectors. Although those expressions are rather cumbersome and not really much needed. Instead we list properties of this operations:

**Proposition 3.11.** Vector additions for subgroups $N$ and $N'$ defined by (3.7)–(3.8) satisfy to the following conditions:

- (i) They are commutative and associative.
- (ii) They are distributive for multiplications (3.3) and (3.4); consequently:
- (iii) They are parabolic rotationally invariant;
- (iv) They are distributive in both ways for the scalar multiplications (3.5) and (3.6) respectively:

\[ a \cdot (w_1 + w_2) = a \cdot w_1 + a \cdot w_2, \quad (a + b) \cdot w = a \cdot w + b \cdot w. \]

To complete the construction we need to define the zero vector and inverse.

**Proposition 3.12.** (N) The zero vector is $(0, 0)$ and consequently the inverse of $(u, v)$ is $(u, 2u^2 - v)$.

(N') The zero vector is $(\infty, -1)$ and consequently the inverse of $(u, v)$ is $(u, -v - 2)$.

Consequently we can check that scalar multiplications by negative reals are given by the same identities (3.5) and (3.6) as for positive ones.

3.3.3. **The Real and Imaginary Parts.** Having the vector addition at hands we may wish to define the real and imaginary parts compatible with it. We can start from the familiar formulae $\frac{1}{2}(w + \bar{w})$ and $\frac{1}{2}(w - \bar{w})$. While such a real part has a reasonable value $(0, |w|)$ (the subgroup $N$ case), the imaginary part suffers from the malformed denominator in (3.7) due to the fact $|w| = |\bar{w}|$.

This should not be seen as a defect of the exotic addition. A moment of reflection reveals that “purely real” dual numbers are naturally defined as having zero argument, e.g. the vertical axis for the subgroup $N$. This seems to agree well with the above real part.

However what are “purely imaginary” dual numbers? The horizontal axis can be the first suggestion, but all its points have different arguments. One may still prefer to choose that the number $(1, 0)$ would be purely imaginary (since $(0, 1)$ is purely real) and all imaginary numbers have the same argument. So purely imaginary numbers

**Definition 3.13.** For both subgroups $N$ and $N'$:

- (i) purely real numbers are defined by the condition $\arg w = 0$.
- (ii) purely imaginary numbers are defined by the condition $\arg w = 1$. 
Since arguments of \( \Re w \) and \( \Im w \) are fixed by this Definition we need only to find their moduli.

**Proposition 3.14.** The natural condition \( w = \Re w + i\Im w \) together with Defn. 3.13 uniquely define \( \Re w \) and \( \Im w \). It is also determined by the identities:

\[
|\Re w| = (1 - \arg w)|w|, \quad |\Im w| = \arg w|w|.
\]

The explicit formulae for the real and imaginary parts in the cases of both subgroup \( N \) and \( N' \) can be found in App. A.

3.3.4. **Linearisation of the exotic form.** Some useful information can be obtained from the transformation between the parabolic unit disk and its linearised model. In such linearised coordinates \( (a, b) \) the addition (3.7)-(3.8) is done in the usual coordinate-wise manner: \( (a, b) + (a', b') = (a + a', b + b') \).

To this end we calculate the value of \( (u, v) = a · (u_1, v_1) + b · (u_2, v_2) \) for \( (u_1, v_1) = (1, 0) \) and \( (u_2, v_2) = (-1, 0) \). For the subgroups \( N \) the transform is given by:

\[
\begin{align*}
    u &= \frac{a - b}{a + b}, & v &= \frac{(a - b)^2}{(a + b)^2} - (a + b), & a &= \frac{u^2 - v}{2}(1 + u), & b &= \frac{u^2 - v}{2}(1 - u).
\end{align*}
\]

For the subgroup \( N' \) such a transformation is:

\[
\begin{align*}
    u &= \frac{a + b}{a - b}, & v &= \frac{(a + b)}{(a - b)^2} - 1, & a &= \frac{u(u + 1)}{2(v + 1)}, & b &= \frac{u(u - 1)}{2(v + 1)}.
\end{align*}
\]

We also note that both norms (3.1) have exactly the same value \( a + b \) in the respective \( (a, b) \)-coordinates.

**Remark 3.15.** The irrelevance of the standard linear structure for parabolic rotations manifests itself in many different ways, e.g. in an apparent “non-conformality” of lengths from parabolic foci, e.g. with the parameter \( \sigma = 0 \) in [9, Prop. 5.10.(iii)]. An adjustment of notions to the proper framework restores the clear picture.

The initial definition of conformality [9, Defn. 5.9] considered the usual limit \( y' \rightarrow y \) along a straight line, i.e. “spoke” in terms of Fig. 1. This is justified in the elliptic and hyperbolic cases. However in the parabolic setting the right “spokes” are vertical lines, see Fig. 2, so the limit should be taken along them [9, Prop. 5.11].

4. **Induced Representations as a Source of Imaginary Units**

As we already mentioned in Rem. 2.1 all three matrix exponents (2.2)-(2.3) are one-parameter subgroups of the group \( \text{SL}_2(\mathbb{R}) \)—the group of \( 2 \times 2 \) matrices with unit determinant. Moreover any one-parameter subgroup of \( \text{SL}_2(\mathbb{R}) \) is a conjugate to either of subgroup \( A \), \( N \) or \( K \), see Rem. 2.1 for their descriptions. Thus our consideration may be applied for construction of *induced representations* of \( \text{SL}_2(\mathbb{R}) \) in an extended meaning.

The general scheme of induced representations is as follows, see [6, § 13.2], [8, § 3.1]. We denote \( \text{SL}_2(\mathbb{R}) \) by \( G \) and let \( H \) be its subgroup. Let \( \Omega = G/H \) be the corresponding homogeneous space and \( s : \Omega \rightarrow G \) be a continuous function [6, § 13.2] which is a left inverse to the natural projection \( G \rightarrow G/H \). In our case we choose:

\[
s : (u, v) \mapsto \frac{1}{\sqrt{v}} \begin{pmatrix} v & u \\ 0 & 1 \end{pmatrix}, \quad (u, v) \in \mathbb{R}^2, \ v > 0.
\]

Then any \( g \in G \) has a unique decomposition of the form \( g = s(\omega)h \) where \( \omega \in \Omega \) and \( h \in H \). We will write:

\[
\omega = s^{-1}(g), \quad h = r(g) := (s^{-1}(g))^{-1}g.
\]
Note that $\Omega$ is a left homogeneous space with the $G$-action defined in terms of $s$ as follows:

$$(4.3) \quad g : \omega \mapsto g \cdot \omega = s^{-1}(g^{-1} * s(\omega)),$$

where $*$ is the multiplication on $G$.

Let $\chi : H \to \mathbb{R}^2$ be a “unitary character” of $H$ in some generalised sense illustrated below. Then it induces a “unitary” representation of $G$, which is very close to induced representations in the sense of Mackey [6, § 13.2]. This representation has the canonical realisation $\rho$ in the space $L_2(\Omega)$ of square integrable $\mathbb{R}^2$-valued functions. It is given by the formula (cf. [6, § 13.2.(7)–(9)):

$$(4.4) \quad [\rho_\chi(g)f](\omega) = \chi_0(\tau(g^{-1} * s(\omega)))f(g \cdot \omega), \quad \chi_0(h) = \chi(h) \left( \frac{d\mu[h \cdot \omega]}{d\mu(\omega)} \right)^{1/2},$$

where $g \in G$, $\omega \in \Omega$, $h \in H$ and $\tau : G \to H$, $s : \Omega \to G$ are maps defined above; $*$ denotes multiplication on $G$ and $\cdot$ denotes the action $(4.3)$ of $G$ on $\Omega$ from the left.

4.1. **Induction from K.** This is the most traditional case in the representation theory. The action $(4.3)$ takes the form:

$$\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) : (u, v) \mapsto \left( \frac{(au + b)(cu + d) + cv^2}{(cu + d)^2 + (cv)^2}, \frac{v}{(cu + d)^2 + (cv)^2} \right).$$

Obviously it preserves the upper-half plane $v > 0$. Moreover with the help of the imaginary unit $i^2 = -1$ it can be naturally represented as a Möbius transformation:

$$\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) : w \mapsto \frac{aw + b}{cw + d}, \quad w = u + iv.$$

Thus it is justified in this particular case to look for complex-valued characters of $K$. They are parametrised by an integer $n \in \mathbb{Z}$:

$$\rho_n(h_t) = e^{int} = (\cos t + i \sin t)^n, \quad \text{where} \quad h_t = \left( \begin{array}{cc} \cos t & -\sin t \\ \sin t & \cos t \end{array} \right).$$

We can also calculate that:

$$\tau(g^{-1} * s(\omega)) = \frac{1}{\sqrt{(cu + d)^2 + (cv)^2}} \left( \begin{array}{cc} cu + d & -cv \\ cv & cu + d \end{array} \right).$$

Taking into account the identity $\frac{|a|}{a} = (\frac{a}{a})^{1/2}$ we obtain such a realisation of (4.4):

$$[\rho_n(g)f](w) = \left( \frac{cw + d}{cw + d} \right)^{1/2} f \left( \frac{aw + b}{cw + d} \right), \quad \text{where} \quad g^{-1} = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right), \quad w = u + iv.$$

4.2. **Induction from A.** In this case the action $(4.3)$ takes the form:

$$\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) : (u, v) \mapsto \left( \frac{(au + b)(cu + d) - cv^2}{(cu + d)^2 - (cv)^2}, \frac{v}{(cu + d)^2 - (cv)^2} \right).$$

This time the map does not preserve the upper-half plane $v > 0$: the sign of $(cu + d)^2 - (cv)^2$ is not determined. To express this map as a Möbius transformation we require the double numbers imaginary unit $\epsilon^2 = 1$:

$$\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) : w \mapsto \frac{aw + b}{cw + d}, \quad w = u + \epsilon v.$$

Remark 4.1. As we can see now the double numbers naturally appear in relation with the group $SL_2(\mathbb{R})$ and thus their introduction in § 1.2 was not “a purely generalistic attempt”, cf. [17, p. 4]. The same is true for dual numbers as can be seen in the next subsection.
Under such conditions it does not make much sense to look for a complex-valued characters of $A$. Instead we will take double number valued characters which are parametrised by a real number $\sigma$:

$$p_{\sigma}(ht) = e^{\sigma t} = (\cosh t + \epsilon \sinh t)^{\sigma},$$

where $ht = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$.

We can also calculate that

$$\tau(g^{-1} * s(\omega)) = \frac{1}{\sqrt{(cu + d)^2 - (cv)^2}} \begin{pmatrix} cu + d & cv \\ cv & cu + d \end{pmatrix}.$$ 

Thus the formula (4.4) becomes:

$$[p_{\kappa}(g)f](w) = \begin{pmatrix} cw + d \\ cw + d \end{pmatrix} \frac{d}{f} \begin{pmatrix} aw + b \\ cw + d \end{pmatrix}, \text{ where } g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad w = u + \epsilon v.$$

4.3. Induction from $N$. We consider here the lower-triangular matrices forming the subgroup $N'$. The action (4.3) takes now the form:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (u, v) \mapsto \begin{pmatrix} au + b \\ cv \end{pmatrix}, \quad w = u + \epsilon v.$$ 

This map preserves the upper-half plane $v > 0$ as the elliptic case of $K$. To express this map as a Möbius transformation we require the dual numbers imaginary unit $\epsilon^2 = 0$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : w \mapsto \begin{pmatrix} aw + b \\ cw + d \end{pmatrix}, \quad w = u + \epsilon v.$$ 

Similarly to the previous hyperbolic case we would look not for complex-valued characters but rather use parabolic rotations described in Section 2.3. A distinction from the hyperbolic case is that they are not given by multiplication of double numbers. Such a “character” parametrised by a real number $\kappa$ is defined by:

$$\rho_{\kappa}(ht) : w \mapsto \frac{(1 + \epsilon \kappa t)w + \kappa t}{(1 - \epsilon \kappa t)} = (1 + 2\epsilon \kappa t)w + (\kappa t + \epsilon \kappa^2 t^2), \quad \text{where } ht = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}.$$ 

Furthermore we calculate that

$$\tau(g^{-1} * s(\omega)) = \begin{pmatrix} 1 & 0 \\ \epsilon v & 1 \end{pmatrix}.$$ 

Thus the formula (4.4) has the following realisation:

$$[p_{\kappa}(g)f](w) = \begin{pmatrix} 1 - \epsilon \frac{2\kappa v}{cu + d} \\ \epsilon v \end{pmatrix} f \begin{pmatrix} aw + b \\ cw + d \end{pmatrix} - \frac{\kappa v}{cu + d} + \epsilon \frac{(kv)^2}{(cu + d)^2}, \quad \text{where } g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad w = u + \epsilon v.$$ 

The vector space of functions, where this representation acts, should be also considered with linear operations defined in §3.3.2.

These three examples will be used to build the corresponding versions of the Cauchy integral formula along the Erlangen Program at Large outlined in [8,9].

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APPENDIX A. OUTPUT OF SYMBOLIC CALCULATIONS

Here are the results of our symbolic calculations. The source code can be obtained from this paper [7] source at \url{http://arXiv.org}. It uses Clifford algebra facilities [12] of the GiNaC library [1]. The source code is written in noweb [18] literature programming environment.

\[
\text{Cayley of the matrix } x: \begin{pmatrix}
-\epsilon (\epsilon + x) & 1 \\
0 & (\epsilon x + 1)
\end{pmatrix}
\]

\[
\text{Rotation by } x: \begin{pmatrix}
1 & x^2 + 2ux + v \\
x & 1 + x^2
\end{pmatrix}
\]

\[
\text{Parabolic norm: } -v + u^2
\]

\[
\text{Real number } x \text{ as a dual number: } 0 - x
\]
Product: \((u + u') - u^2u'^2 + uu'^2 + uu'^3v - v^3v + 2uu' + u^2v' + u^2\)

Product by a scalar: \((u - av + u^2 - u^2a)\)

Real part: \((0 - uv + v - u^2 + u^3)\)

Imag part: \((1 + uv - u^3)\)

Lin comb of two vectors \(a*(1, 0) + b*(-1, 0): (\frac{a - b}{a + b} - \frac{3ab^2 + 2ab + b^3 - a^2 - 3a^2b + a^3}{(a + b)^2})\)

\(P\) is the sum \(\text{Re}(P)\) and \(\text{Im}(P)\): true

The real part of a real dual number is itself: true

Norm is invariant: true

Product is invariant: true

Product is norm squared: true

Product \((u, v) \ast (u_0, v_0)\) is \((u, v)\): true

Add is commutative: true

Add is associative: true

S-mult commutative: true

S-mult associative: true

S-mult distributive 1: true

S-mult distributive 2: true

Product is symmetric (commutative): true

Prod is associative: true

Product is distributive: true

\[
\text{Cayley of the matrix } x: \begin{pmatrix}
-\frac{(e^1 \ast e^0)x + 1}{e^0x} & \frac{0}{e^0x} \\
-\frac{(e^1 \ast e^0)x + 1}{e^0x} & -\frac{(e^1 \ast e^0)x + 1}{e^0x}
\end{pmatrix}
\]

Rotation by \(x: \begin{pmatrix}
-u & -\frac{u^2x^2 - 2ux - v}{1 + u^2x^2 - 2ux} \\
-\frac{u^2x^2 - 2ux - v}{1 + u^2x^2 - 2ux} & -\frac{u^2x^2 - 2ux - v}{1 + u^2x^2 - 2ux}
\end{pmatrix}
\]

Rotation of \((u_0, v_0)\) by \(x: \begin{pmatrix}
-\frac{1}{x} & -\frac{1 + x^2}{x^2} \\
-\frac{1 + x^2}{x^2} & -\frac{1}{x}
\end{pmatrix}
\]

Parabolic norm: \(\frac{u^2}{1 + v}\)

Real number \(x\) as a dual number: \(\infty \begin{pmatrix}
\frac{\infty}{x} & 0 \\
0 & \frac{\infty}{x}
\end{pmatrix}\)

Product: \(\begin{pmatrix}
\frac{uu'}{u + u'} & -\frac{1 + u'^2 - v'^2 - v + 2uu'}{(u + u')^2} \\
-\frac{1 + u'^2 - v'^2 - v + 2uu'}{(u + u')^2} & -\frac{1 + u'^2 - v'^2 - v + 2uu'}{(u + u')^2}
\end{pmatrix}\)

Product by a scalar: \(\begin{pmatrix}
u & -\frac{1 + a - w}{a} \\
-\frac{1 + a - w}{a} & -\frac{1 + a - w}{a}
\end{pmatrix}\)

Real part: \(\infty \begin{pmatrix}
u + \infty^2 + \infty^2u - u^2 \\
\frac{-1 + |1 + u|}{u}
\end{pmatrix}\)

Imag part: \(\begin{pmatrix}1 & -\frac{1 + u - u'}{u} \end{pmatrix}\)

Lin comb of two vectors \(a*(1, 0) + b*(-1, 0): (\frac{a + b}{a - b} - \frac{a + 2ab - b^2 - a^2 + b}{(a - b)^2})\)

\(P\) is the sum \(\text{Re}(P)\) and \(\text{Im}(P)\): true

The real part of a real dual number is itself: true

Norm is invariant: true

Product is invariant: true

Product is norm squared: true

Product \((u, v) \ast (u_0, v_0)\) is \((u, v)\): true

Add is commutative: true

Add is associative: true

S-mult commutative: true

S-mult associative: true

S-mult distributive 1: true

S-mult distributive 2: true

Product is symmetric (commutative): true

Prod is associative: true

Product is distributive: true
Elliptic case of induced representations

map \( r(M) \): 
\[
\begin{pmatrix}
\frac{d^2}{d^2+c^2} & \frac{dc}{d^2+c^2} \\
\frac{dc}{d^2+c^2} & \frac{d^2}{d^2+c^2}
\end{pmatrix}
\]

map \( s^{-1}(M) \): 
\[
\begin{pmatrix}
\frac{1}{d} & \frac{c^2b+c+cd^2b}{d^2} \\
0 & \frac{d^2}{d^2+c^2}
\end{pmatrix}
\]

character: 
\[
\begin{pmatrix}
\frac{2udc+2d^2+u^2c^2}{2udc+2d^2+u^2c^2} & \frac{cv(d+uc)}{cv(d+uc)^2} \\
\frac{2udc+2d^2+u^2c^2}{2udc+2d^2+u^2c^2} & \frac{2udc+2d^2+u^2c^2}{2udc+2d^2+u^2c^2}
\end{pmatrix}
\]

Moebius map: 
\[
\begin{pmatrix}
uucb+db+und+acd & cv^2+uac \\
\frac{2udc+2d^2+u^2c^2}{2udc+2d^2+u^2c^2} & \frac{adv-cvb}{(d+uc)^2}
\end{pmatrix}
\]

Moebius map is given by the imaginary unit: true

Parabolic (N) case of induced representations

map \( r(M) \): 
\[
\begin{pmatrix}
1 & 0 \\
\frac{d}{d} & 1
\end{pmatrix}
\]

map \( s^{-1}(M) \): 
\[
\begin{pmatrix}
\frac{1}{d} & b \\
0 & d
\end{pmatrix}
\]

character: 
\[
\begin{pmatrix}
\frac{1}{d+uc} & 0 \\
\frac{cv}{d+uc} & 1
\end{pmatrix}
\]

Moebius map: 
\[
\begin{pmatrix}
uucb+db+und+acd & cv^2+uac \\
\frac{2udc+2d^2+u^2c^2}{2udc+2d^2+u^2c^2} & \frac{adv-cvb}{(d+uc)^2}
\end{pmatrix}
\]

Moebius map is given by the imaginary unit: true

Hyperbolic case of induced representations

map \( r(M) \): 
\[
\begin{pmatrix}
\frac{d^2}{d^2-c^2} & \frac{dc}{d^2-c^2} \\
\frac{dc}{d^2-c^2} & \frac{d^2}{d^2-c^2}
\end{pmatrix}
\]

map \( s^{-1}(M) \): 
\[
\begin{pmatrix}
\frac{1}{d} & \frac{-c^2b+c+cd^2b}{d^2} \\
0 & \frac{d^2}{d^2-c^2}
\end{pmatrix}
\]

character: 
\[
\begin{pmatrix}
\frac{2udc+2d^2+u^2c^2}{2udc+2d^2+u^2c^2} & \frac{cv(d+uc)}{cv(d+uc)^2} \\
\frac{2udc-c^2v^2+d^2+u^2c^2}{2udc+2d^2+u^2c^2} & \frac{2udc-c^2v^2+d^2+u^2c^2}{2udc+2d^2+u^2c^2}
\end{pmatrix}
\]

Moebius map: 
\[
\begin{pmatrix}
uucb+db+und-acd & cv^2+uac \\
\frac{2udc+2d^2+u^2c^2}{2udc+2d^2+u^2c^2} & \frac{adv-cvb}{(d+uc)^2}
\end{pmatrix}
\]

Moebius map is given by the imaginary unit: true

**APPENDIX B. PROGRAM FOR SYMBOLIC CALCULATIONS**

This is a documentation for our symbolic calculations. You can obtain the program itself from the source files of this paper [7] at arXiv.org; \LaTeX{} compilation of it will produce the file parab-rotation.nv in the current directory. This is a \texttt{C++} code of the program. It uses Clifford algebra facilities [12] of the GiNaC library [1].

This piece of software is licensed under GNU General Public License (Version 3, 29 June 2007) [3].

**B.1. Class dual_number.**

**B.1.1. Public Methods.**

A dual number can be created simply by listing its two components.

```cpp
(Public methods 14) ≡

dual_number(const ex & a, const ex & b);
```

Uses dual_number 29a 29a 29a 29a 29a 29a 29a 29a 29a 29a 29a 29a 29a 29a 30d 31a 31b 31c 31c 32a 32a 32a 32b 32b 32b.
Alternatively you can provide a $1 \times 2$ or $2 \times 1$ matrix, a list, another dual number $P$ or a complex expression with a non-zero imaginary part to give two components. If $P$ does not have two components and is a real-valued expression, it will be embedded into dual numbers with zero argument and norm equal to $P$.

(Public methods 14) \[ + \equiv \]
\[
\text{dual\_number}(\text{const ex \& P});
\]

Uses \text{dual\_number} 29a 29a 29a 29a 29a 29a 29a 29a 29c 30d 31a 31b 31c 31c 32a 32a 32b 32b 32b.

We can also obtain the module and argument of a \text{dual\_number}.

(Public methods 14) \[ + \equiv \]
\[
\text{ex arg()} \text{ const;}
\]
\[
\text{ex norm()} \text{ const;}
\]

We define the conjugate of a \text{dual\_number} by $u + \varepsilon v = -u + \varepsilon v$.

(Public methods 14) \[ + \equiv \]
\[
\text{ex conjugate()} \text{ const \{ return dual\_number(-u\_comp, v\_comp); \}}
\]

Uses \text{dual\_number} 29a 29a 29a 29a 29a 29a 29a 29a 29c 30d 31a 31b 31c 31c 32a 32a 32b 32b 32b.

Real part may be defined out of the formula $\Re(u, v) = \frac{1}{2}((u, v) + \overline{(u, v)})$, but it simply reduces to the value of norm for the \text{dual\_number}.

(Public methods 14) \[ + \equiv \]
\[
\text{ex real\_part()} \text{ const;}
\]
\[
\text{ex imag\_part()} \text{ const;}
\]

Negative of a \text{dual\_number} and its power.

(Public methods 14) \[ + \equiv \]
\[
\text{dual\_number neg()} \text{ const \{ return dual\_number(-u\_comp, -v\_comp + (is\_subgroup\_N? 2*pow(u\_comp, 2) : -2)); \}}
\]

Uses \text{dual\_number} 29a 29a 29a 29a 29a 29a 29a 29a 29c 30d 31a 31b 31c 31c 32a 32a 32b 32b 32b.

We can also convert a \text{dual\_number} to a \text{matrix}.

(Public methods 14) \[ + \equiv \]
\[
\text{matrix to\_matrix()} \text{ const \{ return matrix (1, 2, lst(u\_comp, v\_comp)); \}}
\]

We define the rule for parabolic norm of a sum, see (3.8).

(Public methods 14) \[ + \equiv \]
\[
\text{ex add\_norms(\text{const dual\_number \& P}) const \{ return (norm()\_P.norm()).normal(); \}}
\]

Uses \text{dual\_number} 29a 29a 29a 29a 29a 29a 29a 29a 29c 30d 31a 31b 31c 31c 32a 32a 32b 32b 32b.

Algebraic operations are defined for \text{dual\_numbers} in a way described in § 3.3.2. The standard C++ operators $+, -, \ast, \div$ will be overloaded later in order to permit natural expressions with \text{dual\_numbers}.

(Public methods 14) \[ + \equiv \]
\[
\text{dual\_number add(\text{const dual\_number \& a}) const;}
\]
\[
\text{dual\_number sub(\text{const dual\_number \& a}) const \{ return add(a.neg()); \}}
\]
\[
\text{dual\_number mul(\text{const dual\_number \& a}) const;}
\]
\[
\text{dual\_number mul(\text{const ex \& a}) const \{ return mul(dual\_number(a)); \}}
\]

Uses \text{dual\_number} 29a 29a 29a 29a 29a 29a 29a 29a 29c 30d 31a 31b 31c 31c 32a 32a 32b 32b 32b.
B.2. **Algebraic Subroutines.** We need a couple of global variables which help to write uniformly algebraic rules for both cases of subgroups \( N \) and \( N' \). Firstly, we need to consider separately cases of subgroup \( N \) and \( N' \), the following global variable keeps track on it.

\[
\langle \text{N-Nprime separation 16a} \rangle \equiv (24b) 16b \triangleright
\]

\[
\begin{align*}
\text{bool is\_subgroup}_N; \\
\end{align*}
\]

In the case of the subgroup \( N' \) the reference point lies at infinity, the following \texttt{realsymbol} variable represents it in the symbolic calculations.

\[
\langle \text{N-Nprime separation 16a}\rangle + \equiv (24b) <16a 16c> \\
\text{realsymbol Inf("Inf", \"\infy\")};
\]

We define the “zero angle”: for the subgroup \( N \) it is 0, for \( N' \)—\( \infty \).

\[
\langle \text{N-Nprime separation 16a}\rangle + \equiv (24b) <16b> \\
\text{#define krg0 (is\_subgroup}_N ? ex(0) : ex(Inf))
\]

Defines:
\[
\text{krg0}, \text{used in chunks 21b, 25f, and 30b}.
\]

Here is the set of algebraic procedures representing definitions made in this paper.

**B.2.1. Argument and \( u \).** In the case of \( N \) the value of \( u \) is simply the argument.

\[
\langle \text{Algebraic procedures 16d}\rangle \equiv (24c) 16c \triangleright
\]

\[
\begin{align*}
\text{ex u\_from\_arg}(\text{const ex & a}) \{ \\
\quad \text{if (is\_subgroup}_N) \\
\quad \quad \text{return a;}
\}
\end{align*}
\]

In the case of \( N' \) the value of \( u \) is the inverse to the argument, and we need to treat properly the case of zero...

\[
\langle \text{Algebraic procedures 16d}\rangle + \equiv (24c) <16d 16f> \\
\text{else}
\]

\[
\begin{align*}
\quad & \text{if (a.normal().is\_zero())} \\
\quad & \quad \text{return Inf;}
\end{align*}
\]

…and \( \infty \). We try to replace \( \frac{1}{\infty} \) by 0.

\[
\langle \text{Algebraic procedures 16d}\rangle + \equiv (24c) <16e 17e> \\
\text{else}
\]

\[
\begin{align*}
\quad & \text{try}
\quad & \quad \text{try} \\
\quad & \quad \quad \text{realsymbol t("t");} \\
\quad & \quad \quad \text{return pow(a.normal().subs(t \equiv 0).normal());}
\text{catch (std::exception &p) \{}
\quad & \quad \quad \text{return pow(a, -1);}
\}
\]

**B.2.2. Argument of a point.** The opposite task (finding argument of a point) is solved similarly.

\[
\langle \text{Dual number class further implementation 16g}\rangle \equiv (24c) 17a \triangleright
\]

\[
\begin{align*}
\text{ex dual\_number\_arg() const} \{
\quad & \text{if (is\_subgroup}_N) \\
\quad & \quad \text{return u\_comp;}
\}
\end{align*}
\]

Uses \texttt{dual\_number} 29a 29a 29a 29a 29a 29a 29a 29a 29a 29c 30d 31a 31b 31c 32a 32a 32a 32b 32b 32b.
Again in the case of $N'$ we need to consider cases of $0\ldots$

$\text{else}$ {
  $\text{if} \ (u_{\text{comp}}.\text{normal}().is_{\text{zero}}())$
  $\text{return} \ \text{Inf};$
}

$\ldots$ and $\infty$. We try to replace $\frac{1}{\infty}$ by $0$.

$\text{else}$ {
  $\text{try}$ {
    $\text{realsymbol} \ l("t");$
    $\text{return} \ \text{pow}(u_{\text{comp}}.\text{subs}(\text{Inf} \equiv \text{pow}(t, -1)), -1).\text{normal}().\text{subs}(t \equiv 0).\text{normal}();$
  }$
  $\text{catch} \ (\text{std}::\text{exception} &p) \ {$
    $\text{return} \ \text{pow}(u_{\text{comp}}, -1);$  
  }$
}

$\text{B.2.3. Norm.}$ The corresponding value of the parabolic norm is calculated by the formulae (3.1).

$\text{else}$ {
  $\text{try}$ {
    $\text{realsymbol} \ l("t");$
    $\text{return} \ \text{lsolve}(\text{dual number} (u, l).\text{norm} () \equiv \text{n}, l).\text{normal}();$
  }$
  $\text{catch} \ (\text{std}::\text{exception} &p) \ {$
    $\text{return} \ \text{lsolve}(\text{pow}(\text{dual number} (u, l).\text{norm} (), -1) \equiv \text{pow}(\text{n}, -1), l).\text{normal}();$
  }$
}

$\text{B.2.4. The value of } v \text{ from the argument and norm.}$ We often need to find values of $v$ such that for a given value of argument $A$ point $(A, v)$ will have a given norm.

$\text{else}$ {
  $\text{if} \ (u_{\text{comp}}.\text{is}_{\text{zero}}()) \ {$
    $\text{if} \ ((v_{\text{comp}} + 1).\text{is}_{\text{zero}}())$
    $\text{return} \ 1;$
    $\text{else}$
    $\text{return} \ \text{pow}(\text{Inf}, 2):(v_{\text{comp}} + 1);$
  }$
  $\text{else}$
  $\text{return} \ \text{pow}(\text{Inf}, 2):(v_{\text{comp}} + 1)).\text{normal}();$
}

Uses $\text{dual number} 29a 29a 29a 29a 29a 29a 29a 29a 29a 29a 29a 29a 30d 31a 31b 31c 31c 32a 32a 32b 32b 32b.$

$\text{The case of subgroup } N' \text{ require treatment of infinity.}$
(Algebraic procedures 16a) $\equiv$
\begin{verbatim}
dual_number zero dual_number() {
    return (is_subgroup N ? dual_number(0, 0) : dual_number(Inf, -1));
}
\end{verbatim}

Uses dual_number 29a 29a 29a 29a 29a 29a 29a 29a 29c 30d 31a 31b 31c 31c 32a 32a 32a 32b 32b 32b.

Dual number class further implementation 16g) $\equiv$
\begin{verbatim}
bool dual_number::is_zero() const {
    return is_equal(zero dual_number());
}
\end{verbatim}

Uses dual_number 29a 29a 29a 29a 29a 29a 29a 29a 29c 30d 31a 31b 31c 31c 32a 32a 32a 32b 32b 32b.

B.2.5. Real and Imaginary Parts. See § 3.3.3 for a discussion of the real and imaginary parts of dual numbers.

(Algebraic procedures 16a) $\equiv$
\begin{verbatim}
ex dual_number::real_part() const {
    return dn_from_arg_mod(0, (1-arg())*norm());
}
\end{verbatim}

Uses dual_number 29a 29a 29a 29a 29a 29a 29a 29a 29c 30d 31a 31b 31c 31c 32a 32a 32a 32b 32b 32b.

(Algebraic procedures 16a) $\equiv$
\begin{verbatim}
ex dual_number::imag_part() const {
    return dn_from_arg_mod(1, arg())*norm());
}
\end{verbatim}

Uses dual_number 29a 29a 29a 29a 29a 29a 29a 29a 29c 30d 31a 31b 31c 31c 32a 32a 32a 32b 32b 32b.

B.2.6. Product of Two Points. We define now the product of two points according to the Definition 3.6. We also include a multiplication by a scalar: if a factor is a scalar it is replaced by a vector with the zero argument and norm equal to the scalar.

(Dual number class further implementation 16g) $\equiv$
\begin{verbatim}
dual_number dual_number::mul(const dual_number & P) const {
    ex u=dn_from_arg_mod(0, (1-arg())*norm());
    return dual_number(u, v_from_norm(u, norm())*P.norm());
}
\end{verbatim}

Uses dual_number 29a 29a 29a 29a 29a 29a 29a 29a 29c 30d 31a 31b 31c 31c 32a 32a 32a 32b 32b 32b.

B.2.7. Vector Addition of Two Points. The sum is calculated from the expression (3.7).

(Dual number class further implementation 16g) $\equiv$
\begin{verbatim}
dual_number dual_number::add(const dual_number & a) const {
    ex norms = add_norms(a);
    if (norms.normal().is_zero())
        return zero dual_number();
    else {
        ex us=dn_from_arg((arg()*norm())+a.arg()+a.norm())/norms.normal();
        return dual_number(us, v_from_norm(us, norms));
    }
}
\end{verbatim}

Uses dual_number 29a 29a 29a 29a 29a 29a 29a 29a 29c 30d 31a 31b 31c 31c 32a 32a 32a 32b 32b 32b.
19a (Algebraic procedures 16d) \( \equiv \) dual_number dn_from_arg_mod(const ex & a, const ex & n) {
  ex us = u_from_arg(a).normal();
  return dual_number(us, v_from_norm(us, n));
}

Uses dual_number 29a 29a 29a 29a 29a 29a 29a 29c 30d 31a 31b 31c 31c 32a 32a 32b 32b 32b.

De Moivre’s Identity:
19b (Dual number class further implementation 16g) \( \equiv \) dual_number dual_number::power(const ex & e) const {
  return dn_from_arg_mod(arg() * e, pow(norm(), e));
}

Uses dual_number 29a 29a 29a 29a 29a 29a 29a 29a 29c 30d 31a 31b 31c 31c 32a 32a 32a 32b 32b 32b.

All algebraic routines are defined now.

B.3. Calculation and Tests. This Subsection contains code for calculation of various expression. See [10] or GiNaC info for usage of Clifford algebra functions.

B.3.1. Calculation of Expressions. Firstly, we output the expression of the Cayley transform for a generic element from subgroups \( N \) and \( N' \).

19c (Show expressions 19c) \( \equiv \)

Ex XC = canonicalize_clifford((TC * X * TCI).evalm());
formula_out("Cayley of the matrix x: ", XC.subs(sign \( \equiv \) 0).normal());

Uses formula_out 19e 20e 27d.

Then we calculate Möbius action of those matrix on a point.

19d (Show expressions 19c) \( \equiv \)

dual_number W(clifford_moebius_map(XC, P.to_matrix()).subs(sign \( \equiv \) 0).normal()),
W1 = W.subs(lst(u \( \equiv \) u1, v \( \equiv \) v1));
formula_out("Rotation by x: ", W);

Uses dual_number 29a 29a 29a 29a 29a 29a 29a 29a 29c 30d 31a 31b 31c 31c 32a 32a 32b 32b 32b and formula_out 19c 20e 27d.

Next we specialise the above result to the reference point.

19e (Show expressions 19c) \( \equiv \)

formula_out("Rotation of \((u0, v0)\) by \((x)\): ", W1.subs(lst(u \( \equiv \) u0, v \( \equiv \) v0)).subs(Inf \( \equiv \) pow(y, -1)).normal().subs(y \( \equiv \) 0).normal());

Defines:
formula_out, used in chunks 19, 20, 23, 24a, and 28a.

The expression for the parabolic norm.

19f (Show expressions 19c) \( \equiv \)

formula_out("Parabolic norm: ", P.norm());

Uses formula_out 19e 20e 27d.

Embedding of reals into dual numbers.

19g (Show expressions 19c) \( \equiv \)

formula_out("Real number x as a dual number: ", dual_number(x));

Uses dual_number 29a 29a 29a 29a 29a 29a 29a 29a 29c 30d 31a 31b 31c 31c 32a 32a 32a 32b 32b 32b and formula_out 19e 20e 27d.
The expression for the product of two points.

\[ \langle \text{Show expressions} \rangle \equiv \text{formula_out}("Product: ", P+P1); \]

Uses \text{formula_out} 19e 20e 27d.

The expression of the product of a point and a scalar.

\[ \langle \text{Show expressions} \rangle \equiv \text{formula_out}("Product by a scalar: ", a*P); \]

Uses \text{formula_out} 19e 20e 27d.

Expressions for the real and imaginary parts.

\[ \langle \text{Show expressions} \rangle \equiv \text{formula_out}("Real part: ", P.real_part()); \]
\[ \text{formula_out}("Imag part: ", P.imag_part()); \]

Uses \text{formula_out} 19e 20e 27d.

The expression for a sum of two points is too cumbersome to be printed.

\[ \langle \text{Show expressions} \rangle \equiv \text{formula_out}("Add is: ", (P+P1).normal()); \]

Uses \text{formula_out} 19e 20e 27d.

Linear combination of points \((1, 0)\) and \((-1, 0)\) with coefficients \(a\) and \(b\), for the linearisation presented in § 3.3.4.

\[ \langle \text{Show expressions} \rangle \equiv \text{formula_out}("Lin comb of two vectors a*(1, 0)+b*(-1, 0): ", (a*P+b*P1).subs(lst(u≡1, v≡0, u1≡-1, v1≡0)).normal()); \]

Defines:
\text{formula_out}, used in chunks 19, 20, 23, 24a, and 28a.

B.3.2. Checking Algebraic Identities. In this Subsection we verify basic algebraic properties of the defined operations.

A dual number is the sum of its real and imaginary parts.

\[ \langle \text{Check identities} \rangle \equiv \text{test_out}("P is the sum Re(P) and Im(P): ", P-(ex_to<\text{dual number}>(P.real_part())+ex_to<\text{dual number}>(P.imag_part()))); \]

Defines:
\text{test_out}, used in chunks 20–22, 24a, and 28a.

A dual number made out of a real \(a\) has the norm of real part equal to \(a\).

\[ \langle \text{Check identities} \rangle \equiv \text{test_out}("The real part of a real dual number is itself: ", ex_to<\text{dual number}>(\text{dual number}(a).real_part()).norm()-a); \]

Defines:
\text{test_out}, used in chunks 20–22, 24a, and 28a.

The norm is invariant under parabolic rotations, i.e. they are in agreement with Defn. 3.1.

\[ \langle \text{Check identities} \rangle \equiv \text{test_out}("norm is invariant: ", P.norm()-W.norm()); \]

Uses \text{test_out} 20f 20g 21b 27c.
The product $w_1 \bar{w}_2$ is invariant under rotations, Prop. 3.9(iii).

21a \(\langle \text{Check identities 20f}\rangle \equiv \)

\(\text{test\_out("Product is invariant: ", P*P1.conjugate()-W*W1.conjugate());}\)

Uses test\_out 20f 20g 21b 27e.

Product $w \bar{w}$ is $\langle 0, |w|^2 \rangle$, Prop. 3.9(iv).

21b \(\langle \text{Check identities 20f}\rangle \equiv \)

\(\text{test\_out("Product is norm squared: ", (P*P.conjugate()-dn\_from\_arg\_mod(Arg0, pow(P.norm(), 2))));}\)

Defines:

\(\text{test\_out, used in chunks 20–22, 24a, and 28a.}\)

Uses Arg0 16c.

The reference point is unit under multiplication.

21c \(\langle \text{Check identities 20f}\rangle \equiv \)

\(\text{test\_out("Product \((u, v)\) is \((u, v)\): ", P*P0-P);}\)

Addition is commutative, Prop. 3.11(i).

21d \(\langle \text{Check identities 20f}\rangle \equiv \)

\(\text{test\_out("Add is commutative: ", (P+P1)-(P1+P));}\)

Uses test\_out 20f 20g 21b 27e.

Addition is associative, Prop. 3.11(i).

21e \(\langle \text{Check identities 20f}\rangle \equiv \)

\(\text{test\_out("Add is associative: ", ((P+P1)+P2)-(P+(P1+P2));}\)

Uses test\_out 20f 20g 21b 27e.

Multiplication by a scalar is commutative.

21f \(\langle \text{Check identities 20f}\rangle \equiv \)

\(\text{test\_out("S-mult commutative: ", P*a-a*P);}\)

Uses test\_out 20f 20g 21b 27e.

Multiplication by a scalar is associative.

21g \(\langle \text{Check identities 20f}\rangle \equiv \)

\(\text{test\_out("S-mult associative: ", b*P*a-a*b);}\)

Uses test\_out 20f 20g 21b 27e.

Distributive law \(a(w_1 + w_2) = aw_1 + aw_2\), Prop. 3.11(iv).

21h \(\langle \text{Check identities 20f}\rangle \equiv \)

\(\text{test\_out("S-mult distributive 1: ", a*(P+P1)-(a*P+a*P1));}\)

Uses test\_out 20f 20g 21b 27e.

Distributive law \((a + b)w = aw + bw\), Prop. 3.11(iv).

21i \(\langle \text{Check identities 20f}\rangle \equiv \)

\(\text{test\_out("S-mult distributive 2: ", P*(a+b)-(P*a + P*b));}\)

Uses test\_out 20f 20g 21b 27e.
Product is commutative, Prop. 3.9(i).

(26a) < 21i 22b

Uses test_out 20f 20g 21b 27e.

Product is associative, Prop. 3.9(i).

(26a) < 22a 22c

Uses test_out 20f 20g 21b 27e.

Product and addition are distributive, Prop. 3.11(ii).

(26a) < 22b

Uses test_out 20f 20g 21b 27e.

B.4. Induced Representations. Here we calculate the basic formulae for Section 4.

B.4.1. Encoded formulae. This routine encodes the map $s : \mathbb{R}^2 \rightarrow SL_2(\mathbb{R})$ (4.1).

(24e) < 22d

```cpp
ex s_map(const ex & u, const ex & v) {
    return matrix(2, 2, lst(v, u, 0, 1));
}
```

This routine encodes the map $r : SL_2(\mathbb{R}) \rightarrow H$ (4.2). The first parameter is an element of $SL_2(\mathbb{R})$, the second—is a generic element of subgroup $H$.

(24e) < 22d 22f

```cpp
ex r_map(const ex & M, const ex & K) {
    ex K = K.evalm(), K2;
    lst vars = is_a<symbol> (K.1.op(2)) ? lst(K.1.op(2)) : lst(K.1.op(1));
    if (is_a<symbol> (K.1.op(3))) {
        vars = vars.append(K.1.op(3));
        K2 = K1. subs(lst((M*K1).evalm().op(2)==0), vars)).subs(K.1.op(3)==1);
    } else
        K2 = K1. subs(lst((M*K1).evalm().op(2)==0), vars));
    return pow(K2, -1).evalm();
}
```

This is the inverse $s^{-1}$ of the above map $s$.

(24e) < 22e 23a

```cpp
ex s_inv(const ex & M, const ex & K) {
    ex MK = M * pow(r_map(M, K), -1).evalm();
    ex D = MK. op(3).subs(x==1).normal();
    return matrix(1, 2, lst((MK. op(1). subs(x==1).normal() * D).normal(),
                            (MK. op(0). subs(x==1).normal() * D).normal()));
}
```
This is a matrix form of the above inverse map \( s_{\text{inv}}() \).

\[
\begin{aligned}
\text{ex } s_{\text{inv}} & (\text{const ex } \& M, \text{const ex } \& K) \{
\quad \text{return } (M^\ast \text{pow}(r_{\text{map}}(M,K),-1)).\text{evalm();}
\}
\end{aligned}
\]

B.4.2. Calculation of induced representation formulae. Firstly we define a generic element \( M \) of \( \text{SL}_2(\mathbb{R}) \).

\[
\begin{aligned}
\text{ex } M &= \text{matrix}(2,2, \text{lst}(a,b,c,d)), H;
\end{aligned}
\]

We consider the three cases.

\[
\begin{aligned}
\text{ex } &\text{subgroups}=\text{lst} \text{matrix}(2,2, \text{lst}(x,y,y,x)), \\
&\text{matrix}(2,2, \text{lst}(1,0,0,1)), \\
&\text{matrix}(2,2, \text{lst}(x,y,x,1));
\end{aligned}
\]

Now we run a cycle over the three cases...

\[
\begin{aligned}
\text{for}(\text{int } i=0; i<3; i++) \{ \\
\quad H=\text{subgroups}[i]; \\
\quad \text{cout} \ll \text{cases}[i] \ll \text{" case of induced representations\\"} \ll \text{endl}; \\
\quad \text{//formula out("M*H: ", (M*H).\text{evalm());}
\}
\end{aligned}
\]

Uses cases 23c and formula out 19e 20e 27d.

... and output expression of \( r \) (4.2),...

\[
\begin{aligned}
\text{formula out("map } \backslash (r(M) \backslash }: ", r_{\text{map}}(M,H));
\end{aligned}
\]

Uses formula out 19e 20e 27d.

... matrix form of the inverse \( s^{-1} \) (4.2),...

\[
\begin{aligned}
\text{formula out("map } \backslash (s^{-1}(M) \backslash }: ", s_{\text{inv}}(M,H).\text{subs(a}\equiv(1+b+c)\div d).\text{normal());}
\end{aligned}
\]

Uses formula out 19e 20e 27d.

... expression for the argument of the character in (4.4),...

\[
\begin{aligned}
\text{formula out("character: ", r_{\text{map}}(M+s_{\text{map}}(P),H));}
\end{aligned}
\]

Uses formula out 19e 20e 27d.
...and finally the action (4.3) of $\text{SL}_2(\mathbb{R})$ on the homogeneous space.

\[ \langle \text{Induced representations } 23b \rangle + \equiv \langle 26e, 23h \rangle \]

\[ \text{formula out}("\text{Moebius map: } s_{\text{inv}}(M\cdot s_{\text{map}}(P_{\text{to matrix}})), H)); \]

\[ \text{test out}("\text{Moebius map is given by the imaginary unit: } s_{\text{inv}}(M\cdot s_{\text{map}}(P), H) - \)

\[ \text{clifford}\cdot \text{moebius}\cdot \text{map}(a \cdot \text{one}, b \cdot e_0, c \cdot e_0, d \cdot \text{one}, P_{\text{to matrix}}), e)_{\text{subs}}(\text{sign} \equiv i^{-1}); \]

\[ \text{cout} \ll \langle \text{latexout }"\vspace{2mm}\hrule" : \text{endl}; \]

\} 

Uses \text{formula out} 19e 20e 27d and \text{test out} 20f 20g 21b 27e.

\textbf{B.5. Program Outline.} Here is the outline how we use the above parts.

Routines for \texttt{dual\_number} are collected in a separate library. We start from the definition \texttt{dual\_number} class in the header file.

\[ \langle \text{dualnum.h } 24b \rangle \equiv \]

\[ \langle \text{Initialisation } 24f \rangle \]

\[ \langle \text{N-Nprime separation } 16a \rangle \]

\[ \langle \text{Dual number class declaration } 28b \rangle \]

\[ \langle \text{Additional routines declarations } 28a \rangle \]

Here is the file with the implementation.

\[ \langle \text{dualnum.cpp } 24c \rangle \equiv \]

\[ \#include <\text{dualnum.h}> \]

\[ \langle \text{Algebraic procedures } 16d \rangle \]

\[ \langle \text{Dual number class implementation } 29c \rangle \]

\[ \langle \text{Dual number class further implementation } 16g \rangle \]

\[ \langle \text{Output routines } 27d \rangle \]

\textbf{B.5.1. Test program outline.} Firstly we load \texttt{dual\_number} support.

\[ (* 24d) \equiv \]

\[ \#include <\text{dualnum.h}> \]

The rest of the program makes all checks.

\[ (* 24d) \equiv \]

\[ \langle \text{Definition of variables } 25a \rangle \]

\[ \langle \text{Test routine } 25f \rangle \]

\[ \langle \text{Induced representations routines } 22d \rangle \]

\[ \langle \text{Main procedure } 26b \rangle \]

This is the initialisation part

\[ \langle \text{Initialisation } 24f \rangle \equiv \]

\[ \#include <\text{cycle.h}> \]

\[ \#include <\text{fstream}> \]

\[ \text{using namespace std;} \]

\[ \text{using namespace GiNaC;} \]

\[ \#include "\text{ginac-utils.h}" \]
B.5.2. Variables. These **realsymbols** are used in our calculations.

\[ \text{Definition of variables } 25a \equiv \]
\[ \text{realsymbol } (u^w, v^w, u_1^w, v_1^w, u_2^w, v_2^w), \]
\[ a^w, b^w, c^w, d^w, x^w, y^w, \]
\[ \text{Finally this variable keeps the signature of the metric space. } 25b \equiv \]
\[ \text{sign}(s^w, \sigma^w); \]
\[ \text{This an index used for the definition of Clifford units. } 25c \equiv \]
\[ \text{varidx } \mu(\mu^w, \mu^w), 2; \]
\[ \text{Three generic points which are used in calculations. } 25d \equiv \]
\[ \text{dual number } P(u, v), P_1(u, v), P_2(u, v); \]
\[ \text{Uses dual number } 29a 29a 29a 29a 29a 29a 29a 29a 29a 29a 29a 29a 29a 29c 30d 31a 31b 31c 31c 32a 32a 32a 32b 32b 32b. \]
\[ \text{Here we define a parabolic Clifford units } e_0, e_1. \]
\[ \text{ex } e = \text{clifford_unit}(\mu^w, \text{diag_matrix}(\text{lst}(-1, \text{sign}))), \]
\[ e_0 = e.\text{subs}(\mu^w \equiv 0), \]
\[ e_1 = e.\text{subs}(\mu^w \equiv 1), \]
\[ \text{one } = \text{dirac.ONE(); } 25e \equiv \]

B.5.3. **Test routine.** This routine make the same sequence of checks for both cases of subgroups \(N\) and \(N'\).

First we define the reference point \((u_0, v_0)\).

\[ \text{void parab_rot_sub() { } 25f \equiv } \]
\[ \text{ex } X, \]
\[ u_0=\text{Arg0}, v_0=\text{from_norm}(u_0, 1), \]
\[ P_0=\text{matrix}(1, 2, \text{lst}(u_0, v_0)), \]
\[ \text{Defines: } \text{parab_rot_sub}, \text{ used in chunk } 26d. \]
\[ \text{Uses Arg0 16c. } 25g \equiv \]
\[ \text{These two matrices define the Cayley transform and its inverse. } \]
\[ \text{For the subgroup } N \text{ we consider upper-triangular matrices, for } N'—\text{lower-triangular. } 25h \equiv} \]
\[ \text{if } (\text{is_subgroup } N) \]
\[ X=\text{matrix}(2, 2, \text{lst}(\text{one}, e_0 \times x, 0, \text{one})); \]
\[ \text{else } X=\text{matrix}(2, 2, \text{lst}(\text{one}, 0, e_0 \times x, \text{one})); \]
Common part of test routine.

\[ Δ \text{Test routine} \equiv Δ \text{Show expressions} \equiv Δ \text{Check identities} \]

\[
cout \ll (∉(latexout ? "\vspace{2mm}\hrule" : "----------------------------------------") \ll \text{endl;}
\]

B.5.4. Main procedure. It just calls the test routine, calculates the induced representation and draws a few pictures.

We output formulae in \LaTeX\ mode.

\[
\text{(Main procedure)} ≡ Δ \text{Induced representations} Δ \text{Drawing pictures}
\]

B.6. Drawing Orbits. To draw cycles we use \texttt{cycle} library \cite{12}. Elliptic orbits (circles).

\[
\text{(Drawing pictures)} ≡ Δ \text{Induced representations} Δ \text{Drawing pictures}
\]
Hyperbolic orbits.

\[\text{(Drawing pictures 26g)} \equiv \]
\[
\text{asymptote} \leftarrow \text{"path[] A="};
\]
\[
\text{for(int i=0; i<6; i++)}
\]
\[
\text{cycle2D(lst(0,0),e.subs(sign=1),-i*i+.04)}
\]
\[
.\text{asy.path(asymptote, -1.5, 1.5, -1.5, 2, 0, (i>0));}
\]
\[
\text{asymptote} \leftarrow \text{";"} \leftarrow \text{endl;}
\]

Parabolic orbits, subgroup \( N \).

\[\text{(Drawing pictures 26g)} \equiv \]
\[
\text{asymptote} \leftarrow \text{"path[] N="};
\]
\[
\text{for(int i=0; i<6; i++)}
\]
\[
\text{cycle2D(1, lst(0, numeric(1,2)), numeric(i,2)-1,e.subs(sign=0))}
\]
\[
.\text{asy.path(asymptote, -1.5, 1.5, -2, 2, 0, (i>0));}
\]
\[
\text{asymptote} \leftarrow \text{";"} \leftarrow \text{endl;}
\]
\[
\text{asymptote.close();}
\]

Parabolic orbits, subgroup \( N' \).

\[\text{(Drawing pictures 26g)} \equiv \]
\[
\text{asymptote} \leftarrow \text{"path[] N1="};
\]
\[
\text{for(int i=0; i<5; i++)}
\]
\[
\text{cycle2D(.5*i*i*i+1, lst(0, numeric(1,2)),-1,e.subs(sign=0))}
\]
\[
.\text{asy.path(asymptote, -1.5, 1.5, -1.5, 2, 0, (i>0));}
\]
\[
\text{asymptote} \leftarrow \text{";"} \leftarrow \text{endl;}
\]

B.6.1. Output routines. We use standardised routines to output results of calculations.

\[\text{(Output routines 27d)} \equiv \]
\[
\text{void formula_out(char* S, ex F) {}
\]
\[
\text{cout} \leftarrow S \leftarrow (latexout ? "\textbf{" : "}) \leftarrow F \leftarrow (latexout ? "\\\\\\\" : "\)
\]
\[
\leftarrow \text{endl;}
\}
\]

Defines:

- \text{formula\_out}, used in chunks 19, 20, 23, 24a, and 28a.

This routine is used to check identities.

\[\text{(Output routines 27d)} \equiv \]
\[
\text{void test\_out(char* S, ex T) {}
\]
\[
\text{cout} \leftarrow S \leftarrow (latexout ? "\textbf{" : "}) \leftarrow (is_a<\text{dual\_number}>(T) ? ex\_fo<\text{dual\_number}>(T).normal().is\_zero() : T.evalm().normal().is\_zero\_matrix()) \leftarrow (latexout ? "\\\\\\\" : "\)
\]
\[
\leftarrow \text{endl;}
\}
\]

Defines:

- \text{test\_out}, used in chunks 20–22, 24a, and 28a.

Uses \text{dual\_number} 29a 29a 29a 29a 29a 29a 29a 29a 29c 30d 31a 31b 31c 31c 32a 32a 32a 32a 32b 32b 32b.
Here is declarations of additional routines for the header file.

(Additional routines declarations 28a) \equiv 
bool latexout;
\textit{dual} \textit{number} \textit{dn} \textit{from} \textit{arg} \textit{mod}(\textit{const} \textit{ex} & \textit{a}, \textit{const} \textit{ex} & \textit{n});
ex \textit{v} \textit{from} \textit{norm}(\textit{const} \textit{ex} & \textit{u}, \textit{const} \textit{ex} & \textit{n});
\textit{void} test\_put(\textit{char} S, \textit{ex} T);
\textit{void} formula\_out(\textit{char} S, \textit{ex} F);

Uses \textit{dual} \textit{number} 29a 29a 29a 29a 29a 29a 29a 29a 29c 30d 31a 31b 31c 31c 32a 32a 32b 32b 32b,
\textit{formula}\_out 19e 20e 27d, and \textit{test}\_out 20f 20g 21b 27e.

B.7. Header and Implementation of the dual\_number Class.

B.7.1. \textit{Header File for dual\_number}. We use the standard GiNaC machinery do define \textit{dual\_numbers} as derived of the class \textit{basic}.

(Dual number class declaration 28b) \equiv 
\textit{class} dual\_number : \textit{public} basic
{} 
\textit{GINAC} \textit{DECLARE} \textit{REGISTERED} \textit{CLASS}(dual\_number, basic)
static const tinfo static \_ t return \_ type \_ tinfo static[256];

Uses dual\_number 29a 29a 29a 29a 29a 29a 29a 29a 29a 29c 30d 31a 31b 31c 31c 32a 32a 32b 32b 32b and
\textit{tinfo} static \_ t 29c.

Public methods (constructors, algebraic, etc.)

(Dual number class declaration 28b) \text-ex (Public methods 14) 
\text-ex (Technical methods 28f)

We redefine protected methods for printing only.

(Dual number class declaration 28b) \equiv 
\textit{protected}:
\textit{void} do\_print(\textit{const} \textit{print} context & \textit{c}, \textit{unsigned} \textit{level}) \textit{const};
\textit{void} do\_print\_latex(\textit{const} \textit{print} latex & \textit{c}, \textit{unsigned} \textit{level}) \textit{const};

Private members: two components of a dual\_number.

(Dual number class declaration 28b) \equiv 
\textit{protected}:
ex \textit{u}\_comp;
ex \textit{v}\_comp;
};

The following methods are needed for GiNaC to work properly.

(Technical methods 28f) \equiv 
\textit{dual} \textit{number} \textit{normal}(\textit{const}) \textit{const} \{ \textit{return} \textit{dual} \textit{number}(\textit{u}\_comp.\textit{normal}(), \textit{v}\_comp.\textit{normal}()); \}
dual\_number \textit{subs}(\textit{const} \textit{ex} & \textit{e}, \textit{unsigned} \textit{options} = 0) \textit{const};
bool is\_zero() \textit{const};
bool is\_equal(\textit{const} \textit{ex} & \textit{other}) \textit{const};
size\_t \textit{nops}() \textit{const} \{ \textit{return} 2; \}
ex \textit{op}(\textit{size\_t} \_ \textit{i}) \textit{const};
ex & \textit{let}(\textit{size\_t} \_ \textit{i});

Uses dual\_number 29a 29a 29a 29a 29a 29a 29a 29a 29a 29a 29a 29c 30d 31a 31b 31c 31c 32a 32a 32b 32b 32b.
We overload standard algebraic operations for `dual_number`.

```cpp
29a chapter
(Dual number class declaration 28b) +\equiv (24b) \oplus 29b chapter
const dual_number operator+(const dual_number & lh, const dual_number & rh);
const dual_number operator-(const dual_number & lh, const dual_number & rh);
const dual_number operator*(const dual_number & lh, const dual_number & rh);
const dual_number operator*(const dual_number & lh, const ex & rh);
const dual_number operator:(const ex & lh, const dual_number & rh);
const dual_number operator:(const ex & lh, const dual_number & rh);
const dual_number operator:(const ex & lh, const dual_number & rh);
const dual_number operator:(const ex & lh, const dual_number & rh);
const dual_number operator:(const dual_number & lh, const ex & rh);
const dual_number operator:(const dual_number & lh, const ex & rh);

Defines:
dual_number, used in chunks 14–20, 25d, 27–30, and 33.
```

```cpp
29b chapter
(Dual number class declaration 28b) +\equiv (24b) \oplus 29a chapter

dual_number dn_from_arg_mod(const ex & a, const ex & n);

/ / End of "header"
```

```cpp
29b chapter
Uses dual_number 29a 29a 29a 29a 29a 29a 29a 29a 29c 30d 31a 31b 31c 31c 32a 32a 32a 32b 32b 32b
```

**B.7.2. Standard Implementation Part.** The implementation uses standard GiNaC technique.

```cpp
29c chapter
(Dual number class implementation 29c) \equiv (24c) 29d chapter
GINAC\_IMPLEMENT\_REGISTERED\_CLASS\_OPT(dual_number, basic,
print\_func<print\_context>(\&dual_number::do\_print),
print\_func<print\_latex>(\&dual_number::do\_print\_latex))

const tinfo\_static\_t dual\_number::return\_type\_info\_static[256] = {{}};

DEFAULT\_ARCHIVING(dual\_number)
```

```
29c chapter
Defines:
dual_number, used in chunks 14–20, 25d, 27–30, and 33.
tinfo\_static\_t, used in chunk 28b.
```

**B.7.3. Implementation of Constructors.** Default constructor.

```cpp
29d chapter
(Dual number class implementation 29c) +\equiv (24c) \oplus 29e chapter
dual\_number::dual\_number() : inherited(\&dual\_number::tinfo\_static), u\_comp(0), v\_comp(0)
{
    setflag(status\_flags::not\_shareable);
}

Uses dual\_number 29a 29a 29a 29a 29a 29a 29a 29a 29c 30d 31a 31b 31c 31c 32a 32a 32a 32b 32b 32b
```

**Constrctor from two components.**

```cpp
29e chapter
(Dual number class implementation 29c) +\equiv (24c) \oplus 29d 30a chapter
dual\_number::dual\_number(const ex & a, const ex & b) : inherited(\&dual\_number::tinfo\_static),
    u\_comp(a), v\_comp(b)
{
}

Uses dual\_number 29a 29a 29a 29a 29a 29a 29a 29a 29c 30d 31a 31b 31c 31c 32a 32a 32a 32b 32b 32b
```
Constructor from a single expression. It may contain two components... 

```cpp
dual_number::dual_number(const ex & P) : inherited(&dual_number::tinfo_static)
{
    if (is_a<lst>(P) \text{ OR } is_a<matrix>(P) \text{ OR } is_a<dual_number>(P)) {
        u_comp = P.op(0);
        v_comp = P.op(1);
    }
}
```

Uses `dual_number`. ...if it is a real expression we embed it into `dual_number`.

```cpp
    } else if (P.imag_part().is_zero()) {
        u_comp = Arg0;
        v_comp = v_from_norm(Arg0, P);
    }
```

Uses `Arg0`. ...or if its a complex expression we decompose it into the real and imaginary parts.

```cpp
    } else {
        u_comp = P.real_part();
        v_comp = P.imag_part();
    }
```

Comparison routine.

```cpp
int dual_number::compare_same_type(const basic & other) const
{
    GINAC_ASSERT(is_a<dual_number>(other));
    const dual_number & o = static_cast<const dual_number &>(other);
    int cmpval = u_comp.compare(o.op(0));
    if (cmpval\neq 0) return cmpval;
    return v_comp.compare(o.op(1));
}
```

Defines:

- `dual_number`, used in chunks 14–20, 25d, 27–30, and 33.

Equality of two dual numbers.

```cpp
bool dual_number::is_equal(const ex & other) const
{
    GINAC_ASSERT(is_a<dual_number>(other));
    const dual_number & o = static_cast<const dual_number &>(other);
    return u_comp.is_equal(o.op(0)) \text{ AND } v_comp.is_equal(o.op(1));
}
```

Uses `dual_number`.
B.7.4. Printing. Default printing.

void dual_number::do_print(const print_context & c, unsigned level) const
{
    c.s << "[";
    u_comp.print(c);
    c.s << ",";
    v_comp.print(c);
    c.s << "]";
}

Defines:
    dual_number, used in chunks 14–20, 25d, 27–30, and 33.

\LaTeX\ printing.

void dual_number::do_print_latex(const print_latex & c, unsigned level) const
{
    c.s << \left(\begin{array}{cc}
    u_comp.print(c);
    c.s << \\
v_comp.print(c);
    c.s << \end{array}\right)";
}

Defines:
    dual_number, used in chunks 14–20, 25d, 27–30, and 33.

B.7.5. Overloading algebraic operations. Addition.

const dual_number operator+(const dual_number & lh, const dual_number & rh)
{
    return lh.add(rh);
}

const dual_number operator-(const dual_number & lh, const dual_number & rh)
{
    return lh.sub(rh);
}

Defines:
    dual_number, used in chunks 14–20, 25d, 27–30, and 33.
Multiplication.

\[\text{const dual\_number operator}\times\text{const\_dual\_number & lh, const\_ex & rh)}\]
\[
\text{\{ return lh.mul(rh); }\]
\[
\text{\}}
\]

\[\text{const dual\_number operator}\times\text{const\_ex & lh, const\_dual\_number & rh)}\]
\[
\text{\{ return rh.mul(lh); }\]
\[
\text{\}}
\]

\[\text{const dual\_number operator}\times\text{const\_dual\_number & lh, const\_dual\_number & rh)}\]
\[
\text{\{ return lh.mul(rh); }\]
\[
\text{\}}
\]

Defines:

dual\_number, used in chunks 14–20, 25d, 27–30, and 33.

Division.

\[\text{const dual\_number operator}\div\text{const\_dual\_number & lh, const\_dual\_number & rh)}\]
\[
\text{\{ return lh.mul(rh.power(-1)); }\]
\[
\text{\}}
\]

\[\text{const dual\_number operator}\div\text{const\_ex & lh, const\_dual\_number & rh)}\]
\[
\text{\{ return rh.power(-1)*lh; }\]
\[
\text{\}}
\]

\[\text{const dual\_number operator}\div\text{const\_dual\_number & lh, const\_ex & rh)}\]
\[
\text{\{ return lh.mul(pow(rh, -1)); }\]
\[
\text{\}}
\]

Defines:

dual\_number, used in chunks 14–20, 25d, 27–30, and 33.
B.7.6. Component-related functions.

ex dual_number::op(size_t i) const
{
    GINAC_ASSERT(i<nops());

    switch (i) {
    case 0:
        return u_comp;
    case 1:
        return v_comp;
    default:
        throw(std::invalid_argument("dual_number::op(): requested" 
                                    " operand out of the range (2)");
    }
}

Uses dual_number 29a 29a 29a 29a 29a 29a 29a 29a 29a 29a 29c 30d 31a 31b 31c 31c 32a 32a 32a 32b 32b 32b.

ex & dual_number::let_op(size_t i) const
{
    GINAC_ASSERT(i<nops());

    ensure_modifiable();

    switch (i) {
    case 0:
        return u_comp;
    case 1:
        return v_comp;
    default:
        throw(std::invalid_argument("dual_number::let_op(): requested operand" 
                                    " out of the range (2)");
    }
}

Uses dual_number 29a 29a 29a 29a 29a 29a 29a 29a 29a 29a 29c 30d 31a 31b 31c 31c 32a 32a 32a 32b 32b 32b.
This is the end of **dual_number** implementation.

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